An integral formula for the powered sum of the independent, identically and normally distributed random variables

Tamio Koyama

Abstract

In this paper, we show that an inversion formula for probability measures on the real line holds in a sense of the theory of hyperfunctions. As an application of our inversion formula, we give an integral representation for the distribution function of the powered sum of the independent, identically and normally distributed random variables.

1 Introduction

Let \( \{X_i \}_{i \in \mathbb{N}} \) be a sequence of the independent, identically distributed random variables, and suppose the distribution of each variable \( X_i \) is the standard normal distribution. Fix a positive integer \( d \) and \( n \). In this paper, we give a formula for the distribution function of the powered sum \( \sum_{i=1}^{n} X_i^d \). This distribution function can be written in the form of \( n \)-dimensional integral:

\[
F(c) := P \left( \sum_{i=1}^{n} X_i^d < c \right) = \int_{\sum_{i=1}^{n} x_i^d < c} \frac{1}{(2\pi)^{n/2}} \exp \left( -\frac{1}{2} \sum_{i=1}^{n} x_i^2 \right) \, dx_1 \ldots dx_n
\]

When \( d = 2 \), this integral equals to the distribution function of the chi-squared statistics. In [1] and [7], numerical analysis of the cumulative distribution function of the weighted sum \( \sum_{i=1}^{n} w_i X_i^2 \) (\( w_i > 0 \)) of the squared normal variables are discussed as a generalization of the chi-square random variable.

In [8], Marumo, Oaku, and Takemura proposed a method evaluating the probability density function \( f(c) := F'(c) \) in the case where \( d = 3 \). This method is based on the holonomic gradient method ([9], Chapter 6 of [2]). In [8], they showed that \( f(c) \) is a holonomic function. They also gave an explicit form of a differential equation for \( f(c) \). For the probability density function \( f(c) \), we have a formal equation

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \varphi(t)^n \, dt,
\]

where \( \varphi(t) := E \left( \exp \left( \sqrt{-1}t X_i^d \right) \right) \) is the characteristic function of \( X_i^d \). Note that it is not certain whether the integral in the right-hand side of (1) is convergent or
not. Actuarily, it is hard to prove the convergence of the integral by utilizing the Laplace approximation. The discussion in [5] does not show the convergence of (1) and treats equation (1) by the theory of Schwartz distributions.

Let us give an intuitive discussion on the formal equation (1). Dividing the domain of the formal integral in the right-hand side of (1), it can be written as

\[ \frac{1}{2\pi} \int_{-\infty}^{0} e^{-itx} \varphi(t) dt + \frac{1}{2\pi} \int_{0}^{\infty} e^{-itx} \varphi(t) dt. \]  

Note that the integral in the first term converges when \( \Im x > 0 \) holds, the second term converges when \( \Im x < 0 \). Hence, the first and the second term define holomorphic functions on the upper half plane and the lower half plane respectively. Since these holomorphic functions have analytic continuation, the expression (2) make a sense for \( x \in \mathbb{R} \backslash \{0\} \). Our discussion utilizing theory of Fourier hyperfunctions justifies this intuitive explanation. Theory of hyperfunction introduced in [11], [6] has many applications to linear partial differential equations (see, e.g., [3]). Applications of hyperfunctions to numerical analysis are discussed in [4] and [10]. Our discussion in this paper is an application of the theory of hyperfunctions to statistics.

The organization of this paper is as follows. In Section 2 we briefly review the theory of hyperfunctions of a single variable based on [11] and [5]. In Section 3 we give an inversion formula for probability distributions on the real line. In Section 4 we apply the inversion formula given in Section 3 to the powered sum of the independent, identically and normally distributed random variable. As a result, we have a boundary value representation of the probability density function of the powered sum. In Section 5 we calculate the limit in the boundary value representation given in Section 4 and show the main theorem of this paper.

2 Hyperfunctions of a Single Variable

In this section, we briefly review the theory of hyperfunctions of a single variable based on [11] and [5]. Let \( \mathcal{O} \) be the sheaf of holomorphic functions on the complex plane \( \mathbb{C} \). The sheaf of the hyperfunctions \( \mathcal{B} \) of a single variable is defined as the 1-th derived sheaf \( \mathcal{H}^1_{\mathbb{R}}(\mathcal{O}) \) of \( \mathcal{O} \). The global section \( \mathcal{B}(\mathbb{R}) \) is the inductive limit \( \lim_{\leftarrow U \supset \mathbb{R}} \mathcal{O}(U \backslash \mathbb{R})/\mathcal{O}(U) \) with respect to the family of complex neighborhoods \( U \supset \mathbb{R} \). The elements of \( \mathcal{B}(\mathbb{R}) \) are called hyperfunctions on \( \mathbb{R} \). Any hyperfunction on \( \mathbb{R} \) can be represented as an equivalent class \([F(z)]\) with a representative \( F(z) \in \mathcal{O}(U \backslash \mathbb{R}) \).

Let \( L_{1,\text{loc}}(\mathbb{R}) \) be the spaces of locally integrable functions on \( \mathbb{R} \). There is a natural embedding \( \iota \) from \( L_{1,\text{loc}}(\mathbb{R}) \) to \( \mathcal{B}(\mathbb{R}) \). When an integrable function \( f \in L_1(\mathbb{R}) \) satisfies some condition, we can take a representative \( F(z) \) of \( \iota(f) \) as a holomorphic function on \( \mathbb{C} \backslash \mathbb{R} \).

**Proposition 1** (M. Sato). Let \( f(x) \) be an integrable function on \( \mathbb{R} \) such that the integral \( \int_{-\infty}^{\infty} (x^2 + 1)^{-1} f(x) dx \) is finite. Take a constant \( \alpha \in \mathbb{C} - \mathbb{R} \), and
Then we have $\iota(f) = |F|$.

Proof. See, [11].

We call a measure $\mu$ on $\mathbb{R}$ to be locally integrable when $\mu(K)$ is finite for any compact set $K \subset \mathbb{R}$. We denote by $M_{loc}$ the space of the locally integrable measures on $\mathbb{R}$. Analogous to the case of $L_{1,loc}(\mathbb{R})$, there is a natural embedding $\iota$ from $M_{loc}$ to $\mathcal{B}(\mathbb{R})$, and the following proposition holds:

**Proposition 2** (M. Sato). Let $\mu$ be a measure on $\mathbb{R}$ such that the integral $\int_{\mathbb{R}}(x^2 + 1)^{-1}\mu(dx)$ is finite. Take a constant $\alpha \in \mathbb{C} - \mathbb{R}$, and put

$$F(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \left( \frac{1}{x-z} - \frac{1}{x-\alpha} \right) \mu(dx) \quad (z \in \mathbb{C}\setminus\mathbb{R})$$

Then we have $\iota(\mu) = [F]$.

Let $I$ be a neighborhood of $0 \in \mathbb{R}$, and put $I^* := I \setminus \{0\}$. A function $F(z)$ holomorphic on a tubular domain $\mathbb{R} + \sqrt{-1}I^* := \{z \in \mathbb{C}| \exists z \in I^* \}$ is said to be slowly increasing if for any compact subset $K \subset I^*$ and any $\varepsilon > 0$, there exists $C > 0$ such that

$$|F(z)| \leq Ce^{\varepsilon|\Re z|}$$

uniformly for $z \in \mathbb{R} + \sqrt{-1}K$. A hyperfunction $f$ on $\mathbb{R}$ is said to be slowly increasing if we can take a slowly increasing function $F(z)$ as its representative, i.e., there exists a slowly increasing function $F(z)$ such that $f = [F(z)]$. Put

$$\chi_+(z) := \frac{e^z}{1 + e^z}, \quad \chi_-(z) := \frac{e^{-z}}{1 + e^{-z}}.$$

**Definition 1.** Let a slowly increasing function $F(z)$ on a tubular domain $\mathbb{R} + \sqrt{-1}I^*$ be a representative of a slowly increasing hyperfunction $f \in \mathcal{B}(\mathbb{R})$. Take a positive number $y \in I^*$. We define the Fourier transform of $f = [F(z)]$ as $\mathcal{F}f = [G(w)]$ where

$$G(w) = \begin{cases} \int_{\mathbb{R}} e^{-\sqrt{-1}wz} \chi_-(z)F(z)dz & \text{if } w > 0 \\ -\int_{\mathbb{R}} e^{-\sqrt{-1}wz} \chi_+(z)F(z)dz & \text{if } w < 0 \end{cases}$$

Here, for $y \in \mathbb{R}$ and a function $\varphi(z)$ with a complex variable $z$, we denote the integral $\int_{\mathbb{R}} \varphi(z + \sqrt{-1}y)dz$ by $\int_{\mathbb{R}} \varphi(z)dz$.

For the definition of the Fourier transformation of the hyperfunctions of several variables, see [5]. Our definition is a specialization into the case of a single variable of the general definition in [5].
3 Inversion Formula

Let $X$ be a random variable. The distribution $\mu_X$ of $X$ and the characteristic function $\phi_X(t) = \mathbb{E}(e^{\sqrt{-1}tX})$ of $X$ can be regarded as hyperfunctions. We denote them by $\iota\mu_X$ and $\iota\phi_X$ respectively. In this section, we show the following equation:

$$F(\iota\phi_X) = 2\pi i \mu_X.$$  \hfill (3)

Since the characteristic function can be regarded as the inverse Fourier transformation of the distribution under a suitable condition, the equation (3) is an inversion formula.

In the first, we calculate defining functions of $\iota\mu_X$ and $\iota\phi_X$. Let $\alpha \in \mathbb{C}\setminus\mathbb{R}$ be a constant. Since $\mu_X$ is a probability measure on $\mathbb{R}$, we have

$$\left| \int_{\mathbb{R}} \frac{1}{x^2 + 1} \mu_X(dx) \right| \leq \int_{\mathbb{R}} \frac{1}{|x^2 + 1|} \mu_X(dx) \leq \int_{\mathbb{R}} \mu_X(dx) = 1 < \infty$$

and

$$\left| \int_{\mathbb{R}} \frac{1}{x - \alpha} \mu_X(dx) \right| \leq \int_{\mathbb{R}} \frac{1}{|x - \alpha|} \mu_X(dx) \leq \int_{\mathbb{R}} \frac{1}{|3\alpha|} \mu_X(dx) = \frac{1}{|3\alpha|} < \infty.$$  \hfill (4)

By Proposition 2 we have

$$\iota\mu_X = \left[ \frac{1}{2\pi i} \int_{\mathbb{R}} \left( \frac{1}{x - z} - \frac{1}{x - \alpha} \right) \mu_X(dx) \right] = \left[ \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{1}{x - z} \mu_X(dx) \right] .$$  \hfill (5)

Since the estimation $\left| \int_{-\infty}^{\infty} \phi_X(t)(t^2 + 1)^{-1} dt \right| < \infty$ holds, we can apply Proposition 1 to $\phi_X(t)$. In fact, we have

$$\left| \int_{-\infty}^{\infty} \phi_X(t) \left( \frac{1}{t^2 + 1} \right) dt \right| \leq \int_{-\infty}^{\infty} \left| \phi_X(t) \right| \frac{1}{t^2 + 1} dt \leq \int_{-\infty}^{\infty} \frac{1}{t^2 + 1} dt = \left[ \arctan(t) \right]_{-\infty}^{\infty} = \pi .$$

Hence, we have

$$\iota\phi_X = \left[ \frac{1}{2\pi \sqrt{-1}} \int_{-\infty}^{\infty} \phi_X(t) \left( \frac{1}{t - w} - \frac{1}{t - \alpha} \right) dt \right] .$$

In the second, we calculate the Fourier transformation of $\iota\phi_X$.

**Lemma 1.** For a random variable $X$, the hyperfunction $\iota\phi_X$ corresponding to the characteristic function $\phi_X$ of $X$ is slowly increasing.

**Proof.** For simplicity, we assume $\alpha = \sqrt{-1}$ without loss of generality. The absolute value of the representative of $\iota\phi_X$ satisfies the inequality

$$\left| \frac{1}{2\pi \sqrt{-1}} \int_{-\infty}^{\infty} \phi_X(t) \left( \frac{1}{t - w} - \frac{1}{t - \alpha} \right) dt \right| \leq \frac{|w - \alpha|}{2\pi} \int_{-\infty}^{\infty} \frac{dt}{|t - w||t - \alpha|} .$$  \hfill (5)
Put $c := |\Re w|$, then the inequalities
\[
\begin{align*}
\int_{c+1}^{\infty} \frac{dt}{|t-w||t-\alpha|} &\leq \int_{c+1}^{\infty} \frac{dt}{(t-c)^2} = 1, \\
\int_{-\infty}^{-(c+1)} \frac{dt}{|t-w||t-\alpha|} &\leq \int_{-\infty}^{-c-1} \frac{dt}{(t+c)^2} = 1, \\
\int_{c-1}^{c+1} \frac{dt}{|t-w||t-\alpha|} &\leq \frac{2(c+1)}{|3w|},
\end{align*}
\]
implies that the right hand side of (5) is bounded above by
\[
\frac{|w-\alpha|}{\pi} \left(1 + \frac{|\Re w| + 1}{|3w|}\right).
\]
Hence, $\nu \varphi_X$ is a slowly increasing hyperfunction. \(\square\)

In order to calculate an explicit form of a defining function of the Fourier transformation of $\nu \varphi_X$, we decompose the function $\varphi_X(t)$ as
\[
\varphi_X(t) = (1 - H(t)) \varphi_X(t) + H(t) \varphi_X(t)
\]
where $H(t)$ is the Heaviside function. Embedding the both sides of the above equation into the global section $\mathcal{B}(\mathbb{R})$ of the sheaf of hyperfunctions, we have
\[
\nu \varphi_X = \nu((1 - H) \varphi_X) + \nu(H \varphi_X)
\]
By Proposition 1, defining functions of $\nu((1 - H) \varphi_X)$ and $\nu(H \varphi_X)$ are given by
\[
\begin{align*}
\varphi_1(w) := &\frac{1}{2\pi\sqrt{-1}} \int_{-\infty}^{0} \varphi_X(t) \left(\frac{1}{t-w} - \frac{1}{t-\alpha}\right) dt \\
\varphi_2(w) := &\frac{1}{2\pi\sqrt{-1}} \int_{0}^{\infty} \varphi_X(t) \left(\frac{1}{t-w} - \frac{1}{t-\alpha}\right) dt
\end{align*}
\]
gequation (6) respectively. Then we have
\[
\nu \varphi_X = \lfloor \varphi_1(w) \rfloor + \lfloor \varphi_2(w) \rfloor.
\]
Analogous to Lemma 1, we can show that both $\lfloor \varphi_1(w) \rfloor$ and $\lfloor \varphi_2(w) \rfloor$ are slowly increasing hyperfunctions.

For a function $F(z)$ with a complex variable, we denote by $\int_{-\infty}^{0+} F(z)dz$ the integral along the path described in Figure 1(a). We also denote by $\int_{0}^{0+} F(z)dz$ the integral along the path described in Figure 1(b).

**Lemma 2.** The holomorphic functions
\[
\begin{align*}
\psi_1(z) := &\begin{cases} \int_{-\infty}^{0+} e^{-\sqrt{-1}zw} \varphi_1(w)dw & (z \in \mathbb{H}), \\
0 & (z \in -\mathbb{H}),\end{cases} \\
\psi_2(z) := &\begin{cases} 0 & (z \in \mathbb{H}), \\
\int_{0}^{0+} e^{-\sqrt{-1}zw} \varphi_2(w)dw & (z \in -\mathbb{H}),\end{cases}
\end{align*}
\]
are defining functions of the Fourier transformations of $[\varphi_1(w)]$ and $[\varphi_2(w)]$ respectively, i.e., we have $\mathcal{F}[\varphi_i(w)] = [\psi_i(z)] (i = 1, 2)$.

**Proof.** By Definition 1, a defining function of $\mathcal{F}[\varphi_1(w)]$ is given by

$$\tilde{\psi}_1(z) = \begin{cases} \int_{3w=y} e^{-\sqrt{-1}z w} \chi_-(w) \varphi_1(w) dw - \int_{3w=-y} e^{-\sqrt{-1}z w} \chi_-(w) \varphi_1(w) dw & (3z > 0) \\
- \int_{3w=y} e^{-\sqrt{-1}z w} \chi_+(w) \varphi_1(w) dw + \int_{3w=-y} e^{-\sqrt{-1}z w} \chi_+(w) \varphi_1(w) dw & (3z < 0) \end{cases}.$$  

Since $\varphi_1$ is holomorphic on $C - R_{\leq 0}$, we have

$$\tilde{\psi}_1(z) = \begin{cases} - \int_{-\infty}^{(0+)} e^{-\sqrt{-1}z w} \chi_-(w) \varphi_1(w) dw & (3z > 0) \\
\int_{\infty}^{(0+)} e^{-\sqrt{-1}z w} \chi_+(w) \varphi_1(w) dw & (3z < 0) \end{cases}. $$

Note that the integral $\int_{-\infty}^{(0+)} e^{-\sqrt{-1}z w} \chi_+(w) \varphi_1(w) dw$ defines a holomorphic function on a neighborhood of $R$. Hence,

$$\psi_1(z) = \tilde{\psi}_1(z) - \int_{-\infty}^{(0+)} e^{-\sqrt{-1}z w} \chi_+(w) \varphi_1(w) dw$$

is a defining function of $\mathcal{F}[\varphi_1(w)]$ also.

We can show $\mathcal{F}[\varphi_2(w)] = [\psi_2(z)]$ analogously. \hfill $\square$

**Lemma 3.** Put

$$\psi(z) = \begin{cases} \int_{-\infty}^{0} e^{-\sqrt{-1}z t} \varphi_X(t) dt & (z \in H), \\
- \int_{0}^{\infty} e^{-\sqrt{-1}z t} \varphi_X(t) dt & (z \in -H). \end{cases} $$

Then $\psi(z)$ is a defining function of $\mathcal{F}[\varphi_X]$.
Proof. It is enough to calculate the right hand side of the equation $\mathcal{F} \varphi_X = [\psi_1 + \psi_2]$.

For $z \in -\mathbf{H}$, we have

$$\psi_2(z) = \int_{+\infty}^{(0+)} e^{-\sqrt{-1}tz} \frac{1}{2\pi\sqrt{-1}} \int_{0}^{\infty} \varphi_X(t) \left( \frac{1}{t-w} - \frac{1}{w} \right) dt dw$$

$$= \frac{1}{2\pi\sqrt{-1}} \int_{+\infty}^{(0+)} \int_{0}^{\infty} e^{-\sqrt{-1}tz} \varphi_X(t) \left( \frac{1}{t-w} - \frac{1}{w} \right) dt dw$$

By the assumption $z \in -\mathbf{H}$, we have

$$\int_{+\infty}^{(0+)} \int_{0}^{\infty} \left| e^{-\sqrt{-1}tz} \varphi_X(t) \left( \frac{1}{t-w} - \frac{1}{w} \right) \right| dt dw < \infty.$$ 

By the Fubini-Tonelli theorem and Cauchy’s integral formula, $\psi_2(z)$ equals to

$$\frac{1}{2\pi\sqrt{-1}} \int_{0}^{\infty} \left( e^{-\sqrt{-1}tz} \varphi_X(t) \left( \frac{1}{t-w} - \frac{1}{w} \right) \right) dt$$

$$= -\frac{1}{2\pi\sqrt{-1}} \int_{0}^{\infty} \varphi_X(t) \int_{+\infty}^{(0+)} e^{-\sqrt{-1}tw} \left( \frac{1}{t-w} + \frac{1}{w} \right) dw dt$$

$$= -\int_{0}^{\infty} e^{-\sqrt{-1}zt} \varphi_X(t) dt.$$

For $z \in \mathbf{H}$, we can show $\psi_1(z) = \int_{-\infty}^{0} e^{-\sqrt{-1}tz} \varphi_X(t) dt$ analogously. 

Lemma 4. For $z \in \mathbf{C} \setminus \mathbf{R}$, the following equation holds:

$$\psi(z) = \frac{1}{i} \int_{\mathbf{R}} \frac{1}{x-z} \mu_X(dx). \tag{7}$$

Proof. For $z \in -\mathbf{H}$, we have

$$-\int_{0}^{\infty} e^{-\sqrt{-1}zt} \varphi_X(t) dt = \int_{0}^{\infty} e^{-\sqrt{-1}(z-x)t} \mu_X(dx) dt \tag{8}$$

by the definition of the characteristic function. Since $\Im z$ is negative, we have

$$\int_{0}^{\infty} \int_{\mathbf{R}} e^{-\sqrt{-1}(z-x)t} \mu_X(dx) dt = \int_{0}^{\infty} e^{3zt} dt \leq -\frac{1}{3z} < \infty.$$

By the Fubini-Tonelli theorem, the right hand side of (8) equals to

$$-\int_{\mathbf{R}} \int_{0}^{\infty} e^{-\sqrt{-1}(z-x)t} dt \mu_X(dx)$$

$$= \int_{\mathbf{R}} \frac{1}{\sqrt{-1}(z-x)} \mu_X(dx) = \frac{1}{\sqrt{-1}} \int_{\mathbf{R}} \frac{1}{x-z} \mu_X(dx).$$

For $z \in \mathbf{H}$, we can show the equation (7) analogously. 

\[
\text{7}
\]
Theorem 1. For a random variable $X$, equation (3) holds.

Proof. By Lemma 4 and equation (4), we have

$$F_{\psi X} = [\psi(z)] = \left[ \frac{1}{\sqrt{-1}} \int_{\mathbb{R}} \frac{1}{x-z} \mu_X(dx) \right] = 2\pi i \mu_X.$$

\[ \square \]

4 Powered Sum of the Normal Variables

Fix a positive integer $d \geq 3$. Let $X_1, \ldots, X_n$ be the independent, identically and normally distributed random variables, and put $X := \sum_{k=1}^{n} X_k^d$. We utilize the results in the previous section to give a formula for the density function $f_X$ of $X$. Let $\varphi(t)$ be the characteristic function of $X_1^d$, i.e., put

$$\varphi(t) = \mathbb{E} \left( e^{-\sqrt{-1}tx_1^d} \right) = \int_{\mathbb{R}} e^{\sqrt{-1}tx^d} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx. \quad (9)$$

Then, the characteristic function $\varphi_X$ of the powered sum $X$ equals to $\varphi(t)^n$. By Lemma 4 and Lemma 3 the hyperfunction $\iota \varphi_X$ corresponding to $\varphi_X(t)$ is a Fourier hyperfunction, and its defining function is

$$\psi(z) := \begin{cases} \int_{0}^{\infty} e^{-\sqrt{-1}tx^d} \varphi(t)^n dt & (z \in H), \\ -\int_{0}^{\infty} e^{-\sqrt{-1}tx^d} \varphi(t)^n dt & (z \in -H). \end{cases}$$

By Theorem 1, the corresponding hyperfunction $\iota f_X$ of $f_X$ equals to $(2\pi)^{-1} F_{\iota \varphi_X}$. The boundary-value representation implies

$$f_X(x) = \frac{1}{2\pi} \lim_{y \to 0^+} \left( \int_{-\infty}^{0} e^{-\sqrt{-1}(x+\sqrt{-1}y)t} \varphi(t)^n dt + \int_{0}^{\infty} e^{-\sqrt{-1}(x-\sqrt{-1}y)t} \varphi(t)^n dt \right)$$

for $x \in \mathbb{R}$.

5 Analytic Continuations

In order to calculate the limit in the right hand side of (10), we discuss on the analytic continuations of the each term of (2).

In the first, we consider the characteristic function in (9). The characteristic function can be decomposed as

$$\varphi(t) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \exp \left( \sqrt{-1}tx^d \right) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \exp \left( \sqrt{-1}tx^d \right) dx.$$ 

We are interested in the analytic continuation of each term in the right hand side. In order to calculate them, we consider the following general form of integral:

$$\varphi_\theta(w) := \frac{1}{\sqrt{2\pi}} \int_{\gamma_\theta} \exp \left( \sqrt{-1}wz^d \right) dz \quad (w \in \mathbb{C}), \quad (11)$$
where the integral path $\gamma_\theta$ with parameter $\theta \in \mathbb{R}$ is defined by
\[
\gamma_\theta(t) := te^{\sqrt{-1}\theta} \quad (0 < t < \infty).
\] (12)

Note that we have $\varphi(t) = \varphi_0(t) - \varphi_\pi(t)$ for $t \in \mathbb{R}$.

For $z \in \mathbb{C}$, put
\[
\mathcal{H} := \{zw|w \in \mathcal{H}\}.
\]

Examining the convergence region of the integral (11) for given parameter $\theta \in \mathbb{R}$, we obtain the following lemma:

**Lemma 5.** Suppose that $d \geq 3$. For given parameter $\theta \in \mathbb{R}$, the integral (11) converges on $e^{-\sqrt{-1}td}\mathcal{H}$.

**Proof.** Take any point $w \in e^{-\sqrt{-1}td}\mathcal{H}$. By the straightforward calculation, we have
\[
|\varphi_\theta(w)| \leq \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp \left( \sqrt{-1}we^{-\sqrt{-1}td}t^d - \frac{1}{2}e^{2\sqrt{-1}t^2} \right) e^{\sqrt{-1}t^2} dt
\]

\[
\leq \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp \left( -3(we^{-\sqrt{-1}td})t^d - \frac{\cos(2\theta)}{2} t^2 \right) dt
\]

Since assumption $we^{-\sqrt{-1}td} \in \mathcal{H}$ implies $\Im(we^{-\sqrt{-1}td}) > 0$, there exist $t_0 > 1$ and $\varepsilon > 0$ such that
\[
-3(we^{-\sqrt{-1}td}) - \frac{\cos(2\theta)}{2t^{d-2}} < -\varepsilon
\]
holds for any $t > t_0$. Hence, we have
\[
\int_{t_0}^\infty \exp \left( -3(we^{-\sqrt{-1}td})t^d - \frac{\cos(2\theta)}{2} t^2 \right) dt
\]
\[
\leq \int_{t_0}^\infty \exp (-\varepsilon t^d) dt \leq \int_{t_0}^\infty d \cdot t^d - 1 \exp (-\varepsilon t^d) dt = \frac{\exp(-\varepsilon t_0)}{\varepsilon}
\]

Consequently, the integral (11) converges on $e^{-\sqrt{-1}td}\mathcal{H}$. \(\Box\)

**Lemma 6.** Suppose that $d \geq 3$. If $\cos 2\theta > 0$ holds, then the integral (11) converges on $e^{-\sqrt{-1}td}\overline{\mathcal{H}}$.

**Proof.** For any point $w \in e^{-\sqrt{-1}td}\overline{\mathcal{H}}$, we have $\Im(we^{-\sqrt{-1}td}) \geq 0$, This implies
\[
|\varphi_\theta(w)| \leq \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp \left( -\Im(we^{-\sqrt{-1}td})t^d - \frac{\cos(2\theta)}{2} t^2 \right) dt
\]
\[
\leq \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp \left( -\frac{\cos(2\theta)}{2} t^2 \right) dt < \infty
\]

Example 1. Let $d = 3$. Figures 2 and 3 show the contour of the integral path $\gamma_\theta$ and the region $\exp(-\sqrt{-1}td)\mathcal{H}$ for $\theta = 0$ and $\theta = \pi/9$ respectively.
Lemma 7. If $|\theta - \theta'| < \pi/d$, then $\varphi(\theta) = \varphi(\theta')$ holds for $w \in e^{-\sqrt{-1}\theta d}H \cap e^{-\sqrt{-1}\theta' d}H$.

Proof. Put

$$
\gamma_{1,R}(t) := te^{\sqrt{-1}\theta} \quad (0 \leq t \leq R),
\gamma_{2,R}(t) := te^{\sqrt{-1}\theta'} \quad (0 \leq t \leq R),
\gamma_{3,R}(\theta'') := Re^{-\sqrt{-1}\theta''} \quad (\theta' \leq \theta'' \leq \theta').
$$

By the Cauchy’s integral formula, we have

$$
\int_{\gamma_{1,R}} \Phi dz - \int_{\gamma_{2,R}} \Phi dz + \int_{\gamma_{3,R}} \Phi dz = 0. \quad (13)
$$

Here, we put $\Phi = (2\pi)^{-1/2} \exp \left( \sqrt{-1}wz^d - z^2/2 \right)$. By some calculations, we
have that there exist $M > 0$ and $\varepsilon > 0$ such that

$$\left| \int_{\gamma_3,R} \Phi dz \right| \leq MR \exp \left( -\varepsilon R^d \right).$$

Taking the limit of (13) as $R$ approaches infinity, we have $\varphi_\theta(z) - \varphi_\theta'(z) = 0$. \qed

In the second, we discuss on the analytic continuation of the each term of (12). Let us consider the following integral:

$$\psi(z; \theta, \theta', \phi) := \int_{\gamma_\phi} e^{-\sqrt{-1} w z} (\varphi_\theta(w) - \varphi_\theta'(w))^n dw,$$

where $\theta, \theta', \phi \in \mathbb{R}$, $z \in \mathbb{C}$, and the integral path $\gamma_\phi$ is defined by (12).

**Lemma 8.** Suppose $\Im(e^{\sqrt{-1}(\phi+\theta d)}) \geq 0$ and $\Im(e^{\sqrt{-1}(\phi' d)}) \leq 0$. Then the integral (14) converges for $z \in -\exp(-\sqrt{-1}\phi)H$.

**Proof.** By the binomial expansion, the integral (14) equals to

$$\sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \int_{\gamma_\phi} e^{-\sqrt{-1} w z} \varphi_\theta(w)^k \varphi_\theta'(w)^{n-k} dw.$$

With some calculation, the each term of the above expression can be written as

$$\int_{\gamma_\phi} e^{-\sqrt{-1} w z} \varphi_\theta(w)^k \varphi_\theta'(w)^{n-k} dw = \frac{1}{(2\pi)^{n/2}} \int_0^\infty \int_{\mathbb{R}^n_+} e^{-h_1(z,t)s + h_2(t)} ds dt,$$

where we put $t := (t_1, \ldots, t_n)^\top$, $dt := dt_1 \ldots dt_n$,

$$h_1(z,t) = \sqrt{-1} e^{\sqrt{-1}\phi} \left( z - e^{\sqrt{-1}\theta d} \sum_{\ell=1}^k t_{\ell}^d - e^{\sqrt{-1}\theta' d} \sum_{\ell=k+1}^n t_{\ell}^d \right),$$

$$h_2(t) = -\frac{1}{2} \left( e^{2\sqrt{-1}\theta} \sum_{\ell=1}^k t_{\ell}^2 + e^{2\sqrt{-1}\theta'} \sum_{\ell=k+1}^n t_{\ell}^2 \right).$$

By the Fubini-Tonelli theorem, it is enough to consider the convergence of the following integral:

$$\int_{\mathbb{R}^n_+} \int_0^\infty |\exp(-h_1(z,t)s + h_2(t))| ds dt.$$  \hspace{1cm} (16)

For $z \in -\exp(-\sqrt{-1}\phi)H$, we have $\Re(\sqrt{-1}z e^{\sqrt{-1}\phi}) > 0$. This inequality and the assumption $\Re(-e^{\sqrt{-1}T (\phi+\theta d)}) \geq 0$ and $\Re(-e^{\sqrt{-1}T (\phi' d)}) \geq 0$ imply $\Re h_1(z,t) > 0$. Hence, the integral (16) equals to

$$\int_{\mathbb{R}^n_+} |\exp(h_2(t))| \int_0^\infty \exp(-s \Re h_1(z,t)) ds dt = \int_{\mathbb{R}^n_+} \frac{|\exp(h_2(t))|}{\Re h_1(z,t)} dt.$$ \hspace{1cm} (17)
By the assumptions, the integral (17) is bounded from above by
\[
\frac{1}{\mathcal{R}(\sqrt{-1}e^{\sqrt{-1}\phi})} \int_{\mathbb{R}_0^+} |\exp(h_2(t))| \, dt
\]
\[
= \frac{1}{\mathcal{R}(\sqrt{-1}e^{\sqrt{-1}\phi})} \left( \int_0^\infty e^{-\frac{1}{2}\cos(2\theta)^2} \, dt \right)^k \left( \int_0^\infty e^{-\frac{1}{2}\cos(2\theta')^2} \, dt \right)^{n-k}
\]
\[
= \frac{1}{\mathcal{R}(\sqrt{-1}e^{\sqrt{-1}\phi})} \left( \frac{\pi}{2 \cos(\theta)} \right)^{k/2} \left( \frac{\pi}{2 \cos(\theta')} \right)^{(n-k)/2} < \infty.
\]
Therefore, the integral (14) converges. \qed

Lemma 9. Suppose \( \Im(e^{\sqrt{-1}(\phi_k + \theta d)}) \geq 0 \) and \( \Im(e^{\sqrt{-1}(\phi_k + \theta' d)}) \geq 0 \) for \( k = 1, 2 \). The equation
\[
\psi(z; \theta, \theta', \phi_1) = \psi(z; \theta, \theta', \phi_2)
\]
holds for \( z \in (-\exp(-\sqrt{-1}\phi_1)) \cap (-\exp(-\sqrt{-1}\phi_2)) \).

Proof. Put
\[
\gamma_{1,R}(t) := te^{\sqrt{-1}\phi_1} \quad (0 \leq t \leq R),
\gamma_{2,R}(t) := te^{\sqrt{-1}\phi_2} \quad (0 \leq t \leq R),
\gamma_{3,R}(\phi) := Re^{\sqrt{-1}\phi} \quad (\phi_1 \leq \phi \leq \phi_2).
\]
By the Cauchy’s integral formula, we have
\[
\int_{\gamma_{1,R}} \Phi \, dw - \int_{\gamma_{2,R}} \Phi \, dw + \int_{\gamma_{3,R}} \Phi \, dw = 0, \tag{18}
\]
where we put \( \Phi = e^{-\sqrt{-1}w^2} (\varphi_\theta(w) - \varphi_{\theta'}(w))^n \). By some calculations, we have
\[
\left| \int_{\gamma_{3,R}} \Phi \, dw \right| \leq MR \exp(-R)
\]
for sufficiently large \( M > 0 \). Taking the limit of (18) as \( R \) approaches infinity, we have \( \psi(z; \theta, \theta', \phi_1) - \psi(z; \theta, \theta', \phi_2) = 0 \). \qed

Theorem 2. Let \( \varepsilon > 0 \) be a sufficiently small positive number. When \( d \) is odd, we have
\[
f_X(x) = \begin{cases}
\psi(x; \varepsilon, \pi, -\varepsilon d) - \psi(x; -\varepsilon, \pi, \pi + \varepsilon d) & (x > 0), \\
\psi(x; 0, \pi - \varepsilon, \varepsilon d) - \psi(x; 0, \pi + \varepsilon, \pi - \varepsilon d) & (x < 0).
\end{cases}
\]
When \( d \) is even, we have
\[
f_X(x) = \begin{cases}
\psi(x; \varepsilon, \pi + \varepsilon, -\varepsilon d) - \psi(x; -\varepsilon, \pi - \varepsilon, \pi + \varepsilon d) & (x > 0), \\
\psi(x; 0, \pi, \varepsilon d) - \psi(x; 0, \pi, \pi - \varepsilon d) & (x < 0).
\end{cases}
\]
Proof. When $d$ is odd and $x > 0$, we have

\[
f_X(x) = \frac{1}{2\pi} \lim_{y \to +0} \left( \psi(x - \sqrt{-1}y; 0, \pi, 0) - \psi(x + \sqrt{-1}y; 0, \pi, \pi) \right)
\]

\[
= \frac{1}{2\pi} \lim_{y \to +0} \left( \psi(x - \sqrt{-1}y; \varepsilon, \pi, 0) - \psi(x + \sqrt{-1}y; 0, \pi + \varepsilon, \pi) \right)
\]

\[
= \frac{1}{2\pi} \lim_{y \to +0} \left( \psi(x - \sqrt{-1}y; \varepsilon, \pi, -\varepsilon d) - \psi(x + \sqrt{-1}y; 0, \pi + \varepsilon, \pi + \varepsilon d) \right)
\]

\[
= \frac{1}{2\pi} \left( \psi(x; \varepsilon, \pi, -\varepsilon d) - \psi(x; 0, \pi + \varepsilon, \pi + \varepsilon d) \right)
\]

When $d$ is odd and $x < 0$, we have

\[
f_X(x) = \frac{1}{2\pi} \lim_{y \to +0} \left( \psi(x - \sqrt{-1}y; 0, \pi, 0) - \psi(x + \sqrt{-1}y; 0, \pi, \pi) \right)
\]

\[
= \frac{1}{2\pi} \lim_{y \to +0} \left( \psi(x - \sqrt{-1}y; -\varepsilon, \pi, 0) - \psi(x + \sqrt{-1}y; 0, \pi - \varepsilon, \pi) \right)
\]

\[
= \frac{1}{2\pi} \lim_{y \to +0} \left( \psi(x - \sqrt{-1}y; -\varepsilon, \pi, \varepsilon d) - \psi(x + \sqrt{-1}y; 0, \pi - \varepsilon, \pi - \varepsilon d) \right)
\]

\[
= \frac{1}{2\pi} \left( \psi(x; -\varepsilon, \pi, \varepsilon d) - \psi(x; 0, \pi - \varepsilon, \pi - \varepsilon d) \right)
\]

When $d$ is even and $x > 0$, we have

\[
f_X(x) = \frac{1}{2\pi} \lim_{y \to +0} \left( \psi(x - \sqrt{-1}y; 0, \pi, 0) - \psi(x + \sqrt{-1}y; 0, \pi, \pi) \right)
\]

\[
= \frac{1}{2\pi} \lim_{y \to +0} \left( \psi(x - \sqrt{-1}y; \varepsilon, \pi + \varepsilon, 0) - \psi(x + \sqrt{-1}y; 0, \pi + \varepsilon, \pi) \right)
\]

\[
= \frac{1}{2\pi} \lim_{y \to +0} \left( \psi(x - \sqrt{-1}y; \varepsilon, \pi + \varepsilon, -\varepsilon d) - \psi(x + \sqrt{-1}y; 0, \pi + \varepsilon, \pi + \varepsilon d) \right)
\]

\[
= \frac{1}{2\pi} \left( \psi(x; \varepsilon, \pi + \varepsilon, -\varepsilon d) - \psi(x; 0, \pi + \varepsilon, \pi + \varepsilon d) \right)
\]

When $d$ is odd and $x < 0$, we have

\[
f_X(x) = \frac{1}{2\pi} \lim_{y \to +0} \left( \psi(x - \sqrt{-1}y; 0, \pi, 0) - \psi(x + \sqrt{-1}y; 0, \pi, \pi) \right)
\]

\[
= \frac{1}{2\pi} \lim_{y \to +0} \left( \psi(x - \sqrt{-1}y; -\varepsilon, \pi, 0) - \psi(x + \sqrt{-1}y; 0, \pi - \varepsilon, \pi) \right)
\]

\[
= \frac{1}{2\pi} \lim_{y \to +0} \left( \psi(x - \sqrt{-1}y; -\varepsilon, \pi, \varepsilon d) - \psi(x + \sqrt{-1}y; 0, \pi - \varepsilon, \pi - \varepsilon d) \right)
\]

\[
= \frac{1}{2\pi} \left( \psi(x; -\varepsilon, \pi, \varepsilon d) - \psi(x; 0, \pi - \varepsilon, \pi - \varepsilon d) \right)
\]
References

[1] A. Castaño-Martínez and F. López-Blázquez. Distribution of a sum of weighted noncentral chi-square variables. *TEST*, 14(2):397–415, December 2005.

[2] T. Hibi (ed.). *Gröbner Bases: Statistics and Software System*. Springer, Tokyo, 2013.

[3] U. Graf. *Introduction to Hyperfunctions and Their Integral Transforms*. Birkhäuser Basel, 2010.

[4] I. Imai. *Applied hyperfunction theory*. Kluwer Academic Publishers, 1992.

[5] A. Kaneko. *Introduction to Hyperfunctions*. KTK Scientific Publishers, 1988.

[6] T. Kawai. On the theory of fourier hyperfunctions and its applications to partial differential equations with constant coefficients. *Journal of the Faculty of Science, the University of Tokyo*, 17(3):467–517, December 1970.

[7] T. Koyama and A. Takemura. Holonomic gradient method for distribution function of a weighted sum of noncentral chi-square random variables. *Computational Statistics*, 31(4):1645–1659, December 2016.

[8] N. Marumo, T. Oaku, and A. Takemura. Properties of powers of functions satisfying second-order linear differential equations with applications to statistics. *Japan Journal of Industrial and Applied Mathematics*, 32:553–572, July 2015.

[9] H. Nakayama, K. Nishiyama, M. Noro, K. Ohara, T. Sei, N. Takayama, and A. Takemura. Holonomic gradient descent and its application to the Fisher-Bingham integral. *Advances in Applied Mathematics*, 47:639–658, 2011.

[10] H. Ogata and H. Hirayama. Hyperfunction method for numerical integrations. *Transactions of the Japan Society for Industrial and Applied Mathematics*, 26(1):33–43, 2016. (in Japanese).

[11] M. Sato. Theory of hyperfunctions. *Sūgaku*, 10:1–27, 1958. in Japanese.