HOMOLOGICAL SELECTIONS AND FIXED-POINT THEOREMS

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Abstract. A homological selection theorem for \( C \)-spaces, as well as a finite-dimensional homological selection theorem is established. We apply the finite-dimensional homological selection theorem to obtain fixed-point theorems for usco homologically \( UV^n \) set-valued maps.

1. Introduction

Banakh and Cauty [1, Theorem 8] provided a selection theorem for \( C \)-spaces, which is a homological version of the Uspenskij’s selection theorem [10, Theorem 1.3]. The aim of this paper was to establish a finite-dimensional form of Banach-Cauty theorem, which is the main tool in proving homological analogues of fixed-point theorems for usco maps established in [2], [4] and [7].

All spaces are assumed to be completely regular. Singular homology \( H_n(X; G) \), reduced in dimension 0, with a coefficient group \( G \) is considered everywhere below. By default, if not explicitly stated otherwise, \( G \) is a ring with unit \( e \). Following the notations from [1], for any space \( X \) let \( S_k(X; G) \) be the group of all singular chains with coefficients from \( G \) consisting of singular \( k \)-simplexes and \( S(X; G) \) denote the singular complex of \( X \), so \( S(X; G) \) is the direct sum \( \bigoplus_{k=0}^{\infty} S_k(X; G) \). The groups \( S_k(X; G) \) are linked via the boundary homomorphisms \( \partial : S_k(X; G) \to S_{k-1}(X; G) \).

If \( \sigma : \triangle^k \to X \) is a singular \( k \)-simplex (\( \triangle^k \) is the standard \( k \)-simplex), we denote by \( ||\sigma|| \) the carrier \( \sigma(\triangle^k) \) of \( \sigma \). Similarly, we put \( ||c|| = \bigcup_i ||\sigma_i|| \) for any chain \( c \in S_k(X; G) \), where \( c = \sum_i g_i\sigma_i \) is the irreducible representation of \( c \).

For an open cover \( U \) of \( X \) let \( S(X; U; G) \) stand for the subgroup of \( S(X; G) \) generated by singular simplexes \( \sigma \) with \( ||\sigma|| \subseteq U \) for some

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Let $A \subset B$ be two subsets of a space $X$. We write $A \overset{H_m}{\hookrightarrow} B$ if the embedding $A \hookrightarrow B$ induces a trivial homomorphism $H_m(A; G) \to H_m(B; G)$.

A set-valued map $\Phi : X \to 2^Y$ is called strongly lower semi-continuous (br., strongly lsc) if for each compact subset $K \subset Y$ the set $\{x \in X : K \subset \Phi(x)\}$ is open in $X$. For example, every open-graph set-valued map $\Phi : X \to 2^Y$ is strongly lsc, see [5, Proposition 3.2].

Here is our first homological selection theorem.

**Theorem 1.1.** Let $X$ be a paracompact $C$-space, $Y$ be an arbitrary space and $\Phi_k : X \to 2^Y$, $m = 0, 1, \ldots, n$, be a finite sequence of strongly lsc maps satisfying the following conditions, where $G$ is a fixed ring with unit:

(i) $\Phi_m(x) \overset{H_m}{\hookrightarrow} \Phi_{m+1}(x)$ for every $m = 0, \ldots, n - 1$ and every $x \in X$;
(ii) $H_m(\Phi_n(x); G) = 0$ for all $m \geq n$ and all $x \in X$.

Then there exists an open cover $U$ of $X$ and a chain morphism $\varphi : S(X, U; G) \to S(Y; G)$ such that $\varphi(S(U; G)) \subset S(\Phi_n(x); G)$ for every $U \in \mathcal{U}$ and every $x \in U$.

Let us mention that the Banakh-Cauty result [1, Theorem 8] is a particular case of Theorem 1.1 with $\Phi_m = \Phi_n$ for all $m$. There is also a finite-dimensional analogue of the above theorem.

**Theorem 1.2.** Let $X$, $Y$ and $G$ be as in Theorem 1.1. The same conclusion holds if $\dim X \leq n$ and the sequence of strongly lsc maps $\Phi_m : X \to 2^Y$ satisfies only condition (i).

Theorem 1.2 and [1, Theorem 7] imply the following fixed point theorem for usco (upper semi-continuous and compact-valued) maps:

**Theorem 1.3.** Let $X$ be a paracompact space with $\dim X \leq n$, $Y$ a compact metric AR, $G$ a field and $\Phi : X \to 2^Y$ be a homologically $UV^{n-1}(G)$ usco map. Then for every continuous map $g : Y \to X$ there exists a point $y_0 \in Y$ with $y_0 \in \Phi(g(y_0))$.

The particular case of Theorem 1.3 with $X = Y$ and $g = \text{id}_X$ is also interesting.

**Corollary 1.4.** Let $X$ be a compact metric AR with $\dim X \leq n$, $G$ a field and $\Phi : X \to 2^X$ be a homologically $UV^{n-1}(G)$ usco map. Then there exists a point $x_0 \in X$ with $x_0 \in \Phi(x_0)$. 
Recall that a closed subset $A$ of a metric space $X$ is called $UV^n$ in $X$ if every neighborhood $U$ of $A$ in $X$ contains another neighborhood $V$ such that the inclusion $V \hookrightarrow U$ generates trivial homomorphisms $\pi_k(V) \to \pi_k(U)$ between the homotopy groups for all $k = 0, \ldots, n$. If considering the homology groups $H_k(\cdot; G)$ instead of the homotopy groups $\pi_k(\cdot)$ (i.e. requiring $V \xrightarrow{H_m} U$ for all $m = 0, 1, \ldots, n$), then $A$ is said to be homologically $UV^n(G)$ in $X$. It follows from the universal formula coefficients that every $UV^n$-subset of $X$ is homologically $UV^n(G)$ in $X$ for all groups $G$. Moreover, following the proof of Proposition 7.1.3 from [8], one can show that $A$ is homologically $UV^n(G)$ in a given metric ANR-space $X$ if and only if it is homologically $UV^n(G)$ in any metric ANR-space that contains $A$ as a closed set. We say that $A$ is homotopically $UV^n$ in $X$ instead of $A$ being $UV^n$ in $X$. We also say that a set-valued map $\Phi : X \to 2^Y$ is homologically $UV^n(G)$ if all values $\Phi(x)$ are homologically $UV^n(G)$-subsets of $Y$.

**Theorem 1.5.** Let $X$ be a compact metric AR-space, $G$ a field and $\Phi : X \to 2^X$ be a homologically $UV^n(G)$ usco map. Then there exists a point $x_0 \in Y$ with $x_0 \in \Phi(x_0)$.

Theorem 1.3 was established by Gutev [7] for usco maps with homotopically $UV^n$ values. A homotopical version of Theorem 1.5 is also known, see Corollaries 3.6 and 5.14 from [4], or Theorem 1.3 from [7]. One can also show that if $X$ is a compact metric AR and $\Phi : X \to 2^X$ is a homologically $UV^n(G)$ usco map, then each value $\Phi(x)$ has trivial Čech homology groups with coefficients in $G$. So, in the particular case when $G$ is the group $\mathbb{Q}$ of the rationals, Theorem 1.5 follows from the more general [3, Theorem 7] treating the so-called algebraic AR’s. However, in the framework of usual AR’s Theorem 1.5 provides a very simple proof.

## 2. Homological selection theorems

In this section we prove Theorems 1.1 - 1.2. For any simplicial complex $K$ and an integer $m \geq 0$ let $K^{(m)}$ and $C_m(K; G)$ denote, respectively, the $m$-skeleton of $K$ and the group generated by the oriented $m$-simplexes of $K$ with coefficients in $G$.

We say that a chain morphism $\mu : C(K; G) \to S(A; G)$ (resp., $\mu : S(A; G) \to C(K; G)$), where $K$ is a simplicial complex and $A$ a topological space, is *correct* provided $\mu(v)$ is a singular 0-simplex in
\[ S(A; G) \] for every vertex \( v \in K^{(0)} \) (resp., \( \mu(\sigma) \) is a vertex of \( K \) for every singular 0-simplex \( \sigma \in S_0(A; G) \)).

**Lemma 2.1.** Suppose \( \{A\}_{k=0}^{m+1} \) is a sequence of subsets of \( Y \) with \( A_k \xrightarrow{H_k} A_{k+1}, k = 0, 1, \ldots, m \). Let \( L \) be a simplicial complex of dimension \( m \) and \( K \) be the cone of \( L \). Then every correct chain morphism \( \mu_m : C(L; G) \to S_m(A_m; G) \) such that \( \mu_m(C(L^0); G) \subset S_k(A_k; G) \) for all \( k \leq m \) can be extended to a correct chain morphism \( \mu_{m+1} : C(K; G) \to S_{m+1}(A_{m+1}; G) \) satisfying the following conditions:

- \( \mu_{m+1}(C(K^0); G) \subset S_k(A_k; G) \) for all \( k = 0, 1, \ldots, m+1; \)
- \( \mu_m \circ \partial_{m+1} = \partial_{m+1} \circ \mu_{m+1} \), where \( \mu_m = \mu_{m+1}|(C(K^m); G) \).

**Proof.** We first extend each morphism \( \mu_k = \mu_m|(C(L^k); G) \) to a morphism \( \tilde{\mu}_k : C(K^k); G) \to S_k(A_k; G) \) such that \( \tilde{\mu}_k \circ \partial_{k+1} = \partial_{k+1} \circ \tilde{\mu}_{k+1} \), \( k = 0, 1, \ldots, m-1 \). To this end, denote by \( v_0 \) the vertex of \( K \) and consider the augmentation \( \epsilon : S_0(A_0; G) \to G \) defined by \( \epsilon(\sigma) = e \) for all singular 0-simplexes \( \sigma \in S_0(A_0; G) \). Define \( \tilde{\mu}_0(\{v_0\}) \) to be a fixed singular simplex \( \sigma_0 \in S_0(A_0; G) \) and \( \tilde{\mu}_0(\{v\}) = \epsilon(\{v\}) \) for any \( v \in L^0 \). Then extend \( \mu_0 \) to a homomorphism \( \tilde{\mu}_0 : C(K^0; G) \to S_0(A_0; G) \) by linearity.

If \( \sigma = (v_1, v_2) \) is an 1-dimensional simplex in \( K \), then

\[ \tilde{\mu}_0(\partial_1(\sigma)) = \tilde{\mu}_0(v_2) - \tilde{\mu}_0(v_1). \]

Hence, \( \epsilon(\tilde{\mu}_0(\partial_1(\sigma))) = 0 \). Since \( A_0 \xrightarrow{H_0} A_1 \), there is a singular chain \( \tau_\sigma \in S_1(A_1; G) \) such that \( \tilde{\mu}_0(\partial_1(\sigma)) = \partial_1(\tau_\sigma) \). Letting \( \tilde{\mu}_1(\sigma) = \tau_\sigma \) if \( \sigma \in K^1 \setminus L^1 \) and \( \tilde{\mu}_1(\sigma) = \mu_1(\sigma) \) if \( \sigma \in L^1 \), we define the homomorphism \( \tilde{\mu}_1 \) on every simplex of \( K^{(1)} \). Then extend this homomorphism to \( \tilde{\mu}_1 : C(K^{(1)}; G) \to S_1(A_1; G) \) by linearity.

Because \( A_{k-1} \xrightarrow{H_{k-1}} A_k \), we can repeat the above construction to obtain the homomorphisms \( \tilde{\mu}_k \) for any \( k \leq m \). Then \( \tilde{\mu}_m : C(K^m; G) \to S_m(A_m; G) \). Since \( A_m \xrightarrow{H_m} A_{m+1} \), we can use once more the above arguments to obtain the chain morphism \( \mu_{m+1} : C(K; G) \to S_{m+1}(A_{m+1}; G) \) satisfying the required conditions. \( \square \)

**Lemma 2.2.** Let \( L \) be a simplicial complex with trivial homology groups and \( A \subset B \) be a pair of spaces. Then every correct chain morphism \( \nu : S(A; G) \to C(L; G) \) can be extended to a correct chain morphism \( \tilde{\nu} : S(B; G) \to C(L; G) \).

**Proof.** We are going to define by induction for each \( k \geq 0 \) a homomorphism \( \tilde{\nu}_k : S_k(B; G) \to C_k(L; G) \) extending \( \nu_k : S_k(A; G) \to C_k(L; G) \) such that \( \tilde{\nu}_k(\partial_{k+1}(c)) = \partial_{k+1}(\tilde{\nu}_{k+1}(c)) \) for every singular chain \( c \in S_{k+1}(B; G) \) and \( k \geq 0 \). For every singular 0-simplex \( \sigma \in S_0(B; G) \) we define \( \tilde{\nu}_0(\sigma) = \nu_0(\sigma) \) if \( \sigma \in S_0(A; G) \) and \( \tilde{\nu}_0(\sigma) = v_0 \) if \( \sigma \notin S_0(A; G) \),
where \( v_0 \) is a fixed vertex of \( L \). Then extend this homomorphism over \( S_0(B; G) \) by linearity. Because \( v \) is correct, \( \tilde{\nu}_0(\sigma) \) is a vertex of \( L \) for all singular 0-simplexes \( \sigma \in S(B; G) \).

To define \( \tilde{\nu}_1 \) we consider the augmentation \( \epsilon : C_0(L; G) \to G \) defined by \( \epsilon(v) = e \) for all vertexes of \( L \), see [9]. Thus, \( \epsilon(\tilde{\nu}_0(\partial_1(\sigma))) = 0 \) for every singular 1-simplex \( \sigma \in S_1(B; G) \). Because \( H_0(L; G) = 0 \), \( \partial_1(C_1(L; G)) = \epsilon^{-1}(0) \). Therefore, for every singular simplex \( \sigma \in S_1(B; G) \backslash S_1(A; G) \) there exists a chain \( c_\sigma \in C_1(L; G) \) such that \( \partial_1(c_\sigma) = \tilde{\nu}_0(\partial_1(\sigma)) \). We define \( \tilde{\nu}_1(\sigma) = \nu_1(\sigma) \) if \( \sigma \in S_1(A; G) \) and \( \tilde{\nu}_1(\sigma) = c_\sigma \) if \( \sigma \in S_1(B; G) \backslash S_1(A; G) \), and extend \( \tilde{\nu}_1 \) over \( S_1(B; G) \) by linearity.

Suppose the homomorphism \( \tilde{\nu}_k : S_k(B; G) \to C_k(L; G) \) was already constructed. Then, using that the kernel of the boundary homomorphism \( \partial_k : C_k(L; G) \to C_{k-1}(L; G) \) coincides with the image \( \partial_{k+1}(C_{k+1}(L; G)) \), we can define \( \tilde{\nu}_{k+1} \) extending \( \tilde{\nu}_k \) and satisfying the equality \( \tilde{\nu}_k \circ \partial_{k+1} = \partial_{k+1} \circ \tilde{\nu}_{k+1} \). □

**Proof of Theorem 1.1.** We modify the proof of [1, Theorem 8]. By induction we are going to construct two sequences of locally finite open covers of \( X \), \( \mathcal{V}_m = \{ \mathcal{V}_\alpha : \alpha \in \Gamma_m \} \) and \( \mathcal{W}_m = \{ \mathcal{W}_\alpha : \alpha \in \Gamma_m \} \), \( m \geq 0 \), an increasing sequence \( K_0 \subset K_1 \subset \ldots \) of simplicial complexes and correct chain morphisms \( \mu_m : C(K_m; G) \to S_m(Y; G) \) such that

(1) \( \mathcal{W}_\alpha \subset \mathcal{V}_\alpha \) for all \( \alpha \in \Gamma_m \), \( m \geq 0 \);
(2) \( \dim \mathcal{V}_m = m \);
(3) \( \mu_{m+1} \circ C(K_m; G) = \mu_m \) and \( \partial_{m+1} \circ \mu_{m+1} = \tilde{\mu}_m \circ \partial_{m+1} \), where \( \tilde{\mu}_m = \mu_{m+1} \circ C(K^{(m)}_m; G) \).

Moreover, for every \( m \) and \( \alpha \in \Gamma_m \) we shall assign a finite sub-complex \( L_\alpha \) of \( K_m \) and a set \( \Omega_\alpha = \bigcup_{\sigma \in L_\alpha} ||\mu_m(\sigma)|| \) satisfying the following conditions:

(4) \( \dim L_\alpha = m \) and \( L_\alpha \) is a cone whose base is a sub-complex \( M_\alpha \subset K_{m-1} \) and having a vertex \( \alpha \);
(5) If \( m \leq n \) and \( \alpha \in \Gamma_m \), then \( \Omega_\alpha \subset \Phi_m(x) \) and \( \Omega^{(k)}_\alpha \subset \Phi_k(x) \) for all \( k \leq m-1 \) and all \( x \in \mathcal{V}_\alpha \), where \( \Omega^{(k)}_\alpha = \bigcup_{\sigma \in L_\alpha} ||\mu_m(\sigma^{(k)})|| \);
(6) If \( m > n \) and \( \alpha \in \Gamma_m \), then \( \Omega_\alpha \subset \Phi_n(x) \) for all \( x \in \mathcal{V}_\alpha \).

To start our construction, for every \( x \in X \) we fix a point \( y_x \in \Phi_0(x) \) and consider the set \( O_x = \{ x' \in X : y_x \in \Phi_0(x') \} \). Since \( \Phi_0 \) is strongly lsc, \( O_x \) is open in \( X \). Let \( \mathcal{V}_0 = \{ \mathcal{V}_\alpha : \alpha \in \Gamma_0 \} \) be a locally finite open cover of \( X \) refining the cover \( \{ O_x : x \in X \} \), and choose \( \mathcal{W}_0 = \{ \mathcal{W}_\alpha : \alpha \in \Gamma_0 \} \) to be a locally finite open cover of \( X \) with \( \mathcal{W}_\alpha \subset \mathcal{V}_\alpha \) for all \( \alpha \in \Gamma_0 \). Let the complex \( K_0 \) be the zero-dimensional complex whose set of vertices is \( \Gamma_0 \). For every \( \alpha \in \Gamma_0 \) we set \( L_\alpha = \{ \alpha \} \) and choose \( x_\alpha \in X \) such that \( \mathcal{V}_\alpha \subset O_{x_\alpha} \). Define \( \mu_0 : C(K_0; G) \to S_0(Y; G) \) to be
the homomorphism assigning to each generator corresponding to \( \alpha \) the singular 0-simplex \( y_{x_\alpha} \), and let \( \Omega_\alpha = \{ y_{x_\alpha} \} \). Obviously, \( \mu_0 \) is correct.

Suppose for some \( m < n - 1 \) and all \( k \leq m \) we already performed the construction satisfying conditions (1) – (5). Then for every \( x \in X \) choose an open neighborhood \( G_x \) of \( x \) meeting only finitely many elements of the cover \( \bigcup_{k \leq m} V_k \) such that \( G_x \subset V_\alpha \) for all \( \alpha \in \bigcup_{k=0}^m \Gamma_k \) with \( G_x \cap W_\alpha \neq \emptyset \). Let \( J(x) = \{ \alpha \in \bigcup_{k=0}^m \Gamma_k : G_x \subset V_\alpha \} \) and \( D_x^{(k)} = \bigcup \{ \Omega_\alpha^{(k)} : \alpha \in J(x) \} \), \( k \leq m \). Since \( J(x) \) is finite, all \( D_x^{(k)} \) are compact subsets of \( Y \) with \( D_x^{(k)} \subset D_x^{(m)} = D_x \). Moreover, condition (5) implies \( D_x \subset \Phi_m(x) \). Consider the finite sub-complex \( M_x = \bigcup \{ L_\alpha : \alpha \in J(x) \} \) of \( K_m \) and the cone \( L_x \) with a vertex \( v_x \notin K_m \) and a base \( M_x \). Then, according to the definition of \( \Omega_\alpha \) and condition (5), we have \( \mu_m(C(M_x^{(k)}; G)) \subset S_k(D_x^{(k)}; G) \subset S_k(\Phi_k(x); G) \), \( k \leq m \). Therefore, we can apply Lemma 2.1 to find a correct chain morphism 

\[
\mu_x : C(L_x; G) \rightarrow S_{m+1}(\Phi_{m+1}(x); G)
\]

extending \( \mu_m(C(M_x; G)) \) such that \( \mu_x(C(L_x^{(k)}; G)) \rightarrow S_k(\Phi_k(x); G) \) and \( \partial_{m+1} \circ \mu_x = (\mu_x|C(L_x^{(m)}; G)) \circ \partial_x \), where \( \partial_x : C(L_x; G) \rightarrow C(L_x^{(m)}; G) \) is the boundary homomorphism. Then \( \Omega_x = \bigcup_{\sigma \in L_x} \{ \mu_x(\sigma) \} \) is a compact subset of \( \Phi_{m+1}(x) \) containing \( D_x \). The strong lower semi-continuity of \( \Phi_{m+1} \) yields that \( O_x^m = \{ x' \in G_x : \Omega_x \subset \Phi_m(x') \} \) is an open neighborhood of \( x \). So, there exists a locally finite open cover \( M_{m+1} = \{ V_\alpha : \alpha \in \Gamma_{m+1} \} \) of \( X \) refining the cover \( \{ O_x^m : x \in X \} \), and take a locally finite open cover \( \mathcal{W}_{m+1} = \{ W_\alpha : \alpha \in \Gamma_{m+1} \} \) satisfying condition (1). Now, for every \( \alpha \in \Gamma_{m+1} \) choose \( x_\alpha \in X \) with \( V_\alpha \subset O_x^m \) and let \( L_\alpha \) be the cone with base \( M_{x_\alpha} \) and vertex \( \alpha \). Define \( K_{m+1} \) to be the union \( K_m \cup \bigcup_{\alpha \in \Gamma_{m+1}} L_\alpha \). Identifying the cones \( L_\alpha \) and \( L_{x_\alpha} \), we define the correct morphism

\[
\mu_{m+1} : C(K_{m+1}; G) \rightarrow S_{m+1}(Y; G)
\]

by \( \mu_{m+1}|C(K_{m}; G) = \mu_m \) and \( \mu_{m+1}|C(L_\alpha; G) = \mu_{x_\alpha} \). Finally, let \( \Omega_\alpha = \Omega_{x_\alpha} \). It is easily seen that conditions (1) – (5) are satisfied. Moreover, the definition of \( G_x \) and the inclusion \( O_x^m \subset G_x \) yield that

\[
(7) \quad \text{For every } \beta \in \bigcup_{k=0}^m \Gamma_k \text{ and } \alpha \in \Gamma_{m+1} \text{ with } W_\alpha \cap W_\beta \neq \emptyset \text{ we have } W_\alpha \subset V_\beta, \text{ and thus } L_\beta \subset L_\alpha.
\]

In this way we can perform our construction for all \( m \leq n \). If we substitute \( \Phi_m = \Phi_n \) for all \( m \geq n \), we have also \( \Phi_m(x) \rightarrow \Phi_{m+1}(x) \) because \( H_m(\Phi_n(x); G) = 0 \) for all \( x \in X \). Therefore, following the above arguments, we can perform for all \( m \) the steps from \( m \) to \( m + 1 \) satisfying conditions (1) – (7).

Let \( K = \bigcup_{m=0}^{\infty} K_m \). Then the morphisms \( \mu_m \) define a correct chain morphism \( \mu : C(K; G) \rightarrow S(Y; G) \). Because \( X \) is a \( C \)-space, there
exists a sequence of disjoint open families \( \mathcal{U}_m = \{ U_\lambda : \lambda \in \Lambda_m \} \), \( m \geq 0 \), such that each \( \mathcal{U}_m \) refines \( \mathcal{W}_m \) and the family \( \mathcal{U} = \bigcup_{m=0}^{\infty} \mathcal{U}_m \) covers \( X \).

The final step of the proof is to construct a chain morphism from \( S(X, \mathcal{U}; G) \) into \( C(K; G) \). To this end, let \( \Lambda = \bigcup_{k=0}^{\infty} \Lambda_k \) and \( \Lambda(m) = \bigcup_{k=0}^{m} \Lambda_k \). Consider also the sub-complexes \( S_m = \sum_{\lambda \in \Lambda(m)} S(U_\lambda; G) \) of \( S(X; G) \), \( m \geq 0 \), whose union is \( S(X, \mathcal{U}; G) \). For every \( \lambda \in \Lambda_m \) select an \( \alpha_\lambda \in \Gamma_m \) with \( U_\lambda \subset W_{\alpha_\lambda} \). We are going to construct a correct chain morphism \( \nu : S(X, \mathcal{U}; G) \to C(K; G) \) such that

\[
(8) \quad \nu(S(U_\lambda; G)) \subset C(L_{\alpha_\lambda}; G) \quad \text{for all } \lambda \in \Lambda.
\]

For any \( \lambda \in \Lambda_0 \) the complex \( L_{\alpha_\lambda} \) is a single point. So, we can find a chain morphism \( \nu_\lambda : S(U_\lambda; G) \to C(L_{\alpha_\lambda}; G) \). Since the family \( \mathcal{U}_0 \) is disjoint, \( S_0 \) is the direct sum of all \( S(U_\lambda; G) \), \( \lambda \in \Lambda_0 \). Hence, the chain morphism \( \nu_0 : S_0 \to C(K; G) \) with \( \nu_0|S(U_\lambda; G) = \nu_\lambda \) for all \( \lambda \in \Lambda_0 \) is well defined and \( \nu_0(S(U_\lambda; G)) \subset C(L_{\alpha_\lambda}; G) \).

Suppose that for some \( m \) we have constructed correct chain morphisms \( \nu_k : S_k \to C(K; G) \), \( k \leq m \), such that \( \nu_k \) extends \( \nu_{k-1} \) and \( \nu_k(S(U_\lambda; G)) \subset C(L_{\alpha_\lambda}; G) \) for all \( \lambda \in \Lambda(k) \). Because \( \mathcal{U}_{m+1} \) is a disjoint family, so is the family \( \{ S(U_\lambda; G) : \lambda \in \Lambda_{m+1} \} \). Therefore, to extend \( \nu_m \) over \( S_{m+1} \), it suffices for every \( \lambda \in \Lambda_{m+1} \) to extend \( \nu_m|S(U_\lambda; G) \cap S_m \) over \( S(U_\lambda; G) \). To this end, observe that if \( \lambda \in \Lambda_{m+1} \) and \( \lambda' \in \Lambda(m) \) with \( U_\lambda \cap U_{\lambda'} \neq \emptyset \), then \( W_{\alpha_\lambda} \cap W_{\alpha_{\lambda'}} \neq \emptyset \). Thus, according to condition (7), \( L_{\alpha_{\lambda'}} \subset L_{\alpha_\lambda} \). Consequently, by (8), \( \nu_m(S(U_\lambda; G) \cap S_m) \subset C(L_{\alpha_\lambda}; G) \) for any \( \lambda \in \Lambda_{m+1} \). Since \( L_{\alpha_\lambda} \) is contractible and \( \nu_m \) is correct, we can apply Lemma 2.2 (with \( A = U_\lambda \cap \bigcup_{\lambda' \in \Lambda(m)} U_{\lambda'} \) and \( B = U_\lambda \)) to find a correct chain morphism \( \nu_\lambda : S(U_\lambda; G) \to C(L_{\alpha_\lambda}; G) \) extending \( \nu_m(S(U_\lambda; G) \cap S_m) \). This completes the induction, so the construction of the required chain morphism \( \nu : S(X, \mathcal{U}; G) \to C(K; G) \) is done.

Finally, let \( \varphi : S(X, \mathcal{U}; G) \to S(Y; G) \) be the composition \( \varphi = \mu \circ \nu \).

Then, according to (7) and the definitions of \( \Omega_{\alpha} \), for every \( \lambda \in \Lambda \) we have

\[
\varphi(S(U_\lambda; G)) \subset \mu(C(L_{\alpha_\lambda}; G)) \subset S(\Omega_{\alpha_\lambda}; G).
\]

Since \( U_\lambda \subset W_{\alpha_\lambda} \), conditions (4) and (5) yield that \( \Omega_{\alpha_\lambda} \subset \Phi_n(x) \) for all \( x \in U_\lambda \). Therefore, \( \varphi(S(U_\lambda; G)) \) is contained in \( S(\Phi_n(x); G) \) whenever \( x \in U_\lambda \). \( \square \)

**Proof of Theorem 1.2.** Since the sequence \( \{ \Phi_m \}_{m=0}^{n} \) satisfies condition (i) from Theorem 1.1, we can perform the construction from the proof of Theorem 1.1 for every \( m = 0, 1, \ldots, n \). So, we construct the locally finite covers \( \mathcal{V}_m = \{ V_\alpha : \alpha \in \Gamma_m \} \) and \( \mathcal{W}_m = \{ W_\alpha : \alpha \in \Gamma_m \} \) of \( X \), the complexes \( K_0 \subset K_1 \subset \ldots \subset K_n \), the sets \( \Omega_{\alpha} \) for any \( \alpha \in \bigcup_{k=0}^{n} \Gamma_k \) and the correct chain morphisms \( \mu_m : C(K_m; G) \to S_m(Y; G) \) satisfying conditions (1) – (5) and the particular case of condition (7)
with \( m \leq n - 1 \). Since the complex \( K = \bigcup_{m=0}^n K_m \) is \( n \)-dimensional, \( K^{(m)} = \emptyset \) for all \( m > n \). So, we can suppose that \( \mu_m = \mu_n \) for \( m \geq n \).

In this way we obtains a chain morphism \( \mu : C(K; G) \to S(Y; G) \).

Because \( \dim X \leq n \), according to Corollary 5.3 from [6], for every \( m = 0, 1, \ldots, n \) there exists a disjoint family \( \mathcal{U}_m = \{ U_\lambda : \lambda \in \Lambda_m \} \) such that each \( \mathcal{U}_m \) refines \( \mathcal{W}_n \) and the family \( \mathcal{U} = \bigcup_{m=0}^n \mathcal{U}_m \) covers \( X \). Then, repeating the arguments from the final part of the proof of Theorem 1.1, we construct the required chain morphism \( \varphi : S(X, \mathcal{U}; G) \to S(Y; G) \).

\[ \square \]

3. Fixed-point theorems for homologically \( UV^n(G) \) usco maps

In this section we prove Theorems 1.3 and 1.5. For a set-valued map \( \Phi : X \to 2^Y \) we denote by \( \mathcal{O}(\Phi) \) the family of the open-graph maps \( \Theta : X \to 2^Y \) such that \( \Phi(x) \subset \Theta(x) \) for all \( x \in X \). Next proposition is a homological version (and its proof is a small modification) of [3, Proposition 4.2].

**Proposition 3.1.** Let \( X \) be a paracompact space, \( Y \) be a space and let \( \Phi : X \to 2^Y \) be an usco map such that for every \( x \in X \) and a neighborhood \( U \) of \( \Phi(x) \) there exists a neighborhood \( V \) of \( \Phi(x) \) with \( V \xrightarrow{H_m} U \). Then for every \( \varphi \in \mathcal{O}(\Phi) \) there exists \( \Theta \in \mathcal{O}(\Phi) \) such that \( \Theta(x) \) is open in \( Y \) and \( \Theta(x) \xrightarrow{H_m} \varphi(x) \) for all \( x \in X \).

**Proof.** Let \( \varphi \in \mathcal{O}(\Phi) \). Then the graph \( G(\varphi) \) is open in \( X \times Y \) and contains the compact set \( \{ x \} \times \Phi(x) \) for every \( x \in X \). So, there exist neighborhoods \( W_1(x) \) and \( U(x) \) of \( x \) and \( \Phi(x) \), respectively, with \( W_1(x) \times U(x) \subset G(\varphi) \). Thus, \( \Phi(x) \subset U(x) \subset \varphi(x') \) for all \( x' \in W_1(x) \).

Then \( V(x) \xrightarrow{H_m} U(x) \) for some open neighborhood \( V(x) \) of \( \Phi(x) \). Since \( \Phi \) is upper semi-continuous, we can find a neighborhood \( W(x) \subset W_1(x) \) such that \( x' \in W(x) \) implies \( \Phi(x') \subset V(x) \). Hence, for all \( x' \in W(x) \) we have

\[ \Phi(x') \subset \overline{V(x)} \xrightarrow{H_m} U(x) \subset \varphi(x'). \]

Next, let \( \gamma = \{ P_\alpha : \alpha \in A \} \) be a locally finite closed cover of \( X \) refining the cover \( \{ W(x) : x \in X \} \) (recall that \( X \) is paracompact), and for every \( \alpha \) fix \( x_\alpha \in X \) such that \( P_\alpha \subset W(x_\alpha) \). For every \( x \in X \) the set \( A(x) = \{ \alpha \in A : x \in P_\alpha \} \) is finite, and define \( \Theta(x) = \bigcap\{ V(x_\alpha) : \alpha \in A(x) \} \). One can show that \( \Theta \) is open-graph (see the proof of [3, Proposition 4.2]). Moreover, since \( x \in \bigcap\{ W(x_\alpha) : \alpha \in A(x) \} \), it follows from (9) that

\[ \Phi(x) \subset \overline{V(x_\alpha)} \xrightarrow{H_m} U(x_\alpha) \subset \varphi(x) \]
for all \( \alpha \in A(x) \). This yields \( \Phi(x) \subset \Theta(x) \subset \Theta(x) \overset{H_m}{\hookrightarrow} \varphi(x) \).

**Proof of Theorem 1.3.** Let \( g : Y \to X \) be a continuous (single-valued) map. Without loss of generality, we may assume that \( g(Y) = X \). We need to show that the set-valued map \( \Phi_g = \Phi \circ g : Y \to 2^Y \) has a fixed-point. Suppose this is not true. So \( y \notin \Phi_g(y) \) for all \( y \in Y \), or equivalently \( \Phi(x) \subset Y \setminus g^{-1}(x) \) for all \( x \in X \). Consider the set-valued map \( \varphi : X \to 2^Y \), \( \varphi(x) = Y \setminus g^{-1}(x) \). Then \( \Phi \) is a selection for \( \varphi \) and it is easily seen that \( \varphi \) has an open graph. Because \( \Phi \) is homologically \( UV^{n-1}(G) \), we can apply Proposition 3.1 to find for each \( m = 0, 1, \ldots, n \) a set valued map \( \Theta_m \): \( X \to 2^X \) such that

- \( \Phi(x) \subset \Theta_0(x), x \in X \);
- \( \Theta_m(x) \overset{H_m}{\hookrightarrow} \Theta_{m+1}(x) \) for all \( x \in X \) and \( m = 0, \ldots, n - 1 \);
- Each \( \Theta_n(x) \) is open in \( Y \) and \( \Theta_n(x) \subset \varphi(x), x \in X \).

Then, according to Theorem 1.2, there exists an open cover \( \mathcal{U} \) of \( X \) and a chain morphism \( \phi : S(X, \mathcal{U}; G) \to S(Y; G) \) such that \( \phi(S(U; G)) \subset S(\Theta_n(x); G) \) for every \( U \in \mathcal{U} \) and every \( x \in U \). Consider the open cover \( \mathcal{U}_g = g^{-1}(\mathcal{U}) \) of \( Y \) and the chain morphism \( g_t : S(Y, \mathcal{U}_g; G) \to S(X, \mathcal{U}; G) \) generated by \( g \). Then \( \phi_g = \phi \circ g_t : S(Y, \mathcal{U}_g; G) \to S(Y; G) \) is a chain morphism with

\[
\text{(10) } \phi_g(S(g^{-1}(U); G)) \subset S(\Theta_n(g(y)); G) \quad \text{for all } U \in \mathcal{U} \quad \text{and } y \in g^{-1}(U).
\]

So, we can apply the homological fixed-point theorem \([1, \text{Theorem 7}]\) to conclude that the chain morphism \( \phi_0 \) has a fixed point \( y_0 \in Y \). This means that for any neighborhood \( W \subset Y \) of \( y_0 \) there is a chain \( c_W \in S(W; G) \cap S(Y, \mathcal{U}_g; G) \) such that \( ||\phi_0(c_W)|| \cap W \neq \emptyset \). Choose \( U_0 \in \mathcal{U} \) with \( y_0 \in g^{-1}(U_0) \) and let \( V \subset g^{-1}(U_0) \) be a neighborhood of \( y_0 \). Then \( c_V \in S(V; G) \subset S(g^{-1}(U_0); G) \). Thus, we have

\[
\text{(11) } ||\phi_g(c_V)|| \cap V \neq \emptyset \quad \text{and, by (10), } ||\phi_g(c_V)|| \subset \Theta_n(g(y_0)).
\]

On the other hand, since \( \Theta_n(x_0) \subset \varphi(x_0) \), where \( x_0 = g(y_0) \), we can choose \( V \) to be so small that \( V \cap \Theta_n(x_0) = \emptyset \). The last relation contradicts condition (11).

**Proof of Theorem 1.5.** The arguments from the proof of \([5, \text{Theorem 1.3}]\) work in our situation. For completeness, we provide a sketch. Since \( X \) can be embedded in the Hilbert cube \( Q \) as a retract, we may suppose that \( \Phi : Q \to 2^Q \) is a homologically \( UV^\omega(G) \) usco map. Identifying \( Q \) with the product \( \prod \{ \mathbb{I}_k : k \in \mathbb{N} \} \), where \( \mathbb{I} = [0, 1] \), let \( \pi_n : Q \to \prod \{ \mathbb{I}_k : k \leq n \} \) be the projection onto the cube \( \mathbb{I}^n \) and \( h_n : \mathbb{I}^n \to Q \) be the embedding assigning to every point \( x = (x_1, \ldots, x_n) \in \mathbb{I}^n \) the point \( h(x) \) having the same first \( n \)-coordinates and all other coordinates 0.
For every $n$ consider the homologically $UV^\omega(G)$ usco map $\Phi_n : \mathbb{I}^n \to 2^Q$ defined by $\Phi_n(x) = \Phi(h_n(x))$. Then, according to Theorem 1.3 (with $X = \mathbb{I}^n$, $Y = Q$, $g = \pi_n$ and $\Phi = \Phi_n$), there is a point $x^n \in Q$ with $x^n \in \Phi_n(\pi_n(x^n))$. If $x^0 \in Q$ is the limit point of a convergent subsequence of $\{x^n\}_{n \geq 1}$, one can see that $x^0 \in \Phi(x^0)$. □

4. Fixed point-theorems for homological $UV^n(G)$ and $UV^\omega(G)$ decompositions

In this section we provide some fixed point-theorems for homological $UV^n$ or homological $UV^\omega(G)$ decompositions of compact metric AR’s, where $G$ is a field. Our results are homological analogues of for homotopical $UV^n$ and $UV^\omega$ decompositions, see [2, Theorems 3-4] and [5, Theorems 7.1-7.3]. We follow Gutev’s scheme [5] of proofs applying our Theorem 1.3, Corollary 1.4 and Theorem 1.5 instead of their homotopical versions.

By a homological $UV^n(G)$ (resp., homological $UV^\omega(G)$) decomposition of a compactum $X$ we mean an upper semi-continuous decomposition of $X$ into compact homologically $UV^n(G)$ (resp., homologically $UV^\omega(G)$) sets. The decomposition space is denoted by $X/\sim$ and $\pi : X \to X/\sim$ is the quotient map.

**Theorem 4.1.** Let $X$ be a compact metric AR with $\dim X \leq n$ and $X/\sim$ be a homological $UV^{n-1}(G)$ decomposition of $X$. Then $X/\sim$ has the fixed-point property.

**Proof.** For any map $f : X/\sim \to X/\sim$ the set-valued map $\Phi := \pi^{-1} \circ f \circ \pi : X \to 2^X$ is usc and homologically $UV^{n-1}(G)$. Then, by Corollary 1.4, $x_0 \in \Phi(x_0)$ for some $x_0 \in X$. Hence, $f(\pi(x_0)) = \pi(x_0)$. □

**Theorem 4.2.** Let $X$ be a compact metric AR and $X/\sim$ be a homological $UV^{n-1}(G)$ decomposition of $X$ with $\dim X/\sim \leq n$. Then $X/\sim$ has the fixed-point property.

**Proof.** For any map $f : X/\sim \to X/\sim$ consider the set-valued map $\Phi := \pi^{-1} \circ f \circ \pi : X/\sim \to 2^X$ and apply Theorem 1.3 to find a point $x_0 \in X$ with $x_0 \in \Phi(\pi(x_0))$. The last equality implies $f(\pi(x_0)) = \pi(x_0)$. □

**Theorem 4.3.** Let $X$ be a compact metric AR and $X/\sim$ be a homological $UV^\omega(G)$ decomposition of $X$. Then $X/\sim$ has the fixed-point property.

**Proof.** We repeat the proof of Theorem 4.1 using now Theorem 1.5 instead of Corollary 1.4. □
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