On the Dynamics of the Tavis–Cummings Model

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Abstract—The purpose of this article is to present a comprehensive study of the Tavis–Cummings model from a system-theoretic perspective. A typical form of the Tavis–Cummings model is composed of an ensemble of non-interacting two-level systems (TLSs) that are collectively coupled to a common cavity resonator. The associated quantum linear passive system is proposed, whose canonical form reveals typical features of the Tavis–Cummings model, including $\sqrt{N}$-scaling, dark states, bright states, single-excitation superradiant, and subradiant states. The passivity of this linear system is related to the vacuum Rabi mode splitting phenomenon in Tavis–Cummings systems. On the basis of the linear model, an analytic form is presented for the steady-state output state of the Tavis–Cummings model driven by a single-photon state. Master equations are used to study the excitation properties of the Tavis–Cummings model in the multieexcitation scenario. Finally, in terms of the transition matrix for a linear time-varying system, a computational framework is proposed for calculating the state of the Tavis–Cummings model, which is applicable to the multieexcitation case.

Index Terms—Open quantum systems, quantum control, Tavis–Cummings model, two-level systems (TLSs).

I. INTRODUCTION

In 1954, Robert Dicke calculated [1] that an ensemble of gaseous molecules interacting with a common radiation field could exhibit a coherent spontaneous emission process, during which the molecules act as a giant molecule that shows superradiation—cooperative radiation rate much faster than independent individual radiation rates. This problem was further studied by Tavis and Cummings [2] by means of a model of $N$ identical noninteracting two-level systems (TLSs) coupled to a single-mode quantized radiation field, see Fig. 1 in Section III for an example. An exact solution of the eigenstates of the Hamiltonian for this model is derived. The model proposed in [2] is called the Tavis–Cummings model in the subsequent literature. The Tavis–Cummings model has been physically realized by quite a few experimental platforms, including superconducting circuits [3]–[6], NV spin ensembles [7], and double quantum dots [8]–[10]. In the Tavis–Cummings model consisting of $N$ TLSs equally coupled to a cavity resonator, under certain conditions the collective coupling strength of the ensemble exhibits a $\sqrt{N}$-scaling, which can be experimentally observed from the vacuum Rabi mode splitting of the resonator transmission spectrum of the bright states. In addition to bright states, a Tavis–Cummings model can also have dark states which contain single-excitation subradiant states as a subclass. Applications of the Tavis–Cummings model can be found in [4], [6], [9], [11] and references therein.

In this article, we aim to introduce the Tavis–Cummings model to the quantum control community and show that many of its typical properties can be uncovered by means of systems theory. The main contributions are summarized as follows.

In Section III-B, we propose a quantum linear system that is associated to the Tavis–Cummings model. This linear model reveals the $\sqrt{N}$-scaling of the coupling strength of the atomic ensemble. Moreover, a transfer function is defined for a performance variable for which the system is passive, and the transfer function reflects the vacuum Rabi mode splitting of the Tavis–Cummings model. Finally, the structural decomposition of the linear model shows that the bright states of the Tavis–Cummings model live in the controllable and observable subspace, whereas the dark states reside in the uncontrollable and unobservable subspace of the linear model.
In Section IV-A, we apply the quantum linear systems theory to derive an analytic form of the output single-photon state of the Tavis–Cummings system driven by a single-photon input. The simulations in Fig. 2 show that the input photon tends not to interact with the atoms when the number of atoms is large, and photon-atom interaction is easier when atoms are nonresonant. On the other hand, when only one of the two-level atoms is initially excited and the input field is vacuum, an analytic form of the joint system-field state is given in Section IV-C, which explains several experimental observations including superradiance and subradiance [6].

In Section V, we study the excitations of TLSs by means of master equations. When all the N atoms are initially in the excited state, we prove that eventually they all settle to the ground state and the output field is in an N-photon state, provided that the coupling strengths are identical.

In our preliminary study [12], a computational framework is proposed to calculate the joint system-field state of a general open quantum system. Here, we develop it further in Section VI and apply it to the study of the Tavis–Cummings model. In particular, we derive the exact form of 2- and 3-photon states.

**Notation:** The reduced Planck constant ħ is set to 1. i = \sqrt{-1} is the imaginary unit, δ_{ij} denotes the Kronecker delta function, and δ(t−r) is the Dirac delta function. I is the identity matrix, and 0 is the zero vector or matrix whose dimension can be easily determined from the context. Given a column vector of complex numbers or operators \( X = [x_1, \ldots, x_n]^\top \), the complex conjugate or adjoint operator of \( X \) is denoted by \( X^\dagger \). Clearly, when \( n = 1 \), \( X^\dagger = X \). Let \( t_0 \) be the initial time, i.e., the time when the system and its input start interaction. |\( \varphi \rangle \) and |\( \psi \rangle \) stand for the ground and excited states of a two-level atom, respectively. The commutator between two operators \( A \) and \( B \) is \([A, B] = AB - BA\). Define superoperators \( LX = -i[X, H] + DLX \) and \( Lρ = -i[H, ρ] + DLρ \), where \( DLX = L^\dagger X L - \frac{1}{2}L^\dagger LX - \frac{1}{2}XL^\dagger L \) and \( DLρ = LρL^\dagger - \frac{1}{2}L^\dagger Lρ - \frac{1}{2}LρL^\dagger L \).

### II. Preliminaries

#### A. Quantum Systems and Fields

Consider a quantum system driven by \( m \) input fields. The inputs are optical or microwave fields, which are generated by the annihilation operators \( b_{in,k}^\dagger(t) \) and their adjoints \( b_{in,k}^\ast(t) \) (creation operators), \( k = 1, \ldots, m \). If there are no photons in an input channel, this input is in the vacuum state |\( Φ_0 \rangle \). Annihilation and creation operators satisfy

\[
\begin{align*}
&b_{in,j}(t) |Φ_0 \rangle = 0, \quad [b_{in,j}(t), b_{in,k}(r)] = [b_{in,j}(t), b_{in,k}^\dagger(r)] = 0, \\
&[b_{in,j}(t), b_{in,k}^\dagger(r)] = δ_{jk}(t−r), \quad ∀j, k = 1, \ldots, m, \quad t, r \in \mathbb{R}.
\end{align*}
\]

The integrated annihilation and creation processes are, respectively, \( b_{in,k}^\ast(t) = \int_{−∞}^{t} b_{in,k}^\dagger(r)dr \) and \( b_{in,k}(t) = \int_{−∞}^{t} b_{in,k}^\dagger(r)dr \), which are quantum Wiener processes. Define Itô increments \( dB_{in,k}(t) = B_{in,k}(t + dt) - B_{in,k}(t) \). Denote \( B_{in}(t) = [B_{in,1}(t), \ldots, B_{in,m}(t)]^\top \). In this article, the input fields are assumed to be canonical fields which include the vacuum, coherent, single- and multiphoton fields. Then, \( dB_{in}(t)dB_{in}^\dagger(\tau) = dB_{in}^\#(t)dB_{in}^\dagger(\tau) = dB_{in}^\#(t)dB_{in}^\dagger(\tau) = 0 \) and \( dB_{in}(t)dB_{in}^\dagger(\tau) = Iδ_{t\tau}dt \).

The quantum system can be parametrized by a triple (\( S, L, H \)) [13], [14]. Here, \( H \) is the inherent system Hamiltonian, \( L = [L_1, \ldots, L_m]^\top \) describes how the system is coupled to its environment, and \( S \) is a scattering operator (e.g., a beamsplitter or a phase shifter). In this article, it is assumed that \( S = I \) (the identity operator). The temporal evolution of the quantum system is governed by a unitary operator \( U(t, t_0) \), which is the solution to the following Itô quantum stochastic differential equation (QSDE):

\[
dU(t, t_0) = \left[ -iH_{eff}dt + dB_{in}^\dagger(t)L - L^\dagger dB_{in}(t) \right] U(t, t_0)
\]

under the initial condition \( U(t_0, t_0) = I \). Denote the joint system-field state by \( |Ψ(t)\rangle \). In the Schrödinger picture, \( |Ψ(t)\rangle = (U(t, t_0)|Ψ(t_0)\rangle, then (2) reduces to, [15, Ch. 11]

\[
d|Ψ(t)\rangle = \left[ -iH_{eff}dt + dB_{in}^\dagger(t)L \right]|Ψ(t)\rangle.
\]

On the other hand, in the Heisenberg picture, the time evolution of the system operator \( X \), denoted by \( j_k(X) \equiv X(t) = U^\ast(t, t_0)(X \otimes I_{field})U(t, t_0) \), follows the Itô QSDE:

\[
dj_k(X) = j_k(LX)dt + \sum_{k=1}^{m} dB_{in,k}(t)j_k([X, L_k]) + \sum_{k=1}^{m} j_k([L_k^\dagger, X])dB_{in,k}(t).
\]

Finally, the output field annihilation operators are \( B_{out,k}(t) = U^\ast(t, t_0)B_{in,k}(t)U(t, t_0) \), \( (k = 1, \ldots, m) \), whose dynamical evolution is \( dB_{out}(t) = L(t)dt + dB_{in}(t) \). More discussions on open quantum systems can be found in, e.g., [13]–[16].

#### B. Continuous-Mode Single-Photon States

For each input channel \( k = 1, \ldots, m \), the creation operator \( b_{in,k}^\dagger \) generates a photon from the vacuum. Mathematically, \( \{1_k, t\} \equiv b_{in,k}^\dagger \langle Φ_0 \rangle \) means a photon is generated at time \( t \) in the \( k \)th input channel. By (1), \( \langle 1_k, t | 1_{k', r} \rangle = δ_{kk'}δ(t−r) \). Hence, \( \{1_k, t\} \equiv t \in \mathbb{R} \) is an orthogonal basis of single-photon states for each channel \( k \). Indeed, a single-photon state with temporal pulse shape \( ξ(t) \) in the \( k \)th channel can be viewed as a superposition...
of a continuum of $|1, k, t\rangle$, i.e.,

$$\left|1_\xi\right> = \int_{-\infty}^{\infty} \xi(t)|1, k, t\rangle dt = \int_{-\infty}^{\infty} \xi(t) b_{m,k}^\dagger(t) |\Phi_0\rangle.$$  \hspace{1cm} (4)

Physically, as a quantum state, $|1_\xi\rangle$ can be interpreted in the following way: the probability of finding the photon in the time bin $[t, t + dt]$ is $|\xi(t)|^2 dt$. The normalization condition $(1_\xi|1_\xi\rangle = 1$ requires $\int_{-\infty}^{\infty} |\xi(t)|^2 dt = 1$. As the single-photon state $|1_\xi\rangle$ is parameterized by an $L^2$ integrable function $\xi(t)$ over $\mathbb{C}$, it is called a continuous-mode single-photon state \cite{17–24}.

### III. TAVIS–CUMMINGS MODEL

We first present the Tavis–Cummings model in Section III-A. In Section III-B, assuming that the atoms are initially in the ground state, the cavity is empty and the input field is in the vacuum state, we present an associated linear model. Physical interpretation of the linear model is discussed in Section III-C.

#### A. Tavis–Cummings Model

In the Tavis–Cummings model as shown in Fig. 1, the $N$ TLSs are not directly coupled to each other; instead, they all couple to the common single-mode cavity. The inherent system Hamiltonian of the Tavis–Cummings model is (see\cite{2, (2.1), 3, (1), 4, (1), 5, (1), 9, (C7), 6, (1)})

$$H_{TC} = \omega_r a^\dagger a + \frac{\omega_j}{2} \sigma_{z,j} + \frac{\gamma_j}{2} (a^\dagger \sigma_{z,j} + \sigma_{z,j} a).$$  \hspace{1cm} (5)

Here, $a, a^\dagger$ denote the annihilation and creation operators of the cavity mode satisfying $[a, a^\dagger] = 1$, $\omega_r$ is the frequency detuning between the cavity mode and the input field. The two-level atom $j$ is coupled to the cavity with coupling strength $\gamma_j, j = 1, \ldots, N$, which is assumed to be a real number but can be negative \cite{5, Table 1}, \cite{7, (2)]. The corresponding detuning between the transition frequency of the two-level atom $j$ and the carrier frequency of the input field is denoted by $\omega_j$. The lowering and raising operators of the two-level atom $j$ are $\sigma_{z,j} = |g_j\rangle \langle e_j|$ and $\sigma_{z,j}^+ = |e_j\rangle \langle g_j|$, respectively. The Pauli $Z$ operator is $\sigma_{z,j} = \sigma_{z,j}^+ \sigma_{z,j} - \sigma_{z,j} \sigma_{z,j}^+$. The system exchanges information with its environment by means of absorbing and emitting photons, which is realized by the coupling operator $L = \sqrt{\kappa} a$. By the development in Section II-A, the Itô QSDEs for the Tavis–Cummings model in the Heisenberg picture are

$$\begin{cases}
    d\sigma_{z,1}(t) = -i\omega_1 \sigma_{z,1}(t) dt + i\Gamma_1 \sigma_{z,1}(t) a(t) dt \\
    d\sigma_{z,N}(t) = -i\omega_N \sigma_{z,N}(t) dt + i\Gamma_N \sigma_{z,N} a(t) dt \\
    da(t) = -i \sum_{j=1}^{N} \Gamma_j \sigma_{z,j}^+(t) dt - \sqrt{\kappa} dB_{out}(t) \\
    dB_{out}(t) = \sqrt{\kappa} a(t) dt + dB_{in}(t), \; t \geq t_0
\end{cases}$$  \hspace{1cm} (6)

which is bilinear.

#### B. Corresponding Linear Model

Assume that all the two-level atoms are initially in the ground state and the cavity is in the vacuum state $|0\rangle$. That is, the initial state of the Tavis–Cummings model is

$$|\zeta\rangle = |g_1 g_2 \cdots g_N\rangle \otimes |0\rangle.$$  \hspace{1cm} (7)

Let

$$X(t) = \left[\sigma_{z,1}(t) \sigma_{z,2}(t) \cdots \sigma_{z,N}(t) a(t)\right]^T.$$  \hspace{1cm} (8)

Notice that

$$\sigma_{z,j} |\zeta\rangle = -|\zeta\rangle$$  \hspace{1cm} (9)

for all $j = 1, \ldots, N$. From (6), we get

$$dX(t) |\zeta\rangle = AX(t) |\zeta\rangle dt + BdB_{in}(t) |\zeta\rangle$$

$$dB_{out}(t) |\zeta\rangle = CX(t) |\zeta\rangle dt + dB_{in}(t) |\zeta\rangle$$  \hspace{1cm} (10)

where

$$A = -i \begin{bmatrix} \omega_1 & 0 & \cdots & 0 & \Gamma_1 \\
0 & \omega_2 & \cdots & 0 & \Gamma_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \omega_N & \Gamma_N \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\sqrt{\kappa} \end{bmatrix}^T,$$  \hspace{1cm} (11)

(10) is a linear system. Actually, a linear quantum system of $N$ linear harmonic oscillators $\bar{a} = [a_1, \ldots, a_N, a]^T$ with system Hamiltonian $H = \omega_r a^\dagger a + \sum_{j=1}^{N} [\omega_j a_j a_j^\dagger + \frac{\gamma_j}{2} (a_j^\dagger a_j + a_j a_j^\dagger)]$ and coupling operator $L = \sqrt{\kappa} a$ has the following linear Itô QSDEs:

$$d\bar{a}(t) = A\bar{a}(t) dt + BdB_{in}(t)$$

$$dB_{out}(t) = C\bar{a}(t) dt + dB_{in}(t)$$  \hspace{1cm} (12)

where $A, B, C$ are exactly those in (11). More discussions on linear quantum systems theory can be found in e.g., \cite{25–27}. The transfer function of the linear quantum system (12) is

$$G[s] = 1 + C(sI - A)^{-1}B.$$  \hspace{1cm} (13)

Plugging (11) into (13) yields

$$G[s] = \frac{\sum_{k=1}^{N} \left( \Gamma_k^2 + \frac{1}{N} (s + i \omega_k)(s + i \omega_r - \frac{\kappa}{2}) \right) \prod_{j \neq k} (s + i \omega_j) }{\sum_{k=1}^{N} \left( \Gamma_k^2 + \frac{1}{N} (s + i \omega_k)(s + i \omega_r + \frac{\kappa}{2}) \right) \prod_{j \neq k} (s + i \omega_j) }.$$  \hspace{1cm} (14)

If $\omega_1 = \cdots = \omega_N = \omega_s$, (14) reduces to

$$G[s] = \frac{(\sqrt{N\Gamma})^2 + (s + i \omega_s)(s + i \omega_r - \frac{\kappa}{2})}{(\sqrt{N\Gamma})^2 + (s + i \omega_s)(s + i \omega_r + \frac{\kappa}{2})}.$$  \hspace{1cm} (15)
where $\Gamma \triangleq \sqrt{\frac{k}{N}} \sum_{k=1}^{N} \Gamma_k^2$. Let $\omega_r = \omega_s = 0$, i.e., all atoms are resonant with the cavity resonator. Define $T[s] \triangleq G[s] - 1$. Then

$$|T[i\omega]|^2 = \frac{k^2\omega^2}{[(\sqrt{N}\Gamma)^2 - \omega^2]^2 + \frac{\sigma^2}{4}\omega^2}. \quad (16)$$

Clearly, $|T[0]|^2 = 0$ and $|T[i\omega]|^2$ has two peaks attained at $\omega = \pm \sqrt{N}\Gamma$, respectively. (If $N = 0$, then $|T[i\omega]|^2$ has only one peak attained at $\omega = 0$, which is the empty cavity case.)

Remark 3.1: Suppose energy enters the system via the input field $B_{in}$ and flows out through the output field $B_{out}$. $T[s]$ is related to the energy stored in the system. In fact, define the performance variable $z \triangleq C\tilde{u}$. Then, $T[s]$ is the transfer function from $B_{in}$ to $z$. It is easy to see that

$$\begin{bmatrix} A + A^\dagger + C^\dagger C & B - C^\dagger \\ B^\dagger & 0 \end{bmatrix} = 0. \quad (17)$$

Thus, by the positive real lemma in [28, Th. 3], the system (12) with the performance variable $z$ is passive. In Section III-C, we will show that $T[s]$ also reflects the vacuum Rabi mode splitting of the Tavis–Cummings system (6). Hence, $T[s]$ bridges the passivity of the linear quantum system (12) and the vacuum Rabi mode splitting phenomenon exhibited by the Tavis–Cummings system (6).

In the following, we perform structural decomposition on the linear quantum system (12). We partition $\omega_1, \ldots, \omega_N$ into $M$ groups according to their degeneracies, where the $j$th degenerate frequency is denoted by $\tilde{\omega}_j$, $(j = 1, \ldots, M)$. Let the number of elements be $n_j$ for the group $j$. In particular, if $M = N$, then $\omega_j \neq \omega_k$ for all $1 \leq j < k \leq N$. For convenience, we can arrange the elements of $\tilde{u}(t)$ in (12) so that the matrix $A$ in (11) is of the form

$$\tilde{A} = -i \begin{bmatrix} \tilde{\omega}_1 \cdots 0 \cdots 0 \cdots 0 & \Gamma_{11} \\ \vdots & \ddots \vdots & \ddots \vdots & \vdots \\ 0 \cdots \tilde{\omega}_1 \cdots 0 \cdots 0 & \Gamma_{1n_1} \\ \vdots & \ddots \vdots & \ddots \vdots & \vdots \\ 0 \cdots 0 \cdots \tilde{\omega}_M \cdots 0 & \Gamma_{M1} \\ \vdots & \ddots \vdots & \ddots \vdots & \vdots \\ 0 \cdots 0 \cdots 0 \cdots \tilde{\omega}_M & \Gamma_{Mn_M} \end{bmatrix} \Gamma_{Mn_M} \omega_r - \frac{\sigma_i}{2}$$

In other words, we group the elements of $\tilde{u}(t)$ according to the partition of the detuned frequencies. It is easy to see that matrices $B$ and $C$ remain the same under this rearrangement.

Lemma 3.1: Partition $\omega_1, \ldots, \omega_N$ into groups as described above. There is an orthogonal matrix $\tilde{T}$ that transforms the linear quantum system (12) with system matrices $(A, B, C)$ to another one, denoted $\Sigma$, with system matrices

$$\tilde{A} = \tilde{T}^\dagger \tilde{A} \tilde{T}$$

$$\begin{bmatrix} \tilde{\omega}_1 I_{n_1-1} & 0 & \cdots & 0 \\ 0 & \tilde{\omega}_1 & \cdots & \cdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \tilde{\omega}_M I_{n_M-1} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{\Gamma_1} \\ \vdots \\ 0 \end{bmatrix}$$

$$= -i \begin{bmatrix} 0 & \cdots & 0 & \sqrt{\Gamma_1} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \sqrt{\Gamma_M} \omega_r - \frac{\sigma_i}{2} \end{bmatrix}$$

$$\tilde{B} = \tilde{T}^\dagger B = B, \quad \tilde{C} = C \tilde{T} = C$$

(18)

where $\tilde{\Gamma}_j \triangleq \sum_{k=1}^{n_j} \Gamma_{jk}^2$, $(j = 1, \ldots, M)$.

Proof: The proof is constructive. Define a matrix $\tilde{T}$ as

$$\tilde{T} = \begin{bmatrix} \tilde{T}_1 & 0 & \cdots & 0 \\ 0 & \tilde{T}_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \tilde{T}_M \end{bmatrix}$$

where for each $j = 1, \ldots, M$, $\tilde{T}_j = [T_{j1}, T_{j2}, \cdots, T_{jn_j}] \in \mathbb{R}^{n_j \times n_j}$, in which

$$T_{j1} = \sqrt{\frac{\Gamma_{j1}^2}{\Gamma_{j1}^2 + \Gamma_{j2}^2}} \begin{bmatrix} 1 - \frac{\Gamma_{j1}}{\Gamma_{j2}} & 0 & \cdots & 0 \end{bmatrix}^\dagger$$

$$T_{j2} = \sqrt{\frac{\Gamma_{j2}^2}{\Gamma_{j1}^2 + \Gamma_{j2}^2}} \begin{bmatrix} \Gamma_{j2}^2 \Gamma_{j1}^2 \\ \Gamma_{j1}^2 + \Gamma_{j2}^2 + \Gamma_{j3}^2 \\ \vdots \\ \Gamma_{j2} \Gamma_{j3} \end{bmatrix} \times \begin{bmatrix} \Gamma_{j2} \Gamma_{j3} \cdots \Gamma_{jn_j} \end{bmatrix}$$

$$T_{jn_j} = \sqrt{\frac{1}{\sum_{k=1}^{n_j} \Gamma_{jk}^2}} \begin{bmatrix} \Gamma_{j1} \Gamma_{j2} \cdots \Gamma_{jn_j} \end{bmatrix} \Gamma_{jn_j} \omega_r - \frac{\sigma_i}{2} \end{bmatrix} \begin{bmatrix} \sqrt{\Gamma_1} \\ \vdots \\ \vdots \\ \vdots \\ \sqrt{\Gamma_M} \omega_r - \frac{\sigma_i}{2} \end{bmatrix}$$

It can be easily verified that $\tilde{T}$ is orthogonal. Moreover, simple algebraic manipulations yield that $\tilde{T}^\dagger \tilde{A} \tilde{T} = \tilde{A}$. Q.E.D.

The transformed linear quantum system with system matrices $(\tilde{A}, \tilde{B}, \tilde{C})$ has a nice structure. Denote by $b_j$ the system coordinate corresponding to the row whose last entry is $-i\sqrt{\Gamma_j}$ in the matrix $\tilde{A}$, $j = 1, \ldots, M$. Then, from the structure of $(\tilde{A}, \tilde{B}, \tilde{C})$,
it can be easily seen that this system has a subsystem of the form

\[
\begin{align*}
\dot{b}_1(t) &= -i\omega b_1(t) - di\sqrt{\Gamma_1}a(t)dt \\
\vdots \\
\dot{b}_M(t) &= -i\omega_M b_M(t) - d\sqrt{\Gamma_M}a(t)dt \\
\dot{a}(t) &= -(i\omega_r + \frac{\kappa}{2})a(t)dt \\
-\Gamma_1 \sum_{j=1}^M \sqrt{\Gamma_j} b_j(t) dt - \sqrt{\kappa} d\dot{b}_m(t) \\
\dot{d}\dot{b}_{\text{out}}(t) &= \sqrt{\kappa a(t)} dt + d\dot{b}_m(t).
\end{align*}
\]

The other $M-1$ subsystems are all isolated systems, which are called decoherence-free subsystems (DFSs) in the linear quantum control literature [25]–[27], [29]. Of course, if $n_j = 1$ for some $j = 1, \ldots, M$, there is no such subsystem as can be seen clearly from the matrix $\hat{A}$ in (18). According to quantum linear systems theory, the subsystem (20) is a controllable and observable subsystem.

The following result is an immediate consequence of Lemma 3.1.

**Corollary 3.1:** If $\omega_1 = \cdots = \omega_N = \omega_s$, then $M = 1$ and $n_1 = N$ in (18). Accordingly, the transformation matrix $\tilde{T}$ in (19) reduces to

\[
\begin{bmatrix}
\tilde{T} \\
0 \\
1
\end{bmatrix} = T
\]

which transforms the quantum linear system (12) to a new one with system matrices

\[
\begin{align*}
\hat{A} &= T^\top \hat{A} T = \begin{bmatrix}
\hat{A}_{g\bar{g}} & 0 \\
0 & A_{co}
\end{bmatrix} \\
\hat{B} &= T^\top \hat{B} T = \begin{bmatrix}
\hat{B}_{g\bar{g}} \\
B_{co}
\end{bmatrix}, \\
\hat{C} &= CT = \begin{bmatrix}
\hat{C}_{g\bar{g}} & \hat{C}_{co}
\end{bmatrix}
\end{align*}
\]

where

\[
\begin{align*}
\hat{A}_{g\bar{g}} &= -i\omega_s I_{N-1}, \\
A_{co} &= \begin{bmatrix}
i\omega_s & i\sqrt{N}\Gamma \\
i\sqrt{N}\Gamma & i\omega_r + \frac{\kappa}{2}
\end{bmatrix} \\
\hat{B}_{g\bar{g}} &= 0, \\
B_{co} &= \begin{bmatrix}0 \\
\sqrt{\kappa}
\end{bmatrix}, \\
\hat{C}_{g\bar{g}} &= 0, \\
\hat{C}_{co} &= \begin{bmatrix}0 & 0 \sqrt{\kappa}
\end{bmatrix}
\end{align*}
\]

**Remark 3.2:** The coupling operator $L = \sqrt{\kappa} a$ is called the noise operator in [30], through which the system information leaks irreversibly into its surrounding environment. It can be clearly seen from (22)–(23) that $L$ has no effect on the DFS with system matrices ($\hat{A}_{g\bar{g}}, \hat{B}_{g\bar{g}}, \hat{C}_{g\bar{g}}$), thus the DFS is robust with respect to the cavity decay rate $\kappa$. This is the so-called $\gamma$-robustness in [30, Def. 3]. Moreover, from (23) it can be seen that the DFS is robust with respect to the variations of the atom-cavity coupling strengths $\Gamma_j$ as well. Finally, the DFS is not attractive from [30, Prop. 3], which is for noiseless subsystems and reduces to DFSs when $\mathcal{H}_F \simeq \mathbb{C}$; see [30, Def. 8 and Sec. II-A] for details.

### C. Physical Interpretation

The discussions of the Tavis–Cummings model in the previous subsection in terms of linear quantum systems theory appear purely mathematical; however, these results can indeed reveal several typical features of the Tavis–Cummings model.

First, from (15), it can be seen that the collection of the $N$ atoms act as a giant atom of detuned frequency $\omega_s$ and coupling strength $\sqrt{N}\Gamma$, which reflects the $\sqrt{N}$- scaling of the collective coupling strength; in other words, this giant atom decays $N$ times as fast as a single atom. This is the physical basis of superradiance.

Second, the two peaks of the transfer function $|T[i\omega]|^2$ in (16) at $\omega = \pm \sqrt{N}\Gamma$ echo the vacuum Rabi mode splitting in the Tavis–Cummings model, see, e.g., [4, Fig. 2(b)], [5, Fig. 3], and [6, Fig. 2(b)].

Third, the controllable and observable subsystem echoes the bright states and the uncontrollable and unobservable subsystem echoes the dark states of the Tavis–Cummings model. For simplicity, we look at the simplest case of $\omega_1 = \cdots = \omega_N = \omega_s$. In this case, according to Corollary 3.1, there is an orthogonal matrix $T$, which yields a system with system matrices given in (22). Let $N = 3$ and assume the coupling constants $\Gamma_1 = -\Gamma_2 = \Gamma_3$ (in [5], $\Gamma_1, \Gamma_2, \Gamma_3$ are, respectively, $g_A, g_B, g_C$). As shown in [5, Table 1], the actual values of $\Gamma_1, -\Gamma_2, \Gamma_3$ are not exactly identical, but the discrepancy has negligible effect as can be seen from consistency between the red region (for real data) and the white dashed curves (for theoretical calculation) in [5, Fig. 3.] The orthogonal transformation matrix $T$ can be calculated as $T_1 = \frac{1}{\sqrt{2}} [1 \ 1 \ 0]^\top$, $T_2 = \frac{1}{\sqrt{2}} [1 \ -1 \ 0]^\top$, $T_3 = \frac{1}{\sqrt{2}} [1 \ -1 \ 1]^\top$, and $T_4 = [0 \ 0 \ 0]^\top$. Identify $g$ with 0 and e with 1, respectively. It can be verified that the dark states $[3, 1, d_1] = \frac{1}{\sqrt{2}} ([g, g, g, 0] - \Gamma_1 [g, g, e, 0])$ and $[3, 1, d_2] = \frac{1}{\sqrt{2}} ([g, g, g, 0] + \Gamma_1 [g, g, e, 0])$ in [5] can be expressed as $[3, 1, d_1] = \frac{1}{\sqrt{2}} T_1 - \frac{\sqrt{2}}{2} T_2$, and $[3, 1, d_2] = \frac{1}{\sqrt{2}} T_1 + \frac{\sqrt{2}}{2} T_2$, respectively, while the bright states $[3, 1, \pm] = \frac{1}{\sqrt{2}} [g, g, g, 1] \pm \frac{1}{\sqrt{2}} [g, g, e, 0]$ in [5] can be written as $[3, 1, \pm] = \frac{1}{\sqrt{2}} (T_2 + T_3)$. In other words, the dark states $[3, 1, d_1]$ and $[3, 1, d_2]$ live in the decoherence-free subspace spanned by $T_1$ and $T_2$, while the bright states $[3, 1, \pm]$ live in the controllable and observable subspace spanned by $T_2$ and $T_4$.

Take the case $N = 2$ for another example. Similar correspondence can be found between the eigenstates of the Tavis–Cummings model (6) and the vectors in (21). Regarding the states $[g, e, 0], [e, g, 0], [g, g, 1]$ in [9] as vectors $[0 \ 1 \ 0]^\top$, $[1 \ 0 \ 0]^\top$, and $[0 \ 0 \ 1]^\top$, respectively. In the resonant ($\omega_r = \omega_s$) case, the dark state $[0, r_3] = \frac{1}{\sqrt{N}\Gamma} (\Gamma_1 [g, g, e, 0] - \Gamma_2 [e, g, g])$ in [9, Appendix C-3] is $-T_1$, and the two bright states $[\pm, r_3] = \frac{1}{\sqrt{N}\Gamma} (\Gamma_1 [g, e, 0] + \Gamma_2 [e, g, 0])$ in [9, Appendix C-3] are $\frac{1}{\sqrt{2}} (T_2 + T_3)$. Also, $[0, r_3]$ and $[\pm, r_3]$ are, respectively, $-\Gamma_1 [1, d_1]$ and $\Gamma_1 [1, d_2]$ in [5] when $\Gamma_1 = -\Gamma_2 = \Gamma_3$. Moreover, in the dispersive regime ($\Delta_r = \omega_r - \omega_s \gg \Gamma_j, j = 1, \ldots, N$) considered in [9, Appendix C], the three states $[\pm, r_3], [-\Gamma_3, r_3], [1, r_3]$ given in [9, (C18)] can be expressed by $[\pm, r_3] = -T_1$, $-\Gamma_3 = \frac{1}{\sqrt{N}\Gamma^2 + \Delta_r^2} (\Delta_r T_2 - \sqrt{N}\Gamma T_3) \approx T_2$, and $[1, r_3] = \frac{1}{\sqrt{N}\Gamma^2 + \Delta_r^2} (\sqrt{N}\Gamma T_2 + \Delta_r T_3) \approx T_2$. Thus, in both cases, the dark states live in the space spanned by $T_2$ while the bright states live in the space spanned by $T_2$ and $T_3$. 
Fourth, the dynamics of two remote spin ensembles coupled by a cavity bus are experimentally studied in [7]. The Hamiltonian of the system is given in [7, (2)], which is of the form of $H_{TC}$ for the Tavis–Cummings model. When $\varphi \approx 48.1^\circ$ in [7, Fig. 2(b)], both spin ensembles resonate with the cavity mode. Thus, this particularly interesting case can be analyzed by Corollary 3.1. Indeed, the eigenstates $|\pm\rangle$ in [7, (3)] and the dark mode $|D\rangle$ in [7, (4)] can be obtained by means of the orthonormal matrix $T$ in Corollary 3.1.

Fifth, in Sections IV-A and IV-B, we study how the Tavis–Cummings model (6) responds to a continuous-mode single-photon input state, where the linear model developed in Section III-B plays an essential role.

Finally, the single-excitation superradiant and subradiant states of the Tavis–Cummings model can be analyzed by means of the quantum linear systems theory presented in Section III-B; see Remark 4.2 in Section IV-C.

IV. SINGLE-EXCITATION CASE

In this section, we investigate the dynamics of the Tavis–Cummings model when there is only one excitation.

A. Response to Single-Photon Inputs

In this section, we derive an analytic expression of the steady-state output field state when the Tavis–Cummings model is initialized in the state $|\zeta\rangle$ given in (7), and driven by a single-photon state.

We start with the following lemma, which discusses the controllability (29, Sec. III-B), [25, Def. 1]) of the passive linear quantum system (12).

**Lemma 4.1:** The passive linear quantum system (12) is controllable if and only if $\omega_j \neq \omega_k$ for all $1 \leq j < k \leq N$.

With the aid of Lemma 4.1, the main result of this section can be derived.

**Theorem 4.1:** Assume that the Tavis–Cummings model (6) is initialized in the state $|\zeta\rangle = |g_1g_2\cdots g_N\rangle \otimes |0\rangle$ and driven by a single-photon input state with pulse shape $\xi$. The steady-state $(t \to \infty$ and $t_0 \to -\infty)$ output field state is a single-photon state with the frequency-domain pulse shape

$$\eta[i\omega] = G[i\omega]\xi[i\omega]$$

where the transfer function $G[s]$ is given by (14).

**Proof:** By system (6) and (9), we have

$$\langle \zeta | \Phi_0 \rangle |_{out}(t) = Ce^{A(t-t_0)} \langle \zeta | \Phi_0 \rangle X(t_0) + \int_{t_0}^{t} Ce^{A(t-\tau)} \langle \zeta | \Phi_0 \rangle b_{in}(\tau) d\tau + \langle \zeta | \Phi_0 \rangle b_{in}(t).$$

If $A$ is Hurwitz stable, then $Ce^{A(t-t_0)} \langle \zeta | \Phi_0 \rangle X(t_0) \to 0$ as $t_0 \to -\infty$. If $A$ is not Hurwitz stable, then by Corollary 3.1, the $c\ell$ subsystem does not affect the input–output behavior, while the $co$ subsystem is Hurwitz stable; see Proposition A.1 in the Appendix. Hence, we also have $Ce^{A(t-t_0)} \langle \zeta | \Phi_0 \rangle X(t_0) \to 0$ as $t_0 \to -\infty$. As a result, sending $t_0 \to -\infty$, we get

$$\langle \zeta | \Phi_0 \rangle |_{out}(t) = \int_{-\infty}^{\infty} g_C(t-r) \langle \zeta | \Phi_0 \rangle b_{in}(r) dr$$

where $g_C(t)$ is the impulse response function associated to the transfer function $G[s]$ in (13). As $\rho_{out}(t) = U^*(t, t_0)b_{in}(t)U(t, t_0)$, we have

$$\langle \zeta | \Phi_0 \rangle b_{in}(t) = \int_{-\infty}^{\infty} g_C(t-r) \langle \zeta | \Phi_0 \rangle b^*_{in}(r) (r, -\infty) dr$$

where $b^*_{in}(t) \triangleq U(t, t_0)b_{in}(t)U^*(t, t_0)$. Then following the stable inverse technique in [20, Lemma 1], we can get

$$b^*_{in}(r, -\infty) |\zeta \rangle_{\Phi_0} = \int_{-\infty}^{\infty} g_{g_{-1}}(r-t) b^*_{in}(t) dt |\zeta \rangle_{\Phi_0}$$

and thus

$$\int_{-\infty}^{\infty} \xi(r)b^*_{in}(r, -\infty) |\zeta \rangle_{\Phi_0} dr = \int_{-\infty}^{\infty} \eta(r)b^*_{in}(t) |\zeta \rangle_{\Phi_0} dt$$

where $\eta(t)$ is the time-domain counterpart of $\eta[i\omega]$ in (24). Consequently, the steady-state joint system-field state in [20, Eq. (85)] is

$$\rho_{\infty} = |\zeta \rangle \langle \zeta | \otimes |1\rangle_{\Phi} \langle 1|_{\Phi}.$$  

The steady-state output field state $\rho_{\text{out}}$ is then obtained by tracing over the initial system state, i.e., $\rho_{\text{out}} = Tr_{\text{sys}}[\rho_{\infty}] = |1\rangle \langle 1|_{\Phi}$, which is a pure state $|1\rangle_{\Phi}$.

B. Simulation Results for $N \leq 4$

In this section, we illustrate Theorem 4.1 by a special case of $N \leq 4$, i.e., at most four two-level atoms are coupled to the cavity. The single-photon input state is supposed to have a rising exponential pulse shape

$$\xi(t) = \left\{ \begin{array}{ll} \sqrt{\gamma} e^{\frac{-t}{\tau}}, & t \leq 0 \\ 0, & t > 0 \end{array} \right.$$
where \( \gamma \) denotes the full width at half maximum (FWHM) of the Lorentzian spectrum. The input and output photon probability distributions are shown in Fig. 2. It is worthwhile to notice that the carrier frequency of the single-photon input field is not shown in (30), the reason is that all the frequencies in the system Hamiltonian \( H_{TC} \) in (5) are detuned from this carrier frequency.

In Fig. 2, it can be observed that the emitted photon is more likely to be found when \( t < 0 \) as the number of atoms increases, which means it interacts with the Jaynes–Cummings system (the \( N = 1 \) case) more easily than with the Tavis–Cummings model. On the other hand, the Rabi oscillation at \( t > 0 \) indicates that the photon can be repeatedly absorbed and emitted by the two-level atoms. The oscillation also becomes stronger when there are more atoms, which indicates that the added atoms increase the time for the photon to escape from the cavity. The Rabi oscillation monotonically decays when all the atoms are resonant with each other, while revivals can be observed in the nonresonant case (the blue dotted curve). Moreover, oscillation sustains much longer in the nonresonant case than in the resonant case. Finally, simulation shows that in the resonant case, the shapes of the input and output pulses are quite close to each other when \( N \) is large, showing that the input single photon hardly interacts with the atomic ensemble. This phenomenon is confirmed by \( \lim_{N \to \infty} G[\omega] = 1 \) for any fixed \( \omega \), where \( G[s] \) is given in (15).

### C. Analytic Form of the Superposition State

In this section, an analytic form of the joint system-field state is derived.

**Theorem 4.2**: Assume that the \( N \)-level atoms of the Tavis–Cummings model are resonant with each other, i.e., \( \omega_1 = \cdots = \omega_N = \omega_s \), the \( k \)-th two-level atom is in the excited state, the others are in the ground state, the cavity is empty and the Tavis–Cummings model is driven by the vacuum input state. That is, the initial joint system-field state is \( |\Psi_k(0)\rangle = |g_1 g_2 \cdots e_k \cdots g_N 0\rangle \otimes |\Phi_0\rangle \). Then the joint system-field state is

\[
|\Psi_k(t)\rangle = c_k(t)|g_1 g_2 \cdots e_k \cdots g_N 0\rangle \\
+ \sum_{j \neq k} c_j(t)|g_1 g_2 \cdots e_j \cdots g_N 0\rangle \\
+ c_{N+1,k}(t)\int_0^t \varphi(\tau) dB_{\omega_s}^\alpha(\tau)|g_1 g_2 \cdots g_N 0\rangle \\
+ c_{N+2,k}(t)|g_1 g_2 \cdots g_N 1\rangle 
\]

where

\[
c_k(t) = \frac{\Gamma_{N-1}}{\sqrt{N}T^2} \left[ 2N \sum_{j \neq k} \Gamma^2_j + \frac{\Gamma_k}{2} \left( \lambda_1 e^{-\frac{j}{2}T} - \lambda_2 e^{\frac{j}{2}T} \right) \right] \\
c_j(t) = -\frac{\Gamma_k \Gamma_j \Gamma_{N-1}}{\sqrt{N}T^2} \left[ 2N \sum_{j \neq k} \Gamma^2_j + \frac{\Gamma_k}{2} \left( \lambda_1 e^{-\frac{j}{2}T} - \lambda_2 e^{\frac{j}{2}T} \right) \right],
\]

\( j \neq k \)

\( c_{N+1,k}(t) = \frac{\Gamma_k}{\sqrt{N}T^2} \left( \lambda_1 e^{-\frac{1}{2}T} - \lambda_2 e^{\frac{1}{2}T} \right) \)

\( c_{N+2,k}(t) = \frac{\Gamma_k}{\sqrt{N}T^2} \left( \lambda_1 e^{-\frac{1}{2}T} - \lambda_2 e^{\frac{1}{2}T} \right) \)

The proof of Theorem 4.2 follows the recursive formula for joint system-field states of general quantum systems; see Remark 6.2 in Section VI.

**Remark 4.1**: By Theorem 4.2, it can be seen that the cavity is eventually empty \( (c_{N+2,k}(\infty) = 0) \), which results in a superposition state of the two-level atoms and the output field. Moreover, by (34), the steady-state excitation probabilities of two-level atoms are independent of the cavity resonant frequency \( \omega_r \) as well as the atomic transition frequency \( \omega_s \).

It is well-known in quantum optics that the collective radiation of an ensemble of two-level atoms can be accelerated by superradiance or inhibited by subradiance [1], [2], [6], the following corollary gives the steady state of the Tavis–Cummings model initialized in a superposition state of the superradiant and subradiant states.

**Corollary 4.1**: Assume that the \( N \)-level atoms in the Tavis–Cummings model are resonant with each other, i.e., \( \omega_1 = \cdots = \omega_N = \omega_s \). Let the system be driven by the vacuum input state and initialized in the following superposition state \( (|B_N\rangle + \beta \ |D_N\rangle) \otimes |0\rangle \), where \( |\alpha|^2 + |\beta|^2 = 1 \), and the single-excitation superradiant state \( |B_N\rangle \) and subradiant state \( |D_N\rangle \) are, respectively,

\[
|B_N\rangle = \frac{1}{\sqrt{N}} \sum_{k=1}^N |g_1 \cdots e_k \cdots g_N\rangle \\
|D_N\rangle = \frac{1}{\sqrt{N}} \sum_{k=1}^N e^{-i\phi_k} |g_1 \cdots e_k \cdots g_N\rangle
\]

(35)
with \( \phi_k = \frac{2\pi}{N} k \). Then, the joint system-field state is

\[
|\Psi'(t)\rangle = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} (\alpha + \beta e^{-i\phi_k}) |\Psi_k(t)\rangle
\]  

(36)

where \( |\Psi_k(t)\rangle \) is given in (31). Moreover, in the steady state (\( t = \infty \)), by ignoring the rotating term \( e^{\frac{N}{2} \Gamma_s t} \) in the coefficients in (32) (equivalently, setting \( \omega_s = 0 \)), the superposition state (36) becomes

\[
|\Psi'(\infty)\rangle = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} (\alpha + \beta e^{-i\phi_k}) |\Psi_k(\infty)\rangle
\]  

(37)

where \( |\Psi_k(\infty)\rangle \) is given in (33).

The proof of Corollary 4.1 is omitted.

Let \( \alpha = 0, \beta = 1 \), i.e., the Tavis–Cummings model is initialized in the pure state \( |D_N(0)\rangle \). By Corollary 4.1, the joint system-field steady state is

\[
|\Psi'(\infty)\rangle = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} c'_{k}(\infty)|g_1 \cdots e_k \cdots g_N 0\Phi_0\rangle
\]  

(38)

where

\[
c'_{k}(\infty) = \frac{e^{-i\phi_k}}{\sqrt{\sum_{j \neq k}^{N} \Gamma_j^2 - \Gamma_k \sum_{j \neq k}^{N} e^{-i\phi_j} \Gamma_j}}
\]  

(39)

provided that \( \omega_s = 0 \). Assume further that the atoms are all equally coupled with the cavity, i.e., \( \Gamma_1 = \Gamma_2 = \cdots = \Gamma_N \), then the steady state (38) reduces to

\[
|\Psi'(\infty)\rangle = |D_N(0)\rangle \otimes |\Phi_0\rangle.
\]  

(40)

In other words, in the steady state, the single excitation only exists in the two-level atoms which have the same excitation probability \( \frac{1}{N} \). The Tavis–Cummings system cannot emit a photon into the cavity or the output field. This theoretical result is consistent with the experimental results given in [6, Fig. 2(c), Fig. 4(a)], where small fluctuations of the collective swapping dynamics are due to the inhomogeneity of the coupling strengths. In fact, when \( \Gamma_1 = \Gamma_2 = \cdots = \Gamma_N \) and \( \omega_s = 0 \), by Corollary 3.1, \( \tilde{A}_{eo} = 0 \), and thus the subsystem \( (\tilde{A}_{eo}, \tilde{B}_{eo}, \tilde{C}_{eo}) \) is static. On the other hand, it is easy to show that \( |\Psi(t)\rangle \equiv |\Psi(0)\rangle = |D_N(0)\rangle \otimes |\Phi_0\rangle \) for all \( t \geq 0 \).

Let \( \alpha = 1, \beta = 0 \), i.e., the Tavis–Cummings model is initialized in the pure state \( |B_N(0)\rangle \). In this case, the steady state is

\[
|\Psi'(\infty)\rangle = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} (c'_{k}(\infty)|g_1 \cdots e_k \cdots g_N 0\Phi_0\rangle
\]  

\[
+ c_{N+1,k}(\infty)|g_1 g_2 \cdots g_N 01\varphi\rangle
\]  

(41)

where the pulse shape \( \varphi \) is given in (32), and \( c'_{k}(\infty) = \sum_{j=1}^{N} \Gamma_j^2 - \Gamma_k \sum_{j=1}^{N} e^{-i\phi_j} \Gamma_j \), provided that \( \omega_s = 0 \), and \( c_{N+1,k}(\infty) \) is given in (34). Assume further that the atoms are all equally coupled with the cavity, i.e., \( \Gamma_1 = \Gamma_2 = \cdots = \Gamma_N \), the steady state (41) reduces to

\[
|\Psi'(\infty)\rangle = |g_1 g_2 \cdots g_N 01\varphi\rangle.
\]  

(42)

In other words, due to the existence of cavity decay rate \( \kappa \), the Tavis–Cummings system eventually emits a photon into the field.

Remark 4.2: The superradiant state \( |B_N\rangle \) and subradiant state \( |D_N\rangle \) can be represented by the eigenvectors given in (21), respectively. For simplicity, we assume the coupling strengths \( \Gamma_1 = \cdots = \Gamma_N = \Gamma \). If the state \( |g_1 \cdots e_k \cdots g_N 0\rangle \) is viewed as a column vector \( \begin{bmatrix} 0 & \cdots & 1 & \cdots & 0 \end{bmatrix}^T \) (the \( k \)-th element is 1 and the others are 0), then \( |B_N(0)\rangle \) is exactly the vector \( T_N(0) \) in (21). Moreover, \( A|B_N(0)\rangle = -i\omega_s T_N - i\sqrt{\Gamma} T_{N+1} \), which lives in the linear span of the two vectors \( T_N \) and \( T_{N+1} \) of the controllable/observable subsystem \( \tilde{A}_{eo} \) given in (23). Hence, the single excitation swaps among all the two-level atoms and the cavity. This echoes the collective swapping dynamics shown in [6, Fig. 2(a)]. Moreover, as shown in (42), eventually all the oscillations will die out as the photon is emitted into the external field due to the lossy nature of the cavity. On the other hand, the subradiant state \( |D_N(0)\rangle \) can be represented by

\[
|D_N(0)\rangle = \sum_{j=1}^{N-1} \alpha_j T_j
\]  

(43)

where \( \alpha_j = -\frac{\sqrt{2j+1}}{2N} \left[ e^{-i\phi_{j+1}} + \frac{1}{\sqrt{j+1}} \sum_{k=j+2}^{N} e^{-i\phi_k} \right] \), for \( j = 1, 2, \ldots, N-2 \), and \( \alpha_{N-1} = -\frac{1}{\sqrt{N}} e^{-i\phi_N} \). Therefore, the subradiant state given in (43) is a linear combination of the \( N-1 \) eigenvectors of the DFS given in (23). In this case, the Tavis–Cummings model is neither reachable by its input nor detectable by its output. Thus, the two-level atoms are decoupled from the cavity mode and the external field, and the Tavis–Cummings model initialized in \( |D_N(0)\rangle \) cannot emit a photon into the cavity or even the external field, and remains in the subradiant state (40).

V. MULTIEXCITATION CASE

In Section IV, we studied the single-excitation dynamics of the Tavis–Cummings model. In this section, we present numerical studies of the multixcitation scenario. For ease of representation, we calculate the excitation probabilities of the first two-level atom.

A. Reduced Density Matrix of the First Two-Level Atom

In this section, we present the master equation for the Tavis–Cummings model; we also give the expression of the reduced density of the first two-level atom.

According to [19], the master equation of the Tavis–Cummings model driven by a single-photon input \( |\zeta\rangle \) is

\[
\dot{\rho}^{11}(t) = \mathcal{L}^{*} \rho^{11}(t) + \zeta(t) [\rho^{01}(t), L^{*}] + \xi^{*}(t) [L, \rho^{10}(t)]
\]  

(44)

where the initial states are

\[
\rho^{11}(t_0) = \rho^{00}(t_0) = |\zeta\rangle \langle \zeta|, \quad \rho^{10}(t_0) = \rho^{01}(t_0) = 0
\]  

(45)
Excitation probabilities of the first two-level atom. When the input is in the vacuum state, which is similar to the case when $\Phi = 2\omega $ and $N = \frac{\Gamma}{\omega} = \frac{\Phi}{\omega}$. We use the master equation (47) to prove this result.

Remark 5.1: When the input is in the vacuum state $|\Phi_0\rangle$, (44)-(45) reduce to the commonly used master equation

$$\dot{\rho}^{00}(t) = L^\dagger \rho^{00}(t), \quad \rho^{00}(t_0) = |\zeta\rangle \langle \zeta|, \quad t \geq t_0. \quad (47)$$

Accordingly, the excitation probability of the first two-level atom is given by $\langle e_1^\dagger \rho_A(t) e_1 \rangle$, where $\rho_A(t)$ should be computed via (46) by replacing $\rho_1^{11}(t)$ with $\rho^{00}(t)$.

In the following simulations, we assume that there are three resonant two-level atoms that are equally coupled to the cavity, i.e., $\omega_1 = \omega_2 = \omega_3 = \omega_r = 1$ and $\Gamma_1 = \Gamma_2 = \Gamma_3 = 1$. Also, let $\kappa = 1.5$. When the input field is in the single photon state $|\Phi_1\rangle$, we use a Gaussian pulse shape

$$\xi(t) = \left(\frac{\Omega^2}{2\pi}\right)^{1/4} \exp\left[-\frac{\Omega^2}{4}(t - t_p)^2\right] \quad (48)$$

where $t_p$ is the peak arrival time of the photon, and $\Omega$ is the frequency bandwidth. Set $t_p = 3$ and $\Omega = 2\kappa$. See the green solid curve in Fig. 3 for the plot of $|\xi(t)|^2$. As $|\xi(t)|^2 \approx 0$ when $t = 0$, it is safe to let the initial time $t_0$ be 0.

B. One Initially Excited Atom Plus a Single-Photon Input

Assume that one of the two-level atoms is in the excited state, the cavity is empty, and the system is driven by a single-photon input. Then there are two excitations in the whole system. According to the reduced density matrix (46), the excitation probability of the first two-level atom in this two-excitation case can be calculated as

$$P_{TLS1}(t) = \langle e_1^\dagger \rho_A(t) e_1 \rangle = \langle e_1 g_2 g_3 \rangle \rho^{11}(t) e_1 g_2 g_3 \rangle$$

and

$$+ \langle e_1^\dagger e_2 g_2 \rangle \rho^{11}(t) e_1^\dagger e_2 g_2 \rangle$$

where the term $\langle e_1 g_2 g_3 \rangle \rho^{11}(t) e_1 g_2 g_3 \rangle$ indicates that a photon is in the external field.

We consider the effect of single-photon input state on the excitation probability, which is simulated in Fig. 3. The solid curve (TLS1: $|\psi_1\rangle = |e_1 g_2 g_3 \rangle \otimes |\Phi_0\rangle$); red solid curve (TLS3: $|\psi_2\rangle = |g_1 e_2 g_2 \rangle \otimes |\Phi_0\rangle$); red dashed curve (TLS1+TLS2: $|\psi_3\rangle = |e_1 g_2 g_3 \rangle \otimes |\Phi_1\rangle$); blue dashed curve (TLS2/3+TLS2: $|\psi_4\rangle = |g_1 e_2 g_2 \rangle \otimes |\Phi_1\rangle$).

Fig. 3. Excitation probabilities of the first two-level atom. Green solid curve: $|\xi(t)|^2$; red solid curve (TLS1: $|e_1 g_2 g_3 \rangle \otimes |\Phi_0\rangle$); blue solid curve (TLS3/2: $|g_1 e_2 g_2 \rangle \otimes |\Phi_0\rangle$ or $|g_1 e_2 g_2 \rangle \otimes |\Phi_0\rangle$); red dashed curve (TLS1+TLS2: $|e_1 g_2 g_3 \rangle \otimes |\Phi_1\rangle$); blue dashed curve (TLS2/3+TLS2: $|g_1 e_2 g_2 \rangle \otimes |\Phi_1\rangle$).

with $|\zeta\rangle$ being the initial system state. As we will focus on the excitation probability of the first two-level atom, we use the partial trace to get its reduced density operator

$$\rho_A(t) = Tr_{\text{cav}}[\rho(t)] = \sum \langle z_2 z_3 \ldots z_n n | \rho^{11}(t) | z_2 z_3 \ldots z_n n \rangle \quad (46)$$

where $|z_j\rangle = |g_j\rangle$ or $|e_j\rangle$, $j = 2, 3, \ldots, N$, and $|n\rangle$ is the state of the cavity.

C. Two Initially Excited Atoms Plus Vacuum Input

In the following simulations, we assume that the Tavis–Cummings system is driven by the vacuum input $|\Phi_0\rangle$ and one, two, or even three atoms are initially excited.

We present the following theoretical result first:

Theorem 5.1: Suppose that the Tavis–Cummings system (6) is initialized in the state $|\zeta\rangle = |e_1 \ldots e_N\rangle$ and driven by the vacuum input field. If $\omega_1 = \ldots = \omega_N$ and $\Gamma_1 = \ldots = \Gamma_N$, then the steady-state output field must be in an $N$-photon state.

Proof: We use the master equation (47) to prove this result.

The steady-state system state can be found by solving

$$L^\dagger \rho^{00}(\infty) = 0. \quad (48)$$

Clearly, in the steady state ($t = \infty$), the cavity cannot contain any photon; otherwise the photon leaks out from the lossy cavity ($\kappa \neq 0$). In other words, such a state cannot be a steady state. Consequently, the steady-state system state is of the form $\rho^{00}(\infty) = \rho_A \otimes |0\rangle \langle 0|$, where $\rho_A$ is the steady state of the two-level atoms. By the form of the system Hamiltonian $H_{\text{TC}}$ and the coupling operator $L$ given in Section III-A, (49) is actually

$$\frac{1}{2} \sum \omega_j \langle \sigma_{z,j} | \rho_A | 0 \rangle \langle 0 | + \sum \Gamma_j \left( \sigma_{-j} \rho_A \otimes |1\rangle \langle 0| - \rho_A \sigma_{+j} \otimes |0\rangle \langle 1| \right) = 0.$$
Given \( \omega_1 = \ldots = \omega_N \equiv \omega \) and \( \Gamma_1 = \ldots = \Gamma_N \equiv \Gamma \), we have

\[
\frac{\omega}{2} \sum_j \sigma_{z,j} \rho_A \otimes \langle 0 | 0 \rangle \\
+ \Gamma \sum_j \left( \sigma_{-j} \rho_A \otimes | 1 \rangle \langle 1 | - \rho_A \sigma_{-j} \rho_A \otimes | 1 \rangle \langle 1 | \right) = 0
\]

which yields

\[
\sum_j \sigma_{-j} \rho_A = 0. \tag{50}
\]

In general, \( \rho_A \) is of the form

\[
\sum_{i_1, \ldots, N; j_1, \ldots, j_N} \alpha_{i_1, \ldots, N; j_1, \ldots, j_N} \langle f_{i_1} \ldots f_{i_N} | g_{j_1} \ldots g_{j_N} | \rangle
\]

where \( f_{i_k} \) and \( g_{j_k} \) are either \( e_k \) or \( g_k \). Because \( \omega_1 = \ldots = \omega_N \) and \( \Gamma_1 = \ldots = \Gamma_N \), all the atoms are indistinguishable. As a result, all the coefficients must be identical; i.e.,

\[
\alpha_{i_1, \ldots, N; j_1, \ldots, j_N} = \alpha \text{ for some } \alpha. \quad \text{Consequently}
\]

\[
\rho_A = \alpha \sum_{i_1, \ldots, N; j_1, \ldots, j_N} \langle f_{i_1} \ldots f_{i_N} | g_{j_1} \ldots g_{j_N} | \rangle.
\]

Substituting this form of \( \rho_A \) into (50) we get \( \sigma_{-j} | f_{i_1} \ldots f_{i_N} \rangle = 0 \), for all \( j = 1, \ldots, m \). Clearly, all \( f_{i_k} \) must be \( g_k \). Consequently, \( \rho_A \) contains a single term and is of the form \( \rho_A = | g_1 \ldots g_N \rangle \langle g_1 \ldots g_N | \). In other words, all atoms are in their ground state. Therefore, in the steady state, the output field is in an \( N \)-photon state.

The simulation results are shown in Fig. 4. We have the following observations. 1) The first atom’s excitation probability when it is initialized in the excited state (the red solid curve) is greater than that when either the second atom or the third atom is initialized in the excited state (the blue solid curve). 2) The first atom’s excitation probability when both the second and third atoms are initialized in the excited state (the red dashed curve) is greater than that when both the first and second atoms (or both the first and third atoms) are initialized in the excited state (the blue dashed curve). 3) The red solid and dashed curves have the same final value \( \approx 0.44 \), while the blue solid and dashed curves have the same final value \( \approx 0.11 \). 4) The excitation probability of the first two-level atom settles to 0 when all the three atoms are initially excited, see the green solid curve in Fig. 4. Actually, by Theorem 5.1, all the three atoms eventually settle in their ground state and a 3-photon output state is generated. However, if the coupling strengths are not identical, the atoms may not settle to their ground state, as demonstrated by the green dashed curve in Fig. 4, where \( \Gamma_1 = 1, \Gamma_2 = 1.5, \) and \( \Gamma_3 = 2 \).

Remark 5.2: In Figs. 3 and 4, when the first atom is initialized in the excited state and the other two are in the ground state, the final value of the excitation probability of the first atom is approximately 0.44. By Theorem 4.2, \( |c_1(\infty)|^2 = \frac{4}{9} \), which explains this simulation result. The other final value 0.11 in Figs. 3 and 4 can be explained by \( |c_2(\infty)|^2 = |c_3(\infty)| = \frac{1}{9} \). Moreover, \( |c_k(t)|^2 \to 1 \) and \( |c_j(t)|^2 \to 0 \) (\( j \neq k \)) as \( N \to \infty \). This means that the system tends to remain intact when the number of atoms is sufficiently large. Finally, if \( \Gamma_1 = \ldots = \Gamma_N \), then by (34), we have \( |c_{N+1,k}(\infty)|^2 = \frac{1}{8} \), i.e., only \( 1/N \) of the excitation energy is radiated. This confirms the analysis in the second paragraph above [1, (29)].

Remark 5.3: Although the red solid and dashed curves have identical stationary excitation probability, their crests and troughs are almost symmetrical during the transient process. Similar phenomena can be observed when \( N = 2 \). This is consistent with the experimental result given in [4, Fig. 4(c)]. Specifically, write states \( |e_1, g_2 \rangle \) and \( |g_1, e_2 \rangle \) as \( |\uparrow \rangle \) and \( |\downarrow \rangle \), respectively. In [4, Fig. 4(c)], the solid red (green) circles plot the evolution of \( |\uparrow \rangle \) (\( |\downarrow \rangle \)). The coherent state transfer of the two states emerges and oscillates symmetrically.

VI. GENERAL FORM OF THE JOINT SYSTEM-FIELD STATES

In Section IV, analytical results are presented for the single-excitation Tavis–Cummings model. These results are not applicable to the multiexcitation case discussed in Section V. Motivated by this, in this section, we derive a recursive relation for the computation of the joint system-field state when the system initially contains multiple excitations \( R > 1 \) and is driven by \( m \) input fields initialized in the vacuum state.

A. Modeling

Let \( H \) be the internal system Hamiltonian of a quantum system. As the system Hamiltonian may be tuned by a time-varying classical signal, for example, transition frequencies of superconducting qubits can be tuned via Josephson energy in Josephson-junction-based superconducting circuits [32], we use \( H(t) \) to emphasize the explicit dependence of the system Hamiltonian on time. The coupling between the quantum system and the \( k \)th (\( k = 1, \ldots, m \)) input channel can be described by a coupling operator \( L_k(t) \); again, here we allow explicit time dependence of the coupling as in some quantum systems couplings can be tuned in real time [33].

The system of interest may be an ensemble of TLSs, or resonators, or even an ensemble of TLSs residing in a resonator like the Tavis–Cummings model studied in this paper. If the number of TLSs is finite, the number of photons in the resonator...
is finite, and the system is driven by a finite number of photons, then the number of total excitations \( R \) in the whole system (the system plus the external field) is finite too. In this case, the state of the quantum system has an orthonormal basis of the form
\[
\{ |0_s\rangle, \ldots, |(K-1)_s\rangle \}
\]  
(51)
where the subscript “s” indicates that the basis states \(| j_s \rangle\), \( j = 0, 1, \ldots, K-1 \), are for the system. For our Tavis–Cummings mode in Fig. 1, \( K = 2^{N}(R+1) \) as each of the \( N \) atoms has two basis states and the cavity can contain at most \( R \) photons.

**Remark 6.1:** If the system of interest is coherently driven, then the number of excitations can be arbitrary large as the drive may generate an arbitrary number of excitations. However, if the coherent drive is not very strong and lasts not long, the number of excitations in the system will be upper bounded. Hence, it can still be assumed that the system admits a basis of finitely many elements.

For notational convenience, the following notation will be adopted. For each \( i = 1, \ldots, m \), we use \( t_{1-k_i}^i \) to denote a set of ordered real numbers \( \{ t_{1-k_i}^1, \ldots, t_{1-k_i}^i \} \). Similarly, \( t_{1-k_i}^j \) is a shorthand of \( t_{1-k_i}^1, \ldots, t_{1-k_i}^j \). Let \( j_{1-k_i} \) be the abbreviation for \( t_{1-k_i}^1, \ldots, t_{1-k_i}^j \). Finally, for each given non-negative integer \( n \), \( \sum_{k_1, \ldots, k_m} \) means the summation over all possible combinations of the non-negative integers \( k_1, \ldots, k_m \) that satisfy \( \sum_{j=1}^m k_j = n \).

The temporal photon-number basis of \( n \) photons superposed over \( m \) channels is
\[
\left\{ |1_{t_1^1} \rangle \otimes \cdots \otimes |1_{t_{1-k_1}^{i_1}} \rangle \otimes \cdots \otimes |1_{t_1^m} \rangle \otimes \cdots \otimes |1_{t_{1-k_m}^m} \rangle : \sum_{j=1}^m t_{1-k_j}^j = n \right\}
\]  
(52)
where \( k_1, \ldots, k_m \) are non-negative integers. Particularly, if \( k_i = 0 \), then there are no photons in channel \( i \) and correspondingly the term \( |1_{t_1^1} \rangle \otimes \cdots \otimes |1_{t_{1-k_1}^{i_1}} \rangle \) reduces to \( |\Phi_0 \rangle \). As a common practice in quantum physics, the tensor product state in (52) is often written as \(|1_{t_1^1, \ldots, t_{1-k_1}^{i_1}, \ldots, t_{1-k_m}^m, \ldots, t_{1-k_m}^m} \rangle \) by means of the notation given in the last paragraph.

Using the temporal photon-number basis in (52), an \( m \)-channel \( n \)-photon state can be written as
\[
|\eta_{\xi} \rangle = \sum_{k_1, \ldots, k_m} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \xi(t_{1-k_1}^1, \ldots, t_{1-k_m}^m) |1_{t_{1-k_1}^1} \rangle \cdots |1_{t_{1-k_m}^m} \rangle \]  
(53)
where the \( j \)th channel has \( k_j \) photons, and \( \sum_{j=1}^m k_j = n \) is the total photon number. Clearly, \( \xi(t_{1-k_1}^1, \ldots, t_{1-k_m}^m) \) is the probability amplitude of the component that the \( j \)th channel has photons at time instants \( t_{1-k_1}^j, \ldots, t_{1-k_m}^j \) for all \( j = 1, \ldots, m \).

Due to the indistinguishability of photons in each channel, for each fixed \( j = 1, \ldots, m \), \( \xi(t_{1-k_1}^j, \ldots, t_{1-k_m}^m) \) is permutation-invariant with respect to indices \( \{ t_{1-k_j}^j \} \). Thus, under scaling \( \frac{1}{k_{1-k_1}^1 \cdots k_{1-k_m}^m} \), the state \(|\eta_{\xi} \rangle \) can be rewritten as
\[
|n_{\xi} \rangle = \sum_{k_1, \ldots, k_m} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \xi(t_{1-k_1}^1, \ldots, t_{1-k_m}^m) |1_{t_{1-k_1}^1} \rangle \cdots |1_{t_{1-k_m}^m} \rangle dt_{1-k_1}^1 \cdots dt_{1-k_m}^m.
\]  
(54)
In fact, one can always define an \( n \)-photon state using (54) with an arbitrary multivariate function \( \xi(t_{1-k_1}^1, \ldots, t_{1-k_m}^m) \) (under normalization), then obtain the form in (53) by permuting indices \( \{ t_{1-k_j}^j \} \) for all \( j = 1, \ldots, m \). The form of \( n \)-photon states (54) is more often to see in practice than that in (53).

For example, by applying a coherent drive to a superconducting qubit embedded in a chiral waveguide, the qubit may generate photon states of the form (54). Therefore, in what follows we use (54) to describe \( n \)-photon states.

### B. General Form

In this section, we present a computational procedure that can be used to compute the joint system-field state. Some other computational framework can be found in, e.g., [35, 36].

The basis state of the joint system-field state when the system is at level \( |j_s \rangle \) and channel \( i \) has \( k_i \) photons at time instants \( t_{1-k_i}^i \), is \(|j_s \rangle \otimes |1_{t_{1-k_1}^{i_1}, \ldots, t_{1-k_m}^m} \rangle \). Hence, the joint system-field state is of the general form
\[
|\Psi \rangle = \sum_{j=0}^{K-1} \sum_{n=0}^{\infty} \sum_{k_1, \ldots, k_m} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \xi_j(t_{1-k_1}^1, \ldots, t_{1-k_m}^m) |j_s \rangle
\]  
(55)
where we write informally \( dB_{m,n}^j(t) = b_{m,n}^j(t)dt \) and \( dB_{m,n}^j(t_{1-k_1}^i) \) as the product \( dB_{m,n}^j(t_{1-k_1}) \cdots dB_{m,n}^j(t_{1-k_m}) \). The normalization condition \( \langle \Psi | \Psi \rangle = 1 \) gives
\[
|\xi_j(t_{1-k_1}^1, \ldots, t_{1-k_m}^m)|^2 \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dt_{1-k_1}^1 \cdots dt_{1-k_m}^m = 1.
\]

Denote
\[
|\eta(t_{1-k_1}^1, \ldots, t_{1-k_m}^m) \rangle = \sum_{j=0}^{K-1} \xi_j(t_{1-k_1}^1, \ldots, t_{1-k_m}^m) |j_s \rangle.
\]
It is worthwhile to note that \(|\eta(t_{1-k_1}^1, \ldots, t_{1-k_m}^m) \rangle \) in general is not normalized. The state \(|\Psi \rangle \) in (55) can be rewritten as
\[
|\Psi \rangle = \sum_{n=0}^{\infty} \sum_{k_1, \ldots, k_m} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dB_{m,n}^j(t_{1-k_1}^1, \ldots, t_{1-k_m}^m) |\eta(t_{1-k_1}^1, \ldots, t_{1-k_m}^m) \rangle
\]  
(56)
If the input field is initially in the vacuum state \(|\Phi_0\rangle\), then the initial joint system-field state is \(|\Psi(t_0)\rangle = \sum_{j=0}^{K-1} \xi_j \, |j_s\rangle \otimes |\Phi_0\rangle \equiv |\eta_0\rangle \otimes |\Phi_0\rangle\), where \(\sum_{j=0}^{K-1} |\xi_j|^2 = 1\). Because the photon generation time is from the initial time \(t_0 = 0\) to the present time \(t > 0\), the joint system-field state at time \(t\) is
\[
|\Psi(t)\rangle = \sum_{n=0}^{\infty} \sum_{k_1, \ldots, k_m} \int_0^t \cdots \int_0^t \int_0^{\sum_{n=0}^{\infty} \sum_{k_1, \ldots, k_m}} dB_{m,n}^s(t_{1-k_1}, \ldots, t_{m-k_m}) |\Phi_0\rangle
\]
where
\[
|\eta(t_{1-k_1}, \ldots, t_{m-k_m})\rangle = \sum_{j=0}^{K-1} \xi_j \, |j_s\rangle \otimes |\Phi_0\rangle.
\]
(56)

(Here, the subscript “\(t\)” indicates that the coefficients are time dependent.) In particular, the term in \(|\Psi(t)\rangle\) that corresponds to \(n = 0\) is \(|\eta(t)\rangle \otimes |\Phi_0\rangle = \sum_{j=0}^{K-1} \xi_j \, |j_s\rangle \otimes |\Phi_0\rangle\). In other words, the field is in the vacuum state and the system is in the superposition state \(\sum_{j=0}^{K-1} \xi_j |j_s\rangle \otimes |\Phi_0\rangle\).

In what follows, we aim to derive formulas for computing the joint system-field state \(|\Psi(t)\rangle\). Differentiating both sides of (56) yields
\[
d|\Psi(t)\rangle = \sum_{n=1}^{\infty} \sum_{k_1, \ldots, k_m} \int_0^t \cdots \int_0^t \int_0^t dB_{m,n}^s(t_{1-k_1}, \ldots, t_{m-k_m}) |\Phi_0\rangle
\]
(57)

\[
\frac{d|\eta(t_{1-k_1}, \ldots, t_{m-k_m})\rangle}{dt} = \sum_{j=0}^{K-1} \xi_j \, \hat{\eta}_j |j_s\rangle \otimes |\Phi_0\rangle
\]
and
\[
|\eta(t_{1-k_1}, \ldots, t_{m-k_m})\rangle = \int_0^t dB_{m,n}^s(t_{1-k_1}, \ldots, t_{m-k_m}) |\Phi_0\rangle
\]
(60)

for all \(i = 1, \ldots, m\). Under the basis (51), the effective Hamiltonian \(H_{eff}\) of the system has a matrix representation. Hence, similar to what has been done in [12], define the propagator \(V(t)\) (a matrix function) that solves a system of deterministic homogeneous ODEs \(\dot{V}(t) = -iH_{eff}(t)V(t)\) under the initial condition \(V(0) = I\). Then, one can define the transition matrix
\[
G(t, \tau) \equiv V(t)V(\tau)^{-1}, \quad t, \tau \geq 0.
\]
(61)

Clearly, for the 0 photon state case, (59) and (61) yield
\[
|\eta(t)\rangle = G(t, 0) |\eta_0\rangle, \quad t \geq 0.
\]
(62)

As is well-known in linear systems theory, see e.g. [37, Ch. 4], iteratively using the transition matrix \(G(t, \tau)\) in (61) and (60), we have
\[
|\eta(t_{1-k_1}, t_{2-k_2}, \ldots, t_{m-k_m})\rangle
\]
(63)

\[
\text{Remark 6.2: Theorem 4.2 in Section IV-C can be proved by applying the recursive relation (63) to the special case when } m = 1 \text{ and the initial system-field state is}
\]
\[
|\Psi(0)\rangle = |\Psi_k(0)\rangle = |\eta_0\rangle \otimes |\Phi_0\rangle.
\]
(64)
where $|\eta_0\rangle = |g_1 \cdots e_k \cdots g_N 0\rangle$. In what follows, we sketch the proof. Because there is only one input channel and the number of excitation is 1, i.e., $m = 1$ and $k_1 = 1$, the joint system-field state $|\Psi(t)\rangle$ in (65) is

$$|\Psi(t)\rangle = |\eta_t\rangle \otimes |\Phi_0\rangle + \int_0^t |\eta(t_1)\rangle dB^*_m(t_1) \otimes |\Phi_0\rangle. \quad (65)$$

The first term in (65) indicates that the single excitation exists in the Tavis–Cummings model. In this case, (60) and (62). Theorem 4.2 follows. 

$$\eta_t = \sum_{j=1}^{N} c_j(t)|g_1 \cdots e_j \cdots g_N 0\rangle + c_{N+2,k}(t)|g_1 g_2 \cdots g_N 1\rangle$$

with the initial conditions $c_0(0) = 1$ and $c_j(0) = c_{N+2,k}(0) = 0$ for $1 \leq j \leq N, j \neq k$. On the other hand, the second term in (65) means that the Tavis–Cummings model emits a photon into the output field, which can be rewritten as

$$\int_0^t |\eta(t_1)\rangle dB^*_m(t_1) \otimes |\Phi_0\rangle = c_{N+1}(t) \int_0^t \varphi(t_1) dB^*_m(t_1) |g_1 g_2 \cdots g_N 0\rangle \otimes |\Phi_0\rangle.$$

Here, $\varphi(t_1)$ is the pulse shape of the single-photon output state. When $t \to \infty$, $c_{N+1}(\infty)$ denotes the probability amplitude corresponding to the reduced joint state $|g_1 g_2 \cdots g_N 0\rangle \otimes |\Phi_1\rangle$. By the recursive relation (63), we have $|\eta_1(t)\rangle = G(t, t_1)|\eta(t_1)\rangle$, where $|\eta(t)\rangle$ can be calculated via (62) and (64). Theorem 4.2 follows.

In the following sections, we apply the above theorem to our Tavis–Cummings model. In this case, $m = 1$, $L_s(t)$ in (60) is $L = \sqrt{\kappa_0}$, and $H_{\text{eff}}(t)$ in (59) is $H_{\text{T-C}} = \frac{1}{2} L^* L$, where $H_{\text{T-C}}$ is given in (5). For simplicity, we set $\omega_1 = \cdots = \omega_N = 0$ and $\Gamma_1 = \cdots = \Gamma_N = \kappa_1 = 1$. Assume that the Tavis–Cummings model contains $R$ initially excited two-level atoms, the cavity is initially empty, and the input is in the vacuum state. In such setting, the number of basis states of the Tavis–Cummings model is $K = 2^N(R + 1)$ as discussed in Section VI-A. Finally, for better illustration we plot the symmetric pulse shape in (53), instead of that in (54).

### C. $N = 3$ and $R = 1$

In this case, the total number of basis states is $K = 16$. The joint system-field state can be expressed as $|\Psi(t)\rangle = |\eta_t\rangle \otimes |\Phi_0\rangle + \int_0^t |\eta(t_1)\rangle dB^*_m(t_1) |\Phi_0\rangle$, where by (57), $|\eta_1(t), \ldots, t_k\rangle = \sum_{j=0}^{15} E^t_j(t_1, \ldots, t_k) |j_s\rangle, k = 0, 1$. Let the initial joint system-field state be $|\Psi(0)\rangle = |e_1 g_2 g_3 0\rangle \otimes |\Phi_0\rangle$. By the recursive relation (63), we have

$$\xi^1_t = \frac{2}{\sqrt{3}} \left( e^{- \frac{i}{4} \pi t} - e^{+ \frac{i}{4} \pi t} \right)$$

$$\xi^2_t = \frac{1}{3} \left( \frac{\sqrt{2} e^{+ \frac{i}{4} \pi t} + \sqrt{2} e^{- \frac{i}{4} \pi t}}{2} \right)$$

and all the others are 0. In the steady state $(t = \infty)$, we have $\xi^1_\infty = 1$, which correspond to the basis vectors $|g_1 g_2 e_3 0\rangle, |g_1 g_2 e_3 0\rangle, |e_1 e_2 g_3 0\rangle$, respectively. Moreover, as $\lim_{t \to \infty} \int_0^t \xi^2_t(t) dt = \frac{2}{\sqrt{3}}$, the steady-state joint system-field state is

$$|\Psi(\infty)\rangle = \frac{2}{3} |e_1 g_2 g_3 0\rangle \otimes |\Phi_0\rangle - \frac{1}{3} |g_1 e_2 g_3 0\rangle \otimes |\Phi_0\rangle$$

$$- \frac{1}{3} |g_1 e_2 g_3 0\rangle \otimes |\Phi_0\rangle + \frac{1}{\sqrt{3}} |g_1 g_2 e_3 0\rangle \otimes |1_\eta\rangle. \quad (66)$$

where the pulse shape of the single photon is $\eta(t) = \sqrt{3} \xi^1_\kappa(t)$ up to a global phase. Finally, the square of the probability amplitude $2/3$ in (66) is consistent with the limiting value of the red solid curve in Fig. 4.

### D. $N = 2$ and $R = 2$

In this case, the total number of basis states is $K = 12$. The joint system-field state can be expressed as

$$|\Psi(t)\rangle = |\eta_t\rangle \otimes |\Phi_0\rangle + \int_0^t |\eta(t_1)\rangle dB^*_m(t_1) |\Phi_0\rangle$$

$$+ \int_0^t \int_0^{t_2} |\eta(t_1, t_2)\rangle dB^*_m(t_1) dB^*_m(t_2) |\Phi_0\rangle$$

where $|\eta_1(t_1, \ldots, t_k)\rangle = \sum_{j=0}^{11} E^t_j(t_1, \ldots, t_k) |j_s\rangle, k = 0, 1, 2$. Let the initial joint system-field state be $|\Psi(0)\rangle = |e_1 e_2 0\rangle \otimes |\Phi_0\rangle$. By Theorem 5.1, the steady-state joint system-field state $(t \to \infty)$ is

$$|\Psi(\infty)\rangle = |g_1 g_2 0\rangle \otimes |2_\eta\rangle. \quad (67)$$
Fig. 6. Comparison between probability distributions of the steady-state single-photon output states $|1_{\eta}\rangle$ in (66) and $|1_{\eta,1,2}\rangle$ in (68).

where $|2_{\eta}\rangle$ is the 2-photon output state. By the recursive relation (63), the pulse shape of the 2-photon state $|2_{\eta}\rangle$ is

$$
\eta(t_1, t_2) = \xi_0^{2\eta}(t_1, t_2) = (0.0228787 - 0.007854411)i e^{\mu_1 t_1 - \mu_4 t_2} + (0.319599 + 0.0291387)i e^{\mu_2 t_1 - \mu_4 t_2} - (0.342477 + 0.0212843)i e^{\mu_3 t_1 - \mu_4 t_2} + \text{c.c.},
$$

where “c.c.” means complex conjugate, and $\mu_1 = -0.336506 + 3.79453i$, $\mu_2 = -0.336506 - 1.01065i$, $\mu_3 = -0.076897 + 1.39194i$, and $\mu_4 = 0.25 + 1.39194i$. It can be easily verified that $\int_0^\infty \int_0^\infty |\eta(t_1, t_2)|^2 dt_1 dt_2 = 1$. The probability distribution of the steady-state output two-photon state $|2_{\eta}\rangle$ in (67) is simulated in Fig. 5.

$E. N = 3$ and $R = 2$.

In this case, the total number of basis states is $K = 24$. The joint system-field state can be expressed as

$$
|\Psi(t)\rangle = |\eta_t\rangle \otimes |\Phi_0\rangle + \int_0^t |\eta(t_1)| dB_{in}^\mu(t_1) |\Phi_0\rangle \\
+ \int_0^t \int_0^t |\eta(t_1, t_2)| dB_{in}^\mu(t_1) dB_{in}^\mu(t_2) |\Phi_0\rangle
$$

where $|\eta(t_1, \ldots, t_k)\rangle = \sum_{j=0}^{2^k} \xi_j^{2\eta}(t_1, \ldots, t_k) |j\rangle$, $k = 0, 1, 2$. Let the initial joint system-field state be $|\Psi(0)\rangle = |g_1 g_2 g_3 0\rangle \otimes |\Phi_0\rangle$. According to the recursive relation (63), we have the steady-state joint system-field state

$$
|\Psi(\infty)\rangle = \frac{2}{3} |g_1 g_2 g_3 0\rangle \otimes |1_{\eta,1,2}\rangle + \frac{1}{3} |g_1 g_2 e_3 0\rangle \otimes |1_{\eta,1}\rangle + \frac{1}{3} |g_1 e_2 g_3 0\rangle \otimes |1_{\eta,1}\rangle + \frac{1}{\sqrt{3}} |g_1 g_2 e_3 0\rangle \otimes |0_{\eta,1}\rangle + \frac{1}{\sqrt{3}} |g_1 e_2 g_3 0\rangle \otimes |2_{\eta,1}\rangle + \frac{1}{\sqrt{3}} |g_1 e_2 e_3 0\rangle \otimes |2_{\eta,1}\rangle
$$

(68)

where $\eta^{12\eta}(t) = \frac{2}{\sqrt{3}} \xi_2^{12\eta}(t) = 0.5163975 (e^{\lambda' t} - e^{-\lambda' t})$ with $\lambda' = -0.25 + 0.968246i$, $\eta^0\eta(t) = \eta^3\eta(t) = -\eta^{12\eta}(t)$, and $\color{red}{\eta^{0\eta}(t_1, t_2) = \frac{\sqrt{3}}{3} \xi_2^{0\eta}(t_1, t_2)}$

$\color{red}{= (0.338571 + 0.0140041i) e^{\lambda_1 t_1 - \lambda_4 t_2}}$

$\color{red}{+ (0.0155773 - 0.00341903i) e^{\lambda_2 t_1 - \lambda_4 t_2}}$

$\color{red}{- (0.354149 + 0.0105851i) e^{\lambda_3 t_1 - \lambda_4 t_2} + \text{c.c.}}$.

with $\lambda_1 = -0.301227 + 1.40985i$, $\lambda_2 = -0.301227 + 4.83767i$, $\lambda_3 = -0.147546 + 1.71391i$, and $\lambda_4 = 0.25 + 1.71391i$. The square of the probability amplitude $\frac{2}{3}$ in (68) is consistent with the limiting value of the red dashed curve in Fig. 4. It is clear in Fig. 6 that $|1_{\eta}\rangle$ has more oscillations than $|1_{\eta,1,2}\rangle$.

Both Figs. 5 and 7 show the probability distributions of a two-photon output state. The one generated by the Tavis–Cummings system with three atoms (shown in Fig. 7) oscillates more rapidly than that with two atoms (see Fig. 5). This result indicates that the Rabi oscillation is enhanced by adding more atoms, which is consistent with the discussions in Fig. 2 and the observations in [38, Fig. 5].

$F. N = 3$ and $R = 3$.

In this case, the total number of basis states is $K = 32$ and the joint system-field state can be expressed as

$$
|\Psi(t)\rangle = |\eta_t\rangle \otimes |\Phi_0\rangle + \int_0^t |\eta(t_1)| dB_{in}^\mu(t_1) |\Phi_0\rangle \\
+ \int_0^t \int_0^t |\eta(t_1, t_2)| dB_{in}^\mu(t_1) dB_{in}^\mu(t_2) |\Phi_0\rangle \\
+ \int_0^t \int_0^t \int_0^t |\eta(t_1, t_2, t_3)| dB_{in}^\mu(t_1) dB_{in}^\mu(t_2) dB_{in}^\mu(t_3) |\Phi_0\rangle
$$

$$
+ dB_{in}^\mu(t_4) |\Phi_0\rangle
$$

Fig. 7. Probability distribution of the two-photon output state $|2_{\eta,1,2}\rangle$ in (68).

Fig. 8. Probability distribution of the three-photon output state $|3_{\eta}\rangle$ in (69).
where \( |\psi(t_1, \ldots, t_k)\rangle = \sum_{j=0}^{31} \zeta_j^k |j_1, \ldots, j_k\rangle |j_s\rangle \), \( k = 0, 1, 2, 3 \). Since the three atoms are all initially excited, the initial joint system-field state is \( |\Psi(0)\rangle = |e_1 e_2 e_3 0\rangle \otimes |\Phi_0\rangle \). By Theorem 5.1, the steady-state joint system-field state is
\[
|\Psi(\infty)\rangle = |g_1 g_2 g_3 0\rangle \otimes |3_{n}\rangle
\]
where \( |3_{n}\rangle \) is the three-photon output state, whose probability distribution is shown in Fig. 8.

**VII. CONCLUSION**

In this article, we have studied the Tavis–Cummings model. Specifically, we have applied quantum linear systems theory to reveal typical features of the Tavis–Cummings model, the analytical expression has been derived for the output single-photon state of a Tavis–Cummings system in response to a single-photon input. We have also proposed a computational framework to derive the analytic form of the superposition state of the system and field. Note that the linear quantum systems' approach is only applicable when there is only one excitation. Future studies are to be done on subradiance and superradiance in the multieexcitation case.

**APPENDIX**

As concepts and properties of quantum linear passive systems are use in this article, in particular Section IV, in this appendix, we collect some of them for readability of this paper.

For the quantum linear system (12), from (17), we know the transfer function \( G[s] \) is an all-pass function [39, pp. 357]. Actually, as shown in Remark 3.1, system (12) is passive. If we take expectation on both sides of (12) with respect to the distribution is shown in

\[
\langle \hat{a}(t) \rangle = A \langle \hat{a}(t) \rangle + B \langle b_m(t) \rangle
\]

\[
\langle b_{out}(t) \rangle = C \langle \hat{a} \rangle dt + \langle b_{in}(t) \rangle.
\]

Thus, we can define controllability, observability, and Hurwitz stability for the quantum linear system (12) using those for the classical linear system (70).

**Definition 1.1:** The quantum linear passive system (12) is said to be Hurwitz stable (resp. controllable, observable) if the corresponding classical linear system (70) is Hurwitz stable (resp. controllable, observable).

**Proposition 1.1:** The quantum linear passive system (12) has the following properties.

1) Its Hurwitz stability, controllability, and observability are equivalent to each other.

2) It has only controllable and observable (co) subsystem and uncontrollable and unobservable (co) subsystem. Actually, its co subsystem forms a closed (namely isolated) quantum system; see Corollary 3.1.

3) Its poles corresponding to a co subsystem are all on the imaginary axis, while its poles corresponding to a co subsystem are on the open left half of the complex plane.

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