Algorithm for determining the coefficients of the interpolation polynomial of Newton with separated differences

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Abstract. The article considers the problem of determining the coefficients of the Newton interpolation polynomial with separated differences. To solve this problem, a scheme, a tabular method and an algorithm for determining the coefficients without simplification and disclosure of brackets that arise when constructing polynomials by the Newton method are proposed. Based on this tabular method, approximation and interpolation problems can also be solved.

1. Introduction
Many computational methods are based on the idea of replacing functions with some close or simple structure involved in the formulation of the problem [1].

Interpolation is a broad concept; its meaning is given below.

Suppose, in the interval \([a, b]\) some function is given \(y = f(x)\) or at least the values \(f(x_0), f(x_1), ..., f(x_n)\) are known. We take a class of polynomials \(\{P(x)\}\) convenient for calculation in a given interval. The task of interpolating a given function \(y = f(x)\) in the interval \([a, b]\) is such an approximate replacement by such a function \(P(x)\), while \(P(x)\) at points \(x_0, x_1, ..., x_n\) must take the same values as \(f(x)\), that is \(P(x_i) = f(x_i), i = 0, ..., n\). Moreover, \(x_0, x_1, ..., x_n\) - are called interpolation nodes, \(P(x)\) is an interpolating function [2-4].

If the interpolating functions \(\{P(x)\}\) are power polynomials, then interpolation is called algebraic.

Algebraic interpolation is used in various fields. For example, in integration and differentiation, image recognition, forecasting, etc. [5].

To solve the problem of approximating functions, this article discusses the search for the latest coefficients, a simple and much used, Newton's interpolation formula.

2. Basic concepts and notations
It is known that the coefficients of the Newton interpolation formula are calculated as follows [6-7]:

\[ b_k = \frac{\Delta_k y_0}{k!h^k}. \] (1)
And based on the found coefficients (1), a polynomial is constructed:

\[
F(x) = y_0 + \sum_{i=1}^{N} \prod_{j=0}^{i-1} (x - x_j).
\]  

(2)

To represent function (2) as

\[
y = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0
\]

(3)

need to expand the brackets and simplify the expression.

To find the coefficients of expression (3), a finite difference table method is proposed.

3. Formulation of the problem

To find the coefficients of expression (3), a finite difference table method is proposed.

Let a table of node values and a node function value be given:

**Table 1. Initial data.**

| \(x_0\) | \(x_1\) | \(...\) | \(x_N\) |
| \(y_0\) | \(y_1\) | \(...\) | \(y_N\) |

The coefficients are determined in the following stages.

Stage 1. Define member \(a_n\). At this stage, the final differences of the first order are calculated, and table 1 continues (table 2).

The final differences of the first order are calculated by the following formula:

\[
\Delta^1 y_i = \frac{(y_i - y_0)}{i}, \quad i = 1, N,
\]

here \(y_i\) are the values of the function with the corresponding \(x_i\).

**Table 2. Initial data and finite differences of the first order.**

| \(x_0\) | \(x_1\) | \(x_2\) | \(...\) | \(x_N\) |
| \(y_0\) | \(y_1\) | \(y_2\) | \(...\) | \(y_N\) |
| \(\Delta^1 y_1\) | \(\Delta^1 y_2\) | \(...\) | \(\Delta^1 y_N\) |

If equality \(\Delta^1 y_i = \Delta^1 y_j\) holds for each value \(i = 1, N - 1, j = i + 1, N\), then the set of given numbers expresses the line. At the same time \(a_n = a_{n-1} = ... = a_2 = 0\), \(a_1 = \Delta y_1\), \(a_0 = y_0\), and the process of determining the coefficients stops, otherwise, go to the next stage.

Stage 2. The finite differences of the second order are calculated, and table 2 continues (table 3). Finite second-order differences are calculated by the following formula:

\[
\Delta^2 y_i = \frac{(\Delta^1 y_{i+1} - \Delta^1 y_i)}{i}, \quad i = 1, N - 1.
\]

**Table 3. Initial data and finite differences of the first and second order.**

| \(x_0\) | \(x_1\) | \(x_2\) | \(|x_3|\) | \(...\) | \(x_N\) |
| \(y_0\) | \(y_1\) | \(y_2\) | \(y_3\) | \(...\) | \(y_N\) |
| \(\Delta^1 y_1\) | \(\Delta^1 y_2\) | \(\Delta^1 y_3\) | \(...\) | \(\Delta^1 y_N\) |
| \(\Delta^2 y_1\) | \(\Delta^2 y_2\) | \(\Delta^2 y_3\) | \(...\) | \(\Delta^2 y_{N-1}\) |
If the equality \( \Delta^2 y_i = \Delta^2 y_j \) holds for each value of \( i = 1, N - 1, j = i + 1, N \), then the set of given numbers expresses a parabola. In this case \( a_1 = \Delta y^2, a_0 = y_0 \), table 4 for determining the coefficient \( a_i \) is compiled as follows:

**Table 4. Determination of the coefficient \( a_1 \).**

| \( x_0 \) | \( x_1 \) | \( \ldots \) | \( x_N \) |
|---|---|---|---|
| \( y_0^* = y_0 \) | \( y_1^* = y_1 - a_2 x_1^2 \) | \( \ldots \) | \( y_N^* = y_N - a_2 x_N^2 \) |

If the equality \( \Delta^2 y_i = \Delta^2 y_j \) is not satisfied for each value of \( i = 1, N - 1, j = i + 1, N \), then go to the next stage.

Stage 3. Definition of a cubic parabola. In this case, the finite differences of the third order are calculated, and table 5 continues.

**Table 5. Source data and final differences of the first, second and third order.**

| \( x_0 \) | \( x_1 \) | \( x_2 \) | \( x_3 \) | \( \ldots \) | \( x_N \) |
|---|---|---|---|---|---|
| \( y_0 \) | \( y_1 \) | \( y_2 \) | \( y_3 \) | \( \ldots \) | \( y_N \) |
| \( \Delta^1 y_1 \) | \( \Delta^1 y_2 \) | \( \Delta^1 y_3 \) | \( \ldots \) | \( \Delta^1 y_N \) |
| \( 0 \) | \( \Delta^2 y_1 \) | \( \Delta^2 y_2 \) | \( \ldots \) | \( \Delta^2 y_{N-1} \) |
| \( 0 \) | \( \Delta^3 y_1 \) | \( \Delta^3 y_2 \) | \( \ldots \) | \( \Delta^3 y_{N-2} \) |

If \( \Delta^3 y_i = \Delta^3 y_j \) is performed for each value of \( i = 1, N - 1, j = i + 1, N \), then this set of points expresses a cubic parabola. In this case \( a_2 = \Delta y^3, a_0 = y_0 \), to determine the coefficient \( a_2 \) table 6 is formed.

**Table 6. Determination of the coefficient \( a_2 \).**

| \( x_0 \) | \( x_1 \) | \( \ldots \) | \( x_N \) |
|---|---|---|---|
| \( y_0^* = y_0 \) | \( y_1^* = y_1 - a_3 x_1^3 \) | \( \ldots \) | \( y_N^* = y_N - a_3 x_N^3 \) |

For a table 6, 1-2 stages are performed. If \( \Delta^3 y_i = \Delta^3 y_j \) is not executed for each value of \( i = 1, N - 1, j = i + 1, N \), then it proceeds to the next stage. The remaining stages are performed similarly to stages 1, 2 and 3.

Stage \( k \). Definition of an algebraic polynomial of \( k \)-order. In this case, finite \( k \)-order differences are calculated, and the table continues similarly to the previous stage.

Finite differences of higher orders are calculated by the following formula:

\[
\Delta^k y_i = \frac{\left(\Delta^{k-1} y_{i+1} - \Delta^{k-1} y_i\right)}{i}, \quad i = 1, N - k .
\]

**Table 7. Initial data and finite \( k \)-order differences.**

| \( x_0 \) | \( x_1 \) | \( \ldots \) | \( x_k \) | \( \ldots \) | \( x_N \) |
|---|---|---|---|---|---|
| \( y_0 \) | \( y_1 \) | \( \ldots \) | \( y_k \) | \( \ldots \) | \( y_N \) |
| \( 0 \) | \( \Delta^1 y_1 \) | \( \Delta^1 y_2 \) | \( \ldots \) | \( \Delta^1 y_k \) | \( \ldots \) | \( \Delta^1 y_N \) |
\[
\begin{array}{ccccccc}
0 & 0 & \Delta^2 y_i & \cdots & \Delta^2 y_k & \cdots & \Delta^2 y_{N-1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \Delta^k y_i & \cdots & \Delta^k y_{N-k+1} \\
\end{array}
\]

If \( \Delta^i y_i = \Delta^i y_j \) holds for each value of \( i = 1, N-1, j = i+1, N \), then the given points denote a \( k \)-order polynomial. At the same time \( a_k = \Delta^k y_i, a_0 = y_0 \), table 7 is re-compiled to determine the coefficient \( a_{k-1} \).

**Table 8.** Determination of the coefficient \( a_{k-1} \).

| \( x_0 \) | \( x_1 \) | \( \cdots \) | \( x_N \) |
|-------|-------|-------|-------|
| \( y_0^* = y_0 \) | \( y_1^* = y_1 - a_k x_1^k \) | \( \cdots \) | \( y_N^* = y_N - a_k x_N^k \) |

For the 8-table, the previous stages are repeated. If \( \Delta^i y_i = \Delta^i y_j \) is not executed for each value of \( i = 1, N-1, j = i+1, N \), then it proceeds to the next stage. The following stages are performed as a \( k \)-stage.

4. **Results of computational experiments**

The effectiveness of the proposed algorithm was evaluated on algebraic polynomials of various orders. The experiment took 1,000 different polynomials of different orders generated by the program, which have degrees from 1 to 50. All coefficients of the taken polynomials are fully and correctly determined. At the same time, the program gave a 100% result.

5. **Conclusion**

The article solves the problem of determining the coefficients of the Newton interpolation polynomial with separated differences. To solve this problem, a scheme, a tabular method and an algorithm for determining the coefficients are proposed without simplification and disclosure of brackets.

The proposed method is based on divided differences, existing methods use the difference between neighboring points, and in the present method, the function is constructed by the starting point and some equal initial values can \( y \).

In the Lagrange interpolation method, each term depends on the entire nodes, so when a new node is introduced, it is necessary to rebuild the polynomial. This problem is resolved using the Newton method. The proposed method is based on the method of divided differences, therefore, when constructing an approximate function, the interpolation accuracy is estimated and calculated as in Newton's method.

**References**

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