New moduli components
of rank 2 bundles on projective space

C. Almeida, M. Jardim, A.S. Tikhomirov and S.A. Tikhomirov

Abstract. We present a new family of monads whose cohomology is a stable rank 2 vector bundle on $\mathbb{P}^3$. We also study the irreducibility and smoothness together with a geometrical description of some of these families. These facts are used to construct a new infinite series of rational moduli components of stable rank 2 vector bundles with trivial determinant and growing second Chern class. We also prove that the moduli space of stable rank 2 vector bundles with trivial determinant and second Chern class equal to 5 has exactly three irreducible rational components.

Bibliography: 40 titles.

Keywords: rank 2 bundles, monads, instanton bundles.

§ 1. Introduction

1.1. In [34] Maruyama proved that the rank $r$ stable vector bundles on a projective scheme $X$ with fixed Chern classes $c_1, \ldots, c_r$ can be parametrized by an algebraic quasi-projective variety, denoted by $\mathcal{B}_X(r, c_1, \ldots, c_r)$. Although this result has been known for almost 40 years, there are just a few concrete examples and established facts about such varieties, even for cases like $X = \mathbb{P}^3$ and $r = 2$. For instance, $\mathcal{B}_{\mathbb{P}^3}(2, 0, 1)$ was studied by Barth in [3], $\mathcal{B}_{\mathbb{P}^3}(2, 0, 2)$ by Hartshorne in [19], $\mathcal{B}_{\mathbb{P}^3}(2, -1, 2)$ by Hartshorne and Sols in [22] and by Manolache in [33], while $\mathcal{B}_{\mathbb{P}^3}(2, -1, 4)$ was described by Bânică and Manolache in [1]. This probably happened due to the fact that the questions of irreducibility (solved in [38] and [39]) and smoothness (answered in [28]) of the so-called instanton component of the moduli space $\mathcal{B}_{\mathbb{P}^3}(2, c_2)$ for all $c_2 \in \mathbb{Z}_+$ remained open until 2014.

1.2. In this paper we continue the study of the moduli space $\mathcal{B}_{\mathbb{P}^3}(2, 0, n)$, which we simply denote by $\mathcal{B}(n)$ from now on, with the goal of providing new examples of families of vector bundles, and understanding their geometry. It is more or less clear from the table in [21], §5.3 that $\mathcal{B}(1)$ and $\mathcal{B}(2)$ should be irreducible, while
$B(3)$ and $B(4)$ should have exactly two irreducible components; see [16] and [11], respectively, for the proof of the statements about $B(3)$ and $B(4)$. As for $B(5)$, a description of all its irreducible components had been a challenge since the 1980s. In the paper we give a complete answer to this problem (see Theorem 2).

For $n \geq 5$, two families of irreducible components have been studied, namely the *instanton components*, whose generic point corresponds to an instanton bundle, and the *Ein components*, whose generic point corresponds to a bundle given as the cohomology of a monad of the form

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-c) \to \mathcal{O}_{\mathbb{P}^3}(-b) \oplus \mathcal{O}_{\mathbb{P}^3}(-a) \oplus \mathcal{O}_{\mathbb{P}^3}(a) \oplus \mathcal{O}_{\mathbb{P}^3}(b) \to \mathcal{O}_{\mathbb{P}^3}(c) \to 0,$$

where $b \geq a \geq 0$ and $c > a + b$. In [31] it was proved that Ein components are rational varieties.

All of the components of $B(n)$ for $n \leq 4$ are of either of these types; here we focus on a new family of bundles that appear as soon as $n \geq 5$.

More precisely, we study the set of vector bundles in $B(a^2 + k)$ for each $a \geq 2$ and $k \geq 1$ which arise as cohomologies of monads of the form:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \oplus V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to V_{2k+4} \otimes \mathcal{O}_{\mathbb{P}^3} \to V'_k \otimes \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(a) \to 0, \quad (1.1)$$

which will be denoted by $G(a, k)$. We provide a bijection between such monads and monads of the form:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \sigma \to \widetilde{E} \tau \to \mathcal{O}_{\mathbb{P}^3}(a) \to 0,$$

where $\widetilde{E}$ is a symplectic rank 4 instanton bundle of charge $k$.

1.3. When $k = 1$, these facts are used to prove our first main result.

**Theorem 1.** For each $a \geq 2$ not equal to 3, $G(a, 1)$ is a nonsingular dense subset of a rational irreducible component of $B(a^2 + 1)$ of dimension

$$4 \left( \frac{a + 3}{3} \right) - a - 1.$$

Our second main result provides a complete description of all the irreducible components of $B(5)$.

**Theorem 2.** The moduli space $B(5)$ has exactly three rational irreducible components, namely:

1) the instanton component, of dimension 37, which is nonsingular and consists of those bundles given as the cohomology of monads of the forms

$$0 \to V_5 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to V_{12} \otimes \mathcal{O}_{\mathbb{P}^3} \to V_5 \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0 \quad (1.2)$$

and

$$0 \to V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-2) \to V_3 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \oplus V_3 \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(2) \to 0; \quad (1.3)$$

2) the Ein component, nonsingular of dimension 40, which consists of those bundles given as the cohomology of monads of the form

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-3) \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to \mathcal{O}_{\mathbb{P}^3}(3) \to 0; \quad (1.4)$$
3) the closure of the set \( G(2,1) \), of dimension 37, which consists of those bundles given as the cohomology of monads of the forms

\[
0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \to V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to 0 \quad (1.5)
\]

and

\[
0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3}(-1) \oplus V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1) \\
\to V_2' \otimes \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to 0. \quad (1.6)
\]

1.4. Hartshorne and Rao proved in [21] that every stable rank 2 bundle \( \mathcal{E} \) on \( \mathbb{P}^3 \) with Chern classes \( c_1(\mathcal{E}) = 0 \) and \( c_2(\mathcal{E}) = 5 \) is the cohomology of one of the monads listed above. Rao showed in [36] that bundles given as the cohomology of monads of the form (1.3) lie in the closure of the family of instanton bundles of charge 5, which was first shown to be irreducible by Coandă, Tikhomirov and Trautmann in [13]; see also [38]. The irreducibility of the family of bundles which arise as the cohomology of monads of the form (1.4) was established by Ein in [15].

The fact that the closure of \( G(2,1) \) is an irreducible rational component of \( B(5) \) is the particular case \( a = 2 \) of the main theorem, Theorem 1. Finally, we show that the set of bundles given by monads of the form (1.6) lies in the closure of \( G(2,1) \).

We now give a brief sketch of the contents of the paper. In §2 we recall some general properties of monads and symplectic instanton bundles on \( \mathbb{P}^3 \). We especially treat the rank 4 symplectic instantons of charge 1. Any such bundle \( E \) is described as a middle term of an exact triple with a rank 2 trivial bundle at the left hand and a null correlation rank 2 sheaf at the right hand. In §3 we study the set \( G(a,k) \) of (the isomorphism classes of) the so-called modified instanton bundles which are rank 2 bundles that arise as the cohomology bundles of monads of the form (1.1) with \( a \geq 2 \) and \( k \geq 1 \). We show that each modified instanton appears as the cohomology bundle of a monad of the form

\[
0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \to E \to \mathcal{O}_{\mathbb{P}^3}(a) \to 0, \quad (1.7)
\]

where \( E \) is a rank 4 symplectic instanton of charge \( k \). In the case when \( k = 1 \) this relation will be essential for the further constructions.

In §4 we study the set \( G(a,1) \). We construct three families of symplectic monads of the form (1.7). The first is the universal family, with the base scheme \( S \), of monads with \( E \) splitting as \( E = \mathcal{O}_{\mathbb{P}^3}^{\oplus 2} \oplus N \) where \( N \) is a null correlation bundle. The second is a family, with the base scheme \( \tilde{S} \) containing \( S \) as a dense open subset, of monads with \( E \) a general symplectic rank 4 instanton of charge 1. The third is a family of monads with \( E \) splitting as in the first, but with a new base \( Y \). All the three families inherit universal cohomology sheaves, and it is shown that the images of their corresponding modular morphisms \( L(a^2 + 1) \) have the same closure \( \overline{G(a,1)} \) (see Propositions 7 and 8). In §5 the three families mentioned above are used to prove the first main theorem, Theorem 1.

Sections 6 and 7 are devoted to the study of the monads of the form (1.6). In §6 we show that the cohomology sheaves \( \mathcal{E} \) of those among such monads that are not reduced to monads of the form (1.5) are closely related (by two subsequent elementary transformations; see Proposition 9) to rank 2 reflexive sheaves with Chern classes \( (0,2,2k), 0 \leq k \leq 3 \). A complete classification of the moduli components
of these reflexive sheaves performed in §7 (see Propositions 10 and 11) leads to a dimension estimate, given in Theorem 4, for the subset of the bundles \( E \) specified above. It follows that this subset is not a component of \( B(5) \), and we use this in §8 to prove the second main theorem, Theorem 2.

**Notation and conventions.** In this work,
- \( k \) is an algebraically closed field of characteristic zero;
- \( V_n \), respectively \( U_n \), denotes a \( k \)-vector space of dimension \( n \);
- \( \langle v \rangle \) is the 1-dimensional subspace of \( V_n \) spanned by a nonzero vector \( v \in V_n \);
- \( \mathbf{P}(F) := \text{Proj}(\text{Sym}_k F) \) is the projective spectrum of \( F \), for a coherent \( \mathcal{O}_X \)-sheaf \( F \) on a given scheme \( X \);
- \( \mathcal{O}_{\mathbf{P}(F)}(1) \) is the Grothendieck sheaf on \( \mathbf{P}(F) \);
- \( \mathbf{V}(F) := \text{Spec}(\text{Sym}_k F) \), for \( X \) and \( F \) as above;
- \( \mathbb{P}^3 := P(U_4) \) is the projective 3-space;
- \( \text{Isom}(V_n \otimes \mathcal{O}_X, F) \to X \) is the principal \( \text{GL}(n,k) \)-bundle of frames of a rank \( n \) locally free \( \mathcal{O}_X \)-sheaf \( F \);
- \( \mathbf{X} := \mathbb{P}^3 \times X \), for a given scheme \( X \);
- \( p_X : \mathbf{X} \to X \) is the projection onto the second factor, for \( X \) and \( X \) as above;
- \( f : \mathbf{X} \to \mathbf{Y} \) is the morphism induced by the morphism of schemes \( f : X \to Y \);
- \( F_X := f^* F, \varphi_X := f^* \varphi : F_X \to G_X \) and \( E_X := f^* E \) for a given \( \mathcal{O}_Y \)-sheaf \( F \), a given morphism \( \varphi : F \to G \) of \( \mathcal{O}_Y \)-sheaves, a given \( \mathcal{O}_Y \)-sheaf (or, a complex of sheaves) \( E \), and \( f : X \to Y \) and \( f : \mathbf{X} \to \mathbf{Y} \) as above;
- \( \mathbf{E}(a,0) := \mathbf{E} \otimes \mathcal{O}_{\mathbb{P}^3}(a) \boxtimes \mathcal{O}_X \) for \( X \) and \( E \) as above and \( a \in \mathbb{Z} \);
- \( X \xleftarrow{g_X} X \times_{\mathbf{Z}} Y \xrightarrow{f_Y} Y \) are the projections of the fibre product \( X \times_{\mathbf{Z}} Y \) induced by the morphisms \( X \xrightarrow{f} \mathbf{Z} \xleftarrow{g} Y \);
- \( H^i(F) \) is the \( i \)th cohomology group of the sheaf \( F \) on \( \mathbb{P}^3 \);
- \( \text{Gr}(n, V_k) \) is the Grassmannian variety of \( n \)-dimensional subspaces of \( V_k \).

*Variety* means an integral (that is, reduced and irreducible) scheme.

Since we are working with rank 2 vector bundles on \( \mathbb{P}^3 \), and Gieseker stability is equivalent to \( \mu \)-stability, we make no distinction between these two concepts.

We do not make any distinction between vector bundles and locally free sheaves.

- \([E]\) is the isomorphism class of a given sheaf on \( \mathbb{P}^3 \); in case \( E \) is a rank 2 stable sheaf on \( \mathbb{P}^3 \), \([E]\) is also considered as a point in the moduli space \( M \) of stable rank 2 sheaves on \( \mathbb{P}^3 \).
- \( \Phi_X : X \to M \), \( x \mapsto \mathbf{E}|_{\mathbb{P}^3 \times \{x\}} \) is the morphism defined by the \( \mathcal{O}_X \)-sheaf \( \mathbf{E} \) which is a family of stable rank 2 vector bundles on \( \mathbb{P}^3 \) with base \( X \), for \( M \) as above. We call \( \Phi_X \) the *modular morphism defined by the family \( \mathbf{E} \).*
- \( \mathcal{R}(e,n,m) \) is the set of isomorphism classes of rank 2 reflexive sheaves on \( \mathbb{P}^3 \) with Chern classes \( (c_1,c_2,c_3) = (e,n,m) \).
- \( \ell(Y) := h^0(\mathcal{O}_Y) \) is the length of a 0-dimensional scheme \( Y \).
- \( H^i(E) = \bigoplus_{i \in \mathbb{Z}} H^i(E(i)) \) is the graded cohomology module over the graded ring \( \Gamma_*(\mathcal{O}_{\mathbb{P}^3}) := \bigoplus_{j \geq 0} H^0(\mathcal{O}_{\mathbb{P}^3}(j)) \).
- \( \{s\}_0 := \{x \in X|s(x) = 0\} \) is the scheme of zeros of a section \( s \) of a given vector bundle on a scheme \( X \).
- \( \text{Sp}(E) \) is the spectrum of a vector bundle \([E] \in B(5)\), that is, the nondecreasing sequence of integers \((a_1,a_2,a_3,a_4,a_5)\) uniquely defined by \( E \) (see [5] and [20], §7).
All the commutative diagrams of sheaves below which do not contain monads are assumed to have exact rows and columns. In these diagrams the arrows $F \hookrightarrow G$ and $F \rightarrow G$ are shortenings for $0 \rightarrow F \rightarrow G$ and $F \rightarrow G \rightarrow 0$, respectively.

Acknowledgements. Part of this work was done during C. Almeida’s visit to the University of Barcelona in 2018, and he is grateful to the institution for its hospitality. Part of the work was done during M. Jardim’s visit to the University of Edinburgh in 2018, and he is grateful to that institution. A.S. Tikhomirov also acknowledges the support from the Max Planck Institute for Mathematics in Bonn, where part of this work was done during the winter of 2020.

The authors express their gratitude to the referee for valuable comments and suggestions for improvements.

§ 2. Monads and symplectic instanton bundles

2.1. Recall that a monad is a complex of vector bundles of the form

$$0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0 \quad (2.1)$$

such that $\alpha$ is injective and $\beta$ is surjective. We call the sheaf $E := \ker \beta / \mathrm{im} \alpha$ the cohomology of the monad (2.1). When $\alpha$ is locally left invertible (that is, it is a subbundle morphism), $E$ is a vector bundle.

The notion of monad is important in the study of vector bundles on $\mathbb{P}^3$ because Horrocks proved in [23] that every vector bundle on $\mathbb{P}^3$ is the cohomology of a monad of the form (2.1) with $A$, $B$ and $C$ being sums of line bundles.

For completeness, we include in this section some useful results about monads that will be required in this work. The following lemma gives a relation between isomorphism classes of monads and their cohomology vector bundles; a proof can be found in [35], Lemma 4.1.3.

**Lemma 1.** Let $E$ and $E'$ be, respectively, the cohomology of the following monads:

$$M: \quad 0 \rightarrow A \xrightarrow{a} B \xrightarrow{b} C \rightarrow 0 \quad (2.2)$$

and

$$M': \quad 0 \rightarrow A' \xrightarrow{a'} B' \xrightarrow{b'} C' \rightarrow 0. \quad (2.3)$$

If $\mathrm{Hom}(B, A') = \mathrm{Hom}(C, B') = \mathrm{Ext}^1(C, A') = \mathrm{Ext}^1(B, A') = \mathrm{Ext}^1(C, B') = \mathrm{Ext}^2(C, A') = 0$, then there exists a bijection between the set of all morphisms from $E$ to $E'$ and the set of all morphisms of monads from (2.2) to (2.3).

The following important corollary will be used several times in what follows, and a proof can also be found in [35], Lemma 4.1.3 and Corollary 2.

**Corollary 1.** Consider the monad $M$ and its dual monad $M^\vee$, where

$$M: \quad 0 \rightarrow A \xrightarrow{a} B \xrightarrow{b} C \rightarrow 0$$

and

$$M^\vee: \quad 0 \rightarrow C^\vee \xrightarrow{b^\vee} B^\vee \xrightarrow{a^\vee} A^\vee \rightarrow 0.$$

If these monads satisfy the hypothesis of Lemma 1, and there exists an isomorphism $f: E \rightarrow E^\vee$ between their cohomology bundles such that $f^\vee = -f$, then there are isomorphisms $h: C \rightarrow A^\vee$, and $q: B \rightarrow B^\vee$, such that $q^\vee = -q$, and $h \circ b = a^\vee \circ q$. 
Recall that every locally-free sheaf $E$ on $\mathbb{P}^3$ is the cohomology of a monad of the form [23]:

$$0 \rightarrow \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^3}(a_i) \rightarrow \bigoplus_{j=1}^{s} \mathcal{O}_{\mathbb{P}^3}(b_j) \rightarrow \bigoplus_{k=1}^{t} \mathcal{O}_{\mathbb{P}^3}(c_k) \rightarrow 0. \quad (2.4)$$

In this work we will be interested in rank 2 locally free sheaves with vanishing first Chern class. Under these conditions, we have $E^\vee \simeq E$, and this implies that $t = r$, $s = 2r + 2$, and $\{a_i\} = \{-c_k\}$. In addition, the middle entry of the monad is also self-dual, so that (2.4) reduces to

$$0 \rightarrow \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^3}(a_i) \rightarrow \bigoplus_{j=1}^{r+1} \mathcal{O}_{\mathbb{P}^3}(b_j) \oplus \mathcal{O}_{\mathbb{P}^3}(-b_j) \rightarrow \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^3}(-a_i) \rightarrow 0.$$

Finally, recall also that $r$ coincides with the number of generators of $H^1_*(E) = \bigoplus_{p \in \mathbb{Z}} H^1(\mathbb{P}(p))$ as a graded module over the ring of homogeneous polynomials in four variables, while $a_i$ are the degrees of these generators (cf. [27], Theorem 2.3).

2.2. Instanton bundles are a particularly important class of stable rank 2 vector bundles due to their many remarkable properties and applications in mathematical physics. Besides this, instanton bundles form the only known irreducible component of the moduli space $\mathcal{B}(c)$ for every $c \in \mathbb{N}$.

In the remaining part of this section we present the main results concerning instanton sheaves that will be used below. We start by recalling the definition of instanton sheaves on $\mathbb{P}^3$; see [25], the introduction, for further information on these objects.

**Definition 1.** An instanton sheaf on $\mathbb{P}^3$ is a torsion-free coherent sheaf $E$ with $c_1(E) = 0$ satisfying the following cohomological conditions:

$$h^0(E(-1)) = h^1(E(-2)) = h^2(E(-2)) = h^3(E(-3)) = 0. \quad (2.5)$$

The integer $n := c_2(E)$ is called the charge of $E$. When $E$ is locally free, we say that $E$ is an instanton bundle.

We remark that instanton bundles of rank $r > 2$ and non-locally-free instanton sheaves of rank $r \geq 2$ on $\mathbb{P}^3$ are not $\mu$-semistable in general, and also the vanishing of $h^1(E(-2))$ does not imply the vanishing of $h^2(E(-2))$. The definition above is the right generalization of the usual definition of an instanton vector bundle in the sense that, applying the Beilinson spectral sequence

$$E_1^{pq} = H^q(\mathbb{P}^3(-p-1) \otimes \mathcal{O}_{\mathbb{P}^3}(p+1)) \Rightarrow E_{\infty}^{p+q} = \begin{cases} E, & p + q = 0, \\ 0, & p + q \neq 0 \end{cases} \quad (2.6)$$

(see [35], Ch. II, Theorem 3.1.4), to an arbitrary rank $r$ instanton sheaf $E$ of charge $k$, the vanishing (2.5) yields that $E$ is the cohomology of a monad of the form

$$0 \rightarrow V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow V_{r+k} \otimes \mathcal{O}_{\mathbb{P}^3} \rightarrow V_k' \otimes \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow 0. \quad (2.7)$$

Note that, conversely, the cohomology of a monad as above is an instanton sheaf as defined in Definition 1 (see [25], Theorem 3).
The cokernel $N$ of any monomorphism of sheaves $\mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow \Omega^1_{\mathbb{P}^3}(1)$ is called a **null correlation sheaf**:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{s} \Omega^1_{\mathbb{P}^3}(1) \rightarrow N \rightarrow 0. \quad (2.8)$$

Such sheaves are precisely the rank 2 instanton sheaves of charge 1, and are parameterized by the projective space $\mathbb{P}H^0(\Omega^1_{\mathbb{P}^3}(2)) \simeq \mathbb{P}^5$. If $N$ is locally free, we say that $N$ is a **null correlation bundle**. The set of non-locally-free null correlation sheaves is parameterized by the Grassmannian of lines in $\mathbb{P}^3$: given a line $l \subset \mathbb{P}^3$ the corresponding null correlation sheaf $N_l$ is defined up to isomorphism by the exact sequence

$$0 \rightarrow N_l \rightarrow V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\xi} \mathcal{O}_l(1) \rightarrow 0. \quad (2.9)$$

### 2.3

For the purposes of this paper, it is important to study rank 4 instanton bundles of charge 1. Some of the following facts might be well known, but for lack of a reference we include proofs here.

**Lemma 2.** Every rank 4 instanton bundle $E$ of charge 1 over $\mathbb{P}^3$ fits into an exact sequence:

$$0 \rightarrow V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \rightarrow E \rightarrow N \rightarrow 0, \quad (2.10)$$

where $N$ is a null correlation sheaf. If $N$ is a null correlation bundle, then the sequence (2.10) splits. In addition,

$$h^0(E) = 2, \quad h^i(E) = 0, \quad i \geq 1. \quad (2.11)$$

**Proof.** As observed in §2.2, $E$ can be obtained as the cohomology of a monad (2.7) for $r = 4$ and $k = 1$:

$$M_E: \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow 0. \quad (2.12)$$

Without loss of generality, we can choose homogeneous coordinates $[x : y : z : w]$ in $\mathbb{P}^3$ and a basis in $V_6$ such that the map $\beta$ can be written as

$$\beta := \begin{pmatrix} x & y & z & w & 0 & 0 \end{pmatrix}. \quad (2.13)$$

Hence using the display of the above monad, we have that $E$ fits into the following short exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus \Omega^1(1) \rightarrow E \rightarrow 0. \quad (2.14)$$

From the above short exact sequence we can build up the following commutative diagram

$$\begin{array}{ccc}
V_2 \otimes \mathcal{O}_{\mathbb{P}^3} & \longrightarrow & V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \\
\downarrow & & \downarrow \\
\mathcal{O}_{\mathbb{P}^3}(-1) & \longrightarrow & V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus \Omega^1(1) \\
\downarrow & & \downarrow \\
\mathcal{O}_{\mathbb{P}^3}(-1) & \longrightarrow & \Omega^1(1) \\
\downarrow & & \downarrow \\
& & N \\
\end{array}$$

The rightmost column is the desired sequence (2.10).
If $N$ is locally free, then $\text{Ext}^1(N, \mathcal{O}_{\mathbb{P}^3}) \simeq H^1(N) = 0$, so the sequence in (2.10) splits. The equality (2.11) follows from (2.10). The lemma is proved.

**Remark 1.** Assume that a bundle $E$ is the cohomology bundle of the monad (2.12). Then an easy cohomological computation shows that $E$ is a rank 4 instanton bundle of charge 1.

Note that, substituting $N$ instead of $E$ into the Beilinson spectral sequence (2.6) yields the monad for $N$:

$$M_N: \quad 0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\pi} V_4 \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1) \to 0, \quad N = \frac{\ker \beta}{\text{im} \alpha}, \quad (2.15)$$

included together with the monad (2.12) in the commutative diagram

$$\begin{array}{ccc}
V_2 \otimes \mathcal{O}_{\mathbb{P}^3} & \downarrow & \\
\mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} V_6 \otimes \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{\beta} & \mathcal{O}_{\mathbb{P}^3}(1) \\
\mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} V_4 \otimes \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{\beta} & \mathcal{O}_{\mathbb{P}^3}(1)
\end{array} \quad (2.16)$$

In this diagram the exact middle column is obtained from the exact triple $0 \to V_2 \to V_6 \to V_4 \to 0$ which arises as the cohomology sequence of the exact triple $0 \to V_2 \otimes \Omega_{\mathbb{P}^3} \to E \otimes \Omega_{\mathbb{P}^3} \to N \otimes \Omega_{\mathbb{P}^3} \to 0$ induced by the triple (2.10). In addition, from (2.16) and (2.13) we obtain

$$\overline{\beta} = \begin{pmatrix} x & y & z & w \end{pmatrix}. \quad (2.17)$$

**Proposition 1.** Let $E$ be a rank 4 instanton bundle $E$ of charge 1 over $\mathbb{P}^3$. Then $h^0(S^2E) = 3$, $h^1(S^2E) = 5$ and $h^2(S^2E) = 0$.

**Proof.** Taking the symmetric power of the sequence in (2.14), we obtain that $S^2E$ fits into the following short exact sequence:

$$0 \to V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \Omega \to (S^2V_2 \otimes \mathcal{O}_{\mathbb{P}^3}) \oplus (V_2 \otimes \Omega(1)) \oplus S^2\Omega(2) \to S^2E \to 0.$$ 

From the long exact sequence of cohomology we have

$$0 \to S^2V_2 \to H^0(S^2E) \to \mathbf{k} \to \Lambda^2U_4^\vee \to H^1(S^2E) \to 0,$$

where $W$ is the four-dimensional $\mathbf{k}$-vector space such that $\mathbb{P}^3 = \mathbb{P}(W)$, and

$$0 \to H^2(S^2E) \to 0.$$

We conclude that $H^2(S^2E) = 0$. The map $\mathbf{k} \to \Lambda^2W^\vee$ is given by the skew-form corresponding to the morphism $\mathcal{O}_{\mathbb{P}^3}(-1) \to \Omega(1)$ in the definition of $E$, and in particular is non-zero, which implies that $\mathbf{k} \to \Lambda^2W^\vee$ is injective, and therefore

$$H^0(S^2E) \simeq S^2V_2, \quad H^1(S^2E) \simeq \Lambda^2U_4^\vee / \mathbf{k},$$

from which our result follows. The proposition is proved.
2.4. In the remaining part of this section we discuss the existence of a symplectic structure on an arbitrary rank 4 instanton bundle of charge 1. Recall that a locally-free sheaf $E$ is said to be symplectic if it admits a symplectic structure, that is, there exists an isomorphism $\varphi : E \to E^\vee$, such that $\varphi^\vee = - \varphi$. A symplectic instanton bundle is a pair $(E, \varphi)$ consisting of an instanton bundle $E$ together with a symplectic structure $\varphi$ on it; two symplectic instanton bundles $(E, \varphi)$ and $(E', \varphi')$ are isomorphic if there exists a bundle isomorphism $g : E \cong E'$ such that $\varphi = g^\vee \circ \varphi' \circ g$.

**Proposition 2.** Any rank 4 instanton bundle $E$ of charge 1 admits a symplectic structure. In particular, if $E$ splits as $E = V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus N$ where $N$ is a null correlation bundle, then any symplectic structure $\varphi$ on $E$ splits as $\varphi = \varphi_1 \oplus \varphi_2$ where $\varphi_1$ and $\varphi_2$ are symplectic structures on $V_2 \otimes \mathcal{O}_{\mathbb{P}^3}$ and $N$, respectively.

**Proof.** Let $E$ be an instanton rank 4 bundle. If $E$ splits as $E = V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus N$, where $N$ is a null correlation bundle, then any symplectic structure $\varphi$ on $E$ splits as $\varphi = \varphi_1 \oplus \varphi_2$ where $\varphi_1$ and $\varphi_2$ are symplectic structures on $V_2 \otimes \mathcal{O}_{\mathbb{P}^3}$ and $N$, respectively, say,

$$\varphi_1 : V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \cong V_2^\vee \otimes \mathcal{O}_{\mathbb{P}^3}, \quad \varphi_2 : N \cong N^\vee. \quad (2.18)$$

Then $\varphi = \varphi_1 \oplus \varphi_2 : E \cong E^\vee$ (2.19) is a symplectic structure on $E$.

Since

$$\text{Hom}(V_2 \otimes \mathcal{O}_{\mathbb{P}^3}, N) = \text{Hom}(N, V_2 \otimes \mathcal{O}_{\mathbb{P}^3}) = 0, \quad (2.20)$$

it follows immediately that any symplectic structure on $E$ splits as in (2.19).

Note also that, in view of (2.8),

$$\text{Ext}^i(V_2 \otimes \mathcal{O}_{\mathbb{P}^3}, N) = \text{Ext}^i(N, V_2 \otimes \mathcal{O}_{\mathbb{P}^3}) = 0, \quad i \geq 1. \quad (2.21)$$

Now let $E$ be a non-splitting instanton, that is, $E/V_2 \otimes \mathcal{O}_{\mathbb{P}^3}$ is a null correlation sheaf $N_l$ which is not locally free at the points of the line $l$ given by the equations, say, $\{x = y = 0\}$. This means that the morphism $\overline{\alpha}$ in the monad (2.15) for $N = N_l$ vanishes on $l$, so that

$$\overline{\alpha} = A \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = (\alpha_{ij}), \quad 1 \leq i \leq 4, \quad 1 \leq j \leq 2, \quad (2.22)$$

where $A$ is a $4 \times 2$ matrix of rank 2. The condition that $\overline{\beta} \circ \overline{\alpha}$ in (2.15) be the zero morphism together with (2.22) and (2.17) implies that all the coefficients $\alpha_{ij}$ of the matrix $A$, except $\alpha_{12}$ and $\alpha_{21}$, vanish and $\alpha_{12} + \alpha_{21} = 0$. Thus, taking without loss of generality $\alpha_{12} = 1$, we obtain

$$\overline{\alpha} = \begin{pmatrix} y \\ -x \\ 0 \\ 0 \end{pmatrix}. \quad (2.23)$$

Since the cohomology sheaf of the middle monad in (2.16) is locally free, the morphism $\alpha$ in that diagram is a subbundle morphism. This together with (2.23)
implies, again without loss of generality, that there exists a $2 \times 2$ matrix $C = (c_{ij})$ such that

$$\alpha = \begin{pmatrix} y & -x & 0 & c_{11}x + c_{12}y + z & c_{21}x + c_{22}y + w \\ -x & 0 & c_{11} & c_{12} & 0 \\ 0 & c_{11} & c_{21} & c_{22} & 0 \end{pmatrix}.$$ (2.24)

It now follows from (2.24) and (2.13) that the skew-symmetric $6 \times 6$ matrix $J$ of the $2 \times 2$ block form

$$J = \begin{pmatrix} Q & 0 & -C^t \\ 0 & 0 & -1 \\ C & 1 & 0 \end{pmatrix},$$

where $Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, satisfies the condition $\alpha = J\beta^t$. This means that, taking $-J$ for the matrix of the symplectic form $q: V_6 \to V_6^\vee$ with respect to the above choice of the basis in $V_6$, we obtain that $\alpha$ and $\beta$ as morphisms satisfy the condition $\beta = \alpha^\vee \circ q$. In other words, the monad (2.12) is symplectic. Then by Corollary 1 its cohomology bundle $E$ also admits a symplectic structure. Proposition 2 is proved.

§3. Modified instanton monads

3.1. Now we study monads of the form (1.1), with $a \geq 2$ and $k \geq 1$:

$$0 \to O_{\mathbb{P}^3}(-a) \oplus V_k \otimes O_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} V_{2k+4} \otimes O_{\mathbb{P}^3} \xrightarrow{\beta} O_{\mathbb{P}^3}(a) \oplus V'_k \otimes O_{\mathbb{P}^3}(1) \to 0,$$ (3.1)

which we call modified instanton monads. The set of isomorphism classes of bundles arising as the cohomology of such monads will be denoted by $\mathcal{G}(a, k)$. Note that, so far, $\mathcal{G}(a, k)$ could possibly be empty.

**Proposition 3.** For each $a \geq 2$ and $k \geq 1$, the family $\mathcal{G}(a, k)$ is non-empty and contains stable bundles, while every $[E] \in \mathcal{G}(a, k)$ is $\mu$-semistable. In addition, every $[E] \in \mathcal{G}(a, 1)$ is stable.

**Proof.** Let $F$ be a rank 2 instanton bundle of charge $k$. Let $a \geq 2$ and take $\sigma \in H^0(F(2a))$ such that its zero locus $X := \sigma_0$ is a curve; such $\sigma$ always exists if $F$ is a Hooft instanton bundle, for instance. Let $Y$ be a complete intersection curve given by the intersection of two surfaces of degree $a$ such that $X \cap Y = \emptyset$. According to Lemma 4.8 in [21], there exists a bundle $E$ and a section $\tau \in H^0(E(a))$ such that $(\tau)_0 = Y \cup X$ which is given as the cohomology of a monad of the form (3.1). In addition, since $F$ is stable, $X$ is not contained in any surface of degree $a$, hence neither is $Y \cup X$, and $E$ is also stable.

It is straightforward to check that every $[E] \in \mathcal{G}(a, k)$ satisfies $h^0(E(-1)) = 0$, thus $E$ is $\mu$-semistable.

Now fix $k = 1$, and assume that there is $[E] \in \mathcal{G}(a, 1)$ satisfying $h^0(E) \neq 0$; see [35], Ch. II. Setting $K := \ker \beta$, it follows that $h^0(K) \neq 0$, hence the quotient $K' := K/O_{\mathbb{P}^3}$ fits into the following exact sequence

$$0 \to K' \to V_5 \otimes O_{\mathbb{P}^3} \xrightarrow{\beta'} O_{\mathbb{P}^3}(1) \oplus O_{\mathbb{P}^3}(a) \to 0.$$
By Theorem 2.7 in [8], $K'$ is $\mu$-stable. However, the monomorphism $\alpha: O_{\mathbb{P}^3}(-a) \oplus O_{\mathbb{P}^3}(-1) \to K$ induces a monomorphism $O_{\mathbb{P}^3}(-1) \to K'$; by the $\mu$-stability of $K'$, we should have

$$-1 < \mu(K') = -\frac{a + 1}{3} \implies a < 2,$$

providing the desired contradiction. The proposition is proved.

Remark 2. Note that the space $X$ of monads (3.1) is a locally closed subscheme of the affine space $A = \text{Hom}(O_{\mathbb{P}^3}(-a) \oplus V_k \otimes O_{\mathbb{P}^3}(-1), V_{2k+4} \otimes O_{\mathbb{P}^3}, O_{\mathbb{P}^3}(a) \oplus V_k' \otimes O_{\mathbb{P}^3}(1))$ defined as $X = \{(\alpha, \beta) \in A \mid \alpha \text{ is a subbundle morphism, } \beta \text{ is an epimorphism and } \beta \circ \alpha = 0\}$, and there is the universal cohomology bundle $E$ on $X$. In case $k = 1$, it follows from Proposition 3 that $G(a, 1)$, if $X$ under the modular morphism $\Phi_X: X \to B(a^2 + 1), x \mapsto [E_{|\mathbb{P}^3 \times \{x\}}]$. Thus, $G(a, 1)$ is a constructible set, that is, a disjoint union of locally closed subsets of $B(a^2 + 1)$.

3.2. Next, we provide a cohomological characterization for modified instanton bundles.

Proposition 4. A vector bundle $E$ on $\mathbb{P}^3$ is the cohomology of a monad of the form (3.1) if and only if $H^1(E)$ has one generator in degree $-a$ and $k$ generators in degree $-1$, and its Chern classes are $c_1(E) = 0$ and $c_2(E) = a^2 + k$.

Proof. The ‘only if’ part is straightforward. If $E$ is a self dual vector bundle on $\mathbb{P}^3$ with one generator in degree $-a$ and $k$ generators in degree $-1$, then by Theorem 2.3 in [27], $E$ is the cohomology of a monad of the type:

$$0 \to O_{\mathbb{P}^3}(-a) \oplus V_k \otimes O_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} \bigoplus_{i=1}^{2k+4} O_{\mathbb{P}^3}(k_i) \xrightarrow{\beta} O_{\mathbb{P}^3}(a) \oplus V_k \otimes O_{\mathbb{P}^3}(1) \to 0.$$

Computing the Chern class gives us $c_2(E) = a^2 + k - \sum_{i=1}^{6} k_i^2$; since $c_2(E) = a^2 + k$, we have $k_i = 0$ for all $i$. The proposition is proved.

The modified instanton bundles are also related to the usual instanton bundles of higher rank in a very important way. The precise relationship is outlined in the next couple of lemmas, and then summarized in Proposition 5 below.

Lemma 3. (i) Given a vector bundle $[E] \in G(a, k)$, there exist a rank 4 instanton bundle $E$ of charge $k$, and sections $\sigma \in H^0(E(a))$ and $\tau \in H^0(E^\vee(a))$ such that the complex

$$0 \to O_{\mathbb{P}^3}(-a) \xrightarrow{\sigma} E \xrightarrow{\tau} O_{\mathbb{P}^3}(a) \to 0$$

is a monad whose cohomology coincides with $E$.

(ii) The construction of the monad (3.2) is functorial in the sense that if $E \sim E'$, then the induced isomorphism $E \xrightarrow{\sim} E'$ extends to an isomorphism of monads

$$O_{\mathbb{P}^3}(-a) \xrightarrow{\sigma} E \xrightarrow{\tau} O_{\mathbb{P}^3}(a) \sim f \xrightarrow{g} \sim h \xrightarrow{\tau'} O_{\mathbb{P}^3}(-a) \xrightarrow{\sigma'} E' \xrightarrow{\tau'} O_{\mathbb{P}^3}(a)$$
Proof. (i) Since \(a \geq 2\), there is the canonical subbundle morphism \(i: V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3}(-a) \oplus V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1)\) which, together with the morphisms \(\alpha\) and \(\beta\) from the monad (3.1), yields a subbundle morphism \(\alpha_1 := \alpha \circ i: V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to V_{2k+4} \otimes \mathcal{O}_{\mathbb{P}^3}\) and an epimorphism \(\beta_1 := i^\vee \circ \beta: V_{2k+4} \otimes \mathcal{O}_{\mathbb{P}^3} \to V'_k \otimes \mathcal{O}_{\mathbb{P}^3}(1)\). We thus obtain a new monad of type (2.7) with \(r = 4\):

\[
0 \to V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha_1} V_{2k+4} \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta_1} V'_k \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0,
\]

the cohomology bundle

\[
E = \frac{\ker(\beta_1)}{\text{im}(\alpha_1)}
\]

of which is a rank 4 instanton, according to a remark after (2.7). The monads (3.1) and (3.4) fit into a commutative diagram with exact columns

\[
\begin{array}{ccc}
V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{i} & \mathcal{O}_{\mathbb{P}^3}(a) \\
\downarrow & & \downarrow \\
V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-a) & \xrightarrow{\alpha} & V_{2k+4} \otimes \mathcal{O}_{\mathbb{P}^3} \\
\downarrow & & \downarrow \\
\mathcal{O}_{\mathbb{P}^3}(-a) & \xrightarrow{\beta} & V'_k \otimes \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(a) \\
\downarrow & & \downarrow \\
\mathcal{O}_{\mathbb{P}^3}(1) & \xrightarrow{i^\vee} & V'_k \otimes \mathcal{O}_{\mathbb{P}^3}(1)
\end{array}
\]

Now a standard diagram-chasing argument with diagram (3.6), using (3.5) and the relation \(E = \frac{\ker(\beta)}{\text{im}(\alpha)}\), yields a subbundle morphism \(\mathcal{O}_{\mathbb{P}^3}(-a) \xrightarrow{\sigma} E\) and an epimorphism \(E \xrightarrow{\tau} \mathcal{O}_{\mathbb{P}^3}(a)\) included in the monad (3.2) with the cohomology bundle \(E\).

(ii) Again, since \(a \geq 2\), it follows immediately from (3.4) and (3.5) that

\[
\text{Hom}(\mathcal{O}_{\mathbb{P}^3}(a), E') = \text{Hom}(E, \mathcal{O}_{\mathbb{P}^3}(-a)) = \text{Ext}^1(\mathcal{O}_{\mathbb{P}^3}(a), \mathcal{O}_{\mathbb{P}^3}(-a))
\]

\[
= \text{Ext}^1(E, \mathcal{O}_{\mathbb{P}^3}(-a)) = \text{Ext}^1(\mathcal{O}_{\mathbb{P}^3}(a), E') = \text{Ext}^2(\mathcal{O}_{\mathbb{P}^3}(a), \mathcal{O}_{\mathbb{P}^3}(-a)) = 0
\]

for the rank 4 instanton bundles \(E\) and \(E'\) of charge \(k\). The statement (ii) now follows from Lemma 4.1.3 in [35]. Lemma 3 is proved.

3.3. We next proceed to the relation between the cohomology bundles of the monads (3.1) and (3.2).

Lemma 4. Given a monad (3.2) with \(E\) being a rank 4 instanton bundle of charge \(k\), there is a monad of the form (3.1) whose cohomology coincides with the cohomology of the monad (3.2).

Proof. This is a diagram-chasing argument. Namely, by (2.7), \(E\) is the cohomology of a monad of the form

\[
0 \to V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha_1} V_{2k+4} \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta_1} V'_k \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0.
\]

This monad can be split to the exact triples of bundles

\[
0 \to E \to \text{coker}(\alpha_1) \to V'_k \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0
\]
and
\[ 0 \rightarrow V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha_1} V_{2k+4} \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\varepsilon} \ker(\alpha_1) \rightarrow 0. \quad (3.9) \]

Correspondingly, the monad (3.2) splits into the exact triples
\[ 0 \rightarrow \ker(\tau) \rightarrow E \xrightarrow{\tau} \mathcal{O}_{\mathbb{P}^3}(a) \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-a) \rightarrow \ker(\tau) \xrightarrow{\delta} \mathcal{E} \rightarrow 0, \quad (3.10) \]

where $\mathcal{E}$ is the cohomology bundle of the monad (3.2). The triple (3.8) and the first triple (3.10), together with the vanishing of $\text{Ext}^1(V'_k \otimes \mathcal{O}_{\mathbb{P}^3}(1), \mathcal{O}_{\mathbb{P}^3}(a))$, yields by push-out the exact triple
\[ 0 \rightarrow \ker(\tau) \xrightarrow{\gamma} \mathcal{V}'_k \otimes \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(a) \rightarrow 0 \]

which, together with (3.9), yields a commutative diagram in which we set $K := \ker(\gamma \circ \varepsilon)$:

\[ \begin{array}{ccc}
V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{\gamma} & K \\
\downarrow & & \downarrow \\
V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{\varepsilon} & \ker(\alpha_1) \\
\downarrow \gamma \circ \varepsilon & & \downarrow \gamma \\
V'_k \otimes \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(a) & \xrightarrow{\gamma} & V'_k \otimes \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(a)
\end{array} \]

Similarly, the upper horizontal triple of this diagram, together with the second triple (3.10), yield the exact triple
\[ 0 \rightarrow V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-a) \rightarrow K \rightarrow \mathcal{E} \rightarrow 0 \]

which, being combined with the middle vertical triple in this diagram, yields the monad (3.1) with the cohomology bundle $\mathcal{E}$. The lemma is proved.

Next, we argue that the instanton bundle $E$ obtained in Lemma 3 comes with a natural symplectic structure.

**Lemma 5.** If $E$ is a rank 4 instanton bundle of charge $k$ that fits into a monad of the form (3.2), such that its cohomology sheaf $\mathcal{E}$ is a vector bundle, then $E$ admits a symplectic structure, and $\tau$ is determined by $\sigma$.

**Proof.** Since $\mathcal{E}$ is a rank 2 vector bundle with $c_1(\mathcal{E}) = 0$, there is a (unique up to scaling) symplectic isomorphism $\varphi : \mathcal{E} \xrightarrow{\simeq} \mathcal{E}^\vee$. Now, repeating the proof of Lemma 3, (ii) for $\mathcal{E}' = \mathcal{E}^\vee$, we obtain an isomorphism of monads
\[ \begin{array}{ccc}
\mathcal{O}_{\mathbb{P}^3}(-a) & \xrightarrow{\sigma} & E & \xrightarrow{\tau} & \mathcal{O}_{\mathbb{P}^3}(a) \\
\downarrow g & \simeq & \varphi & \simeq & \downarrow h \\
\mathcal{O}_{\mathbb{P}^3}(-a) & \xrightarrow{\tau^\vee} & E^\vee & \xrightarrow{\sigma^\vee} & \mathcal{O}_{\mathbb{P}^3}(a)
\end{array} \]

such that $\varphi^\vee = -\varphi$, so $(E, \varphi)$ is a symplectic instanton bundle, and $\tau = \sigma^\vee \circ \varphi$. The lemma is proved.

Putting Lemmas 3, 4 and 5 together, we obtain the following statement.
Proposition 5. A rank 2 bundle $E$ belongs to $G(a,k)$, that is, $E$ is the cohomology of a monad of the form (3.1) if and only if it is also the cohomology $E = \mathcal{H}^0(A_{E,\varphi,\sigma})$ of a monad of the form:

$$A_{E,\varphi,\sigma}: \quad 0 \to O_{p3}(-a) \overset{\sigma}{\to} E \overset{\sigma\circ\varphi}{\to} O_{p3}(a) \to 0,$$

where $(E, \varphi)$ is a rank 4 symplectic instanton bundle of charge $k$.

§ 4. The set $G(a,1)$ and related families of sheaves

4.1. We introduce a piece of notation which we will use below. Denote by $I(k)$ the set of isomorphism classes of symplectic rank 4 instanton bundles with $c_2 = k$. As before, let $V_k$ and $V_{2k+4}$ be the fixed vector spaces of dimensions $k$ and $2k + 4$, respectively, and let $(\Lambda^2 V_{2k+4}^\vee)^0$ be an open subset of the vector space $\Lambda^2 V_{2k+4}^\vee$ consisting of nondegenerate symplectic forms on $V_{2k+4}$. Next, for a given morphism $\tilde{\alpha}: V_k \otimes O_{p3}(-1) \to V_{2k+4} \otimes O_{p3}$ we denote by $a$ the homomorphism $V_k \otimes U_4 \to V_{2k+4}$ corresponding to the morphism $\tilde{\alpha}$ under the isomorphism $\text{Hom}(V_k \otimes O_{p3}(-1), V_{2k+4} \otimes O_{p3}) \cong W := \text{Hom}(V_k \otimes U_4, V_{2k+4})$, where $U_4 := H^0(O_{p3}(1))^\vee$. We call $\tilde{\alpha}$ the morphism associated to $a \in W$.

Recall the description of symplectic rank 4 instantons $(E, \varphi)$ in terms of symplectic monads (4.1) below. Namely, for a given point

$$m = (a, q) \in W \times (\Lambda^2 V_{2k+4}^\vee)^0$$

consider the monad (3.7) in which $\alpha_1 = \tilde{\alpha}$ is the morphism associated to the homomorphism $a$, and the morphism $\beta_1 = \tilde{\beta}$ is such that $\tilde{\beta} = \tilde{\alpha}^t(q)$, where $\tilde{\alpha}^t(q)$ is the composition $V_{2k+4} \otimes O_{p3} \xrightarrow{q \otimes \text{id}_{O_{p3}}} V_{2k+4}^\vee \otimes O_{p3} \xrightarrow{\tilde{\alpha}^t(q)} V_k^\vee \otimes O_{p3}(1)$:

$$A_m: \quad 0 \to V_k \otimes O_{p3}(-1) \overset{\tilde{\alpha}}{\to} V_{2k+4} \otimes O_{p3} \overset{\tilde{\alpha}^t(q)}{\to} V_k^\vee \otimes O_{p3}(1) \to 0.$$  

We call $A_m$ a symplectic monad. We also denote by $\mathcal{H}^0(A_m)$ the cohomology bundle of the monad $A_m$.

Consider the set $\mathcal{M}(k)$ of symplectic monads (4.1):

$$\mathcal{M}(k) = \{(a, q) \in W \times (\Lambda^2 V_{2k+4}^\vee)^0 \mid (a, q) \text{ satisfies conditions (i) and (ii)}\},$$  

where:

(i) the morphism $\tilde{\alpha}$ associated to $a$ is a subbundle morphism,

(ii) the composition $\tilde{\alpha}^t(q) \circ \tilde{\alpha}$ is the zero morphism.

Since $W$ is a vector space, and the condition (i) is an open condition on the point $a \in W$, while the condition (ii) is a closed condition on the point $a \in W$, it follows that $\mathcal{M}(k)$ has a natural structure of a locally closed subscheme of the affine space $W \times \Lambda^2 V_{2k+4}^\vee$.

4.2. From now on we restrict to the case $k = 1$. Set $\widetilde{M} := \mathcal{M}(1)$. Note that condition (i) in the definition of $\mathcal{M}(k)$ is void in the case when $k = 1$, since in this case the vanishing of $\Lambda^2 (V_1^\vee \otimes O_{p3}(1))$ clearly implies $\alpha^t(q) \circ \alpha = 0$. Hence $\widetilde{M}$ is a nonempty open (hence dense) subset of the affine space $W \times \Lambda^2 V_6^\vee$, where $W = \text{Hom}(V_1 \otimes U_4, V_6) \simeq k^{24}$. In particular, $\widetilde{M}$ is irreducible and

$$\dim \widetilde{M} = \dim W + \dim \Lambda^2 V_6^\vee = 45.$$  

(4.3)
Proposition 6. Any rank 4 instanton of charge 1 appears as a cohomology bundle of a symplectic monad

\[ A_m : 0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\alpha^t} \mathcal{O}_{\mathbb{P}^3}(1) \to 0 \]  

(4.4)

for some \( m \in \tilde{M} \).

Proof. Let \( E \) be a rank 4 instanton of charge 1. According to Proposition (2), \( E \) admits a symplectic structure \( \varphi : E \xrightarrow{\sim} E^\vee \). It is then known from [10], §3 that under the condition \( h^0(E) = h^1(-2) = 0 \) on a symplectic bundle \( E \), this bundle is a cohomology of a symplectic monad from \( \tilde{M} \). However, the proof given therein works without changes under the slightly weaker conditions (2.5) used in Definition 1. The proposition is proved.

On \( \tilde{M} = \mathbb{P}^3 \times \tilde{M} \) there is the universal symplectic monad

\[ \mathcal{A}_M : 0 \to \mathcal{O}_M(-1,0) \xrightarrow{\alpha} V_6 \otimes \mathcal{O}_M \xrightarrow{\alpha^t} \mathcal{O}_M(1,0) \to 0 \]

with the cohomology sheaf \( \tilde{E} = \ker \alpha^t/\text{im} \alpha \). Here \( \alpha^t = \alpha^\vee \circ q_M \) and \( q_M : V_6 \otimes \mathcal{O}_M \xrightarrow{\sim} V_6^\vee \otimes \mathcal{O}_M \) is the tautological symplectic structure on \( V_6 \otimes \mathcal{O}_M \).

From now on we fix an isomorphism of the monad \( \mathcal{A}_M \) with its dual monad \( \mathcal{A}_M^\vee \) by the following diagram:

\[ \begin{array}{ccc}
\mathcal{A}_M : & \mathcal{O}_M(-1,0) & \xrightarrow{\alpha} V_6 \otimes \mathcal{O}_M \xrightarrow{\alpha^t} \mathcal{O}_M(1,0) \\
 & \downarrow \text{id} \xrightarrow{\sim} & \downarrow q_M \xrightarrow{\sim} \downarrow \text{id} \\
\mathcal{A}_M^\vee : & \mathcal{O}_M(-1,0) & \xrightarrow{(\alpha^t)^\vee} V_6^\vee \otimes \mathcal{O}_M \xrightarrow{\alpha^\vee} \mathcal{O}_M(1,0)
\end{array} \]

This isomorphism induces the symplectic structure

\[ \varphi_M : \tilde{E} \xrightarrow{\sim} \tilde{E}^\vee, \quad \varphi_m = \varphi_M|_{\mathbb{P}^3 \times \{m\}} : E_m \xrightarrow{\sim} E_m^\vee, \]

(4.5)

where \( E_m := \tilde{E}|_{\mathbb{P}^3 \times \{m\}} \), \( m \in \tilde{M} \), that is, \( (E_m, \varphi_m) \) is a symplectic rank 4 instanton of charge 1. Note that, by the universality of the space \( \tilde{M} \), for any symplectic rank 4 instanton \( (E, \varphi) \), there exists a unique point \( m \in \tilde{M} \) such that \( (E, \varphi) = (E_m, \varphi_m) \), where \( E_m \) and \( \varphi_m \) are given by (4.5). It follows from (2.11) and the Base Change that the \( \mathcal{O}_M \)-sheaf \( \tilde{U} := p_M^* \tilde{E} \) is a rank 2 locally-free sheaf and there is an exact triple on \( \tilde{M} \), where \( \text{ev} \) is the canonical morphism:

\[ 0 \to \tilde{U}_M \xrightarrow{\text{ev}} \tilde{E} \to \tilde{N} \to 0, \quad \tilde{N} := \text{coker(\text{ev})}, \]

(4.6)

and, for any \( m \in \tilde{M} \), the restriction of this triple onto \( \mathbb{P}^3 \times \{m\} \) coincides with the triple (2.10) for \( E = E_m \). We thus have a map \( \Psi : \tilde{M} \to \mathbb{P}^5 = P(\wedge^2 V_4^\vee) \), \( m \mapsto [\tilde{N}|_{\mathbb{P}^3 \times \{m\}}] \). The map \( \Psi \) has the following explicit description. Given a point \( m = (a, q) \in \tilde{M} \), consider a homomorphism \( f(a, q) : V_4 \xrightarrow{a} V_6 \xrightarrow{q} V_6^\vee \xrightarrow{a^\vee} V_4^\vee \). It is
clearly skew-symmetric: \( f(a, q) \in \wedge^2 V_q^\vee \). An easy diagram-chasing argument with the display of the monad \( A_{\mathcal{M}|_{\mathbb{P}^3 \times \{m\}}} \) (that is, equivalently, of the monad (4.4)) using (4.6) shows that

\[
\Psi(m) = \langle f(a, q) \rangle \in P(\wedge^2 V_q^\vee),
\]

so that \( \Psi \) is a well-defined morphism. By the universality of the monad \( A_{\mathcal{M}} \), \( \Psi \) is surjective.

\[\text{4.3.} \]

We next consider the set

\[ M := \{ m \in \widetilde{M} \mid \widetilde{N}|_{\mathbb{P}^3 \times \{m\}} \text{ is locally free} \}. \]

From the definition of \( M \) it follows that it is a nonempty open subset of \( \widetilde{M} \), hence it is irreducible since \( \widetilde{M} \) is irreducible. Denote

\[
\begin{align*}
E &:= \bar{E}_{\mathcal{M}}, & \varphi_{\mathcal{M}} &:= (\varphi_{\mathcal{M}})_{\mathcal{M}} : E \rightarrow E^\vee, & U &:= \bar{U}_{\mathcal{M}}, & N &:= \bar{N}_{\mathcal{M}},
\end{align*}
\]

where \( \varphi_{\mathcal{M}} \) is the symplectic structure (4.5). Note that, by Lemma 2, for any \( m \in M \) the triple (4.6) restricted onto \( \mathbb{P}^3 \times \{m\} \) splits:

\[
E_m \cong \mathcal{O}_{\mathbb{P}^3} \oplus N_m, \quad m \in M,
\]

where \( N_m \) is a null correlation bundle. We now show that these splittings globalize to the splitting of the triple \( 0 \rightarrow U \rightarrow E \rightarrow N \rightarrow 0 \) obtained from (4.6) by restriction onto \( \mathcal{M} \):

\[
E = U \oplus N.
\]

Indeed, the last triple considered as an extension is given by an element of \( \text{Ext}^1(N, U) \). By (2.20), (2.21) and the Base Change (see [32], Theorem 1.4), the sheaves \( \text{Ext}^i_{\mathcal{M}|_{\mathbb{P}^3}}(N, U) \), \( i = 0, 1 \), vanish, and the exact sequence relating global and relative Ext (see [32], (1)) yields \( \text{Ext}^1(N, U) = 0 \).

Now, for \( a \geq 2 \) and any \( m \in M \), the triple (2.10) twisted by \( \mathcal{O}_{\mathbb{P}^3}(a) \), in which we set \( E = E_m \), yields:

\[
h^0(E_m(a)) = 4 \left( \frac{a + 3}{3} \right) - a - 2, \quad h^i(E_m(a)) = 0, \quad i > 0.
\]

Formulae (4.5) and (4.11) and the Base Change show that the sheaf

\[
F = p_M(\mathcal{E}(a, 0))
\]

is a locally free \( \mathcal{O}_M \)-sheaf of rank \( r = h^0(E_m(a)) \). Consider the scheme \( T = \mathbf{P}(F^\vee) \). By the above, \( T \) is set-theoretically described as

\[
T = \{(m, \langle \sigma \rangle) \mid m \in M, \ 0 \neq \sigma \in H^0(E_m(a)) \},
\]

and the natural projection \( \rho : T \rightarrow M, (m, \langle \sigma \rangle) \mapsto m \) is a locally trivial \( \mathbb{P}^{r-1} \)-bundle. Note that, since \( M \) is an open subset of the affine space \( W \), it follows that \( T \) is a variety, and from (4.3) and (4.11) we have

\[
\dim T = h^0(E_m(a)) - 1 + \dim M = 4 \left( \frac{a + 3}{3} \right) - a + 42.
\]
4.4. On $T$ and $M$ we have canonical morphisms $F_T^\vee \xrightarrow{ev} L$ and $F_M \xrightarrow{can} E(a,0)$, respectively, where $L = \mathcal{O}_{P(F^\vee)}(1)$ is the Grothendieck sheaf. Consider the composition of morphisms

$$\sigma : \mathcal{O}_{\mathbb{P}^3} \times L^\vee \xrightarrow{ev_T^\vee} F_T \xrightarrow{can_T} E_T(a,0).$$

By definition, for any point $(m, k\sigma) \in T$ the restriction $\sigma|_{\mathbb{P}^3 \times \{(m, k\sigma)\}}$ coincides, up to a twist by $\mathcal{O}_{\mathbb{P}^3}(-a)$, with the morphism $\sigma : \mathcal{O}_{\mathbb{P}^3}(-a) \to E_m$. In view of (4.9) we can represent $\sigma$ as $\sigma = (\sigma_1, \sigma_2)$, where $\sigma_1 \in H^0(\mathcal{O}_{\mathbb{P}^3}(2))$ and $\sigma_2 \in H^0(N_m(a))$.

For the pair $\sigma = (\sigma_1, \sigma_2) \neq (0,0)$, in the sequel, together with the notation $\langle \sigma \rangle$, we adopt the following equivalent notation:

$$[\sigma_1 : \sigma_2] := \langle \sigma \rangle = \{ (\lambda \sigma_1, \lambda \sigma_2) \mid \lambda \in \mathbb{k}^\times \},$$

and also understand $[\sigma_1 : \sigma_2]$ as a point of the projective space $P(H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \oplus H^0(N_m(a)))$. Under this notation, define an open subset $S$ of $T$ as

$$S := \left\{ (m, [\sigma_1 : \sigma_2]) \in T \mid \begin{array}{l}
(\text{i}) \sigma = (\sigma_1, \sigma_2) : \mathcal{O}_{\mathbb{P}^3}(-a) \to E_m \simeq \mathcal{O}_{\mathbb{P}^3}^\oplus \oplus N_m \\
\text{is a subbundle morphism;}

(\text{ii}) \sigma_1, \sigma_2 \neq 0
\end{array} \right\}. $$

(4.17)

The subset $S$ is clearly open in $T$. Moreover, it is nonempty. Indeed, for any point $m \in M$, $E_m$ decomposes as in (4.9). Take any $a \geq 2$. Since the direct summand $N_m$ is a null correlation bundle, it follows quickly from the triple (2.8) for $N = N_m$, twisted by $\mathcal{O}_{\mathbb{P}^3}(a)$, that $N_m(a)$ is generated by global sections. From this it follows easily (cf. the proof of Proposition 1.4 in [19]) that a general section $\sigma_2 \in H^0(N_m(a))$ has one-dimensional zero-locus $\langle \sigma_2 \rangle_0$. Next, since a general section $\sigma_1 \in H^0(\mathcal{O}_{\mathbb{P}^3}^\oplus(2))$ has for its zero locus a complete intersection curve $\langle \sigma_1 \rangle_0 = D_1 \cap D_2$ for two surfaces $D_1$ and $D_2$ of degree $a$, it follows that for general $D_1$ and $D_2$ we have $\langle \sigma_1 \rangle_0 \cap \langle \sigma_2 \rangle_0 = \emptyset$. Hence the section $\sigma = (\sigma_1, \sigma_2) \in H^0(E_m(a))$ has no zeros and therefore defines a subbundle morphism $\sigma : \mathcal{O}_{\mathbb{P}^3}(-a) \to E_m$.

It follows that $S$ is irreducible and dense in $T$ since $T$ is irreducible. The morphism $\sigma_S$ is included in the monad $\mathcal{A} := (\mathcal{A}_M)_S$ on $S$:

$$\mathcal{A} : \quad 0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \boxtimes L^\vee \xrightarrow{\sigma_S^t} E_S \xrightarrow{\sigma_S} \mathcal{O}_{\mathbb{P}^3}(a) \boxtimes L \to 0,$$

(4.18)

where $\sigma_S^t$ is the composition $E_S \xrightarrow{\varphi_S} E_S^\vee \xrightarrow{\sigma_S^\vee} \mathcal{O}_{\mathbb{P}^3}(a) \boxtimes L$. By construction, for any point $(m, \langle \sigma \rangle) \in S$, the restriction of the monad $\mathcal{A}$ onto $\mathbb{P}^3 \times \{(m, \langle \sigma \rangle)\}$ is isomorphic to the monad $A_{Em, \varphi_m, \sigma}$ in (3.11). Hence

$$\mathcal{H}^0(\mathcal{A})|_{\mathbb{P}^3 \times \{(m, \langle \sigma \rangle)\}} = \mathcal{H}^0(A_{Em, \varphi_m, \sigma}), \quad (m, \langle \sigma \rangle) \in S.$$  

(4.19)

4.5. In (4.21)–(4.23) below we extend the constructions (4.12), (4.13) and (4.17)–(4.19) of the data $F$, $T$, $S$, $\mathcal{A}$ and $\mathcal{H}^0(\mathcal{A})$ over $M$ to the constructions of the corresponding data $\tilde{F}$, $\tilde{T}$, $\tilde{S}$, $\tilde{\mathcal{A}}$ and $\mathcal{H}^0(\tilde{\mathcal{A}})$ over $\tilde{M}$. As a consequence, it will follow that

$$F = \tilde{F}_M, \quad T = M \times \tilde{M}, \quad S \xrightarrow{\text{open dense}} \tilde{S},$$

$$\mathcal{A} = (\tilde{\mathcal{A}})_S, \quad \mathcal{H}^0(\mathcal{A}) = (\mathcal{H}^0(\tilde{\mathcal{A}}))_S.$$  

(4.20)
For this, we first set

\[ \tilde{F} := p_M^*(\mathcal{E}(a, 0)), \quad \tilde{T} := P(\tilde{F}^\vee) \]

and remark that formulae (4.11) are still true for any \( m \in \tilde{M} \), so that the sheaf \( \tilde{F} \) is a locally free \( O_{\tilde{M}} \)-sheaf of rank \( r = h^0(E_m(a)) \) given by (4.11), and the scheme \( \tilde{T} := P(\tilde{F}^\vee) \) is set-theoretically described as \( \tilde{T} = \{ (m, \langle \sigma \rangle) \mid m \in \tilde{M}, \ 0 \neq \sigma \in H^0(E_m(a)) \} \). The natural projection \( \tilde{\rho}: \tilde{T} \to \tilde{M}, (m, \langle \sigma \rangle) \mapsto m \), is a locally trivial \( \mathbb{P}^{r-1} \)-bundle, so that, since \( \tilde{M} \) is an open subset of the affine space \( W \), it follows that \( \tilde{T} \) is an irreducible variety of dimension

\[ \dim \tilde{T} = h^0(E_m(a)) - 1 + \dim \tilde{M} = 4\left(\frac{a + 3}{3}\right) - a + 42. \tag{4.22} \]

Here, in accordance with (4.14), \( \tilde{T} \) and \( T \) have the same dimension. Next, we have an open subset \( \tilde{S} \) of \( \tilde{T} \) defined as

\[ \tilde{S} := \{ (m, \langle \sigma \rangle) \in \tilde{T} \mid \sigma: O_{\mathbb{P}^3}(-a) \to E_m \text{ is a subbundle morphism} \}. \]

Since condition (ii) in (4.17) is open, comparing the definition of \( \tilde{S} \) with (4.17) we obtain that \( S \) is an open subset of \( T \cap \tilde{S} \), where the intersection is taken in \( \tilde{T} \).

Since \( S \) is nonempty and \( \tilde{T} \) is irreducible, the inclusion \( S \overset{\text{open dense}}{\subseteq} \tilde{S} \) in (4.20) follows and, moreover, \( \tilde{\rho}_S: S \to M \) coincides with the projection \( \rho \).

Next, we have the extension of the universal monad (4.18) from \( S \) to \( \tilde{S} \):

\[ \tilde{A}: 0 \to O_{\mathbb{P}^3}(-a) \boxtimes L^\vee \overset{\sigma}{\longrightarrow} \tilde{E}_S \overset{\sigma'}{\longrightarrow} O_{\mathbb{P}^3}(a) \boxtimes L \to 0, \]

satisfying a relation similar to (4.19):

\[ H^0(\tilde{A})|_{\mathbb{P}^3 \times \{ (m, \langle \sigma \rangle) \}} = H^0(A_{E_m, \varphi_m, \sigma}), \quad (m, \langle \sigma \rangle) \in \tilde{S}. \tag{4.23} \]

Hence relations (4.20) follow from (4.8), (4.21) and the Base Change.

Consider the modular morphisms

\[ \Phi_S: S \to \mathcal{B}(a^2 + 1), \quad \Phi_{\tilde{S}}: \tilde{S} \to \mathcal{B}(a^2 + 1), \tag{4.24} \]

defined by the families of sheaves \( H^0(\tilde{A}) \) and \( H^0(\tilde{\mathcal{A}}) \), respectively. The relations (4.20), (4.23) and Proposition 5 together with the irreducibility of \( \tilde{S} \) yield the following.

**Proposition 7.** (i) For \( a \geq 2 \), the set \( \mathcal{G}(a, 1) \) of isomorphism classes of the cohomology sheaves of monads (3.1) for \( k = 1 \) is the image of the modular morphism

\[ \Phi_S: \tilde{S} \to \mathcal{B}(a^2 + 1), \quad (m, \langle \sigma \rangle) \mapsto [H^0(\tilde{A})|_{\mathbb{P}^3 \times \{ (m, \langle \sigma \rangle) \}}], \]

defined by the family \( H^0(\tilde{A}) \) of sheaves over \( \tilde{S} \). Its closure \( \overline{\mathcal{G}(a, 1)} \) in \( \mathcal{B}(a^2 + 1) \) is an irreducible scheme.

(ii) The set \( \mathcal{G}(a, 1)_0 := \Phi_S(S) \) is dense in \( \overline{\mathcal{G}(a, 1)} \).
4.6. In the remaining part of this section we construct a new family of monads $A_Y$ on $\mathbb{P}^3$, with base $Y$ and cohomology sheaves belonging to $G(a, 1)$, for which the related modular morphism

$$
\Phi_Y : Y \to B(a^2 + 1), \quad y \mapsto [H^0(A_Y)|_{\mathbb{P}^3 \times \{y\}}],
$$

has $G(a, 1)_0$ as its image (see Proposition 8 below). This family will be used in the next section to prove one of the main results of the paper—the rationality of $G(a, 1)$.

To construct the variety $Y$, consider the moduli space of $B := B(1)$ of locally-free null correlation bundles on $\mathbb{P}^3$. This is well known to be isomorphic to $\mathbb{P}^5 \setminus G(2, 4)$, where $G(2, 4)$ is the Plücker hyperquadric (see [35], Theorem 4.3.4, for example). Moreover, on $B = \mathbb{P}^3 \times B$ there is the universal family $\mathcal{N}$ of null correlation bundles. Consider the vector bundle $\mathcal{E} = V_2 \otimes \mathcal{O}_B \oplus \mathcal{N}$ and denote $E_b = \mathcal{E}|_{\mathbb{P}^3 \times \{b\}}$, $N_b = \mathcal{N}|_{\mathbb{P}^3 \times \{b\}}$, $b \in B$, so that

$$
E_b = V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus N_b, \quad b \in B. \quad (4.25)
$$

By linear algebra, there are canonical isomorphisms $\varphi_{(1)} : V_2 \otimes \mathcal{O}_B \cong V_2^\vee \otimes \wedge^2 V_2 \otimes \mathcal{O}_B$ and $\varphi_{(2)} : \mathcal{N} \cong \mathcal{N}^\vee \otimes \wedge^2 \mathcal{N}$. The sheaf $\mathcal{N}$ fits in the exact triple

$$
0 \to \mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathcal{O}_B(-1) \to \Omega^1_{\mathbb{P}^3}(1) \otimes \mathcal{O}_B \to \mathcal{N} \to 0 \quad \text{globalizing (2.8), so that}
$$

$$
\wedge^2 \mathcal{N} \cong \mathcal{O}_{\mathbb{P}^3} \otimes \mathcal{O}_B(1). \quad (4.26)
$$

Consider the varieties $B_1 := V(\wedge^2 V_2 \otimes \mathcal{O}_B) \setminus \{0\text{-section}\}$ and $B_2 := V(\mathcal{O}_B(-1)) \setminus \{0\text{-section}\}$. Note that the pullback of a line bundle onto its total space with the 0-section removed trivializes this bundle, so we obtain $\pi_1^* \mathcal{O}_B(1) \cong \mathcal{O}_{B_2}$, hence by (4.26), $(\wedge^2 \mathcal{N})|_{B_2} \cong \mathcal{O}_{B_2}$, respectively $(\wedge^2 V_2 \otimes \mathcal{O}_B)|_{B_1} \cong \mathcal{O}_{B_1}$. Thus, we obtain the symplectic structures

$$
\varphi_{B_1} := (\varphi_{(1)})_{B_1} : V_2 \otimes \mathcal{O}_B \cong V_2^\vee \otimes \mathcal{O}_B, \quad \text{and} \quad \varphi_{B_2} := (\varphi_{(2)})_{B_2} : \mathcal{N}_{B_2} \cong \mathcal{N}^\vee_{B_2}. \quad (4.27)
$$

Consider the variety $\tilde{B} := B_1 \times_B B_2$. On $\tilde{B}$ we obtain from $\mathcal{E}$ a vector bundle $\mathcal{E}_{\tilde{B}}$ with the symplectic structure $\varphi_{\tilde{B}}$, where

$$
\mathcal{E}_{\tilde{B}} = V_2 \otimes \mathcal{O}_{\tilde{B}} \oplus \mathcal{N}_{\tilde{B}}, \quad \varphi_{\tilde{B}} = \varphi_{B_1} \oplus \varphi_{B_2} : \mathcal{E}_{\tilde{B}} \to \mathcal{E}_{\tilde{B}}^\vee. \quad (4.27)
$$

and $\varphi_1 := (\varphi_{B_1})_{\tilde{B}} : V_2 \otimes \mathcal{O}_{\tilde{B}} \cong V_2^\vee \otimes \mathcal{O}_{\tilde{B}}$ and $\varphi_2 := (\varphi_{B_2})_{\tilde{B}} : \mathcal{N}_{\tilde{B}} \cong \mathcal{N}_{\tilde{B}}^\vee$. By the above, we have the following description of the varieties $B_1$, $B_2$ and $\tilde{B}$:

$$
B_1 = \{ (b, \varphi_1) \mid b \in B, \varphi_1 : V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \cong V_2^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \text{ is a symplectic structure} \},
$$

$$
B_2 = \{ (b, \varphi_2) \mid b \in B, \varphi_2 : N_b \cong N^\vee_b \text{ is a symplectic structure} \},
$$

$$
\tilde{B} = \{ (b, \varphi_1, \varphi_2) \mid (b, \varphi_i) \in B_i, i = 1, 2 \} = B_1 \times_B B_2. \quad (4.28)
$$
4.7. The following constructions (see (4.30)–(4.35)) are parallel to the constructions (4.17)–(4.19). Twisting the equality (4.25) by $\mathcal{O}_{p^3}(a)$, we obtain as in (4.11):

$$
    h^0(E_b(a)) = 4\left(\frac{a + 3}{3}\right) - a - 2 \quad \text{and} \quad h^i(E_b(a)) = 0, \quad i > 0.
$$

Thus, as in (4.12), the sheaf $F_B = p_{B*}(\mathcal{E}(a,0))$ is a locally free $\mathcal{O}_B$-sheaf of rank $r = h^0(E_b(a))$. Consider the variety $T := \mathbb{P}(F_B^\vee)$. Similarly to (4.13) we have

$$
    T = \{(b, \langle \sigma \rangle) \mid b \in B, \ 0 \neq \sigma \in H^0(E_b(a))\}. \quad (4.29)
$$

For any point $(b, \langle \sigma \rangle) \in T$ in view of (4.25) we may represent $\sigma$ as a pair $\sigma = (\sigma_1, \sigma_2)$, $\sigma_1 \in H^0(V_2 \otimes \mathcal{O}_{p^3}(a))$ and $\sigma_2 \in H^0(N_b(a))$. Thus, using the notation (4.16) we can rewrite (4.29) as $T = \{(b, [\sigma_1 : \sigma_2]) \mid b \in B, [\sigma_1 : \sigma_2] \in P(H^0(E_b(a)))\}$. On the other hand, representing $\sigma$ as a morphism $\sigma : \mathcal{O}_{p^3}(-a) \to E_b$, we see that, when $(b, \langle \sigma \rangle)$ runs through $T$, the morphisms $\sigma$, as in (4.15), globalize to a morphism $\sigma_T : \mathcal{O}_{p^3}(-a) \boxtimes L_T^\vee \to \mathcal{E}_T$ on $T$, where $L_T$ is the Grothendieck sheaf $\mathcal{O}_T/B(1)$. Next, similarly to (4.17), we define an open subset $S$ of $T$ as

$$
    S := \left\{(b, [\sigma_1 : \sigma_2]) \in T \mid \begin{array}{l}
    (i) \, (\sigma_1, \sigma_2) : \mathcal{O}_{p^3}(-a) \to E_b \\
    \quad \quad \text{is a subbundle morphism;} \\
    (ii) \, \sigma_1, \sigma_2 \neq 0
    \end{array} \right\}. \quad (4.30)
$$

Note that $S$ is a nonempty set. (The proof mimics that of nonemptiness of the subset $M$ of $T$ given in the paragraph after (4.17).) By the Base Change, the sheaf $F_B = p_{B*}(\mathcal{E}_B(a,0))$ is isomorphic to the sheaf $(F_B)_B$. Therefore, from the definition of $T$ it follows that the variety $\tilde{Y} := \mathbb{P}(F_B^\vee)$ is isomorphic to $\tilde{B} \times_B T$:

$$
    \tilde{Y} \simeq \tilde{B} \times_B T. \quad (4.31)
$$

Thus by (4.28) and (4.29) we have

$$
    \tilde{Y} = \{(b, \varphi_1, \varphi_2, [\sigma_1 : \sigma_2]) \mid (b, \varphi_1, \varphi_2) \in \tilde{B}, \ [\sigma_1 : \sigma_2] \in P(H^0(E_b(a)))\},
$$

and the natural projection $\tilde{Y} \to \tilde{B}$, $(\beta, \langle \sigma \rangle) \mapsto \beta$, is a locally trivial $\mathbb{P}^{r-1}$-bundle. We now use (4.31) and the open subset $S$ of $T$ to define an open subset $Y$ of $\tilde{Y}$ as

$$
    Y := \tilde{B} \times_B S. \quad (4.32)
$$

Here, $Y$ is a nonempty open subset in $\tilde{Y}$ since $S$ is nonempty. It follows that $Y$ is irreducible and dense in $\tilde{Y}$ since $\tilde{Y}$ is irreducible. In addition, using (4.30) and the above description of $\tilde{Y}$ we obtain:

$$
    Y = \left\{(b, \varphi_1, \varphi_2, [\sigma_1 : \sigma_2]) \in \tilde{Y} \mid \begin{array}{l}
    (i) \, (\sigma_1, \sigma_2) : \mathcal{O}_{p^3}(-a) \to E_b \\
    \quad \quad \text{is a subbundle morphism;} \\
    (ii) \, \sigma_1, \sigma_2 \neq 0
    \end{array} \right\}. \quad (4.33)
$$

The morphism $\sigma_Y : = (\sigma_T)_Y$ is included in the universal monad on $Y$:

$$
    A_Y : \ 0 \to \mathcal{O}_{p^3}(-a) \boxtimes L_Y \xrightarrow{\sigma_Y} \mathcal{E}_Y \xrightarrow{\sigma_Y^i} \mathcal{O}_{p^3}(a) \boxtimes L_Y \to 0, \quad (4.34)
$$
where $L_Y = (L_T)_Y$ and $\sigma^t_Y$ is the composition $E_Y \xrightarrow{\phi_Y} E_Y^\vee \xrightarrow{\sigma_Y^t} \mathcal{O}_{\mathbb{P}^3}(a) \boxtimes L_Y$. By construction, for any point $(\beta, \langle \sigma \rangle) \in Y$, $\beta = (b, \varphi_1, \varphi_2)$, the restriction of the monad $A_Y$ onto $\mathbb{P}^3 \times \{(\beta, \langle \sigma \rangle)\}$ is isomorphic to the monad $A_{E_b, \varphi_1 \oplus \varphi_2, \sigma}$ in (3.11). Hence

$$H^0(A_Y)|_{\mathbb{P}^3 \times \{(\beta, \langle \sigma \rangle)\}} = H^0(A_{E_b, \varphi_1 \oplus \varphi_2, \sigma})$$

(4.35)

4.8. Now consider the rank 2 vector bundle $U$ on $M$ defined in (4.8) and its associated principal frame bundle

$$I := \text{Isom}(V_2 \otimes \mathcal{O}_M, U) \xrightarrow{\xi} M$$

together with the tautological isomorphism $V_2 \otimes \mathcal{O}_I \xrightarrow{\sim} U_I$. Using this isomorphism and applying the functor $\xi^*$ to (4.10) we obtain an isomorphism

$$E_I \cong V_2 \otimes \mathcal{O}_I \oplus N_I.$$  

(4.36)

Besides, by (4.8), we have a symplectic structure $\varphi_I := (\varphi_M)_I: E_I \xrightarrow{\sim} E_I^\vee$ on $E_I$. This symplectic structure in view of (4.36) splits into a direct sum of two symplectic structures

$$\varphi_I = \varphi_{I,1} \oplus \varphi_{I,2}, \quad \varphi_{I,1}: V_2 \otimes \mathcal{O}_I \xrightarrow{\sim} V_2^\vee \otimes \mathcal{O}_I, \quad \varphi_{I,2}: N_I \xrightarrow{\sim} N_I^\vee.$$  

(4.37)

Note that by the description of the morphism $\Psi$ given in (4.7), we have $\Psi(M) = B$. Now, comparing (4.27) and (4.28) with (4.36) and (4.37), we obtain a morphism

$$\Gamma: I \rightarrow \tilde{B}, \quad x \mapsto (b, \varphi_1, \varphi_2), \quad b = \Psi(\xi(x)), \quad \varphi_i = \varphi_{I,i}|_{\mathbb{P}^3 \times \{x\}}, \quad i = 1, 2,$$

(4.38)

such that

$$E_I \cong (\mathcal{E}_B)_I \quad \text{and} \quad \varphi_I \cong (\varphi_B)_I,$$  

(4.39)

and these isomorphisms are compatible with the direct sum decompositions (4.36), (4.37) and (4.27). From (4.38) and the surjectivity of $\Psi$ it follows that $\Gamma$ is also surjective. Set

$$X := I \times_M S, \quad Y \xleftarrow{\Gamma_Y} X \xrightarrow{\xi_S} S, \quad F_I := p_{I*}(\mathcal{E}_I(a, 0)).$$

(4.40)

From (4.12), (4.39), the isomorphism $F_{\tilde{B}} \cong (F_B)_{\tilde{B}}$ and the Base Change we obtain $F_I \cong (F_B)_I$, so that, in view of (4.31) and the equality $T = \mathbb{P}(F_I^\vee)$, the variety $\tilde{X} := \mathbb{P}(F_{\tilde{X}}^\vee)$ satisfies the isomorphisms

$$I \times_M T \cong \tilde{X} \cong I \times_{\tilde{B}} \tilde{Y}.$$  

(4.41)

The definition of $X$ (see (4.40)) and the left isomorphism (4.41) imply that there exists an open embedding $X \hookrightarrow \tilde{X}$ such that $X = \tilde{X} \times_T S$. Therefore, comparing the descriptions (4.33) and (4.17) of $Y$ and $S$ and using the right isomorphism (4.41), we obtain:

$$X \cong I \times_{\tilde{B}} Y.$$  

(4.42)
This together with (4.39) implies that $E_X \cong (E_Y)_X$. Moreover, since $X = I \times_M S$, we have

$$\mathcal{A}_X \cong (\mathcal{A}_Y)_X,$$

where the monads $\mathcal{A}$ and $\mathcal{A}_Y$ were defined in (4.18) and (4.34), respectively. Consider the modular morphisms

$$\Phi_X : X \to B(a^2 + 1) \quad \text{and} \quad \Phi_Y : Y \to B(a^2 + 1),$$

(4.44)
defined by the (families of) sheaves $\mathcal{H}^0(\mathcal{A}_X)$ and $\mathcal{H}^0(\mathcal{A}_Y)$, respectively. From (4.43), (4.42) and (4.40) it follows that $\Phi_X$ factors through $\Gamma_S$ and through $\xi_S$ as:

$$\Phi_X = \Phi_Y \circ \Gamma_Y = \Phi_S \circ \xi_S.$$ Here, $\Phi_S : S \to B(a^2 + 1)$ is the modular morphism (4.24), $\xi_S$ in (4.40) is surjective by the surjectivity of $\xi$, and $\Gamma_Y$ is surjective as $\Gamma$ is surjective. Hence

$$G(a, 1)_0 = \Phi_S(S) = \Phi_Y(Y).$$

(4.45)

On the other hand, by Proposition 7, $G(a, 1)_0$ is dense in $G(a, 1)$. We thus obtain the following.

**Proposition 8.** Let $\Phi_Y : Y \to B(a^2 + 1)$ be the modular morphism defined by the family of sheaves $\mathcal{H}^0(\mathcal{A}_Y)$, where $\mathcal{A}_Y$ is the monad (4.34). Then $\Phi_Y(Y)$ is dense in $G(a, 1)$.

---

§ 5. Series of rational irreducible components of the moduli spaces $B(a^2 + 1)$

In this section we prove the first main result of the paper, Theorem 1, which states the existence of a new infinite series of rational irreducible components of the Gieseker-Maruyama schemes $B(a^2 + 1)$, $a \geq 2$. To do this, we first prove a necessary technical result, Theorem 3, about the properties of the modular morphism $\Phi_Y : Y \to B(a^2 + 1)$ from Proposition 8. The argument leading to Theorem 3 occupies the bulk of this section. For the convenience of the reader, we will divide this argument into three steps which we now describe in detail.

5.1. **Step 1.** At this step we construct a rational variety $G$ (see (5.9) for a definition) and relate it to the variety $S$ defined in (4.30) via a morphism $\tau : S \to G$. This morphism will be obtained as the quotient morphism of a $G$-action on $S$, where $G$ is a certain factor group of the group $\overline{G} = GL(V_2) \times k^\times$.

Consider the variety $Y$. According to the description (4.32), (4.33), any point $y \in Y$ is a collection of data

$$y = (b, \varphi_1, \varphi_2, [\sigma_1 : \sigma_2]),$$

where

(i) $b \in B$;

(ii) $\varphi_1 : V_2 \otimes \mathcal{O}_{P_3} \cong V_2^\vee \otimes \mathcal{O}_{P_3}$ and $\varphi_2 : N_b \cong N_b^\vee$ are symplectic isomorphisms:

$$\varphi_1 \in H^0(\wedge^2 (V_2 \otimes \mathcal{O}_{P_3})^\vee) \setminus \{0\} = \wedge^2 V_2^\vee \setminus \{0\} \cong k^\times,$$

$$\varphi_2 \in H^0(\wedge^2 N_b^\vee) \setminus \{0\} = H^0(\mathcal{O}_{P_3}) \setminus \{0\} \cong k^\times;$$

(5.1)
(iii) $\sigma_1$ and $\sigma_2$ are:
\[
0 \neq \sigma_1 \in H^0(V_2 \otimes O_{\mathbb{P}^3}(a)) = \text{Hom}(V_2^\vee, W), \\
W := H^0(O_{\mathbb{P}^3}(a)), \quad 0 \neq \sigma_2 \in H^0(N_b(a));
\]
\[(5.2)\]

(iv) $\sigma = (\sigma_1, \sigma_2) : O_{\mathbb{P}^3}(-a) \to V_2 \otimes O_{\mathbb{P}^3} \oplus N_b$ is a subbundle morphism.

In $\text{Hom}(V_2^\vee, W)$ consider the open subset
\[
\text{Hom}^\text{in}(V_2^\vee, W) := \{\sigma_1 \in \text{Hom}(V_2^\vee, W) \mid \sigma_1 : V_2^\vee \to W \text{ is a monomorphism}\}.
\]

One can easily see (using the argument in the paragraph after (4.17)) that
\[
\text{Hom}^\text{in}(V_2^\vee, W) = \{\sigma_1 \in \text{Hom}(V_2^\vee, W) \mid \dim(\sigma_1)_0 = 1\}.
\]

In addition, note that the group $\text{GL}(V_2)$ acts naturally on $\text{Hom}^\text{in}(V_2^\vee, W)$ via its action on $V_2^\vee$, and we have an isomorphism
\[
\text{Hom}^\text{in}(V_2^\vee, W)/\text{GL}(V_2) \xrightarrow{\sim} \text{Gr}(2, W)
\]
and the factorization morphism
\[
\tau_1 : \text{Hom}^\text{in}(V_2^\vee, W) \to \text{Gr}(2, W), \quad \sigma_1 \mapsto \text{im}(\sigma_1 : V_2^\vee \hookrightarrow W). \quad (5.4)
\]

Next, as was mentioned in §4 (see the paragraph after (4.17)), the set
\[
H^0(N_b(a))^* := \{\sigma_2 \in H^0(N_b(a)) \mid \dim(\sigma_2)_0 = 1\}
\]

is open dense in $H^0(N_b(a))$. Besides, it is clearly invariant under the action of the group $\text{Aut}(N_b(a)) = k^\times$.

(Recall that the null correlation bundle $N_b$ is stable and therefore simple, that is, $\text{End}(N_b(a)) = k \cdot \text{id}$.) Hence
\[
P(H^0(N_b(a))^*) = H^0(N_b(a))^*/\text{Aut}(N_b(a)) \xrightarrow{\text{open}} P(H^0(N_b(a))) \simeq \mathbb{P}^r,
\]

where $r = 2\left(\frac{a+3}{3}\right) - a - 3$, and we have the factorization morphism
\[
\tau_2 : H^0(N_b(a))^* \to P(H^0(N_b(a)))^*, \quad \sigma_2 \mapsto \langle \sigma_2 \rangle. \quad (5.6)
\]

Now the above condition (iv) imposed on $(\sigma_1, \sigma_2)$ can be rewritten in the form:
\[
(\sigma_1, \sigma_2) \in H_b := \{(\sigma_1, \sigma_2) \in \text{Hom}^\text{in}(V_2^\vee, W) \times H^0(N_b(a))^* \mid (\sigma_1)_0 \cap (\sigma_2)_0 = \emptyset\}.
\]

(5.7)

Clearly, $H_b$ is a dense open subset of $\text{Hom}^\text{in}(V_2^\vee, W) \times H^0(N_b(a))^*$. This subset is invariant under the action of the group $k^\times$ by homotheties. Therefore, denoting $P(H_b) := H_b/k^\times$ and using (5.4) and (5.6), we obtain the factorization morphism
\[
\tau : P(H_b) \to \text{Gr}(2, W) \times P(H^0(N_b(a)))^*, \quad [\sigma_1 : \sigma_2] \mapsto (\tau_1(\sigma_1), \tau_2(\sigma_2)). \quad (5.8)
\]

To globalize the above pointwise (with respect to $b \in B$) constructions over $B$, set $K := p_{B*}(\mathcal{N}(a, 0))$. By the Base Change, $K$ is a locally-free sheaf of rank $h^0(N_b(a)) = \chi(N_b(a))$ over $B$, hence $P(K^\vee)$ is a rational variety which is described pointwise as $P(K^\vee) = \{(b, \langle \sigma_2 \rangle) \mid b \in B, \langle \sigma_2 \rangle \in P(H^0(N_b(a)))\}$. Consider its dense open subset
\[
\Pi := \{(b, \langle \sigma_2 \rangle) \in P(K^\vee) \mid \langle \sigma_2 \rangle \in P(H^0(N_b(a)))^*\}.
\]
and set
\[ G := \text{Gr}(2, W) \times \Pi = \{(b, V, \langle \sigma_2 \rangle) \mid V \in \text{Gr}(2, W), (b, \langle \sigma_2 \rangle) \in \Pi \}. \] (5.9)

Note that since \( \Pi \) is a rational variety, \( G \) is rational as well.

Next, consider the variety \( S \) introduced in (4.30). From (4.30) and (5.7) we obtain
\[ S = \{(b, [\sigma_1 : \sigma_2]) \mid b \in B, \ [\sigma_1 : \sigma_2] \in P(H_b) \}. \]

Thus, by (5.8), we have a well-defined morphism
\[ \tau : S \to G, \quad (b, [\sigma_1 : \sigma_2]) \mapsto (b, \tau_1(\sigma_1), \tau_2(\sigma_2)). \] (5.10)

Consider the group
\[ \overline{G} = \text{GL}(V_2) \times k^\times, \] (5.11)
its normal subgroup \( G' = \{(\rho \cdot \text{id}_{V_2}, \rho) \mid \rho \in k^\times \}, \) and let
\[ G = \overline{G}/G' \] (5.12)
be the factor group. We use the following notation for elements of \( G' \): \( [g_1 : \lambda] := (g_1, \lambda)G' = \{(\rho g_1, \rho \lambda) \mid \rho \in k^\times \}, \) \((g_1, \lambda) \in \overline{G}\). The group \( G \) acts naturally on \( \overline{S} \) as:
\[ a_S : S \times G \to \overline{S}, \quad ((b, [\sigma_1 : \sigma_2]), [g_1 : \lambda]) \mapsto (b, [g_1 \circ \sigma_1 : \lambda \sigma_2]), \] (5.13)
and formulae (5.3)–(5.10) show that \( G = S/G \) and the morphism \( \tau : S \to G \) in (5.10) is the quotient morphism for this action and it is a principal \( G \)-bundle. Therefore in view of (4.11) we have:
\[ \dim G = \dim P(H_b) + \dim B - \dim G = 4 \left( \frac{a + 3}{3} \right) - a - 2. \] (5.14)

5.2. Step 2. Here we construct a new rational variety \( P \) and relate it to the variety \( Y \) via the morphism \( \pi : Y \to P \) defined below.

Consider the variety \( P(\wedge^2 V_2 \otimes O_B \oplus O_B(1)) \) together with the embeddings
\[ P(\wedge^2 V_2 \otimes O_B) \xrightarrow{i_1} P(\wedge^2 V_2 \otimes O_B \oplus O_B(1)) \xrightarrow{i_2} P(O_B(1)) \]
and denote \( P \tilde{B} := P(\wedge^2 V_2 \otimes O_B \oplus O_B(1)) \setminus \{\text{im}(i_1) \cup \text{im}(i_2)\} \). By construction, the natural projection \( P \tilde{B} \to B \) is a locally trivial fibration with fibre
\[ F \simeq \mathbb{P}^1 \setminus \{2 \text{ points}\}. \] (5.15)

Using the second relation (4.26), the description (4.28) of the varieties \( B_1, B_2, \tilde{B} \) and the notation \( [\varphi_1 : \varphi_2] := \{\lambda \varphi_1, \lambda \varphi_2 \mid \lambda \in k^\times \} \) for \((b, \varphi_1, \varphi_2) \in \tilde{B}\), we obtain that
\[ P \tilde{B} = \{(b, [\varphi_1 : \varphi_2]) \mid (b, \varphi_i) \in B_i, \ i = 1, 2\} \]
and that the group \( k^\times \) acts naturally on \( \tilde{B} \) as
\[ \tilde{B} \times k^\times \to \tilde{B}, \quad ((b, \varphi_1, \varphi_2), \lambda) \mapsto (b, \lambda \varphi_1, \lambda \varphi_2). \] (5.16)
We obtain the morphism
\[ (b, \varphi_1, \varphi_2) \mapsto (b, [\varphi_1 : \varphi_2]) \] (5.18)
Consider the varieties
\[ PY := \tilde{P} \tilde{B} \times_B S \] (5.9)
\[ \mathcal{P} := \tilde{P} \tilde{B} \times_B \mathcal{G} \] (5.10)
and
\[ \dim \mathcal{P} = \dim \mathcal{G} + \dim F = 4 \left( \frac{a + 3}{3} \right) - a - 1. \] (5.20)
From the description (4.33) of the variety \( Y \) and of the morphism \( \pi_{\tilde{B}} \) in (5.18) we obtain the morphism
\[ \pi_Y : Y \to PY, \quad (b, \varphi_1, \varphi_2, [\sigma_1 : \sigma_2]) \mapsto (b, [\varphi_1 : \varphi_2], [\sigma_1 : \sigma_2]), \] (5.21)
and from (5.16)–(5.18) it follows that \( \pi_Y \) is a factorization morphism of the following \( k^\times \)-action on \( Y \):
\[ a_Y : Y \times k^\times \to Y, \quad ((b, \varphi_1, \varphi_2, [\sigma_1 : \sigma_2]), \lambda) \mapsto (b, \lambda \varphi_1, \lambda \varphi_2, [\sigma_1 : \sigma_2]). \] (5.22)
Correspondingly, the morphism \( \tau : S \to \mathcal{G} \) defined in (5.10) induces a morphism
\[ \tau_Y : PY \to \mathcal{P}, \quad (b, [\varphi_1 : \varphi_2], [\sigma_1 : \sigma_2]) \mapsto (b, [\varphi_1 : \varphi_2], \tau_1(\sigma_1), \tau_2(\sigma_2)). \] (5.23)
Define the morphism \( \pi : Y \to \mathcal{P} \) as the composition
\[ \pi : Y \xrightarrow{\pi_Y} PY \xrightarrow{\tau_Y} \mathcal{P}, \quad (b, \varphi_1, \varphi_2, [\sigma_1 : \sigma_2]) \mapsto (b, [\varphi_1 : \varphi_2], \tau_1(\sigma_1), \tau_2(\sigma_2)). \] (5.24)
5.3. Step 3. At this step we show that \( \pi \) is the quotient morphism of a well-defined \( \mathcal{G} \)-action on \( Y \) and that the fibres of \( \pi \) are isomorphism classes of symplectic monads \( A_y \) in (5.26). Together with the technical Lemma 6 and Theorem 3, this will conclude the proof of Theorem 1.

**Definition 2.** Introduce on \( Y \) the following equivalence relation:
\[ y = (b, \varphi_1, \varphi_2, [\sigma_1 : \sigma_2]) \sim (\tilde{b}, \tilde{\varphi}_1, \tilde{\varphi}_2, [\tilde{\sigma}_1 : \tilde{\sigma}_2]) = \tilde{y} \] (5.25)
if there is a commutative diagram with rows \( A_y \) and \( A_{\tilde{y}} \):

\[
\begin{array}{ccc}
A_y : & \mathcal{O}_{\mathbb{P}^3}(-a) \xrightarrow{(\sigma_1, \sigma_2)} V_2 \otimes \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{(\sigma_1 \circ \varphi_1, \sigma_2 \circ \varphi_2)} N_b \mathcal{O}_{\mathbb{P}^3}(a) \\
& h_- \xrightarrow{\sim} (g_1, g_2) \xrightarrow{\sim} h_+ & \\
A_{\tilde{y}} : & \mathcal{O}_{\mathbb{P}^3}(-a) \xrightarrow{(\tilde{\sigma}_1, \tilde{\sigma}_2)} V_2 \otimes \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{(\tilde{\sigma}_1 \circ \tilde{\varphi}_1, \tilde{\sigma}_2 \circ \tilde{\varphi}_2)} N_b \mathcal{O}_{\mathbb{P}^3}(a) \\
\end{array}
\] (5.26)
We denote by \([y] = [b, \varphi_1, \varphi_2, [\sigma_1 : \sigma_2]]\) the equivalence class of a point \(y = (b, \varphi_1, \varphi_2, [\sigma_1 : \sigma_2]) \in Y\) under this equivalence relation.

Note that, in diagram (5.26), one has
\[
g_1 \in \text{Isom}(V_2 \otimes O_{P^3}, V_2 \otimes O_{P^3}) \cong \text{GL}(V_2) \tag{5.27}
\]
and \(g_2 \in \text{Isom}(N_b, N_e)\) which in view of the stability of \(N_b\) implies that
\[
b = \tilde{b} \quad \text{and} \quad g_2 = \lambda \cdot \text{id}_{N_b}, \quad \lambda \in \mathbb{k}^*; \tag{5.28}
\]
besides, the isomorphisms \(h_-\) and \(h_+\) are multiplications by some constants \(\mu, \nu \in \mathbb{k}^*\), respectively:
\[
h_- = \mu \cdot \text{id}_{O_{P^3}(-a)} \quad \text{and} \quad h_+ = \nu \cdot \text{id}_{O_{P^3}(a)}. \tag{5.29}
\]
Furthermore, in view of (5.1), (5.27), (5.28) and the symplecticity of \(\varphi_1\) and \(\varphi_2\), we have in (5.26)
\[
\varphi_1 = \lambda_1 \varphi_1 \quad \text{and} \quad \varphi_2 = \lambda_2 \varphi_2, \quad \lambda_1, \lambda_2 \in \mathbb{k}^*. \tag{5.30}
\]
The leftmost square of diagram (5.26) together with (5.29) implies:
\[
\tilde{\sigma}_1 = \frac{1}{\mu} g_1 \circ \sigma_1 \quad \text{and} \quad \tilde{\sigma}_2 = \frac{\lambda}{\mu} \sigma_2. \tag{5.31}
\]
Correspondingly, the rightmost square of diagram (5.26) yields \(\nu \sigma_1^\vee \circ \varphi_1 = \tilde{\sigma}_1^\vee \circ \tilde{\varphi}_1 \circ g_1\) and \(\nu \sigma_2^\vee \circ \varphi_2 = \lambda \tilde{\sigma}_2^\vee \circ \tilde{\varphi}_2\). Substituting (5.29)–(5.31) into the last equalities we obtain the relations \(g_1^\vee \circ \varphi_1 \circ g_1 = \text{det}(g_1) \varphi_1\) and \(g_2^\vee \circ \varphi_2 \circ g_2 = \lambda^2 \varphi_2\), which imply \(\nu = \lambda_1 \text{det}(g_1)/\mu\) and \(\nu = \lambda_2 \lambda^2/\mu\). Whence, \(\lambda_1 \text{det}(g_1) = \lambda_2 \lambda^2\). This relation shows that the \(G\)-action (5.13) on \(S\) lifts to the following \(G\)-action on \(PY\):
\[
a_{PY} : PY \times G \to PY,
((b, [\varphi_1 : \varphi_2, [\sigma_1 : \sigma_2]], [g_1 : \lambda]) \mapsto \left(b, \left[\frac{\varphi_1}{\text{det}(g_1)} : \frac{\varphi_2}{\lambda^2}\right], [g_1 \circ \sigma_1 : \lambda \sigma_2]\right). \tag{5.32}
\]
Thus, \(P = PY/G\) and the morphism
\[
\tau_Y : PY \to P \tag{5.33}
\]
in (5.23) is the quotient morphism for this action. We therefore have a commutative diagram
\[
\begin{array}{ccc}
PY & \xrightarrow{\tau_Y} & P \\
\downarrow \text{pr}_S & & \downarrow \text{pr}_G \\
S & \xrightarrow{\tau} & G
\end{array}
\]
where \(\text{pr}_G\) is a natural projection.
Next, from (5.21), (5.22), (5.32) and (5.33) it follows that the morphism
\[ \pi: Y \to \mathcal{P} \] in (5.24) is the quotient morphism of the following action of the group
\[ G = \text{GL}(V_2) \times \mathbb{k}^\times \] on \( Y \), where \( G \) was introduced in (5.11):
\[ a_Y: Y \times G \to Y, \]
\[ ((b, \varphi_1, \varphi_2, [\sigma_1: \sigma_2]), ([g_1: \lambda], \mu)) \mapsto \left( b, \frac{\mu \varphi_1}{\det(g_1)}, \frac{\mu \varphi_2}{\lambda^2}, [g_1 \circ \sigma_1: \lambda \sigma_2] \right). \] (5.34)

Moreover,
\[ \pi: Y \to \mathcal{P} = Y/G \] is a principal \( G \)-bundle, (5.35)
and computations (5.27)–(5.32) show that the equivalence class \([y] \) of any point \( y \in Y \) is the \( G \)-orbit of \( y \):
\[ [y] = a_Y(\{y\} \times G) = \pi^{-1}(\pi(y)), \quad y \in Y. \] (5.36)

In other words, \( \mathcal{P} \) is the set of equivalence classes of points of \( Y \):
\[ \mathcal{P} = \{[y] \mid y \in Y\}. \] (5.37)

Note that, by Corollary 1, the equality \([y] = [\bar{y}]\), that is, the isomorphism of symplectic monads \( A_y \) and \( A_{\bar{y}} \) in (5.26), is equivalent to the isomorphism of their cohomology rank 2 bundles as symplectic bundles \((H^0(A_y), \psi_y)\) and \((H^0(A_{\bar{y}}), \psi_{\bar{y}})\), that is, to the commutativity of the diagram
\[ \begin{array}{ccc}
H^0(A_y) & \xrightarrow{\psi_y} & H^0(A_y)^\vee \\
\downarrow f & \simeq & \downarrow f^\vee \\
H^0(A_{\bar{y}}) & \xrightarrow{\psi_{\bar{y}}} & H^0(A_{\bar{y}})^\vee 
\end{array} \] (5.38)

Here \( \psi_y \), respectively \( \psi_{\bar{y}} \), is a symplectic isomorphism induced by the symplectic isomorphism of the monad \( A_y \) with its dual \( A_y^\vee \), respectively of \( A_{\bar{y}} \) with \( A_{\bar{y}}^\vee \). Thus, denoting by \([H^0(A_y), \psi_y]\) the isomorphism class of the pair \((H^0(A_y), \psi_y)\), we have:
\[ [y] = [H^0(A_y), \psi_y] = [H^0(A_y)]. \] (5.39)

This together with (5.35)–(5.37) shows that the modular morphism
\[ \Phi_Y: Y \to \mathcal{B}(a^2 + 1), \quad y \mapsto [H^0(A_y)], \]
factors through an injective map \( \Theta: \mathcal{P} \to \mathcal{B}(a^2 + 1) \), that is,
\[ \Phi_Y = \Theta \circ \pi. \] (5.40)

Since \( Y \) is clearly smooth, the map \( \Theta \) is actually a morphism. This results from the following well-known general result. (For the convenience of the reader we give its proof here.)

**Lemma 6.** Let \( X \), \( Y \) and \( Z \) be quasiprojective varieties with \( Y \) smooth, and let \( a: X \to Y \) and \( b: X \to Z \) be morphisms such that \( a \) is surjective and \( b \) is constant on the fibres of \( a \). Then there exists a morphism \( f: Y \to Z \) such that \( b = f \circ a \).
Proof. Consider the morphism \( g : X \to Y \times Z, x \mapsto (a(x), b(x)) \), and let \( Y \xrightarrow{a'} Y \times Z \xrightarrow{b'} Z \) be the projections onto factors so that \( a = a' \circ g \) and \( b = b' \circ g \). Since \( b \) is constant on the fibres of \( p \), it follows that \( \tilde{a} := a'|_{g(X)} : g(X) \to Y \) is a bijection. Therefore, as \( Y \) is smooth, \( \tilde{a} \) is an isomorphism (see, for example, [37], Ch. 2, §4.4, Theorem 2.16). The desired morphism \( f \) is now the composition \( f = b' \circ \tilde{a}^{-1} \). The lemma is proved.

Now Proposition 8, together with (5.20), the fact that \( \mathcal{P} \) is rational, (5.35) and (5.40), yields the following.

**Theorem 3.** There exists an injective morphism \( \Theta : \mathcal{P} \hookrightarrow \mathcal{B}(a^2 + 1) \) such that the modular morphism \( \Phi_Y : Y \to \mathcal{B}(a^2 + 1) \) factorizes as

\[
\Phi_Y : Y \xrightarrow{\pi} \mathcal{P} \xrightarrow{\Theta} \mathcal{B}(a^2 + 1),
\]

where \( \pi : Y \to \mathcal{P} \) is a principal \( \overline{G} \)-bundle with the group \( \overline{G} \) defined in (5.11). The variety \( \mathcal{G}(a,1) \) containing the rational variety \( \mathcal{G}(a,1)_0 = \Theta(\mathcal{P}) \) as a dense subset is rational of dimension \( 4\binom{a+3}{3} - a - 1 \).

**5.4.** We next obtain the following important formula.

**Lemma 7.** For every \([\mathcal{E}] \in \mathcal{G}(a,1)_0 \) with \( a \geq 2 \), we have

\[
h^1(\text{End}(\mathcal{E})) = 4 \binom{a+3}{3} - a - 1 + \varepsilon(a),
\]

where \( \varepsilon(a) = 1 \) when \( a = 3 \), and \( \varepsilon(a) = 0 \) when \( a \neq 3 \).

**Proof.** Since \( \mathcal{E} \) is a self-dual rank 2 bundle, we have \( \text{End}(\mathcal{E}) \simeq S^2 \mathcal{E} \oplus \Lambda^2 \mathcal{E} = S^2 \mathcal{E} \oplus \mathcal{O}_{\mathbb{P}^3} \), thus \( h^1(\text{End}(\mathcal{E})) = h^1(S^2 \mathcal{E}) \). We compute the latter.

By the definition of \( \mathcal{G}(a,1)_0 \) (see Proposition 7, (ii), (4.17) and (4.19)), \( \mathcal{E} \) is the cohomology of a complex \( M^\bullet \) with terms \( M^{-1} = \mathcal{O}_{\mathbb{P}^3}(-a) \), \( M^0 = E \simeq \mathcal{O}_{\mathbb{P}^3} \oplus N \) and \( M^1 = \mathcal{O}_{\mathbb{P}^3}(a) \). We proceed to the double complex \( M^\bullet \otimes M^\bullet \) and to its total complex \( T^\bullet \). The symmetric part of \( T^\bullet \) is the monad \( 0 \to E(-a) \to S^2 E \oplus \mathcal{O}_{\mathbb{P}^3} \to E(a) \to 0 \), whose cohomology sheaf is isomorphic to \( S^2 \mathcal{E} \). Therefore this monad can be broken into two short exact sequences

\[
0 \to K \to S^2 E \oplus \mathcal{O}_{\mathbb{P}^3} \to E(a) \to 0 \quad \text{and} \quad 0 \to E(-a) \to K \to S^2 \mathcal{E} \to 0.
\]

Since \( h^0(E(-a)) = h^0(S^2 \mathcal{E}) = 0 \), it follows that \( h^0(K) = 0 \); in addition, \( h^1(E(a)) = h^2(S^2 E \oplus \mathcal{O}_{\mathbb{P}^3}) = 0 \) (use Proposition 1) implies that \( h^2(K) = 0 \). It then follows in view of the splitting \( E \simeq \mathcal{O}_{\mathbb{P}^3} \oplus N \) that

\[
h^1(S^2 \mathcal{E}) = h^1(K) + h^2(E(-a)) = h^1(K) + \varepsilon(a), \quad \varepsilon(a) := h^2(N(-a)),
\]

since \( h^1(E(-a)) = 0 \) for \( a \geq 2 \).

To complete our calculation, consider the exact sequence

\[
0 \to H^0(S^2 E \oplus \mathcal{O}_{\mathbb{P}^3}) \to H^0(E(a)) \to H^1(K) \to H^1(S^2 E \oplus \mathcal{O}_{\mathbb{P}^3}) \to 0.
\]
Since \( h^0(S^2E \oplus \mathcal{O}_{\mathbb{P}^3}) = 4 \) and \( h^1(S^2E \oplus \mathcal{O}_{\mathbb{P}^3}) = 5 \) by Proposition 1, we conclude that
\[
h^1(K) = h^0(E(a)) + 1 = h^0(N(a)) + 2h^0(\mathcal{O}_{\mathbb{P}^3}(a)) + 1,
\]
which, together with the equality in equation (5.42), yields the desired formula. The lemma is proved.

It is interesting to observe that the right-hand side of the formula in Lemma 7 yields the value of \( h^1(\text{End}(E)) \) expected by the deformation theory when \( a = 2 \) and \( a = 3 \), respectively 37 and 77; when \( a \geq 4 \), one can check that \( 4(a+3) - a - 1 > 8(a^2 + 1) - 3 \).

Noting that, in view of Theorem 3, the dimension of \( \overline{G(a,1)} \) equals \( h^1(\text{End}(E)) \) for \( a = 2 \) and \( a \geq 4 \), as calculated in Lemma 7, and using Proposition 7, we have therefore completed the proof of the first main result of this paper, Theorem 1.

In particular, for the case \( a = 2 \), we conclude that rank 2 bundles given as the cohomology of monads of the form (1.5) yield a dense subset of an irreducible component of \( B(5) \) with expected dimension 37.

§ 6. Cohomology bundle \( \mathcal{E} \) of the monad of type (1.6) and the related reflexive sheaf \( \mathcal{F} \)

6.1. Consider the set
\[
\mathcal{H} = \{ [\mathcal{E}] \in B(5) \mid \mathcal{E} \text{ is the cohomology of a monad of type (1.6)} \}.
\]
It is known that \( \mathcal{H} \neq \emptyset \) (see [21], Table 5.3, \( c_2 = 5 \), Case (2), ii)). Note that the set \( \mathcal{H} \) is a constructible subset of \( B(5) \), as well as \( \mathcal{G}(2,1) \) (see Remark 2 after Proposition 3). The aim of this and the subsequent sections is to prove the following theorem.

**Theorem 4.** The set \( \mathcal{H} \) satisfies the condition
\[
\dim(\mathcal{H} \setminus (\mathcal{G}(2,1) \cap \mathcal{H})) \leq 36.
\]
Its closure in \( B(5) \) does not constitute a component of \( B(5) \).

In this section we relate the vector bundle \( [\mathcal{E}] \in \mathcal{H} \setminus (\mathcal{G}(2,1) \cap \mathcal{H}) \) to a rank 2 reflexive sheaf \( \mathcal{F} \) with Chern classes \( c_1(\mathcal{F}) = 0 \), \( c_2(\mathcal{F}) = 2 \) and \( c_3(\mathcal{F}) = 2k \), \( 0 \leq k \leq 6 \), which appears as the middle cohomology of a left-exact complex \( K^\bullet \) (see (6.25)) induced by the monad of type (1.6) defining \( \mathcal{E} \). This relation will be established in Proposition 9. Then we will use it in §7 to prove Theorem 4.

Let \( [\mathcal{E}] \in \mathcal{H} \setminus (\mathcal{G}(2,1) \cap \mathcal{H}) \) be the cohomology bundle of the monad of the form (1.6):
\[
M^\bullet: \quad 0 \to M^{-1} \xrightarrow{\alpha} M^0 \xrightarrow{\beta} M^1 \to 0,
\]
\[
M^{-1} = \mathcal{O}_{\mathbb{P}^3}(-2) \oplus V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1), \quad M^0 = \mathcal{O}_{\mathbb{P}^3}(-1) \oplus V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1),
\]
\[
M^1 = V'_2 \otimes \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2).
\]

(6.1)
Since the bundle $V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1)$ is a uniquely defined subbundle of the bundle $M^{-1}$ (correspondingly, $V'_2 \otimes \mathcal{O}_{\mathbb{P}^3}(1)$ is a uniquely defined quotient bundle of $M^1$), we obtain a commutative diagram in which $\alpha_0$ and $\beta_0$ are the induced morphisms:

![Diagram](image)

Here the induced monad

$$0 \to V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha_0} M^0 \xrightarrow{\beta_0} V'_2 \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0$$

has the rank 4 cohomology bundle

$$E = \frac{\ker \beta_0}{\text{im} \alpha_0}. \quad (6.4)$$

Mimicking now the argument with diagram (3.6), we obtain that there exist a sub-bundle morphism $\sigma: \mathcal{O}_{\mathbb{P}^3}(-2) \to E$ and an epimorphism $\tau: E \to \mathcal{O}_{\mathbb{P}^3}(2)$ which yield the following monad and its cohomology bundle $\mathcal{E}$:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \xrightarrow{\sigma} E \xrightarrow{\tau} \mathcal{O}_{\mathbb{P}^3}(2) \to 0, \quad \mathcal{E} = \frac{\ker \tau}{\text{im} \sigma}. \quad (6.5)$$

Since there is a uniquely defined (up to a scalar multiple) quotient morphism $M^0 \to \mathcal{O}_{\mathbb{P}^3}(-1)$, we have a well-defined morphism

$$\tilde{\alpha}: V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xleftarrow{\alpha_0} M^0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \quad (6.6)$$

and, dually, a well-defined morphism

$$\tilde{\beta}: \mathcal{O}_{\mathbb{P}^3}(1) \xleftarrow{\beta_0} V'_2 \otimes \mathcal{O}_{\mathbb{P}^3}(1). \quad (6.7)$$

Assume that both $\tilde{\alpha}$ and $\tilde{\beta}$ are nonzero morphisms. Then a standard diagram-chasing argument shows that, in the monad (6.3), one can split out a direct summand $\mathcal{O}_{\mathbb{P}^3}(-1)$ from $V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1)$ and $M^0$ (correspondingly, split out a direct summand $\mathcal{O}_{\mathbb{P}^3}(1)$ from $M^0$ and $V'_2 \otimes \mathcal{O}_{\mathbb{P}^3}(1)$) without changing its cohomology bundle $E$. Thus, the monad (6.3) reduces to a monad

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha'_0} V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta'_0} \mathcal{O}_{\mathbb{P}^3}(1) \to 0, \quad E = \frac{\ker \beta'_0}{\text{im} \alpha'_0}. \quad (6.8)$$

Now by Remark 1 after Lemma 2, $E$ is a rank 4 instanton bundle, so that, by (6.5) and Lemma 4, $\mathcal{E}$ is the cohomology bundle of the monad (3.1) for $a = 2$ and $k = 1$. This means that $\mathcal{E} \in \mathcal{G}(2,1) \cap \mathcal{H}$, contrary to the assumption on $\mathcal{E}$.

We thus may assume that either (a) $\tilde{\alpha} = 0$ and $\tilde{\beta} \neq 0$, or (b) $\tilde{\alpha} = \tilde{\beta} = 0$. (We omit the case $\tilde{\alpha} \neq 0$ and $\tilde{\beta} = 0$, since it is completely similar to the case (a).)
6.2. Case (a): $\tilde{\alpha} = 0$ and $\tilde{\beta} \neq 0$. We are going to show that this case is impossible.

First, note that, since $\tilde{\beta} \neq 0$, we may as above split out a direct summand $O_{P^3}(1)$ from the middle term and the right-hand term of the monad (6.3), without changing its cohomology bundle $E$. Thus, this monad reduces to a monad

$$0 \to V_2 \otimes O_{P^3}(-1) \xrightarrow{\alpha'} O_{P^3}(-1) \oplus V_6 \otimes O_{P^3} \xrightarrow{\beta'} O_{P^3}(1) \to 0, \quad E = \frac{\ker \beta'}{\im \alpha'}. \quad (6.9)$$

Next, the condition $\tilde{\alpha} = 0$ means that the subbundle morphism $\alpha'$ in (6.3) factors through a subbundle morphism $\alpha''$ in the commutative diagram

$$
\begin{array}{c}
V_2 \otimes O_{P^3}(-1) \xrightarrow{\alpha''} V_6 \otimes O_{P^3} \\
\downarrow \quad \downarrow \\
V_2 \otimes O_{P^3}(-1) \xrightarrow{\alpha'} O_{P^3}(-1) \oplus V_6 \otimes O_{P^3} \\
\downarrow \quad \downarrow \\
O_{P^3}(-1) \\
\end{array}
\xrightarrow{\lambda} 
\begin{array}{c}
F_4 \\
\downarrow \quad \downarrow \\
F_5 \\
\downarrow \quad \downarrow \\
O_{P^3}(-1) \\
\end{array}
\quad (6.10)
$$

where $F_4 := \text{coker } \alpha''$ and $F_5 := \text{coker } \alpha'$ are vector bundles of rank 4 and 5, respectively. From this diagram it follows immediately that $O_{P^3}(-1)$ splits out as a direct summand of $F_5$:

$$F_5 \cong O_{P^3}(-1) \oplus F_4. \quad (6.11)$$

The monad (6.9) and the diagram (6.10) yield a commutative diagram

$$
\begin{array}{c}
F_3 \xrightarrow{\eta \circ \lambda} F_4 \\
\downarrow \quad \downarrow \quad \downarrow \\
E \xrightarrow{\nu} F_5 \xrightarrow{\eta} O_{P^3}(1) \\
\downarrow \quad \downarrow \quad \downarrow \\
B \xrightarrow{\mu \circ \nu} O_{P^3}(-1) \xrightarrow{\pi} C \\
\end{array}
\quad (6.12)
$$

where $F_3 := \ker(\eta \circ \lambda)$, $A := F_4/F_3$, $B := E/F_3$ and $C := O_{P^3}(1)/A$. Here $A \neq 0$, since otherwise $C \cong O_{P^3}(1)$, and then $\pi$ is not surjective, contrary to (6.12). Hence $C$ is a torsion sheaf, and $A$, $B$ and $F_3$ are torsion-free sheaves of rank 1, 1 and 4, respectively. Therefore, the diagram (6.12) implies that $c_1(F_4) - c_1(E) = 2c_1(O_{P^3}(1))$. On the other hand, in view of (6.11) we have a well-defined injective morphism $\rho: E \xrightarrow{\nu} F_5 \xrightarrow{pr_2} F_4$ such that, by the Snake Lemma, $Q := \text{coker } \rho \cong A/B$ is a torsion sheaf. In addition, by the above equality, $c_1(Q) = 2c_1(O_{P^3}(1)) \neq 0,$
that is, $Q \neq 0$. However, (6.12) and the Snake Lemma yield a commutative diagram

$$
\begin{array}{cccccc}
E & \xrightarrow{\rho} & F_4 & \xrightarrow{i} & Q \\
\downarrow & & \downarrow i & & \downarrow \bar{i} \\
E & \xrightarrow{\nu} & O_{\mathbb{P}^3}(-1) \oplus F_4 & \xrightarrow{\eta} & O_{\mathbb{P}^3}(1) \\
\downarrow & & \downarrow \nu & & \downarrow \eta \\
O_{\mathbb{P}^3}(-1) & & O_{\mathbb{P}^3}(1)/\bar{i}(Q)
\end{array}
$$

(6.13)

where $i$ is the inclusion of the direct summand and $\bar{i}$ is the induced morphism. But the torsion sheaf $Q$ is not a subsheaf of $O_{\mathbb{P}^3}(1)$, and we obtain a contradiction, as claimed.

Summarizing the above arguments, we see that the bundle $[E] \in \mathcal{H}$ is the cohomology $H^0(M\bullet)$ of a monad $M\bullet$ of the form (1.6) satisfying condition (a): $(\bar{\alpha}, \bar{\beta}) \neq (0, 0)$, then $M\bullet$ is reducible to a monad of the form (1.5), that is, $[E] \in \mathcal{H} \cap \mathcal{G}(2, 1)$. Thus, denoting

$$\mathcal{H}_0 := \{[E] \in \mathcal{H} \mid E = H^0(M\bullet), \text{ where } M\bullet \text{ satisfies condition (b): } \bar{\alpha} = \bar{\beta} = 0\},$$

we obtain

$$\mathcal{H} \setminus (\mathcal{H} \cap \mathcal{G}(2, 1)) \subset \mathcal{H}_0.$$

(6.14)

(6.15)

We thus proceed to the study of the case when $\bar{\alpha} = \bar{\beta} = 0$.

6.3. Case (b): $\bar{\alpha} = \bar{\beta} = 0$. First consider the commutative diagram

$$
\begin{array}{cccccc}
O_{\mathbb{P}^3}(1) & \xrightarrow{j_1} & V_6 \otimes O_{\mathbb{P}^3} \oplus O_{\mathbb{P}^3}(1) & \xrightarrow{i_0} & M^0 & \xrightarrow{g_0} & O_{\mathbb{P}^3}(1) \\
\downarrow & & \downarrow i_0 & & \downarrow g_0 & & \downarrow \ \\
V_6 \otimes O_{\mathbb{P}^3} & \xrightarrow{h_1} & V_6 \otimes O_{\mathbb{P}^3} \oplus O_{\mathbb{P}^3}(-1) & \xrightarrow{i} & V_6 \otimes O_{\mathbb{P}^3} \oplus O_{\mathbb{P}^3}(1) & \xrightarrow{g} & O_{\mathbb{P}^3}(1)
\end{array}
$$

(6.16)

and the exact triples following from (6.3) and (6.4)

$$0 \to V_2 \otimes O_{\mathbb{P}^3}(-1) \xrightarrow{\alpha_0} M^0 \xrightarrow{c_0} C_0 \to 0,$$

$$0 \to E \xrightarrow{d_0} C_0 \xrightarrow{c_0} V_2' \otimes O_{\mathbb{P}^3}(1) \to 0, \quad C_0 := \text{coker } \alpha_0, \quad \beta_0 = e_0 \circ c_0.$$  

(6.17)

(6.18)

The condition $\bar{\alpha} = 0$ implies that there exists a subbundle morphism $0 \to V_2 \otimes O_{\mathbb{P}^3}(-1) \xrightarrow{\alpha_1} V_6 \otimes O_{\mathbb{P}^3} \oplus O_{\mathbb{P}^3}(1)$ such that

$$\alpha_0 = i_0 \circ \alpha_1.$$  

(6.19)
Setting $C := \text{coker}(h_0 \circ \alpha_0)$, $C_1 := \text{coker} \alpha_1$, $\alpha_2 := h_1 \circ \alpha_1$, and $C_2 := \text{coker} \alpha_2$, we obtain from (6.16), (6.17) and (6.19) an induced commutative diagram

$$
\begin{array}{c}
\mathcal{O}_{\mathbb{P}^3}(1) \xrightarrow{j_1} \mathcal{O}_{\mathbb{P}^3}(1) \\
\downarrow \quad \downarrow \\
C_1 \xrightarrow{\tau_1} C_0 \xrightarrow{\tau_0} \mathcal{O}_{\mathbb{P}^3}(-1) \\
\downarrow \quad \downarrow \\
C_2 \xrightarrow{\tau_1} C \xrightarrow{\tau_0} \mathcal{O}_{\mathbb{P}^3}(-1)
\end{array}
(6.20)
$$

and an exact triple

$$
0 \to V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha_2} V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{c_2} C_2 \to 0.
(6.21)
$$

From the condition $\tilde{\beta} = 0$ and diagram chasing it follows that there exists an injective morphism $j: \mathcal{O}_{\mathbb{P}^3}(1) \to E$ such that $\tilde{j}_0 = d_0 \circ j$. From this relation and (6.18), (6.20) and (6.21) by diagram chasing we obtain the following data:

1) an exact triple

$$
0 \to \mathcal{O}_{\mathbb{P}^3}(1) \xrightarrow{j} E \xrightarrow{h} E_3 \to 0, \quad E_3 := \text{coker} j;
(6.22)
$$

2) a commutative diagram

$$
\begin{array}{c}
\mathcal{F} \xrightarrow{\varepsilon} E_3 \xrightarrow{\varepsilon} \mathcal{O}_{\mathbb{P}^3}(-1) \\
\downarrow \quad \downarrow \\
C_2 \xrightarrow{\tau} C \xrightarrow{\tau} \mathcal{O}_{\mathbb{P}^3}(-1) \\
\downarrow \quad \downarrow \\
V_2' \otimes \mathcal{O}_{\mathbb{P}^3}(1) \xrightarrow{\tau} V_2' \otimes \mathcal{O}_{\mathbb{P}^3}(1)
\end{array}
(6.23)
$$

where $d$ and $e$ are the induced morphisms, $\varepsilon := d \circ \tau$ and $\bar{\varepsilon} := e \circ \tau$;

3) a sheaf

$$
\mathcal{F} := \ker \varepsilon
(6.24)
$$

and a left-exact complex

$$
K^\bullet: \quad 0 \to K^{-1} \xrightarrow{\alpha_2} K^0 \xrightarrow{\beta_2} K^1 \to 0, \quad \beta_2 := \bar{\varepsilon} \circ c_2,
(6.25)
$$

$$
K^{-1} = V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1), \quad K^0 = V_6 \otimes \mathcal{O}_{\mathbb{P}^3}, \quad K^1 = V_2' \otimes \mathcal{O}_{\mathbb{P}^3}(1),
$$

such that

$$
\mathcal{H}^0(K^\bullet) = \mathcal{F} \quad \text{and} \quad \mathcal{H}^1(K^\bullet) = \text{coker} \varepsilon.
(6.26)
$$
From (6.5), (6.25) and the vanishing of $\text{Hom}(\mathcal{O}_{\mathbb{P}^3}(1), \mathcal{O}_{\mathbb{P}^3}(-2))$ follows the commutative diagram

$$
\begin{array}{ccc}
\mathcal{O}_{\mathbb{P}^3}(1) & \xrightarrow{j} & \mathcal{O}_{\mathbb{P}^3}(1) \\
\downarrow & & \downarrow \\
\mathcal{O}_{\mathbb{P}^3}(-2) & \xrightarrow{\sigma} & E \\
\downarrow & & \downarrow \\
\mathcal{O}_{\mathbb{P}^3}(-2) & \xrightarrow{h} & \text{coker } \sigma
\end{array}
$$

(6.27)

where $\mathcal{L} := \text{coker}(h \circ \sigma)$ and $j'$ is an induced morphism which is nonzero, hence injective, since coker $\sigma$ is locally free by the exact triple $0 \to \mathcal{E} \to \text{coker } \sigma \to \mathcal{O}_{\mathbb{P}^3}(2) \to 0$ following from (6.5). Since $\mathcal{E}$ is stable by assumption, so that $h^0(\mathcal{E}(-1)) = 0$ (see [35]), the last triple and (6.27) yield a commutative diagram

$$
\begin{array}{ccc}
\mathcal{O}_{\mathbb{P}^3}(1) & \xrightarrow{j'} & \mathcal{O}_{\mathbb{P}^3}(1) \\
\downarrow & & \downarrow \\
\mathcal{E} & \xrightarrow{\sigma} & \text{coker } \sigma \\
\downarrow & & \downarrow \\
\mathcal{E} & \xrightarrow{\sigma} & \mathcal{O}_{\mathbb{P}^3}(2)
\end{array}
$$

(6.28)

where $\mathbb{P}^2 = \text{Supp}(\text{coker } \overline{\beta})$ is a projective plane in $\mathbb{P}^3$. Note that, in this diagram, $\overline{\beta}$ is the composition $\overline{\beta}: \mathcal{O}_{\mathbb{P}^3}(1) \xrightarrow{\text{can}} M^0 \xrightarrow{\beta} M^1$, and $\text{im } \overline{\beta} \hookrightarrow \mathcal{O}_{\mathbb{P}^3}(2)$ since $\overline{\beta} = 0$. Thus, $\mathbb{P}^2$ is uniquely defined by the morphism $\beta$ in the monad $M^\bullet$. In a similar way, since $\overline{\alpha} = 0$, the morphism $\alpha$ in $M^\bullet$ uniquely defines a morphism $\overline{\alpha}: \mathcal{O}_{\mathbb{P}^3}(-2) \to \mathcal{O}_{\mathbb{P}^3}(-1)$, hence a projective plane $\mathbb{P}^2_0 = \text{Supp}(\text{coker } \overline{\alpha})$. For these two planes we use the notation

$$
\mathbb{P}^2 = \mathbb{P}^2(M^\bullet, \beta), \quad \mathbb{P}^2_0 = \mathbb{P}^2(M^\bullet, \alpha).
$$

(6.29)

Consider the lower horizontal triple in (6.28):

$$
0 \to \mathcal{E} \xrightarrow{\theta} \mathcal{L} \xrightarrow{\gamma} \mathcal{O}_{\mathbb{P}^2}(2) \to 0, \quad \mathbb{P}^2 = \mathbb{P}^2(M^\bullet, \beta).
$$

(6.30)

6.4. The properties of the sheaf $\mathcal{L}$ introduced above are summarized in the following lemma.

**Lemma 8.** The sheaf $\mathcal{L}$ in (6.30) is a stable reflexive rank 2 sheaf on $\mathbb{P}^3$, $[\mathcal{L}] \in R(1, 4, 6)$.

**Proof.** We first show that the triple (6.30) does not split. Indeed, otherwise, the lower horizontal triple in (6.27) extends to a commutative push-out diagram

$$
\begin{array}{ccc}
\mathcal{O}_{\mathbb{P}^3}(-2) & \xrightarrow{\sigma} & E_3 \\
\downarrow & & \downarrow \\
\mathcal{O}_{\mathbb{P}^3}(-2) & \xrightarrow{\sigma} & \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^2}(2) \\
\downarrow & & \downarrow \\
\mathcal{O}_{\mathbb{P}^3}(-2) & \xrightarrow{\sigma} & \mathcal{O}_{\mathbb{P}^2}(2)
\end{array}
$$

(6.31)
Thus, by the first equality in (6.35), the inequality
\[
\delta' : \mathcal{O}_{\mathbb{P}^2}(2) \to \mathcal{F}.
\]
On the other hand, (6.25) and (6.26) yield an exact triple
\[
0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \to \ker \beta_2 \to \mathcal{F} \to 0
\]
which, together with (6.32), extends to a push-out diagram similar to (6.31):
\[
\begin{array}{c}
\mathcal{O}_{\mathbb{P}^3}(-2) \ar[r] \ar@{=}[d] & \ker \beta_2 \ar[r] \ar@{>->}[d] & \mathcal{F} \ar@{-->}[d] \\
\mathcal{O}_{\mathbb{P}^3}(-2) \ar[r] & \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^2}(2) \ar[r] & \mathcal{O}_{\mathbb{P}^2}(2)
\end{array}
\]
where \(\delta''|_{\mathcal{O}_{\mathbb{P}^2}(2)}\) is nonzero. However, this is impossible since \(\ker \beta_2\) by definition is torsion free as a subsheaf of a locally-free sheaf \(V_6 \otimes \mathcal{O}_{\mathbb{P}^3}\).

Next, since \(\mathcal{E} \cong \mathcal{E}^\vee\) is locally free and \(\text{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(2), \mathcal{O}_{\mathbb{P}^3}) = \mathcal{O}_{\mathbb{P}^2}(-1)\), then, applying the functor \(\text{Ext}^\bullet(\mathcal{E}, \mathcal{O}_{\mathbb{P}^3})\) to the triple (6.30) we have an exact sequence
\[
0 \to \mathcal{L}^\vee \xrightarrow{\theta^\vee} \mathcal{E} \to \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\varphi} \mathcal{L} \to 0, \quad \mathcal{L} := \text{Ext}^1(\mathcal{L}, \mathcal{O}_{\mathbb{P}^3}).
\]
Let \(d = \dim(\mathcal{L})\). Consider the three possible cases: (a) \(d = 2\); (b) \(d = 1\); and (c) \(d = 0\). We show that the cases (a) and (b) lead to a contradiction.

(a) \(d = 2\). In this case \(\dim \text{Sing} \mathcal{L} = 2\), that is, the torsion subsheaf \(\text{Tors}(\mathcal{L})\) of \(\mathcal{L}\) has dimension 2. This necessarily implies that the composition \(\text{Tors}(\mathcal{L}) \hookrightarrow \mathcal{L} \twoheadrightarrow \mathcal{O}_{\mathbb{P}^2}(2)\) is an isomorphism giving the splitting of the triple (6.30), contrary to the above.

(b) \(d = 1\). In this case \(\mathcal{L} = \mathcal{O}_Z(-1)\) for \(Z\) a subscheme of \(\mathbb{P}^2\) of dimension \(\dim Z = 1\). Hence \(\ker \varphi \hookrightarrow \mathcal{O}_{\mathbb{P}^2}(-1 - k)\) for some \(k \geq 2\). By (6.34) the sheaf \(\ker \varphi\) is the quotient of \(\mathcal{E}\), whence it follows that \(h^0(\mathcal{E}_{\mathbb{P}^2}(-1 - k)) \neq 0\), \(k \geq 1\), and so \(h^0(\mathcal{E}_{\mathbb{P}^2}(-2)) \neq 0\). On the other hand, since \(\mathcal{E}\) is the cohomology of (6.1), by [21], Table 5.3. case 5, (2.ii), it has the spectrum \(\text{Sp}(\mathcal{E}) = (-1, 0, 0, 0, 1)\), and then it follows that
\[
h^1(\mathcal{E}(-3)) = 0 \quad \text{and} \quad h^1(\mathcal{E}(-2)) = 1.
\]
Thus, by the first equality in (6.35), the inequality \(h^0(\mathcal{E}_{\mathbb{P}^2}(-2)) \neq 0\) contradicts the cohomology sequence of the exact triple
\[
0 \to \mathcal{E}(-3) \to \mathcal{E}(-2) \to \mathcal{E}_{\mathbb{P}^2}(-2) \to 0,
\]
as \(h^0(\mathcal{E}(-2)) = 0\) by the stability of \(E\). Note also that we have proved here the equality
\[
H^0(\mathcal{E}_{\mathbb{P}^2}(-2)) = 0 \quad \forall \mathbb{P}^2 \subset \mathbb{P}^3.
\]

(c) \(d = 1\). In this case \(\mathcal{L} = \mathcal{O}_Z(-1)\) for \(Z\) a subscheme of \(\mathbb{P}^2\) of dimension \(\dim Z = 0\), and we have the exact sequence \(0 \to \mathcal{L}^\vee \xrightarrow{\theta^\vee} \mathcal{E} \to \mathcal{I}_{Z, \mathbb{P}^2}(-1) \to 0\), and the equality \(Z = (s)_0, 0 \neq s \in H^0(\mathcal{E}(-1)|_{\mathbb{P}^2})\). Since \(\dim Z = 0\), it follows that \(\text{Ext}^1(\mathcal{I}_{Z, \mathbb{P}^2}(-1), \mathcal{O}_{\mathbb{P}^3}) = \text{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_{\mathbb{P}^3}) = \mathcal{O}_{\mathbb{P}^2}(2)\). Thus, applying the functor \(\text{Ext}^\bullet(\mathcal{E}, \mathcal{O}_{\mathbb{P}^3})\) to the last triple, since \(\mathcal{E} \cong \mathcal{E}^\vee\) we obtain an exact triple.
0 \to E \xrightarrow{\theta} L \xrightarrow{\gamma} O_{\mathbb{P}^2}(2) \to 0. \quad \text{Comparing this triple with (6.30) and taking into account that, by construction, the composition } E \xrightarrow{\theta} L \xrightarrow{\gamma} \text{ coincides with } \theta \text{ we obtain that } L \xrightarrow{\gamma} \text{ is an isomorphism, that is, } L \text{ is reflexive.}

Next, as } c_t(E) = 1 + 5t^2, \text{ formulae for Chern classes of } L \text{ follow from (6.30). In particular, } L \cong L(-1) \text{ has } c_1(L(-1)) = -1, \text{ and since } h^0(E) = 0, \text{ it follows that } h^0(L(-1)) = 0. \quad (6.38)

Thus, } L \text{ is stable by Lemma 3.1 in [20]. Lemma 8 is proved.}

6.5. We now proceed to the closer study of the sheaf } F. \text{ Consider the upper horizontal triple of the diagram (6.23) which extends to an exact sequence:}

\[ 0 \to F \to E_3 \xrightarrow{\varepsilon} O_{\mathbb{P}^2}(-1) \to O_{\overline{Y}}(-1) \to 0, \quad \overline{Y} \subset \mathbb{P}^3. \quad (6.39) \]

**Lemma 9.** The sheaf } F \text{ defined in (6.24) is a reflexive rank 2 sheaf on } \mathbb{P}^3 \text{ included in an exact triple}

\[ 0 \to F \xrightarrow{\zeta} L \to I_{\overline{Y},\mathbb{P}^2}(1) \to 0 \quad (6.40) \]

\text{and in its dual}

\[ 0 \to L(-1) \to F \xrightarrow{\rho} I_{\overline{Z},\mathbb{P}^2}(2) \to 0, \quad (6.41) \]

\text{where } \mathbb{P}^2_0 = \mathbb{P}^2(M^*, \alpha), \overline{Y}, \overline{Z} \subset \mathbb{P}^2_0, \dim \overline{Y} \leq 0, \dim \overline{Z} \leq 0, \text{ and}

\[ \ell(\overline{Y}) + \ell(\overline{Z}) = 6. \quad (6.42) \]

\text{Chern classes of } F \text{ are}

\[ c_1(F) = 0, \quad c_2(F) = 2 \quad \text{and} \quad 0 \leq c_3(F) = 2\ell(\overline{Y}) \leq 12. \]

**Proof.** \text{We first show that } \text{rk} F = 2. \text{ Indeed, if } \varepsilon \text{ in (6.39) is the zero morphism, then the diagram (6.23) and the Snake Lemma yield an epimorphism } V_2' \otimes O_{\mathbb{P}^2}(1) \to O_{\mathbb{P}^2}(-1), \text{ which is impossible. Hence } \varepsilon \neq 0 \text{ and (6.24) implies that } \text{rk} F = 2 \text{ and, moreover, that } \overline{Y} \subset \mathbb{P}^3, \text{ that is, } \overline{Y} \text{ is a proper subscheme of } \mathbb{P}^3. \text{ Note also that, by (6.3) and (6.4), } c_1(E) = 0, \text{ hence } c_1(E_3) = -1 \text{ in view of (6.22). Thus, (6.39) implies that } c_1(F) = c_1(O_{\overline{Y}}(-1)) \geq 0. \]

Next, consider the lower exact triple in (6.27):

\[ 0 \to O_{\mathbb{P}^2}(-2) \xrightarrow{h \sigma} E_3 \to L \to 0. \quad (6.43) \]

If the composition } f := \varepsilon \circ h \circ \sigma \text{ is zero, then (6.39) and (6.43) imply that there exist injective morphisms } O_{\mathbb{P}^2}(-2) \xrightarrow{f_1} F \text{ and } \text{coker}(f_1) \xrightarrow{f_2} L. \text{ Since } \text{rk} F = 2, c_1(F) \geq 0 \text{ and } L \text{ is reflexive by Lemma 8, it follows that } \text{coker}(f_1) \text{ is a torsion-free sheaf with } c_1(\text{coker}(f_1)) \geq 2. \text{ Thus, the injectivity of } f_2 \text{ shows that } h^0(L(-2)) \neq 0, \text{ contrary to the stability of } L \text{ (see Lemma 8). It follows that } f \neq 0, \text{ so that (6.39) and (6.43)}
extend to a commutative diagram

\[
\begin{array}{ccc}
\mathcal{O}_{\mathbb{P}^3}(-2) & \xrightarrow{\epsilon} & \mathcal{O}_{\mathbb{P}^3}(-1) \\
\downarrow{h_0\sigma} & & \downarrow{f} \\
\mathcal{F} & \xrightarrow{\delta} & \mathcal{O}_{\mathcal{V}}(-1)
\end{array}
\]

(6.44)

where \(\mathbb{P}^2_0\) is some projective plane in \(\mathbb{P}^3\). If \(\delta\) is an isomorphism, then \(\text{coker}(\epsilon) \simeq \mathcal{O}_{\mathbb{P}^2_0}(-1)\), so that the diagram (6.23) and the Snake Lemma yield an epimorphism \(V'_2 \otimes \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow \mathcal{O}_{\mathbb{P}^2_0}(-1)\), which is impossible. Hence \(\mathcal{V} \subseteq \mathbb{P}^2_0\), that is, \(\mathcal{V}\) is a proper subscheme of \(\mathbb{P}^2_0\), \(\dim \mathcal{V} \leq 1\), and (6.44) yields an exact triple (6.40).

We show that the case \(\dim \mathcal{V} = 1\) is impossible. Indeed, in this case \(\mathcal{V}\) contains a divisor \(D \subset \mathbb{P}^2_0\) of degree \(k \geq 1\) as a subscheme, and this yields an epimorphism \(\mathcal{O}_{\mathcal{V}}(-1) \xrightarrow{b} \mathcal{O}_D(-1)\). On the other hand, the middle horizontal exact sequence in (6.44), together with diagram (6.23) and the Snake Lemma, yields an epimorphism \(V'_2 \otimes \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow \mathcal{O}_{\mathcal{V}}(-1)\). This epimorphism composed with the above epimorphism \(b\) gives an epimorphism \(V'_2 \otimes \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow \mathcal{O}_D(-1)\) which is impossible, since \(h^0(\mathcal{O}_D(-2)) = 0\), as follows from the cohomology of the exact triple \(0 \rightarrow \mathcal{O}_{\mathbb{P}^2_0}(-2 - k) \rightarrow \mathcal{O}_{\mathbb{P}^2_0}(-k) \rightarrow \mathcal{O}_D(-2) \rightarrow 0\).

Hence \(\dim \mathcal{V} \leq 0\) and therefore, denoting \(^i\mathcal{I} := \mathcal{E}xt^i(\mathcal{I}_{\mathcal{V},\mathbb{P}^2_0}(-1), \mathcal{O}_{\mathbb{P}^3})\), \(i \geq 1\), we obtain \(^1\mathcal{I} = \mathcal{O}_{\mathbb{P}^2}(2)\), \(\dim ^2\mathcal{I} \leq 0\), \(^3\mathcal{I} = 0\). In addition, set \(^i\mathcal{F} := \mathcal{E}xt^i(\mathcal{F}, \mathcal{O}_{\mathbb{P}^3})\), \(^i\mathcal{L} := \mathcal{E}xt^i(\mathcal{L}, \mathcal{O}_{\mathbb{P}^3})\), \(i \geq 1\), and note that, for the reflexive sheaf \(\mathcal{L}\), \(\dim ^1\mathcal{L} = 0\), \(^i\mathcal{L} = 0\), \(i = 2, 3\) (see the proof of Theorem 2.5 in [20]). Now, applying the functor \(\mathcal{E}xt^i(\cdot, \mathcal{O}_{\mathbb{P}^3})\) to (6.40) and using the above relations, we obtain the equalities \(^i\mathcal{F} = 0\), \(i = 2, 3\), and an exact sequence \(0 \rightarrow ^1\mathcal{L} \xrightarrow{\nu} \mathcal{F} \rightarrow ^2\mathcal{I} \rightarrow 0\), where \(\dim ^1\mathcal{F} \leq 0\) and \(\ker \mu \simeq \mathcal{I}_{\mathcal{Z},\mathbb{P}^2_0}(2)\) for some subscheme \(\mathcal{Z}\) of \(\mathbb{P}^2_0\) of dimension \(\dim \mathcal{Z} \leq 0\). We thus obtain an exact triple \(0 \rightarrow ^1\mathcal{L} \xrightarrow{\nu} \mathcal{F} \rightarrow \mathcal{I}_{\mathcal{Z},\mathbb{P}^2_0}(2) \rightarrow 0\) and the relation \(\mathcal{E}xt^1(\mathcal{I}_{\mathcal{Y},\mathbb{P}^2_0}(2), \mathcal{O}_{\mathbb{P}^3}) = \mathcal{O}_{\mathbb{P}^2}(-1)\). Next, applying the functor \(\mathcal{E}xt^i(\cdot, \mathcal{O}_{\mathbb{P}^3})\) to the last triple yields an exact sequence \(0 \rightarrow \mathcal{F} \xrightarrow{\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{\mathbb{P}^3})} \mathcal{L} \xrightarrow{\nu} \mathcal{O}_{\mathbb{P}^2}(-1) \nu\). By Corollary 1.2 in [20] \(\mathcal{F}\) is a reflexive rank 2 sheaf, hence \(\dim \mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{\mathbb{P}^3}) \leq 0\) by Remark 2.7.1 in [20], and therefore \(\ker \nu \simeq \mathcal{I}_{\mathcal{W},\mathbb{P}^2_0}(-1)\) for some subscheme \(\mathcal{W}\) of \(\mathbb{P}^2_0\) of dimension \(\dim \mathcal{W} \leq 0\). Thus the last sequence leads to an exact triple \(0 \rightarrow \mathcal{F} \xrightarrow{\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{\mathbb{P}^3})} \mathcal{L} \rightarrow \mathcal{I}_{\mathcal{W},\mathbb{P}^2_0}(2) \rightarrow 0\) which together with (6.40) fits in a commutative diagram

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{\mathbb{P}^3})} & \mathcal{L} \\
\downarrow{\text{can}} & \downarrow{\text{can}} & \downarrow{c} \\
\mathcal{F} & \xrightarrow{\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{\mathbb{P}^3})} & \mathcal{L}
\end{array}
\]
Moreover, the above-stated relations \( i \mathcal{F} = 0, i = 2, 3, \) \( \dim^1 \mathcal{F} \leq 0 \) show that the sheaf \( \mathcal{F} \) is locally free outside the set of dimension \( \leq 0 \), and this shows that the sheaf \( \kappa = \text{coker}(\mathcal{F} \xrightarrow{\text{can}} \mathcal{F}^\vee \vee) \) has dimension \( \leq 0 \) and by the Snake Lemma \( \kappa \) is a subsheaf of \( \ker c \). However, the sheaf has no subsheaves of dimension 0. Hence \( \kappa = 0 \) and \( \mathcal{F} \xrightarrow{\text{can}} \mathcal{F}^\vee \vee \) is an isomorphism, that is, \( \mathcal{F} \) is reflexive. A standard computation with the triple (6.40) yields the values of Chern classes of \( \mathcal{F} \). The triple (6.41) and the equality (6.42) are obtained by applying to (6.40) the functor \( \mathcal{E} \mathcal{X}(\cdot, \mathcal{O}_{\mathbb{P}^3}) \) and using formulae for Chern classes of \( \mathcal{F} \) and \( \mathcal{L} \). The inequality \( 0 \leq c_3(\mathcal{F}) \leq 12 \) follows from (6.42). Lemma 9 is proved.

**Lemma 10.** The projective planes \( \mathbb{P}^2 \) and \( \mathbb{P}^2_0 \) defined in (6.29) coincide.

**Proof.** The middle horizontal triple \( 0 \to \mathcal{E} \to \ker \sigma \to \mathcal{O}_{\mathbb{P}^3}(2) \to 0 \) in (6.28) is defined as an extension by a nonzero element in \( \text{Ext}^1(\mathcal{E}, \mathcal{O}_{\mathbb{P}^2}(2)) \simeq H^1(\mathcal{E}(-2)) \).

Since \( h^1(\mathcal{E}(-2)) = 1 \) by (6.35), it follows that the sheaf \( \ker \sigma \) is defined by \( \mathcal{E} \) uniquely up to an isomorphism. Since \( h^0(\mathcal{L}(-1)) = 0 \) as \( \mathcal{L} \) is stable by Lemma 8, the middle vertical triple \( 0 \to \mathcal{O}_{\mathbb{P}^3} \to \ker \sigma(-1) \to \mathcal{L}(-1) \to 0 \) twisted by \( \mathcal{O}_{\mathbb{P}^3}(-1) \) in (6.28) shows that \( h^0(\ker \sigma(-1)) = 1 \). Hence \( \mathcal{L} = \mathcal{L}(M^\bullet) \) is uniquely defined up to isomorphism by \( \ker \sigma \) (and therefore by \( \mathcal{E} \)) as

\[
\mathcal{L}(M^\bullet) = (\ker \sigma(-1)/\mathcal{O}_{\mathbb{P}^3})(1). \tag{6.45}
\]

Then the lower horizontal triple in (6.28) shows that the plane \( \mathbb{P}^2 = \mathbb{P}^2(\mathcal{M}^\bullet, \beta) \) is determined uniquely by \( \mathcal{E} \) as

\[
\mathbb{P}^2(\mathcal{M}^\bullet, \beta) = \text{Supp}(\mathcal{L}(M^\bullet)/\mathcal{E}). \tag{6.46}
\]

Next, it follows from (6.29) that

\[
\mathbb{P}^2(\mathcal{M}^\bullet, \alpha) = \mathbb{P}^2(\mathcal{M}^\vee, \beta^\vee), \quad \mathbb{P}^2(\mathcal{M}^\bullet, \beta) = \mathbb{P}^2(\mathcal{M}^\vee, \alpha^\vee), \tag{6.47}
\]

where \( M^\vee: 0 \to (M^1)^\vee \xrightarrow{\beta^\vee} (M^0)^\vee \xrightarrow{\alpha^\vee} (M^{-1})^\vee \to 0 \) is the monad dual to \( M^\bullet \).

The monad \( M^\vee \) defines the monad dual to (6.5): \( 0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \xrightarrow{\tau^\vee} \mathcal{E} \xrightarrow{\sigma^\vee} \mathcal{O}_{\mathbb{P}^3}(2) \to 0 \) with \( \mathcal{E}^\vee = \ker(\sigma^\vee)/\text{im}(\tau^\vee) \), and the argument dual to the above yields the formulae dual to (6.45) and (6.46): \( \mathbb{P}^2(\mathcal{M}^\vee, \beta^\vee) = \text{Supp}(\mathcal{L}(M^\vee)/\mathcal{E}^\vee), \mathcal{L}(M^\vee) = (\ker(\tau^\vee)(-1)/\mathcal{O}_{\mathbb{P}^3})(1) \).

Since \( \mathcal{E}^\vee \simeq \mathcal{E} \) these formulae mean in view of (6.47) that the plane \( \mathbb{P}^2_0 = \mathbb{P}^2(\mathcal{M}^\bullet, \alpha) \) is uniquely defined by \( \mathcal{E} \) via the same construction as above, hence it coincides with \( \mathbb{P}^2 = \mathbb{P}^2(\mathcal{M}^\bullet, \beta) \). The lemma is proved.

**6.6.** Let \( \mathcal{F} \in \mathcal{R}(0, 2, 2k) \) be the reflexive sheaf defined in (6.24), where \( 0 \leq k \leq 6 \) by Lemma 9, that is,

\[
[\mathcal{F}] \in \bigsqcup_{0 \leq k \leq 6} \mathcal{R}_k, \quad \mathcal{R}_k := \mathcal{R}(0, 2, 2k). \tag{6.48}
\]

Formulae (6.14), (6.15) and Lemmas 8, 9 and 10 yield the following.

**Proposition 9.** There is an inclusion

\[
\mathcal{H} \setminus (\mathcal{H} \cap \mathcal{G}(2, 1)) \subset \bigsqcup_{0 \leq k \leq 6} \mathcal{H}_k. \tag{6.49}
\]
where

\[ \mathcal{H}_k = \left\{ [\mathcal{E}] \in \mathcal{B}(5) \mid \mathcal{E} \text{ is obtained from } \mathcal{F}, \text{ where } [\mathcal{F}] \in \mathcal{R}_k, \right. \]

by the two subsequent elementary transformations

\[ 0 \to \mathcal{L}(-1) \to \mathcal{F} \xrightarrow{\mathcal{L}} \mathbb{P}^2(2) \to 0, \]

\[ 0 \to \mathcal{E} \to \mathcal{L} \xrightarrow{\mathcal{L}} \mathcal{O}_{\mathbb{P}^2}(2) \to 0; \]

\[ \mathbb{P}^2 \text{ is some plane in } \mathbb{P}^3, \quad \mathcal{Z} \subseteq \mathbb{P}^2, \quad \dim \mathbb{Z} \leq 0, \quad \ell(\mathcal{Z}) = 6 - k, \]

and \( \mathcal{L} \) is a stable reflexive sheaf from \( \mathcal{R}(1,4,6) \). \hfill (6.50)

§ 7. Geometric properties of sheaves \( \mathcal{F} \) and moduli of the cohomology bundles \( \mathcal{E} \) of monads (1.6)

7.1. In this section we explore in detail the geometry of the reflexive sheaves \( \mathcal{F} \) described in Lemma 9. The main result of this study will be the upper estimates for the dimensions of the moduli space of sheaves \( \mathcal{F} \) and sheaves \( \mathcal{L} \) obtained from \( \mathcal{F} \) by the elementary transformation (6.41). These estimates are obtained in Propositions 10 and 11 below. This will eventually lead to the proof of Theorem 4.

Denote

\[ \mathcal{R}^u_k := \{ [\mathcal{F}] \in \mathcal{R}_k \mid \mathcal{F} \text{ is unstable} \}, \quad \mathcal{R}^s_k := \{ [\mathcal{F}] \in \mathcal{R}_k \mid \mathcal{F} \text{ is stable} \}, \]

\[ \mathcal{H}^u_k := \{ [\mathcal{E}] \in \mathcal{H}_k \mid \mathcal{E} \text{ is obtained from } \mathcal{F} \text{ in (6.50), where } [\mathcal{F}] \in \mathcal{R}^u_k \}, \]

\[ \mathcal{H}^s_k := \{ [\mathcal{E}] \in \mathcal{H}_k \mid \mathcal{E} \text{ is obtained from } \mathcal{F} \text{ in (6.50), where } [\mathcal{F}] \in \mathcal{R}^s_k \}, \]

where \( 0 \leq k \leq 6 \). Thus, \( \mathcal{R}_k = \mathcal{R}^u_k \sqcup \mathcal{R}^s_k \) and (6.15) and (6.49) yield:

\[ \mathcal{H} \setminus (\mathcal{H} \cap \mathcal{G}(2,1)) \subset \bigsqcup_{0 \leq k \leq 6} (\mathcal{H}^u_k \sqcup \mathcal{H}^s_k). \hfill (7.1) \]

The estimate for the dimension of \( \mathcal{H} \setminus (\mathcal{H} \cap \mathcal{G}(2,1)) \) will eventually follow from the computations of dimensions of \( \mathcal{H}^u_k \) and \( \mathcal{H}^s_k \) which we give below. For this, we start with an explicit description of the spaces \( \mathcal{R}^u_k \) and \( \mathcal{R}^s_k \).

7.2. The properties of the sheaf \( \mathcal{F} \) in the unstable case \( [\mathcal{F}] \in \mathcal{R}^u_k \) and, respectively, in the stable case \( [\mathcal{F}] \in \mathcal{R}^s_k \) are summarized in Propositions 10 and 11 below.

**Proposition 10.** (i) \( \mathcal{R}^u_k \neq \emptyset \) only for \( 0 \leq k \leq 3 \), and any sheaf \( \mathcal{F} \) from \( \mathcal{R}^u_k \) fits in an exact triple

\[ 0 \to \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\mathcal{I}} \mathcal{F} \xrightarrow{\mathcal{I}} \mathcal{I}_{C,\mathbb{P}^3} \to 0, \hfill (7.2) \]

where \( C = \text{Sing}(\mathcal{F}/\mathcal{O}_{\mathbb{P}^3}) \) is an l.c.i. curve of degree 2 in \( \mathbb{P}^3 \) and \( \chi(\mathcal{O}_C) = 4 - \frac{1}{2} c_3(\mathcal{F}) = 4 - k \).

(ii) If \( C \) is a reduced curve, then either \( c_3(\mathcal{F}) = 4 \) and \( C \) is a disjoint union \( l_1 \sqcup l_2 \) of two projective lines in \( \mathbb{P}^3 \), or \( c_3(\mathcal{F}) = 6 \), and then \( C \) is a plane conic in \( \mathbb{P}^3 \).

(iii) If \( C \) is a nonreduced curve, then \( C \) is the scheme structure of multiplicity two on a projective line \( l \) in \( \mathbb{P}^3 \) defined by an exact sequence

\[ 0 \to \mathcal{I}_{C,\mathbb{P}^3} \to \mathcal{I}_{l,\mathbb{P}^3} \to \mathcal{O}_l(m) \to 0, \quad -1 \leq m = 2 - k \leq 2. \hfill (7.3) \]
(iv) The moduli spaces $\mathcal{R}^u_k$ are varieties of dimensions
\begin{equation}
\dim \mathcal{R}^u_0 = \dim \mathcal{R}^u_3 = 14, \quad \dim \mathcal{R}^u_1 = \dim \mathcal{R}^u_2 = 13,
\end{equation}
and they are fine.

**Proof.** (i)–(iii). By Lemma 9, we have $c_1(\mathcal{F}) = 0$, $c_2(\mathcal{F}) = 2$. Since $\mathcal{F}$ is unstable, it follows from [20], Lemma 3.1, that $H^0(\mathcal{F}) \neq 0$. In addition, from (6.38) and the triple (6.40) twisted by $\mathcal{O}_{\mathbb{P}^3}(-1)$ we obtain $H^0(\mathcal{F}(-1)) = 0$. Take a section $0 \neq s \in H^0(\mathcal{F})$ and define a subscheme $C$ in $\mathbb{P}^3$ by the ideal sheaf $\mathcal{I}_{C,\mathbb{P}^3} = \text{im}(u: \mathcal{F} \overset{\text{can}}{\rightarrow} \mathcal{F}^\vee \overset{s}{\rightarrow} \mathcal{O}_{\mathbb{P}^3})$. (The canonical isomorphism can: $\mathcal{F} \overset{\cong}{\rightarrow} \mathcal{F}^\vee$ holds since $c_1(\mathcal{F}) = 0$.)

From the equality $H^0(\mathcal{F}(-1)) = 0$, by [20], Theorem 4.1, we obtain that:

(a) $C$ is a Cohen-Macaulay curve in $\mathbb{P}^3$ satisfying the triple (7.2), so that $\deg C = c_2(\mathcal{F}) = 2$;

(b) the triple (7.2) is exact, and the equality $\chi(\mathcal{O}_C) = 4 - k$ follows from this triple and Theorem 2.3 in [20]; moreover, (7.2) defines an extension
\begin{equation}
\xi \in \text{Ext}^1(\mathcal{I}_{C,\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3}) \simeq H^0(\mathcal{E}xt^1(\mathcal{I}_{C,\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3})) \simeq H^0(\mathcal{E}xt^2(\mathcal{O}_C, \mathcal{O}_{\mathbb{P}^3})).
\end{equation}
(Here we use standard isomorphisms relating global Ext-groups and $\mathcal{E}xt$-sheaves; see [20], §4.) If $C$ is a reduced curve, then, since $\deg C = 2$, $C$ is either a disjoint union $L_1 \cup L_2$ of lines, or a conic. If $C$ is nonreduced, then $C$ is the scheme structure of multiplicity two on a projective line $l$ (in the sense of the definition on p. 58 in [12]). Moreover, since $C$ is Cohen-Macaulay, the sheaf $\mathcal{I}_{L,\mathbb{P}^3}/\mathcal{I}_{C,\mathbb{P}^3}$ has no 0-dimensional torsion. Hence, by the claim on p. 59 in [12] the exact triple (7.3) follows and, moreover, $C$ is a locally complete intersection. The triples (7.3) and (7.2) yield the equality $m = 2 - \frac{1}{2}c_3(\mathcal{F}) = 2 - k$. Furthermore, (7.3) and the isomorphism
\begin{equation}
\mathcal{I}_{L,\mathbb{P}^3}|_l \simeq N_{l/\mathbb{P}^3} \simeq \mathcal{O}_l(-1)^{\oplus 2}
\end{equation}
imply $m \geq -1$. Besides, $2 - m = k = \frac{1}{2}c_3(\mathcal{F}) \geq 0$, as $\mathcal{F}$ is reflexive. Thus, $-1 \leq m \leq 2$ and therefore $0 \leq k \leq 3$.

(iv) Consider the varieties
\begin{equation}
\mathcal{C}_k = \{ C \mid C \text{ is an l.c.i. curve of degree 2 in } \mathbb{P}^3, \chi(\mathcal{O}_C) = 4 - k \}, \quad 0 \leq k \leq 3.
\end{equation}
From (i)–(iii) and Remark 1.3 in [12] it follows that the $\mathcal{C}_k$ are rational varieties of dimensions
\begin{equation}
\dim \mathcal{C}_0 = 11, \quad \dim \mathcal{C}_1 = 9 \quad \text{and} \quad \dim \mathcal{C}_2 = \dim \mathcal{C}_3 = 8.
\end{equation}
Note that (7.3) yields an exact triple $0 \rightarrow \mathcal{O}_l(2 - k) \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_l \rightarrow 0, \ k = \frac{1}{2}c_3(\mathcal{F})$. Applying the functor $\mathcal{E}xt^2(\mathcal{O}_l, \mathcal{O}_{\mathbb{P}^3})$ to it and using the relations $\mathcal{E}xt^2(\mathcal{O}_l, \mathcal{O}_{\mathbb{P}^3}) \simeq \det(N_{l/\mathbb{P}^3}) \simeq \mathcal{O}_l(2)$, $\mathcal{E}xt^i(\mathcal{O}_l, \mathcal{O}_{\mathbb{P}^3}) = 0, \ i = 1, 3$ (see [35; pp. 49, 50]), we obtain an exact triple $0 \rightarrow \mathcal{O}_l(2) \rightarrow \mathcal{E}xt^2(\mathcal{O}_C, \mathcal{O}_{\mathbb{P}^3}) \rightarrow \mathcal{O}_l(k) \rightarrow 0$ which, together with (7.5), yields
\begin{equation}
\begin{align*}
\dim \text{Hom}(\mathcal{I}_{C,\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3}) &= 1, \quad \text{Ext}^{\geq 2}(\mathcal{I}_{C,\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3}) = 0, \\
\dim \text{Ext}^1(\mathcal{I}_{C,\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3}) &= h^0(\mathcal{E}xt^2(\mathcal{O}_C, \mathcal{O}_{\mathbb{P}^3})) = k + 4.
\end{align*}
\end{equation}
Now, by (i)–(iii), for \(0 \leq k \leq 3\), the spaces \(\mathcal{R}_k^i\) are described as:
\[
\mathcal{R}_k^i = \{ ([\mathcal{F}], (\xi)) \mid [\mathcal{F}] \in \mathcal{R}_k, \mathcal{F} \text{ fits in } (7.2), \langle \xi \rangle \in \mathbb{P}(\text{Ext}^1(\mathcal{I}_{C,\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3})) \}
\]
\[
= \{ (C, (\xi)) \mid C \in \mathcal{C}_k, \langle \xi \rangle \in \mathbb{P}(\text{Ext}^1(\mathcal{I}_{C,\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3})) \}.
\]
This, together with (7.8), shows that \(\mathcal{R}_k\) is a projective fibration with fibre \(\mathbb{P}^{k+3}\) over \(\mathcal{C}_k\), and (7.7) yields (7.4). Note that there exist universal flat families \(\Gamma \subset \mathcal{C}_k\) of curves \(C\), and in view of (7.8) and Theorem 1.4 in [32] the sheaves \(\text{Ext}^i_{\mathcal{R}_k}^{(i)}(\mathcal{I}_C, \mathcal{O}_{\mathcal{C}_k})\) commute with the Base Change. Hence, by Proposition 4.2 in [32] there exist universal sheaves \(\mathcal{F}\) on \(\mathcal{R}_k^i\), that is, the \(\mathcal{R}_k^i\) are fine moduli spaces. Proposition 10 is proved.

**Proposition 11.** Suppose that \([\mathcal{F}] \in \mathcal{R}_k^3\). Then the following statements hold:

(i) \(\mathcal{R}_k^2 \neq \emptyset\) only for \(0 \leq k \leq 2\);

(ii) \(\dim \mathcal{R}_k^3 = 13\), \(k = 0, 1, 2\);

(iii) for \(0 \leq k \leq 2\) and any \([\mathcal{F}] \in \mathcal{R}_k^3\), \(\dim \text{Ext}^1(\mathcal{F}, \mathcal{F}) = 13\) and \(\text{Ext}^2(\mathcal{F}, \mathcal{F}) = 0\);

(iv) for any \(\mathbb{P}^2 \subset \mathbb{P}^3\) and any \([\mathcal{F}] \in \mathcal{R}_k^3\), \(h^0(\mathcal{F}_{\mathbb{P}^2}(2)) = 10\) and \(h^1(\mathcal{F}_{\mathbb{P}^2}(2)) = 0\).

**Proof.** Statements (i)–(iii) were proved in [12], §2. The equalities in statement (iv) follow from the exact triple \(0 \rightarrow \mathcal{F}(1) \rightarrow \mathcal{F}(2) \rightarrow \mathcal{F}_{\mathbb{P}^2}(2) \rightarrow 0\) and Tables 2.8.1 and 2.12.2 in [12] for \(k = 2, 4\) and, respectively, from [19], §9, for \(k = 0\). The proposition is proved.

### 7.3
We next proceed to a detailed description of the connection between the spaces \(\mathcal{H}_k^i\) and \(\mathcal{R}_k^i\) for \(0 \leq k \leq 3\) and, correspondingly, between the spaces \(\mathcal{H}_k^i\) and \(\mathcal{R}_k^i\) for \(0 \leq k \leq 2\), given by means of the transformations in (6.50). This will eventually lead to formula (7.18).

Denote
\[
\begin{align*}
2S_{\mathcal{F}}^u &:= \{ \mathbb{P}^2 \in \mathbb{P}^3 \mid \dim(C \cap \mathbb{P}^2) = 1, C \subset \mathbb{P}^2, \mathcal{O}_C = \mathcal{O}_{\mathbb{P}^3}/(\mathcal{F}/\mathcal{O}_{\mathbb{P}^3}), [\mathcal{F}] \in \mathcal{R}_3^u, \\
2S_{\mathcal{F}}^u &:= \emptyset, \quad [\mathcal{F}] \in \mathcal{R}_k^u, \quad k \leq 2, \\
1S_{\mathcal{F}}^u &:= \{ \mathbb{P}^2 \in \mathbb{P}^3 \mid \dim(C \cap \mathbb{P}^2) = 1, \mathcal{O}_C = \mathcal{O}_{\mathbb{P}^3}/(\mathcal{F}/\mathcal{O}_{\mathbb{P}^3}), C \not\subset \mathbb{P}^2 \}, \quad [\mathcal{F}] \in \mathcal{R}_k^u, \\
2S_{\mathcal{F}}^s &:= \emptyset, \quad 1S_{\mathcal{F}}^s := \emptyset, \quad [\mathcal{F}] \in \mathcal{R}_k^s, \\
0S_{\mathcal{F}}^u &:= \{ \mathbb{P}^2 \in \mathbb{P}^3 \mid (1S_{\mathcal{F}}^u + 2S_{\mathcal{F}}^u) \mid \text{Sing } \mathcal{F} \cap \mathbb{P}^2 \neq \emptyset \}, \quad [\mathcal{F}] \in \mathcal{R}_k^u, \\
0S_{\mathcal{F}}^s &:= \{ \mathbb{P}^2 \in \mathbb{P}^3 \mid (1S_{\mathcal{F}}^s + 2S_{\mathcal{F}}^s) \mid \text{Sing } \mathcal{F} \cap \mathbb{P}^2 \neq \emptyset \}, \quad [\mathcal{F}] \in \mathcal{R}_k^s,
\end{align*}
\]
\[
-1S_{\mathcal{F}}^u := \mathbb{P}^3 \setminus (0S_{\mathcal{F}}^u + 1S_{\mathcal{F}}^u + 2S_{\mathcal{F}}^u), \\
-1S_{\mathcal{F}}^s := \mathbb{P}^3 \setminus (0S_{\mathcal{F}}^s + 1S_{\mathcal{F}}^s + 2S_{\mathcal{F}}^s),
\]
\[
\begin{align*}
\mathcal{D}_k^u &:= \mathcal{R}_k^u \times \mathbb{P}^3, \quad i\mathcal{D}_k^u := \{(\mathcal{F}, \mathbb{P}^2) \mid \mathbb{P}^2 \in iS_{\mathcal{F}}^u \}, \quad -1 \leq i \leq 2, \\
\mathcal{D}_k^s &:= \mathcal{R}_k^s \times \mathbb{P}^3, \quad i\mathcal{D}_k^s := \{(\mathcal{F}, \mathbb{P}^2) \mid \mathbb{P}^2 \in iS_{\mathcal{F}}^s \}, \quad -1 \leq i \leq 2, \\
\mathcal{D}_k &:= \mathcal{D}_k^u \cup \mathcal{D}_k^s = \bigsqcup_{-1 \leq i \leq 2} i\mathcal{D}_k^u, \quad i\mathcal{D}_k := i\mathcal{D}_k^u \cup i\mathcal{D}_k^s, \quad -1 \leq i \leq 2,
\end{align*}
\]
Clearly, \(i\mathcal{D}_k^u\) (respectively \(i\mathcal{D}_k^s\)) are locally closed in \(\mathcal{D}_k^u\) (\(\mathcal{D}_k^s\)) and
\[
\begin{align*}
\dim i\mathcal{D}_k^u &\leq \dim \mathcal{R}_k^u + 2 - i, \quad \dim i\mathcal{D}_k^s \leq \dim \mathcal{R}_k^s + 2 - i, \quad -1 \leq i \leq 2. \quad (7.9)
\end{align*}
\]
Next, denote
\[
\Pi(\mathcal{F}, \mathbb{P}^2) := \{ (\rho) \in \mathbb{P}(\text{Hom}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^2}(2))) \mid \text{im}(\rho): \mathcal{F} \to \mathcal{O}_{\mathbb{P}^2}(2)) = \mathcal{I}Z_{\mathbb{P}^2}(2)
\]
for a subscheme \( Z \subset \mathbb{P}^2, \dim Z \leq 0, \ell(Z) = 6 - k \}, \quad ([\mathcal{F}], \mathbb{P}^2) \in \mathcal{D}_k,
\]
(7.11)
\[
K([\mathcal{F}], \mathbb{P}^2) := \{ (\rho) \in \Pi(\mathcal{F}, \mathbb{P}^2) \mid [\mathcal{L}_\rho] = (\ker \rho)(1) \in \mathcal{R}(1, 4, 6) \text{ is stable} \},
\]
(\mathcal{F}, \mathbb{P}^2) \in \mathcal{D}_k, \quad \mathcal{Q}_k := \{ ([\mathcal{F}, \mathbb{P}^2], (\rho)) \mid ([\mathcal{F}, \mathbb{P}^2]) \in \mathcal{D}_k, (\rho) \in K([\mathcal{F}, \mathbb{P}^2]) \}, \quad (7.12)
\]
\[
T_k := \{ ([\mathcal{F}, \mathbb{P}^2], (\rho), (\gamma)) \mid ([\mathcal{F}, \mathbb{P}^2], (\rho)) \in \mathcal{Q}_k, (\gamma) \in \Sigma([\mathcal{F}, \mathbb{P}^2], (\rho)) \},
\]
\[
T_k \overset{p_{2k}}{\longrightarrow} \mathcal{Q}_k \text{ is the forgetful map,} \quad p_{2k}^{-1}([\mathcal{F}, \mathbb{P}^2], (\rho)) = \Sigma([\mathcal{F}, \mathbb{P}^2], (\rho)) \quad (7.13)
\]
and
\[
\mathcal{Q}^u_k := p_{1k}^{-1}(\mathcal{D}^u_k \cap p_{1k}(\mathcal{Q}_k)), \quad T^u_k := p_{2k}^{-1}(\mathcal{Q}^u_k \cap p_{2k}(T_k)),
\]
\[
\mathcal{Q}^s_k := p_{1k}^{-1}(\mathcal{D}^s_k \cap p_{1k}(\mathcal{Q}_k)), \quad T^s_k := p_{2k}^{-1}(\mathcal{Q}^s_k \cap p_{2k}(T_k)).
\]
(7.14)
(Here \( 0 \leq k \leq 3 \) in the unstable case and \( 0 \leq k \leq 2 \) in the stable case.) Since \( \mathcal{D}_k = \mathcal{D}^u_k \cup \mathcal{D}^s_k \), it follows that
\[
\mathcal{Q}_k = \mathcal{Q}^u_k \cup \mathcal{Q}^s_k \text{ and} \quad T_k = T^u_k \cup T^s_k, \quad 0 \leq k \leq 3.
\]
(7.15)
(For consistency, in (7.15) and below we set \( \mathcal{Q}^s_3 = T^s_3 = \emptyset \).) Since the stability of the sheaf \( \mathcal{L}_\rho \) is an open property in flat families (see [24], Proposition 2.3.1) it follows that
\[
p_{1k}^{-1}([\mathcal{F}, \mathbb{P}^2]) \overset{\text{open}}{\longrightarrow} \Pi(\mathcal{F}, \mathbb{P}^2), \quad ([\mathcal{F}, \mathbb{P}^2]) \in \mathcal{D}_k.
\]
(7.16)
Take any point \( ([\mathcal{F}, \mathbb{P}^2], (\rho), (\gamma)) \). Since by definition \( [\mathcal{L}_\rho] \in \mathcal{R}(1, 4, 6) \) is stable and \( \mathcal{E} = \ker \gamma \) is a vector bundle, we obtain from the second triple (6.50) that \( [\mathcal{E}] \in \mathcal{R}(0, 5, 0) \) is also stable, that is, \( [\mathcal{E}] \in \mathcal{B}(5) \). Thus, we obtain a natural map
\[
f_k: T_k \to \mathcal{B}(5), \quad ([\mathcal{F}, \mathbb{P}^2], (\rho), (\gamma)) \mapsto [\ker \gamma], \quad (7.17)
\]
and by Proposition 9, \( \mathcal{H}^u_k \subset f_k(T^u_k) \) and \( \mathcal{H}^s_k \subset f_k(T^s_k) \). This, together with (7.1) and the second formula (7.15), yields
\[
\mathcal{H} \setminus (\mathcal{H} \cap \mathcal{G}(2, 1)) \subset \bigsqcup_{0 \leq k \leq 3} f_k(T_k).
\]
(7.18)
It will follow from the computations below that the \( T_k \) are disjoint unions of schemes and the \( f_k \) are morphisms for each of these schemes and all admissible values of \( k \).
7.4. We state some further properties of the sheaf $F$ in the following lemma.

**Lemma 11.** Let $([F], \mathbb{P}^2) \in D_k$ and $\Pi([F], \mathbb{P}^2) \neq \emptyset$. Then

(i) there is an open embedding $j: \Pi(F, \mathbb{P}^2) \xrightarrow{\text{open}} \mathbb{P}(H^0((F_{\mathbb{P}^2})^{\vee}(2)))$ and for any $\langle \rho \rangle \in \Pi(F, \mathbb{P}^2)$ there exists a subscheme $W(\rho)$ of $\mathbb{P}^2$, $\dim W(\rho) \leq 0$, and an exact triple

$$0 \to F_{\mathbb{P}^2}(2) \xrightarrow{\text{can}} (F_{\mathbb{P}^2})^{\vee}(2) \to O_{W(\rho)} \to 0, \quad \ell W(\rho) = k; \quad (7.19)$$

(ii) if $\Sigma_{([F], \mathbb{P}^2, \langle \rho \rangle)} \neq \emptyset$ for $([F], \mathbb{P}^2, \langle \rho \rangle) \in O_k$, then there is an open embedding $\Sigma_{([F], \mathbb{P}^2, \langle \rho \rangle)} \xrightarrow{\text{open}} \mathbb{P}(\hom(L, O_{\mathbb{P}^2}(2))) \simeq \mathbb{P}^{10}$;

(iii) if $([F], \mathbb{P}^2) \in -1D_k$, then $k = 0$, $h^0(F_{\mathbb{P}^2}^{\vee}(2)) = h^0(F_{\mathbb{P}^2}(2)) = 10$;

(iv) if $0D_k \neq \emptyset$ and $([F], \mathbb{P}^2) \in 0D_k$, then $1 \leq k \leq 2$, $h^0(F_{\mathbb{P}^2}(2)) = 10$ and

$$h^0((F_{\mathbb{P}^2})^{\vee}(2)) = 10 + k; \quad (7.20)$$

(v) if $1D_k \cup 2D_k \neq \emptyset$ and $([F], \mathbb{P}^2) \in 1D_k \cup 2D_k$, then the equalities (7.20) hold for $k = 1, 2, 3$ and

$$h^0((F_{\mathbb{P}^2})^{\vee}(2)) = h^0((F_{\mathbb{P}^2})(2)) = 11 \quad \text{if} \ k = 0. \quad (7.21)$$

**Proof.** (i) Take any $\langle \rho \rangle \in \Pi(F, \mathbb{P}^2)$. By the definition of $\Pi(F, \mathbb{P}^2)$ we may consider $\rho$ as a composition $\rho: F \otimes O_{\mathbb{P}^2} \xrightarrow{p} I_{Z, \mathbb{P}^2}(2)$ with $\dim Z \leq 0$, $\ell Z = 6 - k$. As $F$ is reflexive, $F_{\mathbb{P}^2}$ has no torsion as an $O_{\mathbb{P}^2}$-sheaf (see [20], §1). Hence $\ker p$ is a rank 1 torsion-free $O_{\mathbb{P}^2}$-sheaf. Since $c_1(F_{\mathbb{P}^2}) = 0$ and $c_2(F_{\mathbb{P}^2}) = 2$, it follows that $\ker p \simeq I_{W, \mathbb{P}^2}(-2)$, where $\dim W \leq 0$, and there is an exact triple

$$0 \to I_{W, \mathbb{P}^2} \xrightarrow{\theta_\rho} F_{\mathbb{P}^2}(2) \xrightarrow{p} I_{Z, \mathbb{P}^2}(4) \to 0. \quad (7.22)$$

The monomorphism $\theta = \theta_\rho$ in this triple extends to a commutative square

$$\begin{array}{ccc}
I_{W, \mathbb{P}^2} & \xrightarrow{\theta_\rho} & F_{\mathbb{P}^2}(2) \\
\downarrow \text{can} & & \downarrow \text{can} \\
O_{\mathbb{P}^2} & \xrightarrow{\theta_\rho^{\vee}} & (F_{\mathbb{P}^2})^{\vee}(2)
\end{array} \quad (7.23)
$$

and we obtain a morphism $j: \Pi(F, \mathbb{P}^2) \to \mathbb{P}(H^0((F_{\mathbb{P}^2})^{\vee}(2)))$, $\langle \rho \rangle \mapsto \theta_\rho^{\vee}$. To construct the morphism $\psi: (\im j) \to \Pi(F, \mathbb{P}^2)$ inverse to $j$, take any $\tilde{\theta}: O_{\mathbb{P}^2} \to (F_{\mathbb{P}^2})^{\vee}(2) \in \im j$. The morphism $\theta: I_{W, \mathbb{P}^2} \to F_{\mathbb{P}^2}(2)$ such that $\tilde{\theta} = \theta^{\vee}$ is recovered from $\tilde{\theta}$ as $\tilde{\theta} I_{W, \mathbb{P}^2}$, where $I_{W, \mathbb{P}^2} = \text{can}^{-1}(\tilde{\theta}(O_{\mathbb{P}^2}) \cap \text{can}(F_{\mathbb{P}^2}(2)))$. Then $\tilde{\theta}$ defines via $\theta$ a morphism $\overline{\theta}$ as the quotient morphism $F_{\mathbb{P}^2}(2) \to \text{coker} \theta \simeq I_{Z, \mathbb{P}^2}(4)$, and we set $\psi((\tilde{\theta})) := (\overline{\theta} \circ (- \otimes O_{\mathbb{P}^2}))$. The openness of $j$ follows from the openness of the condition that $\rho: F \to O_{\mathbb{P}^2}(2)$ be surjective.

Next, note that in (7.23) the $O_{\mathbb{P}^2}$-sheaf $\text{coker} \theta \simeq I_{Z, \mathbb{P}^2}(4)$ has no torsion, hence there is no nonzero morphism $O_W = O_{\mathbb{P}^2}/I_{W, \mathbb{P}^2} \to \text{coker} \theta$, since $\dim W \leq 0$. Thus, (7.23) and the Snake Lemma yield an exact triple (7.19) with $W(\rho) = W$. 

New moduli components of rank 2 bundles on projective space 1545
(ii) The injection \( \Sigma([\mathcal{F}], \mathbb{P}^2, (\rho)) \hookrightarrow \mathbb{P}(\text{Hom}(\mathcal{L}_\rho, \mathcal{O}_{\mathbb{P}^2}(2))) \simeq \mathbb{P}^{10} \) is an open embedding since the condition that \( \gamma: \mathcal{L} := \mathcal{L}_\rho \to \mathcal{O}_{\mathbb{P}^2}(2) \) be an epimorphism and \( \ker \gamma \) be locally free is open on \( (\gamma) \in \mathbb{P}(\text{Hom}(\mathcal{L}, \mathcal{O}_{\mathbb{P}^2}(2))). \) We thus have to show that \( \dim \text{Hom}(\mathcal{L}, \mathcal{O}_{\mathbb{P}^2}(2)) = 11. \) Consider the epimorphism \( \overline{\gamma} = \gamma|_{\mathbb{P}^2}: \mathcal{L}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(2). \) Since by definition \( [\mathcal{L}] \in \mathcal{R}(1,4,6), \) it follows that \( \ker \overline{\gamma} \simeq \mathcal{I}_{Y, \mathbb{P}^2}(-1) \) for some subsheaf \( Y \) of \( \mathbb{P}^2, \dim Y = 0, \ell_Y = 6. \) This yields an exact triple \( 0 \to \mathcal{I}_{Y, \mathbb{P}^2}(-1) \to \mathcal{L}_{\mathbb{P}^2} \xrightarrow{\overline{\gamma}} \mathcal{O}_{\mathbb{P}^2}(2) \to 0. \) Applying the functor \( \mathcal{E}xt^\bullet_{\mathcal{O}_{\mathbb{P}^2}}(-, \mathcal{O}_{\mathbb{P}^2}(2)) \) to it we obtain an exact triple \( 0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{H}om(\mathcal{L}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(2)) \to \mathcal{O}_{\mathbb{P}^2}(3) \to 0 \) which implies \( \dim \text{Hom}(\mathcal{L}, \mathcal{O}_{\mathbb{P}^2}(2)) = \dim \text{Hom}(\mathcal{L}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(2)) = h^0(\text{Hom}(\mathcal{L}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(2))) = 11. \)

(iii) Since \( ([\mathcal{F}], \mathbb{P}^2) \in -1D_k, (\mathcal{F}_{\mathbb{P}^2})^{\vee \vee} \simeq \mathcal{F}_{\mathbb{P}^2} \) is a locally-free \( \mathcal{O}_{\mathbb{P}^2}\)-sheaf, and (7.19) implies \( k = 0. \) Now, if \( \mathcal{F} \) is unstable, then applying the functor \( - \otimes \mathcal{O}_{\mathbb{P}^2}(2) \) to (7.2) we have an exact triple

\[
0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{F}_{\mathbb{P}^2} \to \mathcal{I}_{Y, \mathbb{P}^2}(2) \to 0, \quad \dim Y = 0, \quad \ell_Y = \deg C = 2, \quad (7.24)
\]

and this triple yields the desired values of \( h^i((\mathcal{F}_{\mathbb{P}^2})^{\vee \vee}(2)) = h^i(\mathcal{F}_{\mathbb{P}^2}(2)). \) If \( \mathcal{F} \) is stable, then these values are given by Proposition 11, (iv).

(iv) Since \( ([\mathcal{F}], \mathbb{P}^2) \in 0D_k \cup 2D_k \neq \emptyset, \) the morphism can in (7.19) is not an isomorphism, hence \( k = \ell_{W(\rho)} \geq 1. \) On the other hand, \( k \leq 3 \) by Propositions 11, (i) and 10, (i). As above, if \( \mathcal{F} \) is unstable, the triple (7.24) is true, which yields the equalities \( h^0(\mathcal{F}_{\mathbb{P}^2}(2)) = 10 \) and \( h^1(\mathcal{F}_{\mathbb{P}^2}(2)) = 0. \) Correspondingly, if \( \mathcal{F} \) is stable, these equalities follow from Proposition 11, (iv). Whence, by (7.19), we have (7.20).

We only have to show that, in case \( \mathcal{F} \) is unstable, \( k \leq 2. \) By the definition of the sets \( iD^u_k, i = 0, 1, \) the condition \( ([\mathcal{F}], \mathbb{P}^2) \in 0D^u_k \) implies that \( \mathbb{P}^2 \nsubseteq 1S^u_k. \) This means that the exact triple (7.24) is true, with \( \dim Y = 0, \ell_Y = \deg C = 2. \) Dualizing this \( \mathcal{O}_{\mathbb{P}^2}\)-triple we easily obtain an inequality \( h^0(\mathcal{E}xt^1(\mathcal{F}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2})) \leq h^0(\mathcal{E}xt^2(\mathcal{O}_Y, \mathcal{O}_{\mathbb{P}^2})) = \ell_Y = 2 \) and an exact triple \( 0 \to \mathcal{O}_{\mathbb{P}^2} \to (\mathcal{F}_{\mathbb{P}^2})^\vee \to \mathcal{I}_{Z, \mathbb{P}^2}(2) \to 0 \) for some scheme \( Z \subset \mathbb{P}^2 \) with \( \dim Z = 0, \ell_Z = 2 - h^0(\mathcal{E}xt^1(\mathcal{F}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2})). \) This triple, together with the (7.24) and the isomorphism \( (\mathcal{F}_{\mathbb{P}^2})^{\vee \vee} \simeq (\mathcal{F}_{\mathbb{P}^2})^{\vee \vee}, \) yields an exact triple \( 0 \to (\mathcal{F}_{\mathbb{P}^2}(2))^{\text{can}} \to (\mathcal{F}_{\mathbb{P}^2})^{\vee \vee}(2) \to K \to 0, \) where \( K \) is an artinian sheaf of length \( h^0(K) = h^0(\mathcal{E}xt^1(\mathcal{F}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2})) \leq 2. \) Comparing this triple with (7.19) we obtain \( K \simeq \mathcal{O}_{W(\rho)} \) and \( k = \ell_{W(\rho)} \leq 2. \)

(v) From the condition \( ([\mathcal{F}], \mathbb{P}^2) \in 1D_k \neq \emptyset \) and Proposition 10 it follows that \( \mathbb{P}^2 \cap \text{Sing} \mathcal{F} = l \) is a line if \( k = 0, 1, 2; \) correspondingly, \( \mathbb{P}^2 \cap \text{Sing} \mathcal{F} = C \) is a conic if \( k = 3. \) Thus, applying the functor \( - \otimes \mathcal{O}_{\mathbb{P}^2}(2) \) to the triples (7.2) and (7.3) and using the resolution \( 0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^2} \to 0, \) we obtain the following exact triples, where \( \dim W = 0 \) and if \( k = 0 \) or 1, then \( W \subset l: \)

\[
0 \to \mathcal{O}_{\mathbb{P}^2}(2) \to \mathcal{F}_{\mathbb{P}^2}(2) \to \mathcal{I}_{C, \mathbb{P}^3}(2)|_{\mathbb{P}^2} \to 0,
\]

\[
0 \to \mathcal{O}_l(3 - k) \to \mathcal{I}_{C, \mathbb{P}^3}(2)|_{\mathbb{P}^2} \to \mathcal{I}_{W, \mathbb{P}^2}(1) \to 0, \quad \ell_W = 3 - k, \quad k \leq 2,
\]

\[
0 \to \mathcal{O}_l \to \mathcal{I}_{C, \mathbb{P}^3}(2)|_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(1) \to 0, \quad k = 3, \quad C \not\subseteq \mathbb{P}^2, \quad \mathbb{P}^2 \cap C = l,
\]

\[
0 \to \mathcal{O}_C(1) \to \mathcal{I}_{C, \mathbb{P}^3}(2)|_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2} \to 0, \quad k = 3, \quad C \subseteq \mathbb{P}^2.
\]

(7.25)

Since \( \mathcal{F} \) is locally free for \( k = 0, \) \( h^i((\mathcal{F}_{\mathbb{P}^2})^{\vee \vee}(2)) = h^i(\mathcal{F}_{\mathbb{P}^2}(2)), \) we obtain (7.21) from (7.25). Correspondingly, for \( k = 1, 2, 3, \) (7.25) and (7.19) imply (7.20). Lemma 11 is proved.
For \(0 \leq k \leq 3\), let \(B \subset \mathbb{P}^3 \times \mathbb{P}^3\) be the graph of incidence, \(\mathcal{O}_B(2) = \mathcal{O}_{\mathbb{P}^3}(2) \boxtimes \mathcal{O}_{\mathbb{P}^3}\), and let \(\text{pr}_1: \mathcal{D}_k^u \to \mathcal{D}_k, \text{pr}_2: \mathcal{D}_k^u \to \mathcal{D}_k^u\) be the projections. For each \(m \geq 0\) consider the set

\[
Y = Y_{k,m}^u := \{([\mathcal{F}], \mathbb{P}^2) \in \mathcal{D}_k^u \mid \dim \text{Hom}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^3}(2)) = m\}, \quad m \geq 0, \tag{7.26}
\]

and set \(Y = \text{pr}_0^{-1}(Y), \quad q_i = \text{pr}_i|_Y, \quad i = 0, 1, 2, \quad L = \text{Ext}_{q_0}(q_1^*\mathcal{F}, q_2^*\mathcal{O}_B(2)), \) where \(\mathcal{F}\) is the universal sheaf on \(\mathcal{R}_k^u\) which exists by Proposition 10, (iv), \(\mathcal{Y} = Y \times_Y \mathcal{P}(L^\vee),\) and let \(\mathcal{P}(L^\vee) \xrightarrow{\lambda} \mathcal{Y} \xrightarrow{\pi} Y\) be the projections. By Satz 3 in [2], \(Y = Y_{k,m}^u\) is locally closed in \(\mathcal{D}_k^u\) and the sheaf \(L\) is a rank \(m\) locally-free sheaf on \(Y\) which commutes with the Base Change, that is, for \(y = ([\mathcal{F}], \mathbb{P}^2) \in Y,\) one has \(L|_y = \text{Hom}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^3}(2)).\) On \(\mathcal{Y}\) there is a universal morphism \(\rho: (q_1 \circ \pi)^*\mathcal{F} \to (q_2 \circ \pi)^*\mathcal{O}_B(2) \otimes \mu^*\mathcal{O}_{\mathcal{P}(L^\vee)}(1).\) Consider the set

\[
X = X_{k,m}^u := \{([\mathcal{F}], \mathbb{P}^2) \in Y_{k,m}^u \mid K([\mathcal{F}], \mathbb{P}^2) \neq \emptyset\}. \tag{7.27}
\]

From this definition it follows that the sheaf \(\text{im} \rho\) is flat over \(\mathcal{P}(L^\vee)\) at any point \(x \in \nu^{-1}(X).\) This implies that \(X\) is an open (possibly empty) subset of \(Y,\) hence it is locally closed in \(\mathcal{D}_k^u.\) Therefore, since in view of Proposition 10, (iv), the \(\mathcal{D}_k^u\) are varieties, the set \(\Phi_k^u = \{m \in \mathbb{Z}_{\geq 0} \mid X_{k,m}^u \neq \emptyset\}\) is finite. By the definitions (7.11), (7.12), (7.14) and (7.27) we have

\[
X_k^u := \bigsqcup_{m \in \Phi_k^u} X_{k,m}^u = p_{1k}(Q_k^u), \quad Q_k^u = p_{1k}^{-1}(X_k^u). \tag{7.28}
\]

Denoting \(iX_k^u = iD_k^u \cap X_k^u, \quad iQ_k^u = p_{1k}^{-1}(iX_k^u), \quad -1 \leq i \leq 2,\) we find from the first equality (7.9) that

\[
Q_k^u = \bigsqcup_{-1 \leq i \leq 2} iQ_k^u. \tag{7.29}
\]

The inclusion (7.16) and Lemma 11, (i) yield that the projection \(p_{1k}: iQ_k^u \to iX_k^u\) decomposes as

\[
p_{1k}: iQ_k^u \xrightarrow{\text{open}} i\tilde{Q}_k^u \xrightarrow{\tilde{p}_{1k}} iX_k^u, \quad -1 \leq i \leq 2, \quad 0 \leq k \leq 3, \tag{7.30}
\]

where \(i\tilde{Q}_k^u, \tilde{p}_{1k}, \tilde{p}_{1k}^{-1}, iX_k^u\) is the projective fibration with fibre \(\mathbb{P}(H^0((\mathcal{F}_{\mathbb{P}^3})^{\vee \vee}(2)))\) over an arbitrary point \(([\mathcal{F}], \mathbb{P}^2) \in iX_k^u.\) Here by (7.28) each \(iX_k^u\) is a disjoint union of schemes. This shows that each \(iQ_k^u\) is a disjoint union of schemes. Since \(iX_k^u \subset D_k^u,\) it follows from (7.10) and Lemma 11, (iii)–(v) that

\[
\dim iQ_k^u \leq \dim i\mathcal{R}_k^u + 11 - i + k, \quad -1 \leq i \leq 2, \quad 0 \leq k \leq 3. \tag{7.31}
\]

Thus, in view of (7.4), we obtain \(\dim iQ_k^u \leq 26\) for all possible \(i\) and \(k,\) hence (7.29) yields

\[
\dim Q_k^u \leq 26, \quad 0 \leq k \leq 3. \tag{7.32}
\]

To obtain a similar estimate for dimensions of \(Q_k^u,\) we define similarly to (7.26) the locally closed subsets \(Y_{k,m}^u := \{([\mathcal{F}], \mathbb{P}^2) \in D_k^u \mid \dim \text{Hom}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^3}(2)) = m\}, \quad m \geq 0, \quad D_k^u.\) Next, note that \(a \text{ priori} \) there is no universal sheaf \(\mathcal{F}\) on \(\mathcal{R}_k^u.\)
However, by Proposition 11, $\text{Ext}^2(\mathcal{F}, \mathcal{F}) = 0$ for any $[\mathcal{F}] \in \mathcal{R}^s_k$, $0 \leq k \leq 2$. This means that the deformation theory for $\mathcal{R}^s_k$ is unobstructed, so there exists an open cover $\mathcal{R}^s_k = \bigcup_{j \in J} \mathcal{U}_j$ and universal sheaves $\mathcal{F}_j$ on $\mathcal{U}_j$ (see, for example, [9], Appendices A1 and A2, [17], Ch. 6). The existence of these local universal sheaves is enough to show that the sets $X_{k,m}^s$ defined similarly to (7.27) as $X_{k,m}^s := \{([\mathcal{F}], \mathbb{P}^2) \in Y_{k,m}^s \mid K([\mathcal{F}], \mathbb{P}^2) \neq \emptyset\}$ are locally closed subsets of $\mathcal{D}^s_k$. We then have, similarly to (7.28) and (7.29), finite disjoint unions of schemes $X_{k,m}^s := \bigcup_{m \in \Phi_k^s} X_{k,m}^s = p_{1k}(Q_k^s)$ and relations $Q_k^s = p_{1k}^{-1}(X_k^s)$. Denoting $iX_k^s = iD_k^s \cap X_k^s$, $iQ_k^s = p_{1k}^{-1}(iX_k^s)$, $-1 \leq i \leq 2$, and mimicking the argument in (7.29)–(7.31) for $s$ in place of $u$, we obtain that $iQ_k^s$ and $Q_k^s$ are disjoint unions of schemes satisfying the inequalities $\dim iQ_k^s \leq \dim iR_k^s + 11 - i + k$, $-1 \leq i \leq 2$, $0 \leq k \leq 2$. These formulae and Proposition 11, (ii), imply the inequalities $\dim Q_k^s \leq 26$, $0 \leq k \leq 2$, which, together with (7.32) and (7.15), yield

$$\dim Q_k \leq 26, \quad 0 \leq k \leq 3. \quad (7.33)$$

(Recall that, as in (7.15), we set $Q_3^s = T_3^s = \emptyset$.) Now from Lemma 11, (ii) and the last formula in (7.13), by analogy with (7.30) we obtain that the $T_k$ are disjoint unions of schemes and the projections $p_{2k}: T_k \to Q_k$ are morphisms which decompose as $p_{2k}: T_k \xrightarrow{\text{open}} \tilde{T}_k \xrightarrow{\tilde{p}_{2k}} Q_k$, where $\tilde{T}_k \xrightarrow{\tilde{p}_{2k}} iQ_k$ is the projective fibration with fibre $\mathbb{P}(H^0((\mathcal{F}_{\mathbb{P}^2})^\vee \vee (2))) \simeq \mathbb{P}^{10}$ over an arbitrary point $([\mathcal{F}], \mathbb{P}^2, (\rho)) \in Q_k$, $0 \leq k \leq 3$. Together with (7.33), this yields

$$\dim T_k \leq 36, \quad 0 \leq k \leq 3. \quad (7.34)$$

Proof of Theorem 4. It is clear from the above that the maps $f_k: T_k \to \mathcal{B}(5)$ defined in (7.17) are morphisms. The inequality $\dim (\mathcal{H} \setminus (\mathcal{G}(2,1) \cap \mathcal{H})) \leq 36$ now follows from (7.18) and (7.34). However, by Remark 3.4.1 in [20], any irreducible component of $\mathcal{B}(5)$ has dimension at least 37. Hence Theorem 4 follows.

§ 8. Components of $\mathcal{B}(5)$

8.1. We finally have at hand all the results needed to complete the proof of our second main result, namely, the characterization of the irreducible components of $\mathcal{B}(5)$ given by Theorem 2. This entire section is devoted to this goal.

Proof of Theorem 2. The first ingredient of the proof is the fact, proved by Hartshorne and Rao, that every bundle in $\mathcal{B}(5)$ is the cohomology of one of the monads (1.2)–(1.6) (see [21], Table 5.3, case 5, (1)–(4)).

Recall that for each stable rank 2 bundle $E$ on $\mathbb{P}^3$ with vanishing first Chern class, the number $\alpha(E) := h^1(E(-2)) \mod 2$ is called the Atiyah-Rees $\alpha$-invariant of $E$ (see [19], Definition on p.237). Hartshorne showed (see [19], Corollary 2.4) that this number is invariant on the connected components of the moduli space of stable vector bundles on $\mathbb{P}^3$. One can easily check that the cohomologies of monads of the forms (1.2) and (1.3) have an $\alpha$-invariant equal to 0, while the cohomologies of the other three types of monads have an $\alpha$-invariant equal to 1.

Rao showed in [36] that the family of bundles obtained as the cohomology of monads of the form (1.3) is irreducible, of dimension 36, and it lies in a unique component of $\mathcal{B}(5)$. Since instanton bundles of charge 5, that is, the cohomologies
of monads of the form (1.2), yield an irreducible family of dimension 37, it follows that the set
\[ \mathcal{I} := \{ [E] \in \mathcal{B}(5) | \alpha(E) = 0 \} \] (8.1)
forms a single irreducible component of $\mathcal{B}(5)$, of dimension 37, whose generic point corresponds to an instanton bundle. In addition, every $[E] \in \mathcal{I}$ satisfies $H^1(\text{End}(E)) = 37$; this was originally proved by Katsylo and Ottaviani for instanton bundles [30], and by Rao for the cohomologies of monads of the form (1.3) (see [36], §3). Therefore, we also conclude that $\mathcal{I}$ is nonsingular. This completes the proof of part 1) of Theorem 2.

8.2. Our next step is to analyse those bundles with Atiyah-Rees $\alpha$-invariant equal to 1. Hartshorne proved in [20], Theorem 9.9, that the family $\mathcal{K}$ of stable rank 2 bundles $E$ with $c_1(E) = 0$ and $c_2(E) = 5$ whose spectrum is $(-2, -1, 0, 1, 2)$ is an irreducible, nonsingular family of dimension 40, and from the definition of the spectrum one has
\[ h^1(\mathcal{E}(-2)) = 3, \quad [\mathcal{E}] \in \mathcal{K}. \] (8.2)
The bundles from $\mathcal{K}$ are precisely those given as cohomologies of monads of the form (1.4) (see [21], Table 5.3, case 5, (4)), which is a particular case of a class of monads studied by Ein in [15]. It is shown in [15] that the closure $\overline{\mathcal{K}}$ of $\mathcal{K}$ in $\mathcal{B}(5)$ is an irreducible component of $\mathcal{B}(5)$ of dimension 40.

We proved in Theorem 1, case $a = 2$, that the bundles arising as the cohomologies of monads of the form (1.5) form a dense subset $\mathcal{G}(2, 1)$ of a rational irreducible component of dimension 37. Consider the set $\mathcal{H}$ of bundles arising as the cohomologies of monads of the form (1.6). Since the bundles from $\mathcal{G}(2, 1) \cup \mathcal{H}$ have the spectrum $(-1, 0, 0, 0, 1)$ by [21], Table 5.3, case 5, (2), in accordance with (6.35) we have
\[ h^1(\mathcal{E}(-2)) = 1, \quad [\mathcal{E}] \in \mathcal{G}(2, 1) \cup \mathcal{H}, \] (8.3)
so that $\alpha(\mathcal{E}) = 1$, and therefore, in view of (8.1), $\mathcal{H} \cap \mathcal{I} = \emptyset$. Since, by Theorem 4, $\mathcal{H}$ does not constitute a component in $\mathcal{B}(5)$, it then follows from the above that $\mathcal{H} \subset \mathcal{G}(2, 1) \cup \mathcal{K}$.

**Proposition 12.** $\mathcal{H} \subset \mathcal{G}(2, 1)$ and $\mathcal{K} = \mathcal{K}$.

**Proof.** We only have to show that $(\mathcal{G}(2, 1) \cup \mathcal{H}) \cap \mathcal{K} = \emptyset$. Suppose for a contradiction that there exists a vector bundle $[\mathcal{E}] \in (\mathcal{G}(2, 1) \cup \mathcal{H}) \cap \mathcal{K}$. By (8.2) and the inferior semi-continuity of the dimension of the cohomology groups of coherent sheaves, one has that $h^1(\mathcal{E}(-2)) \geq 3$, contrary to (8.3). The proposition is proved.

Proposition 12 finally concludes the proof of parts 2) and 3) in Theorem 2. Theorem 2 is proved.

8.3. We summarize all the information from Theorem 2, and the discrete invariants of stable rank 2 bundles with $c_1 = 0$ and $c_2 = 5$ in Table 1.

**Remark 3.** Inspired by the techniques introduced in the present paper, the authors of [40] constructed another infinite series of irreducible components of $\mathcal{B}(0, n)$ whose special point corresponds to a bundle obtained as the cohomology of a monad similar to the one in (1.7), just by substituting a direct sum of two rank 2 instanton bundles for the rank 4 instanton bundle of charge 1 in the middle term.
Table 1. Irreducible components of $B(5)$

| Component          | Dimension | Monads | Spectra          | $\alpha$-invariant |
|--------------------|-----------|--------|------------------|--------------------|
| Instanton          | 37        | (1.2)  | (0, 0, 0, 0, 0)  | 0                  |
|                    |           | (1.3)  | (−1, −1, 0, 1, 1)|                    |
| Ein component      | 40        | (1.4)  | (−2, −1, 0, 1, 2)| 1                  |
| Modified instanton | 37        | (1.5)  | (−1, 0, 0, 0, 1) | 1                  |
|                    |           | (1.6)  |                  |                    |

Bibliography

[1] C. Bănică and N. Manolache, “Rank 2 stable vector bundles on $\mathbb{P}^3(\mathbb{C})$ with Chern classes $c_1 = −1$, $c_2 = −4$”, *Math. Z.* **190**:3 (1985), 315–339.

[2] C. Bănică, M. Putinar and G. Schumacher, “Variation der globalen Ext in Deformationen kompakter komplexer Räume”, *Math. Ann.* **250**:2 (1980), 135–155.

[3] W. Barth, “Some properties of stable rank-2 vector bundles on $\mathbb{P}_n$”, *Math. Ann.* **226**:2 (1977), 125–150.

[4] W. Barth, “Stable vector bundles on $\mathbb{P}_3$, some experimental data”, *Les équations de Yang-Mills*, Seminar E.N.S. 1977–1978, Astérisque, vol. 71-72, Soc. Math. France, Paris 1980, pp. 205–218.

[5] W. Barth and G. Elencwajg, “Concernant la cohomologie des fibres algébriques stables sur $\mathbb{P}_n(\mathbb{C})$”, *Variétés analytiques compactes* (Nice 1977), Lecture Notes in Math., vol. 683, Springer, Berlin 1978, pp. 1–24.

[6] W. Barth and K. Hulek, “Monads and moduli of vector bundles”, *Manuscripta Math.* **25**:4 (1978), 323–347.

[7] W. Barth, K. Hulek, C. A. M. Peters and A. Van de Ven, *Compact complex surfaces*, 2nd ed., Ergeb. Math. Grenzgeb. (3), vol. 4, Springer-Verlag, Berlin 2004, xii+436 pp.

[8] G. Bohnhorst and H. Spindler, “The stability of certain vector bundles on $\mathbb{P}^n$”, *Complex algebraic varieties* (Bayreuth 1990), Lecture Notes in Math., vol. 1507, Springer, Berlin 1992, pp. 39–50.

[9] J. Brun and A. Hirschowitz, “Variété des droites sauteuses du fibré instanton général”, *Compos. Math.* **53**:3 (1984), 325–336.

[10] U. Bruzzo, D. Markushevich and A. S. Tikhomirov, “Moduli of symplectic instanton vector bundles of higher rank on projective space $\mathbb{P}^3$”, *Cent. Eur. J. Math.* **10**:4 (2012), 1232–1245.

[11] Mei-Chu Chang, “Stable rank 2 bundles on $\mathbb{IP}^3$ with $c_1 = 0$, $c_2 = 4$ and $\alpha = 1$”, *Math. Z.* **184**:3 (1983), 407–415.

[12] Mei-Chu Chang, “Stable rank 2 reflexive sheaves on $\mathbb{P}^3$ with small $c_2$ and applications”, *Trans. Amer. Math. Soc.* **284**:1 (1984), 57–89.

[13] I. Coandă, A. Tikhomirov and G. Trautmann, “Irreducibility and smoothness of the moduli space of mathematical 5-instantons over $\mathbb{P}_3$”, *Internat. J. Math.* **14**:1 (2003), 1–45.

[14] L. Costa and R. M. Miró-Roig, “Monads and regularity of vector bundles on projective varieties”, *Michigan Math. J.* **55**:2 (2007), 417–436.

[15] L. Ein, “Generalized null correlation bundles”, *Nagoya Math. J.* **111** (1988), 13–24.
New moduli components of rank 2 bundles on projective space

[16] G. Ellingsrud and S. A. Strømme, “Stable rank 2 vector bundles on \( \mathbb{P}^3 \) with \( c_1 = 0 \) and \( c_2 = 3 \)”, Math. Ann. 255:1 (1981), 123–135.

[17] B. Fantechi, L. Göttsche, L. Illusie, S. L. Kleiman, N. Nitsure and A. Vistoli, Fundamental algebraic geometry. Grothendieck’s FGA explained, Math. Surveys Monogr., vol. 123, Amer. Math. Soc., Providence, RI 2005, x+339 pp.

[18] R. Hartshorne, Algebraic geometry, Grad. Texts in Math., vol. 52, Springer-Verlag, New York–Heidelberg 1977, xvi+496 pp.

[19] R. Hartshorne, “Stable vector bundles of rank 2 on \( \mathbb{P}^3 \)”, Math. Ann. 238:3 (1978), 229–280.

[20] R. Hartshorne, “Stable reflexive sheaves”, Math. Ann. 254:2 (1980), 121–176.

[21] R. Hartshorne and A. P. Rao, “Spectra and monads of stable bundles”, J. Math. Kyoto Univ. 31:3 (1991), 789–806.

[22] R. Hartshorne and I. Sols, “Stable rank 2 vector bundles on \( \mathbb{P}^3 \) with \( c_1 = -1 \), \( c_2 = 2 \)”, J. Reine Angew. Math. 325 (1981), 145–152.

[23] G. Horrocks, “Vector bundles on the punctured spectrum of a local ring”, Proc. London Math. Soc. (3) 14:4 (1964), 689–713.

[24] D. Huybrechts and M. Lehn, The geometry of moduli spaces of sheaves, 2nd ed., Cambridge Math. Lib., Cambridge, Cambridge Univ. Press 2010, xviii+325 pp.

[25] M. Jardim, “Instanton sheaves on complex projective spaces”, Collect. Math. 57:1 (2006), 69–91.

[26] M. Jardim, S. Marchesi and A. Wissdorf, “Moduli of autodual instanton bundles”, Bull. Braz. Math. Soc. (N.S.) 47:3 (2016), 823–843.

[27] M. Jardim and R. V. Martins, “Linear and Steiner bundles on projective varieties”, Comm. Algebra 38:6 (2010), 2249–2270.

[28] M. Jardim and M. Verbitsky, “Trihyperkähler reduction and instanton bundles on \( \mathbb{C}P^3 \)”, Compos. Math. 150:11 (2014), 1836–1868.

[29] P. I. Katsylo, “Rationality of the module variety of mathematical instantons with \( c_2 = 5 \)”, Lie groups, their discrete subgroups, and invariant theory, Adv. Soviet Math., vol. 8, Amer. Math. Soc., Providence, RI 1992, pp. 105–111.

[30] P. I. Katsylo and G. Ottaviani, “Regularity of the moduli space of instanton bundles \( \text{MI}_{\mathbb{P}^3}(5) \)”, Transform. Groups 8:2 (2003), 147–158.

[31] A. A. Kytmanov, A. S. Tikhomirov and S. A. Tikhomirov, “Series of rational moduli components of stable rank two vector bundles on \( \mathbb{P}^3 \)”, Selecta Math. (N.S.) 25:2 (2019), 29, 47 pp.

[32] H. Lange, “Universal families of extensions”, J. Algebra 83:1 (1983), 101–112.

[33] N. Manolache, “Rank 2 stable vector bundles on \( \mathbb{P}^3 \) with Chern classes \( c_1 = -1 \), \( c_2 = 2 \)”, Rev. Roumaine Math. Pures Appl. 26:9 (1981), 1203–1209.

[34] M. Maruyama, “Moduli of stable sheaves. I”, J. Math. Kyoto Univ. 17:1 (1977), 91–126.

[35] Ch. Okonek, M. Schneider and H. Spindler, Vector bundles on complex projective spaces, Progr. Math., vol. 3, Birkhäuser, Boston, Mass. 1980, vii+389 pp.

[36] A. P. Rao, “A family of vector bundles on \( \mathbb{P}^3 \)”, Space curves (Rocca di Papa 1985), Lecture Notes in Math., vol. 1266, Springer, Berlin 1987, pp. 208–231.

[37] I. R. Shafarevich, “Part 1. Basic concepts”, Basic algebraic geometry, 3rd augmented ed., Moscow Center for Continuous Mathematical Education, Moscow 2007, pp. 13–306 (Russian); English transl., I. R. Shafarevich, Book 1. Basic algebraic geometry: Varieties in projective space, 3rd ed., Springer, Heidelberg 2013, xviii+310 pp.
[38] A. S. Tikhomirov, “Moduli of mathematical instanton vector bundles with odd $c_2$ on projective space”, Izv. Ross. Akad. Nauk Ser. Mat. 76:5 (2012), 143–224; English transl. in Izv. Math. 76:5 (2012), 991–1073.

[39] A. S. Tikhomirov, “Moduli of mathematical instanton vector bundles with even $c_2$ on projective space”, Izv. Ross. Akad. Nauk Ser. Mat. 77:6 (2013), 139–168; English transl. in Izv. Math. 77:6 (2013), 1195–1223.

[40] A. S. Tikhomirov, S. A. Tikhomirov and D. A. Vassiliev, “Construction of stable rank 2 bundles on $\mathbb{P}^3$ via symplectic bundles”, Sibirsk. Mat. Zh. 60:2 (2019), 441–460; English transl. in Siberian Math. J. 60:2 (2019), 343–358.

Charles Almeida
Department of Mathematics,
Federal University of Minas Gerais,
Belo Horizonte, Brazil
E-mail: charlesalmeida@mat.ufmg.br

Marcos Jardim
Department of Mathematics,
Institute of Mathematics, Statistics
and Scientific Computing,
Campinas, Brazil
E-mail: jardim@ime.unicamp.br

Alexander S. Tikhomirov
Faculty of Mathematics,
National Research University
Higher School of Economics,
Moscow, Russia
E-mail: astikhomirov@mail.ru

Sergey A. Tikhomirov
Faculty of Physics and Mathematics,
Yaroslavl State Pedagogical University
named after K. D. Ushinsky,
Yaroslavl, Russia
E-mail: satikhomirov@mail.ru