MULTIPLIER IDEALS, V-FILTRATION, AND SPECTRUM

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Abstract. For an effective divisor on a smooth algebraic variety or a complex manifold, we show that the associated multiplier ideals coincide essentially with the filtration induced by the filtration \( V \) constructed by B. Malgrange and M. Kashiwara. This implies another proof of a theorem of L. Ein, R. Lazarsfeld, K.E. Smith and D. Varolin that any jumping coefficient in the interval \((0,1]\) is a root of the Bernstein-Sato polynomial up to sign. We also give a refinement (using mixed Hodge modules) of the formula for the coefficients of the spectrum for exponents not greater than one or greater than the dimension of the variety minus one.

Introduction

Let \( X \) be a smooth complex algebraic variety or a complex manifold, and \( D \) be an effective divisor on \( X \) with a defining equation \( f \). The multiplier ideal \( J(D) \) is a coherent sheaf of ideals of the structure sheaf \( O_X \), and can be defined by using an embedded resolution \( \pi : (X', D') \to (X, D) \), see [8], [10], [17], [21]. This is defined also for the \( \mathbb{Q} \)-divisors \( \alpha D \) with \( \alpha > 0 \), and we get a decreasing family \( \{J(\alpha D)\}_{\alpha \in \mathbb{Q}} \), where \( J(\alpha D) = O_X \) for \( \alpha \leq 0 \). By construction there exist positive rational numbers \( 0 < \alpha_1 < \alpha_2 < \cdots \) such that \( J(\alpha_j D) = J(\alpha D) \neq J(\alpha_{j+1} D) \) for \( \alpha_j \leq \alpha < \alpha_{j+1} \) where \( \alpha_0 = 0 \), see (1.1). These numbers \( \alpha_j (j > 0) \) are called the jumping coefficients (or numbers) of the multiplier ideals associated to \( D \). We define the graded pieces \( G(D, \alpha) \) to be \( J((\alpha - \varepsilon)D) / J(\alpha D) \) for \( 0 < \varepsilon \ll 1 \).

Let \( i_f : X \to Y := X \times \mathbb{C} \) denote the embedding by the graph of \( f \), and \( t \) be the coordinate of \( \mathbb{C} \). Let \( B_f (= O_X \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]) \) denote the direct image of the left \( D_X \)-module \( O_X \) by \( i_f \). B. Malgrange [20] constructed the filtration \( V \) on \( B_f \) and M. Kashiwara [15] did it in a more general case, see also [16]. We index \( V \) decreasingly by rational numbers so that the action of \( \partial_t - \alpha \) on \( \text{Gr}^\alpha V_j f (= V^\alpha B_f / V^{>\alpha} B_f) \) is locally nilpotent. We denote also by \( V \) the induced filtration on \( O_X (= O_X \otimes 1) \). Then

0.1. Theorem. We have \( V^\alpha O_X = J(\alpha D) \) if \( \alpha \) is not a jumping coefficient. In general, \( J(\alpha D) = V^{\alpha + \varepsilon} O_X \) and \( V^\alpha O_X = J((\alpha - \varepsilon)D) \) for any \( \alpha \in \mathbb{Q} \) if \( \varepsilon > 0 \) is sufficiently small.

This is actually an immediate consequence of [24], 3.5 and [23], 3.3.17. Indeed, the normal crossing case was proved in the former (see also (2.3) below) and the general case...
follows from it using the theory of bifiltered direct images developed in the latter, see (3.2) and (3.4) below. By Theorem (0.1) we have

0.2. Corollary. $\text{Gr}_V^\alpha \mathcal{O}_X = \mathcal{G}(D, \alpha)$ for any $\alpha \in \mathbb{Q}$.

This corollary together with Theorem (0.3) below has been conjectured in [3], and proved in [5] using [24], 3.5 together with [24], 2.14 instead of [23], 3.3.17. As a corollary of (0.2), we get another proof of a theorem of L. Ein, R. Lazarsfeld, K.E. Smith and D. Varolin [12] that any jumping coefficient in the interval $(0, 1]$ is a root of the $b$-function (i.e. the Bernstein-Sato polynomial) up to sign.

If $D$ is reduced, let $\rho : \tilde{D} \to D$ be a resolution of singularities. It is well known that $\rho_* \omega_{\tilde{D}} \subset \omega_D$ is independent of $\tilde{D}$ (see [13]). Let $\tilde{\omega}_D = \rho_* \omega_{\tilde{D}}$. Combining (0.2) with [25], we get

0.3. Theorem. If $D$ is reduced, there exists a decreasing filtration $V$ on $\omega_D/\tilde{\omega}_D$ indexed by $0 \leq \alpha \leq 1$ (i.e. $V^{\geq \alpha} \omega_D/\tilde{\omega}_D = \omega_D/\tilde{\omega}_D$, $V^{< \alpha} \omega_D/\tilde{\omega}_D = 0$) such that

$$\text{Gr}_V^\alpha (\omega_D/\tilde{\omega}_D) = \omega_D \otimes \mathcal{G}(D, \alpha) \quad \text{for } 0 < \alpha < 1,$$

and $\text{Gr}_1^V(\omega_D/\tilde{\omega}_D)$ is a quotient of $\omega_D \otimes \mathcal{G}(D, 1)$.

This means that the $\mathcal{G}(D, \alpha)$ measure the non rationality of the singularity of $D$. In the isolated singularity case, (0.3) is related to [18], [19], [29]. For $\alpha = 1$ we have a more precise formula, see (3.5–6) below.

Let $x \in X$. J. Steenbrink ([26], [27]) defined the spectrum $\text{Sp}(f, x)$ by using the monodromy and the mixed Hodge structure on the vanishing cohomology groups $H^j(F_x, \mathbb{C})$, where $F_x$ denotes the Milnor fiber of $f$ around $x$. In this paper we use the normalization such that the spectrum is a fractional polynomial $\sum_{0 < k < n} n_k t^k$ where $n = \dim X$, see (5.2). In [3] and [4], the “localized” graded pieces $\mathcal{G}(D, x, \alpha)$, which are quotients of $\mathcal{G}(D, \alpha)$, were defined. Using a formula for nearby cycles in [9] together with the theory of cyclic covers and Serre duality, the following theorem was proved:

0.4. Theorem [4]. $n_\alpha = \dim \mathcal{G}(D, x, \alpha)$ for $0 < \alpha \leq 1$.

By construction there is a coherent sheaf $K_\alpha$ on the embedded resolution $(X', D')$ of $(X, D)$ such that $\mathcal{G}(D, x, \alpha)$ is the global sections of $K_\alpha$. Here $E := \pi^{-1}(x)_{\text{red}}$ is assumed to be a divisor with normal crossings, and $K_\alpha$ is actually the restriction of $\mathcal{G}(D, x, \alpha)$ to $E$ as an $O$-module (up to the twist by the relative dualizing sheaf). We have also $K'_\alpha$ by twisting $K_\alpha$ with a certain line bundle, see (5.4.1). We prove that these sheaves coincide with the first nonzero piece of the Hodge filtration of the mixed Hodge modules which calculate the vanishing cohomology groups with or without compact supports, see (4.3). This implies

0.5. Theorem. For $j \in \mathbb{Z}$ and $\alpha \in (0, 1]$, there are natural isomorphisms

$$H^j(E, K_\alpha) = F^{n-1} H^{j+n-1}(F_x, \mathbb{C})_{e(-\alpha)},$$

$$H^j(E, K'_\alpha) = F^{n-1} H^{j+n-1}(F_x, \mathbb{C})_{e(-\alpha)}.$$
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where $H^j(F_x, \mathbb{C})_\lambda$ is the $\lambda$-eigenspace of the monodromy and $e(-\alpha) = \exp(-2\pi i \alpha)$.

This gives a more direct proof of (0.4). Using duality, (0.5) implies

0.6. Corollary. $n_\alpha = \chi(E, K'_n-\alpha)$ for $n - 1 < \alpha < n$.

This was also proved in [3]. Note that (0.5) is useful for an explicit calculation of the spectrum. In the case $n = 3$, it is reduced to that of $\dim G(D, x, \alpha)$ and the monodromy.

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In Sect. 1 we recall the definition of multiplier ideal. In Sect. 2 we review the theory of $V$-filtrations, and calculate it in the normal crossing case. In Sect. 3 we review the theory of bifiltered direct images and prove Theorem (0.1). In Sect. 4 we study the localizations of nearby cycle sheaves for the proof of (0.5). In Sect. 5 we recall the definition of spectrum and prove Theorem (0.5).

1. Multiplier Ideals

1.1. Let $X$ be a smooth complex algebraic variety or a complex manifold, and $D$ be an effective divisor on it. Let $\pi : (X', D') \to (X, D)$ be an embedded resolution, i.e. $D' := \pi^*D$ is a divisor with normal crossings. Here $\pi$ is assumed to be projective. Let $D'_i$ be the irreducible components of $D'$ with $m_i$ the multiplicity so that $D' = \sum_i m_i D'_i$. For positive rational numbers $\alpha$, we define

$$J(\alpha D) = \pi_\bullet(\omega_{X'/X} \otimes \mathcal{O}_{X'}(-\sum_i [\alpha m_i] D'_i)),$$

and $J(\alpha D) = \mathcal{O}_X$ for $\alpha \leq 0$. Here $\omega_{X'/X}$ is the relative dualizing sheaf $\omega_{X'} \otimes \pi^*\omega_X$, and $\pi_\bullet$ denotes the sheaf theoretic direct image. They form a decreasing family of coherent subsheaves of $\mathcal{O}_X$, and there exist (at least locally) a strictly increasing sequence of positive rational numbers $\{\alpha_j\}_{j>0}$ (called the jumping coefficients) such that

$$J(\alpha_j D) = J(\alpha D) \neq J(\alpha_{j+1} D) \quad \text{for } \alpha_j \leq \alpha < \alpha_{j+1},$$

where $\alpha_0 = 0$. We define the graded pieces by $\mathcal{G}(D, \alpha) = J((\alpha - \varepsilon)D)/J(\alpha D)$ for $0 < \varepsilon \ll 1$. It is well-known (see [8], [10], [21]) that the higher direct images vanish, i.e.

$$R^j \pi_\bullet(\omega_{X'/X} \otimes \mathcal{O}_{X'}(-\sum_i [\alpha m_i] D'_i)) = 0 \quad \text{for } j > 0.$$

So the $\mathcal{G}(D, \alpha)$ are isomorphic to the direct image of the quotient sheaves.

1.2. Remark. Originally, the multiplier ideal $J(\alpha D)$ is defined by the local integrability of $|g|^2/|f|^{2\alpha}$ for $g \in \mathcal{O}_X$, see [8], [21]. This interpretation justifies the definition of the graded pieces $\mathcal{G}(D, \alpha)$. 
2. $V$-Filtration

2.1. Let $X$ be a smooth complex algebraic variety or a complex manifold of dimension $n$, and $D$ be an effective divisor with a defining equation $f$. Let

\begin{equation}
B_f = \mathcal{O}_X \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]
\end{equation}

as in the introduction. The action of $\mathcal{D}_Y$ on $\mathcal{O}_X \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]$ is given by

\begin{align*}
g(h \otimes \partial_t^i) &= gh \otimes \partial_t^i, \\
\xi(h \otimes \partial_t^i) &= \xi h \otimes \partial_t^i - (\xi f)h \otimes \partial_t^{i+1} \\
\partial_t(h \otimes \partial_t^i) &= h \otimes \partial_t^{i+1}, \\
t(h \otimes \partial_t^i) &= fh \otimes \partial_t^i - jh \otimes \partial_t^{i-1}
\end{align*}

for $g, h \in \mathcal{O}_X, \zeta \in \Theta_X$, i.e. $\mathcal{O}_X \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]$ is identified with $\mathcal{O}_X \otimes_{\mathbb{C}} \mathbb{C}[\partial_t] \Lambda(f-t)$ where $\delta(f-t)$ is the delta function.

The filtration $V$ on $B_f$ (see [15], [20]) is an exhaustive decreasing filtration of coherent $\mathcal{D}_X$-submodules, and is characterized by the following properties:

(i) $t(V^\alpha B_f) \subset V^{\alpha+1} B_f$, $\partial_t(V^\alpha B_f) \subset V^{\alpha-1} B_f$ for any $\alpha \in \mathbb{Q}$,

(ii) $t(V^\alpha B_f) = V^{\alpha+1} B_f$ for $\alpha > 0$,

(iii) the action of $\partial_t t - \alpha$ on $\text{Gr}^\alpha V B_f$ is locally nilpotent.

Here $\text{Gr}^\alpha V = V^\alpha / V > \alpha$ with $V > \alpha = \bigcup_{\beta > \alpha} V^\beta$, and $V$ is indexed discretely by $\mathbb{Q}$ so that for some positive integer $m$

\begin{equation}
V^\alpha = V^{j/m} \text{ for } (j-1)/m < \alpha \leq j/m.
\end{equation}

In this case $m$ can be taken to be the order of the semisimple part of the monodromy on the nearby cycles.

The Hodge filtration $F$ on $B_f$ is defined by the order of $\partial_t$ shifted by $-n$, i.e.

\begin{equation}
F_{p-n} B_f = \bigoplus_{0 \leq j \leq p} \mathcal{O}_X \otimes \partial_t^j.
\end{equation}

This shift comes from the shift of the Hodge filtration $F$ on $\mathcal{O}_X$ which is defined by $\text{Gr}^\alpha_p = 0$ for $p \neq -n$. (Actually this is for the corresponding right $\mathcal{D}$-module $\omega_X$. We use this because the direct image by a closed embedding requires a shift of filtration if we use the usual filtration for left $\mathcal{D}$-modules.) We have

\begin{equation}
F_{-n} \text{Gr}^\alpha V B_f = 0 \text{ for } \alpha \leq 0,
\end{equation}

because $\partial_t : \text{Gr}^{\alpha+1} V (B_f, F) \to \text{Gr}^\alpha V (B_f, F[-1])$ is strictly surjective for $\alpha \leq 0$, see [23], 3.2.1 and 5.1.4. In particular $V^0 \mathcal{O}_X = \mathcal{O}_X$. This is related to [14].

2.2. Normal crossing case. Assume $D = f^{-1}(0)$ is a divisor with normal crossings. Let $(x_1, \ldots, x_n)$ be a local coordinate at $x \in X$ such that $f = \prod_i x_i^{m_i}$ (étale locally in the algebraic case), where $m_i$ are nonnegative integers. For a multi index $\nu = (\nu_1, \ldots, \nu_n) \in$
Let $x^\nu = \prod x_i^{\nu_i}$. We define the filtration $V^\alpha O_{X,x}$ to be the $O_{X,x}$-submodule generated by

\[(2.2.1)\quad x^\nu \quad \text{with} \quad \nu_i + 1 \geq m_i \alpha \quad \text{for any} \quad i.\]

This is generated by one element, and

\[(2.2.2)\quad f(V^\alpha O_{X,x}) = V^{\alpha+1} O_{X,x} \quad \text{for} \quad \alpha > 0.\]

By [24], 3.4, $V^\alpha B_{f,x}$ is generated over $D_{X,x}$ by

\[(2.2.3)\quad V^{\alpha+j} O_{X,x} \otimes \partial_t^j \quad \text{with} \quad j \leq \max\{1 - \alpha, 0\}.

Indeed, this follows from

\[(2.2.4)\quad (\partial/\partial x_i)x_i(x^\nu \otimes \partial_t^j) = (\nu_i + 1 - m_i(s + j))x^\nu \otimes \partial_t^j,

where $s = \partial_t t$.

For $\alpha \in (0, 1]$, let $D(\alpha)$ be the union of the irreducible components $D_i$ whose multiplicity $m_i$ satisfies $m_i \alpha \in \mathbb{Z}$. Then we have

\[(2.2.5)\quad \text{supp Gr}^\alpha_b B_f \subset D(\alpha).

**2.3. Proposition.** With the assumption of (2.2), we have for any $\alpha$

\[(2.3.1)\quad F_{-n} V^\alpha B_{f,x} = V^\alpha O_{X,x} \otimes 1.

**Proof.** See [24], Prop. 3.5.

**2.4. Remark.** The equality (2.3.1) is not a simple corollary of (2.2.3), and we need some calculation. Note that (2.3.1) is equivalent to Theorem (0.1) in the normal crossing case using

\[(2.4.1)\quad \min\{r \in \mathbb{Z} : r + 1 \geq \beta\} = \max\{r \in \mathbb{Z} : r < \beta\},

where $\beta = m_i \alpha$. To deduce the general case from this, we need the theory of bifiltered direct images as explained in the next section.

### 3. Bifiltered Direct Images

**3.1.** Let $\pi : X' \to X$ be a projective morphism of complex manifolds or a proper morphism of smooth complex algebraic varieties. Let $n = \dim X$, $m = \dim X'$, and $f' = f \pi$. Then we can define $B_{f'}$ on $Y' = X' \times \mathbb{C}$ similarly. We factorize $\pi$ by the composition of the embedding by the graph

\[i_\pi : X' \to X'' := X' \times X\]
and the projection \( pr : X'' \to X \). Let \((x_1, \ldots, x_n)\) be a local coordinate system of \( X \), and put \( \partial_i = \partial/\partial x_i \). Let \( h_i \) be the pull-back of \( x_i \) by \( pr \). Then \( B'' := (i_\pi \times id)_\ast B' \) is defined locally on \( X \) by

\[
(3.1.1) \quad B'_\pi \otimes_\mathbb{C} C[\partial_1, \ldots, \partial_n] = \mathcal{O}_{X'}, \otimes_\mathbb{C} C[\partial_1, \ldots, \partial_n, \partial_t],
\]

and the action of a vector field \( \xi \) on \( X' \) is given by

\[
(3.1.2) \quad \xi(g \otimes \partial' \partial_t^j) = (\xi g) \otimes \partial' \partial_t^j - \sum_i (\xi h_i) g \otimes \partial_i \partial_t^j - (\xi f') g \otimes \partial' \partial_t^{j+1}
\]

for \( g \in \mathcal{O}_{X'} \), where \( \nu = (\nu_1, \ldots, \nu_n) \) is a multi index. The filtration \( F \) on \( B'' \) is given by the total order of \( \partial_i, \partial_t \), and \( V \) is generated over \( C[\partial_1, \ldots, \partial_n] \) by \( V \) on \( B' = \mathcal{O}_{X'} \otimes_\mathbb{C} C[\partial_t] \).

Globally we need the twist by the relative dualizing sheaf \([2] \). For example

\[
(3.1.3) \quad F_{-m} B'' = \pi^* \omega_X',
\]

where \( \omega_X' \) is the dual of the line bundle \( \omega_X \).

The bifiltered direct image \((\pi \times id)_\ast (B''_\pi; F, V)\) is then defined to be the direct image of \( B'' \) by \( pr \). It is defined by the sheaf theoretic direct image (using the canonical flasque resolution) of the relative de Rham complex \( DR_{C' \times X} (B''_\pi; F, V) \) whose \( i \)-th component is

\[
(3.2.1) \quad (H^0(\pi \times id)_\ast B''_\pi; F, V) \to (B''_\pi; F, V),
\]

which induces isomorphisms for \( p \in \mathbb{Z}, \alpha > 0 \)

\[
(3.2.2) \quad F_p V^\alpha H^0(\pi \times id)_\ast B''_\pi = F_p V^\alpha B'_\pi
\]

**Proof.** The first assertion follows from [23], 3.3.17. Forgetting \( V \), (3.2.1) is induced by the canonical morphism

\[
(3.2.3) \quad H^0(\pi \ast (\mathcal{O}_{X'}, F) \to (\mathcal{O}_X, F),
\]

whose restriction to \( F_{-n} \) is given by the trace morphism \( \pi_\ast \omega_{X'/X} \to \mathcal{O}_X \). The compatibility with \( V \) follows from the functoriality of the filtration \( V \), see [23], 3.1.5. The kernel
of (3.2.1) is supported on $D$ so that $V^\alpha$ of the kernel vanishes for $\alpha > 0$, see [23], 3.1.3. So we get (3.2.2) because morphisms of Hodge modules are bistrictly compatible with $(F, V)$, see [23], 3.3.3–5.

3.3. Remark. The bistrictness implies that $F_pV^\alpha H^0 = H^0F_pV^\alpha$, see [23], 1.2.13. So the left-hand side of (3.2.2) for $p = -n = -m$ is $\pi_*(\omega_{X'}/X \otimes V^\alpha \mathcal{O}_{X'})$ by (3.1.4). The bistrictness implies also

\[(3.3.1) \quad R^j \pi_*(\omega_{X'/X} \otimes V^\alpha \mathcal{O}_{X'}) = 0 \quad \text{for } j > 0.\]

Here $V^\alpha \mathcal{O}_{X'}$ can be replaced with $\mathcal{O}_{X'}(-\sum |(\alpha - \varepsilon)m_i| D'_i)$ by (2.2–3). This gives another proof of (1.1.3).

3.4 Proof of Theorem (0.1). By (2.3) and (3.2), we have for $\alpha > 0$

\[(3.4.1) \quad F_{-n}V^\alpha \mathcal{B}_f = F_{-n}V^\alpha H^0(\pi \times id)_* \mathcal{B}_f = \pi_*(\omega_{X'/X} \otimes V^\alpha \mathcal{O}_{X'}).\]

The left-hand side of (3.4.1) is $V^\alpha \mathcal{O}_X$ by definition. Using (1.1.1), (2.3.1), this implies

\[V^\alpha \mathcal{O}_X = \mathcal{J}((\alpha - \varepsilon)D) \quad \text{for } \alpha \in \mathbb{Q} \quad \text{and} \quad 0 < \varepsilon \ll 1.\]

This completes the proof of Theorem (0.1).

3.5. Proposition. Assume $D$ reduced, and let $\tilde{\omega}_D = \rho_* \omega_{\tilde{D}} (\subset \omega_D)$ as in the introduction. Then

\[\tilde{\omega}_D = \pi_*(W_1 \omega_{X'}(D'_\text{red}))/\omega_X \subset \omega_D = \omega_X(D)/\omega_X, \quad \omega_D/\tilde{\omega}_D = \omega_X(D)/\pi_*(W_1 \omega_{X'}(D'_\text{red})).\]

Here $\omega_X(D) = \omega_X \otimes \mathcal{O}_X(D)$ and $W$ denotes the weight filtration on the logarithmic forms $\omega_{X'}(D'_\text{red})$ (see [6]) so that $W_1 \omega_{X'}(D'_\text{red}) = (\partial f')\omega_{X'}(D')$ with $(\partial f')$ the ideal generated by the partial derivatives of $f' := \pi^* f$. If the proper transform $D''$ of $D$ is smooth (i.e. if the proper transforms of the irreducible components of $D$ do not intersect each other), then

\[\tilde{\omega}_D = \pi_*(\omega_{X'}(D''))/\omega_X \subset \omega_D = \omega_X(D)/\omega_X, \quad \omega_D/\tilde{\omega}_D = \omega_X(D)/\pi_*(\omega_{X'}(D'')).\]

Proof. If $D''$ is smooth, we may take $\tilde{D} = D''$, and the vanishing of higher direct image $R^1 \pi_* \omega_{X'}$ implies the isomorphisms of (3.5.2). (A similar formula was shown in [3].) In this case we can also show

\[\pi_*(W_1 \omega_{X'}(D'_\text{red})) = \pi_*(\omega_{X'}(D'')),\]

because $\pi_*(\omega_{D'_j}) = 0$ for any irreducible component $D'_j$ of $D'_\text{red}$ such that $\dim \pi(D'_j) < \dim D'_j$. (This is easy if $\pi$ is obtained by iterating blow-ups with smooth centers.) Finally,
we can verify that $\pi_\ast(W_1\omega_{X'}(D'_{\text{red}}))$ is independent of the embedded resolution $(X', D')$. So the assertion follows.

**3.6. Corollary.** Choosing a defining equation $f$ of $D$, we have

$$\text{Gr}^1_V(\omega_D/\omega_D) = V^1\omega_X/\pi_\ast((\partial f)\omega_{X'}),$$

where $V^1\omega_X = V^1\mathcal{O}_X \otimes \omega_X$. If $D''$ is smooth, then

$$\text{Gr}^1_V(\omega_D/\omega_D) = V^1\omega_X/\pi_\ast(\omega_{X'}(D'' - D')).$$

**3.7. Remark.** The formula (3.6.2) is essentially due to [3]. The sheaf $\pi_\ast(\omega_{X'}/\omega_X(D'' - D'))$ is called the adjoint ideal, and has been studied in [28] and [11].

### 4. Localizations of Nearby Cycle Sheaves

**4.1.** With the notation of (2.1) we define the nearby and vanishing cycles of $(\mathcal{O}_X, F)$ and $(\mathcal{B}_f, F)$ by

$$\psi_{f,e(-\alpha)}(\mathcal{O}_X, F) = \psi_{t,e(-\alpha)}(\mathcal{B}_f, F) := \text{Gr}^0_V(\mathcal{B}_f, F[1]) \quad (\alpha \in (0, 1)],$$

$$\varphi_{f,1}(\mathcal{O}_X, F) = \varphi_{t,1}(\mathcal{B}_f, F) := \text{Gr}^0_V(\mathcal{B}_f, F),$$

and $\psi_f(\mathcal{O}_X, F) = \bigoplus_{0<\alpha \leq 1} \psi_{f,e(-\alpha)}(\mathcal{O}_X, F)$ (similarly for $\mathcal{B}_f$). These are the underlying filtered $\mathcal{D}_X$-modules of the nearby and vanishing cycles of mixed Hodge modules $\psi_f(Q_X^H[n])$, $\varphi_{f,1}(Q_X^H[n])$, see [23], [24]. Indeed, these functors correspond to the nearby and vanishing cycles functors [7] for constructible sheaves, which are shifted by $-1$, see [15], [20]. The last shift is needed for the stability of perverse sheaves [1] by these functors.

Assume now that $D$ is a divisor with normal crossings. Let $E, E'$ be reduced divisors on $X$ such that

$$E \cup E' = D_{\text{red}} \quad \text{and} \quad \text{codim } E \cap E' \geq 2.$$

Set $E'' = E \cap E'$. Let $i : E \to X, j : X \setminus E \to X$ denote the inclusion morphisms, and similarly for $i', j'$ with $E$ replaced by $E'$. For $\alpha \in (0, 1]$, let

$$E(\alpha) = D(\alpha) \cap E$$

with $D(\alpha)$ as in (2.2.5), and similarly for $E'(\alpha), E''(\alpha)$. Then

**4.2. Theorem.** With the above notation and the assumption, there are canonical isomorphisms

$$i_\ast i^\ast \psi_f(Q_X^H[n]) \xrightarrow{\sim} j' j'^\ast \psi_f(Q_X^H[n]),$$

$$j' j'^\ast \psi_f(Q_X^H[n]) \xrightarrow{\sim} i_\ast i^\ast \psi_f(Q_X^H[n]).$$
Proof. Since we have canonical morphisms by the property of open direct images, it is enough to show (4.2.1) for the underlying perverse sheaves or $\mathcal{D}$-modules, and the assertion is local.

Let $M = \psi_f \mathcal{O}_X$. Take local coordinates as in (2.2). For $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Q}^n$, let

$$M^\mu = \bigcap_i (\bigcup_j \text{Ker}((\partial_i x_i - \mu_i)^j : M \to M)).$$

Then $M$ is generated by $\bigoplus \mu M^\mu$, and is a completion of $\bigoplus \mu M^\mu$ (i.e. regular singular of normal crossing and quasi unipotent type, see [24], 3.2). So we get an infinite direct sum decomposition by the $M^\mu$, which is compatible with the Hodge filtration $F$.

To simplify the argument, we first show the case $D$ reduced (i.e. semistable) and $f = x_1 \cdots x_n$. In this case $M^\mu = 0$ unless $\mu \in \mathbb{Z}^n$ (i.e. $M$ has unipotent local monodromies). For a subset $I$ of $\{1, \ldots, n\}$, let $M_I = M^\mu$ with $\mu_i = 0$ if $i \in I$ and $\mu_i = 1$ otherwise. For $i \in I$, we have morphisms

$$\partial_i : M_{I \setminus \{i\}} \to M_I, \quad x_i : M_I \to M_{I \setminus \{i\}}.$$

It is well known that $\psi_f \mathcal{O}_X$ is determined by $\{M_I\}$ with the above morphisms. Using the calculation in (2.2), it is easy to see that

(4.2.2) \quad $M_I = \mathbb{C}[s]/\mathbb{C}[s] s^{I}.$

Furthermore, $\partial_i$ is given by the multiplication by $s$, and $x_i$ by the projection.

Let $J, J'$ be the subsets of $\{1, \ldots, n\}$ such that $E = \bigcup_{i \in J} x_i^{-1}$, $E' = \bigcup_{i \in J'} x_i^{-1}$ locally. Let

$$M(E_i) = j_i j^* \psi_f \mathcal{O}_X, \quad M(E'_i) = j'_i j'^* \psi_f \mathcal{O}_X,$$

and define $M(E_i)_I, M(E'_i)_I$ as above. Here $j_i$ is the (regular singular) meromorphic direct image, and $j_i = \mathbb{D} j_* \mathbb{D}$. (Note that $i_* v^1$ and $i_* v^*$ in (4.2.1) are defined by the mapping cone of $j_i j^* \to \text{id}$ and the shifted mapping cone of $\text{id} \to j'_i j'^*$ respectively.) Then

$$M(E_i)_I = \mathbb{C}[s] s^{I \cap J}/\mathbb{C}[s] s^{I}, \quad M(E'_i)_I = \mathbb{C}[s]/\mathbb{C}[s] s^{I \setminus J'},$$

so that we have the bijective morphisms

$$\partial_i : M(E_i)_I \xrightarrow{\sim} M(E_i)_{I \cup \{i\}} \quad \text{for } i \in J \setminus I,$n

$$x_i : M(E'_i)_{I \cup \{i\}} \xrightarrow{\sim} M(E'_i)_I \quad \text{for } i \in J' \setminus I.$$

Thus we get the exact sequence

(4.2.3) \quad $0 \to j_i j^* \psi_f \mathcal{O}_X \to \psi_f \mathcal{O}_X \to j'_i j'^* \psi_f \mathcal{O}_X \to 0,$

and the assertion follows in this case.
In general, we use $M^\mu_I$ instead of $M_I$ where $M^\mu_I = M^{\mu'}_I$ with $\mu' = \mu - 1_I$. Here $\mu \in (0, 1]^n$ and $I \subset \{1, \ldots, n\}$ satisfy $\mu_i = 1$ for $i \in I$, and $1_I \in \mathbb{Z}^n$ is defined by $(1_I)_i = 1$ if $i \in I$ and 0 otherwise. For $M = \psi_f \mathcal{O}_X$, we have

$$M^\mu_I = \mathbb{C}[s]/\mathbb{C}[s]s^{|I|} \quad \text{if } \mu_i + m_i \alpha \in \mathbb{Z} \text{ for any } i,$$

and $M^\mu_I = 0$ otherwise, see [24], 3.3. Here we assume $m_i \neq 0$ for any $i$. Then the argument is similar. This finishes the proof of (4.2).

4.3. Theorem. With the notation and the assumption of (4.1), there are canonical isomorphisms

$$F_{1-n}(i_* i^* \text{Gr}^\alpha_n \mathcal{B}_f) = F_{1-n} \text{Gr}^\alpha_n \mathcal{B}_f \otimes \mathcal{O}_X \mathcal{O}_{E(\alpha)},$$

$$F_{1-n}(j_* j^* \text{Gr}^\alpha_n \mathcal{B}_f) = F_{1-n} \text{Gr}^\alpha_n \mathcal{B}_f \otimes \mathcal{O}_X \mathcal{O}_{E(\alpha)}(-E''(\alpha)).$$

Proof. We first consider the case $D$ reduced as above. Since the Hodge filtration in the normal crossing case is compatible with the decomposition by the $M^\mu$, and the Hodge filtration on $M_I$ is given by the order of $s$, we can verify the first isomorphism of (4.3.1) using (4.2.2). This implies the second isomorphism by (4.2) because the difference between $j'_* j'^* \psi_f \mathcal{O}_X$ and $j'_* j'^* \psi_f \mathcal{O}_X$ is given by the multiplication by $\prod_{i \in J'} x_i$ (see also [24], 3.10 for the difference between the two open direct images in the normal crossing case). The argument in the general case is similar. This finishes the proof of (4.3).

5. Spectrum

5.1. Let $X$ be a complex manifold of dimension $n$, and $f$ be a holomorphic function on $X$. For $x \in X$, let $H^j(F_x, \mathbb{Q})$ denote the vanishing cohomology at $x$ (i.e. $F_x$ denotes the “fiber” of the Milnor fibration around $x$). It has a canonical mixed Hodge structure (see [22], [24]). Using mixed Hodge modules, it is given by

$$H^j(F_x, \mathbb{Q}) = H^{j-n+1} i_x^* \psi_f(\mathbb{Q}_X^H[n]),$$

where $i_x : \{x\} \to X$ denotes the inclusion, and we put $F^p = F_{-p}$. (See (4.1.1) for the underlying filtered $\mathcal{D}$-module of $\psi_f(\mathbb{Q}_X^H[n])$. Note that the functor $\psi$ for mixed Hodge modules is shifted by $-1$.) It has also the action of the semisimple part $T_s$ of the monodromy $T$. Let $H^j(F_x, \mathbb{C})_\lambda$ denote the $\lambda$-eigenspace. Put $e(\alpha) = \exp(2\pi i \alpha)$.

The spectrum $\text{Sp}(f, x)$ is a fractional Laurent polynomial $\sum_{\alpha \in \mathbb{Q}} n_{\alpha} t^\alpha$ such that

$$n_{\alpha} = \sum_j (-1)^{j-n+1} \dim \text{Gr}^p_{F_x} \tilde{H}^j(F_x, \mathbb{C})_{e(-\alpha)} \quad \text{with } p = [n - \alpha],$$

where $\tilde{H}^j(F_x, \mathbb{C})$ denotes the reduced cohomology. This normalization of the spectrum is different from the one in [27] by the multiplication by $t$, and coincides with the one in [4], [25].
5.2. Proposition. $n_\alpha = 0$ if $\alpha \geq n$ or $\alpha \leq 0$.

Proof. The assertion for $\alpha \leq 0$ is equivalent to $\text{Gr}^p H^j(F_x, \mathbb{C}) = 0$ for $p \geq n$. The functor $i_x^*$ is defined by using co-Cech complex of open direct images, and preserves the property that $F_p = 0$ for $p \leq -n$. So the assertion is clear for $\alpha \leq 0$. (Note that the filtration is shifted by 1 in (4.1.1).)

For $\alpha \geq n$ we have the duality isomorphisms (see [24], 2.6)

\begin{equation}
\mathbb{D}(\psi_f Q_\alpha X [n]) = (\psi_f Q_\alpha X [n])(n - 1),
\end{equation}

because $\mathbb{D}(Q_\alpha X [n]) = (Q_\alpha X [n])(n)$. Here $\mathbb{D}$ is the functor which associates the dual. We first show $\text{Gr}^p H^j(F_x, \mathbb{C}) = 0$ for $p \leq 0$. The second isomorphism of (5.2.1) implies

\begin{equation}
\mathbb{D} H^j(F_x, \mathbb{C}) = H^{-j+n-1}i_x^! \varphi_{f,1}(Q_\alpha X [n])(n)
\end{equation}

because $\tilde{H}^j(F_x, \mathbb{C}) = H^{-j+n+1}i_x^* \varphi_{f,1}(Q_\alpha X [n])$, and $\mathbb{D}$ exchanges $i_x^*$ and $i_x^!$. By (2.1.4) we have the vanishing of $F_p$ on the underlying $\mathcal{D}$-module of $\varphi_{f,1}(Q_\alpha X [n])$ for $p < 1 - n$, and the functor $i_x^!$ preserves the property that $F_p = 0$ for $p < 1 - n$ (i.e. $F_p = 0$ for $p > n - 1$). So the assertion for $\alpha \in \mathbb{Z}$ follows from (5.2.2).

Similarly, we can show that $\text{Gr}^p H^j(F_x, \mathbb{C}) = 0$ for $p < 0$, using the first isomorphism of (5.2.1). This completes the proof of (5.2).

5.3. Remarks. (i) In the isolated singularity case, (5.2) is due to Steenbrink [26].

(ii) We can show in general

\begin{equation}
\text{Gr}^p H^j(F_x, \mathbb{C}) = 0 \quad \text{for } p > j.
\end{equation}

Indeed, the pull-back to a divisor with a defining equation $g$ (e.g. a coordinate function) is represented by $\mathcal{C}(\text{can} : \psi_{g,1} \rightarrow \varphi_{g,1})$, and the Hodge filtration is shifted by 1 for $\psi$.

5.4. Let $\pi : (X', D') \rightarrow (X, D)$ be an embedded resolution such that $E := \pi^{-1}(x)_{\text{red}}$ is a divisor with normal crossings. Let $E'$ be the reduced divisor on $X'$ such that $D'_{\text{red}} = E \cup E'$ and $E'' := E \cap E'$ has codimension $\geq 2$. Let $E(\alpha) = E \cap E'(\alpha)$, and similarly for $E'(\alpha), E''(\alpha)$. With the notation of (1.1) we define

\begin{equation}
K_\alpha = \omega_{X'/X} \otimes \mathcal{O}_{X'}(-\sum_i[(\alpha - \varepsilon)m_i]D'_i) \otimes \mathcal{O}_{E(\alpha)} = K_\alpha(\alpha - \varepsilon m_i D'_i) \otimes \mathcal{O}_{E(\alpha)},\end{equation}

\begin{equation}
K'_\alpha = \omega_{X'/X} \otimes \mathcal{O}_{X'}(-\sum_i[(\alpha - \varepsilon)m_i]D'_i) \otimes \mathcal{O}_{E(\alpha)}(-E''(\alpha)),
\end{equation}

for $0 < \varepsilon \ll 1$. In the algebraic case, we have by [4]

\begin{equation}
G(D, x, \alpha) = \Gamma(E(\alpha), K_\alpha).
\end{equation}

5.5. Proof of Theorem (0.5). By (2.3) and (4.1.1) we have canonical isomorphisms

\begin{equation}
F_{1-n}(\psi_{f,e(-\alpha)} \mathcal{O}_{X'}) = F_{1-n} \text{Gr}^\alpha_{\nu} \mathcal{B}_{f'} = \text{Gr}^\alpha_{\nu} \mathcal{O}_{X'} \quad (\alpha \in (0, 1]).
\end{equation}
Let \( E(\alpha), E'(\alpha), E''(\alpha) \) be as in (4.1), and \( i : E \to X' \) be the inclusion morphism. Then by (2.2–3) and (4.3) we get

\[
K_\alpha = \omega_{X'/X} \otimes F_{1-n} \text{Gr}_V^\alpha \mathcal{B}_{f'} \otimes \mathcal{O}_{E(\alpha)}
\]

\[
= \omega_{X'/X} \otimes F_{1-n} (i_* i^* \text{Gr}_V^\alpha \mathcal{B}_{f'}),
\]

\[
K'_\alpha = \omega_{X'/X} \otimes F_{1-n} \text{Gr}_V^\alpha \mathcal{B}_{f'} \otimes \mathcal{O}_{E(\alpha)} (-E''(\alpha))
\]

\[
= \omega_{X'/X} \otimes F_{1-n} (i_* i^! \text{Gr}_V^\alpha \mathcal{B}_{f'}). \]

Taking the cohomology of these coherent sheaves, we get \( F_{1-n} \) of the direct image of \( i_* i^* \text{Gr}_V^\alpha \mathcal{B}_{f'}, i_* i^! \text{Gr}_V^\alpha \mathcal{B}_{f'} \) under \( \pi : X' \to X \) by definition of the direct image of filtered \( \mathcal{D} \)-modules (see [23] and also (3.2) above), because \( F_p = 0 \) for \( p < 1 - n \). We have

\[
(i_x)_* i_x^* \psi_f(Q^H_X[n]) = (i_x)_* i_x^* \pi_* \psi_{f'}(Q^H_{X'}[n]) = \pi_* i_* i^* \psi_{f'}(Q^H_{X'}[n]),
\]

and similarly for \( i_x^! \). So we get the assertion.

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