The massless irreducible representation in E theory and how bosons can appear as spinors.

Keith Glennon and Peter West
Department of Mathematics
King’s College, London WC2R 2LS, UK

Abstract

We study in detail the irreducible representation of E theory that corresponds to massless particles. This has little algebra $I_c(E_9)$ and contains 128 physical states that belong to the spinor representation of $SO(16)$. These are the degrees of freedom of maximal supergravity in eleven dimensions. This smaller number of the degrees of freedom, compared to what might be expected, is due to an infinite number of duality relations which in turn can be traced to the existence of a subalgebra of $I_c(E_9)$ which forms an ideal and annihilates the representation. We explain how these features are inherited into the covariant theory. We also comment on the remarkable similarity between how the bosons and fermions arise in E theory.
1. Introduction

The symmetries of the non-linear realisation of $E_{11} \otimes s l_1$ with respect to the Cartan involution invariant subalgebra of $E_{11}$, denoted by $I_c(E_{11})$, lead to equations of motion at low levels that are precisely those of maximal supergravity provided one discards the dependence on the coordinates of the spacetime that are beyond those we usually consider [1,2,3,4]. In particular the degrees of freedom they contain are those of the familiar graviton and the three form. While there is no complete understanding of the role of all the higher level fields a large class of them are known to provide different field descriptions of the graviton and the three form degrees and freedom to which they are related by invariant duality relations. The higher level fields fields also account for the gauging of the maximal supergravity theories. Indeed the non-linear realisation contains all the maximal supergravity theories in their different dimensions including those that are gauged. For a review see references [5,6].

In a different approach the irreducible representation of the semi-direct product of $I_c(E_{11})$ with its vector representation $l_1$, denoted $I_c(E_{11}) \otimes s l_1$ were formulated [7]. This latter algebra is the analogue of the Poincare group which can be written in the form $SO(1,D − 1) \otimes s T$ where $T^D$ are the translations in $D$ dimensions. The irreducible representations of the Poincare group were found in 1939 by Wigner [8] and one can use a similar method in E theory. One important difference is that while the construction of the irreducible representations of the Poincare algebra begin by considering the possible values of the momentum, those in E theory involve the vector, or first fundamental representation of $E_{11}$ and these include all brane charges [2,9,10,11,12]. Thus while the irreducible representations of the Poincare algebra lead to all possible particles those in E theory lead to all possible point particle and branes, that is, all extended objects in E theory.

The irreducible representation that arises when all the members of the vector representation vanish except for the usual momentum, which was taken to be massless, were studied briefly in reference [7]. It was found that the corresponding little group was $I_c(E_9)$ and it was argued that the representation only contains a finite number of states which are those of the graviton and three form in eleven dimensions and so the degrees of freedom of eleven dimensional supergravity. In this paper we will study this irreducible representation in detail. We will show that one can impose an infinite number of duality equations on the representation that are invariant under $I_c(E_9)$ and reduce the number of independent fields to be 128. These are the fields $h_{i_1i_2} = h_{(i_1,i_2)}$, $h_i^j = 0$ and $A_{i_1i_2i_3} = A_{[i_1i_2i_3]}$, $i_1,i_2,... = 2,...,10$ and they belong to the 128 dimensional spinor representation of $I_c(E_8) = SO(16)$. Corresponding to the duality relations we find that there exist an infinite number of generators of $I_c(E_9)$ which annihilate the representation and these form a subalgebra $I$ that is an ideal. Indeed the Lie algebra $\frac{I_c(E_9)}{I}$ = SO(16).

To understand how it is that the bosonic states, and in particular the graviton, can belong to the spinor representation of an algebra that involves spacetime symmetries we will decompose the physical states into representations of the subalgebra $SO(8) \otimes SO(8)$ of SO(16). We find that they belong to the representations $128 = (8_v,8_v) \oplus (8_c,8_s)$. The graviton belongs to the first representation.

Given an irreducible representation of the Poincare algebra one can embed it in a larger representation to find a Lorentz covariant formulation of the representation. The price is
that the Lorentz covariant fields obey conditions which are the physical state conditions and are subject to gauge transformations. We carry out the same procedure for the massless irreducible representation of $I_c(E_{11}) \otimes l_1$ and make contact with the non-linear realisation of $E_{11} \otimes l_1$ with respect to the Cartan involution invariant subalgebra of $E_{11}$, denoted by $I_c(E_{11})$. Indeed the degrees of freedom of the massless irreducible representation are the ones of the latter theory. This supports the assertion that these are the only degrees of freedom that arise in the non-linear realisation and it explains the origin and structure of the duality relations that arise in the non-linear realisation.

2. The irreducible representation for a massless particle in E theory

The irreducible representations in E theory were discussed in reference [7]. To be more precise this paper studied the irreducible representations of the semi-direct product of $I_c(E_{11})$ and it’s vector ($l_1$) representation, denoted $I_c(E_{11}) \otimes l_1$. This is a natural extension of the method used to find the irreducible representations of the Poincare group [8]. The similarity comes from the fact that the Poincare group is a semi-direct product of the Lorentz group $SO(1,3)$ and the group of translations $T_4$, denoted $SO(1,3) \otimes T_4$. Indeed at the lowest level $I_c(E_{11}) \otimes l_1$ is just the Poincare group in eleven dimensions. For the case of the Poincare group one selects a value for the momentum corresponding to if it is massive or massless and then one computes the little group that preserves this choice. One then takes a representation of this little group and finds a representation of the full Poincare group by boosting, or said more technically, by taking an induced representation. In this way one can find all irreducible representations of the Poincare group.

In E theory the vector ($l_1$) representation contains all the brane charges and so the first step is to select a preferred value of the vector representation and compute the little algebra that preserves this value. One then takes an irreducible representation of this little algebra and boosts it to the full $I_c(E_{11}) \otimes l_1$ algebra. At lowest levels the vector representation contains $P_a, Z^{a_1, a_2}, Z^{a_1...a_5}, ...$ with $a_1, a_2, ... = 0, 1, ..., 10$ where $P_a$ are just the usual momenta and $Z^{a_1, a_2}$ and $Z^{a_1...a_5}$ the well known two form and five form charges that first appeared in the supersymmetry algebra. In reference [7] the irreducible representation that arises when one takes the momentum to take a value corresponding to it being massless with the other components of the vector representation being zero was discussed. We will refer to this irreducible representation as the massless irreducible representation. Taking some of the higher charges to be non-zero one finds the irreducible representations corresponding to branes.

The purpose of this section is to fully elucidate the properties of the massless irreducible representation. To begin the construction we will take the momentum to have the values $p_0 = -m, p_{10} = m$ with all other momenta being zero. It will be more convenient to use light-cone notation, for which the components of a vector $V^a$ are defined as $V^\pm = \frac{1}{\sqrt{2}}(V^{10} \pm V^0), V^i, 1, ..., 9,$ and the Minkowski metric becomes $\eta_{+-} = 1, \eta_{ij} = \delta_{ij}$ so that $V_\pm = \frac{1}{\sqrt{2}}(V_{10} \pm V_0)$. In light-cone notation the massless irreducible representation begins by taking the brane charges to be $p^+ = p_- = \sqrt{2}m$, with all other momenta and other brane charges being set equal to zero. We next seek the subalgebra $H$ of $I_c(E_{11})$ which preserves this choice of brane charges. The brane charges $l_A$ transform under $I_c(E_{11})$ and
so the subalgebra $\mathcal{H}$ is determined by the requirement that
\[
\delta l_A = [\Lambda_\alpha S_\alpha, l_A] = 0
\] (2.1)
where the charges $l_A$ take the above values.

It is straightforward to show that for a massless particle the parameters $\Lambda_\alpha$ must satisfy $\Lambda_{+a} = 0 = \Lambda_{+b} = \Lambda_{+ab} = \ldots$, that is, any parameter $\Lambda_\alpha$ with a lowered $+$ index is zero. The resulting algebra which leaves this choice invariant is [7]
\[
\mathcal{H} = \{J_{+i}, J_{ij}, S_{+ij}, S_{i1i2i3}, S_{+i1...i5}, \ldots\}, \quad i,j,\ldots = 2,\ldots10.
\] (2.2)
The generators possessing a lowered $+$ index satisfy the commutation relations
\[
[J_{+i}, J_{+j}] = 0, \quad [S_{+i1i2}, S_{+j1j2}] = 0, \quad [J_{+i}, S_{+j1j2}] = 0, \quad \ldots
\] (2.3)

In order to obtain a finite-dimensional unitary representation, these commutation relations imply that all generators with a lowered $+$ index be trivially realized in the representation. The remaining generators obey the $I_c(E_{11})$ commutation relations with indices restricted to $i,j = 2,\ldots,10$. The result is that the subalgebra which preserves the above choice of charges is the Cartan involution invariant subalgebra of $E_9$, denoted $\mathcal{H} = I_c(E_9)$, given by the generators [7]
\[
J_{ij}, S_{t_1t_2t_3}, S_{i1...i6}, S_{i1...i8,j}, S_{i1...i9,j1j2j3}, \ldots
\] (2.4)

These steps are similar to those of the irreducible massless representations of the Poincare algebra. In this case of finds that the generators $J_{+i}$ commute and in order to obtain a finite dimensional irreducible representation one takes a representation in which these act to give zero leaving us to take an irreducible representation of the algebra $SO(D-2)$.

The Dynkin diagram of $E_9$ is given by deleting the first two nodes, which correspond to the $+$ and $-$ directions, from the Dynkin diagram of $E_{11}$;

```
\begin{center}
\begin{tikzpicture}
  \node (1) at (0,0) {$1$};
  \node (2) at (1,0) {$2$};
  \node (3) at (2,0) {$3$};
  \node (4) at (3,0) {$4$};
  \node (5) at (4,0) {$5$};
  \node (6) at (5,0) {$6$};
  \node (7) at (6,0) {$7$};
  \node (8) at (7,0) {$8$};
  \node (9) at (8,0) {$9$};
  \node (10) at (9,0) {$10$};
  \node (11) at (10,0) {$11$};

  \node (12) at (1,3) {$\oplus$};
  \node (13) at (2,3) {$\ominus$};
  \node (14) at (3,3) {$\ominus$};
  \node (15) at (4,3) {$\oplus$};
  \node (16) at (5,3) {$\ominus$};
  \node (17) at (6,3) {$\ominus$};
  \node (18) at (7,3) {$\ominus$};
  \node (19) at (8,3) {$\ominus$};
  \node (20) at (9,3) {$\ominus$};
  \node (21) at (10,3) {$\ominus$};

  \draw (1) -- (2);
  \draw (2) -- (3);
  \draw (3) -- (4);
  \draw (4) -- (5);
  \draw (5) -- (6);
  \draw (6) -- (7);
  \draw (7) -- (8);
  \draw (8) -- (9);
  \draw (9) -- (10);
  \draw (10) -- (11);

  \draw (12) -- (13);
  \draw (13) -- (14);
  \draw (14) -- (15);
  \draw (15) -- (16);
  \draw (16) -- (17);
  \draw (17) -- (18);
  \draw (18) -- (19);
  \draw (19) -- (20);
  \draw (20) -- (21);
\end{tikzpicture}
\end{center}
```

Here $\oplus$ indicates that the node has been removed from the Dynkin diagram of $E_{11}$.

In general the theory in $D$ dimensions can then be obtained from the $E_{11}$ Dynkin diagram by deleting node $D$ and analysing the theory with respect to the algebra corresponding to the remaining nodes. Thus to find the theory in eleven dimensions we delete node eleven, that is, the top node and analysis the theory when decomposed into the algebra $GL(11)$, while in $D$ dimensions we decompose into representations of $A_{D-1} \otimes E_{9-D}$ algebra. To find the irreducible representations in $D$ dimensions one carries out the same steps and so for the massless irreducible representation we end up deleting nodes one and two, as explained above, as well as node $D$. For the case of eleven dimensions we should
therefore delete nodes one and two to arrive at $E_9$ and decompose this algebra in terms of representations of the $A_8$ subgroup of $E_9$ as shown in the Dynkin diagram below

\[
\begin{array}{ccccccccccc}
\oplus & - & \oplus & - & \bullet & - & \bullet & - & \bullet & - & \bullet & - & \bullet & - & \bullet & - & \bullet & - & 11
\end{array}
\]

Since the irreducible representations are representations of $I_c(E_{11})$, rather than $E_{11}$ itself, we actually decompose the $I_c(E_9)$ subgroup of $E_9$ into representations of $I_c(A_8) = SO(9)$. Indeed the generators of equation (2.4) are listed in this decomposition as from the beginning of our discussion we had in effect deleted node eleven as we are concentrating on the eleven dimensional case.

The next step is to choose an irreducible representation of $I_c(E_9)$. Such an reducible representation is provided by the Cartan involution odd generators of $E_9$ which are given by [7]

\[
T_{ij} = \eta_{kk}K_i^k + \eta_{jk}K_i^k, \quad T_{i_1 i_2 i_3} = R^{j_1 j_2 j_3}_{i_1 i_2 i_3} \eta_{j_1 i_1} \eta_{j_2 i_2} \eta_{j_3 i_3} + R_{i_1 i_2 i_3},
\]

(2.5)

\[
T_{i_1 \ldots i_6} = R^{j_1 \ldots j_6}_{i_1 \ldots i_6} \eta_{j_1 i_1} \eta_{j_2 i_2} \eta_{j_3 i_3} + R_{i_1 \ldots i_6},
\]

(2.6)

\[
T_{i_1 \ldots i_8, k} = R^{j_1 \ldots j_8, i}_{k} \eta_{j_1 i_1} \eta_{j_2 i_2} \eta_{j_3 i_3} + R_{i_1 \ldots i_8, k},
\]

(2.7)

\[
T_{i_1 \ldots i_9, j_1 j_2 j_3} = R^{m_1 \ldots m_9, h_{123}}_{i_1 i_2 i_3} \eta_{m_1 i_1} \eta_{m_2 i_2} \eta_{m_3 i_3} + R_{i_1 \ldots i_9, j_1 j_2 j_3}, \ldots
\]

(2.8)

These generators provide a linear representation of $I_c(E_9)$ because the involution operator $I_c$ is defined to act on the generators $A, B, \ldots$ of a Lie algebra as $I_c(AB) = I_c(A)I_c(B)$ and so $I_c([\text{even, odd}]) = -[\text{even, odd}]$ which guarantees that the commutator will always be a Cartan involution odd generator. The generators of equation (2.8) are the Cartan involution odd generators of $I_c(E_{11})$ when restricted to their indices taking only the values $i, j = 2, ..., 10$. Clearly one could for the massless case take a different irreducible representation but in this paper we will only consider this case.

As a result we take our representation to consist of fields corresponding to the Cartan involution odd generators

\[
\{h_{i_1}(0) , A_{i_1 i_2 i_3}(0) , A_{i_1 \ldots i_6}(0) , h_{i_1 \ldots i_9, j}(0) , \ldots\}, \text{ with } i, j, \ldots = 2, ..., 10.
\]

(2.9)

The value (0) indicates that these fields are before the boost which takes them to be a representation of $I_c(E_{11}) \otimes_s l_1$.

We recognise $h_{i_1}(0)$ and $A_{i_1 i_2 i_3}(0)$ as the degree of freedom of the graviton and three form respectively. There are however, an infinite number of higher level fields. It was proposed in [7] that these fields are connected to the graviton and three form by certain duality relations. The purpose of this section is to investigate these relations in more detail.

The fields of equation (2.9) live in the representation associated to the Cartan involution odd generators and so we consider the quantity

\[
\nabla = h^i_j T^{ij} + A^{i_1 i_2 i_3}_{i_1 i_2 i_3} T^{i_1 i_2 i_3} + A_{i_1 \ldots i_6} T^{i_1 \ldots i_6} + h_{i_1 \ldots i_8, j} T^{i_1 \ldots i_8, j} + A_{i_1 \ldots i_9, j_1 j_2 j_3} T^{i_1 \ldots i_9, j_1 j_2 j_3} + \ldots
\]

(2.10)
The transformations of the fields follow from that of the Cartan involution odd generators under $I_c(E_9)$. In particular under the level one generator $S_{i_1 i_2 i_3}$ of $I_c(E_9)$ we take

$$
\delta \nabla = [\Lambda^{i_1 i_2 i_3} S_{i_1 i_2 i_3}, \nabla] + [\Lambda^{i_1 \ldots i_6} S_{i_1 \ldots i_6}, \nabla] + [\Lambda^{i_1 \ldots i_8} S_{i_1 \ldots i_8}, \nabla] + [\Lambda^{i_1 \ldots i_8 j} S_{i_1 \ldots i_8 j}, \nabla] \quad (2.11)
$$

Using the $E_{11}$ algebra to evaluate equation (2.11) we find that

$$
\delta h_{ij} = 18 \Lambda (i|k_1 k_2| A_j) k_1 k_2 - 2 \eta_{ij} \Lambda^{k_1 k_2 k_3} A_{k_1 k_2 k_3} - 5! \eta_{ij} \Lambda^{k_1 \ldots k_6} A_{k_1 \ldots k_6} - 9! \Lambda^{k_1 \ldots k_5} (i A_j) k_1 \ldots k_5
$$

$$
+ 7 \cdot 9 \cdot 10 \cdot 16 \left( h_{k_1 \ldots k_8 | i} \Lambda^{k_1 \ldots k_8} A_{j} \right) + 8 h_{i|k_1 \ldots k_7, l} \Lambda_{(j)} k_1 \ldots k_7, l - \eta_{ij} \Lambda^{k_1 \ldots k_8, l} h_{k_1 \ldots k_8, l} \quad (2.12)
$$

$$
\delta A_{i_1 i_2 i_3} = -3 \cdot 2 h_{i_1 k} \Lambda_{i_2 i_3} k + \frac{5!}{2} A_{i_1 i_2 i_3} k_1 k_2 k_3 \Lambda_{k_1 k_2 k_3} + \frac{5!}{2} \Lambda_{i_1 i_2 i_3} k_1 k_2 k_3 A_{k_1 k_2 k_3}
$$

$$
+ \frac{7! \cdot 2}{3} \Lambda^{k_1 \ldots k_6} \left( h_{i_1 i_2 i_3 | k_1 \ldots k_5, k_6} - h_{k_1 \ldots k_6 | i_1 i_2 i_3} \right) \quad (2.13)
$$

$$
\delta A_{i_1 \ldots i_6} = 2 \Lambda_{i_1 i_2 i_3} A_{i_4 i_5 i_6} + 112 h_{i_1 \ldots i_5 k_1 k_2 k_3} \Lambda^{k_1 k_2 k_3} + 112 h_{i_1 \ldots i_5 k_1 k_2 k_3, i_6} \Lambda^{k_1 k_2 k_3}
$$

$$
+ 12 \Lambda_{k_1 \ldots i_5} \delta h_{i_6} \quad (2.14)
$$

$$
\delta h_{i_1 \ldots i_8, j} = 2 \left( \Lambda_{i_1 i_2 i_3} A_{i_4 \ldots i_8} A_{i_1 \ldots i_8} - \Lambda_{j i_1 i_2} A_{i_3 \ldots i_8} \right) - 12 \cdot 3 \Lambda^{k_1 k_2 k_3} (A_{i_1 \ldots i_8 k_1 k_2 k_3 j} + A_{i_1 \ldots i_7 j k_1 k_2 k_4 i_8})
$$

$$
+ 3 \Lambda_{i_1 \ldots i_6} A_{i_7 \ldots i_8} - 3 \Lambda_{i_1 \ldots i_6} A_{i_7 \ldots i_8} \quad (2.15)
$$

In this section we will only use the $\Lambda_{i_1 i_2 i_3}$ variations.

We will now go to postulate duality relations and show they are preserved under the $I_c(E_9)$ symmetry and as a result the number of fields in the representation is radically reduced, indeed, one can take the representation to contain only the graviton and the three form. We first propose a duality relation between the three-form and six-form which is given by

$$
E_{i_1 i_2 i_3} = A_{i_1 i_2 i_3} + c \varepsilon_{i_1 i_2 i_3} j_1 \ldots j_6 A_{j_1 \ldots j_6} = 0 \quad (2.16)
$$

where $c$ is a constant which we will fix by requiring that this relation is part of an infinite set of relations that are, as a collection, left invariant by $I_c(E_9)$. Varying equation (2.16) under $I_c(E_9)$ we find that

$$
\delta E_{i_1 i_2 i_3} = 2 c \Lambda_{j_1 j_2 j_3} \varepsilon_{i_1 i_2 i_3} j_1 \ldots j_6 (A_{j_4 j_5 j_6} + \frac{1}{c} \frac{1}{3! 4!} \varepsilon_{j_4 j_5 j_6} i_{12} i_{3} j_{12} j_{3} A_{i_1 i_2 i_3 j_1 j_2 j_3}) + \ldots \quad (2.17)
$$

where $+ \ldots$ denotes the gravity terms. Clearly we will only recover our original relation if $c = \frac{\pm 1}{12}$ and we choose $c = - \frac{1}{12}$. For this choice the variation takes the form

$$
\delta E_{i_1 i_2 i_3} = \frac{1}{6} \Lambda_{j_1 j_2 j_3} \varepsilon_{i_1 i_2 i_3} j_1 \ldots j_6 E_{j_4 j_5 j_6} - 6 E_{k | i_1} \Lambda^{k}_{i_2 i_3} \quad (2.18)
$$

where

$$
E_{i_1 i_2 i_3} = A_{i_1 i_2 i_3} + \frac{1}{12} \varepsilon_{i_1 i_2 i_3} j_1 \ldots j_6 A_{j_1 \ldots j_6} = 0 \quad (2.19)
$$
Thus we not only recover our original duality relation but find the new duality relation of equation (2.20) which relates the graviton and dual graviton. The irreducibility condition \( h_{[i_1...i_8,j]} = 0 \) implies that for this relation to be consistent, the field \( h_{ij} \) must be traceless, \( h^i_i = 0 \). With this condition \( h_{ij} \) has \( \frac{10\cdot9}{2} - 1 = 44 \) degrees of freedom as it should.

We now vary the duality relation of equation (2.20) to find that

\[
\delta E_{ij} = -\frac{1}{2} \frac{1}{3! \cdot 5!} \varepsilon^{i_1...i_8} (A_{r_1 r_2 r_3} \varepsilon_{r_4...r_8} k_1 k_2 k_3 E_{k_1 k_2 k_3} - \Lambda_{j r_1 r_2} \varepsilon_{r_3...r_8} k_1 k_2 k_3 E_{k_1 k_2 k_3})
+ 12 \cdot 2 \cdot 3 \Lambda^{k_1 k_2 k_3} \varepsilon^{i_1...i_8} (E_{r_1...r_8 k_1 k_2 k_3} - E_{j r_1...r_7 k_1 k_2 k_3 r_8}) = 0
\]

(2.21)

where

\[
E_{i_1...i_9,j_1 j_2 j_3} = A_{i_1...i_9,j_1 j_2 j_3} + \frac{1}{9!} \varepsilon_{i_1...i_9} A_{j_1 j_2 j_3} = 0.
\]

Thus the variation contains the previously-derived duality relation of equation (2.13), and a new duality relation relating the three-form \( A_{i_1 i_2 i_3} \) and nine-three form \( A_{i_1...i_9,j_1 j_2 j_3} \).

These duality relations \( E_{ij} = 0, E_{i_1 i_2 i_3} = 0, E_{i_1...i_9,j_1 j_2 j_3} = 0 \) are the first of an infinite tower of duality relations showing that the fields at levels, two, three, four, etc... can be expressed in terms of the fields

\[
h_{ij}, \ A_{i_1 i_2 i_3}.
\]

The six form is related to the three form by equation (2.19) and the dual graviton field \( h_{i_1...i_9,j} \) is related to the graviton by equation (2.20). All the higher level fields carry the indices of these fields as well a multiple sets of blocks of nine indices and we can expect that in all the higher level duality relations these indices are carried by \( \varepsilon_{i_1...i_9} \)'s in a way that is similar to how they appear in equation (2.22). These indices correspond to the action of the affine generator of \( I_c(E_9) \). As a result it should be possible of show that the complete infinite set of duality relations are invariant under the \( I_c(E_9) \) symmetry.

Hence we have found that the massless irreducible representation contains the fields of equation (2.23) and as a result it contains the \( 44 + 84 = 128 \) bosonic degrees of freedom of eleven-dimensional supergravity. Rather than write them as in equation (2.23) we can write the degrees of freedom in terms of an \( E_8 \) multiplet, or in our case here, an \( I_C(E_8) = SO(16) \) multiplet, namely

\[
h_{i'j'}, A_{i'_1 i'_2 i'_3}, A_{i'_1...i'_6}, \ h_{j'} \equiv \frac{1}{8!} \varepsilon^{i'_1...i'_8} h_{i'_1...i'_8,j'}, \ i', j', \ldots = 3, \ldots, 10
\]

(2.24)

In this paper we will take un-primed indices to range over \( i, j, \ldots = 2, \ldots, 10 \) and primed indices to range over \( i', j', \ldots = 3, \ldots, 10 \). The fields of equation (2.24) have been obtained from those of equation (2.23) by expressing \( A_{i'_1 i'_2} \) in terms of \( A_{i_1...i_6} \) using the duality relation of equation (2.16) and also by expressing \( h_{j'} \) in terms of \( h_{i'_1...i'_8,j'} \) using the duality relation of equation (2.20). The fields of equation (2.24) also give \( 36 + 56 + 28 + 8 = 128 \) degrees of freedom. We note that \( h_{ij} i' + h_9^9 = 0 \) and as we have not included \( h_9^9 \) we do not take \( h_{ij} i' \) equal to zero.
3. Reduction of degrees of freedom from the algebra viewpoint

In the previous section we have seen that the higher level fields can be expressed in terms of the fields $h_{ij}$ and $A_{i_1 i_2 i_3}$ at levels zero and one through the existence of duality relations such as those of equations (2.19), (2.20) and (2.22). In this section we will show that the massless irreducible representation is annihilated by an infinite set of generators and this corresponds to the existence of the infinite set of duality relations. We begin by considering the generator of the form

$$N_{k_1 k_2 k_3} = S_{k_1 k_2 k_3} + c_1 \varepsilon_{k_1 k_2 k_3} r_1 \ldots r_6 S_{r_1 \ldots r_6}$$  \hspace{1cm} (3.1)$$

for a constant $c_1$. Under the action of this generator the fields transform under $\Lambda^{k_1 k_2 k_3} N_{k_1 k_2 k_3}$ as

$$\delta h_{ij} = 18 \Lambda_{(i[k_1 k_2]} A_{j)} k_1 k_2 - 2 \eta_{ij} \Lambda^{k_1 k_2 k_3} A_{k_1 k_2 k_3}$$

$$-5! c_1 (3 \cdot 3 \Lambda^{k_1 k_2 k_3} \varepsilon_{k_1 k_2 k_3} r_1 \ldots r_5 (i A_j) r_1 \ldots r_5 + \eta_{ij} \Lambda^{k_1 k_2 k_3} \varepsilon_{k_1 k_2 k_3} r_1 \ldots r_6 A_{r_1 \ldots r_6})$$  \hspace{1cm} (3.2)$$

$$\delta A_{i_1 i_2 i_3} = -3 \cdot 2 h_{i_1 j} A_{i_2 i_3} + 60 A_{i_1 i_2 i_3} k_1 k_2 k_3 \Lambda_{k_1 k_2 k_3} - 5! c_1 \Lambda^{k_1 k_2 k_3} A_{r_1 r_2 r_3} \varepsilon_{k_1 k_2 k_3 r_1 r_2 r_3 i_1 i_2 i_3}$$

$$- \frac{7!}{3} c_1 \Lambda^{k_1 k_2 k_3} \varepsilon_{k_1 k_2 k_3 r_1 \ldots r_6 A_{r_1 \ldots r_6 [i_1 i_2 i_3] - \varepsilon_{k_1 k_2 k_3 r_1 \ldots r_5 j A_{i_1 i_2 i_3 r_1 \ldots r_5 j}} (3.3)$$

$$\delta A_{i_1 \ldots i_6} = 2 \Lambda_{i_1 i_2 A_{i_3 i_4 i_5 i_6}} + 112 h_{i_1 \ldots i_6 k_1 k_2 k_3} \Lambda_{k_1 k_2 k_3} + 112 h_{i_1 \ldots i_5 |k_1 k_2 k_3 i_6} \Lambda_{k_1 k_2 k_3}$$

$$- 12 c_1 \Lambda^{k_1 k_2 k_3} \varepsilon_{k_1 k_2 k_3 [i_1 \ldots i_5 k_1]} j \Lambda_{k_1 k_2 k_3}$$

$$- 3 \cdot 7! c_1 \Lambda^{k_1 k_2 k_3} \varepsilon_{k_1 k_2 k_3 r_1 \ldots r_6 A_{r_1 \ldots r_6 |i_1 i_4 r_1 \ldots r_6 i_5 i_6]}$$

$$- 15 \cdot 7! c_1 \Lambda^{k_1 k_2 k_3} \varepsilon_{k_1 k_2 k_3 r_1 \ldots r_4 r_5 r_6 A_{r_1 \ldots r_4 [i_1 \ldots i_5 r_5 r_6]}}$$

$$- 4 \cdot 7! c_1 \Lambda^{k_1 k_2 k_3} \varepsilon_{k_1 k_2 k_3 r_1 \ldots r_6 A_{r_1 \ldots r_6 |i_1 i_2 i_3 i_4 i_5 i_6}}$$  \hspace{1cm} (3.4)$$

$$\delta h_{i_1 \ldots i_8 ; j} = 2 (A_{i_1 i_2 A_{i_3 i_4 i_5 i_6}} - \Lambda_{j i_1 i_2 A_{i_3 i_4 i_5 i_6}} - 12 \cdot 8 \cdot 3 \Lambda^{k_1 k_2 k_3} (A_{i_1 \ldots i_8 k_1 k_2 k_3 j + A_{i_1 \ldots i_7 j k_1 k_2 k_3 i_8}})$$

$$+ 3 c_1 \Lambda^{k_1 k_2 k_3} \varepsilon_{k_1 k_2 k_3 [i_1 \ldots i_6 A_{i_7 i_8}] - 3 c_1 \Lambda^{k_1 k_2 k_3} \varepsilon_{k_1 k_2 k_3 [i_1 \ldots i_6 A_{i_7 i_8}]}$$  \hspace{1cm} (3.5)$$

On setting $c_1 = -\frac{1}{3 \cdot 7!}$ in the generator of equation (3.1) becomes

$$N_{k_1 k_2 k_3} = S_{k_1 k_2 k_3} - \frac{1}{3 \cdot 7!} \varepsilon_{k_1 k_2 k_3 r_1 \ldots r_6 S_{r_1 \ldots r_6}}$$  \hspace{1cm} (3.6)$$

With this choice the variations of equations (3.2) to (3.5) can be expressed in terms of the duality relations defined in equations (2.19) (2.20) and (2.22) and become equal to zero

$$\delta h_{ij} = 18 \Lambda_{(i k_1 k_2 E_j) k_1 k_2} - 2 \eta_{ij} \Lambda^{k_1 k_2 k_3} E_{k_1 k_2 k_3} = 0$$  \hspace{1cm} (3.7)$$

$$\delta A_{i_1 i_2 i_3} = \frac{1}{3!} \varepsilon_{i_1 i_2 i_3 k_1 k_2 k_3 r_1 r_2 r_3 \Lambda^{k_1 k_2 k_3} E^{r_1 r_2 r_3} - 6 \Lambda_{[i_1 i_2 k_1 k_2 k_3] E_{i_3}]} k = 0$$  \hspace{1cm} (3.8)$$

$$\delta A_{i_1 \ldots i_6} = \frac{1}{30} \Lambda^{k_1 k_2 k_3} \varepsilon_{k_1 k_2 k_3 [i_1 \ldots i_5] j E_{i_6} j + 28 \Lambda^{k_1 k_2 k_3} \varepsilon_{k_1 k_2 k_3 r_1 \ldots r_6 E_{r_1 \ldots r_6 [i_1 i_2 i_3 i_4 i_5 i_6]}}$$  \hspace{1cm} (3.9)$$
\[-140 \Lambda_{k_1k_2k_3} \varepsilon^{k_1k_2k_3r_1r_4r_5r_6} E_{i_1...i_6[r_1...r_3,r_4...r_6]} = 0 \quad (3.10)\]

In a similar fashion one can identify other generators that annihilate the massless irreducible representation. Indeed one finds that the generator

\[N_{ij} = J_{ij} + \frac{2}{8!} \varepsilon^{i_1...i_8}_{i_1...i_8} S_{r_1...r_8} = 0, \quad (3.11)\]

The variation of the graviton under a transformation of this generator with parameter \(\Lambda^{ij}\) gives

\[\delta h_{ij} = 2\Lambda^k_i E_{kj} + 2\Lambda^k_j E_{ki} = 0, \quad (3.12)\]

It would seem inevitable that one has an infinite number of generators that annihilate the entire representation and we will take this to be the case. At the next level we would expect to find the generator

\[\tilde{N}_{i_1i_2i_3} = S_{i_1i_2i_3} + \frac{1}{9!} \varepsilon^{j_1...j_9} S_{j_1...j_9} i_1i_2i_3 \quad (3.13)\]

where we have used the value of the coefficient that we will find below. The pattern of the higher level generators that annihilate the representation is apparent. To a given generator one adds with a suitable coefficient another generator which possess an additional block of nine anti-symmetric indices which are saturate with the \(\varepsilon\) symbol.

The set of all generators that annihilate the representation must form a subalgebra, denoted \(I_c\) of \(I_c(E_9)\) and we will now show that this is true for such lowest level generators. Indeed we find that

\[\lbrack N^i_{i_1i_2i_3}, N_{j_1j_2j_3} \rbrack = 2N^i_{i_1i_2i_3 j_1j_2j_3} - 9 \cdot 4\delta^{i_1i_2}_{j_1j_2} N^i_{j_3} - \frac{1}{2} \varepsilon^{i_1i_2i_3}_{j_1j_2j_3} k_1k_2k_3 \tilde{N}^i_{k_1k_2k_3} \quad (3.14)\]

where

\[N_{i_1...i_6} = S_{i_1...i_6} + \frac{1}{12} \varepsilon_{i_1...i_6} k_1k_2k_3 S_{k_1k_2k_3} = \frac{1}{12} \varepsilon_{i_1...i_6} k_1k_2k_3 N_{k_1k_2k_3} \quad (3.15)\]

The generators that annihilate the irreducible representation also form an ideal. We recall that an ideal \(I\) of a Lie algebra \(G\) is a subalgebra consisting of elements \(X \in I\) such that such that \([X,Y] \in I\) for all \(Y \in G\). One finds that

\[\lbrack S^i_{i_1i_2i_3}, N_{j_1j_2j_3} \rbrack = -\frac{1}{6} \varepsilon^i_{i_1i_2i_3 j_1j_2j_3} k_1k_2k_3 N^i_{k_1k_2k_3} - 18\delta^i_{j_1j_2} N^i_{j_3} \quad (3.16)\]

and that

\[\lbrack S^i_{i_1i_2i_3}, N_{i_1} \rbrack = 3\eta_{i_1} N_{k_1} \tilde{N}^i_{k_1} + 3\eta_{i_1} N_{i_1} N^i_{k_1} \quad (3.17)\]

Since the commutators of the generator \(S^i_{i_1i_2i_3}\) lead to the whole of \(I_c(E_9)\) it follows that the equations (3.16) and (3.17) together with their higher level analogues, which we have not shown, imply that the generators in the subalgebra \(I\) are an ideal of \(I_c(E_9)\). Starting from the level zero field \(h_{ij}(0)\) of the massless irreducible representation we can find all fields in the representation by the action of \(S^i_{i_1i_2i_3}\). Hence if the generators which annihilate the massless irreducible representation form an ideal in \(I_c(E_9)\) it follows that if all the
elements of $I$ annihilate the lowest field, the graviton, then they will annihilate all fields in the representation.

Given a Lie algebra $G$ (group) which contains an ideal subalgebra (subgroup) $I$ then the coset $G/I$ is also a Lie algebra (group). If $A_1, A_2$ belong to $G$ the corresponding equivalence relation is $A_1 \sim A_2$ means $A_1 = A_2 + i$ for some $i \in I$. Indeed $\frac{I_c(E_9)}{I}$ is a Lie algebra. It is clear from the form of the generators in the ideal that all the generators in $I_c(E_9)$ are related to the generators $J_{i_1i_2}$ and $S_{i_1i_2i_3}$ by the equivalence relations. Hence $\frac{I_c(E_9)}{I}$ contains just these two generators. These generators contain $36 + 84 = 120$ generators and, as we will show in the next next section, they generate $SO(16)$. As a result

\[
\frac{I_c(E_9)}{I} = SO(16)
\]

(3.18)

In this section we have seen that there is a radical reduction in the number of states of the massless irreducible representation can be traced to the existence of a ideal in $I_c(E_9)$. It contains only the graviton and three form. Indeed due to equation (3.18) these states belong to an irreducible representation of the much smaller algebra $SO(16)$. The situation is a bit similar to the existence of highest weight states in representations of the Virasoro algebra and the corresponding reduction in the number of states in the representation.

4. The bosonic states viewed as a spinor

In the section two we found that although the massless irreducible representation of the little group $I_c(E_9)$ contained, at first sight, an infinite number of states it was subject to an infinite set of duality relations that reduced the number of degrees of freedom it contained to 128. These can be listed as in equation (2.24), or as an $I_c(E_8) = SO(16)$ multiplet, in equation (2.24). The $SO(16)$ algebra has two 128 dimensional irreducible representations both of which are spinors. The number of components of a spinor in sixteen dimensions is $2^{16} = 256$. However, in sixteen, effectively Euclidean, dimensions we can have Majorana Weyl spinors which have just 128 component. Hence it must be that our 128 bosonic states belong to a spinor representation of $SO(16)$. As $E_8$ contains the gravity line consisting of the nodes three to ten it contains the $SL(9)$ algebra and as a result $I_c(E_8) = SO(16)$ contains the $SO(9)$ which acts on the spacetime time coordinates. The purpose of this section is to understand how the bosonic states can be assembled into this spinor representation.

We found that for the massless representation the little algebra is $I_c(E_9)$ which arises when we delete node one and two in the $E_{11}$ Dynkin diagram. Further decomposing the $I_c(E_9)$ representations into those of $I_c(E_8)$ corresponds to also deleting node three in addition to nodes one and two in the $E_{11}$ Dynkin diagram as shown below

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The eleven dimensional theory appears when we deleted node eleven and decompose with respect to the remaining $A_{10}$ algebra. In the context of the massless irreducible
representation, which has little group $I_c(E_8)$, this implies that we should decompose with respect to $I_c(A_8) = SO(9)$. Indeed the fields of equation (2.9), and so equation (2.23), appear as representations of $I_c(A_8) = SO(9)$. However, we wish to study the massless irreducible representation from the viewpoint of $I_c(E_8) = SO(16)$ which is the algebra that appears in the above $E_{11}$ Dynkin diagram. As such we have in effect to undo the deletion of node eleven. To better understand the manner in which bosonic fields belong to the 128 dimensional spinor representation we consider the fields from the viewpoint of the $SO(8) \times SO(8)$ subalgebra of $SO(16)$.

The first step is to identify the $SO(8) \times SO(8)$ generators amongst those of $I_c(E_8)$. The latter generators arise in $E_{11}$ as

$$\{J_{i'j'}, S_{i'_1i'_2\ldots i'_6}, S_{i'_1\ldots i'_6j'}\}, \quad \text{with } i', j', \ldots, 3, \ldots, 10 \quad (4.1)$$

These Cartan involution invariant generators appear in equations (2.5-2.8) if we change the signs between the two generators from plus to minus and visa-versa. In equation (4.1) one finds $28 + 56 + 28 + 8 = 120$ generators. The $J_{i'j'}$ generate an $SO(8)$ This subalgebra is the $I_c(A_7)$ that appears as the gravity line consisting of nodes four to ten of the $E_{11}$ Dynkin diagram. As such this $SO(8)$ is just part of the familiar gravity line symmetries that are foremost in E theory discussions and act on the spacetime coordinates. The $SO(8) \times SO(8)$ subalgebra consists of the generators

$$J_{i'j'}^\pm = J_{i'j'} \pm 2\hat{J}_{i'j'}, \quad \text{where } \quad \hat{J}_{i'j'} = \frac{1}{6!}\varepsilon_{i'j'k'_1\ldots k'_6}S_{i'_1\ldots i'_6} \quad (4.2)$$

Using the well known commutators of the $E_{11}$ algebra one finds that they obey the commutation relations

$$[J_{i'j'}^+, J_{k'l'}^-] = 0 \quad , \quad \quad (4.3)$$

$$[J_{i'j'}^+, J_{k'l'}^-] = -8\delta^i_{[k'}J_{j']l'}, \quad [J_{i'j'}^-, J_{k'l'}^-] = -8\delta^j_{[k'}J_{i']l']}. \quad (4.4)$$

which are indeed those of $SO(8) \times SO(8)$. We note that the gravity line $SO(8)$ discussed above arises as the algebra which is the diagonal subalgebra of $SO(8) \times SO(8)$. We note that we have three different $SO(8)$ algebras.

We will now decompose the spinor representation of $SO(16)$, which contains the 128 degrees of freedom, into representations of $SO(8) \times SO(8)$. To begin with rather than consider the fields we will consider the corresponding Cartan involution odd generators of equation (2.5-2.8). While these transform into each other in the usual way under the $SO(8)$ rotations generated by $J_{i'j'}$, under the $S_{i'_1\ldots i'_6}$ generator we find that

$$[S_{i'_1\ldots i'_6}, T^{j'k'}] = -6T_{[i'_1\ldots i'_5}j'k'\delta^{i'_6]i'_6]} - 6T_{[i'_1\ldots i'_5}k'\delta^{j'_6]i'_6]}, \quad (4.5)$$

$$[S_{i'_1\ldots i'_6}, T^{j'j'_2j'_3}] = 60T_{[i'_1\ldots i'_5}j'_2j'_3\delta^{j'_4]}i'_6] + 3T_{[i'_1\ldots i'_6}j'_2j'_3] \quad (4.6)$$

$$[S_{i'_1\ldots i'_6}, T^{j'\ldots j'_6}] = -120T^{k'}k'\delta^{j'_1j'_2j'_3j'_4j'_5j'_6]i'_1\ldots i'_6]} + 1080T_{[i'_1}j'_2j'_3j'_4j'_5j'_6]} \quad (4.7)$$

$$[S_{i'_1\ldots i'_6}, T^{j'\ldots k'}] = 3360(T^{j'_1j'_2j'_3j'_4j'_5j'_6]}k'k' \delta^{j'_1j'_2j'_3j'_4j'_5j'_6]} - T^{j'_1j'_2j'_3j'_4j'_5j'_6]}k'k' \delta^{j'_1j'_2j'_3j'_4j'_5j'_6]}), \quad (4.8)$$

11
Examining these equations we notice that \( T^{i'k'} \) and \( \hat{T}_{i'j'} \equiv \frac{1}{6} \varepsilon_{i'j'}k'_1\ldots k'_6 T_{k'_1\ldots k'_6} \) transform into each other as do \( T^{i'j'} \) and \( T^{i'} \equiv \frac{1}{8} \varepsilon_{k'_1\ldots k'_6} T_{k'_1\ldots k'_6} \).

Motivated by the result just above we define the combination

\[
\hat{T}_{i'j'} = T_{i'j'} + 2\hat{T}_{i'j'} - \frac{1}{6} \delta_{i'j'} T_{k'k'}
\]

which contains \( 8.8 = 64 \) degrees of freedom as \( T_{i'j'} \) and \( \hat{T}_{i'j'} \) are symmetric and antisymmetric respectively. Under \( SO(8) \times SO(8) \) the objects \( \hat{T}_{i'j'} \) transforms as

\[
[J^{i'j'}, \hat{T}_{k'l'}] = -4\delta[i'k'] \hat{T}_{j'l'}, \quad [J^{-i'j'}, \hat{T}_{k'l'}] = -4\delta[i'k'] \hat{T}_{j'l'}
\]

We recognise that the first \( SO(8) \) transforms the first index on \( \hat{T}_{i'j'} \) as a vector and the second \( SO(8) \) transforms the second index on \( \hat{T}_{i'j'} \) as a vector. Thus we recognise that \( \hat{T}_{k'l'} \) transforms as the \((8_v, 8_v)\) representation of \( SO(8) \times SO(8) \) where \( 8_v \) is the vector representation of \( SO(8) \).

We now turn our attention to the objects \( T^{i'j'k'} \) and \( T^{i'} \) which have 56 and 8 degrees of freedom respectively. Under \( SO(8) \times SO(8) \) they transform as

\[
[J^{i'j'}, T_{k'l'}] = 2T_{[i'j']k'l'} + \frac{1}{2} T_{i'j'k'l'}
\]

and

\[
[J^{i'j'}, T_{k'[i'k']}] = 6T_{[k'[i'k'][i'j'][k'_3]} + \frac{1}{6} \varepsilon_{i'j'k'_1k'_2k'_3} T_{l'_1l'_2l'_3} \pm 12\delta_{i'j'} T_{k'_3}
\]

Hence \( T_{k'} \) and \( T_{i'k'_1k'_3} \) and form the \( 56 + 8 = 64 \) dimensional representation of \( SO(8) \times SO(8) \).

We will now identify what representation this is. We observe that the eight dimensional gamma matrices in Euclidean space obey the equations

\[
\gamma^{i'j'k'} = \gamma^{i'j'k'} + 2\delta_{k'[i'j']}
\]

and

\[
\gamma^{i'j'}k'_1k'_2k'_3 = 6\delta_{j'[i'k'_1k'_2k'_3]} + \gamma^{i'j'}k'_1k'_2k'_3 - 6\delta_{i'j'} \gamma^{k'_1k'_2k'_3}
\]

\[
= 6\delta_{j'[i'k'_1k'_2k'_3]} + \frac{1}{3!} \varepsilon_{i'j'k'_1k'_2k'_3l'_1l'_2l'_3} \gamma_{l'_1l'_2l'_3} - 6\delta_{i'j'} \gamma^{k'_1k'_2k'_3}
\]

where

\[
\gamma_9 \equiv \gamma^{1\ldots 8}
\]

Hence if we were to make the identifications

\[
T_{k'} = \gamma_{k'}, \quad T_{i'j'k'} = -2\gamma^{i'j'k'}
\]

then multiplication on the left by \( \gamma^{i'j'} \) has the same result as \( J^{i'j'} \) in the commutators of equations (4.11) and (4.12) provided \( \gamma_9 \) takes the value \(-1\). Similarly one can verify that
right multiplication by \(-\gamma^i j\) has the same result as \(J^i j\) in the commutators of equations (4.11) and (4.12).

As a result we can think of \(T^{ij}_1 j_2 j_3\) and \(T^i\) as belonging to a bi-spinor which is Weyl projected. A bi-spinor takes the form

\[
\Gamma = c_0 I + c_i \gamma^i + \ldots + c_{i_1 \ldots i_8} \gamma^{i_1 \ldots i_8}
\]  

(4.17)

However, if we require \(\gamma_9 \Gamma = \Gamma\) then we find that \(c_0 = \frac{1}{8!} \epsilon^{i_1 \ldots i_8} c_{i_1 \ldots i_8}\) etc and we can take the bi-spinor to be of the form

\[
\Gamma = (1 + \gamma_9)(c_0 I + c_i \gamma^i + \ldots + c_{i_1 \ldots i_8} \gamma^{i_1 \ldots i_8})
\]  

(4.18)

However, we can also demand that \(\Gamma \gamma_9 = -\Gamma\) then find that \(\Gamma\) takes the form

\[
\Gamma = (1 + \gamma_9)(c_i \gamma^i + c_{i_1 \ldots i_8} \gamma^{i_1 \ldots i_8})
\]  

(4.19)

Thus we find the representation carried by \(T^{ij}_1 j_2 j_3\) and \(T^i\) is a bispinor with the above values of Weyl projection. The coefficients \(c_i\) and \(c_{i_1 i_2 i_3}\) correspond to the fields \(h_i = \frac{1}{8!} \epsilon^{i_1 \ldots i_8} h_{i_1 \ldots i_8,i}\) and \(A_{i_1 \ldots i_8}^i\) of equation (2.24). It we denote the eight dimensional Majorana Weyl spinor representation of \(SO(8)\) with Weyl projection +1, that is \(\gamma_9 \epsilon = \epsilon\) by \(8_c\) and the one with Weyl projection \(-1\) by \(8_s\) then the generators \(T^{ij}_1 j_2 j_3\) and \(T^i\) belong to the \((8_c,8_s)\) representation of \(SO(8) \times SO(8)\).

Hence we find that the the spinor representation of \(SO(16)\) which contains the 128 bosonic degrees of freedom of the massless irreducible representation decomposes into the

\[
128 = (8_v,8_v) \oplus (8_c,8_s)
\]  

(4.20)

representations of \(SO(8) \times SO(8)\). The \((8_v,8_v)\) contains the fields \(h_{ij}^i\) and \(A_{i_1 \ldots i_8}^i\) of equation (2.24) while the \((8_c,8_s)\) contains the \(A_{i_1 i_2 i_3}^i\) and \(h_{i_1 \ldots i_8,k}^i\) fields. The duality equations relate these two representations. The higher level fields are related by duality equations to the fields of these two representations and so these duality relations can be thought of arising from the action of the affine operator that takes \(I_c(E_8)\) to \(I_c(E_9)\).

Rather than discuss the transformations of the Cartan involution odd generators we can consider the transformations of the corresponding fields which we will now derive. The Cartan form containing these fields can be written as

\[
\hat{\gamma} = \hat{A}_{i_1 i_2}^{i_3} T^{i_1 i_2} + A_{i_1 i_2 i_3}^{i_4} T^{i_1 i_2 i_3} + A_{i_1 i_2 i_3}^{i_4 \ldots i_8} T^{i_1 i_2 i_3}
\]

(4.21)

In terms of our original generators this takes the form

\[
\hat{\gamma} = \left[\left(\hat{A}_{i_1 i_2}^{i_3} - \frac{1}{6} \hat{A}^{k} k, \delta_{i_1 i_2}^{i_3} \right) T^{i_1 i_2} + A_{i_1 i_2 i_3}^{i_4} T^{i_1 i_2 i_3} + \frac{2}{6!} \epsilon_{i_1 \ldots i_6}^{k_1 k_2} \hat{A}_{k_1 k_2}^{i_3} T^{i_1 \ldots i_8} + \frac{1}{8!} \epsilon_{i_1 \ldots i_8} A_{k}^{i} T^{i_1 \ldots i_8} k, k'\right]
\]

(4.22)
Using the commutation relations (4.10), (4.11), (4.12) that

\[ \delta \hat{\mathcal{V}} = [\Lambda^\pm, \hat{A}_{\pm}^{\prime}] = \frac{\Lambda^\pm}{2} \Lambda^{\prime} \hat{A}_{\mp}^{\prime} \]

(4.23)

\[ \delta_{\Lambda^+ \hat{A}_{\pm}^{\prime}} = 4\Lambda^+ A_{\pm}^{\prime} \hat{A}_{\mp}^{\prime} \]
\[ \delta_{\Lambda^- \hat{A}_{\pm}^{\prime}} = -4\hat{A}_{\mp}^{\prime} \Lambda^- A_{\pm}^{\prime} \]
\[ \delta_{\Lambda^\pm} A_{\pm}^{\prime} = 2\Lambda^\pm A_{\mp}^{\prime} \hat{A}_{\pm}^{\prime} \pm 12\Lambda^\pm A_{\pm}^{\prime} \hat{A}_{\pm}^{\prime} \Lambda^\pm \]

(4.24)

\[ \delta_{\Lambda^\pm} A_{\pm}^{\prime} = \frac{1}{2} \Lambda^\pm A_{\mp}^{\prime} \Lambda^{\prime} \Lambda^\pm + 6\Lambda^\pm [\Lambda^\pm, A_{\mp}^{\prime}] + \frac{1}{6} \epsilon i_1 i_2 i_3 j_1 j_2 j_3 k_1 k_2 k_3 \Lambda^\pm j_1 j_2 k_1 k_2 k_3 \]

(4.25)

It will also be instructive to decompose the 120 dimensional adjoint representation of \( I_c(E_8) = SO(16) \), whose generators are given in equation equation (4.1) into representations of terms of SO(8) × SO(8). The generators not included in SO(8) × SO(8) are \( S_{k'} \equiv \frac{1}{8} \epsilon i_1 \ldots i_8 j_1 \ldots j_8, k' \) and \( S_{k_1 k_2 k_3} \) and their commutation relations with the SO(8) × SO(8) generators are given by

\[ [J^\pm i' j', S_{k'}] = 2S_{i'\pm j'} \Lambda^\pm \pm \frac{1}{2} S_{i'\pm j'} \Lambda^\pm \]

(4.26)

and

\[ [J^\pm i' j', S_{k_1 k_2 k_3}] = 6S_{[k_1 k_2 i' [\delta j_1 j_2 j_3] k_3]} \pm \frac{1}{6} \epsilon i' j' k_1 k_2 k_3 l_1 l_2 l_3 [\delta j_1 j_2 j_3 k_1 k_2 k_3] \pm \frac{1}{6} \epsilon i' j' k_1 k_2 k_3 l_1 l_2 l_3 [\delta j_1 j_2 j_3 k_1 k_2 k_3] \]

(4.27)

Comparing with the commutators of equations (4.11) and (4.12) we see that \( S_{k'} \) and \( S_{k_1 k_2 k_3} \) have the same commutators with SO(8) × SO(8) as \( T_{k'} \) and \( T_{k_1 k_2 k_3} \) except for an opposite sign in the second term on the right-hand side of equation (4.27). Examining the gamma matrix algebra of equations (4.13) and (4.14) we see that we can identify \( S_{k'} \) and \( S_{k_1 k_2 k_3} \) with \( \gamma_{k'} \) and \( -2\gamma_{k_1 k_2 k_3} \) provided we take \( \gamma_9 \) in equation (4.14) to take the value +1. Hence these generators belong to the \( (8_s, 8_c) \) representation of SO(8) × SO(8). Hence the 120 dimensional adjoint representation of SO(16) consists of the

\[ 120 = (28, 1) \oplus (1, 28) \oplus (8_s, 8_c) \]

(4.28)

representation of SO(8) × SO(8). The \( (28, 1) \oplus (1, 28) \) contain the adjoint representation of SO(8) × SO(8). We note that the decompositions we have found are not quite the same as those one finds in certain books.

5. Spinors of \( I_c(E_9) \) decomposed into representations of SO(8) × SO(8)

Having decomposed the 128 dimensional SO(16) spinor representation that contains the bosonic degrees of freedom in terms of representation of SO(8) × SO(8) it will be educational to also analyse the 128 dimensional spinor representation to which the fermionic degrees of freedom belong. Long ago it was shown that the fermionic degrees of freedom appear in maximal supergravity in \( D \) dimensions as a linear representation of the Cartan involution invariant subgroup of the duality group \( E_{11-D} \), for example in four dimensions
this group is $I_c(E_7) = SU(8)$ [16]. It was therefore natural to take the fermions in $E$ theory to be a linear representation of the Cartan involution invariant subalgebra of $I_c(E_{11})$. In fact fermions were first introduced [17-20] in this way in the context of the $E_{10}$ theory and subsequently [21] in the $E_{11}$ theory. The key to these constructions was the realisation that $I_c(E_{10})$ and $I_c(E_{11})$ admit highly unfaithful representations. In particular it was shown that $I_c(E_{11})$ has a representation based on a spinor of $SO(1,10)$, $\epsilon_\alpha$ in which the lowest level generators take the form [2]

$$J_{a_1 a_2} \epsilon_\alpha = -\frac{1}{2}(\gamma_{a_1 a_2})_\alpha^\beta \epsilon_\beta, \quad S_{a_1 a_2 a_3} \epsilon_\alpha = \frac{1}{2}(\gamma_{a_1 a_2 a_3})_\alpha^\beta \epsilon_\beta,$$

$$S_{a_1 \ldots a_6} \epsilon_\alpha = -\frac{1}{4}(\gamma_{a_1 \ldots a_6})_\alpha^\beta \epsilon_\beta, \quad S_{a_1 \ldots a_8, b} = \delta_{[a_1} \epsilon_{a_2 \ldots a_8]} c_1 c_2 \gamma^{c_1 c_2}, \ldots \tag{5.1}$$

The generators at higher levels can be found by substituting the above actions into the $I_c(E_{11})$ commutation relations. Clearly, some parts of the higher level generators are trivially realised and so the representation is unfaithful.

We will be interested in the fermionic degrees of freedom from the viewpoint of the irreducible representations of $I_c(E_{11}) \otimes_s l_1$ and in particular as a representation of $I_c(E_9)$. The $I_c(E_9)$ transformations of the spinor $\epsilon_\alpha$ are given by equation (5.1) with the indices restricted to take the values 2, \ldots, 10. It is straightforward to show that the representation, $\epsilon$ is annihilated by precisely the same generators of equation (3.6) and (3.11), that is,

$$N_{i_1 i_2 i_3} \epsilon = 0 = N_{i_1 i_2} \epsilon \tag{5.2}$$

It must be true that there are an infinite number of such generators that also annihilate $\epsilon$. Thus the spinor $\epsilon$ is annihilated by the same generators as the representation that contains the bosonic degrees of freedom and so they form the same ideal $I$. As such the spinor is really a representation of $\frac{K(E_9)}{I} = SO(16)$.

The generalisation of the unfaithful representation of equation (5.1) to the gravitino which was first done in the context of $E_{10}$ [17-20] and the result for $E_{11}$ [21] is

$$J_{a_1 a_2} \psi_b = -\frac{1}{2}\gamma_{a_1 a_2} \psi_b - 2\eta_{b[a_1} \psi_{a_2]}$$

$$S_{a_1 a_2 a_3} \psi_b = \lambda\{\frac{1}{2}\gamma_{a_1 a_2 a_3} \psi_b - \gamma_{b[a_1 a_2} \psi_{a_3]} + 4\eta_{b[a_1} \gamma_{a_2 a_3]}\}$$

$$S_{a_1 \ldots a_6} \psi_b = -\frac{1}{4}\gamma_{a_1 \ldots a_6} \psi_b - 2\gamma_{b[a_1 \ldots a_5} \psi_{a_6]} + 5\eta_{b[a_1} \gamma_{a_2 \ldots a_3} \psi_{a_6]}$$

$$S_{a_1 \ldots a_8, c} \psi_b = \lambda\{\gamma_{b[a_1 \ldots a_8} \psi_c] - \gamma_{a_1} \psi_{a_8} + 8(\eta_{b[c} \gamma_{a_1 \ldots a_7} \psi_{a_8]} - \eta_{b[c} \gamma_{a_1} \psi_{a_7} \psi_{a_8]}\} - 2\eta_{c[a_1} \gamma_{a_2 \ldots a_8]} \psi_{b] - 28\gamma_{b[a_1 \ldots a_6} \psi_{a_7} \eta_{a_8]}} \tag{5.3}$$

We can take the constant $\lambda$ to take the values $\lambda = -1$ or $\lambda = 1$ and still have a representation of $I_c(E_{11})$. This reflects the way the generator $S_{a_1 a_2 a_3}$ appears in the algebra resulting from its level one character. In reference [21] we took $\lambda = 1$ but in this paper we will find that the other sign is better.
When the indices in equation (5.2) take the range 3 to 10 we can interpret this equation as containing the representation that contains the fermionic degrees of freedom as they occur in the little algebra \( I_c(E_9) \).

In this section we will decompose the spinor representation of equation (5.2) into those of \( \text{SO}(8) \times \text{SO}(8) \) which is generated by \( J^\pm_{i'j'} \) defined in equation (4.2), but we will begin with the representation of equation (5.1). We find that

\[
J^\pm_{i'j'} \epsilon = -\frac{1}{2} \gamma_{i'j'} (1 \mp \gamma_9) \epsilon
\]

where \( \gamma_9 = \gamma_1 \ldots \gamma_8 \). Defining \( \epsilon_{\pm} = \frac{1}{2} (I \pm \gamma_9) \epsilon \) we can rewrite this equation as

\[
J^+_{i'j'} \epsilon_+ = 0, \quad J^-_{i'j'} \epsilon_+ = -\frac{1}{2} \gamma_{i'j'} \epsilon_+; \quad J^+_{i'j'} \epsilon_- = -\frac{1}{2} \gamma_{i'j'} \epsilon_- \quad J^-_{i'j'} \epsilon_- = 0
\]

If we take \( \epsilon_+ \) and \( \epsilon_- \) to be the \( 8_c \) and \( 8_s \) representations of \( \text{SO}(8) \) respectively then the spinor \( \epsilon = \epsilon_+ \oplus \epsilon_- \) is in the \( (I \otimes 8_c) \oplus (8_s \otimes I) \) of \( \text{SO}(8) \times \text{SO}(8) \).

We now wish to decompose the gravitino into representations of \( \text{SO}(8) \times \text{SO}(8) \). The action of the \( \text{SO}(16) \) generators can be read off from equation (5.3) by taking the indices to take the values \( i', j' = 3, \ldots 10 \). The gravitino obeys the condition \( \gamma^i \psi_i = \gamma \cdot \psi + \gamma_9 \psi_9 = 0 \) and so we can eliminate the \( \psi_9 \) component in terms of \( \gamma \cdot \psi \equiv \gamma^i \psi_i \). We find that the \( \text{SO}(8) \times \text{SO}(8) \) generators \( J^\pm_{i'j'} \) act on the gravitino \( \psi_k \) as

\[
J^\pm_{i'j'} \psi_j' = -\frac{1}{2} \gamma_{i'j'} (1 \mp \gamma_9) \psi_j' - 2 \eta_{i'j'} [i' \psi_{i'2}] \pm \frac{1}{3} \gamma_9 \gamma_{i'j'} \gamma \cdot \psi - \frac{2}{3} \gamma_9 \epsilon_9 \eta_{i'j'} [i' \psi_{i'2}] \pm \frac{4}{3} \gamma_9 \eta_{i'j'} [i' \psi_{i'2}]
\]

In deriving this equation we have used the identity

\[
\gamma_{k_1 \ldots k_n} = (-1)^{n(n+1)/2} \frac{1}{m!} \epsilon_{k_1 \ldots k_n} \gamma_{j_1 \ldots j_m} \gamma_9 \gamma_{j_1 \ldots j_m}
\]

where \( n + m = 8 \). While the action of \( J^\pm_{i'j'} \) on \( \gamma \cdot \psi \) follows from equation (5.6) and it is given by

\[
J^\pm_{i'j'} \gamma \cdot \psi = -\frac{1}{2} \gamma_{i'j'} \gamma \cdot \psi \pm \frac{1}{6} \gamma_9 \gamma_{i'j'} \gamma \cdot \psi \pm \frac{4}{3} \gamma_9 \gamma [i' \psi_{i'2}]
\]

We now take the combination

\[
\Psi_i = \psi_i - \frac{1}{2} \gamma_i \gamma \cdot \psi
\]

which transforms as

\[
J^\pm_{i'j'} \Psi_k = -\frac{1}{2} \gamma_{i'j'} (1 \mp \gamma_9) \Psi_k' - 2 \eta_{k' [i'1} (1 \mp \gamma_9) \Psi_{i'2}]
\]
If we define $\Psi^\pm_k = \frac{1}{2}(1 \pm \gamma_9)\Psi_k$ then equation (5.10) can be rewritten as

$$J^+_{i'_1 i'_2} \Psi^+_{i_1} = -\frac{1}{2} \gamma_{i'_1 i'_2} \Psi^+_{i_1} \quad \text{and} \quad J^-_{i'_1 i'_2} \Psi^+_{i_1} = -2\eta_{k[i_1} \Psi^+_{i_2]},$$

(5.11)

As a result we find that $\Psi_k = \Psi^+_k + \Psi^-_k$ belongs to the $(8_c, 8_v) \oplus (8_v, 8_s)$ representation of $\text{SO}(8) \times \text{SO}(8)$. Thus both the fermionic and bosonic degrees of freedom belong to spinor representations of $\text{SO}(16)$ and they have a rather similar decompositions into representations of $\text{SO}(8) \times \text{SO}(8)$. Indeed the bosonic degrees of freedom belong to the $(8_v, 8_v) \oplus (8_c, 8_s)$ and one can interchange the bosons and fermions by interchanging $8_v$ with $8_c$ for the first $\text{SO}(8)$ factor.

It was observed in reference [18] that the fermionic degrees of freedom encoded in the gravitino carried a highly unfaithful representation of the Cartan involution algebra of the relevant algebra and as a result the representation should be annihilated certain generators. This paper also pointed out that this set of generators would form a subalgebra that was an ideal. We will now compute the lowest level generators in the context of irreducible representation, that is $I_c(E_9)$. Using equation (5.3) with the indices taking the values $3, \ldots, 10$, we find, taking $\lambda = -1$ that

$$\left( S_{i_1 i_2 i_3} \Psi_{i_1 i_2} - \frac{2}{6!} \epsilon^{i_1 i_2 i_3 j_1 \ldots j_6} S_{j_1 \ldots j_6} \right) \psi_k = 0 \quad \text{(5.12)}$$

and that

$$\left( J_{i j} + \frac{2}{8!} \epsilon_{[i} r_1 \ldots r_8} S_{r_1 \ldots r_8,|j]} \right) \psi_k = 0 \quad \text{(5.13)}$$

These are precisely the same generators that annihilated the bosonic states. There are no doubt an infinite similar equations at higher levels. One could also take the choice $\lambda = 1$ and find the same ideal if one systematically changed our definition of $\gamma_9$ by introducing a minus sign. Thus seen from the perspective of the irreducible representations of $I_c(E_9)$, the bosonic and fermionic states have a remarkably similar structure. They both belong to 128-dimensional spinor representations of $\text{SO}(16)$, they have the same ideal which annihilates the two representations which carry a representation of the coset of $I_c(E_9)$ with the ideal $I$ whose coset $I_c(E_9)/I$ which is just $\text{SO}(16)$.

However, there is an important difference in the way the bosons and fermions appear. The original irreducible representation for the bosons contains an infinite number of fields whose number is reduced by duality relations, while for the fermions we just have the gravitino. This latter representation was not deduced from some general theory but by hand following the example of the maximal supergravity theories in lower dimensions. It would seem natural to introduce higher level spinors and have these linked to the gravitino by duality relations. One possibility is to introduce the fields

$$\psi_\alpha, \psi_{i_1 \ldots i_8 \alpha}, \psi_{i_1 \ldots i_9 \alpha}, \psi_{i_1 \ldots i_9, j_1 \ldots j_8 \alpha}, \ldots \quad \text{(5.14)}$$
and corresponding duality relations
\[
\psi_{i_1\ldots i_8} = \gamma_{i_1\ldots i_8 j} \psi_j, \ldots
\] (5.15)

It might be straightforward to extend this representation to \( I_c(E_{11}) \) and even \( E_{11} \). This latter step could be possible if one exploited the fact that the algebras SL(D) do admit spinor representation. These have not been popular as they infinite dimensional, but this is what we need in this context. [25].

6. The covariant formulation

In the above section we examined how the bosonic degrees of freedom of supergravity arose as the massless irreducible representation of \( I_c(E_{11}) \otimes_s l_1 \). In particular we saw that they occurred as an irreducible representation of the little group \( I_c(E_9) \) as carried by the fields of equation (2.9). To find the full representation of \( I_c(E_{11}) \otimes_s l_1 \) one has to carry out a boost. This procedure was discussed in reference [7]. In carrying out this last step one does not introduce any additional fields and so even though all the symmetries are present they are not manifest. However, in field theory one usually requires representations that have some of the symmetries realised in a covariant manner. For example, for the irreducible representations of the Poincare algebra one usually requires the Lorentz algebra to be manifest. In our case we want the \( I_c(E_{11}) \) symmetry to be essentially manifest. How to achieve this was very briefly outlined in reference [7] and in this section we will discuss this procedure in detail. We will also examine how the duality relations and annihilation generators which are present in the irreducible representation are inherited into the covariant formulation.

Before we begin we will briefly recall the general theory for obtaining a covariant formulation of an irreducible representation of \( I_c(E_{11}) \otimes_s l_1 \). We will use the same notation as in section four of reference [7]. Let the little algebra be denoted by \( \mathcal{H} \) and then any group element \( g \) of \( I_c(E_{11}) \otimes_s l_1 \) can be written as \( g = e^{\varphi \cdot S} h \) for \( h \in \mathcal{H} \). Before the boost we choose a linear representation \( u_\alpha(0) \) of \( \mathcal{H} \) which we can take to transform as
\[
U(g)u_\alpha(0) = D(g^{-1})_{\alpha \beta} u_\beta(0), \quad L_A u_\alpha(0) = l_A^{(0)} u_\alpha(0)
\] (6.1)

This representation will contain the irreducible representation of the little algebra we are considering. To be more precise, our irreducible representation of the little algebra is embedded in the representation \( u_\alpha(0) \) which, as a result, must be a reducible representation of \( \mathcal{H} \).

The next step is to boost the representation \( u_\alpha(0) \) to find a representation of \( I_c(E_{11}) \otimes_s l_1 \) in much the same way as we boosted the irreducible representation of \( \mathcal{H} \) (see section four of reference [7]). In particular, we take \( u_\alpha(\varphi) \equiv U(e^{\varphi \cdot S}) u_\alpha(0) \) where the symbol \( U \) denotes the action of the group element. However, \( u_\alpha(\varphi) \) does not transform covariantly and so instead we consider the objects
\[
A_\alpha(\varphi) = D(e^{\varphi \cdot S})_{\alpha \beta} u_\beta(\varphi) = D(e^{\varphi \cdot S})_{\alpha \beta} U(e^{\varphi \cdot S}) u_\beta(0)
\] (6.2)

It is straightforward to show that \( U_\alpha(\varphi) \) transforms covariantly, and in particular
\[
U(g)A_\alpha(\varphi) = D(g^{-1})_{\alpha \beta} A_\beta(\varphi')
\] (6.3)
where \( g \in I_c(E_{11}) \) and we have used the relation \( g e^{\varphi \cdot S} = e^{\varphi' \cdot S} h(g, \varphi) \) with \( h(g, \varphi) \in \mathcal{H} \). Although we have embedded the original irreducible representation into a bigger representation of \( \mathcal{H} \) we still have to ensure that we still have the irreducible representation and as a result we have to impose projection conditions and, for the massless case, equivalence relations. How to do this in general has yet to be understood but we will do it for the massless irreducible representation.

For comparison we very briefly give an account of the spin one massless irreducible representation of the Poincaré algebra. We can choose all momenta to vanish except for \( p^+ = p^- = \sqrt{2} m \). The little algebra is \( \text{SO}(D-2) \) since we have to take the generator \( J_{+i} \) to vanish in order to have a unitary representation. The representation of \( \text{SO}(D-2) \) is carried by the fields \( A_{\alpha}(0) \), \( \alpha = 1, \ldots, 10 \). We embed this irreducible representation into the \( \text{SO}(D-2) \) reducible representation \( A^\mu(0) \), \( \mu = 0, 1, \ldots, D-1 \). To get the same fields as in the original irreducible representation we demand that \( A^+_\alpha(0) = 0 \) and require the equivalence relation \( A^-\alpha(0) \sim A^-\alpha(0) + p_\Lambda(0) \). The latter allows us to set \( A^-\alpha(0) = 0 \). To find the effect of these conditions in the covariant formulation in terms of \( A^\mu(0) \) we can impose these conditions in the rest frame fields, as given in equation (6.2), and we recover, in \( x \)-space the familiar gauge fixing condition \( \partial^\alpha A_a(x) = 0 \) and the well known gauge transformation \( A_a(x) \sim A_a(x) + \partial_a \Lambda(x) \).

We will now carry out the above discussion in the context of the massless irreducible representation we studied in the earlier sections. The first step is to embed the fields of the irreducible of \( I_c(E_9) \otimes s l_1 \) contained in equation (2.9) into a larger representation. There is an obvious candidate, namely, we take the fields corresponding to the Cartan involution odd generators of \( E_{11} \) rather than just those of \( E_9 \). As such we consider the fields

\[
A_\alpha(0) = \{ h_{a_1a_2}(0), A_{a_1a_2a_3}(0), A_{a_1}\ldots a_6(0), h_{a_1}\ldots a_8_b(0), \ldots \} \quad a, b, \ldots = 0, \ldots, 10. \quad (6.4)
\]

which carry a linear reducible representation of \( I_c(E_{11}) \). These correspond to the fields \( A_\alpha(0) \) of equation (6.2).

We will now address the issue of the embedding condition that will ensure that we are really still dealing with the same massless irreducible representation. We will first do this before the fields are boosted as in equation (6.2). We take the fields \( A_\alpha(0) \) of equation (6.4) to be subject to two conditions. The first of these is

\[
K^{AB} G_{A, B}^C = K^{AB} (D^\alpha)_B^C \partial_A \tilde{A}^\alpha = 0 \quad (6.5)
\]

where \( K^{AB} \) is the metric on a tangent space which transforms under \( I_c(E_{11}) \) as its indices suggest [13], and the second is the equivalence relation

\[
A^\alpha \sim A^\alpha + (D^\alpha + D^-\alpha)_A^B \partial_B \Lambda^A. \quad (6.6)
\]

The matrices \((D^\alpha)_B^C\) are those of the first fundamental representation and are defined in the equation \([R^\alpha, I_B] = -(D^\alpha)_B^C I_C\). As we will see equation (6.5) eliminates all fields with a lower + index, while the second equation (6.6) eliminating fields which take on a lower − index, leaving us with the fields of equation (2.9), that is, those of the original massless irreducible representation.
When acting on the fields of equation (6.4), the only component of $\partial_A$ which is non-zero is $\partial_- = \frac{\partial}{\partial x^-}$. Thus the only non-zero component of $K^{AB}$ in equation (6.5) is $K^{-+} = \eta^{-+}$, and so equation (6.5) reduces to $(D^\alpha)_+ C \partial_\alpha A = 0$. Since $p_- = \sqrt{2} m$ is a constant we have the condition $(D^\alpha)_+ C \partial_\alpha A = 0$. The above matrix occurs in the equation $[R^\alpha, P_+] = -(D^\alpha)_+ B l_B$ and so our condition can be expressed as $[A_\alpha R^\alpha, P_+] = 0$ which consists of transformations $A_\alpha$, $R^\alpha$ that preserve the $p_+ = 0$ with the only non-zero momentum being $p_-$. By definition this includes the little algebra transformations $E_0$ and so the condition of equation (6.5) places no constraints on the fields in equation (2.9). At levels zero, one and two the condition of equation (6.5) lead to

$$h_{++} = 0 = h_{i+} = h^i_+; \quad A_{+a_1a_2} = 0 = A_{+a_1...a_5} \quad (6.7)$$

At higher levels we find that the commutator $[A_\alpha R^\alpha, P_+] = 0$ contains a factor $\delta^\alpha_+ l_1$ times $A_\alpha$ times the $l_1$ generator at the corresponding level. Hence we find that the condition of equation (6.5) implies that any field $A_\alpha$ with a lower + index is set to zero.

We can recover the same results by directly examining the component equations that follow from equation (6.5). At levels zero, one and three, these are given by [13]

$$\partial^e h^a_e - \frac{1}{2} \partial^e h^e_e = 0 \quad ; \quad \partial^e A_{ea_1a_2} = 0 \quad ; \quad \partial^e A_{ea_1...a_5} = 0 \quad , \quad \ldots \quad (6.8)$$

where we have thrown away derivatives with respect to the higher level coordinates. By taking all the momentum except $p_-$ non-zero in equation (6.8) we recover the above result.

We now consider the equivalence relation (6.6). In the rest frame, that is, for the fields of equation (6.4), this reduces to $(D_\alpha + D_-\alpha)_A^- \partial_- \Lambda^A$ which is proportional to $(D_\alpha + D_-\alpha)_A^- \Lambda^A$. This matrix occurs in the commutator $[R^\alpha + R_\alpha, l_A] = -(D^\alpha + D_\alpha)_A^- P_-$. At level one we have the commutator $[R_{a_1a_2a_3}, Z^{b_1b_2}] = 6\delta_{[a_1a_2}^{b_1b_2} P_{a_3]}$ and so we require $R_{a_1a_2}$ in order to get $P_-$. At higher levels one also finds that this expression will only be non-zero if the index $\alpha$ contains a lower – index. To see this we note that the level zero $P_-$ on the right-hand side arises from the commutator of a level $n l_1$ generator and a level $-n$ level generator of $E_{11}$ with a resulting structure constant that contains a $\delta^\alpha_- \Lambda^A$ factor. Hence all the fields $A_\alpha$ with a lower – index are subject to an equivalence relation and for a suitable choice of $\Lambda^A$ they can be set to zero.

The equivalence relation (6.6) have been worked out at low levels [14] and they lead to the expected gauge transformations;

$$\delta h_{(ab)} = \partial_\alpha \xi_b + \partial_b \xi_\alpha \quad ; \quad \delta A_{a_1a_2a_3} = \partial_{[a_1} \Lambda_{a_2a_3]} \quad ; \quad \delta A_{a_1...a_6} = 2\partial_{[a_1} \Lambda_{a_2...a_6]} \quad , \quad \ldots \quad (6.9)$$

As the only non-zero momenta is $p_-$ we find that at levels zero, one and two we can set to zero $h_{-+}$, $A_{-a_1a_2}$ and $A_{-a_1...a_5}$, ... in agreement with the above discussion.

So far we only considered the effect of the conditions of equation (6.5) and (6.6) in the rest frame, that is, $p_- = \sqrt{2} m$ all other members of the vector representation being zero. To find the analogue of these conditions for the covariant theory we take the conditions to act on $u_\alpha(0)$ in equation (6.2) and then apply the boost and matrix multiplication that this equation contains. Since equation (6.5) contains the $I_c(E_{11})$ covariant expression
K^{AB}(D^\alpha)_B^C \partial_A it retains the same form under the boost and so we can take this equation to hold in the rest frame and carrying out the boost and matrix multiplication, as in equation (6.2), we find that it takes the form as in equation (6.6) but with a parameter \( \Lambda^A(\varphi) = D(e^{-\varphi \cdot S})C^A U(e^{\varphi \cdot S})\Lambda^C(0) \). We recognise this as a gauge transformation of reference [14].

Hence we have shown that the conditions of equations (6.5) and (6.6) do not affect the fields of equation (2.9). However for the fields of equation (6.4) we find that they set all fields with a lower + to zero and we can, using the equivalence relation remove all fields with a lower − index. As a result equations (6.5) and (6.6) ensure that \( A_\alpha \), given in equation (6.4), contains the same fields as occurred in the original massless irreducible representation, given in equation (2.9). Hence it is just the same massless irreducible representation of \( I_c(E_{11}) \otimes_s l_1 \).

In section two we found that the fields of the massless irreducible representation obeyed the duality relations of equation (2.19), (2.20) and (2.22) as well as similar higher level equations. While in section three we found that as a result this representation was annihilated by the generators which belonged to an ideal, for example those in equations (3.6) and (3.11). As the covariant formulation of the irreducible representation is essentially the same as the original irreducible representation we can expect that the generators in equations (3.6) and (3.11), as well as those at higher level, imply the presence of analogous equations which we will not find. Indeed, starting from the equations that the generators in the ideal \( I \) annihilate the fields in the irreducible representation in the rest frame, we can carry out a boost, as in equation (6.2), to find an infinite number of constraints in the covariant theory. However, as the boost contains generators which levels greater than zero it will transform the elements in the ideal with different levels into each other.

Rather than carry out the boost, we will find some covariant operators that agree with elements in the ideal in the rest frame. We begin by considering the operator

\[
\hat{N}_{a_1...a_4} = P_{[a_1} S_{a_2 a_3 a_4]} + \frac{1}{2.6!} \varepsilon_{a_1...a_4} c_1...c_7 P_{e_1} S_{e_2...e_7} + \frac{1}{4} Z^{e_1 e_2} S_{e_1 e_2 a_1...a_4} \\
+ \frac{1}{7.2!} \varepsilon_{a_1...a_4} c_1...c_7 Z^{e_1 e_2} S_{e_1 c_1...c_7, e_2} + \frac{1}{12} Z_{[a_1 a_2} J_{a_3 a_4]} \tag{6.10}
\]

Acting on the fields of the massless irreducible representation of equation (2.9), which are in the rest frame, it obviously vanishes except for \( \hat{N}_{-i_1 i_2 i_3} \) which is equal to \( p_-N_{i_1 i_2 i_3} \), where \( N_{i_1 i_2 i_3} \) is defined in equation (3.6). However, as we have shown in section three \( N_{i_1 i_2 i_3} \) vanishes on the massless irreducible representation of equation (2.9) and as a result \( \hat{N}_{a_1...a_4} \) also vanishes on this representation on the fields the rest frame. The terms in \( \hat{N}_{a_1...a_4} \) that contain higher level elements of \( l_1 \) vanish in the rest frame and the precise coefficients that we have given will be derived below. Clearly, \( \hat{N}_{a_1...a_4} \) is one of the covariant operators resulting from the action of the boost on the annihilation operators in the rest frame as discussed just above.

We now consider if \( \hat{N}_{a_1...a_4} \) really does annihilate the covariant states of equation (6.4). Since the covariant fields are only defined up to an equivalence, or equivalently gauge symmetry, it simplifies the calculation considerably if we act on objects that are
gauge invariant. At the linearised level such objects are
\( \omega_{a,b_1 b_2} = -\partial_{b_1} h(a_2 b_2) + \partial_{b_2} h(a_1 b_1) + \partial_a h[b_1 b_2], \quad G_{a_1 \ldots a_4} = \partial_{[a_1} A_{a_2 a_3 a_4]}, \quad G_{a_1 \ldots a_7} = \partial_{[a_1} A_{a_2 \ldots a_7]}, \ldots \) \hspace{1cm} (6.11)

We find that
\[
\hat{N}^{b_1 b_2 b_3} G_{a_1 a_2 a_3 a_4} = -\frac{1}{4!} P_{a_1} \varepsilon_{a_2 a_3 a_4} b_1 \ldots b_4 c_1 c_2 c_3 c_4 E_{c_1 c_2 c_3 c_4} + 3 \delta^{[b_1 b_2}_{a_1 a_2} P_{a_3} E_{a_4]} b_3 b_4 \hspace{1cm} (6.12)
\]

and
\[
\hat{N}^{b_1 b_2 b_3} \omega_{a_1 a_2} = -4 \eta_{a_1 a_2} P_{a_2} E^{b_1 \ldots b_4} + 4 \eta_{a_1 a_2} P_{a_2} E^{b_1 \ldots b_4} - 90 P_{c_1} \eta_{c_2 a_2} \delta[a_1 a_2] E^{b_1 \ldots b_4]}
-90 P_{c_1} \eta_{c_2 a_2} \delta[a_1 a_2] E^{b_1 \ldots b_4]}

+ \partial_{a_1} \tilde{\Lambda}_{a_2 a_3} b_1 \ldots b_4 \hspace{1cm} (6.13)
\]

where
\[
E_{a_1 \ldots a_4} = G_{a_1 \ldots a_4} - \frac{1}{2} \frac{1}{4!} \varepsilon_{a_1 \ldots a_4} b_1 \ldots b_7 G_{b_1 \ldots b_7} = 0, \hspace{1cm} (6.14)
E_{a_1 b_1 b_2} = \omega_{a_1 b_1 b_2} = -\frac{1}{4} \varepsilon_{b_1 b_2 c_1 \ldots c_9} G_{c_1 \ldots c_9 a} = 0, \hspace{1cm} (6.15)
\]

and
\[
\partial_{a_1} \tilde{\Lambda}_{a_2 a_3} b_1 \ldots b_4 = P_{a_1} [12 \delta_{a_3} [b_1 G_{a_2 b_2 b_3 b_4}] - 12 \delta_{a_2} [b_1 G_{a_3 b_2 b_3 b_4}] + 3 \delta_{a_3} [b_1 P_{a_2} A_{b_2 b_3 b_4}]
-3 \delta_{a_2} [b_1 P_{a_2} A_{b_2 b_3 b_4}] + \frac{3}{8} \varepsilon_{b_1 \ldots b_4 c_1 \ldots c_7} P_{c_1} \eta_{c_2 a_2} A_{a_3 c_3 \ldots c_7} - \frac{3}{8} \varepsilon_{b_1 \ldots b_4 c_1 \ldots c_7} P_{c_1} \eta_{c_2 a_2} A_{a_2 c_3 \ldots c_7]}. \hspace{1cm} (6.16)
\]

We recognise equations (6.14) and (6.15) as the duality relations that occur in the covariant theory arising from the non-linear realisation of \( E_{11} \otimes s l_1 \) with local subalgebra \( I_c(E_{11})[1,2,3,4] \). Thus we find that \( N_{a_1 \ldots a_4} \) will indeed vanish on the covariant fields provided the duality relations hold. The last term in equation (6.13) is defined in equation (6.16) and it just corresponds to the fact that \( \omega_{a_1 b_1 b_2} \) is subject to local Lorentz transformations. We note that \( \tilde{\Lambda}_{a_2 a_3} b_1 \ldots b_4 \) is indeed of the form of a local Lorentz transformation since the \( b_1 \ldots b_4 \) indices can be contracted with a parameter with these indices.

We will now consider the operator
\[
\hat{N}^{b_1 b_2 b_3} = P_{b_1} J_{b_2 b_3} - \frac{1}{8!} \varepsilon_{b_1 b_2} e^{e_1 \ldots e_9} P_{c_1} S_{e_2 \ldots e_9} b_3 + \ldots \hspace{1cm} (6.17)
\]

where + \ldots means terms which contain higher order \( s l_1 \) generators. Acting on the fields of equation (2.9) in the rest frame this operator we find that it obviously vanishes except for \( \hat{N}^{-i_2 l_3} = p_{-} N_{i_2 l_3} \) where \( N_{i_2 l_3} \) is defined in equation (3.11). However, this generator also vanishes on the rest frame states and so \( \hat{N}^{b_1 b_2 b_3} \) vanishes on the rest frame states. As such we would expect it to vanish on the covariant fields of equation (6.4) One finds that
\[
\hat{N}^{b_1 b_2 b_3} \omega_{a_1 a_2 a_3} = \eta_{a_1 a_3} P_{a_2} E^{[b_1 b_2 b_3] - P_{a_2} \delta^{[b_1}_{a_2} E_{a_1]} b_2 b_3} - \eta_{a_1 a_2} P_{a_3} E^{[b_1 b_2 b_3]} \hspace{1cm} (6.18)
\]

22
covariant duality relations are precisely those that occur in the non-linear realisation of massless irreducible states of equation (2.9) in the rest frame lead to an infinite set of $N$ in natural way from the context of the Poincare algebra. For the Poincare algebra, that is, will take a different approach which generalises, in a natural way, the way spinors appear in the context of the Poincare algebra. This is consistent with the observation that the local subalgebra $I(1)$ annihilate the covariant fields one would expect their commutator with the generators of $I_c(E_{11})$ to give them back. At lowest order, and assuming we can find a missing factor of two, we find that

$$\partial_{a_1} \tilde{N}_{a_2a_3} b_1 b_2 b_3 = P_{a_1} \{ \epsilon^{[b_1 b_2]_{r_1\ldots r_9} b_3}_{r_1\ldots r_9} - \epsilon^{b_1 b_2} \hat{a}_{a_3} \hat{b}_3 \}$$

is an expected Lorentz transformation.

Since the operators $\tilde{N}_{a_1\ldots a_4}$ and $\tilde{N}_{a_1 a_2 a_3}$, and presumably their higher level analogues, annihilate the covariant fields one would expect their commutator with the generators of $I_c(E_{11})$ to give them back. At lowest order, and assuming we can find a missing factor of two, we find that

$$[S_{a_1 a_2 a_3}, \tilde{N}_{b_1 b_2 b_3 b_4}] = +9.3 \delta^{[a_1 a_2}_{b_1 b_2} {\tilde{N}_{a_3}}_{b_3 b_4} + \frac{1}{24} \epsilon_{b_1 b_2 b_3 b_4} a_{a_1 a_2 a_3 c_1 c_2 c_3 c_4} \tilde{N}_{c_1 c_2 c_3 c_4}$$

Similar results can be expected for $\tilde{N}_{a_1\ldots a_3}$ and the higher level generators of $I_c(E_{11})$. It would be interesting to compute the commutators of $\tilde{N}_{a_1\ldots a_4}$ and $\tilde{N}_{a_1\ldots a_3}$ with themselves. Given the above result one would expect the result to be proportional to $\tilde{N}$’s times $l_1$ generators. A result which would again annihilate the covariant fields.

The covariant fields are subject to the Casimir condition $L^2 \Psi = K^{AB} L_A L_B \Psi = 0$ as well as higher level conditions [15]. We would expect that the generators $\tilde{N}_{a_1\ldots a_4}$ and $\tilde{N}_{a_1\ldots a_3}$, will commute with these up to terms proportional to themselves. They should also be consistent with the gauge transformation of equation (6.5).

In this section we have found that the generators in the ideal $I$ that annihilate the massless irreducible states of equation (2.9) in the rest frame lead to an infinite set of operators which annihilate the massless irreducible representation when expressed in terms covariant fields provided the covariant fields obey an infinite set of duality relations. These covariant duality relations are precisely those that occur in the non-linear realisation of $I_c(E_{11}) \otimes s l_1$ with local subalgebra $I_c(E_{11})$. This is consistent with the observation that the duality conditions on the covariant fields are just covariant versions of the duality identities that exist in the original massless irreducible representation [7]. Thus the existence of the duality relations which contain the dynamics arise from the irreducible representation and in particular many of them correspond to the action of the affine action of $I_c(E_9)$ on the $I_c(E_8)$ representation of equation (2.24).

7. An alternative formulation of fermions in E theory

The spinors were introduced in E theory by hand in the sense that they do not follow in natural way from the $E_{11}$ algebra but started from the familiar gravitino and insisted that it carry a representation of $I_c(E_{11})$, which was constructed by hand. In this section we will take a different approach which generalises, in a natural way, the way spinors appear in the context of the Poincare algebra. For the Poincare algebra, that is, $SO(1,D-1) \otimes s T^D$
the vector representation \( T^D \) leads to a \( D \) dimensional spacetime on which \( SO(1, D - 1) \) acts. For each coordinate of the spacetime we introduce a \( \gamma^a \) matrix which act on a spinor.

In \( E \) theory we start from the algebra \( I_c(E_{11}) \otimes l_1 \) which is also a semi-direct product. The vector representation \( l_1 \) leads in the nonlinear realisation to the spacetime and we now introduce matrices \( \Gamma^A \)'s which are in one to one correspondence with the vector representation, hence their label \( A \). As a result they are also carry the indices of the spacetime coordinates in \( E \) theory. For example, in eleven dimensions we introduce the matrices

\[
\Gamma^A = \{ \Gamma^a, \Gamma^{a_1 a_2}, \Gamma^{a_1 ... a_5}, \Gamma^{a_1 ... a_7, b}, \Gamma^{a_1 ... a_8}, ... \} \tag{7.1}
\]

These matrices are to obey the equation

\[
\Gamma^A \Gamma^B + \Gamma^B \Gamma^A = 2K^{AB} \tag{7.2}
\]

where \( K^{AB} \) is the \( I_c(E_{11}) \) invariant tangent space metric. We now introduce a spinor \( \Psi \) which carries a representation of the \( \Gamma \) matrices and transforms under \( I_c(E_{11}) \) as

\[
\delta \Psi = U(S^\alpha)\Psi = -S^\alpha \Psi \tag{7.3}
\]

where \( U(S^\alpha) \) is the action of the generator \( S^\alpha \) and the matrix \( S^\alpha \) its effect. We require that

\[
[S^\alpha, \Gamma^A] = \Gamma^B(\tilde{D}^\alpha)_B^A \tag{7.4}
\]

where \( \tilde{D}^\alpha \equiv D^\alpha - D_\alpha \). By its definition the matrix of the vector representation appears in the commutator \( [R^\alpha, l_A] = -(D^\alpha)_A^B l_B \). We recall that \( S^\alpha = R^\alpha - R_\alpha \) and that \( (\tilde{D}^\alpha)_A^C K_{CB} \) is an antisymmetric matrix. As such the commutator of \( S^\alpha \) with the \( \Gamma^A \)'s leads to a transform under \( I_c(E_{11}) \) which is that of the vector representation.

A generalised Dirac equation is given by

\[
\Gamma^A \partial_A \Psi = 0 \tag{7.5}
\]

It is invariant under \( I_c(E_{11}) \) transformations as the derivatives \( \partial_A \equiv \frac{\partial}{\partial x^A} \) transforms as

\[
\delta(\partial_A) = -(\tilde{D}^\alpha)_A^B \partial_B \]

At level zero \( I_c(E_{11}) \) is just \( SO(1,10) \) and so at this level the above discussion just reduces to the standard discussion of the Dirac equation. The Gamma matrices corresponding to the higher level coordinates occur with derivatives with respect to these coordinates. Thus if we neglect the higher level derivatives this is just the familiar Dirac equation. We note that we can not take \( \Gamma^{a_1 a_2}, ... \) to be proportional to the standard gamma matrix \( \gamma^{a_1 a_2}, ... \) as this does not satisfy equation (7.2).

The above applies if we replace \( E_{11} \) by any Kac-Moody algebra and the vector representation by anyone of its representations. To give a simple example we consider the non-linear realisation of \( A_1^{1+++} \otimes l_1 \) which corresponds to gravity in four dimensions. At levels zero and one the generators of the vector representation are \( P_a \) and \( Z^{a} \) and so we introduce the gamma matrices \( \Gamma^A = \Gamma^a, \Gamma^a, ..., \) with \( a = 0, 1, ..., 3 \). The algebra \( I_c(A_1^{1+++}) \) contains the generators \( J_{ab} \) and \( S_{ab} \) at levels zero and one respectively. For an account
of this theory see reference [26] and the earlier references it contains. We can take the
gamma matrices to be
\[ \Gamma^a = \gamma^a \otimes I, \quad \tilde{\Gamma}^a = 2i\gamma^5 \otimes \gamma^a \] (7.6)
where \( \gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab} \) are the usual gamma matrices in four dimensions, \( (i\gamma^5)^2 = I \) and \( \gamma^5 \gamma^b + \gamma^b \gamma^5 = 0 \). Thus this spinor has eight components.

The vector representation of the algebra \( I_{c}(E_{11}) \) contains at level one the coordinates \( x^{a_{1}a_{2}} \) and so the corresponding spinor will carry a representation of the matrices \( \Gamma^{a_{1}a_{2}} \)
which obey
\[ \Gamma^{a_{1}a_{2}} \Gamma_{b_{1}b_{2}} + \Gamma_{b_{1}b_{2}} \Gamma^{a_{1}a_{2}} = 2\delta_{b_{1}b_{2}}^{a_{1}a_{2}} \] (7.7)

Such matrices are often required if we consider other very extended Kac-Moody algebras. If we have the index range in three dimensions then we can define \( \gamma^a = \frac{1}{2} \epsilon^{ab_1 b_2} \Gamma_{b_1 b_2} \) and we can use our usual representation of gamma matrices in three dimensions. So we require the spinor in this sector to have two components. If the index range is in four, for example, Euclidean dimensions then we can define
\[ \Gamma^{+a_{1}a_{2}} \Gamma_{b_{1}b_{2}} + \Gamma_{b_{1}b_{2}} \Gamma^{+a_{1}a_{2}} = 2\delta_{b_{1}b_{2}}^{a_{1}a_{2}} \] (7.8)

which obey the relations
\[ \frac{1}{2} \epsilon^{a_{1}a_{2}b_{1}b_{2}} \Gamma_{b_{1}b_{2}}^{\pm} = \pm \Gamma^{\pm a_{1}a_{2}}, \quad \text{and} \quad \{ \Gamma^{+a_{1}a_{2}}, \Gamma^{-b_{1}b_{2}} \} = \delta_{b_{1}b_{2}}^{a_{1}a_{2}} \] (7.9)

As such we can take the independent matrices to be \( \Gamma^{\pm 12}, \Gamma^{\pm 13} \) and \( \Gamma^{\pm 14} \) and a representation is formed by defining the vacuum \( |> \) to satisfy
\[ \Gamma^{-12}|> = 0 = \Gamma^{-13}|> = \Gamma^{-14}|> \] (7.10)

We can regard the matrices \( \Gamma^{+12}, \Gamma^{+13} \) and \( \Gamma^{+14} \) as creation operators acting on this vacuum, that is,
\[ |>, \Gamma^{+12}|> , \ldots, \Gamma^{+12}\Gamma^{+13}|>, \ldots \] (7.11)

Thus we have \((1 + 1)^3 = 8\) states, or spinor components, due to this sector of the spinor.

Let us also consider the algebra \( I_{c}(E_{8}) \otimes_s l_{1} \). The vector representation of \( E_{8} \) has
dimension 248 and as explained in section four this decomposes into the 120 plus 128
dimensional representations of \( I_{c}(E_{8}) = \text{SO}(16) \). We can take Gamma matrices corre-
sponding to both of these representations or just one of them. If we take just the spinor
128 dimensional representation the we should take the Gamma matrices
\[ \Gamma^i, \Gamma^{i_1 i_2}, \Gamma^{i_1 i_2 i_3}, \Gamma^{(ij)}, \quad \text{where} \quad i, j = 1, 2, \ldots, 8 \] (7.12)

where we have given them in the \( \text{SO}(8) \otimes \text{SO}(8) \) decomposition of section four. The
generalised Dirac equation would have the form
\[ (\Gamma^i \partial_i + \Gamma^{i_1 i_2} \partial_{i_1 i_2} + \ldots)\Psi = 0 \] (7.13)
The Gamma matrices can be written in block diagonal form using the analogue of the $\gamma_5$ matrices as above. The $\Gamma^i$ can be taken to be the usually gamma matrices in the first block and as their are eight of them this part of the spinor will have $2^4 = 16$ components. More generally the spinor which carries a representation can be found by defining annihilation and creation operators. We might expect 64 creation operators and 64 destruction operators and so the corresponding part of the spinor should have $2^{64}$ components.

The above considerations were at the linearised order but it is straightforward to generalise it to the full symmetries of the $E_{11} \otimes_{s} l_1$ by taking

$$\Gamma^A E^M_A (\partial_M + Q_M) \Psi = 0$$

where $E^A_M$ is the vielbein and $Q_M$ the connection found in the non-linear realisation.

We end this section with some speculative remarks. To account for the gravitino we can introduce the object $\Psi_A$ which is a spinor with the vector index $A$. We could take this to obey the on-shell conditions

$$\Gamma^B \partial_B \Psi_A = 0 = \Gamma^A \Psi_A = \partial^A \Psi_A$$

which does indeed contain the correct on-shell states if we restrict to the first part of the spinor $\Psi_A$. We leave it to the future to examine how this fits into the full theory.

We could also introduce a generalised supersymmetry generator $Q^A$ which is a generalised spinor and so with $I_c(E_{11})$ generators it has the relation

$$[S^\alpha, Q] = -S^\alpha Q$$

One might speculate that they obey an anti-commutator of the generic form

$$\{Q, Q\} = \Gamma^A l_A$$

How the generalised spinor introduced in this section fits in detail into $E$ theory and what are its mathematical properties for future study.

8 Discussion

In this paper we have analysed in detail the irreducible representation of $I_c(E_{11}) \otimes_{s} l_1$ corresponding to a massless point particle. The corresponding little algebra, $I_c(E_9) \otimes_{s} l_1$, is an infinite dimensional Lie algebra and as a result one may expect it to contain an infinite number of degrees of freedom. However, the fields are subject to an infinite number of duality relations which are preserved by $I_c(E_9)$ and these reduce the number of independent degrees of freedom to be just 128 which belong to the spinor representation of $I_c(E_8) = SO(16)$. These can be taken to be $h_{ij}$ ($h^{ij} = 0$) and $A_{i_1 i_2 j_3}$ where $i, j = 2, \ldots, 10$. They are indeed the bosonic degrees of freedom of eleven dimensional supergravity. The infinite number of duality equations relate all the other fields in the representation to these fields. A consequence of these duality relations is that the irreducible representation is annihilated by an infinite number of generators of $I_c(E_9)$ which form a subalgebra $I$ which is an ideal. The Lie algebra $\frac{I_c(E_9)}{I}$ is $SO(16)$.
The 128 bosonic independent degrees of freedom belong to the $(8_v, 8_v) \oplus (8_c, 8_s)$ representations when decomposed into the subalgebra $\text{SO}(8) \times \text{SO}(8)$ subalgebra of $\text{SO}(16)$. These two representations contain the fields $(h_{i'j'}, A_{i'j'...i''})$ and $(A_{i'i''j'}^{i''...i''}, h_{i'i''...i''}^{j'})$ respectively where $i', j' = 3, \ldots, 10$. The $\text{SO}(16)$ transformations not in the $\text{SO}(8) \times \text{SO}(8)$ subalgebra change these representations into each other. The remaining fields which appear in the irreducible representation of $I_c(E_9)$ are affine copies of these 128 fields and are related by duality relations to the fields in the $(8_v, 8_v) \oplus (8_c, 8_s)$ representations. Like all such irreducible representations we can formulate the massless irreducible representation in a covariant manner. The infinite number of duality relations become covariant duality equations which contain the dynamics and one finds a corresponding set of operators that annihilate the representation.

The dynamics that results from the non-linear realisation of $E_{11} \otimes sl_1$ with local subalgebra $I_c(E_{11})$ has been found at low levels and the resulting equations of motion agree precisely with those of maximal supergravity if one discards the dependence on the spacetime coordinates beyond those usually considered. The dynamics appears through an infinite set of duality equations that relate an infinite number of the higher level fields to the graviton and three form and it is by taking space time derivatives of these that one can find the standard equations of motion of maximal supergravity. The duality relations that arise in the covariant formulation of the massless irreducible of $I_c(E_{11}) \otimes l_1$ in the covariant formulation are essentially those that arise in the non-linear realisation of $E_{11} \otimes l_1$ with local subalgebra $I_c(E_{11})$ at the linearised level. This provides strong support for the strongly suspected fact that the only degrees of freedom contained in the non-linear realisation are the 128 bosonic degrees of freedom of supergravity. We also see that the infinite number of duality relations that appear in the non-linear realisation are a consequence of the duality relations that occur in the irreducible representation and this allows us to predict the structure of the duality relations in the former theory. In the non-linear realisation there are also higher level fields which have blocks of ten, or eleven, indices and so these fields do not appear in the irreducible representation. These fields obey equations predicted by the non-linear realisation and they lead to the gauged supergravities.

In the irreducible representations in the E theory approach the bosonic and fermionic degrees of freedom appear in a unified way. They belong to the two different spinor representations of $I_c(E_8) = \text{SO}(16)$ which are associated with the two nodes at the end of the $\text{SO}(16)$ Dynkin diagram. Thus swapping these two nodes results in swapping bosons and fermions. One puzzle with E theory is the way it leads to predictions that were usually seen as a result of supersymmetry. Examples are the appearance of the two and five brane charges and the BPS conditions [15]. Supersymmetry was discovered, at least from the Russian viewpoint, by demanding that internal and spacetime symmetries were contained in the same symmetry algebra. This required the introduction of the supersymmetry generators. However, $E_{11}$ achieves the same objective as it does contain the symmetries of spacetime, such as Lorentz symmetry, and also internal symmetries such as the internal exceptional symmetries of the maximal supergravities.

In this paper we have concentrated on the massless irreducible representation of $E_{11} \otimes l_1$ which corresponds to the maximal supergravity theories. The vector representation of
$E_{11}$ contains the brane charges and by taking these to be non-zero we will find other irreducible representations of $E_{11} \otimes s l_1$ corresponding to branes. In a future paper we hope to examine some of these representations in detail and make the connection to the the brane dynamics as also appears in the context of E theory, see for example references [27].

The interesting paper [28] also remarks on the similar properties of bosons and fermions although from a different viewpoint. It noticed that the SO(9) representations to which bosonic $(44 \oplus 84)$ and fermionic $(128)$ degrees of freedom emerge in a very natural way as a solution of the equation $K \Psi = 0$. In this equation $K$ is the Kostant operator which is of the form $K = \sum_a T_a \gamma^a$ where $\gamma^a$ are gamma matrices and $T^a$ are generators of $F_4$ that belong to the coset $\frac{E_8}{SO(9)}$. The algebra $F_4$, dimension 52, can be found from the algebra SO(9), dimension 36, by adding to the latter the 16 generators that belong to the spinor representation of SO(16).

One can speculate that this picture can be generalised to incorporate the features of the irreducible representations found in this paper. The bosonic degrees of freedom belong to one spinor representation of SO(16), while the fermionic degrees of freedom belong to the other spinor representation, both of which have dimension 128. The algebra $E_{8}$ emerges from the SO(16) algebra if we add to the 128 generators belonging to the spinor representation of SO(16) to the 120 generators of SO(16). We can then consider the coset of $\frac{E_8}{SO(16)}$. The corresponding Kostant operator would consist of 128 Gamma matrices multiplied by the generators in the coset. Could it be that the solutions of the equation $K \Psi = 0$ contain the two 128 dimensional spinor representation of SO(16)? We can also wonder what are the higher spin solutions? Speculating even further we can suppose that some of these 128 Gamma matrices can be taken to be the supercharges similar to what took place in the discussion of the $N = 2$ hypermultiplet given in reference [28]. We hope to study these matter further and how they might fit into E theory.

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