Ergodic Property of Markovian Semigroups on Standard Forms of von Neumann Algebras

Yong Moon Park

Department of Mathematics, Yonsei University
Seoul 120-749, Korea
E-mail: ympark@yonsei.ac.kr

Abstract

We give sufficient conditions for ergodicity of the Markovian semigroups associated to Dirichlet forms on standard forms of von Neumann algebras constructed by the method proposed in Refs. [Par1, Par2]. We apply our result to show that the diffusion type Markovian semigroups for quantum spin systems are ergodic in the region of high temperatures where the uniqueness of the KMS-state holds.

Keywords: Standard forms of von Neumann Algebras; Dirichlet forms; Markovian semigroups; ergodicity; quantum spin systems; KMS-states.

1 Introduction

The purpose of this work is to investigate ergodic property of the Markovian semigroups associated to Dirichlet forms on the standard form of a von Neumann algebra \( \mathcal{M} \) acting on a Hilbert space \( \mathcal{H} \) with a cyclic and separating vector \( \xi_0 \). Denote by \( \sigma_t, t \in \mathbb{R} \), the modular automorphism on \( \mathcal{M} \) associated with the pair \( (\mathcal{M}, \xi_0) \) and \( \mathcal{M}_{1/2} \) the dense subset of \( \mathcal{M} \) consisting of every \( \sigma_t \)-analytic element on a domain containing the strip \( \{ z \in \mathbb{C} : |\text{Im}z| \leq 1/2 \} \). Let \( \{x_k : k \in I\} \) be a (finite or countable) family of elements in \( \mathcal{M}_{1/2} \) which generates \( \mathcal{M} \). Let \( (\mathcal{E}, D(\mathcal{E})) \) be the Dirichlet form constructed with \( \{x_k : k \in I\} \) and an admissible function by means of Refs. [Par1, Par2]. For the details, see Section 2. Denote by \( T_t, t \geq 0 \), the
Markovian semigroup associated to \((\mathcal{E}, D(\mathcal{E}))\). Let \(\mathcal{N}\) be the fixed point space of \(T_t\):

\[
\mathcal{N} = \{ \eta \in \mathcal{H} : T_t\eta = \eta, \forall t \geq 0 \}.
\]

We show that \(\mathcal{N} = [\mathcal{Z}(\mathcal{M})\xi_0]\), where \(\mathcal{Z}(\mathcal{M})\) is the center of \(\mathcal{M}\); \(\mathcal{Z}(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}'\), and \([\mathcal{Z}(\mathcal{M})\xi_0]\) is the closure of \(\mathcal{Z}(\mathcal{M})\xi_0\). As a consequence, \(T_t\) is ergodic if and only if \(\mathcal{M}\) is a factor. We apply our result to the translation invariant Markovian semigroups for quantum spin systems \([\text{Par1}]\), and show that the semigroups are ergodic in region of high temperatures where the uniqueness of the KMS-state holds.

Let us describe the background of this study briefly. The need to construct Markovian semigroups on von Neumann algebras, which are (KMS) symmetric with respect to non-tracial states, is clear for various applications to open systems\([\text{Dav}]\), quantum statistical mechanics\([\text{BR}]\) and quantum probability\([\text{Acc, Part}]\). Although on the abstract level we have quite well-developed theory\([\text{Cip1, GL1, GL2}]\), the progress in concrete applications is relatively slow. For construction of Dirichlet forms and associated Markovian semigroups, we refer to \([\text{BKPT, BKP2, MZ1, MZ2, Par1, Zeg}]\) and the references there in.

During the last ten years, systematic methods to construct Dirichlet forms and associated Markovian semigroups of jump and diffusion types have been developed. Nontrivial translation invariant symmetric semigroups of jump type for quantum spin systems have been constructed and the strong ergodicity of the semigroups has been established in Refs. \([\text{MZ1, MZ2}]\). See also \([\text{Zeg}]\) and the references there in. In \([\text{Par1}]\), we gave a general construction method of Dirichlet forms of diffusion type in the framework of the general theory of Dirichlet forms and Markovian semigroups on standard forms of von Neumann algebras developed by Cipriani\([\text{Cip1}]\). The method has been used successfully to construct Dirichlet forms and associated Markovian semigroups for quantum spin systems \([\text{Par1}]\), CCR and CAR algebras with respect to quasi-free states\([\text{BKPT, BKP2, KP}]\), and quantum mechanical
Recently, in [Par2] we have shown that the symmetric embedding of a
general Lindblad type (bounded) generator of a quantum dynamical semigroup sat-
sifying KMS-symmetry can be written in the form of a Dirichlet operator associated
to a Dirichlet form given in [Par1].

The next step in this research area would be the investigation of detailed proper-
ties of Markovian semigroups, such as ergodicity, mixing property and convergence
to the equilibrium, etc. In the case of CCR and CAR algebras with respect to quasi-
free states, the spectrum of the generators of the Markovian semigroups constructed
in [CFL, BKPI, BKP2, KP] has been analyzed. However, in general the detail prop-
erties of the Markovian semigroups associated to Dirichlet forms in [Par1, Par2] are
hard to be established. Thus it would be nice to have a simple criteria whether the
Markovian semigroup under study is ergodic or not in the sense of Cipriani[Cip2].

We organize the paper as follows: In Section 2, we introduce notations and
terminologies, and then list our results(Theorem 2.1 and Corollary 2.1). We also
give comments (Remark 2.1) on possible applications of our results. Section 3 is
devoted to the proof of Theorem 2.1 We first describe the basic ideas of the proof
and then establish some technical lemmas(Lemma 3.1 - Lemma 3.5) which are need
in the sequel. Using the lemmas we complete the proof of Theorem 2.1 In Section
4, we apply our main results to prove that the diffusion type translation invariant
Markovian semigroups for quantum spin systems constructed in [Par1] are ergodic
in the region of high temperature where the uniqueness of the KMS state holds.

2 Notation, Terminologies and Main Results

We first introduce necessary terminologies in the theory of Dirichlet forms and
Markovian semigroups on standard form of von Neumann algebras[Cip1]. Next
we give a brief review on the construction of Dirichlet forms developed in [Par1] and
then list our main results.

Let $\mathcal{M}$ be a $\sigma$-finite von Neumann algebra acting on a complex Hilbert space $\mathcal{H}$ with an inner product $\langle \cdot , \cdot \rangle$ which is conjugate linear in the first and linear in the second variable. Let $\xi_0$ be a cyclic and separating vector for $\mathcal{M}$. We use $\Delta$ and $J$ to denote respectively, the modular operator and the modular conjugation associated with the pair $(\mathcal{M}, \xi_0)$. The associated modular automorphism is denoted by $\sigma_t$: $\sigma_t(A) = \Delta^{it}A\Delta^{-it}, A \in \mathcal{M}$. Finally, $j : \mathcal{M} \to \mathcal{M}'$ is the antilinear $\ast$-isomorphism defined by $j(A) = JAJ, A \in \mathcal{M}$, where $\mathcal{M}'$ is the commutant of $\mathcal{M}$. By the Tomita-Takesaki theorem (Theorem 2.5.14 of [BR]), it follows that $\sigma_t(\mathcal{M}) = \mathcal{M}$ and $j(\mathcal{M}) = \mathcal{M}'$.

The natural positive cone $\mathcal{P}$ associated with the pair $(\mathcal{M}, \xi_0)$ is the closure of the set

$$\{ Aj(A)\xi_0 : A \in \mathcal{M} \}.$$  

By a general result, the closed convex cone $\mathcal{P}$ can be obtained by the closure of the set

$$\{ \Delta^{1/4}AA^*\xi_0 : A \in \mathcal{M} \}$$

and this cone $\mathcal{P}$ is self-dual in the sense that

$$\{ \xi \in \mathcal{H} : \langle \xi, \eta \rangle \geq 0, \ \forall \eta \in \mathcal{P} \} = \mathcal{P}.$$  

For the details we refer to [Ara] and Section 2.5 of [BR].

The form $(\mathcal{M}, \mathcal{H}, \mathcal{P}, J)$ is the standard form associated with the pair $(\mathcal{M}, \xi_0)$. We shall use the fact that $\mathcal{H}$ is the complexification of the real subspace $\mathcal{H}^J = \{ \xi \in \mathcal{H} : \langle \xi, \eta \rangle \in \mathbb{R}, \ \forall \eta \in \mathcal{P} \}$, whose elements are called $J$-real. $\mathcal{H} = \mathcal{H}^J \bigoplus i\mathcal{H}^J$. The cone $\mathcal{P}$ gives rise to a structure of ordered Hilbert space on $\mathcal{H}^J$ (denoted by $\leq$) and to an anti-linear involution $J$ on $\mathcal{H}$, which preserves $\mathcal{P}$ and $\mathcal{H}^J$: $J(\xi + i\eta) = \xi - i\eta, \ \forall \xi, \eta \in \mathcal{H}^J$. Also note that any $J$-real element $\xi \in \mathcal{H}^J$ can be decomposed
uniquely as a difference of two mutually orthogonal, positive and negative part of $\xi$, respectively: $\xi = \xi_+ - \xi_-$, $\xi_+, \xi_- \in \mathcal{P}$ and $\langle \xi_+, \xi_- \rangle = 0$.

A bounded operator $A$ on $\mathcal{H}$ is called $J$-real if $AJ = JA$ and positive preserving if $A\mathcal{P} \subset \mathcal{P}$. A semigroup $\{T_t\}_{t \geq 0}$ is said to be $J$-real if $T_t$ is $J$-real for any $t \geq 0$ and it is called positive preserving if $T_t$ is positive preserving for any $t \geq 0$. A bounded operator $A : \mathcal{H} \to \mathcal{H}$ is called sub-Markovian (with respect to $\xi_0$) if $0 \leq \xi \leq \xi_0$ implies $0 \leq A\xi \leq \xi_0$. $A$ is called Markovian if it is sub-Markovian and also $A\xi_0 = \xi_0$. A semigroup $\{T_t\}_{t \geq 0}$ is said to be sub-Markovian (with respect to $\xi_0$) if $T_t$ is sub-Markovian for every $t \geq 0$. A semigroup $\{T_t\}_{t \geq 0}$ is called Markovian if $T_t$ is Markovian for every $t \geq 0$.

Next, we consider a sesquilinear form on some linear manifold of $\mathcal{H}$: $\mathcal{E}(\cdot, \cdot) : D(\mathcal{E}) \times D(\mathcal{E}) \to \mathbb{C}$. We also consider the associated quadratic form: $\mathcal{E}[\cdot] : D(\mathcal{E}) \to \mathbb{C}$, $\mathcal{E}[\xi] := \mathcal{E}(\xi, \xi)$. A real valued quadratic form $\mathcal{E}[\cdot]$ is said to be semi-bounded (or bounded below) if $\inf \{ \mathcal{E}[\xi] : \xi \in D(\mathcal{E}), \|\xi\| = 1 \} = -b > -\infty$. A quadratic form $(\mathcal{E}, D(\mathcal{E}))$ is said to be $J$-real if $JD(\mathcal{E}) \subset D(\mathcal{E})$ and $\mathcal{E}[J\xi] = \overline{\mathcal{E}[\xi]}$ for any $\xi \in D(\mathcal{E})$. For a semi-bounded quadratic form $\mathcal{E}$, one considers the inner product given by $\langle \xi, \eta \rangle_\lambda := \mathcal{E}(\xi, \eta) + \lambda \langle \xi, \eta \rangle$, for $\lambda > b$. The form $(\mathcal{E}, D(\mathcal{E}))$ is closed if $D(\mathcal{E})$ is a Hilbert space for some of the above inner products. The form $(\mathcal{E}, D(\mathcal{E}))$ is called closable if it admits a closed extension.

Associated to a semi-bounded closed form $(\mathcal{E}, D(\mathcal{E}))$, there are a self-adjoint operator $(H, D(H))$ and a strongly continuous, symmetric semigroup $\{T_t\}_{t \geq 0}$. Each of the above objects determines uniquely the others according to well known relations (see [RS] and Section 3.1 of [BR]).

Let us denoted by $Proj(\xi, Q)$ the projection of the vector $\xi \in \mathcal{H}^J$ onto the closed,
convex cone $Q \subset H^J$. For $\xi, \eta \in H^J$, define
\[
\xi \vee \eta := \text{Proj}(\xi, \eta + \mathcal{P}),
\]
\[
\xi \wedge \eta := \text{Proj}(\xi, \eta - \mathcal{P}).
\]

A $J$-real, real-valued, densely defined quadratic form $(\mathcal{E}, D(\mathcal{E}))$ is called Markovian with respect to $\xi_0 \in \mathcal{P}$ if
\[
\xi \in D(\mathcal{E})^J \text{ implies } \xi \wedge \xi_0 \in D(\mathcal{E}) \text{ and } \mathcal{E}[\xi \wedge \xi_0] \leq \mathcal{E}[\xi],
\]
where $D(\mathcal{E})^J := D(\mathcal{E}) \cap H^J$. A closed Markovian form is called a Dirichlet form.

Next, we collect main results of [Cip1]. Let $(\mathcal{E}, D(\mathcal{E}))$ be a $J$-real, real valued, densely defined closed form. Assume that the following properties hold:
\begin{enumerate}[(a)]
  \item $\xi_0 \in D(\mathcal{E})$, 
  \item $\mathcal{E}(\xi, \xi) \geq 0$ for $\xi \in D(\mathcal{E})$, 
  \item $\xi \in D(\mathcal{E})^J$ implies $\xi_\pm \in D(\mathcal{E})$ and $\mathcal{E}(\xi_+, \xi_-) \leq 0$.
\end{enumerate}
Then $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form if and only if $\mathcal{E}(\xi, \xi_0) \geq 0$ for all $\xi \in D(\mathcal{E}) \cap \mathcal{P}$.

The above result follows from Proposition 4.5 (b) and Proposition 4.10 (ii) of [Cip1].

The following is one of main results (Theorem 4.11) of [Cip1]: Let $(\mathcal{E}, D(\mathcal{E}))$ be a $J$-real, strongly continuous, symmetric semigroup on $H$ and let $(\mathcal{E}, D(\mathcal{E}))$ be the associated densely defined $J$-real, real valued quadratic form. Then the followings are equivalent:
\begin{enumerate}[(a)]
  \item $\{T_t\}_{t \geq 0}$ is sub-Markovian.
  \item $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form.
\end{enumerate}

We refer the reader to [Cip1] for the details.

Next, we describe the construction of Dirichlet forms developed in [Par1]. See also [Par2]. For any $\lambda > 0$, denote by $I_\lambda$ the closed strip given by
\[
I_\lambda = \{z : z \in \mathbb{C}, |Im\ z| \leq \lambda\}. 
\]
Recall that an analytic function $f : D \to \mathbb{C}$ on a domain $D$ containing the strip $I_{1/4}$ is called *admissible* if the following properties hold:

(a) $f(t) \geq 0$ for $\forall t \in \mathbb{R}$, \hspace{1cm} (2.3)
(b) $f(t + i/4) + f(t - i/4) \geq 0$ for $\forall t \in \mathbb{R}$,
(c) there exist $M > 0$ and $p > 1$ such that the bound

$$|f(t + is)| \leq M(1 + |t|)^{-p}$$

holds uniformly in $s \in [-1/4, 1/4]$.

We also consider the function $f_0 : \mathbb{R} \to \mathbb{R}$ given by

$$f_0(t) = 2(e^{2\pi t} + e^{-2\pi t})^{-1}. \hspace{1cm} (2.4)$$

One can see that $f_0$ has an analytic extension, denoted by $f_0$ again, to the interior of $I_{1/4}$.

For any $\lambda > 0$, denote by $\mathcal{M}_\lambda$ the dense subset of $\mathcal{M}$ consisting of every $\sigma_t$-analytic element with a domain containing $I_\lambda$. By Proposition 2.5.21 of [BR], any $A \in \mathcal{M}_\lambda$ is strongly analytic on $I_\lambda$. We denote by $\mathcal{M}_0$ the dense subset of $\mathcal{M}$ consisting of every $\sigma_t$-entire analytic element, i.e., $\mathcal{M} = \bigcap_\lambda \mathcal{M}_\lambda$.

Let $I$ be a finite or countable (index) set. For given family $\{x_k : k \in I\} \subset \mathcal{M}_{1/2}$ of self-adjoint elements in $\mathcal{M}_{1/2}$ and an admissible function $f$ or $f = f_0$, define a sesquilinear form by

$$D(\mathcal{E}) = \{\xi \in \mathcal{H} : \sum_{k \in I} \mathcal{E}_k(\xi, \xi) < \infty\}, \hspace{1cm} (2.5)$$

$$\mathcal{E}(\eta, \xi) = \sum_{k \in I} \mathcal{E}_k(\eta, \xi), \hspace{1cm} (2.6)$$

where for each $k \in I$

$$\mathcal{E}_k(\eta, \xi) \hspace{1cm} (2.7)$$

$$= \int \left\langle (\sigma_{t-i/4}(x_k) - j(\sigma_{t-i/4}(x_k)))\eta, (\sigma_{t-i/4}(x_k) - j(\sigma_{t-i/4}(x_k)))\xi \right\rangle f(t) dt.$$
For each $k \in I$, the above form is positive and bounded. In fact, the form $(\mathcal{E}_k, \mathcal{H})$ is a Dirichlet form for each $k \in I$ by Theorem 3.1 of [Par1]. See also Theorem 2.1 of [Par2] for $f = f_0$. Moreover, if $D(\mathcal{E})$ is dense in $\mathcal{H}$, then the form $(\mathcal{E}, D(\mathcal{E}))$ given in (2.6) is a Dirichlet form by Theorem 5.2 of [Cip1].

Before proceeding further, we would like to make a few remarks. The function $f_0$ given in (2.5) played a special role in [Par2]. The symmetric embedding of a general Lindblad type (bounded) generator of a quantum dynamical semigroup (satisfying KMS-symmetry) on $\mathcal{M}$ can be written as the Dirichlet operator associated to a Dirichlet form in (2.6) with $f = f_0$. Next, we would like to mention that it is not necessary to assume that each $x_k$ in \{ $x_k : k \in I$ \} is self-adjoint if one defines the Dirichlet form $(\mathcal{E}_k, \mathcal{H})$ in (2.7) appropriately as in (2.6) in [Par2]. Note that, by a simple transformation, one can write $\mathcal{E}_k(\eta, \xi)$ as a sum of two Dirichlet forms corresponding two self-adjoint elements. See Remark 2.1 (a) in [Par2]. Thus without loss of the generality, we assume that each $x_k$ is self-adjoint.

A family \{ $x_k : k \in I$ \} is said to generate $\mathcal{M}$ if the $*$-algebra generated by \{ $x_k : k \in I$ \} is dense in $\mathcal{M}$. For given \{ $x_k : k \in I$ \} $\subset \mathcal{M}_{1/2}$ of self-adjoint elements and either an admissible function $f$ or else $f = f_0$, let $(\mathcal{E}, D(\mathcal{E}))$ be the Dirichlet form defined as in (2.5) - (2.7). Denote by $(H, D(H))$ and \{ $T_t$ \}_{t \geq 0} the Dirichlet operator and Markovian semigroup associated to $(\mathcal{E}, D(\mathcal{E}))$, i.e., $T_t = e^{-tH}$. We denote by $\mathcal{N}$ the fixed point space of $T_t$:

$$\mathcal{N} := \{ \eta \in \mathcal{H} : T_t \eta = \eta \} = \{ \eta \in \mathcal{H} : H \eta = 0 \}. \quad (2.8)$$

The following is the main result in this paper:

**Theorem 2.1** For a family \{ $x_k : k \in I$ \} $\subset \mathcal{M}_{1/2}$ of self-adjoint elements and an admissible function $f$ or else $f = f_0$, let $(\mathcal{E}, D(\mathcal{E}))$ be the densely defined Dirichlet
form given as in (2.5) - (2.7). Assume that \{x_k : k \in I\} generates \mathcal{M}. Then the equality
\[ N = [\mathcal{Z}(\mathcal{M})\xi_0] \]
holds, where \( \mathcal{Z}(\mathcal{M}) \) is the center of \( \mathcal{M} \), i.e., \( \mathcal{Z}(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}' \), and \([\mathcal{Z}(\mathcal{M})\xi_0]\) is the closure of \( \mathcal{Z}(\mathcal{M})\xi_0 \).

Recall that a symmetric, strongly continuous, positive preserving semigroup \{\( T_t \)\}_{t \geq 0} on \( \mathcal{H} \) is called \textit{ergodic} if for each \( \xi, \eta \in \mathcal{P}, \), \( \xi, \eta \neq 0 \), there exists \( t > 0 \) such that 
\[ \langle \xi, T_t\eta \rangle > 0 \]  
[Cip2]. Assume that \( \inf \sigma(H) \) is an eigenvalue of the generator \( H \) of \{\( T_t \)\}_{t \geq 0}. Then the ergodicity of \{\( T_t \)\}_{t \geq 0} is equivalent to that \( \inf \sigma(H) \) is a simple eigenvalue of \( H \) with a strictly positive (cyclic and separating) eigenvector (Theorem 4.3 of [Cip2]). As a consequence of Theorem 2.1 we have the following:

**Corollary 2.1** Let \( \mathcal{M} \) be a factor, i.e., \( \mathcal{Z}(\mathcal{M}) = \mathbb{C}1 \). Let \((\mathcal{E}, D(\mathcal{E}))\) be the densely defined Dirichlet form as in Theorem 2.1 and \( T_t \) the associated Markovian semigroup.

Under the assumptions as in Theorem 2.1, \{\( T_t \)\}_{t \geq 0} is ergodic in the sense that zero is a simple eigenvalue of the generator \( H \) of \( T_t \) with eigenvector \( \xi_0 \).

**Proof:** Under the assumptions, \( N = \mathbb{C}\xi_0 \) by Theorem 2.1. Since \( j(\sigma_{t-i/4}(x))\xi_0 = \sigma_{t-i/4}(x)\xi_0 \), it follows from (2.7) that \( \mathcal{E}_k(\xi_0, \xi_0) = 0 \) for each \( k \in I \) and so \( \mathcal{E}(\xi_0, \xi_0) = 0 \), which implies that \( H\xi_0 = 0 \). See also Theorem 3.1 (a) of [Par1]. Hence zero is a simple eigenvalue of \( H \) with eigenvector \( \xi_0 \). \( \Box \)

We will produce the proof of Theorem 2.1 in the next section. Before closing this section, it may be worth to give comments on possible applications of Theorem 2.1.
Remark 2.1 (a) In order to apply Theorem 2.1 (and Corollary 2.1) to concrete models, one has to choose a family \( \{x_k : k \in I\} \subset \mathcal{M}_{1/2} \) which generates \( \mathcal{M} \). Recall that the condition \( x_k \in \mathcal{M}_{1/2} \subset \mathcal{M}_{1/4} \) for each \( x_k \) is needed for \((\mathcal{E}_k, \mathcal{H})\) to be well defined. If \( \mathcal{H} \) is a finite dimensional Hilbert space, then the modular operator \( \Delta \) is bounded and so every element \( x \) of \( \mathcal{M} \) is \( \sigma_t \)-entire analytic. In general, it would be not easy to choose a generating family \( \{x_k : k \subset I\} \) from \( \mathcal{M}_{1/2} \) directly.

(b) For quantum spin systems in the region of high temperatures, every local observable belongs to \( \mathcal{M}_{1/2} \). In this case, the choice of \( \{x_k : k \in I\} \) is easy. See Section 4 for the details.

(c) Let \( \{f_n : n \in \mathbb{N}\} \) be an orthonormal basis for \( L^2(\mathbb{R}^d) \) and let \( a^*(f_n) \) and \( a(f_n), n \in \mathbb{N}, \) be the creation and annihilation operators which generate a CAR algebra \( \mathcal{A} \). Let \( \omega \) be a quasi-free state on \( \mathcal{A} \) and \((\mathcal{H}_\omega, \pi_\omega(\mathcal{A}), \Omega_\omega)\) be the cyclic representation associated to \((\mathcal{A}, \omega)\). Let \( \mathcal{M} = \pi_\omega(\mathcal{A})'' \) and \( \xi_0 = \Omega_\omega \). Then for each \( n \in \mathbb{N}, \pi_\omega(a(f_n)) \) and \( \pi_\omega(a^*(f_n)) \) are \( \sigma_t \)-entire analytic element \([BKP2]\). Thus one can apply Theorem 2.1 and Corollary 2.1 directly in this case.

(d) In applications to open systems \([Dav]\) and quantum statistical mechanics \([BR]\), one may need to construct a Dirichlet form for a given \( \{x_k : k \in I\} \), where each \( x_k \) is unbounded (self-adjoint) operator affiliated with \( \mathcal{M} \). By employing appropriate approximation procedures, one may be able to construct the Dirichlet form associated to \( \{x_k : k \in I\}\) \([BK, BKPT]\) and then extend Theorem 2.1 by modifying the method used in this paper.

3 Proof of Theorem 2.1

Before producing the proof of Theorem 2.1 we first describe the basic ideas used in the proof, and then establish necessary technical lemmas which will be needed in the proof. Using the lemmas, we complete the proof at the last part of this section.
The inclusion \([Z(\mathcal{M})\xi_0] \subset \mathcal{N}\) is easy to check. See the proof of Theorem 2.1. Thus we concentrate to the inclusion \(\mathcal{N} \subset [Z(\mathcal{M})\xi_0]\). Note that \(\eta \in \mathcal{N}\) if and only if \(E[\eta] = \langle H^{1/2}\eta, H^{1/2}\eta \rangle = 0\). Since \(E_k[\eta] \geq 0\) for \(\eta \in \mathcal{H}, k \in I, \eta \in \mathcal{N}\) if and only if \(E_k[\eta] = 0\) for any \(k \in I\). Since \(f\) is an admissible function or else \(f = f_0\), it is easy to show that \(E_k[\eta] = 0\) if and only if

\[
\| (\sigma_{t-i/4}(x_k) - j(\sigma_{t-i/4}(x_k)))\eta \| = 0
\]

for any \(t \in \mathbb{R}\) and \(k \in I\). See Lemma 3.1. The above implies that

\[
\| (\sigma_{-i/4}(x_k) - j(\sigma_{-i/4}(x_k)))\eta \| = 0, \ k \in I. \tag{3.1}
\]

Suppose that \(\eta\) is of the form \(\eta = \Delta^{1/4}Q\xi_0, Q \in \mathcal{M}\). Then the above equality implies that

\[
[x_k, Q]\xi_0 = 0, \ \forall k \in I
\]

and so

\[
[x_k, Q]A'\xi_0 = 0, \ \forall k \in I
\]

for any \(A' \in \mathcal{M}'\). Since \(\mathcal{M}'\xi_0\) is dense in \(\mathcal{H}\), we conclude that \([x_k, Q] = 0\) for \(k \in I\). Since \(\{x_k : k \in I\}\) generates \(\mathcal{M}, Q \in \mathcal{M}'\). Thus \(Q \in \mathcal{M} \cap \mathcal{M}'\) and \(\eta = \Delta^{1/4}Q\xi_0 = Q\xi_0\). However, in general \(\eta \in \mathcal{N}\) can not be written as \(\eta = \Delta^{1/4}Q\xi_0, Q \in \mathcal{M}\).

Note that \(H\) is \(J\)-real, \(JH = HJ\), and so \(J\mathcal{N} = \mathcal{N}\). Any \(\eta \in \mathcal{H}\) can be written as \(\eta = \eta_r + i\eta_i\), where \(\eta_r = (\eta + J\eta)/2\) and \(\eta_i = -(\eta - J\eta)/2\). Thus \(\eta \in \mathcal{N}\) implies \(\eta_r, \eta_i \in \mathcal{N}\). Hence we may suppose that \(\eta \in \mathcal{N} \cap \mathcal{H}^J\). Because of the Dirichlet property 2.1 (c) of \(E[\eta]\), it can be shown that \(\eta \in \mathcal{N} \cap \mathcal{H}^J\) implies \(\eta_+ , \eta_- \in \mathcal{N} \cap \mathcal{P}\) (Lemma 3.2). Thus the problem is reduced to the case \(\eta \in \mathcal{N} \cap \mathcal{P}\).

Any \(\eta \in D(\Delta^{-1/4}) \cap \mathcal{P}\) can be written as \(\eta = \Delta^{1/4}Q\xi_0\) where \(Q\) is positive self-adjoint operator affiliated with \(\mathcal{M}\) (Lemma 3.4). For any \(\eta \in D(\Delta^{-1/4}) \cap (\mathcal{N} \cap \mathcal{P}), \eta = \Delta^{1/4}Q\xi_0\), we use (3.1) to show that

\[
(x_k Q - Q x_k)\xi_0 = 0.
\]
Using the facts that \( \mathcal{M}' \xi_0 \) is dense in \( \mathcal{H} \) and that \( \{x_k : k \in I\} \) generates \( \mathcal{M} \), we will show that \( Q \) is affiliated with \( \mathcal{M}' \). Since \( \Delta \xi = \xi \) for any \( \xi \in [\mathcal{Z}(\mathcal{M})\xi_0] \), we conclude that \( \eta \in [\mathcal{Z}(\mathcal{M})\xi_0] \). Next, we use the fact that \( D(\Delta^{-1/4}) \cap (\mathcal{N} \cap \mathcal{P}) \) is dense in \( (N \cap \mathcal{P})(\text{Lemma 3.3}) \) to complete the proof of Theorem 2.1.

Next, we collect technical lemmas which will be used in the sequel. In the rest of this section, we assume that the conditions in the Theorem hold.

**Lemma 3.1** A vector \( \eta \in \mathcal{H} \) belongs to \( \mathcal{N} \) if and only if the equality

\[
(\sigma_{t-i/4}(x_k) - j(\sigma_{t-i/4}(x_k)))\eta = 0
\]

holds for any \( t \in \mathbb{R} \) and \( k \in I \).

*Proof:* Since

\[
\langle \eta, H\eta \rangle = \mathcal{E}(\eta, \eta)
\]

\[
= \sum_{k \in I} \mathcal{E}_k(\eta, \eta),
\]

and \( \mathcal{E}_k(\eta, \eta) \geq 0 \) for \( \eta \in \mathcal{H} \) and \( k \in I \), \( H\eta = 0 \) if and only if \( \mathcal{E}_k(\eta, \eta) = 0 \). Recall the expression of \( \mathcal{E}_k(\eta, \eta) \) in (2.7). Notice that \( f_0(t) > 0 \) for any \( t \in \mathbb{R} \) by (2.4). If \( f \) is an admissible function, \( f(t) \geq 0 \) by (2.3) (a). Since \( f \) is analytic on a domain containing \( I_{1/4} \), \( f(t) > 0 \) except on a countable set with no accumulation points. Thus the left hand side of the expression in the lemma is zero except on a countable set of \( t \in \mathbb{R} \). Since \( \sigma_{t-i/4}(x_k) \) is strongly continuous with respect to \( t \in \mathbb{R} \), we proved the lemma. \( \square \)

**Lemma 3.2** (a) \( \mathcal{N} \) is a closed subspace of \( \mathcal{H} \).

(b) \( \Delta^t \mathcal{N} = \mathcal{N}, \forall t \in \mathbb{R} \).
\((c)\) \(J\mathcal{N} = \mathcal{N}\).

\((d)\) \(\eta \in \mathcal{N} \cap \mathcal{H}^J\) implies \(\eta_+, \eta_- \in \mathcal{N} \cap \mathcal{P}\).

**Proof:** (a) Since \(H\) is self-adjoint (closed), (a) is obvious.

(b) Notice that
\[
H \Delta^{-is} \eta = \Delta^{-is}(\Delta^{is}H\Delta^{-is})\eta
\]
and \(\Delta^{is}H\Delta^{-is}\) is the Dirichlet operator associated to the Dirichlet form constructed with \(\{\sigma_s(x_k) : k \in I\}\). Note that \(\sigma_{t-i/4}(\sigma_s(x_k)) = \sigma_{t+s-i/4}(x_k)\). Thus, if \(\eta \in \mathcal{N}\), \(\Delta^{-is} \eta \in \mathcal{N}\) by Lemma 3.1. Hence \(\Delta^{-is} \mathcal{N} \subset \mathcal{N}\) for any \(s \in \mathbb{R}\), which also implies \(\mathcal{N} \subset \Delta^{is} \mathcal{N}\) for any \(s \in \mathbb{R}\).

(c) Since each \(E_k\) is \(J\)-real (Theorem 2.1 (b) of [Par2]), it is easy to check that \(H\) is \(J\)-real, \(JH = HJ\). Thus \(HJ\eta = JH\eta = 0\) if \(\eta \in \mathcal{N}\) and so \(J\mathcal{N} \subset \mathcal{N}\). Since \(J^2 = 1\), we also have that \(\mathcal{N} \subset J\mathcal{N}\).

(d) Let \(\eta \in \mathcal{N} \cap \mathcal{H}^J\), and \(\eta = \eta_+ - \eta_-\). Notice that

\[
0 = \mathcal{E}(\eta, \eta) = \mathcal{E}(\eta_+, \eta_+) - 2\mathcal{E}(\eta_+, \eta_-) + \mathcal{E}(\eta_-, \eta_-).
\]

Here we have used the fact that \(\eta \in D(\mathcal{E})\) implies \(\eta_+, \eta_- \in D(\mathcal{E})\). Since \(\mathcal{E}(\eta_+, \eta_-) \leq 0\) by (2.1) (c) (Theorem 2.1 (c) of [Par2]), we have that \(\mathcal{E}(\eta_+, \eta_+) = \mathcal{E}(\eta_-, \eta_-) = 0\), which imply \(H\eta_+ = H\eta_- = 0\). \(\square\)

**Lemma 3.3** (a) For any bounded, positive definite, continuous function \(f : \mathbb{R} \to \mathbb{R}\),
\[
f(\log \Delta)(\mathcal{N} \cap \mathcal{P}) \subset \mathcal{N} \cap \mathcal{P}.
\]

(b) \((\bigcap_{\alpha \in \mathbb{R}} D(\Delta^\alpha)) \cap (\mathcal{N} \cap \mathcal{P})\) is dense in \(\mathcal{N} \cap \mathcal{P}\).
Proof: (a) Let $f$ be a bounded, positive definite, continuous function on $\mathbb{R}$. Then $f$ can be written as

$$f(x) = \int e^{itx}d\mu(t),$$

where $\mu$ is a positive finite Borel measure on $\mathbb{R}$. Thus

$$f(\log \Delta) = \int \Delta^u d\mu(t).$$

The inclusion

$$f(\log \Delta)\mathcal{P} \subset \mathcal{P}$$

holds by the fact that $\Delta^u \mathcal{P} \subset \mathcal{P}$ (Proposition 2.5.26 of [BR]). Due to Lemma 3.2 (b), the inclusion

$$f(\log \Delta)\mathcal{N} \subset \mathcal{N}$$

also holds. This proved the part (a) of the lemma.

(b) Let

$$f_n(x) := e^{-x^2/2n^2}.$$

Then by the part (a) of the lemma,

$$f_n(\log \Delta)(\mathcal{N} \cap \mathcal{P}) \subset \mathcal{N} \cap \mathcal{P}.$$ 

For any $\eta \in \mathcal{N} \cap \mathcal{P}$, $f_n(\log \Delta)\eta \in D(\Delta^\alpha)$ for any $\alpha \in \mathbb{R}$, and $f_n(\log \Delta)\eta \to \eta$ as $n \to \infty$. This proved the part (b). □

Lemma 3.4 Let $\eta \in D(\Delta^{-1/4}) \cap \mathcal{P}$. Then there is a positive self-adjoint operator $Q$ affiliated with $\mathcal{M}$ such that $Q\xi_0 \in D(\Delta^{1/4})$ and $\eta = \Delta^{1/4}Q\xi_0$.

Proof: We use the method similar to that employed in the proof of Proposition 2.5.27(1) of [BR]. Let $\eta \in D(\Delta^{-1/4}) \cap \mathcal{P}$. For any $A \in \mathcal{M}$, $\Delta^{-1/4}j(A^*)j(A)\xi_0 \in \mathcal{P}$
and so
\[ \langle \Delta^{-1/4} j(A^*) j(A) \xi_0, \eta \rangle \geq 0, \ \forall A \in \mathcal{M}, \]
which implies
\[ \langle j(A^*) j(A) \xi_0, \Delta^{-1/4} \eta \rangle \geq 0, \ \forall A \in \mathcal{M}. \]

Define an operator \( \tilde{Q}, D(\tilde{Q}) = \mathcal{M}' \xi_0, \) by
\[ \tilde{Q} j(B) \xi_0 = j(B) \Delta^{-1/4} \eta, \ \forall B \in \mathcal{M}. \]

Then for any unitary \( U' \in \mathcal{M}', \)
\[ U' \tilde{Q} j(B) \xi_0 = U' j(B) \Delta^{-1/4} \eta = \tilde{Q} U' j(B) \xi_0, \]
and so
\[ U'^* \tilde{Q} U' = \tilde{Q}. \]

For any \( A \in \mathcal{M}, \)
\[ \langle j(A) \xi_0, \tilde{Q} j(A) \xi_0 \rangle = \langle j(A) \xi_0, j(A) \Delta^{-1/4} \eta \rangle \]
\[ = \langle \Delta^{-1/4} j(A^*) j(A) \xi_0, \eta \rangle \]
\[ \geq 0. \]

Thus \( \tilde{Q} \) is a positive symmetric operator. Notice that for any unitary \( U' \in \mathcal{M}', \)
\[ U'D(\tilde{Q}) \subset D(\tilde{Q}). \] Let \( Q \) be the Friedrichs extension of \( \tilde{Q}. \) By the uniqueness of Friedrichs extension
\[ U'^* Q U' = Q \]
for any unitary \( U' \in \mathcal{M}'. \) Thus \( Q \) is affiliated with \( \mathcal{M}. \) Since \( Q \xi_0 = \Delta^{-1/4} \eta, \) \( Q \xi_0 \in D(\Delta^{1/4}) \) and \( \eta = \Delta^{1/4} Q \xi_0. \) \( \square \)
Lemma 3.5 Let $\eta \in D(\Delta^{-1/4}) \cap (\mathcal{N} \cap \mathcal{P})$ and $\eta = \Delta^{1/4}Q\xi_0$ as in Lemma 3.4. Then $x_k\xi_0 \in D(Q)$ and $x_kQ\xi_0 = Qx_k\xi_0$ for any $k \in I$.

Proof: Since $\eta \in \mathcal{N}$, it follows from Lemma 3.1 that

$$\sigma_{-i/4}(x_k) - j(\sigma_{-i/4}(x_k))\Delta^{1/4}Q\xi_0 = 0, \ k \in I.$$  

Recall that $M_0$ is the dense subset of $M$ consisting of $\sigma_t$-entire analytic elements. For any $A \in M_0$,

$$0 = \langle \sigma_{i/4}(A)\xi_0, [\sigma_{-i/4}(x_k) - j(\sigma_{-i/4}(x_k))]\Delta^{1/4}Q\xi_0 \rangle$$

$$= \langle \sigma_{i/4}(x_k)\sigma_{i/4}(A)\xi_0, \Delta^{1/4}Q\xi_0 \rangle$$

$$- \langle j(\sigma_{i/4}(x_k))\sigma_{i/4}(A)\xi_0, \Delta^{1/4}Q\xi_0 \rangle. \quad (3.2)$$

For any $A \in M_0$,

$$\sigma_{i/4}(x_k)\sigma_{i/4}(A)\xi_0 = \sigma_{i/4}(x_kA)\xi_0$$

$$= \Delta^{-1/4}x_kA\xi_0 \quad (3.3)$$

and

$$j(\sigma_{i/4}(x_k))\sigma_{i/4}(A)\xi_0 = j(\sigma_{i/4}(x_k))j(\sigma_{-3i/4}(A^*))\xi_0$$

$$= \Delta^{-1/4}j(\sigma_{i/2}(x_k))j(\sigma_{-i/2}(A))\xi_0$$

$$= \Delta^{-1/4}j(\sigma_{i/2}(x_k))A\xi_0. \quad (3.4)$$

Substituting (3.3) and (3.4) into (3.2), we have

$$\langle A\xi_0, [x_k - j(\sigma_{-i/2}(x_k))]Q\xi_0 \rangle = 0$$

for any $A \in M_0$ and $k \in I$. Since $M_0\xi_0$ is dense in $\mathcal{H}$,

$$[x_k - j(\sigma_{-i/2}(x_k))]Q\xi_0 = 0.$$
Since $Q$ is affiliated with $\mathcal{M}$, $j(\sigma_{-i/2}(x_k))Q\xi_0 = Qj(\sigma_{-i/2}(x_k))\xi_0 = Qx_k\xi_0$, and so $x_k\xi_0 \in D(Q)$ and $x_kQ\xi_0 = Qx_k\xi_0$. □

Next, we use Lemma 3.4 and Lemma 3.5 to prove the following result:

**Proposition 3.1** Let $\eta \in D(\Delta^{-1/4}) \cap (\mathcal{N} \cap \mathcal{P})$. Then there is a positive self-adjoint operator $Q$ affiliated with $\mathcal{Z}(\mathcal{M})$ such that $\eta = Q\xi_0$.

**Proof:** Let $\eta = \Delta^{1/4}Q\xi_0$ as in Lemma 3.4. Due to Lemma 3.5

\[ x_kQ\xi_0 = Qx_k\xi_0, \quad \forall k \in I. \]

Since $x_k \in \mathcal{M}_{1/2} \subset \mathcal{M}$ and $Q$ is affiliated with $\mathcal{M}$,

\[ x_kQj(A)\xi_0 = j(A)x_kQ\xi_0 = j(A)Qx_k\xi_0 = Qx_kj(A)\xi_0 \]

for any $A \in \mathcal{M}$, and so

\[ x_kQj(A)\xi_0 = Qx_kj(A)\xi_0, \quad \forall A \in \mathcal{M}. \]

Notice that for any $x_{k_1}$, $x_{k_2} \in \{x_k : k \in I\}$

\begin{align*}
    x_{k_1}x_{k_2}Qj(A)\xi_0 &= x_{k_1}Qx_{k_2}j(A)\xi_0 \\
    &= x_{k_1}Qj(A)j(\sigma_{-i/2}(x_{k_2}))\xi_0 \\
    &= Qx_{k_1}j(A)j(\sigma_{-i/2}(x_{k_2}))\xi_0 \\
    &= Qx_{k_1}x_{k_2}j(A)\xi_0. \tag{3.5}
\end{align*}

Let $\hat{\mathcal{M}}$ be the $*$-algebra generated by $\{x_k : k \in I\}$. Then $\hat{\mathcal{M}}$ is dense in $\mathcal{M}$ by the assumption in Theorem 2.1. The relation (3.5) implies that for any $x \in \hat{\mathcal{M}}$,

\[ xQj(A)\xi_0 = Qxj(A)\xi_0, \quad \forall A \in \mathcal{M}. \]
For given \( x \in \mathcal{M} \), choose a sequence \( x_n \in \tilde{\mathcal{M}} \) such that \( x_n \to x \) strongly. Then

\[
Q x_n j(A) \xi_0 = x_n Q j(A) \xi_0 \to x Q j(A) \xi_0 \quad \text{as} \quad n \to \infty.
\]

Due to the closedness of \( Q \) and the fact that \( x_n j(A) \xi_0 \to x j(A) \xi_0 \) as \( n \to \infty \), we conclude that \( x j(A) \xi_0 \in D(Q) \) and

\[
x Q j(A) \xi_0 = Q x j(A) \xi_0 \quad \text{(3.6)}
\]

for any \( x, A \in \mathcal{M} \).

Denote by

\[
(\mathcal{M} \times \mathcal{M}') \xi_0 := \{ AA' \xi_0 : A \in \mathcal{M}, A' \in \mathcal{M}' \}.
\]

By (3.6), \( (\mathcal{M} \times \mathcal{M}') \xi_0 \in D(Q) \) and for any \( A_1, A_2 \in \mathcal{M} \) and \( A'_1, A'_2 \in \mathcal{M}' \)

\[
A_1 QA_2 A'_2 \xi_0 = A_1 A_2 Q A'_2 \xi_0
\]

\[
= Q A_1 A_2 A'_2 \xi_0 \quad \text{(3.7)}
\]

and

\[
A'_1 QA_2 A'_2 \xi_0 = Q A_2 A'_1 A'_2 \xi_0
\]

\[
= Q A'_1 A_2 A'_2 \xi_0. \quad \text{(3.8)}
\]

Let \( Q_0 \) be the restriction of \( Q \) on \( (\mathcal{M} \times \mathcal{M}') \xi_0 \). Then \( Q_0 \) is a positive symmetric operator. It follows from (3.7) and (3.8) that for any unitary \( U \in \mathcal{M}, U' \in \mathcal{M}' \),

\[
U^* Q_0 U = Q_0, \quad \text{(3.9)}
\]

\[
U''^* Q_0 U' = Q_0.
\]

Notice that \( U \) and \( U' \) leave \( (\mathcal{M} \times \mathcal{M}') \xi_0 \) invariant. Let \( \hat{Q} \) be the Friedrichs of \( Q_0 \).

By the uniqueness of Friedrichs extension,

\[
U^* \hat{Q} U = \hat{Q},
\]

\[
U''^* \hat{Q} U' = \hat{Q}.
\]
for any unitary $U \in \mathcal{M}$, $U' \in \mathcal{M}'$. Thus $\hat{Q}$ is affiliated with $\mathcal{Z}(\mathcal{M})$. By the inclusions $\mathcal{M}'\xi_0 \subset (\mathcal{M} \times \mathcal{M}')\xi_0 \subset D(Q)$ and the uniqueness of the Friedrichs extension, $\hat{Q} = Q$. Since $\Delta \xi = \xi$ for any $\xi \in [\mathcal{Z}(\mathcal{M})\xi_0]$, $\eta = \Delta^{1/4}Q\xi = Q\xi_0$. This completes the proof of the proposition. □

We are ready to prove Theorem 2.1.

**Proof of Theorem 2.1.** The inclusion

$$[\mathcal{Z}(\mathcal{M})\xi_0] \subset \mathcal{N}$$  
(3.10)

is easy to prove as follow: Let $\xi \in [\mathcal{Z}(\mathcal{M})\xi_0]$. Then $\xi = A\xi_0$ for some $A \in [\mathcal{Z}(\mathcal{M})]$. Thus

$$[\sigma_{t-i/4}(x_k) - j(\sigma_{t-i/4}(x_k))]A\xi_0 = A[\sigma_{t-i/4}(x_k) - j(\sigma_{t-i/4}(x_k))]\xi_0 = 0.$$ 

By Lemma 3.1, $\xi \in \mathcal{N}$. Since $\mathcal{N}$ is closed by Lemma 3.2 (a), the closure of $[\mathcal{Z}(\mathcal{M})\xi_0]$ is a subspace of $\mathcal{N}$. This proved the inclusion (3.10).

Next, we prove the inclusion

$$\mathcal{N} \subset [\mathcal{Z}(\mathcal{M})\xi_0].$$  
(3.11)

Any $\eta \in \mathcal{N}$ can be written as $\eta = \eta_r + i\eta_i$, where $\eta_r = (\eta + J\eta)/2$ and $\eta_i = -i(\eta - J\eta)/2$. By Lemma 3.2 (c), $\eta_r, \eta_i \in \mathcal{N}$. Note that $\|\eta\|^2 = \|\eta_r\|^2 + \|\eta_i\|^2$. Thus we may assume that $\eta$ is $J$-real, $\eta \in \mathcal{N} \cap \mathcal{H}$. $\eta$ is decomposed uniquely as $\eta = \eta_+ - \eta_-$, $\eta_+ \in \mathcal{P}$ and $\eta_- \perp \eta_-$. See Proposition 2.5.28 (3) of [BR]. By Lemma 3.2 (d), $\eta_+, \eta_- \in \mathcal{N} \cap \mathcal{P}$. Lemma 3.3 (b) implies that $D(\Delta^{-1/4}) \cap (\mathcal{N} \cap \mathcal{P})$ is dense in $\mathcal{N} \cap \mathcal{P}$. Thus Lemma 3.3 (b) and Proposition 3.1 imply that $\eta_+, \eta_- \in [\mathcal{Z}(\mathcal{M})\xi_0]$, and so $\eta \in [\mathcal{Z}(\mathcal{M})\xi_0]$. This completes the proof of Theorem 2.1. □
4 Ergodicity of Markovian Semigroups for Quantum Spin Systems

In this section, we first describe the translation invariant Markovian semigroups for quantum spin systems constructed in [Par1], and then apply Theorem 2.1 (and Corollary 2.1) to show the ergodicity of the semigroups in region of high temperatures where the uniqueness of KMS-state holds.

Let us describe quantum spin systems briefly. For details, we refer to Section 6.2 of [BR]. Let \( \mathbb{Z}^d \) be a \( d \)-dimensional lattice space and let \( \mathcal{F} \) denote the family of all finite subsets of \( \mathbb{Z}^d \). Let \( \mathcal{A} \) be a \( C^* \)-algebra with norm \( \| \cdot \| \) defined as the inductive limit over a finite-dimensional matrix algebra \( \mathbb{M} \). For any \( X \in \mathcal{F} \), let \( \mathcal{A}_X \) denote the subalgebra localized in \( X \), i.e., the subalgebra in \( \mathcal{A} \) isomorphic to \( \mathbb{M}^X \).

An element \( A \in \mathcal{A} \) will be called \emph{local} if there is some \( Y \in \mathcal{F} \) such that \( A \in \mathcal{A}_Y \). By \( \mathcal{A}_0 \) we denote the subset of all local elements, i.e., \( \mathcal{A}_0 = \bigcup_{X \in \mathcal{F}} \mathcal{A}_X \).

Let \( \Phi := \{ \Phi_X \}_{X \in \mathcal{F}} \) be an interaction, i.e., a family of self-adjoint element in \( \mathcal{A} \).

Suppose that

\[
\| \Phi \|_\lambda := \sup_{i \in \mathbb{Z}^d} \sum_{X \in \mathcal{F} : i \in X} e^{\lambda |X|} \| \Phi_X \| < \infty \tag{4.1}
\]

for some \( \lambda > 0 \), where \( |X| = \text{card}(X) \). Define a derivation \( \delta \) by

\[
D(\delta) = \mathcal{A}_0, \quad \delta(A) = -i \sum_{X \cap \Lambda \neq \emptyset} [\Phi, A], \quad A \in \mathcal{A}_\Lambda. \tag{4.2}
\]

Then \( \mathcal{A}_0 \) is a norm-dense \( * \)-subalgebra of analytic element of the closure \( \overline{\delta} \) of \( \delta \). Thus \( \overline{\delta} \) generates one-parameter group of \( * \)-automorphism \( \tau \) of \( \mathcal{A} \). Let \( \omega \) be a \( \tau \)-KMS state corresponding to the interaction \( \Phi \).

Let \( (\mathcal{H}_\omega, \pi_\omega, \Omega_\omega) \) be the GNS representation of \( (\mathcal{A}, \omega) \). For the standard form, we choose \( \mathcal{H} = \mathcal{H}_\omega, \mathcal{M} = \pi_\omega(\mathcal{A})'' \) and \( \xi_0 = \Omega_\omega \). By the uniqueness of the modular
automorphism (see Theorem 5.3.10 of [BR]), one may identify \( \sigma_t = \tau_t, \ t \in \mathbb{R}, \) on \( \mathcal{M}. \)

In this section, we denote by \( \mathcal{M}_0 \) the algebra of local elements, i.e., \( \mathcal{M}_0 = \pi_\omega(A_0). \) Every element \( A \in \mathcal{M}_0 \) is an analytic element for \( \sigma_t. \) For a given \( \lambda > 0, \) put \( \gamma = \lambda/2\|\Phi\|_\lambda. \) Then for any \( s \in (-\gamma, \gamma) \) the series

\[
\sigma_{is}(A) = \sum_{n=0}^{\infty} (-i s)^n \frac{\delta^n(A)}{n!}, \ A \in \mathcal{M}_0,
\]

converges absolutely, where \( \delta \) is the derivation given (4.2). See the proof of Theorem 5.2.4 of [BR]. From now on, we assume that \( \Phi \) is chosen sufficiently small so that \( \gamma > 1/2. \)

We now turn to Dirichlet form for quantum spin systems [Par1]. Let \( \{\tau_j\}_{j \in \mathbb{Z}^d} \) be the translational automorphism on \( \mathcal{M} \) corresponding to the translation of the lattice by vectors \( j \in \mathbb{Z}^d. \) Let \( x^a \in \pi_\omega(\mathbb{M}) \), \( a = 1, 2, ..., D, \) be a basis of \( \pi_\omega(\mathbb{M}) \) consisting of self-adjoint elements of norm one and let \( x_j^a = \tau_j(x^a), \ j \in \mathbb{Z}^d. \) For the family \( \{x^a_j : j \in \mathbb{Z}^d, a = 1, 2, ..., D\} \) and an admissible function \( f \) (or else \( f_0 \)), let \( (\mathcal{E}, D(\mathcal{E})) \) be the quadratic form defined as in (2.5) - (2.7):

\[
D(\mathcal{E}) = \{\xi \in \mathcal{H} : \sum_{j \in \mathbb{Z}^d} \sum_{a=1}^{D} \mathcal{E}_{a,j}[\xi] < \infty\};
\]

\[
\mathcal{E}[\xi] = \sum_{j \in \mathbb{Z}^d} \sum_{a=1}^{D} \mathcal{E}_{a,j}[\xi], \ \xi \in D(\mathcal{E})
\]

where

\[
\mathcal{E}_{a,j}[\xi] = \int \|(\sigma_{t-i/4}(x^a_j) - j(\sigma_{t-i/4}(x^a_j)))\xi\| f(t) dt.
\]

The following is Theorem 5.1 of [Par1]:

**Theorem 4.1 :** (Theorem 5.1 of [Par1]) Let \( f \) be an admissible function such that \( p \) in (2.3) (c) is greater than \( d + 1, \) i.e., \( p > d + 1. \) Let the interaction \( \Phi \) be of finite range and translation invariant. Then the form \( (\mathcal{E}, D(\mathcal{E})) \) defined as in (4.4) - (4.5) is a densely defined Dirichlet form which generates a translation invariant, symmetric, Markovian semigroup.
Remark 4.1 The strongly decay property of $f$, i.e., $p > d+1$, has been used to show that $D(E)$ is dense in $H$. See the proof of Theorem 5.1 of [Par1]. The function $f_0$ given in (2.4) decays exponentially fast and so the conclusion in Theorem 4.1 holds for $f = f_0$.

In order to describe the main result, we need to replace $\Phi$ by $\beta \Phi$, where $\beta$ is the inverse temperature. Then the condition $\gamma > 1/2$ is equivalent to $(\lambda/2\beta\|\Phi\|_\lambda) > \frac{1}{2}$. This is, $\beta\|\Phi\|_\lambda < \lambda$. The following is the main result in this section:

**Theorem 4.2** Let $f$ be either an admissible function satisfying the decay property in Theorem 4.1 or else $f = f_0$. Let the interaction $\Phi$ be of finite range and translation invariant. For $\{x^a_j : j \in \mathbb{Z}^d, a = 1, 2, ..., D\}$, let $\{T_t\}_{t \geq 0}$ be the translation invariant Markovian semigroup associated to the Dirichlet form defined as in (4.4)-(4.5). Assume that $\beta\|\Phi\|_\lambda$ is sufficiently small so that $(\tau, \beta)$-KMS state for $\Phi$ is unique. Then the Markovian semigroup $\{T_t\}_{t \geq 0}$ is ergodic.

Remark 4.2 The region of high temperatures where the uniqueness of $(\tau, \beta)$-KMS state holds can be given explicitly. For an instance, see Proposition 6.2.25 of [BR]. For one-dimensional models with uniform bounded surface energies, the uniqueness of $(\tau, \beta)$-KMS state is independent of temperature (Theorem 6.2.47 of [BR]). However, we still need the condition $\beta\|\Phi\|_\lambda < \lambda$.

**Proof of Theorem 4.2**. By the condition $\beta\|\Phi\|_\lambda < \lambda$, the series (4.3) converges absolutely on a region containing $[-\beta/2, \beta/2]$. Thus it is easy to see that $\{x^a_j : j \in \mathbb{Z}^d, a = 1, 2, ..., D\} \subset \mathcal{M}_{\beta/2}$. Since the $*$-algebra generated by the family is $\mathcal{M}_0$, which is dense in $\mathcal{M}$, the condition in Theorem 2.1 hold. The uniqueness of the $(\tau, \beta)$-KMS state $\omega$ implies that $\omega$ ia an extremal $(\tau, \beta)$-KMS state, and hence a factor state by Theorem 5.3.30 of [BR]. Thus $\mathcal{M}$ is a factor, and so $\{T_t\}_{t \geq 0}$ is ergodic by Corollary 2.1. □.
Acknowledgements: This work was supported by Korea Research Foundation (KRF-2003-005-C00010), Korean Ministry of Education.

References

[Acc] L. Accardi, Topics in quantum probability, Phys. Rep. 77, 169-192 (1981).

[Ara] H. Araki, Some properties of modular conjugation operator of von Neumann algebras ans noncommutative Radon-Nikodym theorem with chain rule, Pacific J. Math. 50 (2), 309-354 (1974).

[BK] C. Bahn and C. K. Ko, Construction of unbounded Dirichlet forms on standard forms of von Neumann Algebras, J. Korean Math. Soc. 39 (6) 931-951 (2002).

[BKP1] C. Bahn, C. K. Ko and Y. M. Park, Dirichlet forms and symmetric Markovian semigroups on CCR Algebras with quasi-free states, J. Math. Phys., 44, 723-753 (2003).

[BKP2] C. Bahn, C. K. Ko and Y. M. Park, Dirichlet forms and symmetric Markovian semigroups on $\mathbb{Z}_2$-graded von Neumann algebras, Rev. Math. Phys., 15, 823-845 (2003).

[BR] O. Bratteli and D. W. Robinson, Operator algebras and quantum statistical mechanics, Springer-Verlag, New York-Heidelberg-Berlin, vol I 1979, vol. II (1981).

[Cip1] F. Cipriani, Dirichlet forms and Markovian semigroups on standard forms of von Neumann algebras, J. Funct. Anal. 147, 259-300 (1997).

[Cip2] F. Cipriani, Perron Theory for Positive Maps Semigroups on von Neumann Algebras, Canadian Math. Soc., Conference Proceedings, Vol. 2a, 115-123 (2000).
[CFL] F. Cipriani, F. Fagnola and J. M. Lindsay, Spectral Analysis and Feller Properties for Quantum Ornstein-Uhlenbeck Semigroups, Comm. Math. Phys. 210, 85-105 (2000).

[Dav] E. B. Davies, Quantum theory of open systems, Academic Press, London-New York-San Francisco, (1976).

[GL1] S. Goldstein and J. M. Lindsay, KMS-symmetric Markov semigroups, Math. Zeit. 219, 590-608 (1995).

[GL2] S. Goldstein and J. M. Lindsay, Markov semigroups KMS-symmetric for a weight, Math. Ann. 313, 39-67 (1999).

[KP] C. K. Ko and Y. M. Park, Construction of a Family of Quantum Ornstein-Uhlenbeck Semigroups, J. Math. Phys., 45, 609-627 (2004).

[MZ1] A. W. Majewski and B. Zegarlinski, Quantum stochastic dynamics I: Spin systems on a lattice, MPEJ 1, Paper 2 (1995).

[MZ2] A. W. Majewski and B. Zegarlinski, Quantum stochastic dynamics II, Rev. Math. Phys. 8 (5), 689-713 (1996).

[Par1] Y. M. Park, Construction of Dirichlet forms on standard forms of von Neumann algebras, Infinite Dimensional Analysis, Quantum Probability and Related Topics, Vol. 3, No. 1, 1-14 (2000).

[Par2] Y. M. Park, Remark on the Structure of Dirichlet Forms on the Standard Forms of von Neumann Algebras, arXiv. Math-ph/04001, to be appeared in Infinite Dimensional Analysis, Quantum Probability and Related Topics.

[Part] K. R. Parthasarathy, An introduction to quantum stochastic calculus, Birkhäuser, Basel (1992).
[RS] M. Reed and B. Simon, *Method of modern mathematical physics I, II*, Academic press (1980).

[Zeg] B. Zegarlinski, Analysis of Classical and Quantum Interacting Partical Systems, In *Quantum Probability and White Noise Analysis*, Vol. XIV, eds, L. Accard and F. Fagnola, World Scientific, 241-336(2000).