ADIABATIC DECOMPOSITION OF THE ζ-DETERMINANT
AND SCATTERING THEORY.

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1. Introduction and Statement of the Results

Let $D : C^\infty(M,S) \to C^\infty(M,S)$ be a compatible Dirac operator acting on sections of a Clifford bundle $S$ over a closed manifold $M$ of dimension $n$. The operator $D$ is a self-adjoint operator, with discrete spectrum $\{\lambda_k\}_{k \in \mathbb{Z}}$. The $\zeta$-determinant of the Dirac Laplacian $D^2$ is given by the formula

\begin{equation}
\det_\zeta D^2 = e^{-\zeta_{D^2}(0)},
\end{equation}

where $\zeta_{D^2}(s)$ is defined as follows:

\begin{equation}
\zeta_{D^2}(s) = \sum_{\lambda_k \neq 0} (\lambda_k^2)^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left[ \text{Tr} (e^{-tD^2}) - \dim \ker D \right] dt.
\end{equation}

This is a holomorphic function of $s$ for $\Re(s) \gg 0$ and has the meromorphic extension to the complex plane with $s = 0$ as a regular point.

Let us consider a decomposition of $M$ as $M_1 \cup M_2$, where $M_1$ and $M_2$ are compact manifolds with boundaries such that

\begin{equation}
M = M_1 \cup M_2, \quad M_1 \cap M_2 = Y = \partial M_1 = \partial M_2.
\end{equation}

In this paper, we study the adiabatic decomposition of the $\zeta$-determinant of $D^2$, which describes the contributions in $\det_\zeta D^2$ coming from the submanifolds $M_1$ and $M_2$. Throughout the paper, we assume that the manifold $M$ and the operator $D$ have product structures in a neighborhood of the cutting hypersurface $Y$. Hence, there is a bicollar neighborhood $N \cong [-1,1]_u \times Y$ of $Y \cong \{0\} \times Y$ in $M$ such that the Riemannian structure on $M$ and the Hermitian structure on $S$ are products of the corresponding structures over $[-1,1]_u$ and $Y$ when restricted to $N$, so that $D$ has the following form,

\begin{equation}
D = G(\partial_u + B) \quad \text{over} \quad N.
\end{equation}

Here $u$ denotes the normal variable, $G : S|_Y \to S|_Y$ is a bundle automorphism, and $B$ is a corresponding Dirac operator on $Y$. Moreover, $G$ and $B$ do not depend on $u$ and they satisfy

\begin{equation}
G^* = -G, \quad G^2 = -\text{Id}, \quad B = B^* \quad \text{and} \quad GB = -BG.
\end{equation}

To prove the adiabatic decomposition formula of $\det_\zeta D^2$, we follow the original Douglas-Wojciechowski proof of the decomposition formula for the $\eta$-invariant in
However, we face two new problems, not present in the case of the $\eta$-invariant. First, $\det_\zeta D^2$ is a much more non-local invariant than the $\eta$-invariant. This results, for instance, in the fact that the value of $\det_\zeta D^2$ varies with the length of the cylinder. Second, the contribution of $\det_\zeta D^2$ over the cylindrical part is now non-trivial. We still follow the idea of the paper [10] and we stretch our manifold $M$ to separate $M_1$ and $M_2$. For this, let us introduce a manifold $M_R$ equal to the manifold $M$ with $N$ replaced by $N_R \cong [-R, R] \times Y$. By assumption of product structures over $N$, we can extend the bundle $S$ to $M_R$. Furthermore, using (1.3), we can extend $D$ to the Dirac operator $D_R$ over $M_R$. Now we decompose $M_R$ by the hypersurface $\{0\} \times Y$ into two submanifolds $M_{1,R}$, $M_{2,R}$ and we obtain $D_{1,R}$, $D_{2,R}$ by restricting $D_R$ to $M_{1,R}$, $M_{2,R}$ respectively.

To formulate the decomposition formula for the $\zeta$-determinant, we have to describe the invariant on a manifold with boundary which enters the picture at this point. The tangential operator $B$ has discrete spectrum with infinitely many positive and infinitely many negative eigenvalues. Let $\Pi_>$, $\Pi_<$ denote the Atiyah-Patodi-Singer (APS) spectral projections onto the subspaces spanned by the eigensections of $B$ corresponding to the positive, negative eigenvalues respectively. We select two involutions $\sigma_1$, $\sigma_2$ on kernel of $B$, which satisfy $G\sigma_i = -\sigma_i G$ and define $\pi_i = \frac{1-\sigma_i}{2}$ the orthogonal projections onto $-1$ eigenspaces of $\sigma_i$ . We define

\begin{align}
(1.6) \quad P_1 = \Pi_< + \pi_1 \quad , \quad P_2 = \Pi_> + \pi_2 \quad ,
\end{align}

which provides us with the ideal boundary condition introduced by Cheeger in [6], [7]. The projection $P_i$ imposes an elliptic boundary condition for $D_{i,R}$ (see [1]; see [2] for an exposition of the theory of elliptic boundary problems for Dirac operators). This means that the associated operator

\begin{align}
(D_{i,R})_{P_i} = D_{i,R} : \text{dom}(D_{i,R})_{P_i} \to L^2(M_{i,R}, S)
\end{align}

where

\begin{align}
\text{dom}(D_{i,R})_{P_i} = \{ s \in H^1(M_{i,R}, S) \mid P_i(s|_Y) = 0 \}
\end{align}

is a self-adjoint Fredholm operator with $\ker(D_{i,R})_{P_i} \subset C^\infty(M_{i,R}, S)$ and discrete spectrum.

The main concern of this paper is to consider the limit of the following ratio of the $\zeta$-determinants,

\begin{align}
(1.7) \quad \frac{\det_\zeta D_{2,R}^2}{\det_\zeta(D_{1,R})_{P_1} \cdot \det_\zeta(D_{2,R})_{P_2}^2} \quad \text{as} \quad R \to \infty
\end{align}

which we call as the adiabatic decomposition of the $\zeta$-determinant of $D^2$.

The eigenvalues of $D_R$ fall into three different categories as $R \to \infty$. There are infinitely many large eigenvalues ($l$-values) bounded away from 0 and infinitely many small eigenvalues ($s$-values) of the size $O(R^{-1})$ . Besides these, there are finitely many eigenvalues which decay exponentially with $R$ ($e$-values). The number $h_M$ of $e$-values is given by

\begin{align}
(1.8) \quad h_M = \dim \ker_{L^2} D_{1,\infty} + \dim \ker_{L^2} D_{2,\infty} + \dim L_1 \cap L_2
\end{align}
where \( D_{i,\infty} \) is the operator defined from \( D \) in a natural way over the manifold \( M_{i,\infty} \), which is equal to \( M_i \) with the half infinite cylinder \([0, \infty) \times Y \) or \((-\infty, 0] \times Y \) attached. More precisely, the operator \( D_i = D|_{M_i} \) extends in a natural way to the manifold \( M_{i,\infty} \). It has a unique closed self-adjoint extension in \( L^2(M_{i,\infty}, S) \), which we denote by \( D_{i,\infty} \). The subspaces \( L_i \subset \ker B \) are the spaces of limiting values of extended \( L^2 \)-solutions of \( D_{i,\infty} \). The decomposition of the eigenvalues of the operator \( D_R \) into different classes was discussed by Cappell, Lee and Müller (see [5]). The corresponding analysis for the operator \( (D_{i,R})_{P_i} \) was provided by Müller (see [17]). The spectrum of the operators \( (D_{i,R})_{P_i} \) splits in the same way as the spectrum of \( D_R \). The only difference is that the operators \( (D_{i,R})_{P_i} \) do not have nonzero \( e \)-values and the dimension of the space of the solutions of \( (D_{i,R})_{P_i} \) is equal to

\[
(1.9) \quad h_i = \dim \ker(D_{i,R})_{P_i} = \dim \ker_{L^2} D_{i,\infty} + \dim L_i \cap \ker(\sigma_i - 1) .
\]

In the adiabatic limit process, the different types of eigenvalues make their contribution at different time intervals of the integral representation of \( \zeta_{D^2}(s) \) in (1.2). The contribution made by \( l \)-values comes from the time interval \([0, R^{2-\varepsilon}] \), where \( \varepsilon \) is a sufficiently small positive number, and we fix \( \varepsilon \) from now on. More precisely, it is not difficult to show that the \( l \)-values contribution to the adiabatic limit of (1.7) from the time interval \([R^{2-\varepsilon}, \infty] \) disappears as \( R \to \infty \) (see Section 2). The contribution made by \( l \)-values was discussed in [19]. To be more precise, in [19] we discussed the case of the operator \( D_i \), such that \( D_{i,\infty} \) and \( B \) have trivial kernels. These conditions imply that there are no \( e \)-values and \( s \)-values. This allows us to reduce the computation of the quotient in (1.7) to the corresponding quotient on the cylinder, hence one can show that the limit of (1.7) as \( R \to \infty \) is equal to \( 2^{-\varepsilon} n(0) \). Actually, even in the presence of \( e \)-values and \( s \)-values, we are able to show that in the adiabatic limit the contribution of \( l \)-values comes only from the time interval \([0, R^{2-\varepsilon}] \) so that we can reduce to the cylinder as in [19]. The method we use to prove this combines Duhamel’s principle and Finite propagation speed property of the wave operators. Details are presented in Section 2.

The \( s \)-values contribution comes from the time interval \([R^{2-\varepsilon}, R^{2+\varepsilon}] \). The computation of the \( s \)-values contribution is the main achievement of this paper. We follow Müller (see [17]) and use the Scattering theory to get a description of the \( s \)-values. The operators \( D_{i,\infty} \) on \( M_{i,\infty} \) determine scattering matrices \( C_i(\lambda) \).

It turned out that the matrix \( C_{12} = C_1(0) \circ C_2(0) \) on \( \ker B \cap \ker(G+i) \) determines the contribution of \( s \)-values of the operator \( D_R \) in the adiabatic limit. Similarly the finite-dimensional unitary matrix \( S_{\sigma_i} \) on \( \ker(\sigma_i + 1) \), which is defined by the scattering matrix \( C_i(0) \) and the involution \( \sigma_i \), determines the contribution of \( s \)-values of the operators \( (D_{i,R})_{P_i} \). The exact correspondence is stated in Section 3.

Finally we have to discuss \( e \)-values of \( D_R \). The number of \( e \)-values is equal to \( h_M \) which, as remarked above, is constant. On the other hand, the set of zero eigenvalues of \( D_R \), which is a subset of \( e \)-values by definition, is very unstable with respect to \( R \). Hence, without making additional assumptions we are not able to control the adiabatic limit of the determinant of \( D_R^2 \) due to the finite
number of nonzero $e$-values. Hence, we assume that all the $e$-values are zero eigenvalues in order to avoid the technical difficulty of the nonzero $e$-values. One of the important examples of such situation is the case of the operator

$$d_\rho + d_\rho^* : \bigoplus_{i=0}^n \Omega^i(M, V_\rho) \to \bigoplus_{i=0}^n \Omega^i(M, V_\rho)$$

where $V_\rho$ denotes the flat vector bundle defined by the unitary representation $\rho$ of $\pi_1(M)$ (see Proposition 3.9). For the operator $L : W \to W$ acting on a finite dimensional vector space $W$, we denote by $\det^* L$ the determinant of the operator $L$ restricted to the subspace $(\ker L)^\perp$. Now we are ready to formulate the main result of the paper.

**Theorem 1.1.** When all the $e$-values of $\mathcal{D}_R$ are zero eigenvalues, the following formula holds:

$$\lim_{R \to \infty} R^{-2h} \cdot \frac{\det \zeta \mathcal{D}_R^2}{\det \zeta (\mathcal{D}_{1,R})^2_{P_1} \cdot \det \zeta (\mathcal{D}_{2,R})^2_{P_2}} = 2^{-\zeta_{\beta_2}(0) - h_Y + 2h_M \cdot \det^* \left( \frac{2 \text{Id} - C_{12} - C_{12}^{-1}}{4} \right) \prod_{i=1}^2 \det^* \left( \frac{2 \text{Id} - S_{\sigma_i} - S_{\sigma_i}^{-1}}{4} \right)^{-1}$$

where $h = h_M - h_1 - h_2$ and $h_Y = \dim \ker B$.

**Remark 1.2.** In [12] and [11], the reduced normal operators corresponding to our operators $C_{12}$, $S_{\sigma_i}$ were introduced in the framework of $b$-calculus and used in the analysis of $s$-values for the analytic surgery of the $\eta$-invariant and analytic torsion.

To prove Theorem 1.1, we consider the following relative $\zeta$-function and its derivative at $s = 0$,

$$\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left[ \text{Tr} \left( e^{-t \mathcal{D}_R^2} - e^{-t (\mathcal{D}_{1,R})^2_{P_1}} - e^{-t (\mathcal{D}_{2,R})^2_{P_2}} - h \right) \right] dt,$$

which we decompose into two parts,

$$\zeta^R_s(s) = \frac{1}{\Gamma(s)} \int_0^{R^{2-\varepsilon}} (\cdot) \ dt, \quad \zeta^L_s(s) = \frac{1}{\Gamma(s)} \int_{R^{2-\varepsilon}}^\infty (\cdot) \ dt,$$

where $\varepsilon$ is the fixed sufficiently small positive number. The derivatives of $\zeta^R_s(s)$ and $\zeta^L_s(s)$ at $s = 0$ give the small and large time contribution in (1.10).

In Section 2 we deal with the small time contribution and prove that this equal $2^{-\zeta_{\beta_2}(0)}$, which gives the first factor on the right side of (1.10). In Section 3 we explain some basic description of the small eigenvalues. We follow [17] and use scattering theory in order to get a description of the $s$-values of $\mathcal{D}_R$ and $(\mathcal{D}_{i,R})_{P_i}$, which allows us to make a comparison of $s$-values of those operators with the eigenvalues of certain model operators over $S^1$. This is the central part of this paper. In Section 4 we use the results of Section 3 to show that, in the adiabatic limit, the large time contribution to the quotient (1.7) is equal to

$$2^{-h_Y + 2h_M \cdot \det^* \left( \frac{2 \text{Id} - C_{12} - C_{12}^{-1}}{4} \right) \prod_{i=1}^2 \det^* \left( \frac{2 \text{Id} - S_{\sigma_i} - S_{\sigma_i}^{-1}}{4} \right)^{-1} \cdot$$
This is the second factor on the right side of (1.10). The zero eigenvalues make their presence via the factor $R^{-2h}$ on the left side of (1.10).

In Section 5 we review the decomposition formula for the $\eta$-invariant and offer a new proof based on the method developed in order to prove Theorem 1.1. This proof is more complicated than other proofs presented in [4], [9], [12], [18], [23], [3], [13], [16]. However, it is a nice illustration of the differences we encounter when we deal with the $\zeta$-determinant instead of the $\eta$-invariant.

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2. Small Time Contribution

In this section we determine the small time contribution, which is done in two steps. First, we use Duhamel’s principle and Finite propagation speed property of the wave operator to show that we can reduce the problem to computations on the cylinder. Then, we perform the explicit calculations on the cylinder. Both parts are fairly standard. The cylinder contribution has been recently computed in [19]. Therefore, we only discuss the reduction scheme and refer to [19] for the explicit computation on the cylinder.

Let $E_R(t; x, y)$ denote the kernel of the operator $e^{-tD^2_R}$. We introduce the specific parametrix for $E_R(t; x, y)$, which fits our main purpose to localize the contribution coming from the cylinder $[-R, R]_u \times Y$ and the interior of $M_R$. In fact, the interesting point here is that we use $E_R(t; x, y)$ to construct this parametrix. Let $E_u(t; x, y)$ denote the kernel of the operator $e^{-t(-\partial^2_u + B^2)}$ on the infinite cylinder $\mathbb{R} \times Y$. We introduce a smooth, increasing function $\rho(a, b) : [0, \infty) \rightarrow [0, 1]$ equal to 0 for $0 \leq u \leq a$ and equal to 1 for $b \leq u$. We use $\rho(a, b)(u)$ to define

$$\phi_{1,R} = 1 - \rho(\frac{5}{7}R, \frac{6}{7}R), \quad \psi_{1,R} = 1 - \psi_{2,R},$$
$$\phi_{2,R} = \rho(\frac{1}{7}R, \frac{2}{7}R), \quad \psi_{2,R} = \rho(\frac{3}{7}R, \frac{4}{7}R).$$

We extend these functions to symmetric functions on the whole real line. These functions are constant outside the interval $[-R, R]_u$ and we use them to define the corresponding functions on a manifold $M_R$, which are denoted by the same notations. Now, we define $Q_R(t; x, y)$ a parametrix for the kernel $E_R(t; x, y)$ by

$$Q_R(t; x, y) = \phi_{1,R}(x)E_u(t; x, y)\psi_{1,R}(y) + \phi_{2,R}(x)E_R(t; x, y)\psi_{2,R}(y).$$

It follows from the Duhamel’s principle that

$$E_R(t; x, y) = Q_R(t; x, y) + (E_R \ast C_R)(t; x, y),$$

where $E_R \ast C_R$ is the convolution given by

$$(E_R \ast C_R)(t; x, y) = \int_0^t ds \int_{M_R} dz E_R(s; x, z)C_R(t - s; z, y).$$
and the error term $\mathcal{C}_R(t; x, y)$ is given by the formula

$$
\mathcal{C}_R(t; x, y) = -\partial_u^2 \phi_{1,R}(x) \partial_c \mathcal{E}_c(t; x, y) \psi_{1,R}(y) - \partial_u \phi_{1,R}(x) \partial_u \mathcal{E}_c(t; x, y) \psi_{1,R}(y)
$$

$$
- \partial_u^2 \phi_{2,R}(x) \partial_c \mathcal{E}_R(t; x, y) \psi_{2,R}(y) - \partial_u \phi_{2,R}(x) \partial_u \mathcal{E}_R(t; x, y) \psi_{2,R}(y).
$$

The following elementary lemma follows from the construction of $Q_R(t; x, y)$,

**Lemma 2.1.** For a fixed $y$, the support of the error term $\mathcal{C}_R(t; x, y)$ as a function of $x$ is a subset of $([-\frac{n}{2} R, -\frac{n}{2} R]_u \cup [\frac{n}{2} R, \frac{n}{2} R]_u) \times Y$. Moreover it is equal to 0 if the distance between $x$ and $y$ is smaller than $\frac{n}{2}$.\hfill .

Now, following Cheeger, Gromov and Taylor (see [8]; see also Section 3 of [4]), we use the Finite propagation speed property for the wave operator. The technique introduced in [8] allows us to compare the heat kernel of the operator $\mathcal{D}_R^2$ over $M_R$ with the heat kernel of the operator $-\partial_u^2 + B^2$ on the cylinder $\mathbb{R} \times Y$. We describe the case we need in our work. Let $X_1$ and $X_2$ be Riemannian manifolds of dimension $n$ and $S_i$ be spinors bundle with Dirac operators $D_i$ over $X_i$. Assume that there exists a decomposition $X_1 = K_1 \cup U_i$, where $U_i$ is an open subset of $X_i$. Moreover, we assume that there exists an isometry $h : U_1 \to U_2$ covered by the unitary bundle isomorphism $\Phi_h : S_1|_{U_1} \to S_2|_{U_2}$, which intertwines Dirac operators $D_1|_{U_1}$ and $D_2|_{U_2}$. We identify $U \cong U_1 \cong U_2$, so that $X_1$ and $X_2$ have a common open subset $U$. Let $\mathcal{E}_i(t; x, y)$ denote the kernel of the operator $e^{-tD_i^2}$. Then we have the following estimate on the difference of the heat kernels on $U$ as in Lemma 3.6 in [4],

**Proposition 2.2.** For $x, y \in U$ and $t > 0$, there exist positive constants $c_1, c_2$ such that

$$
(2.3) \quad \|\partial_u \mathcal{E}_1(t; x, y) - \partial_u \mathcal{E}_2(t; x, y)\| \leq c_1 e^{-c_2 x^2 / t}
$$

where $j = 0, 1$ and $r = \min(d(x, K_1), d(y, K_1))$.

In our situation, $X_1 = M_R$, $X_2 = \mathbb{R} \times Y$ and $U = [-R, R]_u \times Y$. Note that the heat kernel $\mathcal{E}_c(t; x, y)$ over $\mathbb{R} \times Y$ satisfies the standard estimate. More precisely, for $t > 0$ we have

$$
(2.4) \quad \|\partial_u \mathcal{E}_c(t; (u, w), (v, z))\| \leq c_1 |u - v|^j t^{-j} e^{-c_2 (u - v)^2 / t},
$$

where $j = 0, 1$, $u, v \in \mathbb{R}$ and $w, z \in Y$. This follows from the corresponding estimate for the heat kernel of $B^2$ over the closed manifold $Y$ (see Proposition 4.1 in [21]) and explicit form of the heat kernel of $-\partial_u^2$ over $\mathbb{R}$. We are going to use (2.3) and (2.4) in the following proposition.

**Proposition 2.3.** There exist constants $c_1, c_2 > 0$ such that for any $t$ with $0 < t < R^{2 - \varepsilon}$ and $(u, w), (v, z) \in \text{supp} \mathcal{C}_R(t; \cdot, \cdot)$,

$$
(2.5) \quad \|\mathcal{E}_R(t; (u, w), (v, z))\| \leq c_1 e^{-c_2 t^{R^2 / t}}, \quad \|\mathcal{C}_R(t; (u, w), (v, z))\| \leq c_1 e^{-c_2 t^{R^2 / t}}.
$$

Proof. For \( j = 0, 1 \), we have
\[
\| \partial^j_u \mathcal{E}_R(t; (u, w), (v, z)) \| \leq \| \partial^j_u \mathcal{E}_c(t; (u, w), (v, z)) \|
\]
\[
+ \| \partial^j_u \mathcal{E}_R(t; (u, w), (v, z)) - \partial^j_u \mathcal{E}_c(t; (u, w), (v, z)) \| .
\]

By Lemma 2.1 and (2.3), (2.4), there exist some constants \( c_1, c_2 > 0 \) such that for \((u, w), (v, z) \in \text{supp} \mathcal{C}_R(t, \cdot, \cdot)\), both summands on the right side satisfy the desired estimate. This estimate for \( j = 0 \) \((j = 1)\) implies the first (second) estimate in (2.5).

Now we are ready to prove the following technical result,

**Proposition 2.4.**

\[
(2.6) \quad \lim_{R \to \infty} \int_{R^2} ds \int_0^{s-1} dt \int_{M_R} \text{tr} (\mathcal{E}_R \ast \mathcal{C}_R)(t; x, x) dx = 0 .
\]

**Proof.** By Lemma 2.4 and Proposition 2.3
\[
| \text{tr} (\mathcal{E}_R \ast \mathcal{C}_R)(t; x, x) | \leq \| (\mathcal{E}_R \ast \mathcal{C}_R)(t; x, x) \|
\]
\[
\leq \int_0^t ds \int_{[-\frac{4}{3}R, \frac{4}{3}R] \times Y} \| \mathcal{E}_R(s; x, z) \mathcal{C}_R(t - s; z, x) \| dz
\]
\[
\leq c_1^2 \int_0^t ds \int_{[-\frac{4}{3}R, \frac{4}{3}R] \times Y} e^{-c_2 \frac{t^2}{s^2}} e^{-c_2 \frac{t^2}{s^2}} dz
\]
\[
\leq c_3 R \int_0^t e^{-c_2 \frac{t^2}{s^2}} ds \leq c_3 R \int_0^t e^{-2c_2 \frac{t^2}{s^2}} ds \leq c_3 R \frac{t}{2} e^{-4c_2 \frac{t^2}{s^2}} ,
\]
where the last estimate is a consequence of the elementary inequality
\[
\int_0^t e^{-\frac{t^2}{s}} ds \leq t e^{-\frac{t}{2}} .
\]

Hence we have proved
\[
(2.7) \quad | \text{tr} (\mathcal{E}_R \ast \mathcal{C}_R)(t; x, x) | \leq c_4 R t e^{-c_5 \frac{t^2}{s^2}} .
\]

This allows us to estimate as follows
\[
\left| \frac{1}{\Gamma(s)} \int_0^{R^2} t^{s-1} dt \int_{M_R} \text{tr} (\mathcal{E}_R \ast \mathcal{C}_R)(t; x, x) dx \right|
\]
\[
\leq c_6 R^2 \left| \frac{1}{\Gamma(s)} \int_0^{R^2} |t|^s e^{-c_5 \frac{t^2}{s^2}} dt \right .
\]

As \( R \to \infty \), the function of \( s \) on the right side uniformly converges to zero for \( s \) in any compact set in \( \mathbb{C} \). Hence, the derivative at \( s = 0 \) of the meromorphic function on the left side converges to zero as \( R \to \infty \). This completes the proof. \( \square \)

The corresponding result for the operator \((\mathcal{D}_R)_P^2\) to Proposition 2.4 can be carried out in exactly the same manner. First, as for \( \mathcal{D}_R \) over \( M_R \), we can construct the parametrices for the heat kernels of \( e^{-t(\mathcal{D}_R)_P^2} \) using \( \mathcal{E}_R(t; x, y) \) and the heat kernels of \((G(\partial_u + B))^2_P\) over \([0, \infty)_u \times Y \) or \((-\infty, 0]_u \times Y \). Second, one
can obtain the corresponding estimate to (2.3) using the explicit form of the heat kernel of \((G(\partial_u + B))^2_{P_i}\). Third, one can also have the corresponding estimate to Proposition 2.3 for \((D_{i,R})^2_{P_i}\) since the similar estimate as in Proposition 2.2 holds over the support of the error terms. (see Lemma 3.6 in \[4\]). All these imply that the similar estimate to Proposition 2.4 holds for \((D_{i,R})^2_{P_i}\). Now we are ready to prove the following main result of this section,

**Proposition 2.5.**

\[
\lim_{R \to \infty} \left( \frac{d}{ds} \right)_{s=0} c^R(s) + h(\gamma + (2 - \varepsilon) \cdot \log R) = \zeta_{B^2}(0) \cdot \log 2 .
\]

**Proof.** First we observe

\[
\frac{d}{ds} \left|_{s=0} \frac{1}{\Gamma(s)} \int_0^{R^{2-\varepsilon}} t^{s-1} \text{Tr} \left( e^{-tD^2_R} - e^{-t(D_{1,R})^2_{P_i}} - e^{-t(D_{2,R})^2_{P_i}} \right) dt \right.
\]

By Proposition 2.4 and corresponding results for \((D_{i,R})^2_{P_i}\), it is sufficient to consider the limit as \(R \to \infty\) of

\[
\frac{d}{ds} \left|_{s=0} \frac{1}{\Gamma(s)} \int_0^{R^{2-\varepsilon}} t^{s-1} \text{Tr} \left( Q_R(t; x, x) - Q_{1,R}(t; x, x) - Q_{2,R}(t; x, x) \right) dx \right.
\]

where \(Q_{i,R}(t; x, y)\) denotes the parametrix for \(e^{-t(D_{i,R})^2_{P_i}}\). Now, the interior contributions to the different parametrices, all determined by the kernel \(E_{R,t}(x, y)\), cancel out and we are left only with the cylinder contribution. Hence we have to deal with the limit as \(R \to \infty\) of

\[
\frac{d}{ds} \left|_{s=0} \frac{1}{\Gamma(s)} \int_0^{R^{2-\varepsilon}} t^{s-1} \text{Tr} \left( \psi_{1,R}E_{c}(t; x, x) \right) - \psi_{1,R}E_{c,1}(t; x, x) - \psi_{1,R}E_{c,2}(t; x, x) \right| dx
\]

where \(E_{c,i}(t; x, y)\) denotes the heat kernel of \((G(\partial_u + B))^2_{P_i}\) over the half cylinder. We repeat computations in Section 2 of [19] where we assumed the conditions that \(B\) is invertible and \(Y\) is even dimensional. But we can easily derive the same formula following Section 2 of [19] without these assumptions. So we can show that for \(s\) in a compact subset of \(C\) the integral part in (2.10) uniformly converges to the following function as \(R \to \infty\),

\[
2 \left( \frac{\Gamma(s)}{4} - \frac{\Gamma(s + \frac{1}{2})}{4s\sqrt{\pi}} \right) \zeta_{B^2}(s) .
\]
Hence, we obtain
\begin{equation}
\lim_{R \to \infty} \frac{d}{ds} \left| \frac{1}{\Gamma(s)} \int_{0}^{R^{2}-e} t^{s-1} \text{Tr} \left( e^{-tD^{2}_{R}} - e^{-t(D_{1,R})^{2}_{1}} - e^{-t(D_{2,R})^{2}_{2}} \right) dt \right|_{s=0} = \frac{2}{\Gamma(s)} \left( \frac{\Gamma(s)}{4} - \frac{\Gamma(s + \frac{1}{2})}{4s\sqrt{\pi}} \right) \zeta_{B2}(s) = \zeta_{B2}(0) \cdot \log 2 .
\end{equation}
Combining (2.9) and (2.11) completes the proof.
\[\square\]

3. SMALL EIGENVALUES AND SCATTERING MATRICES

In this section we investigate the relation between the s-values of the operators \(\mathcal{D}_{R}, (\mathcal{D}_{i,R})_{R} \) and the scattering matrices \(C_{i}(\lambda)\) determined by the operators \(\mathcal{D}_{i,\infty}\) on \(M_{\infty}\) for \(i = 1, 2\). We refer to Section 4 and Section 8 in [17] for a more detailed exposition of the elements of Scattering theory that we use in this paper.

Let us recall that \(M_{R}\) has the cylindrical part \(N_{R} = [-R, R]_{u} \times Y\). Hence \(M_{1,R}, M_{2,R}\) have the cylindrical part \([-R, 0]_{u} \times Y, [0, R]_{u} \times Y\) respectively. But, in order to consider \(M_{i,R}\) as a submanifold of \(M_{\infty}\) which is obtained by attaching \([0, \infty)_{u} \times Y\) to \(M_{u}\), we change the variable by \(v = u + R\) or \(v = u - R\) so that the cylindrical part of \(M_{i,R}\) is given by \([0, R]_{u} \times Y\) or \([-R, 0]_{u} \times Y\).

Throughout this section, we will use this convention when it is needed.

For any \(\psi \in \ker B\) and \(\lambda \in \mathbb{C} - (-\infty, -\mu_{1}] \cup [\mu_{1}, +\infty)\) where \(\mu_{1}\) denotes the lowest positive eigenvalue of the tangential operator \(B\), there exists a generalized eigensection \(E(\psi, \lambda)\) of \(\mathcal{D}_{1,\infty}\) over \(M_{1,\infty}\) determined by the couple \((\psi, \lambda)\) (see Section 4 in [17]) in the following sense,
\[\mathcal{D}_{1,\infty}E(\psi, \lambda) = \lambda E(\psi, \lambda) .\]

The section \(E(\psi, \lambda)\) has the following form over \([0, \infty)_{u} \times Y\),
\begin{equation}
E(\psi, \lambda) = e^{-i\lambda u}(\psi - iG\psi) + e^{i\lambda u}C_{1}(\lambda)(\psi - iG\psi) + \theta(\psi, \lambda)
\end{equation}
where \(\theta\) is a square integrable section such that, for each \(v\), \(\theta(\psi, \lambda, (v, \cdot))\) is orthogonal to \(\ker B\). The operator \(C_{1}(\lambda) : \ker B \to \ker B\) is regular and unitary for \(|\lambda| < \mu_{1}\) and equals the \textit{Scattering matrix} such that
\[C_{1}(\lambda)C_{1}(-\lambda) = \text{Id} , \quad C_{1}(\lambda)G = -GC_{1}(\lambda) ,\]
which imply
\[C_{1}(0)^{2} = \text{Id} , \quad C_{1}(0)G = -GC_{1}(0) .\]
Therefore \(C_{1}(0)\) gives a distinguished unitary involution of \(\ker B\). In fact, the space of the limiting values of the extended \(L^{2}\)-solutions of \(\mathcal{D}_{1,\infty}\), \(L_{1} \subset \ker B\) is equal to the \((+1)\)-eigenspace of \(C_{1}(0)\), that is, \(L_{1} = \ker(C_{1}(0) - 1)\). The following proposition is a basic tool to deal with \(E(\psi, \lambda)\),

**Proposition 3.1.** (Maass-Selberg) The following equality holds,
\[\langle E(\phi, \lambda), E(\psi, \lambda) \rangle_{M_{1,R}} = 4R(\phi, \psi)_{Y} - i\langle C_{1}(-\lambda)C_{1}'(\lambda)(\phi - iG\phi), \psi - iG\psi \rangle_{Y} + O(e^{-cR})\]
where \(\phi, \psi \in \ker B\).
Proof. By Green’s formula, we have

\begin{equation}
\tag{3.2}
\begin{aligned}
&\langle i e^{-ih}\langle \phi - iG\phi, \psi - iG\psi \rangle_Y \\
&- i e^{ih}\langle C_1(\lambda + h)\phi - iG\phi, C_1(\lambda)(\psi - iG\psi) \rangle_Y + O(e^{-cR})
\end{aligned}
\end{equation}

We have the following estimates

\begin{equation}
(3.3)
\end{equation}

Now, dividing the right side by $h$ and taking the limit $h \to 0$, we obtain

\begin{equation}
2R\langle \phi - iG\phi, \psi - iG\psi \rangle_Y - i(C_1'(\lambda)\phi - iG\phi, C_1(\lambda)(\psi - iG\psi)) \rangle_Y + O(e^{-cR})
\end{equation}

Comparing this with (3.2) (divided by $h$) completes the proof.

Now we shall analyze the $s$-values of $D_R$ over $M_R$. Let us consider a $s$-value $\lambda = \lambda(R)$ of $D_R$ such that

\[ |\lambda(R)| \leq R^{-\kappa} \text{ for sufficiently large } R \]

where $\kappa$ is a fixed constant with $0 < \kappa < 1$. Let $\Psi_R$ denote a normalized eigensection of $D_R$ corresponding to $s$-value $\lambda$, that is,

\[ D_R\Psi_R = \lambda \Psi_R, \quad ||\Psi_R|| = 1. \]

Over the cylindrical part $[-R, R]_u \times Y$ in $M_R$, the eigensection $\Psi_R$ corresponding to $s$-value $\lambda$ of $D_R$ has the following form,

\begin{equation}
\tag{3.3}
\begin{aligned}
\Psi_R = e^{-i\lambda u}\psi_1 + e^{i\lambda u}\psi_2 + \hat{\Psi}_R
\end{aligned}
\end{equation}

where $\psi_1 \in \ker B \cap \ker(G - i)$, $\psi_2 \in \ker B \cap \ker(G + i)$ and $\hat{\Psi}_R$ is orthogonal to $\ker B$.

Lemma 3.2. We have the following estimates

\[ ||\Psi_R|_{[u]} \times Y||_Y \leq c_1 e^{-c_2 R} \quad \text{for} \quad -\frac{3}{4}R \leq u \leq \frac{3}{4}R \]

where $c_1, c_2$ are positive constants independent of $R$.

The proof of this lemma is same as the one of Lemma 2.1 in \cite{22}. Now we can prove

Proposition 3.3. The zero eigenmode $e^{-i\lambda u}\psi_1 + e^{i\lambda u}\psi_2$ of the eigensection $\Psi_R$ of $s$-value $\lambda(R)$ of $D_R$ is non-trivial.
Proof. We follow the proof of Theorem 2.2 in [22], so we assume that the zero eigenmode of $\Psi_R$ is trivial, which will contradict to the fact $\lambda(R)$ is a $s$-value. Throughout the proof, we regard $M_{1,R}$ as a submanifold of $M_{1,\infty}$ using the change of variable $v = u + R$. We define a section $\Phi_R$ on $M_{1,\infty}$ by

$$
\Phi_R = \begin{cases} 
h_R(x)\Psi_R(x) & \text{for } x \in M_{1,R} \\
0 & \text{for } x \in M_{1,\infty} \setminus M_{1,R}
\end{cases}
$$

where $h_R$ is a smooth function on $M_{1,\infty}$, equal to 1 for $x \in M_1 \cup [0, \frac{R}{2}] \times Y$ and equal to 0 for $x \in [\frac{R}{2}, \infty)_v \times Y$ with $|\frac{\partial h}{\partial v}| \leq CR^{-j}$ for a constant $C > 0$. Let $H^1(M_{1,\infty}, S)$ denote the first Sobolev space. For any $a \geq 0$, we introduce a closed subspace of $H^1(M_{1,\infty}, S)$ by

$$
H^1_a(M_{1,\infty}, S) = \{ \Phi \in H^1(M_{1,\infty}, S) \mid \langle \Phi(v, \cdot), \phi_k \rangle = 0 \text{ for } v \geq a, \ k = 1, \ldots, h_Y \}
$$

where $\phi_1, \ldots, \phi_{h_Y}$ denotes an orthonormal basis of ker $B$. Consider the quadratic form,

$$
Q(\Phi) = \|D\Phi\| \text{ for } \Phi \in H^1_a(M_{1,\infty}, S)
$$

where $D$ denotes the differential operator over $M_{1,\infty}$ whose self adjoint extension is $D_{1,\infty}$. Then this quadratic form is represented by a positive self adjoint operator $H_a$ in the closure of $H^1_a(M_{1,\infty}, S)$ in $L^2(M_{1,\infty}, S)$. Then $H_a$ has pure point spectrum near 0 and ker $H_a = \ker L^2D_{1,\infty}$ for any $a \geq 0$ by Proposition 8.7 in [17]. Following the proof of Proposition 2.4 in [22], we can prove that there exist positive constants $c_1, c_2$ such that

$$
|\langle \Phi_R, s \rangle| \leq c_1 e^{-c_2R}\|s\|
$$

for $s \in \ker H_a$. Now let $\tilde{\Phi}_R := \Phi_R - \sum_{k=1}^{h_1,\infty} (\Phi_R, s_k) s_k$ where $\{s_k\}_{k=1}^{h_1,\infty}$ denotes an orthonormal basis of ker $H_a$ with $h_1,\infty := \dim \ker H_a$. Hence, $\tilde{\Phi}_R$ is orthogonal to ker $H_a$, and by (3.3) there is a positive constant $C$ independent of $R$ such that $\|\tilde{\Phi}_R\| \geq \frac{1}{2}\|\Phi_R\| \geq C > 0$ for sufficiently large $R$. Noting that $\tilde{\Phi}_R \in \dom H_a$, and by the mini-max principle, we have

$$
\langle H_a \tilde{\Phi}_R, \tilde{\Phi}_R \rangle \geq \nu^2 C^2
$$

where $\nu^2$ is the smallest nonzero eigenvalue of $H_a$. Now we have

$$
\lambda(R)^2 = \langle D_R^2 \Psi_R, \Psi_R \rangle = \int_{M_R} \|D_R \Psi_R(x)\|^2 \, dx \\
\geq \int_{M_{1,R}} \|D_R \Psi_R(x)\|^2 \, dx = \int_{M_{1,R}} \|D_R (h_R \Psi_R + (1-h_R) \Psi_R)(x)\|^2 \, dx \\
\geq \int_{M_{1,\infty}} \|H_a \Phi_R(x)\| \, dx - \int_{M_{1,R}} \|D_R (1-h_R) \Psi_R(x)\|^2 \, dx .
$$
By (3.5), the first term has the lower bound \( \nu^2 C^2 \) since \( H_a \Phi_R = H_a \tilde{\Phi}_R \). For the second term, we have
\[
\int_{M_{1,R}} \|D_R (1 - h_R) \Psi_R(x)\|^2 \, dx
= \int_{M_{1,R}} \| (1 - h_R)(x)D_R \Psi_R(x) - G(\partial_a h_R)(x) \Psi_R(x)\|^2 \, dx
\leq 2 \int_{M_{1,R}} \| \lambda(R)(1 - h_R)(x) \Psi_R(x)\|^2 + \| G(\partial_a h_R)(x) \Psi_R(x)\|^2 \, dx.
\]
By applying Lemma 3.2 with \( v = u + R \) to each term of the last line, we have
\[
\int_{M_{1,R}} \| D_R (1 - h_R) \Psi_R(x)\|^2 \, dx \leq c_3 e^{-c_4 R}
\]
for positive constants \( c_3, c_4 \). Hence these inequalities imply that \( \lambda(R)^2 \geq \frac{1}{2} \nu^2 C^2 \) for sufficiently large \( R \). This completes the proof.

By Proposition 3.3, one of \( \phi^1 \) and \( \phi^2 \) in (3.6) is nontrivial. First we assume that \( \phi^1 \) is nontrivial. Now, since \( L_1 = \text{Im}(C_1(0) + 1) \) and \( C_1(0) \) is an involution over \( \ker B \), by Lemma 3.4 we can choose \( \psi \in L_1 \) such that \( \phi^1 = \psi - iG\psi \). Then the generalized eigensection \( E(\psi, \lambda) \) over \( M_{1,\infty} \) associated to \( \psi \) has the following expression
\[
E(\psi, \lambda) = e^{-i\lambda \nu} (\psi - iG\psi) + e^{i\lambda \nu} C_1(\lambda)(\psi - iG\psi) + \theta(\psi, \lambda)
\]
over \( [0, \infty)_v \times Y \). Following [17], we introduce
\[
F = \Psi_R|_{M_{1,R}} - E(\psi, \lambda)|_{M_{1,R}}.
\]
Green’s formula gives
\[ 0 = \langle D_{1, R} F, F \rangle_{M_{1, R}} - \langle F, D_{1, R} F \rangle_{M_{1, R}} = \int_{\partial(M_{1, R})} \langle G F |_{\partial(M_{1, R})}, F |_{\partial(M_{1, R})} \rangle \, dy. \]

On the other hand, Lemma 3.2 shows that
\[ \int_{\partial(M_{1, R})} \langle G F |_{\partial(M_{1, R})}, F |_{\partial(M_{1, R})} \rangle \, dy = -i \| C(\lambda) \phi^1 - \phi^2 \|^2 + O(e^{-c_3 R}) \]
for some positive constant \( c_3 \). This produces the estimate
\[ \| C(\lambda) \phi^1 - \phi^2 \| \leq e^{-c R} \]
for a positive constant \( c \). Therefore, for \( R \gg 0 \), if \( \phi^1 \) is nontrivial, then \( \phi^2 \) is also nontrivial. In the same way, one can show its inverse. Hence we can conclude that both \( \phi^1, \phi^2 \) in (3.6) are nontrivial for \( R \gg 0 \).

Now we want to get the corresponding estimate involving the scattering matrix \( C_2(\lambda) \). For this, we change the variable by \( v = u - R \) and regard the cylindrical part as \([-2R, 0], \times Y \). Then we have the corresponding expression for \( \Psi_R \),
\[ \Psi_R = e^{-i\lambda v} \phi^1 + e^{i\lambda v} \phi^2 + \hat{\Psi}_R \]
where \( \phi^1 = e^{-i\lambda R} \psi_1, \phi^2 = e^{i\lambda R} \psi_2 \). For the given \( \phi^2 \in (\ker B)_- \), using Lemma 3.4 we choose \( \psi \in L_2 = \text{Im}(C_2(0) + 1) \) such that \( \phi^2 = \psi + iG\psi \). The generalized eigensection \( E(\psi, \lambda) \) over \( M_{2, \infty} \) attached to the couple \((\psi, \lambda)\) has the following expression
\[ E(\psi, \lambda) = e^{i\lambda v}(\psi + iG\psi) + e^{-i\lambda v}C_2(\lambda)(\psi + iG\psi) + \theta(\psi, \lambda) \]
over \((-\infty, 0] \times Y\). As above, comparing \( \Psi_R \) and \( E(\psi, \lambda) \), we obtain
\[ \| C_2(\lambda) \phi^2 - \phi^1 \| \leq e^{-c R} \]
for a positive constant \( c \). By definition, we have
\[ \phi^1 = e^{2i\lambda R} \phi^1, \quad \phi^2 = e^{-2i\lambda R} \phi^2. \]
Now, combining (3.7), (3.8) and (3.9), we get
\[ \| e^{4i\lambda R} C_1(\lambda) \circ C_2(\lambda) \phi^1 - \phi^2 \| \leq e^{-c R}. \]

We define the operator \( C_{12}(\lambda) \) by
\[ C_{12}(\lambda) := C_1(\lambda) \circ C_2(\lambda)|_{(\ker B)_-} : (\ker B)_- \rightarrow (\ker B)_- \].

The operator \( C_{12}(\lambda) \) is a unitary operator and is an analytic function of \( \lambda \) for \( \lambda \in (-\delta, \delta) \) for a small \( \delta > 0 \) since the unitary operators \( C_1(\lambda), C_2(\lambda) \) are analytic functions of \( \lambda \) for \( \lambda \in (-\delta, \delta) \). Furthermore, there exist real analytic functions \( \alpha_j(\lambda) \) for \( 1 \leq j \leq 2M \) of \( \lambda \in (-\delta, \delta) \) such that \( \exp(i\alpha_j(\lambda)) \) are the corresponding eigenvalues of \( C_{12}(\lambda) \) and \( \alpha_j(\lambda) \) has the following expansion at \( \lambda = 0 \),
\[ \alpha_j(\lambda) = \alpha_{j0} + \alpha_{j1}\lambda + \alpha_{j2}\lambda^2 + \alpha_{j3}\lambda^3 + \ldots. \]

We now introduce
\[ \Omega(R) := \{ \rho \in \mathbb{R} \setminus \{0\} \mid \det(e^{4i\rho R} C_{12}(\rho) - \text{Id}) = 0, \ |\rho| \leq R^{-\kappa} \} \].

The following theorem is a main result of this section,
Theorem 3.5. Assume that all the e-values of $\mathcal{D}_R$ are zero eigenvalues. Let $\lambda_1(R) \leq \lambda_2(R) \leq \ldots \leq \lambda_{p(R)}(R)$ be the nonzero eigenvalues, counted to multiplicity, of $\mathcal{D}_R$ satisfying $|\lambda_k(R)| \leq R^{-\kappa}$, and let $p_1(R) \leq p_2(R) \leq \ldots \leq p_{m(R)}(R)$ be the nonzero element, counted to multiplicity, of $\Omega(R)$. Then there exist $R_0$ and $c > 0$, independent of $R$, such that for $R \geq R_0$, $p(R) = m(R)$ and

$$|\lambda_k(R) - \rho_k(R)| \leq e^{-cR} \quad \text{for} \quad k = 1, \ldots, p(R).$$

Proof. The proof of this theorem consists of two steps.

Step I: Let $\lambda = \lambda(R)$ be a given s-value with the multiplicity $m(\lambda)$. By Proposition 3.3, we get $m(\lambda)$ linearly independent vectors $\phi_1, \ldots, \phi_{m(\lambda)}$ in $(\ker B)_-$, which satisfies 3.10. Since $C_{12}(\lambda)$ is unitary, the eigenvalues of $e^{4i\lambda R}C_{12}(\lambda) - \text{Id}$ have the form $e^{i\theta} - 1$ for $\theta \in \mathbb{R}$. Let $0 \leq \zeta$ be the smallest eigenvalue of $(e^{4i\lambda R}C_{12}(\lambda) - \text{Id})(e^{4i\lambda R}C_{12}(\lambda) - \text{Id})^*; then

$$\zeta = \min_{\phi \in (\ker B)_-} \frac{\| (e^{4i\lambda R}C_{12}(\lambda) - \text{Id})\phi \|^2}{\| \phi \|^2}.$$ 

Combined with 3.10, this implies that $\zeta \leq e^{-cR}$. Hence $e^{4i\lambda R}C_{12}(\lambda)$ has an eigenvalue $e^{i\theta}$ satisfying $|1 - \cos \theta| \leq e^{-cR}$, and there exists $k \in \mathbb{Z}$ such that $|2\pi k - \theta| \leq e^{-cR}$. Therefore, by definition of $\alpha_j(\lambda)$, the following holds

$$|4\lambda R + \alpha_j(\lambda) - 2\pi k| \leq e^{-cR} \tag{3.13}$$

for pairwise distinct branches $\alpha_1, \ldots, \alpha_{m(\lambda)}$. Now, let us fix $\delta_1$ with $0 < \delta_1 < \delta$ and let

$$m_j = \max_{\lambda \in (-\delta_1, \delta_1)} |\alpha_j(\lambda)|. \tag{3.14}$$

Then the function $f(\lambda) = 4\lambda R + \alpha_j(\lambda)$ is strictly increasing for $|\lambda| < \delta_1$ and $R \geq m_j$. Choose $R_1$ such that $R_1 \geq \max(m_j, \delta_1 \frac{1}{\kappa})$ for any $j = 1, \ldots, \frac{h}{2}$. For $R \geq R_1$ and $k \in \mathbb{Z}$, there exists at most one solution $\rho_{j,k}$ of

$$4\lambda R + \alpha_j(\lambda) = 2\pi k, \quad |\lambda| \leq R^{-\kappa}. \tag{3.15}$$

Let $k_{j,\text{max}}$ be the maximal $k$ for which 3.15 has a solution; then by 3.14,

$$|k_{j,\text{max}}| \leq \frac{2R^{1-\kappa}}{\pi} + C \leq R^{1-\kappa}. \tag{3.16}$$

Then, for $R \geq R_1$, any element $\rho$ in $\Omega(R)$ is given by $\rho = \rho_{j,k}$ for some $1 \leq j \leq \frac{h}{2}$, and $|k| \leq k_{j,\text{max}}$. Therefore, if $R \geq R_1$, for a given $\lambda$ satisfying 3.13 with $|\lambda| \leq R^{-\kappa}$, there is a unique solution $\rho_{j,k}$ of 3.14 such that

$$|\lambda - \rho_{j,k}| \leq e^{-cR}. \tag{3.16}$$

In conclusion, if $R \geq R_1$, for a given s-value $\lambda = \lambda(R)$ of $\mathcal{D}_R$ with the multiplicity $m(\lambda)$ satisfying $|\lambda| \leq R^{-\kappa}$, there exist $m(\lambda)$-number of elements $\rho_{j,k}$’s in $\Omega(R)$ with the relation 3.16, in particular, $p(R) \leq m(R)$.

Step II: To complete the proof, we need to prove that $m(R) \leq p(R)$. For $k$ with $1 \leq k \leq m(R)$, we choose $\psi_k \in (\ker B)_-$ with the following properties,

1. $e^{4i\rho_k R}C_{12}(\rho_k)\psi_k = \psi_k$, $|\rho_k| \leq R^{-\kappa}$,
(2) When \( \rho_k = \rho_{k+1} = \ldots = \rho_{k+l} \), \( \psi_k, \psi_{k+1}, \ldots, \psi_{k+l} \) form an orthonormal system of vectors of \((\ker B)_-\).

For a given pair \((\psi_k, \rho_k)\) for some \( k \), we put

\[
(3.17) \quad \phi_k^1 = e^{-i\rho_k R} C_1(-\rho_k) \psi_k, \quad \phi_k^2 = e^{i\rho_k R} \psi_k.
\]

Now we consider the generalized eigensection \( E(\phi_k^1, \rho_k) \) over \( M_{1,\infty} \) and \( E(\phi_k^2, \rho_k) \) over \( M_{2,\infty} \), which have the following forms,

\[
E(\phi_k^1, \rho_k) = e^{-i\rho_k v} \phi_k^1 + e^{i\rho_k v} C_1(\rho_k) \phi_k^1 + O(e^{-cv}) \quad \text{over } [0, \infty)_v \times Y \subset M_{1,\infty} ;
\]

\[
E(\phi_k^2, \rho_k) = e^{i\rho_k v} \phi_k^2 + e^{-i\rho_k v} C_2(\rho_k) \phi_k^2 + O(e^{-cv}) \quad \text{over } (-\infty, 0)_{v} \times Y \subset M_{2,\infty}.
\]

(Here we use abuse notations for simplicity since the correct notation for \( E(\phi_k^1, \rho_k) \)

is \( E(\hat{\phi}_k^1, \rho_k) \) with \( \hat{\phi}_k^1 = \phi_k^1 + (-1)^i \sqrt{-1} G_k^0 \phi_k^2 \) by Lemma \( \ref{lem:eigenvalue} \)).

Restricting \( E(\phi_k^2, \rho_k) \) to \( M_{1,\infty} \), we obtain sections over \( M_{1,\infty} \).

Let \( f_{1,\infty} \) be the restriction to \( M_{1,\infty} \) of the smooth function \( h_{1,\infty} \) over \( M_{1,\infty} \) defined in the proof of Proposition \( \ref{prop:eigenvalue} \) and \( f_{2,\infty} \) be a smooth function over \( M_{2,\infty} \) defined in a similar way. These functions have the obvious extension over \( M_{R} \). Denoting by \( E_0(\phi_k^1, \rho_k) \) the zero eigenmode of \( E(\phi_k^1, \rho_k) \) and using \( \ref{eq:eigenvalue} \) and \( e^{4i\rho_k R} C_{12}(\rho_k) \psi_k = \psi_k \), we have

\[
(3.18) \quad E_0(\phi_k^1, \rho_k) = e^{-i\rho_k u} e^{-2i\rho_k R} C_1(-\rho_k) \psi_k + e^{i\rho_k u} \psi_k
\]

\[
= e^{i\rho_k u} \psi_k + e^{-i\rho_k u} e^{2i\rho_k R} C_2(\rho_k) \psi_k = E_0(\phi_k^2, \rho_k).
\]

Hence we can see that \( E_0(\phi_k^1, \rho_k) \) and \( E_0(\phi_k^2, \rho_k) \) define a smooth section over \( N_R \), which we denote by \( E_0(\psi_k, \rho_k) \). Let us define

\[
(3.19) \quad \hat{\Psi}_k := f_{1,\infty}(E(\phi_k^1, \rho_k) - \chi_{[-R,0]} u E_0(\phi_k^1, \rho_k))
\]

\[
+ f_{2,\infty}(E(\phi_k^2, \rho_k) - \chi_{[0,R]} u E_0(\phi_k^2, \rho_k)) + \chi_{[-R,R]} u E(\psi_k, \rho_k)
\]

where \( \chi_{[a,b]} \) is the characteristic function of the \( u \)-variable over \([a, b] \times Y \subset N_R \).

By \( \ref{eq:eigenvalue} \), \( \hat{\Psi}_k \) is a smooth section over \( M_{R} \). Put \( \Psi_k := \hat{\Psi}_k / \| \hat{\Psi}_k \| \) and

\[
\Phi_k = \Psi_k - \pi_R \Psi_k, \quad \text{for } k = 1, \ldots, m(R),
\]

where \( \pi_R \) denote the orthogonal projection of \( L^2(M_{R}, S) \) onto \( \ker D_R \). Let us recall that \( \ker D_R \) equals the space spanned by eigensections of \( \epsilon \)-values by our assumption, so that the dimension of this space is constant with respect to \( R \).

Combining this fact and Lemma \( \ref{lem:eigenvalue} \) we have

\[
| \langle \hat{\Psi}_k, \hat{\Psi}_\ell \rangle - \delta_{k\ell} | \leq e^{-cR} \quad \text{for } k, \ell = 1, \ldots, m(R).
\]

From this and \( \ref{eq:eigenvalue} \), it follows that \( \{ \hat{\Psi}_k \}_{k=1}^{m(R)} \) are linearly independent for \( R \gg 0 \).

Now let \( 0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{p(R)} \) denote the nonzero eigenvalues, counted with multiplicity, of \( D_R \), which are \( \leq R^{-2k_1} \). Let \( k_1, \ldots, k_{m(R)} \) be a permutation of \( \{1, \ldots, m(R)\} \) such that \( 0 < \rho_{k_1}^2 \leq \ldots \leq \rho_{k_m(R)}^2 \). By the mini-max principle, we have

\[
\lambda_\ell = \min_{W} \max_{\phi \in W} \frac{\| D_R \phi \|^2}{\| \phi \|^2}
\]
where $W$ runs over all $\ell$-dimensional subspaces of $L^2(M_R, S)$ which are orthogonal to $\ker(D_R)$. Let $W_\ell$ be the subspace of $L^2(M_R, S)$ spanned by $\hat{\Psi}_{k_1}, \ldots, \hat{\Psi}_{k_\ell}$. Then, by Lemma 3.6 we have

$$\lambda_\ell \leq \max_{\phi \in W_\ell} \frac{\|D_R\phi\|^2}{\|\phi\|^2} \leq \rho^2_{k_\ell}(1 + Ce^{-cR})$$

for some constants $C, c > 0$. Hence, there exists $R_2$ such that $m(R) \leq p(R)$ for $R \geq R_2$. Putting $R_0 = \max(R_1, R_2)$, this completes the proof of Theorem 3.5.

**Lemma 3.6.** Assume that all the $e$-values of $D_R$ are zero eigenvalues. Then there exist $c_1, c_2 > 0$ such that

$$|\langle \Psi_k, \Psi_\ell \rangle| \leq c_1 e^{-c_2 R} \quad \text{for} \quad k \neq \ell, \quad k, \ell = 1, \ldots, m(R),$$

$$|\langle \Psi_k, \Psi_\ell \rangle| \leq c_1 e^{-c_2 R} \quad \text{for} \quad k = 1, \ldots, m(R), \text{ and } \Psi \in \ker(D_R) \text{ with } ||\Psi|| = 1.$$

**Proof.** For a couple $(\psi, \rho)$ and $\phi_{k_\ell}$ satisfying (3.17), we put

$$E_k^\perp = E(\phi_{k_1}^1, \rho_k)|_{M_1R} - \chi_{[-R, 0]} E_0(\phi_{k_1}^1, \rho_k) + E(\phi_{k_2}^2, \rho)|_{M_2R} - \chi_{[0, R]} E_0(\phi_{k_2}^2, \rho_k)$$

and $E_{k,0} = E(\psi, \rho)$, where $\chi_{[-R, 0]}$ is the characteristic function of $[-R, 0]$. Now we have

$$\langle \hat{\Psi}_k, \hat{\Psi}_\ell \rangle = \langle f E_k^\perp + \chi E_{k,0}, f E_\ell^\perp + \chi E_{\ell,0} \rangle$$

$$= \langle f E_k^\perp, f E_\ell^\perp \rangle + \langle \chi E_{k,0}, \chi E_{\ell,0} \rangle$$

$$= \langle E_k^\perp - (1 - f) E_k^\perp, E_\ell^\perp - (1 - f) E_\ell^\perp \rangle + \langle \chi E_{k,0}, \chi E_{\ell,0} \rangle$$

$$= \langle E_k^\perp, E_\ell^\perp \rangle - \langle E_k^\perp, (1 - f) E_\ell^\perp \rangle - \langle (1 - f) E_k^\perp, E_\ell^\perp \rangle$$

$$+ \langle (1 - f) E_k^\perp, (1 - f) E_\ell^\perp \rangle + \langle \chi E_{k,0}, \chi E_{\ell,0} \rangle$$

(3.20)

$$= \langle E_k, E_\ell \rangle - \langle E_k^\perp, (1 - f) E_\ell^\perp \rangle - \langle (1 - f) E_k^\perp, E_\ell^\perp \rangle$$

$$+ \langle (1 - f) E_k^\perp, (1 - f) E_\ell^\perp \rangle$$

where $f = f_R, \chi = \chi_{[-R, 0]}$. Since supp$(1 - f_R) \subset [-\frac{R}{2}, \frac{R}{2}] \times Y$, where $E_k^\perp, E_\ell^\perp$ are $O(e^{-cR})$, the last three terms in (3.20) are $O(e^{-cR})$. Now we consider the first term in (3.20), which can be written as

$$\langle E_k, E_\ell \rangle = \langle E(\phi_{k_1}^1, \rho_k), E(\phi_{k_2}^2, \rho_\ell)|_{M_1R} + \langle E(\phi_{k_2}^2, \rho_k), E(\phi_{k_2}^2, \rho_\ell)|_{M_2R} \rangle.$$

When $\rho_k \neq \rho_\ell$, as in the proof of Proposition 3.1, we apply Green formula to each term on the right side of (3.21), then these equal

$$(\rho_k - \rho_\ell)^{-1} \langle GE(\phi_{k_1}^1, \rho_k)|_{\partial(M_1R)}, E(\phi_{k_2}^2, \rho_\ell)|_{\partial(M_2R)} \rangle$$

$$= - (\rho_k - \rho_\ell)^{-1} \langle GE(\phi_{k_2}^2, \rho_k)|_{\partial(M_2R)}, E(\phi_{k_2}^2, \rho_\ell)|_{\partial(M_2R)} \rangle.$$

Now using (3.18), the restrictions of constant terms over $\partial(M_iR)$ cancel each other out and the remaining terms are $O(e^{-cR})$. Hence, in this case, the left side of (3.21) is $O(e^{-cR})$, so all the terms in (3.20) are $O(e^{-cR})$. When $\rho_k = \rho_\ell$, note
that $\langle \phi_k^i, \phi_k^j \rangle = 0$ for $i = 1, 2$, so applying Proposition 3.1 we can see that all the terms are $O(e^{-cR})$ except the following terms,

\begin{equation}
\langle C_1(-\rho_k)C_1'(\rho_k)\phi_k^1, \phi_k^1 \rangle + \langle C_2(-\rho_k)C_2'(\rho_k)\phi_k^2, \phi_k^2 \rangle .
\end{equation}

Using the conditions in (3.17) for $\phi_k^i, \phi_k^j$ and the relation

\begin{equation}
e^{4i\rho_kR}C_2(\rho_k)\psi_k = C_1(-\rho_k)\psi_k, \quad e^{4i\rho_kR}C_2(\rho_k)\psi_\ell = C_1(-\rho_\ell)\psi_\ell,
\end{equation}

one can show that the terms in (3.22) equal

\begin{equation}
\langle e^{4i\rho_kR}C_1'(\rho_k)C_2(\rho_k)\psi_k, \psi_k \rangle + \langle e^{4i\rho_kR}C_1'(\rho_k)C_2(\rho_k)\psi_\ell, \psi_\ell \rangle .
\end{equation}

Now we choose a family of sections $\psi_k(t)$ with $\psi_k(0) = \psi_\ell$ for $t \in (-\epsilon, \epsilon)$ such that

\begin{equation}
a(t)C_1(\rho_k + t)C_2(\rho_k + t)\psi_k(t) = \psi_k(t),
\end{equation}

where $a(0) = e^{4i\rho_kR}$. Taking the derivative of this at $t = 0$, we obtain

\begin{equation}
e^{4i\rho_kR}C_1'(\rho_k)C_2(\rho_k)\psi_k + e^{4i\rho_kR}C_1(\rho_k)C_2'(\rho_k)\psi_k = -a'(0)C_1(\rho_k)C_2(\rho_k)\psi_k - e^{4i\rho_kR}C_1(\rho_k)C_2(\rho_k)\psi_\ell(0) + \psi_\ell(0) .
\end{equation}

Using this, (3.22) and $\langle \psi_k, \psi_\ell \rangle = 0$, we can see that (3.24) equals

\begin{equation}
\langle -e^{4i\rho_kR}C_1(\rho_k)C_2(\rho_k)\psi_k(0) + \psi_\ell(0), \psi_k \rangle = \langle \psi_k(0), \psi_k \rangle - \langle \psi_\ell(0), e^{-4i\rho_kR}C_2(-\rho_k)C_1(\rho_k)\psi_\ell \rangle = 0 .
\end{equation}

Hence, in the case of $\rho_k = \rho_\ell$, all the terms in (3.20) are $O(e^{-cR})$. This completes the proof of the first claim recalling $\Psi_k = \tilde{\Psi}_k/||\tilde{\Psi}_k||$.

For the second claim, let us recall that the eigenspaces of the $e$-values are spanned by the sections defined by gluing (as in (3.14)) the elements in $\ker_{L^2}D_{i,\infty}$ for $i = 1, 2$ or the extended $L^2$-solutions of $D_{i,\infty}$ whose limiting values lying in $L_1 \cap L_2$. By our assumption, this space is the same as $\ker D_R$. For a section $\Psi$ given by gluing elements in $\ker_{L^2}D_{i,\infty}$, the claim follows easily by applying Green’s formula as above. For a section $\Psi$ given by gluing the extended $L^2$-solutions of $D_{i,\infty}$ whose limiting values lying in $L_1 \cap L_2$, we use Theorem 3.8 which implies that such a $\Psi$ is actually given by (3.19) for the couple $(\psi_k, \rho_k)$ with $\rho_k = 0$. Hence, the claim for this case can be proved as in the previous case of $\rho_k \neq \rho_\ell$. This completes the proof of the second claim.

In general, the map $C_{12} := C_{12}(0) : (\ker B)_- \to (\ker B)_-$ does not equal the identity map, but it is not difficult to see that

\begin{equation}
C_1(0) \circ C_2(0)\phi = \phi \quad \text{if and only if} \quad \phi \in (L_1 \cap L_2) \oplus (GL_1 \cap GL_2) .
\end{equation}

Putting

\begin{equation}
I_+ = 1 + iG : \ker B \to (\ker B)_- ,
\end{equation}

we can see that $I_+(L_1 \cap L_2)$ and $I_+(GL_1 \cap GL_2)$ are the same subspace in $(\ker B)_-$. 

**Proposition 3.7.** The map $C_{12}$ equals the identity map when restricted to the subspace $I_+(L_1 \cap L_2)$ and the multiplicity of the eigenvalue $(+1)$ of the operator $C_{12}$ is $\dim(L_1 \cap L_2) = \dim(I_+(L_1 \cap L_2))$. 

Proof. Using the following diagram

$$L_1 \cap L_2 \xrightarrow{I_{\rho}} (\ker B)_-$$

we can easily see that the first claim holds. To complete the proof, it is sufficient to show that if \( C_1(0) \circ C_2(0) \phi = \phi \), then \( \phi \in (L_1 \cap L_2) \oplus (GL_1 \cap GL_2) \). For this, choose \( \phi_+ \in L_1 \), then \( C_1(0) \circ C_2(0) \phi_+ = \phi_+ \) implies \( \phi_+ = C_1(0) \phi_+ = C_2(0) \phi_+ \) since \( C_1(0)^2 = \text{Id} \). Hence, this means that \( \phi_+ \in L_2 \), so \( \phi_+ \in L_1 \cap L_2 \). Repeating the same argument, if \( \phi_- \in GL_1 \) and \( C_1(0) \circ C_2(0) \phi_- = \phi_- \), then \( \phi_- \in GL_1 \cap GL_2 \). Since \( \ker B = L_1 \oplus GL_1 \), this completes the proof.

Now let us consider the eigenvalues \( \lambda(R) \), which correspond to \( \alpha_j(0) = 0 \) and \( k = 0 \) in the following equality equivalent to (3.13),

$$4\lambda R + \alpha_j(\lambda) = 2\pi k + O(e^{-cR}) \ .$$

It is easy to see that such eigenvalues must be \( e \)-values. Hence, by Lemma 8.27 this provides another proof of the following result, originally shown in [5].

**Theorem 3.8.** The space of eigensections corresponding to \( e \)-values, which are not determined by \( \ker_{L_2}(\mathcal{D}_{i,\infty}) \) for \( i = 1,2 \), is given by the space \( L_1 \cap L_2 \).

Now let us consider the following Dirac type operator

$$\mathcal{D}_R = d_\rho + d_\rho^* : \oplus_{i=0}^n \Omega^i(M_R, V_\rho) \to \oplus_{i=0}^n \Omega^i(M_R, V_\rho)$$

where \( V_\rho \) denotes the flat vector bundle defined by a unitary representation \( \rho \) of \( \pi_1(M_R) \). The dimension of \( \ker \mathcal{D}_R \) is constant with respect to \( R \) since \( \ker \mathcal{D}_R \) is the space of the twisted harmonic forms over \( M_R \) and this space is always isomorphic to de Rham cohomology \( H^*(M_R, V_\rho) \) by the Hodge theorem. Moreover, one can show that all the \( e \)-values of the operator \( \mathcal{D}_R \) in (3.25) are the zero eigenvalues using the argument in Section 4 of [11].

**Proposition 3.9.** For the operator \( \mathcal{D}_R \) in (3.25), all the \( e \)-values of \( \mathcal{D}_R \) are the zero eigenvalues.

Proof. First let us observe that \( \mathcal{D}_{i,\infty} \) is self adjoint, so \( \ker_{L^2(\mathcal{D}_{i,\infty})} = \ker_{L^2(\mathcal{D}^2_{i,\infty})} \) and \( L_i \) is also the limiting value of extended \( L^2 \)-solutions of \( \mathcal{D}^2_{i,\infty} \). Let \( \Delta^q_{i,\infty} \) be the restriction of \( \mathcal{D}^2_{i,\infty} \) to \( \Omega^q(M_R, V_\rho) \) and

$$h^q_M := \dim \ker_{L^2} \Delta^q_{1,\infty} + \dim \ker_{L^2} \Delta^q_{2,\infty} + \dim L^q_1 \cap L^q_2$$

where \( L^q_i \) is the limiting values of the extended \( L^2 \)-solutions of \( \Delta^q_{i,\infty} \). Then, it is sufficient to show that \( \beta^q := \dim(\ker \mathcal{D}^2_R \cap \Omega^q(M_R, V_\rho)) \geq h^q_M \) since \( \beta_q \leq h^q_M \) by definition. For this, we use the following Mayer-Vietoris sequence

$$\ldots \to H^{q-1}(Y) \to H^q(M_R) \to H^q_\partial(M_{1,R}) \oplus H^q_\partial(M_{2,R}) \to H^q(Y) \to \ldots$$

where \( H^q_\partial(M_{i,R}) \) denotes the absolute cohomology. (Here, for simplicity, the bundle \( V_\rho \) is dropped in the notation.) The space \( \oplus_{q=0}^n H^q_\partial(M_{i,R}) \) can be identified
with the kernel of the operator $\mathcal{D}_{i,R}$ with the absolute boundary condition. In more detail, the operator $\mathcal{D}_R = d_\rho + d_\rho^*$ has the following form over $N_R$,

$$\mathcal{D}_R = d_\rho + d_\rho^* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left( \partial_u + \begin{pmatrix} 0 & d_Y \\ d_Y^* & 0 \end{pmatrix} \right)$$

with respect to

$$\Omega^*(N_R) \cong (\Omega^*(Y) \oplus \Omega^*(Y)) \otimes C^\infty([-R, R])_u).$$

Here $d_Y$, $d_Y^*$ denote the restricted operator to $Y$ of $d_\rho$ and its adjoint respectively. The operator $\mathcal{D}_{i,R}$ has the same form near the boundary and with respect to $(3.29)$. A section $\Psi$ in $\Omega^*(M_{i,R})$ over the cylinder near the boundary $Y$ has the following form,

$$\Psi = \Psi_0 + \Psi_1 \wedge du$$

where $\Psi_i$ has no factor $du$. Then the absolute boundary condition for $\mathcal{D}_{i,R}$ is given by $\Psi_1 = 0$. Similarly the relative boundary condition for $\mathcal{D}_{i,R}$ is given by $\Psi_0 = 0$. We denote by $\mathcal{D}_{i,R}^a$, $\mathcal{D}_{i,R}^r$ the resulting operators. Now let us recall that the Cauchy data spaces $\mathcal{H}(\mathcal{D}_{i,R})$ of $\mathcal{D}_{i,R}$ are Lagrangian subspaces in $\Omega^*(Y) \oplus \Omega^*(Y)$ with respect to the symplectic form $(G, \cdot)$ where $G = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $(\cdot, \cdot)$ are the inner product over $\Omega^*(Y) \oplus \Omega^*(Y)$. Then, this implies that $\mathcal{H}_0(\mathcal{D}_{i,R}) := \mathcal{H}(\mathcal{D}_{i,R}) \cap (H^*(Y) \oplus H^*(Y))$ are also Lagrangian subspaces in $H^*(Y) \oplus H^*(Y)$. Moreover, the space $\mathcal{H}_0(\mathcal{D}_{i,R})$ has the following decomposition,

$$\mathcal{H}_0(\mathcal{D}_{i,R}) = A_i \oplus R_i
$$

where $A_i, R_i$ the spaces spanned by the boundary values of $\ker \mathcal{D}_{i,R}^a, \ker \mathcal{D}_{i,R}^r$ in $H^*(Y) \oplus H^*(Y)$. Decomposing $A_i = \oplus_{q=0}^{n-1} A_i^q$, $R_i = \oplus_{q=0}^{n-1} R_i^q$ where $A_i^q, R_i^q$ are spaces of $q$-form parts, the Lagrangian subspace property of $\mathcal{H}_0(\mathcal{D}_{i,R})$ in $H^*(Y) \oplus H^*(Y)$ implies

$$H^q(Y) \cong A_i^q \oplus R_i^q.$$

By the exactness of $(3.27)$, we also have

$$H^q(M_R) \cong \text{Im}(H^{q-1}(Y) \to H^q(M_R)) \oplus \text{Im}(H^q(M_R) \to H^q_a(M_{1,R}) \oplus H^q_a(M_{2,R}))
\cong (\text{Im} k^{q-1})^\perp \oplus \ker k^q$$

where $k^q$ is the boundary map from $H^q_a(M_{1,R}) \oplus H^q_a(M_{2,R})$ to $H^q(Y)$. Now we summarize the consequences of the previous considerations. First, by $(3.30)$, we have

$$(\text{Im} k^{q-1})^\perp = (A_i^{q-1} + A_i^{q-1})^\perp = R_1^{q-1} \cap R_2^{q-1}.$$ 

Second, we note that $\ker k^q$ contains the Lagrangian sections whose boundary values are lying in $A_i^q \cap A_i^q$ and the harmonic sections that can be extended as $L^2$-solutions of $\Delta_{i,\infty}$. Hence,

$$\dim \ker k^q \geq \dim (A_i^q \cap A_i^q) + \dim \ker L^2 \Delta_{i,\infty}^q + \dim \ker L^2 \Delta_{i,\infty}^q.$$ 

By these facts and the following equality

$$\dim (L_1^q \cap L_2^q) = \dim (A_i^q \cap A_i^q) + \dim (R_1^{q-1} \cap R_2^{q-1}),$$

...
we can conclude that $\beta^q \geq h_M^q$. This completes the proof.

Let $\Psi_R$ be a normalized eigensection of $(D_{1,R})_{P_1}$, which corresponds to the $s$-value $\lambda = \lambda(R)$ with $|\lambda(R)| \leq R^{-\kappa}$ where $\kappa$ is the fixed constant such that $0 < \kappa < 1$. Then we have

$$D_{1,R} \Psi_R = \lambda \Psi_R, \quad \|\Psi_R\| = 1, \quad P_1(\Psi_R|_{v=R} \times Y) = 0.$$ 

The section $\Psi_R$ can be represented in the following way on $[0, R] \times Y \subset M_{1,R}$

$$\Psi_R = e^{-i\lambda v} \psi_1 + e^{i\lambda v} \psi_2 + \hat{\Psi}_R$$

where $\psi_1 \in (\ker B)_{+}$, $\psi_2 \in (\ker B)_{-}$ and $\hat{\Psi}_R$ is a smooth $L^2$-section orthogonal to $\ker B$. The next result corresponds to Proposition 3.3, which can be proved in the same way as Proposition 8.14 of [17],

**Proposition 3.10.** The zero-eigenmode $e^{-i\lambda v} \psi_1 + e^{i\lambda v} \psi_2$ of the eigensection $\Psi_R$ of $s$-value $\lambda(R)$ of $(D_{1,R})_{P_1}$ is a non-trivial.

Now we define

$$I_{\pm} = 1 \pm iG : \ker B \to (\ker B)_{\mp},$$

$$I_{\sigma_1} = I_{-|_{\ker(\sigma_1 + 1)}} : \ker(\sigma_1 + 1) \to (\ker B)_{+},$$

$$P_{\sigma_1} = \frac{1}{2}(1 - \sigma_1) : \ker B \to \ker(\sigma_1 + 1)$$

and

$$S_{\sigma_1}(\lambda) = -P_{\sigma_1} \circ C_1(\lambda) \circ I_{\sigma_1} : \ker(\sigma_1 + 1) \to \ker(\sigma_1 + 1).$$

For $\psi_1$ in (3.32), by Lemma 3.4, there exists a unique $\phi \in \ker(\sigma_1 + 1)$ such that $\psi_1 = \phi - iG\phi$. As in the derivation of (3.7), we compare $\Psi_R$ with $E(\phi, \lambda)$ and using Proposition 3.10 we obtain

$$\|C_1(\lambda)\psi_1 - \psi_2\| \leq e^{-cR}.$$

By the boundary condition in (3.31), we have

$$e^{-2i\lambda R}P_{\sigma_1}(\psi_1) = -P_{\sigma_1}(\psi_2).$$

Combining this equation and (3.33), we derive

$$\|e^{2i\lambda R}S_{\sigma_1}(\lambda)\phi - \phi\| \leq e^{-cR}$$

for $\phi \in \ker(\sigma_1 + 1)$. We also define

$$I_{\sigma_2} = I_{+|_{\ker(\sigma_2 + 1)}} : \ker(\sigma_2 + 1) \to (\ker B)_{-},$$

$$P_{\sigma_2} = \frac{1}{2}(1 - \sigma_2) : \ker B \to \ker(\sigma_2 + 1)$$

and

$$S_{\sigma_2}(\lambda) := -P_{\sigma_2} \circ C_2(\lambda) \circ I_{\sigma_2} : \ker(\sigma_2 + 1) \to \ker(\sigma_2 + 1),$$

where $C_2(\lambda)$ is the scattering matrix defined from the generalized eigensection over $M_{2,\infty}$. By the same way as above we can derive

$$\|e^{2i\lambda R}S_{\sigma_2}(\lambda)\phi - \phi\| \leq e^{-cR}.$$
for $\phi \in \ker(\sigma_2 + 1)$. Now we introduce
\[
\Omega_i(R) := \{ \rho \in \mathbb{R} - \{0\} \mid \det(e^{2\pi i R} S_{\sigma_i}(\rho) - \text{Id}) = 0, \, |\rho| \leq R^{-\kappa} \}
\]
for $i = 1, 2$. We repeat the argument used in [17] to prove the corresponding result for $s$-values of $(\mathcal{D}_{i,R})_{P_i}$ noting all the argument for the involution $C_i(0)$ in [17] holds for the involution $\sigma_i$, and we obtain

**Theorem 3.11.** For $i = 1, 2$, let $\lambda_1(R) \leq \lambda_2(R) \leq \ldots \leq \lambda_p(R)$ be the nonzero eigenvalues, counted to multiplicity, of $(\mathcal{D}_{i,R})_{P_i}$ satisfying $|\lambda_k(R)| \leq R^{-\kappa}$, and let $\rho_1(R) \leq \rho_2(R) \leq \ldots \leq \rho_m(R)$ be the nonzero elements, counted to multiplicity, of $\Omega_i(R)$. Then there exist $R_0$ and $c > 0$, independent of $R$, such that for $R \geq R_0$, $p(R) = m(R)$ and
\[
|\lambda_k(R) - \rho_k(R)| \leq e^{-cR} \quad \text{for} \quad k = 1, \ldots, p(R).
\]

We now have the following proposition

**Proposition 3.12.** There is a natural isomorphism
\[
\ker(\mathcal{D}_{i,R})_{P_i} \cong \ker L^2 \mathcal{D}_{i,\infty} \oplus \ker(\sigma_i - 1) \cap L_i
\]
for $i = 1, 2$.

**Proof.** Let $\Psi \in \ker(\mathcal{D}_{1,R})_{P_1}$. Then the section $\Psi$ satisfies $G(\partial_v + B)\Psi = 0$ on the cylinder $[0, R]_v \times Y$, and it has the following representation when restricted to this cylinder
\[
\Psi = \phi_0 + \sum_{\mu_j > 0} c_j e^{-\mu_j v} \phi_j
\]
where $(\sigma_i - 1)(\phi_0) = 0$. We use this expansion to extend $\Psi$ to a smooth section $\tilde{\Psi}$ on $M_{1,\infty}$ satisfying $\mathcal{D}_{1,\infty} \tilde{\Psi} = 0$. This means that $\tilde{\Psi}$ belongs to the space of the extended $L^2$-solutions of $\mathcal{D}_{1,\infty}$. Hence $\phi_0$ is an element of $L_1$. Let $E(\phi_0, \lambda)$ be the generalized eigensection attached to $\phi_0$. Then $\tilde{\Psi} - \frac{1}{2} E(\phi_0, 0)$ is square integrable and $\mathcal{D}_{1,\infty}(\tilde{\Psi} - \frac{1}{2} E(\phi_0, 0)) = 0$, and the map
\[
\Psi \rightarrow (\tilde{\Psi} - \frac{1}{2} E(\phi_0, 0), \phi_0)
\]
gives the expected isomorphism.

The restriction of $S_{\sigma_i} := S_{\sigma_i}(0)$ to $\ker(\sigma_i + 1) \cap \ker(C_i(0) + 1)$ is equal to the identity map and
\[
\dim(\ker(\sigma_i + 1) \cap \ker(C_i(0) + 1)) = \dim(\ker(\sigma_i - 1) \cap \ker(C_i(0) - 1)).
\]
It follows from Proposition 3.12 that the number of $(+1)$-eigenspace of $S_{\sigma_i} := S_{\sigma_i}(0)$ is equal to the dimension of the subspace of $\ker(\mathcal{D}_{i,R})_{P_i}$ complementary to the subspace $\ker L^2 \mathcal{D}_{i,\infty}$ for $i = 1, 2$.

Now we define our model operator. Let $C : W \rightarrow W$ denote a unitary operator acting on a $d$-dimensional vector space $W$ with eigenvalues $e^{i\alpha_j}$ for $j = 1, \ldots, d$. We introduce the operator $D(C)$,
\[
D(C) := -i \frac{1}{2} \frac{d}{du} : C^\infty(S^1, E_C) \rightarrow C^\infty(S^1, E_C)
\]
where $E_C$ is the flat vector bundle over $S^1 = \mathbb{R}/\mathbb{Z}$ defined by the holonomy $\overline{C}$, the complex conjugate of $C$. The spectrum of $D(C)$ is equal to

$$\{ \pi k - \frac{1}{2} \alpha_j \mid k \in \mathbb{Z}, j = 1, \ldots, d \}.$$  

(3.35)

The operators $C_{12}$, $S_{\sigma_1}$ and $S_{\sigma_2}$ are the unitary operators acting on finite dimensional vector spaces. Hence we can define self-adjoint, elliptic operators $D(C_{12}), D(S_{\sigma_1}), D(S_{\sigma_2})$ on $S^1$.

**Theorem 3.13.** Assume that all the $e$-values of $D_R$ are zero eigenvalues. Let $\lambda_1(R) \leq \lambda_2(R) \leq \ldots \leq \lambda_{p(R)}(R)$ be the nonzero eigenvalues, counted to multiplicity, of $D_R$ satisfying $|\lambda_k(R)| \leq R^{-\kappa}$, and let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{n(R)}$ be the nonzero eigenvalues, counted to multiplicity, of $D(C_{12})$ satisfying $|\lambda_k| \leq 2R^{1-\kappa}$. Then there exist $R_0$ and $C > 0$, independent of $R$, such that for $R \geq R_0$, $p(R) = n(R)$ and

$$|2R \lambda_k(R) - \lambda_k| \leq CR^{-\kappa} \quad \text{for} \quad k = 1, \ldots, p(R).$$

The similar statement holds for $(D_{i,R})_{P_i}$ and $D(S_{\sigma_i})$ with the relation

$$|R \lambda_k(R) - \lambda_k| \leq CR^{-\kappa} \quad \text{for} \quad k = 1, \ldots, p_i(R)$$

where $p_i(R)$ is the number of $s$-values of $(D_{i,R})_{P_i}$ with $|\lambda_k(R)| \leq R^{-\kappa}$.

**Proof.** First we introduce

$$\Omega^*(R) := \{ \rho \in \mathbb{R} - \{0\} \mid \det(e^{4i\rho R}C_{12} - \text{Id}) = 0, \ |\rho| \leq R^{-\kappa} \}.$$  

By definition, this set consists of the nonzero solution $\rho_{j,k}^*$ of

$$4\lambda R + \alpha_j(0) = 2\pi k \quad \text{with} \quad |\lambda| \leq R^{-\kappa}$$  

(3.36)

where $e^{i\alpha_j(0)}$ for $j = 1, \ldots, \frac{\kappa}{2}$ are the eigenvalues of $C_{12} = C_{12}(0)$. Now, for an element $\rho_{j,k}^*$ in $\Omega(R)$ defined in (3.12), one can show (as near (3.14)) that if $R \gg \omega$ there is the corresponding solution $\rho_{j,k}^*$ of

$$4\lambda R + \alpha_j(0) = 2\pi k \quad \text{with} \quad |\lambda| \leq R^{-\kappa} + R^{-1-\kappa},$$

noting $|\alpha_j(\lambda) - \alpha_j(0)| \leq cR^{-\kappa}$ for a positive constant $c$. Since $|\rho_{j,k}^* - \rho_{j,k}^{*'}| \leq cR^{-1-\kappa}$, this gives a one to one correspondence from $\Omega(R)$ to $\Omega^*(R_0)$ with $R_0^{-\kappa} = R^{-\kappa} + R^{-1-\kappa}$ for $R \gg 0$. Now, let us observe that for any pair of $\rho_{j,k}^* \neq \rho_{j',k'}^{*}$ in $\Omega^*(R)$, $|\rho_{j,k}^* - \rho_{j',k'}^{*}| \geq a_0 R^{-1}$ for a positive constant $a_0$. Hence, for $R \gg 0$, this implies that $\Omega^*(R) = \Omega^*(R_0)$ with $R_0^{-\kappa} = R^{-\kappa} + R^{-1-\kappa}$. In conclusion, there is a one to one correspondence between $\Omega(R)$ and $\Omega^*(R)$ for $R \gg 0$ with the relation

$$|\rho_k - \rho_k^*| \leq cR^{-1-\kappa}$$

where $\rho_1 \leq \ldots \leq \rho_{n(R)} (\rho_1^* \leq \ldots \leq \rho_{p(R)}^*)$ denotes the elements, counted to multiplicity, of $\Omega(R)$ ($\Omega^*(R)$). For $\rho^* \in \Omega^*(R)$, the map $\rho^* \to 2R\rho^*$ gives a one to one correspondence from $\Omega^*(R)$ to the subset of the eigenvalues $\lambda_k$ of $D(C_{12})$ with $|\lambda_k| \leq 2R^{1-\kappa}$. Now, applying Theorem (3.5) completes the proof for $s$-values for $D_R$. The case of $(D_{i,R})_{P_i}$ can be proved in the same way.

\[\square\]
4. LARGE TIME CONTRIBUTION

In this section, we prove the following proposition

**Proposition 4.1.**

\[
\lim_{R \to \infty} \int_{R^2}^{\infty} t^{-1} \left[ \text{Tr} \left( e^{-t D_2^2} - e^{-t(D_1, R)^2} - e^{-t(D_2, R)^2} \right) - h \right] dt - h(\gamma - \varepsilon \cdot \log R)
\]

\[
= \frac{d}{ds} \bigg|_{s=0} \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \left[ \text{Tr} \left( e^{-t D_2^2} - e^{-t(D_1, R)^2} - e^{-t(D_2, R)^2} \right) - h \right] dt
\]

where \( h = \dim(L_1 \cap L_2) - \dim(L_1 \cap \ker(\sigma_1 - 1)) - \dim(L_2 \cap \ker(\sigma_2 - 1)) \).

Recalling

\[ \frac{d}{ds} \bigg|_{s=0} \zeta^R(s) = \int_{R^2}^{\infty} t^{-1} \left[ \text{Tr} \left( e^{-t D_2^2} - e^{-t(D_1, R)^2} - e^{-t(D_2, R)^2} \right) - h \right] dt , \]

Proposition 4.1 immediately implies that the large time contribution to the adiabatic decomposition formula for the \( \zeta \)-determinant is equal to

\[
\frac{\det_\zeta \frac{1}{2} D(C_{12})^2}{\det_\zeta D(S_{\sigma_1})^2 \cdot \det_\zeta D(S_{\sigma_2})^2} .
\]

We start with the following result,

**Proposition 4.2.** The following equality holds,

\[
\lim_{R \to \infty} \left( \frac{d}{ds} \bigg|_{s=0} \frac{1}{\Gamma(s)} \int_0^{R^{-\varepsilon}} t^{s-1} \left[ \text{Tr} \left( e^{-t \frac{1}{4} D(C_{12})^2} - e^{-t D_1^2} - e^{-t D_2^2} \right) - h \right] dt + h(\gamma - \varepsilon \cdot \log R) \right) = 0 .
\]

**Proof.** Recalling the definition of \( D(C) \) in (3.34), we can see that if \( L \) is one of \( D(C_{12})^2 \), \( D(S_{\sigma_1})^2 \), \( D(S_{\sigma_2})^2 \), then

\[
\text{Tr} \left( e^{-t L} \right) \sim \sqrt{\frac{\pi}{t}} \frac{h_2}{2} + O(\sqrt{t}) \quad \text{near} \quad t = 0
\]

since \( h_2 = 2 \dim(\ker B) = 2 \dim \ker(\sigma_1 + 1) \). Hence there exists a constant \( c_1 \) such that

\[
|\text{Tr} \left( e^{-t \frac{1}{4} D(C_{12})^2} - e^{-t D_1^2} - e^{-t D_2^2} \right)| < c_1 \sqrt{t} \quad \text{near} \quad t = 0 .
\]

This allows us to estimate

\[
\left| \frac{d}{ds} \bigg|_{s=0} \frac{1}{\Gamma(s)} \int_0^{R^{-\varepsilon}} t^{s-1} \left[ \text{Tr} \left( e^{-t \frac{1}{4} D(C_{12})^2} - e^{-t D_1^2} - e^{-t D_2^2} \right) \right] dt \right|
\]

\[
\leq c_2 \cdot \int_0^{R^{-\varepsilon}} \frac{dt}{\sqrt{t}} = 2c_2 \cdot R^{-\varepsilon/2} .
\]

Combining this with

\[
\frac{d}{ds} \bigg|_{s=0} \frac{h}{\Gamma(s)} \int_0^{R^{-\varepsilon}} t^{s-1} dt = h(\gamma - \varepsilon \cdot \log R)
\]
completes the proof. □

It follows from Proposition 4.2 that Proposition 4.1 is equivalent to the following equation

\[(4.2) \quad \lim_{R \to \infty} \int_{R^{2-\varepsilon}}^{\infty} t^{-1} \left[ \text{Tr} \left( e^{-tD_R^2} - e^{-t(D_1,R)^2_{P_1}} - e^{-t(D_2,R)^2_{P_2}} \right) - h \right] dt \]
\[= \lim_{R \to \infty} \frac{d}{ds} \mid_{s=0} \Gamma(s) \int_{R^{2-\varepsilon}}^{\infty} t^{s-1} \left[ \text{Tr} \left( e^{-t\frac{1}{2}D(C)}^2 - e^{-tD(S_1)^2} - e^{-tD(S_2)^2} \right) - h \right] dt.\]

Now using change variables we obtain

\[\int_{R^{2-\varepsilon}}^{\infty} t^{-1} \left[ \text{Tr} \left( e^{-tD_R^2} - e^{-t(D_1,R)^2_{P_1}} - e^{-t(D_2,R)^2_{P_2}} - h \right) \right] dt \]
\[= \int_{R^{2-\varepsilon}}^{\infty} t^{-1} \left[ \text{Tr} \left( e^{-tR^2D_R^2} - e^{-tR^2(D_1,R)^2_{P_1}} - e^{-tR^2(D_2,R)^2_{P_2}} - h \right) \right] dt.\]

Then the equality (4.2) is equivalent to

\[(4.3) \quad \lim_{R \to \infty} \int_{R^{2-\varepsilon}}^{\infty} t^{-1} \left[ \text{Tr} \left( e^{-tR^2D_R^2} - e^{-tR^2(D_1,R)^2_{P_1}} - e^{-tR^2(D_2,R)^2_{P_2}} - h \right) \right] dt \]
\[= \lim_{R \to \infty} \frac{d}{ds} \mid_{s=0} \Gamma(s) \int_{R^{2-\varepsilon}}^{\infty} t^{s-1} \left[ \text{Tr} \left( e^{-t\frac{1}{2}D(C)}^2 - e^{-tD(S_1)^2} - e^{-tD(S_2)^2} \right) - h \right] dt.\]

Now we split
\[\text{Tr} \left( e^{-tR^2D_R^2} - e^{-tR^2(D_1,R)^2_{P_1}} - e^{-tR^2(D_2,R)^2_{P_2}} \right) - h\]
into two parts
\[\text{Tr}_{I,R} \left( e^{-tR^2D_R^2} - e^{-tR^2(D_1,R)^2_{P_1}} - e^{-tR^2(D_2,R)^2_{P_2}} \right),\]
\[\text{Tr}_{II,R} \left( e^{-tR^2D_R^2} - e^{-tR^2(D_1,R)^2_{P_1}} - e^{-tR^2(D_2,R)^2_{P_2}} \right)\]

where \(\text{Tr}_{I,R}(\cdot)\) (\(\text{Tr}_{II,R}(\cdot)\)) is the part of the trace restricted to the eigenvalues of \(R^2D_R^2, R^2(D_1,R)^2_{P_1}, R^2(D_2,R)^2_{P_2}\) which are larger (smaller or equal to) than \(R^2\).

The next proposition shows that \(\text{Tr}_{I,R}(\cdot)\) can be neglected as \(R \to \infty\),

**Proposition 4.3.** We have the following estimate

\[\int_{R^{2-\varepsilon}}^{\infty} t^{-1} \text{Tr}_{I,R} \left( e^{-tR^2D_R^2} - e^{-tR^2(D_1,R)^2_{P_1}} - e^{-tR^2(D_2,R)^2_{P_2}} \right) dt \leq c_1 e^{-c_2 R^{\frac{3}{2} - \varepsilon}}\]

for some positive constants \(c_1, c_2\).
Proof. Let $\lambda_{k_0}(R)^2$ denote the smallest large eigenvalue of $D_R^2$, that is, the smallest one with $\lambda_{k_0}(R)^2 > R^{-\frac{3}{2}}$. We now estimate $\text{Tr}_{I,R}(e^{-tR^2D_R^2})$ as follows:

\[
\text{Tr}_{I,R}(e^{-tR^2D_R^2}) = \sum_{\lambda_i > R^{-\frac{3}{2}}} e^{-t\lambda_i^2} \leq e^{-tR^{-\frac{3}{2}}\sum_{\lambda_i > R^{-\frac{3}{2}}} \lambda_i^2} \leq e^{-tR^{-\frac{3}{2}}\lambda_i^2} \leq e^{-tR^{-\frac{3}{2}}\lambda_i^2} \leq e^{-tR^{-\frac{3}{2}}\lambda_i^2} \leq b_1 R e^{-tR^{-\frac{3}{2}}\lambda_i^2} \leq b_1 R e^{-b_2 t\sqrt{R}},
\]

for some positive constants $b_1, b_2$. Hence we have

\[
\int_{R^{-\epsilon}}^\infty t^{-1} \text{Tr}_{I,R}(e^{-tR^2D_R^2}) \, dt \leq \int_{R^{-\epsilon}}^\infty t^{-1} b_1 R e^{-b_2 t\sqrt{R}} \, dt \leq \frac{b_1}{b_2} R^{\frac{1}{2} + \epsilon} \int_{b_2 R^{\frac{1}{2}} - \epsilon}^\infty e^{-v} \, dv \leq b_3 e^{-b_4 R^{\frac{1}{2}} - \epsilon}.
\]

The trace $\text{Tr}_{I,R}(e^{-tR^2(D_{I,R})^2})$ for $i = 1, 2$ can be estimated in the same way. This completes the proof.

We also split $\text{Tr}(e^{-t\frac{1}{4}D(C_{12})^2} - e^{-tD(S_{s_1})^2} - e^{-tD(S_{s_2})^2} - h)$ into two parts

\[
\text{Tr}_{I,R}(e^{-t\frac{1}{4}D(C_{12})^2} - e^{-tD(S_{s_1})^2} - e^{-tD(S_{s_2})^2}) \quad \text{and} \quad \text{Tr}_{II,R}(e^{-t\frac{1}{4}D(C_{12})^2} - e^{-tD(S_{s_1})^2} - e^{-tD(S_{s_2})^2})
\]

where $\text{Tr}_{I,R}(\cdot)$ ($\text{Tr}_{II,R}(\cdot)$) is taken over the nonzero eigenvalues of $\frac{1}{4}D(C_{12})^2$, $D(S_{s_1})^2$, $D(S_{s_2})^2$ which are larger (smaller or equal to) than $R^{\frac{1}{2}}$. The following proposition corresponds to Proposition 4.4 and its proof is essentially the same as the proof of Proposition 4.3.

**Proposition 4.4.** There exist positive constants $c_1, c_2$ such that

\[
\int_{R^{-\epsilon}}^\infty t^{-1} \text{Tr}_{I,R}(e^{-t\frac{1}{4}D(C_{12})^2} - e^{-tD(S_{s_1})^2} - e^{-tD(S_{s_2})^2}) \leq c_1 e^{-c_2 R^{\frac{1}{2}} - \epsilon}.
\]

By Propositions 4.3 and 4.4 we can see that the equality 4.3 is equivalent to

\[
\lim_{R \to \infty} \left( \int_{R^{-\epsilon}}^\infty t^{-1} \text{Tr}_{II,R}(e^{-tR^2D_R^2} - e^{-tR^2(D_{1,R})^2} - e^{-tR^2(D_{2,R})^2}) \, dt \right) = 0.
\]

The equation 4.4 is a consequence of the next result.
Proposition 4.5. For sufficiently large $R$, there exist positive constants $c_1, c_2$ independent of $R$ and $t$, such that

$$| \text{Tr}_{II,R}(e^{-tR^2D_R^2}) - \text{Tr}_{II,R}(e^{-t\frac{1}{4}D(C_{12})^2}) | \leq c_1 t R \frac{1}{2} e^{-c_2 t}$$

$$| \text{Tr}_{II,R}(e^{-tR^2(D_{i,R})^2}) - \text{Tr}_{II,R}(e^{-t\frac{1}{4}D(S_{i,R})^2}) | \leq c_1 t R \frac{1}{2} e^{-c_2 t}$$

for any $t > 0$.

**Proof.** We use the analysis of $s$-values developed in Section 3. We fix $\kappa = \frac{3}{4}$. It follows from Theorem 3.13 that, for any eigenvalue $\lambda(R)$ of $D_R$ with $|\lambda(R)| \leq R^{-\frac{3}{4}}$, there exists an analytic function $\alpha(\lambda)$ such that

$$R \lambda(R) = \lambda_j + \frac{1}{4} \lambda(R) \alpha(\lambda(R)) + O(e^{-cR})$$

where $\lambda_j$ is an eigenvalue of $\frac{1}{2}D(C_{12})$ with $|\lambda_j| \leq R^\frac{1}{4}$. Therefore, there exist functions $c(R), d(R)$ and a constant $C > 0$ such that

$$R^2 \lambda(R)^2 = \lambda_j^2 + \lambda_j \frac{c(R)}{R^\frac{3}{2}} + \frac{d(R)}{R^\frac{3}{2}} \quad \text{with} \quad |c(R)| \leq C, \quad |d(R)| \leq C,$$

for any sufficiently large $R$. We use the elementary inequality $|e^{-\lambda} - 1| \leq |\lambda| e^{\lambda}$ to get

$$|e^{-tR^2 \lambda(R)^2} - e^{-t\lambda_j^2}| = |e^{-t\lambda_j^2}(e^{-t[R^2 \lambda(R)^2 - \lambda_j^2]} - 1)|$$

$$\leq \left(\frac{|\lambda_j| c(R)}{R^\frac{3}{2}} + \frac{|d(R)|}{R^\frac{3}{2}}\right) t e^{-t\left(R^2 \lambda(R)^2 - \lambda_j^2\right)} \leq \frac{C}{R^\frac{3}{2}} t e^{-\frac{1}{2} \lambda_j^2 t}$$

for $R \gg 0$. In the last inequality we used the fact that $|\lambda_j| \leq R^\frac{1}{4}$. Let us fix a sufficiently large $R$. We take the sum over finitely many eigenvalues $\lambda_j^2$ of $\frac{1}{4}D(C_{12})^2$ with $\lambda_j^2 \leq R^\frac{1}{2}$, and obtain

$$|\text{Tr}_{II,R}(e^{-tR^2D_R^2}) - \text{Tr}_{II,R}(e^{-t\frac{1}{4}D(C_{12})^2}) | \leq C \frac{t}{R^\frac{3}{2}} \sum_{\lambda_j^2 \leq R^\frac{1}{2}} e^{-\frac{1}{2} \lambda_j^2 t}.$$

The operator $\frac{1}{4}D(C_{12})^2$ is a Laplace type operator over $S^1$, hence the number of eigenvalues $\lambda_j^2$ of $\frac{1}{4}D(C_{12})^2$ with $\lambda_j^2 \leq R^\frac{1}{2}$ can be estimated by $R^\frac{1}{2}$. Therefore, we have

$$C \frac{t}{R^\frac{3}{2}} \sum_{\lambda_j^2 \leq R^\frac{1}{2}} e^{-\frac{1}{2} \lambda_j^2 t} \leq c_1 \frac{t}{R^\frac{1}{2}} R^\frac{1}{2} e^{-\frac{1}{2} \lambda_1^2 t}$$

where $\lambda_1^2$ denotes the first nonzero eigenvalue of $\frac{1}{4}D(C_{12})^2$. Note that $c_1$ and $\lambda_1^2$ are independent of $R$. This proves the first claim putting $c_2 = \frac{1}{2} \lambda_1^2$. The proof of the second claim goes in the same way.

The proof of Proposition 4.4 is now complete. \qed
Proof of Theorem 1.1: Now Proposition 2.5 and Proposition 4.1 give us the following equality,

\[
\lim_{R \to \infty} \left( \zeta_R'(0) + h(\gamma + (2 - \varepsilon) \cdot \log R) + \zeta_R'(0) - h(\gamma - \varepsilon \cdot \log R) \right)
\]

\[
= \frac{d}{ds} \left. \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left[ \text{Tr} \left( e^{-t \frac{1}{4} D(C_{12})^2} - e^{-t D(S_{\sigma_1})^2} - e^{-t D(S_{\sigma_2})^2} - h \right) \right] \, dt \right|_{s=0}
\]

By an elementary computation (for instance, see Proposition 2.2 in [15]), we can derive

\[\det \zeta \left( \frac{1}{4} D(C_{12})^2 \right) = 2^{h_Y + 2h_M} \det \left( \frac{2 \text{Id} - C_{12} - C_{12}^{-1}}{4} \right) ,\]

(4.5)

\[\det \zeta D(S_{\sigma_i})^2 = 2^{h_Y} \det \left( \frac{2 \text{Id} - S_{\sigma_i} - S_{\sigma_i}^{-1}}{4} \right) .\]

Combining all these equalities provides us with the final formula (1.10) in Theorem 1.1.

5. A Proof of the Decomposition Formula of the \(\eta\)-Invariant

In this section we offer a new proof of the decomposition formula for the \(\eta\)-invariant. This formula has been proved by several authors (see [4], [9], [12], [18], [23], [3], [13], [16]) and the proof we discuss in this section is not the simplest one. Still we believe that it is worthy to present the scattering approach to the decomposition of the \(\eta\)-invariant. The key in our proof is to show that the scattering data provides us with the contribution given by the boundary conditions in the decomposition formula for the \(\eta\)-invariant.

Let us remind the reader that the \(\eta\)-function of a Dirac operator \(\mathcal{D}\) on a closed manifold \(M\), introduced in [1], is defined as

\[
\eta_{\mathcal{D}}(s) = \sum_{\lambda_k \neq 0} \text{sign} (\lambda_k) |\lambda_k|^{-s} ,
\]

where the sum is taken over all nonzero eigenvalues of \(\mathcal{D}\). The \(\eta\)-function is well-defined for \(\Re(s)\) large and it has a meromorphic extension to the whole complex plane and \(s = 0\) is a regular point, hence \(\eta_{\mathcal{D}}(0)\) is well-defined. Following [1] we introduce the \(\eta\)-invariant of \(\mathcal{D}\) as

(5.1) \[\eta(\mathcal{D}) = \frac{1}{2} (\eta_{\mathcal{D}}(0) + \dim \ker \mathcal{D}) .\]

Now, let us assume that we have a decomposition of a closed odd-dimensional manifold \(M\) to \(M_1 \cup M_2\) in the way described in the introduction. For \(\mathcal{D}_i := \mathcal{D}|_{M_i}\), we impose the boundary conditions given by the generalized APS spectral projections \(P_i\) defined in (1.6). Then the \(\eta\)-function of \((\mathcal{D}_i)_{P_i}\) is also well-defined and it has the same properties as the \(\eta\)-function of the Dirac operator on a closed manifold, in particular, the \(\eta\)-function of \((\mathcal{D}_i)_{P_i}\) is regular at \(s = 0\). Hence, we can define the \(\eta\)-invariant of \((\mathcal{D}_i)_{P_i}\) as in (5.1). The following result was proved by several authors as we remarked above,
Theorem 5.1. The following formula holds,

\[ \eta(D) = \eta((D_1)_{P_1}) + \eta((D_2)_{P_2}) + \eta(D; \sigma_1, \sigma_2) \mod \mathbb{Z}, \]

where \( \eta(D; \sigma_1, \sigma_2) \) denotes the \( \eta \)-invariant of the operator \( D = G(\partial_u + B) \) over \( N \simeq [-1,1] \times Y \), subject to the boundary condition \( P_2 \) at \( u = -1 \) and \( P_1 \) at \( u = 1 \).

For the involution \( \sigma_i \) which defines \( P_i \) in (1.6), let us observe that

\[ U = \sigma_1 \sigma_2 : \ker B \to \ker B \]

is the unitary operator, such that \( UG = GU \), \( \det U = 1 \) and \( U^* = \sigma_1 U \sigma_1 \). It follows that the spectrum of \( U \) is invariant under complex conjugation. Moreover, the maps \( U_\pm = U_{|_{(\ker B)_\pm}} : (\ker B)_\pm \to (\ker B)_\pm \) are well-defined. The following result proved in Section 2 of [14] was the key ingredient in the proof of Theorem 5.1.

Proposition 5.2. We have the following formulas

\[ \eta(D; \sigma_1, \sigma_2) = \frac{-1}{2\pi i} \log \det(U_+) \mod \mathbb{Z}. \]

One way to prove the decomposition formula (5.2) is to use the adiabatic analysis we developed in the proof of Theorem 1.1. This analysis easily gives us the following theorem,

Theorem 5.3. The following formula for the \( \eta \)-invariant holds,

\[ \eta(D) - \eta((D_1)_{P_1}) - \eta((D_2)_{P_2}) = \eta(D(C_{12})) - \eta(D(S_{\sigma_1})) - \eta(D(S_{\sigma_2})) \mod \mathbb{Z}. \]

Proof. We repeat the corresponding argument to derive Theorem 1.1 for the \( \eta \)-invariant to obtain the expected formula

\[
\lim_{R \to \infty} \{ \eta(D_R) - \eta((D_{1,R})_{P_1}) - \eta((D_{2,R})_{P_2}) \} = \eta(D(C_{12})) - \eta(D(S_{\sigma_1})) - \eta(D(S_{\sigma_2})) \mod \mathbb{Z}.
\]

Now, we use the fact that \( \eta(D_R), \eta((D_i, R)_{P_i}) \) are independent of \( R \) modulo integer (see Proposition 2.16 of [17]) to complete the proof. \( \square \)

Now we need to show

\[ \eta(D; \sigma_1, \sigma_2) = \eta(D(C_{12})) - \eta(D(S_{\sigma_1})) - \eta(D(S_{\sigma_2})) \mod \mathbb{Z}. \]

For this, we observe the followings: The scattering matrix \( C_i = C_i(0) \) can be represented in the following way,

\[ C_i = \begin{pmatrix} 0 & C(i)_- \\ C(i)_+ & 0 \end{pmatrix} \text{ where } C(i)_\pm C(i)_\mp = \text{Id}, \]

with respect to the decomposition \( \ker B = (\ker B)_+ \oplus (\ker B)_- \). We see that

\[ C_{12} = C(1)_+ C(2)_- : (\ker B)_- \to (\ker B)_-. \]
Similar formulas hold for the involutions \( \sigma_i \) and we have
\[
S_{\sigma_1} = - P_{\sigma_1} \circ C_1 \circ I_{\sigma_1} = - \frac{1}{2} \begin{pmatrix}
\text{Id} & \sigma(1)_- \\
-\sigma(1)_+ & \text{Id}
\end{pmatrix}
\begin{pmatrix}
0 & C(1)_- \\
C(1)_+ & 0
\end{pmatrix}
\begin{pmatrix}
2\text{Id} & 0 \\
0 & 0
\end{pmatrix}
\]
\[
= \begin{pmatrix}
\sigma(1)_- C(1)_+ & 0 \\
-C(1)_+ & 0
\end{pmatrix}.
\]

We can also see that each element of \( \ker(\sigma_1 + 1) \) is represented in the form
\[
\begin{pmatrix}
f \\
-\sigma(1)_+ f
\end{pmatrix}
\]
for some \( f \in (\ker B)_+ \). This allows us to represent the map \( S_{\sigma_1} \) over \( \ker(\sigma_1 + 1) \) as
\[
S_{\sigma_1} \begin{pmatrix}
f \\
-\sigma(1)_+ f
\end{pmatrix} = \begin{pmatrix}
\sigma(1)_- C(1)_+ & 0 \\
-C(1)_+ & 0
\end{pmatrix} \begin{pmatrix}
f \\
-\sigma(1)_+ f
\end{pmatrix} = \begin{pmatrix}
\sigma(1)_- C(1)_+ f \\
-\sigma(1)_+ C(1)_+ f
\end{pmatrix}.
\]

Therefore, from the spectral point of view, the operator \( S_{\sigma_1} \) is equal to the operator
\[
\sigma(1)_- C(1)_+ : (\ker B)_+ \to (\ker B)_+ ,
\]

or equivalently to the operator
\[
C(1)_+ \sigma(1)_- : (\ker B)_- \to (\ker B)_- .
\]

The corresponding analysis for the operator \( S_{\sigma_2} \), implies that \( S_{\sigma_2} \) is equivalent to
\[
C(2)_- \sigma(2)_+ : (\ker B)_+ \to (\ker B)_+ \quad \text{or} \quad \sigma(2)_+ C(2)_- : (\ker B)_- \to (\ker B)_- .
\]

Combining (5.5) and these, we obtain
\[
\frac{\det(C_{12})}{\det(S_{\sigma_1}) \det(S_{\sigma_2})} = \det(\sigma(1)_+ \sigma(2)_-) .
\]

For the operator \( D(C) \) on \( S^1 \) defined by a unitary map \( C \) in (3.34),
\[
\eta(D(C)) = - \frac{1}{2\pi i} \log \det(-C) \mod \mathbb{Z} .
\]

(see Theorem 2.1 and Lemma 2.3 in [14]). If we combine (5.6) and (5.7), we have
\[
\eta(D(C_{12})) - \eta(D(S_{\sigma_1})) - \eta(D(S_{\sigma_2})) = - \frac{1}{2\pi i} \log \det(-\sigma(1)_+ \sigma(2)_-) \mod \mathbb{Z} .
\]

Noting \( \det(-\sigma(1)_+ \sigma(2)_-) = \det(-\sigma(1)_- \sigma(2)_+) \), this and Proposition 5.2 end the proof of the following theorem,

**Theorem 5.4.**
\[
\eta(D; \sigma_1, \sigma_2) = \eta(D(C_{12})) - \eta(D(S_{\sigma_1})) - \eta(D(S_{\sigma_2})) \mod \mathbb{Z} .
\]

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