Schwinger’s Propagator Is Only A Green’s Function

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ABSTRACT

Schwinger used an analytic continuation of the effective action to correctly compute the particle production rate per unit volume for QED in a uniform electric field. However, if one simply evaluates the one loop expectation value of the current operator using his propagator, the result is zero! We analyze this curious fact from the context of a canonical formalism of operators and states. The explanation turns out to be that Schwinger’s propagator is not actually the expectation value of the time-ordered product of field operators in the presence of a time-independent state, although it is of course a Green’s function. We compute the true propagator in the presence of a state which is empty at $x_+ = 0$ where $x_+ \equiv (x^0 + x^3) / \sqrt{2}$ is the lightcone evolution parameter. Our result can be generalized to electric fields which depend arbitrarily on $x_+$.

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1 Introduction

Nearly everyone who has made a serious study of quantum field theory in the past half century is familiar with Schwinger’s treatment of quantum electrodynamics (QED) in the presence of a constant electromagnetic field [1]. A complicating feature of this background is that particle production occurs when the electric field is nonzero, so no state can be stable even in the free theory. Schwinger was well aware of this feature and took pains to work around it by assuming, where ever necessary, that the electromagnetic field was purely magnetic. He first computed the electron propagator then used it to evaluate what would later be known as the in-out effective action at one loop. He computed the rate of particle production for the unstable case of an electric field by inferring the imaginary part of the effective action through an analytic continuation from the stable case of a magnetic field.

Schwinger’s analysis is rightly regarded as one of the great achievements of quantum field theory. However, its generality gives rise to a question. Schwinger actually obtained a form for the electron propagator for any constant electromagnetic field, magnetic or electric. What would result from using his electron propagator to calculate the one loop expectation value of the current operator for the unstable case of a constant electric field? This is how one might begin computing the back-reaction of particle production on the electric field.

The result turns out to be zero. Of course this does not mean that there is no particle creation at one loop! Rather it implies that, for the case of a non-zero electric field, Schwinger’s “propagator” is really only a Green’s function and not the expectation value of the time-ordered product of $\psi(x)\bar{\psi}(x')$ in the presence of any fixed state. Schwinger never said otherwise, and it was presumably to avoid this problem that he employed the circuitous analytic continuation procedure.

The distinction between propagators and Green’s functions can be understood most clearly in the context of the one dimensional harmonic oscillator with mass $m$ and frequency $\omega$. We can use the Heisenberg operator equations to express the position operator $q(t)$ in terms of its initial values $q_0$ and $\dot{q}_0 = p_0/m$,

$$q(t) = q_0 \cos(\omega t) + \frac{p_0}{m\omega} \sin(\omega t).$$

(1)

The time-ordered product of two such operators can be expressed in terms
of the $\mathbb{C}$-number commutator function and the operator anti-commutator,

$$T\{q(t)q(t')\} = \frac{1}{2} \text{sgn}(\Delta t)[q(t), q(t')] + \frac{1}{2}\{q(t), q(t')\},$$

(2)

$$= -\frac{i}{2m\omega} \sin(m|\Delta t|) + \frac{1}{2}\{q(t), q(t')\},$$

(3)

where $\Delta t \equiv t - t'$. A general propagator is the expectation value of this in the presence of some normalized state,

$$i\Delta(t; t') \equiv \langle S|T\{q(t)q(t')\}|S\rangle,$$

(4)

$$= -\frac{i}{2m\omega} \sin(\omega|\Delta t|) + \alpha \cos(\omega t')\cos(\omega t') + \beta \sin[\omega(t + t')] + \gamma \sin(\omega t)\sin(\omega t').$$

(5)

We have expressed the result in terms of three real numbers $\alpha$, $\beta$ and $\gamma$ defined as follows:

$$\alpha \equiv \langle S|q_0^2|S\rangle, \quad \beta \equiv \frac{1}{2m\omega} \langle S|q_0 p_0 + p_0 q_0|S\rangle, \quad \gamma \equiv \frac{1}{m^2\omega^2} \langle S|p_0^2|S\rangle.$$

(6)

The key point is that the uncertainty principle imposes an inequality on $\alpha$ and $\gamma$,

$$\alpha \gamma \geq \left(\frac{1}{2m\omega}\right)^2.$$ 

(7)

But $\alpha$, $\beta$ and $\gamma$ can be any $\mathbb{C}$-numbers if one only requires the Green’s function equation,

$$-m\left(\frac{d^2}{dt^2} + \omega^2\right)i\Delta(t; t') = i\delta(t - t'),$$

(8)

and symmetry under interchange of $t$ and $t'$. For example, $\alpha = \beta = \gamma = 0$ gives a Green’s function but not a propagator.

This paper contains six sections of which this introduction is the first. In Section 2 we show that Schwinger’s propagator gives zero for the expectation value of the current operator at one loop. To see that Schwinger’s “propagator” is really only a Green’s function it is useful to first express it in a diagonal function and spinor basis. This is the work of Section 3. In Section 4 we exhibit an initial value solution for the Heisenberg field operators analogous to the one presented above for the harmonic oscillator. We
then show that there is no fixed state which gives Schwinger’s result. The manner in which it fails also explains the zero current result, and incidentally provides what is probably the simplest picture we shall ever get of particle production. In Section 5 we work out a true propagator in the presence of a natural state. It is significant that we can actually do this for a class of backgrounds which is general enough to include the actual electric field as it evolves under the influence of quantum electrodynamic back-reaction. Our conclusions comprise Section 6.

2 Zero current with Schwinger’s propagator

We begin by representing the propagator as Schwinger did, as the expectation value of a first quantized operator,

\[ iS(x; x') \equiv \langle x \left| \frac{i}{\mathcal{P} - e \mathcal{A}(X) - m + i\epsilon} \right| x' \rangle, \quad (9) \]

\[ = \langle x \left| [\mathcal{P} - e \mathcal{A} + m] \int_0^\infty dse^{is[(P-eA)^2 - \frac{1}{2}eF_{\mu\nu}\sigma^{\mu\nu} - m^2 + i\epsilon]} \right| x' \rangle. \quad (10) \]

The position and momentum operators of the first quantized theory are \( X^\mu \) and \( P^\nu \), respectively. We assign the standard meaning to \( \sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu] \), and we will assume that the vector potential \( A_\mu(X) \) is a linear function of \( X^\nu \). We also follow Schwinger in using the proper time method to regulate expressions involving the propagator,

\[ iS(x; x') = \lim_{s_0 \to 0^+} \int_0^{\infty} dse^{-is(m^2 - i\epsilon)} \times \langle x \left| [\mathcal{P} - e \mathcal{A} + m] e^{-\frac{i}{2}eF_{\mu\nu}\sigma^{\mu\nu}} e^{is[(P-eA)^2]} \right| x' \rangle. \quad (11) \]

Note that if the electric field has magnitude \( E \) then the exponential of the matrix term is,

\[ e^{-\frac{i}{2}eF_{\mu\nu}\sigma^{\mu\nu}} = \cosh(eEs)I + \frac{1}{2E} \sinh(eEs)F_{\mu\nu}\gamma^\mu\gamma^\nu. \quad (12) \]

At one loop order the expectation value of the current operator is just the trace of \( e\gamma^\mu \) up against the coincident propagator,

\[ \langle \Omega | J^\mu(x) | \Omega \rangle = \text{Tr} \left[ e\gamma^\mu iS(x; x) \right], \quad (13) \]
\[
\lim_{s_0 \to 0^+} 4e \int_{s_0}^{\infty} ds e^{-is(m^2 - ie)} \begin{aligned}
&\left\{ \cosh(eEs) \left\langle x \left| [P^\mu - eA^\mu] e^{is(P - eA)^2} \right| x \right\rangle \\
&- \frac{1}{E} \sinh(eEs) F^{\mu\nu} \left\langle x \left| [P_\nu - eA_\nu] e^{is(P - eA)^2} \right| x \right\rangle \right\}. 
\end{aligned}
\tag{14}
\]

Note that the coincidence limit is regulated by the factor of \(e^{is(P - eA)^2}\) as long as \(s \neq 0\). Note also that we can get the factor of \(P^\mu - eA^\mu\) by commutation,

\[
P^\mu - eA^\mu = -\frac{i}{2} \left[ X^\mu, (P - eA)^2 \right]. 
\tag{15}
\]

The commutator of \(X^\mu\) with the exponential gives \(is\) times a special operator ordering of the product of the exponential times with this factor. Because the first quantized bra and ket are the same the original ordering can be restored,

\[
\left\langle x \left| [X^\mu, e^{is(P - eA)^2}] \right| x \right\rangle = 2s \left\langle x \left| (P^\mu - eA^\mu) e^{is(P - eA)^2} \right| x \right\rangle . 
\tag{16}
\]

But the commutator vanishes for the same reason. So the expectation value of the current operator computed using Schwinger’s propagator vanishes.

### 3 Going to lightcone momentum space

For definiteness we assume the electric field has magnitude \(E\) and is directed along the positive \(z\) axis. If we define the separation 4-vector as \(\Delta x^\mu \equiv x^\mu - x'^\mu\) then Schwinger’s result for the electron propagator is \(iS\)

\[
iS(x; x') = \frac{i}{32\pi^2} \exp \left[ -i e \int_{x'}^x d\xi A_\mu(\xi) \right] \int_{s_0}^{\infty} ds e^{-is(m^2 - ie)} \\
\times \exp \left[ \frac{1}{4s} \left( \Delta x^2 + \Delta y^2 + eEs \coth(eEs) [\Delta z^2 - \Delta t^2] \right) \right] \\
\times \left\{ \left[ -\frac{2eE}{s} m + \frac{eE}{s^2} \left( \gamma^1 \Delta x + \gamma^2 \Delta y \right) \right] \left[ \coth(eEs) + \gamma^0 \gamma^3 \right] \\
+ \frac{e^2E^2}{s} \text{csch}^2(eEs) \left[ \gamma^3 \Delta z - \gamma^0 \Delta t \right] \right\}. 
\tag{17}
\]

The special role assumed by the 0 and 3 directions strongly suggests that \(iS(x; x')\) should be expressed in terms of lightcone coordinates. The fact that translation invariance is broken only by the exponential of the line integral of
the vector potential also suggests that the propagator should be transformed to momentum space.

We define the lightcone coordinates and gamma matrices as follows:

$$x_\pm \equiv \frac{1}{\sqrt{2}} (x^0 \pm x^3), \quad \gamma_\pm \equiv \frac{1}{\sqrt{2}} (\gamma^0 \pm \gamma^3).$$  \hspace{1cm} (18)

The other ("transverse") components of $x^\mu$ and $\gamma^\mu$ comprise the 2-vectors $\tilde{x}$ and $\tilde{\gamma}$, and the invariant contraction decomposes as follows,

$$\gamma^\mu x_\mu = \gamma^0 x^0 - \gamma^3 x^3 - \tilde{\gamma} \cdot x = \gamma_+ x_- + \gamma_- x_+ - \tilde{\gamma} \cdot \tilde{x}.$$  \hspace{1cm} (19)

Note that $(\gamma_\pm)^2 = 0$. We follow Kogut and Soper \[2\] in defining lightcone spinor projection operators,

$$P_\pm \equiv \frac{1}{2} (I \pm \gamma^0 \gamma^3) = \frac{1}{2} \gamma_\mp \gamma_\pm.$$  \hspace{1cm} (20)

With these conventions the propagator takes the form,

$$i S(x; x') = -\frac{ie^2 E^2}{32\pi^2} \exp \left[ -ie \int_{x'}^{x} d\xi A_\mu(\xi) \right] \int_0^\infty \frac{ds}{s} \csch^2(eEs) e^{-is(m^2 - ie)}$$

$$\times \exp \left[ \frac{i}{4s} \Delta \tilde{x} \cdot \Delta \tilde{x} - \frac{i}{2} eE \coth(eEs) \Delta x_+ \Delta x_- \right]$$

$$\times \left\{ \left[ \frac{m}{eE} - \frac{\tilde{\gamma} \cdot \Delta \tilde{x}}{2eEs} \right] P_+ \left( e^{2eEs} - 1 \right) + \gamma_+ \Delta x_- + \gamma_- \Delta x_+ + \left[ \frac{m}{eE} - \frac{\tilde{\gamma} \cdot \Delta \tilde{x}}{2eEs} \right] P_- \left( 1 - e^{-2eEs} \right) \right\}.$$  \hspace{1cm} (21)

We define the lightcone components of the vector potential $A_\mu$ as

$$A_\pm \equiv \frac{1}{\sqrt{2}} (A_0 \pm A_3).$$  \hspace{1cm} (22)

Our gauge condition is $A_+ = 0$, and to get $F^{30} = E$ we take $A_- = -E x_+$ with $\tilde{A} = 0$. It is now possible to compute the initial phase. The path is $\xi^\mu(\tau) = x'^\mu + \Delta x^\mu \tau$ so we get

$$- ie \int_{x'}^{x} d\xi A_\mu(\xi) = ie E \Delta x_- \int_0^1 d\tau \left( x'_+ + \Delta x_+ \tau \right),$$

$$= \frac{i}{2} eE \Delta x_- (x'_+ + x'_+).$$  \hspace{1cm} (23)
The propagator is invariant under translations of $x$ and $\tilde{x}$ so those are the variables on which we shall Fourier transform,

$$i\tilde{S}(x_+, x'_+; k_+, \tilde{k}) \equiv \int_{-\infty}^{\infty} d\Delta x_- e^{ik_+ \Delta x_-} \int d^2\Delta \tilde{x} e^{-i\tilde{k} \cdot \Delta \tilde{x}} iS(x; x'), \quad (25)$$

$$= \frac{e^2E^2}{4} \int_0^{\infty} ds \text{csch}^2(eEs) e^{-is(\tilde{\omega} - i\epsilon)} \left\{ \left[ \frac{m - \tilde{\gamma} \cdot \tilde{k}}{eE} \right] P_+ (e^{2eEs} - 1) \right.$$  

$$- i\gamma_+ \frac{\partial}{\partial k_+} + \gamma_- \Delta x_+ + \left[ \frac{m - \tilde{\gamma} \cdot \tilde{k}}{eE} \right] P_- \left( 1 - e^{-2eEs} \right) \}$$  

$$\times \delta \left( k_+ + \frac{1}{2}eE(x_+ + x'_+) - \frac{1}{2}eE \coth(eEs) \Delta x_+ \right), \quad (26)$$

where $\tilde{\omega} = m^2 + \tilde{k} \cdot \tilde{k}$. The delta function can be recast to determine $s$,

$$\delta \left( k_+ + \frac{1}{2}eE(x_+ + x'_+) - \frac{1}{2}eE \coth(eEs) \Delta x_+ \right) = \frac{\delta \left( s - \frac{1}{2eE} \ln \left[ \frac{k_+ + eEx_+}{k_+ + eEx'_+} \right] \right)}{\frac{1}{2}e^2E^2\text{csch}^2(eEs)|\Delta x_+|}. \quad (27)$$

This brings the propagator to the form,

$$i\tilde{S}(x_+, x'_+; k_+, \tilde{k}) = \frac{1}{2|\Delta x_+|} \left\{ \left[ \frac{m - \tilde{\gamma} \cdot \tilde{k}}{eE} \right] \left( \frac{eE \Delta x_+}{k_+ + eEx'_+} \right) P_+ \right.$$  

$$- \gamma_+ \frac{\partial}{\partial k_+} + \gamma_- \Delta x_+ + \left[ \frac{m - \tilde{\gamma} \cdot \tilde{k}}{eE} \right] \left( \frac{eE \Delta x_+}{k_+ + eEx'_+} \right) P_- \}$$  

$$\times \int_0^{\infty} ds e^{-i(\tilde{\omega} - i\epsilon)} \delta \left( s - \frac{1}{2eE} \ln \left[ \frac{k_+ + eEx_+}{k_+ + eEx'_+} \right] \right). \quad (28)$$

At this stage it becomes crucial to recall that the electron charge is negative so $eE < 0$. The delta function can only become singular for $s > 0$ if $k_+ > -eEx_+$ for $\Delta x_+ > 0$, or if $k_+ > -eEx_+$ for $\Delta x_+ < 0$. The integration over $s$ therefore gives,

$$\left[ \frac{k_+ + eEx_+}{k_+ + eEx'_+} \right]^{i(\tilde{\omega} - i\epsilon) / 2eE} \{ \theta(\Delta x_+) \theta(k_+ + eEx_+) + \theta(-\Delta x_+) \theta(-k_+ - eEx_+) \}. \quad (29)$$

Note that the $ie$ in the exponent means that the term in square brackets is raised to a power with a small positive real part. This has the important
consequence that multiplying by $\delta(k_+ + eEx_+)$ gives zero, so we need not worry about the $-i\partial/\partial k_+$ acting on the theta functions. Acting on the power it gives,

$$-i \frac{\partial}{\partial k_+} \left[ \frac{k_+ + eEx_+}{k_+ + eEx'_+} \right] = \frac{1}{2} \tilde{\omega}^2 \Delta x_+ \left[ \frac{k_+ + eEx_+}{(k_+ + eEx_+)(k_+ + eEx'_+)} \right] \frac{\tilde{\omega}^2}{2eE}. \quad (30)$$

The final result for the propagator is,

$$i \tilde{S}(x_+, x'_+; k_+, \tilde{k}) = \text{sgn}(\Delta x_+) \theta [\text{sgn}(\Delta x_+)(k_+ + eEx_+)]$$

$$\times \left[ \frac{k_+ + eEx_+}{k_+ + eEx'_+} \right] \frac{\tilde{\omega}^2}{2eE} \left\{ \left( \frac{m - \tilde{\gamma} \cdot \tilde{k}}{k_+ + eEx'_+} \right) P_+ \right.$$ 

$$+ \left( \frac{m - \tilde{\gamma} \cdot \tilde{k}}{k_+ + eEx'_+} \right) \frac{1}{2} \frac{\tilde{\omega}^2}{\sqrt{2}} P_+$$

$$\left. \right\} \text{sgn}(\Delta x_+)(k_+ + eEx_+). \quad (31)$$

It is more illuminating for the work of the next section to right-multiply this by $\gamma^0 = (\gamma_+ + \gamma_-)/\sqrt{2}$ and slightly re-arrange the order in which the four spinor matrices appear,

$$i \tilde{S}(x_+, x'_+; k_+, \tilde{k}) \gamma^0 = \text{sgn}(\Delta x_+) \theta [\text{sgn}(\Delta x_+)(k_+ + eEx_+)]$$

$$\times \left[ \frac{k_+ + eEx_+}{k_+ + eEx'_+} \right] \frac{\tilde{\omega}^2}{2eE} \left\{ \frac{1}{\sqrt{2}} P_+ + \frac{1}{\sqrt{2}} P_+ \right. \left( \frac{m + \tilde{\gamma} \cdot \tilde{k}}{k_+ + eEx'_+} \right) P_-$$

$$\left. + \left( \frac{m - \tilde{\gamma} \cdot \tilde{k}}{k_+ + eEx'_+} \right) \frac{1}{2} \frac{\tilde{\omega}^2}{\sqrt{2}} P_+ + \frac{1}{2} \frac{\tilde{\omega}^2}{\sqrt{2}} P_- \right\} \theta (k_+ + eEx_+)(k_+ + eEx'_+). \quad (32)$$

### 4 Inferring the state

The point of this section is to understand Schwinger’s propagator in terms of operators and states. Let us start with notation for the Fourier transform (on $x_-$ and $\bar{x}$) of the electron field operator $\psi(x_+, x_-, \bar{x})$,

$$\Psi(x_+, k_+, \tilde{k}) \equiv \int dx_- e^{ik_+x_-} \int d^2\bar{x} e^{-i\bar{k}\bar{x}} \psi(x_+, x_-, \bar{x}). \quad (33)$$
This section is about finding a state \( |S\rangle \) such that,

\[
i S\left(x_+, x'_+; k_+, \tilde{k}\right) \gamma^0 (2\pi)^3 \delta(k_+ - q_+) \delta^2(\tilde{k} - \tilde{q})
= \theta(\Delta x_+) \left\langle S \left| \Psi \left(x_+, k_+, \tilde{k}\right) \Psi^\dagger \left(x'_+, q_+, \tilde{q}\right) \right| S \right\rangle
- \theta(-\Delta x_+) \left\langle S \left| \Psi^\dagger \left(x'_+, q_+, \tilde{q}\right) \Psi \left(x_+, k_+, \tilde{k}\right) \right| S \right\rangle ,
\]

(34)

\[
\equiv \left\langle S \left| X_+ \left\{ \Psi \left(x_+, k_+, \tilde{k}\right) \Psi^\dagger \left(x'_+, q_+, \tilde{q}\right) \right\} \right| S \right\rangle .
\]

(35)

Note that the adjoint is taken after Fourier transforming.

We define the \( \pm \) components of the electron field (and its Fourier transform) using the \( P_\pm \) projectors,

\[
\psi_\pm \left(x_+, x_-, \vec{x}\right) \equiv P_\pm \psi \left(x_+, x_-, \vec{x}\right) .
\]

(36)

By acting \( P_+ \) on the left and right of expression (32) it is easy to see that the state \( |S\rangle \) must obey

\[
\left\langle S \left| X_+ \left\{ \Psi_+ \left(x_+, k_+, \tilde{k}\right) \Psi^\dagger_+ \left(x'_+, q_+, \tilde{q}\right) \right\} \right| S \right\rangle = (2\pi)^3 \delta(k_+ - q_+) \delta^2(\tilde{k} - \tilde{q})
\times \text{sgn}(\Delta x_+) \theta \left[ \text{sgn}(\Delta x_+)(k_+ + eE x_+) \right] \left[ k_+ + eE x_+ \right]^{\frac{-i\omega^2}{2}} \times \frac{1}{\sqrt{2}} P_+ .
\]

(37)

The other components can be recognized similarly,

\[
\left\langle S \left| X_+ \left\{ \Psi_+ \Psi_-^\dagger \right\} \right| S \right\rangle = \text{same} \times \frac{1}{2} P_+ \frac{1}{2} \gamma_+ \left( \frac{m + \vec{\gamma} \cdot \vec{k}}{k_+ + eE x_+} \right) ,
\]

(38)

\[
\left\langle S \left| X_+ \left\{ \Psi_- \Psi_+^\dagger \right\} \right| S \right\rangle = \text{same} \times \left( \frac{m - \vec{\gamma} \cdot \vec{k}}{k_+ + eE x_+} \right)^\frac{1}{2} \gamma_+ \frac{1}{\sqrt{2}} P_+ ,
\]

(39)

\[
\left\langle S \left| X_+ \left\{ \Psi_- \Psi_-^\dagger \right\} \right| S \right\rangle = \text{same} \times \frac{1}{2} \omega^2 \frac{1}{\sqrt{2}} P_-
\frac{1}{k_+ + eE x_+}(k_+ + eE x_+). 
\]

(40)

Of relations (37)-(40) only the first is really independent, the others follow from the equations of motion. To see this consider the Dirac equation for our vector potential,

\[
(\gamma^\mu i\partial_\mu - \gamma^\mu eA_\mu - m) \psi = \left( \gamma_+ i\partial_+ + \gamma_- (i\partial_- + eE x_+) + \vec{\gamma} \cdot i\vec{\nabla} - m \right) \psi .
\]

(41)
Multiplication alternately with $\gamma_-$ and $\gamma_+$ gives

$$
i\partial_+\psi_+(x_+, x_-, \bar{x}) = \left( m + \gamma \cdot i\nabla \right) \frac{1}{2} \gamma_-\psi_-(x_+, x_-, \bar{x}), \quad (42)$$

$$
\left( i\partial_- + eE\bar{x}_+ \right) \psi_-(x_+, x_-, \bar{x}) = \left( m + \gamma \cdot i\nabla \right) \frac{1}{2} \gamma_+\psi_+(x_+, x_-, \bar{x}). \quad (43)
$$

Fourier transforming (42) and multiplying by $(m - \gamma \cdot \bar{k})\gamma_+/\omega^2$ gives,

$$
\Psi_-(x_+, k_+, \bar{k}) = \left( \frac{m - \gamma \cdot \bar{k}}{\omega^2} \right) \gamma_+ i\partial_+ \Psi_+(x_+, k_+, \bar{k}). \quad (44)
$$

It follows that $\Psi_-$ can be eliminated in favor of $\Psi_+$,

$$
\langle S \left| X_+ \left\{ \Psi_+\Psi_+^\dagger \right\} \right| S \rangle = \langle S \left| X_+ \left\{ \Psi_+(-i\partial_+)\Psi_+^\dagger \right\} \right| S \rangle \gamma_+ \left( \frac{m + \gamma \cdot \bar{q}}{\omega^2} \right), \quad (45)
$$

$$
\langle S \left| X_+ \left\{ \Psi_-\Psi_+^\dagger \right\} \right| S \rangle = \left( \frac{m - \gamma \cdot \bar{k}}{\omega^2} \right) \gamma_+ \langle S \left| X_+ \left\{ i\partial_+\Psi_+\Psi_+^\dagger \right\} \right| S \rangle, \quad (46)
$$

$$
\langle S \left| X_+ \left\{ \Psi_-\Psi_+^\dagger \right\} \right| S \rangle = \left( \frac{m - \gamma \cdot \bar{k}}{\omega^2} \right) \gamma_+ \langle S \left| X_+ \left\{ \partial_+\Psi_+\Psi_+^\dagger \right\} \right| S \rangle \times \gamma_+ \left( \frac{m + \gamma \cdot \bar{q}}{\omega^2} \right). \quad (47)
$$

The derivatives with respect to $x_+$ and $x'_+$ can be pulled outside the $x_+$-ordering symbol if we agree that they do not act on $\theta(\pm \Delta x_+)$. The delta functions in (37) set $q_+ = k_+$ and $\bar{q} = \bar{k}$ and the derivatives give,

$$
i\partial_+ \left[ \frac{k_+ + eE\bar{x}_+}{k_+ + eE\bar{x}'} \right]^{-\frac{\omega^2}{2\pi}} = \frac{1}{\pi\omega^2} \left[ \frac{k_+ + eE\bar{x}_+}{k_+ + eE\bar{x}_+} \right]^{-\frac{\omega^2}{2\pi}}, \quad (48)
$$

$$
-i\partial'_+ \left[ \frac{k_+ + eE\bar{x}_+}{k_+ + eE\bar{x}'} \right]^{-\frac{\omega^2}{2\pi}} = \frac{1}{2\pi\omega^2} \left[ \frac{k_+ + eE\bar{x}_+}{k_+ + eE\bar{x}_+} \right]^{-\frac{\omega^2}{2\pi}}. \quad (49)
$$

So relations (38)-(40) indeed follow from (37).

To identify the state $|S\rangle$ we must express the operators as functions of $(x_+, k_+, \bar{k})$ and functionals of the initial value operators. These initial value operators are the only true degrees of freedom of any Heisenberg operator. On the lightcone the “initial value surface” can be taken as $x_+ = 0$ with
Taking $L$ to $-\infty$ gives the following result for QED in this background [3],

\[
\Psi_+ (x_+, k_+, \bar{k}) = \left[ \frac{k_+ + eE x_+ + i\epsilon}{k_+ + i\epsilon} \right]^{i\lambda} \Xi_0 (k_+, \bar{k})
- \theta(k_+) \theta(-eE x_+ - k_+) \left[ \frac{k_+ + eE x_+ + i\epsilon}{i\epsilon} \right]^{i\lambda} \frac{\sqrt{2\pi\lambda}}{\Gamma(1 + i\lambda)} \Xi_\infty (k_+, \bar{k}),
\]

where $\lambda \equiv -\bar{\omega}^2/(2eE)$ and the initial value operators are,

\[
\Xi_0 (k_+, \bar{k}) \equiv \Psi_+ (0, k_+, \bar{k}),
\]

\[
\Xi_\infty (k_+, \bar{k}) \equiv \sqrt{\frac{2\pi}{\lambda}} \left( \frac{m - \gamma \cdot \bar{k}}{-eE} \right) \frac{i}{2} \gamma - \int d^2 \bar{x} e^{-i\bar{k} \cdot \bar{x}} \lim_{x_+ \to -\infty} e^{-ik_+ x_+} \psi_+ (-\frac{k_+}{eE}, x_-, \bar{x}).
\]

The initial value operators anti-commute canonically with their adjoints,

\[
\{ \Xi_0 (k_+, \bar{k}), \Xi_0^\dagger (q_+, \bar{q}) \} = (2\pi)^3 \delta(k_+ - q_+) \delta^2(\bar{k} - \bar{q}) \frac{1}{\sqrt{2}} P_+,
\]

\[
\{ \Xi_\infty (k_+, \bar{k}), \Xi_\infty^\dagger (q_+, \bar{q}) \} = (2\pi)^3 \delta(k_+ - q_+) \delta^2(\bar{k} - \bar{q}) \frac{1}{\sqrt{2}} P_+.
\]

The “0” operators anti-commute with the “\(\infty\)” ones by causality since the two surfaces are spacelike related for all finite $L$.

It is worth pausing at this point to comment on the significant features of our operator solution. First, note that for $k_+ > 0$ the mode functions experience a characteristic drop in amplitude as they evolve through the singular point at $x_+ = -k_+/eE$,

\[
\left| \left[ \frac{k_+ + eE x_+ + i\epsilon}{k_+ + i\epsilon} \right]^{i\lambda} \right| = \begin{cases} 1 & \forall x_+ < -k_+/eE \\ e^{-\pi\lambda} & \forall x_+ > -k_+/eE \end{cases}.
\]

The amplitude lost by $\Xi_0 (k_+, \bar{k})$ passes to $\Xi_\infty (k_+, \bar{k})$ by virtue of the identity [11],

\[
\left| \left[ \frac{k_+ + eE x_+ + i\epsilon}{i\epsilon} \right]^{i\lambda} \right| = \sqrt{\frac{2\pi\lambda}{\Gamma(1 + i\lambda)}} = \sqrt{1 - e^{-2\pi\lambda}},
\]

\[x_- > L \text{ and } x_+ > 0 \text{ with } x_- = L.\]
for $0 < k_+ < -eE x_+$. The physical interpretation of the amplitude drop is particle production. This is a discrete and instantaneous event on the lightcone. As each mode passes through its singularity at $x_+ = -k_+/eE$, the eigenvalue of $-i\partial_+$ changes sign and the operator coefficient switches interpretation from annihilator to creator. Since Heisenberg states are fixed a state which was originally empty in that mode seems, after singularity, to have acquired a particle with probability equal to the square of the post-singular amplitude (55).

The second point worthy of mention about (50) is that we cannot follow the usual lightcone practice, for nonzero mass and/or more than two space-time dimensions, of ignoring the “$\infty$” operators. These are always technically present in lightcone field theory, but they usually remain segregated to sector at $k_+ = 0$. In computing scattering amplitudes one can ignore this sector and recover the zero momentum limit instead by analytic continuation. We cannot get away with this for QED in a constant electric field. The physical momentum is the minimally coupled one, $p_+ \equiv k_+ + eE x_+ + eE x_+$, which reaches the far infrared near singularity. At this point operators from the surface at $x_- = -\infty$ can and do mix with the “$0$” operators in (50).

We should end this digression by noting that part of our operator solution (50) has been obtained in a different context by Srinivasan and Padmanabhan [5, 6]. The mode solution they give corresponds to the term proportional to $\Xi_0(k_+, \tilde{k})$, although without our $i\epsilon$ convention. They do not get the part proportional to $\Xi_\infty(k_+, \tilde{k})$. They employed a WKB approach to evolve around the singularity at $k_+ + eE x_+ = 0$, and they claim this results in a second term proportional to $\Xi_0(-k_+, \tilde{k})$. We have been unable to understand why the WKB technique applies to a first order evolution equation, nor can we reproduce their results.

Our operator solution (50) implies the following anti-commutation relation for $x_+ \neq x'_+$:

$$\{\Psi_+ (x_+, k_+, \tilde{k}) , \Psi^\dagger_+ (x'_+, q_+, \tilde{q}) \} = (2\pi)^3 \delta^3 (k - q) \frac{k_+ + eE x_+}{k_+ + eE x'_+} \frac{\lambda}{\sqrt{2}} P_+ .$$

Comparison with (57) reveals that $|S\rangle$ must obey,

$$\langle S | \Psi_+ (x_+, k_+, \tilde{k}) , \Psi^\dagger_+ (x'_+, q_+, \tilde{q}) | S\rangle = \theta(k_+ + eE x_+) \{\Psi_+ , \Psi^\dagger_+ \} ,$$

(58)
\[ \langle S \mid \Psi_+^\dagger (x', q', \bar{q}') , \Psi_+ (x_+, k_+, \bar{k}) \mid S' \rangle = \theta(-k_+ - eEx_+) \{ \Psi_+ , \Psi_+^\dagger \} . \] (59)

The state \(|S\rangle\) must therefore be annihilated by \(\Psi_+\) for \(k_+ + eEx_+ > 0\) and by \(\Psi_+^\dagger\) for \(k_+ + eEx_+ < 0\). In terms of the fundamental operators this translates to,

\[ \Xi_0 (k_+ + eEx_+ , \bar{k}) \mid S \rangle = 0 = \Xi_0^\dagger (-k_+ - eEx_+ , -\bar{k}) \mid S \rangle , \] (60)

\[ \Xi_\infty (k_+ + eEx_+ , \bar{k}) \mid S \rangle = 0 = \Xi_\infty^\dagger (-k_+ - eEx_+ , -\bar{k}) \mid S \rangle , \] (61)

for all \(k_+ > 0\). It is immediately apparent that \(|S\rangle\) is not a proper Heisenberg state in the sense of remaining fixed. We must rather use a different state for each value of \(x_+\) in order to recover Schwinger’s result. Hence it is only a Green’s function and not a true propagator.

We can also understand the curious result of Section 2 that Schwinger’s “propagator” gives a null result for the expectation value of the current operator at one loop. Recall from our operator solution that the eigenvalue of \(-i\partial_+\) on \(\Psi_+ (x_+, k_+, \bar{k})\) is negative for \(k_+ + eEx_+ > 0\) and positive for \(k_+ + eEx_+ < 0\). This means that at fixed \(x_+\) the electron annihilators are proportional to \(\Psi_+ (x_+, k_+, \bar{k})\), for all \(k_+ + eEx_+ > 0\), and the positron annihilators are proportional to \(\Psi_+^\dagger (x_+, k_+, \bar{k})\), again for \(k_+ + eEx_+ > 0\). Conditions (60-61) guarantee that these operators annihilate \(|S\rangle\), which means that the state is empty at \(x_+\). This is no contradiction with the fact that particle production really happens in this background because it is always possible to find a state which is empty at one particular instant. If the state were held fixed one would see a nonzero (actually infinite) current at later \(x_+\), but that is not what Schwinger’s “propagator” does. As \(x_+\) changes the state also changes to the one which happens to be instantaneously empty at the new value of \(x_+\). So one of course sees zero current. End of mystery.

5 A true propagator

We have been able to solve the Heisenberg operator equations for a vector potential \(A_- (x_+)\) which depends arbitrarily upon \(x_+\). Since there is no

\footnote{At certain points we do assume that \(eA_- (x_+)\) is an increasing function of \(x_+\). This could be avoided at the cost of more complicated expressions.}
significant simplification arising from the assumption that $A_-(x_+) = -Ex_+$, we begin by stating the more general solution. The derivation can be found elsewhere [3].

Modes still undergo particle production when the minimally coupled momentum $k_+ - eA_-(x_+)$ vanishes. We define this value of $x_+$ as $X(k_+)$,

$$k_+ = A_-(X(k_+)) .$$

Since the electric field is no longer necessarily constant we must generalize the definition of $\lambda$,

$$\lambda(k_+, \tilde{k}) \equiv \frac{\tilde{\omega}^2}{2eA'_-(X(k_+))} .$$

The mode functions require a similar generalization,

$$\mathcal{E}[A_-](y_+, x_+; k_+, \tilde{k}) \equiv \exp \left[ -\frac{i}{2} \tilde{\omega}^2 \int_{y_+}^{x_+} \frac{du}{k_+ - eA_-(u) + i\epsilon} \right] .$$

With these conventions the operator solution can be written as follows:

$$\Psi_+(x_+, k_+, \tilde{k}) = \mathcal{E}[A_-](0, x_+; k_+, \tilde{k}) \Xi_0(k_+, \tilde{k})$$

$$-\theta(k_+)\theta(eA_-(x_+) - k_+)\mathcal{E}(X(k_+), x_+; k_+, \tilde{k}) \frac{\sqrt{2\pi\lambda}}{\Gamma(1+i\lambda)} \Xi_\infty(k_+, \tilde{k}) ,$$

where the initial value operators are,

$$\Xi_0(k_+, \tilde{k}) \equiv \Psi_+(0, k_+, \tilde{k}) ,$$

$$\Xi_\infty(k_+, \tilde{k}) \equiv \sqrt{\frac{2\pi}{\lambda(k_+, \tilde{k})}} \left( \frac{m - \tilde{\gamma} \cdot \tilde{k}}{eA'_-(X(k_+))} \right) \frac{i}{2} \gamma \cdot \tilde{k} \right]$$

$$\times \int d^2\bar{x} e^{-ik_+x_+} \lim_{x_+ \to -\infty} e^{-iek_+x_+} \psi_-(X(k_+), x_-, \bar{x}) .$$

The canonical anti-commutation relations (53-54) are unchanged.

\footnote{We have actually quoted a simplified form in which a distributional limit was taken assuming that $k_+$ is somewhat displaced from the singular points at $k_+ = 0$ and $k_+ = eA_-(x_+).$ The more accurate expression must be used in taking derivatives with respect to $x_+.$}
It is natural to study the state that is empty at \( x_+ = 0 \), which means

\[
\Xi_0 (k_+ , \tilde{k}) \ket{\Omega} = 0 = \Xi_0^\dagger (-k_+ , -\tilde{k}) \ket{\Omega} ,
\]  

(68)

for all \( k_+ > 0 \). It is also natural to forbid particles from entering via the surface at \( x_- = -\infty \), which implies

\[
\Xi_\infty^\dagger (k_+ , \tilde{k}) \ket{\Omega} = 0 ,
\]  

(69)

also for \( k_+ > 0 \). The two operator orderings of \( \Psi_+ \) and \( \Psi_+^\dagger \) give,

\[
\langle \Omega | \Psi_+ (x_+, k_+, \tilde{k}) \Psi_+^\dagger (x'_+, q_+ , \tilde{q}) | \Omega \rangle = (2\pi)^3 \delta (k_+ - q_+) \delta^2 (\tilde{k} - \tilde{q}) \frac{P_+}{\sqrt{2}}
\]

\[
\quad \times \theta(k_+) \mathcal{E} (0, x_+ ; k_+, \tilde{k}) \mathcal{E}^* (0, x'_+ ; k_+, \tilde{k}) ,
\]

(70)

\[
\langle \Omega | \Psi_+^\dagger (x'_+, q_+, \tilde{q}) \Psi_+ (x_+, k_+, \tilde{k}) | \Omega \rangle = (2\pi)^3 \delta (k_+ - q_+) \delta^2 (\tilde{k} - \tilde{q}) \frac{P_+}{\sqrt{2}}
\]

\[
\quad \times \left\{ \theta(k_+) \theta (x_+ - X) \mathcal{E} (X, x_+ ; k_+, \tilde{k}) \mathcal{E}^* (X, x'_+ ; k_+, \tilde{k}) \left( e^{\pi \lambda} - e^{-\pi \lambda} \right)
\]

\[
\quad + \theta(-k_+) \mathcal{E} (0, x_+ ; k_+, \tilde{k}) \mathcal{E}^* (0, x'_+ ; k_+, \tilde{k}) \right\} .
\]

(71)

Now note that the conjugated mode function can be re-expressed using the Dirac identity,

\[
\mathcal{E}^* (0, x'_+ ; k_+, \tilde{k}) = \exp \left[ \frac{i}{2} \tilde{\omega}_2 \int_0^{x'_+} \frac{du}{k_+ - eA_-(u) - ie} \right] ,
\]

(72)

\[
\rightarrow \exp \left[ \frac{i}{2} \tilde{\omega}_2 \int_0^{x'_+} du \left\{ \frac{1}{k_+ - eA_-(u) + ie} + 2\pi i \delta (k_+ - eA_-(u)) \right\} \right] ,
\]

(73)

\[
= \exp \left[ \frac{i}{2} \tilde{\omega}_2 \int_0^{x'_+} \frac{d\theta}{k_+ - eA_-(u) + i\theta} \right] e^{-2\pi \lambda \theta (k_+)} .
\]

(74)

We can therefore write the various \( \mathcal{E} \times \mathcal{E}^* \) products as,

\[
\mathcal{E} (0, x_+ ; k_+, \tilde{k}) \mathcal{E}^* (0, x'_+ ; i k_+ , \tilde{k})
\]

\[
\rightarrow \mathcal{E} (x'_+, x_+ ; k_+ \tilde{k}) e^{-2\pi \lambda \theta (k_+)} ,
\]

(75)

\[
\mathcal{E} (X, x_+ ; k_+, \tilde{k}) \mathcal{E}^* (X, x'_+ ; k_+, \tilde{k})
\]

\[
\rightarrow \mathcal{E} (x'_+, x_+ ; k_+ \tilde{k}) e^{-\pi \lambda \theta (k_+)} .
\]

(76)
Assembling all the pieces gives the following result for the expectation value of the $x_+$-ordered product:

\[
\left< \Omega \left| X_+ \left\{ \Psi_+ \left( x_+, k_+, \vec{k} \right) \Psi_+^\dagger \left( x'_+, q_+, \vec{q} \right) \right\} \right| \Omega \right> = (2\pi)^3 \delta^3(k-q) \frac{P_+}{\sqrt{2}}
\]

\[
\times \mathcal{E} \left( x'_+, x_+; k_+ \vec{k} \right) \left\{ \theta(\Delta x_+)\theta(k_+)e^{-2\pi \lambda \theta(x'_+ - X)} - \theta(-\Delta x_+)\theta(-k_+) \\
- \theta(-\Delta x_+)\theta(k_+)\theta(x_+ - X) \left( 1 - e^{-2\pi \lambda} \right) \right\} .
\]

(77)

Expanding the exponential which contains the theta function,

\[
e^{-2\pi \lambda \theta(x'_+ - X)} = \theta(X - x'_+) + \theta(x'_+ - X)e^{-2\pi \lambda},
\]

(78)

and simplifying under the assumption that $eA_-(u)$ is an increasing function gives the following more compact result,

\[
\left< \Omega \left| X_+ \left\{ \Psi_+ \left( x_+, k_+, \vec{k} \right) \Psi_+^\dagger \left( x'_+, q_+, \vec{q} \right) \right\} \right| \Omega \right> = (2\pi)^3 \delta^3(k-q) \frac{P_+}{\sqrt{2}}
\]

\[
\times \mathcal{E} \left( x'_+, x_+; k_+ \vec{k} \right) \left\{ \theta(\Delta x_+)\theta(X - x'_+) - \theta(-\Delta x_+)\theta(X - x_+) \\
+ \theta(x_+ - X)\theta(x'_+ - X)\theta(k_+)e^{-2\pi \lambda} \right\} .
\]

(79)

We can still obtain the minus components from the plus ones,

\[
\Psi_- \left( x_+, k_+, \vec{k} \right) = \left( \frac{m - \vec{\gamma} \cdot \vec{k}}{\omega^2} \right)^{-1} \gamma_i \partial_i \Psi_+ \left( x_+, k_+, \vec{k} \right),
\]

(80)

so that relations (77-77) still determine the other components of the propagator from the ++ ones. So the full 2-point function is,

\[
\left< \Omega \left| X \left\{ \Psi \left( x_+, k_+, \vec{k} \right) \Psi^\dagger \left( x'_+, q_+, \vec{q} \right) \right\} \right| \Omega \right> = (2\pi)^3 \delta^3(k-q)\delta^3(k-q) \frac{\gamma_+ P_+}{\sqrt{2}}
\]

\[
\times \mathcal{E} \left( x'_+, x_+; k_+ \vec{k} \right) \left\{ \theta(\Delta x_+)\theta(X - x'_+) - \theta(-\Delta x_+)\theta(X - x_+) \\
+ \theta(x_+ - X)\theta(x'_+ - X)\theta(k_+)e^{-2\pi \lambda} \right\} \left\{ \frac{1}{\sqrt{2}} P_+ \\
+ \frac{P_+ \gamma_-}{2\sqrt{2}} \left( \frac{m + \vec{\gamma} \cdot \vec{k}}{k_+ - eA_-(x'_+) + i\epsilon} \right) + \frac{m - \vec{\gamma} \cdot \vec{k}}{k_+ - eA_-(x_+) + i\epsilon} \frac{\gamma_+ P_+}{2\sqrt{2}} \\
+ \frac{1}{2} \frac{\gamma_+ P_+}{\sqrt{2} \sqrt{T}} \left( k_+ - eA_-(x_+) + i\epsilon \right) \right\} .
\]

(81)
6 Discussion

We have shown that, if one computes the one loop expectation value of the current using Schwinger’s propagator, the result is zero. The reason for this emerges when one attempts to write the propagator as the expectation value of the time-ordered product of $\psi(x)\overline{\psi}(x')$ in the presence of some state. The only way to do so is with a state which changes as the time parameter ($x_+$) of the $\psi$ field does. At each different value of $x_+$ the appropriate state is the one which happens to be free of particles at that instant, so of course the current is zero.

Because of the special roles played by time and by the direction of the electric field, lightcone coordinates are particularly well suited to this problem. At the level of operators and states this implies a profound departure from the usual picture in which one imagines specifying a state on a surface of constant $x^0$. For a state specified instead on a surface of constant $x_+$ the phenomenon of pair production is a discrete and instantaneous event. For each mode of fixed $k_+$ it occurs when the minimally coupled momentum $p_+ = k_+ - eA(x_+)$ vanishes. This can be understood quite simply by representing the lightcone system in the standard way as the infinite boost limit of a conventional system in which the states are defined on surfaces of constant time \cite{2}. Particles produced with finite $p^3$ in that frame must have $p^3 = -\infty$ in the lightcone frame, and $p^3 = -\infty$ implies $p_+ = 0^+$ for a particle which is on shell \cite{3}.

A remarkable feature of our treatment is that explicit (and quite simple!) mode functions can be obtained for a background in which the electric field depends arbitrarily upon the lightcone coordinate $x_+$\footnote{The simplicity of the lightcone mode functions has been noted previously, for the special case of constant electric field, by Srinivasan and Padmanabhan \cite{5,6}. Note, however, that we do not agree with some aspects of their solution.}. We have in fact worked out the actual electron propagator for such a background in the presence of a state which is empty at $x_+ = 0$. Since neither the photon propagator nor the vertices of QED are affected by the background, expression (81) completes the Feynman rules with which one can compute the quantum electrodynamic back-reaction to any order. It is significant that we can get the propagator for a class of backgrounds general enough to include the actual solution as the electric field evolves under the impact of the current it generates.
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