The Cumulant Bijection and Differential Forms

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Description

Given a graded commutative algebra $A$ there is a canonical (tautologous) map $\tau : SA \to A$ where $SA$ is the graded commutative algebra generated by $A$. The map $\tau$ is given by the formula

$$\tau(x_1 \land x_2 \land \ldots x_n) = x_1 x_2 \ldots x_n$$

$SA = A \oplus A \land A \oplus \ldots$ also has a coproduct structure $\Delta : SA \to SA \land SA$ given by the formula

$$\Delta(x_1 \land x_2 \land \ldots x_n) = \sum \epsilon \cdot x_{\pi_1} \land x_{\pi_2}$$

where $\pi_1 \sqcup \pi_2$ is a partition of the indices into two arbitrary subsets and $\epsilon$ is the sign that arises when the first subset is moved all the way to the left. One knows that any map like $\tau : SA \to A$ lifts to a canonical coalgebra mapping $\tilde{\tau} : SA \to SA$ so that $\pi \circ \tilde{\tau} = \tau$ where $\pi : SA \to A$ is projection onto $A$, the linear summand of $SA$.

We verify that $\tilde{\tau}$ is given by the formula

$$\tilde{\tau}(v) = \tau(v) + \tau \land \epsilon \circ \Delta(v) + \tau^{\land 3} \circ \Delta^2(v) + \ldots$$

(1)

Note that $\tilde{\tau}(v) = v + \text{lower order terms}$. This implies that it is a bijection. We call this coalgebra isomorphism $\tilde{\tau}$ the \textit{cumulant bijection}.

The relation of $\tilde{\tau}$ to cumulants appears when it is used to change coordinates in $SA$, in particular when it is used to conjugate extensions to $SA$ of a linear mapping of $A$ to either a coderivation of $SA$ or to a coalgebra mapping of $SA$.

General coderivations $D : SA \to SA$ have canonical decompositions by orders $D = D_1 + D_2 + \ldots$ where $D_k$ is determined by “Taylor coefficients” $D^k$:

$$D^1 : A \to A$$

$$D^2 : A \land A \to A$$

$$D^3 : A \land A \land A \to A$$

and so on.

The explicit formulae relating $D_k$ to $D^k$ are

$$D_k(x_1 \land \ldots \land x_n) = \sum \epsilon \cdot D^k(x_I) \land \ldots \hat{x}_I \ldots$$
where the sum is taken over all subsets \( I \) of size \( k \) of the indexing set.

A coalgebra mapping \( G : SA \to SA \) is also determined by independent maps (also called the Taylor coefficients)

\[
G^1 : A \to A \\
G^2 : A \wedge A \to A \\
G^3 : A \wedge A \wedge A \to A \\
\text{and so on by the formula}
\]

\[
G(x_1 \wedge x_2 \wedge \ldots) = \sum_{\text{partitions}} \epsilon \cdot G^{j_1}(x_{\pi_{j_1}}) \wedge G^{j_2}(x_{\pi_{j_2}}) \ldots
\]

The canonical extension of a map \( A \to A \) to either a coderivation or a coalgebra mapping, only has a non-zero leading order coefficient. After conjugating the canonical extensions by \( \tilde{\tau} \) they have in general Taylor coefficients of all orders.

In the case of a coalgebra extension \( \tilde{f} \) of \( f : A \to A \) the conjugated \( g = \tilde{\tau}^{-1} \tilde{f} \tilde{\tau} \) has coefficients which measure the deviation of \( f \) from being an algebra homomorphism of \( A \).

The first few Taylor coefficients of \( g \) are

\[
g^1(x) = f(x) \\
g^2(x,y) = f(xy) - f(x)f(y) \\
g^3(x,y,z) = f(xyz) - f(xy)f(z) - f(yz)f(x) - f(zx)f(y) + 2f(x)f(y)f(z)
\]

In the case of a coderivation extension \( \hat{f} \) of \( f \), the first few coefficients of the conjugate \( h = \hat{\tau}^{-1} \hat{f} \hat{\tau} \) are

\[
h^1(x) = f(x) \\
h^2(x,y) = f(xy) - f(x)y - xf(y) \\
h^3(x,y,z) = f(xyz) - f(xy)z + xyf(z) - f(yz)x + yxf(x) - f(zx)y + zxf(y)
\]

In the first case all the higher Taylor coefficients vanish iff \( f(xy) - f(x)f(y) = 0 \), that is \( f \) is an algebra map.

In the second case all higher Taylor coefficients vanish iff \( f(xy) - f(x)y - xf(y) = 0 \), that is \( f \) is a derivation.

1 First Application

Let us now consider the case where \( A \) is a chain complex with a differential \( \partial \) which is not necessarily a derivation of the product structure. Suppose \( i : (C, \partial_C) \to (A, \partial) \) is a sub-complex of \((A, \partial)\) such that \( A \) deformation retracts to \( C \). That is, there is \( I : (A, \partial) \to (C, \partial_C) \) so that \( I \circ i = \text{Id}_C \) and there is a
degree +1 mapping \( s : A \to A \) so that \( \partial s + s \partial = i \circ I - I d_A \). Now let \( \tilde{I} \) and \( \tilde{i} \) be the canonical lifts of \( I \) and \( i \) to coalgebra maps between \( SA \) and \( SC \).

\[
\begin{array}{ccc}
SA & \xrightarrow{\pi} & A \\
\downarrow{\tilde{i}} & & \downarrow{\tilde{I}} \\
SC & \xrightarrow{i} & C
\end{array}
\]

Let \( d \) be the extension of \( \partial \) to \( SA \) and \( \tilde{d} = \tilde{\tau}^{-1} d \tilde{\tau} \). Then by our previous discussion the cumulant bijection \( \tilde{\tau} \) gives an isomorphism between \((SA, d)\) and \((SA, \tilde{d})\).

**Theorem 1.** Let \( C \) be as above. Suppose there is a coderivation \( \partial_\infty \) on \( SC \) and a differential graded coalgebra map \( \iota \) from \((SC, \partial_\infty)\) to \((SA, d)\) which extends \( i \). Then there is an induced isomorphism, “the induced cumulant bijection” \( \tilde{\tau}_C \) between \((SC, \partial_\infty)\) and \((SC, d_C)\) where \( d_C \) is the canonical coderivation extension of \( \partial_C \). \( \tilde{\tau}_C \) is uniquely characterized by the commutativity of the following diagram.

\[
\begin{array}{ccc}
(SA, \tilde{d}) & \xrightarrow{\tilde{\tau}} & (SA, d) \\
\downarrow{\iota} & & \downarrow{\tilde{I}} \\
(SC, \partial_\infty) & \xrightarrow{\tilde{\tau}_C} & (SC, d_C)
\end{array}
\]

**Proof.** As \( \iota \) is an injection and \( \tilde{I} \) is a surjection, \( \tilde{\tau}_C \) must be defined to be equal to \( \tilde{I} \circ \tilde{\tau} \circ \iota \). The fact that \( \tilde{\tau}_C \) is an isomorphism follows as in Lemma 2 from the fact that since \( \iota \) agrees with \( i \) on the linear terms and \( I \circ i \) is identity, \( \tilde{\tau}_C \) induces the identity map on linear terms. Then since \( \tilde{\tau}_C \) is a coalgebra map the inductive step of Lemma 2 holds. For more details see section 3.

**Remark 1.** Note that \( SC \) receives the induced cumulant bijection from the map \( \iota \) and not from a commutative algebra structure on \( C \).

## 2 The missing details, the cumulant bijection in the associative context and more applications

**Definition 1.** A **graded coassociative conilpotent coalgebra** is a graded vector space \( C \) together with a degree zero coassociative coproduct \( \Delta : C \to C \otimes C \) with the property that for all \( v \) in \( C \) there exists an integer \( n \) such that \( \Delta^n(v) = 0 \). If the image of the coproduct is in the graded symmetric tensors then \( C \) is called the **graded symmetric coassociative conilpotent coalgebra**.
Definition 2. The free coassociative conilpotent coalgebra without co-unit generated by a graded vector space $V$ is a nilpotent coalgebra $T^c V$ with a projection map onto $V$ with the following universal property: Given a linear map from a graded nilpotent coassociative coalgebra to $V$ there exists a unique coalgebra map $\tilde{f}$ from $C$ to $T^c V$ such that the following diagram commutes.

\[
\begin{array}{ccc}
C & \xrightarrow{\exists \tilde{f}} & T^c V \\
\downarrow{f} & & \downarrow{\pi} \\
V & & V
\end{array}
\]

(2)

On the space $TV = V \oplus V^\otimes 2 \oplus V^\otimes 3 \ldots$ consider the coproduct $\Delta$ given by the following formula.

\[
\Delta(x_1 \otimes x_2 \otimes \ldots \otimes x_n) = \sum_{i=1}^{n-1} x_1 \otimes \ldots \otimes x_i \otimes \ldots \otimes x_n
\]

$TV$ with the coproduct $\Delta$ is the universal free conilpotent coalgebra associated to $V$. The formula for the lift is given later in Lemma 1.

Definition 3. The free graded coassociative cocommutative nilpotent coalgebra generated by $V$ is a sub-coalgebra $S^c V$ of $T^c V$ such that if $C$ taken as above is also graded cocommutative then the image of $\tilde{f}$ lies in $S^c V$, that is the following diagram commutes.

\[
\begin{array}{ccc}
C & \xrightarrow{\exists \tilde{f}} & S^c V \\
\downarrow{f} & & \downarrow{\pi} \\
V & & V
\end{array}
\]

(3)

\[
S^c V = \bigoplus_{n=1}^{\infty} S^n V
\]

where $S^n V$ is the subspace of $V^\otimes n$ generated by

\[
\sum_{\sigma \in S_n} \pm x_{\sigma(1)} \otimes x_{\sigma(2)} \ldots x_{\sigma(n)}
\]

The signs in the summation are given by the degrees of $x_i$, for example in $S^2 V$ is generated by elements of the form $x \otimes y + (-1)^{|x||y|} y \otimes x$. The coproduct $\Delta$ restricts to a coproduct on $S^c V$. $S^c V$ is linearly isomorphic to $SV = V \oplus V \wedge V \oplus V \wedge V \wedge V \ldots$ by the map which sends $x_1 \wedge x_2 \wedge \ldots x_n$ in $SV$ to $\sum_{\sigma \in S_n} \pm x_{\sigma(1)} \otimes x_{\sigma(2)} \ldots x_{\sigma(n)}$ in $S^c V$. Note the product that $TV$ acquires from its isomorphism to $TV$ does not restrict to the product $S^c V$ acquires from its isomorphism to $SV$. Here onwards we will use $SV$ to denote both, the algebra with the usual product structure and the coalgebra with the induced coproduct structure.
Lemma 1. The map \( \tilde{f} \) in (2) is given by the formula,

\[
\tilde{f}(x) = f(x) + f \otimes f \circ \Delta_C(x) + f^{\otimes 3} \circ \Delta_C^2(x) + \ldots
\]

where \( \Delta_C \) is the reduced coproduct on \( C \). Since \( C \) is nilpotent this sum is finite.

Proof. Note that \( \pi \circ \tilde{f}(x) = f(x) \). To prove the lemma we need to show that

\[
\Delta_T \circ \tilde{f} = \tilde{f} \otimes \tilde{f} \circ \Delta_C
\]

The Sweedler notation for higher powers of \( \Delta_C \) applied to \( x \) is

\[
\Delta^n_C(x) = \sum_{(x)} x^{(1)} \otimes x^{(2)} \otimes \ldots \otimes x^{(n+1)}
\]

So the formula for \( \tilde{f} \) as defined becomes

\[
\tilde{f}(x) = \sum_n \sum_{(x)} f(x^{(1)}) \otimes f(x^{(2)}) \otimes \ldots \otimes f(x^{(n+1)})
\]

and when we apply the coproduct to this we get

\[
\Delta_T \circ \tilde{f}(x) = \sum_n \sum_{(x)} \sum_i f(x^{(1)}) \otimes \ldots \otimes f(x^{(i)}) \otimes \ldots \otimes f(x^{(n+1)})
\]

On the other hand we have

\[
\tilde{f} \otimes \tilde{f} \circ \Delta_C(x) = \sum_{(x)} \tilde{f}(x^{(1)}) \otimes \tilde{f}(x^{(2)})
\]

\[
= \sum_{(x)} \sum_{i,j} f(x^{(1)}) \otimes \ldots \otimes f(x^{(i)}) \otimes \ldots \otimes f(x^{(i+j)})
\]

\[
= \sum_n \sum_{(x)} f(x^{(1)}) \otimes \ldots \otimes f(x^{(i)}) \otimes \ldots \otimes f(x^{(n+1)})
\]

which is what is required.

If \( C \) is graded cocommutative then the image of \( \tilde{f} \) is in \( S^cV \) which we have identified with \( SV \). So the formula in Lemma \( \n \) becomes

\[
\tilde{f}(x) = f(x) + f \wedge f \circ \Delta_C(x) + f^{\wedge 3} \circ \Delta_C^2(x) + \ldots
\]

Suppose \( A \) is a graded commutative associative algebra. Then consider the map \( \tau : SA \rightarrow A \) which takes \( x_1 \wedge x_2 \wedge \ldots x_n \) to \( x_1 x_2 \ldots x_n \). We call this map the tautologous map. Since this is a map from a graded cocommutative coalgebra we can lift it to a map \( \tilde{\tau} : SA \rightarrow SA \) and from Lemma \( \n \) the formula for \( \tilde{\tau} \) is given by the equation \( \n \).
Lemma 2. \( \tilde{\tau} : SA \rightarrow SA \) is a coalgebra isomorphism.

Proof. Let \( F_n = \bigoplus_{i=1}^{n} \wedge^i A \) where \( \wedge^i A \) is the graded symmetric product of \( i \) copies of \( A \). Then \( F_n \) defines an increasing filtration of \( SA \) and \( SA \) is the direct limit of this sequence of sub-coalgebras.

\[ A = F_1 \hookrightarrow F_2 \hookrightarrow \ldots \]

For \( v \in \wedge^n A \), \( \tilde{\tau} \) is given by the formula

\[ \tilde{\tau}(v) = \tau(v) + \tau(\Delta(v)) + \ldots + \tau^{\wedge n}(\Delta^{n-1}(v)) \]

Note that the highest degree term in the formula is \( v \) and so this formula preserves the filtration \( F_n \). We will show by induction that \( \tilde{\tau} \) restricted to \( F_n \) is an isomorphism for each \( n \). Now on \( F_1 \), \( \tilde{\tau} \) is identity hence an isomorphism. Now consider the following short exact sequence

\[ 0 \rightarrow F_{n-1} \hookrightarrow F_n \rightarrow \wedge^n A \rightarrow 0 \]

On \( F_{n-1} \), \( \tilde{\tau} \) is an isomorphism by induction hypothesis and on \( \wedge^n A \) the map induced is the identity map. So \( \tilde{\tau} \) is an isomorphism on \( F_n \). Since it’s restriction to each \( F_n \) is an isomorphism, \( \tilde{\tau} \) is an isomorphism from \( SA \) to \( SA \).

A linear map from a graded commutative algebras \( A \) to itself can be extended to either a coderivation or a coalgebra map from \( SA \) to \( SA \). These can then be conjugated by \( \tilde{\tau} \) as it is an isomorphism. The linear map itself doesn’t have to be an algebra map or a derivation.

Theorem 2. i) Suppose \( d \) is any linear map from a graded commutative algebra \( A \) to \( A \). Then there exists a unique coderivation \( \tilde{d} \) from \( SA \) to \( SA \) such that the following diagram commutes.

\[ \begin{array}{ccc}
SA & \xrightarrow{\tau} & A \\
\downarrow{\tilde{d}} & & \downarrow{d} \\
SA & \xrightarrow{\tau} & A 
\end{array} \]  

The coderivation construction, \( d \) gives \( \tilde{d} \), preserves commutators. This implies that if \( d \) has degree \(-1\) and squares to zero then \( \tilde{d} \) also has degree \(-1\) and squares to zero.

ii) Suppose \( f \) is any linear map between graded commutative algebras \( A \) to \( B \). Then there exists a unique coalgebra morphism \( \tilde{f} \) from \( SA \) to \( SB \) such that the following diagram commutes.

\[ \begin{array}{ccc}
SA & \xrightarrow{\tau_A} & A \\
\downarrow{i} & & \downarrow{f} \\
SB & \xrightarrow{\tau_B} & B 
\end{array} \]
Note that the uniqueness implies that the coalgebra construction, \( f \) gives \( \hat{f} \), respects composition.

iii) \((1)\ and\ (2)\) Suppose \( A \) and \( B \) are graded commutative algebras and \( d_A \) and \( d_B \) are linear maps, not necessarily derivations, of degree \(-1\) and square zero on \( A \) and \( B \) respectively. Suppose \( f \) is a linear map, not necessarily an algebra map, from \( A \) to \( B \) such that \( f \circ d_A = d_B \circ f \). Then \( \hat{f} \) as in ii) is the unique homomorphism of differential graded coalgebras from \((SA, \tilde{d}_A)\) to \((SB, \tilde{d}_B)\) such that the following diagram commutes.

\[
\begin{array}{ccc}
(SA, \tilde{d}_A) & \xrightarrow{\tau_A} & (A, d_A) \\
\downarrow f & & \downarrow f \\
(SB, \tilde{d}_B) & \xrightarrow{\tau_B} & (B, d_B)
\end{array}
\] (6)

**Proof.**

i) A linear map \( d \) from \( A \) to \( A \) can be uniquely extended to a coderivation \( D \) on \( SA \) which commutes with the projection map to \( A \) for the coalgebra structure on \( SA \). That is there exists a unique \( D \) such that the following diagram commutes.

\[
\begin{array}{ccc}
SA & \xrightarrow{\pi} & A \\
\downarrow d & & \downarrow d \\
SA & \xrightarrow{\pi} & A
\end{array}
\] (7)

where \( \pi \) is the projection map. \( D \) is given by the following formula.

\[
D(x_1 \wedge x_2 \wedge \ldots \wedge x_n) = \sum_i \pm x_1 \wedge \ldots \wedge d(x_i) \wedge \ldots x_n
\]

The signs in the sum depend on the degrees of \( x_i \) and the degree of the map \( d \). Consider the map \( \tilde{d} := \tilde{\tau} \circ D \circ \tilde{\tau}^{-1} \). This map is the unique coderivation on \( SA \) which makes (4) commute.

If \( d_1 \) and \( d_2 \) are two linear maps of some degrees from \( A \) to \( A \) and let \( D_1 \) and \( D_2 \) be the coderivations on \( SA \) which commute with the projection maps. Since the bracket of two coderivations is a coderivation, \([D_1, D_2]\) is a coderivation. Besides this is the unique coderivation which extends \([d_1, d_2]\). So, \([\tilde{d}_1, \tilde{d}_2]\) is the unique coderivation with the above property for the map \([d_1, d_2]\). Thus if \( d \) has degree \(-1\) and squares to zero, then \( \tilde{d}^2 = 1/2[d, d] \) is a coderivation which uniquely extends \( d^2 \). Since \( d^2 \) is zero, \( \tilde{d}^2 \) will also be zero. So maps of degree \(-1\) which square to zero give coderivations which square to zero on \( SA \).

ii) A linear map \( f \) between \( A \) and \( B \) can be uniquely extended to a homomorphism of coalgebras \( F \) from \( SA \) to \( SB \) which commutes with the
projection maps. That is there exists a homomorphism of coalgebras such that the following diagram commutes.

\[
\begin{array}{ccc}
SA & \xrightarrow{\pi_A} & A \\
\downarrow F & & \downarrow f \\
SB & \xrightarrow{\pi_B} & B
\end{array}
\] (8)

\(F\) is given by the following formula.

\[F(x_1 \wedge x_2 \wedge \ldots x_n) = f(x_1) \wedge f(x_2) \wedge \ldots f(x_n)\]

The map \(\hat{f} = \tilde{\tau} \circ F \circ \tilde{\tau}^{-1}\) is then the unique coalgebra homomorphism which makes (5) commute.

iii) We only need to check that \(\hat{f} \circ \tilde{d}_A = \tilde{d}_B \circ \hat{f}\). Let \(D_A\) and \(D_B\) be the coderivations on \(SA\) and \(SB\) respectively which commute with the projection maps and \(F\) be the coalgebra homomorphism from \(SA\) to \(SB\) which commutes with the projections. Since \(f \circ d_A = d_B \circ f\), by direct computation we have that \(F \circ D_A = D_B \circ F\). It follows then that \(\hat{f} \circ \tilde{d}_A = \tilde{d}_B \circ \hat{f}\).

Conjugation by \(\tilde{\tau}\) of the lift to a coalgebra map measures the deviation of the original map from being an algebra homeomorphism and that of the lift to a coderivation measures how far the original map was from being a derivation for the algebra structure.

**Proposition 1.** Let \(d\) be a linear map from \(A\) to \(A\) and \(\tilde{d}\) be the coderivation constructed in the previous theorem. Let \(f\) be a linear map from \(A\) to \(B\) and \(\hat{f}\) be homomorphisms constructed from \(f\) in the last theorem.

i) \(d\) is a derivation if and only if

\[\tilde{d}(x_1 \wedge x_2 \wedge \ldots x_n) = \sum_i \pm x_1 x_2 \ldots d(x_i) \ldots x_n\]

ii) \(f\) is a homomorphism of algebras if and only if

\[\hat{f}(x_1 \wedge x_2 \wedge \ldots x_n) = \tau(f(x_1) \wedge f(x_2) \wedge \ldots f(x_n)) = f(x_1)f(x_2)\ldots f(x_n)\]

**Proof.** i) In the proof of the last theorem we defined the coderivation \(D\) such that it satisfies the formula

\[\tau \circ D(x_1 \wedge x_2 \wedge \ldots x_n) = \sum_i \pm x_1 x_2 \ldots d(x_i) \ldots x_n\]

This expression is equal to \(d(x_1 x_2 \ldots x_n) = d \circ \tau(x_1 \wedge x_2 \wedge \ldots x_n)\) if and only if \(d\) is a derivation for the algebra \(A\). Thus we have that \(d \circ \tau = \tau \circ D\) if and only if \(d\) is a derivation for the algebra \(A\). Since \(\tilde{d}\) is the unique coderivation that commutes with \(\tau, \tilde{d} = D\) if and only if \(d\) is a derivation.
ii) Similar to the proof of the first part

\[ \tau \circ F(x_1 \wedge x_2 \ldots x_n) = \tau(f(x_1) \wedge f(x_2) \ldots f(x_n)) = f(x_1)f(x_2)\ldots f(x_n) \]

which is equal to \( f(x_1x_2\ldots x_n) \) if and only if \( f \) is a homomorphism. By uniqueness of \( \hat{f} \) we have that it is equal to \( F \) if and only if \( f \) is a homomorphism.

\[ \square \]

3 More detailed Proof of Theorem \[1\]

Proof. The proof is similar to the proof of Lemma \[2\]. Define \( \hat{\tau}_C \) to be \( \hat{I} \circ \hat{\tau} \circ \iota \). This map is a composition of differential coalgebra maps and hence is itself a differential coalgebra map. Let \( F_n = \bigoplus_{i=1}^{n} \wedge^i C \), where \( \wedge^i C \) is the graded symmetric product of \( i \) copies of \( C \). The \( F_n \) is the filtration on \( SC \). We will use induction on the degree of the filtration to prove that \( \hat{\tau}_C \) is an isomorphism. As \( \iota \) preserves the filtration \( \hat{\tau}_C \) also preserves the filtration. Also as \( \iota \) agrees with \( \iota \) on \( C = F_1 \) and since \( \hat{\tau} \) is identity on \( A \), \( \hat{\tau}_C \) is identity on \( F_1 \). Now suppose \( \hat{\tau}_C \) is an isomorphism when restricted to \( F_{n-1} \). Since \( \hat{\tau}_C \) is a coalgebra mapping, for \( v = x_1 \wedge x_2 \ldots x_n \) in \( \wedge^n C \) we have that \( \hat{\tau}_C^{\wedge n} \circ \Delta^{n-1}(v) = \Delta^{n-1} \circ \hat{\tau}_C(v) = v \). This implies that \( \hat{\tau}_C(v) = v + \) lower order terms. Then consider the short exact sequence

\[ 0 \to F_{n-1} \hookrightarrow F_n \to \wedge^n C \to 0 \]

By induction hypothesis \( \hat{\tau}_C \) is an isomorphism on \( F_{n-1} \) and by the above argument it induces identity on \( \wedge^n C \). Hence it is an isomorphism on \( F_n \) for every \( n \) which implies it is a coalgebra isomorphism of \( SC \).

\[ \square \]

4 Related Work

Our focus on the cumulant bijection was directly inspired by a lecture of Jae Suk Park at CUNY November 2011. The ideas in the lecture have been developed in the two papers \[1\] and \[2\].

The “induced cumulant bijection” is used in \[3\] to set up potential algorithms for computing 3D fluid motion based on differential forms and the integration deformation retract to cochains.

References

[1] Gabriel C. Drummond-Cole, Jae-Suk Park, John Terilla. “Homotopy Probability Theory I”. preprint arXiv February 2013

[2] Gabriel C. Drummond-Cole, Jae-Suk Park, John Terilla. “Homotopy Probability Theory II”. preprint arXiv February 2013

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[3] D. Sullivan. “3D Incompressible Fluids: Combinatorial Models, Eigenspace Models, and a Conjecture about Well-posedness of the 3D Zero Viscosity Limit”. to appear in JDG Hirzebruch Volume 2014