RIGIDITY OF REDUCIBILITY OF GEVREY QUASI-PERIODIC COCYCLES ON $U(n)$

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RIGIDITY OF REDUCIBILITY
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Abstract. — We consider the reducibility problem of cocycles $(\alpha, A)$ on $T^d \times U(n)$ in Gevrey classes, where $\alpha$ is a Diophantine vector. We prove that, if a Gevrey cocycle is conjugated to a constant cocycle $(\alpha, C)$ by a suitable measurable conjugacy $(0, B)$, then for almost all $C$ it can be conjugated to $(\alpha, C)$ in the same Gevrey class, provided that $A$ is sufficiently close to a constant. If $B$ is continuous we obtain that it is Gevrey smooth. We consider as well the global problem of reducibility in Gevrey classes when $d = 1$.

Résumé (Rigidité de réductibilité des cocycles quasi-périodiques de Gevrey sur $U(n)$)

On considère le problème de la réductibilité de cocycles $(\alpha, A)$ sur $T^d \times U(n)$ dans les classes de Gevrey, où $\alpha$ est Diophantien. Si $A$ est proche d'une constante et le Gevrey cocycle $(\alpha, A)$ est conjugué au cocycle constant $(\alpha, C)$ par une conjugaison mesurable $(0, B)$, on montre que pour presque tous $C$ le cocycle peut être conjugué à $(\alpha, C)$ dans la même classe de Gevrey. Si $B$ est continue on obtient qu'elle est Gevrey. On considère aussi le problème de la réductibilité globale dans les classes de Gevrey dans le cas où $d = 1$.

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1. Introduction

This article is concerned with the reducibility of cocycles in Gevrey classes on the unitary group $U(n)$. A cocycle on $U(n)$ is a diffeomorphism of $T^d \times U(n)$, where $T^d$ is the torus $T^d = \mathbb{R}^d / \mathbb{Z}^d$, given by the skew-product $(\alpha, A): T^d \times C^n \to T^d \times C^n$, $(\theta, v) \mapsto (\theta + \alpha, A(\theta)v)$, where $\alpha \in T^d$ and $A: T^d \to U(n)$ is a map. The corresponding dynamics is defined by the iterates of the cocycle by composition $(\alpha, A)^n$, $n \in \mathbb{Z}$. We denote by $C^r(T^d, U(n))$ ($r = 0, 1, \ldots, \omega$) the set of all $C^r$ functions $A$. For any $\rho \geq 1$ and $L > 0$ we denote by $G^\rho_L(T^d, U(n))$ the class of Gevrey functions with an exponent $\rho$ and Gevrey constant $L$. A map $A \in C^\infty(T^d, U(n))$ belongs to that class if it satisfies (2.10) (see Section 2.2). Denote by $SW^\rho_G(T^d, U(n))$ the set of all Gevrey quasi-periodic cocycles on $U(n)$.

The dynamics is particularly simple if $(\alpha, A)$ is a constant cocycle. The cocycle $(\alpha, A)$ is said to be constant if $A$ is a constant matrix. Two cocycles $(\alpha, A), (\alpha, \tilde{A}) \in SW^\rho(T^d, U(n))$ are said to be conjugated if there exists $B: T^d \to U(n)$ such that $Ad(B)(\alpha, A) := (\alpha, B(\cdot + \alpha)^{-1}AB) = (\alpha, \tilde{A})$, which means that $B(\theta + \alpha)^{-1}A(\theta)B(\theta) = \tilde{A}(\theta)$ for any $\theta \in T^d$. The cocycle $(\alpha, A)$ is said to be reducible if it is conjugated to a constant one. We say also that the conjugation or the reducibility is Gevrey, $C^r$, or measurable, if $B$ belongs to the corresponding class of functions.

Reducibility problem of cocycles has been investigated for a long time. The local reducibility problem (the cocycle is close to a constant one) is usually studied using KAM-type iterations. In particular, Eliasson’s KAM method developed in [3] gives full-measure reducibility for generic one-parameter families of cocycles [2, 4, 10, 9, 5, 6]. The global reducibility problem (cocycles are no longer close to a constant one) has been studied by Avila, Krikorian and others. By means of a renormalization scheme Krikorian obtained a global density result for $C^\infty$ cocycles on $SU(2)$ [11] and also results for cocycles on $SL(2, \mathbb{R})$ [1, 12]. Almost reducibility for Gevrey cocycles has been studied by Chavaudret in [2].

The rigidity problem we are interested in, can be formulated as follows. Suppose that a Gevrey cocycle is measurably reducible. Is it also Gevrey reducible? In the case of $C^\infty$ or $C^r$ cocycles the rigidity problem has been investigated in [1, 12, 7, 6].
In this paper, we will focus our attention on the Gevrey case. We will prove a local rigidity result of reducibility in Gevrey classes which can be viewed as a Gevrey analogue of the main result in [7]. To this end we use techniques developed in [17]. When \( d = 1 \), the local result together with Krikorian’s renormalization scheme imply as in [11, 1] a global rigidity result for Gevrey quasi-periodic cocycles on \( T^1 \times U(n) \).

Why are we interested in Gevrey classes? Gevrey classes appear naturally in the KAM theory when dealing with Diophantine frequencies [16, 17]. They provide a natural framework for studying KAM systems, Birkhoff normal forms with an exponentially small reminder terms and the Nekhoroshev theory, and give an inside relation between these theories [14, 15, 16, 17]. One can consider as well the more general Roumieu classes of non-quasi-analytic functions. In the case of Bruno-Rüssmann arithmetic conditions we suggest that similar results hold in appropriate Roumieu spaces.

To formulate the main results we recall certain arithmetic conditions. Given \( \gamma > 0 \) and \( \tau > d - 1 \), we say that \( \alpha \in \mathbb{R}^d \) is \((\gamma, \tau)\)-Diophantine if
\[
|e^{2\pi i \langle k, \alpha \rangle} - 1| > \frac{\gamma^{-1}}{|k|^\tau}, \quad 0 \neq k \in \mathbb{Z}^d,
\]
and we denote by \( \text{DC} (\gamma, \tau) \) the set of all such Diophantine vectors. Hereafter, \( i := \sqrt{-1} \) stands for the imaginary unit. It is well known that \( \text{DC} (\tau) := \bigcup_{\gamma > 0} \text{DC} (\gamma, \tau) \) is a set of full Lebesgue measure. For any given \( \alpha \in \mathbb{R}^d \), we denote by \( \Upsilon (\alpha; \chi, \nu) \) the set of all vectors \((\phi_1, \ldots, \phi_n) \in \mathbb{R}^n\), satisfying
\[
|\langle k, \alpha \rangle + \phi_p - \phi_q - j| \geq \frac{\chi}{(1 + |k|)^\nu}
\]
for any \( p \neq q \in \{1, 2, \ldots, n\} \), \( k \in \mathbb{Z}^d \) and \( j \in \mathbb{Z} \). The set
\[
\Upsilon (\alpha) := \bigcup_{\chi, \nu > 0} \Upsilon (\alpha; \chi, \nu)
\]
has full Lebesgue measure in \( \mathbb{R}^n \). Recall that the Lie group \( U(n) \) consists of all \( A \in GL(n, \mathbb{C}) \) satisfying \( A^* A = I \). Hereafter, \( I \) stands for the identity matrix and \( A^* \) is the adjoint matrix to \( A \) in \( M_n = M_n (\mathbb{C}) \). The corresponding Lie algebra \( u(n) \) is the set of \( X \in gl(n, \mathbb{C}) \) satisfying \( X^* + X = 0 \). Any \( A \in U(n) \) is diagonalizable, and the set of eigenvalues of \( A \), denoted by \( \text{Spec} (A) \), is a subset of \( \{ z \in \mathbb{C} : |z| = 1 \} \). Denote by \( \Sigma (\alpha; \chi, \nu) \) the set of \( A \in U(n) \) with spectrum \( \text{Spec} (A) := \{ \lambda_1, \lambda_2, \ldots, \lambda_n \} \) satisfying
\[
|\lambda_p - \lambda_q e^{2\pi i \langle k, \alpha \rangle}| \geq \frac{\chi}{(1 + |k|)^\nu}
\]