On the ground state of gapless two flavor color superconductors

R. Gatto
Départ. de Physique Théorique, Université de Genève, CH-1211 Genève 4, Suisse

M. Ruggieri
Dipartimento di Fisica, Università di Bari, I-70126 Bari, Italia and
I.N.F.N., Sezione di Bari, I-70126 Bari, Italia
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This paper is devoted to the study of some aspects of the instability of two flavor color superconductive quark matter. We find that, beside color condensates, the Goldstone boson related to the breaking of $U(1)_A$ suffers of a velocity instability. We relate this wrong sign problem, which implies the existence of a Goldstone current in the ground state or of gluonic condensation, to the negative squared Meissner mass of the $8^{th}$ gluon in the g2SC phase. Moreover we investigate the Meissner masses of the gluons and the squared velocity of the Goldstone in the multiple plane wave LOFF states, arguing that in such phases both the chromo-magnetic instability and the velocity instability are most probably removed. We also do not expect Higgs instability in such multiple plane wave LOFF, at least when one considers fluctuations with small momenta. The true vacuum of gapless two flavor superconductors is thus expected to be a multiple plane wave LOFF state.

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presence of Goldstone currents, showing that there is no sign problem for the squared velocity. Such a result is shown to hold also for multiple-plane-wave LOFF. Contrary to one-plane-wave, such states may be valid candidates for the ground state. We discuss the Meissner masses for multiple-plane-waves, arguing that they are of positive squares. Finally we examine the Higgs instability for multiple-plane-wave LOFF, arguing in favor of no Higgs instability at least for small momenta. Our conclusion is in favor of a multiple-plane-wave LOFF for the ground state of gapless two flavor superconductors. A comparison with the phases of condensed gluons, following the treatment depicted in Ref. [12] for the case of the single plane wave LOFF phase, is at this stage of vital importance.

The plan of the paper is as follows: in Sec. II we introduce the model. In Sec. III we derive the parameters of the small momenta action for the Goldstone field $\phi$: the squared decay constant and the squared velocity. In Sec. IV we study the true ground state of the two flavor color superconductor. We first report on the equivalence of the one-plane-wave LOFF state with the Goldstone current. Then we discuss the fluctuations in the cur-g2SC phase. We write down the effective action for the Goldstone boson in the LOFF phase and calculate the Meissner masses in the multiple-plane-wave phases. Finally we briefly discuss the Higgs instability for such phases. In the Appendix we derive the screening Debye and Meissner masses of a fictitious gauge boson. The calculation is then matched to the low energy properties of the $\phi$.

II. THE MODEL

In this paper we consider two flavor superconductive quark matter whose action is given by

$$S = \int d^4x \left[ \bar{\psi}_{i\alpha} \left( i\gamma^\mu \partial_\mu + \mu_{ij}^{\alpha\beta} \gamma_0 \right) \psi_{j\beta} + \langle L \rightarrow R \rangle + \mathcal{L}_\Delta \right] ;$$

in the above equation $\alpha, \beta = 1, 2, 3$ denote color and $i, j = 1, 2$ stem for flavor. The spinor $\psi$ is a left-handed Weyl spinor. The chemical potential matrix $\mu_{ij}^{\alpha\beta}$ is defined as

$$\mu_{ij}^{\alpha\beta} = (\mu_b \delta_{ij} - Q \mu_e) \delta_{\alpha\beta} + (\mu_3 T_3^{\alpha\beta} + \mu_8 T_8^{\alpha\beta}) \delta_{ij} .$$

In Eq. (2) $\mu_b$ is one third of the baryon chemical potential; $\mu_e$ is the electron chemical potential, coupled to quarks via the electric charge matrix $Q_{ij} = \text{Diag}[2/3, -1/3, -1/3]$; finally $\mu_3$ and $\mu_8$ are color chemical potentials related to the conserved charges $Q_3 = \langle \psi^3 T_3 \psi \rangle$ and $Q_8 = \langle \psi^8 T_8 \psi \rangle$. It has been shown in [13] that in the 2SC and the g2SC phases of QCD, to which we are interested in this paper, the introduction of $\mu_3, \mu_8$ is enough in order to properly achieve color neutrality.

Condensation in the quark-quark channel is described by the condensation lagrangian $\mathcal{L}_\Delta$ which is given by

$$\mathcal{L}_\Delta = -\frac{\Delta}{2} \bar{\psi}_{i\alpha}^L C \psi_{j\beta}^R \epsilon^{\alpha\beta^3} \epsilon_{ij} + h.c - \langle L \rightarrow R \rangle ;$$

it can be obtained in the mean field approximation from a local four-fermion interaction. In writing Eq. (3) we are assuming that in the ground state

$$\langle \psi_{i\alpha}^L C \psi_{j\beta}^R \rangle = \langle \psi_{i\alpha}^R C \psi_{j\beta}^L \rangle \propto \Delta \epsilon^{\alpha\beta^3} \epsilon_{ij} \neq 0 ,$$

where the superscripts $L, R$ denote left-handed and right-handed quarks respectively. In the ground state characterized by the above quark condensate one has $\mu_3 = 0$: as a matter of fact, the vacuum expectation value $\langle 0 \rangle$ is neutral with respect to color transformations generated by $T_3$; therefore $Q_3 = 0$ identically, and the color chemical potential associated to $T_3$ vanishes. As for $\mu_8$, it has been shown that it is non-zero but in any case it is negligible (that is $\mu_8 \ll \mu_e$). So from now on we will put $\mu_8 = 0$ and assume that the only difference in chemical potential among quarks arises from the charge chemical potential $\mu_Q = -\mu_e$.

In this paper we adopt the high density effective theory (HDET) approximation in order to derive the quark propagator $S_{LL}$,[44]. This approximation is justified since quarks live at high baryon chemical potential $\mu_b$ and the relevant momenta for the dynamics are those near the Fermi momentum (that in the case of massless quarks coincides with their chemical potential). In this case, negative energy fields are decoupled and suppressed with respect to the positive energy ones. Thus at the leading order in the expansion $1/\mu$ one can describe the system in terms of the positive energy fields only. In HDET one decomposes the quark momenta as

$$p_0 = \ell_0 , \quad p = (\mu + \ell_\parallel) \mathbf{v} + \ell_\perp ,$$
with \( v \) the Fermi velocity of the quarks, \(|v| = 1\) in the massless case considered here. In the above equation
\[ \mu = (\mu_u + \mu_d)/2. \]

The HDET action in momentum space reads
\[ S = \frac{\mu^2}{\pi} \int \frac{dn}{8\pi} \int \frac{d\ell_0 d\ell}{(2\pi)^2} \chi^\dagger \left( \frac{V \cdot \ell + \delta \mu A}{\Delta} V \cdot \ell - \delta \mu A \right) \chi + (L \rightarrow R) \quad (6) \]

where \( \delta \mu = (\mu_d - \mu_u)/2; \chi \) is a left-handed and positive energy velocity dependent Nambu-Gorkov field defined as
\[ \chi = \begin{pmatrix} \Psi(v) \\ C\Psi^*(-v) \end{pmatrix}, \quad (7) \]

and \( \Psi = (\psi_{ur}, \psi_{ug}, \psi_{dr}, \psi_{dg}, \psi_{ub}, \psi_{db}) \). In Eq. (6) we have introduced the matrix \( A = \text{Diag}[-1, -1, 1, 1, -1, 1] \). Finally the gap matrix \( \Delta \) is
\[ \Delta = \Delta \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (8) \]

The poles of \( D(\ell_0, \ell) \) for left and right handed fields can be easily obtained from the above action once we write it in the form
\[ S = \frac{\mu^2}{\pi} \int \frac{dn}{8\pi} \int \frac{d\ell_0 d\ell}{(2\pi)^2} \chi^\dagger D(\ell_0, \ell) \chi + (L \rightarrow R). \quad (9) \]

The transformation in Eq. (13) gives rise, at the leading order, to a three body and a four body interaction term among quarks and \( \phi \), namely
\[ L_3 = -\left( \frac{2 \phi^2}{f^2} \right) \frac{\mu^2}{\pi} \int \frac{dn}{8\pi} \int \frac{d\ell_0 d\ell}{(2\pi)^2} \chi^\dagger \left( \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix} \right) \chi + (L \rightarrow R), \quad (15) \]

\[ L_4 = -\left( \frac{2 \phi^2}{f^2} \right) \frac{\mu^2}{\pi} \int \frac{dn}{8\pi} \int \frac{d\ell_0 d\ell}{(2\pi)^2} \chi^\dagger \left( \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix} \right) \chi + (L \rightarrow R). \quad (16) \]

### III. EFFECTIVE ACTION FOR THE GOLDSTONE FIELD

In order to derive the effective action for the Goldstone boson \( \phi \) associated to the breaking of the \( U(1)_A \) symmetry in the \( 2\text{SC} \) and \( g_2\text{SC} \) phases, we introduce the external field \( U = \exp(i\phi/f) \) by means of the following transformation
\[ \chi = \begin{pmatrix} \Psi(v) \\ C\Psi^*(-v) \end{pmatrix} \rightarrow \begin{pmatrix} U^\dagger \Psi \\ UC\Psi^* \end{pmatrix}, \quad (13) \]

so that the fermion action becomes
\[ S = \frac{\mu^2}{\pi} \int \frac{dn}{8\pi} \int \frac{d\ell_0 d\ell}{(2\pi)^2} \chi^\dagger \left( \begin{pmatrix} V \cdot \ell + \delta \mu A & \Delta e^{2i\phi/f} \\ \Delta e^{-2i\phi/f} & V \cdot \ell - \delta \mu A \end{pmatrix} \right) \chi + (L \rightarrow R). \quad (14) \]
Integration over the fermion fields in the functional integral gives rise to the effective lagrangian which, at the second order in $\phi$, consists of two terms, namely a self-energy and a tadpole action:

$$
\mathcal{L}_{s.e.} = \frac{i}{2} \left( \frac{2\imath \phi}{f} \right)^2 \frac{2\imath \phi}{f^2} \mu^2 \frac{\mu^2}{\pi^2} \int_0^{\infty} d\ell_0 \left[ D \left( \begin{array}{cc} 0 & \Delta \\ -\Delta & 0 \end{array} \right) \right],
$$

$$
\mathcal{L}_{tad} = -i \left( -\frac{2\phi^2}{f^2} \right) \frac{2\phi^2}{f^2} \mu^2 \frac{\mu^2}{\pi^2} \int_0^{\infty} d\ell_0 \left[ D \left( \begin{array}{cc} 0 & \Delta \\ \Delta & 0 \end{array} \right) \right].
$$

Evaluation of the traces gives in momentum space

$$
i\mathcal{L}_{s.e.}(p_0,p) = \frac{\phi(-p)\phi(p)}{f^2} \frac{\mu^2}{\pi^2} \int_0^{\infty} d\ell_0 \frac{d\ell_0}{8\pi} \left\{ \frac{4\Delta^2 \Delta^2 - (\ell_0 + \delta \mu)(\ell_0 + \delta \mu) + \ell_0 (\ell_0 + p \cdot n) - \ell_0 (\ell_0 + p \cdot n) + \Delta^2 + \delta \mu \rightarrow -\delta \mu} {4\Delta^2 (\ell_0 + \delta \mu)(\ell_0 + \delta \mu) + \ell_0 (\ell_0 + p \cdot n) - \ell_0 (\ell_0 + p \cdot n) + \Delta^2 + \delta \mu} \right\}
$$

and

$$\mathcal{L}_{tad} = -\mathcal{L}_{s.e.}(p_0 = 0, p = 0).$$

The last equation implies that $\mathcal{L}_{s.e.}(p_0 = 0, p = 0) + \mathcal{L}_{tad} = 0$, which is equivalent to the vanishing of the mass of $\phi$ as required by the Goldstone theorem.

### A. Decay constant

For the computation of the decay constant $f$ it is enough to evaluate $\mathcal{L}_{s.e.}(p_0, p = 0) + \mathcal{L}_{tad}$. To this end, in order to properly treat the infrared divergences arising from the gapless modes in the fermion propagator, we perform the $\ell_0$ integral at finite temperature, $\int d\ell_0 = 2\pi i T \sum_n$ and $\ell_0 \to i\pi T (2n + 1), p_0 \to i\pi T m$ in the integrand. Once the summation over Matsubara frequencies is performed we take the limit $T \to 0$. The result is

$$\mathcal{L}_{tad} + \mathcal{L}_{s.e.}(p_0, p = 0) = \frac{\phi(-p_0)\phi(p_0)}{f^2} \frac{\mu^2}{\pi^2} \int_0^{\infty} d\ell_0 \frac{d\ell_0}{8\pi} \left[ 1 - \theta(\delta \mu - E(\ell_0)) \right] \left( \frac{8\Delta^2}{2E(\ell_0) + p_0 + i0^+} + \frac{8\Delta^2}{2E(\ell_0) - p_0 - i0^+} - \frac{8\Delta^2}{E(\ell_0)} \right).$$

In the above relation $E(x) = \sqrt{x^2 + \Delta^2}$. In the analytical continuation from imaginary to real boson energy we add a small positive imaginary part to $p_0$. As usual, this gives rise to an imaginary part of the polarization tensor for the boson, which is related to its decay rate. The imaginary part develops when $p_0 > 2\Delta$, the sum of the rest energies of the two quasi-particles. This is equal to what happens in the gapped 2SC phase, when $\delta \mu = 0$. The integral over quark momentum can be performed analytically for each value of $p_0$; at the order $p_0^2/\Delta^2$ one gets

$$\mathcal{L}_{tad} + \mathcal{L}_{s.e.}(p_0, p = 0) = \frac{1}{2f^2} (p_0\phi)(p_0\phi) \frac{4\mu^2}{\pi^2} \left( 1 - \theta(\delta \mu - \Delta) \frac{\sqrt{\delta \mu^2 - \Delta^2}}{\delta \mu} \right).$$

Imposing the canonical normalization of the field $\phi$ in Eq. (22) fixes the value of $f$, namely

$$f^2 = \frac{4\mu^2}{\pi^2} \left( 1 - \theta(\delta \mu - \Delta) \frac{\sqrt{\delta \mu^2 - \Delta^2}}{\delta \mu} \right).$$

When $\delta \mu = 0$ we obtain the well known result of the 2SC phase, $f_{2SC}^2 = 4\mu^2/\pi^2$. In the limit $\delta \mu \gg \Delta$ one has

$$f^2 \approx \frac{2\mu^2}{\pi^2} \frac{\Delta^2}{\delta \mu^2}. $$

B. Squared velocity

Next we turn to the computation of the squared velocity of $\phi$. To this end it is enough to consider $\mathcal{L}_{s.c.}(0, p) + \mathcal{L}_{\text{tad}}$. The integral over quark energy is performed as above at finite temperature, keeping the limit $T \to 0$ after the summation on the Matsubara frequencies is done. We find

$$
\mathcal{L}_{s.c.}(0, p) + \mathcal{L}_{\text{tad}} = \frac{\phi(-p)\phi(p) \mu^2}{f^2} \int \frac{dn}{8\pi} \int_{-\infty}^{+\infty} d\ell \parallel [F(\ell \parallel, p \cdot n) - F(\ell \parallel, 0)] \tag{25}
$$

with

$$
F(x, y) = \frac{4\Delta^2}{2x + y} \frac{2x}{\sqrt{x^2 + y^2 + \Delta^2}} \left(1 - \theta(\delta\mu - \sqrt{x^2 + \Delta^2})\right) + \frac{4\Delta^2}{2x + y} \frac{2(x + y)}{\sqrt{(x + y)^2 + \Delta^2}} \left(1 - \theta(\delta\mu - \sqrt{(x + y)^2 + \Delta^2})\right). \tag{26}
$$

Once the integral over the quark momentum is performed (using the $\theta$–functions) we expand at $O(p^2/\Delta^2)$; we find

$$
\mathcal{L}_{s.c.}(0, p) + \mathcal{L}_{\text{tad}} = -\frac{1}{2f^2} (p_i \phi)(p_i \phi) \frac{4\mu^2}{3\pi^2} \left(1 - \theta(\delta\mu - \Delta)\frac{\delta\mu}{\sqrt{\delta\mu^2 - \Delta^2}}\right)
$$

$$
\equiv -\frac{\nu^2}{2} (p_i \phi)(p_i \phi), \tag{27}
$$

with the squared velocity defined as

$$
\nu^2 = \frac{4\mu^2}{3\pi^2 f^2} \left(1 - \theta(\delta\mu - \Delta)\frac{\delta\mu}{\sqrt{\delta\mu^2 - \Delta^2}}\right) \tag{28}
$$

and $f^2$ given by Eq. (23). For $\delta\mu = 0$ we recover the well known result $\nu^2_{2SC} = 1/3$. With these results at hand we can write the effective action of $\phi$ at small momenta as

$$
S_{eff}[\phi] = \int d^4x \frac{1}{2} ((\partial_0 \phi)^2 - \nu^2 (\nabla \phi \nabla \phi)). \tag{29}
$$

Eq. (29), with $f^2$ and $\nu^2$ given respectively in Eqs. (23) and (28), is one of the results of this paper.

Notice that in the gapless phase we find a negative squared velocity of $\phi$. This instability in the Goldstone sector is directly related to the chromo-magnetic instability of the gapless 2SC phase. As a matter of fact, we see that we can write the relation

$$
\nu^2 \propto m_{M,8}^2 \tag{30}
$$

where $m_{M,8}^2$ is the squared Meissner mass of the 8th gluon calculated in (21)

$$
m_{M,8}^2 = \frac{4\alpha_s \mu^2}{9\pi} \left(1 - \theta(\delta\mu - \Delta)\frac{\delta\mu}{\sqrt{\delta\mu^2 - \Delta^2}}\right) \tag{31}
$$

As far as the gapped phases, the relation between the squared velocity of the Goldstone mode and the squared Meissner mass of a gluon (or, more generally, of a spin 1 gauge field) is not new, see [52, 54, 55, 56] for 2SC and CFL. It is interesting to notice that the $\nu^2$ of $\phi$ is negative only in the gapless regime $\delta\mu > \Delta$, while the 2SC phase presents the chromo-magnetic instability also in the gapped region $\Delta/\sqrt{2} < \delta\mu < \Delta$: in this interval one has $m_{M,8}^2 > 0$ but $m_{M,a}^2 < 0$, with $a = 4, \cdots, 7$. As shown in [38] the instability in the gapped regime can be cured by a gluonic condensate. The method of Ref. 38 however can be applied only in the region $\delta\mu \approx \Delta/\sqrt{2}$, where the gluon condensates are expected to be small, because it relies on the expansion of the expectation values of some of the gluonic fields around zero, see [38] for more details. In the gapless regime $\delta\mu > \Delta$ the instability could be cured either by a gluon condensate or by net baryon and/or meson currents, see [38] for a wide discussion. We expect that the removal of the chromo-magnetic instability for the 8th gluon is enough for the removal of the Goldstone mode instability. Finally, we notice that $\nu^2 < 0$ for the Goldstone mode $\phi$ is similar to the instability found in Ref. 46 for the 8th gluon.
IV. THE GENUINE GROUND STATE

In the previous section we have computed the low energy parameters of the Goldstone mode $\phi$ both in the 2SC and in the g2SC phase. In particular, we have found a negative squared velocity, a signal of an instability of the ground state. Therefore, the g2SC can not be the true vacuum of the model and one has to look for other solutions. In this section we introduce an ansatz for the true ground state, which imitates the LOFF phase of condensed matter. This topic is not new: both in the two flavor and in the three flavor gapless phases of QCD it has been extensively studied [29]. However, it has never been related to the negative squared velocity of one (or more) Goldstone bosons. We cover this topic in this section.

A. The one plane wave LOFF state and its equivalence with the Goldstone current

The starting point is the observation that, because of the wrong sign in the spatial part of the kinetic term in Eq. (29), the expectation value for $\nabla \phi$ (VEV) may be expected to be different from zero. In order to determine the true ground state we decompose the $\phi$ around a mean field part, and study the effective action for the fluctuations around this mean field. Moreover, since $\nabla \phi$ is a vector, the VEV breaks spontaneously the rotational symmetry. The simplest way to introduce the fluctuations is writing $\phi$ as

$$\phi(x) = \Phi \cdot x + h(x) \ ,$$

where $\Phi$ is a constant and homogeneous vector and $h$ is the fluctuation field; in this way one has

$$\nabla \phi = \Phi + \nabla h \ ;$$

assuming $\langle \nabla h \rangle = 0$ we are left with $\langle \nabla \phi \rangle = \Phi$. Then, choosing $\Phi$ appropriately, one has to show that the lagrangian of the fluctuations $h(x)$ does not suffer of the sign problem in the spatial part. Since $\Phi \neq 0$ corresponds to a Goldstone current, we call the ground state cur-g2SC phase.

Introducing Eq. (32) into Eq. (14) one notices that the action resembles that for the two flavor LOFF phase in the one plane wave (1PW) structure,

$$S = \int d^4x \int \frac{dn}{8\pi} \chi^f \left\{ iV \cdot \partial + \delta \mu A \right\} \Delta \exp \left\{ \frac{2i(\Phi \cdot x + h)}{f} \right\} \chi + (L \rightarrow R) \ ,$$

with $\Phi/f = q$ playing the role of the wave vector and the field $h$ being the phonon [28] as well as the Goldstone boson related to the $U(1)_A$. Since the fermion dispersion laws in 1PW and in cur-g2SC are the same, the thermodynamic behavior of the two phases is indistinguishable and in what follows we can refer both as the 1PW and as the cur-g2SC. However we stress that the cur-g2SC phase is built with zero momentum Cooper pairs, and the wave vector $q$ in this context is related to the non-vanishing Goldstone current in the ground state. On the other hand, the 1PW ground state is built with Cooper pairs with a total momentum equal to $2q$; the Goldstone current in this case is vanishing, as we prove in the next section.

The 1PW state can be analyzed exactly, and the effective action for the fluctuation $h$ can be determined without approximations: it is enough to shift the quark momenta by an amount $\pm q$, where the upper and lower signs stem respectively for $u$ and $d$ quarks. In this way the $x$ dependence in the gap term is ruled out in place of a shift $\delta \mu \rightarrow \delta \mu - q \cdot n$ in the quark chemical potentials; but in this case the expressions involved in the calculations of the loop integrals are more complicated than the homogeneous ones. Therefore we prefer to treat this problem by a Ginzburg-Landau (GL) expansion of the fermion propagator in $\Delta/\delta \mu$. In expanding $a$ la Ginzburg-Landau we pay the price of loosing exact expressions, obtaining equations valid only to a fixed order in $\Delta/\delta \mu$, but the formula are easy to handle and to generalize to the case of more complicated crystal structures (see below).

In the GL approximation one formally writes the quark propagator, which can be read from Eq. (35), as

$$D = \sum_{n=1}^{\infty} (-1)^n \left[ D_0 \left( \begin{array}{cc} 0 & \Delta \\ \Delta & 0 \end{array} \right) \right]^n D_0 \ ,$$

where $D_0$ is the propagator at $\Delta = 0$. The zero temperature thermodynamic potential for the quarks in this approximation reads [20]

$$\Omega = \Omega_0 + \frac{\alpha}{2} \Delta^2 + \frac{\beta}{4} \Delta^4 + \frac{\gamma}{6} \Delta^6 + O(\Delta^8) \ ,$$

as
where the coefficients are given by

$$\alpha = -\frac{4\mu^2}{\pi^2} \left(1 - \frac{\delta\mu}{2|q|} \log \left|\frac{|q| + \delta\mu}{|q| - \delta\mu}\right| + \frac{1}{2} \log \frac{\Delta_0^2}{4(|q|^2 - \delta\mu^2)} \right),$$

(37)

$$\beta = \frac{\mu^2}{\pi^2} \frac{1}{|q|^2 - \delta\mu^2}, \quad \gamma = \frac{\mu^2}{8\pi^2} \left|\frac{|q|^2 + \delta\mu^2}{(|q|^2 - \delta\mu^2)^3}\right|,$$

(38)

and \(\Omega_0\) is the free gas contribution. The physical value of \(q\) is obtained as usual by minimization of the thermodynamic potential. At the leading order in \(\Delta/\delta\mu\) one has

$$\frac{\partial\Omega}{\partial|q|}\bigg|_{q=Q} = 0 \Leftrightarrow 1 - \frac{\delta\mu}{2Q} \log \left|\frac{Q + \delta\mu}{Q - \delta\mu}\right| = 0,$$

(39)

which gives the result \(Q \approx 1.2\delta\mu\) well known in the LOFF literature. Since \(\Phi^2 = f^2 q^2\), once we know \(f^2\) we are able to evaluate the Goldstone current in the cur-g2SC state.

## B. Effective action of the fluctuation in the cur-g2SC state

Next we turn to the effective action for the fluctuation field \(h\). Evaluating the traces in Eqs. (17) and (18) at the leading order in \(\Delta/\delta\mu\) one is left with the expression

$$L_{s.c.}(p) + L_{had} = -i \frac{2\Delta^2}{f^2} h(-p) \left[\mathcal{J}(p) - \mathcal{J}(0)\right] h(p),$$

(40)

where \(p = (p_0, \vec{p})\) and the loop integral \(\mathcal{J}(p)\) is defined as

$$\mathcal{J}(p) = -2 \int \frac{d\vec{n}}{4\pi} \int \frac{d^4\ell}{(2\pi)^4} \left[\frac{1}{(V \cdot \ell - \delta\mu + \vec{q} \cdot \vec{n})(V \cdot (\ell + p) - \delta\mu + \vec{q} \cdot \vec{n})}ight] + \delta\mu \to -\delta\mu.$$

(41)

From now on the calculation is similar to the one presented in great detail in [47] for the displacement fields in the three flavor LOFF phase of QCD; therefore in this paper we simply show the main steps of the calculation, referring the interested reader to Ref. [47] for further details. For small external momenta \(p\) one has

$$\mathcal{J}(p) - \mathcal{J}(0) = 2 \int \frac{d\vec{n}}{4\pi} \int \frac{d^4\ell}{(2\pi)^4} \frac{\tilde{V} \cdot p}{V \cdot p} \frac{V \cdot p}{V \cdot (\ell - \delta\mu + \vec{q} \cdot \vec{n})^2} \frac{V \cdot (\ell + p) - \delta\mu + \vec{q} \cdot \vec{n}}{2} + \delta\mu \to -\delta\mu.$$

(42)

The computation of the loop integral is done in the usual way by Wick rotating to imaginary energies \(\ell_0 \to i\ell_4\); since the integral is convergent one can send the ultraviolet cutoff on \(\ell_\parallel\) to infinity, and perform the integral over \(\ell_\parallel\) by residues, followed by integration over \(\ell_4\). This is the same procedure used for the calculation of the coefficients \(\beta, \gamma\) in the GL effective potential [29]. We find

$$\mathcal{J}(p) - \mathcal{J}(0) = -\frac{\mu^2}{4\pi^2} \Re \int \frac{d\vec{n}}{4\pi} \frac{\tilde{V} \cdot p}{(\delta\mu - q \cdot n + i0^+)^2} + \delta\mu \to -\delta\mu.$$

(43)

From Eqs. (40) and (43) one can easily read the low energy parameters of the effective lagrangian for the fluctuation field in the cur-g2SC phase, namely

$$f^2 = -\frac{\Delta^2 \mu^2}{\pi^2} \Re \int \frac{d\vec{n}}{4\pi} \frac{1}{(\delta\mu - q \cdot n + i0^+)^2} + (\delta\mu \to -\delta\mu) = \frac{2\mu^2}{\pi^2} \frac{\Delta^2}{Q^2 - \delta\mu^2},$$

(44)

$$v_x^2 = v_y^2 = -\frac{\Delta^2 \mu^2}{f^2 \pi^2} \Re \int \frac{d\vec{n}}{4\pi} \frac{n_x^2}{(\delta\mu - q \cdot n + i0^+)^2} + (\delta\mu \to -\delta\mu) = 0,$$

(45)
Finally, in configuration space, the lagrangian of the fluctuation reads
\[ \mathcal{L}[h] = \frac{1}{2} \left( (\partial_0 h)^2 - \mathbf{v} \cdot (\nabla h) \mathbf{v} \cdot (\nabla h) \right). \] (47)

By means of Eq. (44) we determine the value of the Goldstone current which minimizes the effective potential,
\[ |(\nabla \phi)|^2 = |\Phi|^2 = \frac{2\mu^2}{\pi^2} \Delta^2 \left( \frac{Q^2}{Q^2 - \delta \mu^2} \right) \approx 0.66\mu^2 \Delta^2, \] (48)
where we have used the relation \( Q \approx 1.2\delta \mu \). Moreover Eqs. (15) and (16) show that the squared velocity of \( h \) is positive along the direction of \( \Phi \) and is zero in the plane orthogonal to \( \Phi \). As can be easily shown by direct calculation, the zero value of the orthogonal velocity is due to the proportionality relation between the integral in Eq. (45) and the derivative \( \partial \Omega / \partial |\mathbf{q}| \) evaluated at the minimum, see Eq. (39). Therefore, the fluctuation \( h \) does not suffer the sign problem of the squared velocity, as anticipated.

C. Effective action of the Goldstone boson in the LOFF phase

In the previous section we have shown that assuming the existence of a Goldstone current in the ground state of the gapless 2SC quark matter, and expanding the Goldstone field around the VEV as in Eq. (32), the lagrangian of the fluctuation does not suffer the sign problem of the squared velocity. The cur-g2SC phase resembles the one plane wave LOFF state in the sense that the breaking of the translational and rotational symmetries due to \( \Phi \neq 0 \) is not distinguishable from the breaking due to a net momentum of the Cooper pair, as can be seen by Eq. (34). Therefore, it is obvious that the effective lagrangian of the Goldstone of \( U(1)_A \) in the 1PW LOFF phase is equal to the lagrangian of the fluctuation found in the previous section. In the 1PW phase the Goldstone does not suffer of the wrong sign problem, and the free energy of the ground state with Goldstone current is equal to the free energy of the 1PW phase.

In the 1PW phase the gap parameter has an explicit spatial dependence of the form
\[ \Delta(r)_{1PW} = \Delta \cdot e^{2i\mathbf{q} \cdot \mathbf{r}}, \] (49)
with \( \mathbf{q} \) being the wave vector (we have dropped for simplicity the color, flavor and Dirac indices), equal to one half of the total momentum of the Cooper pairs. Because of the anisotropy, only a fraction of the Fermi surfaces of the quarks are available for the pairing: this results in \( \Delta \) (and consequently, \( \Omega \)) smaller than the one of the BCS phase. However the free energy of one plane wave LOFF state can be lowered by summing up \( P \) plane waves (PPW) [29],
\[ \Delta(r)_{PPW} = \Delta \cdot \sum_{a=1}^{P} e^{2i\mathbf{q}_a \cdot \mathbf{r}}; \] (50)
the resulting phase is known as a crystalline superconductor, as the behavior of the gap in the configuration space resembles that of a crystal lattice.

Since the free energy of the LOFF phase with order parameter given by Eq. (50) is lower than the single plane wave one, it follows that the 1PW can not be the ground state of a crystalline superconductor. Stated in other words, since the cur-g2SC and the 1PW phases are not distinguishable, the free energy of the cur-g2SC phase is higher than the free energy of a PPW crystalline state. Therefore it seems that the cur-g2SC can not be the ground state of two flavor quark matter, as it can be easily replaced by crystalline phases. This situation is quite different from the three flavor case. As a matter of fact, it has been shown in [40] that a Goldstone current there exists in the ground state, near the onset CFL\( \rightarrow \)gCFL. The curCFL phase considered in [40] is not likely to be replaced by a multiple plane wave LOFF state near the onset since the free energy of the latter, as evaluated in [32], is higher than the gCFL one and therefore still higher than the energy of the curCFL phase.

Since in the crystalline LOFF phases there exists the Goldstone related to the breaking of \( U(1)_A \), the calculation of its effective action in the PPW state becomes of vital importance. Nevertheless, we learn from the 1PW an important lesson: the effective lagrangian, at least in this simple case, does not suffer the sign problem. We wish to verify that the same property is valid for a multiple plane wave crystalline superconductor, which is a better candidate for the ground state.
In the PPW phase the inverse quark propagator can not be inverted, so one is forced to make some approximation in order to write a propagator. As anticipated in the previous section we employ the GL expansion. In this case, at the leading order in $\Delta/\delta\mu$, the value of $|\mathbf{q}|$ which minimizes the thermodynamic potential is again given by Eq. (59), in which now $Q$ represents the equilibrium value of the total momentum of the pairs (in the cur-g$2$SC context it is related instead to the VEV of the Goldstone current).

The calculation of the effective action can be done following the same steps outlined in the previous section, replacing the fluctuation $\hat{h}$ with the Goldstone $\phi$. As shown explicitly in [30], in the PPW state Eq. (45) has to be replaced with

$$\mathcal{J}(p) - \mathcal{J}(0) = -\frac{i}{4\pi^2} \Re \sum_{a=1}^P \int \frac{dn}{4\pi} \frac{\hat{V} \cdot p \cdot V \cdot p}{(\delta\mu - \mathbf{q}_a \cdot \mathbf{n} + i0^+)^2} + \delta\mu \to -\delta\mu \, .$$

(51)

In the above equation the interference terms that mix different $\mathbf{q}_a$ do not appear: this is due to momentum conservation in the loop integral, which forces the wave vector of the two gap insertions to be equals. From Eq. (51) one has for the squared decay constant

$$f^2 = \frac{2\mu^2}{\pi^2} \frac{P\Delta^2}{Q^2 - \delta\mu^2} \, ,$$

(52)

with $Q \approx 1.2\delta\mu$. For what concerns the squared velocity it is convenient to introduce the matrix

$$\mathcal{V}_{ij} = \frac{\Delta^2 \mu^2}{f^2 \pi^2} \Re \sum_{a=1}^P \int \frac{dn}{4\pi} \frac{n_i n_j}{(\delta\mu - \mathbf{q}_a \cdot \mathbf{n} + i0^+)^2} + (\delta\mu \to -\delta\mu) \, ;$$

(53)

the elements of the matrix $\mathcal{V}$ are easily evaluated: following the strategy depicted in [30], for each angular integral in the sum one rotates the $\mathbf{q}_a$ along the $z-$axes by means of the orthogonal matrix $R^a$. In this way one is left with the expression

$$\mathcal{V}_{ij} = \frac{1}{P} \sum_{a=1}^P R^a_{i3} R^a_{j3} \, ,$$

(54)

where we have used Eq. (52) and the fact that at the minimum only the longitudinal integrals are not vanishing, see Eqs. (15) and (46). With this definition the effective action for the Goldstone reads, in momentum space,

$$\mathcal{L}(\phi) = \frac{1}{2} \phi(-p) \left( \hat{p}^2 - \mathcal{V}_{ij} \hat{p}_i \hat{p}_j \right) \phi(p) \, .$$

(55)

The dispersion law $E(p)$ of the Goldstone is the solution of the equation

$$E(p)^2 - \mathcal{V}_{ij} \hat{p}_i \hat{p}_j = 0 \, .$$

(56)

As a consequence, in order to show that the lagrangian does not suffer the sign problem, it is sufficient to show that the matrix $\mathcal{V}$ is semi-definite (or definite) positive, that is for each $\hat{p} \neq 0$ it satisfies the condition $\mathcal{V}_{ij} \hat{p}_i \hat{p}_j > 0$ (or $\mathcal{V}_{ij} \hat{p}_i \hat{p}_j > 0$). This is an easy task: as a matter of fact, from the very definition of the rotation matrices $R^a$ it follows that one can write $\mathcal{V}_{ij} = \sum_a \hat{q}_a^i \hat{q}_a^j / P$, with $\hat{q}_a^i$ a unit vector parallel to $\mathbf{q}_a$; since the tensor $\hat{q}_a^i \hat{q}_a^j$ is semi-definite positive, as it has one eigenvalue 1 and two eigenvalues 0, then $\mathcal{V}_{ij}$ is semi-definite or definite positive, depending on the particular crystalline structure considered. We show this in three interesting cases.

In the first case we consider a crystal with three mutually orthogonal wave vectors (3PW). Each of these vectors can be chosen along one of the three axes,

$$\mathbf{q}_1 = Q(1,0,0) \, , \quad \mathbf{q}_2 = Q(0,1,0) \, , \quad \mathbf{q}_3 = Q(0,0,1) \, .$$

(57)

The second structure we consider is the body centered cube crystal (BCC), defined by the following wave vectors:

$$\mathbf{q}_1 = Q(1,0,0) \, , \quad \mathbf{q}_2 = Q(0,1,0) \, , \quad \mathbf{q}_3 = Q(0,0,1) \, ,$$

$$\mathbf{q}_4 = -\mathbf{q}_1 \, , \quad \mathbf{q}_5 = -\mathbf{q}_2 \, , \quad \mathbf{q}_6 = -\mathbf{q}_3 \, .$$

(58)

Finally we consider the case of the face centered cube structure (FCC), whose wave vectors are given by

$$\mathbf{q}_1 = \frac{Q}{\sqrt{3}}(1,+1,+1) \, , \quad \mathbf{q}_2 = \frac{Q}{\sqrt{3}}(+1,-1,+1) \, , \quad \mathbf{q}_3 = \frac{Q}{\sqrt{3}}(1,-1,+1) \, ,$$

$$\mathbf{q}_4 = -\mathbf{q}_1 \, , \quad \mathbf{q}_5 = -\mathbf{q}_2 \, , \quad \mathbf{q}_6 = -\mathbf{q}_3 \, .$$

(59)
where $S$ is done in the Ginzburg-Landau approximation at the second order in $\Delta$.

The polarization tensor in the HDET approach is given by the sum of two contributions, $\Pi_0$ and $\Pi_4$ integrals, namely

$$\Pi_0 = \int \frac{d^4 \ell}{(2\pi)^4} \frac{T_0}{\ell^2} + \int \frac{d^4 \ell}{(2\pi)^4} \left( \frac{\delta \Pi_0}{\delta \mu} - \frac{\delta \Pi_4}{\delta \mu} \right) T_0,$$

for all of the gluons one is left with two kinds of contributions to compare their free energy with the gluonic phases and/or the gapped 2SC. Indeed, it would be interesting to build electrical and color neutral crystalline phases in two flavor quark matter, and the genuine ground state of two flavor quark matter at high density. A more accurate study is needed at this point: in all of the crystals we find $V_{ij} = \delta_{ij}/3$, which implies that the dispersion law of the Goldstone is

$$E^2(p) = \frac{1}{3} p^2,$$

and the squared velocity does not suffer the wrong sign problem. From the results of $[29, 32]$ we can infer that the BCC and the FCC may exist in a large window of the ratio $\delta \mu/\Delta_0$, both on the left and on the right of the Clogstone limit $\Delta_0/\sqrt{2}$. Since our result shows that in these crystalline phases the Goldstone does not suffer the instability towards the formation of a current, we argue that such phases could be stable and represent a genuine ground state of two flavor quark matter at high density. A more accurate study is needed at this point: indeed, it would be interesting to build electrical and color neutral crystalline phases in two flavor quark matter, and to compare their free energy with the gluonic phases and/or the gapped 2SC.

\section*{D. Meissner masses of the gluons in the PPW LOFF phase}

In this subsection we compute the Meissner masses of the gluons in the multiple plane wave state. The calculation is done in the Ginzburg-Landau approximation at the second order in $\Delta/\delta \mu$. The Meissner tensor can be defined as

$$\left( M^I_{\mu\nu} \right)^2 = -\Pi^I_{\mu\nu}(p_0 = 0, p \to 0),$$

where the polarization tensor in the HDET approach is given by the sum of two contributions, $\Pi^I_{\mu\nu}(p) = S^I_{\mu\nu}(p) + T^I_{\mu\nu}$, where $S^I_{\mu\nu}(p)$ is a self-energy diagram,

$$S^I_{\mu\nu}(p) = i \frac{2 \mu^2}{\pi} \int \frac{d^4 \ell}{(2\pi)^4} \text{Tr} \left[ D(\ell + p) \left( -V^\mu T_0 \begin{array}{cc} 0 & 0 \\ \tilde{V}_\nu T_0^* \end{array} \right) D(\ell) \left( -V^\nu T_0 \begin{array}{cc} 0 & 0 \\ \tilde{V}_\nu T_0^* \end{array} \right) \right],$$

and $T^I_{\mu\nu}$ is a tadpole diagram,

$$T^I_{\mu\nu} = -i \frac{2 \times 2}{\pi} \int \frac{d^4 \ell}{(2\pi)^4} \text{Tr} \left[ D(\ell) \left( \frac{(\mu + \ell_0)^2}{2 \mu + \tilde{V}_\nu T_0 T_0^*} T_0 T_0^* \begin{array}{cc} 0 & 0 \\ 2 \mu + \tilde{V}_\nu T_0 T_0^* \end{array} \right) \right].$$

In the above equations $D(\ell)$ is the fermion propagator, which will be evaluated in the following in the GL approximation. $T_0$ is the $SU(3)$ color generator in the basis $\Psi = (\psi_{ur}, \psi_{ug}, \psi_{dr}, \psi_{dg}, \psi_{ab}, \psi_{db})$, with the normalization $\text{Tr}[T_a T_b] = \delta_{ab}/2$. Finally the projector $P^\mu_\nu$ is defined as

$$P^\mu_\nu = g^\mu_\nu - \frac{1}{2} \left[ V^\mu V^\nu + \bar{V}^\mu \bar{V}^\nu \right].$$

The trace in the diagrams have to be evaluated in NG as well as in color-flavor indices.

Technically, the evaluation of the loop integrals for the Meissner masses is similar to that involved in the calculation of the decay constant and the velocity of the Goldstone boson: for all of the gluons one is left with two kinds of integrals, namely

$$I_{ij} = \sum_{a=1}^P \frac{dn}{4\pi} \int \frac{d^4 \ell}{(2\pi)^4} \frac{n_i n_j}{(V \cdot \ell - \delta \mu)^3 (V \cdot \ell - \delta \mu + 2 q_a \cdot n)} + \delta \mu \to -\delta \mu$$

$$= -i \frac{\text{Re}}{16\pi} \sum_{a=1}^P \frac{dn}{4\pi} \frac{n_i n_j}{(\delta \mu - q_a \cdot n + i0^+)^2} + (\delta \mu \to -\delta \mu).$$

The rotation matrices $R^a$ are trivial for both the case of the 3PW and the BCC crystals. As for the FCC they can be found in the appendix B of [31]. For all of the crystals we find $V_{ij} = \delta_{ij}/3$, which implies that the dispersion law of the Goldstone is

$$E^2(p) = \frac{1}{3} p^2,$$
related to a diagram with two gap insertions on a fermion branch and zero insertions on the other branch, and

\[ J_{ij} = \sum_{a=1}^{P} \int \frac{dn}{4\pi} \int \frac{dE_d dE_a}{(2\pi)^2} \frac{n_in_j}{(V \cdot \ell - \delta \mu)^2 (V \cdot \ell - \delta \mu + 2q_a \cdot n)^2} + \delta \mu \rightarrow -\delta \mu = 2I_{ij}, \]

related to a diagram with one gap insertions on a fermion branch and one gap insertion on the other branch. Because of the proportionality relation between \( I_{ij} \) and \( J_{ij} \), one can express the Meissner tensor for each crystal structure only in terms of \( I_{ij} \). At the leading order in \( \Delta/\delta \mu \) we find

\[
\begin{align*}
\left( M_{11}^{ij} \right)^2 &= \left( M_{22}^{ij} \right)^2 = \left( M_{33}^{ij} \right)^2 = 0, \\
\left( M_{44}^{ij} \right)^2 &= \frac{-i\Delta^2 \mu^2 I_{ij}}{\pi}, \quad a = 4, \ldots, 7, \\
\left( M_{88}^{ij} \right)^2 &= \frac{4}{3} \left( M_{44}^{ij} \right)^2.
\end{align*}
\] (67)

\[
\begin{align*}
\left( M_{aa}^{ij} \right)^2 &= \frac{f^2}{16} \nu_{ij}, \\
\left( M_{88}^{ij} \right)^2 &= \frac{f^2}{12} \nu_{ij},
\end{align*}
\] (70)

where \( \nu_{ij} \) is the coefficient that multiplies the gradient term in the action of the Goldstone boson \( U(1)_A \). Therefore the positivity of the matrix \( \nu_{ij} \) reflects to the positivity of the Meissner tensor. Since we have shown that both in the case of the BCC and of the FCC crystals \( \nu_{ij} \) is indeed defined positive, a real value for all of the Meissner masses follows at the leading order in \( \Delta/\delta \mu \).

We may ask how the higher order corrections can modify the result expressed in Eqs. (68), (69). To this end it is enough to investigate on the typical ratio \( \Delta/\delta \mu \) in a crystalline LOFF state. Unfortunately, as already stressed in the previous sections, the free energy (and the gap parameters) in a generic crystalline structure can be computed only in some approximation. In Ref. 32 a smearing over the cell has been introduced: in this scheme it was found that for the BCC the ratio \( \Delta/\delta \mu \) lies in the interval 0.35 – 0.40; for the FCC the interval is 0.27 – 0.30 (see Tables I and II of 32). Since the corrections to Eqs. (68), (69) are of order \( \Delta^4/\delta \mu^4 \), we see that the next-to-leading order terms are suppressed as \( \Delta^2/\delta \mu^2 \) when compared to the leading order results. At the worst, one has \( \Delta^2/\delta \mu^2 \approx 0.2 \) for the BCC and \( \Delta^2/\delta \mu^2 \approx 0.1 \) for the FCC. Therefore we argue that the next-to-leading order terms should not change in a dramatic way the main result of Eqs. (68), (69), namely positive squared Meissner masses for all of the gluons in a crystalline LOFF phase. Of course this hint should be supported by the calculation of the next-to-leading order correction: unfortunately the computation of such terms in BCC and/or FCC is much more involved than in the single plane wave case. Another strategy would be the calculation using the smearing approach 32 instead of the GL expansion of the Meissner tensor. We will come back to this problem in the future.

E. A brief comment about Higgs stability of the LOFF phases

Before concluding this section we briefly discuss how the Higgs instability could arise in the LOFF phases of more than a single plane-wave, following the treatment discussed in 49. The Higgs (or amplitude) instability is related to inhomogeneous variation of the amplitude of gap parameter \( \Delta \). It has been studied in great detail for the homogeneous g2SC phase 49, 50, 51, where it has been related to the existence of the Sarma instability.

In the LOFF phase, one can introduce a variation of the gap parameter in Eq. 51 by means of the replacement \( \Delta \rightarrow \Delta + H(r) \) with \( H(r) \) real, with Fourier transform given by \( H(k) \):

\[
\begin{align*}
\Delta(r)_{PW} \rightarrow \sum_{a=1}^{P} e^{i2q_a \cdot r} \left[ \Delta + \int \frac{d^3k}{(2\pi)^3} e^{-i k \cdot r} H(k) \right].
\end{align*}
\] (71)

Keeping into account fluctuations in the magnitude of \( \Delta \) and neglecting the phase fluctuations (i.e. the Goldstones) one can write the grand potential, in the Gaussian approximation, as

\[
\Omega = \Omega_0 + \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} H^*(k) \frac{\partial^2 \Omega}{\partial H^*(k) \partial H(k)} H(k),
\] (72)
where $\Omega_0$ is the grand potential evaluated at the minimum and the derivative, evaluated at the minimum of the free energy, is nothing but the self-energy of the fluctuations (see [50] for more details). The momentum integral in Eq. (72) can be divided *grosso modo* into an integral over small momenta (with respect to $q$) and an integral over larger momenta. Following [HUANG], in the low momentum region one can replace, in the thermodynamic limit,

$$\frac{\partial^2 \Omega}{\partial H^*(k) \partial H(k)} \rightarrow \frac{\partial^2 \Omega}{\partial \Delta^2};$$

the derivative on the r.h.s. of the above equation is positive because the LOFF state is in a minimum of the grand potential. Therefore a LOFF phase can not suffer Higgs instability, at least for small momenta.

V. CONCLUSIONS

In this paper we have computed the low energy properties of the Goldstone mode $\phi$ both in the gapped and in the gapless 2SC phase of QCD. We stress that such a Goldstone is related to the axial U(1) current and it is not one of the would-be Goldstones to be eaten up after gauging of color. In the calculation presented in Sec. III we introduce the $\phi$ as an external field. The integration over the fermion fields in the functional integral allows for the computation of the squared decay constant and of the squared velocity of $\phi$. In particular, we find a negative squared velocity in the gapless regime $\delta \mu > \Delta$. The simple proportionality relation between $v^2$ and the squared Meissner mass of the 8th gluon shows that the two instabilities, in the Goldstone and in the gluon sectors, are related and the removal of the second is equivalent to the removal of the first one.

Because of the wrong sign of the squared Goldstone velocity, the gradient of the Goldstone field may take on a non-vanishing expectation value in the vacuum, with consequent breaking of the symmetry under rotations. We call this phase cur-g2SC phase. The Lagrangian for the fluctuations can be constructed in the cur-g2SC phase and does not suffer of the squared-velocity sign problem.

Since the thermodynamics of the cur-g2SC is the same as that of the one-plane-wave LOFF state, its free energy is higher than the energy of a multiple-plane-wave phase. Motivated by this observation we have computed the low energy parameters of the effective lagrangian for the $U(1)_A$ Goldstone in the multiple-plane-wave LOFF phase (PPW); we find that that such crystalline phases do not present instabilities towards formation of currents. Moreover we have calculated the Meissner masses of the gluons in the PPW phases, generalizing the results of [48]: we find chromo-magnetic stability.

The other instability which could appear in in these LOFF phases is the Higgs instability, which has been studied for the homogeneous g2SC and related to the Sarma instability. We have argued that the Higgs instability should be absent in the PPW LOFF phase, at least for small momenta. A complete study of the amplitude instability requires in addition, in the Gaussian approximation, the calculation of the self-energy of $H(k)$ for each value of the 3-momentum.

A multiple-plane-wave LOFF seems in conclusion to be one of the best candidates for the ground state of a two flavor color superconductor, beside the gluonic phases considered in [38]. A comparison of the free energies of the neutral LOFF phases in the multiple plane wave state and of the phases with condensed gluons is therefore crucial for a deeper understanding of the phase diagram of two flavor QCD (in the single plane wave case this study has been performed in [42]).

Beside the comparison of the PPW state with the gluonic phases, a natural prosecution of this work should be the extension to the three flavor superconductive phases of QCD. In this context a lot of work has been done in order to remove the chromo-magnetic instability of the gapless CFL phase, see [39]. Moreover, the three flavor superconductive crystalline phase does not seem to have such an instability, at least when the Meissner masses are computed in the Ginzburg-Landau approximation [52]. It would be interesting to study the Goldstone properties both in the kaon condensed and in the three flavor LOFF phases of QCD, in order to see whether the instability found here is present also in these cases, and if it is completely removed once the gluon sector is cured.

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In the previous sections we have computed the squared decay constant $f^2$ and the squared velocity $v^2$ of the Goldstone boson $\phi$, both in the gapped and in the gapless 2SC phases of QCD. The results are in Eq. (23) and Eq. (25). In particular, we find a decreasing decay constant as the ratio $\delta \mu / \Delta$ is increased in the gapless phase. Moreover, when $\delta \mu > \Delta$ we find a negative squared velocity of $\phi$.

For the gapped 2SC and CFL phases it is possible to compute the low energy parameters directly from the screening masses of the gluons, or more generally of spin 1 gauge fields, see e.g. [53, 54, 55, 56]. We wish to perform such a program also in this case, trying to relate the decay constant and the squared velocity of $\phi$ found in the previous section to the Debye and Meissner masses of a fictitious gauge boson related to the $U(1)_A$ symmetry. In this section we compute these masses.

To begin with we promote the global $U(1)_A$ symmetry to a local one: this is done in the usual textbook way by the introduction of a fictitious gauge field $W_\mu$ whose coupling to the fermions is described via the covariant derivative $D_\mu = \partial_\mu + iQ_A W_\mu$. Here $Q_A$ is the charge of the fermions under $U(1)_A$, $Q_A = \pm 1$ respectively for left and right handed quarks. The calculation of the static screening masses is similar to that presented in the main text for the low energy parameters of the Goldstone mode; therefore here we simply quote the main results. In this case one has to evaluate the polarization tensor $\Pi_{\mu \nu}$ of the gauge boson $W_\mu$. According to Ref [2] we define

$$m_D^2 = \Pi_{\mu 0}(q_0 = 0, \mathbf{q} \rightarrow 0), \quad m_A^2 \delta_{ij} = -\Pi_{ij}(q_0 = 0, \mathbf{q} \rightarrow 0). \quad \text{(A1)}$$

First we consider the Debye mass. After some computation we are left with the expression

$$m_D^2 = \frac{4\mu^2}{\pi^2} \Delta^2 \int_0^{+\infty} \frac{d\ell_{\parallel}}{(\ell_{\parallel}^2 + \Delta^2)^{3/2}} + M_{\text{blue}}^2, \quad \text{(A2)}$$

where the first addendum on the r.h.s. is the contribution of the paired red and green quarks, while the second addendum is the contribution of the unpaired blue quarks given by the standard many-body result

$$M_{\text{blue}}^2 = \frac{2}{\pi^2} \mu^2. \quad \text{(A3)}$$

In the gapless phase $\delta \mu > \Delta$ it is interesting to distinguish, in the loop integral involving paired quarks, the contribution of the quarks whose loop momentum $\ell_{\parallel}$ lies in the blocking region (12) from the contribution of the quarks living in the pairing region (defined as the complementary of the blocking region):

$$\int_0^{+\infty} \frac{d\ell_{\parallel}}{(\ell_{\parallel}^2 + \Delta^2)^{3/2}} = \int_0^{\sqrt{\delta \mu^2 - \Delta^2}} \frac{d\ell_{\parallel}}{(\ell_{\parallel}^2 + \Delta^2)^{3/2}} + \int_{\sqrt{\delta \mu^2 - \Delta^2}}^{+\infty} \frac{d\ell_{\parallel}}{(\ell_{\parallel}^2 + \Delta^2)^{3/2}}$$

$$= \frac{\sqrt{\delta \mu^2 - \Delta^2}}{\Delta^2 \delta \mu} + \left( \frac{1}{\Delta^2} - \frac{\sqrt{\delta \mu^2 - \Delta^2}}{\Delta^2 \delta \mu} \right)$$

$$\equiv M_{BR}^2 + M_{PR}^2, \quad \text{(A4)}$$

where $BR$ and $PR$ stem respectively for Blocking Region and Pairing Region. With these definitions at hand one can write

$$m_D^2 = M_{BR}^2 + M_{PR}^2 + M_{\text{blue}}^2 = 6 \times \frac{\mu^2}{\pi^2}. \quad \text{(A5)}$$

The Debye mass is independent of the value of the ratio $\delta \mu / \Delta$ and is equal to the result of the normal phase. This can be understood by noticing that both the condensate and the fermion fields have $U(1)_A$ charge, so one expects screening both in the gapped and in the gapless phase.

Next we turn to the Meissner mass. We get

$$m_M^2 = \frac{4\mu^2}{3\pi^2} \left( 1 - \theta(\delta \mu - \Delta) \frac{\delta \mu}{\sqrt{\delta \mu^2 - \Delta^2}} \right). \quad \text{(A6)}$$

In the gapped 2SC phase one has $\delta \mu < \Delta$ and the squared Meissner mass is positive and constant. On the other hand in the gapless phase $\delta \mu > \Delta$ the squared Meissner mass is negative and divergent ad the transition point. The divergence is related to the divergence of the density of gapless quasi-particle states.
The calculations of the Debye and Meissner masses which lead to the results in Eqs. (A10) and (A11) are done using the so-called high energy theory [54], that is using a lagrangian written in terms of the quarks degrees of freedom only. It would be interesting to make the same calculation in the so-called low energy theory, that is by using a lagrangian written in terms of the light degrees of freedom, the Goldstone mode and the gapless quarks. The matching of the results obtained by the two procedures should allow to compute $f^2$ and $\nu^2$.

In the gapped CFL phase of QCD the matching procedure is easy to perform since the light degrees of freedom are the Goldstone bosons (an octet plus a singlet), while the high energy theory is defined in terms of the nine gapped quarks. Also in the 2SC phase the paired quarks are gapped, so the low energy degrees of freedom are the singlet $\phi$ and the unpaired blue quarks. In the gapless 2SC phase, on the other hand, one has also gapless paired fermions in the low energy spectrum, so one should consider them in the matching procedure.

Let us consider first the case of the gapped 2SC phase. On the basis of invariance under $U(1)_A$ it is easy to recognize that the effective lagrangian for $U = \exp(\phi/f)$ can be cast in the form

$$\mathcal{L} = \frac{f^2}{2} (\partial_\mu U \partial^\mu U^* - \nu^2 \nabla \nabla U^*) \tag{A7}$$

(with this definition the kinetic term of the field $\phi$ gets the canonical normalization). The gauging of the $U(1)_A$ is made by the replacement of the usual derivative with the covariant one $D_\mu = \partial_\mu + iW_\mu$. After the gauging, a mass term for the fictitious gauge field $W_\mu$ is obtained, namely

$$\delta \mathcal{L} = \frac{f^2}{2} \left(W_0^2 - \nu^2 W \cdot W\right) \equiv \frac{M_D^2}{2} W_0^2 - \frac{M_M^2}{2} W \cdot W, \tag{A8}$$

where we denote by the capital letter $M_{D,M}$ the contribution of the Goldstone to the Debye and the Meissner mass of $W_\mu$. However one has to add the contribution of the blue unpaired quarks to Eq. (A7) in order to properly describe the low energy theory. The blue quarks contribute only to the Debye mass, as they are not superconductive and do not screen static “magnetic” fields $W$. As a consequence, the Debye and Meissner masses in the low energy theory are

$$m_D^2 = M_D^2 + M_{blue}^2, \quad m_M^2 = M_M^2. \tag{A9}$$

On the other hand one can read $m_D^2$ and $m_M^2$ from the high energy calculations, see Eqs. (A5) and (A6). Comparison allows to compute $f^2$ and $\nu^2$,

$$f_{2SC}^2 = \frac{4\mu^2}{\pi^2}, \quad \nu_{2SC}^2 = \frac{1}{3}. \tag{A10}$$

Now we turn to the gapless 2SC phase and consider the Debye mass. In this case the effective lagrangian for $\phi$ has the same form of the gapped phase, Eq. (A7), since it is related only to the symmetries of the ground state and not to the absence/presence of gapless excitations in the spectrum. Therefore the contribution of $\phi$ to the screening masses is described by Eq. (A8) and $m_D^2$ is given in the low energy theory by

$$m_D^2 = M_D^2 + M_{blue}^2 + M_{gapless}^2, \tag{A11}$$

where $M_{gapless}$ is the (till unknown) contribution of the gapless fermions. A direct comparison with Eq. (A9) allows the identification

$$f^2 = \frac{4\mu^2}{\pi^2} - M_{gapless}^2. \tag{A12}$$

At this point, if we knew $M_{gapless}^2$ we should be able to determine $f^2$ in the low energy theory. However this contribution to the Debye mass is not well defined, as to calculate it one should choose an appropriate infrared cutoff and integrate out all quarks whose momenta are above such a cutoff. In this way, the theory would contain only the light degrees of freedom and the computation of $M_{gapless}^2$ would be straightforward.

However, in the gapless regime we can use a different strategy: we already have computed $f^2$ using a completely different approach in the previous section, see Eq. (23). Using this result, the matching condition (A12) can be used not to compute $f^2$ but to evaluate $M_{gapless}^2$, namely

$$M_{gapless}^2 = \frac{4\mu^2}{\pi^2} \theta(\delta\mu - \Delta) \sqrt{\frac{\delta\mu^2 - \Delta^2}{\delta\mu}}. \tag{A13}$$
which leads to the result quoted in Eq. \((28)\) once Eqs. \((A5)\), \((A6)\) and \((A13)\) are used.

This result is equal to \(M_{BR}^2\) in Eq. \((A14)\), which is obtained by integrating in the loop over the quark momenta in the blocking region. This coincidence seems to show that the g2SC phase can be described like a two-component Fermi liquid, with a normal part given by the blocking region and relevant for the low energy dynamics, and a superfluid part living in the pairing region.

We consider now the Meissner mass: this case is instructive since it shows that the aforementioned interpretation of the g2SC as a two-component superfluid leads to meaningful results once one tries the matching. Following our interpretation we argue that the quarks in the blocking region, as they behave as a normal liquid, do not contribute to the Meissner mass. The same is true for the unpaired blue quarks. Finally, the quarks in the pairing regions are superfluid and do not contribute to the low energy dynamics. As a consequence, the only contribution to the Meissner mass in the low energy theory is given by the \(\phi\) as in the 2SC phase, see Eqs. \((A8)\) and \((A9)\). Matching with the high energy result in Eq. \((A8)\) one gets

\[
v^2 = \frac{m^2_f}{f^2} = \frac{m^2_f - M^2_{gapless} - M^2_{blue}}{m^2_f - M^2_{gapless}} ,
\]

which leads to the result quoted in Eq. \((28)\) once Eqs. \((A5)\), \((A8)\) and \((A13)\) are used.

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