Controlled Rough Paths on Manifolds I

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Abstract

In this paper, we build the foundation for a theory of controlled rough paths on manifolds. A number of natural candidates for the definition of manifold valued controlled rough paths are developed and shown to be equivalent. The theory of controlled rough one-forms along such a controlled path and their resulting integrals are then defined. This general integration theory does require the introduction of an additional geometric structure on the manifold which we refer to as a “parallelism.” The transformation properties of the theory under change of parallelisms is explored. Using these transformation properties, we are able to show that the integration of a smooth one-form along a manifold valued controlled rough path is in fact well defined independent of any additional geometric structures. Lastly, we present a theory of push-forwards and show how it is compatible with our integration theory.

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1 Introduction

In a series of papers [20–22], Terry Lyons introduced and developed the far reaching theory of rough path analysis. This theory allows one to solve (deterministically) differential equations driven by rough signals at the expense of “enhancing” the rough signal with some additional information. Lyons’ theory has found numerous applications to stochastic calculus and stochastic differential equations, for example see [4], [5], [6], [8], and the references therein. For some more recent applications, see [1], [19], [18], [9], and [2].

The rough path theory mentioned above has been almost exclusively developed in the context of state spaces being either finite or infinite dimensional Banach spaces with the two exceptions of [7] and [3]. In [7], a version of manifold valued rough paths is developed in the context of “currents,” while in [3] the authors develop a more concrete theory by working with embedded submanifolds.

The purpose of this paper is to define and develop a third interpretation of rough paths on manifolds based on Gubinelli’s [14] notions of “controlled” rough paths. As Gubinelli’s perspective has proved extremely useful in the flat case (most notably see Hairer [15]), it is expected such a theory of controlled rough paths on manifolds can give new insights as well as applications to the existing literature. We now will present a brief summary of the results contained in this paper.

1.1 Summary of Results

Let $M^d$ be a $d$-dimensional manifold, $X_{s,t} := 1 + x_{s,t} + X_{s,t}$ be a geometric rough path in $\mathbb{R}^k$ with $1 \leq p < 3$. A rough path controlled by $X$ on $M$ (see Definition 2.34) is a pair of continuous functions $y : [0, T] \to M$, and $y^\dagger : [0, T] \to \text{L} (\mathbb{R}^k, TM)$ such that (somewhat imprecisely speaking) for all $0 \leq s \leq t \leq T$:

1) $y^\dagger_t : \mathbb{R}^k \to T_{y_t} M$, 2) $\psi (y_s, y_t) = y^\dagger_s x_{s,t} + O \left( |x_{s,t}|^2 \right)$, and 3) $U (y_s, y_t) y^\dagger_t - y^\dagger_s = O \left( |x_{s,t}| \right)$,

where $\psi$ is a “logarithm” on $M$ (see Definition 2.15) and $U$ is a “parallelism” on $M$ (see Definition 2.16). [When $M = \mathbb{R}^d$, one identifies all tangent spaces in which case one typically takes $U (m, n) = \ldots].
I and $\psi(m,n) = n - m$. The pair $\mathcal{G} := (\psi, U)$ is called a gauge. Alternatively one can define controlled rough paths locally via a chart $\phi$ by requiring (see Definition 2.39)

$$\phi(y_t) - \phi(y_s) - d\phi \circ y_t^s x_{s,t} = O \left( |x_{s,t}|^2 \right)$$

and $d\phi \circ y_t^s - d\phi \circ y_s^t = O \left( |x_{s,t}| \right)$.

It is shown in Theorem 2.43 that these two notions of controlled rough paths agree.

Two natural examples of manifold valued controlled rough paths are as follows. 1) If $M^d$ is an embedded submanifold (see subsection 2.6) and the path $x_s \in \mathbb{R}^d$ happens to lie in $M$ (i.e. $x_s \in M$ for all $s$ in $[0, T]$), then $(x_s, P(x_s))$ is an $M$ – valued rough path controlled by $X$ where $P(m)$ is orthogonal projection onto $T_m M$ (see Example 2.54). 2) If $f : \mathbb{R}^k \to M^d \subseteq \mathbb{R}^k$ is smooth, then $(f(x_s), f'(x_s))$ is a rough path controlled by $X$ (see Example 2.55).

Now let $\mathcal{G} = (\psi, U)$ be a gauge, $V$ be a Banach space, and $y = (y, y^t)$ be an $M$ – valued controlled rough path as above. A pair of continuous functions $\alpha : [0, T] \to L(TM, V)$ and $\alpha^* : [0, T] \to L(\mathbb{R}^k \otimes TM, V)$ is a $U$ – controlled (rough) one-form along $y$ with values in a Banach space $V$ provided (see Definition 3.1 for details);

1. $\alpha_s : T_{y_s} M \to V$ for all $s$,
2. $\alpha^*_s : \mathbb{R}^k \otimes T_{y_s} M \to V$ for all $s$,
3. $\alpha_t \circ U(y_t, y_s) - \alpha_s - \alpha^*_s(x_{s,t} \otimes (\cdot)) = O \left( |x_{s,t}|^2 \right)$
4. $\alpha^*_t \circ (I \otimes U(y_t, y_s)) - \alpha^*_s = O \left( |x_{s,t}| \right)$.

To abbreviate notation we write $\alpha_s = (\alpha_s, \alpha^*_s)$. As an example, if $\alpha \in \Omega^1(M, V)$ is a smooth one form on $M$ and $U$ is a parallelism it is shown in Proposition 4.12 how to construct $\alpha^*_s U$ so that $\alpha^*_s U = (\alpha_s := \alpha|_{T_{y_s} M}, \alpha^*_s U)$ is a $U$ – controlled rough path along $y$.

Theorem 3.21 below constructs the integral, $\int (\alpha, dy^\otimes)$, of $\alpha$ along $y = (y, y^t)$. This integral is a standard flat $V$ – valued controlled rough path along $X$ which, as the notation suggests, a priori depends on a choice of gauge, $\mathcal{G} = (\psi, U)$. However, it is shown in Corollary 3.31 that the integral actually only depends on the parallelism, $U$. In Theorem 3.24 (also see Proposition 3.6), we show that the integral, $\int (\alpha, dy^\otimes)$, satisfies a basic but useful associativity property.

In Theorem 3.32 it is shown that there are “natural” transformations relating all of the above structures under change of parallelism, $U \to \tilde{U}$ in such a way that the integral, $\int (\alpha, dy^\otimes)$, is preserved. Consequently, as shown in Theorem 4.3, if $\alpha \in \Omega^1(M, V)$ is a smooth one form on $M$ with values in $V$ and $\alpha^*_s U = (\alpha_s := \alpha|_{T_{y_s} M}, \alpha^*_s U)$ is the associated $U$ – controlled (rough) one-form along $y$, then the resulting integral $\int (\alpha^*_s U, dy^\otimes)$ is in fact independent of both the parallelism, $U$, and the logarithm, $\psi$, used in the construction. Therefore we may simply denote the resulting integral by $\int \alpha(dy)$. A gauge independent formula for this integral using charts is given in Corollary 4.7. Lastly it is shown in Theorem 4.15 that if $\alpha \in \Omega^1(M, V)$ is a smooth one form on $M$ and $f : M \to \tilde{M}$ is a smooth map between two manifolds, then

$$\int f^* \alpha(dy) = \int \alpha(d(f_* y)),$$

where $f_* y = (f \circ y, f_* \circ y^t)$ is the “push-forward” of $y$ by $f$ (see Definition 4.11) and $f^* \alpha \in \Omega^1(\tilde{M}, V)$ is the pull-back of $\alpha$. 3
In a sequel to this paper, we will use the machinery developed here to build a theory of rough differential equations. We will also develop notions of parallel translation along a controlled rough path along with rough version of Cartan’s rolling and unrolling maps in order to characterize all controlled rough paths on \( M \).

## 2 Definitions of Controlled Rough Paths with Examples

### 2.1 Review of Euclidean Space Rough Paths

The presentation here will be brief. For a more thorough development, the reader can refer to many sources, for example [12] or [13].

Throughout this paper, we denote \( W = \mathbb{R}^k \). Let \( 1 \leq p < 3 \) and let \( \Delta_{[S,T]} = \{(s,t) : S \leq s \leq t \leq T \} \).

**Definition 2.1** A control \( \omega \) is a continuous function \( \omega : \Delta_{[0,T]} \to \mathbb{R}_+ \) which is superadditive\(^1\) and such that \( \omega(s,s) = 0 \) for all \( s \in [0,T] \).

**Definition 2.2** Let \( X = (x,X) \) where

\[
x : [0,T] \to W \quad \text{and} \quad X : \Delta_{[0,T]} \to W \otimes W
\]

and are continuous. Then \( X \) is a \( p \)-rough path with control \( \omega \) if

1. The Chen identity holds:
\[
X_{s,u} = X_{s,t} + X_{t,u} + x_{s,t} \otimes x_{t,u}
\]

   for all \( 0 \leq s \leq t \leq u \leq T \) where \( x_{s,t} := x_t - x_s \).

2. For all \( 0 \leq s \leq t \leq T \),
\[
|x_{s,t}| \leq \omega(s,t)^{1/p} \quad \text{and} \quad |X_{s,t}| \leq \omega(s,t)^{2/p}.
\]

Further, we say that \( X \) is weak-geometric if the symmetric part of \( X_{s,t} \) satisfies the relation

\[
sym(X_{s,t}) = \frac{1}{2} x_{s,t} \otimes x_{s,t}.
\]

**Notation 2.3** Let \( F_{s,t} \) and \( G_{s,t} \) be a pair of functions. When it is not important to keep careful track of constants we will often write \( F_{s,t} \approx G_{s,t} \) (for any \( i \in \mathbb{N} \)) to indicate that there exists \( C < \infty \) and \( \delta > 0 \) such that

\[
|F_{s,t} - G_{s,t}| \leq C \omega(s,t)^i \quad \text{for all} \quad 0 \leq s \leq t \leq T \quad \text{with} \quad |t-s| \leq \delta.
\]

In this paper, \( V, \tilde{V}, \) and \( \hat{V} \) will denote Banach spaces.

**Example 2.4** If \( x(t) \in C^\infty([0,T], V) \) is a smooth curve in \( V \) and
\[
X_{s,t} = \int_{s \leq u \leq v \leq t} dx_u \otimes dx_v = \int_s^t x_{s,v} \otimes dx_v,
\]

then \( X = (x,X) \) is a weak-geometric rough path controlled by \( \omega(s,t) = |t-s| \). In this example we could take even take \( p = 1 \).

\(^1\)To say \( \omega \) is superadditive means \( \omega(s,t) + \omega(t,u) \leq \omega(s,u) \) for all \( 0 \leq s \leq t \leq u \leq T \).
We denote \( L(V, \tilde{V}) \) as the bounded linear transformations from \( V \) to \( \tilde{V} \).

**Definition 2.5** Let \( X \) be a \( p \)-rough path on \( W \otimes W^{\otimes 2} \) with control \( \omega \). The continuous pair \( y := (y, y^i) \in C([0, T], V) \times C([0, T], L(W, V)) \) is a \( \tilde{V} \) valued rough path controlled by \( X \) (denoted \( y \in CRP_X(\tilde{V}) \)) if there exists a \( C \) such that

1. \( |y_t - y_s - y^i_s x_{s,t}| \leq C \omega(s, t)^{2/p} \)
2. \( |y^i_t - y^i_s| \leq C \omega(s, t)^{1/p} \)

for all \( s \leq t \) in \([0, T]\).

The approximations in Definition 2.5 are statements which only need to hold locally. For controlled rough paths, a sewing lemma is not difficult to prove:

**Lemma 2.6** Let \( y := (y, y^i) \in C([0, T], V) \times C([0, T], L(W, V)) \) and let \( 0 = t_0 < t_1 < \ldots < t_l = T \) be a partition of \([0, T]\) such that \( y|_{[t_i, t_{i+1}]} \) is a rough path controlled by \( X|_{[t_i, t_{i+1}]} := (x|_{[t_i, t_{i+1}]}, X|_{[t_i, t_{i+1}]}(\Delta_{[t_i, t_{i+1}]})) \) for all \( 0 \leq i \leq l - 1 \). Then \( y \) is a rough path controlled by \( X \).

**Proof.** Let \( C \) with \( 0 \leq i \leq l - 1 \) be such that

\[
|y_t - y_s - y^i_s x_{s,t}| \leq C_i \omega(s, t)^{2/p} \quad \text{and} \quad |y^i_t - y^i_s| \leq C_i \omega(s, t)^{1/p}
\]

whenever \((s, t) \in \Delta_{[t_i, t_{i+1}]}\). Let \( \tilde{C} := \sum_{i=0}^{l-1} C_i \). Then by telescoping and the fact that \( \omega \) is superadditive, it is clear that

\[
|y^i_t - y^i_s| \leq \tilde{C} \omega(s, t)^{1/p}
\]

for all \((s, t) \in \Delta_{[0, T]}\).

Now let \( C = (2l - 1) \tilde{C} \). If \((s, t) \in \Delta_{[0, T]}\) then there exists \( j \) and \( j^* \) such that \( s \in [t_j, t_{j+1}] \) and \( t \in [t_j^*, t_{j^*+1}] \) with \( j \leq j^* \). If \( j = j^* \) then

\[
|y_t - y_s - y^i_s x_{s,t}| \leq C \omega(s, t)^{2/p}
\]

trivially. Otherwise, we have

\[
y_t - y_s - y^i_s x_{s,t} = (y_t - y_{t^*}) + (y_{t^*} - y_s) + \sum_{i=j+1}^{j^* - 1} y^i_{t_{i+1}} - y_i \\
- y^i_s x_{t_{i+1}, t_i} = \sum_{i=j+1}^{j^* - 1} y^i_{t_{i+1}} x_{t_i, t_{i+1}}
\]

\[
= (y_t - y_{t^*} - y^i_{t^*} x_{t^*, t}) + (y_{t_{j+1}} - y_s - y^i_{t_{j+1}} x_{t_{j+1}, t_{j+1}}) + \sum_{i=j+1}^{j^* - 1} y^i_{t^*} - y^i_s x_{t^*, t}
\]

\[
+ \sum_{i=j+1}^{j^* - 1} (y_{t_{i+1}} - y_i - y^i_s x_{t_{i+1}, t_{i+1}}) + \sum_{i=j+1}^{j^* - 1} [y^i_{t^*} - y^i_s] x_{t^*, t_{i+1}}
\]

for all \((s, t) \in \Delta_{[0, T]}\).
Taking absolute values and using the fact that $\omega$ is superadditive, we have that the absolute value of each term on the right (including those within the summations) is bounded by $\tilde{C} \omega(s, t)^{2/p}$. Thus
\[
|y_t - y_s - y^1_s x_{s,t}| \leq (2l - 1) \tilde{C} \omega(s, t)^{2/p} = C \omega(s, t)^{2/p}
\]
Additionally it is clear that
\[
|y^1_t - y^1_s| \leq C \omega(s, t)^{1/p}.
\]

In [14], the following generalization of a result of [22] is proved.

**Theorem 2.7** Let $X$ be a $p$-rough path on $W \oplus W^{\otimes 2}$ with control $\omega$ and let $(y, y^1)$ be an $L(W, V)$-valued rough path controlled by $X$. Then there exists a $z \in C([0, T], V)$ with $z_0 = 0$ and a $C > 0$ such that
\[
|z_t - z_s - y_s x_{s,t} - y^1_s X_{s,t}| \leq C \omega(s, t)^{3/p}
\]
for all $s \leq t$ in $[0, T]$.

We will more commonly refer to the path $z_t$ as $\int^t_0 \langle y_\tau, dX_\tau \rangle$ and its increment, $z_{s,t} := z_t - z_s$, as $\int^t_s \langle y_\tau, dX_\tau \rangle$. Theorem 2.9 below is a generalization of Theorem 2.7, but before we state it, we will make a remark about certain identifications of spaces.

**Remark 2.8** If $V, \hat{V}$, and $\hat{V}$ are vector spaces, we can make the identification
\[
L \left( V, L \left( \hat{V}, \hat{V} \right) \right) \cong L \left( V \otimes \hat{V}, \hat{V} \right)
\]
via the map $\Xi : L \left( V, L \left( \hat{V}, \hat{V} \right) \right) \rightarrow L \left( V \otimes \hat{V}, \hat{V} \right)$ given by
\[
\Xi(\alpha)[v \otimes \hat{v}] = \alpha(v) \langle \hat{v} \rangle.
\]
if $\alpha \in L \left( V, L \left( \hat{V}, \hat{V} \right) \right)$.

**Theorem 2.9** Let $X$ be a $p$-rough path on $W \oplus W^{\otimes 2}$ with control $\omega$, let $(y, y^1)$ be an $V$-valued rough path controlled by $X$ and let $\alpha = (\alpha, \alpha^1)$ be an $L \left( V, \hat{V} \right)$-valued rough path controlled by $X$ where $\alpha^1 \in L \left( W, L \left( \hat{V}, \hat{V} \right) \right) \cong L \left( W \otimes \hat{V}, \hat{V} \right)$. Then there exists a $z \in C([0, T], V)$ with $z_0 = 0$ and a $C > 0$ such that
\[
|z_t - z_s - \alpha_s (y_t - y_s) - \alpha^1_s (I \otimes y^1_s) X_{s,t}| \leq C \omega(s, t)^{3/p}
\]
for all $s \leq t$ in $[0, T]$. Moreover if we let $z^1_s := \alpha_s \circ y^1_s$, then $z_s := (z_s, z^1_s)$ is a $\hat{V}$-valued controlled rough path.

The path $z_t$ in this case will be denoted $\int^t_0 \langle \alpha_\tau, dY_\tau \rangle$ and we will typically summarize Inequality (2.5) by writing
\[
\int^t_s \langle \alpha_\tau, dY_\tau \rangle \approx \langle \alpha_s, y^X_{s,t} \rangle := \alpha_s y_{s,t} + \alpha^1_s (I \otimes y^1_s) X_{s,t}
\]
wherein we let $y^X_{s,t}$ be the increment process defined by,

$$y^X_{s,t} := (y_{s,t}, (I \otimes y^\dagger_s) X_{s,t}). \quad (2.7)$$

Notice that Theorem 2.7 does indeed follow from Theorem 2.9 upon replacing $(\alpha, \alpha^\dagger)$ by $(y, y^\dagger)$ and $(y, y^\dagger)$ by $(x, I_W)$ in Inequality (2.5).

**Remark 2.10 (Motivations)** In order to develop some intuition for the expression appearing on the right side of Eq. (2.6), suppose for the moment that all functions $X$, $(y, y^\dagger)$, and $(\alpha, \alpha^\dagger)$ are smooth so that $X$ is given by Eq. (2.3). In this case we want $z_{s,t}$ to be the usual integral

$$\int_s^t \alpha_{s,\tau} d\tau = \int_s^t \alpha_{s,\tau} y^\dagger_{s,\tau} d\tau = \alpha_s y_{s,t} + \alpha^\dagger_s (I \otimes y^\dagger_s) X_{s,t} + O \left( (t-s)^3 \right).$$

We have the identity:

$$\int_s^t \alpha_{s,\tau} d\tau = \int_s^t [\alpha_s + \alpha_{s,\tau}] y^\dagger_{s,\tau} d\tau = \alpha_s y_{s,t} + \int_s^t \alpha_{s,\tau} y^\dagger_{s,\tau} d\tau. \quad (2.8)$$

The last term on the right hand side is approximated up to an error of size $O \left( (t-s)^3 \right)$ as follows,

$$\int_s^t \alpha_{s,\tau} y^\dagger_{s,\tau} d\tau = \int_s^t \alpha_{s,\tau} y^\dagger_{s,\tau} x^\dagger_{s,\tau} d\tau \quad (2.9)$$

Combining Eq. (2.8) and Eq. (2.9) gives the approximate equality,

$$\int_s^t \alpha_{s,\tau} d\tau = \alpha_s y_{s,t} + \alpha^\dagger_s (I \otimes y^\dagger_s) X_{s,t} + O \left( (t-s)^3 \right).$$

2.2 Manifold Valued Controlled Rough Paths

Let $M = M^d$ be a $d$-dimensional manifold, $TM$ be its tangent space, and $\pi : TM \to M$ be the natural projection map. Throughout, let $X = (x, X)$ be a weak-geometric $p$-rough path on $[0, T]$ with with values in $W \oplus W^\otimes 2$ and control $\omega$. 


Notation 2.11 When \( x \) appears in this paper without a subscript, it will be a point in Euclidean space; otherwise it refers to this path. Occasionally this will also be true for \( y \); it will be clear from the context when this is the case.

Notation 2.12 When \( M = \mathbb{R}^d \) we will identify \( T \mathbb{R}^d \) with \( \mathbb{R}^d \times \mathbb{R}^d \) via
\[
\mathbb{R}^d \times \mathbb{R}^d \ni (m, v) \mapsto v_m := \frac{d}{dt}|_0 (m + tv) \in T_m \mathbb{R}^d
\]
and, by abuse of notation, we let \( |v_m| = |v| \) when \( |\cdot| \) is the standard Euclidean norm.

Notation 2.13 Whenever \( \phi \) is a map, let \( D(\phi) \) and \( R(\phi) \) denote the domain and range of \( \phi \) respectively. If \( \phi \in C^\infty(M, \mathbb{R}^d) \) has open domain, let \( d\phi : TD(\phi) \to \mathbb{R}^d \) be defined by
\[
d\phi(v_m) := \frac{d}{dt}|_0 \phi(\sigma(t)) \in \mathbb{R}^d
\]
where \( \sigma \) is such that \( \sigma(0) = m \in D(\phi) \) and \( \dot{\sigma}(0) = v_m \in T_m M \). Denote \( d\phi_m := d\phi|_{T_m M} \).
If \( f \in C^\infty(M, \tilde{M}) \) where \( \tilde{M} \) is another manifold, we let \( f_* \) be the push-forward of \( f \) so that \( f_* : TD(f) \to TM \) is defined by
\[
f_*(v_m) := \frac{d}{dt}|_0 f(\sigma(t)) \in T_{f(m)} \tilde{M}
\]
where again \( \dot{\sigma}(0) = v_m \). Analogously we let \( f_{*m} = f_*|_{T_m M} \). Note that \( \phi_*(v_m) = (\phi(m), d\phi(v_m)) = [d\phi(v_m)]_{\phi(m)} \).

2.3 Gauges

Definition 2.14 Let \( \mathcal{U} \) be an open set on \( M \). An open set \( \mathcal{D}_U \subseteq M \times M \) is a \( \mathcal{U} \)– diagonal domain if it contains the diagonal of \( \mathcal{U} \), that is \( \Delta U := \bigcup_{m \in U} (m, m) \subseteq \mathcal{D} \). A local diagonal domain is a \( \mathcal{V} \)– diagonal domain for some nonempty open \( \mathcal{V} \subseteq M \).

If \( \mathcal{U} = M \) we write \( \mathcal{D} := \mathcal{D}^M \) and refer to \( \mathcal{D} \) simply as a diagonal domain.

Throughout the paper, \( \mathcal{D} \) will always denote a diagonal domain.

Definition 2.15 A smooth function \( \psi : \mathcal{D} \to TM \) is called a logarithm if:

1. \( \psi(m, n) \in T_m M \)
2. \( \psi(m, m) = 0_m \)
3. \( \psi(m, \cdot)_* |_{T_m M} = I_m \)

We also write \( \psi_m \) for \( \psi(m, \cdot) \).

If the above holds for \( \psi \) defined on a local diagonal domain, we may refer to \( \psi \) as a local logarithm.
If $E$ is a any vector bundle, we will denote the smooth sections of $E$ by $\Gamma(E)$. We define $L(TM, TM)$ as the vector bundle $E$ over the manifold $M \times M$ such that $E_{(n, m)} = L(T_mM, T_nM)$ and

$$E = \bigcup \left\{ E_{(n, m)} : n, m \in M \right\}$$

**Definition 2.16** A smooth section $U \in \Gamma(L(TM, TM))$ with domain $D$ (i.e. $U(n, m) \in L(T_mM, T_nM)$ for all $(n, m) \in D$) is called a parallelism if $U(m, m) = I_m$. If $U$ is only defined on a local diagonal domain, we refer to $U$ as a local parallelism.

Let $g$ be any smooth Riemannian metric on $M$. We write $|v_m|_g := \sqrt{g_m(v_m, v_m)}$.

**Definition 2.17** We call the pair $G = (\psi, U)$ (where $\psi$ and $U$ have common domain $D$) a gauge on the manifold $M$. If $D$ is replaced by a local diagonal domain, we call $G$ a local gauge.

**Example 2.18** If $M = \mathbb{R}^d$, the maps $\psi(x, y) = [y - x]_x$ and $U_{x,y}v_y = v_x$ form the standard gauge on $\mathbb{R}^d$.

**Example 2.19** One natural example of a gauge comes from any covariant derivative $\nabla$ on $TM$. The construction is as follows. Choose an arbitrary Riemannian metric $g$ on $M$. If $m, n \in M$ are “close enough”, there is a unique vector $v_m$ with minimum length such that $n = \exp_m^n(v_m)$. We denote this vector by $\psi^\nabla(m, n) := (\exp_m^n)^{-1}(n)$ or by $\exp_m^{-1}(n)$ if $\nabla$ is clear from the context. We further let

$$U^\nabla(n, m) := U_{n,m}^\nabla := \left\{ t \rightarrow \exp_m(t \exp_m^{-1}(n)) \right\},$$

where, for any smooth curve $\sigma : [0, 1] \rightarrow M$, we let $\int_s^1(\sigma) = \int_s^1 \sigma : T_{\sigma(0)}M \rightarrow T_{\sigma(s)}M$ denote parallel translation along $\sigma$ up to time $s \in [0, 1]$. It is shown in Corollary 2.32 that there is a diagonal domain $D \subset M \times M$ such that $(\psi^\nabla, U^\nabla)$ so defined is a gauge on $D$.

**Remark 2.20** We can also get a covariant derivative from a parallelism. If $U$ is a parallelism, then we can define covariant derivative $\nabla^U$ on $TM$ by

$$\nabla^U_{v_m} Y := \frac{d}{dt} |_{t=0} U(m, \sigma_t) Y(\sigma_t),$$

where $\sigma(0) = v_m$ and $Y$ is a vector field on $M$.

**Remark 2.21** Although the definition of a gauge includes stipulating a $U$, if we have just $\psi$, we can define $U^\psi(n, m) := \psi(n, \cdot)_{, m}$ and set $G^\psi := (\psi, U^\psi)$.

**Remark 2.22** We may make a local gauge out of a chart $\phi$. Indeed, we pull back the flat gauge in Example 2.18 to $M$ to define

$$\psi^\phi(m, n) := (d\phi_m)^{-1}[\phi(n) - \phi(m)]$$

$$U^\phi(n, m) := (d\phi_n)^{-1}d\phi_m.$$  

This is a gauge which is also consistent with Remark 2.21 and $D(\psi^\phi) = D(U^\phi) = D(\phi) \times D(\phi)$.

Before moving on to controlled rough paths on manifolds, let us record the structure of the general gauge on $\mathbb{R}^d$. 

**Notation 2.23** If \((\psi, U)\) is a local gauge on \(\mathbb{R}^d\), then we write \((\bar{\psi}, \bar{U})\) to mean the functions determined by the relations
\[
\psi(x, y) = [\bar{\psi}(x, y)]_x
\]
and
\[
U(x, y)(v_y) = [\bar{U}(x, y)v]_x
\]
so that \(\bar{\psi}(x, y) \in \mathbb{R}^d\) and \(\bar{U}(x, y) \in \text{End}(\mathbb{R}^d)\).

**Theorem 2.24** If \(\mathcal{G} = (\psi, U)\) is a local gauge on \(\mathbb{R}^d\), for every open convex \(V \subseteq \mathbb{R}^d\) such that \(V \times V \subseteq D(\mathcal{G})\), there exists smoothly varying functions \(A(x, y) \in L\left(\mathbb{R}^d \otimes_2 \mathbb{R}^d\right)\) and \(B(x, y) \in L\left(\mathbb{R}^d, \text{End}(\mathbb{R}^d)\right)\) defined on \(V \times V\) such that
\[
\bar{U}(x, y) = I + B(x, y)(y - x)
\]
\[
\bar{\psi}(x, y) = y - x + A(x, y)(y - x)^{\otimes 2}.
\]
The converse holds as well. Moreover, \(B(x, x) = D_2\bar{U}(x, x)\) and \(A(x, x) = \frac{1}{2}(D^2_2\bar{\psi})(x, x)\).

**Proof.** Let \(x, y\) be points in \(V\). Taylor’s theorem with integral remainder applied to the second variable with \(x\) fixed gives,
\[
\bar{U}(x, y) = I + \int_0^1 (D_2\bar{U})(x, x + t(y - x))(y - x)\,dt
\]
and
\[
\bar{\psi}(x, y) = 0 + (D_2\bar{\psi})(x, x)(y - x) + \int_0^1 (D^2_2\bar{\psi})(x, x + t(y - x))(y - x)^{\otimes 2}(1 - t)\,dt
\]
from which the results follows with
\[
B(x, y) = \int_0^1 (D_2\bar{U})(x, x + t(y - x))\,dt \quad \text{and} \quad A(x, x) = \frac{1}{2}(D^2_2\bar{\psi})(x, x)(1 - t)\,dt.
\]
The remaining statements in the proof are now easy to verify. ■

Let \(B_r(x) \subseteq \mathbb{R}^d\) be the open ball of radius \(r\) centered at \(x\).

**Remark 2.25** If \(\bar{\psi}\) and \(\tilde{\psi}\) are local logarithms on \(\mathbb{R}^d\), it is easy to check using Theorem 2.24 that for all \(\tilde{x} \in \mathbb{R}^d\), there exists an \(r > 0\) and \(C > 0\) such that \(|\psi(x, y)| \leq C|\tilde{\psi}(x, y)|\) for all \(x, y \in B_r(\tilde{x})\).

We now wish to transfer these local results to the manifold setting. In order to do this we need to develop some notation for stating that two objects on a manifold are “close” up to some order. Let \(d_g\) be the metric associated to \(g\).
\textbf{Definition 2.26} Let $F,G$ be smooth $TM$ \textit{respectively $L(TM,TM)$} valued functions with $W$ - diagonal domains. The expression

\begin{equation}
F(m,n) =_k G(m,n) \quad \text{on} \quad W \tag{2.11}
\end{equation}

indicates that for every point in $w \in W$, there exists an open $O_w \subseteq M$ containing $w$ such that $O_w \times O_w \subseteq D(F) \cap D(G)$ and a $C > 0$ such that

\begin{equation}
|F(m,n) - G(m,n)|_{g,\|g,\text{op}\|} \leq C(d_g(m,n))^k \tag{2.12}
\end{equation}

for all $m,n \in O_w$. Here $\|\cdot\|_{g,\text{op}}$ is the operator “norm” induced by $\|\cdot\|_g$ on $L(TM,V)$, i.e. if $f_m \in L(T_mM,V)$, then

\[ |f_m|_{g,\text{op}} := \sup \{|f_m(v_m)| : |v_m|_g = 1 \}. \]

Sometimes we will omit the reference to $W$ in which case it we mean the condition \textcolor{red}{2.12} holds where it makes sense to hold.

Note that in \textcolor{red}{2.11}, the reference to $g$ is not explicit. In fact, the definition does not depend on the choice of $g$ as all Riemannian metrics are locally equivalent. [See Corollary 5.4 in the Appendix for precise statement and proof of this standard fact.]

We may also use the $=_k$ notation to make statements in regards to other measures of distance:

\textbf{Corollary 2.27} Let $W$ be an open subset of $M$ and $g$ and $\tilde{g}$ be any two Riemannian metrics on $M$. If $F(m,n) =_k G(m,n)$ on $W$ (so that $F$ and $G$ have $W$-diagonal domains), then for every local logarithm $\psi$ and $w \in W$ such that $(w,w) \in D(\psi)$, there exists an open $O_w \subseteq W$ containing $w$ and $C > 0$ such that

\[ |F(m,n) - G(m,n)|_{\tilde{g},\|\tilde{g},\text{op}\|} \leq C|\psi(m,n)|_{\tilde{g}}^k \quad \forall \quad m,n \in O_w. \]

In particular, using the local logarithm $\psi(m,n) = (d\phi_m)^{-1}[\phi(n) - \phi(m)]$, we have that if $w \in D(\phi) \cap W$, then there exists an $O_w \subseteq D(\phi) \cap W$ and a $C > 0$ such that

\[ |F(m,n) - G(m,n)|_{\tilde{g},\|\tilde{g},\text{op}\|} \leq C|\phi(n) - \phi(m)|_{\tilde{g}}^k \quad \forall \quad m,n \in O_w. \]

\textbf{Proof.} The proof of the Corollary will use Remark 2.25 and Corollary 5.4 in the Appendix. First we simplify matters by assuming that we are working in Euclidean space which may be accomplished by pushing the metric and functions forward using charts. Assuming this, we now derive a local inequality that holds for any two logarithms $\psi$ and $\tilde{\psi}$ when $(w,w) \in D(\psi) \cap D(\tilde{\psi})$. Namely, there exist an open neighborhood, $O_w$, of $w$ such that

\[ |\tilde{\psi}(m,n)|_g \leq C_1 |\psi(m,n)| \leq C_2C_1 |\psi(m,n)|_{\tilde{g}} \leq C_3C_2C_1 |\psi(m,n)|_{\tilde{g}} \quad \forall m,n \in O_w \times O_w, \]

where the first and third inequality follow from Corollary 5.4 with one metric being the standard Euclidean metric and the other metric being $g$ or $\tilde{g}$ respectively, and the second inequality is true by Remark 2.25. Thus, there exists a $\tilde{C}$ such that

\[ |\tilde{\psi}(m,n)|_{\tilde{g}} \leq \tilde{C}|\psi(m,n)|_{\tilde{g}} \]


Now let $\nabla^g$ be the Levi-Civita covariant derivative associated to $g$. By setting $\tilde{\psi}(m,n) = (\exp^g_m)^{-1}(n)$ and shrinking $O_w$ if necessary to ensure that $(\exp^g_m)^{-1}(\cdot)$ is defined and injective on $O_w \times O_w$, we have that

$$
(\exp^g_m)^{-1}(n) \leq \tilde{C} |\psi(m,n)|^g.
$$

In this setting, $d_g(m,n) = |(\exp^g_m)^{-1}(n)|^g$, and since $F(m,n) = k G(m,n)$ on $W$ (by shrinking $O_w$ if necessary), we have

$$
|F(m,n) - G(m,n)|_{g,[g,op]} \leq \hat{C} (d_g(m,n))^k \quad \forall \ m, n \in O_w
$$

for some $\hat{C}$. Thus, we have

$$
|F(m,n) - G(m,n)|_{g,[g,op]} \leq \hat{C} \left( \hat{C} \right)^k |\psi(m,n)|^g.
$$

which is the statement of the Corollary with $C := \hat{C} \left( \hat{C} \right)^k$.

In the sequel, Corollary 2.27 will typically be used without further reference in order reduce the proof of showing $F(m,n) = k G(m,n)$ in the manifold setting to a local statement about functions on convex neighborhoods in $\mathbb{R}^d$ equipped with the standard Euclidean flat metric structures. The first example of this strategy will already occur in the proof of Corollary 2.28 below. For a general parallelism it is not true that $U(n,m)^{-1} = U(m,n)$, yet $U(m,n)$ is a very good approximation to $U(n,m)^{-1}$.

**Corollary 2.28** If $U$ is a parallelism on a manifold, $M$, then

$$
U(n,m)^{-1} =_2 U(m,n).
$$

**Proof.** This is a local statement so we may use Corollary 2.27 to reduce to the case that $M$ is a convex open subset of $\mathbb{R}^d$. We then may use Theorem 2.24 to learn

$$
\hat{U}(n,m)^{-1} = (I + [B(n,m)(m-n)])^{-1} = I + [B(n,m)(n-m)] + O\left(|n-m|^2\right)
$$

while

$$
\hat{U}(m,n) = (I + [B(m,n)(n-m)]).
$$

Subtracting these two equations shows,

$$
\hat{U}(n,m)^{-1} - \hat{U}(m,n) = [B(n,m) - B(m,n)](n-m) + O\left(|n-m|^2\right)
$$

wherein we have used $B(n,m) - B(m,n)$ vanishes for $m = n$ and therefore is of order $|m-n|$.
2.3.1 A Covariant Derivative Gives Rise to a Gauge

Let $\nabla$ be a covariant derivative on $TM$, and $g$ be any fixed Riemannian metric on $M$. Let $G : TM \to M \times M$ be the function on $TM$ defined by

$$ G (v_m) := (m, \exp_m^G (v_m)) \text{ for all } v_m \in D (G), $$

(2.13)

where $D (G)$ is the domain of $G$ defined by

$$ D (G) := \{ v_m \in TM : t \to \exp_m^G (tv_m) \text{ exists for } 0 \leq t \leq 1 \}. $$

We will now develop a subset of $D (G)$ for which $G$ is injective. For each $m \in M$, let $\Lambda_m$ denote the set of $r > 0$ so that $B_r (0_m) \subseteq D (G)$, $\exp_m^G (B_r (0_m))$ is an open neighborhood of $m$ in $M$, and $\exp_m^G : B_r (0_m) \to \exp_m^G (B_r (0_m))$ is a diffeomorphism (here $B_r (0_m)$ is the open ball in $T_m M$ centered at $0_m$ with radius $r$). The fact that $\Lambda_m$ is not empty is a consequence of the inverse function theorem and the fact that $\exp_m^G$ is injective. For each $m \in M$ and $r_m = \sup \Lambda_m$, $B_r (0_m)$ is possible and allowed. A little thought shows that $\exp_m^G (B_r (0_m))$ is open and $\exp_m^G : B_r (0_m) \to \exp_m^G (B_r (0_m))$ is a diffeomorphism, i.e. either $r_m = \infty$ or $r_m \in \Lambda_m$.

Let us now set $C^* := \cup_{m \in M} B_r (0_m) \subseteq TM$ and let $G^* : C^* \to M \times M$ be the map defined by

$$ G^* (v_m) := (m, \exp_m^G (v_m)) \text{ for all } v_m \in C^*. $$

It is easy to verify that $G^*$ is injective.

We will now build our domain $C$ for which $G|_C$ is diffeomorphic onto its range. First we need a simple local invertibility proposition.

**Proposition 2.29** Let $G$ be the function defined in Eq. (2.13). Then for each $m \in M$, there exists open subsets $V_m \subseteq TM$ and $W_m \subseteq M$ such that $0_m \in V_m$, $m \in W_m$, and $G|_{V_m} : V_m \to W_m \times W_m$ is a diffeomorphism.

**Proof.** As this a local result we may assume that $M = \mathbb{R}^d$ and identify $TM$ with $M \times M$. The function $G : TM \to M \times M$ then takes on the form $G (x, v) = \left( x, G (x, v) \right)$ where $G (x, 0) = x$ and $(D_2 G) (x, 0) = I_M$ for all $x \in M$. A simple computation then shows

$$ G' (x, 0) = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix} \text{ for all } x \in M. $$

The result now follows by an application of the inverse function theorem. 

**Notation 2.30** If $W$ is an open subset of $M$ and $\epsilon > 0$, let $U (W, \epsilon)$ be the open subset of $TM$ defined by

$$ U (W, \epsilon) := \{ v \in \pi^{-1} (W) \subseteq TM : |v|_g < \epsilon \}. $$

**Theorem 2.31** Let $C := \cup U (W, \epsilon)$ where the union is taken over all open subsets $W \subseteq M$ and $\epsilon > 0$ such that $U (W, \epsilon) \subseteq D (G)$ and $G|_{U (W, \epsilon)} : U (W, \epsilon) \to G (U (W, \epsilon))$ is a diffeomorphism. Then $C$ is an open subset of $TM$ such that $D := G (C)$ is open in $M \times M$, $G : C \to D$ is a diffeomorphism,

$$ \{ 0_m : m \in M \} \subseteq C \subseteq C^*, \text{ and } \Delta^M = \{ (m, m) : m \in M \} \subseteq D. $$
Proof. According to Proposition 2.29 for each \( m \in M \) there exists an open neighborhood \( W \) of \( m \in M \) and \( \epsilon > 0 \) so that \( U(W,\epsilon) \subseteq D(G) \) and \( G : U(W,\epsilon) \to G(U(W,\epsilon)) \) is a diffeomorphism. From this it follows that \( \{ 0_m : m \in W \} \subseteq C \) and \( U(W,\epsilon) \subseteq C^* \). As \( m \in M \) was arbitrary we may conclude \( \{ 0_m : m \in M \} \subseteq C \subseteq C^* \). It is now easily verified that \( G(C) = \cup G(U(W,\epsilon)) \) is open, \( G : C \to G(C) \) is a surjective local diffeomorphism and hence is a diffeomorphism as \( G|_C \) is injective (since \( G|_C \) is injective).

**Corollary 2.32** Continuing the notation used in Theorem 2.31, we have \( D \) is a diagonal domain and \( \psi := G|^{-1}_C : D \to C \subseteq TM \) is a logarithm. Moreover, if we define

\[
U(m,n) := /1 (\exp^v (\cdot \psi (m,n)))^{-1} : T_n M \to T_m M
\]

for all \( (m,n) \in D \), then \( U \) is a parallelism on \( M \).

**Proof.** The only thing that remains to be proven is that \( U(m,n) \) is smoothly varying. This is a consequence of the fact that solutions to ordinary differential equations depend smoothly on their starting points and parameter in the vector fields. To be more explicit in this case, for \( a \in \mathbb{R}^d \) let \( B^\alpha_x (\mu) = \dot{u} (0) \) where \( u(t) = /1 (\exp^v (\cdot \mu)) \mu \) for \( \mu \) in the frame bundle \( GL(M) \) over \( M \), so that \( B^\alpha_x \) are the \( \nabla^- \) horizontal vector fields. Now suppose that \( \psi \) in \( M \) is given and \( O(m) : \mathbb{R}^d \to T_m M \) is a local frame defined for \( m \) in an open neighborhood \( W \) of \( w \). For \( v \in \pi^{-1} \) \( (W) \cap C \) let \( \gamma(t) = \exp^v (tv) \) and \( u(t) := /1 (\gamma) O(\pi(v)) \). Then we have

\[
\dot{\gamma}(t) = /1 (\gamma) v = u(t) O(\pi(v))^{-1} v \quad \text{and} \quad \frac{\nabla^v u}{dt} = 0 \text{ with } u(0) = O(\pi(v)) .
\]

These equations are equivalent to solving

\[
\dot{u}(t) = B^\nabla_{O(\pi(v))^{-1} v} (u(t)) \quad \text{with} \quad u(0) = O(\pi(v)) \quad (2.14)
\]

in which case \( \gamma(t) = \pi(u(t)) \) where \( \pi \) is the projection onto \( M \). We now define \( F(v) := u(1) \) provided \( v \in \pi^{-1} \) \( (W) \cap C \). It then follows that \( F : \pi^{-1} \) \( (W) \cap C \to GL(M) \) is smooth as the solutions to Eq. \( (2.14) \) depend smoothly on its starting point and parameter. From this we learn for \( (m,n) \in G(\pi^{-1} \) \( (W) \cap C \) that

\[
U(m,n) = F(\psi(m,n)) O(m)^{-1}
\]

is a smooth function of \( (m,n) \).

**2.4 Controlled Rough Paths**

**Notation 2.33** Throughout the remainder of this paper, \( y := (y,y^1) \) denotes a pair of continuous functions, \( y \in C ([0,T],M) \) and \( y^1 \in C ([0,T],L(W,TM)) \), such that \( y^1 \) \( L(W,T_y,M) \) for all \( s \).

**Definition 2.34** Let \( \psi, U \) be a gauge. The pair \( (y_s,y^1_s) \) is \( \psi, U \) --rough path controlled by \( X \) if there exists a \( C > 0 \) and \( \delta > 0 \) such that

1. 

\[
|\psi(y_s,y_t) - y^1_s x_s|_y \leq C \omega(s,t)^{2/p}
\]

and
\[ |U(y_s, y_t) y^\dagger_{s,t} - y^\dagger_{s,t}| \leq C \omega(s, t)^{1/p} \]  \hspace{1cm} (2.16)

hold whenever \( 0 \leq s \leq t \leq T \) and \(|t - s| \leq \delta\). Occasionally we will refer to \( y_s \) as the path and \( y^\dagger_s \) as the derivative process (or Gubinelli derivative).

**Remark 2.35** In Definition 2.34 and in the definitions that follow, we use the convention that the \( \delta \) is small enough to ensure that all of the expressions are well defined (in particular here it is small enough to ensure \((y_s, y_t) \in D\)).

**Remark 2.36** Any path \( z_s \) in Euclidean space naturally gives rise to a two-parameter “increment process,” namely

\[ z_{s,t} = z_t - z_s. \]

If \( \varphi \) is any function such that \( \varphi(z, \tilde{z}) \approx \tilde{z} - z \), then it makes sense to define \( z^\varphi_{s,t} := \varphi(z_s, z_t) \). This is some motivation for the following notation.

**Notation 2.37** Denote \( y^\psi_{s,t} := \psi(y_s, y_t) \) and \( (y^\dagger) U_{s,t} := U(y_s, y_t) y^\dagger_{s,t} - y^\dagger_{s,t} \). These will be referred to as the local increment processes of \((y, y^\dagger)\).

**Remark 2.38** With Notation 2.37, (2.15) becomes

\[ |y^\psi_{s,t} - y^\dagger_s x_{s,t}| \leq C \omega(s, t)^{2/p} \]  \hspace{1cm} (2.17)

and (2.16) becomes

\[ |(y^\dagger) U_{s,t}| \leq C \omega(s, t)^{1/p} \]  \hspace{1cm} (2.18)

Definition 2.34 gives one possible notion of a controlled rough path on a manifold. We can also define such an object without having to provide a metric or gauge by using charts on the manifold.

**Definition 2.39** The pair \( y_s = (y_s, y^\dagger_s) \) is a chart-rough path controlled by \( X \) if for every chart \( \varphi \) on \( M \) and every \([a, b]\) such that \( y([a, b]) \subseteq D(\varphi) \) we have the existence of a \( C_{\varphi,a,b} \geq 0 \) such that

1. \[ |\varphi(y_t) - \varphi(y_s) - d\varphi \circ y^\dagger_{s,t} x_{s,t}| \leq C_{\varphi,a,b} \omega(s, t)^{2/p} \]  \hspace{1cm} (2.17)

and

2. \[ |d\varphi \circ y^\dagger_t - d\varphi \circ y^\dagger_s| \leq C_{\varphi,a,b} \omega(s, t)^{1/p} \]  \hspace{1cm} (2.18)

We will denote \( C_{\varphi,a,b} \) by \( C_{\varphi} \) when it is clear from the context.

**Notation 2.40** If \( (y_s, y^\dagger_s) \) is a chart rough path and \( \varphi \) is a chart as in Definition 2.39, we will abuse notation and write \( \varphi_* \) to mean \( \varphi_* y_s := \varphi_* (y_s, y^\dagger_s) := (\varphi \circ y_s, d\varphi \circ y^\dagger_s) \).

Note that as long as \( y \) remains away from the boundary of \( D(\varphi) \), this is controlled rough path on \( \mathbb{R}^d \). Another way to think of this is that a chart controlled rough path is one which pushes forward to a controlled rough path.
Before moving on, we'll make a few remarks.

**Remark 2.41** If \( y^\dagger \) is any function satisfying the conditions in either of Definitions 2.34 or 2.39, then \( s \rightarrow y^\dagger_s \) is automatically continuous. For example, if \((y_s, y^\dagger_s)\) satisfies the conditions of a \((\psi, U)\)–rough path in Definition 2.34, then the function \( t \rightarrow U(y_s, y_t) y^\dagger_t \) is a continuous at \( s \) and therefore \( t \rightarrow y^\dagger_t = U(y_s, y_t)^{-1} U(y_s, y_t) y^\dagger_t \) is continuous at \( s \).

**Remark 2.42** If \( M = \mathbb{R}^d \) and \( \phi = \text{id} \) then the chart Definition 2.39 reduces to the usual Definition 2.5 of controlled rough paths. However, in this case, note we identify all the tangent spaces with \( \mathbb{R}^d \) and we forget the base point in the derivative process.

### 2.5 Chart and Gauge CRP Definitions are Equivalent

**Theorem 2.43** Let \( y := (y, y^\dagger) \) be a pair of continuous functions as in Notation 2.33, \( M \) be a manifold, and \( G = (\psi, U) \) be any gauge on \( M \). Then \( y \) is a chart controlled rough path (Definition 2.39) if and only if it is a \((\psi, U)\)-controlled rough path (Definition 2.34).

**Corollary 2.44** We have the equality of sets

\[
\{(\psi, U) - \text{rough paths}\} = \left\{ (\tilde{\psi}, \tilde{U}) - \text{rough paths} \right\}
\]

for any gauges \((\psi, U)\) and \((\tilde{\psi}, \tilde{U})\) on \( M \).

**Notation 2.45** Let \( \text{CRP}_X(M) \) be the collection of controlled rough paths in \( M \), i.e. pairs of functions \( y = (y, y^\dagger) \) as in Notation 2.33 which satisfy either (and hence both) of Definitions 2.34 or 2.39.

We will prove Theorem 2.43 after assembling a number of preliminary results that will be needed in the proof and in the rest of the paper.

#### 2.5.1 Results used in proof of Theorem 2.43

Our first result is a local version of Theorem 2.43.

**Theorem 2.46** Let \( G = (\psi, U) \) be a gauge on \( \mathbb{R}^d \), \( z = (z, z^\dagger) \in C([a, b], \mathbb{R}^d) \times C([a, b], L(W, \mathbb{R}^d)) \), and \( W \) be an open convex set such that \( z([a, b]) \subseteq W \) and \( W \times W \subseteq D(G) \). Then \( z \in \text{CRP}_X(\mathbb{R}^d) \) iff \( z \) is a \((\psi, U)\)-rough path controlled by \( X \) with the choice \( \delta := b - a \).

**Proof.** Suppose \( z \in \text{CRP}_X(\mathbb{R}^d) \). By Theorem 2.24

\[
\tilde{\psi}(x, y) = y - x + A(x, y)(y - x)^\otimes 2 \quad \forall \ x, y \in W.
\]

Clearly \( A \) is bounded if it is restricted to \( x, y \) to the convex hull of \( z([a, b]) \) (which is compact and contained in \( W \)). Thus, for all such points, we have there exists a \( C_1 \) such that

\[
|\tilde{\psi}(x, y) - (y - x)| \leq C_1 |y - x|^2.
\]
Taking $y = z_t$ and $x = z_s$ in this inequality shows
\[ |\bar{\psi}(z_s, z_t) - z_{s,t}| \leq C_1 |z_t - z_s|^2. \]
(2.21)

Since $z \in CRP_X(\mathbb{R}^d)$, there exists a $C_2$ such that
\[ |z_{s,t} - z_{s}^s x_{s,t}| \leq C_2 \omega (s, t)^{2/p} \]
(2.22)
\[ |z_{s,t}^s| \leq C_2 \omega (s, t)^{1/p}. \]
(2.23)

By enlarging $C_2$ if necessary we may further conclude,
\[ |z_{s,t}| \leq C_2 \omega (s, t)^{1/p}. \]
(2.24)

Using Eqs. (2.22) and (2.24) in Eq. (2.21) gives the existence of a $C_3 < \infty$ such that
\[ |\bar{\psi}(z_s, z_t) - z_{s}^s x_{s,t}| \leq C_3 \omega (s, t)^{2/p}. \]

By Theorem 2.24 once more, we have
\[ \bar{U}(x, y) = I + B(x, y)(y - x). \]
As was the case for $A$, $B$ is bounded on the convex hull of $z([a, b])$ so that there exists a $C_4$ such that
\[ |\bar{U}(z_s, z_t) z_{t}^s - z_{s,t}^s| \leq C_4 |z_{s,t}| \]
\[ \leq (C_2 + C_4 C_2) \omega (s, t)^{1/p}. \]

Thus $z$ is a $(\psi, U)$-rough path controlled by $X$ with the choice $\delta := b - a$ where our $C := \max\{C_1, C_2 (1 + C_4)\}$.

Conversely if $z$ is a $(\psi, U)$-rough path controlled by $X$ with the choice $\delta := b - a$ as in Definition 2.34, From Eq. (2.20) and the triangle inequality we have
\[ |y - x| \leq C_1 |y - x|^2 + |\bar{\psi}(x, y)|. \]

Taking $x = z_s$ and $y = z_t$ in this inequality and using Definition 2.34 we may find $C_2 < \infty$ such that
\[ |z_{s,t}| \leq C_1 |z_{s,t}|^2 + |\psi(z_s, z_t)| \]
\[ \leq C_1 |z_{s,t}|^2 + C_2 \omega (s, t)^{1/p} \]
for all $s \leq t$ in $[a, b]$. By the uniform continuity of $z$ on $[a, b]$, there exists $\epsilon > 0$ such that $C_1 |z_{s,t}| \leq \frac{1}{2}$ when $|t - s| \leq \epsilon$ which combined with the previous inequality implies
\[ |z_{s,t}| \leq 2C_2 \omega (s, t)^{1/p} \text{ when } |t - s| \leq \epsilon. \]

For general $a \leq s \leq t \leq b$ we may write $z_{s,t}$ as a sum of at most $n \leq (b - a) / \epsilon$ increments whose norms are bounded by $2C_2 \omega (s, t)^{1/p}$ wherein we have repeatedly used the estimate above along with
the monotonicity of $\omega$ resulting from superactivity. Thus we conclude, with $C_3 := 2C_2 (b - a) / \epsilon < \infty$, that

$$|z_{s,t}| \leq C_3 \omega(s,t)^{1/p} \forall s, t \in [a, b].$$

This estimate along with the inequality in Eq. (2.20) gives,

$$|\tilde{\psi}(z_{s,t}) - z_{s,t}| \leq C_1 |z_{s,t}|^2 \leq C_1 C_3^2 \omega(s,t)^{2/p} \forall s, t \in [a, b].$$

The previous inequality along with the assumption that $z$ is a $(\psi, U)$-rough path shows there exists $C_4 < \infty$ such that

$$|z_{s,t} - z^k_s x_{s,t}| \leq |\tilde{\psi}(z_{s,t}) - z^k_s x_{s,t}| \leq C_4 \omega(s,t)^{2/p}.$$ 

By Theorem 2.24 one last time, we have

$$\tilde{U}(x,y) = I + B(x,y)(y-x)$$

so that there exists a $C_5$ such that

$$|z_{s,t}^k| \leq |U(z_{s,t}^k z^k_s - z_{s}^k) + C_5 |z_{s,t}|.$$ 

This inequality along with the assumption that $z$ is a $(\psi, U)$-rough path shows there exists $C_6 < \infty$ such that $|z_{s,t}^k| \leq C_6 \omega(s,t)^{1/p}$ for all $a \leq s \leq t \leq b$. Thus we have shown $z \in CRP^X(\mathbb{R}^d)$.

The rest of this section is now devoted to a number of “stitching” arguments which will be used to piece together a number of local versions of Theorem 2.43 over subintervals as described in Theorem 2.46 into the full global version as stated in Theorem 2.48. For the rest of this section let $X$ be a topological space and $0 \leq S < T < \infty$.

**Lemma 2.47** If $y : [S, T] \to X$ is continuous and $y([S, T]) \subseteq \bigcup_{\alpha \in A} O_\alpha$ where $\{O_\alpha\}_{\alpha \in A}$ is a collection of open subsets of $X$, then there exists a partition of $[S, T]$, $S = t_0 < t_1 < \ldots < t_l = T$, and $\alpha_i \in A$ such that for all $i$ less than $l$, we have

$$y([t_i, t_{i+1}]) \subseteq O_{\alpha_i}.$$ 

**Proof.** Define $T^* := \sup \{t : S \leq t \leq T, \text{ the conclusion of the theorem holds for } [S, t].\}$ Note that trivially $T^* > S$. For sake of contradiction, suppose $T^* < T$. Then there exists an $\epsilon > 0$ such that $T^* + \epsilon < T$, $T^* - \epsilon > S$ and $y(T^* - \epsilon, T^* + \epsilon) \subseteq O_{\alpha^*}$ for some $\alpha^*$. But the condition of the theorem holds for $T^* - \epsilon$ for some partition $P$. By appending $P$ with $T^* + \lambda \epsilon$ with $\lambda \in (-1, 1]$ we have that $T^* \geq T^* + \epsilon$ which is absurd. Thus, we must have that $T^* = T$.

**Definition 2.48** The set $\{a_i, b_i\}_{i=0}^l \subseteq [S, T]$ is an **interlaced cover of** $[S, T]$ if $S = a_0 < a_1 < b_0 < a_2 < b_1 < a_3 < b_2 < \ldots < a_l < b_{l-1} = b_1 = T$. Let $y : [S, T] \to X$. The set $\{a_i, b_i\}_{i=0}^l$ is an **interlaced cover for** $y$ if $\{a_i, b_i\}_{i=0}^l$ is an interlaced cover of $[S, T]$ and $y(a_{i+1}) \neq y(b_i)$ for all $i$ less than $l$.

**Corollary 2.49** Suppose $y : [S, T] \to X$ is continuous and $y([S, T]) \subseteq \bigcup_{\alpha \in A} O_\alpha$ where $\{O_\alpha\}_{\alpha \in A}$ is a collection of open sets $O_\alpha$. There exists an interlaced cover for $y$, $\{a_i, b_i\}_{i=0}^l$ such that $y([a_i, b_i]) \subseteq O_{\alpha_i}$. Note that for such a setup, this implies $y([a_{i+1}, b_i]) \subseteq O_{\alpha_i} \cap O_{\alpha_{i+1}}$.
Lemma 2.50 Let here in more generality to avoid too much indexing notation later.

Proof. The first step will be a technical one to get rid of unnecessary endpoints. Let $t'_i$ and $\alpha'_i$ be as given in Lemma 2.47. Then clearly $y(t'_i) \in \mathcal{O}_{\alpha'_{i-1}} \cap \mathcal{O}_{\alpha'_i}$ for all $1 \leq i < l'$. Starting with $t'_1$, we check if $y([t'_0, t'_1]) \subseteq \mathcal{O}_{\alpha'_1}$. In the case it is, we may renumber our partition after removing $t'_1$ and $\mathcal{O}_{\alpha'_0}$ to get a new set of $t'_j$ and $\alpha'_j$ which still satisfy the result of the lemma. Continuing this process inductively, we may assume that we have such a set \{ $t_i, \alpha_i\}_{i=0}^l$ such that $y([t_i, t_{i+1}])$ is not contained in $\mathcal{O}_{\alpha_{i+1}}$.

To construct the desired interlaced cover, we define $b_i := t_{i+1}$ for all $1 \leq i := l' - 1$ and $a_0 := S$. Note for now that this means $y([b_{i-1}, b_i]) \subseteq \mathcal{O}_{\alpha_i}$. Then we define the lower end stopping time $T_i$ for all $i > 0$ by the formula

$$T_i := \inf \{ t < b_i : y([t, b_i]) \subseteq \mathcal{O}_{\alpha_{i+1}} \}.$$ 

By construction and because we refined our partition, $b_{i-1} \leq T_i < b_i$. It is clear that $y(T_i) \neq y(b_i)$ by the continuity of $y$. Thus, there exists a time $T_i^*$ such that $T_i < T_i^*$ and $y(T_i^*) \neq y(b_i)$. Define

$$a_{i+1} := T_i^*$$

for all $0 < i < n$. Since $y([b_{i-1}, b_i]) \subseteq \mathcal{O}_{\alpha_i}$ and $a_i > b_{i-1}$, we have that $y([a_i, b_i]) \subseteq \mathcal{O}_{\alpha_i}$. □

Since the following patching trick will be used multiple times in later proofs, we will prove it here in more generality to avoid too much indexing notation later.

Lemma 2.50 Let $\omega$ be a control and \{ $a_i, b_i\}_{i=0}^l$ be an interlaced cover of $[S, T]$ such that $\omega (a_{i+1}, b_i) > 0$ for all $i < n$. Let $\theta > 0$ and $F : D \to [0, \infty)$ be a bounded function such that $D \subseteq \Delta_{[S, T]}$ and for each $1 \leq i \leq l$ there exists $C_i < \infty$ such that

$$F(s, t) \leq C_i \omega (s, t)^\theta$$

for all $(s, t) \in \Delta_{[a_i, b_i]} \cap D$.

Then there exists a $\tilde{C} < \infty$ such that

$$F(s, t) \leq \tilde{C} \omega (s, t)^\theta \quad \forall \ (s, t) \in D.$$

(2.25)

Proof. Let

$$m := \min \{ \omega (a_{i+1}, b_i)^\theta : 0 \leq i < n \},$$

$$C := \max \{ C_i : 0 \leq i \leq n \},$$

$$M := \sup \{ F(s, t) : (s, t) \in D \} < \infty$$

and then define $\tilde{C} := \max \{ \frac{M}{m}, C \}$. We claim that Inequality (2.25) holds.

Figure 1: An interlaced cover of $[S, T]$
If there exists an $i$ such that $s, t \in [a_i, b_i] \cap D$, then (2.25) holds trivially. Otherwise, let $i^*$ be the largest $i$ such that $s \in [a_i, b_i]$. Then $s < a_{i^*+1}$ and $t > b_{i^*}$. However this says that 

$$[s, t] \supset [a_{i^*+1}, b_{i^*}]$$

so that

$$F(s, t) \leq M = \frac{M}{m} \leq \hat{C}\omega (a_{i^*+1}, b_{i^*})^{\theta} \leq \hat{C}\omega (s, t)^{\theta}.$$ 

\[ \blacksquare \]

2.5.2 Proof of the Theorem 2.43

The recurring strategy here will be localize appropriately to work in the $\mathbb{R}^d$ case so that we may apply Theorem 2.46. We must choose these localizations carefully so that we may patch the estimates together (with two different strategies) using the lemmas above. One method of patching is a bit more involved than the other; therefore we will present it more formally:

**Remark 2.51 (Proof Strategy)** Let $y : [a, b] \to M$ be the first component of either a $(\psi, U)$ - controlled rough path or chart controlled rough path. Also suppose for each $m \in y([a, b])$, we are given an open neighborhood, $W_m \subseteq M$, of $m$. By Corollary 2.49, there exists an interlaced cover for $y$, \{a_i, b_i\}_{i=1}^l and \{m_i\}_{i=1}^l such that $y([a_i, b_i]) \subseteq W_{m_i}$ and $\omega (a_{i+1}, b_i) > 0$. Thus, if $F : D \to [0, \infty)$ is a bounded function such that $D \subseteq \Delta_{[a,b]}$, then in order to prove that

$$F(s, t) \leq C\omega (s, t)^{\theta} \forall (s, t) \in D,$$

it suffices to prove; for each $1 \leq i \leq l$ there exists $C_i < \infty$ such that

$$F(s, t) \leq C_i\omega (s, t)^{\theta} \text{ for all } (s, t) \in \Delta_{[a_i, b_i]} \cap D.$$

Therefore in attempting to prove an assertion in the form of Inequality (2.26), we may assume, without loss of generality, that $y([a, b]) \subseteq W$ where the $W$ will have nice properties dependent on our setting.

The proof of Theorem 2.43 will consist of two steps:

1. If gauge conditions of (2.15) and (2.16) hold for some $C > 0$ and $\delta > 0$, then the chart conditions of (2.17) and (2.18) hold. We will reduce this to the $\mathbb{R}^d$ case immediately, then use Lemma 2.6 to patch the estimates together.

2. If the chart condition of (2.17) and (2.18) hold, then gauge condition of (2.15) and (2.16) hold for an appropriately chosen $\delta$. Here we will first show which local estimates we need to satisfy to use Remark 2.51 and then reduce to the $\mathbb{R}^d$ case.

In simple terms, step 1 is “localize then patch” and step 2 is “cut nicely, localize, then patch”.

**Proof of Theorem 2.43** Step 1: Definition 2.34 $\Longrightarrow$ Definition 2.39

We’ll first assume that the gauge definition holds, i.e. that there exists a $\delta > 0$ and a $C_1 > 0$ such that

$$|\psi(y_s, y_t) - y^i_\Lambda x_{s,t}|_g \leq C_1\omega (s, t)^{2/p}$$

(2.27)
and
\[ |U(y_s, y_t) y_t^1 - y_s^1|_g \leq C_1 \omega(s, t)^{1/p} \]
hold for all \(0 \leq s \leq t \leq T\) such that \(|t - s| \leq \delta\). Let \(\phi\) be a chart on \(M\) and let \([a, b]\) be such that \(y([a, b]) \subseteq D(\phi)\). If we define
\[ \psi^\phi(x, y) := \phi \circ \psi^{-1}(x), \phi^{-1}(y) \]
\[ U^\phi(x, y) := \phi U(\phi^{-1}(x), \phi^{-1}(y)) \circ (\phi^{-1})_\phi(y) \]
\[ z_s := \phi(y_s) \]
\[ z_s^1 := d\phi \circ y_s^1, \]
then it is clear that there exists a \(C_2\) such that
\[ |\psi^\phi(z_s, z_t) - z_s^1 x_{s,t}| \leq C_2 \omega(s, t)^{2/p} \] (2.28)
\[ |U^\phi(z_s, z_t) z_t^1 - z_s^1| \leq C_2 \omega(s, t)^{1/p} \] (2.29)
for all \(a \leq s \leq t \leq b\) such that \(t - s \leq \delta\) where \((\psi^\phi, U^\phi)\) is a local gauge on \(\mathbb{R}^d\). Thus \((z, z^1)\) is a \((\psi^\phi, U^\phi)\)–rough path controlled by \(X\). In this language, we need to prove that there exists a \(C_{\phi, a,b}\) such that
\[ |z_t - z_s - z_s^1 x_{s,t}| \leq C_{\phi, a,b} \omega(s, t)^{2/p}. \] (2.30)
and
\[ |z_t^1 - z_s^1| \leq C_{\phi, a,b} \omega(s, t)^{1/p} \] (2.31)
for all \(s, t\) such that \(a \leq s \leq t \leq b\).

In light of Lemma 2.6 and Lemma 2.47, we only need to show that for each \(u \in [a, b]\), the inequalities (2.30) and (2.31) hold with \(C_{\phi, a,b}\) replaced with \(C_u\) for all \(s, t \in (u - \delta_u, u + \delta_u) \cap [a, b]\) such that \(s \leq t\) for some \(\delta_u > 0\). We begin with inequality (2.30).

For any \(u \in [a, b]\), let \(W_u\) be an open convex set of \(z_u\) such that \(W_u \times W_u \subseteq D(\psi^\phi)\). We then choose \(\delta_u > 0\) to be such that \((u - \delta_u, u + \delta_u) \cap [a, b] \subseteq W_{z_u}\) and \(2\delta_u \leq \delta\). However, now we are in the setting of Theorem 2.46 and are therefore finished with this step.

**Step 2: Definition 2.39 \implies Definition 2.34**

Suppose that the chart item (2.17) holds, that is for every chart \(\phi\) on \(M\) and \([a, b]\) such that \(y([a, b]) \subseteq D(\phi)\), there exists a \(C_{\phi, a,b} > 0\) such that
\[ |\phi(y_t) - \phi(y_s) - d\phi \circ y_s^1 x_{s,t}| \leq C_{\phi, a,b} \omega(s, t)^{2/p} \] (2.32)
\[ |d\phi \circ y_t^1 - d\phi \circ y_s^1| \leq C_{\phi, a,b} \omega(s, t)^{1/p} \] (2.33)
for all \(a \leq s \leq t \leq b\). We must prove that there exists a \(\delta, C > 0\) such that
\[ |\psi(y_s, y_t) - y_s^1 x_{s,t}|_g \leq C \omega(s, t)^{2/p} \] (2.34)
\[ |U(y_s, y_t) y_t^1 - y_s^1|_g \leq C \omega(s, t)^{1/p} \] (2.35)
for all \(s \leq t\) such that \(|t - s| \leq \delta\).
We choose $\delta$ such that $|t - s| \leq \delta$ for $0 \leq s \leq t \leq T$ implies that both $|\psi(y_s, y_t)|_g$ and $|U(y_s, y_t)|_g$ make sense and are bounded. Around every point $m$ of $y([0, T])$, there exists an open $O_m$ containing $m$ and such that $O_m \times O_m \subseteq \mathcal{D}$. Additionally there exists a chart $\phi^m$ such that $m \in D(\phi^m)$. By considering an open ball around $\phi^m(m)$ in $R(\phi^m)$ and shrinking the radius, we may assume that $V_m := D(\phi^m) \subseteq O_m$ and the range, $W_m := \phi(V_m)$, of $\phi^m$ is convex. Since $\{V_m\}_{m \in M}$ is an open cover of $y([a, b])$, we may use this cover along with $D = \{(s, t): 0 \leq s \leq t \leq T$ and $|t - s| \leq \delta\}$ to employ the proof strategy in Remark 2.51. We will do this twice, with $F(s, t) = |\psi(y_s, y_t) - y_s^\dagger x_{s,t}|_g$ in the first iteration and and $F(s, t) = |U(y_s, y_t)_g y_t^\dagger - y_s^\dagger|_g$ in the second; this will reduce us to considering the case where there exists a single chart $\phi$ and open set $\mathcal{V} \subseteq D(\phi)$ such that $y([0, T]) \subseteq \mathcal{V}$, $\mathcal{V} \times \mathcal{V} \subseteq \mathcal{D}$ and $\phi(\mathcal{V}) = \mathcal{W}$ is convex.

Now that we have reduced to a single chart $\phi$, we may define $(\psi^\phi, U^\phi)$ and the path $(z, z^\dagger)$ as in Step 1. Then $z([0, T]) \subseteq \mathcal{W}$ and $\mathcal{W} \times \mathcal{W} \subseteq D(\psi^\phi) = D(U^\phi)$. However, by Theorem 2.46 we have that the proper estimates hold because $z$ is a $(\psi^\phi, U^\phi)$–rough path controlled by $X$. Therefore, we are finished by patching using Remark 2.51. □

**Remark 2.52** In the proof of Theorem 2.44, we would have been able to show (and did so somewhat indirectly) that Inequality (2.17) implies Inequality (2.19) for some $\delta > 0$. However, it is not true in general that, for a fixed $\delta$, Inequality (2.15) implies Inequality (2.17). See Example 5.7 in the Appendix for a counterexample.

In situations in which we are given a covariant derivative $\nabla$ on a manifold, by Example 2.19 we have an equivalent definition:

**Example 2.53** The pair $(y_s, y_t^\dagger)$ is an element of $\text{CRP}_X(M)$ if and only if there exists a $C$ such that

1. $\left|\left(\exp^\nabla_{y_s}\right)^{-1}(y_t) - y_t^\dagger x_{s,t}\right|_g \leq C\omega(s, t)^{2/p}$ (2.36)

2. $\left|U^\nabla_{y_s, y_t}y_t^\dagger - y_s^\dagger\right|_g \leq C\omega(s, t)^{1/p}$ (2.37)

where $(\exp^\nabla)^{-1}$ and $U^\nabla_{y_s, y_t}$ are defined as in Example 2.19 and the inequalities hold when $(y_s, y_t)$ are in the domain as described in Section 2.3.7. In particular, on a Riemannian manifold we can use this definition with the Levi-Civita covariant derivative.

Before providing yet another equivalent definition of controlled rough paths on manifolds, we will present some examples.

### 2.6 Examples of Controlled Rough Paths

Recall $X = (x, X)$ is a rough path with values in $W \oplus W^\otimes 2$ where $W = \mathbb{R}^k$. The results here will rely on basic approximations found in the Appendix, Section 5.

**Example 2.54** Let $M^d \subseteq W$ be an embedded submanifold and for every $m \in M^d$, let $P(m)$ be the orthogonal projection onto the tangent space $T_mM$. Suppose $x_s \in M^d$ for all $s$ in $[0, T]$. Then $(x_s, P(x_s)) \in \text{CRP}_X(M)$.
Proof. We will use the gauge as given in Example 2.53 where the $\nabla$ is the Levi-Civita covariant derivative from the induced metric from Euclidean space. Verifying that $P(x_s)$ lives in the correct space is trivial.

Next, to show Inequality 2.36 is satisfied, we use item 1 of Lemma 5.2 which says $\exp^{-1}_{m} (\tilde{m}) = P(m)(\tilde{m} - m) + O(|\tilde{m} - m|^3)$ for all $m \in M^d$. Letting $m = x_s$ and $\tilde{m} = x_t$, we are done.

Inequality (2.37) is also satisfied as a result of Lemma 5.2 which says that $U_t \nabla_{m,m} = P_{m}(m) + O(|\tilde{m} - m|)$. Thus

$$P(x_t) - U_{x_s,x_s}^\nabla P(x_s) \approx P(x_t) - P(x_s) P(x_s) = P(x_t) - P(x_s) \approx 0$$

The next example will be proved in more generality in Section 4.2. However, we find it instructive to prove it without charts and in the embedded context where the reader may be more comfortable.

Example 2.55 Let $f$ be a smooth function from $W$ to an embedded manifold $\tilde{M}^d \subseteq \mathbb{R}^{\tilde{k}}$. Then $(f(x_s), f'(x_s)) \in CRP_{X}(\tilde{M})$.

Proof. Again we will use the Levi-Civita covariant derivative $\nabla$ from the embedded metric. First we note that $f'(x_s)$ lives in the correct space as $R(f) \subseteq \tilde{M}^d$.

To show Inequality (2.36) holds one can use the fact that $(f((x_s), f'(x_s))$ is a controlled rough path in the embedded space or Taylor’s Theorem to see that

$$f(x_t) - f(x_s) - f'(x_s)(x_t - x_s) \approx 0.$$ 

Since $P$ is a projection, we have

$$P(f(x_s))[f(x_t) - f(x_s) - f'(x_s)(x_t - x_s)] \approx 0.$$ 

But again by Lemma 5.2

$$P(f(x_s))[f(x_t) - f(x_s) - f'(x_s)(x_t - x_s)] = P(f(x_s))[f(x_t) - f(x_s)] - f'(x_s)(x_t - x_s) \approx 0.$$ 

Thus

$$P(f(x_s))^{-1} (f(x_t)) - f'(x_s)(x_t - x_s) \approx 0.$$ 

Lastly to show Inequality (2.37), we have

$$f'(x_t) - f'(x_s) \approx 0.$$
Since $P(f(x_t))$ is a projection and since
\[ P(f(x_t)) [f'(x_t) - f'(x_s)] = f'(x_t) - P(f(x_t)) f'(x_s) \]
\[ \approx f'(x_t) - U_{f(x_t),f(x_s)} f'(x_s) \]
by Lemma 5.2 we have
\[ f'(x_t) - U_{f(x_t),f(x_s)} f'(x_s) \approx 0. \]

2.7 Smooth Function Definition of CRP

In the spirit of semi-martingales on manifolds [see for example [11] Chapter III or [10], we can define controlled rough paths on manifolds as elements which, when composed with any smooth function, give rise to a one-dimensional controlled rough path on flat space. More precisely we have the following theorem.

**Theorem 2.56** $y = (y, y^\dagger) \in CRP_X(M)$ if and only if for every $f \in C^\infty(M)$,
\[ f \star y = (f(y), df \circ y^\dagger) \in CRP_X(\mathbb{R}). \]

**Proof.** The proof that $y \in CRP_X(M)$ implies that $f \star y \in CRP_X(\mathbb{R})$ for every $f \in C^\infty(M)$ will be deferred to the more general case proved in Proposition 4.10 (in which case we consider the codomain of $f$ to be a manifold $\tilde{M}$).

To prove the converse, we have to prove that for every chart $\phi$ and $a, b$ such that $y([a, b]) \subseteq D(\phi)$, there exists a $C_{\phi,a,b}$ such that
\[ |\phi(y_t) - \phi(y_s) - d\phi \circ y^\dagger x_{s,t}| \leq C_{\phi,a,b} \omega(s,t)^{2/p}. \]
and
\[ |d\phi \circ y^\dagger_t - d\phi \circ y^\dagger_s| \leq C_{\phi,a,b} \omega(s,t)^{1/p}. \]

For a given $\phi, a, b$, we can find an open $O$ such that
\[ y([a, b]) \subseteq O \subseteq D(\phi) \]
such that the closure of $O$ is compact and still contained in $D(\phi)$. Then by using a smoothing function, we can manufacture global functions $f^i \in C^\infty(M)$ which agree with the coordinates $\phi^i$ on $O$. It is then easy to see that the necessary equations hold by the definition of $f^i \star y$ being an element of $CRP_X(\mathbb{R})$. ■

3 Integration of Controlled One-Forms

In the flat case, a controlled rough path with values in an appropriate Euclidean spaces can be integrated against another controlled rough path (see Theorem 2.9) provided their controlling rough path $X$ is the same. The integral in this case is another rough path controlled by $X$. We can do something similar on manifolds, though it will be necessary to add some extra structure. As usual let $y_s = (y_s, y^\dagger_s)$ be a controlled rough path on $M$ controlled by $X = (x, X) \in W \oplus W \otimes 2$. Let $V$ be a Banach space.
3.1 Controlled one-forms along a rough path

Let $U$ be a parallelism on $M$.

**Definition 3.1** The pair $(\alpha_s, \alpha^\dagger_s)$ is a $V$–valued $U$–controlled (rough) one-form along $y_s$ if

1. $\alpha_s \in L(T_{y_s} M, V)$
2. $\alpha^\dagger_s \in L(W \otimes T_{y_s} M, V)$
3. $\alpha_t \circ U(y_t, y_s) - \alpha_s - \alpha_s^\dagger (x_{s,t} \otimes (\cdot)) \approx 0$
4. $\alpha^\dagger_t \circ (I \otimes U(y_t, y_s)) - \alpha^\dagger_s \approx 0$

By items 3 and 4, we mean these hold if $|t - s| < \delta$ for some $\delta > 0$ to ensure the expressions make sense.

**Remark 3.2** For the sake of clarity, by item 3 of Definition 3.1, we mean that if $s, t$ are close, then there exists a $C$ such that

$$\left| \alpha_t \circ U(y_t, y_s) - \alpha_s - \alpha_s^\dagger (x_{s,t} \otimes (\cdot)) \right|_{g,op} \leq C \omega(s, t)^{2/p}.$$

For item 4, we mean for $s, t$ close, there exists a $C$ such that

$$\left| \alpha^\dagger_t \circ (w \otimes U(y_t, y_s)) - \alpha^\dagger_s (w \otimes (\cdot)) \right|_{g,op} \leq C |w| \omega(s, t)^{1/p}$$

for all $w \in W$. By Corollary 5.4, it does not matter which $g$ we choose here.

**Notation 3.3** Let $CRP^U_y(M, V)$ denote those $\alpha_s := (\alpha_s, \alpha^\dagger_s)$ satisfying Definition 3.1. We refer to $CRP^U_y(M, V)$ as a space of $U$–controlled one-forms along $y$.

**Remark 3.4** If $M = \mathbb{R}^d$ and $U = I$ and we identify $T_{y_s} M$ with $\mathbb{R}^d$ then Definition 3.1 reduces to the flat case definition of a $L(\mathbb{R}^d, V)$–valued rough path controlled by $X$.

**Remark 3.5** Note that 3 and 4 of the definition force continuity of the $\alpha_s$ and $\alpha^\dagger_s$.

We can take linear combinations of elements of $CRP^U_y(M, V)$ to form other elements in $CRP^U_y(M, V)$. The following proposition, whose simple proof is left to the reader, shows how to construct more non-trivial examples of elements in $CRP^U_y(M, V)$.

**Proposition 3.6** If $V$ and $\tilde{V}$ are Banach spaces, $\alpha \in CRP^U_y(M, V)$ and

$$f = (f, f^\dagger) \in CRP_X \left( \text{Hom} \left( V, \tilde{V} \right) \right),$$

then

$$(f \alpha)_s := (f_s \alpha_s, f^\dagger_s \alpha_s + f_s \alpha^\dagger_s) \in CRP^U_y \left( M, \tilde{V} \right),$$

where by $f^\dagger_s \alpha_s$ we mean $f^\dagger_s \left( (\cdot) \otimes \alpha_s (\cdot) \right)$.
The idea will be that for each $V$–valued $U$–controlled rough one-form along $y_s$, we get a new controlled rough path which represents an integral. The definition of the integrators and the notion of integration is dependent on a choice of parallelism. However, as shown in Theorem 3.32 below, these different notions of integration are all related by a natural transformation. Before we explain this, we will first prove the existence of an integral of $\alpha_s$ along $y_s$. To do so, we will define an approximation that is close to what our integral will be. We will need one more differential geometric notion before we can state the approximation.

3.2 The Compatibility Tensors

**Definition 3.7** The compatibility tensor, $S^{\tilde{U}, U} \in \Gamma (T^* M \otimes T^* M \otimes TM)$, of two parallelisms $\tilde{U}$ and $U$ on $M$ is the defined by

$$S^{\tilde{U}, U}_m := d \left[ \left( U(\cdot, m)^{-1} \tilde{U}(\cdot, m) \right) \right]_m.$$  

In more detail if $v_m, w_m \in T_m M$, then

$$S^{\tilde{U}, U}_m [v_m \otimes w_m] = v_m \left[ x \to U(x, m)^{-1} \tilde{U} (x, m) w_m \right].$$

**Remark 3.8** There are actually multiple ways to define $S^{\tilde{U}, U}_m$. For example, we have on simple tensors

$$S^{\tilde{U}, U}_m (v_m \otimes w_m) = d \left[ \left( \nabla (U(\cdot, m) - \tilde{U}(\cdot, m)) \right) \right]_m,$$

where $\nabla$ is any covariant derivative on $M$. Similar to the proofs of Corollary 2.28 above and Theorem 3.15 below, the identities in Eq. (3.1) are straightforward to prove by employing charts to reduce them to Euclidean space identities.

**Example 3.9** If $\nabla$ and $\tilde{\nabla}$ are two covariant derivatives on $TM$, $U = U^\nabla$, $\tilde{U} = U^{\tilde{\nabla}}$, and $A \in \Omega^1(\text{End}(TM))$ such that $\nabla = \tilde{\nabla} + A$, then

$$S^{\tilde{U}, U}_m (v_m \otimes w_m) = A(v_m) w_m \in T_m M.$$

Indeed,

$$v_m \left[ U(\cdot, m)^{-1} \tilde{U} (\cdot, m) w_m \right] = \nabla v_m \left[ \tilde{U} (\cdot, m) w_m \right] = \tilde{\nabla} v_m \left[ \tilde{U} (\cdot, m) w_m \right] + A(v_m) \tilde{U} (m, m) w_m = 0 + A(v_m) w_m = A(v_m) w_m.$$

**Example 3.10 (Converse of Example 3.9)** If $U$ and $\tilde{U}$ are two parallelisms on $M$ and $\nabla = \nabla^U$ and $\tilde{\nabla} = \tilde{\nabla}^\tilde{U}$ are the corresponding covariant derivatives on $TM$ (as in Remark 2.20), then

$$\nabla v_m = \tilde{\nabla} v_m + S^{\tilde{U}, U}_m (v_m \otimes (\cdot)) \forall v_m \in T_m M.$$
The verification is as follows. If $Y$ is a vector-field on $M$ and $\sigma_t$ is such that $\dot{\sigma}_0 = v_m$, we have

$$
\nabla v_m Y - \nabla v_m Y := \frac{d}{dt}\bigg|_0 \left[ U (m, \sigma_t) - \bar{U} (m, \sigma_t) \right] Y (\sigma_t)
$$

$$
= \left( \nabla v_m \left[ U (m, \cdot) - \bar{U} (m, \cdot) \right] \right) Y (m) + 0 \cdot \nabla v_m Y
$$

$$
= S^\Omega \Psi (v_m \otimes Y (m))
$$

wherein we have used Eq. \[3.1\] for the last equality.

Lemma 3.11 If $U, \bar{U}$, and $\hat{U}$ are three parallelisms, then

$$
S^\Omega \Psi \Psi + S^\Omega \bar{U} \Psi + S^\Omega \Psi \bar{U} = -S^\Omega \bar{U} \Psi.
$$

Proof. For $v_m, w_m \in T_m M$, an application of the product rules shows

$$
S^\Omega \Psi (v_m \otimes w_m) = v_m \left[ U (\cdot, m)^{-1} \bar{U} (\cdot, m) w_m \right]
$$

$$
= v_m \left[ \left[ U (\cdot, m)^{-1} \bar{U} (\cdot, m) \right] \left[ \bar{U} (\cdot, m)^{-1} \hat{U} (\cdot, m) \right] w_m \right]
$$

$$
= S^\Omega \bar{U} (v_m \otimes w_m) + S^\Omega \Psi (v_m \otimes w_m).
$$

Similarly,

$$
S^\Omega \bar{U} [v_m \otimes (\cdot)] = v_m \left[ \hat{U} (\cdot, m)^{-1} U (\cdot, m) \right]
$$

$$
= v_m \left[ U (\cdot, m)^{-1} \bar{U} (\cdot, m) \right]^{-1}
$$

$$
= -v_m \left[ U (\cdot, m)^{-1} \bar{U} (\cdot, m) \right]
$$

$$
= -S^\Omega \Psi [v_m \otimes (\cdot)].
$$

Notation 3.12 If $G := (\psi, U)$ is a gauge, we let $S^G := S^\psi \Psi$ be the compatibility tensor between $U^\psi$ and $U$, where $U^\psi (m, n) := \psi (m, \cdot)_n$ as in Remark 2.21.

If we have a covariant derivative $\nabla$ on $M$, then as in Example 2.19 we have the choice of gauge $G = (\psi, U) = \left( (\exp^\nabla)^{-1}, U^\nabla \right)$. In this case, the tensor $S^G_m$ is a more familiar object.

Lemma 3.13 If $\psi = (\exp^\nabla)^{-1}$ and $U = U^\nabla$, then

$$
S^G_m = \frac{1}{2} T_m^\nabla
$$

where $T_m^\nabla$ is the Torsion tensor of $\nabla$. 
Proof. By transferring the covariant derivative and functions using charts, we may assume we are working on Euclidean space. In this case, by Eq. (5.10) and Corollary 5.6 we have

\[ S_m^G ((m, v) \otimes (m, w)) = \left( \nabla_{(m,v)} \left[ U^\nabla_{m,\cdot} \right] \right) w \]

\[ = \left[ \partial_{(m,v)} + A_m \langle v \rangle \right] \left[ U^\nabla_{m,\cdot} \right] w \]

\[ = (U^\nabla_{m,\cdot})' (m) [v \otimes w] - \left( \left( \exp_{m}^{\nabla} \right)^{-1} \right) (m) [v \otimes w] + A_m \langle v \rangle w - A_m \langle v \rangle \langle w \rangle \]

\[ = A_m \langle v \rangle \langle w \rangle - \frac{1}{2} A_m \langle v \rangle \langle w \rangle - \frac{1}{2} A_m \langle w \rangle \langle v \rangle \]

\[ = \frac{1}{2} \left[ A_m \langle v \rangle \langle w \rangle - A_m \langle w \rangle \langle v \rangle \right] \]

\[ = \frac{1}{2} T_m ((m, v) \otimes (m, w)). \]

Here is one last example of a gauge and its compatibility tensor.

**Proposition 3.14** Let \( G \) be a Lie group and \( \nabla \) be the left covariant derivative on \( TG \) uniquely determined by requiring the left invariant vector fields to be covariantly constant, i.e. \( \nabla \tilde{A} = 0 \) for all \( A \in g \). Then for \( g \) near \( k \),

\[ U^\nabla (g, k) = \exp (k \log (k^{-1}g)) = L_{gk^{-1}} \]

and

\[ \psi^\nabla (k, g) = (exp_k^{-1}) (g) = k \cdot \log (k^{-1}g) \]

where \( L_g : G \rightarrow G \) is left multiplication by \( g \in G \) and \( \log \) is the local inverse of the map \( A \rightarrow e^A \). Moreover the compatibility tensor for this gauge is given by

\[ S (\xi_g, \eta_g) = -\frac{1}{2} L_{g*} [\theta (\xi_g) \cdot \theta (\eta_g)] \]

for all \( \xi_g, \eta_g \in T_g G \)

where \( \theta \) is the Maurer-Cartan form on \( G \) defined by \( \theta (\xi) := L_{\xi^{-1}k} \xi \in g = T_e G \) for all \( \xi \in T_g G \).

**Proof.** The torsion of \( \nabla \) is given by

\[ T (\tilde{A}, \tilde{B}) = \nabla_{\tilde{A}} \tilde{B} - \nabla_{\tilde{B}} \tilde{A} - \tilde{A} \cdot \tilde{B} \]

or equivalently as

\[ T (\xi_g, \eta_g) = -L_{g*} [\theta (\xi_g) \cdot \theta (\eta_g)] \]

for all \( \xi_g, \eta_g \in T_g G \).

Eq. (3.4) follows from the above formula along with the result in Lemma 3.13

If \( \xi (t) \) is a path \( TG \) above \( \sigma (t) \in G \) it may be written as \( \xi (t) = L_{\sigma(t)*} \theta (\xi (t)) \). Since \( L_{\sigma(t)*} \) is parallel translation, it follows that

\[ \frac{d}{dt} \left( \theta (\xi (t)) \right) = L_{\sigma(t)*} \frac{d}{dt} \theta (\xi (t)) \]

Thus \( \xi (t) \in TG \) is parallel iff \( \theta (\xi (t)) \) is constant for all \( t \). If \( \sigma \) is a general curve in \( G \), we may conclude

\[ // (\sigma |_{s,t}) = L_{\sigma(t)*} L_{\sigma(s)^{-1}s} = L_{\sigma(t) \sigma(s)^{-1}s} \]
Corollary 3.16

If two different parallelisms and two different logarithms on \( U \) are given, then the formula for \( \psi \) in Eq. (3.3) now follows. ■

The last three results of this subsection show how the compatibility tensor allows us to compare two different parallelisms and two different logarithms on \( M \).

**Theorem 3.15** Suppose that \( U \) and \( \tilde{U} \) are two parallelisms on \( M \) and \( \psi \) is a logarithm on \( M \), then

\[
U(m,n)\tilde{U}(m,n)^{-1} = 2 I + S^{\psi U}_{m,n}(\psi(m,n) \otimes (\cdot)).
\] (3.5)

**Proof.** By using charts it suffices to prove the theorem when \( M = \mathbb{R}^d \). By Taylor’s theorem (see Theorem 2.24),

\[
U(m,n) = 2 I + [(D_2 U)(m,m)(n-m)] \quad \text{and}
\]

\[
\tilde{U}(m,n) = 2 I + \left((D_2 \tilde{U})(m,m)(n-m)\right).
\]

and therefore

\[
U(m,n)\tilde{U}(m,n)^{-1} = 2 \left(I + [(D_2 U)(m,m)(n-m)]\right) \left(I - \left((D_2 \tilde{U})(m,m)(n-m)\right)\right)
\]

\[
= 2 I + \left(\left((D_2 U)(m,m) - (D_2 \tilde{U})(m,m)\right)(n-m)\right).
\] (3.6)

However, by Eq. (3.1) we have

\[
S^{\psi U}_{m,n} = (D_2 U)(m,m) - (D_2 \tilde{U})(m,m).
\] (3.7)

Using this identity back in Eq. (3.6) shows

\[
U(m,n)\tilde{U}(m,n)^{-1} = 2 I + S^{\psi U}_{m,n}([n-m]_m \otimes (\cdot))
\]

from which Eq. (3.5) follows because \( \psi(m,n) = [n-m]_m \). ■

**Corollary 3.16** If \( \mathcal{G} = (\psi,U) \) is a gauge on \( M \), then

\[
\psi(n,\cdot)_{sn} = 2 U(n,m) \left[I + S^\mathcal{G}_{m,n}(\psi(m,n) \otimes (\cdot))\right].
\] (3.8)

In particular

\[
\psi(y_t,\cdot)_{ys} \approx 2 U(y_t,y_s) \left[I + S^\mathcal{G}_{ys}(\psi(y_s,y_t) \otimes (\cdot))\right].
\] (3.9)

**Proof.** Theorem 3.15 implies

\[
U(m,n)\psi(m,\cdot)_{sn}^{-1} = 2 I + S^\mathcal{G}_{m,n}(\psi(m,n) \otimes (\cdot))
\]

while Corollary 2.28 shows,

\[
U(m,n)^{-1} = 2 U(n,m) \quad \text{and} \quad \psi(m,\cdot)_{sn}^{-1} = 2 \psi(n,\cdot)_{sn}.
\]

Eq. (3.8) now easily follows from the last two displayed equations. The second statement follows by patching. ■

Lastly we may use the compatibility tensor to compare two logarithms.
Proposition 3.17 Suppose that \( \psi \) and \( \tilde{\psi} \) are two logarithms on a manifold \( M \). Then the compatibility tensor, \( S^{\psi, \tilde{\psi}} \) is symmetric and

\[
\psi(m, n) - \tilde{\psi}(m, n) = \frac{1}{2} S^{\psi, \tilde{\psi}}_m (\psi(m, n) \otimes \psi(m, n)).
\] (3.10)

Proof. As usual it suffices to prove this result when \( M = \mathbb{R}^d \) in which case we omit the base points of tangent vectors. From Eq. (3.7) with \( U(x, y) = \psi_x'(y) \) and \( \tilde{U}(x, y) = \tilde{\psi}_x'(y) \), we see that

\[
S^{\psi, \tilde{\psi}}_x = \psi''_x(x) - \tilde{\psi''}_x(x)
\] (3.11)

which is symmetric since mixed partial derivatives commute. Then by Taylor’s theorem and Eq. (3.11),

\[
\psi(x, y) - \tilde{\psi}(x, y) = \frac{1}{2} \left[ \psi''_x(x) - \tilde{\psi''}_x(x) \right] (y - x)^{\otimes 2} + O \left( |y - x|^3 \right)
\]

wherein we have also used \((y - x)^{\otimes 2} = 3 \psi(x, y)^{\otimes 2}\).

\[\blacksquare\]

Remark 3.18 If \( \nabla \) is any covariant derivative on \( TM \), then

\[
S^{\psi, \tilde{\psi}}_m = \nabla \left( \psi(m, \cdot) - \tilde{\psi}(m, \cdot) \right) = \text{Hess}_{\nabla} \left( \psi - \tilde{\psi} \right)
\]

where \( \text{Hess}_{\nabla} f_m := (\nabla df)_m \). By choosing \( \nabla \) to be Torsion free we again see that \( S^{\psi, \tilde{\psi}}_m \) is a symmetric tensor.

3.3 \( U \) - controlled rough integration

Our next goal is to construct “the” integral, \( \int \langle \alpha, dy \rangle \), where \( y \in CRP_x (M) \) and \( \alpha \in CRP_y (M, V) \). We begin with the following proposition which is meant to motivate the definitions to come.

Proposition 3.19 Assume (in this proposition only) that all functions, \( y_s, \alpha_s, \) and \( x_s \) are smooth, \( p = 1 \), and \( \omega(s, t) = |t - s| \). Further assume \( y \) (respectively \( \alpha \)) still satisfy the estimates of being controlled rough path (along \( y \)).

\[
\int_s^t \alpha_\tau \dot{y}_\tau d\tau = \alpha_s \left[ \psi(y_s, y_t) + S_{y_s}^\psi \left( y_s^i \otimes y_s^j X_{s,t} \right) \right] + \alpha_t^i (I \otimes y_t^i) X_{s,t} + O \left( (t-s)^3 \right).
\] (3.12)

Proof. Our assumptions give,

\[
\psi(y_s, y_t) = y_t^i x_{s,t} + O \left( (t-s)^2 \right) \Rightarrow \dot{y}_s = y_t^i \dot{x}_s,
\]

\[
\alpha_t U(y_t, y_s) = \alpha_s + \alpha_s^i x_{s,t} + O \left( (t-s)^2 \right),
\]

\[
U(y_s, y_t) y_t^i = y_s + O(t-s), \quad \text{and}
\]

\[
\alpha_t^i \left( I \otimes U(y_t, y_s) \right) = \alpha_t^i + O(t-s).
\]

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Putting this all together proves Eq. (3.12).

We start with the identity,

\[
\int_s^t \alpha_s y_{t \tau} d\tau = \int_s^t \alpha_s U(y_{s \tau}, y_s) U(y_{s \tau}, y_s)^{-1} y_{t \tau} d\tau
\]

\[
= \int_s^t \left[ \alpha_s + \alpha_s^1 x_{s \tau} + O\left( (\tau - s)^2 \right) \right] U(y_{s \tau}, y_s)^{-1} y_{t \tau} d\tau
\]

\[
= \int_s^t \alpha_s U(y_{s \tau}, y_s)^{-1} y_{t \tau} d\tau + \int_s^t \alpha_s^1 x_{s \tau} U(y_{s \tau}, y_s)^{-1} y_{t \tau} d\tau + O\left( (t - s)^3 \right)
\]

\[
= \int_s^t \alpha_s U(y_{s \tau}, y_s)^{-1} y_{t \tau} d\tau + \int_s^t \alpha_s^1 x_{s \tau} U(y_{s \tau}, y_s)^{-1} y_{t \tau} d\tau + O\left( (t - s)^3 \right). \quad (3.13)
\]

\[
=: A + B + O\left( (t - s)^3 \right) \quad (3.14)
\]

wherein we have used Corollary [2.28](#2.28) in order to show it is permissible to replace \( U(y_{s \tau}, y_s)^{-1} \) by \( U(y_{s \tau}, y_s) \) above. The \( B \) term is then easily estimated as

\[
B = \int_s^t \alpha_s^1 x_{s \tau} y_{s \tau}^1 \dot{x}_{s \tau} d\tau = \int_s^t \alpha_s^1 x_{s \tau} y_{s \tau}^1 \dot{x}_{s \tau} d\tau
\]

\[
= \int_s^t \alpha_s^1 x_{s \tau} y_{s \tau}^1 \dot{x}_{s \tau} d\tau + O\left( (t - s)^3 \right) = \alpha_s^1 \left( I \otimes y_{s \tau}^1 \right) X_{s \tau} + O\left( (t - s)^3 \right).
\]

The estimate of the \( A \) term to order \( O\left( (t - s)^3 \right) \) requires more care. For this term we use

\[
\frac{d}{dt} \psi(y_s, y_t) = \psi(y_s, \cdot)_{s y_t} y_t \implies \dot{y}_t = \psi(y_s, \cdot)_{s y_t}^{-1} \frac{d}{dt} \psi(y_s, y_t)
\]

and (from Theorem 3.15) that

\[
U(y_s, y_{t \tau}) \psi(y_s, \cdot)_{s y_t}^{-1} = I + S_{y_s}^\sigma \left( \psi(y_s, y_t) \otimes (\cdot) \right)
\]

in order to conclude,

\[
A := \int_s^t \alpha_s U(y_{s \tau}, y_s) \dot{y}_{t \tau} d\tau = \int_s^t \alpha_s U(y_{s \tau}, y_s) \psi(y_s, \cdot)_{s y_t}^{-1} \frac{d}{dt} \psi(y_s, y_{t \tau}) d\tau
\]

\[
= \int_s^t \alpha_s \left[ I + S_{y_s}^\sigma \left( \psi(y_s, y_{t \tau}) \otimes (\cdot) \right) \right] \frac{d}{dt} \psi(y_s, y_{t \tau}) d\tau + O\left( (t - s)^3 \right)
\]

\[
= \alpha_s \left( \psi(y_s, y_t) \right) + \alpha_s \int_s^t S_{y_s}^\sigma \left( \psi(y_s, y_{t \tau}) \otimes \frac{d}{d\tau} \psi(y_s, y_{t \tau}) \right) d\tau + O\left( |t - s|^3 \right)
\]

\[
= \alpha_s \left( \psi(y_s, y_t) \right) + \alpha_s \int_s^t S_{y_s}^\sigma \left( y_{s \tau}^1 x_{s \tau} \otimes y_{s \tau}^1 \dot{x}_{s \tau} \right) d\tau + O\left( |t - s|^3 \right)
\]

Putting this all together proves Eq. (3.12).
Definition 3.20 ((G, y)- integrator) Given a gauge G := (ψ, U) and y ∈ CRPX (M), the (G, y)- integrator is the increment process;

\[ y_{s,t} := (ψ(y_s, y_t) + S_{y_s}^G (y_t^\top \otimes 2 X_{s,t})), (I \otimes y_t^1) X_{s,t} \] \in T_y M \times [W \otimes T_y M].

Moreover, for \( α ∈ CRP^U_y (M, V) \) (see Notation 3.3) let

\[ z_{s,t} := \langle α_s, y_{s,t}^G \rangle = α_s (ψ(y_s, y_t) + S_{y_s}^G (y_t^\top \otimes 2 X_{s,t})) + α_s^1 (I \otimes y_t^1) X_{s,t} \]  \tag{3.15}

which is defined for \((s, t) ∈ \Delta_{[0, T]}\) with \( |t - s| < δ \) for some sufficiently small \( δ > 0 \).

Recall that a two-parameter function \( F : \Delta_{[0, T]} → V \) is an almost additive functional if there exists a \( \theta > 1 \), a control \( \tilde{ω}(s, t) \) and a \( C > 0 \) such that

\[ |F_{s,u} - F_{s,t} - F_{t,u}| ≤ C\tilde{ω}(s, t)^\theta \]

for all \( 0 ≤ s ≤ t ≤ u ≤ T \).

Theorem 3.21 Let \( G := (ψ, U) \) be a gauge, \( α ∈ CRP^U_y (M, V) \), and \( z_{s,t} \) be as in Definition 3.20. Then there exists a unique \( z := (z, z^1) ∈ CRPX (V) \) such that \( z_0 = 0 \), \( z_{s,t} ≡ \tilde{z}_{s,t} \), and \( z_1 = α_s \circ y_t \).

We denote this unique controlled rough path by \( \int \langle α_s dγ^y \rangle \), i.e.

\[ \int_s^t \langle α_s dγ^y \rangle := [\int \langle α_s dγ^y \rangle]_s^t ≡ \langle α_s, y_{s,t}^G \rangle \] and \[ [\int \langle α_s dγ^y \rangle]_s^t = α_s \circ y_t \].

Proof. By Theorem 3.26 below, \( \tilde{z}_{s,t} := \langle α_s, y_{s,t}^G \rangle \) is an almost additive functional and therefore by Lyons \[ 22 \] Theorem 3.3.1 there exists a unique additive functional \( z_{s,t} \) such that \( z_{s,t} ≡ \tilde{z}_{s,t} \).

Moreover,

\[ z_{s,t} ≡ \tilde{z}_{s,t} ≡ α_s (ψ(y_s, y_t)) ≡ α_s (y_t^1 x_{s,t}) \]

which shows that \( z := (z_0 = 0) = \tilde{z}_{s,t} \) is indeed a controlled rough path with values in \( V \).

Example 3.22 In the case that \( U = U^ψ \) so that

\[ α_t (ψ(y_t)) + α_t^1 (x_{s,t} \otimes (\cdot)) \approx 0 \]

we have that \( y_{s,t} := (ψ(y_s, y_t), (I \otimes y_t^1) X_{s,t}) \) and so

\[ \int_s^t \langle α_s dγ^y \rangle ≡ α_s (ψ(y_s, y_t)) + α_s^1 (I \otimes y_t^1) X_{s,t} \].

Example 3.23 If \( G = (\exp^V)^{-1}, U^V \), then by Lemma 3.13 we have that

\[ \int_s^t \langle α_s dγ^y \rangle ≡ α_s (\exp^V_{y_t} (y_t)) + α_s^1 (I \otimes y_t^1) X_{s,t} + α_t \left( \frac{1}{2} T_{y_t} \circ y_{s,t}^\top \otimes 2 X_{s,t} \right) \].
It turns out $f \alpha \in CRP^U_y \left(M, \hat{V}\right)$ (which was defined in Proposition 3.6) associates with $y$ in the way we expect:

**Theorem 3.24 (Associativity Theorem I)** Let us continue the notation in Theorem 3.21. If $f$ and $f \alpha := (f s_0, f \alpha) := \left(f s_0, f \alpha \right)$ are as in Proposition 3.6 and $z = (z, z^\dagger) = \int \langle \alpha, dy^\nabla \rangle$, then

$$\int \langle f, dz \rangle = \int \langle f \alpha, dy^\nabla \rangle,$$

or in other words,

$$\int \langle f, d \int \langle \alpha, dy^\nabla \rangle \rangle = \int \langle f \alpha, dy^\nabla \rangle.$$

**Proof.** We have the approximations

$$\left[\int \langle f \alpha, dy^\nabla \rangle \right]_{s,t}^{1} \approx f_s \alpha_s \left(y_t, y_t^\dagger\right) + \left[\left(f s_0 \left(I \otimes \alpha_s\right) + f \alpha\right) \left(I \otimes y_s^\dagger\right) \mathcal{X}_{s,t}\right]$$

$$= f_s \left(y_t, y_t^\dagger, y^\nabla \mathcal{X}_{s,t}\right) + \alpha_s \left(I \otimes y_s^\dagger\right) \mathcal{X}_{s,t} + f \left(I \otimes \alpha_s y_s^\dagger\right) \mathcal{X}_{s,t}$$

$$\approx f_s \left(z_{s,t}\right) + f \left(I \otimes z_{s,t}\right) \mathcal{X}_{s,t}$$

$$\approx f \left(\int \langle f, dz \rangle \right)_{s,t}^{1}. $$

As the first and last terms of this equation are additive functionals, they must be equal.

Secondly

$$\left[\int \langle f \alpha, dy^\nabla \rangle \right]_{s,t}^{1} = f_s \alpha_s \left(y_t^\dagger\right) = f_s z_{s,t}^* = \left[\int \langle f, dz \rangle \right]_{s,t}^{1}.$$

Thus, the two controlled rough paths are equal. 

**Remark 3.25** The $(\mathcal{G}, y)$-integrator $y^\nabla_{s,t}$ is helpful in easing notation so that the integral is simply written

$$\int_t^s \langle \alpha, dy^\nabla \rangle.$$

However, the $y^\nabla_{s,t}$ does hide a few terms. We could alternatively use the more explicit notation

$$\int_t^s \langle (\alpha, \alpha^\dagger), d \left(y^\nabla, \mathcal{X}\right) \rangle_{y^\nabla}$$

where $S^\nabla_{y^\dagger}(s)$ is the block matrix defined by

$$S^\nabla_{y^\dagger}(s) := \begin{pmatrix} I & S^\nabla Y^\nabla \left(y^\dagger\right) \\ 0 & I \otimes y_s^\dagger \end{pmatrix}$$

and $\langle \cdot, \cdot \rangle_{y^\nabla}$ is the “inner product” given by the matrix $S^\nabla_{y^\dagger}$. When $s$ is close to $t$, we have

$$\int_t^s \langle (\alpha, \alpha^\dagger), d \left(y^\nabla, \mathcal{X}\right) \rangle_{y^\nabla} \approx \alpha_s \alpha_s^\dagger \begin{pmatrix} I & S^\nabla Y^\nabla \left(y^\dagger\right) \\ 0 & I \otimes y_s^\dagger \end{pmatrix} \left(y^\nabla, \mathcal{X}_{s,t}\right)$$

$$= \alpha_s \left(y_t, y_t^\dagger + S^\nabla Y^\nabla \left(y^\dagger\right) \mathcal{X}_{s,t}\right) + \alpha_s^\dagger \left(I \otimes y_s^\dagger\right) \mathcal{X}_{s,t}. $$
3.4 Almost additivity result

The following theorem was the key ingredient in the proof of Theorem 3.21 on the existence of rough path integration in the manifold setting.

**Theorem 3.26** If \( G := (\psi, U) \) is a gauge and \( \alpha \in CRP^U_g(M, V) \), then \( \tilde{z}_{s,t} \in V \) defined as in Definition 3.24 is an almost additive functional.

The proof of Theorem 3.26 will be given after Corollary 3.29 which states that logarithms are “almost additive.” We first need a couple of lemmas. Recall from Definition 2.15 that Definition 3.20 is an almost additive functional.

**Lemma 3.27** We have

\[
S_{\tilde{U},U}^y \circ U(y_t, y_s) \approx_{\frac{1}{2}} U(y_t, y_s) \circ S_{\tilde{U},U}^y
\]

**Proof.** By the usual patching arguments it suffices to prove this lemma for \( M = \mathbb{R}^d \). In the Euclidean space setting the identity is trivial to prove since \( U(n, m) = I \) and \( S_{n,m}^{\tilde{U},U} = S_{m,n}^{\tilde{U},U} \).

**Lemma 3.28** Let \( K \) be a compact, convex set in \( \mathbb{R}^d \). If \( \psi \) is a logarithm with domain \( D \) and \( K \times K \subseteq D \), then there exists a \( C_K \) such that

\[
|\psi'(x) \psi(x,y) + \psi(y,z) - \psi'(x) \psi(x,z)| \leq C_K \max \{|\psi(x,y)|, |\psi(y,z)|, |\psi(x,z)|\}^3
\]

for all \( x, y, z \in K \).

**Proof.** We will use the notation \( |x,y,z| := \max \{|x-y|,|y-z|,|z-x|\} \) and write \( f(x,y,z) =_k g(x,y,z) \) iff \( f(x,y,z) = g(x,y,z) + O(|x,y,z|^k) \). Since \( \psi \) is zero on the diagonal and \( \psi'_y(y) = id \) for all \( y \), it follow from Taylor’s theorem (or see Theorem 2.24) that

\[
\psi'(x) =_{2} id + \psi''(y)(x-y) \quad \text{and} \quad \psi(x,y) =_{3} (y-x) + \frac{1}{2} \psi''(x)(y-x)^{\otimes 2}
\]

\[
=_{3} (y-x) + \frac{1}{2} \psi''(y)(y-x)^{\otimes 2}.
\]

(3.16)

from these approximations we learn,

\[
\psi(x,y) - \psi(x,z) =_{3} y - z + \frac{1}{2} \psi''(y) \left[ (y-x)^{\otimes 2} - (z-x)^{\otimes 2} \right]
\]

and

\[
\psi'(x) \psi(x,y) - \psi'(x) \psi(x,z) =_{3} [id + \psi''(y)(x-y) \otimes (\cdot)] (\psi(x,y) - \psi(x,z))
\]

\[
=_{3} y - z + \frac{1}{2} \psi''(y) \left[ (y-x)^{\otimes 2} - (z-x)^{\otimes 2} \right] + \psi''(y) [(x-y) \otimes (y-z)].
\]

As simple calculation now shows, with \( a = y-x \) and \( b = y-z \), that

\[
\frac{1}{2} \left[ (y-x)^{\otimes 2} - (z-x)^{\otimes 2} \right] + (x-y) \otimes (y-z) = -\frac{1}{2} [b^{\otimes 2} + b \otimes a - a \otimes b].
\]
Using Corollary 3.29 followed by Corollary 3.16 we find

\[
\psi'_{y'} (x) \psi(x, y) - \psi'_{y'} (x) \psi(x, z) = 3 \, y - z - \frac{1}{2} \psi''_{y'} (y) \left| y \right|^2 = - \left[ (z - y) + \frac{1}{2} \psi''_{y'} (y) (z - y) \right] = 3 - \psi(y, z).
\]

The bounds derived above are uniform over a compact set \( K \). Because of Eq. (3.16), we may replace \( O \left( [x, y, z]^3 \right) \) with \( O \left( \max \{ |\psi(x, y)|, |\psi(y, z)|, |\psi(x, z)| \} \right) \).

**Corollary 3.29** If \( (y_s, y_u) \) is a controlled rough path and \( \psi \) is a logarithm, there exists \( C_\psi, \delta_\psi > 0 \) such that if \( 0 \leq s \leq t \leq u \leq T \) and \( u - s \leq \delta_\psi \), then

\[
\left| \psi (y_t, y_u) - \psi (y_t, \cdot)_{y_t} \right|_{y_t} \leq C_\psi \omega (s, u)^{3/p}
\]

**Proof.** Around every point in \( y([0, T]) \), using our usual techniques, we can find a neighborhood \( \mathcal{W} \) such that \( \mathcal{W} \times \mathcal{W} \subseteq \mathcal{D} \) and maps to a convex open set by a chart. We can then use Remark 2.51 with a slightly modified version (which includes three variables instead of two) of Lemma 2.50 to create a global estimate. It is clear that we can choose a \( \delta \) such that \( u - s \leq \delta \) forces the path to lie within one of these sets \( \mathcal{W} \). Therefore, it suffices to prove the estimate locally. However, we can push forward the metric and \( \psi \) to a convex set on Euclidean space. The rest follows from the Lemma 3.28 and the fact that \( |\psi(y, y)| \leq C \omega (t)^{1/p} \) for all \( |t - s| \leq \delta \) for some \( C < \infty \) and \( \delta > 0 \). ■

**3.5 Proof of Theorem 3.26**

**Proof of Theorem 3.26** Let \( 0 \leq s \leq t \leq u \leq T \). Throughout this proof, we will use the notation \( \approx \) with respect to the times \( s \) and \( u \). To prove the statement, we need to show \( \tilde{z}_{s,t} + \tilde{z}_{t,u} \approx \tilde{z}_{s,u} \).

We begin by working on the three terms for \( \tilde{z}_{t,u} \) in the following equation

\[
\tilde{z}_{t,u} = \alpha_t \left( \psi(y_t, y_u) \right) + \alpha_t^I \left( I \otimes y_t^I \right) \mathcal{K} \, t, u + \alpha_t \left( S^G_{y_t} \circ y_t^I \otimes^2 \mathcal{K} \, t, u \right). \tag{3.17}
\]

Using Corollary 3.29 followed by Corollary 3.16 we find

\[
\alpha_t \left( \psi(y_t, y_u) \right) \approx \alpha_t \psi(y_t, \cdot)_{y_t} \left[ \psi(y_s, y_u) - \psi(y_s, y_t) \right] \\
\approx \alpha_t U(y_t, y_s) \left[ I + S^G_{y_t} \left( \psi(y_s, y_t) \otimes (\cdot) \right) \right] \left[ \psi(y_s, y_u) - \psi(y_s, y_t) \right] \\
\approx \left[ \alpha_t + \alpha_t^I x_{s,t} \otimes (\cdot) \right] \left[ I + S^G_{y_t} \left( \psi(y_s, y_t) \otimes (\cdot) \right) \right] \left[ \psi(y_s, y_u) - \psi(y_s, y_t) \right] \\
\approx \alpha_t \left[ I + S^G_{y_t} \left( \psi(y_s, y_t) \otimes (\cdot) \right) \right] \left[ \psi(y_s, y_u) - \psi(y_s, y_t) \right] + \alpha_t^I x_{s,t} \otimes \left[ \psi(y_s, y_u) - \psi(y_s, y_t) \right].
\]

Combining this equation with the estimates

\[
\psi(y_s, y_t) \approx y_t^I x_{s,t} \text{ and } \psi(y_s, y_u) - \psi(y_s, y_t) \approx y_t^I [x_{s,u} - x_{s,t}] = y_t^I x_{t,u},
\]

we find

\[
\tilde{z}_{s,t} + \tilde{z}_{t,u} \approx \tilde{z}_{s,u}.
\]
then shows,
\[
\alpha_t (\psi (y_t, y_u)) \approx \alpha_s [\psi (y_s, y_u) - \psi (y_s, y_t)] + \alpha_s (y_s^\dagger)^2 x_{s,t} \otimes x_{t,u} + \alpha_s^\dagger (I \otimes y_s^\dagger) x_{s,t} \otimes x_{t,u}.
\] (3.18)

By the definitions of \( CRP_X (M) \) and \( CRP^U_y (M, V) \) we have
\[
\alpha_s^\dagger (I \otimes y_s^\dagger) X_{t,u} \approx \alpha_s^\dagger (I \otimes U (y_t, y_s) y_s^\dagger) X_{t,u}
\]
\[
= \alpha_s^\dagger (I \otimes U (y_t, y_s)) (I \otimes y_s^\dagger) X_{t,u} \approx \alpha_s^\dagger (I \otimes y_s^\dagger) X_{t,u}.
\] (3.19)

Lastly by the definitions of \( CRP_X (M) \) and \( CRP^U_y (M, V) \) along with Lemma \ref{lem:3.27} with \( \hat{U} (m, n) = (\psi_m)_{s,n} \), we have and
\[
\alpha_t \left( S^G_{y_t} \circ y_t^\dagger \otimes X_{t,u} \right) \approx \alpha_t \left( S^G_{y_t} \circ U (y_t, y_s) \otimes y_s^\dagger \otimes X_{t,u} \right)
\]
\[
\approx \alpha_t \left( U (y_t, y_s) \circ S^G_{y_t} \circ y_s^\dagger \otimes X_{t,u} \right) \approx \alpha_s (S^G_{y_t} \circ y_s^\dagger \otimes X_{t,u})
\] (3.20)

Adding together Eqs. (3.18) - (3.20) to
\[
\tilde{z}_{s,t} = \alpha_s (\psi (y_s, y_t)) + \alpha_s^\dagger (I \otimes y_s^\dagger) X_{s,t} + \alpha_s (S^G_{y_s} \circ y_s^\dagger \otimes X_{s,t})
\]
while making use Chen’s identity in Eq. (2.1) shows

\[
\tilde{z}_{s,t} + \tilde{z}_{t,u} \approx \alpha_s (\psi (y_s, y_u)) + \alpha_s^\dagger (I \otimes y_s^\dagger) X_{s,u} + \alpha_s (S^G_{y_s} \circ y_s^\dagger \otimes X_{s,u}) = \tilde{z}_{s,u}.
\]

\[\blacksquare\]

### 3.6 A Map From \( CRP^U_y (M, V) \) to \( CRP^\hat{U}_y (M, V) \)

Suppose that \( G = (\psi, U) \) and \( \hat{G} = (\tilde{\psi}, \tilde{U}) \) are two gauges on \( M \). Generally, if \( \alpha := (\alpha, \alpha^\dagger) \in CRP^U_y (M, V) \), there is no reason to expect it also to be an element of \( CRP^\hat{U}_y (M, V) \). However, the main theorem [Theorem \ref{thm:3.32}] of this section shows there is a “natural” bijection between \( CRP^U_y (M, V) \) and \( CRP^\hat{U}_y (M, V) \) which preserves the notions of integration. The following proposition is needed in the proof of Theorem \ref{thm:3.32} and moreover motivates the statement of the theorem.

**Proposition 3.30** If \( G = (\psi, U) \) and \( \hat{G} = (\tilde{\psi}, \tilde{U}) \) are two gauges on \( M \) and \( Y = (y^\dagger, y) \in CRP_X (M) \), then
\[
y^\hat{G}_{s,t} \approx y^G_{s,t} + \left( S^G_{y_s} \left( (y_s^\dagger)^2 X_{s,t} \right), 0 \right),
\] (3.21)

where \( y^G_{s,t} \) and \( y^\hat{G}_{s,t} \) are as in Definition \ref{def:3.26}.

**Proof.** From Proposition \ref{prop:3.17}
\[
\psi (y_s, y_t) - \tilde{\psi} (y_s, y_t) \approx \frac{1}{2} S^\psi \cdot \psi \cdot (\psi (y_s, y_t) \otimes \psi (y_s, y_t))
\]
\[
\approx \frac{1}{2} S^\psi \cdot \psi \cdot ((y_s^\dagger \otimes y_s^\dagger) [x_{s,t} \otimes x_{s,t}]) = S^\hat{\psi} \cdot \hat{\psi} \cdot ((y_s^\dagger)^2 X_{s,t})
\]

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wherein we have used \( S_{\tilde{\psi},\psi}^\theta \) is symmetric and \( X = (x, X) \) is a geometric rough path for the last equality. Making use of this estimate it now follows that

\[
y_{s,t}^\theta - y_{s,t}^\theta = \left( \psi(y_s, y_t) - \tilde{\psi}(y_s, y_t) + \left( S_{y_s}^\theta - S_{y_t}^\theta \right) \left( (y_t)^{\otimes 2} X_{s,t} \right), 0 \right)
\approx \frac{3}{3} \left( \left( S_{y_s}^\theta \right) + S_{y_t}^\theta \right) \left( (y_t)^{\otimes 2} X_{s,t} \right), 0 \right).
\]

(3.22)

On the other hand, by Lemma 3.11,

\[
S_{\tilde{\psi},\psi}^\theta = S_{\tilde{\psi},\tilde{U}} + S_{\psi,\psi}^\theta = S_{\tilde{\psi},\tilde{U}} + S_{\psi,\psi}^\theta
\]

which combined with Eq. (3.22) gives Eq. (3.21).

**Corollary 3.31** The integral, \( \int \langle \alpha, \psi U \rangle \) only depends on the choice of parallelism \( U \) and not on the logarithm used to make the gauge \( G = (\psi, U) \).

**Proof.** From Proposition 3.30 with \( U = \tilde{U} \), it follows that

\[
\int_s^t \langle \alpha, \psi U \rangle \approx \langle \alpha_s, y_{s,t}^\theta \rangle \approx \frac{3}{3} \int_s^t \langle \alpha, \psi U \rangle
\]

from which it follows that the two additive functionals, \( \int \langle \alpha, \psi U \rangle \) and \( \int \langle \alpha, \psi ^\theta \rangle \), must be equal.

If \( \alpha = (\alpha, \alpha^\dagger) \in CRP^U_y (M, V) \) and \( U \neq \tilde{U} \), then

\[
\langle \alpha_s, y_{s,t}^\theta \rangle \approx \langle \alpha_s, y_{s,t}^\theta \rangle \approx \left( \left( S_{\psi,\psi}^\theta \left( (y_t)^{\otimes 2} X_{s,t} \right), 0 \right) \right) = \langle \tilde{\alpha}_s, y_{s,t}^\theta \rangle
\]

(3.23)

where \( \tilde{\alpha}_s \) is defined in Eq. (3.24) below. The identity in Eq. (3.23) suggests the following theorem.

**Theorem 3.32** The map

\[
\alpha_s = (\alpha_s, \alpha_s^\dagger) \rightarrow \tilde{\alpha}_s := (\tilde{\alpha}_s, \tilde{\alpha}_s^\dagger) := \left( \alpha_s, \alpha_s^\dagger + \alpha_s S_{\psi,\psi}^\theta y_s \otimes I \right)
\]

(3.24)

is a bijection from \( CRP^U_y (M, V) \) to \( CRP^\tilde{U}_y (M, V) \) such that

\[
\int \langle \alpha, \psi U \rangle = \int \langle \alpha, \psi U \rangle.
\]

(3.25)

**Proof.** The only thing that is really left to prove here is the assertion that \( \tilde{\alpha} \in CRP^\tilde{U}_y (M, V) \).

First we prove that item 3 of Definition 3.1 holds for \( \tilde{\alpha} \).

From Theorem 3.15 with \( m = y_s \) and \( n = y_t \), we find

\[
U(y_s, y_t) \tilde{U}(y_s, y_t)^{-1} \approx I + S_{\psi,\psi}^\theta \left( \psi(y_s, y_t) \otimes (\cdot) \right)
\]

and then combining this result with Corollary 2.28 shows

\[
\tilde{U}(y_t, y_s) \approx U(y_t, y_s) \left[ I + S_{\psi,\psi}^\theta \left( \psi(y_s, y_t) \otimes (\cdot) \right) \right].
\]

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From this equation and the fact that $\alpha \in \mathcal{C}^\infty \! (M, V)$, we learn

$$
\alpha_t \bar{U} (y_t, y_s) - \alpha_s \approx \alpha_t U (y_t, y_s) \left[ I + S^\bar{U}_{y_s} (\psi (y_s, y_t) \otimes \cdot) \right] - \alpha_s
$$

$$
\approx \frac{1}{2} (\alpha_s + \alpha_t^\perp x_{s,t}) \left[ I + S^\bar{U}_{y_s} (\psi (y_s, y_t) \otimes \cdot) \right] - \alpha_s
$$

$$
\approx \alpha_t^\perp x_{s,t} + \alpha_s S^\bar{U}_{y_s} (y_t^\perp x_{s,t} \otimes \cdot) = \tilde{\alpha}_s^\perp (x_{s,t} \otimes \cdot)
$$

as desired.

Next we check item 4 of Definition 3.1. We are given

$$
\begin{align*}
0 & \approx \alpha_t^\perp \circ (I \otimes U (y_t, y_s)) - \alpha_s^\perp \\
& = \tilde{\alpha}_t^\perp \circ (I \otimes \bar{U} (y_t, y_s)) - \tilde{\alpha}_s^\perp \\
& \quad - \alpha_t \circ S^\bar{U}_{y_s} \circ (y_t^\perp \otimes U (y_t, y_s)) + \alpha_s \circ S^\bar{U}_{y_s} \circ (y_t^\perp \otimes I)
\end{align*}
$$

wherein we have used that $U (y_s, y_t) \approx \bar{U} (y_s, y_t)$. We therefore must show the last line is approximately 0. However, by Lemma 3.27, we have $S^\bar{U}_{y_s} \circ U (y_t, y_s) \otimes 2 \approx U (y_t, y_s) \circ S^\bar{U}_{y_s}$. Thus

$$
\begin{align*}
\alpha_t \circ S^\bar{U}_{y_s} \circ (y_t^\perp \otimes U (y_t, y_s)) & - \alpha_s \circ S^\bar{U}_{y_s} \circ (y_t^\perp \otimes I) \\
\approx \alpha_t \circ S^\bar{U}_{y_s} \circ (U (y_t, y_s) y_t^\perp \otimes U (y_t, y_s)) & - \alpha_s \circ S^\bar{U}_{y_s} \circ (y_t^\perp \otimes I) \\
\approx [\alpha_t \circ U (y_t, y_s) & - \alpha_s] \left[ S^\bar{U}_{y_s} \circ (y_s^\perp \otimes I) \right] \approx 0.
\end{align*}
$$

\section{4 Integrating One-Forms Along a CRP}

\textbf{Lemma 4.1} Let $V$ be a Banach space and $U$ be a parallelism on $M$. If $\alpha \in \Omega^1 (M, V)$ is a $V$-valued smooth one-form on $M$, then

$$
\alpha_n \circ U (n, m) - \alpha_m = _2 \nabla_U^{\psi (m, n)} \alpha
$$

where $\nabla_U$ is the covariant derivative defined in Remark 2.20.

\textbf{Proof.} As this is a local result we may assume $M = \mathbb{R}^d$. Then by Taylor’s theorem,

$$
\begin{align*}
\alpha_n \circ U (n, m) &= \alpha_m + D [\alpha_n \circ U (\cdot, m)] (m) (n - m) + O \left( |n - m|^2 \right) \\
& = \alpha_m + \nabla_{(n-m)}^{U \psi (n, m)} \alpha + O \left( |n - m|^2 \right) \\
& = \alpha_m + \nabla_{(m,n)}^{U \psi (m, n)} \alpha + O \left( |\psi (m, n)|^2 \right).
\end{align*}
$$

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A key point we have used here is Corollary 2.28 since we should have
\[
\nabla_{v_{m}}^{U} \alpha = \frac{d}{dt} \bigg|_{0} \left[ \alpha \circ U \left( m, \sigma_{t} \right)^{-1} \right]
\]
where \( \sigma_{0} = v_{m} \) but according to Corollary 2.28 \( U \left( m, n \right)^{-1} = U \left( n, m \right) \).

Suppose that \( \alpha \in \Omega^{1} \left( M, V \right) \) is a \( V \)–valued one-form and \( U \) is a parallelism on \( M \). We wish to take \( \alpha_{s}^{U} = \alpha_{y_{s}} \). Making use of Lemma 4.1 we find
\[
\alpha_{t}^{U} \circ U \left( y_{t}, y_{s} \right) - \alpha_{s} \approx \frac{1}{2} \nabla_{y_{t}}^{U} \psi \left( y_{s}, y_{t} \right) \alpha \approx \frac{1}{2} \nabla_{y_{t}}^{U} y_{s} \alpha
\]
and this computation suggests the following proposition.

**Proposition 4.2** Suppose that \( \alpha \in \Omega^{1} \left( M, V \right) \) is a \( V \)–valued one-form and \( U \) is a parallelism on \( M \), then
\[
\alpha_{s}^{(y, U)} : = \left( \alpha_{y_{s}}, \alpha_{s}^{(y, U)} \right) := \left( \alpha \big|_{T_{y_{s}}M}, \nabla_{y_{s}}^{U} \alpha \right) \in CRP^{U}_{y} \left( M, V \right).
\]

**Proof.** In light of how \( \alpha_{s}^{(y, U)} \) has been defined and of Eq. 4.1, we need only verify Item 4 in Definition 3.1 is satisfied. To this end, suppose that \( w \in W \), then
\[
\alpha_{t}^{(y, U)} \circ \left( I \otimes U \left( y_{t}, y_{s} \right) \right) \left( w \otimes \cdot \right) = \left( \nabla_{y_{t}}^{U} \alpha \right) U \left( y_{t}, y_{s} \right)
\]
\[
\approx \left( \nabla_{y_{t}}^{U} \left( y_{s}, y_{t} \right) \right) U \left( y_{t}, y_{s} \right) \]
wherein we have used Inequality (2.16) along with Corollary 2.28 in the last line. Since for \( v_{m} \in T_{m}M \) the function \( F \left( n \right) := \left( \nabla_{U \left( n, m \right)}^{U} v_{m} \alpha \right) U \left( n, m \right) \in L \left( T_{m}M, V \right) \) is smooth, it follows by Taylor’s theorem that \( F \left( n \right) = F \left( m \right) \) which translates to
\[
\left( \nabla_{U \left( n, m \right)}^{U} v_{m} \alpha \right) U \left( n, m \right) = 1 \nabla_{v_{m}}^{U} \alpha.
\]
Taking \( m = y_{s}, n = y_{t} \), and \( v_{m} = y_{s}^{I}w \) in this estimates shows
\[
\left( \nabla_{U \left( y_{t}, y_{s} \right)}^{U} y_{s}^{I}w \alpha \right) U \left( y_{t}, y_{s} \right) \approx \nabla_{y_{s}^{I}w \alpha}
\]
which combined with Eq. 4.2 completes the proof. 

**Theorem 4.3** If \( \alpha \in \Omega^{1} \left( M, V \right) \) is a \( V \)–valued one-form, then the integral
\[
\int \left\langle \alpha^{(y, U)}, dy \right\rangle,
\]
is independent of any choice of gauge \( \mathcal{G} = (\psi, U) \) on \( M \). In the future we denote this integral more simply as \( \int \left\langle \alpha, dy \right\rangle \).

**Proof.** Suppose that \( U \) and \( \tilde{U} \) are two parallelisms. According to Theorem 3.32 it suffices to show
\[
\alpha_{s}^{(y, \tilde{U})} = \alpha_{s}^{(y, U)} + \alpha_{y_{s}} \psi_{y_{s}} \left[ y_{s}^{I} \otimes I \right].
\]
We will see that Eq. (4.3) is a fairly direct consequence of Example 3.10 which, when translated to the language of forms \(^2\) rather than vector fields, states
\[ \nabla_{v_m} \alpha = \tilde{\nabla}_{v_m} \alpha - \alpha \circ S^{U,V}_m (v_m \otimes \cdot) \, . \] (4.4)
So for \(w \in W\), we have
\[ \alpha_s^{(y,U)} w = \tilde{\nabla}_{y_s} \alpha = \nabla_{y_s} \alpha + \alpha_{y_s} S^{U,V}_m (y_s^t w \otimes \cdot) \]
\[ = \alpha_s^{(y,U)} w + \alpha_{y_s} S^{U,V}_m (y_s^t w \otimes \cdot) \]
which proves Eq. (4.3). \( \blacksquare \)

Let us now record a number of possible different expressions for computing \(\int_s^t \alpha (dy)\) depending on the choice of gauge we make.

**Proposition 4.4** Let \(G = (\psi, U)\) be a gauge. There exists a \(\delta > 0\) such that for \(s < t\) and \(t - s < \delta\), the approximation
\[ \left[ \int \alpha (dy) \right]_{s,t}^1 \approx \alpha_{y_s} (\psi (y_s, y_t)) + \left[ (\nabla^{U,V}_C)_{y_s} + \alpha_{y_s} S^{U,V}_y \right] \circ y_s^{\otimes 2} X_{s,t} \]
holds.

In the case that we take \(U = U^\psi\), we get a slightly simpler formula.

**Corollary 4.5** Let \(\psi\) be a logarithm. There exists a \(\delta > 0\) such that for \(s < t\) and \(t - s < \delta\), the approximation
\[ \left[ \int \alpha (dy) \right]_{s,t}^1 \approx \alpha_{y_s} (\psi (y_s, y_t)) + \frac{1}{2} \alpha_{y_s} \circ T^{\psi}_{y_s} \circ y_s^{\otimes 2} X_{s,t} \]
holds.

If we have a covariant derivative in hand, we can rewrite our integral approximation slightly.

**Example 4.6** Let \(\nabla\) be a covariant derivative on \(M\). There exists a \(\delta > 0\) such that for \(s < t\) and \(t - s < \delta\), the approximation
\[ \left[ \int \alpha (dy) \right]_{s,t}^1 \approx \alpha_{y_s} \left( \exp^{\nabla}_{y_s} (y_t) \right) + \frac{1}{2} \alpha_{y_s} \circ T^{\nabla}_{y_s} \circ y_s^{\otimes 2} X_{s,t} \]
holds. Indeed this follows immediately from Proposition 4.4, Lemma 3.13, and the fact that
\[ (\nabla \alpha)_{y_s} (v_m, w_m) := v_m [\alpha (W)] - \alpha (\nabla_{v_m} W) \]
\[ = \left( \alpha_{y_s} \circ W (\cdot) \right)_{y_s} (v_m) - \alpha (\nabla_{v_m} W) \]
where \(W\) is any vector field such that \(W (m) = w_m\). Choosing \(W = U (\cdot, m) w_m\), we have
\[ \nabla_{v_m} W = \nabla_{v_m} U (\cdot, m) w_m \]
\[ = 0 \]
by the definition of parallel translation.

\(^2\)Recall that \((\nabla_{v_m} \alpha) (Y (m)) := v_m [\alpha (Y)] - \alpha (\nabla_{v_m} Y)\) and it is this latter negative sign which accounts for the negative sign in Eq. (4.4) above.
4.1 Integration of a One-Form Using Charts

It is easy to see that by independence of gauges, the integral of a one-form along \((y_s, y^*_s)\) is an object which we only need to compute locally. As mentioned in Remark 2.22 we have an example of a local gauge by using a chart. Plugging this formula into the integral approximation from Corollary 4.5, we get the following.

**Corollary 4.7** Let \(\phi\) be a chart on \(M\). For all \(a, b \in [0, T]\) such that \(y[a, b] \subset D(\phi)\), we have the approximation

\[
\left[ \int \alpha \left( d\psi \right) \right]^{1}_{s,t} \approx \alpha_{y_s} \left( (d\phi_{y_s})^{-1} \left( \phi(y_t) - \phi(y_s) \right) \right) + d \left( \alpha_{(\cdot)} \circ (d\phi_{(\cdot)})^{-1} \right) d\psi_{y_s} \circ y^*_s \otimes^2 X_{s,t} \tag{4.5}
\]

holds for all \(s < t \in [a, b]\).

Although this formula looks a bit complicated, it may be reduced to something that makes more sense. First, note that

\[
\alpha_m \circ (d\phi_m)^{-1} = \left( (\phi^{-1})^* \alpha \right)_{\phi(m)}.
\]

Thus we can reduce the right hand side Eq. (4.5) to

\[
\left( (\phi^{-1})^* \alpha \right)_{\phi(y_s)} \left( \phi(y_t) - \phi(y_s) \right) + d \left( (\phi^{-1})^* \alpha \right)_{\phi(y_s)} d\psi_{y_s} \circ y^*_s \otimes^2 X_{s,t}.
\]

Now, if we recall Notation 2.40 we see that this is approximately equal to another rough integral. More precisely

\[
\left[ \int \alpha \left( d\psi \right) \right]^{1}_{s,t} \approx \left[ \int (\phi^{-1})^* \alpha \left( d\phi \psi \right) \right]^{1}_{s,t}.
\]

However, additive functionals are unique up to this order, so in fact

\[
\left[ \int \alpha \left( d\psi \right) \right]^{1}_{s,t} = \left[ \int (\phi^{-1})^* \alpha \left( d\phi \psi \right) \right]^{1}_{s,t},
\]

which is a relation which should hold under any reasonable integral. This is summarized in the following theorem which gives us an alternative way of defining this integral.

**Theorem 4.8** The integral, \(\int \alpha \left( d\psi \right)\), is the unique \(V\)–valued rough path controlled by \(X\) on \([0, T]\) starting at \(0\) determined by

1. \(\left[ \int \alpha \left( d(y, y^*) \right) \right]^{1}_{s,t} = \left[ \int (\phi^{-1})^* \alpha \left( d\phi \psi \right) \right]^{1}_{s,t}\) for any chart and \(s < t \in [0, T]\) such that \(y([s, t]) \subset D(\phi)\)
2. \(\left[ \int \alpha \left( d\psi \right) \right]^{1}_{s} = \alpha_{y_s} \circ y^*_s\)
Theorem 4.9 (Associativity Theorem II) Suppose that $y \in CRP(M), \alpha \in \Omega^1(M,V)$, and $K : M \to L(V,\tilde{V})$ is a smooth function so that $K\alpha \in \Omega^1(M,\tilde{V})$. If $z = \int \alpha(dy) \in CRP(V)$, then

$$\int (K\alpha)(dy) = \int \langle K_\ast(y), dz \rangle =: \int \langle K_\ast(y), d \int \alpha(dy) \rangle,$$

where $K_\ast(y) = (K(y), K_{sy\gamma} y^1) \in CRP_X(Hom(V,V')).$

**Proof. Method 1:** Letting $G = (\psi, U)$ be any gauge, we define $f := (f, f^\dagger) \in CRP_X(Hom(V,\tilde{V}))$ by the formula

$$f_s := K(y_s) \quad \text{and} \quad f^\dagger_s := K_{sy\gamma} y^1_s$$

and $\alpha^{(y,U)}$ as in Proposition 4.2 (see Proposition 4.10 below to see why $f \in CRP_X(Hom(V,\tilde{V})).$

Then by Theorem 3.24, we have

$$\int \langle f\alpha^{(y,U)}, dy^G \rangle = \int \langle f, dz \rangle \quad (4.6)$$

where $z = \int \langle \alpha^{(y,U)}, dy^G \rangle = \int \langle \alpha(dy) \rangle$. The right hand side in Equation (4.6) is simply $\int \langle K_\ast(y), dz \rangle$ while the $f\alpha^{(y,U)}$ term on the left hand side can be recognized as $(K\alpha)^{(y,U)}$. Indeed, by the product rule with $\nabla^U$, we have

$$(K\alpha)^{(y,U)}_s = \left( K(y_s) \alpha_{|T_y M, \nabla^U_y} \right)\right) = \left( K\alpha_{|T_y M, K_{sy\gamma} y^1} + K(y_s) \nabla^U \alpha^{(y,U)}_s \right) = \left( f_s \alpha_{s}, f^\dagger_s \alpha^{(y,U)}_s \right) = f\alpha^{(y,U)}_s.$$

Thus

$$\int (K\alpha)(dy) := \int \langle (K\alpha)^{(y,U)}_s, dy^G \rangle = \int \langle f\alpha^{(y,U)}, dy^G \rangle = \int \langle K_\ast(y), dz \rangle.$$

**Method 2:** By a simple patching argument, this is really a local result and hence using the chart definitions of integration it suffices to check this result in the case $M$ is an open subset of $\mathbb{R}^d$. First we check the derivative processes. From the definitions we have

$$z^\dagger_s = \alpha_{y_s} \circ y^\dagger_s \quad \text{and} \quad \left[ \int (K\alpha)(dy) \right]_s = (K\alpha)_{y_s} \circ y^\dagger_s = K(y_s) \alpha_{y_s} \circ y^\dagger_s = K(y_s) z^\dagger_s.$$

Thus

$$\left[ \int (K\alpha)(dy) \right]_s = K(y_s) z^\dagger_s.$$
On the other hand,

\[ \int (K_s (y) \cdot dx) \bigg|_{s \to t} = [K (\gamma)]_s z^s_s = K (\gamma) z^s_s. \]

Similarly for the paths

\[ z_{s,t} \approx \alpha (y_{s,t}) + \alpha'_{y_s} y_s^{\otimes 2} X_{s,t}. \]

and so

\[
\begin{align*}
\int (K) (dy) \bigg|_{s \to t} &\approx (K) y_{s,t} + (K) y_s^{\otimes 2} X_{s,t} \\
&= K (y_s) y_{s,t} + K (y_s) y_s^{\otimes 2} X_{s,t} + [K' (y_s) y_s^{\otimes 2} X_{s,t}]
\end{align*}
\]

On the other hand

\[
\begin{align*}
\int (K) (dy) \bigg|_{s \to t} &\approx (K) y_{s,t} + (K) y_s^{\otimes 2} X_{s,t} \\
&= K (y_s) y_{s,t} + K' (y_s) y_s^{\otimes 2} X_{s,t}.
\end{align*}
\]

Comparing these expressions completes the proof. 

4.2 Push-forwards of Controlled Rough Paths

Let \( M = M^d \) and \( \tilde{M} = \tilde{M}^d \) be manifolds. Let \( f : M \to \tilde{M} \) be smooth and suppose \( y_s = (y_s, y_s') \) is a controlled rough path in CRP_\( \tilde{M} \). In Definition 4.11 below, we are going to give a definition of the push-forward of \( y \) by \( f \) which generalizes Example 2.55.

Proposition 4.10 The pair \( \left( f (y_s), f_s \circ y_s' \right) \) is an element of CRP_\( \tilde{M} \).

Proof. Suppose \( \tilde{\phi} \) is a chart on \( \tilde{M} \) such that \( f \circ y ([a, b]) \subseteq D (\tilde{\phi}) \). We must show that

\[
\left| \tilde{\phi} \circ f (y_t) - \tilde{\phi} \circ f (y_s) - d \tilde{\phi} \circ f_s y_s^{\otimes 2} x_{s,t} \right| \leq C_{\tilde{\phi}, a, b} \omega (s, t)^{2/p}
\]

(4.7)

and

\[
\left| d \tilde{\phi} \circ f_s y_s^{\otimes 2} x_{s,t} \right| \leq C_{\tilde{\phi}, a, b} \omega (s, t)^{1/p}
\]

(4.8)

hold for some \( C_{\phi, a, b} \) for all \( s \leq t \) in \([a, b]\). We can again use our proof strategy outlined in Remark 2.51 to treat this problem in nice neighborhoods. We leave it to the reader to follow the pattern of earlier proofs to see that we can assume without loss of generality that there is a chart \( \phi \) on \( M \) such that \( y ([a, b]) \subseteq D (\phi) \) and \( R (\phi) \) is convex. Which these simplifications, we note that \( (z_s, z_s') := (\phi (y_s), d \phi \circ y_s') \) is a controlled rough path on \( R (\phi) \) and the function \( F := \tilde{\phi} \circ f \circ \phi^{-1} : R (\phi) \to R (\tilde{\phi}) \) is a map between Euclidean spaces. Therefore Inequalities (4.7) and (4.8) reduce to the fact that the pair \( (F (z_s), F' (z_s) \circ z_s') \) is a controlled rough path in \( \mathbb{R}^d \).
(which is trivial by applying Taylor’s theorem after we check that we get the correct terms); indeed, by a simple computation, we have

\[ F'(z_s) \circ z^1_s = d\tilde{\phi} \circ f_s \circ (d\phi^{-1})_z \circ d\phi_y \circ y^1_s \]

\[ = d\tilde{\phi} \circ f_s \circ (d\phi_y)^{-1} \circ d\phi_y \circ y^1_s \]

\[ = d\tilde{\phi} \circ f_s y^1_s \]

and clearly \( F(z_s) = \tilde{\phi} \circ f(y_s) \).

**Definition 4.11** The **push-forward** of \( y \) denoted by \( f_\ast y \) or \( f_\ast (y, y^1) \) is the rough path controlled by \( X \) with path \( f(y_s) \) and derivative process \( f_\ast y_s \). If \( M = \mathbb{R}^d \), we will abuse notation and write \( f_\ast y_s \) to mean \( (f(y_s), df \circ y^1_s) \) (i.e. we forget the base point on the derivative process).

**Remark 4.12** The push-forward operation on elements in \( \text{CRP}_X(M) \) is clearly covariant, i.e. if \( f : M \rightarrow N \) and \( g : N \rightarrow P \) are two smooth maps of manifolds, \( M, N, \) and \( P \), then \( (g \circ f)_\ast (y) = g_\ast (f_\ast (y)) \).

This definition is consistent with how we defined the integral of a one-form along a controlled rough path in the sense that we have a fundamental theorem of calculus. Let \( V \) be a Banach space.

**Theorem 4.13** Let \( y_s = (y_s, y^1_s) \in \text{CRP}_X(M) \) and \( f \) be a smooth function from \( M \) to \( V \). Then

\[ f(y_s) - f(y_0) = \left[ \int df \, [dy] \right]_{0,s}^1 \]

where \( df \) is interpreted as a one-form. Since we have \( df \circ y^1_s = \left[ \int df \, [dy] \right]_{s,t}^1 \), we have the equality

\[ f_\ast (y, y^1) - (f(y_0), 0) = \int df \, (dy) \]

**Proof.** Although there are ways to do this proof without much machinery, we find it more instructive to work on a Riemannian manifold with the Levi-Civita covariant derivative. Since we have proved that the integral is independent of choice of metric, it does not matter which one we pick. With this in mind, we have the approximation

\[ \left[ \int df \, [dy] \right]_{s,t}^1 \approx df_{y_s} \left( \exp_{y_s}^{-1}(y_t) \right) + (\nabla df)_{y_s} \left[ y^1_s \otimes x_{s,t} \right] \]

and as \( \nabla df \) is symmetric, it follows that

\[ \left[ \int df \, [dy] \right]_{s,t}^1 \approx df_{y_s} \left( \exp_{y_s}^{-1}(y_t) \right) + \frac{1}{2} (\nabla df)_{y_s} \left[ y^1_s \otimes x_{s,t} \otimes x_{s,t} \right] \]

\[ \approx df_{y_s} \left( \exp_{y_s}^{-1}(y_t) \right) + \frac{1}{2} (\nabla df)_{y_s} \left[ \exp_{y_s}^{-1}(y_t) \otimes 2 \right] \]

\[ \approx f(y_t) - f(y_s) \]
The last approximation above follows from Taylor’s Theorem on manifolds (Theorem 5.1 in the Appendix). Note here that $f(y_t) - f(y_s)$ is additive so that

$$\left[ \int df \frac{d}{dy} \right]_{s,t}^1 = f(y_t) - f(y_s).$$

\[ \square \]

**Remark 4.14** If $M \subseteq \mathbb{R}^k = W$, and $(y_s, y^\dagger_s) \in CRP_X(M)$, and $I : M \to W$ denotes the identity (or embedding) map, then denoting $(z_s, z^\dagger_s) := I_*(y_s, y^\dagger_s)$ we have

$$z_s = y_s, \quad z^\dagger_s = \pi_2 \circ y^\dagger_s$$

where $\pi_2$ is the projection of the tangent vector component (i.e. it forgets the base point). We can associate to it a unique rough path $(y, Y)$ in $\mathbb{R}^k$ such that

$$(z_s \otimes z^\dagger_s) \equiv Y_{s,t}.$$ 

In this case, this is a rough path in the embedded sense (see [3]) since

$$[I (y_s) \otimes Q (y_s)] \equiv [I (y_s) \otimes Q (y_s)] [z_s^\dagger \otimes z^\dagger_s] \equiv Y_{s,t} = 0$$

as $Q(y_s) \circ z^\dagger_s = 0$ where $Q = I - P$ and $P(x)$ is orthogonal projection onto the tangent space at $x$.

Lastly, we have a relation between push-forwards of paths and pull-backs of one-forms.

**Theorem 4.15** Let $f : M \to \tilde{M}$, let $y_s = (y_s, y^\dagger_s) \in CRP_X(M)$ and let $\hat{\alpha} \in \Omega^1 \left( \tilde{M}, V \right)$. Then

$$\left[ \int f^* \alpha (dy) \right] = \left[ \int \alpha (d(f, y)) \right].$$

Moreover

$$\int f^* \alpha (dy) = \int \alpha (d(f, y))$$

**Proof.** This is a statement we only have to prove locally. Indeed for each $s \in [0, T]$, there are charts $\phi^*$ and $\hat{\phi}^*$ on $M$ and $\tilde{M}$ respectively such that $y_s \in D(\phi^*)$ and $f(y_s) \in D(\hat{\phi}^*)$ which are open. We take $U_s := f^{-1} \left( D(\hat{\phi}^*) \cap D(\phi^*) \right)$ and shrink it if necessary so that $V_s = \phi(U_s)$ is convex. Thus if we can prove that Eq. (4.9) holds whenever $y([a, b]) \subseteq U$ such that $\phi(U)$ is convex and such that $f(y([a, b])) \subseteq D(\hat{\phi})$, we will be done. We do this now:
By Theorem 4.8 the fact that pull-backs are contravariant, and that push-forwards are covariant, we have

\[
\left[ \int f^* \alpha (dy) \right]_{s,t}^1 = \left[ \int (\phi^{-1})^* f^* \alpha (d\phi_* y) \right]_{s,t}^1
\]

\[
= \left[ \int (f \circ \phi^{-1})^* \alpha (d\phi_* y) \right]_{s,t}^1
\]

\[
= \left[ \int (\tilde{\phi}^{-1} \circ \tilde{\phi} \circ f \circ \phi^{-1})^* \alpha (d\phi_* y) \right]_{s,t}^1
\]

\[
= \left[ \int (\tilde{\phi} \circ f \circ \phi^{-1})^* \left( (\tilde{\phi}^{-1})^* \alpha \right) (d\phi_* y) \right]_{s,t}^1
\]

\[
= \left[ \int (\tilde{\phi}^{-1})^* \alpha \left( d \left( (\tilde{\phi} \circ f \circ \phi^{-1}) \circ \phi_* y \right) \right) \right]_{s,t}^1
\]

where the last step is just Eq. (4.9) on Euclidean space. This is a simple computation (for example, see the appendix of [3]). Thus, we have

\[
\left[ \int f^* \alpha (dy) \right]_{s,t}^1 = \left[ \int (\tilde{\phi}^{-1})^* \alpha \left( d \left( (\tilde{\phi} \circ f \circ \phi^{-1}) \circ \phi_* y \right) \right) \right]_{s,t}^1
\]

\[
= \left[ \int (\tilde{\phi}^{-1})^* \alpha \left( d \left( \tilde{\phi}_* (f\circ y) \right) \right) \right]_{s,t}^1
\]

\[
= \left[ \int \alpha \left( d \left( f\circ y \right) \right) \right]_{s,t}^1
\]

The fact that

\[
\left[ \int f^* \alpha (dy) \right]_{s,t}^1 = \left[ \int \alpha \left( d \left( f\circ y \right) \right) \right]_{s,t}^1
\]

is trivial. ■

5 Appendix

5.1 Taylor Expansion on a Riemannian manifold

Let \((M, g)\) be a Riemannian manifold, \(\nabla\) be the Levi-Civita covariant derivative, \(\exp (tv)\) be the geodesic flow, and \(//_t (\sigma)\) denote parallel translation relative to \(\nabla\). Recall that Taylor’s formula with integral remainder states for any smooth function \(g\) on \([0, 1]\), that

\[
G (1) = \sum_{k=0}^{n} \frac{1}{n!} G^{(k)} (0) + \frac{1}{n!} \int_{0}^{1} G^{(n+1)} (t) (1 - t)^n \, dt. \tag{5.1}
\]
We now apply this result to \( G(t) := f(\exp_m(tv)) \) where \( f \in C^\infty(M) \), \( v \in T_m M \) and \( m \in M \). To this end let \( \sigma(t) := \exp(tv) \) so that \( \nabla \sigma(t)/dt = 0 \). It then follows that

\[
\dot{G}(t) = \frac{d}{dt} \left( \sigma(t) \right) = df_{\sigma(t)}(\dot{\sigma}(t)),
\]

\[
\dot{G}(t) = \frac{d}{dt} df_{\sigma(t)}(\dot{\sigma}(t)) = (\nabla_{\dot{\sigma}(t)} df)(\dot{\sigma}(t)) + df_{\sigma(t)} \left( \frac{\nabla}{dt} \dot{\sigma}(t) \right) = (\nabla_{\dot{\sigma}(t)} df)(\dot{\sigma}(t)) = (\nabla df)(\dot{\sigma}(t) \otimes \dot{\sigma}(t)).
\]

Therefore we may conclude that

\[
f(\exp_m(v)) = G(1) = \sum_{k=0}^n \frac{1}{k!} G^{(k)}(0)
\]

\[
= f(x) + \sum_{k=1}^n \frac{1}{k!} (\nabla^{k-1} df) (v \otimes k) + \frac{1}{n!} \int_0^1 (\nabla^n df)(\dot{\sigma}(t)^{(n+1)})(1-t)^n dt.
\]

Letting \( n = \exp_m(v) \) in this formula then gives the following version of Taylor’s theorem on a manifold.

**Theorem 5.1.** Let \( f \in C^\infty(M) \) and \( m, n \in M \) with \( d_g(m,n) \) sufficiently small so that there exists a unique \( v \in T_m M \) such that \( |v|_g = d_g(m,n) \) and \( n = \exp_m(v) \). Then we have

\[
f(n) = f(m) + \sum_{k=1}^n \frac{1}{k!} (\nabla^{k-1} df) (v \otimes k) + \frac{1}{n!} \int_0^1 (\nabla^n df)(\dot{\sigma}(t)^{(n+1)})(1-t)^n dt
\]

where \( \sigma(t) = \exp_m(tv) \). In particular since \( |\dot{\sigma}(t)|_g = |v|_g = d_g(m,n) \) it follows that

\[
f(n) = f(m) + \sum_{k=1}^n \frac{1}{k!} (\nabla^{k-1} df) (\exp_m^{-1}(n) \otimes k) + O \left( d(m,n)^{n+1} \right).
\]

**Lemma 5.2.** For \( m, n \in M \) with \( m \) close to \( n \) and \( M \) an embedded submanifold of \( W = \mathbb{R}^k \) we have the estimates

1. \( P(m) \left[ \exp_m^{-1}(n) - (n - m) \right] = O \left( |n - m|^3 \right) \).

   Moreover, \( \exp_m^{-1}(n) - (n - m) = O \left( |n - m|^2 \right) \)

2. \( U_{n,m}^\nabla = P(m) + dP(\exp_m^{-1}(n)) + O \left( |n - m|^2 \right) = P(n) + O \left( |n - m|^2 \right) \)
Lastly, in a small neighborhood around \( m = dP \) note that \( \nabla G = 0 \).

**Proof.** We will denote \( v := \exp^{-1}_m (n) \in T_m M \) and \( \sigma(t) = \exp_m (tv) \).

For 1, we have by Taylor expansion on manifolds (Theorem 5.1) that

\[
G(n) = G(m) + dG(v) + \frac{1}{2} (\nabla dG)(v \otimes v) + \frac{1}{2} \int_0^1 (\nabla^2 dG) \left( \dot{\sigma}(t) \right)^3 (1 - t)^2 \, dt
\]

where \( G \in C^\infty(M,W) \). Letting \( G(m) = m \) as a function into \( W \), we have

\[
n = m + \exp_m^{-1}(n) + \frac{1}{2} (\nabla P)(v \otimes v) + O \left( |v|^3 \right).
\]

Rearranging, we have

\[
\exp_m^{-1}(n) - (n - m) = -\frac{1}{2} \nabla (v \otimes v) + O \left( |v|^3 \right)
\]

so that

\[
P(m) \left[ \exp_m^{-1}(n) - (n - m) \right] = -\frac{1}{2} P(m) (\nabla P)(v \otimes v) + O \left( |v|^3 \right).
\]

Note that \( (\nabla P)(v \otimes v) = dP(v) v = dP(v) P(m) v \). Using the identities \( dPQ - PdQ = 0 \) and \( dP = -dQ \), where \( Q = I - P \), we get that \( PdPP = 0 \). Thus we have

\[
P(m) \left[ \exp_m^{-1}(n) - (n - m) \right] = O \left( |v|^3 \right).
\]

Lastly, in a small neighborhood around \( m, |v|_g = |m - n| + o(|m - n|) \) so that

\[
P(m) \left[ \exp_m^{-1}(n) - (n - m) \right] = O \left( |n - m|^3 \right)
\]

The fact that \( \exp_m^{-1}(n) - (n - m) = O \left( |n - m|^3 \right) \) is immediate from Eq. (5.6).

For 3, we use Taylor’s theorem again this time with \( G : M \rightarrow L(W,W) \) defined by \( G(n) := P(n) \) to see that

\[
P(n) - P(m) = dP \left( \exp^{-1}_m(n) \right) + O \left( |v|^2 \right).
\]

As before, this is equivalent to \( P(n) - P(m) = dP \left( \exp^{-1}_m(n) \right) + O \left( |m - n|^2 \right) \).

Lastly for 2, Taylor applied to \( G_m : M \rightarrow L(T_m M, \mathbb{R}^N) \) defined by \( G_m(n) = U_{n,m} \) gives

\[
U_{n,m} - P(m) = dG_m \left( \exp^{-1}_m(n) \right) + O \left( |m - n|^2 \right).
\]

But

\[
dG_m \left( \exp^{-1}_m(n) \right) = \frac{d}{dt} |_{t=0} U_{\sigma(t),m}
\]

\[
= -dQ (\dot{\sigma}(t)) |_0
\]

\[
= -dQ \left( \exp^{-1}_m(n) \right)
\]

\[
= dP \left( \exp^{-1}_m(n) \right).
\]

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Thus we have
\[ U_{n,m} = P(m) + dP \left( \exp^{-1}_m(n) \right) + O \left( |m-n|^2 \right) \]
which is the first equality of 2. The second equality follows trivially from this and 3. ■

5.2 Equivalence of Riemannian Metrics on Compact Sets

**Proposition 5.3** Let \( \pi : E \to N \) be a real rank \( d < \infty \) vector bundle over a finite dimensional manifold \( N \). Further suppose that \( E \) is equipped with smoothly varying fiber inner product \( g \) and let \( S_g := \{ \xi \in E : g(\xi,\xi) = 1 \} \) be a sub-bundle of \( E \). Then for any compact \( K \subseteq N \), \( \pi^{-1}(K) \cap S_g \) is a compact set.

**Proof.** We wish to show that every sequence \( \{ \xi_l \}_{l=1}^\infty \subset \pi^{-1}(K) \cap S_g \) has a convergent subsequence. Since \( \{ \pi(\xi_l) \}_{l=1}^\infty \) is a sequence in \( K \), by passing to a subsequence if necessary we may assume that \( m := \lim_{l \to \infty} \pi(\xi_l) \) exists in \( K \). By passing to a further subsequence if necessary we may assume that \( \{ \xi_l \}_{l=1}^\infty \subset \pi^{-1}(K) \cap S_g \) where \( K_0 \) is a compact neighborhood of \( m \) which is contained in an open neighborhood \( U \) over which \( E \) is trivializable and hence we may now assume that \( \pi^{-1}(U) = U \times \mathbb{R}^d \) and that \( \xi_l = (n_l, v_l) \) where \( \lim_{l \to \infty} n_l = m \in K_0 \).

Let \( S^{d-1} \) denote the standard Euclidean unit sphere inside of \( \mathbb{R}^d \). The function, \( F : U \times S^{d-1} \to (0, \infty) \) defined by \( F(n,v) = g( (n,v) , (n,v) ) \) is smooth and hence has a minimum \( c > 0 \) and a maximum, \( C < \infty \) on the compact set, \( K \times S^{d-1} \). Therefore by a simple scaling argument we conclude that
\[
c |v|^2 \leq g( (n,v) , (n,v) ) \leq C |v|^2 \quad \forall \ n \in K \text{ and } \ v \in \mathbb{R}^d. \tag{5.7}
\]
From the lower bound in Inequality (5.7) and the assumption that \( 1 = g(\xi_l,\xi_l) \) it follows that \( |v_l|_{g,\ell} \leq 1/\sqrt{c} \) for all \( l \) and therefore has a convergent subsequence \( \{ v_{k_l} \}_{k=1}^{\infty} \). This completes the proof as \( \{ \xi_{k_l} = (n_{k_l}, v_{k_l}) \}_{k=1}^{\infty} \) is convergent as well. ■

**Corollary 5.4** If \( g, \bar{g} \) are two Riemannian metrics on \( TM, K \subseteq M \) is compact, then there exists \( 0 < c_K, C_K < \infty \) such that
\[
c_K |v|_{\bar{g},m} \leq |v|_{g,m} \leq C_K |v|_{\bar{g},m} \quad \forall \ v \in \pi^{-1}(K) \tag{5.8}.
\]
In other words, all Riemannian metrics are equivalent when restricted to compact subsets, \( K \subseteq M \).

**Proof.** The function, \( F : TM \to [0, \infty) \), defined by \( F(v) := g(v,v) \) is smooth and positive when restricted to \( S_g \cap \pi^{-1}(K) \) which is compact by Proposition 5.3. Therefore there exists \( 0 < c_K < C_K < \infty \) such that \( c_K \leq g(v,v) \leq C_K \) for all \( v \in S_g \cap \pi^{-1}(K) \) from which Inequality (5.8) follows by a simple scaling argument. ■

5.3 Covariant Derivatives on Euclidean Space

On \( \mathbb{R}^d \) every covariant derivative takes the form \( \nabla_{(z,v)} = \partial_v + A_z \langle v \rangle \) where \( A : \mathbb{R}^d \to L(\mathbb{R}^d, L(\mathbb{R}^d, \mathbb{R}^d)) \). If \( \sigma^v_z(t) = \exp_z(tv) \) where \( \exp = \exp^v \), we have by definition
\[
\partial_{\sigma^v_z(t)} \sigma^v_z = -A_{\sigma^v_z(t)} \langle \sigma^v_z(t) \rangle \partial_{\sigma^v_z(t)} \sigma^v_z(t) \\
\sigma^v_z(0) = v \\
\sigma^v_z(0) = x
\]
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In particular if $f_x = \exp_x (\cdot)$ plugging in at $t = 0$ we get

$$f''_x (0) [v \otimes v] = -A_x (v) v.$$ 

Now if we denote $G_x := \exp_x^{-1} (\cdot)$ and by differentiating $f_x \circ G_x$ twice, we get that

$$G''_x (x) [v \otimes v] = A_x (v) v.$$ 

Indeed we have

$$0 = (f_x \circ G_x)'' (x) = \left[ f''_x (G_x (x)) G'_{G_x (x)} \right] + f'_x (G_x (x)) G''_{G_x (x)}.$$ 

Since $G_x (x) = 0$, $G'_x (x) = I$, and $f'_x (0) = I$ we have

$$f''_x (0) = -G''_x (x).$$ 

Parallel translation $U_{\sigma^v (t), x}^\nabla$ solves

$$\frac{d}{dt} U_{\sigma^v (t), x}^\nabla = -A_{\sigma^v (t)} \langle \sigma^v (t) \rangle U_{\sigma^v (t), x}^\nabla$$

$$U_{\sigma^v, x}^\nabla = I$$

Again, using $t = 0$ we have that if $\tilde G_x = U_{\sigma^v, x}^\nabla$ then

$$\tilde G''_x (x) v = -A_x \langle v \rangle.$$ 

To summarize, we have

$$(\exp_x^{-1})'' (x) [v \otimes v] = A_x (v) v$$

and

$$(U_{\sigma^v, x}^\nabla)' (x) v = -A_x \langle v \rangle.$$ 

Since $(\exp_x^{-1})'' (x)$ is symmetric, we have that

$$(\exp_x^{-1})'' (x) [v \otimes w] = \frac{1}{2} (\exp_x^{-1})'' (x) (v \otimes w + w \otimes v) + \frac{1}{2} (\exp_x^{-1})'' (x) (v \otimes w - w \otimes v)$$

$$= \frac{1}{2} (\exp_x^{-1})'' (x) (v \otimes w + w \otimes v)$$

$$= \frac{1}{2} \left( A_x (v \otimes w + w \otimes v) \right)$$

$$= \frac{1}{2} (A_x \langle v \rangle w + A_x \langle w \rangle v)$$

Another way of saying this is that $(\exp_x^{-1})'' (x)$ equals the symmetric part of $A_x$. By using this fact and Taylor's theorem, we get the following result.
Lemma 5.5 If $\nabla_{(x,v)} = \partial_v + A_x \langle v \rangle$ is a covariant derivative on $\mathbb{R}^d$, then
\[
(\exp_x)^{-1} (y) - (y - x) - \frac{1}{2} A_x \langle y - x \rangle \langle y - x \rangle = O \left( |y - x|^3 \right)
\]
\[
U_{y,x} \nabla - I + A_x \langle y - x \rangle = O \left( |y - x|^2 \right)
\]
where $|x - y|$ is small enough for these terms to make sense.

Corollary 5.6 If $\nabla_{(x,v)} = \partial_v + A_x \langle v \rangle$ is a covariant derivative on $\mathbb{R}^d$, then
\[
U_{y,x} \nabla - I - A_y \langle x - y \rangle = O \left( |y - x|^2 \right)
\]
where $|x - y|$ is small enough for these terms to make sense. In particular, we have
\[
(U_{y,x})' (x) v = A_x \langle v \rangle
\]

Proof. This is immediate after expanding $A_{(\cdot)}$ about $x$ in the direction $y - x$ in Eq. (5.11) with Taylor’s theorem.

5.4 Second order gauge inequality does not imply second order chart inequality.

Example 5.7 Let $x_s$ and $y_s$ be the $C ([0, 2], \mathbb{R})$ paths defined by
\[
y_s = x_s = \begin{cases} 
0 & \text{if } 0 \leq s \leq 1 \\
s^{1/p} - 1 & \text{if } 1 \leq s \leq 2
\end{cases}
\]
and the control $\omega (s, t)$ be defined by
\[
\omega (s, t) = \begin{cases} 
0 & \text{if } t \leq 1 \\
t - (s \vee 1) & \text{if } t \geq 1
\end{cases}
\]
Then it is easy to check that
\[
|x_s, t| \leq \omega (s, t)^{1/p}
\]
Let
\[
y_s^\dagger = \begin{cases} 
2 - 2s & \text{if } 0 \leq s \leq \frac{1}{2} \\
1 & \text{else } \frac{1}{2} \leq s \leq 2
\end{cases}
\]
Then if $t - s \leq 1/2$, $y_{s,t} - y_s^\dagger x_{s,t} = 0$ so that $(y, y^\dagger)$ satisfies Inequality (2.15) with $\delta = 1/2$ and $\psi (x, y) = y - x$. On the other hand if $s = 0$ and $t = 1 + \epsilon$, then
\[
y_{s,t} - y_s^\dagger x_{s,t} = \epsilon^{1/p} - 2 \epsilon^{1/p} = -\epsilon^{1/p}.
\]
Thus
\[
\left| \frac{y_{0,1+\epsilon} - y_0^\dagger x_{0,1+\epsilon}}{\omega (0, 1 + \epsilon)^{2/p}} \right| = \frac{1}{\epsilon^{1/p}}
\]
so that $(y, y^\dagger)$ does not satisfy Inequality (2.17) with the identity chart.
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