COMPOSITION SERIES OF ARBITRARY CARDINALITY
IN ABELIAN CATEGORIES

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Abstract. We extend the notion of a composition series in an abelian category to allow the multiset of composition factors to have arbitrary cardinality. We then provide sufficient axioms for the existence of such composition series and the validity of “Jordan–Hölder–Schreier-like” theorems. We give several examples of objects which satisfy these axioms, including pointwise finite-dimensional persistence modules, Prüfer modules, and presheaves. Finally, we show that if an abelian category with a simple object has both arbitrary coproducts and arbitrary products, then it contains an object which both fails to satisfy our axioms and admits at least two composition series with distinct cardinalities.

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1. Introduction

The Jordan–Hölder Theorem (sometimes called the Jordan–Hölder–Schreier Theorem) remains one of the foundational results in the theory of modules. More generally, abelian length categories (in which the Jordan–Hölder Theorem holds for every object) date back to Gabriel [G73] and remain an important object of study to this day. See e.g. [K14, KV18, LL21].

The importance of the Jordan–Hölder Theorem in the study of groups, modules, and abelian categories has also motivated a large volume work devoted to establishing when a “Jordan–Hölder-like theorem” will hold in different contexts. Some recent examples include exact categories [BHT21, E19+] and semimodular lattices [Ro19, P19+]. In both of these examples, the “composition series” in question are assumed to be of finite length, as is the case for the classical Jordan-Hölder Theorem.

In the present paper, we extend the notion of a composition series in an abelian category so that there is no longer any assumption on the cardinality of the chain of subobjects. More precisely, for an object \( X \) in an abelian category \( \mathcal{A} \), we consider sets \( \Delta \) of subobjects of \( X \) which are totally ordered with respect to inclusion. We then consider “successive subfactors” only for those subobjects in \( \Delta \) which contain an immediate successor or predecessor. When each of these subfactors is simple and \( \Delta \) satisfies certain closure conditions with respect to limits and colimits, we call \( \Delta \) a composition series. See Section 3 for precise definitions and examples.

The consideration of infinite composition series goes back to at least 1934. Indeed, in the paper [B34], Birkhoff shows that if \( \Delta \) and \( \Gamma \) are composition series (in the above sense) for some

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group $G$ and are well-ordered, then their successive subfactors are the same up to permutation and isomorphism. The same paper, however, also shows that there is no hope for a fully general “Jordan–Hölder-like theorem” beyond this. Indeed, for distinct primes $p$ and $q$, we have that

$$0 \subset \cdots \subset p^2 \mathbb{Z} \subset p \mathbb{Z} \subset \mathbb{Z} \quad \text{and} \quad 0 \subset \cdots \subset q^2 \mathbb{Z} \subset q \mathbb{Z} \subset \mathbb{Z}$$

are both composition series for $\mathbb{Z}$ (as a $\mathbb{Z}$-module) in the sense that their successive subfactors are simple. On the other hand, the successive subfactors of the first series are all cyclic of order $p$, while those of the second are cyclic of order $q$.

In this paper, we consider a class of objects in an abelian category which we call (weakly) Jordan–Hölder–Schreier (Definition 4.0.4). We then establish a “Jordan–Hölder-like theorem” for this class of objects. This class includes objects of finite length, but is in general much larger. For example, both Prüfer modules over arbitrary rings and pointwise finite-dimensional persistence modules over arbitrary fields are (at least weakly) Jordan–Hölder–Schreier. On the other hand, every abelian category with both arbitrary coproducts and arbitrary products contains an object which is not (weakly) Jordan–Hölder–Schreier. See Section 5 for details about these and additional examples.

1.1. Motivation. Our first motivation comes from the study of pointwise finite-dimensional persistence modules. As defined by Botnan and Crawley-Boevey [BC-B20], a pointwise finite-dimensional persistence module is a functor from a small category $C$ to the category of finite dimensional vector spaces over an algebraically closed field $k$. Pointwise finite-dimensional persistence modules are primarily studied in topological data analysis via persistent homology. One key property in this setting is that, in many applications, the simple persistence modules are in bijection with the objects in the category $C$. This means there is a well-defined concept of support; that is, for any pointwise finite-dimensional persistence module $M$, there is a set $\{x \in C : M(x) \neq 0\}$ that we call the support of $M$. In Section 5.3 we show that if $C$ is acyclic (for example, when $C$ is a poset category), then the composition factors of a pointwise finite-dimensional persistence module $M$ over $C$ are precisely the simple modules $S$ corresponding to the objects in the support of $M$.

Another source of motivation is Prüfer modules. A module $M$ over an arbitrary ring $R$ is called Prüfer if there exists a locally nilpotent surjective endomorphism $\varphi : M \to M$ so that $\ker(\varphi)$ has finite length. It follows that $M = \bigcup_{n \in \mathbb{N}} \ker(\varphi^n)$ is a filtered colimit of finite length modules, but is not itself of finite length.

In the representation theory of finite-dimensional (associative) algebras, Prüfer modules are closely related to generic modules, and themselves contain information about the category of finitely-generated modules. See for example [Ri09]. It is therefore natural to try to include the Prüfer modules when studying modules of finite length. As we show in Section 5.2 this is possible using our generalized notion of a composition series. More precisely, we show that any filtered colimit of finite length modules is Jordan–Hölder–Schreier, and therefore has a unique multiset of composition factors under our definition.

1.2. Organization and Main Results. The organization of this paper is as follows. In Section 2 we recall background information on subobjects, length categories, and (co)limits in functor categories. In Section 3 we define the notion of a bicomplete subobject chain of an object in an abelian category (Definition 3.1.3) and formalize our notion of a composition series as a special case (Definition 3.2.6). We then prove our first main theorem.

Theorem A (Theorem 3.3.7). Let $X$ be an object of a skeletally small abelian category. Suppose that every totally ordered (under inclusion) set $\Delta$ of subobjects of $X$ has a greatest lower bound $\lim \Delta$ and a least upper bound $\text{colim} \Delta$ which are subobjects of $X$. Then there exists a composition series of $X$. Moreover,

1. A totally ordered set $\Delta$ of subobjects of $X$ is a composition series if and only if it is not a proper subset of a larger totally ordered set of subobjects of $X$. 

(2) If $\Delta$ is a totally ordered set of subobjects of $X$, then there exists a composition series $\Delta'$ of $X$ with $\Delta \subseteq \Delta'$.

**Remark 1.2.1.** We note that our notions of bicomplete subobject chains and composition series also come with “one-sided” variants, where we assume only the existence of either greatest lower bounds or least upper bounds of totally ordered sets of subobjects. We maintain, however, that the “two-sided” variant is the correct generalization due to the existence of “one-sided” composition series $\Delta$ and $\Delta'$ of the same object which satisfy $\Delta \subseteq \Delta'$. See Proposition 3.3.6 and Example 3.3.8.

**Remark 1.2.2.** Theorem A only guarantees the existence of a composition series, but does not imply any notion of uniqueness. Indeed, $\mathbb{Z}$ (as a $\mathbb{Z}$-module) admits a composition series under our definition. See Example 3.2.8.

We begin Section 4 by establishing an equivalence relation on (bicomplete) subobject chains, which we call **subfactor equivalence** (Definition 4.0.1). This generalizes the classical equivalence relation making two chains equivalent if they have the same length and successive subfactors (up to isomorphism and permutation). We then introduce **(weakly) Jordan–Hölder–Schreier** objects in an abelian category (Definition 4.0.4). These can be seen as objects where the least upper bound and greatest lower bound of a totally ordered set of subobjects not only exist, but are well-behaved. (As with subobject chains and composition series, there are again “one-sided” variants of these objects where properties are assumed about either least upper bounds or greatest lower bounds.)

Once establishing the definition and basic properties, Section 4.1 is devoted to proving the main result of this paper.

**Theorem B** (Theorem 4.1.14). Let $X$ be a weakly Jordan–Hölder–Schreier object in a skelletally small abelian category. Then there exists a composition series for $X$ and any two composition series for $X$ are subfactor equivalent.

In light of this theorem, the (multi)set of composition factors associated to a (weakly) Jordan–Hölder–Schreier object $X$, denoted $\text{sf}(X)$, is well-defined.

Our proof of Theorem B is similar to that of the classical (and well-ordered) Jordan–Hölder Theorem given by “conflating” two chains of subobjects (see e.g. [L02, Section I.3], [S09+]). In particular, it relies on a generalization of Schreier’s classical “refinement theorem” (see Theorems 2.2.2 and 4.1.13).

In Section 4.2, we discuss the relationship between (weakly) Jordan–Hölder–Schreier objects and their duals. Unlike with objects of finite length, there exist (weakly) Jordan–Hölder–Schreier objects which are not (weakly) Jordan–Hölder–Schreier in their opposite category (see Example 5.2.3). Nevertheless, categories in which every object satisfies both the definition of (weakly) Jordan–Hölder–Schreier and its dual offer a natural generalization of length categories. In particular, we prove the following.

**Theorem C** (Theorem 4.2.7, simplified). Let $A$ be a skelletally small abelian category so that every object is weakly Jordan–Hölder–Schreier in both $A$ and $A^{op}$. Let $0 \rightarrow Y \xrightarrow{f} X \xrightarrow{g} Z \rightarrow 0$ be a short exact sequence in $A$. Then there is an induced bijection (of multisets) $\text{sf}(Y) \sqcup \text{sf}(Z) \cong \text{sf}(X)$. Moreover, if both $Y$ and $Z$ are Jordan–Hölder–Schreier, then $Z = 0$ if and only if the induced inclusion $\text{sf}(Y) \hookrightarrow \text{sf}(X)$ is an isomorphism.

Finally, in Section 5, we give examples and non-examples of (weakly) Jordan–Hölder–Schreier objects. Our examples include objects of finite length (Section 5.1) and functor categories with a length category as a target (Section 5.3). The latter in particular includes categories of pointwise definite-dimensional persistence modules. Furthermore, when the source category is directed (for example, when the functors are presheaves over some topological space), we are actually able to describe the composition factors explicitly. See Sections 2.3 and 5.3 for precise definitions and the more general version of our final main theorem, which we state in simplified form below.
**Theorem D** (Proposition 5.3.1, Theorem 5.3.2, and Corollary 5.3.3 simplified).

1. Let $C$ be a directed small category and let $k$ be an arbitrary field. Then any pointwise finite-dimensional $C$-persistence module $M : C \to \text{vec}(k)$ is Jordan–Hölder–Schreier (in the category of covariant functors from $C$ to that of finite-dimensional $k$-vector spaces). Moreover, for each object $X$ of $C$, the simple module with support at $X$ is a composition factor of $M$ with multiplicity $\dim_k M(X)$.

2. Let $\mathcal{A}$ be a skeletally small abelian category such that every object in $\mathcal{A}$ is Jordan–Hölder–Schreier. Then any presheaf on a topological space $X$ with values in $\mathcal{A}$ is Jordan–Hölder–Schreier (in the category of contravariant functors from the poset of open sets of $X$ to $\mathcal{A}$).

We conclude Section 5 with two additional examples. First, we show that every object of Igusa and Todorov’s category of representations of $\mathcal{R}$ [IT15] are weakly Jordan–Hölder–Schreier, even though this category contains no simple objects (Section 5.4). Finally, we show that every nonzero abelian category which has both arbitrary coproducts and arbitrary products contains an object which is not weakly Jordan–Hölder–Schreier and, if the category contains a simple object, admits two composition series with different cardinalities (Section 5.5).

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2. **Background**

In this section we fix our notation and recall necessary background information. We note that, in any category, limits commute with limits and colimits commute with colimits, when they exist. Moreover, in an abelian category, finite limits commute with finite colimits. Throughout the present paper, when we say "abelian category" we mean "skeletally small abelian category" unless we explicitly state otherwise. We fix such a category $\mathcal{A}$ for the duration of this paper.

2.1. **Subobjects in abelian categories.** We begin by recalling the definition and basic properties of subobjects in an abelian category. A detailed and well-written treatise for this theory can be found in [M06].

Let $X$ be an object in $\mathcal{A}$. We recall that a subobject of $X$ is a pair $(Y, \iota_Y)$ where $Y$ is also an object in $\mathcal{A}$ and $\iota_Y : Y \to X$ is a monomorphism. The subobjects of $X$ form a category, which we denote $\text{Sub}(X)$. Morphisms in this category are given by

$$\text{Hom}_{\text{Sub}(X)}((Y, \iota_Y), (Z, \iota_Z)) = \{ h \in \text{Hom}_{\mathcal{A}}(Y, Z) | \iota_Y = \iota_Z \circ h \}.$$

Necessarily, we see that if $h \in \text{Hom}_{\text{Sub}(X)}((Y, \iota_Y), (Z, \iota_Z))$, then $h$ is a monomorphism in $\mathcal{A}$. Moreover, we have that $|\text{Hom}_{\text{Sub}(X)}((Y, \iota_Y), (Z, \iota_Z))| \leq 1$.

From now on, we will identify $\text{Sub}(X)$ with a skeleton. This gives $\text{Sub}(X)$ the structure of a poset under the relation $(Y, \iota_Y) \sqsubseteq (Z, \iota_Z)$ if $\text{Hom}((Y, \iota_Y), (Z, \iota_Z)) \neq \emptyset$. If we denote this unique map by $f_{Y,Z}$, then we can view $(Y, f_{Y,Z})$ as an object in $\text{Sub}(Z)$. As a result, we will also use the notation $\sqsubseteq$ to mean "is a subobject of".

In the case that $\mathcal{A}$ is a module category, this relation coincides with the usual notion of containment for submodules. Nevertheless, we have chosen to use the notation $\sqsubseteq$ for this relation to avoid confusion with refinements of (pre-)subobject chains, which are actually subsets (see Definitions 3.3.1 and 3.3.2).

We adopt the common notation of omitting the inclusion map from the description of a subobject when this data is implied. We caution that, as in the module case, this means there may be
subobjects $Y$ and $Z$ of $X$ which are isomorphic in $\mathcal{A}$ but which are not isomorphic in $\text{Sub}(X)$. To distinguish this, we will say that $Y = Z$ when they are isomorphic as subobjects of $X$.

The poset $\text{Sub}(X)$ is known to form a lattice. Given $Y, Z \in \text{Sub}(X)$, their least upper bound is denoted $Y + Z$ and their greatest lower bound is denoted $Y \cap Z$. In module categories, these coincide with the usual notions of sums and intersections. Categorically, we have that $Y \cap Z$ and $Y + Z$ are the kernel and image of the morphism $Y \oplus Z \xrightarrow{[Y \to Z]} X$, respectively. In particular, let $j_Y : Z \cap Y \to Y$ and $J_Z : Z \cap Y \to Z$ be the inclusion maps. Then $Y \cap Z$ and $Y + Z$ are the pullback and pushout of the respective diagrams

$$Z \xrightarrow{j_Z} X \xleftarrow{j_Y} Y \quad \text{and} \quad Z \xleftarrow{J_Z} Z \cap Y \xrightarrow{j_Y} Y.$$

If $\mathcal{A}$ has arbitrary coproducts, we can more generally speak of infinite intersections and sums (and in this case, the lattice $\text{Sub}(X)$ is complete); however, we will not in general assume the existence of infinite coproducts.

It should be noted that there is a dual theory for quotient objects in an abelian category $\mathcal{A}$, which can be viewed as the presented theory in $\mathcal{A}^{\text{op}}$. In particular, pushouts become pullbacks and vice-versa. In Section 12 we will have need of both theories simultaneously, and so we introduce the relevant notation here. For an object $X$ and two quotient maps $q_Y : X \to Y$, $q_Z : X \to Z$, we denote the pushout by $Y \amalg_X Z$. This is also the cokernel object of the map $(\ker q_Y + \ker q_Z) \to X$. Finally, we note that if $Y$ is a subobject of $X$, then there is a one-to-one correspondence between subobjects of $Z$ with $Y \subseteq Z$ and subobjects of $X/Y$ which sends $Z$ to $Z/Y$.

2.2. Composition series and length. Let $X$ be an object in $\mathcal{A}$. Consider

$$\Delta = \{0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_{n-1} \subseteq X_n = X\}$$

a finite sequence of subobjects of $X$. We refer to the objects $X_i/X_{i-1}$ for $1 \leq i \leq n$ as the (successive) subfactors of $\Delta$. If all of these subfactors are simple, then $\Delta$ is called a composition series. We say $\Delta$ is subfactor equivalent to another finite chain of subobjects

$$\Gamma = \{0 = Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_{m-1} \subseteq Y_m = X\}$$

if $m = n$ and there exists a permutation $\sigma$ on $\{1, \ldots, n\}$ so that $X_i/X_{i-1} \cong Y_{\sigma(i)}/Y_{\sigma(i)-1}$ for each $i$. This allows us to state the well-known Jordan–Hölder Theorem for abelian categories.

**Theorem 2.2.1** (Jordan–Hölder Theorem). *Let $X$ be an object in $\mathcal{A}$ and let $\Delta$ and $\Gamma$ be composition series of $X$. Then $\Delta$ and $\Gamma$ are subfactor equivalent.***

Based upon this theorem, if $\Delta$ is a composition series of $X$, then the length of $\Delta$ is referred to as the length of $X$ and the subfactors of $\Delta$ are referred to as the composition factors of $X$. The abelian category $\mathcal{A}$ is then called a length category if every object has finite length (and thus well-defined length and composition factors).

Closely related to the Jordan–Hölder Theorem is the Schreier Refinement Theorem, which says that one may essentially conflate the data of two finite chains of subobjects.

**Theorem 2.2.2** (Schreier Refinement Theorem). *Let $X$ be an object in $\mathcal{A}$ and let $\Delta$ and $\Gamma$ be finite filtrations of $X$. Then there exist filtrations $\Delta'$ and $\Gamma'$ of $X$ such that:

1. Each object that appears in $\Delta$ appears in $\Delta'$ and each object that appears in $\Gamma$ appears in $\Gamma'$.
2. The filtrations $\Delta'$ and $\Gamma'$ are (subfactor) equivalent.*

Indeed, it is common to prove the Schreier Refinement Theorem first and obtain the Jordan–Hölder Theorem as a corollary. See e.g. [L02, Section I.3] in the setting of groups. We will adopt a similar strategy in proving our generalization of the Jordan–Hölder Theorem.
2.3. Functor categories. We now discuss categories of functors with target in an abelian category. We again refer to [M06] for more details.

Let $C$ be a small category and recall that $A$ is an abelian category. Then the category $\text{Fun}(C, A)$ of covariant functors from $C$ to $A$ is once again abelian. This is sometimes referred to as the category of $A$-representations of $C$.

We recall that colimits and limits can be computed “pointwise” in functor categories, in the following sense. Let $\mathcal{D} = (\mathcal{F}, \mathcal{N})$ be a diagram in $\text{Fun}(C, A)$ (with set of objects $\mathcal{F}$ and set of morphisms $\mathcal{N}$). For $X$ an object of $A$, define
\[
C(X) := \text{colim}\{M(X)|M \in \mathcal{F}\}, \{\eta_X|\eta \in \mathcal{N}\}
\]
if this colimit exists. If $C(X)$ exists for all objects $X$, then for each morphism $X \xrightarrow{f} Y$ in $C$, there is a natural morphism $C(X) \xrightarrow{C(f)} C(Y)$. This makes $C$ into a functor from $\mathcal{D}$ to $A$, and we have $C = \text{colim} \mathcal{D}$. The limit of $\mathcal{D}$ is computed similarly (when all of the pointwise limits exist).

To conclude this section, we recall known results about certain indecomposable objects in $\text{Fun}(\mathbb{R}, \text{vec}(k))$ and the morphisms between them. These results are due to Crawley-Boevey [C-B15]. Much of the theory also extends to the case where $\mathbb{R}$ is given a partial order in place of its standard order. See [BC-B20, IRT19+]. In Section 5, we will also consider a certain subcategory of $\text{Fun}(\mathbb{R}, \text{vec}(k))$ which was first studied by Igusa and Todorov [IT15].

For $M \in \text{Fun}(\mathbb{R}, \text{vec}(k))$ and $x \leq y$ in $\mathbb{R}$, we denote by $M(x, y)$ the result of applying $M$ to the unique morphism $x \to y$.

For every (open, closed, or half-open) interval $I \subseteq \mathbb{R}$ and for all $x, y \in \mathbb{R}$ with $x \leq y$, denote
\[
M_I(x) = \begin{cases} 
1 & x \in I \\
0 & x \notin I 
\end{cases} 
M_I(x, y) = \begin{cases} 
1 & x \leq y \in I \\
0 & \text{otherwise}. 
\end{cases}
\]
The functor $M_I$ is often referred to as the interval indecomposable representation associated to $I$. Up to isomorphism, these are precisely the indecomposable objects in $\text{Fun}(\mathbb{R}, \text{vec}(k))$. Moreover, for two intervals $I, J$, there is a monomorphism $M_I \to M_J$ in $(\mathbb{R}, \text{vec}(k))$ if and only if (a) $I \subseteq J$, (b) the right endpoint of $I$ is the same as the right endpoint of $J$, and (c) the intervals are either both open on the right or both closed on the right. Moreover, if these conditions are met then this monomorphism is unique up to scalar multiplication.

Finally, let $\Delta$ be a set of intervals in $\mathbb{R}$ which is totally ordered with respect to the relation $I \leq J$ if there is a monomorphism $M_I \to M_J$. Then we have
\[
\sum_{I \in \Delta} M_I = M_{(\bigcup_{I \in \Delta} I)} \quad \bigcap_{I \in \Delta} M_I = M_{(\bigcap_{I \in \Delta} I)}.
\]

3. Subobject Chains

3.1. Limits and colimits of (pre-)subobject chains. In this section we define (pre-)subobject chains, cocompleteness, completeness, and bicompleteness. Then we study limits and colimits in subobject chains. We recall that $\mathcal{A}$ refers to a skeletally small abelian category.

Definition 3.1.1. Let $X$ be an object in $\mathcal{A}$ and let $\Delta$ be a set of subobjects of $X$. We say that $\Delta$ is a pre-subobject chain of $X$ if

(C1) The set $\Delta$ is totally ordered under the relation $\subseteq$.

We say that a pre-subobject chain is a subobject chain if in addition
(C2) The minimum element $0$ of $\text{Sub}(X)$ is in $\Delta$.
(C3) The maximum element $X$ of $\text{Sub}(X)$ is in $\Delta$.

Let $\Delta$ be a pre-subobject chain of $X$. Given $Y, Y' \in \Delta$ with $Y \subseteq Y'$, we denote by $f_{Y,Y'} : Y \to Y'$ the unique morphism in $\text{Hom}_{\text{Sub}}(X,Y,Y')$. We will also denote by $f_{Y,Y'}$ the morphism $U(f_{Y,Y'})$, where $U : \text{Sub}(X) \to \mathcal{A}$ is the forgetful functor sending $(Y, i_Y)$ to $Y$.

Now consider a pre-subobject chain $\Delta$ of an object $X$ in $\mathcal{C}$. There are two diagrams we may associate to $\Delta$:

1. In the category $\text{Sub}(X)$, we can identify $\Delta$ with the diagram $\mathcal{D}$, with set of objects $(Y, i_Y) \in \Delta$ and morphisms $f_{Y,Y'}$ for $(Y, i_Y) \subseteq (Y', i_{Y'}) \in \Delta$.

2. In the category $\mathcal{A}$, we can identify $\Delta$ with the diagram $\mathcal{D} = U(\mathcal{D}_s)$, with set of objects $U(Y)$, for each $Y \in \Delta$ and morphisms $f_{Y,Y'}$ for $Y \subseteq Y' \in \Delta$.

Let us first consider the limits of these two diagrams. Suppose there exists an object $Z$ of $\mathcal{C}$ with morphisms $g_Y : Z \to U(Y)$ for each $Y \in \Delta$ such that $g_Y = f_{Y,Y'} \circ g_{Y'}$ whenever $Y \subseteq Y'$. Then $\text{im } g_Y \supseteq \text{im } g_{Y'}$ whenever $Y \subseteq Y'$. Now let $G = \text{im } g_Y$ for some $Y \in \Delta$ and let $j : G \to Y$ be the induced inclusion. Then each $g_Y$ factors through $G$, and $(G, i_Y \circ j)$ is in $\text{Sub}(X)$. From this it follows that if $\text{lim } \mathcal{D}$ or $\lim \mathcal{D}_s$ exists then they both exist and $\text{lim } \mathcal{D} \cong U(\text{lim } \mathcal{D}_s)$.

On the other hand, it may be the case that $\text{colim } \mathcal{D} \neq U(\text{colim } \mathcal{D}_s)$. Indeed, $X$ is a solution to $\mathcal{D}$, so there will be a unique morphism $\text{colim } \mathcal{D} \to X$, but this morphism may not be a monomorphism.

As an example: for any left linear topological ring $R$, the category $\mathcal{T}$ of strict and complete topologically finitely generated left $R$-modules is co-Grothendieck [O69 Theorem 5]. Then $\mathcal{T}^{\text{op}}$ is Grothendieck and we know that for any $X$ in $\mathcal{T}^{\text{op}}$,

$$\prod_{i=1}^{\infty} X = \text{lim} \left( \cdots \to \bigoplus_{i=1}^{n} X \to \bigoplus_{i=1}^{n-1} X \to \cdots \to X \to X \to 0 \right) \cong \bigoplus_{i=1}^{\infty} X.$$

In particular, $\bigoplus_{i=1}^{\infty} X$ does not surject onto $\prod_{i=1}^{\infty} X$. Thus, in $\mathcal{T}$ we have the dual situation.

For clarity, we make explicit that by $\text{colim } (\Delta)$ we mean $\text{colim } \mathcal{D}$ and by $\text{lim } (\Delta)$, we mean $\text{lim } \mathcal{D} = U(\text{lim } \mathcal{D}_s)$. We will say that $\text{colim } (\Delta)$ is a subobject of $X$ if the induced morphism $\text{colim } D_1 \to X$ is a monomorphism. In this case, we have $\text{colim } \mathcal{D} = U(\text{colim } \mathcal{D}_s)$.

Moreover, if $\mathcal{A}$ is Grothendieck, then $\text{lim } \mathcal{D}_s = \bigcap_{Y \in \Delta} Y$ and $\text{colim } \mathcal{D}_s = \sum_{Y \in \Delta} Y$. In general, however, the infinite intersection and infinite product may not exist (for example if $\mathcal{A}$ does not have arbitrary coproducts).

**Remark 3.1.2.** Let $X$ be an object in $\mathcal{A}$ and let $\Delta$ be a pre-subobject chain of $X$. Suppose $L = \text{lim } (\Delta)$ exists and let $\iota_L : L \hookrightarrow X$ be the induced inclusion. Then in $\text{Sub}(X)$ we have that if $(Z, i_Z) \subseteq (Y, i_Y)$, for all $(Y, i_Y) \in \Delta$, then $(Z, i_Z) \subseteq (L, \iota_L)$. That is, $(L, \iota_L)$ is the supremum of $\Delta$ in $\text{Sub}(X)$. Simiarly, if $C = \text{colim } (\Delta)$ exists and is a subobject of $X$ with induced inclusion $\iota_C$ then $(C, \iota_C)$ is the infimum of $\Delta$ in $\text{Sub}(X)$.

Based on this discussion, we introduce the following definitions.

**Definition 3.1.3.** Let $X$ be an object of $\mathcal{A}$ and let $\Delta$ be a pre-subobject chain of $X$. We say that $\Delta$ is **cocomplete** if it satisfies the following:

(C4) If $\Delta' \subseteq \Delta$ is an arbitrary subset, then $\text{colim } \Delta'$ exists and is an element of $\Delta$. In particular, $\text{colim } \Delta'$ is a subobject of $X$.

Similarly, we say that $\Delta$ is **complete** if it satisfies the following:

(C5) If $\Delta' \subseteq \Delta$ is an arbitrary subset, then $\text{lim } \Delta'$ exists and is an element of $\Delta$.

Finally, we say that $\Delta$ is **bicomplete** if it is both cocomplete and complete.

**Remark 3.1.4.** We note that, although similar, (C5) is not the dual of (C4). Indeed, the dual of (C4) is a statement about limits of totally ordered sets of quotient objects, not of subobjects.
This is a theme in several of the definitions that will occur throughout this paper. We will discuss how the duality between subobjects and quotients comes into play with these definitions in more detail in Section 4.2.

Example 3.1.5.

1. In any abelian category, any finite subobject chain is bicomplete.
2. Consider \( \mathbb{Z} \) as a \( \mathbb{Z} \)-module. For \( p \) a prime, define \( \Delta_p = \{0\mathbb{Z}\} \cup \{p^\alpha \mathbb{Z} : \alpha \in \mathbb{N}\} \) with the natural inclusion maps. Then \( \Delta_p \) is a bicomplete subobject chain of \( \mathbb{Z} \).
3. Consider the category \( \text{Fun}(\mathbb{R}, \text{vec}(k)) \) for some field \( k \). Now define \( \Delta_1 = \{M(a, 1) \mid a \in [0, 1)\} \cup \{0\} \)
\( \Delta_2 = \{M(a, 1) \mid a \in (0, 1)\} \cup \{0, M(0,1)\} \)
\( \Delta_3 = \Delta_1 \cup \Delta_2 \).

(See Equation 2 for an explanation of the notation \( M(a,1) \) etc.) We note that as posets, \( \Delta_1 \) and \( \Delta_2 \) are isomorphic to \( [0,1] \) and \( \Delta_3 \) is isomorphic to the lexicographical order on \( [0,1] \times \{0,1\} \) without the elements \( (0,0) \) and \( (1,1) \).

Now for \( i \in \{1,2,3\} \) and \( \Delta' \subseteq \Delta_i \), let
\[ C = \bigcup_{M_I \in \Delta'} I, \quad L = \bigcap_{M_I \in \Delta'} I \]
and note that \( \text{colim} \Delta' = M_C \) and \( \text{lim} \Delta' = M_L \) (see Equation 3). Since the intervals defining \( \Delta_1 \) are closed under unions but not intersections, this means \( \Delta_1 \) is a cocomplete subobject chain of \( M(0,1) \) which is not complete. Likewise, \( \Delta_2 \) is a complete subobject chain of \( M(0,1) \) which is not cocomplete, and \( \Delta_3 \) is a bicomplete subobject chain of \( M(0,1) \).

3.2. Subfactor multisets and composition series. In this section, we generalize the definition of a composition series to include subobject chains of arbitrary cardinality. As a starting point, this requires us to precisely describe the subfactors of such a subobject chain. To that end, we fix the following notation.

Notation 3.2.1. Let \( X \) be an object of \( \mathcal{A} \) and let \( \Delta \) be a pre-subobject chain of \( X \). For \( Y \in \Delta \), we denote by
\[
Y^-_\Delta := \text{colim}\{Z \in \Delta \mid Z \not\subseteq Y\},
Y^+_\Delta := \text{lim}\{Z \in \Delta \mid Y \not\subseteq Z\}
\]
when these diagrams are not empty and their colimits and limits exist. Likewise, if \( 0 \in \Delta \), we define \( 0^-_\Delta := 0 \) and if \( X \in \Delta \), we define \( X^+_\Delta := X \). When the pre-subobject chain \( \Delta \) is clear, we will sometimes write \( Y^- \) for \( Y^-_\Delta \) and \( Y^+ \) for \( Y^+_\Delta \).

One can consider \( Y^-_\Delta \) and \( Y^+_\Delta \) as the “predecessor” and “successor” of \( Y \) in \( \Delta \), respectively. For example, suppose \( \Delta \) is bicomplete (and so \( Y^-_\Delta \) and \( Y^+_\Delta \) both exist for every \( Y \in \Delta \)). If \( \Delta \) is well-ordered with respect to \( \subseteq \), then \( Y^+_\Delta \) will truly be the successor of \( Y \) in \( \Delta \). (This is the approach used to prove “Jordan–Hölder-like” and “Schreier-like” theorems for well-ordered subobject chains in \([S09+]\).

The case where \( Y^+_\Delta = Y \) similarly corresponds to when \( Y \) has no successor. Example 3.2.3(3) shows an example of this.

The following definition extends this generalization of predecessors and successors to a generalization of what is meant by the successive subfactors of a subobject. We recall that a multiset is allowed to contain multiple distinct copies of the same element.
Definition 3.2.2.
(1) Let $X$ be an object of a $\mathcal{A}$ and let $\Delta$ be a cocomplete subobject chain of $X$. Then the lower subfactor multiset of $\Delta$ is
\[ \sf^-(\Delta) := \{ Y/Y^-_\Delta \mid Y \in \Delta, Y \neq Y^- \}. \]
(2) Let $X$ be an object of $\mathcal{A}$ and let $\Delta$ be a complete subobject chain of $X$. Then the upper subfactor multiset of $\Delta$ is
\[ \sf^+(\Delta) := \{ Y_\Delta^+ / Y \mid Y \in \Delta, Y \neq Y^+ \}. \]

Example 3.2.3.
(1) Let $X$ have finite length and let $\Delta$ be a composition series of $X$. Then $\Delta$ is subobject bicomplete and $\sf^-(\Delta) = \sf^+(\Delta)$ is the multiset of composition factors of $X$ (with multiplicity).
(2) For $p$ a prime, let $\Delta_p$ be as in Example 3.1.5(2). Then $\sf^+(\Delta_p) = \sf^-(\Delta_p)$ consists of countably many copies of $\mathbb{Z}/p\mathbb{Z}$.
(3) Let $\Delta_1, \Delta_2,$ and $\Delta_3$ be as in Example 3.1.5(3). Now for $a \in (0,1)$, we have
\[
\begin{align*}
(M_{(a,1)})_{\Delta_1} &= \text{colim}\{M_{(b,1)} \mid b \in (a,1)\} = M_{(a,1)}, \\
(M_{(a,1)})_{\Delta_2} &= \text{colim}\{M_{(b,1)}\} \cup \{M_{(a,1)}\} = M_{(a,1)}, \\
(M_{(a,1)})_{\Delta_3} &= \text{colim}\{M_{(b,1)}\} \cup \{M_{(a,1)}\} = M_{(a,1)}.
\end{align*}
\]
This shows that $\sf^+\Delta_1 = \emptyset$, and $\sf^+\Delta_3 = \{M_{[a,a]} \mid a \in (0,1)\}$. A similar argument shows that $\sf^+\Delta_2 = \emptyset$ and $\sf^+\Delta_3 = \sf^-\Delta_3$.

We note that for those subobject chains in Example 3.2.3 which are bicomplete, the upper and lower subfactor multisets coincide. The following shows that this is completely general.

Proposition 3.2.4. Let $X$ be an object of $\mathcal{A}$ and let $\Delta$ be a bicomplete subobject chain of $X$.
(1) If $Y \in \Delta$ and $Y \neq Y^-$, then $(Y^-)^+ = Y$.
(2) If $Y \in \Delta$ and $Y \neq Y^+$, then $(Y^+)^- = Y$.
(3) There is an equality of multisets $\sf^-(\Delta) = \sf^+(\Delta)$.

Proof. (1) Let $\Omega = \{Y' \in \Delta \mid Y^- \subseteq Y'\}$. By assumption, we have that $Y \in \Omega$. Moreover, if $Y'' \in \Delta$ with $Y'' \subseteq Y$, then $Y'' \subseteq Y^-$ by definition. We conclude that $Y = \lim \Omega = (Y^-)^+$. The proof of (2) is similar.
(3) Let
\[ F^- (\Delta) = \{(Y', Y) \in \Delta \times \Delta \mid Y' = Y^- \neq Y\}. \]
Then there is a natural bijection $F^- (\Delta) \to \sf^- (\Delta)$ given by $(Y', Y) \mapsto Y/Y'$. Likewise, define
\[ F^+ (\Delta) = \{(Y, Y') \in \Delta \times \Delta \mid Y = Y^+ \neq Y'\}, \]
so that there is a natural bijection $F^+ (\Delta) \to \sf^+ (\Delta)$ given by $(Y, Y') \mapsto Y'/Y$.

Now given $(Y', Y) = (Y^-, Y) \in F^-(\Delta)$, we have by (1) that $(Y^-)^+ = Y \neq Y'$. This means $(Y', Y) \in F^+ (\Delta)$. Likewise, given $(Y, Y') = (Y, Y^+) \in F^+ (\Delta)$, we have by (2) that $(Y^+)^- = Y \neq Y'$. This means $(Y, Y') \in F^- (\Delta)$. We conclude that $F^+ (\Delta) = F^- (\Delta)$ and thus $\sf^- (\Delta) = \sf^+ (\Delta)$. \qed

Notation 3.2.5. From now on, when the hypotheses of Proposition 3.2.4 are satisfied, we will write $\sf(\Delta)$ instead of $\sf^- (\Delta)$ or $\sf^+ (\Delta)$.

We are now ready to define generalized notions of composition series and filtrations over a subcategory.

Definition 3.2.6. Let $\mathcal{D}$ be a subcategory of $\mathcal{A}$ which is closed under isomorphisms. Let $X$ be an object of $\mathcal{A}$ and let $\Delta$ be a subobject chain of $X$. 


(1) We say that \( \Delta \) is a **lower \( \mathcal{D} \)-filtration** if \( \Delta \) is subobject cocomplete and every object in \( \mathsf{s} \mathsf{f}^- (\Delta) \) is in \( \mathcal{D} \). If an addition every object in \( \mathcal{D} \) is simple in \( \mathcal{A} \), we say that \( \Delta \) is a **lower composition series**.

(2) We say that \( \Delta \) is an **upper \( \mathcal{D} \)-filtration** if \( \Delta \) is subobject complete and every object in \( \mathsf{s} \mathsf{f}^+ (\Delta) \) is in \( \mathcal{D} \). If an addition every object in \( \mathcal{D} \) is simple in \( \mathcal{A} \), we say that \( \Delta \) is an **upper composition series**.

(3) We say that \( \Delta \) is a **\( \mathcal{D} \)-filtration** (respectively a **composition series**) if \( \Delta \) is subobject bicomplete and is a lower \( \mathcal{D} \)-filtration (respectively a lower composition series).

**Remark 3.2.7.** As an immediate consequence of Proposition [3.2.4] we see that a bicomplete subobject chain \( \Delta \) is a \( \mathcal{D} \)-filtration (respectively a composition series) if and only if it is an upper \( \mathcal{D} \)-filtration (respectively an upper composition series). That is, we could replace both instances of “lower” with “upper” in Definition [3.2.6(3)] without actually changing the definitions.

**Example 3.2.8.** All of the subobject chains in Example [3.2.3] are (lower or upper) composition series. In particular, (finite) composition series in the traditional sense are also composition series under Definition [3.2.6]

### 3.3. The existence of composition series

We now turn towards determining when an object \( X \) of \( \mathcal{A} \) admits a composition series. In order to do so, we establish the following definitions.

**Definition 3.3.1.** Let \( X \) be an object of \( \mathcal{A} \). We say that \( X \) is **subobject cocomplete** if (JHS1) For \( \Delta \) a pre-subobject chain of \( X \), the colimit \( \text{colim} \Delta \) exists and is a subobject of \( X \).

Similarly, we say that \( X \) is **subobject complete** if (JHS2) For \( \Delta \) a pre-subobject chain of \( X \), the limit \( \text{lim} \Delta \) exists.

We say that \( X \) is **subobject bicomplete** if it is both subobject cocomplete and subobject complete.

**Definition 3.3.2.** Let \( \Delta \) and \( \Delta' \) be subobject chains of \( X \). We say that \( \Delta' \) is a **refinement** of \( \Delta \) if \( \Delta \subseteq \Delta' \) and that \( \Delta' \) is a **proper** refinement of \( \Delta \) if \( \Delta \subseteq \Delta' \).

We now show that when an object \( X \) is subobject bicomplete (resp. cocomplete, complete), there is a natural way to refine an arbitrary subobject chain of \( X \) into a bicomplete (resp. cocomplete, complete) subobject chain. We first prove the following lemma.

**Lemma 3.3.3.** Let \( X \) be an object of \( \mathcal{A} \) and let \( Y \subseteq X \) be a subobject. If \( X \) is subobject bicomplete (respectively cocomplete, complete), then \( Y \) is bicomplete (respectively cocomplete, complete).

**Proof.** Let \( \Delta \) be a pre-subobject chain of \( Y \) and note that \( \Delta \) is also a pre-subobject chain of \( X \). Now, if \( \text{lim} \Delta \) exists, then it is a subobject of \( Y \) automatically. Moreover, if \( \text{colim} \Delta \) exists and is a subobject of \( X \), then the composition of the inclusion \( Y \subseteq X \) and the natural map \( \text{colim} \Delta \to Y \) must be the inclusion \( \text{colim} \Delta \subseteq X \). We conclude that \( \text{colim} \Delta \) is a subobject of \( Y \). This proves the result.

**Proposition 3.3.4.** Let \( X \) be a subobject bicomplete object in \( \mathcal{A} \), and let \( \Delta \) be a subobject chain of \( X \). Then there exists a canonical bicomplete subobject chain \( \overline{\Delta} \) which is a refinement of \( \Delta \) and so that if a refinement \( \Delta' \) of \( \Delta \) is bicomplete, then \( \Delta' \) is a refinement of \( \overline{\Delta} \). Moreover, the statement is true if one replaces each “bicomplete” either with “cocomplete” or with “complete.”

**Proof.** We prove the result when \( \mathcal{A} \) is bicomplete, as the other two cases follow from similar constructions and arguments. Define

\[
\overline{\Delta} := \{ \text{colim} \Gamma, \text{lim} \Gamma \mid \Gamma \subseteq \Delta \}.
\]

We note that \( \overline{\Delta} \) is a set of subobjects of \( X \) by the axioms [JHS1] and [JHS2]. Moreover, we see that \( \Delta \subseteq \overline{\Delta} \) because for any \( Y \in \Delta \) we have \( Y = \text{colim}\{Y\} = \text{lim}\{Y\} \). In particular, \( \overline{\Delta} \) satisfies (C2) and (C3).
We now show that $\Delta$ satisfies (C1). By the previous paragraph, we know that every subobject in $\Delta$ is either the colimit or the limit of a pre-subobject chain $\Gamma \subseteq \Delta$. Thus let $\Gamma, \Gamma' \subseteq \Delta$ be two such chains.

We first show that $\text{colim} \Gamma$ and $\text{colim} \Gamma'$ are comparable. Indeed, if there exists $Y \in \Gamma$ such that $Y' \subseteq Y$ for all $Y' \in \Gamma'$, then we can see $\Gamma'$ as a pre-subobject chain of $Y$. It then follows from Lemma 3.3.3 and the axiom (JHS1) that $\text{colim} \Gamma' \subseteq Y \subseteq \text{colim} \Gamma$. Otherwise, by symmetry, we may assume that for all $Y' \in \Gamma'$, there exists $Y'' \in \Gamma$ with $Y \subseteq Y''$ and vice versa. In this case, it is straightforward to see that $\text{colim} \Gamma = \text{colim} \Gamma'$.

The argument that $\text{lim} \Gamma$ and $\text{lim} \Gamma'$ are comparable is completely analogous using the axiom (JHS2). Therefore, consider $\text{colim} \Gamma$ and $\text{lim} \Gamma'$. Now if there exist $Y' \in \Gamma'$ and $Y \in \Gamma$ with $Y' \subseteq Y$, then $\text{lim}(\Gamma') \subseteq \text{colim}(\Gamma)$ and we are done. Otherwise, for all $Y' \in \Gamma'$ and $Y \in \Gamma$, we have $Y \subseteq Y'$. Lemma 3.3.3 and the axiom (JHS2) then imply that $\text{colim} \Gamma \subseteq Y'$ for all $Y' \in \Gamma'$. In particular, we have that $\text{colim} \Gamma$ is a solution to the diagram $\Gamma'$, and so there exists a morphism $\text{colim} \Gamma \to \text{lim} \Gamma'$ in $A$. Finally, this morphism must be mono since the composition with the inclusion $\text{lim} \Gamma' \subseteq Y'$ is mono for any $Y' \in \Gamma'$. We conclude that $\Delta$ satisfies (C1).

We have shown that $\Delta$ is a subobject chain of $X$. It remains to show that $\Delta$ is bicomplete; i.e., that $\Delta$ satisfies (C4) and (C5).

Let $\Gamma \subseteq \Delta$. By (JHS1) and (JHS2) we know that $\text{colim} \Gamma$ and $\text{lim} \Gamma$ exist and are subobjects of $X$. Now denote
\[
\Gamma := \{ Y \in \Delta \mid \text{colim} \Gamma \subseteq Y \subseteq \text{lim} \Gamma \}.
\]

First suppose that $\Gamma = \emptyset$. By the previous paragraphs, this means that if there exist pre-subobject chains $\Gamma_1, \Gamma_2 \subseteq \Delta$ with $\text{colim} \Gamma_1, \text{colim} \Gamma_2 \in \Gamma$, then $\text{colim} \Gamma_1 = \text{colim} \Gamma_2$. Likewise, if there exist pre-subobject chains $\Gamma_1, \Gamma_2 \subseteq \Delta$ with $\text{lim} \Gamma_1, \text{lim} \Gamma_2 \in \Gamma$, then $\text{lim} \Gamma_1 = \text{lim} \Gamma_2$. We conclude that $|\Gamma| \leq 2$, and therefore $\text{colim} \Gamma, \text{lim} \Gamma \in \Gamma \subseteq \Delta$.

If $\Gamma \neq \emptyset$, let $\Gamma' = \{ Y \in \Gamma \mid \text{colim} \Gamma \subseteq Y \subseteq \text{lim} \Gamma \}$. By analogous reasoning to before, we see that $|\Gamma'| \leq 2$. If $\Gamma' = \emptyset$, then we have $\text{colim} \Gamma = \text{lim} \Gamma \in \Delta$. Otherwise, we have that $\text{colim} \Gamma \in \Gamma \subseteq \Delta$. The argument that $\text{colim} \Gamma \in \Delta$ is analogous.

We will refer to $\Delta$ as in Proposition 3.3.4 as the bicompletion (respectively cocompletion, completion) of $\Delta$. To avoid possible ambiguity, we will be explicit about for which of these three any use of the notation $\Delta$ refers.

**Example 3.3.5.** In Examples 3.1.5(3) and 3.2.3(3), we have that $\Delta_3$ is the bicompletion of both $\Delta_1$ and $\Delta_2$.

We conclude this section by proving the existence of (lower and upper) composition series and relating composition series to the existence of proper refinements. We first consider the subobject cocomplete and complete cases. We then consider the subobject bicomplete case and prove our first main result (Theorem A in the introduction).

**Proposition 3.3.6.** Let $X$ be an object of $A$ and suppose $X$ is subobject cocomplete. Then there exists a lower composition series of $X$. Moreover, for a pre-subobject chain $\Delta$ of $X$, we have:

1. If $\Delta$ admits no proper refinements, then it is a lower composition series.
2. There exists a lower composition series $\Delta'$ of $X$ which is a refinement of $\Delta$.

Likewise, the result holds if one replaces “cocomplete” with “complete” and “lower composition series” with “upper composition series.”

**Proof.** We prove only the complete case, as the other case is similar. We also note that the existence of a lower composition series follows immediately from (2) since $\{0, X\}$ is a (pre-)subobject chain of $X$.

1. By Proposition 3.3.4, we have that $\Delta = \Delta$ is subobject cocomplete. Now let $Y \in \Delta$ with $Y \neq Y_\Delta$ and let $Z$ be a subobject of $X$ which satisfies $Y_\Delta \subseteq Z \subseteq Y$. Then $\Delta \cup \{Z\}$ is a subobject
chain which is a refinement of $\Delta$, and so $Z \in \Delta$. It follows that either $Z = Y_\Delta^-$ or $Z = Y$. Therefore, $Y/Y_\Delta^-$ must be simple.

(2) We prove the result using Zorn’s Lemma. Let $\mathcal{F}(X)$ be the set of subobject chains of $X$ which refine $\Delta$. Note that $\mathcal{F}(X)$ is nonempty since $\Delta \in \mathcal{F}(X)$. Moreover, if $\{\Delta_\alpha\}_{\alpha \in B}$ is a totally ordered subset of $\mathcal{F}(X)$ (with respect to $\subseteq$), then $\bigcup_{\alpha \in B} \Delta_\alpha$ is a subobject chain of $X$. We conclude that $\mathcal{F}(X)$ contains a maximal element by Zorn’s Lemma. This element is a lower composition series by (1).

\[ \square \]

**Theorem 3.3.7** (Theorem [A].) Let $A$ be a skelletally small abelian category and let $X$ be a subobject bicomplete object of $A$. Then there exists a composition series of $X$. Moreover, for a pre-subobject chain $\Delta$ of $X$ we have:

(1) $\Delta$ is a composition series if and only if it admits no proper refinements.

(2) There exists a composition series $\Delta'$ of $X$ which is a refinement of $\Delta$.

\[ \text{Proof.} \text{ As with Proposition 3.3.6 we note that the existence of a composition series of } X \text{ follows immediately from (2) since } \{0, X\} \text{ is a (pre)-subobject chain of } X. \]

(1) Let $\Delta$ be a composition series of $X$ and let $\Gamma$ be a subobject chain of $X$ with $\Delta \subseteq \Gamma$. Let $Z \in \Gamma$. We will show that $Z \in \Delta$ and so $\Delta = \Gamma$.

Define $\Omega^- = \{Y \in \Delta \mid Y \subseteq Z\}$ and $\Omega^+ = \{Y \in \Delta \mid Z \subseteq Y\}$. Denote $Y_0 = \text{colim} \Omega^-$ and $Y_1 = \lim \Omega^+$, which exist and are subobjects of $X$ by (JHS1) and (JHS2). Since composition series are subobject bicomplete, we then have that $Y_0, Y_1 \in \Delta$ and that $(Y_1)_\Delta = Y_0$. This means the quotient $Y_1/Y_0$ is either 0 or simple. In either case, this implies that either $Z = Y_0$ or $Z = Y_1$, and so $Z \in \Delta$.

The reverse implication follows from an argument analogous to the proof of Proposition 3.3.6(1).

(2) This follows from an argument analogous to the proof of Proposition 3.3.6(2). \[ \square \]

**Example 3.3.8.** We note that, unlike for subobject bicomplete chains, the converse of Proposition 3.3.6(1) does not hold. Indeed, the subobject chains $\Delta_1$ and $\Delta_2$ in Example 3.1.5(3) are lower and upper composition series, respectively, but both admit $\Delta_1 \cup \Delta_2$ as a proper refinement.

4. (Weakly) Jordan–Hölder–Schreier objects

In this section, we address the uniqueness of composition series. More precisely, for any object $X$ in the (skelletally small) abelian category $A$, we wish to know when all of the composition series of $X$ are equivalent in the following sense.

**Definition 4.0.1.** Let $X$ be an object of $A$ and let $\Delta$ and $\Gamma$ be cocomplete subobject chains of $X$. We say that $\Delta$ and $\Gamma$ are \textbf{subfactor equivalent} if there exists a bijection

$$
\Phi : \{Y \in \Delta \mid Y \neq Y_\Delta^-\} \to \{Z \in \Gamma \mid Z \neq Z_\Gamma^-\}
$$

so that $Y/Y_\Delta^- \cong \Phi(Y)/\Phi(Y)_\Gamma^-$ for all $Y$. We define subfactor equivalence for complete and bicomplete subobject chains analogously.

**Remark 4.0.2.** It is an immediate consequence of Proposition 3.2.4 that two bicomplete subobject chains are subfactor equivalent as cocomplete subobject chains if and only if they are subfactor equivalent as complete subobject chains.

**Remark 4.0.3.** We emphasize that the domain of the bijection $\Phi$ in Definition 4.0.1 is generally not all of $\Delta$, but rather in the cocomplete case consists of only those elements of $\Delta$ which have a “predecessor” and in the complete case consists of only those elements of $\Delta$ which have a “successor”. When $\Delta$ and $\Gamma$ are well-ordered and complete (or in particular finite), every element has a “successor”. Therefore, subfactor equivalence implies that $\Delta$ and $\Gamma$ have the same cardinality in this case. Based solely on the definition, however, it is not clear in general whether subobject chains with different cardinalities can be subfactor equivalent.
We now give the set of axioms under which we will prove our “Jordan–Hölder-like” and “Schreier-like theorems”. Given a pre-subobject chain $\Delta$ of $X$ and a subobject $Z \subseteq X$, we denote $\Delta \cap Z = \{Y \cap Z \mid Y \in \Delta\}$ and $\Delta + Z = \{Y + Z \mid Y \in \Delta\}$.

**Definition 4.0.4.** Let $X$ be an object of $\mathcal{A}$. We say that $X$ is **weakly Jordan–Hölder–Schreier** (or **weakly JHS**) if the following hold.

- **(JHS3)** $X$ satisfies [JHS1] and, for all pre-subobject chains $\Delta$ of $X$ and all subobjects $Z \subseteq X$, we have $\text{colim}(\Delta \cap Z) = (\text{colim} \Delta) \cap Z$.
- **(JHS4)** $X$ satisfies [JHS2] and, for all pre-subobject chains $\Delta$ of $X$ and all subobjects $Z \subseteq X$, we have $\lim(\Delta + Z) = (\lim \Delta) + Z$.

We say that $X$ is **Jordan–Hölder–Schreier** (or **JHS**) if in addition

- **(JHS5)** If $X \neq 0$ and $\Delta$ is a composition series of $X$, then $\text{sf}(\Delta) \neq \emptyset$.

**Remark 4.0.5.**

1. We note that there do exist weakly JHS objects which are not JHS. Indeed, in Section 5.4, we study an abelian subcategory of $\text{Fun}(\text{R}, \text{vec}(\mathbb{K}))$ which was first considered by Igusa and Todorov [IT15]. Every object in this category is weakly JHS, but only the zero object is JHS. See Proposition 5.4.4.

2. We stress that being (weakly) JHS makes some reference to the ambient category. For example, suppose $X$ is an object in $\mathcal{A}$ and that every endomorphism of $X$ is an automorphism. Then the full subcategory of $\mathcal{A}$ consisting of only the objects $0$ and $X$ is abelian, and $X$ is JHS in this subcategory. Likewise, it is possible that an object $X$ is JHS in $\mathcal{A}$ but not in some subcategory of $\mathcal{A}$. Again, see Proposition 5.4.4.

3. We recall that if $\mathcal{A}$ has arbitrary coproducts, then Grothendieck’s axiom (AB5) is equivalent to requiring that $(\text{colim} \Delta \cap Z) = (\text{colim} \Delta) \cap Z$ for all objects $Z$ and directed systems $\Delta$. In particular, every (AB5) category satisfies (JHS3). We give an alternative explanation of this fact in Remark 4.2.4.

4. In some sense, the axiom (JHS4) can be seen as generalizing the artinian property. Indeed, if $X$ is a (left) artinian module over any ring, then any pre-subobject chain $\Delta$ of $X$ must contain a minimal object $Y$. We then have $\lim(\Delta + Z) = Y + Z = (\lim \Delta) + Z$.

The following can be deduced immediately from the definition, and will be used throughout the remainder of this paper.

**Proposition 4.0.6.** Let $X$ be a weakly JHS object in $\mathcal{A}$. Then any subobject $Y \subseteq X$ is also weakly JHS in $\mathcal{A}$.

**Remark 4.0.7.** We note that if $X$ is JHS then it is possible a subobject $Y \subseteq X$ is not JHS. For example, let $Y$ be weakly JHS in an abelian category $\mathcal{B}$ and let $Z$ be JHS in an abelian category $\mathcal{C}$. Then $X = Y \oplus Z$ is JHS in $\mathcal{B} \times \mathcal{C}$ and $Y \subseteq (Y \oplus Z)$. Alternatively, suppose $X$ is JHS and a composition series $\Delta$ of an object $X$ has exactly one subfactor $Y/\Delta$ and that $Y/\Delta \neq 0$. Then $Y/\Delta$ is weakly JHS but not JHS.

Before we continue, we prove the following technical lemma, which will be useful in what follows.

**Lemma 4.0.8.** Let $X$ be an object of $\mathcal{A}$ and suppose that for every pre-subobject chain $\Delta$ of $X$ the colimit $\text{colim} \Delta$ exists (in $\mathcal{A}$). Then the following are equivalent.

1. $X$ is subobject cocomplete; i.e, $X$ satisfies [JHS1].
2. If $\{f_\alpha : Y_\alpha \rightarrow Z_\alpha\}_\alpha$ is a directed system in $\text{Sub}(X)$, then $\text{colim}\{f_\alpha\}$ (considered in $\mathcal{A}$) is a monomorphism.
3. If $\{0 \rightarrow Y_\alpha \xrightarrow{f_\alpha} Z_\alpha \xrightarrow{g_\alpha} W_\alpha \rightarrow 0\}_\alpha$ is a directed system of short exact sequences with $\{f_\alpha : Y_\alpha \rightarrow Z_\alpha\}$ a directed system in $\text{Sub}(X)$, then $\text{colim}\{W_\alpha\}$ exists and there is a short
Proof. (1 $\implies$ 2): Let $\{f_\alpha : Y_\alpha \to Z_\alpha\}$ be a directed system in Sub($X$). Note that in particular, each morphism $f_\alpha$ is mono. Now by (1), we have that $Y := \text{colim}\{Y_\alpha\}$ exists and is a subobject of $X$, and likewise for $Z := \text{colim}\ Z_\alpha$. In particular, the composition of $\text{colim}\{f_\alpha\} : Y \to Z$ with the inclusion $Z \subseteq X$ must be the inclusion $Y \subseteq X$. This means that $\text{colim}\{f_\alpha\}$ is a monomorphism.

(2 $\implies$ 3): By (2), we have that $\text{colim}\{f_\alpha\} : \text{colim}\{Y_\alpha\} \to \text{colim}\{Z_\alpha\}$ is a monomorphism. We then have that

$$\text{coker}(\text{colim}\{f_\alpha\}) = \text{colim}\{\text{coker}(f_\alpha)\} = \text{colim}\{g_\alpha\}.\$$

This means that $\text{colim}\{W_\alpha\}$ exists and the desired sequence is exact, as claimed.

(3 $\implies$ 1): Let $\Delta$ be a presubobject chain of $X$. Then by (3), we have that

$$0 \to \text{colim}\ \Delta \to X \to X/\text{colim}\ \Delta \to 0$$

is an exact sequence, and so $\text{colim}\ \Delta$ is a subobject of $X$. \hfill \square

4.1. Conflations and Jordan–Hölder–Schreier-like Theorems. In this section, we formalize the notion of conflating two subobject chains and prove our “Jordan–Hölder–like theorem” (Theorem 3 in the introduction). The argument is similar to the proof of the classical Jordan–Hölder Theorem given in [L02, Section I.3] and of the well-ordered case in [S09+].

Part of the basis for this argument is the “Butterfly Lemma” of Zassenhaus. For convenience, we give a proof of this result here in our notation.

**Lemma 4.1.1** (Zassenhaus Butterfly Lemma). Let $X$ be an object in $\mathcal{A}$ and let $Y, Y^-, Z, Z^-$ be subobjects of $X$ with $Y^- \subseteq Y$ and $Z^- \subseteq Z$. Then

$$\frac{Y^- + (Y \cap Z)}{Y^- + (Y \cap Z^-)} \cong \frac{Z^- + (Z \cap Y)}{Z^- + (Z \cap Y^-)}.$$

**Proof.** Note that

$$Y^- + (Y \cap Z) = (Y^- + (Y \cap Z^-)) + (Y \cap Z).$$

Moreover, we have

$$(Y \cap Z) \cap (Y^- + (Y \cap Z^-)) = (Y^- \cap Z) + (Y \cap Z^-).$$

Therefore by the isomorphism theorems, we have

$$\frac{Y^- + (Y \cap Z)}{Y^- + (Y \cap Z^-)} \cong \frac{Y \cap Z}{(Y^- \cap Z) + (Y \cap Z^-)}.$$

Since the resulting expression is symmetric in $Y$ and $Z$, this proves the result. \hfill \square

**Remark 4.1.2.** If $Y = Y^-$ or $Z = Z^-$, then both sides of Equation 4.1.1 are zero.

We now define a procedure to conflate the data of two subobject chains.

**Definition 4.1.3.** Let $X$ be an object in $\mathcal{A}$, and let $\Delta$ and $\Gamma$ be subobject chains of $X$. We the define the **lower conflation** of $\Delta$ by $\Gamma$ to be

$$\text{con}^- (\Delta, \Gamma) := \Delta \cup \{Y^-_\triangle + (Y \cap Z) \mid (Y, Z) \in \Delta \times \Gamma, Y^-_\triangle \text{ exists}\}.$$

Dually, we define the **upper conflation** of $\Delta$ by $\Gamma$ to be

$$\text{con}^+ (\Delta, \Gamma) := \Delta \cup \{Y^+_\triangle + (Y \cap Z) \mid (Y, Z) \in \Delta \times \Gamma, Y^+_\triangle \text{ exists}\}.$$
**Example 4.1.4.**

1. Let $\mathbb{Z}_{24} = \{0, 1, \ldots, 23\}$ be the cyclic group of order 24 (considered in the category of $\mathbb{Z}$-modules). For each $i \in \mathbb{Z}_{24}$, we denote by $\langle i \rangle$ the subgroup generated by $i$. Now consider the (bicomplete) subobject chains

$$\Delta = \{\langle 0 \rangle, \langle 4 \rangle, \langle 1 \rangle\} \quad \text{and} \quad \Gamma = \{\langle 0 \rangle, \langle 6 \rangle, \langle 1 \rangle\}$$

of $\mathbb{Z}_{24}$. We then have

$$\text{con}^- (\Delta, \Gamma) = \text{con}^+(\Delta, \Gamma) = \{\langle 0 \rangle, \langle 12 \rangle, \langle 4 \rangle, \langle 2 \rangle, \langle 1 \rangle\}$$

$$\text{con}^- (\Gamma, \Delta) = \text{con}^+(\Gamma, \Delta) = \{\langle 0 \rangle, \langle 12 \rangle, \langle 6 \rangle, \langle 2 \rangle, \langle 1 \rangle\}.$$ 

In particular, we note that $\text{con}^- (-, -)$ and $\text{con}^+ (-, -)$ are not symmetric in their arguments.

2. Consider the subobject chains $\Delta_1$ and $\Delta_2$ in Example 3.1.5(3). Then $\text{con}^- (\Delta_1, \Delta_2) = \Delta_1 = \text{con}^+ (\Delta_1, \Delta_2)$, but for different reasons. We have $\text{con}^- (\Delta_1, \Delta_2) = \Delta_1$ because $\Delta_1 = Y$ for all $Y \in \Delta_1$. On the other hand, we have $\text{con}^+ (\Delta_1, \Delta_2) = \Delta_1$ because for $Y \in \Delta_1$, $Y_{\Delta_1}$ is only an element of $\Delta_1$ if $Y = 0$, in which case $Y_{\Delta_1}^+ = Y$.

3. Let $p$ and $q$ be distinct primes and let $\Delta_p$ and $\Delta_q$ be as in Example 3.1.5(2). Note that for $\alpha \in \mathbb{N}$, we have $(p^\alpha \mathbb{Z})_{\Delta_p} = p^{\alpha+1} \mathbb{Z}$. Moreover, for $\alpha, \beta \in \mathbb{N}$, we have

$$p^{\alpha+1} \mathbb{Z} + (p^\alpha \mathbb{Z} \cap q^\beta \mathbb{Z}) = p^{\alpha} \mathbb{Z}.$$ 

We then have that $\text{con}^- (\Delta_p, \Delta_q) = \Delta_p$. We can also see this as a consequence of Proposition 3.3.7(1) and the fact that $\Delta_p$ is a composition series. See Remark 4.1.7 below.

**Remark 4.1.5.** Note that if $\Delta$ is cocomplete, then for $Y \in \Delta$ we can write $Y = Y_{\Delta}^- + Y \cap X$, where $X \in \Gamma$ since $\Gamma$ is a subobject chain. In particular, the “$\Delta \cup$” in the definition of $\text{con}^- (\Delta, \Gamma)$ becomes redundant in this case. The behavior is similar in the complete case. Nevertheless, it is necessary to include this in the definition in order for Proposition 4.1.6 below to hold. Indeed, in Example 4.1.4(2), removing “$\Delta \cup$” from the definition of $\text{con}^+(\Delta_1, \Delta_2)$ would result in $\text{con}^+(\Delta_1, \Delta_2) = \{0\}$, which is not a refinement of $\Delta_1$.

We now prove a series of results about the conflation of two subobject chains. From these, we will deduce our “Schreier-like” and “Jordan–Hölder-like” theorems (Theorems 4.1.13 and 4.1.14, respectively).

**Proposition 4.1.6.** Let $X$ be an object in $\mathcal{A}$, and let $\Delta$ and $\Gamma$ be subobject chains of $X$. Then $\text{con}^- (\Delta, \Gamma)$ and $\text{con}^+ (\Delta, \Gamma)$ are subobject chains of $X$ which are refinements of $\Delta$.

**Proof.** We prove the result only for $\text{con}^- (\Delta, \Gamma)$. The proof for $\text{con}^+ (\Delta, \Gamma)$ is similar.

Since $\Delta \subseteq \text{con}^- (\Delta, \Gamma)$, both (C2) and (C3) are satisfied. Let $W, W' \in \text{con}^- (\Delta, \Gamma)$. We will show that either $W' \subseteq W$ or $W \subseteq W'$, and therefore (C1) holds.

If both $W$ and $W'$ are elements of $\Delta$, then we are done. If $W \notin \Delta$, write $W = Y_{\Delta}^- + (Y \cap Z)$ with $Y \in \Delta$ and $Z \in \Gamma$. Now, there exists $Y' \in \Delta$ so that either $W' = Y'$ or $(Y')_{\Delta}^- \subseteq W' \subseteq Y'$. We see that if $Y' \subseteq Y$ (respectively $Y \subseteq Y'$) then $W' \subseteq W$ (respectively $W' \subseteq W$). Since $\Delta$ is a subobject chain, the only other possibility is that $Y = Y'$. Since $Y$ has a predecessor, this means there exists $Z' \in \Gamma$ so that $W' = Y_{\Delta}^- + (Y \cap Z')$. Then since $\Gamma$ is a subobject chain, we have without loss of generality that $Z' \subseteq Z$ and thus $W' \subseteq W$. 

**Remark 4.1.7.** If $\Delta$ is a composition series, then Propositions 3.3.7 and 4.1.6 immediately imply that $\text{con}^- (\Delta, \Gamma) = \Delta = \text{con}^+(\Delta, \Gamma)$ for any $\Gamma$.

**Lemma 4.1.8.** Let $X$ be an object in $\mathcal{A}$, and let $\Delta$ and $\Gamma$ be subobject chains of $X$.

1. If $X$ satisfies (JHS3) and both $\Delta$ and $\Gamma$ are cocomplete, then $\text{con}^- (\Delta, \Gamma)$ is cocomplete.

2. If $X$ satisfies (JHS4) and both $\Delta$ and $\Gamma$ are complete, then $\text{con}^+ (\Delta, \Gamma)$ is complete.
(3) If $X$ is weakly JHS and both $\Delta$ and $\Gamma$ are bicomplete, then $\operatorname{con}^-(\Delta, \Gamma) = \operatorname{con}^+(\Delta, \Gamma)$ and this subobject chain is bicomplete.

Proof. (1) Let $\Sigma \subseteq \operatorname{con}^-(\Delta, \Gamma)$ and let $\Sigma_{\Delta} = \{Y \in \Delta \mid \exists Z \in \Gamma : Y_\Delta^- + (Y \cap Z) \in \Sigma\}$. Denote $Y_0 = \varprojlim \Sigma_{\Delta}$. Note that since $\Delta$ is cocomplete, we have $Y_0 \in \Delta$.

If $Y_0 \notin \Sigma_{\Delta}$, then for all $Y \subseteq Y_0$ in $\Delta$, there exists $Y' \in \Delta$ and $Z' \in \Gamma$ so that $W := (Y')_\Delta^- + (Y' \cap Z') \in \Sigma$ and $Y \subseteq W \subseteq Y_0$. We conclude that colim $\Sigma = Y_0 \in \Delta \subseteq \operatorname{con}^-(\Delta, \Gamma)$.

Otherwise, let $\Sigma_{\Gamma} = \{Z \in \Gamma \mid (Y_0)_\Delta^- + (Y_0 \cap Z) \in \Sigma\}$. By \textbf{(JHS3)} and Lemma 4.0.8, we then have

$$\colim \Sigma = \colim\{(Y_0)_\Delta^- + (Y_0 \cap Z) \mid Z \in \Sigma_{\Gamma}\} = (Y_0)_\Delta^- + (Y_0 \cap \colim \Sigma_{\Gamma}).$$

Now since $\Gamma$ is cocomplete, we have colim $\Sigma_{\Gamma} \in \Gamma$ and therefore colim $\Sigma \in \operatorname{con}^-(\Delta, \Gamma)$.

(2) Let $\Sigma \subseteq \operatorname{con}^+(\Delta, \Gamma)$ and let $\Sigma_{\Delta} = \{Y \in \Delta \mid \exists Z \in \Gamma : Y + (Y^+ \cap Z) \in \Sigma\}$. Denote $Y_0 = \varinjlim \Sigma_{\Delta}$. Note that since $\Delta$ is complete, we have $Y_0 \in \Delta$.

If $Y_0 \notin \Sigma_{\Delta}$, then for all $Y_0 \subseteq Y$ in $\Delta$, there exists $Y' \in \Delta$ and $Z' \in \Gamma$ so that $W := Y' + ((Y')_\Delta^+ \cap Z') \in \Sigma$ and $Y_0 \subseteq W \subseteq Y$. We conclude that lim $\Sigma = Y_0 \in \Delta \subseteq \operatorname{con}^+(\Delta, \Gamma)$.

Otherwise, let $\Sigma_{\Gamma} = \{Z \in \Gamma \mid Y_0 + ((Y_0)_\Delta^+ \cap Z) \in \Sigma\}$. Since intersections are limits, we then have by \textbf{(JHS4)} that

$$\varinjlim \Sigma = \lim\{Y_0 + ((Y_0)_\Delta^+ \cap Z) \mid Z \in \Sigma_{\Gamma}\} = Y_0 + ((Y_0)_\Delta^+ \cap \varinjlim \Sigma_{\Gamma}).$$

Now since $\Gamma$ is complete, we have that lim $\Sigma_{\Gamma} \in \Gamma$ and therefore lim $\Sigma \in \operatorname{con}^+(\Delta, \Gamma)$.

(3) Let $Y \in \Delta$ and $Z \in \Gamma$. If $Y_{\Delta}^- = Y$, then

$$Y_{\Delta}^- + (Y \cap Z) = Y = Y + Y_{\Delta}^+ \cap 0.$$

Otherwise, we have

$$Y_{\Delta}^- + (Y \cap Z) = Y_{\Delta}^- + ((Y_{\Delta}^-)^+ \cap Z)$$

by Proposition 3.2.4. Therefore in either case, we have that $Y_{\Delta}^- + (Y \cap Z) \in \operatorname{con}^+(\Delta, \Gamma)$. By symmetry, we conclude that $\operatorname{con}^-(\Delta, \Gamma) = \operatorname{con}^+(\Delta, \Gamma)$. This subobject chain is then bicomplete as a consequence of (1) and (2). \hfill \square

\textbf{Example 4.1.9.} We now present an example of a conflation between a bicomplete chain and a cocomplete chain which does not result in a bicomplete chain. Consider the category $\mathbf{Fun}(\mathbb{R}, \mathbf{vec}(k))$ and define the subobject chains $\Delta$ and $\Gamma$:

$$\Delta = \begin{cases} 0, & M_{[1,1]}, M_{[\frac{1}{2},1]}, M_{[0,1]} \end{cases}$$

$$\Gamma = \begin{cases} 0, & M_{[1,1]}, M_{[0,1]} \end{cases} \cup \begin{cases} M_{(a,1)} & a \in \mathbb{R}, 0 < a < 1 \end{cases}.$$

Note that $\Delta$ is bicomplete and $\Gamma$ is cocomplete. Thus it makes sense to take $\operatorname{con}^-(\Delta, \Gamma)$, which is cocomplete by Lemma 4.1.8. We see $\operatorname{con}^-(\Delta, \Gamma)$ is given by

$$\operatorname{con}^-(\Delta, \Gamma) = \begin{cases} 0, & M_{[1,1]}, M_{[\frac{1}{2},1]}, M_{[0,1]} \end{cases} \cup \begin{cases} M_{(a,1)} & a \in \mathbb{R}, 0 < a < 1 \end{cases} = \Delta \cup \Gamma.$$

We see $\Delta \cup \Gamma$ is still missing uncountably many limits and so it is not complete. Thus, $\operatorname{con}^-(\Delta, \Gamma)$ is not bicomplete.

\textbf{Lemma 4.1.10.} Let $X$ be an object in $\mathcal{A}$ which is subobject bicomplete. Let $\Delta$ be a cocomplete subobject chain of $X$ and let $\Gamma$ be a complete subobject chain of $X$.

(1) Suppose $X$ satisfies $\textbf{(JHS4)}$ let $W \in \operatorname{con}^-(\Delta, \Gamma)$, and denote

$$\Omega(W) = \{(Y, Z) \in \Delta \times \Gamma \mid Y_{\Delta}^- + (Y \cap Z) = W\}.$$

Then $\Omega(W)$ contains a minimal element with respect to lexicographical ordering of $\Delta \times \Gamma$. 

(2) Suppose $X$ satisfies\(^{\text{JHS3}}\) let $W \in \mathsf{con}^+(\Gamma, \Delta)$, and denote

$$\Omega(W) = \{(Z, Y) \in \Gamma \times \Delta \mid Z + (Z^+ \cap Y) = W\}.$$ 

Then $\Omega(W)$ contains a maximal element with respect to the lexicographical ordering of $\Gamma \times \Delta$.

**Proof.** We prove only (1) since the proof of (2) is similar. Let $\Sigma_\Delta = \{Y \in \Delta \mid \exists Z \in \Gamma : (Y, Z) \in \Omega(W)\}$. We first claim that $|\Sigma_\Delta| \leq 2$. Indeed, suppose $Y' \subseteq Y$ and there exist $Z, Z' \in \Gamma$ such that $(Y, Z)$ and $(Y', Z')$ are in $\Omega(W)$. Note that $Y^-_\Delta, (Y')^-_\Delta \in \Delta$ since $\Delta$ is cocomplete. Then

$$(Y')^-_\Delta \subseteq (Y')^-_\Delta + (Y' \cap Z') \subseteq Y' \subseteq Y^-_\Delta \subseteq Y^- + (Y \cap Z)$$

implies that $Y' = Y^-_\Delta$, and so $\Sigma_\Delta = \{Y, Y'\}$. Now let $Y_0 = \lim\Sigma_\Delta$ and note that $Y_0 \in \Sigma_\Delta$ since this set contains at most two elements. Define $\Sigma_\Gamma = \{Z \in \Gamma \mid (Y_0, Z) \in \Sigma\}$. Since $\Gamma$ is complete, we note that $Z_0 := \lim\Sigma_\Gamma \in \Gamma$. Then by (JHS4) we have

$$(Y_0^-_\Delta + (Y_0 \cap Z_0), \lim_{\alpha \to \infty} p^{\alpha} \in \mathbb{N} \text{ with } \beta \neq 0, \text{ we have}$$

$$\lim_{\alpha \to \infty} (p^{\alpha}Z + q^\beta Z) = \lim_{\alpha \to \infty} p^{\alpha}Z + q^\beta Z = q^\beta Z.$$ 

Now let $\alpha \neq 0$. Working in $\mathsf{con}^-(\Delta_p, \Delta_q)$, we then have by Example 4.1.4(3) that

$$\Omega(p^{\alpha}Z) = \{(p^{\alpha}Z, q^\beta Z) \mid \beta \neq 0\} \cup \{(p^{\alpha+1}Z, 0Z)\}.$$ 

We note that $\Omega(p^{\alpha}Z)$ has $(p^{\alpha+1}Z, 0Z)$ as its minimal element, but has no maximal element.

**Example 4.1.11.** We note that as a $\mathbb{Z}$-module, $Z$ is subobject bicomplete and satisfies (JHS3). To see that $Z$ does not satisfy (JHS4), let $p$ and $q$ be primes and let $\Delta_p$ and $\Delta_q$ be as in Example 3.1.5. Let $\Delta'_p = \Delta_p \setminus \{0Z\}$. Then for any $\beta \in \mathbb{N}$ with $\beta \neq 0$, we have

$$\lim_{\alpha \to \infty} (\Delta'_p + q^\beta Z) = \lim_{\alpha \to \infty} (p^{\alpha}Z + q^\beta Z) = Z$$

$$\lim_{\alpha \to \infty} (\Delta'_p + q^\beta Z) = \left(\lim_{\alpha \to \infty} p^{\alpha}Z\right) + q^\beta Z = \beta^\beta Z.$$ 

Now let $\alpha \neq 0$. Working in $\mathsf{con}^-(\Delta_p, \Delta_q)$, we then have by Example 4.1.4(3) that

$$\Omega(p^{\alpha}Z) = \{(p^{\alpha}Z, q^\beta Z) \mid \beta \neq 0\} \cup \{(p^{\alpha+1}Z, 0Z)\}.$$ 

We note that $\Omega(p^{\alpha}Z)$ has $(p^{\alpha+1}Z, 0Z)$ as its minimal element, but has no maximal element.

**Lemma 4.1.12.** Let $X$ be a weakly JHS object in $\mathcal{A}$, let $\Delta$ be a cocomplete subobject chain of $X$, and let $\Gamma$ be a complete subobject chain of $X$.

(1) Let $W \in \mathsf{con}^- (\Delta, \Gamma)$ and write $W = Y^-_\Delta + (Y \cap Z)$ with $(Y, Z) \in \Delta \times \Gamma$ minimal in the sense of Lemma 4.1.10(1). Then

$$W_{\mathsf{con}^- (\Delta, \Gamma)} = \begin{cases} W & X \subseteq Y^-_\Delta + (Y \cap Z^-_\Gamma) \setminus Y^-_\Delta \neq Y \end{cases}$$

(2) Let $W \in \mathsf{con}^+ (\Gamma, \Delta)$ and write $W = Z + (Z^+ \cap Y)$ with $(Z, Y) \in \Gamma \times \Delta$ maximal in the sense of Lemma 4.1.10(2). Then

$$W_{\mathsf{con}^+ (\Gamma, \Delta)} = \begin{cases} W & Z^+ = Z \setminus (Z^+ \cap Y^+_{\Delta}) \setminus Z^+ \neq Z \end{cases}$$

**Proof.** We prove only (1) as the proof of (2) is similar. We first observe that since $X$ satisfies (JHS4) and $\Gamma$ is complete, it is indeed possible to write $W = Y^-_\Delta + (Y \cap Z)$ with $(Y, Z) \in \Delta \times \Gamma$ minimal in the sense of Lemma 4.1.10(1).
Now if \( Y_\Delta^- = Y \), then we must have \( Z = 0 \). Then for any \( W' \sqsubseteq W \), we have \( W' \sqsubseteq Y \) and so \( W^-_{\con^-(\Delta, \Gamma)} \sqsubseteq Y \). Moreover, if \( Y' \sqsubseteq Y \), then because \( \Delta \) is cocomplete we have \( Y' = (Y')_\Delta^- + (Y' \cap X) \in \con^- (\Delta, \Gamma) \). In particular, since \( Y' \sqsubseteq Y_\Delta^- \), this means that \( Y \sqsubseteq W^-_{\con^-(\Delta, \Gamma)} \). We conclude that \( W^-_{\con^-(\Delta, \Gamma)} = Y \).

Otherwise, we have \( Y^- \sqsubseteq W \sqsubseteq Y \). In particular, \( Z \neq 0 \). By \([\text{JHS3}]\) and Lemma \(4.0.8\) this means

\[
W^-_{\con^-(\Delta, \Gamma)} = \text{colim}\{Y^-_\Delta + (Y \cap Z') \mid Z' \in \Gamma, Z' \sqsubseteq \Delta\} \\
= Y^-_\Delta + (Y \cap \text{colim}\{Z' \in \Gamma \mid Z' \sqsubseteq \Delta\}) \\
= Y^-_\Delta + (Y \cap Z^-_\Gamma).
\]

\[\square\]

We are now ready to prove our “Schreier-like” theorem.

**Theorem 4.1.13.** Let \( X \) be a weakly JHS object of \( A \) and let \( \Delta \) and \( \Gamma \) be subobject chains of \( X \). Then there exist bicomplete subobject chains \( \Delta' \supseteq \Delta \) and \( \Gamma' \supseteq \Gamma \) which are subfactor equivalent.

**Proof.** Let \( \overline{\Delta} \) and \( \overline{\Gamma} \) be the bicompletions of \( \Delta \) and \( \Gamma \), respectively. We will show that \( \con^- (\overline{\Delta}, \overline{\Gamma}) \) and \( \con^- (\overline{\Gamma}, \overline{\Delta}) \) are subfactor equivalent. We note that these are bicomplete subobject chains which refine \( \Delta \) and \( \Gamma \), respectively, by Proposition \(3.3.4\) and Lemma \(4.1.8\).

Let \( W \in \con^- (\overline{\Delta}, \overline{\Gamma}) \). Since \( X \) satisfies \([\text{JHS4}]\) we can write \( W = Y^-_\Delta + (Y \cap Z) \) with \( (Y, Z) \in \overline{\Delta} \times \overline{\Gamma} \) minimal in the sense of Lemma \(4.1.10\). Now if \( Y_\Delta^- = Y \), then \( W_{\con^- (\overline{\Delta}, \overline{\Gamma})} = W \) by Lemma \(4.1.12\). In particular, this means \( W/W^-_{\con^-(\Delta, \Gamma)} \notin \text{sf}(\con^- (\overline{\Delta}, \overline{\Gamma})) \). Otherwise, again by Lemma \(4.1.12\) we have that

\[
W/W^-_{\con^-(\Delta, \Gamma)} = \frac{Y^-_\Delta + (Y \cap Z)}{Y^-_\Delta + (Y \cap Z^-_\Gamma)}.
\]

Now denote \( V := Z^-_\Gamma + (Z \cap Y) \) and \( V' := Z^-_\Gamma + (Z \cap Y^-_\Delta) \). Then by Lemma \(4.1.1\) we have that \( W/W^-_{\con^-(\Delta, \Gamma)} \in \text{sf}(\con^- (\overline{\Delta}, \overline{\Gamma})) \) if and only if \( V' \neq V \). It follows that if \( W/W^-_{\con^-(\Delta, \Gamma)} \) is a subfactor, then \( (Z, Y) \) is the minimal representative of \( V \) in \( \con^- (\overline{\Gamma}, \overline{\Delta}) \) and that \( V' = V^-_{\con^-(\Gamma, \Delta)} \).

In particular, we have \( V/V' \in \text{sf}(\con^- (\overline{\Gamma}, \overline{\Delta})) \).

By symmetry, there is a bijection

\[
\Phi : \left\{ W \in \con^- (\overline{\Delta}, \overline{\Gamma}) \mid W \neq W^-_{\con^-(\Delta, \Gamma)} \right\} \to \left\{ V \in \con^- (\overline{\Gamma}, \overline{\Delta}) \mid V \neq V^-_{\con^-(\Gamma, \Delta)} \right\}
\]

so that \( W/W^-_{\con^-(\Delta, \Gamma)} \cong \Phi(W)/\Phi(W)^-_{\con^-(\Delta, \Gamma)} \) given by sending the minimal representative \( W = Y^-_\Delta + (Y \cap Z) \) to \( Z^-_\Gamma + (Y \cap Z) \). We conclude that \( \con^- (\overline{\Delta}, \overline{\Gamma}) \) and \( \con^- (\overline{\Gamma}, \overline{\Delta}) \) are subfactor equivalent as claimed. \[\square\]

From this, we deduce our “Jordan–Hölder-like theorem”.

**Theorem 4.1.14 (Theorem 3).** Let \( A \) be a skeletally small abelian category and let \( X \) be a weakly Jordan–Hölder–Schreier object in \( A \). Then there exists a composition series for \( X \) and any two composition series for \( X \) are subfactor equivalent.

**Proof.** The existence of a composition series is shown in Theorem \(3.3.7\). Moreover, if \( \Delta \) and \( \Gamma \) are two pre-subobject chains of \( X \), then there exist \( \Delta' \supseteq \Delta \) and \( \Gamma' \supseteq \Gamma \) which are subfactor equivalent by Theorem \(4.1.13\). However, by Theorem \(3.3.7\) \(1\), it must be that \( \Delta = \Delta' \) and \( \Gamma = \Gamma' \). This proves the result. \[\square\]
Remark 4.1.15. Recall from Example 3.3.8 that an upper or lower composition series may admit a proper refinement. Thus, our proof of Theorem 4.1.14 cannot be readily modified to prove a “Jordan–Hölder-like theorem” for upper or lower composition series. Moreover, we have indeed made use of both the axioms (JHS3) and (JHS4) in our proof of Theorem 4.1.14. For example, both were necessary in deducing Lemma 4.1.12 from Lemmas 4.1.8 and 4.1.10.

4.2. Duality and the Stitching Lemma. In this section, we first explore the relationship between (weakly) JHS objects in $A$ and $A^{op}$. Note that these classes of objects are generally not the same, as we show in Example 5.2.3. We conclude this section by showing how categories $A$ in which every object is weakly JHS in both $A$ and $A^{op}$ allow for a generalization of length categories. In particular, we prove Theorem [C] from the Introduction.

Lemma 4.2.1. Let $X$ be an object of $A$ which satisfies (JHS1). Then $X$ satisfies (JHS2); that is, $X$ satisfies (JHS2) as an object in $A^{op}$.

Proof. Let $\nabla = \{q_\alpha : X \to Z_\alpha\}$ be a directed system of quotient objects of $X$ in $A$. Then $\Delta := \{\ker q_\alpha \mid q_\alpha \in \nabla\}$ is a pre-subobject chain of $X$. By (JHS1) and Lemma 4.0.8 we then have an exact sequence $0 \to \operatorname{colim} \Delta \to X \to \operatorname{colim} \nabla \to 0$.

In particular, colim $\nabla$ exists. \qed

Remark 4.2.2. We emphasize that the converse of Lemma 4.2.1 may not hold. Indeed, if $\nabla = \{q_\alpha : X \to Z_\alpha\}$ is a directed system of quotient objects of $X$ in $A$ and colim $\nabla$ exists, then the kernel of the map $X \to \operatorname{colim} \nabla$ may not be the colimit of $\{\ker q_\alpha \mid q_\alpha \in \nabla\}$. See the co-Grothendieck example in Section 3.1.

Lemma 4.2.3. Let $X$ be an object in $A$ which satisfies (JHS1). Then the following are equivalent.

1. $X$ satisfies (JHS3).
2. If $\Delta$ is a pre-subobject chain of $X$ and $Z \subseteq X$ is a subobject of $X$, then $\colim(\Delta + Z) = (\colim \Delta) + Z$.
3. $X$ satisfies (JHS4) $^{op}$; i.e., $X$ satisfies (JHS4) as an object in $A^{op}$.

Proof. (1 $\implies$ 2): By (JHS1) we have that colim$(\Delta \cap Z)$, colim$(\Delta)$, and colim$(\Delta + Z)$ all exist and are subobjects of $X$. Moreover, Lemma 3.3.3 implies that colim$(\Delta \cap Z)$ is a subobject of both $Z$ and colim$(\Delta)$. In fact, colim$(\Delta + Z)$ can be seen as the colimit over $Y \in \Delta$ of the pushout of $Y \leftarrow Y \cap Z \rightarrow Z$. Thus since pushouts are colimits (and hence commute with other colimits), we have two pushout squares:

\[
\begin{array}{ccc}
\text{(colim} \Delta) \cap Z & \longrightarrow & Z \\
\downarrow & & \downarrow \\
\text{colim} \Delta & \longrightarrow & \text{colim} \Delta + Z \\
\end{array}
\]

where the upper left terms are assumed to be the same. It follows immediately that $\text{colim} \Delta + Z = (\text{colim} \Delta) + Z$.

(2 $\implies$ 1): As before, we have that colim$(\Delta \cap Z)$, colim$(\Delta)$, and colim$(\Delta + Z)$ all exist and are subobjects of $X$. Now note that for $Y \in \Delta$, we have a short exact sequence

0 $\to$ $Y \cap Z$ $\to$ $Z$ $\to$ $Z$ $\xrightarrow{Y + Z_Y}$ 0.

It then follows from Lemma 4.0.8 that there is an exact sequence

0 $\to$ colim$(\Delta \cap Z)$ $\to$ $Z$ $\to$ colim$(\Delta + Z)$ $\xrightarrow{\text{colim} \Delta}$ 0.
Likewise, there is an exact sequence

\[ 0 \to (\text{colim} \Delta) \cap Z \to Z \to \frac{(\text{colim} \Delta) + Z}{\text{colim} \Delta} \to 0. \]

Thus if \( \text{colim}(\Delta + Z) = (\text{colim} \Delta) + Z \), then we must have \( (\text{colim} \Delta) \cap Z = \text{colim}(\Delta \cap Z) \).

(1 \implies 3): We will show that \((\text{JHS}3)^{op}\) implies \((\text{JHS}4)^{op}\). Thus let \( \Delta = \{ f_\alpha : Y_\alpha \to X \} \) be a pre-subobject chain of \( X \) and note that \( \text{lim} \Delta \) exists by the dual of Lemma 4.2.1. Now let \( \nabla = \{ q_\alpha : X \to \text{coker}(f_\alpha) \mid f_\alpha \in \Delta \} \). Let \( Z \subseteq X \) be a subobject of \( X \) with inclusion map \( i_Z \), and let \( r : X \to \text{coker}(i_Z) \) be the corresponding quotient map. Then by \((\text{JHS}3)^{op}\), we have that \( \text{lim} \nabla \cap \text{coker} i_Z = (\text{lim} \nabla) \cap \text{coker} i_Z \), and that these are quotients of \( X \). By the dual of Lemma 4.0.8, the kernels of the corresponding quotient maps are \( \text{lim}(\Delta + Z) \) and \( \text{lim}(\Delta) + Z \), respectively. This proves the result.

(3 \implies 1): We will show that \((\text{JHS}4)^{op}\) and \((\text{JHS}1)^{op}\) imply \((\text{JHS}3)^{op}\).

Let \( \nabla = \{ q_\alpha : X \to Z_\alpha \} \) be a system of quotient objects of \( X \) and let \( r : X \to Y \) be a quotient of \( X \). Let \( \Delta = \{ \ker q_\alpha \mid q_\alpha \in \nabla \} \). By \((\text{JHS}1)^{op}\), the limits \( \text{lim}(\nabla \cap X Y) \) and \( \text{lim}(\nabla) \cap X Y \) exist and are quotients of \( X \). The dual of Lemma 4.0.8 then implies that there are exact sequences

\[ 0 \to \text{lim}(\Delta + \ker r) \to X \to \text{lim}(\nabla \cap X Y) \to 0 \]

\[ 0 \to (\text{lim} \Delta) + \ker r \to X \to (\text{lim} \nabla) \cap X Y \to 0. \]

\((\text{JHS}4)^{op}\) then implies that the left terms of these sequences are the same (as subobjects of \( X \)), and therefore \( \text{lim}(\nabla \cap X Y) = \text{lim}(\nabla) \cap X Y \).

\[ \Box \]

**Remark 4.2.4.** Let \( X \) be an object of an \((\text{AB}5)\) category \( \mathcal{C} \) (for example, the category of all left modules over some ring). We recall that for \( \Delta \) a pre-subobject chain of \( X \), we have \( \text{colim} \Delta = \sum_{Y \in \Delta} Y \). It then follows immediately from 4.2.3 that \( X \) satisfies \((\text{JHS}3)^{op}\).

Putting together the results from this section, we obtain the following result. To avoid confusion, if an object \( X \) is weakly JHS in both \( \mathcal{A} \) and \( \mathcal{A}^{op} \), we denote by \( \text{sf}_\mathcal{A}(X) \) and \( \text{sf}_{\mathcal{A}^{op}}(X) \) the subfactor multisets of \( X \) as an object in \( \mathcal{A} \) and \( \mathcal{A}^{op} \), respectively.

**Proposition 4.2.5.** Let \( X \) be weakly JHS in \( \mathcal{A} \).

(1) Then \( X \) is weakly JHS as an object in \( \mathcal{A}^{op} \) if and only if \( X \) satisfies \((\text{JHS}1)^{op}\).

(2) Suppose \( X \) is weakly JHS as an object in \( \mathcal{A}^{op} \). Then \( \text{sf}_\mathcal{A}(X) = \text{sf}_{\mathcal{A}^{op}}(X) \). In particular, \( X \) satisfies \((\text{JHS}5)^{op}\) if and only if it satisfies \((\text{JHS}5)^{op}\).

**Proof.** (1) Let \( X \) be weakly JHS in \( \mathcal{A} \). It then follows from Lemmas 4.2.1 and 4.2.3 that \( X \) satisfies \((\text{JHS}2)^{op}\) and \((\text{JHS}4)^{op}\). Now suppose \( X \) satisfies \((\text{JHS}1)^{op}\). Since \( X \) satisfies \((\text{JHS}4)^{op}\), Lemma 4.2.3 then implies that \( X \) satisfies \((\text{JHS}3)^{op}\) and is therefore weakly JHS in \( \mathcal{A}^{op} \). The other implication follows immediately from the definition.

(2) Suppose \( X \) is JHS as an object in \( \mathcal{A} \) and weakly JHS as an object in \( \mathcal{A}^{op} \). Let \( \Delta \) be a composition series for \( X \) (in \( \mathcal{A} \)) and note that by Theorem 3.3.7 \( \Delta \) admits no proper refinements. Now denote \( \nabla := \{ X/Y \mid Y \in \Delta \} \) and note that \( \nabla \) is a subobject chain of \( X \) in \( \mathcal{A}^{op} \). Moreover, for \( \nabla' \subseteq \nabla \), denote \( \Delta' = \{ Y \in \Delta \mid X/Y \in \nabla' \} \). Then by Lemma 4.0.8 the axioms \((\text{JHS}3)^{op}\) and \((\text{JHS}3)^{op}\) imply that there are exact sequences

\[ 0 \to \text{colim} \Delta' \to X \to \text{colim} \nabla' \to 0 \]

\[ 0 \to \text{lim} \Delta' \to X \to \text{lim} \nabla' \to 0. \]

In particular, for all \( Y \in \Delta \), we have \( (X/Y)_{\nabla} = X/(Y_{\Delta}) \). This implies that \( \text{sf} \Delta = \text{sf} \nabla \). It then follows that \( \nabla \) admits no proper refinements, and therefore is a composition series by Theorem 3.3.7. \[ \Box \]
In the remainder of this section, we show that categories in which every object satisfies the hypotheses of Proposition 4.2.5 share many properties with length categories. Indeed, every object of finite length is JHS and satisfies \(\text{(JHS1)}\) (see Proposition 5.1.1), making this setting quite natural.

**Lemma 4.2.6** ("The Stitching Lemma"). Suppose every object in \(\mathcal{A}\) is weakly JHS and satisfies \(\text{(JHS1)}\). Let \(\mathcal{D}\) and \(\mathcal{E}\) be subcategories of \(\mathcal{A}\) which are closed under isomorphisms. If an object \(X\) in \(\mathcal{A}\) has a \(\mathcal{D}\)-filtration \(\Delta_{\mathcal{D}}\) and each \(W \in \text{sf}(\Delta_{\mathcal{D}})\) has an \(\mathcal{E}\)-filtration, then \(X\) has an \(\mathcal{E}\)-filtration.

**Proof.** We explicitly construct the \(\mathcal{E}\)-filtration. For each \(W_Y := Y/Y_{\Delta_{\mathcal{D}}} \in \text{sf}(\Delta_{\mathcal{D}})\), let \(\Gamma_Y\) be an \(\mathcal{E}\)-filtration of \(W_Y\). Now given \(j_{Y,V} : V \hookrightarrow W_Y\) in \(\Gamma_Y\), we denote the kernel of the map \(Y \to W_Y \to W_Y/V\) by \(h_{Y,V} : Z_{Y,V} \to Y\). Define \(\iota_{Y,V} := \iota_Y \circ h_{Y,V}\) and

\[
\Delta_{\mathcal{E}} := \left( \bigcup_{W_Y \in \text{sf}(\Delta_{\mathcal{D}})} \{\iota_{Y,V} : Z_{Y,V} \to X \mid V \in \Gamma_Y\} \right) \bigcup \Delta_{\mathcal{D}}.
\]

Whenever \(V \subseteq V' \subseteq W_Y\), we have \(Y_{\Delta_{\mathcal{D}}} \subseteq Z_{Y,V} \subseteq Z_{Y,V'} \subseteq Y\). Thus, \(\Delta_{\mathcal{E}}\) is a subobject chain of \(X\). The completeness and cocompleteness of \(\Delta_{\mathcal{E}}\) follow from the fact that every object in \(\mathcal{A}\) satisfies all of \(\text{(JHS1)}\), \(\text{(JHS2)}\), \(\text{(JHS1)}\), and \(\text{(JHS2)}\) as a result of Proposition 4.2.5. \(\square\)

As a consequence, we obtain the final result of this section, offering a generalization of length categories.

**Theorem 4.2.7** (Theorem C). Let \(\mathcal{A}\) be a skeletally small abelian category in which every object is weakly Jordan–Hölder–Schreier and satisfies \(\text{(JHS1)}\). Let \(Y \subseteq X\) be objects in \(\mathcal{A}\) and consider the exact sequence

\[
0 \to Y \xrightarrow{i} X \xrightarrow{q} X/Y \to 0.
\]

Let \(\Delta_Y\) and \(\Delta_{X/Y}\) be composition series of \(Y\) and \(X/Y\), respectively. Then:

1. There is a composition series of \(X\) given by \(\Delta_Y \cup q^{-1}(\Delta_{X/Y})\). Moreover, this composition series induces a bijection \(\text{sf}(Y) \bigcup \text{sf}(X/Y) \cong \text{sf}(X)\).

2. If \(Y\) and \(X/Y\) are Jordan–Hölder–Schreier, then \(Y = X\) if and only if the induced inclusion \(\text{sf}(Y) \hookrightarrow \text{sf}(X)\) is a bijection.

**Proof.** (1) Let \(\mathcal{D}\) be the full subcategory of \(\mathcal{A}\) consisting of objects isomorphic to either \(Y\) or \(X/Y\) and let \(\mathcal{E}\) be the full subcategory of \(\mathcal{A}\) consisting of simple objects. We then have that \(\{0,Y,X\}\) is a bicomplete \(\mathcal{D}\)-filtration with \(\text{sf}([0,Y,X]) = \{Y,X/Y\}\) and both \(Y\) and \(X/Y\) have \(\mathcal{E}\)-filtrations, which are composition series by Theorem 3.3.7.

We note that \(\{0,Y,X\}\) is a bicomplete subobject chain. Lemma 4.2.6 then implies that there is an \(\mathcal{E}\)-filtration (composition series) of \(X\) which refines \(\{0,Y,X\}\). Moreover, by the construction in the proof of Lemma 4.2.6 we have that this composition series is given by \(\Delta_Y \cup q^{-1}(\Delta_{X/Y})\) and induces the desired bijection on subfactor sets.

(2) Let us have that the induced inclusion \(\text{sf}(Y) \hookrightarrow \text{sf}(X)\) is a bijection if and only if \(\text{sf}(X/Y) = \emptyset\). By the axiom \(\text{(JHS5)}\), this is equivalent to \(X/Y = 0\). \(\square\)

5. Examples and Discussion

In this section, we provide several examples and non-examples of (weakly) Jordan–Hölder–Schreier objects.
5.1. Finite length. We first show that Jordan–Hölder–Shreier objects generalize objects of finite length.

**Proposition 5.1.1.** Let $X$ be an object in $\mathcal{A}$ of finite length. Then $X$ is JHS in both $\mathcal{A}$ and $\mathcal{A}^{\text{op}}$.

**Proof.** We note that $X$ has finite length as an object in either $\mathcal{A}$ or $\mathcal{A}^{\text{op}}$. Therefore, we need only prove that $X$ is JHS in $\mathcal{A}$.

Let $\Delta$ be a pre-subobject chain of $X$. Since $X$ has finite length, it is both noetherian and artinian. Therefore, we know that $\Delta$ is a finite set. This means $\text{colim} \Delta = \text{max} \Delta$, $\text{lim} \Delta = \text{min} \Delta$, and both exist (and are subobjects of $X$). Moreover, the axioms (JHS3) and (JHS4) follow immediately from the fact that finite colimits commute with finite limits in abelian categories. Finally, suppose that $\Delta$ is a composition series of $X$ and that $X \neq 0$. Since $\Delta$ is finite, we have $X_\Delta \neq X$ and thus $\text{sf}(\Delta) \neq \emptyset$. We conclude that $X$ is JHS. $\square$

5.2. Colimits of Jordan–Hölder–Schreier objects. In this section, we consider objects which are themselves colimits of (weakly) Jordan–Hölder–Schreier objects. In particular, this includes colimits of finite length objects.

**Proposition 5.2.1.** Let $X$ be an object which is subobject bicomplete and satisfies (JHS3) in an abelian category $\mathcal{A}$. Suppose there exists a subobject chain $\Gamma$ of $X$ so that $X_\Gamma = X$ and every $Y \in \Gamma \setminus \{X\}$ is weakly JHS. Then $X$ is weakly JHS. Moreover, if there exists $0 \neq Y \in \Gamma \setminus \{X\}$ which is JHS, then $X$ is JHS.

**Proof.** Let $\Delta$ be a pre-subobject chain of $X$ and let $Z \subseteq X$ be a subobject. Now for $Y \in \Gamma \setminus \{X\}$, the fact that $Y$ is weakly JHS and intersections are limits implies that

$$Y \cap ((\text{lim} \Delta) + Z) = \text{lim}(Y \cap \Delta) + Y \cap Z = \text{lim}(Y \cap \Delta + Y \cap Z) = Y \cap \text{lim}(\Delta + Z)$$

To simplify notation, let $\Gamma' = \Gamma \setminus \{X\}$. Since $X$ satisfies (JHS3) we then have

$$(\text{lim} \Delta) + Z = (\text{colim} \Gamma') \cap ((\text{lim} \Delta) + Z) = \text{colim}[\Gamma' \cap ((\text{lim} \Delta) + Z)] = \text{colim}[\Gamma' \cap \text{lim}(\Delta + Z)] = (\text{colim} \Gamma') \cap \text{lim}(\Delta + Z) = \text{lim}(\Delta + Z)$$

We conclude that $X$ satisfies (JHS4) and is therefore weakly JHS. For the moreover part, suppose there exists a nonzero subobject $Y \subseteq X$ that satisfies (JHS5). Now recall from Theorem 3.3.7 that there exists a composition series $\Delta$ of $Y$, and that this composition series admits no proper refinements. Moreover, the same theorem implies that there exists a composition series $\Delta'$ of $X$ which is a refinement of $\Delta \cup \{X\}$. Since $\Delta$ admits no proper refinements, it follows that $(Y')_{\Delta} = (Y')_{\Delta}$, for all $Y' \in \Delta$. In particular, we have $\emptyset \neq \text{sf} \Delta \subseteq \text{sf} \Delta'$. $\square$

As an immediate corollary of Proposition 5.2.1, we obtain the following result.

**Corollary 5.2.2.** Let $R$ be a ring and let $\Delta$ be a set of finite length (left) $R$-modules which is totally ordered under inclusion. Then $\text{colim} \Delta$ is JHS. In particular:

1. Any Prufer module over any ring is JHS.
2. Any countably generated (left) module over any (left) artinian ring is JHS.

**Proof.** We note that $\text{colim} \Delta$ is subobject bicomplete and satisfies (JHS3) since we are working in a module category (see Remark 4.2.4). The result thus follows immediately from Propositions 5.1.1 and 5.2.1. $\square$
We conclude this section with an important example showing that objects which are (weakly) JHS in an abelian category \( \mathcal{A} \) may not be (weakly) JHS in \( \mathcal{A}^{\text{op}} \).

**Example 5.2.3.** Let \( k \) be an arbitrary field and let \( \text{Vec}(k) \) be the category of \( k \)-vector spaces. Then \( \bigoplus_{N} k \) is JHS by Corollary 5.2.2. However, the limit of the pre-quotient chain
\[
\cdots \to k \oplus k \oplus k \to k \oplus k \to k \to 0
\]
of \( \bigoplus_{N} k \) is \( \prod_{N} k \), which is not a quotient object of \( \bigoplus_{N} k \). We conclude that \( \bigoplus_{N} k \) does not satisfy \( \text{(JHS1)} \), and so is not (weakly) JHS in \( \text{Vec}(k)^{\text{op}} \).

### 5.3. Functor categories.
In this section, we consider functor categories in which every object in the target is (weakly) Jordan–Hölder–Schreier. In particular, this includes the category of pointwise finite-dimensional (pwf) persistence modules over any field and small category. For certain small categories, we are further able to describe the composition factors explicitly.

**Proposition 5.3.1.** Let \( \mathcal{C} \) be a small category and let \( \mathcal{A} \) be an abelian category in which every object is weakly JHS. Then:

1. Every object in \( \text{Fun}(\mathcal{C}, \mathcal{A}) \) is weakly JHS.
2. If every object in \( \mathcal{A} \) satisfies \( \text{(JHS1)}^{\text{op}} \), then every object in \( \text{(Fun}(\mathcal{C}, \mathcal{A}))^{\text{op}} \) is weakly JHS.

**Proof.** (1) Let \( M \in \text{Fun}(\mathcal{C}, \mathcal{A}) \) and let \( \Delta \) be a pre-subobject chain of \( M \). For \( x \in \mathcal{C} \), we have that \( \text{colim}(\Delta(x)) \) exists and is a subobject of \( M(x) \) since \( M(x) \in \mathcal{A} \) satisfies \( \text{(JHS1)} \). It follows that \( \text{colim}(\Delta) \) exists and that the natural map \( \text{colim}(\Delta) \to M \) is a monomorphism. Therefore, \( \text{Fun}(\mathcal{C}, \mathcal{A}) \) satisfies \( \text{(JHS1)} \). The proof that \( \text{Fun}(\mathcal{C}, \mathcal{A}) \) satisfies \( \text{(JHS2)} \) is similar.

Now let \( N \subseteq M \) be an arbitrary subfunctor of \( M \). Once again, for \( x \in \mathcal{C} \), we have \( \text{colim}(\Delta(x) \cap N(x)) = \text{colim}(\Delta(x)) \cap N(x) \) since \( M(x) \in \mathcal{A} \) satisfies \( \text{(JHS3)} \). It again follows immediately that \( \text{colim}(\Delta \cap N) = \text{colim}(\Delta) \cap N \) in \( \text{Fun}(\mathcal{C}, \mathcal{A}) \), and so \( \text{Fun}(\mathcal{C}, \mathcal{A}) \) satisfies \( \text{(JHS3)} \). The proof that \( \text{(JHS4)} \) is satisfied is again similar.

(2) We recall that \( \text{Fun}(\mathcal{C}, \mathcal{A})^{\text{op}} \) can be identified with \( \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{A}^{\text{op}}) \). The result thus follows from (1) and Proposition 4.2.5. \( \square \)

We now wish to apply Proposition 5.3.1 to pointwise finite-dimensional persistence modules. To do so, we recall that a small category \( \mathcal{C} \) is called directed if (a) for all objects \( x \neq y \in \mathcal{C} \), at least one of \( \text{Hom}_{\mathcal{C}}(x, y) \) and \( \text{Hom}_{\mathcal{C}}(y, x) \) is empty and (b) for all objects \( x \in \mathcal{C} \), \( \text{Hom}_{\mathcal{C}}(x, x) = \{1_{x}\} \). For example, any poset category is directed.

We also fix the following notation. Given an object \( X \) in an abelian category \( \mathcal{A} \) and an object \( x \) of a directed category \( \mathcal{C} \), we denote \( \tilde{X}_{x} : \mathcal{C} \to \mathcal{A} \) the functor which has \( \tilde{X}_{x}(x) = X \) and \( \tilde{X}_{x}(y) = 0 \) for all other objects \( y \in \mathcal{C} \). We note that \( \tilde{X}_{x} \) is simple in \( \text{Fun}(\mathcal{C}, \text{vec}(k)) \) if and only if \( X \) is simple in \( \mathcal{A} \). Moreover, given a functor \( M : \mathcal{C} \to \mathcal{A} \), we denote by \( \text{supp}(M) \) the set of objects \( x \in \mathcal{C} \) for which \( M(x) \neq 0 \). Finally, recall that the disjoint union of multisets is again a multiset where the multiplicity of elements is additive in the sense of cardinalities.

**Theorem 5.3.2.** Let \( \mathcal{C} \) be a small directed category, let \( \mathcal{A} \) be a skeletally small abelian category in which every object of \( \mathcal{A} \) is JHS, and let \( M \) be a functor in \( \text{Fun}(\mathcal{C}, \mathcal{A}) \). Then \( M \) is Jordan–Hölder–Schreier and satisfies
\[
\text{sf}(M) = \prod_{x \in \text{supp}(M)} \left\{ \tilde{S}_{x} \mid S \in \text{sf}(M(x)) \right\}.
\]

**Proof.** The fact that \( M \) is weakly JHS follows from Proposition 5.3.1. Since \( \text{supp}(M) = \emptyset \) if and only if \( M = 0 \), we thus need only show the subfactors of \( M \) are as described.

We first choose some total order \( \leq \) on the objects of \( \text{supp}(M) \) so that \( \text{Hom}_{\mathcal{C}}(x, y) = \emptyset \) whenever \( y \leq x \) and \( x, y \in \text{supp}(M) \). For each \( x \in \text{supp}(M) \), we then choose a composition series \( \Omega_{x} \) of \( M(x) \).
Now for each $X \in \Omega_x$ we define a functor $M_{x,X}$ in $\text{Fun}(\mathcal{C}, \mathcal{A})$ as follows. For $y$ an object in $\mathcal{C}$, we set

$$M_{x,X}(y) = \begin{cases} M(y) & x \leq y \\ X & y = x \\ 0 & y \leq x, \end{cases}$$

For $f : y \rightarrow y'$ a morphism in $\mathcal{C}$, we then set $M_{x,X}(f) = M(f)|_{M_{x,X}(y)}$. It is straightforward that each $M_{x,X}$ is well defined.

Furthermore, we see that $M_{x,X} \subseteq M_{y,Y}$ if and only if either (a) $y \leq x$ in $\mathcal{C}$, or (b) $x = y$ in $\mathcal{C}$ and $X \subseteq Y$ in $\mathcal{A}$. Now recall that, in a total ordered set, an interval $I$ is closed above (respectively, below) if $y \in I$ and $y \leq z$ (respectively, $z \leq y$) implies $z \in I$. For each interval $I$ of objects in $\text{supp}(M)$ which is closed above (with respect to $\leq$), we define a functor $M_I$ in the following way. For each object $x$ in $\mathcal{C}$, define

$$M_I(y) = \begin{cases} M(y) & y \in I \\ 0 & y \notin I. \end{cases}$$

For each morphism $f : y \rightarrow y'$ in $\mathcal{C}$, we set $M_I(f) = M(f)|_{M_I(y)}$. We note that if $I$ has an infimum (with respect to $\leq$), then there are two possibilities. Either $\inf I \notin I$, and so $M_I = M_{x,0}$, or $\inf I \in I$, and so $M_I = M_{x,M(x)}$. If, on the other hand, $I$ does not have an infimum, then $M_I$ is not equal to $M_{x,X}$ for any object $x$ in $\mathcal{C}$ and $X \in \Omega_x$.

We now see that

$$\Delta = \{M_{x,X} \mid x \in \text{supp}(M), X \in \Omega_X \} \cup \{M_I \mid I \subseteq \text{supp}M \text{ is closed above} \}$$

is a subobject chain of $M$. It remains to show that $\Delta$ is a composition series and that the subfactors of $\Delta$ are as described.

Let $x \in \text{supp}(M)$ and recall that $\Omega_x$ is cocomplete. If $X \in \Omega_x$ has a predecessor $X_\Omega^{-}$, denote $Y := X_\Omega^{-}/X_\Omega^{-}$. Then $(M_{x,X})_\Delta = M_{x,X_\Omega^{-}}$, and so $M_{x,X}/(M_{x,X})_\Delta$ is the simple functor $\tilde{Y}_x$. If $X \in \Omega_x$ does not have a predecessor, then $(M_{x,X})_\Delta = M_{x,X}$. Now suppose that $x$ has an immediate successor $y \in \text{supp}(M)$ under the restriction of $\leq$; that is, $y$ is such that if $x \leq z \leq y$ and $z \in \text{supp}(M)$, then $z = x$. In this case, it follows that $M_{x,0} = M_{y,M(y)}$. Otherwise, we have $(M_{x,0})_\Delta = M_{x,0}$. Finally, suppose that $I$ is an interval of objects in $\text{supp}(M)$ that is closed above such that $M_I \neq M_{x,X}$ for any object $x$ in $\mathcal{C}$ and $X \in \Omega_x$. Note that the intervals that are closed above also form a totally ordered set under inclusion. Then there exists no interval $I'$ such that $I'$ is the predecessor or successor to $I$ by inclusion. Thus, $(M_I)_\Delta = M_I$ and $(M_I)^\perp = M_I$.

It follows from the above paragraph that

$$\text{sf}(M) = \coprod_{x \in \text{supp}M} \left\{ S_x \mid S \in \text{sf}(M(x)) \right\}$$

as claimed. This means $\Delta$ is a composition series if it is bicomplete. To see this, let $\Delta' \subseteq \Delta$ and, for each object $x$ in $\text{supp}(M)$, denote by $I_x$ the interval of objects in $\text{supp}(M)$ which is closed above and has minimal element $x$. Let

$$\mathcal{B} = \{ I \subseteq \text{supp}(M) \text{ closed above} \mid \exists x \in \text{supp}(M), X \in \Omega_x : M_{x,X} \in \Delta', I \subseteq I_x \}$$

and

$$\cup \{ I \subseteq \text{supp}(M) \text{ closed above} \mid \exists J \subseteq \text{supp}(M) \text{ closed above} : M_J \in \Delta', I \subseteq J \}.$$
follows that \( \text{colim} \Delta' = M_{x, \text{colim} \Omega_x} \in \Delta \). We conclude that \( \Delta \) is cocomplete. The argument that \( \Delta \) is complete is similar.

We now specialize Proposition 5.3.1 and Theorem 5.3.2 to obtain Theorem D from the introduction.

**Corollary 5.3.3** (Theorem D).

1. Let \( C \) be a directed small category, let \( k \) be a field, and let \( M : C \to \text{vec}(k) \) be a pointwise finite-dimensional \( C \)-persistence module. Then \( M \) is Jordan–Hölder–Schreier and satisfies

\[
\text{sf}(M) = \prod_{x \in \text{supp} M} \left( \prod_{i=1}^{\text{dim}_k M(x)} \{ k_x \} \right).
\]

2. Let \( A \) be an abelian category such that every object of \( A \) is Jordan–Hölder–Schreier and let \( X \) be a topological space. Then any presheaf on \( X \) with values in \( A \) is Jordan–Hölder–Schreier.

**Proof.** (1) For any field \( k \), every object in \( \text{vec}(k) \) is JHS by Proposition 5.1.1. (2) The source category \( \text{Top}(X)^{op} \) of a presheaf is the opposite category of a directed category, and so itself is directed.

**Remark 5.3.4.** We note that the proof technique for Corollary 5.3.3 will not work for all sheaves \( O \) of a space \( X \) with values in \( A \). Suppose there is a covering \( \{ U_i \} \) of \( U \) in \( \text{Top}(X) \) and a pair \( s_j \in \mathcal{O}(U_j) \) and \( s_k \in \mathcal{O}(U_k) \) such that \( s_j|_{U_j \cap U_k} = s_k|_{U_j \cap U_k} \neq 0 \). Then \( \mathcal{O} \) is not a sheaf. However, it remains an interesting question: if every object of \( A \) is JHS, when does a sheaf \( O \) on a space \( X \) with values in \( A \) have a composition series of sheaves in the sense of Definition 3.2.6?

5.4. **Igusa and Todorov’s representations of \( \mathbb{R} \).** In this section we show that the (nonzero) representations examined by Igusa and Todorov in [IT15] are weakly JHS, but are not JHS. We recall from Section 2.3 that for \( k \) an arbitrary field, we denote by \( \text{Fun}(\mathbb{R}, \text{vec}(k)) \) the category of (covariant) functors from \( \mathbb{R} \), considered as a poset category, to the category of finite-dimensional \( k \)-vector spaces. Moreover, given \( M \in \text{Fun}(\mathbb{R}, \text{vec}(k)) \) and \( x \leq y \in \mathbb{R} \), we denote by \( M(x, y) \) the result of applying \( M \) to the unique morphism \( x \to y \) in \( \mathbb{R} \).

**Definition 5.4.1.** We denote by \( \text{IT}(\mathbb{R}) \) the full subcategory of \( \text{Fun}(\mathbb{R}, \text{vec}(k)) \) consisting of functors \( M : \mathbb{R} \to \text{vec}(k) \) that are JHS.

(IT1) For all \( x \in \mathbb{R} \), we have \( \lim_{y>x} V(y) = V(x) \) and

(IT2) For all but finitely many \( x \in \mathbb{R} \), there exists \( \varepsilon > 0 \) such that \( M(x, x+\varepsilon) \) is an isomorphism.

In [IT15], Igusa and Todorov studied functors \( \mathbb{R}^{op} \to \text{vec}(k) \) that satisfied the symmetric condition to [IT1] and the same condition [IT2] in Definition 5.4.1. We may then use their results.

**Theorem 5.4.2** (Adapted from [IT15]).

1. The category \( \text{IT}(\mathbb{R}) \) is an abelian and extension closed subcategory of \( \text{Fun}(\mathbb{R}, \text{vec}(k)) \).

2. For every interval \( I \) that is closed below and open above, there is an interval indecomposable \( M_I \) in \( \text{IT}(\mathbb{R}) \) given by

\[
M_I(x) = \begin{cases} k & x \in I \\ 0 & x \notin I \end{cases} \quad M_I(x, y) = \begin{cases} 1_k & x \leq y \in I \\ 0 & \text{otherwise}. \end{cases}
\]

3. Every representation in \( \text{IT}(\mathbb{R}) \) is isomorphic to a finite direct sum of interval indecomposables. This direct sum is unique up to permuting the direct summands.

We note that (2) includes intervals of the form \([a, +\infty)\) for every \( a \in \mathbb{R} \). By results in [BC-B20, IRT19+], each object in \( \text{Fun}(\mathbb{R}, \text{vec}(k)) \) is also a direct sum of interval indecomposables, but the intervals may be of any form and the sum may be infinite.
We have already shown in Propositions 5.1.1 and 5.3.1 that each object $M$ in $\text{IT}(\mathbb{R})$ is JHS when considered as an object in $\text{Fun}(\mathbb{R}, \text{vec}(k))$. However, as a subcategory of $\text{Fun}(\mathbb{R}, \text{vec}(k))$, $\text{IT}(\mathbb{R})$ is not closed under subobjects. As a result, the colimits and limits of pre-subobject chains will generally differ depending on which category we are working in. The following lemma describes this relationship precisely. To increase readability in the proof of the lemma, we use $\bigoplus$ for direct sums of objects and $\bigcup$ for sums of morphisms using the abelian structure of $\text{IT}(\mathbb{R})$.

**Lemma 5.4.3.** Let $X$ be an object in $\text{IT}(\mathbb{R})$ and $\Delta$ a pre-subobject chain of $X$ in $\text{IT}(\mathbb{R})$.

1. Let $\text{colim}_{\text{Fun}} \Delta := \bigoplus_{\lambda} M_{I_{\lambda}}$ be the colimit of $\Delta$ in $\text{Fun}(\mathbb{R}, \text{vec}(k))$. Then the colimit of $\Delta$ in $\text{IT}(\mathbb{R})$ is given by $\text{colim}_{\text{IT}} \Delta := \bigoplus_{\lambda} M_{I_{\lambda} \cup (I_{\lambda})}$.
2. The limit $\text{lim}_{\text{Fun}} \Delta$ of $\Delta$ in $\text{Fun}(\mathbb{R}, \text{vec}(k))$ is also the limit of $\Delta$ in $\text{IT}(\mathbb{R})$.

**Proof.** By Proposition 5.3.1, we know $\text{colim}_{\text{Fun}} \Delta$ and $\text{lim}_{\text{Fun}} \Delta$ exist and are subfunctors of $M$. By [IRT194, Theorem 3.0.1], we know that $M$, and thus $\text{colim}_{\text{Fun}} \Delta$ and $\text{lim}_{\text{Fun}} \Delta$, are finitely generated in $\text{Fun}(\mathbb{R}, \text{vec}(k))$. In particular, this means $\text{colim}_{\text{Fun}} \Delta$ and $\text{lim}_{\text{Fun}} \Delta$ are finite direct sums of interval indecomposables. It follows that both $\text{lim}_{\text{Fun}} \Delta$ and $\text{colim}_{\text{Fun}} \Delta$ satisfy (IT2).

To see that $\text{lim}_{\text{Fun}} \Delta$ satisfies (IT1), we note that for all $x \in \mathbb{R}$, we have that $(\text{vec}(k))$

$$\lim_{y>x} (\text{lim}_{\text{Fun}} \Delta(y)) = \lim_{N \in \Delta} \left( \lim_{y>x} N(y) \right).$$

Since each $N \in \Delta$ is in $\text{IT}(\mathbb{R})$, this implies that $\lim_{y>x} (\text{lim}_{\text{Fun}} \Delta(y)) = \text{lim}_{\text{Fun}} \Delta(x)$, as desired.

Now write $\text{colim}_{\text{Fun}} \Delta = \bigoplus_{i=1}^{m} M_{I_{i}}$ and $M_i = \bigoplus_{k=1}^{p} M_{K_{k}}$, where each $M_{I_{i}}$ and $M_{K_{k}}$ is an interval indecomposable. Let $\tau : \text{colim}_{\text{Fun}} \Delta \to M$ be the inclusion map and write $\tau = \sum_{i=1}^{m} \sum_{k=1}^{p} \tau_{i,k} : M_{I_{i}} \to M_{K_{k}}$, with each $\tau_{i,k} : M_{I_{i}} \to M_{K_{k}}$. We claim that each $I_{i}$ is open on the right. Indeed, if $I_{i}$ is closed on the right, then there exists some $K_{k}$ so that $(\tau_{i,k} \sup I_{i}) : M_{I_{i}}(\sup I_{i}) \to M_{K_{k}}(\sup I_{i})$ is nonzero. But this can only happen if $\sup I_{i} = \sup K_{k} \in K_{k}$, a contradiction. In particular, this implies that $\text{colim}_{\text{IT}} \Delta$ is indeed an object in $\text{IT}(\mathbb{R})$.

It remains to show that $\text{colim}_{\text{IT}}(\mathbb{R}) \Delta$ satisfies the universal property of the colimit. To see this, let $N = \bigoplus_{j=1}^{n} M_{J_{j}}$ be an object in $\text{IT}(\mathbb{R})$ and let $g = \sum_{i=1}^{m} \sum_{j=1}^{n} g_{i,j} : M_{I_{i}} \to M_{J_{j}}$ be a morphism in $\text{Fun}(\mathbb{R}, \text{vec}(k))$. Since each $J_{j}$ is closed on the left by assumption, it follows that each $g_{i,j}$ factors through $M_{I_{\cup} I_{\cup I}}$. Thus, $g$ factors through $\text{colim}_{\text{IT}} \Delta$. This proves the result.

**Proposition 5.4.4.** Every nonzero object $X$ in $\text{IT}(\mathbb{R})$ is weakly JHS but not JHS.

**Proof.** We first show every object is weakly JHS and then show that $\text{IT}(\mathbb{R})$ has no simple objects. This will imply that $\text{sf}(M) = \emptyset$ for all objects $M$ of $\text{IT}(\mathbb{R})$, and so no nonzero object in $\text{IT}(\mathbb{R})$ is JHS.

Let $M$ be an object of $\text{IT}(\mathbb{R})$. We note that $M$ satisfies (JHS1) [JHS2] and (JHS4) in $\text{IT}(\mathbb{R})$ as an immediate consequence of Proposition 5.3.1 and Lemma 5.4.3. To see that $M$ satisfies (JHS3), let $\Delta$ be a pre-subobject chain of $M$. Then for $Z$ a subobject of $M$ in $\text{IT}(\mathbb{R})$, Proposition 5.3.1 implies that $\text{colim}_{\text{Fun}}(\Delta \cap Z) = (\text{colim}_{\text{Fun}} \Delta) \cap Z$. Now, for $x \in \mathbb{R}$, the description of colimits in the category $\text{IT}(\mathbb{R})$ given in Lemma 5.4.3 implies that

$$((\text{colim}_{\text{IT}} \Delta) \cap Z)(x) = \lim_{y>x} ((\text{colim}_{\text{Fun}} \Delta) \cap Z)(y) = \lim_{y>x} (\text{colim}_{\text{Fun}}(\Delta \cap Z))(y) = (\text{colim}_{\text{IT}}(\Delta \cap Z))(x).$$

Thus, $M$ satisfies (JHS4) in $\text{IT}(\mathbb{R})$.

Now we show that $\text{IT}(\mathbb{R})$ contains no simple objects. For contradiction, suppose $S$ is simple in $\text{IT}(\mathbb{R})$. Then $S \cong M_{I}$ for some interval $I$ that is closed below and open above. However, this means that $\inf I \neq \sup I$, and so there exists $\varepsilon > 0$ such that $\inf I < \varepsilon + \inf I < \sup I$. Then $M_{[\varepsilon + \inf I, \sup I]}$ is a subobject of $S$, a contradiction.
Example 5.4.5. Let $I = [a, b)$ for a pair of real numbers $a < b$. Then

$$\Delta = \{ M_{[x, b)} \mid a \leq x < b \} \cup \{0\}$$

is a composition series of $M_I$ in $\text{IT}(\mathbb{R})$ which satisfies $\text{sf}(\Delta) = \emptyset$.

5.5. Non-example: infinite products. We conclude by showing that infinite products in general are not (weakly) JHS and can admit two composition series with different cardinalities of subfactor multisets. We note that the following setting example in particular.

Proposition 5.5.1. Let $A$ be an abelian category which is possibly not skeletally small and has both arbitrary coproducts and arbitrary products. Then

1. If $S$ is a nonzero object in $A$, then the infinite product $\prod_{n \in \mathbb{N}} S$ does not satisfy $(\text{JHS}4)$, and is therefore not weakly JHS.

2. If $S$ is simple in $A$, then the infinite product $\prod_{n \in \mathbb{N}} S$ admits composition series $\Delta$ and $\Gamma$ with $|\text{sf}(\Delta)| = |\mathbb{N}| \neq |\text{End}_A(S)|^{|\mathbb{N}|} = |\text{sf}(\Gamma)|$.

Proof. (1) For clarity, denote $X = \prod_{n \in \mathbb{N}} S$. Now for each $n \in \mathbb{N}$, denote $X_n := \prod_{n \in \mathbb{N}\setminus[0,n]}$ and consider the inclusion $X_n \to X$ given by mapping the $i$-th factor to the $i$-th factor. We then obtain a subobject chain $\Delta = \{0, \ldots, X_2, X_1, X\}$ of $X$.

Again for clarity, denote $\Delta_0 = \Delta \setminus \{0\}$. We claim that $\lim \Delta_0 = 0$. To see this, consider any nonzero map $f: M \to X$. Then there must be a smallest $n\in \mathbb{N}$ so that the composition $\pi_j \circ f$ of $f$ with the $j$-th projection map is nonzero. Then $f$ does not factor through the inclusion $X_j \subseteq X$, and so $f$ is not a solution to $\Delta_0$. We conclude that $\lim \Delta_0 = 0$ as claimed.

Now consider the subobject $Y := \bigoplus_{n \in \mathbb{N}} S$ of $X$, with the inclusion map again mapping the $i$-th factor to the $i$-th factor. We note that for $n \in \mathbb{N}$, we have

$$Y + X_n = \left( \prod_{n \in \mathbb{N}\setminus[0,n]} \right) \oplus \left( \bigoplus_{i=0}^n S \right) = X.$$

This implies that that $(\lim \Delta_0) + Y = Y \nsubseteq X = \lim(\Delta_0 + Y)$. This proves the result.

(2) Let $S$ be simple in $A$, and let $X, \Delta, \text{and } \text{and } \Gamma$ be as in the proof of (1). We claim $\Delta$ is then a composition series of $S$.

Indeed, it is clear that $\Delta$ is bicomplete, since any $\Delta' \subseteq \Delta$ contains a maximal element and either contains a minimal element or satisfies $\lim \Delta' = \lim \Delta_0 = 0$. Moreover, we have $X_0 = X_0$ and $X/X_0 \cong S$. Likewise, for $n \in \mathbb{N}$, we have $(X_n)_0 = X_{n+1}$ and $X_n/X_{n+1} \cong S$. We conclude that all of the subfactors of $\Delta$ are simple, and therefore $\Delta$ is a composition series which satisfies $|\text{sf}(\Delta)| = |\mathbb{N}|$.

We now construct a second composition series of $X$ with larger cardinality. To begin, we note that since $S$ is simple, the ring $\text{End}_A(S)$ is either a field or division ring. Thus we have that

$$\text{Hom}_A(S, X) \cong \prod_{n} \text{End}_A(S)^{op}$$

is a free $\text{End}_A(S)^{op}$-module. It is well-known that the $(\text{End}_A(S)^{op})$-dimension of this module is $|\text{End}_A(S)|^{|\mathbb{N}|}$, which in particular is uncountable. See e.g. [J53, Theorem XI.5.2]. Thus choose some basis $B$ of $\text{Hom}_A(S, X)$ and choose a well-order $\leq$ on $B$ for which $B$ contains a minimal element $b_0$ and maximal element $b_1$.

Now for any subset $B' \subseteq B$, denote

$$X_{B'} := \sum_{b \in B'} \text{im}(b).$$

We note that if $B' \subseteq B'' \subseteq B$, then there is a proper inclusion $X_{B'} \hookrightarrow X_{B''}$. Moreover, we have $X_B = X$. Since $B$ is well-ordered, we then have a subobject chain

$$\Gamma = \{0, X_{\{b_0\}}, X_B \} \cup \{ X_{\{b_0, b\}} \mid b \in B \setminus \{b_0\}\}$$

of $X$. Now let $\Gamma' \subseteq \Gamma$. If $0 \in \Gamma'$, then clearly $\lim \Gamma' = 0 \in \Gamma$. Otherwise, define

$$J_0 := \bigcap_{I \subseteq B : X_I \in \Gamma'} I$$

and

$$J_1 := \bigcup_{I \subseteq B : X_I \in \Delta'} I.$$  

It follows that $\lim \Gamma' = X_{J_0} \in \Gamma$ and $\colim \Gamma' = X_{J_1} \in \Gamma$. We conclude that $\Delta$ is bicomplete. Moreover, we have $0_\Gamma = X_{\{b_0\}}$ and $X_{\{b_0\}}/0 \cong S$. Likewise, for $b \in B \setminus \{b_0\}$, we have $(X_{\{b_0, b\}})^\Gamma = X_{\{b_0, b\}}$ and $X_{\{b_0, b\}}/X_{\{b_0\}} \cong S$. We conclude that $\Gamma$ is a composition series and $|\sf(\Gamma)| = |B|$.

\[ \square \]

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