The percolation transition in correlated hypergraphs

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Abstract. Correlations are known to play a crucial role in determining the structure of complex networks. Here we study how their presence affects the computation of the percolation threshold in random hypergraphs. In order to mimic the correlation in real networks, we build hypergraphs from generalized hidden variable ensembles and we study the percolation transition by mapping this problem to the fully connected Potts model with heterogeneous couplings.

Keywords: random graphs, networks, critical phenomena of socio-economic systems, socio-economic networks
1. Introduction

In the last decade the topological properties of networks have attracted great interest [1]–[4], mainly driven by the emergence of novel dynamical effects in random processes defined on them [5, 6]. The impact of this research is wide and has important consequences in the domains of biology, socio-economical theories and technological infrastructure design. It has been shown that complex networks can display a universal behavior which strongly affects the dynamics of statistical models built on them.

A major example is provided by the percolation transition [7]–[10], one of the most famous emergent collective phenomena that can be defined on complex networks. Its dependence on degree distribution, degree correlations and directionality of the links has been extensively studied in the last few years [11]–[13]. In particular, the interplay between topological features and the nature of the percolation transition has been fully investigated within different kinds of random network ensembles [14]–[20]. These constitute null models for networks, each of them being formed by graphs sharing with real complex networks a number of structural features, such as degree distribution and correlations between neighboring components.

Recently, attention has been devoted to the structure of hypergraphs [21]–[23], describing, for example, many on-line social and professional communities which collaborate in order to give a semantic structure to a set of data or different kinds of cellular networks. Biological hypergraphs capture the many-body interactions in the context of protein interactions, or chemical reactions [23]. Among these communities, also named folksonomies, we mention Flickr and CiteULike, whose structure is formed by triplets of users, resources and tags linked together. Such networks show important correlations since the interest of the user and the subject of the resources (for example a picture for Flickr and an article for CiteULike) usually have a strong interdependence.

These correlations are responsible for the build-up of communities, after projecting these networks into networks made only by user–user, resource–resource and tag–tag interactions [22].

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In a recent paper [21], the percolation transition in random uncorrelated hypergraphs was characterized, providing a first approximation to the real hypergraphs properties. In this work we extend these results to more general ensembles of correlated hypergraphs which can give a better description of real social communities. We will show how to build null models for hypergraphs on the basis of a recent parallel construction introduced for networks [19,20]. Moreover, we will derive the percolation threshold of correlated hypergraphs by mapping the problem to the solution of a fully connected Potts model with heterogeneous couplings [24] following the method developed in a recent paper on percolation phase transition in simple networks [25].

2. Correlated random hypergraphs

To mimic the correlation of real hypergraphs we propose to study randomized ensembles of correlated hypergraphs within the theoretical framework developed in [19,20] for networks. In these works it has been shown how it is possible to correctly define statistical microcanonical and canonical network ensembles. The relation between microcanonical and canonical ensembles generalizes the relation existing between the $G(N,L)$ random network ensemble—the collection of all the networks with fixed number of nodes $N$ and links $L$—and $G(N,p)$—the collection of all the networks formed by $N$ nodes, each pair of nodes linked with probability $p$. In the latter case the number of links $L$ is not fixed but fluctuates and it has a Poissonian distribution with mean $p \times N$. It is known that in the thermodynamic limit these two different ensembles share the same statistical properties when the external parameters, $L$ and $p$, satisfy the following relation: $p = L/N$. We call $G(N,L)$ a microcanonical ensemble because it satisfies a hard constraint on the extensive number of links $L$—like the energy constraint for usual microcanonical ensembles, while in the other case the $G(n,p)$ ensemble is canonical, in the sense that the Lagrange multiplier $p$ for the number of links fluctuates, fixing only its average value, at $L = p \times N$. Generalizing this approach to several properties, like the degree sequence or a given partition into different communities, it is possible to define complex microcanonical and canonical network ensembles satisfying more stringent constraints [19,20]. Here we sketch how this general framework can be easily generalized to hypergraphs.

Let us consider a hypergraph formed by nodes of different types $\alpha = 1, \ldots, K$ linked in $K$ groups. For example in Flickr we will have $K = 3$ and $\alpha = 1, 2, 3$ indicating respectively agents, pictures and tags. We assume also, in order to gain in generality, that each node can be associated with a different feature $r^\alpha_i = 1, \ldots, R^\alpha$ indicating a given classification of the nodes. Again, in the case of Flickr the agents can be classified in relation to their interests or age, the tags in relation to their general meaning and the pictures in relation of the type of subject that is represented.

A given random correlated ensemble of hypergraphs can be defined as the set of all hypergraphs which satisfy a number of constraints.

In particular, we choose these constraints to be the number of hyperlinks that each node has and the number of hyperlinks bridging sets of nodes with different features. Following these prescriptions we can construct microcanonical and canonical hypergraph ensembles.
2.1. Microcanonical hypergraph ensembles

Let us define a hypergraph by the tensor \( a_{i_1, i_2, \ldots, i_K} = 1 \) if the nodes \((i_1, i_2, \ldots, i_K)\) are linked together and \( a_{i_1, i_2, \ldots, i_K} = 0 \) otherwise. Let us call \( N_{\alpha} \) the number of nodes of type \( \alpha \). The networks in the microcanonical hypergraph ensemble will then satisfy the following conditions:

\[
\begin{align*}
  k_{i_{\alpha}} &= \sum_{\{i_\gamma\}_{\gamma \neq \alpha}} a_{i_1, \ldots, i_{\alpha}, \ldots, i_K}, \\
  A(x_1, x_2, \ldots, x_K) &= \sum_{\{i_\gamma\}} a_{i_1, \ldots, i_{\alpha}, \ldots, i_K} \prod_{\gamma} \delta(r_{i_\gamma} - x_\gamma) 
\end{align*}
\]  

(1)

with \( k_{i_{\alpha}} \) being the hyperdegree of node \( i_{\alpha} \) and with \( A(x_1, x_2, \ldots, x_K) \) being the number of hyperedges between nodes of features \((x_1, x_2, \ldots, x_K)\).

2.2. Canonical hypergraph ensembles

By using the statistical mechanics approach described in [19,20] it can be shown that networks in the canonical hypergraph ensembles, satisfying on average the constraints (1), can be constructed by assigning a hyperlink \( a_{i_1, i_2, \ldots, i_K} = 1 \) with probability

\[
p_{i_1, i_2, \ldots, i_K} = \frac{\theta_{i_1} \theta_{i_2} \cdots \theta_{i_K}}{1 + \theta_{i_1} \theta_{i_2} \cdots \theta_{i_K}} W(r_{i_1}, r_{i_2}, \ldots, r_{i_K}).
\]

(2)

If the constants \( \theta_{i_{\alpha}} \) and the tensor \( W(r_{i_1}, r_{i_2}, \ldots, r_{i_K}) \) satisfy the following conditions:

\[
\overline{k}_{i_{\alpha}} = \sum_{\{i_\gamma\}_{\gamma \neq \alpha}} p_{i_1, i_2, \ldots, i_K}
\]

(3)

\[
A(x_1, x_2, \ldots, x_K) = \sum_{\{i_\gamma\}} p_{i_1, i_2, \ldots, i_K} \prod_{\gamma=1}^K \delta(r_{i_\gamma} - x_\gamma)
\]

then the hyperdegrees \( k_i \) and the number of hyperedges \( A(x_1, x_2, \ldots, x_K) \)

\[
k_{i_{\alpha}} = \sum_{\{i_\gamma\}_{\gamma \neq \alpha}} a_{i_1, \ldots, i_{\alpha}, \ldots, i_K},
\]

\[
A(x_1, x_2, \ldots, x_K) = \sum_{\{i_\gamma\}} a_{i_1, \ldots, i_{\alpha}, \ldots, i_K} \prod_{\gamma} \delta(r_{i_\gamma} - x_\gamma)
\]

are Poisson distributed [17] with average \( \overline{k}_{i_{\alpha}} \) and \( \overline{A(x_1, x_2, \ldots, x_K)} \) given by (3).

In the limit of hypergraphs with a linking probability independent on the features of the nodes \( r_{i_{\alpha}} \) we obtain that the probability (2) becomes

\[
p_{i_1, i_2, \ldots, i_K} = \frac{\theta_{i_1} \theta_{i_2} \cdots \theta_{i_K}}{1 + \theta_{i_1} \theta_{i_2} \cdots \theta_{i_K}}.
\]

(4)

Moreover, we recover the configuration model for the uncorrelated hypergraphs taking the limit \( \prod_{\alpha} \theta_{i_{\alpha}} \ll 1 \forall \{i_{\alpha}\} \). In this case the hyperedge probability is

\[
p_{i_1, i_2, \ldots, i_K} \simeq \theta_{i_1} \theta_{i_2} \cdots \theta_{i_K} = \frac{k_{i_1} \cdots k_{i_K}}{(K(N))^K(K-1)}.
\]

(5)

and this last expression describes uncorrelated hypergraphs whose properties have been studied in [21].
3. The Potts model and the percolation transition

It is a well known result [26]–[28] of statistical mechanics that the Potts phase transition in a fully connected systems for the number of colors $q \to 1$ describes the bond percolation transition [24,29] in Erdős–Rényi $G(N,p)$ random graphs [30]. Recently a fully connected Potts model with heterogeneous couplings [24,25] was introduced in order to study the percolation transition in random networks with heterogeneous degree distribution and additional structural properties. Here we show that such a method can be extended to correlated canonical hypergraphs.

Instead of a pure Potts model we consider the following generalized Hamiltonian:

\[
H = - \sum_{i_1 i_2 \cdots i_K} J_{i_1 i_2 \cdots i_K} \delta_{s_{i_1} s_{i_2} \cdots s_{i_K}}
\]

where the summation runs over all the $K$-sets that compose the fully connected hypergraph and the variables $\{s_i\}$ are Potts spins taking $q$ values from $[1, 2, \ldots, q]$. We define for later convenience the vector $s_\alpha = \{s_{1_\alpha}, \ldots, s_{N_\alpha}\}$ where $N_\alpha$ is the number of sites of a given type $\alpha$. Introducing an external parameter $\beta$ as the inverse of temperature, the partition function of the Hamiltonian model reads

\[
Z = \sum_{s_1 s_2 \cdots s_K} e^{-\beta H[s_\alpha]},
\]

whose cluster expansion gives the following expression:

\[
Z = \sum_{H \in K(H)} \prod_{i_1 i_2 \cdots i_K \in E} v_{i_1 i_2 \cdots i_K} q^{C(H)}.
\]

The value $C(H)$ is the number of connected components of the hypergraph $H$ and

\[
v_{i_1 i_2 \cdots i_K} = e^{\beta J_{i_1 i_2 \cdots i_K}} - 1 \approx \beta J_{i_1 i_2 \cdots i_K},
\]

where the last expression is valid in the small $\beta$ limit. After identifying the coupling

\[
\beta J_{i_1 i_2 \cdots i_K} = \frac{p_{i_1 i_2 \cdots i_K}}{1 - p_{i_1 i_2 \cdots i_K}} = \theta_{i_1} \theta_{i_2} \cdots \theta_{i_K} W(r_{i_1}, r_{i_2}, \ldots, r_{i_K})
\]

then the sum in (8) is a weighted sum over random hypergraphs $H$ in which each link has probability $p_{i_1 i_2 \cdots i_K}$ reported in equation (2). If the condition (9) holds, the transition of the Hamiltonian model (6) for $q \to 1$ coincides with the percolation transition in the random correlated hypergraph ensemble. In section 3.1 we will solve the Potts model providing the percolation condition for hypergraphs in the ensemble (4) that could be written in the form

\[
\det(\Xi) = 0
\]

with the matrix $\Xi$ to be determined in the following.

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3.1. Solving the Potts model

In this section we solve the Potts model (6), in the generic case, for arbitrary $K$ and $J_{i_1,i_2,...,i_K}$ given by (9). We introduce the order parameter $c_{\theta,r}^\alpha(s)$ indicating the fraction of nodes of type $\alpha = 1, \ldots, K$ associated with the ‘hidden variable’ $\theta$ and the feature $r$, having Potts spin equal to $s$

$$c_{\theta,r}^\alpha(s) = \frac{1}{N_\alpha p^\alpha(\theta, r)} \sum_{i_\alpha} \delta_{s,s_{i_\alpha}} \delta(\theta, \theta_{i_\alpha}) \delta_{r,r_{i_\alpha}}$$

where $p^\alpha(\theta, r)$ is the probability distribution for having a node of $\alpha$-type with local properties described by variables $\theta$ and $r$. We have introduced for convenience the Dirac delta function, with its proper normalization $\int d\theta \delta(\theta) = 1$ and the Kronecker delta $\sum_i \delta_{i,0} = 1$ defined as

$$\delta(\theta, \theta') = \begin{cases} 0 & \theta \neq \theta' \\ \infty & \theta = \theta' \end{cases}, \quad \delta_{s,s'} = \begin{cases} 0 & s \neq s' \\ 1 & s = s'. \end{cases}$$

Noticing the symmetry of the Hamiltonian in terms of $c_{\theta,r}^\alpha(s)$

$$H[\{c_{\theta,r}^\alpha(s)\}] = -\sum_{\gamma} N_\gamma \sum_{s,\{r\}} \prod_{\gamma} \left[ \int d\theta^\gamma p^\gamma(\theta^\gamma, r^\gamma)c_{\theta^\gamma,r^\gamma}^\gamma(s) \right] J(\{\theta\}, \{r\})$$

we can express the partition function as a summation over these new variables

$$Z = \sum_{\{c_{\theta,r}^\alpha(s)\}} e^{-\beta H[\{c_{\theta,r}^\alpha(s)\}]}$$

where the free energy function is given by

$$\beta F[\{c_{\theta,r}^\alpha(s)\}] = \beta H[\{c_{\theta,r}^\alpha(s)\}] + \sum_\alpha N_\alpha \int d\theta^\alpha \sum_r p^\alpha(\theta^\alpha, r^\alpha)c_{\theta^\alpha,r^\alpha}^\alpha(s) \ln c_{\theta^\alpha,r^\alpha}^\alpha(s).$$

The phase transition of the Potts model is determined by the point at which the free energy becomes unstable with respect to variation of the order parameters around the symmetric solution

$$c_{\theta,r}^\alpha = \frac{1}{q}.$$ 

In order to determine the stability of the free energy we evaluate the Hessian $\mathcal{H}$ of components

$$\mathcal{H}(\alpha;\{a\})_{\{\alpha\}}(s, s') = \frac{\partial^2 F}{\partial c_{\{\alpha\}}(s) \partial c_{\{\alpha\}}(s')}$$

where we indicate by $\{a\}(\{a'\})$ the triplets $\{\alpha, \theta^\alpha, r^\alpha\}(\{\alpha, \theta^{\alpha'}, r^{\alpha'}\})$. Making the calculation explicitly, the previous equation (17) reads

$$\mathcal{H}(\alpha;\{a\})_{\{\alpha\}}(s, s') = \delta_{s,s'} \left[ \delta_{\{a\},\{a'\}} \frac{N_\alpha p^\alpha(\theta^\alpha, r^\alpha)}{c_{\theta^\alpha,r^\alpha}^\alpha(s)} - p^\alpha(\theta^\alpha, r^\alpha)p^{\alpha'}(\theta^{\alpha'}, r^{\alpha'}) \right] \times \prod_\gamma N_\gamma \prod_{\gamma \neq \alpha,\alpha'} \left[ \int d\theta^\gamma \sum_r p^\gamma(\theta^\gamma, r^\gamma)c_{\theta^\gamma,r^\gamma}^\gamma(s) \right] \beta J(\{\theta\}, \{r\}).$$

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with the following definition of $\delta_{\{a\},\{a'\}} = \delta_{\alpha,\alpha'}\delta(\theta^\alpha, \theta^{\alpha'})\delta_{r^\alpha, r^{\alpha'}}$. Taking $J(\{\theta\}, \{q\})$ from equation (9), we obtain that the eigenvalue problem associated with the Hessian matrix is

$$[\lambda + N_\alpha q(\theta^\alpha, r^\alpha)q]\mathbf{e}(\{a\}) = \frac{N_\alpha q(\theta^\alpha, r^\alpha)}{q^{K-2}} \sum_{a' \neq a} \left\{ \prod_{\gamma \neq \alpha, \alpha'} \left[ \int d\theta^\gamma \sum_{r^\gamma} N_\gamma p^\gamma(\theta^\gamma, r^\gamma) \right] \times \int d\theta'^\alpha \sum_{r'^\alpha} p'^\alpha(\theta'^\alpha, r'^\alpha)\beta J(\{\theta\}, \{r\})\mathbf{e}(\{a'\}) \right\}$$

(19)

where $\lambda$ and $\mathbf{e}(\{a\})$ are respectively the eigenvalue and the eigenvector of the problem. Equation (19) can be written as

$$\mathbf{e}(\{a\}) = \frac{N_\alpha q(\theta^\alpha, r^\alpha)}{\lambda + N_\alpha q(\theta^\alpha, r^\alpha)q} q^{K-2} \Delta(\{a\})$$

(20)

with the $\Delta(\{a\})$ defined as

$$\Delta(\{a\}) = \sum_{a' \neq a} \left\{ \prod_{\gamma \neq \alpha, \alpha'} \left[ \int d\theta^\gamma \sum_{r^\gamma} N_\gamma p^\gamma(\theta^\gamma, r^\gamma) \right] \times N_{a'} \int d\theta'^{a'} \sum_{r'^{a'}} p'^{a'}(\theta'^{a'}, r'^{a'})\beta J(\{\theta\}, \{r\})\mathbf{e}(\{a'\}) \right\}.$$  

(21)

The symmetric solution becomes unstable when the maximal eigenvalue of the Hessian problem becomes positive. Therefore, in order to determine the critical point of the Potts model for $q \rightarrow 1$ we consider equations (20) and (21) when the eigenvalue vanishes, $\lambda = 0$. If we take into account the explicit form of the coupling constant given by equation (9), we find that the vector $\Delta$ takes the form $\Delta(\{a\}) = \theta^\alpha \mathbf{v}_{a,r^\alpha}$ where the $\mathbf{v}$s satisfy the linear system of equations

$$\Xi \mathbf{v} = 0$$

(22)

with the matrix $\Xi_{\{a,r^\alpha\},\{a',r'^{\alpha'}\}}$ given by

$$\Xi_{\{a,r^\alpha\},\{a',r'^{\alpha'}\}} = -\delta_{\alpha,a'}\delta_{\alpha',r^\alpha r'^{\alpha'}} + [1 - \delta_{\alpha,\alpha'}\delta_{\alpha',r^\alpha r'^{\alpha'}}] \prod_{\gamma \neq \alpha, \alpha'} \left[ \int d\theta^\gamma \sum_{r^\gamma} N_\gamma p^\gamma(\theta^\gamma, r^\gamma)\theta^\gamma \right] \times \int d\theta'^{\alpha'} N_{a'} p'^{\alpha'}(\theta'^{\alpha'}, r'^{\alpha'}) (\theta'^{\alpha'})^2 W(r_1, \ldots, r^\alpha, \ldots, r'^{\alpha'}, \ldots, r_K).$$

(23)

Therefore the condition determining the critical point of the Potts model for $q \rightarrow 1$ comes from the vanishing of the determinant, i.e. $\det \Xi = 0$.

### 3.2. Simplified cases

In the simplified case in which the linking distributions do not depend on the communities $\{r\}$ we have that $\mathbf{v}(\alpha, r^\alpha) = \mathbf{u}_\alpha$. In fact the coupling constant in the Potts model given by (9) becomes simply

$$J_{i_1, i_2, \ldots, i_K} = \theta_{i_1} \theta_{i_2} \cdots \theta_{i_K}$$

(24)

and then the solution $\mathbf{v}(\alpha, r^\alpha)$ can be expressed as

$$\mathbf{v}(\alpha, r^\alpha) = u^\alpha.$$  

(25)

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Using the definition (23) we get a non-null solution for the \( u_s \) if and only if the condition \( \det \Phi = 0 \) is satisfied, with \( \Phi \) defined as
\[
\Phi_{\alpha,\alpha'} = -\delta_{\alpha,\alpha'} + (1 - \delta_{\alpha,\alpha'})N_{\alpha'}(\theta^2)_{\alpha'} \prod_{\gamma \neq \alpha,\alpha'} [N_\gamma(\theta)].
\]
(26)

In the case \( K = 3 \) this condition reduces to the following relation:
\[
2\pi_{12}\pi_{23}\pi_{31} + \pi_{12}\pi_{32} + \pi_{13}\pi_{23} + \pi_{21}\pi_{31} - 1 = 0
\]
(27)

with the proper identification
\[
\pi_{\alpha,\alpha'} = N_{\alpha}(\theta^2)_{\alpha}N_{\alpha'}(\theta)_{\alpha'}.
\]
(28)

The formula (27) is valid in the general case of hypergraphs that show non-trivial correlation between nodes. We recover as a special case the percolation condition in the uncorrelated hypergraphs, just remembering the relation between the variables \( \theta \) and the mean site connectivity \( \theta_i = k_i/\langle k \rangle N \). Therefore, using the previous condition, the \( \pi_s \) are given by
\[
\pi_{\alpha,\alpha'} = \frac{\langle k(k-1) \rangle_{\alpha}}{\langle k \rangle}
\]
(29)

and after some algebra we obtain the condition for the percolation already found in [21]
\[
\frac{\langle k \rangle_1}{\langle k^2 \rangle_1} + \frac{\langle k \rangle_2}{\langle k^2 \rangle_2} + \frac{\langle k \rangle_3}{\langle k^2 \rangle_3} = 2.
\]
(30)

4. Conclusions

Correlations account for the non-trivial structure of complex networks and must play a significant role also in the characterization of hypergraphs describing folksonomies. In this paper we have studied ensembles of correlated hypergraphs which can be used to model the interactions between different kinds of nodes in real complex hypergraphs. We determined the percolation threshold by mapping this problem to a fully connected Potts model with heterogeneous couplings. Our approach extends the present knowledge on percolation in uncorrelated hypergraphs. Future development will link these findings to the study and characterization of real folksonomies and to the analysis of the robustness of the giant component against the removal of nodes or hyperedges.

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