STABILITY CONDITIONS AND THE MIRROR SYMMETRY OF K3 SURFACES IN ATTRACTOR BACKGROUNDS

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Abstract We study the space of stability conditions on K3 surfaces from the perspective of mirror symmetry. This is done in the attractor backgrounds (moduli). We find certain highly non-generic behaviors of marginal stability walls (a key notion in the study of wall crossings) in the space of stability conditions. These correspond via mirror symmetry to some non-generic behaviors of special Lagrangians in an attractor background. The main results can be understood as a mirror correspondence in a synthesis of the homological mirror conjecture and SYZ mirror conjecture.

Key words mirror symmetry; stability conditions; K3 surfaces; attractor mechanism

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1 Introduction

We study in this paper the interaction of the following three topics on K3 surfaces:

(1) Stability conditions on derived categories of coherent sheaves.
(2) Mirror Symmetry.
(3) Some special moduli in the moduli space of complex structures known as attractors.

The study of stability conditions is a generalization of the study of stable bundles. The space of stability conditions on the derived categories of coherent sheaves is a generalization of the space of complexified Kähler moduli. The study of counting stable objects is closely related to enumerative geometry. If we move in the space of stability conditions we might encounter real codimension one walls called marginal stability walls, such that the counting problem regarding stable objects exhibits nontrivial jumps called wall crossings. Moreover, stability conditions are actually defined for general triangulated categories which include scenarios beyond complex geometry.

The stability condition is a key notion which connects the two major programs of mirror symmetry: the SYZ mirror conjecture of Strominger, Yau and Zaslow, and homological mirror conjecture of Kontsevich.

Roughly speaking, the homological mirror conjecture predicts that for a mirror pair of Calabi-Yau varieties \( M \) and \( \hat{M} \), there should be a duality exchanging the complex geometry...
and the symplectic geometry of them. We expect a correspondence of the following form:

\[ DFuk(M) \leftrightarrow D(M). \]  

(1.1)

Here \( DFuk(M) \) is the derived Fukaya category of \( M \). It is obtained by applying some categorical construction to the Fukaya category. Lagrangian submanifolds are objects of the Fukaya category. \( D(M) \) is the derived category of coherent sheaves on \( M \).

On the other hand, the SYZ mirror conjecture predicts that \( M \) carries a special Lagrangian torus fibration structure and that \( \hat{M} \) should be constructed as the dual special Lagrangian fibration (this is just a very rough and imprecise version and we only mention it here as a motivation). Passing from Lagrangians to special Lagrangians means that we need to impose conditions on the central charge \( Z(\gamma, \Omega) \) (see Section 2 for the definition). It turns out that when we define stability conditions, there is a similar procedure involving central charges. We can speculate that there are stability conditions on \( DFuk(M) \) satisfying essentially the same axioms as those in the definition of stability conditions on derived categories of coherent sheaves. Moreover, special Lagrangians should be realized as stable objects.

On the other side of the mirror duality (i.e., \( D(\hat{M}) \)), we will also put stability conditions on \( D(\hat{M}) \) and then consider stable objects. Mirror symmetry should then exchange stable objects of the two sides. In fact, in some cases, one can check the correspondence of special Lagrangians in \( M \) and the stable bundle on \( \hat{M} \); this can be interpreted as a baby version of this correspondence. Since we expect wall crossings when we count stable objects, there should also be a correspondence of wall crossings of the two sides.

Thus far we have explained that we expect a much more ambitious and unified version of mirror symmetry of the following form:

\[ DFuk(M) + \text{stability conditions} \leftrightarrow D(\hat{M}) + \text{stability conditions}, \]  

(1.2)

\[ \text{Stable objects in } DFuk(M) \leftrightarrow \text{Stable objects in } D(\hat{M}), \]  

(1.3)

\[ \text{Space of stability conditions of } DFuk(M) \leftrightarrow \text{Space of stability conditions of } D(\hat{M}), \]  

(1.4)

\[ \text{Wall crossings for } DFuk(M) \leftrightarrow \text{Wall crossings for } D(\hat{M}). \]  

(1.5)

As far as we know, there is very little work about this version of mirror symmetry, because it seems extremely difficult to study stability conditions on derived Fukaya categories. In particular, it seems that there are no results about correspondences of type (1.5).

One of the purposes of this paper is to formulate and prove a few theorems that can be considered as a softened version of a correspondence of type (1.5) for certain K3 surfaces. To explain this we need to introduce the third player of this paper: attractor backgrounds (moduli), which have been introduced by physicists.

Attractor backgrounds are very special and mathematically interesting subsets in the moduli space of complex structures. We will explain the physical motivation very briefly at the end of the paper. If the Calabi-Yau threefold is \( K3 \times T^2 \), one can determine them explicitly. It is then interesting to ask what mirror symmetry can say about them. In this paper we study the mirror symmetry of K3 surfaces coming from the \( K3 \times T^2 \) attractor backgrounds. This

\[ \text{This is meant to be a very short description of the background, so we have deliberately ignored many important details.} \]
is somewhat unconventional, as the attractor moduli points can be far from the large complex limits where most work of mirror symmetry has been done.

How does this problem relate to stability conditions? It turns out that due to the very special property of attractor backgrounds, the central charges of special Lagrangians in $K3$ correct and $K3 \times T^2$ have certain highly non-generic properties. Roughly speaking, if we have a correct framework for defining stability conditions such that special Lagrangians are stable objects, then this non-generic property would mean that there is an unexpected huge number of marginal stability walls containing the attractor background. From the perspective of wall crossings this is certainly very unusual and intricate.

Unfortunately, the right framework for defining stability conditions for derived Fukaya categories does not exist yet. Nevertheless, we can apply the mirror symmetry of $K3$ surfaces and ask if the above property is transformed into some statement which can be proven independently and can be viewed as the corresponding mirror statement under a correspondence of type (1.5). In other words, we want to check if there is a highly non-generic configuration of marginal stability walls for stability conditions on the derived category of coherent sheaves on the mirror $K3$ surface, and if this behavior has nice correspondences (induced by mirror symmetry) with the similar behavior mentioned before for the original $K3$ surface in an attractor background. If this is the case, then we not only have some nontrivial results about mirror symmetry in attractor backgrounds, but can also connect these to marginal stability walls in the space of stability conditions. Then the three topics mentioned in this introduction will finally be on the same page.

In Section 2 we review the attractor backgrounds of $K3$ surfaces given by Moore [13], and also some basic properties of those that were obtained by Shioda and Inose [15]. In Section 3 we review the work of Bridgeland [6] on stability conditions on the derived categories of coherent sheaves on $K3$ surfaces. Then, following the exposition of Dolgachev [8] and Huybrechts [12], we explain the mirror symmetry of $K3$ surfaces. We discuss the non-generic property in an attractor background in Section 5. This was observed by physicists Aspinwall, Maloney and Simons [3]. Our main theorems are formulated and proven in Section 6, where we also finish by providing some speculations.

2 Singular $K3$ Surfaces as Attractors

Let us formulate the mathematical condition proposed by physicists which characterizes attractor backgrounds. For a Calabi-Yau threefold $X$ we fix a cohomology class (called a charge)

$$\gamma \in H^3(X, \mathbb{Z}). \quad (2.1)$$

Let $\tilde{\mathcal{M}}$ be the universal cover of the moduli space $\mathcal{M}$ of complex structures of $X$. Consider the family of the marked Calabi-Yau threefolds $\pi : \tilde{X} \to \tilde{\mathcal{M}}$ and the bundle $\mathcal{L} := R\pi_*\Omega^{3,0}(\tilde{X}) \to \tilde{\mathcal{M}}$. For $\gamma$ and a section $\Omega$ of $\mathcal{L}$, we define a function called the central charge:

$$Z(\gamma, \Omega) := \int_{\tilde{X}} \Omega \wedge \gamma. \quad (2.2)$$
To simplify notations for later, we will just write $Z(\gamma)$ for $Z(\gamma, \Omega)$ when it is clear what the holomorphic 3-form $\Omega$ is.\(^2\)

Attractor backgrounds are determined by charges. We fix a charge $\gamma$. Then, following [13], we consider the attractor equation
\[
\gamma = \gamma^{3,0} + \gamma^{0,3} \in H^{3,0}(X) \oplus H^{0,3}(X).
\]
In other words, the Hodge decomposition of the topological object $\gamma$ has only $(3,0)$ and $(0,3)$ parts. Clearly this is an equation about complex structures. Solutions of the attractor equation are called attractor moduli or attractor backgrounds. The point is that special complex moduli (attractor backgrounds) are determined by cohomology classes (charges).\(^3\)

The study of attractors for general Calabi-Yau threefolds is difficult. In this paper we only consider Calabi-Yau threefolds $X := S \times T^2$ with varying complex moduli. Here $S$ is a $K3$ surface. The torus $T^2$ is topologically described as the quotient of $\mathbb{R}^2$ by $x \sim x + 1, y \sim y + 1$. This can always be done via coordinate changes. Note that we have not said anything about complex structures or the metrics. Due to our description of the torus, we can choose a basis of $H^1(T^2, \mathbb{Z})$ as $\{dx, dy\}$. Since $H^3(X, \mathbb{Z}) = H^2(S, \mathbb{Z}) \otimes H^1(T^2, \mathbb{Z})$, a charge can be written as
\[
\gamma = pdx + qdy.
\]
Here $p, q \in H^2(S, \mathbb{Z})$. Let $\tau$ be the complex modulus of $T^2$ and let $\Omega$ be a holomorphic 2-form on $S$. Then $dz = dx + \tau dy$ is a holomorphic 1-form on $T^2$ and
\[
\Omega^{3,0} := (dx + \tau dy) \wedge \Omega
\]
is a holomorphic 3-form on $X$. In [3] and [13] the attractor equation was solved, with the result that
\[
\tau = \frac{p \cdot q + 1\sqrt{|D_{p,q}|}}{p^2},
\]
\[
\Omega = q - \tau p.
\]
Here $p \cdot q$ is the intersection paring on $H^2(S, \mathbb{Z})$, $p^2 := p \cdot p$ and $D_{p,q} = p^2q^2 - (p \cdot q)^2$. We always assume that $D_{p,q} > 0$.

$H^2(S, \mathbb{Z})$ endowed with the intersection paring is a lattice isomorphic to
\[
\Gamma := 2(\mathbb{Z}E_8) \oplus 3U
\]
where $U$ is called the hyperbolic plane and is defined as the free group of rank two generated by isotropic vectors $\{e_1, e_2\}$ with $e_1 \cdot e_2 = 1$. The signature of $\Gamma$ is $(3, 19)$. The choice of a lattice isomorphism $\phi : H^2(S, \mathbb{Z}) \simeq \Gamma$ is called a marking. Recall the definition of the Neron-Severi lattice $NS(S)$ and transcendental lattice $T_S$ (in what follows, $j : H^2(S, \mathbb{Z}) \to H^2(S, \mathbb{C})$ is the inclusion):
\[
NS(S) := H^{1,1}(S, \mathbb{R}) \cap j(H^2(S, \mathbb{Z})),
\]

\(^2\)This notation suggests that we fix an $\Omega$ and view $Z$ as a function of $\gamma$. Then we can let $\Omega$ (which is determined by the choice of a complex structure) vary and see what happens. We will compare $Z$ with central charges of derived categories via mirror symmetry, and this is also the strategy used there. For derived categories, a central charge is viewed as a map on the Grothendieck group, and then we let the stability condition vary.

\(^3\)There is a link between attractor backgrounds and the central charge. In fact, for a fixed $\gamma$ a stationary point of $|Z|$ in $\tilde{M}$ with nonzero $Z$-value is an attractor background; see [10] and [13] for details.
For an algebraic $K3$ surface $NS(S) = Pic(S)$, the rank $\rho(S)$ of $NS(S)$ is called the Picard number. The intersection paring on $NS(S)$ has the signature $(1, \rho(S) - 1)$. As noticed in [13], the equation (2.7) means that $NS(S)$ is the orthogonal complement of the lattice generated by $p, q$. Therefore $\rho(S)$ is 20. Note that 20 is the largest possible Picard number of a $K3$ surface.

Shioda and Inose studied $K3$ surfaces with $\rho(S) = 20$ (see [15]), these are called singular $K3$ surfaces.\footnote{The word singular here does not mean nonsmooth. Perhaps it is more appropriate to call these attractive $K3$ surfaces because of Theorem 2.2.}

**Theorem 2.1** ([15]) There is a natural one-to-one correspondence between the set of singular $K3$ surfaces and the set of equivalence classes of positive-definite even integral binary quadratic forms with respect to the action of $SL_2(\mathbb{Z})$.

For a singular $K3$ surface, the rank 2 transcendental lattice $T_S$ has a natural orientation which is achieved by requiring that the ratio of two corresponding periods has a positive imaginary part. Letting $\{p,q\}$ be an oriented basis of $T_S$, we can define a quadric form $Q_{p,q}$ by

$$Q_{p,q} := \begin{pmatrix} p^2 & p \cdot q \\ p \cdot q & q^2 \end{pmatrix}.\tag{2.11}$$

This is the quadratic form mentioned in the theorem. Here positive-definite means $D_{p,q} > 0$. The equivalence relation with respect to the action of $SL_2(\mathbb{Z})$ is that $Q_1 \sim Q_2$ iff $Q_1 = r^T Q_2 r$ for some $r \in SL_2(\mathbb{Z})$. Note that $p^2$ and $q^2$ need not be squares of integers.

Conversely, for any positive-definite even integral binary quadratic form, Shioda and Inose constructed a singular $K3$ surface such that the intersection matrix of the rank 2 transcendental lattice is the quadratic form.

Thus we have explained the following basic fact, discovered in [13]:

**Theorem 2.2** ([13]) For $X := S \times T^2$ and a charge specified by the choice of $p, q$ above, the attractor background is determined by (2.6) and (2.7). The $K3$ surface $S$ is the singular $K3$ surface corresponding to $Q_{p,q}$.

From now on, a $K3$ surface in an attractor background is always understood as the singular $K3$ surface corresponding to some given $Q_{p,q}$.

Shioda and Inose also proved that singular $K3$ surfaces always carry elliptic fibration structures.

**Theorem 2.3** ([15]) Every singular $K3$ surface has an elliptic fibration structure with an infinite group of sections.

This theorem implies that every singular $K3$ surface has an infinite group of automorphisms (this is actually a theorem in [15], and Theorem 2.3 is obtained in the proof of it). It is also interesting to know that there might be more than one elliptic fibration structures with sections on a given singular $K3$ surface.
3 Stability Conditions on $K3$ Surfaces

Bridgeland defined stability conditions on a triangulated category based on the work of Douglas. We summarize some results from [6].

**Definition 3.1** A stability condition $\sigma = (Z, \mathcal{P})$ on a triangulated category $\mathcal{D}$ consists of an additive group homomorphism $Z : K(\mathcal{D}) \to \mathbb{C}$ called the central charge (here $K(\mathcal{D})$ is the Grothendieck group of $\mathcal{D}$), and full additive subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$ for each $\phi \in \mathbb{R}$, satisfying the following axioms:

1. if $0 \neq E \in \mathcal{P}(\phi)$, then $Z(E) = m(E) \exp(i \pi \phi)$ for some $m(E) \in \mathbb{R}_{>0}$;
2. for all $\phi \in \mathbb{R}$, $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$;
3. if $\phi_1 > \phi_2$ and $A_j \in \mathcal{P}(\phi_j)$, then $\text{Hom}_\mathcal{D}(A_1, A_2) = 0$;
4. for $0 \neq E \in \mathcal{D}$, there is a finite sequence of real numbers $\phi_1 > \phi_2 > \cdots > \phi_n$.

and a collection of triangles

\[
\begin{array}{cccccc}
0 &=& E_0 & \rightarrow & E_1 & \rightarrow \\
& & A_1 & \rightarrow & A_2 & \rightarrow \\
& & & & \cdots & \rightarrow \\
& & & & E_{n-1} & \rightarrow \\
& & & & A_n & \rightarrow \\
& & & & & E_n = E
\end{array}
\]

with $A_j \in \mathcal{P}(\phi_j)$ for all $j$.

Each subcategory $\mathcal{P}(\phi)$ is abelian, and nonzero objects of it are called semistable objects with phase $\phi$ in $\sigma$. Simple objects of $\mathcal{P}(\phi)$ are called stable objects. Define

\[
\phi_\sigma^+(E) = \phi_1, \quad \phi_\sigma^-(E) = \phi_n. \tag{3.1}
\]

The set of stability conditions on a triangulated category $\mathcal{D}$ is denoted by $\text{Stab}(\mathcal{D})$.\footnote{In fact one imposes a technical condition called local finiteness (for details see [6]).}

This has a natural topology induced by the following generalized metric:

\[
d(\sigma_1, \sigma_2) = \sup_{\sigma \neq E \in \mathcal{D}} \left\{ |\phi_\sigma^-(E) - \phi_{\sigma_1}^-(E)|, \ |\phi_\sigma^+(E) - \phi_{\sigma_1}^+(E)|, \ |\log \frac{m_{\sigma_2}(E)}{m_{\sigma_1}(E)}| \right\}. \tag{3.2}
\]

We will not discuss abstract triangulated categories. A triangulated category in this paper is always the bounded derived category of coherent sheaves $\mathcal{D}(X)$ on a smooth complex projective variety $X$. For two objects $E$ and $F \in \mathcal{D}(X)$ we define the Euler form $\chi(E, F)$ by

\[
\chi(E, F) := \sum_i (-1)^i \dim \text{Hom}^i_X(E, F[i]).
\]

Define $\mathcal{N}(X) = K(X)/K(X)^\perp$, where $K(X)^\perp$ is the radical of the bilinear form on $K(X)$ induced by the Euler form.

From now on we shall always assume that a stability condition $\sigma = (Z, \mathcal{P})$ is numerical in the sense that the central charge $Z : K(X) \to \mathbb{C}$ factors through the quotient group $\mathcal{N}(X)$. As a result, the central charge must be of the form

\[
Z(E) = -\chi(\pi(\sigma), E) \tag{3.3}
\]

for some vector $\pi(\sigma) \in \mathcal{N}(X) \otimes \mathbb{C}$. Note that the above equation defines the map $\pi$. The set of locally finite numerical stability conditions on $\mathcal{D}(X)$ is denoted by $\text{Stab}(X)$ instead of $\text{Stab}(\mathcal{D})$.\footnote{\textcopyright Springer}
From now on, whenever we say “space of stability conditions”, we always mean “space of locally finite numerical stability conditions”.

**Theorem 3.2** ([6]) For each connected component of $\text{Stab}^*(X) \subset \text{Stab}(X)$ there is a linear subspace $V \subset \mathcal{N}(X) \otimes \mathbb{C}$ such that

$$
\pi : \text{Stab}^*(X) \rightarrow \mathcal{N}(X) \otimes \mathbb{C}
$$

is a local homeomorphism onto an open subset of $V$.

Now let $S$ be a projective $K3$ surface. One can show that

$$
\mathcal{N}(S) = \mathbb{Z} \oplus \text{NS}(S) \oplus \mathbb{Z}. 
$$

(3.4)

On the cohomology ring $H^*(S, \mathbb{Z}) = H^0(S, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z})$, we define the Mukai pairing $(\cdot, \cdot)$:

$$(r_1, D_1, s_1), (r_2, D_2, s_2)) := D_1 \cdot D_2 - r_1 s_2 - r_2 s_1. 
$$

(3.5)

For $E \in \mathcal{D}(S)$, the Mukai vector $v(E)$ is

$$
v(E) = (r(E), c_1(E), s(E)) := \text{ch}(E) \sqrt{td(S)}. 
$$

(3.6)

One can show that $v(E)$ is integral and lies in $\mathcal{N}(S)$. Moreover, the condition (3.3) becomes

$$
Z(E) = (\pi(\sigma), v(E))
$$

(3.7)

for some vector $\pi(\sigma) \in \mathcal{N}(S) \otimes \mathbb{C}$.

To study the space of stability conditions on a fixed algebraic $K3$ surface $S$ we first need to find some stability conditions. Take a pair of divisors $B, \omega \in \text{NS}(S) \otimes \mathbb{R}$ such that $\omega$ lies in the ample cone. Define a group homomorphism $Z: \mathcal{N}(S) \rightarrow \mathbb{C}$ by

$$
Z(E) := (\exp(B + i \omega), v(E)). 
$$

(3.8)

Explicitly, this is

$$
Z(E) = \frac{1}{2r}(D^2 - 2rs) + r^2 \omega^2 - (D - rB)^2 + i(D - rB) \cdot \omega. 
$$

(3.9)

Here $(r, D, s) := v(E)$, and we assume that $r \neq 0$. If $r = 0$, then

$$
Z(E) = (D \cdot B - s) + i(D \cdot \omega). 
$$

(3.10)

**Theorem 3.3** ([6]) Suppose that $B, \omega \in \text{NS}(S) \otimes \mathbb{Q}$ and that $\omega$ lies in the ample cone. Suppose that $Z(E)$ does not lie in $\mathbb{R}_{\leq 0}$ for all spherical sheaves $E$ (this holds if $\omega^2 > 2$). Then there is a stability condition on $\mathcal{D}(S)$ with a central charge given by (3.8).

Next we describe the space of stability conditions. Let $\mathcal{P}(S) \subset \mathcal{N}(S) \otimes \mathbb{C}$ be the open set consisting of all elements of $\mathcal{N}(S) \otimes \mathbb{C}$ whose real and imaginary parts span positive definite two-planes in $\mathcal{N}(S) \otimes \mathbb{R}$. Consider the following tube domain:

$$
\{B + i \omega \in \text{NS}(S) \otimes \mathbb{C} : \omega^2 > 0\}. 
$$

It is easy to see that if $B + i \omega$ is in this tube domain, then

$$
\Psi := \exp(B + i \omega) \in \mathcal{P}(S). 
$$

(3.11)

Thus $\mathcal{P}(S)$ contains the space of complexified Kähler deformations (the tube domain).
Let \( P^+(S) \) be the connected component of \( P(S) \) containing the vectors of the form \( \exp(B + i\omega) \) for ample real classes \( \omega \). Define
\[
\Delta(S) := \{ \delta \in \mathcal{N}(S) : (\delta, \delta) = -2 \}.
\]
Note that this contains \((-2)\)-classes of curves of \( S \). For each \( \delta \in \Delta(S) \), we define a hyperplane
\[
\delta^\perp := \{ \Psi \in \mathcal{N}(S) \otimes \mathbb{C} : (\Psi, \delta) = 0 \}.
\]
The hyperplane complement is
\[
P_0(S) = P(S) \setminus \bigcup_{\delta \in \Delta(S)} \delta^\perp.
\]
We also define
\[
P_0^+(S) = P^+(S) \setminus \bigcup_{\delta \in \Delta(S)} \delta^\perp.
\]
A connected component \( \text{Stab}^*(S) \subset \text{Stab}(S) \) is a good component if it contains a point \( \sigma \) such that \( \pi(\sigma) \in P_0(S) \). Let \( \text{Stab}^0(S) \subset \text{Stab}(S) \) be the connected component containing the stability conditions constructed in Theorem 3.3.

**Theorem 3.4** ([6]) The subset \( P_0(S) \subset \mathcal{N}(S) \otimes \mathbb{C} \) is open, and the restriction
\[
\pi : \pi^{-1}(P_0(S)) \to P_0(S)
\]
is a topological covering map. If a connected component \( \text{Stab}^*(S) \) is good, then the image \( \pi(\text{Stab}^*(S)) \) contains one of the two connected components of the open subset \( P_0(S) \). Moreover,
\[
\pi : \text{Stab}^0(S) \to P_0^+(S)
\]
is a topological covering map.

Note that even though in Theorem 3.3 we have assumed that \( B \) and \( \omega \) are rational, we can drop this restriction now as long as \( \exp(B + i\omega) \in P_0(S) \). Theorem 3.4 guarantees that there are stability conditions in the pre-image under the map \( \pi \), and that central charges are still given by (3.8).

There are certain walls in the space of stability conditions that are our main interest in this paper.

Given \( \mathcal{D}(S) \), we say that a set of objects \( M \) has bounded mass in a connected component \( \text{Stab}^*(S) \) if \( \sup\{ m_\sigma(E) : E \in M \} < \infty \) for some \( \sigma \in \text{Stab}^*(S) \). Take such a set \( M \) and fix a compact subset \( B \subset \text{Stab}^*(S) \). Let \( T \) be the set of nonzero objects \( A \) of \( \mathcal{D}(S) \) such that, for some \( \sigma \in B \) and some \( E \in M \), one has \( m_\sigma(A) \leq m_\sigma(E) \). Let \( \{v_i : i \in I\} \) be the finite set of Mukai vectors of objects of \( T \). Here \( I \) is the index set and the finiteness is established in [6]. Let \( \tilde{\Gamma} \) be the set of pairs \( i, j \in I \) such that \( v_i \) and \( v_j \) do not lie on the same real line in \( \mathcal{N}(S) \otimes \mathbb{R} \).

**Definition 3.5** For each \( \tilde{\gamma} \in \tilde{\Gamma} \) we define
\[
W_{\tilde{\gamma}} = \{ \sigma = (Z, \mathcal{P}) \in \text{Stab}^*(S) : Z(v_i)/Z(v_j) \in \mathbb{R}_{>0} \}.
\]
Each \( W_{\tilde{\gamma}} \) is called a marginal stability wall.

A marginal stability wall depends on a pair of charges instead of a single charge. One can also show that each \( W_{\tilde{\gamma}} \) is a real codimension one submanifold of \( \text{Stab}^*(S) \) ([6]).

The following theorem is Proposition 9.3 in [6]:
Theorem 3.6 ([6]) For the finite collection \( \{ W_\gamma : \gamma \in \tilde{\Gamma} \} \) defined above, any connected component

\[
C \subset B \setminus \bigcup_{\gamma \in \tilde{\Gamma}} W_\gamma
\]

has the following property: if \( E \in M \) is semistable in \( \sigma \) for some \( \sigma \in C \), then \( E \) is semistable for all \( \sigma \in C \).

The above theorem means that when we change the stability condition, the set of semistable objects does not change if no marginal stability walls are crossed. If a marginal stability wall is crossed, then the set of semistable objects might change, and determining these changes quantitatively is a subject referred to as wall crossing formulas.

4 Mirror Symmetry of K3 Surfaces

We will be following the expositions of [12] and [8].

First we introduce some period domains. Let \( \Gamma \) be the \( K3 \) lattice \( 2(-E_8) \oplus 3U \) and let \( \Gamma_R = \Gamma \otimes \mathbb{R} \). Define \( \text{Gr}^{\text{po}}_{2,1}(\Gamma_R) \) to be the space of all oriented two dimensional positive subspaces of \( \Gamma_R \). Here “positive” means that the restriction of the bilinear form on \( \Gamma_R \) induced by the lattice paring is positive definite. Similarly, we can define \( \text{Gr}^{\text{po}}_{3,1}(\Gamma_R) \) and \( \text{Gr}^{\text{po}}_{4,1}(\Gamma_R \oplus U_R) \). We also define

\[
\text{Gr}^{\text{po}}_{2,1}(\Gamma_R) := \{(P, \omega) : P \in \text{Gr}^{\text{po}}_{2}(\Gamma_R), \omega \in P^\perp \subset \Gamma_R, \omega^2 > 0\},
\]

and

\[
\text{Gr}^{\text{po}}_{2,2}(\Gamma_R \oplus U_R) := \{(H_1, H_2) : H_1 \in \text{Gr}^{\text{po}}_{2}(\Gamma_R \oplus U_R), H_1 \perp H_2\}. \tag{4.2}
\]

Take a copy of hyperbolic plane \( U \) (which is not to be considered as a sublattice of \( \Gamma \)). Let us fix a standard basis \( \{w, w^*\} \) of \( U \) (so that \( w^2 = (w^*)^2 = 0 \) and \( w \cdot w^* = 1 \)). There is a natural injection \( s : \text{Gr}^{\text{po}}_{2,1}(\Gamma_R) \times \Gamma_R \rightarrow \text{Gr}^{\text{po}}_{2,2}(\Gamma_R \oplus U_R) \) so that

\[
s(((P, \omega), B)) = (H_1, H_2), \tag{4.3}
\]

with

\[
\begin{align*}
H_1 &:= \{x - (x \cdot B)w : x \in P\}, \\
H_2 &:= \mathbb{R}(1/2(\omega^2 - B^2)w + w^* + B) \oplus \mathbb{R}(\omega - (\omega \cdot B)w).
\end{align*}
\]

There is an isomorphism

\[
\varphi : \text{Gr}^{\text{po}}_{3,1}(\Gamma_R) \times \mathbb{R}_{>0} \times \Gamma_R \cong \text{Gr}^{\text{po}}_{4,1}(\Gamma_R \oplus U_R) \tag{4.4}
\]

given by

\[
\varphi : (F, \alpha, B) \rightarrow \Pi := \mathbb{R}B' \oplus F'. \tag{4.5}
\]

Here \( F' := \{f - (f \cdot B)w : f \in F\} \) and \( B' := B + 1/2(\alpha - B^2)w + w^* \). The inverse map \( \psi : \Pi \rightarrow (F, (B')^2, B) \) is defined in the following way: first, define a three dimensional space \( F' := \Pi \cap w^\perp \) and then define \( F := \pi(F') \), where \( \pi \) is the natural projection \( \pi : \Gamma_R \oplus U_R \rightarrow \Gamma_R \). There exists a \( B' \in \Pi \) such that \( \Pi = \mathbb{R}B' \oplus F' \) is an orthogonal splitting. Such a \( B' \) is of course not unique, but we can make it unique by imposing the condition \( B' \cdot w = 1 \). Finally we define \( B := \pi(B') \). We call \( B \) a \( B \)-field.

The spaces defined above have interpretations as period domains.
**Definition 4.1** A marked K3 surface is a pair \((S, \phi)\) such that \(S\) is a K3 surface and \(\phi : H^2(S, \mathbb{Z}) \cong \Gamma\) is an lattice isomorphism (it must respect the parings). Two marked K3 surfaces are equivalent if there is an isomorphism of the underlying K3 surfaces such that the lattice isomorphisms are identical after composing with the induced map on the cohomology ring.

Then we can define the period map \(P_{\text{complex}}\) by sending an equivalence class of marked K3 surfaces \((S, \phi)\) to

\[
P_{\text{complex}}(S, \phi) := \phi(\text{Re } \Omega, \text{Im } \Omega) \in G_r^{\text{po}}(\Gamma_{\mathbb{R}}).
\]  

(4.6)

Here \(\Omega\) is a holomorphic 2-form on \(S\) and \(\langle \text{Re } \Omega, \text{Im } \Omega \rangle\) is the subspace generated by Re \(\Omega\) and Im \(\Omega\).

We can also define a marked complex hyperkähler K3 surface with a \(B\)-field which is a tuple \((S, g, I, B, \phi)\), a marked hyperkähler K3 surface with a \(B\)-field which is a tuple \((S, g, B, \phi)\), and a marked Kähler K3 surface with a \(B\)-field which is a tuple \((S, \omega, B, \phi)\). The meanings of \(S\) and \(\phi\) are the same as before. \(g\) is a hyperkähler metric on \(S\), \(I\) is a compatible complex structure, and \(B \in H^2(S, \mathbb{R})\). \(\omega\) is a Kähler class on \(S\). We can also define the obvious equivalence relations. By Yau’s solution of Calabi’s conjecture, we know that there is a natural bijection between the set of equivalence classes of marked complex hyperkähler K3 surfaces with \(B\)-fields and the set of equivalence classes of marked Kähler K3 surfaces with \(B\)-fields.

We have period maps from the set of equivalence classes of \((S, g, B, \phi)\) and \((S, g, I, B, \phi)\) (or equivalently \((S, \omega, B, \phi)\)) to period domains.

Let \(H^2_2(S, \mathbb{R})\) be the three dimensional space spanned by Re \(\Omega\), Im \(\Omega\) and \(\omega\). This is the cohomology of the space of self-dual 2-forms (see [1]) and is determined by the metric \(g\). Now we define the period maps

\[
P^{4,1}(S, g, B, \phi) := (\phi(H^2_2(X, \mathbb{R})), \omega^2, \phi(B)) \in G_r^{\text{po}}(\Gamma_{\mathbb{R}}) \times \mathbb{R} \times \Gamma_{\mathbb{R}} \cong G_r^{\text{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}),
\]

\[
P^{2,3}(S, g, I, B, \phi) := (P(S, \omega, \phi, \phi(B)) \in G_r^{\text{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}}.
\]

(4.7)

(4.8)

Here we have used the fact that there is a natural bijection between the set of equivalence classes of \((S, g, I, \phi)\) and the set of equivalence classes of \((S, \omega, \phi)\). Here \(P(S, \omega, \phi)\) is defined to be

\[
P(S, \omega, \phi) := (P_{\text{complex}}(S, \phi), \phi(\omega)) \in G_r^{\text{po}}(\Gamma_{\mathbb{R}}).
\]

(4.9)

These period maps are all \(O(\Gamma)\)-equivariant.

We recall the following famous surjectivity theorem [4]:

**Theorem 4.2** \(P_{\text{complex}}\) defined in (4.6) is surjective.

Let us assume that there is a splitting \(\Gamma = \Gamma' \oplus U'\), where \(U'\) is a copy of hyperbolic plane. We fix an isomorphism \(f : U' \cong U\), where \(U'\) is a copy of hyperbolic plane. We fix an isomorphism \(f : U' \cong U\). Let \(\iota : G_r^{\text{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}) \to G_r^{\text{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})\) be the involution given by \(\iota : (H_1, H_2) \to (H_2, H_1)\). Let \(\xi \in O(\Gamma \oplus U)\) be the map which is the identity on \(\Gamma'\) and which swaps \(U\) and \(U'\) via the isomorphism \(f\).

**Definition 4.3** Define the mirror map \(\tilde{\xi}\) on the period domain \(G_r^{\text{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})\) as

\[
\tilde{\xi} := \iota \circ \xi.
\]

(4.10)

This acts on \(G_r^{\text{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})\) and is an involution.
Since there is a natural projection $\pi : Gr^\mathrm{po}_{2,2}(\Gamma_\mathbb{R} \oplus U_\mathbb{R}) \rightarrow Gr^\mathrm{po}_{2,1}(\Gamma_\mathbb{R} \oplus U_\mathbb{R})$ by sending $(H_1, H_2)$ to $H_1 \oplus H_2, \xi$ also naturally acts on $Gr^\mathrm{po}_{2,1}(\Gamma_\mathbb{R} \oplus U_\mathbb{R})$, and coincides with $\xi$. We also recall that $Gr^\mathrm{po}_{2,1}(\Gamma_\mathbb{R}) \times \Gamma_\mathbb{R}$ can be considered as a subspace of $Gr^\mathrm{po}_{2,2}(\Gamma_\mathbb{R} \oplus U_\mathbb{R})$ via a natural injection $s$ defined by (4.3).

**Theorem 4.4 ([12])** Let $\{v, w^*\}$ be a basis of $U'$ that corresponds to $\{w, w^*\}$ of $U$ under $f$. Let $pr$ be the natural projection $pr : \Gamma_\mathbb{R} \rightarrow \Gamma_\mathbb{R}$. Let $((P, \omega), B) \in Gr^\mathrm{po}_{2,1}(\Gamma_\mathbb{R}) \times \Gamma_\mathbb{R}$ such that $\omega, B \in \Gamma_\mathbb{R}' \oplus \mathbb{R}v$; then its image under the mirror map $\tilde{\xi}$ is $((\tilde{P}, \tilde{\omega}), \tilde{B}) \in Gr^\mathrm{po}_{2,1}(\Gamma_\mathbb{R}) \times \Gamma_\mathbb{R}$. This is explicitly given as

$$
\tilde{\Omega} := \frac{1}{(\text{Re} \Omega)} \cdot v (pr(B + i\omega) - 1/2(B + i\omega)^2 v + v^*),
$$

(4.11)

$$
\tilde{B} + i\tilde{\omega} := \frac{1}{(\text{Re} \Omega)} \cdot v (pr(\Omega) - (\Omega \cdot B)v).
$$

(4.12)

Here $P$ is spanned by $\text{Re} \Omega$ and $\text{Im} \Omega$ and we choose $\Omega$ such that $\text{Im} \Omega$ is orthogonal to $v$.

So far the mirror map has been studied only on period domains. To relate it to actual $K3$ surfaces we use the construction of Dolgachev [8].

**Definition 4.5** Let $N$ be a sublattice of $\Gamma$ of signature $(1, r)$ ($r$ can be 0). A marked $N$-polarized $K3$ surface is a marked $K3$ surface $(S, \phi)$ such that $N \subset \phi(Pic(S))$.

For a projective $K3$ surface $S$ we set that

$$
\tilde{\Delta}(S) := \{\delta \in Pic(S) : \delta^2 = -2\}.
$$

(4.13)

$\tilde{\Delta}(S)$ has two components:

$$
\tilde{\Delta}(S) = \tilde{\Delta}(S)^+ \coprod \tilde{\Delta}(S)^-.
$$

(4.14)

Here $\tilde{\Delta}(S)^+ = -\tilde{\Delta}(S)^-$ and $\tilde{\Delta}(S)^+$ consists of effective classes. Define

$$
V(S) := \{x \in H^{1,1}(S) \cap H^2(S, \mathbb{R}) : x^2 > 0\}.
$$

(4.15)

This is a cone consisting of two components. We denote by $V(S)^+$ the component which contains the class of some Kähler form on $S$. Then we set that

$$
C(S) := \{x \in V(S)^+ : (x, \delta) \geq 0 \text{ for any } \delta \in \tilde{\Delta}(S)^+\}.
$$

(4.16)

One can show that the set of interior points $C(S)^+$ of $C(S)$ is the Kähler cone of $S$. We also set that

$$
Pic(S)^+ := C(S) \cap H^2(S, \mathbb{Z}), \quad Pic(S)^{++} := C(S)^+ \cap H^2(S, \mathbb{Z}).
$$

(4.17)

For the sublattice $N$ we set that

$$
V(N) := \{x \in N_\mathbb{R} : x^2 > 0\}.
$$

(4.18)

This has two components. We fix one of these and call it $V(N)^+$. Define that

$$
\tilde{\Delta}(N) := \{\delta \in N : \delta^2 = -2\}.
$$

(4.19)

It is easy to see that $\tilde{\Delta}(N) = \tilde{\Delta}(N)^+ \coprod \tilde{\Delta}(N)^-$ with $\tilde{\Delta}(N)^+ = -\tilde{\Delta}(N)^-$ and that $\tilde{\Delta}(N)^+$ has the following property: if $\delta_1, \cdots, \delta_k \in \tilde{\Delta}(N)^+$ and $\delta$ is a linear combination of these with nonnegative coefficients, then $\delta \in \tilde{\Delta}(N)^+$. Denote by $C(N)^+$ the following set:

$$
C(N)^+ : \{h \in V(N)^+ \cap N : h \cdot \delta > 0 \text{ for any } \delta \in \tilde{\Delta}(N)^+\}.
$$

(4.20)
Definition 4.6  A marked $N$-polarized $K3$ surface $S$ is called a marked pseudo-ample $N$-polarized $K3$ surface if
\[ C(N)^+ \cap \phi(Pic(S)^+) \neq \emptyset. \]  (4.21)

It is called a marked ample $N$-polarized $K3$ surface if
\[ C(N)^+ \cap \phi(Pic(S)^{++] \neq \emptyset. \]  (4.22)

Let $\mathcal{T}_N^{2,2}$ be the set of equivalence classes of marked Kähler $K3$ surfaces with $B$-fields. Let $\mathcal{T}_{N\subset \Gamma}^{2,2} \subset \mathcal{T}_\Gamma^{2,2}$ be the subset consisting of all marked Kähler $K3$ surfaces with $B$-fields $(S, \omega, B, \phi)$ such that $\omega, B \in N_\mathbb{R}$ and $N \subset \phi(Pic(S))$. $\mathcal{T}_{N\subset \Gamma}^{2,2}$ is realized as a subset of $Gr_2^{po}(\Gamma) \times \Gamma_\mathbb{R}$ by the period map. It is shown that the closure $\bar{\mathcal{T}}_{N\subset \Gamma}^{2,2}$ is
\[ \bar{\mathcal{T}}_{N\subset \Gamma}^{2,2} = \{((P, \omega), B) \in Gr_2^{po}(\Gamma) \times \Gamma_\mathbb{R} : B, \omega \in N_\mathbb{R}, P \subset N_\mathbb{R}^+ \}. \]

We assume, additionally, that the orthogonal complement $N^\perp \subset \Gamma$ contains a hyperbolic plane $U'$ (this implies that $r < 19$). Then one can show that there is a splitting $N = N \oplus U'$ for a sublattice $\tilde{N}$ with signature $(1,18-r)$. Moreover, we have that
\[ \Gamma = (U')^+ \oplus U'. \]  (4.23)

It is easy to see that $\tilde{N} = N$.

Then we can introduce $\tilde{\mathcal{T}}_{N\subset \Gamma}^{2,2}$ and $\mathcal{T}_{N\subset \Gamma}^{2,2}$ by replacing $N$ by $\tilde{N}$ in the definition of $\mathcal{T}_{N\subset \Gamma}^{2,2}$ and $\mathcal{T}_{N\subset \Gamma}^{2,2}$.

We define the period domain of marked $N$-polarized $K3$ surfaces as follows:
\[ D_N := \{ \Omega \in Pic(N^\perp \otimes \mathbb{C}) : \Omega \cdot \Omega = 0, \Omega \cdot \tilde{\Omega} > 0 \}. \]  (4.24)

Here $Pic(N^\perp \otimes \mathbb{C})$ is the projective space associated to $N^\perp \otimes \mathbb{C}$. $D_N$ is a subset of
\[ Q_\Gamma := \{ \Omega \in Pic(\Gamma) : \Omega^2 = 0, \Omega \cdot \tilde{\Omega} > 0 \} \cong Gr_2^{po}(\Gamma_\mathbb{R}). \]  (4.25)

We define the following tube domain:
\[ T_N := \{ B + i\omega \in N \otimes \mathbb{C} : \omega^2 > 0 \}. \]  (4.26)

$D_N$ and $T_N$ are interpreted as the space of complex deformations and the space of complexified Kähler deformations, respectively, for a marked $N$-polarized $K3$ surface (but note that in the definition above we do not insist that $\omega$ is Kähler). Similarly, we can define $D_{\tilde{N}}$ and $T_{\tilde{N}}$.

Clearly the map $((P, \omega), B) \to B + i\omega$ defines a surjection $\tilde{\mathcal{T}}_{N\subset \Gamma}^{2,2} \to T_N$. There is also the natural surjective map $\mathcal{T}_{N\subset \Gamma}^{2,2} \to D_N$, achieved by forgetting $\omega$ and $B$.

Theorem 4.7 ([12])  The mirror map $\tilde{\xi}$ induces a bijection
\[ \tilde{\mathcal{T}}_{N\subset \Gamma}^{2,2} \cong \mathcal{T}_{N\subset \Gamma}^{2,2}. \]  (4.27)

We also have the following commutative diagram:
\[ \begin{array}{ccc} \tilde{\mathcal{T}}_{N\subset \Gamma}^{2,2} & \rightarrow & \mathcal{T}_{N\subset \Gamma}^{2,2} \\ \downarrow & & \downarrow \\ T_N & \rightarrow & D_{\tilde{N}}. \end{array} \]  (4.28)
Here the two vertical maps are the two surjective maps defined above. The top horizontal one is the bijection \( T_{N \subset \Gamma}^{2,2} \cong T_{\tilde{N} \subset \Gamma}^{2,2} \), and the bottom horizontal one is an isomorphism \( a \) given by
\[
a(z) := [z - \frac{1}{2} z^2 v + v^*], z \in T_N.
\]
(4.29)

Clearly, the above theorem is still true if we swap \( N \) and \( \tilde{N} \), because \( \tilde{\tilde{N}} = N \).

We also define
\[
\Gamma' := (U')^\perp = N \oplus \tilde{N}
\]
so that \( \Gamma = \Gamma' \oplus U' \). The fact that \( (U')^\perp \) is proven in [12, Section 7].

**Remark 4.8** It is clear that what matters is the splitting \( \Gamma = \Gamma' \oplus U' \), the splitting \( \Gamma_R = N_R \oplus \tilde{N}_R \), and the isomorphism \( f : U' \cong U \). The lattices \( N \) and \( \tilde{N} \) are not important.

We do not expect that we have a bijection \( T_{N \subset \Gamma} \cong T_{\tilde{N} \subset \Gamma} \). To relate period domains to \( K3 \) surfaces we use the results below. As pointed out in [8], the fine moduli space of marked \( N \)-polarized \( K3 \) surface \( K_N \) exists and the period map \( p \) (which is defined as the restriction of the period map of marked Kähler \( K3 \) surfaces) maps it to \( D_N \).

**Theorem 4.9** ([8]) Let \( p \) be the restriction of the period map \( p : K_N \to D_N \) to the subset \( K_{pa}^N \) of equivalence classes of marked pseudo-ample \( N \)-polarized \( K3 \) surfaces. Then \( p \) is surjective. There is a natural bijection between the fiber of the map \( p \) over a point in \( D_N \) and a subgroup of isometries of \( \Gamma \) generated by some reflections (see [8] for details).

This theorem tells us that mirror symmetry (the mirror map) exchanges complex deformations and complexified Kähler deformations associated with marked pseudo-ample \( N \)-polarized \( K3 \) surfaces and marked pseudo-ample \( \tilde{N} \)-polarized \( K3 \) surfaces and vice versa. However, for a given marked pseudo-ample \( N \)-polarized \( K3 \) surface (whose complex structure and complexified Kähler structure are given by \( P \) and \( B + i \omega \) in the triple \( ((P, \omega), B) \)), there is not a unique mirror \( K3 \) surface, even though \( ((\tilde{P}, \tilde{\omega}), \tilde{B}) \) is uniquely determined. This is because the map \( p \) of Theorem 4.4 is not bijective, even though the fiber is discrete. The next theorem improves the situation.

For any \( \delta \in \tilde{\Delta}(N^\perp) \), we define that
\[
H_\delta := \{ z \in N^\perp_\mathbb{C} : z \cdot \delta = 0 \}, \quad D_N^\delta := D_N \setminus \left( \bigcup_{\delta' \in \Delta(N^\perp)} H_{\delta'} \cap D_N \right).
\]
(4.31)

**Theorem 4.10** ([8]) Let \( K_{pa}^N \) be the subset of \( K_N \) consisting of equivalence classes of marked ample \( N \)-polarized \( K3 \) surfaces. The restriction of the period map \( p \) to \( K_{pa}^N \) (also denoted by \( p \)) induces a bijection
\[
p : K_{pa}^N \to D_N^\delta.
\]
(4.32)

### 5 Special Langangians in Attractor Backgrounds

Let \( Y = S \times T^2 \), where \( S \) is a singular \( K3 \) surface. We pick a hyperkähler metric \( g \) such that the underlying complex structure of the singular \( K3 \) surface \( S \) is compatible with \( g \). Such a hyperkähler metric exists and is determined by the choice of a Kähler class on \( S \). The underlying complexified Kähler deformations.

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The integration of $\omega_L$ restriction of $\Omega_Y$ here \( L \) vanishes on \( I \). Thus, in the complex structure of the singular \( K3 \) surface is denoted by \( J \). The set of all compatible complex structures is identified with \( P^1 \) and generated by three complex structures \( I, J, K \) (i.e., it is \( \{ aI + bJ + cK : a^2 + b^2 + c^2 = 1 \} \)). Let $\omega_I, \omega_J, \omega_K$ be the three corresponding Kähler classes, and let $\Omega_I, \Omega_J, \Omega_K$ be the three corresponding holomorphic 2-forms (appropriately normalized). Then we have that

$$ \Omega_I = \omega_J + i\omega_K, \quad \Omega_J = \omega_K + i\omega_I. \quad (5.1) $$

The trick of hyperkähler rotation is that if we consider the compatible complex structure \( I \), then a holomorphic cycle \( L \) in \( J \) becomes a special Lagrangian cycle. In fact, \( \omega_J = \text{Im} \Omega_J \) vanishes on \( L \), since \( L \) is holomorphic in \( J \), and \( \text{Im} \Omega_I = \omega_K = 1/2(\Omega_J + \bar{\Omega}_J) \) becomes 0 on \( L \), while \( \text{Re} \Omega_I = \omega_J \) becomes the volume form on \( L \). This recovers the definition of a special Lagrangian cycle. Thus, in the complex structure \( I \), each class \( l \in \text{Pic}(S) \) is the Poincare dual of a special Lagrangian cycle (but it may not be a smooth special Lagrangian submanifold).

The following theorem is implicitly contained in [3]:

**Theorem 5.1** ([3]) Let \( l_1, l_2 \in \text{Pic}(S) \). Pick a compatible hyperkähler metric \( g \) on \( S \) (the underlying complex structure is \( J \)) and a flat metric \( g' \) on \( T^2 \). Endow \( Y \) with the product metric \( \hat{g} \). Take the hyperkähler rotation on \( S \) by rotating \( J \) to \( I \). Endow the manifold \( Y \) with the product complex structure which is the product of \( I \) and the one on \( T^2 \) corresponding to \( \tau \). Set \( u_i := l_i \wedge dy \). Then \( u_i \) is dual to a special Lagrangian cycle in \( Y \) endowed with the product metric and the product complex structure specified above. Moreover,

$$ \arg Z(u_1) = \pm \arg Z(u_2). \quad (5.2) $$

In fact, we have a stronger result, which implies (5.2), namely,

$$ Z(u_i) = Z_{K3}(l_i) := \omega_J \cdot l_i \in \mathbb{R}. \quad (5.3) $$

**Proof** The holomorphic volume form (up to a normalization) on \( S \) is \( \Omega_I = q - \bar{\tau}p \) (equation (2.7)) while the holomorphic volume form (up to a normalization) on \( T^2 \) is \( dz = dx + \tau dy \). After the hyperkähler rotation on \( S \), the holomorphic volume form is \( \Omega_I = \omega_J + i(q - \text{Re}(\tau)p) \). Thus the normalized holomorphic 3-form on the product \( Y \) with the product complex structure is

$$ \Omega_Y := \Omega_I \wedge dz = (\omega_J + i(q - \text{Re}(\tau)p)) \wedge (dx + \tau dy). \quad (5.4) $$

\( l_i \) is dual to a holomorphic cycle \( L_i \) of \( S \) in \( J \). Then \( u_i \) is dual to a three cycle \( L_i \times S^1 \) in \( Y \). Here \( S^1 \) is a circle in \( T^2 \) dual to \( dy \).

On \( Y \) with the product metric and the product complex structure, we should consider the restriction of \( \Omega_Y \). Clearly \( \text{Im} \Omega_Y \) restricts on \( L_i \times S^1 \) to 0, while \( \text{Re} \Omega_Y \) becomes \( \omega_J \wedge dx |_{L_i \times S^1} \). The integration of \( \omega_J \wedge dx |_{L_i \times S^1} \) on \( L_i \times S^1 \) is its normalized volume in \( \hat{g} \). Thus we know that \( L_i \times S^1 \) is a special Lagrangian cycle in the product \( Y \).

Direct calculation gives that

$$ Z(u_i) = Z(u_i, \Omega_Y) = \omega_J \cdot l_i \in \mathbb{R}. \quad (5.5) $$

Here we have used that \( \int_{T^2} dx \wedge dy = 1 \) and \( p \cdot l_i = q \cdot l_i = 0 \).
Remark 5.2 We have defined $Z_{K3}(l_i)$ to indicate that the central charge is actually associated to the $K3$ surface. This will be related by the mirror symmetry to another central charge $Z(\mu(l_i))$, associated to the $K3$ surface defined in Section 6.

In fact, even if $S$ is just a $K3$ surface, the hyperkähler rotation trick gives us special Lagrangian cycles assuming that $S$ has holomorphic cycles. One (trivial) interesting point about a singular $K3$ is that there are many such cycles. Theorem 5.1 gives us another interesting fact: there are many special Lagrangians with aligned or anti-aligned central charges in an attractor background of $S \times T^2$ (after a hyperkähler rotation on $S$), and if we fix the topological data (the cohomology classes), then we can view the central charge as a function of the complex moduli. The condition of two charges having the same phase for the central charges is a condition which gives a real codimension one locus (a wall) in the moduli space. Thus we have just shown that many codimension one walls intersect at an attractor background, which is quite a special situation.

As explained in the introduction, we expect that there exist stability conditions on derived Fukaya categories such that special Lagrangians are stable objects (see [2, 16, 17] for some hints about this). Then the condition of having aligned central charges will become the definition of marginal stability walls in the space of such stability conditions. Therefore we expect that Theorem 5.1 can be interpreted as the statement that many marginal stability walls intersect at an attractor background.

6 Mirror Symmetry and Stability Conditions of $K3$ Surfaces in Attractor Backgrounds

Let us assume that $S$ is an elliptic $K3$ surface $\pi: S \to P^1$ with a section $\sigma_0$. We denote the cohomology class of a general fiber by $f$ and also denote the section class by $\sigma_0$. Then there is a hyperbolic plane sublattice $U' \subset H^2(S, \mathbb{Z})$ generated by the basis $\{v, v^*\}$ with

$$v := f, \quad v^* := f + \sigma_0.$$  

(6.1)

Here we have used that $f^2 = 0$, that $f \cdot \sigma_0 = 1$ and that $\sigma_0^2 = -2$.

Denote the complex structure of the elliptic $K3$ surface $S$ by $J$. Pick a compatible hyperkähler metric $g$. The set of all compatible complex structures is generated by three complex structures $I, J$ and $K$. The trick of hyperkähler rotation then tells us that if we consider the compatible complex structure $I$, then $\pi: S \to P^1$ becomes a (singular) special Lagrangian torus fibration.

Inspired by the SYZ mirror conjecture we consider the mirror symmetry of $S$ in the complex structure $I$.\^8 Note that we already have $\omega_I \in \Gamma'$. We will discuss below how to pick an $N$-polarization for a singular $K3$ surface. Here let us assume that we already have a marked pseudo-ample $N$-polarization structure. We choose $B \in N_R$. $\Omega_I$ is normalized by

$$\omega_I^2 = (\text{Re} \, \Omega_I)^2 = (\text{Im} \, \Omega_I)^2.$$  

(6.2)

\^8Here we study the mirror of $S_I$ (not $S_J$) for good reasons. From the perspective of attractors, it is natural to consider a singular $K3$ surface with elliptic fibration structure (i.e., $S_J$). From the perspective of the SYZ mirror conjecture (concerning dual special Lagrangian torus fibrations), it is natural to study $S_I \times T^2$, because it contains special Lagrangian cycles. The two natural perspectives are connected via the hyperkähler rotation, which is also a natural operation.
Note that $\text{Im} \, \Omega_I \in \Gamma'_R$.

Under the assumption above, the $\xi$ mirror $(\bar{\Omega}_I, \bar{\omega}_I, \bar{B})$ of $(\Omega_I, \omega_I, B)$ is given by (according to Theorem 4.4)

$$\bar{\Omega}_I = \frac{1}{(\text{Re} \, \Omega_I) \cdot f} (B + i \omega_I - \frac{1}{2} (B + i \omega_I)^2 f + f_0),$$

$$\bar{\omega}_I = \frac{1}{(\text{Re} \, \Omega_I) \cdot f} (\text{Im} \, \Omega_I - ((\text{Im} \, \Omega_I) \cdot B) f),$$

$$\bar{B} = \frac{1}{(\text{Re} \, \Omega_I) \cdot f} (\text{pr}(\text{Re} \, \Omega_I) - ((\text{Re} \, \Omega_I) \cdot B) f).$$

(6.3) (6.4) (6.5)

We now study the mirror symmetry of $K3$ surfaces in an attractor background. We assume that $S$ is a singular $K3$ surface corresponding to $Q_{p,q}$. As before, we choose a hyperkähler metric $g$, such that this complex structure is $J$ in a triple $(I, J, K)$. $S$ carries elliptic fibration structures with sections. Pick and then fix such a structure. Let $f$ be the fiber class of a general fiber and let $\sigma_0$ be the cohomology class of a section.

To specify the mirror map $\xi$ on the period domains, we need a hyperbolic plane $U$ (with basis $\{w, w^*\}$) and an isomorphism $f : U' \cong U$. Here $U'$ is the hyperplane generated by $v := f$ and $v^* := f + \sigma_0$. In Section 4, the lattice $U$ is somewhat abstract. One natural way to geometrically identify $U$ is to identify it with $H^0(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z})$. We extend the intersection paring on $H^2(S, \mathbb{Z}) \cong \Gamma$ to the Mukai paring on $H^*(S, \mathbb{Z}) := H^0(S, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z})$. Thus

$$
\Gamma_R \oplus U_R \cong H^*(S, \mathbb{R}).
$$

(6.6)

We pick a basis $\{w, w^*\}$ of $U$. In terms of the notation of Section 2, we take that

$$w = (0, 0, -1), \quad w^* = (1, 0, 0).$$

(6.7)

Clearly $(w, w^*) = 1$.\(^9\)

As for the isomorphism $f : U' \cong U$, we pick the one that maps $\{f, \sigma_0\}$ to $\{w, (1, 0, 1)\}$. Thus $f$ maps $\{v, v^*\}$ to $\{w, w^*\}$. This respects the Mukai pairing.

Fix any marking $\phi$ of $S$. We need an $N$-polarization which is at least pseudo-ample.

**Theorem 6.1** Assume that $\sqrt{D_{p,q}}$ (which is an invariant of equivalence classes of $Q_{p,q}$) is an integer. Denote by $(S, I)$ the $K3$ surface $S$ with the complex structure $I$, and by $\text{Pic}(S, I)$ its Picard lattice.\(^{10}\) Then there is a marked pseudo-ample $N$-polarization of $((S, I), \phi)$ for some sublattice $N \subset \phi(\text{Pic}(S, I))$ with signature $(1, r)$ ($r \leq \text{rank}(\text{Pic}(S, I)) - 1$). Moreover, $U' \subset N^\perp$.

**Proof** By our assumption about $\sqrt{D_{p,q}}$, the holomorphic 2-form $\Omega_I = q - \bar{\tau}p$ in the complex structure $J$ is rational. After the hyperkähler rotation to $I$ we have a rational Kähler class $\omega_I = \text{Im} \, \Omega_I = \text{Im}(\tau)p$. Therefore $p^2 \omega_I$ (which is integral) is in $\text{Pic}(S, I)$. This guarantees

\(^9\)Here $w = (0, 0, -1)$, because we want to use the Mukai pairing on $H^*$. If we use the standard intersection paring (we can do that), then we should take $w = (0, 0, 1)$ so that $w \cdot w^* = 1$. Using the Mukai pairing as the paring on $U$ introduces a somewhat confusing point that we want to clarify now. In Section 4, the paring on the lattice $U$ is denoted by $x \cdot y$, but now it should be denoted by $(x, y)$. Thus the condition $w \cdot w^* = 1$ in Section 4 now becomes $(w, w^*) = 1$.

\(^{10}\)We should also use $(S, J)$ and replace $\text{Pic}(S)$ by $\text{Pic}(S, J)$, but when there is no danger of confusion, we may keep using $\text{Pic}(S)$. © Springer
the existence of a nonempty sublattice \( N \). For definiteness, we can choose that \( N = \langle y^2 \omega_1 \rangle \).

This \( N \)-polarization is an ample \( N \)-polarization.

Since \( f, \sigma_0 \in \text{Pic}(S, J) \), \( f \cdot \omega_1 = \sigma_0 \cdot \omega_1 = 0 \). Therefore, at least for \( N = \langle y^2 \omega_1 \rangle \), we have \( U' \subset N^\perp \). Of course the choice of \( N \) in Theorem 6.1 need not be unique. \( \square \)

We choose \( B \in N_\mathbb{R} \) after we have chosen \( N \). If we choose \( B = 0 \) (we will do that later), then the choice of \( N \) or \( N_\mathbb{R} \) from Theorem 6.1 is irrelevant (only the existence is required). In fact, since the splitting \( \Gamma = \Gamma' \oplus U' \) is independent of the choice of \( N \) or \( N_\mathbb{R} \), so are equations (6.3), (6.4) and (6.5) when \( B = 0 \).

Let \((\tilde{S}, \Omega_I, \omega_I, B, \tilde{\phi})\) be a mirror of \((S, \Omega_I, \omega_I, B, \phi)\). \((\tilde{S}, \tilde{\phi})\) can be assumed to be a marked pseudo-ample \( \tilde{N} \)-polarized \( K3 \) surface by Theorem 4.9. Note that although \( \tilde{S} \) is projective, it is not guaranteed that \( \tilde{\omega}_I \) is Kähler. Also note that \((\tilde{S}, \tilde{\phi})\) may not be uniquely determined by \((S, \Omega_I, \omega_I, B, \phi)\) if the period point of \( \Omega_I \) is not in \( D_N^0 \). Of course \((\Omega_I, \omega_I, B)\) is uniquely determined by \((S, \Omega_I, \omega_I, B, \phi)\). This guarantees that results in the rest of this section hold, regardless of which \((\tilde{S}, \tilde{\phi})\) we choose from Theorem 4.9.

Recall the definition of the mirror map \( \xi = \iota \circ \xi \), where \( \xi \) exchanges \( U \) and \( U' \) via \( U \cong U' \) and becomes the identity on \( \Gamma_\mathbb{R} \). Clearly we have a lattice isomorphism between two sublattices of \( H^2(S, \mathbb{Z}) \) and \( H^2(\tilde{S}, \mathbb{Z}) \):

\[
j := \tilde{\phi}^{-1} \circ \xi \circ \phi : \phi^{-1}(\Gamma') \to \tilde{\phi}^{-1}(\Gamma'). \tag{6.8}
\]

**Definition 6.2** For a cohomology class \( l \in H^2(S, \mathbb{Z}) \) we define the mirror class \( \mu(l) \in H^*(\tilde{S}, \mathbb{Z}) \) in the following way: if \( l \in \phi^{-1}(\Gamma') \), then

\[
\mu(l) := j(l). \tag{6.9}
\]

If \( l \in \phi^{-1}(U') \), then

\[
\mu(l) := \tilde{\phi}^{-1} \circ f \circ \phi(l). \tag{6.10}
\]

Then we can extend \( \mu \) so as to be defined on \( H^2(S, \mathbb{Z}) \cong \phi^{-1}(\Gamma') \oplus \phi^{-1}(U') \). Here we have extended \( \phi \) to the isomorphism \( H^*(\tilde{S}, \mathbb{Z}) \to \Gamma \oplus U \) by identifying \( H^0 \oplus H^4 \) with \( U \) with the specification of the basis \( \{w, w^*\} \).

**Remark 6.3** This transformation of cohomology classes is not only demanded by the mirror symmetry on the level of period domains (and the non-explicit construction of mirrors by the surjectivity of period maps), but is also compatible with some other approaches to the mirror symmetry of \( K3 \) surfaces (for example see [11] and [14]).

However, we are not able to describe the mirror classes explicitly, as \( \tilde{\phi} \) is obtained by using the surjectivity of period maps, which is not an explicit construction (at least to the author). Nevertheless, since \( \phi, \tilde{\phi} \) are lattice isomorphisms, we can compute the intersection (or Mukai) pairings of classes on \( S \) or \( \tilde{S} \) by going to \( \Gamma \oplus U \), and therefore we omit the markings in the formulas below.

Now let us formulate and prove main theorems of this paper.

**Theorem 6.4** Let \((S, J)\) be the singular \( K3 \) surface corresponding to \( Q_{p,q} \) with a specified elliptic fibration structure with sections. Assume that \( \sqrt{D_{p,q}} \) is an integer. Fix a tuple \((S, \Omega_I, \omega_I, B, \phi)\). Fix a marked pseudo-ample \( N \)-polarization for some \( N \) from Theorem 6.1. Fix the lattices \( U \) and \( U' \) with the basis specified above. Fix a splitting \( \Gamma = \Gamma' \oplus U' \). Fix the isomorphism \( f \) specified above.
For any \( l_1, l_2 \in \text{Pic}(S) \), define \( Z(\mu(l_i)) := (\exp(B + i\omega_I), \mu(l_i)) \) (here \( \mu(l_i) \) is considered as an element of \( \Gamma \oplus U \) via the (extended) marking \( \hat{\phi} \) but we have omitted the makings in our notations). Then
\[
\arg Z(\mu(l_1)) = \pm \arg Z(\mu(l_2)).
\] (6.11)
In fact we have a stronger result that implies (6.11):
\[
Z(\mu(l_i)) \in \mathbb{R}.
\] (6.12)

**Proof** The holomorphic 2-form (up to a normalization) on \( S \) is given by: \( \Omega_I = q - \bar{r}p \).

Therefore,
\[
\omega_I = \text{Im}(\Omega_I) = \text{Im}(\tau)p,
\] (6.13)
\[
\text{Im}(\Omega_I) = \frac{1}{2}(\Omega_I + \bar{\Omega}_I) = q - \text{Re}(\tau)p,
\] (6.14)
\[
\text{Re}(\Omega_I) = \omega_J.
\] (6.15)

\((\bar{\Omega}_I, \bar{\omega}_I, \bar{B})\) are given by
\[
\bar{\Omega}_I = \frac{1}{(\omega_J) \cdot f} (B + i\text{Im}(\tau)p - \frac{1}{2}(B + i\text{Im}(\tau)p)^2 f + f + \sigma_0),
\] (6.16)
\[
\bar{\omega}_I = \frac{1}{(\omega_J) \cdot f} (q - \text{Re}(\tau)p - ((q - \text{Re}(\tau)p) \cdot B)f),
\] (6.17)
\[
\bar{B} = \frac{1}{(\omega_J) \cdot f} (pr(\omega_J) - (\omega_J \cdot B)f).
\] (6.18)

Here \( pr \) is the projection \( pr : \Gamma \to \Gamma' \).

Since \( \Gamma = \Gamma' \oplus U' \), we know that \( l_i = l_i^a + l_i^b \), with \( l_i^a \in \Gamma' \) and \( l_i^b \in U' \). Since \( U' \subset \text{Pic}(S) = \text{Pic}(S, J) \), we see that \( l_i^a, l_i^b \in \text{Pic}(S) \). Therefore, to prove the theorem, it suffices to prove (6.12) for any \( l_i \in \Gamma' \) and for any \( l_i \in U' \).

If \( l_i \in \Gamma' \), then
\[
Z(\mu(l_i)) = (\exp(B + i\omega_I), (0, l_i, 0)).
\]

Here we are computing the pairing in \( \Gamma \oplus U \) as explained before. So
\[
Z(\mu(l_i)) = l_i \cdot B + \bar{u}_i \cdot \bar{\omega}_I.
\] (6.19)

Since \( \Omega_I \cdot l_i = 0 \) (\( l_i \in \text{Pic}(S) \)) and \( l_i \) is real, we get that \( p \cdot l_i = q \cdot l_i = 0 \). Also note that \( f \cdot l_i = 0 \) (because \( l_i \in \Gamma' = (\bar{U}')^\perp \)). Therefore \( l_i \cdot \omega_I = 0 \), i.e., \( Z(\mu(l_i)) \in \mathbb{R} \).

For the case of \( l_i \in U' \), it suffices to show that \( Z(\mu(f)), Z(\mu(\sigma_0)) \in \mathbb{R} \). By (3.9) and (3.10),
\[
Z(\mu(f)) = Z((0, 0, -1)) = 1,
\] (6.20)
\[
Z(\mu(\sigma_0)) = Z((1, 0, 1)) = -\bar{B} \cdot \bar{\omega}_I.
\] (6.21)

We have that \( pr(\omega_I) \cdot f = 0, f^2 = 0 \) and \( f \cdot p = f \cdot q = \sigma_0 \cdot p = \sigma_0 \cdot q = 0 \). We also have that \( \omega_I \cdot \Omega_I = 0 \), which implies \( \omega_I \cdot p = \omega_I \cdot q = 0 \). Thus we see that \( \bar{B} \cdot \bar{\omega}_I = 0 \), which means that \( Z(\mu(\sigma_0)) \) is also real. \( \square \)

We want to show that \( \exp(B + i\omega_I) \in \mathcal{P}_0(\bar{S}) \). In general this is hard to verify and may not be true, but we have some freedom when we choose the data \( (\Omega_I, \omega_I, B) \), so the idea is that maybe we can use the freedom to make sure that \( \exp(B + i\omega_I) \) is away from \( \bigcup_{\delta \in \Delta(S)} \delta^\perp \).
We certainly do not want to change the singular K3 surface \((S,J)\), so we fix \(\Omega_J\). After choosing the \(B\) equation (6.16) tells us that \(\Omega_I\) is fixed up to a real scaling factor. Now \(\Delta(\hat{S})\) depends on the complex structure, so we had better fix the period point \(\hat{\Omega}_I\). This means that after fixing \(B\) once and for all, the only freedom is changing \(\omega_J\), i.e., changing the hyperkähler metric \(g\). This is natural, as the attractor mechanism says nothing about \(\omega_J\). Changing \(\omega_J\) will change \(\hat{B}\).

**Theorem 6.5** We make the assumptions of Theorem 6.4. We choose that \(B = 0\).\(^{11}\) In addition we assume that \(D_{p,q} \neq 2p^2\). Then there exists some Kähler class \(\omega_J\) on \((S,J)\) such that

\[
\exp(\hat{B} + i\hat{\omega}_I) \in \mathcal{P}_0(\hat{S}).
\]  
(6.22)

**Proof** With \(B = 0\), equations (6.16), (6.17) and (6.18) simplify to

\[
\hat{\Omega}_I = \frac{1}{(\omega_J)^{1-f}}(\text{Im}(\tau)p + \frac{1}{2}(\text{Im}(\tau))^2p^2f + f + \sigma_0),
\]  
(6.23)

\[
\hat{\omega}_I = \frac{1}{(\omega_J)^{1-f}}(q - \text{Re}(\tau)p),
\]  
(6.24)

\[
\hat{B} = \frac{1}{(\omega_J)^{1-f}}(pr(\omega_J)).
\]  
(6.25)

We need to check that, for any \(\delta \in \mathcal{N}(\hat{S})\) with \((\delta,\delta) = -2\), we have \((\exp(\hat{B} + i\hat{\omega}_I),\delta) \neq 0\). Here again we are computing the pairing on \(\Gamma \oplus U\) and have omitted the markings in the notations. Such a \(\delta = (r,D,s)\) must belong to one of the following types:

1. \(D^2 = -2, r = 0, s \neq 0\). In this case,

\[
\text{Re}(\exp(\hat{B} + i\hat{\omega}_I),\delta) = D \cdot \hat{B} - s;
\]  
(6.26)

2. \(D^2 - 2rs = -2, r \neq 0, s \neq 0\). In this case,

\[
\text{Re}(\exp(\hat{B} + i\hat{\omega}_I),\delta) = \frac{1}{2r}(-2rs + r^2\hat{\omega}_I^2 - r^2\hat{B}^2 + 2rD \cdot \hat{B});
\]  
(6.27)

3. \(D^2 = -2, r \neq 0, s = 0\). In this case,

\[
\text{Re}(\exp(\hat{B} + i\hat{\omega}_I),\delta) = \frac{1}{2r}(r^2\hat{\omega}_I^2 - r^2\hat{B}^2 + 2rD \cdot \hat{B});
\]  
(6.28)

4. \(\delta = (0,D,0), D^2 = -2\). In this case,

\[
\text{Re}(\exp(\hat{B} + i\hat{\omega}_I),\delta) = D \cdot \hat{B}.
\]  
(6.29)

Pick a reference Kähler class \(\omega_J^0\) in the complex structure \(J\). Let \(\{n_1, \cdots, n_{20}\}\) be a basis of \(Pic(S)\). Let \(\alpha := (\alpha_1, \cdots, \alpha_{20}) \in \mathbb{R}^{20}\). Define \(\|\alpha\| := \max_{1 \leq i \leq 20} \{\|\alpha_i\|\}\). For any given positive number \(\beta\), if \(\|\alpha\|\) is small enough (depending on \(\beta\)), then the class

\[
\omega_J^{\alpha,\beta} := \beta\omega_J^0 + \sum_{k=1}^{20} \alpha_k n_k
\]  
(6.30)

is Kähler. First of all, \(\omega_J^{\alpha,\beta}\) is real and of type \((1,1)\). By our assumptions about \(\beta\) and \(\|\alpha\|\), we can assume that \((\omega_J^{\alpha,\beta})^2 > 0\). Therefore, \(\omega_J^{\alpha,\beta}\) is in a small enough neighborhood of \(\beta\omega_J^0\) in the positive cone \(V(S)^+\). Since the dimension of the positive cone is equal to the dimension of

\(^{11}\) Assumptions made in this theorem may not be optimal.
the Kähler cone, and $\beta \omega_j^\alpha$ is in the interior of the Kähler cone, we know that a small enough neighborhood of $\beta \omega_j^\alpha$ in the positive cone is actually contained in the Kähler cone. Thus $\omega_j^{\alpha,\beta}$ is Kähler.

We have that $\omega_j^{\alpha,\beta} \in \langle p, q \rangle_R^\perp := (\langle p, q \rangle \otimes \mathbb{R})^\perp$, where $\langle p, q \rangle$ is the sublattice generated by $p, q$ (which is the transcendental lattice of the singular K3 surface $S$). We also know that $pr(\omega_j^{\alpha,\beta}) \in (U_R')^\perp$. Moreover, $U_R' \subset \langle p, q \rangle_R^\perp$. Therefore,

$$pr(\omega_j^{\alpha,\beta}) \in \langle p, q, f, \sigma_0 \rangle_R^\perp. \tag{6.31}$$

We set that $\omega_j = \omega_j^{\alpha,\beta}$. Then

$$\hat{\omega}_j^2 - B^2 = \frac{1}{((\omega_j^{\alpha,\beta}) \cdot f)^2}(q^2 + (\text{Re}(\tau))^2 p^2 - 2\text{Re}(\tau)p \cdot q - (pr(\omega_j^{\alpha,\beta}))^2), \tag{6.32}$$

$$D \cdot \hat{B} = \frac{1}{(\omega_j^{\alpha,\beta}) \cdot f}(pr(\omega_j^{\alpha,\beta}) \cdot D). \tag{6.33}$$

When $\delta$ belongs to type 1, 2 or 3, we are going to show that the condition $\text{Re}(\exp(\hat{B} + i\hat{\omega}_1), \delta) = 0$ is too strong to be satisfied by a generic perturbation of $\omega_j^{\alpha,\beta}$. For type 1, $\delta$ is a nonzero integer. If $D \cdot \hat{B}$ happens to be a nonzero integer, then we just redefine $\omega_j^{\alpha,\beta}$ by adding $c\sigma_0$ (with a small enough positive constant $c$) to it. Then, due to (6.31), the numerator of $D \cdot \hat{B}$ does not change, while the denominator is modified by adding a real number $c$. Hence, for such a small perturbation of $\omega_j^{\alpha,\beta}$, the integral property $D \cdot \hat{B} = \eta$ cannot hold. Note that this means that $\text{Re}(\exp(\hat{B} + i\hat{\omega}_1), \delta) = 0$ for all $\delta$’s of type 1.

For type 2 or 3, $s, r, q^2 + (\text{Re}(\tau))^2 p^2 - 2\text{Re}(\tau)p \cdot q$ are fixed and rational. $r\hat{\omega}_1^2 - rB^2 + 2D \cdot \hat{B}$ must be rational. We can assume that $pr(\omega_j^{\alpha,\beta}) \neq 0$. In fact, if this is not the case, we can perturb $\omega_j^{\alpha,\beta}$ along a direction in $(U')^\perp \cap \text{Pic}(S)$. Then we will have that $pr(\omega_j^{\alpha,\beta}) = 0$. Here $(U')^\perp \cap \text{Pic}(S) \neq \emptyset$, because $\rho(S)$ is too large. Since $\langle p, q, f, \sigma_0 \rangle$ has the signature $(3, 1)$, we know that $\langle p, q, f, \sigma_0 \rangle^\perp$ has the signature $(0, 18)$. Thus $(pr(\omega_j^{\alpha,\beta}))^2 = 0$. Then we know that for a generic choice of $\omega_j^{\alpha,\beta}$ such that $(pr(\omega_j^{\alpha,\beta}))^2$ is irrational, $r\hat{\omega}_1^2 - rB^2$ and hence $2D \cdot \hat{B}$ must be nonzero. We also notice that the denominator of $D \cdot \hat{B}$ is not a fixed multiple of the denominator of $\hat{\omega}_1^2$. Therefore the same perturbation of the Kähler class argument (adding $c\sigma_0$) shows that for some small perturbation of $\omega_j^{\alpha,\beta}$, the condition $\text{Re}(\exp(\hat{B} + i\hat{\omega}_1), \delta) = 0$ is not true for all $\delta$’s of type 1, 2 or 3.

We still need to handle type 4, for which the condition $\text{Re}(\exp(\hat{B} + i\hat{\omega}_1), \delta) = 0$ becomes

$$D \cdot \hat{B} = 0. \tag{6.34}$$

The above perturbation of the Kähler class argument can fail in this case. It fails if and only if

$$D \cdot \eta = 0 \text{ for any } \eta \in \langle p, q, f, \sigma_0 \rangle_R^\perp. \tag{6.35}$$

Clearly, if (6.35) is true, then (6.34) always holds, no matter how we perturb $\omega_j^{\alpha,\beta}$. If (6.34) is true and there is an $\eta \in \langle p, q, f, \sigma_0 \rangle^\perp_R$ such that $D \cdot \eta \neq 0$, then we can perturb $\omega_j^{\alpha,\beta}$ by adding $c\eta$ to get that $D \cdot \hat{B} \neq 0$.

Now let us show that when $\delta$ is of type 4, the condition

$$\langle \exp(\hat{B} + i\hat{\omega}_1), \delta \rangle = 0 \tag{6.36}$$

and the condition (6.35) cannot both hold. Clearly this (together with the discussion on type
1, 2 and 3) implies that, for some Kähler class \( \omega^a_\beta \) on \((S, J)\), we have that
\[
(\exp(\hat{B} + i\hat{\omega}_I), \delta) \neq 0, \text{ for any } \delta \in \Delta(\hat{S}).
\] (6.37)

Then the theorem is proven.

Let us suppose that (6.36) and (6.35) both hold for some \( \delta \) of type 4. By computation, (6.36) is equivalent to
\[
\text{Re}(\exp(\hat{B} + i\hat{\omega}_I), \delta) = D \cdot \hat{B} = 0, \tag{6.38}
\]
\[
\text{Im}(\exp(\hat{B} + i\hat{\omega}_I), \delta) = D \cdot \hat{\omega}_I = 0. \tag{6.39}
\]
The first equation is already implied by (6.35). We also have the following obvious relations (since \( D \in \text{NS}(\hat{S}) \)):
\[
\text{Im}(\hat{\Omega}_I) \cdot D = 0, \text{ i.e., } p \cdot D = 0, \tag{6.40}
\]
\[
\text{Re}(\hat{\Omega}_I) \cdot D = 0, \text{ i.e., } \left( \frac{1}{2} \text{Im}(\tau) \right)^2 p^2 f + f + \sigma_0 \cdot D = 0. \tag{6.41}
\]

Since \( \langle p, q, f, \sigma_0 \rangle^\perp \) has signature \((0, 18)\), (6.35) implies that \( D \in \langle p, q, f, \sigma_0 \rangle \). Note that \( U' \) is the orthogonal complement of \( \langle p, q \rangle \) in \( \langle p, q, f, \sigma_0 \rangle \). Then (6.40) and (6.39) together imply that
\[
D = mf + n\sigma_0 \text{ for some integers } m, n. \tag{6.44}
\]
which implies that
\[
n(m - n) = -1. \tag{6.43}
\]
Since \( m, n \) are integers we must have that \( n = \pm 1 \) and that \( m - n = \mp 1 \). Substituting these results back to (6.42), we get that
\[
D_{p,q} = 2p^2, \tag{6.44}
\]
but this violates our assumption, and therefore is a contradiction.

Finally, we want to point out that the condition that \( \sqrt{D_{p,q}} \) is integral and the condition that \( D_{p,q} \neq 2p^2 \) can be satisfied at the same time. The positive definite even integral quadratic form \( Q_{p,q} := \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix} \) is such an example. \( \square \)

Theorem 3.4 and Theorem 6.5 tell us that there is a covering map
\[
\pi^{-1}(\mathcal{P}_0(\hat{S})) \to \mathcal{P}_0(\hat{S}) \tag{6.45}
\]
such that there are stability conditions in \( \pi^{-1}(\mathcal{P}_0(\hat{S})) \) whose central charge is
\[
Z(E) = (\exp(\hat{B} + i\hat{\omega}_I), \nu(E)). \tag{6.46}
\]
Let \( \text{Stab}^*(\hat{S}) \) be the good component containing these stability conditions. We will write \( Z(E) \) as \( Z(l), l = \nu(E) \) to emphasize that the central charge depends only on the Mukai vector \( l \in \mathcal{N}(\hat{S}) \), and not on \( E \) directly. Clearly \( Z(l) \) can be extended by the same formula (6.46) to be defined for all \( l \in H^*(\hat{S}, \mathbb{Z}) \).
Definition 6.6 Let \( v_i, v_j \in H^*(\mathcal{S}, \mathbb{Z}) \) and \( \tilde{\gamma} = (i, j) \). We define the generalized marginal stability wall \( \tilde{W}_{\tilde{\gamma}} \) by

\[
\tilde{W}_{\tilde{\gamma}} = \{ \sigma = (Z, \mathcal{P}) \in \text{Stab}^*(\mathcal{S}) : Z(v_i)/Z(v_j) \in \mathbb{R}_{>0} \}.
\] (6.47)

Although the definition of \( \tilde{W}_{\tilde{\gamma}} \) is similar to the definition of the marginal stability wall \( W_{\gamma} \) in Section 3 there is a difference. In the definition of \( \tilde{W}_{\tilde{\gamma}} \) we say nothing about objects in \( \mathcal{D}(\mathcal{S}) \), while in the definition of \( W_{\gamma} \), we require that the elements \( v_i, v_j \in \mathcal{N}(\mathcal{S}) \) are Mukai vectors of some objects in \( \mathcal{D}(\mathcal{S}) \). Every marginal stability wall is a generalized marginal stability wall. Of course, if \( v_i, v_j \in \mathcal{N}(\mathcal{S}) \) are Mukai vectors of some objects used in the Definition 3.5 then \( \tilde{W}_{\tilde{\gamma}} \) is a marginal stability wall.

Let \( \{ l_1, \cdots, l_{20} \} \) be a basis of \( \text{Pic}(\mathcal{S}) = \text{Pic}(S, J) \). By definition, we have that \( \{ \mu(l_i) \in H^*(\mathcal{S}, \mathbb{Z}) \}, 1 \leq i \leq 20 \). Let \( \tilde{\Gamma} \) be the set of pairs \( (i, j), 1 \leq i, j \leq 20 \) with \( i < j \). Then for any \( \tilde{\gamma} = (i, j) \in \tilde{\Gamma} \), we have the generalized marginal stability wall

\[
\tilde{W}_{\tilde{\gamma}} = \{ \sigma = (Z, \mathcal{P}) \in \text{Stab}^*(\mathcal{S}) : Z(\mu(l_i))/Z(\mu(l_j)) \in \mathbb{R}_{>0} \}.
\] (6.48)

Theorem 6.7 We make the assumptions of Theorem 6.5. Then there is some Kähler class \( \omega_J \) on \( (\mathcal{S}, J) \) such that (6.22) is true, and \( \text{Re}Z(\mu(l_i)) \neq 0 \) for any \( i \). We change \( l_i \) to \( -l_i \) for some \( i \)'s if necessary. Then the projections of generalized marginal stability walls \( \{ \pi(\tilde{W}_{\tilde{\gamma}}) \}, \tilde{\gamma} \in \tilde{\Gamma} \) from the space of stability conditions on \( \mathcal{S} \) to \( \mathcal{N}(\mathcal{S}) \otimes \mathbb{C} \) intersect at \( \tilde{B} + i\tilde{\omega}_I \).

Proof First we show that \( \text{Re}Z(\mu(l_i)) \neq 0 \) for any \( i \).

We already know that \( Z(\mu(f)) = 1 \). For \( \text{Re}Z(\mu(\sigma_0)) \) we see that \( \mu(\sigma_0) = (1, 0, 1) \) is of type 2 in the proof of Theorem 6.5. By the proof of Theorem 6.5, there is an \( \omega_J \) such that both (6.22) and \( \text{Re}Z(\mu(\sigma_0)) \neq 0 \) are true. For \( l_i \in \Gamma' \) (also note that \( l_i \in \text{Pic}(\mathcal{S}) \)), we have that \( \text{Re}Z(\mu(l_i)) = l_i \cdot \tilde{B} \), which is the same as (6.29) with \( D \) replaced by \( l_i \). However, here we know that \( l_i \in \langle p, q, f, \sigma_0 \rangle^\perp \), so the condition (6.35) is not true, but then the proof of Theorem 6.5 tells us that \( \text{Re}Z(\mu(l_i)) \neq 0 \) for some small perturbation of \( \omega_J \). In general, \( l_i = al_i^0 + mf + n\sigma_0 \), where \( l_i^0 \in \Gamma' \) and \( a, m, n \) are integers. Again a perturbation of Kähler class guarantees that \( \text{Re}Z(\mu(l_i)) \neq 0 \) for all \( i \), and (6.22) holds.

By Theorem 6.4, every \( Z(\mu(l_i)) \) is real, i.e., \( Z(\mu(l_i)) = \text{Re}Z(\mu(l_i)) \). Thus after changing some \( l_i \) to \( -l_i \) for some \( i \)'s, we can assume that \( \text{arg}Z(\mu(l_i)) > 0 \) for any \( i \). We know that \( \exp(\tilde{B} + i\tilde{\omega}_I) \in \mathcal{P}_0(\mathcal{S}) \), and therefore is in \( \pi(\text{Stab}^*(\mathcal{S})) \). Clearly, it is in \( \pi(\text{Stab}^*(\mathcal{S})) \) for some connected component \( \text{Stab}^*(\mathcal{S}) \). Therefore we can define the generalized marginal stability walls \( \{ \tilde{W}_{\tilde{\gamma}} \}, \tilde{\gamma} \in \tilde{\Gamma} \). The rest of the theorem is only a restatement of the fact that \( \text{arg}Z(\mu(l_i)) = \text{arg}Z(\mu(l_j)) \) for each pair \( \tilde{\gamma} = (i, j) \). \( \square \)

Remark 6.8 Theorem 6.7 and Theorem 5.1 together form a statement about the correspondence of mirror symmetry in an attractor background. On both sides we have correspondent conditions about central charges (i.e., alignment of many central charges with charges related by the mirror map). On one side of this correspondence we get intersections of generalized marginal stability walls in the space of stability conditions. The only remaining element

\[\text{As explained in the proof of the theorem, we might change some } l_i \text{ to } -l_i \text{ so that we can assume that } \text{arg}Z(\mu(l_i)) > 0 \text{ for any } i.\]

\[\text{As in the proof of Theorem 6.5, the case of } l_i \cdot B \neq 0 \text{ requires a different treatment from other cases, so we have discussed it before the general cases.}\]
required to get a complete correspondence of type (1.5) is defining stability conditions on derived Fukaya categories properly.

**Remark 6.9** We may wonder if we can find a statement of correspondences of type (1.3). This problem is harder. It requires us to verify that stable objects are related by the mirror symmetry (this is much more refined than the homological mirror conjecture), but the mirror symmetry of $K3$ surfaces used in this paper involves mapping cohomology classes to cohomology classes (this is coarser than the homological mirror conjecture). The following question should be answered for the study of correspondences of type (1.3): when is a generalized marginal stability wall in Theorem 6.7 a marginal stability wall?

**Remark 6.10** It is natural to ask if there is a generalization of main theorems to $S \times T^2$. After all, this is where the story begins. The only reason that we do not study this problem is that people do not know how to build stability conditions on Calabi-Yau threefolds (however see [5]).

**Remark 6.11** In this paper we have directly verified that $Z_{K3}(l)$ equals $Z(\mu(l))$ up to a real normalization when the complex moduli for the former and the complexified Kähler moduli for the latter are related by mirror symmetry (of course the former is also assumed to be an attractor background). This kind of statement has been widely used by physicists (for more general settings) and can lead to nontrivial mathematical predictions (see [9]). However, the author feels that it is not clear why this should be true from a purely mathematical perspective; it should be considered as a prediction of mirror symmetry, and requires justifications.

**Remark 6.12** In the study of certain four dimensional black hole solutions obtained by reduction from a ten dimensional string theory, physicists have discovered an interesting fact called the attractor mechanism. We assume that the black hole solution has spherical symmetry so that it makes sense to talk about the radial direction. The black hole solution of course depends on the six dimensional aspect of spacetime. It turns out that the six dimensional part over each value of the radial direction should be a Calabi-Yau threefold and these threefolds can have different moduli. Now the attractor mechanism claims that the underlying supersymmetry of the theory forces the moduli points over the horizon of the black hole to be special. These are called attractor backgrounds in this paper. In fact, there is a flow called the attractor flow in the moduli space and the attractor backgrounds are stationary points.

**Remark 6.13** One can speculate whether a mirror $\hat{S}$ of a $K3$ surface in an attractor background is also in some sort of attractor background. This would make mirror symmetry more perfect. Of course for $\hat{S}$, the moduli must be the complexified Kähler moduli. In fact there does exist an attractor mechanism in the space of complexified Kähler moduli for black holes in another type of string theory. A lot of papers have been written about this attractor mechanism, however the central charges in this attractor mechanism receive corrections that are not under control when we are far from the large volume limits. Thus even though it might be true that the mirror of an attractor background (in the complex moduli) is an attractor background (in the complexified Kähler moduli), it is not clear how to verify this.

**Remark 6.14** The wall crossings we study in this paper are static in the sense that we are sitting on attractor backgrounds where many marginal stability walls intersect. A more dynamical approach would be starting from somewhere away from attractor backgrounds, and
then moving to an attractor background. During this process we might encounter some marginal
stability walls. If we move along the attractor flow (which seems to be the most natural
choice), then we can actually derive some very sophisticated wall crossing formulas. One such
example is [7]. This point of view may be very helpful when we try to formulate and prove the
correspondences of (1.3).

Remark 6.15  Black holes actually do not play a significant role in this paper, all that
we need from them is the motivation to study singular K3 surfaces and the idea that we
should compute the central charge. However, if one wants to use the dynamical approach in
the previous paragraph to study wall crossings of stable objects, there is a way to make the
black hole perspective essential. Instead of just playing in the space of complex moduli (or
complexified Kähler moduli), one can study the existence of black hole solutions and use this to
deduce nontrivial wall crossing information. For an example of this big subject see [7]. Clearly
we have just seen the tip of a huge iceberg. We hope to discuss this topic in the future.

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