Lambert $W$ function and hanging chain revisited

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Abstract

In classical physics, calculating the slack of a hanging chain is a problem that has attracted interest. This study aims to solve this problem through experiment and theory. When the length and distance of both the ends of a hanging chain are given, the length of slack can be expressed by a Lambert $W$ function or an irrational function within a certain distance. Herein, a simple observation of the slack is presented. The result obtained is one of the applications of the Lambert $W$ function in the field of physics. Though the shape of a hanging chain is well-known to be a catenary, the calculation of slack possesses richer aspects of mathematical physics.
I. INTRODUCTION

Experimental demonstration is important in physics. However, there are a few experiments based on the mathematical physics. This paper provides an example of a simple demonstration experiment that exhibits the mathematical elements of classical mechanics.

In classical mechanics, the physics of a hanging chain includes various fields of mathematics, such as infinitesimal calculus, differential equations, and variational method. It is well-known that the shape of a hanging chain (rope or cable) is catenary (hyperbolic cosine function). Theoretically, the curve for which the gravitational potential energy acting on the entire chain is minimized is catenary. In other words, a catenary can be derived from the equilibrium of forces acting on the line segment of the chain. It can be confirmed by observation that the shape of the hanging chain is in accordance with a catenary. Thus, the hanging chain is a good pedagogic example that demonstrates the integration of experiment and theory. Let us study the method to calculate the length of slack. Initially, this seems to be a simple calculation but it is not so. Given the length and distance of both ends of a hanging chain, the length of slack is given by solving transcendental equations, which will be explained later.

In this paper, it is shown that the slack is expressed by a Lambert $W$ function or an irrational function with respect to the distance between both ends of the chain under study. To compare the theoretical results and experimental data, a demonstration observing the slack position has been proposed. Interestingly, the Lambert $W$ function appears in the demonstration experiment of the hanging chain. In mathematics, the Lambert $W$ function appears in combinational theory, delay time differential equations, and iterated exponentials, among others. This function is also widely used to solve problems in physics.

The rest of the paper is organized as follows. Section II briefly outlines the definition of the Lambert $W$ function. Section III describes the physics of the hanging chain and determines the equations for deriving the slack. The horizontal tension of the chain and the asymptotic solutions of the equations are also provided in this section. Section IV compares the theoretical results and experimental data and introduces the demonstration experiment for measuring the slack, and the paper is concluded in Section V.
II. BRIEF REVIEW OF THE LAMBERT $W$ FUNCTION

For $x \in \mathbb{R}$, the Lambert $W$ function is defined by

$$W(x)e^{W(x)} = x. \quad (1)$$

The plot of $W(x)$ is shown in Fig. 1. For $-e^{-1} \leq x < 0$, $W(x)$ has two branches. Then the branches satisfying $W(x) \geq -1$ and $W(x) \leq -1$ are denoted by $W_0(x)$ and $W_{-1}(x)$, respectively. For $x \in \mathbb{C}$, $W_k(x)$ is a multi-branched function labelled by an integer $k$.

The Lambert $W$ function has been widely applied in mathematics, computer science, and engineering\textsuperscript{2,3}. It has also been used in numerical simulation software, such as Maple and Mathematica. Furthermore, the Lambert $W$ function is often used to solve problems in physics; it has applications in general relativity\textsuperscript{4}, quantum mechanics\textsuperscript{4,5}, falling motion of a massive object with air resistance\textsuperscript{6}, calculating the velocity of solar wind\textsuperscript{7}, the transient phenomenon in an electric circuit\textsuperscript{8}, and gravity discharge vessels\textsuperscript{9}, among others. The transcendental equation is useful for solving various problems in physics. The closed-form analytical expression for the solution of the transcendental equation can be obtained by the Lambert $W$ function. If the transcendental equation is given by

$$a^x = x + b, \quad (2)$$

definition (1) leads to the solution $x$:

$$x = -b - \frac{1}{\ln a} W \left( -a^{-b} \ln a \right). \quad (3)$$

III. THEORY OF STATIC HANGING CHAIN

Let us reconsider the shape of static hanging chain in a uniform gravitational field. Suppose that the infinitesimal line segment of the chain is labelled by $s$, where the arc length is measured from the origin, as shown in Fig. 2. Assume that the chain is composed of a material with uniform density. Let $\rho$, $T(s)$, and $\theta(s)$ be the chain’s mass per unit length, tension, and tangential angle of the infinitesimal line segment $s$, respectively. Thus, Newton’s law gives

$$\frac{d}{ds} \left( T(s) \cos \theta(s) \right) = 0, \quad (4)$$

$$\frac{d}{ds} \left( T(s) \sin \theta(s) \right) = \rho g. \quad (5)$$
where $g$ is the gravitational acceleration. Eq. (4) leads to

$$T \cos \theta = T_h,$$

(6)

where $T_h$ denotes the constant horizontal tension. Substituting Eq. (6) into Eq. (5), the equation determining the shape of the chain is given by

$$\frac{d}{ds} \left( \frac{dy}{dx} \right) = \frac{\rho g}{T_h},$$

(7)

where $dy/dx = \tan \theta$. By defining the dimensionless quantity

$$\beta = \frac{\rho g d}{2T_h},$$

(8)

the integration of Eq. (7) leads to the shape of the hanging chain, imposed by the boundary condition of Fig. 2,

$$y(x) = \frac{d}{2\beta} \left\{ \cosh \left( \frac{2x - d}{d} \beta \right) - \cosh \beta \right\}.$$

(9)

The curve is a so-called catenary. The length $l$ of the chain is computed as

$$l = \int_0^d \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = \frac{d}{\beta} \sinh \beta.$$

(10)

The ratio of $d$ and $l$ is

$$\frac{d}{l} = \frac{\beta}{\sinh \beta},$$

(11)

and Eq. (9) leads to the ratio $h$ and $d$:

$$\frac{h}{d} = \frac{1}{\beta} \sinh^2 \left( \frac{\beta}{2} \right).$$

(12)

From Eqs. (11) and (12), the ratio of $h$ and $l$ is given by

$$\frac{h}{l} = \frac{1}{2} \tanh \left( \frac{\beta}{2} \right).$$

(13)

As shown in Eq. (13), the value of slack $h$ can be calculated if the value of $\beta$ is known. However, the aim of this study was to calculate the value of slack $h$ without knowing $\beta$ (the value of $\rho$ and $T_h$). Since the slack $h$ is experimentally measured while varying $d$, $h$ should be expressed as function of $d$.

The relation between $d/l$ and $h/l$ is obtained by eliminating $\beta$ from Eqs. (11) and (13). Eq. (11) is a transcendental equation with respect to $\beta$. Accordingly, Eq. (13) cannot yield
the analytical expression as \( h = h(d) \). However, the ratio \( d/l \) can be expressed as a function of \( h/l \) by substituting Eq. (13) into Eq. (11):

\[
\frac{d}{l} = 1 - \frac{(2h/l)^2}{2h/l} \arctanh \left( \frac{2h}{l} \right). \tag{14}
\]

Eq. (14) is an implicit equation for \( h \) and would be useless in the demonstration experiment. To observe the explicit behavior of \( h \) for \( d \), the asymptotic expression of Eq. (14) is derived with respect to the dimensionless \( \beta \), as defined in Eq. (8).

When \( \beta \ll 1 (\rho gd \ll T_h) \), i.e. the chain is almost straightly stretched, Eqs. (11) and (13) are expanded in power series of \( \beta \)

\[
\frac{d}{l} \sim 1 - \frac{\beta^2}{6}, \tag{15}
\]

\[
\frac{h}{l} \sim \frac{\beta}{4}. \tag{16}
\]

Eliminating \( \beta \), the dimensionless slack \( h/l \) is expressed as an irrational function with respect to the dimensionless distance \( d/l \):

\[
\frac{h}{l} \sim \sqrt{\frac{3}{8} \left( 1 - \frac{d}{l} \right)}. \tag{17}
\]

Furthermore, Eq. (15) yields the ratio of the horizontal tension \( T_h \) and mass \( m \) of the chain as a function of \( d/l \)

\[
\frac{T_h}{mg} \sim \frac{d}{l} \frac{\sqrt{d/l}}{\sqrt{24 \left( 1 - \frac{d}{l} \right)}}, \tag{18}
\]

where \( m = \rho l \). Taking the limit of \( d \to l \), the horizontal tension goes to infinity; this indicates that the chain can be straightly stretched with arbitrarily large tension, unless the chain breaks.

When \( \beta \gg 1 (\rho gd \gg T_h) \), i.e. the chain is loosely stretched, Eq. (11) leads to

\[
\frac{d}{l} \sim 2\beta e^{-\beta}. \tag{19}
\]

From Eq. (1) and (19), \( \beta \) is expressed in terms of the Lambert \( W \) function as

\[
\beta \sim -W_{-1} \left( -\frac{d}{2l} \right), \tag{20}
\]
where $W_{-1}$ is selected because $\beta \to \infty$ as $T_h \to 0$. Substituting Eq. (20) into Eq. (13), the dimensionless slack can be expressed as a function of the dimensionless distance

$$\frac{h}{l} \sim \frac{1}{2} \tanh \left( -\frac{1}{2} W_{-1} \left( -\frac{d}{2l} \right) \right),$$

(21)

and the horizontal tension is given by

$$\frac{T_h}{mg} = \exp \left( W_{-1} \left( -\frac{d}{2l} \right) \right).$$

(22)

Consequently, given the length $l$, the relation between $h$ and $d$ corresponds to Eq. (17) and Eq. (21) for $\beta \gg 1$ and $\beta \ll 1$, respectively.

IV. MEASUREMENT OF THE HANGING CHAIN AND ITS MATHEMATICAL ASPECT

By measuring the slack $h$ while varying the distance $d$ of both ends of the hanging chain, the measurement graph of $d$ and $h$ and the theoretical graph can be compared.

To perform the hanging chain demonstration in a classroom, a simple experiment was prepared, as shown in Fig. 3. One end of a chain was fixed, while the other end was left free to move on the rail; graph paper was placed behind the hanging chain. Since the position of the movable end was $d$, the slack $h$ of the chain could be observed by measuring the position of the middle point of chain. The chain used in the demonstration was an aluminum mantel chain of length $l = 40$ cm. In Fig. 4, the dots are the measurement data (dimensionless $d/l$ and $h/l$). The theoretical curve of Eq. (14) was drawn using Mathematica. Obviously, the theoretical curve agrees with the measurement data. The asymptotic equations, Eqs. (17) and (21), are more suitable than the implicit equation, Eq. (14), when teaching the mathematical physics aspect of the hanging chain in a classroom.

The measurement data and graphs of Eqs. (17) and (21) are shown in Fig. 5. At this stage, the relative difference between the theoretical value and measurement data is taken to be less than approximately 1%. As a result, Eq. (17) and Eq. (21) are valid for $0.76 < d/l \leq 1$ and $0 \leq h/l < 0.63$, respectively. Consequently, the dimensionless slack is
asymptotically given by

\[
\frac{h}{l} = \begin{cases} 
\frac{1}{2} \tanh \left( -\frac{1}{2} W_{-1} \left( -\frac{d}{2l} \right) \right) & 0 \leq \frac{d}{l} < 0.63 \\
\sqrt{\frac{3}{8} \left( 1 - \frac{d}{l} \right)} & 0.76 < \frac{d}{l} \leq 1
\end{cases}
\]  

(23)

Using Eq. (23), a simple demonstration of the mathematical aspect of the hanging chain can be performed. In the experiment of Fig. 3, the movable point was initially placed at the left edge, and then the point was slowly pulled toward the right edge. Then, the trajectory of the middle point of the chain could be observed. On the contrary, the trajectory corresponds to the plot of \( \left( \frac{d}{2}, h \right) \) of Eq. (23). As shown in Fig. 6, by sticking the plots of Eq. (23) (drawn by Mathematica) behind the chain, it can be confirmed that the trajectory of the middle point is in agreement with the theoretical plots. In a physics class demonstration, students can see that the actual trajectory is in agreement with the theoretical curve predicted by Eq. (23).

V. CONCLUSIONS

Herin, it was shown that the problem of calculating the slack of a hanging chain has great mathematical relevance in classical mechanics. Given the length \( l \) of the chain, the slack \( h \) and distance \( d \) can be determined by Eqs. (11) and (13) for all the values of \( \beta \), which depends on the horizontal tension of the chain. The asymptotic solutions for the two equations are found to derive \( h \) as a function of \( d \). When stretching the chain almost straightly, as shown in Eq. (17), the slack is expressed by an irrational function with respect to the distance. On the contrary, as shown in Eq. (21), when stretching the chain loosely, the slack is expressed in terms of the Lambert \( W \) function, which is applied to the solution of the transcendental equation. To determine the range, where the asymptotic behaviors are valid, the slack was measured via the experiment shown in Fig. 3. The results are given by Eq. (23).

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FIG. 1. Plot of the Lambert $W(x)$ function for $x \in \mathbb{R}$. The two branches, $W_0$ and $W_{-1}$, are shown.

FIG. 2. Configuration of the static hanging chain. The positions of both ends are at the same height. The slack $h$ of the chain is depicted as above.
FIG. 3. Experiment to measure the slack $h$ while varying the distance between both ends of the chain.

FIG. 4. For the graph of dimensionless $\frac{h}{l}$ and $\frac{d}{l}$, the dots are the measurement data, and the solid curve is Eq. (14).
FIG. 5. For the graph of dimensionless $\frac{h}{l}$ and $\frac{d}{l}$, the broken curve is Eq. (17), and the solid curve is Eq. (21). The two vertical broken lines correspond to $\frac{d}{l} = 0.63, 0.76$.

FIG. 6. The photographs of the demonstration observing the middle point while pulling the movable point of the chain from left to right. The length of the chain is $l = 40$ cm. The two solid curves correspond to the plot of $\left(\frac{d}{2}, h\right)$ in Eq. (23).