Research of the current distribution in solids using the Gibbs magnetodynamic principle

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Abstract. To find equilibrium distribution of surface and volumetric currents in continuous bodies Gibbs' thermodynamic hypothesis is used, which allows to solve a task using the magnetic principle of virtual work. Variation of the magnetic energy is considered under additional conditions, defining constancy of currents, two of which are of a differential nature, necessary and sufficient to solve the task in case of simply connected body. If the body under consideration is doubly-connected (e.g. a torus, a thick ring), one extra condition is necessary. It is shown in this paper that this condition, which makes the solution unique at the same time, can be the constancy of current flowing through the cross section of the torus or setting of a constant flux of magnetic induction through the torus hole. Setting up a problem, the first method is selected as more visual. The task is solved with the use of the method of Lagrange multipliers. The fundamental result of this paper is that magnetic induction and the volumetric current in a volume are vanished. Thus, the magnetic field with currents are extruded to the surface. Connections between obtained results, the Meissner-Oxenfield effect and the London equitation used in the theory of superconductivity are discussed, as well as the problem of connection of molecular currents and conduction currents.

1. Introduction

A structure of complex system such as magnetic properties of materials has been researched over a long time period, starting from the earliest works by authors such as G. Green (1871) [1] – section «application of the preliminary results to the theory of magnetism» and K.W. Thomson (1872) [2], in which results were collected from most known papers at that time, devoted to the subject nowadays [3-5].

In spite of colossal progress in practical usage of magnetism (i.e. from compass to electronic computer) the nature of this complex structure still remains unclear. It will suffice to mention a problem associated with the fact that it has not been possible yet to reduce laws of Coulomb and Bio-Savart to a single model, underlying the phenomenological approach to magnetism, although it’s clear, that the nature of this phenomena is naturally general.

In this paper the problem of distribution of volumetric and surface constant currents is being explored as well as characteristics of magnetic field in continuous solid body.

Therefore, in our opinion, any task solved in this field can push forward and expand our understanding of the magnetism basics.
2. Calculation of the magnetic energy variation of a doubly-connected structure in a thermostat

To derive the magnetic field variation we consider a doubly-connected body (torus) of volume $V$, with surface $S$, in which volumetric and surface currents with densities $\mathbf{j}$ and $\mathbf{t}$ flow, placed in thermostat with characteristics $V_o, T_o, P_o$. A perfectly conducting shell $S_o$ shields the magnetic field. External volumetric and surface forces $\rho \mathbf{j}, \mathbf{F}$, applied to the body, are given. Let us find the equilibrium body state and get the set of full necessary and sufficient conditions of body equilibrium criterion – thermal, mechanical and magnetic (Figure 1).

![Figure 1](image)

The general boundary value problem is reduced to:

\[
\begin{align*}
\text{rot} \left( \frac{1}{\mu_o \mu} \text{rot} \mathbf{A} (\mathbf{r}) \right) &= \mathbf{j} (\mathbf{r}) \quad \text{rot} \mathbf{A} = \mathbf{B} \quad \mathbf{r} \in V \\
\text{rot} \left( \text{rot} \mathbf{A}_0 (\mathbf{r}) \right) &= 0 \quad \text{rot} \mathbf{A}_0 = \mathbf{B}_0 \quad \mathbf{r} \in V_0
\end{align*}
\]

The Coulomb gauge of vector potentials is an additional condition:

\[
\begin{align*}
\mathbf{A} &= 0 \\
\mathbf{A}_0 &= 0
\end{align*}
\]

It is known that this calibration is used under consideration of non-relativistic magnetostatic problems unlike Lorenz calibration, which is used for dynamic problems.

Boundary conditions are:

\[
A_{ni}|_S = A_{0i}|_S; \quad \left[ \frac{1}{\mu} \text{rot} \mathbf{A} - \text{rot} \mathbf{A}_0 \right], \mathbf{n} = \mu_0 \mathbf{t}
\]

\[
|_{S_o} = 0 \quad \text{condition of ideal conductivity of the shell } S_o.
\]

To exclude variations of unknown values $\delta \mathbf{A}$ and $\delta \mathbf{A}_0$, energy and accordingly its variation will be calculated using two different methods.

\[
W_i = \frac{1}{2} \int_{\mu_i} B_i^2 dV + \frac{1}{2} \int_{S_i} B_i^2 dV \quad W_i = \frac{1}{2} \int_{V_i} (\mathbf{j}, \mathbf{A}) dV + \frac{1}{2} \int_{S_i} (\mathbf{t}, \mathbf{A}) dS
\]

\[
W = W_i = W_i \quad \partial W = 2 \partial W_i - \partial W_i
\]

Transformations lead to the following expression for the magnetic field variation:

\[
\partial W = \frac{1}{2} \mu_0 \int \left\{ \left( \frac{B^2}{\mu} - B_0^2 \right) \delta^* q - 2 B_{0i} \left( \mathbf{B}_0, \delta^* q \right) + 2 B_{si} \left( \mathbf{B}_0, \delta^* \mathbf{q} \right) \right\} dS + \frac{1}{2} \mu_0 \int \left[ \left( \delta^* \mathbf{q} \right) \mathbf{B} + \int \left( \delta \mathbf{j}, \mathbf{A} \right) dV + \oint \left( \delta^* \mathbf{t}, \mathbf{A} \right) dS \right]
\]

(1)

\[
\partial W = \frac{1}{2} \mu_0 \int \left\{ \left( \frac{B^2}{\mu} - B_0^2 \right) \delta^* q - 2 B_{0i} \left( \mathbf{B}_0, \delta^* q \right) + 2 B_{si} \left( \mathbf{B}_0, \delta^* \mathbf{q} \right) \right\} dS + \frac{1}{2} \mu_0 \int \left[ \left( \delta^* \mathbf{q} \right) \mathbf{B} + \int \left( \delta \mathbf{j}, \mathbf{A} \right) dV + \oint \left( \delta^* \mathbf{t}, \mathbf{A} \right) dS \right]
\]

(2)
Where $\delta \vec{j}$ and $\delta \vec{q}$ are density variations of free currents. As this takes place, we should note two circumstances. Intrinsic currents turn out to be excluded and free currents only remain. Variations of volumetric currents $\delta j$ are written in the common form using Euler coordinates system and variations of surface currents $\delta q$ are written using Lagrange variables. The point is that in Euler coordinates system the observation coordinates are strictly fixed and all occurring changes including variations as if “pass by” these points, remaining within body boundaries. But boundary surface is shifted by varying, too, while surface currents go aside with them traversing observation points. Therefore, it’s more physical, as we consider, to bind surface currents to the deformable boundary, which occurs to be the core of Lagrange variables. A similar remark can be applied to $\delta^* \vec{q}$—displacement variations.

3. Gibbs’ thermodynamic principle

Now then, let us consider isolated body-thermostat system, because it’s a system the well-known thermodynamic Gibbs is to be applied to in order to find any equilibrium conditions in any physical and physicochemical situations.

Furthermore, the obtained result is used for recording the magnetic principle of virtual work.

The only necessary and sufficient equilibrium condition, resulting from the magnetic principle of virtual work, is:

$$\delta(U - TS + pV - W) = \int \rho (\vec{f}, \delta \vec{q}) dV + \oint (\vec{F}, \delta \vec{q}) dS$$

(3),

the last term of sum on the left-hand side is, obviously, more properly to connect not with the principle of conservation of energy and the principle of minimum energy following from it, but with the principle of least action. Besides internal energy $U$ and terms $TS$ and $pV$, taking into account the conditions providing maintenance of constant conditions $(T_0, p_0)$ in the thermostat, the full energy of magnetic field $W$ is included under the sign of variation. Significantly, that it includes with the minus sign, not with a plus one, as in the electric field case. Variations of entropy and volume are written as usual:

$$\delta S = \int \rho \delta^* s dV, \quad \delta V = \oint \delta^* q_n dS.$$

Then equilibrium conditions of a solid magnet with intrinsic currents are reduced to the following:

a) thermal equilibrium: $T = T_0$, where:

$$T = \frac{\left( \frac{\partial u}{\partial s} \right)_{f}}{2\mu \mu}, \quad B^2 \left( \frac{\partial u}{\partial s} \right)_{f}$$

- local temperature determination;

b) surface equilibrium:

$$F_i = Q_{in} n_i - \delta^* n_i**, \quad \text{where the total stress tensor} \quad Q_{in} = P_{in} + T_{in} \quad \text{is composed of internal stress tensors} -$$

$$P_{in} = p_{in} - \frac{B^2}{2\mu \mu} \frac{\partial \mu}{\partial q_{in}} (\delta_{in} - q_{in}),$$

and the Maxwell stress tensor -

c) volume equilibrium:

$$\frac{\partial q_{in}}{\partial s_{in}} + \rho f_i = 0$$

Thus, based on the only necessary and sufficient equilibrium condition occurring from the magnetic principle of virtual works considered above, necessary and sufficient conditions of thermal and
mechanical equilibrium are found, which match these ones in the case of intrinsic currents, both for liquid and solid magnets. So the terms related to intrinsic currents will remain in base equation of the magnetic principle of virtual work. Other terms have led to equilibrium conditions with an allowance for intrinsic currents.

4. Conclusion

In conclusion, let us make several important remarks, in our opinion. The assumption of constancy in time of surface and volumetric currents implies the presence of regulated currents in the system, which is quite difficult to take into account, or accepting the assumption of the magnet ideality. Under ideality we understand very low resistance. In this regard the expounded material begins to intersect with the standard exposition of an introduction to the elementary theory of superconductivity. The obtained result with vanished induction of the magnetic field of an ideal conductor is closely related to the Meissner-Osenfeld effect [6-7]. However, in the usual exposition of this effect in most courses of superconductivity the case of an ideal conductor and the case of a superconductor are drawn near. And besides, in the comments to the pictures it is indicated that in the first case the field remains, and "real superconductors behave differently". According to the conclusions of this paper, in both cases there is no field inside. After the external field disconnection in the thickness of the ideal conductor, persistent currents arise, which, as shown above, also leads to vanishing the magnetic field, as the effect with a superconductor. Thus, the artificially created distinction between an ideal conductor and a superconductor is removed. Therefore, the illustration of this phenomenon passing from one book to another, on which the lines of force inside the body are indicated, raises doubts.

Let us note some more circumstances. It is known that in 1935, F. and G. London [8-9] suggested that the relation obtained from Maxwell's equations for a medium with low resistance and which was considered valid for \( \vec{B} : \Delta \vec{B} - \frac{1}{\lambda^2} B = 0 \), where \( \lambda^2 = \frac{m}{n \mu_0 e^2} \) is the London length, should be regarded as an equation for \( \vec{B} \). I.e. from the equality vanishing \( \frac{\partial}{\partial t} \left( \Delta \vec{B} - \frac{1}{\lambda^2} \vec{B} \right) = 0 \) a mathematically illogical assumption was made that the function had vanished. Then, the Meissner effect is explained, taking into account the solution of this equation:

\[
\vec{B} = \vec{B}_e \exp \left( -\frac{z}{\lambda} \right) \quad \text{(one-dimensional case)}
\]

Although this theory has no rigorous substantiation and is mathematically incorrect, it has become generally accepted. In this case, the London equation implies a certain characteristic thickness \( \lambda \), in which the field decreases exponentially.

It is important to note that many researchers perfectly understand all the incorrectness of the derivation of the London equations «...but for our purposes it is sufficient to consider it as an intuitive hypothesis fully justified by its success.» ([10] p.4).

It is interesting to note that a semi-intuitive explanation of this phenomenon is led by R.E. Peierls ([11] p.145):

«What has gone wrong is that we have assumed we are dealing with a number of complete circular orbits. In any finite volume, however, the electrons near the boundary cannot complete the circle. In addition to a number of complete circles, we must then consider also some circular arcs belonging to all those electrons whose orbits intersect the wall. These arcs amount together to a surface current which circles the volume in a sense opposite to that of the individual electron orbit, and which can easily be shown to cancel the effect of the complete orbits».

At first sight, there is no depth of field penetration in our model. In fact, if we take into account that the surface current is nothing more than a current distributed in a thin but finite surface layer, the thickness of this layer can be associated with the London length. Despite the seeming similarity with the Meissner effect about pushing out the magnetic field and currents to the surface, this is about different things. In the classical formulation, apparently, an external field is appeared to be, which is created, for example, by a toroidal coil (see Fig. 3), whereas we consider that fields created by our
own currents.

The second problem is that from all constructions discussed above consideration of so-called molecular currents is excluded, the contribution from which could lead to the creation of a constant magnetic field with spontaneous polarization. However, as it was shown in [12], it is not possible to put together formally the conduction currents and molecular currents in the framework of the complex form of Maxwell's equations, understanding at the same time that the nature of magnetism is naturally general, i.e. there is a physical difficulty in unambiguously dividing the total current flowing in this situation by the conduction current and the molecular magnetizing current. Note that a subject close to the subject matter was considered in a series of works by Schwinger [13-16]. However, both molecular and polarization currents disappeared from his treatment, and the equations were recorded, apparently, for a vacuum, and therefore the characteristics of the medium did not enter, which in our opinion seems not entirely correct.

In addition, the diamagnetic character of the phenomenon under consideration follows from the condition $0 = H = \mu_0 (H + M)$ and $M = M_0 + \chi_\mu H$; the magnetic susceptibility $\chi_\mu = -1$ (allowance for spontaneous conductivity, i.e. permanent magnets, was not carried out: $M_0 = 0$). It is clear that these quantities can't be infinitely large: $M = -H$ and therefore, apparently, as in the case of superconductors of the first kind, for some $H_1$ (for not entirely clear reasons) this dependence is destroyed and the diamagnetic moment of the magnet vanishes.

The obtained result has been successfully applied in the cases of a superconductive ball [12] and a cylindrical conductor [13], and, moreover, the results were different from generally accepted ones: the magnetic induction on the surface (more precisely, on a thickness commensurable with London depth induction value jumps).

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