Propagating Ferrodark Solitons in a Superfluid: Exact Solutions and Anomalous Dynamics

Xiaquian Yu\textsuperscript{1,2,*} and P. B. Blakie\textsuperscript{2}

\textsuperscript{1}Graduate School of China Academy of Engineering Physics, Beijing 100193, China
\textsuperscript{2}Department of Physics, Centre for Quantum Science, and Dodd-Walls Centre for Photonic and Quantum Technologies, University of Otago, Dunedin, New Zealand

Exact propagating topological solitons are found in the easy-plane phase of ferromagnetic spin-1 Bose-Einstein condensates, manifesting themselves as kinks in the transverse magnetization. Propagation is only possible when the symmetry-breaking longitudinal magnetic field is applied. Such solitons have two types: a low energy branch with positive inertial mass and a higher branch solution with negative inertial mass. Both types become identical at the maximum speed, a new speed bound that is different from speed limits set by the elementary excitations. The physical mass, which accounts for the number density dip, is negative for both types. In a finite one-dimensional system subject to a linear potential, the soliton undergoes oscillations caused by transitions between the two types occurring at the maximum speed.

Introduction— The inertial mass (or effective mass) of emergent quasi-particles contains rich information on the dynamics of quantum many body systems [1]. In quantum fluids the inertial mass of a topological soliton is determined by both the kinetic and interaction energies and is a key quantity governing its dynamics. For instance, the one-dimensional (1D) motion of a dark/grey soliton in a superfluid (bosonic or fermionic) can be described by a Newton equation with negative inertial mass [2], leading to oscillations in a harmonic trap [2–4]. The sign of inertial mass also signals the stability of the soliton in a system of higher than one spatial dimension. Indeed, two- or three-dimensional solitons with negative inertial mass typically decay [5] due to the snake instability (growth of transverse deformations) [6–8]. It is a rather general feature for solitons in quantum fluids that the soliton energy decreases with increasing velocity, giving rise to a negative inertial mass. Relevant examples are dark/grey solitons in bosonic and fermionic quantum gases [2], phase domain walls in binary Bose-Einstein condensates (BECs) with strong coherent coupling [9–12], magnetic solitons in both binary [13] and anti-ferromagnetic spin-1 BECs [14, 15]. A soliton with positive inertial mass should be stable in higher dimensions and exhibit anomalous dynamics.

In this Letter we report on the discovery of two types of exact topological solitons that have positive and negative inertial mass, respectively, occurring as kinks in the transverse magnetization of a ferromagnetic spin-1 BEC. We refer to them as ferro-dark solitons (FDSs). In the zero velocity limit the FDSs connect to the stationary magnetic domain walls (MDWs) recently found in Ref. [16]. The FDSs can only propagate at a finite speed in a longitudinal magnetic field which provides a necessary condition for the motion, i.e., breaking the transverse magnetization conversation. In addition, the FDSs exhibit a number of other novel features different from conventional solitons. When traveling, the transverse magnetization is always zero in the core of a FDS and hence there is no magnetic current. The motion arises from a coupling between the magnetization and nematic degrees of freedom caused by the magnetic field. Interestingly, the moving speed is not limited by group velocities of elementary excitations but has a new speed bound, at which the two types of solitons become identical. We study dynamics of the soliton in a hard-wall trapped quasi-1D system with a superimposed linear potential and find transitions between the two types via internal spin currents, leading to an oscillatory motion. While we focus on the exactly solvable case, we would like to emphasize that FDSs exist with the characteristic features revealed by the exact solutions in the whole easy-plane phase.

Spin-1 BECs— The Hamiltonian density of a spin-1 condensate reads

$$\mathcal{H} = \frac{\hbar^2}{2M} \nabla \psi^2 + \frac{g_n}{2} |\psi|^2 + \frac{g_s}{2} S \psi^2 + q \psi S^2 \psi,$$

where the three-component wavefunction $\psi = (\psi_+, \psi_0, -\psi_-)^T$ describes the atomic hyperfine state. $F = 1, m = +1, 0, -1$, $M$ is the atomic mass, $g_n$ > 0 is the density interaction strength, $g_s$ is the spin-dependent interaction strength, $S = (S_x, S_y, S_z)$ with $S_{j=x,y,z}$ being the spin-1 matrices [17], and $q$ denotes the quadratic Zeeman energy. The spin-dependent interaction term allows for spin-mixing collisions between $m = 0$ and $m = \pm 1$ atoms. At the mean-field level, the dynamics of the field $\psi$ is governed by the Gross-Pitaevskii equations (GPEs)

$$i\hbar \frac{\partial \psi_{\pm 1}}{\partial t} = [H_0 + g_s (n_0 + n_{\pm 1} - n_{x1})] \psi_{\pm 1} + g_n \psi_{0}^2 \psi_{\pm 1},$$

$$i\hbar \frac{\partial \psi_0}{\partial t} = [H_0 + g_s (n_{+1} - n_{-1})] \psi_0 + 2g_n \psi_{0}^2 \psi_{-1},$$

where $H_0 = -\hbar^2 \nabla^2 / 2M + g_n n + n_m = |\psi_m|^2$ and $n = \sum n_m$. Spin-1 BECs support magnetic order [18–22], quantified by the order parameter magnetization $\mathcal{F} \equiv \psi^* \mathcal{S} \psi$. This identifies ferromagnetic order $|\mathcal{F}| > 0$ for $g_s < 0$ (sodium) and anti-ferromagnetic order $\mathcal{F} = 0$ for $g_s > 0$ (C\textsuperscript{2}Na).

Quadratic Zeeman driven propagating FDSs— We consider a uniform ferromagnetic ($g_s < 0$) spin-1 BEC with total number density $n_0$. In the presence of a uniform magnetic field along the z-axis ($0 < q < -2g_s n_0$) [23], the uniform ground state with zero longitudinal magnetization ($F_z = n_{+1}^b - n_{-1}^b = 0$) is transversally magnetized (easy-plane phase) [21, 22], characterized by the transverse magnetization $F_x \equiv F_x + iF_y = \sqrt{8} n_{\pm 1} n_0 \psi_{\pm 1}^* \psi_{\pm 1}$, where $n_{\pm 1}^b = (1 - \tilde{q}) n_{\pm 1}/4$ and $n_0^b = n_0 (1 + \tilde{q})/2$ are the component densities, and $\tilde{q} \equiv q / (4g_n n_0 / \hbar^2).$
\(-q/(2g_an_b)\). The SO(3) symmetry is broken by the magnetic field and the system processes the remnant SO(2) symmetry, parameterized by the rotational angle about the \(z\)-axis \(\tau\).

In the following we focus on a 1D system. In the easy plane phase, exact transverse magnetic kink solutions of Eq. (2) are found for a large spin-dependent interaction strength \(g_s = -g_n/2\) and \(0 < q < -2g_n n_b\). There are two types of such traveling kinks and the transverse magnetizations and the total number densities read

\[
F_{\perp}^{\text{II}}(x,t) = -e^{i\tau} n_b \sqrt{\frac{2\hbar^2}{M (g_n n_b - MV^2 + Q)}} \tanh \left( \frac{x - Vt}{\ell_{\perp}^{\text{II}}} \right),
\]

\[
n_{\perp}^{\text{II}}(x,t) = n_b - \frac{g_n n_b - MV^2 + Q}{2g_n} \sech^2 \left( \frac{x - Vt}{\ell_{\perp}^{\text{II}}} \right),
\]

where \(V\) is the moving velocity,

\[
\ell_{\perp}^{\text{II}} = \sqrt{\frac{2\hbar^2}{M (g_n n_b - MV^2 + Q)}},
\]

and

\[
Q = \sqrt{M^2 V^4 + q^2 - 2g_n M n_b V^2}.
\]

The above kink solutions are of Ising-type and connect regions transversely magnetized in opposite directions [26]. Hereafter we refer to them as ferro-dark solitons (FDSs) and the minus (plus) sign in front of \(Q\) specifies type-I (II) FDS. Unless specified, we choose \(\tau = 0\) for convenience. At the core, the transverse magnetization \(F_c\) is zero while the component densities \(n_{\pm} = \pm 1\) do not vanish for finite velocity \(V\). The corresponding wavefunctions at the exactly soluble region are shown in Table I. Recently a \(^7\)Li spin-1 BEC has been prepared in the strong spin interacting regime close to the exactly soluble point [27].

The inequality \(Q^2 \geq 0\) gives rise to the upper bound of the traveling speed [28]

\[
V \leq \sqrt{\frac{g_n n_b}{M}} \sqrt{1 - \sqrt{1 - \left( \frac{q}{g_n n_b} \right)^2}} \equiv c_{\text{FDS}}.
\]

The speed bound Eq. (7) is markedly different from the group velocities of low-lying elementary excitations which normally set the speed limits [29]. In the easy-plane phase, the gapless branches of the elementary excitations involve spin waves of magnetization \(\mathbf{F}\) (dominantly) and mixed waves of \(F_\perp\) and \(n\), with group velocities at long wavelengths \(c_m = \sqrt{q/(2M)}\) and \(c_{mp} = (n_{\parallel}(g_n + g_s))/M\), respectively [30]. Strikingly, for \(1 > q/g_n n_b > \sqrt{3}/2\), \(c_{\text{FDS}} > c_{mp} > c_m\), implying that the FDSs can travel with speed greater than \(c_m\) and \(c_{mp}\). This can happen because a propagating FDS does not involve magnetic currents (see below). Another conspicuous feature is that the soliton profile does not vanish at \(V = c_{\text{FDS}}\) (see Fig. 1). The velocity of grey solitons in scalar BECs is bounded by the speed of sound, and at this velocity the soliton disappears [29]. At the transition point \(q = g_n n_b\), the easy-plane phase becomes unstable, signalled by the divergence of \(\ell_{\perp}^{\text{II}}\).

![FIG. 1. (a)-(d) Transverse magnetizations and densities of FDSs at \(g_s = -g_n/2\) and \(q = 0.5\) for different velocities: \(V/c_{\text{FDS}} = 1\) (solid line); \(V/c_{\text{FDS}} = 0.1\) (dashed line). (e) Excitation energies of FDSs as functions of \(V^2\) evaluated from Eq. (10) and Eq. (12) at \(q = 0.5\). Here \(\zeta = x - Vt\). The inset shows widths of FDSs, where the \(x\)-axis is the same as in (e).](image)

Similar to scalar gray solitons, the density dip of the type-II FDS becomes shallower for greater velocities [Fig. 1(d)]. In contrast, for the type-I FDS the density dip behaves anomalously and deepens with increasing velocity [Fig. 1(b)]. Crucially, at the maximum velocity \(V = c_{\text{FDS}}\), \(Q = 0\) and the two types of FDSs become identical upon a U(1) gauge transformation, namely \(\tilde{\psi}(x,t) = i\tilde{\psi}^\dagger(x,t)\) (see Table I).

When \(q \to 0\), \(c_{\text{FDS}} \to 0\), implying that the propagation is prohibited in the absence of a magnetic field, where the conservation law of magnetization is restored. In this limit, the two types become degenerate and are related via a SO(3) spin-rotation [16]. Clearly, a magnetic field does not automatically induce motion. At \(V = 0\) the FDSs recover stationary MDWs at finite \(q\) [16, 31].

**Currents**— Moving FDSs involve nematic degrees of freedom and internal spin currents. Since the magnetization is zero at the core of a moving FDS, there is no magnetic current, i.e., \(J^r = h/(2M) \tilde{\psi}^\dagger(\tilde{\psi}/\nabla \tilde{\psi} - \text{H.c.}) = 0\). According to the continuity equation

\[
\frac{\partial F_j}{\partial t} + \nabla \cdot J^r = K_{iz},
\]

the time evolution of magnetic domains enclosed by the MDWs is governed by the source term \(K_{iz} = (2q/h)\tilde{K}_{iz}\) [32, 33], where \(\tilde{K}_{iz} = \sum_{\mathbf{p}} \epsilon_{\mathbf{p}} N_{\mathbf{p}i}\), \(N_{\mathbf{p}i}\) is the nematic tensor, \(N_{\mathbf{p}i} = (S_i S_j + S_j S_i)/2\) and \(i, j \in \{x, y, z\}\). For propagating FDSs \(\tilde{K}_{iz} \neq 0\) and \(\tilde{K}_{iz} \to 0\) as \(V \to 0\). At \(q = 0\), \(\tilde{K}_{iz} = 0\), and FDSs must stay still.

The continuity equations for particle number in each spin state read \(\partial n_{\pm}/\partial t + \nabla \cdot J_{\pm} = 0\) and \(\partial n_0/\partial t + \nabla \cdot J_0 + \sum_{m=\pm} J_{m\rightarrow m} = 0\), where \(J_{\pm\rightarrow \pm} = h/(2M) \tilde{\psi}^\dagger_{\pm} \nabla \tilde{\psi}_{\pm} - \text{H.c.}\) are the component number current densities [34], and

\[
J_{\pm\rightarrow \pm} = J_{\pm\rightarrow \pm} = \frac{g_k}{\hbar} \left( \psi_{\pm}^\dagger \psi_{\pm} - \text{H.c.} \right).
\]
TABLE I. Wavefunctions and currents of propagating FDSs in the exactly solvable regime \((g_x = -g_x/2, 0 < q < -2q, n_s)\). The coefficients satisfy the following relations: \((a^0M^I = \delta^0q^III)\) and \((a^0M^I)^2 + (\delta^0q^III)^2 = (\delta^0q^III)^2 = 1\). It is straightforward to check that stationary solutions are obtained when \(V \to 0\) [16, 24]. Here \(K^2 = \sum K_{\alpha}^2\) is SO(2) rotationally invariant [25]. The counter-propagating solution is \(\psi^\ast (x,-t)\).

are the internal spin currents, reflecting the internal coherent spin exchange dynamics: \((00) \leftrightarrow (+1)|-1\) [18–20]. Rewriting Eq. (9) in terms of wavefunction phases \((\theta_{\pm 1,0})\) and densities, we obtain \(J^x_{\pm 1,-0} = (2n^0q_{\pm 1}g_x/\hbar) \sin[2(\theta_{1,0} - \theta_0)]\) which suggests an analogy to Josephson currents [35, 36]. It is important to note that these built-in currents are invariant under SO(2) rotations \((e^{-i\tau_3})\). Table I shows the expressions of currents at the exactly solvable point. Interestingly, \(J^x_{\pm 1} \) and \(J^0_{\pm 1}\) have opposite signs and \(\int dx J_{\pm 1, -0} = 0\), forming a Josephson vortex-like structure near the core of a FDS.

Excitation energy and inertial mass—The excitation energy of FDSs can be obtained by evaluating the difference of grand canonical energies \(\delta K = K_{\text{FDS}} - K_{\text{fg}}\), where \(K_{\text{FDS}} = \int dx (\mathcal{H}(\psi) - \mu)\), \(K_{\text{fg}} = \int dx (\mathcal{H}(\psi_0) - \mu_{\text{fg}})\), \(\psi_0\) is the ground state wavefunction and \(\mu = (g_x + g_z)n_s + q/2\) is the chemical potential. For type-I FDSs, we obtain

\[
\delta K^I(q, V^2) = \frac{\sqrt{2}\hbar (g_x n_b - MV^2 - Q)^{3/2}}{3g_n \sqrt{M}}. \tag{10}
\]

Expanding Eq. (10) around \(V = 0\), we have \(\delta K^I(q, V^2) = \delta K^I(q, 0) + M^I_{\text{in}} V^2/2 + \alpha(V^2)\) where \(\delta K^I(q, 0) = \sqrt{2}\hbar (g_x n_b - Q)^{3/2}/(3g_n \sqrt{M})\) and the inertial mass is

\[
M^I_{\text{in}} \equiv \frac{\delta \delta K^I(q, V^2)}{V^2} \bigg|_{V=0} = \frac{2\sqrt{2}\hbar (g_x n_b - Q)^{3/2}}{g_n \eta q} > 0. \tag{11}
\]

As \(q \to 0\), \(M^I_{\text{in}} \to \infty\) and the FDS becomes infinitely heavy, consistent with the absence of propagation at zero magnetic field due to the conservation of magnetization [16]. In contrast to the normal behavior of grey solitons, the excitation energy \((\delta K^I)\) of the type-I FDS increases monotonically with increasing \(V^2\) [Fig. 1(e)], in accordance with the anomalous behavior of the density [Fig. 1(b)]. It is worth noting that here every component density has a dip (see Table I) and the inertial mass of type-I FDSs being positive is a highly non-trivial nonlinear effect. Following conventional arguments [5] the positive inertial mass explains the stability of MDWs against transverse snake perturbations in 2D [16].

The physical mass is defined as \(M_{\text{phy}} \equiv M_0 N^\ast\), where \(N^\ast = \int dx [\eta(x) - n_0]\). For type-I FDSs, we obtain \(M_{\text{phy}} = -2\hbar^2/(g_n \eta^2) < 0\). In the presence of an external potential \(U\), a soliton with negative physical mass experiences an effective force from the surrounding liquid pointing in the opposite direction to \(-\nabla U\) (similar to buoyant force). For a scalar grey soliton the inertial and the physical masses are both negative and it exhibits normal particle-like behavior, e.g., oscillations in a harmonic potential [2, 4]. Whereas a type-I FDS in a harmonic potential would be expelled, i.e., moves away from the potential minimum.

The excitation energy of the type-II FDS is

\[
\delta K^{II}(q, V^2) = \frac{\sqrt{2}\hbar (g_x n_b - MV^2 + Q)^{3/2}}{3g_n \sqrt{M}} \tag{12}
\]

with \(\delta K^{II}/\partial V^2 < 0\) [Fig. 1(e)]. Expansion of Eq. (12) leads to \(\delta K^{II}(q, V^2) = \delta K^{II}(q, 0) + M^{II}_{\text{in}} V^2/2 + \alpha(V^2)\), where \(\delta K^{II}(q, 0) = \sqrt{2}\hbar (g_x n_b + Q)^{3/2}/(3g_n \sqrt{M})\) and the inertial mass

\[
M^{II}_{\text{in}} \equiv \frac{\delta \delta K^{II}(q, V^2)}{V^2} \bigg|_{V=0} = -\frac{2\sqrt{2}\hbar (g_x n_b + Q)^{3/2}}{g_n \eta q} < 0. \tag{13}
\]

Consistently, \(M^{II}_{\text{in}} \to -M^{II}_{\text{phy}} \to -\infty\) as \(q \to 0\). The physical mass \(M_{\text{phy}}^{II} = -2\hbar^2/(g_n \eta^2) < 0\). Thus, the inertial and physical mass of the type-II FDS is similar to those of ordinary grey/dark solitons. Excitation energies of type-I and type-II FDSs coincide smoothly at the maximum speed [Fig. 1(e)],
making transitions between the two types of FDSs possible under certain circumstances.

**Oscillations between type-I and type-II FDSs** — As discussed earlier the FDS does not vanish as \( V \to c_{\text{FDS}} \), so a natural question is what will happen if it is further accelerated? Let us consider a hard-wall trapped quasi-1D spin-1 BEC subjected to a linear potential whose gradient is along the positive \( x \)-axis. A \( V = 0 \) type-I FDS is initially placed near the left end of the system, and the later dynamic shows, surprisingly, a periodic motion. The FDS accelerates until it reaches the maximum speed (the local value of \( c_{\text{FDS}} [37] \)) at which point it smoothly transforms into a type-II FDS. Due to the sign change of the inertial mass (or more generally \( \partial \mathcal{K}^I / \partial V^2 > 0 \to \partial \mathcal{K}^{II} / \partial V^2 < 0 \)), it starts to accelerate in the opposite direction. After reaching the turning point, the FDS starts to move to the left. It converts back to the type-I FDS and experiences positive acceleration again when gaining the maximum speed. Later it returns to the initial configuration. Note that during the motion there is no sign change of the physical mass. Numerical simulations show that this process continues without decay (see Fig. 2 and a movie [30]).

![Fig. 2](image-url)

**Fig. 2.** Oscillations of a FDS in a hard-wall trapped spin-1 BEC with a superimposed linear potential [38]. The system size is 200\( \xi_\perp \), \( g_s/\bar{n}_0 = -1/2 \) and \( \bar{q} = q/(-2q_\perp \bar{n}_\perp) = 0.3 \). Here \( \bar{n}_0 \) is the average density, \( t_0 = \hbar / g_s \bar{n}_0 \) and \( \xi_\parallel = \hbar / \sqrt{M_{\parallel} \bar{n}_0} \) is the density healing length. Upper and middle panels show spin and density dynamics of a FDS, respectively. The transverse magnetization is always zero at the core (see also Fig. S3 [30]) and the topological characteristic, i.e., the sign of \( F_\parallel \), is kept. Bottom panel shows the velocity of the FDS as a function of time, obtained by taking the derivative of its position with respect to time. The slope refers to the acceleration of the FDS and indicates the sign of the inertial mass (positive: blue; negative: red). The transition between type-I and type-II FDSs occurs when the slope changes sign at the maximum speed. Here \( c_{\text{FDS}} \) is the local speed limit for the (background) density at the position where \( dV_x / dt \) changes sign.

During the motion the total number density profile of the soliton has only minor changes with respect to the local background density (see Fig. 2 and Fig. S3 [30]). However internal oscillations (driven by the gradient of the external potential) between \( m = \pm 1 \) and \( m = 0 \) spin states and the excitation energy for one complete cycle of the motion described in Fig. 2. The black arrows specify the evolution direction. (a1)-(b5) show component densities of the initial state (type-I FDS with zero velocity) [blue], at the maximum velocity [black], at the turning point (type-II FDS with zero velocity) [red], at the negative maximum speed [black], and of the final state (returning the initial state) [blue], respectively. (c) shows analytical predictions (solid lines) for \( n_b = \bar{n}_b \) vs. numerical results (markers) for the mapped uniform system with the same density (see main text). Number labels indicate the stages corresponding to those showing in the upper panels. Note that the total energy [39] is conserved.

![Fig. 3](image-url)

**Fig. 3.** Internal oscillations between \( m = \pm 1 \) and \( m = 0 \) spin states and the excitation energy for one complete cycle of the motion described in Fig. 2. The black arrows specify the evolution direction. (a1)-(b5) show component densities of the initial state (type-I FDS with zero velocity) [blue], at the maximum velocity [black], at the turning point (type-II FDS with zero velocity) [red], at the negative maximum speed [black], and of the final state (returning the initial state) [blue], respectively. (c) shows analytical predictions (solid lines) for \( n_b = \bar{n}_b \) vs. numerical results (markers) for the mapped uniform system with the same density (see main text). Number labels indicate the stages corresponding to those showing in the upper panels. Note that the total energy [39] is conserved.

**Conclusion** — We discover a propagating magnetic kink corresponding to a topological soliton with negative physical mass and positive inertial mass in the easy-plane phase of a ferromagnetic spin-1 BEC. It can convert to its higher energy
counterpart with simplified physical and inertial mass at a novel maximum speed that can be greater than the group velocities of elementary excitations which normally set the speed limits. The transition between the two types induces oscillations in a linear potential [40]. Our findings open up a possibility to explore novel domain wall/soliton dynamics and could be highly relevant to out-of-equilibrium quench dynamics in 1D ferromagnetic superfluids [41, 42]. Advances in engineering optical potential [43–45] and nondestructive spin-sensitive imaging methods [45–47] open the possibility of experimental investigations of the ferrodark soliton dynamics.

Acknowledgment— We thank M. Antonio, L. Qiao, D. Bailie and Y. Yang for useful discussions. We particularly thank J. N. BiGuo for pointing out that the energy expression of type-I FDSs can be simplified to the current form. X.Y. acknowledges support from NSAF (No. U1930403) and NSFC (No. 12175215). P.B.B acknowledges support from the Marsden Fund of the Royal Society of New Zealand.

References

[1] G. D. Mahan, Many-particle physics (Springer Science & Business Media, 2013).

[2] R. G. Scott, F. Dalfovo, L. P. Pitaevskii, and S. Stringari, Phys. Rev. Lett. 106, 185301 (2011).

[3] T. Busch and J. R. Anglin, Phys. Rev. Lett. 84, 2298 (2000).

[4] V. V. Konotop and L. Pitaevskii, Phys. Rev. Lett. 93, 240403 (2004).

[5] A. M. Kamchatnov and L. P. Pitaevskii, Phys. Rev. Lett. 100, 160402 (2008).

[6] E. Kuznetsova and S. Turitsyn, Zh. Eksp. Teor. Fiz 129, 104 (2008).

[7] A. E. Muryshev, H. B. van Linden van Heuvel, and G. V. Shlyapnikov, Phys. Rev. A 60, R2665 (1999).

[8] Long wavelength transverse deformations of a soliton with negative inertial mass will be enhanced and eventually lead to a breakdown of a soliton.

[9] A. Gallelli, L. P. Pitaevskii, S. Stringari, and A. Recati, Phys. Rev. A 100, 023607 (2019).

[10] C. Qu, M. Tylutki, S. Stringari, and L. P. Pitaevskii, Phys. Rev. A 95, 033614 (2017).

[11] S. S. Shamaiov and J. Brand, SciPost Phys. 4, 18 (2018).

[12] In coherently coupled BECs, the Son-Stephanov phase domain wall [48] has positive inertial mass for weak coherent coupling strengths [9–11]. However a long wall fragments into smaller ones due to the bending caused by vortices on the ends of the wall [9, 49].

[13] C. Qu, L. P. Pitaevskii, and S. Stringari, Phys. Rev. Lett. 116, 160402 (2016).

[14] A. Faroli, D. Trypogeorgos, C. Mordini, G. Lamporesi, and G. Ferrari, Phys. Rev. Lett. 125, 030401 (2020).

[15] X. Chai, D. Lao, K. Fujimoto, R. Hamazaki, M. Ueda, and C. Raman, Phys. Rev. Lett. 125, 030402 (2020).

[16] X. Yu and P. B. Blakie, Phys. Rev. Research 3, 023043 (2021).

[17] Generators of the rotational group SO(3):

$$S_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, S_2 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$
Supplemental Material for “Propagating ferrodark solitons in a superfluid: Exact solutions and anomalous dynamics”

ELEMENTARY EXCITATIONS IN THE EASY-PLANE PHASE

Let us denote $\psi_g$ as the ground state wavefunction in the easy-plane phase ($0 < q < -2g_n n_b$). Substituting the perturbed wavefunction $\psi = \psi_g + \delta \psi$ with $\delta \psi = \epsilon [u(x)e^{-i\omega t} + v^*(x)e^{i\omega^* t}]$ into 1D Gross-Pitaevskii equations (Eq. (2) in the main text) and keeping the leading order terms, we obtain the bosonic Bogoliubov-de Gennes (BdG) equations.
where $\epsilon$ is a dimensionless small number, $E = \hbar \omega$, $\mathcal{L}_{\text{GP}} \equiv -\hbar^2 \nabla^2 / 2M + g_n \psi \psi^\dagger + \sum_{i=1}^3 S_i \psi \psi^\dagger S_i + q m^2$, $X \equiv g_s \sum_{i=1}^3 S_i \psi \psi^\dagger S_i + g_n \psi \psi^\dagger$, $\Delta \equiv g_n \psi \psi^\dagger T + g_s \sum_{i=1}^3 (S_i \psi \psi^\dagger S_i) T$ and $\mu = (g_n + g_s) n_b + q/2$ with $n_b = |\psi|^{2}$. Note that $\mathcal{L}_{\text{GP}} \psi = \mu \psi$. Since the system has translational symmetry, it is natural to parameterize the perturbations according to the wave-vector $k_x$: $u(x) = u e^{ik_x x}$ and $v(x) = ve^{ik_x x}$. Solving Eq. (S1), we obtain

$$E_m(k) = \pm \frac{\hbar}{2M} \sqrt{(k_e^2 q^2 + 2M \Gamma k_x^2)},$$

$$E_{\text{mp}}(k) = \pm \frac{g_s}{-2M} (2g_n k_x^2 M n_b + k_x^4 + 2M q^2) + 8g_s M^2 n_b^2 - 2g_n^2 k_x^2 M n_b^2 + 2M \Gamma_k k_x^2),$$

$$E_{\text{gap}}(k) = \pm \frac{g_s}{-2M} (2g_n k_x^2 M n_b + k_x^4 + 2M q^2) + 8g_s M^2 n_b^2 - 2g_n^2 k_x^2 M n_b^2 - 2M \Gamma_k k_x^2),$$

where $\Gamma_k = \sqrt{g_s \left( (g_n n_b + 3 g_s)^2 - q^2 (g_n + 2g_s) \right) k_x^2 + 2g_s M n_b (4g_s^2 n_b^2 - q^2) k_x^2 + g_s M^2 \left( q^2 - 4g_s^2 n_b^2 \right) k_x^2}$. For small $k_x$, $\Gamma_k \approx g_s n_b (g_n + 3 g_s) k_x^2 - 2g_s M (4g_s^2 n_b^2 - q^2)$ and the spectrum of the two gap-less modes read

$$E_m(k_x) \approx \pm c_m \hbar k_x \quad \text{and} \quad E_{\text{mp}}(k_x) \approx \pm c_{\text{mp}} \hbar k_x,$$

where

$$c_m = \sqrt{\frac{q}{2M}} \quad \text{and} \quad c_{\text{mp}} = \sqrt{\frac{n_b (g_n + g_s)}{M}}.$$

The spectrum, and the fluctuations and magnetic currents associated with the gap-less excitations are shown in Fig. S1.

**WAVEFUNCTIONS OF PROPAGATING MDWS**

Fig. S2 shows examples of moving FDS wavefunctions presented in Table I in the main text. The fact that $\text{Im}(\psi_x^1)$ and $\text{Re}(\psi_x^1)$ are constants admit exact solutions at $g_s = -g_n/2$ and $0 < q < -2g_n n_b$. Similarly, for type-II FDSs, $\text{Re}(\psi_x^1)$ and $\text{Im}(\psi_x^1)$ are constants. Away from the exact solvable regime, the constant components develop humps or dips near the domain wall core depending on the value of $g_s$.

![Wavefunctions of type-I FDSs (left) and the type-II FDSs (right) for $V/c_{\text{FDS}} = 0.1$ (blue) and $V/c_{\text{FDS}} = 1$ (red) at $g_s = -g_n/2$ and $\bar{q} = 0.5$. Here $\zeta = x - Vt$.](image-url)
OSCILLATIONS IN A LINEAR POTENTIAL

Mapping to a uniform system

It is possible to extract the FDS energy $\delta K$ from the simulated dynamics to compare with analytical predictions which are valid for a uniform system. We construct a mapping $\tilde{\psi}^m(t) \rightarrow \tilde{\psi}^m(t) = \tilde{\psi}^0 \hat{\psi}^m(t)/\psi^m$ for each spin state ($m = -1, 0, +1$), where $\tilde{\psi}$ is the ground state in the presence of the potentials (linear+hard-wall), $\tilde{\psi}$ is the uniform ground state with density $\bar{n}_b$. The mapped wavefunction $\tilde{\psi}^m(t)$ describes a FDS in a uniform system with density $\bar{n}_b$ and the corresponding excitation energy reads $\delta K[\tilde{\psi}^m(t)] = K[\tilde{\psi}^m(t)] - K[\tilde{\psi}^m]$, where $K[\tilde{\psi}^m(t)] = \int dx \left( \mathcal{H}[(\tilde{\psi}^m(t)] - \mu \tilde{\psi} \hat{\psi} \right)$. $K[\tilde{\psi}^m] = \int dx \left( \mathcal{H}[\tilde{\psi}^m] - \mu \tilde{\psi} \hat{\psi} \right)$

$$\mathcal{H}[\psi] = \frac{\hbar^2 |\nabla \psi|^2}{2M} + \frac{g_n}{2} |\psi|^2 + \frac{g_s}{2} |\psi|^2 S^z \psi + q \psi^1 S^z \psi$$

(S7)

and $\mu = (g_n + g_s)\bar{n}_b + q/2$ is the chemical potential.

Dynamics in components

Here we present further details of oscillations presented in the main text. Fig. S3 shows number densities and the magnetization density at different stages of the oscillation and Fig. S4 shows the internal dynamics of component densities.

Away from the exactly solvable regime

The properties exhibited by the exact solutions have no qualitative change away from the exact solvable regime ($g_s = -g_n/2$, $0 < q < -2g_n\bar{n}_b$). Fig. S5 shows oscillatory dynamics for $g_s/g_n = -0.2$ and $g_s/g_n = -0.6$. 

FIG. S3. A complete circle of the oscillation of a FDS in a linear potential. The parameters are the same as in Fig. 3 in the main text. The black arrows specify the evolution direction. From left to right: densities of initial state [type-I FDS with $V = 0$] (blue), at the maximum velocity (black), at the turning point (red), at the negative maximum speed (back), and of the final state [returning the initial state] (blue).
FIG. S4. Internal oscillations between $m = \pm 1$ and $m = 0$ spin states during the motion described in Fig. 2 in the main text. Top and middle panels show component number densities (subtracting their background values) in $m = \pm 1$ and $m = 0$ spin states, respectively. Bottom panel shows oscillations of number of missing particles (due to the density dip) in $m = \pm 1$ states $\delta N_{\pm 1}$ (solid line) and $\delta N_0$ (dashed line). At the maximum speed, $r_c \delta N_{\pm 1} = \delta N_0$ and transitions between type-I (blue) and type-II (red) FDSs occur. Here $r_c \equiv (\delta N_0/\delta N_{\pm 1})|_{v=V_{FDS}} = 2 \sqrt{[g_s n(x_c) + q]/[g_s n(x_c) - q]}$, $n(x_c)$ is the background density at position $x_c$ where the transition occurs.

FIG. S5. Oscillations away from the exactly solvable regime. The parameters are $g_s/g_n = -0.2$ and $\tilde{q} = q/(2g_s \bar{n}_b) = 0.1$ (left); $g_s/g_n = -0.6$ and $\tilde{q} = q/(2g_s \bar{n}_b) = 0.1$ (right). The other parameters are the same as in Fig. 2 in the main text. Here $\xi_n = \hbar/\sqrt{Mg_n \bar{n}_b}$ is the density healing length, $\bar{n}_b$ is the average density, and $t_0 = \hbar/g_s \bar{n}_b$. 