Spherical maximal functions, variation and oscillation inequalities on Herz spaces

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ABSTRACT
In this work, the boundedness of the spherical maximal function, the mapping properties of the fractional spherical maximal functions, the variation and oscillation inequalities of Riesz transforms on Herz spaces have been established.

1. Introduction
In this paper, the mapping properties of the spherical maximal function and the spherical fractional maximal functions on Herz spaces are obtained. The variation and oscillation inequalities of Riesz transforms on Herz spaces are also established.

The Herz space is a generalization of the Lebesgue space. The Herz spaces have several applications on the studies of partial differential equations, Fourier series and Fourier transform (Ragusa, 2009, 2012; Weisz, 2008; Zhou & Cao, 2014). The mapping properties of a number of important operators from harmonic analysis such as the Hardy-Littlewood maximal function, the fractional integral operator and singular integral operators had been extended to Herz spaces (Li & Yang, 1996; Lu & Yang, 1996; Lu et al., 2008).

In (Ho, 2018e), the Young’s inequality and the Hausdorff-Young inequality had also been extended to the Herz spaces by using the real interpolation method. This motivates us to use real interpolation to further study the mapping properties of some other operators on Herz spaces. In this paper, the mapping properties of the spherical maximal function, the fractional maximal functions, the variation and oscillation operators for Riesz transform are studied. They are all sublinear operators. In order to apply the real interpolation to sublinear operators, the function spaces are required to be a normed Riesz space. Since the Herz spaces are normed Riesz spaces, the estimates on the K-functional are applicable to Herz spaces and, consequently, the real interpolation yields the main results.

This paper is organized as follows. The definitions of Herz spaces, the real interpolation and the normed Riesz spaces are presented in Section 2. The interpolation of the quasilinear operators on normed Riesz spaces is presented in Section 3. The main results for the spherical maximal function, the fractional spherical maximal functions, the variation and oscillation inequalities of Riesz transforms are obtained in Section 4.
Definition 2.1. Let $\alpha \in \mathbb{R}$, $0<p<\infty$ and $0<q<\infty$. The homogeneous Herz space consists of those Lebesgue measurable function $f$ which satisfies

$$||f||_{K^{q,p}_\alpha} = \left( \sum_{k \in \mathbb{Z}} 2^{\alpha k p} ||f|_{L^p}|_{K^q}\right)^{1/p} < \infty$$

where $B_k$ is the ball in $\mathbb{R}^n$ with center at origin and radius $2^k$. Similarly, the inhomogeneous Herz space consists of those Lebesgue measurable function $f$ which satisfies

$$||f||_{K^{q,p}_\alpha} = \left( \sum_{k \in \mathbb{Z}} 2^{\alpha k p} ||f|_{L^p}|_{K^q}\right)^{1/p} < \infty$$

When $0<p\leq r<\infty$, $0<q<\infty$ and $\{a_i\}_{i \in \mathbb{Z}} \subseteq \mathbb{R}$, the inequality \(\sum_{i \in \mathbb{Z}} |a_i|^r \leq \sum_{i \in \mathbb{Z}} |a_i|^s\) guarantees that

$$K^{r,p}_{a_i} \rightarrow K^{s,p}_{a_i}.$$ \hspace{1cm} (2.1)

The reader is referred to (Lu et al., 2008) for the properties of Herz spaces.

Let $\alpha \in \mathbb{R}$ and $0<q<\infty$. The power weighted Lebesgue space $L^q_\alpha$ consists of those Lebesgue measurable function $f$ satisfying

$$||f||_{L^q_\alpha} = \left( \int_{\mathbb{R}^n} |f(x)|^q |x|^\alpha dx \right)^{1/q} < \infty.$$

For any weight function $\omega : \mathbb{R}^n \rightarrow (0, \infty)$, write

$$L^q_{\alpha,\omega} = \left\{ f : ||f||_{L^q_{\alpha,\omega}} = \left( \int_{\mathbb{R}^n} |f(x)|^q \omega(x) dx \right)^{1/p} < \infty \right\}.$$

It is easy to see that $K^{q,p}_{\alpha} = L^q_\alpha$ and $L^q_\alpha = L^q_{\alpha,\omega}$.

The definition of the well known real interpolation method is recalled in the following. A pair of Banach spaces $(X_0, X_1)$ is said to be an interpolation couple if there exists a linear Hausdorff space $Z$ such that $X_0 \rightarrow Z$ and $X_1 \rightarrow Z$.

Next, the definition of $K$-functional is recalled from (Bennett & Sharpley, 1988, Section 3.1) and (Triebel, 1978, Section 1.3.1).

Let $(X_0, X_1)$ be an interpolation couple. For any $f \in X_0 + X_1$, the $K$-functional is defined as

$$K(t, f, X_0, X_1) = \inf \left\{ ||f||_{X_0} + t ||f||_{X_1} : f = f_0 + f_1 \right\},$$

where the infimum is taken over all $f = f_0 + f_1$ for which $f_i \in X_i$, $i = 0, 1$.

Let $0<\theta<1$ and $(X_0, X_1)$ be an interpolation couple. For $0<q<\infty$, $(X_0, X_1)_{\theta,q}$ consists of those $f \in X_0 + X_1$ such that

$$||f||_{(X_0, X_1)_{\theta,q}} = \left( \int_0^\infty \left( \int_{t^{-\theta}}^t K(t, f, X_0, X_1) \right)^q dt \right)^{1/q} < \infty.$$

For the details about real interpolation, the reader is referred to (Triebel, 1978, Section 1.2). The above interpolation method had been generalized in (Ho, 2016, 2018a, 2018c, 2018d) to study the Fourier transform, the $k$-plane transform and the maximal estimate of the solution of some partial differential equations.

In order to present the real interpolation of quasi-linear operators, some notions and notations from vector lattices and Riesz spaces are recalled (Zaanen, 1997).

Let $\mathcal{R}$ be a partially ordered set and $\mathcal{T}$ be a non-empty subset of $\mathcal{R}$. The element $x \in \mathcal{R}$ is called the upper bound of $\mathcal{T}$ if $t \leq x$ for all $t \in \mathcal{T}$. If $x$ is an upper bound of $\mathcal{T}$ such that $x \leq y$ where $y$ is any other upper bound of $\mathcal{T}$, $x$ is the supremum of $\mathcal{T}$. The definition of lower bound and infimum are analogous.

Let $\mathcal{R}$ be a partially ordered set. If every subset consisting of two elements has a supremum and an infimum, then $\mathcal{R}$ is a lattice.

For any lattice $\mathcal{R}$, the supremum and the infimum of $x, y \in \mathcal{R}$ are defined as $x \lor y$ and $x \land y$, respectively.

The real vector space $\mathcal{R}$ is a Riesz space if there is a partial order $\leq$ such that $\mathcal{R}$ is a lattice with respect to $\leq$ and

1. $f \leq g \Rightarrow f + h \leq g + h$ for any $h \in \mathcal{R}$,
2. $0 \leq f \Rightarrow 0 \leq af$ for any $a \in [0, \infty)$.

Let $\mathcal{R}$ be a Riesz space. For any $f \in \mathcal{R}$, write $f^+ = f \lor 0, f^- = (-f) \lor 0$ and $|f| = f \land (-f)$. Define $\mathcal{R}^+ = \{ f \in \mathcal{R} : 0 \leq f \}$. Then $f^+, f^-, |f| \in \mathcal{R}^+$ are valid (Zaanen, 1997, Theorem 5.1).

The following Riesz decomposition theorem for Riesz spaces is given in (Zaanen, 1997, Theorem 6.4).

Let $\mathcal{R}$ be a Riesz space. If $u, z_1, z_2 \in \mathcal{R}^+$ satisfy $u \leq z_1 + z_2$, then there exist $u_1, u_2 \in \mathcal{R}^+$ such that $u_1 \leq z_1, u_2 \leq z_2$ and $u = u_1 + u_2$.

Next, the definition of normed Riesz space is recalled from (Zaanen, 1997, p.85).

Let $\mathcal{R}$ be a Riesz space associated with the partial order $\leq$. A Riesz space $\mathcal{R}$ is said to be a normed Riesz space if $\mathcal{R}$ is equipped with a norm $|| \cdot ||_\mathcal{R}$ satisfying

$$||f|| \leq |g| \Rightarrow ||f||_\mathcal{R} \leq ||g||_\mathcal{R}. \hspace{1cm} (2.2)$$

For any $\alpha \in \mathbb{R}$, $1 \leq q<\infty$, $||f|| \leq |g|$ gives

$$||f||_\mathcal{R} \leq \left( \int_{\mathbb{R}^n} |f(x)|^q |x|^\alpha dx \right)^{1/q}$$

$$\leq \left( \int_{\mathbb{R}^n} |g(x)|^q |x|^\alpha dx \right)^{1/q} = ||g||_\mathcal{R},$$

therefore, $L^q_\alpha$ is a normed Riesz space.
3. Interpolation of quasilinear operators

In this section, a folklore fact on the real interpolation of quasilinear operators (Sagher, 1971, 1972) is presented. It states that whenever the target spaces $Y_0, Y_1$ are normed Riesz spaces, then the real interpolation of quasilinear operator is valid. Even though this is a folklore fact, for completeness, the details of the estimate on $K$-functional and the corresponding interpolation theorem are presented.

Since function spaces on $\mathbb{R}^n$ are considered, for the rest of this paper, the ordering $f \leq g$, $f, g \in \mathcal{M}_0$, is defined as $f(x) \leq g(x)$ a.e. on $\mathbb{R}^n$. Therefore, the vector space of Lebesgue measurable functions endowed with the ordering is a Riesz space, see Definition 2.6.

If $Y_0, Y_1 \subset \mathcal{M}_0$ are Riesz spaces, then $Y_0 + Y_1$ is also a Riesz space.

Next, the interpolation of quasilinear operators is obtained.

Let $0 < \theta < 1$, $0 < p < \infty$, $(X_0, X_1)$ and $(Y_0, Y_1)$ be interpolation couples. Let $T : X_0 + X_1 \to Y_0 + Y_1$ be a quasilinear operator, that is, there is a constant $C > 0$ such that

$$|T(h + g)| \leq C \left( |T(h)| + |T(g)| \right), \quad h, g \in X_0 + X_1$$

$$T(kf) = |k|T(f), \quad k \in \mathbb{R}, f \in X_0 + X_1$$

and $T : X_i \to Y_i$, $i = 0, 1$ are bounded. When $C = 1$, $T$ is called as a sublinear operator.

If $Y_0, Y_1$ are normed Riesz subspaces of $\mathcal{M}_0$, then, there is a constant $C_0 > 0$ such that for any $f \in (Y_0, Y_1)_{\theta, p}$

$$|||T(f)|||_{(Y_0, Y_1)_{\theta, p}} \leq C_0 ||f||_{(X_0, X_1)_{\theta, p}}.$$ 

Suppose that $f = g + h$ where $g \in X_0$ and $h \in X_1$. As $T$ is a quasilinear operator,

$$|T(f)| \leq C \left( |T(g)| + |T(h)| \right),$$

where $|T(g)| \in Y_0$ and $|T(h)| \in Y_1$.

According to Proposition 3.1, $Y_0 + Y_1$ is a Riesz space, the Riesz decomposition theorem, Theorem 2.1, yields $u_0, u_1 \in Y_0 + Y_1$ such that

$$|T(f)| = u_0 + u_1$$

and $0 < u_0 \leq C|T(g)|, 0 < u_1 \leq C|T(h)|$.

As $Y_0$ and $Y_1$ are normed Riesz spaces, the boundeness of $T$ guarantees that

$$||u_0||_{Y_0} \leq C ||T(g)||_{Y_0} \leq C_0 ||g||_{X_0}$$

$$||u_1||_{Y_1} \leq C ||T(h)||_{Y_1} \leq C_0 ||h||_{X_1}$$

for some $C_0 > 0$. Consequently, for any $t \in (0, \infty)$

$$K(t, |T(f)|, Y_0, Y_1) \leq ||u_0||_{Y_0} + t ||u_1||_{Y_1} \leq C_0 \left( ||g||_{X_0} + t ||h||_{X_1} \right).$$

By taking infimum over $f = g + h$ with $g \in X_0$ and $h \in X_1$ on both sides of the above inequalities, the above inequalities yield

$$K(t, |T(f)|, Y_0, Y_1) \leq C_0 K(t, f, X_0, X_1).$$

Therefore, when $0 < p < \infty$,

$$||Tf||_{(Y_0, Y_1)_{\theta, p}}^p \left( \int_0^\infty \left( t^{-\theta} K(t, |Tf|, Y_0, Y_1) \right)^p \frac{dt}{t} \right)^{1/p}$$

$$\leq C \left( \int_0^\infty \left( t^{-\theta} K(t, |f|, X_0, X_1) \right)^p \frac{dt}{t} \right)^{1/p} = C ||f||_{(X_0, X_1)_{\theta, p}}^p.$$ 

The reader is referred to (Ho, 2018d) for the interpolation of sublinear operators by general interpolation functors.

Let $x_0, x_1 \in \mathbb{R}, x_0 \neq x_1$, $0 < \theta < \infty$, $0 < p < \infty$ and $1 < q < \infty$. Suppose that $a = (1 - \theta)x_0 + \theta x_1$ and $T : X_0 + X_1 \to L^q_a + L^q_1$ is a quasilinear operator. If $T : X_i \to L^q_i, i = 0, 1$, are bounded, then there is a constant $C > 0$ such that

$$||T(f)||_{L^q_{x_0 + x_1}} \leq C ||f||_{(X_0, X_1)_{\theta, p}}^p.$$ 

In view of (Ho, 2017, Corollary 3.2),

$$(L^q_a + L^q_1)_{\theta, p} = K^p_q,$$  \hspace{1cm} (3.1)

where $a = (1 - \theta)x_0 + \theta x_1$. Obviously, $L^q_i, i = 0, 1$ are normed Riesz spaces, therefore, Theorem 3.2 and (3.1) yield a constant $C > 0$ such that

$$||T(f)||_{K^p_q} \leq C ||f||_{(X_0, X_1)_{\theta, p}}^p.$$  \hspace{1cm} \hfill \blacksquare

4. Main results

In this section, the interpolation of quasilinear operators from the previous section is used to establish the boundeness of the spherical maximal function, the fractional spherical maximal function, the oscillation operator and the $\rho$-variation operator of the Riesz transform.

4.1. Spherical maximal function

The boundeness of the spherical maximal function on Herz spaces is established in this section. As an application of this result, some estimates of the weak solution for wave equation on Herz spaces are obtained.

Let $d\mu$ be the normalized surface measure on the unit sphere on $\mathbb{R}^n$. For any locally integrable function $f$, the spherical maximal function of $f$ is defined by

$$Mf(x) = \sup_{r > 0} \int_{|y| = r} f(x - ty')d\mu(y'),$$

see (Garcia-Cuerva & Rubio de Francia, 1985, p.571).

(Duandikoetxea & Vega, 1996; Garcia-Cuerva & Rubio de Francia, 1985, Corollary 7.9; Gunawan, 1998) give the following power weighted norm inequalities for the spherical maximal function.

Let $n \geq 3, 1 < q < \infty$ and $1 - n \leq a < q(n - 1) - n$. There exists a constant $C > 0$ such that
\[
\int_{\mathbb{R}^n} |Mf(x)|^q |x|^n \, dx \leq C \int_{\mathbb{R}^n} |f(x)|^q |x|^n \, dx.
\]

Obviously, \( M \) is a sublinear operator on \( L^{q_0,1} + L^{q_1,1} \)
when \( 1 < q < \infty \) and \( 1 - n \leq a_0, a_1 < q(n-1) - n \).

The boundedness of the spherical maximal function on Herz spaces is presented and established in the following theorem.

Let \( n \geq 3, 1 < q < \infty, 0 < p < \infty \) and
\( 1 - n < a < q(n-1) - n \). There is a constant \( C > 0 \) such that
\[
||Mf||_{L^p_{K^n_x}} \leq C ||f||_{L^p_{K^n_x}}.
\]

Select \( 0 < \theta < 1 \) and \( 1 - n < a_0 < a_1 < q(n-1) \) so that
\( a = (1 - \theta)a_0 + \theta a_1 \). In view of Theorem 4.1, \( M \) is bounded on \( L^{q_0/q}_{a_0} \) and \( L^{q_1/q}_{a_1} \), respectively.

Consequently, Theorem 3.3 yields
\[
||Mf||_{L^p_{K^n_x}} = ||Mf||_{L^{q_0/q}_{a_0}} \leq C ||f||_{L^{q_0/q}_{a_0}} = C ||f||_{L^p_{K^n_x}}
\]

for some \( C > 0 \).

The above result gives estimates on the weak solution of the wave equation on Herz spaces because it is well known that the spherical maximal function can provide some estimates for the weak solution of the wave equation, see (Stein, 1976).

The above result is used to study the following classical initial-value problem
\[
\Delta_x u = u_{tt}, \quad (x,t) \in \mathbb{R}^3 \times (0, \infty),
\]
\[
u(x,0) = 0, \quad x \in \mathbb{R}^3,
\]
\[
u_t(x,0) = f(x), \quad x \in \mathbb{R}^3.
\]

For any \( f \in L^q(\mathbb{R}^3), \frac{1}{q} < q < \infty \), the weak solution of the above initial value problem is given by
the Kirchhoff's formula (Evan, 2010)
\[
u(x,t) = ct \int_{|y|<\frac{1}{ct}} f(x-ty') \, d\mu(y').
\]

The above formula gives
\[
||u(x,t)||_{L^p_{K^n_x}} \leq C t Mf(x), \quad \forall x \in \mathbb{R}^3, \ t > 0.
\]

Therefore, Theorem 4.2 yields the following estimate of the weak solution for the wave equation on Herz spaces.

Let \( \frac{1}{q} < q < \infty, 0 < p < \infty \) and \( 1 - n < a < q(n-1) - n \). There exists a constant \( C > 0 \) such that
\[
||u(., t)||_{L^p_{K^n_x}} \leq C t ||f||_{L^p_{K^n_x}}, \quad t > 0.
\]

Some further generalizations of the study of maximal function on Herz spaces are presented. Let \( \phi \in L^1(\mathbb{R}^n) \) be a radical function satisfying Zo's condition
\[
\int_{|x| > 2|y|} \sup_{\delta > 0} |\phi_{\delta}(x-y) - \phi_{\delta}(x)| \, dx \leq C, \quad y \in \mathbb{R}^n.
\]

Define
\[
\text{Define } M_q f(x) = \sup_{\delta > 0} |\phi_{\delta} \ast f(x)|
\]

where \( \phi_{\delta}(x) = \delta^{-n} \phi(\delta^{-1}x), \ \delta > 0 \).

Let \( 0 < p < \infty, 1 < q < \infty, -n < a < q(n-1) \) and \( \phi \in L^1(\mathbb{R}^n) \) be a radical function satisfying Zo's condition. Then, there is a constant \( C > 0 \) such that
\[
||M_q f||_{L^p_{K^n_x}} \leq C ||f||_{L^p_{K^n_x}}.
\]

Select \( 0 < \theta < 1 \) and \( 1 - n < a_0 < a_1 < q(n-1) \) so that
\( a = (1 - \theta)a_0 + \theta a_1 \). According to (García-Cuerva & Rubio de Francia, 1985, Corollary 7.7), \( M_q \) is bounded on \( L^{q_0/q}_{a_0} \) and \( L^{q_1/q}_{a_1} \), respectively. Therefore, our result follows from (3.1).

### 4.2. Fractional spherical maximal functions

Let \( \alpha > 0 \) and \( d\mu \) be the normalized surface measure on the sphere \( B(0,t) = \{x \in \mathbb{R}^n : |x| < t \} \) in \( \mathbb{R}^n \). For any locally integrable function \( f \), the fractional spherical maximal function of \( f \) is defined as
\[
M^\alpha f = \sup_{r > 0} |r^\alpha \mu \ast f|.
\]

For the mapping properties of \( M^\alpha \), the reader may consult (Gunawan, 1998; Oberlin, 1989).

Since the Muckenhoupt weight function is involved to present the weighted norm inequality for \( M^\alpha \), the definition of the Muckenhoupt weight function is recalled from (Grafakos, 2009, Chapter 9). Let \( B \) denote the collection of balls in \( \mathbb{R}^n \).

For \( 1 < p < \infty \), a locally integrable function \( \omega : \mathbb{R}^n \rightarrow [0, \infty) \) is said to be an \( A_p \) weight if
\[
||\omega||_{A_p} = \sup_{D \in B} \left( \frac{1}{|D|} \int_D \omega(x) \, dx \right)^{1/p} < \infty
\]

where \( p' = \frac{p}{p-1} \). A locally integrable function \( \omega : \mathbb{R}^n \rightarrow [0, \infty) \) is said to be an \( A_1 \) weight if
\[
||\omega||_{A_1} = \sup_{D \in B} \left( \frac{1}{|D|} \int_D \omega(y) \, dy \right)^{1} < \infty
\]

Moreover, \( A_\infty \) to be the union of \( A_p \) for all \( p \geq 1 \). That is, \( A_\infty = \cup_{p \geq 1} A_p \).

The following is the weighted norm inequality for the fractional spherical maximal functions.

Let \( \frac{q-1}{q-1} < r < q(n-1) \), \( \alpha = \frac{q}{q-1} - \frac{q}{r} \) and
\max \{0, 1 - \frac{q}{r} \} < \gamma < 1 - \frac{q}{r} \frac{\alpha}{\alpha - 1} \). If \( \omega \in A_1 \) where
\[
s = \frac{r(2\gamma + n - n\gamma - 1) - q}{r\gamma},
\]
then there is a constant \( C > 0 \) such that
\[
||M^\alpha f||_{L^p_{K^n_x}} \leq C ||f||_{L^p_{K^n_x}}.
\]
The reader is referred to (Cowling, García-Cuerva & Gunawan, 2002, Theorem 4.4) for the proof of the preceding result.

The mapping properties for the fractional spherical maximal operators on Herz spaces are established in the following.

Let \(0 < p < \infty\), \(0 < q < (n-1)r\), \(\alpha = \frac{\gamma - \frac{n}{q}}{\gamma + 1} (n \geq 1)\) and \(\max \{0, 1 - \frac{q}{n}\} < \gamma < 1 - \frac{q}{n(n-1)}\). If \(-n < a < n(s-1)\) where \(s = \frac{(2\gamma + n - ny\gamma - 1) - q}{r\gamma}\), then there is a constant \(C > 0\) such that

\[
\|M^\alpha f\|_{K^p_q} \leq C\|f\|_{K^p_q},
\]

In view of (Grafakos, 2009, p.286) and the condition \(-n < a < n(s-1)\), \(\cdot\) \(\in A_s\). Select \(0 < \theta < 1\) and \(-n < a_0 < a_1 < n(s-1)\) such that \(a = (1-\theta) a_0 + \theta a_1\).

Theorem 4.5 yields a constant \(C > 0\) such that for any \(f \in L^{ap}_K\)

\[
\|M^\alpha f\|_{L^{ap}_K} \leq C\|f\|_{L^{ap}_K},
\]

Thus, Theorem 3.3 guarantees that \(M^\alpha : K^{p,q}_r \rightarrow K^{p,q}_r\) is bounded.

The above result shows that weighted norm inequality for Muckenhoupt weighted Lebesgue spaces can be used to generate mapping properties for quasi-linear operators on Herz spaces. This idea is employed in the following section to study variation and oscillation inequalities for Riesz transforms on Herz spaces.

4.3. Variation and oscillation inequalities

For any \(\varepsilon > 0\), let

\[
R_j f(x) = C_n \int_{|x-y| > \varepsilon} \frac{x_i - y_i}{|x-y|^n} f(y) dy, \quad j = 1, \ldots, n,
\]

hence any given \(\{t_i\}_i\) with \(t_i \downarrow 0\), the oscillation operator of the Riesz transform \(R_j f\) is given by

\[
O(R_j f) = \left( \sum_{i=1}^\infty \sup_{t_i, s_i, \ldots, s_i} |R_{j \varepsilon_i} f - R_{j \varepsilon_i} f| \right)^{1/2},
\]

and the \(\rho\)-variation operator is defined as

\[
V_{\rho}(R_j f) = \sup \left( \sum_{i=1}^\infty |R_{j \varepsilon_i} f - R_{j \varepsilon_i} f|^\rho \right)^{1/\rho}
\]

where \(\rho > 0\) and the supremum is taken over all sequence \(\{\varepsilon_i\}_i\) satisfying \(\varepsilon_i \downarrow 0\).

The weighted norm inequalities for the oscillation operators and the \(\rho\)-variation operators for the Riesz transforms are established in (Ma, Torrea & Xu, 2017; Zhang & Wu, 2017).

Let \(\rho > 2\). For every \(1 \leq j \leq n, 1 < p < \infty\) and \(\omega \in A_p\), there exists a constant \(C > 0\) such that

\[
\int |O(R_j f)(x)|^p \omega(x) dx \leq C \int |f(x)|^p \omega(x) dx, \quad (4.1)
\]

\[
\int |V_{\rho}(R_j f)(x)|^p \omega(x) dx \leq C \int |f(x)|^p \omega(x) dx. \quad (4.2)
\]

The reader is referred to (Gillespie & Torres, 2004; Ma et al., 2017; Zhang & Wu, 2017) for the proofs of the above results.

The above inequalities yield the variation and oscillation inequalities for Riesz transforms on Herz spaces.

Let \(1 < q < \infty, 0 < p < \infty\) and \(-n < a < n(s-1)\). There exists a constant \(C > 0\) such that

\[
\|O(R_j f)\|_{K^{p,q}_r} \leq C\|f\|_{K^{p,q}_r}, \quad (4.3)
\]

\[
\|V_{\rho}(R_j f)\|_{K^{p,q}_r} \leq C\|f\|_{K^{p,q}_r}. \quad (4.4)
\]

Take \(0 < \theta < 1\) and \(-n < a_0 < a_1 < n(s-1)\) such that \(a = (1-\theta) a_0 + \theta a_1\). Theorem 4.7 assures that

\[
\|O(R_j f)\|_{L^{ap}_K} \leq C\|f\|_{L^{ap}_K}, \quad i = 0, 1,
\]

\[
\|V_{\rho}(R_j f)\|_{L^{ap}_K} \leq C\|f\|_{L^{ap}_K}, \quad i = 0, 1.
\]

Since \(O(R_j)\) and \(V_{\rho}(R_j)\) are sublinear operators, (4.3) and (4.4) are obtained by Theorem 3.3.

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