The Paradox of Power Loss in a Lossless Infinite Transmission Line

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Abstract—It is shown that the famous paradox of continuous power drain from the source at the input of an otherwise lossless infinite transmission line is successfully resolved when one takes into account both incident and reflected waves in the circuit. The solution of the paradox lies in the realization that while in an open-circuit finite transmission line/ladder network, there is a reflected wave, there may be no such thing in the infinite case. Actually in a lossless finite line, the source does keep on supplying power as an incident wave (even when there are no resistive elements which could consume power as Ohmic losses), but gets it back equally in terms of the reflected wave. Therefore there is no net power loss from the source which is consistent with the circuit comprising only reactive elements presenting zero resistance. It is demonstrated that even a simple, driven LC circuit can be analyzed as an open-circuit 1-block ladder network and from the superposition of incident and reflected waves the input impedance of the LC circuit calculated for all driving frequencies. However in the case of an infinite ladder network or infinite transmission line, assumed to be ideal with no discontinuities enroute, there is no termination to start a reflected wave, and the source while continuously supplying power in the forward direction, does not retrieve it from a reflected wave. Thus there is an apparent net power loss, which ultimately appears as the stored electromagnetic energy in the reactive elements (capacitances and inductances), which initially had no such energy stored within, further down the line as the incident wave travels forward. It is also shown that radiation plays absolutely no role in resolving this paradox.

Index Terms—Circuits and Systems, Transmission Lines, Electrical Engineering Education

I. INTRODUCTION

A transmission line is described by its line parameters $R, L, C, G$, where $R$ is the series resistance per unit length of line (including both wires), $L$ is the series inductance per unit length of line, $C$ is the capacitance between the two conducing wires per unit length of line and $G$ is the shunt leakage conductance between the two conducing wires per unit length of line. For an incremental length $\Delta z$ of the line, the equivalent circuit is shown in Fig. 1. The increments in voltage and current along the line are \[1\], \[2\], \[3\], \[4\].

\[
\Delta V(z) = -I(z)(R + j\omega L)\Delta z \tag{1}
\]

\[
\Delta I(z) = -V(z)(G + j\omega C)\Delta z. \tag{2}
\]

These could be written in limit $\Delta z \to 0$ as,

\[
dV(z)/dz = -I(z)(R + j\omega L) \tag{3}
\]

\[
dI(z)/dz = -V(z)(G + j\omega C). \tag{4}
\]

From \[3\] and \[4\] one gets a general solution for voltage along the line,

\[
V(z) = V_0^r e^{-\gamma z} + V_0^l e^{\gamma z} \tag{5}
\]

where $\gamma$ is the propagation constant. The phasor part is written with an assumed $e^{j\omega t}$ time dependence throughout. Now of the two terms in \[5\], the first one represents a wave travelling along increasing $z$ commencing at $z = 0$, while the second represents a wave travelling towards decreasing $z$ which in case of an infinite line would have to start from $z = \infty$ an infinite time back and thus must be dropped. Therefore the voltage along an infinite transmission line can be written as,

\[
V(z) = V_0 e^{-\gamma z}. \tag{7}
\]

From this one gets for the electric current,

\[
I(z) = (V_0/Z_0) e^{-\gamma z}. \tag{8}
\]

Here $Z_0$, the characteristic impedance of the line given by,

\[
Z_0 = \sqrt{(R + j\omega L) / (G + j\omega C)}. \tag{9}
\]

Equations \[7\] and \[8\] represent an attenuated sinusoidal wave along $z$, with $\alpha$ as the attenuation constant and $\beta = 2\pi/\lambda$ as the wave number.

For an infinite line, the input impedance (at $z = 0$) is calculated from \[7\] and \[8\] as,

\[
Z_i = V(0)/I(0) = Z_0. \tag{10}
\]

In a lossless line, $R = 0$ and $G = 0$, we have $\alpha = 0$ and $\beta = \omega\sqrt{LC}$, i.e., a sinusoidal wave without any attenuation along the line. But we also have $Z_i = Z_0 = \sqrt{L/C}$, i.e., its impedance has a real value. This is a paradox because though the transmission line contains no resistive element so there

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could be no Ohmic losses in the line, yet its input impedance is a pure resistance. That means for an input voltage \( V_0 \), power will be drained from the source at the rate of \( V_0^2/(2\sqrt{L/C}) \) [4]. The questions therefore arise as to why does a pure resistance show up in a circuit comprising only reactivities and where does this energy go? The paradox can be also seen from the Smith chart where the input impedance of a lossless open-circuit line, goes through cycles when its length is varied. Not only does the input impedance not converge to a single unique value when the length of the line is increased indefinitely but also in general it is an imaginary value, i.e., a pure reactance (\( \Omega \), \( [2] \), \( [3] \)) for any length of the line, which contradicts the conclusion that the infinite line presents a real input impedance.

Now a transmission line with distributed parameters is almost identical in behaviour to a ladder network comprising lumped parameters (\( L \), \( C \), \( \omega \)), and the above paradox exists in the infinite ladder network also. A Ladder network of \( n \) blocks, with each block a symmetrical \( T \) section consisting of two \( L/2 \) inductances and a capacitance \( C \), has a characteristic impedance \( Z_0 = \sqrt{L/C - \omega^2 L^2/4} \) [1], [2], [3]. The number \( n \) of blocks could be a finite value, or it could even be infinite (\( n \to \infty \)). Figure 2(a) shows an infinite ladder network while Figure 2(b) shows a finite ladder network, but terminated in its characteristic impedance \( Z_0 = \sqrt{L/C - \omega^2 L^2/4} \).

A solution for the input impedance \( Z_i \) of the infinite network is obtained in the following manner [5], [6], [7], [8]. Since adding another block to the beginning of an infinite ladder network does not change the input impedance (it still remains the same infinite network), \( Z_i \) must equal the impedance of a circuit having a single block terminated in a load impedance equal to \( Z_i \). Therefore we have,

\[
Z_i = \frac{j\omega L}{2} + \frac{(Z_i + j\omega L/2)(1/j\omega C)}{Z_i + j\omega L/2 + 1/j\omega C},
\]

which has a solution,

\[
Z_i = \sqrt{L/C - \omega^2 L^2/4}.
\]

The input impedance of the infinite network equals its characteristic impedance, i.e., \( Z_i = Z_0 \). Now for \( \omega < \omega_0 = 2/\sqrt{LC} \), \( Z_i \) is a real value. This leads to the same paradox as for the infinite transmission line of distributed parameters – how come a circuit containing only purely imaginary impedances has for its input impedance a real value which could absorb energy continuously? And where does this energy go as it could not be dissipated in the inductors and capacitors of the circuit? For this Feynman [4] writes “But how can the circuit continuously absorb energy, as a resistance does, if it is made only of inductances and capacitances? Answer: Because there is an infinite number of inductances and capacitances, so that when a source is connected to the circuit, it supplies energy to the first inductance and capacitance, then to the second, to the third, and so on. In a circuit of this kind, energy is continually absorbed from the generator at a constant rate and flows constantly out into the network, supplying energy which is stored in the inductances and capacitances down the line.”

But there seems to be a catch here too. It was shown [6] that the input impedance in an open-circuit ladder network, consisting of blocks comprising inductors and capacitors, does not converge to a unique fixed value when additional identical blocks are added. For all driving frequencies it seems to yield only imaginary (reactive) input impedance values, even when the number of blocks is increased to infinity. This contradicts Feynman’s observation [5] above that the infinite ladder work has an input impedance which is pure resistance for \( \omega < \omega_0 \). It has been argued [9] that a non-zero real part of impedance appears only if there is a termination in an impedance that has a real part and that a circuit consisting solely of components with purely imaginary impedances has a purely imaginary input impedance. Later the behaviour of infinite ladder network, its convergence and solutions have been analyzed much more thoroughly [10], [11].

Actually while writing (11) for \( Z_i \) one implicitly assumed that the infinite series converges to a unique value and it is only under this existence supposition that a unique solution (12) could be obtained. If the series does not converge, then of course this basic assumption itself breaks down and the solution obtained thereby may not represent a true value. It seems that the infinite ladder networks of type in Fig 2(a) may have different answers for the input impedance, and thereby implying different power consumptions depending upon the method of solution. Hence a paradox exists as one gets different results with different arguments, and a question still remains whether or not does an infinite ladder network converge to a pure resistance drawing continuous power from an input source, and if so where does this energy go. What could be the missing factor, if any, in these arguments?

II. THE ABSENCE OF RADIATION LOSSES

Could the two different modes of thinking be reconciled if this energy were lost into the surrounding medium by the process of radiation, with \( Z_0 = \sqrt{L/C} \) as the radiation resistance? In transmission line or ladder network containing resistive elements, power loss by the source is fully accounted for by the energy dissipation in the circuit, for any value of \( R \) and \( G \). Consider the lossy infinite line (i.e., with \( R \) and \( G \) non-zero), where input power from the source is [4],

\[
P_i = \frac{|V_0^2|}{(2|Z_0|)} \cos(\angle Z_0) = |V_0^2/(2|Z_0|^2)| \text{Re}(Z_0). \tag{13}
\]

On the other hand the power dissipated in an infinitesimal
line element (Fig. 1) is \[ z = (V_0^2/2) \left[ (R/|Z_0|^2) + G \right] e^{-2\alpha z}dz \] 

Hence the total power dissipated in the infinite line is 

\[ P_d = (V_0^2/2) \left[ \left( R/|Z_0|^2 \right) + G \right] \int_0^\infty e^{-2\alpha z}dz \]

Substitution for \(|Z_0|\) and \(\alpha\) shows that \(P_d = P_i\) [4], and all power losses are accounted for without anything going into radiation. This is true for all \(R\) and \(G\), in particular even when in limit \(R \to 0\) and \(G \to 0\). Now it cannot happen that when \(R = 0\) and \(G = 0\) radiation suddenly shows up into picture from somewhere. Further, even in a lossless line, all the power (assumed to be lost by the source) can be consumed by terminating the line in its characteristic impedance, at whatever stage irrespective of the length of the line up to that point. This implies that up to any arbitrarily selected length of the line, the radiation losses had not taken place. Therefore for resolving this paradox there does not seem any scope for radiation hypothesis at all and a satisfactory resolution of the paradox lies elsewhere.

III. Solution of the Paradox – Incident versus Reflected Waves

The solution of the paradox lies in the realization that there is an absence of a reflected wave in an infinite transmission line or infinite ladder network.

A. The case of an infinite ladder network

In a finite open-circuit network (and thereby one not terminated in its characteristic impedance \(Z_0 = \sqrt{L/C - \omega^2L^2/4}\)), there is a reflected wave from its terminated end as it has to match the conditions for a zero net current (implying the electric current out of phase by angle \(\pi\) for the incident and the reflected waves) but the voltage being in phase there, at the reflection (termination) point. When we analyze a finite network, the voltages and currents being considered are the superposition of incident and reflected waves. Therefore the calculated \(Z_i\) may depend upon the length of the line or equivalently the number of blocks in the network as that would determine the relative phases of the incident and reflected waves at the input point AA in Fig. 2.

Now for frequencies below a critical value \(\omega_0 = 2/\sqrt{LC}\), the characteristic impedance can be written as \(Z_0 = (\omega L/2)\sqrt{(\omega_0/\omega)^2 - 1}\), which is a real quantity, meaning a pure resistance. Let us consider the propagation factor \(e^{-\gamma}\) between adjacent blocks calculated [1], [5] by terminating the ladder network in \(Z_0\). This ensures that there is no reflected wave and one is dealing only with the incident wave. In the low frequency (\(\omega < \omega_0\)) case one gets [1], [5],

\[ \frac{V_n'}{V_{n-1}'} = \frac{I_n'}{I_{n-1}'} = \frac{\sqrt{L/C - \omega^2L^2/4 - j\omega L/2}}{\sqrt{L/C - \omega^2L^2/4 + j\omega L/2}} \]

Simplifying (18) we can write,

\[ \frac{V_n'}{V_{n-1}'} = 1 - \omega^2LC/2 - j\sqrt{\omega^2LC\sqrt{1 - \omega^2LC/4}} \]

\[ = 1 - 2(\omega/\omega_0)^2 - j2(\omega/\omega_0)^2\sqrt{1 - (\omega/\omega_0)^2}. \]

A prime (') over voltages and currents is to indicate that these represent an incident wave. From the real and imaginary parts in (20), it can be readily seen that the propagation factor has a unit magnitude and represents a simple phase change \(e^{-j\beta}\), \(\cos \beta - j \sin \beta\), between successive blocks in the network.

For calculating the propagation factor of the circuit we need to isolate the incident wave by terminating this network with its characteristic impedance \(Z_0\). However the propagation properties of the incident wave, that is, the propagation constant to be calculated from (20) of incident wave between two neighbouring blocks (say, \(n - 1\) and \(n\)) does not depend upon this termination. The incident wave has an input impedance everywhere equal to the characteristic impedance \(Z_0\) of the network. By terminating the circuit in its characteristic impedance we are only ensuring the absence of the reflected wave in the circuit. Of course the voltages and currents at any point are decided by the superposition of the incident and reflected waves at that point.

When the input impedance of a network is calculated as in [6], [7], [10], [11], what one gets as the solution is for the superposition of the incident and the reflected wave with their phases duly taken into account. To prove our assertion this indeed is the case in general, we want to calculate input impedance of an open-circuit line, made of any finite number of blocks (say, \(n\)), by evaluating voltage and current at \(z = 0\) due to the sum of the incident and reflected waves, the latter arising from the termination just after the \(n\)th block.

Reflection plays a role in resolving the paradox was briefly mentioned in [11] but without much further elaboration, which we do here much more explicitly, by calculating input impedance of a finite ladder network case by a superposition of incident and reflected waves. For a cascaded network of \(n\) identical blocks, the propagation factor is simply \(e^{-j\beta}\). The angle \(\beta\) here is half of \(\theta\) in (21) of [11]. If the network has a total of \(n\) blocks, then voltage \(V_0\) at \(z = 0\) includes a reflected wave with a phase change of angle \(2n\beta\) from the incident wave, while the current \(I_0\) has a phase change of angle \(2n\beta + \pi\) (an extra phase of angle \(\pi\) in the current wave at the reflection point). Therefore the input impedance is given by,

\[ Z_i = V_0/I_0 = Z_0 \frac{1 + e^{-j2n\beta}}{1 - e^{-j2n\beta}} = -jZ_0 \cot(n\beta). \]

We see that the calculated input impedance is the same what was calculated in an alternative method for a finite open-circuit ladder network [10], [11], which thus proves our assertion that the propagation factor of the incident wave is unaffected by the termination impedance. As \(n\) is increased, \(Z_i\) is always of an imaginary value which goes through cycles, even becoming 0 or \(\infty\), and in general does not converge to a unique value even when \(n \to \infty\).
The propagation factor $e^{-\gamma}$ between adjacent blocks, for a high frequency ($\omega > \omega_0$) case can be written as,
\[
\frac{V_i'}{V_n'} = 1 - 2(\omega/\omega_0)^2 + 2(\omega/\omega_0)^2 \sqrt{1 - (\omega_0/\omega)^2}.
\] (22)

From (22) it can be seen that for $\omega > \omega_0$ the propagation factor, written as $e^{-(\alpha+i\pi)} = -(\cosh \alpha - \sinh \alpha)$, is of magnitude less than unity and is always of a negative value, implying a phase change of angle $\pi$ between successive blocks accompanied by an exponential decrease in amplitude. The voltages and currents do not penetrate too far in the circuit, and there is no continuous transport of energy along $z$. The input impedance at frequencies $\omega > \omega_0$ for a cascade network of $n$ blocks is,
\[
Z_i = V_0/I_0 = Z_0 \frac{1 + e^{-2n\alpha}}{1 - e^{-2n\alpha}} = Z_0 \coth(n\alpha),
\] (23)

which is imaginary, in spite of $\coth(n\alpha)$ being always a real value, because the characteristic impedance $Z_0 = (j\omega L/2) \sqrt{1 - (\omega_0/\omega)^2}$ is imaginary for $\omega > \omega_0$. For $n \to \infty$, $Z_i \to Z_0$, a pure reactance, thus there is no paradox for the $\omega > \omega_0$ case, and we do not pursue it any further.

With sinusoidal excitation, when the voltage source supplies energy to any LC network, the current drawn must be $\pi/2$ out of phase with the voltage, so that there is no power being absorbed from the source. It is interesting that a standing wave in even a stand–alone LC circuit driven at a frequency $\omega$, can be treated as a superposition of an incident and a reflected wave, and the input impedance $j(\omega L - 1/\omega C)$ can be calculated from that (see Appendix A). Thus even in a driven LC circuit, which is the simplest special case of a ladder network, it is the reflected wave that brings the electric power back to the source resulting in zero net power loss from the source.

No radiation losses are involved here. Suppose these were there, and some energy has been radiated up to some block $m$, and then the circuit continues up to a block $n$ ($m < n$) and the circuit is open just after the $n$th block, then all the electric power is reflected at the termination which could not have occurred if part of the energy had already been radiated. Or if just after element $n$ there is a matched resistive load, then all the power supplied from the generator will appear as Ohmic losses in the load which could not have happened if energy had been already been partially lost as radiation. Here $m$ and $n$ could be chosen arbitrarily large. Thus radiation could not provide a resolution to the paradox.

\[Z_i = Z_0 \frac{e^{n\beta} + e^{-n\beta}}{e^{n\beta} - e^{-n\beta}}, \] (26)

In general the input impedance of a line of length $l$ is given by (1).
\[
Z_i = Z_0 \left( \frac{e^{n\beta} + K e^{-n\beta}}{e^{n\beta} - K e^{-n\beta}} \right),
\] (24)

where $K$ (reflected voltage at load/incident voltage at load) is the reflection coefficient,
\[
K = \frac{Z_r - Z_0}{Z_r + Z_0}.
\] (25)

where $Z_r$ is the impedance at the receiving (load) end. The input impedance reduces to $Z_0$ when there is no reflected wave, i.e., when $K = 0$. Now the absence of a reflected wave can be due to three reasons. First, the line is finite but terminates in a load matched to the characteristic impedance of the line, i.e., when $Z_r = Z_0$. Second, the line has small resistance which can cause the incident voltage to die over its long length $l$, i.e., if $\gamma l \to \infty$, so that the amplitude and thence of the reflected wave is zero, then the series does converge to a unique solution (6), (7) which is consistent with $Z_i = Z_0$. Thirdly the line is lossless but truly of infinite extent so that it could be assumed that the incident wave, which assumedly started a finite time back, has not yet reached the termination point to start a reflected wave. In all three cases, the input impedance, which is the ratio of the voltage and current at the input point, is the same as that is not affected by what happens at its termination point, and we obtain the same result for the input impedance, viz. $Z_i = Z_0$.

On the other hand, for an open-circuit line of finite length $l$ ($Z_r = \infty$, $K = 1$), the input impedance is given by,
\[
Z_i = Z_0 \left( \frac{e^{n\beta} + e^{-n\beta}}{e^{n\beta} - e^{-n\beta}} \right).
\] (26)

Now in a lossless line, $\gamma = j\beta = j\omega \sqrt{LC}$, then the input impedance becomes,
\[
Z_i = Z_0 \left( \frac{e^{j\beta l} + e^{-j\beta l}}{e^{j\beta l} - e^{-j\beta l}} \right).
\] (27)

which is a pure reactance, and thereby no net power consumed, and which is similar to the result derived for the ladder network (21). It should be noted that in case of a ladder network, the quantities $\gamma, \alpha, \beta$, or even $L, C$ etc. are specified as per block of the circuit while in the case of a transmission line with distributed parameters all such quantities are defined per unit length of the line. Therefore in (21) it is the phase angle change $n\beta$ over $n$ blocks while in (28) it is the phase angle change $\beta l$ over length $l$ of the line. In fact with increasing $l$, $Z_i/ Z_0$ from (28) is cyclic and is indeed the value read from the Smith chart.

One thing that we notice from (28) is that the input impedance $Z_i$ depends on the length $l$ of the line in terms of wavelength $\lambda$. Thus depending upon $2\pi l/\lambda$, $Z_i$ could be zero, a finite value or even infinity, but always a pure imaginary value, with a zero real part similar to what was seen for the ladder network in IIIA. Here as much amount of power is reflected back to the generator as much it supplies in the incident wave.

In the case where there is only an incident wave, i.e., there is no reflected wave, the current is in fact in phase...
with the voltage, implying power is being drawn from the source. However, if there is a reflected wave as well, then the voltage and current are not in phase everywhere. Thus it is the absence of reflected wave in infinite transmission line that results in a continuous positive energy flux along the line. The relative phases of \( V \) and \( I \) depend upon the reflected wave, which in turn depends upon at how far point the reflection took place. Of course no reflection will ever take place in a uniform infinite line as the incident wave will never reach the termination point which is at infinity. However if we consider the lossless case when there is a reflected wave from an open-circuit termination, then equal power is being returned to the source by the reflected wave and in that case the current is indeed \( \pi/2 \) out of phase with the voltage \( 21, 28 \).

If we take a transmission line without any discontinuity then it will have to be an infinite line and the energy will be getting stored in its elements further and further along the line as electric and magnetic fields. There is no violation of the energy conservation, and when there is no reflected wave to restore the energy to the source, it may be continuously giving away energy, which gets stored in electric and magnetic fields in more and more inductances and capacitances further away along the line. Seen this way there does not appear to be any paradox.

The paradox actually had arisen only because we were comparing two sets of solutions which are for quite different situations. One involves only an incident wave (i.e., without any reflected wave) and then the input impedance \( Z_i = Z_0 \) is a real quantity, and the voltages and currents are in phase everywhere along the circuit, with energy giving apparently “spent” as it is getting stored in the inductors and capacitors down the line as the incident keeps on advancing for ever in an infinite transmission line. The other solution was for the case with a reflected wave, and there the superposition of the incident and reflected waves results in \( Z_i \) to have imaginary value with no net power loss since the source gets the energy back as the reflected wave.

IV. CONCLUSIONS

It was shown that while an open-circuit finite ladder network or a transmission line with distributed network has a characteristic impedance \( z_0 \) which is only reactive (imaginary), an infinite ladder network or an infinite transmission line has a finite real component of the input impedance. It was shown that the famous paradox of power loss in a lossless infinite transmission line is successfully resolved when one takes into account both the incident and reflected waves. The solution of the paradox lies in the realization that there is an absence of a reflected wave in an infinite transmission line. In a finite transmission line or ladder network, the source still keeps on supplying power as an incident wave but gets it equally back in terms of the reflected wave. Therefore there is no further net power transfer from the source which is consistent with the reactive elements presenting zero net resistance. However in the case of an infinite ladder network or an infinite transmission line there is no discontinuity to start a reflected wave, thus the source supplies power in a forward direction, but does not get it back in terms of a reflected wave from the termination point. Therefore there is an apparent net power loss, which actually appears as stored energy in its reactive elements (capacitances and inductances) further down the line. It was also shown that radiation plays absolutely no role in resolving this paradox.

APPENDIX A

INPUT IMPEDANCE OF A DRIVEN LC CIRCUIT COMPUTED FROM A SUPERPOSITION OF INCIDENT AND REFLECTED WAVES

Here we explicitly demonstrate that a driven LC circuit can be treated as an open-circuit 1-block ladder network having incident and reflected waves and from their superposition, the voltages and currents, and in particular, input impedance of the LC circuit can be calculated for all driving frequencies. We denote by \( V_0, I_0 \) and \( V_1, I_1 \) the voltages and currents at the input (AA) and termination (BB) respectively, and which (Fig. 3(a)) are related by \( V_0 - V_1 = j\omega L I_0/2 \), \( I_0 = j\omega CV_1 \), where \( \omega \) is the frequency at which the circuit is being driven by, say, a generator at the input end AA. The input impedance \( Z_i = V_0/I_0 \) is given by,

\[
Z_i = j\omega L/2 + 1/(j\omega C).
\] (29)

Denoting voltages and currents for the incident and reflected waves by \( V', I' \) and \( V'' \), \( I'' \) respectively, the boundary conditions at open end BB in Fig. (3a) imply \( V''_0 = V''_1 \) and \( I''_0 = -I''_1 \), the minus sign arising because the reflected current is out of phase with the incident wave by an angle \( \pi \), so as to make the net current \( I_1 = I'_1 + I''_1 = 0 \). However to evaluate \( I'_1 \), we need to isolate the incident wave and which can be done by terminating the circuit in its characteristic impedance \( Z_0 \) (Fig. 3(b)). The propagation factor for the incident wave from [19] is,

\[
\frac{V'_0}{V'_1} = \frac{I'_0}{I'_1} = 1 - \omega^2 LC/2 - j\sqrt{\omega^2 LC\sqrt{1 - \omega^2 LC}/4},\]

(30)

with \( V'_0/I'_0 = V'_1/I'_1 = Z_0 \). As demonstrated in IIIA, incident wave is not attenuated, irrespective of the termination impedance. The only difference is that there is also a reflected wave in the open circuit case (Fig. 3(a)), while there is no reflected wave when the circuit is terminated in its characteristic impedance \( Z_0 \) (Fig. 3(b)).

For the reflected wave in Fig. 3(a) one can write the propagation factor as,

\[
\frac{V''_0}{V''_1} = \frac{I''_0}{I''_1} = 1 - \omega^2 LC/2 - j\sqrt{\omega^2 LC\sqrt{1 - \omega^2 LC}/4}.
\] (31)
Equation (30) can be rewritten as,
\begin{align*}
\frac{V_0'}{V_1'} &= \frac{I_0'}{I_1'} = 1 - \frac{\omega^2 LC}{2} + j\sqrt{\frac{\omega^2 LC}{1 - \frac{\omega^2 LC}{4}}}.
\end{align*}
(32)

From (31) and (32) we get for the voltage \( V_0 \) and current \( I_0 \) as the superposition of the incident and reflected waves,
\begin{align*}
V_0 &= V_0' + V_0'' = 2V_1'(1 - \frac{\omega^2 LC}{2}) \\
I_0 &= I_0' + I_0'' = 2I_1'j\sqrt{\frac{\omega^2 LC}{1 - \frac{\omega^2 LC}{4}}}
\end{align*}
(33) (34)

Therefore we get the input impedance \( Z_i \) as,
\begin{align*}
Z_i &= \frac{V_0}{I_0} = \frac{V_1'}{I_1'} \frac{1 - \frac{\omega^2 LC}{2}}{j\sqrt{\frac{\omega^2 LC}{1 - \frac{\omega^2 LC}{4}}}}
\end{align*}
(35)

Using \( Z_0 = \sqrt{\frac{L}{C} - \frac{\omega^2 L^2}{4}} = \sqrt{\frac{L}{C} \sqrt{1 - \frac{\omega^2 LC}{4}}} \), we get \( Z_i = j\frac{\omega L}{2} + \frac{1}{(j\omega C)} \), which of course is the expected result (29). The input impedance is imaginary for all driving frequencies.

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