Electrostatic Turbulence in Magnetic Fields with Arbitrary Topology

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November 23, 2021

Abstract

We derive a model equation describing electrostatic plasma turbulence in general (inhomogeneous and curved) magnetic fields by analysing the effect of curved geometry on the polarization drift. The derived nonlinear equation generalizes the Hasegawa-Mima equation governing drift wave turbulence in a straight homogeneous magnetic field. We identify the conserved energy of the system, and obtain conditions on magnetic field topology for conservation of generalized enstrophy. We further show how self-organized steady turbulent states are modified by the curvature of the magnetic field through numerical examples.

1 Introduction

The Hasegawa-Mima equation \cite{1, 2} describes 2-dimensional electrostatic plasma turbulence in a straight homogeneous magnetic field. Mathematically, it is related to the quasi-geostrophic equation for atmospheric dynamics on rotating planetary surfaces \cite{3, 4}, and it reduces to the vorticity equation for a 2-dimensional incompressible fluid in the limit of high electron temperature \cite{5}. As such, the Hasegawa-Mima equation exhibits properties analogous to 2-dimensional fluid turbulence \cite{6, 7, 8}, including inverse cascade of energy \cite{9, 10, 11} associated with the presence of two inviscid invariants, energy and generalized enstrophy \cite{12}. Furthermore, in the inviscid limit the system is endowed with a noncanonical Hamiltonian structure \cite{13, 14}, where energy and generalized enstrophy play the roles of Hamiltonian and Casimir invariant respectively \cite{15}. These properties combined with a relative simplicity make the Hasegawa-Mima equation an effective tool in the study of 2-dimensional fluid and plasma turbulence. Applications include the investigation of turbulence in geophysical flows \cite{16} and magnetically confined plasmas \cite{17, 18}, and more generally the characterization of self-organized turbulent states and zonal flows \cite{19, 20}.

Several generalizations of the Hasegawa-Mima equations exist. On one hand, the Hasegawa-Wakatani system \cite{21, 22} consists of coupled nonlinear equations for the electrostatic potential and the ion density. These equations reveal the interplay between drift wave turbulence and zonal flow mediated by the Kelvin-Helmholtz instability \cite{23}. Hasegawa and Wakatani also included the effect of field curvature in cylindrical geometry to their equations \cite{24}. On the other hand, reduced magnetohydrodynamics \cite{25} and the four-field model \cite{26} take into account the time-evolution of magnetic flux, parallel ion velocity, and electron pressure. Nevertheless, these models are 2-dimensional, any deviation from a straight magnetic field being treated as a higher-order correction in the relevant ordering. Hence, a model of electrostatic plasma turbulence in a general magnetic field is not available at present. The root cause of this difficulty can be ascribed to the rather elusive nature of the polarization drift \cite{27, 28} in nontrivial magnetic topologies. The purpose of
this paper is to fill this gap by deriving an equation describing the evolution of electrostatic turbulence in a general magnetic field for the simplest plasma system consisting of cold ions and a cloud of electrons obeying the Boltzmann distribution.

2 Model equation

Consider a plasma system consisting of ions and electrons. Let \( n_e \) denote the electron density. We assume that \( n_e \) follows a Boltzmann distribution with temperature \( T_e \),

\[
n_e = A_e \exp \left\{ - \frac{q\phi}{k_B T_e} \right\}.
\]  

(1)

Here, \( A_e \) is a positive real constant, \( \phi \) denotes the electric potential, \( q = -e \) the electron charge, and \( k_B \) the Boltzmann constant. It is convenient to introduce the constant

\[
\lambda = \frac{e}{k_B T_e}.
\]  

(2)

Let \( B = B(x) \) denote a static magnetic field. Recall that \( B \) must be solenoidal, \( \nabla \cdot B = 0 \). In the following, we demand that \( B \neq 0 \) throughout the domain of interest \( \Omega \). Let \( n \) denote the ion density. Assuming quasineutrality, \( n_e = Z n \) with \( Z \) the number of protons in the ions. Using \([1]\), the ion continuity equation reads

\[
\lambda \phi_t = -\nabla \cdot \mathbf{v} - \lambda \mathbf{v} \cdot \nabla \phi.
\]  

(3)

Here, \( \mathbf{v} \) denotes the ion fluid velocity. In the guiding center approximation \([27]\), the field \( \mathbf{v} \) is expected to be of the form

\[
\mathbf{v} = \mathbf{v}_\parallel + \mathbf{v}_E + \mathbf{v}_\nabla B + \mathbf{v}_\kappa + \mathbf{v}_{\text{pol}} + \mathbf{v}_{\text{dia}} + \mathbf{v}_F.
\]  

(4)

In this expression, \( \mathbf{v}_\parallel \) is the ion fluid velocity along the magnetic field, \( \mathbf{v}_E \) the \( \mathbf{E} \times \mathbf{B} \) drift, \( \mathbf{v}_\nabla B \) the gradient \( B \) drift, \( \mathbf{v}_\kappa \) the curvature drift, \( \mathbf{v}_{\text{pol}} \) the polarization drift, \( \mathbf{v}_{\text{dia}} \) the diamagnetic drift arising from the pressure \( P \), and \( \mathbf{v}_F \) any additional drift caused by a force \( \mathbf{F} \) affecting the system. In this study we are concerned with ‘drift’ time scales \( t_d = 2\pi/\omega_d \) much longer than the time scale \( t_b = 2\pi/\omega_b \) associated with bouncing motion of charged particles along the magnetic field, \( t_d >> t_b \). This is the case, for example, of the periodic motion associated with the third adiabatic invariant (magnetic flux) \( \Psi \) in an axially symmetric magnetic field, such as a dipole magnetic field. Over the time scale \( t_d \), the ion fluid velocity along the magnetic field effectively averages to zero, \( \mathbf{v}_\parallel = 0 \). The remaining terms can thus be expressed as

\[
\mathbf{v} = \frac{\mathbf{B} \times \nabla \phi}{B^2} + \frac{K_\perp}{Ze} \frac{\mathbf{B} \times \nabla B}{B^3} + 2\frac{K_\parallel}{Ze} \frac{\mathbf{B} \times \kappa}{B^2} + \mathbf{v}_{\text{pol}} + \frac{\mathbf{B} \times \nabla P}{Ze n B^2} + \mathbf{v}_F.
\]  

(5)

The expressions of the polarization drift \( \mathbf{v}_{\text{pol}} \) and the force drift \( \mathbf{v}_F \) will be discussed separately because they require careful treatment. In the equation above, \( K_\parallel \) and \( K_\perp \) are the ion kinetic energies along and across the magnetic field, while \( \kappa = -R_c/R^2 \) is the magnetic field curvature. In the cold ions approximation \( K_\parallel = K_\perp = 0 \). Furthermore, we assume that the mechanical pressure \( P \) is small when compared with the magnetic pressure \( B^2 \). Hence, the diamagnetic drift \( \mathbf{v}_{\text{dia}} \) can be neglected in equation \([5]\). Next, we demand that the electric potential energy is small when compared with the electron kinetic energy and the energy stored in the magnetic field,

\[
\left| \frac{e\phi}{k_B T_e} \right| << 1, \quad \left| \frac{2\mu_0 e\phi}{B^2\Omega} \right| << 1.
\]  

(6)

Here, \( \Omega \) is the total volume occupied by the plasma and \( \mu_0 \) denotes the vacuum permeability. Under these conditions, it will be shown that both \( \mathbf{v}_{\text{pol}} \) and the relevant drift \( \mathbf{v}_F \) are small when compared with \( \mathbf{v}_E \), that is they represent higher-order corrections in the ordering of equation \([6]\). Therefore at leading order the advective derivative can be written as

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla = \frac{\partial}{\partial t} + \mathbf{v}_E \cdot \nabla.
\]  

(7)
We are now ready to derive the expression for the polarization drift $v_{\text{pol}}$. This transient drift arises from the change of the $E \times B$ drift velocity $v_E$ in time. In particular, the acceleration felt by a charged particle must be balanced by the corresponding Lorentz force,

$$m \frac{dv_E}{dt} = Z e v_{\text{pol}} \times B.$$  \hfill (8)

where $m$ is the ion mass. The polarization drift therefore has expression

$$v_{\text{pol}} = -\frac{m}{Ze} \frac{dv_E \times B}{B^2}. \hfill (9)$$

Since $v_E$ scales as $\phi/B$, the polarization drift (9) is a first-order term in the ordering of equation (6). Next, observe that the total energy $h$ of each particle must be constant. At leading order each particle thus feels an effective potential energy

$$V_E = h - \frac{1}{2} m v_E^2. \hfill (10)$$

Hence, the corresponding force $F = -\nabla V_E$ generates a drift

$$v_F = -\frac{m}{2Ze} B \times \nabla^2 v_E. \hfill (11)$$

Again, this drift represents a first-order term in the ordering of equation (6) because it scales as $\phi^2/B^3$. Combining equations (5), (9), (11), and taking into account the assumptions outlined above, the total ion velocity has expression

$$v = v_E - \sigma B \times \left[ v_E \times (\nabla \times v_E) \right] - \sigma \frac{\partial}{\partial t} \nabla \phi \times \left( \nabla \phi \right)$$

$$= \left( 1 - \sigma B \cdot \nabla \times v_E \right) v_E - \frac{\sigma}{\partial t} \nabla \phi \left( \nabla \phi \right), \hfill (12)$$

where we introduced the physical constant

$$\sigma = \frac{m}{Ze}, \hfill (13)$$

and the orthogonal gradient operator

$$\nabla^\perp f = \frac{B \times (\nabla f \times B)}{B^2}, \hfill (14)$$

with $f$ some function. Then, we have

$$\nabla \cdot v = \nabla \cdot \left[ \left( 1 - \sigma B \cdot \nabla \times v_E \right) v_E \right] - \frac{\sigma}{\partial t} \nabla \cdot \left( \nabla \phi \right), \hfill (15)$$

and also

$$\lambda v \cdot \nabla \phi = -\lambda \sigma \frac{\partial}{\partial t} \left( \nabla \phi \right)^2. \hfill (16)$$

Substituting equations (15) and (16) into (3), we obtain

$$\frac{\partial}{\partial t} \left[ \lambda \phi - \frac{\sigma}{2B^2} \left( \nabla \phi \right)^2 \right] - \sigma \frac{B \cdot \nabla \phi}{B^2} \left( \nabla \phi \right) = \nabla \cdot \left[ \left( \frac{\sigma B \cdot \nabla \times v_E}{B^2} - 1 \right) v_E \right]. \hfill (17)$$

The second term on the left-hand side scales as $\lambda \phi^2/B^2$ and it is therefore a second order term that can be neglected. At first order in the ordering (6) we thus arrive at

$$\frac{\partial}{\partial t} \left[ \lambda \phi - \sigma \nabla \cdot \left( \nabla \phi \right) \frac{B^2}{\nabla \phi} \right] = \nabla \cdot \left[ \left( \frac{\sigma B \cdot \nabla \times v_E}{B^2} - 1 \right) v_E \right]. \hfill (18)$$
We suggest that equation (18) is appropriate to describe electrostatic turbulence in magnetic fields with arbitrary topology. It can be verified that under suitable boundary conditions equation (18) is endowed with a preserved energy

$$H_\Omega = \frac{1}{2} \int_\Omega \left( \lambda \phi^2 + \frac{\sigma |\nabla \phi|^2}{B^2} \right) dV,$$

(19)

where $\Omega \subset \mathbb{R}^3$ is a bounded domain and $dV$ the volume element in $\mathbb{R}^3$. Indeed, denoting with $n$ the unit outward normal to the boundary $\partial \Omega$, and with $dS$ the surface element on $\partial \Omega$, one has

$$\frac{dH_\Omega}{dt} = \int_\Omega \left( \lambda \phi \phi_t + \sigma \nabla \cdot \left( \frac{\nabla \phi}{B^2} \right) \right) dV + \int_{\partial \Omega} \sigma \phi \nabla \phi_t \cdot n dS$$

(20)

Hence, the rate of change in $H$ is given by boundary integrals which vanish if, for example, the system is periodic or $\phi = 0$ on $\partial \Omega$.

3 Limit to the Hasegawa-Mima equation and curvature effects

In this section we will show that equation (18) reduces to the Hasegawa-Mima equation [1] when the magnetic field is straight, $B = B_0 \nabla z$ with $B_0 \in \mathbb{R}$. To elucidate how a curved inhomogeneous magnetic field modifies the Hasegawa-Mima equation, it is convenient to study equation (18) in the limit of a magnetic field of the type

$$B = B_0 \nabla z + \epsilon a,$$

(21)

where $\epsilon \ll 1$ is an ordering parameter and $a = (a_x, a_y, a_z)$ a vector field such that $\nabla \cdot a = 0$. For simplicity, we assume that $a_z = a \cdot \nabla z = 0$. Notice that at first order in $\epsilon$ the curvature of the magnetic field (21) is given by

$$\kappa = \frac{B \cdot \nabla (B \cdot \nabla z)}{B_0} = \frac{\epsilon \partial a}{B_0 \partial z}.$$  

(22)

It is convenient to introduce the 2-dimensional gradient and Laplacian operators

$$\nabla_{(x,y)} f = \frac{\partial f}{\partial x} \nabla x + \frac{\partial f}{\partial y} \nabla y, \quad \Delta_{(x,y)} f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2},$$

(23)

where $f = f(x, y, z)$ is some function. Then, at first order in $\epsilon$ we have

$$B^2 = B_0^2,$$

(24a)

$$\nabla \phi = \nabla_{(x,y)} \phi - \frac{\epsilon}{B_0} \left[ (a \cdot \nabla \phi) \nabla z + \frac{\partial \phi}{\partial z} a \right],$$

(24b)

$$v_E = \frac{\nabla z \times \nabla \phi}{B_0} + \frac{\epsilon}{B_0} a \times \nabla \phi,$$

(24c)

$$\nabla \cdot v_E = -\frac{\epsilon}{B_0} \nabla \phi \cdot \nabla \times a,$$

(24d)

$$\nabla \times v_E = \frac{\Delta_{(x,y)} \phi \nabla z + \epsilon}{B_0^2} \nabla \times (a \times \nabla \phi) - \frac{1}{B_0} \nabla_{(x,y)} \phi \frac{\partial \phi}{\partial z}.$$  

(24e)
examine how the Hasegawa-Mima equation is modified when curvature is a leading order term. To this end, that the component of the magnetic field \( \epsilon \) the system. Finally, the last term on the right-hand side results from the polarization drift associated with the magnetic field. Terms involving \( \partial \phi \)

Recalling equation (22), we see that the third term on the left-hand side of (26) arises from the curvature of the magnetic field. Terms involving \( \partial \phi \) describe the effect of the inhomogeneity of the electric potential along the vertical axis. The term including \( \nabla \times B = \epsilon \nabla \times a \) can be ascribed to the presence of electric current in the system. Finally, the last term on the right-hand side results from the polarization drift associated with the component of the magnetic field \( a \). Observe that equation (26) reduces to the Hasegawa-Mima equation when \( \epsilon = 0 \),

The effect of the field curvature (22) can be made explicit by expanding \( a \) in Taylor series around \( z = 0 \), and by considering the dynamics on such plane. At first order in \( z \), one has

where \( a_0 = (a_{0x}, a_{0y}, 0) \) and \( a_1 = (a_{1x}, a_{1y}, 0) \) are vector fields independent of \( z \). Using (28) and setting \( \frac{\partial \phi}{\partial z} = 0 \), all terms in (26) lose the dependence on \( z \), giving a fully 2-dimensional equation

In many applications, the curvature of the magnetic field is not small. It is therefore instructive to examine how the Hasegawa-Mima equation is modified when curvature is a leading order term. To this end, consider a circular magnetic field \( B = \mu r \nabla \phi \), where \( (r, \varphi, z) \) are cylindrical coordinates and \( \mu \in \mathbb{R} \). Notice that \( B^2 = \mu^2 \) is constant, and that the curvature of the magnetic field is given by \( \kappa = -\nabla \log r \). Then, equation (18) can be written as

where we introduced the differential operators

Notice that the field curvature modifies the Hasegawa-Mima equation through the modulus \( \kappa^2 = 1/r^2 \), and that equation (30) is fully 2-dimensional if axial symmetry \( \partial \phi / \partial \varphi = 0 \) is assumed. Furthermore, since we expect the electric potential \( \phi \), the electric field \( -\nabla (\kappa r) \phi \), and the electric charge \( -\Delta (\kappa r) \phi \) to be bounded, the Hasegawa-Mima equation can be recovered in the limit \( r \to \infty \) (\( \kappa \to 0 \)) where field lines become progressively straight. Next, observe that in the Hasegawa-Mima equation (27) steady states are given by the equation \( \frac{\partial t}{\partial t} (\phi, \Delta (\kappa r) \phi) = 0 \), or simply \( \Delta (\kappa r) \phi = -f (\phi) \) with \( f \) some function of \( \phi \). Since the electric charge \( -\Delta (\kappa r) \phi \) should vanish when \( \phi = 0 \), for small \( \phi \) we may set \( f = f_0 \phi \) with \( f_0 \) a positive real constant (the positive sign
physically means that charge density gradients \(-\nabla (x,y) \Delta_{(x,y)} \phi\) are directed in the opposite direction to the electric field \(-\nabla \phi\). Then, considering a 2-dimensional domain \((x,y) \in \Omega = [0,\pi]^2\) such that the electric potential is grounded on the boundary, i.e. \(\phi = 0\) on \(\partial\Omega\), solution of the steady Hasegawa-Mima equation gives self-organized states of the type \(\phi = \phi_0 \sin (m \pi x) \sin (n \pi y)\) with \(\phi_0 \in \mathbb{R}\), \(m,n \in \mathbb{Z}\), and \(f_0 = m^2 + n^2\). Since \(f_0\) corresponds to the ratio between enstrophy and energy in the fluid limit \(\lambda = 0\), at equilibrium the minimum possible value \(f_0 = 2\) corresponding to \(m = n = 1\) is expected to be preferentially selected (see e.g. [29] on this point). Let us see how the situation changes in a curved magnetic field. Since the right-hand side of (30) can be written as

\[
\frac{1}{r} \left[ \phi, -r - \frac{\sigma}{\mu^3} \frac{\partial \phi}{\partial r} + \frac{\sigma}{\mu^3} r \Delta_{(z,r)} \phi \right]_{(z,r)},
\]

steady states in the circular magnetic field \(B = \mu r \nabla \phi\) are described by the equation

\[
\frac{\sigma}{\mu^3} \left( \frac{\partial^2 \phi}{\partial r^2} + \frac{\partial^2 \phi}{\partial z^2} \right) = \frac{1}{\mu} - \frac{f(\phi)}{r}.
\]

(33)

Considering the case \(f = f_0 \phi\) and taking a 2-dimensional toroidal domain with squared cross section \((r,z) \in \Omega = [1,2]^2\) and Dirichlet boundary conditions \(\phi = 0\) on the boundary \(\partial\Omega\), solution of (33) results in a self-organized steady state sustained by the curvature of the magnetic field. Figure 1 shows contour plots of the steady electric potential \(\phi\) and vorticity \(\omega = \nabla \times v \cdot r \nabla \phi\) with \(v\) given by (12) obtained by numerical solution of (33) as compared with the Hasegawa-Mima case. The corresponding guiding center drifts \(v_E\) and \(v = v_E + v_{pol} + v_F\) are given in figure 2.

The effect of vertical inhomogeneity in the potential \(\phi\) can be further examined by setting \(\phi = q + \sin (m \pi z) p\), with \(q(r)\) and \(p(r)\) radial functions and \(m \in \mathbb{Z}\). Then, equation (33) reduces to the system

\[
\frac{\sigma}{\mu^3} \frac{\partial^2 p}{\partial r^2} + \left( \frac{f_0}{r} - \frac{m^2 \sigma}{\mu^3} \right) p = 0,
\]

\[
\frac{\sigma}{\mu^3} \frac{\partial^2 q}{\partial r^2} + \frac{f_0}{r} q - \frac{1}{\mu} = 0.
\]

(34)

A plot of the electric potential \(\phi\) obtained by solution of system (34) for different values of \(m\) is given in figure 3.
Figure 2: (a) and (b): Vector plot of the $E \times B$ drift velocity $v_E$ associated with the steady states of figure 1 and contour plot of the modulus $|v_E|$. (c) and (d): Vector plot of the corresponding total guiding center drift velocity $v = v_E + v_{pol} + v_F$ of (12) and contour plot of the modulus $|v|$.

4 The case of integrable magnetic fields

The Hasegawa-Mima equation is endowed with two inviscid invariants, the total energy and the generalized enstrophy. The generalized enstrophy is an invariant arising from the 2-dimensional nature of the governing equation, which is restricted to the flat $(x, y)$ plane with normal given by the straight vertical magnetic field $B = B_0 \nabla z$. It is useful to consider the conditions under which the same kind of topological invariant persist in general magnetic fields. A necessary condition for the analogy with 2-dimensional vorticity dynamics to apply is that the magnetic field defines the normal direction of a general (not necessarily flat) 2-dimensional surface $\Sigma \subset \mathbb{R}^3$. In this case, the dynamics is restricted to the surface $\Sigma$ because the drift velocity (12) satisfies $B \cdot v = 0$. The geometric condition for a vector field $B$ to locally define the normal of a surface $\Sigma$ is given by the Frobenius integrability condition (35),

$$B \cdot \nabla \times B = 0.$$  

In particular, if the magnetic field $B$ has vanishing helicity density then there exist locally defined functions $\alpha, C$ such that

$$B = \alpha \nabla C.$$  

The magnetic field $B$ thus defines the normal to the surface $C = \text{constant}$. When $\alpha = \alpha (C)$, the magnetic field $B$ becomes a vacuum magnetic field because $\nabla \times B = 0$. In this section, we will be concerned with magnetic fields of the type (36). We will assume that the functions $\alpha$ and $C$ exist in the domain $\Omega$, and that $B \neq 0$ in $\Omega$. Next, recall that the magnetic field $B$ must be solenoidal. Hence, the Lie-Darboux theorem \cite{31, 32} applies: locally there exist functions $\Psi, \theta$ such that

$$B = \nabla \Psi \times \nabla \theta.$$  

Again, we will assume that the functions $\Psi$ and $\theta$ are well defined in the domain $\Omega$. A typical example of vacuum magnetic field is the magnetic field generated by a point dipole. In this case

$$C = -M \frac{z}{(r^2 + z^2)^{3/2}}, \quad \Psi = M \frac{r^2}{(r^2 + z^2)^{3/2}}, \quad \theta = \varphi,$$

(38)
Figure 3: Contour plots of the electric potential $\phi$ obtained by solution of (34) for different values of $m$ and $\mu$. Dirichlet boundary conditions $p(1) = p(2) = 1$ and $q(1) = q(2) = 1$ are used. (a) and (d): The case $m = 1$ for $\mu = 1$ (above) and $\mu = 5$ (below). (b) and (e): The case $m = 3$ for $\mu = 1$ (above) and $\mu = 5$ (below). (c) and (f): The case $m = 6$ for $\mu = 1$ (above) and $\mu = 5$ (below). In this simulation $\sigma = 0.5$ and $f_0 = 1$.

where $(r, \varphi, z)$ are cylindrical coordinates and $M$ a physical constant with units of $Tm^3$. The functions $(C, \Psi, \theta)$ can be used as a system of curvilinear coordinates. The Jacobian determinant of the coordinate transformation is given by

$$J = \nabla C \cdot \nabla \Psi \times \nabla \theta = B^2 \alpha. \quad (39)$$

Given two functions $f, g$ it is convenient to introduce the bracket

$$[f, g]_{(\Psi, \theta)} = \frac{\partial f}{\partial \Psi} \frac{\partial g}{\partial \theta} - \frac{\partial f}{\partial \theta} \frac{\partial g}{\partial \Psi}. \quad (40)$$

Using (40), the derived equation (18) can be written as

$$\frac{\partial}{\partial t} \left[ \lambda \phi - \sigma \nabla \cdot \left( \frac{\nabla \phi}{B^2} \right) \right] = \frac{B^2}{\alpha} \left[ \phi, -\frac{\alpha}{B^2} + \sigma \frac{\alpha^2}{B^4} \nabla \cdot \left( \frac{\nabla \phi}{\alpha} \right) \right]_{(\Psi, \theta)}. \quad (41)$$

One can verify that under appropriate boundary conditions on the surface boundary $\partial \Sigma$ the surface energy

$$H_\Sigma = \frac{1}{2} \int_\Sigma \left( \frac{\nabla \phi}{B^2} \right)^2 \frac{\alpha}{B^2} d\Psi d\theta, \quad (42)$$

is a constant of (41). The condition for conservation of generalized enstrophy,

$$W_\Sigma = \frac{1}{2} \int_\Sigma \left( \frac{\nabla \phi}{B^2} + \sigma \left[ \nabla \cdot \left( \frac{\nabla \phi}{B^2} \right) \right] \right)^2 \frac{\alpha}{B^2} d\Psi d\theta, \quad (43)$$

can be obtained by noting that the second argument of the bracket on the right-hand side of equation (41) must be a function of $\nabla \cdot (B^{-2} \nabla \phi)$ up to a function of $\phi$. The condition is:

$$\alpha = \frac{k}{|\nabla C|^2} = \frac{B^2}{k}, \quad k \in \mathbb{R}. \quad (44)$$
In this case, $\nabla \cdot \mathbf{v}_E = \nabla \phi \cdot \nabla \times (B^{-2} \mathbf{B}) = 0$. Notice also that $\nabla \cdot \mathbf{B} = 0$ implies that configurations of the type (44) must satisfy

$$\nabla \cdot \left( \frac{\nabla C}{|\nabla C|^2} \right) = 0.$$  

(45)

In spherical geometry, denoting with $R$ the spherical radius, a magnetic field satisfying (44) and (43) can be obtained by setting

$$\alpha = \frac{\mu}{R^4}, \quad \mu \in \mathbb{R}, \quad C = \frac{R^3}{3}.$$  

(46)

Similarly, in cylindrical geometry magnetic fields compatible with (44) and (45) include those generated by

$$\alpha = \frac{\mu}{r^2}, \quad \mu \in \mathbb{R}, \quad C = \frac{r^2}{2},$$

(47a)

$$\alpha = \mu r^2, \quad \mu \in \mathbb{R}, \quad C = \phi.$$  

(47b)

Observe that when $\lambda = 0$ equation (41) with either (46) or (47a) gives the usual 2-dimensional vorticity dynamics on a sphere or cylinder respectively. The case of equation (47b) corresponds to a circular magnetic field $\mathbf{B} = \mu r^2 \nabla \phi$ with curvature $\kappa = -\nabla \log r$. Assuming axial symmetry $\phi = \phi(r,z)$, the corresponding form of equation (18) is again fully 2-dimensional,

$$\frac{\partial}{\partial t} \left[ \lambda \phi - \frac{\sigma}{\mu^2} \nabla \cdot (\kappa^2 \nabla_{(z,r)} \phi) \right] = \frac{\sigma}{\mu^2} \kappa \left[ \phi, \nabla \cdot (\kappa^2 \nabla_{(z,r)} \phi) \right]_{(z,r)}. $$  

(48)

5 Noncanonical Hamiltonian structure

A peculiarity of $\mathbf{E} \times \mathbf{B}$ dynamics is that the equations of motion $\dot{\mathbf{x}} = \mathbf{v}_E = B^{-2} \mathbf{B} \times \nabla \phi$ do not define a Hamiltonian system unless the helicity density of the magnetic field is identically zero, i.e. (35) holds. This is because the Jacobi identity for the operator Hamiltonian system unless the helicity density of the magnetic field is identically zero, i.e. (35) holds. This

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(47b)

Observe that when $\lambda = 0$ equation (41) with either (46) or (47a) gives the usual 2-dimensional vorticity dynamics on a sphere or cylinder respectively. The case of equation (47b) corresponds to a circular magnetic field $\mathbf{B} = \mu r^2 \nabla \phi$ with curvature $\kappa = -\nabla \log r$. Assuming axial symmetry $\phi = \phi(r,z)$, the corresponding form of equation (18) is again fully 2-dimensional,

$$\frac{\partial}{\partial t} \left[ \lambda \phi - \frac{\sigma}{\mu^2} \nabla \cdot (\kappa^2 \nabla_{(z,r)} \phi) \right] = \frac{\sigma}{\mu^2} \kappa \left[ \phi, \nabla \cdot (\kappa^2 \nabla_{(z,r)} \phi) \right]_{(z,r)}. $$  

(48)

In this case, $\nabla \cdot \mathbf{v}_E = \nabla \phi \cdot \nabla \times (B^{-2} \mathbf{B}) = 0$. Notice also that $\nabla \cdot \mathbf{B} = 0$ implies that configurations of the type (44) must satisfy

$$\nabla \cdot \left( \frac{\nabla C}{|\nabla C|^2} \right) = 0.$$  

(45)

In spherical geometry, denoting with $R$ the spherical radius, a magnetic field satisfying (44) and (43) can be obtained by setting

$$\alpha = \frac{\mu}{R^4}, \quad \mu \in \mathbb{R}, \quad C = \frac{R^3}{3}.$$  

(46)

Similarly, in cylindrical geometry magnetic fields compatible with (44) and (45) include those generated by

$$\alpha = \frac{\mu}{r^2}, \quad \mu \in \mathbb{R}, \quad C = \frac{r^2}{2},$$

(47a)

$$\alpha = \mu r^2, \quad \mu \in \mathbb{R}, \quad C = \phi.$$  

(47b)

Observe that when $\lambda = 0$ equation (41) with either (46) or (47a) gives the usual 2-dimensional vorticity dynamics on a sphere or cylinder respectively. The case of equation (47b) corresponds to a circular magnetic field $\mathbf{B} = \mu r^2 \nabla \phi$ with curvature $\kappa = -\nabla \log r$. Assuming axial symmetry $\phi = \phi(r,z)$, the corresponding form of equation (18) is again fully 2-dimensional,

$$\frac{\partial}{\partial t} \left[ \lambda \phi - \frac{\sigma}{\mu^2} \nabla \cdot (\kappa^2 \nabla_{(z,r)} \phi) \right] = \frac{\sigma}{\mu^2} \kappa \left[ \phi, \nabla \cdot (\kappa^2 \nabla_{(z,r)} \phi) \right]_{(z,r)}. $$  

(48)

5 Noncanonical Hamiltonian structure

A peculiarity of $\mathbf{E} \times \mathbf{B}$ dynamics is that the equations of motion $\dot{\mathbf{x}} = \mathbf{v}_E = B^{-2} \mathbf{B} \times \nabla \phi$ do not define a Hamiltonian system unless the helicity density of the magnetic field is identically zero, i.e. (35) holds. This is because the Jacobi identity for the operator $e^{-1}B^{-2}\mathbf{B} \times$ acting on the single particle Hamiltonian $\mathbf{e} \phi$ is given by $e^{-2}B^{-4} \mathbf{B} \cdot \nabla \times \mathbf{B} = 0$. We therefore expect equation (18) to define a Hamiltonian system when the magnetic field is integrable $\mathbf{B} = \alpha \nabla C$ and the polarization drift $\mathbf{v}_{pol}$ and the drift $\mathbf{v}_F$ are neglected ($\sigma = 0$) in the fluid velocity (12). Furthermore, if $\mathbf{B} = \alpha \nabla C$ and $\sigma \neq 0$, we conjecture that a necessary condition for equation (41) to define a Hamiltonian system is that the condition for conservation of generalized enstrophy (44), which should appear as a Casimir invariant of the Poisson algebra, is satisfied. Before verifying these facts, we remark that the absence of a Hamiltonian structure in a general magnetic field should not be regarded as a defect of the theory, but rather as a natural consequence of the mathematical reduction leading to $\mathbf{E} \times \mathbf{B}$ dynamics, which does not always define a Hamiltonian system.

In the following, we assume that $\mathbf{B} = \alpha \nabla C$ in $\Omega$. For simplicity, we consider periodic boundary conditions on $\partial \Omega$. The candidate Hamiltonian is given by the energy $H_\Sigma$ of equation (42). It is convenient to introduce the notation $\eta = \lambda \phi - \sigma w$ with $w = \nabla \cdot (B^{-2} \nabla_\bot \phi)$ and the operator $\mathcal{D}$ such that $w = \mathcal{D} \phi$ and $\phi = (\lambda - \sigma \mathcal{D})^{-1} \eta$ (in this equation we assume the solution $\phi$ to be well defined upon suitable characterization of the corresponding function space). Then, it can be verified that equation (41) can be expressed as

$$\frac{\partial \eta}{\partial t} = \{ \eta, H_\Sigma \},$$  

(49)

where the antisymmetric bracket $\{,\}$ is defined by

$$\{ F, G \} = \int_\Sigma \left( \sigma \frac{\alpha}{B^2} w - 1 \right) \left[ \frac{\delta F}{\delta \eta}, \frac{\delta G}{\delta \eta} \right]_{\Psi, \theta} \frac{\alpha}{B^2} d\Psi d\theta,$$

(50)

with $F, G$ functionals of $\eta$. For (50) to define a Poisson bracket, the Jacobi identity must be satisfied. Observing that the bracket $[,]_{\Psi, \theta}$ within the integrand satisfies itself the Jacobi identity, the case $\sigma = 0$
corresponding to pure $\mathbf{E} \times \mathbf{B}$ drift follows immediately. When $\sigma \neq 0$ the Jacobi identity for the bracket \[ \{ F, \{ G, H \} \} + \odot = \int_{\Sigma} \left( \sigma \frac{\alpha}{B^2} w - 1 \right) \left[ \frac{\delta F}{\delta \eta}, (\lambda - \sigma \mathfrak{D})^{-1} \left( \frac{\alpha}{B^2} \left[ \frac{\delta G}{\delta \eta}, \frac{\delta H}{\delta \eta} \right] \right) - \frac{\alpha}{B^2} \left[ \frac{\delta G}{\delta \eta}, \frac{\delta H}{\delta \eta} \right] \right] \frac{\alpha}{B^2} \ d\Psi d\theta + \odot, \] (51)

where $\odot$ indicates summation over even permutations of the functionals $F, G, H$ and the lower index $(\Psi, \theta)$ has been omitted in the bracket $[,]_{(\Psi, \theta)}$. Enforcing the condition for conservation of generalized enstrophy \[ \{ F, \{ G, H \} \} + \odot = \lambda \sigma \frac{k}{3} \int_{\Sigma} w \left[ \frac{\delta F}{\delta \eta}, (\lambda - \sigma \mathfrak{D})^{-1} \left[ \frac{\delta G}{\delta \eta}, \frac{\delta H}{\delta \eta} \right] \right] d\Psi d\theta + \odot. \] (52)

This quantity identically vanishes in the limit $\lambda = 0$ ($T_e = \infty$). When $\lambda \neq 0$, the Jacobi identity is satisfied if the inverse operator $(\lambda - \sigma \mathfrak{D})^{-1}$ commutes with $\frac{\partial}{\partial \Psi}$ and $\frac{\partial}{\partial \theta}$.

6 Concluding remarks

In conclusion, we have derived a model equation \[ (18) \] describing electrostatic plasma turbulence in a general magnetic field. The equation preserves the energy \[ (19) \], and reduces to the Hasegawa-Mima equation in the limit of a straight magnetic field. We have shown that curvature can sustain self-organized steady states, and that conservation of generalized enstrophy holds when the magnetic topology is integrable and compatible with 2-dimensional vorticity dynamics. This latter fact reflects the different origin of the model equation, which is obtained from the continuity equation rather than a fluid momentum equation.

Inhomogeneous electron density can be further introduced into the derived equation \[ (18) \] by appropriate modeling of the relevant guiding center drifts. In particular, for an integrable magnetic field satisfying \[ (44) \] and assuming $|a \nabla \log n_e| << 1$ where $a$ is a pertinent length scale, it is sufficient to add a term $k^{-1} \log A_e, \phi_{(\Psi, \theta)}$ on the right-hand side of equation \[ (41) \] where the quantity $A_e$ of equation \[ (1) \] is now an arbitrary non-negative function. Finally, dissipation can be represented through a diffusion operator taking into account the effect of magnetic field curvature on the underlying transport process.

Acknowledgment

The research of NS was partially supported by JSPS KAKENHI Grant No. 21K13851 and No. 17H01177.

Data Availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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