The principal advantage of the Green–Schwarz heterotic string sigma model with respect to the Neveu–Schwarz–Ramond formulation is its manifest N=1, D=10 space–time supersymmetry. It is this property which renders it particularly suitable for the derivation of the low energy effective superstring theory, i.e. N=1, D=10 Supergravity–Super–Yang–Mills, in superspace. Indeed, the κ–anomaly cancellation mechanism [1] constitutes a systematic approach for the derivation of superspace constraints for the low energy effective theory which are automatically consistent with the Bianchi identities in superspace: the Wess–Zumino consistency condition on the κ–anomalies ensures that they can be cancelled by imposing suitable constraints on the superfields of the low energy theory and that with these constraints the Bianchi identities can be consistently solved.

Along these lines in ref. [2], see also [3], the order α’ one–loop κ–anomalies of the heterotic string sigma–model have been explicitly determined, by a direct perturbative computation, together with the order α’ superspace constraints which give rise to their cancellation.

On the other hand, exact solutions, i.e. to all orders in α’, of the relevant Bianchi identities have been found previously in the literature [4,5,6].

In this letter we point out that the exact constraints found in this way, if truncated to order α’, appear to differ from the ones found through the κ–anomaly cancellation mechanism in ref. [2], the difference not being simply related to the usual ambiguity in the choice of standard constraints for a theory formulated in superspace. We present the solution of this puzzle by showing that the difference between the two sets of constraints is related on one hand to a trivial κ–anomaly and that, on the other hand, this difference can be eliminated by order–α’ superfield redefinitions.

The heterotic string sigma model action is given by

$$I = -\frac{1}{2} \int d^2 \sigma \sqrt{g} \left( V^a_{+a} + V^A_{+} B_{BA} - \psi D^- \psi \right)$$

(1)

where the induced Zehnbein are given by $V^A_i = \partial_i Z^M E^A_M (Z)$ and the index $A = (a, \alpha)$ stands for ten bosonic ($a=1, ..., 10$) and sixteen fermionic ($\alpha = 1, ..., 16$) entries. The two–dimensional light–cone indices are introduced via $V^A_\pm = e^i_\pm V^A_i$ (see [2] for the notation). The components of the two–super form B are given in
Zehnbein basis by $B = \frac{1}{2} E^A E^C B_{CA}$ where $E^A = dZ^M E_M^A(Z)$.

A $p$–superform $\wedge^p$ can be decomposed in sectors with $r$ bosonic ($E^a$) and $s$ fermionic ($E^\alpha$) Zehnbein according to

$$\wedge^p = \frac{1}{p!} E^{A_1} \cdots E^{A_p} \wedge_{A_p} \cdots A_1 = \sum_{r+s=p} \wedge_{(r,s)}.$$  \hspace{1cm} (2)

For what follows it is convenient to introduce the following subspace $\mathcal{H}_4$ of closed four–superforms:

$$\mathcal{H}_4 = \{ \wedge^4 | d\wedge^4 = 0, \wedge_{(0,4)} = 0 = \wedge_{(1,3)} \}. \hspace{1cm} (3)$$

The heterotic fermions $\psi$ stay in the fundamental representation of $SO(32)$ and $D_\pm = e^i_\pm (\partial_i - V^B_i A_B)$ where $A_B$ are the components of the connection one–superform $A = E^B A_B$, with values in the Lie algebra of $SO(32)$.

The action (1) is invariant under $\kappa$–transformations, with transformation parameter $\kappa_+^\alpha$, a space–time spinor, which are induced by

$$\delta_k Z^M = \Delta^\alpha E^M_\alpha,$$

$$\delta_\kappa \psi = \Delta^\alpha A_\alpha \psi,$$

$$\Delta^\alpha \equiv V^\alpha (\Gamma_a)^{\alpha \beta} \kappa_+^\beta = (\Psi_2 - \kappa_+^\alpha)^\alpha. \hspace{1cm} (4)$$

Due to the Virasoro constraint, $V_+^2 = V_-^a V_\alpha V^\alpha_a = 0$, we have that $V_- \Delta = V_+^2 \kappa_+ = 0$. To be more precise, taking the Virasoro constraint into account*, under the transformations (4) the action varies as follows

$$\delta_\kappa I = -\frac{1}{2} \int d^2 \sigma \sqrt{g} \left( 2 V_+^\alpha V_+^\beta \Delta^\gamma + V_-^C V_+^D \Delta^\gamma (dB)_\gamma DC \right). \hspace{1cm} (5)$$

To get (5) we used the standard superspace constraints on the torsion, $T^A = D E^A = dE^A + E^B \Omega_B^A = \frac{1}{2} E^B E^C T_{CB}^A$ where $\Omega_A^B$ is the $D = 10$ Lorentz connection one–form,

* The use of the Virasoro constraint can be avoided by introducing non trivial $\kappa$–transformations for the zweibein $e_+^i$. Here, to avoid a merely technical complication, we prefer to enforce the equation of motion for the world–sheet metric, i.e. $V_-^2 = 0$.  

2
\[ T_{\alpha\beta}^a = 2 \Gamma^a_{\alpha\beta} \]
\[ T_{\alpha a}^b = 0, \]  
(6)

and on the Yang–Mills curvature two form \( F = dA + AA = \frac{1}{2} E^A E^B F_{BA}, \)

\[ F_{\alpha\beta} = 0. \]  
(7)

(5) can now be set to zero by choosing standard constraints for \( H^0 \equiv dB, \)

\[ H_{0\alpha\beta\gamma}^0 = H_{0a\beta}^0 = 0 \]
\[ H_{0\alpha\alpha\beta}^0 = 2(\Gamma_a)_{\alpha\beta}. \]  
(8)

At the quantum level the presence of \( \kappa \)–anomalies may require to modify the constraints (6), (7) and (8). Indeed, on general grounds [1] the \( \kappa \)–anomaly \( A_{\kappa} \)

is structured in three terms, \( A_{\kappa} = A + A_T + A_F; A_T \) and \( A_F \) induce quantum corrections to (6) and (7) respectively which are expected to be absent, at least at one loop i.e. at order \( \alpha' \), while \( A \) induces corrections to (8) which appear already at order \( \alpha' \). Its general one–loop structure is given by

\[ A = -\frac{\alpha'}{8} \int d^2 \sigma \sqrt{g} V^A_+ V^B_+ \Delta_+^\gamma (\omega_{3Y} - \omega_{3L} + X)_{\gamma BA} \]  
(9)

where \( \omega_{3Y} \) and \( \omega_{3L} \) are respectively the Yang–Mills and Lorentz Chern–Simons three superforms satisfying \( d\omega_{3Y} = trFF, d\omega_{3L} = trRR \), while \( X_{CBA} \) are the components of a gauge and Lorentz–invariant three superform \( X = \frac{1}{3!} E^A E^B E^C X_{CBA} \). The Wess–Zumino consistency condition on (9) becomes

\[ \delta_\kappa A = -\frac{\alpha'}{8} \int d^2 \sigma \sqrt{g} V^C_+ V^D_+ \Delta_+^\alpha \Delta_+^\beta (trFF - trRR + dX)_{\beta\alpha DC} = 0, \]  
(10)

which is precisely equivalent to the condition

\[ \alpha'(trFF - trRR + dX) \in \mathcal{H}_4. \]

Since we have an \( \alpha' \) in front, in checking this we can use the classical constraints (6), (7); moreover, following [7] we can impose the classical constraint \( R_{\alpha\beta a}^b = 0 \)

** Our normalizations are determined by defining the effective action as \( \exp(\frac{i}{2\pi\alpha'}\Gamma) = \int \{ \mathcal{D}\varphi \} \exp(\frac{i}{2\pi\alpha'}I), \Gamma = I + \alpha' \Gamma^{(1)} + o(\alpha'^2), A_{\kappa} = \alpha' \delta_\kappa \Gamma^{(1)}. \)
on the Lorentz curvature two–super form
\[ R_{ab}^{} = \frac{1}{2} E^C E^D R_{DCa}^{} = d\Omega_{a}^{} b + \Omega_{a}^{} c \Omega_{c}^{} b. \]
As a consequence of \( F_{\alpha\beta} = 0 \) and \( R_{\alpha\beta a}^{} = 0 \) one has that \( trFF \in \mathcal{H}_4 \) and \( trRR \in \mathcal{H}_4 \) separately. One remains with the condition
\[ dX \in \mathcal{H}_4 \] (11)
at zero–order in \( \alpha' \).
The anomaly can now be cancelled by imposing
\[ \delta_{\kappa} \Gamma = \delta_{\kappa} I + A = 0 \] (12)
which amounts to define the generalized \( B– \) curvature
\[ H = dB + \frac{\alpha'}{4} (\omega_{3YM} - \omega_{3L}) \] (13)
and to impose on it the constraints
\[ H_{\alpha\beta\gamma} = -\frac{\alpha'}{4} X_{\alpha\beta\gamma} \]
\[ H_{\alpha\alpha\beta} = 2(\Gamma_a)_{\alpha\beta} - \frac{\alpha'}{4} X_{\alpha\alpha\beta} \] (14)
\[ H_{\alpha\beta\alpha} = -\frac{\alpha'}{4} X_{\alpha\beta\alpha}. \]
Thanks to (11) the Bianchi identity associated to (13), i.e.
\[ dH = \frac{\alpha'}{4} (trFF - trRR), \] (15)
can be consistently solved with the constraints (14).

An explicit computation of the one–loop \( \kappa– \)anomaly has been performed in [2] and the result was indeed formula (9), but with \( X = 0 \), meaning that the order \( \alpha' \) corrections to the \( H– \)field in (14) are absent.

On the other hand, exact solutions of the Bianchi identity (15), based on the Bonora–Pasti–Tonin (BPT) theorem [4], are known in the literature [5], [6]. These solutions differ among themselves only because of different choices for classical standard constraints, such as (6) and (8). The choice of ref. [6] differs from ours just by a shift of the Lorentz connection \( (\Omega_{a}^{} b \rightarrow \Omega_{a}^{} b + E^c T_{ca}^{} b, \) where \( T_{cab} \) is the vectorial torsion) and so we make our comparison with that reference.
The BPT theorem ensures that one can write $trRR = d\tilde{X} + K$ where $\tilde{X}$ is an invariant three superform and $K \in H_4$. Then (15) can be written as

$$d \left( H + \frac{\alpha'}{4} \tilde{X} \right) = \frac{\alpha'}{4} (trF^2 - K) \in H_4$$

(16)

and the exact solution is found by imposing on $H$ the constraints in (14) with $X \rightarrow \tilde{X}$. However, the $\tilde{X}$ found in [6], when truncated to zero–order in $\alpha'$, is different from zero. Calling $X = \tilde{X}|_{\alpha'=0}$ the authors of ref. [6] got

$$X_{\alpha\beta\gamma} = 0$$
$$X_{\alpha\alpha\beta} = -\frac{1}{3}(\Gamma_a c_1 - c_4)_{\alpha\beta}(R_{c_1 - c_4} + T_{c_1 c_2} \gamma (\Gamma_{c_3 c_4})_{\gamma \delta} \lambda_{\delta})$$

(17)

where $T_{ab}^\alpha$ is the gravitino field strength and $\lambda_{\beta}$ is the gravitello. We do not report the explicit expressions of $X_{(2,1)}$ and $X_{(3,0)}$ since they are completely determined by (17) and by the condition

$$dX \in H_4 \text{ (at zero–order in } \alpha').$$

(18)

Eq. (17) seems thus to be in contrast with the result, $X = 0$, of ref. [2].

To solve this puzzle we translate first the expression (17) in the “missing” contribution to the anomaly

$$A_X = -\frac{\alpha'}{8} \int d^2\sigma \sqrt{g} V^A V^B \Delta^\gamma X_{\gamma B A}$$

(19)

and note then that the three form $X$, determined through (17) and (18), admits the following remarkable decomposition at zero–order in $\alpha'$ (meaning that one can use the equations of pure supergravity to verify it):

$$X = dC + X^{(0)}$$

(20)

where the two–form $C$ is given by

$$C_{\alpha\beta} = 0$$
$$C_{\alpha\alpha} = -\frac{2}{3}(\Gamma_a \Gamma_{bc})_{\alpha\beta} T_{bc}^\beta$$
$$C_{ab} = -\frac{4}{3} R_{[ab]} - \frac{1}{3} T_{cd}^\alpha (\Gamma_{ab}^{cd})_{\alpha} \beta \lambda_{\beta}$$

(21)

and the three form $X^{(0)}$ is given by
\[
\begin{align*}
X_{\alpha\beta\gamma}^{(0)} &= 0 \\
X_{a\alpha\beta}^{(0)} &= -\frac{2}{3} R(\Gamma_a)_{\alpha\beta} \\
X_{a\beta\alpha}^{(0)} &= \frac{1}{3} (\Gamma_{ab})_{\alpha}^\beta D_\beta R \\
X_{a\beta\gamma}^{(0)} &= -\frac{1}{3} R_{abc} + \frac{1}{8} (\Gamma_{abc})_{\alpha}^\beta \left( \lambda_\alpha D_\beta R - \frac{1}{3} D_\alpha D_\beta R \right).
\end{align*}
\]

Here \( R_{ab} \equiv R^c_{\cdot\cdot\cdot\cdot bc} \) and \( R \equiv R^a_a \). Moreover

\[dX^{(0)} \in \mathcal{H}_4.\]

Notice that the property (23) holds actually \( \text{exactly} \) i.e. to all orders in \( \alpha' \), due to the definition (22).

Inserting (20) and (22) in (19) we get for the “missing” anomaly

\[
A_X = -\frac{\alpha'}{8} \int d^2 \sigma \sqrt{g} \left( V_-^C V_+^D \Delta^\gamma (dC)_{\gamma DC} + \frac{1}{3} V_-^a V_+^a \Delta^\alpha D_\alpha R + \frac{2}{3} RV_-^a V_+^a \Delta^\gamma \right),
\]

and it is not difficult to realize that this is actually a trivial cocycle

\[
A_X = \delta_\kappa \left( -\frac{\alpha'}{8} \int d^2 \sigma \sqrt{g} \left( V_-^A V_+^B C_{BA} + \frac{1}{3} V_-^a V_+^a R \right) \right)
\]

which can be eliminated by subtracting from the classical action a local counter term

\[
I \rightarrow I + \frac{\alpha'}{8} \int d^2 \sigma \sqrt{g} \left( V_-^A V_+^B C_{BA} + \frac{1}{3} V_-^a V_+^a R \right).
\]

Equivalently this trivial “anomaly” can be eliminated by the redefinitions

\[
B^* = B + \frac{\alpha'}{4} C \\
E^{a*} = \left( 1 + \frac{\alpha'}{24} R \right) E^a.
\]

This explains that in ref. [2] no non–trivial one–loop anomaly has been lost.

On the other side the exact solution of ref. [6] is based on the identity (BPT theorem)
where \( K \in H_4 \) exactly. However, this decomposition is not unique for two reasons: first, \( \tilde{X} \) is defined only modulo exact forms and, second, if we find a three form \( Z \) such that \( dZ \in H_4 \) the decomposition (27) holds also if we replace \( \tilde{X} \rightarrow \tilde{X} - Z \) and \( K \rightarrow K + dZ \in H_4 \). Now, since (20) holds at zero order we can write the exact relation

\[
\tilde{X} = dC + X^{(0)} + \alpha'X^{(1)}
\]

(28)

for some three–form \( X^{(1)} \). Substituting this in (27) we get

\[
trRR = d \left( \alpha'X^{(1)} \right) + K',
\]

(29)

where \( K' = K + dX^{(0)} \in H_4 \) due to the fact that (23) holds exactly. With (29) the Bianchi identity (15) becomes

\[
d \left( H + \frac{\alpha'^2}{4}X^{(1)} \right) = \frac{\alpha'}{4}(trFF - K') \in H_4
\]

(30)

meaning that with this decomposition there are no order–\( \alpha' \) corrections to the classical superspace constraints of \( H \), see (8), (they start at order \( \alpha'^2 \)) in complete agreement with the results from the \( \kappa \)–anomaly cancellation mechanism.

Equivalently, the solution of ref. [6] can be transformed to a solution where there are no order–\( \alpha' \) corrections to the classical \( H \) constraints by accompanying the redefinitions (26) with

\[
E^{*\alpha} = \left( 1 + \frac{\alpha'}{48}R \right) E^\alpha - \frac{\alpha'}{48} \left( (\Gamma_\alpha)^{\alpha\gamma} D_\gamma R \right) E^\alpha
\]

\[
\Omega^{*\alpha}_b = \Omega^{\alpha}_b - \frac{\alpha'}{24} E^\alpha (\Gamma_{ab})_\alpha^\beta D_\beta R.
\]

(31)

These redefinitions, apart from eliminating \( X^{(0)} \) and \( dC \) from the \( H \)– constraints, are just the ones which preserve (6).

The conclusions of this letter are that there is perfect agreement between results from the \( \kappa \)–anomaly cancellation mechanism and exact solutions to the
$H$--Bianchi identities determined previously, and that there exists an exact solution which at first order in $\alpha'$ becomes rather simple. At first order in $\alpha'$ one has in particular

\begin{align*}
T_{\alpha\beta}^a &= 2\Gamma_{\alpha\beta}^a \\
T_{\alpha\beta}^\gamma &= 2\delta_{(\alpha}^\gamma\lambda_{\beta)} - \Gamma^\gamma_{\alpha\beta}(\Gamma_g)^{\gamma\epsilon}\lambda_\epsilon \\
T_{\alpha\alpha}^b &= 0 \\
T_{\alpha\alpha}^{\beta}\ &= \frac{1}{4}(\Gamma^{bc})_{\alpha}^{\gamma}\Gamma_{\alpha\gamma}^b T_{abc} - \frac{\alpha'}{8}(\Gamma_a)_{\alpha\epsilon}\left(\text{tr}(\chi^\epsilon\chi^\beta) - \text{tr}(T^\epsilon T^\beta)\right) \\
R_{\alpha\beta ab} &= -\frac{\alpha'}{4}(\Gamma_{[a})_{\alpha\epsilon}\left(\text{tr}(\chi^\epsilon\chi^\varphi) - \text{tr}(T^\epsilon T^\varphi)\right)(\Gamma_{b]}_{\varphi\beta}) \\
R_{\alpha\beta bc} &= 2(\Gamma_a)_{\alpha\beta}\Gamma_{bc}^{\beta} + \frac{3\alpha'}{8}(\Gamma_{[a})_{\alpha\beta}\left(\text{tr}(F_{bc})\chi^\beta) - \text{tr}(R_{bc})T^\beta)\right)
\end{align*}

(32)

$F_{\alpha\beta} = 0$

$F_{\alpha\alpha} = 2(\Gamma_a)_{\alpha\beta}\chi^\beta$

$H_{\alpha\beta\gamma} = 0$

$H_{\alpha\alpha\beta} = 2(\Gamma_a)_{\alpha\beta}$

$H_{\alpha\beta\alpha} = 0$

$H_{\alpha\beta c} = T_{\alpha\beta c}$

Here $\chi^\alpha$ is the gluino superfield and $\text{tr}(T^\alpha T^\beta) = T_{ab}^\alpha T^{ba\beta}$ etc. The dilaton $\phi$ satisfies $\lambda_\alpha = D_\alpha \phi$.

Finally we would like to notice that, since with the above parametrizations the Bianchi identities close only up to order $\alpha'$, the $X^{(1)}$ appearing in (29) is clearly non vanishing and there has to be a non trivial contribution of the $X$--type, see (9), to the two--loop $\kappa$--anomaly.

Moreover, the so called “poltergeist”, appearing in N=1, D=10 supergravity theory when the Lorentz–Chern–Simons term is present, which in previous formulations [4,5,6] show up at order $\alpha'$, in the formulation corresponding to (32) are shifted to order $(\alpha')^2$. 
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String $\kappa$–anomalies and $D = 10$ Supergravity
constraints: the solution of a puzzle

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Abstract

The $\kappa$–anomaly cancellation mechanism in the heterotic superstring determines
the superspace constraints for $N=1$, $D=10$ Supergravity–Super–Yang–Mills theory.
We point out that the constraints found recently in this way appear to disagree
with superspace solutions found in the past. We solve this puzzle establishing
perfect agreement between the two methods.

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