Note on integrability of certain homogeneous Hamiltonian systems in 2D constant curvature spaces

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Abstract
We formulate the necessary conditions for the integrability of a certain family of Hamiltonian systems defined in the constant curvature two-dimensional spaces. Proposed form of potential can be considered as a counterpart of a homogeneous potential in flat spaces. Thanks to this property Hamilton equations admit, in a general case, a particular solution. Using this solution we derive necessary integrability conditions investigating differential Galois group of variational equations.

Key words: integrability obstructions; Liouville integrability; Morales–Ramis theory; differential Galois theory; constant curvature spaces

1 Introduction

Integrability of natural Hamiltonian systems of the form
\[ H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + V(q), \quad q = (q_1, \ldots, q_n) \]  
(1.1)
has been intensively investigated during last decades and significant successes were achieved. Here \( q = (q_1, \ldots, q_n) \) and \( p = (p_1, \ldots, p_n) \) are canonical variables in \( \mathbb{C}^{2n} \) considered as a symplectic linear space. It seems that among new methods which have been
invented, the most powerful and efficient are those formulated in the framework of the differential Galois theory. The necessary conditions for the integrability of a Hamiltonian system in the Liouville sense are given in terms of properties of the differential Galois group of variational equations obtained by linearisation of equations of motion in a neighbourhood of a particular solution. The fundamental Morales-Ramis theorem of this approach says that if a Hamiltonian system is meromorphically integrable in the Liouville sense in a neighbourhood of a phase curve \( \Gamma \) corresponding to a particular solution, then the identity component \( G^0 \) of the differential Galois group \( G \) of variational equations along \( \Gamma \) is Abelian, see e.g. [11, 10].

To apply the above mentioned method we have to know a particular solution of the system but, in general, we do not know how to find it. Fortunately, for Hamiltonian systems (1.1) with the potential \( V(q) \) which is homogeneous of degree \( k \in \mathbb{Z} \), we know that in a generic case they admit particular solutions of the form

\[
q(t) = \varphi(t)d, \quad p(t) = \varphi(t)d, \quad \ddot{\varphi} = -\varphi^{k-1}.
\] (1.2)

where \( d \in \mathbb{C}^n \) is a non-zero solution of the non-linear system \( V'(d) = d \). Moreover, variational equations along these particular solutions can be transformed into a system of uncoupling hypergeometric equations depending on the degree of homogeneity \( k \) and eigenvalues \( \lambda_i \), for \( i = 1, \ldots, n \), of the Hessian \( V''(d) \). Since differential Galois group of the hypergeometric equation is well known it was possible to obtain necessary conditions of the integrability of Hamiltonian systems (1.1) in the form of arithmetic restrictions on \( \lambda_i \) that must belong to appropriate sets of rational numbers depending on \( k \), see e.g. [10, 11]. Later it appeared that between \( \lambda_i \) some universal relations exist which improves conditions mentioned in the above papers, for details see e.g. [7, 12, 4].

Successful integrability analysis of Hamiltonian systems with homogeneous potentials in flat Euclidean spaces motivated us to look for systems in curved spaces with similar properties. For natural systems

\[
H = \frac{1}{2} \sum_{i,j=1}^{n} g^{ij} p_i p_j + V(q)
\] (1.3)

defined on \( T^* M^n \), where \( M^n \) is a Riemannian manifold with metric \( g = \{ g_{ij} \} \), there is a good notion of homogeneous functions. In paper [14] we proposed to study the following form of the Hamiltonian

\[
H = T + V, \quad T = \frac{1}{2} r^{m-k} \left( p_r^2 + \frac{p_{\varphi}^2}{r^2} \right), \quad V = r^m U(\varphi),
\] (1.4)

where \( m \) and \( k \) are integers, \( k \neq 0 \) and \( U(\varphi) \) is a meromorphic function. If we consider \( (r, \varphi) \) as the polar coordinates, then the kinetic energy corresponds to a flat singular metric on a plane. This is just an example of a natural system which possesses certain common features with standard Hamiltonian systems with homogeneous potentials in the Euclidean plane \( \mathbb{R}^2 \).
In this paper we propose another class of natural Hamiltonian systems with two degrees of freedom defined on $T^*M^2$ where $M^2$ is a two dimensional manifold with a constant curvature metrics. More specifically, $M^2$ is either sphere $S^2$, the Euclidean plane $\mathbb{E}^2$, or the hyperbolic plane $\mathbb{H}^2$ with curvature parameter $\kappa$ positive, equal to zero or negative, respectively. In order to consider those three cases simultaneously we will proceed as in [5, 13] and we define the following $\kappa$-dependent trigonometric functions

$$C_\kappa(x) := \begin{cases} \cos(\sqrt{\kappa}x) & \text{for } \kappa > 0 \\ 1 & \text{for } \kappa = 0 \\ \cosh(\sqrt{-\kappa}x) & \text{for } \kappa < 0 \end{cases}, \quad S_\kappa(x) := \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}x) & \text{for } \kappa > 0 \\ x & \text{for } \kappa = 0 \\ \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa}x) & \text{for } \kappa < 0 \end{cases}. \quad (1.5)$$

These functions satisfy the following identities

$$C_\kappa^2(x) + \kappa S_\kappa^2(x) = 1, \quad S_\kappa'(x) = C_\kappa(x), \quad C_\kappa'(x) = -\kappa S_\kappa(x). \quad (1.6)$$

Our aim is to study the following natural Hamiltonian systems

$$H = T + V(r, \varphi), \quad T = \frac{1}{2} \left( p_r^2 + \frac{p_\varphi^2}{S_\kappa(r)^2} \right), \quad V(r, \varphi) = S_\kappa^m(r)U(\varphi), \quad (1.7)$$

where $m \in \mathbb{Z}$ and $U(\varphi)$ is a meromorphic function of variable $\varphi$. This is a natural Hamiltonian system defined on $T^*M^2$ for the prescribed $M^2$. Notice that the kinetic energy as well as the potential depends on the curvature $\kappa$. It appears that for such Hamiltonian systems we can find certain particular solutions and we are able to perform successfully differential Galois integrability analysis.

To apply the Morales-Ramis theory we consider the complex version of our system. We assume that there exists $\varphi_0 \in \mathbb{C}$ such that $U'(\varphi_0) = 0$ and $U(\varphi_0) \neq 0$. Under these assumptions we define

$$\lambda := 1 + \frac{U''(\varphi_0)}{mU(\varphi_0)}. \quad (1.8)$$

The main result of this paper is the following theorem that gives necessary conditions for the integrability of Hamiltonian systems governed by Hamiltonian (1.7).

**Theorem 1.1.** If the Hamiltonian system governed by Hamilton function (1.7) with $m \kappa \neq 0$ is meromorphically integrable, then the pair $(m, \lambda)$ belongs to the following list

| case | $m$ | $\lambda$ |
|------|-----|----------|
| 1    | $m$ | $-\frac{(m-2p)(p-1)}{m}$ |
| 2    | $m$ | $-\frac{(m+4p)[m-4(1+p)]}{8m}$ |
| 3    | $-2+4p$ | arbitrary |
| 4    | $m = 2q - 1$ | $-\frac{(-2+3m+12p)[3m-2(5+6p)]}{72m}$ |

Here $p$ and $q$ are arbitrary integers.
Case $\kappa = 0$ is excluded from the above theorem because it is already covered by Theorem 5.1 in [10, 11]. Case $m = 0$ and $\kappa = 0$ was already considered in [3] while case $m = 0$ and $\kappa \neq 0$ needs separate investigations.

Let us notice that conditions for integrability given in the above theorem neither depend on the sign, nor on the value of the curvature. This fact is related to the homogeneity of the system. Using real scaling we can reduce the curvature to three values $\kappa \in \{-1, 0, 1\}$. As we work in complex variables we can always choose variables and time such that $\kappa$ is either zero or one.

2 Proof of Theorem 1.1

Hamilton equations governed by Hamiltonian (1.7) have the form

\begin{align*}
\dot{r} &= p_r, \\
\dot{p}_r &= \frac{p^2_\varphi}{S^2_k(r)} C_k(r) - m S^{m-1}_k(r) C_k(r) U(\varphi), \\
\dot{\varphi} &= \frac{p_\varphi}{S^2_k(r)}, \\
\dot{p}_\varphi &= -S^m_k(r) U'(\varphi).
\end{align*}

(2.1)

If $U'(\varphi_0) = 0$ for a certain $\varphi_0 \in \mathbb{C}$, then system (2.1) has a two dimensional invariant manifold of the form

\[ N = \{ (r, p_r, \varphi, p_\varphi) \in \mathbb{C}^4 \mid \varphi = \varphi_0, \ p_\varphi = 0 \} \]  

(2.2)

Invariant manifold $N$ is foliated by phase curves parametrised by energy $e$

\[ e = \frac{1}{2} p_r^2 + S^m_k(r) U(\varphi_0), \]

that gives us particular solutions. If $[R, \Phi, P_R, P_\Phi]^T$ denote the variations of variables $[r, \varphi, p_r, p_\varphi]^T$, then variational equations along a particular solution lying on $N$ take the form

\begin{equation}
\begin{bmatrix}
R \\
\Phi \\
P_R \\
P_\Phi
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & S^{-2}_k(r) \\
m S^{m-2}_k(r) [m \kappa S^2_k(r) - (m - 1)] U(\varphi_0) & 0 & 0 & 0 \\
0 & -S^m_k(r) U''(\varphi_0) & 0 & 0
\end{bmatrix}
\begin{bmatrix}
R \\
\Phi \\
P_R \\
P_\Phi
\end{bmatrix}.
\end{equation}

(2.3)

Since the motion takes place in the plane $(r, p_r)$ the normal part of variational equations is given by the following closed subsystem

\[ \dot{\Phi} = S^{-2}_k(r) P_\Phi, \quad \dot{P}_\Phi = -S^m_k(r) U''(\varphi_0) \Phi, \]

or rewritten as a one second order equation

\[ \Phi + a(r, p_r) \Phi + b(r, p_r) \Phi = 0, \quad a(r, p_r) = \frac{C_k(r)}{S_k(r) p_r}, \quad b(r, p_r) = S^{m-2}_k(r) U''(\varphi_0). \]
Using the change of independent variable \( t \mapsto z = S_\kappa(r) / \sqrt{\kappa} \) we can transform it into a linear equation with rational coefficients

\[
\Phi'' + p(z)\Phi' + q(z)\Phi = 0,
\]

where \( u \) is defined by relation \( e^{\kappa m/2} = U(\varphi_0)u^m \). Notice that \( u \) is a singular point of the equation which depends on the choice of the energy. In order to analyse effectively cases with arbitrary \( m \) we fix \( u = 0 \). It is equivalent to take zero energy level. The variational equation (2.4) reduces to the following one

\[
\Phi'' + \frac{(m+6)z^2 - 4 - m}{2z(z^2 - 1)}\Phi'' + \frac{k(\lambda - 1)}{2z^2(z^2 - 1)}\Phi = 0.
\]

This equation has three regular singular points at \( z \in \{−1, 0, 1\} \). Using transformation \( z \mapsto y = z^2 \) we transform it into equation

\[
\frac{d^2\Phi}{dy^2} + \left[\frac{m+6}{4y} + \frac{1}{2(y-1)}\right] \frac{d\Phi}{dy} + \left[\frac{m(1-\lambda)}{8y^2} + \frac{m(\lambda - 1)}{8y(y-1)}\right] \Phi = 0,
\]

which is the Riemann P equation

\[
\frac{d^2\Phi}{dy^2} + \left[\frac{1-\alpha - \alpha'}{y} + \frac{1-\gamma - \gamma'}{y-1}\right] \frac{d\Phi}{dy} + \left[\frac{\alpha\alpha'}{y^2} + \frac{\gamma\gamma'}{(y-1)^2} + \frac{\beta\beta' - \alpha\alpha' - \gamma\gamma'}{y(y-1)}\right] \Phi = 0,
\]

see e.g. [15]. For the considered equation the respective differences of exponents at singularities \( y = 0, y = 1 \) and \( y = \infty \) are the following

\[
\rho = \alpha - \alpha' = \frac{1}{4} \sqrt{(m-2)^2 + 8m\lambda}, \quad \tau = \gamma - \gamma' = \frac{1}{2}, \quad \sigma = \beta - \beta' = \frac{m + 4}{4}.
\]

If the system is integrable, then according to the Morales-Ramis theorem, the identity components of the differential Galois group of variational equations (2.3) as well as of normal variational equation (2.5) are Abelian. If the identity component of the differential Galois group is Abelian, then in particular it is solvable. But necessary and sufficient conditions for solvability of the identity component of the differential Galois group for the Riemann P equation are well known and formulated in the Kimura theorem which we recall in Appendix A, for details see [6]. These conditions are expressed in terms of conditions on differences of exponents (2.7). The proof of Theorem 1.1 consists in a direct application of Theorem A.1 to our Riemann P equation (2.5) and taking into account that \( m \) has to be an integer.
The condition A of Theorem A.1 is fulfilled if at least one of the numbers \( \rho + \tau + \sigma \), \(-\rho + \tau + \sigma \), \( \rho - \tau + \sigma \), \( \rho + \tau - \sigma \) is an odd integer. This condition gives the following forms for \( \lambda \):

\[
\lambda = -\frac{(m - 4p)(2p - 1)}{m}, \quad \text{or} \quad \lambda = \frac{2p(m - 2 + 4p)}{m},
\]

where \( p \in \mathbb{Z} \). Let us notice that these two formulae can be obtained from the following one:

\[
\lambda = -\frac{(m - 2p)(p - 1)}{m},
\]

for \( p \) even and odd, respectively.

In Case B of Theorem A.1 the quantities \( \rho \) or \(-\rho\), \( \sigma \) or \(-\sigma\) and \( \tau \) or \(-\tau\) must belong to Table 1, that is called Schwarz’s table. As \( \sigma = \frac{1}{3} \) only items 1, 2, 4, 6, 9, or 14 of Table 1 are allowed and we will analyse them case by case. In calculations below numbers \( p \) and \( q \) are integers.

In item 1 the choice \( \pm \rho = 1/2 + p \) and arbitrary \( \sigma \) gives

\[
\lambda = -\frac{(m + 4p)(m - 4(1 + p))}{8m}, \quad (2.8)
\]

and no obstructions on \( m \). The second possibility in this item that \( \pm \sigma = 1/2 + p \) and \( \rho \) is arbitrary gives \( m = -2 + 4p \) and no obstructions for \( \lambda \).

In item 2, condition \( \pm \rho = 1/2 + p \) gives (2.8) and \( \sigma = 1/3 + q \) leads to non-integer \( m = -8/3 + 4q \).

In item 4 the choice \( \pm \sigma = 1/3 + q \) leads to non-integers \( m = -8/3 + 4q \) and \( m = -16/3 - 4q \). The second possibility in this item: \( \pm \rho = 1/3 + p \) gives

\[
\lambda = -\frac{(-2 + 3m + 12p)(3m - 2(5 + 6p))}{72m}
\]

and conditions \( \pm \sigma = 1/4 + q \) lead to \( m = -3 + 4q \) or \( m = -5 - 4q \), respectively. Elements of both these sets can be written as \( m = 2q - 1 \).

In item 6 the choice \( \pm \sigma = 1/3 + q \) leads to non-integers \( m = -8/3 + 4q \) or \( m = -16/3 - 4q \). Also the second possibility \( \pm \sigma = 1/5 + q \) gives only non-integer \( m = -16/5 - 4q \) or \( m = -24/5 - 4q \).

Similarly in item 9 both possibilities lead to non-integer \( m \). Namely, choice \( \pm \sigma = 1/5 + q \) gives non-integer \( m = -16/5 + 4q \) or \( m = -24/5 - 4q \) and \( \pm \sigma = 2/5 + q \) produces only non-integer \( m = -12/5 + 4q \) or \( m = -28/5 - 4q \).

In item 14 the choice \( \pm \sigma = 2/5 + q \) gives non-integers \( m = -12/5 + 4q \) or \( m = -28/5 - 4q \), respectively. Similarly the second possibility in this item that \( \pm \sigma = 1/3 + q \) leads only to non-integers \( m = -8/3 + 4q \) or \( m = -16/3 - 4q \).

Collecting all admissible forms for \( \lambda \) and \( m \) we obtain list in Eq. (1.9).

### 3 Remarks and examples

Result of Theorem A.1 is not optimal. In the proof we selected a phase curve with zero energy. We did this because for this energy value the normal variational equation reduces
to the Gauss hypergeometric equation for arbitrary $m$. For other energy values normal variational equation remains Fuchsian but the number of singularities depends on $m$ and we did not know how to analyse properties of its differential Galois group for an arbitrary $m$. For small values of $|m|$, one can apply the Kovacic algorithm to test if the identity component of the differential Galois group is solvable. In this way we can show the following.

**Lemma 3.1.** If Hamiltonian system given by (1.2) with $\kappa \neq 0$ and $m = 1$ is integrable, then either $\lambda = 1$, or $\lambda = 0$.

However, for $|m| > 2$ as well as for $m = -1$ there is no an effective way to perform the Kovacic algorithm till the end because its splits into infinitely many cases for which we have to determine if certain systems of algebraic equations have solutions.

If $m = -2$ the Hamiltonian system (1.7) is integrable. The additional first integral has the following form

$$ G = \frac{p_r^2}{2} + U(\varphi). $$

In fact, in this case system is separable in variables $(r, \varphi)$. For such systems one can ask about its superintegrability and in [9] it was shown that the necessary condition is that $\lambda = 1 - s^2$, where $s$ is a non-zero rational number.

It is known, see e.g. [10, 11], that for $m = 2$ and $\kappa = 0$ we cannot deduce any obstruction for the integrability from analysis of variational equations for prescribed straight-line solutions. Our theorem for $m = 2$ also does not give obstructions for integrability because it belongs to case 3 in Table (1.9). However, one can suspect that our weak conclusion is an effect of considering a peculiar energy level. But it is not this case. For $m = 2$ variational equation (2.4) is always solvable with solutions

$$ \Phi_\pm(z) = \frac{1}{z} \sqrt{z^2 + \lambda - 1} \exp \left[ \pm \int \omega(z) dz \right], $$

where

$$ \omega(z)^2 = \frac{\lambda(1 - \lambda)(u^2 + \lambda - 1)}{(z^2 + \lambda - 1)^2[z^4 - (1 + u^2)z^2 + u^2]]. $$

As our first example we consider the Hamiltonian (1.7) with the potential

$$ V(r, \varphi) = S^m_\kappa(r) \cos^m \varphi, $$

for which $\varphi_0 = 0$ and $\lambda = 0$, so the necessary conditions for integrability are fulfilled. In fact the system is integrable, and the additional first integral is

$$ I_\kappa = p_r \sin \varphi + p_\varphi \cos \varphi \frac{C_\kappa(r)}{S_\kappa(r)}, \quad \kappa \neq 0. $$

We notice that it does not depend on $m$. Limit

$$ I_0 = \lim_{\kappa \to 0} I_\kappa = p_r \sin \varphi + r^{-1} p_\varphi \cos \varphi, $$

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gives the first integral for the case $\kappa = 0$. If $\kappa = 0$ and $m = 1$, then there exists additional independent first integral quadratic in momenta

$$I_2 = \left( p_r^2 - \frac{p_\varphi^2}{\kappa^2} \right) \cos \varphi \sin \varphi + r^{-1} p_r p_\varphi \cos(2\varphi) - r \sin \varphi. \quad (3.7)$$

Thus, in this case the system is maximally super-integrable. Failure of direct search of additional first integral up to degree three in momenta for $\kappa \neq 0$ is in accordance with results of paper [13] where authors gave classification of superintegrable systems on $S^2$ and $\mathbb{H}^2$ with additional first integrals quadratic in momenta. In particular in this paper it was shown that for our potential $V = S_\kappa^m(r) \cos^m \varphi$ other terms must be added in order to get superintegrability with quadratic first integrals.

For the potential

$$V(r, \varphi) = S_\kappa^m(r) \sin^m \varphi, \quad (3.8)$$

$\varphi_0 = \frac{\pi}{2}$ and $\lambda = 0$, thus necessary integrability conditions are satisfied. We notice that it is obtained from (3.4) by shifting the angle $\varphi \to \frac{\pi}{2} + \varphi$. Thus we can immediately write integrable cases replacing in the above formulæ (3.5) – (3.7) for first integrals $\sin \varphi \to \cos \varphi$ and $\cos \varphi \to -\sin \varphi$. Potential (3.8) for $m = 2$ corresponds to the special case of the so-called anisotropic Higgs oscillator with $\delta = 0$, see Section 3 in [11] and Section 5 in [2].

**Remark 3.2** Hamiltonian systems described in (1.7) are obtained by a restriction of natural systems in $\mathbb{R}^3$ given by the following Lagrangian

$$L = \frac{1}{2\kappa} \left[ \dot{x}_0^2 + \kappa (\dot{x}_1^2 + \dot{x}_2^2) \right] - W(x_0, x_1, x_2), \quad (3.9)$$

to a quadric

$$\Sigma = \left\{ (x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_0^2 + \kappa(x_1^2 + x_2^2) = 1 \right\}. \quad (3.10)$$

Variables $(r, \varphi)$ are just local coordinates on $\Sigma$. We have

$$x_0 = C_\kappa(r), \quad x_1 = S_\kappa(r) \cos \varphi, \quad x_2 = S_\kappa(r) \sin \varphi,$$

and $V(r, \varphi) = W(x_0, x_1, x_2)$.

It is instructive to take $(x_1, x_2)$ as local coordinates on $\Sigma$. Then the Lagrangian has the form

$$L = \frac{1}{2} (\dot{x}_1^2 + \dot{x}_2^2) + \frac{\kappa(x_1\dot{x}_1 + x_2\dot{x}_2)}{2(1 - \kappa(x_1^2 + x_2^2))} - V(x_1, x_2),$$

where now $V(x_1, x_2) = W(x_0, x_1, x_2)$, with $x_0^2 = 1 - \kappa(x_1^2 + x_2^2)$. Denoting by $(y_1, y_2)$ the momenta conjugate to $(x_1, x_2)$ we obtain

$$y_1 = \dot{x}_1 + \frac{\kappa x_1(\dot{x}_1 + x_2\dot{x}_2)}{1 - \kappa(x_1^2 + x_2^2)}, \quad y_2 = \dot{x}_2 + \frac{\kappa x_2(x_1\dot{x}_1 + \dot{x}_2)}{1 - \kappa(x_1^2 + x_2^2)}.$$
and the Hamilton function of geodesic motion takes the form

\[ H = \frac{1}{2} \left[ y_1^2 + y_2^2 - \kappa(x_1 y_1 + x_2 y_2)^2 \right] + V(x_1, x_2). \]  

(3.11)

Notice that now the potential \((3.4)\) in ambient coordinates is just \(x_1^m\). The first integral \(I_\kappa\) given by \((3.5)\) in ambient variables takes the form

\[ I_\kappa = y_2 \sqrt{1 - \kappa(x_1^2 + x_2^2)}. \]

This function is a first integral for arbitrary potential \(V = f(x_1)\).

![Figure 1: The Poincaré cross section for the Hamiltonian (1.7) with potential \(V = \sin^{-1} r \cosh \varphi\) on energy level \(e = 50\). The cross-section plane is \(r = \pi/2\) with \(p_r > 0\).](image)

As the second example we consider the following family of potentials

\[ V(r, \varphi) = S^m_\kappa(r) \cosh \varphi. \]  

(3.12)

For this potential we have \(\lambda = (m + 1) / m\). Comparing this value with the forms of \(\lambda\) given in the cases 1, 2, 3, 4 of the integrability table (1.9), we obtain two values: \(m = -1\), and \(m = -3\). In order to check if for these values of \(m\) system is integrable we made Poincaré cross-sections. For \(m = -1\), cross-section in Figure 1 made for \(\kappa = 1\) clearly shows that the system is not integrable. For the case \(m = -3\) the answer was not obvious. We repeated our investigations for different choices of energy with the same results, namely the system behaves as a regular one, see in Fig. 2a made for \(\kappa = 1\). Only a big magnification of the region around an unstable periodic solution shows a small region with chaotic behaviour, see in Fig. 2b.
Figure 2: The Poincaré cross section for the Hamiltonian (1.7) with potential $V = \sin^{-3} r \cosh \varphi$ on energy level $e = 50$. The cross-section plane is $r = \pi/2$ with $p_r > 0$

It appears that chaotic region for this system can be detected much easier in the complex part of the phase space. Notice that after the change of variables $\varphi = i \psi$, $p_\varphi = -i p_\psi$, the Hamiltonian takes the form

$$H = \frac{1}{2} \left( p_r^2 - \frac{p_\psi^2}{S_\kappa(r)^2} \right) + S_\kappa''(r) \cos \psi. \quad (3.13)$$

Now non empty constant real energy levels of Hamiltonian contain points $(r, \psi, p_r, p_\psi) \in \mathbb{R}^4$ which are not accessible in original variables. A global cross-section for Hamiltonian (3.13) is presented in Fig. 3a. It shows a small chaotic area in the vicinity of a certain unstable periodic solution, see magnification of this region presented in Figure 3b. It is a few orders bigger in comparison with the chaotic region in Fig. 2b. We observed earlier this phenomenon for the Gross-Neveu systems in [8].

We can conclude that the potential (3.12) with $\kappa \neq 0$ and for $m = -1, -3$, satisfying the necessary integrability conditions, is not integrable. However, in the case of flat space when $\kappa = 0$, and for $m = -3$ the system is integrable with a first integral quartic in momenta of the following form

$$I = p_\varphi^4 + 2r^{-1} p_\varphi^2 \cosh \varphi + 2p_r p_\varphi \sinh \varphi - r^{-2} \sinh^2 \varphi. \quad (3.14)$$

This case admits a generalisation. We can take

$$U(\varphi) = Ae^\varphi + Be^{-\varphi}, \quad (3.15)$$

where $A$ and $B$ are arbitrary constants, and then the quartic first integral has the form

$$I = p_\varphi^4 + 2r^{-1} U(\varphi)p_\varphi^2 + 2U'(\varphi)p_r p_\varphi - r^{-2} (U'(\varphi))^2. \quad (3.16)$$
Our last example is system (1.7) with \( m = 2 \) and with the potential
\[
V(r, \varphi) = S_\kappa(r) U(\varphi), \quad U(\varphi) = c_1 \cos(2\varphi) + c_2 \sin(2\varphi),
\] (3.17)
where \( c_1 \) and \( c_2 \) are arbitrary constants. In this case system is integrable with the following first integral
\[
I = \left[ p_r^2 - \left( p_\varphi \frac{C_\kappa(r)}{S_\kappa(r)} \right)^2 \right] U(\varphi) + p_r p_\varphi \frac{C_\kappa(r)}{S_\kappa(r)} U'(\varphi) + 2(c_1^2 + c_2^2) S_\kappa(r)^2. \quad (3.18)
\]

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A Gauss hypergeometric equation

The Riemann \( P \) equation is the most general second order differential equation with three regular singularities \[15\]. If we place using homography these singularities at \( z \in \{0, 1, \infty\} \), then it has the canonical form
\[
\frac{d^2 \eta}{dz^2} + \left( \frac{1 - \alpha - \alpha'}{z} + \frac{1 - \gamma - \gamma'}{z - 1} \right) \frac{d \eta}{dz} + \left( \frac{\alpha \alpha'}{z^2} + \frac{\gamma \gamma'}{(z - 1)^2} + \frac{\beta \beta' - \alpha \alpha' - \gamma \gamma'}{z(z - 1)} \right) \eta = 0,
\] (A.1)
where \((\alpha, \alpha'), (\gamma, \gamma'), (\beta, \beta')\) are the exponents at the respective singular points. These exponents satisfy the Fuchs relation

\[ \alpha + \alpha' + \gamma + \gamma' + \beta + \beta' = 1. \]

We denote the differences of exponents by

\[ \rho = \alpha - \alpha', \quad \tau = \gamma - \gamma', \quad \sigma = \beta - \beta'. \]

Necessary and sufficient conditions for solvability of the identity component of the differential Galois group of (A.1) are given by the following theorem due to Kimura [6].

\[
\begin{align*}
1 \quad & 1/2 + r \quad 1/2 + s \quad \text{arbitrary complex number} \\
2 \quad & 1/2 + r \quad 1/3 + s \quad 1/3 + p \\
3 \quad & 2/3 + r \quad 1/3 + s \quad 1/3 + p \quad r + s + p \text{ even} \\
4 \quad & 1/2 + r \quad 1/3 + s \quad 1/4 + p \\
5 \quad & 2/3 + r \quad 1/4 + s \quad 1/4 + p \quad r + s + p \text{ even} \\
6 \quad & 1/2 + r \quad 1/3 + s \quad 1/5 + p \\
7 \quad & 2/5 + r \quad 1/3 + s \quad 1/5 + p \quad r + s + p \text{ even} \\
8 \quad & 2/3 + r \quad 1/5 + s \quad 1/5 + p \quad r + s + p \text{ even} \\
9 \quad & 1/2 + r \quad 2/5 + s \quad 1/5 + p \\
10 \quad & 3/5 + r \quad 1/3 + s \quad 1/5 + p \quad r + s + p \text{ even} \\
11 \quad & 2/5 + r \quad 2/5 + s \quad 2/5 + p \quad r + s + p \text{ even} \\
12 \quad & 2/3 + r \quad 1/3 + s \quad 1/5 + p \quad r + s + p \text{ even} \\
13 \quad & 4/5 + r \quad 1/5 + s \quad 1/5 + q \quad r + s + p \text{ even} \\
14 \quad & 1/2 + r \quad 2/5 + s \quad 1/3 + p \\
15 \quad & 3/5 + r \quad 2/5 + s \quad 1/3 + p \quad r + s + p \text{ even}
\end{align*}
\]

Table 1: Schwarz’s table. Here \(r, s, p \in \mathbb{Z}\)

**Theorem A.1.** The identity component of the differential Galois group of the Riemann P equation (A.1) is solvable iff

A. at least one of the four numbers \(\rho + \sigma + \tau, -\rho + \sigma + \tau, \rho + \sigma - \tau, \rho - \sigma + \tau\) is an odd integer, or

B. the numbers \(\rho\) or \(-\rho\) and \(\sigma\) or \(-\sigma\) and \(\tau\) or \(-\tau\) belong (in an arbitrary order) to some of appropriate fifteen families forming the so-called Schwarz’s Table, see Table 1.

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