MULTIVARIATE HASSE–SCHMIDT DERIVATION ON EXTERIOR ALGEBRAS

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ABSTRACT. The purpose of this short note is to consider multi-variate Hasse-Schmidt derivations on exterior algebras and showing as they easily provide remarkable identities holding in the algebra of square matrices, generalizing the classical theorem of Cayley-Hamilton.

1. Introduction

1.1. The purpose of this paper is to propose a multivariate version of the notion of Hasse–Schmidt (HS) derivation on an exterior algebra introduced by Gatto in [2]. To be more precise, if \( z := (z_1, \ldots, z_n) \) is an \( n \)-tuple of indeterminates and \( V \) a \( \mathbb{K} \)-vectorspace, by a multi-variate Hasse–Schmidt derivation of \( \wedge V \) we shall mean a linear map \( D(z) : \wedge V \to \wedge V[[z]] \) such that

\[
D(z)(u \wedge v) = D(z)u \wedge D(z)v
\]

In case \( n = 1 \), i.e. \( D(z) = D(z) := D(z_1) \), one recovers the picture described by Gatto and collaborators e.g. in [2, 3, 4], to whom we refer the reader for additional details. This short note is devoted to extend some elementary properties holding for uni-variate HS derivations on \( \wedge V \) to the multivariate case, notably the property called integration by parts as in [3, Proposition 4.9] or [5, formula (1.3)]. We learn how to associate a HS-derivation to a finite sequence of endomorphisms of \( V \). As an application we obtain the following

MAIN THEOREM. (Cf. Theorem 4.8) For a fixed \( n \geq 1 \), Let \( V = \mathbb{K}^n \), denote by \( e_i \) the canonical basis of \( \mathbb{K}^n \) and let \( A := (A_1, \ldots, A_n) \) be an ordered \( n \)-tuple of \( n \times n \) \( \mathbb{K} \)-valued matrices. Then

\[
\sum_{k=0}^{n} (-1)^k \frac{1}{k!} \sum_{\sigma \in S_n} \tau_{e_{\sigma(1)} + \cdots + e_{\sigma(k)}}(A) \cdot (A_{\sigma(k+1)} \cdot \cdots \cdot A_{\sigma(n)}) = 0,
\]

where \( S_n \) denotes the symmetric group on \( n \) letters.

See Section 4 for the definition of \( \tau(A) \) of an ordered \( k \)-tuple of endomorphisms, suitably replacing the usual notion of traces of one given matrix. A few comments are in order. First of all, if \( A_1 = \cdots = A_n = A \) the theorem

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gives back the Cayley–Hamilton theorem, provided we are in characteristic zero, according which each matrix is a root of its own characteristic polynomial. Secondly, the theorem can be related to some trace identities relations as in [1] and it would be interesting to see how much of the formalism of the present paper can be generalized to deal with the more general situations studied there. The main theorem also suggests a number of amusing corollaries. For instance it shows that if \( A, B \) are \( 2 \times 2 \) matrices, the bilinear map
\[
A_1 \ star A_2 = A_1 A_2 - d_1(A)B + d_{1,2}(A)1_{2 \times 2}
\]
is skew symmetric, i.e.:
\[
(1.1) \quad A_1 \ star A_2 + A_2 \ star A_1 = 0.
\]
However an easy check shows that \( (\mathbb{K}^{2 \times 2}, *) \) is not a Lie algebra. Putting \( A = A_1 = A_2 \) one obtains
\[
0 = A \ star A = 2 \cdot (A^2 - \text{tr}(A)A + \text{det}(A)1_{2 \times 2}) = 0
\]
that explains in which sense formula (1.1) generalizes Cayley-Hamilton theorem for \( 2 \times 2 \) matrices, as announced. See section 4 for the definition of \( d(A) \). Similarly, the tri-linear map
\[
\mathbb{K}^{3 \times 3} \times \mathbb{K}^{3 \times 3} \times \mathbb{K}^{3 \times 3} \rightarrow \mathbb{K}^{3 \times 3}
\]
defined by
\[
A_1 \ star A_2 \ star A_3 = A_1 A_2 A_3 - d_1(A)A_2 A_3 - d_2(A)A_3 A_1 - d_3(A)A_1 A_2
\]
\[
+ d_{1,2}(A)A_3 + d_{2,3}(A)A_1 + d_{1,3}(A)A_2
\]
\[
- d_{1,2,3}(A)1_{3 \times 3}
\]
is skew-symmetric, i.e.
\[
s(A_1) \ star s(A_2) \ star s(A_3) = (-1)^{|s|} A_1 \ star A_2 \ star A_3
\]
for all permutation \( s : \{A_1, A_2, A_3\} \rightarrow \{A_1, A_2, A_3\} \), where by \( |s| \) we have denoted the sign of the permutation. Again \( A \ star A \ star A = 0 \) gives back the content of the Cayley–Hamilton theorem for \( 3 \times 3 \) matrices.

1.2. The paper is organized as follows. In section 2, we recall some preliminaries on HS–derivations on exterior algebra, as in e.g. [2, 3], to keep this note as self contained as possible. There we also recall the main theorem of [5], regarding a Cayley–Hamilton vanishing holding on the exteriors algebra of a vector space. Section 3 sets a minimum of foundational material regarding multi-variate HS–derivations, putting the usual emphasis on the integration by parts formula. Section 4, puts together the main definitions, lemmas and propositions to reach the claim and the proof of the main Theorem above.
2. Preliminaries

We quickly recall here some basic definitions and needed statements. In Section 3, the notion of HS–derivation is generalized on the algebra of formal power series in more than one variable.

Let $V$ be a vector space of dimension $n$ over a field $K$, and let $\{e_1, \ldots, e_n\}$ be one $K$-basis. We denote by $\bigwedge V$, the exterior algebra of $V$, which is a direct sum of exterior powers: $\bigoplus_{j \geq 0} \bigwedge^j V$. Recall that $\bigwedge^0 V = K$, and that if $j \geq 0$, then $\bigwedge^j V$ is a vector space with a basis $e_{i_1} \wedge \cdots \wedge e_{i_j}$ with $i_1 < i_2 < \cdots < i_j$. If $\gamma \in S_j$, the symmetric group of $1, 2, \cdots, j$, then $e_{i_1} \wedge \cdots \wedge e_{i_j} = \operatorname{sgn}(\gamma) e_{\gamma(i_1)} \wedge \cdots \wedge e_{\gamma(i_j)}$. Let $\bigwedge V[[z]]$ be the $\bigwedge V$-valued formal power series with the natural algebra structure.

**Definition 2.1.** A Hasse–Schmidt derivation, HS–derivation for short, on the exterior algebra $\bigwedge V$ is an algebra homomorphism $D(z) : \bigwedge V \rightarrow \bigwedge V[[z]]$.

i.e such that for $u, v \in \bigwedge V$

(2.1) $D(z)(u \wedge v) = D(z)u \wedge D(z)v$

We shall write $D(z)$ in the form $\sum_{i \geq 0} D_i z^i$, where $D_i \in \operatorname{End}_K(\bigwedge V)$

**Proposition 2.2.** Let $\overline{D}(z) \in \operatorname{End}_R(\bigwedge V)$ such that $\overline{D}(z) : D(z) = 1$. Then it is a HS–derivation, said to be the inverse of $D(z)$.

If $D_0$ is invertible in $\operatorname{End}(\bigwedge V)$, then $D(z)$ is invertible in $\bigwedge V[[z]]$. We shall write $D(z) = \sum_{i \geq 0} D_i z^i$ and $\overline{D}(z) = \sum_{i \geq 0} (-1)^i \overline{D}_i z^i$ with $D_i, \overline{D}_i \in \operatorname{End}_K(\bigwedge V)$.

Let $HS(\bigwedge V)$ be the set of all HS–derivations on $\bigwedge V$. It is easily seen that the invertible HS–derivations on $\bigwedge V$ form a group under composition.

Formula (2.1) is equivalent to the fact that each $D_i$ satisfies the Leibniz rule, i.e for all $u, v \in \bigwedge V$ [2, Proposition 4.1.5]

$$D_i(u \wedge v) = \sum_{k=0}^{i} D_k u \wedge D_{i-k} v.$$  

**Proposition 2.3.** Let $f \in \operatorname{End}_K(V)$ and let

$$f(z) = \sum_{i \geq 0} (-1)^i f^i z^i : V \rightarrow \bigwedge V[[z]].$$

Then there is a unique HS-derivation $D_f(z) \in HS(\bigwedge V)$ such that $D_f(z)|_V = f(z)$.

**Proof.** See [3, Page 68].
2.4. Within the same situation of Proposition 2.3, denote by \( D^f(z) \) the inverse of \( Df(z) \). For each \( i \geq 0 \), let \( e_i \) be the eigenvalue of \( D^f_i \) as an element of \( \text{End}_K(\wedge^n V) \).

**Theorem 2.5.** [5, p. 4] If \( f \in \text{End}_K(V) \) and \( D(z) \) is the corresponding HS-derivation on \( \wedge V \), then the equality
\[
D^f_n - e_1 D^f_{n-1} + \cdots + (-1)^n e_n 1 = 0
\]
holds in the whole exterior algebra.

Equality (2.2) above implies the classical theorem by Cayley and Hamilton, by restricting to \( V \), i.e.
\[
0 = (D^f_n - e_1 D^f_{n-1} + \cdots + (-1)^n e_n 1)|_V = f^n - e_1 f^{n-1} + \cdots + (-1)^n e_n f = 0
\]
Indeed one can easily check that the eigenvalues of \( D^f|\wedge^n V \) are precisely the coefficients of the characteristic polynomial.

3. **Multivariate HS derivations on \( \wedge V \)**

3.1. Same setting as the previous section. Let \( V \) be a vector space of dimension \( n \) and \( \wedge V[z] := \wedge V[z_1, \cdots, z_n] \) be the algebra of formal power series in \( n \) formal variables \( z := (z_1, \cdots, z_n) \).

**Definition 3.2.** A \( n \)-multivariate HS-derivation is an algebra homomorphism
\[
D(z) : \wedge V \rightarrow \wedge V[z_1, \cdots, z_n]
\]
in the sense that for every \( u, v \in \wedge V \), \( D(z)(u \wedge v) = D(z)u \wedge D(z)v \),

Denote by \( HS_n(\wedge V) \), the set of all \( n \)-multivariate HS-derivations on \( \wedge V \). Each element \( \mathcal{D}(z) \in HS_n(\wedge V) \) can be written as
\[
\mathcal{D}(z) = \sum_{i_1, \cdots, i_n \geq 0} D_{i_1, \cdots, i_n} z_1^{i_1} z_2^{i_2} \cdots z_n^{i_n}, \quad D_{i_1, \cdots, i_n} \in \text{End}_K(\wedge V).
\]

3.3. The product
\[
\mathcal{D}(z)E(z) = \sum \mathcal{D}(z)E_{i_1, \cdots, i_n} z_1^{i_1} \cdots z_n^{i_n}
\]
where
\[
\mathcal{D}(z)E_{i_1, \cdots, i_n} = \sum_{j_1, \cdots, j_n \geq 0} (D_{j_1, \cdots, j_n} \circ E_{i_1, \cdots, i_n}) z_1^{j_1} \cdots z_n^{j_n}
\]
makes \( HS_n(\wedge V) \) into a multiplicative monoid. The identity is the identity map of \( \wedge V \) thought of as a constant HS-derivation. Indeed \( \mathcal{D}(z)E(z) \) is an element of \( HS_n(\wedge V) \) if \( \mathcal{D}(z) \) and \( E(z) \) are in \( HS_n(\wedge V) \) then for all \( u, v \in \wedge V \), the following holds.
Proof. \((\text{element of End}_K(\wedge V))^n,\) \(e = (e_1, \ldots, e_n)\) is a usual derivation of the exterior algebra.

Lemma 3.4. Let \(u, v \in \wedge V\) and \(D(\overline{z}) = \sum_{i_1, \ldots, i_n \geq 0} D_{i_1, \ldots, i_n} z_1^{i_1} \cdots z_n^{i_n}\) be an element of \(\text{End}_K(\wedge V)[\overline{z}]\). Then the following formulas are equivalent.

1. \(D(\overline{z})(u \wedge v) = D(\overline{z}) u \wedge D(\overline{z}) v\).

2. \(D_{i_1, \ldots, i_n}(u \wedge v) = \left( \sum_{j_1 + l_1 = i_1} D_{j_1, \ldots, j_n} u \wedge D_{l_1, \ldots, l_n} v \right) z_1^{i_1} \cdots z_n^{i_n}\).

Proof. (1) \(\Rightarrow\) (2) is clear since (1) means

\[ D(\overline{z})(u \wedge v) = \sum_{j_1, \ldots, j_n \geq 0} D_{j_1, \ldots, j_n}(u) z_1^{j_1} \cdots z_n^{j_n} \wedge \sum_{l_1, \ldots, l_n \geq 0} D_{l_1, \ldots, l_n}(v) z_1^{l_1} \cdots z_n^{l_n} \]

and so, the coefficient of \(z_1^{i_1} \cdots z_n^{i_n}\) in the first member of (1) is precisely \(\sum_{j_1 + l_1 = i_1} D_{j_1, \ldots, j_n} u \wedge D_{l_1, \ldots, l_n} v\). Conversely, if (2) holds one has

\[ D(\overline{z})(u \wedge v) = \sum_{i_1, \ldots, i_n \geq 0} D_{i_1, \ldots, i_n}(u \wedge v) z_1^{i_1} \cdots z_n^{i_n} \]

\[ = \sum_{i_1, \ldots, i_n \geq 0} \left( \sum_{j_1 + l_1 = i_1} D_{j_1, \ldots, j_n} u \wedge D_{l_1, \ldots, l_n} v \right) z_1^{i_1} \cdots z_n^{i_n} \]

\[ = \sum_{i_1, \ldots, i_n \geq 0} \left( \sum_{j_1 + l_1 = i_1} D_{j_1, \ldots, j_n} u z_1^{j_1} \cdots z_n^{j_n} \wedge D_{l_1, \ldots, l_n} v z_1^{l_1} \cdots z_n^{l_n} \right) \]

\[ = D(\overline{z}) u \wedge D(\overline{z}) v \]

Example 3.5. Let \(e_k\) be the \(k\)-th element of the canonical basis of \(Z^n\). In particular, \((e_1, \ldots, e_n)\) is the canonical basis of \(K^n\). It is easy to see that for every \(0 \leq k \leq n\), \(D_{e_k} := D_{(0, \ldots, 1, \ldots, 0)}\) (1 as \(k\)-th entry) is the coefficient of \(z_k\). Each \(D_{e_k}\) is a usual derivation of the exterior algebra.

3.6. As in the case \(n = 1\), one can show that if

\[ f(\overline{z}) = \sum_{i_1, \ldots, i_n} f_{i_1, \ldots, i_n} z_1^{i_1} \cdots z_n^{i_n} : V \rightarrow V[\overline{z}] \]
is a $K$–vector space homomorphism then there is a unique $D^f(z) \in HS_n(\wedge V)$ such that $D^f(z)|_V = f(z)$.

If $D^f_{0,\ldots,0}$ is invertible, then $D^f(z)$ is invertible in $\text{End}(\wedge V)[z]$ and we shall write

$$\overline{D}^f(z) = \sum_{k \geq 0} \sum_{i_1+\cdots+i_n = k} (-1)^k D^f_{i_1,\ldots,i_n} z_1^{i_1} \cdots z_n^{i_n}.$$ for its inverse.

Clearly $\overline{D}^f_{0,\ldots,0} = D^f_{0,\ldots,0}$, and for every $k$, $\overline{D}^f_{e_k} = D^f_{e_k}$.

**Proposition 3.7.** The inverse $\overline{D}(z)$ of the HS–derivation $D(z)$ is a HS–derivation.

**Proof.** Using equation (3.1), for every $u, v \in \wedge V$, we have

$$\overline{D}(z)(u \wedge v) = \overline{D}(z)(\overline{D}(z)D(z)u \wedge \overline{D}(z)D(z)v)$$

$$= \overline{D}(z)D(z)(\overline{D}(z)u \wedge \overline{D}(z)v)$$

$$= \overline{D}(z)u \wedge \overline{D}(z)v.$$ Therefore, $\overline{D}(z) \in HS_n(\wedge V)$.

**Proposition 3.8** (Integration by parts). The following formulas hold for every $u, v \in \wedge V$:

$$\overline{D}(z)u \wedge v = D(z)(u \wedge \overline{D}(z)v)$$

(3.2)

$$\overline{D}(z)u \wedge v = \overline{D}(z)(u \wedge D(z)v)$$

(3.3)

**Proof.** It is straightforward and works exactly the same as [3, Proposition 4.1.9].

**3.9.** From the fact that each $v \in \wedge^k V$ is a finite linear combination of elements of the form $v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k}$, where $v_{i_j} \in V$, and

$$D(z)(v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k}) = D(z)v_{i_1} \wedge \cdots \wedge D(z)v_{i_k},$$

it follows that $D(z)$ is uniquely determined by its restriction

$$D(z)|_V : V \rightarrow \wedge V[z].$$

to the first degree of the exterior algebra. In this note we shall stick to the case in which $D(z)$ is homogeneous of degree 0 with respect to the exterior algebra graduation, i.e. each coefficient of $D(z)$ maps $\wedge^k V$ to itself. In this case $D(z)|_V \in \text{End}_K(V)[z]$. In the sequel we shall construct a relevant homogeneous multivariate HS-derivation of degree 0.
4. Main result

**Proposition 4.1.** Let \( (A_1, \ldots, A_n) \in \text{End}_k(V)^n \) be endomorphisms of \( V \), chosen once and for all. We shall simply denote by \( \overline{D}(z) \) the unique multivariate HS–derivation such that

\[
\overline{D}(z)|_V = 1 - (A_1z_1 + \cdots + A_nz_n)
\]

Then for all \( u \in \bigwedge^k V \), \( \overline{D}(z)u \in V[z] \), is a polynomial in \( z_1, \ldots, z_n \) of degree at most \( k \).

**Proof.** We use induction. For \( k = 1 \), we have

\[
\overline{D}(z)u = u - (A_1u)z_1 - \cdots - (A_nu)z_n
\]

which is a polynomial of degree one in the indeterminates \( z_1, \ldots, z_n \). Suppose the property holds true for all \( 1 \leq l < k \) and \( u \in \bigwedge^k V \). Without loss of generality, one may assume \( u \) homogeneous and of the form \( u_1 \wedge u_2 \), where \( u_1 \in \bigwedge^1 V = V \) and \( u_2 \in \bigwedge^{k-1} V \). Thus

\[
\overline{D}(z)u = \overline{D}(z)(u_1 \wedge u_2) = \overline{D}(z)u_1 \wedge \overline{D}(z)u_2.
\]

Since \( \overline{D}(z)u_1 \) is a polynomial of degree one, and \( \overline{D}(z)u_2 \) is a polynomial of degree at most \( k - 1 \), by the inductive hypothesis, it follows that \( \overline{D}(z)u \) is polynomial of degree at most \( k \). \( \square \)

**Corollary 4.2.** For all \( i = i_1 + \cdots + i_n > j \), \( \overline{D}_{i_1, \ldots, i_n}|_{\bigwedge^j V} = 0 \). In other words

\[
i_1 + \cdots + i_n > j \quad \Rightarrow \quad \bigwedge^j V \subset \ker(\overline{D}_{i_1, \ldots, i_n}).
\]

**Proof.** Let \( u \in \bigwedge^j V \). Then we know \( \overline{D}_{i_1, \ldots, i_n}u \) is the coefficient of \( z_1^{i_1} \cdots z_n^{i_n} \) in \( \overline{D}(z)u \). Now if \( i = i_1 + \cdots + i_n > j \), by Proposition (4.1) the coefficient of \( z_1^{i_1} \cdots z_n^{i_n} \) in \( \overline{D}(z)u \) is zero. \( \square \)

**Definition 4.3.** With the assumption of the Proposition 4.1, where \( A_i \)'s are distinct, let \( 1 \leq k \leq n \) and consider the integers \( 1 \leq i_1 < \cdots < i_k \leq n \). Recall the notation \( \underline{A} = (A_1, \ldots, A_n) \). We define the trace, which we denote it by \( \tau_{i_1, \ldots, i_k}(\underline{A}) \), to be the scalar satisfying the following equality

\[
\tau_{e_1 + \cdots + e_k}(\underline{A})(u_1 \wedge \cdots \wedge u_n) = \frac{1}{(n - k)!} \sum_{\sigma \in S_n} (-1)^{\sigma(i)} A_{i_1}u_{\sigma(1)} \wedge A_{i_2}u_{\sigma(2)} \wedge \cdots \wedge A_{i_k}u_{\sigma(k)} \wedge \cdots \wedge u_{\sigma(n)}.
\]

where \( u_1 \wedge \cdots \wedge u_n \) is any arbitrary basis of \( \bigwedge^n V \).

**Remark 4.4.** Recall that given \( A_1, \ldots, A_n \) endomorphisms of \( V \), the \( A_i \)'s define linear maps \( A_{i_1} \wedge \cdots \wedge A_{i_k} : \bigwedge^k V \to \bigwedge^k V \) imposing the equality

\[
A_{i_1} \wedge \cdots \wedge A_{i_k}(v_1 \wedge v_2 \wedge \cdots \wedge v_k) = A_{i_1}v_1 \wedge \cdots \wedge A_{i_k}v_k.
\]
for all choices $1 < i_1 \leq \cdots \leq i_k \leq n$ with $1 \leq k \leq n$. It is not hard to see that the usual notion of trace map $\text{tr} : \text{End}_K(\wedge^k V) \to K$ defined in general for endomorphisms of an arbitrary vector space coincides with that given by equality (4.2).

In the case that we have one matrix, say $A = A_1$, then clearly $\tau_{e_1}(A) = \text{tr}(A)$. Moreover, from a practical point of view, as an immediate consequence of definition 4.3, it is easily seen that it can be computed as follows. Identify each $A_j$, $1 \leq j \leq n$, with its matrix with respect to the basis $(e_i)$ of $V$, in such a way that $A_i e_j$ can be identified with the $j$-th column of the matrix $A_i$ in the given basis. It easily follows from definition 4.3 that

$$\tau_{e_1 + \cdots + e_k}(A) = \frac{1}{(n-k)!} \sum_{\sigma \in S_n} (-1)^{\sigma} \det(A_{\sigma(1)}e_1, \cdots, A_{\sigma(k)}e_k, \cdots, e_{\sigma(n)}).$$

i.e. the sums of the determinants of the identity matrix after replacing the columns $e_1, \ldots, e_k$ with the $j_n$-th column of the matrix $A_{ih}$ for $0 \leq h \leq k$.

**Example 4.5.** Let $A_1, A_2, A_3 : \mathbb{K}^3 \to \mathbb{K}^3$. Then

$$\tau_{e_1 + e_2}(A) = \det(A_1 e_1, A_2 e_2, e_3) + \det(e_1, A_1 e_2, A_2 e_3) + \det(A_1 e_1, e_2, A_2 e_3)$$

$$\det(A_2 e_1, A_1 e_2, e_3) + \det(e_1, A_2 e_2, A_1 e_3) + \det(A_2 e_1, e_2, A_1 e_3)$$

**4.6.** Let $(e_1, \ldots, e_n)$ be the canonical basis of $\mathbb{K}^n$. Recall that we have used the notational convention according which $\overline{D}_{e_1 + \cdots + e_k}$ is the coefficient of the monomial $z_{i_1} \cdots z_{i_k}$ in the power series expansion of $\overline{D}(z)$. Recall that $\wedge^n V$ is a common eigenspace of all endomorphism $\overline{D}_{e_1 + \cdots + e_k}$.

**Lemma 4.7.** The following equality holds for all $n \geq m \geq k$:

$$\overline{D}_{e_1 + \cdots + e_k} u_1 \wedge \cdots \wedge u_m = \tau_{e_1 + \cdots + e_k}(A)u_1 \wedge \cdots \wedge u_m$$

**Proof.** First of all we notice that for $m = k$, $\overline{D}_{e_1 + \cdots + e_k} u_1 \wedge \cdots \wedge u_k$ is equal precisely to the second member of (4.3), being the coefficient of $z_{i_1} \cdots z_{i_k}$ of $\overline{D}(z)$ evaluated at $u_1 \wedge \cdots \wedge u_k$. In case $m = k + 1$ one has:

$$\overline{D}(z)(u_1 \wedge \cdots \wedge u_k \wedge u_{k+1}) = \overline{D}(z)(u_1 \wedge \cdots \wedge u_k) \wedge \overline{D}(z) u_{k+1}$$

Now the coefficient of $z_{i_1} \cdots z_{i_k}$ of the second member is precisely given by

$$\tau_{e_{\sigma(1)} + \cdots + e_{\sigma(k)}}(A) u_1 \wedge \cdots \wedge u_k \wedge u_{k+1}$$

$$+ \sum_{j=1}^k (-1)^{j-1} v_j \wedge \sum_{\sigma \in S_{k+1}} \tau_{e_{\sigma(1)} + \cdots + e_{\sigma(j)} + \cdots + e_{\sigma(k+1)}}(A) u_1 \wedge \cdots \wedge \hat{u}_j \wedge \cdots \wedge u_{k+1}$$

$$= \sum_{\sigma \in S_k} \tau_{e_{\sigma(1)} + \cdots + e_{\sigma(k)}}(A) u_1 \wedge u_2 \wedge \cdots \wedge u_{k+1}.$$

The general case follows by a standard inductive argument not worth to be repeated here. \qed

We are now in position to state and prove the main result of this note.
Theorem 4.8. Let $V$ and $A_1, \ldots, A_n \in \text{End}_\mathbb{K}(V)$ as above. Then

$$
\sum_{k=0}^{n} (-1)^k \frac{1}{k!} \sum_{\sigma \in S_n} \tau_{e_{\sigma(1)} + \cdots + e_{\sigma(k)}}(A) \cdot (A_{\sigma(k+1)} \cdot \cdots \cdot A_{\sigma(n)}) = 0
$$

where by convention we set to $\sum_{\sigma \in S_n} A_{\sigma(1)} \cdots A_{\sigma(n)}$ and $\tau_{e_{\sigma(1)} + \cdots + e_{\sigma(n)}}(A)$ the summand corresponding to $k = 0$ and $k = n$ respectively.

Proof. Let

$$
\mathcal{D}(z_1, \ldots, z_n): \bigwedge V \to \bigwedge V[[z_1, \ldots, z_n]]
$$

be the unique multivariate HS–derivation such that

$$
\mathcal{D}(z_1, \ldots, z_n)|_V = 1 - A_1 z_1 - A_2 z_2 - \cdots - A_n z_n.
$$

The restriction to $V$ of its inverse is

$$
\mathcal{D}(z_1, z_2, \ldots, z_n)|_V = \frac{1}{1 - A_1 z_1 - A_2 z_2 - \cdots - A_n z_n} = 1 + A_1 z_1 + \cdots + A_n z_n + (A_1 z_1 + \cdots + A_n z_n)^2 + \cdots
$$

(4.5)

Let $u \in V$ be an arbitrary non null vector. Integration by parts gives

$$
\mathcal{D}(z_1, \ldots, z_n)(\mathcal{D}(z_1, \ldots, z_n)u \wedge v) = u \wedge \mathcal{D}(z_1, \ldots, z_n)v
$$

Corollary 4.2 implies that then the coefficients of $z_1 z_2 \cdots z_n$ of the second member must be zero. Now we compare the coefficients of the monomial $z_1 z_2 \cdots z_n$ occurring on both sides of (4.6) which are all of the form

$$
(-1)^k \mathcal{D}e_{i_1} + \cdots + e_{i_k} (D_{e_{\sigma(k+1)} + \cdots + e_{\sigma(n)}} u \wedge v)
$$

where $(i_1, \ldots, i_k, i_{k+1}, \ldots, i_n)$ is any permutation of $(1, \ldots, n)$. Thus

$$
\frac{1}{k!} \sum_{k=1}^{n} (-1)^k \mathcal{D}e_{\sigma(1)} + \cdots + e_{\sigma(k)} (D_{e_{\sigma(k+1)} + \cdots + e_{\sigma(n)}} u \wedge v) = 0
$$

(4.7)

The coefficient $\frac{1}{k!}$ appears to remove multiplicities in (4.7).

Since $D_{e_{\sigma(k+1)} + \cdots + e_{\sigma(n)}} u \wedge v \in \bigwedge^n V$, it is an eigenvector of $\mathcal{D}e_{\sigma(1)} + \cdots + e_{\sigma(k)}$ with respect to the eigenvalue $\tau_{e_{\sigma(1)} + \cdots + e_{\sigma(k)}}(A)$, as prescribed by Lemma 4.7. On the other hand $D_{e_{\sigma(k+1)} + \cdots + e_{\sigma(n)}}$ is the coefficient of $z_{\sigma(k+1)} \cdots z_{\sigma(n)}$ in $(1 + A_1 z_1 + \cdots + A_n z_n)^{n-k}$, which is precisely (keeping into account that the product of square matrices is not commutative)

$$
\sum_{\gamma \in S_{n-k}} A_{\gamma(\sigma(k+1))} \cdots A_{\gamma(\sigma(n))}
$$
where we are thinking $S_{n-k}$ precisely as the group of bijection of the set \{k + 1, \ldots, n\}. Therefore (4.7) can be rewritten as

\[
(4.8) \quad 0 = \sum_{k=0}^{n} (-1)^{\frac{k}{k!}} \sum_{\sigma \in S_n} \tau_{\sigma(1)} + \cdots + \tau_{\sigma(k)}(A) \cdot (A_{\sigma(k+1)} \cdots A_{\sigma(n)}) u \wedge v
\]

\[
= \left[ \sum_{k=0}^{n} (-1)^{\frac{k}{k!}} \sum_{\sigma \in S_n} \tau_{\sigma(1)} + \cdots + \tau_{\sigma(k)}(A) \cdot A_{\sigma(k+1)} \cdots A_{\sigma(n)} u \right] \wedge v
\]

using the fact that $\tau_{\sigma(1)} + \cdots + \tau_{\sigma(k)}(A) \in \mathbb{K}$ and the multilinearity of the wedge product. Since (4.8) holds for each choice of $v \in \wedge^{n-1} V$, formula (4.4) follows. \qed

**Notation.** Consider linear operators $A_1, \ldots, A_n \in \text{End}_\mathbb{K}(V)$, and fix the basis $e_1 \wedge \cdots \wedge e_n$ for $\wedge^n V$. For each $1 \leq k \leq n$, define $d_{i_1, \ldots, i_k}(A)$ corresponds to $A_{i_1, \ldots, A_{i_k}}$ to be a scalar which satisfies the following equality.

\[
(4.9) \quad d_{i_1, \ldots, i_k}(A)(e_1 \wedge \cdots \wedge e_n) = \sum_{1 \leq j_1 < \cdots < j_k \leq n} e_1 \wedge \cdots \wedge A_{i_1} e_{j_1} \wedge \cdots \wedge A_{i_k} e_{j_k} \wedge \cdots \wedge e_n.
\]

This definition is equivalent to the following equation.

\[
d_{i_1, \ldots, i_k}(A) = \sum_{1 \leq j_1 < \cdots < j_k \leq n} \det(e_1, \ldots, A_{i_1} e_{j_1}, \ldots, A_{i_k} e_{j_k}, \ldots, e_n).
\]

Notice that $d(A)$ is not independent of the choice of basis.

5. **Applications and Examples**

**Example 5.1.** Let $V = \mathbb{K}^2$ be the $\mathbb{K}$-vector space of the columns with 2 components. If $A_1, A_2 : V \to V$ are two linear transformations, consider the unique HIS-derivation on $\wedge \mathbb{K}^2$ and its inverse where

\[
\overline{D}(z_1, z_2)|_V = 1 - A_1 z_1 - A_2 z_2.
\]

Theorem 4.8 and its proof say that

\[
\overline{D}_{1,0} = \tau_{e_1}(A), \quad \overline{D}_{0,1} = \tau_{e_2}(A), \quad \overline{D}_{1,1} = \tau_{e_1 + e_2}(A)
\]

It means if we consider the coefficients of $z_1 z_2$ in (3.3), then we get

\[
(5.1) \quad A_1 A_2 + A_2 A_1 - \tau_{e_1}(A) A_2 - \tau_{e_2}(A) A_1 + \tau_{e_1 + e_2}(A) 1_{2 \times 2} = 0.
\]

In particular, as announced in the introduction, the bilinear form

\[
A_1 \star A_2 = A_1 A_2 - \tau_{e_1}(A) A_2 - \tau_{e_2}(A) A_1 + \tau_{e_1 + e_2}(A) 1_{2 \times 2}
\]

is anti-symmetric. Notice, for sake of being explicit, that if

\[
A_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}
\]
Example 5.2. For $A$, by (5.2)
\[
\tau_{e_1 + e_2}(A) = \det(A_1 e_1, A_2 e_2) + \det(A_1 e_2, A_2 e_1) = \det \begin{bmatrix} a & \beta \\ b & \delta \end{bmatrix} + \det \begin{bmatrix} \alpha & b \\ \gamma & d \end{bmatrix}.
\]
If $A = A_1 = A_2$, then
\[
\det(A e_1, A e_2) = \det(A).
\]
Also,
\[
d_1(A) = \det(A e_1 \wedge e_2) + \det(e_1 \wedge A e_2) = \text{tr}(A).
\]
Therefore, in this case, (5.1) is the usual Cayley-Hamilton theorem, say
\[
A^2 - \text{tr}(A) A + \det(A) I_{2 \times 2} = 0.
\]

Example 5.3. Let $(e_1, e_2, e_3)$ be the canonical basis of $V = \mathbb{K}^3$. Let $A_1, A_2 : V \to V$ be $\mathbb{K}$-endomorphisms. Consider the HS-derivation
\[
\overline{D}(z_1, z_2) : \wedge V \to \wedge V[z_1, z_2]
\]
and its inverse, where
\[
\overline{D}(z_1, z_2) = 1 - A_1 z_1 - A_2 z_2.
\]

Let us look at the coefficients of third degree of (3.8) to an element of $\wedge^3 \mathbb{K}^3$ of the form $u \wedge v$ with $u \in \wedge^1 V = V$ and $v \in \wedge^2 V$. In particular we shall look to the coefficients of $z_1^2 z_2$, in (3.3). Using theorem 4.8 we get
\[
\overline{D}_{1,0} = A_1, \quad \overline{D}_{0,1} = A_2, \quad \overline{D}_{1,1} = \tau_{e_1 + e_2}(A),
\]
where by $\tau_{e_1 + e_2}(A)$ we mean computing formula (4.4) with two equal matrix $A_1$. Which means
\[
\tau_{e_1 + e_2}(A)(u_1 \wedge u_2 \wedge u_3) = \frac{1}{2!} \sum_{\sigma \in S_3} (-1)^{|\sigma|} A_1 u_{\sigma(1)} \wedge A_1 u_{\sigma(2)} \wedge u_{\sigma(3)}.
\]
Since we have two equal matrices here, we face with multiplicity 2!. So we should multiple another $\frac{1}{2!}$ to the formula to remove it. The coefficient $\tau_{e_1 + e_1 + e_2}(A)$ also is computed similarly, using matrices $A_1, A_2$ with order...
\[ A_1 u_{\sigma(1)} \wedge A_1 u_{\sigma(2)} \wedge A_2 u_{\sigma(3)}. \] Therefore, from the integration by parts we conclude that

\[
(A_1^2 A_2 + A_1 A_2 A_1 + A_2 A_1^2) - \tau_{e_1}(A_1 A_2 + A_2 A_1) - \tau_{e_2}(A_1^2 A_2 + A_2 A_1) = 0
\]

(5.6)

Example 5.4. In particular, if in the example 5.2 \( A_1 = A_3 \) then:

\[
A_1 \ast A_1 \ast A_2 + A_2 \ast A_1 \ast A_1 + A_1 \ast A_2 \ast A_1 = 0
\]

(5.7)

which is precisely the content of (5.6). Once again, putting \( A = A_1 = A_2 \) (i.e. \( A = A_1 = A_2 = A_3 \) in (5.3)) in (5.7) one obtains the Cayley–Hamilton theorem for \( 3 \times 3 \) matrices (by clearing a factor 3):

\[
A \ast A \ast A = A^3 - \text{tr}(A) A^2 + \tau_{e_1 + e_2}(A) - \det(A) 1_{3 \times 3} = 0
\]

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