ON EXISTENTIAL DEFINITIONS OF C.E. SUBSETS OF RINGS OF FUNCTIONS OF CHARACTERISTIC 0

RUSSELL MILLER & ALEXANDRA SHLAPENTOKH

Abstract. We extend results of Denef, Zahidi, Demeyer and the second author to show the following.
(1) Rational integers have a single-fold Diophantine definition over the ring of integral functions of any function field of characteristic 0.
(2) Every c.e. set of integers has a finite-fold Diophantine definition over the ring of integral functions of any function field of characteristic 0.
(3) All c.e. subsets of polynomial rings over totally real number fields have finite-fold Diophantine definitions. (These are the first examples of infinite rings with this property.)
(4) Let $K$ be a one-variable function field over a number field and let $p$ be any prime of $K$. Then the valuation ring of $p$ has a Diophantine definition.
(5) Let $K$ be a one-variable function field over a number field and let $\mathcal{S}$ be a finite set of its primes. Then all c.e. subsets of $O_{K,\mathcal{S}}$ are existentially definable. (Here $O_{K,\mathcal{S}}$ is the ring of $\mathcal{S}$-integers or a ring of integral functions.)

1. Introduction

In 1969, building on earlier work by Martin Davis, Hilary Putnam and Julia Robinson, Yuri Matijasevich demonstrated the impossibility of solving Hilbert’s Tenth Problem. In doing so, he also completed the proof of the theorem asserting that Diophantine (or existentially definable in the language of rings) sets and computably enumerable sets of integers were the same. In other words, it was proved that for every positive integer $n$, every computably enumerable subset of $\mathbb{Z}^n$ had a Diophantine definition over $\mathbb{Z}$. We describe the notions of a Diophantine definition and a Diophantine set in a more general setting.

Definition 1.1. Let $R$ be a commutative ring and let $n$ be a positive integer. In this case a set $A \subseteq R^n$ is called Diophantine over $R$ if for some $m > 0$ and some polynomial $f(T_1, \ldots, T_n, X_1, \ldots, X_m) \in R[T, X]$ we have that for all $(t_1, \ldots, t_n) \in R^n$ it is the case that $(t_1, \ldots, t_n) \in A$ if and only if $\exists x_1, \ldots, x_m \in R$ such that $f(t_1, \ldots, t_n, x_1, \ldots, x_m) = 0$.

The polynomial $f(T, X)$ is called a Diophantine definition of $A$ over $R$.

If a set $A$ is Diophantine over $R$ and for every $\bar{t} \in A$ we have that $\bar{x}$ as above is unique, we say that $f(T, X)$ is a single-fold definition of $A$. If for every $\bar{t} \in A$ we have that there are only finitely many $\bar{x}$ as above, we say that $f(T, X)$ is a finite-fold definition of $A$.

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**Question 1.2.** Does every c.e. set of integers have a finite-fold Diophantine definition over \( \mathbb{Z} \)?

The answer to this question, raised by Yuri Matijasevich almost immediately after his solution to Hilbert’s Tenth Problem, is unknown to this day. The issue of finite-fold representation is of more than just esoteric interest because of its connection to many other questions. For an extensive survey of these connections we refer the reader to a paper of Matijasevich ([Mat10]). Here we would like to give just one example which can be considered a generalization of Hilbert’s Tenth Problem.

Let \( \mathcal{M} = \{0, 1, \ldots, \aleph_0\} \) and let \( \mathcal{M} \) be any nonempty proper subset of \( \mathcal{M} \). Let \( \mathcal{P}(\mathcal{M}) \) be the set of polynomials \( P \) with integer coefficients such that the number of solutions to the equation \( P = 0 \) is in \( \mathcal{M} \). Martin Davis showed in [Dav72] that \( \mathcal{P}(\mathcal{M}) \) is undecidable. If we ask whether \( \mathcal{P}(\mathcal{M}) \) is c.e, then the answer is currently unknown. At the same time, if we replace polynomials by exponential Diophantine equations, then we can now answer the question. Craig Smoryński in [Smo77] proved that \( \mathcal{E}(\mathcal{M}) \) is c.e. if and only if \( \mathcal{M} = \{\alpha | \alpha \geq \beta\} \) for some finite \( \beta \). (Here \( \mathcal{E}(\mathcal{M}) \) is a collection of exponential Diophantine polynomials with positive integer coefficients such that if an exponential Diophantine polynomial \( E \in \mathcal{E}(\mathcal{M}) \), then the number of solutions to the equation \( E = 0 \) is in \( \mathcal{M} \).) Smoryński’s proof relied on a result obtained by Matijasevich in [Mat77] that every computably enumerable set has a single-fold exponential Diophantine definition. One would expect a similar result for (non-exponential) Diophantine equations if the finite-fold question is answered affirmatively.

Matijasevich also showed that to show that all c.e. sets of integers have single-fold (or finite-fold) Diophantine definitions it is enough to show that the set of pairs \( \{(a, b) \in \mathbb{Z}_+^2 | b = 2^a\} \) has a single-fold (finite-fold) Diophantine definition. (This will not be surprising to readers familiar with the history of Hilbert’s Tenth Problem.)

Unfortunately, the finite-fold question over \( \mathbb{Z} \) remains out of reach at the moment, as with many other Diophantine questions. In Section 3 of this paper, we take some first timid steps in the investigation of this issue by considering it in a more hospitable environment over function fields of characteristic 0, as described in Section 2. We extend the results of the second author from [Shl06] to show that over any ring of integral functions (otherwise known as the ring of \( \mathcal{P}-\)integers) the rational integers have a single-fold definition (see Theorem 3.8), while every c.e. set of integers has a finite-fold definition (see Theorem 3.10). We also show that over smaller rings inside functions fields of characteristic 0 it is possible to give a single-fold definition to every c.e. set of integers (see Corollary 4.6).

Using our results on finite-fold definability of \( \mathbb{Z} \), following results of Jan Denef from [Den78] and Karim Zahidi from [Zah00], we show in Section 5 that all computably enumerable subsets of a polynomial ring over a ring of integers of a totally real number fields are finite-fold existentially definable. As far as we know, this is the first example of this kind. (See Theorem 5.2.)

In Section 7 we generalize results of Jeroen Demeyer from [Dem10] to show that all c.e. subsets of rings of integral functions over number fields are Diophantine (see Theorem 7.7). In order to do so, we needed to generalize the earlier treatments (given in the papers [MB06] of Laurent Moret-Bailly and [Eis07] of Kirsten Eisentraeger) of definability of integrality at a degree one valuation over a function field of characteristic zero where the constant field is a number field. Both of those papers in turn extend results of H. K. Kim and Fred Roush from [KR95] where the two authors give a Diophantine definition of integrality at a valuation of
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degree 1 over a rational function field with a constant field embeddable into a $p$-adic field. (Such constant fields include all number fields.) The papers of Moret-Bailly and Eisentraeger are primarily concerned with extending results pertaining to Hilbert’s Tenth Problem and so they extend the results of Kim and Roush just enough for their arguments to go through, by showing the following for a function field $K$ over a field of constants $k$ as described above: if $T$ is a non-constant element of $K$ and the pole $q$ of $T$ splits completely into distinct primes in the extension $K/k(T)$, then there exists a Diophantine subset of $K$ such that all rational functions in that subset are integral at $q$.

In contrast, our proof required a Diophantine definition of a set such that all the functions in the set, not just the rational ones, were integral at the prime in question. In order to obtain such a definition we reworked the original construction of Kim and Roush. To keep the technical details to a minimum we considered only the case of the constant field being algebraic over $\mathbb{Q}$, though our approach is extendible to a much larger class of fields. In this paper we show that valuation ring of “almost” any prime is definable over a function field with a constant field algebraic over $\mathbb{Q}$ and embeddable into a $p$-adic field or $\mathbb{R}$. (See Section 6.)

2. Number Fields, Function Fields and Rings

Throughout this paper, by a function field $K$ (of characteristic 0) we will mean a finite extension of a rational function field $k(T)$, where $T$ is transcendental over a field $k$. By the constant field of $K$ we will mean the algebraic closure of $k$ in $K$. (In our case we will often have a situation where the algebraic closure of $k$ in $K$ is equal to $k$ by construction.)

By a prime of $k(T)$, we will mean a prime ideal of $k[t]$ or a prime ideal of $k[1/t]$ generated by $1/t$. The ring of integral functions of $K$ is the integral closure of $k[t]$ in $K$. The primes of $K$ are prime ideals of the ring of integral functions of $K$ or the prime ideals of the integral closure of $k[1/t]$ in $K$ such that their intersection with $k[1/t]$ produces the ideal generated by $1/t$. These ideals (or primes or valuations) are sometimes referred to as “infinite” ones.

For each integral function $f$ and each prime ideal $\mathfrak{p}$ of the ring of integral functions, let $\text{ord}_\mathfrak{p} f = n$ be the non-negative integer such that $f \in \mathfrak{p}^n \setminus \mathfrak{p}^{n+1}$. To define order at $\mathfrak{p}$ for an arbitrary non-zero element of the field, write it as a ratio of two integral elements and extend the definition to the ratio in the natural way. It can be shown that while representing elements of the field as ratios of integral elements can be done in infinitely many ways, each such representation produces the same order. Define the order of 0 to be infinity. The order at infinite valuations is defined in the same manner but with respect to the integral closure of $k[1/t]$ in $K$.

Given a definition of the order at a prime we can now give an alternative description to the rings of integral functions. Unlike rings of algebraic integers within number fields, rings of integral functions are not uniquely determined within a given function field, but depend on the choice of $t$. One can show that describing a ring of integral functions is equivalent to selecting a finite set of primes of $K$ and considering the ring functions which are allowed to have negative order at these primes only. This is the definition we will use below in Notation 2.1.

For the field of constants we will most often select some algebraic extension of $\mathbb{Q}$. When such an extension is finite, the field is called a number field. One can also consider all the possible embeddings of $k$ into $\bar{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$. If all the embeddings are
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contained in \( \mathbb{Q} \cap \mathbb{R} \), then the field is called totally real. This is related to the notion of a formally real field, i.e., a field where \(-1\) is not a sum of squares. Observe that if a field is algebraic over \( \mathbb{Q} \) and has an embedding into \( \mathbb{Q} \cap \mathbb{R} \), then it must be formally real.

**Notation 2.1.** In this paper we will use the following notation.

- Let \( K \) denote a function field of characteristic 0 over the constant field \( k \).
- Let \( \mathcal{S} = \{p_1, \ldots, p_s\} \) be a finite set of primes of \( K \).
- Let \( O_{K,\mathcal{S}} = \{ x \in K \mid (\forall p \not\in \mathcal{S}) \text{ ord}_p x \geq 0 \} \) be the ring of \( \mathcal{S} \)-integers of \( K \).
- For \( d \in O_{K,\mathcal{S}} \) let \( H_{K,d,\mathcal{S}} = \{ x - d^{1/2}y \mid x, y \in O_{K,\mathcal{S}} \land x^2 - dy^2 = 1 \} \).

3. Finite-fold Diophantine Representations of C.E. Sets of Integers over Rings of \( \mathcal{S} \)-integers of Function Fields of Characteristic 0

In this section we describe a finite-fold Diophantine definition of c.e. sets of integers over rings of integral functions. Before we do that, we have to reconsider certain old methods of defining sets to make sure they produce finite-fold definitions. We start with the issue of intersection of Diophantine sets.

3.1. Single-Fold and Finite-Fold Definition of “And”. As long as we consider rings whose fraction fields are not algebraically closed, we can continue to use the “old” method of combining several equations into a single one without introducing extra solutions, as in Lemma 1.2.3 of \[Shl06\]. More specifically we have the following proposition.

**Proposition 3.1.** Let \( R \) be an integral domain and let \( h(T) = a_0 + a_1T + \ldots + T^n \) be a polynomial without roots in the fraction field of \( R \). Let \( k \in \mathbb{Z}_{>0} \) and \( A \subset R^k \) be a Diophantine subset of \( R^k \) with a single (finite) fold definition, saying that \( \bar{x} = (x_1, \ldots, x_k) \in A \) if and only if there exists a unique \( m \)-tuple (finitely many \( m \)-tuples) \( \bar{z} = (z_1, \ldots, z_m) \) such that

\[
 f(\bar{x}, \bar{z}) = 0 \land g(\bar{x}, \bar{z}) = 0,
\]

where \( f, g \in R[X_1, \ldots, X_k, Z_1, \ldots, Z_m] \) are fixed polynomials (for this set \( A \)). In this case, we also have a single (finite) fold definition of \( A \), which says that \( \bar{x} = (x_1, \ldots, x_k) \in A \) if and only if \( a_0f(\bar{x}, \bar{z})^n + a_1f(\bar{x}, \bar{z})^{n-1}g(\bar{x}, \bar{z}) + \ldots + g(\bar{x}, \bar{z})^n = 0 \).

The proof of this proposition is the same as for Lemma 1.2.3 of \[Shl06\].

3.2. Pell Equations over Rings of Functions of Characteristic 0. Next we take a look at the old workhorse of Diophantine definitions: the Pell equation. It turns out that in the context of defining integers over rings of functions this equation produces “naturally” single-fold definitions.

**Lemma 3.2.** (Essentially Lemma 2.1 of \[Den79\], or Lemma 2.2 of \[Shl90\]) Let \( R \) be an integral domain of characteristic not equal to 2. Let \( f, g, s \in R[x], s \not\in R, x \text{ transcendental over } R \). Let \( (f_n(s), g_n(s)) \in R[x] \) be such that \( f_n(s) - (s^2 - 1)^{1/2}g_n(s) = (s - (s^2 - 1)^{1/2})^n \). (In \[Shl90\] there is a typographical error in this equation: the “square” is misplaced on the right-hand side.) In this case

1. \( \deg(f_n) = n \cdot \deg(s), \deg(g_n) = (n - 1) \cdot \deg(s) \),
2. \( \ell \text{ dividing } n \) is equivalent to \( g_\ell \text{ dividing } g_n \),
3. The pairs \( (f_n, g_n) \) are all the solutions to \( f^2 - (s^2 - 1)g^2 = 1 \) in \( R[x] \)

The next lemma is a slight modification of Lemma 2.5 of \[Shl92\].
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**Proposition 3.3.** Assume there exists \( a \in O_{K,\mathcal{S}} \) such that \( \text{ord}_{p_i} a < 0, \text{ord}_{p_i} a = 0, i = 2, \ldots, s \) with \( a^2 - 1 \) not being a square mod \( p_i \). In this case, for \( d = a^2 - 1 \), we have that \( H_{K,d,\mathcal{S}} \) is a group of rank 1 generated by \( a - d^{1/2} \) modulo \( \{ \pm 1 \} \), and that \( a - 1 \) and \( a - \sqrt{a^2 - 1} - b \) (for any \( b \in \mathbb{Z}_{\neq 0} \)) are not units of \( O_{K,\mathcal{S}}[\sqrt{a^2 - 1}] \).

*Proof.* The proof of this modified version is almost identical to the proof of the original version except for the following point. When we consider zeros and poles of any element of \( H_{K,d,\mathcal{S}} \), as in the original version, we conclude that these elements can have zeros and poles at factors of \( p_i \) only. However, in the original version the reason for that was that \( p_2, \ldots, p_s \) were ramified in the extension \( K(\sqrt{a^2 - 1})/K \). In the new version, the reason why each \( p_i \) (\( i > 1 \)) does not have a factor occurring as a zero or a pole of an element of \( H_{K,d,\mathcal{S}} \) is that \( p_i \) does not split in the extension \( K(\sqrt{a^2 - 1})/K \).

**Remark 3.4.** As long as the residue field of each of \( p_2, \ldots, p_s \) contains an element \( a \neq 0 \) such that \( a^2 - 1 \) is not a square in the residue field, the existence of \( a \in O_{K,\mathcal{S}} \) satisfying assumptions of Lemma 3.3 follows by the Strong Approximation Theorem for function fields. If \( k \) is a number field, then all the residue fields are also number fields and the requisite \( a \) exists in all the residue fields. This will also be true for \( k \) algebraic over \( \mathbb{Q} \) as long as non-dyadic primes of \( \mathbb{Q} \) have no factors in any subfield of \( k \) of ramification degree divisible by arbitrarily high powers of 2. (In other words, \( k \) has to be 2-bounded.)

**Lemma 3.5.** (See Lemma 2.3 of [Shl92].) Define \( (u_n, w_n) \in O_{K,\mathcal{S}} \), to be such that \( u_n - d^{1/2}w_n = (a - d^{1/2})^n, n \in \mathbb{Z} \), where \( d = a^2 - 1 \). Then \( w_{-n} = -w_n \) and \( w_n \equiv n \mod (a - 1) \) in the ring \( \mathbb{Z}[a] \). (Note that \( u_n, w_n \in \mathbb{Z}[a] \).)

**Lemma 3.6.** (Essentially Lemma 3.4 of [Shl92].) Let \( R \) be any subring of \( O_{K,\mathcal{S}} \) containing a local subring of \( \mathbb{Q} \). Then there exists a subset \( C \) of \( R \) which contains only constants, includes \( \mathbb{Z} \), and is single-fold Diophantine over \( R \).

*Proof.* Let \( n \) be the size of \( \mathcal{S} \), let \( \pi \) be the product of all non-invertible rational primes (or 1, if \( R \) contains \( \mathbb{Q} \)), and let \( C \subset R \) be the set defined by the following equations over \( R \):

\[
\begin{align*}
j_1(\pi x^2 + 1) &= 1, \\
\vdots \\
j_{n+1}(\pi x^2 + (n + 1)\pi + 1) &= 1
\end{align*}
\]

(3.1)

We claim that System (3.1) has solutions in \( R \) only if \( x \) is a constant, while conversely, if \( x \in \mathbb{Z} \), then these equations have solutions in \( R \). Indeed, if \( x \) is not a constant, neither are \( x^2 + \pi + 1, \ldots, x^2 + (n + 1)\pi + 1 \). Therefore, since they are invertible in \( O_{K,\mathcal{S}} \), they all must have zeros at valuations of \( \mathcal{S} \). However, these \( n + 1 \) elements do not share any zeros, and there are \( n \) valuations in \( \mathcal{S} \) and \( n + 1 \) elements under consideration. This implies that two of them must share the same zero, which is impossible since their differences are constant. The converse is obvious: if \( x \in \mathbb{Z} \), then \( \pi x^2 + \pi + 1 \) is invertible for each \( \pi \in \mathbb{Z} \). Please note that given \( x \in O_{K,\mathcal{S}} \), if System (3.1) has solutions, then these solutions are unique.

**Remark 3.7.** If \( R \) contains \( \mathbb{Q} \), then we can also define the set of non-zero constants containing \( \mathbb{Z}_{\neq 0} \). We just have to add the equation \( j_0 x = 1 \).

**Theorem 3.8.** \( \mathbb{Z} \) has a single-fold Diophantine definition over \( O_{K,\mathcal{S}} \).
The following set has a single-fold definition over Theorem 3.9.

Thus System (3.7), with all the variables ranging over valuation of $K$, 3.3 and consider the following equations and conditions:

\begin{equation}
\tag{3.2}
u^2 - \sqrt{a^2 - 1}w^2 = 1;
\end{equation}

\begin{equation}
\tag{3.3}
c \equiv \frac{u - \sqrt{a^2 - 1}w - 1}{a - \sqrt{a^2 - 1} - 1} \mod (a - \sqrt{a^2 - 1} - 1)
\end{equation}
in $O_{K,\mathcal{S}}[\sqrt{a^2 - 1}]$;

\begin{equation}
\tag{3.4}
c \text{ is a constant.}
\end{equation}

Supposed these equations and assumptions are satisfied. Let $\varepsilon = a - \sqrt{a^2 - 1}$ and observe that by assumption on $a$ we have that $u - \sqrt{a^2 - 1}w = \pm \varepsilon^n$, and $c \equiv \pm \varepsilon^n - 1 \mod (\varepsilon - 1)$ in $O_{K,\mathcal{S}}[\sqrt{a^2 - 1}]$. Thus, $c \equiv \pm n \mod (\varepsilon - 1)$ in $O_{K,\mathcal{S}}[\sqrt{a - 1}]$. Since $a - 1$ is not a unit of $O_{K,\mathcal{S}}$, we have that $\varepsilon - 1$ is not a unit of $O_{K,\mathcal{S}}[\sqrt{a - 1}]$ and therefore $c \pm n$ is a unit with a zero at some valuation of $K$. Hence $c = \pm n$.

Conversely, given $c = n$, set $u - \sqrt{a^2 - 1}w = \varepsilon^n$ and observe that all the equations are satisfied. Note also, that this is the only solution to the equations. \qed

We now give a single-fold Diophantine definition of exponentiation up to a sign.

**Theorem 3.9.** The following set has a single-fold definition over $O_{K,\mathcal{S}}$:

\[\{(b, c, d) \mid b, c, d \in \mathbb{Z} \land |b| = |c||d\}\]

**Proof.** Let $a \in O_{K,\mathcal{S}}$ be again as in Lemma 3.3, and $\varepsilon = a - \sqrt{a^2 - 1}$. Consider the following equations and conditions:

\begin{equation}
\tag{3.5}b, c, d \in \mathbb{Z}, b \neq 0;
\end{equation}

\begin{equation}
\tag{3.6}\exists n \in \mathbb{Z}, x, y \in O_{K,\mathcal{S}}[\sqrt{a^2 - 1}] : \varepsilon^n \pm c = (\varepsilon - b)x \land d \pm \frac{\varepsilon^n - 1}{\varepsilon - 1} = y(\varepsilon - 1).
\end{equation}

From (3.6) we conclude that $(\varepsilon - b)$ divides $\pm b^n - c$ in $O_{K,\mathcal{S}}[\sqrt{a^2 - 1}]$. Since $b \neq 0$, by Lemma 3.3 we conclude that $(\pm \varepsilon - b)$ is not a unit, and therefore $b^n \pm c$ has a zero at a valuation of $K$. Since $b, c$ are integers, we must infer that $\pm b^n = c$, and $n \geq 0$. At the same time, also from (3.6) we have that $d = \pm n$. Conversely, assuming $b, c, d \in \mathbb{Z}, b, c \neq 0$, it is easy to see that (3.6) can be satisfied with only one choice for the sign.

We now rewrite (3.6) over $O_{K,\mathcal{S}}$:

\begin{equation}
\tag{3.7}\begin{cases}
u^2 - (a^2 - 1)w^2 = 1 \text{ (in other words, } \varepsilon^n = u - \sqrt{a^2 - 1}w) \\
u - \sqrt{a^2 - 1}w - c = (a - \sqrt{a^2 - 1} - 1)(x_1 - x_2\sqrt{a^2 - 1}) \text{ (in other words, } \varepsilon^n - c = (\varepsilon - b)x) \\
d(a - \sqrt{a^2 - 1} - 1) - (u - \sqrt{a^2 - 1}w - 1) = (y_1 - \sqrt{a^2 - 1}y_2)(a - \sqrt{a^2 - 1} - 1)^2 \text{ (in other words, } d(\varepsilon - 1) - (\varepsilon^n - 1) = y(\varepsilon - 1)^2)
\end{cases}
\end{equation}

Thus System (3.7), with all the variables ranging over $O_{K,\mathcal{S}}$, is equivalent to Conjunction (3.6). \qed
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To define the non-negative integers we use the four squares theorem, giving us a finite-fold definition of positive integers over $O_{K,S}$. Combining this with a result of Matiyasevich from [Mat77] we now have the following theorem.

**Theorem 3.10.** Every c.e. set of integers has a finite-fold Diophantine representation over $O_{K,S}$.

If we consider just the polynomial rings over $\mathbb{Z}$, we will be able to do better: Corollary 4.6 will show the existence of single-fold Diophantine representations of c.e. sets of integers.

4. **Single-Fold Diophantine Representations of C.E. Sets of Rational Integers over Polynomial Rings of Characteristic 0 not containing a local subring of $\mathbb{Q}$.

We start by investigating single/finite-fold properties of definitions of non-zero integers. We will consider the situation where the ring of constants does not contain $\mathbb{Q}$, meaning that we have at least one rational non-inverted prime. First we need the following basic fact.

**Lemma 4.1.** Let $R$ be a ring of characteristic zero containing a rational number of the form $\frac{a}{b}$ where $a, b$ are non-zero relatively prime integers. In this case $R$ contains $\frac{1}{b}$.

*Proof.* Since $(a, b) = 1$ we have that for some $x_1, x_2 \in \mathbb{Z} \subseteq R$ it is the case that $ax_1 + bx_2 = 1$. Thus $\frac{1}{b} = \frac{ax_1 + bx_2}{b} = x_1 \frac{a}{b} + x_2 \in R$.  

Next we make another easy but very useful observation.

**Lemma 4.2.** Let $R$ be a ring of characteristic 0 such that a rational prime $p$ does not have any inverse in the ring. In this case, if the set $A = \{px - 1 \mid x \in R\} \subset R$, then $0 \notin A$. Moreover, if some $x \in R$ has $px - 1 \in \mathbb{Z}$, then $x \in \mathbb{Z}$.

*Proof.* First of all, if $0 \in A$ then $\frac{1}{p} \in R$ and we have a contradiction. Second, clearly if $px - 1 \in \mathbb{Z}$ then $px = z \in \mathbb{Z}$ and $\frac{z}{p} = x \in R$. If $x \notin \mathbb{Z}$, then $(p, z) = 1$, and $p$ has an inverse in $R$ by Lemma 4.1 in contradiction to our assumptions.  

**Theorem 4.3.** (Essentially Theorem 5.1 of [Shl90]) If $R$ is an integral domain of characteristic 0 and $x$ is transcendental over $R$, then $\mathbb{Z}$ is single-fold Diophantine over $R[x]$.

*Proof.* Consider the following set of equations,

\begin{align*}
(4.8) & \quad (f_i - (ax)^{1/2}g_i) = (ax - (ax)^{1/2})^i, \quad i = 2, 3 \\
(4.9) & \quad f^2 - (a^2x^2 - 1)g^2 = 1, \\
(4.10) & \quad g_3 \mid g, \\
(4.11) & \quad t \mid g_3g_2, \\
(4.12) & \quad t \equiv g \mod g_3^2, \\
(4.13) & \quad ax \mid f, \\
(4.14) & \quad a = t/g_3.
\end{align*}
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We show that these relations can be satisfied with some values of variables

\[ a \neq 0, f, g, f_3, g_3 \in R[x] \]

only if we choose the odd integer. For by Lemma 2.1 we have from (4.13) that \( f = \pm t^m, g = \pm g_m \) for some \( m \in \mathbb{Z}_{>0} \). Next from (4.10) we get that \( m = 3r, r \in \mathbb{Z}_{>0} \) and from (4.13) we obtain that \( f_1/f_m \), implying that \( m \) is odd. Hence, \( r \) is odd. From Lemma 3.2 we have that

\[ g_{3r} = \sum_{r-i \text{ odd}} \binom{r}{i} f_3^i((ax)^2 - 1)^{(r-i-1)/2} g_3^{r-i}. \]

Thus \( g_{3r} \equiv r f_3^{r-1} g_3 \mod g_3^2 \). We also know that \( f_3^2 \equiv 1 \mod g_3^2 \). Since \( r - 1 \) is even, we now deduce \( f_{3r} \equiv r g_3 \mod g_3^2 \). Thus, we conclude using (4.12) that \( t \equiv \pm r g_3 \mod g_3^2 \) or equivalently

\[ g_3^2 | (t \pm r g_3). \]

From (4.11) we have \( t | g_3 r_2 \) so that \( \deg(t) < 2 \deg(g_3) \), and \( \deg(t \pm r g_3) < \deg(g_3^2) \). Therefore (4.15) implies that \( t \pm r g_3 = 0 \), \( a = t/g_3 = \pm r \), that is, \( a \) is an odd integer.

Conversely, suppose \( r \) is an odd integer and let \( a = r \). Set \( (g, h) = (f_3^i | r x), \pm g_3^i | r x) \), where \( "-" \) will correspond to negative \( r \). Now (4.8)–(4.10), and (4.13) are satisfied. Since \( g_2(r x) = 2r x \), letting \( t = r g_3 (r x) \) would imply \( t | g_3 (r x) g_2(r x) \), and (1.11), (4.12) and (4.14) are satisfied. In order to complete our Diophantine definition we can note that \( m \) is a rational integer if and only if \( (2m + 1) \) is a rational odd integer and Lemma 3.2 provides single-fold definition of a subset of \( R \setminus \{0\} \) containing integers if \( R \) does not contain \( \mathbb{Q} \). If \( R \) does contain \( \mathbb{Q} \) then we can simply require inverses for sufficient number of elements of our subset (as in Lemma 3.6).

Using the same argument as in Theorem 3.9 we can obtain the following result.

**Proposition 4.4.** The following set has a single-fold definition over \( R[t] \):

\[ \text{Exp} = \{(b, c, d) \mid b, c, d \in \mathbb{Z} \land \mid b \mid = \mid c \mid \mid d \}. \]

Using this result we can now show the following.

**Theorem 4.5.** The set of non-negative integers has a single-fold Diophantine definition over \( R[t] \), for any ring \( R \) of characteristic zero not containing inverses of any rational primes.

**Proof.** Let \( (b, d^4 + 1, 2d) \in \text{Exp} \) and assume

\[ 2d \equiv b - 1 \mod d^4. \]

In this case we know that \( b = (d^4 + 1)^{2|d|} \) and \( 2d \equiv 2|d| \mod d^4. \) Assuming \( |d| \neq 1 \) and \( d < 0 \), we have that \( d^3 \mid 4, \) leading to a contradiction. \( \square \)

Substituting the definition of powers of 2 from Proposition 4.4 in place of sums of four squares, we obtain the following corollary.

**Corollary 4.6.** All c.e. subsets of \( \mathbb{Z} \) have single-fold Diophantine definitions over \( R[t] \), as long as \( R \) does not contain inverses of rational primes.
So far we have produced finite and single-fold definitions of certain c.e. subsets of a ring. We now construct our first examples of rings where all c.e. sets have finite-fold definitions. To do this we combine the arguments above with the proof of Zahidi from [Zah00], which showed that over a polynomial ring with coefficients in a ring of integers of a totally real number field, all c.e. sets were Diophantine. Zahidi’s result was in turn an extension of a result of Denef from [Den78] in which the coefficients of the polynomial ring came from \( \mathbb{Z} \).

Any discussion of c.e. sets of a polynomial ring and a ring of integral functions, which is to take place later, must of course involve some discussion of indexing of the ring. In other words we will need a bijection from a ring into the positive integers such that given a “usual” presentation of a polynomial (or an integral function in the future) we can effectively compute the image of this polynomial (or this integral function), and conversely, given a positive integer, we can determine what polynomial (or integral function) was mapped to it. For a discussion of an effective indexing map in the case of a rational function field we refer the reader to the paper of Zahidi. A discussion of indexing for function fields can be found in [Shl06]. In this paper we will assume that such an indexing is given and, following Zahidi, will denote it by \( \theta \) going from positive integers to polynomials. Below we describe the rest of our notation and assumptions.

**Notation and Assumptions 5.1.**

- Let \( k \) be a totally real number field.
- Let \( O_k \) be the ring of integers of \( k \).
- Let \( \alpha_1, \ldots, \alpha_r \) be an integral basis of \( O_k \) over \( \mathbb{Z} \).
- Let \( \theta : \mathbb{Z}_{>0} \rightarrow O_k[T] \) be the effective bijection discussed above.
- Define \( P_n(T) := \theta(n) \).
- Let \( (X_n(T), Y_n(T)) \in O_k[T] \) be such that
  \[
  X_n(T) - (T^2 - 1)^{1/2}Y_n(T) = (T - (T^2 - 1)^{1/2})^n.
  \]

As we indicated before, our intention is to follow the plan laid out by Zahidi and Denef, just making sure that all the definitions in that plan are finite-fold. This plan entails showing (a) that all c.e. subsets of \( \mathbb{Z} \) are (finite-fold) Diophantine over the polynomial ring in question and (b) that the indexing is (finite-fold) Diophantine or, in other words, the set

\[
\{(n, P_n)|n \in \mathbb{Z}_{>0}\}
\]

is (finite-fold) Diophantine over \( O_k[T] \). Zahidi provides a brief argument in his paper that we apply to our situation, given that we have a finite-fold way of combining equations, to see that (a) and (b) imply the following theorem.

**Theorem 5.2.** Let \( R \) be the ring of integers in a totally real number field \( k \). Let \( \theta \) be an effective indexing of \( R[T] \); then every \( \theta \)-computably enumerable relation over \( R[T] \) is a finite-fold Diophantine relation over \( R[T] \).

Below we lay out details making sure that the finite-fold argument goes through. Like the earlier authors, we will make use a theorem of Y. Pourchet representing positive-definite polynomials as sums of five squares.
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**Definition 5.3.** If $F$ is a polynomial in $O_k[T]$, then $F$ is positive-definite on $k$ (denoted by $\text{Pos}(F)$) if and only if $|\sigma(F(t))| \geq 0$ for all $t \in k$ and for all real embeddings $\sigma$ of $k$ into its algebraic closure.

**Remark 5.6.**

**Lemma 5.7.** The relation $\text{Pos}$ is finite-fold Diophantine over $O_k[t]$.

**Proof.** We claim that the following equivalence holds: $\text{Pos}(F)$ if and only if there exist $F_1, \ldots, F_5 \in O_k[t], g \in \mathbb{Z} \setminus \{0\}$

\[
(5.16) \quad g^2 F = F_1^2 + \ldots + F_5^2.
\]

If (5.16) holds, then clearly $\text{Pos}(F)$ holds.

Conversely suppose that $F$ is positive definite on $K$. In this case it follows from a theorem of Pourchet (see [Pou71]) that $F$ can be written as a sum of five squares in $k[t]$. Now multiplying by a suitable positive integer constant $g$ these polynomials can be taken to be in $O_k[t]$, and hence (5.16) has solutions. \quad \Box

We now show that for a given $g$ and $F$ there can be only finitely many solutions to (5.16). First of all, the degrees of $F_1, \ldots, F_5$ are bounded by the degree of $F$. Secondly, observe that for any $a \in O_k$ we have that $|\sigma(F_i(a))| \leq g^2|\sigma(F(a))|$ for all $i$ and all embeddings $\sigma$. Since $F_i(a)$ is an algebraic integer of $k$ of bounded height, there are only finitely many $b \in k$ such that $F_i(a)$ can be equal to $b$. Thus, there are only finitely many possible values for the coefficients of each $F_i$.

**Definition 5.5.** The relation $\text{Par}(n, b, c, d, v_1, \ldots, v_r)$ on the rational integers is defined to be the conjunction of the following conditions:

1. $n$ is the enumeration index of a polynomial $P_n \in O_k[T]$;
2. $b, c, d, g \in \mathbb{Z}_{\geq 0}, v_1, \ldots, v_r \in \mathbb{Z}$;
3. $d = \text{deg}(P_n)$;
4. $c$ is the smallest possible positive integer so that $\text{Pos}(Y_{d+2}^2 + c - P_n^2 - 1)$
5. $g$ is the smallest possible positive integer so that there exist $F_1, \ldots, F_5 \in O_k[T]$ satisfying $g^2(Y_{d+2}^2 + c - P_n^2 - 1) = F_1^2 + \ldots + F_5^2$.
6. $\forall x \in \mathbb{Z}: \text{ if } 0 \leq x \leq d \text{ then } Y_{d+2}(x) \leq b$;
7. $P_n(2b + 2c + d) = v_1 \alpha_1 + \ldots + v_r \alpha_r$.

**Remark 5.6.** Observe that $\text{Par}$ is indeed a c.e. relation on integers.

The final piece of proof comes from the lemma below, which is taken essentially verbatim from Zahid’s paper.

**Lemma 5.7.** $F \in O_k[T] \land F = P_n$ is equivalent to $\exists b, c, d, v_1, \ldots, v_r \in O_k[T]$:

1. $\text{Par}(n, b, c, d, g, v_1, \ldots, v_r)$;
2. $g^2(Y_{d+2}^2 + c - F^2 - 1) = F_1^2 + \ldots + F_5^2$ or $Y_{d+2}^2 + c - F^2 - 1 \in \text{Pos}$;
3. $F(2b + 2c + d) = v_1 \alpha_1 + \ldots + v_r \alpha_r$.

**Proof.** Suppose $F = P_n$ for some natural number $n$. Then one can easily find natural numbers $b, c, d, g$ and rational integers $v_1, \ldots, v_r$ such that the relation (1) is satisfied. (2) can be satisfied because $\text{deg}(P_n) < \text{deg}(Y_{d+2})$.

Conversely suppose Conditions (1)-(3) are satisfied for some natural numbers $c, d, n, b$ and integers $v_1, \ldots, v_r$. In this case we have to prove that $F = P_n$. From conditions (2) and (3) it follows that

\[(F - P_n)(2b + 2c + d) = 0.\]
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Thus, if $F \neq P_n$, there is some $S \in O_k[T] \neq 0$ such that

$$F - P_n = (2b + 2c + d - T)S(T).$$

Now by Condition (2), it is the case that $F$ has degree at most $d + 1$, while $P_n$ has degree $d$ (by Condition (1)) and hence $S$ has degree at most $d$. So for some integer $k$ with $0 \leq k \leq d$, we have $S(k) \neq 0$. Now for at least one real embedding $\sigma$ we have

$$|\sigma((F - P_n)(k))| = |(2b + 2c + d - k)||\sigma(S(k))| \geq 2b + 2c,$$

(since $k \leq d$ and the fact that given an algebraic integer $a$ in a totally real number field, $a \neq 0$, there is at least one real embedding such that $|\sigma(a)| \geq 1$). At the same time, by Parts (4) and (6) of the definition of the relation Par, for any real embedding $\sigma$ we have, for all integers $x$ with $0 \leq x \leq d$:

$$|\sigma(F(x))| \leq |\sigma(F(x)^2 + 1)| \leq Y_{d+2}^2(x) + c < b + c,$$

and

$$|\sigma(P_n(x))| \leq |\sigma(P_n^2(x) + 1)| \leq Y_{d+2}^2(x) + c < b + c,$$

and hence

$$|\sigma((F - P_n)(x))| < 2b + 2c,$$

leading to a contradiction.

$\square$

6. Defining Order over Function Fields

In this section we give an existential definition of valuation rings for function fields of characteristic 0 over some classes of fields of constants including all number fields.

Proposition 6.1. Let $k$ be any field of characteristic 0 such that the following form:

$$X^2 - aY^2 - bZ^2 + abW^2$$

is anisotropic over $k$ for some values $a, b \in k$. If $K$ is a function field over $k$, $\mathfrak{F}$ is a prime (or a valuation) of $K$ of degree 1 and $h \in K$ is such that $\text{ord}_\mathfrak{F}h$ is odd, then

$$X^2 - aY^2 - bZ^2 + abW^2 = h$$

has no solution in $K$.

Proof. Using a corollary to Riemann-Roch Theorem, we can assume without loss of generality that functions $X, Y, Z, W, h$ all have a negative order at only one prime of $K$. We denote this prime by $\mathfrak{Q}$ and we can assume $\mathfrak{Q} \neq \mathfrak{F}$. Further,

$$2 \min(\text{ord}_\mathfrak{F}X, \text{ord}_\mathfrak{F}Y, \text{ord}_\mathfrak{F}Z, \text{ord}_\mathfrak{F}W) < \text{ord}_\mathfrak{Q}h$$

and we can find $g \in K$ such that $g$ has a pole at $\mathfrak{Q}$ only and

$$\text{ord}_\mathfrak{Q}g = \min(\text{ord}_\mathfrak{F}X, \text{ord}_\mathfrak{F}Y, \text{ord}_\mathfrak{F}Z, \text{ord}_\mathfrak{F}W).$$

Now divide (6.2) by $g^2$ to get:

$$\left(\frac{X}{g}\right)^2 - a \left(\frac{Y}{g}\right)^2 - b \left(\frac{Z}{g}\right)^2 + ab \left(\frac{W}{g}\right)^2 = \frac{h}{g^2}$$

Next consider (6.3) mod $\mathfrak{F}$, taking into account that $\mathfrak{F}$ is a degree one prime, to get a contradiction with our assumption that the form in (6.1) is anisotropic over $k$. $\square$
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**Remark 6.2.** Below we specialize our discussion by assuming $k$ to be an algebraic extension of $\mathbb{Q}$ which has an embedding into a finite extension of $\mathbb{Q}_p$ for some odd rational prime $p$ or into $\mathbb{R}$ (making $k$ formally real). The method we use, however, is extendible to a much larger class of fields of characteristic 0 and to higher transcendence degree fields of positive characteristic, and we intend to describe these extensions in future papers.

**Lemma 6.3.** Let $k$ be a number field. Let $a, b \in k$ be such that for some prime $p$ of $k$ we have that $\text{ord}_p a = 0$, $a$ is not a square mod $p$ and $\text{ord}_p b$ is odd. Then (6.1) is anisotropic over $k$.

**Proof.** We can rewrite (6.1) as

$$\frac{X^2 - aY^2}{Z^2 - aW^2} = b$$

or

(6.4) $\exists y \in k(a^{1/2}) : b = N_{k(a^{1/2})/k}(y)$.

However, (6.4) does not have a solution $y \in k(a^{1/2})$ because $p$ does not split in the extension $k(a^{1/2})/k$ and therefore any $k(a^{1/2})/k$-norm has even order at $p$. $\square$

**Lemma 6.4.** Let $k, p, a, b$ be as above, but assume $a, b$ are units at $p$. Then in $k_p$ the form is isotropic.

**Proof.** If $a$ is a unit at $p$, then $p$ is unramified in the extension $k(a^{1/2})/k$. Further, since $b$ is a unit at $p$ also, it is a norm in this extension by local class field theory and therefore (6.4) can be solved. $\square$

**Corollary 6.5.** Let $k, p, a, b$ be as above, but assume $a$ is a unit at $p$ while $\text{ord}_p b$ is even. Then in $k_p$ the form is isotropic.

**Proof.** If $\text{ord}_p b$ is even, then we can replace $b$ in the quadratic form by $\hat{b} = \pi^2 b$, where $\text{ord}_p \pi^2 = -\text{ord}_p b$ without changing the status of the form with respect to being isotropic or anisotropic. Observe that $\hat{b}$ is a unit at $p$. $\square$

**Lemma 6.6** (Essentially Eisenstein Irreducibility Criteria). Let $k_p$ be a $p$-adic completion of some number field and let

$$f(T) = a_m T^m + a_{m-1} T^{m-1} + \ldots + a_0$$

be such that $\text{ord}_p a_m = 0$, $\text{ord}_p a_i \geq r > 1$, for $i = 1, \ldots, m-1$, $\text{ord}_p a_0 = r-1$ with $(m, r-1) = 1$. In this case $f(T)$ is irreducible over $k_p$ and adjoining a root of $f(T)$ produces a totally ramified extension of $k_p$.

**Proof.** Let $\pi$ be a local uniformizing parameter at $p$ and let $W = \frac{T}{\pi}$. Now

$$\frac{f(T)}{\pi^m} = a_m \left( \frac{T}{\pi} \right)^m + a_{m-1} \frac{T}{\pi} \left( \frac{T}{\pi} \right)^{m-1} + \ldots + a_0 \frac{T}{\pi^m} = a_m W^m + a_{m-1} \frac{W^{m-1}}{\pi} + \ldots + a_0 \frac{W}{\pi^m} = g(W).$$

Let $\alpha$ be a root of $g(W)$ in the algebraic closure of $k_p$. Let $\mathfrak{p}$ be a prime above $p$ in $k_p(\alpha)$ and consider $\text{ord}_p \alpha$. If $\text{ord}_p \alpha \geq 0$, then

$$(m - r + 1) = (m - \text{ord}_p a_0) = -\text{ord}_p \frac{a_0}{\pi^m} \leq \max_{i=1, \ldots, m-1} (-\text{ord}_p \frac{a_i}{\pi^i}) \leq (m - 1 - r),$$

where
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and we have a contradiction. Thus, \( \text{ord}_p \alpha < 0 \). In this case, \( \text{mord}_p \alpha = e(m - r + 1) \), where \( e \) is the ramification of \( \mathbb{Q} \) over \( p \). Since \((r - 1, m) = 1\), we have that \( e = m \) and the extension must be of degree \( m \). Hence the polynomial must be irreducible, and \( p \) is completely ramified in the extension. \( \square \)

**Proposition 6.7.** Let \( k \) be a field algebraic over \( \mathbb{Q} \) such that \( k \) has an embedding into a finite extension of \( \mathbb{Q}_p \) for an odd rational \( p \), but has no real embeddings. Let \( a, b \in k \) be relatively prime algebraic integers equivalent to \( 1 \) mod \( 4 \) such that, for some non-dyadic prime \( p \) of \( k \) lying above \( p \), we have that \( a \) is not a square at \( p \) and \( \text{ord}_p b \) is odd. (These assumptions can be realized since \( k \) can be embedded into a finite extension of \( \mathbb{Q}_p \) for an odd \( p \).) Let \( T \) be transcendental over \( k \) and let \( K \) be a finite extension of \( k(T) \) so that \( k \) is relatively algebraically closed in \( K \). Let \( q \) be a prime of \( K \) of degree \( 1 \), unramified over \( k(T) \), and assume it is a pole of \( T \) (so that \( \text{ord}_q T = -1 \)). Let \( f \in K \).

1. If \( \text{ord}_q f \) is odd, then for any \( \xi_1 \neq 0, \xi_2 \in \mathbb{Z} \), \( \xi_3 \in k \), the equation

\[
X^2 - aY^2 - bZ^2 + abW^2 = \xi_1 f^3 + \xi_2 T + \xi_3
\]

has no solution in \( K \).

2. If \( f \in k(T) \) and \( \text{ord}_q f \) is even (or in other words \( \text{deg}(f) \) is even), then there exist \( \xi_1 \neq 0, \xi_2 \neq 0, \xi_3 \in k \) such that \((6.5)\) has a solution.

**Proof.** Suppose \( f \in K \) and \( \text{ord}_q f \) is odd. In this case, for any \( \xi_1 \neq 0, \xi_2 \neq 0, \xi_3 \in k \) we know that \( \text{ord}_q (\xi_1 f^3 + \xi_2 T + \xi_3) \) is odd, and by Proposition 6.1 applied to \( h = \xi_1 f^3 + \xi_2 T + \xi_3 \) we conclude that \((6.5)\) has no solutions in \( K \).

Now assume that \( f \in k[T] \backslash k \) and \( \text{ord}_q f \) is even (or in other words \( \text{deg}(f) \) is even). Without loss of generality we can now assume that \( k \) is a number field without any real embeddings. We also remind the reader that \( a, b \) are both equivalent to \( 1 \) mod \( 4 \) and therefore the dyadic primes do not ramify when we adjoin square roots of \( a \) or \( b \). We now show that for some constants \( \xi_1 \neq 0, \xi_2 \neq 0, \xi_3 \in k \) the quadratic form equation \((6.5)\) has solutions in \( k(T) \).

We set \( \xi_2 = 1 \), let \( F(T) = f^3(T) + T \) and describe the first set of conditions on \( \xi_1 \) and \( \xi_3 \). Let \( p \) be a prime of \( k \) such that \( \text{ord}_p a \) is odd and \( b \) is not a square mod \( p \) (in other words, \( p \) is one of finitely many primes where the quadratic form \((6.5)\) is anisotropic locally). Let \( \pi_p \in k \) be an element of order \( 1 \) at \( p \) and of order \( 0 \) at every prime dividing \( a \) or \( b \). (Such an element exists by the Weak Approximation Theorem.) Let \( r \) be smallest non-negative integer such that \( \text{ord}_p \pi_p^{\alpha_i} a_n \geq 2 \) for any coefficient \( a_i, i = 0, \ldots, n - 1 \). Let

\[
F(T) = a_n T^n + \ldots + a_0 = a_n \left( \frac{\pi_p^r T}{\pi_p} \right)^n + a_{n-1} \left( \frac{\pi_p^r T}{\pi_p} \right)^{n-1} + \ldots + a_0
\]

\[
a_n \left( \frac{W}{\pi_p} \right)^n + a_{n-1} \left( \frac{W}{\pi_p} \right)^{n-1} + \ldots + a_0
\]

Now set \( \xi_1 = \frac{\pi_p^n}{a_n} \) and let \( g(W) = \xi_1 F(T) \) with

\[
c_{n-i} = \frac{\pi_p^{rn-(n-i)r}}{a_n} a_{n-i} = \frac{\pi_p^{ir}}{a_n} a_{n-i},
\]

Then

\[
g(W) = W^n + c_{n-1} W^{n-1} + \ldots + c_0,
\]
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where $\text{ord}_p c_i \geq 2$ for $i = 1, \ldots, n - 1$. Next let $\text{ord}_p \xi_3 = 1$ and observe that $h(W) = g(W) + \xi_3$ is irreducible by Lemma [6.6] and adjoining a root $\alpha$ of $h(W)$ to $k$ will ramify $p$ with even ramification degree. This will make the quadratic form in [6.2] isotropic at the single factor above $p$. By adjusting $\xi_1$ and $\xi_3$ we can process all the other primes at which the form is anisotropic locally. Thus, the form will be isotropic in $k(\alpha)$ by the Hasse-Minkowski Theorem. We can now apply Proposition 3 of Pourchet [Pou71] to reach the desired conclusion.

We now consider the case of $k$ algebraic over $\mathbb{Q}$ and with a real embedding. Without loss of generality we can assume that $k$ contains the square root of 2. (Adjoining the square root of 2 will not change the existence of a real embedding.) This assumption ensures that adjoining a square root of $-1$ results in an unramified extension and makes $-1$ a local norm at every prime. However, since $k$ is formally real, the form $X^2 + Y^2 + Z^2 + W^2$ is anisotropic because it is anisotropic at the real completion. At the same time, since $-1$ is a norm locally at every finite prime, the form is isotropic at every non-real or $p$-adic completion. Consequently, we have the following proposition.

**Proposition 6.8.** Let $k$ be an algebraic extension of $\mathbb{Q}$ with a real embedding and containing a square root of 2. Let $T$ be transcendental over $k$ and let $K$ be a finite extension of $k(T)$ so that $k$ is relatively algebraically closed in $K$. Let $q$ be a prime of $K$ of degree 1, unramified over $k(T)$, and assume it is a pole of $T$ (so that $\text{ord}_q T = -1$). Let $f \in K$.

1. If $\text{ord}_q f$ is odd, then for any $\xi_1 \neq 0, \xi_2 \neq 0, \xi_3 \in k$, the equation

$$X^2 + Y^2 + Z^2 + W^2 = \xi_1 f^3 + \xi_2 T + \xi_3$$

has no solution in $K$.

2. If $f \in k[T]$ and $\text{ord}_q f$ is even (or in other words $\deg(f)$ is even), then there exist $\xi_1 \neq 0, \xi_2 \neq 0, \xi_3 \in k$ such that (6.6) has a solution.

**Proof.** If $\text{ord}_q f$ is odd, the argument is the same as in Proposition [6.7]. So assume now that $\deg(f)$ is even and $f$ is not a constant. We choose $\xi_1$ so that the leading coefficient is positive (under the real embeddings). We also choose $\xi_2 = 1$ and $\xi_3 > 0$ and large enough so that $\xi_1 f^3 + T + \xi_3$ has no real roots (under the real embedding). Let $h(T) = \xi_1 f^3(T) + T + \xi_3$ and let $g(T)$ be an irreducible factor of $h(T)$. Then $g(T)$ has no real roots (under the real embeddings) and therefore must be of even degree. Further if we adjoin a root $\alpha$ of $g(T)$ to $k$, the extended field $k(\alpha)$ will have no real embeddings and the left side of (6.6) will become isotropic. Thus we can apply Proposition 3 of Pourchet again to reach the desired conclusion.

To complete our definition, one needs a diophantine definition of a large enough set of constants. For the $p$-adic case this was done in Theorem 5.5 of [Eis07]. For the formally real case, it can be done in a similar fashion using elliptic curves.

We now proceed as follows. Over the field $K$ a function $g$ has a non-negative order at $q$ if and only if $h = T g^2 + T^2$ has even order at $q$. If $h \in k(T)$, then $h$ has even order at $q$ if and only if $h = \frac{f_1}{f_2}$ where $f_1, f_2$ are polynomials of even degree and thus of even order at $q$. The next step is to consider the pole divisor of $T$ in $K$. Let $Q$ be the prime of $k(T)$ below $q$ and let $\Omega = \prod_{i=1}^n q_i$ be the factorization of $\Omega$ in $K$, where $q_1 = q$ and without loss of generality...
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(making a constant field extension, if necessary) all the factors are of degree 1. We can also choose $T$ so that all the factors are distinct, again using a corollary to the Riemann-Roch theorem.

Now if $g$ is an arbitrary function of $K$, then $\text{ord}_{q_i} g \geq 0$ for all $i \leq n$ if and only if $g$ satisfies a monic (irreducible) polynomial with coefficients in $k[T]$, so that all the coefficients of the polynomial have non-negative order at each $q_i$. Thus,

$$(\forall i \leq n \text{ ord}_{q_i} f \geq 0) \iff \exists \xi_{1,i,j}, \xi_{2,i,j}, \xi_{3,i,j} \in k; a_1, \ldots, a_{n-1}, f_1, 1, \ldots, f_{n-1,2}, X_{i,j}, Y_{i,j}, Z_{i,j}, W_{i,j} \in K :$$

$$f^n + a_{n-1}f^{n-1} + \ldots + a_0 = 0 \& \bigwedge_{i=1}^{n-1} T a_i^2 + T^2 = \frac{f_{1,i}}{f_{2,i}} \&$$

$$\bigwedge_{i=1,j=1,2}^{n-1} X_{i,j}^2 - a Y_{i,j}^2 - b Z_{i,j}^2 + ab W_{i,j}^2 = \xi_{1,i,j} f_{i,j}^3 + \xi_{2,i,j} T + \xi_{3,i,j},$$

where $a = b = -1$ if $k$ has a real embedding or as described above otherwise.

To define the valuation ring of $q = q_1$ we need another parameter $S \in K$ besides $T$. This element $S$ has to be such that its pole divisor is a product of distinct factors of degree 1 and $S$ has a pole at $q_2, \ldots, q_n$ and possibly other primes but not $q_1$. (Again a corollary to the Riemann-Roch Theorem and possibly a constant field extension will produce such an $S$.) Using the procedure described above, we can define the subring $R_S \subset K$ of all elements integral with respect to the pole divisor of $S$. Now the valuation ring of $q$ will be the set of all elements $x \in K$ such that there exists $y \in R_S$ with $yx \in R_T$.

The possible necessity of extending the field of constants can be a problem in the case when we are relying on a real embedding. There one can work with primes of odd degree or even of odd ramification degree over the chosen rational field. In any case, if $k$ is a number field, we can define order at any valuation of the function field.

7. Diophantine Definition of C.E. Sets over Rings of Integral Functions

In this section we extend results of J. Demeyer to show that c.e. sets are definable over each ring of integral functions, assuming the constant field is a number field. Since Demeyer showed that such a result holds over polynomial rings over number fields and since rings of integral functions are finitely generated over polynomial rings, it is enough to show that we can give a Diophantine definition of polynomials over the rings of integral functions to achieve the desired result.

7.1. Arbitrary powers of a ring element. In this section we again turn our attention to the rings of $\mathcal{S}$-integers of function fields discussed in Section 2 and consider the case where the field is not necessarily rational. Recall the notation and assumptions we used in that section.

Notation and Assumptions 7.1.

- Let $K$ denote a function field of characteristic 0 over constant field $k$.
- Let $\mathcal{S} = \{p_1, \ldots, p_s\}$ be a finite set of primes of $K$.
- Let $O_{K,\mathcal{S}} = \{x \in K | \text{ord}_{p} x \geq 0 \forall p \notin \mathcal{S}\}$ be the ring of $\mathcal{S}$-integers of $K$.
- Let $a \in O_{K,\mathcal{S}}$ such that $\text{ord}_{p_1} a < 0, \text{ord}_{p_i} a = 0, i = 2, \ldots, s$ with $a^2 - 1$ not being a square mod $p_i$.
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- Let $T = a - \sqrt{a^2 - 1}$.
- Let $q_{\infty}$ be one of the two factors that $p_1$ has in $K(\sqrt{a^2 - 1})$. Assume $q_{\infty}$ is the pole of $T$.
- Let $Q$ be the algebraic closure of $\mathbb{Q}$ in $k$.
- For each positive integer $m$ let $\xi_m$ be a primitive $m$-th root of unity.
- For each positive integer $m$ let $\Phi_m$ be the monic irreducible polynomial of $\xi_m$ over $\mathbb{Q}$.

We refer to polynomials of this form as “cyclotomic” polynomials.

For the results below we need to construct a Diophantine definition of arbitrary powers of a non-constant element of the ring. It turns out that it is more convenient to construct this definition for an element of a quadratic extension of the ring. We will proceed as follows.

1. Let $R$ be the given ring $O_{K,\sqrt{}}$. Let $R'$ be the integral closure of $R$ in a quadratic extension of $K$ without any constant field extension. Observe that every polynomial equation of $R'$ can be translated into a polynomial equation over $R$ so that the first equation has solutions in $R'$ if and only if the second equation has solutions in $R$.

2. Over $R'$, using variables ranging over $R$, we will define existentially arbitrary powers of a specific element $T$ and use them to define existentially a polynomial ring $\mathbb{Z}[T]$ (making use of results of J. Demeyer).

3. At this point, following J. Demeyer’s use of results of J. Denef, we deduce that all c.e. subsets of $\mathbb{Z}[T]$ are Diophantine over $R'$.

4. Let $Y \in O_{K,\sqrt{}}$, $Y \notin k$ and note that $Q(Y,T)/Q(T)$ is a finite extension. Let $m = [Q(Y,T) : Q(T)]$, let $\Omega = \{\omega_i, i = 1, \ldots, m\}$ be any basis of $Q(Y,T)$ over $Q(T)$, and observe that the set of $2m$-tuples $\{a_1(T), b_1(T), \ldots, a_m(T), b_m(T)\} \subset \mathbb{Z}[T]$ such that $b_1(T) \ldots b_m(T) \neq 0$ and $\sum_{i=1}^{m} a_i(T) \omega_i \in \mathbb{Z}[Y]$ is recursive and therefore is Diophantine over $\mathbb{Z}[T], R'$ and $R = O_{K,\sqrt{}}$.

5. If $k = Q$ and is a number field, we can also conclude that all c.e. subsets of $R$ are Diophantine over $R$.

### 7.2. Defining Polynomials over $\mathbb{Z}$ using Special Root-of-Unity Polynomials

In this section we generalize the discussion of root-of-unity polynomials from [Dem10] and adapt it to algebraic extensions. In this paper J. Demeyer defined a set $\mathcal{C}$ of root-of-unity polynomials to be the set of polynomials $F \in \mathbb{Z}[T]$ satisfying one of the following three equivalent conditions:

1. $F$ is a divisor of $T^u - 1$ for some $u > 0$.
2. $F$ or $-F$ is a product of distinct cyclotomic polynomials.
3. $F(0) = \pm 1$, $F$ is squarefree, and all the zeros of $F$ are roots of unity.

We will use a slightly smaller set $\mathcal{D}$ of polynomials that we call special root-of-unity polynomials. We still consider polynomials $F(T) = \pm \prod_i \Phi_{n_i}(T)$, where each $\Phi_{n_i}(T)$ is a cyclotomic polynomial, but we impose on each $n_i$ the condition that $n_i = p_i m_i$, where $p_i$ is a prime number and $m_i | (p_i - 1)$.

We now check to see that our set $\mathcal{D}$ of polynomials has the same properties as the set $\mathcal{C}$ of J. Demeyer. If a positive integer $n$ is of the form $pm$ with $p$ prime and $m | (p - 1)$, we will say that it is of the special form.
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**Lemma 7.2** (Compare to Corollary 2.5 of [Dem10]). Let \( d \in \mathbb{Z}_{>0} \) and \( s \in \{-1, 1\} \). In this case there exist infinitely many positive integers \( n \) of the special form such that

\[
\Phi_n(T) = 1 + sT^d \pmod{T^{2d}}.
\]

**Proof.** Factor \( d \) as \( d = \prod_{i=1}^{u} p_i^{e_i} \) and set \( m = \prod_{i=1}^{u} p_i^{e_i + 1} \). If \( r \) is any square-free number coprime to \( m \), then it follows from Proposition 2.4 of [Dem10] that \( \Phi_{rm}(T) \) is congruent to

\[
1 \pm T^d \pmod{T^{2d}},\]

where the sign of \( T^d \) is determined by the parity of the number of factors in \( r \). If we need \( r \) to have an odd number of factors we can set it to be \( r = p \) for some prime number \( p \) such that \( m|(p - 1) \). Otherwise, we can set \( r = p_1p_2 \) for some prime numbers \( p_1 \) and \( p_2 \) so that \( p_2m|(p_1 - 1) \). \( \square \)

Now we can prove the following proposition using essentially the same argument as in the proof of Proposition 2.7 of [Dem10].

**Proposition 7.3.** Let \( F \in \mathbb{Z}[T] \) with \( F(0) \in \{-1, 1\} \), and let \( d \in \mathbb{Z}_{>0} \). In this case there exists a polynomial \( M \in \mathcal{S} \) such that \( F \equiv M \pmod{T^d} \) in \( \mathbb{Z}[T] \).

In what follows we will need some well-known facts about roots of unity, which the reader will find in the appendix.

**Lemma 7.4.** Suppose \( \alpha \in O_{K(\sqrt{a-1}),\{q_\infty\}} \), \( c, n_i, m_i, p_i, \ell \in \mathbb{Z}_{>0} \) are defined as in Lemma 8.4 and assume that (1)–(5) below hold.

1. \( \alpha\mid(T^\ell - 1) \in O_{K(\sqrt{a-1}),\{q_\infty\}} \).
2. \( \alpha - \prod_{i=1}^{r} \Phi_{n_i}(1) \equiv 0 \pmod{(T - 1)} \) in \( O_{K,\mathcal{S}} \).
3. \( \text{ord}_{q_\infty} \alpha = \text{ord}_{q_\infty}(\sum_{i=1}^{r} \varphi(n_i)) \).
4. \( q_\infty \) is the only pole of \( \alpha \), and \( \text{ord}_{q} \alpha = 0 \) for every \( q \in \mathcal{S} \setminus \{p_1\} \) and also for every \( q \mid p_1 \) such that \( q \neq q_\infty \).
5. There is a some constant \( b \in \mathbb{Z} \) with \( \text{ord}_{p_i} b = \varphi(m_i) \) for all \( i = 1, \ldots, r \), such that \( (\alpha - b) \equiv 0 \pmod{(T - c)} \) in \( O_{K,\mathcal{S}} \).

In this case \( N_{K/Q(T)}(\alpha) = \prod_{i=1}^{r} \Phi_{n_i}(T)^{|K:Q(T)|} \) and \( \alpha \in Q(T) \). Conversely, if \( \alpha = \prod_{i=1}^{r} \Phi_{n_i}(T) \), then (1)–(5) are true.

**Proof.** First observe that \( N_{K/Q(T)}(\alpha) \) is a polynomial of degree \( \sum_{i=1}^{r} |K:Q(T)| \varphi(n_i) \). Second, since \( \alpha\mid(T^\ell - 1) \in O_{K(\sqrt{a-1}),\{q_\infty\}} \) and thus \( N_{K/Q(T)}(\alpha) \mid(T^\ell - 1)\mid(K:Q(T)) \) in \( \mathbb{Z}[T] \) while \( \text{ord}_{q} \alpha = 0 \) for every \( q \in \mathcal{S} \setminus \{p_1\} \) and every \( q \mid p_1 \) such that \( q \neq q_\infty \), we have that all the roots of \( N_{K/Q(T)}(\alpha) \) are of multiplicity at most \( |K:Q(T)| \) and are \( \ell \)-th roots of unity. Since

\[
G(T) = N_{K/k(T)}(\alpha) \in \mathbb{Q}[T],
\]

we now have that

\[
G(T) = u \prod_{j=1}^{\ell} \Phi_j(T)^{a_j},
\]

where \( a_j \in \{0, 1, \ldots, |K:Q(T)|\} \) and \( u \in \mathbb{Q} \). Since \( G(1) = \pm 1 \), we conclude that \( u = \pm 1 \). From Assumption (5), we have that all conjugates of \( \alpha \) over \( Q(T) \) are equivalent to \( b \) modulo
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\[(T - c)\] and thus \(N_{K/Q(T)}(\alpha) = G(T) \cong b^{[K : Q(T)]} \mod (T - c)\). Consequently, for all \(i = 1, \ldots, r\), we have that

\[
\text{ord}_{p_i} G(c) \equiv 0 \mod (\varphi(m_i)[K : Q(T)]).
\]

By Lemmas 8.2, 8.3 and 8.4 we have that \(\text{ord}_{p_i} G(c) = \text{ord}_{p_i} \Phi_n(c)^{\alpha_i} = a_n \varphi(m_i)\). Thus,

\[
[K : Q(T)]^n a_n. \text{ But } 0 \leq a_n \leq [K : Q(T)]. \text{ Hence, } a_n = [K : Q(T)] \text{ and }
\]

\[
G(T) = \pm \prod \Phi_n(T)^{[K : Q(T)]}.
\]

Thus the divisors of \(\alpha\) and \(\prod_{i=1}^{r} \Phi_n(T)\) are the same. Hence \(\alpha \in Q[T]\). Further, since \(\alpha(1) = \prod_{i=1}^{r} \Phi_n(1)\), we have that \(\alpha = \prod_{i=1}^{r} \Phi_n(T)\). It is clear that \(\alpha = \prod_{i=1}^{r} \Phi_n(T)\) satisfies (1) - (5). \(\square\)

We now show that all the conditions in Lemma 7.4 are Diophantine over \(O_{K,\mathcal{S}}\) and therefore the set \(\mathcal{D}\) is Diophantine over \(O_{K,\mathcal{S}}\).

**Lemma 7.5.** \(\mathcal{D}\) is Diophantine over \(O_{K,\mathcal{S}}\).

**Proof.** We convert Conditions (1) - (5) of Lemma 7.4 into a Diophantine definition of the set \(\mathcal{D}\). First consider a recursive subset \(Z\) of \(Z^4\) satisfying the following condition.

\[(\ell, n, c, a) \in Z\]

if and only if

1. \(\exists s \in Z_{\geq 0}: \ell = n_1 \ldots n_s, (n_i, n_j) = 1\) and each \(n_i = p_i^{m_i}\) is of the special form, i.e., \(p_i\) is prime and \(m_i| (p_i - 1)\);
2. \(n = \varphi(n_1) + \ldots + \varphi(n_s)\);
3. \(a = \prod_{i=1}^{s} \Phi_n(1)\); and
4. \(c \equiv \xi_{m_i} \mod p_i^{\varphi(m_i)} \in Q_{p_i}^\times\).

By the MDRP theorem \(Z\) is Diophantine over \(Z\) and therefore over \(O_{K,\mathcal{S}}\). Further, as we noted above, the set \(\{(s, T^s), s \in Z_{\geq 0}\}\) is also Diophantine over \(O_{K,\mathcal{S}}\). Thus Condition (1) is Diophantine. We can replace Condition (2) with

\[
\frac{\alpha - a}{T - 1} \in O_{K,\mathcal{S}}.
\]

Conditions (3) can be replaced with

\[
\text{ord}_{q_\infty} \frac{\alpha}{T^n} \geq 0
\]

and Condition (4) with

\[
\text{ord}_{q} \alpha \geq 0 \text{ for all } q \in \mathcal{S} \setminus \{p_1\} \text{ and for all } q | p_1 \text{ with } q \neq q_\infty.
\]

The order conditions are Diophantine over \(O_{K,\mathcal{S}}\) as explained in Section 6. \(\square\)

We will now use Proposition 7.3 to give an existential definition of all polynomials in \(T\) over \(Z\). We use what elsewhere we called the “Weak Vertical Method” (see [Shl06]). Observe that \(O_{K(\sqrt{a^2 - 1}), \{q_\infty}\}\) is the integral closure of \(Q[T]\) in \(K(\sqrt{a^2 - 1})\), and let \(\beta \in O_{K(\sqrt{a^2 - 1}), \{q_\infty}\}\).
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generate \( K(\sqrt{a^2 - 1}) \) over \( \mathbb{Q}(T) \). Then there exists a polynomial \( D(T) \in \mathbb{Q}(T) \) such that for every \( X \in O_{K(\sqrt{a^2 - 1}), \{q_\infty\}} \) we have that

\[
D(T)X = \sum_{i=0}^{r} a_i(T)\beta^i,
\]

where \( r = \lceil K(\sqrt{a^2 - 1}) : \mathbb{Q}(T) \rceil - 1, a_i \in \mathbb{Q}[T] \). Further, for all \( i \) we have that

\[
\deg(a_i) < C(\beta) \text{ord}_{q_\infty} X,
\]

where \( C(\beta) \) depends on \( \beta \) only.

Suppose now \( Y \in O_{K(\sqrt{a^2 - 1}), \{q_\infty\}}, M \in \mathbb{Q}[T] \) is as in Proposition 7.3, \( \ell > C(\beta) \text{ord}_{q_\infty} Y \), and \( Y \equiv M \mod T^\ell \) in \( O_{K(\sqrt{a^2 - 1}), \{q_\infty\}} \) (or in other words, for all \( K(\sqrt{a^2 - 1}) \)-primes \( p \neq q_\infty \) it is the case that \( \text{ord}_p(Y - M) \geq \ell \cdot \text{ord}_p T \). As above we can write \( D(T)Y = \sum_{i=0}^{r} b_i(T)\beta^i \), where \( b_i(T) \in \mathbb{Q}[T] \) and

\[
D(T)(Y - M) = (b_0 - D(T)M) + b_1(T)\beta + \ldots + b_r\beta^r.
\]

Further,

\[
\frac{D(T)(Y - M)}{T^\ell} = \left( \frac{b_0 - D(T)M}{T^\ell} \right) + \frac{b_1(T)}{T^\ell} \beta + \ldots + \frac{b_r}{T^\ell}\beta^r,
\]

by the discussion above. Hence, for \( i = 1, \ldots, r \) either \( b_i = 0 \) or \( \ell \leq \deg(b_i) \leq C(\beta) \text{ord}_{q_\infty} Y \). The inequality violates our assumptions on \( \ell \) and thus \( D(T)\frac{Y-M}{T^\ell} \in \mathbb{Q}(T) \) with

\[
Y \in O_{K(\sqrt{a^2 - 1}), \{q_\infty\}} \cap \mathbb{Q}(T) = \mathbb{Q}[T].
\]

Now that we know that \( Y \in \mathbb{Q}[T] \), we can use the same argument as in the proof of Theorem 3.1 of [Dem10] to force \( Y \) into \( \mathbb{Z}[T] \). Further, if \( R \) is any polynomial in \( \mathbb{Z}[T] \), then for some \( c \in \mathbb{Z} \) we have that \( R(0) + c = 1 \) and we can set \( Y = R + c \) so that for some \( M \in \mathcal{H} \) we have \( Y \equiv M \mod T^\ell \) in \( O_{K(\sqrt{a^2 - 1}), \{q_\infty\}} \). We have now proved the following results.

**Theorem 7.6.** Let \( K \) be a function field over a field of constants which is a finite extension of \( \mathbb{Q} \), and let \( \mathcal{H} \) be a finite collection of its valuations. Let \( T \) be any non-constant element of \( K \). Then \( \mathbb{Z}[T] \) is Diophantine over \( O_{K,\mathcal{H}} \).

**Theorem 7.7.** Let \( K \) be a function field over a field of constants which is a finite extension of \( \mathbb{Q} \), and let \( \mathcal{H} \) be a finite collection of its valuations. Then every c.e. subset of \( O_{K,\mathcal{H}} \) is Diophantine over \( O_{K,\mathcal{H}} \).

8. Appendix

This section contains some facts about roots of unity collected for the convenience of the reader. Below, for \( r \in \mathbb{Z}_{>0} \), the polynomial \( \Phi_r(X) \in \mathbb{Z}[X] \) denotes the monic irreducible polynomial of a primitive \( r \)-th root of unity \( \xi_r \).

**Lemma 8.1.**

1. For every positive integer \( s \), it is the case that \( (\xi_s - 1)|s \) in \( \mathbb{Z}[^s] \).

2. For every positive integers \( s \), every prime \( p \), and every positive integer \( i \) prime to \( p \), it is the case that \( \Phi_{p^i}(1) = p \) and

\[
\frac{\xi_{p^i}^i - 1}{\xi_{p^i} - 1}.
\]
(3) Let $r = p^s r_1$, where $r, r_1$ are positive integers, $s \geq 0$, $p$ is a rational prime, and $(r_1, p) = 1$. In this case $(\Phi_r(1), p) = 1$.

**Proof.** (1) Note that $\Phi_s(X) \mid (X^{s-1} + \ldots + 1)$. Hence $\Phi_s(1) \mid s$ in $\mathbb{Z}$. At the same time $(\xi_s - 1) \mid \Phi_s(1)$ in $\mathbb{Z}[\xi_s]$ and the first assertion of the lemma follows.

(2) See [Lan70, Chapter IV, §1].

(3) For $s > 0$, observe that

$$\Phi_r(X) \bigg| \frac{X^{r-1} + \ldots + 1}{\Phi_p(X)\Phi_{p^2}(X) \ldots \Phi_{p^r}(X)}.$$

Therefore,

$$\Phi_r(1) \mid \frac{r}{\Phi_p(1)\Phi_{p^2}(1) \ldots \Phi_{p^r}(1)} = \frac{r}{p^s} = r_1.$$ 

If $s = 0$, then $r = r_1$ and $\Phi_r(1) = \Phi_{r_1}(1) \mid r_1$, while by assumption $(r_1, p) = 1$.

**Lemma 8.2.** Let $r, m, p$ be positive integers such that $p$ is prime, $p$ does not divide $m$, and $r \neq m$. If also $(p, \Phi_p(\xi_m)) \neq 1$ in $\mathbb{Z}[\xi_r, \xi_m]$, then for some positive integer $a$ we have that $r = mp^a$, and $m$ divides $p^a - 1$.

**Proof.** Let $r = r_1 dp^a, m = m_1d$ with GCD$(r, m) = d, (r_1d, p) = 1$, and $(m_1, r_1) = 1$, and let $a \in \mathbb{Z}_{\geq 0}$. Fix $s = \text{LCM}(m, r) = m_1r_1dp^a$ and consider the following.

$$\xi_m - \xi_r = \xi_s^{s/m} - \xi_s^{s/r} = \xi_s^{s/r}(\xi_s^{s/r} - 1) = \xi_r(\xi_s^{p^{r_1-m_1}} - 1).$$

By Lemma 8.1 Parts (1) and (3), the necessary condition for a factor of $p$ to divide $(\xi_s^{p^{r_1-m_1}} - 1)$ is for $a > 0$ (i.e. for $r$ to be divisible by $p$), and for $dr_1m_1$ to divide $p^a r_1 - m_1$.

Indeed, suppose $a = 0$, and thus $(s, p) = 1$. Since $|r_1 - m_1| < s$, we have that $\xi_s^{r_1-m_1} = \xi_j$, where $j > 1, j \mid s$ and therefore $j$ is prime to $p$. Finally by Lemma 8.1 Part (1), we conclude that $(\xi_j - 1) \mid j$ and therefore $(\xi_j - 1, p) = 1$.

Suppose now $a > 1$ but $dr_1m_1$ does not divide $p^a r_1 - m_1$. Then $\xi_s^{p^{r_1-m_1}} = \xi_{\ell}$, where $\ell \mid s$ and $\ell = \ell_1p^a, \ell_1 \neq 1, (\ell, p) = 1$. Then using Parts (1) and (2) of Lemma 8.1 we conclude that $(\xi_\ell - 1) \mid \Phi_\ell(1)$ and $(\Phi_\ell(1), p) = 1$.

Since $(m_1, r_1) = 1$, it follows that $m_1 = r_1 = 1$ and $d$ divides $p^a - 1$, or in other words $m = d$ and $m$ divides $p^a - 1$. 

**Lemma 8.3.** If $n = pm$, and $m \mid (p - 1)$, then $\frac{\Phi_n(\xi_m)}{p^{\nu(n)}}$ is an integer relatively prime to $p$ in $\mathbb{Z}[\xi_m]$.

**Proof.** Let $r = \frac{n-1}{m}$ and proceed as in lemma above by considering

$$\xi_m - \xi_{pm} = \xi_{pm}^p - \xi_{pm} = \xi_{pm}(\xi_{pm}^{p-1} - 1) = \xi_{pm}(\xi_p^{r-1} - 1) = \xi_{pm} \frac{\xi_p^{r-1} - 1}{\xi_p - 1}(\xi_p - 1) = \delta(\xi_p - 1),$$
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where \((r, p) = 1\) and \(\delta\) is a unit of the ring of integers of \(\mathbb{Q}[\xi_m]\). Thus,

\[
\Phi_n(\xi_m) = \mu(\xi_p - 1)^{(p-1)\varphi(m)} = \nu p^{\varphi(m)},
\]

where \(\mu\) and \(\nu\) are units of the ring of integers of \(\mathbb{Q}[\xi_m]\).

\[\square\]

**Lemma 8.4.** Let \(\ell = p_1 m_1 \ldots p_r m_r\), where \(m_1, \ldots, m_r\) are positive integers, \(p_1, \ldots, p_r\) are prime integers, and all listed integers are pairwise co-prime. Assume also that \(m_i | (p_i - 1)\). Let \(n_i = p_i m_i\) and let \(\Phi(T) = \Phi_n(T) \ldots \Phi_{n_r}(T)\). In this case there exists \(c \in \mathbb{Z}\) such that \(\text{ord}_{p_i}(\Phi(c)) = \text{ord}_{p_i}(\Phi_n(c)) = \varphi(m_i)\) and \(\text{ord}_{p_i}(\Phi_j(c)) = 0\) for all \(j \ell, j \neq n_i\).

**Proof.** Since \(m_i | (p_i - 1)\), by Hensel’s Lemma we have that \(\xi_m \in \mathbb{Z}_{p_i}\), where \(\mathbb{Z}_{p_i}\) is the ring of integers in the field \(\mathbb{Q}_{p_i}\) of \(p_i\)-adic numbers. Let \(c \in \mathbb{Z}\) be such that \(c \equiv \xi_m \mod p_i^{\varphi(m_i)+1}\) for all \(i = 1, \ldots, r\). (Such a \(c\) exists by the Chinese Remainder Theorem.) In this case, for each \(i\) we have that \(\text{ord}_{p_i}(\Phi_n(\xi_m) - \Phi_n(c)) \geq \varphi(m_i) + 1\) in \(\mathbb{Z}_{p_i}\) and therefore

\[
\text{ord}_{p_i}(\Phi_n(c)) = \text{ord}_{p_i}(\Phi_n(c) - \Phi_n(\xi_m)) + \Phi_n(\xi_m)) = \\
\min(\text{ord}_{p_i}(\Phi_n(c) - \Phi_n(\xi_m))), \text{ord}_{p_i}(\Phi_n(\xi_m)) = \text{ord}_{p_i}(\Phi_n(\xi_m)).
\]

We now consider any positive integer \(j \neq n_i\) dividing \(\ell\) and \(\text{ord}_{p_i}(\Phi_j(\xi_m))\). By Lemma 8.2, \(\text{ord}_{p_i}(\Phi_j(\xi_m)) > 0\) implies \(j = p_i^r m_i\). Since all \(p_i, m_i\) are pairwise co-prime and \(j \neq n_i = p_i m_i\), we deduce that each such \(j\) satisfies \(\text{ord}_{p_i}(\Phi_j(\xi_m)) = 0\).

Thus for any \(j \ell\) with \(j \neq n_i\), we have that

\[
\text{ord}_{p_i}(\Phi_j(c)) = \text{ord}_{p_i}(\Phi_j(c) - \Phi_j(\xi_m)) = \\
\min(\text{ord}_{p_i}(\Phi_j(c) - \Phi_j(\xi_m))), \text{ord}_{p_i}(\Phi_j(\xi_m)) = \text{ord}_{p_i}(\Phi_j(\xi_m)) = 0.
\]

\[\square\]

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DEPARTMENT OF MATHEMATICS
QUEENS COLLEGE – C.U.N.Y.
65-30 Kissena Blvd.
Queens, New York 11367 U.S.A.

PH.D. PROGRAMS IN MATHEMATICS & COMPUTER SCIENCE
C.U.N.Y. GRADUATE CENTER
365 FIFTH AVENUE
New York, New York 10016 U.S.A.

E-mail: Russell.Miller@qc.cuny.edu
Web page: qcpages.qc.cuny.edu/~rmiller

DEPARTMENT OF MATHEMATICS
EAST CAROLINA UNIVERSITY
GREENVILLE, NC 27858 U.S.A.
E-mail: shlapentokha@ecu.edu
Web page: myweb.ecu.edu/shlapentokha