An Attempt to Determine the Largest Scale of Primordial Density Perturbations in the Universe

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Causality constraints require a super-Hubble scale suppression in the primordial power spectrum. This modification is implemented and a three parameter likelihood analysis is performed of the COBE-DMR 4-year data with respect to the amplitude, spectral index, and suppression scale. All suppression length scales larger than $c/H_0$ are consistent with the data, but scales of order $4c/H_0$ are slightly preferred, at roughly the one-sigma level. Many non-inflation models would be consistent with a small suppression length scale, whereas for standard inflation models, it would require small e-folds. Suppression scales smaller than $c/H_0$ are strongly excluded by the anisotropy data.

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Causality prohibits coherence between physical phenomena with superhorizon scale separation. For density perturbations which are generated by local causal processes in a Robertson-Walker universe, the implications of causality have been shown to imply a suppression of the power spectrum which decreases faster than $k^4$ for scales larger than the horizon. In non-inflationary cosmology the horizon and Hubble radius are about the same size, so that no causal mechanism could produce super-Hubble scale perturbations. Inflationary cosmology is characterized by a time period in which the horizon grew exponentially fast while the Hubble radius remained essentially constant. Thus inflation provides the only known causal mechanism from present theory that allows density perturbations to be super-Hubble scale with respect to the present-day Hubble radius $c/H_0 \approx 3000 \, h^{-1} \, Mpc$, where $h^{-1} = 100/H_0 \, km \, sec^{-1} \, Mpc^{-1}$.

For large scale structure, the power spectrum of the primordial scalar density perturbations can be written as

$$P(k) = V A k^n f(k), \quad (1)$$

where $V$ is a large rectangular volume and $f(k)$ is the long wavelength suppression factor, which has been added to satisfy causality constraints. The modification of a suppression scale to the pure power-law power spectrum is mandated by causality. Under very general conditions, the suppression factor $f(k)$ can be obtained from causality to be

$$f(k) = \frac{1}{1 + (k_{\text{min}}/k)^m}, \quad (2)$$
where \( k_{\text{min}} \) is the wavenumber of the suppression scale. Causality places the strict constraint for the suppression index \( m \geq 4 - n \). A suppression factor like eq. (2) also has been found in a model with cosmic strings plus cold or hot dark matter \[3\]. Eq. (3) contains three parameters: the power spectrum amplitude \( A \), the power-law spectral index \( n \), and the super-Hubble suppression scale wavenumber \( k_{\text{min}} \).

There is a second constraint inferable from causality, which is convenient from the point of view of inflationary cosmology. Models of primordial density perturbations are generally classified as either inflationary or non-inflationary. For both types, the long wavelength suppression of the spectrum is a general characteristic which is imposed by causality, although it is typically ignored for inflation models. Since for non-inflationary models the causal horizon is about the same as the present-day Hubble radius, causality constraints also imply that \( k_{\text{min}} \sim \pi H_0 \) in eq. (2). It should be noted that for non-inflationary models, density perturbations can be both primordial and generated after last scattering, whereas eq. (4) strictly is valid for the former type. For many of the latter cases, eq. (4) may still be valid based on \[3\]. The focus here is not on specific models, although this caveat is worth stating. For inflation models \( k_{\text{min}} \) is far less constrained. For such models the largest scale of primordial density perturbations arose from the fluctuations which first crossed the Hubble radius during inflation \[4\]. The factor of expansion for them is given by the parameter \( N \), which is the number of e-folds of cosmic scale growth. In order to solve the horizon problem, inflation’s key purpose, the minimum required expansion places a lower limit of \( N \geq 50 - 70 \). Models of the standard inflationary cosmology offer no convincing reason for \( N \) to be near its lower limit and generally predict it to be several orders of magnitude bigger. Hence, for standard inflation models the super-Hubble suppression length scale generically is expected to be large, \( k_{\text{min}} \sim 0 \).

Previous analyses of the COBE-DMR data have tacitly assumed that the fundamental parameter \( k_{\text{min}} \) was zero. In this Letter we perform a maximum likelihood fit of the 4-year DMR data with the three parameter spectrum in eq. (1). We fix the suppression index \( m \) at various values and determine the most likely values of the amplitude \( A \), and spectral index \( n \), in the presence of a third fundamental parameter, the super-Hubble suppression scale wavenumber, \( k_{\text{min}} \), and we place limits on the allowable range of \( k_{\text{min}} \).

This analysis will consider the standard case of a flat universe \( \Omega_0 = 1 \) with zero cosmological constant \( \Lambda = 0 \). The CMBR temperature fluctuation along the direction unit vector \( \hat{n} \) is

\[
\frac{\delta T(\hat{n})}{T} = \frac{T(\hat{n}) - T}{T} = -\frac{H_0^2}{2V c^2} \sum_k \frac{\delta(k)}{k^2} e^{-ik \cdot y}, \quad (3)
\]

where \( \delta(k) \) is the Fourier amplitude of the density contrast \( \delta(r) \) and \( y \) is a vector of length \( y = 2c/H_0 \), the
distance to the particle horizon in a matter dominated universe, which points in the direction \( \hat{n} \). The power spectrum is defined as

\[
P(k)\delta_{kk'} \equiv \langle \delta(k)\delta(k') \rangle = \langle |\delta(k)|^2 \rangle \delta_{kk'},
\]

where the brackets denote ensemble average. \( P(k) \) will be taken as eqs. (1) and (2).

The CMBR temperature fluctuations on the celestial sphere are usually expressed by spherical harmonics

\[
\delta T(\hat{n})/T = \sum_{lm} a_{lm} Y_{lm}(\hat{n}),
\]

where \( Y_{lm}(\hat{n}) \) are the spherical harmonic functions. Defining a rotationally invariant coefficient \( C_l \equiv 1/(2l+1) \sum_m |a_{lm}|^2 \), one finds from eqs. (3)

\[
C_l = \frac{H_0^4}{2\pi c^4} \int_0^\infty dk \frac{P(k)}{k^2} |j_l(ky)|^2,
\]

where \( j_l(x) \) are the spherical Bessel functions. Using eq. (1), eq. (6) becomes

\[
C_l = \frac{AH_0^4}{2\pi c^4} \int_0^\infty dk \frac{k^{n-2}|j_l(ky)|^2}{1 + (k_{\text{min}}/k)^m}.
\]

By the above conventions, the quadrupole anisotropy is given as \( Q_{\text{rms}} = \sqrt{5}C_2/4\pi T \) where \( T = 2.728 K \) is the mean CMBR temperature.[6]

To simplify this initial analysis, we have not considered temperature fluctuations produced by tensor (gravitational) perturbations. Most theoretical models expect the CMBR anisotropies to be dominated by scalar perturbations. Tensor perturbation will be subject to a long wavelength suppression like eq. (2), although the power spectrum index \( n \) generally should be different.

To check for possible confusion from secondary sources of anisotropy, such as the Integrated Sachs-Wolfe (ISW) effect, we have generated fully processed power spectra with the code CMBFAST for a range of the cosmological parameters \( \Omega_{\text{vac}} \) and \( H \), for which a significant ISW effect is possible. The ISW effect generically increases the power in low-order multipoles and can thus "fill in" some of the suppression generated by a cutoff scale. However, this requires values of \( \Omega_{\text{vac}} \) in excess of \( \sim 0.5 \) to be significant, and in no case can the ISW effect mimic the effects of a small suppression length scale. Thus we do not further consider the ISW effect in this paper since it cannot decrease the significance of any possible detection of a cutoff scale. An alternative possibility is that an anti-alignment of the Galaxy quadrupole with the CMBR quadrupole would suppress the quadrupole power. However, Galaxy modeling indicates that this is not a significant effect.

To fit the parameters of our model power spectra, we will use the pixel based likelihood method (pixel based method) introduced in [5] and used in COBE-DMR studies. This method is predated by Gaussian likelihood fits to the 2-point angular correlation function (2-point method). The primary disadvantage of the 2-point...
method is that the 2-point correlation function is not Gaussian distributed, so the Gaussian likelihood fit is only approximate.

In the pixel based method the covariance matrix is computed between map pixels $i$ and $j$ as

$$M_{ij} \equiv \langle \frac{\delta T_i}{T} \frac{\delta T_j}{T} \rangle = \frac{1}{4\pi} \sum_l (2l+1) W_l^2 C_l \hat{n}_i \cdot \hat{n}_j,$$

where $T_i$ is the temperature in pixel $i$ of a map, $W_l^2$ is the experimental window function that includes the effects of beam smoothening and finite pixel size, $C_l$ is given in eq. (7), $P_l(\hat{n}_i \cdot \hat{n}_j)$ is the Legendre polynomial of order $l$, and $\hat{n}_i$ is the unit vector towards the center of pixel $i$. For pixel temperatures that are Gaussian distributed, the covariance matrix fully specifies the statics of the temperature fluctuations. The probability of observing a map with pixel temperatures $\vec{T}$, given a model $C_l$, is

$$P(\vec{T}|C_l(p)) d\vec{T} = \frac{d\vec{T}}{(2\pi)^{J/2}} e^{-\frac{1}{2} \vec{T}^T \cdot M^{-1}(C_l(p)) \cdot \vec{T}} \sqrt{\det M(C_l(p))},$$

(9)

where $J$ is the number of pixels in the map. Assuming a uniform prior distribution of cosmological model parameters, the probability, or likelihood function, of a given $C_l$ with parameters $p$ and a given map $\vec{T}$ is then

$$L(C_l(p)|\vec{T}) \propto e^{-\frac{1}{2} \vec{T}^T \cdot M^{-1}(C_l(p)) \cdot \vec{T}} \sqrt{\det M(C_l(p))}. \hspace{1cm} (10)$$

For convenience we will denote the likelihood function simply as $L(p)$.

We have evaluated the above likelihood function using the model power spectra in eq. (7) for two cases: sharp cutoff $m = \infty$ and minimal cutoff $m = 4 - n$. The results reported below are from the COBE correlation technique map, which has the best estimate of the high-latitude Galaxy subtracted off [8]. We have tested a case with a map which has no residual Galaxy subtracted and the results are not qualitatively different.

Full details of our tri-parameter likelihood analysis will be reported elsewhere. Here we will present our results in terms of the projected likelihoods $L(k_{\min}, n; Q_{\text{rms}-PS})$, $L(k_{\min}; Q_{\text{rms}-PS}, n)$, and $L(n; k_{\min}, Q_{\text{rms}-PS})$. For a likelihood function $L(p)$, defined by a set of parameters $p = (p_1, p_2)$, the projected likelihood $L(p_1; p_2)$ is defined as $L(p)$ for fixed $p_1$, evaluated at the most likely $p_2$. For $L(k_{\min}, n; Q_{\text{rms}-PS})$ the most likely $(k_{\min}, n)$ are given in table 1 with 68% confidence level (CL) uncertainties with respect to the projected likelihoods $L(k_{\min}; n, Q_{\text{rms}-PS})$ and $L(n; k_{\min}, Q_{\text{rms}-PS})$. Table 2 gives the projected likelihood results for both cutoff models for $L(k_{\min}, Q_{\text{rms}-PS})$ under the constraint $n = 1$. In both tables 1 and 2, the most likely quadrupole anisotropies $Q_{\text{rms}-PS}$, are given with 68% CL uncertainties from the likelihood evaluated
at the most likely values of $k_{\text{min}}$ and $n$ (or fixed $n = 1$ for table 2). These errors reflect the precision of the normalization for the specified models for a fixed shape of the spectrum i.e. $(k_{\text{min}}, n)$. Finally table 3 gives the 99% confidence upper limits on $k_{\text{min}}$ for unconstrained $n$ and $n = 1$. Our results reproduce those given in [11,12] in the pure power-law limit, $k_{\text{min}} = 0$.

Table 3 confirms a detection of a coherence length bigger than the Hubble radius, $c/H_0$, for both cutoff models. Tables 1 and 2 show that all values of the suppression length scale larger than the Hubble diameter ($\lambda_{\text{max}} \sim 2c/H_0$) are consistent with data, although length scales of order $4c/H_0$ are slightly preferred. If the spectral index is fixed at $n = 1$ (Table 2), there is a one sigma exclusion of $k_{\text{min}} = 0$. However with $n$ left unconstrained (Table 1), only the sharp cutoff model is found to exclude $k_{\text{min}} = 0$ at 68% confidence (Fig. 1). In no case was a two sigma exclusion of $k_{\text{min}} = 0$ found.

The quadrupole anisotropies $Q_{rns-Ps}$ in all cases in tables 1 and 2 are smaller than the pure power-law result of $Q_{rns-Ps} = 15.3^{+3.8}_{-2.8} \mu K$ [11,12], because of the low-$\ell$ suppression in the spectrum. Note that our results for $Q_{rns-Ps}$ are comparable to those in [13] where the quadrupole $C_2$ was fit independent of the rest of the spectrum. This suggests that the shape of the likelihood in our current analysis is being driven primarily by the low quadrupole, and the most-likely normalized spectra are such that the mean quadrupole in each case is comparable to the actual quadrupole in our sky.

One can get a feeling for the relative preference between the sharp and minimal cutoffs by examining the likelihood at $k_{\text{min}} = 0$, where both model spectra are common. By this approach the sharp cutoff is slightly preferred to the minimal cutoff by a factor 1.24. Note that the most-likely spectral index $n$ is closer to the scale-invariant limit ($n = 1$) with non-zero $k_{\text{min}}$ than it was with $k_{\text{min}} = 0$ ($n = 1.2 \pm 0.3$ at $k_{\text{min}} = 0$ [11,12]). In summary, a finite super-Hubble suppression length scale is, at best, suggested by the data.

Two features of the power spectrum model should be mentioned. First, for very large suppression length scales, $k_{\text{min}} y < 1$, the model spectra are virtually indistinguishable from a pure power-law spectrum. Therefore this analysis is most powerful at placing upper limits on the suppression scale wavenumber $k_{\text{min}}$. Second, there is some degeneracy between $k_{\text{min}}$ and $n$, particularly for the minimal cutoff model with $m = 4 - n$: decreasing the suppression scale $k_{\text{min}}$ steepens the slope of the power spectrum at low spherical harmonic order $\ell$, which partially mimics the effect of increasing the spectral index $n$. Additional large and medium scale anisotropy data, as expected from the forthcoming satellites MAP and Planck, should allow us to place better limits on $n$ so as to partly constrain this degeneracy.

To cross-check the results of our likelihood analysis, we simulated 1000 pure power-law, scale-invariant skies, $(k_{\text{min}}, n) = (0.0, 1.0)$, to determined what fraction have likelihood functions similar to the data. For the upper
limit on $k_{\text{min}}$, the Monte Carlo results firmly confirm the 99% CL given in table 3 for both types of cutoffs. For the lower limit on $k_{\text{min}}$, the Monte Carlo results indicate that a pure power-law, scale invariant universe has a 20% (33%) chance of spuriously imitating a universe with a super-Hubble suppression scale of the size favored by the data with a sharp (minimal) cutoff. These results are consistent with the likelihood analysis. In particular they verify that the tendency found in the data for a small suppression length scale is not an effect of anomalously large spurious fluctuations. The Monte Carlo analysis also indicates that increasing the signal-to-noise ratio with further data may give a 1-2 sigma greater discrimination of $k_{\text{min}} = 0$, but cosmic variance prohibits much greater significance. Thus, while the evidence for a finite suppression scale is not statistically significant, we find it suggestive enough to consider its implications.

The standard inflation models predict $k_{\text{min}} \sim 0$, which is consistent with, but not preferred by, the data. It is worth noting that the super-Hubble suppression behavior in the basic new inflation [13] and chaotic inflation [14] scenarios is closer to the sharp cutoff form. For a large class of non-inflationary models there is no mechanism for the growth of super-Hubble scale perturbations. In models with a "late time" cosmological phase transition [15], we expect $k_{\text{min}} \gamma \geq 4 \Gamma$. In models with topological defects, in which the perturbations are produced before last-scattering, the super-Hubble suppression scale depends on the dynamics of the defects. For example, in models with cosmic strings plus hot or cold dark matter it was found that $k_{\text{min}} \gamma \sim 2.1 - 7.9$ with a cutoff behavior closer to the minimal form [2]. These two types of non-inflation models are potentially consistent with the COBE data, but subhorizon evolution must be checked. For cosmic strings, recent simulations [16] indicate that subhorizon evolution induces greater power in the low-order multipoles, $C_{\ell}$. If further study supports this finding, it will imply that a small super-Hubble suppression length scale in the COBE data cannot be explained by cosmic strings. Both the super-Hubble suppression scale and cutoff should be noted in all theoretical models of primordial density perturbations.

If further investigation substantiates the suggestion of a small super-Hubble suppression length scale, there are at least two possible explanations. One is that density perturbations produced by non-inflation models are the dominant contributor. The second is that density perturbations are primarily produced during inflation, but that the number of e-folds, $N$, is close to its lower bound. In standard inflation models a small $N$ is viewed as a fine tuning of the theory. Nevertheless, a small $N$ is consistent from two other directions. Firstly, observationally a small $N$ also could consistently explain a nearly-flat to open universe [17]. We plan to study the suppression scale in low density power spectrum models. Secondly, it has been shown from standard Friedmann cosmology [18] that the requirement of small $N$ can be realized by a symmetry breaking phase transition at finite
temperature, during which the universe smoothly goes from an inflation-like stage to the radiation dominated stage without an intermediate reheating period. The naturalness of a small $N$ found in this study \cite{18} motivated the search in this paper for a small super-Hubble suppression length scale. Scenarios in this non-isentropic inflation-like expansion regime, which was named in \cite{18} the big-bang-like inflation regime, avoid requiring both localized fields on ultraflat potential surfaces and an impulsive large scale energy release during a reheating period \cite{4}, which must be isotropic over causally vastly-disconnected regimes. Both these fundamental conundrums have been difficult to resolve in the standard inflation picture. The theory of density perturbations \cite{19} in this intermediate regime of radiation and vacuum energy requires further study of warm inflation \cite{20} and possible other mechanisms before direct comparison is possible with standard inflation models. The increased complexity of this mixed fluid regime poses several theoretical questions which must be understood before firm predictions can be made.

The super-Hubble suppression scale added to the power spectrum in this paper is required on first principle by any causal theory. It is more fundamental than the spectral index and the amplitude and much less model-dependent than other processes that can effect the large-scale anisotropy, such as the Integrated Sachs-Wolfe effect. Moreover, the suppressing effect of causality on the power spectrum can produce a potentially significant effect on large scales that is not readily confused with other processes that generically boost the large-scale anisotropy power. This possibility calls for a reexamination of existing parameter fits to the COBE data.

In conclusion, we modified the primordial power spectrum of density fluctuations from a pure power-law form to a form that includes a super-Hubble suppression scale, $k_{\text{min}}$, so as to properly respect causality constraints. We fit this spectrum to the 4-year COBE-DMR data and find that the data prefers a finite suppression scale, but does not rule out $k_{\text{min}} = 0$. The best fit to the COBE 4-year data is $(n, k_{\text{min}y}, Q_{\text{rms}} - P_S) = (1.07^{+0.32}_{-0.35}, 3.4^{+0.7}_{-2.2}, 10.9 \pm 0.7 \mu K)$, with a slight preference for a sharp super-Hubble cutoff. Upper limits on $k_{\text{min}y}$ have been firmly placed. We conclude that this third fundamental parameter is measurable from the COBE data.

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FIGURE CAPTIONS
Figure 1: Relative likelihood for the suppression scale wavenumber, \(k_{\text{min}}\), projected over \(n\) and \(Q_{\text{rms}}-P_S\) with \(y = 2/H_0\).
super-Hubble cutoff & $k_{\text{min} y}$ & $n$ & $Q_{\text{rms-PS}}$ \\
Minimal ($m = 4 - n$) & $2.9^{+1.8}_{-2.9}$ & $1.06^{+0.48}_{-0.67}$ & $13.0 \pm 0.9$ \\
Sharp ($m = \infty$) & $3.4^{+0.7}_{-2.2}$ & $1.07^{+0.32}_{-0.35}$ & $10.9 \pm 0.7$

Table 1: Maximum likelihood parameter estimates for unconstrained $n$ with 68% CL uncertainties from the respective projected likelihood functions.

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super-Hubble cutoff ($n = 1$) & $k_{\text{min} y}$ & $Q_{\text{rms-PS}}$ \\
Minimal ($m = 4 - n$) & $3.2^{+7.0}_{-2.9}$ & $12.8 \pm 0.9$ \\
Sharp ($m = \infty$) & $3.5^{+0.8}_{-1.8}$ & $10.8 \pm 0.7$

Table 2: Maximum likelihood parameter estimates for constrained $n = 1$ with 68% CL uncertainties from the respective projected likelihood functions.

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super-Hubble cutoff & $k_{\text{min} y}$ & $k_{\text{min} y}|_{(n=1)}$ \\
Minimal ($m = 4 - n$) & $10.5$ & $7.2$ \\
Sharp ($m = \infty$) & $5.0$ & $4.9$

Table 3: 99% CL upper limits on $k_{\text{min} y}$. 

