On a coupled Kadomtsev–Petviashvili system associated with an elliptic curve

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Abstract
The coupled Kadomtsev–Petviashvili system associated with an elliptic curve, proposed by Date, Jimbo, and Miwa [J. Phys. Soc. Jpn., 52:766–771, 1983], is reinvestigated within the direct linearization framework, which provides us with more insights into the integrability of this elliptic model from the perspective of a general linear integral equation. As a result, we successfully construct for the elliptic coupled Kadomtsev–Petviashvili system not only a Lax pair composed of differential operators in $2 \times 2$ matrix form but also multisoliton solutions with phases parametrized by points on the elliptic curve. Dimensional reductions based on the direct linearization, to the elliptic coupled Korteweg–de Vries and Boussinesq systems, are also discussed. In addition, a novel class of solutions is obtained for the $D_\infty$-type Kadomtsev–Petviashvili equation with nonzero constant background as a by-product.

KEYWORDS
DKP, direct linearization, dimensional reduction, elliptic coupled KP, Lax pair, $\tau$-function, nonzero constant background, solitons

1 | INTRODUCTION

It is well known that integrable systems often come in three different classes comprising rational, trigonometric/hyperbolic, and elliptic models. Elliptic models are described by equations where there are parameters that are essentially moduli of elliptic curves, or where the dependent variable appears in the argument of elliptic functions. The recent history of the subject has taught us
the remarkable finding that the theory of integrable systems is intimately linked to that of elliptic functions and curves. One aspect of this connection is the fact that, as far as we know, the richest class of integrable systems are the ones associated with those curves. For example, the Adler equation (i.e., the discrete Krichever–Novikov equation)\(^1\) acts as the master equation among first-order partial difference equations; the elliptic Painlevé equation is on the top in Sakai’s classification\(^2\) of the discrete Painlevé equations.

Here, we focus on a three-component integrable partial differential system given by

\[
\begin{align*}
\partial_3 u &= \frac{1}{4} \partial_1^3 u + \frac{3}{2} (\partial_1 u)^2 + \frac{3}{4} \partial_1^{-1} \partial_2^2 u + 3 g (1 - u w), \\
\partial_3 v &= -\frac{1}{2} \partial_1^3 v - 3 (\partial_1 u - 3 e) \partial_1 v + \frac{3}{2} \partial_1 \partial_2 v + 3 (\partial_2 u) v, \\
\partial_3 w &= -\frac{1}{2} \partial_1^3 w - 3 (\partial_1 u - 3 e) \partial_1 w - \frac{3}{2} \partial_1 \partial_2 w - 3 (\partial_2 u) w,
\end{align*}
\]

where the solutions \(u, v,\) and \(w\) are functions depending on variables \(x_1, x_2,\) and \(x_3.\) The constant parameters \(e\) and \(g\) are moduli of an elliptic curve, given below in (5), and \(\partial_j\) denotes the partial-differential operator

\[
\partial_j [\cdot] \doteq \frac{\partial}{\partial x_j} [\cdot],
\]

and \(\partial_j^{-1}\) is a pseudodifferential operator that can be interpreted as an integral

\[
\partial_j^{-1} [\cdot] \doteq \int_{x_j}^{x} [\cdot] \text{d}x_j
\]

in a standard way, see Ref. 3. Equation (1) is an alternative presentation of an elliptic Kadomtsev–Petviashvili (KP)-type system that was originally proposed by Date, Jimbo, and Miwa, the form of which actually suggests a \((3 + 1)\)-dimensional differential-difference system, see section 3 of Ref. 4. It was also pointed out by those authors that such a system follows from a similar construction of solutions of the fully anisotropic Landau–Lifshitz (LL) equation from a bilinear perspective as in Ref. 5. Thus, one may infer that Equation (1) in a sense is a KP (higher-dimensional) analog of the LL equation. Furthermore, from the form (1) of the elliptic KP system, we can conclude that there is a connection (see Section 7) with the coupled KP system

\[
\begin{align*}
\partial_t U' &= \frac{1}{4} \partial_x^3 U' + \frac{3}{2} V \partial_x U' + \frac{3}{4} \partial_x^{-1} \partial_y^2 U' - 6 \partial_x (V W), \\
\partial_t V &= -\frac{1}{2} \partial_x^3 V - \frac{3}{2} U \partial_x V + \frac{3}{2} \partial_x \partial_y V + \frac{3}{2} (\partial_x^{-1} \partial_y U) V, \\
\partial_t W &= -\frac{1}{2} \partial_x^3 W - \frac{3}{2} U \partial_x W + \frac{3}{2} \partial_x \partial_y W - \frac{3}{2} (\partial_x^{-1} \partial_y U) W
\end{align*}
\]

proposed by Hirota and Ohta (cf. Ref. 6 and also formula (3.94) in Ref. 7), where the solutions \(U', V,\) and \(W\) are functions of the independent variables \(x, y,\) and \(t,\) with partial derivatives \(\partial_x, \partial_y,\) and \(\partial_t,\) respectively, and where \(\partial_x^{-1}\), defined in a similar way as above, is the pseudodifferential operator with respect to \(x.\) As Equation (4) is one of the members in the DKP hierarchy (a KP-type hierarchy
associated with the infinite-dimensional Lie algebra $D_{\infty}$, see Ref. 8) that possesses a rich integrable structure in the theory of integrable systems, we believe that the understanding of elliptic model (1) will certainly yield additional insights into the integrability of many other nonlinear systems. However, this remarkable elliptic integrable system has attracted little attention in the literature since the paper, 4 as far as we are aware. For this reason, we believe that the elliptic coupled KP system (1) deserves reinvestigation to explore further its integrability.

In the present paper, we adopt the direct linearization (DL) method to study the elliptic coupled KP equation (1). Originally proposed by Fokas, Ablowitz, and Santini for constructing a large class of solutions of nonlinear integrable partial differential equations, the method that is based on formal singular linear integral equations 9–11 was subsequently developed into a comprehensive framework to construct (discrete and continuous) integrable systems and study their underlying algebraic structures, see, for example, Refs. 12, 13, and 14, 15 for constructions of integrable discretization of nonlinear partial differential equations, and Refs. 16–20 for the treatment of three-dimensional equations of KP-type. In Refs. 21–23, the connection between the DL and integrable systems on Lie algebras was developed. A powerful tool emerging from DL, first developed in Ref. 12, was an infinite matrix structure in the space of the spectral variable. The associated infinite matrix representation of the linear integral equation allows us to turn the DL into an algebraic method that has been very effective in forging an understanding of the underlying integrability of the nonlinear integrable systems and their interconnections. In Refs. 24 and 25, the notion of elliptic infinite matrix (effectively amounting to an index-relabeling system of infinite matrices) was introduced. This allows us to study integrable equations associated with elliptic curves within the DL framework.

By reparametrizing the time evolution and the Cauchy kernel for the LL equation in the fermionic construction 5 and simultaneously considering a linear integral equation with a skew-symmetric integration measure that was introduced in Refs. 21–23 for the BKP-type equations, we establish in this paper the DL scheme for the elliptic coupled KP system (1). This allows us to study the integrability of the elliptic coupled KP system from a unified perspective. As a result, we successfully derive the nonlinear equation (1) together with a suitable Lax pair from the elliptic infinite matrix structure. Meanwhile, we construct the elliptic soliton solutions of (1) in terms of the $\tau$-function, which possesses a Pfaffian structure, using a Pfaffian version of the well-known Laplace-type expansion formula (see Ref. 26 and also Appendix B), as well as a Pfaffian analog of the famous Frobenius formula for the determinants of elliptic Cauchy matrices (see Ref. 5 and also Appendix C). These results underpin not only the integrability of the elliptic coupled KP system from the viewpoint of solvability, but also induce a new class of solutions of the DKP equation with nonzero constant background as a by-product. In addition, we discuss dimensional reductions of (1), from which we obtain the elliptic coupled Korteweg–de Vries (KdV) and Boussinesq (BSQ) systems together with their respective Lax pairs.

The paper is organized as follows. In Section 2, we introduce the fundamental objects that will be used in construction of elliptic integrable systems, including the notion of elliptic index-raising matrices and elliptic index labels. The DL scheme of the elliptic coupled KP system is established in Section 3 in the language of infinite matrices. Section 4 is concerned with the construction of the elliptic coupled KP system (1) and its Lax pair. In the subsequent section 5, we discuss dimensional reductions to the elliptic coupled KdV and BSQ systems. The formulae of the elliptic soliton solutions to (1) are presented in Section 6. Finally, we explain in Section 7 how the soliton solutions to (1) generate those to the DKP equation with nonzero constant background.
We present an introduction to the fundamental objects that are needed in this paper, including the elliptic curve, infinite matrices, elliptic index-raising operators, and so on. These objects were introduced in Refs. 24, 25 for the DL construction of the so-called discrete and continuous elliptic KdV and KP equations.

The elliptic curve that we consider in this paper is of the form

\[ k^2 = K + 3e + \frac{g}{K}, \]

in which

\[ e \doteq e_1 \quad \text{and} \quad g \doteq (e_1 - e_2)(e_1 - e_3) \]

are the moduli of the curve, for \( e_1, e_2, \) and \( e_3 \) being the branch points of the standard Weierstrass elliptic curve \( z^2 = 4(Z - e_1)(Z - e_2)(Z - e_3) \). The elliptic curve (5) is parametrized by a uniformizing variable \( \kappa \) through the coordinates

\[ k = \frac{1}{2} \frac{\wp'(\kappa)}{\wp(\kappa) - e}, \quad K = \wp(\kappa) - e, \]

where \( \wp \) and \( \wp' \) denote the standard Weierstrass elliptic function and its first-order derivative, respectively.

We consider infinite matrices taking the form of \( U = (U_{i,j})_{\infty \times \infty} \) and infinite column and row vectors \( a = (a_i)_{\infty \times 1} \) and \( a^t = (a_i)_{1 \times \infty} \). We adopt the notations \( t(\cdot) \) for the transpose, \( (\cdot)^{(i,j)} \) for the \((i,j)\)-entry of an infinite matrix, and \( (\cdot)^{(i)} \) for the \(i\)th component of an infinite vector.

**Definition 1.** The index-raising infinite matrix \( \Lambda \) and its transpose \( t^t \Lambda \) are defined by their respective \((i,j)\)-entries

\[ \Lambda^{(i,j)} \doteq \delta_{i+1,j} \quad \text{and} \quad t^t \Lambda^{(i,j)} \doteq \delta_{i,j+1}, \quad \forall i, j \in \mathbb{Z}, \]

where \( \delta_{i,j} \) is the standard Kronecker \( \delta \)-function defined as

\[ \delta_{i,j} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \]

**Remark 1.** The infinite matrices \( \Lambda \) and \( t^t \Lambda \) are entitled index-raising matrices because of the identities

\[ (\Lambda U)^{(i,j)} = U^{(i+1,j)} \quad \text{and} \quad (U^t \Lambda)^{(i,j)} = U^{(i,j+1)}, \quad \forall i, j \in \mathbb{Z}; \]

in other words, the operation of \( \Lambda \) (respectively, \( t^t \Lambda \)) from the left (respectively, right) raises all the row (respectively, column) indices of an infinite matrix by 1. Similarly, we have the identities

\[ (\Lambda a)^{(i)} = a^{(i+1)} \quad \text{and} \quad (a^t \Lambda)^{(i)} = a^{(i+1)}, \quad \forall i \in \mathbb{Z}, \]

for infinite column and row vectors.
Definition 2. The infinite projection matrix $O$ is defined by its $(i, j)$-entries

$$O^{(i,j)} = \delta_{i,0}\delta_{0,j}, \quad \forall i, j \in \mathbb{Z}. \quad (12)$$

Remark 2. It is easily verified that the multiplication between $U$ and $O$ results in identities

$$(OU)^{(i,j)} = \delta_{i,0}U^{(0,j)} \quad \text{and} \quad (OU)^{(i,j)} = U^{(i,0)}\delta_{0,j}, \quad \forall i, j \in \mathbb{Z}. \quad (13)$$

This implies that the projection infinite matrix $O$ plays the role of mapping arbitrary $U$ to an infinite matrix of rank one. Likewise, we have for arbitrary $a$ the identities

$$(Oa)^{(i)} = \delta_{i,0}a^{(0)} \quad \text{and} \quad (aO)^{(i)} = a^{(0)}\delta_{0,i}, \quad \forall i \in \mathbb{Z}. \quad (14)$$

Remark 3. Although Equation (8) suggests that the infinite matrices $\Lambda$ and $'\Lambda$ are each other’s inverse, their role in the structure is such that they are never multiplied together, and appear as distinct symbols separated by the projection matrix $O$ in the algebraic structure of infinite matrices.

Definition 3. We define a special particular unit column vector $e$ and its transpose $'e$ by their respective components

$$e^{(i)} = e^{(i)} = \delta_{i,0}, \quad \forall i \in \mathbb{Z}. \quad (15)$$

Remark 4. The infinite projection matrix $O$ can be written as the multiplication of $e$ and $'e$, namely, $O = e'e$. It is easily verified that $e$ and $'e$ possess the properties

$$(eU)^{(i)} = U^{(0,i)}, \quad (UE)^{(i)} = U^{(i,0)} \quad \text{and} \quad eUe = U^{(0,0)}, \quad \forall i \in \mathbb{Z} \quad (16)$$

for an arbitrary infinite matrix $U$ as well as

$$'e a = a^{(0)} \quad \text{and} \quad 'ae = a^{(0)} \quad (17)$$

for arbitrary infinite column and row vectors $a$ and $'a$.

To deal with the elliptic integrable systems, we introduce the notion of elliptic index-raising matrices for future convenience.

Definition 4. The elliptic index-raising operators $\Lambda$ and $L$ are defined as

$$\Lambda \doteq \frac{1}{2} \frac{\varphi'(\Lambda)}{\varphi(\Lambda)} - e \quad \text{and} \quad L \doteq \varphi(\Lambda) - e, \quad (18)$$

respectively; similarly, their respective transposes are defined by

$$'\Lambda \doteq \frac{1}{2} \frac{\varphi'(\Lambda)}{\varphi(\Lambda)} - e \quad \text{and} \quad 'L \doteq \varphi(\Lambda) - e. \quad (19)$$

These elliptic index-raising operators should be understood as formal series expansions of $\Lambda$ and $'\Lambda$, respectively.
Remark 5. The elliptic index-raising operators $\Lambda$, $L$ and $\tilde{\Lambda}$, $\tilde{L}$ obey the elliptic curve relations

$$\Lambda^2 = L + 3e + \frac{g}{L} \quad \text{and} \quad \tilde{\Lambda}^2 = \tilde{L} + 3e + \frac{g}{\tilde{L}},$$

(20)

respectively, as consequences of formulae (5) and (7). Here, for notational convenience when there is no issue of matrix ordering, we have used the notations $\frac{1}{L}$ and $\frac{1}{\tilde{L}}$ to denote the inverses of $L$ and $\tilde{L}$, respectively.

Remark 6. The formal operators $\Xi$, $t\Xi$, $L$ and $tL$ should be understood in the following way. Since $\Xi$ and $L$ commute, and similarly, $t\Xi$ and $tL$ commute, we can consider a joint set of formal eigenvectors $c$ and $tc$, respectively, obeying the eigenvalue equations

$$\Xi c(\kappa) = kc(\kappa), \quad Lc(\kappa) = Kc(\kappa) \quad \text{and} \quad t\Xi c(\kappa') = k'c(\kappa'), \quad tLc(\kappa') = K'c(\kappa'),$$

(21)

where $(k, K)$ and $(k', K')$ are the points on the elliptic curve (5), parametrized by their respective uniformizing spectral parameters $\kappa$ and $\kappa'$ for $\kappa, \kappa' \in \mathbb{C}$, namely, we have relations

$$k = \frac{1}{2} \frac{\wp'(\kappa)}{\wp(\kappa) - e}, \quad K = \wp(\kappa) - e, \quad k' = \frac{1}{2} \frac{\wp'(\kappa')}{\wp(\kappa') - e}, \quad \text{and} \quad K' = \wp(\kappa') - e.$$ 

(22)

Thus, we can think of the points $(k, K)$ and $(k', K')$ on the curve as the “symbols” of these operators, in some representation defined by the basis of infinite vectors $c(\kappa) = (\kappa^l)_{0<\infty}$ and $t\Xi c(\kappa') = (\kappa'^l)_{1<\infty}$, respectively, composed of the monomials of uniformizing spectral parameters $\kappa$ and $\kappa'$. Furthermore, these variables will play the role of the spectral parameters for the elliptic coupled KP system.

In the sections below, we only deal with the elliptic index-raising operations. For this reason, we introduce elliptic index labels $[i, j]$ and $[i]$ for infinite matrices and infinite vectors, respectively, compared with the above nonelliptic index labels $(i, j)$ and $(i)$.

Definition 5. In the elliptic index-labeling system, the $[i, j]$-entries of the infinite matrix $U$ are defined by the following:

$$U^{[2i, 2j]}(L^iU^j)(0, 0), \quad U^{[2i+1, 2j+1]}(L^iU^j\Lambda)(0, 0),$$

(23a)

$$U^{[2i+1, 2j]}(\Lambda L^iU^j)(0, 0), \quad U^{[2i, 2j+1]}(L^iU^j\Lambda)(0, 0),$$

(23b)

for all $i, j \in \mathbb{Z}$. Likewise, we define the $[i]$th components of the infinite vectors $a$ and $t\tilde{a}$ as follows:

$$a^{[2i]}(L^i\tilde{a})(0, 0), \quad a^{[2i+1]}(L^i\tilde{a})(0, 0),$$

(24a)

$$t\tilde{a}^{[2i]}(L^i\tilde{a})(0), \quad t\tilde{a}^{[2i+1]}(L^i\tilde{a})(0),$$

(24b)

for arbitrary $i \in \mathbb{Z}$.

In the elliptic index-labeling system, $\Lambda$ and $\tilde{\Lambda}$ (respectively, $L$ and $\tilde{L}$) play roles of order 1 (respectively, order 2) index-raising operators. We also comment that in concrete calculation,
sometimes, the curve relations in (20) are useful to reduce the powers of $\Lambda$ and $'\Lambda$. For example, we have

$$(\Lambda^2 U)^{(0,0)} = U^{[2,0]} + 3eU^{[0,0]} + gU^{[-2,0]} \quad \text{and} \quad (\Lambda^3 U)^{(0,0)} = U^{[3,0]} + 3eU^{[1,0]} + gU^{[-1,0]},$$

(25)
because of $\Lambda^2 U = (L + 3e + g/L)U$ and $\Lambda^3 U = \Lambda\Lambda^2 U = \Lambda(L + 3e + g/L)U$.

3 INFINITE MATRIX REPRESENTATION OF THE ELLIPTIC COUPLED KP SYSTEM

The essential ingredients in the construction of the elliptic coupled KP system are the plane wave factor, which defines the dynamics of the system in terms of the independent variables $x_j$, given by

$$\rho_n(\kappa) = \exp \left\{ \sum_{j=0}^{\infty} k^{2j+1} x_{2j+1} + \sum_{j=1}^{\infty} \left[ K^j - \left( \frac{g}{K} \right)^j \right] x_{2j} \right\} \left( \frac{K}{\sqrt{g}} \right)^n,$$

(26)
and the skew-symmetric Cauchy kernel

$$\Omega(\kappa, \kappa') \equiv \frac{K - K'}{k + k'} = \frac{k - k'}{1 - \frac{g}{KK'}}.$$ 

(27)
The plane wave factor (26) and the kernel (27) are reparametrization of those objects first presented in Ref. 5 in the different context of a fermionic construction of the LL equation.

**Definition 6.** For the given plane wave factor (26) and the Cauchy kernel (27), the linear integral equation of the elliptic coupled KP system takes the form of

$$u_n(\kappa) + \int_D d\zeta(\lambda, \lambda') \rho_n(\kappa) \Omega(\kappa, \lambda') \rho_n(\lambda') u_n(\lambda) = \rho_n(\kappa) c(\kappa),$$

(28)
where the infinite column vector $u_n(\kappa)$ is the wave function whose components depend on the independent variables $x_j$ for $j \in \mathbb{Z}^+$ and the spectral parameter $\kappa$, and the integration measure $d\zeta$ and the integration domain $D$ must obey the antisymmetry property

$$d\zeta(\kappa, \kappa') = -d\zeta(\kappa', \kappa), \quad \forall (\kappa, \kappa') \in D.$$ 

(29)

**Remark 7.** At the current stage, we do not specify the form of the integration measure. The only requirement is that the associated homogeneous integral equation of (28) has only zero solution, which guarantees the most general solution space from the perspective of the DL approach.

For the sake of construction of integrable systems, we need the infinite matrix $C_n$ defined as

$$C_n = \int_D d\zeta(\kappa, \kappa') \rho_n(\kappa)c(\kappa) c(\kappa') \rho_n(\kappa'),$$

(30)
in which the integration measure and domain must satisfy the same antisymmetry property (29), and also the infinite matrix $\Omega$ defined by

\begin{equation}
\Omega c(\kappa) \Omega c(\kappa) \doteq \Omega(\kappa, \kappa').
\end{equation}

(31)

From the definitions, it is reasonable to think of $C_n$ and $\Omega$ as the infinite matrix representation of the plane wave factor (26) and the kernel (27). Due to the antisymmetry of the integration measure and the kernel, it is verified that both $C_n$ and $\Omega$ are skew-symmetric, that is, $C_n = -C_n$ and $\Omega = -\Omega$. The key object toward nonlinear integrable systems in the direct linearization is a potential matrix. We give its definition as follows.

**Definition 7.** The potential matrix in the DL is a double integral in terms of the spectral parameters, defined as

\begin{equation}
U_n \doteq \int_D d\xi(\kappa, \kappa') u_n(\kappa) c(\kappa') \rho_n(\kappa'),
\end{equation}

(32)

in which $u_n(\kappa)$ satisfies the linear integral equation (28), and the measure is the same as the one for (28), obeying the antisymmetry property (29).

By following (32) and (31), we can reformulate the linear integral equation (28) as

\begin{equation}
u_n(\kappa) = (1 - U_n \Omega) \rho_n(\kappa) c(\kappa).
\end{equation}

(33)

Performing the operation $\int_D d\xi(\kappa, \kappa')(33) \Omega c(\kappa') \rho_n(\kappa')$, we can further derive

\begin{equation}
U_n = (1 - U_n \Omega) C_n, \quad \text{or alternatively,} \quad U_n = C_n(1 + \Omega C_n)^{-1}.
\end{equation}

(34)

Equation (34) in some sense can be considered as the infinite matrix version of (33).

We now introduce the $\tau$-function associated with the elliptic coupled KP system.

**Definition 8.** The $\tau$-function is formally defined by

\begin{equation}
\tau^2_n \doteq \det(1 + \Omega C_n),
\end{equation}

(35)

where $\Omega$ and $C_n$ are given by (31) and (30), respectively. The determinant should be understood as the formal expansion

\begin{equation}
\det(1 + \Omega C_n) = 1 + \sum_i (\Omega C_n)^{(i,j)} + \sum_{i<j} \left| (\Omega C_n)^{(i,j)} \right| (\Omega C_n)^{(j,i)} + \cdots.
\end{equation}

(36)

We also remark that the determinant satisfies the identity $\ln[\det(1 + \Omega C_n)] = \text{tr}[\ln(1 + \Omega C_n)]$.

Our aim is to derive the dynamical relations of $U_n$ in terms of the continuous variables $x_j$ and the discrete variable $n$. To realize this, we first investigate the evolutions of $C_n$. Observing the identity (21) and the form of the plane wave factor (26), we obtain the following dynamical relations:

\begin{equation}\tag{37a}
\delta_{2j+1} C_n = \Lambda^{2j+1} C_n + C_n \Lambda^{2j+1},
\end{equation}
\[
\begin{align*}
\partial_{2j} C_n &= \left[ L^j - \left( \frac{g}{L} \right)^j \right] C_n + C_n \left[ L^j - \left( \frac{g}{L} \right)^j \right], \\
C_{n+1} \frac{\sqrt{g}}{L} &= \frac{L}{\sqrt{g}} C_n,
\end{align*}
\]  

(37b) (37c)

by differentiating \( C_n \) with respect to \( x_j \) and shifting \( C_n \) with respect to \( n \). Next, from Equations (27) and (31), we are able to derive

\[
\Omega \Lambda + ^t \Lambda \Omega = OL - ^t L O \quad \text{and} \quad \Omega - g^t L^{-1} \Omega L^{-1} = O \Lambda - ^t \Lambda O.
\]  

(38)

In fact, multiplying the two equations in (38), respectively, by \( ^t e(\xi') \) from left and \( e(\xi) \) from the right yield

\[
\Omega(\xi, \xi')k + k' \Omega(\xi, \xi') = K - K' \quad \text{and} \quad \Omega(\xi, \xi') - \frac{g}{K K'} \Omega(\xi, \xi') = k - k',
\]  

(39)

namely (27); in other words, equations in (38) are nothing but the infinite matrix representation of the elliptic Cauchy kernel. With the help of (38), we can further derive the following relations for the infinite matrix \( \Omega \) by mathematical induction:

\[
\begin{align*}
\Omega \Lambda^{2j+1} + ^t \Lambda^{2j+1} \Omega &= O_{2j+1} L - ^t L O_{2j+1}, \\
\Omega \left[ L^j - \left( \frac{g}{L} \right)^j \right] + \left[ L^j - \left( \frac{g}{L} \right)^j \right] \Omega &= O_j \Lambda - ^t \Lambda O_j, \\
\Omega \frac{L}{\sqrt{g}} - \frac{\sqrt{g}}{L} \Omega &= O_2 \frac{L}{\sqrt{g}},
\end{align*}
\]  

(40a) (40b) (40c)

in which

\[
O_j \doteq \sum_{i=0}^{j-1} (-^t \Lambda)^i O \Lambda^{j-1-i}, \quad O_j' \doteq \sum_{i=0}^{j-1} g^{it} L^{-i} O L^{j-1-i}.
\]  

(41)

**Proposition 1.** The infinite matrix \( U_n \) defined by (32) satisfies the following continuous and discrete dynamical evolutions:

\[
\begin{align*}
\partial_{2j+1} U_n &= \Lambda^{2j+1} U_n + U_n \Lambda^{2j+1} - U_n \left( O_{2j+1} L - ^t L O_{2j+1} \right) U_n, \\
\partial_{2j} U_n &= \left[ L^j - \left( \frac{g}{L} \right)^j \right] U_n + U_n \left[ L^j - \left( \frac{g}{L} \right)^j \right] - U_n \left( O_j \Lambda - ^t \Lambda O_j \right) U_n, \\
U_{n+1} \frac{\sqrt{g}}{L} &= \frac{L}{\sqrt{g}} U_n - U_{n+1} O_2 \frac{L}{\sqrt{g}} U_n.
\end{align*}
\]  

(42a) (42b) (42c)

**Proof.** We only prove (42b) and (42c). Differentiating (34) with respect to \( x_{2j} \) gives rise to

\[
\partial_{2j} U_n = (1 - U_n \Omega) (\partial_{2j} C_n) - (\partial_{2j} U_n) \Omega C, \quad \text{namely,} \quad (\partial_{2j} U_n) (1 + \Omega C_n) = (1 - U_n \Omega) (\partial_{2j} C_n).
\]  

(43)
Notice that the infinite matrix $C_n$ obeys the evolution given by (37b). The above equation is reformulated as

$$ (\partial_{2j} U_n)(1 + \Omega C_n) = (1 - U_n \Omega) \left[ L^j - \left( \frac{g}{L} \right)^j \right] C_n + U_n \left[ L^j - \left( \frac{g}{L} \right)^j \right]. \quad (44) $$

We can now replace $\Omega [L^j - (\frac{g}{L})^j]$ by following (40b). This, in turn, implies

$$ (\partial_{2j} U_n)(1 + \Omega C_n) = \left[ L^j - \left( \frac{g}{L} \right)^j \right] C_n + U_n \left[ L^j - \left( \frac{g}{L} \right)^j \right] \left( 1 - U_n \Omega \right) \left[ O' \Lambda - ' \Lambda O' \right] C_n, \quad (45) $$

which immediately results in (42b) by multiplying $(1 + \Omega C_n)^{-1}$ from the right. Equation (42c) is derived by a similar approach. We shift (34) with respect to $n$ and obtain

$$ U_{n+1} \frac{\sqrt{g}}{L} = (1 - U_{n+1} \Omega) \frac{L}{\sqrt{g}} C_n \quad (46) $$

in virtue of (37c). By substituting $\Omega \frac{L}{\sqrt{g}}$ with the help of (40c), this equation turns out to be

$$ U_{n+1} \frac{\sqrt{g}}{L} = \frac{L}{\sqrt{g}} C_n - U_{n+1} O_2 \frac{L}{\sqrt{g}} \Omega \Omega C_n, \quad (47) $$

that is,

$$ U_{n+1} \frac{\sqrt{g}}{L} \left( 1 + \Omega C_n \right) = \frac{L}{\sqrt{g}} C_n - U_{n+1} O_2 \frac{L}{\sqrt{g}} C_n. \quad (48) $$

Multiplying $(1 + \Omega C_n)^{-1}$ from the right, we end up with (42c). Equation (42a) is proven in a similar way.

**Proposition 2.** The infinite matrix $U_n$ satisfies the antisymmetry condition

$$ ^t U_n = -U_n, \quad \text{and consequently} \quad U_{n}^{[i,j]} = -U_{n}^{[j,i]} \quad (49) $$

in terms of the elliptic index labels $[i, j]$ for all $i, j \in \mathbb{Z}$.\[\[\]

**Proof.** Notice that $C_n$ and $\Omega$ are both skew-symmetric. We from (34) obtain

$$ ^t U_n = ^t [C_n(1 + \Omega C_n)^{-1}] = ^t [C_n^{-1} + \Omega]^{-1} $$

$$ = ( ^t C_n^{-1} + ^t \Omega )^{-1} = -( C_n^{-1} + \Omega )^{-1} = -C_n(1 + \Omega C_n)^{-1} = -U_n, \quad (50) $$

and subsequently, $U_{n}^{[i,j]} = -U_{n}^{[j,i]}$ for all $i, j \in \mathbb{Z}$ by following the elliptic index labels defined by (23).\[\[\]

We can also follow the derivation of (42) and construct the dynamical relations for the wave function $u_n(\kappa)$.\[\[\]
**Proposition 3.** The wave function of the linear integral equation (28) obeys dynamical evolutions with respect to the continuous variables $x_j$ and the discrete variable $n$ as follows:

\[
\frac{\partial}{\partial j+1} u_n(\kappa) = \Lambda^{2j+1} u_n(\kappa) - U_n \left( O_{2j+1} - \Omega \right) u_n(\kappa), \tag{51a}
\]

\[
\frac{\partial}{\partial j} u_n(\kappa) = \left[ L^j - \left( \frac{g}{L} \right)^j \right] u_n(\kappa) - U_n \left( O_j' \Lambda - \Omega \right) u_n(\kappa), \tag{51b}
\]

\[
u_{n+1}(\kappa) = \frac{L}{\sqrt{g}} u_n(\kappa) - U_{n+1} O_2 \frac{L}{\sqrt{g}} u_n(\kappa). \tag{51c}
\]

**Proof.** We only present the proof of (51b) and (51c). By differentiating (33) with respect to $x_{2j}$, we obtain

\[
\frac{\partial}{\partial j} u_n(k) = (1 - U_n \Omega) \left[ \frac{\partial}{\partial j} \rho_n(\kappa) \right] c(\kappa) - (\frac{\partial}{\partial j} U_n) \Omega \rho_n(\kappa) c(\kappa). \tag{52}
\]

Equations (21) and (42b) can help us to reformulate the above equation as

\[
\frac{\partial}{\partial j} u_n(k) = \left[ L^j - \left( \frac{g}{L} \right)^j \right] (1 - U_n \Omega) \rho_n(\kappa) c(\kappa)
- U_n \left\{ \Omega \left[ L^j - \left( \frac{g}{L} \right)^j \right] + \left[ L^j - \left( \frac{g}{L} \right)^j \right] \right\} \rho_n(\kappa) c(\kappa)
+ U_n \left( O_j' \Lambda - \Omega \right) U_n \Omega \rho_n(\kappa) c(\kappa). \tag{53}
\]

Replacing $\Omega \left[ L^j - \left( \frac{g}{L} \right)^j \right] + \left[ L^j - \left( \frac{g}{L} \right)^j \right] \Omega$ by $O_j' \Lambda - \Omega \Lambda O_j'$ according to (40b), we reach to

\[
\frac{\partial}{\partial j} u_n(k) = \left[ L^j - \left( \frac{g}{L} \right)^j \right] (1 - U_n \Omega) \rho_n(\kappa) c(\kappa) - U_n \left( O_j' \Lambda - \Omega \right) U_n \Omega \rho_n(\kappa) c(\kappa), \tag{54}
\]

which is nothing but Equation (51b) according to (33). To derive (51c), we shift (33) with respect to $n$. This gives rise to

\[
u_{n+1}(\kappa) = (1 - U_{n+1} \Omega) \rho_{n+1}(\kappa) c(\kappa) = (1 - U_{n+1} \Omega) \frac{L}{\sqrt{g}} \rho_n(\kappa) c(\kappa). \tag{55}
\]

Then (40c) leads this equation to

\[
u_{n+1}(\kappa) = \frac{L}{\sqrt{g}} \rho_n(\kappa) c(\kappa) - U_{n+1} \left( \frac{\sqrt{g}}{L} \Omega + O_2 \frac{L}{\sqrt{g}} \right) \rho_n(\kappa) c(\kappa). \tag{56}
\]

Finally, by substituting $U_{n+1} \frac{\sqrt{g}}{L}$ with the help of (42c), the above equation turns out to be

\[
u_{n+1}(\kappa) = \frac{L}{\sqrt{g}} (1 - U_n \Omega) \rho_n(\kappa) c(\kappa) - U_{n+1} O_2 \frac{L}{\sqrt{g}} (1 - U_n \Omega) \rho_n(\kappa) c(\kappa), \tag{57}
\]

namely, Equation (51c) is proven in virtue of (33). Equation (51a) can be proven through the same procedure. ■
Finally, we present the dynamics of the $\tau$-function in terms of the indices of the infinite matrix $U_n$.

**Proposition 4.** The $\tau$-function satisfies dynamical evolutions

\[
2\partial_{2j+1} \ln \tau_n = \sum_{i=0}^{2j} (-1)^i \left( \Lambda^{2j-i} L U_n^t \Lambda^i - \Lambda^{2j-i} U_n^t \Lambda^i \right)^{(0,0)} \tag{58a}
\]

and

\[
2\partial_{2j} \ln \tau_n = \sum_{i=0}^{j-1} g^i \left( L^{j-1-i} \Lambda U_n^t L^{-i} - L^{j-1-i} U_n^t \Lambda^i L^{-i} \right)^{(0,0)} \tag{58b}
\]

with respect to the continuous arguments $x_j$, as well as

\[
\frac{\tau_{n+1}}{\tau_n} = 1 + g^{-1} U_n^{[3,2]} \quad \text{and} \quad \frac{\tau_{n-1}}{\tau_n} = 1 - U_n^{[1,0]} \tag{58c}
\]

with respect to the discrete argument $n$.

**Proof.** We first prove (58a). Differentiating the logarithm of the $\tau$-function with respect to $x_{2j+1}$ gives rise to

\[
\partial_{2j+1} \ln \tau_n^2 = \partial_{2j+1} \ln \det(1 + \Omega C_n) = \partial_{2j+1} \tr[\ln(1 + \Omega C_n)] = \tr[(1 + \Omega C_n)^{-1} \Omega (\partial_{2j+1} C_n)]. \tag{59}
\]

Replacing $\partial_{2j+1} C_n$ with the help of (37a) and (34), we then obtain

\[
\partial_{2j+1} \ln \tau_n^2 = \tr[(1 + \Omega C_n)^{-1} \Omega (\Lambda^{2j+1} + L \Omega C_n)] = \tr[(\Omega \Lambda^{2j+1} + L \Omega) U_n]. \tag{60}
\]

Recall that $\Omega \Lambda^{2j+1} + L \Omega = O_{2j+1} L - L O_{2j+1}$. We end up with

\[
2\partial_{2j+1} \ln \tau_n = \partial_{2j+1} \ln \tau_n^2 = \tr \left( \left( O_{2j+1} L - L O_{2j+1} \right) U_n \right), \tag{61}
\]

which is nothing but (58a). Equation (58b) follows from a similar derivation. Next, performing the shift operation on (35), we obtain

\[
\tau_{n+1}^2 = \det(1 + \Omega C_{n+1}) = \det \left( 1 + \Omega \frac{L}{\sqrt{g}} C_n \frac{L}{\sqrt{g}} \right) = \det \left[ 1 + \left( O_2 \frac{L}{\sqrt{g}} + \frac{\sqrt{g}}{L} \Omega \right) \frac{L}{\sqrt{g}} \right] C_n \frac{L}{\sqrt{g}} \tag{62}
\]

\[
= \det \left[ 1 + \Omega C_n + (1 + \Omega C_n)^{-1} \frac{L}{\sqrt{g}} O_2 \frac{L}{\sqrt{g}} C_n \right] = \tau_n^2 \det \left[ 1 + (1 + \Omega C_n)^{-1} \frac{L}{\sqrt{g}} O_2 \frac{L}{\sqrt{g}} C_n \right],
\]

where the second and third equalities hold because of (37c) and (40c), respectively. Hence, this equation can further be rewritten as

\[
\frac{\tau_{n+1}^2}{\tau_n^2} = \det \left[ 1 + g^{-1}(1 + \Omega C_n)^{-1} \left( \frac{L}{\sqrt{g}} - \frac{L}{\sqrt{g}} \Lambda \right) O \left( \frac{\Lambda L}{L} \right) \frac{C_n}{\sqrt{g}} \right] \]
\[
= \det \left[ 1 + g^{-1} \left( (1 + \Omega C_n)^{-1} \Omega e, -(1 + \Omega C_n)^{-1} L^t \Lambda e \right) \left( \begin{array}{c} \epsilon e\Lambda C_n \\ \epsilon e\Lambda C_n^t \end{array} \right) \right], \quad (63)
\]

where we have used the identity \( O = e^t e \), namely, \( \tau_{n+1}^2 / \tau_n^2 \) is of the form

\[
\det \left[ 1 + \left( a_1, a_2 \right) \left( \begin{array}{c} b_1 \\ b_2 \end{array} \right) \right]
\]

for infinite column vectors \( a_i \) and infinite row vectors \( b_i \) for \( i = 1, 2 \). Using the rank 2 Weinstein–Aronszajn formula

\[
\det \left[ 1 + \left( a_1, a_2 \right) \left( \begin{array}{c} b_1 \\ b_2 \end{array} \right) \right] = \det \left[ 1 + \left( \begin{array}{c} b_1 a_1 \\ b_2 a_1, b_2 a_2 \end{array} \right) \right]
\]

and also Equation (34), we obtain

\[
\frac{\tau_{n+1}^2}{\tau_n^2} = \det \left[ \begin{array}{cc} 1 + g^{-1} e\Omega L U_n^t \Omega e & -g^{-1} e\Omega L U_n^t L^t \Lambda e \\ g^{-1} e\Omega L U_n^t \Lambda e & 1 - g^{-1} e\Omega L U_n^t L^t \Lambda e \end{array} \right]
\]

\[
= \det \left[ \begin{array}{cc} 1 + g^{-1} (\Omega L U_n^t L)^{(0,0)} & -g^{-1} (\Omega L U_n^t L^t \Lambda)^{(0,0)} \\ g^{-1} (\Omega L U_n^t \Lambda)^{(0,0)} & 1 - g^{-1} (\Omega L U_n^t L^t \Lambda)^{(0,0)} \end{array} \right]
\]

\[
= \det \left[ \begin{array}{cc} 1 + g^{-1} U_n^{[3,2]} & -g^{-1} U_n^{[3,3]} \\ g^{-1} U_n^{[2,2]} & 1 - g^{-1} U_n^{[2,3]} \end{array} \right] = \left( 1 + g^{-1} U_n^{[3,2]} \right)^2,
\]

where the second equality holds because of the third equation in (16), and we have made use of the antisymmetry property (49) and (23) for the last equality. Without loss of generality, we have \( \tau_{n+1} / \tau_n = 1 + g^{-1} U_n^{[3,2]} \). Likewise, by performing the backward shift operation on (35), we derive the other relation in (58c).

Equations (42a), (42b), and (49) together form the infinite matrix representation of the elliptic coupled KP hierarchy, describing possible dynamical evolutions and an algebraic constraint of the potential matrix. Meanwhile, equations in (51) form the infinite vector representation of the Lax pair of the elliptic coupled KP hierarchy. In the subsequent section, we present a closed-form multicomponent nonlinear system composed of certain entries of the infinite matrix \( U_n \) and a linear system based on a particular component of the infinite vector \( u_n(\kappa) \), which, respectively, form the closed-form elliptic coupled KP system and its Lax pair.

### 4 CLOSED-FORM NONLINEAR SYSTEM AND ASSOCIATED LINEAR PROBLEM

We now construct the closed-form elliptic coupled KP system and its Lax pair. Our attention is only paid to the first nontrivial flow in the hierarchy, namely, the equation evolving with respect to the flow variables \( x_1, x_2, \) and \( x_3 \). Our construction of the elliptic coupled KP system is based on new variables as follows:

\[
u_n \doteq U_n^{[2,0]} \equiv -U_n^{[0,2]},
\]

(67a)
\[ v_n = 1 + g^{-1}U_n^{[3,2]} \equiv 1 - g^{-1}U_n^{[2,3]}, \quad (67b) \]
\[ w_n = 1 - U_n^{[1,0,]} \equiv 1 + U_n^{[0,1]}. \quad (67c) \]

From the dynamical equations in (42), we are able to find a closed-form three-component system composed of the variables \( u_n, v_n, \) and \( w_n \). We present the result as the following theorem.

**Theorem 1.** Suppose that \( u_n(x) \) is a solution to the linear integral equation (28) subject to (26), (27), and (29). The variables \( u_n, v_n, \) and \( w_n \) defined by (67) provide solutions to the elliptic coupled KP system

\[
\partial_3 u_n = \frac{1}{4} \partial_1^3 u_n + \frac{3}{2} \partial_1^2 u_n + \frac{3}{4} \partial_1 \partial_2^2 u_n + 3g(1 - v_n w_n), \quad (68a)
\]
\[
\partial_3 v_n = -\frac{1}{2} \partial_1^3 v_n - 3(\partial_1 u_n - 3e)\partial_1 v_n + \frac{3}{2} \partial_1 \partial_2 v_n + 3(\partial_2 u_n)w_n, \quad (68b)
\]
\[
\partial_3 w_n = -\frac{1}{2} \partial_1^3 w_n - 3(\partial_1 u_n - 3e)\partial_1 w_n - \frac{3}{2} \partial_1 \partial_2 w_n - 3(\partial_2 u_n)w_n, \quad (68c)
\]
in which \( e \) and \( g \) are the moduli of the elliptic curve (5). Equation (68) is exactly the same as (1) by ignoring the discrete variable \( n \).

**Proof.** Equations in (68) are verified by direct computation. To realize this, we first of all need the fundamental formulae for the first-order derivatives \( \partial_i U_n^{[i,j]} \) for \( j \in \mathbb{Z}^+ \) and \( i, j \in \mathbb{Z} \). Notice that

\[
\partial_i U_n^{[2i,2j]} = \partial_i(L^i U_n^t L^j)^{(0,0)} = (L^i \partial_i U_n^t L^j)^{(0,0)},
\]
\[
\partial_i U_n^{[2i+1,2j]} = \partial_i(AL^i U_n^t L^j)^{(0,0)} = (AL^i \partial_i U_n^t L^j)^{(0,0)},
\]
\[
\partial_i U_n^{[2i+2,2j+1]} = \partial_i(L^i U_n^t L^j L^k)^{(0,0)} = (L^i \partial_i U_n^t L^j L^k)^{(0,0)},
\]
\[
\partial_i U_n^{[2i+1,2j+1]} = \partial_i(AL^i U_n^t L^j L^k)^{(0,0)} = (AL^i \partial_i U_n^t L^j L^k)^{(0,0)},
\]

(69)
due to (23). Replacing all the \( \partial_i U_n \) with the help of (42a) and (42b) for odd and even \( j \), respectively, we end up with the formulae for all the \( \partial_i U_n^{[i,j]} \) expressed by various algebraic combinations of \( U_n^{[i,j]} \). For instance, when \( i = 2 \) and \( j = 1 \), the simplest formulae are given by

\[
\partial_1 U_n^{[2,0]} = U_n^{[3,0]} + U_n^{[2,1]} - U_n^{[2,0]} U_n^{[2,0]} + U_n^{[2,2]} U_n^{[0,0]} \quad (70)
\]

and

\[
\partial_2 U_n^{[2,0]} = U_n^{[4,0]} - g U_n^{[0,0]} + U_n^{[2,2]} - g U_n^{[2,2]}
- U_n^{[2,0]} U_n^{[3,0]} - U_n^{[2,2]} U_n^{[1,0]} + U_n^{[2,1]} U_n^{[2,0]} + U_n^{[2,3]} U_n^{[0,0]}.
\]

(71)

Then, by iteration, we can further derive the general formulae for the higher-order derivatives of \( U_n^{[i,j]} \) such as \( \partial_i^2 U_n^{[i,j]} \), \( \partial_i^3 U_n^{[i,j]} \), \( \partial_i^4 U_n^{[i,j]} \), and so on, for \( i, j \in \mathbb{Z}^+ \) and \( i, j \in \mathbb{Z} \). These formulae together with the antisymmetry property (49) allow us to verify equations given by (68) in a purely algebraic way. For example, to verify (68a), we need the corresponding formulae for
\[ \partial_1 U^{[2,0]}_n, \partial_1^2 U^{[2,0]}_n, \partial_1^4 U^{[2,0]}_n, \partial_2^2 U^{[2,0]}_n, \text{ and } \partial_3 U^{[2,0]}_n. \] Then it is verified that

\[ -\partial_1 \partial_3 U^{[2,0]}_n + \frac{1}{4} \partial_1^4 U^{[2,0]}_n + 3 \left( \partial_1 U^{[2,0]}_n \right) \left( \partial_1^2 U^{[2,0]}_n \right) + \frac{3}{4} \partial_1^2 U^{[2,0]}_n - 3g \partial_1 \left[ \left( 1 + g^{-1} U^{[1,2]}_n \right) \left( 1 - U^{[1,0]}_n \right) \right] \] (72)

vanishes in virtue of \( U^{[j,i]}_n = -U^{[i,j]}_n \). This essentially means that

\[ \partial_1 \partial_3 u_n = \partial_1 \left( \frac{1}{4} \partial_1^3 u_n + \frac{3}{2} (\partial_1 u_n)^2 \right) + \frac{3}{4} \partial_2^2 u_n - 3g \partial_1 (v_n w_n). \] (73)

Observing the conditions on the asymptotic limits \( u_n \to 0, v_n \to 1 \) and \( w_n \to 1 \) (which follow from (67)), we immediately derive (68a) by integration with respect to \( x_1 \). Equations (68b) and (68c) are verified in the same manner.

Next, we derive the bilinear form of the elliptic coupled KP system. Note that the simplest cases of (58) give us the bilinear transforms

\[ u_n = \partial_1 \ln \tau_n, \quad v_n = \frac{\tau_{n+1}}{\tau_n} \quad \text{and} \quad w_n = \frac{\tau_{n-1}}{\tau_n}. \] (74)

The transformations (74) help us to construct a closed-form bilinear system in terms of the \( \tau \)-function. The result is presented as the theorem below.

**Theorem 2.** The \( \tau \)-function defined by (35) satisfies the following system of bilinear equations:

\[ \left( D_1^4 - 4D_1 D_3 + 3D_2^2 \right) \tau_n \cdot \tau_n = 24g \left( \tau_{n+1} \tau_{n-1} - \tau_n^2 \right), \] (75a)

\[ \left( D_1^3 + 2D_2 - 3D_1 D_2 - 18eD_1 \right) \tau_{n+1} \cdot \tau_n = 0, \] (75b)

\[ \left( D_1^3 + 2D_2 + 3D_1 D_2 - 18eD_1 \right) \tau_{n-1} \cdot \tau_n = 0, \] (75c)

where \( D_j \) stand for the bilinear derivatives (see Appendix A for the definition) with respect to \( x_j \).

**Proof.** These equations are obtained from (68) by using the transformations in (74) and identities (A.3) and (A.4). For instance, the bilinear transformations (74) reformulate (68a) as

\[ \partial_1 \partial_3 \ln \tau_n = \frac{1}{4} \partial_1^3 \ln \tau_n + \frac{3}{2} (\partial_1^2 \ln \tau_n)^2 + \frac{3}{4} \partial_1^2 \ln \tau_n + 3g \left( 1 - \frac{\tau_{n+1} \tau_{n-1}}{\tau_n^2} \right) \] (76)

in terms of the \( \tau \)-function, which is exactly the same as (75a), due to the logarithmic transformations given by (A.3). Equations (75b) and (75c) are derived similarly from (68b) and (68c), respectively, where we need to make use of the bilogarithmic transformations listed in (A.4). We also comment that these bilinear equations can alternatively be verified directly by following the

---

1The last two equations imply that \( v_n \) and \( w_n \) are connected each other through \( v_n w_{n+1} = 1 \). However, our aim is to present the elliptic coupled KP system as a (2 + 1)-dimensional continuous integrable model, where there is no dynamical evolution with respect to the discrete independent variable \( n \). Hence, \( v_n \) and \( w_n \) are treated as two separate variables, as we have done in (68).
same procedure of deriving (68). This is because Equations (58) establish the connection between $\tau_n$ and $U_n^{[i,j]}$.

Remark 8. In fact, the last two equations in the bilinear equations (75) are equivalent to each other, which means either of them can be omitted. However, we would still like to reserve both equations because here the main idea is to present the bilinear elliptic coupled KP system as a $(2 + 1)$-dimensional continuous closed-form system for $\tau_n, \tau_{n+1},$ and $\tau_{n-1}$ in terms of the independent variables $x_1, x_2, x_3,$ and $n$. We also note that equations in (75) are reparametrizations of those bilinear equations in Ref. 4, which, up to the moduli $e$ and $g$, have appeared in Ref. 27 (see also references therein) as the first few members in the bilinear DKP hierarchy.

In addition, by introducing the scalar wave function $\phi_n \equiv u_n^{[0]}(\kappa)$, we can also construct from (51) the associated linear problem for the nonlinear system (68). We conclude the result as the following theorem.

Theorem 3. For an arbitrary solution $u_n(\kappa)$ to the linear integral equation (28), the scalar wave function $\phi_n$ and the variables $u_n, v_n,$ and $w_n$ defined by (67) satisfy the linear system for $\phi_n$ as follows:

\begin{align*}
\partial_2 \phi_n &= -\left[\partial_1^2 + 2\partial_1 u_n - 3e\right] \phi_n + 2\sqrt{g}v_n w_n \phi_{n+1}, \\
\partial_3 \phi_n &= \left[\partial_1^3 + 3(\partial_1 u_n)\partial_1 + \frac{3}{2}(\partial_1^2 u_n - \partial_2 u_n)\right] \phi_n - 3\sqrt{g}v_n(\partial_1 w_n) \phi_{n+1}, \\
\sqrt{g}v_n w_n &\phi_{n+1} + \sqrt{g} \phi_{n-1} \\
&= \left[\partial_1^2 - (\partial_1 \ln w_n)\partial_1 + 2\partial_1 u_n - 3e + \frac{1}{2}(\partial_1^2 \ln w_n + (\partial_1 \ln w_n)^2 + \partial_2 \ln w_n)\right] \phi_n.
\end{align*}

Proof. The idea of the proof is very similar to that of verifying (68). We start with deriving from (51) the algebraic expressions of the derivatives $\partial_1 u_n^{[i]}(\kappa)$. Following the elliptic index labels introduced by (24), we have

\begin{align*}
\partial_1 u_n^{[2]}(\kappa) &= \partial_1[L^i u_n(\kappa)]^{(0)} = [L^i \partial_1 u_n(\kappa)]^{(0)} \\
\partial_1 u_n^{[2+1]}(\kappa) &= \partial_1[\Lambda L^i u_n(\kappa)]^{(0)} = [\Lambda L^i \partial_1 u_n(\kappa)]^{(0)},
\end{align*}

which can be further expressed by algebraic expressions of $U_n^{[i,j]}$ and $u_n^{[i]}$ by replacing $\partial_1 u_n(\kappa)$ with the help of (51a) and (51b). For instance, we have for $i = 0$ the simplest relations as follows:

$\partial_1 u_n^{[0]}(\kappa) = u_n^{[1]}(\kappa) + U_n^{[0,2]} u_n^{[0]}(\kappa),$  

\footnote{These equations are effectively reparametrization of the linear equations listed in Ref. 4. We also remark that a typo in the third equation in Ref. 4 has been fixed here.}

\[ \begin{align*}
\partial_2 u_n^{[0]}(\kappa) &= \left( 1 + U_n^{[0,1]} \right) u_n^{[2]}(\kappa) - U_n^{[0,2]} u_n^{[1]}(\kappa) + U_n^{[0,3]} u_n^{[0]}(\kappa) - g u_n^{-[2]}(\kappa), \\
\partial_1^2 u_n^{[0]}(\kappa) + \partial_1^2 u_n^{[0]}(\kappa) - 2\sqrt{g} v_n w_n u_n^{[0]}(\kappa) + 2 \left( U_n^{[3,0]} + U_n^{[2,1]} - U_n^{[2,0]} U_n^{[1,0]} + U_n^{[2,2]} U_n^{[0,0]} - \frac{3}{2} e \right) u_n^{[0]}(\kappa)
\end{align*} \]

is identically zero in virtue of the antisymmetry condition \( U_n^{[i,j]} = -U_n^{[j,i]} \) as well as the algebraic relations (which helps us to eliminate those shifted variables \( U_n^{[i,j]} \) in terms of \( n \)) that follow from taking \([i, j]-labels\) of (42c). Such an identity is essentially the linear equation (77a) once \( u_n^{[0]}(\kappa) \) and \( U_n^{[i,j]} \) are expressed by \( \phi_n, u_n, v_n, \) and \( w_n \). Linear equations (77b) and (77c) are proven similarly. \( \blacksquare \)

Remark 9. The third equation in (77) allows us to rewrite the other two equations as a two-component linear system composed of

\[ \partial_2 \left( \frac{\phi_n}{\phi_{n+1}} \right) = P \left( \frac{\phi_n}{\phi_{n+1}} \right) \quad \text{and} \quad \partial_3 \left( \frac{\phi_n}{\phi_{n+1}} \right) = Q \left( \frac{\phi_n}{\phi_{n+1}} \right), \]

where \( P \) and \( Q \) are \( 2 \times 2 \) matrix operators given by

\[ \begin{align*}
P &= \begin{pmatrix}
-\partial_1^2 - 2\partial_1 u_n + 3e \\
-2\sqrt{g}
\end{pmatrix}
+ \begin{pmatrix}
2\sqrt{g} v_n w_n \\
\partial_1^2 + 2(\partial_1 \ln v_n) \partial_1 + 2\partial_1 u_n - 3e + \partial_1^2 \ln v_n + (\partial_1 \ln v_n)^2 - \partial_2 \ln v_n
\end{pmatrix}
\]

and

\[ \begin{align*}
Q &= \begin{pmatrix}
\partial_1^3 + 3(\partial_1 u_n) \partial_1 + \frac{3}{2} (\partial_1^2 u_n - \partial_2 u_n) - 3\sqrt{g} v_n (\partial_1 w_n) \\
-3\sqrt{g} (\partial_1 \ln v_n)
\end{pmatrix}
\]

respectively, in which

\[ * = \partial_1^3 + 3(\partial_1 \ln v_n) \partial_1^2 + 3(\partial_1 u_n + \partial_1^2 \ln v_n + (\partial_1 \ln v_n)^2) \partial_1 \\
+ \frac{3}{2} (\partial_1^2 u_n - \partial_2 u_n) + 3(\partial_1 \ln v_n)(2\partial_1 u_n - 3e) \\
+ \frac{3}{2}(\partial_1 + \partial_1 \ln v_n)(\partial_1^2 \ln v_n + (\partial_1 \ln v_n)^2 - \partial_2 \ln v_n). \]
The linear equations in (82) form the Lax pair for the elliptic coupled KP system, namely, the zero curvature equation
\[ \partial_3 P - \partial_2 Q + [P, Q] = 0 \] (85)
with \([P, Q] = PQ - QP\) yields the nonlinear system (68).

5 | Dimensional Reductions

In this section, we further discuss the relevant (1 + 1)-dimensional elliptic integrable models by performing dimensional reductions on the elliptic coupled KP system (68). This is realized by imposing a restriction of the form \(F(\kappa, \kappa') = 0\) on the spectral variables in the linear integral equation (28), such that the effective plane wave factor \(\rho_n(\kappa)\rho_n(\kappa')\) turns out to be independent of a certain argument \(x_j\). Recall that the dynamics of the elliptic coupled KP system completely relies on the effective plane wave factor
\[ \rho_n(\kappa)\rho_n(\kappa') = \exp \left\{ \sum_{j=0}^{\infty} \left( k^{2j+1} + k'^{2j+1} \right) x_{2j+1} + \sum_{j=1}^{\infty} \left[ K^j - \left( \frac{g}{K} \right)^j + K'^j - \left( \frac{g}{K'} \right)^j \right] x_{2j} \right\} \left( \frac{KK'}{g} \right)^n. \] (86)

Thus, we can impose
\[ k^{2j_0+1} + k'^{2j_0+1} = 0 \quad \Rightarrow \quad k + k' = 0 \quad \text{or} \quad \sum_{i=0}^{2j_0} k^{2j_0-i}(-k')^i = 0 \] (87)
and
\[ K^{j_0} - \left( \frac{g}{K} \right)^{j_0} + K'^{j_0} - \left( \frac{g}{K'} \right)^{j_0} = 0 \quad \Rightarrow \quad \left( \frac{KK'}{g} \right)^{j_0} = 1 \quad \text{or} \quad K^{j_0} + K'^{j_0} = 0, \] (88)
to realize \(x_{2j_0+1}\)- and \(x_{2j_0}\)-independence, respectively. However, in practice, we are only allowed to set
\[ F_{2j_0+1}(\kappa, \kappa') \equiv \sum_{i=0}^{2j_0} k^{2j_0-i}(-k')^i = 0 \] (89)
and
\[ F_{2j_0}(\kappa, \kappa') \equiv K^{j_0} + K'^{j_0} = 0 \] (90)
to, respectively, perform \(x_{2j_0+1}\)- and \(x_{2j_0}\)-reductions. This is because either \(k + k' = 0\) or \((KK'/g)^{j_0} = 1\) will lead to trivialities; to be more precise, useful independent variables are also eliminated in these two cases. Below we give the simplest examples including the elliptic coupled KdV and BSQ systems arising from the \(x_2\)- and \(x_3\)-reductions of (68).

To construct the elliptic coupled KdV system, we set
\[ K + K' = 0, \] (91)
namely, the $j_0 = 1$ case of (90) leading to the $x_2$-independence. This, in turn, implies
\[ k^2 + k'^2 = 6e \]
(92)
due to the elliptic curve relation (5). The constraint on the spectral points $(k, K)$ and $(k', K')$ results in $\partial_{4j+2} \rho_n(\varsigma) \rho_n(\varsigma') = 0$. From the definition of $C_n$, that is, (30), we can easily prove $\partial_{4j+2} C_n = 0$ and subsequently $\partial_{4j+2} U_n = 0$ by (34). Recall that the $\tau$-function and the potentials are defined as (35) and (67). We obtain reduction conditions
\[ \partial_{4j+2} \tau_n = \partial_{4j+2} u_n = \partial_{4j+2} v_n = \partial_{4j+2} w_n = 0, \]
(93)
for $j = 0, 1, 2, \ldots$. Performing such reductions on the elliptic coupled KP system (68), we obtain a coupled system
\[ \partial_3 u_n = \frac{1}{4} \partial_1^3 u_n + \frac{3}{2} (\partial_1 u_n)^2 + 3g(1 - v_n w_n), \]
(94a)
\[ \partial_3 v_n = -\frac{1}{2} \partial_1^3 v_n - 3(\partial_1 u_n - 3e) \partial_1 v_n, \]
(94b)
\[ \partial_3 w_n = -\frac{1}{2} \partial_1^3 w_n - 3(\partial_1 u_n - 3e) \partial_1 w_n, \]
(94c)
which we refer to as the elliptic coupled KdV system. The bilinear form of Equation (94) is obtained from (75), which is a multicomponent system given by
\[ (D_1^4 - 4D_1 D_3) \tau_n \cdot \tau_n = 24g(\tau_{n+1} \tau_{n-1} - \tau_n^2), \]
(95a)
\[ (D_1^3 + 2D_3 - 18e D_1) \tau_{n+1} \cdot \tau_n = 0, \]
(95b)
\[ (D_1^3 + 2D_3 - 18e D_1) \tau_{n-1} \cdot \tau_n = 0. \]
(95c)
Notice that $\partial_{4j+2} U_n = 0$ and $u(\varsigma)$ satisfies (33). We can further derive
\[ \partial_{4j+2} \phi_n = (K^{2j+1} - (g/K)^{2j+1}) \phi_n. \]
(96)
This provides us with a reduction on the Lax pair (82). Therefore, the Lax pair of the elliptic coupled KdV system (94) is composed of
\[ P \left( \begin{array}{c} \phi_n \\ \phi_{n+1} \end{array} \right) = (K - g/K) \left( \begin{array}{c} \phi_n \\ \phi_{n+1} \end{array} \right) \text{ and } \partial_3 \left( \begin{array}{c} \phi_n \\ \phi_{n+1} \end{array} \right) = Q \left( \begin{array}{c} \phi_n \\ \phi_{n+1} \end{array} \right). \]
(97)
Here, the Lax matrices are given by
\[ P = \begin{pmatrix} -\partial_1^2 - 2\partial_1 u_n + 3e & 2\sqrt{g} v_n w_n \\ -2\sqrt{g} & \partial_1^2 + 2(\partial_1 \ln v_n) \partial_1 + 2\partial_1 u_n - 3e + \partial_1^2 \ln v_n + (\partial_1 \ln v_n)^2 \end{pmatrix}, \]
(98a)
and
\[ Q = \begin{pmatrix} \partial_1^3 + 3(\partial_1 u_n) \partial_1 + \frac{3}{2}(\partial_1^2 u_n - \partial_2 u_n) - 3\sqrt{g} v_n (\partial_1 w_n) & 2\sqrt{g} v_n w_n \\ -3\sqrt{g} \partial_1 \ln v_n & * \end{pmatrix}, \]
(98b)
respectively, in which
\[
\begin{align*}
\ast = \partial_1^3 &+ 3(\partial_1 \ln v_n)\partial_1^2 + 3(\partial_1 u_n + \partial_1^2 \ln v_n + (\partial_1 \ln v_n)^2)\partial_1 \\
&+ \frac{3}{2} \partial_1^2 u_n + 3(\partial_1 \ln v_n)(2\partial_1 u_n - 3e) + \frac{3}{2}(\partial_1 + \partial_1 \ln v_n)(\partial_1^2 \ln v_n + (\partial_1 \ln v_n)^2).
\end{align*}
\] (99)

For the elliptic coupled BSQ system, we set \(j_0 = 1\) in \((89)\), namely,
\[
k^2 - kk' + k'^2 = 0,
\] (100)
to induce the \(x_3\)-independence. This simultaneously leads to the fact that \(K\) and \(K'\) also obey a restriction in the form of
\[
\left( K + 3e + \frac{g}{K} \right)^2 + \left( K + 3e + \frac{g}{K'} \right) \left( K' + 3e + \frac{g}{K'} \right) + \left( K' + 3e + \frac{g}{K'} \right)^2 = 0,
\] (101)
which follows form the curve relation \((5)\). In this case, the reduction conditions are given by
\[
\partial_{3j} \tau_n = \partial_{3j} u_n = \partial_{3j} v_n = \partial_{3j} w_n = 0 \quad \text{as well as} \quad \partial_{3j} \phi_n = k^{3j} \phi_n
\] (102)
for \(j \in \mathbb{Z}^+\). Therefore, we obtain from \((68)\) the elliptic coupled BSQ system
\[
\begin{align*}
\partial_2^2 u_n &= -\frac{1}{3}\partial_1^3 u_n - 4(\partial_1 u_n)\partial_1^2 u_n + 4g\partial_1(v_n w_n), \\
\partial_1 \partial_2 v_n &= \frac{1}{3}\partial_1^3 v_n + 2(\partial_1 u_n - 3e)\partial_1 v_n - 2(\partial_2 u_n)v_n, \\
\partial_1 \partial_2 w_n &= -\frac{1}{3}\partial_1^3 w_n - 2(\partial_1 u_n - 3e)\partial_1 w_n - 2(\partial_2 u_n)w_n.
\end{align*}
\] (103)

The corresponding bilinear form is a consequence of the \((75)\), taking the form of the following:
\[
\begin{align*}
(D_1^4 + 3D_2^2) \tau_n \cdot \tau_n &= 24g(\tau_{n+1} \tau_{n-1} - \tau_n^2), \\
(D_1^3 - 3D_1 D_2 - 18e D_1) \tau_{n+1} \cdot \tau_n &= 0, \\
(D_1^3 + 3D_1 D_2 - 18e D_1) \tau_{n-1} \cdot \tau_n &= 0.
\end{align*}
\] (104)

The Lax pair of the elliptic coupled BSQ equation \((103)\) is composed of
\[
Q \left( \begin{array}{c} \phi_n \\ \phi_{n+1} \end{array} \right) = k^3 \left( \begin{array}{c} \phi_n \\ \phi_{n+1} \end{array} \right) \quad \text{and} \quad \partial_2 \left( \begin{array}{c} \phi_n \\ \phi_{n+1} \end{array} \right) = P \left( \begin{array}{c} \phi_n \\ \phi_{n+1} \end{array} \right),
\] (105)
where \(P\) and \(Q\) are the same as the ones given by \((83)\).

We note that the third equations in \((95)\) and \((104)\) can be omitted, because they are equivalent to their respective second equations. However, here we still reserve these equations to treat both \((95)\) and \((104)\) as \((1 + 1)\)-dimensional three-component partial-differential systems of \(\tau_n\), \(\tau_{n+1}\), and \(\tau_{n-1}\), instead of \((2 + 1)\)-dimensional differential-difference systems.
To further explore the integrability of the elliptic coupled KP system, we now specify a class of concrete solutions (i.e., elliptic multisoliton solutions) to (68). They are constructed directly from the DL framework by specifying a discrete measure associated with distinct simple poles at values $\kappa_\nu$ for $\nu = 1, \ldots, N$ of the uniformizing spectral parameter, associated with distinct points $(k_\nu, K_\nu)$ on the elliptic curve (5). This will lead the linear integral equation (28) to

$$\mathbf{u}_n(\kappa) + \sum_{\nu,\nu'=1}^{N} c_{\nu,\nu'}(\kappa) \Omega(\kappa, \kappa) \rho_\nu(\kappa, \kappa') \mathbf{u}_\nu(\kappa_\nu) = \rho_\nu(\kappa) \mathbf{c}(\kappa), \quad (106)$$

where the coefficients $c_{\nu,\nu'}$ are skew-symmetric in terms of the indices, that is, $c_{\nu,\nu'} = -c_{\nu',\nu}$. Let us introduce the finite matrices $M = (M_{\mu,\nu})_{N \times N}$ and $R = (R_{\mu,\nu})_{N \times N}$ whose respective entries are defined as

$$M_{\mu,\nu} = \Omega(\kappa_\mu, \kappa_\nu) = \frac{K_\mu - K_\nu}{k_\mu + k_\nu} = \frac{k_\mu - k_\nu}{1 - g K_\mu K_\nu} \quad (107)$$

and

$$R_{\mu,\nu} = c_{\nu,\mu} \rho_\nu(\kappa_\mu) \rho_\nu(\kappa_\nu)$$

$$= c_{\nu,\mu} \exp \left\{ \sum_{j=0}^\infty \left( k_{\mu}^{2j+1} + k_{\nu}^{2j+1} \right) x_{2j+1} + \sum_{j=1}^\infty \left( K_\mu^j - \left( K_\mu \right)^j + K_\nu^j - \left( K_\nu \right)^j \right) x_{2j} \right\} \left( \frac{K_\mu K_\nu}{g} \right)^n. \quad (108)$$

Equations (107) and (108) show that both $M$ and $R$ are skew-symmetric. By setting $\kappa = \kappa_\mu$ for $\mu = 1, 2, \ldots, N$, we can rewrite (106) as

$$\phi_\mu + \sum_{\nu,\nu'=1}^{N} M_{\mu,\nu'} R_{\nu',\nu} \phi_\nu = \mathbf{c}(\kappa_\mu), \quad (109)$$

for the infinite vector $\phi_\mu \equiv \mathbf{u}_n(\kappa_\mu) / \rho_\nu(\kappa_\mu)$, which can, in turn, be reformulated as a linear problem for an $N$-component block vector $(\phi_1, \phi_2, \ldots, \phi_N)$ given by

$$(I + MR) \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix} = \begin{pmatrix} \mathbf{c}(\kappa_1) \\ \mathbf{c}(\kappa_2) \\ \vdots \\ \mathbf{c}(\kappa_N) \end{pmatrix}, \quad (110)$$

namely,

$$(\phi_1, \phi_2, \ldots, \phi_N) = (\mathbf{c}(\kappa_1), \mathbf{c}(\kappa_2), \ldots, \mathbf{c}(\kappa_N))(I + RM)^{-1}, \quad (111)$$
where $I = I_{N \times N}$ denotes the $N \times N$ identity matrix. Meanwhile, the discrete measure also reduces (32) to

$$U_n = \sum_{\mu, \mu'} c_{\mu, \mu'} u_n(\xi_{\mu}) \phi(\xi_{\mu'}) \rho_n(\xi_{\mu'}) = \sum_{\mu, \mu'} \phi_{\mu} R_{\mu' \mu} \phi(\xi_{\mu'})$$

where $I = I_{N \times N}$ denotes the $N \times N$ identity matrix. Meanwhile, the discrete measure also reduces (32) to

$$U_n = \sum_{\mu, \mu'} c_{\mu, \mu'} u_n(\xi_{\mu}) \phi(\xi_{\mu'}) \rho_n(\xi_{\mu'}) = \sum_{\mu, \mu'} \phi_{\mu} R_{\mu' \mu} \phi(\xi_{\mu'})$$

Equation (122) together with (111) allows us to write down the formula of $U_n$ for the elliptic multisoliton solutions as follows:

$$U_n = -(\phi_1, \phi_2, ..., \phi_N) R \begin{pmatrix}
\phi(\xi_1) \\
\phi(\xi_2) \\
\vdots \\
\phi(\xi_N)
\end{pmatrix}$$

(112)

A remark here is that the multiplication in (113) is essentially based on $N \times N$ finite matrices $I$, $R$, and $M$ and $N$-component block vectors with their respective components being infinite vectors $\phi(\xi)$ and $\phi(\xi')$. Thus, the final result is an infinite matrix, which coincides with the form of (32), as we expect. Notice that the variables $u_n$, $v_n$, and $w_n$ of the elliptic coupled KP system purely rely on the entries of $U_n$, see (67). We can therefore derive the general formulae for the Cauchy matrix solutions to (68).

**Theorem 4.** The elliptic coupled KP system (68) possesses Cauchy matrix solutions

$$u_n = \phi_1 (I + RM)^{-1} R k_0,$$

(114a)

$$v_n = 1 - \phi_2 (I + RM)^{-1} R k_2,$$

(114b)

$$w_n = 1 + \phi_3 (I + RM)^{-1} k_0,$$

(114c)

where $M$ and $R$ are $N \times N$ matrices introduced in (107) and (108), and $k_i$ for $i = 0, 1, 2, 3$ are $N$-component column vectors defined as

$$k_0 = \phi(1, 1, ..., 1), \quad k_1 = \phi(k_1, k_2, ..., k_N),$$

$$k_2 = \phi(K_1, K_2, ..., K_N) \quad \text{and} \quad k_3 = \phi(k_1 K_1, k_2 K_2, ..., k_N K_N).$$

(115)

For these solutions, the square of the $\tau$-function (35) is of the form

$$\tau_n^2 = \det(I + MR) = \det(I + RM) = \det\left(\frac{M I}{-I R}\right).$$

(116)
As both $\mathbf{M}$ and $\mathbf{R}$ are skew-symmetric matrices the latter $N \times N$ determinant is a square of a $(2N-1) \times (2N-1)$ Pfaffian, and hence, the $\tau$-function itself can be written as a Pfaffian\(^3\) that

$$\tau_n = \text{pf}(\mathbf{M}|\mathbf{R})$$

$$= \begin{vmatrix} M_{1,2} & M_{1,3} & \cdots & M_{1,N} & 1 & 0 & \cdots & 0 & 0 \\ M_{2,3} & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & M_{2,N} & 0 & 1 & \cdots & 0 & 0 \\ M_{N-1,N} & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 1 \\ R_{1,2} & \cdots & R_{1,N-1} & R_{1,N} & 0 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ R_{N-2,N-1} & R_{N-2,N} & \cdots & R_{N-1,N} \\ R_{N-1,N-1} & R_{N-1,N} \\ \end{vmatrix} \equiv \left( -1 \right)^{N(N-1)/2}. \quad (117)$$

To give an explicit expression for this Pfaffian (a definition is given in Appendix B), we need an expansion formula for the Pfaffian, similar to the expansion for a determinant of the form $\det(1 + \mathbf{M}\mathbf{R})$ in terms of the matrix invariants of the matrix $\mathbf{M}\mathbf{R}$, but now in terms of Pfaffians of $\mathbf{M}$ and $\mathbf{R}$ separately. Such an expansion formula is given by (B.8) in Appendix B. Furthermore, we can make use of the lemma below that amounts to a Pfaffian analog of the Frobenius (i.e., elliptic Cauchy, cf. Ref. 29) determinant formula.

**Lemma 1.** The Pfaffian of an elliptic Cauchy matrix $\mathbf{M}$ with entries $M_{i,j} = \frac{K_i - K_j}{k_i + k_j}$ for $i, j = 1, 2, \ldots, m$, where $(k_i, K_i)$ are distinct points on the elliptic curve (5), is given by

$$\text{pf}(\mathbf{M}) = \frac{g^{m(m-2)/8}}{\left( \prod_{i=1}^{m} K_i \right)^{(m-2)/2}} \prod_{1 \leq i < j \leq m} \frac{K_i - K_j}{k_i + k_j}, \quad (118)$$

for $m$ is even, whereas the Pfaffian vanishes (by definition) for $m$ is odd.

**Remark 10.** Essentially formula (118) in the lemma has appeared in Ref. 5 without a proof. A proof based on the Frobenius determinant formula for the elliptic Cauchy matrix is given in Appendix C.

Together with the expansion formula (B.8), this allows us to write an explicit “Hirota-type” formula\(^4\) for the elliptic $N$-soliton solution of the elliptic coupled KP system. Combining the results of Appendices B and C, we thus obtain the following explicit expression for the $\tau$-function of the $N$-soliton solution to the elliptic coupled KP system. In fact, the expansion formula (B.8) gives a finite sum of terms, each of which is a product of individual sub-Pfaffians of the matrices $\mathbf{M}$ and $\mathbf{R}$. The result is given in the following theorem.

\(^3\) We refer the reader to Ref. 28 for such a notation (i.e., the triangular array) of Pfaffian. Here, the postfactor is chosen to comply with a normalization of the $\tau$-function such that it has the form $\tau_n = 1 + \text{perturbation}$.

\(^4\) Note that taking the square root of the corresponding expansion of the determinant expansion does not necessarily lead to a finite expansion formula, whereas the Pfaffian analog (B.8) does provide one.
Theorem 5. The \( \tau \)-function for the \( N \)-soliton solution to the bilinear elliptic coupled KP system (75) is given by

\[
\tau_n = 1 + \sum_{m \in J} \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq N} \frac{g^{m(m-2)/8}}{\prod_{\gamma=1}^m K_{i_\gamma}} \left( \prod_{1 \leq \nu < \nu' \leq m} \frac{K_{i_\nu} - K_{i_{\nu'}}}{k_{i_\nu} + k_{i_{\nu'}}} \right) \text{pf}(\mathcal{C}_{i_1,\ldots,i_m}) \left( \prod_{\gamma=1}^m \rho_n(\chi_{i_\gamma}) \right)
\]

with \( J = \{ j = 2i | i = 1, 2, \ldots, \lfloor N/2 \rfloor \} \), where \( \mathcal{C}_{i_1,\ldots,i_m} \) denotes the submatrix of the coefficient matrix \( \mathcal{C} = (c_{\mu,\nu})_{N \times N} \) obtained from selecting from it the rows and columns labeled by \( i_1, \ldots, i_m \), and the exponential factors \( \rho_n \) are given in (26). The solutions \( u_n, v_n, \) and \( w_n \) of the elliptic coupled KP system (68) are subsequently inferred from the expression (119) into (74).

Remark 11. Above we present the formula for the elliptic \( N \)-soliton solution parametrized by the curve (5). The degenerate solutions are obtained when any two of the branch points coincide with each other. The situation when \( e_1 = e_2 \) or \( e_1 = e_3 \) corresponds to \( g = 0 \) due to (6). In these two cases, Equation (68) reduces to the scalar KP equation, but simultaneously the elliptic \( N \)-soliton solution turns out to be a trivial one when \( g = 0 \). The third possibility \( e_2 = e_3 \) leads to \( 9e^2 = 4g \), in which case we obtain a degenerate curve

\[
k^2 = K + 3e + \frac{9}{4} \frac{e^2}{K} = \left( \sqrt{K} + \frac{3}{2} \frac{e}{\sqrt{K}} \right)^2
\]

as well as the corresponding degenerate solutions.

Below we give examples for \( N = 2, 3, 4, 5 \) (\( N = 1 \) results in the seed solution \( \tau_n = 1 \)).

\( N = 2 \)

\[
\tau_n = 1 + \frac{K_1 - K_2}{k_1 + k_2} c_{1,2} \rho_n(\chi_1) \rho_n(\chi_2);
\]

\( N = 3 \)

\[
\tau_n = 1 + \sum_{1 \leq i_1 < i_2 \leq 3} \frac{K_{i_1} - K_{i_2}}{k_{i_1} + k_{i_2}} c_{i_1,i_2} \rho_n(\chi_{i_1}) \rho_n(\chi_{i_2});
\]

\( N = 4 \)

\[
\tau_n = 1 + \sum_{1 \leq i_1 < i_2 \leq 4} \frac{K_{i_1} - K_{i_2}}{k_{i_1} + k_{i_2}} c_{i_1,i_2} \rho_n(\chi_{i_1}) \rho_n(\chi_{i_2})
\]

\[
+ \frac{g}{4} \left( \prod_{1 \leq i_1 < i_2 \leq 4} \frac{K_{i_1} - K_{i_2}}{k_{i_1} + k_{i_2}} \right) (c_{1,2}e_{3,4} - c_{1,3}e_{2,4} + c_{1,4}e_{2,3}) \left( \prod_{i=1}^4 \rho_n(\chi_i) \right);
\]

\( n = 1 \) results in the seed solution \( \tau_n = 1 \).
\[ N = 5 \]

\[ \tau_n = 1 + \sum_{1 \leq i_1 < i_2 \leq 5} \frac{K_{i_1} - K_{i_2}}{k_{i_1} + k_{i_2}} c_{i_1,i_2} \rho_n(\xi_{i_1}) \rho_n(\xi_{i_2}) \]

\[ + \sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq 5} \frac{g}{4 \prod_{\nu = 1}^{4} K_{i_{\nu}}} \left( \prod_{1 \leq \nu < \nu' \leq 4} \frac{K_{i_{\nu}} - K_{i_{\nu}'}}{k_{i_{\nu}} + k_{i_{\nu}'}} \right) \left( c_{i_{1},i_{2}} c_{i_{3},i_{4}} - c_{i_{1},i_{3}} c_{i_{2},i_{4}} + c_{i_{1},i_{4}} c_{i_{2},i_{3}} \right) \times \left( \prod_{\nu = 1}^{4} \rho_n(\xi_{i_{\nu}}) \right) \]

\[ (124) \]

We observe from the \( \tau \)-function (119) as well as the examples that the solutions for even \( N \) resemble those of the BKP hierarchy apart from extra terms as well as the elliptic kernel and plane wave factors. For instance, setting \( c_{1,3} = c_{2,4} = c_{1,4} = c_{2,3} = 0 \) and introducing \( d_{i,j} \) (which are effectively arbitrary constants) determined by

\[ \exp(d_{i,j}) \equiv \frac{c_{i,j}}{k_i + k_j} \]

(125)

in the formula for \( N = 4 \) yields a particular solution to the bilinear system (75) as follows:

\[ \tau_n = 1 + \exp(\xi_1 + \xi_2 + d_{1,2}) + \exp(\xi_3 + \xi_4 + d_{3,4}) \]

\[ + \frac{g}{K_1 K_2 K_3 K_4} \frac{(k_1 - k_3)(k_1 - k_4)(k_2 - k_3)(k_2 - k_4)}{(k_1 + k_3)(k_1 + k_4)(k_2 + k_3)(k_2 + k_4)} \exp(\xi_1 + \xi_2 + \xi_3 + \xi_4 + d_{1,2} + d_{3,4}), \]

(126)

in which the plane wave factors are given by

\[ \exp(\xi_i) \equiv \exp \left\{ \sum_{j=0}^{\infty} K_i^{2j+1} x_{2j+1} + \sum_{j=1}^{\infty} \left[ K_i^j - \left( \frac{g}{K_i} \right)^j \right] x_{2j} \right\} \left( \frac{K_i}{\sqrt{g}} \right)^n \]

(127)

for \( i = 1, 2, 3, 4 \). The “two-soliton” solution to the bilinear BKP hierarchy, including its first member

\[ (D_1^6 - 5D_1^3 D_3 - 5D_3^2 + 9D_1 D_5) \tau \cdot \tau = 0, \]

(128)

takes the form of (cf. Ref. 30 and also31)

\[ \tau = 1 + \exp(\xi_1 + \xi_2 + d_{1,2}) + \exp(\xi_3 + \xi_4 + d_{3,4}) \]

\[ + \frac{(k_1 - k_3)(k_1 - k_4)(k_2 - k_3)(k_2 - k_4)}{(k_1 + k_3)(k_1 + k_4)(k_2 + k_3)(k_2 + k_4)} \exp(\xi_1 + \xi_2 + \xi_3 + \xi_4 + d_{1,2} + d_{3,4}) \]

(129)

with plane wave factors

\[ \exp(\xi_i) \equiv \exp \left\{ \sum_{j=0}^{\infty} k_i^{2j+1} x_{2j+1} \right\} \]

(130)

for \( i = 1, 2, 3, 4 \) and arbitrary constants \( d_{1,2} \) and \( d_{3,4} \). The main difference here is that the linear dispersion is parametrized by the elliptic curve (5), and also an elliptic phase shift term (as a
consequence of the significant formula (118)) is involved. They together describe an elliptic-type (which is remarkable from our viewpoint) soliton interaction. The formulae for odd \( N \) are in a sense the respective parameter extensions of those for even numbers \( N - 1 \).

## 7 MULTISOLITON SOLUTIONS TO THE DKP EQUATION WITH NONZERO CONSTANT BACKGROUND

We now discuss the connection between the elliptic coupled KP system (1) and the DKP equation (4), and show how soliton solutions to the DKP equation (4) with nonzero constant background are constructed as a by-product of the result in Section 6.

By introducing new variables

\[
\begin{align*}
\varphi &= 2\partial_1 u + c_1, \\
\psi &= c_2 v \\
\omega &= c_3 w,
\end{align*}
\]

where \( c_1, c_2, \) and \( c_3 \) are constants obeying \( c_1 = -6e \) and \( c_2 c_3 = g \), we are able to reformulate (1) as

\[
\begin{align}
\partial_3 \varphi &= \frac{1}{4} \partial_1^3 \varphi + \frac{3}{2} (\varphi + 6e) \partial_1 \varphi + \frac{3}{4} \partial_1^{-1} \partial_2^2 \varphi - 6 \partial_1 (\psi \omega), \\
\partial_3 \psi &= -\frac{1}{2} \partial_1^3 \psi - \frac{3}{2} \varphi \partial_1 \psi + \frac{3}{2} \partial_1 \partial_2 \psi + \frac{3}{2} (\partial_1^{-1} \partial_2 \varphi) \psi, \\
\partial_3 \omega &= -\frac{1}{2} \partial_1^3 \omega - \frac{3}{2} \varphi \partial_1 \omega - \frac{3}{2} \partial_1 \partial_2 \omega - \frac{3}{2} (\partial_1^{-1} \partial_2 \varphi) \omega.
\end{align}
\]

Then the following transformations between partial-differential operators:

\[
\begin{align*}
\partial_1 &= \partial_x, \\
\partial_2 &= \partial_y, \\
\partial_3 &= \partial_t + 9e \partial_x,
\end{align*}
\]

which effectively follow from a Galilean transformation composed of

\[
\begin{align*}
x &= x_1 + 9e x_3, \\
y &= x_2, \\
t &= x_3,
\end{align*}
\]

lead (132) to the DKP equation (4). Notice that the solutions we have obtained for the elliptic coupled KP system in Section 6 are the ones with background \( u = 0, v = 1, \) and \( w = 1 \). This implies that here we are able to construct soliton solutions to the DKP equation with nonzero constant background with the help of (131) and (134). We conclude this as the theorem below.

**Theorem 6.** The \( N \)-soliton solutions to the DKP equation (4) with nonzero constant background\(^5\)

\[
\begin{align*}
\varphi &= c_1, \\
\psi &= c_2, \\
\omega &= c_3 \quad \text{for} \quad c_2 c_3 \neq 0
\end{align*}
\]

are given by

\[
\begin{align*}
\varphi &= c_1 + 2 \partial_x^2 \ln \tau_n, \\
\psi &= c_2 \frac{\tau_{n+1}}{\tau_n} \quad \text{and} \quad \omega &= c_3 \frac{\tau_{n-1}}{\tau_n},
\end{align*}
\]

\(^5\)The constant \( c_1 \) is not necessarily nonzero, because \( c_1 = 0 \) does not lead to degeneration of the elliptic curve (138).
where the $\tau$-function (cf. (119)) takes the form of

$$
\tau_n = 1 + \sum_{m \in J} \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq N} \frac{(c_2 c_3)^m(m-2)/8}{\prod_{i_1 < y < i_2 \leq m} K_i} \left( \sum_{1 \leq y < y' \leq m} \frac{K_{i_y} - K_{i_y'}}{k_{i_y} + k_{i_y'}} \right) \text{pf}(C_{i_1, \ldots, i_m}) \left( \prod_{\nu=1}^m \rho_n(x_{i_y}) \right)
$$

(137)

with $(k_{i_y}, K_{i_y})$ being the spectral points on the elliptic curve

$$
k^2 = K - \frac{c_1}{2} + \frac{c_2 c_3}{K}
$$

(138)

and $\rho_n(x_{i_y})$ being the plane wave factors determined by

$$
\rho_n(x) = \exp \left\{ kx + \left( K - \frac{c_2 c_3}{K} \right) y + \left( k^2 + \frac{3c_1}{2} k \right) t \right\} \left( \frac{K}{\sqrt{c_2 c_3}} \right)^n.
$$

(139)

**Remark 12.** The special case when the background constants $c_1, c_2,$ and $c_3$ satisfy $c_1^2 = 16c_2c_3$ will lead to the fact that the spectral points in the soliton solutions are parametrized by the degenerate curve

$$
k^2 = K - \frac{c_1}{2} + \frac{c_2^2}{16K} = \left( \sqrt{K} - \frac{c_1}{4} \frac{1}{\sqrt{K}} \right)^2.
$$

(140)

We comment that the multisoliton solutions to the DKP equation discussed here differ from those with zero background $U' = V = W = 0$ in the literature, cf. Refs. 6, 32, 33. In this paper, the constants $c_1, c_2,$ and $c_3$ for the nonzero “seed solutions” will eventually play a role of the moduli in the elliptic curve that parameterizes the spectral points in the multisolitons.

For example, by taking $N = 2$, we obtain the simplest nontrivial $\tau$-function for soliton solutions to the DKP equation (4) with nonzero constant background (135) as follows:

$$
\tau_n = 1 + \exp \left\{ (k_1 + k_2)x + \left( K_1 - \frac{c_2 c_3}{K_1} + K_2 - \frac{c_2 c_3}{K_2} \right) y 
\right.
\left. + \left( k_1^2 + k_2^2 + \frac{3c_1}{2} (k_1 + k_2) \right) t + d_{1,2} \right\} \left( \frac{K_1 K_2}{c_2 c_3} \right)^n,
$$

(141)

where $(k_1, K_1)$ and $(k_2, K_2)$ are points on the elliptic curve (138) and $d_{1,2}$ is an arbitrary constant. In this case, the corresponding components for the “one-soliton” solution to the DKP equation (4) are given by

$$
U = c_1 + 2\beta_1^2 \ln \tau_0, \quad V = c_2 \frac{\tau_1}{\tau_0} \quad \text{and} \quad W = c_3 \frac{\tau_{-1}}{\tau_0},
$$

(142)

where we have fixed $n = 0$ in (136) for simplicity.
We can similarly construct solutions with nonzero constant background given by \( U = c_1, V = c_2, \) and \( W = c_3 \) to the coupled KdV system (which was introduced in Ref. 34 as a generalization of the Hirota–Satsuma equation \(^{35}\))

\[
\begin{align*}
\partial_t U &= \frac{1}{4} \partial_x^3 U + \frac{3}{2} \partial_x V - 6 \partial_x (VW), \\
\partial_t V &= -\frac{1}{2} \partial_x^2 V - \frac{3}{2} \partial_x U, \\
\partial_t W &= -\frac{1}{2} \partial_x^2 W - \frac{3}{2} \partial_x U \partial_x W
\end{align*}
\] (143a, 143b, 143c)

based on \(^{(94)}\). Since the procedure is the same, we omit the detail here.

Note that Galilean transformation \(^{(134)}\) has affected the coefficient of \( t \) in the linear dispersion (compare \(^{(139)}\) with \(^{(26)}\)). Generally, we are not able to directly construct multisoliton solutions to the coupled BSQ system (which follows from the \( t \)-independent reduction of \(^{(4)}\))

\[
\begin{align*}
\partial_x^2 U &= -\frac{1}{3} \partial_x^4 U - 2 U \partial_x^2 U - 2(\partial_x U)^2 + 8 \partial_x^2 (VW), \\
\partial_x \partial_y V &= \frac{1}{3} \partial_x^3 V + U \partial_x V - (\partial_x^{-1} \partial_y U) V, \\
\partial_x \partial_y W &= -\frac{1}{3} \partial_x^3 W - U \partial_x W - (\partial_x^{-1} \partial_y U) W
\end{align*}
\] (144a, 144b, 144c)

with nonzero constant background from those to \(^{(103)}\), unless we discuss a very special case for \( c_1 = 0 \) (which implies that the constant background is given by \( U = 0, V = c_2, \) and \( W = c_3 \)). This is because the \( t \)-independent reduction from \(^{(4)}\) to \(^{(144)}\) requires a constraint in the form of

\[
F(k, k') = k^3 + k'^3 + \frac{3c_1}{2}(k + k') = 0,
\] (145)

which is incompatible with the constraint \( k^2 + kk' + k'^2 = 0 \) (essentially leading to \( k^3 + k'^3 = 0 \)) in the construction of \(^{(103)}\), from the perspective of the DL, cf. Section 5.

8 | CONCLUDING REMARKS

The elliptic coupled KP system \(^{(1)}\) was studied within the DL framework, which provided us with a unified perspective to understand the integrability of the elliptic model. As a consequence, we have constructed its Lax pair and elliptic soliton solutions. The elliptic coupled KdV and BSQ systems were obtained from dimensional reductions of the elliptic coupled KP system, together with their respective Lax pairs. An interesting observation is that from the elliptic coupled KP system, we are able to construct a new class of solutions (i.e., solitons with nonzero constant background) to the DKP equation through a Galilean transformation.

As the DL approach is akin to the Riemann–Hilbert method appearing in the inverse scattering, based on similar singular integral equations, we expect that the present results open the way to a comprehensive study of the initial value problems associated with these elliptic models. Also, we expect that algebro-geometric solutions of higher genus can be treated, as well as other reductions (e.g., pole reductions) leading to possibly novel finite-dimensional integrable systems. Furthermore, the structures emerging from the DL approach in terms of the infinite matrix representation \(^{(42)}\) are evidence for the possibility that these nonlinear systems are also integrable in
the sense of possessing higher order symmetries. However, it remains an open problem to explicitly construct these higher-order symmetries. In particular, we may want to establish a Sato-type scheme for the coupled elliptic KP system through a matrix pseudodifferential operator algebra, and construct recursion operators for the elliptic coupled KdV and BSQ systems.

Further aspects of the elliptic KP family of systems remain to be investigated. So far, we have only constructed the single-component KP-type equation and its dimensional reductions in the present paper. It would be interesting to find the multicomponent (or matrix) KP-type hierarchy in this family, and simultaneously to see how they are related to the LL equation. In addition to the LL equations, there also exist the Krichever–Novikov equation \(^{36}\) and the elliptic analog of the Toda equation.\(^{37}\) Both are elliptic integrable systems that play roles of the master equations in their respective classes. How these particular elliptic systems and their discrete analogs are related to this elliptic family remains a problem for future work. The first step toward this goal might be searching for a discrete analog of the elliptic coupled KP system (1), because the integrability of the discrete (nonelliptic) coupled KP system has been studied in Ref. \(^{38}\) which we believe would bring us insights into the elliptic case.

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In this appendix, we briefly recapitulate the notion of Hirota’s bilinear derivative and the relevant logarithmic transformations. We also refer the reader to the monographs \cite{7,39} for more details regarding bilinear derivatives.

**Definition 9.** Suppose that $F$ and $G$ are differentiable functions of the independent variables $x_j$ for $j \in \mathbb{Z}^+$. The $m$th-order bilinear derivative of $F$ and $G$ with respect to the argument $x_j$ is defined as

$$D_j^m F \cdot G \equiv \left( \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j'} \right)^m F(x)G(x') \bigg|_{x'=x} \quad (A.1)$$

for $x = (x_1, x_2, \ldots)$ and $x' = (x_1', x_2', \ldots)$.

**Remark 13.** We can alternatively define the bilinear derivative by

$$e^{\varepsilon D_j} F \cdot G = F(\ldots, x_{j-1}, x_j + \varepsilon, x_{j+1}, \ldots)G(\ldots, x_{j-1}, x_j - \varepsilon, x_{j+1}, \ldots), \quad (A.2)$$

in which $\varepsilon$ is a parameter. Then we obtain the explicit formulae for $D_j^m F \cdot G$ from the coefficients of $\varepsilon^m$ for $m = 1, 2, \ldots$ in the series expansion.

Bilinear derivatives are closely related to derivatives of logarithmic functions through the so-called logarithmic and bilogarithmic transformations. These transformations are derived from some fundamental identities in terms of exponents of $D$- and $\partial$-operators, see, for example, Ref. \cite{7}. Below we only list transformations that are needed in this paper (which can even be proven by definition of bilinear derivative). The logarithmic transformations include the following:

$$\frac{D_j^2 F \cdot F}{F^2} = 2\partial_j^2 \ln F, \quad (A.3a)$$

$$\frac{D_j^4 F \cdot F}{F^2} = 2\partial_j^4 \ln F + 12\left(\partial_j^2 \ln F\right)^2, \quad (A.3b)$$

$$\frac{D_j^6 F \cdot F}{F^2} = 2\partial_j^6 \ln F + 60\left(\partial_j^2 \ln F\right)\left(\partial_j^4 \ln F\right) + 120\left(\partial_j^2 \ln F\right)^3, \quad (A.3c)$$

$$\frac{D_j D_j F \cdot F}{F^2} = 2\partial_j \partial_j \ln F. \quad (A.3d)$$

These transformations only involve a single function $F$. Instead, bilogarithmic transformations involve two functions $F$ and $G$. The first few of such transformations are as follows:

$$\frac{D_j F \cdot G}{FG} = \partial_j \ln \frac{F}{G}, \quad (A.4a)$$

$$\frac{D_j^2 F \cdot G}{FG} = \left(\partial_j \ln \frac{F}{G}\right)^2 + \partial_j^2 \ln \frac{F}{G} + 2\partial_j^2 \ln G, \quad (A.4b)$$

$$\frac{D_j^3 F \cdot G}{FG} = \left(\partial_j \ln \frac{F}{G}\right)^3 + \partial_j^3 \ln \frac{F}{G} + 3\left(\partial_j \ln \frac{F}{G}\right)\left(\partial_j^2 \ln \frac{F}{G} + 2\partial_j^2 \ln G\right), \quad (A.4c)$$
\[
\frac{D_i D_j F \cdot G}{F^2 G^2} = \left( \delta_i \ln \frac{F}{G} \right) \left( \delta_j \ln \frac{F}{G} \right) + \delta_i \delta_j \ln \frac{F}{G} + 2 \delta_i \delta_j \ln G.
\] (A.4d)

Equations (A.3) and (A.4) together allow us to transfer the bilinear equations in (75) to nonlinear equations in (68). We can also use these formulae reversely, to reformulate the nonlinear system as its corresponding bilinear form.

**APPENDIX B: PFAFFIAN AND AN EXPANSION FORMULA**

Here we remind the reader of a few facts about Pfaffians (see Refs. 7, 28), and present an expansion formula (see Refs. 26, 40) that is effective in expressing the soliton solutions of the elliptic coupled KP system in a concise (Hirota-type) form.

Let \( A \) be an \( N \times N \) skew-symmetric matrix with entries \( a_{i,j} \); hence, \( A \) is of the form

\[
A = \begin{pmatrix}
0 & a_{1,2} & a_{1,3} & \cdots & a_{1,N} \\
-a_{1,2} & 0 & a_{2,3} & \cdots & a_{2,N} \\
-a_{1,3} & -a_{2,3} & 0 & \cdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-a_{1,N} & -a_{2,N} & \cdots & -a_{N-1,N} & 0
\end{pmatrix}.
\] (B.1)

The Pfaffian \( \text{pf}(A) \) associated with the matrix \( A \) can be defined as follows.

**Definition 10.** The \( N \)-th-order Pfaffian associated with \( A \) is the triangular array (see, e.g., Ref. 28 for such a notation)

\[
\text{pf}(A) = \begin{vmatrix}
a_{1,2} & a_{1,3} & \cdots & a_{1,N} \\
a_{2,3} & \cdots & a_{2,N} \\
\vdots & \ddots & \vdots \\
a_{N-1,N} & \cdots & a_{N-1,N}
\end{vmatrix},
\] (B.2a)

which is uniquely defined by the recursion relation

\[
\text{pf}(A) = \sum_{i=2}^{N} (-1)^{i} a_{1,i} \text{pf}\left( A_{i-1,i} \right),
\] (B.2b)

together with the initial values defined as \( | \cdot | = 0 \) and \( |a_{1,2}| = a_{1,2} \) for \( N = 1 \) and \( N = 2 \), respectively.

**Remark 14.** The definition implies that a Pfaffian is only nonzero when \( N \) is even. For example, we have

\[
\begin{vmatrix}
a_{1,2} & a_{1,3} \\
a_{2,3}
\end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix}
a_{1,2} & a_{1,3} & a_{1,4} \\
a_{2,3} & a_{2,4} \\
a_{3,4}
\end{vmatrix} = a_{1,2} a_{3,4} - a_{1,3} a_{2,4} + a_{1,4} a_{2,3}.
\] (B.3)
for $N = 3$ and $N = 4$, respectively. We note the following important relation between the Pfaffians and determinants of skew-symmetric matrices of the form given above:

$$\text{det}(A) = (\text{pf}(A))^2. \quad (B.4)$$

Hence, the Pfaffian of $A$ can be thought of as the square root of a determinant of a skew-symmetric matrix. However, the latter relation does not define the Pfaffian uniquely, whereas the recursion relation does.

What we need, to obtain our explicit form of the soliton solutions, is a Pfaffian analog of the expansion formula for a determinant of the type $\text{det}(I + AB)$ in terms of the matrix invariants of $AB$ (which are the sums of its principal minors). This is given in the lemma below. To express the formula in a compact manner, let us introduce the notation

$$A_{i_1,i_2,\ldots,i_m} \quad (B.5)$$

denoting the submatrix of $A$ by selecting from it the rows and columns labeled by $i_1, i_2, \ldots, i_m$ for $1 \leq i_1 < i_2 < \cdots < i_m < N$. For examples, we have

$$A_{i_1} = a_{i_1,i_1}, \quad A_{i_1,i_2} = \begin{pmatrix} a_{i_1,i_1} & a_{i_1,i_2} \\ a_{i_2,i_1} & a_{i_2,i_2} \end{pmatrix} \quad \text{and} \quad A_{1,2,\ldots,N} = A. \quad (B.6)$$

In this notation, the expansion formula (B.2b) can be rewritten as

$$\text{pf}(A) = \sum_{i=2}^N (-1)^i a_{1,i} \text{pf}(A_{2,\ldots,i-1,i+1,\ldots,N}). \quad (B.7)$$

**Lemma 2.** Let $A$ and $B$ be $N \times N$ skew-symmetric matrices of the form given above for $A$ and a similar form for $B$ (with entries $b_{i,j}$). The following expansion formula holds for the special Pfaffian of the format

$$\text{pf}(A \mid B) \doteq$$

$$\begin{vmatrix} a_{1,2} & a_{1,3} & \cdots & a_{1,N} & 1 & 0 & \cdots & 0 & 0 \\ a_{2,3} & \cdots & a_{2,N} & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{N-1,N} & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ b_{1,2} & \cdots & b_{1,N-1} & b_{1,N} \\ \vdots & \vdots & \vdots & \vdots \\ b_{N-2,N-1} & b_{N-2,N} \\ b_{N-1,N} \end{vmatrix} = (-1)^{N(N-1)/2} \left[ 1 + \sum_{m \in J} (-1)^{m(m-1)/2} \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq N} \text{pf}(A_{i_1,i_2,\ldots,i_m}) \text{pf}(B_{i_1,i_2,\ldots,i_m}) \right], \quad (B.8)$$

for $J = \{ j = 2i \mid i = 1, 2, \ldots, [N/2] \}$.

Formula (B.8) is a special case of a Laplace-type expansion formula for Pfaffian, see proposition 2.3 in Ref. 26 for the general formula and its proof. The crucial upshot of the lemma is that the
terms in the expansion (B.8) are products of separate sub-Pfaffians of $A$ and $B$, respectively. To give an idea how this expansion looks like, let us write them down explicitly for the values of $N = 2, 3, 4, 5$.

$N = 2$

$$\text{pf}(A | B) = -1 + a_{1,2}b_{1,2};$$  \hspace{1cm} (B.9)

$N = 3$

$$\text{pf}(A | B) = -1 + a_{1,2}b_{1,2} + a_{1,3}b_{1,3} + a_{2,3}b_{2,3};$$  \hspace{1cm} (B.10)

$N = 4$

$$\text{pf}(A | B) = 1 - \sum_{1 \leq i_1 < i_2 \leq 4} a_{i_1,i_2}b_{i_1,i_2} + \begin{vmatrix} a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,3} & a_{2,4} & a_{3,4} \\ b_{1,2} & b_{1,3} & b_{1,4} \\ b_{2,3} & b_{2,4} & b_{3,4} \end{vmatrix};$$  \hspace{1cm} (B.11)

$N = 5$

$$\text{pf}(A | B) = 1 - \sum_{1 \leq i_1 < i_2 \leq 5} a_{i_1,i_2}b_{i_1,i_2} + \sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq 5} \begin{vmatrix} a_{i_1,i_2} & a_{i_1,i_3} & a_{i_1,i_4} \\ a_{i_2,i_3} & a_{i_2,i_4} & a_{i_3,i_4} \\ b_{i_1,i_2} & b_{i_1,i_3} & b_{i_1,i_4} \\ b_{i_2,i_3} & b_{i_2,i_4} & b_{i_3,i_4} \end{vmatrix};$$  \hspace{1cm} (B.12)

In the case of the soliton solution (117), the fact that in this expansion we get sums of products of separate Pfaffians is crucial, as we can compute the Pfaffians of the elliptic Cauchy matrix in explicit form, with the formulae given in the next appendix.

**APPENDIX C: APPLYING FROBENIUS FORMULA FOR ELLIPTIC CAUCHY MATRIX**

We introduce the elliptic functions

$$W_1(x) \equiv \Phi(\omega_1(x)e^{-\eta_1x}, W_2(x) \equiv \Phi(\omega_2(x)e^{-\eta_2x} \text{ and } W_3(x) \equiv \Phi(\omega_3(x)e^{-\eta_3x},$$  \hspace{1cm} (C.1)

where $\Phi(\alpha(x)$ is the Lamé function, given in terms of the Weierstrass $\sigma$-function $\sigma(x) = \sigma(x|2\omega_1, 2\omega_2)$ with half-periods $\omega_1, \omega_2 (2\omega_1 \text{ and } 2\omega_2 \text{ being the elementary periods of the period lattice)}:

$$\Phi(\alpha(x) \equiv \frac{\sigma(x+\alpha)}{\sigma(\alpha)\sigma(x)},$$  \hspace{1cm} (C.2)

where $\alpha$ is an arbitrary complex variable not coinciding with any zero of the $\sigma$-function, and $\eta_1 = \zeta(\omega_1), \eta_2 = \zeta(\omega_2), \eta_3 = \zeta(\omega_3)$, where $\omega_3 = -\omega_1 - \omega_2$ (see, e.g., Ref. 41 for the standard notation of Weierstrass elliptic functions).

From the standard addition formulae for the Weierstrass functions $\sigma(x), \zeta(x)$, and $\varphi(x)$, we obtain the relations for the functions $W_1, W_2, \text{ and } W_3$ including the Yang–Baxter-type relation

$$W_1(x)W_2(z) + W_2(y)W_3(x) + W_3(z)W_1(y) = 0, \quad x + y + z = 0.$$  \hspace{1cm} (C.3)
Equation (C.3) follows from the well-known three-term addition formula for the \( \sigma \)-function, which can be written in the compact form of an elliptic partial fraction expansion in terms of \( \Phi \), namely,

\[
\Phi_\alpha(x)\Phi_\beta(y) = \Phi_{\alpha+\beta}(x)\Phi_\beta(y-x) + \Phi_\alpha(x-y)\Phi_{\alpha+\beta}(y).
\] (C.4)

In fact, Equation (C.3) follows directly by setting \( \alpha = \omega_1 \) and \( \beta = \omega_2 \) in (C.4) that implies \( \alpha + \beta = -\omega_3 \) and using the relation \( \Phi_{-\omega_3}(x) = \Phi_{\omega_3}(x)e^{-2\eta_3 x} \) that follows from quasi-periodicity of the \( \sigma \)-function

\[
\sigma(x + 2\omega_1) = -\sigma(x)e^{2\eta_1(x+\omega_1)} \quad \text{and} \quad \sigma(x + 2\omega_2) = -\sigma(x)e^{2\eta_2(x+\omega_2)}.
\] (C.5)

Furthermore, we have the following addition formulae:

\[
W_1^2(x) + e_1 = W_2^2(x) + e_2 = W_3^2(x) + e_3 = \wp(x),
\] (C.6a)

\[
W_1(x)W_2(x)W_3(x) = -\frac{1}{2}\wp'(x),
\] (C.6b)

where \( e_1 = \wp(\omega_1) \), \( e_2 = \wp(\omega_2) \), \( e_3 = \wp(\omega_3) \). Equations (C.6) are essentially reformulations of some well-known addition formulae for the Weierstrass elliptic functions. For example, equations in (C.6a) follow from

\[
\Phi_\alpha(x)\Phi_\alpha(-x) = \wp(\alpha) - \wp(x),
\] (C.7)

by setting \( \alpha = \omega_1, \omega_2, \omega_3 \) successively. We refer the reader to page 397 of the monograph\(^{42}\) for those addition formulae in terms of the \( W \)-functions and their proofs.

Let us now single out one of the half-periods \( \omega_1 \) and the corresponding function \( W_1(x) \), for which we have the relation

\[
W_1(x)W_1(x + \omega_1) = -\frac{e^{2\eta_1 \omega_1}}{\sigma^2(\omega_1)},
\] (C.8a)

and subsequently,

\[
(\wp(x) - e_1)(\wp(x + \omega_1) - e_1) = \frac{e^{2\eta_1 \omega_1}}{\sigma^4(\omega_1)} = (e_1 - e_2)(e_1 - e_3) = \frac{1}{2}\wp''(\omega_1),
\] (C.8b)

We also introduce the corresponding parameters

\[
k = \zeta(x + \omega_1) - \zeta(x) - \eta_1 = -\frac{W_2(x)W_3(x)}{W_1(x)} \quad \text{and} \quad K = \wp(x) - e_1,
\] (C.9)

for which we note that the relations (C.8b), together with (C.6b) and (C.9), lead to the elliptic curve in the rational form (5), that is,

\[
k^2 = K + 3e + \frac{g}{K},
\] (C.10)

where \( e \equiv e_1 = \wp(\omega_1) \) and \( g \equiv (e_1 - e_2)(e_1 - e_3) \). For convenience below, we suppress the suffix 1 from the corresponding functions when we single out \( \omega_1 \), that is, we set \( \omega \equiv \omega_1, \eta \equiv \eta_1 = \zeta(\omega_1), \) and \( W(x) \equiv W_1(x) = \Phi_{\omega_1}(x)e^{-\eta_1 x} \).
The aim is to express the elliptic Cauchy matrix \( \Omega(\kappa_i, \kappa_{i'}) \) in terms of the \( W \)-function that, in turn, allows us to apply the famous Frobenius determinant formula\textsuperscript{29} for elliptic Cauchy matrices. In terms of the function \( W \), we have the following key relation:

\[
\frac{W(\kappa + \kappa')}{W(\kappa)W(\kappa')} = -\frac{k - k'}{K - K'} \quad \text{or equivalently} \quad \frac{W(\kappa - \kappa')}{W(\kappa)W(\kappa')} = \frac{k + k'}{K - K'},
\]

and furthermore, we can identify the Cauchy kernel of the elliptic KP system as follows:

\[
\Omega(\kappa, \kappa') = \frac{K - K'}{k + k'} = -W(\kappa)W(\kappa') \sqrt{g} W(\kappa - \kappa' + \omega).
\]

To compute the determinants and Pfaffians, we note that as the function \( W \) is essentially a Lamé function \( \Phi_{\omega} \), we can apply the Frobenius determinant formula for elliptic Cauchy matrices:

\[
det \begin{pmatrix} \Phi_{\omega}(\kappa_i - \kappa_{i'}) \end{pmatrix}_{i,j = 1, \ldots, N} = \frac{\sigma(\alpha + \sum_{i} (\kappa_i - \kappa_i'))}{\prod_{1 \leq i < j \leq N} \sigma(\kappa_i - \kappa_j')} \prod_{1 \leq i < j \leq N} \sigma(\kappa_i - \kappa_j'),
\]

which vanishes if \( m \) is odd in accordance with the Pfaffian structure. Subsequently, (C.12) tells us that the elliptic Cauchy matrix \( (\Omega(\kappa_i, \kappa_{i'}))_{y, y' = 1, \ldots, m} \) is essentially \( (W(\kappa_i - \kappa_{i'} + \omega))_{y, y' = 1, \ldots, m} \) up to diagonal factors \( \frac{W(\kappa_{i'} + \omega)W(\kappa_i)}{\sqrt{g}} \) that can be converted to factors \( K_{i_{y'}} \) using (C.6) and (C.8). Thus, we end up with

\[
det (\Omega(\kappa_i, \kappa_{i'}))_{y, y' = 1, \ldots, m} = (-1)^{m(m+1)/2} \frac{\sigma((m + 1)\omega)}{\sigma^{m+1}(\omega)} \prod_{i = 1}^{m} \frac{K_{i_y}}{g} \prod_{1 \leq y < y' \leq m, y \neq y'} \frac{1}{W^2(\kappa_{i_y} - \kappa_{i_{y'}})}.
\]

Furthermore, the prefactor satisfies

\[
\frac{\sigma((m + 1)\omega)}{\sigma^{m+1}(\omega)} = \begin{cases} 0, & \text{if } m \text{ odd}, \\ (-1)^{m/2} \gamma_m^2, & \text{if } m \text{ even}, \end{cases}
\]
where \(\gamma_m = g^{m(m+2)/8}\). This can be proven by induction using the quasi-periodicity of the \(\sigma\)-function and the relations (C.8), leading to \(\gamma_{2n+2}/\gamma_{2n} = g^{n+1}\). Therefore, the Pfaffian of the skew-symmetric elliptic Cauchy kernel takes the form

\[
\text{pf}(\Omega(x_\nu, x_{\nu'}) \mid_{\nu, \nu' = 1, \ldots, m}) = \frac{g^{m(m-2)/8}}{\prod_{\nu=1}^{m} K_{\nu}} \prod_{1 \leq \nu < \nu' \leq m} \frac{K_{\nu} - K_{\nu'}}{K_{\nu} + K_{\nu'}},
\]

for \(m\) even, whereas for \(m\) odd the Pfaffian vanishes. In (C.17), we have used (C.11) and (C.6a) to express the fully skew-symmetric product in rational form.

**Remark 15.** As a curiosity, we note that as a consequence of the Frobenius–Stickelberger determinant formula, that is, the elliptic van der Monde-determinant (see Ref. 43), the prefactor can be written as a Hankel determinant in the following form:

\[
\frac{\sigma((m+1)x)}{\sigma(m+1)^2(x)} = \frac{(-1)^{m^2}}{(1!2! \ldots m!)^2} \begin{vmatrix}
\varphi'(x) & \varphi''(x) & \ldots & \varphi^{(m)}(x) \\
\varphi''(x) & \varphi'''(x) & \ldots & \varphi^{(m+1)}(x) \\
\vdots & \vdots & \ddots & \vdots \\
\varphi^{(m)}(x) & \varphi^{(m+1)}(x) & \ldots & \varphi^{(2m-1)}(x)
\end{vmatrix},
\]

see Example 20.21 in Ref. 44 and references therein, which vanishes at \(x = \omega\) when \(m\) is odd, whereas for \(m = 2n\) even yields (up to a sign) a perfect square, namely,

\[
\frac{\sigma((2n+1)\omega)}{\sigma(2n+1)^2(\omega)} = (-1)^n \gamma_m^2,
\]

with

\[
\gamma_m \doteq \frac{1}{1!2! \cdots (2n)!} \begin{vmatrix}
\varphi^{(2)}(\omega) & \varphi^{(4)}(\omega) & \ldots & \varphi^{(2n)}(\omega) \\
\varphi^{(4)}(\omega) & \varphi^{(6)}(\omega) & \ldots & \varphi^{(2n+2)}(\omega) \\
\vdots & \vdots & \ddots & \vdots \\
\varphi^{(2n)}(\omega) & \varphi^{(m+1)}(\omega) & \ldots & \varphi^{(4n-2)}(\omega)
\end{vmatrix},
\]

which miraculously turns out to be a pure power of the modulus \(g\) alone, even though the individual entries depend on \(g\) and \(e\).