Quantum gravitational optics: Effective Raychaudhuri equation

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Abstract

Vacuum polarization in QED in a background gravitational field induces interactions which \textit{effectively} modify the classical picture of light rays, as the null geodesics of spacetime. These interactions violate the strong equivalence principle and affect the propagation of light leading to superluminal photon velocities. Taking into account the QED vacuum polarization, we study the propagation of a bundle of rays in a background gravitational field. To do so we consider the perturbative deformation of Raychaudhuri equation through the influence of vacuum polarization on photon propagation. We analyze the contribution of the above interactions to the optical scalars namely, shear, vorticity and expansion using the Newman-Penrose formalism.

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I. INTRODUCTION

Vacuum polarization is an essential ingredient of QED whose contribution leads to astonishingly precise agreement between predicted and observed values of the electron magnetic moment and Lamb shift. On the other hand being a quantum field theoretic effect it would be interesting to look for its implication in semi-classical gravity as a quantum characteristic of the electromagnetic field coupled to the gravitational field [1]. Effects of QED interactions such as vacuum polarization on the photon propagation in a classical background gravitational field gives rise to a wide range of new phenomena. Simple analysis of null rays in classical general relativity implies that a curved spacetime, compared to the flat case, could be treated as an optical medium with a refractive index [2]. Now adding to that QED effects, in the context of semi-classical gravity, leads to interesting phenomena such as dispersive effects and polarization dependent propagation of photons. One of the main consequences is found to be the light cone modification in such a way that the QED photons do not propagate along the null geodesics of the background geometry. They propagate instead along null geodesics of an effective geometry. In many cases depending on the direction and polarization of photons, superluminal propagation becomes possible. Quantum characteristics of photon propagation in a curved background opens up a whole new field coined quantum gravitational optics (QGO) [3], Started with the pioneering paper by Drummond and Hathrell who considered the effect of vacuum polarization on photon propagation in a curved background and continued with detailed study of the same effect for specific spacetimes [4]-[5]. Coupling of the electromagnetic field with curvature in the QED action introduced in [1] violates the strong equivalence principle (SEP) in a mass scale comparable to the electron mass $m$. The modification of the underlying spacetime geometry felt by photons due to QED interactions has some unexpected results. The Newman-Penrose (NP) formalism has been found to be an elegant way of analyzing the results in the SEP violating cases. Quantum gravitational optics attributes velocity shift and null cone modification at each point to only a single NP scalar for each of the Ricci and Weyl tensors, namely $\Phi_{00}$ and $\Psi_{0}$ respectively [6]. The question is what would be the role of other NP scalars in determining the forms of Ricci and Weyl tensors? or what, if any, would be the effect of the gravitational field on the other photons traveling with the normal speed of light Here we shall study the geometric properties of physical null congruences which are families of physical null rays determined
by the effective metric $G_{\mu\nu}$. These are then distinct from the properties of the geometric null congruences characterized by the spacetime metric $g_{\mu\nu}$. These congruences are special in the sense that the modification of the light cone in a local inertial frame (LIF) can be described as $(\eta_{(a)(b)} + \alpha^2\sigma_{(a)(b)} (R)) k^{(a)}k^{(b)} = 0$, where $\eta_{(a)(b)}$ is the Minkowski metric, $\alpha$ is the fine structure constant and $\sigma_{(a)(b)} (R)$ depends on the Riemann curvature at the origin of the LIF. In one-loop approximation, these photons travel with unit velocity. For on-shell photons, some of the NP scalars play the role of optical scalars and their propagation, along the rays, is governed by the known Raychaudhuri equation. We show that these scalars must be effectively modified to describe the physical null congruences and find that a new set of NP scalars contribute in the definition of these effective optical scalars. The modified Raychaudhuri equation governing their propagation will be discussed. The effective geometry may have features that are seen only by photons when propagating in the geometry. There may be hidden singularities in the form of geodesic incompleteness which can be studied through the implementation of the effective Raychaudhuri equation.

II. QUANTUM GRAVITATIONAL OPTICS

The effect of one-loop vacuum polarization on the photon propagation in a fixed curved background spacetime is represented by the following effective action derived by Drummond and Hathrell [1],

$$\Gamma = \int dx \sqrt{-g} \left[ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{m^2} \left( aRF_{\mu\nu}F^{\mu\nu} + bR_{\mu\nu}F^{\mu\lambda}F^{\nu}_{\lambda} + cR_{\mu\lambda\rho\nu}F^{\mu\lambda}F^{\nu\rho}\right) \right].$$

(1)

Here, $a = -\frac{1}{144} \alpha$, $b = \frac{33}{360} \alpha$ and $c = -\frac{1}{360} \alpha$ where $\alpha$ is the fine structure constant and $m$ is the electron mass. The notable feature in the above action is the direct coupling of the electromagnetic field to the curvature tensor which in effect violates the strong equivalence principle. When we consider the equations of motion derived from this action the Bianchi identity does not change but it gives rise to curvature dependent modifications to the equation of motion in the form

$$D_{\mu}F^{\mu\nu} + \frac{1}{m^2} \left( 2bR^{\mu}_{\lambda}D_{\mu}F^{\lambda\nu} + 4cR^{\mu\nu}_{\lambda\rho}D_{\mu}F^{\lambda\rho}\right) = 0.$$

(2)

There are some approximations under which this equation of motion was obtained. The first one is the low frequency approximation in the sense that the derivation is only applicable to
wavelengths $\lambda > \lambda_c$. By this approximation we ignore terms in the effective action involving higher order field derivatives. The second is a weak field approximation for gravity. This implicitly means that the wavelengths $\lambda << L$ are considered, where $L$ is a typical curvature scale.

The characteristics of the light propagation can be studied by applying the geometric optics approximation. The electromagnetic field is written in the form $A_\mu = A a_\mu e^{i \theta}$, where $A$ is the amplitude and $a_\mu$ is the polarization vector. The amplitude is taken to be slowly-varying in comparison with $\theta$. In each small region of space, we can speak of a direction of propagation, normal to a surface at all of whose points the phase of the wave is constant.

If we now identify the wave vector as $k_\mu = \partial_\mu \theta$, the equation governing the components of this vector field for a photon with spacelike, normalized polarization vector i.e., $a_\mu a^\mu = -1$, can be read from (2) as

$$k^2 - \frac{2b}{m^2} R_{\mu\lambda} k^\mu k^\lambda + \frac{8c}{m^2} R_{\mu\nu\lambda\rho} k^\mu k^\lambda a^\nu a^\rho = 0. \tag{3}$$

Eq. (3) is an effective light cone equation, re-expressed in terms of the Weyl tensor is given by

$$k^2 - \frac{(2b + 4c)}{m^2} R_{\mu\lambda} k^\mu k^\lambda + \frac{8c}{m^2} C_{\mu\nu\lambda\rho} k^\mu k^\lambda a^\nu a^\rho = 0. \tag{4}$$

The corresponding momentum of photon, $p^\mu$, is the tangent vector to the light ray $x^\mu$ i.e., $p^\mu = \frac{d}{ds} x^\mu (s)$. Equation (4) being quadratic and homogenous could be written in the following form

$$G^\mu{}_{\nu} k_\mu k_\nu = 0 \tag{5}$$

This equation implies a nontrivial relation between $k_\mu$ and $p^\mu$, through $G$ by the following argument. Refering to the definitions of $k_\mu$ and $p^\mu$, the former defined as the gradient of a phase, constitutes the component of a one-form and hence belongs to a cotangent space whereas the latter being a tangent vector belongs to the tangent space. In the usual free electromagnetic theory they are related through the metric tensor i.e $k_\mu = g_{\mu\nu} p^\nu$. In the modified theory however, using equations (4) and (5), we arrive at the following non-trivial relation,

$$k_\mu = G_{\mu\nu} p^\nu = p_\mu + \frac{1}{m^2} [(b + 2c) R_{\mu\lambda} p^\lambda - (4c) C_{\mu\lambda\sigma\kappa} a^\lambda a^\sigma p^\kappa] \tag{6}$$

where $G = G^{-1}$. This shows that at this level of approximation, QGO can be characterized as a bimetric theory. The physical light cones are determined by the effective metric $G_{\mu\nu}$,
and are distinct from the geometrical null cones which are fixed by the spacetime metric $g_{\mu\nu}$. (Indices are always raised and lowered using the spacetime metric $g_{\mu\nu}$).

It should be noted that all these equations are manifestly local Lorentz invariant though the presence of the explicit curvature coupling in the effective action means different dynamics in the LIFs at different points in spacetime. In this sense these equations violate the strong principle of equivalence (SEP). Some of their implications including gravitational birefringence and superluminal speed of light have been discussed in [7]. Equation (6) can be employed to examine the SEP violating effects which arise in the physical photon congruence. In the present study we will focus on the modifications produced by the vacuum polarization effects on the propagation of light rays in a curved background.

III. EFFECTIVE OPTICAL SCALARS

The physical congruence is specified by the integral curves $\gamma$ of the vector field $k^\mu$ parametrized by the parameter $u$ and scaled such that $\nabla_k u = 1$. In order to examine the relation between the neighboring curves of the congruence, the connecting vector field $q^\mu$ is introduced [8]. Defined along a particular curve $\gamma$, it characterizes the displacement from a point $P \in \gamma$ to a point $P'$ on a neighboring curve, where $P$ and $P'$ have the same parameter value $u$. Mathematically, this means that $q^\mu$ is being Lie transported along the curve by the vector field $k^\mu$.

$$\mathcal{L}_k q^\mu = 0 \quad \text{that is} \quad \nabla_k q^\mu = \nabla_q k^\mu.$$  (7)

Here the notation $\nabla_X = X^\alpha \nabla_\alpha$ for directional covariant derivative is used. In a SEP violating theory like QGO, it is illuminating to work in the NP tetrad formalism. Following closely the notations used in [7] and [8], the first step is to choose a null tetrad. We choose $l^\mu$ be a null vector along the photon momentum. Let $a^\mu$ and $b^\mu$ be spacelike transverse vectors (in the present case they will be identified with the polarization vectors later) and define the complex null vectors $m^\mu$ and $\overline{m}^\mu$ by $m^\mu = \frac{1}{\sqrt{2}} (a^\mu + ib^\mu)$ and $\overline{m}^\mu = \frac{1}{\sqrt{2}} (a^\mu - ib^\mu)$. Adding to this set another real null vector $n^\mu$, they satisfy the usual NP orthogonality conditions

$$l.m = l.\overline{m} = n.m = n.\overline{m} = 0,$$  (8)

and

$$l.l = n.n = m.m = \overline{m}.\overline{m} = 0,$$  (9)
The normalization conditions
\[ l.n = -m.m = 1, \]  
are also imposed. We assign a vierbein \( e^\mu_{(a)} \) to this tetrad as follows
\[ e_1 = e_2; \quad e_2 = e_3 = -e_4 = m; \quad e_4 = -e_3 = m, \]  
with the frame metric of the form
\[ \eta^{(a)(b)} = e^\mu_{(a)} e^\nu_{(b)} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 
\end{pmatrix}. \]

Weyl tensor being trace free is completely specified in this basis, by the following five complex scalars,
\[ \begin{align*}
\Psi_0 &= -C_{abcd}l^a m^b n^c m^d = -C_{1313} \\
\Psi_1 &= -C_{abcd}l^a n^b l^c m^d = -C_{1213} \\
\Psi_2 &= -C_{abcd}l^a m^b m^c n^d = -C_{1342} \\
\Psi_3 &= -C_{abcd}l^a n^b m^c n^d = -C_{1242} \\
\Psi_4 &= -C_{abcd}n^a m^b n^c m^d = -C_{2424}. 
\end{align*} \]

The ten components of the Ricci tensor are also defined in terms of the following four real and three complex scalars.
\[ \begin{align*}
\Phi_{00} &= -\frac{1}{2} R_{11}; \quad \Phi_{22} = -\frac{1}{2} R_{22}; \quad \Phi_{02} = -\frac{1}{2} R_{33}; \quad \Phi_{20} = -\frac{1}{2} R_{44}; \\
\Phi_{01} &= -\frac{1}{2} R_{13}; \quad \Phi_{10} = -\frac{1}{2} R_{14}; \quad \Phi_{12} = -\frac{1}{2} R_{23}; \quad \Phi_{21} = -\frac{1}{2} R_{24}; \\
\Phi_{11} &= -\frac{1}{4} (R_{12} + R_{34}); \quad \Lambda = \frac{1}{24} R = \frac{1}{12} (R_{12} - R_{34}). 
\end{align*} \]

In tetrad formalism, various quantities called "Ricci rotation coefficients", \( \gamma_{(c)(a)(b)} \), are employed to account for the tetrad covariant differentiation. These quantities can be defined as
\[ e_{(a)\mu} = e^{(c)}_{\mu} \gamma_{(c)(a)(b)} e_{(b)}^{\nu}. \]
In NP formalism, they are called "spin coefficients" and are designated by the following special symbols:

\[ \kappa = \gamma(3)(1)(1); \quad \rho = \gamma(3)(1)(4); \quad \epsilon = \frac{1}{2} \left( \gamma(2)(1)(1) + \gamma(3)(4)(1) \right); \]
\[ \sigma = \gamma(3)(1)(3); \quad \mu = \gamma(2)(4)(3); \quad \gamma = \frac{1}{2} \left( \gamma(2)(1)(2) + \gamma(3)(4)(2) \right); \]
\[ \lambda = \gamma(2)(4)(4); \quad \tau = \gamma(3)(1)(2); \quad \alpha = \frac{1}{2} \left( \gamma(2)(1)(4) + \gamma(3)(4)(4) \right); \]
\[ \nu = \gamma(2)(4)(2); \quad \pi = \gamma(2)(4)(1); \quad \beta = \frac{1}{2} \left( \gamma(2)(1)(3) + \gamma(3)(4)(3) \right). \] (16)

A general rule to obtain the complex conjugate of any quantity is to replace the index (3), wherever it occurs with the index (4) and vice versa.

Using the NP formalism has some advantages, including the fact that entirely scalar quantities are used. Also a considerable economy of notation is achieved when complex spin coefficients are used instead of the Christoffel symbols employed in the conventional coordinate approach. This formalism is most advantageous if the null tetrads introduced could be completely tied to the geometry of the problem. Otherwise some freedom remains in the choice of the basis as a result of subjecting it to the Lorentz transformation. This has the effect that many of the quantities involved in the calculation are of the same nature as gauge quantities whose values transform in certain ways as the basis frame is varied in accordance with the remaining freedom. In QGO, one null direction is singled out and that is the unperturbed photon momentum which is fixed in the direction of the null vector \( l^\mu \).

Therefore the allowed transformation would be those which leave the \( l^\mu \) direction unchanged and preserve the underlying orthogonality and normalization conditions. These are classified as follows:

**Class I:**
\[
\begin{align*}
l & \mapsto l; \quad m \mapsto m + al; \quad m^\mu \mapsto m^\mu + a^\mu l; \quad n \mapsto n + a^\mu m + a m^\mu + a a^\mu l; \\
\end{align*}
\]

**Class II:**
\[
\begin{align*}
l & \mapsto A^{-1} l; \quad n \mapsto A n; \quad m \mapsto e^{i\theta} m; \quad m^\mu \mapsto e^{-i\theta} m^\mu, \\
\end{align*}
\]

where \( a \) is a complex function and \( A \) and \( \theta \) are two real functions on the manifold. We shall now proceed to write down equation (7) in the above tetrad basis. Before doing so we employ all the gauge freedom we have to fix the tetrad basis. As the tetrad frame is transformed, different tetrad components are subject to changes. With \( l \) vectors as the velocity vector along the geometric congruence, we have

\[
\nabla_l l^\mu = (\epsilon + \epsilon^*) l^\mu - \kappa m^\mu - \kappa^* m^\mu \propto l^\mu \implies \kappa = 0, \] (18)
and if they are affinely parametrized

$$\nabla_l l^\mu = 0 \implies \kappa = \epsilon = 0. \quad (19)$$

If $\kappa = 0$, the latter requirement can be met by a class II rotation which will not affect the direction of $l$ nor of initially vanishing of $\kappa$. By a suitable rotation of class I, it could be arranged so that $\pi = 0$. So the gauge we are working in is the one in which $\kappa = \epsilon = \pi = 0$. After such a rotation, the newly oriented vectors $n, m$ and $\overline{m}$ will remain unchanged as they are parallely propagated along $l$. This could be invoked through the following relations

$$\begin{align*}
\nabla_l m^\nu &= \pi^* l^\nu - \kappa n^\nu + (\epsilon - \epsilon^*) m^\nu \\
\nabla_l n^\nu &= \pi^* \overline{m}^\nu + \pi m^\nu - (\epsilon + \epsilon^*) n^\nu
\end{align*} \quad (20)$$

which vanish in the above gauge. Now we have exploited all the gauge freedom we had and ready to write equation (7) in the above gauge. We only consider the states which propagate with a well defined polarization [18]. For two transverse polarizations expressed in terms of the vectors $m^\mu$ and $\overline{m}^\mu$, representing the left and the right handed circular polarizations, we have

$$\nabla_k q^\nu = \nabla_k \left[ l^\nu + \frac{1}{m^2} (b + 2c) R^{\lambda\nu} l_\lambda \mp \frac{2c}{m^2} C^{\nu}_{\lambda\sigma\kappa} l^\sigma \left( m^\lambda \pm \overline{m}^\lambda \right) \left( m^\kappa \pm \overline{m}^\kappa \right) \right]. \quad (21)$$

After multiplying by $e_{\nu}^{(c)}$, we get

$$\nabla_k q^{(c)} = \left( \eta^{(c)(a)} + \frac{1}{m^2} A^{(c)(a)} \right) q^{(b)} \gamma^{(a)(1)(b)} + q^{(a)} \gamma^{(a)(c)(b)} k^{(b)}, \quad (22)$$

where

$$\begin{align*}
A^{(c)(a)} &= (b + 2c) R^{(c)(a)} \mp (2c) M^{(c)(a)}, \\
M^{(c)(a)} &= C^{(c)(3)(a)(3)} + C^{(c)(4)(a)(4)} \pm \left( C^{(c)(3)(a)(4)} + C^{(c)(4)(a)(3)} \right).
\end{align*} \quad (23)$$

The specifically gravitational birefringence shows up here in $M^{(a)(c)}$. (we note that $A^{(c)(a)} = A^{(a)(c)}$). Equation (22) is a system of coupled differential equations describing the propagation of the tetrad components of the connecting vector along the velocity vector $k^\mu$. The second term in (22) appears as a consequence of the null tetrad propagation along the physical ray (with the tangent vector $k^\mu$) i.e $q^\nu \nabla_k e^{(c)}_{\nu}$. We introduce different tetrad components of $q^\mu$ in the following way

$$q^\mu = gl^\mu + \xi \overline{m}^\mu + \bar{\xi} m^\mu + hn^\mu. \quad (24)$$
so that

\[ q^{(1)} = q_{(2)} = g; \quad q^{(2)} = q_{(1)} = h; \quad q^{(3)} = -q_{(4)} = \xi; \quad q^{(4)} = -q_{(3)} = \xi. \]  

(25)

Explicitly for \( c = 4 \), (22) gives

\[
\nabla_k q^{(4)} = \left( \eta^{(4)(3)} + \frac{1}{m^2} A^{(4)(3)} \right) \left[ q^{(2)} \gamma_{(3)(1)(2)} + q^{(3)} \gamma_{(3)(1)(3)} + q^{(4)} \gamma_{(3)(1)(4)} \right]
+ \frac{1}{m^2} A^{(4)(4)} \left[ q^{(2)} \gamma_{(4)(1)(2)} + q^{(3)} \gamma_{(4)(1)(3)} + q^{(4)} \gamma_{(4)(1)(4)} \right]
+ \frac{1}{m^2} A^{(4)(2)} q^{(b)(a)} \gamma_{(a)(b)}
\]

(26)

in which the last term is again the contribution due to the null tetrad propagation along the physical ray discussed below equation (23). To identify the geometrical and physical content of this equation, a "covariant approach" in which any complicated gauge behavior is completely avoided must be chosen. For example, in (26), the term \( A^{(4)(4)} q^{(2)} \gamma_{(4)(1)(2)} \) having such a complicated gauge behavior, transforms into

\[
\left[ A^{(4)(4)} - 2a A^{(4)(2)} + a^2 A^{(2)(2)} \right] q^{(2)} [\gamma_{(4)(1)(2)} + a^* \gamma_{(4)(1)(3)} + a \gamma_{(4)(1)(4)}]
\]

under a class I transformation. To keep the formalism covariant we remove the gauge-dependent terms by applying the extra condition that the neighboring pair of rays to satisfy the following condition,

\[ q.k = 0, \]  

(28)

and call them abreast [8]. This is a primary constraint and it means

\[
O (\alpha^0) : \quad q.l = 0 \Rightarrow h = 0;
\]

\[
O (\alpha^1) : \quad A_{(1)(a \neq 2)} = 0 \Rightarrow \Phi_{00} = \Phi_{01} = \Phi_{10} = \Psi_0 = \Psi_1 = 0.
\]

(29)

Eq. (29) expresses the abreastness constraint which is independent of the parametrization of \( k^\mu \) as could be easily checked by making the substitution \( q^\mu \mapsto q^\mu + \Lambda k^\mu \). It is easy to see that under the new parametrization, \( q.k = O (\alpha^2) \), and therefore could be ignored. Furthermore, like any other constrained system, we need to make sure that the eq. (28) is conserved under the propagation along \( k \). In other words, \( \nabla_k (q.k) = 0 \) does not lead to a secondary constraint. Up to the first order in \( \alpha \), we have

\[
\nabla_k (q^\mu k_\mu) = 0 \implies (\nabla q^\mu) k_\mu + q^\mu (\nabla_k k_\mu) = 0
\]

\[
O (\alpha^0) : \quad (\nabla q^l_\mu) l_\mu + q^\mu (\nabla_l l_\mu) = 0
\]

\[
O (\alpha^1) : \quad \left[ A^{(2)(a)} q^{(b)} + A^{(2)(b)} q^{(a)} \right] \gamma_{(a)(1)(b)} = 0,
\]

(30)

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which leads to the same constraint mentioned above. We emphasize that provided the abreastness constraint (29) is satisfied, the tangent vector $k^\mu$ can be described as

$$
\begin{align*}
  k_\mu(2) &= 1 - \frac{1}{m^2} A_{(1)(2)}, \\
  k_{(\alpha \neq 2)} &= 0, \\
  k^2 &= O(\alpha^2), \\
  \nabla_k k^\mu &= O(\alpha^2).
\end{align*}
$$

(31)

The abreast rays are special physical rays whose effective light cone condition, in a local inertial frame, is given by

$$
\left( \eta(a)(b) + \alpha^2 \sigma(a)(b) (R) \right) k^{(a)} k^{(b)} = 0.
$$

(32)

We recall from (3) that the general light cone modification induced by the effective action (4) is of order $\alpha$.

For abreast rays, the covariant deformation of $\xi$ implies

$$
\nabla_k \xi = - (\bar{\xi} \sigma + \xi \rho) + \frac{1}{m^2} A^{(4)(3)} (\bar{\xi} \sigma + \xi \rho) + \frac{1}{m^2} A^{(4)(4)} (\bar{\xi} \rho^* + \xi \sigma^*) .
$$

(33)

Due to their gauge-dependence and also for our purpose, propagation of the other components of the connecting vector $q$ are of less geometric and physical importance and we just state the results below for the sake of the completeness of the discussion,

$$
\nabla_k h = 0
$$

$$
\nabla_k g = \bar{\xi} (\beta + \alpha^*) + \xi (\alpha + \beta^*) + \frac{1}{m^2} \left[ A^{(1)(3)} (\bar{\xi} \rho + \xi \sigma^*) + c.c \right] .
$$

(34)

Now having the equation (33) we are ready to study the evolution of the cross sections of (physical) bundles of rays, formed by their intersection with the spatial plane spanned by $m^\mu$ and $\overline{m}^\mu$. To do so we employ the so called optical scalars associated with a bundle of rays, namely the expansion, the shear and the twist [10]. Since the transverse distances from a specified ray $\gamma$ are (frame)observer-independent, for abreast rays these scalars bear explicit physical interpretations. To simplify the case we consider the propagation along $k$ of the area of a small triangle formed by the points $a, \xi_1$ and $\xi_2$ (see figure 1), we have

$$
\nabla_k \delta A = \nabla_k \left[ \frac{i}{2} (\xi_1 \xi_2 - \xi_2 \xi_1) \right] = - \left[ \rho - \frac{1}{m^2} \left( A^{(4)(4)} \rho + A^{(3)(3)} \sigma \right) + c.c \right] \delta A
$$

(35)

The scalar quantity

$$
- \text{Re} \left[ \rho - \frac{1}{m^2} \left( A^{(3)(4)} \rho + A^{(3)(3)} \sigma \right) \right]
$$

(36)
FIG. 1: Three abreast rays of a congruence and the small triangle formed by two connecting vectors $\xi_1$ and $\xi_2$, connecting the ray $\gamma$ to the rays $\gamma_1$ and $\gamma_2$ in the $m\bar{m}$-plane.

is called the expansion parameter, $\theta_{\text{eff}}$, whose role as a measure of the pattern convergence (or divergence) is clear from (35). This equation shows that in an effective theory, unlike in the classical case, the local effect of $\sigma$ can also change the area. Another useful parameter, the "luminosity parameter" $L$, is defined along a bundle of rays as $L^2 \propto \delta A$ and is related to $\theta_{\text{eff}}$ as follows (using (35)),

$$\nabla_k L = \theta_{\text{eff}} L.$$  \hfill (37)

In other words, the expansion parameter is the logarithmic derivative of the luminosity parameter. An alternative formula for $\theta_{\text{eff}}$ is given by $\frac{1}{2} k^{\mu} \cdot \mu$.

Setting $\theta_{\text{eff}}$ and the coefficient of $\bar{\xi}$ equal to zero in (33), we obtain

$$\nabla_k \xi = -i \text{Im} \left[ \rho - \frac{1}{m^2} \left( A^{(3)(4)} \rho + A^{(3)(3)} \sigma \right) \right] \xi,$$  \hfill (38)
So it could also be claimed that Im $\left[\rho - \frac{1}{m^2} (A^{(3)(4)}\rho + A^{(3)(3)}\sigma)\right]$ measures the twist in the bundle’s cross section and could be called the effective twist, $\omega_{\text{eff}}$. The fact that this combination of scalars is a measure of twist is consistent with the requirement that it measures the failure of $k_\mu$ to be hypersurface orthogonal. In other words it should vanish for a congruence corresponding to the gradient of a scalar field i.e,

$$k_\mu = \nabla_\mu f \implies \omega_{\text{eff}} = 0$$

(39)

To show this, we note that the left hand side of (39) is equivalent to $\nabla [k_\mu] = 0$. At the leading order, $O (\alpha^0)$, this gives $\gamma_{(a)(1)(b)} = \gamma_{(b)(1)(a)}$, which in turn, using (10), leads to $\text{Im} \rho = 0$ for $a = 3$ and $b = 4$. The subleading term, $O (\alpha^1)$, is given by

$$[(b + 2c) R_{\lambda \nu} - (4c) C_{\nu \rho \lambda \sigma} a^\rho a^\sigma] e^{(a)\lambda} \gamma_{(a)(1)(b)} e_{\mu} = [(b + 2c) R_{\lambda \mu} - (4c) C_{\mu \rho \lambda \sigma} a^\rho a^\sigma] e^{(e)\lambda} \gamma_{(e)(1)(f)} e_{\nu}$$

(40)

Multiplying both sides of (40) by $e^{\mu(d)} e^{\nu(c)} e^{\rho(g)} e^{\sigma(h)}$, we end up with

$$A^{(a)(c)} \gamma_{(a)(1)(b)} \eta^{(b)(d)} = A^{(e)(d)} \gamma_{(e)(1)(f)} \eta^{(f)(c)}$$

(41)

Using the abreastness conditions (29), for $c = 4$ and $d = 3$, we obtain $\text{Im} \left[\rho - \frac{1}{m^2} (A^{(3)(4)}\rho + A^{(3)(3)}\sigma)\right] = 0$, proving our assertion that $\omega_{\text{eff}} = 0$ to the right order.

If $\theta_{\text{eff}} = \omega_{\text{eff}} = 0$, the remaining part of eq. (33), i.e

$$\sigma + \frac{1}{m^2} \left(A^{(4)(3)}\sigma + A^{(4)(4)}\rho^*\right),$$

(42)

can be interpreted as the effective shear, $\sigma_{\text{eff}}$, which is a measure of the distortion in the shape of the bundle’s cross section so that a circular cross section transforms into an elliptic one. Setting $\sigma_{\text{eff}} = se^{2i\theta}$, (33) reads $\nabla_k \xi = -se^{2i\theta}\vec{\xi}$. This shows that, $\nabla_k \xi$ is a real multiple of $\xi$ when $\text{arg} \xi = \theta, \theta + \pi$ or $\theta \pm \frac{1}{2}\pi$, where for $s > 0$, in the first two cases we get contraction towards the origin while in the other two cases it experiences dilation. Thus $s$ is a measure of the pattern’s shear while $\theta$ defines the angle that its minor axis makes with the $\xi$-plane.

The quantities $\omega_{\text{eff}}$ and $\sigma_{\text{eff}}$ can also be defined as

$$\omega_{\text{eff}}^2 = \frac{1}{2} k_{[i;j]} k^{i;j}$$

$$|\sigma_{\text{eff}}|^2 + \theta_{\text{eff}}^2 = \frac{1}{2} k_{(i;j)} k^{i;j}. $$

(43)
IV. PHYSICAL NULL CONGRUENCE AND SPACE TIME CURVATURE: EFFECTIVE RAYCHAUDHURI EQUATION

Since the evolution of the cross section of a bundle of rays is characterized by the quantities $\rho$ and $\sigma$ in the one hand and on the other hand the spacetime curvature affects the geometry of null congruences, we need to study the variation of optical scalars along the physical ray. This can be achieved by examining the second derivative of the connecting vector, $q^\mu$, propagating along the wave vector $k^\mu$ through the operation of $\nabla_k$ on eq. (1),

$$\nabla_k \nabla_k q^\mu = \nabla_k (\nabla_k k^\mu) = \nabla_k R_{\lambda\nu\sigma}^{\phantom{\lambda\nu\sigma}\mu} k^\lambda q^\nu k^\sigma = R_{\lambda\nu\sigma}^{\phantom{\lambda\nu\sigma}\mu} k^\lambda q^\nu k^\sigma + O(\alpha^2). \quad (44)$$

After multiplying by $e^{(c)}_\mu$ and taking the components corresponding to the parallely propagated tetrads we get

$$\nabla_k \nabla_k q^{(c)} = R_{(1)(3)(1)}^{(c)} \bar{\xi} + R_{(1)(4)(1)}^{(c)} \xi + O(\alpha^2). \quad (45)$$

Here, we have neglected the terms including the product of curvature components which would be suppressed by $O\left(\frac{R}{m^2}\right)$. From (33), we obtain the useful $(2 \times 2)$ matrix form as

$$\nabla_k Z = PZ, \quad (46)$$

where

$$Z = \begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix}, \quad P = \begin{pmatrix} \theta_{\text{eff}} - i\omega_{\text{eff}} & \sigma_{\text{eff}} \\ \sigma^*_{\text{eff}} & \theta^*_{\text{eff}} + i\omega^*_{\text{eff}} \end{pmatrix}. \quad (47)$$

For abreast rays (30), the equation (44) gives

$$\nabla_k \nabla_k Z = 0. \quad (48)$$

After differentiating (46) once more and using (48), we get

$$\nabla_k \nabla_k Z = (\nabla_k P) Z + P (\nabla_k Z) = (\nabla_k P + P^2) Z = 0. \quad (49)$$

Since this holds for arbitrary $\xi$, we must have

$$\nabla_k P = -P^2. \quad (50)$$

Written out in full, eq. (50) gives the directional propagation of $\rho$ and $\sigma$ along the physical ray as follows,
\[ \nabla_k \rho = \rho^2 + |\sigma|^2 + \frac{1}{m^2} \left[ A^{(3)(4)} (\rho^2 + |\sigma|^2) + A^{(3)(3)} \rho \sigma + A^{(4)(4)} \rho \sigma^* \right] \]
\[ \nabla_k \sigma = (\rho + \rho^*) \sigma + \frac{1}{m^2} \left[ A^{(3)(4)} (\rho + \rho^*) \sigma + A^{(3)(3)} \sigma^2 + A^{(4)(4)} |\rho|^2 \right] \]  
(51)

Compared with the results in the classical case, the above equations could be called effective Sachs equations. With these equations in hand, we can derive the equations governing the directional propagation of the effective optical scalars as
\[ \nabla_k \sigma_{\text{eff}} = -2 \theta_{\text{eff}} \sigma_{\text{eff}}, \]
(52)
\[ \nabla_k \omega_{\text{eff}} = -2 \theta_{\text{eff}} \omega_{\text{eff}}, \]
(53)
\[ \nabla_k \theta_{\text{eff}} = \omega_{\text{eff}}^2 - \theta_{\text{eff}}^2 - |\sigma_{\text{eff}}|^2. \]
(54)

These equations are the main results of the paper and their physical importance (consequences) is discussed below.

V. DISCUSSION

Here we employed the Newmann-Penrose tetrad formalism to derive the effective Raychaudhuri equation, i.e the Raychaudhuri equation modified by the vacuum polarization effects on the propagation of a bundle of rays. In the derivation of the classical version of the Raychaudhuri equation [11]-[12] the variations of the tangent vector \( k_{\mu\nu} \) is projected [19] on the plane spanned by \( m^\mu \) and \( \overline{m}^\mu \) and then its trace, antisymmetric and symmetric parts are designated as expansion, vorticity and shear respectively. Here in the modified version we chose, through the equation (7), an equivalent method in which the variations of the connecting vector \( q^\mu \) instead of the tangent vector have been considered and then defined the corresponding effective optical scalars. The invariants of the theory are easily seen in this way. For a luminosity parameter, \( L \), we see from (53) that \( \nabla_k (L^2 \omega_{\text{eff}}) = 0 \), in other words here \( q.k \) and \( L^2 \omega_{\text{eff}} \) are the constant geometric quantities along the congruence. For two neighbouring rays of \( \gamma \), whose connecting vectors \( q^\mu \) and \( \tilde{q}^\mu \) independently satisfy (11), a symplectic invariant is attributed as follows
\[ q^\mu \nabla_k \tilde{q}_\mu - \tilde{q}^\mu \nabla_k q_\mu. \]
(55)

These invariants reduce to the classical ones in the limit of zero perturbation and therefore there remains no anomaly in QGO. This is the case since we have applied the symmetries
present at $O(\alpha^0)$ as a set of constraints and exploited them to find the covariant quantum corrections. Since the effective action we started with has been derived in a gauge invariant manner [1], the results obtained are also gauge invariant.

The classical limit of the eq (54) can be compared with that corresponding to the standard (pure general relativistic) Raychaudhuri equation, namely

$$\nabla_k \theta = \omega^2 - \theta^2 - |\sigma|^2 - \Phi_{00}. \tag{56}$$

The last term, i.e. $\Phi_{00} = \frac{1}{2} R_{(1)(1)}$, is negative for all known forms of ponderable matter. Then for an initially contracting congruence, in the absence of rotation and shear ($\rho$ real and $\sigma = 0$), the eq. (56) results in the divergence of the expansion parameter ($\theta \to -\infty$). This is a necessary though not a sufficient condition in singularity theorems [13]. In our case a real $\rho$ and vanishing $\sigma$ is equivalent to $\omega_{\text{eff}} = \sigma_{\text{eff}} = 0$. Using (51), we see that during the propagation along the ray, they differ from zero only by $O(\alpha^2)$,

$$\nabla_k \theta = \nabla_k \theta_{\text{eff}} - \frac{1}{m^2} \nabla_l (A_{(3)(4)} \rho) \quad = \quad -\theta^2 + \frac{1}{m^2} A_{(3)(4)} \theta^2 + O(\alpha^2) \tag{57}$$

On the other hand, the last term in (56) does not appear in our calculations due to the abreastness conditions. However, even if $R_{(1)(1)} = 0$, the correction terms calculated in this paper will contribute and their sign must be taken into account. In QGO, the light cone condition as well as the velocity shift depend on $\Phi_{00}$ and $\Psi_0$ (or $A_{(1)(1)}$ in our notation), while in the propagation of the expansion parameter these are $\Phi_{11}$, $\Lambda$ and $\Psi_2$ (or $A_{(3)(4)}$ in our notation) that play role perturbatively. These may have consequences including existence of singularities in the effective geometry that are seen only by QGO photons. The existence of singularities seen by nonlinear photons has been studied in [14]. The generalization of Drummond-Hathrell results to massless neutrino propagation in a background spacetime [15] can be employed to find the corresponding effective Raychaudhuri equation. On the other hand, the effects originated from QED interactions discussed in this paper, in which local invariance and weak equivalence principle are preserved, are somewhat similar to the phenomenological theories incorporating Lorentz and CPT violating interactions. For example, in a theory describing the propagation of light discussed in [16], the Lorentz violating coupling constant $K_{\mu\nu\lambda\rho}$ plays the role of the Riemann tensor $R_{\mu\nu\lambda\rho}$ in eq. (11).

It is therefore easy to translate the phenomenological aspects of the effective Raychaudhuri
equation discussed here to this class of theories. We recall that the relative shift in velocity or optical scalars would be of $O \left( \frac{\alpha^2 \zeta^2}{L} \right)$ for a typical curvature scale $L$.

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[17] Tetrad indices are enclosed in parantheses to be distinguished from the tensor indices.
[18] A linearly polarized light may be considered as a superposition of a left and a right circularly polarised lights with a phase difference which represents the direction of linear polarization. As the light passes through the birefringent medium, the right and the left circularly polarised
lights propagate with different speeds. So one of the physical manifestation of the birefringence is the rotation of the polarisation plane of the linearly polarized light. However, expecting any SEP violations to be tiny, we are allowed to neglect the rotation and focus on astronomical sources with a well defined polarization.

[19] The projection operator is $P_{\mu\nu} = m_\nu \bar{m}_\mu + \bar{m}_\nu m_\mu = g_{\mu\nu} + l_\mu n_\nu + n_\mu l_\nu$. 