Singularities in fidelity surfaces for quantum phase transitions: a geometric perspective

Huan-Qiang Zhou, Jian-Hui Zhao, Hong-Lei Wang and Bo Li

Centre for Modern Physics and Department of Physics, Chongqing University, Chongqing 400044, People’s Republic of China
E-mail: hqzhou@cqu.edu.cn

Received 15 October 2010, in final form 5 December 2010
Published 21 December 2010
Online at stacks.iop.org/JPhysA/44/042002

Abstract

The fidelity per site between two ground states of a quantum lattice system corresponding to different values of the control parameter defines a surface embedded in a Euclidean space. The Gaussian curvature naturally quantifies quantum fluctuations that destroy orders at transition points. It turns out that quantum fluctuations wildly distort the fidelity surface near a transition point, at which the Gaussian curvature is singular in the thermodynamic limit. As a concrete example, the one-dimensional quantum Ising model in a transverse field is analyzed. We also perform a finite-size scaling analysis for the transverse Ising model of finite sizes. The scaling behavior for the Gaussian curvature is numerically checked and the correlation length critical exponent is extracted, which is consistent with the conformal invariance at the critical point.

PACS numbers: 03.67.−a, 05.70.Fh, 64.60.Ak

Quantum phase transitions (QPTs) have been a research topic subject to intense study, since their significant role was realized in accounting for high-$T_c$ superconductors, fractional quantum Hall liquids, and quantum magnets [1, 2]. Recently, significant advances have been made in an attempt to clarify the connection between quantum many-body physics and quantum information science. This provides a new perspective to investigate QPTs from entanglement [3–7] and fidelity [8–12], basic notions of quantum information science [13], and turns out to be very insightful in our understanding of QPTs in a variety of quantum lattice systems in condensed matter.

Conventionally, orders and fluctuations provide a proper language to study QPTs, with order parameters being the key to quantify quantum fluctuations. Instead, the fidelity approach is based on state distinguishability arising from the orthogonality of different ground states in the thermodynamic limit. In fact, the ground state fidelity for a quantum system may be mapped onto the partition function of the equivalent classical statistical lattice model with the same geometry [11]. Thus, the fidelity per site is well defined in the thermodynamic
For the two ground states control parameter, although the extension to multiple control parameters is straightforward. A control parameter, here we restrict ourselves to discuss the simplest case with one single fidelity surfaces. For a quantum lattice system described by a Hamiltonian $H(\lambda)$, with $\lambda$ a control parameter, we can define a fidelity surface as a surface embedded in a Euclidean space, which in turn is determined by the average fidelity per lattice site between two ground states of a quantum lattice system as a function of control parameters. This makes the whole machinery developed in differential geometry of curves and surfaces available to study QPTs. As is well known, the Gaussian curvature, or equivalently the Ricci scalar curvature for the surfaces embedded in Euclidean spaces, is a fundamental concept used to measure how curved a surface is. Therefore, the Gaussian curvature is expected to naturally quantify quantum fluctuations that destroy orders at transition points. We discuss the global behaviors of the Gaussian curvature. It turns out that quantum fluctuations wildly distort the fidelity surfaces near the transition points. Generically, precursors of QPTs occur in the Gaussian curvature for finite-size systems. In the thermodynamic limit, the Gaussian curvature becomes singular at transition points. The one-dimensional quantum Ising model in a transverse field is exploited to explicitly illustrate the theory. We also perform a finite-size scaling analysis for the Gaussian curvature with different lattice sizes to extract the correlation length critical exponent.

**Fidelity surfaces.** For a quantum lattice system described by a Hamiltonian $H(\lambda)$, with $\lambda$ a control parameter, here we restrict ourselves to discuss the simplest case with one single control parameter, although the extension to multiple control parameters is straightforward. For the two ground states $|\psi(\lambda_1)\rangle$ and $|\psi(\lambda_2)\rangle$ corresponding to different values of the control parameter $\lambda$, the fidelity is defined as $F(\lambda_1, \lambda_2) = |\langle \psi(\lambda_2) | \psi(\lambda_1) \rangle|$. For a large but finite $L$, the fidelity $F$ asymptotically scales as $F(\lambda_1, \lambda_2) \sim d^L(\lambda_1, \lambda_2)$, where the scaling parameter $d(\lambda_1, \lambda_2)$ characterizes how fast the fidelity changes when the thermodynamic limit is approached [10]. Physically, it is the fidelity per site. Here, note that the contribution from each site to $F(\lambda_1, \lambda_2)$ is multiplicative. Following [11], the ground state fidelity for a quantum system is nothing but the partition function of the equivalent classical statistical lattice model with the same geometry, if one utilizes the tensor network representations of the ground state many-body wavefunctions. Therefore, $d(\lambda_1, \lambda_2)$ may be interpreted as the partition function per site [14], which is well defined in the thermodynamic limit:

$$\ln d(\lambda_1, \lambda_2) = \lim_{L \to \infty} \ln F(\lambda_1, \lambda_2)/L.$$  

(1)

The fidelity per site $d(\lambda_1, \lambda_2)$ satisfies the following properties: (1) symmetry under interchange $\lambda_1 \leftrightarrow \lambda_2$; (2) $d(\lambda_1, \lambda_1) = 1$; and (3) $0 \leq d(\lambda_1, \lambda_2) \leq 1$.

For simplicity, let us assume that the system undergoes a QPT at $\lambda_c$. If $|\psi(\lambda_1)\rangle$ and $|\psi(\lambda_2)\rangle$ are in the same phase, then they flow to the same stable fixed point in the sense of renormalization group theory, and their different differences arise from quantum fluctuations depending on the details of the system. On the other hand, if $|\psi(\lambda_1)\rangle$ and $|\psi(\lambda_2)\rangle$ are in different phases, then they flow to two different stable fixed points. Therefore, they possess different orders, although quantum fluctuations originate from the same unstable fixed point $\lambda_c$.1 Imagine that if there were no quantum fluctuations, then $d(\lambda_1, \lambda_2)$ would be simply $1$.

1 Actually, in the fidelity approach [9], we prefer to introduce relevant information and irrelevant information as the counterparts of orders and fluctuations in the conventional theory, since the fidelity, unlike order parameters, is not physical observable, although both may be used to quantify quantum fluctuations.
when $|\psi(\lambda_1)|$ and $|\psi(\lambda_2)|$ are in the same phase; otherwise, when $|\psi(\lambda_1)|$ and $|\psi(\lambda_2)|$ are in different phases, $d(\lambda_1, \lambda_2)$ would take the minimum value corresponding to the two stable fixed points. For continuous QPTs, quantum fluctuations are strong enough such that no orders survive at the transition point, so $d(\lambda_1, \lambda_2)$ is continuous, but displays singularities; whereas for the first-order QPTs, $d(\lambda_1, \lambda_2)$ remains to be discontinuous at the transition point. An interesting observation is to regard the fidelity per site, $d(\lambda_1, \lambda_2)$, as a two-dimensional surface embedded in the three-dimensional Euclidean space, with a Riemannian metric induced from the Euclidean metric. Our aim is to give an intrinsic characterization of singularities in such a fidelity surface in terms of Riemannian geometry.

**Differential geometry of the two-dimensional surfaces embedded in the three-dimensional Euclidean space.** Let us briefly recall the fundamentals of differential geometry of surfaces embedded in Euclidean spaces [15]. For a two-dimensional surface embedded in a three-dimensional Euclidean space: $z = f(\lambda_1, \lambda_2)$, the first fundamental form on the surface is

$$
\mathrm{d}l^2 = g_{ij} \mathrm{d}\lambda^i \mathrm{d}\lambda^j = E(\mathrm{d}u)^2 + 2F(\mathrm{d}u \, \mathrm{d}v) + G(\mathrm{d}v)^2,
$$

(2)

where $g_{ij}$ is the Riemannian metric on the surface: $g_{11} = 1 + f_{\lambda_1}^2$, $g_{12} = g_{21} = f_{\lambda_1} f_{\lambda_2}$, and $g_{22} = 1 + f_{\lambda_2}^2$. Here, the subscripts $\lambda_1$ and $\lambda_2$ denote the partial differentiations with respect to $\lambda_1$ and $\lambda_2$, respectively. In terms of the co-ordinates $u = \lambda_1$ and $v = \lambda_2$, we have $E = g_{11}$, $F = g_{12} = g_{21}$ and $G = g_{22}$. Suppose the surface is given in a parametric form: $r = r(u, v)$. Then, the vector product $r_u \times r_v$ is a non-zero vector perpendicular to the surface at each non-singular point; define $m$ to be a unit vector in the normal direction; then one has $r_u \times r_v = |r_u \times r_v| m$. For a curve $r = r(u(l), v(l))$ on the surface, the projection of the second-order derivative $\dot{r}$ of $r$ with respect to the arc length $l$ on the normal to the surface leads to the second fundamental form as follows:

$$
\langle \dot{r}, m \rangle (\mathrm{d}l)^2 = b_{ij} \mathrm{d}\lambda^i \mathrm{d}\lambda^j = X(\mathrm{d}u)^2 + 2Y \mathrm{d}u \, \mathrm{d}v + Z(\mathrm{d}v)^2,
$$

(3)

if a surface is given in the form $z = f(\lambda_1, \lambda_2)$ with $\lambda_1 = u$, $\lambda_2 = v$, and $r(u, v) = (u, v, f(u, v))$. Therefore, we have $X = b_{11} = f_{\lambda_1\lambda_1}/\sqrt{1 + f_{\lambda_1}^2 + f_{\lambda_2}^2}$, $Y = b_{12} = b_{21} = f_{\lambda_1\lambda_2}/\sqrt{1 + f_{\lambda_1}^2 + f_{\lambda_2}^2}$, and $Z = b_{22} = f_{\lambda_2\lambda_2}/\sqrt{1 + f_{\lambda_1}^2 + f_{\lambda_2}^2}$.

The eigenvalues of the pair of quadratic forms (2) and (3) are the principal curvatures of the surface at the point under investigation. The product of the principal curvatures is the Gaussian curvature $K$ of the surface at the point, and their sum the mean curvature. The principal curvatures $k_1$ and $k_2$ are the solutions of the equation

$$
\det(Q - kG) = 0,
$$

(4)

where $Q = (b_{ij})$ is the matrix of the second fundamental form, and $G = (g_{ij})$. Since the first fundamental form is positive definite, its matrix $G$ is non-singular. Hence $\det(Q - kG) = \det G \det(G^{-1}Q - k \cdot I)$, and we deduce that the Gaussian curvature $K = k_1 k_2 = \det(G^{-1}Q) = \det Q / \det G$ and the mean curvature $M = k_1 + k_2 = \text{tr}(G^{-1}Q)$. Therefore, the Gaussian curvature $K$ and the mean curvature $M$ take the form

$$
K = \frac{f_{\lambda_1\lambda_1} f_{\lambda_2\lambda_2} - f_{\lambda_1\lambda_2}^2}{(1 + f_{\lambda_1}^2 + f_{\lambda_2}^2)^2},
$$

(5)

and

$$
M = \frac{(1 + f_{\lambda_1}^2) f_{\lambda_1\lambda_1} + (1 + f_{\lambda_2}^2) f_{\lambda_2\lambda_2} - 2 f_{\lambda_1} f_{\lambda_2} f_{\lambda_1\lambda_2}}{(1 + f_{\lambda_1}^2 + f_{\lambda_2}^2)^2}.
$$

(6)
respectively. We note that the sign of the Gaussian curvature \( K \) is the same as the sign of the determinant \( f_{\lambda_1,\lambda_2} f_{\lambda_2,\lambda_1} - f_{\lambda_1,\lambda_2}^2 \), i.e., the Hessian of \( z = f(\lambda_1, \lambda_2) \).

It follows that, in contrast with the mean curvature \( M \), the Gaussian curvature \( K \) of a surface may be expressed in terms of the induced metric on the surface alone, and is therefore an intrinsic invariant of the surface [15]. In addition, a two-dimensional surface in a three-dimensional space may also be regarded as a differentiable manifold endowed with a Riemannian metric induced from the Euclidean metric. The Ricci scalar curvature \( R \) is twice the Gaussian curvature \( K : R = 2K \).

**Global behaviors of the Gaussian curvature \( K \) for a fidelity surface.** Now we consider the (logarithmic function of) fidelity per site, \( \ln d(\lambda_1, \lambda_2) \), as a two-dimensional surface embedded in the three-dimensional Euclidean space: \( z = f(\lambda_1, \lambda_2) = \ln d(\lambda_1, \lambda_2) \). The Gaussian curvature \( K(\lambda_1, \lambda_2) \) for such a fidelity surface may be used to quantify how strong quantum fluctuations are in given quantum many-body ground states, thus providing an intrinsic characterization of singularities in the fidelity surface. Indeed, as justified in [9–11], the fidelity per site \( d(\lambda_1, \lambda_2) \) is singular when \( \lambda_1(\lambda_2) \) crosses \( \lambda_c \) for a fixed \( \lambda_2(\lambda_1) \) in the thermodynamic limit. Therefore the Gaussian curvature \( K(\lambda_1, \lambda_2) \) for the fidelity surface is singular at \( \lambda_1 = \lambda_c \) and/or \( \lambda_2 = \lambda_c \) in the thermodynamic limit. Generically, we have (1) \( K(\lambda_1, \lambda_2) > 0 \), and there is a neighborhood of the point throughout which the surface lies on one side of the tangent plane at the points; (2) \( K(\lambda_1, \lambda_2) < 0 \); then the surface intersects the tangent plane at the point arbitrarily close to the point. If the surface is strictly convex, then we say that the Gaussian curvature \( K(\lambda_1, \lambda_2) \) is positive at every point of the surface. That is what happens if \( \lambda_1 \) and \( \lambda_2 \) are away from the transition point. However, if \( \lambda_1 \) and \( \lambda_2 \) are close to the transition point, then the Gaussian curvature \( K(\lambda_1, \lambda_2) \) can be negative.

For finite-size systems, the Gaussian curvature \( K(\lambda_1, \lambda_2) \) remains smooth, although the precursors of QPT’s occur as anomalies in the Gaussian curvature \( K(\lambda_1, \lambda_2) \). The anomalies get more pronounced when the thermodynamic limit is approached. We may take advantage of this fact to perform finite-size scaling to extract the correlation length critical exponent.

**Quantum XY spin-1/2 model.** The quantum XY spin model is described by the Hamiltonian

\[
H = -\sum_{j=1}^{M} \left( \frac{1}{2} \sigma_j^x \sigma_{j+1}^x + \frac{1}{2} \sigma_j^y \sigma_{j+1}^y + \lambda \sigma_j^z \right).
\]

Here, \( \sigma_j^x, \sigma_j^y \), and \( \sigma_j^z \) are the Pauli matrices at the \( j \) th lattice site. The parameter \( \gamma \) denotes an anisotropy in the nearest-neighbor spin–spin interaction, whereas \( \lambda \) is an external magnetic field. The Hamiltonian (7) may be exactly diagonalized [16, 17] for any finite size \( L \) with \( L = 2M + 1 \). In the thermodynamic limit \( L \to \infty \), \( \ln d(\lambda_1, \lambda_2) \) takes the form [9]

\[
\ln d(\lambda_1, \lambda_2) = \frac{1}{2\pi} \int_0^{\pi} \, d\varphi \, \ln \mathcal{F}(\lambda_1, \lambda_2; \alpha),
\]

where \( \mathcal{F}(\lambda_1, \lambda_2; \alpha) = \cos[\vartheta(\lambda_1; \alpha) - \vartheta(\lambda_2; \alpha)]/2 \), with \( \cos \vartheta(\lambda; \alpha) = (\cos \alpha - \lambda)/\sqrt{(\cos \alpha - \lambda)^2 + \gamma^2 \sin^2 \alpha} \).

Now it is straightforward to calculate the Gaussian curvature \( K(\lambda_1, \lambda_2) \) for the fidelity surface of the quantum XY spin chain. In figure 1, we plot \( K(\lambda_1, \lambda_2) = 0.6 \) for the fidelity surface of the quantum XY model (\( \gamma = 1 \) for the upper panel and \( \gamma = 1/2 \) for the lower panel). One observes that \( K(\lambda_1, \lambda_2) \) is divergent as a function of \( \lambda_1 \) at the critical point \( \lambda_c = 1 \) for the infinite-size system \( L = \infty \), indicating that the fidelity surface is wildly

\[\text{The explicit expression for } \ln d(\lambda_1, \lambda_2) \text{ was also given in [9, 10] for the quantum } XY \text{ model of finite sizes, which is needed for the evaluation of the Gaussian curvature } K(\lambda_1, \lambda_2) \text{ for the corresponding fidelity surfaces.}\]
Figure 1. The behavior near the critical point \( \lambda_c = 1 \) is analyzed for the Gaussian curvature \( K(\lambda_1, \lambda_2) \) of the quantum transverse Ising model for various lattice sizes. The curves shown correspond to different lattice sizes \( L = 201, 401, 1201, 2001, \) and \( \infty \). The peaks (dips) get more pronounced in the left (right) side with increasing system size. The Gaussian curvature \( K(\lambda_1, \lambda_2) \) diverges at the critical point \( \lambda_1 = \lambda_c \) for the infinite-size system (\( L = \infty \)).

Upper panel: here \( K(\lambda_1, \lambda_2) \) is regarded as a function of \( \lambda_1 \) for \( \lambda_2 = 0 \) and \( \gamma = 1 \). Lower panel: here \( K(\lambda_1, \lambda_2) \) is regarded as a function of \( \lambda_1 \) for \( \lambda_2 = 0.6 \) and \( \gamma = 1/2 \).

distorted, due to strong quantum fluctuations near the critical point. This is true for any non-zero \( \gamma \), consistent with the fact that the quantum XY model for any non-zero \( \gamma \) belongs to the same universality class as that of the quantum transverse Ising model. That is, there is a critical line \( \gamma \neq 0 \) and \( \lambda_c = 1 \); only one (second-order) critical point \( \lambda_c = 1 \) separates two gapful phases: (spin reversal) \( Z_2 \) symmetry-breaking and symmetric phases.

**Finite-size scaling analysis for the Gaussian curvature \( K \).** We focus on the quantum Ising universality class. The order parameter, i.e. magnetization \( \langle \sigma^x \rangle \), is non-zero for \( \lambda < 1 \), and otherwise zero. At the critical point, the correlation length \( \xi \sim |\lambda - \lambda_c|^{-\nu} \) with \( \nu = 1 \) [17]. In order to analyze how the Gaussian curvature \( K(\lambda_1, \lambda_2) \) behaves near the critical point \( \lambda_c = 1 \), we perform a finite-size scaling analysis for the quantum transverse Ising model.

As already observed, the drastic change of the ground-state wavefunctions makes the Gaussian curvature \( K(\lambda_1, \lambda_2) \) divergent when the system undergoes the second-order QPT at the critical point \( \lambda_c = 1 \) in the thermodynamic limit. However, for finite-size systems, \( K(\lambda_1, \lambda_2) \) remains smooth for the quantum XY model. In figure 1, the numerical results are also plotted for the Gaussian curvature \( K(\lambda_1, \lambda_2) \) with different system sizes, where \( \lambda_2 = 0.6 \) and \( \gamma = 1 \) (upper panel) and \( \lambda_2 = 0.6 \) and \( \gamma = 1/2 \) (lower panel). More precisely, in the
thermodynamic limit, \( K(\lambda_1, \lambda_2) \) (as a function of \( \lambda_1 \) for a fixed \( \lambda_2 \)) diverges at the critical point \( \lambda_1 = \lambda_c \):

\[
K(\lambda_1, \lambda_2) \sim \frac{1}{|\lambda_1 - \lambda_c|^\nu (\ln |\lambda_1 - \lambda_c|)}.
\] (9)

However, there is no divergence for finite-size systems, but there are clear anomalies, featuring two quasi-critical values \( \lambda_{p} \) and \( \lambda_{d} \), one at each side of the critical point. On the left (right) side, the so-called quasi-critical points \( \lambda_{p} \) (\( \lambda_{d} \)) approach the critical value as

\[
\lambda_{p} \approx 1 - 1.6149 L^{1.03531} (\lambda_{d} \approx 1 + 9.691 98 L^{-0.974152}),
\]

with the values at peaks (dips) diverging with the increasing system size \( L \):

\[
K(\lambda_1, \lambda_2) \big|_{\lambda_1 = \lambda_{p(d)}} = k_{p(d)} \frac{L}{(\ln L)^4} + \text{constant}.
\] (10)

Here, the prefactor \( k_{p(d)} \) is non-universal in the sense that it depends on \( \lambda_2 \) and \( \gamma \). We emphasize that equation (10) follows from equation (9), if we take into account the fact that the model is conformally invariant at the critical point. Indeed, on the one hand, from equation (9) and the correlation length \( \xi \sim |\lambda - \lambda_c|^{-\nu} \) with \( \nu = 1 \), we have

\[
K(\lambda_1, \lambda_2) \sim \frac{\xi}{(\ln \xi)^4}.
\]

On the other hand, the conformal invariance requires the scale invariance:

\[
\frac{\xi}{L} = \frac{\xi'}{L'}.
\]

The numerical results are, respectively, plotted for \( K(\lambda_1, \lambda_2) \big|_{\lambda_1 = \lambda_{p(d)}} \) in figure 2 and for \( \lambda_{p(d)} \) in figure 3 with \( \lambda_2 = 0.6 \) and \( \gamma = 1 \). The same is also true for any non-zero \( \gamma \). This shows that, consistent with the exact result, the correlation length critical exponent \( \nu \) equals 1, as long as \( \gamma \) is non-zero.

**Conclusions.** We have shown that singularities in fidelity surfaces may be intrinsically characterized in terms of Riemannian geometry, based on the fidelity description of QPTs. Generically, the Ricci curvature tensor for finite-size systems is analytic and it exhibits singularities at transition points in the thermodynamic limit, as reflected in the Ricci scalar curvature that blows up when the system size tends to \( \infty \). This opens up the possibility of exploiting the theory of Ricci flows [18] to characterizing QPTs in condensed matter theory. The one-dimensional quantum Ising model in a transverse field is exploited as an example to
Figure 3. (a) The positions of the peaks approach the critical point $\lambda_c = 1$ with the increasing system size $L$ as $\lambda_p \approx 1 - 1.61490L^{-1.03531}$. (b) The positions of the dips approach the critical point $\lambda_c = 1$ with the increasing system size $L$ as $\lambda_d \approx 1 + 9.69198L^{-0.974152}$. In both cases, $\lambda_2 = 0.6$ and $\gamma = 1$.

explicitly illustrate the theory\textsuperscript{3}, and a finite-size scaling analysis has been performed for the Ricci scalar curvature with different lattice sizes, and the correlation length critical exponent has been extracted, consistent with the known exact value.

In closing, we emphasize that, in our geometric approach, the Riemannian metric is extrinsic, in the sense that the Riemannian geometry is inherited from embedding a fidelity surface into a Euclidean space, in contrast to a related quantum metric approach [20]. In the latter approach, the quantum metric arises from the fact that the fidelity is essentially a distance measure between two (close) quantum states. The advantage of our approach lies in the robust connection between transition points and singularities on fidelity surfaces, irrespective of the type of orders present in a quantum system, as demonstrated for the Kosterlitz–Thouless transitions [21]. However, the approach discussed in [20] often fails to detect the Kosterlitz–Thouless transitions in various quantum many-body systems. This is simply due to the fact that, in their approach, the metric only retains information encoded in the second-order derivative of the fidelity (between two close states) with respect to the control parameter, which turns out to be not enough to capture the (infinite-order) Kosterlitz–Thouless phase transitions.

Acknowledgments

We thank John Paul Barjaktarević, Sam Young Cho and John Fjaerestad for helpful discussions and comments. The support from the National Natural Science Foundation of China (grant nos 10774197 and 10874252), the Natural Science Foundation of Chongqing (grant no CSTC, 2008BC2023), and Chongqing University Postgraduates Science and Innovation Fund (project no 200911C1A0140330) is acknowledged.

\textsuperscript{3} Although we restrict ourselves to discuss an exactly solvable model in one spatial dimension, the theory in principle applies to any quantum lattice systems in any spatial dimensions [11] by using the newly-developed tensor network algorithms [19].
References

[1] Sachdev S 1999 Quantum Phase Transitions (Cambridge: Cambridge University Press)
[2] Wen X-G 2004 Quantum Field Theory of Many-Body Systems (Oxford: Oxford University Press)
[3] Osborne T J and Nielsen M A 2002 Phys. Rev. A 66 032110
[4] Osterloh A, Amico L, Falci G and Fazio R 2002 Nature 416 608
[5] Vidal G, Latore J I, Rico E and Kitaev A 2003 Phys. Rev. Lett. 90 227902
[6] Korepin K E 2004 Phys. Rev. Lett. 92 096402
[7] Levin M and Wen X-G 2004 Phys. Rev. Lett. 93 266402
[8] Vidal G, Latorre J I, Rico E and Kitaev A 2003 Phys. Rev. Lett. 90 227902
[9] Zhou H-Q, Bartheil T, Fjaerestad J O and Schollwöck U 2006 Phys. Rev. A 74 050305
[10] Verstraete F, Martin-Delgado M A and Cirac J I 2004 Phys. Rev. Lett. 92 087201
[11] Barnum H, Knill E, Ortiz G, Somma R and Viola L 2004 Phys. Rev. Lett. 92 107902
[12] Dur W, Hartmann L, Hein M, Lewenstein M and Briegel H J 2005 Phys. Rev. Lett. 94 097203
[13] Nielsen M A and Chuang I L 2000 Quantum Computation and Quantum Information (Cambridge: Cambridge University Press)
[14] Baxter R J 2005 arXiv:cond-mat/0611167
[15] Dubrovin B A, Fomenko A T and Novikov S P 1985 Modern Geometry—Methods and Applications Part 1 2nd edn (New York: Springer)
[16] Lieb E, Schultz T and Mattis D 1961 Ann. Phys., N.Y. 16 407
[17] Pfeuty P 1970 Ann. Phys. 57 79
[18] Perelman G 2003 arXiv:math/0307245
[19] Vidal G 2002 Phys. Rev. Lett. 89 208101
[20] Zanardi P, Giorda M and Cozzini M 2007 Phys. Rev. Lett. 99 100603
[21] Wang H-L, Zhou J-H, Li B and Zhou H-Q 2009 arXiv:0902.1670