On the Component Factor Group $G/G_0$

of a Pro-Lie Group $G$

Rafael Dahmen and Karl H. Hofmann

Abstract. A pro-Lie group $G$ is a topological group such that $G$ is isomorphic to the projective limit of all quotient groups $G/N$ (modulo closed normal subgroups $N$) such that $G/N$ is a finite dimensional real Lie group. A topological group is almost connected if the totally disconnected factor group $G_0 = G/G_0$ of $G$ modulo the identity component $G_0$ is compact. In this case it is straightforward that each Lie group quotient $G/N$ of $G$ has finitely many components. However, in spite of a comprehensive literature on pro-Lie groups, the following theorem, proved here, was not available until now:

**Theorem.** A pro-Lie group $G$ is almost connected if each of its Lie group quotients $G/N$ has finitely many connected components.

The difficulty of the proof is the verification of the completeness of $G_t$.

Mathematics Subject Classification 2010: 22A05, 22e15, 22e65, 22e99.

Key Words and Phrases: Pro-Lie groups, almost connected groups, projective limits.

Projective Limits of Almost Connected Lie Groups

A notorious problem in the structure theory of pro-Lie groups is the completeness of quotient groups, notably that of the group $G/G_0$ of connected components. In one of the sources on pro-Lie groups, [2], the section following Definition 4.24 on pp.195ff. exhibits some of the characteristic difficulties involving the completeness of quotients of a pro-Lie group $G$ in general and the quotient $G_1 = G/G_0$ in particular. In their entirety, these difficulties remain unresolved today. We shall settle the completeness issue of $G_1$ here for any pro-Lie group whose Lie group quotients have finitely many components.

Existing literature (see [4], Corollary 8.4) provides the following conclusion, which reinforces the independent interest in the result of this note:

An almost connected pro-Lie group $G$ contains a maximal compact subgroup $C$ and a closed subspace $V$ homeomorphic to $\mathbb{R}^J$ for a set $J$ such that $(e, x) \mapsto cx: C \times V \to G$

is a homeomorphism.

So, let $G$ denote a topological group and $\mathcal{N}(G)$ the set of all closed normal subgroups of $G$ for which $G/N$ is a Lie group. With these conventions we formulate a theorem, to be proved subsequently. The proof requires some effort. It is based on information from [2].
Theorem 1. For a pro-Lie group $G$, the following statements are equivalent:

1. $G_t$ is compact,
2. There is a compact totally disconnected subspace $D \subseteq G$ being mapped homeomorphically onto $G_t$ by the quotient map $q_t: G \to G_t$.
3. $(G/N)_t$ is finite for all $N \in \mathcal{N}(G)$.

The proof of the theorem will require the proof of some new lemmas and some references to existing literature. The first one is cited from [4], Main Theorem 8.1, Corollary 8.3.

Lemma 2. Let $G$ be an almost connected pro-Lie group. Then the following conclusions hold:

(i) $G$ contains a maximal compact subgroup $C$, and any compact subgroup of $G$ has a conjugate inside $C$.
(ii) $G = G_0C$.
(iii) $G$ contains a profinite subgroup $D$ such that $G = G_0D$.

For every compact group $K$ there is a compact totally disconnected subspace $D \subseteq K$ such that $(k,d) \mapsto kd : K_0 \times D \to K$ is a homeomorphism (see [3], Corollary 10.38, p. 573). From Lemma 2 we know that $G$ is almost connected iff there is a compact subgroup $K \subseteq G$ such that $G = G_0K$. Write $K = K_0D$ with the topological direct factor $D \subseteq K$ as we just pointed out. Then $G = G_0K_0D = G_0D$ and so $(g,d) \mapsto gd : G_0 \times D \to G$ is readily seen to be a homeomorphism. Thus, in Theorem 1 Condition (1) implies (2), and for (2)$\Rightarrow$(1) there is nothing to prove.

Let us establish that (1)$\Rightarrow$(3):
Assume $G/N$ to be a Lie group quotient of $G$. Then $(G/N)_0 = G_0N/N \cong G_0/(G_0 \cap N)$ (cf. [2], Lemma 3.29, p.152). Let $K$ be a compact subgroup of $G$ such that $G = G_0K$, and let $L = G/N$. Then $L = (G_0N/N)(KN/N) = L_0C$ for the compact Lie group $C = KN/N$. Thus $L_t = L_0C/L_0 \cong C/(C \cap L_0)$ is a compact totally disconnected Lie group and is therefore finite. This proves (3).

There remains a proof of the implication (3)$\Rightarrow$(1):
For the moment let us assume that the following hypothesis is satisfied

(H) $G_t$ is a complete topological group.

By [2], Corollary 3.31, hypothesis (H) implies that $G_t$ is prodiscrete, that is, $G_t = \lim_{N \in \mathcal{N}(G)} G_t/N$ where $G_t/N$ is discrete. Now $G_t/N$ is a Lie group quotient of $G$ and so is finite by (3). Hence $G_t$ is profinite and thus compact. This proves Condition (1).

It therefore remains to prove (H). For this purpose we shall invoke results from [2], pp.195ff.
Firstly, we define $\mathcal{M}(G)$ to be the subset of all $M \in \mathcal{N}(G)$ with the additional property that each open subgroup $N \subseteq M$ from $\mathcal{N}(G)$ has finite index in $M$. We shall then use

**Lemma 3.** If $G$ is a pro-Lie group such that

(*) each Lie group quotient $G/N$, $N \in \mathcal{N}(G)$ is almost connected, then $\mathcal{M}(G)$ is cofinal in $\mathcal{N}(G)$ and thus is a filter basis.

Moreover, $G$ is the strict projective limit of the $G/M$, $M \in \mathcal{M}(G)$.

**Proof.** For the proof see Lemma 4.25 in [2], pp.195 and 196.

We note that Lemma 4.25 in [2] states as hypothesis that $G$ is almost connected which implies (*). But the hypothesis (*) is all that is used in the proof of Lemma 4.25.

Any set $\mathcal{Z}$ of subsets of a set $G$ may be considered as a subbasis of closed sets for a topology. If $G$ is a topological group, and $\mathcal{Z}$ is the set of all cosets $gM$ with $g \in G$ and $M \in \mathcal{M}(G)$, then $\mathcal{Z}$ generates the set of closed sets of a topology on $G$, called the $\mathcal{Z}$-topology.

**Lemma 4.** The $\mathcal{Z}$-topology on a pro-Lie group $G$ satisfying Condition (*) of Lemma 3 is a compact $T_1$-topology.

**Proof.** See Proposition 4.27 of [2], pp. 197–201.

Again we note that Proposition 4.27 in [2] assumes the hypothesis that $G$ is almost connected, but the proof of the conclusion of Lemma 4 only uses Hypothesis (*) of Lemma 3.

We now adjust the proof of Theorem 4.28 on p. 202 of [2] for our purposes.

**Lemma 5.** Let $G$ be a pro-Lie group satisfying hypothesis (*) of Lemma 3. Then $G_\iota$ is complete.

We note right away that Lemma 5 will prove hypothesis (H) and therefore complete the proof of Theorem 1.

**Proof of Lemma 5.** We let $f: G \to G_\iota = G/G_0$ be the quotient morphism and consider a Cauchy filter $\mathcal{C}$ on $G_\iota$. We have to show that $\mathcal{C}$ converges. By Lemma 3, $\mathcal{M}(G)$ is cofinal in $\mathcal{N}(G)$. For each $N \in \mathcal{M}(G)$ let $N^* = \overline{f(N)}$ and let $p_N \cdot : G \to G_\iota/N^*$ be the quotient morphism. Then the image $p_{N^*}(\mathcal{C})$ is a Cauchy filter in the Lie group $G_\iota/N^*$ and thus has a limit $g_N$. Then $(g_N)_{N \in \mathcal{M}(G)} \in \prod_{N \in \mathcal{M}(G)} G_\iota/N^*$ is an element of $\lim_{N \in \mathcal{M}(G)} G_\iota/N^*$; indeed $\mathcal{C}$ has to converge to a point in the completion of $G_\iota$. Now let $F_N = (p_{N^*} \circ f)^{-1}(g_N)$. Then $\{F_N : N \in \mathcal{M}(G)\}$ is a filter basis consisting of cosets modulo $N_* = \overline{G_0N}$ of $G$. We claim that $N_* \in \mathcal{M}(G)$. Indeed we have $N_* \in \mathcal{N}(G)$. Now we let
$M$ be an open subgroup of $N_0 \supset G_0$. Then $G_0 \subseteq M$, so $G_0 N \subseteq MN$ and $MN$ is open-closed in $N_0 = \frac{G_0 N}{M}$. Thus $N_0 = MN$. So $MN/M \cong N/(N \cap M)$ is discrete and then $M \cap N$ is open in $N$. But $N \in \mathcal{M}(G)$ then implies that $N_0/M \cong N/(M \cap N)$ is finite. This shows that $N_0 \in \mathcal{M}(G)$ as claimed. Since $G$ is Z-compact by Lemma 4 we find an element $g \in \bigcap_{N \in \mathcal{M}(G)} F_N$. But then $p_N(f(g)) = g_N$ for all $N \in \mathcal{M}(G)$ which implies that $f(g) = \lim C$. Thus every Cauchy filter in $G_t$ converges showing that $G_t$ is complete.

This completes the proof of Theorem 1.

An inspection of [2] shows that the following questions appear to be unsettled:

**Question 1.** For which pro-Lie groups $G$ is $\mathcal{M}(G)$ cofinal in $\mathcal{N}(G)$?

For each of these groups $G$ we would know that $G$ is (isomorphic to) the strict projective limit $\lim_{M \in \mathcal{M}(G)} G/M$. In [2] this is proved of all almost connected pro-Lie groups.

Test examples are the nondiscrete pro-discrete groups $\mathbb{Z}^{(N)}$ and $\mathbb{Z}^N$ (see e.g. [2], Example 4.4ff., Proposition 5.2).

**Question 2.** For which pro-Lie groups is the Z-topology compact?

In [2], Proposition 4.27, this is shown for almost connected pro-Lie groups, and here we have proved it for those pro-Lie groups $G$ all of whose Lie group quotients are known to be almost connected.

Theorem 1 suggests the following rather general question:

**Question 3.** When is the projective limit of a projective system of almost connected topological groups almost connected?

Theorem 1 says that within the category of pro-Lie groups we have an affirmative answer for the projective system of all Lie group quotients. See also some background information in [2] in and around Theorem 1.27, p. 88.

One remark is in order in the context of the Z-topology discussed in [2] on pp. 197–203:

In Exercise E4.2(i), p. 202, it is pointed out that $\mathcal{M}(\mathbb{Z}) = \{\{0\}, \mathbb{Z}\}$ and that therefore the Z-topology on $\mathbb{Z}$, being the cofinite topology, is compact. Whereas the topology generated by the set of cosets $z + N$, $N \in \mathcal{N}(\mathbb{Z})$, the set of all subgroups of $\mathbb{Z}$, fails to be compact.

Theorem 1 will play a significant role in the authors’ study [1] of weakly complete real or complex topological algebras with identity, which will explore in detail their relation to pro-Lie theory and aims for a systematic treatment of weakly complete group algebras of topological groups and their representation and duality theories.

**Acknowledgment.** The authors thank the referee for his swift, yet thorough contributions to the final form of this note.
References

[1] R. Dahmen and K. H. Hofmann, *Weakly Complete Unital Algebras, Group Algebras, and Pro-Lie Groups*, in preparation.

[2] K. H. Hofmann and S. A. Morris, “The Lie Theory of Connected Pro-Lie Groups—A Structure Theory for Pro-Lie Algebras, Pro-Lie Groups, and Connected Locally Compact Groups,” European Mathematical Society Publishing House, 2006.

[3] K. H. Hofmann and S. A. Morris, “The Structure of Compact Groups,” 3rd Edition, De Gruyter Studies in Mathematics 25, De Gruyter, Berlin, 2013.

[4] K. H. Hofmann and S. A. Morris, *The Structure of Almost Connected Pro-Lie Groups*, J. of Lie Theory 21 (2011), 347–383.