DEDUCTION OF PLANCK’S FORMULA FROM MULTIPHOTON STATES

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Abstract

We obtain the black body radiation formula of Planck by considering independent contributions of multiphoton entities
1. Einstein introduced the light quantum hypothesis in 1905 \[1\] by noticing that the entropy calculated from the Wien formula for the black body radiation

\[ I_W(\nu, T) = a\nu^3 \exp(-b\nu/T), \quad (1) \]

which is the high-frequency, low-density limit of the complete Planck’s formula

\[ I(\nu, T) = \frac{8\pi\hbar\nu^3}{c^3} \frac{1}{e^{(\hbar\nu/kT)} - 1}, \quad (2) \]

this entropy had the same form as the entropy of a “gas” of independent particles, light quanta (later called photons). Viceversa, starting from a gas of independent light quanta one arrives only to Wien’s formula, as shown since 1910 by many people \[6\].

The question arises as how to get the complete radiation formula from the corpuscular concept of the photon. In this letter we present what we think is about the simplest derivation of Planck’s radiation formula, starting from the idea of “multiphoton states” \[2\], i.e. molecules of light compound of \( n \) photons with zero binding energy.

We know of course that the most satisfactory deduction of Planck’s formula is obtained nowadays by incorporating the right B.E. statistics into the corpuscular picture, as first shown by S.N.Bose \[3\] and interpreted and extended by Einstein in 1924/25 \[4\]. Our calculation makes use of well-defined multiphoton states as the natural way of taking into account the identical, indistinguishable character of the photons.

The idea of multiphoton “molecules” arose around 1910, after P. Debye presented \[5\] a simple, ondulatory method to obtain the radiation law; some historical comments are included below in this letter.

2. We start from the photon entity as particles of energy \( \epsilon = h\nu \) (they are truly massless particles of helicity \( \pm 1 \) with energy and momentum \( \epsilon = h\nu = pc, p \), but we want to think and write in the spirit of the 1910s).

We make now this additional simple hypothesis \[2\]: light of definite direction and frequency \( \nu \) presents itself in units (“molecules”) of 0, 1, 2, ..., \( n \), ...photons, with energy \( n\hbar\nu \), that is, with zero binding energy, and they contribute independently to the energy density.

The rationale for this assumption is, of course, in the spirit of the atomistic or democritean point of view, that all the photons of same frequency
are created equal (identical), and that in an occamian, economical description, we should abstain from considering situations in which the photons are individually distinct (indistinguishability); that is, there is a “state” with two photons different, for the counting, from the naive juxtaposition of two distinguishable photons; same thing for the three photon states, etc. So the multiphoton molecules should be thought of as entities only to the effect of counting, and not as claiming for the existence of some force which really binds the photons together.

In this way one gets the correct radiation formula at once. We are unaware of any complete proof of this statement, so we now show the deduction, leaving some historical remarks for later.

For the \( n \)-photon “molecules” of energy \( nh \nu \), \((n = 0, 1, 2, \ldots)\) the probability is of course given by the Boltzmann factor

\[
p(n) = c \exp(-nh \nu / kT)
\]

\[
\sum_{n=0}^{\infty} p(n) = 1 \text{ gives } c = 1 - x, \ x \equiv \exp(-h \nu / kT).
\]

The multiphoton states still have frequency \( \nu \), so the common phase space factor is \( 2(\text{polarization}) \times 4\pi (\text{solid angle}) \times \nu^2 \) (radial 3D factor); hence the density of energy per inverse wavelength is

\[
I(\nu, T) = \frac{8\pi \nu^3}{c^3} \sum_{n=0}^{\infty} nh \nu (1 - x) \exp(-nh \nu / kT)
\]

\[
= \frac{8\pi \nu^3 h}{c^3} \frac{1}{1 - x} \exp(h \nu / kT) - 1
\]

i.e., the correct complete radiation formula.

The two traditional limits of the radiation formula do have a particle interpretation, as already remarked by Wolfke [2]. E.g. at low temperature only the lightest “molecule” is excited, that is, the one-photon mode, of energy \( h \nu \), that leads to the Wien’s limit law (1), which was Einstein’s starting point [1].

On the other hand, for high densities or temperatures all the \( n \)-photon states contribute together and sort of coalesce, giving a scale-invariant power law distribution \( I \propto \nu^2 \), i.e. the classical electromagnetic (Rayleigh-Jeans) limit formula.
3. If we now start from (5) and expand the denominator, we get

\[ I(\nu, T) = 8\pi(\nu^2/c^3)(h\nu) \sum_{n=1}^{\infty} \exp(-nh\nu/kT) \]  

(6)

already obtained by several people [6] in the 1910s; superficially, (6) looks like an additive contribution from each \( n \)-photon state, and it was this analogy which prompted Ehrenfest [7], Wolfke [2], de Broglie [8, 9] and others [6] to advance the “\( n \)-photon molecule” concept; notice however that the literal interpretation of (6) as a cooperative multiphoton formula falls short of the correct derivation in (5) on three accounts:

1. There is no contribution from the zero-photon state
2. The \( n \)-photon state contributes with energy \( h\nu \), instead of the correct \( nh\nu \)
3. Probabilities \( \exp(-nh\nu/kT) \) do not add up to one.

Still, it is remarkable that the expansions (5) and (6) are really identical!

4. Why our simple derivation was not, to the best of our knowledge, obtained and exploited many decades ago, is for the historians to discuss; perhaps light quanta correlations due to indiscernibility was too novel at the time. Recall, however, that the Gibbs paradox was well known at the time, with the proposed solution by Gibbs himself [10]. Notice that this treatment of photons is totally correct even from the modern point of view; in particular the zero-point energy does not show up. Also we are really doing second quantization. Namely “first quantization” just amounts to counting frequency (or wavenumber) modes, because \( \nu = pc/h \); the “second” quantization amounts to the existence of these photon molecules as separate entities. We finish by remarking that zero-energy bound states appear frequently in modern guise as Bogomol’nyi-Prasad-Sommerfield (BPS) states, and moreover uncharged PBS states, like the photon, are massless.
References

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[6] Some historical works contain more detailed references; we mention three: (1) J. Mehra and H. Rechenberg, *The historical Development of Quantum Theory* (Springer, New York 1982-2000, several volumes), esp. Vol I, p.59 ff and p.554 ff. (2) M. Jammer, *The Conceptual development of Quantum Mechanics*, (Mc Graw-Hill, New York 1966), esp. pp.51-53, and (3) E. Whittaker, *A History of the Theories of Aether and Electricity* (Nelson, London; rep. Humanities Press, New York 1953), Vol. II, pp.87-89 and 102.

[7] P.Ehrenfest. *Ann.d.Phys*(4) 36, 91-118 (1911)

[8] L. de Broglie. *J.phys. et rad.*(6) 3, 422-428 (1922)

[9] L. de Broglie. *These*, Paris (25-XI-1924).(Masson et Cie., Paris 1925).

[10] J.W. Gibbs, *Elementary Principles in Statistical Mechanics* (Yale University Press, New Haven (1902)); Gibbs proposal was close to the modern indiscernibility.