Branching Annihilating Random Walks with Long-Range Attraction in One Dimension

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We introduce and numerically study the branching annihilating random walks with long-range attraction (BAWL). The long-range attraction makes hopping biased in such a manner that particle’s hopping along the direction to the closest particle has larger transition rate than hopping against the direction. Still, unlike the Lévy flight, a particle only hops to one of its nearest neighbor sites. The strength of bias takes the form $x^{-\sigma}$ with nonnegative $\sigma$, where $x$ is the distance to the closest particle from a particle to hop. By extensive Monte Carlo simulations, we show that the critical decay exponent $\delta$ varies continuously with $\sigma$ up to $\sigma = 1$ and $\delta$ is the same as the critical decay exponent of the directed Ising (DI) universality class for $\sigma \geq 1$. Investigating the behavior of the density in the absorbing phase, we argue that $\sigma = 1$ is indeed the threshold that separates the DI and non-DI critical behavior. We also show by Monte Carlo simulations that branching bias with symmetric hopping exhibits the same critical behavior as the BAWL.

I. INTRODUCTION

The branching annihilating random-walks model (BAW) \cite{1} is a reaction-diffusion system with (symmetric) diffusion and pair annihilation ($2A \rightarrow 0$) as well as branching $m$ offspring by a particle ($A \rightarrow (m + 1)A$). The competition between pair annihilation and branching results in an absorbing phase transition between an active phase with nonzero steady-state density and an absorbing phase with zero steady-state density. The BAW exhibits rich phenomena in that critical behavior depends on the parity of the number $m$ of offspring \cite{1,2}. It belongs to the directed percolation (DP) universality class for even $m$, whereas it belongs to the directed Ising (DI) universality class for odd $m$. For a review of these two classes, see, e.g., Refs. \cite{10,15}.

When a global hopping bias is introduced to the BAW in such a way that hopping along a predefined direction is preferred (for example, in one dimension hopping to the right has larger transition rate than hopping to the left), this bias in the (asymptotic) field theory is gauged away by a Galilean transformation \cite{19}, which makes critical behavior remain intact even in presence of the global bias. Recently, a local hopping bias is introduced to the BAW \cite{20} in such a manner that a particle prefers hopping toward the closest particle. Since two close particles are likely to get closer by the local bias, this form of interaction associated with the local bias is termed as attraction in Ref. \cite{21}. Unlike the global bias, the local bias does not produce macroscopic current, for hopping along any direction is equally likely on average. In this sense, the Galilean transformation cannot remove the local bias and, in turn, the local bias may be relevant in the renormalization-group sense. Indeed, it was shown that the local bias changes the critical behavior when the system has an even parity ($m$ is an even number) \cite{20,21}.

Unlike a long-range jump (Lévy flight) in models exhibiting absorbing phase transition \cite{22,23}, every particle still hops to one of its nearest neighbors by the local bias. In this sense, one may think of the local bias as short-range interaction. This idea seems to have a support because the BAW with odd number of offspring is not affected by the local bias, while Lévy flight applied to DP models changes critical behavior \cite{24,25}. In Ref. \cite{21}, however, it was shown that the local bias is indeed of long-range feature when the number of offspring is even. Furthermore, it was argued that spontaneous annihilation ($A \rightarrow 0$) arising by combination of branching with pair annihilation ($A \rightarrow 2A \rightarrow 0$) removes the long-range nature of the local bias for odd $m$ \cite{21}.

In Ref. \cite{21}, the strength of the local bias does not depend on distance $x$ between a particle and the closest particle from it. If the strength of the local bias becomes zero when the distance is larger than a certain length $R$, which is termed as the range of attraction \cite{21}, it turned out that the BAW with even number of offspring again belongs to the DI class and the crossover for finite but large $R$ is described by a crossover exponent $\phi$, which is found to be $1.39 \pm 0.04$ \cite{21}.

In this paper, we study the effect of slowly varying strength of the local bias with the distance. To this end, we generalize the model in Ref. \cite{20} such a way that the strength of the bias is a power-law function $x^{-\sigma}$ of the distance $x$, where $\sigma \geq 0$. The case with $\sigma = 0$ will be identical to the model in Ref. \cite{20}. The purpose of this paper is to investigate how the critical behavior is affected by the value of $\sigma$.

The structure of this paper is as follows. In Sec. \textbf{II} we define a model with a local bias due to long-range attraction between particles. As explained above, the strength of the bias becomes a power-law function of the distance between two shortest-distance particles. We will call this model the branching annihilating random walks with long-range attraction (BAWL). In Sec. \textbf{III} we present our simulation results, focusing on the critical decay exponent that is defined in Sec. \textbf{II}. We will also find $\sigma_c$ that separates the DI critical behavior (for $\sigma \geq \sigma_c$) and non-DI critical behavior (for $\sigma < \sigma_c$). In Sec. \textbf{IV} we discuss what happens if branching rather than hopping is biased. Section \textbf{V} summarizes the paper.
II. MODEL AND METHODS

The BAWL is defined on a one-dimensional lattice of size $L$ with periodic boundary conditions. Each site $i$ ($i = 1, 2, \ldots, L$) is characterized by the occupation number $a_i$ that takes either one or zero. If $a_i = 1$, we say that there is a particle at site $i$. If $a_i = 0$, we say that site $i$ is vacant. For later purpose, we define $r_i$ and $l_i$ such that

$$r_i = \min \{ x | a_{i+x} = 1, \ x > 0 \},$$

$$l_i = \min \{ x | a_{i-x} = 1, \ x > 0 \},$$

where we assume that site $j + L$ is identical to site $j$ (periodic boundary condition). In words, $r_i$ ($l_i$) is the distance from site $i$ to the closest particle on the right-hand (left-hand) side. One can regard $a_i$, $r_i$, $l_i$ as random variables.

If there is a particle at site $i$ ($a_i = 1$), it either hops to one of its nearest neighbors at rate $p$ (hopping event) or branches four offspring with rate $1 - p$ (branching event). In the hopping event, it hops to site $i$ with probability $q_{\pm}$, where

$$q_{\pm} = \frac{1}{2} \pm \zeta x^{-\sigma}, \quad x = \min\{r_i, l_i\}, \quad \sigma \geq 0,$$

with $0 \leq \zeta \leq 0.5$.

$$\zeta = \begin{cases} \epsilon, & \text{if } r_i < l_i, \\ -\epsilon, & \text{if } r_i > l_i, \\ 0, & \text{if } r_i = l_i. \end{cases}$$

In a sense, a particle is attracted by the closest particle. In the branching event, its four offspring are placed at sites $i-2$, $i-1$, $i+1$, and $i+2$ ($A \to 5A$). If a particle is to be placed at an occupied site either by hopping or branching, these two particles are annihilated immediately ($2A \to 0$). We summarize the above dynamic rules as follows.

$$1, a_{i+1} \to 0, a_{i+1} \text{ rate } pq_+,$$  \hspace{1cm} (4a)

$$a_{i-1} \to 1, a_{i-1} \text{ rate } pq_-,$$  \hspace{1cm} (4b)

$$1, a_{i+1} a_{i+2} \to 1, a_{i+1} a_{i+2} \text{ rate } 1-p,$$  \hspace{1cm} (4c)

where $1, (0_i)$ means that $a_i$ is one (zero) and $\pi_j = 1 - a_j$. We set $\epsilon = 0.1$ in simulations but other choice of nonzero $\epsilon$ does not change our conclusion.

The BAWL with $\sigma = 0$, which is identical to the model in Ref. [20], does not belong to the DI class, while the BAWL under $\sigma \to \infty$ limit, which is equivalent to the model in Ref. [21] with the range of attraction to be 1, belongs to the DI class. Thus, there should be $\sigma_c$ such that the BAWL with $\sigma \geq \sigma_c$ belongs to the DI class. This paper not only finds $\sigma_c$, but studies the critical behavior for $\sigma < \sigma_c$.

We will study the average density $\rho$ of occupied sites at time $t$ defined as

$$\rho(t) = \frac{1}{L} \sum_{i=1}^{L} \langle a_i \rangle,$$  \hspace{1cm} (5)

where $\langle \cdots \rangle$ stands for average over all ensemble. As an initial condition, we will always use the configuration with $a_i = 1$ for all $i$ in this paper.

At the critical point, $\rho(t)$ shows a power-law behavior with a critical decay exponent $\delta$ such that

$$\rho(t) = A t^{-\delta} \left[1 + B t^{-\chi} + o(t^{-\chi}) \right],$$  \hspace{1cm} (6)

where $t^{-\chi}$ is the leading behavior of corrections to scaling, $o(t)$ stands for all terms that decrease faster than $x$ as $x \to 0$, and $A$, $B$ are constants. We will call $\chi$ the corrections-to-scaling exponent. Our main interest is to figure out how $\delta$ depends on $\sigma$.

To find $\delta$, we study an effective exponent $-\delta_e$ defined as

$$-\delta_e(t, b) = \frac{\ln[\rho(t)/\rho(t/b)]}{\ln b},$$  \hspace{1cm} (7)

where $b$ is a constant. At the critical point, the effective exponent in the long time limit should behave as

$$-\delta_e(t, b) \approx -\delta - \frac{\ln x - 1}{\ln b} t^{-\chi}.$$  \hspace{1cm} (8)

From Eq. (8), it is obvious that at the critical point $-\delta_e$ drawn as a function of $t^{-\chi}$ becomes a straight line for small $t^{-\chi}$. On the other hand, if the system is slightly off the critical point and is actually in the active (absorbing) phase, $-\delta_e$ should eventually veer up (down) as $t^{-\chi}$ becomes smaller. Accordingly, we can find the critical point by observing how $-\delta_e$ behaves. Once we find the critical point, the critical decay exponent can be found by linear extrapolation of $-\delta_e$ vs $t^{-\chi}$ at the critical point.

To estimate $\delta$ accurately, therefore, information of $\chi$ is crucial. To find $\chi$, we analyze a corrections-to-scaling function $Q$ defined as

$$Q(t; b, \chi) = \frac{\ln \rho(t/b^2) + \ln \rho(t) - 2 \ln \rho(t/b)}{(b^\chi - 1)^2},$$  \hspace{1cm} (9)

whose asymptotic behavior at the critical point is $Q \sim B t^{-\chi}$ regardless of the value of $b$ if $\chi$ is correctly chosen. Notice that if $B$ is positive (negative), $-\delta_e$ approaches $-\delta$ from below (above).

For convenience, an $i$-th measurement is performed at time $T_i$ defined as

$$T_i = \begin{cases} i, & 1 \leq i \leq 40, \\ 40 \times 2^{(i-40)/15}, & 41 \leq i \leq 55, \\ 2T_{i-15}, & 56 \leq i, \end{cases}$$  \hspace{1cm} (10)

where $[x]$ is the floor function (greatest integer not larger than $x$). With this choice of measurement time, the effective exponent as well as the corrections-to-scaling function is analyzed using $b = 2^n$ ($n = 1, 2, \ldots$).

III. RESULTS

In this section, we present our simulation results for the critical decay exponent $\delta$ for various values of $\sigma$. To
begin, we analyze the BAWL with $\sigma = 0.1$ and $0.3$. In simulations for these two cases, the system size is $L = 2^3$ and the maximum observation time is $T_{289} \approx 4 \times 10^6$. The number of independent runs is between 80 and 200. We first analyzed the corrections-to-scaling function $Q$ and we found $\chi$ to be $0.3$ and $0.25$ for $\sigma = 0.1$ and $0.3$, respectively (details not shown here). Using $\chi$ we found, we depict in Fig. 1 the effective exponent as a function of $t^{-\chi}$ for $\sigma = 0.1$ [Fig. 1(a)] and $0.3$ [Fig. 1(b)]. Here, the effective exponents are calculated using $b = 16$.

Since middle curves in both panels show linear behaviors, while the other curves eventually veer up or down, we estimate the critical point as $p_c = 0.572375(25)$ for $\sigma = 0.1$ and $p_c = 0.5905(1)$ for $\sigma = 0.3$, where the numbers in parentheses indicate uncertainty of the last digits. By linear extrapolation, we get $\delta = 0.2532(8)$ for $\sigma = 0.1$ and $0.276(1)$ for $\sigma = 0.3$. It is clear that $\delta$ does depend on $\sigma$, which is a typical feature of absorbing phase transitions with long-range jump. Once again we confirm the claim in Ref. [21] that the model with hopping bias in Ref. [20] does not belong to the DI class because the hopping bias is of long-range nature.

We have established that the critical decay exponent varies with $\sigma$. Now we move on to the next task to find $\sigma_c$. Recall that the BAWL with $\sigma \geq \sigma_c$ is supposed to belong to the DI class. To this end, we simulated the system of size $L = 2^3$ for various $\sigma$’s. As we have done in Fig. 1 we first found $\chi$ and $p_c$, then analyzed the effective exponent. In Fig. 2 we depict the resulting effective exponents at the critical point for $\sigma = 0.4, 0.6, 0.8, 1$ against $(T_M/t)^{\delta}$, where $T_M$ is the maximum observation time of each simulation for the corresponding parameter set. For $\sigma \leq 0.8$, $\delta$ of the BAWL is found to be distinct from $\delta$ of the DI class, which is shown as a dotted line in Fig. 2. To reduce statistical error especially for $\sigma = 0.8$, we performed 800 independent runs for this case. The values of $p_c, \chi$, and $\delta$ for various $\sigma$’s are summarized in Table 1. In Fig. 3 we graphically show how $\delta$ and $p_c$ depend on $\sigma$. Our simulation results suggest $\sigma_c = 1$ and our preliminary simulations also showed that $\delta$ does not change for $\sigma > 1$ (details not shown here).

Now we will argue that $\sigma_c$ is indeed one. Since the DI class is intimately related to the annihilation fixed point [13], a necessary condition for a model to belong to the DI class is that the asymptotic behavior of density should be $t^{-0.5}$ in the absorbing phase. In this context, we will analyze how the density of the BAWL with $p = 1$

![FIG. 1. Plots of $-\delta_e$ vs $t^{-\chi}$ for (a) $\sigma = 0.1$ with $\chi = 0.3$ and (b) $\sigma = 0.3$ with $\chi = 0.25$. We set $b = 16$. The straight lines overlapped with the middle curves show the results of linear extrapolation for the critical decay exponent. Clearly, the critical decay exponent $\delta$ varies with $\sigma$.](image)

![FIG. 2. Plots of $-\delta_e$ vs $(T_M/t)^{\chi}$ with $b = 32$ at the critical point for $\sigma = 0.4, 0.6, 0.8$ and 1 (top to bottom), where $T_M$ is the maximum observation time. Here, $T_M = T_{289} \approx 4 \times 10^6$ for $\sigma = 0.4$ and $T_M = T_{309} \approx 10^7$ for other cases. Straight lines are results of linear extrapolation and the dotted horizontal line indicates the critical decay exponent of the DI class.](image)

| $\sigma$ | $p_c$   | $\chi$ | $\delta$  |
|----------|---------|--------|-----------|
| 0.4      | 0.562 142(3) | 0.3   | 0.2303(3) |
| 0.6      | 0.572 375(25) | 0.3   | 0.2532(8) |
| 0.8      | 0.581 85(5)   | 0.3   | 0.2647(7) |
| 1.0      | 0.5905(1)     | 0.25  | 0.276(1)  |
| 1.2      | 0.5983(1)     | 0.25  | 0.2828(8) |
| 1.4      | 0.6112(1)     | 0.35  | 0.2855(5) |
| 1.6      | 0.621 11(1)   | 0.4   | 0.2866(3) |
| 1.8      | 0.628 75(5)   | 0.4   | 0.2872(4) |

* From Ref. [21].
* Data not shown here.
FIG. 3. Plot of $\delta$ vs $\sigma$. The critical decay exponent of the DI class is shown as a horizontal dotted line. The size of the error bar is comparable to the symbol size. (Inset) Plot of $p_c$ vs $\sigma$. The line is for guides to the eyes.

(without branching) behaves in the long time limit.

In the absorbing phase, the density approaches zero in the long time limit. Hence, the asymptotic behavior of the density for the BAWL with $p = 1$ can be understood by studying a random walk model with an attracting center at the origin. In this random walk model, a walker located at site $n$ $(n > 0)$ hops to the right with rate $(1 - vn^{-\sigma})/2$ and to the left with rate $(1 + vn^{-\sigma})/2$. Now we will find the mean first-passage time to the origin, once it starts from site $m$. It is convenient to regard the origin as an absorbing wall.

The analysis starts from writing down the master equation $(n \geq 1)$

$$\frac{\partial}{\partial t} P_n(t) = -P_n(t) + \frac{1 + v(n + 1)^{-\sigma}}{2} P_{n+1}(t) + \frac{1 - v(n - 1)^{-\sigma}}{2} (1 - \delta_{n,1}) P_{n-1}(t),$$  \tag{11}$$

$$\frac{\partial}{\partial t} P_0(t) = \frac{1 + v}{2} P_1(t),$$ \tag{12}$$

where $P_n(t)$ is the probability that the walker is at site $n$ at time $t$. For $n \geq 2$, we rewrite Eq. (11) as

$$\frac{\partial}{\partial t} P_n(t) = -\partial_n \left[-vn^{-\sigma} P_n(t)\right] + \frac{1}{2} \partial_n^2 P_n(t),$$ \tag{13}$$

where $\partial_n f(n) \equiv [f(n + 1) - f(n - 1)]/2$ and $\partial_n^2 f(n) \equiv f(n + 1) + f(n - 1) - 2f(n)$. Taking (naive) continuum limit, we get a Fokker-Planck equation ($n$ is now a continuous variable)

$$\frac{\partial}{\partial t} P(n, t) = -\partial_n \left[-vn^{-\sigma} P(n, t)\right] + \frac{1}{2} \partial_n^2 P(n, t),$$ \tag{14}$$

which is equivalent to the Langevin equation

$$\dot{n} = -vn^{-\sigma} + \xi,$$ \tag{15}$$

where $\xi$ is the white noise with zero mean and unit variance.

Using a mean-field-like approximation $\langle n^{-\sigma} \rangle \approx \langle n \rangle^{-\sigma}$, where $\langle \ldots \rangle$ is the average over noise, we get

$$\langle \dot{n} \rangle \approx -\frac{v}{\langle n \rangle^{\sigma}} \Rightarrow \langle n \rangle \approx m \left[ 1 - \frac{(1 + \sigma)vt}{m^{1+\sigma}} \right]^{1/(1+\sigma)},$$ \tag{16}$$

where $m$ is the initial position of the walker. If we further assume that the white noise makes $P_n$ be a Gaussian with variance $t$, we arrive at

$$P_n(t) \approx \frac{1}{\sqrt{2\pi t}} \exp \left[ -\frac{(n - \langle n \rangle)^2}{2t} \right],$$ \tag{17}$$

for sufficiently large $n$ (and $m$).

To check how good the approximation is, we performed Monte Carlo simulations for the continuous time master equation (11) with $\sigma = 0.2$, $v = 0.2$, and $m = 2000$. In Fig. 4 we depict $P_n(t)$ at $t = 1000, 3000, 5000, 7000$ together with Eq. (17). Our approximation is in an excellent agreement with numerical (exact) result.

If $\sigma < 1$, the mean first-passage time $\tau$ to the origin is obtained by $\langle n \rangle = 0$, which gives $\tau \sim m^{1+\sigma}$. On the other hand, if $\sigma > 1$, the spreading by fluctuation is faster than the deterministic motion. Accordingly, time $\tau$ to arrive at the origin is dominated by diffusion, which gives $\tau \sim m^2$. If we write $\tau \sim m^z$, we find

$$z = \begin{cases} 
1 + \sigma, & \sigma < 1 \\
2, & \sigma \geq 1. 
\end{cases} \tag{18}$$

From Eq. (15) and the scaling argument for the pair annihilation dynamics, we predict that the long time behavior of the density is $t^{-\alpha}$ with

$$\alpha = 1/z = \begin{cases} 
1/(1 + \sigma), & \sigma < 1, \\
1/2, & \sigma \geq 1. 
\end{cases} \tag{19}$$
To confirm the anticipation, we simulated the BAWL with $\epsilon = 0.1$ and $p = 1$ for various $\sigma$’s. We present the behavior of effective exponent $-\alpha_e$ for $\sigma = 0.2$, 0.6, and 1 in Fig. 5 which shows an excellent agreement with the analytic argument.

From the above analysis, the BAWL with $\sigma < 1$ should not belong to the DI class, as we have seen in Fig. 2. Since the BAWL with $\sigma = 1$ belongs to the DI class as shown in Fig. 2 we conclude that the upper bound $\sigma_c$ is indeed 1

![Image](image-url)

FIG. 5. Plots of $-\alpha_e$ vs $t^{-\chi}$ for $\sigma = 0.2$ ($\chi = 0.8$: bottom), $\sigma = 0.6$ ($\chi = 0.3$: middle), and $\sigma = 1$ ($\chi = 1$: top). Dotted line segments indicate the anticipated value of $-\alpha$ from Eq. [19].

IV. DISCUSSION: BRANCHING BIAS

We have shown that the local hopping bias due to long-range attraction with decreasing strength as $x^{-\sigma}$ continuously changes the critical decay exponent of the BAWL when $\sigma \leq 1$. Now, we would like to ask which one determines the critical behavior, hopping bias or bias in itself. To answer this question, we modify the BAWL in such a way that hopping is symmetric but branching is biased. To be concrete, we will now investigate a model with dynamics

$$1_i a_{i \pm 1} \rightarrow 0_i \pi_{i+1} \quad \text{rate } p/2, \quad (20a)$$

$$1_i \prod_{k=1}^{4} a_{i+k} \rightarrow 1_i \prod_{k=1}^{4} \pi_{i+k} \quad \text{rate } (1-p)q_+, \quad (20b)$$

$$1_i \prod_{k=1}^{4} a_{i-k} \rightarrow 1_i \prod_{k=1}^{4} \pi_{i-k} \quad \text{rate } (1-p)q-, \quad (20c)$$

where we use the same notation as in Eq. [1].

Before presenting simulation results, let us ponder on what would happen in this modified model. The driven pair contact process with diffusion (DPCPD) [13] would be a good starting point for our discussion. In the DPCPD, though it has global bias, only presence of bias is an important factor to determine the universality class, as it is immaterial whether hopping or branching is biased [32]. In this regard, one would conclude that bias in itself is relevant (in the renormalization-group sense) and that the critical behavior of the BAWL would not be affected by to which dynamic process the local bias is applied. However, the DPCPD should be considered a system with two independent fields and both the hopping bias and the branching bias in the DPCPD generates a relative bias between the two fields [10, 33]. Since the BAW is described by a single field [13], the discussion about the DPCPD would not give a clear answer to our question.

In the mean time, one may easily come up with an argument that only hopping bias is relevant, because the density of the modified model with $p = 1$ (trivially) behaves as $t^{-0.5}$ for any $\sigma$. This should be compared with the discussion in Sec. III based on the analysis of the BAWL with $p = 1$. However, this argument has a serious flaw; the dynamics at $p = 1$ may not represent the absorbing phase of the modified model. An example in this context is the BAW with one offspring (BAW$_1$). As
in the BAWL, let us denote the branching rate of the BAWL by $1 - p$. If $p = 1$, the density (again trivially) decays as $t^{-0.5}$. If branching rate is turned on, however, a spontaneous annihilation of a single particle by the chain of reactions $A \rightarrow 2A \rightarrow 0$ can occur, which results in an exponential density decay. That is, the BAWL with $p = 1$ cannot capture the main feature (exponential density decay) of its absorbing phase.

Actually, the behavior of the BAWL around $p = 1$ can be described by a scaling function

$$
\rho(t) = t^{-0.5} F[(1 - p)t],
$$

where $F(x)$ is expected to decrease exponentially for large $x$. The reason why $(1 - p)t$ should be a single scaling parameter is clear. The spontaneous annihilation can be crucial only when substantial amount of branching events happen, which is expected when time elapses more than $1/(1 - p)$. In Fig. 6 we show scaling collapse of the BAWL for $p$ close to 1, which confirms the scaling ansatz (21). Here, the system size is $2^{25}$ and average over 8 independent runs for each parameter is taken. As the example of the BAWL reveals, it is possible that $p = 1$ of the modified model is in a sense a singular point and that the modified model in the absorbing phase does not exhibit $t^{-0.5}$ behavior for small $\sigma$.

To obtain the answer, we now resort to Monte Carlo simulations. Using systems of size $L = 2^{24}$, we performed simulations for $\epsilon = 0.5$ and $p = 0.8$. To reduce statistical error, we performed 40 independent runs for each parameter set. Figure 7 shows the behavior of the density for $\sigma = 0, 0.2, 0.6$, and 1 on a double logarithmic scale. Just like the BAWL with $p = 1$, the density decays as $t^{-\alpha}$ with $\alpha$ in Eq. (19). Hence, we expect that the critical behavior is the same regardless of whether hopping or branching is biased. We have checked this anticipation by simulations and our preliminary simulations for $\sigma = 0$ indeed show that the critical behavior of the modified model is the same as the BAWL (though the corrections-to-scaling exponent of the modified model is found to be smaller than the BAWL; details not shown here). This also indirectly confirms that the BAWL with $p = 1$ correctly represents the behavior in the absorbing phase. To conclude this section, we have shown that the presence of the local bias due to long-range attraction is enough to exhibit non-DI critical phenomena, irrespective of which dynamic process the local bias is applied.

\section{SUMMARY}

To summarize, we studied the branching annihilating random walks with long-range attraction (BAWL). The long-range attraction has a power-law feature with exponent $\sigma$; see Eq. (2). We investigated the critical decay exponent $\delta$ that describes how the density behaves at the critical point. We first numerically found that $\delta$ varies continuously with $\sigma$ for $\sigma < 1$ and is the same as the critical decay exponent of the directed Ising universality class for $\sigma \geq 1$. By the analysis of a random walk with an attracting center at the origin together with Monte Carlo simulations for the BAWL with $p = 1$, we argued that $\sigma_c$ should be 1.

We also studied the modified model in which the local bias is applied to the branching process and hopping is unbiased. We found that the absorbing phase of the modified model shows the same asymptotic behavior of the BAWL for the same value of $\sigma$. Therefore, we concluded that it is immaterial which dynamic process, hopping or branching, is biased by the long-range attraction.

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