U-cross Gram matrices and their invertibility

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Abstract. The Gram matrix is defined for Bessel sequences by combining synthesis with subsequent analysis operators. If different sequences are used and an operator $U$ is inserted we reach so called $U$-cross Gram matrices. This can be seen as reinterpretation of the matrix representation of operators using frames. In this paper we investigate some necessary or sufficient conditions for Schatten $p$-class properties and the invertibility of $U$-cross Gram matrices. In particular, we show that under mild conditions the pseudo-inverse of a $U$-cross Gram matrix can always be represented as a $U$-cross Gram matrix with dual frames of the given ones. We link some properties of $U$-cross Gram matrices to approximate duals. Finally, we state several stability results. More precisely, it is shown that the invertibility of $U$-cross Gram matrices is preserved under small perturbations.

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1. Introduction and motivation

Some operator equations, e.g. in acoustics [27] and vibration simulation [10] cannot be treated analytically, but have to be solved numerically. Depending on the problem this can be done using a boundary element method [30] or finite element method [15] approach. Thereby operator equations $Of = b$, are transferred to matrix levels to be able to be treated numerically [30]. A standard approach for that, the Galerkin method [25], is using orthonormal basis (ONB) $\{e_i\}_{i \in I}$ and investigate the matrix $M_{k,l} := \langle Oe_l, e_k \rangle$ [25] solving $Mc = d$ for $d = \{d_l\}_{l \in I} = \{\langle b, e_l \rangle\}_{l \in I}$. More recently frames are used for such a discretization [8,35]. On a more theoretical level, it is well known that operators can be represented by matrices using orthonormal bases [23]. Recently, the theory for frames has been settled for this theoretical approach [6,8,11].
Those matrices are constructed by concatenating the given operator $U$ with the synthesis and the analysis operators. Therefore they can be considered as generalizations of Gram matrices. In this article we study those so called $U$-cross Gram matrices and investigate their invertibility, in particular. The composition and the invertibility of $U$-cross Gram matrices are our main questions in this paper. In addition, it is very natural to ask whether the composition and more intricate and interesting, the inverses of $U$-cross Gram matrices can be stated as $U$-cross Gram matrices. The affirmative answer to these questions will be useful in applied frame theory, mentioned above. Similar questions are studied for frame multipliers, $K$-frame multipliers and fusion frame multipliers in [4, 32, 33, 36, 38] and matrix representations [11, 12, 26].

This paper is built up as follows: In Section 2 we fix the notation and collect results needed. In Section 3 we give the basic definition of $U$-cross Gram matrices, some examples, look at Schatten $p$-class properties and investigate this concept for Riesz sequences. In Section 4 we look at the pseudo-inverses of $U$-cross Gram matrices. In particular, we show under which circumstances this can be written as such a matrix again. In Section 5 we look at sufficient and necessary conditions on the $U$-cross Gram matrix to imply the involved sequences to be approximate duals. And finally in Section 6 we investigate how stable the invertibility of this matrix is regarding the perturbation of the operator or the sequences.

2. Notation

Throughout this paper, $\mathcal{H}$ is a separable Hilbert space, $I$ a countable index set and $I_\mathcal{H}$ the identity operator on $\mathcal{H}$. The orthogonal projection on a subspace $V \subseteq \mathcal{H}$ is denoted by $\pi_V$. We will denote the set of all linear and bounded operators between Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and for $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, it is represented by $\mathcal{B}(\mathcal{H})$. We denote the range and the null spaces of an operator $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ by $\text{R}(U)$ and $\text{N}(U)$, respectively. For a closed range operator $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, the pseudo-inverse of $U$ is the unique operator $U^\dagger \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ satisfying that

$$\text{N}(U^\dagger) = \text{R}(U)^\perp, \text{R}(U^\dagger) = \text{N}(U)^\perp, \text{and } UU^\dagger U = U.$$ 

If $U$ has closed range, then $U^*$ has closed range and $(U^*)^\dagger = (U^\dagger)^*$, see e.g. [19 Lemma 2.5.2].

A sequence $\Phi = \{\phi_i\}_{i \in I}$ in a separable Hilbert space $\mathcal{H}$ is a frame if there exist constants $A_\Phi, B_\Phi > 0$ such that for all $f \in \mathcal{H}$

$$A_\Phi \|f\|^2 \leq \sum_{i \in I} |\langle f, \phi_i \rangle|^2 \leq B_\Phi \|f\|^2. \quad (2.1)$$
The numbers $A_\Phi$ and $B_\Phi$ are called the frame bounds. If $\{\phi_i\}_{i \in I}$ is assumed to satisfy the right hand of (2.1), then it is called a Bessel sequence with Bessel bound $B_\Phi$. We say that a sequence $\{\phi_i\}_{i \in I}$ in $\mathcal{H}$ a frame sequence if it is a frame for $\text{span}\{\phi_i\}_{i \in I}$. For a Bessel sequence $\Phi = \{\phi_i\}_{i \in I}$, the synthesis operator $T_\Phi : \ell^2 \to \mathcal{H}$ is defined by

$$T_\Phi \{c_i\}_{i \in I} = \sum_{i \in I} c_i \phi_i.$$  

Its adjoint operator $T_\Phi^* : \mathcal{H} \to \ell^2$; the so called analysis operator is given by

$$T_\Phi^* f = \{\langle f, \phi_i \rangle\}_{i \in I}.$$  

The operator $S_\Phi : \mathcal{H} \to \mathcal{H}$, which is defined by

$$S_\Phi f = T_\Phi T_\Phi^* f = \sum_{i \in I} \langle f, \phi_i \rangle \phi_i,$$  

for all $f \in \mathcal{H}$, is called the frame operator. For a frame $\Phi$ the operator $T_\Phi$ is onto, $T_\Phi^*$ is one-to-one, and $S_\Phi$ is positive, self-adjoint and invertible [19]. Also, if $B_\Phi$ is the Bessel bound of $\Phi$, then

$$\|T_\Phi c\| \leq \sqrt{B_\Phi} \|c\|,$$  

for every sequence of scalars $c = \{c_i\}_{i \in I} \in \ell^2$. Note that those operators can be defined for any sequence [13] resulting in potential unbounded operators. We call a complete Bessel sequence an upper semi-frame [1, 2].

A dual for a Bessel sequence $\Phi = \{\phi_i\}_{i \in I} \subseteq \mathcal{H}$ is a Bessel sequence $\Psi = \{\psi_i\}_{i \in I}$ in $\mathcal{H}$ such that

$$f = \sum_{i \in I} \langle f, \psi_i \rangle \phi_i, \quad (f \in \mathcal{H}).$$  

For a frame $\Phi$ it is obvious to see that the Bessel sequence $\{S_\Phi^{-1} \phi_i\}_{i \in I}$ is a dual and is itself a frame again. This dual, denoted by $\tilde{\Phi} = \{\tilde{\phi}_i\}_{i \in I}$, is called the canonical dual. Note that this is the only equivalent dual, i.e $\text{R}(T_\Phi^*) = \text{R}(T_{\tilde{\Phi}}^*)$.

Recall that Bessel sequences $\Phi$ and $\Psi$ in $\mathcal{H}$ are called approximate dual frames, if

$$\|T_\Psi T_\Phi - I_{\mathcal{H}}\| < 1 \quad \text{or} \quad \|T_\Phi T_\Psi - I_{\mathcal{H}}\| < 1.$$  

Note that if $\Phi$ and $\Psi$ are approximately dual frames, then the operator $T_\Psi T_\Phi^*$ is invertible, in other words $\Phi$ and $\Psi$ are a reproducing pair [34] or pseudo-dual [28]. Hence each $f \in \mathcal{H}$ has the representation

$$f = (T_\Psi T_\Phi^*)^{-1} T_\Psi T_\Phi^* f = \sum_{i \in I} \langle f, \phi_i \rangle (T_\Psi T_\Phi^*)^{-1} \psi_i.$$  

In particular, $\Phi$ and $(T_\Psi T_\Phi^*)^{-1} \Psi$ are a pair of dual frames [21].

A Riesz basis for $\mathcal{H}$ is a family of the form $\{U e_i\}_{i \in I}$, where $\{e_i\}_{i \in I}$ is an orthonormal basis for $\mathcal{H}$ and $U : \mathcal{H} \to \mathcal{H}$ is a bounded bijective operator. Every Riesz
basis is a frame and has a biorthogonal sequence which is also its unique dual \cite{19}. The following proposition will be used in this manuscript.

**Proposition 2.1.** \cite{13, 19} For a sequence $\Phi = \{\phi_i\}_{i \in I}$ in $\mathcal{H}$, the following conditions are equivalent:

1. $\Phi$ is a Riesz basis for $\mathcal{H}$.
2. $\Phi$ is complete in $\mathcal{H}$ and there exist constants $A, B > 0$ such that
   \[
   A \sum_{i \in I} |c_i|^2 \leq \left\| \sum_{i \in I} c_i \phi_i \right\|^2 \leq B \sum_{i \in I} |c_i|^2,
   \]  
   \hspace{1cm} (2.2)
   for every finite scalar sequence $\{c_i\}_{i \in I}$.
3. $\Phi$ is a frame and $T \Phi$ is one to one.
4. $T^* \Phi$ is onto and $\Phi$ is an upper semi frame.

A sequence $\{\phi_i\}_{i \in I}$ satisfying (2.2) for all finite sequences $\{c_i\}_{i \in I}$ is called a Riesz sequence. Therefore a Riesz basis is a complete Riesz sequence.

For more details of frame theory see \cite{9, 16, 19}.

Recall that if $U$ is a compact operator on a separable Hilbert space $\mathcal{H}$, then there exist orthonormal sets $\{e_n\}_{n \in I}$ and $\{\sigma_n\}_{n \in I}$ in $\mathcal{H}$ such that

\[
Ux = \sum_{n \in I} \lambda_n \langle x, e_n \rangle \sigma_n,
\]
for $x \in \mathcal{H}$, with $\lambda_n \in c_0$, i.e. $\lim_{n \to \infty} \lambda_n = 0$. $\lambda_n$ is called the $n$th singular value of $U$. Given $0 < p < \infty$, we define the Schatten $p$-class of $\mathcal{H}$, denoted $S_p(\mathcal{H})$, as the space of all compact operators $U$ on $\mathcal{H}$ for which singular value sequence $\{\lambda_n\}_{n \in I}$ belongs to $\ell^p$. In this case, $S_p(\mathcal{H})$ is a Banach space with the norm

\[
\|U\|_p = \left( \sum_{n \in I} |\lambda_n|^p \right)^\frac{1}{p}.
\]  
   (2.3)

The Banach space $S_1(\mathcal{H})$ is called the trace class of $\mathcal{H}$ and $S_2(\mathcal{H})$ is called the Hilbert-Schmidt class.

We know that $U \in S_p(\mathcal{H})$ if and only if $\{\|U e_n\|\}_{n \in I} \in \ell^p$, for all orthonormal bases $\{e_n\}_{n \in I}$. For $0 < p \leq 2$ it is even enough to have the property for a single orthonormal basis, i.e. $U \in S_p(\mathcal{H})$ if and only if $\{\|U e_n\|\}_{n \in I} \in \ell^p$, for some orthonormal basis $\{e_n\}_{n \in I}$. It is proved that $S_p(\mathcal{H})$ is a two sided $*$-ideal of $\mathcal{B}(\mathcal{H})$, that is, a Banach algebra under the norm (2.3) and the finite rank operators are dense in $(S_p(\mathcal{H}), \cdot \|\cdot\|_p)$. This can be extended to operators between separate spaces; according to Theorem 7.8(c) \cite{39} if $U_1 \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, then $\|U_1 U_2\|_p \leq \|U_1\| \|U_2\|_p$ and $\|U_2 U_1\|_p \leq \|U_1\| \|U_2\|_p$ for all $U_2 \in S_p(\mathcal{H})$. For more information about these operators, see \cite{29, 31, 39, 41}. 

In the following theorem, the trace norm of bounded operators is computed by orthonormal bases.

**Theorem 2.2.** \[39\] Let $U \in B(H_1, H_2)$. Then $U \in S_p(H_1, H_2)$ if and only if

$$\|U\|_p = \sup \left( \sum_{i \in I} |\langle U e_i, f_i \rangle|^p \right)^{1/p} < \infty,$$

where the supremum is taken over all orthonormal bases $\{e_i\}_{i \in I}$ of $H_1$ and $\{f_i\}_{i \in I}$ of $H_2$.

Finally, recall \[24\] that for every matrix operator $M = (M_{k,l})$ on $\ell^2$ we have the mixed norm

$$\|M\|_{p,q} := \left( \sum_{k \in I} \left( \sum_{l \in I} |M_{k,l}|^q \right)^{p/q} \right)^{1/p}.$$ 

It is called the Frobenius norm when $p = q = 2$.

We will use the following criterion for the invertibility of operators.

**Proposition 2.3.** \[23\] Let $U_1 : \mathcal{H} \rightarrow \mathcal{H}$ be bounded and invertible on $\mathcal{H}$. Suppose that $U_2 : \mathcal{H} \rightarrow \mathcal{H}$ is a bounded operator and $\|U_2 h - U_1 h\| \leq \nu \|h\|$ for all $h \in \mathcal{H}$, where $\nu \in [0, \frac{1}{\|U_1^{-1}\|})$. Then $U_2$ is invertible on $\mathcal{H}$ and $U_2^{-1} = \sum_{k=0}^\infty (U_1^{-1} (U_1 - U_2))^k (U_1)^{-1}$.

### 3. $U$-cross Gram matrices

In this section, we define $U$-cross Gram matrices and introduce their properties.

**Definition 3.1.** Let $\Psi = \{\psi_i\}_{i \in I}$ and $\Phi = \{\phi_i\}_{i \in I}$ be Bessel sequences in Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. For $U \in B(H_1, H_2)$, the matrix $G_{U, \Phi, \Psi}$ given by

$$(G_{U, \Phi, \Psi})_{i,j} = \langle U \psi_j, \phi_i \rangle, \quad (i, j \in I),$$

is called the $U$-cross Gram matrix. If $\mathcal{H}_1 = \mathcal{H}_2$ and $U = I_{\mathcal{H}_1}$, it is called the cross Gram matrix and denoted by $G_{\Phi, \Psi}$. We use $G_{\Phi}$ for $G_{\Phi, \Phi}$; the so called Gram matrix \[19\].

Note this is just another viewpoint to the matrix representation of operators \[6\].

In the next lemma, we rephrase needed results in that paper for the $U$-cross Gram matrices viewpoint.

**Lemma 3.2.** Let $\Phi = \{\phi_i\}_{i \in I}$ and $\Psi = \{\psi_i\}_{i \in I}$ be two Bessel sequences in $\mathcal{H}_2$ and $\mathcal{H}_1$. Also, let $U \in B(H_1, H_2)$. The following assertions hold.

1. $G_{U, \Phi, \Psi} = T_{\Phi}^* U T_{\Psi}$. In particular, the $U$-cross Gram matrix $G_{U, \Phi, \Psi}$ defines a bounded operator on $\ell^2$ and $\|G_{U, \Phi, \Psi}\| \leq \sqrt{B_{\Phi} B_{\Psi}} \|U\|$. \[19\]
(2) \((G_{U,\Phi,\Psi})^* = G_{U^*,\Psi,\Phi}\).

Proof. (1) is shown in [6]. (2) is trivial.

As in [6] we have the representation of an operator \(U\) by

\[U = T_{\Phi^d}G_{U,\Phi,\Psi}T_{\Phi^d}^*,\]

(3.2)

where \(\Phi\) and \(\Psi\) are frames with dual frames \(\Phi^d\) and \(\Psi^d\), respectively.

For any sequence \(\Phi\) in \(\mathcal{H}\), \(G_{\Phi}\) is a bounded operator in \(\ell^2\) if and only if \(\Phi\) is a Bessel sequence [19]. This result, naturally, does not hold for \(G_{\Phi,\Psi}\) and \(G_{U,\Phi,\Psi}\). For example, if \(\Phi = \{ke_k\}_{k \in I}\), \(\Psi = \{\frac{1}{k}e_k\}_{k \in I}\) and \(U = I_\mathcal{H}\), where \(\{e_k\}_{k \in I}\) is an orthonormal basis of \(\mathcal{H}\), then it is easy to see that \(G_{U,\Phi,\Psi}\) is a bounded operator in \(\ell^2\), however, \(\Psi\) is not Bessel sequence. Moreover, if \(U \in \mathcal{B}(\mathcal{H})\) is defined as \(Ue_k = \frac{1}{k^2}e_k\), \(k \in \mathbb{N}\), then \(U\Phi = \Psi\), and therefore,

\[G_{U,\Phi,\Phi} = T_{\Phi^d}UT_{\Phi} = T_{\Phi^d}T_{\Psi} = I_{\ell^2}\]

is bounded, even invertible, but \(\Phi\) is not Bessel sequence (For similar examples see [37]). To find those sequences for which \(G_{\Phi,\Psi}\) (or even \(G_{U,\Phi,\Psi}\)) is invertible, is connected to the concepts of reproducing pairs [34] and pseudo frames [28].

By these examples we see that even for nice operator \(U\) we cannot deduce properties of the sequence \(\Phi\) and \(\Psi\). We will investigate the converse in this paper, which properties of \(U\) can be deduced from those of \(G_{U,\Phi,\Psi}\) for nice sequences \(\Psi\) and \(\Phi\).

Remark 3.3. Let \(\Phi\), \(\Psi\), \(\Theta\) and \(\Xi\) be Bessel sequences in \(\mathcal{H}\). Let \(U_1\) and \(U_2 \in \mathcal{B}(\mathcal{H})\). Then

\[(1) \; G_{U_1,\Phi,\Psi}G_{U_2,\Theta,\Xi} = T_{\Phi^d}U_1T_{\Psi^d}T_{\Phi^d}U_2T_{\Xi^d} = G_{(U_1T_{\Psi^d}U_2)\Phi,\Xi};\]
\[(2) \; G_{U_1,\Phi,\Psi}G_{U_2,\Psi,\Xi} = T_{\Phi^d}U_1T_{\Psi^d}T_{\Psi^d}U_2T_{\Xi^d} = T_{\Phi^d}U_1S_{\Psi^d}U_2T_{\Xi^d} = G_{(U_1S_{\Psi^d}U_2)\Phi,\Xi}.\]

Suppose \(\Psi\) is a frame, \(\Psi^\dagger\) any dual and \(\tilde{\Psi}\) the canonical dual of \(\Psi\). Let \(\Delta = \{\delta_i\}_{i \in I}\) be the standard orthonormal basis of \(\ell^2\), then we obtain [6]

\[(3) \; G_{U_1,\Phi,\Psi}G_{U_2,\Psi^\dagger,\Xi} = G_{U_1,\Phi,\Psi^\dagger G_{U_2,\Psi,\Xi} = G_{(U_1U_2)\Phi,\Xi};\]
\[(4) \; G_{S_{\Psi^\dagger},\Psi^\dagger}G_{S_{\Phi},\Psi} = G_{S_{\Phi},S_{\Psi}^\dagger}G_{S_{\Phi},\Psi} = G_{\Psi};\]
\[(5) \; G_{S_{\Psi^\dagger},\Psi^\dagger}G_{S_{\Phi},\Psi} = G_{S_{\Phi},\Psi};\]
\[(6) \; G_{T_{\Phi^d},\Delta,\Psi} = G_{\Phi,\Psi}.\]

In fact

\[(G_{T_{\Phi^d},\Delta,\Psi})_{i,j} = \langle T_{\Phi^d}\psi_j, \delta_i \rangle \]
\[= \sum_{k \in I} \langle \psi_j, \phi_k \rangle \langle \delta_k, \delta_i \rangle \]
\[= \langle \psi_j, \phi_i \rangle = (G_{\Phi,\Psi})_{i,j}.\]

Let \(\Psi = \{\psi\}_{i \in I}\) be a Riesz basis in \(\mathcal{H}\) then we have
(7) \( G_{T^*_A, \Delta, \tilde{\psi}} = G_{S_A, \tilde{\psi}, \tilde{\psi}} = G_{S^{-1}_A, \psi, \psi} = I \). More precisely, by using (3.1) and the biorthogonality of a Riesz basis and its canonical dual [19, Theorem 5.5.4], we obtain

\[
\left( G_{T^*_A, \Delta, \tilde{\psi}} \right)_{i,j} = \left( \langle T^*_A \tilde{\psi}_j, \delta_i \rangle \right) = \langle \tilde{\psi}_j, \psi_i \rangle = \delta_{i,j}.
\]

The proof of the other statements are obvious by the biorthogonal property.

3.1. Schatten \( p \)-classes

An operator \( O \) is compact if and only if \( \lim_{k \to \infty} \| O e_k \| = 0 \), for all ONBs \( \{ e_k \}_{k \in I} \).

This is true if and only if \( \lim_{k \to \infty} \sum_{l \in I} | \langle O e_k, f_l \rangle |^2 = 0 \), for all orthonormal bases \( \{ e_n \}_{n \in I} \) and \( \{ f_n \}_{n \in I} \). So, using the canonical basis of \( \ell^2 \) for our setting, this means that if \( G_{U, \Phi, \Psi} \) is compact, then \( \lim_{i \to \infty} \sum_{l \in I} | \langle U \psi_i, \phi_l \rangle |^2 = 0 \). As \( O \) is compact, if only \( O^* \) is compact, this is also equivalent to \( \lim_{i \to \infty} \sum_{l \in I} | \langle U^* \phi_i, \psi_l \rangle |^2 = 0 \). In particular, this implies that \( \lim_{i \to \infty} \langle U \psi_i, \phi_i \rangle = 0 \). Naturally, Frobenius matrices correspond to Hilbert-Schmidt operator [7]. Therefore, if \( \sum_{i \in I} \sum_{j \in I} | \langle U \psi_i, \phi_j \rangle |^2 < \infty \), then \( G_{U, \Phi, \Psi} \) is Hilbert-Schmidt, and therefore compact. More generally, this is true if \( \| G_{U, \Phi, \Psi} \|_{p, 2} < \infty \), for \( 1 \leq p < \infty \).

This allows to formulate the following results for Bessel sequences:

**Corollary 3.4.** Let \( U \in \mathcal{B}(\mathcal{H}) \), \( \Phi = \{ \phi_i \}_{i \in I} \) and \( \Psi = \{ \psi_i \}_{i \in I} \) be Bessel sequences in \( \mathcal{H} \). Then the following assertions hold.

1. If the operator \( U \) is compact, the matrix \( G_{U, \Phi, \Psi} \) is also compact. In particular, \( \lim_{i \to \infty} \sum_{l \in I} | \langle U \psi_i, \phi_l \rangle |^2 = 0 \).
2. If the operator \( U \) is Schatten \( p \)-class, the matrix \( G_{U, \Phi, \Psi} \) is Schatten \( p \)-class. In this case \( \left( \sum_{i \in I} | \langle U \psi_i, \phi_i \rangle |^p \right)^{1/p} < \infty \) and \( \| G_{U, \Phi, \Psi} \|_{p, 2} < \infty \). In particular:
   2a. If the operator \( U \) is trace-class, then \( G_{U, \Phi, \Psi} \) is trace-class, if and only if \( \sum_{i \in I} | \langle U \psi_i, \phi_i \rangle | < \infty \).
   2b. If the operator \( U \) is Hilbert-Schmidt, then \( G_{U, \Phi, \Psi} \) is Hilbert-Schmidt, if and only if \( \sum_{i \in I} \sum_{l \in I} | \langle U \psi_i, \phi_l \rangle |^2 < \infty \).

**Proof.** This follows from the ideal property of the considered operator spaces, as \( G_{U, \Phi, \Psi} = T^*_\Phi U T_\Psi \), as well as the above comments. \( \square \)

For frames we can show equivalent conditions:

**Lemma 3.5.** Let \( U \in \mathcal{B}(\mathcal{H}) \), \( \Phi = \{ \phi_i \}_{i \in I} \) and \( \Psi = \{ \psi_i \}_{i \in I} \) be frames in \( \mathcal{H} \). Then the following assertions hold.
The operator $\mathbf{U}$ is compact, if and only if $\mathbf{G}_{\mathbf{U}, \Phi, \Psi}$ is compact. In this case
\[ \lim_{i \to \infty} \sum_{l \in I} |\langle \mathbf{U}\psi_i, \phi_l \rangle|^2 = 0. \]

The operator $\mathbf{U}$ is in the Schatten $p$-class, if and only if $\mathbf{G}_{\mathbf{U}, \Phi, \Psi}$ is Schatten $p$-class. In this case
\[ \left( \sum_{i \in I} |\langle \mathbf{U}\psi_i, \phi_i \rangle|^p \right)^{1/p} < \infty \quad \text{and} \quad \|\langle \mathbf{U}\psi_i, \phi_i \rangle\|_{p,2} < \infty. \]
In particular:

2a. The operator $\mathbf{U}$ is trace-class, if and only if $\mathbf{G}_{\mathbf{U}, \Phi, \Psi}$ is trace-class, if and only if
\[ \sum_{i \in I} |\langle \mathbf{U}\psi_i, \phi_i \rangle| < \infty. \]

2b. The operator $\mathbf{U}$ is Hilbert-Schmidt, if and only if $\mathbf{G}_{\mathbf{U}, \Phi, \Psi}$ is Hilbert-Schmidt, if and only if
\[ \sum_{i, l \in I} |\langle \mathbf{U}\psi_i, \phi_l \rangle|^2 < \infty. \]

Proof. This follows from above, and Corollary 3.4. □

This generalizes result for operators and frames [14]. Note that $\mathbf{U}$ is compact respectively Schatten $p$-class if and only if $\mathbf{U}^*$ is. So, the role of $\mathbf{U}$ and $\mathbf{U}^*$ as well as $\Phi$ and $\Psi$ can be completely switched (for frames).

### 3.2. $\mathbf{U}$-cross Gram matrices and Riesz bases

It is apparent that $\Phi$ is an orthonormal basis if and only if $\mathbf{G}_{\Phi} = I_{\ell^2}$ as this means that $\Phi$ is biorthogonal to itself. In the sequel, we discuss the invertibility of $\mathbf{G}_{\mathbf{U}, \Phi, \Psi}$ when $\Phi$ and $\Psi$ are Riesz bases.

**Proposition 3.6.** Let $U \in \mathcal{B}(H)$, $\Phi = \{\phi_i\}_{i \in I}$ and $\Psi = \{\psi_i\}_{i \in I}$ be two frames in $H$ and $\Phi^d$ be a dual of $\Phi$. Then

1. $\mathbf{G}_{\mathbf{U}, \Phi, \Psi} = I_{\ell^2}$ if and only if $\Phi$ and $\Psi$ are Riesz bases. Also, $\Phi = S_{\Phi}U\Psi$ and $\Psi = S_{\Psi}U^*\Phi$. In this case $U = T_{\Phi}^*T_{\Psi}$ is invertible.

2. If $\mathbf{G}_{\mathbf{U}, \Phi, \Phi^d} = I_{\ell^2}$, then $U = I_H$ and $\Phi^d = \tilde{\Phi}$. The converse is true only if $\Phi$ is a Riesz basis.

**Proof.** If $\mathbf{G}_{\mathbf{U}, \Phi, \Psi} = I_{\ell^2}$, then
\[ \delta_{ij} = (\mathbf{G}_{\mathbf{U}, \Phi, \Psi})_{i,j} = \langle \mathbf{U}\psi_j, \phi_i \rangle. \]

Hence, $\Phi$ has a biorthogonal sequence, and therefore it is a Riesz basis. Also, $\Psi$ is a Riesz basis since $U^*\Phi$ is its biorthogonal sequence. In particular, $\tilde{\Phi} = U\Psi$ by Theorem 5.5.4 of [19]. By (3.2), $U = T_{\Phi}^*T_{\Psi}^*$. This shows (1).

By (1) $\Phi$ is a Riesz basis, and has only one, the canonical dual. Now, the invertibility of $S_{\Phi}$ implies that $U = I_H$. The converse is clear. □

In the next theorem, we study sufficient conditions for the invertibility of the $\mathbf{U}$-cross Gram matrix associated to Riesz sequences.
Theorem 3.7. Let \( U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \), \( \Phi = \{\phi\}_{i \in I} \) and \( \Psi = \{\psi_i\}_{i \in I} \) be two Bessel sequences in \( \mathcal{H}_2 \) and \( \mathcal{H}_1 \), respectively, such that \( G_{U,\Phi,\Psi} \) is invertible. Then \( \Phi \) and \( \Psi \) are Riesz sequences in \( \mathcal{H}_2 \) and \( \mathcal{H}_1 \), respectively. If \( \Phi \) and \( \Psi \) are assumed to be upper semi-frames, \( \Phi \) and \( \Psi \) are Riesz bases and \( U \) is invertible. In this case,

\[
(G_{U,\Phi,\Psi})^{-1} = G_{U^{-1},\tilde{\Phi},\tilde{\Psi}}.
\]

Proof. It is sufficient to show that \( T_{\Phi} \) is bounded below. To see this

\[
\|d\|^2 = \left| \langle G_{U,\Phi,\Psi}^{-1}G_{U,\Phi,\Psi}d, d \rangle \right|
= \left| \langle T_{\Phi}^* UT_{\Psi} G_{U,\Phi,\Psi}^{-1} G_{U,\Phi,\Psi}d, d \rangle \right|
= \left| \langle T_{\Psi} G_{U,\Phi,\Psi}^{-1} G_{U,\Phi,\Psi}d, U^* T_{\Phi} d \rangle \right|
\leq \sqrt{B_{\Psi}} \left\| G_{U,\Phi,\Psi}^{-1} \right\| \left\| d \right\| \left\| U \right\| \left\| T_{\Phi} d \right\|,
\]

for every \( d = \{d_i\}_{i \in I} \in \ell^2 \). This follows that

\[
\frac{\|d\|}{\sqrt{B_{\Psi}} \left\| G_{U,\Phi,\Psi}^{-1} \right\| \left\| U \right\|} \leq \|T_{\Phi} d\|.
\]

To obtain a lower bound for \( \Psi \), an analogue argument can be used.

As \( G_{U,\Phi,\Psi} = T_{\Phi}^* UT_{\Psi} \) it follows that \( U \) is invertible for complete sequences.

Note that, the invertibility of \( G_{U,\Phi,\Psi} \) does not imply that \( \Phi \) and \( \Psi \) are Riesz bases, in general. This is because \( G_{U,\Phi,\Psi} \) can never imply anything about completeness, as the considered space is irrelevant for \( G_{U,\Phi,\Psi} \). For an example assume that \( \{e_i\}_{i=1}^\infty \) is an orthonormal basis for a separable Hilbert space \( \mathcal{H} \) and \( \Phi = \{e_2, e_3, e_4, \ldots\} \). \( \Phi \) is non-complete. Still,

\[
(G_{\Phi})_{i,j} = \langle \phi_j, \phi_i \rangle = \delta_{i,j}.
\]

This is even true if one erases countably many elements, for example only considering \( \{e_2, e_4, e_6, \ldots\} \).

In Theorem 3.7 if \( \Phi \) and \( \Psi \) are Bessel sequences in finite dimensional Hilbert spaces, the invertibility \( G_{U,\Phi,\Psi} \) implies that \( \Phi \) and \( \Psi \) are Riesz bases and \( U \) is invertible operator. This is because the invertibility

\[
G_{U,\Phi,\Psi} = T_{\Phi}^* UT_{\Psi}
\]

yields \( T_{\Phi}^* \) is onto and \( T_{\Psi} \) is one to one. Because \( \mathcal{H} \) is finite dimensional, the operators \( T_{\Phi}^* \) and \( T_{\Psi} \) are invertible, in particular, \( \Phi \) and \( \Psi \) are Riesz basis. As a consequence \( U \) is also invertible.

\footnote{For finite frames see \cite{5,17}}
The next proposition solves the question of how the above result can be generalized to the existence of a left or right inverses.

**Proposition 3.8.** Let $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, $\Phi$ and $\Psi$ be Bessel sequences in $\mathcal{H}_2$ and $\mathcal{H}_1$, respectively. Then the following assertions are hold.

1. If $G_{U,\Phi,\Psi}$ has a right inverse, then $\Phi$ and $U^*\Phi$ are Riesz sequences. Moreover, if $\Phi$ is an upper semi-frame, then $\Phi$ is a Riesz basis and $U\Psi$ is a frame.

2. If $G_{U,\Phi,\Psi}$ has a left inverse, then $\Psi$ and $U\Psi$ are Riesz sequences. Moreover, if $\Psi$ is an upper semi-frame, then $\Psi$ is a Riesz basis and $U^*\Phi$ is a frame.

**Proof.** (1) The assumption shows that $T^*_\Phi UT_\Psi = T^*_U T^*_\Phi T_\Psi$ is surjective, and so $T^*_\Phi$ and $T^*_U = T^*_U T^*_\Phi$ are surjective. Using Proposition 2.1 immediately follows that $\Phi$ and $U^*\Phi$ are Riesz sequences. Moreover, if $\Phi$ is an upper semi-frame, then $T^*_\Phi$ is bijective by Proposition 2.1, and hence

$$T_\Psi = (T^*_\Phi)^{-1} T^*_U T_\Psi = (T^*_\Phi)^{-1} G_{U,\Phi,\Psi}$$

has a bounded right inverse, or equivalently $U\Psi$ is a frame. The proof of the second part is similar. □

4. The pseudo-inverse of $U$-cross Gram matrices

Similar to the case for multipliers [38] we can show that there exist duals that allow the representation of the pseudo-inverse as a matrix of the same class. Note that, from now, we put as an assumption that the $U$-cross Gram matrix has closed range. In Section 4.1 we put some statements about when this occurs.

**Theorem 4.1.** Let $\Psi$ and $\Phi$ be frames in Hilbert space $\mathcal{H}$, $U \in \mathcal{B}(\mathcal{H})$ be an invertible operator and $G_{U,\Phi,\Psi}$ have closed range. Then the following assertions hold:

1. There exists a unique dual $\Phi(U,\Psi)$ of $\Phi$ such that

$$(G_{U,\Phi,\Psi})^\dagger = G_{U^{-1},\bar{\Psi},\Phi(U,\Psi)}.$$

2. There exists a unique dual $\Psi(U,\Phi)$ of $\Psi$ such that

$$(G_{U,\Phi,\Psi})^\dagger = G_{U^{-1},\bar{\Psi}(U,\Phi)}.$$

**Proof.** (1) Note that $G^\dagger := G^\dagger_{U,\Phi,\Psi}$ exists and

$$N\left( G^\dagger \right) = (R(G_{U,\Phi,\Psi}))^\perp = (R(T^*_\Phi U T_\Psi))^\perp = R(T^*_\Phi)^\perp = N(T_\Phi),$$

$$R\left( G^\dagger \right) = (N(G_{U,\Phi,\Psi}))^\perp = (N(T^*_U U T_\Psi))^\perp = (N(T_\Psi))^\perp = R(T^*_\Psi).$$

(4.1) (4.2)
Putting,

\[
\Phi(U, \Psi) = \{ \phi_i(U, \Psi) \}_{i \in I} := \{ UT_\Psi G^{\dagger} \delta_i \}_{i \in I}, \tag{4.3}
\]

where \( \{ \delta_i \}_{i \in I} \) is the canonical orthonormal basis of \( \ell^2 \). Then

\[
T_{\Phi(U, \Psi)} T_{\Phi}^* = UT_\Psi G^{\dagger} T_{\Phi}^*
= T_{\Phi} T_{\Phi}^* UT_\Psi G^{\dagger} T_{\Phi}^* UT_\Psi U^{-1}
= T_{\Phi} G_{U, \Psi} G^{\dagger} G_{U, \Psi} U^{-1}
= T_{\Phi} G_{U, \Psi} U^{-1}
= T_{\Phi} T_{\Phi}^* UT_\Psi T_{\Phi}^* U^{-1} = I_H.
\]

So, \( \Phi(U, \Psi) \) is a dual of \( \Phi \). Note that for all duals \( \Phi^d \) and \( \Psi^d \) of \( \Phi \) and \( \Psi \), respectively, we have

\[
G_{U, \Psi} G_{U^{-1}, \Psi^d, \Phi^d} G_{U, \Psi} = T_{\Phi}^* UT_\Psi T_{\Phi}^* U^{-1} T_{\Phi} T_{\Phi}^* UT_\Psi
= UT_\Psi = G_{U, \Psi}.
\]

Moreover, \( N(T_{\Phi(U, \Psi)}) = N(T_{\Phi}) \). Indeed, by (4.1) and (4.3) we obtain \( N(T_{\Phi}) = N(G^{\dagger}) \subseteq N(T_{\Phi(U, \Psi)}) \). For the reverse inclusion, suppose that \( c = \{ c_i \}_{i \in I} \in N(T_{\Phi(U, \Psi)}) \) and so, \( T_{\Psi} G^{\dagger} c = 0 \). On the other hand, by (4.2) it follows that

\[
G^{\dagger} c = T_{\Psi}^* f, \tag{4.4}
\]

for some \( f \in H \). Then

\[
f = S_{\Psi}^{-1} T_{\Psi} T_{\Psi}^* f
= S_{\Psi}^{-1} T_{\Psi} G^{\dagger} c = 0.
\]

Applying (4.4) and (4.1) we have \( c \in N(G^{\dagger}) = N(T_{\Phi}) \). Furthermore,

\[
N \left( G_{U^{-1}, \Psi^d, \Phi(U, \Psi)} \right) = N \left( T_{\Psi}^* U^{-1} T_{\Phi(U, \Psi)} \right)
= N(T_{\Phi(U, \Psi)})
= N(T_{\Phi})
= N(G^{\dagger}).
\]

Moreover, it follows from (4.2) that

\[
R \left( G_{U^{-1}, \Psi^d, \Phi(U, \Psi)} \right) = R \left( T_{\Psi}^* U^{-1} T_{\Phi(U, \Psi)} \right)
= R(T_{\Psi}^*)
= R(G^{\dagger}).
\]
Hence, $G^\dagger = G_{U^{-1}, \tilde{\psi}, \Phi(U, \Psi)}$. To show the uniqueness, assume that $\Phi^\dagger$ is also a dual of $\Phi$ such that

$$G_{U^{-1}, \tilde{\psi}, \Phi(U, \Psi)}^\dagger = G_{U^{-1}, \tilde{\psi}, \Phi^\dagger}.$$ 

It follows that $U^{-1}T_{\Phi(U, \Psi)} = U^{-1}T_{\Phi^\dagger}$ and hence, $\Phi(U, \Psi) = \Phi^\dagger$.

The proof of (2) is similar, using $\Psi(U, \Phi) = \{\psi_i(U, \Phi)\}_{i \in I} = \{U^*T_{\Phi} (G^\dagger)^* \delta_i\}_{i \in I}.$

We have that

$$\Phi(U, \Psi) = \{U^*T_{\Psi} (G^\dagger)^* \delta_i\}_{i \in I}$$

and $(G^\dagger_{U, \Psi, \Phi})^* = G^\dagger_{U^*, \Phi, \Psi}$.

By comparing $\Phi(U, \Psi)$ and $\Phi(U, \Psi) = \{U T_{\Psi} G^\dagger_{U, \Phi, \Psi} \delta_i\}_{i \in I}$, we obtain that $\Phi(U, \Psi) = \Phi(U^*, \Psi)$.

Using the same arguments we can show

**Corollary 4.2.** Let $\Psi$ and $\Phi$ be frames in the Hilbert spaces $\mathcal{H}_1$ respectively $\mathcal{H}_2$, $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ an invertible operator and $G_{U, \Psi, \Phi}$ has closed range. Then the following assertions hold:

1. There exists a unique dual $\Phi(U, \Psi)$ of $\Phi$ such that

$$\Phi(U, \Psi) = \{U^*T_{\Psi} (G^\dagger_{U, \Phi, \Psi})^* \delta_i\}_{i \in I}$$

2. There exists a unique dual $\Psi(U, \Phi)$ of $\Psi$ such that

$$\Psi(U, \Phi) = \{U T_{\Phi} G^\dagger_{U, \Psi, \Phi} \delta_i\}_{i \in I}$$

Our next goal is to determine $G^\dagger_{U, \Phi, \Psi}$ when the invertibility assumption on $U$ is dropped and it is only assumed to be closed range. In fact, we prove that all results of the above theorem except the uniqueness are true, assuming additionally that $R(U^*) = S_{\Psi}R(U^*)$ or $R(U) = S_{\Phi}R(U)$.

For that we first look at frames for the range of an operator. Naturally if $\Phi$ is a frame, $\pi_{R(U)} \Phi = UU^\dagger \Phi$ is a frame for $R(U)$. Also $U \Phi$ has the same property:

**Corollary 4.3.** Let $U$ have closed range, and $\Psi$ be a frame with bounds $A_{\Psi}, B_{\Psi}$. Then $U \Psi = (U \psi_k)_k$ is a frame for $R(U)$ with frame bounds $m \cdot A_{\Psi}, M \cdot B_{\Psi}$. Here, $m$ is the lower bound of $U$, i.e. $m \|f\|^2 \leq \|U^* f\|^2$ for $f \in N(U^*)^\perp$ and $M = \|U^*\|^2$.

We have that $S_{U\Psi}^{-1} = U^* S_{\Psi}^{-1} U^\dagger.$
Proof. The first part is [19 Proposition 5.3.1].

We have $S_{U\Psi}^{-1} = (S_{U\Psi})^\dagger = \left(\sqrt{S_{\Psi}} U^*\right)^\dagger \left(U \sqrt{S_{\Psi}}\right)^\dagger = U^* \left(\sqrt{S_{\Psi}}\right)^{-\frac{1}{2}} \left(\sqrt{S_{\Psi}}\right)^{-\frac{1}{2}} U = U^* S_{\Psi}^{-1} U^\dagger.$

\[ \square \]

Corollary 4.4. Let $U \in \mathcal{B}(\mathcal{H})$ have closed range, $\Phi$ and $\Psi$ be frames for $R(U)$ and $R(U^*)$, respectively. Then $G_{U, \Phi, \Psi}$ has closed range and

$$\begin{align*} (G_{U, \Phi, \Psi})^\dagger &= G_{U^\dagger, \Phi, \Psi} \Sigma_{U, \Phi, \Psi}, \\ R(U^*) &= S_{\Psi} R(U^*). \end{align*}$$

Proof. It follows immediately by using Corollary 4.2 for the invertible operator $U : R(U^*) \rightarrow R(U)$ and the fact that $U^\dagger_{|R(U)} = (U_{|R(U^*)})^{-1}$.

\[ \square \]

Theorem 4.5. Let $\Psi$ and $\Phi$ be frames in Hilbert space $\mathcal{H}$, $U \in \mathcal{B}(\mathcal{H})$ a closed range operator and $G_{U, \Phi, \Psi}$ have closed range.

1. The following assertions are equivalent:
   1. There exists a dual $\Phi^{(U, \Psi)}$ of $\Phi$ on $R(U)$ such that
      $$(G_{U, \Phi, \Psi})^\dagger = G_{U^\dagger, \Phi^{(U, \Psi)}}.$$
   2. $(G_{U, \Phi, \Psi})^\dagger = G_{U^\dagger, \Phi, \Psi U^\dagger \Phi^{(U, \Psi)}}$.
   3. $R(U^*) = S_{\Psi} R(U^*)$.

2. The following assertions are equivalent:
   1. There exists a dual $\Psi^{(U, \Phi)}$ of $\Psi$ on $R(U^*)$ such that
      $$(G_{U, \Phi, \Psi})^\dagger = G_{U^\dagger, \Psi^{(U, \Phi)}}.$$ 
   2. $(G_{U, \Phi, \Psi})^\dagger = G_{U^\dagger, \Psi^{(U, \Phi)}}$.
   3. $R(U) = S_{\Phi} R(U)$.

Proof. For the first part we have

$$(1 \Leftrightarrow 3) \text{ Putting } G^\dagger := (G_{U, \Phi, \Psi})^\dagger. \text{ Then}$$

$$N \left(G^\dagger\right) = \left(\left(R(G_{U, \Phi, \Psi})^\dagger\right) = \left(R(T_{\Psi}^* U T_{\Psi})\right)^\dagger = \left(R(T_{\Psi}^* U\right)^\dagger = N(U^* T_{\Phi}), \quad (4.5)$$

$$R \left(G^\dagger\right) = \left(N(G_{U, \Phi, \Psi})^\dagger\right) = \left(N(T_{\Psi}^* U T_{\Psi})\right)^\dagger = N(U T_{\Psi}) = R(T_{\Psi}^* U^*). \quad (4.6)$$

Take,

$$\Phi^{(U, \Psi)} = \{\phi_{i}^{(U, \Psi)}\}_{i \in I} := \{UT_{\Psi} G^\dagger \delta_{i}\}_{i \in I}, \quad (4.7)$$
where \( \{ \delta_i \}_{i \in I} \) is the canonical orthonormal basis of \( \ell^2 \). Then \( \Phi(U, \Psi) \) is a Bessel sequence and on \( \mathbb{R}(U) \) we obtain

\[
T_{\Phi(U, \Psi)} T_{\Phi}^* = U T_{\Psi} G^\dagger T_{\Phi}^*
\]

\[
= T_{\Phi}^* U T_{\Psi} G^\dagger T_{\Phi} U T_{\Phi}^* U\dagger
\]

\[
= T_{\Phi}^* G_{U, \Psi} G^\dagger G_{U, \Phi} T_{\Phi} U\dagger
\]

\[
= T_{\Phi}^* G_{U, \Psi} T_{\Phi}^* U\dagger
\]

\[
= U U\dagger = I_{\mathbb{R}(U)},
\]

where \( \Phi^d \) and \( \Psi^d \) are duals of \( \Phi \) and \( \Psi \), respectively. So, \( \Phi(U, \Psi) \) is a dual of \( \Phi \) on \( \mathbb{R}(U) \), in particular a frame on \( \mathbb{R}(U) \). Also,

\[
G_{U, \Phi, \Psi} G_{U^1, \Psi^d, \Phi^d} G_{U, \Phi, \Psi} = T_{\Phi}^* U T_{\Psi} T_{\Phi}^* U^\dagger T_{\Phi}^* U T_{\Psi}
\]

\[
= T_{\Phi}^* U U\dagger U T_{\Psi}
\]

\[
= T_{\Phi}^* U T_{\Psi} = G_{U, \Phi, \Psi}.
\]

Moreover, \( \mathcal{N}(U^\dagger T_{\Phi(U, \Psi)}) = \mathcal{N}(U^* T_{\Phi}) \). Indeed, the equations (4.5) and (4.7) yield

\[
\mathcal{N}(U^* T_{\Phi}) = \mathcal{N}(G^\dagger) \subseteq \mathcal{N}(T_{\Phi(U, \Psi)}) \subseteq \mathcal{N}(U^\dagger T_{\Phi(U, \Psi)}).
\]

For the reverse inclusion, suppose that \( c = \{c_i\}_{i \in I} \in \mathcal{N}(U^\dagger T_{\Phi(U, \Psi)}) \) and so, \( U^\dagger T_{\Phi(U, \Psi)} c = 0 \). The injectivity \( U^\dagger \) on \( \mathbb{R}(U) \) and \( \mathbb{R}(T_{\Phi(U, \Psi)}) \subseteq \mathbb{R}(U) \) imply that \( T_{\Phi(U, \Psi)} c = 0 \). On the other hand, by the fact that \( G^\dagger G_{U, \Phi, \Psi} G^\dagger = G^\dagger \) we have

\[
G^\dagger c = G^\dagger G_{U, \Phi, \Psi} G^\dagger c
\]

\[
= G^\dagger T_{\Phi}^* U T_{\Psi} G^\dagger c
\]

\[
= G^\dagger T_{\Phi}^* T_{\Phi(U, \Psi)} c = 0.
\]

Hence, \( c \in \mathcal{N}(G^\dagger) = \mathcal{N}(U^* T_{\Phi}) \). Therefore,

\[
\mathcal{N}(G_{U^1, \Phi(U, \Psi)}) = \mathcal{N}(T_{\Psi}^* U^\dagger T_{\Phi(U, \Psi)})
\]

\[
= \mathcal{N}(U^\dagger T_{\Phi(U, \Psi)})
\]

\[
= \mathcal{N}(U^* T_{\Phi}) = \mathcal{N}(G^\dagger).
\]
Combining (4.6) and the assumptions we obtain

\[
R\left(G_{U^{\dagger},\tilde{\psi},\Phi(U,\psi)}\right) = R\left(T_{\psi}^{*}U^{\dagger}T_{\Phi(U,\psi)}\right) \\
= R\left(T_{\psi}^{*}U^{\dagger}\right) \\
= R\left(T_{\psi}^{*}U^{*}\right) \\
= R\left(T_{\psi}^{*}S_{\psi}^{-1}U^{*}\right) \\
= R\left(T_{\psi}^{*}U^{*}\right) = R\left(G^{\dagger}\right).
\]

So, \(G^{\dagger} = G_{U^{\dagger},\tilde{\psi},\Phi(U,\psi)}\). Conversely, suppose there is a dual of \(\Phi\) as \(\Phi(U,\psi)\) such that \(G^{\dagger} = G_{U^{\dagger},\tilde{\psi},\Phi(U,\psi)}\). Then

\[
R\left(T_{\psi}^{*}S_{\psi}^{-1}U^{*}\right) = R\left(T_{\psi}^{*}S_{\psi}^{-1}U^{\dagger}\right) \\
= R\left(G_{U^{\dagger},\tilde{\psi},\Phi(U,\psi)}\right) \\
= R\left(G^{\dagger}\right) \\
= R\left(G_{U,\Phi,\psi}\right) \\
= R\left(T_{\psi}^{*}U^{*}T_{\Phi}\right) = R\left(T_{\psi}^{*}U^{*}\right).
\]

This follows that \(R\left(S_{\psi}^{-1}U^{*}\right) = R\left(U^{*}\right)\).

(2 \iff 3) It is easy to see that \(UU^{\dagger}\Phi\) is a dual of \(\Phi\) on \(R\left(U\right)\) and

\[
G_{U,\Phi,\psi}G_{U^{\dagger},\tilde{\psi},UU^{\dagger}\Phi}G_{U,\Phi,\psi} = G_{U,\Phi,\psi}.
\]

Using this fact \(UU^{\dagger}\Phi\) is a frame on \(R\left(U\right)\) and \(UU^{\dagger} = \pi_{R(U)}\) (see Section 4.1) we obtain

\[
N\left(G_{U^{\dagger},\tilde{\psi},UU^{\dagger}\Phi}\right) = N\left(T_{\psi}^{*}U^{\dagger}T_{UU^{\dagger}\Phi}\right) \\
= N\left(U^{\dagger}T_{U\Phi}\right) \\
= N\left(T_{UU^{\dagger}\Phi}\right) \\
= R\left(T_{UU^{\dagger}\Phi}^{*}\right)^{\perp} \\
= R\left(T_{UU^{\dagger}\Phi}\right)^{\perp} \\
= R\left(T_{\Phi}^{*}UU^{\dagger}\right)^{\perp} \\
= R\left(T_{\Phi}^{*}U\right)^{\perp} = N\left(G^{\dagger}\right).
\]
We can see that $R(U^*) = S_{\psi}R(U^*)$ if and only if
\[
R\left(G_{U^*,\tilde{\psi},\tilde{U}^\dagger\Phi}\right) = R\left(T^*_{\tilde{\psi}}U^\dagger T_{\tilde{\psi}}^\dagger\tilde{U}^\dagger\Phi\right) \\
= R\left(T^*_{\tilde{\psi}}U^\dagger\right) \\
= T^*_{\tilde{\psi}}U^\dagger(H) \\
= T^*_{\tilde{\psi}}U^*(H) \\
= R(T^*_{\tilde{\psi}}U^*) = R\left(G^\dagger\right).
\]

Hence, (1) is proved.

For the second part note that

(1 $\iff$ 3) is similar to the first part.

(2 $\iff$ 3) One can see that $\tilde{U}^\dagger U\tilde{\psi}$ is a dual of $\tilde{\psi}$ on $R(U^*)$ and
\[
G_{U,\Phi,\tilde{\psi}}G_{U^*,\tilde{U}^\dagger\tilde{U}\tilde{\psi},\tilde{\Phi}}G_{U,\Phi,\tilde{\psi}} = G_{U,\Phi,\tilde{\psi}}.
\]

Using this fact $R((U^*)^*) = R(U)$, then $S_{\Phi}R(U) = R(U)$ if and only if
\[
N\left(G_{U^*,\tilde{U}^\daggerU\tilde{\psi},\tilde{\Phi}}\right) = N\left(T^*_{\tilde{U}^\daggerU\tilde{\psi}}U^\dagger T_{\tilde{\Phi}}\right) \\
= N(U^\dagger T_{\tilde{\Phi}}) \\
= N(U^\dagger S_{\Phi}^{-1}T_{\Phi}) \\
= R(T_{\Phi}S_{\Phi}^{-1}(U^*)^*)^\perp \\
= R(T_{\Phi}S_{\Phi}^{-1}U) \\
= R(T_{\Phi}U)^\perp \\
= N(U^*T_{\Phi}) = N\left(G^\dagger\right).
\]

Applying this fact $\tilde{U}^\dagger U\tilde{\psi}$ is a frame for $R(U^\dagger)$ we have
\[
R\left(G_{U^*,\tilde{U}^\daggerU\tilde{\psi},\tilde{\Phi}}\right) = R\left(T^*_{\tilde{U}^\daggerU\tilde{\psi}}U^\dagger T_{\tilde{\Phi}}\right) \\
= R\left(T^*_{\tilde{U}^\daggerU\tilde{\psi}}U^\dagger\right) \\
= R(T^*_{\tilde{U}^\daggerU\tilde{\psi}}U^\dagger) \\
= R(T^*_{\tilde{U}^\daggerU\tilde{\psi}}U^\dagger) \\
= R(T^*_{\tilde{U}^\daggerU\tilde{\psi}}U^\dagger) \\
= R(T_{\tilde{\psi}}U^*) = R\left(G^\dagger\right).
\]
\square
The assumptions \( R(U^*) = S_\Psi R(U^*) = R(S_\Psi U^*) \) naturally leads to the question for which operators \( U \) this is fulfilled, leading to questions about invariant subspaces, see e.g. [22], beyond the scope of this paper.

As we have mention before, for a closed range operator \( U \) the uniqueness property of Theorem 4.1 does not hold in general as the next example indicates.

Example 4.6. Let \( \mathcal{H} \) be a Hilbert space with an orthonormal basis \( \{e_i\}_{i \in I} \). Let \( \Psi = \{e_1, e_2, e_3, e_4, \ldots\} \) and \( \Phi = \{e_1, e_2, e_3, e_4, \ldots\} \). It is clear to see that \( \Phi^a = \{e_1, 0, e_2, e_3, \ldots\} \) and \( \Phi^b = \{e_1, \frac{e_2}{2}, e_2, e_3, \ldots\} \) are respective duals. Define \( U \in \mathcal{B}(\mathcal{H}) \) by

\[
Ue_i = e_i, \quad (i \neq 2), \quad Ue_2 = e_1.
\]

Obviously, \( R(U) = \{e_2\}^\perp \) and \( N(U) = \text{span}(e_1 - e_2) \). Hence \( U \) has closed range, so the operator \( G_{U,\Phi,\Psi} \) has also closed range. Moreover, \( U : N(U)^\perp \to R(U) \) is invertible, hence \( U^\dagger \) is given as

\[
U^\dagger e_1 = \frac{e_1 + e_2}{2}, \quad U^\dagger e_2 = 0, \quad \text{and} \quad U^\dagger e_i = e_i, \quad (i \geq 3).
\]

In fact, it is the unique right inverse of \( U \) on \( R(U) \) such that \( R(U^\dagger) = R(U^*) \) where \( U^* \) is determined by

\[
U^* e_1 = e_1 + e_2, \quad U^* e_2 = 0, \quad \text{and} \quad U^* e_i = e_i, \quad (i \geq 3).
\]

Moreover,

\[
T_{\Phi^a}\{c_k\} = c_1e_1 + c_3e_2 + c_4e_3 + ..., \quad T_{\Phi^b}\{c_k\} = c_1e_1 + c_2e_2/2 + c_3e_2/2 + c_4e_3 + ....
\]

Hence, \( U^\dagger T_{\Phi^a} = U^\dagger T_{\Phi^b} \). So

\[
T_{\Psi^d}^* U^\dagger T_{\Phi^a} = T_{\Psi^d}^* U^\dagger T_{\Phi^b},
\]

for every dual \( \Psi^d \) of \( \Psi \).

Corollary 4.7. Let \( U \in \mathcal{B}(\mathcal{H}) \) be an operator with closed range, and \( \Psi \) and \( \Phi \) frames in Hilbert space \( \mathcal{H} \). Then

\[
(G_{U,UU^\dagger\Phi,\Psi})^\dagger = (G_{U,\Phi,U^\dagger U^*\Psi})^\dagger = G_{U^\dagger,\widetilde{U^\dagger U^*},\widetilde{U^\dagger U^*}}.
\]

Proof. By Lemma 4.10 \( G_{U,\Phi,\Psi} \) has closed range. It is easy to see that \( UU^\dagger \Phi \) and \( U^\dagger U^* \Psi \) are frames for \( R(U) \) and \( R(U^*) \), respectively. So, \( S_{UU^\dagger U^*\Psi} R(U) = R(U) \) and \( S_{U^\dagger U^* \Psi} R(U^*) = R(U^*) \). Then the result immediately follows by Theorem 4.3. □
4.1. More on the closed range conditions

In this subsection we present some conditions for a $U$-cross Gram matrix having closed range. For example, if $U$ is a positive invertible and $\Phi$ is a frame, then $\sqrt{U}\Phi$ is a frame, and therefore $T_{\sqrt{U}\Phi}$ has closed range. Using Corollary 2.3 of [20] it follows that

$$G_{U,\Phi,\Psi} = T_{\Phi}^* U T_{\Phi} = T_{\Phi}^* \sqrt{U} \sqrt{U} T_{\Phi} = (T_{\sqrt{U}\Phi})^* (T_{\sqrt{U}\Phi})$$

has closed range.

**Proposition 4.8.** Let $U \in \mathcal{B}(\mathcal{H})$ have closed range, $\Phi$ be a Bessel sequence and $\Psi$ a frame for a Hilbert space $\mathcal{H}$. Then the following are equivalent:

1. $G_{U,\Phi,\Psi}$ has closed range.
2. $U^* \Phi$ is a frame sequence.

**Proof.** Since $\Psi$ is a frame for $\mathcal{H}$ we obtain

$$R(G_{U,\Phi,\Psi}) = R(T_{U^*\Phi}^* T_{\Psi}) = R(T_{U^*\Phi}^*).$$

(4.8)

The synthesis operator of $U^* \Phi$ has closed range [13, 18] if and only if $U^* \Phi$ is a frame sequence. \hfill \square

**Corollary 4.9.** Let $U$ be a surjective operator in $\mathcal{B}(\mathcal{H})$, $\Phi$ a Bessel sequence and $\Psi$ a frame for a Hilbert space $\mathcal{H}$. Then the following are equivalent:

1. $G_{U,\Phi,\Psi}$ has closed range.
2. $\Phi$ is a frame sequence.

**Lemma 4.10.** Let $U \in \mathcal{B}(\mathcal{H})$ have closed range, and $\Phi$ and $\Psi$ be Bessel sequences in a Hilbert space $\mathcal{H}$. The following assertions hold.

1. If $U \Psi$ and $UU^\dagger \Phi$ are frames for $\mathcal{R}(U)$, then

$$R(G_{U,\Phi,\Psi}) = R(T_{UU^\dagger \Phi}^*).$$

2. If $U^\dagger U \Psi$ and $U^* \Phi$ are frames for $\mathcal{R}(U^*)$, then

$$R(G_{U,\Phi,\Psi}) = R(T_{U^* \Phi}^*).$$

In particular, in both cases $G_{U,\Phi,\Psi}$ has closed range.

**Proof.** We have that $UU^\dagger$ is the orthogonal projection on $\mathcal{R}(U)$ [20]. Then

$$G_{U,\Phi,\Psi} = T_{\Phi}^* U T_{\Psi} = T_{UU^\dagger \Phi}^* U T_{\Psi} = T_{UU^\dagger \Phi}^* T_{U^\dagger \Psi}.$$ 

By assumption the considered sequences are frames for $\mathcal{R}(U)$, and so $R(G_{U,\Phi,\Psi}) = R(T_{UU^\dagger \Phi}^*)$. This proves the first part. In order to obtain (2) we have

$$G_{U,\Phi,\Psi} = T_{\Phi}^* U T_{\Psi} = T_{U^\dagger \Phi}^* U T_{U^\dagger \Psi} = T_{U^* \Phi}^* T_{U^\dagger \Psi}.$$ 

\hfill \square
The assumption of Lemma 4.8 are fulfilled, if $\Psi$ and $\Phi$ are frames for $\mathcal{H}$.

**Theorem 4.11.** Let $U \in \mathcal{B}(\mathcal{H})$ have closed range, $\Phi$ and $\Psi$ be frames for $\mathcal{H}$. Then

$$(G_{U,\Phi,\Psi})^\dagger = T_{U\Psi}^*T_{\Phi}^*$$

if and only if $\text{R}(T_{\Phi}^*) = \text{R}(T_{\Phi}^*U)$.

**Proof.** One can see that

$$G_{U,\Phi,\Psi}T_{U\Psi}^*T_{\Phi}G_{U,\Phi,\Psi} = T_{\Phi}^*UT_{\Psi}T_{U\Psi}^*T_{\Phi}^*UT_{\Psi} = T_{\Phi}^*T_{U\Psi}^*T_{\Phi}^*UT_{\Psi} = T_{\Phi}^*UT_{\Psi} = G_{U,\Phi,\Psi}.$$ 

Also,

$$\text{R} \left( T_{U\Psi}^*T_{\Phi}^* \right) = \text{R} \left( T_{U\Psi}^* \right) = \text{R} \left( T_{U\Psi}^* \Phi \right) = \text{R} \left( T_{\Phi}^*U^* \right) = \text{R} \left( G_{U,\Phi,\Psi}^* \right) = \text{R} \left( (G_{U,\Phi,\Psi})^\dagger \right).$$

Now, $\text{R}(T_{\Phi}^*) = \text{R}(T_{\Phi}^*U)$ if and only if

$$\text{N} \left( T_{U\Psi}^*T_{\Phi}^* \right) = \text{N} \left( T_{\Phi}^* \right) = \text{N} \left( T_{\Phi}^* \right) = \text{R} \left( T_{\Phi}^* \right)^\perp = \text{R} \left( T_{\Phi}^*U \right)^\perp = \text{R} \left( G_{U,\Phi,\Psi} \right)^\perp = \text{N} \left( (G_{U,\Phi,\Psi})^\dagger \right).$$

$\square$

**Corollary 4.12.** Let $U$ have closed range, and $\Phi$ and $\Psi$ be frames for $\text{R}(U)$ and $\mathcal{H}$, respectively. Then

$$(G_{U,\Phi,\Psi})^\dagger = T_{U\Psi}^*T_{\Phi}^*. $$

Based on the above results, we have the following theorem:

**Theorem 4.13.** Let $U \in \mathcal{B}(\mathcal{H})$ have closed range, and $\Phi$ and $\Psi$ be frames for $\mathcal{H}$. Then $G_{U,\Phi,\Psi}$ has closed range and

$$G_{U_1,\Phi_1,\Psi_1}^\dagger = G_{(U_1^*U_1U_1^*)^\dagger,\Psi_1,\Phi_1} = G_{(U_1^*U_1U_1^*)^{-1},\Psi_1,\Phi_1},$$

where $U_1 = U \mid_{\text{R}(U^*)}$. 
Proof. Using Corollary 4.3 we have \( S_{U_1^* \Phi}^{-1} = (U_1 U_1^*)^\dagger S_{U_1 * \psi}^{-1} (U_1 U_1^*)^\dagger \). Applying Corollary 4.12 and the fact that \( U_1 \) is invertible we obtain

\[
G_{U_1^* U_1 \Phi, U_1^* \Psi}^\dagger = T_{U_1^* \Phi}^* U_1^* S_{U_1^* \psi}^{-1} U_1 T_{U_1^* \Phi}^* \Psi = T_{U_1^* U_1 \Phi, U_1^* \Psi}^* T_{U_1^* \Phi}^* U_1^* S_{U_1^* \psi}^{-1} U_1 T_{U_1^* \Phi}^* \Psi.
\]

\( \square \)

5. Approximate duals

We can give several conditions for appropriate duality based on the \( U \)-cross Gram matrix. We start with sufficient conditions.

Proposition 5.1. Let \( \Phi \) and \( \Psi \) be frames in \( \mathcal{H} \) with duals \( \Phi^d \) and \( \Psi^d \), respectively. The following assertions are hold.

1. \( \Phi \) and \( \Psi \) are approximate dual frames, if

\[
\| I_{\mathcal{L}^2} - G_{\Psi, \Phi} \| < \frac{1}{\sqrt{B_{\Phi} B_{\Psi}}}.
\]

(5.1)

2. \( \Phi^d \) and \( \Psi^d \) are approximate dual frames, if

\[
\| I_{\mathcal{L}^2} - G_{\Phi, \Psi} \| < \frac{1}{\sqrt{B_{\Phi} B_{\Psi}}}.
\]

(5.2)

3. \( \Phi \) and \( \Psi \) are approximate dual frames, if

\[
\| I_{\mathcal{L}^2} - G_{\Psi, \Phi} \| < \frac{1}{\sqrt{B_{\Phi} B_{\Psi}}}.
\]

(5.3)

4. If \( V \in \mathcal{B}(\mathcal{H}) \) is a right inverse of \( U \) such that

\[
\| I_{\mathcal{L}^2} - G_{U, \Phi} G_{V, \Phi^d, \Phi} \| < \frac{1}{\sqrt{B_{\Phi} B_{\Psi}}},
\]

then \( \Phi \) and \( \Psi \) are approximate dual frames.

Proof. (1) According to the dual property and using (5.1) we have

\[
\| I_{\mathcal{H}} - T_{\Phi} T_{\Psi}^* \| = \| T_{\Phi} (I_{\mathcal{L}^2} - T_{\Psi}^* T_{\Phi}) T_{\Psi}^* \| \leq \sqrt{B_{\Phi} B_{\Phi^d}} \| I_{\mathcal{L}^2} - G_{\Psi, \Phi} \| < 1.
\]

(2) One can see that (5.2) yields

\[
\| I_{\mathcal{H}} - T_{\Phi^d} T_{\Psi^d}^* \| = \| T_{\Phi^d} (T_{\Phi^d} T_{\Psi} - I_{\mathcal{L}^2}) T_{\Psi^d}^* \| \leq \sqrt{B_{\Phi^d} B_{\Psi^d}} \| I_{\mathcal{L}^2} - G_{\Phi, \Psi} \| < 1.
\]
Using (5.3) it is straightforward to see that
\[ \| I_{\mathcal{H}} - T_\Phi T_\Phi^* \| = \| T_\Phi^* (I_{\ell^2} - T_\Phi^* T_\Phi^*) T_\Phi^* \| \leq \sqrt{B_\Phi B_\Psi} \| I_{\ell^2} - G_{\Phi,\Psi} \| < 1. \]

(4) Finally, note that \( G_{U_{\Psi,\Phi}} G_{V_{\Phi^d,\Phi}} = G_{\Psi,\Phi} \). Then the result follows immediately from (5.4) and the first part.

Note that the role in this result of the primal and dual frames, i.e. \( \Psi, \Phi \) and \( \Psi^d, \Phi^d \), can be switched.

We can only give one necessary condition, and this holds only in the Riesz basis case:

**Lemma 5.2.** If \( \Psi \) is an approximate dual for a Riesz basis \( \Phi \), then
\[ \| I_{\ell^2} - G_{\Phi,\Psi} \| < \sqrt{B_\Phi B_\Psi} \sqrt{A_\Phi A_\Psi}. \]

**Proof.** Let \( \Psi \) be a Riesz basis and \( \Phi \) an approximate dual \( \Psi \). Then \( T_\Psi \) and \( G_{\Phi,\Psi} \) are invertible. It follows that
\[ \| (T_\Phi T_\Psi^*)^{-1} \|^{-1} \| I_{\ell^2} - T_\Phi^* T_\Psi \| \leq \| T_\Phi^* T_\Psi (I_{\ell^2} - T_\Phi^* T_\Psi) \| \leq \| T_\Phi^* T_\Psi - T_\Phi^* T_\Phi T_\Phi^* T_\Psi \| \leq \| T_\Phi^* (I_{\mathcal{H}} - T_\Psi T_\Phi^*) T_\Psi \| \leq \sqrt{B_\Phi} \sqrt{B_\Psi}. \]

So,
\[ \| I_{\ell^2} - T_\Phi^* T_\Psi \| \leq \sqrt{B_\Phi} \sqrt{B_\Psi} \| (T_\Phi T_\Psi^*)^{-1} \| \leq \sqrt{B_\Phi} \sqrt{B_\Psi} \| T_\Phi^{-1} \| \| T_\Psi^{-1} \| \leq \sqrt{B_\Phi} B_\Psi A_\Phi A_\Psi. \]

□

6. Stability of \( U \)-cross Gram matrices

In this section, we state some sufficient conditions for the invertibility of \( U \)-cross Gram matrices.

**Proposition 6.1.** Let \( \Phi \) be a Bessel sequence in \( \mathcal{H} \) with Bessel bound \( B_\Phi \). If \( U_1, U_2 \) and \( U_3 \in \mathcal{B}(\mathcal{H}) \) such that \( G_{U_1,\Phi,\Phi} \) is an invertible operator and
\[ \| U_2^* U_1 U_3 - U_1 \| < \frac{1}{\| G_{U_1,\Phi,\Phi}^{-1} \| B_\Phi}, \]
then $G_{U_1, U_2 \Phi, U_3 \Phi}$ is also invertible. Moreover, if $\Phi$ is a frame then $\Phi$ is a Riesz basis, $U_1$ is invertible and

$$G_{U_1, U_2 \Phi, U_3 \Phi}^{-1} = T_\Phi \sum_{k=0}^{\infty} \left( I_H - U_1^{-1} U_2^* U_1 U_3 \right)^k U_1^{-1} T_\Phi.$$ 

Proof. Assumption (6.1) yields

$$\| G_{U_1, U_2 \Phi, U_3 \Phi} G_{U_1, \Phi, \Phi}^{-1} - I_{L^2} \| = \| (G_{U_1, U_2 \Phi, U_3 \Phi} - G_{U_1, \Phi, \Phi}) G_{U_1, \Phi, \Phi}^{-1} \| \leq \| G_{U_2} U_1, \Phi, \Phi - G_{U_1, \Phi, \Phi} \| \| G_{U_1, \Phi, \Phi}^{-1} \| = \| T_\Phi^* (U_2^* U_1 U_3 - U_1) T_\Phi \| \| G_{U_1, \Phi, \Phi}^{-1} \| \leq \| T_\Phi \|^2 \| (U_2^* U_1 U_3 - U_1) \| \| G_{U_1, \Phi, \Phi}^{-1} \| < 1.$$ 

This shows that $G_{U_1, U_2 \Phi, U_3 \Phi} G_{U_1, \Phi, \Phi}^{-1}$ is invertible and hence, $G_{U_1, U_2 \Phi, U_3 \Phi}$ is invertible. Moreover, if $\Phi$ is a frame, then it is also a Riesz basis, $U_1$ is invertible and $G_{U_1, \Phi, \Phi}^{-1} = T_\Phi^{-1} U_1^{-1} (T_\Phi^*)^{-1}$. Due to Proposition 2.3 we obtain

$$G_{U_1, U_2 \Phi, U_3 \Phi}^{-1} = \sum_{k=0}^{\infty} \left( T_\Phi^{-1} U_1^{-1} (T_\Phi^*)^{-1} \right)^k G_{U_1, \Phi, \Phi}^{-1} = \sum_{k=0}^{\infty} \left( T_\Phi^{-1} U_1^{-1} (U_1 - U_2^* U_1 U_3) T_\Phi \right)^k T_\Phi^{-1} U_1^{-1} (T_\Phi^*)^{-1} = \sum_{k=0}^{\infty} T_\Phi^{-1} \left( I_H - U_1^{-1} U_2^* U_1 U_3 \right)^k T_\Phi T_\Phi^{-1} U_1^{-1} (T_\Phi^*)^{-1} = T_\Phi \sum_{k=0}^{\infty} \left( I_H - U_1^{-1} U_2^* U_1 U_3 \right)^k U_1^{-1} T_\Phi.$$ 

Corollary 6.2. Let $\Phi$ be a Bessel sequence in $H$ with Bessel bound $B_\Phi$. If $U_1, U_2 \in B(H)$ such that $G_{U_1, \Phi, \Phi}$ is an invertible operator and

$$\| U_2 - I_H \| < \frac{1}{\| G_{U_1, \Phi, \Phi}^{-1} \| B_\Phi \| U_1 \|}, \quad (6.2)$$

then $G_{U_1, \Phi, U_3 \Phi}$ and $G_{U_1, U_2 \Phi, \Phi}$ are also invertible. Moreover, if $\Phi$ is a frame and $U_1 \in B(H)$ is invertible, then

$$G_{U_1, \Phi, U_2 \Phi}^{-1} = G_{U_1, U_2 \Phi, \Phi}^{-1}.$$
and

\[ G_{U_1, U_2 \Phi, \Phi}^{-1} = G_{U_1^{-1}, \tilde{\Phi}, U_2 \Phi}. \]

**Proof.** By using the assumption (6.2) we obtain

\[ \| G_{U_1, U_2 \Phi, \Phi}^{-1} - I \| \leq \| G_{U_1, U_2 \Phi, \Phi} - G_{U_1, \Phi, \Phi} \| \| G_{U_1, \Phi, \Phi}^{-1} \| \]

\[ \leq \| T_{\Phi} U_1 (U_2 - I_\mathcal{H}) T_{\Phi} \| \| G_{U_1, \Phi, \Phi}^{-1} \| \]

\[ \leq \| T_{\Phi} \| ^2 \| U_1 \| \| U_2 - I_\mathcal{H} \| \| G_{U_1, \Phi, \Phi}^{-1} \| \]

\[ \leq B_{\Phi} \| U_1 \| \| U_2 - I_\mathcal{H} \| \| G_{U_1, \Phi, \Phi}^{-1} \| < 1. \]

Then \( G_{U_1, U_2 \Phi, \Phi}^{-1} \) is invertible and so \( G_{U_1, U_2 \Phi, \Phi}^{-1} \) is invertible. The proof of invertibility \( G_{U_1, U_2 \Phi, \Phi}^{-1} \) is similar. The rest is immediately follows by Theorem 3.7. \( \square \)

**Theorem 6.3.** Suppose that \( \Phi \) and \( \Psi \) are Bessel sequences in \( \mathcal{H} \) such that \( G_{U, \Phi, \Psi}^{-1} \) is invertible.

1. If \( V \in B (\mathcal{H}) \) such that

\[ \| U - V \| < \frac{1}{\| G_{U, \Phi, \Psi}^{-1} \| \sqrt{B_{\Phi} B_{\Psi}}}, \]  

then \( G_{V, \Phi, \Psi} \) is also invertible.

2. If \( \Xi = \{ \xi_i \}_{i \in I} \) is a Bessel sequence in \( \mathcal{H} \) such that

\[ \left( \sum_{i \in I} \| \psi_i - \xi_i \| ^2 \right)^{1/2} < \frac{1}{\| G_{U, \Phi, \Psi}^{-1} \| \sqrt{B_{\Phi}} \| U \|}, \]  

then \( G_{U, \Phi, \Xi} \) is invertible.

3. If \( \Theta = \{ \theta_i \}_{i \in I} \) is a Bessel sequence in \( \mathcal{H} \) such that

\[ \left( \sum_{i \in I} \| \phi_i - \theta_i \| ^2 \right)^{1/2} < \frac{1}{\| G_{U, \Phi, \Psi}^{-1} \| \sqrt{B_{\Psi}} \| U \|}, \]

then \( G_{U, \Theta, \Psi} \) is invertible.
Proof. By assumption (6.3) we have
\[
\left\| I_{\ell^2} - G_{U,\Phi,\Psi}^{-1} G_{V,\Phi,\Psi} \right\| \leq \left\| G_{U,\Phi,\Psi}^{-1} (G_{U,\Phi,\Psi} - G_{V,\Phi,\Psi}) \right\|
\leq \left\| G_{U,\Phi,\Psi}^{-1} \right\| \left\| G_{U,\Phi,\Psi} - G_{V,\Phi,\Psi} \right\|
= \left\| G_{U,\Phi,\Psi}^{-1} \right\| \left\| T_{\Phi}^* UT_{\Psi} - T_{\Phi}^* VT_{\Psi} \right\|
= \left\| G_{U,\Phi,\Psi}^{-1} \right\| \left\| T_{\Phi}^* (U - V) T_{\Psi} \right\|
\leq \left\| G_{U,\Phi,\Psi}^{-1} \right\| \sqrt{B_{\Phi} B_{\Psi}} \|U - V\| < 1,
\]
and \( G_{V,\Phi,\Psi} \) is also invertible. This proves (1). To show (2) note that
\[
\| T_{\Psi} - T_{\Xi} \| \leq \left( \sum_{i \in I} \| \psi_i - \xi_i \|^2 \right)^{1/2}.
\]
Using (6.4) follows that
\[
\left\| I_{\ell^2} - G_{U,\Phi,\Psi}^{-1} G_{U,\Phi,\Xi} \right\| = \left\| G_{U,\Phi,\Psi}^{-1} (G_{U,\Phi,\Psi} - G_{U,\Phi,\Xi}) \right\|
\leq \left\| G_{U,\Phi,\Psi}^{-1} \right\| \left\| G_{U,\Phi,\Psi} - G_{U,\Phi,\Xi} \right\|
= \left\| G_{U,\Phi,\Psi}^{-1} \right\| \left\| T_{\Phi}^* UT_{\Psi} - T_{\Phi}^* UT_{\Xi} \right\|
= \left\| G_{U,\Phi,\Psi}^{-1} \right\| \left\| T_{\Phi}^* U (T_{\Psi} - T_{\Xi}) \right\|
\leq \left\| G_{U,\Phi,\Psi}^{-1} \right\| \sqrt{B_{\Phi}} \|U\| \left( \sum_{i \in I} \| \psi_i - \xi_i \|^2 \right)^{1/2} < 1.
\]
Hence, \( G_{U,\Phi,\Xi} \) is invertible by the invertibility \( G_{U,\Phi,\Psi}^{-1} G_{U,\Phi,\Xi} \). Finally, (3) follows similarly.

\[
\text{Note that the condition } \left( \sum_{i \in I} \| \psi_i - \xi_i \|^2 \right)^{1/2} \text{ is a typical condition for results dealing with the perturbation of frames [18] or 'nearness of sequences' [3, 6].}
\]

**Theorem 6.4.** Let \( \Psi = \{ \psi_i \}_{i \in I} \) be a Bessel sequence and \( \Phi = \{ \phi_i \}_{i \in I} \) a Riesz basis such that
\[
\sum_{i \in I} \| U \psi_i - \phi_i \|^2 < \frac{A_{\Phi}^2}{B_{\Phi}},
\]
where \( A_{\Phi} \) and \( B_{\Phi} \) are lower and upper bounds of \( \Phi \), respectively. Then \( G_{U,\Phi,\Psi} \) is invertible and
\[
G_{U,\Phi,\Psi}^{-1} = \sum_{k=0}^{\infty} (I_{\ell^2} - T_{\Phi}^{-1} U T_{\Psi})^k G_{\Phi}^{-1}.
\]
Proof. Since $\Phi$ is a Riesz basis, we conclude that $G_{\Phi}$ is invertible and
\[
\|G_{\Phi}^{-1}\| = \|T_{\Phi}^{-1}(T_{\Phi}^*)^{-1}\| \leq A_{\Phi}^{-1}.
\]
Therefore,
\[
\|G_{U,\Phi,\Psi} - G_{\Phi}\| = \|T_{\Phi}^*UT_{\Psi} - T_{\Phi}^*T_{\Phi}\|
\leq \sqrt{B_{\Phi}} \|UT_{\Psi} - T_{\Phi}\|^{1/2}
\leq A_{\Phi} \leq \|G_{\Phi}^{-1}\|^{-1}.
\]
Hence, $G_{U,\Phi,\Psi}$ is invertible by Proposition 2.3. Moreover, by Proposition 2.3 we have
\[
G_{U,\Phi,\Psi}^{-1} = \sum_{k=0}^{\infty} \left( G_{\Phi}^{-1} (G_{\Phi} - G_{U,\Phi,\Psi}) \right)^k G_{\Phi}^{-1}
= \sum_{k=0}^{\infty} \left( G_{\Phi}^{-1} T_{\Phi}^* (T_{\Phi} - UT_{\Psi}) \right)^k G_{\Phi}^{-1}
= \sum_{k=0}^{\infty} (I_{\ell^2} - T_{\Phi}^{-1}UT_{\Psi})^k G_{\Phi}^{-1}.
\]
\[\square\]

Now we are ready to state our main result about the stability of $U$-cross Gram matrices.

**Theorem 6.5.** Let $U$ and $V \in B(H)$, $\Phi = \{\phi_i\}_{i \in I}$ and $\Psi = \{\psi_i\}_{i \in I}$ be frames. Let $G_{U,\Phi,\Psi}$ be invertible. If $\Xi = \{\xi_i\}_{i \in I}$ and $\Theta = \{\theta_i\}_{i \in I}$ are Bessel sequences such that
\[
\left\| \sum_{i \in I} c_i (\psi_i - \theta_i) \right\| + \left\| \sum_{i \in I} c_i (\phi_i - \xi_i) \right\| \leq \lambda_1 \left\| \sum_{i \in I} c_i \psi_i \right\| + \lambda_2 \left\| \sum_{i \in I} c_i \phi_i \right\|
+ \lambda_3 \left\| \sum_{i \in I} c_i \xi_i \right\| + \lambda_4 \left\| \sum_{i \in I} c_i \theta_i \right\|,
\]
for all $c = \{c_i\}_{i \in I} \in \ell^2$, and
\[
\|U - V\| < \mu, \quad \mu + 2\|U\|\lambda < \sqrt{A_{\Psi}A_{\Phi}}, \quad \text{and} \quad \lambda \left( 1 + 3\sqrt{\frac{B}{A}} \right) < 1,
\]
where $B = \max\{B_{\Phi}, B_{\Psi}, B_{\Xi}, B_{\Theta}\}$, $\lambda = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$ and $A_{\Psi}, A_{\Phi}$ are lower bounds of $\Psi$ and $\Phi$, respectively. Then $G_{V,\Xi,\Theta}$ is invertible and $\Xi$ and $\Theta$ are Riesz bases.
Proof. First note that from (6.6) it easily follows that
\[ \|T_\Psi - T_\Theta\| + \|T_\Phi - T_\Xi\| \leq \lambda_1 \sqrt{B_\Psi} + \lambda_2 \sqrt{B_\Theta} + \lambda_3 \sqrt{B_\Phi} + \lambda_4 \sqrt{B_\Xi}. \]  
(6.8)

This immediately implies that \( \Xi \) and \( \Theta \) are frames by Theorem 5.6.1 of [19]. On the other hand, Theorem 3.7 implies that \( \Phi \) and \( \Psi \) are Riesz bases and \( U \) is invertible. In particular,
\[ G_{U, \Phi, \Psi}^{-1} = G_{U^{-1}, \bar{\Phi}, \bar{\Psi}}. \]

This shows that
\[ \frac{\sqrt{A_\Psi A_\Phi}}{\|U^{-1}\|} \leq \frac{1}{\|G_{U, \Phi, \Psi}^{-1}\|}, \]  
(6.9)

where \( A_\Psi \) and \( A_\Phi \) are the lower frame bounds of \( \Psi \) and \( \Phi \), respectively. Combining (6.7), (6.8) and (6.9) we obtain
\[ \|G_{V, \Xi, \Theta} - G_{U, \Phi, \Psi}\| = \|G_{V, \Xi, \Theta} - G_{U, \Xi, \Theta} + G_{U, \Xi, \Theta} - G_{U, \Xi, \Psi} + G_{U, \Xi, \Psi} - G_{U, \Phi, \Psi}\| \]
\[ \leq \|T_\Xi (V - UT_\Theta)\| + \|T_\Xi U (T_\Theta - T_\Psi)\| + \|T_\Phi^* (T_\Xi - T_\Psi) U T_\Psi\| \]
\[ \leq \mu \sqrt{B_\Xi B_\Theta} + \sqrt{B_\Xi} \|U\| \|T_\Theta - T_\Psi\| + \|U\| \sqrt{B_\Psi} \|T_\Phi - T_\Xi\| \]
\[ \leq \mu \sqrt{B_\Xi B_\Theta} + (\sqrt{B_\Xi} + \sqrt{B_\Psi}) \|U\| (\lambda_1 \sqrt{B_\Psi} + \lambda_2 \sqrt{B_\Theta} \]
\[ + \lambda_3 \sqrt{B_\Phi} + \lambda_4 \sqrt{B_\Xi}) \]
\[ \leq B (\mu + 2 \|U\| \lambda) \]
\[ \leq \sqrt{\frac{A_\Psi A_\Phi}{\|U^{-1}\|}} \leq \left\| G_{U, \Phi, \Psi}^{-1} \right\|^{-1}. \]

Hence, \( G_{V, \Xi, \Theta} \) is invertible by Proposition 2.3. In particular, \( \Xi \) and \( \Theta \) are Riesz bases by Theorem 3.7. \( \square \)

Corollary 6.6. Let \( U \in \mathcal{B}(\mathcal{H}) \), \( \Phi = \{\phi_i\}_{i \in I} \) and \( \Psi = \{\psi_i\}_{i \in I} \) be frames in \( \mathcal{H} \). \( \Phi^n = \{\phi^n_i\}_{i \in I} \rightarrow \Phi \) and \( \Psi^n = \{\psi^n_i\}_{i \in I} \rightarrow \Psi \) in \( \mathcal{H} \) and \( U_n \rightarrow U \) in \( \mathcal{B}(\mathcal{H}) \), then \( G_{U_n, \Phi^n, \Psi^n} \rightarrow G_{U, \Phi, \Psi} \) in \( \mathcal{B}(\ell^2) \).
Proof. Applying (6.5) and assumptions we have
\[
\| G_{U_n, \Phi_n, \Psi_n} - G_{U, \Phi, \Psi} \| = \| T_{\Phi_n}^* U_n T_{\Psi} - T_{\Phi}^* U T_{\Psi} \|
\leq \| T_{\Phi_n}^* U_n T_{\Psi} - T_{\Phi_n}^* U_n T_{\Psi} \| + \| T_{\Phi_n}^* U_n T_{\Psi} - T_{\Phi_n}^* U_n T_{\Psi} \| \| T_{\Psi} \|
\leq \| T_{\Phi_n}^* U_n \| \| T_{\Psi} - T_{\Psi} \| + \| T_{\Phi_n}^* U_n \| \| T_{\Psi} - T_{\Psi} \|
+ (\| T_{\Phi_n}^* U_n - T_{\Phi}^* U_n \| + \| T_{\Phi_n}^* U_n - T_{\Phi}^* U_n \| ) \| T_{\Psi} \|
\leq \| T_{\Phi_n}^* U_n \| \| T_{\Psi} - T_{\Psi} \|
+ (\| T_{\Phi_n}^* - T_{\Phi}^* \| \| U_n \| + \| T_{\Phi_n}^* \| \| U_n - U \| ) \| T_{\Psi} \| \to 0.
\]
\[\square\]

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