A Bit-Parallel Russian Dolls Search for a Maximum Cardinality Clique in a Graph

Ricardo C. Corrêa, Philippe Michelon, Bertrand Le Cun, Thierry Mautor, Diego Delle Donne

Universidade Federal do Ceará, Departamento de Computação, Campus do Pici, Bloco 910, 60440-554 Fortaleza - CE, Brazil
Université d’Avignon et des Pays du Vaucluse, Laboratoire d’Informatique d’Avignon, F-84911 Avignon Cedex 9, France
Université de Versailles Saint Quentin, 45 Avenue des Etats Unis, 78035, Versailles, France
Sciences Institute, National University of General Sarmiento, J. M. Gutiérrez 1150, Malvinas Argentinas, (1613) Buenos Aires, Argentina

Abstract

Finding the clique of maximum cardinality in an arbitrary graph is an NP-Hard problem that has many applications, which has motivated several approaches to solve it exactly despite its difficulty. In this paper, we propose an exact algorithm for the maximum clique problem based on the Russian Dolls Search method. In comparison to a first implementation of this Russian Dolls method from the literature, several improvements are presented. Some of them are adaptations of techniques already employed successfully to this problem, like the use of approximate coloring for pruning and bit-parallel operations. Three different coloring heuristics are tested: the standard greedy and two others from the literature, namely greedy with recoloring and fractional coloring. Other improvements are directly related to the Russian Dolls scheme: the adoption of recursive calls where each subproblem (doll) is solved itself by a Russian Dolls Search and the application of an elimination rule allowing not to generate a significant number of dolls. Results of computational experiments show that the algorithm outperforms the best exact combinatorial algorithms in the literature for the great majority of the instances tested, being more than twice faster in several cases.

1. Introduction

1.1. Problem Statement

Let $G = (V, E)$ be a simple and undirected graph, with $V$ the set of vertices ($|V| = n, V = \{0, 1, \ldots, n - 1\}$) and $E$ the set of edges (an edge is an unordered pair of distinct vertices). A clique of $G$ is a subset (of $V$) of pairwise adjacent vertices. We consider the CLIQUE problem, which consists in finding in $G$ a clique of maximum size $\omega(G)$, which in turn is called the clique number of $G$. The CLIQUE problem belongs to the class of NP-hard problems \cite{1} and is even hard to approximate \cite{2}. Naturally, if the graph is restricted to have a special structure, then the problem can sometimes be solved in polynomial time. But, in this paper, we deal with exact algorithms for determining the clique number of arbitrary graphs. The CLIQUE problem has many practical applications (see for instance \cite{3} and \cite{4} for the description of some applications). The attractiveness of this problem is further enhanced by the fact it is computationally equivalent to the maximum stable set problem and the minimum vertex cover problem ($S \subseteq V$ is a stable set of $G$ if it is a clique in the complement of $G$ and a vertex cover if every edge in $E$ has at least one endpoint in $S$).

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Email addresses: correa@lia.ufc.br (Ricardo C. Corrêa), philippe.michelon@univ-avignon.fr (Philippe Michelon), blec@prism.uvsq.fr (Bertrand Le Cun), thierry.mautor@prism.uvsq.fr (Thierry Mautor), ddelledo@ungs.edu.ar (Diego Delle Donne)}
Before going into the details of the problem and of its algorithms, let us state some notations.

- \([n] = \{0, 1, 2, \ldots, n - 1\}\).
- \(N(u) = \{v \in V \mid (u, v) \in E\}\) is the neighborhood of a vertex \(u\) in \(G\) whose members are neighbors of \(u\).
- If \(U \subseteq V\), then \(G[U] = (U, E[U])\) denotes the subgraph of \(G\) induced by \(U\).
- If \(v \in V\), then \(U + v\) stands for \(U \cup \{v\}\) and \(U - v\) for \(U \setminus v\).
- An \(\ell\)-coloring of \(G\) is a proper coloring of \(G\) with \(\ell\) colors (assignment of a color from \(\{0, 1, \ldots, \ell - 1\}\) to every vertex of \(G\) such that the endpoints of any edge get different colors). It can be characterized by \(\ell\) disjoint subsets \(C_0, \ldots, C_{\ell - 1}\) such that \(\cup_{i=0}^{\ell-1} C_i = V\) and \(G[C_i]\) is a stable set for all \(i \in [\ell]\).
- \(\chi(G)\) is the chromatic number of a graph \(G\) i.e. the smallest number \(\ell\) of colors such there exists an \(\ell\)-coloring of \(G\).

1.2. Exact Algorithms via Branch and Bound

Several algorithms have been proposed to solve the CLIQUE problem exactly. Mainly, these algorithms are based on a Branch and Bound approach where a key element of it is the use of approximate colorings of selected subgraphs \(G'\) of \(G\) to derive an upper bound for \(\omega(G')\) that is used to possibly prune the enumeration. This bound, first proposed in [5], is based on the following remark:

**Remark 1 (Upper bound from vertex coloring).** If \(G\) admits an \(\ell\)-coloring, then \(\omega(G) \leq \chi(G) \leq \ell\).

Any heuristic that provides a proper coloring (an \(\ell\)-coloring) of a graph gives an upper bound. Generally, as it is applied at each of the Branch and Bound nodes, a very simple and fast heuristic is used (simpler and faster than, for instance, the celebrated DSatur or Welsh-Powell heuristics): a greedy and sequential coloring in which the vertex considered is colored in the smallest possible color and where the key point is the ordering of vertices (usually called list heuristic in the literature).

In this vein, the algorithms proposed in [6] and [7] (algorithms MCQ and MCR) are very representative and very efficient. The greedy coloring is used in each node of the Branch and Bound tree not only as a bounding strategy but also as a branching strategy. The branching is done only on vertices whose colors are greater than the value of the best known solution (best clique) which significantly reduces the height, and so the size, of the B&B tree. Experiments with these algorithms show that they attain a good tradeoff between time spent computing approximate colorings and number of nodes explored in the search tree.

Since then, some improvements in this basic algorithm have been performed in the direction of proposing coloring heuristics not too time-consuming in comparison with the reduction in the search space thereby obtained. In [8], a more judicious ordering of the vertices in the nodes of the search tree is proposed, improving the bounds obtained with the greedy coloring heuristic. In [9] (algorithm MCS) the algorithm is modified in two points related to the coloring heuristic: first, a static order similar to the one proposed in [8] is adopted; second, a color exchange strategy is employed to try to recolor a vertex with a big color into a smaller color that could improve the upper bound. In [10] and [11], an heuristic is applied first: the Iterated Local Search (ILS) heuristic proposed in [12] to obtain an initial high-quality solution that allows to prune early branches in the Branch and Bound tree.

Despite of being published prior to the references mentioned above, an algorithm that deserves to be mentioned in this overview is the one proposed in [13]. The main point to observe is that instead of using simply an \(\ell\)-coloring, that paper introduces a fractional coloring heuristic to be used as an upper bound for pruning purposes. Let the fractional clique number of \(G\) be the linear relaxation of the clique number, i.e.

\[
\omega_f(G) = \max \left\{ \sum_{v \in [n]} x_v \mid \sum_{v \in S} x_v \leq 1, \forall S \in \mathcal{S}, \text{ and } x_v \geq 0, \forall v \in [n] \right\},
\]
where $S$ stands for the family of all maximal stable sets of $G$. Its dual is the so called fractional chromatic number, given by

$$\chi_F(G) = \min \left\{ \sum_{S \in S} y_S \mid \sum_{(S \in S \ni v \in S)} y_S \geq 1, \forall v \in [n], \text{ and } y_S \geq 0, \forall S \in S \right\}.$$ 

Let us define an extension to the $\ell$-coloring: if each vertex belongs to exactly $k \geq 1$ subsets, then $C_0, \ldots, C_{\ell-1}$ (with $\bigcup_{i=0}^{\ell-1} C_i = V$ and $G[C_i]$ is a stable set for all $i \in [\ell]$) is a $k$-fold $\ell$-coloring of $G$. It has been proved that $\chi_F(G)$ corresponds to the smallest ratio $\ell/k$, taken over all $k$ and $\ell$ such that there exists a $k$-fold $\ell$-coloring of $G$ \cite{13}. Thus, we get the following upper bound:

**Remark 2 (Upper bound from fractional vertex coloring).** If $G$ admits a $k$-fold $\ell$-coloring, then $\omega(G) \leq \omega_F(G) \leq \chi_F(G) \leq \ell/k$.

An efficient alternative to the coloring procedure has been proposed in \cite{14} where a Max-Sat based procedure is used for the upper bound.

1.3. Bit-Level Parallelism

Another improvement of the algorithm, by means of the encoding of the graph as a bitmap and the incorporation of bit-parallel operations in the greedy coloring heuristic and in the ordering of vertices used for branching, is described in \cite{15} and improved in \cite{16}. A bitmap is a data structure for set encoding which stores individual elements of the set in a compact form while allows for direct address of each element. Typical applications of bitmaps occur in problems involving the manipulation of sets of vertices or edges of a graph. Still more interesting is its ability to benefit from the potential bit-level parallelism available in hardware to perform collective set operations through fast bit-masking operations (intersection of two sets is a typical example detailed in Section 5). However, to exploit this potential parallelism in practice to improve overall efficiency is not a trivial task since the manipulation of bitmaps turns out to be less efficient when the enumeration of elements is relevant \cite{17}.

1.4. An Exact Algorithm via the Russian Dolls Method

An original and interesting algorithm has been proposed in \cite{18} where the exact method is not based on a pure Branch and Bound but on a Russian Dolls method. The Russian Dolls method has been initially proposed in \cite{19}. It consists in iteratively and optimally solving larger and larger subproblems (also referred to as dolls) until the resolution of the global problem. Each doll is solved using as much as possible the results obtained in the resolution of the previous subproblems (i.e. smaller dolls). From a practical point of view, it generally consists in sorting the variables of the problem and in considering the doll $k$ as the subproblem induced by the $k$ first variables. The optimum value of each doll is stored and subsequently used for pruning purposes when solving larger subproblems.

The Russian Dolls method was originally developed to solve the daily management problem of an earth observation satellite \cite{20}. It has also been applied by the designers of the method to a radio-link frequency assignment problem \cite{21}. Few works have been proposed since but an application to the combinatorial best barbeque problem \cite{22} and an other one to the Steiner triple covering problem \cite{23}.

In the application of the Russian Dolls to the CLIQUE problem by \cite{18}, subproblems are associated with subgraphs $G_i = (V_i, E_i)$, $i \in \{1, \ldots, n\}$, where $V_1 = \{x_1\}$, $V_{i+1} = V_i \cup \{x_{i+1}\}$, $E_i = E[V_i]$, and $G_n = G$ (considering that the vertices are initially sorted and renumbered accordingly). An optimum solution of the subproblem of index $i$ is a clique of maximum size in the associated subgraph $G_i$. An important observation is that searching for a maximum clique in $G_i$ corresponds to decide whether $\omega(G_i) = \omega(G_{i-1})$ or $\omega(G_i) = \omega(G_{i-1}) + 1$. Moreover, $\omega(G_i)$ can be equal to $\omega(G_{i-1}) + 1$ only if the unique vertex $v$ in $V_i \setminus V_{i-1}$ appears in the maximum clique of $G_i$. Equivalently, subproblem of index $i$ leads to the decision subproblem of deciding whether $G[V_{i-1} \cap N(v)]$ contains a clique of size $\omega(G_{i-1})$ or not. This decision is realized by a recursive process leading to the construction of a tree with no lower bound testing but with 3 pruning rules (in which $C_l$ denotes the current clique in construction and $Candidate$ stands for the set of vertices in $V_{i-1}$ that can extend the current clique, i.e. $Candidate = \{v \in V_{i-1} \setminus Cl \mid Cl \subseteq N(v)\}$):
1. \(|Cl| = \omega(G_{i-1}) + 1\): if one solution of value \(\omega(G_{i-1}) + 1\) is found, then set \(\omega(G_i) = \omega(G_{i-1}) + 1\) and go to doll \(G_{i+1}\), if \(i < n\), or stop otherwise.

2. \(|Cl| + |Candidate| \leq \omega(G_{i-1})\): if there are not enough candidates to improve \(\omega(G_{i-1})\), then backtrack.

3. \(|Cl| + \omega(G_{pd}) \leq \omega(G_{i-1})\), where \(V_{pd}\) is maximal with respect to \(Candidates \subseteq V_{pd} \subseteq V_{i-1}\): if there is no clique in the previous maximal doll \(G_{pd}\) that can improve the current best clique, then backtrack.

1.5. Our Contributions

In this paper, we propose a new exact algorithm for the CLIQUE problem based on the Russian Dolls method. The goal is to significantly improve the Russian Dolls method by incorporating various effective procedures where some of them have shown their effectiveness in the Branch and Bound approaches previously described. In fact our goal is to get the best of the two types of approaches - Russian Dolls and Branch and Bound - to make a very effective method. Thus we suggest the following improvements to the initial implementation of Russian Dolls presented in [18]:

- An initial renumbering of the vertices according to the same order used in the MCR algorithm[7].
- The use of approximate colorings as upper bounds - we tested three different versions of coloring: the “standard greedy”[6], the greedy with recoloring[9] and the fractional coloring[13].
- The use of properties that allow to jump a number of iterations (dolls) in the Russian Dolls method that we call elimination rule.
- A bitmap encoding of \(G\) and a bit-parallel implementation of several procedures to benefit from bit parallelism.
- Each subproblem itself is solved by a Russian Dolls search.

Results of computational experiments show that our algorithms outperform the best exact combinatorial algorithms in the literature for graphs with density between 30% and 95%, being more than twice faster in several cases.

The remainder sections are organized as follows. In Section 2, we give an overall description of our Russian Dolls Search algorithm, describing its main elements. The details of the algorithm is the subject of Section 3. This section includes the description of the different improvements and specific features listed above. Finally, experimental results and analysis are presented in Section 4. The paper is closed with some concluding remarks in Section 5.

2. Overall Description of the Algorithm

In this section, we give a general overview of our algorithm. The main elements of the algorithm are the subproblem enumeration procedure and pruning strategies. We postpone the details on how these elements are implemented and connected until Section 3.

2.1. Decision Subproblems Enumeration

Alg. 1 outlines the way in which the decision subproblems are generated and solved recursively with the Russian Dolls Search method. Assume we have a subproblem \(G_i\), its clique number \(\omega(G_i)\), and a vertex \(v \in V \setminus V_i\) to add to \(G_i\) in order to obtain the next subproblem \(G_{i+1}\). This is the situation after line 5 of Alg. 1. It is worth remarking that the choice of \(v\) determines the enumeration order. In [18], for instance, this choice is simply \(v = i + 1\), which implies that the initial numbering of the vertices in that algorithm also determines the order in which they are considered during the search. In Alg. 1 however, \(v\) is chosen to be the smallest vertex not in \(V_i\). It follows that the choice of \(v\) depends on the result of lines 3 and 7, where the elimination rule described in Subsection 3.3 is applied. After executing line 5, we have to decide (with the recursive function Decide) whether the decision subproblem \(G_i[V_i \cap N(v)]\) has a clique of size \(\omega(G_i)\). If this search fails, then \(\omega(G_i + v) = \omega(G_i)\) and we go to the next subproblem \(G_{i+1} = G_i + v\). Otherwise,
Algorithm 1: Overview of the general method

1: \( \textit{MAX} \leftarrow 0 \) /* Size of the maximum clique found so far */
2: \( i \leftarrow 0, V_i \leftarrow \emptyset \)
3: \( \triangleright \) Pre-processing - Apply elimination rule, update \( \textit{MAX} \) and \( i \) appropriately and define the elements of the set \( V_i \) for the new value of \( i \).
4: \( \textbf{while} \ i \leq n \textbf{ do} \)
5: \( \text{Let} \ v \text{ be the smallest vertex in } V \setminus V_i \text{ (it follows that } \omega(G_i + v) \leq \textit{MAX} + 1) \)
6: \( \text{if } \text{DECIDE}(G, V_i \cap N(v), \textit{MAX}, Cl) \text{ then} \)
7: \( \triangleright \) Apply elimination rule, update \( \textit{MAX} \) and \( i \) appropriately, define the elements of the set \( V_i \) for the new value of \( i \) and store \( Cl \)
8: \( \text{else} \)
9: \( V_{i+1} \leftarrow V_i + v, i \leftarrow i + 1 \)

10: \( \textbf{function} \ \text{DECIDE}(G, R, \ell, Cl) \)
11: \( \text{if } R = \emptyset \text{ then} \)
12: \( \text{return } (\ell = 0) \)
13: \( \triangleright \) Check, and possibly apply, pruning
14: \( W \leftarrow \emptyset \)
15: \( \textbf{while} \ W \neq R \textbf{ do} \)
16: \( \text{Let } v \text{ be the smallest vertex in } R \setminus W \)
17: \( \text{if } \text{DECIDE}(G, W \cap N(v), \ell - 1, Cl) \text{ then} \)
18: \( \text{return } \text{TRUE} \)
19: \( W \leftarrow W + v \)
20: \( \text{return } \text{FALSE} \)

there exists a clique of \( G_i + v \) containing \( v \) that gives \( \omega(G_{i+1}) = \omega(G_i + v) = \omega(G_i) + 1 \), which enables the application of the elimination rule. Function \textsc{decide} gets three parameters as input, namely the graph \( G \), a subset of vertices \( R \) (candidates), and an integer \( \ell \). It returns \text{TRUE} if and only if \( G[R] \) contains a clique of size \( \ell \) that is returned by the output parameter \( Cl \) (otherwise, it returns \text{FALSE}). After solving the decision subproblem \( G[V_i \cap N(v)] \) (and possibly applying the elimination rule), a new vertex is chosen (if any remains) to be added to \( G_{i+1} \).

2.2. Pruning

A decision subproblem can be pruned from the search when it can be proved in function \textsc{decide}(G, R, \ell, Cl) that it does not contain any clique of the desired size \( \ell \). The simplest rule that can be applied in this context is

\textit{If } \(|R| < \ell \) \text{ Then Return FALSE} \)

since \( |R| \) is an upper bound for the clique number of the decision subproblem at hand. In [18], the value of \( \omega(G_{pd}) \), where \( pd \) (previous doll) is the biggest index of the elements in \( R \), is used as a tighter upper bound than \( |R| \) for pruning purposes (note that \( G[R] \) is a subgraph of \( G_i \)). This pruning rule could be adapted to our algorithms with extra data structures in order to determine at each call of the function \textsc{decide} what value is taken by \( pd \). We do not do that because handling such extra data structures would require some additional computational effort. On the other hand, tighter upper bounds than \( |R| \) can be derived from Remark 1 (see next Section).

3. New Features

3.1. Bit-Parallelism

Exploiting bit-level parallelism in sets encoded as bitmaps is central in our algorithms due to its ability to speedup some operations that are executed very often during the search. In this subsection, we describe the bitmap data structure and its elementary operations. Its application to exploit bit-level parallelism in greedy coloring heuristics or in elimination rules will be described in the next subsections.
A bitmap $B_n$ is a special encoding of a directly addressed set $B \subseteq [n]$ whose elements are represented as bits in an array. In such an encoding, for every $i \in [n]$, bit indexed $i$ in $B_n$ is 1 if and only if $i$ is an element of $B$. For example, the subset $\{1,3,6\}$ of $[8]$ is encoded as 01010010. Naturally, instead of being viewed as an array of bits, a bitmap is stored as an array of bitmap nodes (or simply nodes), each of the same size (in bits), denoted by $w$. If we take $w = 4$ in the previous example, the bitmap consists of an array of two nodes: node of index 0 in the array has value 0101, and the one of index 1 is 0010. Typically, the size $w$ of a node corresponds to the number of bits of a CPU register. We assume that $w$ is a power of 2 (which is a reasonable assumption since it equals 32 or 64 in nowadays computers). The size, in nodes, of $B_n$ is $\lceil n/w \rceil$ and accessing an element $i$ of the set stored in $B_n$ implies finding first the correct node and then addressing the exact bit in this node. Formulas of bit displacement allow this: for instance $i \gg \log_2 w$ gives the number of node where the bit corresponding to $i$ is.

The use of bitmaps will be there applied to the adjacency matrix of $G$. The neighborhood of a vertex $v$ is the set contained in the bitmap $\text{neig}(G,v)$ composed of $\lceil n/w \rceil$ nodes. Some of the operations performed on these bitmaps involve only one element of the set at hand and cannot benefit from bit-parallelism while classical set operations as difference or intersection take full advantage of this bit parallelism. The operations on bitmaps that we use in our algorithms are the following ones:

- **ADD($B_n,v$)**: adds the element $v < n$ to bitmap $B_n$ (determine the node containing $v$ and set in this node the corresponding bit to 1).
- **REM($B_n,v$)**: removes the element $v < n$ from bitmap $B_n$ (determine the node containing $v$ and set in this node the corresponding bit to 0).
- **BIT($B_n,v$)**: returns 1 if $v < n$ is in the bitmap $B_n$, and 0 otherwise (this corresponds to the value of the $v$th bit of $B_n$).
- **FSB($B_n,i,n$)**: returns the smallest element in $\{i,\ldots,n-1\} \cap B$ (which corresponds to the index of the least significant bit greater than or equal to $i$ that equals 1 in bitmap $B_n$). Such an operation is more time consuming with respect to the previous ones since it could incur a search in several nodes of the bitmap. The search in a node $e$ is done by means of the special function $\text{lsb}(e)$ returning the least significant bit of node $e$ (a negative number is returned if the value of $e$ is zero). In many nowadays processors, such a function is provided by the assembler set of instructions. For details on efficient implementations of $\text{lsb}(e)$, we refer the reader to [15] and references therein.
- **INTER($B_n,B'_n,n$)**: the intersection of the two bitmap encoded sets $B$ and $B'$ is computed (by making a logical & on each pair of corresponding nodes of $B_n$ and $B'_n$) and returned.
- **DIFF($B_n,B'_n,n$)**: the difference $B \setminus B'$ of two bitmap encoded sets is computed (by making the call INTER($B_n,B'_n,n$)) and returned.

### 3.2. Upper Bounds by Vertex Coloring

We have studied three different greedy coloring heuristics which are used in our algorithms for pruning and whose descriptions and bit-parallel implementations are given below.

#### 3.2.1. Greedy Coloring

This first heuristic is the adaptation of the very classical and simple list heuristic where the vertices are considered sequentially, each vertex being colored with the smallest possible color. However, two originalities of our implementation depicted in Alg. 2 have to be mentioned. First, the coloring is done by colors and not by vertices as in classical implementations of this heuristic ([7] for instance). The final coloring is the same but this approach is better suited to bit-parallelism. Second, the algorithm acts to determine at most $d$ ($d$ is an input parameter) disjoint stable sets, so at most $d$ colors. Not all vertices are colored and this represents another way to speedup the procedure. In Alg. 2 bit-parallelism is exploited in the set operations at lines [4] (copy of a set) and [7] (set difference). In addition, vertices still candidates for a color $d'$ are enumerated in an increasing order by the use of function $\text{FSB}$ (called at lines [5] and [11]). The colored vertices are stored in bitmap $C$. If we need to store vertices in each color separately, then $C$ has to be replaced by $C[d']$ at line [10].
More generally, given a graph $G$, let $\ell$ be the number of colors initially assigned to the vertices of $G$. Then, the remaining vertices of $G$ can be colored by trying to assign a lower color to nodes whose initial color is greater than a given value (based on the value of the benefit from bit parallelism.

### 3.2.2. MCS Coloring

MCSCOLORING is a method to improve a greedy coloring successfully employed in [9]. The idea is to try assigning a lower color to nodes whose initial color is greater than a given value (based on the value of the benefit from bit parallelism.

To do so, iteratively for each $v \in R$ we search for a color $i \in [d]$ such that $v$ has only one neighbour — say $u$ in the corresponding color class (it has at least one neighbour, otherwise $v$ would have been colored with color $i$). If such a color does not exist, we skip the current vertex $v$ and go to the next one in $R$. Otherwise, $N(v) \cap C[i] = \{u\}$ and $u$ has at least one neighbor in every color smaller than $i$. Thus, we search for a color $j$, $i < j < d$, that could accommodate $u$. If such a color is found, move $u$ to color $j$ and $v$ to color $i$ and insert $v$ in $R$. So, the number of vertices colored (in one of the $d$ first colors) has been increased. We do not detail the algorithm but, again, the operations FSB, ADD, REM, and INTER have to be applied allowing to benefit from bit parallelism.

### 3.2.3. Fractional Coloring

It follows from Remark 2 that a $k'$-fold $\ell'$-coloring of $G[R]$ gives a tighter upper bound for $\omega(G[R])$ if $\lfloor \ell'/k' \rfloor$ is smaller than the number of colors $\ell$ employed in a greedy coloring $C$. As proposed in [13], one initial step to obtain such a tighter bound is to build a 2-fold coloring starting from $C$. To this end, a “second” color is assigned to as many vertices of $R$ as possible by greedily augmenting every non maximal color of $C$. Then, the remaining vertices of $R$ are colored with new colors in order to obtain a 2-fold $\ell'$-coloring for $G[R]$, with $\ell' \geq \ell$. If the total number of used colors $\ell'$ is such that $\lfloor \ell'/2 \rfloor < \ell$, then we have a tighter upper bound. More generally, given a $k$-fold $\ell$-coloring $C$ for $G[R]$, if some of the colors in $C$ are not maximal in $G[R]$, then we can extend these colors to maximal ones by assigning a $(k+1)$-th color to as many vertices of $R$ as we can and then assigning new colors to the remaining vertices of $R$ in order to build a $(k+1)$-fold $\ell'$-coloring (a remark on this new coloring is that only the new colors can be non maximal). If $\lfloor \ell'/(k+1) \rfloor < \lfloor \ell/k \rfloor$, then we have a tighter upper bound.

With this objective in mind, function FRACCOLORING in Alg. 3 acts on a $k$-fold $\ell$-coloring of $G[R]$ provided as input in array $C$ (where each entry $C[j]$ is a size-$[n/w]$ bitmap encoding of a stable set corresponding to a color class) and builds a $(k + 1)$-fold $\ell'$-coloring of $G[R]$. In addition, FRACCOLORING also gets as input three other parameters, the number $k$ of colors per vertex, the total number of colors $\ell$ and the number of colors $\ell^{old}$ that remain if we delete the last color assigned to each vertex (i.e., the number of colors used by the previous $(k-1)$-fold $\ell^{old}$-coloring from which $C$ was obtained). Parameter $\ell^{old}$ determines that only colors $\{\ell^{old}, \ell^{old}+1, \ldots, \ell-1\}$ are candidates to be non maximal. Each of these color classes is augmented by adding the vertices with no neighbours in the class (line 3 in Alg. 3). After this, the remaining vertices of $R$ are greedily colored with at most $\lfloor l/k \rfloor - 1$ new colors by calling function STACKCOLORING (line 4). As it is a very close variation of GREEDYCOLORING that records the coloring obtained in the array $C$ of bitmaps, we do not detail this function STACKCOLORING. If it succeeds to color all remaining vertices in $R$, then we have a tighter upper bound.

### Algorithm 2 Bit-parallel greedy coloring heuristic

1: function GREEDYCOLORING($G$, $R$, $C$, $d$)
2:     $d' \leftarrow 0$
3:     while $d' < d$ and $R \neq \emptyset$
4:         $S \leftarrow$ a copy of $R$
5:         $v \leftarrow$ FSB($S$, 0, $n$)
6:         while $v \geq 0$
7:             $S \leftarrow$ DIFF($S$, NEIG($G$, $v$), $n$)
8:             REM($S$, $v$)
9:             REM($R$, $v$)
10:            ADD($C$, $v$)
11:            $v \leftarrow$ FSB($S$, $v + 1$, $n$)
12:     $d' \leftarrow d' + 1$

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then the bound given by the fractional coloring is improved. With the goal of finding a good upper bound for \( \omega(G[R]) \), function FRACCOLORING has to be called as long as improvements on the bound are obtained.

**Algorithm 3** Bit-parallel fractional coloring heuristic

1: function FRACCOLORING(G, R, C, k, \( \ell \), \( \ell^{old} \))
2: \( isMaximal \leftarrow TRUE \)
3: for \( j \leftarrow \ell^{old}, \ell^{old}+1, \ldots, \ell-1 \) do
4: \( v \leftarrow \text{FSB}(R, 0, n) \)
5: while \( v \geq 0 \) do
6: if \( \text{bit}(C[j], v) = 0 \) then
7: \( S \leftarrow \text{INTER}(C[j], \text{NEIG}(G, v), n) \)
8: if \( S = \emptyset \) then
9: \( \text{ADD}(C[j], v) \)
10: \( \text{REM}(R, v) \)
11: \( isMaximal \leftarrow FALSE \)
12: \( v \leftarrow \text{FSB}(R, v+1, n) \)
13: if \( isMaximal = FALSE \) then
14: \( \text{STACKCOLORING}(G, R, C, \lceil \ell/k \rceil - 1) \)

3.3. Elimination Rule

In the implementation of the Russian Dolls method presented in [18], the \( n \) iterations were all performed: one iteration for each doll \( G_i \), \( i \in \{1, \ldots, n\} \). We propose to apply an enhanced elimination rule allowing not to generate several dolls and so reducing the number of iterations. This rule is based on the basic property that, in a stable set, at most one vertex can belong to the maximum clique: indeed two vertices that are not linked cannot belong conjointly to a clique. More fundamentally, we use the following property:

**Property 1.** Let \( 0 \leq i < j \leq n \) be such that \( G[V_j \setminus V_i] \) admits an \( \ell \)-coloring, \( \ell \geq 1 \). Then, \( \omega(G_j) \leq \omega(G_i) + \ell \).

The remark \( \square \) is a direct consequence of this proposition by taking \( V_i = \emptyset \).

**Proof.** Since \( V_j = V_i \cup V_j \setminus V_i \), we have \( \omega(G_j) \leq \omega(G_i) + \omega(G_j \setminus G_i) \). In addition, since at most one vertex of each color can be in a clique, \( \omega(G_j \setminus G_i) \leq \ell \) and the result follows. \( \square \)

This elimination rule is applied in lines \( 3 \) and \( 7 \) of Alg. \( 3 \) during the initialization phase (pre-processing) and each time the size of the maximum clique found so far (variable \( MAX \)) is increased. At the beginning of the algorithm, vertices of \( G \) are first renumbered according to the order given by the MCR initial vertex sorting list heuristic \( \square \). This renumbering fixes the order employed in all executions of greedy heuristics, especially coloring heuristics. Then, a maximal clique is built with a greedy and simple heuristic (very similar to the greedy coloring) that gives an initial value to \( MAX \) (we do not apply an Iterated Local Search as in \( \square \)). Finally, a coloring function – one of the heuristics described in the previous subsection – is called with the parameter \( MAX \) as the number of colors (see Alg. \( 4 \)). By taking \( V_i \) as the set of vertices colored by the coloring function employed, Property \( \square \) establishes that \( \omega(V_i) \leq MAX \). These \( i \) first iterations can so be skipped and the Russian Dolls algorithm can advance to iteration \( i+1 \) directly. Let us remark that all the vertices of the maximal clique of size \( MAX \) are not necessarily colored and so not necessarily in \( V_i \).

The same principle is applied when \( MAX \) is improved. It can be noticed that the clique of \( G_{i+1} \) obtained when the call \( \text{DECIDE}(G, V_i \cap N(v), MAX, CI) \) returns TRUE is not necessarily maximal in \( G \). So, first we make it maximal and then we call the greedy coloring on the remaining vertices with \( \ell = MAX - \omega(V_i) \) colors: \( \text{COLORING}(G, V \setminus V_i, C, MAX - \omega(V_i)) \). Again, the vertices that have been colored can be skipped in the Russian Dolls process.

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Algorithm 4 Elimination rule during the pre-processing

1: $\text{MAX} \leftarrow 0$
2: $S \leftarrow$ a copy of $V$
3: $v \leftarrow \text{fsb}(S, 0, n)$
4: while $v \geq 0$ do
5:   $S \leftarrow \text{inter}(S, \text{neig}(G, v), n)$
6:   $\text{MAX} \leftarrow \text{MAX} + 1$
7:   $v \leftarrow \text{fsb}(S, v + 1, n)$
8: coloring($G$, $V$, $C$, $\text{MAX}$)

3.4. Recursive Russian Dolls Searches

Another originality of our implementation is that each subproblem is solved itself by a Russian Dolls
Search. As described in the overview of the general method (Alg. 1), one iteration of the global Russian
Dolls Search consists in solving the subproblem $G_i + v$: determining if the maximum clique of this subproblem
equals $\text{MAX} + 1$. In such a case $v$ belongs to this clique and we need so to verify if $V_i \cap N(v)$ contains a
clique of size $\text{MAX}$. This is done by the function decide that is detailed in Alg. 5.

In special, at line 6 a search for a partial $(\ell - 1)$-coloring of $G[R]$ is executed (moving the colored vertices
from $R$ to $C$). Algorithm coloring may be either greedyColoring, mcsColoring or fracColoring
as detailed in a previous subsection. If $G[R]$ admits an $(\ell - 1)$-coloring (i.e., $R = \emptyset$ after this call), then
the subproblem is pruned, as there cannot exist an $\ell$-size clique on $G[R]$. In the other case, if $R \neq \emptyset$ after this
line, $C$ is taken as the initial subproblem of this execution. It corresponds to the elimination rule performed
during the preprocessing. The vertices colored in the $\ell - 1$ colors may be jumped and the Russian Dolls
process starts after these vertices. Each of the remaining vertices in $R$ constitutes a doll of the recursive
Russian Dolls process. So, at each iteration of the loop starting at line 7 a vertex $v$ is chosen from the set of
remaining vertices to be added to $C$ and a recursive call looks for a clique of size $(\ell - 1)$ on $G[C \cap N(v)]$. If
such a clique is found, then $v$ is added to this clique and the function returns TRUE. If not, the procedure
is repeated until there are no more vertices in $R$.

Algorithm 5 Decision subproblem

1: function decide($G$, $R$, $\ell$, $Cl$)
2: if $R = \emptyset$ then
3:   return $(\ell = 0)$
4: if $|R| < \ell$ then
5:   return FALSE
6: coloring($G$, $R$, $C$, $\ell - 1$)
7: while $R \neq \emptyset$ do
8:   $v \leftarrow \text{fsb}(R, 0, n)$
9:   newR $\leftarrow \text{inter}(C, \text{neig}(G, v), n)$
10: if decide($G$, newR, $\ell - 1$, $Cl$) then
11:   add($Cl$, $v$)
12: return TRUE
13: add($C$, $v$)
14: rem($R$, $v$)
15: return FALSE

By a simple inspection in Alg. 5 the memory space required in the worst case corresponds to two size-$[n/w]$ bitmaps for each of the $n$ subproblems considered in the recursion and an additional size-$[n/w]$ bitmap in function coloring. This gives $2n[n/w] + [n/w]$ bits in the worst case. It can be remarked that the bitmap implementation described in [15] requires, besides a bitmap, two arrays containing a (partial)
coloring that establishes the order in which the vertices of each subproblem are examined. This leads to a
memory space of $\Omega(n^2)$ integers, or $\Omega(n^2 \log n)$ bits, in the worst case.
4. Experimental Results

Computational experiments were carried out with implementations of the bit-parallel Russian Dolls Search algorithm described in the previous sections. Three versions of RD_MAXCLIQUE have been tested to compare the effectiveness of different coloring heuristics, namely: $RD_1$ denotes the version that uses GREEDYCOLORING with no additional subproblem elimination, $RD_2$ and $RD_3$ refer to the versions with additional subproblem elimination via MCSCOLORING and FRACCOLORING, respectively. The results were obtained with experiments on a computer running a 64-bit Linux operating system and gcc as the C compiler (with compiling options -m64 -O3). We ran the three versions of our algorithm with DIMACS challenge benchmark graphs and with random graphs. For the sake of comparison, experimental results presented in [15] (algorithm BB_MAXCLIQUE), [9] (algorithm MCS), and [16] (algorithm BBMCI/BBMCR), were adjusted according to the usual DIMACS challenge methodology [24] (adopted in the three mentioned papers [15], [9], [16]). The average values ($T_1/T_2$, $T_1/T_3$, and $T_1/T_4$) of the user times in [15] ($T_2$), [9] ($T_3$), and [16] ($T_4$) with our user times ($T_1$) are shown in Table 1.

| Instance | $T_1$ | $T_2$ | $T_1/T_2$ | $T_3$ | $T_1/T_3$ | $T_4$ | $T_1/T_4$ |
|----------|-------|-------|-----------|-------|-----------|-------|-----------|
| r100.5   | 0.00  | 0.00  | -         | 0.00  | 0.00      | -     | -         |
| r200.5   | 0.04  | 0.031 | 1.29      | 0.064 | 0.63      | 0.003 | 13.3      |
| r300.5   | 0.39  | 0.234 | 1.67      | 0.562 | 0.69      | 0.203 | 1.92      |
| r400.5   | 2.41  | 1.531 | 1.57      | 3.48  | 0.69      | 1.186 | 2.03      |
| r500.5   | 9.18  | 5.766 | 1.59      | 13.3  | 0.69      | 4.587 | 2.00      |

Table 1: User times used to compute the average factors of $T_1/T_2 = 1.61$, $T_1/T_3 = 0.69$, and $T_1/T_4 = 1.98$ with the benchmark program dfmax and the machine benchmark graphs r300.5, r400.5 e r500.5.

| Instance | $n$ | dens. | $\omega(G)$ | Instance | $n$ | dens. | $\omega(G)$ |
|----------|-----|-------|---------------|----------|-----|-------|---------------|
| brock200_1 | 200 | 0.75  | 21            | MANN_a27  | 378 | 0.99  | 126           |
| brock200_2 | 200 | 0.49  | 12            | MANN_a65  | 1035| 1.00  | 345           |
| brock200_3 | 200 | 0.49  | 15            | p_hat300-1 | 300 | 0.24  | 8             |
| brock200_4 | 200 | 0.65  | 17            | p_hat300-2 | 300 | 0.49  | 25            |
| brock400_1 | 400 | 0.75  | 27            | p_hat300-3 | 300 | 0.74  | 36            |
| brock400_2 | 400 | 0.75  | 29            | p_hat700-1 | 700 | 0.25  | 11            |
| brock400_3 | 400 | 0.75  | 31            | p_hat700-2 | 700 | 0.50  | 44            |
| brock400_4 | 400 | 0.75  | 33            | p_hat700-3 | 700 | 0.75  | 62            |
| brock800_1 | 800 | 0.65  | 23            | p_hat1000-2 | 1000| 0.49  | 46            |
| brock800_2 | 800 | 0.65  | 24            | san200_0.7 | 200 | 0.70  | 18            |
| brock800_3 | 800 | 0.65  | 25            | san200_0.9 | 200 | 0.90  | 70            |
| brock800_4 | 800 | 0.65  | 26            | san200_0.9 | 200 | 0.90  | 60            |
| C250.9    | 250 | 0.90  | 44            | san200_0.9 | 200 | 0.90  | 44            |
| DSJC500.5 | 500 | 0.50  | 13            | san200_0.7 | 200 | 0.69  | 18            |
| hamming8-4 | 256 | 0.64  | 16            | san200_0.9 | 200 | 0.89  | 42            |
| keller4   | 171 | 0.65  | 11            | san400_0.5 | 400 | 0.50  | 13            |
| gen200_0.9_44 | 200 | 0.90  | 44            | san400_0.7 | 400 | 0.70  | 21            |
| gen200_0.9_55 | 200 | 0.90  | 55            |           |     |       |               |

Table 2: Selected DIMACS challenge graphs and their numbers of vertices, densities, and clique numbers.

The selection of DIMACS benchmark graphs is shown in Table 2. This selection was made avoiding instances with running times too small (adjusted running time of [15], [9], or [16] below 0.1 sec.). In Table 3, the selection of random graphs $G(n, p)$ is shown, where $n$ is the number of vertices and $p$ is the probability that each pair of vertices is picked to define an edge. These graphs were generated having between 200 and 5000 vertices, and probabilities from 0.1 up to 0.98. For each configuration, five graphs were generated.

Tables 4 (for DIMACS benchmark graphs) and 5 (for random graphs) show the number of decision subproblems generated and the CPU user times (in seconds) measured in the experiments. With respect to overall performance of RD_MAXCLIQUE, in its three versions, compared with those in [16] and [9], we
measured the average speedup per density as depicted in Figure 1. For each graph instance $G$, the speedup is given by the smallest computation time for $G$ among the implementations of MCS (in [9]) and BBMCI and BBMCR (in [16]), divided by the smallest time among the three versions of RD\_MaxClique. We can conclude from Figure 1 that our algorithm outperforms the best algorithms in the literature for graphs with density between 0.30 and 0.95 (in several cases, RD\_MaxClique is more than twice faster). An exception is density 0.65 due to graphs brock800\_2 and brock800\_4. RD\_MaxClique also succeeded to reduce the number of generated subproblems for many instances due to subproblems elimination and pruning.

In a more accurate analysis, we can observe that brock400\_4, C250\_9, gen200\_0.9\_55, p\_hat300-3, p\_hat700-3, p\_hat1000-2, sanr200\_0.9, and sanr400\_0.7 are instances of DIMACS benchmark graphs whose running times of RD\_MaxClique are more than twice faster, which shows its effectiveness when the density is between 0.5 and 0.9. This conclusion is corroborated by the results with random graphs. In spite of this, instances whose number of subproblems and execution times are very small lead RD\_MaxClique not to outperform other algorithms. This is the case of MANN\_a45, p\_hat300-2, p\_hat700-1, and sanr200\_0.9\_3. This happens because the reduction in the number of generated subproblems is not able to circumvent the initial overload due to the initial vertex renumbering. It should be noticed that brock800\_2 and brock800\_4 are two cases with large running times for which MCS is faster than RD\_MaxClique. For MANN\_a45, our implementation with recoloring has a performance comparable to that of [10].

Finally, it can be observed by a comparison of the three versions of RD\_MaxClique that the use of

| Instance Graph | $\omega(G)$ [15] | $RD$ | Instance Graph | $\omega(G)$ [15] | $RD$ |
|----------------|------------------|------|----------------|------------------|------|
| (200, 0.7)     | 17-18            | 18   | (500, 0.65)    | 17               |
| (200, 0.8)     | 25-26            | 25-27| (500, 0.7)     | 22-23           |
| (200, 0.9)     | 40-42            | 40-44| (1000, 0.1)    | 5-6             |
| (200, 0.95)    | 61-64            | 60-63| (1000, 0.2)    | 7-8             |
| (200, 0.98)    | 90-103           | 93-96| (1000, 0.3)    | 9-10            |
| (300, 0.65)    | 17               |      | (1000, 0.4)    | 12              |
| (300, 0.7)     | 20-21            | 20-21| (1000, 0.5)    | 14-15           |
| (300, 0.8)     | 28-29            | 28-29| (5000, 0.1)    | 7               |
| (500, 0.5)     | 13               | 13-14| (5000, 0.2)    | 9-10            |
| (500, 0.6)     | 17               | 17   | (5000, 0.3)    | 7               |

Table 3: Selected random graphs $G(n, p)$ and their clique numbers reported in [15], [9], and in our experiments. For every configuration, the smallest and greatest clique numbers among the several instances considered are given.
mcsColoring and fracColoring (in RD₂ and RD₃ respectively) leads to the smallest number of generated subproblems in most cases. Indeed, the only exception in our experiments is G(5000, 0.3). In general, the number of subproblems generated by RD₂ and RD₃ are comparable. With respect to running times, RD₁ tends to be faster for graphs with density close to 0.3. The cases for which RD₁ outperforms the other two is due to the overload of mcsColoring and fracColoring with respect to greedyColoring. In these cases, the subproblems generated by RD₁ but pruned by RD₂ and RD₃ are of small size. Thus, their enumeration by RD₁ is faster than pruning with mcsColoring and fracColoring.

| Instance     | Number of subproblems × 10⁻³ | Time (seconds) | \(RD₁\) | \(RD₂\) | \(RD₃\) |
|--------------|-------------------------------|----------------|--------|--------|--------|
| brock200_1   | 295.8                         | 0.805          | 0.618  | 0.434  | 0.457  | 0.450 |
| brock200_4   | 64.7                          | 0.129          | 0.123  | 0.074  | 0.078  | 0.083 |
| brock400_1   | 168768                        | 809.5          | 478.2  | 675.2  | 279.9  | 264.2 |
| brock400_2   | 66381                         | 350.6          | 277.2  | 204.9  | 166.6  | 157.3 |
| brock400_3   | 118561                        | 541.3          | 322.9  | 453.4  | 148.3  | 139.0 |
| brock400_4   | 66670                         | 332.3          | 264.1  | 171.1  | 83.67  | 70.74 |
| brock800_1   | 1.93×10⁶                      | 13846          | 6449   | 4965   | 5016   | 5028 |
| brock800_2   | 1.75×10⁶                      | 12642          | 5774   | 6129   | 6224   | 6237 |
| brock800_3   | 1.10×10⁶                      | 8152           | 3971   | 2067   | 2684   | 2695 |
| brock800_4   | 0.73×10⁶                      | 5753           | 2758   | 3772   | 3802   | 3806 |
| C250.9       | 4.58×10⁵                      | 2247           | 2554   | 1244   | 978.5  | 973.2 |
| DSJC500.5    | 4.74                          | 1.250          | 1.390  | 1.387  |        |
| gen200_0.9_44| 35                            | 0.324          | 0.370  | 0.382  | 0.411  | 0.297 |
| gen200_0.9_55| 112                           | 0.828          | 0.865  | 0.114  | 0.105  | 0.109 |
| MANN_a27     | 38.3                          | 0.908          | 0.552  | 0.370  | 0.744  | 0.658 |
| MANN_a45     | 2953                          | 402.7          | 193.9  | 84.0   | 134.6  | 87.0  |
| p_hat300-2   | 6.2                           | 0.101          | 0.103  | 0.108  |        |
| p_hat300-3   | 590.2                         | 3.191          | 1.725  | 2.475  | 0.493  | 0.505 |
| p_hat700-1   | 30.2                          | 0.129          | 0.093  | 0.347  | 0.352  | 0.379 |
| p_hat700-2   | 714.9                         | 9.08           | 3.86   | 6.95   | 2.37   | 2.31  |
| p_hat700-3   | 247992                        | 4378           | 1651   | 3247   | 637.5  | 512.9 |
| p_hat1000-2  | 25694                         | 438.9          | 152.5  | 370.3  | 71.6   | 61.8  |
| san200_0.9_1 | 110.8                         | 0.491          | 0.152  | 0.186  | 0.078  | 0.077  |
| san200_0.9_2 | 62.8                          | 0.312          | 0.276  | 0.123  | 0.077  | 0.076  | 0.075 |
| san200_0.9_3 | 13.2                          | 0.078          | 0.030  | 0.109  | 0.102  | 0.101 |
| sanr200_0.7  | 130.9                         | 0.301          | 0.235  | 0.248  | 0.137  | 0.143  |
| sanr200_0.9  | 9586                          | 4.48           | 26.3   | 27.5   | 16.3   | 12.1   |
| sanr400_0.5  | 257                           | 0.704          | 0.588  | 0.378  | 0.430  | 0.440 |
| sanr400_0.7  | 54279                         | 215.1          | 124.9  | 180.2  | 59.4   | 61.0   | 61.1 |

Table 4: Comparison of the number of generated subproblems and running times between the implementations described in [15], [9], and [16] with RD₁, RD₂, and RD₃ on instances in the DIMACS benchmark. Execution times of [15], [9], and [16] are adjusted according to the respective factors listed in Table 1. The computation times for [16] are the smallest between those reported for BBMCI and BBMCR. A blank entry means “information not available.”

5. Concluding Remarks

In this paper, we propose a new Russian Dolls Search algorithm, improving another implementation by Östergård [15] in several directions as the use of approximate colorings for subproblems pruning, an effective use of bit-level parallelism, the application of an enhanced elimination rule. These improvements allow the algorithm to further reduce the running times of the faster previously published combinatorial algorithms in several instances. The computational results show the effectiveness of the combination of techniques employed in RD_MAXCLIQUE for hard instances (graphs with an high density). In particular, for graphs of density between 0.5 and 0.9, our algorithm is more than twice faster in almost all graphs tested. These results show that, for some combinatorial optimization problems, the Russian Dolls method can constitute a very interesting alternative to classical Branch and Bound approaches.
Table 5: Comparison of the number of generated subproblems and running times between the implementations described in [15], [9], and [16] with $RD_1$, $RD_2$, and $RD_3$ on randomly generated instances. Execution times of [15], [9] and [16] are adjusted according to the respective factors listed in Table 1. The computation times for [16] are the smallest between those reported for BBMCI and BBMCR. For each case, values presented are averages over 5 random instances. A blank entry means “information not available.”

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