Neutron Skin size dependence of the nuclear binding energy

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The nuclear binding energy is studied using a finite temperature density functional theory. A Skyrme interaction is used in this work. Volume, surface, and symmetry energy contributions to the binding energy are investigated. The case of neutron skin is considered in detail. The ratio of surface symmetry energy to volume symmetry energy of neutron skin dependent part is much larger than the corresponding ratio of neutron skin independent part. This shows that the large part of symmetry energy comes from the different size of neutron and proton distributions.

I. INTRODUCTION

The mass formulae \[1,2\] characterizes the binding energy of nuclei in terms of the proton number \(Z\) and neutron number \(N\) or nucleon number \(A = Z + N\). Volume, surface, Coulomb and pairing terms appear. Of importance and one of the least determined parts of the binding energy expression is the symmetry energy term. The symmetry energy has both a volume symmetry part and a surface symmetry part, similar to the division of the volume and surface terms. The division into volume and surface energies occurs because observed nuclei have finite nucleon number with a maximum value of \(A \sim 250\). The symmetry energy appears in the total binding energy with factors involving both \((N - Z)/A\)^2\(A\) and \(((N - Z)/A)^2\)\(A^{2/3}\) for the volume and surface symmetry terms respectively. For a heavy nucleus with \(A = 200, Z = 80, N = 120\) the \((N - Z)/A = 1/5\) and \(((N - Z)/A)^2 = 1/25\). The isospin dependence of the binding energy arises in part from kinetic energy differences between protons and neutrons and from interaction terms arising from the isospin dependence of the nuclear force between nucleons. This interaction has terms involving the \(\vec{r}_i \cdot \vec{r}_j\) which arises from the exchange of isovector mesons. The surface terms in the mass formulae arise from the loss of binding energy for nucleons near the surface of a nucleus. In a neutron star, the volume term only exists. Thus, of importance for neutron star physics is the extraction of the volume term from properties of known finite nuclei which contain both volume and surface terms. The symmetry term also plays a significant role in heavy ion collisions \[3,4\] in neutron star physics \[5\], in the valley of nuclear stability and associated neutron and proton drip lines, in neutron halo nuclei in isobaric analog states \[6,7\], and in giant isovector dipole states \[8\]. Extensive discussion of the symmetry energy can be found in the work of Danielewicz et al. \[12\]. The role of the symmetry energy in the nuclear surface has also been studied in Ref.\[12\]. An extensive review can be found in Ref.\[8\].

In our previous paper \[13\], we have examined the \(T\) dependence of the expansion coefficients in mass formulae by minimizing the Helmholtz free energy using Skyrme interaction. In there we have assumed \(R_n = R_p\) and thus missed any effects of neutron skin which would exists in asymmetric nuclei. We estimated approximately neutron skin effect thus we will study the neutron skin dependence more fully here by expanding energy with the neutron skin size \(R_n - R_p\).

In this paper we study the neutron skin effect on the volume and surface contributions to the symmetry energy. Our approach is based on a finite temperature density functional theory and we use a Skyrme type of interaction which we develop in the next section. The same Skyrme approach was also used to study phase transition in Ref.\[14\]--\[17\]. The symmetry energy and the neutron skin effects on symmetry energy and other energy coefficients for three cases of Skyrme interactions are studied in Sect.\[14\] and the results are summarized in Sect.\[17\].

II. BINDING ENERGY IN A DENSITY FUNCTIONAL APPROACH.

A density functional theory based on a Skyrme interaction will be used in our investigation which is summarized in Appendix\[A\] The density distribution is taken to be of the Saxon-Wood form:

\[
\rho_q(\vec{r}) = \rho_q(R_q) = \frac{\rho_{qc}}{1 + e^{(r - R_q)/a_q}} \tag{1}
\]

The \(q = p, n\) for protons, neutrons. We will allow the central density \(\rho_{qc}\) and the \(R_q\) for protons and neutrons to be different in general. The diffuseness \(a_q\) are taken to be the same for simplicity since our focus is on the neutron skin size dependence rather than the effects of different diffusness. Two limiting cases can also be considered. These
are: 1. $R_p = R_n$ so that the difference in $N \neq Z$ nuclei is in the central density or $\rho_{pc} \neq \rho_{nc}$ and 2. $R_p \neq R_n$ and $\rho_{pc} = \rho_{nc}$. In case 2 the neutron distribution reflects a neutron halo for $N > Z$.

Using this density functional approach the binding energy of nuclei as a function of mass number $A$, proton number $Z$ and temperature $T$ can be evaluated. The Weizacker semiempirical mass formulae [1], its extension by Myers and Swiatecki [18–21] and also including finite temperature effects [13] is

$$E(A, Z, T) = E_V(T)A + E_S(T)A^{2/3} + S_V(T)I^2A + S_S(T)I^2A^{2/3} + E_C Z^2 A^{1/3} + E_{df} Z^2 A + E_{ex} Z^4 A^{1/3} + c\Delta A^{-1/2}$$

(2)

where $I = (N - Z)/A = (A - 2Z)/A$. The $B = -E_V$ is the usual bulk energy per nucleon. The $E_{df}$ and $E_{ex}$ are the coefficients for the diffuseness correction and the exchange correction to the Coulomb energy. For the pairing correction with constant $\Delta$, $c = +1$ for odd-odd nuclei, 0 for odd-even nuclei, and $-1$ for even-even nuclei. The above formula at $T = 0$ is the well known Weizacker semiempirical mass formula [1] studied extensively by Myers and Swiatecki [18–21]. Early studies excluded the surface symmetry term $S_S$ and only the surface term $E_S$ was included. The values of the coefficients as found in textbooks such as Refs. [2, 4] are $E_V(0) = -B(0) \approx -16$, $E_S(0) \approx 17$, and $S_V(0) \approx 24$ in MeV. The ratio $E_S/B$ of surface to bulk energy at $T = 0$ is very close to unity.

Myers and Swiatecki [18–21] have considered an $A^{1/3}$ curvature term and a higher order $I^4$ term also. However they dropped these two terms in their preliminary and illustrative study of nuclear droplet model with arbitrary shape [20, 21]. From the values in Ref. [19] the curvature term is $7.0A^{1/3}$ which is smaller than 1/6 of the surface term of $18.6A^{2/3}$ for $A > 10$. It should be noted that the so-called nuclear curvature energy puzzle [22] concerns a higher theoretical value of the order of 10 MeV compared to a negligibly small empirical value for the nuclear curvature energy. On the other hand, the volume asymmetry energy is $28.1I^2A - 24.5I^4A$ [19]. However typical values of the isospin asymmetry $I$ are smaller than 1/5. Thus the term of $I^4$ is less than a few percent of $I^2$ term. Furthermore for the energy of Eq. (A1) an $I^4$ term comes directly only from kinetic energy. The simple kinetic energy can be expanded as

$$N^{5/3} + Z^{5/3} \approx \left( \frac{A}{2} \right)^{5/3} \left( 1 + \frac{5}{9} I^2 + \frac{5}{27} I^4 \right)$$

Thus the ratio of the $I^4$ term compared to the $I^2$ term is $I^2/3$ which is about 1/75 for the lead region. Since we are interested in the qualitative study of the energy expansion coefficients within the temperature and neutron skin size dependent part of the nuclear energy we did not include $A^{1/3}$ term and $I^4$ term here.

In our previous paper [13], we have examined the $T$ dependence of the expansion coefficients in Eq. (2) by minimizing the Helmholtz free energy using a Skyrme interaction with the density distribution of Eq. (1). With $R_n = R_p = R$, we can integrate Skyrme interaction analytically to obtain total energy as a function of $R$. Using this energy function, the total energy $E(A, Z, T)$ minimizing free energy is found by varying the value of $R$ for a nucleus with $Z$ protons and $N$ neutrons at temperature $T$. Then we use Eq. (2) for various nuclei to obtain the expansion coefficients. Since we have assumed $R_n = R_p$ in our previous studies, we missed any effects of neutron skin which would exist in asymmetric nuclei. We estimated approximately neutron skin effect for $T^2$ term in kinetic energy, Eq. (A3), by setting the central densities to be the same, $\rho_{nc} = \rho_{pc} = \rho_c/2$ where $\rho_c$ is the total central density obtained with $R_n = R_p$, and found the effect of $R_n \neq R_p$ in kinetic energy is small.

In the present paper we now consider neutron skin effects more fully. Specifically, we study the neutron skin size $t = R_n - R_p$ dependence of total energy $E(A, Z, T, t)$. Then we expand each coefficient $E_i$ of Eq. (2) as

$$E_i(T, t) = E_i(T) + E_{i,sk}(T) \frac{|t|}{A^{1/3}}$$

(3)

Notice here that $E_i(T, t)$, not $E_i(T)$, in Eq. (3) corresponds to $E_i(T)$ in Eq. (2) which is the expansion coefficients of empirical nuclear energy. In Eq. (4), $E_i(T)$ (we use this notation in Eq. (3) just for simplicity) is the expansion coefficient of neutron skin size $t$ independent part of nuclear energy (which is obtained by assuming same neutron and proton distribution size $R_n = R_p = R$) and $E_{i,sk}(T)$ is the expansion coefficient of the first order $t$ dependent part of nuclear energy with $tA^{1/3}$ factor. Since we can integrate a Skyrme interaction analytically for the case of $R_n = R_p = R$, we can obtain the neutron skin $t$ dependence by expanding the integral for the case of $R_n \neq R_p$ around $R_n = R_p = R$. This can be done by expanding the density distribution of Eq. (1) around $R$ which we now discuss. It should also be noted that in Eq. (3) the surface correction appears with the factor $tA^{1/3}$ which originates from the dimensionless factor $t/R$ with $R = r_0A^{1/3}$.

When $R_q \approx R$, the neutron skin size $t/R$ is proportional to $A^2Z = \frac{4}{(1-i)}$ as can be seen in Appendix I. Thus the skin dependence of the energy expansion coefficients have this extra factor of $I$ dependence. On the other hand,
when neutron and proton central densities are the same, the neutron skin size $t/R$ is proportional to $(N - Z)/A$ (see Appendix [3]). Then the skin dependence of the energy expansion coefficients introduce an extra $I = (N - Z)/A$ behavior and the energy expansion of Eq.(2) with Eq.(3) becomes a third power expansion in the isospin factor $I$. That is $t/A^{1/3} \sim I$ since $t/R \sim I$. For such a case, by expanding the empirical nuclear energy for various nuclei together with $I$ and $I^3$ terms included, we may be able to extract the information about the neutron skin size from the odd term in $I$. The odd term in $I$ coming from the skin size $t$ does not break isospin symmetry as we will discuss after Eq.[3]. An odd power of $|I|$ was considered in Ref.[23] also.

The central density $\rho_q$ of the density distribution Eq.(1) for a given value of $R_q$ should be determined to give a fixed number of nucleons $N_q$. This normalization condition gives the expansion of $\rho_q(R_q)$ as given by Eq.(15) in Appendix [5] up to first order in $t_q = (R_q - R)$. Thus the first order correction coefficient to $\rho_q$ due to fixed $N_q$ is the zeroth order term times $-\frac{m}{\pi^3} \frac{3 + \pi^2(n_q^*)^2}{1 + \pi^2(n_q^*)^2}$. This result is independent of which type of particle, neutron or proton. Furthermore if we keep the particle number $N_q$ and the total central density $\rho_c = \rho_{nc} + \rho_{pc}$ to be a constant while varying $R_q$, that is,

$$\rho_c(R) = \rho_{nc}(R) + \rho_{pc}(R_p) = \rho_{nc}(R) + \rho_{pc}(R)$$

then we have

$$t_n = R_n - R = \frac{Z}{A} t, \quad t_p = R_p - R = -\frac{N}{A} t$$

(5)

to lowest order in $t = R_n - R_p = t_n - t_p$ when we expand about $R_q = R$ (see Appendix [5]). For $N > Z$, $t > 0$ with $t_n > 0$ and $t_p < 0$. For $Z > N$, $t < 0$ with $t_n < 0$ and $t_p > 0$. Thus the roles of $t_n$ and $t_p$ are exchanged as $t$ changes sign.

Since the term $E_i(T)_skt/A^{1/3}$ without the absolute value sign in Eq.(3) comes from the expansion term $\sum_q \left( \frac{dE(A,Z,T)}{dR_q} \right)_{R_q=R} t_q$ of the energy $E(A,Z,T)$, exchanging the role of $t_n$ and $t_p$ does not break isospin symmetry of the nuclear energy. Due to the sign in Eq.(3), $E_i(T)_sk$ changes sign as $t$ changes sign keeping the sign of the whole term unchanged. However if we use $|t|$ with absolute value sign explicitly written instead of $t$ itself as used in Eq.(3) then $E_i(T)_sk$ does not change sign as $t$ changes sign. Since we considered here beta stable nuclei only with $N > Z$ we can drop the absolute value sign of $t$ in Eq.(3).

The Fermi density $\rho_q(r)$ is then expanded about $R_q = R$ up to first order in $(R_q - R)$ as given by Eqs.(B13)–(B20). The results of Appendix [3] show that the quantity $\rho^m_n(r) + \rho^m_p(r)$ has a first order correction from skin size $t$ and the first order correction vanishes for $m = 1$. That is the total density $\rho(r) = \rho_n(r) + \rho_p(r)$ is independent of the neutron skin size $t = R_n - R_p$ up to first order. Thus only the explicit $\rho_q$ dependent terms in Skyrme interaction, not the total density $\rho$ dependent terms, depend on the skin size $t$ up to the first order.

Since $F[\rho_q] = \int d^3r f(\rho_q(\vec{r}))$ for a Fermi density $\rho_q(\vec{r}) = \frac{1}{1 + e^{(\vec{r} - \vec{r}_F)/\rho_q}}$, where $f(\rho_q)$ is only a function of single density $\rho_q$, then $F[\rho_q]$ can be integrate exactly. The result is a function $F(R_q)$ of $R_q$ [2], which we can expand easily in terms of $t_q = R_q - R$ around $F(R)$. That is

$$F(R_q) = F(R) + \frac{dF(R)}{dR} t_q + \frac{d^2F(R)}{dR^2} \frac{t_q^2}{2} + \cdots$$

(6)

This procedure is much simpler than using a method of expanding the density in $t$ first and then integrating the results. For fixed $N_q$, the central density $\rho_q(R_q)$ is also a function of $R_q$ as in Eq.(34). The expansion of integral of various power of $\rho_q$ are summarized in Appendix [3].

For the integral of the form of $\int d^3r \rho^m_p$, we need to expand $\rho^m_p(R_q)$ around $R$ first then integrate each term which is now a function of $R$ only. The results for various cases are also given in Appendix [3].

III. NEUTRON SKIN SIZE DEPENDENCE

Here we examine the neutron skin size dependence of nuclear energy using various Skyrme interactions for beta stable nuclei. We used three sets of Skyrme parameters with different values of the effective mass in symmetric nuclear matter which are SLy4 with $m^*/m = 0.69$, SkM* with $m^*/m = 0.79$, and SkM$^*(m^* = m)$ with $m^*/m = 1$. The results for these three cases are given in Table I and summarized in the following equations. The three cases cover a wide range of interaction types in terms of effective mass within the many various Skyrme interactions.
### Table I: Neutron Skin Dependence of Energy coefficient minimizing free energy.

| $T$ (MeV) | SLy4 | SkM* | SkM($m^* = m$) |
|-----------|------|------|----------------|
|           | 0    | 1    | 2    | 3    | 0    | 1    | 2    | 3   |
| $E_C(T)$ | –15.308 | –15.296 | –15.263 | –15.217 | –15.127 | –15.108 | –15.051 | –14.962 | –15.310 | –15.270 | –15.152 | –14.954 |
| $T$-indp | –15.308 | –15.308 | –15.308 | –15.312 | –15.127 | –15.127 | –15.127 | –15.130 | –15.310 | –15.310 | –15.310 | –15.310 |
| $T^2$    | 0.01151 | 0.01145 | 0.01121 | 0.01061 | 0.01925 | 0.01921 | 0.01908 | 0.01868 | 0.03948 | 0.03948 | 0.03950 | 0.03948 |
| Kine     | 28.653 | 28.594 | 28.423 | 28.124 | 25.483 | 25.444 | 25.319 | 25.105 | 19.957 | 19.978 | 20.037 | 20.134 |
| $E_V(T)_sk$ | 5.5202 | 5.5100 | 5.4792 | 5.4274 | 5.5837 | 5.5730 | 5.5411 | 5.4874 | 6.9809 | 6.9701 | 6.9379 | 6.88390 |
| $T$-indp | 5.5202 | 5.5172 | 5.5081 | 5.4927 | 5.5837 | 5.5706 | 5.5177 | 5.5563 | 6.9809 | 6.9701 | 6.9697 | 6.9553 |
| $T^2$    | –0.00722 | –0.00722 | –0.00723 | –0.00725 | –0.00761 | –0.00762 | –0.00763 | –0.00765 | –0.00793 | –0.00793 | –0.00793 | –0.00794 |
| Kine     | –3.8780 | –3.8654 | –3.8269 | –3.7576 | –3.893 - | –0.4995 | –0.5501 | –0.5672 | –0.4253 | –0.4253 | –0.4253 | –0.4253 |

For Coulomb energy, from Eqs. (C8) and (C5) with $a = 0.53$ fm and $R = 1.25A^{1/3}$ fm, $E_C$ part is $0.6912000$, $E_{ex}$ part is $-0.5728064$ and $E_{diff} = -1.430810$ and the skin dependence of Coulomb energy $E_C$ part is $-0.5529600$, $E_{ex}$ part is $0.4222251$ and $E_{diff} = 3.433944$, that is

$$E_C(T,t) = \frac{Z^2}{A^{1/3}} + E_{diff}(T,t) = \frac{Z^2}{A^{1/3}} + E_{ex}(T,t) = \frac{Z^4/3}{A^{1/3}}$$

For Coulomb energy, from Eqs. (C8) and (C5) with $a = 0.53$ fm and $R = 1.25A^{1/3}$ fm, $E_C$ part is $0.6912000$, $E_{ex}$ part is $-0.5728064$ and $E_{diff} = -1.430810$ and the skin dependence of Coulomb energy $E_C$ part is $-0.5529600$, $E_{ex}$ part is $0.4222251$ and $E_{diff} = 3.433944$, that is

$$E_C(T,t) = \frac{Z^2}{A^{1/3}} + E_{diff}(T,t) = \frac{Z^2}{A^{1/3}} + E_{ex}(T,t) = \frac{Z^4/3}{A^{1/3}}$$

in MeV and fm units.

In Table I, the items labeled “$E_i(T)$” and “$E_i(T)_sk$” are corresponding terms of Eq. (3) for the energy expansion coefficients given in Eq. (2). The items labeled by “$T$-indp”, “$T^2$”, and “Kine” under $E_i(T)$ are the values of temperature $T$ independent part, $T^2$ dependent term in kinetic energy (Eq. (A3)), and the total kinetic energy contribution to the energy expansion coefficients $E_i(T)$ respectively. Similarly the items labeled by “$T$-indp”, “$T^2$”, and “Kine” under $E_i(T)_sk$ are the values of temperature $T$ independent part, $T^2$ dependent term in kinetic energy (Eq. (X3)), and the total kinetic energy contribution to the neutron skin dependent part of the energy expansion coefficients $E_i(T)_sk$ respectively.

The results of Table I show the following features for the skin size dependence of various coefficients $E_i(T)_sk$ of Eq. (X4) in the mass formulae which behave as $E_i(T)_sk/t^{1/3}$. One overall feature for all components $E_i(T)_sk$ is the weak dependence on temperature.

The volume binding energy coefficient $E_i(T)_sk = E_V(T)_sk$ of about $5 \sim 7$ MeV/fm has the smallest value of all the $E_i(T)_sk$ terms. Comparing to this value, the skin independent volume energy coefficient $E_V(T)$ is about $-15$ MeV. (Notice here that the $E_i(T)_sk/t^{1/3}$ with an unknown small value for the $t/A^{1/3}$.
factor and not the $E_i(T)_{sk}$ itself should be compared to $E_i(T)$ for the value of energy coefficients.) The surface energy coefficient $E_S(T)_{sk}$ itself has a somewhat larger values of about $-17 \sim -22$ MeV/fm and is negative compared to about 20 MeV for skin independent coefficient $E_S(T)$. The neutron skin size dependence of the symmetry energy terms have the following features. The volume symmetry energy coefficient $S_V(T)_{sk} \sim 60$ MeV/fm while the surface symmetry energy coefficient has the largest magnitude of about $-600 \sim -700$ MeV/fm. Comparing to these the neutron skin size independent coefficients are $S_I(T) \approx 20 \sim 30$ MeV and $S_S(T) \approx 30 \sim 50$ MeV. Some dependence on the effective mass is present for all coefficients as can be seen in comparing SLy4 ($m^*/m = 0.7$), SkM* ($m^*/m = 0.8$), and SkM($m^* = m$). However the dependences of the individual coefficients on the temperatures and on the Skyrme parameters we used are not so sensitive. The magnitude of the neutron skin size dependent coefficients are largely different between different coefficients ranging from about 5 MeV/fm for volume energy coefficient to about 700 MeV/fm for surface symmetry energy coefficient while the magnitudes of the skin size independent parts were of the same order of magnitude ranging from about 15 MeV for volume energy to about 50 MeV for surface symmetry energy coefficient. The neutron skin size dependence of surface symmetry energy is much larger than the skin dependence of volume symmetry energy with $A$ dependence included even for a large $A$ of over 200. The ratio of the surface symmetry energy to the volume symmetry energy $S_S(T)_{sk}/S_V(T)_{sk} \sim 10$ for neutron skin size $t$ dependent part while $S_S(T)/S_V(T) \sim 1.5$ for neutron skin independent part of $R_n = R_p$. (Notice also that $E_S(T)_{sk}/E_V(T)_{sk} \sim 3$ compared to $E_S(T)/E_V(T) \sim 1.2$.) This show that the large part of symmetry energy comes from the different size of neutron and proton distributions. If we assume $t/A^{1/3} = 0.1$, then the ratio of total symmetry energy $\left[ S_S(T) + S_S(T)_{sk}t/A^{1/3} \right]/\left[ S_V(T) + S_V(T)_{sk}t/A^{1/3} \right] \sim 3$. One dimensional semi infinite nuclear matter calculations show that this ratio is $S_S/S_V \approx 1 \sim 4$ for total symmetry energy $[8]$. We can also see that the neutron skin size dependent coefficients for volume energy and surface energy $E_V(T)_{sk}$ and $E_S(T)_{sk}$ have opposite sign compare to the corresponding skin size independent coefficients $E_V(T)$ and $E_S(T)$, while volume symmetry energy and surface symmetry energy $S_V(T)$ and $S_S(T)$ have the same sign for neutron skin size dependent coefficients and independent coefficients.

The kinetic energy contribution to the neutron skin size dependent and independent coefficients have a much more sensitive dependence on the Skyrme parameter set we used. The magnitude of kinetic energy contribution follows somewhat the magnitude of the effective mass of the Skyrme parameter set used. Since the neutron skin size dependent coefficients $E_i(T)_{sk}$ themselves are somewhat insensitive to the Skyrme parameter set used, the potential energy contribution to the coefficients, which are function of density, also are sensitive to the parameter set used. The kinetic energy contribution to the neutron skin size dependence $E_i(T)_{sk}$ are now opposite in sign to the kinetic energy part to the skin size independent coefficients $E_i(T)$ of energy expansion for all the terms. In turn, they are all opposite sign to the skin size independent coefficients $E_i(T)$ themselves. The kinetic energy contribution to the neutron skin size dependent volume symmetry energy coefficient $S_V(T)_{sk}$ has a small magnitude similar to the magnitude of the kinetic energy contribution to the skin size dependent volume energy coefficient $E_V(T)_{sk}$. The kinetic energy contribution to the surface symmetry energy coefficients, both the neutron skin size dependent and independent ones $S_S(T)_{sk}$ and $S_S(T)$, have largest magnitude among the energy expansion coefficients. It is much larger even with $A$ dependence included than skin size dependence of volume symmetry energy $S_V(T)_{sk}$ and other energy expansion coefficients.

When the Helmholtz free energy is minimized, from the values of “$T$-indp” and “$T^{2n}$” for $T = 0$ in Table II the temperature and neutron skin size dependence of the energy at low $T$ becomes

$$E(A, Z, T, t) = \left[ -(15.308 - 0.012T^2)A + (20.008 + 0.548T^2)A^{2/3} ight.$$  
$$+ (31.113 + 0.545T^2)I^2A - (41.035 + 2.577T^2)I^2A^{2/3} \right.$$  
$$+ \left[ (5.520 - 0.007T^2)A - (17.077 - 0.020T^2)A^{2/3} \right.$$  
$$+ (63.203 + 0.031T^2)I^2A - (711.559 + 0.916T^2)I^2A^{2/3} \right] \frac{t}{A^{1/3}}$$  
$$+ E_C(T, t) \frac{Z^2}{A^{1/3}} + E_{dis}(T, t) \frac{Z^2}{A} + E_{exc}(T, t) \frac{Z^{4/3}}{A^{1/3}}$$

(8)

for SLy4 parameter set,

$$E(A, Z, T, t) = \left[ -(15.127 - 0.019T^2)A + (18.756 + 0.537T^2)A^{2/3} ight.$$  
$$+ (29.655 + 0.548T^2)I^2A - (43.596 + 2.560T^2)I^2A^{2/3} \right.$$  
$$+ \left[ (5.584 - 0.008T^2)A - (17.261 - 0.021T^2)A^{2/3} \right.$$  
$$+ (62.787 + 0.030T^2)I^2A - (706.246 + 0.940T^2)I^2A^{2/3} \right] \frac{t}{A^{1/3}}$$

\[ E(A, Z, T, t) = \left( -15.310 - 0.039T^2 \right) A + (18.303 + 0.500T^2) A^{2/3} + E_{d_{ij}(T, t)} T^2 A - (3.572 + 0.022T^2) A^{2/3} + \left( 6.981 - 0.008T^2 \right) A - (21.691 - 0.022T^2) A^{2/3} + (63.366 + 0.031T^2) T^2 A - (611.848 + 0.955T^2) T^2 A^{2/3} \right) \frac{t}{A^{1/3}} \]
dependent volume energy coefficient $E_V(T)_{sk}$ has smallest effect of $T^2$ dependence and the surface symmetry energy $S_S(T)_{sk}$ has a largest effect similar to the $T^2$ dependence of the corresponding skin independent coefficients of energy expansion.

By fitting the values of $E_i(T)_{sk}$ for temperatures which are $T =$ 0, 1, 2, and 3 MeV in Table I the $T$-dependences of the neutron skin dependent coefficients are

$$E_V(T)_{sk} = 5.52016 - 0.00722T^2$$
$$E_S(T)_{sk} = -17.07720 + 0.01959T^2$$
$$S_V(T)_{sk} = 63.20304 + 0.03095T^2$$
$$S_S(T)_{sk} = -711.5588 - 0.91606T^2$$

for SLy4 parameter set. The first expressions given in the square parenthesis are from the values for $T = 0$ in Table I same as in Eqs. (8)-(13) for comparison.

$$E_V(T)_{sk} = 5.58368 - 0.00761T^2$$
$$E_S(T)_{sk} = -17.26147 + 0.02082T^2$$
$$S_V(T)_{sk} = 62.78658 + 0.02954T^2$$
$$S_S(T)_{sk} = -706.2461 - 0.94030T^2$$

for SkM$^*$ parameter set.

$$E_V(T)_{sk} = 6.98087 - 0.00793T^2$$
$$E_S(T)_{sk} = -21.69093 + 0.02201T^2$$
$$S_V(T)_{sk} = 63.36637 + 0.03144T^2$$
$$S_S(T)_{sk} = -611.8476 - 0.95456T^2$$

for SkM($m^* = m$) parameter set.

By fitting the values for $T =$ 0, 1, 2, and 3 MeV in Table I the temperature $T$ and neutron skin size $t$ dependence of the expansion coefficients of the kinetic energy are

$$E_V(T,t)_{K} = (28.65250 + 0.01151T^2) + (-3.87804 - 0.00722T^2)\frac{t}{A^{1/3}}$$
$$E_S(T,t)_{K} = (32.10626 + 0.54833T^2) + (12.32541 + 0.01959T^2)\frac{t}{A^{1/3}}$$
$$S_V(T,t)_{K} = (24.67405 + 0.54454T^2) + (1.33205 + 0.03095T^2)\frac{t}{A^{1/3}}$$

$$S_S(T,t)_{K} = (28.65250 + 0.00801T - 0.01048T^2 - 0.00008T^3)$$
$$+ (-3.87804 + 0.00117T + 0.01064T^2 + 0.00078T^3)\frac{t}{A^{1/3}}$$

$$- (32.10626 + 0.01492T + 0.45071T^2 + 0.00230T^3)$$
$$+ (12.32541 - 0.00188T - 0.03149T^2 - 0.00207T^3)\frac{t}{A^{1/3}}$$

$$- (24.67405 + 0.54454T^2) + (1.33205 + 0.03095T^2)\frac{t}{A^{1/3}}$$
\[
E_V(T, t)_K = (25.48323 + 0.01925T^2) + (-0.48934 - 0.00761T^2) \frac{t}{A^{1/3}}
\]
\[
E_S(T, t)_K = (-26.27463 + 0.53708T^2) + (1.40926 + 0.02082T^2) \frac{t}{A^{1/3}}
\]
\[
S_V(T, t)_K = (-33.63624 + 0.54816T^2) + (-1.33997 + 0.02954T^2) \frac{t}{A^{1/3}}
\]
\[
S_S(T, t)_K = (166.8184 - 2.55977T^2) + (-40.35256 - 0.94030T^2) \frac{t}{A^{1/3}}
\]
\[
E_V(T, t)_K = (19.95733 + 0.03948T^2) + (-0.38673 - 0.00793T^2) \frac{t}{A^{1/3}}
\]
\[
E_S(T, t)_K = (-15.84463 + 0.49952T^2) + (1.12277 + 0.02201T^2) \frac{t}{A^{1/3}}
\]
\[
S_V(T, t)_K = (-6.88128 + 0.42271T^2) + (-0.87085 + 0.03144T^2) \frac{t}{A^{1/3}}
\]
\[
S_S(T, t)_K = (56.50820 - 2.06058T^2) + (-31.54054 - 0.95456T^2) \frac{t}{A^{1/3}}
\]

for SkM parameter set.

\[
E_V(T, t)_K = (-24.67405 - 0.13177T + 0.26686T^2 - 0.02916T^3)
\]
\[
E_S(T, t)_K = (171.0202 - 2.57693T^2) + (-276.2811 - 0.91606T^2) \frac{t}{A^{1/3}}
\]
\[
S_V(T, t)_K = (171.0202 + 1.18275T - 2.36020T^2 + 0.23595T^3)
\]
\[
S_S(T, t)_K = (-276.2811 - 0.18403T + 1.15055T^2 + 0.04358T^3) \frac{t}{A^{1/3}}
\]

for SkM*(m* = m) parameter set.
Comparing the first line (or Eqs. (8)-(13)) and second line of above Eqs. (14) – (37), we can see the temperature $T$ dependence of the neutron skin size dependent coefficients $E_i(T)_{sk}$ comes not only from the $T^2$ term in the kinetic energy, Eq. (A3), but also from the potential energy through the different saturation density for different temperature. Especially, the neutron skin size dependence of kinetic energy expansion coefficients for SLy4 parameter set have opposite sign in their $T$ dependence compared with the $T^2$ term in kinetic energy ($T^2$ dependence in Eqs. (8)-(13)). Table I and Eqs. (14), (25) show that the $T$ dependence of the neutron skin size dependent coefficient is much faster than the $T^2$ term in kinetic energy of Eq. (A3) alone for volume energy coefficient $E_V(T)_{sk}$ and surface energy coefficient $E_S(T)_{sk}$. By contrast this is much slower than the $T^2$ term for volume symmetry energy coefficient $S_V(T)_{sk}$ and surface symmetry energy coefficient $S_S(T)_{sk}$ except for volume symmetry energy $S_V(T)_{sk}$ of the SkM$^*$ parameter set.

Eqs. (26) – (37) show that even the kinetic energy also has an extra $T$ dependence, beside the $T^2$ term in kinetic energy of Eq. (A3), through the $T$ dependence of saturation density. For SLy4 Skyrme interaction, the $T$ dependence of the kinetic energy part of the neutron skin size dependent energy expansion coefficients $E_i(T)_{sk}$ has opposite sign with the $T^2$ term of kinetic energy Eq. (A3). For this interaction the $T$ dependence of the kinetic energy part of the volume symmetry energy coefficient $S_V(T)_{sk}$ is rather linear as compared to a $T^2$ behavior. For SkM$^*$ and SkM$(m^* = m)$ interactions, the kinetic energy part of the neutron skin dependent energy expansion coefficients $E_i(T)_{sk}$ has the same sign in its $T$ dependences with the $T^2$ term in the kinetic energy of Eq. (A3) but has a faster dependence of $T$ compared to the $T^2$ term of Eq. (A3). The neutron skin independent kinetic energy expansion coefficients have a slower $T$ dependence than a $T^2$ term for a kinetic energy, Eq. (A3), except the volume energy coefficient $E_i(T)_{sk}$ for SLy4 and SkM$^*$ parameter sets which have an opposite sign compared to $T^2$ term of kinetic energy Eq. (A3). The neutron skin size independent volume symmetry energy coefficient $S_V(T, t = 0)_K$ for SLy4 parameter set has a linear $T$ dependence comparable order to the $T^2$ dependence.

The kinetic energy expansion at zero $T$, from Eqs. (11–13) or from Eqs. (26–37), are

$$E_K(A, Z, T = 0, t) = \left[28.653A - 32.106A^{2/3} - 24.674t^2A + 171.020t^2A^{2/3}\right]$$

$$+ \left[-3.878A + 12.325A^{2/3} + 1.332t^2A - 276.281t^2A^{2/3}\right] \frac{t}{A^{1/3}}$$

(38)

for SLy4 parameter set,

$$E_K(A, Z, T = 0, t) = \left[25.483A - 26.275A^{2/3} - 33.636t^2A + 166.818t^2A^{2/3}\right]$$

$$+ \left[-0.489A + 1.409A^{2/3} - 1.340t^2A - 40.353t^2A^{2/3}\right] \frac{t}{A^{1/3}}$$

(39)

for SkM$^*$ parameter set, and

$$E_K(A, Z, T = 0, t) = \left[19.957A - 15.845A^{2/3} - 6.881t^2A + 56.508t^2A^{2/3}\right]$$

$$+ \left[-0.387A + 1.123A^{2/3} - 0.871t^2A - 31.541t^2A^{2/3}\right] \frac{t}{A^{1/3}}$$

(40)

for SkM$(m^* = m)$ parameter set. For an infinite nuclear matter, the surface terms disappear and only the volume terms survive. These results show that the surface symmetry energy coefficient has the largest effect from kinetic energy as compared to the other coefficients. Here we can see the surface kinetic energy and the volume symmetry kinetic energy coefficients are negative. However the total kinetic energy and total symmetry kinetic energy including $A$ factors are positive. The neutron skin dependent kinetic energies with $A$ factor included are negative. Compore to result from Fermi gas model,

$$E_K = 12t^2A + 9t^2A^{2/3}$$

(41)

which is good for high $T$ or low density limit without any interaction. Here both the volume and surface symmetry energies are positive. With Skyrme interaction, the isospin dependent part of the effective mass in a finite nuclei may become negative depending on the force parameter and densities of proton and neutron. Thus the signs in Eqs. (38) – (40) result.

IV. CONCLUSION AND SUMMARY

Understanding properties of the symmetry energy is important in many area of nuclear physics as mentioned in the introduction. In this paper we studied properties of the symmetry energy, both volume and surface parts, along
with other terms which appear in the Weizsacker mass formulae. Our investigation was based on a finite temperature density functional approach. In a finite temperature approach, the dependence of various terms on temperature can be obtained. Energetic probes lead to excited nuclei which may be characterized by a hot liquid drop extension of the Weizsacker mass formulae. We used several different interactions of the Skyrme type to examine the dependence of various quantities on the interaction and associated effective masses that appear. Our analysis in the present study emphasized the role of the neutron skin on various terms that appear in the mass formulae. The radii of protons and neutrons were therefore allowed to be different in a Saxon-Wood form for the density distributions of these particles. We then proceeded to calculate various terms using an expansion about the equal radii point. The corrections that arise from a neutron skin are then proportional to the skin thickness \( t/R \) over the radius \( R \) or \( t/R \sim t/A^{1/3} \). The skin thickness \( t/R \) itself can be proportional to the neutron excess fraction \( I = (N - Z)/A \) when the proton and neutron central densities are the same. On the other hand, if \( R_p = R_n = R \) then \( t/R \) is proportional to \( A^2/ZN = 4/(1 - I^2) \). Thus the \( I \) dependences of various terms in the mass formulae are modified.

Table I contains the results for three Skyrme interactions, SLy4, SkM* and SkM(\( m^* = m \)). Results for the volume energy \( E_V(T) \) and \( E_V(T)_{sk} \), surface energy \( E_S(T) \) and \( E_S(T)_{sk} \), volume symmetry energy \( S_V(T) \) and \( S_V(T)_{sk} \), and surface symmetry energy \( S_S(T) \) and \( S_S(T)_{sk} \) are given. The terms with an additional subscript “sk” are the skin coefficients of Eq.(3) and the terms without subscript “sk” are the skin independent part \(( t = 0) \) in Eq.(3). The kinetic energy contributions, labeled “Kine”, to each term are also given. The difference of the total and kinetic term behavior of the kinetic energy expansion coefficients has a factor of 4 for neutron skin independent part. This shows that the large part of symmetry energy comes from the different size of proton and neutron distributions. The surface symmetry kinetic energy coefficients, both the neutron skin size dependent and independent ones, have the largest magnitude of the kinetic energy expansion coefficients. The neutron skin size dependent volume energy coefficient has the smallest magnitude while the surface symmetry energy coefficient has the largest magnitude. The neutron skin size dependence of surface symmetry energy is much larger than the skin size dependence of the volume symmetry energy with \( A \) dependence included. The ratio of the surface symmetry energy coefficient to the volume symmetry energy coefficient is \( S_S(T)_{sk}/S_V(T)_{sk} \sim 10 \) for neutron skin dependent part compared to \( S_S(T)/S_V(T) \sim 1.5 \) for neutron skin independent part. This shows that the large part of symmetry energy comes from the different size of proton and neutron distributions. The surface symmetry kinetic energy coefficients, both the neutron skin size dependent and independent ones, have the largest magnitude of the kinetic energy expansion coefficients. The neutron skin size dependent volume energy coefficient has the smallest temperature dependence and the neutron skin size dependent surface symmetry energy coefficient has the largest temperature dependence similar to the neutron skin independent coefficients. The temperature dependence of the neutron skin size dependent coefficients is smaller than the temperature dependence of the skin size independent coefficients. The temperature dependences of the neutron skin size dependent energy coefficients have a large effect from the different saturation density for different temperature and thus do not follow \( T^2 \) behavior of the explicit \( T^2 \) dependence of kinetic energy.

Considering the neutron skin size dependence \( t/R \) factor with \( R = r_0A^{1/3} \), the neutron skin dependent energy expansion coefficients has a factor of \( t/A^{1/3} \) in the energy expansion. If we relate the dimensionless factor \( t/R \) to the isospin factor \( I = (N - Z)/A \) then the energy expansion including the neutron skin size dependence introduce an extra \( I \) factor. With this extra \( I \) dependence we may be able to extract some information on the neutron skin size by expanding the empirical energy of various nuclei with including odd power of \( I \) up to third order if the neutron and proton central densities are the same.

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Appendix A: Skyrme interaction

The Hamiltonian for a Skyrme interaction is

\[
H(\vec{r}) = H_B(\vec{r}) + H_S(\vec{r}) + H_C(\vec{r})
\]

\[
H_B = \frac{\hbar^2}{2m_p} \tau_p + \frac{\hbar^2}{2m_n} \tau_n + \frac{1}{4} \left( t_1 \left( 1 + \frac{x_1}{2} \right) + t_2 \left( 1 + \frac{x_2}{2} \right) \right) \rho \tau - \frac{1}{4} \left[ t_1 \left( \frac{1}{2} + x_1 \right) - t_2 \left( \frac{1}{2} + x_2 \right) \right] (\rho_p \tau_p + \rho_n \tau_n)
\]
and the Coulomb term is important for the charged proton component. The density gradient term depends on the slope at surface region the ratio of surface to volume kinetic energy may be sensitive to this term. For a Fermi density, the integral of the last term is zero while the ratio of the integral 

\[ T = \frac{1}{36} \text{factor. Furthermore, this term is} \]

\[ \text{important in finite nuclei and the Coulomb term is important for the charged proton component. The} \]

\[ 0 \] 

\[ \tau = \sum_{j\neq q} |\psi_j|^2 = \int \frac{\pi^2}{2} f_q(\vec{r}, \vec{p}) d^3p \text{ and } \rho_q = \sum_{j\neq q} |\psi_j|^2 = \int f_q(\vec{r}, \vec{p}) d^3p. \] 

The gradient terms in Eq. (A1) are important in finite nuclei and the Coulomb term is important for the charged proton component. The \( t_0, t_1, t_2, t_3 \) and \( x_0, x_1, x_2, x_3 \) are parameters. Different choices of these parameters give rise to different Skyrme interactions.

The effective mass \( m_q^* \) is

\[ \frac{m}{m_q^*} = 1 + \frac{2m}{h^2} \left[ \frac{1}{4} \left[ t_1 \left( 1 + \frac{x_1}{2} \right) + t_2 \left( 1 + \frac{x_2}{2} \right) \right] - t_2 \left( \frac{1}{2} + x_2 \right) \right] \rho_q \]

\[ \tau_q(\vec{r}) = \frac{3}{5} \left( \frac{6\pi^2}{\gamma} \right)^{2/3} \left[ \frac{5\pi^2 m_q^2}{3h^4} \left( \frac{\gamma}{6\pi^2} \right)^{4/3} \rho_q^{1/3} T^2 + \cdots \right] \] (A3)

At zero \( T \), extended Thomas Fermi approximation gives

\[ \tau_q(\vec{r}) = \frac{3}{5} \left( \frac{6\pi^2}{\gamma} \right)^{2/3} \rho_q^{5/3} + \frac{1}{36} \left( \nabla \rho_q \right)^2 + \frac{1}{3} \nabla^2 \rho_q \] (A4)

Since the density gradient term depend on the slope at surface region the ratio of surface to volume kinetic energy might be sensitive to this term. For a Fermi density, the integral of the last term is zero while the ratio of the integral of the second term without the numerical factor (1/36) to the integral of the first term including all factors is

\[ \left( \frac{\rho_q}{\pi \alpha} \right)^2 \left[ 3 + 6 \left( \frac{\alpha}{\pi \alpha} \right) + \pi^2 \left( \frac{\alpha}{\pi \alpha} \right)^2 \right] \]

\[ \frac{1}{\pi} \left( \frac{6\pi^2}{\gamma} \right)^{2/3} \rho_q^{5/3} \left[ 1 - 2.28 \left( \frac{\alpha}{\pi \alpha} \right)^2 + 9.10 \left( \frac{\alpha}{\pi \alpha} \right)^2 - 7.81 \left( \frac{\alpha}{\pi \alpha} \right)^3 \right] \]

which is the order of one. Thus the second term of Eq. (A1) is only a few percent of the first term because of the 1/36 factor. Furthermore, this term is \( T \) independent and independent of neutron skin size since it depends only on the slope. Thus the density gradient term would not affect much the relative effect of the \( T \) dependent to \( T \) independent part and also the relative effect of neutron skin size \( t \) to \( t \) independent part. Since we are more interested in the temperature dependence and the neutron skin size dependence of energy expansion coefficients we neglect the gradient dependent terms of Eq. (A3) here in evaluation of total nuclear energy. It is shown that the gradient dependent correction to the coefficient of \( T^2 \) term modifies numerical results only little [24, 25].

Appendix B: Expansion of density

The central density \( \rho_q \) of the density distribution for a given value of \( R_q \) should be determined to give a fixed number of nucleons \( N_q \); 

\[ N_q(R_q) = \int d^3r \rho_q(\vec{r}) = 4\pi \int_0^\infty r^2 dr \frac{\rho_q(R_q)}{1 + e^{(r-R_q)/a}} = \rho_q(R_q) \frac{4\pi R_q^3}{3} \left[ 1 + \pi^2 \left( \frac{a}{R_q} \right)^2 \right] \] (B1)
Thus we get to lowest order in densities $\rho$:

$$dN_q = \frac{d\rho_q}{dR} = \frac{4\pi}{3} R^3 \left[ 1 + \pi^2 \left( \frac{a}{R} \right)^2 \right] dR_q + \rho_q \frac{4\pi}{3} R^2 \left[ 3 + \pi^2 \left( \frac{a}{R} \right)^2 \right] dR_q = 0 \quad (B2)$$

$$d\rho_{qc} = -\rho_q \frac{\rho_q(R)}{R} \left[ 3 + \pi^2 \left( \frac{a}{R} \right)^2 \right] \left[ 1 + \pi^2 \left( \frac{a}{R} \right)^2 \right] R_q = \rho_{qc}(R) \left[ 1 - \left( 3 + \pi^2 \left( \frac{a}{R} \right)^2 \right) \left( \frac{R_q - R}{R} \right) + \cdots \right] \quad (B4)$$

This normalization condition gives up to first order in $\rho_{qc}$ due to fixed $N_q$ is the zeroth order term times $-\rho_q \frac{4\pi}{3} R^2 \left[ 3 + \pi^2 \left( \frac{a}{R} \right)^2 \right]$. This result is independent of which type of particle, neutron or proton.

If we keep the particle number $N_q$ and the total central density $\rho_c = \rho_{nc} + \rho_{pc}$ to be a constant while varying $R_q$, then

$$\rho_c(R) = \rho_{nc}(R_n) + \rho_{pc}(R_p)$$

$$= \rho_{nc}(R) + \rho_{pc}(R) - \left[ 3 + \pi^2 \left( \frac{a}{R} \right)^2 \right] \left[ \rho_{nc}(R) \left( \frac{R_n - R}{R} \right) + \rho_{pc}(R) \left( \frac{R_p - R}{R} \right) \right]$$

$$= \rho_{nc}(R) + \rho_{pc}(R) \quad (B5)$$

Thus we get

$$\rho_{nc} \left( \frac{R_n - R}{R} \right) + \rho_{pc} \left( \frac{R_p - R}{R} \right) = 0 \quad (B6)$$

and

$$R_p - R = -\frac{\rho_{nc}}{\rho_{pc}} (R_n - R)$$

$$t = R_n - R_p = (R_n - R) - (R_p - R) = (R_n - R) \left( 1 + \frac{\rho_{nc}}{\rho_{pc}} \right) = \frac{\rho_c}{\rho_{pc}} (R_n - R) \quad (B7)$$

Finally we have

$$t_n = R_n - R = \frac{\rho_{pc}}{\rho_c} t$$

$$t_p = R_p - R = -\frac{\rho_{nc}}{\rho_c} t \quad (B8)$$

The same result can be obtained by requiring $A = N + Z$ constant with keeping the central densities $\rho_{qc}$ unchanged in the $dR_q$ expansion. When $R_q = R$, $\rho_q(R)/\rho_c(R) = N_q/A$ and thus we get

$$t_n = R_n - R = \frac{Z}{A} t_n$$

$$t_p = R_p - R = -\frac{N}{A} t \quad (B9)$$

This is the same result given in Ref. [21]. Due to Eq. (B8) or (B9), $t_n > 0$ and $t_p < 0$ with $t > 0$ for $N > Z$ while $t_n < 0$ and $t_p > 0$ with $t < 0$ for $Z > N$. Thus the role of $t_n$ and $t_p$ is exchanged as the sign of $t$ changes.

Since the size $R_q$ depends on the central density $\rho_{qc}$ for a given value of particle number $N_q$ as in Eq. (B1), the neutron skin size is related to the particle number $N_q$ and the central density $\rho_{qc}$. Using Eqs. (B1) and (B8) we can obtain following conditions to lowest order in $t$.

$$N - Z = \frac{4\pi}{3} \left[ (R_n^3 + \pi^2 a^2 R_n) \rho_{nc} - (R_p^3 + \pi^2 a^2 R_p) \rho_{pc} \right]$$

$$= \frac{4\pi}{3} \left[ R^3 \left( 1 + \frac{\rho_{pc}}{\rho_c} \frac{t}{R} \right)^3 \rho_{nc} + \pi^2 b^2 R \left( 1 + \frac{\rho_{pc}}{\rho_c} \frac{t}{R} \right) \rho_{nc} \right]$$


Thus for an uniform distribution (diffuseness parameter $\delta = 0$).

Even if Eqs. (B11) and (B15) look different they are the same equation. The proton ratio

$$
\frac{t}{R} \approx \frac{1}{3} A \left( \frac{\rho_p}{\rho_c} + \frac{\rho_{nc}}{\rho_c} \right) \left[ \left( \frac{N}{A} - Z \right) - A \left( \frac{\rho_{nc} - \rho_p}{\rho_c} \right) \right] \left[ 1 + \frac{1}{3} \left( \frac{\rho_p}{\rho_c} \right)^2 \right]
$$

(B10)

where $y_c = \rho_c/\rho_p$ is the proton fraction of the central density. For the case of $\rho_{nc} \approx \rho_p$, $y_c = 1/2 + \epsilon$ and $(1 - y_c) = 1/2 - \epsilon$. Then the factor $[2(1 - y_c) y_c]^{-1}$ becomes

$$
\frac{1}{2(1 - y_c) y_c} = \frac{2}{(1 - 2\epsilon)(1 + 2\epsilon)} \approx 2(1 - 4\epsilon^2)
$$

(B12)

Thus for an uniform distribution (diffuseness parameter $\delta = 0$) with $\rho_{nc} \approx \rho_p$, the neutron skin size $t/R$ of Eq. (B11) becomes, up to first order in $\epsilon$,

$$
\frac{t}{R} = \frac{2}{3} \left( \frac{N - Z}{A} - 2(1 - 2y_c) \right) \approx 2(1 - 4\epsilon^2)
$$

(B13)

which is the result given in Ref. [21]. It is shown that the empirical neutron skin size $t$ is approximately proportional to $I = (N - Z)/A$ [26]. On the other hand, as another form,

$$
\frac{N}{\rho_{nc}} - \frac{Z}{\rho_p} = \frac{4\pi}{3} \left[ (R_n^3 + \pi^2 a^2 R_n) - (R_p^3 + \pi^2 a^2 R_p) \right]
$$

$$
= \frac{4\pi}{3} \left[ R_n^3 \left( 1 + \frac{\rho_p}{\rho_c} \right) t^3 - R_p^3 \left( 1 + \frac{\rho_{nc}}{\rho_c} \right) t^3 + \pi^2 a^2 R \left( 1 + \frac{\rho_p}{\rho_c} \right) - \pi^2 a^2 R \left( 1 + \frac{\rho_{nc}}{\rho_c} \right) \right]
$$

$$
\approx \frac{4\pi}{3} \left[ R_n^3 \left( 1 + 3\frac{\rho_p}{\rho_c} \right) t - R_p^3 \left( 1 - 3\frac{\rho_{nc}}{\rho_c} \right) t + \pi^2 a^2 R \left( 1 + \frac{\rho_p}{\rho_c} \right) - \pi^2 a^2 R \left( 1 - \frac{\rho_{nc}}{\rho_c} \right) \right]
$$

$$
= \frac{4\pi}{3} \left[ R_n^3 \left( \frac{\rho_p}{\rho_c} + \frac{\rho_{nc}}{\rho_c} \right) + \pi^2 a^2 R \left( \frac{\rho_p}{\rho_c} + \frac{\rho_{nc}}{\rho_c} \right) \right] \frac{t}{R} = \frac{4\pi}{3} \left[ R_n^3 + \pi^2 a^2 R \right] \frac{t}{R}
$$

$$
= \frac{3}{A} \left( \frac{\rho_c}{\rho_p} \right) \left[ 1 + \frac{1}{3} \left( \frac{\rho_p}{\rho_c} \right)^2 \right] \frac{t}{R}
$$

(B14)

$$
\frac{t}{R} = \frac{1}{3} A \left( \frac{N}{\rho_{nc}} - \frac{Z}{\rho_p} \right) \left[ 1 + \frac{1}{3} \left( \frac{\rho_p}{\rho_c} \right)^2 \right] = \frac{1}{3} A \left( \frac{N}{1 - y_c} - \frac{Z}{y_c} \right) \left[ 1 + \frac{1}{3} \left( \frac{\rho_p}{\rho_c} \right)^2 \right]
$$

(B15)

Even if Eqs. (B11) and (B15) look different they are the same equation. The proton ratio $y_c$ has the range of $Z/A \leq y_c \leq 1/2$ for finite nuclei where $y_c = Z/A$ when $R_n = R_p$ and $y_c = 1/2$ for $\rho_{nc} = \rho_p$.

For one extreme case of the same central density $\rho_{nc} = \rho_p$, the proton ratio $y = 1/2$ and

$$
\frac{t}{R} \approx \frac{2}{3} \left( \frac{N - Z}{A} \right) \left[ 1 + \frac{1}{3} \left( \frac{\rho_p}{\rho_c} \right)^2 \right] \left[ 1 + \frac{1}{3} \left( \frac{\rho_p}{\rho_c} \right)^2 \right] I
$$

(B16)
Thus the neutron skin size is linearly proportional to the isospin factor $I$ when the proton and neutron central densities are same. For the other extreme case of the same size $R_n = R_p = R$, the proton ratio is $y_c = Z/A$ with $1 - y_c = N/A$, and thus the neutron skin size becomes $t/R = 0$. However when $R_n \approx R_p$ with $y_c = Z/A + \epsilon$, we have $1 - y_c = N/A - \epsilon$ and, from Eq. (B11),

$$
\frac{t}{R} = \frac{1}{3} \left[ \frac{1}{2N/A - \epsilon} \right] \left[ \left( \frac{N - Z}{A} \right) - \left( \frac{N - Z}{A} - \epsilon \right) \right] \left[ \frac{1 + \left( \frac{\epsilon}{R^2} \right)^2}{1 + \frac{1}{3} \left( \frac{\epsilon}{R} \right)^2} \right]
$$

$$
\approx \frac{1}{3} \left[ \frac{1}{2N/A} \right] \left[ \frac{1}{1 + \frac{1}{3} \left( \frac{\epsilon}{R} \right)^2} \right] \left[ \frac{1}{1 + \frac{1}{3} \left( \frac{\epsilon}{R} \right)^2} \right] \epsilon
$$

$\approx \frac{1}{3} \left( A^2 NZ \right) \left[ 1 + \left( \frac{\epsilon}{R} \right)^2 \right] \epsilon = \frac{1}{3} \left( \frac{4}{1 - T^2} \right) \left[ 1 + \left( \frac{\epsilon}{R} \right)^2 \right] \epsilon
$$

(B17)

up to the first order in $\epsilon$.

Using Eq. (B13), the Fermi density Eq. (1) is expanded up to first order in $(R_q - R)$ as

$$
\rho_q(r) = \frac{\rho_{qc}(R_q)}{1 + e^{(r - R_q)/a}} = \frac{\rho_{qc}(R_q)}{1 + e^{(y + (R - R_q)/a)}},
$$

$$
= \rho_{qc}(R) \left[ 1 - \left( \frac{e^y}{1 + e^y} \right) \left( \frac{R - R_q}{a} \right) + \cdots \right] \left[ 1 - \left( \frac{3 + \pi^2 \left( \frac{\epsilon}{R} \right)^2}{1 + \pi^2 \left( \frac{\epsilon}{R} \right)^2} \right) \left( \frac{R_q - R}{R} \right) + \cdots \right]
$$

$$
= \rho_{qc}(R) \left[ 1 - \left( \frac{e^y}{1 + e^y} \right) \left( \frac{R - R_q}{a} \right) - \left( \frac{3 + \pi^2 \left( \frac{\epsilon}{R} \right)^2}{1 + \pi^2 \left( \frac{\epsilon}{R} \right)^2} \right) \left( \frac{R_q - R}{R} \right) + \cdots \right]
$$

(B18)

$$
\rho_q^m(r) = \rho_{qc}(R) \left[ \frac{1}{1 + e^y} \right] \left[ 1 + m \left( \frac{e^y}{1 + e^y} \frac{R}{a} \right) - \left( \frac{3 + \pi^2 \left( \frac{\epsilon}{R} \right)^2}{1 + \pi^2 \left( \frac{\epsilon}{R} \right)^2} \right) \left( \frac{R_q - R}{R} \right) + \cdots \right]
$$

(B19)

where $y = (r - R)/a$. Since $t_n = \frac{Z}{A} t$ and $t_p = -\frac{N}{A} t$ (Eq. (13)) with $\rho_{qc}(R)/\rho_c(R) = N/A$, we have

$$
\rho_n^m(r) + \rho_p^m(r) = \rho_{qc}(R) \left[ \frac{1}{1 + e^y} \right] \left[ \left( \frac{N + Z}{A} \right) + m \left( \frac{e^y}{1 + e^y} \frac{R}{a} - \left( \frac{3 + \pi^2 \left( \frac{\epsilon}{R} \right)^2}{1 + \pi^2 \left( \frac{\epsilon}{R} \right)^2} \right) \left( \frac{N + Z}{A} \right) \right) \left( \frac{R_q - R}{R} \right) + \cdots \right]
$$

(B20)

From this result it is easy to show that the quantity $\rho_n^m(r) + \rho_p^m(r)$ has a first order correction from skin size $t$ and the first order correction vanishes for $m = 1$. That is the total density $\rho(r) = \rho_n(r) + \rho_p(r)$ is independent of the neutron skin size $t = R_n - R_p$ up to first order. Thus only the explicit $\rho_q$ dependent terms in Skyrme interaction, not the total density $\rho$ dependent terms, depend on the skin size $t$ up to the first order.

**Appendix C: Integral of density functional**

Since $F[\rho_q] = \int d^3 r f(\rho_q(r))$ for a Fermi density $\rho_q(r) = \frac{\rho_{qc}(r)}{1 + e^{(r - R_q)/a}}$, where $f(\rho_q)$ is a function of a single density $\rho_q$ only, can be integrated exactly as a function $F(R_q)$ of $R_q$ [13], we can expand $F(R_q)$ easily in terms of $t_q = R_q - R$ or $x_q = -t_q/a$ around $F(R)$. That is

$$
F(R_q) = F(R) + \frac{dF(R)}{dR} t_q + \frac{d^2 F(R)}{dR^2} \left( \frac{t_q}{2} \right) + \cdots
$$

(C1)
This is much simpler than using previous method of expanding in t (Eq.(B13)) first then integrate the results. For fixed \( N_q \), the central density \( \rho_{qc}^m(R_q) \) is expanded as

\[
\rho_{qc}^m(R_q) = \rho_{qc}^m(R) \left[ 1 - m \left( \frac{3 + \pi^2 (\frac{\alpha}{R})^2}{1 + \pi^2 (\frac{\alpha}{R})^2} \right) \left( \frac{R_q - R}{R} \right) + \cdots \right]
\]  

(C2)

Thus

\[
\int d^3 r \rho_q^m(\vec{r}) = \frac{4\pi}{3} R^3 \rho_{qc}^m(R_q) \left( R_q^3 - 3aR_q^2 + \pi^2 a^2 R_q - \pi^2 a^3 \right)
\]

\[
= \frac{4\pi}{3} R^3 \rho_{qc}^m(R) \left[ 1 - 3 \left( \frac{a}{R} \right) + \pi^2 \left( \frac{a}{R} \right)^2 - \pi^2 \left( \frac{a}{R} \right)^3 \right] \left[ 1 - 2 \left( \frac{3 + \pi^2 (\frac{\alpha}{R})^2}{1 + \pi^2 (\frac{\alpha}{R})^2} \right) \left( \frac{R_q - R}{R} \right) \right]
\]

\[
+ \frac{4\pi}{3} R^3 \rho_{qc}^m(R) \frac{a}{R} \left[ 3 - 6 \left( \frac{a}{R} \right) + \pi^2 \left( \frac{a}{R} \right)^2 \left( \frac{R_q - R}{a} \right) \right] + \cdots
\]  

(C3)

\[
\int d^3 r \rho_q(\vec{r}) \nabla^2 \rho_q(\vec{r}) = -\frac{4\pi}{3} R^3 \rho_{qc}^m(R_q) \left[ 1 + \left( \frac{\pi^2}{3} - 2 \right) \left( \frac{a}{R_q} \right)^2 \right]
\]

\[
= -\frac{4\pi}{3} R^3 \rho_{qc}^m(R) \left[ 1 + \left( \frac{\pi^2}{3} - 2 \right) \left( \frac{a}{R} \right)^2 \right] \left[ 1 - 2 \left( \frac{3 + \pi^2 (\frac{\alpha}{R})^2}{1 + \pi^2 (\frac{\alpha}{R})^2} \right) \left( \frac{R_q - R}{R} \right) \right]
\]

\[
+ \frac{4\pi}{3} R^3 \rho_{qc}^m(R) \frac{a}{R} \left[ 3 - 4 \left( \frac{\alpha}{R} \right) + \pi^2 \left( \frac{\alpha}{R} \right)^2 \left( \frac{R_q - R}{a} \right) \right] + \cdots
\]  

(C4)

\[
\int d^3 r \rho_q^{1/3}(\vec{r}) = \frac{4\pi}{3} R^3 \rho_{qc}^{1/3}(R_q) \left[ 1 - 1.335546875 \left( \frac{a}{R_q} \right) + 8.81615625 \left( \frac{a}{R_q} \right)^2 - 5.0303125 \left( \frac{a}{R_q} \right)^3 \right]
\]

\[
= \frac{4\pi}{3} R^3 \rho_{qc}^{1/3}(R) \left[ 1 - 1.335546875 \left( \frac{a}{R} \right) + 8.81615625 \left( \frac{a}{R} \right)^2 - 5.0303125 \left( \frac{a}{R} \right)^3 \right]
\]

\[
\times \left[ 1 - \frac{4}{3} \left( \frac{3 + \pi^2 (\frac{\alpha}{R})^2}{1 + \pi^2 (\frac{\alpha}{R})^2} \right) \left( \frac{R_q - R}{R} \right) \right]
\]

\[
+ \frac{4\pi}{3} R^3 \rho_{qc}^{1/3}(R) \frac{a}{R} \left[ 3 - 2.67109375 \left( \frac{a}{R} \right) + 8.81615625 \left( \frac{a}{R} \right)^2 \left( \frac{R_q - R}{a} \right) \right] + \cdots
\]  

(C5)

\[
\int d^3 r \rho_q^{5/3}(\vec{r}) = \frac{4\pi}{3} R^3 \rho_{qc}^{5/3}(R_q) \left[ 1 - 2.276943 \left( \frac{a}{R_q} \right) + 9.10458 \left( \frac{a}{R_q} \right)^2 - 7.80506 \left( \frac{a}{R_q} \right)^3 \right]
\]

\[
= \frac{4\pi}{3} R^3 \rho_{qc}^{5/3}(R) \left[ 1 - 2.276943 \left( \frac{a}{R} \right) + 9.10458 \left( \frac{a}{R} \right)^2 - 7.80506 \left( \frac{a}{R} \right)^3 \right]
\]

\[
\times \left[ 1 - \frac{5}{3} \left( \frac{3 + \pi^2 (\frac{\alpha}{R})^2}{1 + \pi^2 (\frac{\alpha}{R})^2} \right) \left( \frac{R_q - R}{R} \right) \right]
\]

\[
+ \frac{4\pi}{3} R^3 \rho_{qc}^{5/3}(R) \frac{a}{R} \left[ 3 - 4.53886 \left( \frac{a}{R} \right) + 9.10458 \left( \frac{a}{R} \right)^2 \left( \frac{R_q - R}{a} \right) \right] + \cdots
\]  

(C6)

\[
\int d^3 r \rho_q^{8/3}(\vec{r}) = \frac{4\pi}{3} R^3 \rho_{qc}^{8/3}(R_q) \left[ 1 - 4.07693333 \left( \frac{a}{R_q} \right) + 11.836907 \left( \frac{a}{R_q} \right)^2 - 13.26781 \left( \frac{a}{R_q} \right)^3 \right]
\]

\[
= \frac{4\pi}{3} R^3 \rho_{qc}^{8/3}(R) \left[ 1 - 4.07693333 \left( \frac{a}{R} \right) + 11.836907 \left( \frac{a}{R} \right)^2 - 13.26781 \left( \frac{a}{R} \right)^3 \right]
\]

\[
\times \left[ 1 - \frac{8}{3} \left( \frac{3 + \pi^2 (\frac{\alpha}{R})^2}{1 + \pi^2 (\frac{\alpha}{R})^2} \right) \left( \frac{R_q - R}{R} \right) \right]
\]

\[
+ \frac{4\pi}{3} R^3 \rho_{qc}^{8/3}(R) \frac{a}{R} \left[ 3 - 8.13586666 \left( \frac{a}{R} \right) + 11.836907 \left( \frac{a}{R} \right)^2 \left( \frac{R_q - R}{a} \right) \right] + \cdots
\]  

(C7)

\[
E_C = \frac{3}{5} \frac{Z^2 e^2}{R_p} \left[ 1 - \left( \frac{7\pi^2}{6} \right) \left( \frac{a}{R_p} \right)^2 \right]
\]
\begin{align*}
&= \frac{3}{5} Z^2 e^2 \left[ 1 - \left( \frac{7 \pi^2}{6} \right) \left( \frac{a}{R} \right)^2 \right] - \frac{3}{5} Z^2 e^2 \frac{a}{R} \left[ 1 - \left( \frac{7 \pi^2}{2} \right) \left( \frac{a}{R} \right)^2 \right] \left( \frac{R_q - R}{a} \right) + \cdots \quad (C8)
\end{align*}

In Ref. [3], the Coulomb exchange term is shown only up to 0th order in $a/R$ but the actual calculation included all terms of $a/R$ up to 3. For the term with mixed densities, we need to integrate after expansion. Up to 1st order in $x = -t/a$,

\begin{align*}
\int d^3r \rho_q^a(r) \rho^a(r) &= 4 \pi \rho_q^m(R_q) \rho_c^a(R) \int_0^\infty r^2 dr \left( \frac{1}{1 + e^{(r-R)/a}} \right)^m \left( \frac{\rho_{qc}(R_q)/\rho_c(R)}{1 + e^{(r-R)/a}} + \frac{\rho_{qc}(R)/\rho_c(R)}{1 + e^{(r-R)/a}} \right)^{\alpha} \\
&\approx 4 \pi \rho_q^m(R) \rho_c^a(R) \int_0^\infty r^2 dr \left( \frac{1}{1 + e^{(r-R)/a}} \right)^{\alpha+m} \left[ 1 - m \left( \frac{3 + 3 \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_q - R}{R} \right) \right] \\
&= \frac{4 \pi}{3} \rho_q^m(R) \rho_c^a(R) \int_{-\infty}^\infty dy (ay + R)^2 \left( \frac{\alpha + m}{1 + e^y} \right)^{\alpha+m+1} \left[ 1 - m \left( \frac{3 + 3 \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_q - R}{R} \right) \right] \\
&+ 4 \pi \rho_q^m(R) \rho_c^a(R) \int_{-\infty}^\infty dy (ay + R)^2 \left( \frac{\alpha + m}{1 + e^y} \right)^{\alpha+m+1} \left( \frac{R_q - R}{R} \right) \left( \frac{R_q - R}{a} \right) \quad (C9)
\end{align*}

For SLy4 parameter with $\alpha = 1/6$ and $m = 2$,

\begin{align*}
\int d^3r \rho_q^2(r) \rho^1/6(r) &= \frac{4 \pi}{3} R^3 \rho_q^2 \rho_c^{1/6} \left[ 1 - 3.30669 \left( \frac{a}{R} \right)^2 + 10.331 \left( \frac{a}{R} \right)^2 - 10.7804 \left( \frac{a}{R} \right)^3 \right] \\
&\times \left[ 1 - 2 \left( \frac{3 + \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_q - R}{R} \right) \right] \\
&= \frac{4 \pi}{3} R^3 \rho_q^2 \rho_c^{1/6} \left[ 1 - 3.204466 \left( \frac{a}{R} \right)^2 + 3.443662 \left( \frac{a}{R} \right)^2 \left( \frac{R_q - R}{R} \right) \right] \\
&= \frac{36}{13} a \left[ 1 - 3.204466 \left( \frac{a}{R} \right)^2 + 3.443662 \left( \frac{a}{R} \right)^2 \left( \frac{R_q - R}{R} \right) \right] \left( \frac{R_q - R}{a} \right) \quad (C10)
\end{align*}

Here $36/13 = 3 \times 2/(13/6)$. For SkM$(m^* = m)$ parameter with $\alpha = 1$ and $m = 2$,

\begin{align*}
\int d^3r \rho_q^2(r) \rho(r) &= \frac{4 \pi}{3} R^3 \rho_q^2 \rho_c \left[ 1 - \frac{9}{2} \left( \frac{a}{R} \right) + (3 + \pi^2) \left( \frac{a}{R} \right)^2 - \frac{3 \pi^2}{2} \left( \frac{a}{R} \right)^3 \right] \\
&\times \left[ 1 - 2 \left( \frac{3 + \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_q - R}{R} \right) \right] \\
&= \frac{4 \pi}{3} R^3 \rho_q^2 \rho_c \left[ 1 - 3 \left( \frac{a}{R} \right) + \left( 1 + \frac{\pi^2}{3} \right) \left( \frac{a}{R} \right)^2 \right] \left( \frac{R_q - R}{a} \right) \left( \frac{R_q - R}{a} \right) \quad (C11)
\end{align*}

Here $2 = 3 \times 2/3$. For $T$-independent term in kinetic energy with $\alpha = 1$ and $m = 5/3$,

\begin{align*}
\int d^3r \rho_q^{5/3}(r) \rho(r) &= \frac{4 \pi}{3} R^3 \rho_q^{5/3} \rho_c \left[ 1 - 4.07693333 \left( \frac{a}{R} \right) + 11.836907 \left( \frac{a}{R} \right)^2 - 13.26781 \left( \frac{a}{R} \right)^3 \right] \\
&\times \left[ 1 - \frac{5}{3} \left( \frac{3 + \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_q - R}{R} \right) \right] \\
&= \frac{4 \pi}{3} R^3 \rho_q^{5/3} \rho_c \left[ 15 a \right] \left[ 1 - 2.71797333 \left( \frac{a}{R} \right) + 3.945626667 \left( \frac{a}{R} \right)^2 \right] \left( \frac{R_q - R}{a} \right) \quad (C12)
\end{align*}

Here $(15/8) = 3 \times (5/3)/(8/3)$. For kinetic energy we cannot use this method since it uses numerical integration.

The kinetic energy has term with the form of $\rho_q^m m_q^* = \rho_q^m (1 + \alpha \rho + b \rho_q)$. Since

\begin{align*}
\rho_q^m(r) &= \left( \frac{\rho_{qc}(R)}{1 + e^y} \right)^m \left[ 1 - m \left( \frac{e^y}{1 + e^y} \right) \left( \frac{R_q - R}{R} \right) - m \left( \frac{3 + \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_q - R}{R} \right) + \cdots \right] \quad (C13)
\end{align*}
Since the \( m^* \) factor depends on \( \rho_q \), the first order term in \( x = -t/a \) of \( (\rho_m^m m^* \rho_q^m m^* \rho_p^m m^*) \) does not become zero for any \( m \) for non-zero \( n \). Similarly total kinetic energy \( \tau = \tau_n + \tau_p \) has nonzero first order term in the neutron skin \( t \) in contrast to the total density \( \rho = \rho_n + \rho_p \) which is independent to \( t \) up to first order. Total kinetic energy is

\[
\tau = \frac{3}{5} \left[ \left( \rho_n^{5/3} + \rho_p^{5/3} \right) + \frac{5 n^2}{3 \hbar^2} \left( \frac{\gamma}{6 \pi^2} \right)^{4/3} T^2 \left( \rho_n^{1/3} m_{q^*} + \rho_p^{1/3} m_{p^*} \right) \right]_{x=0}^n
\]

\[
\times \left[ \left( \frac{e^y}{1 + e^y} \right) \left( \frac{R - R_q}{a} \right) + \left( \frac{3 + \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_q - R}{b} \right) \right]_{x=0}^n + \cdots
\]

With

\[
\tau_q = \frac{3}{5} \left[ \left( \rho_q^{5/3} + \rho_p^{5/3} \right) + \frac{5 n^2}{3 \hbar^2} \left( \frac{\gamma}{6 \pi^2} \right)^{4/3} T^2 \rho_q^{1/3} m_{q^*} \right]_{t=0}^n
\]

\[
\times \left[ \left( \frac{e^y}{1 + e^y} \right) \left( \frac{R - R_q}{a} \right) + \left( \frac{3 + \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_q - R}{b} \right) \right]_{t=0}^n + \cdots
\]

\[
\tau_p = \frac{3}{5} \left[ \left( \rho_p^{5/3} + \rho_n^{5/3} \right) + \frac{5 n^2}{3 \hbar^2} \left( \frac{\gamma}{6 \pi^2} \right)^{4/3} T^2 \rho_p^{1/3} m_{p^*} \right]_{t=0}^n
\]

\[
\times \left[ \left( \frac{e^y}{1 + e^y} \right) \left( \frac{R - R_p}{a} \right) + \left( \frac{3 + \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_p - R}{b} \right) \right]_{t=0}^n + \cdots
\]
and

\[
\rho_q^7 q = \frac{3}{5} \left( \frac{6\pi^2}{\gamma} \right)^{2/3} \left[ \frac{5\rho_q^5}{3h^4} \left( \frac{\gamma}{6\pi^2} \right)^{4/3} T^2 \rho_q^{1/3} m_q^2 + \cdots \right]
\]

\[
= \frac{3}{5} \left( \frac{6\pi^2}{\gamma} \right)^{2/3} \left[ \frac{5\rho_q^5}{3h^4} \left( \frac{\gamma}{6\pi^2} \right)^{4/3} T^2 \rho_q^{1/3} m_q^2 \right]_{x=0}
\]

\[
- \rho_0 \frac{3}{5} \left( \frac{6\pi^2}{\gamma} \right)^{2/3} \left[ \frac{8}{3} \frac{5\rho_q^5}{3h^4} \left( \frac{\gamma}{6\pi^2} \right)^{4/3} \rho_q^{1/3} m_q^2 \left( \frac{4}{3} - 2b \rho_q^{1/3} m_q^2 T \right) \right]_{x=0}
\]

\[
\times \left[ \left( \frac{e^\nu}{1 + e^\nu} \right) \left( \frac{R - R_q}{a} \right) + \left( \frac{3 + \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_q - R}{R} \right) \right] + \cdots
\]

\[
= \frac{3}{5} \left( \frac{6\pi^2}{\gamma} \right)^{2/3} \left[ \frac{5\rho_q^5}{3h^4} \left( \frac{\gamma}{6\pi^2} \right)^{4/3} T^2 \rho_q^{1/3} m_q^2 \right]_{t=0}
\]

\[
+ \rho_0 \frac{3}{5} \left( \frac{6\pi^2}{\gamma} \right)^{2/3} \left[ \frac{5\rho_q^5}{3h^4} \left( \frac{\gamma}{6\pi^2} \right)^{4/3} \rho_q^{1/3} m_q^2 T^2 \right]_{t=0}
\]

\[
\times \left[ \left( \frac{e^\nu}{1 + e^\nu} \right) \left( \frac{R_q - R}{a} \right) - \left( \frac{3 + \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_q - R}{R} \right) \right] + \cdots
\]  

(C17)

In Weizacker mass formula it might be better expanding in terms of \((R_n - R_p)/R\) rather than \((R_n - R_p)/a\) since the first one is independent of \(A\) while the second one is dependent on \(A\).


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