Another Critique of the Replica Trick

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Abstract

Kamenev and Mezard, and Yurkevich and Lerner, have recently shown how to reproduce the large-frequency asymptotics of the energy level correlations for disordered electron systems, by doing perturbation theory around the saddles of the compact nonlinear σ model derived from fermionic replicas. We present a critical review of their procedure and argue that its validity is limited to the perturbative regime of large frequency. The miraculous exactness of the saddle-point answer for $\beta = 2$ (unitary symmetry) in the universal limit, is shown to be a special feature due to the Duistermaat-Heckman theorem.

1 Introduction

Suppose we are given a disordered statistical mechanical system with free energy $f = -\ln Z$, and we are to compute the disorder average $\langle f \rangle$. To tackle that task, it is popular among theoretical physicists to follow a recipe called the replica trick. Instead of attempting to calculate $\langle \ln Z \rangle$ directly, one computes the disorder average $f_n \equiv \langle Z^n \rangle$ for all positive integers $n = 1, 2, \ldots \infty$, which is be done by introducing $n$ copies, or replicas, of the system. Then, recalling the identity $\ln Z = \lim_{n \to 0} (Z^n - 1)/n$, and assuming some extension of the discrete set $(f_n)_{n \in \mathbb{N}}$ to an analytic function $f(u)$ on $\mathbb{C}$, one hopes that $f'(0)$ equals $-\langle \ln Z \rangle$. An obvious problem with this trick is the non-uniqueness of the analytic continuation. Indeed, if $f(u)$ satisfies $f(u)|_{u=n} = f_n$, then so does $f(u) + c/\Gamma(-u)$, where $\Gamma(u)$ is the Euler gamma function and $c$ is an arbitrary constant. Notice that the increment $c/\Gamma(-u)$ vanishes for all positive integers $u = n$ but has finite derivative at $u = 0$. Thus we do not know whether the answer for $-\langle \ln Z \rangle$ is $f'(0)$ or $f'(0) - c$. To determine the unknown constant, which may be a function of temperature, disorder strength etc., one must have some control on $\langle Z^n \rangle$, for example a bound on its asymptotic behavior as $u \to i\infty$. Sadly, such control is not always available.

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Some time ago, Verbaarschot and the author (VZ) addressed the above concern in the context of disordered electron systems, i.e., the problem of computing averages of retarded and advanced Green’s functions $G^\pm(E) = (E \pm i\epsilon - H)^{-1}$ of a disordered single-particle Hamiltonian $H$. For such systems, one implements the replica trick by starting from a generating function $Z_{m,n}(\omega, J_1, J_2) = \langle \text{Det}^m(E + \omega + i\epsilon - H + J_1) \text{Det}^n(E - i\epsilon - H + J_2) \rangle$ depending on matrix sources $J_1, J_2$. Angular brackets denote averaging over disorder. For positive integers $m$ and $n$, this generating function can be expressed as a Gaussian integral over anticommuting complex fields (“fermionic replicas”), and for negative integers over commuting fields (“bosonic replicas”). VZ focused on the spectral two-point function, $S_2(\omega) = \langle \text{Tr}(E + \omega + i\epsilon - H)^{-1} \text{Tr}(E - i\epsilon - H)^{-1} \rangle$ ($\epsilon > 0$), for the simple case of Wigner-Dyson statistics for $\beta = 2$, i.e., the Gaussian ensemble of complex Hermitian matrices with unitary symmetry (GUE). By following the standard procedure [2, 3, 4] of mapping the problem on a nonlinear $\sigma$ model, and then doing the natural analytic continuation $m = n \to 0$, VZ found that fermionic and bosonic replicas give different answers, and both answers differ from the exact result. What is reproduced correctly by the replica trick is the asymptotics of $S_2(\omega)$ as $\omega \to \infty$. This finding led to a general consensus that the replica trick for disordered electron systems is limited to those regions of parameter space where the nonlinear $\sigma$ model can be evaluated perturbatively.

This consensus is now being challenged by recent proposals of Kamenev and Mezard [5, 6], and of Yurkevich and Lerner [7]. A partial list of their claims is as follows: i) According to Kamenev and Mezard [5], past work using the replica trick for disordered electron systems suffers from the tacit assumption of the absence of “spontaneous breaking of replica symmetry”. ii) When replica-symmetry broken saddle points are included, the “standard” nonlinear $\sigma$ model derived from fermionic replicas gives the correct oscillatory asymptotics of the level correlation functions. iii) Perturbation theory in $\omega^{-1}$ yields a “systematic“ expansion for the amplitudes of the oscillatory terms. iv) The fermionic replica trick with replica-symmetry breaking is “nonperturbative” and reproduces the results of the supersymmetric method. v) Yurkevich and Lerner speculate that if limits are taken in the proper order (first $n \to 0$, then $\omega \to 0$), the fermionic replica trick may reproduce also the singular part of the two-point function.

These claims and speculations add up to a substantial revision of the accepted picture. They provoke some sort of response, and it is the purpose of the present paper to offer clarification. Section 2 calculates in some detail the one-point function, Section 3 reviews the two-point function, Section 4 explains why replicas can be manipulated to yield the exact answer for the spectral correlations of the GUE, and Section 5 summarizes the conclusions.
2 Fermionic replicas: one-point function

Let us start from the beginning and consider first the spectral one-point function
\( S_1(z) = (\text{Tr} (z - H)^{-1}) \). The fundamental difficulty plaguing fermionic replicas for disordered electron systems is already visible in this case.

On elementary grounds, the function \( S_1(z) \) has a branch cut along the real axis (or the line of spectral support of the Hamiltonian \( H \)), and its discontinuity across the cut yields the average density of states, \( \rho(E) \):

\[
\rho(E) = \frac{1}{2\pi i} \lim_{\epsilon \to 0^+} \left( S_1(E - i\epsilon) - S_1(E + i\epsilon) \right) = \frac{1}{\pi} \lim_{\epsilon \to 0^+} \text{Im} S_1(E - i\epsilon) .
\]

The replica trick, instead of attempting to calculate \( S_1 \) directly, looks at the discrete family of functions

\[
f_n(z) = \left\langle \text{Tr} (z - H)^{-1} \text{Det}^n(z - H) \right\rangle \quad (z \in \mathbb{C})
\]

for \( n \in \mathbb{Z} \). The member \( f_0(z) \) of this family is the desired quantity \( S_1(z) \). In the fermionic version of the replica trick, one computes \( f_n(z) \) for all positive integers \( n = 1, 2, \ldots \infty \), and hopes to infer \( S_1(z) \) by setting \( n = 0 \) at the end of the calculation. On brief reflection, however, this hope must appear ill-founded. Indeed, it is easy to see that, if all moments \( \left\langle \langle \text{Tr} H^p \rangle \right\rangle_{p=1,\ldots,n-1} \) exist, the function \( f_n(z) \) is analytic in \( z \) for all positive integers \( n \in \mathbb{N} \), which means in particular that \( f_n(z) \) is continuous across the real axis, and

\[
\rho_n(E) = \frac{1}{2\pi i} \lim_{\epsilon \to 0^+} \left( f_n(E - i\epsilon) - f_n(E + i\epsilon) \right) = \frac{1}{\pi} \lim_{\epsilon \to 0^+} \text{Im} f_n(E - i\epsilon) = 0
\]

is zero, for all \( E \in \mathbb{R} \) and \( n \in \mathbb{N} \). This is beginning to look bizarre. Given that all imaginary parts \( \rho_n(E) \) vanish identically for \( n = 1, 2, \ldots \infty \), how can we hope to predict, based on these data only, the nonzero value of the average density of states \( \rho(E) = \rho_n(E)|_{n=0} \)?

Let us postpone the answer for a moment and jump to the computation of \( f_n(z) \) (\( n \geq 1 \)) using fermionic replicas. For simplicity we focus on the case of \( N \times N \) random matrices with \( \beta = 2 \) (unitary symmetry) and take the probability measure for the complex Hermitian matrix \( H \) to be Gaussian with weight function \( P(H) = \text{const} \times \exp(-N\text{Tr}H^2/2w^2) \). By a standard sequence of transformations \( \mathfrak{B} \), the function \( f_n(z) \) can then be cast in the form of an integral over \( u(n) \), the Lie algebra of the unitary group \( U(n) \):

\[
f_n(z) = \frac{N}{n} \int_{u(n)} \text{Tr} (z - Q)^{-1} \text{Det}^N(z - Q) e^{N\text{Tr}Q^2/2w^2} dQ ,
\]

where \( dQ \) is the flat measure on \( u(n) \) normalized by \( \int_{u(n)} e^{N\text{Tr}Q^2/2w^2} dQ = 1 \). The elements of \( u(n) \) are skew-Hermitian matrices, whence \( \text{Tr} Q^2 = -\text{Tr} Q^\dagger Q \leq 0 \) and the integral exists. Note that the integrand is invariant under the adjoint action \( Q \mapsto UQU^{-1} \) of \( U(n) \) on its Lie algebra.
For large $N$, the integral over $Q$ is saturated by the solutions of the saddle point equation
\[ \frac{\delta}{\delta Q} \left( \text{Tr} \frac{Q^2}{2w^2} + \text{Tr} \ln(z - Q) \right) = 0 = \frac{Q}{w} + (Q - z)^{-1}. \]

This equation is of real type. Its solutions therefore come in Hermitian conjugate pairs: if $Q$ is a solution for $z = E \in \mathbb{R}$, then so is $Q^\dagger$. In particular, for $|E| < 2w$ there exists a solution
\[ Q = w e^{i\theta} \times 1_n, \quad e^{i\theta} = \frac{E}{2w} + i \sqrt{1 - \left(\frac{E}{2w}\right)^2}, \]
which is proportional to the unit matrix, and is paired with $Q^\dagger = w e^{-i\theta} \times 1_n$. These two solutions are unique in that they are stable under the adjoint action of $U(n)$. For $n \geq 2$, there exist also solutions with less symmetry, which organize into smooth saddle-point manifolds. They can be constructed by starting from a diagonal solution with $n - p$ (resp. $p$) matrix elements equal to $w e^{i\theta}$ (resp. $w e^{-i\theta}$), and then applying the adjoint action of the symmetry group $U(n)$.

The point to be made here is this: proper application of the saddle-point method to the integral representation for $f_n(z)$ requires us to sum over pairs of conjugate saddles. It is clear that the contribution from a saddle point (or saddle-point manifold) $Q$ is precisely the complex conjugate of the contribution from its Hermitian conjugate $Q^\dagger$, if $z = E \pm i0$. Therefore, we can be sure that the result for $f_n(z)$ on the real energy axis is real:
\[ f_n(E) = \varphi_n(E) + \bar{\varphi}_n(E) \in \mathbb{R}, \]
and $\rho_n(E) = \pi^{-1} \lim_{\epsilon \to 0} f_n(E - i\epsilon)$ is zero for all $n = 1, 2, \ldots \infty$. This, of course, was to be expected in view of what we saw at the beginning of the section.

We now return to the question posed before. Since $\rho_n(E)$ vanishes identically for all $n \in \mathbb{N}$, how can we manage to produce a reasonable result for $\rho(E) = \rho_n(E)|_{n=0}$? Clearly, the answer is that some additional input must be injected. What is missing from the formalism with fermionic replicas is the entire information about causality, i.e., the distinction between retarded and advanced Green’s functions ($z = E + i\epsilon$, versus $z = E - i\epsilon$). This information is present in $f_n(z)$ only for $n \leq 0$. In the bosonic replica trick, where one evaluates $f_n(z)$ for $n = -1, -2, \ldots -\infty$, causality is reflected in the $Q$-integral representation as a singularity of the integrand, which restricts the number of saddles that are accessible by continuous deformation of the integration contour $\nabla$. It turns out that for all $n < 0$, there exists just one saddle point which is accessible, and all others are inaccessible. On the other hand, in the rigorous formulation of $f_0(z)$ by the supersymmetric method $\nabla$, there are four saddle points, two accessible and two inaccessible ones. Both the supersymmetric formula and bosonic replicas will be discussed in more detail below.

2.1 The recipe

As we have seen, the reason for the vanishing of $\rho_n(E)$ ($n \in \mathbb{N}$) is the pairwise appearance of saddles as conjugate pairs $(Q, Q^\dagger)$. It is therefore clear that,
in order to assist the fermionic replica trick and extrapolate $\rho_n(E)$ to a non-vanishing answer for $n = 0$, we must come up with a recipe for breaking the symmetry between the members of conjugate pairs. Let us denote by $Q_p$ the diagonal matrix with the first $n - p$ entries equal to $we^{i\theta}$ and the last $p$ entries equal to $we^{-i\theta}$. If $z = E \in \mathbb{R}$, the matrix $Q_p^*$ lies on the saddle-point manifold $UQ_{n-p}U^{-1}$ ($U \in U(n)$), so the pairing of saddles is $p \leftrightarrow n - p$.

For definiteness, let us fix $z = E - i0$, in which case $\text{Im } S_1(z) \geq 0$. According to the saddle-point equation $(z - Q)^{-1} = Q/w^2$, the expression

$$\text{Im } (E - we^{i\theta})^{-1} = \text{Im } e^{i\theta}/w = w^{-1}\sqrt{1 - (E/2w)^2}$$

is positive for $|E| < 2w$, which is known \[10\] to be the range of the energy spectrum in the limit $N \to \infty$. Since the integral representation for $f_n(z)$ contains the factor $\text{Tr } (z - Q)^{-1}$, the “good” saddle point, contributing to $f_n(z)$ with an imaginary part of the desired positive sign, is $Q_0 = we^{i\theta} \times 1_n$. By the same token, $Q_n = we^{-i\theta} \times 1_n$ is a “bad” saddle point. Alternatively, we could say that $Q_0$ is “causal”, whereas $Q_n$ is “acausal”. By heritate, or continuity, the natural recipe now is to declare the saddle-point manifolds $UQ_pU^{-1}$ with $p = 0, 1, 2, \ldots$ to be “causal”, and their conjugates $p = n, n - 1, n - 2, \ldots$ to be “acausal”. (Clearly, this only makes sense if there is some mechanism that terminates the sequences before they overlap.) Anticipating the emergence of causality for $n = 0$, we then retain only the contributions from the “causal” saddles, while throwing away the others. The resulting fake expressions for $f_n(E - i\epsilon)$ have nonzero imaginary parts, and extrapolation to $n = 0$ can now give a finite density of states. The result so obtained for $|E| < 2w$ reads \[11\]

$$\rho = (N/\pi w) \sin \theta + O(N^0) + \ldots$$

$$- (4\pi w \sin^2 \theta)^{-1} \cos \left( N(2\theta - \sin 2\theta) \right) + O(N^{-1}) + \ldots \quad (1)$$

The leading term is Wigner’s famous semicircle law, $\sin \theta = \sqrt{1 - (E/2w)^2}$, and stems from the invariant saddle point $p = 0$. The subleading (oscillatory) term originates from the saddle-point manifold $p = 1$, which is generated by applying the adjoint action of $U(n)$ to $Q_1$, the diagonal matrix with $n - 1$ entries $we^{i\theta}$ and one entry $we^{-i\theta}$. The stability group of $Q_1$ in $U(n)$ is $U(n - 1) \times U(1)$. Hence the saddle-point manifold $p = 1$ is isomorphic to the quotient $U(n)/U(n - 1) \times U(1)$. More generally, the saddle-point manifold with index $p$ is isomorphic to the Grassmannian $G_{n,p} = U(n)/U(n - p) \times U(p)$, which is a symmetric space of type AIII in Cartan’s classification \[13\]. Carrying out the saddle-point approximation requires integrating over the directions transverse to the saddle-point manifold approximately, and over the saddle-point manifold itself exactly. The latter integral produces a volume factor $\text{vol}(G_{n,p})$. For $p = 1$, this volume goes essentially as $\Gamma(n)^{-1}$ (see for example the appendix of \[5\]), which combines with the prefactor $n^{-1}$ of the integral representation for $f_n(z)$ to produce a finite limit $n^{-1}\Gamma(n)^{-1} \to 1$ for $n = 0$. More generally, “causal” analytic continuation \[5\] of the volume of $G_{n,p}$ yields a power $n^p$ near $n = 0$. The contributions from the saddle-point manifolds $p \geq 2$ therefore extrapolate to zero at $n = 0$.\[5\]
Remarkably, the above $N^{-1}$ expansion of the average density of states agrees with the asymptotic behavior that can be inferred from an exact representation in terms of orthogonal polynomials. In view of the rather ad hoc nature of the derivation, mathematicians will throw up their hands in horror or despair, while physicists are much intrigued. In any case, putting aside questions of mathematical rigor, it is definitely desirable to gain a better understanding of why the procedure works.

### 2.2 Supersymmetry

One avenue towards some understanding is to make a comparison with the rigorous representation \[9, 12\] of $S_1(z)$ as an integral over a $2 \times 2$ supermatrix $Q = \begin{pmatrix} x & \xi \\ \eta & iy \end{pmatrix}$:

$$S_1(z) = \frac{N}{2} \int DQ \text{Tr} (z - Q)^{-1} \text{SDet}^{-N} (z - Q) e^{-\text{NSTr} Q^2/2w^2}.$$  

Here $DQ$ stands for the flat Berezin form $DQ = (2\pi)^{-1} dx dy \partial^2 / \partial \xi \partial \eta$, and the integral is over $x \in \mathbb{R}$ and $iy \in i\mathbb{R}$. The symbols STr and SDet denote the supertrace and the superdeterminant.

The above Berezin integral for $S_1(z)$ has a total of four saddle points, obtained by going through the various sign choices in $Q = w \text{diag}(e^{\pm i\theta}, e^{\pm i\theta})$. Two of these are “bad”, as they cannot be reached by continuous deformation of the integration contour for $x$ without crossing the $N$-th order pole of $\text{SDet}^{-N} (z - Q)$ at $z = x$. As before, let us fix $z = E - i0$. Then the accessible saddle points are $Q_0 = w \text{diag}(e^{i\theta}, e^{-i\theta})$ and $Q_1 = w \text{diag}(e^{i\theta}, e^{-i\theta})$. The first one gives rise \[9\] to Wigner’s semicircle, and on comparing with the fermionic replica trick, one begins to suspect that perturbation theory around the second one might yield the oscillatory contribution to the asymptotic result \[1\]. I have checked that this is precisely the case.

In hindsight, the coincidence is not all that surprising. At the non-invariant saddle point $Q_1 = w \text{diag}(e^{i\theta}, e^{-i\theta})$, both $\xi$ and $\eta$ are Goldstone fermions of the supersymmetric integrand $\text{SDet}^{-N} (z - Q) \exp -\text{NSTr} Q^2/2w^2$. On the other hand, the $p = 1$ saddle-point manifold $U(n)/U(n-1) \times U(1)$ of the fermionic replica trick is isomorphic to the complex projective space $\mathbb{CP}_{n-1}$, which is a manifold of real dimension $2(n-1)$. At $n = 0$, this has real dimension $-2$, and has to be understood as a space generated by two anticommuting degrees of freedom, say $\xi$ and $\eta$. Hence, integrating over $\mathbb{CP}_{n-1} \simeq U(n)/U(n-1) \times U(1)$ and then setting $n = 0$ is just a complicated way of taking two derivatives $\partial^2 / \partial \xi \partial \eta$. The latter is what the Berezin integral formula does. The agreement between the asymptotic expansions to leading order motivates us to propose the following conjecture: the fermionic replica trick, augmented by the plausible recipe of retaining only the saddles $p = 0$ and $p = 1$, is *perturbatively equivalent* to the supersymmetric integral representation for $S_1(z)$. In other words, pushing the asymptotic expansion around the two saddles $Q_0$ and $Q_1$ to higher order, we expect to get agreement in every order of $N^{-1}$. 

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Can this conjecture be put on a more solid footing? With the goal of justifying better the asymptotic expansion obtained from the fermionic replica trick, we are now going to take a more close-up look at the procedure of analytic continuation in \( n \).

2.3 Analytic continuation?

To keep the calculations as simple as possible, we start from the generating function

\[
F_n(z) = \langle \text{Det}^n(z - H) \rangle,
\]

taking aim to eventually apply \( \frac{n^{-1}}{dz} \) and analytically continue to \( n = 0 \), so as to extract \( S_1(z) \). As we have seen, the values of \( F_n(z) \) at the positive integers \( n \) do not contain sufficient information to permit analytic continuation. Moreover, it is not known how to compute \( F_n(z) \) for \( n \notin \mathbb{Z} \), at least not directly. We do have access, however, to \( F_n(z) \) at the negative integers \( n \), by means of the bosonic replica trick. Let us see what additional information we can collect from there.

By some elementary transformations, we arrive at the identity

\[
F_{-n}(z) = \int_{i u(n)} \text{Det}^{-N}(z - Q) e^{-N \text{Tr} Q^2/2w^2} dQ \quad (n \in \mathbb{N}).
\]

We are now integrating over the space of Hermitian matrices, viewed as \( i u(n) \). The saddle-point equation governing the large-\( N \) limit still reads \( (z - Q)^{-1} = Q/w^2 \) and, as before, the invariant \( p = 0 \) saddle point yields Wigner’s semicircle. Other saddle-point manifolds with index \( p = 1, 2, \ldots, n \) exist, too. The difference from before is that these are now disconnected from the integration domain \( i u(n) \) by the singularities of the function \( \text{Det}^{-N}(z - Q) \). In view of what happen for fermionic replicas, one might think that one should ignore disconnectedness and incorporate the other saddles. Surely, moving across the \( N \)-th order pole of \( \text{Det}^{-N}(z - Q) \) just produces some residue, and one could argue that this is schematically given by \( \frac{\delta^{N-1}}{\delta Q^{N-1}} \exp(-N \text{Tr} Q^2/2w^2)|_{Q=\pm z} \), which is exponentially small and hence negligible, for large \( N \). However, we are on the wrong track. A little thought shows that the saddles \( p = 0, 1, \ldots, n \) are aligned transversely to the contour of integration for \( Q \). This makes it impossible to deform the contour so as to pass through more than one member of the sequence of saddles without backtracking. (In contrast, for fermionic replicas the alignment is longitudinal, and a constant shift \( Q \to Q + E/2 \) puts all the saddles right on the contour.) Moreover, if one proceeds anyway and computes the contribution from \( p = 1 \), one obtains an expression containing the exponential factor

\[
\exp\left( nN(i\theta - e^{i\theta}/2) - iN(2\theta - \sin 2\theta) \right).
\]

(again, this is for \( z = E - i0 \). Near \( n = 0 \) this can be seen to increase without bound as \( z \) moves away from the real axis toward \( z = -i\infty \). Such behavior is incompatible with analyticity of the Green’s function at infinity. For all these
reasons, it is safe to say that \( F_n(z) \) at the negative integers \( n \) is exhausted by a single saddle point, \( p = 0 \). Evaluation of the corresponding integral in Gaussian approximation gives

\[
F_n(2w \cos \theta - i0) = w^n N \frac{\exp \left( nN \left( \frac{1}{2} e^{2i\theta} - i \theta \right) \right)}{\sqrt{1 - e^{2i\theta} n^2}} + O(N^{-1}) \quad (-n \in \mathbb{N}) . \tag{2}
\]

This result will place a strong constraint on the analytic continuation of \( F_n \).

Next we return to fermionic replicas, and carry out a more complete saddle-point evaluation of

\[
F_n(z) = \int_{u(n)} \text{Det}^N (z - Q) e^{N \text{Tr} Q^2 / 2w^2} dQ
\]

for \( n \in \mathbb{N} \). As before, let \( Q_p \) denote a diagonal matrix with \( n - p \) entries \( we^{i\theta} \) and \( p \) entries \( we^{-i\theta} \). By acting with the symmetry group \( U(n) \) on \( Q_p \) we get the saddle-point manifold \( G_{n,p} = U(n)/U(n-p) \times U(p) \). To perform the saddle-point approximation, we need to factor the integration domain near \( G_{n,p} \) in a suitable manner. For this purpose, let \( u(n) \) be decomposed into two linear subspaces: \( u(n) = V + W \), where the elements of \( V = u(n-p) + u(p) \) commute, while those of \( W \) anticommute, with \( Q_p \). We then introduce adapted coordinates by

\[
Q = U(Q_p + X)U^{-1} ,
\]

where \( UQ_pU^{-1} \) parametrizes \( G_{n,p} \), and \( X \in V \) spans the directions transverse to \( G_{n,p} \). By an elementary calculation, the volume element \( dQ \) under this factorization transforms into

\[
\text{Det ad}(Q_p + X) |_W dX dU .
\]

Here, \( dU \) is a \( U(n) \)-invariant measure for \( G_{n,p} \), \( dX \) is a flat measure on \( V \), and \( \text{ad}(Y) |_W \) denotes the commutator (or adjoint) action of \( Y \in V + iW \) on the elements of \( W + iW \). By expanding the integrand \( \text{Det}^N (z - Q) \exp N \text{Tr} Q^2 / 2w^2 \) w.r.t. \( X \) and doing the integral over \( X \) in Gaussian approximation, we obtain

\[
F_n \simeq \sum_{p=0}^n \text{Det}^{-N}(Q_p/w^2) \exp \left( N \text{Tr} Q_p^2 / 2w^2 \right) \text{Det ad}(Q_p) |_W dU \int_V e^{(N/2w^2) \text{Tr} (X^2 - w^{-2} X Q_p X Q_p)} dX
\]

\[
= w^n N \sum_{p=0}^n \text{vol}(G_{n,p}) (N/2\pi)^{p(n-p)} \frac{(e^{i\theta} - e^{-i\theta})^{2p(n-p)}}{\sqrt{1 - e^{2i\theta} (n-p)^2} \sqrt{1 - e^{-2i\theta} p^2}} \times \exp \left( nN \left( \frac{1}{2} e^{2i\theta} - i \theta \right) + ipN(2\theta - \sin 2\theta) \right) \quad (n \in \mathbb{N}) . \tag{3}
\]

An analogous expression for the derivative \( n^{-1} dF_n / dz \) was obtained in [8]. We now face the task of finding an analytic continuation of \( F_n \) to \( n \in \mathbb{C} \) which respects the results (2) and (3) for \( n \in \mathbb{Z} \). Kamenev and Mezard [8] have
suggested that the solution to this problem is to analytically continue \( \text{vol}(G_{n,p}) \) as
\[
\text{vol}(G_{n,p}) = (2\pi)^{p(n-p)} \prod_{j=1}^{p} \frac{\Gamma(j)}{\Gamma(n-j+1)},
\]
and extend the upper limit on the sum over \( p \) in (3) to infinity. The latter does not change the value of \( F_n \) for \( n \in \mathbb{N} \), as \( \Gamma(n-p+1)^{-1} \) vanishes for \( p \geq n + 1 \). For the same reason, the terms \( p \geq 1 \) vanish at the negative integers \( n \), in agreement with the constraint posed by (2). So far so good. Unfortunately, when \( n \) is not an integer, the infinite sum over \( p \) is ill-behaved. The culprit is the rapid growth with \( p \) of the (analytically continued) volume factors:
\[
(2\pi)^{2p-n-1} \frac{\text{vol}(G_{n,p})}{\text{vol}(G_{n,p-1})} = \frac{\Gamma(p)}{\Gamma(n-p+1)} = \Gamma(p)\Gamma(p-n) \pi^{-1} \sin \pi(p-n),
\]
which makes the sum over \( p \) diverge. There appears to exist no obvious remedy for this difficulty, and we have to concede that extension of the sum over \( p \) to infinity does not define \( F_n(z) \) away from the integers. Thus we have so far failed in our attempt to perform a bona fide analytic continuation, and the recipe of Section 2.1 still awaits justification.

Nevertheless, there seems to be some truth to the proposal by Kamenev and Mezard. If we boldly ignore the divergence issue, and formally apply \((\pi n)^{-1} d/dz\) at \( n = 0 \), then all the terms for \( p \geq 2 \) disappear, and the answer from \( p = 0 \) and \( p = 1 \) can be shown to agree with the large-\( N \) asymptotics extracted from the exact formula for \( S_1(z) \) as a Berezin integral. Upon further thought, the failure of the proposed analytic continuation can be interpreted as follows. Recall that the failure is caused by the late terms in the sum over \( p \), which in turn stem from our insisting that the analytic continuation reproduce the result (3) for all \( n \in \mathbb{N} \). But at fixed \( N < \infty \), the saddle-point approximation leading to that expression breaks down when \( n \) becomes large. Indeed, for \( n \gg N \) the \( Q \)-integral formula for \( F_n(z) \) is inappropriate, and it is better to return to the original representation
\[
F_n(z) = \int \text{Det}^n(z-H) e^{-NTrH^2/2w^2} dH.
\]
By evaluating this integral in saddle-point approximation, one gets a an answer formally similar to (3). A major difference is that the sum over saddle-point manifolds \( p \) now terminates at \( N \), not \( n \). Thus, making \( n \) bigger no longer extends the range of \( p \). The lesson to be learnt from this is that the approximate result (3) must not be trusted for large values of \( n \). It is likely that the divergence of the sum over \( p \) is fake and is somehow cut off when a more correct approximation for large \( n \) is used. When \( N \) is big enough, this cutoff will have a negligible effect on the extrapolation to \( n = 0 \). On the other hand, for small values of \( N \) the cutoff will probably lead to some unknown correction.

We could have anticipated the existence of such corrections already from our simplified presentation in Section 2.1. Surely, the recipe of throwing away
half of the saddles makes sense only in the perturbative domain of isolated saddles. When the saddles are not well separated (as is the case for small $N$), their asymptotic series in some sense “interfere”, which is likely to give rise to corrections of exponentially small type $\sim e^{-\text{const} \times N}$. For the one-point function, the existence of such corrections is of minor concern, as we are usually satisfied with knowing the large-$N$ asymptotics. As we shall see, however, the situation is less favorable for the two-point function. Note also that the question of what happens beyond the saddle-point approximation or, more precisely, whether a non-analytic function can be reconstructed from its asymptotic series (by Borel resummation, for example), is a very difficult issue.

Let me finish the section with a word on vocabulary. In my opinion it is inappropriate to call the appearance of the $p=1$ saddle-point manifold an instance of “spontaneous breaking of replica symmetry”. Spontaneous symmetry breaking, a phenomenon which may occur in physical systems with infinitely many degrees of freedom, means that a symmetry of the Hamiltonian is not manifest in the (ground) state of the system. The symmetry group of the present problem is $U(n)$. The saddle-point manifold $\mathbb{C}P_{n-1} \cong U(n)/U(n-1) \times U(1)$ is a beautiful symmetric (and replica-symmetric) space, whose geometry is invariant under the action of $U(n)$. It is true, of course, that an individual point on the manifold is not invariant under $U(n)$. However, our job is to integrate, and therefore the issue is not invariance of individual points but invariance of the domain of integration. The integration manifold $\mathbb{C}P_{n-1} \cong U(n)/U(n-1) \times U(1)$ definitely does not break $U(n)$ replica symmetry, just as the familiar two-sphere in Euclidean 3-space, $S^2 = SO(3)/SO(2)$, does not break SO(3) rotational symmetry.

### 3 Fermionic replicas: two-point function

Let $G$ be one of the compact matrix groups $\text{Sp}(2m+2n)$, $\text{U}(m+n)$, or $\text{O}(2m+2n)$. Acting with $G$ on a fixed matrix $\Sigma_3$ by conjugation, we get an orbit of elements we denote by $Q = U \Sigma_3 U^{-1}$ ($U \in G$). For $G = \text{Sp, U, O}$ we take $\Sigma_3 = \text{diag}(1_{2m}, -1_{2n})$; and for $G = \text{U}$, $\Sigma_3 = \text{diag}(1_m, -1_n)$. To keep the dimensions explicit, we write $X_{m,n}$ for the orbit of $G$ on $\Sigma_3$. The stability group of $\Sigma_3$ is a subgroup, $H$, of $G$. Letting $G$ act on $\Sigma_3$ by conjugation amounts to taking the quotient of $G$ by $H$, so $X_{m,n}$ is isomorphic to the coset space $G/H$ (which is moreover a symmetric space).

Now let $dQ$ be a $G$-invariant measure on $X_{m,n}$, and consider the generating function

$$Z_{m,n}(\omega) = \int_{X_{m,n}} \exp \left( \frac{i\omega}{2\alpha} \text{Tr} \Sigma_3 Q \right) dQ .$$

This is the zero-dimensional limit of the nonlinear $\sigma$ model of Wegner and Efetov [2,4], truncated to the compact sector arising from replicated fermionic fields $\oplus^3$. The generating function for $G = \text{Sp, U, O}$ is intended to describe the universal low-frequency limit of a disordered electron system with symmetry index $\beta = 1, 2, 4$ (i.e., orthogonal, unitary, or symplectic symmetry), in this order. The dimensionless parameter $\omega$ is a frequency measured in units of the
mean level spacing divided by $\pi$, and $\alpha = 2, 1, 1$ for $\beta = 1, 2, 4$. For future use, note that the integrand is invariant under the action of the little group $H$.

The transformation $Q \mapsto -Q$ is a natural isomorphism from $X_{m,n}$ to $X_{n,m}$. Using it in the expression for $Z_{m,n}(\omega)$ we obtain the symmetry relation

$$Z_{m,n}(\omega) = Z_{n,m}(-\omega).$$

This symmetry does not come as a surprise but reflects the fact that $Z_{m,n}(\omega)$ is a low-frequency approximation to

$$\langle \text{Det}^m(E - i\epsilon + \omega/2 - H) \text{Det}^n(E + i\epsilon - \omega/2 - H) \rangle.$$

Because this is an analytic function of $\omega/2 - i\epsilon$ for $m, n \in \mathbb{N}$, nothing is lost by setting $\epsilon = 0$. We then see at a glance that the generating function is invariant under the combined operation of exchanging $m \leftrightarrow n$ and sending $\omega \to -\omega$.

Recall the definition of the two-point function

$$S_2(u, v) = \langle \text{Tr}(u - H)^{-1} \text{Tr}(v - H)^{-1} \rangle.$$

We are interested in the case where the arguments of $S_2$ lie on opposite sides of the real energy axis: $u = E_1 - i\delta$, $v = E_2 + i\delta$, and their difference is a fixed multiple $\omega/\pi$ of the mean level spacing. The resulting function is still denoted by $S_2(\omega)$.

In the fermionic replica trick for the two-point function, one hopes to extract $S_2(\omega)$ from the knowledge of $Z_{m,n}(\omega)$ for all positive integers $m$ and $n$, by applying two derivatives $-(mn)^{-1} \partial^2 / \partial \omega^2$ and setting $m = n = 0$. The bad news is that this hope is frustrated by the symmetry relation $Z_{m,n}(\omega) = Z_{n,m}(-\omega)$. If the replica limit exists in a naive sense, then it cannot matter whether we first set $m = 0$ and then $n = 0$, or the other way around. Another option is to set $m - n = 0$ first and $m + n = 0$ afterwards. If we proceed in the latter fashion, the symmetry of the generating function reduces to $Z_{n,n}(\omega) = Z_{n,n}(-\omega)$. Thus, $Z_{n,n}(\omega)$ is an even and hence real function of $\omega$. By naive extrapolation to $n = 0$, we would conclude that $S_2(\omega)$ is an even and real function of its argument. But this clearly is nonsense, for $S_2$ is neither even nor real. For example, if $\text{Im}\omega < 0$ the universal answer for the case $\beta = 2$ is $S_2(\omega) = 1 - 2i\omega^{-2}e^{-i\omega} \sin \omega$.

The reason for the nonsensical answer produced by our use of fermionic replicas is the same as for the one-point function: the analyticity in $\omega$ of the function $Z_{n,n}(\omega)$ means that the fermionic replica trick suffers from the deficiency of being entirely ignorant of causality, or the distinction between retarded and advanced Green’s functions. To get a reasonable answer for $S_2(\omega)$ and other quantities, we need some recipe for adding the missing information. Such a recipe is available in the perturbative domain of large $\omega$, where the integral for $Z_{m,n}(\omega)$ can be evaluated by stationary-phase approximation around the saddle points of the integrand. As in the case of the one-point function, the recipe is to select those saddle points that contribute with the behavior dictated by causality. These are the “good” or “causal” saddle points, and they are retained. The remaining ones are “bad” or “acausal” and are thrown away.
expressions for \(S\) (causal) analytic continuation of the volume of these manifolds. The asymptotic

\[ m \]

method \[4\]. It is reasonable to expect that this agreement is not accidental but

\[ R \]

the two-level correlation function \(R_2(\omega)\), have been worked out in Refs. \[4 - 6\]. It was found that only a small number of saddle-point manifolds \((p = 0, 1\) for \(\beta = 1, 2\) and, in addition, \(p = 2\) for \(\beta = 4\)) survive in the limit \(m = n = 0\). The mechanism for termination is again the dependence on \(m + n\). Growth would be incompatible with the analytic properties of the Green’s function.) By the same token, the saddle point \(Q = -\Sigma_3\) is bad and is thrown away. When the sign of \(\Im \omega\) is changed, the roles of \(Q = \Sigma_3\) and \(Q = -\Sigma_3\) are reversed. Note that the two matrices \(Q = \pm \Sigma_3\) are special in that they are invariant under the action of the symmetry group \(H\), which means they really are isolated saddle points.

Other solutions \(Q_1, Q_2, \ldots\) of the saddle-point equation are obtained by starting from the good \(Q_0 = \Sigma_3\) and exchanging the positions of a small number \(p\) (or \(2p\)) of the \(m\) (or \(2m\)) entries \(+1\) with the same portion of the \(n\) (or \(2n\)) entries \(-1\). Since they descend from a causal parent, the saddles so obtained are expected to be still “causal” and are retained. Their negatives \(-Q_1, -Q_2, \ldots\) are thrown away. Unlike \(Q_0\), the diagonal matrices \(Q_1, Q_2, \ldots\) are not stable under the action of the symmetry group \(H\) but belong to saddle-point manifolds or orbits generated by \(H\). Once again, these manifolds belong to the category of symmetric spaces. For example, the orbit of \(H = U(m) \times U(n)\) on \(Q_1\) for \(\beta = 2\) is

\[
\left( \frac{U(m)}{U(m-1)} \times U(1) \right) \times \left( \frac{U(n)}{U(n-1)} \times U(1) \right).
\]

The contributions of all these saddles to the large-\(\omega\) asymptotics of \(S_2(\omega)\), or

\[ \beta = 1:\ 1 - 2 \omega^{-2} + \omega^{-4} e^{-2i\omega} \]

\[ \beta = 2:\ 1 - \omega^{-2} + \omega^{-2} e^{-2i\omega} \]

\[ \beta = 4:\ 1 - \frac{1}{2} \omega^{-2} + (\pi/2) \omega^{-1} e^{-2i\omega} + (2\omega)^{-4} e^{-4i\omega}. \]

The real parts of these expressions agree with the asymptotic limits that are known, for \(R_2(\omega)\), from orthogonal polynomials \[1\] or the supersymmetric method \[8\]. It is reasonable to expect that this agreement is not accidental but extends to all orders in the asymptotic expansion in \(\omega^{-1}\). In other words, we conjecture that the fermionic replica trick in its recent elaboration is perturbatively equivalent to the supersymmetric method. Also, note that the expression for \(\beta = 2\) is not just asymptotic but is exact! We will elaborate on this later.

What about the bosonic replica trick? For the one-point function we saw that there is a basic asymmetry between bosonic and fermionic replicas. The same
is true here. While the use of fermionic replicas for $Z_{m,n}(\omega)$ leads to a compact symmetric space $X_{m,n}$, bosonic replicas for $Z_{m,-n}(\omega)$ lead to a noncompact analog space, $Y_{m,n}$ (which is still a Riemannian symmetric space, albeit of noncompact type.) The pairing of compact spaces with their noncompact analogs is given in the following list:

- $\beta = 1$: $\text{Sp}(2m + 2n)/\text{Sp}(2m) \times \text{Sp}(2n)$
- $\beta = 2$: $\text{U}(m + n)/\text{U}(m) \times C\text{U}(n)$
- $\beta = 4$: $\text{O}(2m + 2n)/\text{O}(2m) \times \text{O}(2n)$

The integral representation for $Z_{m,-n}(\omega)$ in the universal low-frequency limit is still of the form

$$Z_{m,-n}(\omega) = \int_{Y_{m,n}} \exp \left( -\frac{i\omega}{2\alpha} \text{Tr} Q \right) \, dQ.$$  

Moreover, the integration manifold $Y_{m,n}$ is still generated by acting with the noncompact large group $G$ on the diagonal matrix $\Sigma_3$ by conjugation: $Q = g\Sigma_3 g^{-1}$ ($g \in G$), and we still have the invariant saddle point $Q_0 = \Sigma_3$. The large-$\omega$ asymptotic expansion around this saddle point of $Y_{m,n}$ is known \[3\] to give the same results as the expansion around $Q_0 = \Sigma_3$ for $X_{m,n}$, at $m = n = 0$. What is different is that $Q_0$ now is the only saddle point. There are no others that lie directly on $Y_{m,n}$, although there do exist saddle-point manifolds that can be reached from $Y_{m,n}$ by moving a finite distance along an imaginary direction of the real manifold $Y_{m,n}$. For example, for $\beta = 2$ we set $A = E_{1,m+1} + E_{m+1,1} \in \text{Lie U}(m,n)$ – where $E_{p,q}$ is the matrix whose entries are zero everywhere except on the intersection of the $p$-th row with the $q$-th column where the entry is unity – and follow the “imaginary” orbit $Q(it) = e^{itA}\Sigma_3 e^{-itA}$. At $t = \pi/2$ we reach $Q(i\pi/2) = Q_1$. By applying the symmetry group $H = \text{U}(m) \times \text{U}(n)$ to $Q_1$, we arrive at an analog of the $p = 1$ saddle-point manifold of the fermionic variant. If one evaluates the integrand on this saddle-point manifold, one gets a factor $e^{-i(m+n-2)\omega}$. For $m = n = 0$, this grows as $\omega$ moves into the lower half of the complex plane, which is unphysical. Hence, this saddle-point manifold must be discarded, and we are back to the statement \[4\] \[5\] that the large-$\omega$ limit of the function $Z_{m,n}(\omega)$ for negative $m$ and $n$ is exhausted by a single saddle point ($p = 0$). In summary, while the fermionic replica trick suffers from an excess of saddle points, the bosonic version has “too few”.

Although the recipe of selecting good saddle points in the fermionic formulation appears to work, it does need further justification. A more proper procedure is to combine the information from bosonic and fermionic replicas and write down a bona fide analytic continuation for $Z_{m,n}(\omega)$ away from the integers. The proposal made in \[4\] \[5\] is again to extend the sum over saddle-point manifolds $p$ all the way up to infinity. Doing so, however, we run into the same difficulty we analysed in considerable detail for the one-point function. We shall not repeat this analysis here, but only summarize the facts: for noninteger $m$ and/or $n$, the coefficients in the sum over $p$ grow in a factorial manner, so that the sum diverges and does not define an analytic continuation of $Z_{m,n}(\omega)$ to
In the absence of a well-defined analytic continuation we, of course, have no mathematical control on the limit \( -\lim_{m,n \to 0} (mn)^{-1} \partial^2 Z_{m,n}/\partial \omega^2 \).

My interpretation of the divergence is the same as before. For a large but fixed value of \( \omega \), the saddle-point approximation to \( Z_{m,n}(\omega) \) eventually breaks down (or at least it does for \( \beta = 1, 4 \)) when \( m \) or \( n \) becomes too large. Indeed, for large \( m = n \), we should introduce standard polar coordinates on \( X_{m,n} \) to cast \( Z_{m,n}(\omega) \) in the form of a Coulomb gas partition function, and look for a mean field of the Coulomb gas. Analytic continuation in the particle number \( m = n \) of the Coulomb gas looks benign. We therefore expect the divergent sum over saddle-point manifolds \( p \) to get cut off at large values of \( m, n \). The existence of such a cutoff implies that there exists a nonperturbative correction which we are missing when writing the sum over saddle-point manifolds \( p \). For large \( \omega \), this correction will not affect the extrapolation to \( m = n = 0 \). On the other hand, for small \( \omega \) the correction may cause an uncontrollable error.

The remaining question is: what happens beyond perturbation theory? Is it reasonable to expect that replicas will penetrate the nonperturbative small-\( \omega \) regime? This is hard to answer, but one warning can be issued with certainty. A series expansion obtained by saddle-point approximation is almost always asymptotic, which means the series diverges. What happens is that for a fixed value of the expansion parameter, \( 1/\omega \) in our case, the series initially becomes a better approximation with increasing order, but eventually turns away and explodes. Without knowing the exact answer, it is often difficult to locate the turning point. For this reason, asymptotic expansions hardly deserve to be called “systematic”. The only safe way of using an asymptotic series is to fix the order of approximation and then lower the expansion parameter \( 1/\omega \) accordingly, so as to make the error term negligible. This limits the usefulness of asymptotic series in practice. To do better, one needs to establish Borel summability and carry out Borel resummation. This is already nontrivial for integrals with one saddle point, and becomes much more difficult when two or more saddle points are involved. On the other hand, not all saddle-point approximations are asymptotic, as is demonstrated by the case of the two-point function for \( \beta = 2 \). We will explain in the next section what is special about that example.

The nonperturbative ambiguities of the replica trick for disordered electron systems are not restricted to the zero-dimensional limit but also affect the replica field theory for \( d \)-dimensional systems. Let me finish the section with an example demonstrating this.

Disordered two-dimensional electrons in a strong magnetic field display the (integer) quantum Hall effect. A replica field theory for such systems was proposed long ago by Pruisken [16]. The theory is a compact nonlinear \( \sigma \) model with Lagrangian

\[
L = \frac{\sigma_{xx}}{8} \text{Tr} \partial_\mu Q \partial_\mu Q + \frac{\sigma_{xy}}{8} \epsilon^{\mu\nu} \text{Tr} Q \partial_\mu Q \partial_\nu Q ,
\]

where the field \( Q = U \Sigma U^{-1} \) parametrizes \( U(m+n)/U(m) \times U(n) \). Without loss in the zero replica limit, we may take \( m = n \). The coupling constants

\( m, n \notin \mathbb{Z} \).
σ_{xx} and σ_{xy} are interpreted as the conductivities (dissipative and Hall) of the noninteracting electron gas. The Hall conductivity σ_{xy} determines the current response transverse to an applied electric field. In particular, it determines the orientation of the current flow around the boundary of a finite sample [17].

Now comes a disaster. Use of the isomorphism \( Q \mapsto -Q \) in Pruisken’s Lagrangian changes the sign of the term \( \epsilon^{\mu\nu}\text{Tr}Q \partial_\mu Q \partial_\nu Q \). Thus the field theory coupling σ_{xy} is defined only up to a sign. For \( m = n = 1 \), where the above field theory (with \( \sigma_{xy} = S \)) is known [18] to describe antiferromagnetic quantum spin-S chains, this indeterminacy makes sense. Antiferromagnets carry no sense of orientation. But for \( n = 0 \), σ_{xy} is supposed to be the physical Hall conductivity, so the field theory makes the nonsensical prediction that the orientation of the current response of a quantum Hall sample is ill-determined?!

Of course, this is just another manifestation of the causality problem we have been emphasizing all along. Can it be cured? Pruisken’s field theory is known to flow to the strong coupling (or small σ_{xx}) regime, where the dominant field configurations form a dense gas of interacting instantons and anti-instantons. In such a nonperturbative soup of saddle points and almost-saddle points, it seems quite hopeless to try and disentangle the good configurations from the bad ones.

4 Semiclassical exactness for \( \beta = 2 \)

We have seen that the fermionic replica trick, augmented by an inspired recipe for selecting good saddle points, for \( \beta = 2 \) already gives the correct answer for the universal limit of \( S_2(\omega) \) when the saddle-point approximation is carried out to leading order in the small parameter \( 1/\omega \). Apparently, the saddle-point approximation in this case is exact, not just approximate! We call integrals where this miracle happens “semiclassically exact”. In the present section we will explain the mathematical basis underlying the phenomenon of semiclassical exactness: the Duistermaat-Heckman theorem. This is a celebrated result in symplectic geometry [19–22] and is included here for the convenience of the targeted reader. A few references congenial to physicists are [23, 24, 25].

Let \( M \) be a symplectic manifold of dimension \( 2f \). In physics such a manifold is called a phase space with \( f \) degrees of freedom. Simple examples are the two-sphere \( S^2 \) or the two-torus \( T^2 \). The example we are particularly concerned with is the coset space \( U(m+n)/U(m) \times U(n) \), which has dimension \( 2f = 2mn \). A symplectic manifold \( M \) comes with a symplectic structure, \( i.e., \) a closed and nondegenerate two-form \( \Omega \). For the two-sphere, \( \Omega \) is the solid angle, expressed in spherical polar coordinates \( \theta, \phi \) by \( \sin \theta d\theta \wedge d\phi \). For the space \( U(m+n)/U(m) \times U(n) \) parametrized by \( Q \), the symplectic structure is \( (8i)^{-1}\text{Tr} Q dQ \wedge dQ \). By Darboux’s theorem, there exist local coordinate systems made up of positions \( q^i \) and momenta \( p_i \), such that \( \Omega \) takes the canonical form \( \Omega = \sum_{i=1}^f dp_i \wedge dq^i \). Let \( \Omega^f = \prod_{i=1}^f dp_i \wedge dq^i \) denote the Liouville form on \( M \). In the case of interest, \( \Omega^f \) agrees with the volume element \( dQ \) up to a multiplicative constant.

The Duistermaat-Heckman theorem in the formulation given below makes a
statement about integrals of the form

\[ Z(\omega) = \int_M \Omega^f e^{i\omega H}, \]

where the function \( H : M \rightarrow \mathbb{R} \) is required to possess the following property. Viewing \( H \) as a Hamiltonian function, we have the canonical equations of motion of Hamilton mechanics:

\[ \dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}. \]

The solutions of these equations are conveniently assembled into a map \( M \times \mathbb{R} \rightarrow M, (x, t) \mapsto \psi_t(x) \), called the phase flow of \( H \). The crucial property we shall require of \( H \) is that there exist some Riemannian metric on \( M \) which is preserved by the flow \( \psi_t \).

Before stating the Duistermaat-Heckman theorem, let us look at a few examples where this condition is satisfied. The simplest one is provided by the Hamiltonian for a classical spin in a magnetic field \[23\]. The phase space \( M \) in this case is the two-sphere \( S^2 \), and the motion is precession of the spin around the axis of the magnetic field. The frequency of precession, the Larmor frequency, does not depend on the tilt angle between the magnetic field axis and the axis of the spin. Thus the phase flow \( \psi_t \) is simply uniform rotation around the field axis, with the rotation angle being a linear function of time. Clearly, this flow preserves the natural metric \( d\theta^2 + \sin^2 \theta d\phi^2 \) of \( S^2 \).

The classical spin in a magnetic field is but one of a large family of integrable Hamiltonian systems with the same property. A brief sketch is as follows. Let \( G \) be a compact semisimple Lie group with maximal abelian subgroup (or maximal torus) \( T \). We fix some regular element \( A \) of the Lie algebra \( \text{Lie}(T) \), and consider the adjoint orbit \( \text{Ad}(G)A \) consisting of elements \( UAU^{-1} \). The adjoint orbit comes with a natural Hermitian structure, i.e., a symplectic structure \( \Omega \) as well as a metric tensor \( g \), which we refrain from writing down here as this would lead us just a little too far (for these details consult Appendix 2 of \[26\]). Note that we get the phase space \( S^2 \) of the classical spin by putting \( G = \text{SU}(2) \) and \( A = \sigma_3 \). Now we pick a quadratic form, say \( \langle X, Y \rangle = \text{Tr} XY \), on the Lie algebra \( \text{Lie}(G) \), and we fix another regular element \( B \) of \( \text{Lie}(T) \) to form the function \( H = \text{Tr} BU AU^{-1} \). We may view the adjoint orbit \( \text{Ad}(G)A \) as a phase space, with \( H \) being a Hamiltonian function on it. The phase flow of this Hamiltonian turns out to be \( \psi_t : UAU^{-1} \mapsto e^{tB} UAU^{-1} e^{-tB} \), and this flow preserves the Riemannian metric \( g \) of the adjoint orbit. We are mentioning this class of examples because it will be seen to give rise to the Itzykson-Zuber formula.

Our example of interest is the function \( H = \text{Tr} \ hQ/2 \), with

\[ h = \text{diag}(h_+, h_-) = \text{diag}(h_{+1}, \ldots, h_{+m}; h_{-1}, \ldots, h_{-n}) \]

a diagonal real matrix, on the symplectic manifold \( \text{U}(m+n)/\text{U}(m) \times \text{U}(n) \). (This example can actually be regarded as a degenerate limit of the above general
To construct its phase flow in an elementary fashion, we express $Q$ as

$$Q = \left( \frac{1 - 2BB^\dagger}{2B^\dagger(1 - BB^\dagger)} \quad 2B\sqrt{1 - B^\dagger B} \right),$$

where $B$ is a complex $m \times n$ matrix with range $0 \leq B^\dagger B \leq 1$. The symplectic structure and the Hamiltonian for this choice of coordinates take the form

$$\Omega = (8i)^{-1}\text{Tr} QdQ \wedge dQ = i^{-1}\text{Tr} dB \wedge dB^\dagger,$$

$$H = \text{Tr} hQ/2 = \text{const} - \text{Tr} h_+ BB^\dagger + \text{Tr} h_- B^\dagger B.$$

The phase flow $\psi_t$ is then readily seen to be $B \mapsto e^{i\text{th} + Be^{-i\text{th}}}$, or in the original representation, $\psi_t : Q \mapsto e^{i\text{th} Qe^{-i\text{th}}}$.

Clearly, this preserves the natural Riemannian metric $\text{Tr} dQ^2$.

We now come to the Duistermaat-Heckman theorem. The theorem is rooted in equivariant cohomology, and its best version relies on no more than Stokes’ formula for differential forms that are equivariantly closed for a compact group action. In particular, no reference to any metric structure is needed. Nevertheless, to avoid mathematical overload, we shall permit ourselves the luxury of using a metric.

**Theorem.** Let $(M, \Omega, H)$ be a Hamiltonian system with compact $2f$-dimensional phase space $M$, and let its phase flow preserve a Riemannian metric on $M$. Then the integral $\int_\mathbb{R}^f \Omega^f e^{iH}$ localizes on the critical set of $H$.

**Remark.** What the theorem is saying is that the integral is completely determined by the values of the function $H$, and a finite number of derivatives thereof, on the set of solutions of $dH = 0$ (the saddle points or saddle-point manifolds). In other words, the stationary-phase approximation is exact.

**Proof.** We will employ a method of proof originally due to Bismut [27] (see also Refs. [28, 29]), which uncovers and takes advantage of a hidden supersymmetry. Let $x^i \ (i = 1, \ldots, 2f)$ be a system of local coordinates of $M$, in which the symplectic structure is expressed by $\Omega = \frac{1}{2} \Omega_{jk}(x) dx^j \wedge dx^k$ (summation convention). We supersymmetrically extend the phase space $M$ by introducing for every $x^i$ a corresponding anticommuting coordinate $\xi^i$. Consider then the expression

$$\int \exp \left( -\frac{1}{2} \Omega_{jk}(x) \xi^j \xi^k + iH(x) \right)$$

where $\int$ means integration with the flat Berezin form $dx^1 \ldots dx^{2f} \partial_{\xi^1} \ldots \partial_{\xi^{2f}}$. By using canonical coordinates, it is not difficult to see that the Fermi integral, i.e., differentiation with respect to the anticommuting variables and multiplication by $dx^1 \ldots dx^{2f}$, produces the Liouville form $\Omega^f$. Hence we have

$$\int \exp \left( -\frac{1}{2} \Omega_{jk} \xi^j \xi^k + iH \right) = \int_M \Omega^f e^{iH}.$$
Next, using the closedness of $\Omega$ ($\xi^j \xi^k \partial \Omega_{jk} / \partial x^l = 0$), one verifies that the exponent \( S \equiv -\frac{1}{2} \Omega_{jk} \xi^j \xi^k + iH \) has the property of being annihilated by 

\[
D = \xi^j \frac{\partial}{\partial x^j} + i\Omega_{jk} \frac{\partial H}{\partial x^j} \frac{\partial}{\partial \xi^k}.
\]

\( D \) is a first-order differential operator of odd type, and is analogous to the BRST operator that plays a central role in the functional integral quantization of nonabelian gauge theories \[30\]. The key step now is to deform the integral by a parameter \( t \):

\[
\int_M \Omega^f e^{iH} = \int e^S = \int e^{S + tD\lambda},
\]

where the function \( \lambda \) is constrained by \( D^2 \lambda = 0 \) and will be specified presently. The Berezin integral on the right-hand side has the crucial property of being independent of \( t \). Indeed, by Taylor expanding,

\[
\int e^{S + tD\lambda} = \int e^S \left( 1 + tD\lambda + \frac{1}{2} t^2 (D\lambda)^2 + \ldots \right),
\]

and using \( D^2 \lambda = 0 \) and partial integration in conjunction with \( DS = 0 \), one sees that the terms of linear and higher order in \( t \) all vanish. To reap full benefit from the \( t \)-independence of the integral, we pick a well-chosen Riemannian metric \( g \) and set

\[
\lambda = -ig^{jk} \Omega_{kl} \frac{\partial H}{\partial x^j} \xi^l.
\]

It can be shown that this expression satisfies the condition \( D^2 \lambda = 0 \) if and only if the metric \( g \) is invariant under the phase flow of \( H \). By assumption, such a metric \( g \) exists. The last step is to take the parameter \( t \) to infinity. Since the number part of

\[
D\lambda = -g^{jk} \frac{\partial H}{\partial x^j} \frac{\partial H}{\partial x^k} + O(\xi \xi) = -|dH|^2 + O(\xi \xi)
\]

is negative definite, this limit localizes the integral \( \int_M \Omega^f e^{iH} = \int e^{S + tD\lambda} \) onto the critical set of \( H \), i.e., the solutions of the saddle-point equation \( dH = 0 \). This concludes the proof.

If the saddle points of \( H \) are isolated, one can easily push the calculation further and write down an explicit formula for the integral as a discrete sum:

\[
\int_M \Omega^f e^{iH} = (2\pi)^f \sum_{x: d_x H = 0} i^{\text{sgn}(\text{Hess}_x H)} \frac{e^{iH(x)}}{\sqrt{\text{Det}(\text{Hess}_x H)}} \sqrt{\text{Det}(\text{Hess}_x H)}
\]

where the phase of the contribution from each saddle point is determined by the signature of the Hessian of \( H \) at \( x \). For the classical spin in a magnetic field, this gives

\[
\int e^{in B} d^2 \mathbf{n} = 2\pi \frac{e^{i|B|} - e^{-i|B|}}{i|B|},
\]

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the correctness of which is easily verified by direct computation. Less trivial integral formulas result on taking for the Hamiltonian system \((M, \Omega, H)\) the adjoint orbit of a compact Lie group \(G\), with the Hamiltonian being \(H = \text{Tr} AUBU^{-1}\). If both \(A\) and \(B\) are regular elements in a Cartan subalgebra of \(G\), one obtains

\[
\int_G e^{\text{Tr} AUBU^{-1}} dU = \left(p(A)p(B)\right)^{-1} \sum_{w \in W_G} (-1)^{|w|} e^{\text{Tr} A w B},
\]

where \(p(A) = \prod_{\alpha > 0} \alpha(A)\) is a product over positive roots, the sum runs over the Weyl group \([11]\) of \(G\), and \(|w|\) denotes the parity of \(w\). (Regularity of \(A\) means that \(p(A)\) is nonzero.) \(dU\) is a suitably normalized Haar measure of \(G\). Note that the integral can and should actually be understood as being over the symplectic quotient \(G/T\). That the factor of \(i\) has disappeared from the exponent is of no concern. Looking back at the proof of the Duistermaat-Heckmann theorem, we see that \(i\) didn’t play any role: the underlying principle is localization on the critical set, not stationary phase.

The above formula is due to Harish-Chandra (1957) who proved it by a quite different method, namely by computing the radial parts of \(G\)-invariant differential operators \([31]\). Specializing to the case \(G = U(N)\) (or \(SU(N)\), it doesn’t matter) and setting \(A = \text{diag}(A_1, \ldots, A_N)\), \(B = \text{diag}(B_1, \ldots, B_N)\) we get

\[
\int_{U(N)} e^{\text{Tr} AUBU^{-1}} dU = \frac{\text{Det}(e^{A_i B_j})_{i,j=1,\ldots,N}}{\prod_{i<j}(A_i - A_j)(B_i - B_j)^2},
\]

which is known in physics as the Itzykson-Zuber formula.

We finally understand the semiclassical exactness of the replica integral for the \(\beta = 2\) two-point function in the universal limit. The nonlinear \(\sigma\) model manifold \(U(m + n)/U(m) \times U(n)\) is a \(2mn\)-dimensional phase with symplectic structure \((8\pi)^{-1}\text{Tr} Q dQ \wedge dQ\), and the flow of the “Hamiltonian” \(H = \omega \text{Tr} \Sigma_3 Q/2\) preserves a Riemannian metric \(g = \text{Tr} dQ^2\). Therefore, the Duistermaat-Heckman localization principle applies, and the integral \(\int \exp(i\omega \text{Tr} \Sigma_3 Q/2) dQ\) is evaluated exactly by saddle-point approximation.

## 5 Conclusion

The replica trick for spectral correlations of disordered electron systems is mathematically ill-founded because analytic continuation of the generating function

\[
Z_{m,n}(u, v) = \left< \text{Det}^m(u - H) \text{Det}^n(v - H) \right>
\]

away from the integers \(m, n\) is not unique. In benign cases, such as the famous Selberg integral, uniqueness is guaranteed by a boundedness property as spelled out by Carlson’s theorem or Carleman’s theorem \([22]\). It appears that no such property is available in the present case. (However, it has been suggested to me
that this problem might be overcome by considering a generating function of the positive real form

\[
\left\langle \text{Det}^m \left( (E_1 - H)^2 + \epsilon^2 \right) \text{Det}^n \left( (E_2 - H)^2 + \epsilon^2 \right) \right\rangle.
\]

For positive \(m, n\) (the fermionic replica trick) the generating function is analytic in \(u\) and \(v\), which means that the entire information about causality of Green’s functions is missing. (Of course, the enlightened user knows about causality, but fermionic replicas by themselves do not.) In recent work by Kamenev and Mezard, and by Yurkevich and Lerner, causality was introduced into the formalism by selecting a well-chosen set of saddle points of the compact nonlinear \(\sigma\) model based on fermionic replicas. In this way, the known large-frequency asymptotics of the level correlation functions for disordered electron systems was reproduced for all cases \(\beta = 1, 2, 4\). This motivates the conjecture that the fermionic replica trick (augmented by the selection of “causal” saddle points) is perturbatively equivalent to the supersymmetric method.

Close inspection of Refs. [5, 6, 7] reveals that the procedure used there is mathematically uncontrolled, as it involves a nonexistent analytic continuation. The fact of the matter is that the sum over saddle-point manifolds \(p = 0, 1, \ldots \infty\) diverges for noninteger \(m, n\). We have argued that this divergence is likely to be cut off by a nonperturbative mechanism. Although such a cutoff is immaterial for the analytic continuation to \(m = n = 0\) for large frequency \(\omega\), it may introduce uncontrollable errors for small \(\omega\). In any case, the method as it stands is limited to the perturbative regime, as it relies on stationary-phase evaluation of integrals, which requires a small parameter \(1/\omega\).

A miraculous exception is Wigner-Dyson statistics for \(\beta = 2\), where a hidden supersymmetry, namely equivariant cohomology and the localization principle underlying the Duistermaat-Heckman theorem, localizes the nonlinear \(\sigma\) model integrals on the saddle points. Put differently, the leading-order stationary-phase approximation in this case is exact, regardless of whether \(\omega\) is large or small. The same mechanism is at work in the supersymmetric formulation.

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