Feedback cooling, measurement errors, and entropy production

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Abstract. The efficiency of a feedback mechanism depends on the precision of the measurement outcomes obtained from the controlled system. Accordingly, measurement errors affect the entropy production in the system. We explore this issue in the context of active feedback cooling by modeling a typical cold damping setup as a harmonic oscillator in contact with a heat reservoir and subjected to a velocity-dependent feedback force that reduces the random motion. We consider two models that distinguish whether the sensor continuously measures the position of the resonator or directly its velocity (in practice, an electric current). Adopting the standpoint of the controlled system, we identify the ‘entropy pumping’ contribution that describes the entropy reduction due to the feedback control and that modifies the second law of thermodynamics. We also assign a relaxation dynamics to the feedback mechanism and compare the apparent entropy production in the system and the heat bath (under the influence of the controller) to the total entropy production in the super-system that includes the controller. In this context, entropy pumping reflects the existence of hidden degrees of freedom and the apparent entropy production satisfies fluctuation theorems associated with an effective Langevin dynamics.

Keywords: stochastic particle dynamics (theory), stationary states

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1. Introduction

Active feedback cooling is a well-established technique which is used to reduce the effective noise temperature of mechanical oscillators well below their operating temperature. It is now used in a wide variety of nano-electromechanical systems, and is a key ingredient for measuring force and mass with very high sensitivity, for limiting thermal noise in gravitational wave detectors, and for reaching the quantum regime of mechanical motion [1]. A commonly used procedure named cold damping consists in measuring the resonator displacement in real time and then applying through a feedback loop a velocity-proportional external force that increases the damping rate. As a result, the Brownian motion of the resonator (for instance, the mirror of an interferometric detector [2] or the cantilever of an AFM [3]) is reduced. Ultimately, the feedback cooling efficiency (that is, the minimum achievable temperature) is bounded by the noise of the detector (that is, by the errors in the measurements).

Such a feedback loop thus plays the role of an external agent that detects the microscopic state of the system and then acts to modify its dynamical evolution. It is therefore natural to wonder how the information acquired through the measurement modifies the thermodynamics of the system, in particular the entropy balance equation and the second law. This issue, which is at the crossroads of information theory and nonequilibrium statistical mechanics, has attracted much attention recently [4]–[15], in
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relation to significant advances in single-molecule manipulations and new fundamental developments in the stochastic thermodynamics of small systems [16]–[18]. Within this framework, nonequilibrium relations such as Jarzynski equality [19] and fluctuation theorems (FT) [20] have been generalized to systems under discrete feedback control by taking into account the mutual information between the state of the system and the measurement outcome. Mutual information quantifies the entropy reduction due to the interaction with the external agent (hereafter also called the controller) and provides a lower bound to the entropy production (EP) in the system. Measurement errors decrease mutual information, and thus limit the entropy reduction and the efficiency of the feedback control.

These results, however, are not directly applicable to cold damping. First, measurements and actuation are performed continuously in this process, i.e., repeated with a period shorter than the characteristic time scales of the system dynamics. In practice, the motion of the feedback-cooled resonator in the vicinity of a resonant frequency is faithfully described by an underdamped Langevin dynamics. Second, the feedback controller is not a genuine Maxwell’s demon that only exchanges information with the system. Indeed, the feedback force performs work on the system and the energetics (i.e. the first law) is thus modified. One expects the formulation of the second law to be modified as well, even for error-free measurements (in such a situation, mutual information diverges since observables are continuous [11, 15]). This issue was first explored in [21], where it was shown that entropy is indeed continuously pumped out of the system by the external agent in the nonequilibrium steady state (NESS). This can be ascribed to the contraction of momentum phase space due to the additional damping. This entropy pumping modifies the second law, Jarzynski equality, and fluctuation theorems [22]. The fact that the feedback force is velocity-dependent makes the formulation of the detailed FT somewhat peculiar, as stressed in a previous work [23].

The goal of this paper is to include the effect of measurement errors in this description. How is the entropy pumping in the NESS affected by the detector noise? Although the influence of noise on the EP in a cold damping process has been explored in a recent work [24], this question has not been addressed [25]. Indeed, the quantity studied in [24] is the EP in the ‘super-system’, which includes the Brownian entity (i.e. the resonator), the heat bath, and the feedback controller. The contribution of the controller to the EP is interpreted as resulting from the measurement process and is obtained by taking an appropriate continuous limit of a discrete series of noisy measurements. In the super-system, the usual second law is obeyed and entropy pumping plays no role. Furthermore, the EP diverges in the limit of error-free measurements and the results of [21, 22] are thus not recovered. This clearly shows that the level of description of the system is different. To generalize the analysis of [21]–[23], one must instead consider the EP from the viewpoint of the controlled system: entropy pumping (like mutual information) then describes the entropy reduction due to the interaction of the system with the controller [12]. This is also

3 Throughout this paper, the term ‘feedback controller’ denotes both the sensor that measures the state of the system and the actuator that modifies its dynamics.
4 The relationship between a single noisy measurement of the velocity and the violation of the fluctuation dissipation theorem in a cold damping process is investigated in [11].
5 In this respect, the use of the term ‘super-system’ in our previous work [23] was inappropriate.

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the viewpoint adopted in the information-theoretic approach of control systems [26]. At this level of description, one is neither interested in knowing the actual energy dissipation inside the controller nor in evaluating the ‘thermodynamic’ cost of measurement [27] (see also [15] and references therein)\(^6\).

This important distinction between the two levels of description becomes even clearer when the controller has its proper dynamics. The super-system is then characterized by several (slow) degrees of freedom and adopting the standpoint of the controlled system means that the degree(s) of freedom of the controller is (are) projected out. In other words, the super-system is coarse-grained, which in turn changes the definition of the EP. This issue of coarse graining has also attracted much attention recently [29]–[37], including on the experimental side [38]. As a rule, one expects that an incomplete description of the dynamics, due to the existence of hidden degrees of freedom or to a low resolution measuring apparatus, results in an underestimation of the EP in the full system.

In this work, we shall indeed assign a relaxation dynamics to the controller. Physically, this may account for the fact that the feedback circuit cannot follow instantaneously the dynamics of the system when the resonator frequency is high (say above 1 MHz). This makes the feedback control non-Markovian and degrades the cooling performance [1]. On the other hand, this procedure allows us to study the EP in a typical cold damping setup in which the position of the resonator (for instance a cantilever [3], [39]–[41], a suspended mirror in a cavity [2], or an optically trapped microsphere [42]) is continuously monitored. In this case, the velocity-proportional feedback force also includes the derivative of the detector noise\(^7\), which is obviously problematic when the noise spectrum is flat (i.e. the noise is white), as is usually assumed (see e.g. [2, 3, 41]). In this case, the relaxation time plays the role of a cutoff that regularizes divergent quantities. On the other hand, no regularization is needed when the measured observable is directly the velocity (in practice, the current in a RLC electric circuit), a situation that corresponds to the electronic feedback cooling schemes described in [43]–[45]. The theoretical analysis of this second setup is simpler and the Markovian limit of the feedback control is well defined (this is actually the case considered in [11, 24]).

The rest of the paper is organized as follows. In section 2, we specify the two models (hereafter called P for position and V for velocity) that are investigated and that correspond to the two experimental situations that have just been described. In section 3, we review the results of [21]–[23] in order to make the whole account of the paper self-explanatory and coherent. In section 4, measurement errors are included and in section 4.1 model V is first studied in the Markovian limit. This is the simplest case for which the expression of entropy pumping can be generalized. The full versions of models V and P are then successively studied in sections 4.2 and 4.3. We summarize the main points and conclude in section 5. Some additional (but important) calculations are detailed in three appendices.

\(^6\) More generally, one does not gain much insight into the physics of the problem by investigating the EP in the super-system unless one has a precise description of the actual mechanisms that take place inside the controller. The case of genuine Maxwell’s demons is different and the problem can be investigated on a case-by-case basis (see e.g. [28]).

\(^7\) See e.g. figure 22 in [1] for a schematic diagram of a cold damping setup.

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The physical systems studied in this paper are described by the underdamped one-dimensional Langevin equation

\[ m \ddot{x} + \gamma \dot{x} + kx = F_{\text{th}}(t) + F_{\text{fb}}(t) \]  

(1)

where \( x(t) \) denotes the position of the resonator as a function of time, \( m \) is an effective mass, \( k \) is a spring constant, and \( \gamma \) is a linear damping. As usual, the resonator dynamics can be also described in terms of the angular resonance frequency \( \omega_0 = \sqrt{k/m} \) and the intrinsic quality factor \( Q_0 = \omega_0 \tau_0 = \sqrt{mk/\gamma} \), where \( \tau_0 = m/\gamma \) is the viscous relaxation time. Equation (1) correctly describes the small displacements of nanomechanical systems around the resonance frequency of a normal mode \[1\]. Alternatively, it may also describe a RLC electrical circuit: \( x(t) \) then represents the charge of the capacitor whereas the velocity \( v(t) \equiv \dot{x}(t) \) is the current through the inductor (with the resistor \( R \), inductor \( L \), and capacitor \( C \) such that \( \gamma = R/L, \omega_0^2 = 1/LC \) and \( m = L \)). One may also simply regard equation (1) as modeling the dynamics of a Brownian particle confined by a harmonic potential.

\( F_{\text{th}}(t) = \sqrt{2\gamma T} \xi(t) \) represents the thermal random force generated by the surrounding medium at temperature \( T \), and \( \xi(t) \) is a Gaussian white noise with zero mean and correlation \( \langle \xi(t)\xi(t') \rangle = \delta(t-t') \) (for notational simplicity, Boltzmann’s constant is adsorbed in the temperature throughout this paper). \( F_{\text{fb}}(t) \) is the velocity-proportional control force which is applied to the resonator via the feedback loop. We consider the following two models.

(i) Model V, in which the velocity \( v(t) \) is continuously measured and the output signal of the detector is \( v'(t) = v(t) + v_n(t) \), where \( v_n(t) \) is the measurement noise. The feedback force is then given by

\[ F_{\text{fb}}(y) = -\gamma' y(t) \]  

(2)

where \( \gamma' = g\gamma \) (\( g \) is the variable gain of the feedback loop), and \( y(t) \) obeys the differential equation

\[ \dot{y} = -\frac{1}{\tau}[y - v'] \]  

(3)

where \( \tau \) is the relaxation time of the feedback circuit.

(ii) Model P, in which the displacement \( x(t) \) is the observable and the output signal of the detector is \( x'(t) = x(t) + x_n(t) \), where \( x_n(t) \) is the measurement noise. In this case

\[ F_{\text{fb}}(\dot{y}) = -\gamma' \dot{y}(t) \]  

(4)

where

\[ \dot{y} = -\frac{1}{\tau}[y - x'] \]  

(5)

Since it is generally observed that the power spectral density of the measurement noise is flat in the frequency band of interest, we assume that \( v_n(t) \) and \( x_n(t) \) are delta-correlated in time:

\[ v_n(t) = \sqrt{S_{v_n}} \eta(t) \]

\[ x_n(t) = \sqrt{S_{x_n}} \eta(t) \]

(6)

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where \( \eta(t) \) is a Gaussian white noise with zero mean and correlation \( \langle \eta(t) \eta(t') \rangle = \delta(t-t') \). We moreover assume that the two noises \( \eta(t) \) and \( \xi(t) \) are uncorrelated and that the spectral densities \( S_{\eta} \) and \( S_{\xi} \) do not depend on the gain \( g \), which is a reasonable approximation\(^8\).

Model V is thus defined by the two coupled linear Langevin equations

\[
\begin{align*}
\dot{m} \ddot{x} + \gamma \ddot{x} + k x + \gamma' y &= \sqrt{2\gamma T} \xi(t) \quad (7a) \\
\tau \dot{y} + y - \dot{x} &= \sqrt{S_{\eta}} \eta(t), \quad (7b)
\end{align*}
\]

whereas model P is defined by

\[
\begin{align*}
\dot{m} \ddot{x} + \gamma \ddot{x} + k x + \gamma' y &= \sqrt{2\gamma T} \xi(t) \quad (8a) \\
\tau \dot{y} + y - x &= \sqrt{S_{\eta}} \eta(t), \quad (8b)
\end{align*}
\]

or better

\[
\begin{align*}
\dot{m} \ddot{x} + \gamma \ddot{x} + k x + \frac{\gamma'}{\tau} (x - y) &= \sqrt{2\gamma T} \xi(t) - \frac{\gamma'}{\tau} \sqrt{S_{\eta}} \eta(t) \quad (9a) \\
\tau \dot{y} + y - x &= \sqrt{S_{\eta}} \eta(t), \quad (9b)
\end{align*}
\]

which is the form under which the model can be numerically studied.

It is worth noting that the physical processes described by the above equations are Markovian if both \( x(t) \) and \( y(t) \) are observed, whereas the effective dynamics of \( x(t) \) obtained by solving equations \( (7b) \) or \( (9b) \) for \( y(t) \) and inserting the result into \( (7a) \) or \( (9a) \) is non-Markovian\(^9\). This transformation is discussed in detail in [47, 48] and more recently in [35] for a system of coupled linear Langevin equations quite similar to equations (7). These equations were originally regarded as modeling the irreversible dynamics of a massive tracer in a granular fluid [49], which is a quite different physical situation from the one considered here. However, the discussion in [35] about the influence of cross-correlations among different degrees of freedom on the entropy production is relevant to the present work.

A Markovian description is of course recovered in the limit \( \tau \to 0 \) as \( y(t) \to v'(t) \) and \( y(t) \to x'(t) \) in model V and P, respectively (note that the measurement itself is always Markovian since the measurement outcomes \( v' \) and \( x' \) do not depend on the state of the system at previous times). The motion of the resonator is then simply described by

\[
\begin{align*}
\dot{m} \ddot{x} + (\gamma + \gamma') \dot{x} + k x &= \sqrt{2\gamma T} \xi(t) - \gamma' v_n(t) \quad (10)
\end{align*}
\]

in model V, and by

\[
\begin{align*}
\dot{m} \ddot{x} + (\gamma + \gamma') \dot{x} + k x &= \sqrt{2\gamma T} \xi(t) - \gamma' \dot{x}_n(t) \quad (11)
\end{align*}
\]

\(^8\) In experiments, the detector noise is usually obtained by fitting the measured spectral density with a theoretical expression such as equation (A.11b) or equation (A.13b) (with \( \tau = 0 \)). It is found that the noise parameters do not depend significantly on the feedback gain \( g \) (see e.g. [3] in the case of a mechanical resonator and [45] in the case of an electrical resonator).

\(^9\) The influence of memory terms on the behavior of feedback-controlled harmonic oscillators is also considered in [46] in reference to feedback-cooled electromechanical oscillators, such as the gravitational wave detector AURIGA [43, 44]. In that study, however, the detector noise is not taken into account and the stochastic thermodynamics of the system is not investigated.
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in model P. Equation (10) with \( k = 0 \) is the equation studied in [11, 24]. It also corresponds to a simplified version of the model considered in [45] that describes the feedback cooling of a macroscopic electromechanical oscillator. On the other hand, equation (11) is typically used to describe the feedback cooling of a cantilever [3]. It is clear, however, that this equation is ill-defined mathematically if \( x_n(t) \) is a white noise. Therefore, as we already pointed out, the introduction of a finite relaxation time \( \tau \) may also be regarded as a way to circumvent this problem without having to describe the frequency dependence of the measurement noise, which is generally unknown. In practice, for the cooling to be efficient, \( \tau \) must be much smaller than \( m/(\gamma + \gamma') \), the effective viscous relaxation time of the feedback-controlled oscillator, so that \( y(t) \) (in model V) and \( \dot{y}(t) \) (in model P) follow \( v(t) \) fast enough. In other words, the dynamics of the controller must be much faster than the dynamics of the system, otherwise the information about the instantaneous velocity is lost.

3. Entropy production with error-free measurements: a reminder

Entropy production in a cold damping setup in the absence of measurement errors was first investigated in [21, 22] and revisited by us in a previous work [23]. The feedback force is given by

\[
F_{fb}(v) = -\gamma'v(t)
\]

and the system is thus described by the Langevin equation

\[
m\ddot{x} + (\gamma + \gamma')\dot{x} + kx = \sqrt{2\gamma T}\xi(t)
\]

which is also obtained from equations (10) and (11) for \( x_n(t) = x_m(t) = 0 \). Accordingly, the probability distribution function \( p_t(x, v) \) at the ensemble level satisfies the Fokker–Planck (FP) equation

\[
\partial_t p_t(x, v) = -\frac{1}{m} \partial_v \left[ v p_t(x, v) + \frac{\gamma T}{m} \partial_v p_t(x, v) \right] + J_t(x, v)
\]

which is conveniently rewritten as

\[
\partial_t p_t(x, v) = -\partial_x [vp_t(x, v)] - \frac{1}{m} \partial_v [-(kx + \gamma'v)p_t(x, v) + J_t(x, v)],
\]

where

\[
J_t(x, v) = -\gamma \left[ v + \frac{T}{m} \partial_v \ln p_t(x, v) \right] p_t(x, v)
\]

is a probability current (in what follows, time is often put as an index for better readability). Although one can directly derive the entropy balance equation at the ensemble level [21], it is helpful to consider the various thermodynamic quantities at the level of an individual stochastic trajectory, as done in [22, 23], in order to better understand the origin of entropy pumping. The ensemble average is then taken in a second stage.

Let \( \{x_s\}_{s \in [0,t]} \) denote a trajectory generated by equation (13) during the time interval \( 0 \leq s \leq t \) with an initial state drawn from some probability distribution \( p_0(x, v) \). Within the framework of stochastic energetics [16], the energy balance equation (or first law) is obtained by multiplying equation (13) by \( \dot{x}_t \) and integrating over the time interval \( [0, t] \).
This yields\(^\text{10}\)
\[
\Delta E = w[\{x_s\}] - q[\{x_s\}]
\]
where
\[
\Delta E = \int_0^t ds \left[ m\ddot{x}_s + kx_s \right] \circ \dot{x}_s = \frac{m}{2} \left[ v_t^2 - v_0^2 \right] + \frac{k}{2} \left[ x_t^2 - x_0^2 \right]
\]
is the change in the internal energy of the (Brownian) system,
\[
w[\{x_s\}] = \int_0^t ds \, F_{\text{fb}}(\dot{x}_s) \circ \dot{x}_s
\]
\[
= -\gamma' \int_0^t ds \, \dot{x}_s^2
\]
is the work done by the feedback force on the system, and
\[
q[\{x_s\}] = \int_0^t ds \left[ \gamma \dot{x}_s - \sqrt{2\gamma T} \xi_s \right] \circ \dot{x}_s
\]
\[
= -\int_0^t ds \left[ m\ddot{x}_s + \gamma' \dot{x}_s + kx_s \right] \circ \dot{x}_s
\]
is the heat dissipated into the thermal environment\(^\text{11}\), which can also be identified with an entropy increase in the medium \(\Delta s_m[\{x_s\}] = q[\{x_s\}]/T\). The crucial point is that this quantity can also be written in the form \(^\text{23}\)
\[
\Delta s_m[\{x_s\}] = \ln \frac{P_+[\{x_s\} | x_0, v_0]}{P_-[\{\dot{x}_s\} | \dot{x}_0, \dot{v}_0]} - \frac{\gamma'}{m} t
\]
where
\[
P_+[\{x_s\} | x_0, v_0] \propto \exp \left[ \frac{\gamma + \gamma'}{2m} t - \frac{1}{4\gamma T} \int_0^t ds \left( m\ddot{x}_s + (\gamma + \gamma') \dot{x}_s + kx_s \right)^2 \right]
\]
is the conditional weight of the path \(\{x_s\}_{s \in [0,t]}\) given the initial point \((x_0, v_0)\) and
\[
P_-[\{\dot{x}_s\} | \dot{x}_0, \dot{v}_0] \quad \text{(defined by} \quad \dot{x}_s \equiv x_{t-s}, \dot{x}_s \equiv -\dot{x}_{t-s})\]
\[
\Delta s_{\text{pu}} = -\frac{\gamma'}{m} t
\]
is interpreted as an ‘entropy pumping’ arising from the contraction of momentum phase space due to the feedback force\(^\text{12}\). This is a unique feature of a velocity-dependent feedback control \(^\text{21, 22}\). As stressed in \(^\text{23}\), the fact that \(\gamma'\) must be treated as an odd variable under time reversal in order to relate \(\Delta s_m[\{x_s\}]\) to the microscopic irreversibility of trajectories is not harmless: it implies that there is no steady state with the conjugate dynamics when \(\gamma' > \gamma\), which is the common situation encountered in cold damping setups.

\(^{10}\) Throughout this paper, products of stochastic variables and stochastic integrals are defined within the Stratonovich interpretation and denoted by \(\circ\).

\(^{11}\) By convention, we assign a positive sign to \(q[\{x_s\}]\) if the energy is dissipated into the bath.

\(^{12}\) To trace back the origin of the term \((\gamma + \gamma')/\sqrt{2m}\) in the path probability \(P_+[\{x_s\} | x_0, v_0]\), see e.g. \([53]\)–\([55]\). The normalization factor cancels out when taking the ratio of the two probabilities.
Introducing the stochastic entropy of the Brownian system [50]–[52],

\[ s_{\text{sys}}(t) = -\ln p_t(x_t, v_t) \] (24)

where \( p_t(x_t, v_t) \) is the solution of the Fokker–Planck equation evaluated along the trajectory, the entropy production for each realization of the stochastic process in the time interval \([0, t]\) is then defined as

\[ \sigma[\{x_s\}] \equiv \ln \frac{\mathcal{P}_+[\{x_s\}]}{\mathcal{P}_-[\{\dot{x}_s\}]} = \Delta s_{\text{sys}} + \Delta s_m[\{x_s\}] - \Delta s_{\text{pu}} \] (25)

where

\[ \Delta s_{\text{sys}} = \ln \frac{p_0(x_0, v_0)}{p_t(x_t, v_t)}. \] (26)

As a direct consequence of equations (21) and (26), \( \sigma[\{x_s\}] \) obeys the integral fluctuation theorem (IFT)

\[ \langle e^{-\sigma[\{x_s\}]} \rangle = 1 \] (27)

where \( \langle \cdots \rangle \) denotes a functional average over all paths \( \{x_s\}_{s \in [0, t]} \) generated by equation (13) that start from \((x_0, v_0)\), as well as averages over initial and final positions and velocities. One can also derive a detailed FT which has a nontrivial interpretation in the NESS because of the change of sign of \( \gamma' \), as discussed in [23]. From Jensen inequality, it follows from equation (27) that the average of \( \sigma[\{x_s\}] \) is a non-negative quantity and thus

\[ \langle \Delta s_{\text{sys}} + \Delta s_m[\{x_s\}] \rangle \geq \Delta s_{\text{pu}}. \] (28)

In other words, the total variation of entropy in the system plus the bath may be negative on average (which may look as a violation of the second law of thermodynamics if the role of the external agent is ignored) but this quantity is always bounded from below by \(- (\gamma/m)t\). In this sense, \(- \Delta s_{\text{pu}} \) plays a role similar to mutual information in the generalization of the second law to systems under feedback control [14].

Upon averaging, \( \Sigma \equiv \langle \sigma[\{x_s\}] \rangle \), \( Q \equiv \langle q[\{x_s\}] \rangle \), and \( \Delta S_m \equiv Q/T \equiv \langle \Delta s_m[\{x_s\}] \rangle \) become the nonequilibrium thermodynamic quantities defined at the ensemble level, and the corresponding averaged rates are given by [21]

\[ \dot{\Sigma}(t) = \frac{1}{\gamma T} \int dx \, dv \frac{J_t^2(x, v)}{p_t(x, v)}, \] (29)

\[ \dot{\mathcal{S}}_m(t) \equiv \frac{\dot{Q}}{T} = -\frac{1}{T} \int dx \, dv \, v J_t(x, v) = \frac{\gamma}{T} \left[ \langle v^2 \rangle_t - \frac{T}{m} \right], \] (30a)

where \( \langle v^2 \rangle_t \equiv \int dx \, dv \, v^2 p_t(x, v) \). The entropy balance equation (or generalized second law) then takes the form

\[ \dot{\Sigma}(t) = \dot{\Sigma}_{\text{sys}}(t) + \dot{\mathcal{S}}_m(t) - \dot{\mathcal{S}}_{\text{pu}} \geq 0 \] (31)

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with

$$\dot{S}_{pu} = -\frac{\gamma'}{m}. \quad (32)$$

This equation can also be derived directly by taking the time derivative of the system Gibbs–Shannon entropy

$$S_{sys}(t) \equiv \langle s_{sys}(t) \rangle_t = -\int dx \, dv \, p_t(x, v) \ln p_t(x, v) \quad (33)$$

and inserting the Fokker–Planck equation (15).

Note that $J_{t}(x, v)$, as defined by equation (16), is actually the irreversible component of the total probability current [56], since $\gamma'$ is odd under time reversal. Therefore, equations (29) and (30) are in agreement with the general definitions of the non-negative irreversible entropy production and of the heat flow in a stochastic system with odd and even variables (see e.g. [55]).

The rates $\dot{\Sigma}$ and $\dot{S}_{m}$ have simple expressions in the NESS where the solution of the FP equation has the form of an equilibrium Gibbs distribution

$$p_{st}(x, v) = \frac{\sqrt{km}}{2\pi T_{\text{eff}}} e^{-(kx^2 + mv^2)/2T_{\text{eff}}} \quad (34)$$

with an effective temperature lower than $T$

$$T_{\text{eff}} = m\langle v^2 \rangle_{st} = \frac{\gamma}{\gamma + \gamma'} T. \quad (35)$$

Then

$$\dot{\Sigma} = Q \left( \frac{1}{T} - \frac{1}{T_{\text{eff}}} \right) \quad (36a)$$

$$= \frac{\gamma'^2}{m(\gamma + \gamma')}, \quad (36b)$$

and

$$\dot{S}_{m} = \dot{Q} = \frac{\gamma}{m} \frac{T_{\text{eff}} - T}{T} \quad (37a)$$

$$= -\frac{\gamma\gamma'}{m(\gamma + \gamma')} \quad (37b)$$

Hence, heat flows from the reservoir to the system on average, and equation (36a) merely describes the entropy flux between two objects at temperature $T$ and $T_{\text{eff}}$.

4. Entropy production and entropy pumping with measurement errors

We now build on the analysis of the preceding section to study the EP in models V and P described by equations (7) and (9), respectively. Thanks to the linearity of the Langevin equations, the probability distributions and the power spectral densities can be explicitly calculated in the NESS and their expressions are given in appendix A.

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4.1. Model V in the limit of a Markovian feedback

We first study model V in the limit $\tau = 0$, that is, when the feedback force is directly proportional to the output signal of the detector

$$F_{\text{fb}}(y) = -\gamma' y(t)$$

$$= -\gamma' [\dot{x}(t) + \sqrt{S_{\text{env}}(t)}]$$ (38)

and the Langevin equation reads

$$m \ddot{x} + \gamma \dot{x} + \gamma' [\dot{x} + \sqrt{S_{\text{env}}(t)}] + kx = \sqrt{2\gamma T} \xi(t).$$ (39)

The first question that arises is whether the previous analysis at the level of an individual stochastic trajectory $\{x_s\}_{s \in [0,t]}$ can be generalized. The key feature is that the probability of the trajectory only contains the total noise acting on the system. Indeed, since the sum of two independent Gaussian noises is also a Gaussian noise, the trajectories generated by equation (39) are also generated by the Langevin equation

$$m \ddot{x} + (\gamma + \gamma') \dot{x} + kx = \sqrt{2(\gamma T + \gamma'T')} \rho(t),$$ (40)

where $T' = \gamma'S_{\text{env}}/2$ has the dimension of temperature and $\rho(t)$ is a zero mean, delta-correlated noise $\langle \rho(t) \rho(t') \rangle = \delta(t - t')$. As a result, the conditional path probability $P_+([x_s] | x_0, v_0)$ is given by equation (22) with $T$ replaced by $(\gamma T + \gamma'T')/\gamma$. It follows that

$$\ln \frac{P_+([x_s])}{P_-([x_s])} = \Delta s_{\text{sys}} - \frac{\gamma}{\gamma T + \gamma'T'} \int_0^t \left[ m \ddot{x}_s + \gamma' \dot{x}_s + kx_s \right] \delta_s + \frac{\gamma'}{m} t$$ (41)

(noting that the product $\gamma'T' = \gamma'^2 S_{\text{env}}/2$ is no affected by the change $\gamma'$ to $-\gamma'$). The problem is that this log-ratio cannot be considered as a sensible definition of the entropy production $\sigma_\text{m}([x_s])$ along the trajectory. In the first place, the second term in the right-hand side does not identify with $\Delta s_{\text{m}}([X_s]) = q([X_s]) / T$, the entropy change in the medium, where the exchanged heat is defined in the usual way as\textsuperscript{13}

$$q([X_s]) = \int_0^t ds \left[ \gamma \dot{x}_s - \sqrt{2\gamma T} \xi_s \right] \circ \dot{x}_s$$

$$= - \int_0^t ds \left[ m \ddot{x}_s + \gamma' y_s + k x_s \right] \circ \dot{x}_s$$ (42)

where $y_s = \dot{x}_s + \sqrt{S_{\text{env}}(t)} \eta_s$. As indicated, $q$ and $\Delta s_{\text{m}}$ are now functionals of $\{X_s\}_{s \in [0,t]} = ([x_s], \{y_s\})_{s \in [0,t]}$, or functionals of the two noises $\{\xi_s\}$ and $\{\eta_s\}$. In the second place, the average of equation (41) in the NESS does not depend on the measurement noise. Indeed, since the stationary probability distribution is again given by equation (34) with

\textsuperscript{13}In this paper, following [16, 57], we define the heat as the work done by the ‘reaction’ force $\gamma \dot{x} - \sqrt{2\gamma T} \xi$, on the surrounding fluid due to the motion of the Brownian entity. This definition does not change in presence of a measurement noise.
an effective temperature

\[ T_{\text{eff}} \equiv m \langle v^2 \rangle_{\text{st}} = \frac{\gamma}{\gamma + \gamma'} T + \frac{\gamma'}{\gamma + \gamma'} T'. \]

Equation (41) yields

\[ \frac{1}{t} \left\langle \ln \left( \frac{\mathcal{P}_+\{x_s\}}{\mathcal{P}_-\{x_s\}} \right) \right\rangle_{\text{st}} = -\frac{\gamma \gamma'}{\gamma T + \gamma' T'} \langle v^2 \rangle_{\text{st}} + \frac{\gamma'}{m} \]

\[ = \frac{\gamma^2}{m (\gamma + \gamma')} , \]

which is the same as equation (36b) for an error-free measurement\(^{14}\). Therefore, defining the entropy production from the microscopic irreversibility of the trajectories \( \{x_s\}_{s \in [0,t]} \) is not pertinent in the present context (alternatively, one could consider the probability of \( \{X\}_{s \in [0,t]} \), but this amounts to changing the level of description of the system since \( y_s \) is no longer a ‘hidden’ variable, as will be discussed below and in more detail in section 4.2).

To bypass this difficulty and still define the EP and the entropy pumping for \( T' > 0 \), there is no other choice than to work at the ensemble level from the outset, as was done originally in \([21]\). To this end, the different terms in the Fokker–Planck equation must be rearranged appropriately. From equation (40), this equation reads

\[ \partial_t p_t(x,v) = -\partial_x [v p_t(x,v)] + \frac{1}{m} \partial_v \left[ kx + (\gamma + \gamma') v \right] p_t(x,v) + \frac{\gamma T + \gamma' T'}{m} \partial_v p_t(x,v) , \]

which is conveniently rewritten as

\[ \partial_t p_t(x,v) = -\partial_x [v p_t(x,v)] - \frac{1}{m} \partial_v \left[ -kx + \tilde{F}_{\text{fb}}(v,t) \right] p_t(x,v) + J_t(x,v) \]

\[ \text{where } J_t(x,v) \text{ is defined by equation (16) and} \]

\[ \tilde{F}_{\text{fb}}(v,t) = -\gamma' \left[ v + \frac{T'}{m} \partial_v \ln p_t(x,v) \right] \]

plays the role of an effective (or apparent) feedback force. In this form, the FP equation is quite similar to equation (15) for \( T' = 0 \) (that is, for an error-free measurement). In particular, the probability current \( J_t(x,v) \) keeps the same definition, which is justified by the fact that equation (30a) still gives the correct result for the average heat \( Q = T \Delta S_m = \langle q[\{x_s\}, \{y_s\}] \rangle \), where the average is taken over all possible realizations \( \{x_s\} \) and \( \{y_s\} \) of the noises in the time interval \([0,t]\).\(^{15}\)

Accordingly, the entropy balance equation obtained by taking the time derivative of the Gibbs–Shannon entropy is formally the same as for \( T' = 0 \),

\[ \dot{\Sigma}(t) = \dot{S}_{\text{sys}}(t) + \dot{S}_m(t) - \dot{S}_{\text{pu}}(t) \geq 0 , \]

\(^{14}\)Since the probability of a trajectory is Gaussian in the NESS, it is of course crucial to change the sign of \( \gamma' \) when time is reversed. If not, the log-ratio would just predict a vanishing entropy production rate. This special property of linear Langevin equations will be encountered again in the next section and is discussed in detail in \([35]\) in the overdamped case.

\(^{15}\)This is because the Gaussian noises \( \xi(t) \) and \( \eta(t) \) are independent. The average of the product \( \xi_s \circ \dot{x}_s \) in the first line of equation (42) is then equal to \( \sqrt{2 \gamma T} / (2m) \), as can be shown by using Novikov’s theorem \([58]\) for instance. Therefore equation (30b) remains true and this result is also obtained from equation (30a).
Feedback cooling, measurement errors, and entropy production

where $\dot{\Sigma}(t)$, the non-negative apparent entropy production rate\textsuperscript{16}, is given by equation (29), $\dot{S}_m(t)$ is given by equation (30), and

$$
\dot{\Sigma}(t) = \frac{1}{m} \int dx \; dv \; p_t(x,v) \frac{\partial \tilde{F}_{fb}(v,t)}{\partial v} \\
= -\frac{\gamma'}{m} \left[ 1 + \frac{T'}{m} \int dx \; dv \; p_t(x,v) \frac{\partial^2 \ln p_t(x,v)}{\partial v^2} \right].
$$

Equation (49) generalizes the entropy pumping in the presence of measurement errors and is the main result of this section (as it must be, the results of [21] recalled in section 3 are recovered for $T' = 0$, with $\dot{\Sigma} \to \dot{\Sigma}$). In general, the entropy pumping is time-dependent and the physical meaning of the second term in the right-hand side of equation (49) is not transparent. Things become more intelligible in the NESS, as the probability distribution $p_{st}(x,v)$ has again the form of a Gibbs–Boltzmann distribution with $T_{\text{eff}}$ given by (43). The apparent feedback force (47) is then proportional to the instantaneous velocity

$$
\tilde{F}_{fb,\text{st}}(v) = -\tilde{\gamma}' v
$$

with an apparent damping coefficient

$$
\tilde{\gamma}' = \gamma' \frac{T_{\text{eff}} - T'}{T_{\text{eff}}} = \gamma \frac{T - T_{\text{eff}}}{T_{\text{eff}}} = \gamma \gamma' \frac{T - T'}{\gamma T + \gamma' T'}.
$$

Equations (29), (30), and (49) then yield

$$
\dot{\Sigma} = \dot{Q} \left( \frac{1}{T} - \frac{1}{T_{\text{eff}}} \right) = \frac{\gamma \gamma'^2}{m(\gamma + \gamma')} \frac{(T - T')^2}{(\gamma T + \gamma' T')T},
$$

$$
\dot{S}_m \equiv \frac{\dot{Q}}{T} = \frac{\gamma}{m} (T_{\text{eff}} - T) = -\frac{\gamma \gamma'}{m(\gamma + \gamma')} \frac{T - T'}{T},
$$

and

$$
\dot{S}_{pu} = -\frac{\tilde{\gamma}'}{m} = -\frac{\gamma \gamma'}{m} \frac{T - T'}{\gamma T + \gamma' T'}.
$$

Hence, the picture of heat exchange between two objects at temperature $T$ and $T_{\text{eff}}$ is still pertinent in the NESS, with the measurement noise only increasing the value of $T_{\text{eff}}$ and

\textsuperscript{16}The use of the term ‘apparent’ borrowed from [38] will be justified below in section 4.2.
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Figure 1. (a) Resonator temperature $T_{\text{eff}}/T$ as a function of the feedback gain $g = \gamma'/\gamma$ for a signal-to-noise ratio $\text{SNR} = 2T/(\gamma S_{\text{nu}}) = 1000$. (b) Different contributions to the entropy production rate according to equations (52b), (53b), and (54b) (in units $k_B \tau_0^{-1}$): $\hat{\Sigma}$ (black solid line), $\hat{S}_m$ (red solid line), $\hat{S}_{\text{pu}}$ (blue solid line). The dashed line is the total entropy production rate given by equation (59b) when the Langevin equation (39) is viewed as describing a Brownian particle coupled to two heat reservoirs.

Thus reducing the efficiency of the cooling [1]. Equation (51a) shows that this can also be interpreted as a weakening of the apparent feedback force, which in turn decreases the average heat flow coming from the reservoir and the apparent EP. We also remark that $-\dot{S}_{\text{pu}}$ decreases as $T'$ increases\(^\text{17}\). Since $\dot{S}_m - \dot{S}_{\text{pu}} \geq 0$ from equation (48) (as $\dot{S}_{\text{sys}} = 0$ in the NESS by definition), the standard formulation of the second law is less and less ‘violated’ as the measurement error increases, which again shows that entropy pumping plays a role similar to mutual information in the generalization of the second law\(^\text{18}\).

As shown in figure 1, the apparent EP rate $\hat{\Sigma}$ has an interesting behavior as a function of the feedback gain $g = \gamma'/\gamma$, in relation to the behavior of the resonator temperature $T_{\text{eff}}$. As is well known in the theory of feedback cooling [1, 3] and is indeed observed experimentally (see for instance [45] for a setup related to model V), the temperature $T_{\text{eff}}$ reaches a minimum at a certain value $g_{\text{min}}$ of the gain for a given signal-to-noise ratio $\text{SNR} = 2T/(\gamma S_{\text{nu}}) = 1000$.\(^{\text{17}}\) Equation (54b) can be written as $-\dot{S}_{\text{pu}} = \gamma'/m - (\gamma'/m)(\gamma + \gamma')/(\gamma T + \gamma'T')T'$.

\(^{\text{17}}\)Note that equations (32)–(54) are not specific to a harmonic model and are more universally valid (e.g., if one adds an additional quartic term in the potential) as long as the feedback force is proportional to the output signal of the detector, as given by equation (38). The stationary probability distribution is then still a Gibbs–Boltzmann distribution and the effective temperature is defined by equation (43). In particular, the apparent feedback force keeps the suggestive form of equation (50).
ratio (SNR)\(^{19}\). Above \(g_{\text{min}}\), too much detector noise is fed back to the resonator and \(T_{\text{eff}}\) starts to increase. In particular, \(T_{\text{eff}} = T\) for \(g = 2T/(\gamma S_{v_0}) = \text{SNR}\) (which corresponds to \(T' = T\)): the resonator is then at equilibrium with its environment, as if there were no feedback. For even larger values of \(g\), the resonator is heated. Interestingly, equations (52) tell us that the apparent EP rate is maximal when \(T_{\text{eff}}\) is minimal and, as can be seen in figure 1, the essential contribution to the EP around \(g_{\text{min}}\) comes from entropy pumping (\(|\dot{S}_{\text{pu}}/\dot{S}_{m}| |_{g=g_{\text{min}}} \approx \sqrt{\text{SNR}/2}\)). Loosely speaking, one may say that the optimal cooling is achieved when the controller extracts the maximal information about the state of the system via the measurement. This can also be put in relation with the behavior of the spectral density \(S_{v'v'}(\omega)\) of the output signal \(v' = v + v_n\), as noticed in [45]. Finally, we stress that the entropy pumping vanishes for \(T = T'\) like the heat flow from the reservoir, so that there is no EP on average, as the heating due to the detector noise exactly compensates the cooling due to the extra friction.

At this stage, it is interesting to compare the above results to those obtained when equation (39) is regarded as a Langevin equation describing a Brownian particle coupled to two thermal environments at temperatures \(T\) and \(T'\). This is a model (with or without the harmonic trap) which is often discussed in the literature [32], [59]–[64] (see also [16] and the recent experimental work described in [65]) as it is probably the simplest example of heat conduction. In this interpretation, the heat flowing from each thermostat to the harmonic trap) which is often discussed in the literature [32], [59]–[64] (see also [16] and the recent experimental work described in [65]) as it is probably the simplest example of heat conduction. In this interpretation, the heat flowing from each thermostat to the particle drives the system out of equilibrium, and to correctly distinguish the two heat flows the Fokker–Planck equation (45) is written as

\[
\partial_t p_t(x,v) = -\partial_x [v p_t(x,v)] - \frac{1}{m} \partial_v [v k p_t(x,v) + J_t(x,v)]
\]

where

\[
J_t'(x,v) = -\gamma'\left[v + \frac{T'}{m} \partial_v \ln p_t(x,v)\right] p_t(x,v).
\]

This leads to the entropy balance equation (see e.g. [32])

\[
\dot{\Sigma}(t) = \dot{\hat{S}}_{\text{sys}}(t) + \dot{\hat{S}}_{\hat{m}}(t) + \dot{\hat{S}}_{\hat{m}'}(t),
\]

where \(\dot{\hat{S}}_{\hat{m}'}(t)\), the entropy flow to the second reservoir, is defined like \(\dot{\hat{S}}_{\hat{m}}(t)\) (with \(T\) and \(J_t(x,v)\) replaced by \(T'\) and \(J'_t(x,v)\), respectively), and

\[
\dot{\Sigma}(t) = \frac{1}{\gamma T} \int dx \int dv \frac{J_t^2(x,v)}{p_t(x,v)} + \frac{1}{\gamma T'} \int dx \int dv \frac{J'_t^2(x,v)}{p_t(x,v)}.
\]

In the steady state, this yields

\[
\dot{\Sigma} = Q \left( \frac{1}{T} - \frac{1}{T'} \right)
\]

\[
= \frac{\gamma \gamma'}{m(\gamma + \gamma')} \frac{(T - T')^2}{TT'}
\]

\(^{19}\)Equation (43) yields \(g_{\text{min}} = \sqrt{1 + \text{SNR} - 1}\) and thus \(T_{\text{eff},\text{min}} = 2T(\sqrt{1 + \text{SNR} - 1})\), where SNR \(\equiv S_{v_0}^2(\omega_0)/S_{v_n}\) is the ratio of the original thermal noise peak (i.e., without feedback) to the detector noise floor. From equation (A.11a) with \(\tau = 0\), SNR \(\equiv 2T/(\gamma S_{v_n}).\) Hence \(T'/T = g/\text{SNR}.\)
where
\[ \dot{Q} = T \dot{S}_m = \frac{\gamma}{m} (T_{\text{eff}} - T) \]
\[ = -T' \dot{S}_{m'} = -\frac{\gamma'}{m} (T_{\text{eff}} - T'). \]
\[ (60) \]

Equation (59b) is the usual thermodynamic expression for the average EP rate associated with a steady heat flux between two reservoirs mediated by a device with thermal conductivity \( \gamma' / [m(\gamma + \gamma')] \) (see e.g. [32, 59, 60]). This expression is quite different from the average apparent EP rate given equation (52b) (whereas \( \dot{Q} \) does not change). Indeed, equation (52a) does not have the same physical content as equation (59a) since only the two temperatures \( T \) and \( T_{\text{eff}} \) come into play in the former case and \( T' \) is only introduced for convenience (changing \( T' \) changes the value of \( T_{\text{eff}} \)). In other words, at the level of description corresponding to equations (52), the entropy exchange with the external agent is not associated with another heat flow but only manifests itself in the form of entropy pumping. This makes a big difference, since the average EP rate given by equation (52b) remains finite when the measurement becomes error-free (\( T' \to 0 \)), whereas \( \dot{S}_{m'} = \dot{Q}/T' \) and thus \( \Sigma \) given by equation (59b) diverge\(^{20}\) (see [28] for a closely related discussion in the case of a true Maxwell demon). Moreover, as a function of \( g \), \( \dot{\Sigma} \) does not display any extremum at \( g = g_{\text{min}} \), which shows that this quantity does not reflect the interesting physics of the problem. More fundamentally, there is no rationale for considering \( T' \) as the genuine temperature of another reservoir in the context of cold damping. It is worth noting, however, that equation (59b) is also the expression of the average EP resulting from the analysis performed in [24], as can be readily checked.

In this work, the Langevin equation (39) (with \( k = 0 \)) is obtained by taking a suitable continuous-time limit of a discrete series of independent measurements of the velocity (as originally considered in [11])\(^{21}\). A quantity that identifies with \( \Delta S_{m'} \) (the change in entropy of the second ‘heat’ reservoir) is then interpreted as the ‘entropy production due to the measurement process’ and contributes to the total EP. In the light of the above discussion, it is clear that this corresponds to another level of description of the system. We shall come back to this issue in the next section and in appendix C.

Now that we have defined the average (apparent) EP rate associated with the Fokker–Planck equation (46), let us again consider individual trajectories and see whether we can define a corresponding (apparent) EP functional. The main problem is to relate the entropy pumping to a momentum phase space contraction like in the case \( T' = 0 \). To simplify the discussion, and also because this is the normal regime of a cold damping setup, we only consider stationary trajectories. Then, equation (50) immediately suggests to replace the original Langevin equation (39) by the effective equation
\[ m \ddot{x} + (\gamma + \gamma') \dot{x} + kx = \sqrt{2 \gamma T} \xi(t) \]
\[ (61) \]
\(^{20}\)More generally, the EP rate given by equation (29), which leads to equation (52b) in the NESS, is always smaller than the EP given by equation (58), since the contribution involving \( J^0_t(x, v)/p_t(x, v) \) is not present. In the NESS, the difference between the two EP rates become asymptotically zero in the large \( g \) limit, as can be seen in figure 1.

\(^{21}\)Specifically, the conditional probability for obtaining the measurement outcome \( y(t) \) from the velocity \( v(t) \) is given by a Gaussian distribution with variance \( \Delta^2 \delta t \), where \( \Delta \) quantifies the error and \( \delta t \) is the infinitesimal time step. The limit \( \Delta \to \infty \) and \( \delta t \to 0 \) is then taken such that \( \Delta^2 \delta t \to 2 \alpha_0 \) finite. Hence \( 2 \alpha_0 \) identifies with the noise spectral density \( S_{v_m} \) in equation (39).
which can be rewritten, using equation (51a), as
\[
m\ddot{x} + \gamma \left( \frac{T}{T_{\text{eff}}} \right) x + kx = \sqrt{2\gamma T} \xi(t).
\]
By construction, this equation leads to a NESS with the same probability distribution \( p_{\text{st}}(x,v) \) as equation (39). On the other hand, the individual stochastic trajectories are different. Since only the Langevin thermal noise \( \xi(t) \) appears in equation (62), we are led back to the problem treated in [21]–[23] and recalled in section 3. Hence, the apparent entropy change \( \Delta s_{\text{app}}(\{x_s\}) \equiv \tilde{q}(\{x_s\})/T \), where \( \tilde{q}(\{x_s\}) \) is the apparent heat dissipated in the environment defined by
\[
\tilde{q}(\{x_s\}) = -\int_0^t ds \left[ m\dot{x}_s + \gamma' \dot{x}_s + kx_s \right] \circ \dot{x}_s,
\]
also satisfies equation (21) with \( \gamma' \) replaced by \( \dot{\gamma}' \) everywhere. It follows that the apparent path-dependent EP
\[
\tilde{\sigma}(\{x_s\}) \equiv \ln \frac{\tilde{P}_+(\{x_s\})}{\tilde{P}_-(\{\dot{x}_s\})} = \Delta s_{\text{sys}} + \Delta \tilde{s}_{\text{app}}(\{x_s\}) + \tilde{\gamma}' \frac{m}{T} t,
\]
where \( \tilde{P}_+(\{x_s\}) \) (resp. \( \tilde{P}_-(\{x_s\}) \)) is the probability of a path \( \{x_s\}_{s\in[0,t]} \) generated by equation (62) in the NESS (resp. by the conjugate equation with \( \dot{\gamma}' \) replaced by \( -\dot{\gamma}' \)), satisfies the IFT \( <e^{-\tilde{\sigma}(\{x_s\})}>_{\text{eff, st}} = 1 \). A proper detailed fluctuation theorem can be also obtained following [23]. By construction, \( (1/t)\langle \tilde{\sigma}(\{x_s\}) \rangle_{\text{eff, st}} \) and \( (1/t)\langle \tilde{q}(\{x_s\}) \rangle_{\text{eff, st}} \) identify with the average apparent EP rate and the average heat flow \( \tilde{Q} \) given by equations (52b) and (53b), respectively.

Let us finally remark that the effective Langevin equation can also be derived from the original Langevin equation (39) by replacing the measurement noise \( \eta(t) \) by its projection onto the space spanned by the stochastic variables \( x \) and \( v \), which is defined as
\[
\hat{\eta}_{\text{st}}(x,v) = \frac{\langle \eta \circ x \rangle}{\langle x^2 \rangle_{\text{st}}} x + \frac{\langle \eta \circ v \rangle}{\langle v^2 \rangle_{\text{st}}} v.
\]
Since \( \langle \eta \circ x \rangle = 0 \) and \( \langle \eta \circ v \rangle = -\sqrt{2\gamma T}/(2m) \), one has \( \hat{\eta}_{\text{st}} = \sqrt{2\gamma T}/(2T_{\text{eff}}) v \) and equation (62) is recovered by inserting this result into equation (39). This procedure will be useful to derive similar effective Langevin equations in what follows.

4.2. Model V for \( \tau > 0 \)

We now generalize the above analysis to the case \( \tau > 0 \). Whereas equations (7) describe the coupled dynamics of the two processes \( x(t) \) and \( y(t) \), we are interested in the apparent entropy production associated with \( x(t) \) only. Like in the overdamped case considered in [35], the coupling makes the effective dynamics of \( x(t) \) no longer Markovian, as we have already noticed. However, since the Langevin equations are linear, the path probability \( \tilde{P}_+(\{x_s\}) \) is Gaussian in the NESS, and the ratio \( \tilde{P}_+(\{x_s\})/\tilde{P}_-(\{\dot{x}_s\}) \) can be easily computed by going to Fourier space, as shown in appendix B. It turns out that the average of this quantity is independent of the measurement error, like for \( \tau = 0 \). Therefore, again, we cannot define an apparent EP functional from the microscopic irreversibility of the trajectories \( \{x_s\} \) and we need to first derive the entropy balance equation at the ensemble level.
Feedback cooling, measurement errors, and entropy production

We thus consider the time evolution of the Shannon entropy

\[ S_{\text{sys}}(t) = -\int dx \, dv \, p_t(x, v) \ln p_t(x, v), \]

where \( p_t(x, v) \) is the marginal of the joint distribution \( p_t(X) \equiv p_t(x, v, y) \) which obeys FP equation (A.2). Specifically, for model V,

\[
\partial_t p_t(X) = -\partial_x [v p_t(X)] - \frac{1}{m} \partial_v [-kx + \gamma y] p_t(X) + J_t(X) \\
- \frac{1}{\tau} \partial_y \left[(v - y) p_t(X) - \frac{T'}{\gamma'} \partial_y p_t(X) \right]
\]

(66)

where

\[
J_t(X) = -\gamma \left[v + \frac{T}{m} \partial_v \ln p_t(X)\right] p_t(X)
\]

(67)

and \( T' = \gamma' S_{vn}/2 \) like before. Integrating over \( y \) yields the FP equation for \( p_t(x, v) \),

\[
\partial_t p_t(x, v) = -\partial_x [v p_t(x, v)] - \frac{1}{m} \partial_v [-kx + \tilde{F}_{fb}(x, v, t)] p_t(x, v) + J_t(x, v)
\]

(68)

where the current

\[
J_t(x, v) = \int dy \, J_t(X) = -\gamma \left[v + \frac{T}{m} \partial_v \ln p_t(x, v)\right] p_t(x, v)
\]

(69)

has the same definition as before, and the apparent feedback force is now defined as

\[
\tilde{F}_{fb}(x, v, t) = -\gamma' \tilde{y}(x, v, t)
\]

(70)

where

\[
\tilde{y}(x, v, t) = \int dy \, y p_t(y|x, v) = \frac{1}{p_t(x, v)} \int dy \, y p_t(X).
\]

(71)

Therefore, the only difference with the preceding calculations for \( \tau = 0 \) lies in the definition of \( \tilde{F}_{fb}(x, v, t) \). By taking the time derivative of \( S_{\text{sys}}(t) \), we thus again obtain equation (48) with \( \dot{\Sigma}(t) \) and \( \dot{S}_m(t) \) given by equations (29) and (30), respectively, whereas the entropy pumping is now given by

\[
\dot{S}_{pu}(t) = \frac{1}{m} \int dx \, dv \, p_t(x, v) \frac{\partial \tilde{F}_{fb}(x, v, t)}{\partial v}
\]

(72a)

\[
= -\frac{\gamma'}{m} \int dx \, dv \, p_t(x, v) \partial_v \tilde{y}(x, v, t).
\]

(72b)

The physical meaning of this contribution is again more transparent in the NESS, where \( p_{st}(X) \) is given by the Gaussian distribution (A.3) and the effective (kinetic) temperature is

\[
T_{\text{eff}}^{(v)} \equiv m \sigma_{2,2} = \frac{\gamma_{\text{eff}} - \gamma' T}{\gamma_{\text{eff}}} + \frac{\gamma' T'}{\gamma_{\text{eff}}}
\]

(73)

with \( \gamma_{\text{eff}} \) defined by equation (A.6). A straightforward calculation then yields

\[
\tilde{y}_{st}(x, v) = \frac{\langle xy \rangle_{st}}{\langle x^2 \rangle_{st} x} + \frac{\langle vy \rangle_{st}}{\langle v^2 \rangle_{st} v}
\]

\[
= \frac{\sigma_{1,3}}{\sigma_{1,1}} x + \frac{\sigma_{2,3}}{\sigma_{2,2}} v
\]

(74)

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where the elements of the covariance matrix $\sigma$ are given by equations (A.5). The apparent feedback force defined by equation (70) thus includes an additional contribution proportional to the instantaneous position of the resonator. However, from equations (72), only the viscous part of $\tilde{F}_{fb, st}(x, v)$ contributes to the entropy pumping and the rate can be again expressed as $\dot{S}_{pu} = -\tilde{\gamma}'/m$ with an apparent friction coefficient

$$
\tilde{\gamma}' \equiv \gamma' \frac{\sigma_{2,3}}{\sigma_{2,2}} \quad (75a)
$$

$$
= \gamma' \frac{T - T_{eff}^{(v)}}{T_{eff}} \quad (75b)
$$

where equation (A.7a) has been used to go from equation (75a) to (75b) (note that $\tilde{\gamma}' \neq \gamma'(T_{eff}^{(v)} - T')/T_{eff}^{(v)}$, in contrast with equation (51a)). Specifically, we obtain

$$
\dot{\tilde{\Sigma}} = \dot{Q} \left( \frac{1}{T} - \frac{1}{T_{eff}^{(v)}} \right) \quad (76a)
$$

$$
= \frac{\gamma' \gamma^2}{m \gamma_{eff}} \left( T - T' \right)^2 \quad (76b)
$$

$$
\dot{S}_m \equiv \frac{\dot{Q}}{T} = -\frac{\gamma' \gamma^2}{m \gamma_{eff}} \frac{T - T'}{T} \quad (77)
$$

and

$$
\dot{S}_{pu} = -\frac{\gamma' \gamma^2}{m} \frac{T - T'}{(\gamma_{eff} - \gamma')T + \gamma'T'} \quad (78)
$$

which generalize equations (52), (53), and (54), respectively. Since $\gamma_{eff} \to \gamma + \gamma'$ when $\tau \to 0$, these equations are recovered in this limit, as it must be. Furthermore, the apparent EP still reaches a maximum as a function of the gain $g = \gamma'/\gamma$ when $T_{eff}^{(v)}$ is minimal.

Following the same line of reasoning as for $\tau = 0$, we can introduce an effective Langevin equation that reproduces the same average heat flow $\dot{Q}$ and the same average EP rate $\dot{\tilde{\Sigma}}$ in the steady state as the original model while allowing one to properly define corresponding fluctuating quantities. The Fokker–Planck equation (68) suggests simply replacing the actual feedback force by the apparent one defined by equation (70), that is to replacing $y$ by $\tilde{y}_{st}(x, v)$ in equation (7a). From equation (74) and (75a), the effective Langevin equation thus reads

$$
m \ddot{x} + (\gamma + \tilde{\gamma}') \dot{x} + \left( k + \gamma' \frac{\sigma_{1,3}}{\sigma_{1,1}} \right) x = \sqrt{2\gamma T} \xi(t), \quad (79)
$$

which can be rewritten, using equations (A.7), as

$$
m \ddot{x} + \gamma \left( \frac{T}{T_{eff}^{(v)}} \right) \dot{x} + k \left( \frac{T_{eff}^{(v)}}{T_{eff}^{(x)}} \right) x = \sqrt{2\gamma T} \xi(t) \quad (80)
$$

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where $T^{(x)}_{\text{eff}} \equiv k(x^2)_{st} = k\sigma_{1,1}$ is another effective temperature characterizing the motion of the Brownian entity\textsuperscript{22} (see below section 4.3). By construction, this equation leads to the same marginal distribution $p_{st}(x,v)$ as equations (7). The apparent heat dissipated in the environment is then given by

$$\bar{q}[(x_s)] \equiv T\Delta\hat{s}_m[(x_s)] = -\int_0^t ds [m\ddot{x}_s + \gamma'\tilde{y}_{st}(x_s, \dot{x}_s) + kx_s] \circ \dot{x}_s,$$

(81)

and the corresponding apparent EP functional $\hat{\sigma}[(x_s)] \equiv \Delta\hat{s}_{\text{sys}} + \Delta\hat{s}_m[(x_s)] + (\gamma'/m)t$ gives back equations (76) upon averaging. The reason for using the term ‘apparent’ like in [38] should now be clear. Apart from the presence of the entropy pumping contribution, which is specific to the present problem, the definition of the path-dependent apparent EP is the same as the one introduced in [38] for an overdamped dynamics\textsuperscript{23}: the total force acting on the observed particle, which depends on the ‘hidden’ degrees of freedom (here, $y$), is replaced by its conditional expectation, that is by its projection on the subspace spanned by the accessible degrees of freedom (here, $x$ and $v$)\textsuperscript{24}. By definition, one has $\hat{\sigma}[(x_s)] = \ln \mathcal{P}_+[(x_s)]/\mathcal{P}_-[(\dot{x}_s)] \neq \ln \mathcal{P}_+(\{x_s\})/\mathcal{P}_-(\{\dot{x}_s\})$, which implies that the apparent EP functional obeys a FT with the trajectories $\{x_s\}_{s\in[0,t]}$ generated by the effective Langevin equation instead of the actual trajectories generated by equations (7)\textsuperscript{25} (here $\hat{P}_-\{\dot{x}_s\}$ is the probability of the time-reversed path generated by the Langevin equation conjugate to equation (79) in which the apparent friction coefficient $\tilde{\gamma}'$ is replaced by $-\gamma'$ and the other terms are unchanged).

Of course, a more standard picture is recovered if one considers the EP along trajectories $\{X_s\}_{s\in[0,t]} \equiv \{(x_s), \{y_s\}\}_{s\in[0,t]}$ in the ‘super-system’. As already stressed, this is not the viewpoint adopted in this work, in contrast with [24]. Moreover, this EP is rather artificial in the present context since equations (7) do not describe the actual physical processes inside the controller. On the other hand, the comparison with the apparent EP defined above may be interesting from the perspective of the influence of coarse graining on entropy production, in particular in the light of the analysis carried out in [35]. Indeed, if one forgets the harmonic potential $kx^2/2$, model V is identical to the two-temperatures underdamped model considered in [35] (appendix B) that describes the irreversible dynamics of a massive tracer in a granular fluid [47]-[49]. For completeness, and also because this sheds some light on the analysis of [24], the calculation of the EP in the super-system (the ‘total’ EP) is detailed in appendix C. Note in particular equation (C.15),

\textsuperscript{22} Note from equations (A.5) that $T^{(v)}_{\text{eff}} = T^{(x)}_{\text{eff}} = T$ when $T' = T$. The Brownian entity is then at equilibrium with the environment. One also has $T^{(v)}_{\text{eff}} = T^{(x)}_{\text{eff}}$ for $\tau = 0$ so that equation (80) gives back equation (62), as it must be.

\textsuperscript{23} The present system, however, is quite different from the one studied in [38], which involves two interacting Brownian particles and two NESSs that can be controlled independently. Here, there is only one Brownian entity and a single steady state.

\textsuperscript{24} $\bar{y}(x,v,t)$, as defined by equation (71), is in fact the minimum mean-squared error (MMSE) estimator of $y$ for given $x$ and $v$, that is the Bayes estimator that minimizes the mean-squared error ($\langle (\bar{y} - y)^2 \rangle$, where the expectation is taken over $x$, $v$ and $y$. Since all variables are jointly Gaussian in the NESS, this estimator is a linear function of $x$ and $v$, as shown by equation (74).

\textsuperscript{25} This is also true for the apparent EP defined in [38]. This trajectory-dependent functional does not obey a FT if the dynamics is described by the original Langevin equation (only FT-like symmetries may be preserved, depending on the experimental parameters). On the other hand, if one defines an effective dynamics (which generates different stochastic trajectories) by replacing the actual force acting on the observed particle by its conditional expectation computed with the full stationary probability distribution, the apparent EP then obeys a FT with this effective dynamics.
which states that the average total EP rate is always larger than the average apparent EP rate. This is consistent with the general expectation that an incomplete description of a system results in an underestimation of the actual dissipation. However, in the present case, since \( \hat{\sigma}(\{x_s\}) \neq \ln P_{+}(\{x_s\})/P_{-}(\{\dot{x}_s\}) \), this inequality does not follow from the general argument that a Kullback–Leibler divergence (or relative entropy) always decreases upon coarse graining [66] (see also [31]).

4.3. Model P

Since it would be tedious to repeat everything for model P, we only point out the main differences from model V and give the main results. We first recall that the model described by equations (9) is ill-defined for \( \tau = 0 \) because the measurement noise on the resonator position \( x_n(t) = \sqrt{\mathcal{S}_{xx}} \eta(t) \) is approximated by a Gaussian white noise. This implies that some quantities diverge in the limit \( \tau \to 0 \) (see equations (A.8)), in particular the effective kinetic temperature \( T_{\text{eff}}^{(v)} \) in the NESS, which is given by

\[
T_{\text{eff}}^{(v)} = \frac{\gamma_{\text{eff}} - \gamma'}{\gamma_{\text{eff}}} T \left( 1 + \frac{\gamma'_{\text{eff}} - \gamma - \gamma'}{\gamma} \right) T',
\]

where \( T' \equiv \gamma'_{\text{eff}} S_{xx}/(2\tau^2) \) has the dimension of a temperature (the important new feature is that \( T' \) diverges for \( \tau \to 0 \), in contrast with the corresponding quantity in model V). On the other hand, the other effective temperature \( T_{\text{eff}}^{(x)} \), which is the one usually considered in experiments [3], remains finite in this limit[26],

\[
T_{\text{eff}}^{(x)} = \frac{\gamma_{\text{eff}} - \gamma'_{\text{eff}}}{\gamma_{\text{eff}}} T \left( \frac{Q_0 \tau}{\tau_0} \right)^2 \frac{\gamma'}{\gamma_{\text{eff}}} T' \left( \frac{T}{1 + g} + \frac{k\omega_0}{2Q_0 \left( 1 + g \right)} S_{xx} \right),
\]

which is equation (5) in [3] (with \( k_B = 1 \) and the two-sided convention for the spectral densities, see footnote 32 in appendix A). As pointed out in appendix A (see equations (A.13)), this can be traced back to the behavior of \( S_{xx}(\omega) \), the power spectral density (PSD) of \( x \), at large frequencies: in the limit \( \tau \to 0 \), the integral of \( S_{xx}(\omega) \) over \( \omega \) is finite whereas the integral of \( S_{vv}(\omega) = \omega^2 S_{xx}(\omega) \) diverges, which is also the case for the integral of \( S_{x'x'}(\omega) \), the PSD of the measured displacement \( x' = x + x_n \) (i.e. the PSD of the detector output) [27].

Another significant difference with model V is the fact that there is no positive value of the feedback gain \( g = \gamma'/\gamma \) for which the resonator is at equilibrium with the environment. This can be readily seen from the above equations, which show that \( T_{\text{eff}}^{(v)} \) and \( T_{\text{eff}}^{(x)} \) cannot be simultaneously equal to the heat bath temperature \( T \). In other words, the detailed balance condition is never satisfied and there is always dissipation in the system[28].

26 Here, to facilitate the comparison with experiments, we use the parameters \( \tau_0 = m/\gamma \), \( \omega_0 = \sqrt{k/m} \) and \( Q_0 = \sqrt{mk}/\gamma \) to describe the resonator instead of \( k, m, \) and \( \gamma \). The temperature \( T_{\text{eff}}^{(v)} \) is denoted \( T_{\text{mode}} \) in [3].

27 In the analysis of the experimental spectra, the integration over \( \omega \) is actually performed in a limited band around the resonance frequency \( \omega_0 \) (see for instance the discussion in [67] for the LIGO interferometer). In model P, \( \tau \) sets the minimal accessible time scale and in principle the upper limit of the integrals must be of the order of \( 2\pi/\tau \). This limit can be extended to \(+\infty\) if the integrals converge.

28 There exists a value of \( g \) for which \( T_{\text{eff}}^{(v)} = T \), but the corresponding kinetic temperature \( T_{\text{eff}}^{(x)} \) is then larger than \( T \). In practice, this value of \( g \) is very large (see figure 3(a) for instance) and this situation has not been observed experimentally to the best of our knowledge.
Using the same method as previously, we first define the apparent EP at the ensemble level. The Fokker–Planck is somewhat more complicated than in model V and reads

\[
\begin{align*}
\frac{\partial}{\partial t} p_t(X) &= -\partial_x [vp_t(X)] + \frac{1}{m} \partial_v \left[ \left( k + \frac{\gamma'}{\tau} \right) x + \gamma v - \frac{\gamma'}{\tau} y \right] p_t(X) \\
&\quad + \frac{1}{\tau} \partial_y [(y - x)p_t(X)] + \frac{\gamma T + \gamma' T'}{m^2} \frac{\partial^2}{\partial v^2} p_t(X) \\
&\quad + \frac{T'}{\gamma' \partial_y} p_t(X) - \frac{2T'}{m} \frac{\partial^2}{\partial v \partial y} p_t(X),
\end{align*}
\]

(84)

the cross derivatives arising from the fact that the noises in the rhs of equations (9a) and (9b) are correlated. However, these terms do contribute after integrating over \( y \), and the FP equation for the marginal probability distribution \( p_t(x, v) = \int dy p_t(x, v, y) \) can be written as

\[
\begin{align*}
\frac{\partial}{\partial t} p_t(x, v) &= -\partial_x [vp_t(x, v)] - \frac{1}{m} \partial_v \left[ - \left( k + \frac{\gamma'}{\tau} \right) x + \tilde{F}_{fb}(x, v, t) \right] p_t(x, v) + J_t(x, v) \quad (85)
\end{align*}
\]

where the current \( J_t(x, v) \) has the same expression as in model V (see equation (69)), and the apparent feedback force is now defined as

\[
\tilde{F}_{fb}(x, v, t) = \frac{\gamma'}{\tau} \left[ \tilde{y}(x, v, t) - \frac{\tau T'}{m} \partial_v \ln p_t(x, v) \right]
\]

(86)

with \( \tilde{y}(x, v, t) \) given by equation (71). This is the main difference from model V, and \( \dot{\Sigma} \) and \( \dot{S}_m \) are again given by equations (29) and (30) in the entropy balance equation (48), whereas the entropy pumping is

\[
\begin{align*}
\dot{S}_{pu}(t) &= \frac{1}{m} \int dx \, dv \, p_t(x, v) \partial_v \tilde{F}_{fb}(x, v, t) \quad (87a) \\
&= \frac{\gamma'}{\tau m} \int dx \, dv \, p_t(x, v) \partial_v \left[ \tilde{y}(x, v) - \frac{\tau T'}{m} \partial_v \ln p_t(x, v) \right]. \quad (87b)
\end{align*}
\]

Equation (87b) is in general different from equation (72b), but the entropy pumping rate in the NESS, after using equations (74) and (A.9b), can be again expressed as \( \dot{S}_{pu} = -\gamma'/m \) with an apparent friction coefficient \( \gamma'/m \) given by equation (75b). Similarly, \( \dot{\Sigma} \) and \( \dot{S}_m \) are given by equations (36a) and (37a), respectively. Hence, all the terms in the entropy balance equation are formally the same as in model V and only the value of the effective kinetic temperature changes. Explicitly, we obtain

\[
\dot{\Sigma} = \dot{Q} \left( \frac{1}{T} - \frac{1}{T_{eff}^{(v)}} \right) = \frac{\gamma'^2}{m \gamma_{eff}^2} \frac{\left[ \gamma T - (\gamma_{eff} - \gamma - \gamma') T' \right]^2}{\gamma (\gamma_{eff} - \gamma') T + \gamma' (\gamma_{eff} - \gamma - \gamma') T'}, \quad (88)
\]

\[
\dot{S}_m = \dot{Q} \left[ \frac{\gamma T - (\gamma_{eff} - \gamma - \gamma') T'}{T} \right], \quad (89)
\]

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and

\[ \dot{S}_{\text{pu}} = -\frac{\gamma \gamma'}{m} \left( \gamma T - (\gamma_{\text{eff}} - \gamma - \gamma') T' \right) \]

with \( \gamma_{\text{eff}} \) given by equation (A.6). As it must be, the results of [21] recalled in section 3 are recovered by setting \( T' = 0 \) (i.e. \( S_{x_n} = 0 \)) and \( \tau = 0 \). On the other hand, if \( S_{x_n} \neq 0 \), both \( \dot{S} \) and \( \dot{S}_{m} \) diverge like \( 1/\tau \) as \( \tau \to 0 \) (whereas the entropy pumping rate stays finite), as a consequence of the divergence of the noise temperature \( T' \).

To derive an effective Langevin equation in the NESS, we cannot simply replace \( y \) by \( \dot{y}_{\text{st}}(x, \dot{x}) \) in equation (8a), as this would introduce an effective mass in the problem. What must be done is to project both \( y \) and the noise \( \eta \) on the subspace spanned by the variables \( x \) and \( v \). We thus define

\[ \tilde{\eta}_{\text{st}}(x, v) = \frac{\langle \eta \circ x \rangle}{\langle x^2 \rangle_{\text{st}}} x + \frac{\langle \eta \circ v \rangle}{\langle v^2 \rangle_{\text{st}}} v = -\frac{1}{2T_{\text{eff}}(v)} \sqrt{2\gamma' T' v}, \]

where we have used \( \langle \eta \circ x \rangle = 0 \) and \( \langle \eta \circ v \rangle = -1/(2m)\sqrt{2\gamma' T'} \) to derive the second equality. Then, replacing \( y \) by \( \tilde{y}_{\text{st}}(x, \dot{x}) \) and \( \eta \) by \( \tilde{\eta}_{\text{st}}(x, \dot{x}) \) into equation (9a) yields the effective Langevin equation

\[ m\ddot{x} + (\gamma + \gamma') \dot{x} + \left( k + \frac{\gamma'}{\tau} - \frac{\gamma' \sigma_{1,3}}{\sigma_{1,1}} \right) x = \sqrt{2\gamma T} \xi(t), \]

which can be exactly rewritten, using equations (74), (A.9b), and (A.9c), as equation (80) in model V. The apparent heat dissipated in the environment is defined as

\[ \tilde{q}[\{x_s\}] \equiv T \Delta \tilde{s}_m[\{x_s\}] = -\int_0^t ds \left[ m \dot{x}_s + \gamma' \dot{x}_s + \left( k + \frac{\gamma'}{\tau} - \frac{\gamma' \sigma_{1,3}}{\sigma_{1,1}} \right) x_s \right] \circ \dot{x}_s, \]

and the corresponding apparent EP functional \( \tilde{\sigma}[\{x_s\}] \equiv \Delta s_{\text{sys}} + \Delta \tilde{s}_m[\{x_s\}] + (\gamma'/m)t \) obeys fluctuation theorems with the trajectories generated by equation (92) and the conjugate equation (where \( \gamma' \) is replaced by \( -\gamma' \) but \( \gamma' \) in the coefficient of \( x \) is not changed), while giving back equation (88) upon averaging.

Finally, we can again compare the apparent EP to the (total) EP in the super-system that contains the full statistical information on the degrees of freedom \( x \) and \( y \). The calculation is more complicated that the one performed in appendix C for model V because the noises coming into play in equations (9) are correlated. Here, for brevity, we only give the expression of the entropy balance equation in the NESS,

\[ \dot{\Sigma} = \dot{S}_{m} + \dot{S}_{m'} = \frac{\gamma}{m} \frac{T''}{T} + \frac{1}{\tau} \frac{T(v) - T(x)}{T'}, \]

\[ = \frac{\gamma }{m_{\text{eff}}} \frac{\left( \gamma T + \gamma' T' \right) T + \left( \gamma_{\text{eff}} - \gamma - \gamma' \right) T'^2}{TT'}, \]

29 This is not surprising since the effective Langevin equation, by construction, must yield the same marginal probability distribution \( p_{\text{st}}(x, v) \) as equations (9), and this quantity has formally the same expression in models V and P, and only the effective temperatures \( T'' \) and \( T(x) \) are different.
Feedback cooling, measurement errors, and entropy production

Figure 2. Spectral density $S_{xx'}(\omega)$ (in Å$^2$ Hz$^{-1}$) given by equation (A.13) for different values of the feedback gain $g$. The resonator parameters correspond to the cantilever 1 studied in [3] (see text). The cantilever is cooled from a base temperature $T = 4.2$ K. From top to bottom: $g = 0, 544, 1321, 5690$. Black dashed lines: $\tau = 0$; red solid lines: $\tau = 0.1$ ms.

where equations (A.9) are used to obtain the last expression. This quantity does not vanish for $T' = T$ and it can be easily checked that it is always larger than $\dot{\Sigma}$. We also notice that $T\dot{S}_m + T'\dot{S}_{m'} = (\gamma'/m)T'$. Therefore, if one associates $T'\dot{S}_{m'}$ with the heat flow $\dot{Q}'$ from a second reservoir at temperature $T'$, one has $\dot{Q} + \dot{Q}' \neq 0$. Accordingly, $\dot{\Sigma} \neq \dot{Q}(1/T - 1/T')$, in contrast with model V.

To illustrate the above equations, let us consider the feedback cooling of the fundamental mechanical mode of a cantilever. As an example, we take the ultrasoft silicon cantilever studied in [3], with intrinsic quality factor $Q_0 = 44200$, resonant frequency $\omega_0 = 3.9$ kHz, and spring constant $k = 86$ µN m$^{-1}$. This cantilever is cooled from a base temperature of 4.2 K, and the spectral density of the measurement noise is $\sqrt{S_{x_n}} \approx 10^{-2}$ Å Hz$^{-1/2}$, as estimated from fits of the measured spectra. Specifically, we take $S_{x_n} = 4 \times 10^{-4}$ Å$^2$ Hz$^{-1}$, which from equation (83) yields $T_{\text{eff}}^{(z)}(g = 544) = 8.3$ mK for $\tau = 0$, in agreement with the value indicated in figure 3 of [3]. Finally, we choose the value $\tau = 0.1$ ms for the relaxation time of the feedback mechanism. This choice is rather arbitrary and is mainly done to illustrate the model behavior (the actual relaxation time in the experiment described in [3] is certainly much smaller; see the discussion at the end of the section). Note that this value is much smaller than the effective momentum relaxation time $\tau_0/(1 + g)$, even for the largest value of $g$ considered in the experiments (and moreover $2\pi/\tau \gg \omega_0$). This guarantees that the resonator is still efficiently cooled in the vicinity of its resonant frequency.

As shown in figure 2, this small but finite value of $\tau$ slightly modifies the measured spectral density $S_{xx'}(\omega)$ of the cantilever computed from equation (A.13). In particular, the resonant frequency is now dependent on the feedback gain$^{30}$. Note that the peak in $S_{xx'}(\omega)$ changes into a dip in the high gain regime, as the detector noise sent back to the

$^{30}$The shape of the PSD may significantly change if $\tau$ is large, especially if the quality factor of the oscillator is small. Such effects are discussed in [46]. Since this is not the main purpose of our work, we have not performed a systematic study of the model behavior as a function of the different parameters.

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Figure 3. (a) Resonator temperature $T_{\text{eff}}^{(x)}$ as a function of the gain $g$ for $\tau = 0$ (black dashed line) and $\tau = 0.1$ ms (black solid line). The red line is the kinetic temperature $T_{\text{eff}}^{(v)}$ for $\tau = 0.1$ ms. (b) Different contributions to the entropy production rate for $\tau = 0.1$ ms according to equations (88)–(90): $\dot{\Sigma}$ (black solid line), $\dot{S}_m$ (red solid line), $\dot{S}_{pu}$ (blue solid line). The dashed line is the total entropy production rate $\dot{\Sigma}$ in the super-system given by equation (94).

Brownian system dominates and acts to heat the mechanical device, an effect known as ‘noise squashing’ [2, 3, 45]. Accordingly, integrating the observed spectrum $S_{xx'}(\omega)$ over $\omega$ leads to an underestimation of the actual resonator temperature. The true resonator motion $S_{xx}(\omega)$ can be recovered by using the theoretical expression, equation (A.13a), or can be directly measured by adding a second transducer outside the feedback loop [68, 69].

As could be expected, in the presence of a finite relaxation time, the feedback cooling becomes less effective, and the minimal achievable temperature $T_{\text{eff}}^{(x)}$ increases, as shown in figure 3(a), albeit very slightly (from 4.3 mK for $\tau = 0$ to 4.6 mK for $\tau = 0.1$ ms). On the other hand, for this value of $\tau$, the two temperatures $T_{\text{eff}}^{(x)}$ and $T_{\text{eff}}^{(v)}$ are very close one to each other around the minimum. Accordingly, the apparent EP rate $\dot{\Sigma}$ displayed in figure 3(b) is maximal when $T_{\text{eff}}^{(x)}$ is close to its minimum. We also notice that the essential contribution to $\dot{\Sigma}$ comes from the entropy pumping since $\dot{S}_m$ is very small in the whole range of $g$ that is experimentally explored. This was also observed in model V for $\tau = 0$ (see figure 1). This picture changes for much larger values of the gain (typically $g > 10^6$), as the resonator is heated instead of cooled. The main contribution to $\dot{\Sigma}$ then comes from the heat dissipated into the environment and entropy pumping is negligible. $\dot{\Sigma}$ thus reaches a minimum for some large value of $g$ (beyond the scale of figure 3(b) and unrelated to the minimum observed for the total EP rate $\dot{\Sigma}$), but in contrast with model
V, this minimum is positive since there is always dissipation in the system, as we already stressed.

Finally, let us again emphasize that the feedback relaxation time $\tau$ is expected to play a role only for resonance frequencies in the MHz range onwards. In this respect, the value $\tau = 0.1$ ms chosen here for illustrative purposes only is certainly too large. Taking a much smaller value would drastically change the qualitative picture in figure 3, since $\dot{\Sigma}$ and $\dot{S}_m$ both diverge as $\tau \to 0$ (whereas $\dot{S}_{pu}$ is finite), as already noticed. This spurious behavior of the model comes from the (commonly made) assumption that the measurement noise is white. Introducing some cutoff at large frequencies would suppress these divergences and allow us to take a more realistic value of $\tau$. This, however, would add another (rather arbitrary) parameter in the model and somewhat complicate the description.

5. Conclusion

In this paper, we have generalized the results of [21]–[23] to take into account the effect of measurement errors (or detector noise) on the entropy production (EP) in a cold damping process. This has led us to consider two models (called P and V) that distinguish whether the position of the resonator or its velocity (in practice an electric current) is the observable, as the two situations actually occur in experimental setups. We also have assigned a finite relaxation time to the feedback mechanism, which may play a role at high frequencies, but is also required to regularize model P when the detector noise is white. This makes the feedback control non-Markovian.

To define the EP, we have adopted the viewpoint of the controlled system, as in [21]–[23] and in most recent studies of the thermodynamic behavior of feedback-controlled systems. In this framework, we have defined and computed in the nonequilibrium steady state the entropy pumping that describes the entropy reduction in the system due to its interaction with the external agent that manipulates the feedback control (which in this case is not a genuine Maxwell’s demon). For error-free measurements, the entropy pumping can be ascribed to the momentum phase space contraction induced by the additional damping force. The situation is more complicated in the presence of noise, as one can no longer relate the heat dissipated along a stochastic trajectory of the system to time irreversibility (more precisely, this identification would lead to an unphysical result, independent of the measurement noise). A proper relationship only exists if one considers the super-system that also includes the external agent, as done in [24]. Accordingly, in the presence of measurement errors, the entropy pumping cannot be simply associated with a contraction of momentum phase space. This has led us to define the entropy pumping rate and the non-negative EP rate at the ensemble level by using the (coarse-grained) Fokker–Planck equation to derive the average entropy balance equation (i.e., the generalized second law). This is the proper generalization of the results of [21]. In particular, the EP rate (called the apparent EP rate as in [38]) in the nonequilibrium steady state remains finite in the limit of error-free measurements whereas the total EP rate in the super-system diverges. Moreover, we have shown that the behavior of the apparent EP as a function of the feedback gain is consistent with the expected behavior of the dissipation in a cold damping setup: it is maximal when the feedback cooling is the most efficient. In the cooling regime, it is found that the main contribution to the EP comes from the entropy pumping, whereas the average heat flow coming from the bath.

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plays a negligible role. Measurement errors decreases the entropy pumping (in absolute value), and the dissipation is in turn reduced. It would be interesting to generalize these observations to transient regimes, for instance when the cooling is abruptly switched on or off, as considered in [2].

Finally, we have shown that trajectory-dependent functionals can be defined in the nonequilibrium steady state by replacing the original Langevin dynamics by an effective dynamics. The so-defined apparent EP functional then obeys fluctuations theorems with the new trajectories generated by this effective dynamics.

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Appendix A. Stationary probability distributions and power spectral densities (PSD) for models V and P

In this appendix we compute the stationary probability distributions and the power spectral densities in models V and P. These quantities are easily obtained by noting that equations (7) and (9) describe multivariate Ornstein–Uhlenbeck processes so that standard textbooks expressions can be used. To this aim, we rewrite these equations in the form

\[
\frac{dX(t)}{dt} = -\Gamma X(t) + \Phi(t)
\]

where \( X \) is the three-dimensional vector \([x, v, y]\), \( \Gamma \) is a 3 \times 3 damping matrix, and \( \Phi \) is a three-variate Gaussian process with zero mean and symmetric covariance matrix \( \langle \phi_i(t)\phi_j(t') \rangle = 2D_{ij}\delta(t - t') \). The explicit expressions of \( \Gamma \) and \( D \) are given below. The corresponding Fokker–Planck equation then reads

\[
\partial_t p_t(X) = -\sum_i \partial_x J^{(i)}_t(X) = \sum_{i,j} \partial_x x_i J_{ij}_t p_t(X) + D_{ij} \partial_x x_j p_t(X),
\]

and the stationary solution is given by the Gaussian distribution [56, 70]

\[
p_{st}(X) = (2\pi)^{-3/2}[\text{Det } \sigma]^{-1/2} \exp(-\frac{1}{2}X\sigma^{-1}X),
\]

where the covariance matrix \( \sigma \) is the solution of the algebraic matrix equation

\[
2D = \Gamma \sigma + \Gamma \sigma^T = \Gamma \sigma + \sigma \Gamma^T.
\]

In model V, the 3 \times 3 damping matrix \( \Gamma \) and the covariance matrix \( D \) are

\[
\Gamma = \begin{pmatrix}
0 & -1 & 0 \\
\gamma/m & \gamma/m & \gamma'/m \\
0 & -1/\tau & 1/\tau
\end{pmatrix}
\]

\[
D = \begin{pmatrix}
0 & 0 & 0 \\
\gamma T/m^2 & 0 & 0 \\
0 & 0 & \gamma'/\tau^2
\end{pmatrix}
\]
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where $T' \equiv \gamma' S_{v_0}/2$. By solving equation (A.4) we then obtain the following expressions for the elements of the covariance matrix $\sigma$:

\[
\begin{align*}
\langle x^2 \rangle_{st} &\equiv \sigma_{1,1} = \frac{\gamma_{\text{eff}} - \gamma' (1 + \tau/\tau_0) T}{\gamma_{\text{eff}}} + \frac{\gamma' (1 + \tau/\tau_0) T'}{\gamma_{\text{eff}}}, \\
\langle v^2 \rangle_{st} &\equiv \sigma_{2,2} = \frac{\gamma_{\text{eff}} - \gamma' T}{\gamma_{\text{eff}}} + \frac{\gamma' T'}{m}, \\
\langle y^2 \rangle_{st} &\equiv \sigma_{3,3} = \frac{\gamma T}{\gamma_{\text{eff}}} + \frac{\gamma_{\text{eff}} - \gamma'(\tau/\tau_0)}{\gamma'(\tau/\tau_0)} T', \\
\langle x v \rangle_{st} &\equiv \sigma_{1,2} = \sigma_{2,1} = 0, \\
\langle x y \rangle_{st} &\equiv \sigma_{1,3} = \sigma_{3,1} = \frac{\tau}{m \gamma_{\text{eff}}} (T - T'), \\
\langle v y \rangle_{st} &\equiv \sigma_{2,3} = \sigma_{3,2} = \frac{1}{m} \frac{\gamma}{\gamma_{\text{eff}}} (T - T').
\end{align*}
\]  
\tag{A.5}

where

\[
\gamma_{\text{eff}} \equiv (\gamma + \gamma') \left(1 + \frac{\tau}{\tau_0}\right) + \frac{k \tau^2}{\tau_0} \\
= (\gamma + \gamma') \left(1 + \frac{\tau}{\tau_0}\right) + \gamma \left(Q_0 \frac{\tau}{\tau_0}\right)^2
\]  
\tag{A.6}

is an effective friction coefficient (recall that $\tau_0 = m/\gamma$ and $Q_0 = \sqrt{mk}/\gamma = \omega_0 \tau_0$). Note the useful relations\(^{31}\)

\[
\begin{align*}
\gamma \sigma_{2,2} + \gamma' \sigma_{2,3} &= \frac{\gamma}{m} T, \tag{A.7a} \\
m \sigma_{2,2} - k \sigma_{1,1} &= \gamma' \sigma_{1,3}. \tag{A.7b}
\end{align*}
\]

Similarly in model P,

\[
\begin{align*}
\Gamma &= \begin{pmatrix} 0 & -1 & 0 \\ (k + \gamma'/\tau)/m & \gamma/m & -\gamma'/m \\ -1/\tau & 0 & 1/\tau \end{pmatrix}, \\
D &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & (\gamma T + \gamma' T')/m^2 & -T'/m \\ 0 & -T'/m & T'/\gamma' \end{pmatrix}.
\end{align*}
\]

\(^{31}\)Note also that the most probable value of $y$ for a given value of $v$ is not $v$, which is a consequence of the non-Markovian character of the feedback control.

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where \( T' \equiv (\gamma'/\tau^2)S_{xn}/2 \). This yields
\[
\begin{align*}
\langle x^2 \rangle_{st} &\equiv \sigma_{1,1} = \frac{\gamma_{\text{eff}} - \gamma'(1 + \tau/\tau_0) T}{k} + \left( \frac{Q_0 \tau}{\tau_0} \right)^2 \frac{\gamma'}{\gamma_{\text{eff}}} \frac{T'}{k} \\
\langle v^2 \rangle_{st} &\equiv \sigma_{2,2} = \frac{\gamma_{\text{eff}} - \gamma' T}{m} + \frac{\gamma' \gamma_{\text{eff}} - \gamma - \gamma' T'}{m} \\
\langle y^2 \rangle_{st} &\equiv \sigma_{3,3} = \frac{\gamma}{\gamma_{\text{eff}}} \left( \frac{1 + \tau}{\tau_0} \right) \frac{T}{k} + \frac{T}{\tau_0} \frac{Q_0^2 \gamma \gamma_{\text{eff}} - \gamma'(\tau/\tau_0) T'}{\gamma_{\text{eff}}} \frac{1}{k} \\
\langle xy \rangle_{st} &\equiv \sigma_{1,2} = \sigma_{2,1} = 0 \\
\langle x y \rangle_{st} &\equiv \sigma_{1,3} = \sigma_{3,1} = \frac{\gamma}{\gamma_{\text{eff}}} \left( \frac{1 + \tau}{\tau_0} \right) \frac{T}{k} - \left( \frac{Q_0 \tau}{\tau_0} \right)^2 \frac{\gamma}{\gamma_{\text{eff}}} \frac{T'}{k} \\
\langle v y \rangle_{st} &\equiv \sigma_{2,3} = \sigma_{3,2} = -\frac{\tau}{\tau_0} \left( \frac{T}{\gamma_{\text{eff}}} + \frac{\gamma' T'}{\gamma_{\text{eff}}} \right).
\end{align*}
\] (A.8)

Note again the useful relations
\[
\begin{align*}
\gamma \sigma_{1,1} + \gamma' \sigma_{1,3} &= \gamma \frac{T}{k} \quad \text{(A.9a)} \\
\gamma \sigma_{2,2} - \gamma' \sigma_{2,3} &= \gamma \frac{T}{m} + \gamma' \frac{T'}{m} \quad \text{(A.9b)} \\
\left( k + \gamma' \frac{T}{\tau} \right) \sigma_{1,1} - \gamma' \frac{T}{\tau} \sigma_{1,3} &= m \sigma_{2,2}.
\end{align*}
\] (A.9c)

The power spectral densities of \( v \) (resp. \( x \)), the actual velocity (resp. displacement) of the resonator, and \( v' = v + v_n \) (resp. \( x' = x + x_n \)), the observed velocity (resp. displacement) are obtained by using the expression for the spectrum matrix of a multivariate Ornstein–Uhlenbeck process in the stationary state [70],
\[
S(\omega) = (\Gamma + i\omega \mathbf{1})^{-1}(2\mathbf{D})(\Gamma^T - i\omega \mathbf{1})^{-1}.
\] (A.10)

For model V, this yields
\[
\begin{align*}
S_{vv}(\omega) &= \left[ \frac{1 + \tau^2 \omega^2}{|D(\omega)|^2} \right] S_{F_{th}} + \left[ \frac{g^2 \omega^2 / \tau_0^2}{|D(\omega)|^2} \right] S_{v_n} \quad \text{(A.11a)} \\
S_{v'v'}(\omega) &\equiv (1 + \tau^2 \omega^2) S_{y,y}(\omega) \\
&= \left[ \frac{1 + \tau^2 \omega^2}{|D(\omega)|^2} \right] S_{F_{th}} + \left[ \frac{(1 + \tau^2 \omega^2)(\omega_0^2 - \omega^2)^2 + \omega^2 / \tau_0^2}{|D(\omega)|^2} \right] S_{v_n} \quad \text{(A.11b)}
\end{align*}
\]
where
\[
D(\omega) = \left[ \omega_0^2 - \left( 1 + \frac{\tau}{\tau_0} \right) \omega^2 \right] + i \omega \left[ \frac{1 + g}{\tau_0} + \tau (\omega_0^2 - \omega^2) \right]
\] (A.12)
and \( S_{F_{th}} \equiv 2\gamma T \) is the white spectral density of the thermal noise force\(^{32}\).

\(^{32}\) In this work we use the two-sided convention for a spectral density, i.e. \( S_{\alpha,\beta}(\omega) \equiv \int_{-\infty}^{+\infty} e^{i\omega t} \phi_{\alpha,\beta}(t) \, dt \), where \( \phi_{\alpha,\beta}(t) \) is a time-translational invariant correlation function in the stationary state. Hence, \( S_{F_{th}} = 2\gamma T \) for the PSD of the Langevin thermal force. On the other hand, the one-sided convention is often used in experimental papers, for instance in [3] (accordingly, \( S_{xn} \), the spectral density of the measurement noise in [3] is two times larger than our \( S_{xn} \)).
Similarly, for model P:

\[
S_{xx}(\omega) = \left[\frac{(1 + \tau^2 \omega^2)/m^2}{|D(\omega)|^2}\right] S_{\text{Fih}} + \left[\frac{g^2 \omega^2 / \tau_0^2}{|D(\omega)|^2}\right] S_{\text{xn}} \tag{A.13a}
\]

\[
S_{x'x'}(\omega) = (1 + \tau^2 \omega^2) S_{y,y}(\omega)
\]

\[
= \left[\frac{(1 + \tau^2 \omega^2)/m^2}{|D(\omega)|^2}\right] S_{\text{Fih}} + \left[\frac{(1 + \tau^2 \omega^2)[(\omega_0^2 - \omega^2)^2 + \omega^2 / \tau_0^2]}{|D(\omega)|^2}\right] S_{\text{xn}}. \tag{A.13b}
\]

The only difference with the PSDs of model V is the absence of the factor \(\omega^2\) in the terms proportional to \(S_{\text{Fih}}\). Equations (A.13) reduce to equations (3) and (4) of [3] for \(\tau = 0\). One can also check that \(\tilde{T}_{\text{eff}}(x) \equiv k \langle x^2 \rangle_{st} = 1/(2\pi) \int_{-\infty}^{\infty} S_{xx}(\omega) d\omega\) and \(T_{\text{eff}}(x) \equiv m \langle v^2 \rangle_{st} = 1/(2\pi) \int_{-\infty}^{\infty} S_{vv}(\omega) d\omega\), which shows that the divergence of the kinetic temperature as \(\tau \to 0\) is related to the behavior of \(S_{vv}(\omega) = \omega^2 S_{xx}(\omega)\) at high frequencies.

**Appendix B. Log-ratio of the path probabilities in the NESS (model V)**

In this appendix we compute the quantity \(\ln \mathcal{P}_+([x_s]) / \mathcal{P}_-([\hat{x}_s])\) in the NESS for model V.

In general, the path probability \(\mathcal{P}_+([x_s])\) can be obtained via two different routes. First, one can start from the conditional probability \(\mathcal{P}_+([X_s] | X_0)\) of the path \([X_s]_{s[t]} = ([x_s], \{y_s\})_{s[t]}\) given by equation (C.1) in appendix C and perform the path integral over \([y_s]\). Second, one can consider the Langevin equation with memory and colored noise obtained by inserting the integrated expression of \(y(t)\) in equation (7a). This is the procedure used in [35] for a very similar (but overdamped) model [35]. In the NESS, however, one can simply use the fact that \(\mathcal{P}_+([x_s])\) is Gaussian and given in the Fourier (frequency) domain by

\[
\mathcal{P}_+([x_s]) \propto \exp\left[-\frac{1}{4} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} x(\omega) S_{xx}(\omega)^{-1} x(-\omega)\right] \tag{B.1}
\]

where \(S_{xx}(\omega) = S_{vv}(\omega)/\omega^2\) is the power spectrum distribution of the position \(x\) (the normalization factor does not play any role in what follows). We stress that the influence of the initial conditions is neglected when going to the frequency domain, which is correct as long as one only considers expectation values [34]. Using the expression of \(S_{vv}(\omega)\) given by equation (A.11a) and replacing \(S_{vv}\) by \(2T / \gamma'\), we obtain after some simple manipulations

\[
\mathcal{P}_+([x_s]) \propto \exp\left[-\frac{m^2}{4 \gamma' T} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} x(\omega) |D(\omega)|^2 \frac{|D(\omega)|^2}{(\gamma T + \gamma' T')/\gamma T + \omega_0^2 \tau^2} x(-\omega)\right] \tag{B.2}
\]

where \(D(\omega)\) is given by equation (A.12). The probability \(\mathcal{P}_-([\hat{x}_s])\) for the time-reversed trajectory is then obtained by replacing \(x(\omega)\) by \(x(-\omega)\) and changing \(\gamma'\) to \(-\gamma'\). This

\[33\] As explained in [35], this route is only valid if the initial condition \(y_0\) is chosen from a specific random distribution. Hence, this route is useful in the asymptotic long-time limit only.

\[34\] Indeed, we have shown in [23] that the so-called ‘boundary’ terms may have a dramatic effect on large fluctuations, a problem that occurs when the position and velocity of the particle are unbounded; this issue was already pointed out in [71]. To obtain the correct expression of the boundary terms, there is no other choice than performing the functional integration of \(\mathcal{P}_+([X_s])\), which is a workable but tedious calculation.

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readily yields

$$\ln \frac{\mathcal{P}_+[\{x_s\}]}{\mathcal{P}_-[\{\hat{x}_s\}]} = -\frac{\gamma'}{\gamma T} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \omega^2 x(\omega) \frac{\gamma + k\tau - m\tau\omega^2}{(\gamma T + \gamma' T')/\gamma T + \omega^2 \tau^2} x(-\omega).$$  \tag{B.3}$$

This log-ratio depends on the measurement noise via the presence of $T'$ in the denominator. However, this dependence disappears when performing the average. Indeed, since $S_{x^2}(\omega) \equiv \langle x(\omega)x(-\omega) \rangle_{st}$ by definition, we obtain

$$\frac{1}{T} \left( \ln \frac{\mathcal{P}_+[\{x_s\}]}{\mathcal{P}_-[\{\hat{x}_s\}]} \right)_{st} = -\frac{\gamma'}{\gamma T} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \omega^2 \frac{\gamma + k\tau - m\tau\omega^2}{(\gamma T + \gamma' T')/\gamma T + \omega^2 \tau^2} S_{x^2}(\omega)$$

$$= -\frac{2\gamma'}{m^2} \int_{0}^{\infty} \frac{d\omega}{\pi} \omega^2 (\gamma + k\tau - m\tau\omega^2) D(\omega)^2.$$

This integral can be computed analytically and, after some tedious but elementary algebra, we obtain the very simple result

$$\frac{1}{T} \left( \ln \frac{\mathcal{P}_+[\{x_s\}]}{\mathcal{P}_-[\{\hat{x}_s\}]} \right)_{st} = \frac{\gamma^2}{m\gamma_{\text{eff}}}$$  \tag{B.4}$$

with $\gamma_{\text{eff}}$ given by equation (A.6). This result generalizes equation (44) for $\tau > 0$, but cannot be taken as a pertinent definition of the EP rate since it is independent of the measurement noise.

**Appendix C. Entropy production in the super-system (model V)**

In this appendix, we compute the entropy production in model V when the full statistical information on the microscopic degrees of freedom is available, that is when the two trajectories $\{x_s\}_{s \in [0,t]}$ and $\{y_s\}_{s \in [0,t]}$ can be observed. The EP along the trajectory $\{X_s\}_{s \in [0,t]}$ can then be defined in the standard way from the log-ratio of the probabilities of the forward and reverse paths. Since the two noises $\xi(t)$ and $\eta(t)$ are independent and Gaussian distributed, the probability of the path $\{X_s\}$, conditioned on the initial state $X_0 = (x_0, v_0, y_0)$, is given by

$$\mathcal{P}[\{X_s\}|X_0] \propto \exp \left[ -\frac{1}{4\gamma T} \int_{0}^{t} ds \left( m\dddot{x}_s + \gamma\ddot{x}_s + \gamma' y_s + k x_s \right)^2 \right.$$  

$$- \frac{1}{2\gamma T} \int_{0}^{t} ds \left( \tau \dot{y}_s + y_s - \dddot{x}_s \right)^2 \right].$$  \tag{C.1}$$

The backward path $\{\hat{X}_s\}_{s \in [0,t]}$ is then defined by the time-reversal operation $\hat{x}_s \equiv x_{t-s}, \hat{\dddot{x}}_s \equiv -\dddot{x}_{t-s}, \hat{y}_s \equiv y_{t-s}, \hat{\dot{y}}_s \equiv -\dot{y}_{t-s}$, which yields

$$\mathcal{P}[\{\hat{X}_s\}|\hat{X}_0] \propto \exp \left[ -\frac{1}{4\gamma T} \int_{0}^{t} ds \left( m\dddot{x}_s - \gamma\ddot{x}_s + \gamma' y_s + k x_s \right)^2 \right.$$  

$$- \frac{1}{2\gamma T} \int_{0}^{t} ds \left( -\tau \dot{y}_s + y_s + \dddot{x}_s \right)^2 \right].$$  \tag{C.2}$$
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Hence

\[
\ln \frac{\mathcal{P}([X_s]|X_0)}{\mathcal{P}([X_s]|X_0)} = \Delta s_m([X_s]) + \Delta s_{m'}([X_s]) \tag{C.3}
\]

where

\[
\Delta s_m([X_s]) = -\frac{1}{T} \int_0^t ds \left( m\ddot{x}_s + \gamma' y_s + k x_s \right) \circ \dot{x}_s \tag{C.4a}
\]

\[
\Delta s_{m'}([X_s]) = -\frac{2}{S_{vn}} \int_0^t ds \left[ \tau \dot{y}_s - \dot{x}_s \right] \circ y_s. \tag{C.4b}
\]

The total entropy production along the trajectory is then given by

\[
\sigma([X_s]) \equiv \ln \frac{\mathcal{P}([X_s])}{\mathcal{P}([X_s])} = \Delta s_{\text{sys}} + \Delta s_m([X_s]) + \Delta s_{m'}([X_s]) \tag{C.5}
\]

where \(\Delta s_{\text{sys}} = \ln p_0(X_0)/p_t(X_t)\). By construction, \(\sigma([X_s])\) satisfies the standard fluctuation theorems [18]. Note that \(y\) has been treated as an even variable under time reversal in order to derive equation (C.3) (despite the fact that it has the dimension of a velocity). This is indeed essential for recovering the correct expression of \(\Delta s_m([X_s]) \equiv q([X_s])/T\), where \(q([X_s])\) is the heat exchanged with the environment at temperature \(T\), defined by equation (42) in the main text.35

If one ignores the contribution of the boundary terms in equation (C.5), which are not extensive in time and vanish on average in the NESS, the entropy production for long times is given by

\[
\sigma([X_s]) \approx \gamma' \left( \frac{1}{T'} - \frac{1}{T} \right) \int_0^t ds \, y_s \circ \dot{x}_s \tag{C.6}
\]

where \(T' = \gamma'S_{vn}/2\) (this corresponds to equation (B7) in [35]).

At this stage, the quantity \(\Delta s_{m'}([X_s])\) has no definite physical meaning. However, if \(T'\) is the actual temperature of a second heat bath coupled to the Brownian particle, and equation (7b) is rewritten as

\[
(\gamma' \tau) \dot{y} + \gamma' (y - \dot{x}) = \sqrt{2\gamma' T'} \eta(t), \tag{C.7}
\]

then one can identify \(\Delta s_{m'}([X_s])\) with \(q'([X_s])/T'\), where

\[
q'([X_s]) = \int_0^t ds \left[ \gamma' y_s - \sqrt{2\gamma' T'} \eta_s \right] \circ y_s \tag{C.8a}
\]

\[
= \gamma' \int_0^t ds \left[ -\tau \dot{y}_s + \dot{x}_s \right] \circ y_s \tag{C.8b}
\]

is the heat exchanged with this second reservoir. Interestingly, equation (C.6) is also the result obtained in [24] for the EP associated with equation (39) when the contribution

35In fact, the correct expression of \(\Delta s_m([X_s])\) can also be recovered with \(y\) odd and \(\gamma'\) changed to \(-\gamma'\), but this does not give a sensible result for \(\Delta s_{m'}([X_s])\) since this quantity then vanishes for \(\tau = 0\). Here \(y\) plays the same role as the auxiliary variables \(v_i\) in [48] that appear when mapping a generalized Langevin equation with exponential memory kernel and colored noise to a set of coupled Markovian equations. These variables are indeed even under time reversal (the same is true for the variable \(U\) defined by equations (B1) in [35]).

\[\text{doi:10.1088/1742-5468/2013/06/P06014}\]
of the feedback controller is included (indeed, note that the above equations have a well-defined limit for \( \tau = 0 \)). In this case, as noted in section 4.1, \( \Delta s_{\text{sy}}[\{X_s\}] \) corresponds to the time-continuous limit [18] of the quantity \( \Delta s_p \), which is interpreted in [24] as the entropy production due to the measurement process. From the expression of \( \Delta s_p \) (see equation (7) in [24]), one can easily see that the definition of the (discrete) reverse process proposed in [24] amounts to treating \( y \) as an even variable under time reversal in the continuous-time limit. In this respect, it is not surprising that the EP of the full system computed in [24] obeys the detailed FT.

Upon averaging, one recovers from equation (C.5) the balance equation obtained from the time derivative of the Shannon entropy. The Fokker–Planck equation (66) is then written as

\[
\partial_t p_t(X) = -\partial_x [vp_t(X)] - \frac{1}{m} \partial_v [-(kx + \gamma' y)p_t(X) + J_t(X)] - \frac{1}{\gamma'/\tau} \partial_y [\gamma' vp_t(X) + J'_t(X)]
\]

where \( J_t(X) \) and \( J'_t(X) \) are the irreversible components of the probability currents defined as

\[
J_t(X) = -\gamma \left[ v + \frac{T}{m} \partial_v \ln p_t(X) \right] p_t(X)
\]

\[
J'_t(X) = -\gamma' \left[ y + \frac{T'}{\gamma'\tau} \partial_y \ln p_t(X) \right] p_t(X).
\]

(Note that \( J'_t(X) \) is indeed the time-antisymmetric component of \( J^{(y)}_t = -\gamma'[(y-v)p_t(X) + T'/(\gamma'\tau)\partial_y p_t(X)] \) because \( y \) is even under time reversal.) One then finds

\[
\dot{\Sigma}(t) = \dot{S}_{\text{sys}}(t) + \dot{S}_{m}(t) + \dot{S}_{\text{sy}}(t)
\]

where \( \dot{S}_{m}(t) = -(1/T) \int dX v J_t(X) = -(1/T) \int dx dv v J_t(x,v) \) is again given by equation (30),

\[
\dot{S}_{\text{sy}}(t) = -\frac{1}{T'} \int dX y J'_t(X)
\]

\[
= \frac{\gamma'}{T'} \left\{ \langle y^2 \rangle_t - \frac{T'}{\gamma'\tau} \right\},
\]

and

\[
\dot{\Sigma}(t) = \frac{1}{\gamma T} \int dX \frac{[J_t(X)]^2}{p_t(X)} + \frac{1}{\gamma' T} \int dX \frac{[J'_t(X)]^2}{p_t(X)}.
\]

This latter expression is in agreement with the general expression of the EP rate in the full phase space when even and odd variables are present [55]. Furthermore, using the inequality\(^{36}\)

\[
\int dy \frac{[J_t(X)]^2}{p_t(X)} \geq \left[ \int dy J_t(X) \right]^2 \left[ \int dy p_t(X) \right] = \frac{[J_t(x,v)]^2}{p_t(x,v)},
\]

\(^{36}\)This is the continuous version of the inequality given in [32] in the case of a discrete set of numbers (see also [72]).

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one finds that
\[ \dot{\Sigma}(t) \geq \dot{\Sigma}(t) + \frac{1}{\gamma^\prime T^\prime} \int dX \frac{|J_t(X)|^2}{p_t(X)} \geq \dot{\Sigma}(t) \] (C.15)
where \[ \dot{\Sigma}(t) = \frac{1}{(\gamma T) \int dx dv [J_t(x, v)]^2 / p_t(x, v) } \] is the apparent EP rate considered in section 4.2.

Finally, in the steady state, one obtains
\[ \dot{\Sigma} = \dot{\mathcal{Q}} \left( \frac{1}{T} - \frac{1}{T^\prime} \right) \]
\[ = \frac{\gamma \gamma^\prime}{m \gamma_{\text{eff}}} \frac{(T - T^\prime)^2}{T T^\prime}, \] (C.16)
which generalizes equations (59) to \( \tau > 0 \) as the heat flow \( \dot{\mathcal{Q}} \) is still given by equation (60) (with \( T_{\text{eff}} \) replaced by \( T_{\text{eff}}^{(v)} \)).

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\langle (\xi(t) F) \xi(t') \rangle = \sum_i \int dt \langle (\xi_i(t) \xi_i(t')) \delta F \delta \xi_i(t') \rangle,
\]
where \( \{ \xi_i \} \) is a set of Gaussian noise variables, \( F \) is a functional of these noise variables, and \( \delta F \delta \xi_i(t') \) is a functional derivative.
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