Noise-induced Synchronization in Small World Network of Phase Oscillators

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A small-world network (SW) of similar phase oscillators, interacting according to the Kuramoto model is studied numerically. It is shown that deterministic Kuramoto dynamics on the SW networks has various stable stationary states. This can be attributed to the defect patterns in a SW network which is inherited to it from deformation of helical patterns in its parent regular one. Turning on an uncorrelated random force, causes the vanishing of the defect patterns, hence increasing the synchronization among oscillators for intermediate noise intensities. This phenomenon which is called stochastic synchronization generally observed in some natural networks like brain neuronal network.

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INTRODUCTION

Noise is usually considered as a source of disturbance against the main signals in laboratory as well as natural systems. Nevertheless, the interplay between the randomness, created by the noise, and non-linearities may lead to the enhancement of regular behaviour in some dynamical systems [1]. Stochastic resonance [2], coherence resonance [3], noise-induced transport [4], noise-induced transition [5] and noise-induced collective firing in excitable media [6] are examples of such a novel phenomena. Being noisy, nature takes the advantages of these mechanisms in order to employ the random fluctuation as agent of self-organization. This is the main reason that why living systems work so reliable in spite of the presence of various sources of noise. Brain neurons are examples of biological systems in which the source of random fluctuations is the background synaptic noises caused by highly fluctuating inputs coming from thousands of other neurons connected to a given neuron [7]. However, this noise plays a constructive role in regular spiking of the individual neurons and also increasing the synchronization among the clusters of connecting neurons [8]. Synchronous spiking among a subset of neurons plays an important role in more efficient propagation of activities from a group of neurons to another [9]. Furthermore, there are also some controversial idea on encoding of information about stimuli thorough synchrony in oscillatory activity of neurons [10]. Another phenomena in which noise-induced synchronization takes place, is gene regulatory processes in systems such as quorum-sensing bacteria, in which noise originates from the small number of molecules involved in the related biochemical reactions [11]. Collective dynamical behaviours, like synchronization, can be simulated by systems of coupled non-linear oscillators. One of such models has been proposed by Kuramoto, which consists of a set of oscillators with fixed amplitude (phase oscillators) mutually coupled with sine of their phases difference [12]. The stochastic Kuramoto model has been studied on the globally connected [13] and also on scale-free (SF) and Erdős-Rényi (ER) random networks [14]. Analytical results on the all-to-all network show that for a given distribution of intrinsic frequencies of oscillators, a minimum value of coupling is needed for them to become synchronized. Deriving the synchronized system by an uncorrelated white noise, causes the synchrony between oscillators declines monotonically by increasing the noise strength. The same results have been found on numerical integrations of the stochastic Kuramoto model on the ER and SF networks. The difference is that the synchronized state in SF networks persists more against applying the noise with respect to the ER and all-to-all networks [14]. In an effort to search for the reason of synchronization between crickets, Watts and Strogatz found out that many systems in nature possess the properties of the small-world (SW) networks [15, 16]. Short mean path between the nodes and high degree of clustering are the two main features of SW networks. Former is a characteristic of random, while the latter is feature of regular networks. It has been found that the presence of random short-cuts, may lead to noise driven ordering phenomena such as stochastic resonance [17] and coherence resonance [18] in SW networks. Motivated by recent discoveries revealing SW topology of brain neural networks [19] and also noise induced regulatory behaviours in such a networks [8], we study the effect of random force on dynamics of SW network of a set of similar phase oscillators coupled to each other based on Kuramoto model. We will show that in this system, for intermediate noise strength, the synchronization among the oscillators is increased. The rest of the paper is organized as follows. In Sec. II, we present the results of numerical integration of deterministic Kuramoto model on regular and SW networks. Investigation of Stochastic Kuramoto model driven by uncorrelated white noise is done in Sec. III and Sec. IV is devoted to summary and concluding remarks.
KURAMOTO MODEL ON COMPLEX NETWORKS

In this section we introduce the Kuramoto model and numerically investigate its steady state solutions on ER, SF and SW networks. Consider a set of phase oscillators, residing on the top of the nodes of a network. Their phases and intrinsic oscillation frequencies are given by $\theta_i$ and $\omega_i$, respectively. According to the Kuramoto model the dynamics of these phase oscillators is given by the following set of coupled differential equations:

$$\dot{\theta}_i = \omega_i + K \sum_{j=1}^{N} a_{ij} \sin(\theta_j - \theta_i), \quad i = 1, \ldots, N,$$

where $K$ is the coupling strength, $N$ is the number of nodes and $a_{ij}$ is the element of adjacency matrix ($a_{ij} = 1$ if nodes $i$ and $j$ are connected and $a_{ij} = 0$ otherwise).

The synchronization of Kuramoto model on SW networks, for random distribution of $\omega_i$ has already been studied by Hong et al. [20]. They showed that small fraction of shortcuts is enough for both phase and frequency synchronizations on SW networks, in spite of absence of any synchronization on regular ones. In our work, we assume that all the intrinsic frequencies are the same ($\omega_i = \omega_0$), therefore moving to a reference frame in which $\omega_0 = 0$, simplifies Eq. (1) to:

$$\dot{\theta}_i = K \sum_{j=1}^{N} a_{ij} \sin(\theta_j - \theta_i), \quad i = 1, \ldots, N. \quad (2)$$

For comparison of the solutions of Kuramoto model in these three types of network, we need to construct them with equal number of nodes and edges. For building a SF network with average connectivity $\langle k \rangle = 2m$, we use the Barabási-Albert algorithm [21]. Starting from $m_0$ initial connected nodes, one attaches a newly entering node to $m \leq m_0$ elder ones with probability proportional to the degree of the present nodes. An ER random network with $N$ nodes and the same average degree per node ($\langle k \rangle = 2m$), is simply produced by connecting randomly chosen pair of nodes with $Nm$ edges [22]. To construct the SW network, we use Watts-Strogatz (WS) algorithm [15]. Starting from a regular network with $N$ nodes and $k = 2m$ edges for each node, we rewire each edge randomly with probability $p$. Choosing $0.005 \lesssim p \lesssim 0.05$, this process converts the initial regular lattice to a complex network with a small mean path length and large clustering coefficient, characteristics of SW networks. Starting from a randomly distributed initial phases $\theta_i(0)$ (which is selected from a box distribution in the interval $[-\pi, \pi]$), The set of coupled differential Eqs. (2) are integrated from $t = 0$ to a given time $t$ with the time step $dt$, using Euler method. This method enable us to compute $\theta_i(t)$ and to determine the synchrony among the oscillators at any time, we define the following complex order parameter:

$$re^{i\psi} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j(t)}, \quad (3)$$

where $0 \leq r(t) \leq 1$ indicates the degree of synchronization in the network and $\psi$ is the phase of the order parameter.

Fig. (1) shows the temporal variations of the $r(t)$ on the three types of network with $N = 1000$ and $\langle k \rangle = 10$.

**FIG. 1**: (Colour on-line) Order parameter ($r$) versus time for SW, SF and ER networks with $N = 1000$ and $\langle k \rangle = 10$.

**FIG. 2**: (Colour on-line) Time dependence of the order parameter ($r$) for a SW network with $N = 1000$ and $\langle k \rangle = 10$. Different curves denote different initial conditions.
steady states of SW networks with helical patterns of regular network and corresponding steady states of SW networks for five different initial conditions. As can be seen from this figure, the steady state is highly dependent on the initial phase distribution.

To obtain these plots, time step is set to $dt = 0.01$ and averaging is done over $100$ realizations of initial phase distributions for a fixed network of each type. The rewiring probability for constructing the SW network out of regular one is chosen to be $p = 0.04$. As can be seen, the oscillators on ER and SF networks immediately reach to a fully synchronized state ($r = 1$), while in the case of SW network they more slowly go toward a partially synchronized state with $r \approx 0.7$. These results show that in contrast to ER and SF networks, the structure of steady states of the Kuramoto model on SW networks are more complex. To get more insight on the nature of steady solutions, we check the dependence of order parameter on the initial phase distributions. Fig. (2) shows temporal variation of $r$ on a SW network for five different initial conditions. As can be seen from this figure, the steady state is highly dependent on the initial phase distribution in such a way the $r(\infty)$ reaches several values between 0 and 1. In what follows, we discuss that the sensibility of dynamics to initial conditions is indeed inherited to SW networks from their regular network parents.

It is easy to show that the stable stationary solutions of Eqs. (2) have to satisfy the following conditions:

$$\sum_{i=1}^{N} \sin \theta_i = \sum_{i=1}^{N} \cos \theta_i = 0,$$

provided that the phase difference between any two adjacent oscillators be less than $\pi/2$ (i.e., $\Delta \theta_{ij} = \theta_i - \theta_j < \pi/2$ if $a_{ij} = 1$). These solutions can be put in two categories: 

1. Full synchronized state with $r = 1$ ($\Delta \theta_{ij} = 0$ for any $i,j$);
2. Phase-locked state with regular arrangement of phases around the phase circle with non-zero phase difference $\Delta \theta$, for which $r = 0$. 

The phase-locked states represent helical wave phase modulations and their number depends on $N$ and $k$. For instance, in the case of $N = 1000$ and $k = 10$, there are 10 of such states with nearest neighbour phase differences $\Delta \theta_{nn} = 2\pi/\lambda^n$, in which $\lambda^n = 20, 25, 40, 50, 100, 125, 200, 250, 500, 1000$, are the wavelengths of the helical states indicated by $\alpha = 1, 2, \cdots 10$, respectively.

The stationary phase configuration of all nodes, corresponding to the initial conditions in Fig. (2), are plotted in Fig. (3), both for the regular and its offspring SW networks. This plots corresponds to the helical patterns with phase differences $\lambda = 1000, 250, 100, 50$, denoted in Fig. (4) by indices b, c, d and e, respectively. This figure shows, rewiring a regular network with phase-locked state, deforms its helical pattern to an inhomogeneous state in the subsequent SW one. Therefore, a SW network possesses various stable stationary states whose number equals the number of helical patterns in its parent regular network.

The local structure of the steady state can be better clarified by the correlation matrix $D$ defined as:

$$D_{ij} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{t_r}^{t_r+\Delta t} \cos(\theta_i(t) - \theta_j(t))dt,$$

in which $t_r$ is the time needed for reaching to stationary state. The matrix element $-1 \leq D_{ij} \leq 1$ is a measure of coherency between each pair of nodes. In the case of full synchrony between $i$ and $j$ ($\theta_i = \theta_j$) the correlation matrix element is $D_{ij} = 1$ and in the case of anti-phase locking ($\theta_i - \theta_j = \pi$), the value of matrix element is $D_{ij} = -1$. Fig. (4) represents the density plot of correlation matrix elements for the four steady states of regular and SW networks corresponding to Figs. (2) and (3). This plots clearly show the inhomogeneous structure of the helical patterns before and after rewiring of the regular network. The correlation matrix represents strip structures in its density plot for helical states in regular network and the width of the strips are proportional to the wavelength of the helices. One can also observe from this plots that converting the regular network to SW, the helical patterns are substantially affected, provided $\lambda$ being large.
FIG. 4: (Colour on-line) Density plot of correlation matrix elements ($D_{ij}$) for 4 helical states of regular network and corresponding stationary states in SW network with $N = 1000$ and $\langle k \rangle = 10$. (b) $\lambda = 1000$, (c) $\lambda = 250$, (d) $\lambda = 100$ and (e) $\lambda = 50$. $\lambda$ is the wavelength of helical states in regular network.

The strip structure of matrix $D$ is almost preserved for small wavelengths, indicating the small wavelength helical patterns, despite of little deformations, are stable against rewiring of network. For large wavelength patterns of regular network, the majority of nodes in the corresponding SW phase configurations are synchronized with each other, however there are some isolated nodes in Anti-phase locking with the rest. These isolated nodes are topological defects and induce spiral phase textures around them, in such a way that the phase of surrounding oscillators varies continuously from 0 to $\pi$, by getting away from these nodes. The number of these defects increases by decreasing the wavelength of corresponding helical pattern. For example, it can be seen from Fig. 4(b) that for $\lambda = 1000$ there is one while for $\lambda = 250$ there are four point defects. Once the structure of the steady states of deterministic Kuramoto model on SW network is known, it would be interesting to investigate the effect of noise on such states.

FIG. 5: (Colour on-line) Order parameter versus noise intensity for SW, SF and random networks. Number of nodes and mean degree for the three networks are $N = 10000$ and $\langle k \rangle = 10$, respectively.

EFFECT OF RANDOM FORCE

A network of oscillators could be plagued by some external random forces. The effect of such forces may be modelled by an uncorrelated white noise ($\eta_i(t)$) applying to all nodes. Adding this noise to Eq. (2), we have:

$$\dot{\theta}_i = K \sum_{j=1}^{N} a_{ij} \sin(\theta_j - \theta_i) + \eta_i(t), \quad i = 1, \ldots, N,$$

where $\langle \eta(t) \rangle = 0$, $\langle \eta_i(t) \eta_j(t') \rangle = 2D \delta(t - t') \delta_{ij}$ with $D$ being the variance or intensity of the noise. In our numerical work, we choose a box distribution in the interval $[-w/2, w/2]$ for $\eta$, so that its variance is equal to $D = w^2/24$. It can be shown that by proper rescaling of the time variable, the effect of parameters $D$ and $K$ can be included in a single parameter $g^2 = D/K$ [14], converting the dynamical equations to:

$$\frac{d\theta_i}{d\tau} = \sum_{j=1}^{N} a_{ij} \sin(\theta_j - \theta_i) + g\xi_i(\tau)$$

where $\tau = Kt$ is the rescaled time variable and $\xi_i(\tau) := \eta_i(t)/g$ is a random variable in the interval $[-1/2, 1/2]$.

The numerical integration of Eq. (7) is carried out by employing Euler method for its deterministic part and Ito’s algorithm [25] for the stochastic part. Fig. 6 represents the variations of stationary order parameters ($r(\infty)$) versus rescaled noise intensity $g$, for the three network types SF, ER and SW. In obtaining these graphs,
we retained the networks fixed and averaged over 100 different realizations of noise.

By inspecting this figure one can extract two essential results: (i) While The order parameter begins from a value less than 1 for SW network its variation is much smaller by increasing noise intensity, relative to ER and SF networks. Hence, at small noise intensities, the partially synchronized state in SW network is more robust against applying the noise than fully synchronized ones in the ER and SF networks. However, at large noise intensities the situation is vice versa. The critical coupling \( g_c \) at which the synchrony disappears among the oscillators is the greatest for SF and the smallest for SW network. Therefore in this region, instead of destroying the synchronization, noise promote the synchrony among oscillators.

The noise-induced synchronization is also called \textit{stochastic synchronization} and its occurrence in SW networks can be explained in terms of defect patterns in the steady states of Kuramoto model. Fig. (6) represents the evolution of correlation matrix density plots versus reduced noise intensity, \( g \), for a specific steady state of SW network with four topological defects. It can be seen in this figure that turning the noise on, the defects resist related to the existence of few nodes with very large number of connections (hubs) in this type of networks [14].

(ii) Increasing the noise intensity, the coherency among the population of phase oscillators destroys monotonically in SF and ER networks. Nevertheless, in SW network, there is an interval of reduced noise intensity i.e \( 5 \lesssim g \lesssim 7 \), within which the synchronization enhances by increasing noise strength. Therefore in this region, instead of destroying the synchronization, noise promote the synchrony among oscillators.

The noise-induced synchronization is also called \textit{stochastic synchronization} and its occurrence in SW networks can be explained in terms of defect patterns in the steady states of Kuramoto model. Fig. (6) represents the evolution of correlation matrix density plots versus reduced noise intensity, \( g \), for a specific steady state of SW network with four topological defects. It can be seen in this figure that turning the noise on, the defects resist
against the noise up to $g \sim 4$ and for $g > 4$ they begin to disappear until $g \sim 6$ where they vanish completely. Disappearance of defects enhances the homogeneity in the system and so the synchrony among the oscillators. This is more apparent in probability distribution of correlation matrix elements ($p(D_{ij})$) shown in Fig. [10]. As can be seen in this figure, $p(D)$ has two peaks at $D = 1, -1$ for $g = 0$. Increasing the noise intensity the two peaks move toward each other and at the onset of stochastic synchronization, $g \sim 6$, they emerge in one peak. At this point the variance of $p(D_{ij})$ reaches to its minimum and again rises by increasing the noise strength.

**CONCLUSION**

In summary, we found that a SW network of similar phase oscillators communicating with each other by Kuramoto coupling shows novel behaviours. Unlike ER and SF networks, this system fails to reach full synchronized state. Moreover, driving it by an uncorrelated with noise, reveals the occurrence of stochastic synchronization, a phenomenon through which a random force induces synchrony among the oscillators. We discussed that the reason for this phenomenon is lied in the stable helical patterns in the regular networks from which the SW ones is built. Rewiring of a regular network of similar phase oscillators with periodic helical pattern, ends to complex inhomogeneous states in the resulting SW network. The existence of such stable inhomogeneous patterns in SW network, appearing some times as topological point defects and also as aperiodic helical patterns, prevents the network from reaching to full synchrony. These patterns persist against applying the noise for small noise intensities. However the external random forces with moderate strengths are able to destroy these patterns in favour of more homogeneous states, hence enhance the synchronization among oscillators. At the end, we hope that our finding that such a simple model representing such a novel behaviours may shed light on the fact that why SW networks are so ubiquitous in natural systems.

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