Database Aggregation

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Abstract

Knowledge can be represented compactly in a multitude ways, from a set of propositional formulas, to a Kripke model, to a database. In this paper we study the aggregation of information coming from multiple sources, each source submitting a database modelled as a first-order relational structure. In the presence of an integrity constraint, we identify classes of aggregators that respect it in the aggregated database, provided all individual databases satisfy it. We also characterise languages for first-order queries on which the answer to queries on the aggregated database coincides with the aggregation of the answers to the query obtained on each individual database. This contribution is meant to be a first step on the application of techniques from rational choice theory to knowledge representation in databases.

1 Introduction

Aggregating information coming from multiple sources is a long-standing problem in both knowledge representation and the study of multi-agent systems (see, e.g., [25]). Depending on the chosen representation for the incoming pieces of knowledge or information, a number of competing approaches has seen the light in these literatures. Belief merging [21, 20, 19] studies the problem of aggregating propositional formulas coming from a number of different agents into a set of models, subject to an integrity constraint. Judgment and binary aggregation [11, 9, 16] asks individual agents to report yes/no opinions on a set of logically connected binary issues, called the agenda, to take a collective decision. Social welfare functions, the cornerstone problem in social choice theory (see, e.g., [2]), can also be viewed as mechanisms merging conflicting information, namely the individual preferences of voters expressed in the form of linear orders over a set of alternatives. Other examples include graph aggregation [12], which has applications in multi-agent argumentation [4, 5, 6] and clustering aggregation [14], as well as ontology merging [24].

In this work we take a general perspective and we represent individual knowledge coming from multiple sources as a profile of databases, modelled as finite relational
structures [1][22]. Our aim is to reconcile two possibly conflicting views of the problem of information fusion. On the one hand, the study of information merging (typically knowledge or beliefs) in knowledge representation has focused on the design of rules that guarantee the consistency of the outcome, with the main driving principles inspired from the literature on belief revision[1]. On the other hand, social choice theory has focused on agent-based properties, such as fairness and representativity of an aggregation procedure, paying attention as well on possible strategic behaviour by either the agents involved in the process or an external influencing source. While there already have been several attempts at showing how specific merging or aggregation frameworks could be simulated or subsumed by one another (see, e.g., [15, 7, 17, 13]), we believe that a more general perspective is needed to reconcile the two views described above. Perhaps the closest approach to ours is the work of Baral et al. [3]. In their paper, the authors consider the problem of merging information represented in the form of a first-order theory, taking a syntactic rather than a semantic approach, and focuses on finding maximally consistent sets of the union of the individual theories received. In doing so, however, the authors privilege the knowledge representation approach, and have no control on the set of agents supporting a given maximally consistent set rather than another.

Our starting point is a set of finite relational structures on the same signature, coming from a set of agents or sources, and our research problem is how to obtain a collective databases summarising the information received. Virtually all of the settings mentioned above (beliefs, graphs, preferences, judgments...) can be represented as databases, showing the generality of our framework. We propose a number of rules for database aggregation, inspired by existing ones proposed in the literature on computational social choice, and we evaluate them axiomatically. We privilege computationally friendly aggregators, for which the time to determine the collective outcome is polynomial in the time spent reading the individual input received.

When integrity constraints are present, we study how to guarantee that a given aggregator “lifts” the integrity constraint from the individual to the collective level, i.e., the aggregated databases satisfies the same constraints as the individual ones. We first analyse the problem of lifting first-order formulas in database aggregation theoretically, comparing the results obtained with the literature on lifting propositional constraints in binary aggregation. We provide characterisation results for a number of natural restricted languages, and we investigate which of the rules we introduced lift classical integrity constraints from database theory: functional dependencies, referential integrity constraints, and value constraints.

Since databases are typically queried using formulas in first order logic, a natural question to ask in a multi-agent setting is whether the aggregation of the individual answers to a query coincides with the answer to the same query on the aggregated database. We provide a partial answer to this important problem, by identifying sufficient conditions on the first-order query language for both the intersection and the union operator.

The paper is organised as follows. In Section[2][3] we introduce the basic definitions

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1 Albeit we acknowledge the work of [10][23], which aggregate individual beliefs, modelled as plausibility orders, in an “Arrovian” fashion.
of databases and integrity constraints. In Sections 3 and 4 we introduce a number of database aggregation procedures, and we propose axiomatic properties for their studies. Sections 5, 6, and 7 contain our main results on the lifting of integrity constraints and aggregated query answering. Section 8 concludes the paper.

2 Preliminaries on Databases

In this section we introduce basic notions on databases that we will use in the rest of the paper. In particular, we adopt a relational perspective [1] and define a database as a finite relational structure over a database schema:

Definition 1 (Database Schema). A (relational) database schema \( \mathcal{D} \) is a finite set \( \{ P_1/q_1, \ldots, P_n/q_n \} \) of relation symbols \( P \) with arity \( q \in \mathbb{N} \).

In the following we assume a countable domain \( U \) of elements \( u, u', \ldots \), for the interpretation of relation symbols in a database schema \( \mathcal{D} \).

Definition 2 (Database Instance). Given domain \( U \) and database schema \( \mathcal{D} \), a \( \mathcal{D} \)-instance over \( U \) is a mapping \( D \) associating each relation symbol \( P \in \mathcal{D} \) with a finite \( q \)-ary relation over \( U \), i.e., \( D(P) \subset \text{fin } U^q \).

By Def. 2 a database instance is a finite (relational) model of a database schema. The active domain \( \text{adom}(D) \) of an instance \( D \) is the set of all individuals in \( U \) occurring in some tuple \( \vec{u} \) of some predicate interpretation \( D(P) \), that is, \( \text{adom}(D) = \bigcup_{P \in \mathcal{D}} \{ u \in U \mid u = u_i \text{ for some } \vec{u} \in D(P) \} \). Observe that, since \( \mathcal{D} \) contains a finite number of relation symbols and each \( D(P) \) is finite, so is \( \text{adom}(D) \). We denote the set of all instances on \( \mathcal{D} \) and \( U \) as \( \mathcal{D}(U) \). Clearly, the formal framework for databases we adopt is quite simple, but still it is powerful enough to cover practical cases of interest [22]. Here we do not discuss in details the pros and cons of the relational approach to database theory and refer to the literature for further details [1].

To specify the properties of databases, we make use of first-order logic with equality and no function symbols. Let \( V \) be a countable set of individual variables.

Definition 3 (FO-formulas over \( \mathcal{D} \)). Given a database schema \( \mathcal{D} \), the formulas \( \varphi \) of the first-order language \( \mathcal{L}_D \) are defined by the following BNF:

\[
\varphi ::= \: x = x' \: | \: P(x_1, \ldots, x_q) \: | \: \neg \varphi \: | \: \varphi \rightarrow \varphi \: | \: \forall x \varphi
\]

where \( P \in \mathcal{D}, x_1, \ldots, x_q \) is a \( q \)-tuple of terms and \( x, x' \) are terms.

We assume “=” to be a special binary predicate with fixed obvious interpretation. By Def. 3 \( \mathcal{L}_D \) is a first-order language with equality over the relational vocabulary \( \mathcal{D} \) and with no function symbols. In the following we use the standard abbreviations \( \exists, \land, \lor, \neg \). Also, free and bound variables are defined as standard. For a formula \( \varphi \in \mathcal{L}_D \), we write \( \varphi(x_1, \ldots, x_r) \), or simply \( \varphi(\vec{x}) \), to list explicitly in arbitrary order all free variables \( x_1, \ldots, x_r \) of \( \varphi \). A sentence is a formula with no free variables. Notice that the only terms in our language \( \mathcal{L}_D \) are individual variables. We can add constant
for individuals with some minor technical changes to the definitions and results in the paper. However, these do not impact on the theoretical contribution and we prefer to keep notation lighter.

To interpret FO-formulas on database instances, we introduce assignments as functions \( \sigma : V \mapsto U \). Given an assignment \( \sigma \), we denote by \( \sigma^+_u \) the assignment such that (i) \( \sigma^+_u(x) = u \); and (ii) \( \sigma^+_u(x') = \sigma(x') \), for every \( x' \in V \) different from \( x \). We can now define the semantics of \( L \).

**Definition 4** (Satisfaction of FO-formulas). Given a \( D \)-instance \( D \), an assignment \( \sigma \), and an FO-formula \( \varphi \in L_D \), we inductively define whether \( D \) satisfies \( \varphi \) under \( \sigma \), or \( (D, \sigma) \models \varphi \), as follows:

\[
\begin{align*}
(D, \sigma) &\models P(x_1, \ldots, x_q) \quad \text{iff} \quad \langle \sigma(x_1), \ldots, \sigma(x_q) \rangle \in D(P) \\
(D, \sigma) &\models x = x' \quad \text{iff} \quad \sigma(x) = \sigma(x') \\
(D, \sigma) &\models \neg \varphi \quad \text{iff} \quad (D, \sigma) \not\models \varphi \\
(D, \sigma) &\models \varphi \to \psi \quad \text{iff} \quad (D, \sigma) \not\models \varphi \text{ or } (D, \sigma) \models \psi \\
(D, \sigma) &\models \forall x \varphi \quad \text{iff} \quad \text{for every } u \in \text{edom}(D), (D, \sigma^+_u) \models \varphi
\end{align*}
\]

A formula \( \varphi \) is true in \( D \), written \( D \models \varphi \), if \((D, \sigma) \models \varphi \) for all assignments \( \sigma \).

Observe that we adopt an active-domain semantics, that is, quantified variables range only over the active domain of \( D \). This is standard in database theory [1], where \( \text{edom}(D) \) is assumed to be the “universe of discourse”.

**Constraints.** It is well-known that several properties and constraints on databases can be expressed as FO-sentences. Here we consider some of these for illustrative purposes.

**Definition 5** (Functional Dependency). A functional dependency is an expression of type \( n_1, \ldots, n_k \to n_{k+1}, \ldots, n_q \). A database instance \( D \) satisfies a functional dependency \( n_1, \ldots, n_k \to n_{k+1}, \ldots, n_q \) for predicate symbol \( P \) with arity \( q \) iff for every \( q \)-ple \( \vec{u}, \vec{u}' \) in \( D(P) \), whenever \( u_i = u'_i \) for all \( i \leq k \), then we also have \( u_i = u'_i \) for all \( k+1 \leq i \leq q \). If \( k = 1 \), we say that it is a key dependency.

Clearly, any database instance \( D \) satisfies a functional dependency \( n_1, \ldots, n_k \to n_{k+1}, \ldots, n_q \) iff it satisfies the following:

\[
\forall \vec{x}, \vec{y} \left( P(\vec{x}) \land P(\vec{y}) \land \bigwedge_{i \leq k} (x_i = y_i) \to \bigwedge_{k+1 \leq i \leq q} (x_i = y_i) \right)
\]

**Definition 6** (Value Constraint). A value constraint is an expression of type \( n_k \in D(P_v) \), where \( D(P_v) \) contains a list of admissible values. A database instance \( D \) satisfies a value constraint \( n_k \in P_v \) for predicate symbol \( P \) with arity \( q \geq k \) iff for every \( q \)-ple \( \vec{u} \) in \( D(P) \), \( u_k \in D(P_v) \).

Also for value constraints, it is easy to check that an instance \( D \) satisfies constraint \( n_k \in P_v \) for symbol \( P \) iff it satisfies the following:

\[\forall x_1, \ldots, x_q (P(x_1, \ldots, x_q) \to P_v(x_k))\]
Definition 7 (Referential Integrity Constraint). A referential integrity constraints enforces the foreign key of a predicate \( P_1 \) to be the primary key of predicate \( P_2 \). A database instance satisfy a referential integrity constraint on the last \( k \) attributes, and we denote it \( (P_1 \rightarrow P_2, k) \), if for all \( q_1\)-tuple \( \vec{u} \in D(P_1) \), there exists a \( q_2\)-tuple \( \vec{u}' \in D(P_2) \) such that for all \( 1 \leq i \leq k \) we have that \( u_{q_1-i} = u'_{i} \).

A referential integrity constraint can also be translated in a first-order formula as follows:

\[
\forall \vec{x}[P_1(\vec{x}) \rightarrow \exists \vec{y}(P_2(\vec{y}) \land \bigwedge_{i=1}^{k} (x_{q_1-i} = y_j))] \]

3 Aggregators

The main research question we investigate in this paper regards how to define an aggregated database instance from the instances of \( N = \{1, \ldots, n\} \) agents. This question is typical in social choice theory, where judgements, preferences, etc., are aggregated according to some notion of rationality that will be introduced in Section 5.

For the rest of the paper we fix a database schema \( D \) over a common domain \( U \), and consider a profile \( \vec{D} = (D_1, \ldots, D_n) \) of \( n \) instances over \( D \) and \( U \). Then, we can define an aggregation procedure on such instances.

Definition 8 (Aggregation Procedure). Given database schema \( D \) and domain \( U \), an aggregation procedure \( F : D(U)^n \rightarrow D(U) \) is a function assigning to each tuple \( \vec{D} \) of instances for \( n \) agents an aggregated instance \( F(\vec{D}) \in D(U) \). Let \( F \) be the class of all aggregation procedures.

We use \( N_{\vec{D}}(P) := \{i \in N \mid \vec{u} \in D_i(P)\} \) to denote the set of agents accepting tuple \( \vec{u} \) for symbol \( P \), under profile \( \vec{D} \). Notice that considering a unique domain \( U \) is not really a limitation of the proposed approach: instances \( D_1, \ldots, D_n \), each on a possibly different domain \( U_i \), for \( i \leq n \), can all be seen as instances on \( \bigcup_{i \in N} U_i \).

Hereafter we illustrate and discuss some examples of aggregation procedures:

Union (or nomination): for every \( P \in D \), \( F(\vec{D})(P) = \bigcup_{i \leq n} D_i(P) \). Intuitively, every agent is seen as having partial but correct information about the state of the world. Union can be considered a good aggregator if databases represent the agents’ knowledge bases (certain information).

Intersection (or unanimity): for every \( P \in D \), \( F(\vec{D})(P) = \bigcap_{i \leq n} D_i(P) \). Here every agent is supposed to have a partial and possibly incorrect vision of the state of the world.

Quota rules: a quota rule is an aggregation rule \( F \) defined via functions \( q_P : U^q \rightarrow \{0, 1, \ldots, n + 1\} \), associating each symbol \( P \) and \( q \)-uple with a quota, by stipulating that \( \vec{u} \in F(\vec{D})(P) \) iff \( |\{i \mid \vec{u} \in D_i(P)\}| \geq q_P(\vec{u}) \). \( F \) is called uniform whenever \( q \) is a constant function for all tuples and symbols. Intuitively, if a tuple \( \vec{u} \) appears in at least \( q(\vec{u}) \) of the initial databases, then it is accepted. The (strict) majority rule is a quota rule for \( q = \lceil (n+1)/2 \rceil \); while union and intersection are quota rule for \( q = 1 \) and...
q = n respectively. We call the uniform quota rules for q = 0 and q = n + 1 trivial rules.

**Distance-based function:** The symmetric distance can be used to measure dissimilarity between databases, obtaining the following definition:

\[
F(\tilde{D})(P) = \arg\min_{A \subseteq U^+} \sum_{i \in N} (|D_i(P) \setminus A| + |A \setminus D_i(P)|)
\]

Intuitively, the symmetric distance minimizes the “distance” between the aggregated database \(F(\tilde{D})\) and each \(D_i\), defined as the number of tuples in \(D_i\) but not in \(F(\tilde{D})\) but not in \(D_i\), calculated across all \(i \in N\).

**Dictatorship of agent** \(i^* \in N\): we have that \(F(\tilde{D}) = D_{i^*}\), i.e., the dictator \(i^*\) completely determines the aggregated database.

**Oligarchy of coalition** \(C^* \subseteq N\): for every \(P \in D\), \(F(\tilde{D})(P) = \bigcap_{i \in C^*} D_i(P)\). Oligarchy reduces to dictatorship for singletons, and to intersection for \(C^* = N\).

Quota rules are inspired by their homonyms in judgment aggregation [8], introduced as a generalisation of the classic majority rule. The union and the intersection rules are well-known in the area of modal epistemic logic, corresponding, respectively, to distributed knowledge and “everybody knows that” [18]. Distance-based procedures have been widely studied and axiomatised in the area of logic-based belief merging [20], while dictatorships and oligarchies are classical notions from social choice theory. Obviously, different aggregation procedures can be thought of. We chose to focus on those above in the following, as they are well-studied in the literature and have nice computational properties such as being computable in polynomial time.

### 4 The Axiomatic Method

Aggregation procedures are best characterised by means of axioms. In particular, we consider the following properties, where relation symbols \(P, P' \in D\), profiles \(\tilde{D}, \tilde{D}' \in D(U)^n\), tuples \(\vec{u}, \vec{u}' \in U^+\) are all universally quantified.

**Independence** (I): if \(N_{\vec{u}}(\tilde{D})(P) = N_{\vec{u}}(\tilde{D}')(P)\) then \(\vec{u} \in F(\tilde{D})(P)\) iff \(\vec{u} \in F(\tilde{D}')(P)\).

Intuitively, if the same agents accept (resp. reject) a tuple in two different profiles, then the tuple is accepted (resp. rejected) in both aggregated instances. The axiom of independence is a widespread requirement from social choice theory, and is arguably the main cause of most impossibility theorems, such as Arrow’s seminal result [2]. From a computational perspective, independent rules are typically easier to compute than non-independent ones. Clearly, quota rules satisfy independence; while neither dictatorship nor oligarchies do.

**Unanimity** (U): \(F(\tilde{D})(P) \supseteq \bigcap_{i \in N} D_i(P)\).

That is, a tuple accepted by all agents, also appears in the aggregated database (for the relevant relation symbol). In particular, all rules in Section 3 satisfy unanimity.

**Groundedness** (G): \(F(\tilde{D})(P) \subseteq \bigcup_{i \in N} D_i(P)\).

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By groundedness any tuple appearing in the aggregated database must be accepted by some agent. All rules from Section 3, with the exception of the distance-based rule, satisfy this property.

**Anonymity (A):** for every permutation \( \pi : \mathcal{N} \rightarrow \mathcal{N} \), we have \( F(D_1, \ldots, D_n) = F(D_{\pi(1)}, \ldots, D_{\pi(n)}) \).

Here the identity of agents is irrelevant for the aggregation procedure. Clearly, this is the case for all aggregators in Section 3 but dictatorship and oligarchy.

**Positive Neutrality (N\(+\)):** if \( N_u^D(P) = N_u^{D'}(P) \) then \( \vec{u} \in F(\vec{D})(P) \) iff \( \vec{u}' \in F(\vec{D})(P) \).

**Negative Neutrality (N\(-\)):** if \( N_u^D(P) = \mathcal{N} \setminus N_u^{D'}(P) \) then \( \vec{u} \in F(\vec{D})(P) \) iff \( \vec{u}' \notin F(\vec{D})(P) \).

Observe that both versions of neutrality differs from independence as here we consider two different tuples in the same profile, while independence deals with the same tuple in two different profiles. We can easily see that all aggregators introduced in Section 3 but dictatorship and oligarchies satisfy positive neutrality and, with the exception of most quota rules (see below), negative neutrality as well.

**Systematicity (S):** if \( N_u^D(P) = N_u^{D'}(P') \) then \( \vec{u} \in F(\vec{D})(P) \) iff \( \vec{u}' \in F(\vec{D})(P') \).

Observe that systematicity is equivalent to the conjunction of neutrality and independence.

**Permutation-Neutrality (N\(^\rho\)):** Given a permutation \( \rho : U \rightarrow U \) over domain \( U \), and its straightforward lifting to a profile \( \vec{D} \), then \( F(\rho(\vec{D})) = \rho(F(\vec{D})) \).

Again, all aggregators but dictatorship and oligarchies satisfy permutation-neutrality.

**Monotonicity (M):** if \( \vec{u} \in F(\vec{D})(P) \) and for every \( i \in \mathcal{N} \), either \( D_i(P) = D'_i(P) \) or \( D_i(P) \cup \{ \vec{u} \} \subseteq D'_i(P) \), then \( \vec{u} \in F(D')(P) \).

Intuitively, a monotonic aggregators keeps on accepting a given tuple if the support for that tuple increases.

Combinations of the axioms above can be used to characterise some of the rules that we defined in Section 3. Some of these results, such as the following, lift to databases known results in judgement (propositional) aggregation.

**Lemma 1.** An aggregation procedure satisfies A, I, and M iff it is a quota rule.

**Proof.** The implication from right to left follows from the fact that quota rules satisfy independence I, anonymity A, and monotonicity M, as we remarked above.

For the implication from left to right, observe that, to accept a given tuple \( \vec{u} \) in \( F(\vec{D})(P) \), an independent aggregation procedure will only look at the set of agents \( i \in \mathcal{N} \) such that \( \vec{u} \in D_i(P) \). If the procedure is also anonymous, then acceptance is based only on the number of individuals admitting the tuple. Finally, by monotonicity, there will be some minimal number of agents required to trigger collective acceptance. That number is the quota associated with the tuple and the symbol in hand.

If we add neutrality (both positive and negative), then we obtain the class of uniform quota rules. If we furthermore impose unanimity and groundedness, then this excludes the trivial quota rules.
Lemma 2. If the number of individuals is odd and $|D| \geq 2$, an aggregation procedure $F$ satisfies $A$, $N^-$, $N^+$, $I$ and $M$ on the full domain $D(U)^n$ if and only if it is the majority rule.

Proof. By neutrality the quota must be the same for all tuples and all relation symbols. By negative-neutrality the two sets $N^\vec{D}(P)$ and $N \setminus N^\vec{D}(P)$ must be treated symmetrically. Hence, the only possibility is to have a uniform quota of $(n + 1)/2$. \hfill \Box

The corresponding versions of these results have been shown in judgment and graph aggregation [8, 12]. Notice however that there are some notable differences w.r.t. the literature. For instance, the axiom of neutrality is here split into a positive and a negative part.

We conclude this section by showing the following equivalence between majority and distance-based rules.

Lemma 3. In the absence of integrity constraints, and for an odd number of agents, the distance-based rule coincides with the majority rule.

Proof. By the definition of the distance based rule, we have that

$$F(\vec{D})(P) = \arg\min_{A \subseteq U^n} \sum_{i \in \mathcal{N}} (|D_i(P) \setminus A| + |A \setminus D_i(P)|)$$

With a slight abuse of notation, if $A \subseteq U^m$ let $A(\vec{u})$ be its characteristic function. Since the minimisation is not constrained, and all structures are finite, this is equivalent to:

$$F(\vec{D})(P) = \arg\min_{A \subseteq U^n} \sum_{i \in \mathcal{N}} \sum_{\vec{u} \in U^p} |D_i(P)(\vec{u}) - A(\vec{u})|$$

Therefore, for each $\vec{u}$, if for a majority of the individuals in $\mathcal{N}$ we have that $\vec{u} \in D_i(P)$, then $\vec{u} \in A$ minimises the overall distance, and symmetrically for the case in which a majority of individuals are such that $\vec{u} \notin D_i(P)$. \hfill \Box

5 Lifting Constraints

In this section we analyse further the properties of the aggregation procedures introduced in Section 3. Specifically, we present a notion of collective rationality that aims to capture the appropriateness of a given aggregator $F$ w.r.t. some constraint $\varphi$ on the input instances $D_1, \ldots, D_n$. Hereafter let $\varphi$ be a sentence in the first-order language $L_D$ associated to $D$, interpreted as a common constraint that is satisfied by all $D_1, \ldots, D_n$. Here we are interested in the following notion:
Definition 9 (Collective Rationality). A constraint $\varphi$ is lifted by an aggregation procedure $F$ if whenever $D_i \models \varphi$ for all $i \in \mathcal{N}$, then also $F(D) \models \varphi$.

An aggregation procedure $F : \mathcal{D}(U)^n \to \mathcal{D}(U)$ is collectively rational (CR) with respect to $\varphi$ iff $F$ lifts $\varphi$.

Intuitively, an aggregator is CR w.r.t. constraint $\varphi$ iff it lifts, or preserves, $\varphi$.

Example 1. We now provide an illustrative example of first-order collective (ir)rationality with the majority rule. Consider agents 1 and 2 with database schema $\mathcal{D} = \{ P/1, Q/2 \}$. Two database instances are given as $D_1 = \{ D(a), Q(a, b) \}$ and $D_2 = \{ D(a), Q(a, c) \}$. Clearly, both instances satisfy the integrity constraint $\varphi = \forall x (P(x) \rightarrow \exists y Q(x, y))$. However, their aggregate $D = F(D_1, D_2) = \{ D(a) \}$, obtained by the majority rule, does not satisfy $\varphi$. This example, which can be considered a paradox in the sense of [16], shows that not every constraint in the language $\mathcal{L}_{\mathcal{D}}$ is collective rational w.r.t. majority, thus obtaining a first, simple negative result.

One natural question to ask about lifting of constraints is the following.

Question 4. Given an axiom $AX$, what is the class of constraints that are lifted by all aggregators $F$ satisfying $AX$?

To make this question more precise, consider the following definition.

Definition 10. Given a language $\mathcal{L} \subseteq \mathcal{L}_{\mathcal{D}}$, define $CR[\mathcal{L}]$ as the class of aggregation procedures that lift all $\varphi \in \mathcal{L}$:

$$CR[\mathcal{L}] := \{ F : \mathcal{D}(U)^n \to \mathcal{D} \mid F \text{ is CR for all } \varphi \in \mathcal{L} \}$$

Moreover, an aggregator $F$ satisfies a set $AX$ of axioms w.r.t. language $\mathcal{L}$, if $F$ satisfies the axioms in $AX$ on set $\{ D \in \mathcal{D}(U) \mid D \models \varphi \}$ for all constraints $\varphi \in \mathcal{L}$. The class of all such aggregators is given as:

$$F_{\mathcal{L}}[AX] := \{ F : \mathcal{D}(U)^n \to \mathcal{D}(U) \mid F \text{ satisfies } AX \text{ on } \{ D \in \mathcal{D}(U) \mid D \models \varphi \} \text{ for all } \varphi \in \mathcal{L} \}$$

The following Lemmas extend results in [16] to the case of database aggregation. Hereafter, for a language $\mathcal{L}$ and operator $\bullet$, $\mathcal{L}^\bullet$ is the language obtained by closing formulas in $\mathcal{L}$ under $\bullet$. The proofs are immediate, so we omit them. We only remark that point (3) follows from the fact that the constraints $\varphi \in \mathcal{L}$ are assumed to be sentences.

Lemma 5. For every language $\mathcal{L} \subseteq \mathcal{L}_{\mathcal{D}}$:

1. $CR[\mathcal{L}^\wedge] = CR[\mathcal{L}^\exists] = CR[\mathcal{L}]$
2. $CR[\mathcal{L} \cup \{ \top \}] = CR[\mathcal{L} \cup \{ \bot \}] = CR[\mathcal{L}]$

Moreover,

3. $CR[\mathcal{L}^\forall] = CR[\mathcal{L}^\exists] = CR[\mathcal{L}]$
By Lemma 5 an aggregator $F$ is CR w.r.t. a language $L$ iff it is CR w.r.t. the closure of $L$ under either conjunction, or coimplication, or universal or existential quantification. Also, adding either $\top$ or $\bot$ does not change collective rationality.

Furthermore, the following result extends Lemma 7 in [16]. Also in this case, proofs are immediate and therefore omitted.

**Lemma 6.** For all languages $L_1, L_2 \subseteq L_D$,

1. If $L_1 \subseteq L_2$ then $CR[L_2] \subseteq CR[L_1]$
2. $CR[L_1 \cup L_2] = CR[L_1] \cap CR[L_2]$

By Lemma 6 collective rationality is anti-monotone w.r.t. language inclusion, and an aggregator $F$ is CR w.r.t. the union of languages iff it is CR w.r.t. each language separately.

The next results, which extend Lemma 8 in [16], relate collective rationality with axioms.

**Lemma 7.** For all languages $L_1, L_2 \subseteq L_D$,

1. If $L_1 \subseteq L_2$ then $F_{L_2}[AX] \subseteq F_{L_1}[AX]$
   In particular, if $\top \in L$ then $F_{L}[AX] \subseteq F_{\{\top\}}[AX]$
2. $F_{L}[AX_1, AX_2] = F_{L}[AX_1] \cap F_{L}[AX_2]$

**Proof.** As regards (1), if $F$ satisfies $AX$ on $\{D \in D(U) \mid D \models \varphi\}$, for all $\varphi \in L_2$, and $L_1 \subseteq L_2$, then in particular it satisfies $AX$ on $\{D \in D(U) \mid D \models \varphi\}$, for all $\varphi \in L_1$. Then, (2) follows immediately from (1), as $\{\top\} \subseteq L$. As for (3), $F$ satisfies $AX_1$ and $AX_2$ on $\{D \in D(U) \mid D \models \varphi\}$, for all $\varphi \in L$, if both $F$ satisfies $AX_1$ and $F$ satisfies $AX_2$.

However, not all results available at the propositional level extend to the first order. In particular, the following result means that Lemma 6 in [16] does not lift to the first order.

**Lemma 8.** There exists languages $L_1$ and $L_2$, both containing $\top$ and $\bot$, such that $L_1 \neq L_2$ but $CR[L_1] = CR[L_2]$.

**Proof.** Consider languages $L_1 = \{\bot, \top\}$ and $L_2 = L_1 \cup \{\forall x P(x)\}$ on $D = \{P/1\}$. By Lemma 6(1), $CR[L_2] \subseteq CR[L_1]$. Now, suppose that $F \in CR[L_1]$ and consider a profile $\bar{D}$ such that $D_i \models \forall x P(x)$ for all $i \in N$. By definition, $F(\bar{D}) \in D(U)$. We consider two alternatives: either $F(\bar{D})$ is empty and then $F(\bar{D}) \models \forall x P(x)$ trivially; or $F(\bar{D})$ is not empty, then $F(\bar{D})(P) \subseteq U$ and $F(\bar{D}) \models \forall x P(x)$ as well. As a result, $CR[L_1] \subseteq CR[L_2]$.

By Lemma 8 the operator $CR[-]$ from languages to sets of aggregators is not injective in general.

Symmetrically, we introduce an operator $LF[-]$ from sets of aggregators to languages.
Definition 11 (Lifted Language). Given a set \( \mathcal{G} \) of aggregation procedures, let \( LF[\mathcal{G}] \) be the language of the constraints that are lifted by all \( F \in \mathcal{G} \):

\[
LF[\mathcal{G}] := \{ \varphi \in L_D \mid F \text{ is CR w.r.t. } \varphi, \text{ for all } F \in \mathcal{G} \}
\]

Clearly, \( LF[\mathcal{G}] \) is the intersection of all \( LF[\{F\}] \), for \( F \in \mathcal{G} \).

Lemma 8 has an impact on the following result, which correspond to Proposition 9 in [16]. In particular, while in [16] we have equality for item (1), here we only have inclusion.

Proposition 9. Let \( L \) a language containing \( \top \) and \( \bot \), and \( \mathcal{G} \) a class of aggregators. Then,

1. \( L \subseteq LF[CR[L]] \), and this inclusion is strict for some languages.
2. \( \mathcal{G} \subseteq CR[LF[\mathcal{G}]] \), and this inclusion is strict for some classes.

Proof. As regards (1), inclusion \( L \subseteq LF[CR[L]] \) is an immediate consequence of the definitions of \( CR \) and \( LF \). On the other hand, consider languages \( L_1 = \{ \bot, \top \} \) and \( L_2 = \mathcal{L}_1 \cup \{ \forall x P(x) \} \) in the proof of Lemma 8. We have \( CR[L_1] = CR[L_2] \), and therefore \( LF[CR[L_1]] = LF[CR[L_2]] \), but \( L_1 \subset L_2 \), and therefore \( L_1 \subseteq LF[CR[L_1]] \).

As for (2), inclusion \( \mathcal{G} \subseteq CR[LF[\mathcal{G}]] \) is also an immediate consequence of the definitions of \( CR \) and \( LF \). Further, in [16] Proposition 9, it is given a class (basically, \( \mathcal{G} \) does not contain generalised dictatorships) for which this inclusion is strict. \( \square \)

To conclude, the relationship between operators \( CR[-] \) and \( LF[-] \) can be represented as in Fig. 1. The two operators are inverse one to the other, but they do not commute.

6 Characterisation Results

In this section we show some correspondences between axiomatic properties and restrictions to the first order language in which integrity constraints can be expressed, in line with previous work by Grandi and Endriss [16]. We then focus on the database-specific constraints introduced in Section 2, showing sufficient and necessary conditions for collective rationality of quota rules.

To state the next result we consider a set \( Con \subseteq U \) of constants, interpreted as themselves in each \( D_i \), that is, \( \sigma(c) = c \) for every \( c \in Con \). Then, let \( lit^+ \subseteq L_D \) be
some language containing only positive literals of form $P(c_1,\ldots,c_q)$, for $P \in \mathcal{D}$ and constants $c_1,\ldots,c_q$.

**Theorem 10.** $F_{\text{lit}^+}[U] \subseteq CR[\text{lit}^+]$, and $F_{\text{lit}^+}[U] \supseteq CR[\text{lit}^+]$ only if Con contains all individuals in the domain of $F$.

**Proof.** As to inclusion $\subseteq$, we see that if all instances $D_1,\ldots,D_n$ satisfy formulas $P(c_1,\ldots,c_q)$ in $\text{lit}^+$, then $\bar{c} \in D_i(P)$ for every $i \in \mathcal{N}$. By unanimity we have that $\bigcap_{i \in \mathcal{N}} D_i(P) \subseteq F(\bar{D})(P)$, and therefore $\bar{c} \in F(\bar{D})(P)$. Hence, $F$ is collectively rational on $\text{lit}^+$.

As to $\supseteq$, suppose that $F \in CR[\text{lit}^+]$ and choose a profile $D_1,\ldots,D_n$ with $\bar{u} \in \bigcap_{i \in \mathcal{N}} D_i(P)$, that is, for every $i \in \mathcal{N}$, $D_i \models P(u_1,\ldots,u_q)$. Since we assumed that Con contains all individuals in the domain of $F$, individuals $u_1,\ldots,u_q$ belong to Con and formulas $P(u_1,\ldots,u_q)$ are in $\text{lit}^+$. Further, $F$ is CR on $D_1,\ldots,D_n$ and therefore $F(\bar{D}) \models P(u_1,\ldots,u_q)$, that is, $\bar{u} \in F(\bar{D})(P)$, which mean that $F$ is unanimous. \(\square\)

By Theorem 10 an aggregator $F$ is collectively rational on a language $\text{lit}^+$ with positive literals only if it is unanimous on the class of instances satisfying the very same positive literals.

A symmetric result holds for the axiom of groundedness and any language $\text{lit}^- \subseteq \mathcal{L}_D$ containing only negative literals of form $\neg P(c_1,\ldots,c_q)$. The proof is similar, so we omit it.

**Theorem 11.** $F_{\text{lit}^-}[G] \subseteq CR[\text{lit}^-]$, and $F_{\text{lit}^-}[G] \supseteq CR[\text{lit}^-]$ only if Con contains all individuals in the domain of $F$.

From Theorem 10 and 11 we immediately obtain the following corollary by the lemmas in section 5 where $\text{lit} = \text{lit}^+ \cup \text{lit}^-$.

**Corollary 12.** $F_{\text{lit}}[U,G] \subseteq CR[\text{lit}]$, and $F_{\text{lit}}[U,G] \supseteq CR[\text{lit}]$ only if Con contains all individuals in the domain of $F$.

**Proof.** As to inclusion $\subseteq$, by Lemma 7(2), $F_{\text{lit}}[U,G] = F_{\text{lit}^+}[U] \cap F_{\text{lit}^-}[G]$, and by Lemma 7(1) $F_{\text{lit}^+}[U] \cap F_{\text{lit}^-}[G] \subseteq F_{\text{lit}^+}[U] \cap F_{\text{lit}^-}[G]$. Then, by Theorem 10 and 11 $F_{\text{lit}^+}[U] \cap F_{\text{lit}^-}[G] \subseteq CR[\text{lit}^+] \cap CR[\text{lit}^-]$. Finally, by Lemma 6(1) $CR[\text{lit}^+] \cap CR[\text{lit}^-] \subseteq CR[\text{lit}]$. The other inclusion is proved similarly. \(\square\)

Notice that, differently from the propositional case [16] Theorem 10], here we need both axioms of unanimity and groundedness to preserve both positive and negative literals, while for propositional literals unanimity suffices. Hence, also simple results do not transfer immediately from the propositional to the first-order setting.

Next, define $\mathcal{L}_{\leftrightarrow}$ as the language of equivalences $\forall \bar{x}\bar{x}'(P(\bar{x}) \leftrightarrow P'(\bar{x}'))$ for relation symbols $P,P' \in \mathcal{D}$. We show the following:

**Theorem 13.** $CR[\mathcal{L}_{\leftrightarrow}] = F_{\leftrightarrow}[N^+]$

**Proof.** As for inclusion $\supseteq$, pick an equivalence $\forall \bar{x}\bar{x}'(P(\bar{x}) \leftrightarrow P'(\bar{x}'))$. This defines a database in which relation symbols $P$ and $P'$ share the same pattern of acceptance/rejection, and since aggregator $F$ is neutral over issues, we get $F(\bar{D}) \models
\( \forall \vec{x}, \vec{x}' (P(\vec{x}) \leftrightarrow P'(\vec{x}')) \). Therefore, the constraint given by the initial equivalence is lifted.

As for inclusion \( \subseteq \), suppose that a profile \( \vec{D} \) is such that \( N_{\vec{u}}^{\vec{D}(P)} = N_{\vec{u}'}^{\vec{D}'(P')} \). This implies that for every \( i \in N \), \( \vec{D}_i \models \forall \vec{x} \exists \vec{x} (P(\vec{x}) \leftrightarrow P'(\vec{x}')) \), and since \( F \) is in \( CR(L_{\leftrightarrow}) \), \( \vec{u} \in F(D)(P) \) iff \( \vec{u}' \in F(D')(P') \). This holds for every such profile \( \vec{D} \), proving that \( F \) is neutral.

By Theorem 13 an aggregator \( F \) is collectively rational on language \( L_{\leftrightarrow} \) iff it is positively neutral on the class of instances satisfying all formulas in \( L_{\leftrightarrow} \).

Let us now define the following class:

**Definition 12** (Generalised dictatorship). An aggregation procedure \( F : D(U)^n \rightarrow D(U) \) is a generalised dictatorship if there exists a map \( g : D(U)^n \rightarrow N \) such that for every \( \vec{D} \in D(U)^n \), \( F(\vec{D}) = D_{g(\vec{D})} \). Let GDIC be the class of all generalised dictatorships.

Generalised dictatorships include classical dictatorships, but also more interesting procedures known as most representative voters rules, which selects the individual input that best summarises a given profile. Clearly, since each single instance satisfies the given set of constraints, a generalised dictatorship is collectively rational with respect to the full first-order language.

**Theorem 14.** GDIC \( \subset CR(L_P) \)

Observe that while for binary aggregation the theorem above is an equality [?], [Theorem 16] GrandiEndrissAIJ2013, this is not the case for database aggregation. This is due to the fact that the first-order language cannot specify uniquely a given database instance. The proof of this fact is rather immediate: consider a dictatorship of the first agent, modified by permuting all the elements in \( U \). That is, \( F(\vec{D}) = \rho(D_1) \) where \( \rho : U \rightarrow U \) is any permutation. Clearly, \( D_1 \neq \rho(D_1) \), but all constraints that were satisfied by \( D_1 \) are also satisfied by \( \rho(D_1) \). Hence, this aggregator is collectively rational with respect to the full first-order language \( L_P \), but is not a generalised dictatorship.

We now turn our attention to integrity constraints proper to databases. We begin with functional dependencies.

**Proposition 15.** A quota rule lifts a functional constraint iff \( q_P > \frac{n}{2} \) for all relation symbols \( P \) occurring in the functional constraint.

**Proof.** By assumption, every instance \( D_i \) satisfies the constraint. That is for every tuple \( (u_1, \ldots, u_k) \), either there is a unique \( (u_{k+1}, \ldots, u_q) \) such that \( (u_1, \ldots, u_q) = \vec{u} \in D_i(P) \), or there is none. Suppose now that the constraint is falsified by the collective outcome. That is, there are \( \vec{u} \neq \vec{u}' \) such that both \( \vec{u} \in F(\vec{D})(P) \) and \( \vec{u}' \in F(\vec{D})(P) \), and \( \vec{u} \) and \( \vec{u}' \) coincide on the first \( k \) coordinates. By definition of quota rules, this means that at least \( q_P \) voters are such that \( \vec{u} \in D_i(P) \), and at least \( q_P \) possibly different voters had \( \vec{u}' \in D_i(P) \). Since each individual can have either \( \vec{u} \) or \( \vec{u}' \) in \( D_i(P) \), by the pigeonhole principle this is possible if and only if the quota \( q_P \leq \frac{n}{2} \).

\( \square \)
As immediate applications of Prop. 15, the intersection rule clearly lifts any functional dependency, while the union lifts none. To see the latter, it is sufficient to consider two database instances that associates different tuples to the same primary key.

**Proposition 16.** An aggregation procedure $F$ lifts a value constraint if $F$ is grounded.

**Proof.** Let $n_k \in D(P_v)$ be a value constraint, where for all $i, j \in \mathcal{N}$, we have that $D_i(P_v) = D_j(P_v)$. A grounded aggregation procedure is such that $F(\vec{D})(P_v) \subseteq \bigcup_{i \in \mathcal{N}} D_i(P_v)$. Hence, for all $\vec{u} \in F(\vec{D})(P_v)$, there exists an $i \in \mathcal{N}$ such that $\vec{u} \in D_i(P_v)$. Since all individual databases satisfy the value constraint, we have that $u_k \in D_i(P_v)$, and therefore $u_k \in F(\vec{D})(P_v) \subseteq \bigcup_{i \in \mathcal{N}} D_i(P_v)$, showing that also $F(\vec{D})(P_v)$ satisfies the value constraint. □

The converse of the Prop. 16 is not true in general, since a non-grounded aggregator could be easily devised while still satisfying a given value constraint.

The last result in this section concerns again quota rules.

**Proposition 17.** A quota rule lifts a referential constraint $(P_1 \rightarrow P_2, k)$ iff $q_{P_2} = 1$.

**Proof.** Let $\vec{u} \in F(\vec{D})(P_1)$. Since all the individual databases satisfy the integrity constraint, we know that for every $i \in \mathcal{N}$ there exists an $\vec{u}_i \in D_i(P_2)$ such that its first $k$ coordinates coincides with the last $k$ coordinates of $P_1$. Since all $\vec{u}_i$ are possibly different, they may be supported by one single individual each. Therefore, the referential constraint is lifted if and only if the quota relative to $P_2$ is sufficiently small, i.e., $q_{P_2} = 1$. □

As immediate application of Prop. 17 intersection and union rules are included in the results above, since they are quota rules. As regards distance-based rules, we only remark that they lift all integrity constraint by their definition, provided that the minimisation is restricted to consistent databases.

## 7 Aggregation and Query Answering

In this section we analyse one of the most common operation on databases, i.e., querying, to the light of (rational) aggregation. Observe that any open formula $\varphi(x_1, \ldots, x_\ell)$, with free variables $x_1, \ldots, x_\ell$, can be thought of as a query [1]. Evaluating $\varphi(x_1, \ldots, x_\ell)$ on a database instance $D$ returns the set $\text{ans}(D, \varphi)$ of tuples $\vec{u} = (u_1, \ldots, u_\ell)$ such that the assignment $\sigma$, with $\sigma(x_i) = u_i$ for $i \leq \ell$, satisfies $\varphi$, that is, $(D, \sigma) \models \varphi$. Hereafter, with an abuse of notation, we often write simply $(D, \vec{u}) \models \varphi$. Given the relevance of query answering in database theory, the following question is of obvious interest.

**Question 18.** What is the relationship between the answer $\text{ans}(F(\vec{D}), \varphi)$ to query $\varphi$ on the aggregated database $F(\vec{D})$, and the answers $\text{ans}(D_1, \varphi), \ldots, \text{ans}(D_n, \varphi)$ to the same query on each instance $D_1, \ldots, D_n$?
Clearly, given a query $\varphi$, every aggregator $F$ on database instances induces an aggregation procedure $F^*$ on the query answers, as illustrated by the following diagram, where $D = F(D)$:

\[
\begin{array}{c}
D_1, \ldots, D_n \\
\downarrow \varphi \\
\text{ans}(D_1, \varphi), \ldots, \text{ans}(D_n, \varphi) \\
\end{array}
\xrightarrow{F^*} 
\begin{array}{c}
\text{ans}(D, \varphi) \\
\end{array}
\]

\[
\begin{array}{c}
\xrightarrow{F} \\
\downarrow \varphi \\
D \\
\end{array}
\]

Hereafter we consider some examples to illustrate this question.

**Example 2.** If we assume intersection as the aggregation procedure, it is easy to check that in general the answer to a query in the aggregated database is not the intersection of the answers for each single instance. To see this, let $D_1(P) = \{(a, b)\}$ and $D_2(P) = \{(a, d)\}$ and consider query $\varphi = \exists y P(x, y)$. Clearly, $\text{ans}(D_1 \cap D_2, \varphi)$ is empty, while $\text{ans}(D_1, \varphi) \cap \text{ans}(D_2, \varphi) = \{a\}$. Hence, in general $\bigcap_{i \in N} \text{ans}(D_i, \varphi) \not\subseteq \text{ans}(\bigcap_{i \in N} D_i, \varphi)$. The converse can also be the case. Consider instances $D_1, D_2$ such that $D_1(P) = \{(a, a), (a, b)\}$, $D_1(R) = \{c\}$, and $D_2(P) = \{(a, a), (a, b)\}$, $D_2(R) = \{d\}$, with query $\varphi = \forall y P(x, y)$. The intersection $\text{ans}(D_1, \varphi) \cap \text{ans}(D_2, \varphi)$ of answers is empty. However the answer w.r.t. the intersection of databases is $\text{ans}(D_1 \cap D_2, \varphi) = \{a\}$, since the active domain of the intersection only includes elements $a$ and $b$. As a result, in general $\text{ans}(\bigcap_{i \in N} D_i, \varphi) \not\subseteq \bigcap_{i \in N} \text{ans}(D_i, \varphi)$.

Similar arguments can be used to show that the union of answers is in general different from the answer on the union of instances.

These examples shows that it is extremely difficult to find aggregators that commute for any first-order query $\varphi \in L_D$. Hence, they naturally raise the question of syntactic restrictions on queries such that the aggregation procedure $F^* = \varphi \circ F \circ \varphi^{-1}$ on answers can be expressed explicitly in terms of $F$ (e.g., the intersection of answers is the answer to the query on the intersection):

**Question 19.** Given aggregation procedures $F$ and $F^*$, is there a restriction of the query language for $\varphi$ such that the diagram above commute?

This problem is related to the following, more general question.

**Question 20.** Given an aggregation procedure $F$ and a query language $L$, what is the aggregation procedure $F^*$? Can $F^*$ be represented explicitly?

The following result provides a first, partial answer to Question 19 in the case $F$ and $F^*$ are unions.

**Lemma 21** (Existential Fragment). Consider the positive existential fragment $L^+_{D}$ of first-order logic defined as follows:

$$
\varphi ::= P(x_1, \ldots, x_q) \mid \varphi \lor \varphi \mid \exists x \varphi
$$

The language $L^+_{D}$ is lifted by unions, that is, for $F$ and $F^*$ equal to set-theoretical union, the diagram commutes for the query language $L^+_{D}$. 

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Proof. The proof is by induction on the structure of query $\varphi$. For atomic $\varphi = P(x_1, \ldots, x_q)$, $\vec{u} \in \text{ans}(\bigcup_{i \in \mathcal{N}} D_i, \varphi)$ iff $(\bigcup_{i \in \mathcal{N}} D_i, \vec{u}) \models \varphi$, iff for some $i \in \mathcal{N}$, $(D_i, \vec{u}) \models \varphi$, iff $\vec{u} \in \text{ans}(D_i, \varphi)$ for some $i \in \mathcal{N}$.

For $\varphi = \psi \lor \psi'$, $\vec{u} \in \text{ans}(\bigcup_{i \in \mathcal{N}} D_i, \varphi)$ iff $(\bigcup_{i \in \mathcal{N}} D_i, \vec{u}) \models \psi$ or $(\bigcup_{i \in \mathcal{N}} D_i, \vec{u}) \models \psi'$, iff for some $i, j \in \mathcal{N}$, $(D_i, \vec{u}) \models \psi$ or $(D_j, \vec{u}) \models \psi'$ by induction hypothesis. In particular, we have both $(D_i, \vec{u}) \models \psi \lor \psi'$ and $(D_j, \vec{u}) \models \psi \lor \psi'$, that is, $\vec{u} \in \bigcup_{i \in \mathcal{N}} \text{ans}(D_i, \varphi)$. On the other hand, $\vec{u} \in \bigcup_{i \in \mathcal{N}} \text{ans}(D_i, \varphi)$ iff $\vec{u} \in \text{ans}(D_i, \varphi)$ for some $i \in \mathcal{N}$, iff $(D_i, \vec{u}) \models \psi$ or $(D_i, \vec{u}) \models \psi'$. In both cases, by induction hypothesis $(\bigcup_{i \in \mathcal{N}} D_i, \vec{u}) \models \varphi$, that is, $\vec{u} \in \text{ans}(\bigcup_{i \in \mathcal{N}} D_i, \varphi)$.

For $\varphi = \exists x \psi$, $\vec{u} \in \text{ans}(\bigcup_{i \in \mathcal{N}} D_i, \varphi)$ iff $(\bigcup_{i \in \mathcal{N}} D_i, \vec{u}) \models \psi$, iff for some $u \in \text{dom}(\bigcup_{i \in \mathcal{N}} D_i)$, $(\bigcup_{i \in \mathcal{N}} D_i, \vec{u} \cdot u) \models \psi$, and therefore for some $i, j \in \mathcal{N}$, $u \in \text{dom}(D_j)$ and $(D_i, \vec{u} \cdot u) \models \psi$. Notice that if $(D_i, \vec{u} \cdot u) \models \psi$, then $u \in \text{dom}(D_i)$ as well, as $\varphi$ belongs to the positive (existential) fragment of first-order logic. Hence, for some $i \in \mathcal{N}$, $u \in \text{dom}(D_i)$ and $(D_i, \vec{u} \cdot u) \models \psi$, that is, $\vec{u} \in \text{ans}(D_i, \varphi)$ for some $i \in \mathcal{N}$. On the other hand, $\vec{u} \in \bigcup_{i \in \mathcal{N}} \text{ans}(D_i, \varphi)$ iff $\vec{u} \in \text{ans}(D_i, \varphi)$ for some $i \in \mathcal{N}$, iff $u \in \text{dom}(D_i)$ and $(D_i, \vec{u} \cdot u) \models \psi$, that is, $u \in \text{dom}(\bigcup_{i \in \mathcal{N}} D_i)$ and $(\bigcup_{i \in \mathcal{N}} D_i, \vec{u} \cdot u) \models \psi$ by induction hypothesis. Hence, $\vec{u} \in \text{ans}(\bigcup_{i \in \mathcal{N}} D_i, \varphi)$. □

By Lemma 21, queries in $\mathcal{L}_+^\varphi$ are preserved whenever both $F$ and $F^*$ are unions. Obviously, it would be of interest to find what is the largest fragment $\mathcal{L}'$ of first-order logic preserved by unions. By the results in this section we know that $\mathcal{L}_+^\varphi \subseteq \mathcal{L}' \subseteq \mathcal{L}_\mathcal{D}$.

Further, we may wonder whether a result symmetric to Lemma 21 holds for intersections and the positive universal fragment $\mathcal{L}_+^\forall$ of first-order logic defined as follows:

$$\varphi := P(x_1, \ldots, x_q) \mid \varphi \land \varphi \mid \forall x \varphi$$

Unfortunately, in Example 2 we provided a formula $\varphi = \forall y P(x, y)$ in $\mathcal{L}_+^\forall$ and instances $D_1, D_2$ such that $\text{ans}(D_1 \cap D_2, \varphi) \subseteq \text{ans}(D_1, \varphi) \cap \text{ans}(D_2, \varphi)$. Hence, for $F$ and $F^*$ equal to set-theoretical intersection, the diagram above does not commute for the query language $\mathcal{L}_+^\forall$.

Nonetheless, we are able to prove a weaker but still significant result related to Question 20. Specifically, the next lemma shows that if in the diagram above $F$ is the intersection and the query language is $\mathcal{L}_+^\forall$, then $F^*$ is unanimous, in the sense that $\bigcap_{i \in \mathcal{N}} \text{ans}(D_i, \varphi) \subseteq \text{ans}(\bigcap_{i \in \mathcal{N}} D_i, \varphi)$.

**Lemma 22.*** Let the aggregator $F$ be the intersection and let the query language be $\mathcal{L}_+^\forall$. Then, the lifted aggregator $F^*$ is unanimous.

**Proof.*** We prove that $\bigcap_{i \in \mathcal{N}} \text{ans}(D_i, \varphi) \subseteq \text{ans}(\bigcap_{i \in \mathcal{N}} D_i, \varphi)$. So, if $\vec{u} \in \bigcap_{i \in \mathcal{N}} \text{ans}(D_i, \varphi)$ then for every $i \in \mathcal{N}$, $(D_i, \vec{u}) \models \varphi$. We now prove by induction on $\varphi \in \mathcal{L}_+^\forall$ that if for every $i \in \mathcal{N}$, $(D_i, \vec{u}) \models \varphi$, then $(\bigcap_{i \in \mathcal{N}} D_i, \vec{u}) \models \varphi$. As to the base case for $\varphi = P(x_1, \ldots, x_q)$ atomic, $(D_i, \vec{u}) \models P(x_1, \ldots, x_q)$ iff $\vec{u} \in D_i(P)$ for every $i \in \mathcal{N}$. In particular, $\vec{u} \in \bigcap_{i \in \mathcal{N}} D_i(P)$ as well, and therefore $(\bigcap_{i \in \mathcal{N}} D_i, \vec{u}) \models P(x_1, \ldots, x_q)$. As to the inductive case for $\varphi = \psi \land \psi'$, suppose that $(D_i, \vec{u}) \models \varphi$, that is, $(D_i, \vec{u}) \models \psi$ and $(D_i, \vec{u}) \models \psi'$ for every $i \in \mathcal{N}$. By induction hypothesis we obtain that $(\bigcap_{i \in \mathcal{N}} D_i, \vec{u}) \models \psi$ and $(\bigcap_{i \in \mathcal{N}} D_i, \vec{u}) \models \psi'$, i.e., $(\bigcap_{i \in \mathcal{N}} D_i, \vec{u}) \models \varphi$. Finally, if $(D_i, \vec{u}) \models \forall x \psi$ for every $i \in \mathcal{N}$, then for all $v \in \text{dom}(D_i)$, $(D_i, \vec{u} \cdot v) \models \psi$. 

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In particular, for all \( v \in \text{dom}(\bigcup_{i \in \mathcal{N}} D_i) \), \( (D_i, \bar{u} \cdot v) \models \psi \) for every \( i \in \mathcal{N} \), and by induction hypothesis, for all \( v \in \text{dom}(\bigcup_{i \in \mathcal{N}} D_i) \), \( (\bigcup_{i \in \mathcal{N}} D_i, \bar{u}) \models \psi \), i.e., \( (\bigcup_{i \in \mathcal{N}} D_i, \bar{u}) \models \forall x \psi \). As a result, \( \bar{u} \in \text{ans}(\bigcup_{i \in \mathcal{N}} D_i, \varphi) \). \( \square \)

A result symmetric to Lemma 22 holds for language \( L^+ \) and unions:

**Lemma 23.** Let the aggregator \( F \) be the union and let the query language be \( L^+ \). Then, the lifted aggregator \( F^+ \) is grounded.

**Proof.** We prove that \( \text{ans}(\bigcup_{i \in \mathcal{N}} D_i, \varphi) \subseteq \bigcup_{i \in \mathcal{N}} \text{ans}(D_i, \varphi) \). So, if \( \bar{u} \in \text{ans}(\bigcup_{i \in \mathcal{N}} D_i, \varphi) \) then \( (\bigcup_{i \in \mathcal{N}} D_i, \bar{u}) \models \varphi \). We now prove by induction on \( \varphi \in L^+ \) that if \( (\bigcup_{i \in \mathcal{N}} D_i, \bar{u}) \models \varphi \), then for some \( i \in \mathcal{N} \), \( (D_i, \bar{u}) \models \varphi \). As to the base case for \( \varphi = P(x_1, \ldots, x_q) \) atomic, \( (\bigcup_{i \in \mathcal{N}} D_i, \bar{u}) \models P(x_1, \ldots, x_q) \) if and only if \( \bar{u} \in \bigcup_{i \in \mathcal{N}} D_i(P) \), iff \( \bar{u} \in D_i(P) \) for some \( i \in \mathcal{N} \). In particular, \( (D_i, \bar{u}) \models P(x_1, \ldots, x_q) \) as well, and therefore \( \bar{u} \in \bigcup_{i \in \mathcal{N}} \text{ans}(D_i, \varphi) \). As to the inductive case for \( \varphi = \psi \lor \psi' \), suppose that \( (\bigcup_{i \in \mathcal{N}} D_i, \bar{u}) \models \psi \), that is, \( (\bigcup_{i \in \mathcal{N}} D_i, \bar{u}) \models \psi \lor \psi' \). In the first case, by induction hypothesis we have that \( \bar{u} \in \bigcup_{i \in \mathcal{N}} \text{ans}(D_i, \psi) \), i.e., for some \( i \in \mathcal{N} \), \( (D_i, \bar{u}) \models \psi \), and therefore \( (D_i, \bar{u}) \models \varphi \). Hence, \( \bar{u} \in \text{ans}(D_i, \varphi) \) for some \( i \in \mathcal{N} \), that is, \( \bar{u} \in \bigcup_{i \in \mathcal{N}} \text{ans}(D_i, \varphi) \). The case for \( (\bigcup_{i \in \mathcal{N}} D_i, \bar{u}) \models \psi' \) is symmetric. Finally, if \( (\bigcup_{i \in \mathcal{N}} D_i, \bar{u}) \models \exists x \psi \), then for some \( v \in \text{dom}(\bigcup_{i \in \mathcal{N}} D_i) \), \( (\bigcup_{i \in \mathcal{N}} D_i, \bar{u} \cdot v) \models \psi \). In particular, by induction hypothesis, \( \bar{u} \cdot v \in \bigcup_{i \in \mathcal{N}} \text{ans}(D_i, \psi) \), that is, \( (D_i, \bar{u} \cdot v) \models \psi \) for some \( i \in \mathcal{N} \). Further, since \( \psi \) is a positive formula, \( v \in \text{dom}(D_i) \), and therefore, \( (D_i, \bar{u} \cdot v) \models \varphi \), i.e., \( \bar{u} \in \bigcup_{i \in \mathcal{N}} \text{ans}(D_i, \varphi) \). \( \square \)

To conclude this section we discuss the results obtain so far. We said that Lemma 21 can be seen as a (partial) answer to Question 19. Similarly, Lemma 22 and 23 are related to Question 20. Results along the lines of Lemmas 21-23 may find application in efficient query answering: it might be that in some cases, rather than querying the aggregated database \( F(D) \), it is more efficient to query the individual instances \( D_1, \ldots, D_n \) and then aggregate the answers. In such cases it is crucial to know which answers are preserved by the different aggregation procedures. The results provided in this section aimed to be a first, preliminary step in this direction.

## 8 Conclusions and Related Work

In this paper we have proposed a framework for the aggregation of conflicting information coming from multiple sources in the form of finite relational databases. We proposed a number of aggregators inspired by the literature on social choice theory, and adapted a number of axiomatic properties. We then focused on two natural questions which arise when dealing with the aggregation of databases. First, we studied what languages for integrity constraints are lifted by some of the rules we proposed, i.e., what constraints are true in the aggregated database supposing that all individual input satisfies the same constraints. Second, we investigated first-order query answering in the aggregated databases, characterising some languages for which the aggregation of the answers to the individual databases corresponds to the answer to the query on the aggregated database.
Our initial results shed light on the possible use of choice-theoretic techniques in the database merging and integration, and opens multiple interesting directions for future research. In particular, the connections to the literature on aggregation and merging can be investigated further. First, section 6 showcased results for which database aggregation behaves similarly to binary aggregation with integrity constraints (see [16]), but pointed out at some crucial differences. In particular, there are natural classes of integrity constraints used in databases for which the equivalent in propositional logic, the language of choice for binary aggregation, would be tedious and lengthy. We were able to provide initial results on their preservation through aggregation. Second, the recent work of Endriss and Grandi [12] is also strongly related to our contribution. Since graphs are a specific type of relational structures, our work directly generalise their graph aggregation framework to relations of arbitrary arity. However, the specificity of their setting allows them to obtain very powerful impossibility results, which are yet to be explored in the area of database aggregation. Third, to the best of our knowledge the problem of aggregated query answering is new in the literature on aggregation, albeit a similar problem has been studied in the aggregation of argumentation graphs [6], a setting closer to that of graph aggregation. Also this direction deserves further investigation.

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