A simple strong solution to non-linear HJB PDEs: an application to the portfolio model

Abstract. We develop a simple and general method for solving non-linear Hamilton-Jacobi-Bellman partial differential equations HJB PDEs. We apply our method to the portfolio model.

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1 Introduction

This paper overcomes major obstacles in the area of stochastics and mathematical finance. We provide simple strong solutions to the (non-linear) Hamilton-Jacobi-Bellman partial differential equation HJB PDE (and possibly other PDEs). In so doing, we transform the HJB PDE into an ordinary differential equation ODE. It is well-known that stochastic-system based HJBs are generally non-smooth and that the existing methods rely on the methods of viscosity and minimax weak solutions (see, for example, Crandall and Lions (1983), among others).

We apply our method to the portfolio model (the stochastic factor model). It is well known that a criticism of the portfolio model is that the optimal portfolio does not depend on wealth for an exponential utility function (the most common utility in finance). Consequently, in this paper, we resolve this issue. That is, we show that the optimal portfolio depends on wealth, regardless of the functional form of the utility.
2 The method

We consider this function $V(x, y)$; it can be expressed as $V(\beta x, y)$, where $\beta$ is a shift parameter with an initial value equal to one (see Alghalith (2008)). Define $g \equiv \beta x$; differentiating $V(g, y)$ with respect to $\beta$ and $x$, respectively, yields

$$V_\beta = V_g x,$$

$$V_x = V_g \beta.$$  

Thus

$$\frac{V_x}{V_\beta} = \frac{\beta}{x} \Rightarrow V_x = \frac{\beta V_\beta}{x}. \quad (1)$$

The second order derivatives of $V(g, y)$ with respect to $\beta$ and $x$, respectively, are

$$V_{\beta \beta} = V_{gg} x^2,$$

$$V_{xx} = V_{gg} \beta^2.$$  

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Therefore
\[ \frac{V_{xx}}{V_{\beta \beta}} = \frac{x^2}{\beta^2} \Rightarrow V_{xx} = \frac{x^2 V_{\beta \beta}}{\beta^2}. \] (2)

Similarly, we can rewrite \( V(x, y) \) as \( V(x, \beta y) \). Define \( f \equiv \beta y \); differentiating \( V(x, f) \) with respect to \( \beta \) and \( y \), respectively, yields

\[
\begin{align*}
V_\beta & = V_f y, \\
V_y & = V_f \beta.
\end{align*}
\]

Thus
\[ V_y = \frac{\beta V_\beta}{y}. \] (3)

The second order derivatives of \( V(x, f) \) with respect to \( \beta \) and \( y \), respectively, are

\[
\begin{align*}
V_{\beta \beta} & = V_{ff} y^2, \\
V_{yy} & = V_{ff} \beta^2.
\end{align*}
\]
Therefore
\[
\frac{V_{yy}}{V_{\beta \beta}} = \frac{\beta^2}{y^2} \implies V_{yy} = \frac{\beta^2 V_{\beta \beta}}{y^2}.
\]  
(4)

Differentiating (1) with respect to \(y\) yields
\[
V_{xy} = \frac{\beta V_{\beta y}}{x} \implies V_{\beta y} = \frac{x V_{xy}}{\beta}.
\]  
(5)

Differentiating (3) with respect to \(y\) yields
\[
V_{yy} = \frac{\beta [\beta y V_{\beta y} - V_{\beta}]}{y^2}.
\]  
(6)

Substituting (5) into (6), we obtain
\[
V_{xy} = \frac{(y^2/\beta) V_{yy} + \beta V_{\beta}}{xy} = \frac{\beta (V_{\beta \beta} + V_{\beta})}{xy}.
\]  
(7)

3 Practical example: the portfolio model

We provide a brief description of the portfolio model (see, for example, Detemple (2012), Alghalith (2009) and Castaneda-Leyva and Hernandez-
Hernandez (2006), among others). Thus, we have a two-dimensional Brownian motion \( \left\{ \left( W_s^1, W_s^{(2)} \right), \mathcal{F}_s \right\}_{t \leq s \leq T} \) defined on the probability space \((\Omega, \mathcal{F}, \mathcal{F}_s, P)\), where \( \mathcal{F}_s \) is the augmentation of filtration. The risk-free asset price process is

\[
S_0 = e^{\int_0^T r(Y_s) ds},
\]

where \( r(Y_s) \in C^2_b \) is the rate of return and \( Y_s \) is the economic factor.

The risky asset price process is given by

\[
dS_s = S_s \left\{ \mu(Y_s) ds + \sigma(Y_s) dW_s^1 \right\},
\]

where \( \mu(Y_s) \) and \( \sigma(Y_s) \) are the rate of return and the volatility, respectively. The economic factor process is given by

\[
dY_s = b(Y_s) ds + \rho dW_s^1 + \sqrt{1 - \rho^2} dW_s^{(2)}, Y_t = y,
\]

where \( |\rho| < 1 \) is the correlation factor between the two Brownian motions and \( b(Y_s) \in C^1 \) with a bounded derivative.

The wealth process is given by

\[
X_T^\pi = x + \int_t^T \left\{ r(Y_s) X_s^\pi + \left[ \mu(Y_s) - r(Y_s) \right] \pi_s \right\} ds + \int_t^T \pi_s \sigma(Y_s) dW_s^1,
\]
where $x$ is the initial wealth, $\{\pi_s, \mathcal{F}_s\}_{t \leq s \leq T}$ is the portfolio process with $E \int_t^T \pi_s^2 ds < \infty$. The trading strategy $\pi_s \in A(x, y)$ is admissible.

The investor’s objective is to maximize the expected utility of terminal wealth

$$V(t, x, y) = \sup_{\pi_t} E\left[u(X_T) \mid \mathcal{F}_t\right], \quad (11)$$

where $V(.)$ is the value function and $u(.)$ is a continuous, bounded and strictly concave utility function.

The corresponding Hamilton-Jacobi-Bellman PDE is (suppressing the notations)

$$V_t + rxV_x + bV_y + \frac{1}{2} V_{yy} + \sup_{\pi_t} \left\{ \frac{1}{2} \pi_t^2 \sigma^2 V_{xx} + \pi_t (\mu - r) V_x + \rho \sigma \pi_t V_{xy} \right\} = 0,$$

$$V(T, x, y) = u(x). \quad (12)$$

Thus we obtain the following well-known HJB PDE

$$V_t + rxV_x + (\mu - r) \pi^*_t V_x + \frac{1}{2} \sigma^2 \pi^*_t^2 V_{xx} + bV_y + \frac{1}{2} V_{yy} + \rho \sigma \pi^*_t V_{xy} = 0, \quad (13)$$
where the asterisk denotes the optimal value. Substituting (1), (2), (4), (3) and (7) into (13) yields

\[ V_t + r \beta V_\beta + (\mu - r) \pi_t^* \frac{\beta V_\beta}{x} + \frac{1}{2} \frac{\sigma^2 \pi_t^* x^2 V_\beta^2}{\beta^2} + \frac{b \beta V_\beta}{y} + \frac{1}{2} \frac{\beta^2 V_\beta}{y^2} + \rho \sigma \pi_t^* \frac{\beta (V_\beta + V_\beta)}{xy} = 0. \]

Using the above procedure, we can easily show that \( V_t = \beta V_\beta / t \) and thus the above equation becomes

\[ \frac{\beta V_\beta}{t} + r \beta V_\beta + (\mu - r) \pi_t^* \frac{\beta V_\beta}{x} + \frac{1}{2} \frac{\sigma^2 \pi_t^* x^2 V_\beta^2}{\beta^2} + \frac{b \beta V_\beta}{y} + \frac{1}{2} \frac{\beta^2 V_\beta}{y^2} + \rho \sigma \pi_t^* \frac{\beta (V_\beta + V_\beta)}{xy} = 0. \]

The solution to the above equation is equivalent to solving an ODE. Since \( V(.) \) is differentiable with respect to the shift parameter \( \beta \), the solution is classical (strong), even if \( V(.) \) is not smooth in \( x, y \) or \( t \).

Setting \( \beta \) at its initial value, we obtain

\[ \frac{V_\beta}{t} + r V_\beta + (\mu - r) \pi_t^* \frac{V_\beta}{x} + \frac{1}{2} \frac{\sigma^2 \pi_t^* x^2 V_\beta^2}{y^2} + \frac{b V_\beta}{y} + \frac{1}{2} V_\beta + \rho \sigma \pi_t^* \frac{(V_\beta + V_\beta)}{xy} = 0. \]
Thus the optimal portfolio is given by

\[
\pi_t^* = \frac{(\mu - r) V_{\beta}}{\sigma^2 x^3 V_{\beta \beta}} - \frac{\rho \sigma (V_{\beta \beta} + V_{\beta})}{\sigma x^3 y V_{\beta \beta}}.
\]

We note that, in contrast to the previous literature, the optimal portfolio depends on wealth, regardless of the functional form of the utility, even for an exponential utility function. Clearly, this is a more realistic result.

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