Forward-Backward Squeezing Propagator

Jamil Daboul

Physics Department, Ben Gurion University of the Negev
84105 Beer Sheva, Israel

Abstract

I show that a usual propagator cannot be defined for the pseudo -diffusion equation of the Q- functions. Instead, a forward-backward propagator is defined, which motivated a generalization of Cahill-Glauber interpolating operator. An algorithm is also given for squeezing Q functions directly, using one-dimensional diffusion propagators.

1 Introduction

In previous papers [1, 2], it was shown that the Q functions for an operator $A$ (see definition in Eq. (13) below) obeys the following partial differential equation

$$\nabla(p, q; \sigma) Q(A : p, q; \sigma) := \left[ \frac{\partial}{\partial \sigma} - \frac{1}{4} \left( \frac{\partial^2}{\partial q^2} - \frac{1}{\sigma^2} \frac{\partial^2}{\partial p^2} \right) \right] Q(A : p, q; \sigma) = 0 , \quad \text{where} \quad \sigma := e^{2y} ,$$

(1)

where $y$ is the squeezing parameter, as defined in (14). (Note that $\sigma$ here is the inverse of the $\lambda := e^{-2y}$ in [1, 2], so that the roles of $p$ and $q$ get exchanged.) This equation describes how the Q functions $Q(p, q; \sigma)$ get changed in phase space $(p, q)$ as the squeezing parameter $\sigma$ is increased. If the Q function belongs to a density operator $\rho$, then $Q(\rho : p, q; \sigma)$ is a probability distribution function, which remains normalized to 1 for all $\sigma$, i.e. if its integral is carried out over the whole phase space. Eq. (1) was called pseudo-diffusion equation [1, 2], because (a) it resembles the diffusion equation in 2 dimensions [3], where the parameter $\sigma$ plays the role of time, and (b) the coefficients of $\frac{\partial^2}{\partial p^2}$ and $\frac{\partial^2}{\partial q^2}$ in (1) have opposite signs. (Since $\sigma$ is a monotonically increasing function of $y$, I shall use the time analogy when referring to either $\sigma$ or $y$, and hope that this will not lead to any confusion). Therefore, this equation describes a diffusive process in the $q$ variable and an infusive one in the $p$ variable for all $\sigma$. In this way a thin distribution along the $p$-axis get continuously deformed into a thin distribution along the $q$-axis, as $\sigma$ is increased from 0 to $\infty$.

The (symplectic) Fourier transform of the above partial differential equation (1), with respect to the phase-space variables $(p, q)$, yields an ordinary differential equation in $\sigma$:

$$\left[ \frac{\partial}{\partial \sigma} + \frac{1}{4} \left( k^2 - \frac{1}{\sigma^2} x^2 \right) \right] \tilde{Q}(k, x; \sigma) = 0 ,$$

(2)

where the symplectic Fourier transform is defined by

$$\tilde{Q}(k, x; \sigma) := \iint_{\mathbb{R}^2} \frac{dp dq}{2\pi} e^{-i[xp-kq]} Q(p, q; \sigma) .$$

(3)

It is easy to check that the “kernel”

$$K(k, x; \sigma, \mu) := e^{-\frac{1}{4}k^2(\sigma-\mu)} e^{-\frac{1}{4}x^2(\sigma^{-1}-\mu^{-1})} ,$$

(4)

$E-mail$: daboul@bguvms.bgu.ac.il
is a solution of (2). Hence, the inverse Fourier transform of the products \(K \tilde{Q}\) yields the solution of the pseudo-diffusion equation (4), for any given initial \(Q\) function \(Q(p, q; \mu)\) with \(\mu \leq \sigma\):

\[
Q(p, q; \sigma) = \int \frac{dk}{2\pi} \int \frac{dx}{2\pi} e^{i[xp-kq]} K(k, x; \sigma, \mu) \tilde{Q}(k, x; \mu).
\]

\[
= \int \frac{dk}{2\pi} \int \frac{dx}{2\pi} e^{i[xp-kq]} K(k, x; \sigma, \mu) \int \int \frac{dp'}{2\pi} \frac{dq'}{2\pi} e^{-i[xp'+kq']} Q(p', q'; \mu).
\]

Eq. (3) shows that in order to get the solutions \(Q(p, q; \sigma)\) we must perform two double integrations, first over \((p', q')\) and then over \((k, x)\). Therefore, it would have been much more efficient, if we are allowed to exchange the order of two double integrations, and carry out the integration over the variables \((k, x)\) first: the resulting integral, if it existed, would have been equal to the Fourier transform of the kernel \(K\):

\[
G_0(p-p', q-q'; \sigma, \mu) := \int \frac{dk}{2\pi} \int \frac{dx}{2\pi} e^{i[x(p-p')-k(q-q')]} K(k, x; \sigma, \mu)
\]

\[
= \int \frac{dk}{2\pi} \int \frac{dx}{2\pi} e^{i[x(p-p')-k(q-q')]} e^{-\frac{1}{4}[k^2(\sigma-\mu)+x^2(\sigma^{-1}-\mu^{-1})]},
\]

and would have enabled us to calculate the squeezed distributions (from unsqueezed or less squeezed ones) directly by a single double integration, as follows

\[
Q(p, q; \sigma) = \int dp' dq' G_0(p-p', q-q'; \sigma, \mu) Q(p', q'; \mu), \quad \text{for} \quad \sigma > \mu.
\]

Unfortunately, the integral (3) does not exist, as we shall see in section 3. Nevertheless, I shall give an algorithm for calculating squeezed \(Q\) functions which is based on separation of variables and uses 1-dimensional diffusion propagators.

Next, I show that it is more natural to define 2-sided propagators (16) for the pseudo-diffusion equation (4); these propagators depend on three squeezing parameters, the present \(\sigma\), the past \(\mu \leq \sigma\) and the future \(\lambda \geq \sigma\). Therefore, I shall refer to these 2-sided propagators as forward-backward (or future-past) squeezing propagators. A special case of these propagators connects \(Q\) functions to their Wigner counterparts. This makes it possible to give a new interpretation of the Wigner function. In turn, this interpretation makes it sensible to define formally generalized \(Q\) functions and \(Q\) operators which depend on two squeezing parameters, \(\sigma_p\) and \(\sigma_q\). On a subdomain of these parameters, which can be expressed in terms of an ordinary squeezing parameter \(\sigma\) and a “thermal parameter” \(\tau\), the above generalized \(Q\) operators can be identified with thermalized and squeezed Wigner operators (I hope to give details elsewhere). Partial differentiation of these operators with respect to \(\sigma\) and \(\tau\) yields the generalized pseudo-diffusion equation (4) and the diffusion equation (17), respectively.

In this paper, I review in Sec. 2 basic properties of the Wigner and the \(Q\) functions. Then I explain in Sec. 3 why the Fourier transform of the kernel \(K\) cannot exist. In Sec. 4 I give an algorithm for calculating squeezed \(Q\) functions. In Sec. 5 I define the two-sided propagators for the pseudo-diffusion equation. In Sec. 6 I give the new interpretation of the Wigner function. In Sec. 7 I define and study generalized \(Q\) functions and \(Q\) operators. Finally, in Sec. 8 I give a summary.

## 2 Wigner and \(Q\) functions

The Wigner representation of an operator \(A\) is defined formally by

\[
W(A : p, q) := 2 \int_{-\infty}^{\infty} \langle q-x | A | q+x \rangle e^{2ixp} dx = 2 \text{Tr} ( A W(p, q) ).
\]
where I use the round kets to denote eigenstates of the position operator, \( Q | x \rangle = x | x \rangle \), to distinguish them from the number states \( | n \rangle \). The operators (one for each \((p, q)\))

\[
W(p, q) := \int_{-\infty}^{\infty} |q + x)(x - q| e^{2ixp} \ dx ,
\]

were called Wigner operators by Dahl. Obviously,

\[
W(0, 0) = \int_{-\infty}^{\infty} x(-x| dx ,
\]

is the parity operator. Therefore, the Wigner operators can be interpreted as displaced parity operators \([5-8]\), because

\[
W(p, q) = D(p, q)W(0, 0)D^{\dagger}(p, q) = D(2p, 2q)W(0, 0) ,
\]

where

\[
D(p, q) := \exp \left[ i(pQ - qP) \right] = \exp \left[ i\frac{pq}{2} \right] \exp \left[ -iqP \right] \exp \left[ ipQ \right]
\]

is called the displacement operator, because \( D(p, q) | x \rangle = e^{ip(x+q/2)} | x + q \rangle \). Since \( W(0, 0) \) is unitary and Hermitian, it follows immediately from \( (11) \) that also the displaced parity operators are unitary and Hermitian: \( W^{-1}(p, q) = W^{\dagger}(p, q) = W(p, q) \), so that \( W^2(p, q) = I \). This means that the \( W(p, q) \) are observables with eigenvalues \( \pm 1 \), as was pointed out by Dahl and Bishop and Vourdas \([7]\).

The Wigner representation \([8]\) yields functions of two variables, \( p \) and \( q \), which may be looked upon as phase-space variables. These ‘Wigner functions’ have interesting properties and are useful for various calculations \([8]\). They are often referred to as quasi-probability functions, because they can take negative values even when \( A \) is a positive operator, \( A \geq 0 \), such as a density operator \( \rho \).

In contrast, the Q representation yields nonnegative functions for positive operators \( A \): These functions are defined as follows \([3, 4]\)

\[
Q(A : p, q; \zeta) = \langle pq; \zeta | A | pq; \zeta \rangle = \text{Tr} \left( A \Pi(pq; \zeta) \right) , \quad \text{where} \quad \Pi(p, q; \zeta) := | pq; \zeta \rangle \langle pq; \zeta | \quad (13)
\]

are projection operators on the squeezed states \( | pq; \zeta \rangle \), which are defined by \([1, 3]\)

\[
| pq; \zeta \rangle = D(p, q)S(\zeta)|0\rangle , \quad \text{where} \quad \zeta := ye^{ip} \quad (-\infty < y < \infty) \quad (14)
\]

and \( |0\rangle \) is the ground state of a specific harmonic oscillator, \( a|0\rangle = 0 \). (i.e. \( a \) is the annihilation operator with a definite frequency \( \omega_0 \); In this paper, we set \( \hbar = m = \omega_0 = 1 \). In \([14]\) \( D(p, q) \) is the displacement operator \([12]\), which generates coherent states when applied to \( |0\rangle \), and

\[
S(\zeta) = \exp \left[ \frac{1}{2} \left( \zeta a^2 - \zeta^* a^2 \right) \right] , \quad (a := \frac{Q + iP}{\sqrt{2}}) \quad (15)
\]

is the squeezing operator, where the squeeze parameter \( y \) vanishes in the coherent-state limit.

If \( A \) is a density matrix \( \rho \), then its Q function \( Q(\rho : p, q; \zeta) \) can naturally be interpreted as a probability distribution. To emphasize this fact, the Q functions were denoted by \( P \) in \([1, 3]\), instead of \( Q \) here.

For simplicity, I shall from now on discuss only squeezings which are pure boosts, without rotation, i.e. with \( \varphi \equiv 0 \) in \([14]\), and use the squeezing parameter \( \sigma := e^{2y} \) instead of \( y \).
3 Non-existence of a 1-sided Propagator

I shall now explain why $G_0$, the Fourier transform of the kernel $K$, can not exist. First, we note that the $G_0$ in (3), if it exists, would be a special case, for $\mu = \lambda$ (!), of the following product of two $G_1$ factors:

$$G(p - p', q - q'; \sigma - 1 - \lambda - 1, \sigma - \mu) := G_1(p - p'; \sigma - 1 - \lambda - 1) G_1(q - q'; \sigma - \mu),$$

where

$$G_1(p - p'; \sigma - 1 - \lambda - 1) := \int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{ix(p-p')} e^{-\frac{x^2}{\pi}(\sigma - 1 - \lambda - 1)} = \frac{1}{\sqrt{\pi(\sigma - 1 - \lambda - 1)}} e^{-\frac{(p-p')^2}{\sigma - 1 - \lambda - 1}}, \quad \text{only for} \quad \lambda > \sigma,$$

$$G_1(q - q'; \sigma - \mu) := \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(q-q')} e^{-\frac{k^2}{4}(\sigma - \mu)} = \frac{1}{\sqrt{\pi(\sigma - \mu)}} e^{-\frac{(q-q')^2}{\sigma - \mu}}, \quad \text{only for} \quad \mu < \sigma.$$  

Note that $G_1(x - x'; \tau)$ is the propagator of the one-dimensional diffusion equation

$$\left[ \frac{\partial}{\partial \tau} - \frac{1}{4} \frac{\partial^2}{\partial x^2} \right] G_1(x - x'; \tau) = 0, \quad \text{for} \quad \tau > 0.$$  

The integral (17) exists only for $\sigma < \lambda$, and then it yields the $p$-propagator, but it diverges for $\sigma > \lambda$. In contrast, the corresponding integral (18) exists only for $\sigma > \mu$, but diverges for $\sigma < \mu$. Thus, their product (16) cannot exist simultaneously, if $\mu = \lambda$. Consequently, the desired propagator $G_0$, as a Fourier transform of $K$, cannot exist.

4 An algorithm for squeezing Q functions

To overcome the above difficulty we recall [2] that the solutions of the pseudo-diffusion equation (1) can be obtained by the method of separation of variables: Writing the solution as a product of two functions, $Q(p, q; \sigma) = \theta(p, \sigma) \psi(q, \sigma)$, where $\theta$ depends only on $p$ and $\sigma$, and $\psi$ depends only on $q$ and $\sigma$, we get

$$0 = \frac{1}{Q} \nabla Q = \frac{1}{\theta \psi} \left( \frac{\partial}{\partial \sigma} - \frac{1}{4} \left[ \frac{\partial^2}{\partial q^2} - \frac{1}{\sigma^2} \frac{\partial^2}{\partial p^2} \right] \right) \theta \psi$$

$$= \frac{1}{\psi} \left( \frac{\partial}{\partial \sigma} - \frac{1}{4} \frac{\partial^2}{\partial q^2} \right) \psi(q; \sigma) - \frac{1}{\theta} \left( \frac{\partial}{\partial \sigma} - \frac{1}{4\sigma^2} \frac{\partial^2}{\partial p^2} \right) \theta(p; \sigma).$$

Since the first term in (21) depends only on $q$ and $\sigma$, while the second term in (21) depends only on $p$ and $\sigma$, we conclude that each of them must be equal to a function of $\sigma$ only, which we denote by $f(\sigma)$. But I shall consider here only the case $f(\sigma) \equiv 0$, since it turns out [2] that $f(\sigma) \neq 0$ yields no new solutions of (1). For $f(\sigma) \equiv 0$, Eq. (20) yields the following two equations:

$$\left( \frac{\partial}{\partial \sigma} - \frac{1}{4\sigma^2} \frac{\partial^2}{\partial p^2} \right) \psi(q; \sigma) = 0 \quad (21)$$

$$\left( - \frac{\partial}{\partial \sigma} - \frac{1}{4\sigma^2} \frac{\partial^2}{\partial p^2} \right) \theta(p; \sigma) = \frac{1}{\sigma^2} \left( \frac{\partial}{\partial \sigma - 1} - \frac{1}{4\sigma^2} \frac{\partial^2}{\partial p^2} \right) \theta(p; \sigma) = 0.$$
where we used $\frac{\partial}{\partial \sigma} = -\frac{1}{\sigma^2} \frac{\partial}{\partial q^2}$ in (22). We see that $\psi$ obeys a 1-dimensional diffusion equation in $q$, where $\sigma$ plays the role of time. Similarly, $\theta$ obeys a diffusion equation in $p$, but with $\sigma^{-1}$ playing the role of time.

Let us for definiteness discuss the case $\sigma > 1$ (the $\sigma < 1$ case can be dealt with by similar arguments): For $\sigma > 1$ we can obtain the solution $\psi(q, \sigma)$ by using the propagator (18). The action of the propagator for $\sigma > 1$ can also be written formally as a differential operator $Y_1(\sigma - 1, q)$, acting on the unsqueezed function $\psi(q, 1)$:

$$\psi(q, \sigma) = \int dq' \frac{1}{\sqrt{\pi(\sigma - 1)}} e^{-\frac{(q - q')^2}{\sigma - 1}} \psi(q', 1)$$

$$= \exp \left[ \frac{(\sigma - 1)}{4} \frac{\partial^2}{\partial q^2} \right] \psi(q, 1) =: Y_1(\sigma - 1, q) \psi(q, 1) .$$

(23)

(24)

The propagator in (23) will blow up for $\sigma < 1$, and the integral in (23) will not exist, even if $\psi(q, 1)$ is a strongly decaying function, such as $\psi(q, 1) = \exp[-q^2/\alpha]$, for $1 - \alpha < \sigma < 1$: This is because, the coefficient of $q^2$ in the exponent in the integrand of (23) would be positive: $\frac{1}{1 - \sigma} - \frac{1}{\alpha} = \frac{\alpha^{-1} + \sigma}{\alpha(1 - \sigma)} > 0$.

In contrast, if we first evaluate the integral (23) using $\sigma > 1$, then we can continue the resulting function $\psi(q, \sigma)$ to $1 - \alpha < \sigma < 1$, as we shall illustrate below.

This suggests the following algorithm for calculating the squeezed $p$ factor: First, calculate the solution $\theta(p, \sigma)$ of the diffusion equation (22) by using the $p$-propagator for $1/\sigma > 1$. Then continue the resulting solution to $1/\sigma < 1$. One could also calculate the solution $\theta(p, \sigma)$ directly for $1/\sigma < 1$, by using the power series expansion of the differential operator $Y_1(\sigma^{-1} - 1, p)$:

$$\theta(p, \sigma^{-1}) = Y_1(\sigma^{-1} - 1, p) \theta(p, 1) = \exp \left[ (\sigma^{-1} - 1) \frac{\partial^2}{\partial p^2} \right] \theta(p, 1) .$$

(25)

As an example, let us squeeze the Q function of thermal light [2], whose density matrix is given by:

$$\rho_{th} = \frac{1}{\pi + 1} \sum_{n=0}^{\infty} \left( \frac{\pi}{\pi + 1} \right)^n \langle n | , \text{ where } \pi \equiv \frac{1}{e^\beta - 1} ,$$

(26)

where $\beta = \hbar \omega/k_B T$ and $\pi$ is the mean number of photons. The (coherent) Q function of (26) is factorizable:

$$Q(\rho_{th} : p, q; 1) = \frac{1}{\pi + 1} \exp \left[ -\frac{p^2 + q^2}{2(\pi + 1)} \right] = \psi(q, 1) \theta(p, 1) ,$$

(27)

where each factor turns out to be proportional to the diffusion propagator:

$$\psi(q, 1) = \theta(q, 1) = \frac{1}{\sqrt{\pi + 1}} \exp \left[ -\frac{q^2}{2(\pi + 1)} \right] = \sqrt{2\pi} G_1(q, 2\pi + 2) .$$

(28)

Since the unsqueezed factors are the same, $\psi(q, 1) = \theta(q, 1)$, we only need to evaluate one integral, by using the propagator for $\sigma > 1$:

$$\psi(q, \sigma) = \int dq' \frac{1}{\sqrt{\pi(\sigma - 1)}} e^{-\frac{(q - q')^2}{\sigma - 1}} \psi(q', 1) = \sqrt{2\pi} \int dq' G_1(q - q', \sigma - 1) G_1(q', 2\pi + 2)$$

$$= \sqrt{2\pi} G_1(q, \sigma + 2\pi + 1) = \sqrt{\frac{2}{\sigma + 2\pi + 1}} \exp \left[ -\frac{q^2}{\sigma + 2\pi + 1} \right] .$$

(29)

We see that $\psi(q; \sigma)$ is well defined for $-(2\pi + 1) < \sigma < 1$. Since $\theta(p, 1) = \psi(p, 1)$, we get $\theta(p, \sigma) = \psi(p, \sigma^{-1})$, so that the squeezed Q function becomes $Q(\rho_{th} : p, q; \sigma) = \psi(p, \sigma^{-1}) \psi(q, \sigma)$. Thus,
the squeezing of \(Q(\rho_{th} : p,q;1)\) required only a single integration, which in this specific case was carried out by noting that the successive action of two propagators is equivalent to the action of one propagator, whose ‘time variable’ is the sum of the individual times: \((\sigma - 1) + (2\pi + 2) = \sigma + 2\pi + 1\).

5 Forward-Backward Squeezing Propagators

The products in \([16]\) will exist and be solutions of the pseudo-diffusion equation, if we relax the condition \(\mu = \lambda\) and demand instead only \(\mu < \sigma < \lambda\). Actually, these products yield a different solution for each 4-tupel \((p', q', \lambda, \mu)\). Since the heart operator \(\heartsuit\) is linear, any superposition of these solutions is also a solution. In particular, if we fix the squeezing parameters \(\mu\) and \(\lambda\) and integrate only over \(p'\) and \(q'\), we get solutions of the form

\[
f(p, q; \sigma^{-1}, \sigma) = \int \int dp' dq' G(p-p', q-q'; \lambda^{-1} - \sigma^{-1}, \sigma - \mu) f(p', q'; \lambda^{-1}, \mu), \quad \text{for} \quad \mu < \sigma < \lambda,
\]

for any given function \(f(p, q; \lambda^{-1}, \mu)\), provided that the integrals \([31]\) exist. We see that \(G\) in \([30]\) is acting as a propagator, which provides a solution of the pseudo-diffusion equation \([\mathbb{I}]\) for the squeezing parameter \(\sigma\) in the range \(\mu < \sigma < \lambda\). I shall therefore call these \(G\) functions \emph{two-sided or forward-backward squeezing propagators} of the pseudo-diffusion equation \([\mathbb{I}]\), since the two squeezing parameters, \(\mu\) and \(\lambda\), lie on opposite sides of \(\sigma\). These \(G\) solutions have the proper limit, which one expects from a propagator, if \(\sigma\) is approached from opposite directions:

\[
\lim_{\mu \to \sigma-\epsilon, \lambda \to \sigma+\epsilon} G(p-p', q-q'; \lambda^{-1} - \sigma^{-1}, \sigma - \mu) = \delta(p-p') \delta(q-q').
\]

An extreme case of the squeezing propagators \([16]\) is obtained by choosing \(\mu = 0\) and \(\lambda = \infty\). These squeezing parameters correspond to the values \(-\infty\) and \(+\infty\) of the \(y = \frac{1}{2} \ln \sigma\) variable, respectively:

\[
G(p-p', q-q'; \sigma^{-1}, \sigma) = \frac{1}{\pi} \exp \left[-\sigma(p-p')^2 - \sigma^{-1}(q-q')^2\right], \quad \text{for} \quad 0 < \sigma < \infty.
\]

For the choice \(\mu = 0\) and \(\lambda = \infty\) in \([31]\), \(\sigma\) can now take any positive value. Moreover, the square-root factors in the two propagators cancel out. Thus, with the propagator \([32]\), the relation \([30]\) becomes simply

\[
f(p, q; \sigma^{-1}, \sigma) = \int \int \frac{dp' dq'}{\pi} \exp[-\sigma(p-p')^2 - \sigma^{-1}(q-q')^2] f(p', q'; 0, 0), \quad \text{for} \quad \sigma > 0.
\]

6 New Interpretation of the Wigner Function

The Wigner representation of the projection operator \(\Pi(p, q; \sigma)\) is given by

\[
\Pi(p, q; \sigma) = \int \int \frac{dp' dq'}{2\pi} \text{Tr}[\Pi(p, q; \sigma) W(p', q')] W(p', q')
\]

\[
eq 2 \int \int \frac{dp' dq'}{\pi} \exp[-\sigma(p-p')^2 - \sigma^{-1}(q-q')^2] W(p', q').
\]

where we used

\[
\text{Tr}[\Pi(p, q; \sigma) W(p', q')] = \langle p, q; \sigma | W(p', q') | p, q; \sigma \rangle = \langle p, q; \sigma | D(p', q') W(0, 0) D\dagger(p', q') | p, q; \sigma \rangle
\]

\[
= \langle p-p', q-q'; \sigma | p'-p, q'-q; \sigma \rangle = \exp[-\sigma(p-p')^2 - \sigma^{-1}(q-q')^2].
\]
By multiplying \(\{34\}\) by an operator \(A\) and taking the trace, we get a similar relation between the \(Q\) function and its Wigner counterpart \(\{35\}\):

\[
Q(A : p, q; \sigma) = \int \int \frac{dp'dq'}{\pi} \exp[-\sigma(p-p')^2 - \sigma^{-1}(q-q')^2] W(A : p', q'), \tag{36}
\]

Comparing this relation with \(\{33\}\), we realize that the \(Q\) function is a propagated Wigner function via the above special 2-sided squeezing propagator \(\{32\}\). This led me to define an interpolating function \(Q\) for two different squeezing parameters, \(\sigma_p\) and \(\sigma_q\). Since the Wigner functions correspond to infinite past and infinite future, we can use the 2-sided propagators to bring them into finite past \(\sigma_q\) and finite future \(\sigma_p\) (or “two different points of the present”), as follows

\[
Q(A : p, q; \frac{1}{\sigma_p}, \sigma_q) := \int \int \frac{dp'dq'}{\pi} G_1(p-p'; \frac{1}{\sigma_p}) G_1(q-q'; \sigma_q) Q(A : p', q'; 0, 0) \tag{37}
\]

\[
= \sqrt{\frac{\sigma_p}{\sigma_q}} \int \int \frac{dp'dq'}{\pi} \exp[-\sigma_p(p-p')^2 - \sigma^{-1}_q(q-q')^2] W(A : p', q') , \tag{38}
\]

where \(W(A : p, q) = Q(A : p, q; 0, 0)\) was used. Note that, for now, we assume \(\sigma_q, \sigma_p > 0\) in \(\{38\}\), but \(\sigma_p\) may be larger or smaller than \(\sigma_q\). Later, we shall extend the range of \(\sigma_q\) and \(\sigma_p\).

Instead of studying \(Q\) functions, it is more useful to define and study generalized \(Q\) operators, which yield \(Q\) functions by taking the trace \(\text{Tr}(AQ)\). Similar to \(\{38\}\), we generalize the operator relation \(\{34\}\), as follows:

\[
Q(p, q; \sigma^{-1}_p, \sigma_q) := 2 \exp\left[\frac{1}{4} \left(\frac{1}{\sigma_p} \frac{\partial^2}{\partial p^2} + \sigma_q \frac{\partial^2}{\partial q^2}\right)\right] W(p, q) \tag{39}
\]

\[
= 2 \sqrt{\frac{\sigma_p}{\sigma_q}} \int \int \frac{dp'dq'}{\pi} \exp[-\sigma_p(p-p')^2 - \sigma^{-1}_q(q-q')^2] W(p', q') \tag{40}
\]

\[
= \int \int \frac{dk dx}{2\pi} \exp\left[-\frac{1}{4} \left(\frac{1}{\sigma_p} x^2 + \sigma_q k^2\right) + i(px - qk)\right] D(k, x) , \tag{41}
\]

where \(\{41\}\) follows from \(\{40\}\) by substituting the Fourier transform of the Wigner operator \(\{36\}\):

\[
W(p, q) = \frac{1}{2} \int \int \frac{dk dx}{2\pi} e^{i(px - qk)} D(k, x) . \tag{42}
\]

So far we assumed that \(\sigma_p\) and \(\sigma_q\) are positive. However, we can formally extend the definition \(\{39\}\) to real (and even complex) values of \(\sigma_p\) and \(\sigma_q\).

From \(\{34\}\) or \(\{11\}\) we see immediately that the \(Q\) operators (and hence also the generalized \(Q\) functions via the trace) satisfy the following partial differential equation

\[
\left[\frac{\partial}{\partial \mu} - \frac{1}{4} \left(\frac{\partial \sigma_q}{\partial \mu} \frac{\partial^2}{\partial q^2} - \frac{1}{\sigma^2_p} \frac{\partial \sigma_p}{\partial \mu} \frac{\partial^2}{\partial p^2}\right)\right] Q(p, q; \frac{1}{\sigma_p(\mu)}, \sigma_q(\mu)) = 0 , \tag{43}
\]

7 Generalized \(Q\) functions

The above interpretation suggests the following definition of generalized \(Q\) functions which depend on two different squeezing parameters, \(\sigma_q\) and \(\sigma_p\): Since the Wigner functions correspond to infinite past and infinite future, we can use the 2-sided propagators to bring them into finite past \(\sigma_p\) and finite future \(\sigma_q\) (or “two different points of the present”), as follows

\[
Q(A : p, q; \frac{1}{\sigma_p}, \sigma_q) := \int \int \frac{dp'dq'}{\pi} G_1(p-p'; \frac{1}{\sigma_p}) G_1(q-q'; \sigma_q) Q(A : p', q'; 0, 0) \tag{37}
\]

\[
= \sqrt{\frac{\sigma_p}{\sigma_q}} \int \int \frac{dp'dq'}{\pi} \exp[-\sigma_p(p-p')^2 - \sigma^{-1}_q(q-q')^2] W(A : p', q') , \tag{38}
\]

where \(W(A : p, q) = Q(A : p, q; 0, 0)\) was used. Note that, for now, we assume \(\sigma_q, \sigma_p > 0\) in \(\{38\}\), but \(\sigma_p\) may be larger or smaller than \(\sigma_q\). Later, we shall extend the range of \(\sigma_q\) and \(\sigma_p\).

Instead of studying \(Q\) functions, it is more useful to define and study generalized \(Q\) operators, which yield \(Q\) functions by taking the trace \(\text{Tr}(AQ)\). Similar to \(\{38\}\), we generalize the operator relation \(\{34\}\), as follows:

\[
Q(p, q; \sigma^{-1}_p, \sigma_q) := 2 \exp\left[\frac{1}{4} \left(\frac{1}{\sigma_p} \frac{\partial^2}{\partial p^2} + \sigma_q \frac{\partial^2}{\partial q^2}\right)\right] W(p, q) \tag{39}
\]

\[
= 2 \sqrt{\frac{\sigma_p}{\sigma_q}} \int \int \frac{dp'dq'}{\pi} \exp[-\sigma_p(p-p')^2 - \sigma^{-1}_q(q-q')^2] W(p', q') \tag{40}
\]

\[
= \int \int \frac{dk dx}{2\pi} \exp\left[-\frac{1}{4} \left(\frac{1}{\sigma_p} x^2 + \sigma_q k^2\right) + i(px - qk)\right] D(k, x) , \tag{41}
\]

where \(\{41\}\) follows from \(\{40\}\) by substituting the Fourier transform of the Wigner operator \(\{36\}\):

\[
W(p, q) = \frac{1}{2} \int \int \frac{dk dx}{2\pi} e^{i(px - qk)} D(k, x) . \tag{42}
\]

So far we assumed that \(\sigma_p\) and \(\sigma_q\) are positive. However, we can formally extend the definition \(\{39\}\) to real (and even complex) values of \(\sigma_p\) and \(\sigma_q\).

From \(\{34\}\) or \(\{11\}\) we see immediately that the \(Q\) operators (and hence also the generalized \(Q\) functions via the trace) satisfy the following partial differential equation

\[
\left[\frac{\partial}{\partial \mu} - \frac{1}{4} \left(\frac{\partial \sigma_q}{\partial \mu} \frac{\partial^2}{\partial q^2} - \frac{1}{\sigma^2_p} \frac{\partial \sigma_p}{\partial \mu} \frac{\partial^2}{\partial p^2}\right)\right] Q(p, q; \frac{1}{\sigma_p(\mu)}, \sigma_q(\mu)) = 0 , \tag{43}
\]
if \( \sigma_q \) and \( \sigma_p \) are varied along a curve \((\sigma_q(\mu), \sigma_p(\mu))\) in the \((\sigma_q, \sigma_p)\) plane. We see that \((43)\) yields a generalized pseudo-diffusion (diffusion) equation, if \(\frac{\partial \sigma_q}{\partial \mu} \) and \(\frac{\partial \sigma_p}{\partial \mu}\) have the same (opposite) signs. In particular, on the subdomain of \(\mathbb{R}^2\), defined by

\[
\sigma_p = \sigma \tau^{-1}, \quad \sigma_q = \sigma \tau, \quad \text{where} \quad \sigma > 0, \quad \tau > -1 .
\] (44)

the operator \(Q(p, q; \tau \sigma^{-1}, \tau \sigma)\) yields a generalization of the Cahill-Glauber interpolation operator \([10]\), which corresponds to zero squeezing \(\sigma \equiv 1\):

\[
Q(p, q; \tau \sigma^{-1}, \tau \sigma) = \int \int \frac{dk dx}{2\pi} \exp \left[ -\frac{\tau}{4} (\sigma^{-1} x^2 + \sigma k^2) + i(px - qk) \right] D(k, x) .
\] (45)

We see that the parameter \(s\) of \([11]\) corresponds to our \(-\tau\) here. I shall call \(\tau\) the thermal variable, since \(Q(0, 0; \tau, \tau)\) is equal to the thermal density operator \(\rho_T\) in \([29]\), if we make the identification \(\tau = \coth(\beta/2)\). Hence, \(Q(0, 0; \tau, \tau)\) is equal to the thermalized parity operator of \([11]\) and to the generalized parity operator of \([8]\), which was introduced in order to obtain the parity operator \(W(0, 0)\) as a limit of trace-class operators.

From equation \((43)\) we see that \(Q(p, q; \tau \sigma^{-1}, \tau \sigma)\) obeys the following two interesting partial differential equations:

\[
\triangledown(p, q; \tau, \sigma) \; Q(p, q; \tau \sigma^{-1}, \tau \sigma) := \left[ \frac{\partial}{\partial \sigma} - \frac{\tau}{4} \left( \frac{\partial^2}{\partial q^2} - \frac{1}{\sigma^2} \frac{\partial^2}{\partial p^2} \right) \right] Q(p, q; \tau \sigma^{-1}, \tau \sigma) = 0 , \] (46)

\[
\left[ \frac{\partial}{\partial \tau} - \frac{1}{4} \left( \sigma \frac{\partial^2}{\partial q^2} + \frac{1}{\sigma} \frac{\partial^2}{\partial p^2} \right) \right] Q(p, q; \tau \sigma^{-1}, \tau \sigma) = 0 . \] (47)

if \(\sigma\) is varied along ‘isothermal curves’ \((\sigma_q/\sigma_p = \tau^2 = \text{const.})\), or \(\tau\) varied along the ‘isosqueeze curves’ \((\sigma_q \sigma_p = \sigma^2 = \text{const.})\), respectively.

Eq.\((44)\) is a generalized pseudo-diffusion equation, since \(\triangledown(p, q; \tau, \sigma)\) now depends also on \(\tau\) and reduces to the original heart operator in \([1]\) for \(\tau = 1\). For \(\tau = 0\) equation \((46)\) tells us that the \(Q\) operator is not affected by squeezing: \(Q(p, q; 0, 0) = 2W(p, q)\) for all \(\sigma\).

Eq.\((47)\) is a diffusion equation in 2 dimensions \([3]\), with different diffusion constants in the \(p\) and \(q\) directions, if \(\sigma \neq 1\).

8 Summary

I reviewed the 3-steps Fourier-transform procedure for calculating squeezed \(Q\) functions \(Q(A : p, q; \sigma)\) from given unsqueezed or less squeezed ones \(Q(A : p, q; \mu)\), where \(\mu < \sigma\) \([2]\): First calculate \(\tilde{Q}(A : k, x; \mu)\), the Fourier transform of \(Q(A : p, q; \mu)\), then multiply \(\tilde{Q}\) by the special kernel \(K\) and finally take the inverse Fourier transformation of the product \(K\tilde{Q}\) and you get \(Q(A : p, q; \sigma)\). Instead, I gave an algorithm for squeezing \(Q\) functions directly, by using one-dimensional diffusion operators.

I also explained why one-sided propagators cannot be defined for the pseudo-diffusion equation \([1]\) and showed that two-sided squeezing propagators \([13]\) are more appropriate for this equation.

I noted that the \(Q\) functions are related to their Wigner counterparts by the extreme two-sided propagator \([14]\), and concluded that the Wigner functions can be looked upon as generalized \(Q\) functions \(Q(A : p, q; \sigma_p^{-1}, \sigma_q)\), which are squeezed forwards \((\sigma_p = \infty)\) in \(p\) variable and backwards \((\sigma_q = 0)\) in the \(q\) variable.

These \(Q\) operators were defined formally in \([39]\) in terms of Wigner operators. On a subdomain of the parameters \((\sigma_p, \sigma_q)\), which is defined in terms of \(\tau\) and \(\sigma\) in \((44)\), these \(Q\) operators yield
the ‘thermalized and squeezed Wigner operator’ $\mathcal{Q}(p, q; \tau \sigma^{-1}, \tau \sigma)$, which yields the interpolating operator of Cahill and Glauber [10] for zero squeezing ($\sigma \equiv 1$). However, we emphasize that even $\mathcal{Q}(p, q; \tau \sigma^{-1}, \tau \sigma)$ for $\sigma \geq 0$ is less general than our generalized $\mathcal{Q}$ operator $\mathcal{Q}(p, q; \sigma_p^{-1}, \sigma_q)$: For example, $\mathcal{Q}(p, q; 0, \sigma_q)$ does not correspond to a thermalized and squeezed Wigner operator.

I also showed that $\mathcal{Q}(p, q; \tau \sigma^{-1}, \tau \sigma)$ obeys the generalized pseudo-diffusion equation (46) for constant $\tau$, and the 2-dimensional diffusion equation (47) for constant $\sigma$. This diffusion equation (47) explains intuitively why each of the $P$, the Wigner and the $\mathcal{Q}$ distributions, become smoother than its predecessor, as $\tau$ is increased from $\tau = -1$ to $\tau = 0$ to $\tau = 1$, for any $\sigma = \text{const}$.

This smoothing process is expected to continue as $\tau$ is increased from 1 to $\infty$. Indeed, it can be shown that the generalized $\mathcal{Q}$ functions $Q(\rho : p, q; \tau \sigma^{-1}, \tau \sigma) = \text{Tr} [\rho \mathcal{Q}(p, q; \tau \sigma^{-1}, \tau \sigma)]$ of density matrices $\rho$ are nonnegative for $\tau \geq 1$, and thus they yield probability distributions and not merely quasi-probability distributions. Consequently, their Wehrl entropy [12]

$$S(\tau, \sigma) := -\iint \frac{dp dq}{2\pi} Q(\rho : p, q; \tau \sigma^{-1}, \tau \sigma) \ln Q(\rho : p, q; \tau \sigma^{-1}, \tau \sigma), \quad \text{for} \quad \tau \geq 0, \quad (48)$$

is well defined and should increase if we vary both $\tau$ and $\sigma$ simultaneously in such a way that both $\sigma_q$ and $\sigma_p$ increase. Hence, further clarification of the physical meaning of the observable $\mathcal{Q}(p, q; \tau \sigma^{-1}, \tau \sigma)$ would be useful.

References

[1] S. S. Mizrahi and J. Daboul, Physica A 189, 635 (1992).
[2] J. Daboul, M. A. Marchiolli and S. S. Mizrahi, J. of Physics A 28, 4623 (1995).
[3] D. V. Widder, The Heat Equation, Academic Press, London, 1975.
[4] Y. S. Kim and M. E. Noz, Phase Space Picture of Quantum Mechanics, World-Scientific, Singapore, 1991.
[5] J. P. Dahl, Physica Scripta 25, 499 (1982).
[6] A. Grossmann, Commun. Math. Phys. 48, 191 (1976); A. Royer, Phys. Rev. A 15, 449 (1977); 43, 44 (1991); 45, 793 (1992); B-G Englert, J. of Physics A 22, 625 (1989).
[7] R. F. Bishop and A. Vourdas, Phys. Rev. A 50, 4488 (1994).
[8] M. G. Benedict, and A. Czirják, J. of Physics A 28, 4599 (1995).
[9] M. M. Nieto, What are squeezed states really like?, in: Frontiers in Nonequilibrium Statistical Physics, Eds. G. T. Moore and M. O. Scully, NATO ASI Series, vol. 135 (Plenum, New York, 1988).
[10] K. E. Cahill and R. J. Glauber, Phys. Rev. 177, 1857 and 1882 (1969).
[11] A. Vourdas and R. F. Bishop, Phys. Rev. A 50, 3331 (1994).
[12] A. Wehrl, Rev. Mod. Phys. 50, 221 (1978); Rep. Math. Phys. 16, 353 (1979).