A renormalization fixed point for Lorenz maps

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Received 6 September 2009, in final form 15 March 2010
Published 29 April 2010
Online at stacks.iop.org/Non/23/1291

Recommended by B Eckhardt

Abstract

A Lorenz map is a Poincaré map for a three-dimensional Lorenz flow. We describe the theory of renormalization for Lorenz maps with a critical point and prove that a restriction of the renormalization operator acting on such maps has a hyperbolic fixed point. The proof is computer assisted and we include a detailed exposition on how to make rigorous estimates using a computer as well as the implementation of the estimates.

Online supplementary data available from stacks.iop.org/Non/23/1291/mmedia

Mathematics Subject Classification: 37E20

1. Introduction

The theory of renormalization has, since its introduction about 30 years ago, become a central tool in the study of dynamical systems. It is used, roughly speaking, to analyse maps having the property that the first-return map to some small part of the phase space resembles the original map itself. This property is usually associated with maps which lie at the ‘boundary of chaos’, like the prototypical example of a unimodal map at the end of a period-doubling cascade. Such period-doubling cascades have been observed for maps as well as for flows, but most renormalization results so far have been for one-dimensional maps (unimodal and circle maps, see e.g. [5, 16]) with some results for higher dimensional maps (Hénon maps, see [1, 3]). This paper contains new results for a class of so-called Lorenz flows whose dynamics can be described using the theory of renormalization.

A Lorenz flow is a three-dimensional flow possessing a singularity of saddle type with a one-dimensional unstable manifold intersecting the two-dimensional stable manifold. If the Poincaré map to a surface straddling the stable manifold can be foliated in such a way that the leaves are invariant and contracted exponentially by the Poincaré map, then the dynamics of the flow is determined by the one-dimensional map induced by the action of the Poincaré map on the leaves (see [7, chapter 14]). Such one-dimensional Lorenz maps increase with a jump discontinuity at the point corresponding to the stable manifold. They have been studied
extensively under the additional assumption that they be expanding (see in particular [6, 15]), but a much wider variety of dynamics is exhibited if there is also some contraction present in the form of a critical point (see [8, 12]) and this is the situation we consider.

The main result of this paper is that a restriction of the renormalization operator on the space of Lorenz maps has a hyperbolic fixed point, which is proved using the contraction mapping theorem on an associated operator. We use a computer to rigorously compute estimates that shows that this associated operator is indeed a contraction. This method was pioneered by Lanford [10] (see also [11]) when he proved the existence of a fixed point of the period-doubling operator on unimodal maps. However, Lanford’s paper only gives a brief outline of the method he employs without an actual proof, so we have gone through quite a lot of pain to include all the missing details in this paper (many of which were borrowed from [9]).

This paper contains the theory and statements of all results. The supplementary data (available from stacks.iop.org/Non/23/1291/mmedia) contains the implementation of the computer estimates used in this paper.

The arrangement of this article is as follows: in section 2 we give all the necessary definitions to state the renormalization conjecture and the main theorem, and then we go on to prove several consequences of the main theorem in section 3. In section 4 we describe the method used to prove the main theorem and in section 5 we give the proof.

In the supplementary data we give exact details on how the estimates needed to prove the main theorem are implemented on a computer. The literature on this type of computer assisted proof seems to have a tradition of never including these details, most likely because it would require an order of thousands of lines of source code. We make a conscious break from this tradition and show how to implement all estimates in only 166 lines of source code\(^1\). The key behind this reduction in size is the use of a functional programming language since it allows us to program in a declarative style: we specify what the program does, not how it is accomplished. This also has the benefit that functions cannot have side-effects (the output from a function only depends on its input) which makes it easier to reason about the source code. In our context this is extremely important since it means that we can check the correctness of each function in complete isolation from the rest of the source code (and a typical function is only one or two lines long which simplifies the verification of individual functions). To further minimize the risk of programming errors we choose a strongly typed language since these are good at catching common programming errors during compilation.

We take this opportunity to advocate the programming language Haskell for tasks similar to the one at hand—it has all the benefits mentioned above and more, but at the same time manages to produce code which runs very fast (thanks to the GHC compiler). Unfortunately, many readers will probably have had little prior exposure to Haskell and for this reason we have, in appendix E of the supplementary data, included a brief overview of Haskell as well as a table highlighting its syntax to aid the reader in understanding the source code.

2. Statement of the main result

In this section we state the main result, but in order to do so we first need quite a few definitions.

**Definition 2.1.** A Lorenz map \( f \) (see figure 1) on a closed interval \( I = [l, r] \) \( (l < 0 < r) \) is a monotone increasing continuous function from \( I \setminus \{0\} \) to \( I \) such that \( f(0^-) = r, f(0^+) = l \) (i.e. \( f \) has a jump discontinuity at \( 0 \))\(^2\).

\(^1\) This includes: definition of the main operator and its derivative (40 lines), an interval arithmetic library (30 lines), a library for computing with analytic functions (65 lines) and a linear equation solver (15 lines).

\(^2\) The notation \( f(0^-) \) is shorthand for \( \lim_{x \to 0^-} f(x) \) and this limit is assumed to exist; \( f(0^+) \) is defined analogously as the right-hand limit.
We require that \( f(x) = \phi(|x|^\rho) \) for all \( x \in (l, 0) \), where \( \phi \) is a symmetric\(^3\) analytic map defined on some complex neighbourhood of \([l, 0]\), and similarly \( f(x) = \psi(|x|^\rho) \) for \( x \in (0, r) \), where \( \psi \) is a symmetric analytic map defined on some complex neighbourhood of \([0, r]\). The maps \( \phi \) and \( \psi \) are called the analytic parts of \( f \). The constant \( \rho > 1 \) is called the critical exponent of \( f \) (and is independent of \( f \)).

Assume \( f \) is defined on \([-1, r] \) and let \( g \) be a Lorenz map on \([-1, r'] \) with analytic parts \((\phi', \psi')\). We define a metric on the set of Lorenz maps by

\[
\|f - g\| = \max\{\|\phi - \phi'\|, \|\psi - \psi'\|\},
\]

where \( \|\cdot\| \) denotes the usual sup-norm on analytic functions. (For Lorenz maps with different domains we first perform a linear coordinate change to ensure that their domains are of the above form, then apply the above formula for the metric.)

**Remark 2.2.** The condition \( \rho > 1 \) ensures that \( Df(x) \to 0 \) as \( x \to 0 \) (from the left or the right) and for this reason we call 0 the critical point. Because of the discontinuity at 0 there are two critical values, namely \( l \) and \( r \).

The smoothness required in our definition of Lorenz maps is not essential for a satisfactory renormalization theory, but our current results are only in this category (which is not a big restriction since they can most likely be extended to \( C^r \) for \( r \geq 3 \) along the lines of \([4, 5]\)). For a discussion on what to expect when the minimum smoothness threshold is approached from below, see [2].

**Definition 2.3.** A branch of \( f^n \) is a maximal open interval \( B \) on which \( f^n \) is monotone (here maximality means that if \( A \) is an open interval which properly contains \( B \), then \( f^n \) is not monotone on \( A \)).

To each branch \( B \) of \( f^n \) we associate a word \( \omega(B) = [\sigma_0, \ldots, \sigma_{n-1}] \) on two symbols by

\[
\sigma_j = \begin{cases} 
0 & \text{if } f^j(B) \subset (l, 0), \\
1 & \text{if } f^j(B) \subset (0, r), 
\end{cases}
\]

for \( j = 0, \ldots, n - 1 \).

\(^3\) Here 'symmetric' means \( \phi(\bar{z}) = \bar{\phi}(z) \).
Definition 2.4. A Lorenz map \( f \) on \( I \) is renormalizable if there exists a maximal interval \( U \subsetneq I \) containing 0 such that the first-return map \( \tilde{f} \) to \( U \) is a Lorenz map on \( U \). In this situation we define the renormalization \( Rf \) of \( f \) as the first-return map rescaled via the increasing linear map \( h : I \to U \) which takes 0 to itself and the left endpoint of \( I \) to the left endpoint of \( U \):

\[
Rf = h^{-1} \circ \tilde{f} \circ h.
\]

The operator \( R \) is called the renormalization operator.

Remark 2.5. When defined on the space of Lorenz maps with analytic branches the renormalization operator is differentiable and its derivative is a compact linear operator. This follows from the fact that \( Rf \) only evaluates \( f \) on a strict subset of the domain of \( f \) (see sections 4.4 and 4.5 of the supplementary data). On the other hand, if we only were to demand \( C^r \)-smoothness for the branches of our Lorenz maps then \( R \) would no longer be differentiable (see [5] and [13, chapter VI.1.1]).

Definition 2.6. Let \( f \) be a renormalizable Lorenz map with associated first-return map \( \tilde{f} \) and return interval \( U \). Then there exist (unique) integers \( a, b \geq 2 \) such that \( \tilde{f}|_L = f^a \) and \( \tilde{f}|_R = f^b \), where \( L = U \cap (L, 0) \) and \( R = U \cap (0, r) \). The interval \( L \) is contained in a branch \( A \) of \( f^a \) with associated word \( \alpha = \omega(A) \), and similarly \( R \subset B \) for a branch \( B \) of \( f^b \) with \( \beta = \omega(B) \). The pair of words \( (\alpha, \beta) \) is called the type of renormalization.

The notation \( R_{\alpha, \beta} \) will be used to denote the restriction of \( R \) to the set of Lorenz maps which have renormalizations of type \( (\alpha, \beta) \).

Remark 2.7. In the kneading theory for unimodal maps a finite part of the itinerary of a point does not contain enough information to recover the exact ordering of the corresponding points on the real line. This leads to the introduction of ‘unimodal permutations’ to describe the combinatorics of unimodal renormalization. Since Lorenz maps are strictly increasing and have no fixed points the ordering of points of a finite part of an orbit is completely determined by the corresponding kneading information so there is no need for this extra complication. However, we may still ask which pairs of words \( (\alpha, \beta) \) give rise to valid types of renormalization. The answer to this question is given in [14]; we will not go into details as this would require more definitions, but suffice to say that there is a simple admissibility condition stated in terms of the shift operator acting on words on two symbols.

Definition 2.8. Let \( f \) be a Lorenz map such that \( R^n f \) is defined for every positive integer \( n \). In this situation we say that \( f \) is infinitely renormalizable. The combinatorial type of \( f \) is the sequence of words \( \{ (\alpha_0, \beta_0), (\alpha_1, \beta_1), \ldots \} \), where \( f \) has renormalization of type \( (\alpha_0, \beta_0) \), \( Rf \) has renormalization of type \( (\alpha_1, \beta_1) \) and so on\(^5\). If the length of the words \( \alpha_k \) and \( \beta_k \) is bounded in \( k \) then \( f \) is said to be of bounded combinatorial type.

Remark 2.9. Let \( f \) be of combinatorial type \( \{ \sigma_0, \sigma_1, \ldots \} \), where \( \sigma_k = (\alpha_k, \beta_k) \). Then \( Rf \) has combinatorial type \( \{ \sigma_1, \sigma_2, \ldots \} \). In other words, \( R \) shifts the combinatorial type to the left.

We are now ready to state the main result, but before doing so we mention the renormalization conjecture for Lorenz maps (the statement is taken from the corresponding result for unimodal maps, see [5]):

\(^4\) The choice of rescaling is somewhat arbitrary (as long as it is affine)—we have chosen it so that the critical point and the left endpoint of the domain of \( f \) are fixed under renormalization, whereas the right endpoint may move. Another natural choice is to fix the endpoints of the domain of \( f \) but then the critical point may move.

\(^5\) When talking about the combinatorial type we implicitly assume that it is admissible in line with the discussion of remark 2.7.
Conjecture 2.10 (Renormalization horseshoe). The limit set of $\mathcal{R}$ acting on the space of Lorenz maps of bounded combinatorial type is a hyperbolic Cantor set where $\mathcal{R}$ is conjugate to the full shift in a finite number of symbols, for every critical exponent $\rho > 1$.

The above theorem represents an ultimate goal for the theory of renormalization. However, our results are much more modest in that we only prove that locally at one point in the limit set of $\mathcal{R}$ the above conjecture holds:

Theorem 2.11 (Main theorem). Let $\alpha = \{0, 1\}$ and $\beta = \{1, 0, 0\}$. The restricted renormalization operator $\mathcal{R}_{\alpha, \beta}$ acting on the space of Lorenz maps with critical exponent $\rho = 2$ has a hyperbolic fixed point.

Proof. This is a direct consequence of the estimates in theorem 5.3 and the discussion in section 4.1. □

We would like to address the question as to how far towards the renormalization conjecture our method of proof can take us. Unfortunately the answer is ‘not very’. Theoretically, given a critical exponent $\rho > 1$ and any periodic combinatorial type $\{\sigma_0, \ldots, \sigma_n, \sigma_0, \ldots, \sigma_n, \ldots\}$ we could check estimates similar to those of theorem 5.3 in order to deduce the existence of a hyperbolic fixed point. However, implementing these checks even for the simple combinatorial type at hand requires a substantial effort, so this does not really have any practical significance. Despite these shortcomings we still think that our current result is an important first step in the theory of renormalization of Lorenz maps.

It is also interesting to ask if any of the methods from the theory of renormalization of unimodal maps can be used to prove the renormalization conjecture for Lorenz maps. We do not know the answer to this question but it seems unlikely since the unimodal theory is based on complex analytic methods that do not have any obvious generalization to Lorenz maps (since Lorenz maps have a point of discontinuity). For this reason the renormalization theory for Lorenz maps poses new and significant difficulties. However, we can use some results from unimodal renormalization as the following remark shows.

Remark 2.12. The fixed point of theorem 2.11 is the simplest non-unimodal fixed point of $\mathcal{R}$. By this we mean that if $\alpha$ and $\beta$ both have length 2 (i.e. $\alpha = \{0, 1\}$, $\beta = \{1, 0\}$) then the fixed point of the period-doubling operator on unimodal maps corresponds to a fixed point for $\mathcal{R}_{\alpha, \beta}$ as follows.

Let $g : [-1, 1] \to [-1, 1]$ be the fixed point for the period-doubling operator normalized so that $g(0) = 1$. Then $g$ is an even map that satisfies the Cvitanović–Feigenbaum functional equation

$$g(x) = -\lambda^{-1} g^2(\lambda x), \quad \lambda = -g(1).$$

Now define a Lorenz map $f$ by $f|_{[-1, 0]} = g$ and $f|_{[0, 1]} = -g$. It is easy to check that the first-return map $\tilde{f}$ to $U = [-\lambda, \lambda]$ is $\tilde{f} = f^2$ and that $U$ is maximal. Thus

$$\mathcal{R} f(x) = \lambda^{-1} \tilde{f}(\lambda x) = \begin{cases} -\lambda^{-1} g^2(\lambda x) = g(x) & \text{if } x < 0, \\ \lambda^{-1} g^2(\lambda x) = -g(x) & \text{if } x > 0, \end{cases}$$

which shows that $f$ is a fixed point of $\mathcal{R}_{\alpha, \beta}$. 
3. Consequences of the main result

The existence of a hyperbolic renormalization fixed point has very strong dynamical consequences, some of which we will give a brief overview of here. Throughout this section let $R$ denote the restricted renormalization operator $R_{\alpha, \beta}$, where $\alpha = \{0, 1\}$ and $\beta = \{1, 0, 0\}$, and let $f_*$ denote the fixed point of theorem 2.11.

**Corollary 3.1 (Stable manifold).** There exists a local stable manifold $W_{s_{\text{loc}}}$ at $f_*$ consisting of maps in a neighbourhood of $f_*$ which under iteration of $R$ remain in this neighbourhood and converge with an exponential rate to $f_*$. The local stable manifold extends to a global stable manifold $W_s$ consisting of maps which converge to $f_*$ under iteration of $R$. If $f \in W_s$ then $f$ is infinitely renormalizable.

**Proof.** The existence of a stable manifold is a direct consequence of the stable and unstable manifold theorem. If $f$ converges to $f_*$, then $R^n f$ is defined for all $k > 0$, which is the same as saying that $f$ is infinitely renormalizable. □

We now turn to studying the dynamical properties of maps on the stable manifold. Let $f \in W_s$, then the times of closest return $(a_n, b_n)$ are given by the recursion

$$
\begin{align*}
    a_{n+1} &= a_n + b_n, & a_1 &= 2, \\
    b_{n+1} &= 2a_n + b_n, & b_1 &= 3.
\end{align*}
$$

These determine the first-return interval $U_n = \text{cl}(L_n \cup R_n)$ for the $n$th renormalization by

$$
L_n = (f^{b_n}(0^+), 0), \quad R_n = (0, f^{a_n}(0^-)).
$$

In other words the first-return map $\hat{f}_n$ to $U_n$ is given by $\hat{f}_n(x) = f^{a_n}(x)$ if $x \in L_n$ and $\hat{f}_n(x) = f^{b_n}(x)$ if $x \in R_n$.

Define

$$
\begin{align*}
    L_k^n &= f^k(L_n), & k &= 0, \ldots, a_n - 1, \\
    R_k^n &= f^k(R_n), & k &= 0, \ldots, b_n - 1.
\end{align*}
$$

The collection of these intervals (over $k$) form a pairwise disjoint collection for each $n$, called the intervals of generation $n$ (see figure 2).

**Theorem 3.2 (Cantor attractor).** If $f \in W_s$ then the closure of the critical orbits of $f$ is a measure zero Cantor set $\Lambda_f$ which attracts almost every point in the domain of $f$.

**Proof.** The critical orbits form the endpoints of the dynamical intervals $\{L_n^k \cup R_n^k\}$, so $\Lambda_f$ is contained in

$$
\bigcap_n \text{cl}(E_n \cup F_n),
$$

where $E_n = \bigcup_k L_n^k$ and $F_n = \bigcup_k R_n^k$. \hfill (1)

Note that $E_{n+1} \subset E_n$ and $F_{n+1} \subset F_n$. 

---

![Figure 2. Illustration of the dynamical intervals of generations 0, 1, 2 for renormalization of type $\alpha = \{0, 1\}$, $\beta = \{1, 0, 0\}$.

First assume that \( f = f_* \). Then the first-return maps \( \tilde{f}_n \) are all equal to \( f \) itself (up to a linear change of coordinates) so the total lengths of \( E_n \) and \( F_n \) shrink with an exponential rate (the position of \( U_{n+1} \) inside \( U_n \) is the same for all \( n \), so we can apply the macroscopic and infinitesimal Koebe principles as in the proof of the ‘real bounds’ in [13, chapter VI.2]). Hence the intersection in (1) is a measure zero Cantor set and consequently \( \Lambda_f \) is as well.

Next, if \( f \) is an arbitrary map in \( W^s \) then \( \mathcal{R}^n f \) converges to \( f_* \). In other words, the first-return maps \( \{ \tilde{f}_n \} \) converge to \( f_* \) (up to a linear change of coordinates). Now use the same arguments as above.

For a proof that \( \Lambda_f \) is an attractor of full measure, see [14]. □

**Theorem 3.3 (Rigidity).** If \( f, g \in W^s \) then there exists a homeomorphism \( h : \Lambda_f \to \Lambda_g \) conjugating \( f \) and \( g \) on their respective Cantor attractors. If furthermore \( f, g \in W^s_{\text{loc}} \), then \( h \) extends to a \( C^{1+\alpha} \) diffeomorphism on the entire domain of \( f \).

**Proof.** Define \( h(f^n(0^−)) = g^n(0^−) \) and \( h(f^n(0^+)) = g^n(0^+) \). This extends continuously to a map on \( \Lambda_f \) as in the proof of [13, proposition VI.1.4].

If \( f, g \in W^s_{\text{loc}} \) then there exist \( C > 0 \) and \( \lambda < 1 \) such that \( d(f^n, g^n) < C\lambda^n \) so we can use an argument similar to that in [13, theorem VI.9.4] to prove the second statement. □

**Remark 3.4 (Universality).** The second conclusion of theorem 3.3 is a strong version of what is known as ‘metric universality’: the small scale geometric structure of the Cantor attractor does not depend on the map itself (only on the combinatorial type and the critical exponent). That is, if we take two maps \( f, g \in W^s_{\text{loc}} \) and zoom in around the same spot on their Cantor attractors then their structures are almost identical since a differentiable map (i.e. the extended \( h \)) is almost linear if one zooms in closely enough.

For example, the limit of \( |L_{n+1}|/|L_n| \) as \( n \to \infty \) exists and is independent of \( f \) (it equals the ratio \( |L_2(f_*)|/|L_1(f_*)| \) for \( f_* \)). More generally, the multifractal spectrum (and Hausdorff measure in particular) of \( \Lambda_f \) does not depend on \( f \) (only on \( f_* \)).

We also want to mention another type of universality called ‘universality in the parameter plane’ where the unstable eigenvalues of \( DT_{f_*} \) govern the structure of the parameter plane for families of Lorenz maps. However, in order to present the details we would need more definite information on the structure of the spectrum of \( DT_{f_*} \) so we will have to return to this discussion in another paper.

4. Outline of the computer assisted proof

In this section we give a brief outline of the method of proof and how to calculate rigorous estimates with a computer.

4.1. Method of proof

Given a Fréchet differentiable operator \( T \) with compact derivative on a Banach space \( X \) of analytic functions we would like to prove that \( T \) has a hyperbolic fixed point. The main tool is the following consequence of the contraction mapping theorem:

**Proposition 4.1.** Let \( \Phi \) be a Fréchet differentiable operator on a Banach space \( X \), let \( f_0 \in X \) and let \( B_r(f_0) \subset X \) be the closed ball of radius \( r \) centred on \( f_0 \). If there are positive numbers
\( \varepsilon, \theta \) such that
\[ (1) \| D\Phi f \| < \theta, \text{ for all } f \in B_r(f_0), \]
\[ (2) \| \Phi f_0 - f_0 \| < \varepsilon, \]
\[ (3) \varepsilon < (1 - \theta)r, \]
then there exists \( f_* \in B_r(f_0) \) such that \( \Phi f_* = f_* \) and \( \Phi \) has no other fixed points inside \( B_r(f_0) \). Furthermore, \( \| f_* - f_0 \| < \varepsilon / (1 - \theta) \).

Our strategy is to find a good approximation \( f_0 \) of a fixed point of \( T \) and then use a computer to verify that the conditions on \( r, \varepsilon, \theta \) hold if \( r \) is chosen small enough. Unfortunately, this is not possible for \( T \) itself since in our case it is not a contraction, so first we have to turn \( T \) into a contraction without changing the set of fixed points. This is done by using Newton’s method to solve the equation \( Tf - f = 0 \), which results in the iteration
\[ f \mapsto f - (DTf - I)^{-1}(Tf - f), \]
where \( I \) denotes the identity operator on \( X \). The operator we use is a slight simplification of this, namely
\[ \Phi f = f - (\Gamma - I)^{-1}(Tf - f) = (\Gamma - I)^{-1}(\Gamma - T)f, \]
where \( \Gamma \) is a finite-rank linear approximation of \( DTf_0 \) (chosen so that \( \Gamma - I \) is invertible). The operator \( \Phi \) is a contraction if \( f_0 \) and \( \Gamma \) are chosen carefully.\(^6\) Note that \( \Phi f = f \) if and only if \( Tf = f \), so once we verify that the conditions of proposition 4.1 hold for \( \Phi \) it follows that \( T \) has a fixed point.

To prove hyperbolicity we need to do some extra work. The derivative of \( \Phi \) is
\[ D\Phi f = (\Gamma - I)^{-1}(\Gamma - DTf). \]
At this stage we would have already checked that the norm of this is bounded from above by 1. By strengthening this estimate to
\[ \| (\Gamma - e^{it}I)^{-1}(\Gamma - DTf) \| < 1, \quad \forall t \in \mathbb{R}, \quad \forall f \in B_r(f_0), \]
we also get that \( DTf_* \) is hyperbolic at the fixed point \( f_* \). To see this, assume that \( e^{it} \) is an eigenvalue of \( DTf_* \) with eigenvector \( h \) normalized so that \( \| h \| = 1. \) Then
\[ \| (\Gamma - e^{it}I)^{-1}(\Gamma - DTf_*)h \| = \| (\Gamma - e^{it}I)^{-1}(\Gamma - e^{it}I)h \| = \| h \| = 1, \]
which is impossible. Since \( DT \) was assumed to be compact we know that the spectrum is discrete, so the lack of eigenvalues on the unit circle implies hyperbolicity.

4.2. Rigorous computer estimates

In order to verify the above estimates on a computer we are faced with two fundamental problems: (i) arithmetic operations on real numbers are carried out with finite precision which leads to rounding problems, (ii) the space of analytic functions is infinite dimensional so any representation of an analytic function needs to be truncated.

The general idea to deal with these problems is to compute with sets which are guaranteed to contain the exact result instead of computing with points: real numbers are replaced with intervals, analytic functions are replaced with rectangle sets \( A_0 \times \cdots \times A_k \times \{ C \} \) in \( \mathbb{R}^n \) representing all functions of the form
\[ a_0 + \cdots + a_k z^k + e^h(z) \mid a_j \in A_j, \quad j = 0, \ldots, k, \quad \| h \| \leq C, \]
\(^6\) Here is how to choose \( f_0 \) and \( \Gamma \): use the Newton iteration on polynomials of some fixed degree to determine \( f_0 \) and set \( \Gamma = DTf_0 \). The hardest part is finding an initial guess such that the iteration converges.
where \( \{ A_j \} \) are intervals. This takes care of the truncation problem and the rounding problem is taken care of roughly by ‘rounding outwards’ (lower bounds are rounded down, upper bounds are rounded up). Once these set representations have been chosen we lift operations on points to operations on sets. Since the form of these sets are most likely not preserved by such operations, this lifting involves finding bounds by sets of the chosen form (e.g. if \( F \) and \( G \) are rectangle sets of analytic functions and we want to lift composition of functions, then we have to find a rectangle set which contains the set \( \{ f \circ g \mid f \in F, g \in G \} \)).

Sections 2 and 4 of the supplementary data contain all the details for computing with intervals and rectangle sets of analytic functions, respectively.

Let us make one final remark concerning the evaluation of the operator norm of a linear operator \( L \) on the space of analytic functions. In order to get good enough bounds on the estimate of the operator norm we will use the \( \ell^1 \)-norm on the Taylor coefficients of analytic functions. The reason for this is that estimating the operator norm with

\[
\| L \| = \sup_{ \| f \|_1 \leq 1} \| Lf \|
\]

will usually result in really bad estimates. With the \( \ell^1 \)-norm, if we think of \( L \) as an infinite matrix (in the basis \( \{ z^k \} \)), the operator norm is found by taking the supremum over the norms of the columns of this matrix, that is

\[
\| L \| = \sup_{k \geq 0} \| L\xi_k \|, \quad \xi_k(z) = z^k.
\]

Evaluating the norms of columns gives much better estimates and for this reason we choose this norm. See section 4.11 of the supplementary data (available from stacks.iop.org/Non/23/1291/mmedia) for the specifics.

5. Proof of the main theorem

First we restate the definition of the restricted renormalization operator, then we change coordinates and restate the main theorem.

5.1. Definition of the operator

From now on we fix the domain of our Lorenz maps to some interval \([-1, r]\). The right endpoint cannot be fixed since it generally changes under renormalization (we will soon change coordinates so that the domain is fixed).

Instead of dealing with functions with a discontinuity we represent a Lorenz map \( F \) by a pair \( (f, g) \), with \( f : [-1, 0] \to [-1, r] \), \( f(0) = r \), and \( g : [0, r] \to [-1, r] \), \( g(0) = -1 \).

With this notation, the first-return map to some interval \( U \) will be of the form \( (F^a, F^b)|_U \) if \( F \) is renormalizable. For the type \( \alpha = \{0, 1\}, \beta = \{1, 0, 0\} \), we can be more precise: in this case \( a = 2, b = 3 \) and the first-return map is of the form \( (g \circ f, f \circ f \circ g)|_U \) if it is renormalizable.

Let \( T \) denote the restricted renormalization operator \( \mathcal{R}_{a, \beta} \), and fix the critical exponent \( \rho = 2 \). If \( T(f, g) = (\hat{f}, \hat{g}) \) then \( T \) is defined by

\[
\hat{f}(z) = \lambda^{-1} g \circ f(\lambda z),
\]

\[
\hat{g}(z) = \lambda^{-1} f \circ f \circ g(\lambda z),
\]

\[
\lambda = -f^2(-1).
\]
5.2. Changing coordinates

To ensure the correct normalization \((g(0) = -1)\) and the correct critical exponent \((\rho = 2)\) we make two coordinate changes and calculate how the operator \(T\) transforms. We will also carefully choose the domain of \(T\) so that all compositions are well defined (e.g. \(\lambda z\) is in the domain of \(f\), etc). This is checked automatically by the computer (and also shows that \(T\) is differentiable with compact derivative, since \(f\) and \(g\) are analytic). Finally, it is important to realize that the choice of coordinates may significantly affect the operator norm of the derivative; not every choice will give a good enough estimate.

The domain of \(T\) is chosen to be contained in the set of Lorenz maps \((f, g)\) with representation
\[
\begin{align*}
f(z) &= \phi(\lambda^2 z^2), \\
g(z) &= \psi(\lambda^2 z^2),
\end{align*}
\]
where \(\phi\) and \(\psi\) have domains \(\{z : |z-1| < s\}\) and \(\{z : |z| < t\}\), respectively (the constants \(s\) and \(t\) will soon be specified). Rewriting \(T\) in terms of \(\phi\) and \(\psi\) gives
\[
\begin{align*}
\hat{\phi}(z) &= \lambda^{-1} \psi(\phi(\lambda^2 z^2)), \\
\hat{\psi}(z) &= \lambda^{-1} \phi(\psi(\lambda^2 z^2)), \\
\lambda &= -\phi(\phi(1)^2).
\end{align*}
\]
This coordinate change ensures the correct critical exponent.

The next coordinate change is to fix the normalization and also to bring the domain of all functions to the unit disk. Fixing the normalization has the benefit that the error involved in the evaluation of \(\lambda\) is minimized (since we only need to evaluate \(f\) close to \(z = 0\), see section 4.8 of the supplementary data). Changing all domains to the unit disk simplifies the implementation of the computer estimates.

**Definition 5.1.** Define \(X\) to be the Banach space of symmetric (with respect to the real axis) analytic maps on the unit disk with finite \(\ell^1\)-norm. That is, if \(f \in X\) then \(f(z) = \sum a_k z^k\) with \(a_k \in \mathbb{R}\) and \(\|f\| = \sum |a_k| < \infty\).

**Definition 5.2.** Define \(Y = X \times X\) with the norm \(\|(f, g)\|_Y = \|f\|_X + \|g\|_X\) and with linear structure defined by \(\alpha(f, g) + \beta(f', g') = (\alpha f + \beta f', \alpha g + \beta g')\). Clearly \(Y\) is a Banach space (since \(X\) is).

Change coordinates from \(\phi, \psi\) to \((f, g) \in Y\) (note that \(f\) and \(g\) are not the same as above) as follows:
\[
\begin{align*}
\phi(z) &= f([z - 1]/s), \\
\psi(z) &= -1 + z \cdot g(z/t),
\end{align*}
\]
where we will choose \(s = 2.2\) and \(t = 0.5\). Rewriting \(T\) in terms of \(f\) and \(g\) gives
\[
\begin{align*}
\hat{f}(w) &= \lambda^{-1} \left\{-1 + f \left(\lambda^2 \left[w + \frac{1}{s}\right] - \frac{1}{s}\right)^2 \cdot g \left(\frac{1}{t} \cdot f \left(\lambda^2 \left[w + \frac{1}{s}\right] - \frac{1}{s}\right)^2\right)\right\}, \\
\hat{g}(w) &= \frac{1}{tw} \left\{1 + \lambda^{-1} f \left(\frac{1}{s} \cdot f \left(\lambda^2 tw \cdot [\lambda^2 tw + 2\lambda^2 t w^2 - 2]\cdot \left[w + \frac{1}{s}\right] - \frac{1}{s}\right)^2\right)\right\}, \\
\lambda &= -f([f(0)^2 - 1]/s).
\end{align*}
\]
This is the final form of the operator that will be studied.
5.3. Computing the derivative

In order to simplify the computation of the derivative of $T$ we break the computation of $T$ into several steps as follows:

$$
p_f(w) = \lambda^2 \cdot (w + s^{-1}) - s^{-1} \quad p_g(w) = \lambda^2 w
$$

$$
f_1 = f \circ p_f \quad g_1 = g \circ p_g
$$

$$
f_2 = f_1^2 \quad g_2 = t \cdot p_g \cdot g_1
$$

$$
f_3 = f_2 / t \quad g_3 = g_2 \cdot (g_2 - 2) / s
$$

$$
f_4 = g \circ f_3 \quad g_4 = f \circ g_3
$$

$$
f_5 = -1 + f_2 \cdot f_4 \quad g_5 = (g_4^2 - 1) / s
$$

$$
f_6 = f_5 / \lambda \quad g_6 = f \circ g_5
$$

$$
g_7 = g_6 / \lambda \quad g_8(w) = (g_7(w) + 1) / (t \cdot w)
$$

With this notation we have that $T(f, g) = (f_6, g_8)$. Note that the result of $g_7(w) + 1$ is a function with zero as constant coefficient, so in the implementation of $g_8$ we will not actually divide by $w$, instead we will ‘shift’ the coefficients to the left.

It is now fairly easy to derive expressions for the derivative. If $f$ is perturbed by $\delta f$ and $g$ is perturbed by $\delta g$, then the above functions are perturbed as follows:

$$
\delta p_f(w) = 2 \cdot \lambda \cdot \delta \lambda \cdot (w + s^{-1}) \quad \delta p_g(w) = 2 \cdot \lambda \cdot \delta \lambda \cdot w
$$

$$
\delta f_1 = Df \circ p_f \cdot \delta p_f + \delta f \circ p_f \quad \delta g_1 = Dg \circ p_g \cdot \delta p_g + \delta g \circ p_g
$$

$$
\delta f_2 = 2 f_1 \delta f_1 \quad \delta g_2 = t \cdot (\delta p_g \cdot g_1 + p_g \cdot \delta g_1)
$$

$$
\delta f_3 = \delta f_2 / t \quad \delta g_3 = \delta g_2 \cdot (g_2 - 2) / s + g_2 \cdot \delta g_2 / s
$$

$$
\delta f_4 = Dg \circ f_3 \cdot \delta f_3 + \delta g \circ f_3 \quad \delta g_4 = Df \circ g_3 \cdot \delta g_3 + \delta f \circ g_3
$$

$$
\delta f_5 = \delta f_3 \cdot f_4 + f_2 \cdot \delta f_4 \quad \delta g_5 = 2 \cdot g_4 \cdot \delta g_4 / s
$$

$$
\delta f_6 = \delta f_5 / \lambda - f_5 \cdot \delta \lambda / \lambda^2 \quad \delta g_6 = Df \circ g_5 \cdot \delta g_5 + \delta f \circ g_5
$$

$$
\delta g_7 = \delta g_6 / \lambda - g_6 \cdot \delta \lambda / \lambda^2 \quad \delta g_8(w) = \delta g_7(w) / (t \cdot w)
$$

With this notation we have that $DT(f, g)(\delta f, \delta g) = (\delta f_6, \delta g_8)$.

5.4. New statement of the main theorem

We now state the main theorem in the form it will be proved. The discussion in section 4.1 shows how this result can be used to deduce theorem 2.11.

Theorem 5.3. There exists a Lorenz map $F_0$ and a matrix $\Gamma$ such that the simplified Newton operator $\Phi = (\Gamma - I)^{-1}(\Gamma - T)$ is well defined and satisfies:

1. $\|D\Phi_F\| < 0.2$, for all $\|F - F_0\| \leq 10^{-7}$.
2. $\|\Phi F_0 - F_0\| < 5 \times 10^{-9}$.
3. $\|(\Gamma - e^{tI})^{-1}(\Gamma - DT_F)\| < 0.9$, for all $t \in \mathbb{R}$, $\|F - F_0\| \leq 10^{-7}$.

Proof. The supplementary data (available from stacks.iop.org/Non/23/1291/mmedia) is dedicated to rigorously checking the first two estimates with a computer. The third estimate is
verified by covering the unit circle with small rectangles and using the same techniques as in
the first two estimates to get rigorous upper bounds on the operator norm. However, we have
left out the source code for this estimate to keep the page count down and also because the
running time of the program went from a few seconds to several hours (we had to cover the
unit circle with 50 000 rectangles in order for the estimate to work).

**Remark 5.4.** The approximate fixed point $F_0$ and approximate derivative $\Gamma$ at the fixed point
are found by performing a Newton iteration eight times on an initial guess (which was found by
trial and error). We will not spend too much time talking about these approximations but they
could potentially be used to compute, e.g., the Hausdorff dimension of the Cantor attractor of
maps on the local stable manifold.

We did, however, compute the eigenvalues of $\Gamma$ and it turns out that $\Gamma$ has two simple
expanding eigenvalues $\lambda_s \approx 23.36530$ and $\lambda_w \approx 12.11202$, and the rest of the spectrum is
strictly contained in the unit disk. Since $\Gamma$ is a good approximation of $DTf$, and both operators
are compact it seems clear that the spectrum of $DTf$ also must have exactly two unstable
eigenvalues.

Lanford [11] claims that in the case of the period-doubling operator if an analog of the
third estimate of theorem 5.3 holds and ‘if $\Gamma$ has spectrum inside the unit disk except for a
single simple expanding eigenvalue, then the same will be true for $DTf$’. It seems plausible
that a similar statement holds in the present situation with two simple expanding eigenvalues
but have not yet managed to prove this (it is easy to see that if $\Gamma$ and $DTf$ were both diagonal
then the third estimate would imply that they have the same number of unstable eigenvalues).

**References**

[1] De Carvalho A, Lyubich M and Martens M 2005 Renormalization in the Hénon family: I. Universality but
non-rigidity J. Stat. Phys. **121** 61–69
[2] Chandramouli V V M S, Martens M, de Melo W and Tresser C P 2009 Chaotic period doubling Ergod. Theory
Dyn. Syst. **29** 381–418
[3] Collet P, Eckmann J-P and Koch H 1981 Period doubling bifurcations for families of maps on $R^n$. J. Stat. Phys. **25** 1–14
[4] Davie A M 1996 Period doubling for $C^{2+\epsilon}$ mappings Commun. Math. Phys. **176** 261–72
[5] de Faria E, de Melo W and Pinto A 2006 Global hyperbolicity of renormalization for $C^r$ unimodal mappings
Ann. Math. (2) **164** 731–824
[6] Guckenheimer J and Williams R F 1979 Structural stability of Lorenz attractors Publ. Math. de L'IHÉS **50** 59–72
[7] Hirsch M W, Smale S and Devane y R L 2004 Differential Equations, Dynamical Systems, and an Introduction
to Chaos 2 edn (Pure and Applied Mathematics) vol 60) (Amsterdam: Elsevier/Academic)
[8] Keller G and St Pierre M 2001 Topological and measurable dynamics of Lorenz maps Ergodic Theory, Analysis,
and Efficient Simulation of Dynamical Systems (Berlin: Springer) pp 333–61
[9] Koch H, Schenkel A and Wittwer P 1996 Computer-assisted proofs in analysis and programming in logic: a
case study SIAM Rev. **38** 565–604
[10] Lanford O E III 1982 A computer-assisted proof of the Feigenbaum conjectures Bull. Am. Math. Soc. **6** 427–34
[11] Lanford O E III 1984 Computer-assisted proofs in analysis Physica A **124** 465–70
[12] Martens M and de Melo W 2001 Universal models for Lorenz maps Ergod. Theory Dyn. Syst. **21** 833–60
[13] de Melo W and van Strien S 1993 One-dimensional dynamics Ergebnisse der Mathematik und ihrer Grenzgebiete
(3) (Results in Mathematics and Related Areas (3) vol 25) (Berlin: Springer)
[14] St Pierre M 1999 Topological and measurable dynamics of Lorenz maps Dissertationes Math. (Rozprawy Mat.)
**382** 134
[15] Williams R F 1979 The structure of Lorenz attractors Publ. Math. de L'IHÉS **50** 73–99
[16] Yampolsky M 2003 Renormalization horseshoe for critical circle maps Commun. Math. Phys. **240** 75–96