Fast and Simple Computation of All Longest Common Subsequences

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Abstract

This paper shows that a simple algorithm produces the all-prefixes-LCSs-graph in \(O(mn)\) time for two input sequences of size \(m\) and \(n\). Given any prefix \(p\) of the first input sequence and any prefix \(q\) of the second input sequence, all longest common subsequences (LCSs) of \(p\) and \(q\) can be generated in time proportional to the output size, once the all-prefixes-LCSs-graph has been constructed. The problem can be solved in the context of generating all the distinct character strings that represent an LCS or in the context of generating all ways of embedding an LCS in the two input strings.

Keywords: longest common subsequences, edit distance, shortest common supersequences

1 Background and Terminologies

Let \(A = a_1a_2\ldots a_m\) and \(B = b_1b_2\ldots b_n\) with \(m \leq n\) be two sequences over an alphabet \(\Sigma\). Any sequence that can be obtained by deleting some symbols of another sequence is referred to as a subsequence of the original sequence. A common subsequence of \(A\) and \(B\) is a subsequence of both \(A\) and \(B\). The longest common subsequence (LCS) problem is to find a common subsequence of greatest possible length.

A pair of sequences may have many different LCSs. In addition, a single LCS may have many different embeddings, i.e., positions in the two strings to which the characters of the LCS correspond. We may pick out a distinguished embedding for each distinct LCS, e.g., the canonical embedding has been defined to be the one in which each character, starting from the beginning of the LCS, is assigned matching positions in both sequences as small as possible \(^1\). It is more convenient in this paper to distinguish embeddings in which the matching positions are chosen as large as possible (starting from the end of the LCS); let us call these anticanonical embeddings. Figure 1 shows an example pair of strings and the various LCS embeddings and anticanonical embeddings. (The matrix in the figure will be explained later.)

A few other terminologies and notations that will be useful are as follows. We use \(A_i\) to represent the prefix \(a_1a_2\ldots a_i\) of \(A\) and similarly for \(B\). When \(a_i = b_j\), we refer to the pair \([i, j]\) as a match; otherwise it is a clash.

\(^1\) It is reasonable to assume that the alphabet size \(|\Sigma|\) is at most \(m\), since the actual value of symbols not present in the shorter string is irrelevant. Extraneous symbols can be culled efficiently if space usage is not of concern, or hashing can be used to obtain a good expected time with little space usage.
Figure 1: Listed are the seven different embeddings and three anticanonical embeddings (corresponding to the three distinct LCSs) for the strings $A = \text{bilabial}$ and $B = \text{balaclava}$. (The naive method of generating all LCSs for this pair of strings would produce a list of length 100, because there would be many duplications.) In the matrix, the $[i, j]$ entry shows the rank $L[i, j]$ as per (1). The matches are circled and are organized into contours as shown by the connecting lines. If a match is dominant, its circle is bold, and if the match is antidominant, its rank is bold. (Note that a match may be both dominant and antidominant.)

The standard “naive” method of computing the length of an LCS is a “bottom-up” dynamic programming approach (as in [21]) based on the following recurrence for the length of an LCS of $A_i$ and $B_j$:

$$L[i, j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ L[i-1, j-1] + 1 & \text{if } i, j > 0 \text{ and } a_i = b_j \\ \max\{L[i-1, j], L[i, j-1]\} & \text{otherwise} \end{cases} \quad (1)$$

(Sankoff [21] may be the first to have published this recurrence, based on the work of Needleman and Wunsch [10]). In $O(mn)$ time, one may fill an array with all the values of $L[i, j]$ for $0 \leq i \leq m$ and $0 \leq j \leq n$, and the length $L$ of an LCS is read off from $L[m, n]$. The same time bound also suffices to produce a single LCS by a “backtracing” approach starting from position $[m, n]$ of the array. At each stage we just step from position $[i, j]$ to a position $[i-1, j-1]$, $[i-1, j]$, or $[i, j-1]$ that is responsible for the setting of $L[i, j]$ as per (1); each match encountered generates a character of the LCS (in reverse order).

A few other terminologies are useful for discussing some alternative solution techniques. Figure 1 shows the matrix of $L$ values as per (1) for a sample pair of input strings, and we will refer to the value of $L[i, j]$ as the rank of $[i, j]$. It is well known and easy to see that the matches can be partitioned by rank so as to form contours as illustrated by the zig-zag lines in Figure 1. Starting from the lower left match on a contour, motion along a contour proceeds monotonically in both dimensions, i.e., the next match is at or above the level of the previous match, and at or to the right of the vertical position of the previous match. Different contours never cross or touch. Each contour may be completely specified by the dominant matches in the upper left corners of the contours, i.e., those matches $[i^*, j^*]$ for which there is no other match $[i', j']$ on the same contour with $i' = i^* \land j' < j^*$ or $j' = j^* \land i' < i^*$. For discussion of the algorithm to be presented in Section 2 we also introduce the notion of antidominant matches, i.e., those matches $[i^*, j^*]$ for which there is no other match $[i', j']$ on the same contour with $i' = i^* \land j' > j^*$ or $j' = j^* \land i' > i^*$. Note also that two matches $[i^*, j^*]$ and $[i', j']$ with $i^* \leq i'$ can belong to the same common subsequence if and only if $i^* < i' \land j^* < j'$. Thus, the problem of finding an LCS can be expressed as finding a longest sequence of matches that is strictly increasing in both dimensions.
The best known upper bound on the time to find an LCS with general inputs (i.e., with the time expressed only in terms of \( m \) and \( n \)) is essentially \( O(mn/\lg n) \) with a finite alphabet or slightly more with an infinite alphabet \([13]\), only a small improvement over the naive method. Several other methods have been proposed to reduce the time under such circumstances as small alphabet, short LCS, or few dominant matches, e.g., \([13] 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\). For all of these algorithms, however, there are still inputs that require \( \Omega(m^2) \) time or more.\(^2\) Thus, the naive method remains a reasonable approach for finding one LCS, particularly in light of its simplicity.

Relatively little attention has been given to the problem of finding all LCS embeddings or all distinct LCSs. (In the latter case, different embeddings of the same character sequence would not be counted as different LCSs.) The naive approach to generate all LCS embeddings \([1]\) would be to extend the backtracing method. At each step, we would consider three possibilities (and continue recursively): from position \([i, j]\), we could add a character to the LCS and move to \([i-1, j-1]\) if \([i, j]\) is a match, and we could move to \([i-1, j]\) or \([i, j-1]\) if the \( L \) value there equals \( L[i, j] \) (without adding a character to the LCS and regardless of whether \([i, j]\) is a match). One could obviously then remove multiple embeddings of the same LCS to obtain a list of all distinct LCSs. This naive approach to generating all LCS embeddings or all distinct LCSs could, however, be painfully inefficient. The naive method may traverse exponentially many paths through the \( L \) matrix even when only one LCS embedding exists. Furthermore, any method of generating distinct LCSs that begins by generating all LCS embeddings could have a run time exceeding the output size by a factor of approximately \( 3/\pi^2 \cdot 2.598^n \) as per the maximum number of different embeddings a single LCS could have in two sequences of length \( n \) \([10]\).

Rick \([3]\) gives a method to produce a compact representation of all LCS embeddings, the LCSs-graph, from which all LCS embeddings can be listed in time proportional to the output size. He also notes that an extra processing stage can prune the compact representation to one that gives only distinct LCSs. The time complexity of his algorithm for constructing the LCSs-graph \( G \) is \( O(|\Sigma|n + T + |G|) \), where \( T \) is the time of any algorithm that determines the dominant matches. Thus, the run time of his algorithm is sometimes better than \( \Theta(mn) \) but certainly could require \( \Theta(mn) \) for some inputs. Furthermore, the number of distinct LCSs could be as large as approximately \( 1.442^n \) for two sequences of length \( n \) \([11]\), so actually listing all distinct LCSs (or all LCS embeddings) may well erase any gain from constructing the LCSs-graph in less than \( \Theta(mn) \) time. Baeza-Yates \([5]\) provides another construction from which the LCSs-graph could be produced but with a potentially longer time of at least \( \Theta(|\Sigma|n \lg n) \). \( O(mn) \) algorithms for creating a structure akin to the LCSs-graph have also been proposed by Gotoh \([8]\) and Altschul and Erickson \([2]\) but with much greater complication than the approach to be presented here.

This paper shows that a much simpler approach than in prior work can be used to perform the pre-processing phase in \( O(mn) \) time. Furthermore, the result of this preprocessing phase is a more versatile structure, the all-prefixes-LCSs-graph. From the all-prefixes-LCSs-graph, we can list, for any prefix \( A_i \) of the first input string and any prefix \( B_j \) of the second input string, all LCSs of \( A_i \) and \( B_j \) in time proportional to the size of the output. The all-prefixes-LCSs-graph can be constructed either for distinct LCSs or for all LCS embeddings. The more interesting case of distinct LCSs is discussed in Section \( \ref{sec:distinct} \). The simpler case of all LCS embeddings is discussed in Section \( \ref{sec:all} \).

## 2 Finding All Distinct Longest Common Subsequences

The basic methodology employed here for enabling efficient computation of all LCSs of any prefixes \( A_i \) and \( B_j \) of the input sequences is a variation on the idea of creating a directed acyclic graph in which every path from the vertex corresponding to \([i, j]\) represents a different LCS of \( A_i \) and \( B_j \). The naive backtracing approach mentioned in Section \( \ref{sec:naive} \) can be thought of as directing edges from \([i, j]\) to some or all of \([i-1, j-1]\),\(^2\) An example with many dominant matches is when one input string contains repeated occurrences of the pattern abc and the other contains repeated occurrences of the pattern cba.
\([i - 1, j]\), and \([i, j - 1]\) (according to ranks of the vertices and whether \([i, j]\) is a match). Then every path from \([i, j]\) would represent an LCS. But, as mentioned before, many paths may be traversed for the same LCS or even the same LCS embedding; also, throughout the enumeration of paths, many steps may be taken in which no match is added to the current LCS.

Rick’s compact representation of all LCS embeddings in \(A\) and \(B\), the LCSs-graph, may be defined as follows (though it is constructed in a much more efficient fashion than this definition suggests). Find the transitive closure of the naive graph, remove all but those vertices that are matches belonging to some LCS, and remove all edges except those connecting a retained vertex to a retained vertex of next lower rank. (Rick actually reverses the direction of every edge, but declining to do so leaves us in a framework more analogous to the naive backtracking approach.) It is easy to see that the LCSs-graph can be used to list all LCS embeddings in time proportional to the output size. Furthermore, Rick notes that a breadth first search on the graph can be used to eliminate certain vertices, leaving a representation of only canonical embeddings.

As noted before, Rick’s construction of the LCSs-graph is relatively complex, and it is predicated upon having first found all dominant matches. Furthermore, it is a compact representation only of the LCSs of \(A\) and \(B\) rather than of the LCSs for each of the \(mn\) pairs of prefixes \(A_i\) and \(B_j\).

We show here a simple construction of the all-prefixes-LCSs-graph, which can be initially thought of as being similar to Rick’s LCSs-graph but without restricting attention to vertices that are matches belonging to an LCS of \(A\) and \(B\). Furthermore, we show how to prune the edges on the fly so that only the single anticanonical embedding is represented for each distinct LCS. Though the number of edges in the all-prefixes-LCSs-graph may exceed \(\Theta(mn)\), we can still produce essentially an adjacency list representation in \(O(mn)\) time, due to a heavy degree of sharing among the adjacency lists of different vertices.

The precise definition of the all-prefixes-LCSs-graph is as follows. Every vertex \([i, j]\) has an edge pointing to each match \([i^*, j^*]\) of the same rank that is antidominant when considering the input strings \(A_i\) and \(B_j\) (i.e., such that \(i^* \leq i \land j^* \leq j\), and there is no other match \([i', j']\) of the same rank with \(i' = i^* \land j \geq j' > j^*\) or \(j' = j^* \land i \geq i' > i^*\)). In Rick’s graph, vertices point to vertices of one lower rank, but the algorithm presented here to generate the all-prefixes-LCSs-graph is simplified by having vertices point to vertices of equal rank. It is still easy to use essentially the same backtracing method to list the (reversed) LCSs corresponding to any starting point \([i, j]\); we simply need to augment the explicit edges of the graph with the notion whenever we include a match in the LCS, we take a diagonal step (subtracting one from each matrix coordinate).

Before continuing, we should verify that the backtraces from \([i, j]\) in the all-prefixes-LCSs-graph will actually represent each distinct LCS once.

**Theorem 1** Considering all paths from \([i, j]\) in the all-prefixes-LCSs-graph (augmented with diagonal steps from match nodes) provides a one-to-one correspondence with distinct LCSs of \(A_i\) and \(B_j\).

**Proof.** This is easiest to see by recalling that finding all LCS embeddings (in reverse order) corresponds to finding all longest sequences of matches (in the submatrix defined by \(A_i\) and \(B_j\)) that are strictly decreasing in both dimensions.

The restricted use of matches that we incorporated into the all-prefixes-LCSs-graph will not make us lose any LCSs, by the following reasoning. If we consider a backtrace in which we go from position \([\hat{i}, \hat{j}]\) to a match \([i^*, j^*]\) such that there exists another match \([\hat{i}', \hat{j}']\) of the same rank with \(\hat{i}' = i^* \land \hat{j} \geq \hat{j}' > j^*\) or \(\hat{j}' = j^* \land \hat{i} \geq i' > i^*\), then we can just replace \([i^*, j^*]\) with \([\hat{i}', \hat{j}']\) and get the same LCS. (\([i^*, j^*]\) and \([\hat{i}', \hat{j}']\) must match on the same character, since they are in the same row or column.)

We also will not duplicate any LCSs, by the following reasoning. To get the same LCS twice, there would need to be a position \([\tilde{i}, \tilde{j}]\) from which two edges in the all-prefixes-LCSs-graph proceed to two matches on the same character, say \([i^*, j^*]\) and \([\tilde{i}, \tilde{j}]\) with \(\tilde{i} \geq i^*\). Then \([\tilde{i}, \tilde{j}]\) would be another match on the same character and of the same rank, implying that \([i^*, j^*]\) and \([\tilde{i}, \tilde{j}]\) are not antidominant for \(A_i\) and \(B_j\).}

Now, each adjacency list in the all-prefixes-LCSs-graph can be thought of as being based upon a linked list of the antidominant matches along one of the contours; let us call such a linked list a *contour list*. That
is, the antidominant matches for \( A \) and \( B \) are nearly sufficient to characterize all distinct LCSs for each \( A_i \) and \( B_j \) (whereas the dominant matches do not suffice, as is illustrated by Rick [18]). We need only add to the adjacency list for \( [i, j] \) at most two matches that are not antidominant for \( A \) and \( B \), by considering the extreme lower left and upper right matches \([i^*, j^*]\) on the relevant contour satisfying \( i^* \leq i \land j^* \leq j \). Thus, each adjacency list in the all-prefixes-LCSs-graph is a portion of a contour list, with the possible addition of a different head and/or tail node. An adjacency list that has a separate head node and then jumps into the midst of a contour list can easily share all but a constant amount of its storage with the contour list; see Figure 2. When adjacency lists digress to incorporate a separate tail node, however, we require a small digression from the standard linked list representation. It suffices to maintain, for each adjacency list, a see Figure 2. When adjacency lists digress to incorporate a separate tail node, however, we require a small digression from the standard linked list representation. It suffices to maintain, for each adjacency list, a

\begin{align*}
\text{head}[i', j'] & \quad \text{pretail}[i', j'] \\
\text{head}[i, j] & \quad \text{pretail}[i, j] \\
\text{tail}[i, j] & \quad \text{tail}[i', j']
\end{align*}

Figure 2: Adjacency lists of different nodes in the all-prefixes-LCSs-graph might share portions of a contour list. In this example, the contour list could be specified by the central line of “next node” pointers, while the adjacency lists of \([i, j]\) and \([i', j']\) are largely just excerpts. The two thin diagonal lines are not actual pointers present in the data structure; rather they show the implicit relationships that

\begin{align*}
\text{head}[i', j'] & \quad \text{pretail}[i', j'] \\
\text{head}[i, j] & \quad \text{pretail}[i, j] \\
\text{tail}[i, j] & \quad \text{tail}[i', j']
\end{align*}

With the above representation in mind, we can adapt the naive method of calculating LCS length based on \([i, j]\) to also set up the appropriate \( \text{head}[i, j] \), \( \text{pretail}[i, j] \), and \( \text{tail}[i, j] \) pointers based on the information already computed at positions \([i - 1, j - 1]\), \([i - 1, j]\), and \([i, j - 1]\). We always use \( \text{tail}[i, j] \) to point to the last node on the adjacency list at position \([i, j]\) or assign \( \text{null} \) for an empty adjacency list. Any other nodes in the adjacency list appear in an ordinary linked list beginning at \( \text{head}[i, j] \) and terminating at \( \text{pretail}[i, j] \), with \( \text{head}[i, j] \) being \( \text{null} \) if there are no such nodes.

The algorithm to construct the all-prefixes-LCSs-graph in \( O(mn) \) time is given in Figure 3. We proceed through positions \([i, j]\) in a row-by-row fashion, with trivial handling for matches in line 3 and handling of a clash in Lines 4 to 7.

When \([i, j]\) is a match: the corresponding vertex in the all-prefixes-LCSs-graph just points to itself; any other match of the same rank must have a higher row or column index. Consistent with the adjacency list approach described above, we use \( \text{tail}[i, j] \) for the pointer to self and leave \( \text{head}[i, j] \) \( \text{null} \) to indicate there is nothing else in the adjacency list of \([i, j]\).

When \([i, j]\) is a clash, backtraces in the naive graph need only proceed through whichever of \([i - 1, j]\) and \([i, j - 1]\) are of the same rank as \([i, j]\). To avoid following duplicate paths that uncover the same embedding or uncovering multiple embeddings of a single LCS, we perform an appropriate merging of information computed at \([i - 1, j]\) and \([i, j - 1]\). The adjacency lists are always maintained so that the nodes are in order along a contour from lower left to upper right, and the adjacency lists at \([i - 1, j]\) and \([i, j - 1]\) are identical (if \([i - 1, j]\) and \([i, j - 1]\) are of the same rank) except for possibly a different head and/or tail. Once we conditionally set the tail of the adjacency list for position \([i, j]\) from the information at \([i - 1, j]\) in line 6, we need only look for additional information at position \([i, j - 1]\) in lines 8 to 17.

If a null \( \text{tail}[i, j] \) was obtained by looking at position \([i - 1, j]\), then the adjacency list for \([i, j]\) contains at most one node; in that case \( \text{tail}[i, j] \) is copied from position \([i, j - 1]\), and our work is done. Otherwise, we
for $i = 1$ to $m$ do for $j = 1$ to $n$ do
  Create a linked list node $p[i, j]$ to represent vertex $[i, j]$ in adjacency lists.
  Set $head[i, j]$, $pretail[i, j]$, and $tail[i, j]$ to NULL.
  Compute $L[i, j]$ as per \([4]\).
  if $[i, j]$ is a match then $tail[i, j] \leftarrow p[i, j]$
  if $[i, j]$ is a clash and $L[i, j] > 0$ then
    if $L[i, 1, j] = L[i, j]$ then $tail[i, j] \leftarrow tail[i - 1, j]$
    if $L[i, j - 1] = L[i, j]$ then
      if $tail[i, j] = NULL$ then $tail[i, j] \leftarrow tail[i, j - 1]$
    else
      if $tail[i, j - 1]$ and $tail[i, j]$ point to same row then $pretail[i, j] \leftarrow pretail[i, j - 1]$
      else $pretail[i, j] \leftarrow tail[i, j - 1]$
      if $pretail[i, j]$ and $tail[i, j]$ point to same column then $tail[i, j] \leftarrow pretail[i, j]$
      else
        Set “next” pointer of $pretail[i, j]$ to $tail[i, j]$
        $head[i, j] \leftarrow head[i, j - 1]$
        if $head[i, j] = NULL$ then $head[i, j] \leftarrow pretail[i, j]$
  endfor endfor

Figure 3: The algorithm to create the all-prefixes-LCSs-graph representing all distinct LCSs.

can tentatively form the adjacency list of $[i, j]$ by just following the adjacency list of $[i, j - 1]$ with $tail[i, j]$. But we need lines \([11]-[13]\) to ensure that each LCS is only represented once; these lines strip from this tentative adjacency list any matches that are duplicative or are not antidominant. If the condition of line \([11]\) is satisfied, then $tail[i, j - 1]$ must be stripped out; in any case, $pretail[i, j]$ can be given the correct value for the case that the adjacency list of $[i, j]$ contains more than just $tail[i, j]$. If the condition of line \([13]\) is satisfied, the tentative value of $tail[i, j]$ must be stripped out; since it was not stripped out when considering position $[i - 1, j]$, $tail[i, j]$ must be in row $i$, and only a single node remains in the adjacency list for $[i, j]$, so that our work is again done. Otherwise, we set the next linked list node of $pretail[i, j]$ to $tail[i, j]$ in line \([15]\), and we set $head[i, j]$ in lines \([14]\) to \([17]\). (The explicit link from $pretail[i, j]$ to $tail[i, j]$ will not be used when processing the adjacency list of $[i, j]$; it may be overwritten when a higher value of $j$ is considered, or it may then become relevant due to $pretail$ advancing along the relevant contour list. As written, the algorithm does not guarantee full construction of all contour lists, but it constructs the portions needed in the adjacency lists of the all-prefixes-LCSs-graph.)

3 Finding All Longest Common Subsequence Embeddings

If we wish to find all LCS embeddings, without duplicating the same embedding, but including multiple embeddings of the same LCS, we can follow an approach similar to that of Section \([2]\) but with some simplification. We can now have each vertex $[i, j]$ in the all-prefixes-LCS-graph point to all matches $[i^*, j^*]$ of the same rank with $i^* \leq i$ and $j^* \leq j$. It is no longer necessary to incorporate $pretail$ information to specify the adjacency lists; rather, each adjacency list is simply an excerpt of a contour list, and each can be specified with a $head$ and $tail$ pointer.

It is now relatively easy to see that this version of the all-prefixes-LCSs-graph can be constructed in $O(mn)$ time as per the algorithm in Figure \([4]\). We now use line \([8]\) to set self-pointers for a match. Then, regardless of whether $[i, j]$ is a match, we merge the information there with information at whichever of $[i - 1, j]$ and $[i, j - 1]$ is of the same rank. In lines \([10]-[11]\) the adjacency list for $[i, j]$ becomes that of $[i - 1, j]$.
for $i = 1$ to $m$ do for $j = 1$ to $n$

Create a linked list node $p[i, j]$ to represent vertex $[i, j]$ in adjacency lists.

Set $\text{head}[i, j]$, $\text{pretail}[i, j]$, and $\text{tail}[i, j]$ to NULL.

Compute $L[i, j]$ as per (1).

if $[i, j]$ is a match then $\text{head}[i, j] \leftarrow p[i, j]$ and $\text{tail}[i, j] \leftarrow p[i, j]$

if $L[i, j] > 0$ then

if $L[i - 1, j] = L[i, j]$ then

if $\text{head}[i, j] = \text{NULL}$ then $\text{head}[i, j] = \text{head}[i - 1, j]$

else Set “next” pointer of $\text{head}[i, j]$ to $\text{head}[i - 1, j]$;

$\text{tail}[i, j] \leftarrow \text{tail}[i - 1, j]$

if $L[i, j - 1] = L[i, j]$ then

if $\text{tail}[i, j]$ is in row $i$ then set “next” pointer of $\text{tail}[i, j - 1]$ to $\text{head}[i, j]$

$\text{head}[i, j] \leftarrow \text{head}[i, j - 1]$

if $\text{tail}[i, j] = \text{NULL}$ then $\text{tail}[i, j] \leftarrow \text{tail}[i, j - 1]$

endfor endfor

Figure 4: The algorithm to create the all-prefixes-LCSs-graph representing all LCS embeddings.

with $[i, j]$ possibly added on front. In lines 12–14, the adjacency list for $[i, j - 1]$ (known to be nonempty due to the test in line 11) is merged onto the front of the adjacency list for $[i, j]$ (taking into account possible overlap).

4 Conclusion

Simple $O(mn)$ algorithms have been presented to produce the all-prefixes-LCS-graph in either the context of representing all distinct LCSs or of representing all LCS embeddings. Once this graph is constructed, we can list all the LCSs (or LCS embeddings) of prefixes $A_i$ and $B_j$ of the two input strings in time proportional to the output size. A C language implementation closely following the presentation given in this paper can be found at http://www.cs.luc.edu/~rig/lcs. Also included is an implementation of the naive backtracing method followed by removal of duplicate LCSs (or duplicate embeddings). Results for many examples are included, with the same final list of LCSs (or LCS embeddings) resulting from the algorithms in this paper or the naive method followed by removal of duplicates.

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