Interpreting a concurrent \(\lambda\)-calculus in differential proof nets - Extended version

Yann Hamdaoui
IRIF, Univ. Paris Diderot
yann.hamdaoui@irif.fr

Abstract

In this paper, we show how to interpret a language featuring concurrency, references and replication into proof nets, which correspond to a fragment of differential linear logic. We prove a simulation and adequacy theorem. A key element in our translation are routing areas, a family of nets used to implement communication primitives which we define and study in detail.

2012 ACM Subject Classification Dummy classification

Keywords and phrases linear logic concurrency simulation

Digital Object Identifier [10.4230/LIPIcs.CVIT.2016.23]

1 Introduction

The distinctive feature of Linear Logic [5] (LL) is to be a resource-aware logic, which explains its success as a tool to study computational processes. The native language for LL proofs - or rather, programs - are proof nets, a graph representation endowed with a local and asynchronous cut-elimination procedure. The fine-grained computations and the explicit management of resources in LL make it an expressive target to translate various computational primitives. Girard provided two translations in its original paper, later clarified as respectively a call-by-value and call-by-name translation of the \(\lambda\)-calculus [11]. PCF [9] has also been considered. Other works have tackled the intricate question of modeling side-effects. State has been considered in a \(\lambda\)-calculus with references [16]. Another direction which has been explored is concurrency and non-determinism. The extension of LL with dual structural rules, Differential Linear Logic [4], happens to be powerful enough to accommodate non-determinism as demonstrated by the encoding of a \(\pi\)-calculus without replication [3].

These translations allow to relate and compare different computational frameworks. They benefit from the consequent work on the semantic and the dynamic of LL which has been carried for thirty years. They are also of practical interest: the proof net representation naturally leads to parallel implementations [8, 13, 15] and forms the basis for concrete operational semantics in the form of token-based automata [2, 7]. In this regard, one may consider LL as a “functional assembly language” with a diverse collection of semantics and abstract machines.

While \(\lambda\)-calculus is a fundamental tool in the study and design of functional programming languages, mainstream programming languages are pervaded with features that enable productive development such as support for parallelism and concurrency, communication primitives, imperative references, etc. Most of them imply side-effects, which are challenging to model, to compose, and to reason about. While some have been investigated individually through the lens of LL, our goal is to go one step further by modeling a language featuring at the same time concurrency, references and replication. The constructs involved in the translation are inspired by the approach in [3] for concurrency and non-determinism coupled with a monadic translation [13] for references. Our goal is to exploit the ability of proof nets
to enable independent computations to be done in parallel without breaking the original operational semantic. The translation we propose can be seen as a compilation from a global shared memory model to a local message passing one, in line with proof nets philosophy.

**Routing areas**

In a concurrent imperative language, references are a means to exchange information between threads. In order to implement communication primitives in proof nets, we use and extend the concept of communication areas introduced in [3]. A communication area is a particular proof net whose external interface, composed of wires, is split between an equal number of inputs and outputs. Inputs and outputs are grouped by pairs representing a plug on which other nets can be connected. They are simple yet elegant devices, whose role is similar to the one of a network switch which connects several agents. Connecting two communication areas yields a communication area again: this key feature enables their use as modular blocks that can be combined into complex assemblies. In this paper, we introduce routing areas, which allows a finer control on the wiring diagram. They are parametrized by a relation which specifies which inputs and outputs are connected. Extending our network analogy, communication areas are rather hubs: they simply broadcast every incoming message to any connected agent. On the other hand, routing areas are more like switches: they are able to choose selectively the recipients of messages depending on their origin. Routing areas are subject to atomic operations that decompose the operation of connecting communication areas. These operations also have counterparts on relations.

We show that routing areas are sufficient to actually describe all the normal forms of the fragment of proof nets composed solely of structural rules. The algebraic description of routing areas then provides a semantic for this fragment.

**A Concurrent λ-calculus**

We consider a paradigmatic concurrent lambda-calculus with higher-order references – λC below – which has been introduced by Amadio in [1]. It is a call-by-value λ-calculus extended with:

- a notion of threads and an operator ‖ for parallel composition of threads,
- two terms set(r, V) and get(r), to respectively assign a value to and read from a reference,
- special threads r ⇐ V, called stores, accounting for assignments.

In this language, the stores are global and cumulative: their scope is the whole program, and each assignment adds a new binding that does not erase the previous ones. Reading from a store is a non deterministic process that chooses a value among the available ones. References are able to handle an unlimited number of values and are understood as a typed abstraction of possibly several concrete memory cells. This feature allows λC to simulate various other calculi with references such as variants with dynamic references or communication [10]. The language is endowed with an appropriate type and effects system ensuring termination. The translation is presented on an explicit substitutions version of λC, introduced to serve as an intermediate representation. This intermediate language is presented and studied in details in [6].

**Contributions.**

The contributions of this paper are:
We introduce and study routing areas which are flexible devices for implementing communication in proof nets. From routing areas we derive a semantic for a fragment of proof nets.

We illustrate the use of routing areas by translating a concurrent \( \lambda \)-calculus to proof nets. Routing areas are the building block of this translation.

We prove a simulation and adequacy theorem for this translation.

Plan of the paper.

- In Section 2, we detail our proof nets setting and state its properties.
- We go on to define routing areas and study them more in detail in Section 3.
- In Section 4, we introduce the source language of our translation, the concurrent \( \lambda \)-calculus \( \lambda C \).
- Section 5 is devoted to the translation. We explain its underlying principles and give some representative cases.
- Section 6 gives the simulation, termination and adequacy theorems together with their proof.

2 Proof nets

In this section, we detail the proof nets that we are considering in the rest of the paper. We do not attempt to give a full treatment of proof nets. We recall the important notions and specify the system that we use. The interested reader may find more details in [13] for example. Proof nets are a representation of proofs as multigraphs, where edges - called wires - correspond to the formulas, and nodes - cells - correspond to the rules. We can decompose our system into three different layers:

Multiplicative The multiplicative fragment is composed of the conjunction \( \otimes \) and the dual disjunction \( \exists \). These connectors can express the linear implication \( \rightarrow \) and are thus able to encode a linear \( \lambda \)-calculus, where all bound variables must occur exactly once in the body of an abstraction.

Exponential The exponentials enable structural rules to be applied on particular formulas distinguished by the \( ! \) modality (its dual being \( ? \)). Structural rules correspond to duplication (contraction) and erasure (weakening) : the multiplicative exponential fragment is the typical setting to interpret the \( \lambda \)-calculus.

Differential Non determinism is expressed by using two rules from Differential LL: cocontraction and coweakening. Semantically, contraction is thought as a family of diagonal morphisms \( \text{cntr}_A : !A \rightarrow !A \otimes !A \), each one taking a resource \( !A \) and duplicate it into a pair \( !A \otimes !A \). Dually, cocontraction is a morphism going in the opposite direction, packing two resources of the same type into one : \( \text{cocntr}_A : !A \otimes !A \rightarrow !A \). What happens when the resulting resource is to be consumed ? Two incompatible possibilities: either the left one is used and the right one is erased, or vice-versa. This corresponds to the rule \( \therefore \) (see Table 2) in proof nets: the reduction produces the non-deterministic sum of the two outcomes. Cocontraction will be used as an internalized non-deterministic sum. While weakening \( \text{weak}_A : !A \rightarrow 1 \) erases a resource, the dual coweakening produces a resource ex nihilo: \( \text{coweak}_A : 1 \rightarrow !A \). This resource can be duplicated or erased, but any attempt to consume it will turn the whole net to 0. Coweakening produces a Pandora box with a 0 inside. It is the neutral element of cocontraction.
Interpreting a concurrent λ-calculus in differential proof nets - Extended version

One can define a correctness criterion to discriminate nets that are well-behaved - the ones that are the representation of a valid proof - ensuring termination and confluence of the reduction. We will not require it here (cf Section 7). Without the correctness criterion, the full fledged reduction of (differential) LL is not even confluent, let alone terminating. We add constraints in order to recover a suitable system that is confluent and verifies a termination property (Theorem 3). Let us now give the definition of proof nets and their reduction:

**Notation 1.** We recall some vocabulary of rewriting theory that we use in the following.

- **A term of a language or a proof net** $t$ is
  - *A normal form* if it can’t be reduced further
  - **Weakly normalizing** if there exists a normal form $n$ such that $t \rightarrow^* n$
  - **Strongly normalizing** if it has no infinite reduction sequence
  - **Confluent** if for all $u, u'$ such that $u \leftarrow^* t \rightarrow^* u'$, there exists $v$ such that $u \rightarrow^* v \leftarrow^* u'$

- A rewriting relation is confluent if all terms are.

**Definition 1.** Proof nets

Given a countable set, whose elements are called ports, a proof net is given by

1. A finite set of ports
2. A finite set of cells. A cell is a finite non-empty sequence of pairwise distinct ports, and two cells have pairwise distinct ports. The first port of a cell $c$ is called the principal port and written $p_c$, and the $(i + 1)$th the $i$th auxiliary port noted $p_i(c)$. The number of auxiliary ports is called the *arity of the cell*. A port is free if it does not occur in a cell.
3. A labelling of cells by symbols amongst $\{1, \otimes, \exists, ?, !\}$. We ask moreover that the arity respects the following table:

   | Symbol | Arity |
   |--------|-------|
   | $\exists$ | 2     |
   | $\otimes$ | 2     |
   | $?$      | 0, 1 or 2 |
   | $!$      | 0, 1 or 2 |
   | $!_p$    | $i \geq 1$ |

4. A partition of its ports into pairs called wires. A wire with one (resp. two) free port is a free (resp. floating) wire.
5. A labelling of wires by MELL formulas: $F ::= 1 | \bot | F \otimes F | F \otimes F | !F | ?F$. A wire $(p_1, p_2)$ labelled by $F$ is identified with the reversed wire $(p_2, p_1)$ labelled by $F^\bot$. We will abusively confuse ports with the wire attached to them.
6. A mapping from $!_p$ nodes - called exponential boxes - to proof nets with no floating wires and $n \geq 1$ free wires. The first one is labelled (assuming orientation toward the free port) by $A$ and the remaining ones by $!B_1, \ldots, !B_{n-1}$. The corresponding $!_p$ node must have $!A$ as principal port and $!B_1, \ldots, !B_{n-1}$ as auxiliary ports.

The different kind of cells are illustrated in Figure 1. We directly represent $!_p$ cells as their associated proof net delimited by a rectangular shape. We impose that labels of wires connected to a cell respect the one given in Figure 1 i.e that nets are well-typed.
2.1 Reduction

The reduction rules are illustrated in Table 2. Notice that the exponential reductions are required to operate on closed boxes (boxes without auxiliary doors). \( \epsilon \rightarrow \) and \( e \rightarrow \) may be performed at any depth. All the other rules are only allowed at the surface: reduction inside boxes is prohibited.

We quotient the proof nets by associativity and commutativity of contraction and cocontractions. We also include in the relation the neutrality of (co)weakening for (co)contraction (Table 3).

We denote by \( \rightarrow \) the full reduction, extended to sums of nets in the same way as in [4].

\[ \text{Theorem 2. Confluence} \]

Theorem 2. Confluence

The reduction \( \rightarrow \) is confluent.

Proof. The reduction is similar in spirit to an orthogonal reduction in a term rewriting system. In the same approach as the proof of confluence of the \( \lambda \)-calculus, we can define a parallel reduction \( \Rightarrow \), which allow to perform an arbitrary number of steps in parallel (meaning that we can’t reduce redexes that are created by other steps). This parallel reduction verifies \( \Rightarrow \subseteq \Rightarrow \subseteq \Rightarrow^{*} \). Then we prove that \( \Rightarrow \) has the diamond property, and the previous expression ensures that \( \Rightarrow^{*} \subseteq \Rightarrow^{*} \), thus \( \Rightarrow^{*} \) is confluent.

2.2 Termination

\[ \text{Theorem 3. A net is weakly normalizing if and only if it is strongly normalizing.} \]
To prove Theorem 3, we proceed in two steps. First, we consider the system where all reductions are restricted to the surface. This reduction is constrained enough so that it verifies the diamond property, which is known to implies the Theorem 3. Then, we consider the non-surface $\rightarrow$ reductions, that we write $\Rightarrow$, and show that adding it to $\rightarrow$ does not invalid the theorem. In this subsection, we denote by $\rightarrow$ all the reductions rules of Figure 2 applied at the surface. We have the decomposition $\rightarrow = \rightarrow + \Rightarrow$. The following lemma shows that $\rightarrow$ and $\Rightarrow$ have a commutation property:

**Lemma 4. Swapping**

Let $S \Rightarrow R$ and $R \Rightarrow R'$. Then there exists $S'$ such that

$$
\begin{align*}
  R & \Rightarrow R' \\
  \Rightarrow & \\
  S & \Rightarrow S'
\end{align*}
$$

**Proof. Lemma 4**

By induction on both the length of the reduction $S \Rightarrow^* R$ and $R \Rightarrow^* R'$. We consider the fundamental case where both reductions are one-step:

$$
\begin{align*}
  R & \rightarrow R' \\
  \Rightarrow & \\
  S & \rightarrow S'
\end{align*}
$$

We make the following remarks:

- As $\Rightarrow$ reduction happens inside boxes, and $\rightarrow$ outside, we can put in a one-to-one correspondence the boxes at depth 0 - and thus the $\Rightarrow$-redexes - of $R$ and $S$.
- If the reduction in $R$ does not involve a box, it does not interact with $\Rightarrow$, and the diagram can be closed as a square with one step reduction in every direction.
- If the reduction in $R$ erases a box, we can erase the corresponding one in $S$, which may delete the $\Rightarrow$-redex involved reduced in $S$. In any case we can reduce $S$ to $S'$ in one step by erasing this box and $S'$ to $R'$ by at most one $\Rightarrow$ step.
- If the reduction in $R$ opens a box, we can open the corresponding one in $S$. If the $\Rightarrow$-redex was at depth 1 in the same box, then it becomes a $\Rightarrow$-redex as it now appears at the surface. Otherwise, it can be performed in one $\Rightarrow$ step. In any case, we can close the diagram by performing either one $\Rightarrow$ step followed by one $\Rightarrow$ step, or two $\Rightarrow$ steps.

In the cases listed above, the length of the reduction between $R'$ and $S'$ never exceeds one, hence we can always fill the following diagram:

$$
\begin{align*}
  R & \rightarrow R' \\
  \Rightarrow & \\
  S & \rightarrow^+ S'
\end{align*}
$$

Where the bottom line does not involve duplication ($S' \Rightarrow R'$ if $S' = R'$ or $S' \Rightarrow R'$).

- Duplication is the only $\rightarrow$ reduction that can create new $\Rightarrow$ redexes, but the bottom $\rightarrow$ reduction requires exactly one step in $S$. This corresponds to the following diagram:
Let us now prove the lemma. We consider the two cases separately: let us assume that $\mathcal{R} \Rightarrow \mathcal{R}'$ is not a duplication. We perform an induction on the length of the reduction $\mathcal{S} \Rightarrow ^k \mathcal{R}$. The induction hypothesis is that we can always fill the following diagram:

$$
\begin{array}{c}
\mathcal{R} \rightarrow \mathcal{R}' \\
\vdots \quad \quad \vdash \\
\mathcal{S} \rightarrow \mathcal{S}'
\end{array}
$$

With $k' \leq k$, and where the bottom reduction does not contain any duplication step. The base case $k = 0$ is trivially true. For the induction step, if we have $\mathcal{S} \Rightarrow ^k \mathcal{T} \Rightarrow \mathcal{R}$, we use the first diagram of the remarks to get the middle line $\mathcal{T} \Rightarrow ^* \mathcal{T}_p$. Then, we apply the induction hypothesis on each reduction step $\mathcal{T}_i \rightarrow \mathcal{T}_{i+1}$, which we can do precisely because our IH states that the lengths $k_i$ of each reduction $\mathcal{S}_i \Rightarrow ^{k_i} \mathcal{T}_i$ verify $k_i \leq k_{i-1} \leq \ldots \leq k$. We paste all the diagrams and get

$$
\begin{array}{c}
\mathcal{R} \rightarrow \mathcal{R}' \\
\vdots \quad \quad \vdash \\
\mathcal{T} \rightarrow \mathcal{T}_1 \rightarrow \ldots \rightarrow \mathcal{T}_p \\
\vdash \quad \vdash \quad \vdash \\
\mathcal{S} \rightarrow \mathcal{S}_1 \rightarrow \ldots \rightarrow \mathcal{S}_p
\end{array}
$$

The case of duplication is simpler as the step $\mathcal{R} \Rightarrow \mathcal{R}'$ is reflected by just one step $\mathcal{S} \rightarrow \mathcal{S}'$ in the second diagram of the remarks. Our IH is now that we can fill the diagram, without further assertion on the length of the $\Rightarrow$ reduction. Indeed, if $\mathcal{S} \Rightarrow ^k \mathcal{T} \Rightarrow \mathcal{R}$, we use the one-step diagram to get $\mathcal{T}'$ and apply the IH to fill the bottom part with $\mathcal{S}'$:

$$
\begin{array}{c}
\mathcal{R} \rightarrow \mathcal{R}' \\
\vdots \quad \quad \vdash \\
\mathcal{T} \rightarrow \mathcal{T}' \\
\vdash \quad \vdash \\
\mathcal{S} \rightarrow \mathcal{S}'
\end{array}
$$

Finally, we can perform a second induction on the length of the reduction $\mathcal{R} \Rightarrow ^+ \mathcal{R}'$ to get the final result.

Writing a reduction $\mathcal{R} \Rightarrow ^+ \mathcal{S}$ as blocks $\mathcal{R} \Rightarrow ^* \mathcal{R}_1 \Rightarrow ^* \mathcal{R}_2 \Rightarrow ^* \ldots \Rightarrow ^* \mathcal{R}_n$, we can iterate Lemma 4 to form a new reduction sequence with only two distinct blocks:
Lemma 5. Standardization

Let $R \rightarrow^* S$. Then $R \rightsquigarrow^* R' \rightsquigarrow^* S$. Moreover, if the original reduction contains at least one $\rightarrow$ step, then $R \rightsquigarrow^+ R'$.

Proof. Lemma 5

This follows from the previous lemma: we decompose the reduction $R \rightarrow^* S$ as blocks $R \rightsquigarrow^* R_1 \rightarrow^* R_2 \rightsquigarrow^* \ldots \rightarrow^* R_n$, and iterate 4 to gather the reductions into only two distinct blocks.

Lemma 6 follows from the observation that as $\rightarrow$ only acts on surface and $\rightsquigarrow$ inside boxes, the latter can not interact with the redexes of the former.

Lemma 6. Neutrality of $\rightsquigarrow$

$\rightsquigarrow$ does not create nor erase $\rightarrow$-redexes. In particular, if $R \rightsquigarrow^* R'$, then $R$ is $\rightarrow$-normal if and only if $R'$ is.

We also need some properties about termination of the reduction $\rightarrow$ and $\rightsquigarrow$:

Lemma 7. Strong normalization for $\rightsquigarrow$

$\rightsquigarrow$ is strongly normalizing.

Proof. It suffices to note that opening or deleting a box strictly decreases the total number of boxes in the net.

Lemma 8. Theorem 3 is true when replacing $\rightarrow$ with $\rightsquigarrow$.

Proof. The surface reduction satisfies the diamond property, which excludes the existence of a weakly normalizing term with an infinite reduction.

We can finally prove Theorem 3:

Proof. Theorem 3

We prove two auxiliary properties:

(a) $\rightarrow$-weak normalization implies $\rightsquigarrow$-weak normalization

Let $R$ be $\rightarrow$-weakly normalizing, $R \rightarrow^* N$ a reduction to its normal form. By 3, we can write $R \rightarrow^* S \rightsquigarrow^* N$. $N$ being a $\rightarrow$-normal form, it is also a $\rightarrow$-normal form, and by 6 so is $S$. $R$ is thus a $\rightarrow$-weakly normalizing.

(b) an infinite $\rightarrow$-reduction gives an infinite $\rightsquigarrow$-reduction

Let $R$ be a net with an infinite reduction, written $R \rightarrow^* \infty$. We will prove by induction that for any $n \geq 0$, there exists $R_n$ such that $R \rightarrow^n R_n \rightarrow^* \infty$.

Case $n = 0$

We just take $R_0 = R$.

Inductive case

If $R \rightarrow^n R_n \rightarrow^* \infty$, we take any infinite reduction starting from $R_n$. If the first step is $R_n \rightarrow S$, then we take $R_{n+1} = S$. Otherwise, the first step is a $\rightsquigarrow$ step, and we take the maximal block of $\rightsquigarrow$ reduction starting from $R_n$. By 4 this block must indeed be finite and we can write $R_n \rightsquigarrow^* S \rightarrow^* S' \rightarrow^* \infty$. By 4 we can swap the two blocks such that $R_n \rightarrow R' \rightarrow^* R'' \rightsquigarrow^* S'$, and we take $R_{n+1} = R'$.

From these two points follows that $\rightarrow$-weak normalization implies $\rightarrow$-strong normalization. If a net is $\rightarrow$-weakly normalizing, then it is $\rightarrow$-weakly normalizing by (a). By 8 it is also $\rightarrow$-strongly normalizing. But by (b) it must be also $\rightarrow$-strongly normalizing.

The next section focuses on a specific family of proof nets that play a key role in the expression of communication primitives inside proof nets.
3 Routing Areas

Let us now define and study a special kind of nets: the routing areas. It is a generalization of the construction of communication areas introduced in [3]. The approach is similar: we aim at constructing nets to be used as building blocks to implement communication primitives. We shall see that this seemingly restricted class of nets is actually the set of normal forms of a fragment of proof nets (Theorem 16). Routing areas are composed only of structural rules: contraction, weakening, cocontraction and coweakening. These basic components act as resource dispatchers (a resource designates a closed exponential box in the following):

- A free wire acts as the identity. It passively forwards a resource that is connected on the input (the left port) to the output (the right port).

- A contraction is a broadcaster with one input and two outputs. A resource connected on the left will be copied to both outputs on the right. A weakening is a degenerate case of a broadcaster with zero outputs as broadcasting something to no one is the same as erasing it.

- Dually, a cocontraction is a packer with two inputs and one output. A packer aggregates its two inputs non deterministically. When a dereliction is connected to the output to consume two packed resources, a non-deterministic sum of the two possible choices for the one to be provided is produced. Similarly, coweakening is seen as a degenerate packer with no inputs.

A routing area can be seen as a router, or a circuit, between inputs and outputs. Inputs are connected to contractions which broadcast resources they receive to cocontractions. Cocontractions may gather resources from multiple such sources. The conclusion of these cocontractions form the outputs. A routing area is then described by a slight generalization of a relation between sets, a multirelation. Its role is to define the wiring diagram which specifies which inputs and outputs are connected. Let us first introduce multirelations:

**Multirelation** Let $A$ and $B$ be two sets, a multirelation $R$ between $A$ and $B$ is a multiset of elements of $A \times B$, or concretely a map $R : A \times B \to \mathbb{N}$. For $k \in \mathbb{N}$, we write $x R_k y$ if $R(x, y) = k$.

**Relations and multirelations** A relation $R$ between $A$ and $B$ can be seen as a multirelation by taking its characteristic function $1_R$. Conversely, we can forget the multiplicity of a multirelation $S$ and recover a relation by taking the subset of $A \times B$ defined by $\{(x, y) \in A \times B \mid S(x, y) \geq 1\}$.

**Composition** Multirelations enjoy a composition operation that computes all the ways to go from an element to another with multiplicities. For multirelations $R, S$ respectively between $A$ and $B$, and $B$ and $C$,

\[(S \circ R)(x, z) = \sum_{y \in B} R(x, y)S(y, z)\]

This composition is associative, coincides with the usual one for relations, and has the identity relation (seen as a multirelation) for neutral. This is in fact the matrix multiplication, seeing a multirelation between $A$ and $B$ as a $|B| \times |A|$ matrix with integer coefficients $R(i, j)$ (identifying finite sets with their cardinal).
The FMRel Category Finite sets and multirelations between them form a category FMRel, with the category FRel of finite sets and relations as a subcategory. FMRel has finite coproducts, extending the one of FRel, and corresponding to the direct sum of matrices.

A routing area is described by a multirelation between its inputs and its outputs. Its value at the pair \((i, o)\) indicates how many times the input \(i\) is connected to the output \(o\).

We are now ready to construct the routing area defined by a multirelation.

▶ Definition 9. Routing area
Let \(L_i, L_o\) be two finite sets called the input labels and the output labels, and a multirelation \(R\) between \(L_i\) and \(L_o\). A routing area \(R\) associated to the triplet \((L_i, L_o, R)\) is a net constructed as follows:

- It has \(|L_i| + |L_o|\) free wires partitioned into \(|L_i|\) inputs and \(|L_o|\) outputs. Each input is labelled by a distinct element of \(L_i\), while outputs are labelled by distinct elements of \(L_o\).
- Each input (resp. output) is connected to the main port of a contraction (resp. cocontraction) tree. Then, for every \((i, o) \in L_i \times L_o\), we connect the tree of the input \(i\) to the tree of output \(o\) with exactly \(R(i, o)\) wires.

We represent them as rectangular boxes, with the inputs appearing on the left and outputs on the right.

▶ Definition 10. Arity
Let \((L_i, L_o, R)\) be a routing area. For an input \(i \in L_i\), we define its arity as the number of leafs of the associated contraction tree given by \(\text{ar}(i) = \sum_{o \in L_o} R(i, o)\). Similarly, the arity of an output \(o \in L_o\) is defined by \(\text{ar}(o) = \sum_{i \in L_i} R(i, o)\). The set of outputs (resp. inputs) connected to an input \(i\) (resp. output \(o\)) is defined as \(\text{co}(i) = \{o \in L_o | R(i, o) \geq 1\}\) (resp. \(\text{co}(o) = \{i \in L_i | R(i, o) \geq 1\}\)). In general, for an input or an output \(x\), \(\text{ar}(x) \leq |\text{co}(x)|\). This is an equality for all \(x\) if and only if \(R\) is a relation.

Routing areas may be combined in two ways such that the resulting proof net reduces to a new routing area. The multirelation describing the result can be computed directly from the initial multirelations of routing areas involved, giving a way of building complex circuits from small components.

Operations
The first operation, juxtaposition, amounts to put side by side two routing areas. The result is immediately seen as a routing area itself, described by the coproduct of the two multirelations:

▶ Definition 11. Juxtaposition
Let \(R = (L_i, L_o, R)\) and \(S = (L'_i, L'_o, S)\), we define the juxtaposition \(R + S\) by \((L_i + L'_i, L_o + L'_o, R + S)\). The corresponding net is obtained by juxtaposing the nets of \(R\) and \(S\):
The second one is more involved: the trace operation consists in connecting an input to an output given that they are not related to begin with, to avoid the creation of a cycle. Doing so, we remove this output and input from the external interface, and create new internal paths between remaining inputs and outputs. If we reduce the resulting net to a normal form, we obtain a new area, whose multirelation can be computed from the initial one.

Definition 12. Trace
Let \( R = (\mathcal{L}_i, \mathcal{L}_o, R) \) be a routing area, and \((i, o) \in \mathcal{L}_i \times \mathcal{L}_o\). The trace at \((i, o)\) of \( R \) is obtained by connecting the input \( i \) with the output \( o \) of \( R \) and reducing this net to its normal form.

Property 1. Trace is a routing area
Let \( R = (\mathcal{L}_i, \mathcal{L}_o, R) \) be a routing area, and \((i, o) \in \mathcal{L}_i \times \mathcal{L}_o\) such that \( \text{ar}(i) = \text{ar}(o) = 0 \). Then the trace at \((i, o)\) of \( R \) is a routing area \( T: R \rightarrow \ast \)

Where \( T = (\mathcal{L}_i - \{ i \}, \mathcal{L}_o - \{ o \}, T) \) is defined by the multirelation:

\[
T(x, y) = R(x, y) + R(x, o)R(i, y)
\] (1)

The formula 1 expresses that in the resulting routing area \( T \), the total number of ways to go from an input \( x \) to an output \( y \) is the number of direct paths \( R(x, y) \) from \( x \) to \( y \) that were originally in \( R \), plus all the ways of going from \( x \) to \( o \) times the ways of going from \( i \) to \( y \). Indeed, any pair of such paths yields a new distinct path in the trace once \( i \) and \( o \) have been connected.

Proof. Property 1
If \( \text{ar}(i) = \text{ar}(o) = 0 \), then we connected a coweakening to a weakening and we can erase them to recover the desired area where \( T \) is just the restriction of \( R \) to \((\mathcal{L}_i - \{ i \}) \times (\mathcal{L}_o - \{ o \})\), which agrees with the formula of 1 as the product \( R(x, o)R(i, y) \) is always zero.

Now, assume that \( \text{ar}(o) = 0 < \text{ar}(i) \). We can reduce the introduced redex as follow :

These coweakening are connected to the trees of the outputs \( \text{co}(i) \). These are either connected to a wire, or a cocontraction tree and we can eliminate superfluous coweakening using the equivalence relation. Once again, we didn’t create new paths and recover an area whose relation is the restriction of \( R \), still agreeing with the formula as \( \text{ar}(i) = 0 \) implies \( R(x, o)R(i, y) \) being zero again. The dual case \( \text{ar}(i) = 0 < \text{ar}(o) \) is treated the same way.

The general case relies on the commutation of contractions and cocontractions trees that can be derived by iterating the \( \text{ba} \) rule. We can apply the following reduction on the trees of \( i \) and \( o \) that have been connected :
where wires $i_1, \ldots, i_p$ are connected to the trees of the inputs in $\text{co}(o)$ and $o_1, \ldots, o_q$ to the trees of the outputs of $\text{co}(i)$. The reduced net now has the shape of a routing area. As before, the direct paths between $x$ and $y$ when $x \neq i$ and $y \neq o$ are left unchanged. But any couple of paths in $R$ between $x$ and $o$ arriving at some $i_k$ and between $i$ and $y$ arriving at some $o_l$ yields exactly one new path between $x$ and $y$ in the new area (see Fig. below). By definition, there are $R(x, o)$ paths connecting $x$ to $o$ and $R(i, y)$ connecting $i$ to $y$: hence there are $R(x, o)R(i, y)$ such couples.

These two operations are sufficient to implement composition which is the connection of an output of an area to an input of another area. Composition is a fundamental feature of routing areas. This is what makes them modular, allowing to build routing areas by composing simple blocks. To connect an output $o$ of $R$ to an input $i$ of $S$, we first perform the juxtaposition followed by a trace at $(i, o)$. This is similar both in form and in spirit, to the composition of Game Semantic or Geometry of Interaction whose motto is “composition = parallel composition plus hiding”.

\textbf{Corollary 13. Composition}

Let $R = (L_i, L_o, R)$ and $S = (L'_i, L'_o, S)$ be two routing areas, $o \in L_o$ and $i \in L'_i$. Then the net resulting from connecting the output $o$ to the input $i$ can be reduced to a new routing area $T = (L_i + L'_i - \{i\}, L_o + L'_o - \{o\}, T)$

\[ i \quad R \quad i \quad S \quad i \quad \rightarrow^* \quad i \quad T \quad i \]

\textbf{Remark.} This operation can be generalized to the connection of $n$ outputs of $R$ to $n$ inputs of $S$. When $n = |L_o| = |L'_i|$, the multirelation $T$ describing the resulting routing area is the composed $S \circ R$.

The following property gives the high level operational behavior of a routing area. It supports our interpretation of routing areas as dispatchers of exponential boxes. Given a closed exponential box, we connect it to the auxiliary port of a cocontraction to obtain a module which can then be connected to an input $i$ of an area. Through reduction, the box will traverse the area and be duplicated $R(i, o)$ times to each output $o$. The role of the additional cocontraction is to preserve the area and allow future connections to the same input. We would get a similar transit property connecting directly the exponential box to $i$, but the process is destructive as it erases the input wire of $i$ and prevents any future use.

\textbf{Property 2. Transit}

Let $\sigma$ be a closed exponential box, $R = (L_i, L_o, R)$ a routing area, $i \in L_i$. Let $\{o_1, \ldots, o_p\} = \text{co}(i)$ and for $1 \leq k \leq p$, $c_k = R(i, o_k)$. Then :

\[ \sigma \quad \rightarrow^* \quad R \quad \text{co}(i) \]

\[ \rightarrow^* \quad R \quad o_k \]

\[ \sigma \quad \rightarrow^* \quad R \quad o_k \]

\[ \sigma \quad \rightarrow^* \quad R \quad o_k \]
The property is straightforward application of reduction rules and the net equivalence.

3.1 The Routing Semantic

Routing areas do not only fulfill practical needs. They are general enough to be the language of normal forms of routing nets. Let us first give a precise definition. A correct net is a net satisfying the correctness criterion defined in [4]:

\[ \text{Definition 14. Routing nets} \]

A routing net \( R \) is a correct net composed only of weakenings, coweakenings, contractions, cocontractions, and possibly floating wires. Moreover, we ask that all wires are labelled with the same formula \( !A \), fixing de facto their orientation.

Paths will also be of interest in the rest of this subsection.

\[ \text{Definition 15. Paths} \]

Let \( R \) be a net. We recall that for a cell \( c \) of \( R \), we write \( p_i(c) \) for its \( i \)-th auxiliary port (if it exists) and \( p(c) \) for its main port. For a wire \( w \), \( s(w) \) designates the source port of \( w \) while \( e(w) \) is its end port. Let \( R \) be a net without exponential boxes, we construct the associated undirected graph \( G(R) \) with ports as vertices and:

- **Wire edges** For any wire \( w \) of \( R \), we add an edge between \( s(w) \) and \( e(w) \).

- **Cell edges** For every auxiliary port \( p_i(c) \) of a cell, we add an edge between \( p_i(c) \) and \( p(c) \). A path \( p \) in \( R \) is a finite sequence \( (p_1, e_1, p_2, e_2, \ldots, e_n, p_{n+1}) \) such that \( p_1 \) is a port of \( R \), \( e_i \) an edge of \( G(R) \) linking \( p_i \) and \( p_{i+1} \) and such that \( e_i \) and \( e_{i+1} \) are of distinct nature (cell/wire edge). We extend \( s \) and \( e \) to operate on path, defined by \( s(p) = p_1 \) and \( e(p) = p_{n+1} \).

Paths whose starting and ending edges are wire edges can also be described as a sequence of corresponding wires \( (w_1, \ldots, w_m) \) as internal ports and cell edges can be recovered. We note \( P(R) \) the set of paths in \( R \) and \( P_f(R) \) the paths starting and ending on free ports.

Albeit closed to switching paths (the ones involved in the correctness criterion), they do not match exactly. A path in a switching graph can arrive at an auxiliary port of a cocontraction and bounce back in the other, while the paths we have defined here must continue via the principal port. However, a correct (routing) net do not contain cyclic paths, which means that switching acyclicity implies acyclicity:

\[ \text{Property 3. Acyclicity of correct nets} \]

A routing net is acyclic, that is there is no path \( p \) such that \( s(p) = t(p) \).

**Proof.**

We prove that the existence of a cycle implies the existence of a cycle in a switching graph. Assume that a cycle exist and take a one of minimal length. Assume that the two auxiliary ports of a cell (contraction or cocontraction) are both visited by the cycle. By the definition of paths, when the cycle visits one of these auxiliary port, it must continue in the principal port. But then a smaller cycle would be derivable, by taking the sub path starting at the principal port of this cell and stopping as long as it comes back by any of the auxiliary ports. Thus a minimal cycle visits at most once of the auxiliary port of any cell, and is also a cycle in a switching graph.

We now state a fundamental property relating reduction to paths:

\[ \text{Property 4. Path preservation} \]

Let \( R \) be a routing net, \( p_1 \) and \( p_2 \) be free ports. We write \( P_f(R, p_1, p_2) = \{ p \in P_f(R) \mid s(p) = p_1, e(p) = p_2 \} \). If \( R \to R' \), then \( P_f(R, p_1, p_2) = P_f(R', p_1, p_2) \). In particular, \( P_f(R) = \sum P_f(R, p_1, p_1') = P_f(R') \).
Proof. The proof is simple in the case of routing nets. As a path in $P_f(R, p_1, p_2)$ must begin and end in a free port, it can’t go through a weakening or a coweakening, and the corresponding reductions leave such paths unchanged. The only relevant reduction is the $\Rightarrow ba$ rule:

Let us define the application $\tau : P_f(R, p_1, p_2) \rightarrow P_f(R', p_2, p_2)$. For a path $p$ which does not cross the redex, $\tau(p) = p$. Otherwise, we replace any subsequence in the left column by the one in the right column:

| Subsequence | Image by $\tau$ |
|-------------|------------------|
| $i_1, c, o_1$ | $i_1, c_1, o_1$ |
| $i_1, c, o_2$ | $i_1, c_2, o_2$ |
| $i_2, c, o_1$ | $i_2, c_3, o_1$ |
| $i_2, c, o_2$ | $i_2, c_4, o_2$ |

We omitted the four other possibilities which can be deduced from this table by reversing both the subsequence and its image. It is easily seen that $\tau$ is an bijection.

\[\text{Theorem 16. Routing area characterization}\]

The normal form of a routing net $S$ is a routing area $R = (L_i, L_o, R)$. For a routing net $S$, we can define the application $[\_] : S \mapsto R$ that maps $S$ to the multirelation $R$ describing its normal form. By unicity of normal forms, this application is invariant by reduction and is thus a semantic for routing nets. It has the following properties:

- **Soundness** The multirelation only depends on the normal form.
- **Adequacy** Two routing nets with the same denotation have the same normal form because a multirelation defines a routing area uniquely.
- **Full completeness** Any multirelation between finite sets is realized by the associated routing area.
- **Compositionnality** We can compute the semantic of a net in a compositional way from the semantic of its smaller parts through juxtaposition and trace.

Proof. Theorem 16

We prove the result by induction on the number $n$ of cells of $R$.

- $(n = 0)$ $R$ is only composed of free wires. We take $L_i = \{ s(w) \mid w \text{ wire} \}$, $L_o = \{ e(w) \mid w \text{ wire} \}$ and $R$ is the relation defined by $i R o \iff \exists w, i = s(w), o = e(w)$. 


\[\text{The following theorem establishes the link between routing areas and normal forms of routing nets. We propose two different intuitive explanations of why the theorem holds:}\]

1. The basic components of routing nets, (co)contractions, wires and (co)weakenings, are routing areas. Then, juxtaposition and trace operations are general enough to combine them into an arbitrary routing net that reduces to a routing area according to Property 1.
2. In a routing net, the $\Rightarrow ba$ rule allows to commute all contractions and cocontractions. Then, by equivalence, we can erase weakenings and coweakenings that are not connected to a free wire. At the end of this process, the resulting net must have the shape of a routing area.
(induction step) Let take any node $N$ of $\mathcal{R}$. We call $\mathcal{R}'$ the subnet obtained by removing $N$ and replacing its ports by free ports.

By induction, $\mathcal{R}'$ can be reduced to a routing area $\mathcal{R}'$.

**Weak**ing If $N$ is a weakening or coweakening, it is a routing area and can be composed with $\mathcal{R}'$, and reduced to a new routing area $\mathcal{R}$ according to [13]. Thus we can reduce the whole net to $\mathcal{R}$.

**Contraction** If $N$ is a contraction or a cocontraction, it can still be seen as a routing area and we juxtapose it to $\mathcal{R}'$. What remains to do is to perform three traces operations to recover the original net and reduce the whole net to a routing area $\mathcal{R}$. However, we must ensure that the input and the output we connect at each step are not already connected in the routing area. The first operation is always legal, as it is actually a merge operation of previously disjoint areas. The following traces are also valid, as reduction does not create cycles, as implied by [1]

For a routing net $S$, we can define the application $[.] : S \mapsto R$ that maps $S$ to the multirelation $R$ describing its normal form. By unicity of normal forms, the application is invariant by reduction and is thus a semantic for our routing nets, with the following remarkable properties:

**Sound** The multirelation only depends on the normal form, this is invariant by reduction.

**Adequate** Two routing nets with the same denotation have the same normal form, as a multirelation define a routing area uniquely.

**Fully complete** Any multirelation on finite sets is realised by the associated routing area.

**Compositionnal** We can compute the semantic of a net in a compositional way from the semantic of its smaller parts, through juxtaposition and trace.

This semantic can be defined without resorting reduction, by counting paths.

**Theorem 17. Path semantic**

Let $S$ be a routing net. Let $\mathcal{L}_i$ be the set of free ports of $S$ which are the source point of a wire, and $\mathcal{L}_o$ the ones that are the end point. Then $[S]$ is the multirelation between $\mathcal{L}_i$ and $\mathcal{L}_o$ given by

$$[S](i,o) = |P_f(S,i,o)|$$

**Proof.** [14]

For normal forms, this follows from the very definition of a routing area associated to the multirelation $[S] : [S](i,o)$ counts precisely the number of paths starting at port $i$ and ending at port $o$. 

CVIT 2016
Otherwise, by path preservation [4], this quantity is invariant by reduction. We conclude by induction on the length of the longest reduction of $S$ to its normal form.

\[ \text{\textcircled{\textup{\textbullet}}} \]

\textbf{Remark. Communication Areas}

The communication areas defined in [3] are a special case of routing areas: for $n \leq 1$, the $n$-communication area is the routing area $\{(1, \ldots, n), \{1, \ldots, n\}, R\}$ where $x R y \iff x \neq y$.

\textbf{Remark. Canonicity of multirelations}

The multirelation defining an area is actually not unique from a set-theoretic point of view: indeed, for $R = (L_i, L_o, R)$, then all multirelations in $\{\tau^{-1} \circ R \circ \sigma : E \to F \mid \sigma : E \to L_i, \tau : F \to L_o, \sigma \text{ and } \tau \text{ bijective} \}$ describe the same area. Even though this object is a proper class, all these relations are considered isomorphic (for example, in the arrow category of $\text{FMRel}$). After all, even finite deterministic automata suffer this kind of subtlety, which is not relevant in practice.

We are now armed to encode communication primitives. This is illustrated in Section 5 by the translation of the $\lambda_C$ calculus, succinctly described in the following section.

\section{The Concurrent $\lambda$-calculus $\lambda_C$}

We present the $\lambda_C$ calculus, following the presentation in Madet’s Ph.D thesis [10].

The $\lambda_C$ calculus is a call-by-value $\lambda$-calculus, equipped with references that abstract the notion of global memory cells. The calculus is enriched with a parallel composition operator $\parallel$ for modeling concurrency.

\subsection{Syntax}

Variables are denoted with $x, y, \ldots$ while references are denoted with $r, s, \ldots$. The language consists of values, terms, stores and programs. A store is a top-value set of associations between references and values, while a program is a store together with a set of terms. The terms in a program can be regarded as threads running in parallel, while the stores represent the state of the global memory. Programs are the objects of interest in $\lambda_C$.

\begin{center}
\begin{tabular}{ll}
-\textit{values} & $V ::= x \mid \ast \mid \lambda x. M$

-\textit{terms} & $M ::= V \mid M M \mid \text{get}(r) \mid \text{set}(r, V) \mid M \parallel M$

-\textit{stores} & $S ::= r \Leftarrow V \mid (S \parallel S)$

-\textit{programs} & $P ::= M \mid S \mid (P \parallel P)$
\end{tabular}
\end{center}

The constant value $\ast$ stands for the return value of a reference assignation: it carries no particular information. The primitives $\text{set}(r, V)$ and $\text{get}(r)$ respectively writes a value $V$ to and reads from a given reference $r$. While assignments can be only performed on values, a more general $\text{set}(r, M)$ can be encoded as $(\lambda x. \text{set}(r, x)) M$ for an arbitrary term $M$. Once reduced, a $\text{set}(r, V)$ produces the special kind of thread $r \Leftarrow V$ at top level in a store, adding $V$ as the possible values available for the reference $r$. Parallelism is accounted for using the operator $\parallel$. Terms, programs and stores can be placed in parallel. For stores, this simply means that all the corresponding associations reference/value are available for substitution of reference in the threads.
The two assignment can be reduced to stores:

\[ P \parallel P' = P \parallel P' \]

\[ (P \parallel P') \parallel P'' = P \parallel (P' \parallel P'') \]

**Table 4** Structural Rules

\[
\begin{align*}
(\beta_v) & \quad C[E[(\lambda x.M) V]] \rightarrow C[E[M[V/x]]] \\
(\text{get}) & \quad C[E[\text{get}(r)]] \parallel r \leftarrow V \rightarrow C[E[V]] \parallel r \leftarrow V \\
(\text{set}) & \quad C[E[\text{set}(r,V)]] \rightarrow C[E[s]] \parallel r \leftarrow V
\end{align*}
\]

**Table 5** Evaluation Contexts

4.2 Reduction

The calculus is endowed with the usual structural rules for the parallel operator, namely associativity and commutativity (Table 4). The rewrite rules for the language are found in Table 6. Together with the \( \beta_v \) rule are two new reductions, one that turns an assignment to a store and one that turns a get to a value. The rules use the evaluation contexts defined in Table 5 to handle congruence. The context \( E \) denotes a weak call-by-value weak reduction which is neither right-to-left or left-to-right. The context \( C \) allows reduction to occur in any thread of a program.

The substitution of a reference is a non-deterministic operation. Reference must be seen as an abstraction for a set of typed memory cells that can hold many values. Let \( \text{proj}_1 = \lambda xy.x \) and \( \text{proj}_2 = \lambda xy.y \) be the Church projections, and consider the term

\[ P = (\lambda x.V_1.V_2) \text{get}(r) \parallel (\lambda y.\text{set}(r,y)) \text{get}(s) \parallel \text{set}(s,\text{proj}_1) \parallel \text{set}(s,\text{proj}_2) \]

The two assignment can be reduced to stores:

\[ P \rightarrow^* (\lambda x.V_1.V_2) \text{get}(r) \parallel (\lambda y.\text{set}(r,y)) \text{get}(s) \parallel * \parallel * \parallel s \leftarrow \text{proj}_1 \parallel s \leftarrow \text{proj}_2 \]

for some distinct values \( V_1 \) and \( V_2 \). Here, \( \text{get}(s) \) have essentially two incompatible ways to reduce: either it is replaced by \( \text{proj}_1 \) or by \( \text{proj}_2 \). In the first case,

\[ P \rightarrow^* (\lambda x.V_1.V_2) \text{get}(r) \parallel \text{set}(r,\text{proj}_1) \parallel P' \]

\[ \rightarrow^* (\lambda x.V_1.V_2) \text{get}(r) \parallel * \parallel r \leftarrow \text{proj}_1 \parallel P' \]

\[ \rightarrow^* (\lambda x.V_1.V_2) \text{proj}_1 \parallel * \parallel r \leftarrow \text{proj}_1 \parallel P' \]

\[ \rightarrow^* V_1 \parallel * \parallel r \leftarrow \text{proj}_1 \parallel P' \]

where \( P' = * \parallel * \parallel s \leftarrow \text{proj}_1 \parallel s \leftarrow \text{proj}_2 \). However if \( \text{get}(s) \) is replaced by \( \text{proj}_2 \), we get that \( P \rightarrow^* V_2 \parallel * \parallel r \leftarrow \text{proj}_2 \parallel P' : P \) has two distinct normal forms.

Despite the \( \parallel \) operator being a static constructor, it can be embedded in abstractions and thus dynamically liberated or duplicated. For example, the term \( (\lambda f.f \parallel * \parallel f \parallel *) \) act like a fork operation: if applied to \( M \), it generates two copy of its argument in two parallel threads \( M \parallel M \parallel * \). The next section is devoted to detailing how terms of \( \lambda C \) are translated to proof nets described in 2 thanks to the areas introduced in 3.

5 Translation

To implement the translation, we make use of two specific routing areas that we introduce below. In the following, \( E_i = \{1, \ldots, i\} \) and \( R_i \) is the binary relation defined on \( E_i \) by

\[ k R_i l \iff k \neq l. \]
The $\gamma$ area is defined by $(E_3, E_3, R_\gamma = R_3)$. $\gamma$ is actually a communication area, composed of 3 pairs of input and outputs grouped by label. Each such pair represents a plug to which translated terms will be connected. The definition of $R_\gamma$ expresses that the input and the output of a plug are not connected, as a component should not receive the data it sent himself: this would be the analog of a short-circuit. All others inputs and output are connected.

The $\delta$ area is an analog structure with 4 plugs: $(E_4, E_4, R_\delta)$. It is designed to handle the application $M \ N$ which includes three potential sources of effects:

1. The effects $e_1$ produced by reducing $M$ to $\lambda x. M'$
2. The effects $e_2$ produced by reducing $N$ to value $V_N$
3. The effects $e_3$ produced by reducing $M'[V_N/x]$ to the final result $V$

The reduction of $\lambda C$ imposes that $e_1$ and $e_2$ happen before $e_3$, while $e_1$ and $e_2$ may happen concurrently. For $1 \leq i \leq 3$, the plug $(i, i)$ of $\delta$ corresponds to the effects $e_i$. The last one is the external interface for future connections. We easily accommodate $\delta$ to implements the sequentiality constraint by removing the couples $(3, 1)$ and $(3, 2)$ from $R_\delta$ to form $R_\delta$. Indeed $e_1$ and $e_2$ happens before $e_3$ thus can not observe any assignment made by the latter. Thus we just cut the corresponding wires. We see that the formalism of routing areas allows us to easily encode the order of effects.

### 5.1 Translating types and effects

Before translating terms, we need to translate the types from the type and effects system for $\lambda C$ to plain LL formulas. We use the approach of [16], a monadic translation, explained in the following. Before translating to LL, let us first try to take a type with effects and translate it to a pure simple type. Let $e = \{r_1, \ldots, r_n\}$ be an effect (a finite set of references), assume we can assign a simple type $R_i$ to each reference $r_i$. We can type a store $S_e = R_1 \times \cdots \times R_n$ representing the current state of the memory. We transform a term $M$ of type $A$ producing effects to a pure term which takes the initial state of the store, and returns the value it computes together with the new state of the store after this computation. Using curryfication for the arrow type, we define the translation:

$$
T_e(\alpha) = S \to S \times \alpha \\
T_e(A \to \alpha) = A \to (S \to (S \times \alpha)) \cong A \times S \to S \times \alpha
$$

From there, we go to LL types by implementing the pair type $A \times B$ as $!A \otimes !B$, and the usual call-by-value translation for the arrow $(A \to B)^* = !(A^* \to B^*)$ [11]. We still have to determine each $R_i$ first. Using the previous formula, we may associate an LL type variable $X_{r_i}$ to each reference and plug everything together to obtain the following equations (where $A_i$ is the type given to $r_i$ by the reference context):

$$
\text{Unit}^* = !1 \\
(A^{(a_1, \ldots, a_m)} \to \alpha)^* = !((A^* \otimes X_{a_1} \ldots \otimes X_{a_m}) \to (X_{a_1} \otimes \cdots \otimes X_{a_m} \otimes \alpha^*)) \\
X_{r_i} = A_i^*
$$

This system is solvable precisely because the type system is stratified [16], and we can thus translates all the types of $\lambda C$ to plain LL types. The behavior type $B$ will be translated to types of the form $A_1 \gamma \ldots \gamma A_n$ as the translation remembers the types of each threads.
5.2 Translating terms

The general form of the translation of a term $x_1 : A_1, \ldots, x_n : A_n \vdash M : (\alpha, \{r_1, \ldots, r_k\})$ is given by

$$M \cdot \alpha \cdot !X_{r_1} \cdots !X_{r_k}$$

We distinguish three different types of free wires:

**Output wire** The right wire, labelled by $\alpha^*$, corresponds to the result of the whole term.

**Variable wires** Each wire on the left corresponds to a variable of the context. The (explicit) substitution of a variable $x$ for a term $V^*$ is obtained by connecting the output wire of $V^*$ to the wire of $x$.

**References wires** The wires positioned at the top are input wires corresponding to references and have a similar role as variable wires, while the wires at the bottom corresponds dually to the output. References wires will be connected by routing areas.

We present some representative cases of the translation: get$(r)$ and set$(r, V)$ for reference management, the abstraction to show how effects are thunked in a function’s body following the monadic translation, and the application that shows the usage of routing areas to handle non-trivial effects scheduling. We start with reference operations, which serve as switch from and to reference wires:

**Set (Figure 1)**

A set$(r, V)$ connects the output of the translation of $V$ to the output reference wire corresponding to $r$. As other assignments are not relevant a weakening is connected on the input wire to ignore any incoming resource. In $\lambda C$, a set reduces to $*$ as it does not compute anything valuable. Consequently the output is the conclusion of a banged 1 which is the translation of $*$. One important remark is that an additional exponential layer is added around the translation of $V$. In a call-by-value language, the non determinism is strict in the sense that non-deterministic term must be evaluated before any copy. For example, the term $(\lambda fx.f x x)$ get$(r) \parallel r \leftarrow V_1 \parallel r \leftarrow V_2$ can reduce either to $f V_1 V_1$ or $f V_2 V_2$ but not to $f V_1 V_2$. Differential LL rather implements the latter call-by-name semantic as hinted at by the $\Rightarrow_{ba}$ rule which expresses that duplication and non-determinism should commute. The mismatch is due to two different usages we want to make of the $!$:

- The first one allows to discriminate what proof nets can be the target of structural rules, which implements substitution. In call-by-value, the only terms that can be substituted are values. The $!$ is introduced by the translations of values, using $!_p$, and eliminated at usage - when applied to another term - for each copy by a dereliction.
- The second usage relates to the differential part. The bang denotes resources that may be packed non deterministically by a cocontraction. The choice is made when a dereliction is met.

But as we noted, these two usages are in contradiction: a non-deterministic packing should not be allowed to be substituted. Technically, the dereliction corresponding to the
place of usage and the dereliction corresponding to the non-deterministic choice should not be the same. This is the reason of the additional ! layer introduced by an exponential box around \( V^* \). The corresponding dereliction is found in the translation of \( \text{get}(r) \).

Get (Figure 2)

The \( \text{get}(r) \), dual of the set, takes a resource from the corresponding input reference wire and redirects it to the output wire. It outputs a coweakening on the reference wire as it does not produce any assignment. As mentioned in the previous case, a dereliction is added on the input wire to force the non-deterministic choice and strip the exponential layer added by the set.

Abstraction (Figure 3)

The abstraction thunks the potential effects of the body \( M \) in the pure term \( \lambda x.M \). Following the monadic translation, the input effects are tensorized with the bound variable, and the output effects with the output of \( M \). Finally the whole term is put in an exponential box as it is a value.

Application (Figure 4)

Finally, the application put the routing area at use. Using the same terminology as in the introduction of this section, we see the effects \( e_1 \) and \( e_2 \) coming respectively from the evaluation of \( M \) and \( N \), and \( e_3 \), liberated by the body of the function being applied, plugged on the \( \delta \) area.
5.3 Full translation of $\lambda_{cES}$ into proof nets

We presented some interesting cases of the translation of $\lambda_C$ to proof nets to provide the reader with some intuition. However the complete translation is rather operating on the intermediate language $\lambda_{cES}$. A translation and a simulation theorem between $\lambda_C$ and $\lambda_{cES}$ are given in [6], completing the picture.

| Typing derivation | Translation |
|-------------------|-------------|
| $\Gamma, x : A \vdash x : (A, \emptyset)$ | (var) |
| $\Gamma \vdash \text{Unit} : (\text{Unit}, \emptyset)$ | (unit) |
| $x : A, \Gamma \vdash M : (\alpha, e)$ | (lam) |
| $\Gamma \vdash \lambda x. M : (A \rightarrow \alpha, \emptyset)$ | |
| $\Gamma \vdash : \lambda x. M : (\alpha, e)$ | |
| $\Gamma \vdash M \cdot N : (\alpha, e = e_1 \cup e_2 \cup e_3)$ | (app) |
We proceed to give the properties satisfied by the translation.

6 Properties of the translation

6.1 Simulation

The first result is the simulation theorem. The formulation precises that a deterministic step in \( \lambda_C \) is mapped to a deterministic reduction in nets, and only the reduction of \( \text{get}(r) \) produces a non-deterministic sum.

\[ \text{Theorem 18. Simulation} \]

Let \( \vdash M : (\alpha, e) \) be a closed well-typed term of \( \lambda_C \). Then

- If \( M \rightarrow N \) by (\( \beta_v \)) or (\( \text{set} \)) then \( M^* \rightarrow^* N^* \)
- If \( M \rightarrow N_i \) by (\( \text{get} \)) then \( M^* \rightarrow^* R + \sum_i N_i^* \)

The presence of the additional term \( R \) is linked to the local nature of non-determinism in proof nets. When facing a get reduction, the proof net can either select one of the available
assignment, or drop them all and wait for an hypothetical future one, which corresponds to this \( R \) net. This is better understood when stating the simulation theorem on \( \lambda_{cES} \):

**Theorem 19. Simulation for \( \lambda_{cES} \)**

Let \( \vdash M : (\alpha,e) \) be a closed well-typed term of \( \lambda_{cES} \). If \( M \rightarrow^* N \), then \( M^* \rightarrow^* N^* \).

Here, the term \( R \) is totally internalized in \( \lambda_{cES} \) and we get a clean simulation theorem. To derive this result, we start by defining a notion of typed context and a translation from contexts to nets.

**Definition 20. Hole typing rule**

\[
\frac{R \vdash (\alpha,e)}{R,[\cdot] : (\alpha,e) \vdash [\cdot] : (\alpha,e)} \quad \text{(hole)}
\]

\([\cdot]\) can be seen just as a special kind of variable in the typing derivation that can be substituted by something else than a value. We use the usual typing rules to build derivations of typed contexts. A context can then be translated to a net, with an interface determined by \( \alpha \) and \( e \), labelling free ports. This is what we call net contexts. The substitution of nets consists in plugging the translation of a term with a matching type in this interface.

**Definition 21. Hole translation**

We define the translation of a typed hole \( R \vdash [\cdot] : (\alpha,e) \) as

\[
\frac{R \vdash (\alpha,e)}{R,[\cdot] : (\alpha,e) \vdash [\cdot] : (\alpha,e)} \quad \text{(hole)}
\]

We can then carry on and use the usual term translation to build the translation of a context \((C[E])^*\), which is a net of the form

![Diagram of \((C[E])^*\)](image)

The substitution of \( M^* \), or of any net with a compatible interface for that matter, in \((C[E])^*\) is defined by just connecting the free wires of the substituted net to the corresponding free wire of the context hole:
The fundamental property of net contexts and net substitutions is that the substitution commutes with the translation, in the following sense:

\begin{itemize}
  \item Property 5. Nets substitution
\end{itemize}

Let $\vdash \cdot : (\alpha, e)$ be a typed hole, $\vdash C[E] : (\beta, e')$ a typed context, and $\vdash M : (\alpha, e)$ a term. Then:

$$(C[E[M]]^*) = (C[E])^*[M^*]$$

The very definition of net reduction immediately entails that if $M^* \rightarrow^* N^*$ then $(C[E])^*[M^*] \rightarrow^* (C[E])^*[N^*]$. Together with Property 5, this ensures that we can focus on the case where $C = E = \cdot$, as the general case follows seamlessly.

From here, we check that each reduction rule of $\lambda_{ES}$ can be simulated on the net side, relying on the definition of the reduction and the behavior of routing areas.

### 6.1.0.1 Variable substitutions reductions

Let us show the simulation for (\texttt{subst}) rules, involving mainly duplication. Thanks to the previous theorem, we can assume that contexts are empty without loss of generality. We consider only closed terms. Indeed, since reduction contexts $S, C, E$ do not bind variables, all the terms appearing in the premise of a rule are thus closed terms, and we can omit their context. For each rule, we write the translation of the premise followed by its reduction in nets, which matches the conclusion.

The fundamental rule is the variable one.

\begin{itemize}
  \item \texttt{(subst\_var)} $\sigma(x)$ undefined
  \item \texttt{(subst\_var)} $\sigma(x)$ defined
  \item \texttt{(subst\_unit)}
  \item \texttt{(subst\_app)}
  \item \texttt{(subst\_subst-r)}
  \item \texttt{(subst\_subst-r')}
\end{itemize}

When reaching a get(r) or a $\ast$, the substitution simply vanishes.

\begin{itemize}
  \item \texttt{(subst\_unit)}
  \item \texttt{(subst\_app)}
  \item \texttt{(subst\_subst-r)} and \texttt{(subst\_subst-r')} perform a duplication and propagate the substitution inside reference substitutions.
\end{itemize}
The `\texttt{subst}_{||}` just duplicate the variable substitution to the two threads.
Finally, the \(\text{subst}_\text{merge}\) distributes the outer substitution to both the term and the inner substitution.

6.1.0.2 Downward reference substitutions reductions

The propagation of references substitutions relies on the behavior of routing area, and especially Property 2. The fundamental case is the non deterministic reduction happening when reducing a get(\(r\)) whose redex is

Then for each \(V\) in the image of \(\mathcal{V}\), there will be exactly one summand of the following form
The remaining term does indeed reduce to the translation of \( \text{get(r)} \):

\[
\begin{align*}
\forall \quad & \quad \rightarrow^* \\
\forall \quad & \quad \rightarrow^*
\end{align*}
\]

\((\text{subst-r}_{\text{val}}), (\text{subst-r}_{||})\) and \((\text{subst-r}_{\text{app}})\) are just direct application of the Property 2.
(\textit{subst-r} \_app) →∗ (\textit{subst-r} \_merge) and (\textit{subst-r} \_subst-r') amount to nothing in nets, as the translation already identifies the redex and the reduct of these rules.
6.1.0.3 Upward reference substitutions reduction

As for downward substitutions, the main ingredient is the Transit lemma applies to our specific routing area $\delta$ and $\gamma$.

\[(\text{subst-}r'_1)\]
6.2 Termination

The second result states that the translation of a term is strongly normalizing:

▶ Theorem 22. Termination
The translation of a well typed term of $\lambda_C$ terminates.

The theorem that we will actually prove is rather the following:

▶ Theorem 23. Termination of $\lambda_{cES}$
The translation of the normal form of a well-typed term of $\lambda_{cES}$ terminates.

While $\lambda_{cES}$ is closer to nets that $\lambda_C$, unfortunately a normal form of $\lambda_{cES}$ is not translated to a normal form in proof nets. The corresponding net can still perform some reductions. But we will see that these are unessential and limited: in a few steps, a normal form is reached in nets.

We proceed in two stages: first, we extend $\lambda_{cES}$ to a language $\lambda_{cES}+$ that is able to do just a little more reductions than $\lambda_{cES}$. The extension of the translation and the simulation theorem for $\lambda_{cES}+$ are straightforward. Then, we show that $\lambda_{cES}+$ also terminates, give an explicit grammar for its normal forms, and show that the translation of these normal forms are strongly normalizing proof nets.

6.2.0.1 $\lambda_{cES}+$

We add a labelled reference substitution variable substitution to $\lambda_{cES}$: $M[|V|]_↓ \mid M[|x/N|]$. The first one, $M[|V|]_↓$, correspond to a downward reference substitution where $V$ has no free variables. The second one corresponds to a variable substitution of a term that is not a value. Both can appear during the reduction in proof nets but are not accounted for in $\lambda_{cES}$. $M[|x/N|]$ can’t be reduced, either in $N$ or by (subst) rules (whether $N$ is a value or not). The labelled reference forbids any reduction under it, except for the labelled reference substitutions. We extend the reduction rules (subst-r) to the labelled substitution $[|V|]_↓$. They can be performed anywhere (included under a variable substitution) except under abstraction, in a context defined by the following grammar: $J ::= [\_] \mid J\ M \mid M\ J \mid J[|V|] \_ \mid J[|σ|] \mid J[|x/N|]$.

We also add a $β$-rule to fire such labelled substitution:

$((β_ν)(λx.M))\ N[|V|]_λ \to M[|V|]_↓[|x/N|]$.

This corresponds to the additional reduction nets can do. Indeed a $β$ redex can always be fired in nets even if the argument is not a value. But then, the corresponding term do need to be reduced before any duplication, erasure or substitution. Let us now show termination and describe the normal forms of $\lambda_{cES}+$:

▶ Definition 6.1. $F$-normal forms
Let $M$ be a normal form of the form of $\lambda_{cES}$. It belongs to the grammar $M_{norm}$ (cf [6]). Then $M$ reduces to a sum of terms in $\lambda_{cES}+$ that belongs to the following grammar of $F$-normal forms:

$F_{norm} ::= get(r) \mid F_{norm}\ V[|V|]_λ \mid F_{norm}\ F_{norm}[|V|]_λ\ M[|x/F_{norm}|]$.

where the $M$ is in $\lambda_{cES}$(contains no labelled substitution).
Proof. By induction on the structure of $M$, with the additional hypothesis that only the additional rules are performed (no upward substitution):

- $M = \text{get}(r) : \text{ok}$
- $M = M_{\text{norm}} V[V]\lambda$ : by induction, $M_{\text{norm}}$ reduces to a sum of $F_{\text{norm}}$. Take one such summand $N$, then $M \rightarrow^* N_{\text{norm}} V[V]\lambda$ (because no rule (subst-r') was used)
- $M = M_{\text{norm}} M'_{\text{norm}}[V]\lambda$ : we proceed as the previous case
- $M = V M_{\text{norm}}[V]\lambda$ : by induction, $M_{\text{norm}}$ reduces to some $N$ in $F_{\text{norm}}$ grammar, so $M \rightarrow^* V N[V]\lambda$. Then, by inversion of typing rules, $V$ is of the form $\lambda x.P$ and we can apply the new $\beta$-reduction to get $P[[V]]_i[[x/N]]$. By pushing down $[[V]]_i$ in $P$, we can reduce it to a sum of $P_i$s where each $P_i$ do not contain labelled substitutions. Then $M \rightarrow^* \sum_i P_i[x/N]$

Lemma 24. F-normal forms are normal.

Proof. By induction :

- $M = \text{get}(r) : \text{ok}$
- $M = F_{\text{norm}} V[V]\lambda$ : by induction, $F_{\text{norm}}$ is normal, and values are not $F$-normal form, so the application can’t create any $\beta$-redex.
- $M = M_{\text{norm}} M'_{\text{norm}}[V]\lambda$ : same as the previous case
- $M = M[[x/F_{\text{normal}}]]$ : the explicit substitution prevents any reduction except (subst-r) ones for labelled reference substitution, but by definition, $M$ does not contain any.

6.2.0.2 F-normal forms and proof nets

We will see in the following that the translation of a $F$-normal form has not many possible reductions left. However, a few steps may remain to eventually reach a normal form. The first step is to collect and merge all the routing areas that are created and connected during the translation. Doing so, we separate the net between a part that closely follows the translated term structure, and a big routing area which connects various subterms to enable communication through references between them. Once this is done, a few starving reads may interact with the routing area, but nothing more. The following definition give the shape obtained after the merging of routing:

Definition 25. Separability

Let $R$ be the translation of a $\lambda_{\text{ES}}$ term, then $R$ is say to be separable if it can be reduced to the following form :

$$\begin{array}{c}
\text{R} \\
\downarrow \\
\text{S} \\
\downarrow \\
\text{O}
\end{array}$$

where $R$ is a routing area and $S$ a net with free wires labelled by $i_1, \ldots, i_n, o_1, \ldots, o_m, O$, satisfying :

(a) There is no redex in $S$
(b) $i_1, \ldots, i_n$ are either connected to the auxiliary port of a $\otimes$ cell, to the auxiliary port of a cocontraction or to the principal port of a dereliction
(c) $o_1, \ldots, o_m, O$ are either connected to a $\Upsilon$ cell, or to the principal door of an open box
The translation of term with at least one free variable is separable. The reason is that constructors such as application, substitution, parallel composition, etc. preserve separability. Moreover, the translation of values with free variables are obviously separable. The weakenings, which correspond to free variables, materialize as auxiliary doors of exponential boxes, thus blocking further reduction.

**Lemma 26. Open terms separability**
Let \( \Gamma, x : A \vdash M : (\alpha, e) \) a \( \lambda\text{ES} \) well-typed term, then its translation \( M^* \) is separable.

**Proof.**
We proceed by induction on terms.

**Values** As stated above, we can first observe that the translation of \( * \), of a variable \( x \) and an abstraction all satisfy the separability conditions. Indeed, they are composed of a box with at least one auxiliary door, and the inside of the box is a normal form (by induction for abstraction and trivially for others). As they are pure terms, \( m = n = 0 \).

**Get** It is almost the same as values, except that the output \( o \) corresponding to the reference of \( \text{get}(r) \) is connected to a dereliction, which is allowed in (b).

**Application** By induction, \( M^* \) and \( N^* \) are separable, thus can be decomposed in the following way:

![Diagram](image)

We can merge the 3 routing areas \( R, R_1 \) and \( \delta \). All the inputs or outputs previously connected to one of the small areas immediately satisfy the conditions (b,c) by IH. The two remaining wires \( i \) and \( o \) are respectively connected to the auxiliary port of a par and the auxiliary port of a cocontraction, thus satisfy (b,c). \( S \) and \( S_1 \) are normal by IH, and the translation of \( V \) is easily seen as normal. \( O \) is the conclusion of a par, hence satisfies (b,c). It only remains to see that the whole net excepted the routing areas is normal, but the only redex that could appear is the connection of a dereliction to \( M^* \). By (c), the output of \( M^* \) is either the conclusion of an open box or a par which do not form a redex. The condition (a) is verified.

**Parallel** Similar to application

**Variable substitution** As for application, we apply the IH on \( M^* \) and \( V^* \), merge the routing areas, and just check that the connection of \( V^* \) to \( M^* \) can not create new redexes using (b) on the output wire of \( V^* \).

**Reference substitutions** Again, the same technique is applied.
Finally, the main result we rely on as explained above is the following one:

**Lemma 27. Normal form separability**

*The translation of a $F$-normal form is separable.*

It is proved as Lemma 26 by induction on the syntax of $F$-normal forms. We can eventually prove from this last lemma:

**Lemma 28. Termination of $\lambda_{ES^+}$**

*The translation of a $F$-normal form is strongly normalizing.*

From which we deduce:

**Corollary 29. Strong normalization**

1. The translation of a closed well-typed term of $\lambda_{ES^+}$ is strongly normalizing.
2. The translation of a closed well-typed term of $\lambda_{ES}$ is strongly normalizing (Theorem 23).

**Proof.** Lemma 28

We apply 27 to reduce the translation of a $F$-normal form $F$ to a routing area $R$ connected to $S$ satisfying separability conditions. $S$ and $R$ are normal. The potential redexes must involve a wire at the interface of $R$ and $S$. The inputs of $R$ are connected to $S$, either to a par or to the conclusion of an open box and thus can’t form any redex with (co)weakenings and (co)contractions of $R$.

The outputs of $R$ are either connected to the auxiliary port of a tensor, the auxiliary port of a cocontraction or to a dereliction. Only the latter may form a new redex. If the output of $R$ is a coweakening, then everything reduces to 0. If it is just a wire, then the dereliction is connected through this wire to an input of $R$ which again can’t be part of any redex. The only remaining case is when the output of $R$ is a cocontraction tree. Then we can perform the non-deterministic $\Rightarrow$ reductions, and we go back to exactly the two previous cases as the dereliction is finally connected to a leaf of the tree of an input.

Hence, after we performed finitely many $\Rightarrow$ reductions, we finally get a normal form, which is either 0, or a sum of the previous net $S$ connected to simpler routing area $R_i$. ◀

Simulation and termination ensure that if a term $M$ reduces to $T = V_1 \parallel \ldots \parallel V_n$, then:

- By simulation, the translation $M^\bullet$ can be reduced to a non deterministic sum in which one summand will be $T^\bullet$.
- By termination, this does not depend on the path of reduction we chose: any reduction will converge to a normal form whose summands contain the net $T^\bullet$.

The normal form of $M^\bullet$ must contain the summand $T$. However, this does not tell us anything about what are the other summands of $M^\bullet$. Let us pretend the support for integers in $\lambda_C$. Assume that a term $M$ have 2 as only normal form. Because of this additional term $R$ appearing in the simulation theorem, our results do not prevent $M^\bullet$ to reduce to $1^\bullet + 2^\bullet + 3^\bullet$. In such a case, one may argue that the proof nets do not reflect faithfully the language as they may have a lot more possible outcomes.

### 6.3 Adequacy

The adequacy theorem states that the summands of the normal form of $M^\bullet$ are either the translation of a normal form $T = V_1 \parallel \ldots \parallel V_n$ that is a reduct of the original $M$, or garbage, that is a non correct net that corresponds to execution paths which deadlocked. We can recognize this garbage, thus eliminate it: with this additional operation, the summands of the normal form of $M^\bullet$ coincide with the values that are reachable by $M$. 
Theorem 30. Adequacy
Let $M$ be a well-typed term of $\lambda_C$. We write $Val(M) := \{ T = T, V_i \mid M \rightarrow^* T \}$. Similarly, for a net $N$ with normal form $N$, we define $Val(N) = \{ S \mid N = S + S', S \text{ is a value net} \}$. Then

$$Val(M) = Val(M^*)$$

We first need to prove adequacy between $\lambda_C$ and $\lambda_{ES}$:

Theorem 31. Adequacy for $\lambda_{ES}$
Let $P$ be a term of $\lambda_C$ and $M = \tilde{P}$ its translation in $\lambda_{ES}$, if $M \rightarrow^* \sum_i M_i$ a normal form, then $\forall i, P \rightarrow^* P_i$ such that $M_i \subseteq \tilde{P}_i$. In other words, any term appearing in the normal form of the translation of $P$ is bounded by the translation of a reduct of $P$.

In particular, applied to values, this gives the sought property for $\lambda_{ES}$:

Corollary 32. Let $P$ be a term of $\lambda_C$, $M = \tilde{P}$ its translation in $\lambda_{ES}$ such that $M \rightarrow^* M' + M''$ where $M'$ is a normal form.
- If $M' = V \parallel N$, then $P \rightarrow^* U \parallel Q$ with $V = \tilde{U}$.
- In particular, if $M' = || V_i$, then $P \rightarrow^* || U_i$ with $V_i = \tilde{U}_i$.

The only problematic case is the non-deterministic reduction ($\text{subst-r}_{\text{get}}$) which creates new summands. The proof consists in showing that these summands are actually limited in what they can do. Formally, they are bounded by the initial term that is being reduced, in the sense of the preorder $\sqsubseteq$ defined in [6]. The following lemma states that indeed the case of deterministic reduction is trivial:

Lemma 33. Values preservation
Let $M$ be a term of $\lambda_C$, and $M \rightarrow^* M'$ without using ($\text{subst-r}_{\text{get}}$). We define $NF(M) = \{ T \mid T \text{ normal and } M \rightarrow^* T \}$. Then $NF(M) = NF(M')$.

Proof. If we do not use ($\text{subst-r}_{\text{get}}$), the two reduction (full and non-deterministic) coincide and we use the confluence in $\lambda_C$. ▲

Lemma 34. Let $M = \tilde{P}$ be the translation of a $\lambda_C$ term, such that $M \rightarrow M'$. Then there exists $M'', N''$, such that $M' \rightarrow^* M'' \subseteq N''$ and $N \rightarrow^{(0,1)} N'$, with all reductions from $M'$ to $M''$ not being ($\text{subst-r}_{\text{get}}$).

Proof. The only redexes in $M$ are either premises of ($\text{subst-r}'$), ($\beta_v$) or ($\text{subst-r}_{\text{get}}$). In the first case, it corresponds to a reducible set whose reduction can be carried on in $M'$ by pushing the upward substitution to the top and pushing down the corresponding generated downward substitutions, to obtain the translation of $N'$ (which is $N$ where we the set is reduced). We can proceed the same way with ($\beta_v$) : this corresponds to a $\beta$-redex in $N$, where $N'$ is the result of reducing it, and $M''$ is obtained by pushing down the generated substitutions (variable and references).

Finally, if the reduction rule is ($\text{subst-r}_{\text{get}}$), then either it chose one of the available values and it corresponds exactly to a get reduction $N \rightarrow N'$, or it threw away available values, in which case $N = N'$, $M' = M''$ and clearly $M'' \subseteq N''$. ▲

Proof. Theorem 31
We proceed by induction on $\eta(M)$, the length of the longest reduction starting from $M$. If $\eta(M) = 0$, ie $M$ is a normal form, this is trivially true.
To prove the induction step, consider a reduct $M'$ of $M$. We use Lemma 34 to get $M' \rightarrow^* M'' \subseteq \tilde{P}'$ for some reduct $P'$ of $P$, such that the reduction to $M''$ doesn't use ($\text{subst-r}_{\text{get}}$). Thus, by
33 NF(M'') = NF(M'). By induction on M'', ∀T ∈ NF(M''), ∃Q, P →* Q and T ⊑ Q̃. But this is true for any reduct M', and we have NF(M) = ⋃M→M NF(M'), hence this is true for M.

From there, we get the result combining Theorem 31 and Theorem 19. One can extract from the proof of Lemma 27 that the translation of a summand that is not a parallel of values either reduces to 0, or to a net which is not a translation of a value.

7 Conclusion

In this paper, we presented a translation of a λ-calculus with higher order references and concurrency inside a fragment of differential proof nets. While several translations of effectful languages have been proposed in the literature, none supports this combination of features to our knowledge. We introduced a generalization of communication areas, routing areas, which turned out to be a useful device to encode references. More generally, we think that routing areas can be used to express various kind of non-deterministic, concurrent communications.

Modeling concurrency comes at a price, as terms such as (λx.set(r, x)) get(r) ∥ (λx.set(r, x)) get(r) translates to a net that do not respect the differential proofs nets correctness criterion [4]. Inside each thread, the set(r, x) depends on the get(r) whose value will replace the x variable. But of these get(r) may also depend from the set(r, x) of the other thread, creating a seemingly circular dependence which breaks the acyclicity required for correctness. This ambiguity is avoided at execution, as the first get(r) to reduce will be forced to chose an available assignment: these set-get dependencies are in fact mutually exclusive. But proof nets seems unable to express this subtlety. Differential LL seems to suffer from more fundamental limitations as a model of concurrency as pointed out by Mazza [12]. It is yet to be clarified how these results apply to the fragment presented here. While seriously limiting what can be modeled in proof nets, this might not be an obstacle when aiming for practical parallel or distributed implementations.

The enrichment of the source language, such as switching to a more realistic erase-on-write semantic for stores, or the addition of new effects and features (synchronization operations, sum types, divergence either by fixpoint or references, etc.) is the main focus of future work.

References

1 Roberto M. Amadio. On stratified regions. In Zhenjiang Hu, editor, Programming Languages and Systems: 7th Asian Symposium, APLAS 2009, Seoul, Korea, December 14-16, 2009. Proceedings, pages 210–225, Berlin, Heidelberg, 2009. Springer Berlin Heidelberg. doi:10.1007/978-3-642-10672-9_16.

2 Vincent Danos and Laurent Regnier. Reversible, irreversible and optimal λ-machines: Extended abstract. Electronic Notes in Theoretical Computer Science, 3(Supplement C):10 – 60, 1996. Linear Logic 96 Tokyo Meeting. URL: http://www.sciencedirect.com/science/article/pii/S1571066105804025 doi:https://doi.org/10.1016/S1571-0661(05)80402-9.

3 Thomas Ehrhard and Olivier Laurent. Interpreting a finitary pi-calculus in differential interaction nets. Information and Computation, 208(6):606 – 633, 2010. Special Issue: 18th International Conference on Concurrency Theory (CONCUR 2007). URL: http://www.sciencedirect.com/science/article/pii/S0890540110000155 doi:https://doi.org/10.1016/j.ic.2009.06.005.
4 Thomas Ehrhard and Laurent Regnier. Differential interaction nets. *Theoretical Computer Science*, 364(2):166–195, November 2006. 30 pages. URL: [https://hal.archives-ouvertes.fr/hal-00150274](https://hal.archives-ouvertes.fr/hal-00150274), doi:10.1016/j.tcs.2006.08.003.

5 Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50(1):1 – 101, 1987. URL: [http://www.sciencedirect.com/science/article/pii/0304397587900454](http://www.sciencedirect.com/science/article/pii/0304397587900454), doi: [https://doi.org/10.1016/0304-3975(87)90045-4](https://doi.org/10.1016/0304-3975(87)90045-4).

6 Yann Hamdaoui and Benoît Valiron. An interactive proof of termination for a concurrent \(\lambda\)-calculus with references and explicit substitutions. [http://yago.gb2n.org/papers/explicit-substs.pdf](http://yago.gb2n.org/papers/explicit-substs.pdf), 2018. [Online].

7 Ugo Dal Lago, Claudia Faggian, Benoît Valiron, and Akira Yoshimizu. Parallelism and synchronization in an infinitary context. In *Proceedings of the 2015 30th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, LICS ’15, pages 559–572, Washington, DC, USA, 2015. IEEE Computer Society. URL: [http://dx.doi.org/10.1109/LICS.2015.58](http://dx.doi.org/10.1109/LICS.2015.58).

8 Ian Mackie. *Applications of the Geometry of Interaction to language implementation*. PhD thesis, Univ. of London, 1994.

9 Ian Mackie. The geometry of interaction machine. In *Proceedings of the 22Nd ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*, POPL ’95, pages 198–208, New York, NY, USA, 1995. ACM. URL: [http://doi.acm.org/10.1145/199448.199483](http://doi.acm.org/10.1145/199448.199483).

10 Antoine Madet. *Complexité Implicite de Lambda-Calculs Concurrents*. Theses, Université Paris-Diderot - Paris VII, December 2012. URL: [https://tel.archives-ouvertes.fr/tel-00794977](https://tel.archives-ouvertes.fr/tel-00794977).

11 John Maraist, Martin Odersky, David N. Turner, and Philip Wadler. Call-by-name, call-by-value, call-by-need, and the linear lambda calculus. *Electronic Notes in Theoretical Computer Science*, 1(Supplement C):370 – 392, 1995. MFPS XI, Mathematical Foundations of Programming Semantics, Eleventh Annual Conference. URL: [http://www.sciencedirect.com/science/article/pii/S1571066104000222](http://www.sciencedirect.com/science/article/pii/S1571066104000222), doi: [https://doi.org/10.1016/S1571-0661(04)00022-2](https://doi.org/10.1016/S1571-0661(04)00022-2).

12 Damiano Mazza. The true concurrency of differential interaction nets. *Mathematical Structures in Computer Science*, FirstView:1–29, 11 2016. doi: [10.1017/S0960129516000402](10.1017/S0960129516000402).

13 Michele Pagani. Visible acyclic differential nets, part i: Semantics. *Ann. Pure Appl. Logic*, 163(3):238–265, 2012.

14 Marco Pedicini and Francesco Quaglia. Pelcr: Parallel environment for optimal lambda-calculus reduction. *ACM Trans. Comput. Logic*, 8(3), July 2007. URL: [http://doi.acm.org/10.1145/1243996.1243997](http://doi.acm.org/10.1145/1243996.1243997), doi: [10.1145/1243996.1243997](10.1145/1243996.1243997).

15 Jorge Sousa Pinto. Parallel implementation models for the lambda-calculus using the geometry of interaction. In *Proceedings of the 5th International Conference on Typed Lambda Calculi and Applications*, TLCA’01, pages 385–399, Berlin, Heidelberg, 2001. Springer-Verlag. URL: [http://dl.acm.org/citation.cfm?id=1754621.1754653](http://dl.acm.org/citation.cfm?id=1754621.1754653).

16 Paolo Tranquilli. Translating types and effects with state monads and linear logic. 14 pages, January 2010. URL: [https://hal.archives-ouvertes.fr/hal-00465793](https://hal.archives-ouvertes.fr/hal-00465793).