GAPS BETWEEN ZEROS OF $\zeta(s)$ AND THE DISTRIBUTION OF ZEROS OF $\zeta'(s)$

MAKSYM RADZIWILL

Abstract. We settle a conjecture of Farmer and Ki in a stronger form. Roughly speaking we show that there is a positive proportion of small gaps between consecutive zeros of the zeta-function $\zeta(s)$ if and only if there is a positive proportion of zeros of $\zeta'(s)$ lying very closely to the half-line. Our work has applications to the Siegel zero problem. We provide a criterion for the non-existence of the Siegel zero, solely in terms of the distribution of the zeros of $\zeta'(s)$. Finally on the Riemann Hypothesis and the Pair Correlation Conjecture we obtain near optimal bounds for the number of zeros of $\zeta'(s)$ lying very closely to the half-line. Such bounds are relevant to a deeper understanding of Levinson’s method, allowing us to place one-third of the zeros of the Riemann zeta-function on the half-line.

1. Introduction.

The inter-relation between the horizontal distribution of zeros of $\zeta(s)$ (denoted $\rho = \beta + i\gamma$) and the horizontal distribution of the zeros of $\zeta'(s)$ (denoted $\rho' = \beta' + i\gamma'$) is the basis of Levinson’s method [12] allowing us to place one third of the zeros of $\zeta(s)$ on the critical line.

Recently it has been understood that the horizontal distribution of the zeros of $\zeta'(s)$ is also related to the vertical distribution of zeros of $\zeta(s)$. As an first attempt at capturing such a relationship we have the following conjecture of Soundararajan [16].

Note: Throughout we assume the Riemann Hypothesis. We recall that $\beta' \geq \frac{1}{2}$ for all non-trivial zeros of $\zeta'(s)$ (see [18]) and that this is equivalent to the Riemann Hypothesis.

Conjecture 1 (Soundararajan [16]). We have

(A) $\liminf_{\gamma \to \infty} (\gamma^+ - \gamma) \log \gamma = 0$

with $\gamma^+$ the ordinate succeeding $\gamma$, if and only if

(B) $\liminf_{\gamma' \to \infty} (\beta' - \frac{1}{2}) \log \gamma' = 0$

Zhang [19] shows that $A \implies B$ (see also [8] for a partial converse). Ki [11] obtained a necessary and sufficient condition for the negation of $B$. Ki’s result shows that zeros $\rho'$ with $(\beta' - \frac{1}{2}) \log \gamma' = o(1)$ arise not only from small gaps between zeros of $\zeta(s)$ but also, for example, from clusters of regularly spaced zeros of $\zeta(s)$. Therefore given our current knowledge about the zeros of $\zeta(s)$ it is possible for $B$ and the negation of $A$ to co-exist. The assertion $A$ is arithmetically very interesting, since, following an idea of Montgomery (made explicit by Conrey and Iwaniec in [2]) if there are many small gaps between consecutive zeros of $\zeta(s)$ then the class number of $\mathbb{Q}(\sqrt{-d})$ is large and there are no Siegel zeros.

2010 Mathematics Subject Classification. Primary: 11M06, Secondary: 11M26.

The author is partially supported by a NSERC PGS-D award.
A more recent attempt at capturing the relation between the distribution of zeros of $\zeta(s)$ and $\zeta'(s)$ is due to Farmer and Ki [4]. Let $w(x)$ be the indicator function of the unit interval. Following Farmer and Ki we introduce two distribution functions,

$$m'(\varepsilon) := \liminf_{T \to \infty} \frac{2\pi}{T \log T} \sum_{T \leq \gamma' \leq 2T} w\left(\frac{(\beta' - \frac{1}{2}) \log T}{\varepsilon}\right)$$

$$m(\varepsilon) := \liminf_{T \to \infty} \frac{2\pi}{T \log T} \sum_{T \leq \gamma \leq 2T} w\left(\frac{(\gamma' - \gamma) \log T}{\varepsilon}\right).$$

These are indeed distribution functions, since in a rectangle of length $T$, both $\zeta(s)$ and $\zeta'(s)$ have asymptotically $N(T) \sim (T/2\pi) \log T$ zeros (see [1]), and it is conjectured that $m'(v) \to 1$ as $v \to \infty$, whereas it is known that $m(v) \to 1$ as $v \to \infty$ (see [16], [7]).

Zhang shows in [19] that if $m(\varepsilon) > 0$ for all $\varepsilon > 0$, then $m'(\varepsilon) > 0$. An analogue of Soundararajan’s conjecture would assert that $m(\varepsilon) > 0$ for all $\varepsilon > 0$ if and only if $m'(\varepsilon) > 0$ for all $\varepsilon > 0$. As explained by Farmer and Ki in [4] if for example the zeros are well-spaced with sporadic large gaps, something we cannot rule out at present, then in principle $m'(\varepsilon) > 0$ is not enough to imply $m(\varepsilon) > 0$. Farmer and Ki propose the following alternative conjecture.

**Conjecture 2** (Farmer and Ki [4]). If $m'(\varepsilon) \gg \varepsilon^v$ with a $v < 2$ as $\varepsilon \to 0$ then $m(\varepsilon) > 0$ for all $\varepsilon > 0$.

This is a realistic conjecture since we expect that $m'(\varepsilon) \sim (8/9\pi)\varepsilon^{3/2}$ as $\varepsilon \to 0$ (see [3]). Farmer and Ki comment “we intend this as a general conjecture, applying to the Riemann zeta-function but also to other cases such as a sequence of polynomials with all zeros on the unit circle” and that “stronger statements should be true for the zeta function”. Our main result is a proof of Conjecture 2 in a stronger and quantitative form for the Riemann zeta-function.

**Main Theorem.** Let $A, \delta > 0$ be given.

- If $m'(\varepsilon) \gg \varepsilon^A$ as $\varepsilon \to 0$ then $m(\varepsilon^{1/2}) \gg \varepsilon^{A+\delta}$ for all $\varepsilon \leq 1$.
- If $m(\varepsilon^{1/2}) \gg \varepsilon^A$ as $\varepsilon \to 0$ then $m'(\varepsilon) \gg \varepsilon^{A+\delta}$ for all $\varepsilon \leq 1$.

We conjecture that $m'(\varepsilon) \asymp m(\varepsilon^{1/2})$ provided that one of $m(\varepsilon)$ or $m'(\varepsilon)$ is $\gg \varepsilon^A$ for some $A > 0$. This is consistent with the expectation that $m(\varepsilon) \sim (\pi/6)\varepsilon^3$ and $m'(\varepsilon) \sim (8/9\pi)\varepsilon^{3/2}$ as $\varepsilon \to 0$ (see [3]). Our Main Theorem could be restated as saying that

$$\log m(\varepsilon) \sim \log m'(\varepsilon^{1/2})$$

as $\varepsilon \to 0$ provided that one of $m(\varepsilon)$ or $m'(\varepsilon)$ is greater than $\varepsilon^A$. As a consequence of the Main Theorem we obtain estimates for $m'(\varepsilon)$ assuming the Pair Correlation Conjecture.

**Corollary 1.** Assume the Pair Correlation Conjecture. Let $\delta > 0$. Then

$$\varepsilon^{3/2+\delta} \ll m'(\varepsilon) \ll \varepsilon^{3/2-\delta}$$

as $\varepsilon \to 0$.

An assumption on the zero distribution in Corollary 1 is inevitable, since $m'(\varepsilon) \to 0$ implies that almost all the zeros of $\zeta(s)$ are simple. Corollary 1 allows one to quantify the loss in Levinson’s method coming from the zeros of $\zeta'(s)$ lying closely to the half-line. Unfortunately
Corollary 1 is a conditional result, and as such it cannot be used to put a greater proportion of the zeros of $\zeta(s)$ on the half-line (see [5] for related work).

A final consequence of our work is a criterion for the non-existence of the Siegel zero in terms of the zeros of $\zeta'(s)$. We state it only for completeness since a stronger result has been obtained by Farmer and Ki [4].

**Corollary 2.** Let $A > 0$. If $m'(\varepsilon) \gg \varepsilon^A$, for all $\varepsilon > 0$, then for primitive characters $\chi$ modulo $q$,

$$L(1; \chi) > (\log q)^{-18},$$

for all $q$ sufficiently large.

**Proof.** If $m'(\varepsilon) \gg \varepsilon^A$ for every $\varepsilon > 0$ then $m(1/4) > 0$ by our Main Theorem, hence $L(1; \chi) > (\log q)^{-18}$ for all $q$ sufficiently large by Theorem 1.1 of Conrey-Iwaniec, [2].

With some care it is possible to turn the above Corollary into an effective result. By Dirichlet’s formula Corollary 2 also implies that the class number of $\mathbb{Q}(\sqrt{-d})$ is at least as large as $c \sqrt{d} (\log d)^{-18}$ with $c$ constant.

Farmer and Ki show that if $m'(\varepsilon) \gg \exp(-\varepsilon^{-1/2+\delta})$ as $\varepsilon \to 0$, for some $\delta > 0$, then there are $N(T) / \log \log T$ ordinates of zeros of $\zeta(s)$ lying in $[T; 2T]$ and such that $(\gamma^+ - \gamma) \log \gamma = \sigma(1)$ as $T \to \infty$. Using the result of Conrey and Iwaniec [2] this is enough to rule out the existence of Siegel zeros. It is an interesting question to determine whether, given the current technology, one can increase the exponent $\frac{1}{2}$ in Farmer and Ki’s assumption $m'(\varepsilon) \gg \exp(-\varepsilon^{-1/2+\delta})$ and still guarantee the non-existence of Siegel zeros.

## 2. Main Ideas

The first part of our Main Theorem follows from the stronger Theorem 1 below.

**Theorem 1.** Let $A, \delta > 0$. There is a constant $C = C(\delta, A)$ such that if $0 < \varepsilon < C$ and $m'(\varepsilon) \geq c \varepsilon^A$ then $m(\varepsilon^{1/2-\delta}) \geq (c/8) \varepsilon^A$.

The approximate value of $C(\delta, A)$ is $(B \delta / A)^{32A/\delta}$ with $B$ an absolute constant. Theorem 1 follows from two technical Propositions which we now describe. Given a zero $\rho' = \beta' + i \gamma'$ of $\zeta'(s)$ we denote by $\rho_c = \frac{1}{2} + i \gamma_c$ the zero of $\zeta(s)$ lying closest to $\rho'$. If there are two choices of $\rho_c$ then we pick the one lying closer to the origin. For any ordinate $\gamma$ of a zero of $\zeta(s)$ we denote by $\gamma^+$ the ordinate succeeding $\gamma$ and by $\gamma^-$ the ordinate preceding $\gamma$. We denote by $\gamma^\pm$ the ordinate closest to $\gamma$. Theorem 1 follows quickly from the following Proposition.

**Proposition 1.** Let $0 < \delta, \varepsilon < 1$. Let $S_{\varepsilon, \delta}(T)$ be a set of zeros $\rho' = \beta' + i \gamma'$ of $\zeta'(s)$ such that $T \leq \gamma' \leq 2T$, $\beta' - \frac{1}{2} \leq \varepsilon / \log T$ and

$$|\gamma_c - \gamma_c^\pm| > \varepsilon^{1/2-\delta} / \log T.$$  

There is a $C = C(\delta, A)$ such that if $0 < \varepsilon < C$ then $|S_{\varepsilon, \delta}(T)| \leq \varepsilon^A \cdot T \log T$.

The proof of Proposition 1 rests on a Proposition describing the structure the roots of $\zeta'(s)$ lying close to the half-line. The Proposition which we are about to state complements with a corresponding upper bound the classical lower bound,
valid for all $\rho' = \beta' + i\gamma'$ (see [16]). It might be of independent interest.

**Proposition 2.** Let $0 < \delta < 1$, $0 < \varepsilon < c$ with $c > 0$ an absolute constant. Let $T$ be large and $\mathcal{Z} := \mathcal{Z}_{\varepsilon, \delta}(T)$ be a set of $\delta / \log T$ well-spaced ordinates of zeros $\rho'$ such that $\rho' \neq \rho$, $\beta' - \frac{1}{2} \leq \varepsilon / \log T$ and $T \leq \gamma' \leq 2T$. If $|\mathcal{Z}_{\varepsilon, \delta}(T)| \gg \varepsilon^A \cdot T \log T$ then, for any given $\kappa > 0$, all but $\kappa |\mathcal{Z}|$ elements $\rho' \in \mathcal{Z}$ satisfy the inequality,

$$\sqrt{\frac{\beta' - \frac{1}{2}}{\log T}} \ll |\rho' - \rho_c| \ll \sqrt{A \log(\kappa \delta)^{-1}} \cdot \sqrt{\frac{\beta' - \frac{1}{2}}{\log T}}.$$

The proof of the converse part of our Main Theorem builds on ideas of Zhang, and follows from the following more precise statement valid for any fixed $\varepsilon > 0$.

**Theorem 2.** Let $A, \delta > 0$. There is a $C = C(\delta, A)$ such that if $0 < \varepsilon < C$ and $m(\varepsilon^{1/2}) \geq c \varepsilon^A$ then $m'(\varepsilon) \geq (c/4) \varepsilon^A + \delta$.

The paper is organized as follows. Most of the paper, all the way until section 7, is devoted to the proof of the propositions above and the deduction of Theorem 1 from them. Following section 7 we prove Theorem 2 and Corollary 1.

### 3. Lemma on Dirichlet polynomials

Define,

$$A_N(s) := \sum_{n \leq N} \frac{\Lambda(n) W_N(n)}{n^s}$$

with

$$W_N(n) = \begin{cases} 1 & \text{for } 1 \leq n \leq N^{1/2} \\ \log(N/n) / \log N & \text{for } N^{1/2} < n \leq N \end{cases}$$

The lemma below is due to Selberg.

**Lemma 1.** Let $\sigma = \frac{1}{2} + 2 / \log N$, with $N \leq T$. Then for $T \leq t \leq 2T$,

$$\sum_{\rho} \frac{\sigma - \frac{1}{2}}{(\sigma - \frac{1}{2})^2 + (t - \gamma)^2} \ll |A_N(s)| + \log T$$

**Proof.** This is equation (2.2) in [15].

Using the explicit formula we obtain an upper bound for the number of zeros in a small window $[t - 2\pi K / \log t; t + 2\pi K / \log t]$, in terms of the Dirichlet polynomial

$$B_N(s) := \sum_{n \leq N} \frac{\Lambda(n)}{n^s} \cdot \left(1 - \frac{\log n}{\log N}\right).$$

We have the following lemma.

**Lemma 2.** For $T \leq t \leq 2T$ and $N \leq T$,

$$N(t + \frac{\pi}{\log N}) - N(t - \frac{\pi}{\log N}) \ll \frac{\log T}{\log N} + \frac{|B_N(\frac{1}{2} + it)|}{\log N}$$
Proof. Let
\[ F_\Delta(v) = \left( \frac{\sin \pi \Delta v}{\pi \Delta v} \right)^2 \]
be the Fejer kernel. The Fourier transform of \( F_\Delta(v) \) is for \(|x| < \Delta\)
\[ \hat{F}_\Delta(x) := \int_{-\infty}^{\infty} F_\Delta(t) e^{-2\pi i tx} dt = \frac{1}{\Delta} \left( 1 - \frac{x}{\Delta} \right) \]
and \( \hat{F}_\Delta(x) = 0 \) for \(|x| > \Delta\). By the explicit formula (see Lemma 1 in [9]),
\begin{align*}
\sum_\gamma F_\Delta(\gamma - t) &= O(e^{\pi \Delta/2} \cdot t^{-2} + 1/\Delta) + \frac{1}{2\pi} \int_{-\infty}^{\infty} F_\Delta(u - t) \cdot \Re \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} + \frac{iu}{2} \right) du \\
&- \frac{1}{2\pi} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \left( \hat{F}_\Delta \left( \frac{\log n}{2\pi} \right) + \hat{F}_\Delta \left( -\frac{\log n}{2\pi} \right) \right)
\end{align*}
The integral over \( u \) is bounded by \( \ll (\log t)/\Delta \). On the other hand the prime sum is bounded by,
\[ \left| \frac{1}{2\pi \Delta} \sum_{n \leq e^{2\pi \Delta}} \frac{\Lambda(n)}{n^{1/2+it}} \cdot \left( 1 - \left| \frac{\log n}{2\pi \Delta} \right| \right) \]
Finally \( (\pi/2)^2 \sum_\gamma F_\Delta(\gamma - t) \) is an upper bound for the number of zeros in the interval going from \( t - 1/(2\Delta) \) to \( t + 1/(2\Delta) \). If \( T \leq t \leq 2T \) we choose \( 2\pi \Delta = \log N \) and we are done. \( \square \)

In order to understand the average behavior of the Dirichlet polynomials \( A_N(s) \) and \( B_N(s) \) we use a version of the large sieve.

Lemma 3. Let \( A(s) \) be a Dirichlet polynomial with positive coefficients and of length \( x \). Let \( s_r = \sigma_r + it_r \) be points with \( T \leq t_r \leq 2T \) and \( 0 \leq \sigma_r - \frac{1}{2} \leq \varepsilon/\log T \) for some small \( \varepsilon > 0 \). Suppose that \( |t_i - t_j| \geq \delta/\log T \) for \( i \neq j \), with \( 100\varepsilon < \delta < 1 \). Then, for \( x^k \leq T \),
\[ \sum_{s_r} |A(s_r)|^{2k} \leq \frac{20 \log T}{\delta} \int_{-2T}^{2T} |A(\frac{1}{2} + it)|^{2k} dt \]
Proof. Let \( D(s) = A(s)^k \). For any \( s \) we have, with \( C \) a circle of radius \( \delta/(2\log T) \) around \( s \),
\[ |D(s)|^2 \leq \frac{4(\log T)^2}{\pi \delta^2} \int_C |D(x + iy)|^2 dxdy \]
Summing over all \( s = s_r \), since the circles are disjoint we obtain,
\[ \sum_{s_r} |D(s_r)|^2 \leq \frac{4(\log T)^2}{\pi \delta^2} \int_{\frac{1}{2} + \frac{\delta/\log T}{2}}^{\frac{1}{2} + \frac{\delta/\log T}{2} + 1} \int_{T-1}^{T+1} |D(\sigma + it)|^2 dtd\sigma \]
Since the coefficients of \( D(s) \) are positive, and \( D \) is of length at most \( T \), by a majorant principle (see Chapter 3, Theorem 3 in [13]), the inner integral is bounded by
\[ \leq 3e^{2\delta} \int_{-2T}^{2T} |D(\frac{1}{2} + it)|^2 dt \]
Since in addition \( \delta < 1 \), the claim follows (we obtain a constant of \( 8e^2/\pi < 20 \)). \( \square \)
Combining the above lemma with Chebyshev’s inequality allows us to understand the average size of the Dirichlet polynomials $A_N(s)$ and $B_N(s)$.

**Lemma 4.** Let $s_r = \sigma_r + it_r$ be a set of well-spaced points as appearing in the statement of Lemma 3. Suppose that $N^k \leq T/\log T$. The number of points $s_r$ for which we have

$$|A_N(s_r)| > (k/e)\log N \quad \text{or} \quad |B_N(s_r)| > (k/e)\log N$$

is bounded above by $\ll (e^{-k}/\delta)T\log T$.

**Proof.** Let $L_N(s)$ be either $A_N(s)$ or $B_N(s)$. Let

$$D_N(s) = \sum_{n \leq N} \frac{\Lambda(n)}{n^s}.$$ 

By a majorant principle (see Chapter 3, Theorem 3 in [13]) we have,

$$\int_{-2T}^{2T} |L_N(\frac{1}{2} + it)|^{2k}dt \leq 3 \int_{-2T}^{2T} |D_N(\frac{1}{2} + it)|^{2k}dt$$

By Soundararajan’s lemma 3 in [17], for $N^k \leq T/\log T$ we have,

$$\int_{-2T}^{2T} |D_N(\frac{1}{2} + it)|^{2k}dt \ll k!T(\log N)^{2k}$$

Therefore, for $N^k \leq T/\log T$, by the previous lemma,

$$\sum |L_N(s_r)|^{2k} \ll \frac{k!}{\delta} \cdot T \log T(\log N)^{2k}$$

It follows that for $N^k \leq T/\log T$, the number of points $s_r$ for which $|L_N(s_r)| > B\log N$ is less than

$$\ll \left( \frac{k}{B} \right)^k \cdot (T/\delta) \log T$$

Choosing $B = k/e$ we conclude that the number of points for which $|L_N(s_r)| > k/e$ is bounded by $(e^{-k}/\delta)T\log T$ as desired. \hfill \Box

**Lemma 5.** Let $0 < c < 1$. Uniformly in $T \leq t \leq 2T$ and $N \leq T$,

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{|s-\rho|<c/\log T} \frac{1}{s-\rho} + O\left( \frac{\log T}{e} \cdot \mathcal{E}_{T,N}(s) \right)$$

where

$$\mathcal{E}_{T,N}(s) := \frac{1}{\log N} \cdot (|A_N(s)| + |B_N(\frac{1}{2} + it)|) + \frac{\log T}{\log N}.$$ 

Furthermore, if $s_r$ is a set of well-spaced points as in Lemma 3, and $N^k \leq T/\log T$, then

$$\sum |\mathcal{E}_{T,N}(s_r)|^{2k} \ll (k^{2k}/\delta)T\log T.$$
Proof. Selberg shows in [14] (see equation (14) on page 4) that
\[ \frac{\zeta'(s)}{\zeta(s)} = \sum_{|s-\rho|<(\log T)^{-1}} \frac{1}{s-\rho} + O\left( \frac{\log T}{\log N} \cdot |A_N(s)| + \frac{\log^2 T}{\log N} \right) \]

It suffices to notice that the contribution of the zeros \( \rho \) with \( c(\log T)^{-1} < |s-\rho| < (\log T)^{-1} \) is bounded above by
\[ \ll \frac{\log T}{c} \cdot \left( N\left(t + \frac{\pi}{\log N}\right) - N\left(t - \frac{\pi}{\log N}\right) \right) \ll \frac{\log T}{c} \cdot \left( \frac{\log T}{\log N} + \frac{1}{\log N} \cdot |B_N(\frac{1}{2} + it)| \right) \]

Combining the above two equations we obtain the first part of the lemma. Now it remains to estimate the moments of \( E_{T,N} \). We have,
\[ \sum_{s_r} |E_{T,N}(s_r)|^{2k} \ll \left( \frac{C}{\log N} \right)^{2k} \cdot \left( \sum_{s_r} |A_N(s_r)|^{2k} + \sum_{s_r} |B_N(s_r)|^{2k} \right) + \left( \frac{(Ck)^{2k}}{\delta} \right) T \log T \]
with \( C > 0 \) an absolute constant. Using Lemma 3 and proceeding as in Lemma 4 we find that the \( 2k \)-th moments of the Dirichlet polynomials \( A_N \) and \( B_N \) is bounded above by \( (k!/\delta) T \log T (\log N)^{2k} \). Hence we conclude that the \( 2k \)-th moment of \( E_{N,T} \) is bounded above by \( ((Ck)^{2k}/\delta) T \log T \).

\[ \square \]

4. PROOF OF PROPOSITION 2

The proof of Proposition 2 rests on the following classical lemma.

**Lemma 6.** If \( \rho' \neq \rho \) then,
\[ \frac{1}{2} \cdot \log \gamma' = \sum_{\rho} \frac{\beta' - \frac{1}{2}}{\left( \beta' - \frac{1}{2} \right)^2 + (\gamma' - \gamma)^2} + O(1). \]

**Proof.** See Zhang [19], Lemma 3. \( \square \)

We will show that on average the zero \( \rho = \rho_c \) dominates, the claim then follows shortly. In order to simplify the notation we define, as in the previous section,
\[ A_N(s) := \sum_{n \leq N} \frac{\Lambda(n)W_N(n)}{n^s} \]
with \( W_N(n) \) the same smoothing as defined in the previous section. We also define
\[ B_N(s) := \sum_{n \leq N} \frac{\Lambda(n)}{n^s} \cdot \left( 1 - \frac{\log n}{\log N} \right). \]

On average both Dirichlet polynomials are of size \( \log N \).

**Proof of Proposition 2.** Let \( N \leq T \) to be fixed later. In the formula
\[ \frac{1}{2} \cdot \log \gamma' = \sum_{\rho} \frac{\beta' - \frac{1}{2}}{\left( \beta' - \frac{1}{2} \right)^2 + (\gamma' - \gamma)^2} + O(1) \]
The contribution of the $\rho$’s for which $|\gamma - \gamma'| < \pi (\log N)^{-1}$ is bounded above by

\[
(5) \quad \ll \left( N\left(\gamma' + \frac{\pi}{\log N}\right) - N\left(\gamma' - \frac{\pi}{\log N}\right) \right) \cdot \frac{\beta' - \frac{1}{2}}{|\rho_c - \rho'|^2}
\]

\[
\ll \left( \frac{\log T}{\log N} + \frac{1}{\log N} \cdot |B_N(\frac{1}{2} + i\gamma')| \right) \cdot \frac{\beta' - \frac{1}{2}}{|\rho_c - \rho'|^2}.
\]

by Lemma 2. On the other hand, to bound the contribution of the $\rho$’s for which $|\gamma - \gamma'| > \pi (\log N)^{-1}$ we notice that if $|\gamma' - \gamma| > \pi (\log N)^{-1}$ then

\[
(\beta' - \frac{1}{2})^2 + (\gamma' - \gamma)^2 \gg (2/ \log N)^2 + (\gamma' - \gamma)^2.
\]

Therefore the contribution of the $\rho$’s with $|\gamma - \gamma'| > \pi (\log N)^{-1}$ to (4) is bounded above by

\[
(6) \quad \ll (\beta' - \frac{1}{2}) \log N \cdot \left( \sum_{\rho} \frac{2/ \log N}{(2/ \log N)^2 + (\gamma' - \gamma)^2} \right)
\]

\[
\ll (\beta' - \frac{1}{2}) \log N \cdot \left( \log T + \left| A_N \left( \frac{1}{2} + \frac{1}{\log N} + i\gamma' \right) \right| \right)
\]

by Lemma 1. Combining (4), (5) and (6) we conclude that

\[
(7) \quad \log T \ll \left( \frac{\log T}{\log N} + \frac{1}{\log N} \cdot |B_N(\frac{1}{2} + i\gamma')| \right) \cdot \frac{\beta' - \frac{1}{2}}{|\rho_c - \rho'|^2} +
\]

\[
+ (\beta' - \frac{1}{2}) \log N \cdot \left( \log T + \left| A_N \left( \frac{1}{2} + \frac{1}{\log N} + i\gamma' \right) \right| \right)
\]

Suppose that $N^k \leq T/ \log T$ with a $k$ to be fixed later and $N$ the largest integer such that $N^k < T/ \log T$. By Lemma 4 the number of $\rho' \in \mathbb{Z}_{\varepsilon, \delta}$ for which $|B_N(\frac{1}{2} + i\gamma')| > (k/ e) \log N$ is bounded above by $c(e^{-k}/ \delta)T \log T$ with $c$ a constant. Similarly the number of $\rho' \in \mathbb{Z}_{\varepsilon, \delta}$ for which $|A_N(\frac{1}{2} + 1/ \log N + i\gamma')| > (k/ e) \log N$ is also bounded by above by $c(e^{-k}/ \delta)T \log T$. Choose $k$ so that $ce^{-k}/ \delta T \log T \leq (k/2)|Z_{\varepsilon, \delta}|$. Since $|Z_{\varepsilon, \delta}| \geq c_1 \varepsilon^A T \log T$ we can take $k$ to be the closest integer to $c_2 A \log(\varepsilon \delta)^{-1}$ with $c_2$ an absolute constant. Choose $N$ to be the largest integer such that $N^k \leq T/ \log T$. With this choice of $k$ and $N$ it follows that for at most $\kappa|Z_{\varepsilon, \delta}|$ elements $\rho' \in \mathbb{Z}_{\varepsilon, \delta}$ we have $|B_N(\frac{1}{2} + i\gamma')| \geq (k/ e) \log N$ or $|A_N(\frac{1}{2} + i\gamma')| \geq (k/ e) \log N$. It follows that for all but at most $\kappa|Z_{\varepsilon, \delta}|$ of the $\rho' \in \mathbb{Z}_{\varepsilon, \delta}$ we have,

\[
\frac{c \log T}{k} \leq \frac{\beta' - \frac{1}{2}}{|\rho_c - \rho'|^2} + \frac{1}{k} \cdot (\beta' - \frac{1}{2}) \cdot (\log T)^2
\]

with $c > 0$ an absolute constant. If $\varepsilon$ is choosen so that $\varepsilon < (c/2)$ then (since $\beta' - \frac{1}{2} < \varepsilon/ \log T$) we obtain

\[
(c/2) \log T \leq k \cdot \frac{\beta' - \frac{1}{2}}{|\rho_c - \rho'|^2}
\]

hence $|\rho_c - \rho'|^2 \leq (k(\beta' - \frac{1}{2})/ \log T)^{1/2}$ which gives the desired bound for all but at most $\kappa|Z_{\varepsilon, \delta}|$ elements $\rho' \in \mathbb{Z}_{\varepsilon, \delta}$. (Recall that $k \ll A \log(\varepsilon \delta \kappa)^{-1}$).
5. Proof of Proposition 1

The lemma below is critical, in that it allows us to produce a sufficiently dense well-spaced sequence of zeros of $\zeta'(s)$.

**Lemma 7** (Soundararajan [16]). Suppose that $\rho_1 = \frac{1}{2} + i\gamma_1$ and $\rho_2 = \frac{1}{2} + i\gamma_2$ are two consecutive zeros of $\zeta(s)$ with $T \leq \gamma_1 < \gamma_2 \leq 2T$ for large $T$. Then the box,

$$\{s = \sigma + it : \frac{1}{2} \leq \sigma < \frac{1}{2} + 1/\log T, \gamma_1 < t < \gamma_2\}$$

contains at most one zero (counted with multiplicity) of $\zeta'(s)$.

**Proof.** The only way that $\rho'$ can lie on the critical line is if $\rho' = \rho$. Since $\gamma_1 < t < \gamma_2$ this possibility is excluded. As for the box $\frac{1}{2} < \sigma < \frac{1}{2} + 1/\log T$ we know by Soundararajan’s work [16] (see Proposition 6) that the box $\frac{1}{2} < \sigma < \frac{1}{2} + 1/\log T$ with $t$ in $[\gamma_1, \gamma_2]$ can contain at most one zero of $\zeta'(s)$, counted with multiplicity. □

We are now ready to prove Proposition 1.

**Proof of Proposition 1.** Suppose that $S = S_{\epsilon, \delta}(T) > \varepsilon^A \cdot T \log T$. We will show that this leads to a contradiction when $0 < \varepsilon < C(\delta, A)$ with $C(\delta, A)$ some explicit constant depending only on $\delta$ and $A$ (for example we could take $C(\delta, A) = (C(\delta, A)^{2A}/\delta$ with $c > 0$ an absolute constant). Since each $\rho' \in S$ satisfies $\gamma^c_\epsilon \leq \gamma' \leq \gamma^+_c$ and $|\gamma_c - \gamma^c_\epsilon| > \varepsilon^{1/2-\delta}/\log T$ by the above lemma for each $\rho' \in S$ there is at most one zero of $\zeta'(s)$ in $[\gamma^c_\epsilon, \gamma_c]$ and at most one zero of $\zeta'(s)$ in $[\gamma^c_\epsilon, \gamma^c_\epsilon]$.

We construct a subset $S'$ of $S$ by skipping every second element in $S$. This produces a subset of at least $(1/2)|S|$ elements, with the property that the ordinates of elements of $S'$ are $\varepsilon^{1/2-\delta}/\log T$ well-spaced, because $|\gamma_c - \gamma^c_\epsilon| \geq \varepsilon^{1/2-\delta}/\log T$ for each $\rho' \in S$.

By Proposition 2, we have for at least half of the $\rho' \in S'$,

$$|\gamma' - \gamma_c| \leq |\rho' - \rho_c| \leq \frac{C\sqrt{A\varepsilon \log(\varepsilon)^{-1}}}{\log T},$$

with $C > 0$ an absolute constant. We call $S''$ the subset of $S'$ satisfying the above inequality. Since $|\gamma^c_\epsilon - \gamma_c| > \varepsilon^{1/2-\delta}/\log T$ for each $\rho' \in S''$ the interval $|\gamma' - t| \leq \varepsilon^{1/2-\delta}/\log T$ contains exactly one zero of a order of a zero of $\zeta(s)$ (namely $\gamma_c$) once $\varepsilon$ is choosen so small so as to make the right-hand side of (3) less than $\varepsilon^{1/2-\delta}/\log T$ (for example $\varepsilon < (\delta/CA)^{2/\delta}$ would suffice).

Using Lemma 5, we have at $s = \rho' \in S$,

$$\sum_{|s-\rho| < \varepsilon/\log T} \frac{1}{s-\rho} \ll \frac{\log T}{c} \cdot |E_{T,N}(s)|$$

Choose $s = \rho' \in S''$, $c = \varepsilon^{1/2-\delta}$ and $N$ the largest integer such that $N^k \leq T/\log T$ with a $k$ to be fixed later (ultimately $k = \lfloor (A+1)/\delta \rfloor$). By our previous remark the left-hand side of the above expression consists of only one term $(\rho' - \rho_c)^{-1}$. Raising the above expression to the $2k$-th power and then summing over all $\rho' \in S''$ we obtain

$$\sum_{\rho' \in S''} \frac{1}{|\rho' - \rho_c|^{2k}} \ll \varepsilon^{-k+2k\delta} \cdot (C \log T)^{2k} \sum_{\rho' \in S''} |E_{T,N}(\rho')|^{2k}$$

$$\ll \varepsilon^{-k+2k\delta} \cdot ((Ck)^{2k}/\varepsilon^{1/2-\delta}) \cdot (\log T)^{2k+1}$$
by Lemma 5, with $C > 0$ an absolute constant (not necessarily the same in each occurrence). Since for each $\rho' \in S''$ we have,
\[ |\rho' - \rho_c| \ll \frac{\sqrt{A\varepsilon \log(\varepsilon)^{-1}}}{\log T} \]
the left-hand side of (9) is at least
\[ \sum_{\rho' \in S'} \frac{1}{|\rho' - \rho_c|^{2k}} \gg |S''| \cdot (C/A)^k \cdot \varepsilon^{-k}(\log(\varepsilon)^{-1})^{-k}(\log T)^{2k} \]
(11)
\[ \gg (C/A)^k \cdot \varepsilon^{A-k} \cdot (\log(\varepsilon)^{-1})^{-k} \cdot T(\log T)^{2k+1} \]
(12)
since $|S''| \gg \varepsilon T \log T$. Combining the upper bound (9) and the lower bound (11) we get
\[ \varepsilon^{A-k}(\log(\varepsilon)^{-1})^{-k} \leq \varepsilon^{-k-1/2+(2k+1)\delta} \cdot (CAk)^{2k} \]
with $C > 0$ an absolute constant. The above inequality simplifies to
\[ \varepsilon^{A+1/2} \leq (CAk)^{2k} \cdot \varepsilon^{2k+1\delta} \cdot (\log(\varepsilon)^{-1})^{k}. \]
Using the inequality $(\log x) \leq x^\delta / \delta$ we obtain
\[ \varepsilon^{A+1/2} \leq (CAk/\delta)^{2k} \cdot \varepsilon^{k\delta} \]
Choosing $k$ to be the smallest integers with $k\delta > A + 1$ we obtain a contradiction once $\varepsilon < (2C\delta^2/A^2)^{10A/\delta}$ with $C$ an absolute constant. (Note: We have certainly not tried to optimize the constant $C(\delta, A)$). \hfill \square

6. PROOF OF THEOREM [1]

Let $T$ be large. By assumption each interval $[T; 2T]$ contains at least $c\varepsilon^A N(T)$ ordinates $T \leq \gamma' \leq 2T$ with $\beta' - 1/2 < \varepsilon / \log T$. If $\rho' = \rho$ for more than half of these zeros of $\zeta'(s)$, then we have $\geq (c/2)\varepsilon^A N(T)$ zeros $\rho$ with $\gamma^+ = \gamma$ and so we are done.

Thus we can assume that there are $\geq (c/2)\varepsilon^A N(T)$ zeros $\rho'$ with $T \leq \gamma' \leq 2T$, $\rho' \neq \rho$ and $\beta' - 1/2 < \varepsilon / \log T$. We call the set of such $\rho'$ by $S$. By Lemma 7 between any two consecutive zeros of $\zeta(s)$ there is at most one $\rho' \in S$. For each $\rho' \in S$ consider two possibilities

1. $|\gamma_c^+ - \gamma_c^-| \leq \varepsilon^{1/2-\delta} / \log T$
2. $|\gamma_c^+ - \gamma_c^-| > \varepsilon^{1/2-\delta} / \log T$

Call $S_2$ the subset of $S$ for which the second possibility holds. If the second possibility holds for at least one half of the elements in $S$ then $|S_2| \geq (c/2)\varepsilon^A T \log T$. But this is impossible by Proposition 1 once $\varepsilon$ is less than $(c/4)C(\delta, A + 1)$, with $C(\delta, A)$ as in the statement of Proposition 1. Therefore the second possibility can hold for at most one half of the elements in $S$. Hence the first possibility holds for at least a half of the elements in $S$. Call $S_1$ the subset of $S$ for which the first possibility holds.

By Lemma 7, there are no two $\rho' \in S_1$ lying between the same tuple of consecutive zeros of $\zeta'(s)$. Every $\rho' \in S_1$ lies either between $[\gamma_c^-, \gamma_c]$ or $[\gamma_c; \gamma_c^+]$ and moreover one of these intervals is of length $\leq \varepsilon^{1/2-\delta} / \log \gamma_c$. Skipping every second $\rho' \in S_1$ we make sure that no two $\rho_1 \in S_1$ and $\rho_2 \in S_1$ lie between the same set of consecutive zeros. Therefore every second $\rho' \in S_1$ gives rise to one (new) zero $\gamma$ (namely $\gamma_c$ or $\gamma_c^+$) with $(\gamma^+ - \gamma) \log \gamma \leq \varepsilon^{1/2-\delta}$. Thus we have at least $(1/2)|S_1| \geq (c/8)\varepsilon^A \cdot T \log T$ zeros $T \leq \gamma \leq 2T$ such that $(\gamma^+ - \gamma) \log \gamma \leq \varepsilon^{1/2-\delta}$. 


Lemma 8 (Gonek [10]). If \( x = a/b \neq 1 \) and \( a, b \leq N \), then,
\[
\sum_{T \leq \gamma \leq 2T} x^{i\gamma} \ll N \log^2 T
\]

Proof. As noted by Ford and Zaharescu (Lemma 1, [6]), it follows from Gonek’s work that,
\[
\sum_{T \leq \gamma \leq 2T} x^{1/2+i\gamma} = -\frac{\Lambda(n_x)}{2\pi} \frac{e^{iT \log(x/n_x)} - 1}{i \log(x/n_x)} + O\left(x \log^2(2xT) + \frac{\log 2T}{\log x}\right).
\]

Since \( x \) is not an integer we have \( x \neq n_x \). Therefore the closest that \( |x/n_x| = |a/(bn_x)| \) can be to 1 is when \( bn_x \) is equal to \( a \pm 1 \). This shows that \( |\log(x/n_x)| \gg a^{-1} \gg N^{-1} \).

Therefore the main term in the above equation is bounded by \( N \log T \). This gives a bound of \( \sum_{T \leq \gamma \leq 2T} x^{i\gamma} \ll N/\sqrt{T} \log T + \sqrt{x} \log^2 T \) for \( x > 1 \). For \( x < 1 \) this bound is reversed to \( \sqrt{x} N \log T + \log^2 T/\sqrt{x} \). In either case the final bound is \( \ll N \log^2 T \) because \( N^{-1} \leq |x| \leq N \).

An quick consequence of the above lemma is a bound for Dirichlet polynomials.

Lemma 9. Let \( B_N(s) \) be as in Lemma 2. If \( N^k \leq \sqrt{T} \) then,
\[
\sum_{T \leq \gamma \leq 2T} |B_N(\frac{1}{2} + i\gamma)|^{2k} \ll (Ck)^k \cdot T \log T \cdot (\log N)^{2k}
\]
for some absolute constant \( C > 0 \).

Proof. First notice that for \( T \leq t \leq 2T \)
\[
\sum_{p^k \leq N, k > 1} \frac{\log p}{p^{k/2+it}} \cdot \left(1 - \frac{\log p^k}{\log N}\right) \ll \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{\zeta'(s) + 1 + 2it}{s^2 \log \sqrt{N}} ds + O(1)
\]
\[
= -\frac{N^{-it}}{2t^2 \log N} + \frac{\zeta'}{\zeta}(1 + 2it) + O\left(1 + \frac{\log T}{\log N} \cdot N^{-1/8}\right)
\]
and that the above expression is less than \( \ll \log \log T \) by a classical estimate for the size of \( \zeta'/\zeta \) on the Riemann Hypothesis. Therefore,
\[
\sum_{T \leq \gamma \leq 2T} |B_N(\frac{1}{2} + i\gamma)|^{2k} \ll C^k \sum_{T \leq \gamma \leq 2T} \left|\sum_{p \leq N} \frac{\log p}{p^{1/2+i\gamma}} \cdot \left(1 - \frac{\log p}{\log N}\right)^k\right|^{2k} + T \log T \cdot (C \log T)^{2k}.
\]
with \( C > 0 \) some absolute constant. We denote the coefficients of the Dirichlet polynomial over primes by \( a(p) \). We have,
\[
\sum_{T \leq \gamma \leq 2T} \left|\sum_{p \leq N} a(p)p^{-i\gamma}\right|^{2k} = \sum_{p_1, \ldots, p_k \leq N, q_1, \ldots, q_k \leq N} a(p_1) \ldots a(p_k) a(q_1) \ldots a(q_k) \sum_{T \leq \gamma \leq 2T} \left(p_1 \ldots p_k q_1 \ldots q_k\right)^{i\gamma}.
\]
The diagonal terms \( p_1 \ldots p_k = q_1 \ldots q_k \) contribute at most
\[
\ll T \log T \cdot k! \left( 2 \sum_{p \leq N} |a(p)|^2 \right)^k \ll (Ck)^k T \log T \cdot (\log N)^{2k}
\]
because given \( q_1, \ldots, q_k \) all the solutions to the equation \( p_1 \ldots p_k = q_1 \ldots q_k \) are obtained by
pairing together each prime \( p_i \) with some other prime \( q_j \), and there is at most \( k! \) such pairings.

To bound the off-diagonal terms \( p_1 \ldots p_k \neq q_1 \ldots q_k \) we notice that \( p_1 \ldots p_k \leq N^k \leq \sqrt{T} \) and
similarly that \( q_1 \ldots q_k \leq N^k \leq \sqrt{T} \). Therefore by Gonek’s lemma
\[
\sum_{T \leq \gamma \leq 2T} \left( \frac{p_1 \ldots p_k}{q_1 \ldots q_k} \right)^{i\gamma} \leq \sqrt{T} \log^2 T.
\]
Since \( \sum_{p \leq N} a(p) \ll \sqrt{N} \) it follows that the off-diagonal terms contribute at most \( CkN^k \cdot \sqrt{T} \log^2 T \ll CkT \log^2 T \), which is less than the main term as soon as \( k > 0 \)
\( \square \)

An immediate consequence of the above lemma is the following.

**Lemma 10.** Let \( T \leq t \leq 2T \). Then,
\[
\sum_{T \leq \gamma \leq 2T} \left| N\left( \gamma + \frac{2\pi}{\log T} \right) - N\left( \gamma - \frac{2\pi}{\log T} \right) \right|^{2k} \ll (Ck)^2 \cdot T \log T.
\]
with \( C > 0 \) an absolute constant.

**Proof.** Let \( N \) be the largest integer such that \( N^k \leq \sqrt{T} \). We have
\[
N\left( \gamma + \frac{2\pi}{\log T} \right) - N\left( \gamma - \frac{2\pi}{\log T} \right) \leq N\left( \gamma + \frac{\pi}{\log N} \right) - N\left( \gamma - \frac{\pi}{\log N} \right)
\leq \frac{\log T}{\log N} \cdot \frac{|B_N(\frac{1}{2} + i\gamma)|}{\log N}
\]
by Lemma 2. Raising the above expression to the \( 2k \)-th power and then summing over all \( T \leq \gamma \leq 2T \) we obtain
\[
\sum_{T \leq \gamma \leq 2T} \left| N\left( \gamma + \frac{2\pi}{\log T} \right) - N\left( \gamma - \frac{2\pi}{\log T} \right) \right|^{2k} \ll (Ck)^2 \cdot T \log T + \frac{C^{2k}}{(\log N)^{2k}} \sum_{T \leq \gamma \leq 2T} \left| B_N(\frac{1}{2} + i\gamma) \right|^{2k}
\]
with \( C > 0 \) an absolute constant. By the previous lemma the sum over \( T \leq \gamma \leq 2T \) is
\( \ll (Ck)^2 \cdot T \log T \cdot (\log N)^{2k} \) and so the claim follows. \( \square \)

**Corollary 3.** Let \( A > 0 \) and \( \delta > 0 \) be given. If \( 0 < \varepsilon < C(\delta, A) \), with \( C(\delta, A) \) depending only on \( \delta \) and \( A \), then,
\[
\# \left\{ T \leq \gamma \leq 2T : N\left( \gamma + \frac{2\pi}{\log T} \right) - N\left( \gamma - \frac{2\pi}{\log T} \right) > \varepsilon^{-\delta} \right\} \leq \varepsilon^{A+1} \cdot T \log T.
\]

**Proof.** By the previous lemma we have for \( k > 1 \),
\[
\sum_{T \leq \gamma \leq 2T} \left| N\left( \gamma + \frac{2\pi}{\log T} \right) - N\left( \gamma - \frac{2\pi}{\log T} \right) \right|^{2k} \ll (Ck)^2 \cdot T \log T
\]
with \( C > 0 \) a positive absolute constant. Therefore the number of \( T \leq \gamma \leq 2T \) for which the interval \([\gamma - 2\pi/\log T; \gamma + 2\pi/\log T]\) contains more than \( \varepsilon^{-\delta} \) zeros is bounded above by \( \varepsilon^{2k\delta}(Ck)^{2k} \cdot T \log T \). Choose \( k = [A/\delta] \). Then \( \varepsilon^{2k\delta}(Ck)^{2k} \leq \varepsilon^A \) provided that \( \varepsilon \leq (cA/\delta)^{-4/\delta} \) with \( c > 0 \) an absolute constant. \( \square \)

8. Proof of Theorem 2

We will require the following two lemma.

**Lemma 11** (Zhang [19]). Let \( \varepsilon < 1 \). If \( \rho = \frac{1}{2} + i\gamma \) is a zero of \( \zeta(s) \) such that \( \gamma \) is sufficiently large and \((\gamma^+ - \gamma) \log \gamma < \varepsilon \) then there exists a zero \( \rho' \) of \( \zeta'(s) \) such that

\[
|\rho' - \rho| \leq \frac{2\varepsilon}{\log \gamma}.
\]

**Lemma 12** (Soundararajan [16]). We have,

\[
|\rho' - \rho_c|^2 \geq \frac{2(\beta' - \frac{1}{2})}{\log \gamma'}.
\]

We are now ready to prove Theorem 2.

**Proof of Theorem 2.** Suppose that there are at least \( c\varepsilon^A \cdot T \log T \) zeros \( T \leq \gamma \leq 2T \) such that \((\gamma^+ - \gamma) \log \gamma \leq \varepsilon^{1/2} \). Call this set \( S \). If \( \gamma^+ = \gamma \) for at least a half of the elements in \( S \) then \( \rho = \rho' \) and hence \( \beta' = \frac{1}{2} \) for at least \( (c/2)\varepsilon^A \cdot T \log T \) zeros.

Hence suppose that \( \gamma^+ > \gamma \) for at least half of the elements in \( S \) and call the subset of such elements \( S_1 \). By Corollary 3, the number of \( T \leq \gamma \leq 2T \) such that the interval \([\gamma - 2\pi/\log T; \gamma + 2\pi/\log T]\) contains more than \( \varepsilon^{-\delta} \) zeros is \( \leq (c/4)\varepsilon^A \cdot T \log T \), provided that \( \varepsilon \) is small enough with respect to \( \delta \) and \( A \). Therefore there is a subset \( S_2 \) of \( S \) of cardinality \( \geq (c/4)\varepsilon^A \cdot T \log T \) with the properties that \( 0 < (\gamma^+ - \gamma) \log \gamma < \varepsilon^{1/2} \) and the number of zeros in the interval \([\gamma - 2\pi/\log T, \gamma + 2\pi/\log T]\) is less than \( \varepsilon^{-\delta} \).

By Lemma 10 each \( \rho \in S_2 \) gives rise to a zero \( \rho' \) such that \( |\rho' - \rho| \leq 2\sqrt{\varepsilon}/\log T \). By Lemma 11 the zero \( \rho' \) satisfies \((\beta' - \frac{1}{2}) \log \gamma \leq \varepsilon \). Furthermore the interval \(|t - \gamma| < 2\sqrt{\varepsilon}/\log T\) contains at most \( \varepsilon^{-\delta} \) zeros. Therefore striking out at most \( \varepsilon^{-\delta} \) zeros from \( S_2 \) we obtain each time a new and distinct zero \( \rho' \) of \( \zeta'(s) \). It follows that \( \varepsilon^\delta |S_2| \) is a lower bound for the number of zeros \( \rho' \) with \((\beta' - \frac{1}{2}) \log \gamma \leq \varepsilon \). Hence \( m'(\varepsilon) \geq (c/4)\varepsilon^{A+\delta} \), as desired. \( \square \)

9. Proof of Corollary 1

The Pair Correlation Conjecture asserts that the number of zeros \( T \leq \gamma_1, \gamma_2 \leq 2T \) for which \( 2\pi\alpha/\log T < \gamma_1 - \gamma_2 \leq 2\pi\beta/\log T \) is asymptotically

\[
N(T) \cdot \int_{\alpha}^{\beta} \left( 1 - \left( \frac{\sin(\pi u)}{\pi u} \right)^2 + \delta(u) \right) du
\]

with \( \delta \) denoting Dirac’s delta function. Here we derive a simple consequence of the Pair Correlation Conjecture for small gaps between consecutive zeros of the Riemann zeta-function. The lower bound is not optimal but sufficient for our needs.

**Lemma 13.** Assume the Pair Correlation Conjecture. Let \( \delta > 0 \) be given. Then \( \varepsilon^{3+\delta} \ll m(\varepsilon) \ll \varepsilon^3 \) provided that \( 0 < \varepsilon < C(\delta) \) with \( C(\delta) \) a constant depending only on \( \delta \).
Proof. The Pair Correlation Conjecture asserts that the number of distinct zeros \( T \leq \gamma_1, \gamma_2 \leq 2T \) for which \( 0 \leq \gamma_1 - \gamma_2 \leq 2\pi\alpha/\log T \) is asymptotically \( N(T) \cdot f(\alpha) \) with \( f(\alpha) \) such that \( f(\alpha) \sim c \cdot \alpha^3 \) as \( \alpha \to 0 \). The number of \( T \leq \gamma \leq 2T \) such that \((\gamma^+ - \gamma) \log \gamma \leq \varepsilon \) is less than the number of distinct \( T \leq \gamma_1, \gamma_2 \leq 2T \) for which \( 0 \leq \gamma_1 - \gamma_2 \leq 2\pi\varepsilon/\log T \) therefore \( m(\varepsilon) \leq f(\varepsilon) \ll \varepsilon^3 \).

Now consider the set of \( T \leq \gamma_1, \gamma_2 \leq 2T \) for which \( 0 \leq (\gamma_1 - \gamma_2) \log \gamma_1 \leq \varepsilon \). Call \( S \) the set of \( T \leq \gamma_1 \leq 2T \) for which the interval \([\gamma_1 - 2\pi/\log T; \gamma_1 + 2\pi/\log T]\) contains at most \( \varepsilon^{-\delta} \) zeros. By Corollary 3, the zero \( T \leq \gamma_1 \leq 2T \) with \( \gamma_1 \notin S \) have cardinality \( \leq \varepsilon^A \cdot T \log T \) provided that \( 0 < \varepsilon < C(\delta, A) \) (we choose \( A = 100 \) for example). We have

\[
\sum_{\substack{\gamma_1 \in S \\ \gamma_1 \notin S}} \frac{1}{1 + \sum_{\gamma_1 \in S} \sum_{\gamma_2 \in S} 1} = \sum_{\substack{T \leq \gamma_1, \gamma_2 \leq 2T \\ \varepsilon/2 \leq (\gamma_1 - \gamma_2) \log \gamma_1 \leq \varepsilon}} 1
\]

Since \( \gamma_1 \in S \) there can be at most \( \varepsilon^{-\delta} \) zeros \( \gamma_2 \) satisfying \( \varepsilon/2 \leq |\gamma_1 - \gamma_2| \log \gamma_1 \leq \varepsilon \). Therefore the first sum is bounded by

\[
\sum_{\substack{\gamma_1 \in S \\ (\gamma_1 - \gamma_2) \log \gamma_1 \leq \varepsilon}} \varepsilon^{-\delta} \ll \varepsilon^{-\delta} \cdot m(\varepsilon) \cdot T \log T
\]

because for each \( \gamma_1 \in S \) the inner sum over \( \gamma_2 \) is \( \leq \varepsilon^{-\delta} \) if \((\gamma_1^+ - \gamma_1) \log \gamma_1 \leq \varepsilon \) and is 0 otherwise. On the other hand the second sum is by Cauchy-Schwarz less than,

\[
|S|^{1/2} \cdot \left( \sum_{T \leq \gamma_1 \leq 2T} \left( \sum_{T \leq \gamma_2 \leq 2T \varepsilon/2 \leq (\gamma_1 - \gamma_2) \log \gamma_1 \leq \varepsilon} 1 \right)^2 \right)^{1/2} \leq |S|^{1/2} \cdot \left( \sum_{T \leq \gamma_1 \leq 2T} \left( N\left(\gamma_1 + \frac{2\pi}{\log T}\right) - N\left(\gamma_1 - \frac{2\pi}{\log T}\right) \right)^2 \right)^{1/2} \ll \varepsilon^{A/2} \cdot T \log T
\]

by Lemma 9. By the Pair Correlation Conjecture the left-hand side of (13) is asymptotically \( C \cdot N(T) \cdot \varepsilon^3 \) for some absolute constant \( C > 0 \). Combining the above three equations we get \( C \varepsilon^3 \leq m(\varepsilon) \varepsilon^{-\delta} + C_1 \varepsilon^{A/2} \) for some absolute constant \( C, C_1 > 0 \). Therefore if \( \varepsilon \) is small enough then \( \varepsilon^{3+\delta} \ll m(\varepsilon) \).

We are now ready to prove Corollary 1.

Proof of Corollary 1. By the previous lemma, on the Pair Correlation, we have \( m(\varepsilon^{1/2}) \gg \varepsilon^{3/2+\delta} \) for all \( C(\delta) > \varepsilon > 0 \). Therefore by the second part of our Main Theorem we get \( m'(\varepsilon) \gg \varepsilon^{3/2+\delta} \) for all \( C(\delta) > \varepsilon > 0 \). Now suppose to the contrary that there is a \( \eta > 0 \) and a sequence of \( \varepsilon \to 0 \) such that \( m'(\varepsilon) \gg \varepsilon^{3/2-\eta} \). Then, by Theorem 1 on the same subsequence of \( \varepsilon \to 0 \) we have \( m(\varepsilon^{1/2-\delta}) \gg \varepsilon^{3/2-\eta} \). However by the Pair Correlation Conjecture we have \( \varepsilon^{3/2-3\delta} \ll m(\varepsilon^{1/2-\delta}) \gg \varepsilon^{3/2-\eta} \). Choosing \( 0 < \delta < (1/3)\eta \) and letting \( \varepsilon \to 0 \) along the subsequence, we obtain a contradiction. \( \square \)

10. Acknowledgments

I would like to thank Prof. Farmer, Prof. Ki and Prof. Soundararajan for comments on an early draft of this paper.
REFERENCES

[1] Bruce C. Berndt. The number of zeros for $\zeta^{(k)}(s)$. J. London Math. Soc. (2), 2:577–580, 1970.
[2] B. Conrey and H. Iwaniec. Spacing of zeros of Hecke $L$-functions and the class number problem. Acta Arith., 103(3):259–312, 2002.
[3] Eduardo Dueñez, David W. Farmer, Sara Froehlich, C. P. Hughes, Francesco Mezzadri, and Toan Phan. Roots of the derivative of the Riemann-zeta function and of characteristic polynomials. Nonlinearity, 23(10):2599–2621, 2010.
[4] David W. Farmer and Haseo Ki. Landau–Siegel zeros and zeros of the derivative of the Riemann zeta function. Adv. Math., 230(4-6):2048–2064, 2012.
[5] Shaoji Feng. A note on the zeros of the derivative of the Riemann zeta function near the critical line. Acta Arith., 120(1):59–68, 2005.
[6] Kevin Ford and Alexandru Zaharescu. On the distribution of imaginary parts of zeros of the Riemann zeta function. J. Reine Angew. Math., 579:145–158, 2005.
[7] Akio Fujii. On the zeros of Dirichlet $L$-functions. II. Trans. Amer. Math. Soc., 267(1):33–40, 1981. With corrections to: “On the zeros of Dirichlet $L$-functions. I” [Trans. Amer. Math. Soc. 196 (1974), 225–235; MR 50 #2096] and subsequent papers.
[8] M. Z. Garaev and C. Y. Yıldırım. On small distances between ordinates of zeros of $\zeta(s)$ and $\zeta'(s)$. Int. Math. Res. Not. IMRN, (21):Art. ID rnm091, 14, 2007.
[9] D. A. Goldston and S. M. Gonek. A note on $S(t)$ and the zeros of the Riemann zeta-function. Bull. Lond. Math. Soc., 39(3):482–486, 2007.
[10] S. M. Gonek. An explicit formula of Landau and its applications to the theory of the zeta-function. In A tribute to Emil Grosswald: number theory and related analysis, volume 143 of Contemp. Math., pages 395–413. Amer. Math. Soc., Providence, RI, 1993.
[11] Haseo Ki. The zeros of the derivative of the Riemann zeta function near the critical line. Int. Math. Res. Not. IMRN, (16):Art. ID rnm064, 23, 2008.
[12] Norman Levinson. More than one third of zeros of Riemann’s zeta-function are on $\sigma = 1/2$. Advances in Math., 13:383–436, 1974.
[13] Hugh L. Montgomery. Ten lectures on the interface between analytic number theory and harmonic analysis, volume 84 of CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1994.
[14] A. Selberg. On the value distribution of the derivative of the Riemann zeta-function. Unpublished manuscript available at http://publications.ias.edu/selberg/section/2483.
[15] Atle Selberg. On the remainder in the formula for $N(T)$, the number of zeros of $\zeta(s)$ in the strip $0 < t < T$. Avh. Norske Vid. Akad. Oslo. I., 1944(1):27, 1944.
[16] K. Soundararajan. The horizontal distribution of zeros of $\zeta'(s)$. Duke Math. J., 91(1):33–59, 1998.
[17] Kannan Soundararajan. Moments of the Riemann zeta function. Ann. of Math. (2), 170(2):981–993, 2009.
[18] Andreas Speiser. Geometrisches zur Riemannschen Zetafunktion. Math. Ann., 110(1):514–521, 1935.
[19] Yitang Zhang. On the zeros of $\zeta'(s)$ near the critical line. Duke Math. J., 110(3):555–572, 2001.