Effects of fluctuations and color-neutrality in a finite volume

Christian Spieles\textsuperscript{1}, Marcus Bleicher\textsuperscript{2} and Carsten Greiner\textsuperscript{2}
\textsuperscript{1} Frankfurt Institute for Advanced Studies, Ruth-Moufang-Strasse 1, D-60438 Frankfurt am Main, Germany
\textsuperscript{2} Institut für Theoretische Physik, Goethe-Universität, Max-von-Laue-Strasse 1, D-60438 Frankfurt am Main, Germany

We investigate properties of strongly interacting matter in a schematic model, based on the combined degrees of freedom of a non-interacting hadronic phase and a non-interacting deconfined phase. It is found that in a finite system both phases contribute to the thermodynamic state due to fluctuations and that signatures of critical behaviour like the divergence of statistical quantities are damped. The constraint of color-neutrality leads to a volume-dependent shift of the effective critical temperature, which follows a scaling law, independent of the baryochemical potential. According to the model, observable baryon-number susceptibilities at a given $T$ and $\mu_B$ strongly depend on the system size. Finally, we compare hadronization conditions from the model with hadrochemical fits to experimental collider data, where a qualitatively similar system size dependence is extracted.

I. INTRODUCTION

The phase-structure of strongly interacting matter, described by Quantum Chromodynamics, has long been a particular focus of theoretical and experimental research (see Ref.\textsuperscript{1} for a review of basic concepts). Within the last two decades, tremendous progress has been made and is still under way, see, e. g., Refs.\textsuperscript{2,3} and Ref.\textsuperscript{4}, respectively. However, it is found that two main difficulties still impede a comprehensive theoretical understanding of the whole phase-structure backed by experimental evidence: One is the fact that rigorous calculations solving QCD for the baryon-rich regime are still not feasible. The other challenge is posed by the limited spatial and temporal scales of any experiment probing strongly-interacting matter under extreme conditions. Furthermore, local fluctuations of energy density within single heavy-ion collisions may play a decisive role for the signatures under investigation (see [5]). In the following, we revisit a schematic model of strongly interacting matter,\textsuperscript{6,7} which allows to explore some possibly relevant features of finite matter, even at $\mu_B \gg 0$. We also report some new findings in Secs. III and IV.

II. THE MODEL

The schematic two-phase model of strongly interacting matter presumes coexistence of two microscopically uncorrelated phases which are connected only via the macroscopic configuration, namely the volume fraction $\xi$ of one of the two phases (we choose the hadronic phase, i. e., $V_h = \xi V$ and $V_q = (1 - \xi)V$). In such a simplified set-up, any macroscopic configuration $\xi$ contributes with a probability $p(\xi) \sim \exp[-\Phi(\xi)/T]$ to the total system, $\Phi(\xi)$ being the grand canonical potential of the system for this particular configuration.\textsuperscript{8}

Since the partition function of the total system factorizes into the partition functions of the two individual phases for any fixed $\xi$, the grand canonical potential $\Phi$ of the total system in configuration $\xi$ can be expressed as

$$\Phi_\xi(T, \mu_B, V) = [\varphi_h(T, \mu_B, \xi)V]_\xi + \varphi_q(T, \mu_B, (1 - \xi)V)(1 - \xi)V,$$

where $\varphi_h$ and $\varphi_q$ are the densities of the grand canonical potential of the hadron gas and the quark-gluon phase, respectively. Any intensive thermodynamic quantity $A(T, \mu_B, V)$ describing the total system is then given as an expectation value according to the weight of all possible configurations:

$$A(T, \mu_B, V) = \int_0^1 \left[ p(\xi; T, \mu_B, V) \right] A_h(T, \mu_B, \xi V) \xi \, d\xi + A_q(T, \mu_B, (1 - \xi)V)(1 - \xi),$$

Note that for a system of infinite volume, the schematic model renders the Gibbs equilibrium condition according to which the two phases only coexist at $T_C$, where the pressures of the individual phases coincide, $p_h = p_q$. The phase transition in this case is of first order.\textsuperscript{9} For finite volumes, in contrast, the model equation (2) necessarily implies a smooth crossover of thermodynamic quantities which is not characteristic of a first order phase transition. However weak, fluctuations must lead to the presence of both phases for any value of $T$ and $\mu_B$, i. e. $0 < \langle \xi \rangle < 1$.

The model equation of state of the hadronic phase is based on an ideal relativistic quantum gas of well-established non-strange baryon and meson resonances up to masses of 2 GeV. Its density of the grand canonical potential is

$$\varphi_h = -\sum_i g_i \frac{g_i}{6\pi^2} \int_0^\infty dp \frac{p^4}{E_i \exp((E_i - \mu_i)/T) \pm 1},$$

\footnote{For sufficiently high values of the baryochemical potential this may well be in accordance with the true properties of strongly interacting matter. At $\mu_B = 0$, however, lattice QCD does not exhibit a first order phase transition.}
where "+" stands for fermions and "−" for bosons, \( g_i \) denotes the degeneracy of particle species \( i \). \( E_i = \sqrt{p_i^2 + m_i^2} \) is the energy of particle species \( i \) and \( \mu_i \) its chemical potential. All thermodynamic quantities are then corrected by the Hagedorn factor \( 1/(1 + \epsilon/4B) \) [9], where \( \epsilon \) is the ideal gas energy density and \( B \) is the bag pressure. The deconfined phase is thought of as an ideal relativistic quantum gas of massless quarks and gluons in a cavity, held together by the bag pressure. In the case of two quark flavors and with the constraint of color-neutrality and fixed total momentum, the corresponding of two quark flavors and with the constraint of color-neutrality and fixed total momentum, the corresponding density of the grand canonical potential for a spherical droplet of volume \( V \) can be approximated according to [10]:

\[
\varphi_q = -T/V \left[ \ln \left( \frac{1}{2V} \frac{1}{3\pi} C^{-4} D^{-3/2} \right) + X - Y \right] + B , \tag{4}
\]

where

\[
X = \pi^2 VT^3 \times \left[ \frac{37}{90} + \left( \frac{\mu_q}{\pi T} \right)^2 + \frac{1}{2} \left( \frac{\mu_q}{\pi T} \right)^4 \right] \\
Y = \pi \left( \frac{3V}{4\pi} \right)^{1/3} \frac{38}{9} + 2 \left( \frac{\mu_q}{\pi T} \right)^2 . \tag{5}
\]

Note, that \( Z_0 = \exp (X - Y) \) is the grand partition function without overall constraints. The remaining parameters for the color and momentum constraints in (4) are

\[
C = 2VT^3 \left[ \frac{4}{3} + \left( \frac{\mu_q}{\pi T} \right)^2 \right] + \frac{20}{3\pi} \left( \frac{3V}{4\pi} \right)^{1/3} T \tag{6}
\]

and

\[
D = 2X - \frac{1}{3} Y . \tag{7}
\]

The bag constant is set to \( B^{1/4} = 215 \text{ MeV} \). For infinite volumes, this corresponds to a critical temperature of \( T_C^\infty \approx 155 \text{ MeV} \) at \( \mu_B = 0 \) in the two-phase coexistence model [2].

### III. VOLUME-DEPENDENCE OF THE EFFECTIVE CRITICAL TEMPERATURE

As was shown in Ref. [6], the color-singlet constraint imposed on the model equation-of-state leads to a volume dependent shift of the "effective critical temperature", \( \Delta T_C(V) = T_C^\infty(V) - T_C^\infty \). In the present study, we define this temperature as the point where both phases contribute — on the average — with equal probability to the macroscopic state of the system, i.e. \( \langle \xi(T_C^\infty(V)) \rangle = 1/2 \). The reason for the shift is that the quark-gluon phase in a finite volume is associated with a smaller grand canonical potential density than in the infinite volume limit. The effective number of degrees of freedom in the quark-gluon phase is reduced in small systems, since only color-neutral combinations of microstates are taken into account in the partition function.

It is expected that the volume dependence of the shift of the critical temperature exhibits a universal scaling behaviour \( \Delta T_C(V) \sim V^{-\lambda} \), which characterizes fundamental properties of the physical system. E. g., [12] have advocated a value of \( \lambda \approx 1 \) as a signature of a first order phase transition. Fig. 1 shows an analysis of \( \Delta T_C(V) \) for different values of the baryochemical potential according to the schematic model. In fact, although the respective scenarios reflect very different physical conditions, a universal scaling exponent of \( \lambda \approx 0.75 \) can be extracted from the numerical analysis. Note that not only the values of the scaling exponent coincide but that also the absolute temperature shifts are very close. Furthermore, the scaling law also holds for different values of the bag constant \( B \) — the one system parameter that physically governs the interdependence of the two phases, which are microscopically uncorrelated.

We contrast our result with findings of [11] where the

---

2 This value agrees with the the chiral transition temperature derived from lattice calculations (see Ref. [2] and references within).
deconfining phase transition of pure SU(3) lattice gauge theory has been investigated by Markov-chain Monte Carlo simulations. In their study, the authors have calculated small volume corrections for the pseudo-transition temperature in a scenario of cold boundary conditions. We show the resulting temperature shifts in Fig. 1 in order to compare it with the scaling law of our model. As can be seen, the absolute temperature shift in the pure SU(3) lattice gauge simulation for a given system size differs considerably from the schematic two-phase model. However, the volume dependence indicates a scaling exponent in the range of $0.75 < \lambda < 1$, which is compatible with the model result. In this context, we would like to draw the attention to the work of [13] where a similar schematic two-phase coexistence model than the one presented in Ref. [6] has been the basis for an analysis of finite-size effects and scaling exponents in the deconfinement phase transition (at vanishing net-baryon density). In their approach, the authors extract a temperature shift scaling exponent of $\lambda = 0.876 \pm 0.041$.

### IV. Hadronization of a Finite Quark-Gluon Plasma

The hadrochemical composition of the final state of any type of high energy collision has been successfully described in terms of thermal models, see, e.g., Refs. [13, 14]. In a recent work, the charged particle multiplicity from a wide range of system sizes has been systematically analyzed using different statistical ensembles [16]. It turns out that the most rigorous theoretical ansatz reviewed by the authors, the grand canonical ensemble with exact conservation of strangeness, baryon number and electrical charge, exhibits an interesting feature, namely an apparent system size dependence of the extracted chemical freeze-out temperature. This is shown in Fig. 2 where results of Ref. [16] are reproduced: The freeze-out temperature corresponding to final states of collisions with low charged particle multiplicity is significantly higher than for high charged particle multiplicities. This is the behaviour we expect from the schematic two-phase coexistence model, if the transition temperature of the quark-gluon plasma created in a high energy collision, determines the hadrochemical freeze-out state. In Fig. 2 we have plotted the effective critical temperature from the two-phase coexistence model as a function of the charged particle multiplicity. The corresponding system volumes are determined by the freeze-out radii from Ref. [16]. The width of the resulting temperature band reflects the error bars of the extracted freeze-out radii. Interestingly, analyzing the volume scaling of the hadrochemical fits to the experimental data, one obtains $\lambda \approx 1$.

![Fig. 2](image-url)

**FIG. 2.** The relevant critical temperature from the two-phase model as a function of charged particle multiplicity for proton-proton, proton-nucleus and nucleus-nucleus collisions at TeV energies (shown as a red band). Also shown are the hadrochemical fits to the same experimental data within a canonical description with exact conservation of baryon number, strangeness and electrical charge [16].

### V. Susceptibilities of Baryon Number in the Finite System

In order to probe the phase structure of strongly interacting matter, event-by-event fluctuations and correlations in relativistic heavy-ion collisions have been proposed as a promising approach (see, e.g., Ref. [4]). Measured cumulants of conserved quantities can be compared with corresponding susceptibilities from lattice QCD calculations [2, 3]. They are believed to provide rather robust signatures of the underlying thermodynamics. The susceptibilities of the baryon number can be calculated as

$$\chi_{B}^{i} = \frac{-\partial^2 \hat{\phi}}{\partial \bar{\mu}_{B}^{2}} ,$$

from the dimensionless density of the grand canonical potential $\hat{\phi} = \Phi(T, \mu_{B}, V)V^{-1}T^{-4}$, where $\bar{\mu}_{B} = \mu_{B}/T$ is the reduced baryochemical potential. The second order susceptibility is proportional to the variance of the net-baryon number:

$$\chi_{B}^{ii}(V)VT^3 = \sigma_{B}^2 = \langle (\delta N_{B})^2 \rangle > .$$

The ratio of the fourth to second order susceptibility is of particular interest, since the volume and temperature
terms in the definition of the susceptibilities (8) cancel out, when the ratio is used. This ratio can be measured experimentally as

$$\frac{\chi_B^4}{\chi_B^2} = \kappa_B \sigma_B^2 \quad ,$$  \hspace{1cm}   (10)

where the excess kurtosis is given by

$$\kappa_B = \frac{\left< (\delta N_B)^4 \right>}{\left< (\delta N_B)^2 \right>} - 3 \quad .$$  \hspace{1cm}   (11)

We begin our analysis of baryon number susceptibilities in finite volumes by employing an equation-of-state for the deconfined phase without explicit volume dependence due to color-neutrality, i.e. we use $\phi_q (V = \infty)$ according to (4) in [1]. By this, we want to isolate the effect of fluctuations of the two-phase composition in a finite system. Although suppressed exponentially, the presence of the quark-gluon phase below the critical temperature $T_C^\infty$ and the presence of the hadronic phase above $T_C^\infty$ has a finite probability. Figure 3 shows the fourth to second order baryon number susceptibility ratio $\chi_B^4 / \chi_B^2$ as function of temperature at $\mu_B = 0$ for different system sizes. Since the two-phase coexistence model represents a physical system with a first-order phase transition in the infinite-volume limit, we expect a critical behaviour of thermodynamic quantities. Indeed, for a large volume, $V = 10^4 \text{ fm}^3$, the two-phase model exhibits a strong divergence of the susceptibility ratio at $T_C$. This corresponds to extreme net-baryon number fluctuations on an event-by-event basis in a small temperature range. However, as shown in Fig. 3, the divergence of $\chi_B^4 / \chi_B^2$ is damped for smaller volumes by orders of magnitude.

This is plausible since a finite system implies a maximum correlation length. The absolute effect of the system fluctuating between the two phases is limited by the boundaries of the finite volume. Note that in all cases, the value of $\chi_B^4 / \chi_B^2$ approaches 1 (expected for an ideal hadron gas) for $T \ll T_C$ and $2/(3\pi^2)$ (expected for a gas of free, massless $u/d$ quarks and gluons) for $T \gg T_C$. We conclude that in real experiments of colliding nuclei, theoretically revealing signatures of the phase structure like "critical fluctuations" might be strongly smeared out and suppressed due to the limited reaction sizes.

In the following, we drop the unrealistic simplification of a quark-gluon phase without color-singlet constraint. As was discussed in Sec. III, the model exhibits a volume-dependent shift of the effective critical temperature independent of the baryochemical potential. This effect is naturally reflected in the baryon number susceptibilities. Fig. 4 shows the fourth to second order baryon-number susceptibility ratio $\chi_B^4 / \chi_B^2$ as a function of temperature for $V = 50 \text{ fm}^3$ for different values of $\mu_B$, contrasted with the respective infinite volume case. Even for $V = \infty$, the susceptibility ratios exhibit an interesting feature due to the properties of the individual phases: The hadronic resonance gas with Hagedorn-correction shows strongly reduced values of $\chi_B^4 / \chi_B^2$ with increasing baryochemical potential at temperatures close to the phase transition. The susceptibility ratio of the pure quark-gluon phase, on the other hand, is virtually independent of $\mu_B$ and $T$.
As a consequence, for low values of $\mu_B < 300$ MeV, the susceptibility ratio of the hadronic phase is significantly higher than that of the quark-gluon phase at $T_C$, while the contrary is true for $\mu_B > 300$ MeV. Now we compare the described characteristics of the infinite-volume equations-of-state with the result of the two-phase coexistence model in a finite volume, $V = 50$ fm$^3$. The shift of the effective critical temperature by $\Delta T_C \approx 25$ MeV, represents a significant "superheating" of the hadronic phase. This means, that the thermodynamic properties of the hadronic phase prevail in the temperature range, where the susceptibility ratios of the hadronic phase have been found to be strongly dependent on $\mu_B$, changing from values of $\approx +1$ to $\approx -1$ within $\Delta \mu_B \approx 300$ MeV. For this reason, according to the two-phase coexistence model, the susceptibility ratios in a finite volume at low baryochemical potential are affected contrarily to the same system at high baryochemical potential, when compared to infinite matter. This may point to complications in experimental studies, where fundamental properties of strongly interacting matter are supposed to be extracted from observable statistical observables.

VI. SUMMARY

We have demonstrated that in the schematic two-phase coexistence model of strongly interacting matter, the shift of the effective critical temperature as a function of system size follows a scaling law independent of the baryochemical potential. The extracted scaling exponent $\lambda \approx 0.75$ is found to be compatible with pure SU(3) lattice gauge calculations. We find further qualitative support for the model in recent hadrochemical analyses of charged particle multiplicities within a canonical formalism with conserved quantum numbers.

ACKNOWLEDGMENTS

This work was supported by the Helmholtz International Center for FAIR within the framework of the LOEWE program launched by the State of Hesse. The computational resources were provided by the Center for Scientific Computing (CSC) of the Goethe University Frankfurt. This work has been supported by COST Action THOR (CA15213).

[1] H. Meyer-Ortmanns, Rev. Mod. Phys. 68, 473 (1996).
[2] A. e. a. Bazavov, Phys. Rev. D 95, 054504 (2017).
[3] J. N. Guenther, R. Bellwied, S. Borsanyi, Z. Fodor, S. Katz, A. Pasztor, C. Ratti, and K. K. Szabo, PoS CPoD2017, 032 (2018).
[4] X. Luo and N. Xu, Nucl. Sci. Tech. 28, 112 (2017).
[5] M. Bleicher, L. Gerland, C. Spieles, A. Dumitru, S. Bass, M. Belkacem, M. Brandstetter, C. Ernst, L. Neise, S. Sof, H. Weber, H. Stöcker, and W. Greiner, Nucl. Phys. A 638, 391 (1998).
[6] C. Spieles, H. Stöcker, and C. Greiner, Phys. Rev. C 57, 908 (1998).
[7] C. Spieles, M. Bleicher, and C. Greiner, J. Phys. G: Nucl. Part. Phys. 46, 025101 (2019).
[8] L. D. Landau, E. M. Lifshitz, and L. P. Pitaevskiy, Theoretical Physics, Vol. 5 Statistical Physics, Sec. 146 (Nauka, Moscow, 1976).
[9] R. Hagedorn and J. Rafelski, Phys. Lett. B 97, 136 (1980).
[10] H. T. Elze and W. Greiner, Phys. Lett. B 179, 385 (1986).
[11] B. A. Berg and H. Wu, Phys. Rev. D 88, 074509 (2013).
[12] K. Binder and D. P. Landau, Phys. Rev. B 30, 1477 (1984).
[13] M. Ladrem and A. Ait-El-Djoudi, Eur. Phys. J. C 44, 257 (2005).
[14] A. Andronic, P. Braun-Munzinger, K. Redlich, and J. Stachel, Nature 561, 321 (2018), arXiv:nucl-th/1710.09425.
[15] R. Stock, F. Becattini, M. Bleicher, and J. Steinheimer, Nucl. Phys. A 982, 827 (2019).
[16] N. Sharma, J. Cleymans, B. Hippolyte, and M. Parazda, Phys. Rev. C 99, 044914 (2019).

$^3$ Note that the critical temperature $T_C$ itself depends on the baryochemical potential. However, this is qualitatively irrelevant for the effect discussed here.