Logistic map as a Fourier’s series expansion: numerical analysis

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Abstract. Logistic map is one of the simplest, and at the same time the most commonly used
dynamic system, which is characterized by chaos. The article presents approximations of the
logistic map through its extension into the Fourier’s series. Obtained in such way dynamical
systems were analyzed, among others for the Lyapunov exponent and bifurcation diagrams.
Furthermore the issue of the density of the iterated variable and some applications in chaos
based cryptography were commented.

1. Introduction
Making data classified is basic task in everyday life. One of the forms of making data classified
is cryptography based on chaos theory.

Chaotic cryptography is using properties of chaotic maps such as behavior similar to
randomness and determinism to construct appropriate algorithms. In literature two chaotic
models are often used - tent map and logistic map. The first map is a dynamic system which is
build from two lines making it look like a tent [1, 2]. The logistic map is defined by formula:

\[ x_{k+1} = ax_k(1 - x_k), \]  

where \( a \in [0, 4] \) and \( x \in [0, 1] \).

Despite the common use of logistic map in chaos based cryptography, it’s characterized
by properties that make it a weak point in the constructed algorithms [3]. These properties
contains the so called periodic windows of value of parameter \( a \) for which the map (1) generates
stable solutions. Another reason is that the distribution of the iterated variable \( x \) is not a flat
distribution and only in specific conditions (e.g. \( a = 4 \)) can be defined by formulas. Among
others for these reasons it’s necessary to improve the properties of the map in subject to chaos
based cryptography. These works are based on applying modifications of logistic map (1) by
using additional parameters [1, 4, 5], which increase the interval in which the chaos exists.

Another approach to this issue is approximation of map (1) by expansion into series. An
example of such approximation may be given by expansion of (1) into Fourier’s series. This
article presents approximation of logistic map (1) by expansion into Fourier’s series. Furthermore
numerical analysis with use of bifurcation diagrams and Lyapunov exponent is shown.
2. Expansion of logistic map into Fourier’s series

Expansion of periodic function \( f(x) \) on interval \([x_0, x_0 + P]\) into Fourier’s series is given by formula [6]:

\[
f(x) = \frac{a_0}{2} + \sum_{i=1}^{\infty} \left( a_i \cos \left( \frac{2\pi ix}{P} \right) + b_i \sin \left( \frac{2\pi ix}{P} \right) \right),
\]

where \( P \) is the period of function \( f(x) \).

Factors \( a_i \) and \( b_i \) are defined by equations

\[
a_i(x) = \frac{2}{P} \int_{x_0}^{x_0+P} f(x) \cos \left( \frac{2\pi ix}{P} \right) dx \quad \text{for} \quad i \geq 0,
\]

\[
b_i(x) = \frac{2}{P} \int_{x_0}^{x_0+P} f(x) \sin \left( \frac{2\pi ix}{P} \right) dx \quad \text{for} \quad i > 0.
\]

Treating the logistic map (1) as periodic function on interval \([0, 1]\) we obtain the following equation:

\[
f(x) = \frac{a}{6} - \sum_{i=1}^{\infty} \frac{a}{(\pi i)^2} \cos (2\pi ix),
\]

which is the expansion of (1) into Fourier’s series.

Therefore the approximation of logistic map is given by equation:

\[
x_{k+1} = \frac{a}{6} - \sum_{i=1}^{n} \frac{a}{(\pi i)^2} \cos (2\pi ix_k),
\]

3. Analysis of selected dynamical systems

Analysis of selected approximations of logistic map (1) for expansion into Fourier’s series was made with use of bifurcation diagram and Lyapunov exponent given by formula:

\[
\lambda = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \ln |f'(x_i)|,
\]

where

\[
x_{k+1} = f(x_k)
\]

is the analyzed dynamical system. Positive value of (7) is a necessary condition for dynamical system (8) to generate chaotic behavior.

The results are shown on Figures 2–10, on which (from the left side) the model of mapping is shown. Next the density for parameter \( a = 4 \), Lyapunov exponent and bifurcation diagram are presented. The mentioned values for logistic map (1) are visible on Figure 1. Further Figures 2–10 shows the properties of map (6) for different values of parameter \( n \). The presented results suggest that the higher value of \( n \) (the more values are in the expansion into Fourier’s series of logistic map) the more accurate is the approximation to (1). On the other hand, the bigger value of \( n \), the more computational operations are necessary.

Very interesting issue occurs for first values of parameter \( n \), which is even number: the density for value \( a = 4 \) is not defined with continuous function.
In order to cryptographic point of view the presented maps are similar to logistic map (1) in case of range of values of parameter $a$, for which the dynamical systems are chaotic (positive Lyapunov exponent value). The invariant density, which for logistic map (1) with $a = 4$ is defined by unimodal function, with the growth of parameter $n$ convergent to the density of (1) for $a = 4$.

Figure 1. Logistic map and its properties. From the left side: dynamical system, invariant density for $a = 4$ (blue color); Lyapunov exponent, bifurcation diagram.

Figure 2. Map (6) with $n = 1$ and its properties. From the left side: dynamical system, invariant density for $a = 4$ (blue color); Lyapunov exponent, bifurcation diagram.

Figure 3. Map (6) with $n = 2$ and its properties. From the left side: dynamical system, invariant density for $a = 4$ (blue color); Lyapunov exponent, bifurcation diagram.

4. Summary
This article presents and characterize dynamical systems which are based on expansion of logistic map into Fourier’s series. For those maps the Lyapunov exponent and bifurcation diagram were counted. The results for first extension of presented logistic map shows that they can be applicable into cryptography based on chaos theory.
Figure 4. Map (6) with $n = 3$ and its properties. From the left side: dynamical system, invariant density for $a = 4$ (blue color); Lyapunov exponent, bifurcation diagram.

Figure 5. Map (6) with $n = 4$ and its properties. From the left side: dynamical system, invariant density for $a = 4$ (blue color); Lyapunov exponent, bifurcation diagram.

Figure 6. Map (6) with $n = 5$ and its properties. From the left side: dynamical system, invariant density for $a = 4$ (blue color); Lyapunov exponent, bifurcation diagram.

Figure 7. Map (6) with $n = 10$ and its properties. From the left side: dynamical system, invariant density for $a = 4$ (blue color); Lyapunov exponent, bifurcation diagram.

References
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Figure 8. Map (6) with $n = 15$ and its properties. From the left side: dynamical system, invariant density for $a = 4$ (blue color); Lyapunov exponent, bifurcation diagram.

Figure 9. Map (6) with $n = 20$ and its properties. From the left side: dynamical system, invariant density for $a = 4$ (blue color); Lyapunov exponent, bifurcation diagram.

Figure 10. Map (6) with $n = 30$ and its properties. From the left side: dynamical system, invariant density for $a = 4$ (blue color); Lyapunov exponent, bifurcation diagram.

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