INFINITELY MANY EXCLUDED MINORS FOR FRAME MATROIDS AND FOR LIFTED-GRAPHIC MATROIDS

RONG CHEN AND JIM GEELEN

Abstract. We present infinite sequences of excluded minors for both the class of lifted-graphic matroids and the class of frame matroids.

1. Introduction

A matroid $M$ is a frame matroid if there is a matroid $M'$ with a basis $V$ such that $M = M'\setminus V$ and, for each $e \in E(M)$, the unique circuit in $V \cup \{e\}$ has size at most 3. A matroid $M$ is lifted-graphic if there is a matroid $M'$ with $E(M') = E(M) \cup \{e\}$ such that $M' \setminus e = M$ and $M'/e$ is graphic. The classes of lifted-graphic matroids and frame matroids were introduced by Zaslavsky [9] who proved that they are minor-closed.

We dispel the widespread belief that these classes would likely have only finitely many excluded minors.

Theorem 1.1. There exist infinitely many pairwise non-isomorphic excluded minors for the class of frame matroids.

Theorem 1.2. There exist infinitely many pairwise non-isomorphic excluded minors for the class of lifted-graphic matroids.

Our excluded-minors are based on constructions introduced by Chen and Whittle [3]. DeVos, Funk, and Pivotto [4, 5] characterised the non-3-connected excluded-minors for the class of frame matroids.

The existence of an infinite set of excluded minors does not necessarily prevent us from describing a class explicitly; see, for example, Bonin’s excluded minor characterization for the class of lattice-path matroids [1]. We believe that the excluded-minors for both the class...
of frame matroids and the class of lifted-graphic matroids are highly structured, and that it may be possible to obtain an explicit characterization of the sufficiently large excluded minors. Towards this end we pose the following conjectures.

**Conjecture 1.3.** There exists an integer $k$ such that all excluded minors for the class of lifted-graphic matroids have branch-width at most $k$.

**Conjecture 1.4.** There exists an integer $k$ such that all excluded minors for the class of frame matroids have branch-width at most $k$.

The class of quasi-graphic matroids, introduced in [6], contains both the lifted-graphic matroids and the frame matroids. The infinitely many excluded minors given in this paper for the class of frame matroids and the class of lifted-graphic matroids are quasi-graphic. In contrast to Theorems 1.1 and 1.2 we remain confident that the class of quasi-graphic matroids admits a finite excluded-minor characterization.

**Conjecture 1.5.** There are, up to isomorphism, only finitely many excluded-minors for the class of quasi-graphic matroids.

In support of this conjecture, Chen [2] recently proved that there are only two 8-connected excluded minors for the class of quasi-graphic matroids; namely $U_{3,7}$ and $U_{4,7}$.

## 2. Preliminaries

We assume that the reader is familiar with matroid theory and we follow the terminology of Oxley [7].

Recall that a circuit-hyperplane of a matroid $M$ is a set $C$ that is both a circuit and a hyperplane, and that we can obtain a new matroid $M'$ by relaxing a circuit-hyperplane $C$ of $M$; see [7] Proposition 1.5.14. More specifically, $\mathcal{B}(M') = \mathcal{B}(M) \cup \{C\}$ where $\mathcal{B}(M)$ is the set of bases of $M$.

The reverse operation was introduced by Chen and Whittle [3]. A free basis of a matroid $M$ is a basis $B$ such that $B \cup \{e\}$ is a circuit for each $e \in E(M) - B$. If $B$ is a free basis of $M$ then $(E(M), \mathcal{B}(M) - \{B\})$ is a matroid (see [3]); we say that $(E(M), \mathcal{B}(M) - \{B\})$ is obtained by tightening $B$.

Let $G$ be a graph. For $v \in V(G)$ we let $\delta_G(v)$ denote the set of edges incident with $v$. For any $U \subseteq V(G)$ and $F \subseteq E(G)$, let $G[U]$ be the induced subgraph of $G$ defined on $U$, and let $G[F]$ be the subgraph of
G with F as its edge set and without isolated vertices. A cycle of a
graph is a connected 2-regular subgraph.

We assume that the reader is familiar with bias graphs; see Zaslavsky [8]. Let (G, B) be a bias graph. The cycles in B are called balanced and a subgraph H of G is balanced if each of the cycles in H is balanced. A set F ⊆ E(G) is balancing if G\F is balanced.

For this paper it is more convenient to use the bias graph definition of frame matroids, which is in fact the way that they were originally defined by Zaslavsky [9]. Let (G, B) be a bias graph. We define FM(G, B) to be the matroid with ground set E(G) such that I ⊆ E(G) is independent if and only if G[I] has no balanced cycles and for each component H of G[I] we have |E(H)| ≤ |V(H)|. Henceforth we will call a matroid M a frame matroid if and only if M = FM(G, B) for some bias graph (G, B); Zaslavsky [10] proved that this definition is equivalent to the geometric definition stated in the introduction.

Let M be a matroid. If (G, B) is a biased graph such that M = FM(G, B), then B is implicitly determined by G (and M). Hence we refer to the graph G, itself, as a frame representation of M, and given a frame representation of a matroid we will refer to its cycles as balanced or non-balanced accordingly.

As with frame matroids, it is also more convenient to use the bias graph definition of lifted-graphic matroids; see Zaslavsky [9]. We define LM(G, B) to be the matroid with ground set E(G) such that I ⊆ E(G) is independent if and only if G[I] has at most one cycle and, should it exist, that cycle is non-balanced. Henceforth we will call a matroid M a lifted-graphic matroid if and only if M = LM(G, B) for some bias graph (G, B); Zaslavsky [11] showed that this new definition is equivalent to the earlier definition stated in the introduction.

Let M be a matroid. If (G, B) is a biased graph such that M = LM(G, B), then B is implicitly determined by G (and M). Hence we refer to the graph G, itself, as a lifted-graphic representation of M, and given a lifted-graphic representation of a matroid we will refer to its cycles as balanced or non-balanced accordingly.

One well-known way to construct a bias graph is via a group-labelled graph (also known as a gain graph and a voltage graph). Here we use only the group of integers under addition, which we denote by Z, and the group of non-zero real numbers under multiplication, which we denote by R×. For an abelian group Γ, a Γ-labelled graph is a pair (⃗G, γ) where ⃗G is an oriented graph and γ : E(⃗G) → Γ. Let (⃗G, γ) be a Γ-labelled graph and let G be the underlying graph of ⃗G. A cycle C of G is balanced if the group-product of the labels on “clockwise”
oriented edges is equal to the group-product of the labels on “counter-clockwise” oriented edges; this is independent of the direction on $C$ we choose as clockwise. If $B$ is the set of balanced cycles of $G$ then $(G, B)$ is a biased graph.

The following construction, due to Zaslavsky [12], builds an $\mathbb{R}$-representable frame matroid from an $\mathbb{R}^+$-labelled graph $(\vec{G}, \gamma)$. We will assume that $(\vec{G}, \gamma)$ has no loops. Let $A$ be a $V(\vec{G}) \times E(\vec{G})$ matrix over $\mathbb{R}$ where, for a vertex $v$ and edge $e$, we have $A_{v,e} = 1$ if $v$ is the tail of $e$, $A_{v,e} = -\gamma(e)$ if $v$ is the head of $e$, and $A_{v,e} = 0$ otherwise. Then $M(A) = FM(G, B)$ where $(G, B)$ is the bias graph associated with $(\vec{G}, \gamma)$.

Zaslavsky [12] also showed how to build an $\mathbb{R}$-representable lifted-graphic matroid from a $\mathbb{Z}$-labelled graph $(\vec{G}, \gamma)$. Again will assume that $(\vec{G}, \gamma)$ has no loops. Let $B$ be the signed incidence matrix of $\vec{G}$; thus $B \in \{0, \pm 1\}^{V(\vec{G}) \times E(\vec{G})}$ where $B$ is $1$ or $-1$ when $v$ is the head or tail, respectively, of $e$. Now construct a matrix $A$ by appending the vector $\gamma \in \mathbb{Z}^{E(\vec{G})}$ as a new row to $B$. Then $M(A) = LM(G, B)$ where $(G, B)$ is the bias graph associated with $(\vec{G}, \gamma)$.

A cocircuit $C^*$ of a matroid $M$ is non-separating if $M \setminus C^*$ is connected. If $C^*$ is a non-separating cocircuit of a matroid $M$ and $M = FM(G, B)$, then either $C^*$ is a balancing set of $(G, B)$ or $C^* = \delta_G(v)$ for some vertex $v \in V(G)$.

3. Frame matroids

In this section we prove Theorem 1.1. Let $k \geq 7$ be an odd integer. (The condition that $k \geq 7$ is to simplify the proof; $k \geq 3$ suffices.) Let $(\vec{G}_k, \gamma)$ be the $\mathbb{R}^+$-labelled graph defined in Figure 1 and let $G_k$ denote its underlying undirected graph. Let $B$ denote the balanced cycles of $(\vec{G}_k, \gamma)$ and let $N_k = FM(G_k, B)$.

Let $P = \{a_1, \ldots, a_k\} \cup \{d_1, \ldots, d_k\}$ and $Q = \{b_1, \ldots, b_k\} \cup \{c_1, \ldots, c_k\}$. Note that $P \cup \{e_1, e_2\}$ and $Q \cup \{e_1, e_2\}$ are free bases of $N_k$; let $M_k^F$ be the matroid obtained from $N_k$ by tightening $P \cup \{e_1, e_2\}$ and $Q \cup \{e_1, e_2\}$. Thus $P \cup \{e_1, e_2\}$ and $Q \cup \{e_1, e_2\}$ are circuits of $M_k^F$.

We will prove that $M_k^F / \{e_1, e_2\}$ is an excluded minor. We start with the easier task of showing that proper minors of $M_k^F / \{e_1, e_2\}$ are frame matroids.

Lemma 3.1. For each $e \in P \cup Q$, both $M_k^F / e$ and $M_k^F \setminus e$ are frame matroids.

Proof. Let $M_P$ and $M_Q$ denote the matroids obtained from $M_k^F$ by relaxing the circuit hyperplanes $P \cup \{e_1, e_2\}$ and $Q \cup \{e_1, e_2\}$ respectively.
Figure 1. The graphs $G_k$ and $G_k / \{e_1, e_2\}$, respectively, with their $\mathbb{R}^x$-labellings; unlabelled edges have group-label 1, where $a_i, b_i, c_i, d_i$ are not group labels but edge names.

For each $e \in P$, we have $M_{PF}^k \setminus e = M_P \setminus e$ and $M_{PF}^k / e = M_Q / e$. Similarly, for each $e \in Q$, we have $M_{PF}^k / e = M_P / e$ and $M_{PF}^k \setminus e = M_Q \setminus e$. So it suffices to prove that $M_P$ and $M_Q$ are frame matroids.

Note that $M_P$ and $M_Q$ are obtained from $N_k$ by tightening the free bases $Q \cup \{e_1, e_2\}$ and $P \cup \{e_1, e_2\}$ respectively. Since $G_k$ is a frame representation of $N_k$ and $G_k [Q \cup \{e_1, e_2\}]$ is a cycle in $G_k$, we have that $G_k$ is a frame representation of $M_P$; so $M_P$ is indeed a frame matroid. Let $G'_k$ be the graph obtained from $G_n \setminus \{e_1, e_2\}$ by adding $e_1$ connecting $s_1$ to $t_2$ and adding $e_2$ connecting $s_2$ to $t_1$. It is straightforward to verify that $G'_k$ is a frame representation of $N_k$ (since $\{e_1, e_2\}$ is a series pair in $N_k$). Finally, since $G'_k [P \cup \{e_1, e_2\}]$ is a cycle in $G'_k$, we have that $G'_k$ is a frame representation of $M_Q$; so $M_Q$ is indeed a frame matroid.

Now it remains to show that $M_{PF}^k / \{e_1, e_2\}$ itself is not a frame matroid.

**Lemma 3.2.** $M_{PF}^k / \{e_1, e_2\}$ is not a frame matroid.

**Proof.** Assume to the contrary that $H$ is a frame representation of $M_{PF}^k / \{e_1, e_2\}$. Let $C_i = \{a_i, b_i, c_i, d_i\}$ for each $i \in \{1, \ldots, k\}$ and let $G = G_k / \{e_1, e_2\}$. Figure 1 depicts $G$ with a group labelling encoding the...
balanced cycles with respect to $N_k/\{e_1,e_2\}$. From this group labelled
graph we see that:

(i) each cocircuit in $N_k/\{e_1,e_2\}$ (and hence also in $M_k^F/\{e_1,e_2\}$)
has size at least 4,

(ii) for each 4-element cycle $C$ of $G$, the set $E(C)$ is a circuit in
$N_k/\{e_1,e_2\}$ and, hence, also in $M_k^F/\{e_1,e_2\}$,

(iii) for each $i \in \{1, \ldots, k\}$, the set $C_i$ is a non-separating cocircuit
in $N_k/\{e_1,e_2\}$ and, hence, also in $M_k^F/\{e_1,e_2\}$, and

(iv) for each $v \in V(G)$, the set $\delta_G(v)$ is a 4-element non-separating

cocircuit in $N_k/\{e_1,e_2\}$ and, hence, also in $M_k^F/\{e_1,e_2\}$.

3.2.1. $H$ is a simple connected 4-regular graph.

Subproof. Since $N_k/\{e_1,e_2\}$ is connected, so is $M_k^F/\{e_1,e_2\}$. Then $H$
is connected. Since $|E(H)| = |P \cup Q| = 4k = 2|V(H)|$, we see that $H$
has average degree 4. It follows from (i) that $H$ is 4-regular. It remains
to show that $H$ is simple; suppose otherwise and let $C$ be a cycle of
length at most 2. At least $3k-6$ of the non-separating cocircuits
described in (iii) and (iv) are disjoint from $E(C)$. Since $k \geq 7$ we have
$3k-6 > |V(H)|$ and hence one of these non-separating cocircuits is
balancing. But then $C$ is a circuit of $M_k^F/\{e_1,e_2\}$, a contradiction to
the fact that $M_k^F/\{e_1,e_2\}$ is a simple matroid.

For each 4-element circuit $C$ of $M_k^F/\{e_1,e_2\}$, since $H$ is simple, $C$ is
a cycle of $H$. In particular, for each 4-cycle $C$ of $G$, since $E(C)$ is a circuit
of $M_k^F/\{e_1,e_2\}$, the set $E(C)$ is a cycle of $H$. Since $C_1$, \{a_1,b_1,a_2,c_2\},
and \{a_1,c_1,a_k,b_k\} are cycles of the simple 4-regular graph $H$, the sets
\{a_1,d_1\} and \{b_1,c_1\} are matchings in $H$. Repeating the analysis
$k$-times, it is routine to show that $H$ is isomorphic to $G$ and, moreover, that

(a) there is an isomorphism that fixes $C_1, \ldots, C_k$ set-wise, and

(b) for each $i \in \{1, \ldots, k\}$, the sets \{a_i,d_i\} and \{b_i,c_i\} are matchings
in $H$.

Now, since $k$ is odd, one of $H[P]$ and $H[Q]$ is a cycle while the other
is the union of two vertex-disjoint cycles. However $P$ and $Q$ are both
circuits in $M_k^F/\{e_1,e_2\}$ which contradicts the fact that $H$ is a frame
representation.

4. Lifted-graphic matroids

In this section we prove Theorem 1.2. Let $k \geq 3$ be an odd integer.
Let $(\tilde{G}_k, \gamma)$ be the $\mathbb{Z}$-labelled graph defined in Figure 2 and let $G_k$
denote its underlying undirected graph. Let $\mathcal{B}$ denote the balanced
cycles of $(\tilde{G}_k, \gamma)$ and let $N_k = LM(G_k, \mathcal{B})$. 

Let $P = \{a_1, \ldots, a_k\} \cup \{d_1, \ldots, d_k\}$ and $Q = \{b_1, \ldots, b_k\} \cup \{c_1, \ldots, c_k\}$. Note that $P \cup \{e_1, e_2\}$ and $Q \cup \{e_1, e_2\}$ are circuit-hyperplanes of $N_k$; let $M^L_k$ be the matroid obtained from $N_k$ by relaxing $P \cup \{e_1, e_2\}$ and $Q \cup \{e_1, e_2\}$.

We will prove that $M^L_k / \{e_1, e_2\}$ is an excluded minor. We start by showing that proper minors of $M^L_k / \{e_1, e_2\}$ are lifted-graphic matroids; this is almost a carbon copy of the proof of Lemma 3.1.

**Lemma 4.1.** For each $e \in P \cup Q$, both $M^L_k / e$ and $M^L_k \setminus e$ are lifted-graphic matroids.

**Proof.** Let $M_P$ and $M_Q$ denote the matroids obtained from $M^L_k$ by tightening the free bases $P \cup \{e_1, e_2\}$ and $Q \cup \{e_1, e_2\}$ respectively. For each $e \in P$, we have $M^L_k \setminus e = M_P \setminus e$ and $M^L_k / e = M_Q / e$. Similarly, for each $e \in Q$, we have $M^L_k / e = M_P / e$ and $M^L_k \setminus e = M_Q \setminus e$. So it suffices to prove that $M_P$ and $M_Q$ are lifted-graphic matroids.

Note that $M_P$ and $M_Q$ are obtained from $N_k$ by relaxing the circuit-hyperplanes $Q \cup \{e_1, e_2\}$ and $P \cup \{e_1, e_2\}$ respectively. Since $G^L_k$ is a lifted-graphic representation of $N_k$ and $G^L_k[Q \cup \{e_1, e_2\}]$ is a cycle in $G^L_k$, we have that $G^L_k$ is a lifted-graphic representation of $M_P$; so $M_P$ is indeed a lifted-graphic matroid. Let $G'_{k}$ be the graph obtained from
representation of $G_k \setminus \{e_1, e_2\}$ by adding $e_1$ connecting $s_1$ to $t_2$ and adding $e_2$ connecting $s_2$ to $t_1$. It is straightforward to verify that $G'_k$ is a lifted-graphic representation of $N_k$ (since $\{e_1, e_2\}$ is a series pair in $N_k$). Finally, since $G'_k[\mathcal{P} \cup \{e_1, e_2\}]$ is a cycle in $G'_k$, we have that $G'_k$ is a lifted-graphic representation of $M_Q$; so $M_Q$ is indeed a lifted-graphic matroid. □

Now it remains to show that $M^L_k / \{e_1, e_2\}$ itself is not a lifted-graphic matroid.

**Lemma 4.2.** $M^L_k / \{e_1, e_2\}$ is not a lifted-graphic matroid.

*Proof.* Assume to the contrary that $H$ is a lifted-graphic representation of $M^L_k / \{e_1, e_2\}$. Since $N_k / \{e_1, e_2\}$ is connected, $M^L_k / \{e_1, e_2\}$ is connected. Since identifying two vertices in different components of a bias graph does not change its lifted-graphic matroid, we may assume that $H$ is connected. Let $A_1 = \{a_1, b_1, a_k, c_k\}$, $A_2 = \{c_1, d_1, b_k, d_k\}$, $B_1 = \{a_1, b_1, b_k, d_k\}$, $B_2 = \{c_1, d_1, a_k, c_k\}$, $C_1 = \{a_1, b_1, c_1, d_1\}$, and $C_2 = \{a_k, b_k, c_k, d_k\}$. Let $G = G_k / \{e_1, e_2\}$; Figure 2 depicts $G$ with a $\mathbb{Z}$-labelling encoding its balanced cycles with respect to $N_k / \{e_1, e_2\}$. From this $\mathbb{Z}$-labelled graph we see that:

(i) each cocircuit in $N_k / \{e_1, e_2\}$ (and hence also in $M^L_k / \{e_1, e_2\}$) has size at least 4, and

(ii) The only 4-element cocircuits of $N_k / \{e_1, e_2\}$ (and hence also of $M^L_k / \{e_1, e_2\}$) are the sets $\delta_h(v)$ for $v \in V(G_k) - \{s_1, s_2, t_1, t_2\}$ and the sets $A_1, A_2, B_1, B_2, C_1,$ and $C_2$.

**4.2.1.** $H$ is a loopless connected 4-regular graph.

*Subproof.* Since $|E(H)| = |P \cup Q| = 4k = 2|V(H)|$, we see that $H$ has average degree 4. It follows from (i) that $H$ is 4-regular and loopless. □

We will call a set $X \subseteq E(H)$ *vertical* if there exists $v \in V(H)$ such that $X = \delta_h(v)$. Now each of the $2k$ vertical sets is a 4-element cocircuit of $M^L_k / \{e_1, e_2\}$ and each element in $P \cup Q$ is in exactly two vertical sets. We have listed all of the 4-element cocircuits of $M^L_k / \{e_1, e_2\}$ in (ii). Note that the elements in $\{a_2, a_3, \ldots, a_{k-1}\} \cup \{d_2, d_3, \ldots, d_{k-1}\}$ are each in exactly two 4-element cocircuits. It follows that, for each $v \in V(G_k) - \{s_1, s_2, t_1, t_2\}$, the set $\delta_{G_k}(v)$ is vertical. There are three possibilities for the pair of remaining vertical sets, namely, $(A_1, A_2)$, $(B_1, B_2)$, and $(C_1, C_2)$.

First suppose that $C_1$ and $C_2$ are both vertical. Then $H[\{a_1, c_1, a_k, b_k\}]$ is the union of two edge-disjoint cycles. So $\{a_1, c_1, a_k, b_k\}$ is dependent in $M^L_k / \{e_1, e_2\}$ and hence also in $N_k / \{e_1, e_2\}$. However, from the definition of $N_k$, the set
\{e_1, e_2, a_1, c_1, a_k, b_k\} is independent. From this contradiction we have that the remaining pair of vertical sets is either \((A_1, A_2)\) or \((B_1, B_2)\).

Now, since \(k\) is odd, one of \(H[P]\) and \(H[Q]\) is a cycle while the other is the union of two vertex-disjoint cycles. However \(P\) and \(Q\) are both independent in \(M^L_k/\{e_1, e_2\}\) which contradicts the fact that \(H\) is a lifted-graphic representation. \(\square\)

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\textbf{References}

[1] J. Bonin, Lattice path matroids: The excluded minors, J. Combin. Theory Ser. B 100 (2010) 585-599.
[2] R. Chen, The 8-connected excluded minors for the class of quasi-graphic matroids, submitted (2017).
[3] R. Chen, G. Whittle, On recognizing frame and lifted-graphic matroids, J. Graph Theory 87 (2018) 72-76.
[4] M. DeVos, D. Funk, I. Pivotto, On excluded minors of connectivity 2 for the class of frame matroids, European. J. Combinatorics 61 (2017) 167-196.
[5] D. Funk, On excluded minors and biased graph representations of frame matroids, Ph. D. dissertation, Simon Fraser University (2015).
[6] J. Geelen, B. Gerards, G. Whittle, Quasi-graphic matroids, J. Graph Theory 87 (2018) 253-264.
[7] J. Oxley, \textit{Matroid theory, second ed.}, Oxford University Press, New York, (2011).
[8] T. Zaslavsky, Biased graphs. I. Bias, balanced, and gains. J. Combin. Theory Ser. B 47 (1989) 32-52.
[9] T. Zaslavsky, Biased graphs. II. The three matroids. J. Combin. Theory Ser. B, 51 (1991) 46-72.
[10] T. Zaslavsky, Frame matroids and biased graphs, European J. Combin. 15 (1994) 303-307.
[11] T. Zaslavsky, Supersolvable frame-matroids and graphic-lift lattices, European. J. Combinatorics 15 (2001) 119-133.
[12] T. Zaslavsky, Biased graphs. IV. Geometrical realizations. J. Combin. Theory Ser. B, 89 (2003) 231-279.

Center for Discrete Mathematics, Fuzhou University, Fuzhou, P. R. China

\textit{E-mail address: rongchen@fzu.edu.cn}

Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Canada

\textit{E-mail address: jim.geelen@uwaterloo.ca}