Pseudo-Riemannian Lie groups of modified $H$-type

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Abstract. We define a class of Riemannian and pseudo-Riemannian 2-step nilpotent Lie groups with nondegenerate centers (Definition 2.2) that generalize the $H$-type groups of Kaplan [8, 9, 10]. Examples are given and geometric properties are investigated.

1 Introduction

Let $N$ be a 2-step nilpotent Lie group with Lie algebra $n$. Let $[,]$ denote the Lie bracket on $n$, and write $n = \mathfrak{z} \oplus \mathfrak{v}$, where $\mathfrak{z}$ is the center of $n$. Let $\langle , \rangle$ denote both a pseudo-Riemannian inner product on $n$ making $\mathfrak{z}$ nondegenerate, and the induced left-invariant metric tensor on $N$.

The adjoint representation of $n$ on itself, $\text{ad} : n \times n \to n$, is given by $\text{ad}_x = [x, ]$. For every $z \in \mathfrak{z}$ define a skew-symmetric linear transformation $j(z)$ on $\mathfrak{v}$ by

$$j(z) = \text{ad}^*_z z,$$

where $\text{ad}^*$ is the adjoint of $\text{ad}$ with respect to $\langle , \rangle$. That is,

$$\langle \text{ad}_x y, z \rangle = \langle y, j(z)x \rangle$$

for all $x, y \in \mathfrak{v}$ and all $z \in \mathfrak{z}$.

These $j$-maps simultaneously encode the algebraic and geometric information about the (pseudo-) Riemannian 2-step nilpotent Lie group $(N, \langle , \rangle)$. There is a special family of groups, originally studied by Kaplan [9], for which these maps act like scaled anti-involutions.

Definition 1.1 A 2-step nilpotent Lie group $N$ with left-invariant Riemannian metric $\langle , \rangle$ is said to be of $H$-type if and only if for each $z \in \mathfrak{z}$,

$$\langle j(z)x, j(z)y \rangle = \|z\|^2 \langle x, y \rangle$$

for all $x, y \in \mathfrak{v}$. Alternatively, for each $z \in \mathfrak{z}$, $j(z)^2 = -\|z\|^2 \text{Id}|_{\mathfrak{v}}$. 

\[\Box\]
Definition 1.2 A pseudo-Riemannian 2-step nilpotent Lie group $N$ with nondegenerate center is said to be of *pseudo-H-type* if and only if for each $z \in \mathfrak{z}$,
\[
\langle j(z)x, j(z)y \rangle = \|z\|^2 \langle x, y \rangle
\]
for all $x, y \in \mathfrak{v}$. Alternatively, for each $z \in \mathfrak{z}$, $j(z)^2 = -\|z\|^2 \text{Id}|_{\mathfrak{v}}$.

\[\diamondsuit\]

2 Definition and Examples

Definitions 1.1 and 1.2 can be generalized by replacing the quadratic form determined by $\langle \ , \ \rangle_{\mathfrak{z}}$ with another quadratic form on $\mathfrak{z}$. J. Lauret [11] had this idea in the Riemannian case, and gave the following definition.

Definition 2.1 A Riemannian 2-step nilpotent Lie group is said to be a *modified H-type group* if for any nonzero $a \in \mathfrak{z}$, $j(a)^2 = \lambda(a)\text{Id}|_{\mathfrak{z}}$ for some $\lambda(a) < 0$.

\[\diamondsuit\]

A priori, the $\lambda$ in this definition is just a function on $\mathfrak{z}$. However, the properties of the $j$-maps actually imply that $\lambda$ is a nondegenerate, negative-definite quadratic form on $\mathfrak{z}$.

We generalize this definition by not only considering pseudo-Riemannian metrics on $N$, but also allowing quadratic forms on $\mathfrak{z}$ of the most general type. That is, we allow that the associated symmetric bilinear form be indefinite and/or degenerate.

Definition 2.2 Let $(N, \langle \ , \ \rangle)$ be a 2-step nilpotent Lie group with left-invariant metric $\langle \ , \ \rangle = \langle \ , \ \rangle_{\mathfrak{z}} + \langle \ , \ \rangle_{\mathfrak{v}}$ making $\mathfrak{z}$ nondegenerate, and let $\varphi$ be a quadratic form on $\mathfrak{z}$. We shall say that $(N, \langle \ , \ \rangle, \varphi)$ is of *modified H-type* if and only if for each $z \in \mathfrak{z}$, the map $j(z) \in \text{End}(\mathfrak{v})$ satisfies
\[
\langle j(z)x, j(z)y \rangle_{\mathfrak{v}} = \varphi(z) \langle x, y \rangle_{\mathfrak{v}}
\]
for all $x, y \in \mathfrak{v}$. Alternatively, since the $j$-maps are skew-adjoint this property is characterized by
\[
j(z)^2 = -\varphi(z) \text{Id}|_{\mathfrak{v}}
\]
for all $z \in \mathfrak{z}$.

\[\diamondsuit\]

Remark 2.1 Even when the metric is Riemannian, Definition 2.2 is more general (although not necessarily more interesting) than Definition 2.1. This is illustrated below by Example 2.4.

\[\diamondsuit\]
Example 2.1 If \( \langle \cdot, \cdot \rangle \) is a left-invariant Riemannian metric on \( N \) and \( \varphi = \| \cdot \|_j^2 \), then \( (N, \langle \cdot, \cdot \rangle) \) is an \( H \)-type group in the sense of Kaplan [9]. If \( \langle \cdot, \cdot \rangle \) is a left-invariant pseudo-Riemannian metric on \( N \) and \( \varphi = \| \cdot \|_j^2 \), then \( (N, \langle \cdot, \cdot \rangle) \) is a pseudo-\( H \)-type group in the sense of Ciatti [1].

Example 2.2 Consider the Heisenberg algebra \( h_3 \) spanned by the vectors \( \{e_1, e_2, e_3\} \) with nontrivial bracket \([e_1, e_2] = e_3\). Clearly the center is \( z = \text{span} \{e_3\} \). Consider the Lorentzian inner product on \( h_3 \) given on this basis by
\[-\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = \langle e_3, e_3 \rangle = 1.\]
Now let \( (H_3, \langle \cdot, \cdot \rangle) \) be the 3-dimensional Heisenberg group with the left-invariant Lorentz metric induced by this inner product on \( h_3 \). The \( j \)-maps on this group satisfy
\[j(z)^2 = \|z\|_j^2 \text{Id}_v,\]
so this group is not of pseudo-\( H \)-type. However, taking \( \varphi = -\| \cdot \|_j^2 \) one sees that it is of modified \( H \)-type. Therefore the class of modified \( H \)-type metrics is strictly larger than the class of pseudo-\( H \)-type metrics.

Example 2.3 In the Riemannian case, the generalised Heisenberg groups satisfy
\[j(z)^2 = -4c \|z\|_j^2 \text{Id}_v\]
for all \( z \in \mathfrak{z}, c \in (0, \infty) \). See [14, 6], for example. These groups are of modified \( H \)-type with \( \varphi = 4c \| \cdot \|_j^2 \), but they are only of \( H \)-type when \( c = \frac{1}{4} \). Thus the class of modified \( H \)-type metrics is strictly larger than the class of \( H \)-type metrics.

This example may be extended straightforwardly to pseudo-Riemannian groups to obtain the pseudo-Riemannian generalised Heisenberg groups.

Each of the previous examples constructs modified \( H \)-type metrics on groups that are already known to admit \( H \)-type or pseudo-\( H \)-type metrics. The next examples show that the class of groups admitting modified \( H \)-type metrics is strictly larger than those admitting (pseudo-) \( H \)-type metrics.

Consider the 4-dimensional group obtained by a trivial central extension of the Heisenberg group, \( N = H_3(\mathbb{R}) \times \mathbb{R} \). The Lie algebra \( \mathfrak{n} = \mathfrak{h}_3 \oplus \mathbb{R} \) has basis \( \{e_0, e_1, e_2, e_3\} \) with nontrivial bracket \([e_1, e_2] = e_3\). Thus \( \mathfrak{z} = \text{span} \{e_0, e_3\} \) and \( \mathfrak{v} = \text{span} \{e_1, e_2\} \). The following examples show that there are modified \( H \)-type metrics on \( N \) of each possible signature.
Moreover, this group does not admit an $H$-type or pseudo-$H$-type metric. Indeed, it does not admit an $H$-type metric because it is singular, and it does not admit a pseudo-$H$-type metric because the minimum dimension of a pseudo-$H$-type group with 2-dimensional center is six; viz. $\Box$.

Example 2.4 Consider the Riemannian metric $g_0$ on $N = H_3(\mathbb{R}) \times \mathbb{R}$ making the basis $\{e_0, e_1, e_2, e_3\}$ orthonormal. That is,

$$\langle e_i, e_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$  

The $j$-maps on this group are given by

$$j(e_0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad j(e_3) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

so that their squares satisfy $j(e_0)^2 = 0$ and $j(e_3)^2 = -\text{Id}|_\mathfrak{z}$. The maps $j(e_0 + e_3) = j(e_3)$ and $j(e_0 - e_3) = -j(e_3)$, so that the $j(e_0 + e_3)^2 = j(e_0 - e_3)^2 = -\text{Id}|_\mathfrak{z}$. Let $\varphi_0$ be the quadratic form on $\mathfrak{z}$ satisfying

$$\varphi_0(e_0) = 0, \quad \varphi_0(e_3) = \varphi_0(e_0 + e_3) = \varphi_0(e_0 - e_3) = 1.$$  

The induced symmetric bilinear form on $\mathfrak{z}$ satisfies

$$(e_0, e_3)_{\varphi_0} = \frac{1}{4}(\varphi_0(e_0 + e_3) - \varphi_0(e_0 - e_3)) = \frac{1}{4}(1 - 1) = 0,$$

so that its matrix representation is

$$\Phi_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$  

This bilinear form is exactly what one should expect since $N$ is a trivial extension of $H_3(\mathbb{R})$. We see that $j(z)^2 = -\varphi_0(z)\text{Id}|_\mathfrak{z}$ for every $z \in \mathfrak{z}$. Therefore $(N, g_0, \varphi_0)$ is a modified $H$-type group with degenerate quadratic form $\varphi_0$. $\Diamond$

Example 2.5 Now consider the left-invariant Lorentzian metric $g_1$ on $N$ given by

$$\langle e_0, e_3 \rangle = \langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = 1,$$

with all other combinations zero. The $j$-maps for this group are

$$j(e_0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad j(e_3) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$
so that \( j(e_0)^2 = -\text{Id}|_{\mathfrak{z}} \) and \( j(e_3)^2 = 0 \). Taking a pseudo-orthonormal basis of \( \mathfrak{z} \),
\[
  u_1 = \frac{1}{\sqrt{2}} (e_0 + e_3), \quad u_2 = \frac{1}{\sqrt{2}} (e_0 - e_3),
\]
we obtain \( j(u_1) = j(u_2) = \frac{1}{\sqrt{2}} j(e_0) \), so that \( j(u_1)^2 = j(u_2)^2 = -\frac{1}{2} \text{Id}|_{\mathfrak{z}} \).
Moreover, \( u_1 + u_2 = \sqrt{2} e_0 \) and \( u_1 - u_2 = -\sqrt{2} e_3 \), so that \( j(u_1 + u_2)^2 = -2 \text{Id}|_{\mathfrak{z}} \) and \( j(u_1 - u_2)^2 = 0 \). Let \( \varphi_1 \) be the quadratic form on \( \mathfrak{z} \) satisfying
\[
  \varphi_1(u_1) = \varphi_1(u_2) = \frac{1}{2}, \quad \varphi_1(u_1 + u_2) = 2, \quad \text{and} \quad \varphi(u_1 - u_2) = 0.
\]
The induced symmetric bilinear form on \( \mathfrak{z} \) satisfies
\[
  (u_1, u_2)_{\varphi_1} = \frac{1}{4} (\varphi_1(u_1 + u_2) - \varphi_1(u_1 - u_2)) = \frac{1}{4} (2 - 0) = \frac{1}{2}.
\]
Therefore its matrix representation is
\[
  \Phi_1 = \begin{pmatrix} 
  1 & \frac{1}{2} \\
  \frac{1}{2} & \frac{1}{2}
  \end{pmatrix},
\]
and \( \varphi_1 \) is degenerate. However, \( j(z)^2 = -\varphi_1(z) \text{Id}|_{\mathfrak{z}} \) for all \( z \in \mathfrak{z} \), so \((N, g_1, \varphi_1)\) is indeed a modified \( H \)-type group.

\[\text{Example 2.6} \quad \text{Next consider the left-invariant neutral metric } g_2 \text{ on } N \text{ given on the Lie algebra } \mathfrak{n} \text{ by} \]
\[
  \langle e_0, e_3 \rangle = \langle e_1, e_2 \rangle = 1,
\]
with all other combinations zero. Before computing the \( j \)-maps, change the basis of \( \mathfrak{n} \) to
\[
  u_1 = \frac{1}{\sqrt{2}} (e_0 + e_3), \quad u_2 = \frac{1}{\sqrt{2}} (e_0 - e_3), \quad v_1 = \frac{1}{\sqrt{2}} (e_1 + e_2), \quad v_2 = \frac{1}{\sqrt{2}} (e_1 - e_2).
\]
The basis \( \{u_1, u_2, v_1, v_2\} \) is pseudo-orthonormal with
\[
  \langle u_1, u_1 \rangle = -\langle u_2, u_2 \rangle = 1, \quad \text{and} \quad \langle v_1, v_1 \rangle = -\langle v_2, v_2 \rangle = 1.
\]
The Lie bracket is now given by \([v_1, v_2] = \frac{1}{\sqrt{2}} (u_1 - u_2)\), \(\mathfrak{z} = \text{span}\{u_1, u_2\}\), and \(\mathfrak{v} = \text{span}\{v_1, v_2\}\). The \(j\)-maps are given with respect to these bases by

\[
j(u_1) = j(u_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},
\]

so that

\[
j(u_1 + u_2) = \sqrt{2} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.
\]

We see that \(j(u_1)^2 = j(u_2)^2 = \frac{1}{2} \text{Id}|_\mathfrak{v}\), and \(j(u_1 + u_2)^2 = 2 \text{Id}|_\mathfrak{v}\). Let \(\varphi_2\) be the quadratic form on \(\mathfrak{z}\) satisfying

\[
\varphi_2(u_1) = \varphi_2(u_2) = -\frac{1}{2}, \quad \text{and} \quad \varphi_2(u_1 + u_2) = -2.
\]

Then the associated bilinear form \((\ , \ )_{\varphi_2}\) satisfies

\[
(u_1, u_2)_{\varphi_2} = \frac{1}{2} (\varphi_2(u_1 + u_2) - \varphi_2(u_1) - \varphi_2(u_2)) = \frac{1}{2} (-2 + \frac{1}{2} + \frac{1}{2}) = -\frac{1}{2},
\]

and its matrix representation is

\[
\Phi_2 = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.
\]

We see that \(j(z)^2 = -\varphi_2(z) \text{Id}|_\mathfrak{v}\) for all \(z \in \mathfrak{z}\), so that \((N, g_2, \varphi_2)\) is a modified \(H\)-type group.

\[\Box\]

**Example 2.7** Consider the left-invariant metric \(g_3\) given on \(\mathfrak{n}\) by

\[
\langle e_0, e_0 \rangle = \langle e_3, e_3 \rangle = \langle e_1, e_2 \rangle = 1.
\]

The \(j\)-maps with respect to the basis \(\{e_1, e_2\}\) are

\[
j(e_0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad j(e_3) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

so that \(j(e_0)^2 = 0\) and \(j(e_3)^2 = \text{Id}|_\mathfrak{v}\). Notice that \(j(e_0 + e_3) = j(e_3)\), so that \(j(e_0 + e_3)^2 = \text{Id}|_\mathfrak{v}\) as well. Let \(\varphi_3\) be the quadratic form on \(\mathfrak{z}\) satisfying

\[
\varphi_3(e_0) = 0, \quad \varphi_3(e_3) = \varphi_3(e_0 + e_3) = -1.
\]

The bilinear form \((\ , \ )_{\varphi_3}\) satisfies

\[
(e_0, e_3)_{\varphi_3} = \frac{1}{2} (\varphi_3(e_0 + e_3) - \varphi_3(e_0) - \varphi_3(e_3)) = \frac{1}{2} (-1 - 0 + 1) = 0,
\]

so its matrix representation is

\[
\Phi_3 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Therefore \((N, g_3, \varphi_3)\) is a modified \(H\)-type group.

\[\Box\]
3 Connection and Curvatures

In this section we give all relevant curvature formulas for modified $H$-type groups. Let $(N, \langle \cdot, \cdot \rangle, \varphi)$ be a modified $H$-type Lie group. Let $(\cdot, \cdot)_\varphi$ be the symmetric bilinear form on $\mathfrak{z}$ obtained by polarizing the quadratic form $\varphi$. Then for all $e, e' \in \mathfrak{v}$ and all $z, z' \in \mathfrak{z}$,

\begin{align}
\langle j(z)e, j(z)e' \rangle_0 &= \varphi(z) \langle e, e' \rangle_0, \\
\langle j(z)e, j(z')e \rangle_0 &= (z, z')_\varphi \langle e, e \rangle_0, \quad \text{and} \\
j(z)j(z') + j(z')j(z) &= -2 (z, z')_\varphi \text{Id}_\mathfrak{v}.
\end{align}

(3) (4) (5)

These identities as well as the following formulas for the Levi-Civita connection and curvatures may be deduced from [4]. We prove only the curvature formulas below that are unique to modified $H$-type groups.

**Theorem 3.1** Let $(N, \langle \cdot, \cdot \rangle)$ be a 2-step nilpotent Lie group with left-invariant pseudo-Riemannian metric making the center nondegenerate. For all $e, e' \in \mathfrak{v}$ and $z, z' \in \mathfrak{z}$, the Levi-Civita connection on $N$ is given by

\begin{align}
\nabla_z z' &= 0, \\
\nabla_z e &= \nabla_e z = -\frac{1}{2} j(z)e, \\
\nabla_e e' &= \frac{1}{2} [e, e'].
\end{align}

(6) (7) (8)

**Theorem 3.2** Let $(N, \langle \cdot, \cdot \rangle)$ be a 2-step nilpotent Lie group with left-invariant pseudo-Riemannian metric making the center nondegenerate. For all $e, e', e'' \in \mathfrak{v}$ and $z, z', z'' \in \mathfrak{z}$, the Riemann curvature tensor on $N$ is given by

\begin{align}
R(z, z')z'' &= 0, \\
R(z, z')e &= \frac{1}{4} [z, z'] e, \\
R(z, e)z' &= \frac{1}{4} j(z)j(z')e, \\
R(z, e)e' &= \frac{1}{4} [e, j(z)e'], \\
R(e, e')z &= -\frac{1}{4} ([e, j(z)e'] + [j(z)e, e'])), \\
R(e, e')e'' &= \frac{1}{4} (j ([e, e''])e' - j ([e', e'']) e) + \frac{1}{2} j ([e, e')] e''.
\end{align}

(9) (10) (11) (12) (13) (14)
Theorem 3.3 Let $(N, \langle \, , \rangle, \varphi)$ be a modified $H$-type group, and suppose that $e, e' \in \mathfrak{v}$, $z, z' \in \mathfrak{z}$ are pseudo-orthonormal. The sectional curvature is given by

\begin{align*}
K(z, z') &= 0, \quad (15) \\
K(z, e) &= \frac{1}{4} \varepsilon_z \varphi(z), \quad (16) \\
K(e, e') &= -\frac{3}{4} \varepsilon_e \varepsilon_{e'} \| [e, e'] \|_2^2, \quad (17)
\end{align*}

where $\varepsilon_\bullet = \| \bullet \|_2^2 = \pm 1$.

Proof: Only equation (16) differs from the usual ones in [4]. Let $z \in \mathfrak{z}$, $e \in \mathfrak{v}$, such that $\varepsilon_z = \| z \|_\mathfrak{z}^2 = \pm 1$ and $\varepsilon_e = \| e \|_\mathfrak{v}^2 = \pm 1$. Then by (3),

\begin{align*}
K(z, e) &= \frac{1}{4} \varepsilon_z \varepsilon_e \langle j(z)e, j(z)e \rangle_\mathfrak{v} \\
&= \frac{1}{4} \varepsilon_z \varepsilon_e \varphi(z) \| e \|_\mathfrak{v}^2 \\
&= \frac{1}{4} \varepsilon_z \varphi(z).
\end{align*}

$\square$

Let $\{z_1, \ldots, z_p\}$ be a pseudo-orthonormal basis for the center $(\mathfrak{z}, \langle \, , \rangle_\mathfrak{z})$, and let $\{e_1, \ldots, e_m\}$ be a pseudo-orthonormal basis for $(\mathfrak{v}, \langle \, , \rangle_\mathfrak{v})$. Define

$$\xi := \sum_{k=1}^p \| z_k \|_\mathfrak{z}^2 \varphi(z_k).$$

(18)

This number does not depend on the choice of basis of $\mathfrak{z}$ as it is the trace of the operator $\mathcal{K} : z \mapsto \| z \|_\mathfrak{z}^2 \varphi(z)$.

Theorem 3.4 Let $(N, \langle \, , \rangle, \varphi)$ be a modified $H$-type group, and let $e, e' \in \mathfrak{v}$, $z, z' \in \mathfrak{z}$. The Ricci curvature is given by

\begin{align*}
\text{Ric}(e, z) &= 0, \quad (19) \\
\text{Ric}(e, e') &= -\frac{\xi}{2} \langle e, e' \rangle_\mathfrak{v}, \quad (20) \\
\text{Ric}(z, z') &= \frac{m}{4} \langle z, z' \rangle_\varphi. \quad (21)
\end{align*}

Proof: Equation (19) is true for all pseudo-Riemannian 2-step nilpotent groups with nondegenerate centers. Using the pseudo-orthonormal bases of
and letting \(z, z' \in \mathfrak{z}, e, e' \in \mathfrak{v}\), we compute

\[
\text{Ric}(e, e') = -\frac{1}{2} \sum_{k=1}^{p} \| z_k \|^2_\mathfrak{z} \langle j(z_k)e, j(z_k)e' \rangle_\mathfrak{v}
= -\frac{1}{2} \sum_{k=1}^{p} \| z_k \|^2_\mathfrak{z} \varphi(z_k) \langle e, e' \rangle_\mathfrak{v}
= -\frac{1}{2} \langle e, e' \rangle_\mathfrak{v} \sum_{k=1}^{p} \| z_k \|^2_\mathfrak{z} \varphi(z_k)
= -\varepsilon \langle e, e' \rangle_\mathfrak{v},
\]

and

\[
\text{Ric}(z, z') = \frac{1}{4} \sum_{i=1}^{m} \| e_i \|^2_\mathfrak{v} \langle j(z)e_i, j(z')e_i \rangle_\mathfrak{v}
= \frac{1}{4} \sum_{i=1}^{m} \| e_i \|^2_\mathfrak{v} \langle z, z' \rangle_\varphi \langle e_i, e_i \rangle_\mathfrak{v}
= \frac{m}{4} \langle z, z' \rangle_\varphi.
\]

The Ricci operator \(\text{Rc} : \mathfrak{h} \to \mathfrak{h}\) is defined by

\[
\langle \text{Rc} x, y \rangle = \text{Ric}(x, y)
\]

for all \(x, y \in \mathfrak{h}\). By (19), the Ricci operator preserves the splitting \(\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}\). By (21), the Ricci operator restricted to \(\mathfrak{z}\) is determined by

\[
\langle \text{Rc} z, z' \rangle_\mathfrak{z} = \frac{m}{4} \langle z, z' \rangle_\varphi.
\]

Denote by \(\text{Id}|^\dagger_\varphi\), the adjoint of the identity map on \(\mathfrak{z}\) with respect to the metrics \(\langle , \rangle_\varphi\) and \(\langle , \rangle_\mathfrak{z}\). This adjoint is defined by the equation

\[
\langle \text{Id}|^\dagger_\varphi z, z' \rangle_\mathfrak{z} = \langle z, \text{Id} z' \rangle_\varphi.
\]

Comparing (22) and (23), \(\text{Rc}|_\mathfrak{z}\) is proportional to \(\text{Id}|^\dagger_\varphi\). We deduce the following theorem regarding the Ricci operator of modified \(H\)-type groups.
Theorem 3.5  Let \((N, \langle , \rangle, \varphi)\) be a modified \(H\)-type group. The Ricci operator preserves the splitting \(n = \mathfrak{z} \oplus \mathfrak{v}\), and is given on each factor by

\[
\begin{align*}
\operatorname{Rc}|_{\mathfrak{z}} &= -\frac{\xi}{2} \operatorname{Id}|_{\mathfrak{z}}, \\
\operatorname{Rc}|_{\mathfrak{v}} &= \frac{m}{4} \operatorname{Id}|_{\varphi}.
\end{align*}
\]

(24) \hspace{1cm} (25)

Remark 3.1  In order to better understand the operator \(\operatorname{Id}|_{\varphi}^{\dagger}\), it is useful to work with matrix representations. Let \(E\) be the matrix representing \(\langle , \rangle_{\mathfrak{z}}\) with respect to the basis \(\{z_1, \ldots, z_p\}\), so that \(E = \operatorname{diag}\{\varepsilon_1, \ldots, \varepsilon_p\}\). Let \(\Phi\) be the matrix representation of \((, \varphi)(, \varphi)\) with respect to the same basis. In general \(\Phi\) is symmetric, but may be singular.

Equation (22) may be written in matrix form as \((\operatorname{Rc}|_{\mathfrak{z}} z)^{T} Ez' = \frac{m}{4} z^{T} \Phi z'\). Taking advantage of the facts that \(E^{T} = E^{-1}\) and \(\Phi^{T} = \Phi\), basic matrix manipulations yield

\[
\begin{align*}
(\operatorname{Rc}|_{\mathfrak{z}} z)^{T} E &= \frac{m}{4} z^{T} \Phi \\
(\operatorname{Rc}|_{\mathfrak{z}} z)^{T} &= \frac{m}{4} z^{T} \Phi E^{-1} \\
(\operatorname{Rc}|_{\mathfrak{z}} z)^{T} &= \left[\frac{m}{4}(\Phi E^{-1})^{T} z\right]^{T}
\end{align*}
\]

This implies that \(\operatorname{Rc}|_{\mathfrak{z}} = \frac{m}{4}(E^{-1})^{T} \Phi E^{-1}\). That is, the matrix representation of \(\operatorname{Rc}|_{\mathfrak{z}}\) with respect to the orthonormal basis \(\{z_1, \ldots, z_p\}\) is

\[
\operatorname{Rc}|_{\mathfrak{z}} = \frac{m}{4} E^{-1} \Phi.
\]

(26)

Notice that \(\xi := \sum \varepsilon_i \varphi(z_i)\) is the trace of the matrix \(E^{-1} \Phi\).

If \(N\) is of (pseudo-) \(H\)-type, then \(\Phi = E\), \(E^{-1} \Phi = I_p\), and we recover all of the previously known results about the Ricci operator. \(\diamondsuit\)

Remark 3.2  Clearly \(\operatorname{Rc}(z) = 0\) implies that either \(z \in \ker \Phi\) or \(\Phi z\) is null with respect to the metric on \(\mathfrak{z}\). This fact will become useful later. \(\diamondsuit\)

Scalar curvature is the trace of the Ricci curvature operator.

Theorem 3.6  Let \((N, \langle , \rangle, \varphi)\) be a modified \(H\)-type group. Since the metric \(\langle , \rangle\) is left-invariant, the scalar curvature is constant on \(N\), and is given by

\[
S = -\frac{m \xi}{4}.
\]

(27)
Proof: Using the pseudo-orthonormal bases of \([18]\), we compute

\[
S = -\frac{1}{4} \sum_{i=1}^{m} \sum_{k=1}^{p} \|z_k\|_j^2 \|e_i\|_o^2 \langle j(z_k)e_i, j(z_k)e_i \rangle_o
\]

\[
= -\frac{1}{4} \sum_{i=1}^{m} \sum_{k=1}^{p} \|z_k\|_j^2 \varphi(z_k)
\]

\[
= -\frac{m\xi}{4}.
\]

\[\square\]

3.1 Examples

We now return to the examples of Section 2. We compute the constant \(\xi\) and the matrix representation of the Ricci operator, and compute its eigenvalues for each group.

Example 3.1 Consider the group \((N, g_0, \varphi_0)\) of Example 2.4. The matrices representing \(g_0\) and \((, \varphi_0)\) on \(\mathfrak{z}\) are

\[
E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \Phi_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Thus \(E^{-1}\Phi_0 = \Phi_0\), \(\xi = \text{tr}(\Phi_0) = 1\), and the scalar curvature of \((N, g_0)\) is \(S = -\frac{1}{2}\). The Ricci operator is given by

\[
\text{Rc}|_o = -\frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and}
\]

\[
\text{Rc}|_z = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

It has eigenvalues \(-\frac{1}{2}\) on \(v\) and \(0, \frac{1}{2}\) on \(\mathfrak{z}\).

\[\diamond\]

Example 3.2 Consider the modified \(H\)-type group \((N, g_1, \varphi_1)\) of Example 2.5. The matrix representation of \(g_1\) on \(\mathfrak{z}\) with respect to the ordered basis \(\{u_1, u_2\}\) is

\[
E = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]
while the matrix representation of \((\ ,\ )_{\varphi_1}\) with respect to the same basis is

\[
\Phi_1 = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2}
\end{pmatrix}.
\]

The matrix \(E^{-1}\Phi_1\) is then

\[
E^{-1}\Phi_1 = \frac{1}{2} \begin{pmatrix}
1 & 1 \\
-1 & -1
\end{pmatrix}.
\]

Clearly \(\xi = \text{tr} (E^{-1}\Phi_1) = 0\), so that \((N, g_1)\) is scalar flat. This also implies that the Ricci operator restricted to \(\mathfrak{v}\) is zero. Moreover, the Ricci operator on \(\mathfrak{z}\), given by the matrix

\[
\text{Rc}|_{\mathfrak{z}} = \frac{1}{4} \begin{pmatrix}
1 & 1 \\
-1 & -1
\end{pmatrix}.
\]

is 2-step nilpotent. Therefore the Ricci operator on the entire Lie algebra \(\mathfrak{n}\) has only one eigenvalue, \(\lambda = 0\).

The space \((N, g_1)\) was also found to be scalar flat but not Ricci flat in [5].

\[\diamond\]

**Example 3.3** Next consider the group \((N, g_2, \varphi_2)\) of Example 2.6. The matrix representations of \(g_2\) and \((\ ,\ )_{\varphi_2}\) with respect to the basis \(\{u_1, u_2\}\) are

\[
E = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}, \quad \text{and} \quad \Phi_2 = -\frac{1}{2} \begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}.
\]

The matrix \(E^{-1}\Phi_2\) is then

\[
E^{-1}\Phi_2 = -\frac{1}{2} \begin{pmatrix}
1 & 1 \\
-1 & -1
\end{pmatrix}.
\]

As in Example 3.2, we see that \(\xi = \text{tr} (E^{-1}\Phi_2) = 0\), so the space \((N, g_2)\) has zero scalar curvature. This also implies that the Ricci operator vanishes on \(\mathfrak{v}\). Again in analogy with Example 3.2, the Ricci operator restricted to \(\mathfrak{z}\) is 2-step nilpotent,

\[
\text{Rc}|_{\mathfrak{z}} = -\frac{1}{4} \begin{pmatrix}
1 & 1 \\
-1 & -1
\end{pmatrix}.
\]

Therefore \(\text{Re}\) has only one eigenvalue, \(\lambda = 0\).
Example 3.4 Consider the group \((N, g_3, \varphi_3)\) of Example 2.7. The matrices for \(g_3\) and \((, )_{\varphi_3}\) on \(\mathfrak{z}\) are given by

\[
E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \Phi_3 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Thus \(E^{-1}\Phi_3 = \Phi_3\), and \(\xi = \text{tr}(\Phi_3) = -1\). Therefore the scalar curvature of \((N, g_3)\) is \(S = +\frac{1}{2}\). The Ricci operator on \(\mathfrak{n}\) is given by

\[
\text{Rc}|_{\mathfrak{v}} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \text{Rc}|_{\mathfrak{z}} = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.
\]

The eigenvalues of \(\text{Rc}\) are then \(\frac{1}{2}\) on \(\mathfrak{v}\) and \(0, -\frac{1}{2}\) on \(\mathfrak{z}\).

4 Geometry of Modified \(H\)-Type Groups

In this section we investigate the geometric consequences of Definition 2.2. Most of these properties can be stated in terms of the Ricci operator of Theorem 3.5.

4.1 Isometry groups

Let \(\mathfrak{n}^C = \mathfrak{z}^C \oplus \mathfrak{v}^C\) be the complexification of the Lie algebra \(\mathfrak{n}\). Since the Ricci operator of Theorem 3.5 respects the splitting \(\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}\), then its complexification respects the induced splitting of \(\mathfrak{n}^C\).

Recall the following lemma of [5].

Lemma 4.1 ([5], Lemma 2) Let \((N, \langle , \rangle)\) be a 2-step nilpotent Lie group such that \(\langle , \rangle\) is a pseudo-Riemannian left-invariant metric for which the center is nondegenerate. Assume

\[
\mathfrak{v}^C = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_j}, \\
\mathfrak{z}^C = V_{\lambda_{j+1}} \oplus \cdots \oplus V_{\lambda_s},
\]

for the different eigenvalues \(\lambda_1, \ldots, \lambda_s\) of the Ricci operator \(\text{Rc}\), where \(V_{\lambda_i}\) are the eigenspaces corresponding to \(\lambda_i\). Then every isometry of \(N\) preserves the splitting \(TN = \mathfrak{v}N \oplus \mathfrak{z}N\); that is, \(\text{Iso}^{\text{spl}}(N) = \text{Iso}(N)\).
Since \( \text{Iso}^{\text{aut}}(N) = \text{Iso}^{\text{spl}}(N) \) for all groups with nondegenerate centers, this actually implies that 
\( \text{Iso}^{\text{aut}}(N) = \text{Iso}^{\text{spl}}(N) = \text{Iso}(N) \) for all such groups satisfying the hypotheses of the lemma.

For modified \( H \)-type groups, the restriction of \( R_c \) to \( v \) has only one eigenvalue, \(-\xi^2\). As long as \(-\xi^2\) is not also an eigenvalue of \( R_c|_z \), then the eigenspace decomposition of \( n \) with respect to the Ricci operator respects the splitting \( n = z \oplus v \). That is, if \( V_{\lambda_i} \) are the eigenspaces for the different eigenvalues \( \lambda_i \) of \( R_c|_z \), then
\[
\begin{align*}
\mathfrak{z}^C &= V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_s}, \\
\mathfrak{v}^C &= V_{-\xi^2}.
\end{align*}
\]

Invoking Lemma 4.1 and recalling that modified \( H \)-type groups have non-degenerate centers by definition, we obtain the following theorem about the isometry groups of modified \( H \)-type groups.

**Theorem 4.2** Let \((N, \langle , \rangle, \varphi)\) be a modified \( H \)-type group. If \(-\xi^2\) is not an eigenvalue of \( R_c|_z \), then the following isometry groups coincide: 
\( \text{Iso}^{\text{aut}}(N) = \text{Iso}^{\text{spl}}(N) = \text{Iso}(N) \).

**Remark 4.1** The groups \((N, g_1)\) and \((N, g_2)\) of Examples 2.5, 3.2, 2.6, and 3.3 do not satisfy the hypothesis of the theorem; viz. [5].

### 4.2 Totally geodesic subgroups

Let \( z \in \mathfrak{z} \) and \( x \in v \) be non-zero vectors. In \( H \)-type groups, the space \( n' := \text{span} \{z, x, j(z)x\} \) is always a totally geodesic subalgebra of \( n \), making \( N' := \exp n' \) a totally geodesic subgroup of \( N \). Modified \( H \)-type groups have the following analogue.

**Theorem 4.3** Let \((N, \langle , \rangle, \varphi)\) be a modified \( H \)-type Lie group. Let \( z \in \mathfrak{z}, \ x \in v \) be non-zero vectors with \( \|x\|_v^2 \neq 0 \), and consider the subspace \( n' = \text{span} \{z, x, j(z)x\} \). The submanifold \( N' = \exp n' \) is a totally geodesic subgroup of \( N \) if and only if \( z \) is not in the kernel of \( R_c|_z \) but is an eigenvector of \( R_c|_z \).

**Proof:** The Levi-Civita connection on \( n' \) is given by
\[
\begin{align*}
\nabla_z x &= \nabla_x z = -\frac{1}{2}j(z)x, \\
\nabla_z j(z)x &= \frac{1}{2} \varphi(z)x, \quad \text{and} \\
\nabla_x j(z)x &= \frac{1}{2} [x, j(z)x].
\end{align*}
\]
Thus, we must show that $z' := [x, j(z)x]$ is proportional to $z$ if and only if $z$ is an eigenvector of $Rc|_\mathfrak{z}$. For $a \in \mathfrak{z}$, we compute

$$
\langle [x, j(z)x], a \rangle_{\mathfrak{z}} = \langle \text{ad}_x j(z)x, a \rangle_{\mathfrak{z}} \\
= \langle j(z)x, j(a)x \rangle_{\mathfrak{z}} \\
= \|x\|^2_{\mathfrak{v}} \langle z, a \rangle_{\phi}. 
$$

(31)

Therefore, $z' = \|x\|^2_{\mathfrak{v}} \text{Id}|^\uparrow_{\mathfrak{v}} z$. By (25), $z' = \|x\|^2_{\mathfrak{v}} \frac{1}{m} Rc z$, and $z'$ is proportional to $z$ if and only if $z$ is an eigenvector of $Rc$.

□

Corollary 4.4 Let $(N, \langle , \rangle, \phi)$ be a modified $H$-type Lie group. Let $z \in \mathfrak{z}$ and $x \in \mathfrak{v}$ be non-zero vectors. Suppose that either $z \in \ker Rc|_\mathfrak{z}$ or $\|x\|^2_{\mathfrak{v}} = 0$ (or both), and consider the subspace $\mathfrak{n}' = \text{span} \{z, x, j(z)x\}$. The submanifold $N' = \exp \mathfrak{n}'$ is totally geodesic, but not a subgroup of $N$.

Proof: By (30), \(\nabla_z x, \nabla_x z, \text{ and } \nabla_z j(z)x\) are all in $\mathfrak{n}'$. By (31), \(\nabla_x j(z)x = 0\). Therefore $N'$ is totally geodesic in $N$. However, $\mathfrak{n}'$ is not a subalgebra since $[x, j(z)x] = 0$.

4.3 Geodesics

In this section we give explicit formulas for the geodesics of a modified $H$-type group in terms of the $j$-maps, quadratic form $\phi$, and the Ricci operator on $\mathfrak{z}$. We follow the calculations of [4], making particular use of one result that is stated here as a lemma. First, some preliminaries.

Suppose $N$ is a connected, simply connected 2-step nilpotent Lie group. Let $I$ be a real interval containing zero, and let $\gamma : I \to N$ be a geodesic such that $\gamma(0) = 1 \in N$ and $\dot{\gamma}(0) = z_0 + x_0 \in \mathfrak{z} \oplus \mathfrak{v} = \mathfrak{n}$. Since the exponential map is a diffeomorphism for simply connected nilpotent Lie groups, $\gamma(t) = \exp(z(t) + x(t))$.

Let $J = j(z_0)$ denote the skew adjoint transformation of $\mathfrak{v}$ determined by the initial condition, and let $\mathfrak{v}^1 = \ker J$. The skew-adjointness of $J$ implies that $\mathfrak{v} = \mathfrak{v}^1 \oplus \mathfrak{v}^2$ is an orthogonal direct sum, where $\mathfrak{v}^2 = \mathfrak{v}/\mathfrak{v}^1$. Notice that $J$ is invertible on $\mathfrak{v}^2$. Decompose $x_0 = x_1 + x_2$, with $x_i \in \mathfrak{v}^i$.

Now let $\{\theta_1, \ldots, \theta_k\}$ be the distinct nonzero eigenvalues of $J^2$. Decompose $\mathfrak{v}^2$ as the orthogonal direct sum $\bigoplus_{j=1}^k \mathfrak{w}_j$, where $J$ leaves each $\mathfrak{w}_j$ invariant and $J^2|_{\mathfrak{w}_j} = \theta_j \text{Id}|_{\mathfrak{w}_j}$. Write $x_2 = \sum_{j=1}^k w_j$ with each $w_j \in \mathfrak{w}_j$.

Under these assumptions, we state the aforementioned result of [4].
Lemma 4.5 ([4], Prop 4.15) If $N$ is a 2-step nilpotent Lie group with non-degenerate center, and $J^2$ diagonalizes, then

\begin{align}
  z(t) &= t z_1(t) + z_2(t), \\
  x(t) &= t x_1(t) + (e^{tJ} - I) J^{-1} x_2,
\end{align}

where

\begin{align}
  z_1(t) &= z_0 + \frac{1}{2} \left[ x_1, (e^{tJ} + I) J^{-1} x_2 \right] + \frac{1}{2} \sum_{j=1}^{k} \left[ J^{-1} w_j, w_j \right], \\
  z_2(t) &= \left[ x_1, (I - e^{tJ})(J^{-2} x_2) \right] + \frac{1}{2} \left[ e^{tJ} J^{-1} x_2, J^{-1} x_2 \right] \\
  &\quad - \frac{1}{2} \sum_{i \neq j} \frac{1}{\theta_j - \theta_i} \left( [e^{tJ} w_i, e^{tJ} J^{-1} w_j] - [e^{tJ} w_i, e^{tJ} w_j] \right) \\
  &\quad + \frac{1}{2} \sum_{i \neq j} \frac{1}{\theta_j - \theta_i} \left( [J w_i, J^{-1} w_j] - [w_i, w_j] \right).
\end{align}

For modified $H$-type groups, the situation is simplified considerably. The map $J^2$ is diagonalizable for all $z_0 \in \mathfrak{z}$ and has only one eigenvalue, $-\varphi(z_0)$. This eliminates the need for any $\mathfrak{w}$-decomposition, thus eliminating many of the terms in $z_1$ and $z_2$ of the lemma. Further, the maps $J^{-1}$ and $e^{tJ}$ may each be written simply in terms of $\varphi(z_0)$ and $j(z_0)$. The following lemma is straightforward to prove.

Lemma 4.6 Let $(N, \langle \ , \ \rangle, \varphi)$ be a modified $H$-type Lie group. Let $z_0 \in \mathfrak{z}$ such that $\varphi_0 := \varphi(z_0) \neq 0$, and denote by $J$ the transformation $j(z_0) \in \text{End}(\mathfrak{v})$. Then

\begin{align}
  J^{-1} &= -\frac{1}{\varphi_0} J, \quad \text{and} \\
  e^{tJ} &= \begin{cases} 
  \cos \left( t \sqrt{|\varphi_0|} \right) I + \frac{1}{\sqrt{|\varphi_0|}} \sin \left( t \sqrt{|\varphi_0|} \right) J & \text{if } \varphi_0 > 0, \\
  \cosh \left( t \sqrt{|\varphi_0|} \right) I + \frac{1}{\sqrt{|\varphi_0|}} \sinh \left( t \sqrt{|\varphi_0|} \right) J & \text{if } \varphi_0 < 0,
\end{cases} \\
  e^{tJ} J^{-1} &= \begin{cases} 
  -\cos \left( t \sqrt{|\varphi_0|} \right) \frac{\varphi_0}{\sqrt{|\varphi_0|}} J + \frac{1}{\sqrt{|\varphi_0|}} \sin \left( t \sqrt{|\varphi_0|} \right) I & \text{if } \varphi_0 > 0, \\
  -\cosh \left( t \sqrt{|\varphi_0|} \right) \frac{\varphi_0}{\sqrt{|\varphi_0|}} J + \frac{1}{\sqrt{|\varphi_0|}} \sinh \left( t \sqrt{|\varphi_0|} \right) I & \text{if } \varphi_0 < 0.
\end{cases}
\end{align}
If \( \varphi(z_0) = 0 \), then \( \ker J = v \) so that \( x_1 = x_0 \) and \( x_2 = 0 \). The formulas of Lemma 4.5 reduce to

\[
\begin{align*}
  z(t) &= tz_0, \\
  x(t) &= tx_0.
\end{align*}
\]

If \( \varphi(z_0) \neq 0 \), then \( \ker J = 0 \) so that \( x_1 = 0 \) and \( x_2 = x_0 \). As mentioned above, this \( x_2 \) does not need to be decomposed any further since \( J^2 \) has only one eigenvalue. In this case, the formulas of Lemma 4.5 reduce to

\[
\begin{align*}
  z(t) &= t\left(z_0 + \frac{1}{2} [J^{-1}x_0, x_0]\right) + \frac{1}{2} \left[e^{tJ}J^{-1}x_0, J^{-1}x_0\right], \\
  x(t) &= (e^{tJ} - I)J^{-1}x_0.
\end{align*}
\]

Carefully applying the formulas of Lemma 4.6 replacing \( J \) with \( j(z_0) \), and recalling from the proof of Theorem 4.3 that \([x_0, j(z_0)x_0] = \frac{4}{m} \|x_0\|^2 \text{Rc}_z x_0\), one obtains the following theorem.

**Theorem 4.7** Suppose \((N, \langle , \rangle, \varphi)\) is a modified \( H \)-type Lie group. Let \( \gamma : I \to N, \gamma(t) = \exp(z(t) + x(t)) \), be a geodesic with \( \gamma(0) = 1 \in N \) and \( \dot{\gamma}(0) = z_0 + x_0 \in n \), and suppose that both \( z_0 \) and \( x_0 \) are nonzero. Let \( \varphi_0 = \varphi(z_0) \).

If \( \varphi(z_0) = 0 \), then \( \gamma \) is given by

\[
\begin{align*}
  z(t) &= tz_0, \\
  x(t) &= tx_0; \quad (34)
\end{align*}
\]

if \( \varphi(z_0) > 0 \), then \( \gamma \) is given by

\[
\begin{align*}
  z(t) &= t\varphi_0 + \left(\frac{4 \sin(t\sqrt{\varphi_0}) - 2t\sqrt{\varphi_0}}{m(\varphi_0)^{3/2}}\right)\|x_0\|^2 \text{Rc}_z x_0, \\
  x(t) &= \left(\frac{1 - \cos(t\sqrt{\varphi_0})}{\varphi_0}\right) j(z_0)x_0 + \left(\frac{\sin(t\sqrt{\varphi_0})}{\sqrt{\varphi_0}}\right)x_0; \quad (37)
\end{align*}
\]
and if \( \varphi(z_0) < 0 \), then \( \gamma \) is given by

\[
    z(t) = t z_0 + \left( \frac{4 \sinh \left( t \sqrt{|\varphi_0|} \right) - 2t \sqrt{|\varphi_0|}}{m \varphi_0 \sqrt{|\varphi_0|}} \right) \|x_0\|_0^2 \text{Rc}|_{z_0}, \tag{38}
\]

\[
    x(t) = \left( \frac{1 - \cosh \left( t \sqrt{|\varphi_0|} \right)}{\varphi_0} \right) j(z_0)x_0 + \left( \frac{\sinh \left( t \sqrt{|\varphi_0|} \right)}{\sqrt{|\varphi_0|}} \right) x_0. \tag{39}
\]

Recalling that it is possible for \( \|x_0\|_0^2 \text{Rc}|_{z_0} \) to vanish while neither \( x_0 \) nor \( z_0 \) are zero, we obtain the following Corollary.

**Corollary 4.8** Suppose \( (N, \langle , \rangle, \varphi) \) is a modified H-type Lie group. Let \( \gamma : I \to N, \gamma(t) = \exp(z(t) + x(t)), \) be a geodesic with \( \gamma(0) = 1 \) and \( \dot{\gamma}(0) = z_0 + x_0 \in \mathfrak{n} \), supposing that both \( z_0 \) and \( x_0 \) are nonzero. If \( \|x_0\|_0^2 = 0 \) or \( z_0 \in \ker(\text{Rc}|_{z_0}) \), then the 3-component of \( \gamma \) is simply given by \( z(t) = t z_0 \). \( \square \)

Just by looking at the formulas for the geodesics in Theorem 4.7 it is clear that every geodesic lives in a submanifold \( M = \exp m \), where \( m = \text{span} \{ z_0, x_0, j(z_0)x_0, \text{Rc}(z_0) \} \). If \( z_0 \) is an eigenvector of \( \text{Rc} \) corresponding to a non-zero eigenvalue and \( x_0 \) is non-null, then by the previous section \( M \) is a totally geodesic subgroup that is isomorphic to the Heisenberg group \( H_3 \).

If either \( x_0 \) is null or \( \text{Rc}(z_0) = 0 \), then by Corollary 4.4 \( [x_0, j(z_0)x_0] = 0 = \text{Rc}(z_0), m = \text{span} \{ z_0, x_0, j(z_0)x_0 \} \), and \( M \) is totally geodesic but not a subgroup. In fact, it is a copy of \( \mathbb{R}^3 \).

If \( x_0 \) is non-null and \( z_0 \) is not in \( \ker(\text{Rc}) \) but is also not an eigenvector of \( \text{Rc} \), then \( M \) is a subgroup isomorphic to \( H_3(\mathbb{R}) \times \mathbb{R} \). Indeed, by the previous section, \( [x_0, j(z_0)x_0] = \frac{4}{m} \|x_0\|_0^2 \text{Rc}z_0 \). Thus \( z_0 \) plays the role of the central extension.

### 4.4 Sectional curvatures of semi-central planes

In the Riemannian case, it is shown in [14] that the generalized Heisenberg groups of Example 2.3 are characterized among nilpotent Lie groups by the following property:

Every plane \( \Pi \) spanning one central and one non-central direction has sectional curvature \( K(\Pi) = c \).
In particular, every semi-central plane $\Pi$ in an $H$-type group has sectional curvature $K(\Pi) = \frac{1}{4}$. In this section we investigate the modified $H$-type analogue to this result.

Let $(N, \langle \ , \ \rangle, \varphi)$ be a modified $H$-type Lie group, and $\Pi = \text{span} \{z, x\}$ be a nondegenerate plane spanned by $z \in \mathfrak{z}$, $x \in \mathfrak{v}$. Since $\Pi$ is assumed to be nondegenerate, we may assume that $\|z\|^2 = \varepsilon_z = \pm 1$ and $\|x\|^2 = \varepsilon_x = \pm 1$. The sectional curvature of $\Pi$ is given in (16) by

$$K(\Pi) = \frac{1}{4}K(z) = \frac{1}{4}\varepsilon_z \varphi(z).$$

If $\varphi(z) = \varepsilon_z 4c$, $c \neq 0$, as in Example 2.3, then every nondegenerate semi-central plane has curvature $K(\Pi) = c$, as in the Riemannian case. Indeed,

$$K(\Pi) = \frac{1}{4}\varepsilon_z \varphi(z) = \frac{1}{4}\varepsilon_z^2 4c = c. \quad (40)$$

In particular, nondegenerate semi-central planes in pseudo-$H$-type groups have sectional curvature $K(\Pi) = \frac{1}{4}$.

The converse also holds.

**Theorem 4.9** Let $(N, \langle \ , \ \rangle)$ be a 2-step nilpotent Lie group with left-invariant metric $\langle \ , \ \rangle$ making the center nondegenerate. If the sectional curvature of every nondegenerate semi-central plane $\Pi$ is constant, $K(\Pi) = c$, then $(N, \langle \ , \ \rangle)$ is a modified $H$-type group with quadratic form $\varphi = 4c \|\cdot\|^2|_\mathfrak{z}$.

**Proof:** Let $z \in \mathfrak{z}$ and $x \in \mathfrak{v}$ such that $\varepsilon_z = \|z\|^2|_\mathfrak{z} = \pm 1$ and $\varepsilon_x = \|x\|^2|_\mathfrak{v} = \pm 1$, and consider the plane $\Pi = \text{span} \{z, x\}$. The sectional curvature of $\Pi$ is given in (16) by

$$K(z, x) = \frac{1}{4}\varepsilon_z \varepsilon_x \langle j(z)x, j(z)x \rangle_\mathfrak{v}. $$

Solving for $\langle j(z)x, j(z)x \rangle_\mathfrak{v}$ yields

$$\langle j(z)x, j(z)x \rangle_\mathfrak{v} = 4\varepsilon_z \varepsilon_x K(z, x).$$

Recalling that $\varepsilon_x = \langle x, x \rangle_\mathfrak{v}$, the assumption $K(\Pi) = c$ implies that

$$\langle j(z)x, j(z)x \rangle_\mathfrak{v} = 4c \varepsilon_z \langle x, x \rangle_\mathfrak{v}$$

for all unit vectors $z \in \mathfrak{z}$, $x \in \mathfrak{v}$. We deduce that $j(z)^2 = -4c \varepsilon_z \text{Id}|_\mathfrak{v}$ for all unit vectors in $\mathfrak{v}$. By linearity, since $\mathfrak{z}$ is nondegenerate, we obtain that $j(z)^2 = -4c \|z\|^2|_\mathfrak{z} \text{Id}|_\mathfrak{v}$ for all $z \in \mathfrak{z}$. Therefore $(N, \langle \ , \ \rangle, 4c \|\cdot\|^2|_\mathfrak{z})$ is a modified $H$-type group. \square
Corollary 4.10 The $H$-type and pseudo-$H$-type groups are characterized, among 2-step nilpotent Lie groups, by the following property: Every nondegenerate semi-central plane has sectional curvature $K = \frac{1}{4}$.

In general, it is possible for modified $H$-type groups to have semi-central planes that are flat. Indeed, if $\varphi(z') = 0$ but $\|z'\|^2 \neq 0$, then any semi-central plane in the direction of $z'$ has $K(\Pi) = 0$.

Remark 4.2 The sectional curvature of nondegenerate semi-central planes depends only on the central direction $z$ and the quadratic form $\varphi$, $K(z, x) = \frac{1}{4}z \varphi(z)$. Therefore, the quadratic form $\varphi$ is given by

$$\varphi(z) = 4 \|z\|^2 K(z, \overline{v})$$

for all non-null vectors in $\mathfrak{z}$, where $\overline{v}$ represents all non-null vectors in $\mathfrak{v}$. This further illustrates the close relationship between the quadratic form $\varphi$ and the sectional curvature of $N$.

4.5 Nilsolitons

A pseudo-Riemannian metric on a 2-step nilpotent Lie group is said to be a nilsoliton if there exists a constant $c \in \mathbb{R}$ such that the operator

$$D := Rc + c \cdot Id$$

is a derivation of $\mathfrak{n}$. That is, $D$ must satisfy

$$D [x, y] = [Dx, y] + [x, Dy]$$

for all $x, y \in \mathfrak{n}$. By Theorem 3.5, the Ricci operator on a modified $H$-type group $(\mathbb{N}, \langle , , \rangle, \varphi)$ is given on $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$ by

$$Rc = \begin{pmatrix} \frac{m}{4}Id_{\mathfrak{z}} & 0 \\ 0 & -\frac{\xi}{2}Id_{\mathfrak{v}} \end{pmatrix}.$$ 

For $D$ to be a derivation, the following equation must be satisfied for all $x, y \in \mathfrak{v}$.

$$Rc|_\mathfrak{z}([x, y]) = (-\xi + c) [x, y]$$

This occurs if and only if the derived algebra $\mathfrak{n}' = [\mathfrak{n}, \mathfrak{n}] = [\mathfrak{v}, \mathfrak{v}]$ is contained in a single eigenspace of $Rc|_\mathfrak{z}$. Indeed, if $\lambda$ is the corresponding eigenvalue of $Rc|_\mathfrak{z}$, then put $c = \lambda + \xi$.

We have proved the following.
Theorem 4.11 Let \((N, \langle , \rangle, \varphi)\) be a modified \(H\)-type group. The metric \(\langle , \rangle\) is a nilsoliton if and only if the derived algebra \([\mathfrak{v}, \mathfrak{v}]\) is contained in a single eigenspace of the Ricci operator \(Rc|_{\mathfrak{j}}\).

As a corollary we recover a well known result about (pseudo-) \(H\)-type groups [12, 13].

Corollary 4.12 Every \(H\)-type and pseudo-\(H\)-type metric is a nilsoliton.

Proof: The Ricci operator for an \(H\)-type or pseudo-\(H\)-type group is given by
\[
Rc = \begin{pmatrix}
\frac{m}{4} \text{Id}|_{\mathfrak{j}} & 0 \\
0 & -\frac{2}{x} \text{Id}|_{\mathfrak{v}}
\end{pmatrix},
\]
so that every \(z \in \mathfrak{j}\) is an eigenvector of \(Rc|_{\mathfrak{j}}\). □

Given a nilsoliton of modified \(H\)-type, we can construct many.

Corollary 4.13 If a modified \(H\)-type group admits a nilsoliton, then every trivial central extension also admits a nilsoliton.

Proof: Let \((N, \langle , \rangle, \varphi)\) be a modified \(H\)-type group such that \(\langle , \rangle\) is a nilsoliton. Let \(m\) be a real vector space endowed with an inner product \(\langle , \rangle|_m\) of any signature, and consider the trivial central extension of \(n\) given by \(\bar{n} = n \oplus m\). The simply connected Lie group \(\bar{N} = \exp \bar{n}\) endowed with the left-invariant metric \(\langle , \rangle + \langle , \rangle|m\) is a modified \(H\)-type algebra with \(\varphi(m) = 0\). The Ricci operator on the center remains unchanged on \(\mathfrak{j}\), so \([\mathfrak{v}, \mathfrak{v}]\) remains contained in a single eigenspace of \(Rc\). □

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