FIXED-POINT LOCALIZATION FOR $\mathbb{RP}^{2m} \subset \mathbb{CP}^{2m}$

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Abstract. We derive a fixed-point formula for integrals on moduli spaces of stable maps to projective spaces of even dimension. This gives a formula for the equivariant open Gromov-Witten invariants of $\mathbb{RP}^{2m} \subset \mathbb{CP}^{2m}$, and the structure constants of the equivariant $A_\infty$ algebra $\text{End}_{Fuk(\mathbb{CP}^{2m})}(\mathbb{RP}^{2m})$ with bulk deformations. The formula involves contributions from Givental’s correlators for the closed theory and the descendent integrals of discs, and specializes to give a new expression for the Welschinger count of real rational curves in the plane passing through some real and conjugation invariant pairs of points.

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References

1. Introduction

1.1. Overview. In this paper we derive a fixed-point formula for the equivariant open Gromov-Witten theory of $\mathbb{RP}^{2m} \subset \mathbb{CP}^{2m}$, as a special case of a more general integration technique on moduli spaces of discs.

The action of a torus group $T$ on the moduli space of closed stable maps $\overline{M}_{0,l}(\mathbb{CP}^n, d)$ can be used to reduce the computation of an integral on $\overline{M}_{0,l}(\mathbb{CP}^n, d)$ to tractable calculations near the fixed points of the $T$-action \[4, 15\]. This technique has become a general, powerful tool which has found many applications.

Fixed-point localization has also been applied to open Gromov-Witten theory, which is concerned with integrals on the moduli spaces parameterizing stable maps of discs. Some notable examples include Katz and Liu \[13\], Pandharipande, Solomon and Walcher \[19\], Georgieva and Zinger \[7\] and Tehrani and Zinger \[26\]. Typically, moduli spaces of discs have boundary, so the geometric setup in these works is constrained by the requirement that the boundary contributions vanish.

Our approach is different, in that the boundary contributions are not required to vanish but are rather canceled by boundary contributions from integrals on additional spaces. One novel feature is that this allows one to introduce boundary constraints. The construction of these additional spaces is closely related to the homological perturbation lemma for $A_\infty$ algebras.

On the same note, let $\text{End}_{\text{Fuk}^c(\mathbb{CP}^n)}(\mathbb{RP}^{2m})$ denote the quantum deformation of the De Rham differential graded algebra $(\Omega(\mathbb{RP}^{2m}), d, \wedge)$ over the Novikov ring with bulk and equivariant deformation parameters included. This algebra admits a natural rigid minimal model and our formula computes the structure constants of this model explicitly.

1.2. Equivariant open Gromov-Witten invariants. Fix some $m \geq 1$ and let $(X, L) = (\mathbb{CP}^{2m}, \mathbb{RP}^{2m})$. We review the definition of equivariant open Gromov-Witten invariants for $L \subset X$ from \[30\]. See \[3.1\] for more details.

For $b = (k, l, \beta) \in \mathbb{Z}^3_{\geq 0}$ such that $k + 2l + 3\beta$ is an odd integer $\geq 3$, we construct a sequence of orbifolds with corners and maps $$(\tilde{\mathcal{M}}_b^r \to \mathcal{M}_b^r \to \mathcal{M}_b^r)_{r \geq 0}$$

We have $$\tilde{\mathcal{M}}_b^0 = \mathcal{M}_b^0 = \tilde{\mathcal{M}}_b^0 = \overline{M}_{0,k,l}(X, L, \beta),$$
the moduli space of stable discs of degree $\beta \in H_2(X, L)$ with $k$ boundary marked points and $l$ interior marked points. More generally, $\mathcal{M}_b^r$ is a disjoint union of products of moduli spaces, modeled on trees with $r$ oriented edges. The endpoints

\[1\] The last two works study real invariants, but these are closely related to disc invariants, as we will see.
of the edges correspond to $2r$ additional boundary marked points. $\tilde{M}_b^r$ is obtained from $M_b^r$ by forgetting the markings corresponding to the tails of the edges, and $\widetilde{M}_b^r$ is obtained from $M_b^r$ by blowing up the locus where two endpoints of an edge map to the same point in $L$. All of the spaces and maps come equipped with a natural action of the rank $m$ torus $T = U(1)^m$.

We use these spaces to define an integration map

$$\int_b : \Omega_b \rightarrow \mathbb{R} [\lambda_1, \ldots, \lambda_m]$$

Here $(\Omega_b, D)$ is the complex of extended forms: an extended form is specified by a sequence $(\tilde{\omega}_r)_{r \geq 0}$, where each $\tilde{\omega}_r$ is a $T$-invariant $\mathbb{R}[\lambda_1, \ldots, \lambda_m]$-valued form on $\tilde{M}_b^r$, subject to some coherence conditions. The Cartan-Weil differential

$$D = d - \sum_{j=1}^{m} \lambda_j \xi_j$$

acts on $\Omega_b$ level-wise. By definition,

$$\int_b \omega = \sum_{r \geq 0} \frac{1}{r!} \int_{\tilde{M}_b^r} \tilde{\omega}_r \cdot \Lambda^{\otimes r}$$

where $\Lambda$ is an equivariant form on the diagonal blow up

$$\tilde{L} \times L = (L \times L \setminus \Delta) \bigcup_{S(N(\Delta) \times (0, \infty)} S(N(\Delta)) \times [0, \infty),$$

representing a homotopy retraction, and $\tilde{\omega}_r$ is the pullback of $\tilde{\omega}_r$ to $\tilde{M}_b^r$. The main result about this construction is Stokes’ theorem,

$$\int_b D \omega = 0.$$

For $k, \beta \in \mathbb{Z}_{\geq 0}$ and $l = (l_0, \ldots, l_{2m}) \in \mathbb{Z}_{\geq 0}^{2m+1}$ with $\sum l = l$, we define the equivariant open Gromov-Witten invariant

$$I_m(k, l, \beta) = \int_b \omega,$$

where $\omega$ is given by pulling back equivariant closed forms along the interior and boundary evaluation maps,

$$\tilde{\omega}_r = \prod_{0 \leq d \leq 2m} \prod_{1 \leq s \leq [d+1]} (\text{ev}_s^b)^* \mathcal{H}^d \cdot \prod_{1 \leq s \leq k} (\text{ev}_s^r)^* p_0^L.$$

Here we’ve set $l[:d] := (\sum_{j=0}^{d-1} l_j)$ and let $\mathcal{H}$ and $p_0^L$ denote equivariant forms representing the hyperplane class in $\mathbb{CP}^{2m}$ and the point class in $\mathbb{RP}^{2m}$, respectively (see 2.1).

If we assume $m = 1$, $\deg \tilde{\omega} = \dim \tilde{M}_b^0$, and we take $\mathcal{H}$ to be conjugation invariant, we can show the contribution of $\tilde{M}_b^r$ for $r > 0$ vanishes, and the contribution of $\tilde{M}_b^0$ matches the invariants defined by Solomon [24]. By a reflection principle argument [21], these invariants are shown to be twice Welschinger’s signed count [27] of real rational planar curves through a generic configuration of $k$ real points and $l$ conjugate pairs of complex points. In sum, we have

$$W_{k,l} = \frac{1}{2} \sum_{l_0} (k, (0,0,l), \frac{k + 2l + 1}{3},$$

so the Welschinger invariants are instances of the open Gromov-Witten invariants.
1.3. Statement of main results. Our geometric result can be stated as follows.

**Theorem 1.** Let \( \omega \in \Omega_b \) satisfy \( D\omega = 0 \). Then

\[
\int_b \omega = \sum_{\phi \in W} \frac{\xi_\phi}{\text{Aut} \, \phi} \int_{\tilde{F}_\phi} e(\tilde{N}_\phi)
\]

Here \( \bigcup_{\phi \in W} \tilde{F}_\phi \) is a finite cover of the \( T \)-fixed points \( C^T \) of a clopen component \( C \subset \bigcup_{r=0} \mathcal{M}^r_b \) parameterizing configurations where all the edges connect an odd degree disc to an even degree disc. \( e(\tilde{N}_\phi) \) is an equivariant Euler form for the normal bundle to \( \tilde{F}_\phi \to \mathcal{M}^r_{b,r}(\phi) \), subject to certain boundary conditions. \( \xi_\phi \) represents the contribution of \( \Lambda_{scr}^r \). See Theorem \([29]\) and Remark \([31]\) for more details.

We turn now to a purely algebraic statement, succinctly expressing the equivariant open Gromov-Witten invariants in terms of known generating functions. Hereafter, when a formal variable is introduced we use a number in overbrace to denote its \( \mathbb{Z}_r \)-grading. Let \( \mathbb{C} [\lambda] = \mathbb{C} \left[ \frac{2}{\lambda_1}, ..., \frac{2}{\lambda_m} \right] \) denote the equivariant \( T \)-cohomology of a point. Let

\[
R = \mathbb{C} [\lambda] \left[ \frac{2-2i}{\eta_0, ..., \eta_l, ..., \eta_{2m}} \right].
\]

We define a generating function

\[
F_{OGW} \in R \left[ \frac{2m+1}{q}, \frac{1-2m}{s} \right]
\]

for the equivariant open Gromov-Witten invariants by

\[
F_{OGW} = \sum_{k, l, \beta} I(k, l, \beta) \left( \sqrt{-1} \right)^{(m+1)w_1(\beta)} q^\beta \eta_0^l s^k k!
\]

where sum ranges over all \( k, \beta \in \mathbb{Z}_{>0} \) and \( l \in \mathbb{Z}_{2m+1} \), \( \eta_0^l := \frac{\eta^{l_0, ..., \eta^{2m}_{l_0, ..., l_{2m}}} }{l_0! l_{2m}!} \), and

\[
w_1(\beta) = \beta \mod 2 \in \{0,1\}.
\]

\( F_{OGW} \) is supported in degree \( 3-2m \).

Let

\[
F_0^a \in \mathbb{R} \left[ \frac{1}{s_0, t_0, t_1, ..., t_i}, \frac{2-2i}{s, s_0, t_0, t_1, ...} \right]
\]

denote the generating function for the descending integrals of discs, developed in \([21]\) (we review the definition in \([22]\)). We introduce auxiliary formal variables \( u \) and \( \nu_{a,d} \) for \( a = 1, ..., 2m \) and \( d > 0 \) a positive integer.

We set \( \alpha_0 = 0 \) and, for \( 1 \leq i \leq m \), \( \alpha_i = \lambda_i \) and \( \alpha_{2m+1-i} = -\lambda_i \). We denote by \( Z_a(\mathbb{C}) \) the value of the quantum correlator with constraints, see \([22]\) which encapsulates the Gromov-Witten theory of closed curves in \( \mathbb{C} \mathbb{P}^{2m} \).

**Theorem 2.** Let

\[
E \in \mathbb{R} \left[ \left[ u^{-1}, u, \nu_{a,d}, q, \eta_0, ..., \eta_{2m}, s, s_0, t_0, t_1, ... \right] \right]
\]
be defined by

\[ E = \exp \left( \sum_{a \neq 0} \sum_{d \geq 0} Z_a \left( \frac{\alpha_a}{d} \right) \partial_{\nu_a,d} \right) \]

\[
\exp \left( \left( -1 \right)^{1+m} \sum_{a \neq 0} \sum_{d \text{ odd}} w^d \nu_a,d \frac{d^{d-1}}{d! (2\alpha_a)^d} \exp \left( \frac{d}{2} \sum_{i=0}^{\infty} \prod_{0 \leq l < d} \frac{d}{d - \alpha_a} \left( \frac{2d - d}{d} \alpha_a - \alpha_b \right) \right) \right)
\]

\[
\exp \left( \sum_{a \neq 0} \sum_{d \geq 0} q^{2d} \nu_a,d \frac{(-1)^d d^{2d-1} \left( \prod_{0 \leq l < d} \alpha_a \right) \left( \sum_{i \in \mathbb{Z}_{\geq 0}} \left( -\frac{d}{\alpha_a} \right)^{i+1} \partial_{\lambda_i} \right) \right)}{(2d)^2 \alpha_a^d \prod_{0 \leq l < d} b \alpha_a,0 \left( \frac{2d - d}{d} \alpha_a - \alpha_b \right) \right) \right)
\]

\[
\exp \left( \frac{u}{\lambda_1 \cdots \lambda_m} \exp \left( s \cdot \lambda_1 \cdots \lambda_m \cdot \partial_{\nu_a,d} \right) \exp \left( \eta_0 \partial_{\nu_a,d} \right) F_0 \right)
\]

Here \( \sum_{a \neq 0} \) means \( a \) ranges over \( 1, \ldots, 2m \).

The OGW-generating function is given by

\[ F_{\text{OGW}} = \partial_a |_{a=0} \log \left( E|_{\nu_a,d=0,\nu_d=0,\ldots,\nu_t=0,...} \right). \]

Here the logarithm is defined by Taylor expansion around 1:

\[ \log x = (x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \cdots. \]

The proof of this theorem is given in [3.4].

The formulae in the theorem are nothing more than a compact description of a diagrammatic sum; the four factors comprising \( E \) generate four types of labeled vertices, with the inner exponents generating edges of various types incident to these vertices. The substitution \( E|_{\nu_a,d=0,\nu_d=0,\ldots,\nu_t=0,...} \) places some restrictions on the labels, and \( \partial_a |_{a=0} \log (\cdot) \) replaces the sum over all diagrams with a sum over tree diagrams.

As a special case, this gives a combinatorial expression for the Solomon-Welschinger invariants [5]. We’ve used a computer to compute all of the invariants with \( k + 2l \leq 17 \) and checked the results agree with values computed using WDVV, see [9]. For example, we have

\[ W_{17,0} = 2,845,440. \]

Another validation for this formula comes from the obvious fact

\[ \int_b \omega = 0 \text{ when } \deg \omega < \dim X_b, \]

though the individual fixed-point contributions to Theorem 1 need not vanish. In [13] we show that such relations are in fact sufficient to determine the intersection theory of discs. Simple examples of both types of sanity checks are given in [8].

1.4. The failure of naive localization. Corollary 14 reproves the classical fixed-point formula of Atiyah and Bott [1]. It shows that if \( \mathcal{X} \) is a closed orbifold and \( \omega \) is a closed equivariant form, then

\[ \int_{\mathcal{X}} \omega = \int_{F} \omega|_F e(N), \]

where \( F \subset \mathcal{X} \) denotes the fixed-point suborbifold and \( e(N) \) is the equivariant Euler to the normal bundle of \( N \).
Fixed-Point Localization for $\mathbb{R}^{2m} \subset \mathbb{C} \mathbb{P}^{2m}$

Set $m = 1$, so $(X, L) = (\mathbb{C} \mathbb{P}^2, \mathbb{R} \mathbb{P}^2)$, and consider

$$1 = W_{2,0} = \frac{1}{2} I_1 (2, (0, 0, 0), 1),$$

the number of lines through a pair of distinct points in the plane. This invariant is given by an integral on $\overline{\mathcal{M}}_{0,2,0} (1)$. $S^1$ acts on $(\mathbb{C} \mathbb{P}^2, \mathbb{R} \mathbb{P}^2)$ by

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{pmatrix}
$$

and thus on $\overline{\mathcal{M}}_{0,2,0} (1)$, but it is easy to see that $\overline{\mathcal{M}}_{0,2,0} (1)$ has no fixed points, which would seem to imply $W_{2,0}$ vanishes. In reality, the fixed points fail to “see” the line because

$$\partial \overline{\mathcal{M}}_{0,2,0} (1) = \overline{\mathcal{M}}_{0,1,0} (1) \times L \overline{\mathcal{M}}_{0,0,3} (0) \neq \emptyset.$$ 

Theorem 1 recovers the correct answer by incorporating the fixed points of the resolution $\mathcal{M}^1 = \overline{\mathcal{M}}_{0,0,0} (1) \times \overline{\mathcal{M}}_{0,5,0} (0)$ of $\partial \overline{\mathcal{M}}_{0,2,0} (1)$. These fixed points can be schematically represented by

$$
(1) \rightarrow (0):
$$

where

- $(1)$ depicts a fixed point of $\overline{\mathcal{M}}_{0,0,0} (1)$, represented by a degree one disc with no markings, whose boundary maps to the line at infinity $\mathbb{R} \mathbb{P}^1 \subset L$ (there are two such fixed-points)
- $\rightarrow$ depicts an oriented edge, which gives the propagator factor $\xi_\phi$ in $(\phi)$
- $(0)$ depicts a fixed point of $\overline{\mathcal{M}}_{0,0,3} (0)$, represented by a degree zero disc carrying the two original markings as well as a marking for the head of the edge $\rightarrow$. Such a disc must map to the origin $p_0 \in L$.

See Example 39 for the full computation. More generally, even though we can choose $\omega$ so

$$W_{k,l} = \int_b \omega = \int_{\mathcal{M}^1_b} \omega_0,$$

with contributions only from the $r = 0$ term in 1, the fixed-point formula (10) for $W_{k,l}$ will involve contributions from $\mathcal{M}^1_b$ with $r$ arbitrarily large, representing contributions of the $r$-codimension corners $\partial^r \overline{\mathcal{M}}_{0,k,1} (\beta)$.

1.5. **Outline of proof.** The structure of the paper is as follows. In Section 2 we review and adapt to our needs some related theories. In particular we discuss quantum constraint correlators and the intersection theory of discs.

In Section 3 we discuss the resolution fixed points, and use this to formulate Theorem 1 more precisely as Theorem 29. We then compute some examples, and deduce the generating function statement, Theorem 4.

The next three sections are devoted to the proof of Theorem 1. Consider

$$I (t) = \int_b e^{t \cal{D} \bar{\eta}} \omega = \sum_{r \geq 0} \frac{1}{r!} \int_{\mathcal{M}^1_b} e^{t \cal{D} \bar{\eta}} \bar{\omega}_r \Lambda^r \hat{\eta}.$$ 

On the one hand, by Stokes’ theorem 31, this is independent of $t$. On the other hand, for suitable $\eta$ we show that

$$\int_b \omega = I (0) = \lim_{t \to \infty} I (t) = \sum_{r \geq 0} \frac{1}{r!} \int_{\mathcal{M}^1_b} e^{t \cal{D} \bar{\eta} (2)} \cdot \hat{\rho}^*(\bar{\omega}_r \otimes \Lambda^r),$$

where
Here $\tilde{N}_b^r$ is a blow up of the normal bundle $N_b^r$ to $F_b^r \to M_b^r$, where $F_b^r = (M_b^r)^{\mathbb{T}} \times M_b^r$ is the inverse image of the fixed points of $M_b^r$. Using a tubular neighborhood, $\tilde{N}_b^r$ becomes an open subset of $\tilde{M}_b^r$, $\tilde{\eta}_b^{(2)}$ is the quadratic part of $\tilde{\eta}_b$, and $\tilde{\rho}$ is a certain retraction. The computation of the limit $\tilde{\rho}$ uses steepest descent analysis, but the singularities of $\Lambda$ demand special attention. We discuss a fairly general formalism for such computations in Section 4 (as a special case, we obtain the Atiyah-Bott localization formula for closed orbifolds). The derivation of (10) is carried out in Section 5.

The final step in the proof of Theorem 1 is carried out in Section 6. Note that (10) already reduces $\int_b \omega$ to local quantities near the fixed points of $\bigcup_r M_b^r$, but the integrals are hard to evaluate in this form (see Section 4 for a detailed explanation). This is related to the singularities of $(\tilde{\rho})^{*} \Lambda^g$ near the diagonal, which we now discuss. Consider a fixed-point component

$$\tilde{F}_T = \prod_{v \in T_0} (\tilde{M}_b^r)^{\mathbb{T}_{v_0,l_v}} (\beta_v)^{\mathbb{T}} \subset (\tilde{M}_b^r)^{\mathbb{T}}$$

corresponding to a tree $T = (T_0, T_1)$ with vertex set $T_0$ and $r$ oriented edges $T_1$. Consider $(v_{\text{tail}}, v_{\text{head}}) \in T_1$. We must have $\beta_{v_{\text{head}}} = 0$ mod 2 whereas $\beta_{v_{\text{tail}}}$ may be odd or even, so

$$T_1 = T_{\text{odd-even}} \bigsqcup T_{\text{even-even}}.$$  

The two boundary markings corresponding to an odd-even edge $e \in T_{\text{odd-even}}$ are separated in $L$. In contrast, both endpoints of an even-even edge map to the unique real fixed point $p_0 \in L$. It follows that

$$\pm \tilde{\rho}^{*} \Lambda^g = (\tilde{\rho}^{*} \Lambda^g_{T_{\text{odd-even}}}) \boxtimes (\tilde{\rho}^{*} \Lambda^g_{T_{\text{even-even}}}) = R \boxtimes S$$

where $R = \tilde{\rho}^{*} \Lambda^g_{T_{\text{odd-even}}}$ is regular, in the sense that it is pulled back along the blow up map $\tilde{N}_b^r \to N_b^r$, while

$$S = \tilde{\rho}^{*} \Lambda^g_{T_{\text{even-even}}} = \theta_0^{g_{T_{\text{even-even}}}}$$

is singular: $\theta_0$ is an angular form for

$$S(T_{p_0}, L) \simeq S(N_{\Delta/L \times L} |_{p_0}),$$

giving the rescaled limit of $\Lambda$ near $(p_0, p_0) \in L \times L$.

Every $e \in T_{\text{even-even}}$ corresponds to a boundary component $\partial^T \tilde{F}_T$, of some other fixed-point component $F_{T'}$. $T'$ is obtained from $T$ by contracting $e$. Let us fix some tree $T$ with only odd-even edges, and consider the total contribution $C_T$ to (11) of the fixed-point components $\{F_T \mid \text{cnt } T = T\}$ of trees $T$ whose odd-even skeleton, obtained by contracting all even-even edges, is $T$. The central result of Section 6, Theorem 57, says that $C_T$ can be regularized. Roughly speaking, the integrals on $\tilde{N}_b^r|_{\tilde{F}_T}$ are replaced by integrals on

$$\partial^T \tilde{F}_T \times [0, 1]^{T_{\text{even-even}}},$$

where $\partial^T \tilde{F}_T$ is the codimension $|T_{\text{even-even}}|$ corner of $\tilde{F}_T$ corresponding to $T$. Thus we find that

$$C_T = \int_{P_T} \Upsilon$$

where the domain

$$P_T = \bigcup_{c \leq 0} \partial^c \tilde{F}_T \times \text{Sym}(c) [0, 1]^c.$$
is obtained by gluing $\partial \tilde{F}_T \times [0,1]$ to $\tilde{F}_T$ along $\partial \tilde{F}_T \times \{0\}$, then filling in the missing squares at the codimension two corners, then the missing cubes at codimension 3 corners, and so on. The integrand $\Upsilon$ satisfies boundary conditions, making it a well-defined cohomology class relative to $\partial P_T$. In particular, $C_T$ is invariant under $\omega \to \omega + Dc$ and the choice of $\Lambda$. Theorem 44 allows us to replace the sum $\sum_T C_T$ over odd-even trees $\sum_T$ appearing on the right hand side of (10), with the right hand side of (10), and conclude the proof of Theorem 1.

Some definitions and basic results about orbifolds with corners are given in the Appendix.

1.6. Quantum $A_\infty$ deformations and the superpotential. Though we will not use the $A_\infty$ formalism in this paper, it is closely related to what we do, and in this subsection we briefly outline this relationship. The equivariant open Gromov-Witten invariants $\{I_m(k,l,\beta)\}$ encode the structure constants of a natural, rigid minimal model for the Fukaya endomorphism algebra of $\mathbb{RP}^{2m} \subset \mathbb{CP}^{2m}$ with bulk and equivariant deformations. The question of whether or not the Calabi-Yau structure is given by the standard pairing on this minimal model requires further study. We will mention a condition on $\Lambda$ that should guarantee this.

Consider the $(\mathbb{Z} \oplus \mathbb{Z}/2)$-graded $R$-module

$$C = \frac{(\mathbb{Z},0)}{\Omega} \left[ \mathbb{RP}^{2m}, \mathbb{R} \otimes \mathbb{Z} \oplus \text{Or}(\mathbb{RP}^{2m}) \right].$$

so the $\mathbb{Z}$ component of the grading, denoted $\text{cd} \ x$, is the total grading of differential forms with values in the graded ring $R = \mathbb{R}[\lambda][[\eta]]$, where now $\deg \lambda_i = (2,0)$ and $\deg \eta_i = (2-2i,0)$.

$C$ is equipped with a pairing

$$(x, y) = (-1)^{\epsilon(x,y)} \int x \wedge y, \quad \epsilon(x, y) = \text{cd} x + \text{cd} x \cdot \text{cd} y.$$ 

For $(k,\beta) \in \mathbb{Z}_+^2 \setminus \{1,0\}$ we define an operation

$$m_{k,\beta} : C^\otimes k \to C[2-k - \mu(\beta), \mu(\beta) \mod 2], \quad \mu(\beta) = (2m+1) \cdot \beta$$

by

$$m_{k,\beta}(x_1, \ldots, x_k) := \sum_{\ell \in \mathbb{Z}_{+}^{2m+1}} \frac{\eta^\ell}{\ell!} \left(\sqrt{-1}\right)^\ast \times \,
\ev^\ast_{0, \tau} (k, \Sigma t, \beta) \left( \ev^\ast_0 x_1 \ev^\ast_0 x_2 \cdots \ev^\ast_0 x_k \prod_{d=0}^{2m} \prod_{0 \leq d \leq d+1} \ev^\ast_0 H^d \right)$$

where $\sqrt{-1} = (-1)^{\sum_{t \in \mathbb{Z}_+} (\text{cd} x_t + 1)}$, $\beta = 0 \mod 2$ if $\beta = 1 \mod 2$, $H$ is an equivariant hyperplane form $\Xi_{\tau}$ and the pushforward is defined using the local system map $\tilde{\eta}^0_{(k, \Sigma t, \beta)}$. We set

$$m_{1,0} = D = d - \sum_{j=1}^m \lambda_j ^{\xi_j},$$
Note (11) induces a pairing on $HC$. A unital cyclic twisted $A_\infty$ algebra structure on $HC$ is specified by a collection of operations
\[ \{ \mu_{k,\beta} : HC \otimes^k \to HC [2 - k - \mu(\beta), \mu(\beta) \mod 2] \} . \]
Following Fukaya [6], we define the superpotential
\[ \Phi (\{ \mu_{k,\beta} \}) := \sum_{k \geq 0, \beta \geq 0} \frac{1}{k + 1} \left( \mu_{k,\beta} \left( p^L_0, \ldots, p^L_k \right), p^L_0 \right) \eta^{s^2} s^{k+1} . \]
\[ \Phi (\{ \mu_{k,\beta} \}) \in s \cdot R[[q,s]] \] is supported in degree $3 - 2m$.

**Proposition 3.** (a) The superpotential induces a bijection
\[ \{ \text{unital cyclic } A_\infty \text{ structures } \{ \mu_{k,\beta} \} \} \to \left\{ \Phi \in s \cdot R[[q,s]] \mid \Phi \mod s^2 \in m \text{ and } \deg \Phi = 3 - 2m \right\} . \]

(b) if $\{ f_{k,\beta} : (HC, \{ \mu_{k,\beta} \}) \to (HC, \{ \mu'_{k,\beta} \}) \}$ is a unital cyclic isomorphism between two such algebras, then $f_{k,\beta} = \text{id}$.

**Proof.** This follows from elementary considerations which use the definition of cyclic symmetry and unitality directly.

In other words, the homotopy category of unital cyclic $A_\infty$ algebra structures on $HC$ is discrete and parameterized by the superpotential, which is just the generating function for the nontrivial structure constants. These structure constants are not subject to any constraints. Here we consider the structure constants which are forced to vanish, as well as $m_{2,0}(p^L_0, 1) = m_{2,0}(1, p^L_0) = p^L_0$, to be trivial.

Consider the unital cyclic minimal model $(HC, \{ m^{\text{can}}_{k,\beta} \})$, which is homotopy equivalent to $(C, \{ m_{k,\beta} \})$ through a unital cyclic morphism. We can also consider the unital cyclic $A_\infty$ algebra $(HC, \{ m^{\text{can}}_{k,\beta} \})$ whose structure constants are given by the open Gromov-Witten invariants, so
\[ \Phi (HC, \{ m^{\text{can}}_{k,\beta} \}) = F^{\text{OWG}} = F_{\text{OWG}} - (F_{\text{OWG}} \mod s) . \]

**Proposition 4.** There exists a unital (but not necessarily cyclic) $A_\infty$ isomorphism
\[ (HC, m^{\text{can}}_{k,\beta}) \xrightarrow{\text{f}} (HC, m^{\text{can}}_{k,\beta}) . \]

Before we prove this proposition, we make a few remarks. First, note that $\{ m^{\text{can}}_{k,\beta} \}$ is well-defined independent of all choices: invariance on choice of the forms $p_0^L$ and $\mathcal{H}$ follows from Stokes’ theorem [34], and the more subtle invariance on $\Lambda$ follows from the fixed-point formula (this is Corollary [30]).

---

2 This is defined similarly to the case $R$ is discrete, except we require only that $m_{0,0} \in m$ in place of $m_{0,0} = 0$; this suffices to establish convergence in the proof of the homological perturbation lemma. See [29] for more details.

3 Note we do not consider an inhomogeneous term, and below we omit the Gromov-Witten invariants $I_n(0, 1, \beta)$ with $k = 0$, to streamline the presentation.
The notion of Calabi-Yau structure \cite{10} \S 10.2 should extend to the twisted algebra case, and then Proposition\footnote{Proposition 2 implies \( \{ m_{k,\beta}^{\mathrm{can}} \} \) is also well-defined independent of all choices. The question of whether \( m_{k,\beta}^{t} = m_{k,\beta}^{\mathrm{can}} \) depends on whether we can represent a unital cyclic homotopy retraction by a kernel \( \Lambda \) on \( \tilde{L} \times L \), as we now explain. Consider the inclusion \( i : HC \to C \) and the projection \( \pi : C \to HC \), defined using the unit and point classes \( 1, p_0^t \in C \) and the pairing \( (\cdot, \cdot) \) in the obvious way. In order to transfer the unital cyclic structure to \( HC \), we need an operator \( h : C \to C [-1] \) satisfying

\[
\langle h, x, y \rangle = (-1)^{cd(x,y)} \langle x, y \rangle, \tag{12}
\]

\[
Dh + Dh = d_C - i \circ \pi \quad \text{and} \quad h^2 = 0, \quad hi = 0, \quad \pi h = 0. \tag{13}
\]

By construction, \( -\Lambda|_{S(N_\Lambda)} \) is an equivariant angular form for the sphere bundle of the normal bundle to the diagonal \( N_\Lambda \), and \( DA = \tilde{p}_i^\ast \Delta \), where for \( i = 1, 2 \), \( \tilde{p}_i : \tilde{L} \times \tilde{L} \to L \times L \to L \) denotes the projection. It follows that if we define \( h_\Lambda : C \to C [-1] \) by

\[
\langle h_\Lambda x, y \rangle = \langle \tilde{p}_2^\ast \left( (\Lambda + \tau^\ast \Lambda) \tilde{p}_1^\ast x \right), y \rangle, \tag{15}
\]

where \( \tau : \tilde{L} \times \tilde{L} \to \tilde{L} \times \tilde{L} \) denotes the swap, then \( h = h_\Lambda \) satisfies (12,13), but we do not know if we can make it satisfy (14) also. The modified operator

\[
h = h_\Lambda D h_\Lambda
\]

satisfies all of the required conditions (12,13,14), and we use it to prove the existence of \( \{ m_{k,\beta}^{\mathrm{can}} \} \) (of course it may not be of the form (15)).

To some extent, Proposition 4 guided our construction of the resolutions \( \mathcal{M}'_k \), their orientations, and the integration map. Thus, the reader may find that the proof below explains and motivates the constructions discussed in 3.1.

\textit{Proof of Proposition 4 (Sketch).} Since \( h_\Lambda \) satisfies (14), we can use it to construct a minimal model \( (HC, \{ m_{k,\beta}^{\prime} \}) \) for \( (C, m_{k,\beta}) \). This is done as in \cite{5}, except we need to adjust the signs for the twisted case. In fact, \( (HC, \{ m_{k,\beta}^{\prime} \}) \) is cyclic, since \( h_\Lambda \) is self-adjoint (12) (see \cite{5} Lemma 10.3)), but the equivalence

\[
(\mathcal{M}', \{ m_{k,\beta}^{\prime} \}) \xrightarrow{f'} (C, \{ m_{k,\beta} \})
\]

may not respect the pairing, i.e. it may not be a cyclic morphism in the sense of \cite{29} Definition 19. We have \( (h_\Lambda, x) = (1, h_\Lambda x) = 0 \) since the De Rham degree of \( h_\Lambda x \) is less than \( 2m \), so \( (HC, \{ m_{k,\beta}^{\prime} \}) \) and \( f' \) are unital. It follows that the roof

\[
(\mathcal{M}', \{ m_{k,\beta}^{\prime} \}) \xrightarrow{f'} (C, \{ m_{k,\beta} \}) \xleftarrow{f_{\text{can}}} (HC, \{ m_{k,\beta}^{\mathrm{can}} \})
\]

Note without the side conditions (14), we cannot work with the geometric series representation of the bar coalgebra differential, and must use summation over trees. Compare \cite{29} Remark 29.
is represented by a unital isomorphism \( f \). To complete the proof, we must show that \( m'_{k,\beta} = m^l_{k,\beta} \). It is enough to check this for the non-trivial structure constants, so we will now prove that for all \( k,\beta \geq 0 \) with \( (k + 1) + \beta = 1 \mod 2 \) we have

\[
\langle m'_{k,\beta} (p^L_0, ..., p^L_0), p^L_0 \rangle = \langle m^l_{k,\beta} (p^L_0, ..., p^L_0), p^L_0 \rangle.
\]

(16)

Note that \( m'_{k,\beta} \) is given by a sum over labeled ribbon trees \( \Gamma \) with \( k + 1 \) external edges, one of which is called the root, and whose vertices \( \Gamma_0 \) are labeled by integers \( \beta_v, l_v \geq 0 \) so that \( \sum_{v \in \Gamma_0} \beta_v = \beta \) and \( k_v + 2l_v + 3\beta_v \geq 3 \), where \( k_v \) is the valency of \( v \). It follows from the projection formula that the contribution \( C_\Gamma \) of \( \Gamma \) to

\[
\langle m'_{k,\beta} (p^L_0, ..., p^L_0), p^L_0 \rangle
\]

is given by an integral over the blow up of

\[
\prod_{v \in \Gamma_0} \widehat{M}^\text{main}_{0,k_v,l_v} (\beta_v)
\]

(more precisely, over \((L \times L) \Gamma \times (L \times L)^r \prod_{v \in \Gamma_0} \widehat{M}^\text{main}_{0,k_v,l_v} (\beta_v)\)) where \( \widehat{M}^\text{main}_{0,k_v,l_v} (\beta_v) \subset \widehat{M}_{0,k_v,l_v} (\beta_v) \) is the component of the moduli space where the markings are required to appear around the boundary of the disc in the cyclic order specified by \( \Gamma \). Since \( (k + 1) + \beta = 1 \mod 2 \), there’s a natural orientation on the edges \( \Gamma_1 \) so that \( k^\text{in}_v + \beta_v = 1 \mod 2 \), where \( k^\text{in}_v \) denotes the number of incoming edges. The contribution of trees \( \Gamma \) with vertices which become unstable when we forget the tails of edges cancel in pairs (this is just a property of \( m_{2,0} \), the analog of (39)).

Fix some set \( \Gamma \) of ribbon trees which differ only by the cyclic ordering of the edges near the vertices. We can think of \( \Gamma \) as a non-ribbon labeled rooted tree. Pick an arbitrary numbering of the \( r = |\Gamma| \) edges, an arbitrary partition of \( \{1, ..., l\} \) into parts of sizes \( \{l_v\} \), and an arbitrary labeling of the external edges by \( \{1, ..., k + 1\} \) so that 1 is the root. This gives a labeled tree \( T \in \mathcal{T}_b \), \( b = (k + 1, l = \sum l_v, \beta) \) in the sense of Definition (13) and it is not hard to see that

\[
\frac{1}{k + 1} \cdot \sum_{\Gamma \in \mathcal{T}} C_\Gamma = \frac{1}{(k + 1)!} \frac{1}{r!} \int_{\mathcal{M}} \hat{\omega}_r \Lambda^\text{gr},
\]

at least up to signs. Here \( \hat{\omega}_r \) is given by (11), and \( \mathcal{M} \subset \mathcal{M}_b \) is the clopen component corresponding to \( T \), see [33, 1.2]. The local system map \( J_b \) was computed recursively so that (38) holds. Using this and its equivariance w.r.t. the \( \text{Sym}(r) \) action (40) and the behavior under permutation of boundary markings near a vertex [30, Eq (29), (31)], we find that the signs are in fact the same. Equation (16) follows. \( \square \)

We remark that originally, we used \( \Phi (HC, \{m_{k,\beta}^{\text{can}}\}) \) as the definition of open Gromov-Witten invariants, motivated by the work of Solomon and Tukachinsky [25] on the odd-dimensional projective spaces. They define invariants for \( RP^{2m+1} \subset CP^{2m+1} \) by constructing recursively a weak bounding cochain directly on \( (C, \{m_{k,\beta}\}) \).

One way to construct such a weak bounding cochain is to take the image of the point class \( \mathcal{P} \) under the \( \Lambda_m \) morphism \( (HC, \{m_{k,\beta}^{\text{can}}\}) \to (C, \{m_{k,\beta}\}) \). This way, the choice of operator \( h \) fixes all the choices involved in the recursive construction of the weak bounding cochain.

\footnote{strictly speaking, since a weak bounding cochain has to have degree one, we need to take \( b = p^L_0, \epsilon^{2m+1} \) where \( \epsilon \) is a generator of the Novikov ring, see [29].}
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2. Related Theories

2.1. Equivariant cohomology of $\mathbb{R}P^{2m} \subset \mathbb{C}P^{2m}$. We set some notation. The unitary group $U(2m + 1)$ acts on $X = \mathbb{C}P^{2m}$. The action of the subgroup of diagonal matrices, $T^c = U(1)^{2m+1} \subset U(2m + 1)$, is given by

$$X = \mathbb{C}P^{2m} = \mathbb{P}(V_0 \oplus V_1 \oplus \cdots \oplus V_{2m}).$$

where $(u_0, ..., u_{2m})$ acts on $V_i$ by $z \mapsto u_i \cdot z$. $T^c$-equivariant cohomology is defined over the ring

$$H_{T^c}^* = \mathbb{H}_{T^c}^*(pt) = \mathbb{R}[\alpha_0, ..., \alpha_{2m}], \quad \deg \alpha_i = 2.$$

Let $\mathcal{H} = c_1(O(1)) \in \Omega(X; \mathbb{R}[i])^+\mathbb{C}$ denote an equivariant form representing the Chern class of the dual to the tautological line bundle, so

$$H_{T^c}^*(X) = \mathbb{R}[[\mathcal{H}], \alpha_0, ..., \alpha_{2m}] / \left( \prod_{i=0}^{2m} (\mathcal{H} - \alpha_i) \right).$$

Let $c: \mathbb{C}^{2m+1} \to \mathbb{C}^{2m+1}$ be the involution given by

$$z \mapsto (\overline{z}_0, \overline{z}_2, ..., \overline{z}_{2m-1}, \overline{z}_1).$$

It induces involutions $c_X: X \to X$ and $c_U: U(2m + 1) \to U(2m + 1)$, the latter given by conjugation. We define $O(2m + 1)$ and $T$ to be the $c_U$-fixed subgroups of $U(2m + 1)$ and $T^c$, respectively. We identify $U(1)^m \approx T$ by

$$U(1)^m \ni (u_1, ..., u_m) \mapsto \text{diag}(1, u_1, ..., u_m, \overline{u}_1, ..., \overline{u}_m) \in T \subset T^c \subset U(2m + 1)$$

so that the homomorphism $T \to T^c$ corresponds to the map

$$H_{T^c}^* \cong \mathbb{R}[\alpha_0, ..., \alpha_{2m}] \overset{c^*}{\longrightarrow} H^*_T = \mathbb{R}[\lambda_1, ..., \lambda_m]$$

given by $\alpha_0 \mapsto 0$, $\alpha_i \mapsto \lambda_i$ and $\alpha_{2m+1-i} \mapsto -\lambda_i$, for $1 \leq i \leq m$. Let $W_0 = \mathbb{R}$ denote the trivial representation of $T$, and let $W_i = \mathbb{R}^2$ denote the real representations of $T$, on which $(e^{\sqrt{-1}\lambda_1}, ..., e^{\sqrt{-1}\lambda_m})$ acts by

$$\begin{pmatrix} \cos t_i & -\sin t_i \\ \sin t_i & \cos t_i \end{pmatrix}.$$

As $\mathbb{Z}/2 \times T$ representations we have

$$V_0 = \mathbb{C} \otimes W_0$$

and for $1 \leq i \leq m$,

$$V_i \oplus V_{2m+1-i} = \mathbb{C} \otimes W_i,$$

where the $\mathbb{Z}/2$ action is the usual conjugation action on the $\mathbb{C}$ factor. We have

$$L = X^{\mathbb{Z}/2} = \mathbb{P}(W_0 \oplus W_1 \oplus \cdots \oplus W_m).$$

The induced action of $O(2m + 1)$ on $L$ is transitive.

We identify $H_2(X; \mathbb{Z}) = \mathbb{Z}$ using the complex structure and $H_2(X, L) = \mathbb{Z}$ so that $H_2(X) \to H_2(X, L)$ corresponds to multiplication by $+2$. 
We let
\( R = S^{-1} \mathbb{R} [\lambda_1, \ldots, \lambda_m] \)
(respectively, \( R^C = S_C^{-1} \mathbb{R} [\alpha_0, \ldots, \alpha_{2m}] \)) denote the localization of \( H^*_C \) (resp., \( H^*_C \)) by the multiplicative subset of nonzero homogeneous polynomials.

2.2. Constraint correlators. The information in the closed Gromov-Witten theory of \( \mathbb{C} P^{2m} \) relevant for the computation of the equivariant OGW invariants is encapsulated by constraint correlators. These are generating functions fashioned after the quantum correlators used by Givental [8]. Our approach follows [20], with two main differences. First, we allow any number of constrained marked points, by introducing a formal variable \( \eta_i \) that records the number of marked points carrying \( H^i \). Second, although the recursion specifying the correlators is computed over \( R \) to avoid non-rigid maps (see Remark [8]), we work with partially localized rings that admit an extension of the ring homomorphism \( \mathbb{R} [\alpha_0, \ldots, \alpha_{2m}] \to \mathbb{R} [\lambda_1, \ldots, \lambda_m] \).

Consider the two rings
\[
R^C_{\text{rat}} = M^{-1} \mathbb{R} \left[ \alpha_0, \ldots, \alpha_{2m}, \frac{2i}{h} \left[ \frac{4m+2}{Q}, \eta_0, \ldots, \eta_i, \ldots, \eta_{2m} \right] \right],
\]
\[
R^C_{\text{an}} = \mathbb{R} \left[ \alpha_0, \ldots, \alpha_{2m}, h \right] \left[ [Q, \eta, h^{-1}] \right].
\]
Here \( M \subset \mathbb{R} [\alpha, h] \) is the multiplicative subset generated by \( h \) and \( h = \frac{\alpha_i - \alpha_j}{n} \) for all \( i \neq j \) and \( n \geq 1 \). We think of \( R^C_{\text{rat}} \) as a subring of \( R^C_{\text{an}} \) using Taylor expansion in powers of \( h^{-1} \). Recall \( S \subset \mathbb{R} [\alpha] \) denotes the multiplicative subset of homogeneous nonzero polynomials. For \( i \neq j \) and \( n \geq 1 \), the map
\[
h \mapsto \frac{\alpha_i - \alpha_j}{n}
\]
extends to an injective, degree-preserving ring homomorphisms \( S^{-1} R^C_{\text{rat}} \to R^C [[Q, \eta]] \).

This is the motivation for considering the subring \( R^C_{\text{rat}} \subset R^C_{\text{an}} \).

For non-negative integers \( d, n \) with \( d + n \geq 3 \) we let
\[
\overline{\mathcal{M}}_{0, n} (X, d)
\]
denote the moduli space parameterizing stable maps to \( X \) of genus 0 and degree \( d \), with \( n \) marked points. \( \pi^C \) acts on \( \overline{\mathcal{M}}_{0, n} (d) \) by translating maps.

By abuse of notation, for \( 0 \leq a \leq 2m \) we let \( p_a = \mathbb{P}^C (V_a) \in X \) denote the \( \pi^C \)-fixed-point, as well as the equivariant Poincare dual form
\[
p_a = \prod_{i=a} (H - \alpha_i)
\]

**Definition 5.** For \( 0 \leq a \leq 2m \) the constraint correlator \( Z^C_a (h) \in R^C_{\text{an}} \) is defined by
\[
Z^C_a (h) = Z^0_a (h) + \sum_{d \geq 0} Q^d \int_{\overline{\mathcal{M}}_{0, n+1} (X, d)} \frac{1}{n!} \prod_{i=1}^{n} \left( \psi_{i, 1}^{2m} \sum_{j=0}^{\psi_{i, 1}} H^j \right) \cdot \frac{e_{\psi_{n+1}} (2m \psi_{n+1} p_a)}{h - \psi_{n+1}}
\]
where \( \psi_{n+1} = c^C_1 (L_{n+1}) \), the first Chern class of the cotangent line to the last marked point, and we set
\[
Z^0_a (h) = \sum_{n \geq 0} h^{1-n} \frac{1}{n!} \left( \sum_{i=0}^{2m} \eta_i \alpha_a^i \right)^n.
\]
We use closed localization to compute the image of $Z_a^C(h)$ in $S^{-1}R_{ran}^C$.

**Proposition 6.** We have $Z_a^C(h) \in S^{-1}R_{rat}^C \subset S^{-1}R_{ran}^C$, and the following equations hold for $0 \leq a \leq 2m$.

\begin{equation}
(24) \quad Z_a^C(h) = Z_a^0(h) + \sum \frac{1}{n!} \left( \sum_{i=0}^{2m} \eta_i \alpha_i \right)^n \prod_{b \neq a} (\alpha_b - \alpha_a)^s \left( \frac{h - s}{s!} \prod_{r=1}^s \frac{d_r}{\alpha_a - \alpha_{j_r}} \right) \left( \frac{d_1}{\alpha_a - \alpha_{j_1}} + \ldots + \frac{d_s}{\alpha_a - \alpha_{j_s}} + h^{-1} \right)^{s+n-2} \times \\
\times \prod_{r=1}^s \frac{(-1)^d d_1^2 d_2^2 \ldots \ldots d_{d-1}^2 d_{r}^2}{(\alpha_a - \alpha_{j_r})^2} \prod_{0 \leq w \leq d, k \neq j_r} \left( \frac{d_1 \alpha_a + d_r \alpha_{j_r} - \alpha_k}{d_r} \right) Z_{j_r}^C \left( \alpha_{j_r} - \alpha_a \right),
\end{equation}

where the undecorated sum ranges over $s, n \in \mathbb{Z}_{\geq 0}$ satisfying $s + n + 1 \geq 3$, and over pairs of $s$-tuples $(j_1, \ldots, j_s) \in \{0, \ldots, s, 2m\}^s$ and $(d_1, \ldots, d_s) \in \mathbb{Z}_{>0}^s$.

In other words, $Z^C = \left( Z_a^C(h) \right)_{a=0}^{2m}$ is the unique fixed point of a map

\[ f : R^C_k \left[ [Q, \eta_0, \ldots, \eta_{2m}] \right] \rightarrow \left( R^C_k \left[ [Q, \eta_0, \ldots, \eta_{2m}] \right] \right)^{2m+1} \]

which is $S^{-1}R_{rat}^C \langle Q, \eta \rangle$-adically contracting, so $Z^C = \lim_{n \to \infty} f^n(0)$.

**Proof.** This is a simple variation of the arguments given in [20 Section 3], to accommodate additional marked points with constraints. \hfill \box

In practice, using the recursion $f$ is inefficient. One can use Sym($s$) invariance to reduce the computation of a coefficient of $Z_a^C$ to a sum over certain isomorphism types of labeled trees, in a straightforward manner. Of course, the same sum can be obtained by directly applying the fixed-point formula to (22); we’re just using algebra to concisely summarize some well-known computations, see for example [10 Chapter 27].

The following result says that $Z_a^C(h)$ is integral, so we can change coefficients $R[\alpha] \xrightarrow{\mathcal{P}_a} R[\lambda]$.

**Lemma 7.** We have $Z_a^C(h) \in R_{rat}^C \subset S^{-1}R_{rat}^C$.

**Proof.** Fix some monomial $Q^d \eta_0^r \ldots \eta_{2m}^r$ and consider the coefficient $C$ of this monomial in $Z_a^C(h)$. By definition, $C \in \mathbb{R}[\alpha, h][[h^{-1}]]$, and by Proposition 6 it is in $S^{-1}M^{-1}R[\alpha, h]$. We thus reduce to proving that the intersection of these two rings in $S^{-1}M^{-1}R[\alpha, h][[h^{-1}]]$ satisfies

\[ M^{-1}R[\alpha, h] = S^{-1}M^{-1}R[\alpha, h] \cap \mathbb{R}[\alpha, h][[h^{-1}]]. \]

Indeed, the right-hand-side consists of elements $C \in \mathbb{R}[\alpha, h][[h^{-1}]]$ such that there exists some $q \in M$ so that $p := q \cdot C \in S^{-1}R[\alpha, h]$; but then $p \in \mathbb{R}[\alpha, h]$ and $C = p/q \in M^{-1}R[\alpha, h]$. \hfill \box
Consider

$$R_{\text{rat}} = \hat{M}^{-1}\mathbb{R}[\lambda_1, \ldots, \lambda_m, h] \left[ \frac{2m+1}{q}, \eta \right]$$

where $\hat{M}^{-1} \subset \mathbb{R}[\lambda, h]$ is the multiplicative subset generated by $h + \rho_T \left( \frac{\alpha_i - \alpha_j}{n} \right)$ for all $i \neq j$ and $n \geq 1$ (i.e., by $h, h \pm \frac{\lambda_i}{n}, h \pm 2\frac{\lambda_i}{n}$ and $h \pm \frac{\lambda_i + \lambda_j}{n}$ for all $i \neq j$ and $n \geq 1$). We denote by

$$R_{\text{rat}}^{C} \xrightarrow{\rho} R_{\text{rat}}$$

the unique extension of $\rho_T : \mathbb{R}[\alpha] \rightarrow \mathbb{R}[\lambda]$ that sends $Q \mapsto q^2, \eta_i \mapsto \eta_i$ and $h \mapsto h$.

**Definition 8.** We define $Z_a (h) = \hat{\rho}_T \left( Z_a^C (h) \right)$.

**Remark 9.** Theorem 2 uses $Z_a (h)$, but the full $\mathbb{T}^C$ action is useful for the computation of the correlators $Z_a (h)$, as Proposition 3 shows.

Although $Z_a^C (h) \in R_{\text{rat}}^{C}$ so we can make sense of $\hat{\rho}_T \left( Z_a^C (h) \right)$, individual summands in the recursion (24) will not in general lie in $R_{\text{rat}}^{C}$. Indeed, we see that some of the recursion denominators may vanish if we apply $\hat{\rho}_T$; also, the substitution $Z_a (\rho_T \left( \frac{a_i - a_j}{m} \right))$ need not be well-defined.

On the other hand, the $h^{-1}$-Taylor expansion of $Z_a (h)$ is given by a sum of equivariant $\mathbb{T}$-integrals, obtained by applying $\rho_T$ to the integrand of (22), and one can compute these integrals directly using $\mathbb{T}$-localization. What is happening is that different $\mathbb{T}^C$-fixed-point components of $\hat{M}_{0,n+1} (X, d)$ may be part of a single fixed-point component for the $T$-action. For a prototypical example, let $F_1$ and $F_2$ be two isolated $\mathbb{T}^C$ fixed-points in $\hat{M}_{0,2} (\mathbb{CP}^2, 2)$. $F_1$ is represented by a holomorphic map $\mathbb{CP}^1 \rightarrow \mathbb{CP}^2$ which is a degree 2 cover of the line $\overline{p_1p_2}$, branched over $p_1$ and $p_2$. $F_2$ is represented by an injective holomorphic map $\Sigma \rightarrow \mathbb{CP}^2$ where $\Sigma$ has two irreducible components covering the lines $\overline{p_1p_0}$ and $\overline{p_0p_2}$, respectively. For both $F_1, F_2$ we let the marked points 1 and 2 map to $p_1$ and $p_2$, respectively. There is a single $\mathbb{T}$-fixed-point component $F_{12} \subset \hat{M}_{0,2} (\mathbb{CP}^2, 2)$ containing $F_1$ and $F_2$.

Consider some equivariant integral $\int_{\hat{M}_{0,2} (\mathbb{CP}^2, 2)} \omega$. If we try to apply $\rho_T$ to $C_1 = \int_{F_1}^{\mathbb{T}} \frac{\omega}{\epsilon_{F_1}}$, the $\mathbb{T}^C$-localization contribution of $F_1$, we see that $\rho_T$ sends the Euler factor in the denominator corresponding to the tangent direction $T_{F_1} F_{12} \subset T_{F_1} \hat{M}_{0,2} (\mathbb{CP}^2, 2)$ to zero, and a similar singularity appears in the contribution $C_2$ of $F_2$. On the other hand, $\rho_T (C_1 + C_2)$ is well-defined, since $C_1 + C_2$ can be interpreted as applying $\mathbb{T}^C$- localization to evaluate $\int_{F_{12}}^{\mathbb{T}} \frac{\omega}{\epsilon_{F_{12}}}$, the $\mathbb{T}$- fixed-point contribution of $F_{12}$.

**Lemma 10.** $Z_a \left( \frac{\rho_T (a_i)}{d} \right)$ is a well-defined element of $\hat{M}^{-1}\mathbb{R}[\lambda, h]$ for all $a \neq 0, d \geq 1$.

**Proof.** This follows from inspection of the denominators in Proposition 6. \qed

### 2.3. Descendent integrals of discs

We now review some aspects of the open intersection theory of discs, as defined and computed in [21]. We reformulate the results in terms of differential forms rather than multisectiions.

Let $k$ and $l$ be finite sets, such that $|k| + 2|| \geq 3$. Let $\overline{M}_{0,k,l}$ denote the moduli space of stable discs with boundary and interior marked points, in bijection with $k$
and \( l \), respectively. It is a smooth manifold with corners. For \( i \in I \) let \( \mathbb{L}_i \to \mathcal{M}_{0,k,l} \) denote the cotangent space to the interior point marked \( i \); \( \mathbb{L}_i \) is a complex vector bundle of rank one. Consider

\[
E = \bigoplus_{i \in I} \mathbb{L}^{a_i}
\]

for some \( a_i \geq 0 \). It is a rank \( n = 2 \cdot (\sum_{i \in I} a_i) \) vector bundle, canonically oriented by the complex orientation. Assume that

\[
(25) \quad n = \dim \mathcal{M}_{0,k,l} = 2 ||k|| - 3
\]

Let \( \pi : S(E) \to \mathcal{M}_{0,k,l} \) denote the associated sphere bundle. Recall (see [2]) a form \( \theta \in \Omega^{n-1} (S(E)) \) is called an angular form for \( E \) if \( \pi^* \theta = 1 \) and \( d\theta = -\pi^* e \) for some form \( e \in \Omega^n (\mathcal{M}_{0,k,l}) \), called the Euler form associated with \( \theta \).

We now introduce boundary conditions on \( \theta \). We have

\[
\partial \mathcal{M}_{0,k,l} = \bigsqcup \mathcal{M}_{0,k',l'} \times \mathcal{M}_{0,k''l''}
\]

where the disjoint union ranges over all pairs of partitions \( k = k' \sqcup k'' \) and \( l = l' \sqcup l'' \); such that \( |k'| + 1 + 2|l'| \geq 3 \), \( |k''| + 1 + 2|l''| \geq 3 \), and \( |k'| \) is odd. We call a pair of partitions as above a boundary specification.

Fix a boundary specification and let \( B = \mathcal{M}_{0,k',l'} \times \mathcal{M}_{0,k''l''} \) denote the corresponding clopen component of \( \partial \mathcal{M}_{0,k,l} \). Since \( |k'| \) is odd we must have \( |k'| + 1 + 2|l'| \geq 4 \) so there’s a well-defined forgetful map

\[
(26) \quad B = \mathcal{M}_{0,k',l'} \times \mathcal{M}_{0,k''l''} \xrightarrow{\text{For}_B} B_\downarrow \mathcal{M}_{0,k',l'} \times \mathcal{M}_{0,k''l''}
\]

For \( x \in l' \) (respectively, \( x \in l'' \)) we denote by \( \mathbb{L}'_x \) (resp. \( \mathbb{L}''_x \)) the pullback to \( B_\downarrow \mathcal{M}_{0,k,l} \) of the corresponding cotangent line on \( \mathcal{M}_{0,k',l'} \) (resp. \( \mathcal{M}_{0,k''l''} \)). We define a complex vector bundle on \( B_\downarrow \mathcal{M}_{0,k,l} \) by

\[
(27) \quad E_\downarrow = \bigoplus_{x \in l'} (\mathbb{L}'_x)^{a_x} \oplus \bigoplus_{x \in l''} (\mathbb{L}''_x)^{a_x}
\]

There’s a natural isomorphism

\[
E|_B \to \text{For}_B^* E_\downarrow
\]

Taking the disjoint union over all \( B \) we obtain a cartesian map of vector bundles

\[
(28) \quad \partial E \to E' \text{ over } \partial \mathcal{M}_{0,k,l} \to \mathcal{M}_{0,k,l}'
\]

with \( \dim \mathcal{M}_{0,k,l}' < \dim \partial \mathcal{M}_{0,k,l} \).

**Definition 11.** An angular form \( \theta \in \Omega^{n-1} (S(E)) \) will be called canonical if there exists some \( \theta' \in \Omega^{n-1} (S(E')) \) such that

\[
\left( i_{\partial \mathcal{M}_{0,k,l}} \right)^* \theta = f^* \theta'.
\]

**Claim 12.** A canonical angular form exists.

**Proof.** This is similar to the proof of Proposition [30] and is omitted. The corresponding statement for multisections is proven in [21]. \( \square \)

**Definition 13.** Let \( k, l \) and \( a_1, \ldots, a_l \) be non-negative integers. If \( (25) \) holds, the open descendant integral is defined by

\[
(29) \quad \langle \tau_{a_1} \cdots \tau_{a_l} \sigma^k \rangle^0 = 2 \int_{\mathcal{M}_{0,\{k\}}} e
\]
where \( e \) is any Euler form associated with a canonical angular form \( \theta \in \Omega^{n-1}(S(E)) \). Otherwise, we set \( \{\tau_{a_1} \cdot \cdots \tau_{a_l}\sigma^k\}_0^o = 0 \).

The language of multisections was developed in [3], and used in [21]. The following lemma allows us to translate results to the language of differential forms.

**Lemma 14.** Let \( \pi : E \to M \) be an oriented rank \( n \) vector bundle over a compact oriented manifold with corners of the same dimension \( n \). Let \( \sigma \) be a multi-valued section of \( E \), transverse to the zero section. Let \( \theta \) be an angular form for \( E \) with corresponding Euler form \( e \). Let \( \#Z(s) \) denote the weighted signed count of the oriented zero set \( Z(s) \) of \( s \). We have

\[
\#Z(s) = \int_M e - \int_{\partial M} \sigma^* \theta
\]

**Proof.** Similar to the proof of [2, Theorem 11.16]. \( \square \)

**Corollary 15.** Definition [13] agrees with [21, Definition 3.4], given in terms of multisections. In particular, the open descendent integrals are integers uniquely specified by the following two relations.

(i) \[
\{\tau_{a_1} \cdot \cdots \tau_{a_l}\sigma^k\}_0^o = \sum_j \left\{ \tau_{a_j}^{-1} \prod_{i \neq j} \tau_{a_i} \sigma^k \right\}_0^o
\]

(ii) In case \( a_i \geq 1 \) for all \( i \),

\[
\{\tau_{a_1} \cdot \cdots \tau_{a_l}\sigma^k\}_0^o = \frac{(1 + \sum_{i=1}^l (2a_i - 1))!}{\prod_{i=1}^l (2a_i - 1)!},
\]

where for an odd integer \( r \) we set \( r!! = r \cdot (r-2) \cdot \ldots \cdot 1 \).

**Proof.** By Lemma 14 to establish the first claim it suffices to show that if \( \sigma \) is a canonical multisection (see [21, Definition 3.1]) we have

\[
\int_{\partial M_{B_{[k],[r]}}} \sigma^* \theta = 0.
\]

This is obvious, since for each \( B = \mathcal{M}_{0,k',\{\ast'\},\ast' \times \mathcal{M}_{0,k''\{\ast''\},\ast''} \cdot (\sigma^* \theta) |_B \) is pulled back from a lower dimensional space, namely \( B_\ast \). The relations are quoted from [21, Theorems 1.2, 1.4]. \( \square \)

**Remark 16.** In [21], the open descendent integrals are shown to satisfy a plethora of relations. Fixed point localization in \((\mathbb{C}P^{2m}, \mathbb{R}P^{2m})\) produces many additional relations, involving also the closed invariants. In fact, the relations from \((\mathbb{C}P^2, \mathbb{R}P^2)\) can be used to produce a recursive algorithm for computing the open genus zero descendent integrals, independently of Corollary 15. This is discussed in [18].

**Definition 17.** Let \( s_0, t_0, t_1, \ldots \) be formal variables. Write \( \gamma = \sum_{i=0}^\infty t_i \tau_i \) and \( \delta = s_0 \sigma \).

Define the (open, genus zero) free energy function by

\[
F_0^0 (s_0, t_0, t_1, \ldots) = \sum_{n=0}^\infty \frac{2^{\frac{1}{2} (n+1)}}{n!} \gamma^n \delta^k (\gamma^n \delta^k)_0^o
\]

\[(30)\]
3. Fixed-point Formula for Extended Forms

In this section we state the fixed-point formula for extended forms, Theorem 29. This is more general and more computationally effective than Theorem 2 but requires more geometric preparation. After stating the theorem in §3.3, we will give explicit formula for the fixed-point contributions to the equivariant open Gromov-Witten invariants §3.4, and compute a few examples §3.5. We will conclude this section with a proof of Theorem 2 using Theorem 29. The proof of Theorem will be taken up in the next sections.

3.1. Review of resolutions. We briefly review some notation and construction from [30 §2].

3.1.1. Moduli spaces. We let \( \mathbb{N} = \{1, 2, \ldots\} \). For any \( S \subset \mathbb{N} \), \( \ast_S' = \{ \ast'_i \}_{i \in S} \), similarly for \( \ast''_S \) and \( \ast''_S, \ast''_S, \ast''_S, \ldots \).

Hollow stars will be used to denote markings related to boundary nodes, and solid stars will be used for interior nodes.

Definition 18. A pre-moduli specification \( b \) is a 3-tuple \( (k, l, \beta) \) where
- \( k \subset \mathbb{N} \sqcup \ast'_N \) and \( l \subset \mathbb{N} \) are finite subsets. Elements of \( k \) and of \( l \) are called orienting labels and interior labels, respectively.
- \( \beta \) is a non-negative integer, the degree, which we think of as an element of \( H_2(X, L) \).

A basic moduli specification is a pre-moduli specification \( b = (k, l, \beta) \) that is
- stable, meaning \( k + 2l + 3\beta \geq 3 \); henceforth we use standard Roman letters to denote the sizes of sets labeled by the corresponding Serif letters, so \( k = |k| \) and \( l = |l| \).
- orientable, meaning

\[
k + \beta = 1 \mod 2.
\]

A moduli specification \( \ast \) is a pair \( \ast = (b, \sigma) \) where \( b = (k, l, \beta) \) is an orientable pre-moduli specification and \( \sigma \subset \ast'_N \) is a finite subset such that \( k + |\sigma| + 2l + 3\beta \geq 3 \). We call \( \sigma \) the superfluous (boundary) labels, and \( \tilde{k} = k \sqcup \sigma \) the boundary labels.

A moduli specification \( \ast = (b, \sigma) \) is called sturdy if \( b \) is stable and wobbly otherwise.

Let \( \ast = (b, \sigma) \) be a moduli specification. If \( \ast \) is sturdy, then \( b \) is a basic moduli specification; if it is wobbly, it is necessarily of the form

\[
((k, \varnothing, 0), \sigma) \quad \text{with} \quad |k| = 1 \quad \text{and} \quad |\sigma| \geq 2.
\]

Either way, the combined moduli specification \( \tilde{\ast} := (\tilde{k}, l, \beta) \) is a basic moduli specification.

A 3-tuple of non-negative integers \( b = (k, l, \beta) \) with \( k + 2l + 3\beta \geq 3 \) and \( k + \beta = 1 \mod 2 \) may be used in place of a basic moduli specification, taking \( b = ([k], [l], \beta) \) where we denote \( [k] = \{1, 2, \ldots, k\} \).

Let \( T = U(1)^m \) and \( \mathbb{T}_b = T \times \text{Sym}(k) \times \text{Sym}(l) \). If \( b \) is a basic moduli specification, the moduli space of stable disc maps

\[
\mathcal{M}_b = \overline{\mathcal{M}}_{0,k,l}(\beta)
\]
is a $T_b$-orbifold with corners. See the appendix for what we mean by this and related notions. Henceforth, all orbifolds will have corners unless explicitly mentioned otherwise.

The construction of $\mathcal{M}_b$ is summarized in the following diagram (see \cite{30} §2.1):

\begin{equation}
\mathcal{M}_b \xrightarrow{\sigma} \mathcal{M}_b^* \xrightarrow{\mathcal{M}^{\mathbb{Z}_2/2}_{c,\beta}} \mathcal{M}^{\mathbb{Z}_2/2}_{c,\beta} \xrightarrow{B} \mathcal{M}^{\mathbb{Z}_2/2}_{c,\beta}. 
\end{equation}

Here $\mathcal{M}_{c,\beta} = \mathcal{M}_{0,k+2l}^c(X,\beta)$ is the moduli space of stable genus zero curves marked by $\mathcal{C} := k \coprod (l \times \{1,2\})$, and $\mathcal{M}^{\mathbb{Z}_2/2}_{c,\beta}$ denotes its homotopy fixed points under an antiholomorphic involution which conjugates the map, fixes $c$ and swaps $l \times \{1\}$ and $l \times \{2\}$. $B$ is a kind of blowup, which replaces a neighbourhood of each point $p$ of a real simple normal crossings divisor $W \xrightarrow{D} \mathcal{M}^{\mathbb{Z}_2/2}_{c,\beta}$ by a $2^c$-orthants, where $c = |D^{-1}(p)|$. $\sigma$ is a 2-cover corresponding (over interior points) to a choice of orientation for the boundary of the domain disc, and $s$ is a clopen component forming a section of the quotient map which identifies $(x,1) \sim (x,2)$ for $x \in I$. $T_b$ acts on $\mathcal{M}_b$ by translating maps and permuting labels. This close relationship between the open and closed moduli spaces turns out to be useful for computations and to avoid some technical obstacles in the construction of orbifold tubular neighborhoods for the $T$-action fixed points.

For a sturdy moduli specification $s = (b,\sigma) = ((k,l,\beta),\sigma)$ and $S \subset \mathbb{N}$ we set

\[ \mathcal{M}_s^S = \mathcal{M}_{(k \coprod (\sigma \cap \sigma'),1,\beta)} \]

with $\mathcal{M}_s = \mathcal{M}_s^\emptyset$ and $\mathcal{M}_s = \mathcal{M}_s^{\{\emptyset\}}$. We denote by $\text{For}_s : \mathcal{M}_s \to \mathcal{M}_s$ the map that forgets the superfluous markings $\sigma$. It is a $b$-fibration (see the appendix). For any $S \subset \mathbb{N}$ we have a decomposition

\[ \mathcal{M}_s \xrightarrow{\text{For}_s^S} \mathcal{M}_s^S \xrightarrow{\text{For}_s^S} \mathcal{M}_s \]

where $\text{For}_s^S$ (respectively, $\text{For}_s^S$) is the map that forgets the markings $\sigma \setminus \sigma'$ (resp. $\sigma \cap \sigma'\$).

We have evaluation maps

\[ \text{ev}_x^S : \mathcal{M}_s \to X \text{ for } x \in \mathbb{N} \]

\[ \text{ev}_x^S : \mathcal{M}_s \to L \text{ for } x \in k \coprod \mathbb{N} \]

3.1.2. \textit{Resolutions.} Let $b = (k,l,\beta)$ be a basic moduli specification. Let $T^r_b = T_b \times \text{Sym}(r)$, the product of a torus group with the group of permutations of $r$ elements. Our approach for defining invariants and proving the fixed-point formula involves the construction of resolutions, which are sequences of $T^r_b$-orbifolds $\mathcal{M}^r_b, \mathcal{M}^r_b$ defined for $r = 0,1,\ldots$ with

\[ \mathcal{M}^r_b = \mathcal{M}^r_b = \mathcal{M}_b \]

and $\mathcal{M}^r_b = \mathcal{M}^r_b = \emptyset$ for sufficiently large $r$. The idea is simple. Recall we have

\begin{equation}
\partial \mathcal{M}_b = \coprod \mathcal{M}_s \xrightarrow{\text{ev}} \mathcal{M}_s \xrightarrow{\text{ev}} \mathcal{M}_s
\end{equation}

where the disjoint union is taken over all pairs of moduli specifications $s' = ((k',l',\beta'),\sigma' = \{\sigma'\}$) and $s'' = ((k'' = k'' \coprod s''_{1,l''},\beta''),\sigma'' = \emptyset)$ for

\begin{equation}
k = k' \coprod k'' \text{ for } l = l' \coprod l'' \text{ and } \beta = \beta' + \beta'',
\end{equation}
and where $s'_1, s''_1$ denote two new boundary markings (representing the special boundary points identified by the node). Note how (31) specifies the order of the fiber factors.

We take $\mathcal{M}_b^1 = \coprod_i \mathcal{M}_i \times \mathcal{M}_i^\sigma$, replacing the fiber products in (34) by products. This amounts to allowing the two discs in the nodal configuration to move independently of each other. The space $\mathcal{M}_b^1$ is obtained from $\mathcal{M}_b^1$ by forgetting the superfluous marking $s'_1 \in \sigma'$. We denote by $\text{For}_b^1 : \mathcal{M}_b^1 \to \mathcal{M}_b^1$ the corresponding forgetful map. The space $\mathcal{M}_b^2$ is obtained in a similar fashion, replacing fiber products with products for the clopen component

$$\partial_+ \mathcal{M}_b^1 \subset \partial \mathcal{M}_b^1$$

lying over $\partial^1 \mathcal{M}_b^1$. We will see that the complementary part of the boundary, $\partial_- \mathcal{M}_b^1$, is insignificant in the sense that it admits an orientation-reversing involution $\tau_b^1 : \partial_- \mathcal{M}_b^1 \to \partial_- \mathcal{M}_b^1$. More generally, $\mathcal{M}_b^r$ is a disjoint union of products of spaces modeled on labeled trees, and $\mathcal{M}_b^r$ is obtained by forgetting $s'_r$. The precise statement is as follows (cf. [60, Lemma 17]).

**Definition 19.** A $(b, r)$-labeled tree $T$ is a tree with oriented labeled edges $T_1 = \{e_1, \ldots, e_r\}$, labeled vertices $T_0 = \{v_1, \ldots, v_{r+1}\}$ and a map $\sigma_T$ which assigns to each $v_i$ a sturdy moduli specification $\sigma_T(v_i) = ((k_T(v_i), l_T(v_i), \beta_T(v_i)), \sigma_T(v_i))$.

These are required to satisfy the following properties.

(a) the head (respectively, the tail) of the edge $e_j \in T_1$ is the vertex $v_i$ if and only if $s''_j \in k_T(v_i)$ (resp., iff $s'_j \in \sigma_T(v_i)$).

(b) we have

$$\prod_{i=1}^{r+1} k_T(v_i) = k \prod_{i=1}^{r+1} s'_r, \prod_{i=1}^{r+1} l_T(v_i) = l, \sum_{i=1}^{r+1} \beta_T(v_i) = \beta, \text{ and } \prod_{i=1}^{r+1} \sigma_T(v_i) = \sigma \prod_{i=1}^{r+1} s'_r$$

(c) Let $1 \leq a \leq r$. Remove the edges $\{e_a, e_{a+1}, \ldots, e_r\}$ from $T$, and let $T', T''$ denote the connected components of the resultant forest that are joined by $e_a$. Then if $v_i$ is a vertex of $T'$ and $v_j$ is a vertex of $T''$ then $i < j$.

We denote the set of $(b, r)$-labeled trees by $\mathcal{B}_T^r$.

For $S \in \mathbb{N}$ we define spaces

$$\hat{\mathcal{M}}_b^{r, S} = \coprod_{T \in \mathcal{B}_T^r} \hat{\mathcal{M}}_T^S, \text{ for } \mathcal{M}_b^{r, S} = \coprod_{i=1}^{r+1} \hat{\mathcal{M}}_T^S(v_i),$$

with $\mathcal{M}_b^r := \mathcal{M}_b^{r, \mathbb{N}}$ and $\hat{\mathcal{M}}_b^{r, \sigma} := \hat{\mathcal{M}}_b^{r, S}$. Note that the products are ordered according to the vertex labels.

We define a $b$-fibration

$$\text{For}_b^r : \mathcal{M}_b^r \to \hat{\mathcal{M}}_b^r$$

by setting $\text{For}_b^r := \coprod_{T \in \mathcal{B}_T^r} \prod_{i=1}^{r+1} \text{For}_{T(v_i)}$. This defines a decomposition of the boundary

$$\partial \mathcal{M}_b^r = \partial_+ \mathcal{M}_b^r \coprod \partial_- \mathcal{M}_b^r$$

into “horizontal” and “vertical” components, respectively, with induced maps

$$(\text{For}_b^r)_+ : \partial_+ \mathcal{M}_b^r \to \partial \hat{\mathcal{M}}_b^r$$

and

$$(\text{For}_b^r)_- : \partial_- \mathcal{M}_b^r \to \partial \hat{\mathcal{M}}_b^r$$
(see [10], §2.1). For any \( S \subset \mathbb{N} \) we have a factorization

\[
\text{For}_b = \text{For}_{b}^r \circ \text{For}_{b}^{-r, S}
\]

through \( b \)-fibrations

\[
\text{For}_{b}^{r, -r, S} = \prod_{T \in \mathcal{T}_b} \prod_{i=1}^{r+1} \text{For}_{b}^{S} (-r, (v_i)); \text{For}_{b}^{-r, S} = \prod_{T \in \mathcal{T}_b} \prod_{i=1}^{r+1} \text{For}_{b}^{S} (-r, (v_i)).
\]

We denote by \( \hat{\pi}_b^r : \hat{\mathcal{M}}_b^C \rightarrow \hat{\mathcal{D}}_b^C \) the locally constant map which sends \( \hat{\mathcal{M}}_T \) to \( \mathcal{T}_b^C \). \( \text{Sym}(r) = \text{Sym}([-r]) \) acts on \( \hat{\mathcal{D}}_b^C \) by relabeling the edges \( e_i \). This action lifts to \( \mathcal{M}_b^C \) by relabeling the special points \( s'_i \) accordingly, and then relabeling the vertices so that condition (c) is preserved. \( \text{Sym}(r) \) also acts on \( \hat{\mathcal{M}}_b^C \) making the maps \( \text{For}_b \) and \( \hat{\pi}_b^r \text{Sym}(r) \)-equivariant. The \( \text{Sym}(r) \) action commutes with the \( T_b \) action and makes all the maps \( T_b^C \)-equivariant. We define an action of \( \text{Sym}([-r] \cap S) \times \text{Sym}([-r] \setminus S) \) on \( \hat{\mathcal{M}}_b^C \) in a similar fashion, and this makes the maps into and out of \( \hat{\mathcal{M}}_b^C \) (\( \text{Sym}([-r] \cap S) \times \text{Sym}([-r] \setminus S) \text{Sym}(r) \))-equivariant.

3.1.3. Additional structures and properties. We have evaluation maps

\[
e v_{x, -r, S} : \hat{\mathcal{M}}_b^C \rightarrow L \text{ for } x \in k \prod_i s''_i [\prod_i s'_i] S
\]

and

\[
e v_{x, r, -r, S} : \hat{\mathcal{M}}_b^C \rightarrow X \text{ for } x \in l.
\]

We set \( ev_{b, -r} = ev_{b, r, N} \) and \( ev_{b, r} = ev_{b, r, S} \) and similarly for the interior evaluation maps, evi. We construct local system maps

\[
\mathcal{F}_b^C : \text{Or}(T \mathcal{M}_b^C) ightarrow \text{Or}(T \hat{\mathcal{M}}_b^C)
\]

and

\[
\mathcal{J}_b^C : \text{Or}(T \hat{\mathcal{M}}_b^C) \rightarrow \text{Or}(T L)^{\otimes (k+r)}
\]

lying over \( \text{For}_b^r \) and \( \prod_{x \in k \prod_i s''_i} e v_{x, -r} : \hat{\mathcal{M}}_b^C \rightarrow L^{k+r} \), respectively. We set

\[
\mathcal{J}_b^C := \mathcal{J}_b^T \circ \mathcal{F}_b^C : \text{Or}(T \mathcal{M}_b^C) ightarrow \text{Or}(T L)^{\otimes (k+r)}.
\]

Writing

\[
ed_{b, r} = ev_{s''_i} \times ev_{s'_i} : \mathcal{M}_b^C \rightarrow L \times L,
\]

there's a cartesian square

\[
\begin{array}{ccc}
\partial \mathcal{M}_b^C & \xrightarrow{g_{b, r}^{r+1}} & \mathcal{M}_b^r \\
\downarrow & & \downarrow \\
L & \xrightarrow{\Delta_L} & L \times L
\end{array}
\]

which induces a local system map

\[
\mathcal{G}_b^{r+1} : \text{Or}(T \partial \mathcal{M}_b^C) \rightarrow \text{Or}(T \mathcal{M}_b^{r+1}) \otimes \left( ev_{s''_i}^{b, r, +1} \right)^{-1} \text{Or}(T L)
\]

lying over \( g_{b, r}^{r+1} \). Let

\[
\left( \mathcal{J}_b^{r+1} \right)^{-1} : \mathcal{M}_b^{r+1} \otimes \left( ev_{s''_i}^{b, r, +1} \right)^{-1} \text{Or}(T L) \rightarrow \text{Or}(T L)^{\otimes (k+r)}
\]
be the local system map derived from $\mathcal{F}^r_\partial : \mathcal{M}^r_\partial \to \text{Or} (T\partial \mathcal{M}_\partial)$. $\mathcal{F}^r_\partial$ is coherent, in the sense that

$$ (\mathcal{F}^r_\partial)^{-1} \circ \mathcal{G}^r_\partial = \mathcal{F}^r_\partial \circ \iota^\partial_{\mathcal{M}_\partial} |_{\partial \mathcal{M}_\partial} \circ \partial_\mathcal{M}_\partial = \mathcal{F}^r_\partial \circ \iota^\partial_{\mathcal{M}_\partial} |_{\partial \mathcal{M}_\partial} $$.  

where

$$ \iota^\partial_{\mathcal{M}_\partial} : \text{Or} (T\partial \mathcal{M}_\partial) \to \text{Or} (T\mathcal{M}_\partial) $$

is the local system map defined by the outward normal orientation convention.

We have a $T_\partial$-equivariant involution $\tau^r_\partial : \partial_\mathcal{M}_\partial \to \partial_\mathcal{M}_\partial$ which reverses the $\mathcal{F}^r_\partial$-orientation

$$ \mathcal{J}^r_\partial \circ (\iota^\partial_{\mathcal{M}_\partial} |_{\partial_\mathcal{M}_\partial}) \circ \text{Or} (d\tau^r_\partial) = (\text{Or} (d\tau^r_\partial)) \circ \mathcal{J}^r_\partial $$

and commutes with $\text{For}^r_\partial$. To construct it, write

$$ \partial_\mathcal{M}_\partial = \bigoplus_{\tau \in T_\partial} \bigoplus_{i=1}^{r+1} \left( \bigoplus_{j=1}^{r+1} \mathcal{M}_{\mathcal{T}_\partial (v_i)} \times \partial_\mathcal{M}_{\mathcal{T}_\partial (v_i)} \times \bigoplus_{j=i+1}^{r+1} \mathcal{M}_{\mathcal{T}_\partial (v_i)} \right) $$

with

$$ \partial_\mathcal{M}_{(k,l,\sigma)} = \bigoplus_{\sigma'}^{\mathcal{T}_\partial} \left( \left( \mathcal{M}_{(k',\sigma',\sigma)} \times \bigoplus_{\sigma''}^{\mathcal{T}_\partial} \mathcal{M}_{(k'' \cup \sigma'' \cup \sigma)} \right) \times \mathcal{L} \mathcal{M}_{(k',\sigma',\sigma)} \right) $$

where the disjoint union ranges over all partitions (33) $\mathcal{T}_\partial$ and $\sigma = \sigma' \bigcup \sigma''$, such that $s_0 = ((k_0,0,0),\sigma_0)$ is wobbly (see 32) for $s_0 = s'$ or $s_0 = s''$, but not both. We let $\tau^r_\partial$ act by relabeling the last two elements of $\sigma_0$.

The local system maps $\mathcal{F}^r_\partial$, $\mathcal{J}^r_\partial$ and $\mathcal{F}^r_\partial$ are $T_\partial$-equivariant. If we let $\tau$, denote the action of $\tau \in \text{Sym} (r) \subset T_\partial$ then

$$ \mathcal{J}^r_\partial \circ \text{Or} (d\tau) \circ \mathcal{J}^r_\partial = \text{sgn} (\tau) \circ \mathcal{J}^r_\partial $$

$$ \mathcal{F}^r_\partial \circ \text{Or} (d\tau) \circ \mathcal{F}^r_\partial = \text{sgn} (\tau) \circ \mathcal{F}^r_\partial $$

$$ \mathcal{F}^r_\partial \circ \text{Or} (d\tau) = \text{Or} (d\tau) \circ \mathcal{F}^r_\partial $$

where $\text{sgn} (\tau) \in \{ \pm 1 \}$ is the sign of $\tau$.

3.1.4. Extended forms and integration. For each $\mathcal{T}_\partial \in \mathcal{J}^{r+1}_\partial$ there’s a unique $\mathcal{T} \in \mathcal{J}^r_\partial$ such that

$$ (\mathcal{g}^{r+1})^{-1} (\mathcal{M}_{\mathcal{T}_\partial}) = \partial^r \mathcal{M}_{\mathcal{T}} \subset \partial \mathcal{M}_{\mathcal{T}} $$

this defines a map $\text{cnt}^r_\partial : \mathcal{J}^{r+1}_\partial \to \mathcal{J}^r_\partial$, which one may think of as contracting $e_{r+1}$ to some vertex $v_i \in \mathcal{T}_\partial$. We have

$$ \partial_\mathcal{M}_{\mathcal{T}} = \bigoplus_{\mathcal{T}_\partial | \text{cnt}^r_\partial (\mathcal{T}_\partial) = \mathcal{T}} \partial^r \mathcal{M}_{\mathcal{T}}. $$

Consider some $S \subset [r]$. We set

$$ \partial^r \mathcal{M}^S = \left( \text{For}^{r,S} \right) (\partial^r \mathcal{M}_{\mathcal{T}}) \subset \partial \mathcal{M}^S. $$

Note that if $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{J}^{r+1}_\partial$ differ by moving $s_i$, $i \notin S$, from the head of $e_{r+1}$ to its tail, or vice-versa, then $\partial^r_\mathcal{M}^S = \partial^r_\mathcal{M}^S$. Generalizing (37), we find that there’s a map

$$ \mathcal{g}^S_{\mathcal{T}_\partial} : \partial^r_\mathcal{M}^S \to \mathcal{M}^{S \cup (r+1)} $$
sitting in a cartesian square

\[
\begin{array}{ccc}
\partial^T \mathcal{M}_T & \xrightarrow{g^r_\bullet} & \mathcal{M}_T \\
\downarrow \quad \Phi_{or}^r & & \downarrow \Phi_{or}^r \\
\partial^T \hat{\mathcal{M}}_T & \xrightarrow{\hat{g}^r_{T_\bullet}} & \hat{\mathcal{M}}_T^{r+1}
\end{array}
\]

We define \( \hat{g}^r_{T_\bullet} : \partial^T \hat{\mathcal{M}}_T \to \hat{\mathcal{M}}_T \) by \( \hat{g}^r_{T_\bullet} = \Phi_{or}^{r+1} \circ \hat{g}^r_{T_\bullet} \).

**Definition 20.** An extended form \( \omega \) for \( b \) is a sequence

\[
\omega = \left\{ \tilde{\omega}_r \in \Omega \left( \mathcal{M}_b^r, \mathcal{E}_b^r \otimes \mathbb{R} [\lambda] \right)^r \right\}_{r \geq 0}
\]

for \( \mathcal{E}_b^r := \bigotimes_{x \in \mathbb{R}} \left( (\mathbb{C}^m b^r_i) \right) \ast \text{Or} (TL) \),

satisfying the following two conditions:

- (coherence) for all \( \mathcal{T}_r \in \mathcal{T}_{b}^{r+1} \) we have

\[
\left( \hat{g}^r_{T_\bullet} \right)^* \tilde{\omega}_r = (\hat{g}^r_{T_\bullet})^* \tilde{\omega}_{r+1}
\]

- (symmetry) \( \tilde{\omega}_r \) is \( \text{Sym} (r) \)-invariant.

The set of extended forms is a complex, \((\Omega_b, D)\), with \( D \) acting level-wise by the Cartan-Weil differential \( D = d - \sum_{j=1}^{m} \lambda_j \xi_j \), where for \( 1 \leq j \leq m \), \( \xi_j \) denotes contraction with \( \xi_j \), the image under the action map of the generator \( \sqrt{-1} u_j \partial / \partial u_j \) of the Lie algebra of \( U(1)^m \).

We construct a blow up of \( \mathcal{M}_b^r \),

\[
\overline{\mathcal{M}}_b^r \overset{\text{bu}_b^r}{\longrightarrow} \mathcal{M}_b^r,
\]

as follows. Let

\[
N_{\Delta} \xrightarrow{\gamma_{\Delta}} L \times L
\]

be a tubular neighborhood\(^6\) for the diagonal \( L \xrightarrow{\Delta} L \times L \).

We use this to construct the blow up of \( L \times L \) at the diagonal,

\[
\widetilde{L \times L} := \{ \Delta \} \times [0, \infty) \cup (L \times L \Delta) \overset{\text{bu}_{\Delta}}{\longrightarrow} L \times L.
\]

The map

\[
ed_b^r := \text{ev}^b_i \ast \cdots \ast \text{ev}^b_r : \mathcal{M}_b^r \to (L \times L)^r
\]

is transverse to \( \Delta^r \subset (L \times L)^r \), and we define \( \overline{\mathcal{M}}_b^r \) by the cartesian square

\[
\begin{array}{ccc}
\overline{\mathcal{M}}_b^r & \xrightarrow{\text{bu}_b^r} & \mathcal{M}_b^r \\
\downarrow \quad \overline{ed}_b^r & & \downarrow \text{ed}_b^r \\
(L \times L)^r & \xrightarrow{\text{bu}_{\Delta}} & (L \times L)^r
\end{array}
\]

\(^6\) in terms of Definition 78 we're taking \( j = \text{id} \).
\( \mathcal{J}_b^r \) induces a local system map
\[
\tilde{\mathcal{J}}_b^r : \text{Or} (T \bar{\mathcal{M}}_b^r) \to \text{Or} (T L) \mathbb{Z}^{(k \Omega^*(\mathcal{U}) ).
\]
We use this to define an \( \mathbb{R} [\tilde{\lambda}] \)-linear, \( \text{Sym} (k) \times \text{Sym} (l) \)-invariant integration map
\[
\int_b : \Omega_b \to \mathbb{R} [\tilde{\lambda}]
\]
as the finite sum of integrals
\[
\int_b \omega = \sum_{r \geq 0} \frac{1}{r!} \int_{\bar{\mathcal{M}}_b^r} (\text{For}_b^r \circ \text{bu}_b^r)^* \omega_r \cdot (\tilde{\mathcal{c}}_b^r)^* \Lambda^{\text{br}}.
\]
Here \( \Lambda \in \Omega \left( \bar{L} \times \bar{\mathcal{L}}; \bar{\text{pr}}_2^* \left( \text{Or} (T L) \mathbb{Z} \mathbb{R} [\tilde{\lambda}] \right) \right) \) is an equivariant homotopy kernel,
\[
\Lambda = \sigma \left( \text{pr}^S_{N(\Delta)} \right)^* \theta_{\Delta} + \text{bu}_b^* \Upsilon
\]
for \( \theta_{\Delta} \) an equivariant angular form\(^7\) for \( S (N_{\Delta}) \), \( \sigma : [0, \infty) \to [0, 1] \) a smooth, compactly supported cutoff function with \( \sigma (0) = +1 \), and \( \Upsilon \) chosen so that
\[
D \Lambda = -\bar{\text{pr}}_2^* p_0^L
\]
for
\[
p_0^L \in \Omega \left( L ; \text{Or} (T L) \mathbb{Z} \mathbb{R} [\tilde{\lambda}] \right) \)
a form representing the equivariant Poincare dual to \( p_0 \) in \( L \). Note we have
\[
\left( i_{L \times L}^\partial \right)^* \Lambda = \theta_{\Delta} + \left( s_{\Delta}^S (N_{\Delta}) \right)^* \Upsilon |_{\Delta}.
\]
The \( r = 0 \) summand in (45) is just
\[
\int_{\mathcal{M}_0(k,\iota)(X,L,\beta)} \omega_0;
\]
one may think of the summands for \( r \geq 1 \) as corrections accounting for the the boundary and corners of \( \mathcal{M}_0(k,\iota)(X,L,\beta) \).

A key property of the integral is that it satisfies Stokes’ theorem:
\[
\int_b D \nu = 0
\]
for any \( \nu \in \Omega_b \).

3.1.5. Sorted odd-even trees.

**Definition 21.** (a) A labeled tree \( T \in \mathcal{J}_b^r \) will be called an odd-even tree if all the moduli specifications \( ((k,1,\beta),\sigma) = \sigma_T (v) \) for a vertex \( v \in T_0 \) satisfy the following condition: if \( \beta = 0 \) mod 2 then \( \sigma = \emptyset \) and if \( \beta = 1 \) mod 2 then \( k = \emptyset \) (note in particular this means the tree is bipartite with respect to the partition into odd degree and even degree vertices, which explains the name). We denote the set of odd-even trees by \( \mathcal{J}_b^{r,0} \).

(b) An odd-even tree \( T \in \mathcal{J}_b^r \) will be called sorted if the graph spanned by the edges \( 1, \ldots, a \) is connected for every \( 1 \leq a \leq r \), and such that if \( \{ \sigma_i', \sigma_j' \} \in \sigma_T (v) \) for some \( i < j \) and \( v \in T_0 \) then \( \sigma_i' \in \sigma_T (v) \) for all \( i \leq a \leq j \).

\(^7\text{We review the definition above Theorem \([29]\).}\)
Proposition 22. Let \( b = (k, l, \beta) \) be a basic moduli specification.

(a) For every \( T \in \mathcal{F}_b^{r,0} \) there exists at least one \( \tau \in \text{Sym}(r) \) such that \( \tau.\mathcal{T} \) is sorted.

(b) For a sorted odd-even \( T \in \mathcal{F}_b^{\tau,0} \) we have
\[
\mathcal{J}_b^{\tau}|_{\mathcal{M}_T} = (-1)^{(1+m)} (\prod \mathcal{J})_b
\]
where \( o \) is the number of odd vertices, \( (\prod \mathcal{J})_b^o|_{\mathcal{M}_T} \) is the composition
\[
\text{Or} \left( T \left( \bigotimes_{i=1}^{r+1} \mathcal{M}_{\tau(v_i)} \right) \right) \xrightarrow{\cong} \text{Or} \left( T\mathcal{M}_{\tau(v_i)} \right) \xrightarrow{\otimes_{i=1}^{r+1} \mathcal{J}_{\tau(v_i)}} \text{Or} (TL)^{(k+r)},
\]
and \( (\prod \mathcal{F})_b^o|_{\mathcal{M}_T} \) is similarly induced from \( \otimes_{i=1}^{r+1} \mathcal{F}_{\tau(v_i)} \) using the order of the vertices.

(c) if \( T \in \mathcal{F}_b^{r,0} \) is sorted and \( \beta_T(v_i) = 1 \mod 2 \) and \( \beta_T(v_j) = 0 \mod 2 \) for some \( 1 \leq i, j \leq r+1 \) then \( i < j \).

Warning. In this paper \( \mathcal{F}_b^{\tau} \) and \( \mathcal{J}_b^{\tau} \) are both twisted by \((-1)^{(\tau)}\) relative to their definition in [30]. This cleans up some signs, compare for instance part (b) of Proposition [22] and [30 Proposition 29]. Their composition \( \mathcal{J}_b^{\tau} \) is of course the same.

3.2. The open fixed points. A local orbifold (or l-orbifold for short) \( \mathcal{X} = \bigsqcup_{n \geq 0} \mathcal{X}_n \) is a disjoint union of orbifolds with corners of dimension \( \mathcal{X}_n = n \). When considering a vector bundle over a local orbifold we will always assume that the total space is an (ordinary) orbifold of constant dimension \( N \), so a vector bundle \( E \to \mathcal{X} \) is given by a vector bundle \( E_n \to \mathcal{X}_n \) of rank \( N - n \) for each \( n \geq 0 \).

Let \( b = (k, l, \beta) \) be a basic moduli specification. Let \( \mathcal{F}_b^{\tau} = k \bigsqcup \{1, 2\} \). We use the following diagram to construct an l-orbifold with corners \( \mathcal{M}_b^{\tau} \) together with a closed embedding \( \mathcal{M}_b^{\tau} \to \mathcal{M}_b \) representing the fixed-point stack (see the appendix).

The bottom row is just [33]. The top row is constructed from right to left. The vertical maps are the fixed-point structure maps, and are closed embeddings (cf. Definition [74]). The unlabeled horizontal maps on the top row are the maps of \( \mathbb{T} \)-fixed points induced by the maps lying below them. The slanted map is the blow up of \( \mathcal{M}_b^{\tau} \to \mathcal{M}_b^{\tau} \), making the adjacent trapezoid 2-cartesian. The map labeled \( \cong \) is the natural diffeomorphism \( (\mathcal{M}_b^{\tau})^T \cong (\mathcal{M})^T \) (cf. [30 §2.1]). Let \( f_b : \mathcal{M}_b \to \mathcal{M}_b^{\tau} \) denote the composition of the maps on the bottom row so \( f_b^\tau \) is the induced map of fixed points. Taking normal bundles of the vertical maps we find that
\[
\text{N}_b \cong (f_b^\tau)^* \text{N}_b^{\tau/2}.
\]
We have immersive closed gluing maps

\[ \mathcal{M}_1 \sqcup \ast \times \mathcal{M}_2 \sqcup \ast_2 \to \mathcal{M}_1 \sqcup \ast_1 + \ast_2, \]

with normal bundle \( \mathbb{L}_1 \ast \otimes \mathbb{L}_2 \ast \), where \( \mathbb{L}_i \ast \) denotes the tangent line at \( \ast_i \). These gluing maps are associative. In particular, we have \( \mathbb{Z}/2 \)-equivariant maps

\[ \mathcal{M}(l, \times (1)) \sqcup \ast, \ast_2 \times X \mathcal{M}_1 \sqcup \ast, \ast_2 \to \mathcal{M}(l, \times (2)) \sqcup \ast, \ast_2 + \ast_2 \simeq \mathcal{M}(l, \times (1), \ast_2, \ast_2). \]

Here \( l^C = k \sqcup \{1, 2\} \) and \( l_\alpha \) are finite, disjoint subsets and \( \beta, \beta_+ \) non-negative integers, subject to the usual stability conditions.

Since the boundary divisor is normal crossing, we find that the real codimension one locus \( D_{k, 1, \beta} \) used to define the blow up \( \mathcal{M}^{2/2}_{k, \beta} \xrightarrow{B} \mathcal{M}^{2/2}_{k, \beta} \), has transversal self-intersection relative to \( \mathcal{M}^{2/2}_{k, \beta} \) (cf. [30, §2.1]). It follows that we have induced open gluing maps

\[ \mathcal{M}(l, \ast, \beta) \times X \mathcal{M}_1 \ast \times X \mathcal{M}_2 \ast \to \mathcal{M}(l, \ast, \beta) \times X \mathcal{M}_1 \ast \times X \mathcal{M}_2 \ast, \]

which are a closed embedding with normal bundle

\[ (49) \quad N_\gamma = \mathbb{L}_1 \ast \otimes \mathbb{L}_2 \ast. \]

The normal bundle to

\[ \mathcal{M}(l, \ast, \beta) \times X \mathcal{M}_1 \ast \times X \mathcal{M}_2 \ast \to \mathcal{M}(l, \ast, \beta) \times X \mathcal{M}_1 \ast \times X \mathcal{M}_2 \ast \]

is \( \text{ev}_\ast, TX \). There’s a natural map of local systems

\[ (50) \quad \text{Or} \left( T \mathcal{M}(l, \ast, \beta) \right) \xrightarrow{\Gamma} \text{Or} \left( T \mathcal{M}(l, \ast, \beta) \right) \]

lying over \( \gamma \), defined using the complex orientations for \( TX, \mathbb{L}_1 \ast \otimes \mathbb{L}_2 \ast \) and the tangent spaces to the closed moduli spaces. We have (see [24])

\[ J(k, 1, \ast, \beta) \circ \Gamma = J(k, 1, \ast, \beta). \]

We can combine the closed and open gluing maps in different ways, and the order of gluing does not matter. In particular, we obtain maps

\[ \left( \mathcal{M}(l, \ast, \beta) \times X \right) \times \left( \mathcal{M}(l, \ast, \beta) \right) \to \mathcal{M}(l, \ast, \beta) \times \mathcal{M}(l, \ast, \beta) \]

where subscript \( \text{Sym}(s) \) denotes the stack quotient. Since the gluing maps are \( T \)-equivariant they induce maps of fixed points.

Let \( s = (b, \sigma) \) be a sturdy moduli specification. We write

\[ \bar{F}_s := \mathcal{M}_s^T := \mathcal{M}_s^T. \]

**Proposition 23.** (a) The fiber product \( F_s := \text{For}_s^{-1}(\bar{F}_s) \) is an orbifold with corners and the map \( F_s \to \mathcal{M}_s \) is a closed embedding.

(b) If \( \beta \) is odd, the induced map \( F_s \to \bar{F}_s \) is a proper submersion.

**Proof.** We prove (a). Using atlases this reduces to showing that if \( X \xrightarrow{\Gamma} Y \) is a closed embedding of manifolds with corners and \( Y' \xrightarrow{\Gamma'} Y \) is a submersion, then

(i) the fiber product \( X' = X_\ast \times Y' \) exists in \( \text{Man}^{\ast} \) and

(ii) the pullback \( X' \xrightarrow{\Gamma'} Y' \) is a closed embedding.
Claim (i) follows from (iii) of [30] Lemma 37. We prove claim (ii). By assumption, there exists a cartesian square

\[
\begin{array}{ccc}
X & \xrightarrow{i} & Y \\
\downarrow & & \downarrow h \\
0 & \xrightarrow{f} & \mathbb{R}^N
\end{array}
\]

It follows that the square

\[
\begin{array}{ccc}
X' & \xrightarrow{i'} & Y' \\
\downarrow & & \downarrow (h \circ f) \\
0 & \xrightarrow{f} & \mathbb{R}^N
\end{array}
\]

is cartesian and \( h \circ f \) is a \( b \)-submersion, so \( i' \) is a closed immersion by Remark 72. The condition of being a homeomorphism onto a closed image is clearly stable under pullbacks, completing the proof of claim (ii).

We prove (b). Suppose \( \beta \) is odd, and let \( p = [\Sigma, \nu, u, \kappa, \lambda] \in \tilde{F}_z \). Since the map \( H_2(X) \to H_2(X, L) \) is identified with \( \mathbb{Z} \xrightarrow{2} \mathbb{Z} \), there is at least one disc connected component \( \Sigma^0 \subset \Sigma \) with

\[
\beta^0 := u_\ast [\Sigma^0, \partial \Sigma^0]
\]

a positive (odd) integer. Such a configuration covers \( \mathbb{P}(W_j \oplus W_0) \) for some \( 1 \leq j \leq m \), and in particular does not pass through the only fixed point \( p_0 \in L \), so there can be no special points on \( \partial \Sigma^0 \). Since \( \partial \Sigma^0 \nu \) is connected in genus zero, we conclude that \( \Sigma^0 \) is the only disc component (the other connected components of \( \Sigma \) are biholomorphic to \( S^2 \)) and \( k = \emptyset \). We call \( \Sigma^0 \) (respectively, \( \beta^0 \)) the intersection disc (resp., intersection degree) of the configuration \( p \).

Let \( \tilde{p} = [\tilde{\Sigma}, \tilde{\nu}, \tilde{u}, \tilde{\kappa}, \tilde{\lambda}] \in F_\ast \) satisfy \( \text{For}_\ast (\tilde{p}) = p \), so \( \Sigma = \Sigma \amalg \Sigma', \nu = \nu \amalg \nu', \tilde{\nu} = \tilde{\nu} \amalg \tilde{\nu}' \) and \( \tilde{\lambda} = \lambda \), with \( \Sigma' \) a possibly disconnected configuration connected by boundary nodes to \( \Sigma \) and \( \nu' \) a locally constant map. Using the infinitesimal transversality of the \( O(2m + 1) \) action we see that any tangent vector \( \nu \) at \( p \) can be lifted to a tangent vector \( \nu' \) at \( p' \), by modifying \( (\Sigma, \nu, u) \) according to \( \tilde{\nu} \) and modifying the other components so that \( \tilde{\nu} \) stays \( \tilde{\nu} \) invariant. Note \( p' \in TS^\ast (M_\ast) \), the immersed suborbifold of points of depth \( c \) in \( M_\ast \), iff \( \tilde{\nu} \) has \( c \) boundary nodes. By construction, \( \nu' \in TS^\ast (M_\ast) \), which shows that \( \text{For}_\ast \) is a submersion. \( \text{For}_\ast \) is proper since \( M_\ast \) is compact. \( \Box \)

3.2.1. Fixed-point profiles. When discussing fixed points, it is convenient to restrict the possible labeling of edges of trees, as follows.

**Definition 24.** Let \( \mathcal{F}_b^{r,r'} \subset \mathcal{F}_b^{r+r'} \) be the subset of trees such that:

- for \( 1 \leq e \leq r \), \( \beta_{\text{tail}}(e) = 1 \mod 2 \),
- for \( r + 1 \leq e \leq r + r' \), \( \beta_{\text{tail}}(e) = 0 \mod 2 \) and
- for \( 1 \leq e \leq r + r' \), \( \beta_{\text{head}}(e) = 0 \mod 2 \).

A labeled tree \( T \in \mathcal{R}_b \) is called *separated* if \( T \in \mathcal{F}_b^{r,r'} \) for some \( r, r' \geq 0 \). The edges \( 1 \leq e \leq r \) are called *odd-even edges* and the edge \( r + 1 \leq e \leq r + r' \) are called *even-even* edges. Note \( \mathcal{F}_b^{r,0} \) is the set of odd-even trees.
Note if \( \tilde{M}_T^0 \neq \emptyset \), then \( \tau, \mathcal{T} \) is separated for some \( \tau \in \Sym(r + r') \). Define \( \mathcal{M}_b^{r,r'} \subset \mathcal{M}_b^{r+rr'} \) by \( \mathcal{M}_b^{r,r'} = \bigcup_{\tau \in \mathcal{T}} \mathcal{M}_b^{r,r'} \). Similarly, we define clopen components \( \mathcal{M}_b^{r,r'} \subset \mathcal{M}_b^{r+rr',S} \) for \( S \subset [r] \), \( \mathcal{M}_b^{r,r'} \subset \mathcal{M}_b^{r+rr'} \) by restricting to separated trees.

Let \( \mathcal{T} \in \mathcal{T}_b^{r,r'} \) be a separated tree. A \( \textit{(fixed-point) profile} \) for \( \mathcal{T} \) is some discrete information associated with each connected component of \( \mathcal{M}_T^0 \subset (\mathcal{M}_b^0)^\mathcal{T} \), describing its behavior near \( L \subset X \). We first explain how to determine the profile of a given fixed point, and then use gluing to reverse the process.

Write \( (\kappa_i, \lambda_i, \nu_i, \mu_i) = \mathfrak{s}_T(v_i) \) for \( 1 \leq i \leq r + 1 \). Let \( q \in \mathcal{M}_T^0 \). We can write

\[
q = (q_1, \ldots, q_{r+1}) \in \prod_{i=1}^{r+1} F_{\kappa_i, \lambda_i, \nu_i, \mu_i},
\]

where each \( q_i \) is represented by a \( \mathcal{T} \)-invariant \( (\kappa_i, \lambda_i, \nu_i) \)-disc configuration

\[
(\Sigma_i, \kappa_i, \lambda_i, \nu_i, \mu_i).
\]

Elementary arguments yield the following.

**Lemma 25.** If \( \beta_i = 1 \mod 2 \) we have \( \kappa_i = \emptyset \), and there’s a unique decomposition

\[
\Sigma_i = D_i \cup \Sigma_i^1
\]

such that \( D_i \) is a connected component biholomorphic to a disc, and \( \Sigma_i^1 \) is either the unique fixed point of \( D_i \) or a closed (possibly disconnected) Riemann surface which does not intersect \( D_i \).

If \( \beta_i = 0 \mod 2 \) there’s a decomposition

\[
\Sigma_i = D_i \coprod \bigcup_{j=1}^{s_i} (P_i^j \cup \Sigma_i^j)
\]

where

- \( D_i \) is the maximal clopen component of \( \Sigma_i \) satisfying (i) \( \partial \Sigma_i \subset D_i \), (ii) \( D_i/\nu \) is connected, and (iii) \( u(D_i) = p_0 \).
- For each \( 1 \leq j \leq s_i \), \( P_i^j \cong \mathbb{C}\mathbb{P}^1 \) is connected by a node to \( D_i \).
- \( \Sigma_i^j \) is either the unique non-nodal fixed point of \( P_i^j \), or a clopen component connected to \( P_i^j \) by a node. In the latter case we require \( \Sigma_i^j/\nu \) to be connected.

The decomposition is unique up to renumbering the \( P_i^j \cup \Sigma_i^j \) components.

Consider some \( 1 \leq i \leq r + 1 \) with \( \beta_i \) even. We set \( l_0^i := \lambda^{-1}(D_i) \), and for \( 1 \leq j \leq s_i \), we define a positive integer by

\[
d_{ij}^i = u_* [P_i^j] \in H_2(X);
\]

we specify \( a_{ij}^i \in \{1, \ldots, 2m\} \) by requiring that \( p_0, p_{a_{ij}^i} \) are the fixed points of \( u \left( P_i^j \right) \); we set \( d_{ij}^i = u_* [\Sigma_i^j] \in H_2(X) \) and \( l_i^j = \lambda^{-1}(\Sigma_i^j) \). We call \( \phi_i = \left( \kappa_i, \lambda_i, \nu_i, \mu_i, \left( d_{ij}^i, a_{ij}^i, d_{ij}^i, l_i^j \right)_{j=1}^{s_i}, \sigma_i \right) \) the profile of \( q_i \). Note \( \Sym(s_i) \) acts on \( \left( d_{ij}^i, a_{ij}^i, d_{ij}^i, l_i^j \right)_{j=1}^{s_i} \), yielding equivalent profiles for \( p_i \).

For \( 1 \leq i \leq r + 1 \) with \( \beta_i \) odd, the profile of \( q_i \) is defined to be \( \phi_i = \left( \beta_i^0, a_i, \tilde{d}_{ij}^i, \sigma_i \right) \) with \( \beta_i^0 = u_* \left( [D_i, \partial D_i] \right) \), \( \tilde{d}_{ij}^i = \frac{d_{ij}^i - \beta_i^0}{2} \), \( \Sigma_i^j \) and \( p_{a_{ij}^i} \in u_i(D_i) \) as in §3.

We call the data
the profile of \(q\). The profile of \(p\) is unique up to the obvious \(\prod_{i=1}^{r+1} \text{Sym}(s_i)\) group action. We denote by \(T(\phi)\) the sturdy tree associated with \(\phi\).

**Definition 26.** (a) For \(s = ((k,i,\beta),\sigma)\) a sturdy moduli specification, the set of (fixed-point) profiles for \(s\), \(\mathcal{P}(s)\), is defined as follows.

Suppose \(\beta = 0 \mod 2\). For \(s \geq 0\) we define \(\mathcal{P}_s(s)\) to be the set of tuples \((\ell, (d', a', \tilde{d}', \ell')_{j=1}^s)\) where: (i) \(\tilde{d}', ..., \tilde{d}'\) are non-negative integers and \(d', ..., d'\) are positive integers such that \(\beta = 2\sum_{j=1}^{s} (d' + \tilde{d}')\), (ii) we have \(l = \bigcup_{j=0}^{s} \ell\) and \(2(|l| + s) + |k| \geq 3\), and (iii) \(a' \in \{1,...,2m\}\) for \(1 \leq j \leq s\). We set \(\mathcal{P}(s) = \bigcup_{s=0}^{\infty} \mathcal{P}_s(s)\).

Suppose \(\beta = 1 \mod 2\). For \(k \neq \emptyset\) we set \(\mathcal{P}(s) = \emptyset\), otherwise we let \(\mathcal{P}(s)\) be the set of pairs \((\beta^0, a, \sigma)\) with \(\beta^0\) an odd integer and \(a \in \{1,...,2m\}\). For notational convenience we set \(l' = l + 1\) and \(\tilde{d}' = \frac{\beta - \beta^0}{2}\) in this case.

(b) For labeled tree \(T \in \mathcal{T}_r\), the set of profiles \(\mathcal{P}(T)\) for \(T\) is

\[
\mathcal{P}(T) = \prod_{i=1}^{r+1} \mathcal{P}(s_T(v_i)).
\]

There’s an obvious \(\text{Sym}(s) := \prod_{i=n_{\text{odd}}+1}^{r+1} \text{Sym}(s_i)\) action on \(\mathcal{P}(s)(T) \subset \mathcal{P}(T)\), where

\[
\mathcal{P}_s(T) := \prod_{i=1}^{n_{\text{odd}}} \mathcal{P}(s_T(v_i)) \times \prod_{i=n_{\text{odd}}+1}^{r+1} \mathcal{P}(s_T(v_i)).
\]

(c) We set \(\mathcal{P}_r' = \bigcup_{T \in \mathcal{T}_r} \mathcal{P}(T)\), \(\mathcal{P}_r^{r,20} = \bigcup_{r'\geq 20} \mathcal{P}_{r'}\) and \(\mathcal{P}_r = \bigcup_{r'\geq 20} \mathcal{P}_{r'}\) (so we only consider profiles with \(T(\phi)\) separated sturdy trees).

We will write \(n_{\text{odd}}(\phi), r(\phi)\) and \(r'(\phi)\) for the number of odd vertices, even-even edges, and odd-even edges, respectively. We denote by \((\beta^0_i(\phi), a_i(\phi), d_i(\phi), \sigma_i(\phi))\), \(1 \leq i \leq n_{\text{odd}}(\phi)\) the odd-vertex components of the fixed-point profile, similarly for the even vertices. For notational convenience, for \(1 \leq i \leq n_{\text{odd}}\) we set \(s_i(\phi) = 1\), \(a^1_i(\phi) = a_i(\phi)\), and \(d^1_i(\phi) = \beta^0_i(\phi)/2\) (this is not an integer). If \(\phi\) is clear from the context we may abbreviate \(\beta^0_i = \beta^0(\phi)\) etc.

The inclusion \(\mathcal{T}_r' \subset \mathcal{T}_r\) is \(\text{Sym}(r) \times \text{Sym}(r') < \text{Sym}(r + r')\) equivariant, and there’s an obvious action of \(\text{Sym}(r) \times \text{Sym}(r')\) on \(\mathcal{P}_{r'}\) compatible with the action on \(\mathcal{T}_r\).

Consider some \(T \in \mathcal{T}_r\) and \(\phi = (\phi_1,...,\phi_{r+1}) \in \mathcal{P}(T)\). We introduce some notation for the associated components.

Assume first \(\beta = 0 \mod 2\). We set

\[
(54) \quad \mathcal{M}_{D}(\phi) := \mathcal{M}_{k,\ell,\eta,\mu(s_1),...,(s_{r+1}),0}^{\{\phi_1,...,\phi_{r+1}\}}(pt) \times L
\]

\[
F_{D}(\phi) := \mathcal{M}_{D}^\tau(\phi) = \mathcal{M}_{k,\ell,\eta,\mu(s_1),...,(s_{r+1}),0}^{\{\phi_1,...,\phi_{r+1}\}}(pt).
\]

Hereafter, we sometimes write \(\mathcal{M}_{k,\ell,\eta,\mu(s_1),...,(s_{r+1}),0}^{\{\phi_1,...,\phi_{r+1}\}}(pt)\) for the moduli of discs, in place of \(\mathcal{M}_{k,\ell,\eta,\mu(s_1),...,(s_{r+1}),0}^{\{\phi_1,...,\phi_{r+1}\}}(pt)\) as in \(\mathcal{M}_{k,\ell,\eta,\mu(s_1),...,(s_{r+1}),0}^{\{\phi_1,...,\phi_{r+1}\}}(pt)\) to avoid confusion. We denote \(\mathcal{M}_{P_{i'}}(\phi) = \mathcal{M}_{k,\ell,\eta,\mu(s_1),...,(s_{r+1}),0}^{\{\phi_1,...,\phi_{r+1}\}}(pt)\) and let

\[
F_{P_{i'}}(\phi) := pt_{\eta,\mu(s_1),...,(s_{r+1})} \subset \mathcal{M}_{P_{i'}}(\phi)
\]
be the unique fixed point with smooth domain that corresponds to a degree $d_i^j$ branched cover of the line $p_0 p_{a_i}$. We also set
\[
\mathcal{M}_{\Sigma_i^j}(\phi) = M_{\Sigma_i^j} \cup_{\Phi_i^i} M_{\Sigma_i^j}^{d_i^j}
\]
\[
F_{\Sigma_i^j}(\phi) = \text{ev}_{\Phi_i^i}^{-1} \left( p_{a_i} \right)^T \subset M_{\Sigma_i^j}^{d_i^j}.
\]

Hereafter we adopt the ad hoc convention that in case $3d_i^j + |l_i| + 1 < 3$,
\[
M_{\Sigma_i^j} \cup \{\Phi_i^i\} \cdot d_i^j = X.
\]

with the evaluation maps given by the identity map to $X$. This corresponds to the cases in Lemma 25 where $\Sigma_i^j = \text{pt}$, and allows us to write formulas that treat these cases on equal footing with the other cases. We set
\[
F_{\Sigma_i^j}(\phi) = \left( F_{D_i}(\phi) \times \prod_{j=1}^{s_i} \left( F_{p_{a_i}}(\phi) \times F_{\Sigma_i^j}(\phi) \right) \right)
\]

Now consider $1 \leq i \leq r + 1$ such that $\beta_i = 1 \mod 2$. We set $M_{D_i}(\phi) = M_{\Phi_i^i \cdot, \cdot \cdot, \beta_i^0}$ and let
\[
F_{D_i}(\phi) = \prod_{j=1}^{s_i} M_{D_i}(\phi)^T
\]

be the unique fixed point with smooth domain passing through $p_{a_i}$. We set $M_{\Sigma_i^j}(\phi) = M_{\Phi_i^i \cdot, \cdot \cdot, \beta_i^0}$ using the same unstable convention \([55]\), and $F_{\Sigma_i^j}(\phi) = \text{ev}_{\Phi_i^i}^{-1} \left( p_{a_i} \right)^T \subset M_{\Phi_i^i \cdot, \cdot \cdot, \beta_i^0}$. We set
\[
F_{\Sigma_i^j}(\phi) = F_{D_i}(\phi) \times F_{\Sigma_i^j}(\phi).
\]

Finally, we set
\[
\tilde{F}_{\phi} = \prod_{i=1}^{r+1} \tilde{F}_{\Sigma_i^j} (\phi).
\]

Lemma 27. The open gluing maps of \([32]\) induce an isomorphism
\[
\left( \tilde{M}_{\mathcal{T}}^T \right) = \prod_{s} \left( \prod_{\phi \in \mathcal{T}} \tilde{F}_{\phi} \right)_{\text{Sym}(s)}
\]
3.3. Fixed-point formula for extended Forms. For \( \phi \in \mathcal{P}^{r,r'}_b \), we set \( F_{\phi} = \left( \text{For}_b^{r+1,\ldots,r+r'} \right)^{-1} (\hat{F}_{\phi}) \) (so we remember only the tails of the even-even edges) and \( \hat{F}_{\phi} = \left( \text{For}_b^{r+1,\ldots,r+r'} \right)^{-1} (\hat{F}_{\phi}) \). We let \( F_{\phi} \xrightarrow{\text{For}_b} \hat{F}_{\phi} \) be the pullback of \( \text{For}_b^{r+1,\ldots,r+r'} \). It follows from Proposition [23] that \( \hat{F}_{\phi} \) and \( F_{\phi} \) are suborbifolds with corners of \( \mathcal{M}_b^{r,r'} \) and of \( \mathcal{M}_b^{r,1,\ldots,r+r'} \), respectively, and that the pullback

\[
\hat{F}_{\phi} \xrightarrow{\text{For}_b} F_{\phi}
\]

of \( \text{For}_b^{r+1,\ldots,r+r'} \) is a proper submersion. We use \( \mathcal{F}_b^{r} \) to orient the fibers of \( \text{For}_b^r \), and define

\[
\xi_{\phi} = (-1)^{r'} r(\phi) \left( \text{For}_b^r \right) (\text{cd}_{b}^{r,r'} \times \cdots \times \text{cd}_{r}^{r,r'})_{\mathcal{P}_{\phi}} (\Lambda |_{L \times L(\Lambda)})^{gr}.
\]

Since \( \Lambda \) has odd degree, \([12]\) implies that

\[
(\xi_{\phi})_{\mathcal{T}(\phi)} = \text{sgn}(\mathcal{T}) \cdot \xi_{\phi}.
\]

A fixed point profile \( \phi \) is called odd-even if \( r'(\phi) = 0 \). An odd-even fixed-point profile is called sorted if \( T(\phi) \) is sorted.

**Lemma 28.** For \( \phi \in \mathcal{P}_{b,0}^{r} \) a sorted odd-even fixed point profile we have

\[
\xi_{\phi} = \prod_{j=1}^{n_{\text{odd}}} \left( \frac{\beta_j^0(\phi)}{2} \cdot \frac{\lambda_1 \cdots \lambda_m}{\alpha_{v_j}} \right)^{\deg(v_j)} \cdot \nu_0^{gr},
\]

where \( \alpha_0 \) is a local section of \( \text{Or}(TL) \) near \( p_0 \) chosen so \( \rho_0|_{p_0} = \lambda_1 \cdots \lambda_m \cdot \alpha_0 \), and \( \deg(v_j) \) is the number of edges incident to \( v_j \) in \( T = T(\phi) \). In particular, \( \xi_{\phi} \) is independent of the choice of \( \Lambda \).

Moreover, for any \( r' \geq 0 \) and \( \phi \in \mathcal{P}_{b}^{r,r'} \) we have \( \xi_{\phi} = \xi_{\text{cut},r'} \), so if \( \text{cut},r' \) \( \phi \) is sorted, \( \xi_{\phi} \) is given by the same formula.

**Proof.** For \( 1 \leq j \leq n_{\text{odd}} = n_{\text{odd}}(\phi) \), Let \( b_j = \min(a_j, 2m + 1 - a_j) \), so \( \rho_{a_j} \in \mathcal{P}(W_{b_j} \otimes \mathbb{C}) \).

If \( \sigma_{v_j} \in \sigma_T(v_j) \) the map

\[
\text{ed}_{i}^{b,r} |_{\mathcal{P}} : E \to L \times L
\]

factors through the inclusion

\[
M_j := \mathcal{P}(W_{b_j}) \times \mathcal{P}(W_0) \to L \times L,
\]

so the Cartan-Weil differential satisfies \( D|_{M_j} = -\lambda_{b_j} \iota_{v_j} \) (here \( \iota_{v_j} \) denotes contraction with the infinitesimal generator of the \( T \) action on \( L \times L \)). We have

\[
D\Lambda|_{M_j} = -\text{pr}_{a_j}^{*} \rho_0|_{M_j} = -\lambda_1 \cdots \lambda_m \text{pr}_{a_j}^{*} \alpha_0
\]

and so

\[
\Lambda|_{M_j} = \lambda_1 \cdots \lambda_m \cdot \frac{d\theta_{a_j}}{4\pi} \cdot \text{pr}_{a_j}^{*} \alpha_0
\]

where \( d\theta_{a_j} \) is an angular form for \( \mathcal{P}(W_{b_j}) \), oriented as the boundary of the unit disc in \( V_{2m+1-a_j} \in \mathcal{P}_C(W_{b_j} \otimes \mathbb{C}) \). Simple degree considerations show that \( \xi_{\phi} \) has de Rham degree zero, and is a locally constant form given by integration on the top dimensional strata of the fiber of \( \hat{F}_{\phi} \).
form $\prod_{j=1}^n ((\partial \Sigma_j (T) \cdot \deg(v_j))$. Pulling back $\prod A_{M_j}^{\deg(v_j)}$ and integrating (cf. Proposition 22), we obtain the result.

We check that for $\phi \in \mathcal{P}_{b}^{r'+ r'}$, $\xi_\phi = \xi_{cnt^{r'} \phi}$. There’s an obvious identification of the fiber of $For_{r'}$ and $For_{r+\text{cnt}^{r'} \phi}$ which respects the integrand, so it suffices to check that the orientations are the same. This follows from coherence of $\mathcal{F}_{r'}$; see [30, Proposition 21]. □

For $\phi \in \mathcal{P}_{b}^{r'+ r'}$ and $S \subset \{r+1, ..., r+r'\}$, let $F^S_{\phi} := \left(\text{For}_{b}^{r, r+ r', S}\right)^{-1} \mathcal{F}_{\phi}$ (cf. Proposition 23) and let $N^S_{\phi} \xrightarrow{\imath^S_{\phi}} F^S_{\phi}$ be the normal l-bundle associated with the closed embedding $F^S_{\phi} \rightarrow \mathcal{M}^S_{T(\phi)}$. We denote $N_{\phi} = N^1_{\phi} \cup \ldots \cup N^{r+1, \ldots, r+r'}_{\phi}$ and $\tilde{N}_{\phi} = N^S_{\phi}$. Consider the short exact sequence associated with smoothing the $(i, j, i', j')$ nodes (cf. (49))

$$0 \rightarrow M^S_{\phi} \xrightarrow{\mu^S_{\phi}} N^S_{\phi} \xrightarrow{\nu^S_{\phi}} S^S_{\phi} \rightarrow 0,$$

where

$$S^S_{\phi} = \bigoplus_{i=n_{odd}+1}^{+1} \bigoplus_{i,j=1}^{r+1} \mathbb{L}^{\nu}_{i,j} \oplus \bigoplus_{i,j=1}^{r+1} \mathbb{L}^{\nu}_{i,j} \delta^S_{\phi} \left( \begin{array}{c} - \alpha_{i,j} \\ d_{i,j} \end{array} \right).$$

Let $\gamma^S_{\phi} : N^S_{\phi} \rightarrow N^S_{\phi} T(\phi)$ and $\tilde{f}^S_{\phi} : N^S_{\phi} \rightarrow \tilde{N}_{\phi}$ denote the linearizations of $g^S_{T(\phi)}$ and $\text{For}_{\phi}^S$ respectively. The sequences (59) and the decomposition (60) are natural with respect to the forgetful maps and attaching maps. In particular, for every $\phi_i \in \mathcal{P}_{b}^{r'+ r'+1}$ with $\text{cnt}(\phi_i) = \phi \in \mathcal{P}_{b}^{r'+ r'}$ we have maps of short exact sequences

$$0 \rightarrow \partial^\phi \tilde{M}_{\phi} \rightarrow \partial^\phi \tilde{N}_{\phi} \rightarrow \partial^\phi \tilde{S}_{\phi} \rightarrow 0$$

$$0 \rightarrow M^{(R)}_{\phi} \xrightarrow{\mu^{(R)}_{\phi}} N^{(R)}_{\phi} \xrightarrow{\nu^{(R)}_{\phi}} S^{(R)}_{\phi} \rightarrow 0$$

$$0 \rightarrow \tilde{M}_{\phi} \rightarrow \tilde{N}_{\phi} \rightarrow \tilde{S}_{\phi} \rightarrow 0$$

where $R = r + r' + 1$. We set $\tilde{\gamma}_{\phi} := \tilde{f}^{(R)}_{\phi} \gamma^S_{\phi} \gamma_{\phi_i}, \tilde{\gamma}_{M_{\phi}} = \tilde{f}^{(R)}_{M_{\phi}} \gamma^S_{\phi}, \text{ and } \tilde{\gamma}_{S_{\phi}} = \tilde{f}^{(R)}_{S_{\phi}} \gamma^S_{\phi}$. Let

$$N_0 := (N_\Delta)_{(p_0, p_0)} \simeq T_{p_0} L,$$

be the fiber of the normal bundle to the diagonal $L \xrightarrow{\Delta} L \times L$, and for $r+i \in S$ let

$$\delta^{i+S}_{\phi} : N_0 \rightarrow N_0 \times F^{S}_{\phi}$$

be the linearization of $\text{cd}_i^{b, r+r'} \circ \text{For}_{\phi}^{r, S}$. We set $\delta^{i+S}_{M_{\phi}} = \delta^{i+S}_{\phi} \mu^{i+S}_{\phi}$. Since $M_{\phi}$ corresponds to nodal configurations, for which the even degree discs map to a point, there exists a map

$$\tilde{\delta}^{i+S}_{M_{\phi}} = \delta^{i+S}_{M_{\phi}} : M_{\phi} \rightarrow N_0 \times \tilde{F}_{\phi}$$
such that \( \delta^i S \) is the pullback of \( \delta^i M_\phi \). To be precise, \( (\text{id} \times \delta^i M_\phi) \delta^i S = \delta^i M_\phi \delta^i S \).

It follows from that there’s a cartesian map

\[
h_{M_\phi} : \partial^\phi S \rightarrow \ker(\delta^i M_\phi + 1)
\]

such that \( i_{\ker(\delta^i M_\phi + 1)} h_{M_\phi} = i_{\delta^i M_\phi + 1} \gamma_{M_\phi} \). We emphasize that for general \( \phi \), \( \delta^i \phi \) is not the pullback of any map \( \tilde{N}_\phi \rightarrow N_0 \times \tilde{F}_\phi \).

Recall that if \( E \rightarrow B \) is a \( T \)-equivariant bundle, with \( S(E) \rightarrow B \) the associated sphere bundle, a form

\[
\theta \in \Omega \left(S(E) \cdot \pi_S^* \text{Or}(E) \otimes \mathbb{R} \left[\hat{\lambda}\right]\right)^T
\]

is an equivariant angular form if \( \pi_S \theta = 1 \) and \( D\theta = -\pi_S^* e \) for some form \( e \) on \( B \) which we call the equivariant Euler form associated with \( \theta \) (see [29]; the discussion extends to the case where the base is an orbifold; here we also allow the rank of the bundle \( E \), and the degrees of the corresponding forms, to vary). If the action on \( B \) is trivial and the action on \( E \backslash B \) is fixed-point free, then \( e \) is invertible.

Consider some odd-even \( \phi \in \mathcal{P}_b^{r,0} \). We say an equivariant Euler \( e(M_\phi) \) for \( M_\phi = \tilde{M}_\phi \) is canonical if it is associated with an equivariant angular form \( \theta_M \) on \( S(M_\phi) \) such that for all \( \phi_+ \) with \( \text{cnt} \phi_+ = \phi \), we have

\[
\left(i_{\delta^i S(M_\phi)}^\phi \right) \theta_M \in \text{Im} S \left(h_{M_\phi} \right)^*.
\]

Here \( S \left(h_{M_\phi} \right) : \partial^\phi S(M_\phi) \rightarrow S \left(\ker(\delta^i M_\phi + 1)\right) \) is the map of sphere bundles induced from the cartesian map \( h_{M_\phi} \). We say an equivariant Euler form \( e(S_\phi) \) for \( S_\phi \) is canonical if it is associated with an angular form \( \theta_S \) on \( S(S_\phi) \) such that for all \( \phi_+ \) with \( \text{cnt} \phi_+ = \phi \) we have

\[
\left(i_{\delta^i S(S_\phi)}^\phi \right) \theta_S \in \text{Im} S \left(\tilde{\gamma}_{S_\phi} \right)^*.
\]

**Theorem 29.** Let \( b \) be a basic moduli specification and let \( \omega = \{\tilde{\omega}_r\} \in \Omega_b \) be an extended form with \( D\omega = 0 \). For every \( r \geq 0 \) and \( \phi \in \mathcal{P}_b^{r,0} \), canonical Euler forms \( e(M_\phi) \) and \( e(S_\phi) \) exist, and for any choice of such forms we have

\[
\int_b \omega = \sum_{r \geq 0} \sum_{\phi \in \mathcal{P}_b^{r,0}} \oint_{r \cdot S(\phi)} \xi_{\phi} \int_{F_\phi} e(M_\phi)^{-1} e(S_\phi)^{-1} \tilde{\omega}_r.
\]

where \( s(\phi) = \prod S_i(\phi) \).

**Corollary 30.** The integral \( \int_b \omega \) is independent of the choice of equivariant homotopy kernel \( \Lambda \).

**Remark 31.** We can rewrite (66) as a sum over isomorphism types of fixed-point profiles. More precisely, choose a subset \( W \subset \mathcal{P}_b \) of an even-odd profiles such that, for every \( r \geq 0 \), \( T \in \mathcal{J}_b^{r,0} \) with \( n_{\text{odd}} \) odd vertices, and \( s = (s_{n_{\text{odd}} + 1},...,s_{r + 1}) \in \mathbb{Z}^{r-n_{\text{odd}}+1} \), \( W \) intersects every orbit of the Sym \((r) \times \text{Sym}(s)\) action on \( \mathcal{P}_b(T) \) in precisely one element. By (41) and (58) we can rewrite (66) as

\[
\int_b \omega = \sum_{\phi \in W} \xi_{\phi} \int_{F_\phi} e(M_\phi)^{-1} e(S_\phi)^{-1} \tilde{\omega}_r.
\]
where Aut $\phi < \text{Sym}(r) \times \prod \text{Sym}(s_i(\phi))$ is the stabilizer subgroup of $\phi$. This has obvious computational advantages.

3.4. Computing equivariant open Gromov-Witten invariants. In this section we apply Theorem 29 to compute the equivariant open Gromov-Witten invariants. We write an explicit formula for the fixed-point contributions in Proposition 32 use this to compute a few examples, and finally derive the generating function expression presented in Theorem 2.

3.4.1. Fixed-point contributions. For $\omega$ given by (??) the contributions to (66) can be computed in terms of the closed correlators and descendant integrals, as follows. If $f$ is a power series in $x, y, \ldots$ we use the notation $f[x^a y^b \ldots] = (\partial_x^a \partial_y^b f)_{x=0, y=0, \ldots}$. This gives the coefficient of $\prod_i x_i^{a_i} y_i^{b_i} \ldots \prod_j z_j^{c_j} \ldots$ for $f(x, y, \ldots)$.

Computing equivariant open Gromov-Witten invariants.

**Proposition 32.** If $\phi \in P_b^+$ is sorted, we have

\[(67) \quad \xi_\phi \int_{\mathcal{P}_b} \epsilon (M_\phi)^{-1} \epsilon (S_\phi)^{-1} \bar{\omega}_\tau = \]

\[\left( -1 \right)^{\binom{n_{\text{odd}}}{2} (1 + m)} \prod_{i=n_{\text{odd}}+1}^{r+1} \frac{(\lambda_1 \cdots \lambda_m)^{k_i}}{\lambda_1 \cdots \lambda_m} \sum_{b \in C_{0,1}} \prod_{j=1}^{s_i} \left( -\frac{d_{i,j}^0}{\alpha_{i,j}} \right)^{b_{i,j}+1} F_0 [t_{b_1} \cdots t_{b_{s_i}} | 0]\right) \times \]

\[\left( \prod_{j=1}^{s_i} \frac{(\beta^0_j)^{\beta^0_{i,j}-1}}{(2 \alpha_{i,j})^{\beta^0_{j}}} \prod_{a' \neq \alpha_{i,j}, 0 \leq c \leq d_{i,j}} \left( \frac{c}{\alpha_{i,j} - \alpha_{a'}} \right) \prod_{a' \neq \alpha_{i,j}, 0 \leq c \leq d_{i,j}} \frac{\prod_{i=1}^{n_{\text{odd}}} (\beta^0_j)^{\beta^0_{i,j}-1}}{(2 \alpha_{i,j})^{\beta^0_{j}}} \prod_{a' \neq \alpha_{i,j}, 0 \leq c \leq d_{i,j}} \left( \frac{c}{\alpha_{i,j} - \alpha_{a'}} \right) \right) \times \]

\[\times \prod_{1 \leq i \leq r + 1} (2 d_{i,j})^{\beta^0_{i,j}-1} Z_{\alpha_{i,j}} \left( h = \frac{\alpha_{i,j}}{d_{i,j}} \right) \left[ q^{2 d_{i,j}} \prod_{x \in \mathcal{T}(\phi)} \eta_{x} \right] \]

Here $\deg(v_i)$ is the number of edges of $\mathcal{T}(\phi)$ incident to $v_i$. $\tilde{k}_i := k_i \setminus \{s_i'\}$, so $k = \prod_{i=1}^{n_{\text{odd}}} \tilde{k}_i$.

**Proof.** We discuss orientations. The sign $(-1)^{\binom{n_{\text{odd}}}{2} (1 + m)}$ comes from Proposition 22 and allows us to treat each moduli factor $\mathcal{M}_s$ separately. Consider some $s = ((k, \lambda, \beta), \sigma)$. If $\beta = 1 \mod 2$ then $\mathcal{J}_s$ determines an orientation for $\text{Or}(TM_s)$. If $\beta = 0 \mod 2$ we use the local section $\epsilon_0$ of $\text{Or}(TL) |_{\mathbb{P}_s}$ to orient $\bigotimes_{x \in \mathcal{T}(\phi)} \text{ev}_x^{-1} \text{Or}(TL) |_{\mathbb{P}_s}$ and then $\mathcal{J}_s$ determines an orientation for $\text{Or}(TM_s)$. The fixed-point components
$F_{p_i}, F_{S_i}$ and $F_{D_i}$ for $1 \leq i \leq n_{odd}$ are oriented by the complex structure or as zero-dimensional orbifolds. The even degree disc fixed-point components, $F_{D_i}$ for $n_{odd} + 1 \leq i \leq r + 1$, are oriented by [21] Lemma 2.22. [60] is naturally oriented by the complex structure. These conventions determine an orientation for $N_\phi$, and $M_\phi$ (see (52)). Thus we consider $e(M_\phi), e(S_\phi)$ as taking values in the trivial local system. Since (50) commutes with $\check{J}_\star$, we can use $\check{J}_\star$ to orient just the normal bundles $N_{D_i}(\phi)$ associated with $F_{D_i} \rightarrow M_{D_i}$ for $1 \leq i \leq r + 1$, using the complex orientation for everything else. For $n_{odd} + 1 \leq i \leq r + 1$ we have $N_{D_i}(\phi) \cong T_{p_i}L$ and the orientation $\check{J}_\star$ agrees with orientation $o_0$.

We compute $e(S_\phi)$. Consider (60). Note that if we forget the $T$-action, $L_{\ast_{i,j}}^\phi(\frac{-\alpha_{i,j}}{d_i}) = L_{\ast_{i,j}}^\phi$. Consider some $\phi_+ \in P^{r,1}$ with $\text{cut} \phi_+ = \phi$. Write $T_\ast = T(\phi_+)$. Let $v_i, v_j \in (T_\ast)_0$ denote the endpoints of the $(r + 1)$st edge $e$. We have $\beta_{T_\ast}(v_i) = \beta_{T_\ast}(v_j) = 0 \text{ mod } 2$, so the orientation of $e$ is determined by the parity of $|k_{T_\ast}(v_i)|$ and so the boundary condition $\gamma_{S_{\phi_+}}$ agrees with the map used to define the descendant integrals of discs (cf. [2,23]). Since $\gamma_{S_{\phi_+}}$ respects the direct sum decomposition (60) we can construct a canonical angular form $\theta_S$ as the join $\theta_S = \star_{i,j} \theta_{i,j}$ of $T$-invariant canonical angular 1-forms $\theta_{i,j}$ for $L_{\ast_{i,j}}^\phi$. It follows that

$$e(S_\phi) = \prod_{i=1}^{n_{odd}} \prod_{j=1}^{r+1} \left( \frac{-\alpha_{i,j}}{d_i} - \psi_{i,j} \right).$$

We turn our attention to $e(M_\phi)$. We denote

$$R_\phi := \prod_{i=1}^{n_{odd}}(\phi) \times \prod_{i=1}^{n_{odd}} \mathcal{M}_{\mathcal{T}(\phi)(v_i)} \times \prod_{i=1}^{n_{odd}+1} \mathcal{M}_{\mathcal{L}_{\mathcal{T}^\ast_{i,j}}(\phi_+)} \times \prod_{i=1}^{n_{odd}+1} \mathcal{M}_{\mathcal{L}_{\mathcal{T}^\ast_{i,j}}(\phi_+)}$$

(this parameterizes everything but the energy zero disc components). Let $N_{R_\phi}$ be the normal bundle to the fixed points $R^T_\phi \rightarrow R_\phi$. By (64) the normal bundle to $F_{D_i}(\phi) \rightarrow M_{D_i}(\phi)$ for $n_{odd} + 1 \leq i \leq r + 1$ is $TL$, and we obtain a short exact sequence

$$0 \rightarrow M_\phi \rightarrow T_{p_i}L_{n_{even}}^\phi \oplus N_{R_\phi} \rightarrow (T_{p_i}X)^N \rightarrow 0,$$

of bundles over $F_{T}(\phi)$ where $N = \sum_{i=1}^{n_{odd}+1} s_i$. For every $\phi_+$ with $\text{cut} \phi_+ = \phi$, we have a map of short exact sequences

$$0 \rightarrow \partial \phi, M_\phi \rightarrow T_{p_i}L_{n_{even}}^\phi \oplus N_{R_\phi} \phi \rightarrow (T_{p_i}X)^N \rightarrow 0,$$

$$0 \rightarrow \ker \phi, M_\phi \rightarrow T_{p_i}L_{(n_{even}+1)}^\phi \oplus N_{R_\phi \phi} \phi \rightarrow (T_{p_i}X)^N \rightarrow 0.$$

9 Let $E_0, E_1$ be vector bundles over $B$ of even rank. We have a map

$$S(E_0) \times B S(E_1) \times [0,1], \quad q \rightarrow S(E_0 \oplus E_1),$$

and we define $\theta_0 \ast \theta_1$ by

$$q^*(\theta_0 \ast \theta_1) = (1-t) d\theta_0 \theta_1 + t \theta_0 d\theta_1 + \theta_0 \theta_1 dt.$$

The case of $n$ direct summands can be defined recursively (a more symmetric definition can also be given along these lines).
where the middle vertical map is the direct sum of (i) the diagonal inclusion \( T_{p_0} \mathcal{L} \rightarrow T_{p_0} \mathcal{L} \) corresponding to the edge \( r + 1 \) and (ii) an isomorphism

\[ N_{\mathcal{R}_\phi} \cong N_{\mathcal{R}_{\phi^+}} \]

induced by the obvious identification \( \mathcal{R}_\phi \cong \mathcal{R}_{\phi^+} \). Use the \( GL(2m+1, \mathbb{C}) \) action on \( \mathcal{R}_\phi \) to fix a map \( \bigoplus_{i,j} \mathbb{C} \rightarrow \mathcal{R}_\phi \) so \( \alpha_\phi \circ (\sigma, 0) = \text{id} \). By (69) \( (\sigma, 0) \) defines a section of \( \alpha_\phi \), compatible with (68). It follows that there exists a canonical equivariant angular form \( \theta_M \) for \( M_\phi \) with

\[ e(M_\phi) = \frac{e_{\mathcal{R}} \cdot \left( \prod_{j=1}^m \lambda_j \right)^{n_{\text{even}}}}{\left( \prod_{j=1}^{2m} \alpha_j \right)^N} \]

where \( e_{\mathcal{R}} \) is an equivariant Euler form for \( N_{\mathcal{R}_\phi} \), pulled back from the orbifold without boundary \( \mathcal{R}_{\phi^+} \).

The computation of \( e_{\mathcal{R}} \) is standard. See, for example, [10, Chapter 27]. We use a computation similar to [19, Lemma 6] to fix the weights for the normal bundle to the odd disc moduli (this determines the sign of the term on the fourth line of (67)). We stop computing after we express \( e_{\mathcal{R}} \) in terms of the equivariant Euler forms to the normal bundle \( F_{\Sigma_j}(\phi) \rightarrow M_{\Sigma_j}(\phi) \).

We integrate. The term \( \sum_{\mathcal{R}_{\phi}} \prod_{j=1}^m \left( -\frac{d_l}{\alpha_{l,j}} \right)^{b_j+1} F_0 \left[ t_{b_1} \cdots t_{b_{n_{\text{odd}}}} s_{0}^N \right] \) comes from expanding \( e(S\phi)^{-1} \) as a power series in \( \psi_{i,j} \) and integrating on \( F_{D, i}(\phi) \). The closed correlator \( Z_{\Sigma_j}(h = \alpha_{a_{l,j}}/d_l \left[ q^{2d_l} \prod_{x \in \Sigma_j} \eta_{c_x} \right] \) comes from summing over the contributions of \( F_{\Sigma_j}(\phi) \subset M_{\Sigma_j}(\phi) \) (keeping the convention (55) in mind). The factor \( \prod_{i=1}^{n_{\text{odd}}} \left( -\frac{\beta_i \lambda_i}{2} \right)^{\deg(v_i)} \) comes from Lemma 28. \( \square \)
Proof of theorem. Recall that $E \in \mathcal{R} := \mathcal{R} \left[ \left[ u^{-1}, u, \nu_{a,d}, q, \eta_0, ..., \eta_{2m}, s, s_0, t_0, t_1, ..., \right] \right]$ was given by

$$E = \exp \left( \sum_{a \geq 0} \sum_{d > 0} Z_a \left( \frac{\alpha_a}{d} \right) \frac{C}{\partial_{\nu_{a,d}}} \right) \exp \left( \sqrt{-1} \frac{1}{d} \sum_{a \geq 0} \sum_{d > 0} \nu_{a,d} \right) \exp \left( \sum_{i \geq 0} \left( - \frac{d}{\alpha_a} \right)^{i+1} \frac{A}{\partial_i} \right) \exp \left( \frac{B}{\partial_{\nu_{a,d}}} \right)$$

Expanding the exponents and the generating functions, we can express $E$ as a sum over diagrams with vertices of type I, II, III or IV and edges of type A, B or C, in accordance with the labeling above. The vertices and edges of a given type are numbered, and assigned additional data as necessary to specify their contribution to $E$.

$E - 1$ belongs to the ideal $I \triangleleft \mathcal{R}$ generated by $q, \nu_{a,d}, \eta_0, ..., \eta_{2m}, s, s_0, t_0, t_1, ..., $ and if we let $E_0 \in I$ denote the sum over nonempty connected diagrams, then $(E - 1)|_{\nu_{a,d}=0} = \sum_{a \geq 1} \frac{1}{a!} \left( E_0 |_{\nu_{a,d}=0} \right)^a$. It follows that $\partial_n |_{\nu_{a,d}=0} \log \left( E |_{\nu_{a,d}=0, t=0, s=0} \right)$ is given by a sum over diagrams whose underlying graph is a tree. The contribution of each tree diagram is easily identified with the contribution of some (non-unique) fixed-point profile $\phi \in \mathcal{P}_b$ to $\mathcal{G}$. This many-to-many relation between fixed-point profiles and tree diagrams can be turned into a bijection of sets by adding labelings to both the profiles and to the tree diagrams. In other words, we have a correspondence

$$\mathcal{P}_b \leftrightarrow \{ \text{common denominator diagrams} \} \to \{ \text{tree diagrams} \}$$

and equation (7) follows by computing the size of the fiber of the two maps in this correspondence.

3.5. Examples and applications. Let $m = 1$ and consider invariants $I_1 (k, (0, l), \beta)$ with $\deg I_1 (k, (0, l), \beta) = 0$. Let $b = (k, l, \beta)$ be a basic moduli specification with $|k| = k, || = l$. As discussed in [30] (see Remark 3 and the discussion at the end of §1) we have

$$I_1 (k, (0, 0, l), \beta) = \int_{\mathcal{M}_b \omega_0},$$
so in this case, the contributions of $\mathcal{M}_b^r$ with $r \geq 1$ in [45] vanish. Moreover, by introducing non-equivariant deformations of $\omega_0$ this integral can be interpreted as a (signed) count of discs or twice Welschinger’s count of rational real curves [27], with suitable constraints.

**Example 33.** Consider

$$I = I_1 (2, (0, 0, 0), 1) = 2,$$

 twice the number of lines through a pair of points in the plane. Let $b = \{1, 2\}, \emptyset, 1\}$ be the corresponding basic moduli specification. Note that $\mathcal{M}_b$ has no fixed points, but $\partial \mathcal{M}_b \neq \emptyset$, see Remark [10]. The only odd-even tree for $b$ is

$$\Gamma = (1) \rightarrow (0) \ldots$$

with $\mathfrak{g}_\Gamma (v_1) = ((\emptyset, \emptyset, 1), s'_1)$, $\mathfrak{g}_\Gamma (v_2) = ((\{1\}, 1, 2), \emptyset, \emptyset)\}.$

$$\mathcal{P} (\Gamma) = \{ \phi_1, \phi_2 \}$$

where $\phi_i$ is specified by the map of the $v_1$ disc, $a_1 (\phi_i) = i$. $F_\Gamma (\phi_i)$ is an isolated fixed point, and

$$\int_{F_\Gamma (\phi_i)} \sigma_{\Gamma} \cdot \tilde{\omega}_1 \cdot \xi^A = \frac{\lambda^2}{\lambda_1} \frac{\prod_{\alpha_i} \alpha_i^3}{\prod_{\alpha_i} \alpha_i^2} \cdot \frac{1}{\theta} \left( \prod_{\alpha_i} \frac{1}{\alpha_i} \right) \cdot \left( \frac{1}{\alpha_i} \right) \cdot \left( \frac{1}{\alpha_i} \right) \cdot Z_{\alpha_i} (h = 2\alpha_{a_i}) [1] = 1$$

so

$$I = \int_{\mathcal{M}_b^r} \omega_0 = 2,$$

as expected.

**Example 34.** Consider $I_m (1, \emptyset, 2) \in \mathbb{R} [\lambda_1, \ldots, \lambda_m]$. Since

$$\deg (\omega_r - \lambda_r \beta_r) - \dim \mathcal{M}_b^r = 2m - [(2m + 1) \cdot 2 + (2m - 3) + 1] = -4m < 0$$

we find that $\int_b \omega = 0$. On the other hand, this can be expressed as the sum of contributions from the following two $\text{Sym} (r)$ orbits (cf. Remark [31]) of odd-even diagrams:

$$\Gamma_1 : (2) \ldots$$

$$\Gamma_2 : (1) - (0) \ldots (1)$$

Note $\Gamma_1$ has $(-1)^{(m+1)(m)} = 1$, whereas $\Gamma_2$ has $(-1)^{(m+1)(m)} = (-1)^{m+1}$ and $|\text{Aut} \Gamma_2| = 2$. In sum we have

$$0 = \sum_{a_1 \in \{1, \ldots, 2m\}} \prod_{a_i = 0, a_1}^{2m} \alpha_{a_i} \prod_{a_i = 0, a_1}^{2m} \left( \alpha_{a_i} - \alpha_{a_i'} \right) \left( -\alpha_{a_i} \right)$$

$$\frac{1}{2} \cdot (-1)^{m+1} \cdot 2 \times \sum_{a_2 \in \{1, \ldots, 2m\}} \prod_{\alpha_i' \in \mathcal{M}_b^r \{2m+1-a_2\}} \frac{1}{\alpha_{a_i} - \alpha_{a_i'}} \left( -\alpha_{a_i} - \alpha_{a_i'} \right) \left( \frac{1}{2\alpha_{a_2}} \right).$$

We used a computer to verify that this relation holds for $1 \leq m \leq 8$. 
4. Fixed-point localization by steepest descent

In this section we use steepest descent to compute the limit of certain integrals on an orbifold $X$ in terms of the asymptotic behavior near a 1-orbifold $Z \subset X$. The limit is expressed using the equivariant Segre form, whose relation to the equivariant Euler of the normal bundle to $Z \subset X$ and other properties will be discussed in §3.1.

The classical localization formula for $\partial X = \emptyset$ is then derived as an easy corollary.

Let $X$ be a compact $T$-orbifold with corners, $Z = \bigsqcup_{n \geq 0} Z_n$ a local $T$-orbifold with corners, and $i : Z \hookrightarrow X$ an equivariant closed embedding. Let $\pi : N \to Z$ denote the associated normal 1-bundle $N = i^*TX/TZ$, with $\mu_n : N \to N$ scalar multiplication by $a \in \mathbb{R}_{>0}$. Let $\varepsilon = \sum_{s < -\infty} s^2 \mu_{s-1}^{\star} \eta = (i \circ \pi)^* \eta$ and

$$
\varepsilon := \lim_{s \to \infty} s^k \mu_{s-1}^{\star} \left( \gamma^{\star} \eta - \sum_{0 \leq j < k} \eta^{(j)} \right),
$$

(70)

then $\eta^{(k)}$ is independent of the choice of tubular neighbourhood.

**Definition 35.** Let $Z \hookrightarrow X$ be a $T$-equivariant closed embedding of compact $T$-orbifolds with corners. A form $\eta \in \Omega \left( X; \mathcal{R} \right)^T = \Omega \left( X; S^{-1} \mathcal{R} [\lambda] \right)^T$ will be called localizing to $Z$ if

(a) $\eta = \sum_{i=1}^N \eta_i$ for invariant 1-forms $\eta_i \in \Omega^1 (X; \mathcal{R})^T$ and nonzero tuples $\zeta_i = (\zeta_i^1, \ldots, \zeta_i^m) \in \mathbb{Q}^m \setminus \{0\}$, where $\zeta_i, \lambda \in \sum_{j=1}^m \zeta_i^j \lambda_j$.

(b) $f := \sum \lambda_j \lambda_i \eta$ is a real-valued, non-negative function with $f^{-1} (0) = Z$.

(c) $f (0), f^{(1)}$ vanish and $f^{(2)} := \lim_{s \to \infty} \mu_{s-1}^{\star} \left( s^2 f \right) |_N$ is a positive definite quadratic form on each fiber of $N$.

**Definition 36.** We say a $T$-orbifold with corners $X$ has regular fixed points if the map $X^T \to X$ is a closed embedding (cf. Definition 77).

All the orbifolds we’ll encounter have regular fixed points, see §3.2. If $X$ has regular fixed points a form $\eta$ localizing to $X^T$ is called a fixed-point localizing form.

We let $i^\partial_X : \partial X \to X$ and $i^\partial : \partial^2 X \to \partial X$ denote the structure maps. Note that $\partial^2 X$ is equipped with a $\mathbb{Z}/2$ action.

**Proposition 37.** Suppose $X$ has regular fixed points. Given a fixed-point localizing form $\eta_0 \in \Omega \left( \partial X; \mathcal{R} \right)^T$ such that $\left( i^\partial_X \right)^* \eta_0$ is $\mathbb{Z}/2$-invariant, there exists a fixed-point localizing form $\eta \in \Omega \left( X; \mathcal{R} \right)^T$ with $\left( i^\partial \right)^* \eta = \eta_0$.
Proof. Fix an equivariant tubular neighborhood

\[ N > \mathcal{V} \xrightarrow{\gamma} \mathcal{X} \]

for \( \mathcal{X}^\mathbb{C} \to \mathcal{X} \) (cf. Lemma 80). We construct an atlas \( \pi : U \to \mathcal{X} \) for \( \mathcal{X} \), with \( U = \bigsqcup U_p \) where \( p \) ranges over a set of representatives for the points \( pt \to \mathcal{X} \) of \( \mathcal{X} \), as well as a provisional form

\[ \eta'' = \bigsqcup \eta''_p \in \Omega \left( \bigsqcup U_p; \mathbb{R} \right). \]

We divide the construction into cases.

Case 1: \( p \) is an interior point which is not a fixed point. There exists a basis \( e'_1, \ldots, e'_m \) over an open \( U \subset \mathbb{R}^m \) with the property that the corresponding vector fields \( \xi'_1, \ldots, \xi'_m \) on \( \mathcal{X} \) satisfy the following condition at \( T_p \mathcal{X} \), for some \( 1 \leq r \leq m \):

\[ \xi'|_p, \ldots, \xi'|_p \text{ are linearly independent, and for } j \in \{ r+1, \ldots, m \}, \xi'|_p = 0. \]

Choose 1-forms

\[ \gamma'_1, \ldots, \gamma'_r \in \Omega^1 \left( \mathcal{X}; \mathbb{R} \right) \]

that satisfy \( \gamma'_i \left( \xi'_j \right)|_p = \delta_{ij} \) for \( 1 \leq i, j \leq r \). Let \( \lambda'_1, \ldots, \lambda'_m \) be a basis for \( \mathcal{H}^2_\mathbb{R} \) dual to \( e'_1, \ldots, e'_m \). We see that \( f := \sum_{j=1}^{m} \lambda'_j \xi_j \left( \sum_{i=1}^{r} \gamma'_i \right) \) is a real-valued function on \( \mathcal{X} \) with \( f \left( p \right) = r \), so it remains positive in an open neighborhood \( U_p \) of \( p \) and we set

\[ \eta''_p = \left( \sum_{i=1}^{r} \gamma'_i \right)|_p. \]

Case 2: \( p \) is an interior fixed point of the \( T \)-action. Choose a trivialization

\[ N|_U \simeq U \times \bigoplus_{i=1}^{b} C_{\mathcal{X}_i} \]

over an open \( p \in U \subset \mathcal{X}^\mathbb{C}. \) (71) becomes

\[ U \times \bigoplus_{i=1}^{b} C_{\mathcal{X}_i} \supset U_p \subset \mathcal{X} \]

Here \( C_{\mathcal{X}_i} \) denotes the rank one complex \( T \)-representation with fractional weight \( \mathcal{X}_i \). Using polar coordinates \( z_i = r_i \cdot e^{\sqrt{-1} \theta_i} \) for \( C_{\mathcal{X}_i} \), we can set

\[ \eta''_p = \sum_{i=1}^{b} r_i^2 \frac{d \theta_i}{(\xi_i, \lambda)} \]

on \( U_p \). Clearly \( f = \sum_{j=1}^{m} \lambda'_j \xi_j \eta''_p = \sum_{i=1}^{b} r_i^2 \) vanishes only on \( U \times 0 = \mathcal{X}^\mathbb{C} \times U_p \), and induces a positive definite quadratic form on the normal bundle.

Before we deal with points of positive depth, we need the following. Write \( \eta_\theta \left( \mathcal{X}^\mathbb{C} \right) \) as in Definition 33 and fix 1-forms \( \tilde{\eta}_i \in \Omega^1 \left( \mathcal{X}; \mathbb{R} \right) \) with \( \check{f} \tilde{\eta}_i = \eta_{\theta, i} \) (cf. Lemma 52) and set \( \tilde{\eta} = \sum_{i=1}^{N} \left( \tilde{\eta}_i \right|_{G} \) \in \Omega \left( \mathcal{X}; \mathbb{R} \right) \) and \( \check{f} := \sum \lambda_j \xi_j \tilde{\eta} \).

Case 3: \( p \) has depth \( c \geq 1 \), \( p \) is not a fixed point. In this case, there’s an open neighborhood \( p \in U_p \subset \mathcal{X} \) with \( \check{f}|_{U_p} > 0 \), and we take \( \eta''_p = \tilde{\eta}|_{U_p} \).

Case 4: \( p \) has depth \( c \geq 1 \), \( p \) is a fixed point. \( p \) has depth \( c \) in \( \mathcal{X}^\mathbb{C} \) also, and the normal bundle to \( \left( \partial \mathcal{X} \right)^\mathbb{C} \to \partial \mathcal{X} \) is identified with \( \left( i_{\mathcal{X}}^* \right)^* \left( \mathcal{H}^2_\mathbb{R} \right) \). \( \left( i_{\mathcal{X}}^* \right)^* \tilde{\eta} = \eta_\theta \) has positive definite quadratic form at \( p \), the same is true for an open neighborhood \( p \in U \subset \mathcal{X}^\mathbb{C} \) and we take \( U_p = N|_U \cap \mathcal{V} \) and \( \eta''_p = \tilde{\eta}|_{U_p} \).

We construct a groupoid \( X_1 \to X_0 \) representing \( \mathcal{X} \) with \( X_0 = \bigsqcup U_p \). Since \( \mathcal{X} \) is compact there exists a compactly supported function \( \rho : X_0 \to [0, 1] \) which
satisfies $t \cdot s^* \rho \equiv 1$ (such a function is called a partition of unity for $X$). We define $\eta' \in \Omega(X; \mathcal{R})$ by $\eta' = t \cdot s^* (\rho \eta'')$, and $\eta \in \Omega(X; \mathcal{R})$ by averaging out the $\mathbb{T}$-action on $\eta'$ using a Haar measure on $\mathbb{T}$. This $\eta$ is seen to be a fixed-point localizing form with $(i^0_X)^* \eta = \eta|_{\partial}$.

We now describe a setup for steepest descent analysis.

**Definition 38.** Let $\mathcal{X}$ be a compact $\mathbb{T}$-orbifold, $\mathcal{Z} = \bigsqcup \mathcal{Z}_n$ a compact local $\mathbb{T}$-orbifold, and $\mathcal{Z} \to \mathcal{X}$ a $\mathbb{T}$-equivariant closed embedding with associated normal bundle $N \to \mathcal{Z}$. A $\mathcal{Z}$-localizable morphism consists of

- An equivariant tubular neighborhood $N \xrightarrow{j} \mathcal{V} \to \mathcal{X}$ for $\mathcal{Z}$,
- a $\mathbb{T}$-equivariant diagram of $\mathbb{T}$-orbifolds with corners

\[
\begin{array}{c}
\xymatrix{ \tilde{N} \ar[d]_{\tilde{j}} & \tilde{\mathcal{V}} \ar[d]_{\tilde{\gamma}} & \tilde{\mathcal{X}} \ar[d]_{\tilde{j}} \\
N & \mathcal{V} & \mathcal{X} 
}\end{array}
\]

with both squares cartesian, $\tilde{\mathcal{X}}$ compact, and $\tilde{j}, \tilde{j}$ relatively oriented, and

- A smooth map $\tilde{N} \times \mathbb{R}_{\geq 0} \xrightarrow{\tilde{\mu} = (\tilde{\mu}_j)_{j \in \mathbb{N}}} \tilde{N}$ such that $\tilde{j} \circ \tilde{\mu} = \mu \circ (\tilde{j} \circ \tilde{j})$ where $\mu : N \times \mathbb{R}_{\geq 0} \to N$ denotes scalar multiplication.

We’ll abbreviate and write “$\tilde{j} : \tilde{\mathcal{X}} \to \mathcal{X}$ is a $\mathcal{Z}$-localizable morphism”, keeping the additional data implicit. A simple diagram chase shows that $\tilde{\mu}^{-1} (\tilde{\mathcal{V}}) \xrightarrow{\tilde{j}} \tilde{N}$ is a diffeomorphism and $\tilde{\rho}_0 := \tilde{j}^* \tilde{\rho}_0 (\tilde{\mu}_0^*)^{-1} : \tilde{N} \to \tilde{\mathcal{V}}$

lies over $\rho_0$, the retraction $N \to \mathcal{Z} \to \mathcal{V}$.

**Proposition 39.** Let $\tilde{j} : \tilde{\mathcal{X}} \to \mathcal{X}$ be a $\mathcal{Z}$-localizable morphism, with the additional data denoted as in Definition 38. Let $\eta \in \Omega(X; \mathcal{R})$ be $\mathcal{Z}$-localizing, and denote $\tilde{\eta} = \tilde{j}^* \eta$. Then for any $\tilde{\omega} \in \Omega\left(\tilde{\mathcal{X}}; \text{Or}_1 (\tilde{\mathcal{V}})^\vee \otimes \mathcal{R}\right)$, we have

\[
\lim_{s \to \infty} \int_{\tilde{\mathcal{X}}} e^{s^2 D\tilde{\eta}} \tilde{\omega} = \int_{\tilde{N}} \tilde{j}^* \left( e^{D\eta (\tilde{\omega})} \right) (\tilde{\rho}_0)^* \tilde{\omega}.
\]

Let us explain what we mean by $(\ref{eqn:prop39})$. Writing $D\tilde{\eta} = -\tilde{\tilde{f}} + d\tilde{\eta}$ for $\tilde{\tilde{f}} = \sum_{j=1}^m \lambda_j t_{\xi_j} \tilde{\eta}$ a real-valued function, we define

\[
e^{s^2 D\tilde{\eta}} := e^{-s^2 \tilde{\tilde{f}}} \sum_{i=0}^\infty \frac{(s^2 \tilde{d}\tilde{\eta})^i}{i!},
\]

where the sum has only finitely many nonzero terms since $d\tilde{\eta}$ is nilpotent, similarly for $e^{D\eta (\tilde{\omega})}$. It follows that there are finitely many linearly independent rational functions $q_j \in \mathcal{R}, 1 \leq j \leq M$ such that $\int_{\tilde{N}} \beta^*_N \left( e^{D\eta (\tilde{\omega})} \right) \tilde{\rho}_0^* (\tilde{\omega} \tilde{\gamma})$ and $\int_{\tilde{\mathcal{X}}} e^{s^2 D\tilde{\eta}} \tilde{\omega}$ (for all $s$) are elements of the finite dimensional vector space $\mathbb{R} \langle q_1, ..., q_M \rangle$. The limit in $(\ref{eqn:prop39})$ is interpreted with respect to the usual topology on $\mathbb{R}^M$. Note that there
are $\bar{a}_1, \ldots, \bar{a}_M \in \mathbb{R}^m$ belonging to the common domain of definition of $\{q_j\}$ such that $[q_{j_1}(\bar{a}_j)]_{j_1,j_2}$ is an invertible matrix, so it is enough to check that

$$\lim_{s \to \infty} \int_{\tilde{X}} \left( e^{s^2 D\tilde{\eta}} \tilde{\omega} \right)_{\text{top}}^{\text{top}} = \int_{\mathcal{N}} \tilde{\rho}_0^* \left( \tilde{\omega} |_{\mathcal{V}} \right)_{\text{top}}^{\text{top}}$$

for all $1 \leq j \leq M$, were $\text{top}$ denotes the top form component of the form obtained by substituting $\tilde{a} = \bar{a}_j$. This is what we'll now do.

**Proof (of Proposition 39).** Fix a smooth $\mathbb{T}$-invariant inner product on $N$, with $\nu : N \to \mathbb{R}_{\geq 0}$ the associated norm. Let $B(r) = \gamma \left( \nu^{-1} \left( (0,r) \right) \right)$, which is defined for sufficiently small $r$. Let $f = \sum_{j=1}^n \lambda_j \xi_j \eta$, and write $f |_{\mathcal{V}} = f^{(2)} + R$ for some real valued function $R : \mathcal{V} \to \mathbb{R}$. By the non-degeneracy assumption, $f^{(2)} \geq 2c^2$ for some $c > 0$. Because $Z$ is compact we can take $\delta > 0$ sufficiently small such that $B(\delta) \subset \mathcal{V}$ and $|R|_{B(\delta)} < c^2$, so $f |_{B(\delta)} \geq c^2$. Since $f$ vanishes only on $\mathcal{Z}$ and $\mathcal{X} \setminus B(\delta)$ is compact, there exists some $C > 0$ such that $f \geq C$ on $\mathcal{X} \setminus B(\delta)$.

We denote $\bar{B}(s) = b^s \bar{B}(s)$. Consider some $1 \leq j \leq M$, and set $\bar{a} = \bar{a}_j$. We will prove (74) by showing that

$$\lim_{s \to \infty} \int_{\tilde{X}} \left( I_{\tilde{X} \setminus B(s^{-1/2})} e^{s^2 D\tilde{\eta}} \tilde{\omega} \right)_{\text{top}}^{\text{top}} = 0$$

and

$$\lim_{s \to \infty} \int_{\tilde{X}} \left( I_{\tilde{X} \setminus B(s^{-1/2})} e^{s^2 D\tilde{\eta}} \tilde{\omega} \right)_{\text{top}}^{\text{top}} = \int_{\mathcal{N}} \left( e^{D\tilde{\eta}^{(2)}} \tilde{\rho}_0^* \left( \tilde{\omega} |_{\mathcal{V}} \right) \right)_{\text{top}}^{\text{top}}.$$  

Here $I_W$ denotes indicator function for $W$.

We begin with (75). There exist smooth top forms $\{\tilde{\omega}_i \in \Omega^{\dim \tilde{X} \setminus (\mathcal{X} \setminus \mathcal{B}(\delta))} \}_{i=0}^n$ (which depend on $\tilde{\omega}$, $d\eta$ and $\bar{a}$) such that

$$\left( e^{s^2 D\tilde{\eta}} \tilde{\omega}_i \right)_{\text{top}}^{\text{top}} = \sum_{i=0}^n e^{-s^2 \bar{f}} s^{2i} \tilde{\omega}_i.$$

For $s$ sufficiently large, we have $\tilde{X} \setminus B(s^{-1/2}) \subset b^{-1} (\mathcal{X} \setminus B(\delta))$, which means

$$\int_{\tilde{X} \setminus B(s^{-1/2})} e^{-s^2 \bar{f}} s^{2i} \tilde{\omega}_i \leq e^{-s^2 C} s^{2i}.$$

The right-hand side tends to zero, and the limit (74) is obtained.

To establish (76) we first change variables by $\left( \tilde{\gamma} \circ j^{-1} \circ \tilde{\mu}_{s^{-1}} \right)$, which we'll denote simply by $\tilde{\mu}_{s^{-1}}$, to obtain:

$$\int_{\tilde{X}} \left( I_{\tilde{X} \setminus B(s^{-1/2})} e^{s^2 D\tilde{\eta}} \tilde{\omega} \right)_{\text{top}}^{\text{top}} = \int_{\mathcal{N}} \left( I_{\tilde{X} \setminus B(s^{-1/2})} e^{s^2 D\tilde{\eta}^{(2)}} \tilde{\rho}_0^* \left( \tilde{\omega} |_{\mathcal{V}} \right) \right)_{\text{top}}^{\text{top}} \tilde{\omega}_{s^{-1}}$$

We have $\left( I_{\tilde{X} \setminus B(s^{-1/2})} e^{s^2 D\tilde{\eta}^{(2)}} \tilde{\rho}_0^* \left( \tilde{\omega} |_{\mathcal{V}} \right) \right)_{\text{top}}^{\text{top}}$ pointwise in $\tilde{N}$. We use Eq (77) to write

$$\left( I_{\tilde{X} \setminus B(s^{-1/2})} e^{s^2 D\tilde{\eta}^{(2)}} \tilde{\rho}_0^* \left( \tilde{\omega} |_{\mathcal{V}} \right) \right)_{\text{top}}^{\text{top}} = \sum_{i=0}^n \int_{\tilde{X} \setminus B(s^{-1/2})} e^{-s^2 \bar{f}} s^{2i} \tilde{\omega}_i.$$  

If we fix some reference $\mathbb{T}$-invariant volume form $\text{vol}$ on $\tilde{N}$ we can write

$$\tilde{\mu}_{s^{-1}} \tilde{\omega}_{i \text{top}} = g_{i,s} \cdot \text{vol}, \quad e^{-s^2 \bar{f}} s^{2i} \tilde{\omega}_{i \text{top}} = h_{i,s} \cdot \text{vol}.$$
for some $\mathbb{R}$-valued functions $g_{i,s}$ and $h_{i,s} = e^{-s^2} \mu_{s-1} f s^2 y_{i,s}$. It is not hard to see that there’s some constant $C_1$ such that $|g_{i,s}| \leq C_1$ for all $0 \leq i \leq n$ and $s \geq 1$. We have $\mu_s f \geq \mu_s \left(ax^2\right) = cs^2 \nu^2$ so

$$|h_{i,s}| \leq e^{-cs^2 \nu^2} s^{2i} C_1$$

which shows that the forms are uniformly bounded by an integrable multiple of $vol$, and we can apply Lebesgue’s dominant convergence theorem. This completes the proof of (76) and of the proposition. \hfill \Box

4.1. Equivariant Segre forms.

**Lemma 40.** Let $\mathcal{X}$ be a local $T$-orbifold and let $\eta \in \Omega(\mathcal{X}; \mathbb{R})^T$ be any form such that $f := \sum_{j=1}^m \lambda_{j} \xi_{j} \eta$ is a real valued function, $f \in \Omega(\mathcal{X}; \mathbb{R})^T$. There’s a well-defined form $P(\eta) \in \Omega(\mathcal{X}; \mathbb{R})^T$ given by

$$P(\eta) = \frac{\eta}{D \eta} (e^{D \eta} - 1) := \frac{\dim \mathcal{X}}{\sum_{i=0}^{\dim \mathcal{X}} \eta^{(i)} (-f) \left(\frac{\partial \eta}{\partial \eta}\right)^i}$$

where $\eta^{(i)}$ denotes the $i$'th derivative of the smooth function $h(a) := \frac{a^{n-1}}{a}$. The form $P(\eta)$ satisfies

$$DP(\eta) = e^{D \eta} - 1.$$

Moreover, for any $f : \mathcal{Y} \to \mathcal{X}$ we have $P(f^* \eta) = f^* P(\eta)$.

**Proof.** Straightforward. \hfill \Box

If $E = \coprod_{n} E_n$ is any $T$-equivariant $l$-vector bundle over a local $T$-orbifold $\mathcal{Z} = \coprod_{n \geq 2} \mathcal{Z}_n$, we say a form $q \in \Omega(E; \mathbb{R})^T$ is quadratic if $q^{(2)} = q$. A quadratic form $q$ will be called localizing if it is localizing to the zero section of $E$. There are bundles $E$ which admit no forms localizing to the zero section (quadratic or otherwise).

**Definition 41.** Suppose $q$ is a quadratic localizing form for $E$. The equivariant Segre form

$$\sigma \in \Omega\left(\mathcal{Z}; Or(E) \otimes \mathbb{R}[\lambda]\right)^T$$

of $E$ associated with $q$ is

$$\sigma = \pi_* \left(e^{D q}\right),$$

where $\pi : E \to \mathcal{Z}$ is the structure map and $\pi_*$ denotes integration on the fiber.

A boundary condition for $E$ consists of a vector bundle $E' \to Z'$ sitting in a cartesian square

$$\begin{array}{ccc}
\partial E & \xrightarrow{h} & E' \\
\partial \pi & \downarrow & \downarrow \pi' \\
\partial \mathcal{Z} & \xrightarrow{g} & Z'
\end{array}$$

where the total dimension of $E'$ is strictly less than the total dimension of $\partial E$ (so $g$ maps $\partial \mathcal{Z}_n$ to $\mathcal{Z}_n$). We say $q$ satisfies the boundary condition if $q|_{\partial E}$ is pulled back along $h$.

Note the degree of $\sigma$ and the rank of $E$ may only be locally constant, with $\deg \sigma = -\text{rk} E$. 
Lemma 42. Let $Z = \bigsqcup Z_n$ be a compact local $\mathbb{T}$-orbifold, and let $E \to Z$ be a vector bundle over $Z$. Let $q,q' \in \Omega(E;\mathbb{R})$ be quadratic localizing forms for the vector bundle $E$, with $\sigma = \pi_* (e^{Dq}), \sigma' = \pi_* (e^{Dq'})$ the associated equivariant Segre forms. Write $Dq = -f + dq$ and $Dq' = -f' + dq'$ for real-valued functions $f,f'$ which are positive definite quadratic forms on each fiber. The form

$$P_2(q,q') = \frac{q-q'}{Dq-Dq'} (e^{Dq}-e^{Dq'}) := (q-q') \sum_{i,j \geq 0} h_2^{(i,j)} (-f,-f') \cdot \frac{(dq)^i (dq')^j}{i! j!},$$

where $h_2^{(i,j)} (a,b) := \partial^i_a \partial^j_b (x,y) = (a,b) \frac{x^i y^j}{x^i y^j}$, satisfies

$$DP_2(q,q') = e^{Dq} - e^{Dq'}.$$

Moreover, $P_2(q,q')$ is rapidly decaying at infinity, so

$$\epsilon = \pi_* (P_2(q,q'))$$

is well-defined, and we have

$$\sigma - \sigma' = D\epsilon.$$

If $q,q'$ satisfy the boundary condition $[79]$, then $\epsilon|_{\partial Z}$ is pulled back along $g : \partial Z \to Z'$.

Proof. Establishing the properties of $P_2(q,q')$ is straightforward; we then compute

$$\sigma - \sigma' = \pi_* (e^{Dq} - e^{Dq'}) = \pi_* (DP_2(q,q')) = D\epsilon,$$

where in the last equality we use the fact $P_2(q,q')$ is rapidly decaying at infinity. Clearly, if $q|_{\partial E}, q'|_{\partial E}$ are pulled back along $f$ so is $P_2(q,q')$ and the last claim follows from the identity $(\partial^* \pi)_* h^* = g^* \pi^*.$

Remark 43. We also have $e^{Dq} - e^{Dq'} = D(P(q) - P(q'))$, but $P(q) - P(q')$ may not be rapidly decaying.

We now show the equivariant Segre form is, up to an exact form, the inverse to the equivariant Euler form. Let $S(E) \xrightarrow{\pi} Z$ be the sphere bundle associated with $E \xrightarrow{\pi} Z$.

Lemma 44. Suppose $E$ admits a localizing quadratic form $q$, with $\sigma$ the corresponding Segre form. Let $\phi \in \Omega(S(E),\pi^*_S \Omega(E)^* \otimes_Z \mathbb{R}[\lambda]^\mathbb{R}$ be an equivariant angular form for $E$ with $e$ the corresponding Euler form, $D\phi = -\pi^*_S e$ (note the degrees of $e$ and $\phi$ varies with the local rank of $E$). There exists a form $\epsilon \in \Omega(Z;\mathbb{R})$ such that

$$\sigma \cdot e = 1 + D\epsilon$$

If $q$ satisfies the boundary condition $[79]$ and $\phi|_{\partial S(E)}$ is pulled back along $S(\partial E) \xrightarrow{S(h)} S(E')$, then we can choose $\epsilon$ pulled back along $g$.

Proof. We blow up the zero section of $E$, to obtain the following setup:

$$\xymatrix{ S(E) \times \{0\} \ar[r]^i & S(E) \times [0,\infty) \ar[r]^\beta & E \ar@/^/[d]^\pi \ar@/_/[d]_\pi \ar[r]_\pi & Z \ar@/^/[l]^\pi \ar@/_/[l]_\pi \ar@/^/[u]^\pi \ar@/_/[u]_\pi}$$

$$\xymatrix{ S(E) \times \{0\} \ar[r]^i & S(E) \times [0,\infty) \ar[r]^\beta & E \ar@/^/[d]^\pi \ar@/_/[d]_\pi \ar[r]_\pi & Z \ar@/^/[l]^\pi \ar@/_/[l]_\pi \ar@/^/[u]^\pi \ar@/_/[u]_\pi}$$
Let \( p_S \) denote the projections of \( S(E) \times [0, \infty) \) on \( S(E) \). Since \( \beta \) has degree one, we have

\[
e \pi_*(\exp(Dq)) = \pi_*(\pi^*e \cdot \exp(Dq)) = \]

\[
\pi_*(p_S^*\pi^*e \cdot \beta^*\exp(Dq)) = \pi_*(D(p_S^*\phi \cdot \beta^*\exp(Dq)))
\]

Since \( \exp(Dq) \) is rapidly decaying at infinity we have

\[
\pi_*(D(p_S^*\phi \cdot \beta^*\exp(Dq))) = \pi_{S*}(i^*(p_S^*\phi \cdot \beta^*\exp(Dq))) = \pi_{S*}(\phi \cdot \exp(0)) + D\epsilon = 1 + D\epsilon
\]

as claimed.

If \( \phi|_{(h S(E))} = S(h)^*\phi' \) and \( q|_{\partial E} = h^*q' \) then \( p_S^*\phi \cdot \beta^*\exp(Dq) \) is pulled back along \( \tilde{h} : \partial S(E) \times [0, \infty) \to S(E') \times [0, \infty) \) and the last claim follows from \( \tilde{h}, \tilde{h}^* = g^*\pi_* \).

4.2. The fixed-point formula for closed orbifolds. We give a proof the fixed-point formula for closed \( T \)-orbifolds (see [1]). Though the result is well-known, this proof helps motivate the setup above and brings into focus the difficulty in applying the fixed-point formula to spaces with boundary.

**Corollary 45.** Let \( \mathcal{X} \) be a compact \( T \)-orbifold without boundary. Let \( F \to \mathcal{X} \) denote the fixed-point suborbifold. For any \( \omega \in \Omega (\mathcal{X}; \text{Or}(T\mathcal{X})^\vee \otimes \mathbb{Z}[\lambda_1, \ldots, \lambda_m]) \) with \( D\omega = 0 \) we have

\[
\int_{\mathcal{X}} \omega = \int_F \frac{\omega|_F}{e(N)}
\]

where \( e(N) \) denotes the equivariant Euler form of the normal bundle \( N \) to \( F \to \mathcal{X} \).

**Proof.** We have

\[
\lim_{s \to \infty} \int_{\mathcal{X}} e^{s^2D\eta} \omega = \int_F \pi_*(e^{D\eta(2)})\omega|_F = \int_F \frac{\omega|_F}{e(N)}
\]

where the first equality uses Proposition [29] with \( \beta = \text{id}_\mathcal{X} \) the identity \( F \)-localizable morphism, and the projection formula; the second equality uses Lemma [31].

On the other hand, by Lemma [30] we have \( (e^{s^2D\eta} - 1)\omega = D(P(s^2\eta))\), so by Stokes’ theorem

\[
\int_{\mathcal{X}} e^{s^2D\eta} \omega = \int_{\mathcal{X}} \omega
\]

for all \( s \). Combining [31] and [32] gives [30].

**Remark 46.** If \( \partial \mathcal{X} \neq \emptyset \), the two sides of \( [32] \) differ by \( \int_{\partial \mathcal{X}} P(s^2\eta)\omega \). Example [33] shows that this can be nonzero.

5. Singular Fixed-point Formula

5.1. A singular fixed-point formula by steepest descent. In this subsection we apply the results of Section 4 to obtain a singular fixed-point formula for integrals of extended forms [1].

Fix some basic moduli specification \( b \). The space \( \mathcal{M}_b^r \) has regular fixed points for every \( r \geq 0 \). We denote \( \mathcal{F}_b^r := (\mathcal{M}_b^r)^\vee \) and let \( i_b^r : \mathcal{F}_b^r \to \mathcal{M}_b^r \) denote the associated closed embedding, with \( \mathcal{N}_b^r = N_i_b^r \) the associated normal l-bundle.
**Proposition 47.** There exists an extended localizing form $\eta = \{\bar{\eta}_r\} \in \Omega_b^R := \mathcal{R}_{\bar{\eta}_b[X]} \Omega_b$, that is, an extended form such that $\bar{\eta}_r$ is fixed-point localizing for every $r \geq 0$.

The proof of this proposition appears in §5.2. We fix an extended localizing form $\eta \in \Omega_b^R$. By Proposition 23 $F^r = (F^r_b)^{-1}$ is an $L$-orbifold. Since $F^r_b$ is a $b$-submersion and $i^r_b$ is a closed embedding, the induced map $N^r_b \to \bar{N}^r_b$ is cartesian. In particular, $\eta_r := (F^r_b)^{-1} \bar{\eta}_r$ is $F^r_b$-localizing for all $r \geq 0$.

Recall $\int_b \omega$ was defined using a blow up $\bar{M}^r_b \xrightarrow{bu^r_b} M^r_b$ (cf. §3.1.4). We will see in §5.3 that the map $bu^r_b : \bar{M}^r_b \to M^r_b$ is $F^r$-localizable. More precisely:

**Proposition 48.** There exists a $\mathbb{T}$-equivariant commutative diagram

\[
\begin{array}{cccc}
\bar{N}^r_b & \xrightarrow{j_b} & \bar{M}^r_b \\
\text{bu}^r_{N^r_b} & & & \text{bu}^r_{M^r_b} \\
N^r_b & \xrightarrow{j_b} & M^r_b
\end{array}
\]

and a smooth family of $\mathbb{T}$-equivariant maps $\{\tilde{\mu}_a : \bar{N}^r_b \to \bar{N}^r_b\}_{a \in \mathbb{R}_{\geq 0}}$ satisfying the conditions of Definition 58.

**Theorem 49.** Let $b$ be a basic moduli specification, and let $\omega = \{\bar{\omega}_r\} \in \Omega_b$ be an extended form with $D\omega = 0$. Let $\eta = \{\bar{\eta}_r\} \in \Omega_b^R$ be an extended localizing form. Then, for $\eta_r = (F^r_b)^{-1} \bar{\eta}_r$, and $\omega_r = (F^r_b)^{-1} \bar{\omega}_r$, we have

\[
\int \omega = \sum_{r \geq 0} \frac{1}{r!} \int_{N^r_b} \text{bu}^r_{N^r_b} e^{D\eta^r_b(\bar{\omega})} (\tau^r_b)^r (\text{bu}^r_b)^r (\bar{\omega}_r) \Lambda^{\bar{\omega}_r}.
\]

**Proof.** Consider $I(s) := \int_b e^{s^2 D\eta} \omega = \sum_{r \geq 0} \frac{1}{r!} \int_{\bar{M}^r_b} \text{bu}^r_{\bar{M}^r_b} (e^{s^2 D\eta})^r (\bar{\omega}_r) \Lambda^{\bar{\omega}_r}$.

First we claim that

\[
I(s) = I(0)
\]

for all $s \geq 0$. To see this, consider the extended form $\epsilon_s \omega = \{\tilde{\epsilon}_{s,r} \tilde{\omega}_r\} \in \Omega_b^R$ given by

\[
\tilde{\epsilon}_{s,r} \tilde{\omega}_r = \frac{\bar{\eta}_r - \eta^r_b}{D\eta^r_b} (e^{s^2 D\eta^r_b} - 1) \tilde{\omega}_r,
\]

where the right hand side is interpreted as in (78). The collection $\{\tilde{\epsilon}_{s,r} \tilde{\omega}_r\}$ is coherent and $\text{Sym}(r)$-invariant since $\{\tilde{\eta}_b\}$ and $\{\tilde{\omega}_r\}$ are. We have $D(\epsilon_s \omega) = (e^{s^2 D\eta^r} - 1) \omega$, so (85) follows from Stokes’ theorem, $\int_b D(\epsilon_s \omega) = 0$.

Now take the limit as $s \to \infty$, and apply Proposition 49.

### 5.2. The group $E^r_k$ and localizing forms

For the constructions in this section and the next, we need to consider a group $E^r_k$ which acts on $\bar{M}^r_b$, extending the action of $\text{Sym}(r)$. Because the self-diffeomorphisms (“autoequivalences”) of a stack do not form a group, we first define these groups as groups of symmetries of a discrete object, and only then let them act on $\bar{M}^r_b$.

Let $T^r_k$ denote the set of trees on vertices $\{v_1, \ldots, v_{r+1}\}$ with oriented edges $\{e_1, \ldots, e_r\}$ as well as inward-pointing half-edges labeled $\{h_x\}_{x \in k}$. A tree $T \in T^r_k$
can be represented by a pair of partitions $\left( \left( k_\mathcal{T}(v_i) \right)_{i=1}^{r+1}, \left( \sigma_{\mathcal{T}}(v_i) \right)_{i=1}^{r+1} \right)$, so that $s'_r = \bigsqcup_{i=1}^{r+1} \sigma_{\mathcal{T}}(v_i)$ and $s''_r \cup \{1, \ldots, k\} = \bigsqcup_{i=1}^{r+1} k_\mathcal{T}(v_i)$.

Consider the subgroup

$$\mathcal{R}_k^r < \text{Sym}\left( T_k^r \times (s''_r \sqcup k) \right)$$

consisting of $\rho \in \text{Sym}\left( T_k^r \times (s''_r \sqcup k) \right)$ such that

1. there exists an element $\alpha^\rho \in \text{Sym}(T_k^r)$ and a collection $\left\{ F_{\mathcal{T}}^\rho \in \text{Sym}\left( s''_r \sqcup k \right) \right\}_{\mathcal{T} \in T_k^r}$ such that
   $$\rho(\mathcal{T}, x) = \left( \alpha^\rho(\mathcal{T}), F_{\mathcal{T}}^\rho(x) \right),$$
2. we have
   $$F_{\mathcal{T}}^\rho(k_\mathcal{T}(v_i)) = k_{\alpha^\rho(\mathcal{T})}(v_i)$$

for all $\mathcal{T} \in T_k^r \ 1 \leq i \leq r + 1$.

Note that $\text{Sym}(v_1, \ldots, v_{r+1})$ acts on $T_k^r$ by relabeling the vertices, and thus forms another subgroup

$$\text{Sym}(v_1, \ldots, v_{r+1}) \times \text{id} < \text{Sym}(T_k^r) \times \text{Sym}\left( s''_r \sqcup k \right) \times \text{Sym}(T_k^r \times (s''_r \sqcup k)) \times \text{Sym}(T_k^r \times (s''_r \sqcup k))$$

The conjugation action of $\text{Sym}(v_1, \ldots, v_{r+1})$ preserves $\mathcal{R}_k^r$, and we take $\mathcal{R}_k^r < \mathcal{R}_k^r$ to be the fixed-points of this action.

**Lemma 50.** Let $b = (k, l, \beta)$ be any basic moduli specification. Then the finite group $\mathcal{R}_k^r$ acts on $\delta_b$.

**Proof.** Let $\rho \in \mathcal{R}_k^r$. The action of $\rho$ takes two steps.

**step 1:** there's an obvious forgetful map that sends $\mathcal{T} \in \mathcal{F}_b^r$ to $\mathcal{T} \in T_k^r$, and we map

$$\delta_b = \prod_{i=1}^{r+1} \delta_v(v_i) \to \prod_{i=1}^{r+1} \delta_v((k_{\delta_b}(v_i), l_{\delta_b}(v_i), \beta_{\delta_b}(v_i), \sigma_{\delta_b}(v_i)))$$

by using $F_{\mathcal{T}}^\rho_k(v_i)$ to relabel the boundary markings.

**step 2:** There's a unique $w = w_{\mathcal{T}} \in \text{Sym}(v_1, \ldots, v_{r+1})$ which orders the vertices so that condition (c) in Definition 19 is satisfied, so that

$$\prod_{i=1}^{r+1} \delta_v((k_{\delta_b}(w(v_i)), l_{\delta_b}(w(v_i)), \beta_{\delta_b}(w(v_i)), \sigma_{\delta_b}(\mathcal{T}(v_i)))) = \mathcal{M}_{\mathcal{T}'}$$

for some $\mathcal{T}' \in \mathcal{F}_b^r$.

The composition of (86) and (87) defines an automorphism of $\mathcal{M}_b^r$. Associativity follows from the assumption that $\text{Sym}(v_1, \ldots, v_{r+1})$ commutes with $\mathcal{R}_k^r$. \qed

Now we'll define a special subgroup of $\mathcal{R}_k^r$. First, consider the following elements of $\mathcal{R}_k^r$.

- Any $\tau \in \text{Sym}(e_1, \ldots, e_r)$ defines an element of $\mathcal{R}_k^r$. We set $\alpha^\tau \in \text{Sym}(T_k^r)$ to be the map which relabels the edges according to $\tau$ and $F_{\mathcal{T}}^\tau \in \text{Sym}(s''_r \sqcup k)$, which is independent of $\mathcal{T}$ in this case, permutes $s''_r$ and fixes $k$. 

• $\text{mv}_{x,e_b} \in \mathcal{R}_k^r$, for $1 \leq b \leq r$, $x \in s'_{\{1,\ldots,b,\ldots,r\}}$, which acts by “moving the tail of $e_a$ along $e_b$”. More precisely, consider some $\mathcal{T} \in T_k^r$, and suppose $e_b$ is incident to $\{v_i, v_j\}$. If $x \in \sigma_{\mathcal{T}}(v_i)$, we define $\alpha_{\text{mv}_{x,e_b}} \mathcal{T}$ by removing $x$ from $\sigma_{\mathcal{T}}(v_i)$ and adding it to $\sigma_{\mathcal{T}}(v_j)$. If $x \notin \sigma_{\mathcal{T}}(v_i) \cup \sigma_{\mathcal{T}}(v_j)$, we set $\alpha_{\text{mv}_{x,e_b}} \mathcal{T} = \mathcal{T}$. In either case we set $F^{\text{mv}_{x,e_b}}_\mathcal{T} = \text{id}_{\mathcal{T}_{v_i}^T} \cup k$.

• $\text{mv}_{x,e_b} \in \mathcal{R}_k^r$ for $1 \leq b \leq r$, $x \in k \bigcup s'_{\{1,\ldots,b,\ldots,r\}}$, which acts by “moving $x$ along $e_b$ and flipping $e_a$”. More precisely, consider some $\mathcal{T} \in T_k^r$, and suppose $e_b$ is incident to $(v_i, v_j)$ where $v_i$ is the tail of $e_b$. If $x \in k_{\mathcal{T}}(v_i)$, we define $\mathcal{T}' = \alpha_{\text{mv}_{x,e_b}} \mathcal{T}$ so that

\[
\begin{align*}
\sigma_{\mathcal{T}'}(v_i) &= \sigma_{\mathcal{T}}(v_i) \setminus \{s'_b\} \\
k_{\mathcal{T}'}(v_i) &= k_{\mathcal{T}}(v_i) \setminus \{x\} \cup \{s''_b\} \\
\sigma_{\mathcal{T}'}(v_j) &= \sigma_{\mathcal{T}}(v_j) \cup \{s''_b\} \\
k_{\mathcal{T}'}(v_j) &= k_{\mathcal{T}}(v_j) \cup \{x\} \forall k \neq \{i,j\}
\end{align*}
\]

for $k \neq \{i,j\}$ we set $\sigma_{\mathcal{T}'}(v_k) = \sigma_{\mathcal{T}}(v_k)$ and $k_{\mathcal{T}'}(v_k) = k_{\mathcal{T}}(v_k)$, and $F^{\text{mv}_{x,e_b}}_{\mathcal{T}'}$ the permutation the swaps $(s''_b, x)$. If $x \notin k_{\mathcal{T}}(v_i)$, we set $\mathcal{T}' = \mathcal{T}$ and $F^{\text{mv}_{x,e_b}}_{\mathcal{T}'} = \text{id}$.

We define $E^r_{k} < \mathcal{R}_k^r$ to be the subgroup generated by all of these elements.

In the next section we’ll also need the subgroup $E^{r,r'}_{k} < \mathcal{R}_k^{r+r'}$ obtained by “restricting the action to the even-even edges $\{r+1,\ldots, r+r'\}$”. More precisely, we have the following.

**Definition 51.** For $r, r' \geq 0$, we let $E^{r,r'}_{k}$ be the group generated by $\text{Sym}(e_{r+1}\ldots, e_{r+r'}) < \text{Sym}(e_1\ldots, e_{r+r'})$ and the elements $\text{mv}_{x,e_b}$ for $r+1 \leq b \leq r+r'$ and $x \in s'_{\{1,\ldots,b,\ldots,r\}} \bigcup k \bigcup s'_{\{1,\ldots,b,\ldots,r\}}$.

It is not hard to check that the action of this subgroup preserves $M^{r,r'}_b$.

**Proof of Proposition [4].** We construct a sequence of forms $\{\tilde{\eta}_r \in \Omega(M^{r,r'}_b; \mathbb{R}[\tilde{\alpha}])\}_{r \geq 0}$ satisfying the following properties, for every $r \geq 0$.

(a) for every $\mathcal{T}_r \in \mathcal{F}_b^{R+1}$ with $\text{cnt} \mathcal{T}_r = \mathcal{T}$, $T^{\mathcal{T}_r}_{\tilde{\alpha}} \tilde{\eta}_r = (\tilde{g}_{\mathcal{T}_r})^* \tilde{\eta}_{r+1}$,

(b) $\tilde{\eta}_r$ is $F^r_b$-localizing, and

(c) $\tilde{\eta}_r$ is $E^r_{k}$-invariant.

The construction proceeds by backward induction. Suppose for $R \geq 0$, we have constructed equivariant forms $\{\tilde{\eta}_r\}_{r \geq R+1}$ satisfying (a) - (c) for $r \geq R+1$. Note that this assumption holds vacuously for $R$ so large that $M^{R+1}_b = \emptyset$. We will now construct $\tilde{\eta}_R$ satisfying (a) - (c).

For $\mathcal{T}_r \in \mathcal{F}_b^{R+1}$ let $\tilde{\eta}_{R,\mathcal{T}_r} = (\tilde{g}_{\mathcal{T}_r})^* \tilde{\eta}_{r+1}$. Consider the covering map

$$\coprod_{\mathcal{T}_r \in \mathcal{F}_b^{R+1}} \partial^R \mathcal{T}_r \tilde{M}_{\text{cnt} \mathcal{T}_r} \to \partial M^{R+1}_b.$$ 

Because $\tilde{\eta}_{R+1}$ is invariant under $\text{mv}_{x,e_{R+1}}$ for $x \in s'_{\{R\}}$, $\coprod_{\mathcal{T}_r} \tilde{\eta}_{R,\mathcal{T}_r}$ is the pullback of some form $\tilde{\eta}_0$ on $\partial M^{R+1}_b$.

We show $\tilde{\eta}_0$ is $E^R_b$-invariant. Let $\mathcal{T}_0 \in \mathcal{F}_b^{R+1}$ and consider a generator $\text{mv}_{x,e_b} \in E^R_b$. We have

$$e. \tilde{g}_{\mathcal{T}_0} = \tilde{g}_{\mathcal{T}_0} \partial^R_{\text{mv}_{x,e_b}}.$$
for \( e \in \{ \text{mv}_v, \text{mv}_x, \text{mv}_c \circ \text{mv}_v, \text{mv}_x, \text{mv}_c \circ \text{mv}_x, \text{mv}_c \circ \text{mv}_c \} \subset E^{R+1}_k \). Since \( \tilde{g}_r \) is also \( \text{Sym}(e_1, \ldots, e_R) \rightarrow \text{Sym}(e_1, \ldots, e_{R+1}) \)-equivariant, \( E^{R+1}_k \)-invariance of \( \tilde{\eta}_\partial \) follows from \( E^{R+1}_k \)-invariance of \( \tilde{\eta}_{R+1} \).

Next we want to argue that \( \tilde{\eta}_\partial \) can be extended inward. For this it suffices to check that \( \left( i^\partial_{\partial \mathcal{M}_b^k} \right)^* \tilde{\eta}_\partial \) is \( \mathbb{Z}/2 \)-invariant with respect to the action that switches the local boundary components at a corner. For \( \mathcal{T}_1 \in \mathcal{R}^{R+1}_b \) and \( \mathcal{T}_0 = \text{cnt} \mathcal{T}_1 \), the decomposition into horizontal and vertical boundaries with respect to \( \tilde{g}_\mathcal{T}_1 \),

\[
\partial \partial^\mathcal{T}_1 \mathcal{M}_{\mathcal{T}_0} = \partial^\mathcal{T}_1 \partial^\mathcal{T}_1 \mathcal{M}_{\mathcal{T}_0} \coprod \partial^\mathcal{T}_1 \partial^\mathcal{T}_1 \mathcal{M}_{\mathcal{T}_0} = (\tilde{g}_\mathcal{T}_1)^{-1} \left( \partial^\mathcal{T}_1 \partial^\mathcal{T}_1 \mathcal{M}_{\mathcal{T}_1} \right),
\]

induces a decomposition \( \partial^2 \mathcal{M}_b^R = \partial^2 \mathcal{M}_b^R \coprod \partial^2 \mathcal{M}_b^R \). We deal with these two components separately.

First, we have

\[
\partial^2 \mathcal{M}_b^R \leftarrow \bigcup_{\mathcal{T}_1 \in \mathcal{R}^{R+1}_b} \partial^\mathcal{T}_2 \partial^\mathcal{T}_1 \mathcal{M}_{\mathcal{T}_0} \xrightarrow{h_-} \bigcup_{\mathcal{T}_1 \in \mathcal{R}^{R+1}_b} \mathcal{M}_{\mathcal{T}_1},
\]

where \( \mathcal{T}_1 \subsetneq \text{cnt} \mathcal{T}_1 \), \( \partial^\mathcal{T}_2 \partial^\mathcal{T}_1 \mathcal{M}_{\mathcal{T}_0} \) is \( \tilde{g}_\mathcal{T}_1 \) equivariant, whereas \( \partial^\mathcal{T}_2 \partial^\mathcal{T}_1 \mathcal{M}_{\mathcal{T}_0} \) is \( \tilde{g}_\mathcal{T}_1 \) equivariant. The diagram is \( \mathbb{Z}/2 \)-equivariant where \( \mathbb{Z}/2 \) acts via \( \text{Sym}(e_{R+1}, e_{R+2}) \rightarrow E^{R+2}_k \). In other words, \( \mathbb{Z}/2 \)-invariance of \( \tilde{\eta}_\partial |_{\partial^2 \mathcal{M}_b^R} \) follows from \( \text{Sym}(R+2) \)-invariance of \( \tilde{\eta}_\partial |_{\partial^2 \mathcal{M}_b^R} \).

Consider next

\[
\partial^2 \mathcal{M}_b^R \leftarrow \bigcup_{\mathcal{T}_1 \in \mathcal{R}^{R+1}_b} \partial_+ \partial^\mathcal{T}_1 \mathcal{M}_{\mathcal{T}_0} \xrightarrow{h_+} \bigcup_{\mathcal{T}_1 \in \mathcal{R}^{R+1}_b} \mathcal{M}_{\mathcal{T}_1},
\]

where \( \partial_+ \) is the covering map and \( h_+ := \tilde{g}_\mathcal{T}_1 \). Since \( \text{Im} h_+ \) is exhausted by clopen components where \( a \in \text{K} \coprod \text{S}_{R+1}^b \) bubbles off on a ghost disc together with \( \text{S}_{R+1}^b \), we find \( \partial_+ \) and \( h_+ \) are \( \mathbb{Z}/2 \)-equivariant, where we act on the domain and codomain of \( h_+ \) by \( \mathbb{Z}/2 \simeq \langle \text{mv}_{a, e_{R+1}} \rangle \subset E^{R+1}_k \). In other words, \( \mathbb{Z}/2 \)-invariance of \( \tilde{\eta}_\partial |_{\partial^2 \mathcal{M}_b^R} \) follows from \( \text{Sym}(\text{mv}_{a, e_{R+1}}) \)-invariance of \( \tilde{\eta}_\partial |_{\partial^2 \mathcal{M}_b^R} \).

By Proposition 37 there exists a form \( \tilde{\eta}_R \) which satisfies conditions \( (a_R) \) and \( (b_R) \). By averaging, we ensure that \( \tilde{\eta}_R \) also satisfies \( (c_R) \). This completes the proof of the inductive step. Clearly, \( \eta = \{ \tilde{\eta}_k \} \in \Omega_b^k \) is an extended localizing form. \( \square \)

5.3. Blow up map is localizable. In this subsection we prove Proposition 48 and derive a slightly more explicit form of 48. Focus on some \( \mathcal{T} \in \mathcal{R}^{R,r'}_b \). Let

\[
ed^r_{\mathcal{T}} := \prod_{i=1}^r \text{ed}^{b,r}_i : \mathcal{M}_{\mathcal{T}} \to (L \times L)^r
\]

and

\[
ed_{\mathcal{T}} = \prod_{i=r+1}^{r+r'} \text{ed}^{b,r}_i : \mathcal{M}_{\mathcal{T}} \to (L \times L)^{r'},
\]

so

\[
ed_{\mathcal{T}} |_{\partial \mathcal{M}_{\mathcal{T}}} = \text{ed}^r_{\mathcal{T}} \times \text{ed}_{\mathcal{T}}.
\]

Let \( i_{\mathcal{T}} : \mathcal{F}_{\mathcal{T}} \to \mathcal{M}_{\mathcal{T}} \) be the pullback of \( \tilde{F}_b^r \to \tilde{\mathcal{M}}^r_c \). It follows from the description of the fixed points in 32 that \( \text{ed} \circ i_{\mathcal{T}} \) factors through \( (p_0, p_0) \rightarrow L \times L \) whereas \( \text{ed}^r \circ i_{\mathcal{T}} \) factors through the open inclusion \( L \times L \setminus \Delta \rightarrow L \times L \). In particular, we have
We construct a diagram of cartesian squares which is cartesian by a simple diagram chase. The maps \( \psi \) commute, where \( \psi \) comes from the linearization of \( \text{ed}_T \).

**Proof of Proposition 48.** Consider \( T \in T_b \), and let \( \nu : N_0 \to L \times L \) be the composition \( N_0 \to N_\Delta \to L \times L \). We define \( \nu_{U_0} \) and \( \tilde{\nu} \) by the following cartesian square

\[
\begin{array}{ccc}
N_0 & \to & \tilde{N}_0 = S(N_0) \times [0, \infty) \\
\downarrow_{\nu_0} & & \downarrow_{\nu_{\tilde{N}_0} = \text{ed}_T} \\
N_0 & \to & L \times L
\end{array}
\]

We construct a diagram of cartesian squares

\[
\begin{array}{ccc}
\tilde{N}_0' & \to & \tilde{V}_0' = (U \times \tilde{N}_0)' \\
\downarrow_{\nu_{\tilde{N}_0}'} & & \downarrow_{\nu_{\tilde{V}_0}' = \text{ed}_T} \\
N_0' & \to & (L \times L)'
\end{array}
\]

Build a diagram modeled on a row of two cubes, by gluing along their bottom row. The remaining vertices of the diagram are the pullbacks

\[
\begin{array}{ccc}
\tilde{N}_T = N_T \times N_0' & \to & \tilde{V}_T = \nu_T \times \nu_{\tilde{V}_0}' \tilde{V}_0' \\
\downarrow_{\text{ed}_T' \times \text{ed}_T} & & \downarrow_{\text{ed}_T' \times \text{ed}_T} \\
\tilde{M}_T = \text{M}_T \times (L \times L)'' & \to & \text{M}_T
\end{array}
\]

and we obtain maps \( \tilde{N}_T \to \tilde{V}_T \to \tilde{M}_T \) as the pullback of \( N_T \to V_T \to M_T \). One face of the row of cubes is the diagram

\[
\begin{array}{ccc}
\tilde{N}_T & \to & \tilde{V}_T \\
\downarrow & & \downarrow \\
N_T & \to & V_T \\
\downarrow & & \downarrow \\
M_T & \to & M_T
\end{array}
\]

which is cartesian by a simple diagram chase. The maps \( \tilde{\mu}_a : \tilde{N}_T \to \tilde{N}_T \) for \( a \geq 0 \) are defined by the linear structure on \( N_T \) and the obvious rescaling of
\[ \tilde{N}_0^{r'} = (S(N_0) \times [0, \infty))^{r'}. \] Smoothness of \( \tilde{\mu} : \tilde{N}_T \times \mathbb{R}_{\geq 0} \to \tilde{N}_T \) follows by elementary arguments from the fact the map
\[ [0, \infty) \times \mathbb{R}_{\geq 0} \to [0, \infty) : (x, y) \mapsto x \cdot y \]
is smooth, see [11, Example 2.3, (vii)].

For \( T \in \mathcal{T}_b^{r=r'} \setminus \mathcal{T}_b^{r,r'} \) with \( F_T \neq \emptyset \), we use an element of \( \text{Sym} (r + r') \) to transfer the construction from some \( T' \in \mathcal{T}_b^{r,r'} \). Taking a disjoint union over all such \( T \) with \( F_T \neq \emptyset \) we obtain the data specified in the statement of the proposition; the conditions of Definition 33 are readily verified.

Let us now take a closer look at the singular localization formula. For \( \phi \in \mathcal{P}_b \) a fixed-point profile, let \( \tilde{N}_\phi \xrightarrow{\text{bu}_\phi} N_\phi \) be defined by the cartesian square
\[
\begin{array}{ccc}
\tilde{N}_\phi & \xrightarrow{\tilde{\pi} \circ \iota} \tilde{N}'_0 \\
\text{bu}_\phi \downarrow & & \downarrow \text{bu}_0' \\
N_\phi & \xrightarrow{\delta_\phi = \Pi \delta_\phi} N'_0
\end{array}
\]

**Corollary 53.** We have
\[
(90) \quad \int_b \omega = \sum_{r,r' \geq 0} \frac{1}{r! \cdot r'!} \sum_{\phi \in \mathcal{P}_b^{r,r'}} (-1)^{r'} \frac{\xi_\phi}{s(\phi)!} \int_{\tilde{N}_\phi} \text{bu}_{N_\phi}^* (e^{D_{\eta \phi} \pi_\phi^*(\omega_\phi)}) \cdot \tilde{\pi}^*(\theta_0)^{r+r'}
\]

Hereafter, \( \eta_\phi = f_\phi^* \tilde{\eta}_\phi \) with \( \tilde{\eta}_\phi = \eta_{r,r'}|_{N_\phi} \), \( \omega_\phi = \tilde{F}_\phi^* \omega_{r,r'}|_{\tilde{F}_\phi} \), and \( -\theta_0 = -\theta_{\Delta|_{\rho_0}} \), which we consider as a form on \( \tilde{N}_0 = S(N_0) \times [0, \infty) \).

**Proof.** We have \( \mathcal{M}_b^R = \bigsqcup_{T \in \mathcal{T}_b^r} \mathcal{M}_T \). The \( \text{Sym} (R) \) orbits of \( \bigsqcup_{r+r'=R} \mathcal{T}_b^{r,r'} \) cover \( \mathcal{T}_b^r \), and we use Lemma 27 to decompose further into fixed-point profiles:
\[
\frac{1}{R!} \int_{\tilde{N}_\phi} \mathcal{I} = \sum_{r + r' = R} \frac{1}{r! \cdot r'!} \sum_{\tau \in \mathcal{T}_b^{r,r'}} \int_{\tilde{N}_\tau} \mathcal{I} = \sum_{r + r' = R} \frac{1}{r! \cdot r'!} \sum_{\phi \in \mathcal{P}_b^{r,r'}} \frac{1}{s(\phi)!} \int_{\tilde{N}_\phi} \mathcal{I},
\]
where \( \tilde{N}_\phi \xrightarrow{\pi_{\phi'}} \tilde{F}_\phi \) is the normal bundle associated with \( \tilde{F}_\phi \to \mathcal{M}_T(\phi) \) (cf. 33), and \( \tilde{N}_\phi \xrightarrow{\text{bu}_{\phi'}} \tilde{N}_0 \) is the pullback of \( \text{bu}_{N_0} \).

We have
\[
\text{bu}_b^* \gamma_b^r \rho_0 = \gamma_b^r \rho_0 \text{bu}_{N_0}^r
\]
and
\[
\tilde{\gamma}_b^r \rho_0 |_{N_\phi} = \tilde{\gamma}_b^r \rho_0 |_{\tilde{F}_\phi} \text{bu}_{N_\phi} \times E^r \tilde{\gamma}_b^r \rho_0
\]
where \( E \) is the retraction \( \tilde{N}_0 \to S(N_0) \to \tilde{N}_0 \), \( \tilde{F}_\phi \) is defined by the cartesian square
\[
\tilde{F}_\phi \text{bu}_{N_\phi} = \text{bu}_b \tilde{F}_\phi \text{bu}_{N_b}, \quad \text{and } (\gamma_b^r \rho_0)|_{\tilde{N}_\phi}
\]
denotes the pullback of \( \gamma_b^r \rho_0 \) along the étale map \( \tilde{N}_\phi \to \tilde{N}_T \to \tilde{N}_b \).
Now we compute
\[
\int_{\tilde{N}_c} \left[ b_{\tilde{N}} e^{D_{\tilde{M}}^2} \cdot (\tilde{\gamma}_b \tilde{\rho}_3) \ast \left( (b_{\tilde{N}})^* \ast \left( \tilde{\omega}_{\tilde{M}} \ast (\tilde{\omega}_b)^* \Lambda^{\tilde{M}_{\tilde{M}}} \right) \right) \right]_{\tilde{N}_c} = (-1)^{r - r'} \int_{\tilde{N}_c} \left\{ \left( j_{\phi} \ast b_{\tilde{N}} \ast j_{\phi}^* e^{D_{\tilde{M}}^2} \ast (\pi_b)^* \ast \tilde{\omega}_{\tilde{M}} \ast (\tilde{\omega}_b)^* \Lambda^{\tilde{M}_{\tilde{M}}} \right) \right\} = \xi_{\phi} \int_{\tilde{N}_c} b_{\tilde{N}} \ast (j_{\phi} e^{D_{\tilde{M}}^2} \ast (\pi_b)^* \ast \tilde{\omega}_{\tilde{M}} \ast (\tilde{\omega}_b)^*) \Lambda^{\tilde{M}_{\tilde{M}}} ,
\]
where in the last equality we applied the projection formula (see [29, Lemma 63] for our sign conventions) for \( \int_{\tilde{N}_c} \) using the push-pull property for the cartesian commutative square \( \tilde{N}_c \), Eq \( [91] \) follows.

In the Section \( 5.4 \) we'll explain how to regularize \( [91] \) to derive the general fixed-point formula \( [66] \).

### 5.4. Tubular neighbourhood construction.

**Definition 54.** Let \( \mathcal{M} \) be an orbifold with corners and let \( V \) be a vector space.

(a) A vector bundle map \( \delta : T \mathcal{M} \to V \times \mathcal{M} \) is strongly surjective if \( \delta \circ d\tilde{\omega}_M : T \mathcal{M} \times V \times \mathcal{M} \to V \times \mathcal{M} \) is fiberwise surjective for every \( c \geq 0 \).

(b) Let \( \mathcal{M} \) be an orbifold with corners. A vector field on \( \mathcal{M} \) is a section \( v : \mathcal{M} \to T \mathcal{M} \) of the map \( T \mathcal{M} \to \mathcal{M} \). A vector field \( v \) will be called \( b \)-tangent if \( (i_{\mathcal{M}}^\partial)^{-1} (v) : \partial \mathcal{M} \to (i_{\mathcal{M}}^\partial)^* T \mathcal{M} \) factors through \( d\tilde{\omega}_M : T \partial \mathcal{M} \to (i_{\mathcal{M}}^\partial)^* T \mathcal{M} \) for all \( c \geq 0 \). More generally if \( v : \mathcal{E} \to \mathcal{M} \) is a vector bundle over \( \mathcal{M} \) we say a map \( \mathcal{E} \to T \mathcal{M} \) is \( b \)-tangent if \( v \circ s \) is \( b \)-tangent for every section \( s : \mathcal{M} \to \mathcal{E} \).

**Lemma 55.** Let \( \mathcal{M} \) be a \( T \)-orbifold with corners and \( V \) a \( T \)-representation of finite rank. Let \( \delta : T \mathcal{M} \to V \times \mathcal{M} \) a strongly surjective \( T \)-equivariant map. Then there exists a \( T \)-equivariant \( b \)-tangent map \( \sigma : V \times \mathcal{M} \to T \mathcal{M} \) with \( \delta \sigma = \text{id} \).

**Proof.** Consider first the case \( T \) acts trivially on \( V = \mathbb{R} \) and \( \mathcal{M} = \mathbb{R}^n \). In terms of local coordinates \( x = (x_1, ..., x_n) \) for \( \mathbb{R}^n \) with \( x_i \geq 0 \) for \( 1 \leq i \leq c \), we can write \( \delta = a_1 (x) \cdot dx_1 + \cdot + a_n (x) \cdot dx_\cdot \) for some functions \( a_i : \mathbb{R}^n_c \to \mathbb{R} \) such that \( a_1 (x), ..., a_n (x) \neq 0 \) everywhere and \( (a_1 (x), ..., a_1 (x), ..., a_n (x)) \neq 0 \) for all \( x \) with \( x_i = 0 \). We take \( \sigma = \sum_{i=1}^{n} \frac{1}{\sum_{i=1}^c a_i^2 + \sum_{i=c+1}^n a_i^2} \cdot \left\{ \begin{array}{ll}
 j a_j & \text{if } 1 \leq j \leq c \\
 a_j & \text{if } c+1 \leq j \leq n .
 \end{array} \right. \)

The general case follows, using a partition of unity and averaging with respect to a Haar measure on \( \mathbb{T} \).

**Proof of Lemma 55.** Use Lemma 50 to fix some \( T \)-equivariant tubular neighbourhood for \( F_T \subset \Delta_T := (\text{ed}_T)^{-1} (\Delta') \).

We thus reduce to proving the following.

**Claim.** There exists an open \( T \)-invariant open neighbourhood \( U_1 \),
\[ p_0^{x'} U_1 \subset U_0^{x'} \subset L^{x'} \to (L \times L)^{x'} , \]
so that the closed immersion
\[ U_T := \text{ed}_T^{-1}(U_1) \xrightarrow{f} \mathcal{M}_T \]
given by the composition \( U_T \to \Delta_T \to \mathcal{M}_T \) admits a \( \mathbb{T} \)-equivariant tubular neighbourhood
\[ N_f = N_0' \times U_T \xrightarrow{j_f} \mathcal{V}_f \xrightarrow{\gamma_f} \mathcal{M}_T \]
making the following diagram commute
\[ \begin{array}{ccc}
N_0' \times U_T & \xrightarrow{j_f} & \mathcal{V}_f & \xrightarrow{\gamma_f} & \mathcal{M}_T \\
\downarrow{\text{pr}_1} & & \downarrow & & \downarrow{\text{ed}_T} \\
N_0' & \xrightarrow{j} & \mathcal{V}_0' & \xrightarrow{\gamma} & (L \times L)' \end{array} \]

We prove the claim. Set \( W_0 = \text{ed}_T^{-1}\left(\gamma_0^{-1}(\mathcal{V}_0')\right) \subset \mathcal{M}_T \), so that
\[ Q := (\gamma_0^{-1})_{r'} \circ \text{ed}_T : W_0 \to N_0' \]
is well-defined, and let
\[ \delta_1 := P \circ dQ : TW_0 \to N_0' \times W_0 \]
be the linearization of \( Q \) composed with the tautological “parallel transport to
\( 0 \in N_0' \)” map \( P : TN_0' \to T_0N_0' = N_0' \).

For every \( c > 0 \) the map \( \text{ed}_T \circ i^c_{\mathcal{M}_T} \) is transverse to the diagonal \( \Delta' \to (L \times L)' \).
It follows that there exists a possibly smaller open neighbourhood \( \text{ed}_T^{-1}(U_1, p_0) \subset W_1 \subset W_0 \) such that the restriction \( \delta_1 := TW_1 \to N_0' \times W_1 \) of \( \delta_0 \) is strongly surjective.

Apply Lemma 55 to construct a b-tangent map \( \sigma_1 : N_0' \times W_1 \to TW_1 \) with \( \delta_1 \sigma_1 = \text{id} \). Exponential flow along \( \sigma_1 \), defined for some sufficiently small suborbifold
\( \mathcal{V}_f \) of \( N_0' \times U_T \) containing \( 0 \times U_T \), gives the desired \( \gamma_f \). This completes the proof of Lemma 52.

**Remark 56.** If we want the flow to be defined uniformly, so that \( \mathcal{V}_f = U_T \times V_1 \) for
some open neighbourhood \( 0 \in V_1 \subset N_0' \), we can restrict \( U_T \) further to a precompact open subset of \( U_T \).

6. **Resummaton**

For \( \phi \in \mathcal{P}^{r,r_0}_b \), we denote \( \mathcal{P}^{r,r}_b := \{ \phi \in \mathcal{P}^{r,r} \mid \text{cnt}^{r'}(\phi) = \underline{\phi} \} \), and set
\[ \text{Cont}_{\underline{\phi}} := \sum_{r \geq 0} \frac{1}{r!} \sum_{\phi \in \mathcal{P}^{r,r}_b} \sum_{s} \left( \text{cnt}^{r'}(\phi) \right)^{s} N_{\phi} \frac{\xi_{\phi}}{s} \int_{N_{\phi}} \text{b}_{\phi} (e^{D\eta_{\phi}} \pi_{\phi}^* (\omega_{\phi})) \cdot \delta_{\phi} (\theta_0) \omega^{r'} \]

so that, by Corollary 53
\[ \int_{b} \omega = \sum_{r \geq 0} \frac{1}{r!} \sum_{\phi \in \mathcal{P}^{r,r_0}_b} \text{Cont}_{\underline{\phi}} \omega^{r} \]

The main result of this section is the following.
Proposition 57. For $\phi \in \mathcal{P}_b^{r,0}$ we have

$$\text{Cont}_{\phi} = \frac{\zeta_{\phi}}{s(\phi)!} \int_{F_{\phi}} e_M^{-1} \cdot e_S^{-1} \cdot \tilde{\omega}_{\phi}$$

where $e_M, e_S$ are canonical equivariant Euler forms for $M_{\phi}, S_{\phi}$, respectively (cf. (64) and (65)).

Theorem 29 immediately follows from this.

6.1. The difficulty with the singular formula. This subsection tries to explain the difficulty in computing $\text{Cont}_{\phi}$, and gives an indication of how we overcome this problem. The subsections that follow are logically independent of this discussion.

6.1.1. $r' (\phi) > 0$ integrals don’t vanish. One might expect the terms in (61) with $r' (\phi) > 0$ to vanish. Indeed, $\eta_{\phi} = f^*_\phi \tilde{\eta}_{\phi}$ is pulled back from a space of strictly lower dimension - but $\tilde{\delta}_{\phi}$ does not factor through $f_\phi$. The next example shows that these contributions can be nonzero.

Example 58. Consider the basic moduli specification $b = (\{1,2,3\}, \{1\}, 2)$ for $(\mathbb{C}P^2, \mathbb{R}P^2)$, and let $\phi_1 \in \mathcal{P}_b^{0,1}$ be a fixed-point profile with $\mathcal{T} = \mathcal{T} (\phi_1)$ given by $\mathcal{T}_0 = \{v_1, v_2\}$, $s_\mathcal{T} (v_1) = (((\{1\}, \{1\}, 2), s_1')$ and $s_\mathcal{T} (v_2) = (((s_1', 2, 3), \emptyset, 0), \emptyset).$ We have

$$F_{\phi_1} = F_{D_1} (\phi_1) \times F_{\Sigma_1} (\phi_1) \times F_{D_2} (\phi_1) = [0, 2\pi] \times \text{pt} \times \text{pt}$$

where $\alpha \in [0, 2\pi] = F_{D_1} (\phi_1)$ is represented by the unit disc with the markings $\ast_{1,1}, 1, s_1'$ at 0, 1, $e^{\sqrt{-1} \alpha} \in \mathbb{C}$, respectively. $\tilde{F}_{\phi_1} = \text{pt}$ and we have an equality of vector spaces

$$\tilde{N}_{\phi_1} = E \oplus \mathbb{C}_s \oplus (T_{p_0} L)^2.$$ 

where we’ve used the evaluation maps at $s_1', s_1''$ together with the translation action of the lie algebra of $O (3) \times O (3)$ to split off $(T_{p_0} L)^2 \simeq \mathbb{C}^2 \cong \mathbb{R}^4_{x_1, y_1}$ and its natural section which keeps the $v_2$ disc fixed and smooths the $s_1''$-node so the boundary of the $v_1$ disc is a circle centered at $p_0 \in \mathbb{R}P^2$.

We have

$$\delta_{\phi_1} : N_{\phi_1} = E \times \mathbb{C} \times (T_{p_0} L)^2 \times [0, 2\pi] \to N_0 = T_{p_0} L$$

$$(e, s, x, y, \alpha) \mapsto y - x - s \cdot e^{\sqrt{-1} \alpha}.$$ 

We can take

$$\eta_{\phi_1} = \eta_E + \eta' = \eta_E + |s|^2 \text{d} \arg s + x_1 dx_2 - x_2 dx_1 + y_1 dy_2 - y_2 dy_1.$$
and then
\[ \int_{\tilde{N}_{\Phi}} e^{Dn_{\Phi}} \pi^*_{\Phi} \omega_{1} \tilde{\delta}_{\Phi} (\theta_0) = \tilde{\omega}_1 (\tilde{F}_{\Phi}) \cdot e_{E}^{-1} \times \left[ \int_{C_x(T_y L)^2 \times [0, 2\pi]} e^{-(|x|^2 - |y|^2 + \lambda_1)} \cdot \arg \left( y - x - s \cdot e^{\sqrt{-1} \alpha} \right) \right] = \frac{\tilde{\omega}_1 (\tilde{F}_{\Phi}) \cdot e_{E}^{-1}}{\lambda_1} \int e^{-x_1^2 - x_2^2 - y_1^2 - y_2^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2} \, dx_1 \, dx_2 \, dy_1 \, dy_2 = \frac{\tilde{\omega}_1 (\tilde{F}_{\Phi}) \cdot e_{E}^{-1}}{\lambda_1} = 0. \]

6.1.2. \( r' (\Phi) = 0 \) integral is ill-defined. A closely related problem is that the contribution of the unique summand to \( 91 \),
\[ \frac{\xi_{\Phi}}{s (\phi)} \int_{\tilde{N}_{\Phi}} e^{Dn_{\Phi}} \pi^*_{\Phi} \omega_{\Phi} = \frac{\xi_{\Phi}}{s (\phi)} \int_{\tilde{F}_{\Phi}} \sigma \omega_{\Phi}, \]
seems to depend on the choice of \( \eta_{\Phi} \). Since this integral is non-singular (i.e. it contains no \(-\theta_0\) factors), we could express it as an integral of a Segre form \( \sigma \) on \( \tilde{F}_{\Phi} = \tilde{F}_{\Phi} \). But \( \sigma \) does not satisfy any boundary condition. More precisely, although
\[ \eta_{\Phi} |_{\partial_{\Phi^{\ast} N_{\Phi}}} = \left( f_{\Phi^{\ast}} \circ \gamma_{\Phi^{\ast}} \right)^{\ast} \eta_{\Phi}, \]
for every \( \Phi \in P^1_{\Phi} \), this does not constitute a boundary condition since
\[ \dim \tilde{N}_{\Phi} > \dim \partial_{\Phi^{\ast} N_{\Phi}}. \]
This problem is also related to the fact \( \delta_{\Phi} \) does not factor through \( f_{\Phi} \); indeed, otherwise \( \eta_{\Phi} |_{\partial_{\Phi^{\ast} N_{\Phi}}} \) would be pulled back from
\[ K_{\Phi} = \ker \left( \tilde{N}_{\Phi} \rightarrow N_0 \right) \]
and we would have had
\[ \sigma |_{\partial_{\Phi^{\ast} F_{\Phi}}} = \text{For}^{\ast}_{\Phi} \left( K_{\Phi} \rightarrow \tilde{F}_{\Phi} \right) \cdot e^{Dn_{\Phi}}. \]

6.1.3. \( 61 \) doesn’t split. Consider the diagram \( 61 \). The maps \( \delta^{\prime^{r+1}}_{M_{\Phi}} = \delta^{\prime^{r+1}, (r' + 1)} \)
and \( \delta^{\prime^{r+1}}_{\Phi} = \delta^{\prime^{r+1}, (r' + 1)} \)
are the cokernels of the fiberwise-injective maps \( \gamma_{M_{\Phi}} \) and \( \gamma_{\Phi} \), respectively (the map \( \gamma_{S_{\Phi}} \) is cartesian). Since \( \delta^{\prime^{r+1}} \) does not factor through \( \tilde{F}_{\Phi} \), it is not possible to have compatible splittings for the three short exact sequences in the diagram \( 61 \) (that is, splittings so that \( 61 \) with the horizontal arrows reversed by the splitting, also commutes). This may be seen as a third problem with the integrals appearing in the singular localization formula (though of course, it is closely related to the previous two).

6.1.4. Resolution. In \( 6.2 \) we will construct sections \( \tilde{N}_{\Phi} \rightarrow \tilde{M}_{\Phi} \), whose pullback we denote by \( \zeta_{\Phi} : N_{\Phi} \rightarrow M_{\Phi} \). The l-bundle map
\[ \delta_{\Phi} = \delta_{M_{\Phi}} \circ \zeta_{\Phi} : N_{\Phi} \rightarrow N_0^{r'} \times F_{\Phi} \]
is a kind of correction for \( \delta \phi \), with the effect of \( S \phi \) pushing on the tail markings \( s'_r \) switched off. Indeed, \( \delta \phi \) (more precisely, \( (\text{id} \times \text{For} \phi) \circ \delta \phi \)) factors through the forgetful map \( f_\phi \). We will see that \( \delta \phi \) and \( \delta \phi \) admit a common section

\[
s_\phi : N_0^{r'} \times F_\phi \to N_0
\]

so the isomorphism

\[
A_\phi = \text{id} + s_\phi (\delta_\phi - \delta \phi)
\]

is isotopic to the identity and satisfies \( \delta_\phi A_\phi = \delta \phi \). Setting \( \hat{\delta} \phi := A_\phi^{-1} \delta_\phi \) we find that \( \hat{f_\phi} \hat{\delta} \phi \) factors through a map \( \partial \phi N_{\text{cnt}} \phi \cong \ker \delta \phi \to \ker \delta M_\phi \) with respect to the splittings induced by \( \zeta_\phi \hat{\delta} \phi \). So the corrections \( \hat{\delta} \phi \) address all of the problems mentioned above. In §6.3 we will use the isotopies \( \phi \) resum \( \text{Cont} \).

6.2. Splitting short exact sequences. For \( \phi \in P_b^{r,r'} \), we set \( \delta M_\phi = \prod \delta_{M_{\phi_i}} \). We denote \( F_{b}^{r,r'} = \text{Prod}_{P_b^{r,r'}} F_\phi \) and similarly for other fixed-point local constructions, e.g. \( N_{b}^{r,r'} = \text{Prod}_{P_b^{r,r'}} N_\phi \), \( \delta_{b}^{r,r'} := \text{Prod}_{P_b^{r,r'}} \delta_\phi \).

There’s a natural action of the group \( E_k^{r,r'} \) (see Definition 51) on \( P_b^{r,r'} \), preserving the node labels. The action of \( E_k^{r,r'} \) on \( N_{b}^{r,r'} \) then induces an action on \( N_{b}^{r,r'} \), \( F_{b}^{r,r'} \), etc., compatible with the forgetful map to \( P_b^{r,r'} \). Similarly, \( \text{Sym} (r') \) acts on \( F_{b}^{r,r'} \).

 considered as a groupoid, \( P_b^{r,r'} \) acts on \( F_{b}^{r,r'} \) in a natural way. The map \( \hat{f_\phi} \) is the quotient map for this action. This action lifts to \( F_{b}^{r,r'}, N_{b}^{r,r'}, \text{M}_{b}^{r,r'} \), and so on, and the maps \( \delta_{M_{b}^{r,r'}}, \delta_{M_{b}^{r,r'}}, \ldots \) respect it, as does the \( E_k^{r,r'} \) action. We will tacitly assume all of the constructions below continue to respect this action, by averaging.

\( \text{Sym} (r') \) acts on \( N_0^{r} \times F_{b}^{r,r'} \) and on \( N_0^{r} \times F_{b}^{r,r'} \) diagonally. For \( N_0^{r} \times F_{b}^{r,r'} \), this extends to an action of \( E_k^{r,r'} \) as follows. For \( 1 \leq b \leq r' \) and \( x \in s([r+r',{r+b}]) \), we set

\[
\text{mv}_{x,c_b} ((y_1, \ldots, y_r), f) = ((y'_1, \ldots, y'_r), f')
\]

where \( f = \text{mv}_{x,c_b} \cdot f \) and, letting \( v_i \) and \( v_j \) denote the tail and head of \( e_{r+b} \) in \( \pi_{r+r'}^{-1} (f) \), we consider the following cases

- if \( x \in k \cup s''[r,r'] \) and \( x \in \sigma_T (v_i) \), we set \( y'_i = -y_b, y'_i = y_i \) for \( i \neq b \)
- if \( x = s''_{r+a} \) for \( a \neq b \), and \( x \in k \cup \sigma_T (v_i) \), we set \( y'_b = -y_b, y'_a = y_a + y_b, y'_i = y_i \) for \( i \neq a, b \).
- if \( x = s''_{r+a} a + b \) and \( x \in \sigma_T (v_i) \), we set \( y'_a = y_a + y_b \) and \( y'_i = y_i \) for \( i \neq a \)
- if \( x = s''_{r+a} a \neq b \) and \( x \in \sigma_T (v_i) \), we set \( y'_a = y_a - y_b \) and \( y'_i = y_i \) for \( i \neq a \)
- in all other cases we set \( y'_i = y_i \) for all \( i \).

The maps \( \delta_{M_{b}^{r,r'}} = \text{Prod}_{P_b^{r,r'}} \delta_{M_{b}^{r,r'}} \) and \( \prod_{(\phi)_{r'(r')=r'}} \delta_\phi \) are naturally \( E_k^{r,r'} \) and \( \text{Sym} (r') \) equivariant, respectively.

\[\text{Concretely, this means that if we choose a representative } \phi_\alpha \text{ for each equivalence class } \phi_\alpha = \text{Sym} (s(\phi_\alpha)) \phi_\alpha < P_b^{r,r'} \text{ then we have an action of } \text{Sym} (s(\phi_\alpha)) \text{ on } \prod_{\phi \in \phi_\alpha} \hat{f_\phi}.\]
Lemma 59. For every \( r, r' \geq 0 \) there exists a \( \mathbb{T} \times E^{r,r'}_k \)-equivariant linear map

\[
N_0' \times F^{r,r'}_b \xrightarrow{s'_{M_b}} \bar{M}^{r,r'}_b
\]

such that the following properties hold.

(a) \( s'_{M_b} \) is a section of \( \bar{M}^{r,r'}_b \), \( \bar{M}^{r,r'}_b \circ s'_{M_b} = \text{id} \),

(b) if we let \( s_{M_b} : N_0' \times F^{r,r'}_b \to \bar{M}^{r,r'}_b \) be the pullback defined by

\[
\bar{f}_{M_b} \circ s_{M_b}'(a) = \bar{f}_{M_b} \circ s_{M_b}'(b) \text{ follows from (a) by functoriality of the pullback:}
\]

then the composition

\[
N_0' \times F^{r,r'}_b \xrightarrow{s'_{M_b}} \bar{M}^{r,r'}_b \to N_0' \times F^{r,r'}_b
\]

we denote \( s'_{M_b} \), is a \( \text{Sym}(r') \)-equivariant section of \( \bar{f}_{M_b} \).

Proof. The \( O(2m + 1)^{r+r'+1} \)-action on \( \bar{M}^{r,r'}_b \) induces a canonical map \( (T_{\mathcal{P}} \mathcal{L})^{r+r'+1} \times F^{r,r'}_b \to \bar{M}^{r,r'}_b \), which is \( \mathbb{T} \times E^{r,r'}_k \)-equivariant, where \( E^{r,r'}_k \) acts on \( (T_{\mathcal{P}} \mathcal{L})^{r+r'+1} \) by reordering the factors according to \( w_{\mathcal{P}} \mathcal{L} \) (see the proof of Lemma 50). The map \( \bar{f}_{M_b} \circ \alpha \) is surjective, and we choose a vector bundle map

\[
N_0' \times F^{r,r'}_b \xrightarrow{\beta} (T_{\mathcal{P}} \mathcal{L})^{r+r'+1} \times F^{r,r'}_b
\]

over \( \text{id}_{F^{r,r'}_b} \) which satisfies \( \bar{f}_{M_b} \circ \alpha \circ \beta = \text{id} \). By averaging, we may assume without loss of generality that \( \beta \) is \( \mathbb{T} \times E^{r,r'}_k \) equivariant, and we set \( \bar{s}_{M_b} = \alpha \circ \beta \); clearly,

\( \bar{s}_{M_b} \) is a \( \mathbb{T} \times E^{r,r'}_k \)-equivariant section as claimed.

(b) follows from (a) by functoriality of the pullback: \( \bar{s}_{M_b} \circ \mu_{M_b} \) is the pullback of \( \bar{s}_{M_b} \). \( \square \)

We decompose \( \bar{s}_{M_b} \) as \( \bigoplus_{i=1}^{r+r'+1} \bar{s}_{M_b}^i \) for \( \bar{s}_{M_b}^i : N_0 \times F^{r,r'}_b \to \bar{M}^{r,r'}_b \).

Let \( \phi_+ \in \mathcal{P}^{r,r'+1} \). To avoid excessive labels, we use the following notation

\[
\delta_{\mathcal{P}}^{r+r'+1} \quad \text{for} \quad N_0^{r+r'+1} \to F^{r+r'+1}_\phi
\]

and

\[
\delta_{\mathcal{P}}^{r+r'+1} \quad \text{for} \quad N_0^{r+r'+1} \to F^{r+r'+1}_\phi
\]

for the pullbacks of \( \bar{s}_{M_b} \) and \( \bar{s}_{M_b} \).

Proposition 60. Let \( b \) be a basic moduli specification. There exist \( \mathbb{T} \times E^{r,r'}_k \)-equivariant sections of \( \mu_{M_b}^{r,r'} \)

\[
\tilde{\zeta}_{M_b}^{r,r'} : \tilde{N}^{r,r'}_b \to \bar{M}^{r,r'}_b
\]

for \( r, r' \geq 0 \), such that the following modified-coherence property holds.

For \( \phi_+ \in \mathcal{P}^{r,r'+1} \) let \( \tilde{\zeta}_{\phi_+} : \tilde{N}^{r+r'+1}_{\phi_+} \to \bar{M}^{r+r'+1}_{\phi_+} \) be the pullback, defined by

\[
\tilde{f}^{r+r'+1}_{M_b} \circ \tilde{\zeta}_{\phi_+} = \tilde{f}^{r+r'+1}_{M_b} \circ \tilde{\zeta}_{\phi_+}
\]

Setting \( \phi = \text{cnt} (\phi_+) \) we have
\[
\hat{\phi}_{\Phi_+} \cdot \hat{\phi}_{\partial^m \mathcal{N}} = \left( \text{id} - \hat{s}^{r' + 1}_{M_{\Phi_+}} \hat{\delta}^{r' + 1}_{M_{\Phi_+}} \right) \hat{\phi}, \hat{\phi}_{\partial^m \mathcal{N}}.
\]

**Proof.** Fix \( r \). The proof is by reverse induction on \( R \geq r \): assume that we’re given \( T \times E^{r', r'}_{k} \)-equivariant sections \( \hat{\phi}_{\partial^m \mathcal{N}} \) of \( M_{\theta}^{r'} \) defined for all \( r' \geq R - r + 1 \) such that (92) holds for all \( \Phi_+ \in \mathcal{P}^{r, R-r-1}_b \). Note that for sufficiently large \( R \), this assumption holds vacuously. We will now prove that there exists a \( T \times E^{r', R-r-1}_{k} \)-equivariant section \( \hat{\phi}_{\partial^m \mathcal{N}} : N_{\Phi_+}^{r', R-r} \to M_{\Phi_+}^{r', R-r} \) so that (92) holds for all \( \Phi_+ \in \mathcal{P}^{r, R-r-1}_b \).

Set \( r' = R - r \). First, for each \( \Phi_+ \in \mathcal{P}^{r, R-r-1}_b \) we define \( \hat{\phi}_{\partial^m \mathcal{N}} \in \text{Hom} \left( \partial^m \Omega^\phi, \partial^m \mathcal{N}_{\Phi_+} \right) \) by (92). Since \( \hat{\phi}_{\partial^m \mathcal{N}} \) is injective on fibers, such \( \hat{\phi}_{\partial^m \mathcal{N}} \) is unique, and it exists since \( \hat{\delta}^{r' + 1}_{M_{\Phi_+}} \left( \text{id} - \hat{s}^{r' + 1}_{M_{\Phi_+}} \hat{\delta}^{r' + 1}_{M_{\Phi_+}} \right) = 0 \) (compare with (60.1.4)).

Consider the étale map \( \prod_{\Phi_+ \in \mathcal{P}^{r, R-r-1}_b} \partial^m \mathcal{N}_{\Phi_+} \to \partial^m \mathcal{N}_{\Phi_+} \). The invariance of \( \left( \text{id} - \hat{s}^{r' + 1}_{M_{\Phi_+}} \hat{\delta}^{r' + 1}_{M_{\Phi_+}} \right) \hat{\phi}_{\partial^m \mathcal{N}} \) under \( \{ \text{mv}_{x, e} \}_{1 \leq x, e, r, r' \leq R} \in E^{r', R-r-1}_{k} \) (and the \( \mathcal{P}^{r, R-r-1}_b \)-groupoid action) implies that \( \prod_{\Phi_+} \hat{\phi}_{\partial^m \mathcal{N}} \) is the \( q \)-pullback of a section \( \partial^m \mathcal{N}^{r', R-r} \to \partial^m \mathcal{N}_{\Phi_+}^{r', R-r} \).

A straightforward computation shows \( \hat{\phi}_{\partial^m \mathcal{N}} = \text{id} \). We show \( \hat{\phi}_{\partial^m \mathcal{N}} \) is \( T \times E^{r', r'}_{k} \)-equivariant. Clearly, \( T \times \text{Sym} (e^r_{x, 1}, \ldots, e^r_{x, r'}) \)-equivariance follows from equivariance of \( \left( \text{id} - \hat{s}^{r' + 1}_{M_{\Phi_+}} \hat{\delta}^{r' + 1}_{M_{\Phi_+}} \right) \hat{\phi}_{\partial^m \mathcal{N}} \). Now let \( \Phi_+ \in \mathcal{P}^{r, R-r-1}_b \) and consider a generator \( \text{mv}_{x, e} \in E^{r, r'} \).

We have
\[
(93) \quad \alpha \cdot \hat{\phi}_{\partial^m \mathcal{N}} = \hat{\phi}_{\partial^m \mathcal{N}} \left( \text{mv}_{x, e} \right)
\]
where \( \alpha : N_{\Phi_+}^{\{R-1\}} \to N_{\Phi_+}^{\{R-1\}} \) satisfies
\[
\hat{\phi}_{\partial^m \mathcal{N}} \left( \text{mv}_{x, e} \right) = \text{id} \cdot \hat{\phi}_{\partial^m \mathcal{N}} \left( \text{mv}_{x, e} \right)
\]
for some \( e \in \{ \text{mv}_{x, e}, \text{mv}_{x, e} \circ \text{mv}_{x, e, r+R+1}, \text{mv}_{x, e} \circ \text{mv}_{x, e, r+R+1} \} \in E^{r', R-r-1}_{k} \). Similarly,
\[
(93) \quad \alpha' \cdot \hat{\phi}_{\partial^m \mathcal{N}} = \hat{\phi}_{\partial^m \mathcal{N}} \left( \text{mv}_{x, e} \right)
\]
where \( \alpha' : M_{\Phi_+}^{\{R-1\}} \to M_{\Phi_+}^{\{R-1\}} \) is a \( \hat{\phi}_{\partial^m \mathcal{N}} \)-lift of \( e \). We have
\[
\alpha' \left( \text{id} - \hat{s}^{r' + 1}_{M_{\Phi_+}} \hat{\delta}^{r' + 1}_{M_{\Phi_+}} \right) \hat{\phi}_{\partial^m \mathcal{N}} = \left( \text{id} - \hat{s}^{r' + 1}_{M_{\Phi_+}} \hat{\delta}^{r' + 1}_{M_{\Phi_+}} \right) \hat{\phi}_{\partial^m \mathcal{N}} \alpha
\]
which implies that \( \hat{\phi}_{\partial^m \mathcal{N}} \) is invariant under \( \text{mv}_{x, e} \), and hence under \( E^{r, r'} \).

We want to show there exists an extension \( \hat{\phi}_{\partial^m \mathcal{N}} \) of \( \hat{\phi}_{\partial^m \mathcal{N}} \) inward. For this it suffices to check that \( \partial^m \Omega^\phi : \partial^2 \mathcal{N}^{r', r'} \to \partial^2 M^{r', r'} \) is \( \mathbb{Z}/2 \)-equivariant with respect to the action that switches boundary components. The argument is similar to the one given in the proof of Proposition 37. We have a decomposition \( \partial^2 F^r_{k} = \partial^2 \tilde{F}^{r, r'}_{k} \parallel \partial^2 \tilde{F}^r_{k} \) induced from the decomposition \( \partial^2 \tilde{M}^{r', r'}_{b} = \partial^2 \tilde{M}^{r', r'}_{b} \parallel \partial^2 \tilde{M}^{r', r'}_{b} \). For any 2-flag \( (\Phi_0, \Phi_1, \Phi_2), \Phi_2 \in \mathcal{P}^{r, r'+2}_b \), \( \phi_{-1} = \text{cut} \phi_2 \), there are maps \( \partial^2 \phi_2 \partial^2 \phi_1, \tilde{N}_{\phi_2} \to \tilde{N}_{\phi_1} \) and \( \partial^2 \phi_2 \partial^2 \phi_1 \tilde{M}_{\phi_2} \to \tilde{M}_{\phi_1} \) induced by \( g \), and we have
\[
\gamma_{\phi_2} \gamma_{\phi_1} \tilde{N}_{\phi_2} = \left( \text{id} - \hat{s}^{r' + 1}_{M_{\Phi_2}} \hat{\delta}^{r' + 1}_{M_{\Phi_2}} - \hat{s}^{r' + 2}_{M_{\Phi_2}} \hat{\delta}^{r' + 2}_{M_{\Phi_2}} \right) \hat{\phi}_{\partial^m \mathcal{N}} \gamma_{\phi_2} \gamma_{\phi_1},
\]
so $\mathbb{Z}/2$-equivariance of $\partial \tilde{\zeta}_b|_{\partial^2 F_{b}^{r',r}}$ follows from Sym $(e_{R+1}, e_{R+2})$-equivariance of $\rho$. $\mathbb{Z}/2$-equivariance of $\partial \tilde{\zeta}_b|_{\partial^2 F_{b}^{r',r}}$ follows from $\langle m v_{a,b,R+1} \rangle$-equivariance of $\left(id - \delta_{M_1 + \delta_{M_2}}^r \delta_{M_2}^r \right) \phi_{s_0}$. We conclude that an extension $\tilde{\zeta}_b^{r,m}$ of $\tilde{\zeta}_b$ exists.

By averaging, we can ensure that $\tilde{\zeta}_b^{r,m}$ is $T \times E_{b}^{r',r}$-equivariant. This completes the inductive step and the proof of the proposition. $\square$

For $\phi \in D_b^{r,r'}$, define $\zeta_\phi : N_\phi \to M_\phi$ by requiring $f_{M_\phi} \zeta_\phi = \tilde{\zeta}_b^{r,m} f_{\phi}$ and set

$$\delta_\phi = \prod_{i=1}^{r'} \delta_i = \delta_{M_\phi} \circ \zeta_\phi.$$  

6.3. Regularizing and resumming. A flag $\Phi_\bullet$ of length $l (\Phi_\bullet) = l$ and depth $r'(\Phi_\bullet) = r'$ is a tuple

$$(\phi_{l}, \phi_{l-1}, ..., \phi_{l}) \in \mathcal{D}^{r,r'} \times \cdots \times \mathcal{D}^{r,r'+l},$$

such that $\phi_{l-i} = \text{cnt} \phi_i$ for $1 \leq i \leq l$. We say $\Phi_\bullet$ lies over $\phi = \text{cnt} \phi_{0} \in \mathcal{D}^{r,0}$. We fix some $\phi \in \mathcal{D}^{r,0}$ and denote by $\Psi^{r,l}$ the set of flags of depth $r'$ and length $l$ over $\phi$.

$\text{Sym}(r') \times \text{Sym}(l) < \text{Sym}(r + r' + l)$ acts on $\Psi^{r,l}$ by relabeling the edges of $\phi_1$ (a flag $\phi_\bullet$ is uniquely specified by $\phi_1$ and the length). Each flag $\phi_\bullet$ specifies a clopen corner component $F_{\phi_\bullet} = \partial^1 F_{\phi_0} \subset \partial^1 F_{\phi_0}$ whose local boundary faces correspond, in this order, to the edges $(r + r' + 1, ..., r + r' + l)$ of $T(\phi_1)$. We use similar notation for corner components of the other spaces, bundles and maps, e.g.

$$\partial \phi \cdot N_{\phi_0} = \partial^1 \phi \cdot N_{\phi_0} \subset \partial^1 N_{\phi_0}.$$  

The action of $\text{Sym}(r') \times \text{Sym}(l)$ lifts to $\prod_{\phi_{0}, \psi^{r,l}} F_{\phi_\bullet}$.

We set $N_{\phi_\bullet} := N_{\phi_0} |_{F_{\phi_\bullet}} = \partial^1 \phi \cdot N_{\phi_0} \to F_{\phi_\bullet}$.

Let $I = [0,1]$ be the unit interval. For each $\phi_{0} \in \Psi^{r,l}$ we define

$$A_{\phi_\bullet} (T, t_1, ..., t_l) : N_{\phi_\bullet} \times I \times I \times I \times I \to N_{\phi_\bullet} \times I \times I$$

by

$$\gamma_{\phi_\bullet} A_{\phi_\bullet} (T, t) := \left( id + (1 - T) \left[ \sum_{i=1}^{r'} s_{\phi_1}^i (\delta_{\phi_1}^i - \delta_{\phi_1}^i) + \sum_{i=1}^{l} t_i \cdot s_{\phi_1}^{r'+i} \delta_{\phi_1}^{r'+i} \right] \right) \gamma_{\phi_\bullet}. $$

Lemma 61. $A_{\phi_\bullet}$ satisfies the following properties.

(a) Let $\phi_{\bullet} = (\phi_0, ..., \phi_1)$ be a flag of length $l$ and depth $r'$. We can write $A_{\phi_\bullet} (T, t)$ as a product of commuting operators

$$A_{\phi_\bullet} (T, t) = \prod_{i=1}^{r'} A_{\phi_{\bullet}, i} (T) \cdot \prod_{j=r'+1}^{r'+l} A_{\phi_{\bullet}, j} (T, t_j)$$

with

$$A_{\phi_{\bullet}, i} (T) := id + (1 - T) s_{\phi_1}^i (\delta_{\phi_1}^i - \delta_{\phi_1}^i)$$

for $1 \leq i \leq r'$ and

$$A_{\phi_{\bullet}, j} (T, t_j) := id + (1 - T) t_j \cdot s_{\phi_1}^j \delta_{\phi_1}^j$$

for $r' + 1 \leq j \leq r' + l$. The operators $A_{\phi_{\bullet}, i} (T)$ for $1 \leq i \leq r'$ are pulled back along $f_{\phi_\bullet} := \partial \phi \cdot f_{\phi_0}$.

(b) Let $\phi_{\bullet} = (\phi_0, ..., \phi_{l+1})$ be a flag of length $l + 1$, and let $\phi_{\bullet}^{2l} = (\phi_1, ..., \phi_{l+1})$ and $\phi_{\bullet}^{2l} = (\phi_0, ..., \phi_1)$ be its truncations. Then
$A_{\phi_*}(T, t_1, ..., t_l, t_{l+1} = 0) = \partial^{\phi_{l+1}} A_{\phi_{l+1}}(T, t_1, ..., t_l)$

and

$\partial^{\phi_{l+1}} \gamma_{\phi}(A_{\phi_*}(T, t_1 = 1, t_2, ..., t_{l+1})) = [A_{\phi_{l+1}}(T, t_2, ..., t_{l+1})] \partial^{\phi_{l+1}} \gamma_{\phi}$.

(c) Let $\phi_* \in \Psi^{r', l}$. For all $1 \leq i \leq r'$ we have

$\partial^{\phi_{l+1}} \gamma_{\phi_0} \circ A_{\phi_*}(0, t_1, ..., t_l) = \partial^{\phi_{l+1}} \gamma_{\phi_0}$.

Proof. Straightforward. \qed

We set $\delta_{\phi_*} = \partial^{\phi_{l+1}} \gamma_{\phi_0}$ and define $\bar{\phi}_{\phi_*}$ and $\beta_{\phi_*}$ so that there’s a cartesian diagram

$\begin{array}{ccc}
\bar{N}_{\phi_*} & \xrightarrow{\beta_{\phi_*}} & N_{\phi_*} \\
\downarrow{\bar{\phi}_{\phi_*}=\Pi_{r', l}\phi_*} & & \downarrow{\delta_{\phi_0}} \\
\bar{N}_{0}' \times F_{\phi_*} & \xrightarrow{\beta_{0}'} & N_{0}' \times F_{\phi_*}
\end{array}$

Setting

$\omega_{\phi_*} = \gamma_{\phi_*}^* \omega_{r+1} \in (\kappa_{F_{\phi_*}}^*)^* \omega_{r+1}$ and $\eta_{\phi_*} = \gamma_{\phi_*}^* \eta_{\phi_0} = (\kappa_{\bar{N}_{\phi_*}}^*)^* \eta_{\phi_0}$,

we define an equivariant form on $\bar{N}_{\phi_*} \times I \times I'$ by

(95) $\mathcal{I}_{\phi_*} = \xi_{\phi_0}^* \beta_{\phi_*}^* (\varphi_{\pi_{\phi_*}^* \omega_{\phi_*}}^*)^{r+1} \eta_{\phi_*} \eta_{\phi_0}^* \delta_{\phi_0}.$

The following proposition is the heart of the resummation result, replacing $\text{Cont}_{\phi_*}$’s original sum of singular integrands corresponding to flags of increasing depth by a sum of regular, depth zero integrands indexed by flags of increasing length, whose domain of integration are corners of $\bar{N}_{\phi_*}$.

In what follows $\bar{N}_{\phi_*}$ is oriented using the outward normal orientation, and $\bar{N}_{\phi_*} \times I \times I'$ is oriented as a product of oriented spaces.

**Proposition 62.** We have

$\text{Cont}(\phi) = \sum_{r \geq 0} \frac{1}{r!} \sum_{\phi_{r'} \in \pi_{r'} \phi_0} \frac{1}{s(\phi_0)!} \int_{\bar{N}_{\phi_*}} \mathcal{I}_{\phi_*}.$

(96) $= \sum_{l \geq 0} \frac{1}{l!} \sum_{\phi_0 \in \Psi_{0,l}} \frac{1}{s(\phi_0)!} \int_{\bar{N}_{\phi_*} \times 0 \times I'} \mathcal{I}_{\phi_*}.$

Proof. The first equality is immediate (the minus sign appears because the orientation on $1 \in \partial I$ is negative). By Lemma 63 below,

$$0 = \sum_{\phi_{r'} \in \Psi_{r,l}} \frac{1}{r! \cdot l! \cdot s(\phi_1)!} \int_{\bar{N}_{\phi_*} \times I \times I'} D \mathcal{I}_{\phi_*} = \sum_{\phi_0 \in \Psi_{0,l}} \frac{1}{s(\phi_0)!} \int_{\partial(\bar{N}_{\phi_*} \times I \times I')} \mathcal{I}_{\phi_*}.$$

We have

$$\partial(\bar{N}_{\phi_*} \times I \times I') = \left(\partial_{\bar{N}_{\phi_*}} \bigcup_{i=1}^r \partial_{\bar{N}_{\phi_*}} \bigcup_{i=1}^r \bigcup_{i=1}^r \left(\delta_{\phi_*}^{-1} \left(S(N_0)\right)\right) \times I \times I' \right) \bigcup \left(\bar{N}_{\phi_*} \times \{0\} \times I'\right) \bigcup \left(\bar{N}_{\phi_*} \times \{1\} \times I'\right) \bigcup \left(\bar{N}_{\phi_*} \times I \times \partial(I')\right).$$
Here $\partial_1 \tilde{N}_{\phi_*}$ denotes the $\beta_{\phi_*}$-inverse image of the decomposition $\partial N_{\phi_*} = \partial^{l_1}_1 N_{\phi_*} \bigcup \partial^{l_1}_t N_{\phi_*}$ into horizontal and vertical components w.r.t. $f_{\phi_*}$. By Lemma 62 we have $\int_{\partial_1 \tilde{N}_{\phi_*} \times I \times I} \mathcal{I}_{\phi_*} = 0$, and by Lemmas 65, 66 the contributions of $\tilde{N}_{\phi_*} \times I \times \partial (I^l) \times \partial_1 \tilde{N}_{\phi_*}$ and $\bigcup_{l=1}^{r'} (\tilde{\delta}_{\phi_*})^{-1} (S (N_0))$ cancel out.

$\int_{\tilde{N}_{\phi_*} \times (0) \times I^l} \mathcal{I}_{\phi_*} = 0$ unless $l = 0$, since $A_{\phi_*} (T = 1, t_1, \ldots, t_l) = id$ is independent of $t_1, \ldots, t_l$ so $\mathcal{I}_{\phi_*} |_{T = 1}$ is pulled back along the projection $\tilde{N}_{\phi_*} \times \{(1) \} \times I^l \rightarrow \tilde{N}_{\phi_*}$. The $l = 0$ contributions give the middle expression in (98), without the minus sign.

Indeed, using Lemma 61(c) we find that

$$\int_{\tilde{N}_{\phi_*} \times (0) \times I} \mathcal{I}_{\phi_*} = \int_{\tilde{N}_{\phi_*}} A_{\phi_*} (0, t)^* \mathcal{I}_{\phi_*}$$

where $\tilde{N}_{\phi_*} = A_{\phi_*} (0, t)^{-1} \tilde{N}_{\phi_*}$ sits in following diagram where both squares are cartesian

$$\begin{array}{ccc}
\tilde{N}_{\phi_*} & \longrightarrow & N_{\phi_*} \\
\downarrow & & \downarrow \delta_{\phi_*} \\
N_0' \times F_{\phi_*} & \longrightarrow & N_0' \times F_{\phi_*}
\end{array}$$

$A_{\phi_*} (0, t)^* \mathcal{I}_{\phi_*}$ is pulled back along $\tilde{N}_{\phi_*} \times (0) \times I^l \rightarrow \tilde{N}_{\phi_*} \times \{(0) \} \times I^l$. For $r' > 0$, $\dim \tilde{N}_{\phi_*} < \dim N_{\phi_*}$. The $r' = 0$ contributions give the right expression in (98), and this concludes the proof of the proposition.

Let $\phi_* = (\phi_0, \ldots, \phi_{l+1})$ be a flag of length $l + 1$ and let $\phi_{s^1} = (\phi_1, \ldots, \phi_{l+1})$ and $\phi_{s^1} = (\phi_0, \ldots, \phi_l)$ be its truncations. By Lemma 61(b),

$$\left( t^{\phi_{s^1}}_{N_{\phi_{s^1}}} \right)^* \mathcal{I}_{\phi_{s^1}} = \mathcal{I}_{\phi_*} |_{(t_{l+1} = 0)}.$$

Factor $\beta_{\phi_{s^1}}$ as

$$\tilde{N}_{\phi_{s^1}} \xrightarrow{\beta_{\phi_{s^1}}} N_{\phi_{s^1}} \xrightarrow{\beta_{\phi_{s^1}}} N_{\phi_{s^1}}$$

where $\beta_{s^1} = \left( \delta_{s^1} \right)^{-1} \left( \beta_{s^1} \right)$ denotes the partial blow up and $\beta_{s^1}$ is defined similarly. Set

$$\mathcal{I}^{s^1}_{\phi_{s^1}} := \left( \beta_{s^1} \right)^* \left( e^{-DA_{\phi_{s^1}} (T, t_2, \ldots, t_{l+1})} (t_{l+1})^* (\delta_{s^1})^* (-\theta_0)^r \right),$$

so that

$$\mathcal{I}_{\phi_{s^1}} = (-1)^r \left( \beta_{s^1} \right)^* \mathcal{I}^{s^1}_{\phi_{s^1}} \cdot (\delta_{s^1})^* (-\theta_0).$$

There’s a map $\tilde{N}_{\phi_*} \xrightarrow{\delta_{\phi_{s^1}}} \tilde{N}_{\phi_{s^1}}$ induced from $\partial^{s^1}_{\phi_{s^1}} \gamma_{\phi_{s^1}}$, and using Lemma 61(b) again, we have

$$\left( \partial^{s^1}_{\phi_{s^1}} \gamma_{\phi_{s^1}} \right)^* \mathcal{I}^{s^1}_{\phi_{s^1}} = \mathcal{I}_{\phi_*} |_{(t_{l+1} = 1)}.$$

**Lemma 63.** We have

$$\int_{N_{\phi_*} \times I \times I} DA_{\phi_*} = 0$$
Proof. $D \left( e^{D \phi'(T, t_1, \ldots, t_l)} \tau_{\phi'} \pi_{\phi'} \omega_{\phi'} \right) = 0$ and $D \theta_0$ is a constant. Using Sym ($r'$) symmetry we reduce to showing

$$\int_{N_{\phi' \times I \times I}} T_{\phi'} = \int_{N_{\phi' \times I \times I}} A_{\phi'} (T, t_1, \ldots, t_l) T_{\phi'}$$

vanishes. But $A_{\phi'} (T, t_1, \ldots, t_l) T_{\phi'} \in \text{Im} \left( f_{\phi'}^* \right)$ and the codomain of $f_{\phi'}^*$ has lower dimension than its domain, so the integral does vanish. 

\[ \square \]

Lemma 64. We have

$$\int_{\partial N_{\phi' \times I \times I}} T_{\phi'} = 0.$$

Proof. The involution $\tau_{\phi'}^* : \partial_+ M_+^r \to \partial_+ M_+^r$ (cf. (39)) induce an orientation-reversing automorphism $\tau_{\phi'}$ of $\partial_+ N_{\phi'} \times I \times I'$ which commutes with $f_{\phi'}$ and $\delta_{\phi'}$, and so lifts to an orientation-reversing automorphism $\bar{\tau}_{\phi'}$ of $\partial_+ \bar{N}_{\phi'} \times I \times I'$, such that $\beta_{\phi'} \bar{\tau}_{\phi'} = \tau_{\phi'} \beta_{\phi'}$, and $\tilde{\delta}_{\phi'} \bar{\tau}_{\phi'} = \tilde{\delta}_{\phi'}$. These properties of $\tau_{\phi'}$ and $\bar{\tau}_{\phi'}$ imply $\bar{\tau}_{\phi'}^* T_{\phi'} = T_{\phi'}$, and the claim follows. 

\[ \square \]

Lemma 65. For $r', l \geq 0$ we have

\[ \sum_{\phi_{\psi'} \in \Psi_{r', l}} \frac{1}{l!} \cdot s (\phi_l)! \int_{\partial N_{\phi' \times I \times I}} T_{\phi'} = \sum_{\phi_{\psi'} \in \Psi_{r', l+1}} \frac{1}{(l + 1)!} \cdot s (\phi_{l+1})! \sum_{j=1}^{l+1} \int_{N_{\phi' \times I \times I \times (0) \times I'}} T_{\phi'}. \]

Proof. We can use (39) to replace the left hand side of (100) by

\[ \sum_{\phi_{\psi'} \in \Psi_{r', l+1}} \frac{1}{l!} \cdot s (\phi_{l+1})! \int_{\bar{N}_{\phi' \times I \times I}} T_{\phi'}. \]

To see this, consider the maps

$$\prod_{\phi_{\psi'} \in \Psi_{r', l}} \partial_+ F_{\phi'} \xrightarrow{f_{\phi'}} \partial_{l+1} F_{+}^{r' r'} \quad \prod_{\phi_{\psi'} \in \Psi_{r', l+1}} F_{\phi'} \xrightarrow{f_{\phi'}} \partial_{l+2} F_{-}^{r' r'}$$

for every point pt $\partial_+ \partial_{l+1} F_{+}^{r' r'}$, the essential fibers $f^{-1} (p)$ and $f_{-1}^{-1} (p)$ are either both empty or have sizes $s (\phi_l)!$ and $s (\phi_{l+1})!$, respectively. Finally, (101) is equal to the right hand side of (100) by Sym ($l + 1$)-equivariance. 

$$\square$$

Lemma 66. For $r', l \geq 0$, we have

\[ \sum_{\phi_{\psi'} \in \Psi_{r', l+1}} \frac{1}{l!} \cdot s (\phi_l)! \sum_{j=1}^{l+1} \int_{\bar{N}_{\phi' \times I \times I \times (0) \times I'}} T_{\phi'} = \sum_{\phi_{\psi'} \in \Psi_{r', l}} \frac{1}{l!} \cdot s (\phi_l)! \sum_{j=1}^{l+1} \int_{N_{\phi' \times I \times I \times (1) \times I'}} T_{\phi'}. \]
Proof. We can write every \( \phi_0 \in \Psi^{r+1} \) uniquely as \( \phi_0^{x} \) for \( \phi_0^{x} \in \Psi^{x+1} \), and use \( \text{Sym}(r+1) \) invariance and Eqs (98,99) to express the left hand side of (??) as
\[
\sum_{\phi_0^{x} \in \Psi^{x+1}} \frac{1}{r!} s(\phi_0^{x}) \int_{(\tilde{\beta}_0^{x})} (S(N_0)) \cdot T_{\phi_0^{x}}^{1} \cdot (\tilde{\theta}_0^{x})^{*} (-\theta_0) =
\]

\[
= \sum_{\phi_0^{x} \in \Psi^{x+1}} \frac{1}{r!} s(\phi_0^{x}) \int_{N_{\phi_0^{x}} \times I^{1}} (\tilde{\phi}_0^{x})^{*} T_{\phi_0^{x}}^{1} = \sum_{\phi_0^{x} \in \Psi^{x+1}} \frac{1}{r!} s(\phi_0^{x}) \int_{N_{\phi_0^{x}} \times I^{1}} T_{\phi_0^{x}}^{1},
\]

which by \( \text{Sym}(l) \)-equivariance is equal to the right hand side of (??).

We turn now to resumming the right hand side of (96). Consider \( \phi_0 \in \mathcal{P}_0^{r-1} \) with \( \text{cnt} \phi_0 = \phi_0 \). Since \( (\text{id}_N \times \text{For} \phi_0) \delta \phi_0 = \delta \phi_0 f \phi_0 \) we get an induced cartesian map \( \partial \phi_0, N_{\phi_0} \rightarrow \text{ker} \delta \phi_0 \) lying over \( \text{For} \phi_0 \), sitting in a map of short exact sequences
\[
0 \rightarrow \partial \phi_0, N_{\phi_0} \rightarrow N_{\phi_0} \rightarrow N_{0 \times F \phi_0} \rightarrow 0.
\]

where \( A = A(\phi_0) \) (0). We have \( \zeta \phi_0, \gamma \phi_0, \partial \phi_0, \zeta \phi_0 \) and \( f \phi_0, \zeta \phi_0 = \zeta \phi_0 f \phi_0 \) (in contrast to (61,103) so
\[
\eta \phi_0 = h_{M \phi_0} \oplus \zeta \phi_0,
\]

for \( h_{M \phi_0}, \zeta \phi_0 \) as in (43) and (65).

Proposition 67. We have
\[
\sum_{\phi_0 \in \Psi^{x+1}} \frac{1}{r!} s(\phi_0) \int_{N_{\phi_0} \times I^{1}} e^{D \phi_0 \psi} \eta \phi_0 \pi \phi_0 \omega \phi_0 = \frac{1}{s(\phi_0)} \int_{F_{\phi_0}} e_{M}^{-1} \cdot e_{S}^{-1} \cdot \omega \phi_0
\]

where \( e_{M}, e_{S} \) are canonical Euler forms for \( M_{\phi}, S_{\phi} \).

Proof. Using a similar argument as in the proof of Proposition 60 construct quadratic localizing forms \( q_{M} \) and \( q_{S} \) for \( M_{\phi} \) and \( S_{\phi} \) respectively, with \( q_{M} | S_{\phi}, M_{\phi} \in \text{Im}(h_{M \phi}) \) and \( q_{S} | S_{\phi}, S_{\phi} \in \text{Im}(\gamma \phi_0) \) for every \( \phi_0 \in \mathcal{P}_0^{r-1} \) s.t. \( \text{cnt} \phi_0 = \phi_0 \). By (103) \( q = q_{M} \oplus q_{S} \) is a quadratic localizing form for \( N_{\phi} = M_{\phi} \oplus S_{\phi} \) which satisfies the hypothesis of Lemma 65 below. On the other hand, we have
\[
\int_{N_{\phi}} e_{D \phi} \pi \phi \omega \phi_0 = \int_{F_{\phi}} \pi \phi \omega \phi_0 = \int_{F_{\phi}} e_{M}^{-1} \cdot e_{S}^{-1} \cdot \omega \phi_0,
\]

where the first equality uses the projection formula and the second uses Lemma 44. This completes the proof.

Lemma 68. We have
\[
\sum_{\phi_0 \in \Psi^{x+1}} \frac{1}{r!} s(\phi_0) \int_{N_{\phi_0} \times I^{1}} e^{D \phi_0 \psi} \eta \phi_0 \pi \phi_0 \omega \phi_0 = \frac{1}{s(\phi_0)} \int_{F_{\phi_0}} e_{D \phi} \pi \phi \omega \phi_0
\]

where \( q \) is any quadratic localizing form on \( N_{\phi} \) such that \( q | S_{\phi}, N_{\phi} \in \text{Im}(h_{N \phi}) \) for every \( \phi_0 \) with \( \text{cnt} \phi_0 = \phi_0 \).
Proof. Consider
\[ 0 = \sum_{\phi \in \Phi_{0,1}} \frac{1}{l! s(\phi)!} \int_{N_{\phi} \times I_u \times I^j (t_1, \ldots, t_j)} D \left[ e^D \left( 1-u \right) A^{-1}_{\phi} \right] \eta_{\phi} + u \left( \frac{\partial}{\partial s} \right)^* \pi^{\omega}_{\phi} \omega_{\phi} \]
\[ = \sum_{\phi, \in \Phi_{0,1}} \frac{1}{l! s(\phi)!} \int_{\partial(N_{\phi} \times I_v \times I^j (t_1, \ldots, t_j))} e^D \left( 1-u \right) A^{-1}_{\phi} \right] \eta_{\phi} + u \left( \frac{\partial}{\partial s} \right)^* \pi^{\omega}_{\phi} \omega_{\phi} \]
Write
\[ \partial \left( N_{\phi} \times I_u \times I^j (t_1, \ldots, t_j) \right) = \left( \partial_+ N_{\phi} \times \bigcup \partial_- N_{\phi} \times I^{+1} \right) \bigcup \]
\[ \bigcup \left( N_{\phi} \times \{0\} \times I^l \bigcup N_{\phi} \times \{1\} \times I^l \bigcup N_{\phi} \times I \times \partial I^l \right) \]
as in the proof of Proposition \[62\], the contribution of \( \partial_+ N_{\phi} \) vanishes and the contribution of \( \bigcup_{\phi} \partial_- N_{\phi} \) cancels with the contribution of
\[ \bigcup_{\phi, \in \Phi_{0,1}} \left( N_{\phi} \times I \times \{1\} \right) \]
vanishes, since the integrand is pulled back along \( h_{\phi,} \), which decreases the dimension of the total space. Similarly, the contribution of \( \bigcup_{\phi, \in \Phi_{0,1}} \left( N_{\phi} \times I \times I^{+l} \times \{1\} \times I^{l-j} \right) \)
vanesishes.

The contribution of \( \bigcup_{\phi, \in \Phi_{0,1}} \left( N_{\phi} \times \{0\} \times I^l \right) \) gives the left hand side of \[105\].

The contribution of \( \bigcup_{\phi, \in \Phi_{0,1}} \left( N_{\phi} \times \{1\} \times I \right) \) vanishes if \( l > 0 \) since the integrand is independent of \( t_1, \ldots, t_l \). If \( l = 0 \) the contribution is
\[ -\frac{1}{s(\phi)!} \int_{N_{\phi}} e^D \pi^{\omega}_{\phi} \omega = -\int_{\phi} \sigma \cdot \omega \].
The claim immediately follows. \( \square \)

Proof of Proposition \[67\]. This follows immediately from Propositions \[62\] and \[64\]. \( \square \)

7. Appendix: Orbifolds with Corners

7.1. Review of notions from \[30\]. We begin by a review of the appendix of \[30\], where more details can be found. Let \( \text{Man}^c \) denote the category of manifolds with corner\[11\] with smooth maps in the sense of \[11\]. Let \( \text{PEG} \) be the category of proper étale groupoids in \( \text{Man}^c \). A morphism \( (X_1 \to X_0) \to (Y_1 \to Y_0) \) in \( \text{PEG} \) is a refinement if it is fully faithful and essentially surjective, and the maps \( R_0 : X_0 \to Y_0, R_1 : X_1 \to Y_1 \) of the object and morphisms spaces are étale. The category \( \text{Orb} \) of orbifolds with corners is the 2-localization of \( \text{PEG} \) by refinements. We usually denote orbifolds by caligraphic letters \( X, Y, M \). They are given by proper étale groupoids. Maps \( X \to Y \) are given by fractions \( F_1/R_1 \) with \( X_1 \to X_0 \) a refinement and \( X_1 \to Y_1 \) a smooth functor. We refer the reader to \[22\] for further details,

\[11\]by corners we always mean “ordinary” corners, modeled on \( [0, \infty)^k \times \mathbb{R}^{n-k} \); though we base our discussion on \[22\], we will not use manifolds with generalized corners in this paper.
including the definition of the 2-cells, the composition operations, etc. We say \( f, f' : \mathcal{X} \to \mathcal{Y} \) are homotopic if there exists a 2-cell \( f \Rightarrow f' \).

**Definition 69.** We say \( f \) is weakly smooth, smooth, strongly-smooth, étale, interior, b-normal, submersive, b-submersive, simple or perfectly simple if \( F_0 \) has the corresponding property as a map of manifolds with corners. It is easy to check that these properties are preserved by 2-cells (and thus are properties of the homotopy class of \( f \)). The map \( f \) is called a \( b \)-fibration if it is b-normal and b-submersive (cf. [11] Definition 4.3).

For \( i = 1, 2 \) let \( f^i : \mathcal{X}^i \to \mathcal{Y} \) be a strongly smooth map given by the fraction \( F^i|R^i \). \( f_1, f_2 \) are said to be transverse (respectively, strongly transverse) if \( F^1_0, F^2_0 \) are transverse (resp. strongly transverse) in the sense of [12] Definitions 6.1, 6.10).

A map \( f : \mathcal{X} \to \mathcal{Y} \) is called a **diffeomorphism** if it is an equivalence in \( \text{Orb} \).

A manifold with corners \( T \) defines an orbifold \( \mathcal{T} \) given by the trivial groupoid \( T \rightrightarrows T \). This gives a full and faithful functor \( \text{Man}_T \to \text{Orb} \) and we say a stack \( \mathcal{X} \) “is a manifold” if it lies in the essential image of this functor. For a fixed orbifold \( \mathcal{X} = X_1 \rightrightarrows X_0 \) the category of arrows \( \mathcal{T} \to \mathcal{X} \) forms a geometric stack, where the map \( X_0 \to \mathcal{X} \) is an atlas for this stack:

**Definition 70.** Let \( \mathcal{X} \) be an orbifold with corners. An **atlas for** \( \mathcal{X} \) is a map \( p : \underline{M} \to \mathcal{X} \) where \( M \) is some manifold with corners, such that for any other map \( f : \underline{N} \to \mathcal{X} \) from a manifold with corners, \( M \times_\mathcal{X} N \) is a manifold with corners and the projection \( M \times_\mathcal{X} N \xrightarrow{\rho'} N \) is surjective and étale.

Conversely, any such atlas defines an orbifold equivalent to \( \mathcal{X} \).

If \( G \) is a Lie group we define the 2-category of \( G \)-orbifolds, \( G \)-equivariant maps, and \( G \)-2-cells as in [23] Definition 2.1]. We define the stack of fixed points \( \mathcal{X}^G \) and the quotient stack \( \mathcal{X}_G / G \) as in [24] Definition 2.3] (sometimes we refer to these as the homotopy fixed points and the stacky quotient, respectively). If \( \mathcal{M} \) is a \( G \)-orbifold, we denote the action of \( g \in G \) by \( g^\bullet : \underline{M} \to \mathcal{M} \) or just \( g : \underline{M} \to \mathcal{M} \).

To make the paper readable, we abuse notation and refer to maps which are canonically isomorphic as equal. For example we may write \( g.h. = (gh) \). The same goes for orbifolds which are canonically equivalent (that is, with a given equivalence, or with an equivalence which is specified up to a unique 2-cell). For example we may write

\[
(M_1 \times_L M_2) \times_L M_3 = M_1 \times_L (M_2 \times_L M_3).
\]

The notion of a sheaf on an orbifold \( \mathcal{X} \) is the same as the notion of a sheaf on the underlying topological orbifold (see [17] [22]). A vector bundle \( E \) on an orbifold with corners \( \mathcal{X} = X_1 \rightrightarrows X_0 \) is given by \((E_0, \phi)\) where \( E_0 \) is a smooth vector bundle on \( X_0 \) and

\[
\phi : s^* E_0 \to t^* E_0
\]

is an isomorphism satisfying some obvious compatibility requirements with the groupoid structure. The sections of \((E_0, \phi)\) form a sheaf over \( \mathcal{X} \). A **local system** on an orbifold \( \mathcal{X} \) is a sheaf which is locally isomorphic to the constant sheaf \( \underline{Z} \). We extend the conventions set forth in [29] §1.1, §6.1] to proper étale groupoids with corners in the obvious way[12]. In particular, for every vector bundle \( E \) on \( X^\bullet \), there's

---

[12] There we worked with \( \mathbb{C} \)-valued local systems, whereas here we work with \( \mathbb{Z} \)-valued local systems.
a local system \( \text{Or}(E) \) on \( X_\bullet \). The orientation local system of \( X_\bullet \) is \( \text{Or}(TX_\bullet) \). We have a local system isomorphism

\[
i_\partial : \text{Or}(T\partial X_\bullet) \rightarrow \text{Or}(TX_\bullet)
\]

lying over \( i_\partial : \partial X_\bullet \rightarrow X_\bullet \), defined by appending the outward normal vector at the beginning of the oriented base. Maps of local systems are always assumed to be cartesian, so to specify a local system map \( L_1 \rightarrow L_2 \) over \( X_1 \rightarrow X_2 \) is equivalent to giving an isomorphism \( L_1 \rightarrow f^{-1}L_2 \).

Differential forms on \( X \) are the global sections of the sheaf of sections of \( \bigwedge T^* X \); in particular they are invariant under the local action of the isotropy group at each point.

### 7.2. Additional notions for manifolds with corners.

**Definition 71.** A map \( f : X \rightarrow Y \) of manifolds with corners is called a **closed immersion** if for every \( p \in X \) there exists an open neighbourhood \( p \in U \subset X \), an open neighbourhood \( f(U) \subset V \subset Y \), and a submersion \( h : V \rightarrow \mathbb{R}^N \) for some integer \( N \geq 0 \) such that the following square is cartesian

\[
\begin{array}{ccc}
U & \xrightarrow{f|_U} & V \\
\downarrow & & \downarrow h \\
0 & \rightarrow & \mathbb{R}^N
\end{array}
\]

(it follows that \( N = \dim Y - \dim X \)). Note the fiber product exists by [30] Lemma 37 since \( h \) is (vacuously) b-normal, and \( 0 \rightarrow \mathbb{R}^N \) is strongly smooth and interior.

**Remark 72.** Any b-submersion to a manifold without boundary is automatically a strongly smooth submersion, so it suffices to assume that \( h \) is a b-submersion.

**Definition 73.** A map \( f : X \rightarrow Y \) of manifolds with corners is called a **closed embedding** if it is a closed immersion, has a closed image, and induces a homeomorphism on its image.

**Definition 74.** A map \( f : X \rightarrow Y \) of manifolds with corners is an **open embedding** if it is étale and injective.

**Lemma 75.** If \( i : X \rightarrow Y \) and \( f : W \rightarrow Y \) are smooth maps of manifolds with corners, with \( i \) either a closed or an open embedding, and \( f(W) \subset i(X) \), then there is a unique smooth map \( g : W \rightarrow X \) with \( f = i \circ g \).

**Proof.** This is Corollary 4.11 of [II].

If we assume in addition that \( f \) is a closed or an open embedding, and that \( f(W) = i(X) \), then \( g \) is a diffeomorphism.

### 7.3. Additional notions for orbifolds with corners.

**Definition 76.** A map \( F|R : \mathcal{X} \rightarrow \mathcal{Y} \) of orbifolds with corners is a **closed immersion** if \( F_0 \) is a closed immersion. In this case, the same holds for any map homotopic to \( F|R \).

**Definition 77.** A map \( f : \mathcal{X} \rightarrow \mathcal{Y} \) of orbifolds with corners is a **closed** (respectively, **open**) **embedding** if for some (hence any) atlas \( p : M \rightarrow \mathcal{Y} \), the 2-pullback \( \overline{M}_p \times_f \mathcal{X} \) is a manifold with corners and the map \( \overline{M}_p \times_f \mathcal{X} \rightarrow \overline{M} \) is a closed (resp. open) embedding of manifolds with corners.
If \( f: \mathcal{X} \to \mathcal{Y} \) is a closed embedding we may refer to \( \mathcal{X} \) as a suborbifold of \( \mathcal{Y} \).

**Definition 78.** Let \( f: \mathcal{X} \to \mathcal{Y} \) be a closed immersion of orbifolds with corners, with normal bundle \( N_f := f^* T\mathcal{Y}/T\mathcal{X} \). A tubular neighbourhood for \( f \) consists of a pair of open embeddings

\[
N_f \xleftarrow{i_0} V \xrightarrow{\gamma} \mathcal{Y}
\]

such that (i) there exists \( i'_0: \mathcal{X} \to V \) such that \( i_0 = j \circ i'_0 \) for \( i_0: \mathcal{X} \to N \) the closed embedding of the zero section, and \( f = \gamma \circ i'_0 \).

(ii) the map

\[
N = (i_0)^* T\mathcal{N}/T\mathcal{X} = (i'_0)^* T\mathcal{V}/T\mathcal{X} \xrightarrow{d\gamma} f^* T\mathcal{Y}/T\mathcal{X} =: N
\]

is the identity.

If we assume in addition that \( f: \mathcal{X} \to \mathcal{Y} \) is a \( T \)-equivariant map, an equivariant tubular neighbourhood for \( f \) is a tubular neighbourhood for \( f \) such that \( j \) and \( \gamma \) are equivariant.

**Definition 79.** If \( \mathcal{X} = \bigsqcup \mathcal{X}_i \) is an l-orbifold with corners (cf. §3.2) and \( \mathcal{Y} \) is an orbifold with corners of dimension \( r \), a closed embedding \( f: \mathcal{X} \to \mathcal{Y} \) is just a coproduct of closed embeddings \( \bigsqcup \mathcal{f}_i \). The total space of the normal l-bundle \( \bigsqcup \mathcal{N}_j \mathcal{f}_i \) is an ordinary orbifold with corners \( N \) and a tubular neighborhood for \( f \) is given by a pair of maps of orbifolds with corners \( N \xleftarrow{j} \mathcal{Y} \xrightarrow{\gamma} \mathcal{Y} \), satisfying the conditions above locally.

**Lemma 80.** Any closed embedding admits a tubular neighbourhood; a \( T \)-equivariant embedding admits a \( T \)-equivariant tubular neighbourhood.

**Proof.** This is proven using geodesic flow along a b-metric, which may be taken to be \( T \)-invariant, see [16] (note that our notion of a closed embedding of manifolds with corners corresponds to what Melrose calls “interior p-submanifold”).

The following lemma is useful for extending constructions from the boundary inward. Let \( \mathcal{X} \) be an orbifold with corners given by a proper étale groupoid \( \mathcal{X} = X_1 \rightrightarrows X_0 \). Let \( E \) be a vector bundle over \( \mathcal{X} \) given by \( (E_0, s^* E_0 \xrightarrow{\phi} t^* E_0) \). A section of \( E \) is given by a section \( \sigma: X_0 \to E_0 \) with \( t^* \sigma = \phi \circ s^* \sigma \). We can pull back bundles and sections along maps of groupoids.

Let \( i_1 \) (respectively, \( i_2 \)) be the map \( \partial^2 \mathcal{X} \to \partial \mathcal{X} \) that remembers the first (respectively, second) local boundary component (so \( i_1 = i^\partial_0_{\partial \mathcal{X}} \)).

**Lemma 81.** Let \( \sigma_0 \) be a section of \( (i^\partial_1_{\mathcal{X}})^* E \) such that \( i_1^* \sigma_0 = i_2^* \sigma_0 \). Then there exists a section \( \sigma \) of \( E \) with \( (i^\partial_1_{\mathcal{X}})^* \sigma = \sigma_0 \).

If we assume that, in addition, \( \mathcal{X} \) is a \( T \)-orbifold, \( E \) is a \( T \)-vector bundle, and \( \sigma_0 \) is a \( T \)-equivariant section, then there exists a \( T \)-equivariant section \( \sigma \) with \( (i^\partial_1_{\mathcal{X}})^* \sigma = \sigma_0 \).

**Proof.** Using a partition of unity, i.e. a smooth function \( \rho: X_0 \to \mathbb{R} \) with \( t_* s^* \rho = 1 \), we reduce to the case \( \mathcal{X} = X^0_0 \) (a manifold with corners). The claim for this case follows from Whitney’s theorem [28], though the argument we have is lengthy. Since the result seems to be well-known to experts (see [11] Proposition 4.41(b)), we omit it.

The second claim follows from the first by averaging out the \( T \)-action on any extension \( \sigma \) of \( \sigma_0 \).
Lemma 82. Let $X$ be a $T$-orbifold. If $\theta_0 \in \Omega(\partial X, \mathbb{R})^T$ satisfies $i^* \theta_0 = i^* \theta_0$ then there exists $\theta \in \Omega(X, \mathbb{R})^T$ with $i^* \theta = \theta_0$.

Proof. As in the proof of Lemma 81, we may assume without loss of generality that

$$X = \mathbb{R}^n_k = \{(x_1, ..., x_n) \mid x_j \geq 0 \text{ for } 1 \leq j \leq k\}$$

and $T$ acts trivially. Moreover, we may focus on some $J = \{j_1, ..., j_r\} \subset [n]$, and consider the coefficient $f$ of $dx^j$ in $\theta$. Let $D_0 \subset 2^{|J|}$ be defined by

$$D_0 = \{I \subset [k] \exists i \in [k] \setminus J \mid I \subset [k] \setminus \{i\}\}$$

This represents the faces of $\mathbb{R}^n_k$ for which the value of $f$ is specified by $\theta_0$: the subset $S \subset [k]$ represents the face $F_S = \{x_a = 0 \forall a \in [k] \setminus S\}$. Define recursively

$$D_{i+1} = \{I \subset [k] \forall a \in I \setminus \{a\} \in D_i\},$$

and prove by induction on $i$ that these are closed under taking subsets and (using the previous lemma) that $f$ extends to $D_{i+1} \setminus D_i$. We have

$$\emptyset \in D_\infty \text{ and } D_\infty = \{I \subset [k] \forall a \in I \setminus \{a\} \in D_\infty\}$$

which implies $D_\infty = 2^{|J|}$ (consider $S \in 2^{|J|} \setminus D_\infty$ with $|S|$ minimal). This shows that $f$ extends to $\mathbb{R}^n_k$ and completes the proof. $\square$

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