Tunnelling of topological line defects in strongly coupled superfluids

Uwe R. Fischer

Institut für Theoretische Physik der Eberhard-Karls-Universität Tübingen
Auf der Morgenstelle 14, D-72076 Tübingen, Germany
uwe.fischer@uni-tuebingen.de

Received 24 February 2000, accepted 17 April 2000 by U. Eckern

Abstract. The geometric theory of vortex tunnelling in superfluid liquids is developed. Geometry rules the tunnelling process in the approximation of an incompressible superfluid, which yields the identity of phase and configuration space in the vortex collective co-ordinate. To exemplify the implications of this approach to tunnelling, we solve explicitly for the two-dimensional motion of a point vortex in the presence of an ellipse, showing that the hydrodynamic collective co-ordinate description limits the constant energy paths allowed for the vortex in configuration space. We outline the experimental procedure used in helium II to observe tunnelling events, and compare the conclusions we draw to the experimental results obtained so far. Tunnelling in Fermi superfluids is discussed, where it is assumed that the low energy quasiparticle excitations localised in the vortex core govern the vortex dynamical equations. The tunnelling process can be dominated by Hall or dissipative terms, respectively be under the influence of both, with a possible realization of this last intermediate case in unconventional, high-temperature superconductors.

Keywords: Tunnelling, Superfluids, Vortex

PACS: 66.35.+a, 67.40.Vs, 67.40.Hf

1 Introductory considerations

The primary notion we have of a superfluid is that it shows no dissipation of the flow, under certain, well-defined conditions. Friction can be caused by the creation of elementary quasiparticle excitations in the superfluid, that is, irreversible energy transfer from the coherently moving superfluid state to incoherent degrees of freedom. This can occur if the superflow (relative to some reference frame) exceeds the Landau critical velocity, creating the excitations, and thereby reducing the superfluid current. A different kind of dissipation can be caused by a topological excitation, the quantized vortex, representing a travelling defect structure in the order parameter of the superfluid. The dissipation mechanism is then represented by the vortex crossing the streamlines of the flow, diminishing the superfluid current by reducing the superfluid phase difference between points on a line perpendicular to the vortex motion. To cause current reduction, a vortex first has to be generated. The task of this paper is to develop a formalism describing a quantized vortex entering a superfluid at the absolute zero of temperature by the quantum mechanism of tunnelling.

The nucleation theory of quantized vortices in the Bose superfluid helium II has
been an elusive subject ever since the existence of quantized vortices was conjectured by Lars Onsager in 1949 [1]. The difficulty to grasp their coming into existence in a quantitative manner from first principles has one fundamental reason: There is no microscopic theory of this dense superfluid. We do not know how to describe the motion of a vortex on atomic scales, where this motion is governed by the full quantum many-body structure of the superfluid. One can even go further, and state that we even do not know precisely what a vortex should be on these small scales. Its very definability as a stable topological object essentially depends on the usage of a large scale approach.

Before we describe the problems inherent in vortex nucleation theory, to clarify terminology we use here and further in this work, it is advisable to fix some notions. The treatment of a fluid will be called hydrodynamic or large scale, with the frequency of excitations of the fluid approaching zero for large wavelengths, if the underlying atomic structure of the fluid is not relevant for the phenomena under investigation. The fluid is reasonably well, on these large scales, approximated to consist of structureless, pointlike particles. In particular, a flow field can be defined, with the prescription that a volume element moving with a certain flow velocity contains enough particles for the hydrodynamic formulation to make sense.

The definition of semiclassicality is related to the existence of quantum ‘fluctuations’ or, better, the fact that a quantum mechanical variable is indeterminate in its value with respect to the outcome of different measurements. If quantum fluctuations are small against the expectation value of a quantum operator, we speak of the semiclassical limit. The notions of hydrodynamic and semiclassical in the dense superfluid helium II are mutually corresponding to each other, and a hydrodynamic treatment on macroscopic scales has semiclassical accuracy. A mean field theory asserts that it is reasonable to replace the field operator by its expectation value, in particular within expressions of order higher than the second in the operator and its conjugate (an example is to replace two of the field operators in the quartic interaction term of the second quantized Hamiltonian by their expectation value). It therefore makes the assumption that the system can be treated on any scale by making use of such a replacement. This procedure certainly cannot be used in the case of helium II. Though, on large scales, this dense superfluid is, to a good approximation, described by an order parameter function $\phi$, on the atomic scales of order the coherence length $\xi$, it is necessary to solve the full second quantized problem. There the indeterminacy of, e.g., the operator particle density is as large as its expectation value.

A measure of the applicability of mean field theory can be obtained if we consider the effective strength $g$ of interaction. It can, in the limit of long wavelengths, be defined to be the spatial integral of the two-body interaction of $^4$He atoms, and is related to the $s$-wave scattering length $a$ [2], the speed of sound $c_s$, and the bulk density $\rho_0$ by $g = 4\pi \hbar^2 a/m = mc_s^2/\rho_0$ (repulsive interaction, $a > 0$, [3]). A mean field treatment is useful if $\rho_0 a^3 \ll 1$ [4], which implies $a = \pi (mc_s^2/\hbar)^2/\rho_0 \ll d$, with $d \equiv \rho_0^{-1/3}$ the interparticle spacing. Using the data of [5], we have in helium II (at $p \simeq 1$ bar), $d \simeq 3.58 \, \text{Å}, c_s \simeq 238 \, \text{m/s}$ and thus $a \simeq 8 \, \text{Å}$, increasing further to approximately double this value at solidification pressure. The scattering length, that is, the effective range of the scattering potential of the atoms for long wavelength excitations, is larger than the interparticle spacing, and the condition for the applicability of mean field
theory is violated. This fact entails that one has to consider helium II as a strongly coupled system, which has caused quite formidable computational efforts trying to understand its behaviour on sub-a scales [32]–[37]. Based on the above considerations, we will understand a dense, strongly coupled superfluid as one fulfilling the condition \( a \gtrsim d \) and, correspondingly, a dilute, weakly coupled superfluid as one having \( a \ll d \), the Bose-Einstein condensed atomic vapours [3] being examples of this kind of superfluid. The condition \( a \gtrsim d \) implies that, in strongly coupled superfluids, the coherence length is of order the interparticle spacing.

We have already mentioned at the beginning that there are two classes of excitations related to dissipation, the first being Landau quasiparticles, the second one defects in the order parameter. There arises the question if these different branches of excitations merge in a certain manner, approximately for a wavevector \( 2\pi/d \) corresponding to the interparticle spacing. The roton, i.e. the excitation at the local minimum of the excitation spectrum of superfluid helium II, has a wavevector \( k_r \) which is very close to this value, \( k_r \approx 2\pi/d \). The idea that the roton could correspond to a vortex ring of the smallest possible size (namely, such that just one atom can pass through it), goes back to the seminal papers of Feynman [8]–[10]. Evidence for such an identification would indeed provide justification to speak of some kind of, at least, similarity on the scale of the interparticle distance, of the two types of excitations. There is still a debate going on about this possibility (recent contributions are found, e.g., in [32],[35]) In this work, we will, however, not enter into such a discussion, for it is evidently still a long way to a complete understanding of the microscopic dynamical behaviour of vortices, as even their stationary microscopic character (provided that a vortex ring of atomic size is properly definable at all), is not yet completely clear. We will take the (classic) point of view that a vortex is a topological object, well-defined as a defect structure in the order parameter. Then, a vortex ring of atomic size is termed virtual, because it does not constitute a topological object on this scale.

The lack of a microscopic idea of vortex motion makes it necessary to resort to a hydrodynamic theory of the motion of a vortex, be this motion in real or imaginary time, the latter of interest for the tunnelling processes we intend to investigate. In a hydrodynamic treatment, the existence of the vortex as a semiclassical object has to be assumed \textit{ab initio}. No details of the underlying microscopic dynamics, i.e. of the actual nucleation event, are to be described in such a theory, but only the laws which rule vortex motion on curvature scales well beyond the atomic one. The microscopic dynamical behaviour of the vortex is, in such a description, bound to appear only in cutoff parameters determining the borderline to the microscopic realm. We will consequently consider the vortex motion as it results from the Lagrangian in terms of a collective co-ordinate for the vortex, which is useful as long as the curvature radii of the line described by this co-ordinate are much larger than \( \xi \). Such an approach in terms of a collective co-ordinate makes sense and is physically meaningful, if we additionally assume that the potential barriers, through which the vortex has to tunnel, have themselves effective curvature radii well beyond a scale \( O(\xi) \). If tunnelling events can be actually observable, then, depends for a given superfluid at a given temperature \( T \) on the ratios \( \xi/d, T^*/T \), where \( k_BT^* \equiv \hbar^2/(md^2) \) is a characteristic quantum energy of the quantum fluid constituted by particles of mass \( m \) and \( T^* \) an associated temperature; for helium II, \( T^* \approx 1 \text{ K} \). This dependence can qualitatively be understood
as follows [1]. Consider a vortex ring of radius $R_0$, which has in the bulk an energy $E(R_0) = (m\rho_0\kappa^2/2)R_0 \ln(R_0/R_c)$ (see section 2.5), where $R_c = O(\xi)$ is the vortex core radius. The relevant quantity to compare this energy with, is the thermal energy $k_B T$. If $E(R_0) \gg k_B T$, quantum tunnelling is exponentially suppressed. Writing the energy in terms of the above quantities, $E(R_0)/k_B T = (T^*/T)(R_0/d) \ln(R_0/R_c)$ (barring a factor of $1/4$ for pair-correlated superfluids). The ratio $\xi/d$ then effectively enters, because the smallest possible value of the radius of the vortex appearing in the fluid is $R_0 \gtrsim R_c = O(\xi)$. Considering the fact that thus simultaneously $T \sim O(T^*)$ and $R_0 \sim O(d)$ are to be fulfilled, helium II is the most promising candidate for quantum tunnelling to happen. The conventional superconductors and $^3$He, with their large $T^*/T$ as well as $\xi/d$ are, already on this ground, ruled out. Hence the only possible candidates remaining for an observation of quantum tunnelling of vortices are, save for helium II, high-$T_c$ superconductors.

Overview

To give the reader a concise impression of what follows, we provide here an overview of the principal directions to be pursued, and ideas to be developed in the three sections of this work to follow. The theory of quantum tunnelling is developed in the next section. It is shown that in the limit of long wavelengths, which we are required to be using in a dense superfluid, the probability of quantum tunnelling is predominantly determined by external geometrical quantities connected to the geometry of the flow.

The tunnelling exponent is separated into a dominant volume contribution solely associated with the tunnelling path of the vortex, which is a contribution independent from the fact that the fluid is compressible or incompressible, and a subdominant area contribution associated with that same path and the fact that the fluid is compressible. The dependence of the dominant volume term in the tunnelling exponent on geometrical quantities is exemplified by the analytically solvable problem of a point vortex in the presence of an ellipse, where the long wavelength, collective co-ordinate treatment is shown to impose strict constraints on the motions allowed for the vortex. In the third section the experimental procedure to observe the temperature independent quantum mechanical triggering of vortex generation, which we wish to explain, is demonstrated. We discuss the data obtained in these experiments, with particular emphasis on the applicability of our predictions in this work. The fourth section describes some aspects of pair-correlated, charged Fermi superfluids. We discuss, in particular, the role which might be played by the existence of bound quasi-particle states for observable vortex tunnelling phenomena in superconductors. The high-$T_c$ superconductors, on account of their small coherence lengths, play a prominent role in these considerations, as emphasised above. It is explained that, even for very low temperatures, in the case of unconventional ($d$-wave) high-$T_c$ superconductors of practically feasible purity, the tunnelling phenomenon is not adequately described by the theory of sections one and two, as this were the case for conventional ($s$-wave), extreme type II superconductors, on a length scale well below that of the magnetic penetration depth.
2 Quantum tunnelling of vortices

2.1 General introduction

The quantum mechanical phenomenon of tunnelling has attracted attention from researchers in theoretical and experimental condensed matter physics, field theory and other areas. It belongs to the most remarkable properties solely pertaining to systems obeying the laws of quantum mechanics. Essentially, quantum mechanical tunnelling in a condensed matter system is described by the motion of a few degrees of freedom (subsumed in what follows into the term ‘particle’) under a potential barrier in configuration space. The particle is assumed to have less energy than represented by the height of the barrier (setting the bottom of the potential equal to the zero of energy). Because of wave-particle duality, a wave function can be associated with the particle. This wave function is damped under the barrier or, in other words, the particle travels with imaginary momentum there. To some extent and with some probability, the particle is thus located under the barrier. It can then even completely penetrate it, if the damping is small enough, getting with some nonzero probability from one side of the barrier to the other, whereas classically this is a completely forbidden process.

There are different means to describe the tunnelling motion. The canonical way is to calculate wave functions for the problem, solving, e.g., the Schrödinger equation for the potential of interest with appropriate boundary conditions imposed \[12, 13\]. The most popular formal means to investigate tunnelling, however, is provided by the calculation of the Euclidean action of the system along the tunnelling trajectory \[6, 14, 15\]. The Euclidean action is obtained from the Minkowski action by rotating to purely imaginary times \(S_e = -iS[t \rightarrow -it_e]\). The time on the imaginary axis of the complex \(t\)-plane will be denoted \(t_e\). If the action is dominated by the classical path in Euclidean time, the tunnelling probability can be calculated in the semiclassical approximation. The corresponding solution of the second order Euclidean classical (field) equations of motion with finite action is called instanton. The name stems from the fact that the instanton is a particle-like object localised in Euclidean time. It exists, so to speak, just long enough for the actual particle to tunnel.

In the semiclassical limit, the tunnelling probability for a given energy \(E\) is taking the form

\[
P(E) = A(E) \exp \left[ -\frac{S_e(E)}{\hbar} \right].
\]

In this relation, \(S_e(E)\) is the Legendre transform of the Euclidean action \(S_e(T_e) \gg \hbar\) as a function of \(T_e\), the Euclidean period of motion \[3\]:

\[
S_e(E) = S_e(T_e) - \frac{\partial S_e}{\partial T_e} T_e = S_e(T_e) - ET_e.
\]

The quantity \(A(E)\), frequently called prefactor, represents the influence of fluctuations around the classical path. It is essentially given by the inverse determinant of the coefficients of the second order deviations from the classical path in the action \[3, 16\].
2.2 The collective co-ordinate action of the vortex

It is a quite commonly accepted wisdom that any complex condensed matter problem remains intractable if we do not single out certain central, collective degrees of freedom, termed in general ‘collective co-ordinates’. This is possible because there are conservation laws and symmetries governing the behaviour of the system as a whole: We can actually describe essential features while not referring explicitly to the $10^{20}$ to $10^{23}$ particles and their interaction.

The obvious choice for the vortex collective co-ordinate is its center $X^i(t, \sigma)$, which also indicates the center of topological stability and thus conserved topological charge. That this co-ordinate represents the vortex sufficiently accurately requires that we consider scales much larger than the vortex core size of order the coherence length $\xi$.

Furthermore, we assume that there is a canonical collective vortex momentum related to this central co-ordinate. The action (2) then is

$$S_e(E) = -i \int_0^{T_e} dt_e \, d\sigma \, \dot{X} \cdot P = -i \oint d\sigma \oint dX \cdot P = -i \oint d\sigma \oint dK \cdot P, \tag{3}$$

where we defined the imaginary differential co-ordinate vector $dK = -i dX$ of the vortex. The parameter $\sigma$ labels points on the vortex string, and $P$ is the canonical momentum per $\sigma$-length.

Note that the co-ordinate differential vectors in (3) are no function of $\sigma$, as the co-ordinate position vectors themselves of course are. The closed time integral indicates that we take the integral over a full period of the motion. That such a periodic motion exists, is of course a highly nontrivial assumption for arbitrary dimension of the phase space. Only in an effectively one-dimensional problem (one spatial dimension), respectively for multidimensional systems separable into such one-dimensional problems (cf. [2] §48), closed phase space trajectories necessarily exist.

2.2.1 Contributions in the tunnelling action

In two dimensions and in a conventional, electrically uncharged superfluid, vortices and charged particles have identical dynamical equations in the hydrodynamic limit. According to the three-dimensional extension of this duality, i.e. as a vortices–charged strings analogy, the vortex Hamiltonian takes the form [17]

$$H_V = \oint d\sigma \sqrt{\gamma} \left[ M_0 c^2 e_\sigma + \frac{1}{2\gamma M_0} (P - qa)^2 + \frac{M_0 \alpha^2}{2\gamma} Q^2 \right] + q \oint d\sigma \left( \frac{1}{2} a_C^0 + d^0_u \right). \tag{4}$$

The first integral represents the self energy of the vortex, which has static, kinetic, and elastic contributions, respectively. The arc length of the line is written as $\sqrt{\gamma} d\sigma$, and the vector $Q$ is a perturbation of some equilibrium string configuration perpendicular to the tangent vector $X'$ of the line. The rest frame mass $M_0$ is given by $M_0 = E_0/c^2$.

$^aK$ is not to be confused with a wave vector. We could have chosen as well to incorporate the $-i$ into the (imaginary) momentum $P$. Crucial is only that $S_e$ is a real quantity.

\[ \text{Ann. Phys. (Leipzig) 9 (2000) 1} \]
where $E_0$ is the logarithmically divergent static self energy of the vortex (per unit length). In helium II this energy reads

$$E_0 = \left[ \frac{N_v^2 \kappa^2 m \rho_0}{(4\pi)} \right] \ln \left( \frac{8R_c/\xi C}{\xi} \right),$$

with $\rho_0$ the bulk number density, $m$ the helium mass, $\kappa = h/m$ the velocity circulation quantum, $R_c$ the infrared cutoff (the local curvature radius of the line), $\xi$ the ultraviolet cutoff (the core size), and $C$ a constant characterising the core structure. The second integral represents the ‘electrostatic’ energy, in which we split the scalar potential $a^0$ into a Coulomb contribution from the interaction with other vortices, $a^0_C$, and the interaction with a nonvortical background flow, $a^0_u$. The vectorial generalization of the stream function $\psi$ of classical hydrodynamics \[27\], is related to $a^0$ by

$$a^0 = \rho_0 \psi \cdot X'.$$

The gauge potentials are derived from the external flow field at the position of the line element at $\sigma$ by a gauge invariant duality relation \[17\]. In its nonrelativistic form needed here, this reads

$$\text{rot} \ a = - \rho X', \quad \text{(5)}$$

$$\partial_t a + \nabla a^0 = X' \times j, \quad \text{(6)}$$

where $j = \rho v$ is the conserved particle current, $\partial_t \rho + \text{div} j = 0$. The quantity of crucial importance in the Euclidean action of constant energy \[3\], determining the tunnelling exponent, is the canonical momentum

$$P = P^{\text{inc}} + P^{\text{kin}} = qa + M_0 \sqrt{\gamma} \dot{X}. \quad \text{(7)}$$

It consists of a contribution $P^{\text{inc}}$, which is related to the Magnus force acting on the vortex, and a second contribution $P^{\text{kin}}$ related to the existence of a nonzero vortex mass, that is, to a finite compressibility and thus finite $c_s$. It follows by integration of \[3\] that the ratio of the momentum contributions $P^{\text{kin}}, P^{\text{inc}}$ is in order of magnitude $\approx (N_v \kappa)/(c_s |X|)(|X|/c_s)$ (neglecting the vortex energy logarithm). Hence for large scales (large curvature radii), and small velocities, that is in the hydrodynamic limit, $P^{\text{kin}}$ is dominated by $P^{\text{inc}}$.

The Euclidean action is given by

$$S_e(T_e) = \int_0^{T_e} dt_e d\sigma \gamma(t_e, \sigma) M_0 c_s^2 \left\{ 1 + \frac{1}{2c_s^2} \dot{Q}^2 + \frac{1}{2\gamma} Q^2 \right\} + q \int_0^{T_e} dt_e d\sigma \left[ a^0 - i a \cdot \dot{X} \right]. \quad \text{(8)}$$

In case that a tunnelling process of constant energy is under consideration, the quantity of interest is the action as a function of constant energy \[3\]. According to the relation \[3\], this action consists of a part related to the vector potential and another part

---

\[b\] In pair-correlated Fermi superfluids, $q = N_v h/2$, where the number density in \[3\] and \[3\] is understood to refer to the “elementary” particles constituting the superfluid, and not to the Cooper pairs.
related to the vortex effective mass ($\dot{X} = \dot{Q}$):

\[ S_e(E) = S_e^{\text{inc}}(E) + S_e^{\text{kin}}(E) = \int_0^{T_e} \oint dt_e d\sigma \dot{X} \cdot [P^{\text{inc}} + P^{\text{kin}}] \]

\[ = \int_0^{T_e} \oint dt_e d\sigma \dot{X} \cdot \left[ -iqa + M_0 \sqrt{g} \dot{X} \right]. \tag{9} \]

The Euclidean action splits into a part $S_e^{\text{inc}}$, due to the interaction of the vortex with an (approximately) incompressible background superfluid, and a part $S_e^{\text{kin}}$ which can be ascribed to the kinetic ('vortex matter') term in the vortex momentum. We will now show that $S_e^{\text{inc}}(E)$ is given by a volume associated with the path the vortex line traces out in configuration space, whereas $S_e^{\text{kin}}$ is connected with an area associated with that path.

To demonstrate this, we integrate relation (5), multiplied with $q$, to obtain ($\rho = \rho_0$)

\[ -\oint \oint P^{\text{inc}}_a dX^a d\sigma = N_v h\rho_0 \int \int \sqrt{g} dX^1 dX^2 d\sigma, \tag{10} \]

where $g$ is the determinant of the coordinate basis on the line (unity for a triad). The closed surface with surface elements of magnitude $dX^a d\sigma$ ($a = 1, 2$ is the index of the two $Q$-directions), encloses the total volume with local element $\sqrt{g} dX^1 dX^2 d\sigma$ traced out by the line on its path. Further, using the gauge freedom for the momentum, we can express the action $S_e^{\text{inc}}(E)$ by the volume integral

\[ \frac{S_e^{\text{inc}}(E)}{\hbar} = 2\pi N_v \rho_0 \int \int \sqrt{g} dZ^1 dZ^2 d\sigma, \tag{11} \]

wherein the co-ordinate differentials are defined to be

\[ dZ^1 = \cos \alpha dK^1 + \sin \alpha dK^2 = -i \left( \cos \alpha dX^1 + \sin \alpha dX^2 \right), \]

\[ dZ^2 = -\sin \alpha dX^1 + \cos \alpha dX^2. \tag{12} \]

The angle $\alpha(\sigma)$ parameterizes in these differentials the local (co-ordinate) gauge freedom for the momentum, of rotations about the line tangent $X'$. It expresses the fact that one degree of freedom is still available, namely that for the direction of the local gauge dependent momentum $P^{\text{inc}}(\sigma)$, even after a local basis on the string has been chosen. For the components of $P^{\text{inc}}$ in the two $Q$-directions $e_1$ and $e_2$ the relation

\[ \partial_2 P^{\text{inc}}_1 - \partial_1 P^{\text{inc}}_2 = N_v h\rho_0 \sqrt{g} \tag{13} \]

obtains \cite{24}, cf. relation (3). The simplest example is a rectilinear line in $z$-direction, for which the local momentum can rotate in the $x$-$y$ plane. The gauge invariant quantity is the integral $\oint \oint P^{\text{inc}}_a dX^a d\sigma$, which is left unchanged by the rotation freedom. The relation (13) implies that in the hydrodynamic limit of $|P^{\text{kin}}/|P^{\text{inc}}| \to 0$, phase space and configuration space become indistinguishable, since the momentum components then become functions of the co-ordinates alone, and are no longer independent variables \cite{18}.

It is illuminating to go back to our ‘electrodynamic’ quantities and rewrite

\[ S_e^{\text{inc}}(E) = iq \int \int B_\sigma \sqrt{g} dX^1 dX^2 d\sigma, \tag{14} \]
where $B_\sigma = -\rho_0$ is the (nonrelativistic) ‘magnetic’ field, pointing antiparallel to the direction of the line tangent. The part $S^\text{inc}_e(E)$ is thus the Aharonov-Bohm type Berry phase $\mathbf{8}$ in the Euclidean wave function of the adiabatically moving quantum object vortex.

The part of the action $S^\text{kin}_e$ explicitly involves the vortex dynamics, and thus can not be calculated by knowledge of the vortex path geometry alone, as this was possible for the part $S^\text{inc}_e$. Treating the influence of the mass as a small perturbation on the vortex path, the ratio of the actions is of the same order. More exactly,

$$
S^\text{kin}_e(E) = \oint \oint P^\text{kin}_a dX^\sigma d\sigma = \oint \oint M_0 \sqrt{\gamma} \dot{Q}_a dX^\sigma d\sigma \\
= \frac{N_v \hbar \rho_0}{4\pi} \Gamma_s \oint \oint \ln(\cdots) \frac{(\dot{Q}_a/c_s)}{\sqrt{\gamma}} dX^\sigma d\sigma \\
\simeq \hbar \rho_0 \ln(\cdots) \xi \oint \oint (\dot{Q}_a/c_s) \sqrt{\gamma} dX^\sigma d\sigma \quad (N_v = 1, \text{He II}).
$$

The last line is valid for a unit circulation vortex in helium II, in which the approximate equality

$$
\kappa \simeq c_s 2\xi \quad \text{(in helium II)}
$$

holds.$\mathbf{9}$ The logarithm is assumed to vary only slightly during the tunnelling process, so that it is written in front of the integral. In the length scale domain of interest, $\ln(\cdots)/2\pi = O(1)$. The dots, $\cdots$, indicate an average over the argument of the vortex energy logarithm.

Essentially, $\mathbf{8}$ tells us that the kinetic part of the Euclidean action in units of $\hbar$ depends on an area (element) multiplied by the (local) velocity of the vortex in units of the speed of sound. The coherence length $\xi$ times this area gives a volume, which multiplied by the bulk number density finally yields a dimensionless action. The full Euclidean action $\mathbf{3}$ thus takes the schematic form (for simplicity again displayed in the unit circulation case of helium II)

$$
\frac{S_e(E)}{\hbar} = \rho_0 \left( 2\pi \Omega^{(d)} + \ln(\cdots) \xi \Sigma^{(d)} (\partial X/(c_s \partial t)) \right).
$$

The volume $\Omega^{(d)} = -i \iiint \sqrt{g} dX^4 dX^\sigma d\sigma$. The effective surface

$$
\Sigma^{(d)} = \oint \oint (\dot{Q}_a/c_s) \sqrt{\gamma} dX^\sigma d\sigma
$$

obtained by integrating over the surface enclosing $\Omega^{(d)}$ is a functional of the vortex velocity scaled by the speed of sound ($d$ indicates the spatial dimension).

To summarize, in a dense, strongly coupled superfluid, the first term dominates the second term in (17) for the following reasons:

$\mathbf{9}$This relation assumes that $\sigma_{LJ} < \xi < a$, with $\xi$ nearer to the lower bound ($a$ is the interparticle spacing, $\sigma_{LJ} = 2.556\,\text{Å}$ the Lennard-Jones parameter of the atomic helium interaction), which is consistent with quantum many-body and density-functional calculations [33, 34, 36, 37].
a. The large scale, collective co-ordinate limit requires that the scales to be considered are much larger than $\xi$. The corresponding volumina and areas have to be very much larger than $\xi^3$ and $\xi^2$, respectively, and hence $\Omega^{(d)} \gg \xi \Sigma^{(d)}$.

b. The area contribution of $S_{\text{kin}}^{\text{inc}}$ is additionally suppressed by the vortex velocity divided by the speed of sound, i.e. $\Sigma^{(d)} \ll \partial \Omega^{(d)}$, where $\partial \Omega^{(d)}$ is the proper surface area enclosing $\Omega^{(d)}$.

This dominance of $S_{\text{inc}}^{\text{inc}}$ over $S_{\text{kin}}^{\text{inc}}$ is in contrast to the case of relativistic (string) objects moving with speeds of order $c_s\ [21, 22]$. Under this circumstance, $S_{\text{kin}}^{\text{inc}}$ is of the same order as $S_{\text{inc}}^{\text{inc}}$ and the action is of order $S_{\text{kin}}^{\text{inc}} / \hbar \approx \rho_0 \ln(\cdots) \xi \partial \Omega^{(d)}$. If one neglects the dependence of the logarithm on the co-ordinates, this is essentially the Nambu action (20), in units of $\hbar$, up to a factor of order unity.

We assumed that the vortex path in phase space is closed. As a consequence, the number of particles in the effective volume on the right-hand side of (17) is quantized according to

$$S_c(E) = (N^{(d)} + \alpha)\hbar \quad \Leftrightarrow \quad S_c(E) / \hbar = 2\pi (N^{(d)} + \alpha). \quad (19)$$

The number of particles in the effective volume (including the small kinetic contribution on the right-hand side of (17)), plays the part of the Bohr-Sommerfeld quantum number in semiclassical quantization. The number $\alpha$ is of the order one and signifies the onset of the microscopic quantum regime. In the semiclassical approximation, $N^{(d)} \gg \alpha$ must hold, so that $N^{(d)} \gg 1$ gives, as usual, a direct measure of semiclassicality.

### 2.3 Geometry of Quantum Tunnelling

#### 2.4 Galilean invariance violation

At the absolute zero of temperature, a homogeneous nonrelativistic superfluid has Galilean invariance, that is, physical contents are invariant under co-ordinate transformations to any reference frame moving at constant velocity. If we approach absolute zero, which is what is actually realized in experiment, we expect the tunnelling rate to make no abrupt change as the temperature is lowered. Thus the result for the rate we obtain at $T = 0$ should also be valid for temperatures slightly above zero (we will estimate the temperatures, for which this is no longer the case, later on).

Because we can always transform to the rest frame comoving with the superfluid, the tunnelling probability at $T = 0$ equals zero if Galilean invariance remains unbroken: In the rest frame there is a tunnelling barrier of infinite height, the logarithmically diverging vortex self energy. Hence it is necessary to explicitly include the violation of Galilean invariance by a flow obstacle into any calculation of tunnelling rates for Galilean invariant superfluids at absolute zero. The necessity of invariance violation for tunnelling to be energetically allowed thus stems in the real superfluid from the fact that it is possible to invariantly transform to the rest frame of the superfluid, for any allowed velocity of flow.

Having thus shown that it is strictly required that Galilean invariance be violated, we are in demand of constructing an explicit solution of the following problem. Given
a vortex in the presence of an invariance breaking flow obstacle, we have to calculate the Hamiltonian energy $H_V$ of the vortex in the Hamiltonian \(H\), as a function of the coordinates, by solving one of the equations of motion corresponding to this Hamiltonian respectively the Euclidean action \(S_e\). We then set $H_V$ equal to a constant $E$, to calculate the vortex trajectory of constant energy, which finally yields the action \(S_e\), respectively \(S_e^{\text{inc}}\), in the form of \(S_{\text{inc}}^{\text{pol}}\). It is obvious that without a very high degree of symmetry, this is a task necessitating a quite formidable computational effort.

We have seen that the problem of determining the dominant contribution $S_e^{\text{inc}}$ is a geometrical problem, because the phase space co-ordinates are functionally dependent on the configuration space co-ordinates. We will therefore restrict the discussion in what follows to cases which elucidate in particular the geometric nature of quantum tunnelling, that is, concentrate on the behaviour of $S_e^{\text{inc}}$, which is a functional of geometrical quantities.

2.5 The vortex half-ring case

To flesh out the discussion which has been so far quite abstract, we now give some quantities relevant in the tunnelling problem for a vortex. Imagine, for concreteness, a singly quantized vortex half-ring of radius $R$ with its circulation axis $e_\phi$ standing perpendicular on the plane $y = 0$. Over the plane is flowing liquid with a velocity $u$ at infinity from right to left (in the negative $x$-direction). Consider, first, the case that the invariance-breaking asperities on the surface are small-scale (it will become clear in a moment what ‘small-scale’ means). Neglecting the elastic and kinetic terms in the Hamiltonian, as well as perturbations of the flow generated by the asperities, the conserved vortex energy is simply

$$H_V = \hbar \rho_0 \int_0^\pi d\phi \left\{ \frac{\kappa}{4\pi} R \ln \left( \frac{8R}{\xi e^C} \right) - \frac{1}{2} uR^2 \right\}.$$  \hspace{1cm} (20)

If we normalize the energy by

$$\tilde{H}_V \equiv \frac{4\pi m}{\hbar^2 \rho_0} H_V,$$ \hspace{1cm} (21)

and solve for the path of constant total energy $\tilde{E}_0$, we get the relation

$$R = \frac{\kappa}{2\pi u} \ln \left[ \frac{8R/\xi}{\exp(C + E_0/(\pi R))} \right].$$  \hspace{1cm} (22)

This equation has two solutions for the radius $R$. One of them is, in the case of $u \ll \kappa/2\pi \xi$ and small $E_0/\pi R \ll C$, located far away from the surface $x = 0$, at

$$R_0 \simeq \frac{\kappa}{2\pi u} \ln \left( \frac{8\kappa/(2\pi u \xi)}{\exp(C)} \right).$$  \hspace{1cm} (23)

The other solution is a half-ring of radius in the order of the coherence length, $R_\xi = O(\xi)$ (which will not be of further interest here; it signifies the path of the (virtual) vortex trapped at the flat boundary).

The closed path in phase space, needed to evaluate \(S_e\), can be obtained as follows. Assume that the small asperities are approximately of the shape of oblate rotation
ellipsoids with half-axes $b$ perpendicular and $a$ parallel to the flow and define the
\textit{sharpness} $\beta = b/a > 1$. Then, the half-ring trajectory over such a small ellipsoid,
collinear with the ring axis, were given by
\begin{equation}
R^2 = b^2 + \beta^2 K^2, \tag{24}
\end{equation}
using the complex co-ordinate $K = -iZ$, and provided that the vortex exactly follows
the surface (up to a constant $O(\xi)$). This trajectory hits the one far away from the
surface, given by $R_0 = \text{const.}$, at $K_0 \approx R_0/\beta$ (provided that $b \ll R_0$). The closed phase
space trajectory thus begins at the ellipsoid top ($K = 0, R \approx b$), then propagates along
the line given by (24), meets the constant $R_0$ at $K_0$, then follows this constant to $-K_0$,
and finally follows the branch of (24) for negative $K$ back to the ellipsoid top.

The tunnelling exponent, using the gauge $P_X = (1/2)\hbar \rho_0 R^2$ for the momentum, is
thereby
\begin{equation}
S_{\text{inc}}(\tilde{E}_0) = 2\hbar \rho_0 \int_0^{\pi} d\phi \int_0^{K_0} dK \frac{1}{2} (R_0^2 - \beta^2 K^2)
= \hbar \rho_0 \frac{2\pi}{3} R_0^3 / \beta = \hbar \rho_0 \Omega^{(3)}. \tag{25}
\end{equation}
The tunnelling volume and thus the exponent may consequently be reduced and the tunnelling probability enhanced if a surface with sufficiently sharp peaks perpendicular
to the flow is present. In particular, if we were able to reduce the small half-axis
to $a \gtrsim \xi$, and still have $b \gg a$, we could reduce the tunnelling volume to $\Omega^{(3)} \approx
(2\pi/3)R_0^3 \xi(R_0/b)$. It is worthwhile to point out that this reduction is not due to an
enhancement of the flow velocity at the ellipsoid top: For any value of $\beta$, because of
cylindrical symmetry, the velocity at the top is exactly $2u$, just as for the half-sphere.

The trajectory (24) is correct only in lowest order of $\beta^{-1}$. In particular, for $\beta = 1$
($a = b$), the half-sphere, the result for the tunnelling exponent (25) is exact in the
low velocity limit. This can be shown by solving the problem for the sphere exactly,
which has been done in [25]. The result for the stream function part of the potential
in the Hamiltonian (4), $a_0 = -\rho_0 \psi$, corresponding to Stokes’ stream function $\psi$ [27],
has been checked by the author, and is in Coulomb gauge
\begin{equation}
\frac{1}{2} \psi_C + \psi_a = R \frac{\kappa}{4\pi} Q_{1/2}(w) - \frac{1}{2} u R^2 \left( 1 - \left( \frac{a}{\sqrt{R^2 + Z^2}} \right)^3 \right), \tag{26}
\end{equation}
with $w \equiv 1 + (1/2)(Z^2 + R^2 - a^2)/(aR)^2$. The function $Q_{1/2}$ is a Legendre function of
the second kind ([23], No. 8.821), a solution of the Poisson equation $\Delta \psi = \kappa \delta^{(2)}(x-X)$
for the combined spherical-cylindrical symmetry of the problem at hand.

The problem of a undeformed, massless half-ring situated at a half-sphere, with
their axes coinciding, is actually the only nontrivial problem in three spatial dimensions
solvable with reasonable effort analytically. It has just the maximal symmetry in three
dimensions. It does not, however, contain the crucial ingredient for a full geometrical
analysis of quantum tunnelling: The curvature of the flow obstacle is constant. The
result that, for the sphere, the vortex half-ring follows exactly the surface (up to a
constant distance of order $\xi$), has led in [25] to the assumption that it is admissible
to continue this result to the ellipsoid case, in the manner of (24). We will show
in the subsection to follow that this is incorrect. The analytic continuation of the sphere result to the ellipsoid case of arbitrary \( \beta \) is incompatible with the geometric requirements imposed by a hydrodynamic, collective co-ordinate treatment.

2.6 Analytical solution in two dimensions

A treatment of our (boundary) problem in two spatial dimensions \([26]\) is advantageous for the following reasons:

- **i.** The additional spacelike co-ordinate, namely the arc length parameter, complicates the analysis enormously because of the locality of any variable in \( \sigma \) and the existence of elastic energy.

- **ii.** Even if we treat an undeformed ring or half-ring, i.e. neglect elastic energy and locality altogether, any 3d problem which has not the maximal symmetry described above, requires a multitude of image vortices and gets quite intractable.

- **iii.** In two dimensions, we have available the tools of conformal transformation, allowing for comparatively simple calculations of analytical solutions.

- **iv.** The geometrical features, which are dominant in the hydrodynamic, collective co-ordinate limit, as expounded at length above, are most clearly seen in two dimensions.

2.6.1 The solution for the circle

The basic solution from which we start is that for a vortex in the presence of a half-circle of radius \( d \) at an otherwise flat boundary (cf. Fig. 1). The complex plane of this original solution is called the \( Z \)-plane (the uppercase letter does here not imply that a vortex position is meant). The imaginary part of the complex potential \([27]\) gives the stream function \( \psi = \Im[w] \), whereas the real part is the usual velocity potential. It follows that a single vortex at \( Z_1 \) has complex potential \( w(Z) = -i(\kappa/2\pi) \ln[Z - \bar{Z}_1] \).

The boundary condition to be fulfilled is obviously that there be no flow into the surface consisting of the line \( Y = 0 \) and the half-circle. This amounts to the requirement that the line \( Y = 0 \) (for \( |Y| > d \)), \( Y = \sqrt{d^2 - X^2} \) (for \( |Y| \leq d \)) is a streamline of constant \( \psi \equiv 0 \). Such a requirement can be met by using the technique of image charges quite familiar from electrostatics: Our vortex problem is completely equivalent to that for a ‘charge’ situated near a perfectly conducting surface, with no tangential ‘electric’ field.

The complex potential generated by the image vortices and acting on the vortex at \( Z_1 \) is then given by

\[
   w_i(Z_1) = -i \frac{\kappa}{2\pi} \ln \left[ \frac{(Z_1 - \bar{Z}_1) \left( d^2/Z_1 - \bar{Z}_1 \right) }{d^2/Z_1 - Z_1} \right].
\]

The first factor in the numerator stems from the image vortex at the plane \( Y = 0 \) with complex potential \( w(Z) = -i(\kappa/2\pi) \ln[Z - \bar{Z}_1] \) (which has to be present even without the circle), the second one is obtained by the circle theorem \([27]\) as the image of the original vortex at the circle. Finally, the potential of the remaining \( +\kappa \)-circulation
vortex inside the circle, contributing in the denominator of the logarithm, completes the image vortex system, again by the circle theorem. The first term in the denominator of the logarithm is incorporated into the static self energy of the vortex, $E_{self} = (m \rho_0 \kappa^2/4\pi) \ln \left( |Z_1 - \bar{Z}_1|/\xi \right)$, which is cut off by $\xi$ and equal to half the energy of a vortex pair separated by $|Z_1 - \bar{Z}_1|$. The expression for the potential of (4) is thus

$$\psi_C = -\left( \kappa/2\pi \right) \ln \left( \left| (d^2/Z_1 - \bar{Z}_1)/(d^2/Z_1 - Z_1) \right| \right), \quad (28)$$

which is just the counterpart of $\psi_C$ in (26) in two dimensions. The complex counterpart of $\psi_u$, in turn, is

$$w_u(Z) = u \left( Z + \frac{d^2}{Z} \right), \quad (29)$$

giving the complex potential of flow from right to left (in the negative $X$-direction).

The full energy of the point vortex, neglecting compressibility effects, and using the normalisation $[21]$, is thence given by the expression

$$\tilde{E}(Z_1) = \ln \left| \frac{(Z_1 - \bar{Z}_1)}{\xi} \frac{(d^2/Z_1 - \bar{Z}_1)}{(d^2/Z_1 - Z_1)} \right| - \frac{4\pi u}{\kappa} \Re \left( Z_1 + \frac{d^2}{Z_1} \right). \quad (30)$$
2.6.2 Conformal transformation

A conformal transformation is a co-ordinate transformation leaving angles invariant, that is, the metric is multiplied by a conformal factor which is the transformation’s Jacobian determinant. What we want to do is to map by a conformal transformation the region outside a boundary surface with varying curvature radius, lying in the (target) $z$-plane, to the domain outside the circle, which is in the (original) $Z$-plane. Any such transformation can be written as a holomorphic function of $Z$ (save for singular points, such as vortex centers), in the form

$$z = a_0 Z + \sum_{n=0}^{\infty} b_n Z^{-n}, \quad (31)$$

where $a_0, b_n$ are some coefficients and $Z = d \exp(i\chi)$ is on the circle (we omitted a constant, indicating a change of $z$-plane origin).

2.6.3 The ellipse solution

We would like to invert relation (31) to obtain the solution for the boundary surface directly from that for the circle, which we have already obtained above. Easiest to perform is this inversion if we let $b_0 = 0, b_n = 0$ for $n > 1$. Furthermore, if $a_0 \equiv 1$ by proper normalizing choice of scale, we are led to the celebrated Joukowski transformation

$$z = Z - l^2/4Z, \quad (32)$$

which maps the outside of an ellipse with half-axes $a, b$ (where $a < b$) to the outside of a circle of radius $d = (a + b)/2$. The parameter $l$ is defined by $l^2 = b^2 - a^2$. The inversion of this transformation is cast into the form

$$Z = \frac{1}{2} \left( z + \sqrt{z^2 + l^2} \right) = \frac{l}{2} \left( \sinh \zeta + \cosh \zeta \right) = \frac{l}{2} \exp[\zeta]. \quad (33)$$

with the aid of elliptic co-ordinates, defined by $(\zeta = \chi + i\eta)$

$$z = l \sinh \zeta = l (\sinh \chi \cos \eta + i \cosh \chi \sin \eta) = x + iy. \quad (34)$$

The lines of constant $\eta$ and $\chi$ are confocal ellipses and hyperbolae, as follows from

$$\frac{y^2}{l^2 \sin^2 \eta} - \frac{x^2}{l^2 \cos^2 \eta} = 1, \quad \frac{x^2}{l^2 \sinh^2 \chi} + \frac{y^2}{l^2 \cosh^2 \chi} = 1 \quad (35)$$

The co-ordinate basis $e_\chi = \partial/\partial \chi, e_\eta = \partial/\partial \eta$ is an ortho-basis with the conformal metric $g_{ij} = l^2 (\cosh^2 \chi \cos^2 \eta + \sinh^2 \chi \sin^2 \eta) \delta_{ij}$. The normalized energy $\tilde{E}$ as a function of the elliptic vortex co-ordinates $\chi_1, \eta_1$ takes the form

$$\tilde{E}(\chi_1, \eta_1) = \ln \left[ \frac{a + b}{\xi} \exp(\chi_1 - \chi_0) |\sin \eta_1| \sinh(\chi_1 - \chi_0) \right] \left[ \frac{\exp(\chi_1 - \chi_0) |\sin \eta_1| \sinh(\chi_1 - \chi_0)}{(\sinh^2(\chi_1 - \chi_0) + \sin^2 \eta_1)^{1/2}} \right] - \frac{4\pi u(a + b)}{\kappa} |\sin \eta_1|, \quad (36)$$
where $\chi_0 = \text{artanh}(a/b)$ is the co-ordinate specifying the (half-)ellipse surface at the boundary, i.e. the half-axes of the ellipse are given by $a = l \sinh \chi_0$, $b = l \cosh \chi_0$.

For direct comparison with the half-ring case treated in subsection 2.5, which had a quite simple geometrical meaning, we will investigate the paths of constant energy $\tilde{E}(\chi_1, \eta_1) \equiv \tilde{E}_0$ mainly in the case of small velocity $u$. The notion of ‘small’ requires some more care in two dimensions as compared to the cylindrically symmetric half-ellipsoid case in three dimensions. Whereas in the latter the velocity enhancement at the top is always $2u$, in the 2d case it is $2(b/a)u$. Hence it is required that we restrict the velocity at infinity to

$$u \ll \frac{a}{2b} v_L,$$

such that the velocity at the top is well below the critical velocity

$$v_L \equiv \frac{\kappa}{2\pi \xi},$$

which gives a measure of the onset of many-body quantum physics in the atomic superfluid helium II. For Fermi superfluids, the corresponding critical velocity is the pair-breaking velocity of Cooper pairs. We will henceforth refer to $v_L$ as ‘Landau velocity’ and generally scale velocities with it. Numerically, the actual Landau critical velocity of roton creation $\simeq 59 \text{ m/s}$ at $p \simeq 1$ bar equals $v_L$ if $\xi \simeq 2.7 \text{ Å}$ is taken.

The approach of the vortex to the ellipse surface is closest at the top ($\eta = \pi/2$). Let us calculate the distance at the top as a function of the constant energy $\tilde{E}_0$. In general, a small distance interval $\delta s$ is given by

$$\delta s = \sqrt{g_{\chi\chi}(\chi_0)} \delta \chi = (a^2 \sin^2 \eta + b^2 \cos^2 \eta)^{1/2} \delta \chi.$$

We define a quantity $s$ by

$$s = a \delta \chi_c \equiv a \delta \chi(\eta = \pi/2).$$

At the ellipse top, $s = \delta s(\eta = \pi/2)$, provided that $\delta \chi_c \equiv \chi_1(\eta = \pi/2) - \chi_0 \ll \chi_0$. The actual distance of the vortex to the top is somewhat different if this inequality does not hold. We do not further dwell on this difference here for the sake of simple argument and take (39) as a definition of the quantity of distance $s$.

Evaluating (36) at $\eta = \pi/2$ for small velocities, and assuming that $\tilde{E}_0$ is small enough for $\delta \chi_c \ll 1$ to hold, we get $\delta \chi_c \simeq \xi \exp[\tilde{E}_0]/(a + b)$ and therefore

$$s \simeq \frac{a}{a + b} \exp[\tilde{E}_0] \xi \simeq \frac{a}{b} \exp[\tilde{E}_0] \xi \quad (\eta = \pi/2, u \text{ small}).$$

On the other hand, far away from the half-ellipse, at the flat boundary, the distance is, in the low velocity limit, given by $\delta s \simeq (1/2) \xi \exp[\tilde{E}_0]$. We observe that the vortex comes closer to the ellipse by a ratio $2a/b$ (for $b \gg a$) than it was far away from the ellipse. If $a = b$, that is, in the case of the circle, the distance remains the same.

\(^{d}\)It should, however, be pointed out that the very notion of (classical) velocity becomes questionable on $\xi$-scales. On these scales, it is more appropriate to refer to (mean) current densities.
2.6.4 Geometric restrictions

Now, if we are bound to remain in the realm of the hydrodynamic description of a collective co-ordinate vortex, we have to impose that the total distance $\Delta s$ of the vortex to the ellipse always exceeds a quantity $O(\xi)$:

$$\Delta s = \int_{\chi_0}^{\chi_1} \delta s \geq O(\xi). \quad (41)$$

The quantity $O(\xi)$ means ‘a value in the order of $\xi$ by definition’, as there can obviously be no sharp distinction between the ‘inside’ and ‘outside’ of a quantum vortex. Additionally, the choice for the lower limit value of the total distance $\int \delta s$ in units of $\xi$ depends on the value of the core constant $C_0 = O(1)$, parameterizing the many-body core structure in the vortex energy logarithm $\ln[R/(\xi \exp[C_0])]$. For the point vortex considered, which has $R = 2Y, \exp{C_0} = 1$; for a ring vortex we chose the parameterization $8 \exp[C_0] = \exp[C]$, cf. [24].

It is apparent from (40) that if the energy of the vortex is sufficiently small (in particular, if it is zero) and the ratio $b/a$ is sufficiently large, the condition (41) will be violated and a hydrodynamic collective co-ordinate description invalid. Thus, there is a minimum vortex energy required for the whole formalism we employ here to retain its validity. Expressing this energy as a function of the parameters involved, we have, for general velocities ($\delta \chi_c \equiv \delta \chi(\eta = \pi/2)$):

$$\tilde{E}_0 = \ln \left[ \frac{a + b}{\xi} \exp{\delta \chi_c \tanh{\delta \chi_c}} \right] - \frac{4\pi u(a + b)}{\kappa} \sinh{\delta \chi_c} \quad (42)$$

$$\cong \ln \left[ \left( 1 + \frac{b}{a} \right) \frac{s}{\xi} \right] - \frac{2u}{v_L} \left( 1 + \frac{b}{a} \right) \frac{s}{\xi}. \quad (43)$$

the last line valid provided that we can approximate, with reasonable accuracy, e.g., $\tanh{\delta \chi_c} \simeq \delta \chi_c$. The energy $\tilde{E}_0$ is the energy needed by the vortex to remain completely describable in terms of a collective co-ordinate on its way along the ellipse. The quantity $s$ is defined in (39), and has a lower bound related to the prescription (41).

We have depicted the normalised potentials corresponding to different velocities in units of $v_L$ and ratios $s/a$ in Figs. 2 and 3. The potentials in these figures correspond to the real space ellipse shown in Fig. 4 which has $b/a \simeq 5.7$ and where $s/a \approx 2$. For clarity, we have additionally displayed the shape of the barriers in the direction of the $y$-axis (i.e. the $e_\chi$-axis at $\eta = \pi/2$) in Fig. 4. Whereas for $u = 0.04$ the semiclassical approximation is still applicable, for $u = 0.08$ this is no longer the case. It is approximately for this velocity that the barrier tends to zero and the tunnelling distance approaches $\xi$. One can see here explicitly that $v_L$ indeed represents a sensible measure for a critical velocity, because the local velocity at the top is $\simeq v_L$, if the velocity at infinity takes the value $u \simeq 0.088$.

The situation we encounter in real space is shown in Figure 5. We visualize the quantum core size as the shaded area around the vortex center, having the $O(\xi)$-radius used in (41). The vortex on path 1 is not able to pass the ellipse without some part of this shaded area covering the ellipse, but the vortex on path 2 with energy $\tilde{E}_0$
Fig. 2  Shape of the potential barrier \( \chi \) with the choice \( \chi_0 = 0.175 \) for the velocity \( u = 0.04 \) in units of \( u_L = \kappa/2\pi\xi \). The corresponding real space ellipse with \( b/a \approx 5.7 \) is shown in Fig. \( \text{[1]} \). The ratios \( a/\xi = 2 \), \( s/\xi = 1 \) and thus \( s/a = 1/2 \). The zero of this normalised potential energy is shifted by \( \tilde{E}_0 \), defined in (42). The contour lines of the vortex paths of constant energy shown are those for \( \tilde{E} = \tilde{E}_0 \).

avoids the ellipse surface completely. That the validity of the hydrodynamic approach enforces that we introduce another geometrical quantity, \( s \), the vortex distance of closest approach, which implies a lower bound of vortex energy, \( \tilde{E}_0(s) \), is an observation of general character. It is of relevance for any attempt to describe tunnelling in a realistic, non-spherical geometry, \( i.e. \) when the boundary and thus the path of the tunnelling object near it is not of \( S^n \) symmetry. A hydrodynamic collective co-ordinate description is valid only if the quantum core structure of the tunnelling object is not touched upon during its motion along the boundary. A pinning potential for the vortex moving in the superfluid stems in general from some flow obstacle, in our case the ellipse. Any phenomenological ansatz for a pinning potential usually employed in tunnelling calculations, which has curvature perpendicular to the applied flow larger than parallel to the flow will have to take into account that the object can approach the surface within its core size.

We now come back to the determination of the two paths of constant energy we need to construct the closed path in Euclidean phase space. Assuming \( \delta\chi \gg 1 \), we get the path far away from the ellipse to which the vortex has to tunnel. It is, with
decreasing magnitude of $u$, given by the relations (as a reminder, $b \gg a$):

\begin{align}
    l \exp[\chi_1] \sin \eta_1 &= \frac{\kappa}{2\pi u} \ln \left[ \frac{l}{\xi \exp[\tilde{E}_0]} \exp[\chi_1] \sin \eta_1 \right] \\
    2Y_1/\xi &= \frac{v_L}{u} \ln \left[ \frac{2Y_1}{\xi \exp[\tilde{E}_0]} \right] \\
    2Y_0/\xi &= \frac{v_L}{u} \ln \left[ \frac{v_L}{u \exp[\tilde{E}_0]} \right],
\end{align}

The relevant path is the solution which has $Y_1 \gg b$ (the second solution of the equations above is just the vortex trapped at the boundary, far from the ellipse). The last relation is exactly analogous to (23), which is valid in the low velocity limit of three dimensions.

The path at the ellipse with energy $\tilde{E}_0$ does have approximately constant $\delta \chi \ll 1$ if $s \ll a$. The shape of this path can be obtained (to lowest order in $\delta \chi$) by solving the equation

\begin{equation}
    (a + b) \sin \eta_1 = \frac{\kappa}{4\pi u \delta \chi} \ln \left[ \frac{(a + b) \delta \chi}{\xi \exp[\tilde{E}_0]} \right]
\end{equation}

Around the ellipse top, within a large range of $\eta$-values, the solution is approximately given by $\delta \chi_c \simeq s/a$. This can also be inferred from Figure 2.
2.6.5 The tunnelling area

To describe the motion of the vortex in momentum space, we choose a gauge for the momentum which is most appropriate to the symmetry of our problem. In Cartesian co-ordinates, this is $P_X = h\rho_0 Y$, i.e. the gauge momentum equals the physical momentum (i.e. Kelvin momentum) for an unconstrained vortex in the bulk superfluid. We will evaluate the action in the limit of small velocities. In this limit the path to which the vortex has to tunnel (named in what follows $Y_E$), is given by a constant, $Y_0$ in (46). We have also seen at the end of the preceding section that the vortex remains approximately on the same ellipse, having the elliptic co-ordinate \( \chi_1 = \chi_0 + \delta\chi_c(\pi/2) \simeq \chi_0 + s/a \), while it is moving around the ellipse (path 2 in Figure 5). Calling this path $Y_E$, we can write
\[
Y_E^2 = l^2 \cosh^2(\chi_0 + \delta\chi) + \tanh^{-2}(\chi_0 + \delta\chi)K^2
\simeq l^2 \cosh^2(\chi_0 + \delta\chi_c) + \tanh^{-2}(\chi_0 + \delta\chi_c)K^2,
\]
using the imaginary co-ordinate $K = -iX$. Then, the integral in (3) takes the form
\[
\frac{S_{\text{inc}}}{\hbar} = 2\rho_0 \int_0^{K_m} (Y_N - Y_E) dK,
\]
where $K_m$ is the point at which the trajectories $Y_N$ and $Y_E$ meet in complex phase space. This point, determined in the low velocity limit by the solution of $Y_E = Y_0$, can be shown to be
\[
K_m \simeq \tanh(\chi_0 + \delta\chi_c) Y_0
\simeq \left(\frac{a}{b} + \frac{s}{a}\right) \frac{v_L}{u} \ln \left[\frac{v_L}{u \exp[-E_0]}\right],
\]
(50)
neglecting the first term in (48), which is possible if $Y_0 \gg l \cosh(\chi_0 + \delta\chi) \equiv b_\chi$. The last line is valid provided that both $\chi_0, \delta\chi_c \ll 1$.

Now, the integral (49), with $a_\chi = a(\chi_0 + \delta\chi) = l \sinh(\chi_0 + \delta\chi)$, takes the form
\[
\frac{S_{\text{inc}}}{\hbar} \simeq 2\rho_0 \left(K_mY_0 - \frac{b_\chi^2}{2a_\chi} K_m^2\right) = \rho_0 \tanh(\chi_0 + \delta\chi_c) Y_0^2.
\]
(51)
Fig. 5 Two vortex paths of constant energy near the ellipse. Whereas the vortex on path 1 with approximately zero energy, $E \approx 0$, does not manage to pass by without coming closer than $O(\xi)$, the second one, having energy $E = E_0$, defined in (42), is able to do so. The velocity $u$ is to be understood that of the flow ‘at infinity’.

The above integral corresponds to the shaded area in Figure 5. Neglecting the small cusp at the bottom reducing slightly this area, as we did in the third line of (51), leads to a volume law of the form

$$\frac{S_e^{\text{inc}}(E)}{\hbar} = \rho_0 \Omega^{(d)} = \rho_0 \beta V_N^{(d)}.$$  

(52)

Here, the effective sharpness $\beta = \tanh(\chi_0 + \delta \chi_c)$ is a measure of the maximal eccentricity $e = \sqrt{1 - \beta^2} \simeq 1 - (1/2)\beta^2$ the vortex path is allowed to have, under the condition that the vortex should remain completely within the hydrodynamic, collective co-ordinate domain.

The area of Fig. 5 thus does have a lower limit. In general, $\beta$ characterizes the effective dimension of the vortex escape path, i.e. the relative degree to which this path is confined to $n$ dimensions by the presence of an asperity which is effectively $n$-dimensional. The tunneling volume $V_N^{(d)}$ is that for a vortex escape path of $O(d-1)$

\footnote{We refer to the contribution of $S_e^{\text{inc}}$ as a volume contribution in general, according to \cite{[17]}. In two dimensions, of course, a volume is an area and an area is a length in conventional terms.}
symmetry, which is the highest possible symmetry if one preferred direction, namely that of the external current, is given. In the $d = 2$ case treated here at length, we have $n \geq 1$, $V^{(2)}_N = Y_0^2$ for a single vortex and correspondingly $V^{(2)}_{N} = 2Y_0^2$ for a vortex pair. In three dimensions ($d = 3$) we have $V^{(3)}_N = (2\pi/3)R_0^3$ for a half-ring with radius $R_0$ and double this value for a full ring. The effective sharpness will be reduced (i.e. the value of $\beta$ larger) for the 3d case (cf. the equation (25)), in analogy to our analytical findings for the vortex in the plane. In order of magnitude, the sharpness will in any dimension be given by the product of the ratios of the curvature radii parallel and perpendicular to the flow of the allowed vortex path.

We conclude with an estimation of the kinetic contribution in the action, using the last line in (15). We will also use that in lowest order of perturbation theory the velocity of the vortex equals the local flow velocity $v_s$ in the incompressible superfluid (the Kelvin-Helmholtz theorem). The velocity maximum (at the ellipse top) we set $v_{\text{max}} = \max(|v_s|)$. Then,

$$S_{\text{kin}}^E = \frac{h\rho_0}{2\pi} \ln(\cdots) \xi \int (\dot{Q}_a/c_s) dX^a$$

$$\ll 2h\rho_0 \frac{\ln(\cdots)}{2\pi} \xi \frac{v_{\text{max}}}{c_s} K_m$$
Tunnelling of defects in superfluids

\[ T = \ln(\cdots) \frac{\xi}{Y_0} \frac{v_{\text{max}}}{c_s} S_{\text{kin}}^c(E). \]  

(53)

The contribution of the vortex mass is thus at least suppressed by two small factors: \( \xi/Y_0 \) and \( v_{\text{max}}/c_s \), contributing both in equal measure to the fact that \( S_{\text{kin}}^c \) is negligible as compared to \( S_{\text{inc}}^c \).

2.6.6 Summary

We now summarize this extended investigation of the analytically soluble 2d problem, emphasising in particular its crucial outcome.

Starting from the 'electrostatic' problem of a single point vortex situated near a half-circle at a boundary, we derived the vortex energy by conformally transforming via the (inverse) Joukowski transformation to the half-ellipse solution. We concluded from the general expression for the vortex energy (36), that it is necessary to introduce the geometric constraint (41), expressing the limits of the hydrodynamic collective coordinate formalism under consideration. Calculating the tunnelling volume (area), we have seen that it assumes the general form (52). The tunnelling volume (area) \( \Omega(2) \) cannot be reduced to a lower bound given by the sharpness of the ellipse, \( \tanh \chi_0 \), but has to be larger, \( \beta = \tanh(\chi_0 + \delta \chi) \), if we are bound to remain within the domain of the approach which has been employed.

It is to be noted that we expressed the energy of the vortex in units of the energy \( \hbar^2 \rho_0/(4\pi m) = m\rho_0 k^2/4\pi \simeq 0.82 \, \text{K/Å} \) (at \( p \simeq 1 \) bar and in three dimensions). For realistic values of the parameters in (42), the values of \( E_0 \) cover the same range as the phonon-maxon-roton spectrum. Energywise, the trapped small scale vortex thus cannot be distinguished from an elementary excitation of the superfluid. It could have been excited thermally and remained trapped at a pinning center during the cool-down of the superfluid to very low temperatures.

From the above analysis it thus follows that it is semantically in general not quite appropriate to employ the widely used term 'vortex nucleation' for vortex tunnelling investigated within a hydrodynamic, collective coordinate theory. If we define 'nucleation' to mean creation from the zero of energy, we have seen that nucleation is not amenable to such a description under general circumstances. Experimentally, it will be impossible to distinguish the tunnelling of a small energy vortex at a rough boundary from the true nucleation event of a nascent vortex there, if no direct means to control the microscopic dynamics can be provided. It remains, of course, to be explained how a vortex of such small energy and size (i.e. one with small distance of \( O(\xi) \) to the boundary), can be defined and described quantitatively. The present approach can only fix the tunnelling rates of a vortex if one presupposes that the vortex can be described by a collective coordinate along its whole path. This, however, will be true if the size of the vortex is sufficiently larger than \( O(\xi) \). Then, the dissipation-free motion of the vortex with constant energy and the purely geometric nature of the problem lead to strict bounds for the possible values of the semiclassical tunnelling exponent, directly related to geometrical quantities.
2.7 The prefactor

Any discussion of quantum tunnelling is incomplete without at least an estimation of the prefactor $A(E)$ in (1). If the quantum fluctuations, or better, the indeterminacies of the vortex position and momentum vanish, the tunnelling probability does the same, because the very process of quantum tunnelling stems from this quantum uncertainty of the vortex in phase space. The crucial advantage of our WKB-like investigation lies in the fact that the behaviour of the exponent, the Euclidean action of the instanton, dominates any dependencies of the prefactor on observable quantities as long as $S_e(E)/\hbar \gg 1$, i.e. as long as we are in the semiclassical limit. We will write (1) in the form

$$P(E) = \exp \left[ -\frac{S_e(E)}{\hbar} + \ln A(E) \right],$$

(54)
to compare $-S_e(E)/\hbar$ with the dimensionless quantity $\ln A(E)$ (the prefactor $A(E)$ is originally in units of Hz).

2.7.1 Estimations

Apart from the considerable difficulties in evaluating prefactors in general, an accurate calculation of $A$ in a dense superfluid like helium II is in principle not possible at present, due to the lack of a microscopic theory. It is, however, feasible to get an idea about the value of this prefactor within about two orders of magnitude.

Let us begin with a physical picture for the origin of the prefactor in the semiclassical limit. The simplest possible idea about the prefactor is gained by considering the frequency $\omega_a$ of a particle oscillating in a metastable well. Then, within about one order of magnitude, $A \sim \omega_a$ [2]. In the thermal activation limit, i.e., in the Arrhenius law case we have more exactly $P = (\omega_a/2\pi) \exp[-U/k_B T]$, where $\omega_a$ is the frequency of oscillations at the metastable well bottom against a barrier of height $U$. The frequency $\nu_a = \omega_a/2\pi$ can generally be understood as a measure of the number of times per second the vortex bounces against the potential barrier, trying to get free.

We have no possibility to describe the vortex state (at the boundary) quantitatively, but we are able to conclude on the order of magnitude of $\omega_a$, if we take into account that there exists a surface layer of vorticity, of width $\xi$: Because the superfluid density goes to zero at the boundary and heals back within $\xi$, the energy needed for the activation of vortices vanishes within this distance [28, 29]. The frequency of motion of these vortices should then be of order

$$\omega_0 = \frac{\kappa}{\pi \xi^2} = 4.87 \cdot 10^{11} \text{sec}^{-1} \xi^{-2} \left( \sigma_{LJ}^{-2} \right) \quad \text{(helium II),}$$

(55)

which is the cyclotron frequency of vortex motion [3]. We scaled $\xi$ with the Lennard-Jones parameter $\sigma_{LJ} = 2.556$ Å of the $^4$He atomic interaction. The frequency $\omega_0$ is the natural vortex frequency associated with the scale $\xi$ alone.

If we wish to connect $\omega_a$ with the quantity relating the strength of interatomic forces and compressibility, namely the speed of sound $c_s$, which is involved in the vortex kinetic term, the following phenomenological treatment is of use. The expression for the vortex self energy is logarithmically divergent. If we consider $\xi$-scales, this may be cured in a heuristic manner by regularizing the logarithm under the condition that
the vortex energy be zero at the boundary (that is, at \( Y = 0 \)) \[41\]. This leads to the following total energy of a point vortex in the laboratory frame, using the normalisation \( 21 \),

\[
\tilde{E}(Y, \dot{X}) = \frac{1}{2} \left( (\dot{X}/c_s)^2 + \ln \left( 1 + (Y/\xi)^2 \right) - 8\pi uY/\kappa \right).
\] (56)

The minima of the potential, existing for low enough velocity, are situated at

\[
Y_{\text{min}} = \frac{\kappa}{8\pi u} \left( 1 - \sqrt{1 - 16 (u/v_L)^2} \right) \simeq \frac{\kappa}{\pi u} \left( \frac{u}{v_L} \right)^2 = 2\xi \frac{u}{v_L},
\] (57)

where in the final result we used an approximation for \( u \ll v_L \). The curvature of the potential is \( \xi^{-2} \left( 1 - (Y/\xi)^2 \right) / \left( 1 + (Y/\xi)^2 \right) \) and hence at the minimum, for \( u \ll v_L \), approximately \( 1/\xi^2 \). We have thus found that the spring constant of an ‘elastic object’ vortex should scale with a quantity of order \( 1/\xi^2 \). Because the mass is \( 1/c_s^2 \) in the same units, the frequency of oscillation is therefore

\[
\omega_s = \frac{c_s}{\xi} = 9.4 \cdot 10^{11} \text{Hz} \frac{c_s[240 \text{m/s}]}{\xi[\sigma_{LJ}]}
\] (58)

The ratio of the two estimates \( 55 \), \( 58 \) is given by \( \omega_0/\omega_s = (\kappa/c_s)/(\pi \xi) \). They thus coincide in order of magnitude because the relation \( 16 \), valid in helium II, holds. It is quite obvious that both estimates can only give a rough idea about the actual attempt frequency in the dense superfluid, whereas at least the estimate \( 54 \) should be quite accurate in dilute superfluids (for ‘nonrelativistic’ vortex velocities \( 31 \)). However, by argument of continuity, we do not expect the analytical relation for the frequency \( \omega_0 \) as a function of the parameters \( \xi, c_s \) to make abrupt changes, if we increase the density of the superfluid. We suspect that the density increase does not cause a change of more than, say, one order of magnitude in \( \omega_0 \), than that predicted by the above estimates.

Now, relying on the estimate \( 54 \) and scaling the frequency with \( 1.13 \cdot 10^{12} \text{Hz} \) (the roton frequency at \( p \approx 1 \text{ bar} \)), so that \( P(E) \equiv P(E)\text{[Hz]} \) and \( (c_s/\xi)_n \equiv (c_s/\xi) [1.13 \cdot 10^{12} \text{Hz}] \), by use of \( 14 \), we arrive at a tunnelling probability having the appearance of \( 54 \):

\[
P(E) = \exp \left\{ -\rho_0 \left( 2\pi \Omega^{(d)} + \ln(\cdots) \xi \Sigma^{(d)} \right) + 27.75 + \ln[(c_s/\xi)_n] \right\}
\]

\[
\simeq \exp \left\{ -\rho_0 \beta Y_0^2 \left( 2\pi + 2\ln(\cdots) \frac{v_{\text{max}}}{Y_0} \frac{\xi}{c_s} \right) + 27.75 + \ln[(c_s/\xi)_n] \right\}
\]

\[
\simeq \exp \left\{ -2\pi N^{(d)} \left( 1 + \frac{1}{\pi} \frac{\xi}{Y_0} \frac{v_{\text{max}}}{c_s} \right) + 27.75 + \ln[(c_s/\xi)_n] \right\},
\] (59)

where in the second line we inserted the results of the 2d half-ellipse problem in equations \( 52 \) and \( 53 \), and the third line employs the Bohr-Sommerfeld quantization of \( 13 \).
3 Comparison to experimental results

Having determined what we can expect for the magnitudes of tunnelling probabilities we now come to discuss how experimental arrangements to measure such tunnelling events are constructed, and what the findings of such experiments are. In most of the measurements (cf. [45] – [50]), the apparatus to measure phase-slip events is structured like that shown in Figure 7. We note in passing that such an apparatus with an effective torus geometry can be used to measure external rotation (e.g., that of the Earth) quite sensitively (see Refs. [51] – [53]).

Fig. 7 The schematic arrangement of the ZAV (Zimmermann-Avenel-Varoquaux) oscillator [4]. A membrane, generating a current $I_t$ by an (electrical) driving force $F$, has mechanical parameters mass $m_d$, stiffness $k$ and damping constant $\eta_d$. This current divides into a current $I_w = I_t(L_k/(L_w + L_k))$ through a micrometer-sized small hole and a current $I_k = I_t(L_w/(L_w + L_k))$ through a long channel. Here, $L_{w,k} = l_{w,k}/(m_0 \rho S_{w,k})$ are the hydrodynamic kinetic inductances, with $l_{w,k}$ the lengths and $S_{w,k}$ the cross-section areas of the micro-orifice and the long channel, respectively. The circulation threading the hole and channel, $I_q$, is quantized. The whole cavity is filled with He II.

Essentially, it is observed in these experiments that at a well-defined value of the amplitude of the diaphragm, driving the current through the micro-orifice (the small hole on the left-hand side of Figure 7), there is an instantaneous (on the scale of the driving frequency) breakdown of the diaphragm amplitude, which is quantized. This quantized dissipation event is associated with a vortex generated at the orifice walls, subsequently crossing all the streamlines of the flow, thereby causing a phase slip event, which draws a quantized amount of energy from the flow. We will first discuss these phase slips [54], which give the crucial physical argument for the interpretation of the
3.1 Phase slips

The picture of phase slips is based on the fact that the phase and particle number are canonically conjugate, that is,

\[ [N, \theta] = i. \]  

The quantum mechanical equation of motion of the phase is hence given by

\[ \dot{\theta} = \frac{1}{i\hbar}\{\theta, H\} = -\frac{1}{\hbar}\frac{\partial H}{\partial N}. \]  

Taking the thermodynamic average of this equation, we see that the time rate of change of the phase equals the (negative) local chemical potential (defined by \( \tilde{\mu} = \mu + (1/2)m v_{c}^{2} \), where \( \mu \) is the chemical potential in the superfluid rest frame), divided by \( \hbar \), a relation which became popular under the name ‘Josephson-Anderson equation’ [54, 55]. We now consider the time average of this thermodynamic average over a long time span \( \tau \) and take the difference of the results for two points A,B in the superfluid:

\[ -\langle (\tilde{\mu}(r_{A}) - \tilde{\mu}(r_{B})) \rangle_{t} = \left\langle h \left( \frac{d(\theta_{A} - \theta_{B})}{dt} \right) \right\rangle_{t} \]

\[ = \lim_{\tau \to \infty} \left[ \frac{h}{\tau} \int_{0}^{\tau} dt \frac{d}{dt} \left( \int_{C_{AB}} \nabla \theta \cdot ds \right) \right] \]

\[ = h \left\langle \frac{dn_{w}}{dt} \right\rangle_{t}. \]  

The last relation tells us that the the negative chemical potential difference between the points A and B, divided by Planck’s quantum of action, is equal to the number of vortices crossing the line \( C_{AB} \), joining A and B, per unit time, \( dn_{w}/dt \). A single phase slip process, caused by one migrating vortex, may be visualized as shown in Figs. 8 and 9. The Figs. 8, 9 represent a pronounced simplification of the actual vortex motion process. We visualize in these pictures a point vortex moving on a straight line across the superflow through the orifice, which gives a highly symmetric view of the process. The real process of vortex half-ring motion is investigated in Refs. [47, 13]. The principal global (topological) features of importance however remain untouched and independent of the actual vortex motion trajectory: The phase difference between two stationary states (times \( t \ll t_{1} \) and \( t \gg t_{3} \) in Figs. 8 and 9) is exactly \( 2\pi \) and the process always sucks the same amount of quantized energy from the flow, given by \( \Delta E = m\rho_{0}\kappa S_{m}v_{c} = \kappa J_{c} \), with \( v_{c} \) being the (mean) critical velocity of flow through the micro-orifice, at which the vortex migration process sets in, and \( J_{c} \) the corresponding mass current (see for a derivation below).

In Figures 8 and 9, we represent a quantized vortex crossing the micro-orifice designated with quantities \( S_{m}, l_{w} \) in Figure 7. The lines emanating from the vortex

This is true in the hydrodynamic limit, after averaging over a cell much larger than the atomic size. In the microscopic domain, we encounter consistency problems related to the general problem of the existence of quantum mechanical phase operators (see, in particular, [57]; also [58]).
Fig. 8 The early and intermediate stage of a phase slip process in the orifice [55].

center (black dot) in Figures 8 and 9 are lines of constant phase, standing perpendicular on the orifice wall (which is a streamline). For ease of representation we have chosen the branch cut of the phase to be exactly parallel to the direction of motion of the vortex. The whole phase slip process is then maximally symmetric. The shaded areas represent the walls of the orifice $S_w, I_w$ on the left hand side of Figure 7. The points A, B are chosen to lie sufficiently far away from the orifice, as indicated by the dots. Initially, at $t = t_1$ (first drawing), when the vortex starts on the left side of the orifice, the phase difference between A and B, $\theta_A - \theta_B$, is zero. When the vortex reaches the line joining A and B, the phase difference is $\pi$ (second picture in Figure 8). Finally, as shown above, the vortex has crossed all the streamlines through the orifice, flowing, as indicated in the first Figure, in the vertical direction, and disappears to the right. The drawings above give a representation of the effect of migration of the order parameter...
zero, the vortex, across the matter flow through the orifice, by means of the phase of the order parameter. The physical result of this migration is invariantly given by the energy change, expressed in terms of the critical mass current \( J_c \) for triggering the generation of the vortex, \( \Delta E = \kappa J_c \), and does not depend on a representation in terms of the phase (or, for that matter, on the choice of the branch cut), i.e. is gauge invariant; it is also invariant under changes of the location of the points A and B (as long as they are situated far away from the micro-orifice). Multiplying the thermodynamic Josephson relation with the critical number current through the orifice, results in a time rate of energy decrease of the external flow driven by the oscillating membrane in Figure 7, \( \dot{E} = (\kappa J_c / 2\pi) \dot{\theta} \), from which the energy change for one phase slip, necessarily causing \( \theta \) to change by exactly \( 2\pi \), follows by time integration.

3.2 Principal findings

In the experiments, the critical velocity \( v_c = v_c(T) \) as a function of temperature \( T \) is measured. The critical amplitude of the diaphragm, corresponding to \( v_c \), is that for which there is a diminished amplitude in the next half-cycle of oscillation. The Figure 10 shows a typical experimental run of measured resonator amplitudes. An important feature of \( v_c \) is that it has a statistical distribution, which has also been recorded. The results, obtained from the statistical analysis of the series of phase slips

\[ A \]

\[ \theta_A = +\pi - \epsilon \]

\[ \theta_B = -\pi + \epsilon \]

\[ (t_3 > t_{1,2}) \]

\[ \theta = +\pi \]

\[ \theta = -\pi \]

\[ +3\pi/4 \]

\[ +3\pi/4 \]

\[ +\pi/2 \]

\[ +\pi/2 \]

\[ -\pi/4 \]

\[ -\pi/4 \]

\[ -\pi/2 \]

\[ -\pi/2 \]

\[ \vec{v} \]

\[ \vec{E} \]

\[ c \]

\[ \cdot \]

\[ (t_1) \]

\[ (t_2 > t_1) \]

\[ (t_3 > t_1, 2) \]

Fig. 9 Final stage of the phase slip process.

\[ \text{Fig. 9} \quad \text{Final stage of the phase slip process.} \]
Elementary phase slip processes in two different runs \( [39] \). The resonator amplitude of the diaphragm, which is ramped up in time by applying a voltage bias on the coated diaphragm, is recorded every half-cycle (triangles pointing up and down, respectively). A phase slip occurs if there is a breakdown of the amplitude to the next half-cycle.

like those in Fig. 10 are shown in Figs. 11, 12, 13 (received from E. Varoquaux and reproduced here with kind permission). The salient results are that the mean critical velocity first rises linearly with temperature and then saturates at \( T_0 \simeq 150 \text{ mK} \). Correspondingly, the statistical width decreases linearly and saturates at approximately the same temperature.

### 3.3 Interpretation

According to the phase slip picture we have developed above, a possible interpretation of the experimental data is as follows. A vortex half-ring, standing with its axis antiparallel to the flow \( [28] \), is generated at the orifice wall and expands under the influence of the diverging flow field through the orifice. During this process, it crosses all the streamlines through the orifice, completing the phase slip, and is finally transported away from the orifice by the flow. The fact that there is a certain critical amplitude of the diaphragm for this procedure to happen, can be associated with the fact that there is a potential barrier opposing the process. Furthermore, the fact that the critical amplitude (\textit{viz.} the critical velocity \( v_c \)), has a statistical distribution, supports the idea that the existence of a phase slip critical velocity has the statistical origin of barrier crossing events. Additional support is provided by the linearity of the amplitude respectively its distribution with temperature, a signature of thermal activation over barriers \( [3] \). It is also measured that the (average) flow velocity through the orifice as a function of temperature, needed to trigger the phase slips, as well as its statistical distribution, saturates at a temperature of \( \simeq 150 \text{ mK} \). This can be ascribed to a non-thermal process of surmounting the existing barrier: The possible explanation of the observed behaviour is quantum tunnelling of half-ring vortices at boundaries.

Within a phenomenological approach \( [44, 42] \), a model of half-rings with axis antiparallel to the applied flow, and standing perpendicular to the walls of the orifice, has been developed. The Hamiltonian is essentially that in \( [50] \), \textit{save for} the kinetic term. It turns out that, to make this model conform with the available measurements, it is necessary to postulate, a) that the coherence length increases to \( \xi \simeq 9 \cdots 10 \text{ Å} \) at
boundaries and b) that the vortex half-ring be described by a collective co-ordinate on $O(\xi)$-scales. This approach must consequently be understood as the parameter-fitting of a simplified model to the available data. From a more fundamental point of view, it does not describe crucial, indeed salient features of the actual problem:

i. The model does not incorporate Galilean invariance violation. We have seen that this is a necessary requisite for any (hydrodynamic) formalism making, in particular, use of the notion of velocity, to describe the quantum tunnelling of vortices at temperatures close to absolute zero.

ii. It cannot be reasonably expected from a description of the entity vortex on $\xi$-scales to make sense for the tunnelling exponent beyond crude order of magnitude estimates, like in the case of the prefactor (where these estimates are sufficient). For the semiclassical tunnelling exponent, all nontrivial dependence on coherence length (respectively microscopic) physics, in whichever form, should be excluded, such a dependence only being allowed in the form of an ultraviolet cutoff.

iii. It is certainly not permissible to neglect the dynamic influence of the kinetic energy of the vortex in the tunnelling exponent, if we approach scales of order
Let us now further analyse the experimental outcome. First of all, we rely on the hydrodynamic relation (23) to deduce the scale of materialisation of the nascent vortex half-ring. Scaling the velocity with 10 m/sec, the order of magnitude of the (local) flow velocity, we get

$$R_0 = \frac{1.59 \text{ nm}}{u[10\text{m/sec}]} \ln \left( \frac{9.89}{\xi[\sigma_{LI}] u[10\text{m/sec}]} \right),$$

(63)

where use was made of the Roberts-Grant result [24] for $C = 1.615$, valid within Gross-Pitaevskii theory [59]-[63]. In the experiments, the measured velocity through the hole is 5-10 m/sec. This can only be measured as an average over the cross-section of the orifice, locally the velocity can of course be higher. Nevertheless, it can be concluded that the radius $R_0$ should be of the order of nanometers. This small mesoscopic scale makes it difficult to decide if a hydrodynamic formalism is applicable in a rigorous sense (one should also bear in mind that the formula above is strictly valid only in the low-velocity limit). In particular, the value of the coherence length is not exactly known under the circumstances considered. The neglect of any large density variations ($\delta \rho/\rho_0 = O(1)$) in the formalism makes it necessary at least to assume that $R_0 \gg \xi$, so that the knowledge of $\xi$ is crucial indeed.

Next we consider the value of the cross-over temperature $T_0 \simeq 150 \text{ mK}$. (Ref. [49] reports a value $T_0 \simeq 200 \text{ mK}$.) The crossover temperature between thermal and
Fig. 13  Magnified portion of the temperature region below $\simeq 200$ mK. The figure shows the influence of minute impurity concentrations of $^3$He, given in ppb, on the critical velocity of vortex phase slip for very low temperatures (for clarity, the curves are shifted downward with respect to one another by 200 instrument units). The dashed curves are a fit to the data, according to a phenomenological theory that $^3$He atoms, bound to the vortex core at the tunnelling site, reduce the critical velocity. The dash-dotted curves are the high-temperature linear fits shown in Fig. 11. For the purest sample (0.9 ppb $^3$He), the critical velocity is temperature independent within experimental accuracy down to the lowest temperature $\simeq 15$ mK.

Quantum behaviour gives in general a measure of the equality of quantum-mechanical and thermal energies of the ‘particle’, trying to surmount the barrier with the aid of these energies. For zero damping (dissipation) the crossover temperature is given by $\hbar \omega_b = 2\pi k_B T_0$, where $\omega_b$ is the frequency of oscillation at the top of the barrier, connected to the trivial solution of the Euclidean equations of motion of the ‘particle’ sitting at the top (the bottom of the inverted potential). This leads to

$$\omega_b = 8.2 \cdot 10^{10} \text{Hz} \cdot T_0[100 \text{mK}]. \quad (64)$$

If we compare the value of $\omega_b$ in (64) with the frequencies (55), (58), we see that it is smaller by a relatively large factor, up to one order of magnitude, provided we assume the coherence length to have its bulk value, which is about $\sigma_{LJ}$. However, a direct comparison of the experimental value and these estimates cannot give more than an order of magnitude agreement. This has several reasons. First of all, these estimates...
can give only a very approximate idea about the true dynamical behaviour of a quantum many-body vortex near the boundary. It is conceivable, for example, that the effective ‘spring constant’ of the vortex against deformations is lowered compared to the semiclassical estimate in (55), because of the many-body quantum uncertainty of its position. Then, the prefactor is in general a function of driving velocity \( u \) and temperature \( T \) [65], and at the measured crossover temperature in terms of the critical velocity not necessarily equal to its value at zero temperature. Finally, as already mentioned, we do not know the (effective) value of the coherence length at the boundary. It is conceivable that the value of \( \xi \) is enlarged as compared to the bulk, because of boundary conditions, i.e. the depleted superfluid density [66]. What one can thus definitely claim to have observed from Fig. 13, is that there exists a crossover temperature from a temperature dependent to a temperature independent régime, whose energy equivalent \( k_B T_0 \) is in order of magnitude agreement with the estimates for the quantum oscillator energies \( \hbar \omega_0 \) and \( \hbar \omega_s \). To conclude this section, we give an idea about the number of particles involved in the tunnelling procedure, therein following the statement of equation (19) and the scaled tunnelling probability in eq. (59). Assuming from the above discussion that the prefactor can vary in its order of magnitude between \( A \approx 10^{10} \cdot \cdots \cdot 10^{12} \) Hz, its logarithm \( \ln A \approx 23 \cdot \cdots \cdot 28 \). Corresponding to these conceivable values of the prefactor, the total number of particles in the tunnelling volume should then be somewhere in the range \( N^{(3)} = 4 \cdot \cdots \cdot 6 \), say, for tunnelling events to be observable within a reasonable span of experimental time. Again, like the value for the materialisation radius \( R_0 \) in (63), this number indicates a rather small scale of tunnelling.

3.4 Concluding evaluation of the experiment

We have seen that the available data on critical velocities of phase slips can be interpreted to be in phenomenological accordance with the picture of the quantum tunnelling of vortices at boundaries below some temperature \( T_0 \). But the very fact of tunnelling at boundaries certainly needs further proof, so that the predictions of tunnelling theory can be compared to that of classical instability mechanisms, which can be temperature independent as well. Theoretical investigations in this classical direction are found under Refs. [67]–[70] (vortex nucleation as a process of classical flow instability in experiments using \(^3\)He-B was discussed in Ref. [1]). Provided that the hydrodynamic, large scale picture we developed here is applicable with sufficient predictive power for the actual materialisation scales of the vortex, one such proof could consist in the comparison of critical velocities for chemically identical orifices of equal global sizes, having different (microscopic respectively mesoscopic) surface structures. If the result of such measurements is negative, i.e. there is no reproducible difference in critical velocities, there is no quantum process taking place describable by hydrodynamic means of semiclassical tunnelling at irregular boundaries.
4 Aspects of vortex tunnelling in Fermi superfluids

The analysis so far has been concerned with dense superfluids which are uncharged, and in which the fundamental constituents, i.e. the particles carrying the superfluid current, bear no internal degrees of freedom like spin. The spinless elementary bosons, which are $^4$He atoms, form the only hitherto known example of such a superfluid. There exists, however, a large number of superfluids which are constituted by elementary fermionic particles. By far the most of these superfluids are charged: The charge carriers in superconductors represent a charged superfluid. Besides He II, the only other charge neutral dense superfluid known in laboratories on Earth is its isotope $^3$He. In the following, we give an overview of some general features of these Fermi superfluids, with particular emphasis on the vortex dynamical equations. The main intention of this section is to set the complications arising in the treatment of vortex motion in Fermi superfluids along the comparatively elementary hydrodynamic problem of unpaired bosons in He II.

4.1 Introduction

For all Fermi superfluids, there has to exist a mechanism binding the fermions into Cooper pairs [71], which constitute the bosons of the superfluid condensate. In distinction from the elementary $^4$He bosons, these effective particles, arising from paired fermions, have in general the internal degrees of freedom spin and angular momentum. According to the symmetry of the order parameter, these superfluids are classified to be $s$-, $p$- or $d$-wave superfluids. In the case of isotropic superfluids, the value of the internal angular momentum $L = 0, 1, 2$ of the Cooper pairs corresponds to $s$-, $p$- or $d$-wave, respectively. A relatively weak effective interaction binding the fermions together leads to coherence lengths $\xi \gg k_F^{-1}$, where $k_F$ signifies the Fermi momentum. The corresponding Cooper-pairs are then objects bound together over distances by far exceeding the microscopic scales relevant for many-body quantum mechanics. One consequence is that, e.g., the Ginzburg-Landau or mean-field levels of description, not sufficient to describe scales of order $\xi$ in He II, are indeed useful for such paired Fermi superfluids on these scales [3]. The notable exception are high-$T_c$ superconductors, where $k_F\xi \gtrsim O(1)$. More properly, this is to be written as $\xi \gtrsim d_L$, where $d_L$ is the lattice spacing, because the Fermi surface is in general not a single continuous surface for the high-$T_c$ materials.

The fact that the Cooper pairs have internal angular momentum and/or spin leads to a richly structured order parameter, which can support symmetries much more complex than the global and local U(1) symmetries associated with particle conservation and the electromagnetic field. In particular, as one of the most important consequences of these enriched symmetries, there can occur nodes of the energy gap in momentum space for quasiparticle excitations above the superfluid ground state [72].

4.2 Vortex motion in Fermi superfluids

Around a vortex line, there exists a potential well for quasiparticles, i.e. the pair potential is position dependent, $\Delta = \Delta(x)$ (we designate $\Delta_\infty$ to be the bulk value
of the gap, infinitely far away from the vortex line). This leads to the existence of bound quasi-particle states in the vortex core, of extension $O(\xi)$. These bound states are obtained by solving the Bogoliubov-deGennes mean-field equations for the wave functions in particle-hole space, in which $\Delta(x)$ plays the role of the potential [4, 78]. They lead to a profound alteration of the low-energy dynamical behaviour of vortices because, among these bound states, there exist the so-called zero modes crossing the zero of energy as a function of a component of the quasiparticle wave vector (the other core bound states excitation branches have energies at least of $O(\Delta_{\infty})$, i.e. of order the bulk energy gap). These zero modes lead to a exchange of quasiparticle momentum between the superfluid vortex vacuum, moving with velocity $\partial X/\partial t$, and the quasiparticle heat bath, moving with velocity $v_n$, if the relative velocity $\partial X/\partial t - v_n$ is non-vanishing: The quasiparticles are driven by the effective (electric-like) field, stemming from this relative velocity, from the occupied negative energy levels to those of positive energy (energies are counted from the Fermi energy $E_F$), thereby transferring momentum to the quasiparticle bath [80]. The motion of the quasiparticles on the zero mode branch is a process for which the notion of spectral flow has been coined. A contribution to spectral flow is stemming only from chiral particles (cf. [72], chapter 6). Due to the momentum exchange between superfluid vacuum and quasiparticle bath, there results an additional transverse force on the vortex, to be added to the usual contributions which occur in superfluids without zero mode bound states [77, 81].

The equation of motion for the vortex in the time domain is in general regions of the parameter space non-local. In frequency space, the equation of motion is local, because the convolutions of Green’s functions with the vortex co-ordinates in the time domain become products after Fourier transformation. The equation of motion may then be written in a form which satisfies Galilean invariance, which implies that only relative velocities of line and heat bath (the lattice) as well as line and superflow are to occur in the equation of motion. We thus assume isotropy and the existence of only one charge carrier, for simplicity of representation. The force balance equation between Magnus and dissipative as well as transverse forces on the vortex is then expressed by [79, 81, 82]

$$F_M = \frac{1}{2} \hbar \rho_s \left( v_s(\omega) - \dot{X}(\omega) \right) \times X' = D(\omega) \left( \dot{X}(\omega) - v_n(\omega) \right) + D'(\omega) X' \times \left( \dot{X}(\omega) - v_n(\omega) \right).$$

(65)

It is assumed that the vortices are singly quantized, with a constant circulation vector, the factor of 1/2 in the Magnus force taking account of the paired nature of the superfluid. For the point vortex we will deal with, $X' = \partial X/\partial \sigma = \pm e_z$, the upper/lower sign valid for a positive/negative circulation vortex. The first line contains the conventional Magnus force (where $\rho_s = \rho_s(T)$ is the superfluid density), and the second line the dissipative and reactive mutual friction forces from momentum exchange between vortex line and quasiparticle bath. We further assume in what follows for the sake of simplicity that the normal component is clamped, i.e. $v_n = 0$ (this, of course, destroys the property of Galilean invariance afforded by (65)). Then, a convenient writing for

$$F_M = \frac{1}{2} \hbar \rho_s \left( v_s(\omega) - \dot{X}(\omega) \right) \times X' = D(\omega) \left( \dot{X}(\omega) - v_n(\omega) \right) + D'(\omega) X' \times \left( \dot{X}(\omega) - v_n(\omega) \right).$$

(65)
the above equation of motion is
\[ \frac{h \rho_s}{2} v_s(\omega) \times X' = D(\omega) \dot{X}(\omega) + \left( D'(\omega) - \frac{1}{2} h \rho_s \right) X' \times \dot{X}(\omega), \] (66)
that is, there is a coefficient \( D(\omega) \) of the force linear in the velocity of the vortex, the damping force, and a coefficient \( D'(\omega) - h \rho_s/2 \) of a force perpendicular to the vortex velocity, which represents the Magnus force part proportional to vortex velocity (the Hall term). The driving term \((1/2) h \rho_s v_s \times X'\) (the superfluid current part of the Magnus force) is proportional to the circulation vector and the superflow current density \( j_s = \rho_s v_s \).

4.2.1 The \( s \)-wave case
We now consider the simple case of a singly quantized vortex in a two-dimensional isotropic superfluid, for which the quasiparticle momentum has components \( k = (k_r, k_\phi) \), and the zero mode branch
\[ E(k_\phi) = -\omega_0 k_\phi. \] (67)
The level spacing \( \omega_0 \) is of order \( \hbar / (m \xi^2) \sim \Delta^2_c / E_F \), where \( m \) is the (effective) fermion mass. The energy levels correspond to those of electrons on anomalous Landau levels, which are linear in momentum [80].

On the approximation level of a single relaxation-time in the quasiparticle kinetic equation, the coefficients are then given by [79, 80, 82]
\[ D(\omega) = \frac{i h C_0 \omega_0}{4} \left( \frac{1}{\omega - \omega_0 + i \tau^{-1}} + \frac{1}{\omega + \omega_0 + i \tau^{-1}} \right) \]
\[ = \frac{(h C_0/2)(\tau^{-1} - i \omega)/\omega_0}{1 + ((\omega_0 \tau)^{-1} - i \omega/\omega_0)^2}, \] (68)
for the longitudinal coefficient in equation (66) and, for the transverse coefficient,
\[ D'(\omega) - \frac{1}{2} h \rho_s = \frac{h C_0 \omega_0}{4} \left( \frac{1}{\omega - \omega_0 + i \tau^{-1}} - \frac{1}{\omega + \omega_0 + i \tau^{-1}} \right) \]
\[ = \frac{-h C_0 / 2}{1 + ((\omega_0 \tau)^{-1} - i \omega/\omega_0)^2}. \] (69)

\(^h\)We remark that in the literature the setting \( \hbar \equiv 1 \) is frequently taken, which means that Planck’s quantum of action \( \hbar \equiv 2\pi \).
In these equations, the parameter density $C_0 = k_F^2/2\pi$ (in three dimensions: $C_0 = k_F^3/3\pi^2$) is the normal state density. The parameter $\tau$ is a constant relaxation time in the collision term of the kinetic equation. The relations above hold provided that the conditions $\omega_0 < (h\beta)^{-1} = k_BT/\hbar$, $\tau \gg h\beta$ and $h\omega < k_BT_c$, $T \ll T_c$ are met. The first condition means the thermal population of the levels represented by $E_k$, such that the sums over the bound state levels in the general expressions for $D, D'$ can be converted into integrals. This condition is not very stringent, and allows for a use of the formulas above for actually quite low temperatures, because $\omega_0 \sim T_c(\Delta_\infty(T)/\Delta_\infty(T = 0))(\Delta_\infty(T)/\hbar\epsilon_F) \ll T_c$. The second condition implies that the broadening of levels by scattering is much less than the temperature. Finally, the condition $h\omega \ll k_BT_c$ restricts the energy equivalent of the vortex motion frequency, $\hbar\omega$, to be much less than the bulk gap $\Delta_\infty \sim k_BT_c$.

It is important to point out that the validity of (63) and (64) relies on the transverse and longitudinal coefficients in the low energy limit being determined by the core level spacing $\omega_0$ and the scattering frequency $\tau^{-1}$ alone. The relations for the Hall and longitudinal coefficients are then given irrespective of an electric charge of the particles carrying the superfluid current.

Two limits of the equations of motion are particularly well known. The first is provided by $\omega_0\tau \gg 1$, $\omega \ll \omega_0$ and corresponds to massive vortex motion under influence of the superfluid Magnus force ($X'(\omega) = -i\omega X'(\omega)$):

$$M_s \dot{X} = \frac{\hbar}{2} \rho_s X' \times \left( \dot{X} - v_s \right).$$

(70)

At the low temperatures considered, we neglected the small contribution of $(\hbar/2) (C_0 - \rho_s) X' \times \partial X / \partial t \approx (\hbar/2) \rho_s X' \times \partial X / \partial t$, where $\rho_s$ is the normal density in the superfluid state. This last expression represents the Iordanskii force (remember that we fixed $v_s = 0$), which is also present in Bose superfluids and not related to spectral flow (73).

The Magnus force dominates in the above limit of "slow" vortex motion with $\omega \ll \omega_0$ the massive term (the Hall term on the right-hand side is larger by a factor of $\omega_0/\omega$ than the inertial term). Still, the vortex core dynamical mass $M_c = \hbar/C_0/(2\omega_0) \sim mC_0\pi\xi^2$ is much larger than the mass arising from the compressible superflow outside the vortex core. Their ratio is of order $M_0/M_c \sim (d/\xi)^2$, where the quantity $d$ signifies the interparticle spacing $d \ll \xi$, so that $M_0$ may be neglected in the equation of motion (70).

Dissipative motion prevails in the limit $\omega_0\tau \ll 1$, $\omega\tau \ll 1$:

$$\frac{\hbar}{2} \rho_s v_s \times X' = \frac{\hbar}{2} C_0(\omega_0\tau) \dot{X} + \frac{\hbar}{2} C_0(\omega_0\tau)^2 \ddot{X} \times X'.$$

(71)

The vortex motion is overdamped, with friction coefficient given by the expression $\eta = (1/2) \hbar C_0^2 \omega_0\tau$. It is observed that time inversion invariance is spoilt by the fact

1. The parameter $C_0$ is in general a measure of the density at the location of gap nodes; it is zero if no gap nodes are present. In our case, the gap nodes are at the position of the vortex line, other possible occurrences of nodes are in the bulk of $p$- or $d$-wave superfluids.
2. We do not show the hydrodynamic and transverse mass terms on the right-hand side of (73), which are small compared to the friction term. The transverse mass $C_{\parallel}$, relating vortex velocity and momentum in different directions, can be defined dynamically from the equations of motion (74), like $M_c$. 

that the first term on the right-hand side of the equation above does not have the factor $X'$, which changes sign under time inversion. The dissipation is in the given limit of ohmic nature with a longitudinal conductivity independent of the driving frequency $\omega$. The Hall part of the Magnus force is suppressed by $\omega_0\tau$ relative to the friction term, which is the result of the spectral flow phenomenon discussed above. Spectral flow is made possible because the minigap $\omega_0$ between zero mode levels is broadened by a large quasiparticle collision frequency $\tau^{-1} \gg \omega_0$. On the other hand, in the case of (70), spectral flow is impeded by the presence of the minigap (save for tunnelling events between the levels [72]), and the Magnus force obtains. It is to be mentioned that in a superconductor the energy levels above the gap are also quantized into Landau levels with the interlevel distance $\hbar\omega_c$, where $\omega_c$ is the usual magnetic cyclotron frequency. For sufficiently small magnetic field, however, the spectrum is quasi-continuous and no change of the above results applies [81].

4.2.2 The d-wave case

For $p$- and $d$-wave superconductors, the gap has nodes not only at the location of the vortex itself, but also in the bulk superfluid. The $d$-wave case distinguishes itself by the fact that these nodes occur on lines in momentum space, whereas in the $p$-wave case they only occur at points [72]. The lines of gap nodes in $d$-wave superconductors lead to a profound alteration of the effective vortex dynamical equations [83, 84, 85].

To explain the essential features of these changes, we observe, first of all, that the minigap $\omega_0 = \omega_0(\alpha)$ is a function of the angle $\alpha$, which indicates the position of the gap nodes in momentum space, where the gap modulus $\Delta = \Delta_0 \sin(2\alpha)$. The average minigap $\Omega_0 \equiv \langle \omega_0(\alpha) \rangle$ is of the same order $\Delta_0^2/E_F$ as the constant $\omega_0$ in the $s$-wave case, $\Omega_0 = O(\omega_0)$. There exists, however, an additional energy scale, the true quantum-mechanical interlevel distance in the vortex core $E_0$, which is defined according to a Bohr-Sommerfeld type of quantization prescription for the canonically conjugate quantum variables $k_\phi(\alpha)$ and $\alpha$. In a $\alpha$-dependent version of (77), $k_\phi(\alpha) = -E/\omega_0(\alpha)$, the Bohr-Sommerfeld quantization reads $\oint k_\phi(\alpha) d\alpha = h(m + \gamma)$ (with $\gamma = 1/2$), so that the true interlevel distance is given by $E_0^{-1} = h^{-1} \int_0^{2\pi} d\alpha \omega_0^{-1}(\alpha)$ (the integral is rendered finite by the existence of a magnetic field [84]). There are several different regimes of vortex motion corresponding to the ratio of the relaxation rate $\tau^{-1}$ and the frequency of vortex motion $\omega$, not only to the (average) minigap $\Omega_0$, as in the $s$-wave case discussed above, but also to $E_0 = h\sqrt{\Omega_0 \omega_c} \ll h\Omega_0^\ast$.

There is a parameter region which yields a comparatively simple and unique result. This is the case of $\Omega_0\tau \ll 1$, which implies $E_0\tau \ll h$. Assuming we have in addition $\omega\tau \ll 1$, we are led back to an equation of motion of the type (71). On the other hand, if $\Omega_0\tau \gg 1$ (and $\omega\tau \ll 1$), there is a ‘universal region’ [83, 84], which is realized if $E_0\tau \ll h$: The dissipative and Hall coefficients are independent of the relaxation time for very low temperatures and small magnetic fields and have the same order of magnitude. Finally, if $\Omega_0\tau \gg 1$ and $E_0\tau \gg h$, the equation of motion is of the type (74), with dominating Magnus force. This is a regime which will presumably

---

Footnotes:

[1] There is a further complication, which we do not take into account here because of the low temperatures $T \ll T_c$ considered. If $T \ll T_c$, the relevant energy scale approaches the cyclotron level spacing $\hbar\omega_c$. Furthermore, the results have in general to be written in a form taking into account both particle- and holelike Fermi surface parts, see [84].
not be realizable with respect to practically achieved sample purities, whereas the universal regime should be observable. It appears useful at this point to insert a short treatise on the terminology in the literature. Superconductors are classified as being
‘superclean’, ‘clean’, ‘moderately clean’ and ‘dirty’, according to the value of the ratio $l/\xi$, where $l = v_F \tau$ is the quasiparticle mean free path. The superclean limit $\omega_0 \tau \gg 1$ (or $\Omega_0 \tau \gg 1$) corresponds to $l \gg \xi(E_F/\Delta_\infty)$, representing a much stricter condition than its clean counterpart $l \gg \xi$. The moderately clean and dirty limits correspond to $l \gg \xi$ and $l \ll \xi$, respectively. We will agree to call a superconductor ‘moderately clean’ if it has $\omega_0 \tau \ll 1$ (but still simultaneously $l \gg \xi$). Because of the above discussed $d$-wave peculiarities, there is yet another notion of ‘extremely clean’, which ought to be introduced. This corresponds to the extreme limit $\Omega_0 \tau \gg 1$ and $E_0 \tau \gg \hbar$, or $l \gg \xi(E_F/\Delta_\infty)$.

The discussion of $s$-wave vortex motion to follow is thus valid for the $d$-wave case in a straightforward sense only if we are in the moderately clean region $\Omega_0 \tau \ll 1$, $E_0 \tau \ll 1$, and have additionally $\omega \tau \ll 1$, with the local form of the equation of motion in \[71\].

4.2.3 Nonlocal motion in the time domain

In order to obtain the real time motion of the vortex, we rewrite the general equation in (71).

4.4.2 Case of $d$-wave peculiarities

Calculating $\text{Res}(z_1) + \text{Res}(-\bar{z}_1)$, they are given by ($\Delta t \equiv t - t'$):

$$K^D(\Delta t) = \theta(\Delta t) \frac{hC_0}{2} \omega_0^2 \exp \left[ -\frac{\Delta t}{\tau} \right] \left( \sin(\omega_0 \Delta t) + \frac{1}{\omega_0 \tau} \cos(\omega_0 \Delta t) \right) ;$$

$$K^H(\Delta t) = \theta(\Delta t) \frac{hC_0}{2} \omega_0^2 \exp \left[ -\frac{\Delta t}{\tau} \right] \left( \cos(\omega_0 \Delta t) - \frac{1}{\omega_0 \tau} \sin(\omega_0 \Delta t) \right) .$$
The gradient of the stream function \( \psi \) describes the superflow perpendicular to the vortex, \( \nabla \psi = \mathbf{v}_s \times \mathbf{X}' \).

Making use of the phase angle \( \Phi = \arctan[(\omega_0 \tau)^{-1}] \) and standard addition theorems, we can cast the action into the suggestive form

\[
S[\mathbf{X}(t)] = \frac{\hbar \rho_s}{2} \int dt \psi[\mathbf{X}(t)] + \frac{\hbar C_0}{2} \omega_0 \sqrt{\omega_0^2 + \omega_T^2} \int dt \int dt' \exp[-\omega_T \Delta t] \times \left\{ \mathbf{X}(t) \cdot \mathbf{X}(t') \sin(\omega_0 \Delta t + \Phi) - \mathbf{X}' \cdot (\mathbf{X}(t) \times \mathbf{X}(t')) \cos(\omega_0 \Delta t + \Phi) \right\}.
\]

The factors multiplying the dot- and cross-products of \( \mathbf{X}(t) \) and \( \mathbf{X}(t') \) in this nonlocal Lagrangian are for any value of \( \omega_0 \tau \) and thus \( \Phi \) just \( 3\pi/2 \) out of phase. In case that \( \omega_0 \tau \ll 1, \Phi \approx \pi/2 \), the first term with the dot-product dominates, whereas if \( \omega_0 \tau \gg 1 \), \( \Phi \approx 0 \), the second one involving the cross-product does.

### 4.3 Euclidean vortex motion

For a description of tunnelling motion, we have to use the Euclidean action in the interval \([-\hbar \beta/2, \hbar \beta/2]\). Performing the Wick rotation in \((73)\) through the replacement \( t \to -it_e \), gives the Euclidean action \( S_e[\mathbf{X}(t_e)] = -iS[\mathbf{X}(t \to -it_e)] \):

\[
S_e[\mathbf{X}(t_e)] = - (\hbar \rho_s/2) \int_{-\hbar \beta/2}^{\hbar \beta/2} dt_e \psi[\mathbf{X}(t_e)]
+ \int_{-\hbar \beta/2}^{\hbar \beta/2} dt_e \int_{-\hbar \beta/2}^{t_e} dt'_e \left[ -K^D(t_e - t'_e) \mathbf{X}(t_e) \cdot \mathbf{X}(t'_e) + K^H(t_e - t'_e) \mathbf{X}' \cdot \{ \mathbf{X}(t_e) \times \mathbf{X}(t'_e) \} \right]
\]

where, under the condition that the real frequency \( \omega \ll \omega_0, \omega_T \), the kernels are approximately given as

\[
\begin{pmatrix}
K^D(t_e - t'_e) \\
K^H(t_e - t'_e)
\end{pmatrix} = \frac{h C_0 \omega_0}{2\pi} \int_0^\infty d\omega e^{-\omega |t_e - t'_e|} \frac{\omega}{\omega_0^2 + \omega_T^2} \begin{pmatrix}
\omega_T \\
-\omega_0
\end{pmatrix}
+ \frac{1}{2\pi} \frac{\omega_0}{(t_e - t'_e)^2} \begin{pmatrix}
\omega_0 \\
-\omega_T
\end{pmatrix},
\]

and include nonlocality in lowest order. The dissipative kernel is of the Caldeira-Leggett-form \((94)\) for ohmic dissipation, \( K^D = (\eta/\pi)(\Delta t_e)^{-2} \), with the friction
Table 1  Comparison of different approaches to quantum tunnelling with respect to dominant contributions employed in the calculation of the Euclidean tunnelling action (shown by an entry with a checkmark, ✓). The Magnus column indicates if a linear coupling of the vortex velocity to the superfluid background, generating the vortex velocity part of the Magnus force, has been used. In the potential \( V_g(q) \), \( \epsilon \ll 1 \) means that the case of a near critical potential has been treated. In Ref. [93], this was done by an analytical method, in [92] by a numerical one. In the case of entry [105], both limits of dominant Magnus and dissipation contributions have been considered, but not the intermediate case of equal strengths. This intermediate case has been examined numerically in [103], far off and near criticality.

 coefficient \( \eta = (1/2)\hbar C_0(\omega_0/\omega_\tau)/\omega_0^2 + \omega_\tau^2 \). The dissipation due to bound states is at a maximum if \( \omega_0/\omega_\tau = O(1) \) and vanishes in the limits \( \omega_0/\omega_\tau \to \infty \) and \( \omega_0/\omega_\tau \to 0 \). The nonlocality of the Hall term in the action is of the same importance as that of the friction term if \( \omega_0 \simeq \omega_\tau \).

4.4 Different regions of parameter space

It is rather obvious that the motion of a vortex for arbitrary competing contributions in the action (74) can be quite complicated. In principle, the following contributions in the action are conceivable. In addition to the terms appearing in (74), there can be contributions arising from the self-interaction, like in (8), that is, the hydrodynamic mass and elasticity terms (the self-energy is usually absorbed into the potential). The hydrodynamic mass was argued to be in general completely negligible as compared to the dynamic core mass in paired Fermi superfluids. The elasticity arises from the generalization of a 2d or rectilinear vortex to one of arbitrary shape and the additional self-energy this creates. Hence, in order to actually describe an imaginary time motion and thus evaluate the tunnelling process probability, only certain classes of metastability problems have been investigated.

The (stream function) potential is, for reasons of (analytical) solvability, frequently represented as a quadratic plus cubic potential for a one-dimensional generalized coordinate \( q \). This potential is conventionally parameterized with two or three quantities, a height \( h \) and width \( w \) of the potential, and possibly with the additional parameter
of closeness to criticality $\epsilon$. The potential thus has the general representation

$$-\psi = V_g(q) = 3V_0 \left[ \epsilon \frac{q}{q_0} - \frac{2}{3} \frac{q}{q_0}^3 \right],$$

(76)

where $\epsilon = \sqrt{1 - v_s/v_{cb}}$ measures the closeness to a critical external velocity $v_{cb}$, for which the barrier vanishes. A measure of the typical curvature radius of the potential is $q_0$. The zeros of the potential are at $q = 0$ and $q = (3/2)\epsilon q_0$, so that the width of the potential may be defined to be $w = (3/2)\epsilon q_0$. The maximum is at $q_{\text{max}} = \epsilon q_0$ and its height equals $h = V_0 \epsilon^3$. The different cases and approximations investigated in a selection of recently published papers on quantum tunnelling are brought together for comparison in Table 1. For the detailed results and methods used, the reader is referred to the cited works.

In relation to what we have found in the preceding section, we can make the following observations. The core level spacing is of order $\omega_0 \sim \hbar/(m\xi^2)$, and the Magnus force dominates over the mass term in (70) if the vortex motion frequency $\omega \ll \omega_0$. The collective co-ordinate approach implies that the curvature radius of the potential $q_0 \gg \xi$, because the motion of a massless vortex in the potential is of typical frequency $\omega \sim \hbar/(m q_0^2)$, the ‘cyclotron’ frequency associated with $q_0$. The condition for the collective co-ordinate approach, $q_0 \gg \xi$, is thus equivalent to the dominance of the Magnus contribution over the core mass term in the superclean limit $\omega_0 \tau \gg 1$.

The fundamental hydrodynamic analysis of the last section, treating the Magnus force as dominant, therefore remains valid for tunnelling in the case of a superclean $s$-wave fermionic superfluid. The case of a $d$-wave superfluid is, as already argued, more intricate. The dominance of the Magnus force only obtains in the extremely clean limit, whereas in a superclean limit, which has $E_0 \tau \ll \hbar$, the dissipative and Hall force components are of comparable magnitude. This necessitates a complete treatment of the tunnelling phenomenon in two dimensions for this ‘universal’ parameter regime, even for a point vortex, because, in the plane, one has to solve two coupled differential equations of motion. We will further discuss this case and its possible occurrence in high-$T_c$ superconductors below. In the extremely clean limit respectively for large enough cyclotron level spacing, and if the temperature goes to zero, i.e. is less than any of the energy scales associated with the average and true minigap $\Omega_0 = \langle \omega_0 \rangle$ and $E_0$, the superfluid Magnus force is the only remaining nondissipative force on the vortex, and the analysis of tunnelling in the last section is valid.

### 4.5 Quantum tunnelling in high-$T_c$ superconductors

In conventional superconductors, the dissipative component in the vortex equations of motion usually is very significant, so that quantum tunnelling is largely suppressed; the temperature region above zero, in which temperature independent quantum tunnelling is of importance, is exceedingly small. There is, however, the intensely investigated class of high-$T_c$ superconductors, which can very well be in a clean or even superclean limit. In this latter case the Hall angle can approach the value $\pi/2$
(the vortex then moving more with the local superflow, rather than perpendicular to it) \(^8^7\). In addition, and even more important, in contrast to conventional superconductors these materials can exhibit quite small coherence lengths in the order of the lattice spacing, and a ratio \(T^*/T_c \sim O(1 \cdots 10)\) (cf. the energy barrier considerations relevant for vortex tunnelling in the introductory considerations. These facts lead to the possibility that in some of these superconductors, quantum tunnelling of flux lines might be observable at low temperatures. The ratio of the crossover temperature \(T_0\) from thermally activated to quantum behaviour for the flux line depinning to the critical temperature \(T_c\) was measured in very different materials \((109 \cdots 111)\). It is found, depending on the material, that \((T_0/T_c)_{\text{high--}T_c} \sim 0.03 \cdots 0.09\), where \(T_0\) is defined in these measurements to be the temperature at which flux motion deviates from the one expected for purely thermal activation. \(^8^7\) We compare this with He II, where \(T_0 \sim 150\) mK and \(T_c \sim 2.2\ K\), so that \((T_0/T_c)_{\text{He II}} \sim 0.07\). Considering the fact that the ratio of the critical temperatures for these high-\(T_c\) superconductors and helium II can be up to a factor of 50, the values of \(T_0/T_c\) are comparatively close, differing at most by a factor of two. This suggests a common physical origin of the deviation from thermal activation behaviour in helium II and high-\(T_c\) superconductors, be it quantum tunnelling or some other, classical, flow instability mechanism. This is only natural from the point of view that both systems represent, on a fundamental level, strongly coupled superfluids.

There are high-\(T_c\) superconductors presumably belonging to the class of \(d\)-wave superconductors \(^7^4\ \(7^5\), or some variety of this pairing symmetry with small deviations from pure \(d\)-wave. The considerations of section \(4.2\) for the \(d\)-wave case then apply, provided we assume a quasiclassical, low energy treatment of vortex motion in linear response to be valid, at least qualitatively, in these superconductors and on the scales of tunnelling.

The \(d\)-wave superconductor, for practically achieved maximal sample purities and sufficiently small magnetic fields, will not be in an ‘extremely clean’ region (Magnus force dominating), but rather in the superclean ‘universal’ region (for the exact conditions on \(H\) and \(T\), see \(8^3\)). This implies that there is a magnetic field region in which, as already explained, the vortex tunnelling motion is not dominated either by the dissipative or the Magnus force (the Hall term), even for very low temperatures, but is governed by both forces. There are indications that in clean high-\(T_c\) materials an intermediate regime between purely dissipative and Hall tunnelling may indeed be realized \(10^9\ \(1^1\). The intermediate case thus clearly needs further investigation, because the (measurable) temperature dependence of the Euclidean action can be different from that expected for purely dissipative or Hall motion, and the transition from quantum to thermally activated depinning of the vortices be of first or second order \(10^9\ \(1^1\ \(111\). A first step in this direction has been made in \(10^3\), where the problem of quantum tunnelling was investigated with the static (low frequency) versions of the formulas for the \(s\)-wave case, \((8^8\) and \((8^9\). It is found that the tunnelling rate displays a minimum for \(\omega_0 \tau \sim O(1)\). This is in accordance with the fact that dissipation due to spectral flow is at a maximum for values of the parameter \(\omega_0 \tau\) which are of order unity. Whereas the nonlocality of the ohmic dissipation was considered there \(\sim\)

\(^{8^7}\)In Ref. \(11^2\), however, quite large ratios up to \((T_0/T_c)_{\text{high--}T_c} \sim 0.22\) have been reported.
la Caldeira-Leggett, that of the Hall term, however, was not taken into account. It is apparent from (75) that this is not justifiable in the intermediate case of interest, in which $\omega_0 \sim \omega_\tau = \tau^{-1}$. Within the formalism we presented, the Hall term can be treated locally in the action only in the limits $\omega_0 \tau \rightarrow \infty$ and $\omega_0 \tau \rightarrow 0$. Apart from this objection, the vortex dynamical behaviour in $d$-wave superconductors is in general more complicated than in conventional $s$-wave superconductors, as we have already pointed out. For example, under certain conditions there can be resonances in the vortex response, if vortex tunnelling frequencies are near $(2k + 1)E_0/\hbar$, with $k$ an integer number \cite{84}. A complete treatment of the tunnelling problem for a $d$-wave system in the intermediate range then necessitates an incorporation of nonlocality in the longitudinal and transverse vortex response, as well as possible resonances with collective modes induced by the moving vortex.

5 Concluding remarks

The present work has treated the consequences and limitations of a large scale description of the motion and tunnelling generation of quantized vortices in dense superfluids.

We may summarize the salient assumptions, pertaining to this treatment, in a compact way as follows. The vortex object cannot be described in its *genesis*, because we do not know precisely in which way a vortex should be represented on curvature scales of order the coherence length. The intrinsic nucleation process, in all its quantum many-body subtleties, happens on these scales. Applying the formalism used in this treatise, we thus have to assert that the vortex somehow comes into a topologically ensured existence. We can, then, assign a collective co-ordinate to the singular center of topological stability. Furthermore, if we wish to describe the vortex as a string (in three dimensions), or point object (in two dimensions), which is tunnelling through a potential barrier, we have to adopt the point of view that we are allowed to quantize vortex position and momentum, in a canonical manner. These two assumptions and their validity lie at the heart of our treatment of the problem of vortex tunnelling in a superfluid at the absolute zero of temperature. There is, then, no theory of vortex *nucleation* in a dense, real life condensed matter system, because we drastically reduce the number of (quantum) degrees of freedom actually relevant for nucleation. There is, though, a consistent theory of vortex quantum tunnelling in the large scale domain, which we represented here in its formal requirements and geometric implications.

The necessity of employing the long wavelength limit affects the contributions of different origin in the tunnelling exponent. The volume contribution, associated with the incompressible superfluid, always dominates over the area contribution. This latter contribution is associated with the vortex effective mass, and is thus depending on the detailed dynamical behaviour of the vortex on the tunnelling path. The dominant volume contribution, in contrast, depends on the shape of the vortex path in configuration space only.

At absolute zero, the Galilean invariance of the bulk superfluid is required to be broken for tunnelling to be energetically possible. This breaking of invariance can be attained by considering vortex motion in the presence of obstacles. The geometrical implications imprinted by a specific obstacle chosen play an important part in our analysis and give a central result. Namely, if the vortex moves near the boundary, trapped by
the pinning potential generated by the obstacle, only those paths are allowed in which the vortex center remains at least within a distance of order the coherence length from the obstacle surface. If the flow obstacle, then, has curvature perpendicular to the flow passing at infinite distance over the obstacle much larger than parallel to this flow, we have shown that the (constant) energy of the tunnelling vortex cannot be less than a given minimal energy. This minimal energy is needed by the vortex to be completely describable by the collective co-ordinate with which we have equipped it, because else it would come within a distance less than the coherence length to the obstacle boundary. This result is generalizable to the case of relativistic vortices in spacetime, where the necessary breaking of (local) Lorentz invariance for timelike currents will lead to the same kind of prediction.

The theory we have developed can claim to make exact predictions on observable tunnelling probabilities in the semiclassical limit, as long as the collective co-ordinate approach makes (geometrical) sense and the Magnus force contribution is dominating the tunnelling action. We have seen, by considering equation (63), that the tunnelling scales in He II, using the available data, will be of order nanometers. Correspondingly, the number of particles participating in the tunnelling event, i.e. those contained in the volume determining the tunnelling action, is comparatively small. It is of order $N^{(3)} = 4 \cdots 6$, given physically realistic estimates of the prefactor and the tunnelling rates to be expected. Hence, the applicability of our theory, for the actual physical conditions encountered in He II, is restricted, in the sense that it can only give lower bounds for tunnelling rates, valid on large enough scales. It is, first of all, not entirely obvious that the dominance of the Magnus force still holds on the scales relevant for tunnelling. Second, we have no really direct means to compare the tunnelling rates observed by varying experimental conditions with the predictions of the theory. This, of course, stems from the very nature of the theory as a geometric theory. The predictions it actually makes concern primarily the variation of tunnelling rates with the geometrical parameters of flow obstacles. Thus the only conceivable possibility of checking the validity of the theory is the observation of a variation of tunnelling rates with micro-orifice surface roughness; such an experiment, in a reproducible fashion, has not been carried out yet. In order to make further progress in relating the experimental findings to a suitable theory, it appears from these arguments that one is required to go beyond the Magnus force dominance in the tunnelling action and consider the modifications of this dominance in the mesoscopic scale domain, in particular by the interaction of the vortex with the elementary excitation spectrum of the superfluid. This interaction should, on these scales, change the relevant effective forces acting on the vortex, leading to measurable effects on the tunnelling probability. The same considerations essentially apply to the possible observation of tunnelling of flux quanta (i.e., of the magnetization) in high-$T_c$ superconductors, however with a considerable amount of complications, caused by different parameter domains, as mentioned in the third section. In addition, in the general case, the requirement that proper electromagnetic gauge invariance is to be satisfied should play an important role.

Future research, along the directions we have been alluding to above, is needed to shed further light on the intrinsic process of the genesis of quantized vortices in superfluids.
References

[1] L. Onsager, in the section Discusione et Osservazioni of Nuovo Cimento Suppl. 6 (1949) 249, also cf. a footnote in Ref. [3].
[2] L. D. Landau, E. M. Lifshitz, Quantum Mechanics, Pergamon Press, Second Edition 1965
[3] L. P. Pitaevskii, Bose-Einstein condensation in magnetic traps. Introduction to the theory, Physics-Uspekhi 41,6 (1998) 569 [Usp. Fiz. Nauk 168 (1998) 641]
[4] P.-G. de Gennes, Superconductivity of Metals and Alloys, Translation from French, W. A. Benjamin, New York, 1966
[5] A. L. Fetter, J. D. Walecka, Quantum Theory of Many-Particle Systems, McGraw-Hill, 1971
[6] S. Coleman, Aspects of Symmetry, Cambridge University Press, 1985
[7] R. J. Donnelly, Experimental Superfluidity, University of Chicago Press, 1967
[8] R. P. Feynman, Atomic Theory of the λ Transition in Helium, Phys. Rev. 91 (1953) 1291
[9] R. P. Feynman, Atomic Theory of the Two-Fluid Model of Liquid Helium, Phys. Rev. 94 (1954) 262
[10] R. P. Feynman, Application of Quantum Mechanics to Liquid Helium, in Progress in Low Temperature Physics I, ed. C. Gorter, North-Holland, Amsterdam, 1955, Chapter 2
[11] U. Parts et al., Single-Vortex Nucleation in Rotating Superfluid 3He-B, Europhys. Lett. 31 (1995) 449
[12] A. Schmid, Quasiclassical Wave Function in Multidimensional Quantum Decay Problems, Ann. Phys. (N.Y.) 170 (1986) 333
[13] U. Eckern, A. Schmid, The Decay of a Metastable State in a Multidimensional Configuration Space, in “Quantum Tunnelling in Condensed Media”, Yu. Kagan and A. J. Leggett (Eds.), Chapter 3, Elsevier 1992
[14] S. Coleman, Fate of the false vacuum: Semiclassical Theory, Phys. Rev. D 15 (1977) 2929
[15] C. G. Callan, S. Coleman, Fate of the false vacuum. II. First quantum corrections, Phys. Rev. D 16 (1977) 1762
[16] H. Kleinert, Path Integrals in Quantum Mechanics, Statistics, and Polymer Physics, World Scientific, 1990
[17] U. R. Fischer, Motion of Quantized Vortices as Elementary Objects, Ann. Phys. (N.Y.) 278 (1999) 62
[18] L. Onsager, Statistical Hydrodynamics, Nuovo Cimento Suppl. 6 (1949) 279
[19] M. V. Berry, Quantal phase factors accompanying adiabatic changes, Proc. Roy. Soc. London A 392 (1984) 45
[20] Y. Nambu, Strings, monopoles, and gauge fields, Phys. Rev. D 10 (1974) 4262
[21] R. L. Davis, Quantum Turbulence, Phys. Rev. Lett. 64 (1990) 2519
[22] H.-c. Kao, K. Lee, Quantum Nucleation of vortex string loops, Phys. Rev. D 52 (1995) 6050
[23] I. S. Gradshteyn, I. M. Ryzhik, Table of Integrals, Series, and Products, Fourth Edition, Academic Press, 1965; Corrected and Enlarged Edition, 1980
[24] P. H. Roberts, J. Grant, Motions in a Bose condensate I. The structure of the large circular vortex, J. Phys. A: Gen. Phys. 4 (1971) 55
[25] G. E. Volovik, Quantum-Mechanical Formation of Vortices in a superfluid liquid, JETP Lett. 15 (1972) 81
[26] U. R. Fischer, Geometric Laws of Vortex Quantum Tunneling, Phys. Rev. B 58 (1998) 105
[27] L. M. Milne-Thomson, Theoretical Hydrodynamics, Fifth Edition, Macmillan, 1968
[28] E. B. Sonin, Critical velocities at very low temperatures, and the vortices in a quantum Bose fluid, JETP 37 (1973) 494
[29] E. B. Sonin, Nucleation and creep of vortices in superfluids and clean superconductors, Physica B 210 (1995) 234
[30] R. J. Donnelly, *Quantized Vortices in Helium II*, Cambridge University Press, Cambridge, 1991
[31] D. P. Arovas, J. A. Freire, *Dynamical Vortices in Superfluid Films*, Phys. Rev. B **55** (1997) 1068
[32] V. Apaja, M. Saarela, *Current patterns in the phonon-maxon-roton excitations in $^4$He*, Phys. Rev. B **57** (1998) 5358
[33] G. Ortiz, D. M. Ceperley, *Core structure of a Vortex in Superfluid $^4$He*, Phys. Rev. Lett. **75** (1995) 4642
[34] S. A. Vitiello, L. Reatto, G. V. Chester, M. H. Kalos, *Vortex line in superfluid $^4$He: A variational Monte Carlo calculation*, Phys. Rev. B **57** (1998) 5358
[35] D. E. Galli, E. Cecchetti, L. Reatto, *Rotons and Roton Wave Packets in Superfluid $^4$He*, Phys. Rev. Lett. **77** (1996) 5401
[36] M. Sadd, G.V. Chester, L. Reatto, *Structure of a Vortex in Superfluid $^4$He*, Phys. Rev. Lett. **79** (1997) 2490
[37] F. Dalfovo, A. Lastrisch, L. Pricauenko, S. Stringari, J. Treiner, *Structural and dynamical properties of superfluid helium: A density-functional approach*, Phys. Rev. B **52** (1995) 1193
[38] O. Avenel, E. Varoquaux, *Observation of Singly Quantized Dissipation Events obeying the Josephson Frequency Relation in the Critical Flow of Superfluid $^4$He through an Aperture*, Phys. Rev. Lett. **55** (1985) 2704
[39] O. Avenel, E. Varoquaux, *Josephson Effect and Phase Slippage in Superfluids*, Jpn. J. Appl. Phys. **26**, Supplement 26-3 (1987) 1798
[40] O. Avenel, E. Varoquaux, *Josephson Effect and Quantum Phase Slippage in Superfluids*, Phys. Rev. Lett. **60** (1988) 416
[41] E. Varoquaux, G. G. Ihas, O. Avenel, R. Aarts, *Dissipative Flow of Superfluid $^4$He through a Small Orifice by Quantum and Thermal Nucleation of Vortices*, J. Low Temp. Phys. **89** (1992) 207
[42] G. G. Ihas et al., *Quantum Nucleation of Vortices in the Flow of Superfluid $^4$He through an Orifice*, Phys. Rev. Lett. **69** (1992) 327
[43] E. Varoquaux, G. G. Ihas, O. Avenel, R. Aarts, *Vortex Nucleation in Superfluid $^4$He Probed by $^3$He Impurities*, Phys. Rev. Lett. **70** (1993) 2114
[44] O. Avenel, G. G. Ihas, E. Varoquaux, *The Nucleation of Vortices in Superfluid $^4$He: Answers and Questions*, J. Low Temp. Phys. **93** (1993) 1031
[45] S. Burkhardt, M. Bernhard, O. Avenel, E. Varoquaux, *Scenario for a Quantum Phase Slip in Superfluid $^4$He*, Phys. Rev. Lett. **72** (1994) 380
[46] P. Hakonen, O. Avenel, E. Varoquaux, *Evidence for Single-vortex Pinning and Unpinning Events in Superfluid $^4$He*, Phys. Rev. Lett. **81** (1998) 3451
[47] K. W. Schwarz, *Fluid Dynamics of a Quantized Vortex Filament in a Hole*, J. Low Temp. Phys. **93** (1993) 1019
[48] K. W. Schwarz, *Phase Slip and Phase-Slip Cascades in $^4$He Superflow through a Small Orifice*, Phys. Rev. Lett. **71** (1993) 259
[49] J. C. Davis et al., *Evidence for Quantum Tunneling of Phase-Slip Vortices in Superfluid $^4$He*, Phys. Rev. Lett. **69** (1992) 323
[50] A. Amar, Y. Sasaki, R. J. Lozes, J. C. Davis, R. E. Packard, *Quantized Phase Slippage in Superfluid $^4$He*, Phys. Rev. Lett. **68** (1992) 2624
[51] O. Avenel, E. Varoquaux, *Detection of the Earth Rotation with a Superfluid Double-hole Resonator*, Czech. J. Phys. **46**, Suppl. S6 (1996) 3319
[52] K. Schwab, N. Bruckner, R. E. Packard, *Detection of the Earth’s rotation using superfluid phase coherence*, Nature **386** (1997) 585
[53] O. Avenel, P. Hakonen, E. Varoquaux, *Detection of the Rotation of the Earth with a Superfluid Gyrometer*, Phys. Rev. Lett. **78** (1997) 3602
[54] B. D. Josephson, *Potential Differences in the Mixed State of Type II Superconductors*, Phys. Lett. **16** (1965) 242
Tunnelling of defects in superfluids

[55] P. W. Anderson, Considerations on The Flow of Superfluid Helium, Rev. Mod. Phys. 38 (1966) 298
[56] E. R. Huggins, Energy-Dissipation Theorem and Detailed Josephson Equation for Ideal Incompressible Fluids, Phys. Rev. A 1 (1970) 332
[57] H. Fröhlich, A contradiction between quantum hydrodynamics and the existence of particles, Physica 34 (1967) 47
[58] D. T. Pegg, S. M. Barnett, Unitary Phase Operator in Quantum Mechanics, Europhys. Lett. 6 (1988) 483
[59] L. P. Pitaevskii, Vortex lines in an imperfect Bose gas, JETP 13 (1961) 451
[60] E. P. Gross, Classical Theory of Boson Wave Fields, Ann. Phys. (N.Y.) 4 (1958) 57
[61] E. P. Gross, Quantum Theory of Interacting Bosons, Ann. Phys. (N.Y.) 9 (1960) 292
[62] E. P. Gross, Structure of a Quantized Vortex in Boson Systems, Nuovo Cimento 20 (1961) 454
[63] E. P. Gross, Hydrodynamics of a Superfluid Condensate, J. Math. Phys. 4 (1963) 195
[64] I. Affleck, Quantum-Statistical Metastability, Phys. Rev. Lett. 46 (1981) 388
[65] D. A. Gorokhov, G. Blatter, Quantum depinning of a pancake vortex from a columnar defect, Phys. Rev. B 57 (1998) 3586
[66] A. A. Sobyanin, A. A. Stratonnikov, Surface tension of helium II and extrapolation length for the order parameter, JETP Lett. 45 (1987) 613
[67] E. A. Kuznetsov, J. Juul Rasmussen, Self-focusing instability of two-dimensional solitons and vortices, JETP Lett. 62 (1995) 105
[68] C. Josserand, Y. Pomeau, Generation of Vortices in a Model of Superfluid 4He by the Kadomtsev-Petviashvili Instability, Europhys. Lett. 30 (1995) 43
[69] P. I. Soi ninen, N. B. Kopnin, Stability of superflow, Phys. Rev. B 49 (1994) 12087
[70] M. Stone, A. Srivastava, Boundary Layer Separation and Vortex Creation in Superflow Through Small Orifices, J. Low Temp. Phys. 102 (1996) 445
[71] J. Bardeen, L. N. Cooper, J. R. Schrieffer, Theory of Superconductivity, Phys. Rev. 108 (1957) 1175
[72] G. E. Volovik, Exotic Properties of superfluid 3He, World Scientific, 1992
[73] U. Weiss, Quantum Dissipative Systems, World Scientific, 1993
[74] D. A. Wollman et al., Experimental Determination of the Superconducting Pairing State in YBCO from the Phase Coherence of YBCO-Pb dc SQUIDs, Phys. Rev. Lett. 71 (1993) 2134
[75] C. C. Tsuei et al., Pure $d_{x^2-y^2}$ order-parameter symmetry in the tetragonal superconductor $Tl_2Ba_2CuO_6+\delta$, Nature 387 (1997) 481
[76] E. B. Sonin, The Magnus force in superfluids and superconductors, Phys. Rev. B 55 (1997) 485
[77] G. E. Volovik, Three nondissipative forces on a moving vortex line in superfluids and superconductors, JETP Lett. 62 (1995) 65
[78] C. Caroli, P. G. de Gennes, J. Matricon, Bound States on a Vortex Line in a Type II Superconductor, Phys. Lett. 9 (1964) 307
[79] A. van Otterlo, M. Feigel'man, V. Geshkenbein, G. Blatter, Vortex Dynamics and the Hall Anomaly: A Microscopic Analysis, Phys. Rev. Lett. 75 (1995) 3736
[80] M. Stone, Spectral flow, Magnus force, and mutual friction via the geometric optics limit of Andreev reflection, Phys. Rev. B 54 (1996) 13222
[81] N. B. Kopnin, G. E. Volovik, U. Parts, Spectral Flow in Vortex Dynamics of 3He-B and Superconductors, Europhys. Lett. 32 (1995) 651
[82] N. B. Kopnin, Theory of mutual friction in superfluid 3He at low temperatures, Physica B 210 (1995) 267-286
[83] N. B. Kopnin, G. E. Volovik, Flux Flow in d-Wave Superconductors: Low Temperature Universality and Scaling, Phys. Rev. Lett. 79 (1997) 1377
[84] N. B. Kopnin, Resonant absorption at the vortex core states in d-wave superconductors, Phys. Rev. B 57 (1998) 11775
[85] Yu. G. Makhlin, *Spectral flow in vortex dynamics of d-wave superconductors*, Phys. Rev. D **56** (1997) 11872

[86] G. Blatter, V. B. Geshkenbein, N. B. Kopnin, *From microscopic theory to Boltzmann kinetic equation: Application to vortex dynamics*, Phys. Rev. B **59** (1999) 14663

[87] J. M. Harris et al., *Hall Angle Evidence for the Superclean Regime in 60 K YBa$_2$Cu$_3$O$_{6+y}$*, Phys. Rev. Lett. **73** (1994) 1711

[88] J.-M. Duan, *Mass of a vortex line in superfluid $^4$He: Effects of gauge-symmetry breaking*, Phys. Rev. B **49** (1994) 12381

[89] N. B. Kopnin, V. M. Vinokur, *Dynamic Vortex Mass in Clean Fermi Superfluids and Superconductors*, Phys. Rev. Lett. **81** (1998) 3952

[90] A. O. Caldeira, A. J. Leggett, *Quantum tunneling with dissipation*, JETP Lett. **37** (1983) 383

[91] A. Schmid, *Diffusion and Localization in a Dissipative Quantum System*, Phys. Rev. Lett. **51** (1983) 1506
[111] D. Monier, L. Fruchter, *Thermal-to-quantum crossover of the flux-line dynamics in Bi$_2$Sr$_2$CaCu$_2$O$_8$*, Phys. Rev. B 58 (1998) R8917

[112] T. Stein et al., *Quantum Creep in Y$_{1-x}$Pr$_x$Ba$_2$Cu$_3$O$_{7-\delta}$ Crystals: Magnetic Relaxation and Transport*, Phys. Rev. Lett. 82 (1999) 2955