Weak and strong error analysis for mean-field rank based particle approximations of one dimensional viscous scalar conservation law

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Abstract

In this paper, we analyse the rate of convergence of a system of $N$ interacting particles with mean-field rank based interaction in the drift coefficient and constant diffusion coefficient. We first adapt arguments by Kolli and Shkolnikhov [18] to check trajectori al propagation of chaos with optimal rate $N^{-1/2}$ to the associated stochastic differential equations nonlinear in the sense of McKean. We next relax the assumptions needed by Bossy [3] to check convergence in $L^1(\mathbb{R})$ with rate $O\left(\frac{1}{\sqrt{N}} + h\right)$ of the empirical cumulative distribution function of the Euler discretization with step $h$ of the particle system to the solution of a one dimensional viscous scalar conservation law. Last, we prove that the bias of this stochastic particle method behaves in $O\left(\frac{1}{N} + h\right)$. We provide numerical results which confirm our theoretical estimates.

Keywords: Propagation of chaos, mean-field interaction, rank-based model, weak-error analysis.

AMS Subject Classification (2010): 65C35, 65C30.

1 Introduction

The order of weak convergence in terms of the number $N$ of particles for the approximation of diffusions nonlinear in the sense of McKean solving

$$X_t = X_0 + \int_0^t \zeta(s, X_s, \mu_s) \, dW_s + \int_0^t \vartheta(s, X_s, \mu_s) \, ds$$

with $\mu_s$ denoting the probability distribution of $X_s$, by the systems of $N$ interacting particles

$$\dot{X}_t^i = X_0^i + \int_0^t \zeta(s, X_s^i, \mu_s^N) \, dW_t^i + \int_0^t \vartheta(s, X_s^i, \mu_s^N) \, ds, \quad i \in \{1, \ldots, N\}$$

has been recently investigated in several papers [19] [2] [5] [6]. Here $(W_t)_{t \geq 0}$ is a $d$-dimensional Brownian motion independent random vector $X_0$, $(W_t, X_0^i)_{i \geq 1}$ are i.i.d. copies of $(W, X_0)$, $\zeta : [0, T] \times \mathbb{R}^n \times \mathcal{P}(\mathbb{R}^n) \to \mathbb{R}^{n \times d}$ and $\vartheta : [0, T] \times \mathbb{R}^n \times \mathcal{P}(\mathbb{R}^n) \to \mathbb{R}^n$ with $\mathcal{P}(\mathbb{R}^n)$ denoting the space of Borel probability distributions on $\mathbb{R}^n$. Typically, under some regularity assumptions, the bias is of order $N^{-1}$ while it is well known since [22] that the strong error is of order $N^{-1/2}$. From a numerical perspective, this implies that simulating $N$ independent copies of the system with $N$ particles leads to a bias and a statistical error both of order $N^{-1}$ which is also the order of the global error resulting from one single simulation of the system with $N^2$ particles. When the computation time of the interaction is quadratic, then the cost of these $N$ copies is of order $N^3$ compared to the order $N^4$ of the computation cost of the system with $N^2$ particles.

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In Theorem 6.1 \cite{19}, Mischler, Mouhot and Wennberg prove that for $\zeta$ uniformly elliptic and not depending on the time and measure arguments, $\sup_{t \in [0, T]} \mathbb{E} \left[ \left| \varphi(X_{t}^{1, N}) - \int_{\mathbb{R}^n} \varphi(x) \mu_t(dx) \right| \right] \leq \frac{C}{N}$ when $\varphi$ is Lipschitz and has some Sobolev regularity and $\vartheta(t, x, \mu) = Ax + \int U(x - y) \mu(dy)$ for some constant matrix $A$ and some function $U$ with Sobolev regularity. In \cite{2}, we consider the case of interaction through moments: 

\[
\left( \frac{\zeta}{\theta} \right)(s, x, \mu) = \left( \frac{\sigma}{b} \right) \left( s, \int_{\mathbb{R}^n} \alpha(x) \mu(dx), x \right).
\]

When $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^p$, $\sigma : [0, T] \times \mathbb{R}^p \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d}$, $b : [0, T] \times \mathbb{R}^p \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ are twice continuously differentiable with bounded derivatives and Lipschitz second order derivatives and $\sigma \sigma^*$ is globally Lipschitz, we obtain

\[
\exists C < \infty, \forall h \in [0, T], \forall N \in \mathbb{N}^*, \sup_{t \in [0, T]} \mathbb{E} \left[ \varphi \left( X_{t}^{1, N, h} \right) - \int_{\mathbb{R}^n} \varphi(x) \mu_t(dx) \right] \leq C \left( \frac{1}{N} + h \right)
\]

where $\tilde{X}_{t}^{i, N, 0}$ denotes the particle system \cite{11} and $\tilde{X}_{t}^{i, N, h}$ its Euler discretization with step $h$ when $h > 0$. When $n = d$, in Theorem 2.17 \cite{5}, Chassagneux, Szpruch and Tse prove the expansion of the bias

\[
\mathbb{E} \left[ \Phi \left( \mu_T^N \right) - \Phi \left( \mu_T \right) \right] = \sum_{j=1}^{k-1} \frac{C_j}{N^j} + \mathcal{O} \left( \frac{1}{N^k} \right),
\]

for time-homogeneous coefficients $\zeta$ and $\vartheta$, $(2k + 1)$-times differentiable with respect to both the spatial coordinates and the probability measure argument (for the notion of lifted differentiability introduced by Lions in his lectures at the Collège de France) with $\zeta$ bounded and $X_0$ admitting a finite moment of order $2k + 1$. They assume the same regularity on the test function $\Phi$ on the space of probability measures on $\mathbb{R}^d$ which is possibly nonlinear: $\Phi(\mu)$ is not necessarily of the form $\int_{\mathbb{R}^d} \varphi(x) \mu(dx)$. In Theorem 3.6 \cite{6}, under uniform ellipticity, Chaudru de Raynal and Friikha prove $\mathbb{E} \left[ \left| \Phi \left( \mu_T^N \right) - \Phi \left( \mu_T \right) \right| \right] \leq \frac{C}{N}$ when $\Phi$ has two bounded and Hölder continuous linear functional derivatives and $\zeta \zeta^*$ and $\vartheta$ are bounded and globally Hölder continuous with respect to the spatial variables and have two bounded and Hölder continuous linear functional derivatives with respect to the measure argument. Notice that the existence of a linear functional derivative requires less regularity than the Fréchet differentiability of the lift since the lifted derivative is the gradient of the linear functional derivative with respect to the spatial variables.

Our aim in the present paper is to check that the $\mathcal{O} \left( \frac{1}{N} + h \right)$ behaviour of the weak error for the Euler discretization with step $h$ of the system with $N$ particles generalizes to a stochastic differential equation with an even discontinuous drift coefficient. This SDE is one-dimensional ($n = d = 1$) and has a constant diffusion coefficient $\zeta(s, x, \mu) = \sigma$ for $\sigma > 0$. The drift coefficient writes $\vartheta(s, x, \mu) = \lambda(\mu((-\infty, x]))$ where $\mathbb{R} \times \mathcal{P}(\mathbb{R}) \ni (x, \mu) \mapsto \mu((-\infty, x])$ is not even continuous and $\lambda$ is the derivative of a $C^1$ function $\Lambda : [0, 1] \rightarrow \mathbb{R}$:

\[
(X_t = X_0 + \sigma W_t + \int_0^t \lambda(F(s, X_s)) \, ds, \quad t \in [0, T])
\]

\[
F(s, x) = \mathbb{P}(X_s \leq x), \quad \forall(s, x) \in [0, T] \times \mathbb{R}.
\]

We denote by $\mu$ the probability distribution of $X_0$ and by $F_0$ its cumulative distribution function. According to Section 2 in the paper \cite{4} specialized to the case $\lambda(\mu) = u^2/2$ and Proposition 1.2 and Theorem 2.1 \cite{8} for a general function $\Lambda$, weak existence and uniqueness hold for the SDE \cite{12}. By \cite{23}, it actually admits a unique strong solution. For $t > 0$, by the Girsanov theorem, the law $\mu_t$ of $X_t$ admits a density $p(t, x)$ with respect to the Lebesgue measure (see Lemma 3.1 below). The function $p(t, x)$ is a weak solution to the Fokker-Planck equation $\partial_t p(t, x) + \partial_x \left( \lambda(F(t, x)) p(t, x) \right) = \frac{\sigma^2}{2} \partial_{xx} p(t, x)$. By integration with respect to the spatial variable $x$, we deduce that $F(t, x)$ is a weak solution to the following viscous conservation law:

\[
\begin{aligned}
\partial_t F(t, x) + \partial_x \left( \lambda(F(t, x)) \right) = \frac{\sigma^2}{2} \partial_{xx} F(t, x),
F_0(x) = m \left((-\infty, x] \right).
\end{aligned}
\]

The corresponding particle dynamics is

\[
\tilde{X}_{t}^{i, N} = X_0 + \sigma W_{t}^{i} + \int_0^t \lambda \left( \frac{1}{N} \sum_{j=1}^{N} \mathbb{1}_{\{X_{j}^{i, N} \leq X_{k}^{i, N} \}} \right) \, ds, \quad 1 \leq i \leq N, \quad t \in [0, T].
\]
As for the initial positions \((X^i_0, i \geq 1)\), we will consider both cases of the random initialization \((X^i_0 = X^i_0)\) i.i.d. according to \(m\) and an optimal deterministic initialization which will be made precise in Section 2. In fact, for \(1 \leq i \leq N\), the coefficient \(\lambda(i/N)\) is close to

\[
\lambda^N(i) = N\left(\Lambda\left(\frac{i}{N}\right) - \Lambda\left(\frac{i-1}{N}\right)\right)
\]

so that the dynamics is close (see Corollary 2.2 for a precise statement) to the one introduced in [12]:

\[
X^{i,N}_t = X^i_0 + \sigma W^i_t + \int_0^t \lambda^N \left( \sum_{j=1}^N 1_{\{X^{j,N}_s \leq X^{i,N}_s\}} \right) ds, \quad 1 \leq i \leq N, \ t \in [0, T].
\]

We denote by \(\mu^N_t = \frac{1}{N} \sum_{i=1}^N \delta_{X^{i,N}_t}\) the empirical measure and by \(F^N(t, x) = \frac{1}{N} \sum_{i=1}^N 1_{\{X^{i,N}_t \leq x\}}\) the empirical cumulative distribution function at time \(t\) of this second particle system. Both dynamics are so-called rank-based models since the drift (and the diffusion) coefficient only depend on the rank of the \(i\)-th particle in the system. We call them mean-field rank-based since the interaction between the particles is also of mean-field type. The ability of rank-based models to reproduce stylized empirical properties observed on stock markets [7] has motivated their mathematical study [1]. By the Girsanov theorem, the stochastic differential equations (1.4) and (1.6) admit a unique weak solution and, according to [24], they actually admit a unique strong solution. Under concavity of \(\Lambda\), Jourdain and Malrieu [12] prove propagation of chaos with optimal rate \(N^{-1/2}\) and study the long-time behaviour of the particle system (1.6) and its mean-field limit (1.2). For the particle system (1.4), this study is extended by Jourdain and Reygner [15] when the diffusion coefficient is no longer constant but also of mean-field rank-based type. For this more general model and without the concavity assumption, Kolli and Shkolnikhov [18] recently proved propagation of chaos with optimal rate \(N^{-1/2}\) and convergence of the associated fluctuations when the initial probability measure \(m\) admits a bounded density with respect to the Lebesgue measure. We choose to focus on the modified dynamics (1.6) because when \(y^1 < y^2 < \ldots < y^N\), then the distribution derivative of \(x \mapsto \Lambda\left(\frac{1}{N} \sum_{i=1}^N 1_{\{y^i \leq x\}}\right)\) is \(x \mapsto \frac{1}{N} \sum_{i=1}^N \lambda^N(i) 1_{\{y^i \leq x\}}\) and not (when \(\Lambda\) is not affine) \(x \mapsto \frac{1}{N} \sum_{i=1}^N \lambda(i/N) 1_{\{y^i \leq x\}}\). For this reason, it is more closely connected to the PDE (1.3). As our error analysis is based on a comparison of the mild formulation of the PDE (1.3) and the perturbed mild formulation satisfied by empirical cumulative distribution function of the Euler discretization of the particle system, we concentrate on (1.6), for which no extra error term appears in this perturbed version. But we will also explain how our results extend to (1.4). Let us also introduce the Euler discretization with time-step \(h \in (0, T]\) of (1.6):

\[
X^{i,N,h}_t = X^i_0 + \sigma W^i_t + \int_0^t \lambda^N \left( \sum_{j=1}^N 1_{\{X^{j,N,h}_s \leq X^{i,N,h}_s\}} \right) ds, \quad 1 \leq i \leq N, \ t \in [0, T] \text{ where } \tau^h_s = \lfloor s/h \rfloor h.
\]

The empirical cumulative distribution function of \(\mu^N_{0} = \frac{1}{N} \sum_{i=1}^N \delta_{X^{i,N}_0}\) is \(F^N_{0}(t, x) := \frac{1}{N} \sum_{i=1}^N 1_{\{X^{i,N}_0 \leq x\}}\). It is natural and convenient to consider that \(\tau^h_s = s\) and \((\bar{X}^{i,N,h}_t)_{t \in [0, T], 1 \leq i \leq N} = (\bar{X}^{i,N}_t)_{t \in [0, T], 1 \leq i \leq N}\). Using these notations, we then have by convention that \(F^{N,h}_{0}(t, x) = F^N(t, x)\). Moreover, we will refer to the empirical cumulative distribution function \(F^N_{0}(t, x) = F^N(t, x)\). Consequently, we will refer to the empirical cumulative distribution function \(F^N_{0}(t, x) = F^N(t, x)\) when choosing positions that are i.i.d. according to \(m\) and by \(F^N_{0}(t, x) = F^N(t, x)\) when choosing optimal deterministic initial positions. Finally let us define \((\tilde{X}^{i,N,h}_t)_{t \in [0, T], 1 \leq i \leq N}\) like \((\bar{X}^{i,N}_t)_{t \in [0, T], 1 \leq i \leq N}\) by replacing \(\lambda^N(k)\) by \(\lambda(k/N)\) in (1.7) and set \(\tilde{\mu}^N_{t} = \frac{1}{N} \sum_{i=1}^N \delta_{\tilde{X}^{i,N,h}_t}\) and \(\tilde{F}^{N,h}_{0}(t, x) := \frac{1}{N} \sum_{i=1}^N 1_{\{\tilde{X}^{i,N,h}_0 \leq x\}}\). The paper is organized as follows. In Section 2, we state our results. Taking advantage of the constant diffusion coefficient to adapt arguments given in [18], we obtain propagation of chaos with optimal rate \(N^{-1/2}\).
for the non time-discretized particle systems (1.6) and (1.4) without any assumption on the initial probability measure \( m \). Then we state that the strong rate of convergence of \( \mu_t^{N,h} \) to \( \mu_t \) for the Wasserstein distance with index one (or equivalently of \( F^{N,h}(t,\cdot) \) to \( F(t,\cdot) \) for the \( L^1 \) norm) is \( O\left(\frac{1}{\sqrt{h}} \right) \), a result already obtained long ago by Bossy \([3]\) under more regularity assumptions on the initial probability measure \( \Lambda \). Our main result is that the weak rate of convergence is \( O\left(\frac{1}{h} \right) \). In Section 3, we introduce the reordered particle system and establish the mild formulation of the PDE (1.3) satisfied by \( F(t,x) \) and the perturbed version satisfied by \( F^{N,h}(t,x) \). Section 4 is dedicated to the proofs of the results in Section 2. In Section 5, we study the initial error for both the random and the optimal deterministic initializations. We finally provide numerical experiments in Section 6 to illustrate our results. Beforehand, we introduce some additional notation.

**Notation:**

- We denote by \( L_\Lambda = \sup_{u \in [0,1]} |\lambda(u)| \) the Lipschitz constant of \( \Lambda \). When \( \lambda \) is also assumed to be Lipschitz continuous, we denote similarly its Lipschitz constant by \( L_\lambda \).
- For \( 1 \leq p < \infty \), we denote by \( L^p(\mathbb{R}) \) the space of measurable real valued functions which are \( L^p \)-integrable for the Lebesgue measure i.e. \( f \in L^p \) if \( \| f \|_{L^p} = \left( \int_{\mathbb{R}} |f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty \).
- The space \( L^\infty(\mathbb{R}) \) refers to the space of almost everywhere bounded measurable real valued functions endowed with the norm \( \| f \|_{L^\infty} = \inf\{ C \geq 0 : |f(x)| \leq C \text{ for almost every } x \in \mathbb{R} \} \).
- We denote the positive part of \( y \in \mathbb{R} \) by \( y^+ = \max(y,0) \).
- We denote by \( \Gamma \) the Gamma function defined by \( \Gamma(x) = \int_0^{+\infty} y^{x-1} \exp(-y) \, dy \) for \( x \in (0,\infty) \).
- For notational simplicity, when a function \( g \) defined on \( [0,T] \times \mathbb{R} \) and \( x \in \mathbb{R} \), we may use sometimes the notation \( g_0(x) := g(0,x) \).

## 2 Main results

Kolli and Shkolnikov \([18]\) prove a quantitative propagation of chaos result at optimal rate \( N^{-1/2} \) and convergence of the associated fluctuations for the particle system without time-discretization in the much more general and difficult case when the diffusion coefficient is also mean-field rank based. Taking advantage of the constancy of the diffusion coefficient, we are going to relax their assumptions on \( \lambda \) and \( m \) to prove the following result.

**Theorem 2.1.** Let the initial positions \( X_0^i \) be i.i.d. according to \( m \) and \( \left( X_t^i \right)_{t \geq 0} \) denote the solution to the stochastic differential equation nonlinear in the sense of McKean \([1,2]\) starting from \( X_0^i \) and driven by \( \left( W_t^i \right)_{t \geq 0} \). If \( \lambda \) is Lipschitz continuous, then

\[
\sup_{t \in [0,T]} \left| X_t^i - X_t^{i,N} \right|^\rho + \sup_{t \in [0,T]} \left| X_t^i - \bar{X}_t^i,N \right|^\rho \leq C N^{-\rho/2}.
\]

The estimation \( \sup_{t \in [0,T]} \left| X_t^i - \bar{X}_t^{i,N} \right|^\rho \leq C N^{-\rho/2} \) follows from Theorem 1.6 \([18]\) when \( \lambda \) is differentiable with an Hölder continuous derivative and \( m \) has a bounded density w.r.t. the Lebesgue measure and a finite moment of order \( 2 + \varepsilon \) for some \( \varepsilon > 0 \). An immediate consequence of Theorem 2.1 is to quantify the proximity of the two particles dynamics (1.4) and (1.6).

**Corollary 2.2.** Assume that the initial positions \( X_0^i \) are i.i.d. according to \( m \) and that \( \lambda \) is Lipschitz continuous. Then:

\[
\sup_{t \in [0,T]} \left| X_t^{i,N} - \bar{X}_t^{i,N} \right|^\rho \leq C N^{-\rho/2}.
\]
In the remaining of this section, we give the main results concerning the convergence of the empirical cumulative distribution function $F^{N,h}_{1}$ of the Euler discretization with time-step $h$ of the system with $N$ interacting particles towards its limit $F$. We will make an intensive use of the interpretation of the $L^1$-norm of their difference as the Wasserstein distance with index 1 between $\mu_{1}^{N,h} = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_{i}^{N,h}}$ and the law $\mu_{t}$ of $X_{t}$.

The Wasserstein distance of index $\rho \geq 1$ between two probability measures $\mu$ and $\nu$ on $\mathbb{R}^d$ is defined by
\[
\mathcal{W}_{\rho}^{\mu}(\mu, \nu) = \inf \{ \mathbb{E} \|X - Y\|^\rho ; \text{Law}(X) = \mu, \text{Law}(Y) = \nu \}.
\]

In dimension $d = 1$, the Hoeffding-Fréchet or comonotone coupling given by the inverse transform sampling is optimal:
\[
\mathcal{W}_{1}^{\mu}(\mu, \nu) = \int_{\mathbb{R}} |F_{\mu}(u) - F_{\nu}(u)| \, du.
\]

We will also take advantage of the dual formulation of the $\mathcal{W}_{1}$ distance which holds whatever $d \in \mathbb{N}^*$:
\[
\mathcal{W}_{1}^{\mu}(\mu, \nu) = \sup_{\varphi \in \mathcal{L}} \left( \int_{\mathbb{R}^d} \varphi(x) \mu(dx) - \int_{\mathbb{R}^d} \varphi(x) \nu(dx) \right)
\]
where $\mathcal{L}$ denotes the set of all 1-Lipschitz function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$.

The initial positions $(X_{0}^{i})_{i \geq 1}$ of the particles are either deterministic or random variables.

- When choosing a random initialization, we denote by $\hat{F}_{0}^{N}(x) = \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{\{X_{0}^{i} \leq x\}}$ and $\hat{\mu}_{0}^{N} = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_{0}^{i}}$ the empirical cumulative distribution function and the empirical measure of the $N$ first random variables in the sequence $(X_{0}^{i})_{i \geq 1}$ i.i.d. according to $m$.

- When choosing a deterministic initialization, we seek to construct a family $x_{1}^{N} \leq x_{2}^{N} \leq \ldots \leq x_{N}^{N}$ of initial positions minimizing the $L^1$ norm of the difference between the piecewise constant function $\hat{F}_{0}^{N}(x) = \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{\{x_{i}^{N} \leq x\}}$ and $F_{0}$. According to (2.1),
\[
\int_{\mathbb{R}} |\hat{F}_{0}^{N}(x) - F_{0}(x)| \, dx = \sum_{i=1}^{N} \int_{x_{i}^{N}}^{x_{i+1}^{N}} |x_{i}^{N} - F_{0}^{-1}(u)| \, du.
\]

Since, as remarked in [14], for $i \in [1, N]$, $y \mapsto N \int_{x_{i}^{N}}^{y} \|y - F_{0}^{-1}(u)\| \, du$ is minimal for $y$ equal to the median $F_{0}^{-1}(\frac{2i-1}{2N})$ of the image of the uniform law on $[\frac{i-1}{N}, \frac{i}{N}]$ by $F_{0}^{-1}$, we choose $x_{i}^{N} = F_{0}^{-1}(\frac{2i-1}{2N})$.

We denote by $\hat{\mu}_{0}^{N} = \frac{1}{N} \sum_{i=1}^{N} \delta_{F_{0}^{-1}(\frac{2i-1}{2N})}$ the associated empirical measure.

The next proposition, proved in Section 5.2 discusses assumptions under which the $L^1$-norm of the difference between $F_{0}$ and $\hat{F}_{0}^{N}$ or $\bar{F}_{0}^{N}$ is of order $N^{-1/2}$. 


Proposition 2.3. We denote for simplicity \( \int \lambda |x|^{2+\varepsilon} m(dx) < \infty \) and \( \int |x|^{2-\varepsilon} m(dx) < \infty \) the fact that \( \int |x|^{2-\varepsilon} m(dx) < \infty \) for each \( \varepsilon \in (0, 2] \). We have the following results concerning the \( O(N^{-1/2}) \) behaviour of the errors:

\[
\sup_{N \geq 1} \frac{\sqrt{N}}{N} \mathbb{E} \left[ W_1 \left( \hat{\mu}^N_1, m \right) \right] < \infty
\]

\[
\int |x|^{2+\varepsilon} m(dx) < \infty \quad \Rightarrow \quad \int \sqrt{F_0(x)(1 - F_0(x))} \, dx < \infty \quad \Rightarrow \quad \int |x|^{2-\varepsilon} m(dx) < \infty
\]

\[
\sup_{x \geq 1} x \int_x^{\infty} (F_0(y) + 1 - F_0(y)) \, dy < \infty \quad \Rightarrow \quad \int |x|^{2-\varepsilon} m(dx) < \infty
\]

\[
\sup_{N \geq 1} \frac{\sqrt{N}}{N} \mathbb{E} \left[ W_1 \left( \hat{\mu}^N_1, m \right) \right] < \infty
\]

Moreover, none of the implications is an equivalence and there exists a probability measure \( m \) such that \( \int |x|^{2-\varepsilon} m(dx) < \infty \) and \( \lim_{N \to \infty} \sqrt{N} W_1 \left( \hat{\mu}^N_1, m \right) = \infty \).

Concerning the weak error, since the empirical cumulative distribution function of i.i.d. samples is unbiased, \( \mathbb{E} \left[ \hat{F}_0^N(x) \right] = F_0(x) \) for all \( N \geq 1 \) and \( x \in \mathbb{R} \) and

\[
\int \mathbb{E} \left[ \hat{F}_0^N(x) - F_0(x) \right] \, dx = 0.
\]

As for the deterministic initialization, we have that:

\[
\int_\mathbb{R} \left| \hat{F}_0^N(x) - F_0(x) \right| \, dx = \int_{-\infty}^{F_0^{-1}(\frac{1}{N})} F_0(x) \, dx + \sum_{i=1}^{N-1} \int_{F_0^{-1}(\frac{i}{N})}^{F_0^{-1}(\frac{i+1}{N})} \left| F_0(x) - \frac{i}{N} \right| \, dx + \int_{F_0^{-1}(\frac{N}{N})}^{+\infty} (1 - F_0(x)) \, dx
\]

where the integrand is not greater than \( 1/2N \). When \( \exists - \infty < c \leq d < \infty \) such that \( m((c, d]) = 1 \), the integrand vanishes outside the interval \([c, d]\). One then easily deduces the next proposition proved in [14] by using the alternative formulation:

\[
\int_\mathbb{R} \left| \hat{F}_0^N(x) - F_0(x) \right| \, dx = \int_{0}^{1} \left( \hat{F}_0^N \right)^{-1}(u) - F_0^{-1}(u) \right| \, du = \sum_{i=1}^{N} \int_{\frac{i}{N}}^{\frac{i+1}{N}} \left( F_0^{-1}(u) - F_0^{-1} \left( u - \frac{1}{2N} \right) \right) \, du.
\]

Proposition 2.4. When \( m \) is compactly supported i.e. \( \exists - \infty < c \leq d < \infty \) such that \( m((c, d]) = 1 \), then

\[
W_1 \left( \hat{\mu}^N_0, m \right) \leq \frac{d - c}{2N}.
\]

Let us now state our estimation of the strong error which is proved in Section 4.2

Theorem 2.5. Assume that for some \( \rho > 1 \), \( \int |x|^\rho m(dx) < \infty \) and assume either that the initial positions are optimal deterministic or the initial positions are i.i.d. according to \( m \). Then

\[
\exists C < \infty, \forall N \in \mathbb{N}^*, \sup_{t \leq T} \mathbb{E} \left[ W_1 \left( \mu_i^{N,0}, \mu_i \right) \right] \leq C \left( \mathbb{E} \left[ W_1 \left( \mu_i^N, m \right) \right] + \frac{1}{\sqrt{N}} \right).
\]

Moreover, if \( \lambda \) is Lipschitz continuous then:

\[
\exists C < \infty, \forall N \in \mathbb{N}^*, \forall h \in (0, T], \sup_{t \leq T} \mathbb{E} \left[ W_1 \left( \mu_i^{N,h}, \mu_i \right) \right] \leq C \left( \mathbb{E} \left[ W_1 \left( \mu_i^N, m \right) \right] + \frac{1}{\sqrt{N}} + h \right).
\]
Combining the theorem with Proposition 2.8, we have the following corollary:

**Corollary 2.6.** Assume that the initial positions are

- either i.i.d. according to \( m \) and
  \[
  \int_{\mathbb{R}} \sqrt{F_0(x)(1 - F_0(x))} \, dx < \infty,
  \]

- or optimal deterministic and
  \[
  \sup_{x \geq 1} x \int_{x}^{+\infty} (F_0(-y) + 1 - F_0(y)) \, dy < \infty.
  \]

Then:

\[
\exists C < \infty, \forall N \in \mathbb{N}^*, \sup_{t \leq T} \mathbb{E} \left[ W_1 \left( \mu_{t}^{N,0}, \mu_t \right) \right] \leq \frac{C}{\sqrt{N}}.
\]

Moreover, if \( \lambda \) is Lipschitz continuous then:

\[
\exists C < \infty, \forall N \in \mathbb{N}^*, \forall h \in (0, T], \sup_{t \leq T} \mathbb{E} \left[ W_1 \left( \mu_{t}^{N,h}, \mu_t \right) \right] \leq C \left( \frac{1}{\sqrt{N}} + h \right).
\]

Let us now state our main result, proved in Section 4.3, concerning the weak error: the \( L^1 \)-weak error between the empirical cumulative distribution function \( F_{N,h} \) of the Euler discretization with time-step \( h \) of the system with \( N \) interacting particles and its limit \( F \) is \( O \left( \frac{1}{N} + h \right) \). We denote by \( \mathbb{E} \left[ \mu_{t}^{N,h} \right] \) the probability measure on \( \mathbb{R} \) defined by

\[
\int_{\mathbb{R}} \varphi(x) \mathbb{E} \left[ \mu_{t}^{N,h} \right] (dx) = \mathbb{E} \left[ \int_{\mathbb{R}} \varphi(x) \mu_{t}^{N,h}(dx) \right] = \mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^{N} \varphi(X_{i,t}^{N,h}) \right]
\]

for each \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \) measurable and bounded. The cumulative distribution function of \( \mathbb{E} \left[ \mu_{t}^{N,h} \right] \) is equal to \( \mathbb{E} [F_{N,h}(t, x)] \) and \( \mathbb{W}_1 \left( \mathbb{E} \left[ \mu_{t}^{N,h} \right], \mu_t \right) = \int_{\mathbb{R}} \mathbb{E} \left[ F_{N,h}(t, x) - F(t, x) \right] \, dx \).

**Theorem 2.7.** Assume that \( \lambda \) is Lipschitz continuous and the initial positions are

- either i.i.d. according to \( m \) and
  \[
  \int_{\mathbb{R}} \sqrt{F_0(x)(1 - F_0(x))} \, dx < \infty,
  \]

- or optimal deterministic and
  \[
  \sup_{x \geq 1} x \int_{x}^{+\infty} (F_0(-y) + 1 - F_0(y)) \, dy < \infty.
  \]

Then:

\[
\exists C_b < \infty, \forall N \in \mathbb{N}^*, \forall h \in [0, T], \sup_{t \leq T} \mathbb{W}_1 \left( \mathbb{E} \left[ \mu_{t}^{N,h} \right], \mu_t \right) \leq C_b \left( \mathbb{W}_1 \left( \mathbb{E} \left[ \mu_{0}^{N} \right], m \right) + \left( \frac{1}{N} + h \right) \right).
\]

Combining the theorem with (2.4) and Proposition 2.4, we obtain the following corollary:

**Corollary 2.8.** Assume that \( \lambda \) is Lipschitz continuous and the initial positions are

- either i.i.d. according to \( m \) and
  \[
  \int_{\mathbb{R}} \sqrt{F_0(x)(1 - F_0(x))} \, dx < \infty,
  \]

- or optimal deterministic with \( m \) compactly supported.

Then:

\[
\exists C_b < \infty, \forall N \in \mathbb{N}^*, \forall h \in [0, T], \sup_{t \leq T} \mathbb{W}_1 \left( \mathbb{E} \left[ \mu_{t}^{N,h} \right], \mu_t \right) \leq C_b \left( \frac{1}{N} + h \right).
\]

Using the dual formulation (2.4) of the Wasserstein distance, we deduce that if \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \) is Lipschitz continuous with constant \( \text{Lip}(\varphi) \) then

\[
\forall N \in \mathbb{N}^*, \forall t, h \in [0, T], \left| \mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^{N} \varphi(X_{i,t}^{N,h}) \right] - \mathbb{E} [\varphi(X_t)] \right| \leq C_b \text{Lip}(\varphi) \left( \frac{1}{N} + h \right).
\]
Remark 2.9. For the dynamics (1.3) with initial positions deterministic and given by $x_i^N = F_0^{-1}(\frac{i}{N})$ when $i = 1, \ldots, N-1$ and $x_N^N = F_0^{-1}(1 - \frac{1}{N})$, Bossy [3] proved an estimation also dealing with the supremum of the expected error between $\hat{F}^{N,h}(t, x)$ and $F(t, x)$ similar to the last statement in Corollary 2.6.

$\exists C < \infty, \forall N \in \mathbb{N}^*, \forall h \in (0, T),$ \begin{align*}
\sup_{t \leq T} \mathbb{E} \left[ W_1 \left( \mu^{N,h}_{t}, \mu_{t} \right) \right] + \sup_{(t,x) \in [0,T] \times \mathbb{R}} \mathbb{E} \left[ |\hat{F}^{N,h}(t, x) - F(t, x)| \right] \leq C \left( \frac{1}{\sqrt{N}} + h \right).
\end{align*}$

She assumes additional regularity on the coefficient $\Lambda$, namely that $\Lambda$ is $C^3$, and on the initial measure $\mu$, namely that $F_0$ is $C^2$ bounded with bounded first and second order derivatives in $x$ and that $\exists M, \beta > 0, \alpha \geq 0$ such that $|\partial_x F_0(x)| \leq \alpha \exp \left( -\beta x^2/2 \right)$ when $|x| > M$. Her proof is based on the regularity of the backward Kolmogorov PDE associated with the generator of the diffusion (1.3). By contrast, our approach is based on a comparison of the mild formulation of the forward in time PDE (1.2) satisfied by $F(t, x)$ and the perturbed mild formulation satisfied by $F^{N,h}(t, x)$. In fact, all the above results hold with $\mu^{N,h}_{t}$ replaced by $\hat{\mu}^{N,h}_{t}$. For those concerning $\sup_{t \leq T} \mathbb{E} \left[ W_1 \left( \mu^{N,h}_{t}, \mu_{t} \right) \right]$, we just need to add the assumption that $\lambda$ is Hölder continuous with exponent $1/2$ to ensure that $\sup_{i \leq N} \sqrt{N} |\lambda'(i) - \lambda(i/N)| < \infty$. Notice that under Lipschitz continuity of $\lambda$, we even get $\sup_{1 \leq i \leq N} N |\lambda'(i) - \lambda(i/N)| < \infty$. See Remark 4.4 below, where we outline how to adapt the proofs.

3 Dynamics of the reordered particle system and mild formulations

The reordering of mean-field rank based particle systems without time discretization has been first introduced in [10] and has proved to be a very useful tool in the study of the limit $N \to \infty$ with vanishing viscosity (the parameter $\sigma$ depends on $N$ and tends to 0 as $N \to \infty$) [11, 13] (the latter when the driving Brownian motions are replaced by symmetric $\alpha$-stable Lévy processes with $\alpha > 1$), the long time behaviour of both the particle system and its mean-field limit [13] and the small noise limit $\sigma \to 0$ of the particle system [10]. Before deriving the dynamics of the reordering of the Euler discretization (1.7), let us check the existence of the density $p(t, x)$ of $X_t$ for $t > 0$, which guarantees that, in the sense of distributions, $\partial_x A(F(t, x)) = \lambda(F(t, x)) p(t, x)$ so that $F(t, x)$ is a weak solution of the viscous scalar conservation law (1.3).

Lemma 3.1. For $t > 0$, $X_t$ admits a density $p(t, x)$ with respect to the Lebesgue measure.

Proof. We recall (1.2): $X_t = X_0 + \sigma W_t + \int_0^t \lambda(F(s, X_s)) \, ds$. The Brownian motion $(W_t)_{t \geq 0}$ is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(\mathcal{F}_t)_{t \geq 0}$ denote the usual augmentation of the natural filtration with respect to $(W_t)_{t \geq 0}$, and let $\mathbb{Q}$ be the measure equivalent to $\mathbb{P}$ defined, using the boundedness of $\lambda$, by: $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = \exp \left\{ -\int_0^t \lambda(F(s, X_s)) \, dw_s - \frac{1}{2\sigma^2} \int_0^t \lambda^2(F(s, X_s)) \, ds \right\}$. Then by Girsanov’s Theorem, $(\frac{1}{\sigma} (X_t - X_0))_{t \geq 0}$ is a $\mathbb{Q}$-Brownian motion independent of $X_0$. This means that for any measurable and bounded function $g$, we have:

$$\mathbb{E}[g(X_t)] = \mathbb{E} \left[ g(X_0 + \sigma W_t) \exp \left\{ -\int_0^t \lambda(F(s, X_0 + \sigma W_s)) \, ds - \frac{1}{2\sigma^2} \int_0^t \lambda^2(F(s, X_0 + \sigma W_s)) \, ds \right\} \right].$$

Let $A$ be a Borel set of null Lebesgue measure. We choose $g \equiv 1_A$ and $t > 0$. With the above equation and since $X_0 + \sigma W_t$ has a density w.r.t. the Lebesgue measure, we have $\mathbb{P}(X_t \in A) = 0$. Therefore, $X_t$ admits a density w.r.t. the Lebesgue measure.

Let for each $t \geq 0$, $\eta_t$ be a permutation of $\{1, \ldots, N\}$ such that $X_{t}^{\eta(1),N,h} \leq X_{t}^{\eta(2),N,h} \leq \ldots \leq X_{t}^{\eta(N),N,h}$ and $Y_{t}^{i,N,h} = X_{t}^{\eta(i),N,h}$ denote the increasing reordering also called order statistics of $(X_{t}^{i,N,h})_{i \in \{1, N\}}$. Even if the empirical measures of the reordered and original positions do not coincide in general at the level of sample-paths, one has, for each $t \geq 0$, $\frac{1}{N} \sum_{i=1}^{N} \delta_{X_{t}^{i,N,h}} = \frac{1}{N} \sum_{i=1}^{N} \delta_{Y_{t}^{i,N,h}}$ and therefore
\[ F^{N,h}(t, x) = \frac{1}{N} \sum_{i=1}^{N} 1\{Y_{i^{*},N,h} \leq x\}. \]

By the Girsanov Theorem, reasoning like in the proof of Lemma 3.1, we show that for \( t > 0 \), the vector \( X_{t}^{1,N,h}, X_{t}^{2,N,h}, \ldots, X_{t}^{N,N,h} \) admits a density with respect to the Lebesgue measure on \( \mathbb{R}^N \) and therefore

\[ \forall t > 0, \text{a.s.}, \text{the original (resp. reordered) particles have distinct positions}. \]

(3.1)

We are going to check that the function \( F(t, x) \) solves a mild formulation of the PDE (1.3) and \( F^{N,h}(t, x) \) solves a perturbed version of this mild formulation. To do so, it is convenient to obtain the dynamics of the reordered positions \( Y_{t}^{i,N,h} \). Let \( \tau^h_s = [s/h]h \) denote the discretization time right after \( s \). We recall that \( \eta_k = [s/h]h \) denotes the discretization time right before \( s \). We set \( t_k = kh \) for \( k \in \mathbb{N} \). For \( s \in (t_k, t_{k+1}) \), \( \tau^h_s = t_k \) and, for \( s \in (t_k, t_{k+1}) \), \( \tau^h_s = t_{k+1} \). For \( t > 0 \), let \( \eta^{-1}_t \) denote the inverse of the permutation \( \eta \). By (3.1), a.s., for each \( k \in \mathbb{N}^* \), the positions \( \{X_{t_k}^{i,N,h}\}_{1 \leq i \leq N} \) are distinct and for \( t \) in the time-interval \([t_k, t_{k+1})\), \( X_{t_k}^{i,N,h} \) evolves with the drift coefficient \( \lambda^N(\eta^{-1}_k(i)) = \lambda^N(\eta^{-1}_k(i)) \). To obtain the same expression of the drift coefficient on the first time interval \([0, t_1)\) we will use from now on the convention

\[ \eta^{-1}_0(i) = \sum_{j=1}^{N} 1\{i \leq j\} \quad \text{for} \quad 1 \leq i \leq N. \]

(3.2)

With this convention, which is consistent with the usual definition of the inverse of a permutation only if the initial positions are distinct, we have

\[ dX_{t}^{i,N,h} = \sigma dW_t^i + \lambda^N(\eta^{-1}_t(i)) \, dt, \quad 1 \leq i \leq N. \]

By Girsanov theorem, we may define a new probability measure equivalent to the original one on each finite time horizon under which the processes \( \{X_{t}^{i,N,h} - X_{0}^{i} = W_t^i + \int_0^t \lambda^N(\eta^{-1}_s(i)) \, ds\}_{t \geq 0, 1 \leq i \leq N} \) are independent Brownian motions. Applying Lemma 3.7 [22], which states that under this probability measure, the reordered positions evolve as a \( N \)-dimensional Brownian motion normally reflected at the boundary of the simplex, we deduce that

\[ dY_{t}^{i,N,h} = \sum_{j=1}^{N} 1\{Y_{t}^{i*,N,h} = X_{t}^{j,N,h}\} \left( \sigma dW_t^j + \lambda^N(\eta^{-1}_t(j)) \, dt \right) + (\gamma^i_t - \gamma^{i+1}_t) \, d|K|, \quad 1 \leq i \leq N \]

where the process \( K \) with coordinates \( K_t^i = \int_0^t (\gamma^i_s - \gamma^{i+1}_s) \, d|K|_s \) is an \( \mathbb{R}^N \)-valued continuous process with finite variation \( |K| \) such that:

(3.3)

\[ d|K|_s \text{ a.e.}, \quad \gamma^1_t = \gamma^{N+1}_t = 0 \text{ and for } 2 \leq i \leq N, \quad \gamma^i_t \geq 0 \text{ and } \gamma^i_t (Y_{t}^{i,N,h} - Y_{t}^{i-1,N,h}) = 0. \]

Defining a \( N \)-dimensional Brownian motion \( \{\beta^1, \ldots, \beta^N\} \) by \( \beta^i_t = \sum_{j=1}^{N} \int_0^t 1\{Y_{s}^{j,N,h} = X_{s}^{j,N,h}\} \, dW_s^j \) and using the definition of \( \eta \) and (3.1), we have

(3.4)

\[ dY_{t}^{i,N,h} = \sigma d\beta^i_t + \lambda^N(\eta^{-1}_t(i)) \, dt + (\gamma^i_t - \gamma^{i+1}_t) \, d|K|, \quad 1 \leq i \leq N. \]

Denoting by \( G_t(x) = \exp\left(-\frac{x^2}{2 \sigma^2 t}\right) \sqrt{2\pi \sigma^2 t} \) the probability density function of the normal law \( \mathcal{N}(0, \sigma^2 t) \), we are now ready to state the mild formulation of the PDE (1.3) satisfied by \( F(t, x) \) and the perturbed version satisfied by \( F^{N,h}(t, x) \).
Proposition 3.2. For each $t \geq 0$ and each $h \in [0, T]$ , we have $dx$ almost everywhere:

\begin{equation}
F(t, x) = G_t * F_0(x) - \int_0^t \partial_x G_{t-s} * \Lambda(F(s, \cdot))(x) \, ds,
\end{equation}

a.s. $F^{N,h}(t, x) = G_t * F_0^{N,h}(x) - \int_0^t \partial_x G_{t-s} * \Lambda(F^{N,h}(s, \cdot))(x) \, ds - \frac{\sigma}{\sqrt{N}} \sum_{i=1}^N \int_0^t G_{t-s}(X_s^{i,N,h} - x) \, dW_s^i$

\begin{equation}
+ \frac{1}{N} \sum_{i=1}^N \int_0^t G_{t-s}(Y_s^{i,N,h} - x) \left[ \lambda^N(i) - \lambda \left( \frac{1}{\sqrt{N}} \eta_i^{-1}(\eta_s(i)) \right) \right] \, ds.
\end{equation}

Remark 3.3. When $h = 0$, one should notice that the third term in $F^{N,h}(t, x)$ is null so that:

$F^{N,0}(t, x) = G_t * F_0^{N,0}(x) - \int_0^t \partial_x G_{t-s} * \Lambda(F^{N,0}(s, \cdot))(x) \, ds - \frac{\sigma}{\sqrt{N}} \sum_{i=1}^N \int_0^t G_{t-s}(X_s^{i,N,0} - x) \, dW_s^i$.

Remark 3.4. Let similarly for each $t \geq 0$, $\bar{\eta}_t$ be a permutation of $\{1, \ldots, N\}$ such that $\bar{X}_t^{\bar{\eta}_t(1),N,h} \leq \bar{X}_t^{\bar{\eta}_t(2),N,h} \leq \ldots \leq \bar{X}_t^{\bar{\eta}_t(N),N,h}$ and $(\bar{Y}_t^{i,N,h} = \bar{X}_t^{\bar{\eta}_t(i),N,h})_{i \in \{1,N\}}$. Let also $\bar{\eta}_t^{-1}$ denote the inverse of the permutation $\bar{\eta}_t$ for $t > 0$ and $\bar{\eta}_0^{-1} = \eta_0^{-1}$. Reasoning like in the proof of Proposition 3.2, we may derive the perturbed mild equation satisfied by the associated empirical cumulative distribution function $F^{N,h}(t, x)$:

$F^{N,h}(t, x) = G_t * F_0^{N}(x) - \int_0^t \partial_x G_{t-s} * \Lambda(F^{N,h}(s, \cdot))(x) \, ds - \frac{\sigma}{\sqrt{N}} \sum_{i=1}^N \int_0^t G_{t-s}(X_s^{i,N,h} - x) \, dW_s^i$

\begin{equation}
+ \frac{1}{N} \sum_{i=1}^N \int_0^t G_{t-s}(Y_s^{i,N,h} - x) \left[ \lambda^N(i) - \lambda \left( \frac{1}{\sqrt{N}} \eta_i^{-1}(\eta_s(i))/N \right) \right] \, ds.
\end{equation}

Using the estimation

$\int_\mathbb{R} \frac{1}{N} \sum_{i=1}^N \int_0^t G_{t-s}(Y_s^{i,N,h} - x) \left[ \lambda^N\left( \frac{1}{\sqrt{N}} \eta_i^{-1}(\eta_s(i)) \right) - \lambda \left( \frac{1}{\sqrt{N}} \eta_i^{-1}(\eta_s(i))/N \right) \right] \, dx \, ds \leq t \max_{1 \leq j \leq N} |\lambda^N(j) - \lambda(j/N)|$

of the additional error term in comparison with (3.9), we may adapt all proofs to check the statements at the end of Remark 3.2.

Proof. Let $t > 0$, $f$ be a $C^1$ and compactly supported function on $\mathbb{R}$ and $\varphi(s, x) = \int_\mathbb{R} 1_{\{x \leq y\}} G_{t-s} * f(y) \, dy$ be the convolution of $G_{t-s}$ with $x \mapsto \int_x^{+\infty} f(y) \, dy$ for $(s, x) \in [0, t) \times \mathbb{R}$ and $\varphi(t, x) = \int_\mathbb{R} 1_{\{x \leq y\}} f(y) \, dy$. The function $\varphi(s, x)$ is continuously differentiable w.r.t. to $s$ and twice continuously differentiable w.r.t. to $x$ on $[0, t] \times \mathbb{R}$ and solves

\begin{equation}
\partial_s \varphi(s, x) + \frac{\sigma^2}{2} \partial_{xx} \varphi(s, x) = 0 \quad \text{for} \quad (s, x) \in [0, t] \times \mathbb{R}.
\end{equation}

Computing $\varphi(t, X_t)$ where $(X_s)_{s \geq 0}$ solves (1.2) and using (3.7), we obtain that:

$\varphi(t, X_t) = \varphi(0, X_0) - \sigma \int_0^t G_{t-s} * f(X_s) \, dW_s - \int_0^t \lambda(F(s, X_s)) G_{t-s} * f(X_s) \, ds$

Since, on $[0, t] \times \mathbb{R}$, $G_{t-s} * f(x)$ is bounded by the supremum of $|f|$, the expectation of the stochastic integral is zero. By Fubini’s theorem and since $G_t$ is even, the expectations of $\varphi(t, X_t)$ and $\varphi(0, X_0)$ are respectively equal to $\int_\mathbb{R} \int_\mathbb{R} 1_{\{x \leq y\}} \mu(dx)f(y) \, dy = \int_\mathbb{R} F(t, y)f(y) \, dy$ and $\int_\mathbb{R} F_0(y)G_t * f(y) \, dy = \int_\mathbb{R} G_t * F_0(y)f(y) \, dy$. 

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Using Fubini’s theorem, the equality \( G_{t-s} \ast f(x) = -\int_\mathbb{R} \mathbf{1}_{\{x \leq y\}} \partial_y G_{t-s} \ast f(y) \, dy \), the fact that, by the chain rule for continuous functions with finite variation, \( \int_\mathbb{R} \mathbf{1}_{\{x \leq y\}} \lambda(F(s,x)) p(s,x) \, dx = \Lambda(F(s,y)) - \Lambda(0) \), the equality \( \int_\mathbb{R} \partial_y G_{t-s} \ast f(y) \, dy = 0 \) and the oddness of \( \partial_y G_{t-s} \), we obtain that the expectation of the last term in the right-hand side is equal to

\[
\int_0^t \int_\mathbb{R} \mathbf{1}_{\{x \leq y\}} \lambda(F(s,x)) p(s,x) \, dx \partial_y G_{t-s} \ast f(y) \, dy \, ds = \int_0^t \int_\mathbb{R} (\Lambda(F(s,y)) - \Lambda(0)) \partial_y G_{t-s} \ast f(y) \, dy \, ds
\]

\[= - \int_0^t \partial_y G_{t-s} \ast \Lambda(F(s,.))(y) f(y) \, dy \, ds.
\]

Exchanging the time and space integrals by Fubini’s theorem, we deduce that

\[
\int_\mathbb{R} F(t,x)f(x)dx = \int_\mathbb{R} G_t \ast F_0(x)f(x) \, dx - \int_\mathbb{R} f(x) \int_0^t \partial_x G_t \ast \Lambda(F(s,.))(x) \, ds \, dx.
\]

Since \( f \) is arbitrary, we conclude that \( F \) satisfies the mild formulation (3.5).

Let us now establish that \( F^{N,h} \) satisfies a perturbed version of this equation. By computing \( \varphi(t, Y_t^{i,N,h}) \) by Itô’s formula, using (3.7) and summing over \( i \in \{1, \ldots, N\} \), we obtain

\[
(3.8) \quad \sum_{i=1}^N \int_{[Y_t^{i,N,h} \leq y]} f(y) \, dy = \sum_{i=1}^N \int_{[Y_t^{i-1,N,h} \leq y]} G_t \ast f(y) \, dy - \sum_{i=1}^N \int_0^t G_{t-s} \ast f(Y_s^{i,N,h}) \left( \sigma d\beta_s^i + \lambda^N \left( \eta_s^{-1}(\eta_s(i)) \right) \right) ds
\]

\[+ \sum_{i=1}^N \int_0^t \partial_x \varphi(s, Y_s^{i,N,h})(\gamma^i_s - \gamma^{i+1}_s) d\|K\|_s.
\]

By summation by parts and (3.3),

\[
\sum_{i=1}^N \int_0^t \partial_x \varphi(s, Y_s^{i,N,h})(\gamma^i_s - \gamma^{i+1}_s) d\|K\|_s = \sum_{i=2}^N \int_0^t (\partial_x \varphi(s, Y_s^{i-1,N,h}) - \partial_x \varphi(s, Y_s^{i,N,h})) \gamma^i_s d\|K\|_s = 0.
\]

Since the empirical cumulative distribution functions of the original and the reordered systems at time \( t \) coincide and the function \( G_t \) is even, the left-hand side and the first term in the right-hand side are respectively equal to \( N \int_\mathbb{R} F^{N,h}(t,y) f(y) \, dy \) and \( N \int_\mathbb{R} G_t \ast F^{N,h}(0,y) f(y) \, dy \). The definition of the Brownian motion \( \beta \) and (3.1) imply that

\[
\sum_{i=1}^N \int_0^t G_{t-s} \ast f(Y_s^{i,N,h}) \, ds = \sum_{j=1}^N \int_0^t G_{t-s} \ast f(X_s^{j,N,h}) \, dW_s^j.
\]

Dividing (3.8) by \( N \), we deduce that

\[
\int_\mathbb{R} F^{N,h}(t,y) f(y) \, dy = \int_\mathbb{R} G_t \ast F^{N,h}(0,y) f(y) \, dy - \frac{\sigma}{N} \sum_{i=1}^N \int_0^t G_{t-s} \ast f(X_s^{i,N,h}) \, dW_s^i
\]

\[\quad - \frac{1}{N} \sum_{i=1}^N \int_0^t G_{t-s} \ast f(Y_s^{i,N,h}) \lambda^N \left( \eta_s^{-1}(\eta_s(i)) \right) \, ds.
\]

We are going to add and subtract

\[
\frac{1}{N} \sum_{i=1}^N G_{t-s} \ast f(Y_s^{i,N,h}) \lambda^N(i) = - \int_\mathbb{R} \partial_y G_{t-s} \ast f(y) \sum_{i=1}^N \mathbf{1}_{\{Y_s^{i,N,h} \leq y\}} (\Lambda(i/N) - \Lambda((i-1)/N)) \, dy
\]

\[= - \int_\mathbb{R} \partial_y G_{t-s} \ast f(y) (\Lambda(F^{N,h}(s,y)) - \Lambda(0)) \, dy = \int_\mathbb{R} f(y) \partial_y G_{t-s} \ast \Lambda \left( F^{N,h}(s,.)(y) \right) \, dy.
\]
On the other hand, since \( f \) is square integrable and with the use of Young’s inequality for the product and the estimate \((A.7)\) from Lemma \(A.2\), we have that:

\[
\int_{\mathbb{R}} \left\{ \int_{0}^{t} \left| G_{t-s}(X_{s}^{i,N,h} - x)f(x) \right|^2 \, ds \right\}^{1/2} \, dx = \int_{\mathbb{R}} \left| f(x) \right| \left\{ \int_{0}^{t} \left| G_{t-s}(X_{s}^{i,N,h} - x) \right|^2 \, ds \right\}^{1/2} \, dx
\]

\[
\leq \frac{1}{2} \int_{\mathbb{R}} f^2(x) \, dx + \frac{1}{2} \int_{\mathbb{R}} \int_{0}^{t} G_{t-s}^2(X_{s}^{i,N,h} - x) \, ds \, dx
\]

\[
= \frac{1}{2} \int_{\mathbb{R}} f^2(x) \, dx + \frac{1}{2\sigma} \sqrt{\frac{t}{\pi}} < \infty.
\]

For that reason, we can use a stochastic Fubini theorem stated by Veraar \[23\] and recalled in Lemma \(A.1\) to deduce that \( \frac{\sigma^N}{N} \sum_{i=1}^{N} \int_{0}^{t} G_{t-s} \ast f(X_{s}^{i,N,h}) \, dW_{s} = \frac{\sigma^N}{N} \sum_{i=1}^{N} \int_{\mathbb{R}} f(x) \left\{ \int_{0}^{t} G_{t-s}(X_{s}^{i,N,h} - x) \, dW_{s} \right\} \, dx \).

Therefore

\[
\int_{\mathbb{R}} F^{N,h}(t,x) f(x) \, dx = \int_{\mathbb{R}} G_t \ast F^{N,h}(0,x) f(x) \, dx - \frac{\sigma^N}{N} \sum_{i=1}^{N} \int_{\mathbb{R}} f(x) \left\{ \int_{0}^{t} G_{t-s}(X_{s}^{i,N,h} - x) \, dW_{s} \right\} \, dx
\]

\[
- \int_{\mathbb{R}} f(x) \partial_x G_{t-s} \ast \Lambda \left( F^{N,h}(s,.) \right) (x) \, dx \, ds
\]

\[
+ \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{t} G_{t-s}(Y_{s}^{i,N,h} - x) f(x) \, dx \left\{ \lambda^N(i) - \lambda^N(\eta_{s}^{-1}(\eta_s(i))) \right\} \, ds.
\]

Since \( f \) is bounded and \( \Lambda \) is bounded on the interval \([0,1]\), using \((A.4)\), we check that we can apply Fubini’s theorem to interchange the space and time integrals in the two last terms of the right-hand side. Since \( f \) is arbitrary, we conclude that \((3.6)\) holds a.s. \( dx \) a.e..

\[\Box\]

### 4 Proofs of Section 2

#### 4.1 Quantitative propagation of chaos result

The proof of Theorem \(2.1\) relies on the following Lemma which estimates for \( t > 0 \) the \( L^\infty \)-norm of the density \( p(t,x) \) of \( X_t \) solution to \((1.2)\) which exists according to Lemma \(3.1\).

**Lemma 4.1.**

\[\forall T \in (0, +\infty), \exists C_{\infty,T} < \infty, \forall t \in (0, T], \quad \|p(t,.)\|_{L^\infty} \leq C_{\infty,T} t^{-1/2}.\]

**Proof.** Reasoning like at the beginning of the proof of Proposition \(3.2\) but with the function \( \varphi(s,x) \) equal to \( G_{t-s} \ast f \) and not its antiderivative, we easily check that \( p(t,x) \) satisfies the mild formulation:

\[
\forall t > 0, \; dx \text{ a.e.,} \quad p(t,x) = G_t \ast m(x) - \int_{0}^{t} \partial_x G_{t-s} \ast (\lambda(F(s,.))p(s,.)) (x) \, ds.
\]

Since for \( t > 0 \), \( \|G_t \ast m\|_{L^\infty} \leq \|G_t\|_{L^\infty} = (2\pi\sigma^2 t)^{-1/2} \), it is enough to check that the estimation holds for the time integral in the mild formulation. By Jensen’s inequality then \((A.3)\) and \((A.6)\), \((4.1)\) implies that, for \( t > 0 \),

\[
\|p(t,.)\|_{L^2} \leq \|G_t\|_{L^2} + \int_{0}^{t} \|\partial_x G_{t-s}\|_{L^2} \|\lambda(F(s,.))p(s,.))\|_{L^1} \, ds \leq \frac{1}{\sqrt{2\sigma(\pi t)^{1/4}}} + \frac{2L\lambda^{1/4}}{\sigma^{3/2}(\pi t)^{1/4}}
\]

\[12\]
where the right-hand side is not greater than \( \left( \frac{1}{\sqrt{2\pi(t)}} + \frac{2\lambda T^{1/2}}{\sigma^3/2\pi t^{-1/4}} \right) t^{-1/4} \) for \( t \in (0, T] \). With the boundedness of \( \lambda \) and Young’s inequality for convolutions, we deduce that for \( t \in (0, T] \),

\[
\|p(t, \cdot)\|_{L^\infty} \leq \|G_t\|_{L^\infty} + L^t_0 \|\partial_s G_{t-s}\|_{L^2} \|\lambda(F(s, \cdot))p(s, \cdot)\|_{L^2} ds \\
\leq (2\pi\sigma^2t)^{-1/2} + \frac{L}{2\sigma^3/2\pi t^{-1/4}} \left( \frac{1}{\sqrt{2\sigma T^{1/4}}} + \frac{2L\lambda T^{1/2}}{\sigma^3/2\pi t^{-1/4}} \right) \int_0^t ds (t-s)^{3/4}s^{-1/4}.
\]

Since \( \int_0^t ds (t-s)^{3/4}s^{-1/4} ds = \int_0^1 \frac{du}{(1-u)^3/4u^{1/4}} \leq \int_0^1 \frac{du}{(1-u)^3/4u^{1/4}} T^{1/2} s^{-1/2} \), we easily conclude. 

We are now ready to prove Theorem 2.1 by adapting the proof of Theorem 1.6 [18]. Since, by Jensen’s inequality, the conclusion with \( \rho = 1 \) implies the conclusion with \( \rho \in (0, 1) \), we suppose without loss of generality that \( \rho \geq 1 \). Lemma 1 implies the following estimation of the Lipschitz constant of \( x \mapsto \lambda(F(t, x)) \):

\[
(4.2) \quad \forall t \in (0, T], \quad L_{\lambda,F(t,\cdot)} \leq C_{\infty,T}L^{-1/2}.\]

We deduce that for a finite constant \( C \) changing from line to line and depends on \( T \) but not on \( N \):

\[
sup_{s \in [0,t]} \|X^i_s - \tilde{X}^{i,N}_s\|^\rho \leq \left( \int_0^t \lambda(F(u, X^i_u)) - \lambda(F^N(u, \tilde{X}^{i,N}_u)) \right)^{\rho} du \\
= \left( \int_0^t u^{-\rho/2} \left( \lambda(F(u, X^i_u)) - \lambda(F^N(u, \tilde{X}^{i,N}_u)) \right) du \right)^{\rho} \\
\leq \left( \int_0^t u^{-1/2} du \right)^{\rho-1} \int_0^t u^{(\rho-1)/2} \left( \lambda(F(u, X^i_u)) - \lambda(F(u, \tilde{X}^{i,N}_u)) \right) du \\
\leq C \int_0^t \left( u^{-1/2} \|X^i_u - \tilde{X}^{i,N}_u\|^\rho + u^{(\rho-1)/2} \|\lambda(F(u, X^i_u)) - \lambda(F^N(u, \tilde{X}^{i,N}_u))\|^\rho \right) du,
\]

where we used H"older’s inequality for the second inequality. Using exchangeability of \( (\tilde{X}^{1,N}, \ldots, \tilde{X}^{N,N}) \), denoting by \( \tilde{Y}^1_N \leq \tilde{Y}^{2,N} \leq \ldots \leq \tilde{Y}^N_N \) (resp. \( \tilde{Y}^1_u \leq \tilde{Y}^{2,u} \leq \ldots \leq \tilde{Y}^N_u \)) the increasing reordering of \( (\tilde{X}^{1,N}, \ldots, \tilde{X}^{N,N}) \) (resp. \( (\tilde{X}^{1}_u, \ldots, \tilde{X}^{N}_u) \)) and using that \( \|\tilde{X}^i - \tilde{X}^j\|^\rho \) and its proof generalizes to the particle system [4] then (4.2), we obtain that

\[
\mathbb{E} \left[ \left| \lambda(F(u, \tilde{X}^{i,N}_u)) - \lambda(F^N(u, \tilde{X}^{i,N}_u)) \right|^\rho \right] = \mathbb{E} \left[ \frac{1}{N} \sum_{j=1}^N \left| \lambda(F(u, \tilde{Y}^j_N)) - \lambda(F^N(u, \tilde{Y}^j_N)) \right|^\rho \right] \\
= \mathbb{E} \left[ \frac{1}{N} \sum_{j=1}^N \left| \lambda(F(u, \tilde{Y}^j_N)) - \lambda(F(u, \tilde{Y}^j_N)) \right|^\rho \right] \\
= \mathbb{E} \left[ \frac{1}{N} \sum_{j=1}^N \left| \lambda(F(u, \tilde{Y}^j_N)) - \lambda(F(u, \tilde{Y}^j_N)) \right|^\rho \right] \\
\leq C \left( u^{-\rho/2} \mathbb{E} \left[ \frac{1}{N} \sum_{j=1}^N \left| \tilde{Y}^j_N - \tilde{Y}^j_u \right|^\rho \right] + \mathbb{E} \left[ \frac{1}{N} \sum_{j=1}^N \left| \tilde{F}(u, \tilde{Y}^j_u) - \tilde{F}(u, \tilde{X}^j_u) \right|^\rho \right] \right).
\]

Since \( F(u, \tilde{Y}^1_u) \leq F(u, \tilde{Y}^2_u) \leq \ldots \leq F(u, \tilde{Y}^N_u) \) is the increasing reordering of the random variables \( (F(u, \tilde{X}^i_u))_{1 \leq i \leq N} \) which are i.i.d. according to the uniform law on \([0, 1]\), according to the proof of Theorem 1.6 [18], the second expectation in the right-hand side is bounded from above by \( CN^{-\rho/2} \). On the other hand, by (2.1),

\[
\frac{1}{N} \sum_{j=1}^N \left| \tilde{X}^i - \tilde{X}^j \right|^\rho = W^\rho \left( \sum_{i=1}^N \delta_{\tilde{X}^i - \tilde{X}^j} \right) \leq \frac{1}{N} \sum_{i=1}^N \left| X^i - \tilde{X}^i \right|^\rho. \]

We deduce that for all \( t \in [0, T] \),

\[
\mathbb{E} \left[ \sup_{s \in [0,t]} \|X^{i}_s - \tilde{X}^{i,N}_s\|^\rho \right] \leq C \left( N^{-\rho/2} + \int_0^t u^{-1/2} du \right) \mathbb{E} \left[ \left( X^i - \tilde{X}^i \right)^\rho \right].
\]
Performing the change of variable \( v = \sqrt{t} \) in the integral and setting \( f(t) = E \left[ \sup_{s \in [0,t]} |X_s^t - \tilde{X}_s^N|^\rho \right] \), we deduce that \( \forall t \in \left[ 0, \sqrt{T} \right], f(t) \leq C \left( N^{-\rho/2} + \int_0^t f(v) dv \right) \). Since, by boundedness of \( \lambda \), the function \( f \) is locally bounded, we conclude using Gronwall’s lemma that \( E \left[ \sup_{s \in [0,t]} |X_s^t - \tilde{X}_s^N|^\rho \right] \leq CN^{-\rho/2} \).

Remarking that \( X_t^i - X_0^i - \sigma W_t^i = -\int_0^t \lambda (F^N(s, X_s^i)) \) \( ds \leq \frac{L_A}{2N} \), we may adapt the arguments to deal with the particle system (1.6).

### 4.2 Rate of convergence of the strong \( L^1 \)-error

To prove Theorem 2.3, we need the following lemmas.

#### Lemma 4.2.

\( \forall 0 \leq s \leq t \leq T, \forall \rho \geq 1, h \in [0, T], N \in \mathbb{N}^* \sum_{j=1}^N |Y_j^{j,N,h} - Y_j^{j,N,h}|^\rho \leq \sum_{j=1}^N |X_t^j - X_{j,N,h}|^\rho. \)

**Proof.** By (2.1), we have \( \mathcal{W}_1^\rho \left( \mu_t^{N,h}, \mu_t^{N,h} \right) = \frac{1}{N} \sum_{j=1}^N |Y_t^j - Y_t^{j,N,h}|^\rho. \) Since \( \frac{1}{N} \sum_{j=1}^N \delta_{(x_t^j - x_t^{j,N,h})} \) is a coupling measure on \( \mathbb{R}^2 \) with first marginal \( \mu_t^{N,h} \) and second marginal \( \mu_t^{N,h} \), we conclude that

\[
\frac{1}{N} \sum_{j=1}^N |Y_t^j - Y_t^{j,N,h}|^\rho \geq \mathcal{W}_1^\rho \left( \mu_t^{N,h}, \mu_t^{N,h} \right) = \frac{1}{N} \sum_{j=1}^N |Y_t^j - Y_t^{j,N,h}|^\rho.
\]

This second lemma ensures the local integrability of \( t \mapsto E \left[ \mathcal{W}_1 \left( \mu_t^{N,h}, \mu_t \right) \right]. \)

#### Lemma 4.3.

\( \forall t, h \in [0, T], \forall N \in \mathbb{N}^*, \quad E \left[ \mathcal{W}_1 \left( \mu_t^{N,h}, \mu_t \right) \right] \leq 2 \sigma \sqrt{\frac{2t}{\pi}} + 2 L_A t + E \left[ \mathcal{W}_1 \left( \mu_0^{N,h}, m \right) \right]. \)

If \( \int_{\mathbb{R}} |x| n(dx) < \infty \), then for each \( N \in \mathbb{N}^* \) and each \( h \in [0, T], t \mapsto E[\mathcal{W}_1(\mu_t^{N,h}, \mu_t)] \) is locally integrable on \( \mathbb{R}_+ \).

**Proof.** Using the triangle inequality, we have:

\[
E \left[ \mathcal{W}_1 \left( \mu_t^{N,h}, \mu_t \right) \right] \leq E \left[ \mathcal{W}_1 \left( \mu_t^{N,h}, \mu_0^{N,h} \right) \right] + E \left[ \mathcal{W}_1 \left( \mu_0^{N,h}, m \right) \right] + \mathcal{W}_1(m, \mu_t).
\]

Since

\[
E \left[ \mathcal{W}_1(\mu_t, m) \right] \leq E \left\| X_t - X_0 \right\| \leq E \left\| \sigma W_t \right\| + E \left\| \int_0^t \lambda(F(s, X_s)) ds \right\| \leq \sigma \sqrt{\frac{2t}{\pi}} + L_A t,
\]

\[
E \left[ \mathcal{W}_1(\mu_t^{N,h}, \mu_0^{N,h}) \right] \leq \frac{1}{N} \sum_{i=1}^N E \left[ |\sigma W_t^i + \int_0^t \lambda^N \left( 1 \{ |X_s^i - X_0^i| \leq X_t^{i,N,h} \} \right) ds \right] \leq \sigma \sqrt{\frac{2t}{\pi}} + L_A t.
\]

then by injecting (4.3) in (4.3), we obtain the upper-bound of \( E \left[ \mathcal{W}_1 \left( \mu_t^{N,h}, \mu_t \right) \right]. \) Moreover, by Lemma 5.1 below, the finiteness of the first order moment implies the finiteness of \( E \left[ \mathcal{W}_1 \left( \mu_0^{N,h}, m \right) \right] \) and therefore the local integrability of \( t \mapsto E \left[ \mathcal{W}_1 \left( \mu_t^{N,h}, \mu_t \right) \right]. \)

The third lemma gives a control of the moments of order \( \rho \geq 1 \) of \( X_t^{i,N,h}, \forall i \in [1, N] \).
Lemma 4.4. If $\int_{\mathbb{R}} |x|^\rho m(dx) < \infty$ for some $\rho \geq 1$, then $\forall N \in \mathbb{N}^*, \forall h \in [0,T]$, 
\[
\sup_{t \leq T} \mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^{N} |X_{t,i,N,h}^i|^\rho \right] \leq M := \left( 2 \left( \int_{\mathbb{R}} |x|^\rho m(dx) \right)^{1/\rho} + \sqrt{2\sigma^2 T} \left( \frac{1}{\sqrt{\pi}} \Gamma \left( \frac{\rho+1}{2} \right) \right)^{1/\rho} + L_A T \right)^{\rho}.
\]

Proof. By Minkowski’s inequality,
\[
\mathbb{E} \left[ |X_{t,i,N,h}^i|^\rho \right] \leq \left( \mathbb{E} \left[ \mathbb{E} \left[ |X_{0,i}^i|^\rho \right] \right]^{1/\rho} + \sigma \mathbb{E} \left[ |W_{t}^i|^\rho \right] \right)^{1/\rho} + \mathbb{E} \left[ \sum_{j=1}^{N} \mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^{N} 1_{\{X_{t,i,N,h}^i \leq X_{t,j,N,h}^j\}} \right] \right]^{\rho} \left( \frac{1}{\rho} \right)^{\rho}.
\]

Since $\mathbb{E} \left[ |W_{t}^i|^\rho \right]^{1/\rho} = \sqrt{2t} \left( \frac{1}{\sqrt{\pi}} \Gamma \left( \frac{\rho+1}{2} \right) \right)^{1/\rho}$, one easily concludes when the initial conditions are i.i.d. according to $m$. When they are optimal deterministic, we sum over $i \in \{1, \ldots, N\}$, divide by $N$ and use the second assertion in Lemma 3.10. \qed

4.2.1 Proof of Theorem 2.5

Defining for all $t, h \in [0,T], N \in \mathbb{N}^*$,
\[
R_{N,h}^n(t, x) = \frac{\sigma}{N} \sum_{i=1}^{N} \int_{0}^{t} G_{t-s}(X_{s,i,N,h}^i - x) dW_s^i,
\]
\[
E_{N,h}^n(t, x) = \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{t} G_{t-s}(Y_{s,i,N,h}^i - x) \left[ \lambda_N^N(i) - \lambda_N^N \left( \frac{1}{\rho} \sum_{s=1}^{\rho} (\eta_{x,h}^N) \right) \right] ds,
\]
we deduce from Proposition 3.2 that:
\[
F_{N,h}^n(t, x) = F(t, x) - \int_{0}^{t} \partial_x G_{t-s} \left( \Lambda(F_{N,h}^n(s, .)) - \Lambda(F(s, .)) \right) ds + R_{N,h}^n(t, x) + E_{N,h}^n(t, x).
\]

Using the triangle inequality and taking expectations, we deduce that:
\[
\mathbb{E} \left[ \|F_{N,h}^n(t, .) - F(t, .)\|_{L^1} \right] \leq \mathbb{E} \left[ \|G_{t} \ast \left( F_{0,N,h}^n - F_0 \right)(x) \|_{L^1} \right] + \mathbb{E} \left[ \|R_{N,h}^n(t, .)\|_{L^1} \right] + \mathbb{E} \left[ \|E_{N,h}^n(t, .)\|_{L^1} \right] + \mathbb{E} \left[ \int_{0}^{t} \partial_x G_{t-s} \left( \Lambda(F_{N,h}^n(s, .)) - \Lambda(F(s, .)) \right) ds \right] \|_{L^1} \right].
\]

Using the estimate (A.4) from Lemma A.2 and setting $A = \frac{L_A}{\sigma} \sqrt{\frac{2}{\pi}}$, we obtain:
\[
\mathbb{E} \left[ \int_{0}^{t} \partial_x G_{t-s} \left( \Lambda(F_{N,h}^n(s, .)) - \Lambda(F(s, .)) \right) ds \right] \leq \int_{0}^{t} \partial_x G_{t-s} \|_{L^1} L_A \mathbb{E} \left[ \|F_{N,h}^n(s, .) - F(s, .)\|_{L^1} \right] ds
\]
\[
= A \int_{0}^{t} \frac{1}{\sqrt{t-s}} \mathbb{E} \left[ \|F_{N,h}^n(s, .) - F(s, .)\|_{L^1} \right] ds.
\]

Therefore,
\[
\mathbb{E} \left[ \|F_{N,h}^n(t, .) - F(t, .)\|_{L^1} \right] \leq \mathbb{E} \left[ \left\| F_{0,N,h}^n - F_0 \right\|_{L^1} \right] + \mathbb{E} \left[ \left\| R_{N,h}^n(t, .)\right\|_{L^1} \right] + \mathbb{E} \left[ \left\| E_{N,h}^n(t, .)\right\|_{L^1} \right] + A \int_{0}^{t} \frac{1}{\sqrt{t-s}} \mathbb{E} \left[ \|F_{N,h}^n(s, .) - F(s, .)\|_{L^1} \right] ds.
\]

The next lemma states that the random variable $R_{N,h}^n(t, x)$ is centered and provides an upper-bound for $\mathbb{E} \left[ \left\| R_{N,h}^n(t, .)\right\|_{L^1} \right]$. 15
Lemma 4.5. We have \( \forall N \in \mathbb{N}^* \), \( \forall h, t \in [0, T] \), \( \| E \left[ R^{N,h}(t, \cdot) \right] \|_{L^1} = 0 \). Moreover, if for some \( \rho > 1 \),
\[
\int_{\mathbb{R}} |x|^\rho m(dx) < \infty,
\]
then:
\[
\exists R < \infty, \ \forall N \in \mathbb{N}^*, \ \forall h \in [0, T], \ \sup_{t \in [0, T]} E \left[ \| R^{N,h}(t, \cdot) \|_{L^1} \right] \leq \frac{R}{\sqrt{N}}.
\]

Proof. We have that
\[
\int_{\mathbb{R}} E \left[ \int_0^t G^2_{t-s}(X^{i,N,h}_s - x) \, ds \right] \, dx = E \left[ \int_0^t \int_0^t G^2_{t-s}(X^{i,N,h}_s - x) \, ds \, dx \right] \leq \frac{1}{\sqrt{t}} \int_0^t < \infty
\]
according to the estimate (A.7) from Lemma A.2. Therefore, \( E \left[ R^{N,h}(t, x) \right] = 0 \) dx a.e.. Moreover, denoting
\[
I_\rho = \int_{\mathbb{R}} \frac{dx}{1 + |x|^\rho}
\]
and using the Itô isometry for the first equality then Young’s inequality for the second inequality and last the estimate (A.7) from Lemma A.2 we obtain:

\[
E \left[ \| R^{N,h}(t, \cdot) \|_{L^1} \right] \leq \frac{R}{\sqrt{N}} \int_{\mathbb{R}} E^{1/2} \left[ \left( \frac{\rho}{N} \sum_{i=1}^N \int_0^t G_{t-s}(X^{i,N,h}_s - x) \, dW^i_s \right)^2 \right] \, dx
\]

\[
= \frac{\rho}{\sqrt{N}} \int_{\mathbb{R}} E^{1/2} \left[ \frac{\rho}{N} \sum_{i=1}^N \int_0^t G^2_{t-s}(X^{i,N,h}_s - x) \, ds \right] \, dx
\]

\[
= \frac{\rho}{\sqrt{N}} \int_{\mathbb{R}} E^{1/2} \left[ \frac{\rho}{N} \sum_{i=1}^N \int_0^t G^2_{t-s}(X^{i,N,h}_s - x) \, ds (1 + |x|^\rho) \right] \frac{dx}{\sqrt{1 + |x|^\rho}}
\]

\[
\leq \frac{\rho}{2 \sqrt{N}} \int_{\mathbb{R}} \left( \frac{1}{1 + |x|^\rho} + E \left[ \frac{\rho}{N} \sum_{i=1}^N \int_0^t G^2_{t-s}(X^{i,N,h}_s - x) \, ds (1 + |x|^\rho) \right] \right) \, dx
\]

\[
= \frac{\rho}{2 \sqrt{N}} \int_{\mathbb{R}} \frac{\rho}{N} \sum_{i=1}^N \int_0^t G^2_{t-s}(X^{i,N,h}_s - x) \, ds (1 + |x|^\rho) \, dx + \int_{\mathbb{R}} |x|^\rho G^2_{t-s}(X^{i,N,h}_s - x) \, dx \, ds
\]

\[
= \frac{\rho}{2 \sqrt{N}} \int_{\mathbb{R}} \frac{\rho}{N} \sum_{i=1}^N \int_0^t \frac{|X^{i,N,h}_s - y|^\rho}{\sigma} \left( \frac{t-s}{\sigma} \right) G_{t-s/2}(y) \, dy \, ds
\]

\[
= \frac{\rho}{2 \sqrt{N}} \int_{\mathbb{R}} \left( \frac{\rho}{N} \sum_{i=1}^N \int_0^t \frac{|X^{i,N,h}_s|^\rho}{\sigma} \left( \frac{t-s}{\sigma} \right) G_{t-s/2}(y) \, dy \, ds + \frac{\rho-1}{\sqrt{2\pi}} \Gamma \left( \frac{\rho+1}{2} \right) \int_0^t (t-s)^{\rho-1}/2 \, ds \right)
\]

With the use of Lemma 4.4 we conclude by setting
\[
R = \frac{1}{2} \left( \sigma I_\rho + \sqrt{\frac{t}{\pi}} \left( 1 + \frac{2^{\rho-1} \sigma^\rho}{\rho + 1} \left( \frac{\rho + 1}{2} \right) \sqrt{\frac{\pi}{\sigma}} \right) + \frac{2^{\rho-1}}{\sqrt{2\pi}} \int_0^t E \left[ \frac{1}{N} \sum_{i=1}^N \left| X^{i,N,h}_s \right|^\rho \right] \right)
\]

Therefore, Inequality (4.8) becomes:

\[
E \left[ \| F^{N,h}(t, \cdot) - F(t, \cdot) \|_{L^1} \right] \leq E \left[ \| F^{N,h}_0 - F_0 \|_{L^1} \right] + \frac{R}{\sqrt{N}} + E \left[ \| E^{N,h}(t, \cdot) \|_{L^1} \right] + A \int_0^t \frac{1}{\sqrt{t-s}} E \left[ \| F^{N,h}(s, \cdot) - F(s, \cdot) \|_{L^1} \right] \, ds.
\]

- One should notice that for \( h = 0 \), \( E^{N,0}(t, x) = 0 \ \forall t \in [0, T], N \in \mathbb{N}^*, x \in \mathbb{R} \). Therefore, to control the
from Equation (4.9) and Proposition 4.6, we have that:

\[
\mathbb{E} \left[ \| F^{N,0}(t, \cdot) - F(t, \cdot) \|_{L^1} \right] \leq \left( 2A\sqrt{t} + 1 \right) \left( \mathbb{E} \left[ \| F_0^{N,h} - F_0 \|_{L^1} \right] + \frac{R}{\sqrt{N}} \right) + A^2 \int_0^t \mathbb{E} \left[ \| F^{N,0}(r, \cdot) - F(r, \cdot) \|_{L^1} \right] \int_r^t \frac{ds}{\sqrt{t-s} \sqrt{s-r}} dr.
\]

Since \( \int_0^t \frac{ds}{\sqrt{t-s} \sqrt{s-r}} = \pi \) and with the use of Lemma 4.3, we can apply Gronwall’s lemma to deduce that

\[
\forall N \in \mathbb{N}^+, \sup_{t \leq T} \mathbb{E} \left[ W_1 \left( \mu_{t,0}^N, \mu_t \right) \right] \leq C \left( \mathbb{E} \left[ W_1 \left( \mu_0^N, m \right) \right] + \frac{1}{\sqrt{N}} \right) \text{ where } C = \max(1, R) \left( 2A\sqrt{T} + 1 \right) \exp \left( A^2\pi T \right).
\]

This concludes the proof of Theorem 2.5 when \( h = 0 \).

- When \( h > 0 \), we need to estimate \( \mathbb{E} \left[ \| E^{N,h}(t, \cdot) \|_{L^1} \right], h \in (0, T] \).

**Proposition 4.6.** We assume that for some \( \rho > 1 \), \( \int |x|^{\rho} m(dx) < \infty \) and that the function \( \lambda \) is Lipschitz continuous. Then \( \exists \Omega < \infty, \forall N \in \mathbb{N}^+, \forall \alpha \in (0, T], \forall t \in [0, T] \),

\[
\mathbb{E} \left[ \| E^{N,h}(t, \cdot) \|_{L^1} \right] \leq Z \left( \frac{1}{\sqrt{N}} + h \right) \mathbb{E} \left[ \| F^{N,h}(t, \cdot) - F(t, \cdot) \|_{L^1} \right] + \int_0^t \frac{1}{2\sqrt{t-s}} \mathbb{E} \left[ \| F(s, \cdot) - F^{N,h}(s, \cdot) \|_{L^1} \right] ds.
\]

Proposition 4.6 will be proved in Section 4.2.2.

From Equation (4.9) and Proposition 4.6, we have that:

\[
\left( 1 - \frac{Z}{\sqrt{N}} \right) \mathbb{E} \left[ \| F^{N,h}(t, \cdot) - F(t, \cdot) \|_{L^1} \right] \leq \mathbb{E} \left[ \| F_0^{N,h} - F_0 \|_{L^1} \right] + \frac{Z + R}{\sqrt{N}} + Z h + \left( A + \frac{Z}{2} \right) \int_0^t \frac{1}{2\sqrt{t-s}} \mathbb{E} \left[ \| F^{N,h}(s, \cdot) - F(s, \cdot) \|_{L^1} \right] ds.
\]

Hence, if we denote \( J = 2(Z + R) \) and \( K = 2A + Z \) then:

\[
2 \left( 1 - \frac{Z}{\sqrt{N}} \right) \mathbb{E} \left[ \| F^{N,h}(t, \cdot) - F(t, \cdot) \|_{L^1} \right] \leq \mathbb{E} \left[ \| F_0^{N,h} - F_0 \|_{L^1} \right] + J \left( \frac{1}{\sqrt{N}} + h \right) + \int_0^t \frac{K}{\sqrt{t-s}} \mathbb{E} \left[ \| F^{N,h}(s, \cdot) - F(s, \cdot) \|_{L^1} \right] ds.
\]

(4.10)

- When \( h \leq \frac{1}{12\pi} \), Equation (4.10) implies:

\[
\mathbb{E} \left[ \| F^{N,h}(t, \cdot) - F(t, \cdot) \|_{L^1} \right] \leq \mathbb{E} \left[ \| F_0^{N,h} - F_0 \|_{L^1} \right] + J \left( \frac{1}{\sqrt{N}} + h \right) + \int_0^t \frac{K}{\sqrt{t-s}} \mathbb{E} \left[ \| F^{N,h}(s, \cdot) - F(s, \cdot) \|_{L^1} \right] ds.
\]

We iterate this inequality to obtain:

\[
\mathbb{E} \left[ \| F^{N,h}(t, \cdot) - F(t, \cdot) \|_{L^1} \right] \leq \left( 1 + 2K\sqrt{T} \right) \left( \mathbb{E} \left[ \| F_0^{N,h} - F_0 \|_{L^1} \right] + J \left( \frac{1}{\sqrt{N}} + h \right) \right) + K^2\pi \int_0^t \mathbb{E} \left[ \| F^{N,h}(r, \cdot) - F^N(r, \cdot) \|_{L^1} \right] dr.
\]

With the use of Lemma 4.3, we can apply Gronwall’s Lemma and deduce that:

\[
\forall t \in [0, T], \quad \mathbb{E} \left[ \| F^{N,h}(t, \cdot) - F(t, \cdot) \|_{L^1} \right] \leq \left( 1 + 2K\sqrt{T} \right) \exp \left( K^2\pi t \right) \left( \mathbb{E} \left[ \| F_0^{N,h} - F_0 \|_{L^1} \right] + J \left( \frac{1}{\sqrt{N}} + h \right) \right).
\]

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• When $h > \frac{1}{4T}$, by Lemma 4.3 and 2.2,

$$
\mathbb{E} \left[ \left\| F^{N,h}(t,\cdot) - F(t,\cdot) \right\|_{L^1} \right] \leq 2\sigma \sqrt{\frac{2t}{\pi}} + 2L_N t + \mathbb{E} \left[ \left\| F^{N,h}_0 - F_0 \right\|_{L^1} \right]
$$

$$
\leq 4Z^2 h \left( 2\sigma \sqrt{\frac{2t}{\pi}} + 2L_N t \right) + \mathbb{E} \left[ \left\| F^{N,h}_0 - F_0 \right\|_{L^1} \right].
$$

We choose $C = \max \left( \max(1, J) \left( 1 + 2K \sqrt{T} \right) \exp \left( K^2 \pi T \right), 4Z^2 \left( 2\sigma \sqrt{\frac{2T}{\pi}} + 2L_N T \right) \right)$ and conclude that:

$$
\forall N \in \mathbb{N}^*, \forall h \in (0, T], \forall t \in [0, T], \sup_{t \leq T} \mathbb{E} \left[ W_1 \left( \mu_t^N, \mu_t \right) \right] \leq C \left( \mathbb{E} \left[ W_1 \left( \mu_0^N, m \right) \right] + \frac{1}{\sqrt{N}} + h \right).
$$

Let us now prove Proposition 4.6 in the following section.

### 4.2.2 Proof of Proposition 4.6

We recall the expression of $E^{N,h}(t, x)$:

$$
E^{N,h}(t, x) = \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{t} G_{t-s} \left( Y_{s,i,N,h} - x \right) \left[ \lambda^N(i) - \lambda^N \left( \eta^{-1}_{\tau^h_i}(\eta(i)) \right) \right] ds
$$

We do not know how to estimate the difference of values of $\lambda^N$ between the brackets. For $s > 0$, we are going to take advantage of the permutation $\eta^{-1}_{\tau^h_s} \circ \eta_{\tau^h_s}$ (because of the convention 3.2), this is not necessarily a permutation for $s = 0$ and $\eta^{-1}_{\tau^h_s} \circ \eta_{\tau^h_s}$ is equal to the identity permutation for $s \geq h$ but not necessarily for $s \in (0, h)$) to change indices and obtain the same value multiplied by a difference of values of the smooth function $G_{t-s}$. Using this permutation for the first equality then that $Y_{s,i_{\tau^h_s}(j),N,h} = X_{s,j,N,h}$ for $s > 0$ and $1 \leq j \leq N$ for the second one, we obtain that

$$
E^{N,h}(t, x) = \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{t} G_{t-s} \left( Y_{s,i,N,h} - x \right) \lambda^N(i) - G_{t-s} \left( Y_{s,i_{\tau^h_s}(j),N,h} - x \right) \lambda^N \left( \eta^{-1}_{\tau^h_s}(\eta(i)) \right) ds
$$

$$
= \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{t} \left\{ \left( G_{t-s} \left( Y_{s,i,N,h} - x \right) - G_{t-s} \left( X_{s,i_{\tau^h_s}(j),N,h} - x \right) \right) \right\} \lambda^N(i) ds,
$$

(4.11)

$$
+ \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{t \wedge h} G_{t-s} \left( X_{s,\eta_0(i),N,h} - x \right) \left( \lambda^N(i) - \lambda^N \left( \eta_{\tau^h_s}(\eta(i)) \right) \right) ds.
$$

Subtracting $G_{t-\tau^h_s} \left( Y_{s,i,N,h} - x \right) - G_{t-\tau^h_s} \left( X_{s,i_{\tau^h_s}(j),N,h} - x \right) = 0$ in the brace in the first term of the right-hand side makes apparent that this term is not too large since $\tau^h_s$ is close to $s$. Computing $G_{t-s} \left( Y_{s,i,N,h} - x \right) - G_{t-\tau^h_s} \left( Y_{s,i,N,h} - x \right)$ and $G_{t-s} \left( X_{s,i_{\tau^h_s}(j),N,h} - x \right) - G_{t-\tau^h_s} \left( X_{s,i_{\tau^h_s}(j),N,h} - x \right)$ by Itô’s formula, we obtain the following new expression of $E^{N,h}(t, x)$:

**Lemma 4.7.** The process $\tilde{\beta} = \left( \tilde{\beta}^1, \ldots, \tilde{\beta}^N \right)$ where $\tilde{\beta}^i_j = \sum_{j=1}^{N} \int_{0}^{t} 1(\eta_s(j) = j) dW_s^j$ is a $N$-dimensional Brownian motion and we can express $E^{N,h}(t, x)$ as $E^{N,h}(t, x) = \sum_{p=0}^{N} e^{N,h}_p(t, x)$ where:

- $e^{N,h}_0(t, x) = \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{t \wedge h} G_{t-s} \left( X_{s,\eta_0(i),N,h} - x \right) \left( \lambda^N(i) - \lambda^N \left( \eta_{\tau^h_s}(\eta(i)) \right) \right) ds,$
\[ e_1^{N,h}(t, x) = \frac{1}{N} \sum_{i=1}^{N} \int_0^t \left( t \wedge \tau_s^h - s \right) \left( \lambda^N(i) - \lambda^N(i-1) \right) \partial_x G_{t-s} \left( Y_s^{i,N,h} - x \right) \gamma_s^i \, d|K|_{s}, \]

\[ e_2^{N,h}(t, x) = \frac{\sigma}{N} \sum_{i=1}^{N} \int_0^t \left( t \wedge \tau_s^h - s \right) \lambda^N(i) \partial_{x^2} G_{t-s} \left( Y_s^{i,N,h} - x \right) \, d\beta_s^i, \]

\[ e_3^{N,h}(t, x) = -\frac{\sigma}{N} \sum_{i=1}^{N} \int_0^t \left( t \wedge \tau_s^h - s \right) \lambda^N(i) \partial_x G_{t-s} \left( X_s^{\eta^h(i),N,h} - x \right) \, d\beta_s^i, \]

\[ e_4^{N,h}(t, x) = \frac{1}{N} \sum_{i=1}^{N} \int_0^t \left( t \wedge \tau_s^h - s \right) \lambda^N(i) \lambda^N \left( \eta_s^{-1} \left( \eta_s^h(i) \right) \right) \partial_x G_{t-s} \left( Y_s^{i,N,h} - x \right) \, ds, \]

\[ e_5^{N,h}(t, x) = -\frac{1}{N} \sum_{i=1}^{N} \int_0^t \left( t \wedge \tau_s^h - s \right) \lambda^N(i) \lambda^N \left( \eta_s^{-1} \left( \eta_s^h(i) \right) \right) \partial_x G_{t-s} \left( X_s^{\eta^h(i),N,h} - x \right) \, ds. \]

Notice that in the definition of \( e_5^{N,h}(t, x) \), \( \lambda^N \left( \eta_s^{-1} \left( \eta_s^h(i) \right) \right) = \lambda^N(i) \) for \( s \geq h \), but because of the convention \( \lambda^{N,0} \equiv 0 \), this equality does not necessarily hold for \( s \in [0, h) \).

**Proof.** For \( 1 \leq i, k \leq N \) and \( t \geq 0 \), one has

\[ \left( \tilde{\beta}^i, \tilde{\beta}^k \right)_t = \sum_{j=1}^{N} \int_0^t 1_{\{ \eta_s^h(i) = j \}} 1_{\{ \eta_s^h(k) = j \}} \, ds = \int_0^t 1_{\{ \eta_s^h(i) = \eta_s^h(k) \}} \, ds = 1_{\{ i = k \}} t, \]

since \( \eta_s^h \) is a permutation for each \( s \geq 0 \). One deduces that \( \tilde{\beta} \) is a Brownian motion by applying Lévy’s characterization. By (4.11) and the equality \( G_{t-s} \left( Y_{\tau_s^h}^{i,N,h} - x \right) - G_{t-s} \left( Y_{\tau_s^h}^{i,N,h} - x \right) = 0 \), it is enough to check that

\[ \frac{1}{N} \sum_{i=1}^{N} \int_0^t \left( G_{t-s} \left( Y_{\tau_s^h}^{i,N,h} - x \right) - G_{t-s} \left( Y_{\tau_s^h}^{i,N,h} - x \right) \right) \lambda^N(i) \, ds = \sum_{p=1}^{5} e_p^{N,h}(t, x). \]

We are going to compute the two differences in the right-hand side by applying Itô’s formula. To do so, let us recall the dynamics of \( X_u^{\eta^h(i),N,h} \) for \( u \in [\tau_s^h, \tau_s^h] \):

\[ dX_u^{\eta^h(i),N,h} = \sigma d\tilde{\beta}^i_u + \lambda^N \left( \eta_s^{-1} \left( \eta_s^h(i) \right) \right) \eta_s^h \left( \eta_s^h(i) \right) \, du. \]

We then have:

\[ G_{t-s} \left( X_{\tau_s^h}^{\eta^h(i),N,h} - x \right) = G_{t-s} \left( X_{\tau_s^h}^{\eta^h(i),N,h} - x \right) + \sigma \int_{\tau_s^h}^{t} \partial_x G_{t-u} \left( X_u^{\eta^h(i),N,h} - x \right) \, d\tilde{\beta}^i_u + \frac{\sigma^2}{2} \partial_{xx} G_{t-u} \left( X_u^{\eta^h(i),N,h} - x \right) \, ds. \]

Since \( \partial_u G_{t-u} = -\partial_t G_{t-u} \), by the heat equation (A.1) from Lemma (A.2) we have:

\[ (4.12) \]

\[ \int_0^t G_{t-s} \left( X_u^{\eta^h(i),N,h} - x \right) - G_{t-s} \left( X_{\tau_s^h}^{\eta^h(i),N,h} - x \right) \, ds = \sigma \int_0^t \int_{\tau_s^h}^{t} \partial_x G_{t-u} \left( X_u^{\eta^h(i),N,h} - x \right) \, d\tilde{\beta}^i_u \, ds + \int_0^t \int_{\tau_s^h}^{t} \lambda^N \left( \eta_s^{-1} \left( \eta_s^h(i) \right) \right) \partial_x G_{t-u} \left( X_u^{\eta^h(i),N,h} - x \right) \, du \, ds. \]
Let us suppose that $t > 0$ and treat each term of the right-hand side of the above equation. For $x \in \mathbb{R}$, The function $u \mapsto \partial_x G_{t-u} \left( X_u^{\eta_{\tau}^{(i)},N,h} - x \right)$ is continuous on $[0, t)$. Since $\left( X_t^{1,N,h}, \ldots, X_t^{N,N,h} \right)$ admits a density, as stated after the proof of Lemma 3.1 $P \left( X_t^{\eta_{\tau}^{(i)},N,h} = x \right) \leq \sum_{j=1}^{N} P \left( X_t^{j,N,h} = x \right) = 0$ a.s.. Therefore, a.s. the previous function has a vanishing limit as $u \to t$ and is therefore bounded on the interval $[0, t]$. We can then apply Fubini’s theorem to obtain:

$$
\int_0^t \int_{\tau_u^h}^{s} \lambda^N \left( \eta_{\tau_u^{(1)}}^{1} (\tau_u^{(i)}) \right) \partial_x G_{t-u} \left( X_u^{\eta_{\tau}^{(i)},N,h} - x \right) du ds
$$

$$
= \int_0^t \left( t \wedge \tau_u^h - u \right) \lambda^N \left( \eta_{\tau_u^{(1)}}^{1} (\tau_u^{(i)}) \right) \partial_x G_{t-u} \left( X_u^{\eta_{\tau}^{(i)},N,h} - x \right) du \text{ a.s.}
$$

Secondly, with the use of Young’s inequality and the same arguments of density of $\left( X_t^{1,N,h}, \ldots, X_t^{N,N,h} \right)$, we get:

$$
\int_0^t \left( \int_{\tau_u^h}^{s} \left( X_u^{\eta_{\tau}^{(i)},N,h} - x \right)^2 \right) \frac{du}{2} \left( \frac{2\sigma^5 \sqrt{\pi} (t-u)^{5/2}}{G(t-u)^2} \right) \frac{1}{2} ds
$$

$$
\leq \frac{t}{2} + \frac{1}{2} \int_0^t \int_{\tau_u^h}^{s} \left( X_u^{\eta_{\tau}^{(i)},N,h} - x \right)^2 \frac{du}{2} \left( \frac{2\sigma^5 \sqrt{\pi} (t-u)^{5/2}}{G(t-u)^2} \right) \frac{1}{2} ds < \infty \text{ a.s.}
$$

Therefore, we can apply the stochastic Fubini Lemma 4.1 and obtain:

$$
\int_0^t \int_{\tau_u^h}^{s} \partial_x G_{t-u} \left( X_u^{\eta_{\tau}^{(i)},N,h} - x \right) ds = \int_0^t \left( t \wedge \tau_u^h - u \right) \partial_x G_{t-u} \left( X_u^{\eta_{\tau}^{(i)},N,h} - x \right) ds.
$$

Equation 4.12 becomes:

$$
(4.13)
$$

$$
\int_0^t \left( G_{t-s} \left( Y_u^{\eta_{\tau}^{(i)},N,h} - x \right) - G_{t-s} \left( Y_{\tau_u^h}^{\eta_{\tau}^{(i)},N,h} - x \right) \right) ds
$$

$$
= \sigma \int_0^t \left( t \wedge \tau_u^h - s \right) \partial_x G_{t-s} \left( X_u^{\eta_{\tau}^{(i)},N,h} - x \right) ds + \int_0^t \left( t \wedge \tau_u^h - s \right) \lambda^N \left( \eta_{\tau_u^{(1)}}^{1} (\tau_u^{(i)}) \right) \partial_x G_{t-s} \left( X_u^{\eta_{\tau}^{(i)},N,h} - x \right) ds.
$$

Now, let us apply Itô’s formula to $G_{t-s} \left( Y_u^{\eta_{\tau}^{(i)},N,h} - x \right)$. Using once again the property A.1 of the kernel $G_1(x)$ from Lemma A.2 and the dynamics of $Y_u^{\eta_{\tau}^{(i)},N,h}$ given by (3.3), we have:

$$
G_{t-s} \left( Y_u^{\eta_{\tau}^{(i)},N,h} - x \right) - G_{t-s} \left( Y_{\tau_u^h}^{\eta_{\tau}^{(i)},N,h} - x \right) = \sigma \int_{\tau_u^h}^{s} \partial_x G_{t-u} \left( Y_u^{\eta_{\tau}^{(i)},N,h} - x \right) d\beta_u^i
$$

$$
+ \int_{\tau_u^h}^{s} \partial_x G_{t-u} \left( Y_u^{\eta_{\tau}^{(i)},N,h} - x \right) \lambda^N \left( \eta_{\tau_u^{(1)}}^{1} (\tau_u^{(i)}) \right) d\eta_u^i + \int_{\tau_u^h}^{s} \partial_x G_{t-u} \left( Y_u^{\eta_{\tau}^{(i)},N,h} - x \right) \left( \gamma_u^i - \gamma_u^{i+1} \right) d[K U]u.
$$

We use the same reasoning as for $X_u^{\eta_{\tau}^{(i)},N,h}$ to treat the integrals from 0 to $t$ of the first two terms:

$$
\int_0^t \int_{\tau_u^h}^{s} \partial_x G_{t-u} \left( Y_u^{\eta_{\tau}^{(i)},N,h} - x \right) d\beta_u^i ds = \int_0^t \left( t \wedge \tau_u^h - s \right) \partial_x G_{t-s} \left( Y_u^{\eta_{\tau}^{(i)},N,h} - x \right) ds
$$

$$
\int_0^t \int_{\tau_u^h}^{s} \partial_x G_{t-u} \left( Y_u^{\eta_{\tau}^{(i)},N,h} - x \right) \lambda^N \left( \eta_{\tau_u^{(1)}}^{1} (\tau_u^{(i)}) \right) ds = \int_0^t \left( t \wedge \tau_u^h - s \right) \partial_x G_{t-s} \left( Y_u^{\eta_{\tau}^{(i)},N,h} - x \right) \lambda^N \left( \eta_{\tau_u^{(1)}}^{1} (\tau_u^{(i)}) \right) ds.
$$
As for the last term \( \int_0^t \int_{\tau_{i+1}}^\tau \partial_x G_{t-u} (Y_u^{i,N,h} - x) \left( \gamma_u^i - \gamma_{u+1}^i \right) d[K]_u ds \), we sum over \( i \in [1, N] \) after multiplying by \( \lambda^N (i) \) then we apply Fubini’s theorem. Using the property (3.3), we finally obtain:

\[
\frac{1}{N} \sum_{i=1}^N \lambda^N (i) \int_0^t \int_{\tau_{i+1}}^\tau \partial_x G_{t-u} (Y_u^{i,N,h} - x) \left( \gamma_u^i - \gamma_{u+1}^i \right) d[K]_u ds
\]

\[
= \frac{1}{N} \sum_{i=2}^N \int_0^t (t \wedge T_s^i - s) \left( \lambda^N (i) \partial_x G_{t-s} (Y_s^{i,N,h} - x) - \lambda^N (i-1) \partial_x G_{t-s} (Y_s^{i-1,N,h} - x) \right) \gamma_s^i d[K]_s
\]

\[
= \frac{1}{N} \sum_{i=2}^N \int_0^t (t \wedge T_s^i - s) \left( \lambda^N (i) - \lambda^N (i-1) \right) \partial_x G_{t-s} (Y_s^{i,N,h} - x) \gamma_s^i d[K]_s.
\]

Therefore,

\[
\frac{1}{N} \sum_{i=1}^N \int_0^t \lambda^N (i) \left( G_{t-s} (Y_s^{i,N,h} - x) - G_{t-\tau_{i+1}} \right) d[K]_s
\]

\[
= \frac{1}{N} \sum_{i=1}^N \lambda^N (i) \left\{ \int_0^t (t \wedge T_s^i - s) \partial_x G_{t-s} (Y_s^{i,N,h} - x) d\beta_s^i + \int_0^t (t \wedge T_s^i - s) \lambda^N (\eta^{-1}_{s+i} (\eta_s(i))) \partial_x G_{t-s} (Y_s^{i,N,h} - x) dK_s \right\}
\]

\[
+ \frac{1}{N} \sum_{i=2}^N \int_0^t (t \wedge T_s^i - s) \left( \lambda^N (i) - \lambda^N (i-1) \right) \partial_x G_{t-s} (Y_s^{i,N,h} - x) \gamma_s^i d[K]_s.
\]

We conclude by combining this equality and the sum over \( i \in [1, N] \) of (1.13) multiplied by \( \lambda^N (i)/N \). □

Now that we got rid of the difference of \( \lambda^N \) in the term \( E^{N,h}(t,x) \), we can control the mean of the \( L^1 \)-norm of this term. We present a succession of lemmas that will estimate each \( \mathbb{E} \left[ \left\| e^{N,h}(t,x) \right\|_{L^1} \right] \) for \( p \in [0, 5] \).

Since \( G_{t-s} \) is a probability density, \( \mathbb{E} \left[ \left\| e^{N,h}(t,x) \right\|_{L^1} \right] \leq \frac{1}{N} \sum_{i=1}^N \int_0^t \lambda^N (i) - \lambda^N (\eta^{-1}_{s+i} (\eta_s(i))) d\tau_s \). Therefore, we obtain the following result concerning the term \( e^{N,h}(t,x) \):

**Lemma 4.8.**

\[ \forall N \in \mathbb{N}^+, \forall h \in (0, T], \quad \sup_{t \leq T} \mathbb{E} \left[ \left\| e^{N,h}(t,x) \right\|_{L^1} \right] \leq \frac{1}{N} \sum_{i=1}^N \int_0^T \lambda^N (i) - \lambda^N (\eta^{-1}_{s+i} (\eta_s(i))) d\tau_s \leq 2L \lambda h. \]

We remark that the terms \( e^{4,N,h}(t,x) \) and \( e^{5,N,h}(t,x) \) are of the same nature.

**Lemma 4.9.** For \( r \in \{4, 5\} \):

\[ \exists C_{4,5} < \infty, \forall N \in \mathbb{N}^+, \forall h \in (0, T], \quad \sup_{t \leq T} \mathbb{E} \left[ \left\| e^{r,N,h}(t,x) \right\|_{L^1} \right] \leq \sup_{t \leq T} \mathbb{E} \left[ \left\| e^{N,h}(t,x) \right\|_{L^1} \right] \leq C_{4,5} h. \]

**Proof.** Let us treat the term \( e^{5,N,h}(t,x) \).

We have, using the estimate (A.2) from Lemma (A.2) for the second inequality, then the opposite monotonicities of the functions \( s \mapsto \frac{1}{\sqrt{t-s}} \) and \( s \mapsto (t \wedge t_{k+1} - s) \) on the time interval \([t \wedge t_k, t \wedge t_{k+1}]\) for the third inequality.
that:

\[
\|E \left[ e_{5,t}^{N}(t,\cdot) \right] \|_{L^1} \leq E \left[ \| e_{5,t}^{N}(t,\cdot) \|_{L^1} \right] \leq \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{t} \left( \lambda^N(i) \left| \lambda^N \left( \eta^N_s^{-1}(\eta^N_t(t)) \right) \right| \| \partial_x G_{t-s} \|_{L^1} \right) \, ds
\]

\[
\leq \frac{L_A^2}{\sigma} \sqrt{\frac{2}{\pi}} \sum_{k \in \mathbb{N}, t_k < t} \int_{t_k}^{t \wedge t_{k+1}} \frac{(t \wedge t_{k+1} - s)}{\sqrt{t - s}} \, ds
\]

\[
\leq \frac{L_A^2}{\sigma} \sqrt{\frac{2}{\pi}} \left\{ \sum_{k \in \mathbb{N}, t_k < t} \frac{1}{t \wedge t_{k+1} - t_k} \int_{t_k}^{t \wedge t_{k+1}} (t \wedge t_{k+1} - s) \, ds \right\}
\]

\[
\leq \frac{L_A^2}{\sigma} \sqrt{\frac{2}{\pi}} \left( \frac{h}{2} \sum_{k \in \mathbb{N}, t_k < t} \int_{t_k}^{t \wedge t_{k+1}} \frac{ds}{\sqrt{t - s}} \right) = \frac{L_A^2}{\sigma} \sqrt{\frac{2h}{\pi}}.
\]

The term \( e_{4,t}^{N}(t,x) \) can be estimated in the same way and the conclusion holds with \( C_{4,5} = \frac{L_A^2}{\sigma} \sqrt{\frac{2T}{\pi}}. \)

We remark that the terms \( e_{2,t}^{N}(t,x) \) and \( e_{3,t}^{N}(t,x) \) are of the same nature as well.

**Lemma 4.10.** For \( r \in \{2, 3\}, \forall N \in \mathbb{N}^*, \forall h \in (0, T], \| E \left[ e_{r,t}^{N,h}(t,\cdot) \right] \|_{L^1} = 0. \) Moreover, if \( \int_{\mathbb{R}} |x|^p \, m(dx) < \infty \) for some \( \rho > 1, \) then:

\[
\exists C_{2,3} < \infty, \sup_{t \leq T} E \left[ \| e_{r,t}^{N,h}(t,\cdot) \|_{L^1} \right] \leq C_{2,3} \sqrt{N}
\]

**Proof.** Let us treat the term \( e_{3,t}^{N,h}(t,x). \)

We have that

\[
\int_{\mathbb{R}} E \left[ \left( \int_{0}^{t} (\partial_x G_{t-s})^2 \left( X_s^{\eta_s^{-1}(i),N,h} - x \right) \, ds \right) \right] \, dx = E \left[ \int_{0}^{t} (\partial_x G_{t-s})^2 \left( X_s^{\eta_s^{-1}(i),N,h} - x \right) \, ds \right] < \infty
\]

according to the estimate \( A.2 \) from Lemma \( A.2. \) Therefore, \( E \left[ e_{3,t}^{N,h}(t,x) \right] = 0 \) dx a.e.. Moreover, using the estimate \( A.3 \) from Lemma \( A.2, \) the Itô isometry for the first equality, Young’s inequality for the second inequality and last Fubini’s theorem, we obtain:

\[
E \left[ \| e_{3,t}^{N,h}(t,\cdot) \|_{L^1} \right] \leq \int_{\mathbb{R}} E^{1/2} \left[ \left( \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{t} (t \wedge t_{s} - s) \lambda^N(i) \partial_x G_{t-s} \left( X_s^{\eta_s^{-1}(i),N,h} - x \right) \, ds \right)^{2} \right] \, dx
\]

\[
= \frac{\sigma}{\sqrt{N}} \int_{\mathbb{R}} E^{1/2} \left[ \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{t} (t \wedge t_{s} - s) \lambda^N(i)^2 (\partial_x G_{t-s})^2 \left( X_s^{\eta_s^{-1}(i),N,h} - x \right) \, ds \right] \, dx
\]

\[
= \frac{\sigma}{\sqrt{N}} \int_{\mathbb{R}} E^{1/2} \left[ \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{t} (t \wedge t_{s} - s) \lambda^N(i)^2 (\partial_x G_{t-s})^2 \left( X_s^{\eta_s^{-1}(i),N,h} - x \right) \, ds \right] \, dx
\]

\[
\leq \frac{\sigma}{2\sqrt{N}} \int_{\mathbb{R}} \left( 1 + |x|^\rho \right) + L_A^2 \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \int_{0}^{t} (t \wedge t_{s} - s) (\partial_x G_{t-s})^2 \left( X_s^{\eta_s^{-1}(i),N,h} - x \right) \, ds \left( 1 + |x|^\rho \right) \, dx
\]

\[
= \frac{\sigma I_\rho}{2\sqrt{N}} + \frac{L_A^2}{4\sigma^4 \sqrt{N} \pi} E \left[ \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{t} (t \wedge t_{s} - s)^2 \left( \int_{0}^{t} (X_s^{\eta_s^{-1}(i),N,h} - x)^2 G_{(t-s)/2} \left( X_s^{\eta_s^{-1}(i),N,h} - x \right) \, dx \right) \, ds \right]
\]

\[
= \frac{\sigma I_\rho}{2\sqrt{N}} + \frac{L_A^2}{4\sigma^4 \sqrt{N} \pi} E \left[ \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{t} (t \wedge t_{s} - s)^2 \left( \int_{0}^{y^2 G_{(t-s)/2}(y)} dy + \int_{y}^{\infty} y^2 G_{(t-s)/2}(y) \, dy \right) \, dx \right]
\]

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\[
\frac{\sigma I_\rho}{2\sqrt{N}} + \frac{L_\Lambda^2}{4\sigma^4\sqrt{N}\pi} E \left[ \frac{1}{N} \sum_{i=1}^{N} \int_0^t \frac{(t \wedge \tau_s - s)^2}{(t-s)^{3/2}} \left( \left( 1 + 2^{\rho-1} |X_s^{n,h} (i)_{N,h} |^\rho \right) \frac{\sigma^2(t-s)}{2} \right) + 2^{\rho-1} \left( \frac{\sigma \sqrt{t-s}}{2} \right)^{2+\rho} \right] ds \\
= \frac{1}{2\sqrt{N}} \left( \sigma I_\rho + \frac{L_\Lambda^2}{4\sigma^2\sqrt{\pi}} \left\{ \int_0^t \frac{(t \wedge \tau_s - s)^2}{(t-s)^{3/2}} ds + 2^{\rho-1} \int_0^t \frac{(t \wedge \tau_s - s)^2}{(t-s)^{3/2}} E \left[ \frac{1}{N} \sum_{i=1}^{N} |X_s^{n,h} (i)_{N,h} |^\rho \right] ds \right. \\
+ 2^{\rho-2}\sigma^\rho \left( \frac{\rho + 3}{2} \right) \int_0^t \frac{(t \wedge \tau_s - s)^2}{(t-s)^{3/2}} ds \right\} \right). \\
\]

With the use of Lemma 4.1, we obtain:
\[
E \left[ \| e_N^{n,h} (t,.) \|_{L^1} \right] \leq \frac{1}{2\sqrt{N}} \left( \sigma I_\rho + \frac{L_\Lambda^2}{4\sigma^2\sqrt{\pi}} \left\{ \left( 1 + 2^{\rho-1} M \right) \int_0^t \frac{(t \wedge \tau_s - s)^2}{(t-s)^{3/2}} ds \right. \\
+ 2^{\rho-2}\sigma^\rho \left( \frac{\rho + 3}{2} \right) \int_0^t \frac{(t \wedge \tau_s - s)^2}{(t-s)^{3/2}} ds \right\} \right). \\
\]

When \( t \geq h \), \[
\int_0^t \frac{(t \wedge \tau_s - s)^2}{(t-s)^{3/2}} ds \leq h^2 \int_0^{t-h} \frac{ds}{(t-s)^{3/2}} + \int_{t-h}^{t-h} \frac{\sqrt{t-s} ds}{(t-s)^{3/2}} \leq \frac{8}{3} h^{3/2} \] and \( \int_0^t \frac{(t \wedge \tau_s - s)^2}{(t-s)^{3/2}} ds \leq \frac{2}{3} h^{3/2} \). Then \( \int_0^t \frac{(t \wedge \tau_s - s)^2}{(t-s)^{3/2}} ds \leq \frac{2}{3} h^{3/2} \). The term \( e_N^{n,h} (t, x) \) can be estimated in the same way and the conclusion holds with
\[
C_{2,3} = \frac{1}{2} \left( \sigma I_\rho + \frac{L_\Lambda^2 T^{3/2}}{4\sigma^2\sqrt{\pi}} \left( \frac{8}{3} \left( 1 + 2^{\rho-1} M \right) \right) \times \frac{2^{\rho-2}\sigma^\rho \left( \frac{\rho + 3}{2} \right)}{(\rho - 1)(\rho + 3)} \right). \\
\]

Now, we finally treat the term \( e_1^{n,h} (t, x) \) in the lemma below:

**Lemma 4.11.** If \( \lambda \) is Lipschitz continuous with constant \( L_\Lambda \) then \( \exists C_1 < \infty, \forall N \in \mathbb{N}^+, \forall h \in (0, T], \forall t \in [0, T], \)
\[
E \left[ \| e_1^{n,h} (t,.) \|_{L^1} \right] \leq C_1 \left( h + \sqrt{h}E \left[ \| F^{n,h} (t,.) - F(t,.) \|_{L^1} \right] + 1_{(t \geq h)} \int_0^{t-h} \frac{h}{2(t-s)^{3/2}} E \left[ \| F^{n,h} (s,.) - F(s,.) \|_{L^1} \right] ds \right). \\
\]

The proof of this assertion relies on the following results.

**Lemma 4.12.** We have:
\[
\exists Q < \infty, \forall 0 \leq s \leq t \leq T, \quad \| F(t, .) - F(s, .) \|_{L^1} \leq Q \left( \sqrt{t} - \sqrt{s} \right) - L_\Lambda (t-s) \ln(t-s). \\
\]

**Proof.** Let \( 0 \leq s \leq t \leq T \). We recall that:
\[
F(t, x) - F(s, x) = (G_t - G_s) * F_0(x) - \int_0^t \partial_x G_{t-u} * \Lambda (F(u, .)) (x) du + \int_0^s \partial_x G_{s-u} * \Lambda (F(u, .)) (x) du. \\
\]

Using Equality (A.1) and the estimates (A.4) and (A.5) from Lemma (A.2) as well as the fact that \( t-s \leq
The conclusion holds with Lemma 4.13.

Proof. Let us start by proving the estimation of $\|G_t - G_s\|_{L^1}$. Since

$$\|G_t - G_s\|_{L^1} + \left\| \int_0^t (G_{t-u} - G_{s-u}) \partial_x \Lambda \left( F(u, \cdot) \right) \, du \right\|_{L^1} + \left\| \int_s^t G_{t-u} \partial_x \Lambda \left( F(u, \cdot) \right) \, du \right\|_{L^1}
$$

we have

$$\leq \left\| \int_s^t (\partial_x G_u \ast F_0) \, du \right\|_{L^1} + \left\| \int_0^t \int_{s-u}^{t-u} \partial_x G_r \ast \partial_x \Lambda \left( F(u, \cdot) \right) \, dr \, du \right\|_{L^1} + \left\| \int_s^t \|G_{t-u} \ast (\lambda (F(u, \cdot)) p(u, \cdot))\|_{L^1} \, du \right\|_{L^1}
$$

\begin{align*}
\leq & \frac{\sigma^2}{2} \int_s^t \| (\partial_x G_u \ast m) \|_{L^1} \, du + \frac{\sigma^2}{2} \int_0^t \int_{s-u}^{t-u} \| \partial_x G_r \ast (\lambda (F(u, \cdot)) p(u, \cdot)) \|_{L^1} \, dr \, du + L \Lambda (t-s) \\
\leq & \frac{\sigma^2}{2} \int_s^t \sqrt{\frac{2}{\pi \sigma^2 u}} \, du + L \Lambda \int_0^t \ln \left( \frac{t-u}{s-u} \right) \, du + L \Lambda (t-s) \\
= & \left( \sqrt{\frac{2}{\pi}} + 2L \Lambda \sqrt{T} \right) \left( \sqrt{t} - \sqrt{s} \right) + L \Lambda \int_s^t (1 + \ln(x)) \, dx - L \Lambda (t-s) \ln(t-s) \\
\leq & \left( \sqrt{\frac{2}{\pi}} + 2L \Lambda \sqrt{T} \right) \left( \sqrt{t} - \sqrt{s} \right) + \frac{L \Lambda}{2} (t-s) + L \Lambda (t-s) \ln(t-s) \\
\leq & \left( \sqrt{\frac{2}{\pi}} + 2L \Lambda \sqrt{T} (1 + T) \right) \left( \sqrt{t} - \sqrt{s} \right) - L \Lambda (t-s) \ln(t-s).
\end{align*}

The conclusion holds with $Q = \sigma \sqrt{2/\pi} + 2L \Lambda \sqrt{T} (1 + T)$. \qed

The next lemma provides two different estimations of the term $\mathbb{E} \left[ \int_s^t \gamma_u^i \, d[K|_u] \right]$. They are both useful to prove Lemma 4.11.

Lemma 4.13. \( \forall N \in \mathbb{N}^*, \forall i \in \{2, N\}, \forall h \in (0, T], \forall 0 \leq s \leq t \leq T, \)

\begin{equation}
\mathbb{E} \left[ \left( \int_s^t \gamma_u^i \, d[K|_u] \right)^2 \right] \leq 9N^2 \left( \sigma^2 + L \Lambda T \right) (t-s), \tag{4.14}
\end{equation}

and

\begin{equation}
\mathbb{E} \left[ \int_s^t \gamma_u^i \, d[K|_u] \right] \leq N \left( \mathbb{E} \left[ \left. \left\| F^{N,h}(t, \cdot) - F^{N,h}(s, \cdot) \right\|_{L^1} \right| + L \Lambda (t-s) \right) \right). \tag{4.15}
\end{equation}

Proof. Let $2 \leq i \leq N$. Since $\gamma_u^{N+1} = 0$, we have

$$\int_s^t \gamma_u^i \, d[K|_u] = \int_s^t \sum_{j=1}^{N} (\gamma_u^j - \gamma_u^{j+1}) \, d[K|_u]$$

with the dynamics (3.4) of $Y^{j,N,h}$, we deduce that

$$\int_s^t \gamma_u^i \, d[K|_u] = \int_s^t \sum_{j=1}^{N} (\gamma_u^j - \gamma_u^{j+1}) \, d[K|_u] = \sum_{j=1}^{N} \left\{ \left( Y_t^{j,N,h} - Y_s^{j,N,h} \right) - \sigma \left( \beta_t^j - \beta_s^j \right) - \int_s^t \lambda^N \left( \sigma_{\tau_u}^{-1} (\eta_u(j)) \right) \, du \right\}. $$

Let us start by proving the estimation of $\mathbb{E} \left[ \left( \int_s^t \gamma_u^i \, d[K|_u] \right)^2 \right]$. With the use of Jensen's inequality and
Lemma 4.2 for $\rho = 2$, we obtain:

$$
\mathbb{E} \left[ \left( \int_s^t \gamma_u^i d[K] | u \right)^2 \right] \leq 3N \left( \sum_{j=1}^N \mathbb{E} \left[ \left| Y_{j,t}^{i,N,h} - Y_{j,s}^{i,N,h} \right|^2 \right] + \sum_{j=1}^N \left( \int_s^t \lambda^N \left( \sigma^{-1}_{\tau^i_u} (\eta_u(j)) \right) | du \right)^2 + \sigma^2 \sum_{j=1}^N \mathbb{E} \left[ |\beta^j - \beta^i_s|^2 \right] \right)
$$

$$
\leq 3N \left( \sum_{j=1}^N \mathbb{E} \left[ X_{t}^{i,N,h} - X_{s}^{i,N,h} \right]^2 \right) + NL^2_\lambda (t-s)^2 + N\sigma^2 (t-s)
$$

$$
\leq 3N \left( \sum_{j=1}^N 2\mathbb{E} \left[ \sigma^2 |W_t^j - W_s^j|^2 + L^2_\lambda (t-s)^2 \right] + NL^2_\lambda (t-s)^2 + N\sigma^2 (t-s) \right)
$$

$$
\leq 9N^2 (\sigma^2 + L^2_\lambda T) (t-s).
$$

Notice that because of the latter contribution of $\mathbb{E} \left[ |W_t^j - W_s^j|^2 \right]$, it was not useful to take advantage of the independence of the Brownian motions $\beta^i$ which ensures $\mathbb{E} \left[ \sum_{j=1}^N (\beta^j - \beta^i_s)^2 \right] = (N + 1 - i)(t-s)$. Let us now prove the second estimation of $\mathbb{E} \left[ \int_s^t \gamma_u^i d[K] | u \right]$. To do so, we use that, according to (4.11) and (4.12),

$$
\frac{1}{N} \sum_{i=1}^N \left| Y_{t}^{i,N,h} - Y_{s}^{i,N,h} \right| = \mathbb{W}_1 \left( \mu^N_i, \mu^N_{s,h} \right) = \int_\mathbb{R} \left| F^{N,h}(t,x) - F^{N,h}(s,x) \right| dx = \| F^{N,h}(t,.) - F^{N,h}(s,.) \|_{L^1}.
$$

to obtain:

$$
\mathbb{E} \left[ \int_s^t \gamma_u^i d[K] | u \right] \leq \sum_{j=1}^N \mathbb{E} \left[ \left| Y_{t}^{j,N,h} - Y_{s}^{j,N,h} \right| \right] + \sum_{j=1}^N \int_s^t \lambda^N \left( \sigma^{-1}_{\tau^i_u} (\eta_u(j)) \right) | du \leq N\mathbb{E} \left[ \| F^{N,h}(t,.) - F^{N,h}(s,.) \|_{L^1} \right] + NL_\lambda (t-s).
$$

□

Let us now prove Lemma 4.11

**Proof.** We recall that $e_1^{N,h}(t,x) = \frac{1}{N} \sum_{i=2}^N \int_0^t \frac{1}{\sqrt{t-s}} (\lambda^N (i) - \lambda^N (i-1)) \partial_x G_{t-s} \left( Y_{s}^{i,N,h} - x \right) \gamma_s^i d[K]_s$.

For $i \in [2, N]$, we have $|\lambda^N (i) - \lambda^N (i-1)| = \left| N \int_0^N \left( \lambda(i) - \lambda \left( u - \frac{1}{N} \right) \right) du \right| \leq \frac{L_\lambda}{N}$. Using the estimate (A.4) from Lemma A.2 and the property (3.3), we have:

$$
\mathbb{E} \left[ \left\| e_1^{N,h}(t,.) \right\|_{L^1} \right] \leq \frac{L_\lambda}{\sigma N} \sqrt{\frac{2}{\pi}} \left\{ \frac{1}{N} \sum_{i=2}^N \mathbb{E} \left[ \int_0^t \frac{1}{\sqrt{t-s}} \gamma_s^i d[K]_s \right] \right\}.
$$

- For $t \leq h$, since $\frac{(t-s)}{\sqrt{t-s}} = \sqrt{t-s} \leq \sqrt{h}$, we deduce from (4.14) that $\mathbb{E} \left[ \left\| e_1^{N,h}(t,.) \right\|_{L^1} \right] \leq \frac{3L_\lambda}{\sigma} \sqrt{\frac{2(\sigma^2 + L^2_\lambda T)}{\pi}} h$.

- For $t \geq h$, we decompose the right-hand side of inequality (4.10) onto the sub-intervals $[0, t-h]$ and $[t-h, t]$ for a better control. Therefore,

$$
\mathbb{E} \left[ \left\| e_1^{N,h}(t,.) \right\|_{L^1} \right] \leq \frac{L_\lambda h}{\sigma N^2} \sqrt{\frac{2}{\pi}} \sum_{i=2}^N \left( \mathbb{E} \left[ \int_0^{t-h} \frac{1}{\sqrt{t-s}} \gamma_s^i d[K]_s \right] + \frac{1}{\sqrt{h}} \mathbb{E} \left[ \int_{t-h}^t \gamma_s^i d[K]_s \right] \right).
$$
As for the first term of the right-hand side of the above inequality, we introduce \( A_s = -\int_s^t \gamma_u^i d|K|_u \) and apply Fubini's theorem to obtain:

\[
\int_0^{t-h} A_s \frac{ds}{2(t-s)^{3/2}} = \int_0^{t-h} \left(A_0 + \int_0^s dA_r\right) \frac{ds}{2(t-s)^{3/2}} = A_0 \left(\frac{1}{\sqrt{h}} - \frac{1}{\sqrt{t}}\right) + \int_0^{t-h} \frac{dA_r}{2(t-s)^{3/2}} ds
\]

Consequently, we obtain that:

\[
\mathbb{E} \left[ \int_0^{t-h} \frac{1}{\sqrt{t-s}} \gamma_u^i d|K|_u \right] + \frac{1}{\sqrt{h}} \mathbb{E} \left[ \int_0^{t-h} \gamma_u^i d|K|_u \right] = \mathbb{E} \left[ \int_0^{t-h} \gamma_u^i d|K|_u \right] + \mathbb{E} \left[ \int_0^{t-h} \frac{1}{2(t-s)^{3/2}} \int_s^t \gamma_u^i d|K|_u ds \right].
\]

We shall use the estimate (4.14) and the estimate (4.15) from Lemma 4.13 for respectively the first term and the second term of the right-hand side of the following inequality:

\[
\mathbb{E} \left[ \left\| e_{1,N}^{h,t}(\cdot,\cdot) \right\|_{L_1} \right] 
\leq \frac{L\sqrt{\sigma N^2}}{\pi} \frac{N \sum_{t=2}^{N}}{\sqrt{2}} \left\{ \frac{1}{\sqrt{t}} \mathbb{E} \left[ \int_0^{t-h} \frac{1}{2(t-s)^{3/2}} \mathbb{E} \left[ \int_s^t \gamma_u^i d|K|_u \right] ds \right] \right\}
\leq \frac{L\sqrt{\sigma N^2}}{\pi} \frac{N \sum_{t=2}^{N}}{\sqrt{2}} \left\{ \frac{3N \sqrt{(\sigma^2 + L^2 \Lambda T)} + N \int_0^{t-h} \frac{L\Lambda}{2 \sqrt{t-s}} ds + N \int_0^{t-h} \frac{1}{2(t-s)^{3/2}} \mathbb{E} \left[ \left\| F^{N,h}(t,\cdot) - F^{N,h}(s,\cdot) \right\|_{L_1} \right] ds \right\}
\leq \frac{L\sqrt{\sigma N^2}}{\pi} \frac{N \sum_{t=2}^{N}}{\sqrt{2}} \left\{ 3 \sqrt{\sigma^2 + L^2 \Lambda T} + L\Lambda \sqrt{t} \right\} \left( \frac{1}{\sqrt{t}} \mathbb{E} \left[ \left. \int_0^{t-h} \frac{1}{2(t-s)^{3/2}} \mathbb{E} \left[ \left\| F^{N,h}(t,\cdot) - F^{N,h}(s,\cdot) \right\|_{L_1} \right] ds \right|_{s=0} \right) \right\}
\leq \frac{L\sqrt{\sigma N^2}}{\pi} \frac{N \sum_{t=2}^{N}}{\sqrt{2}} \left\{ 3 \sqrt{\sigma^2 + L^2 \Lambda T} + L\Lambda \sqrt{t} \right\} \left( \frac{1}{\sqrt{t}} \mathbb{E} \left[ \left. \int_0^{t-h} \frac{1}{2(t-s)^{3/2}} \mathbb{E} \left[ \left\| F^{N,h}(t,\cdot) - F^{N,h}(s,\cdot) \right\|_{L_1} \right] ds \right|_{s=0} \right) \right\}
\]

Since

\[
\mathbb{E} \left[ \left\| F^{N,h}(t,\cdot) - F^{N,h}(s,\cdot) \right\|_{L_1} \right] \leq \mathbb{E} \left[ \left\| F^{N,h}(t,\cdot) - F^{t,\cdot} \right\|_{L_1} \right] + \mathbb{E} \left[ \left\| F(t,\cdot) - F(s,\cdot) \right\|_{L_1} \right] + \mathbb{E} \left[ \left\| F(s,\cdot) - F^{N,h}(s,\cdot) \right\|_{L_1} \right],
\]

using Lemma 4.12 we obtain:

\[
\int_0^{t-h} \frac{h}{2(t-s)^{3/2}} \mathbb{E} \left[ \left\| F^{N,h}(t,\cdot) - F^{N,h}(s,\cdot) \right\|_{L_1} \right] ds \leq \sqrt{h} \mathbb{E} \left[ \left\| F^{N,h}(t,\cdot) - F(t,\cdot) \right\|_{L_1} \right]
\]

\[
+ h \int_0^{t-h} \mathbb{E} \left[ \left\| F(t,\cdot) - F^{N,h}(s,\cdot) \right\|_{L_1} \right] ds \mathbb{E} \left[ \left. \left\| F^{N,h}(t,\cdot) - F^{t,\cdot} \right\|_{L_1} \right|_{s=0} \right] + \mathbb{E} \left[ \left\| F(t,\cdot) - F^{N,h}(s,\cdot) \right\|_{L_1} \right] \cdot \mathbb{E} \left[ \left. \left\| F^{N,h}(t,\cdot) - F^{t,\cdot} \right\|_{L_1} \right|_{s=0} \right].
\]

To treat the last term of the right-hand side of the above inequality, we will use the fact that \( \sup \{ \sqrt{x} (2 - \ln(x)) \} = 2 \).

\[
\int_0^t \left( Q \sqrt{t} \sqrt{\frac{t}{2(t-s)^{3/2}}} - \frac{L\Lambda \ln(t-s)}{2} \right) ds = Q \int_0^1 \frac{1 - \sqrt{x}}{2(1-x)^{3/2}} dx - L\Lambda \left( \sqrt{t} \ln(t) - 2 \sqrt{t} \right)
\]

\[
= Q \left( \left[ (1-x)^{-1/2} (1 - \sqrt{x}) \right]_1^1 + \int_0^1 \frac{dx}{2 \sqrt{x} \sqrt{1-x}} \right) + L\Lambda \sqrt{t} (2 - \ln(t))
\]

\[
= Q \left( \frac{\pi}{2} - 1 \right) + L\Lambda \sqrt{t} (2 - \ln(t))
\]

\[
\leq Q \left( \frac{\pi}{2} - 1 \right) + 2L\Lambda.
\]
Therefore,
\[
\mathbb{E} \left[ \left\| e_1^{N,h}(t,\cdot) \right\|_{L^1} \right] \leq C_1 \left( h + \sqrt{h} \mathbb{E} \left[ \left\| F^{N,h}(t,\cdot) - F(t,\cdot) \right\|_{L^1} \right] + \int_0^{t-h} \frac{h}{2(t-s)^{3/2}} \mathbb{E} \left[ \left\| F(s,\cdot) - F^{N,h}(s,\cdot) \right\|_{L^1} \right] ds \right)
\]
where \( C_1 = \frac{L_A}{\sigma} \sqrt{\frac{t}{2}} \left[ 1 \vee \left( 3\sqrt{(\sigma^2 + L_A^2)T} + L_A(2 + \sqrt{T}) + Q \left( \frac{\pi}{2} - 1 \right) \right) \right] \).

Using Lemmas 4.7, 4.8, 4.9, 4.10 and 4.11 and the fact that for \( s \in [t-h,t] \), \( \frac{h}{2(t-s)^{3/2}} \leq \frac{1}{2\sqrt{t-s}} \), we conclude the proof of Proposition 4.10 for the choice \( Z = 2 \max(L_A + C_1/2 + C_{4.5}, C_{2.3}) \).

**Remark 4.14.** In Lemma 4.15 we provide two estimations of \( \mathbb{E} \left[ \int_s^t \gamma_t^* d|K|_u \right] \). If we only use the first estimation \( 4.14 \) in the proof of Lemma 4.17, we obtain, using a decomposition that we will detail in Section 4.3.2, a rough estimation of \( \mathbb{E} \left[ \left\| e_1^{N,h}(t,\cdot) \right\|_{L^1} \right] \) where we lose a \( \ln(h) \) factor.

### 4.3 Estimation of the bias

We recall Equation (4.7):
\[
F^{N,h}(t, x) - F(t, x) = G_t * \left( F_0^{N,h} - F_0 \right)(x) - \int_0^t \partial_x G_{t-s} * \left( \Lambda(F^{N,h}(s,\cdot)) - \Lambda(F(s,\cdot)) \right) dx ds + R^{N,h}(t, x) + E^{N,h}(t, x),
\]
and we shall use the expression of \( E^{N,h}(t, x) \) proved in Lemma 4.7. The next lemma provides an upper-bound of \( \mathbb{E} \left[ \left\| E^{N,h}(t,\cdot) \right\|_{L^1} \right] \).

**Lemma 4.15.** Assume that \( \lambda \) is Lipschitz continuous and the initial positions are

- either i.i.d. according to \( m \) and \( \int \sqrt{F_0(x)(1 - F_0(x))} \, dx < \infty \),
- or optimal deterministic and sup \( x \geq 1 \int_x^{+\infty} (F_0(-y) + 1 - F_0(y)) \, dy < \infty \).

Then
\[
\exists Z_b < \infty, \forall N \in \mathbb{N}^*, \forall h \in (0, T], \sup_{t \in [0,T]} \mathbb{E} \left[ \left\| E^{N,h}(t,\cdot) \right\|_{L^1} \right] \leq Z_b \left( \sqrt{\frac{h}{N}} + h \right).
\]

**Proof.** To estimate \( \mathbb{E} \left[ \left\| E^{N,h}(t,\cdot) \right\|_{L^1} \right] \), we estimate each \( \mathbb{E} \left[ e_p^{N,h}(t,\cdot) \right]_{L^1}, p \in [0, 5] \). From Lemmas 4.8, 4.9 and 4.10 we have \( \mathbb{E} \left[ \left\| E^{N,h}(t,\cdot) \right\|_{L^1} \right] \leq \mathbb{E} \left[ \left\| e_1^{N,h}(t,\cdot) \right\|_{L^1} \right] + 2L_A h + 2C_{4.5} h \). By Lemma 4.11 and Corollary 2.6, we have:
\[
\mathbb{E} \left[ \left\| e_1^{N,h}(t,\cdot) \right\|_{L^1} \right] \leq C_1 h + C_1 \sqrt{h} \mathbb{E} \left[ \left\| F^{N,h}(t,\cdot) - F(t,\cdot) \right\|_{L^1} \right] + C_1 1_{\{t \geq h\}} \int_0^{t-h} \frac{h}{2(t-s)^{3/2}} \mathbb{E} \left[ \left\| F(s,\cdot) - F^{N,h}(s,\cdot) \right\|_{L^1} \right] ds
\]
\[
\leq C_1 h + C_1 \sqrt{h} C \left( \frac{1}{\sqrt{N}} + h \right) + C_1 1_{\{t \geq h\}} \int_0^{t-h} \frac{h}{2(t-s)^{3/2}} C \left( \frac{1}{\sqrt{N}} + h \right) ds
\]
\[
\leq C_1 h + C_1 Ch^{3/2} + C_1 C \sqrt{\frac{h}{N}} + C_1 C \left( \frac{1}{\sqrt{N}} + h \right) \sqrt{h}
\]
\[
\leq \left( C_1 + 2C_1 C \sqrt{T} \right) h + 2C_1 C \sqrt{\frac{h}{N}}.
\]

Then the conclusion holds with \( Z_b = \max(2L_A + 2C_{4.5} + C_1 + 2C_1 C \sqrt{T}, 2C_1 C) \).
The proof of Theorem 2.7 relies on the following Proposition that we will prove in Section 4.3.2.

**Proposition 4.16.** Assume that \( \int_\mathbb{R} |x|m(dx) < \infty \) and \( \lambda \) is Lipschitz continuous. Then:

\[
\exists M_b < \infty, \forall N \in \mathbb{N}^*, \forall h \in (0, T], \sup_{t \leq T} \mathbb{E} \left[ \left\| F^{N,h}(t, \cdot) - F(t, \cdot) \right\|_{L^2}^2 \right] \leq M_b \left( \frac{1}{N} + h \right).
\]

### 4.3.1 Proof of Theorem 2.7

Taking the expectation of Equation (4.7) and using Lemma 4.5, we obtain that

\[
\mathbb{E} \left[ F^{N,h}(t, x) - F(t, x) \right] = G_t * \mathbb{E} \left[ F_0^N(x) - F_0(x) \right] - \int_0^t \partial_s G_{t-s} * \mathbb{E} \left[ (\Lambda(F^{N,h}(s, \cdot)) - \Lambda(F(s, \cdot))) \right] (x) \, ds
\]

Besides, using Taylor-Young’s inequality, we have that:

\[
\left| \Lambda \left( F^{N,h}(s, \cdot) \right) - \Lambda(F(s, \cdot)) \right| \leq \mathbb{E} \left[ F^{N,h}(s, \cdot) - F(s, \cdot) \right]
\]

which implies:

\[
\| \mathbb{E} \left[ \Lambda(F^{N,h}(s, \cdot)) \right] - \Lambda(F(s, \cdot)) \|_{L^1} \leq \| \Lambda(F(s, \cdot)) \|_{L^\infty} \| \mathbb{E} \left[ F^{N,h}(s, \cdot) - F(s, \cdot) \right] \|_{L^1} + \frac{L_\Lambda}{2} \mathbb{E} \left[ \| F^{N,h}(s, \cdot) - F(s, \cdot) \|_{L^2}^2 \right].
\]

Therefore, using the fact that \( G_t \) is a probability density and the estimate (4.4) from Lemma A.2, we obtain:

\[
\| \mathbb{E} \left[ F^{N,h}(t, \cdot) \right] - F(t, \cdot) \|_{L^1} \leq \| \mathbb{E} \left[ F_0^N \right] - F_0 \|_{L^1} + \sqrt{\frac{2}{\pi \sigma^2}} \int_0^t \frac{1}{\sqrt{t-s}} \left\{ L_\Lambda \| \mathbb{E} \left[ F^{N,h}(s, \cdot) \right] - F(s, \cdot) \|_{L^1} \right\} ds
\]

Using Lemma 4.15 and Proposition 4.16, then Young’s inequality, we deduce that:

\[
\| \mathbb{E} \left[ F^{N,h}(t, \cdot) \right] - F(t, \cdot) \|_{L^1} \leq \| \mathbb{E} \left[ F_0^N \right] - F_0 \|_{L^1} + \sqrt{\frac{2}{\pi \sigma^2}} \int_0^t \frac{1}{\sqrt{t-s}} \left\{ L_\Lambda \| \mathbb{E} \left[ F^{N,h}(s, \cdot) \right] - F(s, \cdot) \|_{L^1} \right\} ds
\]

We iterate this inequality and use that \( \int_0^t \frac{\sqrt{s}}{\sqrt{t-s}} \, ds = \frac{t^2}{2} \) to obtain:

\[
\| \mathbb{E} \left[ F^{N,h}(t, \cdot) \right] - F(t, \cdot) \|_{L^1} \leq \left( 1 + \frac{\sqrt{2}}{\sigma} \right) Z_t + \frac{L_\Lambda M_b (1 + \sqrt{2}) Z_t}{\sigma} \sqrt{\frac{2\pi}{\sigma}} + \frac{L_\Lambda L_\Lambda M_b}{\sigma^2} \left( \frac{1}{N} + h \right)
\]

By Lemma 4.3, the application \( t \mapsto \| \mathbb{E} \left[ F^{N,h}(t, \cdot) \right] - F(t, \cdot) \|_{L^1} \) is locally integrable \( \forall h \in [0, T], N \in \mathbb{N}^* \). Therefore, we can apply Gronwall’s lemma and choosing

\[
C_b = \max \left( 1 + \frac{L_\Lambda}{\sigma} \sqrt{\frac{8T}{\pi}}, 1 + \frac{\sqrt{2}}{\sigma} Z_t + \frac{L_\Lambda M_b (1 + \sqrt{2}) Z_t}{\sigma} \sqrt{\frac{2\pi}{\sigma}} + \frac{L_\Lambda L_\Lambda M_b}{\sigma^2} \left( 2 \frac{L_\Lambda^2}{\sigma^2} T \right) \right)
\]

concludes the proof of the theorem.
4.3.2 Proof of Proposition 4.16

For all $t, h \in [0, T], N \in \mathbb{N}^*$, we use Jensen’s inequality upon Equation (4.17) and obtain:

\[
E \left[ \| F^{N,h}(t,.) - F(t,.) \|_{L^2}^2 \right] \leq 4 \int_\mathbb{R} E \left[ G_t \ast \left( F^{N,h}_0 - F_0 \right)^2(x) \right] dx + 4 \int_\mathbb{R} E \left[ R^{N,h}(t, x) \right] dx + 4 \int_\mathbb{R} E \left[ E^{N,h}(t, x) \right] dx \\
+ 4 \int_\mathbb{R} E \left[ \left( \int_0^t \partial_x G_{t-s} \ast \left( \Lambda(F^{N,h}(s, .)) - \Lambda(F(s, .)) \right)(x) ds \right)^2 \right] dx.
\]

On the one hand, we have using the definition (4.5) of $R^{N,h}(t, x)$, Itô’s isometry and the estimate (A.7) from Lemma A.2 that:

\[
\int_\mathbb{R} E \left[ R^{N,h}(t, x) \right] dx = \frac{\sigma^2}{N^2} \sum_{i=1}^N \int_\mathbb{R} E \left[ G_{t-s}^2 (X_{s,N,h} - x) \right] dx ds = \frac{\sigma}{N} \sqrt{\frac{t}{\pi}}.
\]

On the other hand, using Minkowski’s, Young’s and Cauchy-Schwarz inequalities in addition to the estimate (A.4) from Lemma A.2 we get:

\[
\int_\mathbb{R} E \left[ \left( \int_0^t \partial_x G_{t-s} \ast \left( \Lambda(F^{N,h}(s, .)) - \Lambda(F(s, .)) \right)(x) ds \right)^2 \right] dx \\
= E \left[ \left\| \int_0^t \partial_x G_{t-s} \ast \left( \Lambda(F^{N,h}(s, .)) - \Lambda(F(s, .)) \right) ds \right\|_{L^2}^2 \right] \\
\leq E \left[ \left( \int_0^t \| \partial_x G_{t-s} \|_{L^1} \| \Lambda(F^{N,h}(s, .)) - \Lambda(F(s, .)) \|_{L^2} ds \right)^2 \right] \\
\leq E \left[ \left( \int_0^t \sqrt{\frac{2L_x}{\pi \sigma^2 (t - s)}} \| F^{N,h}(s, .) - F(s, .) \|_{L^2} ds \right)^2 \right] \\
\leq \frac{2L_x^2}{\pi \sigma^2} \int_0^t \frac{du}{\sqrt{t - u}} \int_0^t \frac{1}{\sqrt{t - s}} E \left[ \| F^{N,h}(s, .) - F(s, .) \|_{L^2}^2 \right] ds \\
= \frac{4L_x^2 \sqrt{t}}{\pi \sigma^2} \int_0^t \frac{1}{\sqrt{t - s}} E \left[ \| F^{N,h}(s, .) - F(s, .) \|_{L^2}^2 \right] ds.
\]

Therefore, Inequality (4.17) becomes:

\[
E \left[ \| F^{N,h}(t, .) - F(t, .) \|_{L^2}^2 \right] \leq 4 \int_\mathbb{R} E \left[ \left( F^{N,h}_0 - F_0 \right)^2(x) \right] dx + \frac{4 \sigma}{N} \sqrt{\frac{t}{\pi}} + 4 \int_\mathbb{R} E \left[ E^{N,h}(t, x) \right] dx \\
+ \frac{16L_x^2 \sqrt{t}}{\pi \sigma^2} \int_0^t \frac{1}{\sqrt{t - s}} E \left[ \| F^{N,h}(s, .) - F(s, .) \|_{L^2}^2 \right] ds.
\]

As for the initialization term, when choosing either initial positions that are i.i.d. according to $m$ or optimal deterministic, according to Lemma 5.1 and Remark 5.2 we have $\int_\mathbb{R} E \left[ \left( F^{N,h}_0 - F_0 \right)^2(x) \right] dx \leq \frac{1}{N} \int_\mathbb{R} |x|m(dx)$.

With Lemma 4.17 below which provides an estimation of the term $\int_\mathbb{R} E \left[ E^{N,h}(t, x) \right] dx$, we deduce that:

\[
E \left[ \| F^{N,h}(t, .) - F(t, .) \|_{L^2}^2 \right] \leq \frac{4}{N} \int_\mathbb{R} |x|m(dx) + \frac{4 \sigma}{N} \sqrt{\frac{t}{\pi}} + 4Qb + \frac{16L_x^2 \sqrt{t}}{\pi \sigma^2} \int_0^t \frac{1}{\sqrt{t - s}} E \left[ \| F^{N,h}(s, .) - F(s, .) \|_{L^2}^2 \right] ds.
\]
Iterating the previous inequality, we obtain:
\[
\mathbb{E} \left[ \left\| F^{N,h}(t,\cdot) - F(t,\cdot) \right\|^2_{L^2} \right] \leq 4 \left( 1 + \frac{32L^2t}{\pi \sigma^2} \right) \left( \frac{1}{N} \int_\mathbb{R} |x| m(dx) + \frac{\sigma}{\sqrt{\pi}} \sqrt{t} + Q_b \right) + \frac{256L^4t}{\pi \sigma^4} \int_0^t \mathbb{E} \left[ \left\| F^{N,h}(r,\cdot) - F(r,\cdot) \right\|^2_{L^2} \right] \, dr
\]

By Lemma 4.13 and since \( |F^{N,h}(t,\cdot) - F(t,\cdot)| \leq 1 \), the function \( t \mapsto \mathbb{E} \left[ \left\| F^{N,h}(t,\cdot) - F(t,\cdot) \right\|^2_{L^2} \right] \) is locally integrable for all \( h \in [0,T], \ N \in \mathbb{N}^* \). We use Gronwall’s lemma once again and conclude for the choice
\[
M_b = 4 \max \left\{ Q_b \left( 1 + \frac{32L^2T}{\pi \sigma^2} \right), \left( 1 + \frac{32L^2t}{\pi \sigma^2} \right) \left( \int_\mathbb{R} |x| m(dx) + \frac{\sigma}{\sqrt{\pi}} \sqrt{T} \right) \right\} \exp \left( \frac{256L^4T^2}{\pi \sigma^4} \right).
\]

**Lemma 4.17.** Assume that \( \lambda \) is Lipschitz continuous. Then
\[
\exists Q_b, \forall h \in (0,T], \forall t \in [0,T), \quad \int_\mathbb{R} \mathbb{E} \left[ E^{N,h}(t,x)^2 \right] \, dx \leq Q_b \, h.
\]

**Proof.** We have that \( \int_\mathbb{R} \mathbb{E} \left[ E^{N,h}(t,x)^2 \right] \, dx \leq 6 \sum_{p=0}^5 \int_\mathbb{R} \mathbb{E} \left[ e^{N,h}(t,x)^2 \right] \, dx \). For this reason, we shall estimate, in what follows, each \( \int_\mathbb{R} \mathbb{E} \left[ e^{N,h}(t,x)^2 \right] \, dx, \ p \in [0,5] \).

On the one hand, we have using Itô’s isometry and the estimate \( A.3 \) from Lemma A.2 that \( \forall h \in (0,T) \):
\[
\int_\mathbb{R} \mathbb{E} \left[ e^{N,h}_2(t,x)^2 \right] \, dx = \frac{\sigma^2}{N^2} \sum_{i=1}^N \int_0^t (t \wedge \tau_h^i - s)^2 \left( \lambda^N(i)-\lambda^N(i-1) \right)^2 \mathbb{E} \left[ \int_\mathbb{R} (\partial_x G_{t-s} (Y^{i,N,h}_s-x))^2 \, dx \right] \, ds \leq \frac{2L^2}{3\sigma^2} \frac{h^{3/2}}{\sqrt{\pi}} N,
\]
where the last inequality has already been derived at the end of the proof of Lemma 4.10. The same estimation can be derived in the same way for \( \int_\mathbb{R} \mathbb{E} \left[ e^{N,h}_3(t,x)^2 \right] \, dx \). The Cauchy-Schwarz inequality then a similar reasoning implies that, for \( \{4,5\} \), \( \int_\mathbb{R} \mathbb{E} \left[ e^{N,h}_4(t,x)^2 \right] \, dx \leq \frac{2L^4T}{3\sigma^2} \frac{h^{3/2}}{\sqrt{\pi}} \). As for the term \( e^{N,h}_0 \), we have using the estimate \( A.6 \) that \( \int_\mathbb{R} \mathbb{E} \left[ e^{N,h}_0(t,x)^2 \right] \, dx \leq \frac{4L^2}{\sigma^2} \frac{h^{3/2}}{\sqrt{\pi}} \).

On the other hand, using Cauchy-Schwarz inequality twice, \( A.3 \) and the estimation \( A.14 \), we obtain:
\[
\int_\mathbb{R} \mathbb{E} \left[ e^{N,h}_1(t,x)^2 \right] \, dx = \mathbb{E} \left[ \int_\mathbb{R} \left( \frac{1}{N} \sum_{i=2}^N \int_0^t \left( t \wedge \tau_h^i - s \right)^2 \left( \lambda^N(i)-\lambda^N(i-1) \right) \partial_x G_{t-s} (Y^{i,N,h}_s-x)^2 \, dx \right) \, ds \right] \leq \frac{2L^2}{\sigma^2} \frac{h^{3/2}}{\sqrt{\pi}} N^2 \sum_{i=2}^N \left( \int_0^t \left( t \wedge \tau_h^i - s \right)^2 \gamma_i^1 \, d|K|_s \right)^2 \left( \int_0^t \gamma_1^i \, d|K|_r \right) \, dr \leq \frac{L^2}{\sigma^2} \frac{T}{\sqrt{\pi}} \sum_{i=2}^N \left( \int_0^t \left( t \wedge \tau_h^i - s \right)^2 \gamma_i^1 \, d|K|_s \right)^2 \left( \int_0^t \gamma_1^i \, d|K|_r \right)^2 \leq 3L^2 \frac{T}{\sigma^2} \frac{h^{3/2}}{\sqrt{\pi}} \sum_{i=2}^N \left( \int_0^t \left( t \wedge \tau_h^i - s \right)^2 \, d|K|_s \right)^2 \left( \int_0^t \gamma_1^i \, d|K|_r \right)^2.
\]
We denote by $m = [\log_2 (t/h)]$ and rewrite the integral the following way:

$$
\int_0^t \frac{(t \wedge t^k - s)^2}{(t-s)^{3/2}} \gamma_s^i d|K|_s = \sum_{k=0}^{m-1} \int_{t-2^{k+1}}^{t-2^k} \frac{(t \wedge t^k - s)^2}{(t-s)^{3/2}} \gamma_s^i d|K|_s + \int_{t-2^m}^{t} \frac{(t \wedge t^k - s)^2}{(t-s)^{3/2}} \gamma_s^i d|K|_s.
$$

Therefore,

$$
\mathbb{E}^{1/2} \left[ \left( \int_0^t \frac{(t \wedge t^k - s)^2}{(t-s)^{3/2}} \gamma_s^i d|K|_s \right)^2 \right] 
\leq \sum_{k=0}^{m-1} \frac{h^2}{(t-2^{k+1})^{3/2}} \mathbb{E}^{1/2} \left[ \left( \int_{t-2^k}^{t-2^{k+1}} \gamma_s^i d|K|_s \right)^2 \right] + \sqrt{n} \mathbb{E}^{1/2} \left[ \left( \int_{t-2^m}^{t} \gamma_s^i d|K|_s \right)^2 \right] 
\leq 3N \sqrt{(\sigma^2 + L_0^2 T)} h^2 \sum_{k=0}^{m-1} \left( \frac{t}{2^{k+1}} \right)^{-1} (h + t/2^m) = 3N \sqrt{(\sigma^2 + L_0^2 T)} (2h^2 t (2^m - 1) + h).
$$

We then have

$$
\int_\mathbb{R} \mathbb{E} \left[ e^{N \hat{h}(t, x)^2} \right] dx \leq \frac{45L_0^2 (\sigma^2 + L_0^2 T)}{4\sigma^2} \sqrt{\frac{T}{\pi}} h.
$$

The conclusion holds for the choice $Q_0 = \frac{1}{\sigma} \sqrt{\frac{T}{\pi}} \left( \frac{16L_0^2}{3} \left( 1 + \frac{L_0^2 T}{4\sigma^2} \right) + \frac{45L_0^2}{4} \left( 1 + \frac{L_0^2 T}{\sigma^2} \right) \right).$ \hfill \Box

## 5 Particle initialization

In this section, we are interested in the strong and weak initialization errors $\mathbb{E} \left[ W_1 \left( \mu_0^N, m \right) \right]$ and $\mathbb{E} \left[ \hat{W}_1 \left( \mu_0^N, m \right) \right].$ For initial positions i.i.d. according to $m$, $\mathbb{E} \left[ \hat{\mu}_0^N \right] = m$ and the weak error is zero. For optimal deterministic initial positions, both are equal to $W_1 \left( \hat{\mu}_0^N, m \right).$ In Section 5.1, we check that the strong error is bounded iff $m$ has a finite first order moment. Section 5.2 is devoted to the proof of Proposition 2.3 which gives conditions for the strong initialization error to be of order $N^{-1/2}$. Last, in Section 5.3, we state necessary conditions for the optimal deterministic initialization error to be of order $N^{-1}$ and study the asymptotic behaviour of moments under this initialization as $N \to \infty$.

### 5.1 Finite Wassertein distance

The following lemma gives a necessary and sufficient condition for the finiteness of $\sup_{N \in \mathbb{N}^*} \mathbb{E} \left[ W_1 \left( \mu_0^N, m \right) \right]$ and $\sup_{N \in \mathbb{N}^*} N \mathbb{E} \left[ \| F_0^N - F_0 \|^2_{L^2} \right].$

**Lemma 5.1.** For the optimal deterministic initial positions, we have

$$
\forall N \in \mathbb{N}^*, \ W_1 \left( \hat{\mu}_0^N, m \right) \leq \int_\mathbb{R} |x| m(dx) \text{ and } \| \hat{F}_0^N - F_0 \|^2_{L^2} \leq \frac{1}{2N} \int_\mathbb{R} |x| m(dx).
$$

For initial positions i.i.d. according to $m$, we have

$$
\forall N \in \mathbb{N}^*, \mathbb{E} \left[ W_1 \left( \hat{\mu}_0^N, m \right) \right] \leq 2 \int_\mathbb{R} F_0(x)(1 - F_0(x)) dx \text{ and } \mathbb{E} \left[ \| \hat{F}_0^N - F_0 \|^2_{L^2} \right] = \frac{1}{N} \int_\mathbb{R} F_0(x)(1 - F_0(x)) dx.
$$

Conversely, the existence of $N \in \mathbb{N}^*$ such that $W_1 \left( \hat{\mu}_0^N, m \right)$ or $\mathbb{E} \left[ W_1 \left( \hat{\mu}_0^N, m \right) \right]$ is finite implies the finiteness of $\int_\mathbb{R} |x| m(dx)$. 


Remark 5.2. Notice that \( \int_{\mathbb{R}} F_0(x)(1 - F_0(x)) \, dx \leq \int_{-\infty}^{0} F_0(x) \, dx + \int_{0}^{+\infty} (1 - F_0(x)) \, dx = \int_{\mathbb{R}} |x|m(dx) \) (see (5.6) below for a short proof of this well-known equality). On the other hand, since

\[
\int_{\mathbb{R}} F_0(x)(1 - F_0(x)) \, dx \geq \frac{1}{2} \int_{-\infty}^{F_0^{-1}(1/2)} F_0(x) \, dx + \frac{1}{2} \int_{F_0^{-1}(1/2)}^{+\infty} (1 - F_0(x)) \, dx \geq \frac{1}{2} \left( \int_{\mathbb{R}} |x|m(dx) - |F_0^{-1}(1/2)| \right),
\]

\[
\int_{\mathbb{R}} F_0(x)(1 - F_0(x)) \, dx \text{ and } \int_{\mathbb{R}} |x|m(dx) \text{ are simultaneously finite (or infinite).}
\]

Proof. We recall that \( F_0(x) = m(-\infty, x] \) is the cumulative distribution function of the probability measure \( m \) on the real line and \( F_0^{-1}(u) = \inf \{ x \in \mathbb{R} : F_0(x) \geq u \}, u \in (0, 1) \) its quantile function.

\( \blacktriangleright \) Since for \( i \in \{1, \ldots, N\} \), \( x_i^N = F_0^{-1} \left( \frac{2i-1}{2N} \right) \) minimizes \( \mathbb{R} \ni y \rightarrow \int_{i-1}^{i} |y - F_0^{-1}(u)| \, du \), we have that:

\[
\mathcal{W}_1 \left( \hat{\mu}_0^N, m \right) = \sum_{i=1}^{N} \int_{x_i^N}^{\hat{F}_0^{-1}(u)} \left| x_i^N - F_0^{-1}(u) \right| \, du \leq \sum_{i=1}^{N} \int_{x_i^N}^{\hat{F}_0^{-1}(u)} |F_0^{-1}(u)| \, du = \int_{\mathbb{R}} |x|m(dx).
\]

Since \( \hat{F}_0^N - F_0 \) is not greater than \( 1/2N \), we deduce that

\[
\left\| \hat{F}_0^N - F_0 \right\|_{L^2}^2 \leq \frac{\left\| \hat{F}_0^N - F_0 \right\|_{L^1}}{2N} \cdot \mathcal{W}_1 \left( \hat{\mu}_0^N, m \right) \leq \frac{1}{2N} \int_{\mathbb{R}} |x|m(dx).
\]

On the other hand,

\[
\int_{\mathbb{R}} |x|m(dx) = \int_{0}^{1} \left| F_0^{-1}(u) \right| \, du \leq \int_{0}^{1} \left| F_0^{-1}(u) - \left( \hat{F}_0^N \right)^{-1}(u) \right| \, du + \int_{0}^{1} \left| \left( \hat{F}_0^N \right)^{-1}(u) \right| \, du
\]

\[
= \mathcal{W}_1 \left( \hat{\mu}_0^N, m \right) + \frac{1}{N} \sum_{i=1}^{N} \left| \left( \frac{2i-1}{2N} \right) \right|.
\]

Since the last sum is finite, the finiteness of \( \mathcal{W}_1 \left( \hat{\mu}_0^N, m \right) \) implies that \( \int_{\mathbb{R}} |x|m(dx) \) is finite.

\( \blacktriangleright \) When choosing initial positions i.i.d. according to \( m \), we first have the following results:

- Since \( \hat{F}_0^N(x) = \frac{1}{N} \sum_{i=1}^{N} 1_{\{x_i \leq x\}} \) where the random variables \( 1_{\{x_i \leq x\}} \) are i.i.d. according to the Bernoulli law with parameter \( F_0(x) \) and variance \( F_0(x)(1 - F_0(x)) \),

\[
\mathbb{E} \left[ \left| \hat{F}_0^N(x) - F_0(x) \right| \right] \leq \mathbb{E}^{1/2} \left[ \left( \hat{F}_0^N(x) - F_0(x) \right)^2 \right] = \sqrt{\frac{F_0(x)(1 - F_0(x))}{N}}.
\]

The equality ensures that \( \mathbb{E} \left[ \left\| \hat{F}_0^N - F_0 \right\|^2_{L^2} \right] = \frac{\int_{\mathbb{R}} F_0(x)(1 - F_0(x)) \, dx}{N} \).

- When \( F_0(x) \leq \frac{1}{N} \), since \( N \hat{F}_0^N(x) \) is distributed according to the binomial law with parameter \( (N, F_0(x)) \) and expectation \( NF_0(x) \), one has

\[
\mathbb{E} \left[ \left| \hat{F}_0^N(x) - F_0(x) \right| \right] = F_0(x) \mathbb{P} \left( F_0^N(x) = 0 \right) + \sum_{k=1}^{N} \left( \frac{k}{N} - F_0(x) \right) \mathbb{P} \left( F_0^N(x) = \frac{k}{N} \right)
\]

\[
= F_0(x) \left( \mathbb{P} \left( F_0^N(x) = 0 \right) - (1 - \mathbb{P} \left( F_0^N(x) = 0 \right)) \right) + \mathbb{E} \left[ F_0^N(x) \right]
\]

\[
= 2F_0(x) \mathbb{P} \left( F_0^N(x) = 0 \right)
\]

(5.2)

\[
= 2F_0(x)(1 - F_0(x))^N.
\]
• When \( F_0(x) \geq 1 - \frac{1}{N} \), we obtain in a symmetric way that:

\[
(5.3) \quad \mathbb{E} \left[ \left| \hat{F}_0^N(x) - F_0(x) \right| \right] = 2 \left( 1 - F_0(x) \right) (F_0(x))^N.
\]

Using these results for the first inequality then the fact that when \( 1/N < F_0(x) < 1 - 1/N \), then

\[
\frac{1}{\sqrt{N}} \leq \sqrt{2F_0(x)(1 - F_0(x))}
\]

for the second one, we obtain that

\[
\begin{align*}
\mathbb{E} \left[ W_1 \left( \hat{\mu}_0^N, m \right) \right] &\leq \int_{\mathbb{R}} 1_{\{1/N < F_0(x) < 1 - 1/N\}} \sqrt{\frac{F_0(x)(1 - F_0(x))}{N}} \, dx \\
&\quad + \int_{\mathbb{R}} 1_{\{F_0(x) \leq 1/N\}} + 1_{\{F_0(x) \geq 1 - 1/N\}} 2F_0(x)(1 - F_0(x)) \, dx \\
&\leq \int_{\mathbb{R}} 1_{\{1/N < F_0(x) < 1 - 1/N\}} \sqrt{2F_0(x)(1 - F_0(x))} \, dx \\
&\quad + \int_{\mathbb{R}} 1_{\{F_0(x) \leq 1/N\}} + 1_{\{F_0(x) \geq 1 - 1/N\}} 2F_0(x)(1 - F_0(x)) \, dx \\
&\leq 2 \int_{\mathbb{R}} F_0(x)(1 - F_0(x)) \, dx.
\end{align*}
\]

On the other hand, using once again Equations (5.2) and (5.3), we have that:

\[
\mathbb{E} \left[ W_1 \left( \hat{\mu}_0^N, m \right) \right] \geq 2 \left( 1 - \frac{1}{N} \right)^N \left( \int_{-\infty}^0 1_{\{F_0(x) \leq 1/N\}} F_0(x) \, dx + \int_{0}^{+\infty} 1_{\{F_0(x) \geq 1 - 1/N\}} (1 - F_0(x)) \, dx \right).
\]

With the inequality

\[
\int_{-\infty}^0 1_{\{F_0(x) > 1/N\}} F_0(x) \, dx + \int_{0}^{+\infty} 1_{\{F_0(x) < 1 - 1/N\}} (1 - F_0(x)) \, dx
\]

\[
\leq \int_{-\infty}^0 1_{\{F_0(x) \geq 1/N\}} \, dx + \int_{0}^{+\infty} 1_{\{F_0(x) < 1 - 1/N\}} \, dx = (F_0^{-1}(1/N)^-) + (F_0^{-1}(1 - 1/N)^+) < \infty,
\]

we conclude that \( \infty > \mathbb{E} \left[ W_1 \left( \hat{\mu}_0^N, m \right) \right] \) implies that \( \infty > \int_{-\infty}^0 F_0(x) \, dx + \int_{0}^{+\infty} (1 - F_0(x)) \, dx = \int_{\mathbb{R}} |x| m(dx). \)

When \( m(dx) = \delta_y(dx) \) for some \( y \in \mathbb{R} \), then \( F_0^{-1} \left( \frac{2i - 1}{2N} \right) = y \) for all \( 1 \leq i \leq N \) and \( W_1(\hat{\mu}_0^N, m) = 0 \) for each \( N \geq 1 \). Otherwise, \( \int_{\mathbb{R}} (F_0(x) \wedge (1 - F_0(x))) \, dx > 0 \) and, according to the next Lemma, the strong error of the optimal deterministic initialization error cannot behave better than \( O(N^{-1}) \).

**Lemma 5.3.**

\[
(5.4) \quad \forall N \geq 1, \quad N \int_{\mathbb{R}} \left| \hat{F}_0^N(x) - F_0(x) \right| \, dx + (N + 1) \int_{\mathbb{R}} \left| \hat{F}_0^{N+1}(x) - F_0(x) \right| \, dx \geq \frac{1}{2} \int_{\mathbb{R}} (F_0(x) \wedge (1 - F_0(x))) \, dx.
\]

As \( \int_{\mathbb{R}} |x| m(dx) = \int_{-\infty}^0 F_0(x) \, dx + \int_{0}^{+\infty} (1 - F_0(x)) \, dx \), \( \int_{\mathbb{R}} |x| m(dx) < \infty \iff \int_{\mathbb{R}} (F_0(x) \wedge (1 - F_0(x))) \, dx < \infty. \) So the above statement can be seen as a refinement of the necessary condition in Lemma 5.1.

**Proof.** Formula (2.5) rewrites:

\[
(5.5) \quad \int_{\mathbb{R}} \left| \hat{F}_0^N(x) - F_0(x) \right| \, dx = \frac{1}{N} \int_{\mathbb{R}} \min_{j \in \mathbb{N}} |NF_0(x) - j| \, dx.
\]

For \( v \in (0, 1) \),
• Either $\lfloor Nv \rfloor \leq Nv < (N+1)v \leq \lfloor Nv \rfloor + 1$ which implies that $(Nv - \lfloor Nv \rfloor) \vee (\lfloor Nv \rfloor + 1 - (N+1)v) \geq \frac{1-v}{2}$ while $\lfloor Nv \rfloor + 1 - Nv = \lfloor Nv \rfloor + 1 - (N+1)v + v \geq v$ and $(N+1)v - \lfloor Nv \rfloor = Nv - \lfloor Nv \rfloor + v \geq v$ so that
\[
\min_{j \in \mathbb{N}} |Nv - j| \vee \min_{j \in \mathbb{N}} |(N+1)v - j| \geq v \wedge \frac{1-v}{2}.
\]

• Or $\lfloor Nv \rfloor \leq Nv < \lfloor Nv \rfloor + 1 \leq (N+1)v$, which implies that $(\lfloor Nv \rfloor + 1 - Nv) \vee ((N+1)v - (\lfloor Nv \rfloor + 1)) \geq \frac{v}{2}$ while $Nv - \lfloor Nv \rfloor = (N+1)v - (\lfloor Nv \rfloor + 1) + 1 - v \geq 1 - v$ and $|Nv| + 2 - (N+1)v = \lfloor Nv \rfloor + 1 - Nv + 1 - v > 1 - v$ so that
\[
\min_{j \in \mathbb{N}} |Nv - j| \vee \min_{j \in \mathbb{N}} |(N+1)v - j| \geq \frac{v}{2} \wedge (1-v).
\]

Synthesising the two cases and remarking that the inequality still holds for $v \in \{0, 1\}$, we deduce that:
\[
\forall v \in [0, 1], \forall N \geq 1, \quad \min_{j \in \mathbb{N}} |Nv - j| \vee \min_{j \in \mathbb{N}} |(N+1)v - j| \geq \frac{v \wedge (1-v)}{2}.
\]

Inserting this inequality with $v = F_0(x)$ into (5.3), we conclude that for each $N \geq 1$,
\[
N \int_{\mathbb{R}} \left| \hat{F}_0^N(x) - F_0(x) \right| dx + (N+1) \int_{\mathbb{R}} \left| \hat{F}_0^{N+1}(x) - F_0(x) \right| dx \geq \frac{1}{2} \int_{\mathbb{R}} (F_0(x) \wedge (1 - F_0(x))) \, dx.
\]

\[\square\]

5.2 Strong errors of order $N^{-1/2}$

In this section, we shall prove each implication in Proposition 5.2. Since $W_1(\bar{\mu}_0^N, m) = \left\| \hat{F}_0^N(\cdot) - F_0(\cdot) \right\|_{L^1}$, the equivalence concerning the strong random initialization error $E[W_1(\bar{\mu}_0^N, m)]$ is a direct consequence of the following lemma.

**Lemma 5.4.** One has:
\[
\sqrt{N} \mathbb{E} \left[ \left\| \hat{F}_0^N(\cdot) - F_0(\cdot) \right\|_{L^1} \right] \leq \int_{\mathbb{R}} \sqrt{F_0(x)(1 - F_0(x))} \, dx,
\]
\[
\lim_{N \to \infty} \sqrt{N} \mathbb{E} \left[ \left\| \hat{F}_0^N(\cdot) - F_0(\cdot) \right\|_{L^1} \right] = \frac{2}{\sqrt{\pi}} \int_{\mathbb{R}} \sqrt{F_0(x)(1 - F_0(x))} \, dx.
\]

**Remark 5.5.** Let us illustrate by an example how to derive slower rates of convergence under weaker integrability conditions. For the Pareto law $m(dx) = 1_{\{x \geq 1\}} \frac{\alpha \, dx}{x^{\alpha+1}}$ with $\alpha > 0$, $(1 - F_0(x)) = 1_{\{x \geq 1\}} x^{-\alpha}$. With (5.3) and (5.1), we deduce that for $N \geq 2$ so that $2 \left( 1 - \frac{1}{N} \right) N^{-1} \leq 1$,
\[
\int_{\mathbb{R}} \mathbb{E} \left[ \left| \hat{F}_0^N(x) - F_0(x) \right| \right] dx \leq \int_{\mathbb{R}} \left( 1_{\{F_0(x) < 1 - \frac{1}{N} \}} \sqrt{\frac{1 - F_0(x)}{N}} + 1_{\{F_0(x) \geq 1 - \frac{1}{N} \}} (1 - F_0(x)) \right) dx
\]
\[
= \int_{1}^{N^{-1/\alpha}} x^{-\alpha/2} \sqrt{N} \, dx + \int_{N^{-1/\alpha}}^{+\infty} x^{-\alpha} \, dx
\]
\[
= \frac{2(N^{-1+1/\alpha} - N^{-1/2})}{\alpha - 1} + \frac{N^{-1+1/\alpha}}{\alpha - 1}
\]
\[
= \frac{\alpha N^{-1+1/\alpha}}{(2 - \alpha)(\alpha - 1)} + \frac{2N^{-1/2}}{\alpha - 2}.
\]

We conclude that, for $\alpha \in (1, 2)$, $\int_{\mathbb{R}} \mathbb{E} \left[ \left| \hat{F}_0^N(x) - F_0(x) \right| \right] dx = \mathcal{O}\left( N^{1+1/\alpha} \right)$ using that its lower bound
\[
\int_{\mathbb{R}} \left| \hat{F}_0^N(x) - F_0(x) \right| dx also is $\mathcal{O}\left( N^{1+1/\alpha} \right)$ according to Remark 2.2 [13].
Let us now prove Lemma 5.4.

Proof. The first assertion is an immediate consequence of (5.1). Moreover, the central limit theorem implies that, for each \( x \in \mathbb{R} \), \( \mathbb{E} \left[ \sqrt{N} \left| F_0^N(x) - F_0(x) \right| \right] \) converges to \( \sqrt{2}F_0(x)(1 - F_0(x)) / \pi \) as \( N \to \infty \). When \( \int_\mathbb{R} \sqrt{F_0(x)(1 - F_0(x))} \, dx = +\infty \), one concludes by applying Fatou’s lemma to the spatial integral with respect to \( dx \). Otherwise, one concludes by Lebesgue’s theorem, using domination deduced from the inequality (5.1).

Let us now check the implications involving the finiteness of \( \int_\mathbb{R} \sqrt{F_0(x)(1 - F_0(x))} \, dx \).

Lemma 5.6.

\[
\int_\mathbb{R} |x|^{2+\epsilon} m(dx) < \infty \Rightarrow \int_\mathbb{R} \sqrt{F_0(x)(1 - F_0(x))} \, dx < \infty \Rightarrow \int_\mathbb{R} x^2 m(dx) < \infty.
\]

The following examples show that the assertions are not equivalent.

Example 5.7. Let \( m(dx) = 1_{\{x \geq 2\}} \frac{c}{x^3 \ln^2(x)} \, dx \) where \( \frac{1}{c} = \int_2^{+\infty} \frac{dx}{x^3 \ln^2(x)} \).

Since \( \int_2^{+\infty} \frac{dy}{y^2 \ln^2(y)} = \int_{\ln(2)}^{+\infty} \frac{dy}{y \ln^2(y)} = \frac{1}{\ln(2)} < \infty \), one has \( \int_\mathbb{R} x^2 m(dx) < \infty \). On the other hand, for \( x \geq 2 \), one has, by integration by parts, \( \frac{1 - F_0(x)}{c} = \int_x^{+\infty} \frac{dy}{y^3 \ln^2(y)} = \frac{1}{2} \int_0^{+\infty} \frac{\ln(y)}{y^3} dy - \int_x^{+\infty} \frac{dy}{y^3 \ln^2(y)} \). Since \( 0 \leq \int_x^{+\infty} \frac{dy}{y^3 \ln^2(y)} \leq \frac{1}{\ln(x)} \int_x^{+\infty} \frac{dy}{y^3 \ln^2(y)} \), we deduce that, as \( x \to +\infty \), \( \frac{1 - F_0(x)}{c} \sim \frac{1}{2x^2 \ln^2(x)} \) so that \( \sqrt{F_0(x)(1 - F_0(x))} \sim \frac{\sqrt{c/2}}{x \ln(x)} \). We conclude that \( \int_\mathbb{R} \sqrt{F_0(x)(1 - F_0(x))} \, dx = +\infty \).

For \( m(dx) = 1_{\{x \geq 2\}} \frac{c}{x^3 \ln^2(x)} \, dx \) where \( \frac{1}{c} = \int_2^{+\infty} \frac{dx}{x^3 \ln^2(x)} \), one has, by similar computations, \( \frac{1 - F_0(x)}{c} \sim \frac{1}{2x^2 \ln^2(x)} \) as \( x \to +\infty \) so that \( \int_\mathbb{R} \sqrt{F_0(x)(1 - F_0(x))} \, dx < \infty \) whereas \( \int_\mathbb{R} |x|^{2+\epsilon} m(dx) = +\infty \) for each \( \epsilon > 0 \).

For the sake of completeness, we are going to reproduce the short proof of the first implication in Lemma 5.6 given in Remark 2.2 [14]. The proof of the second implication relies on the following result, the proof of which is postponed.

Lemma 5.8. One has

\[
\forall y \in \mathbb{R}, \quad \frac{1}{2} \int_\mathbb{R} (x - y)^2 m(dx) \leq \left( \int_{-\infty}^{y} \sqrt{F_0(x)} \, dx \right)^2 + \left( \int_{y}^{+\infty} \sqrt{1 - F_0(x)} \, dx \right)^2.
\]

Let us prove Lemma 5.6.

Proof. Since for \( y = F_0^{-1}(1/2), \) \( \sqrt{2} \int_\mathbb{R} \sqrt{F_0(x)(1 - F_0(x))} \, dx \geq \int_{-\infty}^{y} \sqrt{F_0(x)} \, dx + \int_{y}^{+\infty} \sqrt{1 - F_0(x)} \, dx \), the second implication is easily deduced from Lemma 5.8.

Let \( \epsilon \geq -1 \). By Fubini’s theorem,

\[
\int_{-\infty}^{0} |x|^{1+\epsilon} F_0(x) \, dx + \int_{0}^{+\infty} |x|^{1+\epsilon}(1 - F_0(x)) \, dx
\]

\[
= \int_{0}^{+\infty} |x|^{1+\epsilon} \int_{-\infty}^{0} 1_{\{y \leq x\}} m(dy) \, dx + \int_{0}^{+\infty} |x|^{1+\epsilon} \int_{0}^{+\infty} 1_{\{y > x\}} m(dy) \, dx
\]

\[
= \frac{1}{2 + \epsilon} \int_\mathbb{R} |y|^{2+\epsilon} m(dy).
\]
By Cauchy-Schwarz inequality,
\[
\left( \int_{\mathbb{R}} \sqrt{F_0(x)}(1 - F_0(x)) \, dx \right)^2 \leq \int_{\mathbb{R}} \frac{dx}{1 + |x|^{1+\varepsilon}} \int_{\mathbb{R}} (1 + |x|^{1+\varepsilon}) \, F_0(x) \, (1 - F_0(x)) \, dx,
\]
where the first integral in the right-hand side is finite when \( \varepsilon > 0 \) and, according to (5.6), the second one is finite when \( \int_{\mathbb{R}} |x|^{2+\varepsilon} m(dx) < \infty \).

The proof of Lemma 5.8 relies on the following integral formulas for the square roots of the cumulative distribution function and the survival function.

**Lemma 5.9.** Let \( m \in \mathcal{P}(\mathbb{R}) \) with cumulative distribution function \( F_0(x) \), \( x \in \mathbb{R} \). Then \( \forall x \in \mathbb{R} \),

\[
\sqrt{F_0(x)} = \int_{(0,1]} \frac{1_{\{z \geq x\}}}{2\sqrt{F_0(z^{-})} + u(F_0(z) - F_0(z^{-}))} m(dz) \, du,
\]

(5.7)

\[
\sqrt{1 - F_0(x)} = \int_{(0,1]} \frac{1_{\{z > x\}}}{2\sqrt{1 - F_0(z^{-})} - u(F_0(z) - F_0(z^{-}))} m(dz) \, du.
\]

(5.8)

Let us show (5.7) (5.8 is obtained by a symmetric reasoning) before checking Lemma 5.8.

**Proof.** If \( F_0(x) = 0 \), then \( m((\infty, x]) = 0 \) and although \( \int_{(0,1]} 2\sqrt{F_0(z^{-})} + u(F_0(z) - F_0(z^{-})) \, m(dz) \, du = +\infty \) for \( z \in (\infty, x] \), the integral in the right-hand side of (5.7) is equal to 0 by the usual convention in measure theory. If the limit \( x_0 := F_0^{-1}(0+) \) of the left-continuous function \( F_0^{-1} \) at point 0 is larger than \( -\infty \) and such that \( m(\{x_0\}) > 0 \) then since \( F_0(x_0^{-}) = m((\infty, x_0]) = 0 \), one has:

\[
\sqrt{F_0(x_0)} = \sqrt{F_0(x_0^{-})} - \sqrt{F_0(x_0^{-})} = m(\{x_0\}) \int_{(0,1]} \frac{du}{2\sqrt{F_0(x_0^{-})} + u(F_0(x) - F_0(x_0^{-}))},
\]

with the right-hand side equal to the one of (5.7) with \( x = x_0 \). So, as soon as \( x_0 > -\infty \), (5.7) holds for \( x = x_0 \).

It is enough to deal with the case \( x > x_0 \) to conclude the proof. Let now \( \varphi : [0,1] \to \mathbb{R} \) be C\(^1\). The chain rule for càdlàg functions with finite variation (see for instance Proposition 4.6 Chapter 0 [21]) writes

\[
d\varphi(F_0(z)) = \varphi'(F_0(z^{-})) dF_0(z) + \varphi(F_0(z)) - \varphi(F_0(z^{-})) - \varphi'(F_0(z^{-}))(F_0(z) - F_0(z^{-}))
\]

\[
= 1_{\{F_0(z^{-}) = F_0(z)\}} \varphi'(F_0(z^{-})) dF_0(z) + 1_{\{F_0(z^{-}) < F_0(z)\}} \frac{\varphi(F_0(z)) - \varphi(F_0(z^{-}))}{F_0(z) - F_0(z^{-})} dF_0(z)
\]

where \( dF_0(z) = m(dz) \). Since

\[
\int_{(0,1]} \varphi'(F_0(z^{-}) + u(F_0(z) - F_0(z^{-}))) \, du = 1_{\{F_0(z^{-}) = F_0(z)\}} \varphi'(F_0(z^{-})) + 1_{\{F_0(z^{-}) < F_0(z)\}} \frac{\varphi(F_0(z)) - \varphi(F_0(z^{-}))}{F_0(z) - F_0(z^{-})},
\]

we deduce that:

\[
d\varphi(F_0(z)) = \int_{(0,1]} \varphi'(F_0(z^{-}) + u(F_0(z) - F_0(z^{-}))) \, du \, m(dz).
\]

Let \( x > x_0 \) and \( y \in (x_0, x) \). By definition of \( x_0 \), one has \( 0 < \sqrt{F_0(y)} \leq \sqrt{F_0(x)} \) and, by choosing some C\(^1\) function \( \varphi \) which coincides with the square root on \( [F_0(y), +\infty) \), we deduce that:

\[
\sqrt{F_0(x)} = \sqrt{F_0(y)} + \int_{(0,1]} \int_{\mathbb{R}} \frac{1_{\{y < z < x\}}}{2\sqrt{F_0(z^{-})} + u(F_0(z) - F_0(z^{-}))} \, m(dz) \, du
\]

We conclude by letting \( y \) decrease to \( x_0 \) in this inequality using monotone convergence to deal with the integral and using the right-continuity of \( F_0 \) together with (5.7) for \( x = x_0 \) when \( x_0 > -\infty \).

Let us now prove Lemma 5.8.
Proof. With Fubini’s theorem, one deduces from the first equality in Lemma 5.9 that

(5.9) \[
\int_{-\infty}^{y} \sqrt{F_0(x)} \, dx = \int_{u=0}^{1} \int_{\mathbb{R}} \frac{1_{\{z \leq y\}}(y-z)}{2\sqrt{F_0(z-)} + u(F_0(z) - F_0(z-))} m(dz) \, du.
\]

By the monotonicity of \( F_0 \), \( \forall z < y, \ \int_{-\infty}^{y} \sqrt{F_0(x)} \, dx \geq \int_{-\infty}^{y} \sqrt{F_0(y)} \, dx \geq \sqrt{F_0(y)}(y-z) \) which implies that
\[\forall z < y, \ \forall u \in [0,1], \ \int_{\mathbb{R}} \frac{1_{\{z \leq y\}}(y-z)^2}{2\sqrt{F_0(z-)} + u(F_0(z) - F_0(z-))} m(dz) \, du \leq \frac{1}{2} \int_{\mathbb{R}} 1_{\{z \leq y\}}(y-z)^2 m(dz). \]

With (5.9), one deduces that
\[\int_{-\infty}^{\infty} \sqrt{F_0(x)} \, dx \geq \frac{1}{2} \int_{\mathbb{R}} 1_{\{z \leq y\}}(y-z)^2 m(dz). \]

One concludes by summing this inequality with
\[\int_{y}^{+\infty} \sqrt{1-F_0(x)} \, dx \geq \frac{1}{2} \int_{\mathbb{R}} 1_{\{z \geq y\}}(y-z)^2 m(dz), \]
obtained in a symmetric way by using the second equality in Lemma 5.9.

Let us now deal with the implication in Proposition 2.3 concerning the finiteness of \( \sup_{N \geq 1} \sqrt{N} \int_{\mathbb{R}} |\hat{F}_N(x) - F_0(x)| \, dx \).

Lemma 5.10. If \( \sup_{x \geq 1} x \int_{x}^{+\infty} (F_0(-y) + 1 - F_0(y)) \, dy < \infty \), then \( \sup_{N \geq 1} \sqrt{N} \int_{\mathbb{R}} |\hat{F}_N(x) - F_0(x)| \, dx < \infty \).

Moreover, \[
\int_{\mathbb{R}} |x|^2 m(dx) < \infty \Rightarrow \sup_{x \geq 1} x \int_{x}^{+\infty} (F_0(-y) + 1 - F_0(y)) \, dy < \infty \Rightarrow \int_{\mathbb{R}} |x|^2 m(dx) < \infty.
\]

Remark 5.11. By [4] p.4975, if for some \( y \in \mathbb{R} \), the restriction of \( m \) to \( [y, +\infty) \) (resp. \( (-\infty, y] \)) has a positive non-increasing (resp. non-decreasing) density with respect to the Lebesque measure then for \( N \) large enough, \( \lim \inf_{N \to \infty} \frac{1}{\sqrt{N}} \int_{\mathbb{R}} |\hat{F}_N(x) - F_0(x)| \, dx \geq \lim sup_{N \to \infty} \frac{F_0^{-1}(1-1/4N)}{\sqrt{N}} \) (resp. \( \lim sup_{N \to \infty} |F_0^{-1}(1/4N)| / \sqrt{N} \)).

Hence \( \sup_{N \geq 1} \sqrt{N} \int_{\mathbb{R}} |\hat{F}_N(x) - F_0(x)| \, dx < \infty \) implies the existence of \( C \in (0, +\infty) \) such that for \( N \) large enough, \( F_0^{-1}(1-1/4N) \leq C \sqrt{N} \) and \( \frac{1}{4N} = 1 - F_0(F_0^{-1}(1-1/4N)) \geq 1 - F_0(C \sqrt{N}) \). Since for \( x \in [C \sqrt{N}, C \sqrt{N} + 1] \), \( \frac{C^2(N+1)}{4x^2} \leq \frac{C^2(N+1)}{4x^2 \sqrt{N}} \leq C^2 \), we deduce that \( 1 - F_0(x) \leq \frac{C^2}{2x^2} \) (resp. \( F_0(-x) \leq \frac{C^2}{2x^2} \)) for \( x \) large enough so that \( \sup_{x \geq 1} x \int_{x}^{+\infty} (1 - F_0(y)) \, dy < \infty \) (resp. \( \sup_{x \geq 1} x \int_{x}^{+\infty} F_0(y) \, dy < \infty \)).

Before proving the lemma, let us exhibit a measure \( m \) such that \( \int_{\mathbb{R}} |x|^2 m(dx) < \infty \) and \( \sup_{N \geq 1} \sqrt{N} \int_{\mathbb{R}} |\hat{F}_N(x) - F_0(x)| \, dx = \infty \) so that, by the first assertion in Lemma 5.10, \( \sup_{x \geq 1} x \int_{x}^{+\infty} (F_0(-y) + 1 - F_0(y)) \, dy < \infty \). We first exhibit a measure \( m \) such that \( \sup_{x \geq 1} x \int_{x}^{+\infty} (F_0(-y) + 1 - F_0(y)) \, dy < \infty \) and \( \int_{\mathbb{R}} |x|^2 m(dx) = \infty \).

Example 5.12. If \( m(dx) = 1_{\{x \geq 1\}} \frac{2}{x^2} dx \), then \( \int_{\mathbb{R}} |x|^2 m(dx) = +\infty \) whereas \( \forall x \geq 1, F_0(-x) = 0 \) and \( 1 - F_0(x) = \frac{1}{x^2} \) so that \( x \int_{x}^{+\infty} (1 - F_0(y)) \, dy = 1 \).

Let now \( m(dx) = 1_{\{x \geq 1\}} \frac{c \ln(x)}{x^3} dx \) where \( 1/c = \int_{1}^{+\infty} \frac{\ln(x)}{x^3} dx \). One has \( \int_{\mathbb{R}} |x|^2 \varepsilon m(dx) < \infty \) for all \( \varepsilon \in [0,2) \). Let us check that \( \lim_{N \to \infty} \sqrt{N} \int_{F_0^{-1}(1-1/2N)}^{+\infty} (1 - F_0(x)) \, dx = +\infty \), which, in view of (2.30), implies that
\[
\lim_{N \to \infty} \sqrt{N} \int_{\mathbb{R}} |\hat{F}_0^N(x) - F_0(x)| \, dx = +\infty \quad \text{and, by the first assertion in Lemma 5.11, that} \quad \sup_{x \geq 1} \int_x^{+\infty} (F_0(-y) + 1 - F_0(y)) \, dy = +\infty.
\]
Using the integration by parts formula like in Examples 5.7, we check that, as \( x \to +\infty, \)
\[
1 - F_0(x) \sim \frac{c\ln(x)}{2x^2} \quad \text{and} \quad \int_x^{+\infty} (1 - F_0(y)) \, dy \sim \frac{c\ln(x)}{2x}. \quad \text{Since} \quad \lim_{u \to +\infty} F_0^{-1}(u) = +\infty, \quad \text{one has, for} \quad N \quad \text{large enough,}
\]
\[
\frac{c\ln(F_0^{-1}(1 - 1/2N))}{F_0^{-1}(1 - 1/2N)^2} \geq 1 - F_0(F_0^{-1}(1 - 1/2N)) = \frac{1}{2N} \quad \text{and} \quad \int_{F_0^{-1}(1 - 1/2N)}^{+\infty} (1 - F_0(x)) \, dx \geq \frac{c\ln(F_0^{-1}(1 - 1/2N))}{4F_0^{-1}(1 - 1/2N)} \geq \frac{c\ln(F_0^{-1}(1 - 1/2N))}{4\sqrt{2N}}.
\]

**Proof.** Let us first assume the existence of \( C \in (0, +\infty) \) s.t. \( \forall x \geq 1, \int_{-\infty}^{-x} F_0(y) \, dy + \int_{x}^{+\infty} (1 - F_0(y)) \, dy < \frac{C}{x} \).
We have:
\[
\int_{F_0^{-1}(1/2N)}^{+\infty} (1 - F_0(x)) \, dx \leq \int_{F_0^{-1}(1/2N)}^{+\infty} (1 - F_0(x)) \, dx + \int_{x}^{+\infty} (1 - F_0(x)) \, dx \leq \frac{c\ln(F_0^{-1}(1/2N))}{2N} + \frac{C}{\sqrt{N}}.
\]
where, for the second inequality, we used that \( 1 - F_0(x) \leq 1/2N \) for \( x \geq F_0^{-1}(1 - 1/2N) \) and \( F_0^{-1}(1 - 1/2N) \geq F_0^{-1}(1/2) \) to deal with the first integral and the hypothesis to deal with the second one. In a symmetric way, we check that
\[
\int_{-\infty}^{F_0^{-1}(1/2N)} F_0(x) \, dx \leq \frac{1}{2N} \leq \frac{1}{2N} - F_0(F_0^{-1}(1 - 1/2N)) \geq \frac{1}{2N} \text{ so that} \quad F_0^{-1}(1 - 1/2N) \leq 1 - \frac{1}{\frac{1}{2N} - F_0(F_0^{-1}(1 - 1/2N))} \geq 2 \left( 1 + \sqrt{2CN} \right). \quad \text{By a symmetric reasoning,}
\]
\[
F_0(x) \leq \frac{4C}{x^2} \quad \text{for} \quad x \leq 2 \quad \text{and} \quad F_0^{-1}(1/2N) \geq -2 \left( 1 + \sqrt{2CN} \right). \quad \text{Since the integrand in (5.10) is smaller than}
\]
\[
1/2N, \quad \text{combining these bounds on} \quad F_0^{-1}(1/2N) \quad \text{and} \quad F_0^{-1}(1 - 1/2N) \quad \text{with (5.10) and the symmetric estimation, we deduce that}
\]
\[
\int_{\mathbb{R}} \left| \hat{F}_0^N(x) - F_0(x) \right| \, dx \leq \frac{2 \left( 1 + \sqrt{2CN} \right)}{N} + \frac{\sqrt{N} - F_0^{-1}(1/2)}{2N} + \frac{2C}{\sqrt{N}},
\]
which concludes the proof of the first assertion.

Since, according to (5.0) for \( \varepsilon = 0 \),
\[
\int_{\mathbb{R}} y^2 m(dy) = 2 \left( \int_{0}^{+\infty} y(1 - F_0(y)) \, dy - \int_{-\infty}^{0} yF_0(y) \, dy \right), \quad \text{one has}
\]
\[
\forall x > 0, \quad \int_{-\infty}^{-x} F_0(y) \, dy + \int_{x}^{+\infty} (1 - F_0(y)) \, dy \leq \frac{1}{2x} \int_{\mathbb{R}} y^2 m(dy).
\]
Let us last check that the existence of \( C \in (0, +\infty) \), \( \forall x \geq 1, \int_{-\infty}^{-x} F_0(y) \, dy + \int_{x}^{+\infty} (1 - F_0(y)) \, dy \leq C \)
implies that \( \int_{\mathbb{R}} |x|^{2-\varepsilon} m(dx) < \infty \) for all \( \varepsilon \in (0, 1) \) (and therefore all \( \varepsilon \in (0, 2) \)). For \( \varepsilon = 1 \), we have
\[
\int_{\mathbb{R}} |x| \, m(dx) = \int_{0}^{+\infty} (F_0(-y) + (1 - F_0(y))) \, dy \leq 1 + \int_{1}^{+\infty} (F_0(-y) + (1 - F_0(y))) \, dy \leq 1 + C.
\]
Let now \( \varepsilon \in (0,1) \). Using Fubini’s theorem then the integration by parts formula and the fact that
\[
\lim_{y \to +\infty} y^{1-\varepsilon} \int_{-y}^{+\infty} (1 - F_0(z)) \, dz = 0,
\]
we conclude that
\[
\frac{1}{2 - \varepsilon} \int_{\mathbb{R}} x^{2-\varepsilon} m(dx) \leq (1 - \varepsilon) \int_{0}^{1} y^{-\varepsilon} \int_{0}^{+\infty} (1 - F_0(z)) \, dz \, dy \leq (1 - \varepsilon) \int_{0}^{1} y^{-\varepsilon} \int_{y}^{+\infty} (1 - F_0(z)) \, dz \, dy.
\]

Combining this inequality with the symmetric one then using the above estimation of \( \int_{0}^{+\infty} (F_0(-y) + (1 - F_0(y))) \, dy \), we conclude that
\[
\frac{1}{2 - \varepsilon} \int_{\mathbb{R}} x^{2-\varepsilon} m(dx) \leq (1 - \varepsilon) \int_{0}^{1} y^{-\varepsilon} \int_{0}^{+\infty} (1 - F_0(z)) \, dz \, dy + (1 - \varepsilon) C \int_{1}^{+\infty} y^{-1-\varepsilon} \, dy \leq 1 + \frac{C}{\varepsilon}.
\]

\( \blacksquare \)

### 5.3 Further properties of the optimal deterministic initialization

According to Proposition 2.4 when \( m \) is compactly supported, then \( \sup_{N \geq 1} N \int_{\mathbb{R}} |\hat{F}_0^N(x) - F_0(x)| \, dx < \infty \).

The next proposition states that for the latter property to hold when \( m \) has a density with respect to the Lebesgue measure, then the Lebesgue measure of the set where this density is finite must be finite.

**Proposition 5.13.** If \( m(dx) = f(x) \, dx \), then
\[
\liminf_{N \to \infty} N \int_{\mathbb{R}} |\hat{F}_0^N(x) - F_0(x)| \, dx \geq \frac{1}{4} \int_{\mathbb{R}} 1_{\{f(x) > 0\}} \, dx.
\]

**Remark 14.** Lemma 2.1 [4] refines the statement when \( f \) is positive on a bounded interval \([c,d]\) and equal to 0 outside by stating that \( \lim_{N \to \infty} N \int_{\mathbb{R}} |\hat{F}_0^N(x) - F_0(x)| \, dx = \frac{d - c}{4} \).

**Proof.** Let us suppose that \( m(dx) = f(x) \, dx \). The continuity of \( F_0 \) implies that \( F_0(F_0^{-1}(v)) = v \) for each \( v \in (0,1) \). Using this equation in the first equality then the inverse transform sampling for the second equality and last that \( F_0^{-1}(v) \leq y \iff v \leq F_0(y) \), we have, for \( u \in \left( \frac{1}{2N}, 1 \right) \),
\[
\int_{u - \frac{1}{2N}}^{u} \frac{dv}{f(F_0^{-1}(v))} = \int_{0}^{1} 1_{\left\{ u - \frac{1}{2N} \leq F_0(F_0^{-1}(v)) < u \right\}} \frac{dv}{f(F_0^{-1}(v))} = \int_{0}^{1} 1_{\left\{ u - \frac{1}{2N} \leq F_0(x) < u \right\}} \frac{m(dx)}{f(x)} = \int_{0}^{1} 1_{\left\{ F_0^{-1}(u - \frac{1}{2N}) \leq x < F_0^{-1}(u) \right\}} 1_{\{f(x) > 0\}} \, dx
\]
\[
= F_0^{-1}(u) - F_0^{-1} \left( u - \frac{1}{2N} \right) - \int_{0}^{1} 1_{\left\{ u - \frac{1}{2N} \leq F_0(x) < u \right\}} 1_{\{f(x) = 0\}} \, dx.
\]

With (2.0), we deduce that
\[
\int_{\mathbb{R}} |\hat{F}_0^N(x) - F_0(x)| \, dx \geq \sum_{i=1}^{N} \int_{\frac{i-1}{2N}}^{\frac{i+1}{2N}} \int_{u - \frac{1}{2N}}^{u} \frac{dv}{f(F_0^{-1}(v))} \, du = \sum_{i=1}^{N} \int_{\frac{i-1}{2N}}^{\frac{i+1}{2N}} \left( v - \frac{i-1}{N} \right) \wedge \left( v - \frac{i+1}{N} \right) \frac{dv}{f(F_0^{-1}(v))} = \frac{1}{N} \int_{0}^{1} \min_{j \in \mathbb{N}} |Nv - j| \frac{dv}{f(F_0^{-1}(v))},
\]
(5.11)
where we used Fubini’s theorem for the first equality. Let $K \in (0, +\infty)$ be some cutoff parameter and $\varphi_K(v) = \frac{1}{f(F_0^{-1}(v))} \wedge K$. For $\varepsilon > 0$, by density of the continuous functions in the space of integrable functions on $[0, 1]$ endowed with the Lebesgue measure, there exists a continuous function $\varphi_{K, \varepsilon}$ such that $\int_0^1 |\varphi_{K, \varepsilon}(v) - \varphi_K(v)| \, dv \leq \varepsilon$. One has

$$\left| \int_0^1 \varphi_K(v)(4 \min_{j \in \mathbb{N}} |Nv - j| \, dv - \int_0^1 \varphi_K(v) \, dv \right| \leq \int_0^1 |\varphi_K(v) - \varphi_{K, \varepsilon}(v)| \, dv$$

$$+ \left| \int_0^1 4 \min_{j \in \mathbb{N}} |Nv - j| \varphi_{K, \varepsilon}(v) \, dv - \int_0^1 \varphi_{K, \varepsilon}(v) \, dv \right| + \int_0^1 |\varphi_{K, \varepsilon}(v) - \varphi_K(v)| \, dv.$$

Since for each $N \geq 1$, $\sup_{v \in [0, 1]} 4 \min_{j \in \mathbb{N}} |Nv - j| \leq 2$, the sum of the first and third terms in the right-hand side is smaller than $3\varepsilon$. On the other hand, the second term goes to 0 as $N \to \infty$, since the probability measures with densities $41_{\{0 \leq v \leq 1\}} \min_{j \in \mathbb{N}} |Nv - j|$ with respect to the Lebesgue measure converge weakly to the uniform distribution on $[0, 1]$. Hence for each $K \in (0, +\infty)$, $\lim_{N \to \infty} \int_0^1 \varphi_K(v)4 \min_{j \in \mathbb{N}} |Nv - j| \, dv = \int_0^1 \varphi_K(v) \, dv$. We deduce that $\liminf_{N \to \infty} \int_0^1 4 \min_{j \in \mathbb{N}} |Nv - j| \frac{dv}{f(F_0^{-1}(v))} \geq \int_0^1 \varphi_K(v) \, dv$ where, by monotone convergence, the right-hand side converges to $\int_0^1 \frac{dv}{f(F_0^{-1}(v))}$ as $K \to \infty$. With (5.11) and the inverse transform sampling, we conclude that

$$\lim_{N \to \infty} \int_{\mathbb{R}} \left| \hat{F}_0^N(x) - F_0(x) \right| \, dx \geq \frac{1}{4} \int_{\mathbb{R}} \frac{1}{f(x)} \, dm(dx) = \frac{1}{4} \int_{\mathbb{R}} 1_{\{f(x) > 0\}} \, dx.$$

Let us finally check that the moments of the empirical measure of the optimal deterministic initial positions converge to those of $m$ as $N \to \infty$.

**Lemma 5.15.** For $\rho > 0$, one has $\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^N \left| F_0^{-1} \left( \frac{2i - 1}{2N} \right) \right|^\rho = \int_\mathbb{R} |x|^\rho \, dm(dx)$ and

$$\forall N \in \mathbb{N}^*, \frac{1}{N} \sum_{i=1}^N \left| F_0^{-1} \left( \frac{2i - 1}{2N} \right) \right|^\rho \leq 2 \int_\mathbb{R} |x|^\rho \, dm(dx).$$

**Proof.** One has $\frac{1}{N} \sum_{i=1}^N \left| F_0^{-1} \left( \frac{2i - 1}{2N} \right) \right|^\rho = \int_0^1 \left| F_0^{-1} \left( \frac{2i - 1}{2N} \right) \right|^\rho \, du$. Since for $u \in (0, 1)$, $\lim_{N \to \infty} \frac{2i}{2N} = u$ and the set of discontinuities of the non-decreasing function $F_0^{-1}$ is at most countable, we deduce from Fatou’s lemma that $\liminf_{N \to \infty} \frac{1}{N} \sum_{i=1}^N \left| F_0^{-1} \left( \frac{2i - 1}{2N} \right) \right|^\rho \geq \int_0^1 |F_0^{-1}(u)|^\rho \, du = \int_\mathbb{R} |x|^\rho \, dm(dx)$, where the last equality follows from the inverse transform sampling. This concludes the proof of the first assertion when $\int_0^1 |F_0^{-1}(u)|^\rho \, du = \infty$. Since

$$\forall i \in \{1, \ldots, N-1\}, \quad F_0^{-1} \left( \frac{2i - 1}{2N} \right) \geq 0 \Rightarrow \frac{|F_0^{-1} \left( \frac{2i - 1}{2N} \right)|^\rho}{N} \leq \int_{\frac{2i - 1}{2N}}^{\frac{2i + 1}{2N}} |F_0^{-1}(u)|^\rho \, du,$$

$$\forall i \in \{2, \ldots, N\}, \quad F_0^{-1} \left( \frac{2i - 1}{2N} \right) \leq 0 \Rightarrow \frac{|F_0^{-1} \left( \frac{2i - 1}{2N} \right)|^\rho}{N} \leq \int_{\frac{2i - 1}{2N}}^{\frac{2i + 1}{2N}} |F_0^{-1}(u)|^\rho \, du,$$
one has
\[
\frac{1}{N} \sum_{i=1}^{N} \left| F_0^{-1} \left( \frac{2i - 1}{2N} \right) \right|^\rho \leq \frac{1}{N} \sum_{i=1}^{N} \left| \frac{F_0^{-1} \left( \frac{1}{2N} \right)^\rho}{\frac{1}{2N}} + \int_{\frac{1}{2N}}^{\frac{2N-1}{2N}} |F_0^{-1}(u)|^\rho \, du + \int_{\frac{2N-1}{2N}}^{1} |F_0^{-1}(u)|^\rho \, du \right|
\]
\[
\leq \int_0^1 |F_0^{-1}(u)|^\rho \, du + \int_0^1 |F_0^{-1}(u)|^\rho \, du + \int_{\frac{2N-1}{2N}}^{1} |F_0^{-1}(u)|^\rho \, du,
\]
from which, we deduce the second assertion. When \( \int_0^1 |F_0^{-1}(u)|^\rho \, du < \infty \), by Lebesgue’s theorem,
\[
\lim_{N \to \infty} \left( \int_0^1 |F_0^{-1}(u)|^\rho \, du + \int_{\frac{2N-1}{2N}}^{1} |F_0^{-1}(u)|^\rho \, du \right) = 0.
\]
We deduce that \( \lim \sup_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \left| F_0^{-1} \left( \frac{2i - 1}{2N} \right) \right|^\rho \leq \int_0^1 |F_0^{-1}(u)|^\rho \, du \), which concludes the proof. \( \square \)

6 Numerical experiments for the Burgers equation

In order to confirm our theoretical estimates for the strong and the weak \(L^1\)-error between \(F^{N,h}\) and its limit \(F\), we consider, for the choice \(\Lambda(u) = (1-u)^2/2\) and the initial condition \(F(0,x) = 1_{\{x \geq 0\}}\), the following equation:
\[
\begin{cases}
\partial_t F(t,x) - \partial_x F(t,x) \left(1 - F(t,x)\right) = \frac{\sigma^2}{2} \partial_{xx} F(t,x), \\
F(0,x) = 1_{\{x \geq 0\}}.
\end{cases}
\]
We can notice that the function \((1 - F(t,.))\) is solution of the Burgers equation that was also used in [4].

The Cole-Hopf transformation yields the following closed-form expression of \(F\):
\[
F(t,x) = 1 - \frac{\mathcal{N} \left( \frac{x}{\sigma \sqrt{t}} \right)}{\mathcal{N} \left( \frac{1-x}{\sigma \sqrt{t}} \right) + \exp \left( \frac{2x-1}{2\sigma^2 t} \right) \mathcal{N} \left( \frac{x}{\sigma \sqrt{t}} \right)},
\]
where \(\mathcal{N}(x) = \int_{-\infty}^{x} \frac{\exp(y^2/2)}{\sqrt{2\pi}} dy\).

The drift coefficient of the \(i^{th}\) particle in the increasing order is then equal to \(\lambda^N(i) = 1 - \frac{2i-1}{2N}\) and the Euler discretization with step \(h \in [0,T]\) of the particle system is:
\[
dX_{i}^{N,h} = \sigma dW_{t}^{i} + \left(1 + \frac{1}{2N} - \frac{1}{N} \sum_{j=1}^{N} 1_{\{X_{i}^{j,N,h} \leq X_{i}^{k,N,h}\}} \right) dt, \quad 1 \leq i \leq N, \quad t \in [0,T].
\]
As \(F_0\) is the cumulative function of the Dirac mass centered at zero, we place the \(N\) particles at zero for their initialisation.

We seek to observe the dependence of the strong \(L^1\)-error \(\mathbb{E} \left[ \mathcal{W}_1 \left( \mu_T^{N,0}, \mu_T \right) \right]\) and the weak \(L^1\)-error \(\mathcal{W}_1 \left( \mathbb{E} \left[ \mu_T^{N,h} \right], \mu_T \right)\) at time \(T\) on the number \(N\) of particles and on the time step \(h\). We recall (2.1) and (2.2) where the Wasserstein distance between a probability measure \(\nu\) and \(\mu_T\) can be expressed either using the quantile functions or the cumulative distribution functions:
\[
\mathcal{W}_1 (\nu, \mu_T) = \int_0^1 \left| F_\nu^{-1}(u) - F_T^{-1}(u) \right| \, du \]
\[
= \int_{\mathbb{R}} \left| F_\nu(x) - F(T,x) \right| \, dx.
\]
We choose to use the second expression because we have an explicit formula for $F(T, \cdot)$ unlike the inverse $F_T^{-1}(\cdot)$ (which can still be numerically estimated but this is costly and induces additional numerical error).

When $\nu$ is an empirical measure of the form $\frac{1}{N} \sum_{i=1}^{N} \delta_{x^i}$, we choose to approximate the $\mathcal{W}_1$ distance not using a grid in the following way. For $(y^1)_{1 \leq i \leq N}$ denoting the increasing reordering of $(x^i)_{1 \leq i \leq N}$, we have:

$$\mathcal{W}_1 \left( \frac{1}{N} \sum_{i=1}^{N} \delta_{x^i}, \mu_T \right) = \Psi (y_1, y_2, \ldots, y_N)$$

where

$$\Psi (y_1, y_2, \ldots, y_N) = \sum_{i=1}^{N-1} \frac{1}{2} \left( y^{i+1} - y^i \right) \left( \left| F(T,y^{i+1}) - \frac{i}{N} \right| + \left| F(T,y^i) - \frac{i}{N} \right| \right).$$

Therefore, for the strong $L^1$-error, $(Y_{i,N,h}^{t})_{i \in [1,N]}$ being the increasing reordering of the particles positions $(X_{i,r}^{t,N,h})_{i \in [1,N]}$, $t > 0$ in the $r^{th}$ run out of $R$ Monte-Carlo runs, we obtain the following approximation:

$$\mathbb{E} \left[ \mathcal{W}_1 \left( \mu_T^{N,h}, \mu_T \right) \right] \simeq \frac{1}{R} \sum_{r=1}^{R} \Psi \left( Y_{T,r}^{1,N,h}, \ldots, Y_{T,r}^{N,N,h} \right).$$

We also define the precision of this estimation as half the width of the 95% confidence interval of the empirical error i.e. Precision $= 1.96 \times \sqrt{\text{Variance} / R}$ where Variance denotes the empirical variance over the runs of the empirical error over the particles.

Concerning the weak $L^1$-error, we approximate $\mathbb{E} \left[ \mu_T^{N,h} \right]$ by $\frac{1}{R \times N} \sum_{r=1}^{R} \sum_{i=1}^{N} \delta_{X_{i,T}^{r,N,h}}$. But as $R \times N$ will be as big as $10^8$ in our simulations, rather than using the previous grid free approximation, we use the grid $(F^{-1}_{T} (\frac{k}{K}))_{1 \leq k \leq K-1}$ ($K$ will be chosen equal to 5000) to compute the $\mathcal{W}_1$ distance. For $k \in [0, K-1]$ and $x \in [F_{T}^{-1}(\frac{k}{K}), F_{T}^{-1}(\frac{k+1}{K})]$, we make the following approximation $F(T, x) \simeq \frac{2k+1}{2K}$. We also define the function $\varphi$ as:

$$\varphi (u_0, u_1, \ldots, u_{K-1}) = \sum_{k=1}^{K-2} \left[ u_k - \frac{2k+1}{2K} \right] \left( F_{T}^{-1} \left( \frac{k+1}{K} \right) - F_{T}^{-1} \left( \frac{k}{K} \right) \right) + 2 \left[ u_{K-1} - \frac{1}{2K} \right] \left( F_{T}^{-1} \left( 1 - \frac{1}{2K} \right) - F_{T}^{-1} \left( 1 - \frac{1}{2K} \right) \right) + 2 \left[ u_{K-1} - \left( \frac{1}{2K} \right) \right] \left( F_{T}^{-1} \left( \frac{1}{K} \right) - F_{T}^{-1} \left( \frac{1}{2K} \right) \right).$$

Therefore, we can approach the weak $L^1$-error by $\mathcal{W}_1 \left( \mathbb{E} \left[ \mu_T^{N,h} \right], \mu_T \right) \simeq \varphi \left( \frac{1}{R} \sum_{r=1}^{R} F_{r,T}^{N,h} (T, F_{T}^{-1} (\frac{2k+1}{2K})) \right)_{0 \leq k \leq K-1}$. We divide the $R$ runs into $B$ batches of $M = R / B$ independent simulations in order to estimate the associated precision. Indeed, we estimate the empirical variance over the batches while estimating the weak error for each independent simulation over the batches. And by the delta method, we may expect that:

$$\sqrt{R} \left[ \varphi \left( \frac{1}{R} \sum_{r=1}^{R} F_{r,T}^{N,h} (T, F_{T}^{-1} (\frac{2k+1}{2K})) \right)_{0 \leq k \leq K-1} \right] - \varphi \left( \mathbb{E} \left[ F_{T}^{N,h} (T, F_{T}^{-1} (\frac{2k+1}{2K})) \right]_{0 \leq k \leq K-1} \right) \xrightarrow{L} \mathcal{N} \left( 0, \nabla \varphi^T \left( \mathbb{E} \left[ F_{T}^{N,h} (T, F_{T}^{-1} (\frac{2k+1}{2K})) \right]_{0 \leq k \leq K-1} \right), \text{Cov} \left( \left( F_{T}^{N,h} (T, F_{T}^{-1} (\frac{2k+1}{2K})) \right)_{0 \leq k \leq K-1} \right) \right) \nabla \varphi \left( \mathbb{E} \left[ F_{T}^{N,h} (T, F_{T}^{-1} (\frac{2k+1}{2K})) \right]_{0 \leq k \leq K-1} \right) \right).$$

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where $\rho \to +\infty$. Applying this result with $\rho = M$ and $\rho = R$, one expects that $1/B$ times the empirical variance of $\left( \frac{1}{M} \sum_{r=(b-1)M+1}^{bM} F_r^{N,h}(T, F_T^{-1}(\frac{2k+1}{2K})) \right)_{0 \leq k \leq K-1}$ over the batches provides an estimator of the variance of $\varphi \left( \frac{1}{R} \sum_{r=1}^{R} F_r^{N,h}(T, F_T^{-1}(\frac{2k+1}{2K})) \right)_{0 \leq k \leq K-1}$. So the precision is computed as $1.96$ times the square root of this estimator.

For both of the errors, we fix the time horizon $T = 1$ and the diffusion coefficient $\sigma^2 = 0.2$.

6.1 Strong $L^1$-error behaviour

We present numerical estimates of $E \left[ \mathcal{W}_1(M^{N,h}, \mu_T) \right]$, computed as described above.

- **Dependence on $N$:**

  We fix the time-step $h = 0.002$ small enough in order to observe the effect of the number $N$ of particles on the error. The simulation is done with $R = 100$ Monte-Carlo runs. We obtain the following results for the estimation of the error and the associated precision:

  | Number of particles $N$ | Estimation | Precision | Ratio of decrease |
  |-------------------------|------------|-----------|------------------|
  | 250                     | 0.03312361 | 0.00290442| ×                |
  | 1000                    | 0.01598253 | 0.00133181| 2.07             |
  | 4000                    | 0.00841976 | 0.00077491| 1.90             |
  | 16000                   | 0.00358799 | 0.00028319| 2.35             |
  | 64000                   | 0.00193416 | 0.00016111| 1.86             |

  We observe that the ratio of successive estimations $\frac{\text{Estimation}(N/4)}{\text{Estimation}(N)}$ is around 2 when we multiply $N$ by 4, which means that the strong $L^1$-error is roughly proportional to $N^{-1/2}$.

- **Dependence on $h$:**

  We apply the same strategy to study the dependence of the error on $h$ by choosing a large number $N = 150000$ of particles. The following table presents numerical estimates of the $L^1$-norm of the error and its associated precision for $R = 100$ runs.

  | Time-step $h$ | Estimation | Precision | Ratio of decrease |
  |--------------|------------|-----------|------------------|
  | 1/2          | 0.07963047 | $1.1036 \times 10^{-4}$ | ×                |
  | 1/4          | 0.03545634 | $9.6417 \times 10^{-5}$ | 2.25             |
  | 1/8          | 0.01689939 | $9.3112 \times 10^{-5}$ | 2.10             |
  | 1/16         | 0.00824982 | $1.0960 \times 10^{-4}$ | 2.05             |
  | 1/32         | 0.00415826 | $9.8371 \times 10^{-5}$ | 1.98             |
  | 1/64         | 0.00226494 | $1.0239 \times 10^{-4}$ | 1.84             |
  | 1/128        | 0.00150670 | $1.0647 \times 10^{-4}$ | 1.50             |
  | 1/256        | 0.00126318 | $9.9860 \times 10^{-5}$ | 1.19             |
  | 1/512        | 0.00140416 | $1.1432 \times 10^{-4}$ | 0.99             |

  We observe that when the time step $h$ between $1/2$ and $1/64$ is divided by 2, the ratio of decrease $\frac{\text{Estimation}(h)}{\text{Estimation}(h/2)}$ is approximately equal to 2. But when $h$ becomes small, the error starts to seem constant because for so small discretization steps the effect of $N$ cannot be neglected unless $N$ is extremely large.
### 6.2 Weak $L^1$-error behaviour

We present numerical estimates of $W_1\left(E\left[\mu_N^{T,h}\right],\mu_T\right)$, computed as described above.

**Dependence on $N$:**

We fix the time-step $h = 0.002$ small enough once again to observe the effect of the number $N$ of particles on the weak error. The estimation is done with $B = 100$ batches of $M = 200$ independent simulations for a total of $R = 20000$ Monte-Carlo runs and $K = 5000$. The results are shown in the following table:

| Number of particles $N$ | Estimation | Precision | Ratio of decrease |
|-------------------------|------------|-----------|------------------|
| 100                     | 0.01018160 | $5.6947 \times 10^{-4}$ | $\times$          |
| 200                     | 0.00483151 | $3.8455 \times 10^{-4}$ | 2.11             |
| 400                     | 0.00248807 | $2.0485 \times 10^{-4}$ | 1.94             |
| 800                     | 0.00136491 | $1.4707 \times 10^{-4}$ | 1.82             |
| 1600                    | 0.00077723 | $1.0822 \times 10^{-4}$ | 1.76             |
| 3200                    | 0.00038285 | $4.9747 \times 10^{-5}$ | 2.03             |

We observe that multiplying the number of particles by 2 implies a division of the error estimation by approximately 2 which proves that the weak $L^1$-error is roughly proportional to $N^{-1}$.

**Dependence on $h$:**

Once again, we do the same to study the dependence of the weak error on $h$ by choosing a large number $N = 100000$ of particles, $B = 20$ batches of $M = 50$ independent simulations for a total of $R = 1000$ Monte-Carlo runs and $K = 5000$.

| Time-step $h$ | Estimation | Precision | Ratio of decrease |
|---------------|------------|-----------|------------------|
| 1/2           | 0.07954397 | $4.7356 \times 10^{-6}$ | $\times$          |
| 1/4           | 0.03546112 | $4.7932 \times 10^{-6}$ | 2.24             |
| 1/8           | 0.01681185 | $4.0437 \times 10^{-6}$ | 2.11             |
| 1/16          | 0.00816986 | $4.1616 \times 10^{-6}$ | 2.06             |
| 1/32          | 0.00407191 | $3.9306 \times 10^{-6}$ | 2.01             |
| 1/64          | 0.00199744 | $3.0719 \times 10^{-6}$ | 2.04             |
| 1/128         | 0.00096767 | $5.0404 \times 10^{-6}$ | 2.06             |
| 1/256         | 0.00048294 | $3.6172 \times 10^{-6}$ | 2.00             |

We observe that dividing the time step $h$ by 2 implies a ratio of decrease $\frac{\text{Estimation}(h)}{\text{Estimation}(h/2)}$ greater or equal to 2 which proves an $L^1$-weak error roughly proportional to $h$. 
A Appendix

The first lemma gives a condition under which we can interchange a Lebesgue and a stochastic integral. It is called the stochastic Fubini theorem and is a consequence of Theorem 2.2 proved by Veraar in [23].

Lemma A.1. Let $V : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ be a progressively measurable function. If $\int_\mathbb{R} \left( \int_0^T |V(t, x)|^2 dt \right)^{1/2} dx < \infty$ almost surely then one has:

$$\forall t \in [0, T], \quad a.s., \quad \int_\mathbb{R} \left( \int_0^t V(s, x) dW_s \right) dx = \int_\mathbb{R} \left( \int_0^t V(s, x) dx \right) dW_s.$$ 

For $t > 0$, let $G_t$ denote the probability density function of the normal law $\mathcal{N}(0, \sigma^2 t)$:

$$G_t(x) = \exp \left( -\frac{x^2}{2\sigma^2 t} \right) \sqrt{\frac{2}{\pi \sigma^2 t}}.$$

The following lemma provides a set of estimates that are very useful:

Lemma A.2. The function $G_t(x)$ solves the heat equation:

$$\partial_t G_t(x) - \frac{\sigma^2}{2} \partial_{xx} G_t(x) = 0, \quad (t, x) \in [0, +\infty) \times \mathbb{R}. \tag{A.1}$$

We can express the square of the first order spatial derivative as:

$$(\partial_x G_t)^2(x) = \frac{x^2}{2\sigma^5 t^{5/2} \sqrt{\pi}} G_{t/2}(x), \tag{A.2}$$

and deduce the $L^1$-norm of $\partial_x G^2$:

$$\int_\mathbb{R} (\partial_x G_t)^2(x) dx = \frac{1}{4\sigma^3 t^{3/2} \sqrt{\pi}}. \tag{A.3}$$

Moreover, we have estimates of the $L^1$-norm of the spatial derivatives of $G$:

$$\|\partial_x G_t\|_{L^1} = \sqrt{\frac{2}{\pi \sigma^2 t}}, \tag{A.4}$$

$$\|\partial_{xx} G_t\|_{L^1} \leq \frac{2}{\sigma^2 t}. \tag{A.5}$$

We also have estimates of the $L^1$-norm of $G^2$:

$$\int_\mathbb{R} G_t^2(x) dx = \frac{1}{\sqrt{2\pi} \sigma t}, \tag{A.6}$$

which implies that for every measurable function $y : [0, T] \to \mathbb{R}$,

$$\int_\mathbb{R} \int_0^t G_{t-s}^2(y(s) - x) \, ds \, dx = \frac{1}{\sigma \sqrt{\pi}} t. \tag{A.7}$$

Proof. The second estimate is obtained by rewriting $\partial_{xx} G_t(x)$ as $\partial_{xx} G_t(x) = -\frac{1}{\sigma^2 t} G_t(x) + \frac{1}{\sigma t} (-x \partial_x G_t(x))$. We apply an integration by parts for the second term and obtain:

$$\int_\mathbb{R} \partial_{xx} G_t(x) dx \leq \frac{2}{\sigma^3 t} \|G_t\|_{L^1} = \frac{2}{\sigma^2 t}. \tag{A.8}$$

As for the estimates of $\|G_t^2\|_{L^1}$ and $\int_\mathbb{R} \int_0^t G_{t-s}^2(y(s) - x) \, ds \, dx$, we use the fact that $G_t^2(x) = G_{t/2}(x)/2\sigma \sqrt{\pi t}$.
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