Research Article

Infinitely Many Solutions for a Generalized Periodic Boundary Value Problem without the Evenness Assumption

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In this paper, we investigate infinitely many solutions for the generalized periodic boundary value problem

\[
\begin{align*}
-\lambda x'' - B_0(t)x + B_1(t)x &= \lambda V(t,x) \text{ a.e. } t \in [0, 1], \\
x(1) &= Mx(0), x'(1) = Nx'(0),
\end{align*}
\]

(1)

where \( \lambda > 0, \ B_0(t), B_1(t) \in L^{\infty}([0, 1], \mathbb{R}^n) = \{B(t) = (b_{jk})_{n \times n} | b_{jk}(t) = b_{kj}(t), t \in [0, 1], b_{jk}(t) \in L^{\infty}([0, 1])\}, \ M, N \in \text{GL}(n) = \{A = (a_{jk})_{n \times n} | a_{jk} \in \mathbb{R} \text{ and det } (A) \neq 0\} \) with \( M \ N^T = I_n, \ I_n \) is the unit matrix of order \( n, \) \( \nabla V(t,x) \) denotes the gradient of \( V(t,x) \) for \( x \in \mathbb{R}^n, \) and \( V(t,x) \) satisfies \((H_0),\) that is, \( V(t,x) \) is continuously differentiable in \( x \) for a.e. \( t \in [0, 1] \) and measurable in \( t \) for every \( x \in \mathbb{R}^n, \) and there exist \( a(x) \in C(\mathbb{R}^n, \mathbb{R}^+) \) and \( b(t) \in L^1([0, 1], \mathbb{R}^+) \) such that

\[
|V(t,x)| + |\nabla V(t,x)| \leq a(|x|)b(t),
\]

(2)

for all \( x \in \mathbb{R}^n \) and a.e. \( t \in [0, 1], \) where \( \mathbb{R}^+ = [0, +\infty). \)

Note that if \( M = N = I_n, B_0(t) \equiv 0, B_1(t) \) is 1-periodic and \( V(t,x) \) is 1-periodic in \( t; \) then, the solutions of problem (1) are the 1-periodic solutions of second-order Hamiltonian systems.

In his pioneer paper [1] of 1978, Rabinowitz studied for the existence of periodic solutions for Hamiltonian systems via the critical point theory. From then on, with the aid of the critical point theory, the existence of infinitely many periodic solutions for Hamiltonian systems has been extensively investigated in some papers (see [2–17]) and the excellent books (see [18–20]).

For second-order Hamiltonian systems, under various conditions, the authors in [3–5, 7, 8, 14–17] obtained infinitely many periodic solutions under the evenness assumption of \( V(t,x), \) Without the evenness assumption of \( V(t,x), \) the authors in [2, 6, 9–13] also obtained infinitely many periodic solutions for first- (or second-) order Hamiltonian systems under the potential function \( V(t,x) \in C^2(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}). \) In this paper, we are interested in the potential function \( V(t,x) \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), \) and without the evenness assumption. We study the existence of infinitely many solutions of...
Theorem 1. Assume that $\theta_i \subset (B_0, 0), \nu_0 \neq 0, B_i(t)$ is positive definite; i.e., there exists $b_0 > \beta > 0$ such that $\mathcal{B}_0 \left[ \frac{\sqrt{< \beta >}}{\beta} \right] \geq (B_1(t) \times x, x) \geq b \left| x \right|^2$ for all $x \in \mathbb{R}^n$. In addition, assume that $V(t, x)$ satisfies $(H_1)$ and $(H_2)$: 

$$\liminf_{\|x\|_\infty \to \infty} \frac{\int_0^1 \sup_{x \in \mathbb{R}^n} V(t, x) dt}{\|x\|_\infty^2} \geq \frac{1}{2k_0^2 b_0} \sup_{x \in \mathbb{R}^n} \left( \frac{1}{\|x\|_\infty} \right) \limsup_{\|x\|_\infty \to \infty} \frac{\int_0^1 V(t, \dot{x}) dt}{\|x\|_\infty^2},$$

(3)

for all $\bar{x} \in \ker (\lambda - B_0)$, where $(\lambda x)(t) = -x''(t), \quad k_0 = \delta_0 \left( \sup \{1, b \} \right)^{-\left(1/2\right)}$, and $\delta_0$ is a constant of the compact imbedding $Z \to \mathbb{L}_0^\infty$ (or see (30) in Section 2).

Then, for each $\lambda \in (\lambda_1, \lambda_2)$, problem (1) possesses infinitely many solutions, where

$$\lambda_1 = \frac{b_0}{2 \limsup_{\|x\|_\infty \to \infty} \frac{\int_0^1 V(t, \dot{x}) dt}{\|x\|_\infty^2}}$$

(4)

and

$$\lambda_2 = \frac{1}{4k_0^2 \liminf_{\|x\|_\infty \to \infty} \frac{\int_0^1 V(t, x) dt}{\|x\|_\infty^2}}.$$ 

Theorem 2. The conclusion of Theorem 1 still holds if we replace $(H_2)$ with $(H_2')$:

$$\liminf_{\|x\|_\infty \to \infty^*} \frac{\int_0^1 \sup_{x \in \mathbb{R}^n} V(t, x) dt}{\|x\|_\infty^2} \geq \frac{1}{2k_0^2 b_0} \sup_{x \in \mathbb{R}^n} \left( \frac{1}{\|x\|_\infty} \right) \limsup_{\|x\|_\infty \to \infty^*} \frac{\int_0^1 V(t, \dot{x}) dt}{\|x\|_\infty^2}.$$ 

(6)

We postpone the proofs to the next section and turn to applications to second-order Hamiltonian systems. For systematic researches of second-order Hamiltonian systems, we refer to the excellent books (see [18–20]).

As the special case, we consider the periodic solution problem:

$$\begin{aligned}
&-x''(t) + B_1(t)x = \lambda V(t, x) \text{ a.e. } t \in [0, 1], \\
x(1) - x(0) = x'(1) - x'(0) = 0,
\end{aligned}$$

(7)

where $\lambda > 0, B_1(t) \in L^\infty([0, 1], L_1(\mathbb{R}^n))$, and $B_1(t)$ and $V(t, x)$ are 1-periodic. After a simple calculation, we have $\lim \sigma (\lambda) = 0, \quad \lim \nu_1 = 0, \quad \lim \nu_2 = 0, \quad \ker (\lambda) = \mathbb{R}^n, \quad \delta_0 = 1/2\sqrt{3},$ and $\|x\| = \|x\|_\infty$ for all $x \in \mathbb{R}^n$. Therefore, the following corollaries are immediately obtained from Theorems 1 and 2.

Corollary 3. Assume that $B_1(t)$ is positive definite and $V(t, x)$ satisfies $(H_0), (H_1)$, and $(H_2)$:

$$\liminf_{\|x\|_\infty \to \infty} \frac{\int_0^1 \sup_{x \in \mathbb{R}^n} V(t, x) dt}{\|x\|_\infty^2} \geq \frac{1}{2k_0^2 b_0} \sup_{x \in \mathbb{R}^n} \left( \frac{1}{\|x\|_\infty} \right) \limsup_{\|x\|_\infty \to \infty} \frac{\int_0^1 V(t, \dot{x}) dt}{\|x\|_\infty^2},$$

(8)

for all $\bar{x} \in \mathbb{R}^n$, where $k_0 = 1/2\sqrt{3} \left( \sup \{1, b \} \right)^{-1/2}$.

Then, for each $\lambda \in (\lambda_1, \lambda_2)$, problem (7) possesses infinitely many 1-periodic solutions, where $\lambda_1, \lambda_2$ are given by (34).

Corollary 4. The conclusion of Corollary 3 still holds if we replace $(H_2)$ with $(H_2')$:

$$\liminf_{\|x\|_\infty \to \infty^*} \frac{\int_0^1 \sup_{x \in \mathbb{R}^n} V(t, x) dt}{\|x\|_\infty^2} \geq \frac{1}{2k_0^2 b_0} \sup_{x \in \mathbb{R}^n} \left( \frac{1}{\|x\|_\infty} \right) \limsup_{\|x\|_\infty \to \infty^*} \frac{\int_0^1 V(t, \dot{x}) dt}{\|x\|_\infty^2}.$$ 

(9)

Next, an example of problem (7) is given below.

Example 5. Let $B_1(t) \equiv I_n$ and

$$\alpha_m = \frac{m! (m + 2)! - 1}{2 (m + 1)!}, \quad \beta_m = \frac{m! (m + 2)! - 1}{2 (m + 1)!},$$ 

(10)

for every $m \in \mathbb{N} \setminus \{0\} = \mathbb{N}^*$. Define the continuous function $V(t, x): [0, 1] \times \mathbb{R}^n \to \mathbb{R}$ as follows:

$$V(t, x) = (a_0, x) \cdot \sin (2\pi t) + V_1(\|x\|) \cdot |\sin (\pi t)|,$$

(11)

where $a_0 \in \mathbb{R}^n \setminus \{0\}$ and

$$V_1(\|x\|) = \begin{cases} \frac{4(m + 1)! (2m + 1)! (m + 1)!}{4(m + 1)!} \left( \frac{1}{\|x\|} - \frac{m(m + 2)}{2} \sqrt{\frac{1}{4(m + 1)!}} \right) \left( \frac{1}{\|x\|} - \frac{m(m + 2)}{2} \right)^2 + \frac{(m + 1)! (2m + 1)! (m + 1)!}{4(m + 1)!} \sin^2 \left( \frac{m(m + 2)}{2} \right) \sin \left( \frac{m(m + 2)}{2} \right) + \frac{(m + 1)! (2m + 1)! (m + 1)!}{4(m + 1)!} \sin \left( \frac{m(m + 2)}{2} \right) + \frac{(m + 1)! (2m + 1)! (m + 1)!}{4(m + 1)!} \sin \left( \frac{m(m + 2)}{2} \right), & \text{if } |x| \leq |a_0| \left( \sin |\pi t| \right), \\
0, & \text{if } |x| > |a_0| \left( \sin |\pi t| \right) \\
\end{cases}$$

(12)
For $s \geq 0$, by

$$V'_1(s) = \begin{cases} \frac{8((m+1))!}{\pi} \left[ \frac{(m+1)!^2 - (ml)!^2}{(m+1)!^2} \right] \cdot \left( \frac{1}{4(m+1)!^2} - \left( \frac{s - ml(m+2)}{2} \right)^2 \right), & \text{if } s \in \bigcup_{m \in \mathbb{N}} [a_m, b_m], \\ 0, & \text{otherwise,} \end{cases}$$

it is easy to see that the conditions (H$_0$) and (H$_1$) are satisfied. Noticing that

$$\lim_{|x| \to \infty} \frac{|(a_0, x)|}{|x|^2} = 0,$$

$$\lim_{m \to \infty} \frac{V_1(a_m)}{a_m} = 0,$$

$$\lim_{m \to \infty} \frac{V_1(b_m)}{b_m} = 4,$$

we have

$$0 = \liminf_{|x| \to \infty} \frac{\int_0^1 \sup_{|y| \leq |x|} V(t, y) dt}{|x|^2} < \frac{1}{2\gamma_0^2 \beta_0} \limsup_{|x| \to \infty} \frac{\int_0^1 V(t, y) dt}{|x|^2} = \frac{24}{\pi},$$

via a simple computation. This shows that (H$_2$) holds. By Corollary 3, there exist infinitely many 1-periodic solutions for problem (7), for each $\lambda \in (\pi/48, +\infty)$. In particular, by $1 \in (\pi/48, +\infty)$, we can see that the following second-order Hamiltonian systems in $\mathbb{R}^n$

$$\begin{cases} -x'' + x = (\sin(2\pi t))a_0 + \frac{V'_1(|x|)}{|x|} \cdot |\sin(\pi t)| \cdot x, \\ x(1) - x(0) = x'(1) - x'(0) = 0 \end{cases}$$

also have infinitely many 1-periodic solutions.

Remark 6. In Example 5, we discard the evenness assumption of potential function $V(t, x)$, which means that Example 5 does not satisfy the assumptions in [3–5, 7, 8, 14–17]. Noticing that the potential function $V(t, x) \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ in Example 5, we can see that Example 5 does also not satisfy the assumptions in [2, 6, 9–13]. Therefore, our result is also new even in the case of periodic solutions for second-order Hamiltonian systems.

2. Variational Setting and Proof of the Main Result

In this section, we first recall the multiple critical point theorem due to [21, 22] and some conclusions of index theory due to [23, 24], respectively.

Lemma 7 ([21], Theorem 2.1; [22], Theorem 2.5). Let $\Phi$ be a reflexive real Banach space, $\Phi : Z \to \mathbb{R}$ a (strongly) continuous, coercive, sequentially weakly lower semicontinuous and Gâteaux differentiable functional, and $\Psi : Z \to \mathbb{R}$ a sequentially weakly upper semicontinuous and Gâteaux differentiable functional. For all $r > \inf \Phi$, let

$$\varphi(r) = \inf_{r - \Phi(x) < \infty} \sup_{y \in \Phi^{-1}(\Phi(x))} \Psi(y) - \Psi(x),$$

$$\gamma = \liminf_{r \to +\infty} \varphi(r),$$

$$\delta = \limsup_{r \to +\infty} \varphi(r).$$

Then,

(a) For every $r > \inf \Phi$ and every $\lambda \in (0, 1/\varphi(r))$, the restriction of the functional $\Phi - \lambda \Psi$ to $\Phi^{-1}(\Phi(x))$ admits a global minimum, which is a critical point (local minimum) of $\Phi - \lambda \Psi$ in $Z$.

(b) If $\gamma < +\infty$, then, for each $\lambda \in (0, 1/\gamma)$, the following alternative holds: either

(b$_1$) $\Phi - \lambda \Psi$ possesses a global minimum or

(b$_2$) there is a sequence $\{x_n\}$ of critical points (local minima) of $\Phi - \lambda \Psi$ such that $\lim_{n \to +\infty} \Phi(x_n) = +\infty$.

(c) If $\delta < +\infty$, then, for each $\lambda \in (0, 1/\delta)$, the following alternative holds: either

(c$_1$) there is a global minimum of $\Phi$ which is a local minimum of $\Phi - \lambda \Psi$ or

(c$_2$) there is a sequence of pairwise distinct critical points (local minima) of $\Phi - \lambda \Psi$, with $\lim_{n \to +\infty}$.
\( \Phi(x_n) = \inf_x \Phi \) which weakly converges to a global minimum of \( \Phi \).

Index theory in [23] deals with a classification of \( L^\infty([0,1], \mathcal{L}_x(R^n)) \) associated with the following system:

\[
-x'' - B(t)x = 0, \quad (18)
\]

\[
x(1) = Mx(0), \quad x'(1) = Nx'(0), \quad (19)
\]

where \( B(t) \in L^\infty([0,1], \mathcal{L}_x(R^n)) \), \( M \in GL(n) \), and \( MN^T = I_n \).

Let \( X = L^2([0,1], R^n) \). Set \((Ax)(t) = -x''(t) \) with \( D(A) = \{ x \in H^2([0,1], R^n) : x \) satisfies (19)\}. By Section 2.4 in [23], we see that \( A \) is self-adjoint and \( \sigma(A) = \sigma_d(A) \subseteq [0, \infty) \). In particular, if \( M = N = I_n \), then \( \sigma(A) = \sigma_d(A) = (2k + 1)^2 \mathbb{Z} \) for any \( k \in \mathbb{N} \), and if \( M = N = -I_n \), then \( \sigma(A) = \sigma_d(A) = \{ (2k - 1)^2 \mathbb{Z} \) for any \( k \in \mathbb{N} \). By Corollary 2.12 in [25], we know that \( Z = D(|A|^{1/2}) = \{ x \in H^1([0,1], R^n) : x(1) = Mx(0) \} \) is a Hilbert space with the norm \( \|x\|_Z = (\int_0^1 |x'(t)|^2 + |x(t)|^2 \, dt)^{1/2} \) for each \( x \in Z \), and the embeddings \( Z \hookrightarrow L^2 = X \), \( Z \hookrightarrow L^p (p \geq 1) \), and \( Z \hookrightarrow L^\infty \) are compact.

For any \( B(t) \in L^\infty([0,1], \mathcal{L}_x(R^n)) \), we define a bilinear form as follows:

\[
q_B(x, y) = \int_0^1 \left( \left( x', y' \right) - (B(t)x, y) \right) \, dt, \quad \forall x, y \in Z. \quad (20)
\]

**Proposition 8** ([24], Proposition 7.2.1). For any \( B(t) \in L^\infty([0,1], \mathcal{L}_x(R^n)) \), the space \( Z \) has an orthogonal decomposition:

\[
Z = Z^+(B) \oplus Z^0(B) \oplus Z^-(B), \quad (21)
\]

such that \( q_B \) is positive definite, null, and negative definite on \( Z^+(B), Z^0(B), \) and \( Z^-(B) \), respectively. Moreover, \( Z^+(B) \) and \( Z^-(B) \) are finite-dimensional.

**Definition 9** ([23], Definition 2.4.1; [24], Definition 7.1.3). For any \( B(t) \in L^\infty([0,1], \mathcal{L}_x(R^n)) \), we define

\[
\nu_M(B) = \text{dim} \ker(A - B), \quad i_M(B) = \sum_{k \geq 0} \nu_M(B + kI_n). \quad (22)
\]

We call \( \nu_M(B) \) and \( i_M(B) \) the nullity and index of \( B \) with respect to the bilinear form \( q_B(\cdot, \cdot) \), respectively.

**Proposition 10** ([24], Proposition 7.2.2). For any \( B(t) \in L^\infty([0,1], \mathcal{L}_x(R^n)) \), we have \( \nu_M(B) = \text{dim} Z^0(B) \) and \( i_M(B) = \text{dim} Z^-(B) \).

**Proposition 11** ([23], Proposition 2.4.2 (1); [24], Corollary 7.2.2 (i)). For any \( B(t) \in L^\infty([0,1], \mathcal{L}_x(R^n)) \), we have that \( Z^0(B) \) is the solution subspace of systems (18) and (19), and \( \nu_M(B) \in \{ 0, 1, 2, \ldots, 2n \} \).

**Remark 12** ([23], Example 2.4.3; [24], Remark 7.1.3). Let \( a_1 \leq a_2 \leq \ldots \leq a_n \) be the eigenvalues of a constant \( n \times n \) symmetric matrix \( B \). For \( \zeta \in \mathbb{R} \setminus \{ 0 \} \) with \( \zeta_0 = \arccos (2\zeta^-1 + \zeta) \), we have

\[
i_{\zeta_0 + \zeta} = \sum_{k \geq 0} \left\{ \left\lfloor \frac{\zeta_0}{2k + 1} \right\rfloor \right\} a_k + \sum_{k \geq 0} \left\{ \left\lfloor \frac{\zeta_0}{2k - 1} \right\rfloor \right\} a_k, \quad (23)
\]

\[
\nu_{\zeta_0 + \zeta} = \sum_{k \geq 0} \left\{ \left\lfloor \frac{\zeta_0}{2k + 1} \right\rfloor \right\} a_k + \sum_{k \geq 0} \left\{ \left\lfloor \frac{\zeta_0}{2k - 1} \right\rfloor \right\} a_k, \quad (24)
\]

where \( S^\delta \) denotes the number of elements in set \( S \). In particular, formulae (23) and (24) when \( \zeta = 1 \) were given first by Mawhin and Willem in [19].

Next, we establish the variational setting for problem (1). It is known that the operator \( \Lambda - B_0 \) is also self-adjoint and \( \sigma(\Lambda - B_0) = \sigma_d(\Lambda - B_0) \) is bounded from below. Noticing that \( i_M(B_0) = 0 \) and \( \nu_M(B_0) \neq 0 \), by Definition 9 and Proposition 10, we know that the operator \( \Lambda - B_0 \) has a sequence of eigenvalues:

\[
0 = \min \sigma(\Lambda - B_0) = \lambda_0 = \lambda_1 = \ldots \lambda_{n_0} < \lambda_{n_0 + 1} \leq \lambda_{n_0 + 2} \leq \ldots \leq \lambda_{n_0 + n} \leq \ldots \rightarrow \infty, \quad (25)
\]

and the system of eigenfunctions \( \{ e_n : n \in \mathbb{N} \} \) corresponding to \( \{ \lambda_n : n \in \mathbb{N} \} \) forming an orthogonal basis in \( L^2 = X \). Hence, we can define another inner product:

\[
(x, y) = q_{B_0}(x, y) + \int_0^1 (x(t), y(t)) \, dt, \forall x, y \in Z, \quad (26)
\]

with the corresponding norm \( \|x\| = (q_{B_0}(x, x) + \|x\|^2_{Z^2})^{1/2} \), \( \forall x \in Z \). Clearly, \( \| \cdot \| \) is equivalent to \( \| \cdot \|_Z \). Put

\[
\|x\|_{B_0} = \left( q_{B_0}(x, x) + \int_0^1 (B_1(t)x(t), x(t)) \, dt \right)^{1/2}, \quad \forall x \in Z. \quad (27)
\]

Since \( B_1(t) \) is positive definite, there exist \( b_0 > b > 0 \) such that

\[
\int_0^1 b_0 |x|^2 \, dt \geq \int_0^1 (B_1(t)x, x) \, dt \int_0^1 b |x|^2 \, dt, \quad (28)
\]
for all \( x \in Z \). So, we have
\[
\min \left\{ 1, b_0 \right\} \|x\|_B^2 \leq \|x\|_{B_1}^2 \leq \max \left\{ 1, b_0 \right\} \|x\|_B^2, \tag{29}
\]
for all \( x \in Z \).

Noticing the compactness of the embedding \( Z \rightarrow L^\infty \), from (29), we know that there is an embedded constant \( \delta_0 > 0 \) such that
\[
|x| \leq \|x\|_\infty \leq \delta_0 \|x\|_B, \tag{30}
\]
for all \( x \in Z \), where \( k_0 = \delta_0 (\min \{ 1, b \})^{(-1/2)} \) and \( \|x\|_\infty \) is the norm of \( L^\infty ([0, 1], \mathbb{R}^n) \).

Now, we define
\[
I_\lambda(x) = \frac{\|x\|_B^2}{2} - \lambda \int_0^1 V(t, x)dt, \quad \forall x \in Z. \tag{31}
\]

From the assumption (H4) and Theorem 1.2 in [19], it is easy to verify that \( I_\lambda \in C^1 (Z, R) \) is weakly lower semicontinuous on \( Z \) and \( I'_\lambda \) is weakly continuous with
\[
I'_\lambda(x)y = q_{B_0}(x, y) + \int_0^1 (B_1(t)x, y)dt - \lambda \int_0^1 (V(t, x), y)dt, \tag{32}
\]
for all \( x, y \in Z \). If \( I'_\lambda(x) = 0 \), we easily find that the critical points of \( I \) correspond to the solutions of problem (1) and omit the details.

Finally, we give the proofs of Theorems 1 and 2.

For convenience, put
\[
\Gamma := \liminf_{\|x\|_\infty \to \infty} \frac{\int_0^1 \sup_{\|y\|_{B_1} \leq 1} V(t, x)dt}{\|x\|_\infty^2}, \tag{33}
\]
\[
Y = \limsup_{\|x\|_\infty \to \infty} \frac{\int_0^1 V(t, x)dt}{\|x\|_\infty^2}, \tag{34}
\]
\[
\lambda_1 := \frac{b_0}{2Y}, \quad \lambda_2 := \frac{1}{4k_0^2 \Gamma}. \tag{35}
\]

Since \( V(t, \theta) = 0 \), we have \( \Gamma \geq 0 \). So, if \( \Gamma = 0 \), we put \( \lambda_1 = +\infty \), and if \( \Gamma = +\infty \), we put \( \lambda_1 = 0 \).

**Proof of Theorem 1.** Set that
\[
\Phi(x) = \frac{\|x\|_{B_1}^2}{2}, \tag{36}
\]
\[
\Psi(x) = \int_0^1 V(t, x)dt, \quad \forall x \in Z. \tag{37}
\]

Obviously, \( \Phi \) is (strongly) continuous, coercive, and Gâteaux differentiable, \( \Psi \) is sequentially weakly upper semicontinuous and Gâteaux differentiable, and \( \inf_{x} \Phi = 0 \). On the other hand, the critical points of \( \Phi - \lambda \Psi \) in \( Z \) are the solutions of problem (1).

Take that \( \lambda \in (\lambda_1, \lambda_2) \). Pick a sequence \( \{ \tilde{x}_n \} \subset \ker (\Lambda - B_0) \) such that \( \lim_{n \to +\infty} \|\tilde{x}_n\| = +\infty \) and \( \lim_{n \to +\infty} \int_0^1 \sup_{\|y\|_{B_1} \leq 1} V(t, x)dt/\|\tilde{x}_n\| = \Gamma \). Set \( r_n = \|\tilde{x}_n\|_{B_1}/2k_0 \). By (30), we have \( \|x(t)\| \leq \|\tilde{x}_n\|_{B_1} \) for any \( x \in Z \) with \( \Phi(x) = (\|x\|_{B_1}/2) < r_n \). From definition of \( \varphi(\cdot) \) and (3) of \( H_2 \), we obtain
\[
\varphi(r_n) = \inf_{\|x\|_B \leq 2r_n} \frac{\sup_{\|y\|_{B_1} \leq 1} \int_0^1 V(t, y)dt - \int_0^1 V(t, x)dt}{r_n - \left( \|x\|_{B_1}/2 \right)} \leq \frac{\sup_{\|y\|_{B_1} \leq 1} \int_0^1 V(t, y)dt}{\|\tilde{x}_n\|_{B_1}}, \tag{38}
\]
which shows that
\[
y = \liminf_{n \to +\infty} \varphi(r_n) \leq 2k_0^2 \Gamma < +\infty. \tag{39}
\]

Next, we prove that \( \Phi - \lambda \Psi \) in \( Z \) is unbounded from below.

Again, pick a sequence \( \{ \tilde{y}_n \} \subset \ker (\Lambda - B_0) \) such that \( \lim_{n \to +\infty} \|\tilde{y}_n\| = +\infty \) and
\[
\lim_{n \to +\infty} \frac{\int_0^1 V(t, \tilde{y}_n)dt}{\|\tilde{y}_n\|_{B_1}^2} = Y. \tag{40}
\]

Noticing that \( \{ \tilde{y}_n \} \subset \ker (\Lambda - B_0) \), by Proposition 8, we have
\[
\Phi(\tilde{y}_n) - \lambda \Psi(\tilde{y}_n) = \frac{1}{2} q_{B_0}(\tilde{y}_n, \tilde{y}_n) + \frac{1}{2} \int_0^1 (B_1(t)\tilde{y}_n, \tilde{y}_n)dt - \lambda \int_0^1 V(t, \tilde{y}_n)dt = \frac{1}{2} \int_0^1 (B_1(t)\tilde{y}_n, \tilde{y}_n)dt - \lambda \int_0^1 V(t, \tilde{y}_n)dt. \tag{41}
\]

If \( Y < +\infty \), put \( \varepsilon \in (0, Y - (b_0/2\lambda)) \). By (34), we know that there exists \( N_\varepsilon > 0 \) such that
\[
\int_0^1 V(t, \tilde{y}_n)dt > (Y - \varepsilon)\|\tilde{y}_n\|_{B_1}^2, \quad \forall n > N_\varepsilon. \tag{42}
\]
This implies that
\[
\Phi(\tilde{y}_n) - \lambda\Psi(\tilde{y}_n) \leq \frac{b_0}{2} \int_0^1 |\tilde{y}_n|^2 dt - \lambda \int_0^1 V(t, \tilde{y}_n) dt \\
\leq ||\tilde{y}_n||^2_\infty \left( \frac{b_0}{2} - \lambda(Y - \varepsilon) \right). \tag{41}
\]
Choosing small enough \(\varepsilon,\) from (34) and \(\lambda > \lambda_1,\) we have
\[
\lim_{n \to \infty} \Phi(\tilde{y}_n) - \lambda\Psi(\tilde{y}_n) = -\infty. \tag{42}
\]
If \(Y = +\infty,\) put \(M > (b_0/2\lambda).\) From (38), there exists \(N_M > 0\) such that
\[
\int_0^1 V(t, \tilde{y}_n) dt > M||\tilde{y}_n||^2_\infty, \quad \forall n > N_M. \tag{43}
\]
This shows that
\[
\Phi(\tilde{y}_n) - \lambda\Psi(\tilde{y}_n) \leq ||\tilde{y}_n||^2_\infty \left( \frac{b_0}{2} - \lambda M \right) \longrightarrow -\infty \text{ as } n \longrightarrow +\infty. \tag{44}
\]
Noting that \((\lambda_1, \lambda_2) \in (0, 1/\gamma),\) from (b) of Lemma 7, we know that \(\Phi - \lambda\Psi\) admits a sequence \(\{x_n\}\) of critical points with \(\lim_{n \to \infty} \Phi(x_n) = +\infty.\) The proof is complete.

Furthermore, from (c) of Lemma 7, we can prove Theorem 2 similar to Theorem 1. Here, we omit it.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**Authors’ Contributions**

All authors typed, read, and approved the final manuscript.

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