HOLOMORPHIC FAMILIES OF LONG $C^2$'S

FRANC FORSTNERIČ

(Communicated by Mei-Chi Shaw)

Abstract. We construct a holomorphically varying family of complex surfaces $X_s$, parametrized by the points $s$ in any Stein manifold, such that every $X_s$ is a long $C^2$ which is biholomorphic to $C^2$ for some but not all values of $s$.

1. The main result

A complex manifold $X$ of dimension $n$ is a long $C^n$ if $X = \bigcup_{j=1}^{\infty} X^j$, where $X^1 \subset X^2 \subset X^3 \subset \ldots$ is an increasing sequence of open domains exhausting $X$ such that each $X^j$ is biholomorphic to $C^n$. Clearly every long $C$ is biholomorphic to $C$. On the other hand, for every $n > 1$ there exists a long $C^n$ which is not a Stein manifold, and in particular is not biholomorphic to $C^n$. Such manifolds have been constructed recently by E. F. Wold [12] using his example of a non-Runge Fatou-Bieberbach domain in $C^2$ [11], thereby solving a problem posed by J. E. Fornæss [3].

Previously Fornæss [2] used Wermer’s example of a non-Runge embedded polydisc in $C^3$ [10] to construct for every $n \geq 3$ an $n$-dimensional non-Stein complex manifold that is exhausted by biholomorphic images of the polydisc.

Recently L. Meersseman asked in a private communication whether it is possible to holomorphically deform the standard $C^n$ to a long $C^n$ that is not biholomorphic to $C^n$. This question arose naturally in certain problems concerning deformations of foliations that he had been considering. Here we give a positive answer and show that the behavior of long $C^n$’s in a holomorphic family can be rather chaotic.

Theorem 1.1. Fix an integer $n > 1$. Assume that $S$ is a Stein manifold, $A = \bigcup_j A_j$ is a finite or countable union of closed complex subvarieties of $S$, and $B = \{b_j\}$ is a countable set in $S \setminus A$. Then there exists a complex manifold $X$ and a holomorphic submersion $\pi: X \to S$ onto $S$ such that

(i) the fiber $X_s = \pi^{-1}(s)$ is a long $C^n$ for every $s \in S$,
(ii) $X_s$ is biholomorphic to $C^n$ for every $s \in A$, and
(iii) $X_s$ is non-Stein for every $s \in B$.

In particular, for any two disjoint countable sets $A, B \subset C$ there is a holomorphic family $\{X_s\}_{s \in C}$ of long $C^2$’s such that $X_s$ is biholomorphic to $C^2$ for all $s \in A$ and is non-Stein for all $s \in B$. This is particularly striking if the sets $A$ and $B$ are chosen to be everywhere dense in $C$. 

Received by the editors January 18, 2011 and, in revised form, February 17, 2011.
2010 Mathematics Subject Classification. Primary 32E10, 32E30, 32H02.
Key words and phrases. Stein manifold, Fatou-Bieberbach domain, long $C^2$.

The author was supported by grants P1-0291 and J1-2152 from ARRS, Republic of Slovenia.

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The conclusion of Theorem 1.1 can be strengthened by adding to the set $B$ a closed complex subvariety of $X$ contained in $X \setminus A$. We do not know whether the same holds if $B$ is a countable union of subvarieties of $X$.

Several natural questions appear:

**Problem 1.2.** Given a holomorphic family $\{X_s\}_{s \in S}$ of long $\mathbb{C}^n$'s for some $n > 1$, what can be said about the set of points $s \in S$ for which the fiber $X_s$ is (or is not) biholomorphic to $\mathbb{C}^n$? Are these sets necessarily a $G_\delta$, an $F_\sigma$, of the first, resp. of the second category, etc.?

A more ambitious project would be to answer the following question:

**Problem 1.3.** Is there a holomorphic family $X_s$ of long $\mathbb{C}^2$'s, parametrized by the disc $D = \{s \in \mathbb{C} : |s| < 1\}$ or the plane $\mathbb{C}$, such that $X_s$ is not biholomorphic to $X_{s'}$ whenever $s \neq s'$?

We do not know of any criteria to distinguish two long $\mathbb{C}^n$'s from each other, except if one of them is the standard $\mathbb{C}^n$ and the other one is non-Stein. Apparently there is no known example of a Stein long $\mathbb{C}^n$ other than $\mathbb{C}^n$. It is easily seen that any two long $\mathbb{C}^n$'s are smoothly diffeomorphic to each other, so the gauge-theoretic methods do not apply.

To prove Theorem 1.1 we follow Wold’s construction of a non-Stein long $\mathbb{C}^2$ [12], but doing all the key steps with families of Fatou-Bieberbach maps depending holomorphically on the parameter in a given Stein manifold $S$. (The same proof applies for any $n \geq 2$.) By using the Andersén-Lempert theory [1, 4, 8, 9] we insure that in a holomorphically varying family of injective holomorphic maps $\phi_s : \mathbb{C}^2 \hookrightarrow \mathbb{C}^2$ ($s \in S$) the image domain $\phi_s(\mathbb{C}^2)$ is Runge for some but not all values of the parameter. In the limit manifold $X$ we thus get fibers $X_s$ that are biholomorphic to $\mathbb{C}^2$, as well as fibers that are not holomorphically convex, and hence non-Stein.

## 2. Constructing holomorphic families of long $\mathbb{C}^n$’s

Let $S$ be a complex manifold that will be used as the parameter space. We recall how one constructs a complex manifold $X$ and a holomorphic submersion $\pi : X \to S$ such that the fiber $X_s = \pi^{-1}(s)$ is a long $\mathbb{C}^n$ for each $s \in S$. (This is a parametric version of the construction in [2] or [12, §2].)

Assume that we have a sequence of injective holomorphic maps

$$\Phi^k : X^k = S \times \mathbb{C}^n \hookrightarrow X^{k+1} = (S \times \mathbb{C}^n, \Phi^k(s, z) = (s, \phi^k_s(z)), $$

where $s \in S$, $z \in \mathbb{C}^n$, and $k = 1, 2, \ldots$. Set $\Omega^k = \Phi^k(X^k) \subset X^{k+1}$. Thus for every fixed $k \in \mathbb{N}$ and $s \in S$ the map $\phi^k_s : \mathbb{C}^n \hookrightarrow \mathbb{C}^n$ is biholomorphic onto its image $\phi^k_s(\mathbb{C}^n) = \Omega^k_s \subset \mathbb{C}^n$ and it depends holomorphically on the parameter $s \in S$. In particular, if $\Omega^k_s$ is a proper subdomain of $\mathbb{C}^n$, then $\phi^k_s$ is a Fatou-Bieberbach map. Let $X$ be the disjoint union of all $X^k$ for $k \in \mathbb{N}$ modulo the following equivalence relation. A point $x \in X'$ is equivalent to a point $x' \in X^k$ if and only if one of the following hold:

- (a) $i = k$ and $x = x'$,
- (b) $k > i$ and $\Phi^{k-1} \circ \cdots \circ \Phi^i(x) = x'$, or
- (c) $i > k$ and $\Phi'^{i-1} \circ \cdots \circ \Phi^k(x') = x$. 


For each $k \in \mathbb{N}$ we have an injective map $\Psi^k: X^k \hookrightarrow X$ onto the subset $\tilde{X}^k = \Psi^k(X^k) \subset X$ which sends any point $x \in X^k$ to its equivalence class $[x] \in X$. Denoting by $i^k: \tilde{X}^k \hookrightarrow \tilde{X}^{k+1}$ the inclusion map, we have

\begin{equation}
\label{2.2}
i^k \circ \Psi^k = \Psi^{k+1} \circ \Phi^k, \quad k = 1, 2, \ldots .
\end{equation}

The inverse maps $(\Psi^k)^{-1}: \tilde{X}^k \xrightarrow{\sim} X^k = S \times \mathbb{C}^n$ provide local charts on $X$. It is easily verified that this endows $X$ with the structure of a Hausdorff, second countable complex manifold. Since each of the maps $\Phi^k$ respects the fibers over $S$, we also get a natural projection $\pi: X \to S$ which is clearly a submersion. For every $s \in S$ the fiber $X_s$ is the increasing union of open subsets $\tilde{X}^k_s$ biholomorphic to $\mathbb{C}^n$. Observe that we get the same limit manifold $X$ by starting with any term of the sequence $\{X_s\}$.

The next lemma follows from the Andersén-Lempert theory [1]; cf. [12, Theorem 1.2].

**Lemma 2.1.** Let $\pi: X \to S$ be as above. Assume that for some $s \in S$ there exists an integer $k_s \in \mathbb{N}$ such that for every $k \geq k_s$, the domain $\Omega^k_s = \phi^k_s(\mathbb{C}^n) \subset \mathbb{C}^n$ is Runge in $\mathbb{C}^n$. Then $X_s$ is biholomorphic to $\mathbb{C}^n$.

**Proof.** The main point is that any biholomorphic map $\mathbb{C}^n \xrightarrow{\sim} \Omega$ onto a Runge domain $\Omega \subset \mathbb{C}^n$ can be approximated, uniformly on compact sets, by holomorphic automorphisms of $\mathbb{C}^n$. This observation allows one to renormalize the sequence of biholomorphisms $(\Psi^k_s)^{-1}: \tilde{X}^k_s \xrightarrow{\sim} \mathbb{C}^n$ for $k \geq k_s$ so that the new sequence converges uniformly on compact sets in $X_s$ to a biholomorphic map $X_s \xrightarrow{\sim} \mathbb{C}^n$; we leave out the straightforward details. \hfill \Box

### 3. Entire families of holomorphic automorphisms

Let $\mathfrak{K}_O(X)$ denote the complex Lie algebra of all holomorphic vector fields on a complex manifold $X$.

A vector field $V \in \mathfrak{K}_O(X)$ is said to be $\mathbb{C}$-complete, or completely integrable, if its flow $\{\phi_t\}_{t \in \mathbb{C}}$ exists for all complex values $t \in \mathbb{C}$, starting at an arbitrary point $x \in X$. Thus $\{\phi_t\}_{t \in \mathbb{C}}$ is a complex one-parameter subgroup of the holomorphic automorphism group $\text{Aut} X$. The manifold $X$ is said to enjoy the (holomorphic) density property if the Lie subalgebra $\mathfrak{L}(X)$ of $\mathfrak{K}_O(X)$, generated by the $\mathbb{C}$-complete holomorphic vector fields, is dense in $\mathfrak{K}_O(X)$ in the topology of uniform convergence on compact sets in $X$ (see Varolin [8, 9]). More generally, a complex Lie subalgebra $\mathfrak{g}$ of $\mathfrak{K}_O(X)$ enjoys the density property if $\mathfrak{g}$ is densely generated by the $\mathbb{C}$-complete vector fields that it contains. This property is very restrictive on open manifolds. The main result of the Andersén-Lempert theory [1] is that $\mathbb{C}^n$ for $n > 1$ enjoys the density property; in fact, every polynomial vector field on $\mathbb{C}^n$ is a finite sum of complete polynomial vector fields (the shear fields).

Varolin proved [8] that any domain of the form $(\mathbb{C}^*)^k \times \mathbb{C}^l$ with $k + l \geq 2$ and $l \geq 1$ enjoys the density property; we shall need this for the manifold $\mathbb{C}^* \times \mathbb{C}$. (Here $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.)

**Lemma 3.1.** Assume that $X$ is a Stein manifold with the density property. Choose a distance function $\text{dist}_X$ on $X$. Let $\psi_1, \ldots, \psi_k \in \text{Aut} X$ be such that for each $j = 1, \ldots, k$ there exists a $C^2$ path $\theta_{j,t} \in \text{Aut} X$ ($t \in [0, 1]$) with $\theta_{j,0} = \text{Id}_X$ and $\theta_{j,1} = \psi_j$. Given distinct points $a_1, \ldots, a_k \in \mathbb{C}^*$, a compact set $K \subset X$ and a
number $\epsilon > 0$, there exists a holomorphic map $\Psi: \mathbb{C} \times X \to X$ satisfying the following properties:

(i) $\Psi_\zeta = \Psi(\zeta, \cdot) \in \text{Aut } X$ for all $\zeta \in \mathbb{C}$,
(ii) $\Psi_0 = \text{Id}_X$,
(iii) $\sup_{x \in K} \text{dist}_X(\Psi(a_j, x), \psi_j(x)) < \epsilon$ for $j = 1, \ldots, k$.

A holomorphic map $\Psi$ satisfying property (i) will be called an entire curve of holomorphic automorphisms of $X$. Here $\text{Id}_X$ denotes the identity on $X$.

Proof. Consider a $C^2$ path $[0, 1] \ni t \mapsto \gamma_t \in \text{Aut } X$. Pick a Stein Runge domain $U \subset X$ containing the set $K$. Then $U_t = \gamma_t(U) \subset X$ is Runge in $X$ for all $t \in [0, 1]$. By [1] or, more explicitly, by (the proof of) [4] Theorem 1.1 there exist finitely many complete holomorphic vector fields $V_1, \ldots, V_m$ on $X$, with flows $\theta_j(t)$, and numbers $c_1 > 0, \ldots, c_m > 0$ such that the composition $\theta_{m,c} \circ \cdots \circ \theta_1,c_1 \in \text{Aut } X$ approximates the automorphism $\psi = \gamma_1$ within $\epsilon$ on the set $K$. (The proof in [4] is written for $X = \mathbb{C}^n$, but it applies in the general case stated here. We first approximate $\gamma_t: U \to U_t$ by compositions of short time flows of globally defined holomorphic vector fields on $X$; here we need the Runge property of the sets $U_t$. Since $X$ enjoys the density property, these vector fields can be approximated by Lie combinations (using sums and commutators) of complete holomorphic vector fields. This approximates $\gamma_t$ for each $t \in [0, 1]$, uniformly on $K$, by compositions of flows of complete holomorphic vector fields on $X$.)

Consider $t^j = (t_1, \ldots, t_m)$ as complex coordinates on $\mathbb{C}^m$. The map

$C^m \ni (t_1, \ldots, t_m) \mapsto \Theta_1(t_1, \ldots, t_m) = \theta_{m,t_m} \circ \cdots \circ \theta_{1,t_1} \in \text{Aut } X$

is entire, its value at the origin $0 \in \mathbb{C}^m$ is $\text{Id}_X$, and its value at the point $(c_1, \ldots, c_m)$ is an automorphism that is $\epsilon$-close to $\psi = \gamma_1$ on $K$.

Using this argument we find for every $j = 1, \ldots, k$ an integer $m_j \in \mathbb{N}$ and an entire map $\Theta_j: C^{m_j} \to \text{Aut } X$ such that $\Theta_j(0) = \text{Id}_X$ and $\Theta_j(c_1^{(j)}, \ldots, c_{m_j}^{(j)})$ is $\epsilon$-close to $\psi_j$ on $K$ at some point $c_j = (c_1^{(j)}, \ldots, c_{m_j}^{(j)}) \in C^{m_j}$. Let $t = (t^1, \ldots, t^k)$ be the complex coordinates on $C^M = C^{m_1} \oplus \cdots \oplus C^{m_k}$, where $t^j = (t_1^j, \ldots, t_{m_j}^j) \in C^{m_j}$. The composition

$C^M \ni t \mapsto \Theta(t^1, \ldots, t^k) = \Theta^k(t^k) \circ \cdots \circ \Theta^1(t^1) \in \text{Aut } X$

is an entire map satisfying $\Theta(0) = \text{Id}_X$ such that $\Theta(0, \ldots, 0, c_j, 0, \ldots, 0)$ is $\epsilon$-close to $\psi_j$ on $K$ for each $j = 1, \ldots, k$.

Choose an entire map $g: \mathbb{C} \to C^M$ with $g(a_j) = (0, \ldots, c_j, \ldots, 0)$ for $j = 1, \ldots, k$ and $g(0) = 0$. Then the map $\mathbb{C} \ni \zeta \mapsto \Psi(\zeta) = \Theta(g(\zeta)) \in \text{Aut } X$ satisfies the conclusion of the lemma.

4. Proof of Theorem 1.1

We shall need the following result from [11] §2. This construction is due to Stolzenberg [6]; see also [7, pp. 392–396].

**Lemma 4.1.** There exists a compact set $Y \subset \mathbb{C}^* \times \mathbb{C}$ (a union $Y = D_1 \cup D_2$ of two embedded, disjoint, polynomially convex discs) such that

(i) $Y$ is $O(\mathbb{C}^* \times \mathbb{C})$-convex,
(ii) the polynomial hull $\tilde{Y}$ contains the origin $(0, 0) \in \mathbb{C}^2$, and
(iii) for any nonempty open set \( U \subset C^* \times C \) there exists a holomorphic automorphism \( \psi \in \text{Aut}(C^* \times C) \) such that \( Y \subset \psi(U) \).

Property (iii) is [11, Lemma 3.1]: Since \( C^* \times C \) enjoys the density property according to Varolin [8], the isotopy that shrinks each of the two discs \( D_1, D_2 \subset Y \) to a point in \( U \) can be approximated by an isotopy of automorphisms of \( C^* \times C \) by using the methods in [4].

Proof of Theorem 1.1. We give the proof for \( n = 2 \). Let \( B = \{ b_1, b_2, \ldots \} \) be as in the theorem. Choose a set \( Y \subset C^* \times C \) satisfying Lemma 1.1. Pick a closed ball \( K \subset C^2 \) (or any compact set with nonempty interior).

We shall inductively construct a sequence of injective holomorphic maps \( \Phi^k: S \times C^2 \mapsto S \times C^2 \) (\( k = 1, 2, \ldots \)) of the form

\[
\Phi^k(s, z) = (s, \phi^k_s(z)), \quad s \in S, \ z \in C^2,
\]

such that, setting

\[
(4.1) \quad \phi^k_s = \phi_s^k \circ \phi_{s}^{k-1} \circ \cdots \circ \phi_s^1, \quad K^k_s = \overline{\phi_s^k(K)} \subset C^2,
\]

the following properties hold for all \( k \in \mathbb{N} \):

(i) \( \Omega^k := \Phi^k(S \times C^2) \subset S \times (C^* \times C) \),

(ii) the fiber \( \Omega_s^k = \phi^k_s(C)^2 \) is Runge in \( C^2 \) for all \( s \in A_1 \cup \cdots \cup A_k \), and

(iii) \( Y \subset \text{Int} K^k_s \) for each \( s \in \{ b_1, \ldots, b_k \} \). In particular, the polynomial hull of the set \( K^k_s \) contains the origin for every such \( s \).

Suppose for the moment that we have such a sequence. Let \( X \) denote the limit manifold and let \( \Psi^k: X^k = S \times C^2 \mapsto \mathbb{X}^k \subset X \) be the induced inclusions (see 2). If \( s \in \bigcup A_k = A \), then property (ii) insures, in view of Lemma 2.1, that the fiber \( X_s \) is biholomorphic to \( C^2 \).

Suppose now that \( s = b_j \) for some \( j \in \mathbb{N} \). Property (iii) shows that for every integer \( k \geq j \) the polynomial hull of the set \( K^k_s \) contains the origin of \( C^2 \); in particular, \( K^k_s \) is not contained in \( \Omega_s^k \subset C^* \times C \). For the corresponding subsets of the limit manifold \( X_s \) we get in view of (2.2) that

\[
\Psi^{k+1}_s(K^k_s) \not\subset \mathbb{X}^k_s, \quad k = j, j + 1, \ldots,
\]

where the hull is with respect to the algebra of holomorphic functions on the domain \( \mathbb{X}^k_s \) in the fiber \( X_s \).

Let \( K_s = \Psi^1_s(K) \) denote the compact set in \( X_s \) determined by \( K \); note that \( K_s \subset \mathbb{X}^1_s \) and \( K_s = \Psi^{k+1}_s(K^k_s) \) for any \( k \in \mathbb{N} \) according to (2.2) and (4.1). The above display then gives

\[
(K_s)_{\mathcal{O}(\mathbb{X}^{k+1})} \not\subset \mathbb{X}^k_s, \quad k = 1, 2, \ldots.
\]

Since \( \mathbb{X}^k_s \) is a domain in \( X_s \), we trivially have \( (K_s)_{\mathcal{O}(\mathbb{X}^{k+1})} \subset (K_s)_{\mathcal{O}(X_s)} \); hence the hull \( (K_s)_{\mathcal{O}(X_s)} \) is not contained in \( \mathbb{X}^k_s \) for any \( k \in \mathbb{N} \). As the domains \( \mathbb{X}^k_s \) exhaust \( X_s \), this hull is noncompact. Hence \( X_s \) is not holomorphically convex (and therefore not Stein) for any \( s \in B \).

This proves Theorem 1.1 provided that we can find a sequence with the stated properties.

We begin with some initial choices of domains and maps. Pick a Fatou-Bieberbach map \( \theta: C^2 \xrightarrow{\cong} D \subset C^* \times C \) whose image \( D = \theta(C^2) \) is Runge in
For each $k = 1, 2, \ldots$, we choose a holomorphic function $f_k : S \to \mathbb{C}$ such that $f_k = 0$ on the subvariety $A_1 \cup \cdots \cup A_k$ of $S$ and $f_k(b_j) = j$ for $j = 1, \ldots, k$. If the set $B \subset X \setminus A$ also contains a closed complex subvariety $B'$ of $X$ of positive dimension, we let $f_k = 1$ on $B'$.

We now construct the first map $\Phi^1(s, z) = (s, \phi^1_s(z))$. Lemma 4.1 furnishes an automorphism $\psi \in \text{Aut}(\mathbb{C}^* \times \mathbb{C})$ such that $Y \subset \psi(\theta(U))$. By Lemma 3.3 there exists an entire curve of automorphisms $\Psi_\zeta \in \text{Aut}(\mathbb{C}^* \times \mathbb{C})$ ($\zeta \in \mathbb{C}$) such that $\Psi_0 = \text{Id}_{\mathbb{C}^* \times \mathbb{C}}$ and $\Psi_1$ approximates $\psi$ close enough on the compact set $\theta(K)$ so that $Y \subset \Psi_1(\theta(U))$. Hence $(0, 0) \in Y \subset \Psi_1(\theta(K))$. Set 

$$ \phi^1_s(z) = \Psi_{f_1(s)}(\theta(z)), \quad s \in S, \quad z \in \mathbb{C}^2. $$

If $s \in A_1$, then $f_1(s) = 0$ and hence $\phi^1_s(z) = \Psi_0(\theta(z)) = \theta(z)$, so $\phi^1_s = \theta$. If $s = b_1$, then $f_1(s) = 1$ and hence $\phi^1_s = \Psi_1 \circ \theta$. Thus $Y \subset \phi^1_{b_1}(U)$ and the polynomial hull $\hat{\phi}^1_{b_1}(K)$ contains the origin of $\mathbb{C}^2$. This gives the initial step.

Suppose that we have found maps $\Phi^1, \ldots, \Phi^k$ satisfying conditions (i)–(iii) above; we now construct the next map $\Phi^{k+1}$ in the sequence. Recall that $\hat{\phi}^k : \mathbb{C}^2 \to \mathbb{C}^2$ is the map defined by (4.11). Let 

$$ U^k_s = (\theta \circ \hat{\phi}^k_s)(U), \quad s \in S; $$

this is a nonempty open set contained in the compact set $\theta(K^k_s) \subset \mathbb{C}^* \times \mathbb{C}$. Lemma 4.1 gives for each $j = 1, \ldots, k+1$ an automorphism $\psi_j \in \text{Aut}(\mathbb{C}^* \times \mathbb{C})$ such that $Y \subset \psi_j(U^k_{b_j})$. By Lemma 3.3 there exists an entire curve of automorphisms $\Psi_\zeta \in \text{Aut}(\mathbb{C}^* \times \mathbb{C})$ ($\zeta \in \mathbb{C}$) such that $\Psi_0 = \text{Id}_{\mathbb{C}^* \times \mathbb{C}}$ and $\Psi_j$ approximates $\psi_j$ for every $j = 1, \ldots, k+1$. If the approximation is close enough on the compact set $\theta(K^k_{b_j})$, then $Y \subset (\Psi_j \circ \theta)(K^k_{b_j})$ and hence the origin $(0, 0) \in \mathbb{C}^2$ is contained in the polynomial hull of $(\Psi_j \circ \theta)(K^k_{b_j})$. Set 

$$ \phi^{k+1}_s(z) = \Psi_{f_{k+1}(s)} \circ \theta(z), \quad s \in S, \quad z \in \mathbb{C}^2. $$

If $s \in A_1 \cup \cdots \cup A_{k+1}$, then $f_{k+1}(s) = 0$ and hence $\phi^{k+1}_s = \theta$. If $s = b_j$ for some $j = 1, \ldots, k+1$, then $f_{k+1}(b_j) = j$ and hence $\phi^{k+1}_{b_j} = \Psi_j \circ \theta$; therefore the polynomial hull of the set $\hat{\phi}^{k+1}_{b_j}(K^k_{b_j})$ contains the origin. Taking $\hat{\phi}^{k+1}_s$ as the next map in the sequence and setting 

$$ \hat{\phi}^{k+1}_s = \phi^{k+1}_s \circ \hat{\phi}^k_s, \quad K^{k+1}_s = \phi^{k+1}_s(K^k_s) $$

we see that properties (i)–(iii) hold also for $k+1$. The induction may continue.

This completes the proof of Theorem I.1. \hfill \Box

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Faculty of Mathematics and Physics, University of Ljubljana, and Institute of Mathematics, Physics and Mechanics, Jadranska 19, 1000 Ljubljana, Slovenia

E-mail address: franc.forstneric@fmf.uni-lj.si