Stability analysis and stabilization of LPV systems with jumps and (piecewise) differentiable parameters using continuous and sampled-data controllers

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Abstract
Linear Parameter-Varying (LPV) systems with jumps and piecewise differentiable parameters is a class of hybrid LPV systems for which no tailored stability analysis and stabilization conditions have been obtained so far. We fill this gap here by proposing an approach relying on the reformulation of the considered LPV system as an extended equivalent hybrid system that will incorporate, through a suitable state augmentation, information on both the dynamics of the state of the system and the considered class of parameter trajectories. Two stability conditions are established using a result pertaining on the stability of hybrid systems and shown to naturally generalize and unify the well-known quadratic and robust stability criteria together. The obtained conditions being infinite-dimensional semidefinite programming problems, a relaxation approach based on sum of squares programming is used in order to obtain tractable finite-dimensional conditions. The conditions are then losslessly extended to solve two control problems, namely, the stabilization by continuous and sampled-data gain-scheduled state-feedback controllers. The approach is finally illustrated on several examples from the literature.

Keywords. LPV systems; hybrid systems; dwell-time; sum of squares

1 Introduction
Linear Parameter-Varying (LPV) systems \[1,2\] are an important class of linear systems that can be used to model linear systems that intrinsically depend on parameters \[3\] or to approximate nonlinear systems in the objective of designing gain-scheduled controllers \[4\]; see \[2\] for a classification attempt. They have been, since then, successfully applied to a wide

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variety of real-world systems such as automotive suspensions systems [5], aircrafts [6,7], etc. The field has also been enriched with very broad theoretical results and numerical tools [2,8,17].

As discussed in [2,16], LPV systems are often separated into two categories depending on whether the parameter trajectories are continuously differentiable or vary arbitrarily fast (possibly including discontinuities) - a very strong dichotomy as those classes of trajectories are quite far apart. As a consequence, this classification seems rather unadapted to parameter trajectories admitting sporadic discontinuities while being differentiable in between. Indeed, such trajectories cannot be considered as differentiable but, at the same time, considering them to be arbitrarily varying is also very restrictive since we would ignore the fact that, in between discontinuities, the parameter trajectories are differentiable. This remark motivated the consideration of LPV systems with piecewise constant parameters in [16] in order to illustrate the fact that considering a more accurate description of the parameter trajectories can, in fact, predict the stability of certain systems for which other methods would have proven inconclusive. As the parameters were piecewise constant, it was possible to readily adapt methods developed for switched/impulsive systems [18–20] to obtain sufficient stability conditions for such LPV systems through the introduction of dwell-time constraints.

The objective of the paper (and that of its conference version [21]) is to extend those results to the case of impulsive LPV systems with piecewise differentiable parameters. Such parameter trajectories may arise when an (impulsive) LPV system is used to approximate a nonlinear impulsive system or in linear systems where the parameters naturally have such a behavior [22]. Finally, they can also be used to approximate parameter trajectories that exhibit intermittent very fast, yet differentiable, variations. Despite being similar to the case of piecewise constant parameters, the fact that the parameters are time-varying between discontinuities leads to additional difficulties which prevent from straightforwardly extending the approach developed in [16]. To overcome this, we propose a different, more general, approach based on hybrid systems theory [23]. Firstly, we equivalently reformulate the considered LPV system into a hybrid system that captures in its formulation both the dynamics of the state of the system and the considered class of parameter trajectories. Recent results from hybrid systems theory [23] combined with the use of quadratic Lyapunov functions are then applied in order to derive sufficient stability conditions for the considered hybrid systems and, hence, for the associated LPV system and class of parameter trajectories. We then prove that the obtained stability conditions generalize and unify the well-known quadratic stability and robust stability conditions, which can be recovered as extremal/partial cases. The obtained stability conditions stated in terms of infinite-dimensional semidefinite programming problems are reminiscent of those obtained in [18–20], a fact that strongly suggests the connection between those approaches. To make these conditions computationally verifiable, we propose a relaxation approach based on sum of squares programming [24–26]; i.e. semidefinite programming [27,28]. The analysis part is validated on two relevant systems from the literature, one of them being known to be not quadratically stable.

Unlike in the conference version [21] of this paper, we extend here the approach to the
case of LPV systems with jumps and to the design of continuous-time and sampled-data gain-scheduled state-feedback controllers. The problem of designing continuous-time controllers for LPV systems with piecewise differentiable parameters has never been addressed until now. On the other hand, the problem of designing sampled-data gain-scheduled state-feedback controllers for standard LPV systems has been solved in [29] using a discretization approach (assuming the parameters are piecewise constant), in [30] using the input-delay approach and in [31] using looped-functionals. The advantage of the proposed approach is that it relies on an exact representation of the problem into an impulsive system and the use of clock-dependent Lyapunov functions which have been shown to be a suitable approach for the design of controllers for linear hybrid systems or their analysis when subject to uncertainties; see e.g. [18,32–35] and references therein. Unlike functionals, this class of Lyapunov functions naturally leads to convex design conditions; see e.g. the discussion in [33].

Outline. The structure of the paper is as follows: in Section 2 preliminary definitions and results are given. Section 3 develops the main stability results of the paper which are then extended to continuous and sampled-data state-feedback control in Section 4 and Section 5, respectively. The examples are treated in the related sections.

Notations. The set of nonnegative and positive integers are denoted by $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_{> 0}$, respectively. The set of symmetric matrices of dimension $n$ is denoted by $\mathbb{S}^n$ while the cone of positive (semi)definite matrices of dimension $n$ is denoted by $(\mathbb{S}^n_{\geq 0}) \mathbb{S}^n_{> 0}$. For some $A, B \in \mathbb{S}^n$, the notation that $A > (\succeq) B$ means that $A - B$ is positive (semi)definite. The maximum and the minimum eigenvalue of a symmetric matrix $A$ are denoted by $\lambda_{\text{max}}(A)$ and $\lambda_{\text{min}}(A)$, respectively. For a square matrix $A$, we define $\text{Sym}[A] = A + A^T$. Given a vector $v \in \mathbb{R}^d$ and a closed set $A \subset \mathbb{R}^d$, the distance of $v$ to the set $A$ is denoted by $|v|_A$ and is defined by $|v|_A := \inf_{y \in A} |v - y|$. For any differentiable function $f(x, y)$, the partial derivatives with respect to the first and second argument evaluated at $(x, y) = (x^*, y^*)$ are denoted by $\partial_x f(x^*, y^*)$ and $\partial_y f(x^*, y^*)$, respectively.

2 Preliminaries

2.1 Hybrid systems

Let us consider here the hybrid system $\mathcal{H} = (F, G, C, D, \mathbb{R}^n)$ given by

\[
\begin{align*}
\dot{\chi}(t) & \in F(\chi(t)) \text{ if } \chi(t) \in C \\
\chi(t^+) & \in G(\chi(t)) \text{ if } \chi(t) \in D
\end{align*}
\] (1)

where $\chi(t) \in \mathbb{R}^d$, $C \subset \mathbb{R}^d$ is closed, $D \subset \mathbb{R}^d$ is closed and $G(D) \subset C$. The flow map and the jump map are the set-valued maps $F : C \rightrightarrows \mathbb{R}^n$ and $G : D \rightrightarrows C$, respectively, which are assumed to satisfy the hybrid basic conditions; see [23, Assumption 6.5]. When considering the usual continuous-time domain, the trajectories of the above system are left-continuous with right-handed limit $\chi(t^+) = \lim_{s \downarrow t} \chi(s)$. Let a hybrid arc $\phi(t,j)$ be a solution of the
system $\mathcal{H}$, $\phi(0,0) = \phi_0$ be the initial condition and $\text{dom} \phi \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ be the hybrid time domain of $\phi(t,j)$ where $t$ is the continuous-time and $j$ is a counter for the number of jumps that has occurred until time $t$. It is assumed here that the solutions are complete (i.e. $\text{dom} \phi$ is unbounded). For more details about hybrid time domains and solutions of hybrid systems see [23].

### 2.2 LPV systems with jumps

LPV systems with jumps are dynamical systems which can be described as

\[
\begin{align*}
\dot{x}(t) &= A(\rho(t))x(t), \quad t \neq t_k \\
x(t_k^+) &= J(\rho(t_k))x(t_k), \quad k \in \mathbb{Z}_{>0} \\
x(0) &= x_0, \quad t_0 = 0
\end{align*}
\]

where $x, x_0 \in \mathbb{R}^n$ are the state of the system and the initial condition, respectively. The matrix-valued functions $A(\cdot), J(\cdot) \in \mathbb{R}^{n \times n}$ are assumed to be continuous. The parameter vector trajectory $\rho : \mathbb{R}_{\geq 0} \to \mathcal{P} \subset \mathbb{R}^N$, $\mathcal{P}$ compact and connected, is assumed to be piecewise differentiable with derivative in $\mathcal{D} \subset \mathbb{R}^N$, where $\mathcal{D}$ is also compact and connected. When the parameters are independent of each other, then we can assume that $\mathcal{P}$ is a box and that the following decompositions hold

$\mathcal{P} =: \mathcal{P}_1 \times \ldots \times \mathcal{P}_N$ where $\mathcal{P}_i := [\bar{\rho}_i, \underline{\rho}_i], \bar{\rho}_i \leq \underline{\rho}_i$ and $\mathcal{D} =: \mathcal{D}_1 \times \ldots \times \mathcal{D}_N$ where $\mathcal{D}_i := [\nu_i, \bar{\nu}_i], \nu_i \leq \bar{\nu}_i$. We also define the set of vertices of $\mathcal{D}$ as $\mathcal{D}^v$; i.e. $\mathcal{D}^v := \{\nu_1, \bar{\nu}_1\} \times \ldots \times \{\nu_N, \bar{\nu}_N\}$. Note, however, that these sets are only considered here to simplify the exposition of the results but, as it will be shown in the examples, other sets can also be considered.

### 3 Stability analysis

The objective of this section is to present the main stability analysis result of the paper. Section 3.1 presents the main result pertaining on the stability analysis under a minimum dwell-time condition. This result is then connected to existing ones in Section 3.2. Finally, some computational discussions are provided in Section 3.3.

#### 3.1 Stability under minimum dwell-time

Let us consider now the family of piecewise differentiable parameter trajectories given by

\[
\mathcal{R}^\mathcal{P} := \left\{ \rho : \mathbb{R}_{\geq 0} \mapsto \mathcal{P} \left| \begin{array}{l}
\dot{\rho}(t) \in \mathcal{Q}(\rho(t)), t \in [t_k, t_{k+1}) \\
T_k \geq \bar{T}, \rho(t_k) \neq \rho(t_k^+), k \in \mathbb{Z}_{\geq 0}
\end{array} \right. \right\}
\]
where $T_k := t_{k+1} - t_k$, $\bar{T} > 0$ and $t_0 = 0$ (we again assume that no discontinuity can occur at $t_0$). The corresponding hybrid system is given by

$$
\begin{align*}
\begin{cases}
\dot{x}(t) = A(\rho(t))x(t) \\
\dot{\rho}(t) \in Q(\rho(t)) \\
\dot{\tau}(t) = 1 \\
\dot{T}(t) = 0 \\
x(t^+) = J(\rho(t))x(t) \\
\rho(t^+) \in \mathcal{P} \\
\tau(t^+) = 0 \\
T(t^+) \in [\bar{T}, \infty)
\end{cases}
\end{align*}
$$

where

$$
C = \mathbb{R}^n \times \mathcal{P} \times E_1, \quad D = \mathbb{R}^n \times \mathcal{P} \times E_2
$$

$$
\begin{align*}
E_1 &= \{ \varphi \in \mathbb{R}_{\geq 0} \times [\bar{T}, \infty) : \varphi_1 \leq \varphi_2 \}, \\
E_2 &= \{ \varphi \in \mathbb{R}_{\geq 0} \times [\bar{T}, \infty) : \varphi_1 = \varphi_2 \}.
\end{align*}
$$

The above system contains the unconventional dwell-time state $T$ as opposed to the model in [23] where $\dot{\tau} \in (0, 1]$ for all $\tau \in [0, \bar{T}]$ and $\tau^+ = 0$ when $\tau = \bar{T}$, or to a modification of the model in [36] where $\dot{\tau} = -1$ for all $\tau \in \mathbb{R}_{\geq 0}$ and $\tau^+ \in [\bar{T}, \infty)$ when $\tau = 0$. While for stability analysis all these models are equivalent, albeit the proposed one is more complex, they cease to be so when one wants to design controllers depending on the timer variable. In this case, the timer variable should be accessible to the controller and should then be computable from known past or current information such as the current time and the time of the last jump. This is the case in the proposed model where the timer $\tau$ can be directly computed using the formula $\tau = t - t_k$ for $t \in [t_k, t_{k+1}]$. On the other hand, the timer $\tau$ cannot be computed from known past or current information in the other formulations since we would need to know the value of the current dwell-time or, equivalently, the value of the next jump instant, which is not directly accessible.

Additionally, the use of this extra state allows one to avoid the use of time-varying flow and jump sets. Indeed, removing that state would require the use of the flow set $C_j = \mathbb{R}^n \times \mathcal{P} \times [0, T_j]$ and the jump set $D_j = \mathbb{R}^n \times \mathcal{P} \times \{T_j\}$, where $j$ is the current value of the discrete time. It will shown later that the presence of this extra state does not yield any increase of complexity in the analysis conditions.

This leads to the following result:

**Theorem 1 (Minimum dwell-time)** Let $\bar{T} \in \mathbb{R}_{>0}$ be given and assume that there exist a bounded continuously differentiable matrix-valued function $S : [0, \bar{T}] \times \mathcal{P} \mapsto S_{\geq 0}^n$ and a scalar $\varepsilon > 0$ such that the conditions

$$
\begin{align*}
\partial_\rho S(\bar{T}, \theta)\mu + \text{Sym}[S(\bar{T}, \theta)A(\theta)] + \varepsilon I &\preceq 0 \\
\partial_\tau S(\bar{T}, \theta) + \partial_\rho S(\bar{T}, \theta)\mu + \text{Sym}[S(\bar{T}, \theta)A(\theta)] + \varepsilon I &\preceq 0
\end{align*}
$$

and

$$
J(\eta)^T S(0, \theta) J(\eta) - S(\bar{T}, \eta) \preceq 0
$$

are satisfied.
hold for all \( \theta, \eta \in \mathcal{P} \), \( \mu \in \mathcal{D}^v \) and all \( \tau \in [0, \bar{T}] \). Then, the LPV system (2) with parameter trajectories in \( \mathcal{P}_{\tau} \) is asymptotically stable.

Proof: The main tool here is Proposition 3.27 in [23] on the stability of hybrid systems under persistent flowing. Let us first define the (closed) set \( \mathcal{A} = \{0\} \times \mathcal{P} \times E_1 \) and note that the LPV system with parameter trajectories in \( \mathcal{P}_{\tau} \) is asymptotically stable if and only if \( \mathcal{A} \) is asymptotically stable for the system (4). We prove now that for the choice of the Lyapunov function

\[
V(x, \tau, \rho) = \begin{cases} 
    x^T S(\tau, \rho)x & \text{if } \tau \leq \bar{T}, \\
    x^T S(\bar{T}, \rho)x & \text{if } \tau > \bar{T},
\end{cases}
\]

(9)

where \( S(\tau, \rho) > 0 \) for all \( \tau \in [0, \bar{T}] \) and all \( \rho \in \mathcal{P} \), the feasibility of the conditions of Theorem 1 implies the feasibility of those in Proposition 3.27 in [23].

Since \( S(\tau, \rho) > 0 \) for all \( \tau \in [0, T_{\max}] \) and all \( \rho \in \mathcal{P} \), then \( V(x, \tau, \rho) = 0 \) if and only if \( x = 0 \) and, hence, the condition (3.7a) in [23] are verified with the functions

\[
\alpha_1((x, \tau, \rho)|_A) = \min_{(\tau, \rho) \in [0, \bar{T}] \times \mathcal{P}} \lambda_{\min}(S(\tau, \rho)) ||x||_2^2 \\
\alpha_2((x, \tau, \rho)|_A) = \max_{(\tau, \rho) \in [0, \bar{T}] \times \mathcal{P}} \lambda_{\max}(S(\tau, \rho)) ||x||_2^2
\]

(10)

are valid \( \mathcal{K}_\infty \) functions (note that \( ((x, \tau, \rho)|_A = ||x|| \) for all \( (x, \rho, \tau) \in C \cup D \)).

To prove that the feasibility of the condition (6) and (7) imply that of the condition (3.7b) in [23], let \( \Psi_2(\bar{T}, \rho, \mu) \) and \( \Psi_1(\tau, \rho, \mu) \) be the matrices on the left-hand side of (6) and (7) when \( \varepsilon = 0 \), respectively. Using the linearity in \( \mu \), it is immediate to get that \( \Psi_1(\tau, \rho, \mu) + \varepsilon I_n \preceq 0 \) for all \( (\tau, \rho, \mu) \in [0, \bar{T}] \times \mathcal{P} \times \mathcal{D}^v \) if and only if \( \Psi_1(\tau, \rho, \mu) + \varepsilon I_n \preceq 0 \) for all \( (\tau, \rho, \mu) \in [0, \bar{T}] \times \mathcal{P} \times \mathcal{D} \). The same statement holds for \( \Psi_2(\bar{T}, \rho, \mu) \). Defining

\[
\Psi(\tau, \rho, \mu) := \begin{cases} 
    \Psi_1(\tau, \rho, \mu) & \text{if } \tau \in [0, \bar{T}) \\
    \Psi_2(\bar{T}, \rho, \mu) & \text{if } \tau \geq \bar{T}
\end{cases}
\]

(11)

we get that the conditions (6) and (7) imply that

\[
\langle \nabla V(x, \tau, \rho), f \rangle = x^T \Psi(\tau, \rho, \mu)x \leq -\varepsilon \|x\|_2^2
\]

(12)

for all \( (x, \tau, \theta, \mu) \in \mathbb{R}^n \times [0, \bar{T}] \times \mathcal{P} \times \mathcal{D} \). Therefore, the satisfaction of the conditions (6) and (7) imply that of the condition (3.7b) in [23].

The condition of statement (c) reads

\[
V(g) - V(x) = x^T (J(\eta)^T S(0, \theta)J(\eta) - S(\bar{T}, \eta))x \leq 0
\]

(13)

and must hold for all \( (x, \theta, \eta) \in \mathbb{R}^n \times \mathcal{P} \times \mathcal{P} \). This is equivalent to the condition (3.8) in Proposition 3.24 in [23].

Finally, we need to check the hybrid time domain condition of Proposition 3.8 in [23]. First, note that the solutions to the system (4) are defined on the hybrid time domain given by

\[
\text{dom} \phi = \bigcup_{j=0}^\infty ([t_j, t_j + T_j], j), \ t_0 = 0
\]

(14)
which, together with $t + j \geq \tilde{T}$ for some $\tilde{T} > 0$, implies that $t + 1 + t/\bar{T} \geq \tilde{T}$ since $j \leq 1 + t/\bar{T}$ for all $(t, j) \in \text{dom} \phi$. Hence, $t \geq (1 + \bar{T}^{-1})(\tilde{T} - 1)$ and, as a result, the time-domain condition of Proposition 3.8 in [23] is satisfied and the set $A$ is asymptotically stable for the system (4). This proves the result.

\[\text{♦} \]

3.2 Connection with quadratic and robust stability

Following the same lines as in [16], it can be shown that the minimum dwell-time result stated in Theorem 1 naturally generalizes and unifies the quadratic and robust stability conditions for LPV systems with $J(\rho) = I_n$ through the concept of minimum dwell-time. This is further explained in the results below:

Proposition 2 (Quadratic stability) When $\bar{T} = 0$ and $t_k \to \infty$ as $k \to \infty$, then the conditions of Theorem 1 are equivalent to saying that there exists a matrix $P \in S^{\bar{n}}_{>0}$ such that

\[ A(\theta)^T P + PA(\theta) < 0 \] (15)

for all $\theta \in \mathcal{P}$.

Proof: First note that since $\bar{T} = 0$ and $t_k \to \infty$ as $k \to \infty$, then there is no accumulation point in the sequence of jumping instants and, therefore, the solution of the hybrid system is complete. We now analyze what happens to the condition around 0. A Taylor expansion of (8) around $\bar{T} = 0$ yields

\[ S(0, \theta) - S(\epsilon, \theta) = -\epsilon \partial_\tau S(0, \theta) + o(\epsilon), \]

\[ S(0, \theta) - S(\epsilon, \eta) = S(0, \theta) - S(0, \eta) - \epsilon \partial_\tau S(0, \eta) + o(\epsilon), \]

\[ S(0, \eta) - S(\epsilon, \theta) = S(0, \eta) - S(0, \theta) - \epsilon \partial_\tau S(0, \theta) + o(\epsilon) \] (16)

where it is assumed that $\eta \neq \theta$ and where $o(\epsilon)$ is the little-o notation. From (8), we get that the first equation implies that $\partial_\tau S(0, \theta) \succeq 0$. Therefore, for the second expression to be negative semidefinite for any arbitrarily small $\epsilon > 0$, we need $S(0, \theta) - S(0, \eta)$ to be negative semidefinite. However, this contradicts the last one and hence we need that $S(0, \eta) = S(0, \theta)$; i.e. $S$ is independent of $\rho$. Finally, since $\epsilon > 0$ is arbitrarily small, then both (16)-(7) can be satisfied with a matrix-valued function $S$ that is independent of $\tau$. Hence, we need that $S(\tau, \theta) := P > 0$ and substituting it in (6)-(7) yield the quadratic stability condition (15).

\[\text{♦} \]

Proposition 3 (Robust stability) When $\tilde{T} \to \infty$, then the conditions of Theorem 1 are equivalent to saying that there exists a differentiable matrix-valued function $P : \mathcal{P} \mapsto S^{\bar{n}}_{>0}$ such that

\[ \partial_\rho P(\theta) \mu + A(\theta)^T P(\theta) + P(\theta) A(\theta) < 0 \] (17)

for all $\theta \in \mathcal{P}$ and all $\mu \in \mathcal{D}_v$.  

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Proof: Clearly, when $\bar{T} \to \infty$, then there are no jumps anymore and the timer variable becomes unnecessary. Hence, we can pick $S(\tau, \theta) = P(\theta)$ and ignore the condition (8) as jumps never occur. The conditions (6)-(7) both reduce to (17). ♦

3.3 Computational considerations

Box 1: SOS program associated with Theorem 1

Find polynomial matrices $S, \Gamma, \Gamma^i, \Omega^i : [0, \bar{T}] \times \mathcal{P} \mapsto \mathbb{S}^n$, $\Theta_1, \Theta_2 : \mathcal{P} \times \mathcal{P} \mapsto \mathbb{S}^n$, $\Upsilon, \Upsilon^i : [0, \bar{T}] \times \mathcal{P} \times \mathcal{D}^v \mapsto \mathbb{S}^n$, $i = 1, \ldots, M$, such that

- $\Gamma, \Gamma^i, \Theta_1^i, \Theta_2^i, \Upsilon^i, \Omega^i, i = 1, \ldots, M$, are SOS for all $\mu \in \mathcal{D}^v$,
- $S(\tau, \theta) - \sum_{i=1}^M \Gamma^i(\tau, \theta) g_i(\theta) - \Gamma(\tau, \theta) \tau(\bar{T} - \tau) - \varepsilon I_n$ is SOS
- $-\partial_\mu S(\tau, \theta) \mu - \partial_\tau S(\tau, \theta) - \text{Sym}[S(\tau, \theta) A(\theta)] - \sum_{i=1}^M \Upsilon^i(\tau, \theta, \mu) g_i(\theta) - \Upsilon(\tau, \theta, \mu) \tau(\bar{T} - \tau)$ is SOS for all $\mu \in \mathcal{D}^v$
- $S(\bar{T}, \eta) - J(\eta)^T S(0, \theta) J(\eta) - \sum_{i=1}^N \Theta_1^i(\theta, \eta) g_i(\theta) - \sum_{i=1}^M \Theta_2^i(\theta, \eta) g_i(\eta) - \varepsilon I_n$ is SOS
- $-\partial_\mu S(\bar{T}, \theta) \mu - \text{Sym}[S(\bar{T}, \theta) A(\theta)] - \sum_{i=1}^M \Omega^i(\theta, \mu) g_i(\theta) - \varepsilon I_n$ is SOS for all $\mu \in \mathcal{D}^v$

The conditions formulated in Theorem 1 are infinite-dimensional semidefinite programs which can not be solved directly. To make them tractable, we propose to consider an approach based on sum of squares programming [25] that will result in an approximate finite-dimensional semidefinite program which can then be solved using standard solvers such as SeDuMi [28]. The conversion to a semidefinite program can be performed using the package SOSTOOLS [26] to which we input the SOS program corresponding to the considered conditions. We illustrate below how an SOS program associated with some given conditions can be obtained. The set $\mathcal{P}$ defined in Section 2.2 can be described as

$$\mathcal{P} =: \{\theta \in \mathbb{R}^N : g_i(\theta) \geq 0, i = 1, \ldots, M\} \quad (18)$$

for some polynomials $g_i : \mathbb{R}^N \mapsto \mathbb{R}$, $i = 1, \ldots, M$ and further note that

$$[0, \bar{T}] = \{\tau \in \mathbb{R} : f(\tau) := \tau(\bar{T} - \tau) \geq 0\}.$$

In what follows, we say that a symmetric polynomial matrix $\Theta(\cdot)$ is a sum of squares matrix (SOS matrix) or is SOS, for simplicity, if there exists a polynomial matrix $\Xi(\cdot)$ such that $\Theta(\cdot) = \Xi(\cdot)^T \Xi(\cdot)$. The following result provides the sum of squares formulation of Theorem 1.
Proposition 4 Let $\varepsilon, \bar{T} > 0$ be given and assume that the sum of squares program in Box 7 is feasible. Then, the conditions of Theorem 1 hold with the computed polynomial matrix $S(\tau, \theta)$ and the system (2) is asymptotically stable for all $\rho \in \mathcal{P}_{\bar{T}}$.

Remark 5 When the parameter set $\mathcal{P}$ is also defined by equality constraints $h_i(\theta) = 0$, $i = 1, \ldots, M'$, these constraints can be simply added in the sum of squares programs in the same way as the inequality constraints, but with the particularity that the corresponding multiplier matrices be simply symmetric instead of being SOS matrices.

3.4 Examples

We consider now two examples. The first one is a 2-dimensional toy example considered in [37] whereas the second one is a 4-dimensional system considered in [3] and inspired from an automatic flight control design problem. The numerical calculations have been performed using the package SOSTOOLS [26] and the semidefinite solver SeDuMi [28] on a PC equipped with 12GB of RAM and a processor Intel i7-950 @ 3.07Ghz.

3.4.1 Example 1

Let us consider here the system (2) with the matrices $J(\rho) = I_n$ and $A(\rho) = \begin{bmatrix} 0 & 1 \\ -2 - \rho & -1 \end{bmatrix}$, with $|\dot{\rho}| \leq \nu$ using an SOS approach with polynomials of degree 4.

Figure 1: Evolution of the computed minimum upper-bound on the minimum stability-preserving minimum dwell-time using Theorem 1 for the system (2)-(20) with $|\dot{\rho}| \leq \nu$ using an SOS approach with polynomials of degree 4.
where the time-varying parameter $\rho(t)$ takes values in $\mathcal{P} = [0, \bar{\rho}]$, $\bar{\rho} > 0$. It is known [37] that this system is quadratically stable if and only if $\bar{\rho} \leq 3.828$ but it was later proven in the context of piecewise constant parameters [16] that this bound can be improved provided that discontinuities do not occur too often. We now apply the conditions of Theorem 1 in order to characterize the impact of parameter variations between discontinuities. To this aim, we consider that $|\dot{\rho}(t)| \leq \nu$ with $\nu \geq 0$ and that $\bar{\rho} \in \{0, 0.1, \ldots, 10\}$. For each value for the parameter upper-bound $\bar{\rho}$ in that set, we solve for the conditions of Theorem 1 to get estimates (i.e. upper-bounds) for the minimum dwell-times. We use here $\epsilon = 0.01$ and polynomials of degree 4 in the sum of squares programs. Note that we have, in this case, $M = 1$, $M' = 0$ and $g_1(\theta) = \theta(\bar{\rho} - \theta)$. The complexity of the approach can be evaluated here through the number of primal and dual variables of the semidefinite program which are 2409 and 315, respectively. The average preprocessing and solving times are given by 6.04sec and 1.25sec, respectively. The results are Fig. 1 where we can see that the obtained minimum values for the dwell-times increase with the rate of variation $\nu$ of the parameter, which is an indicator of the fact that increasing the rate of variation of the parameter tends to destabilize the system and, consequently, the dwell-time needs to be increased in order to preserve the overall stability of the system.

3.4.2 Example 2

Let us consider now the system [2] with the matrices $J(\rho) = I_n$ and [3] p. 55:

$$A(\rho) = \begin{bmatrix}
\frac{3}{4} & 2 & \rho_1 & \rho_2 \\
0 & \frac{1}{2} & -\rho_2 & \rho_1 \\
-3v\rho_1/4 & v(\rho_2 - 2\rho_1) & -v & 0 \\
-3v\rho_2/4 & v(\rho_1 - 2\rho_2) & 0 & -v
\end{bmatrix}$$

(21)

where $v = 15/4$ and $\rho \in \mathcal{P} = \{z \in \mathbb{R}^2 : ||z||_2 = 1\}$. It has been shown in [3] that this system is not quadratically stable but was proven to be stable under minimum dwell-time equal to 1.7605 when the parameter trajectories are piecewise constant [16]. We propose now to quantify the effects of smooth parameter variations between discontinuities. Note, however, that the set $\mathcal{P}$ is not a box as considered along the paper but, as the next calculations show, this is not a problem since a proper set for the values of the derivative of the parameters can be defined. To this aim, let us define the parametrization $\rho_1(t) = \cos(\beta(t))$ and $\rho_2(t) = \sin(\beta(t))$ where $\beta(t)$ is piecewise differentiable. Differentiating these equalities yields $\dot{\rho}_1(t) = -\dot{\beta}(t)\rho_2(t)$ and $\dot{\rho}_2(t) = \dot{\beta}(t)\rho_1(t)$ where $\dot{\beta}(t) \in [-\nu, \nu]$, $\nu \geq 0$, at all times where $\beta(t)$ is differentiable. In this regard, we can consider $\dot{\beta}$ as an additional parameter that enters linearly in the stability conditions and hence the conditions can be checked at the vertices of the interval, that is, for all $\dot{\beta} \in \{-\nu, \nu\}$. Note that, in this case, we have $M = 0$, $M' = 1$ and $h_1(\rho) = \rho_1^2 + \rho_2^2 - 1$.

We now consider the conditions of Theorem 1 and we get the results gathered in Table 1 where we can see that, as expected, when $\nu$ increases then the minimum dwell-time has to increase to preserve stability. Using polynomials of higher degree allows to improve the numerical results at the expense of an increase of the computational complexity. As a final
Table 1: Evolution of the computed minimum upper-bound on the minimum dwell-time using Theorem 1 for the system (2)-(21) with $|\dot{\beta}| \leq \nu$ using an SOS approach with polynomials of degree $d$. The number of primal/dual variables of the semidefinite program and the preprocessing/solving time are also given.

| $\nu$ | primal/dual vars. | time (sec) |
|-------|-------------------|------------|
|       |                   |            |
| $\nu = 0$ | 9820/1850 | 20/27 |
| $\nu = 0.1$ | 43300/4620 | 212/935 |
| $\nu = 0.3$ | 26.1883 | 11.6859 |
| $\nu = 0.5$ | 6.4539 | 4.6317 |
| $\nu = 0.8$ | 2.9466 | 2.2561 |
| $\nu = 0.9$ | 2.7282 | 2.9494 |

Comment, it seems important to point out the failure of the semidefinite solver due to too important numerical errors when $d = 4$ and $\nu = 0.9$.

4 Stabilization using continuous-time LPV controllers

We consider in this section the following extension for the system (2)

$$
\dot{x}(t) = A(\rho(t))x(t) + B(\rho(t))u(t), \quad t \neq t_k
$$

$$
x(t_k^+) = J(\rho(t_k))x(t_k), \quad k \in \mathbb{Z}_{>0}
$$

$$
x(0) = x_0, \quad t_0 = 0
$$

where $x, x_0 \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the state of the system, the initial condition and the control input, respectively. The same assumptions as for the system (2) are made for the above system. We also consider the following class of state-feedback controllers

$$
u(t_k + \tau) = \begin{cases} 
K(\tau, \rho(t_k + \tau))x(t_k + \tau), & \tau \in [0, \bar{T}] \\
K(\bar{T}, \rho(t_k + \tau))x(t_k + \tau), & \tau \in (\bar{T}, T_k],
\end{cases}
$$

(23)

where $K(\cdot, \cdot) \in \mathbb{R}^{m \times n}$ is the controller gain which can be determined using the following result:

**Theorem 6 (Minimum dwell-time)** Let $\bar{T} \in \mathbb{R}_{>0}$ be given and assume that there exist a bounded continuously differentiable matrix-valued function $R : [0, \bar{T}] \times \mathcal{P} \mapsto \mathbb{S}^n_{>0}$, a matrix-valued function $U : [0, \bar{T}] \times \mathcal{P} \mapsto \mathbb{R}^{m \times n}$ and a scalar $\varepsilon > 0$ such that the conditions

$$
- \partial_\rho R(\bar{T}, \theta)\mu + \text{Sym}[\Phi(\bar{T}, \theta)] + \varepsilon I_n \preceq 0,
$$

(24)

$$
- \partial_\tau R(\tau, \theta) - \partial_\rho R(\tau, \theta)\mu + \text{Sym}[\Phi(\tau, \theta)] + \varepsilon I_n \preceq 0
$$

(25)

and

$$
J(\eta)R(\bar{T}, \eta)J(\eta)^T - R(0, \theta) \preceq 0
$$

(26)

hold for all $\theta, \eta \in \mathcal{P}$, $\mu \in \mathcal{D}_v$ and all $\tau \in [0, \bar{T}]$ where $\Phi(\tau, \theta) := A(\theta)R(\tau, \theta) + B(\theta)U(\tau, \theta)$.

Then, the LPV system (22)-(23) with parameter trajectories in $\mathcal{P}_{\bar{T}}$ is asymptotically stable with the controller gain $K(\tau, \theta) = U(\tau, \theta)R(\tau, \theta)^{-1}$. \triangle
Proof : The state matrix of the continuous-time part of the closed-loop system (22)-(23) is given by

\[
A_{cl}(\tau, \theta) = \begin{cases} 
A(\theta) + B(\theta)K(\tau, \theta), & \tau \in [0, \bar{T}] \\
A(\theta) + B(\theta)K(\bar{T}, \theta), & \tau \in (\bar{T}, T_k]. 
\end{cases}
\] (27)

Substituting this matrix in the conditions of Theorem 1 and performing a congruence transformation with respect to the matrix

\[
R(\tau, \theta) = S(\tau, \theta) - 1
\]

yields the conditions (24)-(25) where we have used the change of variables

\[
U(\tau, \theta) = K(\tau, \theta) R(\tau, \theta)
\]

and the facts that

\[
-\partial_\tau R(\tau, \theta) = R(\tau, \theta) \partial_\tau S(\tau, \theta) R(\tau, \theta) \\
-\partial_\rho R(\tau, \theta) = R(\tau, \theta) \partial_\rho S(\tau, \theta) R(\tau, \theta).
\]

Finally, the condition (26) is obtained from (8) using two successive Schur complements. This proves the desired result.

As for the previously obtained results, the conditions in Theorem 6 can be checked using sum of squares and convex programming since the conditions are convex in the decision variables \( R \) and \( U \). It should be also mentioned that the clock variable \( \tau \) is explicitly used in the control law and, hence, this variable should be computable/measurable. This was the motivation for considering the system (4) instead of a variation of the system in [23, Example 2.13] (see the discussion after the definition of the system (4)-(5)). It is important to stress that, due to the potential lack of knowledge in the initial value of the timer, there may be an initial mismatch in the actual value of the timer \( \tau \) and the one implemented in the controller. In this regard, there may be a finite transient phase where the system is unstable until the first jump occurs. Only then, both the timers of the system and the controller will synchronize and the controller will start stabilizing the system. However, since we are only interested in asymptotic stabilization, this issue is not of great importance here. This will become a problem if one is interested in exponential stabilization with prescribed convergence rate.

The example below illustrates this result:

Example 7 Let us consider back the example from [16]

\[
\dot{x} = \begin{bmatrix} 3 - \rho & 1 \\
1 - \rho & 2 + \rho
\end{bmatrix} x + \begin{bmatrix} 1 \\
1 + \rho
\end{bmatrix} u, \quad J = I_n
\] (28)

where \( P = [0, 1] \) and \( D = [-\nu, \nu] \). It was proved in [16] that this system cannot be stabilized quadratically. This latter property makes it a perfect example to illustrate the proposed approach since neither quadratic nor robust stabilization results can be used here. Applying then Theorem 6 with \( \bar{T} = 0.05 \), we find that the conditions are feasible for \( \nu \in \{0, 0.1, 0.3\} \) for \( d = 2 \) and \( \nu \in \{0.5, 0.7, 0.9, 1, 2\} \) for \( d = 3 \). When \( d = 2 \) the number of primal/dual variables is 834/180 whereas, when \( d = 3 \), this number is 2414/315. Finally, when \( d = 2 \), it takes roughly 2.62sec to solve the problem whereas, in the case \( d = 3 \), it takes around 6.31sec. For simulation purposes, we consider the parameter trajectory

\[
\rho(t_k + \tau) = \frac{1 + \sin(2\nu(t_k + \tau) + \varphi_k)}{2}, \quad k \in \mathbb{Z}_{\geq 0}
\] (29)
where $\varphi_k$ is a uniform random variable taking values in $[0, 2\pi]$ and we generate a random sequence of instants satisfying the minimum dwell-time condition. At each of these time instants, we draw a new value for the random variable $\varphi_k$, which introduces a discontinuity in the trajectory. Note, however, that between discontinuities we have that $|\dot{\rho}(t_k + \tau)| \leq \nu$ for all $\tau \in (0, T_k]$, $k \in \mathbb{Z}_{\geq 0}$. We then obtain the results depicted in Fig. 2 for the case $\nu = 1$ where we can see that stabilization is indeed achieved for this system.

![Figure 2: Trajectories of the closed-loop system (28)-(23)-(29) with $\nu = 1$.](image)

5 Stabilization using sampled-data LPV controllers

Let us consider now the case of sampled-data controllers. In this case, it seems reasonable to consider the following class of continuously differentiable parameters:

$$\mathcal{P}_\infty := \left\{ \rho : \mathbb{R}_{\geq 0} \mapsto \mathcal{P} \mid \dot{\rho}(t) \in \mathcal{Q}(\rho(t)), t \geq 0 \right\}.$$  

(30)

Discontinuities in the parameter trajectories have been removed in this context as having discontinuities occurring at the same time as control updates does not motivate the use of any gain-scheduled controller because the controller would be scheduled with an incorrect parameter value right at the beginning of the sampling interval. Hence, the use of gain-scheduled controllers is, in this case, not beneficial over that of robust controllers from a stabilization perspective. Note, however, that parameter discontinuities that would occur at different times than control updates can be incorporated via the introduction of an extra clock/timer variable, at the expense of an increase of the model complexity and computational burden. This case is not be treated here for brevity.
5.1 A preliminary stability result

We are interested here in deriving a stability result under a range dwell-time constraint for the sequence of jumping instants, that is, for all sequences of jumping instants in the set

\[ \mathcal{T} := \left\{ \{ t_k \}_{k \in \mathbb{Z}_0} \middle| t_{k+1} - t_k \in [T_{\text{min}}, T_{\text{max}}], t_0 = 0, k \in \mathbb{Z}_0 \right\} \]

for some \( 0 \leq T_{\text{min}} \leq T_{\text{max}} < \infty \). The corresponding hybrid system is given by

\[
\begin{cases}
\dot{x}(t) = A(\rho(t))x(t) \\
\dot{\rho}(t) \in Q(\rho(t)) \\
\dot{\tau}(t) = 1 \\
\dot{T}(t) = 0 \\
x(t^+) = J(\rho(t))x(t) \\
\rho(t^+) = \rho(t) \\
\tau(t^+) = 0 \\
T(t^+) \in [T_{\text{min}}, T_{\text{max}}]
\end{cases}
\]

where

\[
C = \mathbb{R}^n \times \mathcal{P} \times E_1, \\
D = \mathbb{R}^n \times \mathcal{P} \times E_2,
\]

\[
E_1 = \{ \varphi \in \mathbb{R}_{\geq 0} \times [T_{\text{min}}, T_{\text{max}}]: \varphi_1 \leq \varphi_2 \}, \\
E_2 = \{ \varphi \in \mathbb{R}_{\geq 0} \times [T_{\text{min}}, T_{\text{max}}]: \varphi_1 = \varphi_2 \}.
\]

With these definitions in mind, the we can now state the main stability result of the section:

**Theorem 8 (Range dwell-time)** Let the scalars \( 0 < T_{\text{min}} \leq T_{\text{max}} < \infty \) be given and assume that there exist a bounded continuously differentiable matrix-valued function \( S : [0, T_{\text{max}}] \times \mathcal{P} \rightarrow \mathbb{S}_{\geq 0} \) and a scalar \( \varepsilon > 0 \) such that the conditions

\[
- \partial_\tau S(\tilde{\tau}, \theta) + \partial_\rho S(\tilde{\tau}, \theta)\mu + \text{Sym}[S(\tilde{\tau}, \theta)A(\theta)] \preceq 0
\]

and

\[
J(\theta)S(\sigma, \theta)J(\theta) - S(0, \theta) + \varepsilon I_n \preceq 0
\]

hold for all \( \theta \in \mathcal{P} \), all \( \mu \in \mathcal{D}_\nu \), all \( \tilde{\tau} \in [0, T_{\text{max}}] \) and all \( \sigma \in [T_{\text{min}}, T_{\text{max}}] \).

Then, the LPV system \([2]\) with parameter trajectories in \( \mathcal{P}_\infty \) is asymptotically stable under the range dwell-time condition \([T_{\text{min}}, T_{\text{max}}]\); i.e. for all sequences of jumping instants in \( \mathcal{T} \).

**Proof:** Define then the set \( \mathcal{A} = \{0\} \times \mathcal{P} \times E_1 \). The proof of this result follows the same lines as the proof of Theorem 1 with the difference that we consider here the Lyapunov function

\[
V(x, \tau, \rho, T) = x^T S(T - \tau, \rho)x
\]

14
where \( S(T - \tau, \rho) \) is positive definite for all admissible values of its arguments and we use \([23, \text{Proposition 3.24}]\). A sufficient condition for the feasibility of the flow condition in \([23, \text{Proposition 3.24}]\)

\[
-\partial_{\tau} S(T - \tau, \theta) + \partial_{\rho} S(T - \tau, \theta) \mu + \text{Sym}[S(T - \tau, \theta)A(\theta)] \preceq 0
\]

holds for all \( \theta \in \mathcal{P} \), all \( \mu \in \mathcal{D}^v \), \( \tau \in [0, T] \) and all \( T \in [T_{\text{min}}, T_{\text{max}}] \), which is equivalent to the condition \((34)\) using the change of variables \( \bar{\tau} = T - \tau \). The feasibility of the condition \((35)\) implies that of the jump condition in \([23, \text{Proposition 3.24}]\). The solutions to the system \((32)\) are defined on the hybrid time domain given

\[
\text{dom} \phi = \bigcup_{j=0}^{\infty} ([t_j, t_j + T_j], j), \ t_0 = 0
\]

which, together with \( t + j \geq \bar{T} \) for some \( \bar{T} > 0 \), implies that \((j + 1)T_{\text{max}} + j \geq \bar{T} \) since \( t \leq (j + 1)T_{\text{max}} \) for all \((t, j) \in \text{dom} \phi\). Hence, \( j \geq (1 + T_{\text{max}})^{-1}(\bar{T} - T_{\text{max}}) \) and, as a result, the set \( \mathcal{A} \) is asymptotically stable for the system \((32)\) and the result follows. This completes the proof. \( \diamond \)

### 5.2 Main stabilization result

We consider here the following class of state-feedback controllers

\[
u(t_k + \tau) = K_1(\rho(t_k))x(t_k) + K_2(\rho(t_k))u(t_k)
\]

where \( \tau \in (0, T_k), T_k \in [T_{\text{min}}, T_{\text{max}}] \) and where \( K_1(\cdot) \in \mathbb{R}^{m \times n} \) and \( K_2(\cdot) \in \mathbb{R}^{m \times m} \) are the gains of the controller to be determined. The sequence of update times for the control input is assumed to satisfy a range dwell-time condition; i.e. \( \{t_k\}_{k \in \mathbb{Z}_{>0}} \in \mathcal{I} \). Before stating the stabilization result, it is necessary to reformulate the closed-loop system \([22]+(38)\) as a hybrid system, which is given below:

\[
\begin{cases}
  \dot{x}(t) = A(\rho(t))x(t) + B(\rho(t))u(t) \\
  \dot{u}(t) = 0 \\
  \dot{\rho}(t) \in Q(\rho(t)) \\
  \dot{\tau}(t) = 1 \\
  \dot{T}(t) = 0 \\
  x(t^+) = J(\rho(t))x(t) \\
  u(t^+) = K_1(\rho(t))x(t) + K_2(\rho(t))u(t) \\
  \rho(t^+) = \rho(t) \\
  \tau(t^+) = 0 \\
  T(t^+) \in [T_{\text{min}}, T_{\text{max}}]
\end{cases}
\]

if \((z(t), \rho(t), \tau(t), T(t)) \in \mathcal{C}\)

\[
\begin{cases}
  \dot{x}(t) = A(\rho(t))x(t) + B(\rho(t))u(t) \\
  \dot{u}(t) = 0 \\
  \dot{\rho}(t) \in Q(\rho(t)) \\
  \dot{\tau}(t) = 1 \\
  \dot{T}(t) = 0 \\
  x(t^+) = J(\rho(t))x(t) \\
  u(t^+) = K_1(\rho(t))x(t) + K_2(\rho(t))u(t) \\
  \rho(t^+) = \rho(t) \\
  \tau(t^+) = 0 \\
  T(t^+) \in [T_{\text{min}}, T_{\text{max}}]
\end{cases}
\]

if \((z(t), \rho(t), \tau(t), T(t)) \in \mathcal{D}\)
where \( z = (x, u) \) and
\[
C = \mathbb{R}^{n+m} \times \mathcal{P} \times E_1,
\]
\[
D = \mathbb{R}^{n+m} \times \mathcal{P} \times E_2,
\]
(40)
where \( E \) and \( F \) are defined in (33). Note that the state of the system has been augmented to contain the piecewise constant control input. For conciseness, we also define the following matrices:
\[
\tilde{A}(\rho) := \begin{bmatrix} A(\rho) & B(\rho) \\ 0 & 0 \end{bmatrix}, \tilde{J}(\rho) := \begin{bmatrix} J(\rho) & 0 \\ 0 & 0 \end{bmatrix}, \tilde{B} := \begin{bmatrix} 0 \\ I_m \end{bmatrix}
\]
(41)
and \( \tilde{K}(\rho) := \begin{bmatrix} K_1(\rho) & K_2(\rho) \end{bmatrix} \).

We can now state the stabilization result of the section:

**Theorem 9** Let \( \bar{T} \in \mathbb{R}_{>0} \) be given and assume that there exist a bounded continuously differentiable matrix-valued function \( R : [0,T_{\max}] \times \mathcal{P} \mapsto \mathbb{S}_{>0} \), a matrix-valued function \( U : \mathcal{P} \mapsto \mathbb{R}^{m \times (n+m)} \) and a scalar \( \varepsilon > 0 \) such that the conditions
\[
\partial_\tau R(\tilde{\tau}, \theta) - \partial_\rho R(\tilde{\tau}, \theta) \mu + \text{Sym}[\tilde{A}(\theta) R(\tilde{\tau}, \theta)] + \varepsilon I_n \preceq 0
\]
(42)
and
\[
\begin{bmatrix} R(\sigma, \theta) & \tilde{J}(\theta)R(0, \theta) + \tilde{B}U(\theta) \\ \ast & -R(0, \theta) \end{bmatrix} \preceq 0
\]
(43)
hold for all \( \theta \in \mathcal{P} \), all \( \mu \in \mathcal{D}_v \), all \( \tilde{\tau} \in [0,T_{\max}] \) and all \( \sigma \in [T_{\min}, T_{\max}] \).

Then, the sampled-data LPV system (22)-(38) with parameter trajectories in \( \mathcal{P}_\infty \) is asymptotically stable under the range dwell-time condition \([T_{\min}, T_{\max}]\) (i.e. for all sequences of jumping instants in \( \mathcal{T} \)) with the controller gain \( \tilde{K}(\theta) = U(\theta)R(\bar{T}, \theta)^{-1} \).

\( \triangle \)

**Proof**: The proof follows from similar lines as the proof of Theorem 6 with the difference that we use here the range dwell-time stability conditions of Theorem 8. The details are omitted for brevity.

\( \diamond \)

### 5.3 Examples

To illustrate the interest of the approach, we consider here three examples from the literature.

**Example 10** Let us consider back the system (28) with the difference that we now aim at stabilizing it with a gain-scheduled sampled-data state-feedback controller of the form (23). Solving for the sum of squares conditions associated with the conditions stated in Theorem 9 with \( d = 2 \) and \( \nu = 1 \) yields the controller gain in (44). Computational-wise, the underlying semidefinite program has 3078/525 primal/dual variables and is solved in 7.88sec. The trajectory of the closed-loop system is depicted in the top panel of Figure 3 for the parameter trajectory \( \rho(t) = (1 + \sin(2\nu t))/2 \) and initial condition \( x_0 = (-1, 1) \), \( u_0 = 0 \).
Figure 3: Trajectories of the closed-loop system (28)-(23) (top) and (45)-(23) (bottom).

\[ K(\theta) = \frac{1}{\text{den}(\theta)} \begin{bmatrix} 3.01 - 2.00\theta + 5.52\theta^2 - 2.43\theta^3 - 0.59\theta^4 + 0.69\theta^5 + 0.04\theta^6 \\ -0.74 - 0.29\theta + 0.77\theta^2 - 1.13\theta^3 + 0.10\theta^4 + 0.24\theta^5 + 0.08\theta^6 \\ -0.002 + 0.014\theta + 0.029\theta^2 - 0.46\theta^3 + 1.10\theta^4 - 0.95\theta^5 + 0.28\theta^6 \end{bmatrix}^T \]

\[ \text{den}(\theta) = -0.32 + 0.56\theta - 1.20\theta^2 + 0.45\theta^3 + 1.15\theta^4 - 1.18\theta^5 + 0.23\theta^6. \]

Example 11 Let us consider the system

\[ \dot{x} = \begin{bmatrix} 2\rho \\ -2.2 + \rho \\ -3.3 \end{bmatrix} x + \begin{bmatrix} 2\rho \\ 0.1 + \rho \end{bmatrix} u \]  

(45)

where \( \rho(t) = \sin(0.2t) \), hence \( \mathcal{P} = [-1, 1] \) and \( \mathcal{D} = [-0.2, 0.2] \). It was shown in [30] that this system could be stabilized at least up to \( T_{\text{max}} = 0.6 \) using an input-delay model for the zero-order hold and a parameter-dependent Lyapunov-Krasovskii functional. Using polynomials of order 4 in the SOS conditions, we can solve the SOS program associated with Theorem 9 and find a controller that makes the closed-loop system stable for all \( T_k \in [0.001, 0.6] \). The program has 9618/966 primal/dual variables and is solved in 36.26 sec. The simulation results are depicted in Figure 3.

Example 12 Let us consider now the system

\[ \dot{x} = \begin{bmatrix} 0 \\ 0.1 \\ 0.4 + 0.6\rho \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \] 

(46)

where \( \rho(t) = \sin(\nu t) \). Hence, \( \mathcal{P} = [-1, 1] \) and \( \mathcal{D} = [-\nu, \nu] \). Using a looped-functional approach, it was shown in [31] that, for \( T_{\text{min}} = 0.001 \), this system could be stabilized up to
$T_{\text{max}} = 1.264$ when $\nu = 0.2$ and up to $T_{\text{max}} = 0.8$ when $\nu = 1$. Using Theorem 9 with $d = 4$, we can show that, for both $\nu = 0.2$ and $\nu = 1$, we can find a controller that stabilizes the system for all $T_k \in [0.001, 1.3]$ in approximately 25sec. The number of primal/dual variables is 9618/966. For simulation purposes, we set $T_{\text{max}} = 0.4$ for both $\nu = 0.2$ and $\nu = 1$, and we design controllers using Theorem 9 with $d = 2$ (in this case, the number of primal/dual variables is given by 3078/525 and the problem is solved in 7sec). Using the initial conditions $x_0 = (-1, 1)$, $u_0 = 0$ and we get the trajectories depicted in Figure 4. Note that, as in [31], using a controller designed for $T_{\text{max}} = 1.3$ would result in a very slow response for the closed-loop system which is not desirable.

![Figure 4: Trajectories of the closed-loop system (46)-(23).](image)

### 6 Future Works

Potential extensions include the consideration of different types of Lyapunov functions of polyhedral or homogeneous types, the consideration of additional clocks in order to consider multiple types of discrete events (such as control update and parameter discontinuities events) and the consideration of switched impulsive LPV systems. Converse results along the lines of [23,38] for this class of systems could also be very interesting to obtain. Finally, the derivation of convex stabilization conditions for the design of dynamic output-feedback controller is a problem which is worth investigating.
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