Density estimates for the solutions of backward stochastic differential equations driven by Gaussian processes

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Abstract. The aim of this paper is twofold. Firstly, we derive upper and lower non-Gaussian bounds for the densities of the marginal laws of the solutions to backward stochastic differential equations (BSDEs) driven by fractional Brownian motions. Our arguments consist of utilising a relationship between fractional BSDEs and quasilinear partial differential equations of mixed type, together with the profound Nourdin-Viens formula. In the linear case, upper and lower Gaussian bounds for the densities and the tail probabilities of solutions are obtained with simple arguments by their explicit expressions in terms of the quasi-conditional expectation. Secondly, we are concerned with Gaussian estimates for the densities of a BSDE driven by a Gaussian process in the manner that the solution can be established via an auxiliary BSDE driven by a Brownian motion. Using the transfer theorem we succeed in deriving Gaussian estimates for the solutions.

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1 Introduction

The problem of density estimates for solutions of stochastic equations has been extensively studied in recent years, see e.g. the monograph [27] and references therein. Remarkably, the celebrated Bouleau-Hirsch criterion (see [27, Theorem 2.1.2]) provides a sufficient condition for a random variable possessing a density. Moreover, in [26], Nourdin and Viens derive a formula for a Malliavin differentiable random variable to admit a density with lower and upper Gaussian estimates. These results have been further extended and applied to solutions of stochastic differential equations (SDEs) and stochastic partial differential equations (SPDEs), among which let us just mention, for examples, the works by Debussche and Romito [8], Delarue, Menozzi and Nualart [10], Millet and Sanz-Solé [23], Mueller and Nualart [24], Nualart and Quer-Sardanyons [28], and the references therein.
On the other hand, in the seminal paper Pardoux and Peng initiated the theory of nonlinear backward stochastic differential equations (BSDEs), which is of increasing importance in stochastic control and mathematical finance (see, e.g., and most recently ). This class of equations is of the following form

\[ y_t = \xi + \int_t^T f(s, y_s, z_s)ds - \int_t^T z_s dB_s, \quad t \in [0, T] \]  

(1.1)
on a given filtered probability space \((\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t \in [0, T]})\), where \(T > 0\) is arbitrarily fixed, \(\xi\) is a \(\mathcal{F}_T\)-measurable random variable, the generator \(f : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\) is a jointly measurable map, and \(B = (B_t)_{t \geq 0}\) is a Brownian motion adapted to \(\{\mathcal{F}_t\}_{t \in [0, T]}\) or simply \(\{\mathcal{F}_t\}_{t \in [0, T]}\) is taken as the natural filtration of \(B\). Recall that a solution to the BSDE (1.1) is a pair of predictable processes \((y, z)\) with suitable integrability conditions such that (1.1) holds \(\mathbb{P}\)-a.s.. To date, there is a wealth of existence and uniqueness results under various assumptions on the generators \(f\) including, for instance, the cases of Lipschitz or (super-)quadratic growth \([11, 12, 18, 29, 30]\). When dealing with applications such as the numerical approximation of the solutions, one needs to investigate the existence and regularity of densities for the marginal laws of \((y, z)\). As far as we know, there are comparably only a few works to study this problem. The first results have been derived by Antonelli and Kohatsu-Higa \([3]\), in which they study the existence and the estimates of the density for \(y_t\) at a fixed time \(t \in [0, T]\) via the Bouleau-Hirsch criterion. Then, based on the Nourdin-Viens formula, Aboura and Bourguin \([1]\) have proved the existence of the density for \(z_t\) under the condition that the generator \(f\) is linear with respect to its \(z\) variable, and further obtained the estimates on the densities of the laws of \(y_t\) and \(z_t\). Recently, Mastrolia, Possamaï and Réveillac in \([21]\) have studied the existence of densities for marginal laws of the solution \((y, z)\) to (1.1) with a quadratic growth generator, and derived the estimates on these densities. Afterwards, Mastrolia \([20]\) has extended the results to the case of non-Markovian BSDEs.

One of the main objective of the present paper concerns the problem of density estimates for the following BSDE

\[ y_t = h(\eta_T) + \int_t^T f(s, \eta_s, y_s, z_s)ds - \int_t^T z_s dB_s^H, \quad t \in [0, T], \]  

(1.2)
where \(\eta_t = \eta_0 + \int_0^t b_s ds + \int_0^t \sigma_s dB_s^H\) with \(\eta_0, b_s\) and \(\sigma_s\) being respectively a constant and deterministic functions, and \(B^H\) is a fractional Brownian motion with Hurst parameter \(H \in (0, 1)\), the stochastic integral is the divergence-type integral. Precise assumptions on the (deterministic and joint measurable) generator \(f : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\) and \(h : \mathbb{R} \rightarrow \mathbb{R}\) will be specified in later sections. Let us recall that, \(B^H\) with Hurst parameter \(H \in (0, 1)\) is a center Gaussian process with covariance

\[ R_H(t, s) := \mathbb{E}(B^H_t B^H_s) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}), \quad t, s \in [0, T]. \]

This implies that for each \(p \geq 1\), there holds \(\mathbb{E}(|B^H_t - B^H_s|^p) = C(p)|t - s|^{pH}\). Then \(B^H\) is \((H - \epsilon)\)-order Hölder continuous for any \(\epsilon > 0\) and is an \(H\)-self similar process. This, together with the fact that \(B^{1/2}\) is a Brownian motion, converts fractional Brownian motion into a natural generalization of Brownian motion and leads to many applications in modelling physical phenomena and finance behaviours.
We mention that there are several papers concerning the existence and uniqueness results of solutions for (1.2). Biagini, Hu, Øksendal and Sulem in [6] first studied linear fractional BSDE with $H \in (1/2, 1)$ which are based on fractional Clark-Ocone formula and the Girsanov transformation. In the spirit of the four step scheme introduced by Ma, Protter and Yong [19] for BSDEs perturbed by a standard Brownian motion, Bender [4] constructed an explicit solutions for a kind of linear fractional BSDEs with $H \in (0, 1)$ via the solution of some PDE and fractional Itô formula. In the case of nonlinear fractional BSDEs with $H \in (1/2, 1)$, Hu and Peng [17] first proved the existence and uniqueness of the solution through the notion of quasi-conditional expectation introduced in [16]. Then, based on [17], Maticiuc and Nie in [22] made some improvements of analysis and extended to the case of fractional backward stochastic variational inequalities, and Fei, Xia and Zhang in [13] generalized the investigation to BSDEs driven by both standard and fractional Brownian motions. In a multivariate setting where each of the components is an independent fractional Brownian motion $B^{H_i}$ with $H_i \in (1/2, 1)$, Hu, Ocone and Song in [15] solved fractional BSDEs by using their relation to PDEs, and they further derived a comparison theorem.

In the present paper, with the help of the connection between the solution to BSDE (1.2) and the solution to its associated PDE of mixed type, we shall give some sufficient conditions to ensure the existence of densities for marginal laws of the solution $(y,z)$ to BSDE (1.2). Moreover, we will derive non-Gaussian tail estimates of densities. To the best of our knowledge these kind of estimates for BSDE (1.2) are not available in the existing literature. When

$$f(s, \eta_s, y_s, z_s) = \alpha_s + \beta_s y_s + \gamma_t z_s,$$

i.e., BSDE (1.2) is linear, the Gaussian bounds for the densities and the tail probabilities of solutions will be derived with a direct and simpler arguments by their explicit expressions in terms of the quasi-conditional expectation.

Our paper is also dedicated to obtaining Gaussian bounds for the densities of the solution to BSDE

$$y_t = h(X_T) + \int_t^T f(s, X_s, y_s, z_s) dV(t) - \int_t^T z_s d\diamond X_s,$$  

(1.3)

where $X$ is a centered Gaussian process with a strictly increasing continuous variance function $V(t) = \text{Var}X_t$, the stochastic integral is the Wick-Itô integral defined via the $S$-transformation and the Wick product. When $X$ is a Brownian motion, the above Wick-Itô integral coincides with the classical Itô integral. In [5] Bender shows the existence and uniqueness results and then obtains a strict comparison theorem for BSDE (1.3) using the transfer theorem which can transfer the concerned problems to an auxiliary BSDE driven by a Brownian motion. In [5] the author also compares this type of equations with other BSDEs driven by Gaussian non-semimartingales, especially BSDEs driven by fractional Brownian motion $B^H$ with $H \in (1/2, 1)$. The final objective of the present paper is to deepen the investigation of BSDE (1.3). We study the Gaussian bounds for marginal laws of the solution $(y,z)$ to BSDE (1.3) via the transfer theorem.

The rest of our paper is structured as follows. Section 2, the next section, presents some basic elements of stochastic calculus with respect to fractional Brownian motion which are needed in later sections. We investigate the non-Gaussian bounds for the densities of the nonlinear fractional BSDEs in Section 3. Section 4 is devoted to the derivation of the Gaussian bounds for the densities and the tail probabilities of linear fractional BSDEs.
Finally in Section 5, we provide the Gaussian bounds for the densities of BSDEs driven by Gaussian processes.

## 2 Preliminaries

In this section, we shall give some basic elements of stochastic calculus with respect to fractional Brownian motion. For a deeper and detailed discussion, we refer the reader to [2, 7, 9] and [27].

Let $\Omega$ be the canonical probability space $C_0([0, T], \mathbb{R})$, i.e., the Banach space of continuous functions on $[0, T]$ vanishing at time 0, equipped with the supremum norm, and $\mathcal{F}$ is taken to be the Borel $\sigma$-algebra. Let $\mathbb{P}$ be the unique probability measure on $\Omega$ such that the canonical process $B^H = (B^H_t)_{t \in [0, T]}$ is a fractional Brownian motion with Hurst parameter $H \in (0, 1)$.

Let $\mathcal{E}$ be the space of step functions on $[0, T]$, and $\mathcal{H}$ the closure of $\mathcal{E}$ with respect to the following scalar product determined by the covariance $\mathcal{R}_H$ of $B^H$.

$$\langle I_{[0,t]}, I_{[0,s]} \rangle_{\mathcal{H}} = R_H(t, s).$$

By the bounded linear transformation theorem, the mapping $I_{[0,t]} \mapsto B^H_t$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space $H^1$ associated with $B^H$. Denote this isometry by $\phi \mapsto B^H(\phi)$.

For $p \geq 1$, we define the Sobolev space $\mathbb{D}^{1,p}_H$ as the completion of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p}^p = \mathbb{E}|F|^p + \mathbb{E}\|D^H F\|^p_{\mathcal{H}}.$$

Next, let $F$ be in $\mathbb{D}^{1,2}_H$ and write $D^H F := \Phi_F(B^H)$, where $\Phi_F : \mathbb{R}^H \to \mathcal{H}$ is a measurable mapping. Set

$$g_F(x) = \int_0^\infty e^{-\theta} \mathbb{E} \left[ \mathbb{E}' \left[ \langle \Phi_F(B^H), \Phi'_F(B^H) \rangle_{\mathcal{H}} \right] | F - \mathbb{E} F = x \right] d\theta, \quad x \in \mathbb{R}$$

(2.2)
where $\Phi^\theta(B^H) := \Phi(e^{-\theta B^H} + \sqrt{1 - e^{-2\theta} B^{'H}})$ with $B^H$ an independent copy of $B^H$ such that $B^H$ and $B^{'H}$ are defined on the product probability space $(\Omega \times \Omega', F \times F', P \times P')$.

We recall the following result, cf. [26, Theorem 3.1 and Proposition 3.7], which presents a criterion for a Malliavin differentiable random variable to have a density with Gaussian bounds based on the above function $g$.

**Proposition 2.1** The law of $F$ has a density $\rho_F$ with respect to the Lebesgue measure if and only if $g_F(F - EF) > 0$ a.s. In this case, $\text{Supp}(\rho_F)$ is a closed interval of $\mathbb{R}$ and for all $z \in \text{Supp}(\rho_F)$, there holds

$$
\rho_F(z) = \frac{\mathbb{E}|F - EF|}{2g_F(z - EF)} \exp \left(- \int_0^{z - EF} \frac{u \, du}{g_F(u)} \right).
$$

Furthermore, if there exist constants $c_1, c_2 > 0$ such that

$$
c_1 \leq g_F(x) \leq c_2, \quad \mathbb{P} - \text{a.s.},
$$

then the law of $F$ has a density $\rho$ satisfying, for almost all $x \in \mathbb{R}$,

$$
\frac{\mathbb{E}|F - EF|}{2c_2} \exp \left(- \frac{(x - EF)^2}{2c_1} \right) \leq \rho(x) \leq \frac{\mathbb{E}|F - EF|}{2c_1} \exp \left(- \frac{(x - EF)^2}{2c_2} \right).
$$

Besides, by [26, Theorem 4.1] (see also [25, Proposition 2.2]) we then have the following tail estimates.

**Proposition 2.2** Let $F \in D^{1,2}_H$ with $EF = 0$. If $0 < g_F(x) \leq a_1 x + a_2$, a.s. for some $a_1 \geq 0$ and $a_2 > 0$, then

$$
\mathbb{P}(F \geq x) \leq \exp \left(- \frac{x^2}{2a_1 x + 2a_2} \right) \quad \text{and} \quad \mathbb{P}(F \leq -x) \leq \exp \left(- \frac{x^2}{2a_2} \right), \quad x > 0.
$$

### 3 BSDEs driven by fractional Brownian motions

The objective of this section is to study the non-Gaussian densities estimates for the solution of the following BSDE driven by fractional Brownian motion

$$
\begin{aligned}
dy_t &= f(t, \eta_t, y_t, z_t)dt - z_t dB^H_t, \\
y_T &= h(\eta_T),
\end{aligned}
$$

with

$$
\eta_t := \eta_0 + \int_0^t b_s ds + \int_0^t \sigma_s dB^H_s.
$$

Here $\eta_0$ is a given constant, $b$ and $\sigma$ are bounded deterministic functions such that $\sigma(t) \neq 0$ for all $t \in [0, T], H \in (\frac{1}{4}, 1)$. A pair of $F_t$-adapted stochastic processes $(y, z)$ is called a solution to the equation (3.1) if

$$
y_t = h(\eta_T) + \int_t^T f(s, \eta_s, y_s, z_s) ds - \int_t^T z_s dB^H_s, \quad t \in [0, T].
$$

We begin with the assumption (H1) below.
(i) $f : [0, T] \times \mathbb{R}^3 \to \mathbb{R}$ is differentiable with respect to the third component and there exists a nonnegative constant $K$ such that, for all $t \in [0, T], x, y_1, y_2, z_1, z_2 \in \mathbb{R}$,

$$|f(t, x, y_1, z_1) - f(t, x, y_2, z_2)| + |f_y(t, x, y_1, z_1) - f_y(t, x, y_2, z_2)| \leq K(|y_1 - y_2| + |z_1 - z_2|).$$

(ii) $h : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and of polynomial growth.

Due to [15, Theorem 3.4], the condition (H1) ensures that there exists a unique solution $(y, z)$ to BSDE (3.1), which is given by $y_t = u(t, \eta_t)$ and $v_t = \sigma_t u(t, \eta_t)$ via some deterministic function $u : [0, T] \times \mathbb{R} \to \mathbb{R}$, where $u_x(t, x) := \frac{\partial}{\partial x} u(t, x)$. The argument in [15] is based on a connection between this equation and a quasilinear PDE of mixed type. In the remaining part of this section, we assume (H1) holds and moreover the unique solution is of the above form. We aim to show the non-Gaussian densities estimates for the marginal laws of $(y, z)$ at a fixed time $t \in (0, T)$. To this end, we let

$$\varrho_t = C_H \int_0^t \int_0^t \sigma_u \sigma_v |u - v|^{2H-2} du dv, \quad p_{\varrho_t}(x) = \frac{1}{\sqrt{2\pi \varrho_t}} e^{-\frac{x^2}{2\varrho_t}}$$

and for each $h \in L^0(\mathbb{R})$, define

$$\overline{h} := \inf \{ \gamma > 0 : \limsup_{|x| \to +\infty} \frac{|h(x)|}{|x|^\gamma} < \infty \}, \quad \underline{h} := \inf \{ \gamma > 0 : \liminf_{|x| \to +\infty} \frac{|h(x)|}{|x|^\gamma} < \infty \}.$$

Clearly, the above $\overline{h}$ and $\underline{h}$ can describe the asymptotic growth of $h$ in the neighborhood of $+\infty$ and $-\infty$.

Our main result of this section reads as follows

**Theorem 3.1** Let $t \in (0, T)$. Suppose that $0 < u(t, \cdot) < +\infty, u_x(t, \cdot) < +\infty$ and there exist positive constants $L, \lambda$ satisfying $u_x(t, \cdot) \geq \frac{1}{L(1+|\cdot|^{\lambda})}$. Then, the law of $y_t = u(t, \eta_t)$ has a density $\rho_{y_t}$, and for any $\epsilon, \delta > 0$ there exist positive constants $C_{\epsilon,t}$ and $C_{\delta,t}$ depending on $\epsilon, t$ and $\delta, t$, respectively, such that

$$\frac{\mathbb{E}|y_t - \mathbb{E}y_t|}{2C(\epsilon, \delta) \varrho_t \left( 1 + |z|^{\epsilon} \right) \left( 1 + |z|^{\epsilon} + \overline{\varrho}_t \right)} \exp \left( -\frac{1}{C(\delta) \varrho_t} \int_0^{z-\mathbb{E}y_t} u \left( 1 + |u + \mathbb{E}y_t|^{2\lambda} \right) du \right) \leq \rho_{y_t}(z)$$

$$\leq \frac{\mathbb{E}|y_t - \mathbb{E}y_t|}{2C(\epsilon, \delta) \varrho_t \left( 1 + |z|^{\epsilon} \right)} \exp \left( -\frac{1}{C(\epsilon, \delta) \varrho_t} \int_0^{z-\mathbb{E}y_t} u du \left( 1 + |u + \mathbb{E}y_t|^{\delta} \right) \left( 1 + |u + \mathbb{E}y_t|^{\delta} + \overline{\varrho}_t \right) \right),$$

where

$$C(\epsilon, \delta) := \sup_{t \in [0, T]} \left[ C_{\epsilon,t}^2 \left( 1 + C_{\delta,t}^{\epsilon - 1} \right) \cdot \left( 1 + 3^{(\epsilon - 1)} |\eta_0 + \int_0^t b_s ds|^{\epsilon} + \frac{C_{\delta,t}^{\epsilon - 1} \eta_0^{(\epsilon - 1)} + \left( \frac{1 + \epsilon}{2} \right) \overline{\varrho}_t}{1 + \epsilon} \right) \right]$$

and

$$\overline{C}(\delta) := \frac{1}{3^{(\lambda - 1)} + 2L^2(1 + (2(2\lambda - 1) + C_{\delta,t}^{2\lambda}))} \cdot \sup_{t \in [0, T]} \int_{\mathbb{R}} \frac{p_{\varrho_t}(z)}{1 + |\eta_0 + \int_0^t b_s ds|^{\lambda} + |z|^{\lambda}} dz$$

with $\epsilon := u_x(t, \cdot) + \epsilon$ and $\delta := u^{-1}(t, \cdot) + \delta$. 

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Proof. By the fact that \( u(t,\cdot) \) is continuous and increasing, we easily verify that \( y_t \) has a density \( \rho_{y_t} \). In order to prove (3.2), we rely heavily on Proposition 2.1.

Notice first that for \( 0 < u \leq t \leq T \), we have \( D_u^H y_t = u_x(t, \eta_t) \sigma_u \). Then, it follows that

\[
\Phi_{y_t}(B^H) = u_x(t, \eta_t) \sigma.
\]

and

\[
\Phi_{y_t}(B^H) = \Phi_{y_t} \left( e^{-\theta} B^H + \sqrt{1 - e^{-2\theta}} B^H \right)
\]

\[
= u_x \left( t, \left( 1 - e^{-\theta} \right) \left( \eta_0 + \int_0^t b_s ds \right) + e^{-\theta} \eta_t + \sqrt{1 - e^{-2\theta}} \int_0^t \sigma_s dB_s^H \right) \sigma.
\]

Hence, using (2.2) and (2.1), we deduce that for \( \theta > 0 \),

\[
g_{y_t}(y) = \int_0^\infty e^{-\theta} E \left[ \mathbb{E}' \left[ (\Phi_{y_t}(B^H), \Phi_{y_t}(B^H))_{\mathcal{H}} \right] |y_t - \mathbb{E} y_t = y \right] d\theta
\]

\[
= \varrho \int_0^\infty e^{-\theta} E \left[ u_x(t, \eta_t)
\right.
\]

\[
\mathbb{E}' u_x \left( t, \left( 1 - e^{-\theta} \right) \left( \eta_0 + \int_0^t b_s ds \right) + e^{-\theta} \eta_t + \sqrt{1 - e^{-2\theta}} \int_0^t \sigma_s dB_s^H \right) \left| \eta_t = u^{-1}(t, y + \mathbb{E} y_t) \right. \right] d\theta
\]

\[
= \varrho u_x(t, u^{-1}(t, y + \mathbb{E} y_t)) \cdot \int_0^\infty e^{-\theta} \int_\mathbb{R} u_x \left( t, \left( 1 - e^{-\theta} \right) \left( \eta_0 + \int_0^t b_s ds \right) + e^{-\theta} u^{-1}(t, y + \mathbb{E} y_t) + \sqrt{1 - e^{-2\theta}} z \right) \rho_{y_t}(z) dz d\theta.
\]

(3.3)

With the relation (3.3) in hand, we shall follow the strategy designed in [21] to obtain upper and lower bounds for \( g_{y_t} \).

Upper bound. Since \( u(t,\cdot) \in (0,\infty) \), we know from [21] Lemma 5.2 that \( u^{-1}(t,\cdot) \in (0,\infty) \). Taking into account the definitions of \( u_x(t,\cdot) \) and \( u^{-1}(t,\cdot) \), we have for each \( \epsilon > 0 \)

\[
(0 <) u_x(t, z) \leq C_{\epsilon,t}(1 + |z|^{u_x(t,z) + \epsilon}), \quad \forall z \in \mathbb{R},
\]

(3.4)

and for each \( \delta > 0 \)

\[
|u^{-1}(t, z)| \leq C_{\delta,t}(1 + |z|^{-u^{-1}(t,z) + \delta}), \quad \forall z \in \mathbb{R},
\]

(3.5)

where \( C_{\epsilon,t} \) and \( C_{\delta,t} \) are both constants depending on \( \epsilon, t \) and \( \delta, t \) respectively. For the convenience of the notation, we let \( \bar{c} = u_x(t,\cdot) + \epsilon \) and \( \bar{\delta} = u^{-1}(t,\cdot) + \delta \). Then
plugging the inequalities (3.4) and (3.5) and resorting to the $C_r$-inequality, we have

\[
g_{yt}(y) 
\leq \frac{\varrho_t}{L^2(1 + |u^{-1}(t, y + \mathbb{E}y_t)|^\lambda)} \cdot \int_0^\infty e^{-\theta} \int_{\mathbb{R}} 1 + \left(1 - e^{-\theta}\right) \eta_0 + \frac{\varrho_t}{L^2(1 + |u^{-1}(t, y + \mathbb{E}y_t)|^\lambda)} d^\lambda\sqrt{1 - e^{-2\varrho t}} \cdot \frac{1}{L^2(1 + |u^{-1}(t, y + \mathbb{E}y_t)|^\lambda)} \cdot \left(\left|\eta_0 + \int_0^t b_s ds\right| + \frac{C_{\delta,t}^{\frac{3}{2}(\epsilon-1)+}}{1 + \epsilon} \left|u^{-1}(t, y + \mathbb{E}y_t)\right|^\epsilon + \frac{C_{\delta,t}^{3(\epsilon-1)+}}{1 + \epsilon} \left|u^{-1}(t, y + \mathbb{E}y_t)\right|^\epsilon + \frac{\Gamma(\frac{1 + \epsilon}{2})}{\sqrt{\pi}} \vartheta_t^{1 + \epsilon} \right) \cdot \left(\left|\eta_0 + \int_0^t b_s ds\right| + \frac{C_{\delta,t}^{3(\epsilon-1)+}}{1 + \epsilon} \left|u^{-1}(t, y + \mathbb{E}y_t)\right|^\epsilon + \frac{C_{\delta,t}^{3(\epsilon-1)+}}{1 + \epsilon} \left|u^{-1}(t, y + \mathbb{E}y_t)\right|^\epsilon + \frac{\Gamma(\frac{1 + \epsilon}{2})}{\sqrt{\pi}} \vartheta_t^{1 + \epsilon} \right) \leq C(\epsilon, \delta)\varrho_t \left(1 + |y + \mathbb{E}y_t|^\epsilon \vartheta_t^{1 + \epsilon} + \vartheta_t^{1 + \epsilon} \right)\tag{3.6}
\]

where

\[
C(\epsilon, \delta) = \sup_{t \in [0, T]} \left[ C_{\delta,t}^{3(\epsilon-1)+} \left(1 + \left|y + \mathbb{E}y_t\right|^\epsilon \right) \left(1 + \left|y + \mathbb{E}y_t\right|^\epsilon + \frac{C_{\delta,t}^{3(\epsilon-1)+}}{1 + \epsilon} \right) \right].
\]

**Lower bound.** Due to the condition on $u_x(t, \cdot)$, the $C_r$-inequality and (3.5), we obtain

\[
g_{yt}(y) \geq \frac{\varrho_t}{L^2(1 + |u^{-1}(t, y + \mathbb{E}y_t)|^\lambda)} \cdot \int_0^\infty e^{-\theta} \int_{\mathbb{R}} 1 + \left(1 - e^{-\theta}\right) \eta_0 + \frac{\varrho_t}{L^2(1 + |u^{-1}(t, y + \mathbb{E}y_t)|^\lambda)} d^\lambda\sqrt{1 - e^{-2\varrho t}} \cdot \frac{1}{L^2(1 + |u^{-1}(t, y + \mathbb{E}y_t)|^\lambda)} \cdot \left(\left|\eta_0 + \int_0^t b_s ds\right| + \frac{C_{\delta,t}^{3(\epsilon-1)+}}{1 + \epsilon} \left|u^{-1}(t, y + \mathbb{E}y_t)\right|^\epsilon + \frac{C_{\delta,t}^{3(\epsilon-1)+}}{1 + \epsilon} \left|u^{-1}(t, y + \mathbb{E}y_t)\right|^\epsilon + \frac{\Gamma(\frac{1 + \epsilon}{2})}{\sqrt{\pi}} \vartheta_t^{1 + \epsilon} \right) \cdot \left(\left|\eta_0 + \int_0^t b_s ds\right| + \frac{C_{\delta,t}^{3(\epsilon-1)+}}{1 + \epsilon} \left|u^{-1}(t, y + \mathbb{E}y_t)\right|^\epsilon + \frac{C_{\delta,t}^{3(\epsilon-1)+}}{1 + \epsilon} \left|u^{-1}(t, y + \mathbb{E}y_t)\right|^\epsilon + \frac{\Gamma(\frac{1 + \epsilon}{2})}{\sqrt{\pi}} \vartheta_t^{1 + \epsilon} \right) \leq C(\epsilon, \delta)\varrho_t \left(1 + |y + \mathbb{E}y_t|^\epsilon \vartheta_t^{1 + \epsilon} + \vartheta_t^{1 + \epsilon} \right)\tag{3.6}
\]

where

\[
C(\epsilon, \delta) = \sup_{t \in [0, T]} \left[ C_{\delta,t}^{3(\epsilon-1)+} \left(1 + \left|y + \mathbb{E}y_t\right|^\epsilon \right) \left(1 + \left|y + \mathbb{E}y_t\right|^\epsilon + \frac{C_{\delta,t}^{3(\epsilon-1)+}}{1 + \epsilon} \right) \right].
\]

\[
\frac{1}{3(\lambda-1)+L^2} \cdot \int_{\mathbb{R}} 1 + \left|\eta_0 + \int_0^t b_s ds\right|^\lambda + \left|y + \mathbb{E}y_t\right|^\lambda \left(1 + \left|u^{-1}(t, y + \mathbb{E}y_t)\right|^\lambda + \frac{C_{\delta,t}^{3(\epsilon-1)+}}{1 + \epsilon} \left|u^{-1}(t, y + \mathbb{E}y_t)\right|^\epsilon + \frac{C_{\delta,t}^{3(\epsilon-1)+}}{1 + \epsilon} \left|u^{-1}(t, y + \mathbb{E}y_t)\right|^\epsilon + \frac{\Gamma(\frac{1 + \epsilon}{2})}{\sqrt{\pi}} \vartheta_t^{1 + \epsilon} \right) \cdot \left(\left|\eta_0 + \int_0^t b_s ds\right| + \frac{C_{\delta,t}^{3(\epsilon-1)+}}{1 + \epsilon} \left|u^{-1}(t, y + \mathbb{E}y_t)\right|^\epsilon + \frac{C_{\delta,t}^{3(\epsilon-1)+}}{1 + \epsilon} \left|u^{-1}(t, y + \mathbb{E}y_t)\right|^\epsilon + \frac{\Gamma(\frac{1 + \epsilon}{2})}{\sqrt{\pi}} \vartheta_t^{1 + \epsilon} \right) \leq C(\delta)\varrho_t \left(1 + |y + \mathbb{E}y_t|^\epsilon \vartheta_t^{1 + \epsilon} + \vartheta_t^{1 + \epsilon} \right)\tag{3.7}
\]
where \( \tilde{C}(\delta) = \frac{1}{2^{1+\gamma_1} + 2^L (1+2^{1+\gamma_1} + \gamma_1 C_{\delta,\lambda})} \cdot \sup_{t \in [0,T]} \int_{\mathbb{R}} \frac{p_{y_t}(z)}{1+|y_{t0} + \int_0^t b_s ds|^\lambda + |z|^\lambda} dz. \)

Therefore, applying Proposition 2.1 together with (3.6) and (3.7), we complete the proof. \( \square \)

**Remark 3.2** Let us have a close look at the lower bound of \( \rho_{y_t}. \) In fact, a direct computation shows that

\[
\int_0^{z-E_{y_t}} u \left(1 + |u + E_{y_t}|^{2\lambda\delta}\right) du = \frac{1}{2} |z - E_{y_t}|^2 + \frac{1}{2(1 + \lambda\delta)} \left(|z|^{2(1 + \lambda\delta)} - |E_{y_t}|^{2(1 + \lambda\delta)}\right)
- \frac{|E_{y_t}|}{1 + 2\lambda\delta} \left(\sgn(zE_{y_t})|z|^{1+2\lambda\delta} - |E_{y_t}|^{1+2\lambda}\right) =: \chi(z, E_{y_t}).
\]

Then, it yields the following

\[
\rho_{y_t}(z) \geq \frac{E|y_t - E_{y_t}|}{2C(\epsilon, \delta)\varrho_t} \left(1 + |z|^{\epsilon\delta}\right) \left(1 + |z|^{\lambda\delta} + \varrho_t^{\frac{\delta}{2}}\right) \exp\left(-\frac{\chi(z, E_{y_t})}{\tilde{C}(\delta)\varrho_t}\right).
\]

As for the upper bound in Theorem 3.1, we get the following result.

**Corollary 3.3** Under the assumptions in Theorem 3.1, there exists \( z_0 > 0 \) such that

\[
\rho_{y_t}(z) \leq \frac{E|y_t - E_{y_t}|}{2\tilde{C}(\delta)\varrho_t} \left(1 + |z|^{2\lambda\delta}\right) \exp\left(-\frac{|z - E_{y_t}|^{2(1 - \epsilon\delta)} - |\sgn z \cdot z_0 - E_{y_t}|^{2(1 - \epsilon\delta)}}{4(1 - \epsilon\delta)C(\epsilon, \delta)\varrho_t}\right)
\]

holds for all \( |z| > z_0. \)

**Proof.** We first observe that by Theorem 3.1 our problem can be reduced to show that the following inequality

\[
\int_0^{z-E_{y_t}} \frac{u}{(1 + |u + E_{y_t}|^{\epsilon\delta}) \left(1 + |u + E_{y_t}|^{\lambda\delta} + \varrho_t^{\frac{\delta}{2}}\right)} du \geq \frac{1}{4(1 - \epsilon\delta)} \left(|z - E_{y_t}|^{2(1 - \epsilon\delta)} - |\sgn z \cdot z_0 - E_{y_t}|^{2(1 - \epsilon\delta)}\right)
\]

holds for each \( |z| > z_0. \) In order to prove (3.9), we start by noticing that

\[
\lim_{u \to +\infty} \frac{u}{(1 + |u + E_{y_t}|^{\epsilon\delta}) \left(1 + |u + E_{y_t}|^{\lambda\delta} + \varrho_t^{\frac{\delta}{2}}\right)} = \frac{1}{u |u|^{\epsilon\delta}} = 1,
\]

and then for some sufficiently large \( u_0 > 0, \) we have for \( u \geq u_0 \)

\[
\frac{u}{(1 + |u + E_{y_t}|^{\epsilon\delta}) \left(1 + |u + E_{y_t}|^{\lambda\delta} + \varrho_t^{\frac{\delta}{2}}\right)} \geq \frac{u}{2|u|^{2\epsilon\delta}}.
\]

Consequently, there exists \( z_0 > 0 \) large enough such that (3.9) holds for any \( |z| \geq z_0. \)

Indeed, when \( z \leq -z_0 \) (the choosing of \( z_0 \) depends on the above argument and moreover
satisfies \( z_0 - |E_{yt}| > 0 \), we get

\[
\int_0^{z-E_{yt}} ud\frac{1}{1 + |u + E_{yt}|^{\delta} \left(1 + |u + E_{yt}|^{\delta} + \frac{1}{\delta_t^2}\right)} = \int_0^0 \frac{-udu}{1 + |u + E_{yt}|^{\delta} \left(1 + |u + E_{yt}|^{\delta} + \frac{1}{\delta_t^2}\right)} \\
\geq \int_{z-E_{yt}}^{z_0-E_{yt}} \frac{-udu}{1 + |u + E_{yt}|^{\delta} \left(1 + |u + E_{yt}|^{\delta} + \frac{1}{\delta_t^2}\right)} \\
\geq \int_{z-E_{yt}}^{z_0-E_{yt}} \frac{-udu}{2|u|^{2\delta}} = \frac{1}{4(1 - \epsilon \delta)} \left((z + E_{yt})^{2(1-\epsilon \delta)} - (z_0 + E_{yt})^{2(1-\epsilon \delta)}\right) \\
= \frac{1}{4(1 - \epsilon \delta)} \left(|z - E_{yt}|^{2(1-\epsilon \delta)} - |\text{sgn} z \cdot z_0 - E_{yt}|^{2(1-\epsilon \delta)}\right).
\]

Along the same lines as above, we can easily check the case \( z \geq z_0 \). This completes our proof. \( \square \)

Notice that \( z_t = \sigma_t u_x(t, \eta_t) \) and then \( D^H u z_t = \sigma_t u_x(t, \eta_t) \sigma_t \), following exactly the same line as the proof of Theorem 3.4 then yields a result for \( z_t \), which we state as follows

**Theorem 3.4** Let \( t \in (0, T] \). Suppose that \( 0 < u_x(t, \cdot) < +\infty, u_{xx}(t, \cdot) < +\infty \) and there exist positive constants \( L, \lambda \) satisfying \( u_{xx}(t, \cdot) \geq \frac{1}{L(1 + |.|)} \). Then the law of \( z_t \) has a density \( \rho_{z_t} \), and for any \( \epsilon, \delta > 0 \) there exists positive constants \( C_1 \) and \( C_2 \) such that

\[
\frac{E|z_t - E_{z_t}|}{C_1 \sigma_t^2 \delta_t \lambda (1 + |z|^{\delta} \left(1 + |z|^{\delta} + \frac{1}{\delta_t^2}\right)} \exp \left(-\frac{1}{C_2 \sigma_t^2 \delta_t \lambda (1 + |u + E_{zt}|^{2\lambda \delta} \right) du \leq \rho_{z_t}(z)
\]

\[
\leq \frac{E|z_t - E_{z_t}|}{C_2 \sigma_t^2 \delta_t \lambda (1 + |z|^{\delta} \left(1 + |z|^{\delta} + \frac{1}{\delta_t^2}\right)} \exp \left(-\frac{1}{C_1 \sigma_t^2 \delta_t \lambda (1 + |u + E_{zt}|^{2\lambda \delta} \right) du \leq \rho_{z_t}(z)
\]

Similar to Remark 3.2 and Corollary 3.3, we have the following estimates.

**Corollary 3.5** With the same preamble as in Theorem 3.4, then the density \( \rho_{z_t} \) fulfills the following bounds

\[
\rho_{z_t}(z) \geq \frac{E|z_t - E_{z_t}|}{C_1 \sigma_t^2 \delta_t \lambda (1 + |z|^{\delta} \left(1 + |z|^{\delta} + \frac{1}{\delta_t^2}\right)} \exp \left(-\frac{\chi(z, E_{zt})}{C_2 \sigma_t^2 \delta_t \lambda} \right), \ z \in \mathbb{R}
\]

and

\[
\rho_{z_t}(z) \leq \frac{E|z_t - E_{z_t}|}{C_2 \sigma_t^2 \delta_t \lambda (1 + |z|^{\delta} \left(1 + |z|^{\delta} + \frac{1}{\delta_t^2}\right)} \exp \left(-\frac{|z - E_{zt}|^{2(1-\epsilon \delta)} - |\text{sgn} z \cdot z_0 - E_{zt}|^{2(1-\epsilon \delta)}}{4(1 - \epsilon \delta)C_1 \sigma_t^2 \delta_t \lambda} \right), \ |z| > z_0
\]

with some positive constant \( z_0 \).
Remark 3.6 (i) Comparing to the relevant results on BSDE driven by the standard Brownian motion \((H = \frac{1}{2})\) proved in [21, Theorem 5.6], it is clear to see that our results apply to more general BSDEs since we replace \(B_t^1\) with \(\eta_t = \eta_0 + \int_0^t b_s ds + \int_0^t \sigma_s dB_s^H\) and treat the case of fractional Brownian motion with arbitrary \(H \in (\frac{1}{2}, 1)\) as driving process. Furthermore, an explicit low bound for the density of the solution \(y_t\) without any restriction for the variable \(z\) is shown in our Remark 3.2.

(ii) The advantage of the above method of estimating densities allows us to obtain non-Gaussian type lower and upper bounds. The drawback is that Theorem 3.1 and Theorem 3.4 must be completed by an analysis of the following PDE

\[
\begin{aligned}
&u_t(t, x) + \frac{1}{2} \sigma^2_t u_{xx}(t, x) + b(t) u_x(t, x) + f(t, x, u(t, x), \sigma_t u_x(t, x)) = 0, \\
&u(T, x) = h(x),
\end{aligned}
\]

which is studied in [15]. In the next two sections, we will present the Gaussian type densities estimates results where the only assumptions are those put on the data of BSDEs.

4 Linear BSDEs driven by fractional Brownian motions

In the previous section, we have presented the non-Gaussian type densities estimates for the fractional BSDE (3.1), which are indeed obtained by using the relation between this type of equation and a quasilinear PDE of mixed type. While the BSDE (3.1) is linear, we are able to prove the Gaussian type bounds and the tail probabilities with a direct and simpler method based on the explicit expression of the solutions in terms of the quasi-conditional expectation.

Consider the following linear BSDE

\[
\begin{aligned}
&d y_t = - (\alpha_t + \beta_t y_t + \gamma_t z_t) dt - z_t dB_t^H, \\
y_T = \xi,
\end{aligned}
\]  

(4.1)

where \(\alpha_t, \beta_t, \gamma_t\) are given as continuous and adapted processes.

Notice that BSDE (4.1) admits a unique solution under the condition (4.2) below. More specifically, set \(\varsigma_t := \gamma_t + \int_0^t D_t^H \beta_s ds, B_t^H := B_t^H + \int_0^t \varsigma_s ds\) and

\[
R_t := \exp \left[ - \int_0^t \left( K_H^{-1} \int_0^t \varsigma_r dr \right) (s) dW_s - \frac{1}{2} \int_0^t \left( K_H^{-1} \int_0^t \varsigma_r dr \right)^2 (s) ds \right].
\]

It follows from the Novikov condition, i.e.

\[
\mathbb{E} \exp \left[ \frac{1}{2} \int_0^T \left( K_H^{-1} \int_0^t \varsigma_r dr \right)^2 (s) ds \right] < \infty,
\]  

(4.2)

that \((R_t)_{t \in [0, T]}\) is an exponential martingale and then the Girsanov theorem implies that \((B_t^H)_{t \in [0, T]}\) is a fractional Brownian motion under the probability measure \(Q := R_T d\mathbb{P}\). Let \(\rho_t := \exp \left\{ \int_0^t \beta_s ds \right\}\). Applying the fractional integration by parts formula to \(\rho_t y_t\) yields that

\[
d(\rho_t y_t) = -\alpha_t \rho_t dt - \rho_t z_t dB_t^H.
\]  

(4.3)
Then, there is a unique solution for BSDE (4.1), and moreover
\[ y_t = \rho_t^{-1} \mathbb{E}^Q \left[ \rho_T \xi + \int_t^T \alpha_s \rho_s ds | \mathcal{F}_t \right], \tag{4.4} \]
where $\mathbb{E}$ stands for the quasi-conditional expectation. More details can be found in \cite{17} Theorem 5.1 or \cite{31}.

Next, we want to show the existence of densities for the marginal laws of the solution $(y, z)$ to BSDE (4.2) and then to derive the Gaussian bounds for them via the relation (4.4) and Proposition 2.1. To this end, let $\alpha_t, \beta_t, \gamma_t$ be given continuous and deterministic functions and $\xi = h(\eta_T)$, in which $\eta$ is defined in the previous section. Put
\[ p_t(x) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}, \]
and further denote
\[ P_t g(x) := \int \rho_t(x-y)g(y) dy. \]

We first state the following useful lemma concerning the representation of the quasi-conditional expectation.

**Lemma 4.1** Assume that $g : \mathbb{R} \to \mathbb{R}$ is a measurable function of polynomial growth, then the following holds
\[ \mathbb{E}(g(\eta_T)|\mathcal{F}_t) = \mathbb{E}(\eta_0 + \int_0^T b_s ds + \int_0^T \sigma_s dB_s^H) g(\tilde{\eta}_t). \]

**Proof.** Though the proof is similar to the one proposed in \cite{17} Theorem 3.8, yet we give a justification for the convenience of the reader. For $t \in [0, T]$, let $\tilde{\eta}_t := \eta_0 + \int_0^T b_s ds + \int_0^T \sigma_s dB_s^H$. Applying the Itô formula (see \cite{17} Theorem 2.3 or \cite{22} Corollary 35) to $P_{\|\sigma\|^2 - \|\tilde{\sigma}\|^2} g(\tilde{\eta}_t)$, we get
\[ g(\tilde{\eta}_t) = P_{\|\sigma\|^2} g(\tilde{\eta}_0) + \int_0^t \frac{\partial}{\partial x} P_{\|\sigma\|^2 - \|\tilde{\sigma}\|^2} g(\tilde{\eta}_s) \sigma_s dB_s^H. \tag{4.5} \]
Choosing $t = T$ in the above equation and then taking the quasi-conditional expectation with respect to $\mathcal{F}_t$, we have
\[ \mathbb{E}(g(\tilde{\eta}_T)|\mathcal{F}_t) = P_{\|\sigma\|^2} g(\tilde{\eta}_0) + \int_0^t \frac{\partial}{\partial x} P_{\|\sigma\|^2 - \|\tilde{\sigma}\|^2} g(\tilde{\eta}_s) \sigma_s dB_s^H. \tag{4.6} \]
By the semigroup property of $P_t$, it is easy to verify that, for $0 \leq s \leq t \leq T$,
\[ \frac{\partial}{\partial x} P_{\|\sigma\|^2 - \|\tilde{\sigma}\|^2} g(x) = P_{\|\sigma\|^2 - \|\tilde{\sigma}\|^2} \frac{\partial}{\partial x} P_{\|\sigma\|^2 - \|\tilde{\sigma}\|^2} g(x). \]
Hence, this, together with (4.6) and (4.5), yields the desired result. \hfill \Box

From now on, let us suppose the following
\[(H2) h : \mathbb{R} \to \mathbb{R} \text{ is twice differentiable and } 0 < c \leq h' \leq C, 0 < \tilde{c} \leq h'' \leq \tilde{C}, \text{ where } c, C, \tilde{c} \]
and $\hat{C}$ are constants.

Besides, we set
\[
\vartheta_1(t) := g_t \exp \left[ 2(T - t) \inf_{s \in [0,T]} \beta_s \right]
\]
and
\[
\vartheta_2(t) := g_t \exp \left[ 2(T - t) \sup_{s \in [0,T]} \beta_s \right].
\]

Recall that $g_t$ is defined in Section 3.

We are now in the position to state our main result of this section.

**Theorem 4.2** Assume that (H2) holds. Then, for each $t \in (0, T]$, $y_t$ and $z_t$ possess densities $p_{y_t}$ and $p_{z_t}$, respectively. Moreover, for almost all $x \in \mathbb{R}$, $p_{y_t}$ and $p_{z_t}$ satisfy, respectively, the following bounds
\[
\frac{\mathbb{E}|y_t - E_{y_t}|}{2C^2\vartheta_2(t)} \exp \left( -\frac{(x - E_{y_t})^2}{2c^2\vartheta_1(t)} \right) \leq p_{y_t}(x) \leq \frac{\mathbb{E}|y_t - E_{y_t}|}{2C^2\vartheta_2(t)} \exp \left( -\frac{(x - E_{y_t})^2}{2C^2\vartheta_1(t)} \right),
\]
and
\[
\frac{\mathbb{E}|z_t - E_{z_t}|}{2C^2\vartheta_2(t)\sigma_t^2} \exp \left( -\frac{(x - E_{z_t})^2}{2c^2\vartheta_1(t)\sigma_t^2} \right) \leq p_{z_t}(x) \leq \frac{\mathbb{E}|z_t - E_{z_t}|}{2C^2\vartheta_2(t)\sigma_t^2} \exp \left( -\frac{(x - E_{z_t})^2}{2C^2\vartheta_2(t)\sigma_t^2} \right).
\]

**Proof.** By (4.4), we obtain $y_t = e^{\int_0^t \beta_s ds}E^Q[\xi | F_t] + \rho_t^{-1} \int_0^T \alpha_s d\rho_s$. Hence, we have
\[
D^H_y y_t = e^{\int_0^t \beta_s ds}D^H_u \left( E^Q[\xi | F_t] \right). \tag{4.7}
\]

On the other hand, from Lemma 4.1, we get
\[
E^Q[\xi | F_t] = E^Q[h(\eta_T) | F_t] = E^Q \left[ h \left( \eta_0 + \int_0^T b_s ds - \int_0^T \sigma_s \gamma_s ds + \int_0^T \sigma_s dB^H_s \right) \right] | F_t
\]= P_{||\sigma||^2_2 - ||\sigma||^2_t} h \left( \eta_0 + \int_0^T b_s ds - \int_0^T \sigma_s \gamma_s ds + \int_0^T \sigma_s dB^H_s \right) \\
= P_{||\sigma||^2_2 - ||\sigma||^2_t} h \left( \eta_0 + \int_0^T b_s ds - \int_t^T \sigma_s \gamma_s ds + \int_0^t \sigma_s dB^H_s \right) \tag{4.8}
\]

Then, for $u \in [0, t]$
\[
D^H_u \left( E^Q[\xi | F_t] \right) = \sigma_u P_{||\sigma||^2_2 - ||\sigma||^2_t} h' \left( \eta_0 + \int_0^T b_s ds - \int_t^T \sigma_s \gamma_s ds + \int_0^t \sigma_s dB^H_s \right). \tag{4.9}
\]

This allows us to deduce from (4.7) that, for $u \in [0, t]$
\[
D^H_u y_t = \sigma_u e^{\int_0^T \beta_s ds} P_{||\sigma||^2_2 - ||\sigma||^2_t} h' \left( \eta_0 + \int_0^T b_s ds - \int_t^T \sigma_s \gamma_s ds + \int_0^t \sigma_s dB^H_s \right). \tag{4.10}
\]

Notice that
\[
\Phi_{y_t}(B^H) = D^H y_t = \sigma_e e^{\int_0^T \beta_s ds} P_{||\sigma||^2_2 - ||\sigma||^2_t} h' \left( \eta_0 + \int_0^T b_s ds - \int_t^T \sigma_s \gamma_s ds + \int_0^t \sigma_s dB^H_s \right),
\]
13
On the other hand, by the fractional Clark-Ocone formula (see [14] and [16]) on \( \bar{\mathcal{H}} \) we arrive at the following bound

\[
\Phi_{\eta_t}(B^H) = \Phi_{\eta_t}(e^{-\theta}B^H + \sqrt{1 - e^{-2\theta}}B'^H)
\]

Thus, according to (2.1), we have

\[
\langle \Phi_{\eta_t}(B^H), \Phi_{\eta_t}(B^H) \rangle_{\bar{\mathcal{H}}} = \varphi(t, \theta),
\]

where

\[
k(t, \theta) = e^{2\int_t^T \beta_s \sigma_s d\mathbb{P} [\|\sigma_s\|^2_2 - \|\sigma_s\|^2_t] \frac{h'}{h}} \left( \eta_0 + \int_0^T b_s ds - \int_t^T \sigma_s \gamma_s ds + \int_0^t \sigma_s dB_s^H + \sqrt{1 - e^{-2\theta}} \int_0^t \sigma_s dB'^H_s \right).
\]

Next, we devote to computing \( D^H z_t \) and then estimating \( g_z \). From (4.3), we know

\[
y_0 - \rho_T \xi - \int_0^T \alpha_t \rho_t dt = \int_0^T \rho_t z_t dB_t^H.
\]

On the other hand, by the fractional Clark-Ocone formula (see [14] and [16]) on \( (\bar{\mathcal{H}}, \mathbb{Q}) \), we can write

\[
y_0 - \rho_T \xi - \int_0^T \alpha_t \rho_t dt = \int_0^T \mathbb{E}_Q^\mathbb{H} \left[ D_t^H \left( y_0 - \rho_T \xi - \int_0^T \alpha_t \rho_t dt \right) \right] d\bar{B}_t^H
\]

Then, combining (4.12) with (4.13) yields the following

\[
z_t = -\frac{\rho_T}{\rho_t} \mathbb{E}_Q^\mathbb{H} \left[ D_t^H \xi \right] = -\sigma_t e^{\int_t^T \beta_s \sigma_s d\mathbb{P} [\|\sigma_s\|^2_2 - \|\sigma_s\|^2_t] \frac{h'}{h}} \left( \eta_0 + \int_0^T b_s ds - \int_t^T \sigma_s \gamma_s ds + \int_0^t \sigma_s dB_s^H \right),
\]

Similar to (4.10) and (4.11), we obtain

\[
D^H z_t = -\sigma_t e^{\int_t^T \beta_s \sigma_s d\mathbb{P} [\|\sigma_s\|^2_2 - \|\sigma_s\|^2_t] \frac{h'}{h}} \left( \eta_0 + \int_0^T b_s ds - \int_t^T \sigma_s \gamma_s ds + \int_0^t \sigma_s dB_s^H \right),
\]

and

\[
c^2 \varphi_1(t) \sigma_t^2 \leq g_z \leq c^2 \varphi_2(t) \sigma_t^2.
\]

Finally, applying Proposition (2.1) to (4.11) and (4.15) respectively, we end up with the desired results and the proof is complete. \( \square \)
Corollary 4.4 Suppose (H2). Then there hold, for all \( t \) \( \in [0,T] \),

\[
\theta_u^Q \left( \mathbb{E}[h(\eta_T)|\mathcal{F}_t] \right) = \mathbb{E}[D_u^H h(\eta_T)|\mathcal{F}_t] = \sigma_u^I[0,t](u) \mathbb{E}[h'(\eta_T)|\mathcal{F}_t]
\]

\[
= \sigma_u^I[0,t](u) P_{\|\sigma\|^2_{\mathbb{P}}-\|\sigma\|^2_{\mathbb{Q}}^2} \left( \eta_0 + \int_0^T b_s ds - \int_t^T \sigma_s \gamma_s ds + \int_0^t \sigma_s dB^H_s \right)
\]

where the last equality is similar to (4.8).

Remark 4.3 We can alternatively derive (4.9) in the above proof, by the following

\[
D_u^H \left( \mathbb{E}[h(\eta_T)|\mathcal{F}_t] \right) = \mathbb{E}[D_u^H h(\eta_T)|\mathcal{F}_t] = \sigma_u^I[0,t](u) \mathbb{E}[h'(\eta_T)|\mathcal{F}_t]
\]

\[
= \sigma_u^I[0,t](u) P_{\|\sigma\|^2_{\mathbb{P}}-\|\sigma\|^2_{\mathbb{Q}}^2} \left( \eta_0 + \int_0^T b_s ds - \int_t^T \sigma_s \gamma_s ds + \int_0^t \sigma_s dB^H_s \right)
\]

In view of the proof of Theorem 4.2 one can derive the following tail estimates for the probability laws of \( y_t \) and \( z_t \).

Corollary 4.4 Suppose (H2). Then there hold, for all \( x > 0 \),

\[
\mathbb{P}(y_t - \mathbb{E}y_t \geq x) \leq \exp \left( -\frac{x^2}{2C^2 \varphi_2(t)} \right), \quad \mathbb{P}(y_t - \mathbb{E}y_t \leq -x) \leq \exp \left( -\frac{x^2}{2C^2 \varphi_2(t)} \right), \quad (4.16)
\]

and

\[
\mathbb{P}(z_t - \mathbb{E}z_t \geq x) \leq \exp \left( -\frac{x^2}{2C^2 \varphi_2(t) \sigma_t^2} \right), \quad \mathbb{P}(z_t - \mathbb{E}z_t \leq -x) \leq \exp \left( -\frac{x^2}{2C^2 \varphi_2(t) \sigma_t^2} \right). \quad (4.17)
\]

Proof. Noticing first that \( g_F(x) = g_{F-E_F}(x) \). Then the relation (4.16) follows by (4.11) and Proposition 2.2 with \( a_1 = 0 \) and \( a_2 = C^2 \varphi_2(t) \). (4.17) can be verified similarly. \( \square \)

5 BSDEs driven by Gaussian processes

In this section, we consider the following BSDE driven by a centered Gaussian process \( \{X_t\}_{t \in [0,T]} \)

\[
\begin{align*}
\text{d}y_t &= -f(t, X_t, y_t, z_t) \text{d}V(t) + z_t \text{d}^0 X_t, \\
y_T &= h(X_T),
\end{align*}
\]

where \( \text{Var}X_t = V(t), t \in [0,T] \), is a strictly increasing, continuous function with \( V(0) = 0 \), the stochastic integral is the Wick-Itô integral defined by the \( S \)-transformation and the Wick product.

As stated in [5, Section 4.4], BSDE (5.1) covers a wide class of Gaussian processes as driving processes including fractional Brownian motion with \( H \in (0,1) \) and fractional Wiener integral and so on, wherein a comparison with BSDE (3.1) is also given. Moreover, we notice that Bender [5] shows the existence and uniqueness results for BSDE (5.1) under the Lipschitz or even superquadratic growth conditions via the transfer theorem which can transfer the concerned problems to an auxiliary BSDE driven by a Brownian motion. Recall that the auxiliary BSDE is of the following form

\[
\begin{align*}
\text{d}\tilde{y}_t &= -f(U(t), \tilde{W}_t, \tilde{y}_t, \tilde{z}_t) \text{d}t + \tilde{z}_t \text{d}\tilde{W}_t, \\
\tilde{y}_V(T) &= h(\tilde{W}_V(T)),
\end{align*}
\]

where \( \{\tilde{W}_t\}_{t \in [0,V(T)]} \) is a standard Brownian motion, \( U(t), t \in [0,V(T)] \), is the inverse of \( V \) defined as

\[
U(t) := \inf\{s \geq 0 : V(s) \geq t\}, \quad t \in [0,V(T)].
\]
In this part, we aim to investigate the existence of densities and then derive their Gaussian estimates for the marginal laws of the solution \((y, z)\) to BSDE (5.1). For this, we start by adopting the following set of conditions from \([1, 5]\): (H3)

(i) \( h \in C^2_b(\mathbb{R}) \) and \( \inf_{x \in \mathbb{R}} h''(x) > 0; \)

(ii) \( \mathbb{E} \int_0^T |f(t, X_t, 0, 0)|^2 dV(t) < \infty, \) and for each \( t \in [0, T], f(t, \cdot, \cdot, \cdot) \in C^2_b(\mathbb{R}^3) \) with all positive derivatives.

We state our final main result as follows

**Theorem 5.1** Assume (H3) holds true. Then, for all \( t \in [0, T], \)

(1) \( y_t \) has a density \( \rho_{y_t}, \) and there exist two strictly positive constants \( c_1 < c_2 \) such that for any \( z \in \mathbb{R}, \)

\[
\frac{\mathbb{E}|y_t - \mathbb{E}y_t|}{c_2 V(t)} \exp \left( -\frac{(z - \mathbb{E}y_t)^2}{c_1 V(t)} \right) \leq \rho_{y_t}(z) \leq \frac{\mathbb{E}|y_t - \mathbb{E}y_t|}{c_1 V(t)} \exp \left( -\frac{(z - \mathbb{E}y_t)^2}{c_2 V(t)} \right), \quad (5.3)
\]

(2) if the generator \( f(t, x, y, z) \) has a linear dependence on the \( z \) component, \( z_t \) possesses a density \( \rho_{z_t}, \) and furthermore if \( f \) only depends on the components \((t, y), \) there exist two strictly positive constants \( c_3 < c_4 \) such that for any \( z \in \mathbb{R}, \)

\[
\frac{\mathbb{E}|z_t - \mathbb{E}z_t|}{c_3 V(t)} \exp \left( -\frac{(z - \mathbb{E}z_t)^2}{c_3 V(t)} \right) \leq \rho_{z_t}(z) \leq \frac{\mathbb{E}|z_t - \mathbb{E}z_t|}{c_4 V(t)} \exp \left( -\frac{(z - \mathbb{E}z_t)^2}{c_4 V(t)} \right). \quad (5.4)
\]

**Proof.** By (H3), it follows by \([12, \text{Theorem 4.1}]\) that the auxiliary BSDE (5.2) admits a unique solution \((\bar{y}, \bar{z})\) which has the following representation

\[
\bar{y}_t = \phi(t, \bar{W}_t), \quad \bar{z}_t = \psi(t, \bar{W}_t), \quad t \in [0, V(T)], \quad (5.5)
\]

where both \( \phi, \psi : [0, V(T)] \times \mathbb{R} \to \mathbb{R} \) are deterministic functions.

Then, in light of \([5, \text{Theorem 4.2 and Theorem 4.4}]\) or the transfer theorem asserted in \([5, \text{Theorem 3.1}]\), we conclude that BSDE (5.2) has a unique solution \((y, z)\) which can be written as

\[
y_t = \phi(V(t), X_t), \quad z_t = \psi(V(t), X_t), \quad t \in [0, T]. \quad (5.6)
\]

Therefore, taking into account the fact that \( \text{Var} X_t = V(t) \) we deduce that the law of \( y_t \) (resp. \( z_t \)) is the same as that of \( \bar{y}_{V(t)} \) (resp. \( \bar{z}_{V(t)} \)).

On the other hand, by \([1, \text{Theorem 3.3}]\) it follows that \( \bar{y}_{V(t)} \) possesses a density \( \rho_{\bar{y}_{V(t)}}, \) and moreover there exist some strictly positive constants \( c_1 < c_2 \) such that for all \( z \in \mathbb{R}, \)

\[
\frac{\mathbb{E}|\bar{y}_{V(t)} - \mathbb{E}\bar{y}_{V(t)}|}{c_2 V(t)} \exp \left( -\frac{(z - \mathbb{E}\bar{y}_{V(t)})^2}{c_1 V(t)} \right) \leq \rho_{\bar{y}_{V(t)}}(z) \leq \frac{\mathbb{E}|\bar{y}_{V(t)} - \mathbb{E}\bar{y}_{V(t)}|}{c_1 V(t)} \exp \left( -\frac{(z - \mathbb{E}\bar{y}_{V(t)})^2}{c_2 V(t)} \right), \quad (5.7)
\]

where both \( \phi, \psi : [0, V(T)] \times \mathbb{R} \to \mathbb{R} \) are deterministic functions.
which then yields our first claim.

As for $z_t$, provided that the generator $f$ has a linear dependence on the $z$ component, owing to [1, Theorem 4.3], we conclude that $ar{z}_{V(t)}$ has a law which is absolutely continuous with respect to the Lebesgue measure. Moreover, if $f$ depends only on $(t, y)$ components, then applying [1, Theorem 4.6], we obtain the following Gaussian bounds for the density of $ar{z}_{V(t)}$

$$
\frac{\mathbb{E}|\bar{z}_{V(t)} - \mathbb{E}\bar{z}_{V(t)}|}{c_4 V(t)} \exp \left( - \frac{(z - \mathbb{E}\bar{z}_{V(t)})^2}{c_3 V(t)} \right) \leq \rho_{\bar{z}_{V(t)}}(z)
$$

$$
\leq \frac{\mathbb{E}|\bar{z}_{V(t)} - \mathbb{E}\bar{z}_{V(t)}|}{c_3 V(t)} \exp \left( - \frac{(z - \mathbb{E}\bar{z}_{V(t)})^2}{c_4 V(t)} \right) \leq \mathbb{E}|\bar{z}_{V(t)} - \mathbb{E}\bar{z}_{V(t)}| \exp \left( - \frac{(z - \mathbb{E}\bar{z}_{V(t)})^2}{c_3 V(t)} \right), \tag{5.8}
$$

where $c_3 < c_4$ are two strictly positive constants.

We therefore obtain the other assertion.



\textbf{Remark 5.2} (i) If we choose Brownian motion as the driving Gaussian process, namely $X_t = B^+_t$, then our estimates (5.3) and (5.4) coincide with the estimates of [1, Theorem 3.3 and Theorem 4.6]. Note that the equation (5.1) that we considered is more general than that of [1] since we allow $X$ to be a wide class of Gaussian processes which includes fractional Brownian motion with arbitrary $H \in (0, 1)$ as special case. Therefore, the results stated in Theorem 5.1 cover that of [1].

(ii) If we take $X_t = \eta_t$, then the equation (5.1) is of the same form as (3.1) and (4.1), which we considered in Section 3 and 4, respectively. By simple calculus we know that the result of Theorem 3.1 combined with Remark 3.2 and Corollary 3.3 is more elaborate than that of Theorem 5.1. Similarly, we also note that the estimates in Theorem 4.2 are better than the estimates (5.3) and (5.4) in Theorem 5.1. Indeed, the derivation of Theorem 5.1 is mainly based on the transfer theorem which is a time change type transformation allowing us to represent the equation (5.1) in terms of a class of BSDE driven by Brownian motion, while the arguments used in Section 3 and 4 focus on the structures of the original equations.

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\textbf{References}

[1] O. Aboura and S. Bourguin, \textit{Density estimates for solutions to one dimensional backward SDE’s}, Potential Anal. 38 (2013), 573–587.

[2] E. Alòs, O. Mazet and D. Nualart, \textit{Stochastic calculus with respect to Gaussian processes}, Ann. Probab. 29 (2001), 766–801.

[3] F. Antonelli and A. Kohatsu-Higa, \textit{Densities of one-dimensional backward SDEs}, Potential Anal. 22 (2005), 263–287.

[4] C. Bender, \textit{Explicit solutions of a class of linear fractional BSDEs}, Systems Control Lett. 54 (2005), 671–680.
[5] C. Bender, *Backward SDEs driven by Gaussian processes*, Stochastic Process. Appl. 124 (2014), 2892–2916.

[6] F. Biagini, Y. Hu, B. Øksendal and A. Sulem, *A stochastic maximum principle for processes driven by fractional Brownian motion*, Stochastic Process. Appl. 100 (2002), 233–253.

[7] F. Biagini, Y. Hu, B. Øksendal and T. Zhang, *Stochastic Calculus for Fractional Brownian Motion and Applications*, Springer-Verlag, London, 2008.

[8] A. Debussche and M. Romito, *Existence of densities for the 3d Navier-Stokes equations driven by Gaussian noise*, Probab. Theory Related Fields 158 (2014), 575–596.

[9] L. Decreusefond and A. S. Üstünel, *Stochastic analysis of the fractional Brownian motion*, Potential Anal. 10 (1998), 177–214.

[10] F. Delarue, S. Menozzi and E. Nualart, *The Landau equation for Maxwellian molecules and the Brownian motion on SO_N(\mathbb{R})*, Electron. J. Probab. 20 (2014), 1–39.

[11] F. Delbaen, Y. Hu and X. Bao, *Backward SDEs with superquadratic growth*, Probab. Theory Related Fields 150 (2010), 145–192.

[12] N. El Karoui, S. Peng and M. C. Quenez, *Backward stochastic differential equations in finance*, Math. Finance 7 (1997), 1–71.

[13] W. Fei, D. Xia and S. Zhang, *Solutions to BSDEs driven by both standard and fractional Brownian motions*, Acta Math. Appl. Sin. Engl. Ser. 29 (2013), 329–354.

[14] Y. Hu, *Integral tranformations and anticipative calculus for fractional Brownian motions*, Mem. Amer. Math. Soc. 175 (2005), no. 825.

[15] Y. Hu, D. Ocone and J. Song, *Some results on backward stochastic differential equations driven by fractional Brownian motions*, in: T. Zhang, X.Y. Zhou (Eds.), Stochastic Analysis and Applications to Finance, World Sci. Publ., Hackensack, NJ, 2012, 225-242.

[16] Y. Hu and B. Øksendal, *Fractional white noise calculus and applications to finance*, Infinite Dim. Anal. Quantum Probab. Related Topics 6 (2003), 1–32.

[17] Y. Hu and S. Peng, *Backward stochastic differential equation driven by fractional Brownian motion*, SIAM J. Control Optim. 48 (2009), 1675–1700.

[18] M. Kobylanski, *Backward stochastic differential equations and partial differential equations with quadratic growth*, Ann. Probab. 28 (2000), 558–602.

[19] J. Ma, P. Protter and J. Yong *Solving forward-backward stochastic differential equations explicitly—a four step scheme*, Probab. Theory Related Fields 98 (1994), 339–359.

[20] T. Mastrolia, *Density analysis of non-Markovian BSDEs and applications to biology and finance*, Stochastic Process. Appl. 128 (2018), 897–938.

[21] T. Mastrolia, D. Possamaï and A. Réveillac, *Density analysis of BSDEs*, Ann. Probab. 44 (2016), 2817–2857.

[22] L. Maticic and T. Nie, *Fractional backward stochastic differential equations and fractional backward variational inequalities*, J. Theor. Probab. 28 (2015), 337–395.
[23] A. Millet and M. Sanz-Solé, *A stochastic wave equation in two space dimension: smoothness of the law*, Ann. Probab. 27 (1999), 803–844.

[24] C. Mueller and D. Nualart, *Regularity of the density for the stochastic heat equation*, Electron. J. Probab. 13 (2008), 2248–2258.

[25] T. D. Nguyen, N. Privault and G. L. Torrisi, *Gaussian estimates for the solutions of some one-dimensional stochastic equations*, Potential Anal. 43 (2015), 289–311.

[26] I. Nourdin and F. G. Viens, *Density formula and concentration inequalities with Malliavin calculus*, Electron. J. Probab. 14 (2009), 2287–2309.

[27] D. Nualart, *The Malliavin Calculus and Related Topics*, Second edition, Springer-Verlag, Berlin, 2006.

[28] E. Nualart and L. Quer-Sardanyons, *Gaussian estimates for the density of the nonlinear stochastic heat equation in any space dimension*, Stochastic Process. Appl. 122 (2012), 418–447.

[29] E. Pardoux and S. Peng, *Adapted solution of a backward stochastic differential equation*, Systems Control Lett. 14 (1990), 55–61.

[30] A. Richou, *Markovian quadratic and superquadratic BSDEs with an unbounded terminal condition*, Stochastic Process. Appl. 122 (2012), 3173–3208.

[31] H. Zhang, *Properties of solution of fractional backward stochastic differential equation*, Applied Mathematics and Computation 228 (2014), 446–453.

[32] J. Zhang, *Backward Stochastic Differential Equations – From Linear to Fully Nonlinear Theory*, Probability Theory and Stochastic Modelling 86, Springer-Verlag, New York, 2017.