A REMARK ON CONTRACTIBLE BANACH ALGEBRAS OF OPERATORS

MAYSAM MAYSAMI SADR

Abstract. For a Banach algebra $A$, we say that an element $M$ in $A \otimes \gamma A$ is a hyper-commutator if $(a \otimes 1)M = M(1 \otimes a)$ for every $a \in A$. A diagonal for a Banach algebra is a hyper-commutator which its image under diagonal mapping is 1. It is well-known that a Banach algebra is contractible if it has a diagonal. The main aim of this note is to show that for any Banach subalgebra $A \subseteq \mathcal{L}(X)$ of bounded linear operators on infinite-dimensional Banach space $X$, which contains the ideal of finite-rank operators, the image of any hyper-commutator of $A$ under the canonical algebra-morphism $\mathcal{L}(X) \otimes \gamma \mathcal{L}(X) \to \mathcal{L}(X \otimes \gamma X)$, vanishes.

AMS 2020 Subject Classification: Primary 46H20; Secondary 47L10.

Key words: Banach algebra, Contractibility, Diagonal, Amenability.

1. INTRODUCTION

A Banach algebra $A$ is called contractible (super-amenable) \cite{8,10}, if every bounded derivation from $A$ into any Banach $A$-bimodule, is inner. Contractibility is a strong version of the notion of amenability. The concept of amenability (for Banach algebras) has been formulated by Johnson in his seminal paper \cite{4} on Hochschild cohomology of Banach algebras. For various notions of amenability in Theory of Banach Algebras, see \cite{7,8,10}. It is known that any finite-dimensional contractible Banach algebra is a finite direct sum of full matrix algebras \cite[Theorem 4.1.4]{10}. Until now, the only known contractible Banach algebras are of this form. Indeed, it is a longstanding question that whether every contractible Banach algebra is finite-dimensional \cite[p. 224]{8}. Also, the following special case of this question has not been answered yet \cite[8, p. 224]{8}: does for any Banach space $X$, the contractibility of the Banach algebra $\mathcal{L}(X)$ of all bounded linear operators on $X$, imply that $X$ is finite-dimensional? For information on these questions see \cite[§4.1 & p. 196]{10} and \cite{12,5}. We must remark that the chance that there exist infinite-dimensional contractible Banach algebras is not very small: For a long time it was a common belief that for infinite-dimensional Banach spaces $X$, $\mathcal{L}(X)$ can
not be amenable. But, in 2009, Argyros and Haydon \[1\] found out a specific infinite-dimensional Banach space $E$ which its dual is $\ell^1 = E^*$ and has The Scaler-Plus-Compact Property. For such a Banach space $E$, as it has been pointed out by Dales, $\mathcal{L}(E)$ is an amenable Banach algebra; see \[9\].

In this note, we introduce the notion of hyper-commutator for Banach algebras. It is well-known that a Banach algebra is contractible iff it is unital and has a diagonal. By definition, a diagonal of a Banach algebra is a hyper-commutator which its image under the diagonal mapping is 1. The main aim of this note is to prove the following property of hyper-commutators:

For any infinite-dimensional Banach space $X$, and any Banach subalgebra $A$ of $\mathcal{L}(X)$ which contains the ideal of finite-rank operators, the image of any hyper-commutator of $A$, under the canonical algebra-morphism,

$$A \otimes^\gamma A \hookrightarrow \mathcal{L}(X) \otimes^\gamma \mathcal{L}(X) \rightarrow \mathcal{L}(X \otimes^\gamma X),$$

vanishes. For the proof we use the famous Kadec-Snobar’s estimate \[2\] (Theorem 6.28) on operator-norms of projections.

Since our results are mainly concerned about contractibility of $\mathcal{L}(X)$, some known results on contractibility are organized in §2 for contractible central Banach algebras. (So, there is nothing special new in §2.) In §3, we prove our main result and give some new remarks on contractibility of $\mathcal{L}(X)$.

2. SOME KNOWN RESULTS ON CONTRACTIBILITY

For preliminaries on contractibility, we refer the reader to Runde’s books \[8, 10\]. (All results in this section are well-known or are simple variations of the results of \[5, 8, 10, 12\].) The topological dual of a Banach space $X$ is denoted by $X^*$. The completed projective tensor product of Banach spaces $X, Y$ is denoted by $X \otimes^\gamma Y$. The projective norm is denoted by $\| \cdot \|_\gamma$.

The Banach space of bounded linear operators from $X$ into $Y$ is denoted by $\mathcal{L}(X, Y)$. For Banach algebras $A, B$, the Banach space $A \otimes^\gamma B$ is a Banach algebra with the multiplication given by $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$ for $a, a' \in A, b, b' \in B$. The diagonal mapping $\Delta : A \otimes^\gamma A \rightarrow A$ for $A$ is the unique bounded linear operator defined by $a \otimes b \mapsto ab$. A diagonal for a unital Banach algebra $A$ is an element $M \in A \otimes^\gamma A$ satisfying

$$\Delta(M) = 1, \quad (c \otimes 1)M = M(1 \otimes c), \quad (c \in A).$$

It is well-known that a Banach algebra is contractible iff it is unital and has a diagonal: Suppose that $A$ is contractible. Let $E$ denote the Banach $A$-bimodule with the underlying Banach space $A$ and, left and right module-operations $ax := ax$ and $xa := 0$ for $a \in A, x \in E$. Then $id : A \rightarrow E$ is a derivation and hence inner. Thus $A$ has a right unit. Similarly, it is proved that $A$ has a left unit and hence $A$ is unital. Now, consider the derivation $D : A \rightarrow \ker(\Delta)$ defined by $a \mapsto (1 \otimes a) - (a \otimes 1)$. (Note that $A \otimes^\gamma A$ is canonically a Banach $A$-bimodule with module-operations given by $c(a \otimes b) := (c \otimes 1)(a \otimes b)$ and $(a \otimes b)c := (a \otimes b)(1 \otimes c)$, and $\Delta$ is a bimodule-morphism.) $D$ must be
inner and thus there is \( N \in \ker(\Delta) \) with the property \( aN - Na = D(a) \); hence \( M := N + (1 \otimes 1) \) is a diagonal for \( A \). Conversely, suppose that \( M \),

\[
M = \sum_{n=1}^{\infty} a_n \otimes b_n, \quad \sum_{n=1}^{\infty} \|a_n\| \|b_n\| \leq \infty, \quad (a_n, b_n \in A)
\]
is a diagonal for \( A \). If \( D : A \to X \) is a bounded derivation, then it can be checked that for the element \( z := \sum_{n=1}^{\infty} a_n D(b_n) \) of \( X \) we have \( D(a) = az - za \). Thus \( D \) is inner.

**Lemma 2.1.** Let \( A \) be a contractible Banach algebra and let \( E, F \) be unital Banach left \( A \)-modules. Then any diagonal for \( A \) gives rise to a bounded projection \( \Phi = \Phi_{E,F} \) from \( \mathcal{L}(E,F) \) onto \( _A \mathcal{L}(E,F) \).

**Proof.** Let \( M \) be a diagonal for \( A \) of the form (1). For \( T \in \mathcal{L}(E,F) \) let

\[
\Phi(T) : E \to F, \quad x \mapsto \sum_{n=1}^{\infty} a_n T(b_n x), \quad (x \in E).
\]

Then it can be checked that \( \Phi(T) \) is well-defined and belongs to \( _A \mathcal{L}(E,F) \). Also, it is easily verified that \( T \mapsto \Phi(T) \) is a bounded linear projection. \( \square \)

**Proposition 2.2.** Let \( A \) be a contractible Banach algebra. Then any diagonal for \( A \) gives rise to a canonical bounded linear operator \( \Psi : A \to \mathcal{Z}(A) \) with \( \Psi(1) = 1 \).

**Proof.** Consider \( A \) as a Banach left \( A \)-module in the canonical fashion. For any \( c \in A \), let \( \ell_c : A \to A \) denote the left multiplication operator by \( c \). By the notations of Lemma 2.1,

\[
\Phi_{\ell_c} : A \to A, \quad x \mapsto \sum_{n=1}^{\infty} a_n cb_n x
\]
is a left module-morphism and hence there is a \( \tilde{c} \in A \) such that \( \Phi_{\ell_c} = r_{\tilde{c}} \) where \( r_{\tilde{c}} : A \to A \) denotes the right multiplication operator by \( \tilde{c} \). It is clear that \( \tilde{c} = \sum_{n=1}^{\infty} a_n cb_n \) and \( \tilde{c} \in \mathcal{Z}(A) \). We let \( \Psi \) to be defined by \( c \mapsto \tilde{c} \). \( \square \)

The following result is a variation of [5, Proposition 5.1].

**Proposition 2.3.** Let \( A \) be a contractible central Banach algebra. Then any diagonal for \( A \) gives rise to a canonical bounded linear functional \( \psi \in A^* \) with \( \psi(1) = 1 \).

**Proof.** We have \( \mathcal{Z}(A) = C1 \). With the notations of Proposition 2.2, \( \psi \) is defined by

\[
\psi(c) = \sum_{n=1}^{\infty} a_n cb_n = \psi(c)1, \quad (c \in A).
\]

\( \square \)

**Theorem 2.4.** Let \( A \) be a contractible central Banach algebra. Then \( A \) has a unique maximal (two-sided) ideal \( \mathcal{M}_A \).
Proof. Let
\[ M_A := \text{closed linear span of } \{ c \in A : c \text{ belongs to a proper ideal of } A \}. \]
It is clear that \( M_A \) is an ideal of \( A \) which contains every proper ideal of \( A \). With \( \psi \) as in Proposition 2.3, for any \( c \in A \) which is contained in a proper ideal \( J \) of \( A \), we must have \( \psi(c) = 0 \), because otherwise we must have \( 1 = \psi(c)^{-1} \sum_{n=1}^{\infty} a_n c b_n \in J \), a contradiction. Thus \( M_A \subseteq \ker(\psi) \) and hence \( M_A \) is a proper ideal of \( A \). □

A closed linear subspace \( F \) of a Banach space \( E \) is called topologically complemented if there is a closed linear subspace \( F' \) of \( E \) such that \( E = F \oplus F' \). In this case \( F' \) is called a topological complement for \( F \). \( F \) is topologically complemented in \( E \) iff there is a bounded linear projection from \( E \) onto \( F \).

Lemma 2.5. Let \( A \) be a contractible Banach algebra and let \( E \) be a unital Banach left \( A \)-module. Suppose that \( E \) is compactly generated i.e. there exists a norm-compact subset \( K \) of \( E \) such that every \( x \in E \) is of the form \( x = ay \) for some \( a \in A, y \in K \). Suppose that \( E \) has approximation property. Then \( E \) is finite-dimensional.

Proof. Let \( M \) be a diagonal for \( A \) of the form (1). We can suppose that \( \|b_n\| \to 0 \) and \( \sup_{n \geq 1} \|a_n\| < \infty \). Continuity of the module-operation implies that the set \( \cup_{n \geq 1} b_n K \subset E \) is contained in a compact subset of \( E \). Let \( \Phi_{E,E} \) be defined as in the proof of Lemma 2.1. The approximation property for \( E \) means that there exists a net \( (S_\lambda) \) of finite-rank operators in \( \mathcal{L}(E) \) such that \( S_\lambda \to \text{id}_E \) uniformly on compact subsets of \( E \). The above assumptions imply that \( \Phi_{E,E}(S_\lambda) \) is a net of compact operators on \( E \) such that converges uniformly to \( \text{id}_E \) on \( K \). Now, since \( \Phi_{E,E}(S_\lambda) \)'s are module-morphisms and \( K \) generates \( E \), we have \( \Phi_{E,E}(S_\lambda) \to \text{id}_E \) in operator-norm. Thus, \( \text{id}_E \) is a compact operator. Hence, \( E \) is finite-dimensional. □

Note that any Banach space \( X \) considered as a unital Banach left \( \mathcal{L}(X) \)-module in the canonical fashion, is generated by any of its nonzero vectors. Also, for any unital Banach algebra \( A \) and any closed left ideal \( J \) of \( A \), the quotient Banach left \( A \)-module \( A/J \) is generated by the class of 1 in \( A/J \).

Lemma 2.6. Let \( A \) be a contractible Banach algebra and let \( E \) be a unital Banach left \( A \)-module. Suppose that \( F \subset E \) is a closed submodule which is (as a Banach space) topologically complemented in \( E \). Then \( F \) has a topological complement in \( E \) which is also a closed submodule.

Proof. Let \( p \) be a bounded linear projection from \( E \) onto \( F \). By Lemma 2.1, \( \Phi_{E,F}(p) \) is a module-morphism from \( E \) into \( F \). It is easily verified that \( \Phi_{E,F}(p) \) is also a projection from \( E \) onto \( F \). Thus \( \ker(\Phi_{E,F}(p)) \) is the desired complement for \( F \). □

If \( A, A' \) are contractible Banach algebras with diagonals \( M, M' \) of the forms as in (1), then \( \sum_{n,m=1}^{\infty} a_n \otimes a'_m \otimes b_n \otimes b'_m \) is a diagonal for \( A \otimes A' \). Also,
\[ \sum_{n=1}^{\infty} b_n \otimes a_n \] is a diagonal for \( A^{\text{op}} \), the opposite algebra of \( A \). Thus if \( A \) is contractible then \( A \otimes \gamma A^{\text{op}} \) is contractible.

**Lemma 2.7.** The analogue of Lemma 2.6 is satisfied for bimodules: Let \( A \) be a contractible Banach algebra and \( E \) a unital Banach \( A \)-bimodule. If \( F \) is a closed sub-bimodule of \( E \) which is topologically complemented, then it has a complement in \( E \) which is also a sub-bimodule.

**Proof.** Any unital Banach \( A \)-bimodule \( E \) may be considered as unital Banach left \( A \otimes \gamma A^{\text{op}} \)-module with module operation given by \((a \otimes b) x := axb \) \((a \in A, b \in A^{\text{op}}, x \in E)\). In this fashion, any \( A \)-bimodule-morphism is a left \( A \otimes \gamma A^{\text{op}} \)-module-morphism. The converses of this facts are also satisfied. Now, the desired result follows from Lemma 2.6. \( \square \)

**Theorem 2.8.** Let \( A \) be a contractible central Banach algebra. Suppose that \( J \) is a closed, proper, and nonzero ideal of \( A \). (Note that the existence of \( J \) implies that \( A \) is infinite-dimensional. Indeed, it follows from [10, Theorem 4.1.2] that any contractible central Banach algebra of finite dimension is isomorphic to a full matrix algebra.) The following statements hold.

(i) \( J \) is not topologically complemented in \( A \).

(ii) \( A/J \) has not approximation property.

(iii) If \( J \) is compactly generated as left (resp. right) \( A \)-module, then \( J \) has not approximation property.

**Proof.** (i): If \( J \) is topologically complemented in \( A \), then by Lemma 2.7 there is a closed, proper, and nonzero ideal \( J' \) such that \( A = J \oplus J' \). Thus we have \( J, J' \subseteq M_A \), a contradiction. (Note that (i) may be concluded from centrality of \( A \). Indeed, if \( A = J \oplus J' \), then there exist orthogonal nonzero central idempotents \( e \in J, e' \in J' \) with \( e + e' = 1 \).) (ii): If \( A/J \) has approximation property, then by Lemma 2.5, \( A/J \) is finite-dimensional and hence \( J \) is topologically complemented in \( A \), a contradiction with (i). (iii) follows from Lemma 2.5 similarly. \( \square \)

The following corollary follows from the above results.

**Corollary 2.9.** Let \( X \) be an infinite-dimensional Banach space. If \( \mathcal{L}(X) \) is contractible then, (i) \( X \) has not approximation property; (ii) \( \mathcal{L}(X) \) has a unique maximal ideal \( M \); (iii) \( M \) is not topologically complemented; and (iv) \( \mathcal{L}(X)/M \) has not approximation property.

A contractible Banach algebra \( A \) is called symmetrically contractible if \( A \) has a symmetric diagonal; that is, a diagonal \( M \) satisfying \( \mathcal{F}_A(M) = M \) where \( \mathcal{F}_A : A \otimes \gamma A \to A \otimes \gamma A \) denotes flip i.e., the unique bounded linear mapping defined by \((a \otimes b) \mapsto (b \otimes a)\). The matrix algebra \( M_n \) is symmetrically contractible. Indeed, it is well-known that \( M_n \) has the unique diagonal \( n^{-1} \sum_{i,j=1}^{n} \delta_{ij} \otimes \delta_{ji} \) where \( \delta_{ij} \)'s denote the standard basis of \( M_n \). Thus any finite-dimensional contractible Banach algebra is symmetrically contractible.

The following results are variations of [5, Proposition 5.3].
THEOREM 2.10. Let $A$ be a symmetrically contractible Banach algebra. Then any symmetric diagonal of $A$ gives rise to a bounded normalized $Z(A)$-valued trace for $A$. If $A$ is central, then $A$ has a normalized trace $\psi \in A^*$.

PROOF. Let $M$ be a symmetric diagonal for $A$ of the form $(1)$. We saw in Proposition 2.2 that the assignment $c \mapsto \sum_{n=1}^{\infty} a_n c b_n$ defines a bounded linear mapping $\Psi : A \rightarrow Z(A)$ with $\Psi(1) = 1$. For every $c, c' \in A$ we have $\sum_{n=1}^{\infty} b_n \otimes a_n c c' = \sum_{n=1}^{\infty} c b_n \otimes a_n c' c$ and hence $\sum_{n=1}^{\infty} a_n c c' \otimes b_n = \sum_{n=1}^{\infty} a_n c' \otimes c b_n$. Thus we have

$$\Psi(c c') = \Delta \left( \sum_{n=1}^{\infty} a_n c c' \otimes b_n \right) = \Delta \left( \sum_{n=1}^{\infty} a_n c' \otimes c b_n \right) = \Psi(c' c).$$

$\psi$ is given as in Proposition 2.3. \hfill $\square$

For the matrix algebra $M_n$, the unique diagonal of $M_n$ gives rise to the ordinary trace.

3. A NULL-PROPERTY OF DIAGONALS

Let $X$ be a Banach space. Consider the unique bounded linear operator

$$\Upsilon : \mathcal{L}(X) \otimes \gamma \mathcal{L}(X) \rightarrow \mathcal{L}(X \otimes \gamma X),$$

defined by

$$[\Upsilon(T \otimes S)](x \otimes y) = T(x) \otimes S(y), \quad (T, S \in \mathcal{L}(X), x, y \in X).$$

Then $\Upsilon$ is an algebra-morphism between Banach algebras. We denote the image under $\Upsilon$ of any element $N \in \mathcal{L}(X) \otimes \gamma \mathcal{L}(X)$, by $N^{\text{op}}$. It follows from properties of projective tensor product, that $\|\Upsilon\| = 1$ and hence $\|N^{\text{op}}\| \leq \|N\|_{\gamma}$. Note that, in general, $\Upsilon$ is not one-to-one. (This fact can be concluded from the fact that the canonical mapping from $X^* \otimes \gamma X^*$ onto the space of nuclear bilinear functionals on $X \times X$ is not necessarily one-to-one [III §2.6].)

PROPOSITION 3.1. Let $\Lambda \in \mathcal{L}(X \otimes \gamma X)$ be such that for every one-rank operator $T \in \mathcal{L}(X)$,

$$(T \otimes 1)^{\text{op}} \Lambda = \Lambda (1 \otimes T)^{\text{op}}.$$

Then there is a unique operator $\Gamma$ in $\mathcal{L}(X)$ such that $\Lambda = (1 \otimes \Gamma)^{\text{op}} F_X$.

PROOF. Let $y$ be a nonzero vector in $X$, and let $f \in X^*$ be such that $f(y) = 1$. Let $T \in \mathcal{L}(X)$ to be defined by $x \mapsto f(x)y$. For $x \in X$ we have

$$\begin{align*}
(T \otimes 1)^{\text{op}} \Lambda(x \otimes y) &= \Lambda(x \otimes y).
\end{align*}$$

$X$ has the decomposition $\langle y \rangle \supset \ker(f)$ where $\langle y \rangle$ denotes the subspace generated by $y$. There exist $z \in \ker(f) \otimes \gamma X$ and $w \in X$ such that

$$\Lambda(x \otimes y) = y \otimes w + z.$$

It follows from $(2)$ that $\Lambda(x \otimes y) = y \otimes w$. Since the mapping $x \mapsto \Lambda(x \otimes y)$ is linear and bounded, there is $\Gamma_y \in \mathcal{L}(X)$ such that $\Lambda(x \otimes y) = y \otimes \Gamma_y(x)$. Now, suppose that $y, y' \in X$ are linearly independent. We have

$$\Lambda(x \otimes (y + y')) = y \otimes \Gamma_y(x) + y' \otimes \Gamma_{y'}(x),$$

where...
Suppose that $\Lambda(x \otimes (y + y')) = (y + y') \otimes \Gamma_{y+y'}(x)$. Thus $\Gamma_y = \Gamma_{y'}$. Also, it can be checked that for every nonzero scalar $\lambda$ we have $\Gamma_{\lambda y} = \lambda \Gamma_y$. Thus there exists $\Gamma \in \mathcal{L}(X)$ such that $\Lambda(x \otimes y) = y \otimes \Gamma(x)$ for every $x, y \in X$. The proof is complete. \hfill \Box

COROLLARY 3.2. Let $M$ be an element of $\mathcal{L}(X) \otimes \gamma \mathcal{L}(X)$ that satisfies
\begin{equation}
(T \otimes 1)M = M(1 \otimes T), \quad (T \in \mathcal{L}(X) \text{ of rank one}).
\end{equation}
Then there exists $\Gamma \in \mathcal{L}(X)$ such that $M^\text{op} = (1 \otimes \Gamma)^\text{op} \mathcal{F}_X$. Moreover, if $M$ is symmetric (i.e. $\mathcal{F}_{\mathcal{L}(X)}(M) = M$) then there exists a scalar $\lambda$ such that $M^\text{op} = \lambda \mathcal{F}_X$.

PROOF. The first part follows directly from Proposition 3.1. Suppose that $M$ is symmetric. It follows from the identity $[\mathcal{F}_{\mathcal{L}(X)}(M)]^\text{op} = \mathcal{F}_X M^\text{op} \mathcal{F}_X$, that
$$\mathcal{F}_X (1 \otimes \Gamma)^\text{op} = (1 \otimes \Gamma)^\text{op} \mathcal{F}_X.$$ Thus for every $x, y \in X$ we have $\Gamma(y) \otimes x = y \otimes \Gamma(x)$. This means that $\Gamma$ is a scalar multiple of identity. The proof is complete. \hfill \Box

Let $Y, Y', Z$ be finite-dimensional Banach spaces. Similar to the mapping $\Upsilon$ above, we denote by $\Upsilon : N \mapsto N^\text{op}$ the unique bounded linear mapping
$$\mathcal{L}(Y, Z) \otimes \gamma \mathcal{L}(Z, Y') \to \mathcal{L}(Y \otimes \gamma Z, Z \otimes \gamma Y'),$$
given by
$$(T \otimes S)^\text{op}(y \otimes z) = (T(y) \otimes S(z)).$$
We know that this is a linear isomorphism.

LEMMA 3.3. With the above assumptions, suppose that $\dim(Y) = \dim(Y')$. Suppose that $T : Y \to Y'$ is a linear isomorphism. For every finite-dimensional Banach space $Z$, let the linear mapping $\tilde{T}_Z$ be given by
$$\tilde{T}_Z : Y \otimes \gamma Z \to Z \otimes \gamma Y', \quad (y \otimes z) \mapsto (z \otimes T(y)).$$
There is a numerical positive constant $c$ such that $c$ is independent from $Z$ (independent from norm and dimension of $Z$) and such that:
$$\|\Upsilon^{-1}(\tilde{T}_Z)\|_\gamma \geq c^{-1} \dim(Z).$$

PROOF. Suppose that $y_1, \ldots, y_k$ and $z_1, \ldots, z_m$ are vector basis respectively for $Y$ and $Z$, and let $y'_i = T(y_i)$. Let the linear operators
$$S_{ij} : Y \to Z, \quad S'_{ji} : Z \to Y', \quad (1 \leq i \leq k, 1 \leq j \leq m)$$
be given by
$$S_{ij}(y_i) = z_j, S_{ij}(y_q) = 0, \quad (q \neq i), \quad S'_{ji}(z_j) = y'_i, S'_{ji}(z_q) = 0, \quad (q \neq j).$$
Let $N := S_{ij} \otimes S'_{ji}$. Then $N^\text{op} = \tilde{T}_Z$ and hence $\Upsilon^{-1}(\tilde{T}_Z) = N$.

Let $\nu$ denote the linear functional on $\mathcal{L}(Y, Y')$ that associates to any operator $Y \to Y'$, the normalized trace of its matrix in the bases $y_1, \ldots, y_k$ and
\[ y_1', \ldots, y_k', \text{ of } Y \text{ and } Y'. \] Suppose that \( c \) denotes the functional-norm of \( \nu \). It is clear that \( c \neq 0 \). Consider the bilinear functional
\[
\mu : \mathcal{L}(Y, Z) \times \mathcal{L}(Z, Y') \to \mathbb{C}, \quad (P, Q) \mapsto \nu(QP).
\]
Then we have \( \|\mu\| \leq c \) and hence \( \|c^{-1}\mu\| \leq 1 \). Now, it follows from the properties of projective tensor-norm that
\[
\|N\|_\gamma \geq |c^{-1}\mu(N)| = c^{-1}m.
\]

**Proposition 3.4.** Let \( X \) be an infinite-dimensional Banach space. Let \( M \in \mathcal{L}(X) \otimes^\gamma \mathcal{L}(X) \) be an element that satisfies (3). Then \( M \in \ker(\Upsilon) \).

**Proof.** Suppose that \( \Gamma \in \mathcal{L}(X) \) is as in Corollary 3.2. Suppose that \( M^{\text{op}} \neq 0 \) and hence \( \Gamma \neq 0 \). Let \( y, y' \) be two nonzero vectors in \( X \) such that \( \Gamma(y) = y' \). Suppose that \( Y, Y' \) denote the one-dimensional subspaces of \( X \) generated respectively by \( y, y' \), and suppose that \( T : Y \to Y' \) is defined by \( T(y) = y' \). Let \( Z \) be an arbitrary finite-dimensional subspace of \( X \). Suppose that \( E_Y : Y \to X \) and \( E_Z : Z \to X \) denote the embedding-maps and \( P_{Y'} : X \to Y' \) is an arbitrary continuous projection from \( X \) onto \( Y' \). By Kadec-Snobar’s Theorem \([2], \text{Theorem 6.28}\) we know that there exists a continuous projection \( P_Z : X \to Z \), from \( X \) onto \( Z \), such that \( \|P_Z\| < 1 + \sqrt{\dim(Z)} \). Let
\[
N := (P_Z \otimes P_{Y'})M(E_Y \otimes E_Z) \in \mathcal{L}(Y, Z) \otimes^\gamma \mathcal{L}(Z, Y').
\]
We have
\[
\|N\|_\gamma \leq \|P_Z\|\|P_{Y'}\|\|M\|_\gamma, \quad N^{\text{op}} = \hat{T}_Z,
\]
where \( \hat{T}_Z \) is as in Lemma 3.3. Now, by Lemma 3.3 we have
\[
\frac{\dim(Z)}{c\|P_{Y'}\|(1 + \sqrt{\dim(Z)})} < \|M\|_\gamma.
\]
This implies that \( \|M\|_\gamma = \infty \), a contradiction. Thus, we have \( M^{\text{op}} = 0 \). \( \square \)

**Definition 3.5.** Let \( A \) be a Banach algebra and let \( M \in A \otimes^\gamma A \). We say that \( M \) is a hyper-commutator for \( A \) if
\[
aM = Ma \quad (a \in A).
\]
By definition, diagonals of contractible Banach algebras are hyper-commutator elements. Following the discussion of \([11]\) the next question is very natural.

**Question 3.6.** Does there exist an infinite-dimensional Banach algebra with a nonzero hyper-commutator? 

The next theorem which is the main result of this note, establishes a null-property of hyper-commutators.
THEOREM 3.7. Let $X$ be an infinite-dimensional Banach space. Let $A \subseteq \mathcal{L}(X)$ be a Banach subalgebra such that contains the ideal of finite-rank operators. Then the image of any hyper-commutator of $A$ under the canonical algebra-morphism

$$A \otimes \gamma A \rightarrow \mathcal{L}(X) \otimes \gamma \mathcal{L}(X),$$

vanishes.

PROOF. It follows directly from Proposition 3.4. □

Note that for any Banach algebra $A$ as in Theorem 3.7, we have

$$Z(A) = 0 \quad \text{or} \quad Z(A) = 1 \mathbb{C}.$$

Remark 3.8. Suppose that $\mathcal{L}(X)$ is contractible. By Theorem 3.7, to prove that $X$ is finite-dimensional, it is enough to prove that at least one of the diagonals of $\mathcal{L}(X)$ is invertible as a member of the Banach algebra $\mathcal{L}(X) \otimes \gamma \mathcal{L}(X)$. Note that for the unique diagonal $M$ of $M_n$ we have $n^2M^2 = 1 \otimes 1$ in the Banach algebra $M_n \otimes \gamma M_n$.

Remark 3.9. Suppose that $X$ is an infinite dimensional Banach space for which the canonical mapping $\Upsilon$ is one-to-one. Then, by Theorem 3.7 any Banach subalgebra of $\mathcal{L}(X)$ containing the ideal of finite-rank operators, is not contractible.

Remark 3.10. Let $X$ be an infinite dimensional Banach space. If $A = \mathcal{L}(X)$ has at least two maximal ideals, then by Corollary 2.9 we know that $A$ is not contractible. (See [6] for some examples of such Banach spaces.) Suppose that $A$ has only one maximal ideal $J$. To prove that $A$ is not contractible it is enough to show that the closer $\tilde{J}$ of the ideal $(J \otimes A) + (A \otimes J) \subseteq A \otimes \gamma A$ is a maximal ideal of $A \otimes \gamma A$: Indeed, if $A$ is contractible then $A \otimes \gamma A$ is contractible and since $A \otimes \gamma A$ is central (this fact can be checked by considering projections onto finite-dimensional subspaces of $X$ similar to the first part of the proof of Lemma 3.3) then it must have a unique maximal ideal. Thus we must have $\ker(\Upsilon) \subseteq \tilde{J}$ and hence for any diagonal $M$ of $A$, $M$ belongs to $\tilde{J}$. Therefore, we have $1 = \Delta(M) \in \Delta(\tilde{J}) \subseteq J$ that contradicts properness of $J$.

References

[1] S.A. Argyros, R.G. Haydon, A hereditarily indecomposable $L_\infty$-space that solves the scalar-plus-compact problem, Acta Math. 206 (2011), 1–54. (arXiv:0903.3921 [math.FA])

[2] M. Fabian, P. Habala, P. Hajek, V. Montesinos, V. Zizler, Banach Space Theory: The Basis for Linear and Nonlinear Analysis, CMS Books in Mathematics, Springer, New York, 2011.

[3] N. Gronbaek, Various notions of amenability, a survey of problems, In Proceedings of 13th International Conference on Banach Algebras in Blaubeuren, pp. 535–547, Walter de Gruyter, Berlin 1998.

[4] B.E. Johnson, Cohomology in Banach algebras, Mem. Amer. Math. Soc. 127 (1972), 1–96.

[5] B.E. Johnson, Symmetric amenability and the nonexistence of Lie and Jordan derivations, Math. Proc. Cambridge Phil. Soc. 120 (1996), 455–473.
[6] N.J. Laustsen, *Maximal ideals in the algebra of operators on certain Banach spaces*, Proc. Edinb. Math. Soc. **45** (2002), 523–546.

[7] O.T. Mewomo, *Various notions of amenability in Banach algebras*, Expo. Math. **29** (2011), 283–299.

[8] V. Runde, *Lectures on Amenability*, Springer-Verlag, Berlin, Heidelberg, 2002.

[9] V. Runde, *Non-amenability of $B(E)$*, Banach Center Publ. **91** (2010), 339–351. ([arXiv:0909.2828 [math.FA]])

[10] V. Runde, *Amenable Banach Algebras*, Springer Monographs in Mathematics, Science+Business Media, Berlin, 2020.

[11] R.A. Ryan, *Introduction to Tensor Products of Banach Spaces*, Springer Monographs in Mathematics, Springer-Verlag, London, 2002.

[12] N. Ozawa, *A remark on contractible Banach algebras*, Kyushu J. Math. **67** (2013), 51–53. ([arXiv:1110.6216 [math.FA]])

Maysam Maysami Sadr  
Institute for Advanced Studies in Basic Sciences  
Department of Mathematics  
Zanjan 45137-66731, Iran  
sadr@iasbs.ac.ir