Extensions of the Duflo map and Chern-Simons expectation values

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The Duflo map is a valuable tool for operator ordering in contexts in which Kirillov-Kostant brackets and their quantizations play a role. A priori, the Duflo map is only defined on the subspace of the symmetric algebra over a Lie algebra consisting of elements invariant under the adjoint action. Here we discuss extensions to the whole symmetric algebra, as well as their application to the calculation of Chern-Simons theory expectation values.

1 Introduction

The Duflo-map [1], a generalization of the Harish-Chandra isomorphism [2], is a map from the symmetric algebra \( \text{Sym}(g) \) over a Lie algebra \( g \) to its universal enveloping algebra \( U(g) \), with marvelous properties. More precisely, it is an algebra isomorphism

\[
Q_D : \text{Sym}(g)^G \rightarrow Z(U(g))
\] 

(1)

between the subalgebra of invariant elements of \( \text{Sym}(g) \) and the center of \( U(g) \). In [3], it was observed that \( Q_D \) can be used as an ordering prescription for invariant functions on a Lie group that preserves all the classical relations.

Using this idea, one of us introduced a new way of calculating Chern-Simons (CS) expectation values of Wilson loops in [4, 5]. The approach consists of two crucial steps: First one uses the fact that under the CS path integral the curvature of the connection can be replaced by a functional derivative with respect to the connection. Secondly, one applies

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a non-Abelian version of Stokes’ theorem [6] to identify holonomies of the connection with surface ordered exponentials of the curvature. Combining these two steps one can thus calculate CS expectation values of traces of holonomies via

\[ \langle W_{\partial S}[A] \rangle = \int_{\mathcal{A}} W_{S}[\exp(iS_{CS}[A])]\mu_{AL}[A], \]

where \( W_{\partial S}[A] = \text{tr}(h_{\partial S}[A]) \) and the operator \( W_{S} \) is (in the notation of [3]) given by the formula

\[ W_{S} = \text{tr} \left[ \mathcal{P} \exp \left( \frac{2\pi i}{\sqrt{\epsilon}} T_{j} h_{j}^{k} \frac{\delta}{\delta A_{i}} \right) \right]. \]

The holonomy \( h \) connects the points of \( S \) with a base point on \( \partial S \), \( (T_{i}) \) is a basis of the Lie algebra \( \mathfrak{g} \), \( \kappa \) denotes the Cartan-Killing metric of \( \mathfrak{g} \), and \( c \) is a \( \mathfrak{g} \)-dependent constant.

If one wants to regard the Duflo map as a quantization map, it is useful to introduce the Kirillov-Kostant bracket which defines a Poisson structure on \( \text{Sym}(\mathfrak{g}) \). To this end we regard \( \text{Sym}(\mathfrak{g}) \) as polynomial functions over \( \mathfrak{g}^{*} \). Given \( a \in \mathfrak{g} \), the corresponding function is \( F_{a}(z) = z(a) \), and the bracket reads

\[ \{ F_{a}(z), F_{b}(z) \} := F_{[a,b]}(z). \]

This bracket extends to all of \( \text{Sym}(\mathfrak{g}) \). \( Q_{D} \) can then be viewed as a quantization map on the subspace of invariant functions. This raises the question whether \( Q_{D} \) can be extended in a natural way to all of \( \text{Sym}(\mathfrak{g}) \). The aim of this work is to consider the merits of several such extensions. These in turn can be used to generalise [4, 5] to expectation values of (untraced) holonomies. While this is interesting in its own, it is also a step towards finding a preferred ordering for a quantisation of (an exponentiated version of) the isolated horizon condition in LQG. Such a generalisation involves in particular finding an extension of the Duflo map to terms which are not gauge invariant, since the results of [4] provide evidence that the Duflo map yields the correct ordering for the products of flux operators occurring in the series expansion of \( W_{S} \). In a first attempt we will use the explicit formula for the Duflo map used in [4] also on gauge variant terms. This calculation is the content of section[2]
Like us, the authors of [7] were also confronted with the problem of finding an ordering prescription for products of flux operators, albeit in a different context. Their work concerns the 2+1 LQG framework and they considered a noncommutative connection, which is a linear combination of the standard LQG connection and a flux operator. When calculating holonomies of this new connection they also encounter products of flux operators, which require a choice of ordering. They choose a variation of the Duflo map, which leads to the simplest possible result. Namely, they recover the Kauffman bracket [8], a link invariant known from knot theory. This work was later generalized to the presence of a cosmological constant in [9]. In section 3 we will apply our results from section 2 to their problem and compare the results.

In section 4 we then compute the image of a very special element of the symmetric algebra, essentially the exponential map. For this calculation, we specialize to the spin-$\frac{1}{2}$-representation of $\mathfrak{su}(2)$. We apply the results to the computation of expectation values of traces of holonomies in $\mathfrak{su}(2)$ CS theory, and discuss the possibility to derive skein relations for the related link invariants in section 5. Our findings are discussed in section 6.

2 An extension of the Duflo map

The explicit formula for the Duflo map $Q_D$ on $S(\mathfrak{g})^0$ used in [4, 5] is given by

$$Q_D = Q_S \circ j^{\frac{1}{2}} (\partial), \quad (5)$$

where $Q_S$ denotes symmetric quantisation (i.e. the Poincaré-Birkhoff-Witt isomorphism) and $j^{\frac{1}{2}} (\partial)$ is an infinite order differential operator obtained by inserting $x^I T_I = x = \partial = T_I \frac{\partial}{\partial x^I}$ into the formula

$$j^{\frac{1}{2}}(x) = \sqrt{\det \left( \sinh \frac{\text{ad}_x}{2} \right)} \quad (6)$$

with $\text{ad}_x$ denoting the adjoint action of $x$. For $\mathfrak{g} = \mathfrak{su}(2)$ we can rewrite this expression as

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\[ j^\frac{1}{2}(x) = \sqrt{\det \left( \sum_{N=0}^{\infty} \frac{1}{(2N+1)!} \left[ \frac{\text{ad}_x}{2} \right]^{2N} \right)} \]

\[ = \sqrt{\left( \sum_{N=0}^{\infty} \frac{1}{(2N+1)!} \left[ -|x|^2 \right]^N \right)^2} \]

\[ = \sum_{N=0}^{\infty} \frac{1}{(2N+1)!} \left[ \frac{|x|^2}{8} \right]^N \]

\[ = \sum_{N=0}^{\infty} \frac{1}{(2N+1)!} 8^{N} \kappa_{m_1n_1} \ldots \kappa_{m_Nn_N} x^{m_1} x^{n_1} \ldots x^{m_N} x^{n_N} \]

(7)

with \( \kappa_{mn} = -2\delta_{mn} \) denoting the components of the Killing metric on \( \text{su}(2) \). Here, in the first line we used the series expansion of \( \sinh \) and in the second line the fact that the determinant is given by the product of the eigenvalues. Then we made use of the relationship \( |x|^2 := \delta_{mn} x^m x^n = \frac{1}{2} \kappa_{mn} x^m x^n =: \frac{1}{2} \|x\|^2 \) and in the last line we also used that the series in the line before corresponds to the function \( \frac{\sinh(y)}{y} \), which is positive everywhere and hence the absolute value can be ignored.

Now we want to extend this map from terms of the form \( \|E\|^{2n} \) to terms of the form \( \|E\|^{2n} E_i \). The latter are not gauge invariant and hence they are not in the domain of the Duflo map (as described by the formula above). However, in this section we will (very naively) use the formula given in equation (5) also for this type of terms. Thus we compute the action of the differential operator \( j^\frac{1}{2}(\partial) \) on this type of terms. Since \( \|\partial\|^{2n} = \|\partial\|^n \) we first calculate

\[ \|\partial\|^2 \left[ \|E\|^{2k} E_i \right] = \kappa_{mn} \kappa_{i_1j_2} \ldots \kappa_{i_{2k-1}j_{2k}} \partial^m \partial^n E_{i_1} \ldots E_{i_{2k}} E_i \]

\[ = \kappa_{mn} \kappa_{i_1j_2} \ldots \kappa_{i_{2k-1}j_{2k}} \partial^m \left[ (2k) \delta_{i_1}^m E_{i_2} \ldots E_{i_{2k}} E_i + \delta_i^m E_{i_1} \ldots E_{i_{2k}} \right] \]

\[ = \kappa_{mn} \kappa_{i_1j_2} \ldots \kappa_{i_{2k-1}j_{2k}} (2k) \delta_{i_1}^m \delta_{i_2}^m \delta_{i_3}^m \ldots E_{i_{2k}} E_i \]

\[ + \kappa_{mn} \kappa_{i_1j_2} \ldots \kappa_{i_{2k-1}j_{2k}} (2k) \delta_{i_1}^m \delta_{i_2}^m E_{i_3} \ldots E_{i_{2k}} E_i \]

\[ + \kappa_{mn} \kappa_{i_1j_2} \ldots \kappa_{i_{2k-1}j_{2k}} (2k) \delta_{i_1}^m \delta_{i_2}^m \delta_{i_3}^m \ldots E_{i_{2k}} \]

\[ + \kappa_{mn} \kappa_{i_1j_2} \ldots \kappa_{i_{2k-1}j_{2k}} (2k) \delta_{i_1}^m \delta_{i_2}^m E_{i_3} \ldots E_{i_{2k}} \]

\[ = (2k + 3) (2k) \|E\|^{2(k-1)} E_i \]

whence we obtain for \( n \leq k \)
\[ ||\partial||^{2n} [||E||^{2k} E_i] = \prod_{m=k-n+1}^{k} (2m + 3)(2n) ||E||^{2(k-n)} E_i = \frac{(2k + 1)!}{(2k - 2n + 1)!} \frac{2k + 3}{2k - 2n + 3} ||E||^{2(k-n)} E_i \]

and \( ||\partial||^{2n} [||E||^{2k} E_i] = 0 \) for \( n > k \). The result of the action of the infinite order differential operator \( j^{\frac{1}{2}} (\partial) \) on this particular type of terms is thus given by

\[ j^{\frac{1}{2}} (\partial) [||E||^{2k} E_i] = \sum_{N=0}^{k} \frac{1}{(2N + 1)!} \frac{1}{8N} \frac{(2k + 1)!}{(2k - 2N + 1)!} \frac{2k + 3}{2k - 2N + 3} ||E||^{2(k-N)} E_i. \]

Since \( Q_S \) is a linear map, in order to compute \( Q_D \) we need the action of \( Q_S \) again on terms of the form \( ||E||^{2n} E_i \). However, we can reduce this to the problem of calculating \( Q_S(||E||^{2(n+1)}) \) via

\[ Q_S(||E||^{2(k+1)}) = \kappa^{i_1 i_2} \ldots \kappa^{i_{2k+1} i_{2k+2}} \hat{E}_{i_1} \ldots \hat{E}_{i_{2k+2}} \]
\[ = \kappa^{i_1 i_2} \ldots \kappa^{i_{2k+1} i_{2k+2}} \frac{1}{2k + 2} \sum_{l=1}^{2k+2} \hat{E}_{i_1} \ldots \hat{E}_{i_{l-1}} \hat{E}_{i_{l+1}} \ldots \hat{E}_{i_{2k+2}} \hat{E}_{i_l} \]
\[ = \kappa^{i_1 i_2} \ldots \kappa^{i_{2k+1} i_{2k+2}} \frac{1}{2k + 2} \sum_{l=1}^{2k+2} \hat{E}_{i_1} \ldots \hat{E}_{i_{2k+1}} \hat{E}_{i_{2k+2}} \]
\[ = Q_S(||E||^{2k} E_{i_{2k+1}}) \kappa^{i_{2k+1} i_{2k+2}} \hat{E}_{i_{2k+2}} \]

where \( \hat{E}_i := Q_S(E_i) \) and we used the definition of \( Q_S \) in the first line and the definition of symmetrisation in the second line. In the third line we then relabelled the dummy indices \( i_l \) and \( i_{2k+2} \) in each term of the sum and used the total symmetry of the first \( 2k + 1 \) indices to restore the original kappas in front of the sum. Thus in the fourth line all terms in the sum are the same and we can express the result in terms of \( Q_S(||E||^{2n} E_{i_{2k+1}}) \). Since \( Q_S(||E||^{2n} E_{i_{2k+1}}) \) has to be proportional to \( \hat{E}_{i_{2k+1}} \) we can thus write \( Q_S(||E||^{2n} E_{i_{2k+1}}) \) in terms of \( Q_S(||E||^{2(k+1)}) \) as

\[ Q_S(||E||^{2n} E_{i_{2k+1}}) = \frac{Q_S(||E||^{2(n+1)})}{\Delta_{\text{sym}(2)}} \hat{E}_{i_{2k+1}}, \]
where $\Delta_{su(2)} := \kappa^{ij} \hat{E}_i \hat{E}_j$ denotes the generator of the center of $U(su(2))$. Now we only need to evaluate $Q_S$ on terms of the form $||E||^{2n}$, which is given in [10] as

$$Q_S(r^{2n}) = \frac{(-1)^{n-1}}{4^n} \sum_{k=0}^{n} \binom{2n + 1}{2k} B_{2k} (4^k - 2) (1 - 4C)^{n-k}$$

(13)

with $r = |E|$, $C = Q_S(r^2)$ and $B_{2k}$ denoting the $2k$-th Bernoulli number. This formula translates to our notation as

$$Q_S(||E||^{2k}) = -\frac{1}{8^k} \sum_{m=0}^{k} \binom{2k + 1}{2m} B_{2m} (2^{2m} - 2) \left(1 + 8\Delta_{su(2)}\right)^{k-m}.$$  

(14)

Combining equations (10), (12) and (14) will lead to a rather lengthy expression. In the spin-$\frac{1}{2}$-representation, however, things simplify drastically. More precisely, we have $\Pi^{(1/2)}(E_i) = \tau_i = -\frac{i}{2} \sigma_i$, with $\sigma_i$ denoting the Pauli matrices, and therefore we obtain $\Pi^{(1/2)}(\Delta_{su(2)}) = \kappa^{ij} \tau_i \tau_j = \frac{3}{8}$, which leads to

$$\Pi^{(1/2)}(Q_S(||E||^{2k})) = -\frac{1}{8^k} \sum_{m=0}^{k} \binom{2k + 1}{2m} B_{2m} (2^{2k} - 2^{2k-2m+1})$$

$$= -\frac{1}{2^k} \sum_{m=0}^{2k} \binom{2k + 1}{m} B_m + \frac{1}{8^k} \sum_{m=0}^{2k} \binom{2k + 1}{m} B_m 2^{2k-m+1}$$

$$= -\frac{1}{2^k} \delta_{k,0} + \frac{1}{8^k} (2k + 1) \sum_{m=0}^{2k} \binom{2k}{m} B_m \frac{2^{2k-m+1}}{2k - m + 1}$$

$$= -\delta_{k,0} + \frac{1}{8^k} (2k + 1) (1 + \delta_{k,0}) = \frac{1}{8^k} (2k + 1),$$

(15)

where we used a basic property of Bernoulli numbers to get the third line and a specific version of Faulhaber’s formula in the first equality of the last line. Now inserting this result, together with equations (10) and (12), into equation (5) we finally get
\[\Pi^{(1/2)}(Q_D(||E||^{2k}E_i)) =\]
\[= \sum_{N=0}^{k} \frac{1}{(2N+1)!} \frac{1}{8^N} \frac{2k+3}{2k-2N+3} \frac{(2k+1)!}{(2k-2N+1)!} \Pi^{(1/2)}(Q_S(||E||^{2(k-N)}E_i))\]
\[= \sum_{N=0}^{k} \frac{1}{(2N+1)!} \frac{1}{8^N} \frac{2k+3}{2k-2N+3} \frac{(2k+1)!}{(2k-2N+1)!} \cdot \frac{1}{8} \cdot \frac{2(2k-N+1)+1}{N} \tau_i\]
\[= \frac{1}{3 \cdot 8^k} \sum_{N=0}^{k} (2k+3) \frac{(2k+1)!}{(2N+1)!(2k-2N+1)!} \tau_i\]
\[= \frac{2}{3} k + 1 \sum_{N=0}^{k} \frac{1}{2k+2} \frac{1}{2N+1} \tau_i\]
\[= \frac{1}{2k} \frac{2}{3} k + 1 \tau_i.\]

(16)

For the sake of completeness we also add the image of terms of the form \(||E||^{2k}\) under the Duflo map, which is given by

\[Q_D(||E||^{2k}) = \left[Q_D(||E||^2)^k \right] = \left[\Delta_{\text{su}(2)} + \frac{1}{8} \right]^k\]

(17)

and in the spin-\(\frac{1}{2}\)-representation simplifies to

\[\Pi^{(1/2)}(Q_D(||E||^{2k})) = \frac{1}{2k}.\]

(18)

### 3 Comparison of different extensions

In this section we want to compare the two different extensions of the Duflo map from the previous section. We therefore apply them to the situation discussed in [7]. They use the Duflo map themselves, but they state a different formula (seemingly copied from [3]) for it. Since the difference crucially influences the result, we redo their calculation here with our formula based on the one given in [4, 1]. The complete expression Noui et al. consider in [7] is rather complicated and fortunately not needed here. The relevant part consists of terms of the form
\[
\frac{z^p}{p!} \tau^{i_1} \ldots \tau^{i_p} \otimes Q(E_{i_1} \ldots E_{i_p}),
\]  

(19)

where \( z \) is some (purely imaginary) constant, \( \tau^i = -\frac{i}{2} \sigma^i \) denote the generators of \( \mathfrak{su}(2) \) in the spin-\( \frac{1}{2} \)-representation and \( Q : S(\mathfrak{g}) \to U(\mathfrak{g}) \) is a quantisation map, i.e. in our case either \( Q_S \) or \( Q_D \). Since the domain of the map \( Q \) is \( S(\mathfrak{g}) \), we know that \( Q(E_{i_1} \ldots E_{i_p}) \) has to be symmetric in all indices and hence we can equivalently write

\[
\frac{z^p}{p!} \tau^{(i_1} \ldots \tau^{i_p)} \otimes Q(E_{i_1} \ldots E_{i_p})
\]

\[
= \frac{z^p}{p!} \left\{ \begin{array}{ll}
\frac{1}{2^k} \{ \tau^{i_1}, \tau^{i_2} \} \ldots \{ \tau^{i_{2k-1}}, \tau^{i_{2k}} \} \otimes Q(E_{i_1} \ldots E_{i_{2k}}) & \text{if } p = 2k \\
\frac{1}{2^{k+1}} \{ \tau^{i_1}, \tau^{i_2} \} \ldots \{ \tau^{i_{2k}}, \tau^{i_{2k+1}} \} \otimes Q(E_{i_1} \ldots E_{i_{2k+1}}) & \text{if } p = 2k + 1
\end{array} \right.
\]

(20)

Hence we are down to evaluating the type of terms we already considered in the previous section. Since we are working in the spin-\( \frac{1}{2} \)-representation here, we can use equations (16) and (18) from the previous section to obtain

\[
\sum_{p=0}^{\infty} \frac{z^p}{p!} \tau^{i_1} \ldots \tau^{i_p} \otimes Q_D(E_{i_1} \ldots E_{i_p}) = 
\]

\[
= \sum_{k=0}^{\infty} \left[ \frac{z^{2k}}{(2k)!} \frac{1}{2^{2k}} \otimes 1 + \frac{z^{2k+1}}{(2k+1)!} \frac{2}{2^{2k+1}} \frac{2}{3} \frac{2k+3}{2k+2} \tau^i \otimes \tau_i \right]
\]

(21)

\[
= \cosh \left( \frac{z}{2} \right) 1 \otimes 1 + \frac{4}{3} \left[ \sinh \left( \frac{z}{2} \right) + \frac{\cosh \left( \frac{z}{2} \right) - 1}{\frac{z}{2}} \right] \tau^i \otimes \tau_i
\]

\[
= \cos \left( \frac{oh \lambda}{2} \right) 1 \otimes 1 - \frac{4i}{3} \left[ \sin \left( \frac{oh \lambda}{2} \right) + \frac{\cos \left( \frac{oh \lambda}{2} \right) - 1}{\frac{oh \lambda}{2}} \right] \tau^i \otimes \tau_i.
\]

This expression is more complicated than the one obtained by the authors of [7] and doesn’t lead to their appealing result. Since both our and their version of the Duflo map appear in the literature, it may be illuminating to further investigate the two formulas.
The difference seems to have its foundation in different formulas for \( j^{\frac{1}{2}}(x) \), which is given by

\[
j^{\frac{1}{2}}(x) = \sqrt{\det \left( \frac{\sinh \text{ad}_x}{\text{ad}_x} \right)}
\]

(22)

in [1], whereas the authors of [2] use

\[
\tilde{j}^{\frac{1}{2}}(x) = \det \left( \frac{\sin \text{ad}_x}{\text{ad}_x} \right)
\]

(23)

instead (at least in the arXiv version of their paper). However, the precise nature of the difference between \( Q_D \) and \( \tilde{Q}_D^{NPP} \) when applied to gauge invariant terms is still under investigation.

4 Quantized exponential map

As we will explain in section 5 for the applications we have in mind, the element

\[
Q \left[ \exp \left( -\frac{8\pi i}{k} \kappa^{ij} E_i T_j \right) \right]
\]

(24)

in \( \mathfrak{u}(\mathfrak{u}(2)) \) is of particular importance. Note that this can be understood as a “quantization of the exponential map”, so it might also be interesting from a purely mathematical standpoint.

We will calculate (24) using the various extensions proposed in the previous text. For comparison we will give the results for \( Q = Q_D, Q = \tilde{Q}_D, Q = \tilde{Q}_D^{NPP} \) and \( Q = Q_S \), where \( Q_D \) denotes the Duflo map as defined in eqn. [3], \( Q_S \) denotes symmetric quantisation as above, \( Q = \tilde{Q}_D \) coincides with \( Q_D \) on terms of the form \( ||E||^{2n} \) but is continued via \( \tilde{Q}_D(||E||^{2n} E_i) = \tilde{Q}_D(||E||^{2n}) \tilde{Q}_D(E_i) \) and \( \tilde{Q}_D^{NPP} \) is the Duflo map used by the authors of [7], i.e. \( \tilde{Q}_D^{NPP}(||E||^{2n}) = \frac{1}{8^n} \) and is continued in the same way as \( \tilde{Q}_D \).

As a first step we expand the exponential as a series yielding
\[\exp\left(-\frac{8\pi i}{k} \kappa^{ij} E_i T_j\right) = \cos \left(\frac{4\pi}{\sqrt{2k}} ||E||\right) \mathbb{1}_2 - \frac{8\pi i}{k} \sin \left(\frac{4\pi}{\sqrt{2k}} ||E||\right) \kappa^{ij} E_i T_j\]

\[= \sum_{m=0}^{\infty} (-1)^m \left(\frac{4\pi}{\sqrt{2k}}\right)^{2m} \frac{||E||^{2m} \mathbb{1}_2}{(2m)!} - \frac{8\pi i}{k} \sum_{m=0}^{\infty} (-1)^m \left(\frac{4\pi}{\sqrt{2k}}\right)^{2m} \frac{||E||^{2m} \kappa^{ij} E_i T_j}{(2m + 1)!} \]  \hspace{1cm} (25)

We will now consider the application of the different quantization maps and for further comparison we also express the results in the basis used in [7].

For \(Q = Q_D\) we obtain

\[
Q_D \left[ \exp\left(-\frac{8\pi i}{k} \kappa^{ij} E_i (T_j)^A_D\right)\right]^C_B
= \cos \left(\frac{2\pi}{k}\right) \delta^A_D \delta^C_B + \frac{4i}{3} \sin \left(\frac{2\pi}{k}\right) - \frac{1 - \cos \left(\frac{2\pi}{k}\right)}{2\pi} \sum_i (T_i)^A_D (T_i)^C_B
= \left[ \cos \left(\frac{2\pi}{k}\right) - i \sin \left(\frac{\pi}{k}\right) \right] \delta^A_B \delta^C_D
- \left[ \cos \left(\frac{2\pi}{k}\right) + i \sin \left(\frac{\pi}{k}\right) \right] \epsilon^{AC} \epsilon_{BD} \]  \hspace{1cm} (26)

In the case \(Q = \tilde{Q}_D\) we have

\[
\tilde{Q}_D \left[ \exp\left(-\frac{8\pi i}{k} \kappa^{ij} E_i (T_j)^A_D\right)\right]^C_B
= \cos \left(\frac{2\pi}{k}\right) \delta^A_D \delta^C_B + 2i \sin \left(\frac{2\pi}{k}\right) \sum_i (T_i)^A_D (T_i)^C_B
= \left[ \cos \left(\frac{2\pi}{k}\right) - \frac{i}{2} \sin \left(\frac{2\pi}{k}\right) \right] \delta^A_B \delta^C_D - \left[ \cos \left(\frac{2\pi}{k}\right) + \frac{i}{2} \sin \left(\frac{2\pi}{k}\right) \right] \epsilon^{AC} \epsilon_{BD}. \]  \hspace{1cm} (27)
Using $Q = \tilde{Q}_D^{NPP}$ we are left with

$$\tilde{Q}_D^{NPP} \left[ \exp \left( -\frac{8\pi i}{k} \kappa \gamma_j E_i (T_j)^A_D \right) \right]_B^C = \cos \left( \frac{\pi}{k} \right) \delta^A_B \delta^C_D + 4i \sin \left( \frac{\pi}{k} \right) \sum_i (T_i)^A_D (T_i)^C_B$$

$$= \left[ \cos \left( \frac{\pi}{k} \right) - \frac{\pi}{k} \sin \left( \frac{\pi}{k} \right) \right] \delta^A_B \delta^C_D - \left[ \cos \left( \frac{\pi}{k} \right) + \frac{2i}{3} \sin \left( \frac{\pi}{k} \right) \right] \epsilon^{AC} \epsilon_{BD}$$

$$= e^{-\frac{i\pi}{k}} \delta^A_B \delta^C_D - e^{\frac{i\pi}{k}} \epsilon^{AC} \epsilon_{BD}. \quad (28)$$

The choice $Q = Q_S$ results in

$$Q_S \left[ \exp \left( -\frac{8\pi i}{k} \kappa \gamma_j E_i (T_j)^A_D \right) \right]_B^C = \left[ \cos \left( \frac{\pi}{k} \right) - \frac{\pi}{k} \sin \left( \frac{\pi}{k} \right) \right] \delta^A_B \delta^C_D - \left[ \cos \left( \frac{\pi}{k} \right) + \frac{2i}{3} \sin \left( \frac{\pi}{k} \right) \right] \epsilon^{AC} \epsilon_{BD}$$

$$= \left[ \cos \left( \frac{\pi}{k} \right) - i \sin \left( \frac{\pi}{k} \right) \right] \delta^A_B \delta^C_D - \left[ \cos \left( \frac{\pi}{k} \right) + \frac{2i}{3} \sin \left( \frac{\pi}{k} \right) \right] \epsilon^{AC} \epsilon_{BD}$$

$$= e^{-\frac{i\pi}{k}} \delta^A_B \delta^C_D - e^{\frac{i\pi}{k}} \epsilon^{AC} \epsilon_{BD}. \quad (29)$$

Lastly, let us also consider $Q = \tilde{Q}_S$, which denotes the continuation of $Q_S$ analogous to $\tilde{Q}_D$. The expression then reads

$$\tilde{Q}_S \left[ \exp \left( -\frac{8\pi i}{k} \kappa \gamma_j E_i (T_j)^A_D \right) \right]_B^C = \left[ \cos \left( \frac{\pi}{k} \right) - i \frac{\pi}{k} \sin \left( \frac{\pi}{k} \right) \right] \delta^A_B \delta^C_D + \frac{2i}{3} \sin \left( \frac{\pi}{k} \right) \sum_i (T_i)^A_D (T_i)^C_B$$

$$= \left[ \cos \left( \frac{\pi}{k} \right) - i \frac{\pi}{k} \sin \left( \frac{\pi}{k} \right) \right] \delta^A_B \delta^C_D + \left[ \cos \left( \frac{\pi}{k} \right) + i \sin \left( \frac{\pi}{k} \right) \right] \epsilon^{AC} \epsilon_{BD}$$

$$= e^{-\frac{i\pi}{k}} \delta^A_B \delta^C_D - e^{\frac{i\pi}{k}} \epsilon^{AC} \epsilon_{BD}. \quad (30)$$

Since the Duflo map is supposed to be a deformed version of symmetric quantisation, it is interesting to note that, while $\tilde{Q}_D^{NPP}$ and $Q_S$ both produce $\frac{2\pi}{k}$ as argument of the occurring sin and cos functions, our version yields $\frac{\pi}{k}$ instead. Additionally, the fact that $\tilde{Q}_D^{NPP}$ leads to a similarly simple result as in [17] indicates that $\tilde{Q}_D^{NPP}$ might be the best choice to define an ordering for the quantization of products of flux operators.
5 Application to quantum Chern Simons theory and black holes

In [4, 5], the Duflo map was used to calculate certain expectation values in CS theory. In that application, the expectation values were calculated in a piecemeal fashion, turning one loop $\partial S$ into an operator $W_S$ at a time, and calculating its action under the path integral. Now that we have extensions of the Duflo map at our disposal, we can aim for skein relations among the expectation values. The argument goes as follows.

We consider the path integral expectation value of the traces of holonomies along the components of a link $L$:

$$\langle F_L \rangle_{CS} = \hat{A} \exp(iS_{CS}[A]) F_L[A] d\mu_{AL}[A]. \tag{31}$$

Consider two holonomy strands passing each other as in figure 1(i). As the expectation value does not depend on smooth deformations of $L$, we can deform the one strand in the manner shown in fig. 1(ii). By applying the non-abelian Stokes theorem (for details see [4]), we can replace the curved section of the deformed strand by a certain ordered exponential integral $I_S$ of the curvature of $A$ over a surface $S$ bounded by the curved section,

\begin{figure}
\centering
\begin{tikzpicture}
\draw (0,0) -- (2,2);
\draw (0,2) -- (2,0);
\node at (1,1) {\small S};
\end{tikzpicture}
\caption{Manipulation of a crossing of two holonomy strands, using the operators $W_S$}
\end{figure}

$$\langle F_L \rangle_{CS} = \int \exp(iS_{CS}[A]) (\tilde{F}_L)^I J[A] (I_S)^J I[A] d\mu_{AL}[A], \tag{32}$$

see also (iii). $(\tilde{F}_L)^I J$ is obtained from the original functional by removing the holonomy along $\partial S$. In the next step, $I_S$ can be replaced by a functional differential operator acting on the action term. For the action

\begin{itemize}
\item[\textsuperscript{1}] Strictly speaking, the deformation depicted in (ii) is smooth only so long as the circle around the other holonomy strand does not get closed completely. If it is not completely closed, however, the replacement in step (iii) (see below) is only an approximation. This approximation can be made arbitrarily good, classically, and we will assume in the following that this is also true in the quantum theory.
\end{itemize}
\[ S_{CS} = \frac{k}{4\pi} \int_M \text{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \]  
(33)

it holds that

\[ \frac{\delta}{\delta A^I(x)} e^{iS_{CS}[A]} = \frac{ick}{2\pi} \kappa_{ij} F^I(x) e^{iS_{CS}[A]} \]  
(34)

c is a Lie algebra dependent constant (c = 1/4 for A an su(2) connection). Thus

\[ \langle F_L \rangle_{CS} = \int (W_S)^I J \left[ \exp(iS_{CS}[A]) \right] (\tilde{F}_L)^J I A \]  
(35)

\( W_S \) is an operator obtained from \( T_S \) by substituting \( 34 \). These functional derivatives can be rigorously defined and they do not commute with each other. Hence an ordering is needed and is provided by (an extension of) the Duflo map.

In the next step, partial functional integration gives

\[ \langle F_L \rangle_{CS} = \int (W_S)^I J \left[ \exp(iS_{CS}[A]) \right] (\tilde{F}_L)^J I A \]  
(36)

It turns out, that \( W_{S}^\dagger \) acts only at intersection points of \( S \) with holonomy loops \( 4 \). In the situation at hand, there is only one intersection. In that case, the action is given by inserting the “quantized exponential map” into the remaining holonomy strand,

\[ (W_S)^A B \left[ (\tilde{F}_L)^D A [A] \right] = Q \left[ \exp \left( \frac{-8\pi i k}{8} \kappa^{ij} E_i (T_j)^A_D \right) \right]^C B (\tilde{F}_L)^D A B C [A]. \]  
(37)

The added pair of indices on \( \tilde{F}_L \) is due to the fact that a strand was cut at the intersection point with the surface \( S \). This leads to a coupling between the two strands by an intertwiner, as in (iv) of fig. 1. In graphical notation, we can write

\[ Q \left[ \exp \left( \frac{-8\pi i k}{8} \kappa^{ij} E_i (T_j)^A_D \right) \right]^C B \overset{A}{\Rightarrow} \overset{\bullet}{\Rightarrow} c. \]  
(38)
Then, expanding the resulting intertwiner in a suitable basis, we obtain an expression that can be compared to the skein relation of knot invariants.

In section 4 we had calculated the quantized exponential map in the spin-$\frac{1}{2}$-representation. The space of intertwiners in this case is 2 dimensional. There are two relevant bases,

\[
\hat{\psi}^{(A)} = \delta^{AC} \delta^{BD}, \quad \hat{\eta}^{(A)} = \epsilon^{AC} \epsilon^{BD} \quad (39)
\]

and

\[
\hat{\psi}^{(A)} = \delta^{AD} \delta^{CB}, \quad \hat{\eta}^{(A)} = \delta^{AD} \delta^{CB}. \quad (40)
\]

They are adapted for comparison to the skein relations for the Kauffman bracket

\[
\langle \quad \rangle = A \langle \quad + A^{-1} \quad \rangle \quad (41)
\]

and the Jones polynomial

\[
- t^{-1} \langle \quad \rangle + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \langle \quad \rangle + t \langle \quad \rangle = 0. \quad (42)
\]

It is well known since [11] that the CS expectation values are closely related to both invariants. While the expectation values are framing dependent, the invariants are not. The Jones polynomial is obtained from the expectation values in standard framing\(^2\), while the bracket contains an additional factor with the writhe as an exponent, making it a regular isotopy invariant. Let us introduce the shortcut

\[
\exp E := \exp \left( \frac{-8\pi i}{k} \kappa^{ij} E_i (T_j) \right). \quad (43)
\]

When $\tilde{Q}_{D}^{NPP}$ is expanded in basis (39), we obtain

\[
\tilde{Q}_{D}^{NPP} [\exp E] = e^{-i\pi} \langle \quad + e^{i\pi} \quad \rangle \quad (44)
\]

\(^2\text{Standard framing is the framing obtained from the consideration of a Seifert surface for the link.}\)
whereas $Q_D$ gives

$$Q_D[\exp(E)] = \left[ \cos\left(\frac{2\pi}{k}\right) - \frac{i}{3} \sin\left(\frac{\pi}{k}\right) + \frac{i}{3} \frac{1 - \cos\left(\frac{2\pi}{k}\right)}{\frac{2\pi}{k}} \right] \langle
$$

$$+ \left[ \cos\left(\frac{2\pi}{k}\right) + \frac{i}{3} \sin\left(\frac{2\pi}{k}\right) - \frac{i}{3} \frac{1 - \cos\left(\frac{2\pi}{k}\right)}{\frac{2\pi}{k}} \right] \langle \langle .
$$

Using

$$\langle \langle = \langle \times - \rangle \langle (46)$$

we can also expanding in basis (40):

$$\tilde{Q}^{NPP}_D[\exp(E)] = \left( e^{-\frac{i\pi}{k}} - e^{\frac{i\pi}{k}} \right) \angle + e^{\frac{i\pi}{k}} \times \langle (47)$$

and

$$Q_D[\exp(E)] = -\frac{2i}{3} \left[ \sin\left(\frac{2\pi}{k}\right) + \cos\left(\frac{2\pi}{k}\right) - 1 \right] \langle 
$$

$$+ \left[ \cos\left(\frac{2\pi}{k}\right) + \frac{i}{3} \sin\left(\frac{2\pi}{k}\right) + \frac{i}{3} \frac{\cos\left(\frac{2\pi}{k}\right) - 1}{\frac{2\pi}{k}} \right] \langle \langle . (48)$$

It is clear and remarkable that $\tilde{Q}^{NPP}_D$ reproduces the skein relation of the Kauffman bracket with

$$A = e^{\frac{i\pi}{k}}. (49)$$

It is equally clear that $Q_D$ gives no direct relationship to either the Jones polynomial or the Kauffman bracket.

On the other hand, it was shown in [4, 5] that $Q_D$ successfully reproduces the relation (see figure 2)

$$\langle \circ \cup L \rangle = q^{-\frac{3}{2}}(q + q^{-1})(L) (50)$$

15
for the Jones polynomial, if one uses the relation
\[ \bigcirc = \text{tr} \left( Q_D \left[ \exp E \right] \right) \] (51)
directly, without recourse to skein relations, whereas one can see from the results presented above, that this is not the case for \( \tilde{Q}^{NPP}_D \).

\[ \Delta := q^{-\frac{3}{4}}. \] (52)

This does not solve the problem of interpreting (48) in terms of a standard skein relation.

6 Conclusion & Outlook

In the present work, we have considered different extensions of the Duflo map to \( \text{Sym}(\mathfrak{g}) \), as well as some variants of the Duflo map. Explicit calculations have been given for \( \mathfrak{g} = \mathfrak{su}(2) \), in particular the image of the element
\[ \exp \left( -\frac{8\pi i}{k} \kappa^{ij} E_i (T_j) \right) \] (53)
in the spin-1/2-representation.

Interpreting the Duflo map as a quantization map, we have applied it and its variants to the calculation of CS expectation values according to a prescription detailed in [4, 5]. The results are very interesting, but not straightforward to interpret: Using the variant \( Q^{NPP}_D \) of [12] one can reproduce the skein relation of the Kauffman bracket. This was already observed in [12]. The calculation we have presented here is in a substantially different
setting though, and thus serves to emphasize the importance and versatility of \( \tilde{Q}^{NPP} \) as a quantization map. Surprisingly, \( Q_D \) does not seem to be able to reproduce any skein relation. This is in contrast to the results [4, 5] that show that certain relations among CS expectation values can be reproduced correctly by \( Q_D \). Those same calculations seem to fail, however, for \( \tilde{Q}^{NPP} \).

We can only speculate about the reasons for these incongruent results. One reason might be that we are missing something in the translation between the mathematical results of the Duflo map (section 3) and the CS expectation values. Another potential source of problems is the fact that we are using the classical recoupling theory in equations like (46), where the recoupling theory of \( U_q(\mathfrak{su}(2)) \) might be expected. It would also be desirable to understand better what distinguishes \( \tilde{Q}^{NPP} \) mathematically, and how it is related to \( Q_D \). We will come back to these questions in the future.

Finally, it is intriguing that the structure relevant for the application to CS theory takes the form \( Q[\exp(E)] \), i.e., a quantization of the exponential map. We would like to further analyze what kind of deformation of \( G \) this object might represent.

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