Improved Hartree–Fock resummations
and spontaneous symmetry breaking

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ABSTRACT

The standard Hartree–Fock approximation of the $O(N)$–invariant $\phi^4$ model suffers from serious renormalization problems. In addition, when the symmetry is spontaneously broken, another shortcoming appears in relation to the Goldstone bosons: they fail to be massless in the intermediate states. In this work, within the framework of out–of–equilibrium Quantum Field Theory, we propose a class of systematic improvements of the Hartree–Fock resummation which overcomes all the above mentioned difficulties while ensuring also exact Renormalization–Group invariance to one loop.

PACS: 11.10.Ef, 11.10.Gh, 11.10.Pg

1 Introduction, summary and outlook

The research in out–of–equilibrium dynamics of quantum fields has received, in recent years, a great impulse by cosmology as well as by particle and condensed matter physics. Indeed, a first–principle theoretical treatment of many important phenomena, such as the reheating of the universe after inflation or the thermalization of the quark gluon plasma in the ultra–relativistic heavy–ion colliders (RHIC, LHC), is required for a good qualitative and quantitative understanding of the late time and strongly coupled evolution of quantum field systems.

In particular, this necessity has encouraged the study of nonperturbative approaches to Quantum Field Theory (QFT) that could provide all–orders partial resummations of Feynman diagrams \[1,2\]. In fact standard perturbation theory does not yield satisfactory results, except for very short times, when non–equilibrium conditions are involved.

Mean–field approximations such as Leading–order large–N expansion \[3,4,5,6\] and Hartree, or Hartree–Fock (HF) variational method \[7,8,9,10\] are the simplest and most studied \[11,12,13,14,15,16\] resummation schemes. Their main features are well

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known: they do provide a backreaction term on the evolution of quantum fluctuations that stabilize dynamics after parametric amplifications or spinodal instabilities, but they fail to reproduce an important property of late time dynamics such as thermalization.

A more powerful tool is the 2PI (or 2PPI) effective action [17, 18] whose expansions at two (or more) loops order or at next–to–leading order in 1/N provide resummations that go beyond mean field approaches [20, 21, 22, 19] and yield, indeed, approximate numerical thermalization at strong coupling.

More formal aspects of resummed approximations, such as their renormalization properties, have been studied as well [23, 24]. Recently a systematic method has been put forward [25, 26, 27, 29] that removes divergences in the \( \Phi \)-derivable approximations by applying a BPHZ subtraction procedure to diagrams with resummed propagators.

In [30] we considered the simple HF approximation of the \( O(N) \, \phi^4 \) model in the unbroken symmetry phase. It is known that it cannot be consistently renormalized by the usual renormalization of bare coupling and mass [8, 23, 24]. We showed that this non-renormalizability is due to the absence of leading logarithmically divergent contributions coming from diagrams which do not have a “daisy” topology and therefore are not present in the standard HF resummation. The inclusion of these contributions together with suitably chosen finite parts led us to the definition of a renormalized and Renormalization–Group invariant version of HF equations.

In the present paper we further develop this analysis, making it more systematic and transparent, and then apply it to the subtler case of spontaneously broken \( O(N) \) symmetry. In this case, together with the shortcomings as far as pure renormalizability is concerned, the HF approximation shows an unphysical non–gapless behaviour. Namely, the effective resummed propagators of the Goldstone bosons, as defined by the HF equation of motion, are not massless. This is a well known problem which has been cured by defining suitably modified approximations [31, 32]. In this paper we define a systematic improvement of the standard HF approach which ensures gaplessness together with renormalizability and RG-invariance. In practice we include all leading logarithmically divergent contributions needed to render finite the resummed propagator masses (which are \( \log \Lambda \) dependent in standard HF) while choosing the finite parts in such a way to have massless transverse degrees of freedom.

More in detail, we proceed as follows. First, by analysing the diagrammatic resummation performed by the HF approximation, we identify the missing leading logarithmically divergent contributions that cause the non-renormalizability. As will be apparent below, this analysis closely parallels that in [30] since, after all, the UV behavior of the theory is the same regardless of the breaking of the \( O(N) \) symmetry. Then we construct our modified HF approximation, including the missing contributions, while choosing the corresponding finite parts by means of the following recipe

- determine the general features of the HF approach that must be shared by the modified approximation, such as mean–field structure, \( O(N) \) Ward identities, leading–log structure and more, and parameterize the class of resummations having such properties.

- identify amid this family of approximations the one(s) with suitable properties of
renormalizability, RG–invariance, gaplessness and infrared finiteness.

One important difference from [30] is that, when the $O(N)$ symmetry is spontaneously broken, the above procedure does not lead to a unique result except for the $N = 1$ case ($Z_2$ symmetry). When $N > 1$ we find a whole class of approximations, corresponding to different choices of finite parts, that share the required characteristics. Like in [30] the improved equations have, by construction, a mean field structure but, unlike the standard ones, are nonlocal in space and time.

In sec. 2 we review some general concepts. We introduce the CTP formulation of out–of–equilibrium problems and define the HF approximation as resummation of bubble diagrams recalling the corresponding, well known, equations of motion [see eqs. (2.4) and eqs. (2.6)]. Some important general features of the mean field approximations are pointed out.

In sec. 3 we study, as an illuminating example, the case of spontaneously broken $Z_2$ symmetry. Subsec. 3.1 is dedicated to a general analysis of the approximation. In subsec. 3.2 we apply the standard renormalization procedure and point out its shortcomings. In subsec. 3.3 we define our modified HF approximation which is renormalizable and RG–invariant. As we already said in this case we are led to a unique result [see eqs. (3.38) and eqs. (3.42)]. Further remarks on the freedom of choosing various initial conditions without spoiling renormalization and RG–invariance are made in subsec. 3.4.

In sec. 4 we generalize to $N > 1$, that is to a continuous $O(N)$ spontaneous symmetry breaking, by first analysing some general features [subsec. 4.1] of the standard HF approximation and its renormalization properties [subsec. 4.2] and then defining our modified approximation by the general recipe given above [subsec. 4.3]. The final results will be a set of constraints that should be satisfied by renormalizable and RG–invariant Hartree–Fock–like approximations together with two examples [see eqs. (4.38) and eqs. (4.39)].

There are several possible developments along the lines of this work. First of all it would be interesting to study other properties of the different improved approximations that we have constructed in the $N > 1$ case and, hopefully, identify further constraints that would allow to single out a unique result. Secondly, a numerical study of the modified HF field equations found in this paper and in [30] would be needed to investigate how the space-time nonlocalities affect the time evolution as compared to the standard HF approximation, which is known to fail even qualitatively at late times. Another challenging task is the extension of our approach to the full two–loop 2PI effective action, through the inclusion of the nonlocal sunset diagram which is absent by definition in any mean–field approximation. In fact one should expect that, even if such an inclusion allows to recover renormalizability as compared to the conventional HF approximation, the two–loop 2PI self-consistent equations still lack RG invariance with the two–loop beta function, since the 2PI effective action does not contain all the diagrams which contribute to the next–to–leading ultraviolet divergences.
2 Generalities

To study the real–time dynamics from a given set of initial conditions we need to evaluate matrix elements of the form

\[ \langle \Psi(t_0) | O(t_1, t_2, \ldots) | \Psi(t_0) \rangle \]

where \( |\Psi\rangle \) is a generic state prepared at a initial time \( t_0 \) and \( O \) is some operator depending on the field evaluated at times \( t_i > t_0 \). Out–of–equilibrium QFT provides the general setup for the calculation such matrix elements, as well as more general expectation values in statistical mixtures of pure states such as \( |\Psi\rangle \). In this section we briefly review some generalities about non–equilibrium QFT, define the Hartree–Fock approximation and point out some properties that will be useful to derive the forthcoming results. In doing this, we restrict to the case of interest for this paper: scalar field theory in 3 + 1 dimensions with quartic interaction and spontaneously broken \( O(N) \) symmetry.

The field variables are \( \varphi_i(x) \) where \( x = (x_1, x_2, x_3) \) are space coordinates and \( i = 1, \ldots, N \) is the \( O(N) \) index. Classical dynamics is defined by an action, functional of trajectories \( \varphi_i(x) \), with \( x = (x, t) \), in the form

\[ S[\varphi] = \int d^4x \left\{ \frac{1}{2} \partial_{\mu} \varphi_i(x) \partial^\mu \varphi_i(x) - \frac{m^2}{2} \varphi_i(x) \varphi_i(x) - \frac{\lambda}{4!} [\varphi_i(x) \varphi_i(x)]^2 \right\} \]

where \( m^2 \) and \( \lambda \) are the squared mass (negative in the case of spontaneously broken symmetry ) and the coupling constant respectively.

In QFT, the general approach to non–equilibrium dynamics was developed by Keldish and Schwinger and is known as closed time path (CTP) formalism. It allows to use standard functional methods (see \[ 17, 18, 35, 36 \]) by introducing path integrals on a time path going from \( t = 0 \) to \( t = +\infty \) and back. Field integration variables are then doubled and subdivided into (+)–components, for the path integral forward in time, and (−)–components for the backward piece. Given an initial state defined by the functional \( \Psi \) of the field configurations \( \varphi(x) \), one writes down the functional integral

\[ e^{iW[j_+, j_-]} = \int D\varphi_+ D\varphi_- \Psi[\varphi_+] \Psi[\varphi_-] e^{iS[\varphi_+] - iS[\varphi_-] + i(j_+|\varphi_+| - j_-|\varphi_-|)} \]

where we have used the short-hand notation

\[ \langle a|M|b \rangle \equiv \int d^3x d^3y \ a_j(x) M_{jk}(x,y) \ b_k(y) \]

which will be useful later on. Integration in eq. (2.1) is on trajectories from \( t = 0 \) to \( t = +\infty \) (with the condition \( \varphi_+ = \varphi_- \) at \( t = +\infty \) and \( \varphi_\pm \) in the wave functional is the \( t = 0 \) section of \( \varphi_\pm \). Notice that, in the action functional \( S \) that enters in the path integral in eq. (2.1), the constant parameters \( \lambda, m^2 \) are to be substituted with the bare (cut-off dependent) ones \( \lambda_0, m_0^2 \). In fact the theory should be thought as regularized with
an UV cut-off $\Lambda$ and then renormalized to remove the divergent dependence on $\Lambda$. In the general setting, field renormalization should be included as well. Here we ignore it since, as we will see, it is absent in the HF approximation of the scalar theory.

By construction $W[j_+, j_-]$ is the generating functional of connected Green functions

$$G_{++\ldots-\ldots} = \left. \frac{(-i)^{n+m} \delta^{n+m} W}{\delta j_+(x_1) \ldots \delta j_+(x_n) \delta j_-(y_1) \ldots \delta j_-(y_m)} \right|_{j_+ = 0 \atop j_- = 0}$$

where $\mathcal{T}$ and $\mathcal{T}$ define time ordered and inverse ordered products, respectively, and internal indices have been omitted for ease of notation. The effective action $\Gamma_{\text{1PI}}[\phi_+, \phi_-]$, which is the generator of 1PI vertex functions, is the Legendre transform of $W[j_+, j_-]$ from the currents $j_\pm$ to the fields $\phi_\pm$. The equation of motion for the background field $\phi(x) = \langle \Psi | \varphi(x) | \Psi \rangle$ then reads

$$\frac{\delta \Gamma_{\text{1PI}}}{\delta \phi_+ (x)} \bigg|_{\phi_+ = \phi_+ = \phi} = 0 \quad \text{(2.2)}$$

Notice that the functional $W$ as well as the 1PI effective action parametrically depend on the initial state $\Psi$. In our present discussion we consider an initial wave functional having the following Gaussian form

$$\Psi[\varphi] = \mathcal{N} \exp \left\{ i \langle \phi(0) | \varphi(x) \rangle - \langle \varphi - \phi(0) | \left[ \frac{1}{4} \mathcal{G}^{-1} + i \mathcal{S} \right] | \varphi - \phi(0) \rangle \right\}$$

whose free parameters are the $t = 0$ background field $\phi(x, 0)$, the $t = 0$ background momentum $\dot{\phi}(x, 0)$, the real symmetric positive kernel $G_{ij}(x, y)$ and the real symmetric kernel $S_{ij}(x, y)$. We can see that $\Gamma_{\text{1PI}}$ now depends parametrically only on the kernels $G$ and $S$, while the $t = 0$ background fields $\phi(0)$ and $\dot{\phi}(0)$ enter instead as initial conditions for the equation eq. (2.2).

The perturbative diagrammatic expansion in the CTP formalism proceeds as in vacuum QFT. The bare propagators are

$$G_{++}(x, y) = G^{(b)}_{++}(x, y) = -i \langle \Psi | \mathcal{T} \varphi(x) \varphi(y) | \Psi \rangle_{\text{conn}} |\text{free}\rangle$$

$$G_{--}(x, y) = G^{(b)}_{--}(x, y) = -i \langle \Psi | \mathcal{T} \varphi(x) \varphi(y) | \Psi \rangle_{\text{conn}} |\text{free}\rangle$$

$$G_{+-}(x, y) = G^{(b)}_{+-}(y, x) = -i \langle \Psi | \varphi(y) \varphi(x) | \Psi \rangle_{\text{conn}} |\text{free}\rangle$$

Here the notation $|\text{free}\rangle$ indicates that, in accordance to the tree level of the theory in the broken symmetry phase, the Heisenberg expectation values have to be calculated for a free scalar field theory with mass matrix $m_{b,ij}^2 = \frac{1}{3} \lambda_0 v_i v_j$, where $v_i$ is the nonvanishing
vacuum expectation value. The bare vertices are

\[
\begin{align*}
\text{(a)} & \quad i \lambda_0 \tau_{ijkm} = i \lambda_0 \tau_{ijkm} \\
\text{(b)} & \quad i \lambda_0 \tau_{ijkm} v_m = i \lambda_0 \tau_{ijkm} v_m \\
\end{align*}
\]

However, in order to obtain sensible results in a generic out--of--equilibrium contexts or even at equilibrium with nonzero temperature, it is known that calculations should go beyond plain perturbation and perform (partial) resummations to all orders in the coupling constant. A very successful resummation method is provided by the introduction of the 2PI effective action (see [17, 18]). It is defined as the double Legendre transform of the \( \mathcal{W} \) generating functional with respect to the usual current one--point \( j_\pm \) and to the two--points current \( K_{\alpha\beta}(x, y) \) coupled through the term

\[
\frac{1}{2} \int d^4x \int d^4y \ K_{\alpha\beta}(x, y) \varphi_\alpha(x) \varphi_\beta(x)
\]

where \( \alpha, \beta = \pm \). \( \Gamma_{2PI} \) is a functional of the classical fields \( \phi_\alpha \) and of the propagators \( G_{\alpha\beta} \). It yields two equations of motions

\[
\left. \frac{\delta \Gamma_{2PI}}{\delta \phi_\alpha(x)} \right|_* = 0 \ , \quad \left. \frac{\delta \Gamma_{2PI}}{\delta G_{\alpha\beta}(x, y)} \right|_* = 0 \quad (2.3)
\]

Here the notation \( |_* \) indicates that, by their physical meaning, the \((\pm)\)--component fields and propagators have to satisfy, on the solutions of motion, the following relations

\[
\begin{align*}
\phi_- (x) &= \phi_+(x) = \phi(x) \\
G_F(x, y) &= G_+(y, x) \theta(x_0 - y_0) + G_+(x, y) \theta(y_0 - x_0) \\
G_F(x, y) &= G_+(x, y) \theta(x_0 - y_0) + G_+(y, x) \theta(y_0 - x_0) \\
\end{align*}
\]

Hence the system eq. (2.3) reduces to two coupled equations for \( \phi \) and \( G_{++} \) only. Moreover, any initial Gaussian state may be absorbed in the \( t = 0 \) term for the \( j_\alpha \) and \( K_{\alpha\beta} \) currents, so that, by the double Legendre transform the initial Gaussian state disappears from the effective action, but fixes the initial conditions on \( \phi \) and \( G_{++} \). The role of \( \phi(0) \) and \( \dot{\phi}(0) \) is immediate, while for the kernels we have

\[
G_{++}(x, y) \big|_{x_0 = y_0 = 0} = G(x, y) \quad \frac{\partial}{\partial y_0} G_{++}(x, y) \big|_{x_0 = y_0 = 0} = 2i [G\mathcal{S}](x, y) + \frac{1}{2} \delta^3(x - y)
\]

Given \( \Gamma_{2PI} \) at a certain perturbative loop order, if we solve the second equation in eqs. (2.3) for a generic \( \phi \) and substitute the result \( G[\phi] \) into the first one we obtain the background
equations of motion corresponding to a resummed diagrammatic approximation of the 1PI effective action $\Gamma_{1PI}$.

In the present scalar theory $\Gamma_{2PI}$ has the general form

$$
\Gamma_{2PI}[\phi, G] = S[\phi] + \frac{i}{2} \text{Tr} \log G + \frac{i}{2} \text{Tr} [G_0^{-1} G] + \Gamma_2[\phi, G]
$$

where $S$ is the complete classical action of the double time path (i.e. $S = S_+ - S_-$). Traces are taken over all indices $i$, $\alpha$ and $x$. $G_0^{-1}$ is the second derivative of the action in a $\phi$ background, $\Gamma_2$ is the sum of all vacuum 2PI diagrams with $G$ propagators and vertices defined by the classical action in a $\phi$ background. To two loops level the diagrams contributing to the $\Gamma_2$ are the “8” and “sunset” diagrams

Now we can introduce the Hartree–Fock approximation as obtained considering only the first contribution (i.e. the “8” graph) to $\Gamma_2$.

$$
\Gamma_2 = \frac{i}{8} \lambda_0 \left[ G^0_P(x, x) - G^2_P(x, x) \right]
$$

Notice that “8” is the only 2PI diagram made of “product” of loops corresponding to a mean field contribution to the mass. In the 1PI framework this corresponds to a resummation of all vacuum 1PI diagrams with daisy and superdaisy topologies of the form

By explicitly using this form of $\Gamma_2$ in eqs. (2.4), setting $G(x, y) = \frac{i}{2} [G_+(x, y) + G_{+}(y, x)]$ and observing that the antisymmetric combination decouples, one obtains

\[
\begin{align*}
\left\{ \left[ \Box + m_0^2 + \frac{i}{8} \lambda_0 \phi_k(x)\phi_k(x) \right] \delta_{ij} + \frac{1}{2} \lambda_0 \tau_{ijkm} G_{km}(x, x) \right\} \phi_j(x) &= 0 \\
\left\{ \left[ \Box + m_0^2 \right] \delta_{ij} + \frac{1}{2} \lambda_0 \tau_{ijkm} \left[ \phi_k(x)\phi_m(x) + G_{km}(x, x) \right] \right\} G_{ij}(x, y) &= 0
\end{align*}
\]

(2.4)

where the tensor $\tau$ is defined as

$$
\tau_{ijkm} = \frac{1}{4} (\delta_{ij}\delta_{km} + \delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk})
$$

(2.5)

By introducing an equivalent formulation of the eqs. (2.4) in terms of mode functions $u_{ka}$ ($k$ is the wave vector and $a$ the $O(N)$ polarization), that will be the one used throughout the paper. For simplicity let us suppose that the initial ($t = 0$) kernels
are translationally invariant (while the background field and momentum may be point dependent). We can then write

\[ G_{ij}(x, y) = \int \frac{d^3k}{(2\pi)^3} \tilde{G}_{ij}(k) e^{ik \cdot x}, \quad S_{ij}(x, y) = \int \frac{d^3k}{(2\pi)^3} \tilde{S}_{ij}(k) e^{ik \cdot x} \]

Next, we introduce the \( t = 0 \) mode functions by

\[ u_{k,ai}(x, 0) = \left[ \tilde{G}(k)^{1/2} \right]_{ai} e^{ik \cdot x}, \quad \dot{u}_{k,ai}(x, 0) = \left[ -\frac{i}{2} \tilde{G}(k)^{-1} + 2 \tilde{S}(k) \right]_{ij} u_{k,aj}(x, 0) \]

Then, one can easily verify that, by the identification

\[ G_{ij}(x, y) = \text{Re} \int \frac{d^3k}{(2\pi)^3} u_{k,ai}(x) \bar{u}_{k,aj}(y) \]

eqs. (2.4) are equivalent to the following equations of motion

\[
\begin{align*}
\left\{ \Box + m_0^2 + \frac{1}{6} \lambda_0 \phi_k(x) \phi_k(x) \right\} \delta_{ij} + \frac{1}{2} \lambda_0 \tau_{ijkm} & \int \frac{d^3p}{(2\pi)^3} u_{p, bm}(x) \bar{u}_{p, bn}(x) \phi_m(x) \phi_n(x) + \int \frac{d^3p}{(2\pi)^3} u_{p, bm}(x) \bar{u}_{p, bn}(x) \right\} \phi_j(x) = 0 \\
\left\{ \Box + m_0^2 \right\} \delta_{ij} + \frac{1}{2} \lambda_0 \tau_{ijmn} & \int \frac{d^3p}{(2\pi)^3} u_{p, bm}(x) \bar{u}_{p, bn}(x) \phi_m(x) \phi_n(x) + \int \frac{d^3p}{(2\pi)^3} u_{p, bm}(x) \bar{u}_{p, bn}(x) \right\} u_{k,aj}(x) = 0
\end{align*}
\]

These are the HF equations of motions that will be used in the rest of the paper.

We conclude this section by introducing a different representation of the CTP formalism, known as the physical representation (see [36]) which allows us to derive some results useful below. We introduce the field redefinitions (omitting again the internal indices to simplify notation)

\[ \phi_\Delta = \phi_+ - \phi_-, \quad \phi_c = \frac{1}{2} (\phi_+ + \phi_-) \]

and write the 1PI effective action as a functional of these new fields, \( \Gamma_{1PI} = \Gamma[\phi_\Delta, \phi_c] \). By calculating vertex functions, one then finds

\[
\left. \frac{\delta^n \Gamma}{\delta \phi_c(x_1) \ldots \delta \phi_c(x_n)} \right|_{\phi_\Delta=0} = 0
\]

and

\[
\left. \frac{\delta^{n+m} \Gamma}{\delta \phi_c(x_1) \ldots \delta \phi_c(x_n) \delta \phi_\Delta(y_1) \ldots \delta \phi_\Delta(y_m)} \right|_{\phi_\Delta=0} = 0
\]

if the time component of anyone of the \( x \)'s is larger than the time component of all the \( y \)'s. Let us also remark that, by the definition of CTP generating functional, all time coordinates in the vertex functions are supposed to be positive so we can set, as well,
these functions to be zero for any negative time. Using eq. \((2.7)\) together with eq. \((2.2)\) one can write the equations of motions in the form

\[
\frac{\delta \Gamma}{\delta \phi_\Delta} \bigg|_{\phi_\Delta=0, \phi_c=\phi} = 0
\]

(2.9)

Notice that \(2n\)–legs vertex functions with one \(\phi_\Delta\) leg and \(2n-1\) \(\phi_c\) legs are the only ones contributing to these equations of motion. Then eq. \((2.8)\) guarantees that all the terms nonlocal in time in eq. \((2.9)\) do satisfy causality.

Perturbative calculations by diagrammatic expansion in the physical representation are based on the bare propagators

\[
G^{(b)}_{\Delta}(x, y) \equiv G^{(b)}_A(x, y) = -i\theta(y_0 - x_0) \langle \Psi | [\phi(x), \phi(y)] | \Psi \rangle |_{\text{free}}
\]

\[
G^{(b)}_{\Delta c}(x, y) \equiv G^{(b)}_R(x, y) = -i\theta(x_0 - y_0) \langle \Psi | [\phi(x), \phi(y)] | \Psi \rangle |_{\text{free}}
\]

(2.10)

The bare retarded and advanced Green functions \(G^{(b)}_A\) and \(G^{(b)}_R\) do not depend on the initial state and are translational invariant. The bare correlation function \(G^{(b)} = \frac{i}{4}G^{(b)}_{\Delta\Delta}\), instead, does depend on \(|\Psi\rangle\). The vertices are

\[
\begin{aligned}
\text{solid lines: } \phi_c \text{ legs while dotted lines: } \phi_\Delta \text{ legs.}
\end{aligned}
\]

where solid lines represent \(\phi_c\) legs while dotted lines represent \(\phi_\Delta\) legs.

As stated above, the HF approximation consists in the resummation of diagrams with daisy and superdaisy topologies. By considering this diagrammatic resummation it is easy to verify that the corresponding effective action has the following general structure

\[
\Gamma_{\text{HF}}[\phi_\Delta, \phi_c] = -\langle \phi_\Delta | \Box | \phi_c \rangle - \mathcal{F}[\xi, \chi, \eta]
\]

where \(\mathcal{F}\) is a functional of the following composite matrix fields

\[
\xi_{ij}(x) = \phi_{c,i}(x)\phi_{c,j}(x), \quad \chi_{ij}(x) = \phi_{c,i}(x)\phi_\Delta,j(x), \quad \eta_{ij}(x) = \phi_\Delta,i(x)\phi_\Delta,j(x)
\]

We now introduce, for ease of notation, the new object

\[
\mathcal{F}'[\xi]_{ij}(x) = \frac{1}{2} \frac{\delta \mathcal{F}}{\delta \chi_{ij}(x)} \bigg|_{\chi=\eta=0}
\]

9
which is a functional of $\xi_{ij}(x) = \phi_{c,i}(x)\phi_{c,j}(x) = \phi_i(x)\phi_j(x)$ only. Then the equation of motion in the HF approximation takes the form

$$\left. \frac{\delta \Gamma}{\delta \phi_{\Delta}} \right|_{\phi_{\Delta}=0, \phi_c=\phi} = \{\Box \delta_{ij} + 2 F'_{ij}[\xi]\} \phi_j = 0$$

We will consider this as a general form for mean-field-type background field equations and it will be that base for our definition of a modified HF approximation.

3 The case $N = 1$

We begin by considering a single scalar field theory with spontaneously broken $\mathbb{Z}_2$ symmetry. The study of this simpler case provides valuable insight into the general features of the diagrammatic resummation performed by the HF approximation. In particular, this allows to understand the origin of the HF shortcomings with respect to renormalizability and RG–invariance and to determine the general recipe for the definition of a modified renormalizable and RG-invariant mean field approximation. Many of the results of this section will hold true also in the more general case of a theory with $N$ scalar fields.

3.1 Analysis of the HF approximation

For the $N=1$ case the Hartree-Fock equations of motion [see eqs. (2.6)] reduce to

$$\left\{ \Box + m_0^2 + \frac{1}{6} \lambda_0 \xi(x) + \frac{1}{2} \lambda_0 \int_{p^2<\Lambda} \frac{d^3p}{(2\pi)^3} |u_p(x)|^2 \right\} \phi(x) = 0$$

$$\left\{ \Box + m_0^2 + \frac{1}{2} \lambda_0 \xi(x) + \frac{1}{2} \lambda_0 \int_{p^2<\Lambda} \frac{d^3p}{(2\pi)^3} |u_p(x)|^2 \right\} u_k(x) = 0$$

(3.1)

where we recall that $\xi(x) = \phi^2(x)$. Notice that, so far, these are still the equations of the regularized theory, written in terms of bare parameters ($\lambda_0, m_0^2$) and explicitly dependent on the sharp cut-off $\Lambda$.

The phase with spontaneously broken symmetry, which is the subject of this paper, is defined by assuming the existence of a static homogeneous vacuum solution of eqs. (3.1) with nonzero background field $\phi(x) = v$. The corresponding mode functions, that will be named $u^{(\text{vac})}$, have the following plane wave form

$$u^{(\text{vac})}_k(x) = \frac{1}{\sqrt{2 \omega_k}} e^{i(k \cdot x - \omega_k t)} , \quad \omega_k^2 = k^2 + m^2$$

(3.2)

where $m^2 = \frac{1}{3} \lambda_0 v^2$ and the vacuum expectation value $v$ of the background should satisfy the gap equation

$$0 = m_0^2 + \frac{1}{6} \lambda_0 v^2 + \frac{1}{2} \lambda_0 \int_{p^2<\Lambda} \frac{d^3p}{(2\pi)^3} \frac{1}{2 \omega_k}$$

(3.3)
The values of bare parameters for which this equation admits a nonzero solution \( v \) are those corresponding to spontaneously broken symmetry and are those we are considering here.

We now introduce the mean field \( V \) according to

\[
V(x) = m_0^2 + \frac{1}{6} \lambda_0 v^2 + \frac{1}{2} \lambda_0 \Delta \xi(x) + \frac{1}{2} \lambda_0 \int_{p^2 < \Lambda} \frac{d^3 p}{(2\pi)^3} |u_p(x)|^2
\]

(3.4)

where \( \Delta \xi = \xi - v^2 \). Notice that \( V = 0 \) on the static solution \( \phi = v \) and \( u = u^{(\text{vac})} \). This definition allows us to rewrite eqs. (3.1) in the form

\[
\begin{align*}
\{ \Box - \frac{1}{3} \lambda_0 \Delta \xi(x) + V(x) \} \phi(x) &= 0 \\
\{ \Box + m^2 + V(x) \} u_k(x) &= 0
\end{align*}
\]

(3.5)

Recalling the definition of the background field equation in terms of the 1PI effective action

\[
\frac{\delta \Gamma_{1\text{PI}}}{\delta \phi_\Delta} \bigg|_{\phi_\Delta = \phi} = (\Box + 2 \mathcal{F}[\xi]) \phi = 0
\]

(3.6)

and comparing it with the first of eqs. (3.5), we read out

\[
\mathcal{F}'[\xi] = \frac{1}{2} V - \frac{1}{6} \lambda_0 \Delta \xi
\]

(3.7)

Notice that \( V \) is regarded here as a functional of \( \xi = \phi^2 \). The implicit dependence on \( \xi \) is determined by solving the second equation in (3.5) with a generic background. To obtain a self-consistent equation for the functional \( V[\xi] \) from the HF equations of motion some preliminary definitions are required.

- We introduce the free mode functions \( u_k^{(0)} \) defined as solutions of the free equation with mass \( m^2 = \frac{1}{3} \lambda_0 v^2 \) and with the same initial conditions of the exact mode functions, that is

\[
(\Box + m^2) u_k^{(0)}(x) = 0 , \quad u_k^{(0)}(x,0) = u_k(x,0) , \quad \dot{u}_k^{(0)}(x,0) = \dot{u}_k(x,0)
\]

- In terms of these we introduce the free correlation

\[
G^{(0)}(x,y) = \text{Re} \int \frac{d^3 p}{(2\pi)^3} u_p^{(0)}(x) u_p^{(0)*}(y)
\]

(3.8)

- Finally we define the free retarded and advanced Green functions as

\[
G_R^{(0)}(x-y) = G_A^{(0)}(y-x) = \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}
\]
By using the above definitions we can cast the equation of motion for the mode functions into a convenient integral form

\[ u_k(x) = u_k^{(0)}(x) + \int d^4y \, G_R^{(0)}(x-y) \mathcal{V}(y) \, u_k(y) \]

In this equation, and everywhere else from now on, all field (\(\phi\), mode functions, \(\mathcal{F}'\), \(\mathcal{V}\), etc...) are to be thought as defined only for positive times (initial conditions are at the limit point \(t = 0^+\)) and all time integrations are restricted to positive values, as appropriate in an initial value problem. Moreover it is convenient to introduce a more compact operator notation with implicit space–time integration, that is

\[ u_k = u_k^{(0)} + G_R^{(0)} \hat{\mathcal{V}} u_k \] \hspace{1cm} (3.9)

where the caret \(\hat{\cdot}\) turns a vector of the functional space into a multiplication operator

\[ \hat{v}(x) \rightarrow \hat{v}(x, y) = v(x) \, \delta^{(4)}(x - y) \]

From now on we shall use this notation throughout the paper.

Eq. (3.9) can be formally solved for \(u_k\)

\[ u_k = [1 - G_R^{(0)} \hat{\mathcal{V}}]^{-1} u_k^{(0)} \]

where \(1\) stands for the space time delta function \(\delta^{(4)}(x-y)\). Then the cutoffed correlation

\[ G(x, y) = \text{Re} \int_{|p| < \Lambda} \frac{d^3p}{(2\pi)^3} \, u_p(x) \, \overline{u}_p(y) \]

can be written as

\[ G = \mathcal{G}[\mathcal{V}] = [1 - G_R^{(0)} \hat{\mathcal{V}}]^{-1} G^{(0)} [1 - \hat{\mathcal{V}} G_A^{(0)}]^{-1} \] \hspace{1cm} (3.10)

in terms of \(\mathcal{V}\), of the free retarded and advanced Green functions and of the free correlation function. In conclusion, by renaming the correlation at coincident points as

\[ I[\mathcal{V}](x) = \mathcal{G}[\mathcal{V}](x, x) \] \hspace{1cm} (3.11)

and substituting into the definition of \(\mathcal{V}\) in eq. (3.4) we obtain the sought self–consistent equation

\[ \mathcal{V} = m_0^2 + \frac{1}{6} \lambda_0 v^2 + \frac{1}{2} \lambda_0 \left( \Delta \xi + I[\mathcal{V}] \right) \] \hspace{1cm} (3.12)

Notice that eq. (3.12) depends parametrically on the initial kernels (i.e. the mode functions initial conditions) through the explicit form of \(G^{(0)}\). Now, before going any further in the discussion, we fix a particular choice for these kernels by considering the HF vacuum as initial state for the quantum fluctuations. That is to say that we start from equilibrium initial conditions for the mode functions

\[ u_k(x, 0) = u_k^{(\text{vac})}(x, 0) \quad \text{and} \quad \dot{u}_k(x, 0) = \dot{u}_k^{(\text{vac})}(x, 0) \]

which corresponds to the following choice of the initial kernels

\[ \hat{\mathcal{G}}(k) = \frac{1}{2 \omega_k} \quad \text{and} \quad \hat{S}(k) = 0 \]

By this choice some simplifying properties follow
• The free mode functions coincide with the vacuum mode functions $u^{(\text{vac})}$ at every time.

• By the self–consistent equation we have $\mathcal{V} = 0$ at the point $\xi(x) = v^2$

• The free correlation function defined in eq. (3.8) turns to be translationally invariant in spacetime. Explicitly

$$G^{(0)}(x - x') = \int_{k^2 < \Lambda^2} \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_k} \cos[\omega_k(t - t') - \mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')]$$

Finally let us stress that, in spite of this choice, we are still considering an out–of–equilibrium problem since we allow for generic initial conditions for the background field. The use of different initial kernels will be discussed later.

At this point some observations on the integral term $I$ defined by eq. (3.11) and eq. (3.10) are in order.

1. $I$ depends on the free retarded (and advanced) Green function and on the free correlation function. These propagators have the same form of their bare correspondents [see eq. (2.10)] but their mass is $m^2 = \frac{1}{2}\lambda_0 v^2$ rather than $m_b^2 = m_0^2 + \frac{1}{2}\lambda_0 v^2$. By using the gap eq. (3.3), the relation between the two sets of propagators reads

$$G^{(0)}_R = G^{(b)}_R + \frac{1}{2}\lambda_0 G^{(0)}_R \hat{I}^{(1)} G^{(b)}_R$$

$$G^{(0)} = [1 - \frac{1}{2}\lambda_0 G^{(0)}_R \hat{I}^{(1)}] G^{(b)} [1 - \frac{1}{2}\lambda_0 \hat{I}^{(1)} G^{(0)}_A]$$

where $I^{(1)}$ is the spacetime constant tadpole of the $G^{(0)}$ propagator

$$I^{(1)} = G^{(0)}(x, x)$$

By recursively solving eq. (3.13) for the free propagators we can see that their definition corresponds to the resummation of all tadpole corrections. As we will see they play the role of internal dressed propagators in the diagrammatic resummation performed by the Hartree-Fock approximation.

2. By expanding $I$ in powers of $\mathcal{V}$ we can see that the term proportional to the $n$-th power contains loop with $n + 1$ free propagators. In particular the expansion up to the linear term reads

$$I[\mathcal{V}](x) = \left[ G^{(0)} + G^{(0)}_R \hat{\mathcal{V}} G^{(0)} + G^{(0)} \hat{\mathcal{V}} G^{(0)}_A \right](x, x) + \ldots$$

which, for later convenience, can be rewritten in the compact form

$$I[\mathcal{V}] = I^{(1)} + I^{(2)} \mathcal{V} + \ldots$$

in terms of the tadpole $I^{(1)}$ and of the two propagators loop

$$I^{(2)}(x - y) = +2 G^{(0)}_R(x, y) G^{(0)}(x, y)$$
3. One can easily realize that $I^{(1)}$ diverges as $\Lambda^2$ and $\log \Lambda$, $I^{(2)}$ diverges as $\log \Lambda$ and the loops with more than two propagators are convergent. It is therefore useful to introduce a new functional $J$ containing only the convergent part of $I$ according to

$$J[\mathcal{V}] = I[\mathcal{V}] - I^{(1)} - I^{(2)} \mathcal{V}$$

(3.14)

The previous observations lead us to rewrite the self-consistent eq. (3.12) in a particular “quasi-renormalized” form, more suitable to analyze the diagrammatic resummation of the effective action and determine its divergent graphs and subgraphs structure. The “quasi-renormalized” form is obtained by expanding the self-consistent eq. (3.12) around the point $\xi(x) = v^2$ ($\mathcal{V} = 0$), explicitly solving the linear terms and writing a self-consistent equation for the higher $\Delta \xi$ powers dependence of $\mathcal{V}$.

First we substitute eq. (3.14) into eq. (3.12), applying the gap eq. (3.3) in order to simplify the constant terms. Then, by solving the terms proportional to $\mathcal{V}$ we obtain

$$\mathcal{V} = \frac{1}{2} \theta \left[ \Delta \xi + J[\mathcal{V}] \right] , \quad \theta = \lambda_0 \left[ 1 - \frac{1}{2} \lambda_0 I^{(2)} \right]^{-1}$$

(3.15)

or, equivalently

$$\mathcal{V} = \frac{1}{2} \theta \Delta \xi + \Delta \mathcal{V} , \quad \Delta \mathcal{V} = \frac{1}{2} \theta J \left[ \frac{1}{2} \theta \Delta \xi + \Delta \mathcal{V} \right]$$

(3.16)

In conclusion we rewrite eq. (3.17) in terms of $\Delta \mathcal{V}$ and of the explicit linear term as

$$\mathcal{F}' = \frac{1}{2} \Omega^{(2)} \Delta \xi + \frac{1}{2} \Delta \mathcal{V} , \quad \Omega^{(2)} = \frac{1}{2} \theta - \frac{1}{3} \lambda_0 \mathbf{1}$$

(3.17)

We can now use eqs. (3.16) and (3.17) to calculate the vertex functions relevant for the background equation of motion. In the physical representation of the CTP formalism these functions have the form

$$\Gamma^{(n)}(x_1, \ldots, x_n) = \left. \frac{\delta \Gamma_{1PI}[\phi_\delta, \phi_c]}{\delta \phi_\Delta(x_1) \delta \phi_c(x_2) \ldots \delta \phi_c(x_n)} \right|_{\phi_\Delta = 0, \phi_c = v}$$

(3.18)

with one $\phi_\Delta$ leg at the point $x_1$ and $n - 1$ $\phi_c$ legs at the points $x_i$ ($i > 1$).

First of all, from the general form of the background equation (3.6) we can see that the vertex functions (3.18) are built from the functional derivatives of $\mathcal{F}'$ w.r.t. $\xi$ at the point $\xi = v^2$. To ease notation we rename such variations by

$$\Omega^{(k)}(x_1, \ldots, x_k) = 2 \left. \frac{\delta \mathcal{F}'[\xi](x_1)}{\delta \xi(x_2) \ldots \delta \xi(x_k)} \right|_{\xi = v^2}$$

Some examples of the expressions of the vertex functions in terms of the $\Omega^{(k)}$ are

$$\Gamma^{(1)}(x) = \Omega^{(1)}(x) v , \quad \Gamma^{(2)}(x_1, x_2) = \left[ \Box + \Omega^{(1)}(x_1) \right] \delta_{1,2} + 2 \Omega^{(2)}_{12} v^2$$

$$\Gamma^{(3)}(x_1, x_2, x_3) = 2 \Omega^{(2)}_{12} \delta_{12} v + 2 \Omega^{(2)}_{13} \delta_{13} v + 2 \Omega^{(2)}_{12} \delta_{13} v + 4 \Omega^{(3)}_{123} v^3$$

$$\Gamma^{(4)}(x_1, x_2, x_3, x_4) = 2 \left\{ \Omega^{(2)}_{13} \delta_{12} \delta_{34} + \Omega^{(2)}_{12} \delta_{14} \delta_{23} + \Omega^{(2)}_{12} \delta_{13} \delta_{24} \right\} + 8 \Omega^{(4)}_{1234} v^4 + \ldots$$

$$\ldots + 4 \left\{ \Omega^{(3)}_{134} \delta_{12} + \Omega^{(3)}_{124} \delta_{13} + \Omega^{(3)}_{123} \delta_{14} + \Omega^{(3)}_{124} \delta_{23} + + \Omega^{(3)}_{123} \delta_{34} + \Omega^{(3)}_{123} \delta_{24} \right\} v^2$$

(3.19)
where we have introduced the shorthand notations
\[ \delta_{ik} = \delta^{(4)}(x_i - x_k), \quad \Omega^{(2)}_{i...k} = \Omega^{(2)}(x_i, \ldots, x_k) \]
From eq. (3.17) we can see that \( \Omega^{(1)} = 0 \). So that \( \Gamma^{(1)} = 0 \), which is just the statement that \( v \) is the vacuum solution of the background equation. Moreover, by eq. (3.17) we have
\[ \Omega^{(2)} = \frac{1}{2} \theta - \frac{1}{3} \lambda_0 \mathbf{1} = \frac{1}{2} \lambda_0 \mathbf{1} - \frac{1}{2} \lambda_0^2 \mathbf{I}^{(2)} [1 - \frac{1}{2} \lambda_0 \mathbf{I}^{(2)}]^{-1} \quad (3.20) \]
Expanding the above expression in powers of \( \lambda_0 \) we see that \( \Omega^{(2)} \) is the sum of a classical term \( \frac{1}{6} \lambda_0 \) plus the resummation of all the chains of \( \mathbf{I}^{(2)} \) loops.

The \( \Omega^{(k)} \) with \( k > 2 \) are obtained by using eq. (3.17) and the self–consistent equation for \( \Delta \mathcal{V} \) [eq. (3.16)]. One can easily see that they are built up with loops having three or more propagators and are therefore finite in the limit \( \Lambda \to \infty \). The loops are attached to each other and to the external legs by the effective vertex \( \theta \) [see eq. (3.15)]. Some graphical examples are

The vertex \( \theta \), by its explicit definition in eq. (3.15), is the sum of the classical term \( \lambda_0 \) plus chains of \( \mathbf{I}^{(2)} \) integrals
\[ \theta = \lambda_0 \mathbf{1} - \frac{1}{2} \lambda_0^2 \mathbf{I}^{(2)} [1 - \frac{1}{2} \lambda_0 \mathbf{I}^{(2)}]^{-1} \quad (3.21) \]

### 3.2 Renormalization

Let us now apply the standard renormalization procedure to the HF approximation of the effective action. As we will see it fails to render all the vertex functions finite.

The first renormalization condition is simply the request that the v.e.v. \( v \) has a physical cutoff–independent value. Then the gap eq. (3.3) must be regarded as defining the bare mass parameter \( m_0^2 \) as a function of \( \lambda_0 \), \( v \) and the cut-off \( \Lambda \)
\[ m_0^2(\lambda_0, v, \Lambda) = -\frac{1}{6} \lambda_0 v^2 - \frac{1}{2} \lambda_0 \int_{p^2 < \Lambda} \frac{d^3p}{(2\pi)^3} \frac{1}{2 \omega_k} \quad (3.22) \]
Substituting the above function for the constant \( m_0^2 \) in the equation of motion turns to be enough to remove all the \( \Lambda^2 \) dependence in the vertex functions.

The second renormalization condition follows by fixing the value of the quartic coupling at some fixed energy scale. In order to do this we introduce the Fourier transform of the vertex functions [see eq. (3.18)], namely
\[ \tilde{\Gamma}^{(m)}(p_1, \ldots, p_{m-1}) = \int \left[ \prod_{n=2}^{m} d^4 x_n \right] \Gamma^{(m)}(x_1, \ldots, x_m) e^{i \sum_n p_n \cdot (x_1 - x_n)} \]
where translational invariance, which is a consequence of our choice of initial conditions, has been assumed. For later convenience we define also the Fourier transform of $\Omega^{(k)}$

$$\tilde{\Omega}^{(k)}(p_1, \ldots, p_{k-1}) = \int \left[ \prod_{n=2}^{k} d^4 x_n \right] \Omega^{(k)}(x_1, \ldots, x_k) e^{i \sum_n p_n \cdot (x_1 - x_n)}$$

We recall that, as appropriate to a causal initial value problem, all Fourier transforms are analytic in the upper complex $p_0$–halfplane.

Now, in the present out–of–equilibrium context, we can define the symmetric point at the scale $s$ by setting the momenta entering the $\phi_c$ legs ($p_2, p_3, p_4 = -p_1 - p_2 - p_3$) to the same, purely spatial, value

$$-\frac{1}{3} p_1 = p_2 = p_3 = p_4 = \frac{1}{2} q_s, \quad q_s \equiv (0, q), \quad |q| = s \quad (3.23)$$

with an arbitrary direction $\hat{q}$. The usual coupling renormalization condition is obtained by evaluating $\Gamma^{(4)}$ at this point and requiring it to be equal to the renormalized coupling constant $\lambda$

$$\lambda = 6 \tilde{\Omega}^{(2)}(q_s) \quad (3.24)$$

Now, recalling the general expression of $\Gamma^{(4)}$ in terms of the $\Omega^{(k)}$ in eq. (3.19), and performing the Fourier transform, we can rewrite eq. (3.24) as

$$\lambda = 6 \tilde{\Omega}^{(2)}(q_s) + 4 \{ \tilde{\Omega}^{(3)}(\frac{1}{2} q_s, \frac{1}{2} q_s) + 2 \tilde{\Omega}^{(3)}(-\frac{3}{2} q_s, \frac{1}{2} q_s) + \tilde{\Omega}^{(3)}(-q_s, \frac{1}{2} q_s) + \tilde{\Omega}^{(3)}(\frac{1}{2} q_s, -q_s) \} v^2 + 8 \tilde{\Omega}^{(4)}(-\frac{3}{2} q_s, \frac{1}{2} q_s, \frac{1}{2} q_s) v^4 \quad (3.25)$$

However, in our case we prefer to slightly change this standard procedure in order to avoid lengthy calculations and obtain a better comparison with the unbroken symmetry case. We therefore substitute eq. (3.25) simply with

$$\lambda = 6 \tilde{\Omega}^{(2)}(q_s) \quad (3.26)$$

We have omitted in this way the contributions to the quartic coupling originating from the superficially convergent $\Omega^{(3)}$ and $\Omega^{(4)}$. Hence the change from eq. (3.25) to eq. (3.26) is to be regarded as a legitimate finite part redefinition of the coupling constant.

Now, by the HF expression for $\Omega^{(2)}$, eq. (3.20), the renormalized coupling constant explicitly reads

$$\lambda = \lambda_0 \frac{1 + \lambda_0 \tilde{I}^{(2)}(q_s)}{1 - \frac{1}{2} \lambda_0 \tilde{I}^{(2)}(q_s)} \quad (3.27)$$

This has to be compared with the corresponding formula of unbroken symmetry case in ref. [30]. Notice that in ref. [30] we have adopted the opposite sign convention for $I^{(2)}$. Eq. (3.27), once inverted, defines $\lambda_0$ as a function of $\lambda$, $v$ and (the logarithm of) $\Lambda$.

As already stated above, the renormalization procedure just outlined fails to define a sensible renormalized theory. In particular we can individuate the following shortcomings.

There is clearly a pathological behavior of the effective quartic coupling $\lambda$ as a function of the bare parameter $\lambda_0$ at fixed cut-off $\Lambda$. In fact $\lambda$ has the correct 1–loop $\lambda_0^2$ term...
dictated by perturbation theory, but certainly fails at higher orders, since it exhibits an unphysical behaviour, growing to a maximum value at \( \lambda_0 = \lambda_0^{\text{max}} \), then decreasing to zero and to even more unphysical negative values (returning positive only for very large values of \( \lambda_0 \)). This implies the breakdown of the HF approximation for values of \( \lambda_0 \) greater than \( \lambda_0^{\text{max}} \) in a theory at fixed cut-off, as compared, for instance, with the standard 1–Loop–Renormalization–Group improved relation which reads

\[
\frac{1}{\tilde{\lambda}} = \frac{1}{\lambda_0} + \frac{3}{16\pi^2} \log \Lambda + \ldots
\]  

(3.28)

where the dots stand for some suitable choice of the renormalization scale and of the finite parts. The relation in eq. (3.28) is monotonically increasing with \( \lambda_0 \), at fixed \( \Lambda \), to an asymptotic plateau value \( \lambda_{\text{max}} \). As a consequence it can be inverted determining a single bare coupling \( \lambda_0 \) value for any \( \Lambda \) and \( \lambda < \lambda_{\text{max}} \). A full trajectory of \( \lambda_0 \) as a function of \( \Lambda \), up to the vertical asymptote at the Landau Pole value \( \Lambda_{\text{LP}} \), corresponds to a single renormalized theory describing the same dynamics at momentum scales much smaller than \( \Lambda \). On the other hand, in the Hartree–Fock approximation, the non–monotonic behaviour of the relation in eq. (3.27) spoils this one–to–one correspondence between bare and renormalized parameters (at fixed cut–off) which holds true only for small coupling.

Moreover, we see that imposing a finite value to \( \lambda \) at the chosen scale \( s \) fails to render finite the running coupling (i.e. the generalization of eq. (3.26) to any value of momentum)

\[
\lambda(p) = 6 \tilde{\Omega}^{(2)}(p)
\]

(3.29)
at any momentum \( p \neq q_s \) and even at any \( q_{s'} \) with \( s' \neq s \).

For what concerns the higher order terms \( \Omega^{(k)} \) with \( k > 2 \) we can see that a logarithmic dependence on the cut–off persists even after imposing the renormalization condition in eq. (3.26). That is they do not parametrically depend solely on \( v \) and \( \lambda \), but also on \( \lambda_0 \) and therefore on \( \log \Lambda \). This happens for two specific reasons

1. The internal propagators \( G_R^{(0)} \), \( G_A^{(0)} \) and \( G^{(0)} \) introduce an explicit and non removable dependence on \( \lambda_0 \) due to the presence of the resummed mass \( m^2 = \frac{1}{3} \lambda_0 v^2 \). Notice that, unlike the unbroken symmetry case, here the resummed contribution to the internal propagators differs from those to the external propagator defined as the functional inverse of the two–legs vertex function. To the latter contribute also diagrams as

\[
\text{---} + \text{---} + \ldots
\]
as can be seen from its explicit form [see eq. (3.19)]

\[
\tilde{\Gamma}^{(2)}(p) = -p^2 + \frac{1}{3} \lambda(p) v^2 = -p^2 + m^2 - \frac{1}{2} \lambda_0^2 \tilde{\Gamma}^{(2)}(p) [1 - \frac{1}{2} \lambda_0 \tilde{\Gamma}^{(2)}(p)]^{-1} v^2
\]

(3.30)

This is finite at momentum \( q_s \) while, according with what we said above about \( \lambda(p) \), logarithmic divergences appear as \( p \neq q_s \).
2. In the HF definition of $\Omega^{(k)}$ with $k > 2$ there appears the effective vertex $\theta$. By the relation [see eq. (3.21)]
\[
\hat{\theta}(p) = \frac{1}{3} \lambda(p) + \frac{2}{3} \lambda_0
\]
we can see that, after imposing the renormalization condition, an unresolved $\lambda_0$ dependence persists even when $p = q_s$. For $p \neq q_s$ further cut-off dependences appear due to the problems concerning the renormalized running coupling in eq. (3.29).

We conclude that the HF approximation cannot be renormalized by the standard renormalization procedure: there is an unphysical bare–to–renormalized coupling relation and a plain failure to eliminate divergences in the subgraphs of the resummation. In the next subsection we will define a modified HF resummation by explicitly requiring renormalizability and a 1–loop–renormalization–group improved relation between $\lambda$ and $\lambda_0$ [see eq. (3.28)]. By comparing the two resummations we will recognize the cause of the renormalization problems of the Hartree-Fock approximation in the incomplete resummation of Leading Logarithms of the cut-off.

### 3.3 Improved HF approximation

Our recipe for the improvement of the HF resummation consists in two fundamental steps

1. Fix some features, proper of the HF approximation, that we want to maintain throughout the modification since they provide a minimal definition of a mean field resummation. The results of sec. 3.1 provide a parameterization of the class of approximations having such defining features.

2. Require explicitly renormalizability and RG–invariance in order to fix the form of the arbitrary parameters.

In conclusion, once the result is obtained, we will be able to establish a diagrammatic interpretation of our modified approximation.

The main features of the HF approximation are encoded in

- The general mean field form of the background equation of motion
\[
\frac{\delta \Gamma_{1PI}}{\delta \phi_{\Delta}(x)} \bigg|_{\phi_{\Delta}=0} = \left( \square + 2 \mathcal{F}'[\xi](x) \right) \phi(x) = 0 \tag{3.31}
\]

- The self consistent definition of $\mathcal{F}'$ (through the mean field $\mathcal{V}$)
\[
\mathcal{F}' = \left[ \frac{1}{2} \Omega^{(2)} - \frac{1}{4} \theta \right] \Delta \xi + \frac{1}{2} \mathcal{V} \quad , \quad \mathcal{V} = \frac{1}{2} \theta \Delta \xi + \frac{1}{2} \theta \ J[\mathcal{V}] \tag{3.32}
\]

where we recall the definition of $J$ by eq. (3.14) and eq. (3.11).
As we have seen in sec. 3.1 the relations in eqs. (3.31) and (3.32) provide a general recipe for building up all vertex functions using \( \Omega^{(2)} \), the effective vertex \( \theta \) and the free propagators. Moreover, the self–consistent eqs. (3.32) for \( F' \) correspond to the following general form of the equation of motion for the mode functions

\[
\{ \Box + m^2 + \mathcal{V}(x) \} u_k(x) = 0 , \quad \mathcal{V} = \frac{1}{2} \theta \{ \Delta \xi + I - \bar{I}^{(1)} - I^{(2)} \mathcal{V} \}
\]

with

\[
I(x) = \int_{p^2 < \Lambda} \frac{d^3p}{(2\pi)^3} |u_p(x)|^2
\]

We now regard \( \Omega^{(2)} \), \( \theta \) and the effective mass \( m^2 \) of the free propagators as tunable parameters defining a class of approximations that share the same general diagrammatic structure. In other words, we abandon the HF definitions for the parameters, that read

\[
m^2 = \frac{1}{3} \lambda_0 v^2 , \quad \theta = \lambda_0 \left[ 1 - \frac{1}{2} \lambda_0 I^{(2)} \right]^{-1} , \quad \Omega^{(2)} = \frac{1}{2} \theta - \frac{1}{3} \lambda_0 \mathbf{1} \quad (3.33)
\]

and look instead for new definitions that ensure proper renormalizability and RG–invariance.

We can actually further specify the form of the tunable parameters. By looking at the HF definitions in eq. (3.33) of \( \Omega^{(2)} \) and \( \theta \) we see that they must have the general leading log structure

\[
\Omega^{(2)} = \lambda_0 F_1(\lambda_0 I^{(2)}) , \quad \theta = \lambda_0 F_2(\lambda_0 I^{(2)}) \quad (3.34)
\]

in terms of two functions, \( F_1 \) and \( F_2 \), of a single variable (the evaluation of these functions on the operator \( I^{(2)} \) is obvious if we consider the Fourier transform). We assume that the same holds true for \( m^2 \). This can be understood looking at the explicit form of the two–legs function in eq. (3.30). We consider \( m^2 \) as a function of \( I^{(2)} \) evaluated at zero momentum since we don’t want to introduce any new mass scale dependence, that is

\[
m^2 = \lambda_0 F_3(\lambda_0 \bar{I}^{(2)}(0)) v^2 \quad (3.35)
\]

Notice that changing \( m^2 \) corresponds to changing the equilibrium solution according to eq. (3.2).

To conclude we observe that the HF approximation resums correctly the 1–loop perturbative order. To maintain this feature we should require that \( \Omega^{(2)} \) matches at tree and 1–loop order while \( \theta \) and \( m^2 \) match at tree level. Explicitly we have the following conditions on the \( F \)’s

\[
F_1(x) = \frac{1}{6} + \frac{1}{4} x + O(x^2) , \quad F_2(x) = 1 + O(x) , \quad F_3(x) = \frac{1}{3} + O(x) , \quad (3.36)
\]

Now we have to define our modified HF approximation by fixing new explicit definitions for the functions \( F \)’s. We obtain renormalizability with the correct 1–loop beta function, by assuming the following logarithmic dependence for the bare coupling \( \lambda_0 \)

\[
\frac{\partial \lambda_0}{\partial \log \Lambda} = \frac{3}{16\pi^2} \lambda_0^3 + O(\Lambda^{-1})
\]
and requiring that the parameters $\Omega^{(2)}$, $\theta$ and $m^2$ do not depend on log $\Lambda$. Hence, differentiating eqs. \((3.34)\) and \((3.35)\) with respect to log $\Lambda$, yields

\[
(1 - \frac{3}{2} x) F'_I(x) + \frac{3}{2} F_I(x) = 0 \Rightarrow F_I(x) = \frac{A_I}{1 - \frac{3}{2} x} \quad I = 1, 2, 3
\]  \(3.37\)

where $A_I$ are integration constants and we have used

\[
\frac{\partial I^{(2)}}{\partial \log \Lambda} = -\frac{1}{8\pi^2} 1 + O(\Lambda^{-1})
\]

Eqs. \((3.37)\) have a unique solution that fulfill the matching constraints in eqs. \((3.36)\).

In terms of the parameters $\Omega^{(2)}$, $\theta$, $m^2$

\[
\theta = 6 \Omega^{(2)} = \lambda_0 [1 - \frac{3}{2} \lambda_0 I^{(2)}]^{-1} \quad , \quad m^2 = \frac{1}{3} \lambda_0 [1 - \frac{3}{2} \lambda_0 \bar{J}^{(2)}(0)]^{-1} v^2
\]  \(3.38\)

These are the explicit forms of the parameters in our improved HF approximation.

Now applying the renormalization condition in eq. \((3.26)\) with the new definition of $\Omega^{(2)}$ we obtain the following bare–to–renormalized relation

\[
\lambda = \frac{\lambda_0}{1 - \frac{3}{2} \lambda_0 I^{(2)}(q_s)}
\]  \(3.39\)

Comparing this with eq. \((3.28)\) we can see that our procedure has reproduced the correct 1–loop–renormalization–group improved behaviour while at the same time fixing the finite parts. Substituting eq. \((3.39)\) into eqs. \((3.34)\) and eq. \((3.34)\) we obtain a manifestly finite form for the parameters

\[
\theta = 6 \Omega^{(2)} = \lambda [1 - \frac{3}{2} \lambda J^{(2)}]^{-1} \quad , \quad m^2 = \frac{1}{3} \lambda [1 - \frac{3}{2} \lambda \bar{J}^{(2)}(0)]^{-1} v^2
\]  \(3.40\)

in terms of $\lambda$ and of the subtracted integral $J^{(2)}$

\[
J^{(2)} = I^{(2)} - \bar{I}^{(2)}(q_s) 1
\]

In particular, recalling the natural definition of the running coupling constant in eq. \((3.29)\), we see that now

\[
\bar{\theta}(p) = \lambda(p) = \frac{\lambda}{1 - \frac{3}{2} \lambda \bar{J}^{(2)}(p)}
\]  \(3.41\)

Now, for what concerns RG–invariance, one verifies from eq. \((3.39)\) that the parameterization of $\lambda_0$ does not depend on the renormalization scale. In fact, renormalizing at scale $s$ with constant $\lambda = \lambda(q_s)$ or at scale $s'$ with constant $\lambda' = \lambda(q_{s'})$ indeed defines the same $\lambda_0$:

\[
\frac{1}{\lambda_0} = \frac{1}{\lambda} + \frac{3}{2} \bar{I}^{(2)}(q_s) = \frac{1}{\lambda'} + \frac{3}{2} \bar{I}^{(2)}(q_{s'})
\]
Therefore eqs. (3.38) can be regarded as manifestly RG–invariant definitions of the parameters.

Finally let us observe that, since $I^{(2)}$ depends on $m^2$, we have here an implicit definition of the physical mass by the finite self–consistent relation in eq. (3.40), rather than an explicit definition, such as the tree–level $m^2 = \frac{1}{3} \lambda v^2$, which can be recovered only if $s = 0$.

We give now a brief diagrammatic interpretation of the results just derived. First of all, with the new form of $m^2$ in eq. (3.38), the free propagators are no longer defined as the resummation of tadpole corrections. In fact are included contributions from graphs as in fig.. These were actually present in the HF definition of the external propagator. Moreover contributions from graphs as

\[ \ldots + \quad + \quad \ldots \]

are included in such a way that the log $\Lambda$ dependence of $m^2$ (at fixed $\lambda_0$) corresponds to the correct 1–loop–renormalization–group improved series. Notice that corrections of this type, if included completely, would define a momentum dependent self–energy. On the other hand it is implicit in a mean–field approximation that the internal propagators have a momentum–independent self–energy. Thus the contributions of these diagrams are to be included in a “local” fashion (i.e. with no momentum passing through the loops).

Now, for what concerns the new form of $\Omega^{(2)}$ in eq. (3.38), we can see that it corresponds to the inclusion, in addition to the chain diagrams of the pure HF approximation, of all the Leading Log contributions from diagrams as

\[ \ldots + \quad + sym \quad \ldots + \quad + sym \quad \ldots \]

As before their log $\Lambda$ dependence is taken while their finite part is fixed by our procedure to be the same of $I^{(2)}$.

In conclusion, the modification of the effective vertex $\theta$ in eq. (3.38) corresponds to include, in the diagrammatic resummation that defines the $\Omega^{(k)}$ ($k > 2$), Leading Logarithmic contributions from graphs of the form

The finite parts of these diagrams are chosen in such a way to maintain the main Hartree–Fock–like features of the effective vertex, namely the single channel structure and its form as a function of $I^{(2)}$. 21
Let us make some final observations on the obtained result. The equations of motion of this modified Hartree-Fock approximation read

\[
\{\Box + \mathcal{V}(x) - \frac{1}{3} \theta \Delta \xi(x)\} \phi(x) = 0 , \quad \{\Box + \frac{1}{3} \tilde{\theta}(0) v^2 + \mathcal{V}(x)\} u_k(x) = 0 \tag{3.42}
\]

\[
\mathcal{V} = \frac{1}{2} \theta \left\{ \Delta \xi + I - \tilde{I}^{(1)} - I^{(2)} \mathcal{V} \right\}
\]

As we can see, in order to obtain renormalizability and RG–invariance, we had to introduce space time non-locality in the equations of motion. Causality in this nonlocal evolution is guaranteed by the analyticity in the upper halfplane of \(\tilde{I}^{(2)}(p)\). Actually in the third equation in eqs. (3.42), the definition of \(\mathcal{V}\) is implicit and it should be solved for \(\mathcal{V}\) in order to have a manifestly causal form. This can be done paying the price of losing manifest finiteness

\[
\mathcal{V} = \frac{1}{2} \lambda \left[ 1 - \lambda \left( I^{(2)} - \frac{3}{2} \tilde{I}^{(2)}(q_s) \right) \right]^{-1} \left\{ \Delta \xi + I - \tilde{I}^{(1)} \right\}
\]

### 3.4 Other initial states

We already pointed out, in sec. 3.1, that eqs. (3.42), with vacuum initial kernels and generic initial conditions on the background field, already describe an out–of–equilibrium problem. In this section we study whether and how we can choose different initial conditions for the mode functions without spoiling the properties of renormalizability and RG invariance. We already considered this problem in [30] treating the unbroken symmetry case. The results in this section will be very similar.

As an example consider an initial state of the same form of the HF vacuum

\[
\tilde{G}(k) = \frac{1}{2\Omega_k} , \quad \tilde{S}(k) = 0 , \quad \Omega_k^2 = k^2 + M^2
\]

but with a mass \(M\) different from the equilibrium mass \(m\) defined in eq. (3.38) with \(M \neq m\). The corresponding free mode functions have no longer the plane wave form \(u^{(0)} = u^{(\text{vac})}\) [see eq. (3.2)]

\[
u^{(0)}_k(x) = \frac{e^{ik \cdot x}}{2\sqrt{2\Omega_k}} \left\{ \left[ 1 + \frac{\Omega_k}{\omega_k} \right] e^{-i\omega_k t} + \left[ 1 - \frac{\Omega_k}{\omega_k} \right] e^{i\omega_k t} \right\}
\]

and the free correlation function is not translationally invariant in time. Then the integral term in eqs. (3.42)

\[
I - \tilde{I}^{(1)} - I^{(2)} \mathcal{V}
\]

in spite of the subtractions still contains the superficially divergent contribution

\[
\int d^3k \, \frac{m^2 - M^2}{4\omega_k^3} \cos 2\omega_k t
\]

that indeed diverges with the cut–off when \(t = 0\). These initial time singularities can be removed by a Bogoliubov transformation on the initial state as first observed in [35].
This transformation redefines the initial kernel in such a way that the leading terms of an high-momentum expansion are the same as at equilibrium

\[ \hat{G}(k) \sim \frac{1}{2\sqrt{k^2}} + \frac{m^2}{4(k^2)^{3/2}} + \ldots \]

Initial singularities as well as any other divergence turn to be absent for any choice of kernel having the above large \( k \) expansion. A simple interpretation is that the renormalization procedure ensures finiteness for any initial Gaussian state belonging to the same Fock space of the HF vacuum. Extrapolating from this simple example a generic condition on the short-distance behaviour of the initial state kernel, we have

\[ \hat{G}(x, y) \simeq \frac{1}{4\pi^2|x - y|} + \frac{m^2}{8\pi^2} \log |x - y| + \ldots \]

This ensures the cancellation of all divergent terms are guaranteed by mass and coupling constant renormalization.

4 The case \( N > 1 \)

We are now ready to consider the more general case of a scalar field theory with spontaneously broken \( O(N) \) symmetry. In doing this we proceed following closely sec. 3.

Before we begin we fix some notational conventions that will help us to handle \( O(N) \) index structures while keeping formulas simple and similar to those in sec. 3. In particular we will use the standard matrix notation for objects with one and two indices

\[ [M V]_i = M_{ij} V_j , \quad [M^{(1)} M^{(2)}]_{ij} = M^{(1)}_{ik} M^{(2)}_{kj} \]

\[ \Pi_{ijkm} = \frac{1}{2} (\delta_{ik} \delta_{jm} + \delta_{im} \delta_{jk}) , \quad T^{-1} T = T T^{-1} = \Pi \]

where the last relation is regarded as restricted to tensors with the symmetry \( T_{ijkm} = T_{ijmk} = T_{jikm} \). Moreover we will use also the following definitions

\[ \hat{1}(x, y) = 1 \delta^{(4)}(x - y) , \quad \hat{\Pi}(x, y) = \Pi \delta^{(4)}(x - y) \]

4.1 Analysis of the HF approximation

The Hartree-Fock equations of motion for a scalar \( O(N) \) theory are shown in eqs. (2.6).

As a first example of the use of the notational conventions in eq. (4.1) and eq. (4.2) we
The mean field of the broken symmetry phase.

different static solutions that correspond to the equivalent distinct vacua of the theory in

\[ \xi \]

where we recall that \( \xi(x) = \phi(x)\phi^T(x) \) and the definition of \( \tau \) in eq. (2.5).

Exactly as in sec. 3.1 we select the values of bare parameters corresponding to the existence of a vacuum solution with non-zero constant and uniform background field, \( \phi(x) = v \). Notice that here \( v \) is a \( O(N) \). The direction \( \hat{v} = v/\sqrt{v^2} \) of the vacuum background field provides the definition of longitudinal and transverse projectors

\[ P_L = \hat{v}\hat{v}^T, \quad P_T = 1 - P_L \]

The vacuum mode functions now are

\[ u_k^{(\text{vac})}(x) = (2\omega_k)^{-1/2} \exp[i(k \cdot x)] \]

\[ = [(2\omega_{k,L})^{-1/2} e^{-i\omega_{k,L}t} P_L + (2\omega_{k,T})^{-1/2} e^{-i\omega_{k,T}t} P_T] e^{i k \cdot x} \]

where in the first line the power and the exponentiation are operations on matrices and \( \omega, \omega_L, \) and \( \omega_T \) are defined as

\[ \omega_k^2 = \omega_{k,L}^2 P_L + \omega_{k,T}^2 P_T = k^2 \mathbf{1} + M^2, \quad M^2 = m_L^2 P_L + m_T^2 P_T \]

The longitudinal mass has the same value as in the \( N = 1 \) case, namely \( m_L^2 = \frac{1}{3} \lambda_0 v^2 \), while the transverse squared mass \( m_T^2 \), as a function of \( \lambda_0, v^2 \) and \( \Lambda \), is obtained by solving the self-consistent equation

\[ 0 = m_T^2 + \frac{1}{3} \lambda_0 \int_{p^2 < \Lambda} \frac{d^3p}{(2\pi)^3} \left\{ \frac{1}{2\omega_{k,L}} - \frac{1}{2\omega_{k,T}} \right\} \]

The vacuum expectation value \( v \) satisfies a gap equation which generalizes the one in eq. (4.4)

\[ 0 = m_0^2 + \frac{1}{6} \lambda_0 v^2 + \frac{1}{2} \lambda_0 \int_{p^2 < \Lambda} \frac{d^3p}{(2\pi)^3} \left( \frac{1}{2\omega_{k,L}} - \frac{1}{2\omega_{k,T}} \right) + \frac{1}{6}(N - 1) \lambda_0 \int_{p^2 < \Lambda} \frac{d^3p}{(2\pi)^3} \left( \frac{1}{2\omega_{k,T}} \right) \]

Notice that the gap equation involves only \( v^2 \). In fact varying the direction of \( \hat{v} \) we obtain different static solutions that correspond to the equivalent distinct vacua of the theory in the broken symmetry phase.

The mean field \( \mathcal{V} \) [see eq. (3.1)] is now an object with two \( O(N) \) indices, defined by

\[ \mathcal{V}(x) = m_0^2 \mathbf{1} - M^2 + \frac{1}{2} \lambda_0 \tau [\xi(x) + \int_{p^2 < \Lambda} \frac{d^3p}{(2\pi)^3} |u_p(x)|^2] \]
Notice that again $V = 0$ on the vacuum solution. In terms of $V$ eqs. (4.3) read
\[
\begin{align*}
\{ \Box + M^2 - \frac{1}{2} \lambda_0 \xi + V \} \phi(x) &= 0 \\
\{ \Box + M^2 + V \} u_k(x) &= 0
\end{align*}
\]
Recalling the general mean field form of the background field equation ($F'$ here is a $O(N)$ matrix)
\[
\left. \frac{\delta \Gamma_{1PI}}{\delta \phi_\Delta} \right|_{\phi_\Delta = 0} = (\Box + 2 F'[\xi]) \phi = 0 \quad (4.6)
\]
we have
\[
F'[\xi] = \frac{1}{2} V + \frac{1}{2} M^2 - \frac{1}{6} \lambda_0 \xi
\]
which should be compared with eq. (3.7) in sec. 3.1.

In order to derive a self–consistent equation for $V$ like the one in eq. (3.12), as in sec. 3.1 we introduce the free mode functions
\[
(\Box + M^2) u_k^{(0)}(x) = 0 \quad , \quad u_k^{(0)}(x, 0) = u_k(x, 0) \quad , \quad u_k^{(0)}(x, 0) = \dot{u}_k(x, 0)
\]
and define the free propagators
\[
\begin{align*}
G_R^{(0)}(x, y) &= G_{R,L}^{(0)}(x, y) P_L + G_{R,T}^{(0)}(x, y) P_T = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{(p^2 + i\epsilon p_0) 1 - M^2} \\
G_{A,ij}^{(0)}(x, y) &= G_{R,ji}^{(0)}(y, x) \\
G_{ij}^{(0)}(x, y) &= G_L^{(0)}(x, y) P_L,ij + G_T^{(0)}(x, y) P_T,ij = \text{Re} \int_{|p| < \Lambda} \frac{d^3p}{(2\pi)^3} u_{p,ia}^{(0)}(x) \overline{u}_{p,aj}^{(0)}(y)
\end{align*}
\]
Performing the same manipulations on the mode function equations as in sec. 3.1 we can rewrite the cutoffed correlation, that now reads
\[
G_{ij}(x, y) = \text{Re} \int_{|p| < \Lambda} \frac{d^3p}{(2\pi)^3} u_{p,ia}(x) \overline{u}_{p,aj}(y)
\]
as a functional of $V$
\[
G = G[\mathcal{V}] = [\hat{1} - G_R^{(0)} \hat{\mathcal{V}}]^{-1} G^{(0)} [\hat{1} - \hat{\mathcal{V}} G_A^{(0)}]^{-1}
\]
In conclusion we have the self–consistent equation for $\mathcal{V}$
\[
\mathcal{V} = m_0^2 \mathbf{1} - M^2 + \frac{1}{2} \lambda_0 \tau (\xi + I[\mathcal{V}]) \quad , \quad I[\mathcal{V}](x) \equiv G[\mathcal{V}](x, x) \quad (4.8)
\]
This should be compared with eq. (3.11) and eq. (3.12).

We keep on following closely the sec. 3.1 by fixing vacuum initial conditions for the mode functions. By this choice follows that $u^{(0)} = u^{(\text{vac})}$ for all times and the free correlation is translationally invariant.
Moreover we can repeat, with little changes, the observations made in sec. 3.1 about the structure of the integral $I[\mathcal{V}]$. We can still interpret the free Green functions as effective internal propagators obtained by resumming all tadpole corrections to the bare ones. We can introduce the tadpole integral

$$I^{(1)} = G^{(0)}(x, x) = I^{(1)}_L P_L + I^{(1)}_T P_T$$

and the two propagators loop integral

$$I^{(2)}_{ijkm}(x - y) = \frac{1}{2} I^{(2)}_{TT}(x - y) P_{T,ik} P_{T,jm} + \frac{1}{2} I^{(2)}_{LL}(x - y) P_{L,ik} P_{L,jm} + \frac{1}{2} I^{(2)}_{TL}(x - y) (P_{L,ik} P_{T,jm} + P_{T,ik} P_{L,jm})$$

(4.9)

In terms of these we can define a functional $J$ by subtracting to $I$ its constant part and the term linear in $\mathcal{V}$. By the same arguments used in sec. 3.1 we can see that $J$ contains only the cut-off convergent part of $I$. The definition of the $J$ functional is formally unchanged respect to eq. (3.14) [here $O(N)$ indices contraction according to the rules in eq. (4.2) is understood]

$$J[\mathcal{V}] = I[\mathcal{V}] - I^{(1)} - I^{(2)} \mathcal{V}$$

(4.10)

Notice also that, by using the two propagator loop integral of eq. (4.9), we can cast the self–consistent definition of $m^2_T$ in eq. (4.4) into a more compact form

$$m^2_T = -\frac{1}{3} \lambda_0 \bar{I}^{(2)}_{LL}(0) m^2_T \left[1 - \frac{1}{3} \lambda_0 \bar{I}^{(2)}_{TL}(0)\right]^{-1}$$

(4.11)

The “quasi-renormalized” form is obtained by manipulations identical to those in sec. 3.1

$$\mathcal{V} = \frac{1}{2} \theta \Delta \xi + \Delta \mathcal{V}, \quad \Delta \mathcal{V} = \frac{1}{2} \theta J[\Delta \mathcal{V} + \frac{1}{2} \theta \Delta \xi], \quad \mathcal{F}' = \frac{1}{2} \Omega^{(1)} + \frac{1}{2} \Omega^{(2)} \Delta \xi + \frac{1}{2} \Delta \mathcal{V}$$

The matrix $\Omega^{(1)}$ and the four–indices objects $\Omega^{(2)}$ and $\theta$ are defined as

$$\Omega^{(1)} = m^2_T P_T, \quad \theta = \lambda_0 \left[\bar{\Pi} - \frac{1}{2} \lambda_0 I^{(2)}\right]^{-1}, \quad \Omega^{(2)} = \frac{1}{2} \theta - \frac{1}{3} \lambda_0 \bar{\Pi}$$

(4.12)

The vertex functions contributing to the equation of motion [see eq. (5.18)] are obtained from the variation of eq. (4.6) with respect to $\phi_c$ at $\phi = v$. They can be expressed in terms of functional derivatives of $\mathcal{F}'$

$$\Omega^{(k)}_{i_1j_1...i_kj_k}(x_1, \ldots, x_k) = 2 \frac{\delta \mathcal{F}'[\xi](x_1)}{\delta \xi_{i_1j_1}(x_2) \ldots \delta \xi_{i_kj_k}(x_k)} \bigg|_{\xi = v^2}$$

as in eq. (3.19). Notice that $\Omega^{(1)}$ is completely transverse [see eq. (4.12)] which implies that $\Gamma^{(1)} = \Omega^{(1)} v = 0$, namely the statement that the vector $v$ is the vacuum static solution.

Of course all the diagrammatic interpretations made in sec. 3.1 still hold true, now with diagrams carrying $O(N)$ indices.
Before going further, we should introduce some notations that will be useful later on. In fact we will have to deal with four–indices tensors which have some particular symmetry properties and are functions of $v$ (and of other scalar quantities)

$$T_{ijkm} = T_jikm = T_{ijmk} = T_{kijm}$$

as, for example, $\theta$ and $\Omega^{(2)}$. Such tensorial objects admit a general decomposition

$$T_{ijkm}(v) = T_\alpha(v^2) \, t^\alpha_{ijkm}(\hat{v}) \quad \alpha = 1, \ldots, 5$$

in terms of the five elementary tensors

$$
\begin{align*}
  t^1_{ijkm} &= \hat{v}_i\hat{v}_j\hat{v}_k\hat{v}_m, \\
  t^2_{ijkm} &= \frac{1}{2} (P_{T,ik} P_{T,jm} + P_{T,im} P_{T,jk}) \\
  t^3_{ijkm} &= P_{T,ij} P_{T,km}, \\
  t^5_{ijkm} &= (P_{L,ij} P_{T,km} + P_{T,ij} P_{L,km}), \\
  t^6_{ijkm} &= \frac{1}{2} (P_{L,ik} P_{T,jm} + P_{L,im} P_{T,jk} + P_{T,ik} P_{L,jm} + P_{T,im} P_{L,jk})
\end{align*}
$$

Notice that the coefficients of the decomposition are functions of $v^2$ alone. As an example, by eq. (4.12) the coefficients $\theta_\alpha$ of the decomposition of $\theta$ are

$$
\begin{align*}
  \theta_1 &= \lambda_0 \left[ 1 - \frac{1}{9} (N + 2) \lambda_0 I^{(2)}_{TT} \right] \left\{ 1 - \frac{1}{6} (N + 1) \lambda_0 I^{(2)}_{LL} \right\}^{-1} \\
  \theta_2 &= \frac{2}{3} \lambda_0 \left[ 1 - \frac{1}{3} \lambda_0 I^{(2)}_{TT} \right]^{-1} \\
  \theta_3 &= \frac{2}{3} \lambda_0 \left[ 1 - \frac{1}{3} \lambda_0 I^{(2)}_{LL} \right]^{-1} \\
  \theta_5 &= \frac{1}{3} \theta_4 \left[ 1 - \frac{1}{3} \lambda_0 I^{(2)}_{TT} \right]^{-1}, \\
  \theta_3 &= \theta_5 \left[ 1 - \frac{1}{3} \lambda_0 I^{(2)}_{LL} \right] \left[ 1 - \frac{1}{3} \lambda_0 I^{(2)}_{TT} \right]^{-1}
\end{align*}
$$

Notice that using eqs. (4.14) the compact form of the self–consistent definition of $m^2_7$ in eq. (4.11) can be rewritten as

$$m^2_7 = \frac{1}{3} \lambda_0 v^2 - \frac{1}{2} \hat{\theta}_4(0) v^2$$

The coefficient of $\Omega$ can be determined from those of $\theta$ by eq. (4.12) and the decomposition of the $\Pi$ tensor

$$
\begin{align*}
  \Omega^{(2)}_\alpha &= \frac{1}{2} \theta_\alpha - \frac{1}{3} \lambda_0 \mathbf{1}, \quad \alpha = 1, 2, 4 \\
  \Omega^{(2)}_\alpha &= \frac{1}{2} \theta_\alpha, \quad \alpha = 3, 5
\end{align*}
$$

### 4.2 Renormalization

Now we apply the standard renormalization procedure as we did in sec. 3.2 for the $N = 1$ case. We will see that the same problems are present also in this case and some others will arise due the presence of the spontaneously broken continuous $O(N)$ symmetry.

By fixing the equilibrium value of the background field we provide the first renormalization condition that defines the bare mass $m^2_0$ as a function of $\lambda_0$, $v$ and $\Lambda$

$$m^2_0(\lambda_0, v, \Lambda) = -\frac{1}{6} \lambda_0 v^2 - \frac{1}{2} \lambda_0 \int_{p^2 < \Lambda} \frac{d^3p}{(2\pi)^3} \frac{1}{2 \omega_{k,L}} + \frac{1}{6} (N - 1) \lambda_0 \int_{p^2 < \Lambda} \frac{d^3p}{(2\pi)^3} \frac{1}{2 \omega_{k,T}}$$
We recall that $m_L^2 = \frac{1}{3} \lambda_0 v^2$ and $m_T^2$ is given (in function of $\lambda_0$, $v$ and $\Lambda$) as solution eq. (4.4). Again, enforcing this condition is enough to remove all $\Lambda^2$ dependence from the vertex functions.

The second renormalization condition is conventionally obtained by evaluating at the symmetric point [see eq. (3.23)] the “all–longitudinal” component of the four–legs vertex function $\Gamma^{(4)}$ and requiring it to be equal to the renormalized coupling $\lambda$

$$
\lambda = \tilde{\Gamma}^{(4)}_{ijklm}(-\frac{3}{2} q_s, \frac{1}{2} q_s, \frac{1}{2} q_s) \hat{v}_i \hat{v}_j \hat{v}_k \hat{v}_m
$$

Explicitly expressed in terms of the variations of $F$ the above expression involves the (longitudinal components) of $\Omega^{(2)}$, $\Omega^{(3)}$ and $\Omega^{(4)}$. As in sec. 3.2 [see eq. (3.26)] we drop the contributions from $\Omega^{(3)}$ and $\Omega^{(4)}$ obtaining a simpler definition

$$
\lambda = 6 \tilde{\Omega}^{(2)}_{ijklm}(q_s) \hat{v}_i \hat{v}_j \hat{v}_k \hat{v}_m = 6 \tilde{\Omega}^{(2)}_1(q_s)
$$

(4.17)

or, thanks to eq. (4.12),

$$
\lambda = \frac{3 \lambda_0 [1 - \frac{N+2}{9} \lambda_0 \tilde{\Gamma}^{(2)}_{TT}(q_s)]}{1 - \frac{N+1}{6} \lambda_0 \tilde{\Gamma}^{(2)}_{TT}(q_s) - \frac{1}{2} \lambda_0 \tilde{\Gamma}^{(2)}_{LL}(q_s) [1 - \frac{N+2}{9} \lambda_0 \tilde{\Gamma}^{(2)}_{TT}(q_s)]} - \frac{1}{3} \lambda_0
$$

(4.18)

This has to be compared with the corresponding relation of the $N = 1$ case in eq. (3.27).

First of all eq. (4.18) shows the same pathological dependence of $\lambda$ on $\lambda_0$, at fixed $\Lambda$ and $v$, that we found in sec. 3.2. As already discussed this prevents a consistent map between bare parameters and renormalized ones unless we restrict to a small coupling regime. The correct 1–Loop–Renormalization–Group improved relation, instead, reads

$$
\frac{1}{\Lambda} = \frac{1}{\lambda_0} + \frac{N + 8}{48\pi^2} \log \Lambda + \ldots
$$

(4.19)

where again the dots stand for some suitable choice of the renormalization scale and of the finite parts.

For what concerns the cut–off dependence of $\Omega^{(2)}$ we can see that things are even worse than in the $N = 1$ case. In fact, not only the log $\Lambda$ dependence is not removed by the renormalization condition from $\tilde{\Omega}^{(2)}(p)$ for $p \neq q_s$, but also, for components $\Omega_\alpha$ with $\alpha \neq 1$ logarithmic divergences already appear at $p = q_s$ as can be verified by the explicit definitions in eqs. (4.16).

Regarding the cut–off dependence of the $\Omega^{(k)}$ with $k > 2$ the same observations of sec. 3.2 still hold in this case. In particular we still have log $\Lambda$ dependence in the masses of the free propagators. This is immediate for $m_L^2 = \frac{1}{3} \lambda_0 v^2$ and can be easily verified for $m_T^2$ by considering its self–consistent definition in eq. (4.4). Moreover, as in the $N = 1$ case, log $\Lambda$ divergences persist in the effective vertex $\theta$ after imposing the renormalization condition [eq. (4.17)].

Besides these renormalization problem analogous to the $N = 1$ case, another very important aspect, which is peculiar of the $N > 1$ case, must be pointed out. First of all,
we may verify the formal $O(N)$ invariance of the HF approximation also in the current out–of–equilibrium context. In fact, eqs. \textit{[13]} are manifestly $O(N)$ symmetric and so it should be for the effective action

$$\Gamma_{1PI}[\phi, \phi_c] = \Gamma_{1PI}[R \phi, R \phi_c], \quad \forall R \in O(N)$$

Hence for infinitesimal $R = 1 + \epsilon = 1 - \epsilon^T$, to first order we must have

$$0 = \int d^4x \left\{ \phi_\Delta^T(x) \epsilon \left( \frac{\delta}{\delta \phi_\Delta(x)} \Gamma_{1PI}[\phi, \phi_c] \right) + \phi_c^T(x) \epsilon \left( \frac{\delta}{\delta \phi_c(x)} \Gamma_{1PI}[\phi, \phi_c] \right) \right\}$$

Upon variation with respect to $\phi_\Delta$ at $\phi_\Delta = 0$, $\phi_c = v$ and recalling that $\Gamma^{(1)} = 0$, we conclude

$$\int d^4x v_i \epsilon_{ij} \tilde{\Gamma}^{(2)}_{jk}(x-y) = 0 \implies \tilde{\Gamma}^{(2)}_T(0) = 0$$

This Ward identity, stating the masslessness of the external transverse propagator (the inverse of $\tilde{\Gamma}^{(2)}_T(p)$), is indeed satisfied by the HF approximation

$$\Gamma^{(2)} = \Gamma^{(2)}_L P_L + \tilde{\Gamma}^{(2)}_T P_T$$

$$\tilde{\Gamma}^{(2)}_L(p) = -p^2 + 2 \tilde{\Omega}^{(2)}_1(p) v^2$$

$$\tilde{\Gamma}^{(2)}_T(p) = -p^2 + \Omega^{(1)}_4 + \tilde{\Omega}^{(2)}_4(p) v^2 = -p^2 + m_T^2 + \left[ \frac{1}{2} \tilde{\theta}_4(p) - \frac{1}{2} \lambda_0 \right] v^2$$

thanks to the self–consistent definition of the transverse mass, eq. \textit{[14,15]}. But the transverse mass itself, which enters the internal transverse propagators $G^{(0)}_{R,T}$ and $G^{(0)}_T$ does not vanish at all (it actually diverges as log $\Lambda$), preventing a consistent interpretation of the transverse modes as Goldstone bosons. This is a well known problem of the HF approximation (see for instance ref. \textit{[32]}).

### 4.3 A class of improved HF approximations

We shall now try and improve the HF approximation to recover the correct properties of Renormalizability and RG-invariance. We shall also require that this improved resummation is gapless; that is, we shall impose that the internal transverse propagators are massless. To this end we follow closely the procedure of sec. \textit{[33]} while stressing some important new features which appear due to the continuous $O(N)$ symmetry and the presence of two kinds of fields (transverse and longitudinal) with different masses. Let us anticipate that in this case our procedure does not single out a unique solution but rather an extended class of resummations that share all the required properties.

As in sec. \textit{[33]} the first step consists in fixing some fundamental properties of the HF approximation that we want to preserve. First of all the general structure of the diagrams resummation which is encoded in the general mean field form of the background equations of motion:

$$\left. \frac{\delta \Gamma_{1PI}}{\delta \phi_{\Delta,i}(x)} \right|_{\phi_{\Delta} = 0, \phi_c = \phi} = (\Box 1 + 2 \mathcal{F}^{[\xi]}(x))_{ij} \phi_j(x) = 0$$

(4.21)
and in the self–consistent definition of $\mathcal{F}'$

$$\mathcal{F}' = \frac{1}{2} \Omega^{(1)} + \left[ \frac{1}{2} \Omega^{(2)} - \frac{1}{4} \theta \right] \Delta \xi + \frac{1}{2} \mathcal{V} \quad , \quad \mathcal{V} = \frac{1}{2} \theta \Delta \xi + \frac{1}{2} \theta \mathcal{J} \mathcal{V}$$  \hspace{1cm} (4.22)

which imply the following mean–field–type equations of motion for the mode functions

$$\left[ \Box + M^2 + \mathcal{V} \right] u_k = 0 \quad , \quad \mathcal{V} = \frac{1}{2} \theta \left\{ \Delta \xi + I - I^{(1)} - I^{(2)} \right\}$$  \hspace{1cm} (4.23)

As in sec. 3.3 we now regard $\Omega^{(1)}$, $\Omega^{(1)}$, $\theta$ and the free propagators masses $m^2_L$ and $m^2_T$ (in matrix form $M^2$) as tunable parameters that we are going to change w.r.t. to their HF definitions

$$\theta = \lambda_0 \left[ \hat{\pi} - \frac{1}{2} \lambda_0 I^{(2)} \right]^{-1} \quad , \quad \Omega^{(1)} = m^2_T P_T \quad , \quad \Omega^{(2)} = \frac{1}{2} \theta - \frac{1}{3} \lambda_0 \hat{\pi}$$ \hspace{1cm} (4.24)

The first important difference with the $N = 1$ case is that now $\Omega^{(2)}$, $\theta$ and $M$ are not independent, but must fulfill certain $O(N)$ symmetry constraints. In fact the form of the equations of motion in eqs. (4.21) and (4.23) is not manifestly $O(N)$ symmetric for generic tunable parameters. More precisely they are covariant under contemporaneous rotations of the field $\phi$, of the mode functions $u_p$ and of the vacuum expectation value $v$. The latter, in fact, enters in the definitions the propagators (see the decomposition in eq. (4.7)) and of $\Omega^{(2)}$ and $\theta$ (see the general decomposition rules in eq. (4.13)). We have to explicitly require symmetry under rotations of $\phi$ and $u_p$ alone or, equivalently, invariance under rotations of $v$. In other words we must impose the proper $O(N)$ Ward identities.

Let us therefore define the operator generating the infinitesimal variation under an $O(N)$ rotation of $v$

$$\delta = \epsilon_{ij} v_i \frac{\delta}{\delta v_j}$$

where $\epsilon$ is an arbitrary antisymmetric matrix. It is immediate to verify that it annihilate any invariant function.

First of all, to require invariance in the mode functions equations (4.23), we have to impose

$$\delta [M^2 + \mathcal{V}] = 0$$  \hspace{1cm} (4.25)

Solving explicitly for $\mathcal{V}$ the second equation in eqs. (4.23) we can rewrite the above condition as

$$0 = \delta M^2 + \delta \left\{ [1 + \frac{1}{2} \theta I^{(2)}]^{-1} \theta \left[ \Delta \xi + I - I^{(1)} \right] \right\}$$  \hspace{1cm} (4.26)

Notice that $\xi$ and $u_k$ are independent and by hypothesis invariant variables (i.e. they must depend only on $v^2$). Then eq. (4.26) implies

$$\delta \left\{ [1 + \frac{1}{2} \theta I^{(2)}]^{-1} \theta \right\} = 0 \quad , \quad 0 = \delta M^2 + [1 + \frac{1}{2} \theta I^{(2)}]^{-1} \theta \left[ -v^2 \delta P_L - \delta I^{(1)} \right]$$

By some easy algebraic manipulations we can rewrite the first equation above as

$$\delta \theta = \frac{1}{2} \theta \delta I^{(2)} \theta$$
which has a simple and natural diagrammatic interpretation. Making the tensorial structure explicit, we obtain the following conditions on the coefficients $\theta_{\alpha}$

\[
\begin{align*}
\theta_2 &= \theta_4 \left[1 - \frac{1}{2} \left(I_{TT}^{(2)} - I_{TL}^{(2)} \right) \right]^{-1} \\
\theta_5 &= \left[\theta_1 - \theta_4 + \frac{1}{2} \theta_4 \theta_1 \left(I_{LL}^{(2)} - I_{TL}^{(2)} \right) \left[1 + \frac{1}{2} \theta_4 \left(I_{TT}^{(2)} - I_{TL}^{(2)} \right) \right]^{-1} \right. \\
\theta_3 &= \theta_5 \left[1 + \frac{1}{2} \theta_4 \left(I_{LL}^{(2)} - I_{TL}^{(2)} \right) \left[1 + \frac{1}{2} \theta_4 \left(I_{TT}^{(2)} - I_{TL}^{(2)} \right) \right]^{-1} \right]
\end{align*}
\]

(4.27)

Notice that these relations uniquely fix the form of $\theta$ once $\theta_1$ and $\theta_4$ have been specified.

Now, using these relations and the following rules for the explicit variations

\[
\begin{align*}
\delta M^2 &= \left(m_L^2 - m_T^2 \right) \delta P_L \\
\delta I^{(1)} &= \left(I_L^{(1)} - I_T^{(1)} \right) \delta P_L = \left(m_L^2 - m_T^2 \right) I_{TL}^{(2)}(0) \delta P_L
\end{align*}
\]

we can rewrite the second equation in eqs. (4.3) as

\[
m_L^2 - m_T^2 = \frac{1}{2} v^2 \bar{\theta}_4(0)
\]

(4.28)

In the second step, we impose invariance of the background field equation (4.21) by requiring

\[
\delta \mathcal{F}' = 0
\]

(4.29)

This can be written, by using eqs. (4.22) and (4.25) and the independence of $\xi$ and $u$, as

\[
\delta \Omega^{(2)} = \frac{1}{2} \delta \theta', \quad \delta \Omega^{(1)} - \delta M^2 - (\Omega^{(2)} - \frac{1}{2} \theta) v^2 \delta P_L
\]

Explicit tensorial calculation for the first equation yields

\[
\begin{align*}
\Omega_2^{(2)} &= \Omega_4^{(2)} + \frac{1}{2} \left[\theta_2 - \theta_4 \right] \\
\Omega_3^{(2)} &= \Omega_1^{(2)} - \Omega_4^{(2)} + \frac{1}{2} \left[\theta_3 - \theta_1 + \theta_4 \right] \\
\Omega_5^{(2)} &= \Omega_1^{(2)} - \Omega_4^{(2)} + \frac{1}{2} \left[\theta_5 - \theta_1 + \theta_4 \right]
\end{align*}
\]

(4.30)

While for the second we obtain

\[
\Omega^{(1)} = -\bar{\Omega}_4^{(2)}(0) v^2 P_T
\]

(4.31)

In conclusion, according to eqs. (4.27), (4.28), (4.30) and (4.31), the actual independent parameters are $\theta_1$, $\theta_4$, $m_L^2$, $\Omega_1^{(2)}$ and $\Omega_4^{(2)}$. The integrated versions of eqs. (4.25) and (4.29) are obtained by writing $M^2 + V$ and $\mathcal{F}'$ in a manifestly $O(N)$ symmetric form

\[
\begin{align*}
M^2 + V &= \mu^{(1)} \mathbf{1} + \frac{1}{2} \gamma^{(1)} [\xi + I] \\
\mathcal{F}' &= \mu^{(2)} \mathbf{1} + \frac{1}{2} \gamma^{(2)} \xi + \frac{1}{2} \left[M^2 + V \right]
\end{align*}
\]

\[
\gamma_{ijk}^{(\alpha)} = \gamma_1^{(\alpha)} \delta_{ij} \delta_{km} + \frac{1}{2} \gamma_2^{(\alpha)} \left[\delta_{ik} \delta_{jm} + \delta_{im} \delta_{jk} \right], \quad \alpha = 1, 2
\]
with \( \gamma^{(\alpha)} \) and \( \mu^{(\alpha)} \) defined in terms of the free parameters as follows

\[
\gamma_1^{(1)} = [\theta_1 - \theta_4 - \frac{1}{2} \theta_4 \theta_1 (I_{LL}^{(2)} - I_{TT}^{(2)})] [1 + \frac{1}{2} \theta_4 (I_{TT}^{(2)} - I_{LL}^{(2)})][1 - \frac{1}{2} \theta_4 I_{TT}^{(2)}]^{-1} \Delta^{-1} \\
\Delta = [1 + \frac{1}{2} \theta_1 (I_{TT}^{(2)} - N I_{TT}^{(2)})][1 + \frac{1}{2} \theta_1 I_{LL}^{(2)}] - \frac{N-1}{2} \theta_1 I_{TT}^{(2)}[1 + \frac{1}{2} \theta_4 I_{TT}^{(2)}]^{-1} \\
\gamma_2^{(1)} = \theta_4 [1 + \frac{1}{2} I_{TT}^{(2)}]^{-1} \\
\gamma_1^{(2)} = \frac{1}{2} \Omega_3^{(2)} - \frac{1}{4} \theta_3 \theta_1 \theta_4, \quad \gamma_2^{(2)} = \frac{1}{2} \Omega_4^{(2)} - \frac{1}{4} \theta_4 \\
\mu^{(1)} = m_L^2 - \frac{1}{2} \gamma_2^{(1)} (0) v^2 + I_L^{(1)} - \frac{1}{2} \gamma_1^{(1)} (0) [v^2 + I_L^{(1)} + (N - 1) I_T^{(1)}] \\
\mu^{(2)} = -\frac{1}{2} m_L^2 - \frac{1}{2} \Omega_1^{(2)} (0) v^2 + \frac{1}{4} \theta_1 (0) v^2 \\
\] (4.32)

One can verify that the standard HF definitions of the tunable parameters indeed satisfy eqs. (4.27), (4.28), (4.30) and (4.31) and that the corresponding parameters of the manifestly \( O(N) \) symmetric form are

\[
\gamma_2^{(1)} = 2 \gamma_1^{(1)} = 4 \gamma_2^{(2)} = 2 \frac{3}{2} \lambda_0 \theta_1 \, , \quad \gamma_1^{(2)} = 0 \, , \quad \mu^{(1)} = \frac{1}{2} m_L^2 \, , \quad \mu^{(2)} = 0 \\
\] (4.33)

Now, as in sec. 3.3, we require that \( \theta \) and \( \Omega^{(2)} \) have the same general leading log structure characteristic of their HF definitions [see eqs. (4.14) and eqs. (4.16)], namely

\[
\Omega_1^{(2)} = \lambda_0 F_1(\lambda_0 I_{LL}^{(2)}, \lambda_0 I_{TT}^{(2)}, N) \, , \quad \theta_1 = \lambda_0 F_2(\lambda_0 I_{LL}^{(2)}, \lambda_0 I_{TT}^{(2)}, \lambda_0 I_T^{(2)}, N) \\
\Omega_4^{(2)} = \lambda_0 F_3(\lambda_0 I_{LL}^{(2)}, \lambda_0 I_{TT}^{(2)}, \lambda_0 I_T^{(2)}, N) \, , \quad \theta_4 = \lambda_0 F_4(\lambda_0 I_{LL}^{(2)}, \lambda_0 I_{TT}^{(2)}, \lambda_0 I_T^{(2)}, N) \\
\] (4.34)

where the \( F_A(x, y, z; N) \) are generic functions of commuting arguments since the operators \( I_{LL}^{(2)}, I_{TT}^{(2)} \) and \( I_T^{(2)} \) are diagonal in Fourier space.

Notice that the same structure is inherited by all the others components of \( \theta \) and \( \Omega^{(2)} \) as can be verified by eqs. (4.27) and eqs. (4.30). We assume a similar structure also for \( m_L^2 \) as we did in eq. (4.34) for the \( N = 1 \) case

\[
m_L^2 = \lambda_0 v^2 F_5(\lambda_0 I_{LL}^{(2)}(0), \lambda_0 I_{TT}^{(2)}(0), \lambda_0 I_T^{(2)}(0); N) \\
\] (4.35)

Notice that, by means of eq. (4.28), the same structure is inherited by \( m_T^2 \) which is consistent with its HF definition [see eqs. (4.24)].

We have thus reduced our parameterization freedom to five functions \( F_A \) of three variables. We still have to require that these functions fulfill some further properties that hold true in the HF approximation.

Hartree–Fock matches perturbatively at 1–loop order. This matching requires that \( \theta \) and \( m_L^2 \) match at tree level and that \( \Omega^{(2)} \) matches at tree level and 1–loop order. Actually \( \Omega_4^{(2)} \) must match only at 1-loop level since the tree level term never appears in the vertex functions. Explicitly we have the following conditions on the \( F’s \)

\[
F_1(x, y, z; N) = \frac{1}{6} + \frac{1}{4} x + \frac{1}{36} (N - 1) z + \ldots \, , \quad F_2(x, y, z; N) = 1 + \ldots \\
F_3(x, y, z; N) = c_0 + \frac{1}{3} y + \ldots \, , \quad F_4(x, y, z; N) = \frac{2}{3} + \ldots \\
F_5(x, y, z; N) = \frac{1}{3} + \ldots \\
\]
where $c_0$ is a purely numerical arbitrary constant. One can easily check that the above conditions, together with the symmetry relations in eqs. (4.27), (4.28), (4.30) and (4.31), are enough to guarantee the matching at 1–loop order of all the vertex functions.

In the $N \to \infty$ limit the HF resummation reproduces correctly the equations of the usual large $N$ approximation. This can be done as follows. First we restrict to the special case of a background field which maintains a fixed direction (i.e. the direction of the vacuum expectation value $v$), that is $\phi_i(x) = \phi(x) \hat{v}_i$. Then we can reduce the equations of motion into a projected form by setting

$$u_k = u_{kL} P_L + u_{kT} P_T , \quad V = V_L P_L + V_T P_T$$

By substituting into eqs. (4.21), (4.22), (4.23) and projecting one obtains

$$[\Box + V_L + (\Omega_1^{(2)} - \frac{1}{2} \theta_1) \Delta \xi] \phi = 0$$

$$[\Box + m_T^2 + V_T] u_{kT} = 0 , \quad [\Box + m_L^2 + V_L] u_{kL} = 0$$

$$V_L = \frac{1}{2} \theta_1 [\Delta \xi + J_L] + \frac{1}{2} (N - 1) \theta_5 J_T , \quad V_T = \frac{1}{2} \theta_5 [\Delta \xi + J_L] + [\frac{1}{2} (N - 1) \theta_3 + \frac{1}{2} \theta_2] J_T$$

where $\xi = \phi^2$ and $J_L$ and $J_T$ have the obvious meaning [see eq. (4.10)]. Now let us rescale the coupling, the background field and the vacuum expectation value as prescribed by the standard large $N$ procedure

$$\lambda_0 \to \lambda_0 / N \quad \phi(x) \to \sqrt{N} \phi(x) \quad v^2 \to N v^2$$

By taking the limit and using the symmetry conditions one can see that the correct large $N$ equations

$$[\Box + V_T] \phi = 0 , \quad [\Box + V_T] u_{kT} = 0$$

$$V_T = \frac{1}{6} \lambda \Delta \xi + \frac{1}{6} \lambda \int_{|p|<\Lambda} \frac{d^3p}{(2\pi)^3} \left[ u_p(x) u_p^\dagger(y) - \frac{1}{2} \omega_{p,T}^2 \right]$$

$$\lambda^{-1} = \lambda_0^{-1} - \frac{1}{6} \tilde{I}_T^{(2)}(q_s)$$

are recovered, provided the limits of the free parameters satisfy the following relations

$$\frac{1}{2} v^2 \tilde{\theta}_4(0) - m_L^2 \to 0$$

$$N (\theta_1 - \theta_4) \to \frac{1}{3} \lambda_0 \left[ 1 - \frac{1}{6} \lambda_0 \tilde{I}_T^{(2)} \right]^{-1}$$

$$N \Omega_1^{(2)} \to \frac{1}{6} \lambda_0 \left[ 1 - \frac{1}{6} \lambda_0 \tilde{I}_T^{(2)} \right]^{-1}$$

Suitable conditions on the functions $F_A$ then follow from eqs. (4.33). Notice that in the integrals $I_T^{(2)}$ in eqs. (4.35) and (4.36) we have $m_T^2 = 0$, as consistent with the limits of the parameters.

To conclude, in the HF approximation $\Omega_1^{(2)}$ and the $\theta_\alpha$ fulfill positivity conditions. More precisely they are real and positive when evaluated at the purely spatial value of the momentum $q_s$ [see eq. (3.23)].
We now proceed in defining our modified HF approximation by making some further sensible requirements that are not satisfied by the HF approximation.

First of all we require our approximation to be gapless. That is to say we require that the transverse mass of the internal propagator is zero. As a consequence by eq. (4.28) we have

\[ m^2_L = \frac{1}{2} v^2 \theta_4(0) \Rightarrow F_5(x, y, z; N) = \frac{1}{2} F_4(x, y, z; N) \]  

(4.37)

Due to this condition the transverse internal propagators are now massless and therefore the Goldstone bosons loop integral \( I^{(2)}_{TT} \) is logarithmically IR–divergent. The symmetry relations in eq. (4.37) and eq. (4.31) define the longitudinal mass \( m^2_L \) and \( \Omega^{(1)} \) in terms of the zero momentum values of \( \theta_4 \) and \( \Omega^{(2)}_4 \) respectively. To avoid IR divergences we then require that \( \theta_4 \) and \( \Omega^{(2)}_4 \) do not depend at all on \( I^{(2)}_{TT} \). Thus we can write

\[ F_3(x, y, z; N) = K_3(x, y; N) \quad \text{,} \quad F_4(x, y, z; N) = K_4(x, y; N) \]

where \( K_A(x, y; N) \ (A = 3, 4) \) are arbitrary functions of two variables only (plus \( N \)). The same argument apply to the parameters \( \mu^{(1)} \) and \( \mu^{(2)} \) of the manifestly symmetric form in eq. (4.32): we require that \( \frac{1}{2} \theta_1 - \Omega^{(2)}_1 \) and \( \gamma^{(2)}_2 \) do not depend on \( I^{(2)}_{TT} \), which in turn implies

\[ F_2(x, y, z; N) = \frac{K_4(x, y; N)}{1 + \frac{1}{2} K_4(x, y; N)(y - x)} \quad \text{and} \quad F_1(x, y, z; N) = \frac{1}{2} \frac{K_4(x, y; N)}{(N - 1) \frac{1 + \frac{1}{2} K_4(x, y; N)(y - z)}{1 + \frac{1}{2} K_4(x, y; N)(y - z)}} + K_1(x, y; N) \]

where, again, \( K_A(x, y; N) \ (A = 1, 2) \) are arbitrary functions of two variables (and \( N \)).

Now, as in sec. 3.3 we require the renormalizability with the 1–loop beta function. That is, we assume the following RG equation for the bare coupling \( \lambda_0 \)

\[ \frac{\partial \lambda_0}{\partial \log \Lambda} = \frac{N + 8}{24\pi^2} \lambda_0^2 + O(\Lambda^{-1}) \]

and ask that the free parameter function \( K_A \) do not depend on \( \log \Lambda \). By the same procedure of sec. 3.3 and using

\[ \frac{\partial I_{LL}^{(2)}}{\partial \log \Lambda} = \frac{\partial I_{TT}^{(2)}}{\partial \log \Lambda} + O(\Lambda^{-1}) = \frac{\partial I_{TT}^{(2)}}{\partial \log \Lambda} + O(\Lambda^{-1}) = -\frac{1}{8\pi^2} 1 + O(\Lambda^{-1}) \]

we obtain the general forms

\[ K_j(x, y; N) = \frac{f_j(\alpha(x, y); N)}{1 - \frac{1}{6}(N + 8) x} \quad , \quad j = 1, 3, 4 \]

\[ K_2(x, y; N) = f_2(\alpha(x, y); N) \]
where
\[ \alpha(x, y) = \frac{x - y}{1 - \frac{1}{6}(N + 8)x} \]

Recalling that \( x = \lambda_0 I_{LL}^{(2)} \) and \( y = \lambda_0 I_{TT}^{(2)} \) we see that the Fourier transform of \( \alpha \) positive definite for purely spatial value of the momentum. Notice also that \( \alpha \) vanishes when \( v \to 0 \) since \( I_{LL}^{(2)} \) and \( I_{TT}^{(2)} \) coincide in this limit.

Notice that now, in contrast to the \( N = 1 \) case studied in sec. 3.3 [see eqs. (3.38)], the requirement of renormalizability does not fix completely the form of the free parameters. This occurs because for \( N > 1 \) there exist three distinct finite parts associated with the logarithmic cutoff divergence of \( I_{LL}^{(2)} \), \( I_{TT}^{(2)} \) and \( I_{LL}^{(2)} \). So now we are left with four arbitrary functions \( f_A \) of one variable (and \( N \)) to determine.

It is convenient to rewrite all the constraints previously found in terms of this new parametrization. After some straightforward, albeit rather involved calculations, one obtains

- **Perturbative matching:**
  \[ f_1(\alpha; N) = \frac{1}{36} \left[ (N - 1) - 3 f_2'(0; N) - 9 f_4'(0; N) \right] \alpha + O(\alpha^2) \]
  \[ f_4(\alpha; N) = 2/3 + O(\alpha), \quad f_3(\alpha; N) = \frac{2}{3(N + 8)} - \frac{1}{3} \alpha + O(\alpha^2) \]
  \[ f_2(\alpha; N) = -(N + 1) + O(\alpha) \]

- **Matching in the large-\( N \) limit:**
  \[ f_1(\alpha/N; N) \to 0, \quad f_4(\alpha/N; N) \to f(\alpha), \quad f_2(\alpha/N; N) + N \to 1 - \frac{6 - \alpha}{2} f(\alpha) \]
  where the arbitrary function \( f \) enters only the decoupled longitudinal sector.

- **Positivity constraints on \( \Omega_{11}^{(2)} \) and \( \theta_\alpha \):** on the real interval \( 0 \leq \zeta \leq 6/(N + 8) \)
  \[ 0 < f_4(\zeta; N) < \frac{2}{\zeta}, \quad f_2(\zeta; N) + N - 1 < 0, \quad f_1(\zeta; N) > 0 \]

Actually these conditions are slightly stronger than strictly necessary. In any case they also guarantee the positivity of all the \( \theta_\alpha \) parameters (which holds true in the usual HF approximation as well).

One last requirement is that the \( N = 1 \) case should be recovered for all the parameters that have meaning also in this case. These are \( \Omega_{11}^{(2)}, \theta_1 \) and \( \theta_4 \) (that, evaluated at zero momentum, determines the longitudinal mass according to eq. (4.37)). When \( N = 1 \) they should have the following form [see eqs. (3.38)]
\[ \theta_1 = 12 \Omega_{11}^{(2)} = \frac{3}{2} \theta_4 = \lambda_0 \left[ 1 - \frac{3}{2} \lambda_0 I_{LL}^{(2)} \right]^{-1} \]
that is, in terms of the functions \( f \)'s,
\[ f_4(\alpha; 1) = \frac{2}{3}, \quad f_2(\alpha; 1) = \frac{2 - \frac{2}{3} \alpha}{1 - \alpha}, \quad f_1(\alpha; 1) = \frac{\alpha}{9 - 3\alpha} \]
Notice also that all the possible choices of free parameters coincide when $v \to 0$, since $\alpha$ vanishes in this limit and the perturbative constraints uniquely fix the form of the $f$’s when $\alpha = 0$. Moreover one can verify that the resulting $v = 0$ improved HF coincide with the massless limit of the improved approximation in the unbroken symmetry phase defined in ref. \[30\].

In conclusion all possible forms of the functions $f$’s that fulfill the above matching constraints define improved Hartree–Fock resummations with the required features of gaplessness and renormalizability. One simple choice for the $f$’s is

$$
f_4(\alpha; N) = \frac{2}{3} \quad , \quad f_2(\alpha; N) = -(N + 1) \frac{1 - \frac{1}{3}\alpha}{1 - \alpha} + (N - 1) \alpha
$$

As already remarked, this choice is not unique. For example we can consider a second form

$$
f_4(\alpha; N) = \frac{2/3}{1 + \frac{N-1}{6} \alpha} \quad , \quad f_2(\alpha; N) = -(N + 1) \frac{1 - \frac{1}{3}\alpha}{1 - \alpha} + \frac{4(N-1)}{3} \alpha \frac{1 + \frac{N+8}{12} \alpha}{1 + \frac{N+8}{6} \alpha}
$$

As already explained in sec. \[33\], our modified resummation adds leading logarithm contributions of diagrams that are not present in the usual HF approximation; then the two forms just provided of the free parameters, as well as all the other possible ones, correspond to different choices of the associated finite parts.

Given one specific choice for the free parameters, we can proceed in applying the coupling constant renormalization condition in eq. \[4.17\]. Notice that the consistence of this renormalization requires that $\Omega_1^{(2)}$ is monotonically growing with $\lambda_0$ for any given purely spatial momentum. We have omitted to include this requirement in the previous general discussion since it would lead in general to rather complicated constraints. It holds true for the two simple examples given in eqs. \[4.38\] and eqs. \[4.39\], as one can explicitly check. Moreover it holds true in general (i.e. for any improved HF resummation) for scales such that $s^2 \gg v^2$, since we have in this limit

$$
\tilde{\Omega}_1^{(2)}(q_s) = \frac{\frac{1}{6} \lambda_0}{1 - \frac{1}{6} (N + 8) \lambda_0 I_1^{(2)}(q_s)} + O(m^2/s^2)
$$

where $I^{(2)}$ stands for anyone of $I_{LL}^{(2)}$, $I_{TL}^{(2)}$ and $I_{TT}^{(2)}$. The renormalization condition thus defines the bare–to–renormalized relation

$$
\lambda = \frac{\lambda_0}{1 - \frac{N+8}{6} \lambda_0 I_1^{(2)}(q_s)} + O(m^2/s^2)
$$

which is the usual 1 loop RG–invariant relation up to $O(m^2/s^2)$ terms. This shows that the direct coupling renormalization condition is approximately scale invariant for high
To obtain complete scale invariance it is enough to slightly modify the bare–to–renormalized parameterization by changing the renormalization condition in the following way

\[ 6 \tilde{\Omega}_1^{(2)}(q_s) = 6 \lambda_0 F_1(\lambda_0 \tilde{I}_{LL}(q_s), \lambda_0 \tilde{I}_{TL}(q_s), \lambda_0 \tilde{I}_{TT}(q_s); N) \]

\[ = 6 \lambda F_1(\lambda \tilde{J}_{LL}^{(2)}(q_s), \lambda \tilde{J}_{TL}^{(2)}(q_s), \lambda \tilde{J}_{TT}^{(2)}(q_s); N) \]

\[ = \lambda + O(m^2/s^2) \quad (4.40) \]

where

\[ \tilde{J}^{(2)}_{LL}(p) = I^{(2)}_{LL} - \tilde{I}^{(2)}_{LL}(q_s)1 \], \quad \tilde{J}^{(2)}_{TL}(p) = I^{(2)}_{TL} - \tilde{I}^{(2)}_{LL}(q_s)1 \], \quad \tilde{J}^{(2)}_{TT}(p) = I^{(2)}_{TT} - \tilde{I}^{(2)}_{LL}(q_s)1 \]

are the properly subtracted finite loops (notice that the subtraction term is always the purely longitudinal \( I^{(2)}_{LL}(q_s) \)). Then, using the parameterization of \( F_1 \) in terms of the \( f_i(\alpha; N) \), one can verify that the bare–to–renormalized relation reads exactly

\[ \lambda = \frac{\lambda_0}{1 - \frac{N+8}{6} \lambda_0 \tilde{I}^{(2)}_{LL}(q_s)} \quad (4.41) \]

Comparing this with eq. (4.19) we can see that this has the correct 1–loop-RG improved behaviour with a specific choice of the finite parts. Then all the free parameters take the following manifestly finite forms

\[ \Omega_1^{(2)} = \lambda F_1(\lambda J_{LL}^{(2)}, \lambda J_{TL}^{(2)}, \lambda J_{TT}^{(2)}; N) \quad \theta_1 = \lambda F_2(\lambda J_{LL}^{(2)}, \lambda J_{TL}^{(2)}, \lambda J_{TT}^{(2)}; N) \]

\[ \Omega_4^{(2)} = \lambda F_4(\lambda J_{LL}^{(2)}, \lambda J_{TL}^{(2)}, \lambda J_{TT}^{(2)}; N) \quad \theta_4 = \lambda F_4(\lambda J_{LL}^{(2)}, \lambda J_{TL}^{(2)}, \lambda J_{TT}^{(2)}; N) \]

Moreover, consistently with the renormalization condition in eq. (4.40), we can define the running coupling constant by means of the following equation

\[ \lambda(p) F_1(\lambda(p) \tilde{J}_{LL}^{(2)}(p), \lambda(p) \tilde{J}_{TL}^{(2)}(p), \lambda(p) \tilde{J}_{TT}^{(2)}(p); N) = \tilde{\Omega}_1^{(2)}(p) \]

whose solution is simply the extension of eq. (4.41) to arbitrary momentum

\[ \lambda(p) = \frac{\lambda}{1 - \frac{N+8}{6} \lambda \tilde{J}_{LL}^{(2)}(p)} \]

In term of this and of the subtracted integrals we can write the renormalized form of the Fourier transform of \( \alpha \):

\[ \tilde{\alpha}(p) = \lambda(p) [ \tilde{J}_{TT}^{(2)}(p) - \tilde{J}_{LL}^{(2)}(p) ] \]

Notice that \( \alpha \) becomes small for large (but smaller than the Landau pole) spatial momentum \( q_s \). More precisely it is of order \( O(\lambda(q_s) v^2/s^2) \). Because of the Landau pole, the cut–off of the theory cannot be removed but should be fixed to values suitably smaller than the pole (i.e such that \( \lambda(q_s) \) is of order one). The mass scale of the theory, that is \( v \), is much smaller than the cut–off itself. Therefore for spatial momenta with values

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near the cut–off $\alpha$ is small. Then the perturbative (in $\alpha$) matching conditions assure that the different allowed choices of the free parameters give the same results to order $O(\lambda(q_\Lambda) v^2/\Lambda^2)$. In this sense we can say that all the class of improved HF approximations shares the same UV behaviour.

One can easily check that if we renormalize at scale $s$ with coupling constant $\lambda$ and at scale $s'$ with coupling constant $\lambda' = \lambda(q_{s'})$ we define the same bare coupling constant. That is to say that the bare–to–renormalized relation in eq. (4.41) is RG–invariant. As a consequence the expressions of the parameters in terms of $\lambda_0$ and $I^{(2)}$ in eqs. (4.33) can be thought as manifestly scale invariant definitions.

5 Conclusions

In this paper we extended to the case of spontaneously broken symmetry the improvement of the HF approach to the $O(N)$ scalar theory that was proposed in ref. [30]. In contrast to the standard HF approximation, our improved one is renormalizable, RG–invariant and correctly gapless in the Goldstone sector. However, it is not unique, except for $N = 1$, because the mass difference between the longitudinal sector and the transverse Goldstone sector allows for a richer structure of renormalization finite parts that cannot be fully restricted by the requirements of renormalizability and RG–invariance. As a consequence, an entire class of improvements is identified, parametrized by the four functions $f_A(\alpha; N)$ which must satisfy some further constraints explicitly written in the previous section. It is important to stress that, albeit of functional type, the remaining freedom is much smaller than what simple dimensional analysis would allow even when simple momentum-independent observables are concerned (the momentum dependences in our approximation are anyway fixed by construction to be of mean–field type). This makes it difficult, although not a priori impossible, to identify consistent schemes to further constrain the functions $f_A(\alpha; N)$ by requiring agreement in suitable calculations beyond one loop. Work on this direction is in progress.

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