THE LOG-CONCAVITY CONJECTURE ON SEMIFREE SYMPLECTIC S¹-MANIFOLDS WITH ISOLATED FIXED POINTS

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Abstract. Let \((M, \omega)\) be a closed 2n-dimensional semifree Hamiltonian \(S¹\)-manifold with only isolated fixed points. We prove that a density function of the Duistermaat-Heckman measure is log-concave. Moreover, we prove that \((M, \omega)\) and any reduced symplectic form satisfy the Hard Lefschetz property.

1. Introduction

Let \(T\) be a connected abelian Lie group, i.e. a torus. Let \((M, \omega)\) be a closed 2n-dimensional Hamiltonian \(T\)-manifold with the moment map \(\mu: M \to t^*\), where \(t\) is the Lie algebra of \(T\). A Liouville measure \(m_L\) on \(M\) is defined by

\[
m_L(U) := \int_U \frac{\omega^n}{n!}
\]

for any open set \(U \subset M\). Then the push-forward measure \(m_{DH} := \mu_* m_L\), called the Duistermaat-Heckman measure, can be regarded as a measure on \(t^*\) such that for any Borel subset \(B \subset t^*\), \(m_{DH}(B) = \int_{\mu^{-1}(B)} \frac{\omega^n}{n!}\) tells us that how many states of our system have momenta in \(B\). Due to [DH], the Duistermaat-Heckman measure is absolutely continuous with respect to the Lebesgue measure on \(t^*\) and the corresponding density function, denoted by DH and called the Duistermaat-Heckman function, is a polynomial on any regular open set. More precisely, for any regular value \(\xi \in t^*\),

\[
DH(\xi) = \int_{M_\xi} \frac{1}{(n - l)!} \omega_\xi^{n-l}
\]

where \(l\) is a dimension of \(T\), \(M_\xi\) is the symplectic reduction at \(\xi\), and \(\omega_\xi\) is the corresponding reduced symplectic form on \(M_\xi\).

In 1996, W.Graham [Gr] proved that if the Hamiltonian \(T\)-action on a Kähler manifold \((M, \omega, J)\) is holomorphic, then the Duistermaat-Heckman measure is log-concave, i.e. \(\log DH\) is a concave function on the image of the moment map.

Example 1.1. Consider a compact symplectic toric manifold \((M, \omega, T)\) with the moment map \(\mu: M \to t^*\). Since any symplectic toric manifold can be obtained by the Kähler reduction, \(M\) is a smooth toric variety and the given \(T\)-action is holomorphic with some \(\omega\)-compatible complex structure on \(M\). By Atiyah-Guillemin-Sternberg convexity theorem, the image of \(\mu\) is a rational convex polytope satisfying the non-singularity condition.

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Choose any circle subgroup $S^1 \subset T$. Then the moment map of the circle action on $M$ is a composition map $\mu_{S^1} : M \xrightarrow{\mu} t^* \xrightarrow{\pi} s^*$ where $\pi$ is a dual map of $i : s \to t$, the inclusion of Lie algebra induced by $S^1 \hookrightarrow T$. The corresponding Duistermaat-Heckman function $DH(t)$ is just $C \cdot \text{vol}(\mu(M) \cap \pi^{-1}(t))$ with respect to the Lebesque measure on $t^*$, which is log-concave by Graham's theorem [Gr]. Here, $C$ is a global constant depending on the embedding $S^1 \hookrightarrow T$. Therefore, for any given Delzant polytope $\triangle$ and any height function $h : \triangle \to \mathbb{R}$, the corresponding slice volume function $g : h(\triangle) \to \mathbb{R}$ defined by $g(t) := \text{vol}(h^{-1}(t))$ is log-concave.

The following figure is the case when $M = S^2 \times S^2 \times S^2$, $\omega = 2\omega_{FS} \oplus 3\omega_{FS} \oplus 6\omega_{FS}$, and a Hamiltonian circle action on $M$ is given by $t \cdot (z_1, z_2, z_3) = (tz_1, tz_2, tz_3)$ for any $t \in S^1$.

Note that any holomorphic Hamiltonian $S^1$-manifold $(M, \omega, J)$ with an $S^1$-invariant Kähler structure $\omega$ satisfies the followings.

(1) $\omega$ satisfies the Hard Lefschetz property,
(2) any reduced symplectic form \( \omega_t \) is Kähler, and hence it satisfies the Hard Lefschetz property, and

(3) the Duistermaat-Heckman measure is log-concave. Hence it is natural to ask that when a Hamiltonian \( S^1 \)-manifold \((M, \omega)\) and the symplectic reduction \((M_t, \omega_t)\) satisfy the Hard Lefschetz property, and when the corresponding Duistermaat-Heckman measure is log-concave.

In this paper, we focus on the case when \((M, \omega)\) is a closed \(2n\)-dimensional symplectic manifold with a semifree Hamiltonian \(S^1\)-action whose fixed point set \(M^{S^1}\) consists of isolated points. Let \(\mu : M \to \mathbb{R}\) be a corresponding moment map which satisfies \(i_X \omega = -d\mu\). As noted above, the Duistermaat-Heckman function \(DH : \mu(M) \to \mathbb{R}\) is defined by

\[
DH(t) = \int_{M_t} \frac{1}{(n-1)!} \omega_t^{n-1}
\]

where \(M_t\) is the reduced space \(\mu^{-1}(t)/S^1\) and \(\omega_t\) is the reduced symplectic form on \(M_t\). Now we state our main results.

**Theorem 1.2.** Let \((M, \omega)\) be a closed symplectic manifold with a semifree Hamiltonian \(S^1\)-action whose fixed point set \(M^{S^1}\) consists of isolated points. Then the Duistermaat-Heckman measure is log-concave.

**Theorem 1.3.** Let \((M, \omega)\) be a closed semifree Hamiltonian \(S^1\)-manifold whose fixed points are all isolated, and let \(\mu\) be the moment map. Then \(\omega\) satisfies the Hard Lefschetz property. Moreover, the reduced symplectic form \(\omega_t\) satisfies the Hard Lefschetz property for every regular value \(t\).

In Section 2, we briefly review the Tolman and Weitsman’s work [TW1] which is very powerful to analyze the equivariant cohomology of the Hamiltonian \(S^1\)-manifold with isolated fixed points. Especially we use the Tolman-Weitsman’s basis of the equivariant cohomology \(H^*_S(M)\) which is constructed by using the equivariant version of the Morse theory [TW2]. In Section 3, we express the Duistermaat-Heckman function explicitly in terms of the integration of some cohomology class on the reduced space. And then we compute the integration by using the Jeffrey-Kirwan residue formula [JK]. Consequently, we will show that the log-concavity of the Duistermaat-Heckman measure is completely determined by the set of pairs \(\{(\mu(F), m_F)| F \in M^{S^1}\}\), where \(\mu(F)\) is the image of the moment map of \(F\) and \(m_F\) is the product of all weights of the \(S^1\)-representation on \(T_PM\).

In Sections 4, we will prove Theorem 1.2. And we will prove Theorem 1.3 in Section 5.

**Remark 1.4.** As noted in [O2], [K] and [Lin], Ginzburg and Knudsen conjectured that for any closed Hamiltonian \(T\)-manifold, the corresponding Duistermaat-Heckman measure is log-concave. Note that the log-concavity conjecture is motivated by a concavity property of the Boltzmann’ entropy in statistical mechanics. (See [O3] for the detail.) The log-concavity problem of the Duistermaat-Heckman measure is proved by A.Okounkov [O1] when \(M\) is a co-adjoint orbit of the classical Lie groups of type \(A_n\), \(B_n\), or \(C_n\) with the maximal torus \(T\)-action, by using the log-concavity of the multiplicities of the irreducible representation of each Lie groups. But the counter-example was found by Y.Karshon [K].
By using the Lerman’s symplectic cutting method, she constructed a closed 6-dimensional semifree Hamiltonian $S^1$-manifold with two fixed components such that the Duistermaat-Heckman measure is not log-concave. Later, Y. Lin [Lin] generalized the construction of 6-dimensional Hamiltonian $S^1$-manifolds not satisfying the log-concavity of the Duistermaat-Heckman measure, and also proved the log-concavity conjecture for any Hamiltonian $T^{n-2}$-manifold $(M^{2n}, \omega)$ whose reduced space has $b_1 = 1$. 

2. TOLMAN-WEITSMAN BASIS OF THE EQUIVARIANT COHOMOLOGY $H^*_S(M; \mathbb{Z})$

In this section, we briefly review Tolman and Weitsman’s results in [TW1]. Throughout this section, we assume that $(M^{2n}, \omega)$ is a closed semifree Hamiltonian $S^1$-manifold whose fixed points are isolated. Note that for each fixed point $p \in M^{S^1}$, the index of $p$ is the Morse index of the moment map at $p$ which is the same as the twice of the number of negative weights of the tangential $S^1$-representation at $p$.

**Proposition 2.1.** [TW1] Let $N_k$ be the number of fixed points of index $2k$. Then $N_k = \binom{n}{k}$. Hence $N_k$ is the same as the one of the standard diagonal circle action on $(S^2 \times \cdots \times S^2, \omega_1 \oplus \cdots \oplus \omega_n)$, where $\omega_i$ is the Fubini-Study form on $S^2$ of $i$-th factor.

**Theorem 2.2.** [TW1] Let $2^n$ be the power set of $\{1, \cdots, n\}$. Then there exist a bijection $\phi : M^{S^1} \to 2^n$ such that

1. For each index-2k fixed point $x \in M^{S^1}$, $|\phi(x)| = k$.
2. Let $u$ be the generator of $H^*(BS^1, \mathbb{Z})$. For each index-2k fixed point $x \in M^{S^1}$, there exists a unique cohomology class $\alpha_x \in H^{2k}_{S^1}(M; \mathbb{Z})$ such that for any $x' \in M^{S^1}$,
   
   $\begin{align*}
   \alpha_{x|x'} &= u^k \text{ if } \phi(x) \subset \phi(x'), \\
   \alpha_{x|x'} &= 0 \text{ otherwise}.
   \end{align*}$

Moreover $\{\alpha_x | x \in M^{S^1}\}$ forms a basis of $H^*_S(M, \mathbb{Z})$.

Applying Theorem 2.2 to $(S^2 \times \cdots \times S^2, \omega_1 \oplus \cdots \oplus \omega_n)$ with the diagonal semifree Hamiltonian circle action, we have a bijection $\psi : (S^2 \times \cdots \times S^2)^{S^1} \to 2^n$ and there is a basis $\{\beta_y | y \in (S^2 \times \cdots \times S^2)^{S^1}\}$ of $H^*_S(S^2 \times \cdots \times S^2; \mathbb{Z})$ satisfies the conditions in Theorem 2.2. Hence we have an identification map

$$\psi^{-1} \circ \phi : M^{S^1} \to (S^2 \times \cdots \times S^2)^{S^1}$$

and $\psi^{-1} \circ \phi$ preserves the indices of the fixed points.

Note that $\psi^{-1} \circ \phi$ gives an identification between $H^*_S(M; \mathbb{Z})$ and $H^*_S(S^2 \times \cdots \times S^2; \mathbb{Z})$ as follow. Let $a_i = \alpha_{\psi^{-1}(i)} \in H^2_{S^1}(M; \mathbb{Z})$ and $b_i = \beta_{\psi^{-1}(i)} \in H^2_{S^1}(S^2 \times \cdots \times S^2; \mathbb{Z})$.

**Lemma 2.3.** [TW1] For each $x \in M^{S^1}$, $\alpha_x = \prod_{j \in \phi(x)} a_j$. Similarly, we have $\beta_y = \prod_{j \in \psi(y)} b_j$ for each $y \in (S^2 \times \cdots \times S^2)^{S^1}$.

**Proof.** For an inclusion $i : M^{S^1} \hookrightarrow M$, we have a natural ring homomorphism $i^* : H^*_S(M) \to H^*_S(M^{S^1}) \cong H^*(M^{S^1}) \otimes H^*(BS^1)$. Kirwan injectivity implies that $i^*$ is injective. Hence it is enough to show that $\alpha_{x|z} = (\prod_{j \in \phi(x)} a_j)|_z$ for all $x, z \in M^{S^1}$. Here, the operation
“|z|” is the restriction $H^*_S(M^S) \to H^*_S(z) \cong H^*(BS^1) = \mathbb{R}[u]$ induced by the inclusion $z \hookrightarrow M^S$. For any $x, z \in M^S$ with $\text{Ind}(x) = 2k$,
- $\alpha_x|_z = u^k$ if $\phi(x) \subset \phi(z)$.
- $\alpha_x|_z = 0$ otherwise.

On the other hand, $(\prod_{j \in \phi(x)} a_j)|_z = \prod_{j \in \phi(x)} a_j|_z$. Since $a_j|_z = u$ if and only if $j \in \phi(z)$, we have
- $(\prod_{j \in \phi(x)} a_j)|_z = u^k$ if $\phi(x) \subset \phi(z)$.
- $(\prod_{j \in \phi(x)} a_j)|_z = 0$ otherwise.

Therefore, $\alpha_x = \prod_{j \in \phi(x)} a_j$ by the Kirwan injectivity theorem. The proof of the second statement is similar.

Hence the $H^*(BS^1)$-module isomorphism $f : H^*_S(M; \mathbb{Z}) \to H^*_S(S^2 \times \cdots \times S^2; \mathbb{Z})$ which sends $\alpha_x$ to $\beta_{\psi^{-1}_{0}(\phi(x))}$ for each $x \in M^S$ is in fact a ring isomorphism by the lemma \[2.3\]. To sum up, we have the following corollary.

**Corollary 2.4.** \[TW1\] There is a ring isomorphism $f : H^*_S(M; \mathbb{Z}) \to H^*_S(S^2 \times \cdots \times S^2; \mathbb{Z})$ which sends $\alpha_x$ to $\beta_{\psi^{-1}_{0}(\phi(x))}$. Moreover, for any $\alpha \in H^*_S(M; \mathbb{Z})$ and any fixed point $x \in M^S$, we have $|\alpha|_z = f(\alpha)|_{\psi^{-1}_{0}(\phi(x))}$.

3. **The Duistermaat-Heckman function and the residue formula**

Let $(M, \omega)$ be a $2n$-dimensional closed Hamiltonian $S^1$-manifold with the moment map $\mu : M \to \mathbb{R}$. We may assume that 0 is a regular value of $\mu$ such that $\mu^{-1}(0)$ is non-empty. Choose two consecutive critical values $c_1$ and $c_2$ of $\mu$ so that the open interval $(c_1, c_2]$ consists of regular values of $\mu$ and contains 0. By the Duistermaat-Heckman’s theorem \[DH, [\omega] = [\omega_0] - et\] where $e$ is the Euler class of $S^1$-fibration $\mu^{-1}(0) \to M_0$, where $M_0$ is the symplectic reduction at 0 with the induced symplectic form $\omega_0$. Hence we have

$$DH(t) = \int_{M_0} \frac{1}{(n-1)!} ([\omega_0] - et)^{n-1}$$

on $(c_1, c_2) \subset \text{Im} \mu$.

Note that a continuous function on an open interval $g : (a, b) \to \mathbb{R}$ is **concave** if $g(tc + (1-t)d) \geq tg(c) + (1-t)g(d)$ for any $c, d \in (a, b)$ and for any $t \in (0, 1)$. We remark the basic property of a concave function as follow.

**Remark 3.1.** Let $g$ be a continuous, piecewise smooth function on a connected interval $I \subset \mathbb{R}$. Then $g$ is concave on $I$ if and only if the derivative of $g$ is decreasing, i.e. $g''(t) \leq 0$ for every smooth point $t \in I$ and $g'_+(c) - g'_-(c) < 0$ for every singular point $c \in I$, where $g'_+(c) = \lim_{t \to c, t > c} g'(t)$ and $g'_-(c) = \lim_{t \to c, t < c} g'(t)$.

Note that Duistermaat and Heckman proved that $DH$ is a polynomial on a connected regular open interval $U \subset \mu(M)$. The following formula due to Guillemin, Lerman and Sternberg describes the behavior of $DH$ near the critical value of $\mu$. In particular, it implies
that DH is $k$-times differentiable at a critical value $c \in \mu(M)$ if and only if $\mu^{-1}(c)$ does not contain a fixed component whose codimension is less than $4 + 2k$.

**Theorem 3.2.** [GLS] Assume that $c$ is a critical value which corresponds to the fixed components $C_i$'s. Then the jump of $DH(t)$ at $c$ is given by

$$DH_+ - DH_- = \sum_i \frac{\text{vol}(C_i)}{(d-1)!} \prod_j a_j (t-c)^{d-1} + O((t-c)^d)$$

where the sum is over the components $C_i$ of $M^{S^1} \cap \mu^{-1}(c)$, $d$ is half the real codimension of $C_i$ in $M$, and the $a_j$'s are the weights of the $S^1$-representation on the normal bundle of $C_i$.

If $c$ is a critical value which is not an extremum, then the codimension of the fixed point set in $\mu^{-1}(c)$ is at least 4. Therefore the theorem 3.2 implies that $DH(t)$ is continuous at non-extremal critical values and $DH'(t)$ jumps at $c$ when $d$ equals 2. In the case when $d = 2$, the two nonzero weights must have opposite signs, so the jump in the derivative is negative, i.e. $DH'(t)$ decreases when it passes through the critical value with $d = 2$. Since $DH$ is continuous, the jump in $\frac{d}{dt} \ln DH(t) = \frac{DH'(t)}{DH(t)}$ is negative at $c$. Combining with the lemma 3.1, we have the following corollary.

**Corollary 3.3.** Let $(M, \omega)$ be a closed Hamiltonian $S^1$-manifold with the moment map $\mu : M \to \mathbb{R}$. Then the corresponding Duistermaat-Heckman function $DH$ is log-concave on $\mu(M)$ if $(\log DH(t))'' \leq 0$ for every regular value $t \in \mu(M)$.

Note that $(\log DH(t))'' = \frac{DH(t) \cdot DH''(t) - DH'(t)^2}{DH(t)^3}$. Therefore $(\log DH(t))'' \leq 0$ is equivalent to $DH(t) \cdot DH''(t) - DH'(t)^2 \leq 0$. The equation (1) implies that

\begin{align*}
(2) & \quad DH(t) \cdot DH''(t) = (n-1)(n-2) \int_{M_0} e^{2[\omega_t]}^{n-3} \cdot \int_{M_0} [\omega_t]^{n-1} \\
(3) & \quad DH'(t)^2 = (n-1)^2 \left( \int_{M_0} e^{[\omega_t]}^{n-2} \right)^2
\end{align*}

To compute the integrals appeared in the equations (2) and (3), we need the following procedure: For an inclusion $\iota : \mu^{-1}(0) \hookrightarrow M$, we have a ring homomorphism $\kappa : H^*_\mathbb{Z}(M; \mathbb{R}) \to H^*_\mathbb{Z}(\mu^{-1}(0); \mathbb{R}) \cong H^*(M_0; \mathbb{R})$ which is called the Kirwan map. Due to the Kirwan surjectivity [K3], $\kappa$ is a ring surjection.

Consider a 2-form $\tilde{\omega} := \omega - d(\mu \theta)$ on $ES^1 \times M$ where $\theta$ is the connection form on $ES^1$. We denote by $x = \pi^* u \in H^2_S(M; \mathbb{Z})$ where $\pi : M \times S^1 \to ES^1 \to BS^1$ and $u$ is a generator of $H^*(BS^1; \mathbb{Z})$ such that the Euler class of the Hopf bundle $ES^1 \to BS^1$ is $-u$. Some part of the following two lemmas are given in [Am], but we give the complete proofs here.

**Lemma 3.4.** $\tilde{\omega}$ is $S^1$-invariant and closed, and $ix \tilde{\omega} = 0$ so that $\tilde{\omega}$ represents a cohomology class in $H^*_\mathbb{Z}(M; \mathbb{R})$. Moreover, for any fixed component $F \in M^{S^1}$, we have $\kappa([\tilde{\omega}]) = [\omega_0]$ and $[\tilde{\omega}]_F = [\omega]_F + \mu(F)u$. In particular, if $F$ is isolated, then $[\tilde{\omega}]_F = \mu(F)u$. 
Proof. For the first statement, it is enough to show that \( i_X \widetilde{\omega} \) and \( L_X \widetilde{\omega} \) vanish. Note that

\[
i_X \widetilde{\omega} = i_X \omega - i_X d(\mu \theta) = -d \mu + di_X(\mu \theta) - L_X(\mu \theta) \]

by Cartan’s formula. Since \( i_X(\mu \theta) = \mu \) and \( \mu \theta \) is invariant under the circle action, we have \( i_X \widetilde{\omega} = -d \mu + d \mu = 0 \). Moreover, it is obvious that \( \widetilde{\omega} \) is closed by definition. Hence \( L_X \widetilde{\omega} = 0 \) by Cartan’s formula again.

To prove the second statement, consider the following diagram.

\[
\begin{array}{ccc}
\mu^{-1}(0) \times ES^1 & \hookrightarrow & M \times ES^1 \\
\downarrow & & \downarrow \\
\mu^{-1}(0) \times S^1 \times ES^1 & \hookrightarrow & M \times S^1 \times ES^1 \\
\downarrow & \Downarrow \sim & \downarrow \\
\mu^{-1}(0) / S^1 & \cong & M_{red}
\end{array}
\]

Since \( d \mu \) is zero on the tangent bundle \( \mu^{-1}(0) \times ES^1 \), the pull-back of \( \widetilde{\omega} = \omega - d \mu \wedge \theta - d \mu \theta \) to \( \mu^{-1}(0) \times ES^1 \) is the restriction \( \omega|_{\mu^{-1}(0) \times ES^1} \). The lift-down of \( \omega|_{\mu^{-1}(0) \times ES^1} \) to \( \mu^{-1}(0) / S^1 \) is just a reduced symplectic form at the level 0. Hence \( \kappa(\widetilde{\omega}) = [\omega] \).

To show the last statement, consider \( [\widetilde{\omega}]_F = [\omega - d(\mu \theta)]_F = [\omega - d(\mu \theta) - d \mu \theta]_F \). Since the restriction \( d \mu|_F \times ES^1 \) vanishes, we have \( [\omega]_F = [\omega]_F - \mu(F) \cdot d \mu|_F = [\omega]_F + \mu(F)u \). If \( F \) is isolated, then we have \( [\omega]_F = \mu(F)u \).

\( \square \)

Lemma 3.5. Consider a 2-form \( d \theta \) on \( ES^1 \times M \) where \( \theta \) is a connection 1-form on \( ES^1 \). Then we can lift \( d \theta \) down to \( ES^1 \times S^1 \) so that \( d \theta \) represents a cohomology class in \( H^*_S(M; \mathbb{R}) \). Moreover, \( [d \theta] = -x \) and \( \kappa([d \theta]) = -\kappa(x) = e \) where \( e \) is the Euler class of the \( S^1 \)-fibration \( \mu^{-1}(0) \to M_{red} \).

Proof. Note that \( i_X d \theta = L_X \theta - d i_X \theta = 0 \). Hence we can lift \( d \theta \) down to \( ES^1 \times S^1 \). For any fixed point \( x \in M_{S^1} \), the restriction \( [d \theta]|_x \) is the Euler class of \( ES^1 \times x \to BS^1 \). Hence \( [d \theta] = -u \cdot 1 = -x \). Here, the multiplication \( \cdot \) comes from the \( H^*(BS^1) \)-module structure on \( H^*_S(M) \). By the diagram in the proof of the Lemma 3.4, \( \kappa([d \theta]) \) is just the Euler class of the \( S^1 \)-fibration \( \mu^{-1}(0) \to \mu^{-1}(0) / S^1 \). Therefore \( \kappa([d \theta]) = -\kappa(x) = e \). \( \square \)

Combining the equations (2), (3), Lemma 3.4, and Lemma 3.5, we have the following corollary.

Corollary 3.6. \( DH(0) \cdot DH''(0) - DH'(0)^2 \leq 0 \) if and only if

\[
(n - 2) \int_{M_0} \kappa([d \theta][\widetilde{\omega}]^3) \cdot \int_{M_0} \kappa([\widetilde{\omega}]^{n-1}) - (n - 1) \left( \int_{M_0} \kappa([d \theta][\widetilde{\omega}]^{n-2}) \right)^2 \leq 0.
\]

To compute the above integrals \( \int_{M_0} \kappa([d \theta][\widetilde{\omega}]^3) \), \( \int_{M_0} \kappa([\widetilde{\omega}]^{n-1}) \), and \( \int_{M_0} \kappa([d \theta][\widetilde{\omega}]^{n-2}) \), we need the residue formula due to Jeffrey and Kirwan. (See [JK] and [J]).

Theorem 3.7. \( \textbf{[JK]} \) Let \( \nu \in H^*_S(M; \mathbb{R}) \). Then

\[
\int_{M_0} \kappa(\nu) = \sum_{F \in M_{S^1}, \mu(F) > 0} \text{Res}(\nu|_F e_F).
\]

Here \( e_F \) is the equivariant Euler class of the normal bundle to \( F \), and \( \text{Res}(f) \) is a residue of \( f \).
Now, let’s compute \( \int_{M_0} \kappa([d\theta]^2 [\tilde{\omega}]^{n-3}) \). By Theorem 3.7,
\[
\int_{M_0} \kappa([d\theta]^2 [\tilde{\omega}]^{n-3}) = \sum_{F \in M_{S^1}, \mu(F) > 0} \text{Res}\left(\frac{[d\theta]^2 [\tilde{\omega}]^{n-3}}{e_F}\right)
\]

Since \([\tilde{\omega}]_z = \mu(z)u \) and \([d\theta]_z = -u \) by lemma 3.4 and 3.5 we have
\[
\int_{M_0} \kappa([d\theta]^2 [\tilde{\omega}]^{n-3}) = \sum_{F \in M_{S^1}, \mu(F) > 0} \text{Res}\left(\frac{\mu(F)^{n-3}u^{n-1}}{e_F}\right)
\]
\[
= \sum_{F \in M_{S^1}, \mu(F) > 0} \text{Res}\left(\frac{\mu(F)^{n-3}u^{n-1}}{m_F u^n}\right)
\]
where \( m_F \) is the product of all weights of tangential \( S^1 \)-representation at \( F \). Similarly, if \( \xi \in \mathbb{R} \) is a regular value of \( \mu \), then we put \( \tilde{\mu} = \mu - \xi \) be the new moment map. By the same argument, we have the following lemma

**Lemma 3.8.** For a regular value \( \xi \) of the moment map \( \mu \),

1. \[
\int_{M_\xi} \kappa([d\theta]^2 [\tilde{\omega}]^{n-3}) = \sum_{F \in M_{S^1}, \mu(F) > \xi} \frac{1}{m_F} (\mu(F) - \xi)^{n-3}.
\]
2. \[
\int_{M_\xi} \kappa([d\theta][\tilde{\omega}]^{n-2}) = \sum_{F \in M_{S^1}, \mu(F) > \xi} \frac{-1}{m_F} (\mu(F) - \xi)^{n-2}.
\]
3. \[
\int_{M_\xi} \kappa([\tilde{\omega}]^{n-1}) = \sum_{F \in M_{S^1}, \mu(F) > \xi} \frac{1}{m_F} (\mu(F) - \xi)^{n-1}.
\]

Combining the corollary 3.6 and the lemma 3.8 we have the following proposition.

**Proposition 3.9.** Let \((M, \omega)\) be a closed Hamiltonian \( S^1 \)-manifold with the moment map \( \mu : M \to \mathbb{R} \). Assume that \( M_{S^1} \) consists of isolated fixed points. Then a density function of the Duistermaat-Heckman measure with respect to \( \mu \) is log-concave if and only if

\[
\sum_{F \in M_{S^1}, \mu(F) > \xi} \frac{1}{m_F} (\mu(F) - \xi)^{n-3} \cdot \sum_{F \in M_{S^1}, \mu(F) > \xi} \frac{1}{m_F} (\mu(F) - \xi)^{n-1} - \left( \sum_{F \in M_{S^1}, \mu(F) > \xi} \frac{1}{m_F} (\mu(F) - \xi)^{n-2} \right)^2 \leq 0
\]

for all regular value \( \xi \in \mu(M) \), where \( m_F \) is the product of all weights of the \( S^1 \)-representation on \( T_F M \). In particular, the log-concavity of the Duistermaat-Heckman measure is completely determined by the set \( \{ (\mu(F), m_F) \mid F \in M_{S^1}\} \).
Corollary 3.10. Let \((M^{2n}, \omega)\) and \((N^{2n}, \sigma)\) be two closed Hamiltonian \(S^1\)-manifold with the moment map \(\mu_1\) and \(\mu_2\) respectively. Assume there exist a bijection \(\phi : M^{S^1} \to N^{S^1}\) which satisfies

1. for each \(F \in M^{S^1}\), \(m_F = m_{\phi(x)}\), and
2. for each \(F \in M^{S^1}\), \(\mu_1(F) = \mu_2(\phi(F))\).

where \(m_F\) is the product of all weights of the tangential \(S^1\)-representation at \(F\). If \(N\) satisfies the log-concavity of the Duistermaat-Heckman measure with respect to \(\mu_2\), then so does \(M\) with respect to \(\mu_1\).

Remark 3.11. The integration formulae (1) and (3) in the lemma are proved by Wu by using the stationary phase method. See Theorem 5.2 in \([Wu]\) for the detail.

4. Proof of the Theorem 1.2

As noted in the introduction, if a Hamiltonian \(S^1\)-action on the Kähler manifold is holomorphic, then the corresponding Duistermaat-Heckman function is log-concave by \([Gr]\). Let \((M^{2n}, \omega)\) be a closed semifree Hamiltonian circle action with the moment map \(\mu\). Assume that all fixed points are isolated. Let \(DH\) be the corresponding Duistermaat-Heckman function with respect to \(\mu\). We will show that there is a Kähler form \(\omega_1 \oplus \cdots \oplus \omega_n\) on \(S^2 \times \cdots \times S^2\) with the standard diagonal holomorphic semifree circle action such that a bijection \(\psi^{-1} \circ \phi : M^{S^1} \to (S^2 \times \cdots \times S^2)^{S^1}\) given in Section 2 satisfies the conditions in the corollary which implies the log-concavity of \(DH\). Now we start with the lemma below.

Lemma 4.1. Let \((M^{2n}, \omega)\) be a closed semifree Hamiltonian circle action with the moment map \(\mu\). Assume that all fixed points are isolated. Then \(\{\mu(F), m_F\}_{F \in M^{S^1}}\) is completely determined by \(\mu(p_0), \mu(p_1), \ldots, \mu(p_n)\), where \(p_k\)'s are the fixed points of index \(2k\) for \(j = 1, \ldots, n\).

Proof. Consider an equivariant symplectic 2-form \(\tilde{\omega}\) on \(ES^1 \times_s M\) which is given in Section 3. Because \(x, a_1, \ldots, a_n\) form a basis of \(H^2_s(M; \mathbb{Z})\), we may let \([\tilde{\omega}] = m_0x + m_1a_1 + \cdots + m_na_n\) for some real numbers \(m_i\)'s. (See Section 2: the definition of \(x, a_1, \ldots, a_n\).) By lemma, we have \([\tilde{\omega}]_{p_0} = \mu(p_0)u\). On the other hand, the right hand side is \((m_0x + m_1a_1 + \cdots + m_na_n)_{p_0} = m_0u\), since every \(a_i\) vanishes on \(p_0\). Hence \(m_0 = \mu(p_0)\). Similarly, \([\tilde{\omega}]_{p_1} = \mu(p_1)u\) and \((m_0x + m_1a_1 + \cdots + m_na_n)_{p_1} = m_0u + m_1u\). Hence we have \(m_i = \mu(p_i) - m_0 = \mu(p_i) - \mu(p_0)\) for each \(i = 1, \ldots, n\). Therefore \(\mu(p_0), \mu(p_1), \ldots, \mu(p_n)\) determine the coefficients \(m_i\)'s of \([\tilde{\omega}]\). For \(p_k\), \(\tilde{\omega}|_{p_k} = \mu(p_k)u\) and \((m_0x + m_1a_1 + \cdots + m_na_n)_{p_k} = m_0u + \sum_{i \in \phi(p_k)} m_iu\). Hence for fixed \(k\), the set \(\{(\mu(p_k), m_{p_k})\}_{j=1}^n\) is just \(\{(m_0 + m_{i_1} + \cdots + m_{i_k})_{i_1, \ldots, i_k} (1)_{i_1, \ldots, i_k} \} \subseteq \{1, 2, \ldots, n\}\). Hence \(\{\mu(F), m_{F}\}_{F \in M^{S^1}}\) is completely determined by \(\mu(p_0), \mu(p_1), \ldots, \mu(p_n)\). \(\square\)

Now we are ready to prove the theorem.
Theorem 5.1. [TW2] Let $S^1$ act on a compact symplectic manifold $(M, \omega)$ with moment map $\mu : M \rightarrow \mathbb{R}$. Assume that all fixed points are isolated and 0 is a regular value. Let $M^{S^1}$ denote the set of fixed points. Define $K_+ := \{ \alpha \in H^*_S(M; \mathbb{Z}) | \alpha_{F_+} = 0 \}$ where $F_+ := M^{S^1} \cap \mu^{-1}(0, \infty)$ and $K_- := \{ \alpha \in H^*_S(M; \mathbb{Z}) | \alpha_{F_-} = 0 \}$ where $F_- := M^{S^1} \cap \mu^{-1}(-\infty, 0)$. Then there is a short exact sequence:

$$0 \longrightarrow K \longrightarrow H^*_S(M; \mathbb{Z}) \xrightarrow{\kappa} (M_{\text{red}}; \mathbb{Z}) \longrightarrow 0$$

where $\kappa : H^*_S(M; \mathbb{Z}) \rightarrow H^*(M_{\text{red}}; \mathbb{Z})$ is the Kirwan map.

Proof of Theorem 5.1. Let $\kappa_M : H^*_S(M; \mathbb{R}) \rightarrow H^*(M_{\text{red}}; \mathbb{R})$ be the Kirwan map for $(M, \omega)$ and let $\kappa$ be the one for $(S^2 \times \cdots \times S^2, \sigma)$, where $\sigma := \omega_1 \oplus \cdots \oplus \omega_n$ is chosen in the proof of Theorem 1.2 in Section 4. As in the proof of Theorem 1.2, we proved that there exists a semifree holomorphic Hamiltonian $S^1$-manifold $(S^2 \times \cdots \times S^2, \sigma)$ with the moment map $\mu$ such that $\psi^{-1} \circ \phi : M^{S^1} \rightarrow (S^2 \times \cdots \times S^2)^{S^1}$ preserves their indices, weights, and the values of the moment map. Hence $\psi^{-1} \circ \phi$ identifies $K^M_+$ with $K^{S^2 \times \cdots \times S^2}_+$ and $K^M$ with $K^{S^2 \times \cdots \times S^2}_M$. The ring isomorphism $f : H^*_S(M; \mathbb{Z}) \rightarrow H^*_S(S^2 \times \cdots \times S^2; \mathbb{Z})$ given in Corollary 2.4 satisfies $f(\alpha)|_x = f(\alpha)|_{\psi^{-1} \circ \phi(x)}$ for any $\alpha \in H^*_S(M; \mathbb{Z})$ and any fixed point $x \in M^{S^1}$. Hence if $\alpha \in K^M_+$, then $f(\alpha) \in K^{S^2 \times \cdots \times S^2}_+$. Similarly for any $\alpha \in K^M$, we have $f(\alpha) \in K^{S^2 \times \cdots \times S^2}_M$. Therefore $f$ preserves the kernel of the Kirwan map $\kappa$ by Theorem 5.1.

Note that $\kappa(f([\omega]))$ is the reduced symplectic class of $S^2 \times \cdots \times S^2$ at 0. Since the Kähler quotient of the holomorphic action is again Kähler, $\kappa(f([\omega]))$ satisfies the hard Lefschetz property. Now, assume that $\omega_0$ does not satisfy the hard Lefschetz property. Then there exists a positive integer $k(< n)$ and some nonzero $\alpha \in H^k(M_{\text{red}}; \mathbb{R})$ such that $\alpha \cdot [\omega_0]^{n-k} = 0$ in $H^{2n-k}(M_{\text{red}})$. By the Kirwan surjectivity, we can find $\tilde{\alpha} \in H^*_S(M; \mathbb{R})$ with $\kappa(\tilde{\alpha}) = \alpha$. Then $\tilde{\alpha} \cdot [\tilde{\omega}]^{n-k}$ is in ker $\kappa$ and hence the image $f(\tilde{\alpha} \cdot [\tilde{\omega}]^{n-k})$ is in ker $\kappa$, since $f$ maps ker $\kappa$ of $M$ to ker $\kappa$ of $S^2 \times \cdots \times S^2$. It implies that $f(\tilde{\alpha}) = 0$ by the hard Lefschetz condition for $f([\tilde{\omega}])$. It contradicts that $f$ is an isomorphism.
It remains to show that \((M, \omega)\) satisfies the Hard Lefschetz property. Remind that 
\(\psi^{-1} \circ \phi: M^S \to (S^2 \times \cdots \times S^2)^S\) induces an isomorphism
\[
f: H^S_{\text{si}}(M; \mathbb{R}) \to H^S_{\text{si}}(S^2 \times \cdots \times S^2; \mathbb{R}),
\]
which sends the equivariant symplectic class \([\bar{\omega}]\) to \([\bar{\sigma}]\) as we have seen in Section 4. Here, 
\(\bar{\sigma}\) is an equivariant symplectic form induced by \(\bar{\sigma} - d(\mu' \theta)\) in 
\(H^S_{\text{si}}(S^2 \times \cdots \times S^2; \mathbb{R})\). Since 
f is a \(H^*(BS^1; \mathbb{R})\)-isomorphism, it induces a ring isomorphism
\[
f_u : \frac{H^S_{\text{si}}(M; \mathbb{R})}{u \cdot H^S_{\text{si}}(M; \mathbb{R})} \to \frac{H^S_{\text{si}}(S^2 \times \cdots \times S^2; \mathbb{R})}{u \cdot H^S_{\text{si}}(S^2 \times \cdots \times S^2; \mathbb{R})}
\]
Moreover, the quotient map \(\pi_M: H^S_{\text{si}}(M; \mathbb{R}) \to \frac{H^S_{\text{si}}(M; \mathbb{R})}{u \cdot H^S_{\text{si}}(M; \mathbb{R})} \cong H^*(M; \mathbb{R})\) \((\pi_{S^2 \times \cdots \times S^2}\) respectively) is just a ring homomorphism which comes from an inclusion 
\(M \hookrightarrow M \times_{S^1} ES^1\) as a fiber. Therefore \(\pi_M([\bar{\omega}]) = [\omega]\) and \(\pi_{S^2 \times \cdots \times S^2}([\bar{\sigma}]) = [\sigma]\). It means that 
\(f_u: H^*(M; \mathbb{R}) \to H^*(S^2 \times \cdots \times S^2; \mathbb{R})\) is a ring isomorphism which sends \([\omega]\) to \([\sigma]\). Since \(\sigma\) is a Kähler form, it satisfies the Hard Lefschetz property. Hence so \(\omega\) does.

\[\square\]

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