HYPERBOLIC GEOMETRY FOR NON-DIFFERENTIAL TOPOLOGISTS

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Abstract. A soft presentation of hyperbolic spaces, free of differential apparatus, is offered. Fifth Euclid’s postulate in such spaces is overthrown and, among other things, it is proved that spheres (equipped with great-circle distances) and hyperbolic and Euclidean spaces are the only locally compact geodesic (i.e., convex) metric spaces that are three-point homogeneous.

1. Introduction

Hyperbolic geometry of a plane is known as a historically first example of a non-Euclidean geometry; that is, geometry in which all Euclid’s postulates are satisfied, apart from the fifth, called parallel, which is false. Discovered in the first half of the 19th century, is one of the greatest mathematical achievements of those times. Although it merits special attention and mathematicians all over the world hear about it sooner or later, there are plenty of them whose knowledge on hyperbolic geometry is greedy and far from formal details. This sad truth concerns also topologists which do not specialise in differential geometry. One of reasons for this state is concerned with the extent of differential apparatus that one needs to learn in order to get to know and understand hyperbolic spaces. It was the main sake for us to propose and prepare an introduction to hyperbolic spaces (and geometry) that will be self-contained, free of differential language and accessible to ‘everyone.’

The other story about hyperbolic spaces concerns their free mobility (or, in other words, absolute metric homogeneity). A metric space is absolutely homogeneous if all its partial isometries (that is, isometries between its subspaces) extend to global (bijective) isometries. According to a deep result from the 50’s of the 20th century, known almost only by differential/Riemannian geometerst, hyperbolic spaces, beside Euclidean spaces and Euclidean spheres, are (in a very strong sense) the only connected locally compact metric spaces that have this property. One may even assume less about a connected locally compact metric space—that only partial isometries between 3-point and 2-point subspaces extend to global isometries. Then such a space is ‘equivalent’ to one of the aforementioned Riemannian manifolds and therefore is automatically absolutely homogeneous (the equivalence we speak here about does not imply that spaces are isometric, but is much stronger than a statement that they are homeomorphic; full details about classification of all such spaces up to isometry can be found in Section 5 below—see Theorem 5.1). So, high level of metric homogeneity makes hyperbolic spaces highly exceptional. This is another reason for putting special attention on them.

The paper is organised as follows. In Section 2 we introduce (one of possible models of) hyperbolic spaces and prove that their metrics satisfy the triangle inequality and are equivalent to Euclidean metrics. In the next, third, section we

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show that hyperbolic spaces are absolutely homogeneous and admit no dilations other than isometries (for dimension greater than 1). Fourth section is devoted to Euclid’s postulates, where it is proved that parallel postulate is false in hyperbolic spaces, whereas all other postulates are satisfied. To make the presentation self-contained, we also define there straight lines and their segments as well as angles, and show that one-dimensional hyperbolic space is isometric to the ordinary real line. The last, fifth, section is devoted to the classification (up to isometry) of all connected locally compact metric spaces that are 3-point homogeneous. All proofs are included, apart from the proof of a deep and difficult result due to Freudenthal [5] (see Theorem 5.3). This result is applied only once in this paper—to classify 3-point homogeneous metric spaces described above.

The reader interested in Riemannian geometry may consult, e.g., [7] or [4].

Notation and terminology. In this paper the term metric means a function (in two variables) that assigns to a pair of points of a fixed set their distance (so, metric does not mean Riemannian metric, a common and important notion in differential geometry).

For a pair of vectors \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) in \( \mathbb{R}^n \), we denote by

\[
\langle x, y \rangle = \sum_{k=1}^{n} x_k y_k
\]

their standard inner product (that is, \( \langle x, y \rangle = \sum_{k=1}^{n} x_k y_k \)), whereas

\[
\|x\| = \sqrt{\langle x, x \rangle}
\]

is the Euclidean norm of \( x \). The Euclidean metric (induced by \( \| \cdot \| \)) will be denoted as \( d_e \). To simplify further arguments and statements, we introduce the following notation:

\[
[x] = \sqrt{1 + \|x\|^2}.
\]

By an isometric map between metric spaces we mean any function that preserves the distances, that is,

\[
f : (X, d_X) \to (Y, d_Y)
\]

is isometric if

\[
d_Y(f(p), f(q)) = d_X(p, q)
\]

for any \( p, q \in X \). The term isometry is reserved for surjective isometric maps. Additionally, the above map \( f \) is a dilation if there is a constant \( c > 0 \) such that

\[
d_Y(f(p), f(q)) = cd_X(p, q)
\]

for any \( p, q \in X \) (we do not assume that dilations are surjective). We will denote by \( \text{Iso}(X, d_X) \) the full isometry group of \( (X, d_X) \); that is,

\[
\text{Iso}(X, d_X) = \{ u : (X, d_X) \to (X, d_X), \ u \text{ isometry} \}.
\]

A metric space \( (X, d_X) \) is said to be geodesic (or convex) if for any two distinct points \( a \) and \( b \) of \( X \) there exists a dilation \( \gamma : [0, 1] \to (X, d_X) \) such that \( \gamma(0) = a \) and \( \gamma(1) = b \) (we do not assume the uniqueness of such \( \gamma \)).

For the reader’s convenience, let us recall that the inverse hyperbolic cosine is defined as

\[
\cosh^{-1}(t) = \log(t + \sqrt{t^2 - 1})
\]

for \( t \geq 1 \) (in whole this paper ‘log’ stands for the natural logarithm).

2. Hyperbolic distance

Below we introduce one of many equivalent models of hyperbolic spaces—the one most convenient for us.

2.1. Definition. The \( n \)-dimensional (real) hyperbolic space is a metric space

\[
(H^n(\mathbb{R}), d_h)
\]

where \( H^n(\mathbb{R}) = \mathbb{R}^n \) and \( d_h \) is a metric (called hyperbolic) given by

\[
d_h(x, y) = \cosh^{-1}(\|[x][y] - \langle x, y \rangle\|) \quad (x, y \in \mathbb{R}^n)
\]

(see (1-1)).

The above formula has its origin in differential/Riemannian geometry (most often it is defined as the length of a geodesic arc—so, to get it one needs to find geodesics and compute integrals related to them), see, e.g., [5]. We underline here—at the
very beginning of our presentation—that establishing the triangle inequality for $d_h$ using elementary methods is undoubtedly the most difficult part in whole this approach.

2.2. Remark. As we will see in Corollary 1.2 the metric space $(H^1(\mathbb{R}), d_h)$ is isometric to $(\mathbb{R}, d_e)$. This contrasts with all other cases, as for $n > 1$ the metric space $(H^n(\mathbb{R}), d_h)$ admits no dilations other than isometries (see Theorem 3.5 below). In particular, for any integer $n > 1$ and positive real $r \neq 1$ the metric spaces $(H^n(\mathbb{R}), d_h)$ and $(H^n(\mathbb{R}), r d_h)$ are different (i.e., they are non-isometric). Each of the metrics $r d_h$ (with fixed $r > 0$) may serve as a ‘standard’ hyperbolic metric. Actually, everything that will be proved in this paper about the metric spaces $(H^n(\mathbb{R}), d_h)$ remains true when the metric is replaced by $r d_h$.

Note also that, similarly as practiced with Euclidean spaces, for $j < k$ the space $H^j(\mathbb{R})$ can naturally be considered as the subspace $\mathbb{R}^j \times \{0\}^{k-j}$ of $H^k(\mathbb{R})$. Under this identification, the hyperbolic metric of $H^j(\mathbb{R})$ coincides with the metric induced from the hyperbolic one of $H^k(\mathbb{R})$. This is the main reason why we ‘forget’ the dimension $n$ in the notations ‘$d_h$’ and ‘$d_e$’.

The aim of this section is to show that $d_h$ is a metric on $\mathbb{R}^n$ equivalent to $d_e$.

2.3. Lemma. For any $x, y \in H^n(\mathbb{R})$, $d_h(x, y)$ is well defined, non-negative and $d_h(x, y) = d_h(y, x)$. Moreover, $d_h(x, y) = 0$ iff $x = y$.

Proof. It follows from the Schwarz inequality that
\[
(2.2)\quad ||x||^2 ||y||^2 + ||x - y||^2 \geq \langle x, y \rangle^2 .
\]
Equivalently, $||x||^2 ||y||^2 + ||x||^2 - 2 \langle x, y \rangle + ||y||^2 \geq \langle x, y \rangle^2$, which gives $1 + ||x||^2 + ||y||^2 + ||x||^2 ||y||^2 \geq 1 + 2 \langle x, y \rangle + \langle x, y \rangle^2$ and hence $(1 + ||x||^2)(1 + ||y||^2) \geq (1 + \langle x, y \rangle)^2$.

Taking square roots from both sides shows that $|x||y| - \langle x, y \rangle \geq 1$ and thus $d_h(x, y)$ is well defined (and, of course, non-negative). Further, we see from (2.2) that $|x||y| - \langle x, y \rangle = 1$ iff $x = y$, which gives the last claim of the lemma. Symmetry is trivial.

To establish the triangle inequality, first we show its special case, which will be used later in the proof of a general case.

2.4. Lemma. For any $x, y, z \in H^n(\mathbb{R})$, $d_h(x, y) \leq d_h(x, z) + d_h(z, y)$, and equality appears iff either $x = ty$ or $y = tx$ for some $t \leq 0$.

Proof. Observe that
\[
d_h(x, 0) + d_h(0, y) = \cosh^{-1}(|x|) + \cosh^{-1}(|y|) \\
= \log(|x| + ||x||) + \log(|y| + ||y||) = \log(|x||y| + |x||y| + ||y|| + ||x|| ||y||).
\]
On the other hand,
\[
d_h(x, y) = \cosh^{-1}(|x||y| - \langle x, y \rangle) \leq \cosh^{-1}(|x||y| + ||x|| ||y||)
\]
and equality holds in the above iff $\langle x, y \rangle = -||x|| ||y||$, or, equivalently, if either $x = ty$ or $y = tx$ for some $t \leq 0$. So, to complete the whole proof, we only need to check that
\[
[x||y| + |x||y| + ||x|| ||y|| = \exp(\cosh^{-1}(|x||y| + ||x|| ||y||)),
\]
which is left to the reader as an elementary exercise.

To get the triangle inequality for general triples of elements of the hyperbolic space, we will use certain one-dimensional perturbations of the identity map, which turn out to be isometries (with respect to the hyperbolic distance). They are introduced in the following
2.5. Definition. For any \( y \in H^m(\mathbb{R}) \) let a map \( T_y : H^n(\mathbb{R}) \rightarrow H^n(\mathbb{R}) \) be defined by
\[
T_y(x) = x + \left( [x] + \frac{\langle x, y \rangle}{|y| + 1} \right) y \quad (x \in H^n(\mathbb{R})).
\]

2.6. Lemma. For any \( y \in H^n(\mathbb{R}) \) the map \( T_y \) is bijective and fulfills the equation:
\[
d_y(T_y(a), T_y(b)) = d_y(a, b) \quad (a, b \in H^n(\mathbb{R})).
\]
Moreover, \( T_{-y} \) is the inverse of \( T_y \).

**Proof.** We start from computing \( [T_y(u)] \) for \( u \in H^n(\mathbb{R}) \):
\[
[T_y(u)]^2 = 1 + \left\| u + \left( [u] + \frac{\langle u, y \rangle}{|y| + 1} \right) y \right\|^2
\]
\[
= 1 + \left\| [u] + 2 \left\| [u] + \frac{\langle u, y \rangle}{|y| + 1} \right\| \langle u, y \rangle + \left( [u] + \frac{\langle u, y \rangle}{|y| + 1} \right) \right\|^2 |y|^2
\]
\[
= [u]^2 + \left( 2[u] \langle u, y \rangle + \frac{2 \langle u, y \rangle^2}{|y| + 1} \right) + [u]^2 |y|^2 + \left( 2[u] \langle u, y \rangle + \frac{\langle u, y \rangle^2}{|y| + 1} \right) \frac{|y|^2 - 1}{|y| + 1}
\]
\[
= [u]^2 |y|^2 + 2[u] \langle u, y \rangle |y| + \langle u, y \rangle^2
\]
and thus
\[
[T_y(u)] = [u] |y| + \langle u, y \rangle.
\]

For simplicity, put \( \alpha(x) = [x] + \frac{\langle x, y \rangle}{|y| + 1} \) (then \( T_y(x) = x + \alpha(x)y \)). Observe that (2-3) is equivalent to
\[
[T_y(a)] [T_y(b)] - (T_y(a), T_y(b)) = [a] [b] - \langle a, b \rangle,
\]
which can easily be transformed to an equivalent form:
\[
[T_y(a)] [T_y(b)] = [a] [b] + \alpha(a) \langle a, y \rangle + \alpha(a) \langle b, y \rangle + \alpha(a) \alpha(b) |y|^2.
\]

The right-hand side expression of the above equation can be transformed as follows:
\[
[a] [b] + \alpha(a) \langle a, y \rangle + \alpha(a) \langle b, y \rangle + \alpha(a) \alpha(b) |y|^2
\]
\[
= [a][b] + [b] \langle a, y \rangle + \frac{\langle b, y \rangle}{|y| + 1} (a, y) + [a] (b, y) + \frac{\langle a, y \rangle}{|y| + 1} (b, y) + [a][b] (|y|^2 - 1)
\]
\[
+ \left( [b] \langle a, y \rangle + [a] (b, y) \right) \frac{|y|^2 - 1}{|y| + 1} + \frac{\langle a, y \rangle (b, y)}{|y| + 1} (|y|^2 - 1)
\]
\[
= [a][b] |y|^2 + [b] \langle a, y \rangle |y| + [a] (b, y) |y| + \langle a, y \rangle (b, y)
\]
\[
= ([a][b] + \alpha(a) \langle a, y \rangle)(b, y) + \langle b, y \rangle),
\]
which equals \( [T_y(a)] [T_y(b)] \), by (2-4). So, (2-3) is proved and, combined with Lemma 2.5, implies that \( T_y \) is one-to-one. To finish the whole proof, it suffices to check that \( T_{-y} \circ T_y \) coincides with the identity map on \( H^n(\mathbb{R}) \) (because then, by symmetry, also \( T_y \circ T_{-y} \) will coincide with the identity map). To this end, we fix \( x \in H^n(\mathbb{R}) \), and may and do assume that \( y \neq 0 \). Then there exist a unique vector \( z \) and a unique real number \( \beta \) such that \( \langle z, y \rangle = 0 \) and \( x = z + \beta y \). Note that \( T_y(x) = z + ([x] + \beta [y])y \) and, consequently,
\[
T_{-y}(T_y(x)) = z + ([x] + \beta [y])y - \left( [T_y(x)] - ([x] + \beta [y]) [y] - 1 \right) y
\]
\[
= z - [T_y(x)]y + ([x] + \beta [y]) [y] y.
\]

Finally, an application of (2-4) enables us continuing the above calculations to obtain:
\[
T_{-y}(T_y(x)) = z - ([x] [y] + \langle x, y \rangle) y + ([x] [y] + \beta [y]^2) y = z - \beta ([y]^2 - [y]^2) y = x.
\]
We are now able to prove the main result of this section.

2.7. Theorem. The function \( d_h \) is a metric equivalent to \( d_e \).

Proof. To show the triangle inequality, consider arbitrary three points \( x, y \) and \( z \) of \( H^n(\mathbb{R}) \). Since \( T_y(0) = y \), it follows from Lemmas 2.6 and 2.4 that
\[
d_h(x, z) = d_h(T_z(y), T_z(z)) \leq d_h(T_z(y), 0) + d_h(0, T_z(z)) = d_h(x, y) + d_h(y, z).
\]
Further, to establish the equivalence of the metrics, observe that
\[
d_h(x, 0) = \cosh^{-1}([x])
\]
and that both \( T_y \) and \( T_{-y} \) are continuous with respect to the Euclidean metric (and thus they are homeomorphisms in this metric). Thus, for an arbitrary sequence \( (x_n)_{n=1}^{\infty} \) of elements of \( \mathbb{R}^n \) and any \( x \in \mathbb{R}^n \) we have (note that \( T_{-x}(x) = 0 \)):
\[
x_n \xrightarrow{d_e} x \iff d_h(x_n, x) \to 0 \iff d_h(T_{-x}(x_n), 0) \to 0 \iff [T_{-x}(x_n)] \to 1
\]
\[
\iff T_{-x}(x_n) \xrightarrow{d_h} 0 \iff x_n \xrightarrow{d_h} T_x(0) = x.
\]
\[\square\]

The following is an immediate consequence of Theorem 2.7 (and the fact that the collections of all closed balls around 0 with respect to \( d_h \) and \( d_e \), respectively, coincide—only radii change when switching between \( d_h \) and \( d_e \)). We skip its simple proof.

2.8. Corollary. For each \( n \), the hyperbolic space \( H^n(\mathbb{R}) \) is locally compact and connected and the metric \( d_h \) is proper (that is, all closed balls are compact) and, in particular, complete.

3. Absolute (metric) homogeneity

3.1. Definition. A metric space \((X, d)\) is said to be absolutely (metrically) homogeneous if any isometric map \( f_0: (X_0, d) \to (X, d) \) defined on a non-empty subset \( X_0 \) of \( X \) extends to an isometry \( f: (X, d) \to (X, d) \).

In this section we will show that hyperbolic spaces are absolutely homogeneous. According to Theorem 5.3 (see Section 5 below), this property makes them highly exceptional among all connected locally compact metric spaces.

The following is a reformulation of Lemma 2.0

3.2. Corollary. For any \( y \in H^n(\mathbb{R}) \), the map \( T_y: (H^n(\mathbb{R}), d_h) \to (H^n(\mathbb{R}), d_h) \) is an isometry.

3.3. Lemma. For a map \( u: A \to H^n(\mathbb{R}) \) where \( A \) is a subset of \( H^n(\mathbb{R}) \) containing the zero vector the following conditions are equivalent:

(i) \( u \) is isometric (with respect to \( d_h \)) and \( u(0) = 0 \);
(ii) \( \langle u(x), u(y) \rangle = \langle x, y \rangle \) for all \( x, y \in A \).

Proof. Assume (i) holds and fix \( x, y \in A \). Then \( [u(x)] = \cosh(d_h(u(x), u(0))) = \cosh(d_h(x, 0)) = [x] \). Similarly, \( [u(y)] = [y] \) and thus \( (u(x), u(y)) = [u(x)] [u(y)] = \cosh(d_h(x, y)) = \langle x, y \rangle \). Conversely, if (ii) holds and \( x, y \in A \), then \( (u(x), u(x)) = (x, x) \) and hence \( [u(x)] = [x] \) (and, similarly, \([u(y)] = [y]\)) and \( u(0) = 0 \). But then easily \( d_h(u(x), u(y)) = d_h(x, y) \) and we are done.
\[\square\]

3.4. Theorem. For any \( n \) the hyperbolic space \((H^n(\mathbb{R}), d_h)\) is absolutely homogeneous.
Proof. Fix an isometric map \( v: A \to H^n(\mathbb{R}) \) where \( A \) is a non-empty subset of \( H^n(\mathbb{R}) \). Take any \( a \in A \), put \( B = T_{-a}(A) \) and \( u = T_{-v(a)} \circ v \circ T_a: B \to H^n(\mathbb{R}) \), and observe that \( 0 \in B \), \( u(0) = 0 \) and \( u \) is isometric (with respect to \( d_h \)). So, the map \( u \) satisfies condition (ii) from Lemma 3.3. It is well known (and easy to show) that each such a map extends to a linear map \( U: \mathbb{R}^n \to \mathbb{R}^n \) that corresponds (in the canonical basis of \( \mathbb{R}^n \)) to an orthogonal matrix. The last property means precisely that also \( U \) fulfills the equation from condition (ii) of Lemma 3.3. Hence, \( U \) is isometric with respect to \( d_h \). Then \( T_{v(a)} \circ U \circ T_{-a} \) is an isometry that extends \( v \).

The above proof enables to describe all isometries of the hyperbolic space \( H^n(\mathbb{R}) \): all of them are of the form \( u = T_a \circ v \) where \( a \) is a vector and \( U: \mathbb{R}^n \to \mathbb{R}^n \) is an orthogonal linear map (both \( a \) and \( U \) are uniquely determined by \( u \)). In particular, the isotropy groups \( \text{stab}(x) = \{ u \in \text{Iso}(H^n(\mathbb{R}), d_h): u(x) = x \} \) of elements \( x \in H^n(\mathbb{R}) \) are pairwise isomorphic and \( \text{stab}(0) \) is precisely the group \( O_n \) of all orthogonal linear transformations of \( \mathbb{R}^n \). The group \( O_n \) also coincides with the isotropy group of the zero vector with respect to the isometry group of the \( n \)-dimensional Euclidean space. So, hyperbolic spaces are quite similar to Euclidean. However, the last spaces have many (bijective) dilations, which contrasts with hyperbolic geometry, as shown by

3.5. Theorem. For \( n > 1 \), every dilation on \( (H^n(\mathbb{R}), d_h) \) is an isometry.

Proof. Assume \( u: H^n(\mathbb{R}) \to H^n(\mathbb{R}) \) satisfies \( d_h(u(x), u(y)) = d_h(x, y) \) for all \( x, y \in H^n(\mathbb{R}) \) and some constant \( c > 0 \). Our aim is to show that then \( c = 1 \). (That is all we need to prove, since any global isometric map on \( H^n(\mathbb{R}) \) is onto—its inverse map is extendable to an isometry which means that actually it is an isometry.)

Replacing \( u \) by \( T_{-u(0)} \circ u \), we may and do assume that \( u(0) = 0 \). Then it follows from (3-5) that for any \( x, y \in H^n(\mathbb{R}) \):

\[
\|f(x)\| = \|f(y)\| \iff \|x\| = \|y\|.
\]

Further, \( d_h(x, 0) = d_h(-x, 0) \) and \( d_h(x, -x) = d_h(x, 0) + d_h(0, -x) \). Consequently, \( d_h(u(x), 0) = d_h(u(-x), 0) \) and \( d_h(u(x), u(-x)) = d_h(u(x), 0) + d_h(0, u(-x)) \). So, we infer from Lemma 2.4 (and from (2.1)) that \( u(x) = -u(x) \) for all \( x \in H^n(\mathbb{R}) \).

Fix arbitrary two vectors \( x, y \in H^n(\mathbb{R}) \) such that \((x, y) = 0\). Then

\[
\cosh(d_h(x, y)) = \|x\|\|y\| = \|x\|\|-y\| = \cosh(d_h(x, -y)),
\]

thus \( d_h(u(x), u(y)) = d_h(u(x), u(-y)) = d_h(u(x), -u(y)) \). This implies that \( (u(x), u(y)) = 0 \).

Now for arbitrary \( t \geq 1 \) choose two vectors \( x, y \in H^n(\mathbb{R}) \) such that \( [x] = [y] = t \) and \((x, y) = 0\). Then also \( (u(x), u(y)) = 0 \) and (by (2.1)) \( [u(y)] = [u(x)] = \cosh(d_h(u(x), 0)) = \cosh(c \cosh^{-1}(t)) \). It follows from the former property that \( c \cosh^{-1}([x] [y]) = c \cosh(x, y) = d_h(u(x), u(y)) = \cosh^{-1}([u(x)] [u(y)]) \), which combined with the latter yields

\[
\cosh(c \cosh^{-1}(t^2)) = (\cosh(c \cosh^{-1}(t)))^2.
\]

The above equality is valid for every \( t \geq 1 \) only if \( c = 1 \). Although this is a well-known fact, for the reader’s convenience we give its brief proof. Observe that

\[
\cosh(c \cosh^{-1}(t^2)) = \frac{1}{2} \left( (t^2 + \sqrt{t^4 - 1})^c + (t^2 + \sqrt{t^4 - 1})^{-c} \right)
\]

and

\[
(\cosh(c \cosh^{-1}(t)))^2 = \frac{1}{4} \left( (t + \sqrt{t^2 - 1})^c + (t + \sqrt{t^2 - 1})^{-c} \right)^2.
\]
As a consequence, \( \lim_{t \to \infty} \frac{\cosh(\cosh^{-1}(t^2))}{t^2} = 2e \), whereas
\[
\lim_{t \to \infty} \frac{(\cosh(\cosh^{-1}(t)))^2}{t^2} = 2e^{-1}.
\]
So, if (3.2) holds for all \( t \geq 1 \), then \( 2e = 2e^{-1} \) and \( e = 1 \). \( \square \)

We underline that the claim of Theorem 3.5 is false for \( n = 1 \) (see Corollary 4.2 below).

As an immediate consequence of the above result we obtain

3.6. **Corollary.** For any \( n > 1 \), the metric spaces \((H^n(\mathbb{R}), d_h)\) and \((\mathbb{R}^n, d_e)\) are non-isometric.

4. **Hyperbolic geometry**

In this section we show that in the hyperbolic space \( H^2(\mathbb{R}) \) all the Euclid’s postulates are fulfilled, apart from the fifth which is false. These properties made hyperbolic geometry iconic.

Although the fifth postulate of Euclidean geometry matters only in the 2nd dimension, our considerations will take place in all hyperbolic spaces.

The first two Euclid’s postulates deal with straight lines and their segments. Recall that a **straight line** in a metric space is an isometric image of the real line, and a **straight line segment** is an isometric image of the compact interval in \( \mathbb{R} \). Additionally, for simplicity, we call three points \( a, b, c \) in a metric space \((X, d)\) **metrically collinear** if there are \( x, y, z \) such that \( d(x, z) = d(x, y) + d(y, z) \) and the sets \( \{x, y, z\} \) and \( \{a, b, c\} \) coincide.

To avoid confusions, straight lines in \((\mathbb{R}^n, d_e)\) (that is, one-dimensional affine subspaces) will be called **Euclidean** lines, whereas straight lines in \((H^n(\mathbb{R}), d_h)\) will be called **hyperbolic** lines. We will use analogous naming for other geometric notions—e.g., we will speak about Euclidean and hyperbolic spheres (as sets of all points equidistant from a given point in Euclidean, resp. hyperbolic, metric).

The first Euclid’s postulate says that any two points of the space can be joint by a straight line segment. In the modern terminology, it is equivalent for the metric to be geodesic (that is, convex). The second postulate is about extending straight line segments to line segments. Both these axioms in the hyperbolic spaces are fulfilled in a very strict form, as shown by

4.1. **Theorem.** Let \( a \) and \( b \) be two distinct points of \( H^n(\mathbb{R}) \).

(A) The metric segment \( I(a, b) = \{x \in H^n(\mathbb{R}) : d_h(a, x) + d_h(x, b)\} \) is a unique straight line segment in \( H^n(\mathbb{R}) \) that joins \( a \) and \( b \).

(B) The set \( L(a, b) \) of all \( x \in H^n(\mathbb{R}) \) such that \( x, a, b \) are metrically collinear is a unique hyperbolic line passing through \( a \) and \( b \).

(C) Every isometric map \( \gamma : \mathbb{R} \to H^n(\mathbb{R}) \) that sends 0 to \( a \) is of the form \( \gamma(t) = T_a(\sinh(t)z) \) where \( z \in H^n(\mathbb{R}) \) is such that \( ||z|| = 1 \).

**Proof.** First assume \( a = 0 \). All the assertions of the theorem in this case will be shown in a few steps.

For any \( z \in H^n(\mathbb{R}) \) with \( ||z|| = 1 \) denote by \( \gamma_z : \mathbb{R} \to H^n(\mathbb{R}) \) a map given by \( \gamma_z(t) = \sinh(t)z \). This map is isometric, which can be shown by a straightforward calculation:

\[
d_h(\gamma(s), \gamma(t)) = \cosh^{-1}(|\sinh(s)z| \sinh(t)z) - \sinh(s) \sinh(t))
= \cosh^{-1}(\sqrt{(\sinh(s)^2 + 1)(\sinh(t)^2 + 1)} - \sinh(s) \sinh(t))
= \cosh^{-1}(\cosh(s) \cosh(t) - \sinh(s) \sinh(t)) = \cosh^{-1}(\cosh(s - t)) = |s - t|.
\]
Observe that the image of $\gamma_z$ coincides with the linear span of the vector $z$. Thus, every Euclidean line passing through the zero vector is also a hyperbolic line.

Now let $x, y, z \in H^n(\mathbb{R})$ satisfy $d_h(x, z) = d_h(x, y) + d_h(y, z)$ and let $0 \in \{x, y, z\}$. We claim that $x, y, z$ lie on a Euclidean line. Indeed, if $y = 0$, it suffices to apply Lemma 2.4 and otherwise we may assume, without loss of generality, that $x = 0$.

In that case we proceed as follows. The linear span $L$ of $y$ is a hyperbolic line (by the previous paragraph). So, there exists an isometry $q: L \to L$ that sends $y$ to 0. By absolute homogeneity established in Theorem 3.4 there exists an isometry $Q: H^n(\mathbb{R}) \to H^n(\mathbb{R})$ that extends $q$. So, $Q(L) = L$, $Q(y) = 0$ and $d_h(Q(x), Q(z)) = d_h(Q(x), Q(y)) + d_h(Q(y), Q(z))$. Again, Lemma 2.4 implies that $Q(z) = tQ(x)$ for some $t \leq 0$ (as $Q(x) \neq 0 = Q(y)$). But $Q(x) = Q(0) \in Q(L) = L$ and therefore also $Q(z) \in L$. So, $x, y, z \in L$ and we are done.

Now we prove items (A)–(C) in the case $a = 0$. Let $z \in H^n(\mathbb{R})$ be a vector such that $\|z\| = 1$ and $b \in \gamma_z(\mathbb{R})$. It follows from the last paragraph that each element $x \in H^n(\mathbb{R})$ such that $x, 0, b$ are metrically collinear belongs to $\gamma_z(\mathbb{R})$. We also know that $\gamma_z(\mathbb{R})$ is a hyperbolic line. This shows that $L(0, b) = \gamma_z(\mathbb{R})$ and proves (B), from which (A) easily follows. Finally, if $\gamma: \mathbb{R} \to H^n(\mathbb{R})$ is an isometric map such that $\gamma(0) = 0$, then for any $t \in \mathbb{R}$, the points $0, \gamma(1)$ and $\gamma(t)$ are metrically collinear, so $\gamma(t) \in L(0, \gamma(1))$. It follows from the above argument that $L(0, \gamma(1)) = \gamma_z(\mathbb{R})$ for some unit vector $z \in H^n(\mathbb{R})$. Then $v = \gamma^{-1} \circ \gamma: \mathbb{R} \to \mathbb{R}$ is an isometric map sending 0 to 0. Thus $v(t) = t$ or $v(t) = -t$. In the former case we get $\gamma = \gamma_z$, whereas in the latter we have $\gamma = \gamma_{-z}$, which finishes the proof of (C).

Now we consider a general case. When $a$ is arbitrary, $b \neq a$ and $\gamma: \mathbb{R} \to H^n(\mathbb{R})$ is isometric and sends 0 to $a$, it is easy to verify that $T_{-a}(I(a, b)) = I(0, T_{-a}(b))$, $T_{-a}(L(a, b)) = L(0, T_{-a}(b))$ and $T_{-a} \circ \gamma$ is an isometric map from $\mathbb{R}$ into $H^n(\mathbb{R})$ that sends 0 to 0. So, the first part of the proof implies that $I(a, b)$ and $L(a, b)$ are a unique straight line segment joining $a$ and $b$, and—respectively—a unique hyperbolic line passing through $a$ and $b$. Similarly, $T_{-a} \circ \gamma = \gamma_z$ for some unit vector $z$. Then $\gamma = T_a \circ \gamma_z$ and we are done. \qed

It follows from the above result that two hyperbolic lines either are disjoint or have a single common point, or coincide.

As a consequence of Theorem 4.1 we obtain the following result, which is at least surprising when one compares the formulas for $d_h$ and $d_e$.

4.2. **Corollary.** The map $(\mathbb{R}, d_e) \ni t \mapsto \sinh(t) \in (H^1(\mathbb{R}), d_h)$ is an isometry.

The above result implies, in particular, that $H^1(\mathbb{R})$ admits many non-isometric dilations. Its assertion is the reason for considering (in almost whole existing literature) only hyperbolic spaces of dimension greater than one.

Another immediate consequence of Theorem 4.1 reads as follows.

4.3. **Corollary.** Hyperbolic spaces are geodesic.

We return to Euclid’s postulates. The third of them is about the existence of circles, which in the metric approach reduces to the statement that for any center $a$ and a radius $r > 0$ the set, called a sphere, of points whose distance from $a$ equals $r$ is non-empty (one can require more—e.g., that each sphere disconnects the space). Hyperbolic spaces satisfy the third Euclid’s postulate in a very strict way: any hyperbolic sphere around 0 of radius $r > 0$ coincides with the Euclidean sphere around 0 of radius $\sinh(r)$ (which easily follows from the formulas for $d_h$ and $d_e$); and—by the absolute homogeneity of $H^n(\mathbb{R})$—any other sphere is the image of a sphere around 0 by a global (hyperbolic) isometry. So, all of them disconnect $H^n(\mathbb{R})$, are pairwise homeomorphic, have homeomorphic complements etc.

The fourth axiom of Euclidean geometry says that all right angles are congruent (i.e., any of them is the image of any other by a global isometry). It deals with
angles and as such is most difficult (among all Euclid’s postulates) to be adapted in the realm of metric spaces. Knowing that hyperbolic spaces satisfy first two Euclid’s postulates, the following seems to be most intuitive definition of angles and related notions (cf. [2]).

4.4. Definition. A (hyperbolic) angle is an ordered triple \((R_1, a, R_2)\) where \(R_1\) and \(R_2\) are closed hyperbolic half-lines in \(H^n(\mathbb{R})\) issuing from \(a\). The point \(a\) is called the vertex of the angle \((R_1, a, R_2)\).

Two angles \((R_1, a, R_2)\) and \((M_1, b, M_2)\) are congruent if there exists an isometry \(u: R_1 \cup R_2 \to M_1 \cup M_2\) such that \(u(R_1) = M_1\), \(u(R_2) = M_2\) and \(u(a) = b\). Whenever \((R_1, a, R_2)\) is a hyperbolic angle, the half-lines \(R_1\) and \(R_2\) can uniquely be enlarged to hyperbolic lines \(L_1\) and \(L_2\), respectively. Denote by \(R'_1\) and \(R'_2\) the half-lines \((L_1 \setminus R_1) \cup \{a\}\) and \((L_2 \setminus R_2) \cup \{a\}\), respectively. We call the angle \((R_1, a, R_2)\) right if the angles \((R_1, a, R'_2)\), \((R'_2, a, R_1)\), \((R_2, a, R'_1)\) and \((R'_1, a, R'_2)\) are pairwise congruent.

Fourth Euclid’s postulate in hyperbolic spaces reads as follows.

4.5. Proposition. Let \(n > 1\).

(a) Among angles with vertex at \(0\), hyperbolic right angles coincide with Euclidean right angles.

(b) For any two hyperbolic right angles \((R_1, a, R_2)\) and \((M_1, b, M_2)\) in \(H^n(\mathbb{R})\) there exists a global isometry \(u: H^n(\mathbb{R}) \to H^n(\mathbb{R})\) such that \(u(R_1) = M_1\), \(u(R_2) = M_2\) and \(u(a) = b\).

Proof. First of all, note that in both hyperbolic and Euclidean spaces, congruency of two angles can be witnessed by a global isometry (since all these spaces are absolutely homogeneous). Having this in mind, both the claims follow from the following three facts (first two of which have already been established, whereas the last is classical):

- among straight lines passing through \(0\), hyperbolic lines coincide with Euclidean lines;
- among maps that leave \(0\) fixed, hyperbolic isometries coincide with Euclidean isometries;
- Euclidean right angles are congruent (with respect to \(d_e\)).

Finally, we arrived at the fifth Euclid’s postulate. Its original statement is about two lines that intersect a third in a way that the sum of the inner angles on one side is less than two right angles. As it is about measuring angles, more convenient for us will be one of its equivalent statements, known as Playfair’s axiom, which says that through any point not lying on a given straight line there passes a unique straight line disjoint from this given one. As we announced, this axioms is false in hyperbolic geometry. Below we present a detailed proof of this fact in its full generality. However, only the case \(n = 2\) is interesting in this matter.

4.6. Theorem. Let \(n > 1\), \(L\) be a hyperbolic line in \(H^n(\mathbb{R})\) and \(a \notin L\). There are infinitely many hyperbolic lines that pass through \(a\) and are disjoint from \(L\).

Proof. Thanks to the metric homogeneity of \(H^n(\mathbb{R})\), we may and do assume that \(a = 0\). So, \(L\) is a hyperbolic line that does not pass through \(0\). We claim that then there exist two linearly independent vectors \(a\) and \(b\) such that

\[
(L = \{\sinh(t)a + \cosh(t)b: t \in \mathbb{R}\}).
\]

Indeed, we know that \(L = \gamma(\mathbb{R})\) where \(\gamma = T_y \circ \gamma_z\), \(y \neq 0\), \(\|z\| = 1\) and \(\gamma_z\) is given by \(\gamma_z(t) = \sinh(t)z\). Then \(\gamma(t) = \sinh(t)z + ([\sinh(t)z] + \frac{\sinh(t)\|z\|}{|y|+1})y =\)
We equip necessary notions. Connected locally compact 3-point homogeneous metric spaces, let us introduce necessary notions. (cf. (3-2)). If \( a \) and \( b \) were not such, then \( L \) would be a subset of the linear span \( Y \) of \( y \). But \( Y \) is a hyperbolic line and therefore we would obtain that \( L = Y \) and hence \( 0 \in L \). This proves (4-1).

Now let \( \mu \in \mathbb{R} \) be such that \( |\mu| > 1 \). Put \( c = \mu a + b \) and let \( L_\mu \) be the linear span of \( c \). We claim that \( L_\mu \) is disjoint from \( L \). Indeed, let \( s \) and \( t \) be real and assume, on the contrary, that \( sc = \sinh(t)a + \cosh(t)b \) (cf. (4-1)). It follows from the linear independence of \( a \) and \( b \) that \( \sinh(t) = s\mu \) and \( \cosh(t) = s \). Consequently, \( |\sinh(t)| = |\cosh(t)|s > |\cosh(t)| \), which is impossible.

So, for any real \( \mu \) with \( |\mu| > 1 \) the set \( L_\mu \) is a hyperbolic line disjoint from \( L \) and passing through 0. Since \( a \) and \( b \) are linearly independent, these sets \( L_\mu \) are all different.

The reader interested in establishing further geometric properties by means of the metric is referred to [2].

5. Absolute vs. 3-point homogeneity

Looking at a complicated formula for the hyperbolic metric it is a natural temptation to search for a 'simpler' realisation of non-Euclidean (hyperbolic) geometry.

Even if it is possible, a new model will not be as 'perfect' as the hyperbolic space described in this paper: its metric will not be geodesic or else this space will not be 3-point homogeneous. (Recall that, for a positive integer \( n \), a metric space \((X, d)\) is metrically \( n \)-point homogeneous if any isometric map \( u: (A, d) \to (X, d) \) defined on a subset \( A \) of \( X \) that has at most \( n \) elements is extendable to an isometry \( v: (X, d) \to (X, d) \).) The above statement is a consequence of deep achievements (which we neither discuss here in full details nor give their proofs) of the 50’s of the 20th century due to Wang [10], Tits [9] and Freudenthal [5]. They classified all connected locally compact metric spaces that are 2-point homogeneous. All that follows is based on Freudenthal’s work [5]; we also strongly recommend his paper [6] where main results of the former article are well discussed (see, e.g., subsection 2.21 therein).

To formulate the main result on the classification, up to isometry, of all connected locally compact 3-point homogeneous metric spaces, let us introduce necessary notions.

For any positive integer \( n \) let \( S^n \) stand for the Euclidean unit sphere in \( \mathbb{R}^{n+1} \):

\[
S^n = \{ x \in \mathbb{R}^{n+1}; \|x\| = 1 \}.
\]

We equip \( S^n \) with metric \( d_s \) of the great-circle distance, given by

\[
d_s(x, y) = \frac{1}{\pi} \arccos((x, y)) = \frac{1}{\pi} \arccos \left( \frac{2 - d_e(x, y)^2}{2} \right).
\]

It is well-known (and easy to prove) that \( d_s \) is a metric equivalent to \( d_e \) (restricted to \( S^n \)) such that the metric space \((S^n, d_s)\) has the following properties:

- it is geodesic and absolutely homogeneous;
- it has diameter 1;
- any two points of \( S^n \) whose \( d_s \)-distance is smaller than 1 (that is, which are not antipodal) can be joint by a unique straight line segment.

Further, for any subinterval \( I \) of \([0, \infty)\) containing 0 let \( \Omega(I) \) be the set of all continuous functions \( \omega: I \to [0, \infty) \) that vanish at 0 and satisfy the following two conditions for all \( x, y \in I \):

1. \( x < y \implies \omega(x) < \omega(y) \);
2. \( \omega(x + y) \leq \omega(x) + \omega(y) \) provided \( x + y \in I \).
Finally, let $\Omega = \Omega_\infty = \Omega([0, \infty))$ and $\Omega_l = \Omega([0, 1])$. For any $\omega \in \Omega$ there exists $\lim_{t \to \infty} \omega(t) \in [0, \infty]$, which will be denoted by $\omega(\infty)$.

After all these preparations, we are ready to formulate the main result of this section.

5.1. **Theorem.** Each connected locally compact 3-point homogeneous metric space having more than one point is isometric to exactly one metric space having the following condition:

- $X = \mathbb{R}^n, \ d = \omega \circ d_e$ where $\omega \in \Omega$ and $\omega(1) = \min(1, \frac{1}{2} \omega(\infty))$.
- $X = \mathbb{S}^n, \ d = \omega \circ d_s$ where $\omega \in \Omega_l$.
- $X = H^n(\mathbb{R}), \ d = \omega \circ d_h$ where $n > 1$ and $\omega \in \Omega$.

In particular:

- all connected locally compact 3-point homogeneous metric spaces are absolutely homogeneous, and each of them is homeomorphic either to a Euclidean space or to a Euclidean sphere (unless it has at most one point);
- each locally compact geodesic 3-point homogeneous metric space having more than one point is isometric to exactly one of the spaces: $(\mathbb{R}^n, d_e), (\mathbb{S}^n, \omega d_s)$ (where $r > 0$), $(H^n(\mathbb{R}), \omega d_h)$ (where $n > 1$ and $r > 0$).

5.2. **Remark.** Busemann’s [3] and Birkhoff’s [11, 2] results have a similar spirit. However, both of them assume, beside metric convexity, also a sort of uniqueness of straight line segments joining sufficiently close two points, and their statements are less general.

It seems that the assertion of Theorem 5.1 (in this form) has never appeared in the literature. Nevertheless, we consider this result as Freudenthal’s theorem—not ours. Such a thinking is justified by the fact that Theorem 5.1 easily follows from Freudenthal’s theorem, stated below, proved in [5].

Denoting by $P^n(\mathbb{R})$ the $n$-dimensional real projective space (realised as the quotient space of $\mathbb{S}^n$ obtained by gluing antipodal points) equipped with a geodesic metric $d_p(u, v) = \frac{1}{2} \arccos(\langle u, v \rangle)$ (where $u, v \in \mathbb{S}^n$ and $\bar{u} = \{u, -u\}$ and $\bar{v} = \{v, -v\}$ denote their equivalence classes belonging to $P^n(\mathbb{R})$), we can formulate Freudenthal’s Hauptsatz IV from [5] as follows.

5.3. **Theorem.** Let $(Z, \lambda)$ be a connected locally compact metric space that satisfies the following condition:

| There are two positive reals $\gamma$ and $\gamma'$ such that: |
|---|
| (a) $\gamma < \lambda(v, w)$ for some $v, w \in Z$; |
| (b) $\gamma' < \lambda(a, b)$ for some $a, b \in Z$ for which there is $c \in Z$ with $\lambda(a, c) = \lambda(b, c) = \gamma$; |
| (c) any isometric map $u: (A, \lambda) \to (Z, \lambda)$ defined on an arbitrary subset $A$ of $Z$ of the form: |
| - $A = \{x, y\}$ where $\lambda(x, y) = \gamma$, or |
| - $A = \{x, y, z\}$ where $\lambda(x, y) = \lambda(y, z) = \gamma$ and $\lambda(x, z) = \gamma'$ is extendable to an isometry $v: (Z, \lambda) \to (Z, \lambda)$. |

Then there is a unique space $(M, \varrho)$ among $(\mathbb{R}^n, d_e)$ $(n > 0)$, $(\mathbb{S}^n, d_s)$ $(n > 0)$, $(H^n(\mathbb{R}), d_h)$ $(n > 1)$, $(P^n(\mathbb{R}), d_p)$ $(n > 1)$ and a homeomorphism $h: Z \to M$ such that:

$$h \circ u \circ h^{-1}: u \in \text{Iso}(Z, \lambda) = \text{Iso}(M, \varrho).$$

We prove Theorem 5.1 in few steps, each formulated as a separate lemma. All of them are already known but—for the reader’s convenience—we give their short proofs.
5.4. Lemma. For $n > 1$ the space $(P^n(\mathbb{R}), d_p)$ is 2-point, but not 3-point, homogeneous.

Proof. We use here the notion introduced in the paragraph preceding the formulation of Theorem 5.3.

Let $(u, v)$ and $(x, y)$ be two pairs of points of $S^n$ such that $d_p(u, v) = d_p(x, y)$. This means that there is $\varepsilon \in \{1, -1\}$ such that $d_4(u, v) = d_4(x, y)$. Consequently, there exists $A \in O_{n+1}$ (which is automatically an isometry with respect to $d_4$) such that $A(u) = x$ and $A(v) = \varepsilon y$. Every member $B$ of $O_{n+1}$ naturally induces an isometry $\tilde{B}: P^n(\mathbb{R}) \to P^n(\mathbb{R})$ that satisfies $\tilde{B}(\varepsilon) = B(z)$ for any $z \in S^n$. Moreover, the assignment

$$O_{n+1} \ni B \mapsto \tilde{B} \in \text{Iso}(P^n(\mathbb{R}), d_p)$$

is surjective (this is classical, but non-trivial). So, $\tilde{A} \in \text{Iso}(P^n(\mathbb{R}), d_p)$ satisfies $\tilde{A}(u) = \tilde{x}$ and $\tilde{A}(v) = \tilde{y}$, which shows that $P^n(\mathbb{R})$ is 2-point homogeneous.

To convince oneself that $P^n(\mathbb{R})$ is not 3-point homogeneous for $n > 1$, it is enough to consider two triples $(x, y, z_1)$ and $(x, y, z_2)$ of points of $S^n$ with $x = (1, 0, 0, 0)$, $y = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0, 0)$, $z_1 = (\frac{1}{4}, \frac{1}{2}, \frac{\sqrt{2}}{2}, 0)$ and $z_2 = (\frac{1}{4}, -\frac{3}{4}, \frac{\sqrt{2}}{2}, 0)$ where $\vec{0}$ denotes the zero vector in $\mathbb{R}^{n-2}$ (which has to be omitted when $n = 2$). Observe that $(x, z_1) = (x, z_2)$ and $(y, z_1) = - (y, z_2)$. Consequently, $d_p(\tilde{w}, \tilde{z}_1) = d_p(\tilde{w}, \tilde{z}_2)$ for any $w \in \{x, y\}$. If $P^n(\mathbb{R})$ was 3-point homogeneous, there would exist an isometry of $P^n(\mathbb{R})$ that fixes $x$ and $y$ and sends $z_1$ onto $z_2$, which is impossible. To see this, assume—on the contrary—there exists such an isometry. It then follows from the surjectivity of (5.2) that there exists $B \in O_{n+1}$ such that $B(x) = \pm x$, $B(y) = \pm y$ and $B(z_1) = \pm z_2$. Then, replacing if needed $B$ by $-B$, we may and do assume $B(x) = x$ and thus, since $(B(x), B(u)) = (x, u)$ for any $u \in \mathbb{R}^{n+1}$, also $B(y) = y$ and $B(z_1) = z_2$. But then $(y, z_2) = (B(y), B(z_1)) = (y, z_1)$ which is false. $\square$

5.5. Lemma. Let $(M, g)$ be a 2-point homogeneous geodesic metric space having more than one point and $I$ denote $g(M \times M)$, and let $f: I \to [0, \infty)$ be a one-to-one function such that $g_f = f \circ g$ is a metric on $M$ equivalent to $g$. Then:

(v0) $I$ is an interval and $f \in \Omega(I)$;
(v1) $(M, g_f)$ is 2-point homogeneous;
(v2) $\text{Iso}(M, g_f) = \text{Iso}(M, g)$;
(v3) $(M, g_f)$ is absolutely homogeneous iff so is $(M, g)$;
(v4) $(M, g_f)$ is geodesic iff $f(t) = ct$ for some positive constant $c$ (and all $t \in I$).

Proof. Since $f$ is one-to-one, any map $u: A \to M$ (where $A \subset M$) is isometric with respect to $g_f$ if and only if it is isometric with respect to $g$. This implies (v1), (v2) and (v3). It follows from 1-point homogeneity of $(M, g)$ that $I = \{g(a, x): x \in M\}$ where $a$ is arbitrarily fixed element of $M$. But this, combined with continuity of $g_f$ with respect to $g$, yields that $f$ is continuous at 0. Further, since $M$ is geodesic, $I$ is an interval and for any $x, y \in I$ with $x + y \in I$ there are points $a, b, c \in M$ such that $g(a, c) = x + y$, $g(a, b) = x$ and $g(b, c) = y$. Then $f(x + y) = g_f(a, c) \leq g_f(a, b) + g_f(b, c) = f(x) + f(y)$. So, (ω2) holds. This implies that $|\omega(x) - \omega(y)| \leq \omega(|x - y|)$ for any $x, y \in I$. Thus $f$, being continuous at 0, is continuous (at each point of $I$). Finally, a continuous one-to-one function vanishing at 0 satisfies (ω1) and therefore $f \in \Omega(I)$.

It remains to show that if $g_f$ is geodesic, then $f$ is linear. Since $f \in \Omega(I)$, it is a bijection between $I$ and $J = f(I)$, and $J$ is an interval. Moreover, $g = f^{-1} \circ g_f$. So, assuming that $g_f$ is geodesic, it follows from the first part of the proof that also $f^{-1} \in \Omega(J)$. But if $f \in \Omega(I)$ and $f^{-1} \in \Omega(J)$, then $f(x + y) = f(x) + f(y)$ for all $x, y \in I$ with $x + y \in I$. The last equation implies that $f$ is linear (for $f$ is continuous). $\square$
5.6. Lemma. Assume \((Z, \lambda)\) and \((M, \varrho)\) are two 2-point homogeneous metric spaces having more than one point and satisfying the following three conditions:

- \((M, \varrho)\) is geodesic;
- there exists \(R \in \{1, \infty\}\) such that \(\varrho(M \times M) = \{ r \in \mathbb{R} : 0 \leq r \leq R\}\);
- there exists a homeomorphism \(h : Z \to M\) such that (5.1) holds.

Then there exists a unique \(\omega \in \Omega_R\) such that \(\lambda = \omega \circ \varrho \circ (h \times h)\). Moreover, \((Z, \lambda)\) is 3-point or absolutely homogeneous iff so is \((M, \varrho)\).

Proof. Recall that \(h \times h : Z \times Z \to M \times M\) is given by \((h \times h)(x, y) = (h(x), h(y))\). Further, for simplicity, denote \(I = \varrho(M \times M)\). It follows from our assumptions that \(I = [0, 1]\) (if \(R = 1\)) or \(I = [0, \infty)\) (if \(R = \infty\)).

For any four points \(a, b, x, y\) in an arbitrary 2-point homogeneous metric space \((Y, p)\) the following equivalence holds:

\[
p(a, b) = p(x, y) \iff \exists \Phi \in \text{Iso}(Y, p) : \Phi(a) = x \text{ and } \Phi(b) = y.
\]

The above condition, applied for both \((Z, \lambda)\) and \((M, \varrho)\), combined with (5-1) yields that

\[
\lambda(a, b) = \lambda(x, y) \iff \varrho(h(a), h(b)) = \varrho(h(x), h(y)) \quad (a, b, x, y \in Z).
\]

We infer that there exists a one-to-one function \(\omega : I \to [0, \infty)\) such that \(\lambda = \omega \circ \varrho \circ (h \times h)\). Since \(\omega \circ \varrho = \lambda \times (h^{-1} \times h^{-1})\) is a metric equivalent to \(\varrho\), we conclude from Lemma 5.5 that \(\omega \in \Omega_R\). The uniqueness of \(\omega\) is trivial.

Finally, the remainder of the lemma (about 3-point or absolute homogeneity) follows from (5.3). Indeed, this condition implies that a map \(u : (A, \lambda) \to (Z, \lambda)\) (where \(A \subset Z\)) is isometric iff the map \(h \circ u \circ h^{-1}|_{h(A)} : (h(A), \varrho) \to (M, \varrho)\) is isometric.

\[\square\]

Proof of Theorem 5.1. First of all, since each of the spaces

\[(\mathbb{R}^n, d_e) \ (n > 0), \ (S^n, d_h) \ (n > 0), \ (H^n(\mathbb{R}), d_h) \ (n > 1)\]

is absolutely homogeneous, it follows from Lemma 5.3 that any space \((X, d)\) listed in the statement of the theorem is also absolutely homogeneous (and, of course, connected, locally compact and has more than one point). Lemma 5.5 enables us recognizing those among them that are geodesic.

Further, if \((Z, \lambda)\) is a connected locally compact 3-point homogeneous metric space having more than one point, we infer from Theorem 5.3 that there are a metric space \((M, \varrho)\) and a homeomorphism \(h : Z \to M\) such that (5.1) holds and either \((M, \varrho)\) is listed in (5.4) or it is \((P^n(\mathbb{R}), d_p)\) with \(n > 1\). However, since \((M, \varrho)\) is 2-point homogeneous (cf. Lemma 5.1), it follows from Lemma 5.4 that it is also 3-point homogeneous (because \((Z, \lambda)\) is so). Thus, Lemma 5.3 implies that \((M, \varrho)\) is listed in (5.4), and we conclude from Lemma 5.6 that there is \(\tilde{\omega} \in \Omega(I)\) (where \(I = \varrho(M \times M)\)) for which

\[(5-5) \quad \lambda = \tilde{\omega} \circ \varrho \circ (h \times h).
\]

Now if \((M, \varrho)\) is not a Euclidean space, put \(\omega = \tilde{\omega}\) and observe that \((X, d) = (M, \omega \circ \varrho)\) is listed in the statement of the theorem and \(h : (Z, \lambda) \to (X, d)\) is an isometry, thanks to (5.4).

In the remaining case—when \((M, \varrho) = (\mathbb{R}^n, d_e)\)—we proceed as follows. There is \(\alpha > 0\) such that \(\tilde{\omega}(\alpha) = \min(1, \frac{1}{\alpha} \tilde{\omega}(\infty))\). We define \(\omega \in \Omega\) by \(\omega(t) = \tilde{\omega}(at)\). Observe that then \(\omega(\infty) = \tilde{\omega}(\infty)\) and hence \(\omega(1) = \min(1, \frac{1}{\alpha} \tilde{\omega}(\infty))\). So, \((X, d) = (\mathbb{R}^n, \omega \circ d_e)\) is listed in the statement of the theorem. What is more, the map

\[(Z, \lambda) \ni z \mapsto \frac{1}{\alpha} h(z) \in (X, d)
\]

is an isometry, again by (5-5) (and the definition of \(\omega\)).
The last thing we need to prove is that all the metric spaces \((X, d)\) listed in the statement of the theorem are pairwise non-isometric. To this end, let \((M_1, \varrho_1)\) and \((M_2, \varrho_2)\) be two spaces among listed in \([\ref{9}]\), \(\omega_1\) and \(\omega_2\) be two functions such that \((X, d) = (M_j, \omega_j \circ \varrho_j)\) for \(j \in \{1,2\}\) is listed in the statement of the theorem; and let \(h: (M_1, \omega_1 \circ \varrho_1) \to (M_2, \omega_2 \circ \varrho_2)\) be an isometry. Then

\[
\text{Iso}(M_1, \omega_1 \circ \varrho_1) = \text{Iso}(M_1, \varrho_1) = \{h^{-1} \circ u \circ h: u \in \text{Iso}(M_2, \varrho_2)\}.
\]

So, we infer from the uniqueness in Theorem \([\ref{8}]\) that \((M_1, \varrho_1) = (M_2, \varrho_2)\). To simplify further arguments, we denote \((M, \varrho) = (M_1, \varrho_1)\). Thus, \(h\) is an isometry from \((M, \omega_1 \circ \varrho)\) onto \((M, \omega_2 \circ \varrho)\). We conclude that:

\[
g(a, b) = g(x, y) \iff g(h(a), h(b)) = g(h(x), h(y)) \quad (a, b, x, y \in M)
\]

(because both \(\omega_1\) and \(\omega_2\) are one-to-one). The above condition implies that there is a one-to-one function \(f: I \to I\) where \(I = g(M \times M)\) such that \(g \circ (h \times h) = f \circ g\).

Since both \(g\) and \(g \circ (h \times h)\) are geodesic, it follows from Lemma \([\ref{5}]\) that there is a constant \(c > 0\) such that \(f(t) = ct\). So, \(h\) is a dilation on \((M, \varrho)\). Now we consider two cases. First assume that \((M, \varrho)\) is not a Euclidean space. Then \(c = 1\), which for hyperbolic spaces follows from Theorem \([\ref{5}]\) and for spheres is trivial (just compare diameters). Thus, \(h \in \text{Iso}(M, \varrho)\) and therefore \(\omega_1 \circ \varrho = \omega_2 \circ \varrho\). Consequently, \(\omega_1 = \omega_2\).

Finally, assume \((M, \varrho) = (\mathbb{R}^n, d_e)\). We have already known that \(g(h(x), h(y)) = c\varrho(x, y)\) for all \(x, y \in \mathbb{R}^n\). Simultaneously, \(\omega_2(g(h(x), h(y))) = \omega_1(\varrho(x, y))\) for any \(x, y \in \mathbb{R}^n\). Both these equations imply that \(\omega_2(ct) = \omega_1(t)\) for any \(t \geq 0\). In particular, \(\omega_2(\infty) = \omega_1(\infty)\) and thus \(\omega_2(e) = \omega_1(1) = \min(1, \frac{1}{2}\omega_1(\infty)) = \omega_2(1)\). Since \(\omega_2\) is one-to-one, we get \(c = 1\) and hence \(\omega_2 = \omega_1\).

\[\square\]

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