Replica Bethe ansatz derivation of the Tracy–Widom distribution of the free energy fluctuations in one-dimensional directed polymers

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Received 28 April 2010
Accepted 16 June 2010
Published 9 July 2010

Online at stacks.iop.org/JSTAT/2010/P07010
doi:10.1088/1742-5468/2010/07/P07010

Abstract. The distribution function of the free energy fluctuations in one-dimensional directed polymers with a δ-correlated random potential is studied by mapping the replicated problem to the N-particle quantum boson system with attractive interactions. We find the full set of eigenfunctions and eigenvalues of this many-body system and perform the summation over the entire spectrum of excited states. It is shown that in the thermodynamic limit the problem is reduced to the Fredholm determinant with the Airy kernel yielding the universal Tracy–Widom distribution, which is known to describe the statistical properties of the Gaussian unitary ensemble as well as many other statistical systems.

Keywords: cavity and replica method, disordered systems (theory)
1. Introduction

Directed polymers in a quenched random potential have been the subject of intense investigations during the past two decades (see e.g. [1]). Diverse physical systems such as domain walls in magnetic films [2], vortices in superconductors [3], wetting fronts on planar systems [4], or Burgers turbulence [5] can be mapped to this model, which exhibits numerous non-trivial features deriving from the interplay between elasticity and disorder. In the most simple one-dimensional case we deal with an elastic string directed along the \( \tau \)-axis within an interval \([0, L]\). Randomness enters the problem through a disorder potential \( V[\phi(\tau), \tau] \), which competes against the elastic energy. The problem is defined by the Hamiltonian

\[
H[\phi(\tau), V] = \int_0^L d\tau \left\{ \frac{1}{2} [\partial_\tau \phi(\tau)]^2 + V[\phi(\tau), \tau] \right\};
\]

where in the simplest case the disorder potential \( V[\phi, \tau] \) is Gaussian distributed with a zero mean \( \overline{V(\phi, \tau)} = 0 \) and the \( \delta \)-correlations:

\[
\overline{V(\phi, \tau)V(\phi', \tau')} = u\delta(\tau - \tau')\delta(\phi - \phi').
\]

Here the parameter \( u \) describes the strength of the disorder. Historically, the problem of central interest was the scaling behavior of the polymer mean squared displacement, which in the thermodynamic limit (\( L \to \infty \)) is believed to have a universal scaling form.

doi:10.1088/1742-5468/2010/07/P07010
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\[ \langle \phi^2 \rangle (L) \propto L^{2\zeta} \] (where \( \langle \cdots \rangle \) and \( \langle \cdot \cdot \cdot \rangle \) denote the thermal and the disorder averages), with \( \zeta = 2/3 \), the so-called wandering exponent. A more general problem for all directed polymer systems of the type, equation (1), is the statistical properties of their free energy fluctuations. Besides the usual extensive (linear in \( L \)) self-averaging part \( f_0 L \) (where \( f_0 \) is the linear free energy density), the total free energy \( F \) of such systems contains a disorder dependent fluctuating contribution \( \tilde{F} \), which is characterized by non-trivial scaling in \( L \). It is generally believed that in the limit of large \( L \) the typical value of the free energy fluctuations scales with \( L \) as \( \tilde{F} \propto L^{1/3} \) (see e.g. [6]–[9]). In other words, in the limit of large \( L \) the total (random) free energy of the system can be represented as

\[ F = f_0 L + c L^{1/3} f \] (3)

where \( c \) is a parameter which depends on the temperature and the strength of disorder, and \( f \) is a random quantity which in the thermodynamic limit \( L \to \infty \) is described by a non-trivial universal distribution function \( P_\ast(f) \). The derivation of this function for the system with a \( \delta \)-correlated random potential, equations (1) and (2) is the central issue of the present work.

For the string with the zero boundary conditions at \( \tau = 0 \) and at \( \tau = L \) the partition function of a given sample is

\[ Z[V] = \int_{\phi(0)=0}^{\phi(L)=0} D[\phi(\tau)] e^{-\beta H[\phi,V]} \] (4)

where \( \beta \) denotes the inverse temperature. On the other hand, the partition function is related to the total free energy \( F[V] \) via

\[ Z[V] = \exp(-\beta F[V]). \] (5)

The free energy \( F[V] \) is defined for a specific realization of the random potential \( V \) and thus represents a random variable. Taking the \( N \)th power of both sides of this relation and performing the averaging over the random potential \( V \) we obtain

\[ Z^N[V] = Z[N, L] = \exp(-\beta NF[V]) \] (6)

where the quantity in the lhs of the above equation is called the replica partition function. Substituting here, \( F = f_0 L + c L^{1/3} f \), and redefining

\[ Z[N, L] = \tilde{Z}[N, L] e^{-\beta N f_0 L} \] (7)

we get

\[ \tilde{Z}[N, L] = \exp(-\lambda N f) \] (8)

where \( \lambda = \beta c L^{1/3} \). The averaging in the rhs of the above equation can be represented in terms of the distribution function \( P_L(f) \) (which depends on the system size \( L \)). In this way we arrive to the following general relation between the replica partition function \( \tilde{Z}[N, L] \) and the distribution function of the free energy fluctuations \( P_L(f) \):

\[ \tilde{Z}[N, L] = \int_{-\infty}^{+\infty} df P_L(f) e^{-\lambda N f}. \] (9)

Of course, the most interesting object is the thermodynamic limit distribution function \( P_\ast(f) = \lim_{L \to \infty} P_L(f) \), which is expected to be a universal quantity. The above equation
is the bilateral Laplace transform of the function $P_L(f)$, and at least formally it allows one to restore this function via the inverse Laplace transform of the replica partition function $\tilde{Z}[N,L]$. In order to do so one has to compute $\tilde{Z}[N,L]$ for an arbitrary integer $N$ and then perform analytical continuation of this function from integer to arbitrary complex values of $N$. In Kardar’s original solution [9], after mapping the replicated problem to interacting quantum bosons, one arrives at the replica partition function for positive integer parameters $N > 1$. Assuming a large $L \to \infty$ limit, one is tempted to approximate the result by the ground state contribution only, as for any $N > 1$ the contributions of excited states are exponentially small for $L \to \infty$. However, in the analytic continuation for arbitrary complex $N$ the contributions, which are exponentially small at positive integer $N > 1$, can become essential in the region $N \to 0$, which defines the function $P(f)$ (in other word, the problem is that the two limits $L \to \infty$ and $N \to 0$ do not commute [10, 11]). As a consequence, the resulting distribution function $P_*(f)$ exhibits severe pathologies, such as the vanishing of its second moment, which assumes that the distribution function is not positively defined. Nevertheless, in terms of this approximation (assuming the universal scaling $L^{1/3}$ of the free energy fluctuations) one can derive the left tail of the distribution function $P_*(f)$, which is given by the asymptotics of the Airy function, $P_*(f \to -\infty) \sim \exp(-\frac{2}{3}|f|^{3/2})$ [12]. For the first time, the form of the right tail of this distribution function, $P_*(f \to +\infty) \sim \exp[-(\text{const})f^3]$, has been derived in terms of the optimal fluctuation approach [13], where it has been demonstrated that both asymptotics (left and right) of the function $P_*(f)$ are consistent with the Tracy–Widom (TW) distribution [14]. Originally, the TW distribution was derived in the context of the statistical properties of the Gaussian Unitary Ensemble, while at present it is well established to describe the statistics of fluctuations in various random systems [15]–[20], which were widely believed to belong to the same universality class as the present model [13, 21, 22]. Finally, quite recently, the one-point distribution of the solutions of the KPZ-equation (which is equivalent to the present model) has been derived for an arbitrary value of $L$, and it has been shown that in the limit $L \to \infty$ this distribution turns into the Tracy–Widom distribution [23, 24].

In a recent paper [25] (which appeared on the web archive in January 2010), in terms of the replica approach, an attempt has been made to derive the free energy distribution function $P_*(f)$ via the calculation of the replica partition function $Z[N,L]$ in terms of the Bethe ansatz solution for quantum bosons with attractive $\delta$-interactions which involved the summation over the entire spectrum of excited states. The paper contains a detailed description of the methodological and technical aspects of the approach, but unfortunately the final result obtained for the function $P_*(f)$ turned out to be wrong (it was not the TW distribution). The mistake was hidden in the treatment of the pre-exponential multiple paired products of the replica partition function (see equation (32) below) and in the subsequent summations of formally divergent series. Besides this, further analytic continuation of the obtained replica partition function $Z(N,L)$ was found to be ambiguous. In parallel, an independent attempt at the solution of the same problem by Calabrese, Le Doussal and Rosso appeared on the web archive in February 2010. Here the result for the distribution function $P_*(f)$ was different from the one of [25], but still it was not the TW distribution (the trouble here was again due to another hidden mistake in the treatment of the same multiple paired products). Finally, by the end of March 2010, these two independent lines of research had converged to the same correct solution of the problem.
The central point of this solution was that the multiple paired products of the replica partition function was possible to express in a sufficiently simple determinant form. Brief reports of the replica derivation of the TW distribution for the function $P_a(f)$ appeared on the web archive within an interval of three days, and at present these two papers are published in EPL [26, 27].

The motivation for writing the present paper is to present the detailed and systematic derivation (from the very beginning until the very end) of the TW distribution in 1D directed polymers by the replica method within one self-contained article. Section 2 is devoted to the reformulation of the considered problem in terms of a one-dimensional $N$-particle system of quantum bosons with attractive $\delta$-interactions. In section 3 it is shown how one can bypass the problem of the analytic continuation of the replica partition function $Z(N,L)$ to noninteger values of $N$ by introducing an integral representation of the distribution function $P^*_a(f)$ which involves the summation over integer $N$'s. Here the replica partition function is calculated by summing over the entire spectrum of excited states of the corresponding $N$-particle bosonic system, and it is shown that in the thermodynamic limit the problem is reduced to the Fredholm determinant with the Airy kernel yielding the universal TW distribution. Finally, appendices A and B contain detailed analysis of the properties of $N$-particle wavefunctions of quantum bosons with both repulsive and attractive $\delta$-interactions. Furthermore, appendix C explains in simple terms the connection between the Fredholm determinant with the Airy kernel and the explicit form of the TW distribution defined in terms of the solution of the Painlevé II differential equation.

2. Mapping to quantum bosons

Explicitly, the replica partition function, equation (6), of the system described by the Hamiltonian, equation (1), is

$$Z(N,L) = \prod_{a=1}^{N} \int_{\phi_a(0)=0}^{\phi_a(L)=0} \mathcal{D}\phi_a(\tau) \exp \left[ -\frac{1}{2} \int_0^L d\tau \sum_{a=1}^N \left\{ \frac{1}{2} [\partial_\tau \phi_a(\tau)]^2 + V[\phi_a(\tau),\tau] \right\} \right].$$

(10)

Since the random potential $V[\phi,\tau]$ has a Gaussian distribution the disorder averaging $\langle \cdots \rangle$ in the above equation is very simple:

$$\exp \left[ -\beta \int_0^L d\tau \sum_{a=1}^N V[\phi_a(\tau),\tau] \right] = \exp \left[ \frac{\beta^2}{2} \int_0^L d\tau \int_0^L d\tau' \sum_{a,b=1}^N V[\phi_a(\tau),\tau] V[\phi_b(\tau'),\tau'] \right].$$

(11)

Using equation (2) we find:

$$Z(N,L) = \prod_{a=1}^{N} \int_{\phi_a(0)=0}^{\phi_a(L)=0} \mathcal{D}\phi_a(\tau)$$

$$\times \exp \left[ -\frac{1}{2} \beta \int_0^L d\tau \left\{ \sum_{a=1}^N [\partial_\tau \phi_a(\tau)]^2 - \beta u \sum_{a,b=1}^N \delta[\phi_a(\tau) - \phi_b(\tau)] \right\} \right].$$

(12)
It should be noted that the second term in the exponential of the above equation contains formally divergent contributions proportional to $\delta(0)$ (due to the terms with $a = b$). In fact, this is just an indication that the continuous model, equations (1) and (2), is ill defined as short distances and requires proper lattice regularization. Of course, the corresponding lattice model would contain no divergences, and the terms with $a = b$ in the exponential of the corresponding replica partition function would produce an irrelevant constant $\frac{1}{2} L^2 u N \delta(0)$ (where the lattice version of $\delta(0)$ has a finite value). Since the lattice regularization has no impact on the continuous long distance properties of the considered system this term will just be omitted in our further study.

Introducing the $N$-component scalar field replica Hamiltonian

$$H_N[\phi] = \frac{1}{2} \int_0^L d\tau \left( \sum_{a=1}^N \left[ \partial_\tau \phi_a(\tau) \right]^2 - \beta u \sum_{a \neq b}^N \delta[\phi_a(\tau) - \phi_b(\tau)] \right)$$

for the replica partition function, equation (12), we obtain the standard expression

$$Z(N, L) = \prod_{a=1}^N \int_{\phi_a(0) = 0}^{\phi_a(L) = 0} D\phi_a(\tau) e^{-\beta H_N[\phi]}$$

where $\phi \equiv \{\phi_1, \ldots, \phi_N\}$. According to the above definition this partition function describes the statistics of $N\delta$-interacting (attracting) trajectories $\phi_a(\tau)$ all starting (at $\tau = 0$) and ending (at $\tau = L$) at zero: $\phi_a(0) = \phi_a(L) = 0$.

In order to map the problem to one-dimensional quantum bosons, let us introduce a more general object

$$\Psi(x; t) = \prod_{a=1}^N \int_{\phi_a(0) = 0}^{\phi_a(t) = x_a} D\phi_a(\tau) e^{-\beta H_N[\phi]}$$

which describes $N$ trajectories $\phi_a(\tau)$ all starting at zero ($\phi_a(0) = 0$), but ending at $\tau = t$ in arbitrary given points $\{x_1, \ldots, x_N\}$. One can easily show that instead of using the path integral, $\Psi(x; t)$ may be obtained as the solution of the linear differential equation

$$\partial_t \Psi(x; t) = \frac{1}{2\beta} \sum_{a=1}^N \partial_{x_a}^2 \Psi(x; t) + \frac{1}{2} \beta^2 u \sum_{a \neq b}^N \delta(x_a - x_b) \Psi(x; t)$$

with the initial condition

$$\Psi(x; 0) = \Pi_{a=1}^N \delta(x_a).$$

One can easily see that equation (16) is the imaginary-time Schrödinger equation

$$-\partial_t \Psi(x; t) = \hat{H} \Psi(x; t)$$

with the Hamiltonian

$$\hat{H} = -\frac{1}{2\beta} \sum_{a=1}^N \partial_{x_a}^2 - \frac{1}{2} \beta^2 u \sum_{a \neq b}^N \delta(x_a - x_b)$$

which describes $N$ Bose-particles of mass $\beta$ interacting via the attractive two-body potential $-\beta^2 u \delta(x)$. The original replica partition function, equation (14), is then
obtained via a particular choice of the final-point coordinates,

\[ Z(N, L) = \Psi(0; L). \]  

(20)

The eigenfunctions of the \( N \)-particle system of repulsive bosons \((u < 0)\), equation (19), have been derived by Lieb and Liniger in 1963 [28] (for details see appendix A, as well as [29,30]). The spectrum and some properties of the eigenfunctions for an attractive \((u > 0)\) one-dimensional quantum boson system have been derived by McGuire [31] and by Yang [32] (see also [33,34]). The detailed structure and properties of these wavefunctions are described in appendix B. A generic time dependent solution \( \Psi(x, t) \) of the Schrödinger equation (16) with the initial conditions, equation (17), can be represented in the form of a linear combination of the eigenfunctions \( \Psi^{(M)}_{q,n}(x) \) (appendix B.4, equation (B.51)):

\[ \Psi(x, t) = \sum_{M=1}^{N} \sum_{n} \int \mathcal{D}q \Psi^{(M)}_{q,n}(x) \Psi^{(M)\ast}_{q,n}(0) \exp[-E_{M}(q, n)t] \]  

(23)

where

\[ E_{M}(q, n) = \frac{1}{2\beta} \sum_{\alpha=1}^{M} n_{\alpha}q_{\alpha}^{2} - \frac{\kappa^{2}}{24\beta} \sum_{\alpha=1}^{M} (n_{\alpha}^{3} - n_{\alpha}) \]  

(21)

where

\[ \kappa = \beta^{3}u. \]  

(22)

A general time dependent solution \( \Psi(x, t) \) of the Schrödinger equation (16) with the initial conditions, equation (17), can be represented in the form of a linear combination of the eigenfunctions \( \Psi^{(M)}_{q,n}(x) \) (appendix B.4, equation (B.51)):

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(21)

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(23)

where

\[ E_{M}(q, n) = \frac{1}{2\beta} \sum_{\alpha=1}^{M} n_{\alpha}q_{\alpha}^{2} - \frac{\kappa^{2}}{24\beta} \sum_{\alpha=1}^{M} (n_{\alpha}^{3} - n_{\alpha}) \]  

(21)

where

\[ \kappa = \beta^{3}u. \]  

(22)
3. Replica partition function and the free energy distribution function

To bypass the problem of the analytic continuation in the replica parameter \( N \) to noninteger values, instead of the free energy distribution function \( P_*(f) \) (introduced in equations (8) and (9)) one can study its integral representation

\[
W(x) = \int_x^\infty df P_*(f) \tag{25}
\]

which gives the probability to find a fluctuation \( f \) greater than a given value \( x \). Indeed, formally, the function \( W(x) \) can be defined as follows:

\[
W(x) = \lim_{\lambda \to \infty} \sum_{N=0}^{\infty} \frac{(-1)^N}{N!} \exp(\lambda Nx) \tilde{Z}^N = \lim_{\lambda \to \infty} \exp[-\exp(\lambda(x-f))] = \theta(f-x). \tag{26}
\]

On the other hand, the probability function \( W(x) \) can be computed in terms of the replica partition function \( \tilde{Z}[N,L] \), equation (8), by summing over all replica integers

\[
W(x) = \lim_{\lambda \to \infty} \sum_{N=0}^{\infty} \frac{(-1)^N}{N!} \exp(\lambda Nx) \tilde{Z}[N,L]. \tag{27}
\]

Using the explicit form of the wavefunctions \( \Psi_{q,n}^{(M)}(x) \), equation (B.28), the expression in equation (24) for the replica partition function can be reduced to (appendix B.4, equations (B.62) and (B.63))

\[
Z(N, L) = e^{-\beta NLf_0} \tilde{Z}(N, \lambda) \tag{28}
\]

where \( f_0 = \frac{1}{24} \beta^4 u^2 - (1/\beta L) \ln(\beta u) \) is the linear (self-averaging) free energy density (cf equation (7)), and

\[
\tilde{Z}(N, L) = N! \left\{ \int_{-\infty}^{+\infty} \frac{dq}{2\pi \kappa N} \exp \left[ -\frac{L}{\beta^2} Nq^2 + \frac{\kappa^2 L}{24\beta} N^3 \right] 
+ \sum_{M=2}^{N} \frac{1}{M!} \prod_{\alpha=1}^{M} \sum_{n_{\alpha}=1}^{\infty} \int_{-\infty}^{+\infty} \frac{dq_{\alpha}}{2\pi \kappa n_{\alpha}} \delta \left( \sum_{\alpha=1}^{M} n_{\alpha}, N \right) 
\times \prod_{\alpha<\beta} \frac{q_{\alpha} - q_{\beta} - (i\kappa/2)(n_{\alpha} - n_{\beta})^2}{q_{\alpha} - q_{\beta} - (i\kappa/2)(n_{\alpha} + n_{\beta})^2} 
\times \exp \left[ -\frac{L}{\beta^2} \sum_{\alpha=1}^{M} n_{\alpha} q_{\alpha}^2 + \frac{\kappa^2 L}{24\beta} \sum_{\alpha=1}^{M} n_{\alpha}^3 \right] \right\}. \tag{29}
\]

The first term in the above expression is the contribution of the ground state \((M=1)\), while the next terms \((M \geq 2)\) are the contributions of the rest of the energy spectrum.

The terms cubic in \( n_{\alpha} \) in the exponential of equation (29) can be linearized with the help of the Airy function, using the standard relation

\[
\exp\left( \frac{1}{3} \lambda^3 n^3 \right) = \int_{-\infty}^{+\infty} dy \text{Ai}(y) \exp(\lambda n). \tag{30}
\]
Using the exponential representation of this determinant we get
\[
Z \equiv \frac{1}{2} \left( \frac{L}{\beta} \right)^{1/3} = \frac{1}{2} (\beta^3 u^2 L)^{1/3}
\] (31)
after shifting the Airy function parameters of integration \( y_a \to y_a + p_a^2 \) the expression for \( Z(N, \lambda) \) becomes sufficiently compact:
\[
\dot{Z}(N, \lambda) = N! \int \int_{-\infty}^{+\infty} \frac{dy \, dp}{4\pi \lambda N} \text{Ai}(y + p^2) e^{\lambda N y}
+ N! \sum_{M=2}^{N} \frac{1}{M!} \left[ \prod_{\alpha=1}^{M} \sum_{n_n=1}^{\infty} \int_{-\infty}^{+\infty} \frac{dy_n \, dp_n}{4\pi \lambda N^2} \text{Ai}(y_n + p_n^2) e^{\lambda N n_n y_n} \right] \\
\times \prod_{\alpha<\beta}^{M} \frac{\lambda(n_a - n_\beta) - i(p_a - p_\beta))^2}{\lambda(n_a + n_\beta) - i(p_a - p_\beta))^2} \delta \left( \sum_{\alpha=1}^{M} n_n, N \right).
\] (32)
Now, using the Cauchy double alternant identity
\[
\frac{\prod_{\alpha<\beta}^{M} (a_{\alpha} - a_{\beta})(b_{\alpha} - b_{\beta})}{\prod_{\alpha,\beta=1}^{M} (a_{\alpha} - b_{\beta})} = (-1)^{M(M-1)/2} \det \left[ \frac{1}{a_{\alpha} - b_{\beta}} \right]_{\alpha,\beta=1,...,M}
\] (33)
the product term in equation (32) can be represented in the determinant form:
\[
\prod_{\alpha<\beta}^{M} \frac{\lambda(n_{\alpha} - n_\beta) - i(p_{a} - p_\beta))^2}{\lambda(n_{\alpha} + n_\beta) - i(p_{a} - p_\beta))^2} = \left[ \prod_{\alpha=1}^{M} (2\lambda n_\alpha) \right] \det \left[ \frac{1}{\lambda n_\alpha - ip_\alpha + \lambda n_\beta + ip_\beta} \right]_{\alpha,\beta=1,...,M}
\] (34)
Substituting now the expression for the replica partition function \( \dot{Z}(N, \lambda) \) into the definition of the probability function, equation (27), we can perform summation over \( N \) (which would lift the constraint \( \sum_{\alpha=1}^{M} n_\alpha = N \)) and obtain:
\[
W(x) = \lim_{\lambda \to \infty} \left\{ 1 + \sum_{M=1}^{\infty} \frac{(-1)^{M}}{M!} \int_{-\infty}^{+\infty} \frac{dy \, dp}{2\pi} \text{Ai}(y + p^2) \sum_{n_n=1}^{\infty} (-1)^{n_n-1} e^{\lambda n_n (y + x)} \right\} \\
\times \det \left[ \frac{1}{\lambda n_\alpha - ip_\alpha + \lambda n_\beta + ip_\beta} \right].
\] (35)
The above expression is nothing else but the expansion of the Fredholm determinant
\[
\det(1 - \hat{K}) \quad \text{(see e.g. [35])}
\] with the kernel
\[
\hat{K} \equiv K[(n,p);(n',p')] = \left[ \int_{-\infty}^{+\infty} dy \, \text{Ai}(y + p^2) (-1)^{n-1} e^{\lambda n (y + x)} \right] \frac{1}{\lambda n - ip + \lambda n' + ip'}.
\] (36)
Using the exponential representation of this determinant we get
\[
W(x) = \lim_{\lambda \to \infty} \exp \left[ - \sum_{M=1}^{\infty} \frac{1}{M} \text{Tr} \hat{K}^M \right]
\] (37)
The function \( F \) and shifting the integration parameters, \( L \) is the so-called Airy kernel. This proves that in the thermodynamic limit, where by definition it is assumed that \( \omega \to 0 \) and \( q \to +\infty \) taking the thermodynamic limit, \( L \to \infty \) (which according to equation (31) is equivalent to \( \lambda \to \infty \) ) we obtain

\[
\lim_{\lambda \to \infty} \sum_{n=1}^{\infty} (-1)^{n-1} e^{\lambda n z} = \lim_{\lambda \to \infty} e^{\lambda z} = \theta(z) \tag{40}
\]

and shifting the integration parameters, \( y_\alpha \to y_\alpha - x + \omega_\alpha + \omega_\alpha^{-1} \) and \( \omega_\alpha \to \omega_\alpha + x/2 \), after taking the thermodynamic limit, \( L \to \infty \) (which according to equation (31) is equivalent to \( \lambda \to \infty \) ) we obtain

\[
\lim_{\lambda \to \infty} \frac{1}{\lambda n_\alpha - i p_\alpha + \lambda n_{\alpha+1} + i p_{\alpha+1}} = \int_0^{\infty} d\omega_\alpha \exp\left[-\left(\lambda n_\alpha - i p_\alpha + \lambda n_{\alpha+1} + i p_{\alpha+1}\right)\omega_\alpha\right] \tag{39}
\]

where by definition it is assumed that \( \omega_0 \equiv \omega_M \). Using the Airy function integral representation, and taking into account that it satisfies the differential equation, \( Ai'(t) = tAi(t) \), one can easily perform the following integrations:

\[
\int_0^{\infty} dy \int_{-\infty}^{+\infty} \frac{dp}{2\pi} Ai(y + p^2 + \omega + \omega')e^{ip(\omega - \omega')} = 2^{-1/3} \int_0^{\infty} dy Ai(2^{1/3} \omega + y)Ai(2^{1/3} \omega' + y)
\]

\[
= \frac{Ai(2^{1/3} \omega)Ai'(2^{1/3} \omega') - Ai'(2^{1/3} \omega)Ai(2^{1/3} \omega')}{\omega - \omega'} \tag{42}
\]

Redefining \( \omega_\alpha \to \omega_\alpha 2^{-1/3} \) we find

\[
\lim_{\lambda \to \infty} \frac{1}{\lambda n_\alpha - i p_\alpha + \lambda n_{\alpha+1} + i p_{\alpha+1}} = \int_0^{\infty} d\omega_\alpha \int_{-x/2^{1/3}}^{\infty} d\omega_2 \cdots d\omega_M K_A(\omega_1, \omega_2)K_A(\omega_2, \omega_3) \cdots K_A(\omega_M, \omega_1) \tag{43}
\]

where

\[
K_A(\omega, \omega') = \frac{Ai(\omega)Ai'(\omega') - Ai'(\omega)Ai(\omega')}{\omega - \omega'} \tag{44}
\]

is the so-called Airy kernel. This proves that in the thermodynamic limit, \( L \to \infty \), the probability function \( W(x) \), equation (25), is defined by the Fredholm determinant,

\[
W(x) = \det[1 - \hat{K}_A] \equiv F_2(-x/2^{2/3}) \tag{45}
\]

where \( \hat{K}_A \) is the integral operator on \([-x/2^{2/3}, \infty) \) with the Airy kernel, equation (44). The function \( F_2(s) \) is the Tracy–Widom distribution [14]

\[
F_2(s) = \exp\left(-\int_s^{\infty} dt (t - s)q^2(t) \right) \tag{46}
\]

where the function \( q(t) \) is the solution of the Painlevé II equation, \( q'' = t q + 2q^3 \) with the boundary condition, \( q(t \to +\infty) \sim Ai(t) \) (see appendix C).
4. Conclusions

The Tracy–Widom distribution, equation (46), was originally derived as the probability distribution of the largest eigenvalue of a $n \times n$ random Hermitian matrix in the limit $n \to \infty$. At present there exists an appreciable list of statistical systems (which do not always look similar) in which the fluctuations of the quantities which play the role of ‘energy’ are described by the same distribution function $F_2(s)$. These systems are: the polynuclear growth (PNG) model [15], the longest increasing subsequences (LIS) model [16], the longest common subsequences (LCS) model [17], the oriented digital boiling model [18], the ballistic decomposition model [19], and the zero-temperature lattice version of the directed polymers with an exponential and geometric site-disorder distribution [20]. Now we can add to this list the continuous one-dimensional directed polymers with Gaussian $\delta$-correlated random potential.

Acknowledgments

The main part of this work was done thanks to numerous illuminating discussions with Gianni Blatter. I am grateful to H Spohn, B Derrida, M Mezard, A Lebedev, S Nechaev, Vladimir Dotsenko, S Korshunov, and V Geshkenbein for fruitful discussions on various aspects of this work. I would also like to acknowledge financial support from the Center for Theoretical Studies at ETH Zurich and the Swiss National Foundation.

Appendix A. Quantum bosons with repulsive interactions

A.1. Eigenfunctions

The eigenstates equation for an $N$-particle system of one-dimensional quantum bosons with $\delta$-interactions is

$$\frac{1}{2} \sum_{a=1}^{N} \frac{d^2}{dx_a^2} \Psi(x) + \frac{i}{2} \kappa \sum_{a \neq b} \delta(x_a - x_b) \Psi(x) = -\beta E \Psi(x)$$

(A.1)

(where $\kappa = \beta^3 u$). Due to the symmetry of the wavefunction with respect to permutations of its arguments it is sufficient to consider it in the sector

$$x_1 < x_2 < \cdots < x_N$$

(A.2)

as well as at its boundary. Inside this sector the wavefunction $\Psi(x)$ satisfy the equation

$$\frac{1}{2} \sum_{a=1}^{N} \frac{d^2}{dx_a^2} \Psi(x) = -\beta E \Psi(x)$$

(A.3)

which describes $N$ free particles, and its generic solution is a linear combination of $N$ plane waves characterized by $N$ momenta $\{q_1, q_2, \ldots, q_N\} \equiv q$. Integrating equation (A.1) over the variable $(x_{i+1} - x_i)$ in a small interval around zero, $|x_{i+1} - x_i| < \epsilon \to 0$, and assuming that the other variables $\{x_j\}$ (with $j \neq i, i+1$) belong to the sector, equation (A.2), one easily finds that the wavefunction $\Psi(x)$ must satisfy the following boundary conditions:

$$(\partial_{x_{i+1}} - \partial_{x_i} + \kappa) \Psi(x)|_{x_{i+1} - x_{i+0} = 0} = 0.$$  

(A.4)
Functions satisfying both equation (A.3) and the boundary condition equation (A.4) can be written in the form

\[ \Psi_{q_1\ldots q_N}(x_1, \ldots, x_N) \equiv \Psi^{(N)}_q(x) = C \left( \prod_{a<b} [\partial_{x_a} - \partial_{x_b} + \kappa] \right) \det[\exp(i q_c x_d)]_{(c,d)=1,\ldots,N} \]  

(A.5)

where \( C \) is a normalization constant to be defined later. First of all, it is evident that being a linear combination of plane waves, the above wavefunction satisfies equation (A.3). To demonstrate in which way this function satisfies the boundary conditions, equation (A.4), let us check it, as an example, for the case \( i = 1 \). According to equation (A.5), the wavefunction \( \Psi^{(N)}_q(x) \) can be represented in the form

\[ \Psi^{(N)}_q(x) = -(\partial_{x_2} - \partial_{x_1} - \kappa) \tilde{\Psi}^{(N)}_q(x) \]  

(A.6)

where

\[ \tilde{\Psi}^{(N)}_q(x) = \left( \prod_{a=3}^N [\partial_{x_1} - \partial_{x_a} + \kappa][\partial_{x_2} - \partial_{x_a} + \kappa] \right) \left( \prod_{3 \leq a < b} [\partial_{x_a} - \partial_{x_b} + \kappa] \right) \times \det[\exp(i q_c x_d)]_{(c,d)=1,\ldots,N}. \]  

(A.7)

One can easily see that this function is \textit{antisymmetric} with respect to the permutation of \( x_1 \) and \( x_2 \). Substituting equation (A.6) into (A.4) (with \( i = 1 \)) we get

\[ -[(\partial_{x_2} - \partial_{x_1})^2 - \kappa^2] \tilde{\Psi}^{(N)}_q(x)|_{x_2=x_1} = 0. \]  

(A.8)

Given the antisymmetry of the lhs expression with respect to the permutation of \( x_1 \) and \( x_2 \) the above condition is indeed satisfied at the boundary \( x_1 = x_2 \).

Since the eigenfunction \( \Psi^{(N)}_q(x) \) satisfying equation (A.1) must be \textit{symmetric} with respect to permutations of its arguments, the function, equation (A.5), can be easily continued beyond the sector, equation (A.2), to the entire space of variables \( \{x_1, x_2, \ldots, x_N\} \in R_N \),

\[ \Psi^{(N)}_q(x) = \left( \prod_{a<b} [-i(\partial_{x_a} - \partial_{x_b}) + i \kappa \sgn(x_a - x_b)] \right) \det[\exp(i q_c x_d)]_{(c,d)=1,\ldots,N} \]  

(A.9)

where, by definition, the differential operators \( \partial_{x_a} \) act only on the exponential terms and not on the \( \sgn(x) \) functions, and for further convenience we have redefined \( i^{N(N-1)/2} C \rightarrow C \). Explicitly the determinant in the above equation is

\[ \det[\exp(i q_c x_d)]_{(c,d)=1,\ldots,N} = \sum_P (-1)^{|P|} \exp \left[ i \sum_{a=1}^N q_{pa} x_a \right] \]  

(A.10)

where the summation goes over the permutations \( P \) of \( N \) momenta \( \{q_1, q_2, \ldots, q_N\} \) over \( N \) particles \( \{x_1, x_2, \ldots, x_N\} \), and \( |P| \) denotes the parity of the permutation. In this way the eigenfunction, equation (A.9), can be represented as follows

\[ \Psi^{(N)}_q(x) = C \sum_P (-1)^{|P|} \left( \prod_{a<b} [-i(\partial_{x_a} - \partial_{x_b}) + i \kappa \sgn(x_a - x_b)] \right) \exp \left[ i \sum_{a=1}^N q_{pa} x_a \right]. \]  

(A.11)
Taking the derivatives, we obtain

\[ \Psi(q)(x) = C \sum_P (-1)^{|P|} \left( \prod_{a<b} [q_{pa} - q_{pb} + i\kappa \text{sgn}(x_a - x_b)] \right) \exp \left[ i \sum_{a=1}^{N} q_{pa} x_a \right]. \tag{A.12} \]

It is evident from these representations that the eigenfunctions \( \Psi(q)(x) \) are \textit{antisymmetric} with respect to permutations of the momenta \( q_1, \ldots, q_N \).

Finally, substituting the expression for the eigenfunctions, equation (A.5) (which is valid in the sector, equation (A.2)), into (A.3) for the energy spectrum we find

\[ E = \frac{1}{2\beta} \sum_{a=1}^{N} q_a^2. \tag{A.13} \]

\section*{A.2. Orthonormality}

Now one can easily prove that the above eigenfunctions with different momenta are orthogonal to each other. Let us consider two wavefunctions \( \Psi(q)(x) \) and \( \Psi(q')(x) \) where it is assumed that

\[ q_1 < q_2 < \cdots < q_N \]
\[ q_1' < q_2' < \cdots < q_N'. \tag{A.14} \]

Using the representation, equation (A.11), for the overlap of these two functions we get

\[
\begin{align*}
\overline{\Psi(q')(x)\Psi(q)(x)} & = |C|^2 \sum_{P,P'} (-1)^{|P|+|P'|} \int_{-\infty}^{+\infty} d^N x \left\{ \prod_{a<b} [i(\partial_{x_a} - \partial_{x_b}) - i\kappa \text{sgn}(x_a - x_b)] \right\} \exp \left[ -i \sum_{a=1}^{N} q_{pa} x_a \right] \\
& \times \left\{ \prod_{a<b} [-i(\partial_{x_a} - \partial_{x_b}) + i\kappa \text{sgn}(x_a - x_b)] \right\} \exp \left[ i \sum_{a=1}^{N} q_{pa} x_a \right].
\end{align*}
\]

Integrating by parts we obtain

\[
\begin{align*}
\overline{\Psi(q')(x)\Psi(q)(x)} &= |C|^2 \sum_{P,P'} (-1)^{|P|+|P'|} \int_{-\infty}^{+\infty} d^N x \exp \left[ -i \sum_{a=1}^{N} q_{pa'} x_a \right] \\
& \times \left\{ \prod_{a<b} [-i(\partial_{x_a} - \partial_{x_b}) - i\kappa \text{sgn}(x_a - x_b)] \right\} \left\{ [-i(\partial_{x_a} - \partial_{x_b}) + i\kappa \text{sgn}(x_a - x_b)] \right\} \exp \left[ i \sum_{a=1}^{N} q_{pa} x_a \right].
\end{align*}
\]

\[ \text{doi:10.1088/1742-5468/2010/07/P07010} \]
or

\[ \Psi_q^{(N)^*}(x) \Psi_q^{(N)}(x) = |C|^2 \sum_{P,P'} (-1)^{|P|+|P'|} \int_{-\infty}^{+\infty} d^N x \]
\[ \times \exp \left[ -i \sum_{a=1}^N q_{p_a} x_a \left( \prod_{a,b} (q_{p_a} - q_{p_b})^2 + \kappa^2 \right) \right] \exp \left[ i \sum_{a=1}^N q_{p_a} x_a \right] . \]  
\[ \text{(A.17)} \]

Taking the derivatives and performing the integrations we find

\[ \Psi_q^{(N)^*}(x) \Psi_q^{(N)}(x) = |C|^2 \sum_{P,P'} (-1)^{|P|+|P'|} \left( \prod_{a,b} (q_{p_a} - q_{p_b})^2 + \kappa^2 \right) \]
\[ \times \int_{-\infty}^{+\infty} d^N x \exp \left[ i \sum_{a=1}^N (q_{p_a} - q'_{p_a}) x_a \right] \]
\[ = |C|^2 \sum_{P,P'} (-1)^{|P|+|P'|} \left( \prod_{a,b} (q_{p_a} - q_{p_b})^2 + \kappa^2 \right) \left( \prod_{a=1}^N (2\pi) \delta(q_{p_a} - q'_{p_a}) \right) . \]  
\[ \text{(A.18)} \]

Taking into account the constraint, equation (A.14), one can easily note that the only terms which survive in the above summation over the permutations are \( P = P' \), all contributing equal value. Thus, we finally get

\[ \Psi_q^{(N)^*}(x) \Psi_q^{(N)}(x) = |C|^2 N! \left( \prod_{a,b} (q_a - q_b)^2 + \kappa^2 \right) \left( \prod_{a=1}^N (2\pi) \delta(q_a - q'_a) \right) . \]  
\[ \text{(A.19)} \]

With the normalization constant

\[ |C(q)|^2 = \frac{1}{N! \prod_{a,b} (q_a - q_b)^2 + \kappa^2} \]  
\[ \text{(A.20)} \]

we conclude that the set of the eigenfunctions, equation (A.11) or (A.12), are orthonormal. The proof of completeness of this set is given in [30]. It should be noted that the above wavefunctions present the orthonormal set of eigenfunctions of the problem, equation (A.1), for any sign of the interactions \( \kappa \), i.e. both for the repulsive, \( \kappa < 0 \), and for the attractive, \( \kappa > 0 \), cases. However, only in the case of repulsion is this set complete, while in the case of attractive interactions, \( \kappa > 0 \), in addition to the solutions, equation (A.11), which describe the continuous free particle spectrum, one finds a whole family of discrete bound eigenstates (which do not exist in the case of repulsion).

Appendix B. Quantum bosons with attractive interactions

B.1. Ground state

The simplest example of the bound eigenstate defined by equation (A.1) (with \( \kappa > 0 \)) is the one in which all \( N \) particles are bound into a single ‘cluster’:

\[ \Psi_q^{(1)}(x) = C \exp \left[ iq \sum_{a=1}^N x_a - \frac{i}{2} \kappa \sum_{a,b=1}^N (x_a - x_b) \right] \]  
\[ \text{(B.1)} \]
where $C$ is the normalization constant (to be defined below) and $q$ is the continuous momentum of free center of mass motion. Substituting this function into equation (A.1), one can easily check that this is indeed the eigenfunction with the energy spectrum given by the relation

$$E = -\frac{1}{2\beta} \sum_{a=1}^{N} \left[ i q - \frac{i}{2} \kappa \sum_{b=1}^{N} \text{sgn}(x_a - x_b) \right]^2 \tag{B.2}$$

where it is assumed (by definition) that $\text{sgn}(0) = 0$. Since the result of the above summations does not depend on the mutual particle positions, for simplicity we can order them according to equation (A.2). Then, using the well known relations

$$\sum_{a=1}^{N} \text{sgn}(x_a - x_b) = -(N + 1 - 2a) \tag{B.3}$$

$$\sum_{a=1}^{N} a = \frac{1}{2} N(N + 1) \tag{B.4}$$

$$\sum_{a=1}^{N} a^2 = \frac{N}{6}(N + 1)(2N + 1) \tag{B.5}$$

for the energy spectrum, equation (B.2), we get

$$E = \frac{N}{2\beta} q^2 - \frac{\kappa^2}{24\beta}(N^3 - N) \equiv E_1(q, N). \tag{B.6}$$

The normalization constant $C$ is defined by the orthonormality condition

$$\Psi_q^{(1)^*}(x) \Psi_q^{(1)}(x) \equiv \int_{-\infty}^{+\infty} dx_1 \cdots dx_N \Psi_q^{(1)^*}(x) \Psi_q^{(1)}(x) = (2\pi)^N \delta(q - q'). \tag{B.7}$$

Substituting here equation (B.1) we get

$$\begin{align*}
\Psi_q^{(1)^*}(x) \Psi_q^{(1)}(x) &= |C|^2 \int_{-\infty}^{+\infty} dx_1 \cdots dx_N \exp \left[ i(q - q') \sum_{a=1}^{N} x_a - \frac{i}{2} \kappa \sum_{a,b=1}^{N} |x_a - x_b| \right] \\
&= |C|^2 N! \int_{-\infty}^{+\infty} dx_1 \int_{x_1}^{+\infty} dx_2 \cdots \int_{x_{N-1}}^{+\infty} dx_N \\
&\quad \times \exp \left[ i(q - q') \sum_{a=1}^{N} x_a + \kappa \sum_{a=1}^{N} (N + 1 - 2a) x_a \right] \tag{B.8}
\end{align*}$$

where for the ordering, equation (A.2), we have used the relation

$$\frac{1}{2} \sum_{a,b=1}^{N} |x_a - x_b| = -\sum_{a=1}^{N} (N + 1 - 2a) x_a. \tag{B.9}$$
Integrating first over $x_N$, then over $x_{N-1}$, and proceeding until $x_1$, we find

$$
\Psi_q^{(1)^*}(x)\Psi_q^{(1)}(x) = |C|^2 N! \left( \prod_{r=1}^{N-1} \frac{1}{r((N-r)\kappa - i(q - q'))} \right) \int_{-\infty}^{+\infty} dx_1 \exp[iN(q - q')x_1] \\
= |C|^2 N! \left( \prod_{r=1}^{N-1} \frac{1}{r(N-r)\kappa} \right) (2\pi) \delta(N(q - q')) \\
= |C|^2 N\kappa \left( \frac{N\kappa}{N!\kappa N} \right) (2\pi) \delta(q - q').
$$

According to equation (B.7) this defines the normalization constant

$$
C = \sqrt{\frac{\kappa N!}{\kappa N}} \equiv C^{(1)}(q).
$$

Note that the eigenstate described by the considered wavefunction, equation (B.1), exists only in the case of attraction, $\kappa > 0$, otherwise this function is divergent at infinity and consequently it is not normalizable.

It should be noted that the wavefunction, equation (B.1), can also be derived from the general eigenfunction structure, equation (A.12), by introducing (discrete) imaginary parts for the momenta $q_a$. We assume again that the positions of the particles are ordered according to equation (A.2), and define the particle momenta according to the rule

$$
q_a = q - \frac{i}{2}\kappa(N + 1 - 2a).
$$

Substituting this into equation (A.12) we get

$$
\Psi_q^{(1)}(x_1 < x_2 < \cdots < x_N) \propto \sum_P (-1)^{|P|} \left( \prod_{a<b}^{N} (q - \frac{i}{2}\kappa(N + 1 - 2P_a)) \right) \exp \left[ iq \sum_{a=1}^{N} x_a + \frac{\kappa}{2} \sum_{a=1}^{N} (N + 1 - 2P_a)x_a \right] \\
\times \sum_P (-1)^{|P|} \left( \prod_{a<b}^{N} [P_b - P_a + 1] \right) \exp \left[ iq \sum_{a=1}^{N} x_a + \frac{\kappa}{2} \sum_{a=1}^{N} (N + 1 - 2P_a)x_a \right].
$$

Here one can easily note that due to the presence of the product $\prod_{a<b}^{N} [P_b - P_a + 1]$ in the summation over permutations, only the trivial one, $P_a = a$, gives a non-zero contribution (if we permute any two numbers in the sequence 1, 2, \ldots, $N$ then we can always find two numbers $a < b$, such that $P_b = P_a - 1$). Thus

$$
\Psi_q^{(1)}(x_1 < x_2 < \cdots < x_N) \propto \exp \left[ iq \sum_{a=1}^{N} x_a + \frac{\kappa}{2} \sum_{a=1}^{N} (N + 1 - 2a)x_a \right].
$$

Taking into account the relation, equation (B.9), we recover the function, equation (B.1), which is symmetric with respect to its $N$ arguments and therefore can be extended.
Replica Bethe ansatz derivation of the free energy fluctuations in one-dimensional directed polymers

beyond the sector, equation (A.2), for arbitrary particle positions. Finally, substituting
the momenta, equation (B.12), into the general expression for the energy spectrum,
equation (A.13), we get

\[ E = \frac{1}{2\beta} \sum_{a=1}^{N} \left[ q - \frac{i}{2}\kappa(N + 1 - 2a) \right]^2. \] (B.16)

Performing here simple summations (using equations (B.4), (B.5)) one recovers
equation (B.6).

B.2. Eigenfunctions

A generic eigenfunction of attractive bosons is characterized by \( N \) momenta parameters
\( \{q_a\} (a = 1, 2, \ldots, N) \) which may have imaginary parts. It is convenient to group these
parameters into \( M \) (\( 1 \leq M \leq N \)) ‘vector’ momenta,

\[ q^\alpha_r = q_\alpha - \frac{i}{2}\kappa(n_\alpha + 1 - 2r) \] (B.17)

where \( q_\alpha (\alpha = 1, 2, \ldots, M) \) are the continuous (real) parameters, and the discrete
imaginary components of each ‘vector’ are labeled by an index \( r = 1, 2, \ldots, n_\alpha \). With the
given total number of particles equal to \( N \), the integers \( n_\alpha \) have to satisfy the constraint

\[ \sum_{\alpha=1}^{M} n_\alpha = N \] (B.18)

In other words, a generic eigenstate is characterized by the discrete number \( M \) of complex
‘vector’ momenta, by the set of \( M \) integer parameters \( \{n_1, n_2, \ldots, n_M\} \equiv \mathbf{n} \) (which are the
numbers of imaginary components of each ‘vector’) and by the set of \( M \) real continuous
momenta \( \{q_1, q_2, \ldots, q_M\} \equiv \mathbf{q} \).

The general expression for the eigenfunctions is given in equations (A.9)–(A.12). To
understand the structure of the determinant of the \( N \times N \) matrix \( \exp(iq_ax_b) \), which defines
these wavefunctions, the \( N \) momenta \( q_a \), equation (B.17), can be ordered as follows:

\[ \{q_a\} \equiv \{q^\alpha_r\} = \{q^1_1, q^1_2, \ldots, q^1_{n_1}; q^2_1, q^2_2, \ldots, q^2_{n_2}; \ldots; q^M_1, q^M_2, \ldots, q^M_{n_M}\}. \] (B.19)

By definition,

\[ \text{det}[\exp(iq_ax_c)]_{(c,d)=1,\ldots,N} = \sum_P (-1)^{[P]} \exp \left[ i \sum_{a=1}^{N} q_{pa} x_a \right] \] (B.20)

where the summation goes over the permutations of \( N \) momenta \( \{q_a\} \), equation (B.19),
over \( N \) particles \( \{x_1, x_2, \ldots, x_N\} \), and \([P]\) denotes the parity of the permutation. For a
given permutation \( P \), a particle number \( a \) is attributed a momentum component \( q^\alpha_r(a) \).
Particles getting momenta with the same \( \alpha \) (having the same real part \( q_\alpha \)) will be called
as belonging to a cluster \( \Omega_\alpha \). For a given permutation \( P \), particles belonging to the
same cluster are numbered by an ‘internal’ index \( r = 1, \ldots, n_\alpha \). Thus, according to
equation (A.11),

$$
\Psi_{\mathbf{q,n}}^{(M)}(x) = C_{\mathbf{q,n}}^{(M)} \sum_P (-1)^{|P|} \left( \prod_{a<b}^{N} \left[ -i(\partial_{x_a} - \partial_{x_b}) + i\kappa \text{sgn}(x_a - x_b) \right] \right) \exp \left[ i \sum_{c=1}^{N} q_{r(c)} x_c \right]
$$

(B.21)

where $C_{\mathbf{q,n}}^{(M)}$ is a normalization constant to be defined later. Substituting here equation (B.17) and taking derivatives we get

$$
\Psi_{\mathbf{q,n}}^{(M)}(x) = C_{\mathbf{q,n}}^{(M)} \sum_P (-1)^{|P|} \prod_{a<b}^{N} \left[ \left( q_{a(a)} - \frac{i\kappa}{2} [n_{a(a)} + 1 - 2r(a)] \right) \right]

- \left( q_{a(b)} - \frac{i\kappa}{2} [n_{a(b)} + 1 - 2r(b)] \right) + i\kappa \text{sgn}(x_a - x_b) \right] \times \exp \left[ i \sum_{c=1}^{N} q_{a(c)} x_c + \frac{\kappa}{2} \sum_{c=1}^{N} (n_{a(c)} + 1 - 2r(c)) x_c \right].
$$

(B.22)

The pre-exponential product in the above equation contains two types of term: pairs of points $(a, b)$ which belong to different clusters $(\alpha(a) \neq \alpha(b))$, and pairs of points which belong to the same cluster $(\alpha(a) = \alpha(b))$. In the latter case, the product $\Pi_\alpha$ over the pairs of points which belong to a cluster $\Omega_\alpha$ reduces to

$$
\Pi_\alpha \propto \prod_{a<b \in \Omega_\alpha} \left[ r(b) - r(a) - \text{sgn}(x_a - x_b) \right].
$$

(B.23)

As for the ground state wavefunction equations (B.14) and (B.15), one can easily note that due to the presence of this product in the summations over $n_{\alpha}$! ‘internal’ (inside the cluster $\Omega_\alpha$) permutations $r(a)$ only one permutation gives a non-zero contribution. To prove this statement, we note that the wavefunction $\Psi_{\mathbf{q,n}}^{(M)}(x)$ is symmetric with respect to permutations of its $N$ arguments $\{x_a\}$; it is then sufficient to consider the case where the positions of the particles are ordered, $x_1 < x_2 < \cdots < x_N$. In particular, the particles $\{x_{a_k}\}(k = 1, 2, \ldots, n_\alpha)$ belonging to the same cluster $\Omega_\alpha$ are also ordered $x_{a_1} < x_{a_2} < \cdots < x_{a_{n_\alpha}}$. In this case

$$
\Pi_\alpha \propto \prod_{k<l}^{n_\alpha} [r(l) - r(k) + 1].
$$

(B.24)

Now it is evident that the above product is non-zero only for the trivial permutation, $r(k) = k$ (since if we permute any two numbers in the sequence $1, 2, \ldots, n_\alpha$, we can always find two numbers $k < l$, such that $r(l) = r(k) - 1$). In this case

$$
\Pi_\alpha \propto \prod_{k<l}^{n_\alpha} [l - k + 1].
$$

(B.25)

Including the values of all these ‘internal’ products, equation (B.25), into the redefined normalization constant $C_{\mathbf{q,n}}^{(M)}$, for the wavefunction, equation (B.22) (with $x_1 < x_2 < \cdots <
$x_N$), we obtain

$$\Psi_{q,n}^{(M)}(x) = C_{q,n}^{(M)} \sum_P (-1)^{|P|} \prod_{a<b}^{N} \left[ \left( q_{\alpha(a)} - \frac{i\kappa}{2} n_{\alpha(a)} \right) \right. $$

$$- \left( q_{\alpha(b)} - \frac{i\kappa}{2} n_{\alpha(b)} \right) + i\kappa(r(a) - r(b) - 1) \left. \right] $$

$$\times \exp \left[ \sum_{c=1}^{N} q_{\alpha(c)} x_c + \frac{\kappa}{2} \sum_{c=1}^{N} (n_{\alpha(c)} + 1 - 2r(c)) x_c \right]$$

(B.26)

where the product goes now only over the pairs of particles belonging to different clusters, and the symbol $\sum_{P'}$ means that the summation goes only over the permutations $P$ in which the ‘internal’ indices $r(a)$ are ordered inside each cluster. Note that although the positions of particles belonging to the same cluster are ordered, the mutual positions of particles belonging to different clusters could be arbitrary, so that geometrically the clusters are free to ‘penetrate’ each other. In other words, the name ‘cluster’ does not assume geometrically compact particle positions.

Now, taking into account the symmetry of the wavefunction $\Psi_{q,n}^{(M)}(x)$ with respect to the permutations of its arguments, the expression in equation (B.26) can be easily continued beyond the sector $x_1 < x_2 < \cdots < x_N$ for the entire coordinate space $R_N$. Using the relations

$$\sum_{a \in \Omega_a} (n_{\alpha} + 1 - 2r(a)) x_a = \sum_{k=1}^{n_{\alpha}} (n_{\alpha} + 1 - 2k) x_{ak} = -\frac{1}{2} \sum_{k,l=1}^{n_{\alpha}} |x_{ak} - x_{al}|$$

(B.27)

(where $x_{a_1} < x_{a_2} < \cdots < x_{a_{n_{\alpha}}}$), for the wavefunction $\Psi_{q,n}^{(M)}(x)$ with arbitrary particle positions we get the following sufficiently compact representation (cf equation (B.21)):

$$\Psi_{q,n}^{(M)}(x) = C_{q,n}^{(M)} \sum_P (-1)^{|P|} \prod_{a<b}^{N} \left[ -i(\partial_{x_a} - \partial_{x_b}) + i\kappa \text{sgn}(x_a - x_b) \right] $$

$$\times \exp \left[ i \sum_{\alpha=1}^{M} q_{\alpha} \sum_{c \in \Omega_\alpha} x_c - \frac{\kappa}{4} \sum_{\alpha=1}^{M} \sum_{a, c, c' \in \Omega_\alpha} |x_c - x_{c'}| \right].$$

(B.28)

Finally, substituting equations (B.17) and (B.18) into equation (A.13), for the energy spectrum one easily obtains:

$$E_M(q,n) = \frac{1}{2\beta} \sum_{\alpha=1}^{M} \sum_{r=1}^{n_{\alpha}} (q_{\alpha}^2) = \frac{1}{2\beta} \sum_{\alpha=1}^{M} n_{\alpha} q_{\alpha}^2 - \frac{\kappa^2}{24\beta} \sum_{\alpha=1}^{M} (n_{\alpha}^3 - n_{\alpha}).$$

(B.29)

### B.3. Orthonormality

We define the overlap of two wavefunctions characterized by two sets of parameters, $(M,n,q)$ and $(M',n',q')$ as

$$Q_{n,n'}^{(M,M')}(q,q') \equiv \int_{-\infty}^{+\infty} d^N x \Psi_{q,n}^{(M')\ast}(x) \Psi_{q,n}^{(M)}(x).$$

(B.30)
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Substituting here equation (B.28) we get

\[
Q_{n_n'}^{(M,M')} (q, q') = C_{q_n} C_{q_n'} \sum_P \sum_{P'} (-1)^{|P| + |P'|} \int_{-\infty}^{+\infty} d^N x \times \\
\left( \prod_{\alpha \leq \beta, \alpha' \neq \alpha'(b)} \left[ i(\partial_{x_a} - \partial_{x_b}) - i\kappa \text{sgn}(x_a - x_b) \right] \right) \times \\
\exp \left[ -i \sum_{\alpha = 1}^{M'} q'_\alpha \sum_{c \in \Omega'_\alpha} n'_\alpha c - \frac{\kappa}{4} \sum_{\alpha = 1}^{M'} \sum_{c, c' \in \Omega'_\alpha} n'_\alpha \right] \times \\
\left( \prod_{\alpha \leq \beta, \alpha' \neq \alpha'(b)} \left[ -i(\partial_{x_a} - \partial_{x_b}) + i\kappa \text{sgn}(x_a - x_b) \right] \right) \times \\
\exp \left[ i \sum_{\alpha = 1}^{M} q_\alpha \sum_{c \in \Omega_\alpha} n_\alpha c - \frac{\kappa}{4} \sum_{\alpha = 1}^{M} \sum_{c, c' \in \Omega_\alpha} n_\alpha \right] \times \\
\left( \prod_{\alpha \leq \beta, \alpha' \neq \alpha'(b)} \left[ -i(\partial_{x_a} - \partial_{x_b}) - i\kappa \text{sgn}(x_a - x_b) \right] \right) \times \\
\exp \left[ i \sum_{\alpha = 1}^{M} q_\alpha \sum_{c \in \Omega_\alpha} n_\alpha c - \frac{\kappa}{4} \sum_{\alpha = 1}^{M} \sum_{c, c' \in \Omega_\alpha} n_\alpha \right]. \tag{B.32}
\]

where \( \{\Omega_\alpha\} \) and \( \{\Omega'_\alpha\} \) denote the clusters of the permutations \( P \) and \( P' \) correspondingly. Integrating by parts we obtain

\[
Q_{n_n'}^{(M,M')} (q, q') = C_{q_n} C_{q_n'} \sum_P \sum_{P'} (-1)^{|P| + |P'|} \int_{-\infty}^{+\infty} d^N x \times \\
\left( \prod_{\alpha \leq \beta, \alpha' \neq \alpha'(b)} \left[ i(\partial_{x_a} - \partial_{x_b}) + i\kappa \text{sgn}(x_a - x_b) \right] \right) \times \\
\exp \left[ i \sum_{\alpha = 1}^{M} q_\alpha \sum_{c \in \Omega_\alpha} n_\alpha c - \frac{\kappa}{4} \sum_{\alpha = 1}^{M} \sum_{c, c' \in \Omega_\alpha} n_\alpha \right] \times \\
\left( \prod_{\alpha \leq \beta, \alpha' \neq \alpha'(b)} \left[ i(\partial_{x_a} - \partial_{x_b}) - i\kappa \text{sgn}(x_a - x_b) \right] \right) \times \\
\exp \left[ i \sum_{\alpha = 1}^{M} q_\alpha \sum_{c \in \Omega_\alpha} n_\alpha c - \frac{\kappa}{4} \sum_{\alpha = 1}^{M} \sum_{c, c' \in \Omega_\alpha} n_\alpha \right]. \tag{B.32}
\]

First, let us consider the case when the integer parameters of the two functions coincide, \( M = M' \), \( n = n' \), and for the moment let us suppose that all these integer parameters \( \{n_\alpha\} \) are different, \( 1 \leq n_1 < n_2 < \cdots < n_M \). Then, in the summations over the permutations in equation (B.32), we find two types of terms:

(A) the ‘diagonal’ ones in which the two permutations coincide, \( P = P' \);
(B) the ‘off-diagonal’ ones in which the two permutations are different, \( P \neq P' \).

doi:10.1088/1742-5468/2010/07/P07010
The contribution of the ‘diagonal’ ones reads

\[ Q_{n,n}^{(M,M)} (q, q') = C_{q,n}^{(M)} C_{q', n}^{(M)*} \sum_P \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathrm{d}x_1 \cdots \mathrm{d}x_N \]

\[ \times \exp \left[ -i \sum_{\alpha=1}^{M} q'_{\alpha} \sum_{c \in \Omega_{\alpha}} x_c - \frac{K}{4} \sum_{a=1}^{M} \sum_{c,c' \in \Omega_{\alpha}} |x_c - x_{c'}| \right] \]

\[ \times \left( \prod_{a < b} \Omega_{\alpha(a) \neq \alpha(b)} \right) \left[ -(\partial x_a - \partial x_b)^2 + \kappa^2 \right] \]

\[ \times \exp \left[ i \sum_{\alpha=1}^{M} q_{\alpha} \sum_{c \in \Omega_{\alpha}} x_c - \frac{K}{4} \sum_{a=1}^{M} \sum_{c,c' \in \Omega_{\alpha}} |x_c - x_{c'}| \right]. \tag{B.33} \]

It is evident that all permutations \( \alpha(a) \) in the above equation give the same contribution and therefore it is sufficient to consider only the contribution of the ‘trivial’ permutation which is represented by equation (B.19). The cluster ordering given by this permutation we denote by \( \alpha_0(a) \). For this particular configuration of clusters we can redefine the particle numbering, so that instead of a ‘plane’ index \( a = 1, 2, \ldots, N \) the particles would be counted by two indices \( (\alpha, r) \): \( \{x_a\} \rightarrow \{x_{\alpha}^r\} (\alpha = 1, \ldots, M) (r = 1, \ldots, n_{\alpha}) \) indicating to which cluster \( \alpha \) a given particle belongs and what is its ‘internal’ cluster number \( r \). Due to the symmetry of the integrated expression in equation (B.33) with respect to the permutations of the particles inside the clusters, we can introduce the ‘internal’ particle ordering for every cluster: \( x_1^\alpha < x_2^\alpha < \cdots < x_{n_{\alpha}}^\alpha \). In this way, using the relation, equation (B.27), we get

\[ Q_{n,n}^{(M,M)} (q, q') = C_{q,n}^{(M)} C_{q', n}^{(M)*} \frac{N!}{n_1! n_2! \cdots n_M!} \]

\[ \times \left[ \prod_{\alpha=1}^{M} \left( n_{\alpha}! \int_{-\infty}^{+\infty} \mathrm{d}x_1^\alpha \int_{x_1^\alpha}^{+\infty} \mathrm{d}x_2^\alpha \cdots \int_{x_{n_{\alpha}-1}^\alpha}^{+\infty} \mathrm{d}x_{n_{\alpha}}^\alpha \right) \right] \]

\[ \times \exp \left[ -i \sum_{\alpha=1}^{M} q'_{\alpha} \sum_{r=1}^{n_{\alpha}} x_r^\alpha + \frac{K}{2} \sum_{\alpha=1}^{M} \sum_{r=1}^{n_{\alpha}} (n_{\alpha} + 1 - 2r) x_r^\alpha \right] \]

\[ \times \left( \prod_{\alpha < \beta} \prod_{r=1}^{n_{\alpha}} \prod_{r'=1}^{n_{\beta}} \left[ -(\partial x_r^\alpha - \partial x_r^\beta)^2 + \kappa^2 \right] \right) \]

\[ \times \exp \left[ i \sum_{\alpha=1}^{M} q_{\alpha} \sum_{r=1}^{n_{\alpha}} x_r^\alpha + \frac{K}{2} \sum_{\alpha=1}^{M} \sum_{r=1}^{n_{\alpha}} (n_{\alpha} + 1 - 2r) x_r^\alpha \right]. \tag{B.34} \]

where the factor \( N!/n_1! \cdots n_M! \) is the total number of permutations of \( M \) clusters over \( N \) particles. Taking the derivatives and reorganizing the terms we obtain

\[ Q_{n,n}^{(M,M)} (q, q') = C_{q,n}^{(M)} C_{q', n}^{(M)*} \frac{N!}{n_1! n_2! \cdots n_M!} \left( \prod_{\alpha < \beta} \prod_{r=1}^{n_{\alpha}} \prod_{r'=1}^{n_{\beta}} \left( q_{\alpha} - \frac{i\kappa}{2} n_{\alpha} \right) - \left( q_{\beta} - \frac{i\kappa}{2} n_{\beta} \right) + i\kappa (r - r' - 1) \right) \]

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\[
\times \prod_{\alpha=1}^{M} \left\{ \int_{-\infty}^{+\infty} dx_1^\alpha \int_{x_1^\alpha}^{+\infty} dx_2^\alpha \cdots \int_{x_{n\alpha-1}}^{+\infty} dx_{n\alpha}^\alpha \right\} \times e^{i(q_{\alpha} - q_{\alpha}')} \sum_{\alpha=1}^{n_{\alpha} + \kappa} (n_{\alpha} + 1 - 2r(a)) x_a^\alpha \}. \tag{B.35}
\]

Simple integrations over \( x_a^\alpha \) yield (cf equations (B.8)–(B.10))

\[
Q^{(M,M)(A)}_{n,n}(q,q') = |C^{(M)}_{q,n}|^2 N! \left( \prod_{\alpha=1}^{M} \prod_{\alpha < \beta} \prod_{r=1}^{n_{\alpha} + n_{\beta} + \kappa} \left( q_{\alpha} - \frac{i\kappa}{2} n_{\alpha} \right) \right.
\]

\[\left. - \left( q_{\beta} - \frac{i\kappa}{2} n_{\beta} \right) + i\kappa (r - r') - 1 \right| \prod_{\alpha=1}^{M} \left[ \frac{n_{\alpha} K_{n_{\alpha}}}{(n_{\alpha}!)^2} (2\pi) \delta(q_{\alpha} - q_{\alpha}') \right]. \tag{B.36}
\]

Now let us prove that the ‘off-diagonal’ terms of equation (B.32), in which the permutations \( P \) and \( P' \) are different, give no contribution. Here we can also chose one of the permutations, say the permutation \( P \), to be the ‘trivial’ one represented by equation (B.19) with the cluster ordering denoted by \( \alpha_0(a) \). Given the symmetry of the wavefunctions it will be sufficient to consider the contribution of the sector \( x_1 < x_2 < \cdots < x_N \). According to equation (B.32), we get

\[
Q^{(M,M)(B)}_{n,n}(q,q') \propto \sum'_{P'} (-1)^{|P'|} \int_{x_1 < \cdots < x_N} d^N x
\]

\[\times \exp \left[ -i \sum_{\alpha=1}^{M} q_{\alpha}' \sum_{a \in \Omega_{\alpha}'} x_a - \frac{\kappa}{4} \sum_{\alpha=1}^{M} \sum_{a,b \in \Omega_{\alpha}'} |x_a - x_b| \right]
\]

\[\times \left( \prod_{a < b \atop \alpha_0(a) \neq \alpha_0(b)} [-i(\partial_{x_a} - \partial_{x_b}) + i\kappa \text{sgn}(x_a - x_b)] \right) \]

\[\times \left( \prod_{a < b \atop \alpha'(a) \neq \alpha'(b)} [-i(\partial_{x_a} - \partial_{x_b}) - i\kappa \text{sgn}(x_a - x_b)] \right) \]

\[\times \exp \left[ i \sum_{\alpha=1}^{M} q_{\alpha} \sum_{a \in \Omega_{\alpha}^2} x_a + \frac{\kappa}{2} \sum_{\alpha=1}^{M} \sum_{a \in \Omega_{\alpha}^2} (n_{\alpha} + 1 - 2r(a)) x_a \right]. \tag{B.37}
\]

Here the symbols \( \{\Omega^-_{\alpha}\} \) denote the clusters of the trivial permutation \( \alpha_0(a) \). Since \( P' \neq P \), some of the clusters \( \Omega_{\alpha}' \) must be different from \( \Omega_{\alpha}^- \). As an illustration, let us consider a particular case of \( N = 10 \), with three clusters \( n_1 = 5 \) (denoted by the symbol ‘\( \bigcirc \)’), \( n_2 = 2 \) (denoted by the symbol ‘\( \times \)’) and \( n_3 = 3 \) (denoted by the symbol ‘\( \triangle \)’):

| Particle number \( a \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|--------------------------|---|---|---|---|---|---|---|---|---|----|
| Permutation \( \alpha_0(a) \) | ○ | ○ | ○ | ○ | ○ | × | × | Δ | Δ | Δ |
| Permutation \( \alpha'(a) \) | ○ | ○ | ○ | Δ | ○ | × | × | ○ | Δ | Δ |

Here in the permutation \( \alpha'(a) \) the particle \( a = 4 \) belongs to the cluster \( \alpha = 3 \) (and not to the cluster \( \alpha = 1 \) as in the permutation \( \alpha_0(a) \)), and the particle \( a = 8 \) belongs to the

doi:10.1088/1742-5468/2010/07/P07010
cluster \( \alpha = 1 \) (and not to the cluster \( \alpha = 3 \) as in the permutation \( \alpha_0(a) \)). Now let us look carefully at the structure of the products in equation (B.37). Unlike the first product, which contains no ‘internal’ products among particles belonging to the cluster \( \Omega \), the second product does. Furthermore, the signs of the differential operators \( (\partial_{x_a} - \partial_{x_b}) \) in the second product are opposite to the ‘normal’ ones in the first product (cf equations (B.23)–(B.25)). It is these two factors (the presence of the ‘internal’ products and the ‘wrong’ signs of the differential operators) which makes the ‘off-diagonal’ contributions, equation (B.37), become zero. Indeed, in the above example, the second product contains the term

\[
\Pi'_{4,5} \equiv [-i(\partial_{x_4} - \partial_{x_5}) + i\kappa] \exp \left[ i \sum_{\alpha=1}^{3} q_{\alpha} \sum_{a \in \Omega_{\alpha}} x_a + \frac{\kappa}{2} \sum_{\alpha=1}^{3} \sum_{a \in \Omega_{\alpha}} (n_{\alpha} + 1 - 2r(a)) x_a \right]
\]  

(B.38)

(we recall that the particles in the clusters \( \Omega_{\alpha} \) are ordered, and in particular \( x_4 < x_5 \)). Taking the derivatives, we get

\[
\Pi'_{4,5} \propto \left[ -\left( i q_1 + \frac{\kappa}{2} (n_1 + 1 - 2r(4)) \right) - i q_1 + \frac{\kappa}{2} (n_1 + 1 - 2r(5)) \right] + \kappa \right] 
\approx [r(4) - r(5) + 1] = 0
\]  

(B.39)

since in the first cluster \( r(a) = a \).

One can easily understand that the above example reflects the general situation. Since all the cluster sizes \( n_\alpha \) are supposed to be different, whatever the permutation \( \alpha'(a) \) is, we can always find a cluster \( \Omega_{\alpha'} \) such that some of its particles belong to the same cluster number \( \alpha \) in the permutation \( \alpha'(a) \) while the others do not. Then one has to consider the contribution of the product of two neighboring number points

\[
\Pi'_{k,k+1} \equiv [-i(\partial_{x_k} - \partial_{x_{k+1}}) + i\kappa] \exp \left[ i \sum_{\alpha=1}^{M} q_{\alpha} \sum_{a \in \Omega_{\alpha}} x_a + \frac{\kappa}{2} \sum_{\alpha=1}^{M} \sum_{a \in \Omega_{\alpha}} (n_{\alpha} + 1 - 2r(a)) x_a \right]
\]  

(B.40)

where in the permutation \( \alpha'(a) \) the particle \( k \) belongs to the cluster number \( \alpha \) and the particle \( (k+1) \) belongs to some other cluster. Taking the derivatives one gets

\[
\Pi'_{k,k+1} \propto [r(k) - r(k+1) + 1] = 0
\]  

(B.41)

as \( r(a) \) is the ‘internal’ particle number in the cluster \( \Omega_{\alpha} \), where \( r(k+1) = r(k) + 1 \) (cf equations (B.23)–(B.25)).

Thus, the only non-zero contribution to the overlap, equation (B.30), of the two wavefunctions \( \Psi^{(M)}_{q,n}(x) \) and \( \Psi^{(M)}_{q',n}(x) \) (having the same number of clusters \( M \) and characterized by the same set of the integer parameters \( 1 \leq n_1 < n_2 < \cdots < n_M \)) comes from the ‘diagonal’ terms, equation (B.36):

\[
Q^{(M,M)}_{n,n}(q,q') = |C^{(M)}_{q,n}|^2 N! \prod_{\alpha=1}^{M} \left[ \frac{n_{\alpha} \kappa}{(n_{\alpha})!^2 \kappa^{n_{\alpha}}} \right] \left( \prod_{\alpha<\beta} \prod_{r=1}^{n_{\alpha}} \prod_{r'=1}^{n_{\beta}} \left( q_{\alpha} - \frac{i\kappa}{2} n_{\alpha} \right) \right) - \left( q_\beta - \frac{i\kappa}{2} n_\beta \right) + i\kappa (r - r' - 1) \right| \prod_{\alpha=1}^{M} (2\pi) \delta(q_{\alpha} - q'_{\alpha}).
\]  

(B.42)

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The situation when there are clusters which have the same numbers of particles \( n_{\alpha} \) is somewhat more complicated. Let us consider the overlap between two wavefunctions \( \Psi^{(M)}_{q',n}(x) \) and \( \Psi^{(M)}_{q,n}(x) \) (which, as before have the same \( M \) and \( n \)) such that in the set of \( M \) integers \( n_1, n_2, \ldots, n_M \) there are two \( n_{\alpha} \)'s which are equal, say \( n_{\alpha_1} = n_{\alpha_2} \) (where \( \alpha_1 \neq \alpha_2 \)). In the eigenstate \((q',n)\) these two clusters have the center of mass momenta \( q'_{\alpha_1} \) and \( q'_{\alpha_2} \), and in the eigenstate \((q,n)\) they have the momenta \( q_{\alpha_1} \) and \( q_{\alpha_2} \) correspondingly. According to the above discussion, the non-zero contributions in the summation over the cluster permutations \( \alpha(a) \) and \( \alpha'(a) \) in equation (B.32) appear only if the clusters \( \{\Omega_{\alpha}\} \) of the permutation \( \alpha(a) \) totally coincide with the clusters \( \{\Omega'_{\alpha}\} \) of the permutation \( \alpha'(a) \). In the case when all \( n_{\alpha} \) are different this is possible only if the permutation \( \alpha(a) \) coincides with the permutation \( \alpha'(a) \). In contrast to that, in the case when we have \( n_{\alpha_1} = n_{\alpha_2} \), there are two non-zero options. The first one, as before, is given by the ‘diagonal’ terms with \( \alpha(a) = \alpha'(a) \) (so that the clusters \( \{\Omega_{\alpha}\} \) and \( \{\Omega'_{\alpha}\} \) are just the same), and this contribution is proportional to \( \delta(q_{\alpha_1} - q'_{\alpha_1}) \delta(q_{\alpha_2} - q'_{\alpha_2}) \). The second (‘off-diagonal’) contribution is given by such a permutation \( \alpha'(a) \) in which the cluster \( \Omega'_{\alpha_1} \) (of the permutation \( \alpha'(a) \)) coincides with the cluster \( \Omega_{\alpha_2} \) (of the permutation \( \alpha(a) \)) and the cluster \( \Omega'_{\alpha_2} \) (of the permutation \( \alpha'(a) \)) coincides with the cluster \( \Omega_{\alpha_1} \) (of the permutation \( \alpha(a) \)), while the rest of the clusters of these two permutations are the same, \( \Omega'_\alpha = \Omega_\alpha (\alpha \neq \alpha_1, \alpha_2) \). Correspondingly, this last contribution is proportional to \( \delta(q_{\alpha_1} - q'_{\alpha_1}) \delta(q_{\alpha_2} - q'_{\alpha_2})(-1)^{n_{\alpha_1}} \). In fact this situation with two equivalent contributions is the consequence of the symmetry of the wavefunction \( \Psi^{(M)}_{q',n}(x) \): the permutation of two momenta \( q_{\alpha_1} \) and \( q_{\alpha_2} \) belonging to the clusters which have the same numbers of particles, \( n_{\alpha_1} = n_{\alpha_2} \), produces the factor \((-1)^{n_{\alpha_1}}\). This is evident from the general expression for the wavefunction, equation (A.9), where the permutation of any two momenta \( q_{\alpha_1} \) and \( q_{\alpha_2} \) belonging to clusters which have the same numbers of particles corresponds to the permutation of \( n \) columns of the matrix \( \text{exp}(iq_{\alpha}x_b) \). Therefore considering clusters with equal numbers of particles as equivalent and restricting the analysis to the sectors \( q_{\alpha_1} < q_{\alpha_2} ; q'_{\alpha_1} < q'_{\alpha_2} \) we find that the second contribution, \( \delta(q_{\alpha_1} - q'_{\alpha_1}) \delta(q_{\alpha_2} - q'_{\alpha_2}) \), is identically equal to zero, thus returning to the above resulting equation (B.42).

A generic eigenstate \((q,n)\) with \( M \) clusters could be specified in terms of the following set of parameters:

\[
(q,n) = \{(q_{s_1,1}, m_1), \ldots, (q_{s_1,s_1}, m_1); (q_{s_1+s_2,1}, m_2), \ldots, (q_{s_1+s_2, s_2}, m_2); \ldots; (q_{s_k+\ldots+s_{k-1}+1,1}, m_k), \ldots, (q_{s_k+\ldots+s_{k}+1,1}, m_k)\}
\]  \hspace{1cm} (B.43)

where \( s_1 + s_2 + \ldots + s_k = M \) and \( k \) integers \( \{m_i\} \) (\( 1 \leq k \leq M \)) are all supposed to be different:

\[
1 \leq m_1 < m_2 < \ldots < m_k.
\]  \hspace{1cm} (B.44)

Here the integer parameter \( k \) denotes the number of different cluster types. For a given \( k \)

\[
\sum_{\alpha=1}^{M} n_{\alpha} = \sum_{i=1}^{k} s_i m_i = N.
\]  \hspace{1cm} (B.45)
Due to the symmetry with respect to the momenta permutations inside the subsets of equal ns it is sufficient to consider the wavefunctions in the sectors

\[
q_1 < q_2 < \cdots < q_{s_1}; \\
q_{s_1+1} < q_{s_1+2} < \cdots < q_{s_1+s_2}; \\
\cdots \cdots \cdots \\
q_{s_1+\cdots+s_{k-1}+1} < q_{s_1+\cdots+s_{k-1}+2} < \cdots < q_{s_1+\cdots+s_{k-1}+s_k}.
\]

(B.46)

In this representation we again recover the above resulting equation (B.42).

Finally, let us consider the overlap of two eigenstates described by two different sets of integer parameters, \(n' \neq n\). In fact this situation is quite simple because if the clusters of the two states are different from each other, it means that in the summation over the pairs of permutations \(P\) and \(P'\) in equation (B.32) there exist no two permutations for which these two sets of clusters \(\{\Omega_a\}\) and \(\{\Omega'_a\}\) would coincide. Which, according to the above analysis, means that this expression is equal to zero. Note that the condition \(M' \neq M\) automatically implies that \(n' \neq n\).

Thus we have proved that

\[
Q^{(M,M')}_{n,n'}(q, q') = |C_{q,n}^{(M)}|^2 \delta(M, M') \left( \prod_{\alpha=1}^{M} \delta(n_\alpha, n'_\alpha) \right) \left( \prod_{\alpha=1}^{M} (2\pi)\delta(q_\alpha - q'_\alpha) \right)
\]

\[
\times N! \left[ \frac{n_\alpha n_\beta}{(n_\alpha !)^2} \right] \frac{M}{n_\alpha n_\beta} \prod_{\alpha < \beta} \prod_{r=1} \prod_{r'=1} \left| q_\alpha - \frac{i\kappa}{2} n_\alpha \right|^2 - \left( q_\beta - \frac{i\kappa}{2} n_\beta + \frac{1}{2} \kappa(r - r' - 1) \right|^2
\]

(B.47)

where the integer parameters \(\{n_\alpha\}\) and \(\{n'_\alpha\}\) are assumed to have the generic structure represented in equations (B.43)–(B.45), and the momenta \(\{q_\alpha\}\) and \(\{q'_\alpha\}\) of the clusters with equal numbers of particles are restricted in the sectors, equation (B.46). According to equation (B.47), the orthonormality condition defines the normalization constant

\[
|C^{(M)}(q, n)|^2 = \frac{1}{N!} \left[ \frac{M}{n_\alpha n_\beta} \right] \prod_{\alpha=1}^{M} n_\alpha n_\beta
\]

\[
\times \prod_{\alpha < \beta} \prod_{r=1} \prod_{r'=1} \left| (q_\alpha - (i\kappa/2)n_\alpha) - (q_\beta - (i\kappa/2)n_\beta) + \kappa(r - r' - 1) \right|^2.
\]

(B.48)

In other words, the wavefunctions, equation (B.28), form an orthonormal set. Although, at present we are not able to prove that this set is complete, the suggestion of the completeness (which assumes that there exist no other eigenstates besides those described above) looks quite natural.

**B.4. Propagator**

The time dependent solution \(\Psi(x, t)\) of the imaginary-time Schrödinger equation

\[
\beta \partial_t \Psi(x, t) = \frac{1}{2} \sum_{a=1}^{N} \partial^2_{x_a} \Psi(x, t) + \frac{1}{2} \kappa \sum_{a \neq b}^{N} \delta(x_a - x_b) \Psi(x, t)
\]

(B.49)
with the initial condition
\[ \Psi(x; 0) = \Pi_{a=1}^{N} \delta(x_a) \]  
(B.50)
can be represented in terms of a linear combination of the eigenfunctions \( \Psi_{q,n}^{(M)}(x) \), equation (B.28):
\[ \Psi(x, t) = \sum_{M=1}^{N} \sum_{n} \int Dq \Psi_{q,n}^{(M)}(x) \Psi_{q,n}^{(M)*}(0) \exp[-E_M(q, n)t] \]  
(B.51)
where the energy spectrum \( E_M(q, n) \) is given by equation (B.29). The summations over \( n_a \) are performed here in terms of the parameters \( \{ s_i, m_i \} \), equations (B.43)–(B.45):
\[ \sum' \equiv \sum_{k=1}^{\infty} \sum_{s_1 \ldots s_k=1}^{\infty} \delta \left( \sum_{i=1}^{k} s_i, M \right) \delta \left( \sum_{i=1}^{k} s_i m_i, N \right) \]  
(B.52)
where \( \delta(n, l) \) is the Kronecker symbol, and for simplicity (due to the presence of these Kronecker symbols) the summations over \( m_i \) and \( s_i \) are extended to infinity. The symbol \( \int Dq \) in equation (B.51) denotes the integration over \( M \) momenta \( q_a \) in the sectors, equation (B.46).

The replica partition function \( Z(N, L) \) of the original directed polymer problem is obtained via a particular choice of the final-point coordinates,
\[ Z(N, L) = \Psi(0; L) = \sum_{M=1}^{N} \sum_{n} \int Dq |\Psi_{q,n}^{(M)}(0)|^2 \exp[-E_M(q, n)L]. \]  
(B.53)
According to equation (B.28), for \( M \geq 2 \),
\[ \Psi_{q,n}^{(M)}(0) = C_{q,n}^{(M)} \sum_{P} (-1)^{|P|} \prod_{a<b}^{N} \left[ \left( q_{\alpha(a)} - \frac{i\kappa}{2} n_{\alpha(a)} + 1 - 2r(a) \right) \right. \]
\[ \left. - \left( q_{\alpha(b)} - \frac{i\kappa}{2} n_{\alpha(b)} + 1 - 2r(b) \right) \right] \]
\[ = C_{q,n}^{(M)} \frac{N!}{n_1! n_2! \ldots n_M!} \prod_{a<b}^{M} \prod_{r=1}^{n_{\alpha}} \prod_{r'=1}^{n_{\beta}} \left[ \left( q_a - \frac{i\kappa}{2} n_{a} \right) - \left( q_{\beta} - \frac{i\kappa}{2} n_{\beta} \right) + i\kappa (r - r') \right]. \]  
(B.54)
Substituting here the value of the normalization constant, equation (B.48), we get
\[ |\Psi_{q,n}^{(M)}(0)|^2 = \frac{N! \kappa^N}{\prod_{a=1}^{M} \kappa n_{a}} \prod_{a<b}^{M} \]
\[ \times \frac{\prod_{r=1}^{n_{a}} \prod_{r'=1}^{n_{\beta}} \left| (q_a - (i\kappa/2)n_{a}) - (q_{\beta} - (i\kappa/2)n_{\beta}) + i\kappa (r - r') \right|^2}{\prod_{r=1}^{n_{a}} \prod_{r'=1}^{n_{\beta}} \left| (q_a - (i\kappa/2)n_{a}) - (q_{\beta} - (i\kappa/2)n_{\beta}) + i\kappa (r - r' - 1) \right|^2}. \]  
(B.55)
This expression can be essentially simplified. Shifting the product over \( r' \) in the denominator by 1 we obtain

\[
|\Psi^{(M)}_{q,n}(0)|^2 = \frac{N!\kappa^N}{(\prod_{\alpha=1}^{M} \kappa n_{\alpha})} \prod_{\alpha < \beta}^{M} \prod_{r=1}^{n_{\alpha}} \frac{|(q_{\alpha} - (i\kappa/2)n_{\alpha}) - (q_{\beta} - (i\kappa/2)n_{\beta}) + i\kappa(r - 1)|^2}{|q_{\alpha} - (i\kappa/2)n_{\alpha}) - (q_{\beta} - (i\kappa/2)n_{\beta}) + i\kappa(r - n_{\beta} - 1)|^2}.
\]

(B.56)

Redefining the product parameter \( r \) in the denominator, \( r \rightarrow n_{\alpha} + 1 - r \), and changing the obtained expression (under the modulus square) by its complex conjugate we get

\[
|\Psi^{(M)}_{q,n}(0)|^2 = \frac{N!\kappa^N}{(\prod_{\alpha=1}^{M} \kappa n_{\alpha})} \prod_{\alpha < \beta}^{M} \prod_{r=1}^{n_{\alpha}} \frac{|(q_{\alpha} - (i\kappa/2)n_{\alpha}) - (q_{\beta} - (i\kappa/2)n_{\beta}) + i\kappa(r)|^2}{|q_{\alpha} - (i\kappa/2)n_{\alpha}) - (q_{\beta} - (i\kappa/2)n_{\beta}) + i\kappa(r)|^2}.
\]

(B.57)

Shifting now the product over \( \alpha \) in the numerator by 1 we finally obtain

\[
|\Psi^{(M)}_{q,n}(0)|^2 = \frac{N!\kappa^N}{(\prod_{\alpha=1}^{M} \kappa n_{\alpha})} \prod_{\alpha < \beta}^{M} \prod_{r=1}^{n_{\alpha}} \frac{|(q_{\alpha} - q_{\beta} - (i\kappa/2)(n_{\alpha} - n_{\beta})|^2}{|q_{\alpha} - q_{\beta} - (i\kappa/2)(n_{\alpha} + n_{\beta})|^2}.
\]

(B.58)

For \( M = 1 \), according to equations (B.1) and (B.11),

\[
|\Psi^{(1)}_{q}(0)|^2 = \frac{\kappa^N N!}{\kappa N}.
\]

(B.59)

Since the function \( f(q,n) = |\Psi^{(M)}_{q,n}(0)|^2 \exp[-E_M(q,n)l] \) in equation (B.53) is symmetric with respect to permutations of all its \( M \) pairs of arguments \((q_{\alpha}, n_{\alpha})\) the integrations over \( M \) momenta \( q_{\alpha} \) can be extended beyond the sector defined in equation (B.46) for the whole space \( R^M \). As a consequence, there is no longer any need to distinguish equal and different \( n_{\alpha} \)’s, and instead of equation (B.52), we can sum over \( M \) integer parameters \( n_{\alpha} \) with the only constraint, equation (B.18) (note that this kind of simplifications holds only for the specific ‘zero final-point’ object \( \Psi(0,t) \), equation (B.53), and not for the general propagator \( \Psi(x,t) \), equation (B.51) containing \( N \) arbitrary coordinates \( x_1, \ldots, x_N \)). Thus, instead of equation (B.53) we get

\[
Z(N, L) = \sum_{M=1}^{N} \sum_{\alpha}^{M} \prod_{n=1}^{\infty} \left[ \int_{-\infty}^{+\infty} \frac{dq_{\alpha}}{2\pi\kappa n_{\alpha}} \frac{1}{M!} \int \sum_{\alpha=1}^{M} \sum_{n_{\alpha}=1}^{\infty} \delta \left( \sum_{\alpha=1}^{M} n_{\alpha}, N \right) |\Psi^{(M)}_{q,n}(0)|^2 \exp \left[ -\frac{L}{2\beta} N q^2 + \frac{\kappa^2 L}{24\beta} (N^3 - N) \right] \right].
\]

(B.60)

Substituting here equations (B.29), (B.58) and (B.59) we get the following sufficiently compact representation for the replica partition function:

\[
Z(N, L) = N!\kappa^N \left\{ \int_{-\infty}^{+\infty} \frac{dq}{2\pi\kappa N} \exp \left[ -\frac{L}{2\beta} N q^2 + \frac{\kappa^2 L}{24\beta} (N^3 - N) \right] \right. \\
+ \sum_{M=2}^{N} \left. \frac{1}{M!} \prod_{\alpha=1}^{M} \sum_{n_{\alpha}=1}^{\infty} \int_{-\infty}^{+\infty} \frac{dq_{\alpha}}{2\pi\kappa n_{\alpha}} \right. \\
\times \prod_{\alpha < \beta}^{M} \frac{|q_{\alpha} - q_{\beta} - (i\kappa/2)(n_{\alpha} - n_{\beta})|^2}{|q_{\alpha} - q_{\beta} - (i\kappa/2)(n_{\alpha} + n_{\beta})|^2} \\
\times \exp \left[ -\frac{L}{2\beta} \sum_{\alpha=1}^{M} n_{\alpha} q_{\alpha}^2 + \frac{\kappa^2 L}{24\beta} \sum_{\alpha=1}^{M} (N^3 - n_{\alpha}) \right] \right\}.
\]

(B.61)
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The first term in the above expression is the contribution of the ground state \((M = 1)\), while the next terms \((M \geq 2)\) are the contributions of the rest of the energy spectrum. After simple algebra the above replica partition function can be represented as follows:

\[
Z(N, L) = e^{-\beta NL f_0} \tilde{Z}(N, \lambda) \tag{B.62}
\]

where
\[
f_0 = \frac{1}{24} \beta^4 u^2 - (1/\beta L) \ln(\beta^3 u),
\]
and
\[
\tilde{Z}(N, L) = N! \left\{ \int_{-\infty}^{+\infty} \frac{dq}{2\pi \kappa N} \exp \left[ \frac{L}{2\beta N} q^2 + \frac{\kappa^2 L}{24\beta N^3} \right] \right. \\
+ \sum_{M=2}^{N} \frac{1}{M!} \left[ \prod_{\alpha=1}^{M} \sum_{n=1}^{\infty} \int_{-\infty}^{+\infty} \frac{dq_\alpha}{2\pi \kappa n_\alpha} \right] \delta \left( \sum_{\alpha=1}^{M} n_\alpha, N \right) \\
\times \prod_{\alpha<\beta} \frac{|q_\alpha - q_\beta - (i\kappa/2)(n_\alpha - n_\beta)|^2}{|q_\alpha - q_\beta - (i\kappa/2)(n_\alpha + n_\beta)|^2} \\
\left. \times \exp \left[ -\frac{L}{2\beta} \sum_{\alpha=1}^{M} n_\alpha q_\alpha^2 + \frac{\kappa^2 L}{24\beta \sum_{\alpha=1}^{M} n_\alpha^3} \right] \right\}. \tag{B.63}
\]

Appendix C. Fredholm determinant with the Airy kernel and the Tracy–Widom distribution

In this appendix the original derivation of Tracy and Widom [14] will be repeated in simple terms to demonstrate that the function \(F_2(s)\) defined as the Fredholm determinant with the Airy kernel can be expressed in terms of the solution of the Painlevé II differential equation, namely

\[
F_2(s) \equiv \det[1 - \hat{K}_A] = \exp \left[ -\int_{s}^{\infty} dt \left( t - s \right) q^2(t) \right] \tag{C.1}
\]

where \(\hat{K}_A\) is the integral operator defined on the semi-infinite interval \([s, \infty)\) with the Airy kernel,

\[
K_A(t_1, t_2) = \frac{\text{Ai}(t_1)\text{Ai}'(t_2) - \text{Ai}'(t_1)\text{Ai}(t_2)}{t_1 - t_2} \tag{C.2}
\]

and the function \(q(t)\) is the solution of the Painlevé II equation,

\[
q'' = t q + 2q^3 \tag{C.3}
\]

with the boundary condition, \(q(t \to +\infty) \sim \text{Ai}(t)\).

Let us introduce a new function \(R(t)\) such that

\[
F_2(s) = \exp \left[ -\int_{s}^{\infty} dt \, R(t) \right] \tag{C.4}
\]

or, according to the definition, equation (C.1),

\[
R(s) = \frac{d}{ds} \ln[\det(1 - \hat{K}_A)]. \tag{C.5}
\]

doi:10.1088/1742-5468/2010/07/P07010
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Here the logarithm of the determinant can be expressed in terms of the trace:

\[
\ln[\det(1 - \hat{K}_A)] = -\sum_{n=1}^{\infty} \frac{1}{n} \operatorname{Tr} \hat{K}_A^n = -\sum_{n=1}^{\infty} \frac{1}{n} \int_{s}^{\infty} dt_1 \int_{s}^{\infty} dt_2 \cdots \int_{s}^{\infty} dt_n K_A(t_1, t_2) K_A(t_2, t_3) \cdots K_A(t_n, t_1). \quad (C.6)
\]

Taking the derivative of this expression we get

\[
R(s) = -\int_{s}^{\infty} dt (1 - \hat{K}_A)^{-1}(s, t) K_A(t, s)
\]

\[
\equiv -K_A(s, s) - \sum_{n=2}^{\infty} \int_{s}^{\infty} dt_1 \int_{s}^{\infty} dt_2 \cdots \int_{s}^{\infty} dt_{n-1} \times K_A(s, t_1) K_A(t_1, t_2) \cdots K_A(t_{n-1}, s). \quad (C.7)
\]

Substituting here the integral representation of the Airy kernel, equation (C.2),

\[
K_A(t_1, t_2) = \int_{0}^{\infty} dz \, \text{Ai}(t_1 + z) \, \text{Ai}(t_2 + z)
\]

(C.8)

after some efforts in simple algebra one gets

\[
R(s) = \int_{s}^{\infty} dt_1 \int_{s}^{\infty} dt_2 \, \text{Ai}(t_1) \, (1 - \hat{K}_A)^{-1}(t_1, t_2) \, \text{Ai}(t_2).
\]

(C.9)

Taking the derivative of this expression and applying some more efforts in slightly more complicated algebra, we obtain

\[
\frac{d}{ds} R(s) = -q^2(s) \quad (C.10)
\]

where

\[
q(s) = \int_{s}^{\infty} dt \, (1 - \hat{K}_A)^{-1}(s, t) \, \text{Ai}(t).
\]

(C.11)

According to equation (C.10),

\[
R(s) = \int_{s}^{\infty} dt \, q^2(t). \quad (C.12)
\]

Let us introduce two more functions

\[
v(s) = \int_{s}^{\infty} dt_1 \int_{s}^{\infty} dt_2 \, \text{Ai}(t_1) \, (1 - \hat{K}_A)^{-1}(t_1, t_2) \, \text{Ai}'(t_2)
\]

(C.13)

\[
p(s) = \int_{s}^{\infty} dt \, (1 - \hat{K}_A)^{-1}(s, t) \, \text{Ai}'(t).
\]

(C.14)
Taking derivatives of the above three functions $q(s)$, $v(s)$ and $p(s)$, equations (C.11), (C.13) and (C.14), after somewhat painful algebra one finds the following three relations:

\[ q' = p - Rq \]  \hspace{1cm} (C.15)

\[ p' = sq - pR - 2qv \] \hspace{1cm} (C.16)

\[ v' = -pq. \] \hspace{1cm} (C.17)

Taking the derivative of the combination $(R^2 - 2v)$ and using equations (C.10) and (C.17), we get

\[ \frac{d}{ds}(R^2 - 2v) = 2q(p - Rq). \] \hspace{1cm} (C.18)

On the other hand, multiplying equation (C.15) by $2q$ we find

\[ \frac{d}{ds}q^2 = 2q(p - Rq). \] \hspace{1cm} (C.19)

Comparing equations (C.18) and (C.19) and taking into account that the values of all the above functions at $s \to \infty$ are zero, we obtain the following relation

\[ R^2 - 2v = q^2. \] \hspace{1cm} (C.20)

Finally, taking the derivative of equation (C.15) and using equations (C.10), (C.15), (C.16) and (C.20) we easily find

\[ q'' = 2q^3 + sq \] \hspace{1cm} (C.21)

which is the special case of the Painlevé II differential equation \[36,37\]. Thus, substituting equation (C.12) into (C.4) we obtain equation (C.1).

The function $F_2(s)$ gives the probability that a random quantity $t$ described by a probability distribution function $P_{TW}(t)$ has a value greater than a given parameter $s$:

\[ F_2(s) = \int_s^\infty dt P_{TW}(t). \] \hspace{1cm} (C.22)

Taking the derivative of this relation and substituting here the result, equation (C.1), we find

\[ P_{TW}(s) = \exp\left[-\int_s^\infty \int_s^\infty dt (t - s)q^2(t)\right] \times \int_s^\infty dt q^2(t). \] \hspace{1cm} (C.23)

In the limit $s \to \infty$ the function $q(s)$, according to its definition, equation (C.11), must go to zero, and in this case equation (C.21) turns into the Airy function equation, $q'' = sq$. Thus

\[ q(s \to \infty) \simeq \text{Ai}(s) \sim \exp[-\frac{2}{3}s^{3/2}]. \] \hspace{1cm} (C.24)

It can be proved \[38\] that in the opposite limit, $s \to -\infty$, the asymptotic form of the solution of the Painlevé equation (C.21) (which has the right tail Airy function limit, equation (C.24)) is

\[ q(s \to -\infty) \simeq \sqrt{-\frac{1}{2} s}. \] \hspace{1cm} (C.25)
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Substituting the above two asymptotics into equation (C.23), we can estimate the asymptotic behavior for the right and the left tails of the TW probability distribution function:

\[ P_{TW}(s \to +\infty) \sim \exp[-\frac{4}{3}s^{3/2}] \]  \hspace{1cm} (C.26)

\[ P_{TW}(s \to -\infty) \sim \exp[-\frac{1}{12}|s|^3]. \]  \hspace{1cm} (C.27)

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doi:10.1088/1742-5468/2010/07/P07010