Mathematical abstraction: students’ concept of triangles

E E Hutagalung*, E Mulyana and T R Pangaribuan
Departemen Pendidikan Matematika, Universitas Pendidikan Indonesia, Jl. Dr. Setia Budhi No. 229, Bandung 40154, Indonesia

*Corresponding author’s email: chutagalung62@gmail.com

Abstract. Mathematical abstraction is a cognitive process when students construct their mathematics knowledge and the result of the process itself. This study explores the mathematical abstraction of students related to the concept and the area of triangles. For this reason, firstly, the researchers developed four questions. This study was conducted in one junior high school in Bandung. Twenty-three students, grade 7 (12-14 years old), were involved in this study. Data were collected through observation of the test results and followed by students interviews. Finally, data were analyzed qualitatively. The researchers found that the low-achievement students tend to understand the triangles which consist of the interior area bounded by three straight lines and the lines themselves; they also tend to understand the area of triangles as the result of the pattern of a base times height divided into two. Besides, there are indications that low-achievement students need help from teachers so that they can achieve a higher level of abstraction.

1. Introduction
It is widely acknowledged that Mathematics has an abstract object, such as the sequence of functions. Besides that, students complain about the subject is “too abstract” in learning advanced mathematics. Here, we will discuss what abstract is in this introduction.

Wilensky asserts that whether an object is abstract or concrete does not depend on the attribute of the objects but rather on the property of an individual’s relationships to the objects [1]. A child, about 6 years old, can do 2+8, 8+7, 15+6, 22+3, and so forth, without help. But he has difficulty when asked to add 3+17, 2+15, even 6+15. It is because he adds the first number as much as the leap in the second number from zero. For example, 2+4=6 is obtained by sorting numbers from 2, 3, … up to four leaps. His failure to add 6+15 because the working memory who can’t accommodate up to fifteen leaps. In this case, 15+6 is concrete, but 6+15 is an abstract thing. This child is not familiar to the additional properties, such as commutative, so that 6+15 becomes abstract.

Another example, when readers read this paper, the first thing presented is abstract. After reading the abstract, readers will get a description of this paper. Then, the abstract will become clearer (the less abstract) when readers read this paper completely. Those examples prove Hazzan’s and Zazkis’s statement who notes that “the closer a person to an object and the more connections he/she has formed to it, the more concrete he/she feels about it.”[1,2].

Abstraction comes from the Latin abstractum which means drawn out [3]. Some writers state that abstraction is a process and some others state that abstraction as an object. But, we found nothing about the writers who state that the abstraction is a process to deny the abstraction as an object and vice versa. We agree with the notion that abstraction has two meanings, namely as a process and as a concept (the
abstraction) output by its process [4]. Because, first, according to Dreyfus, “an abstraction, to most mathematicians, is an object which incorporates a structure–elements and relationships between them–common to many instances appearing in diverse context.” [2]. Second, according to Tall, the students’ abstraction arises by observed the characteristics of the object. Thus, in this paper, the abstraction seen as a process and also as an output of its process [5].

Abstraction related to vertical mathematisation by Hans Freudenthal. He writes that “…in other one symbols are shaped, reshaped, and manipulated, mechanically, comprehendingly, reflectingly; this is vertical mathematisation.” [6]. In mathematics learning, students recognize several objects, situations, or experiences based on similarities that distinguish from others [3]. These similarities were formed into a new scheme in the student’s mind. When the students are forming a new scheme, they are actually shaping, reshaping, or modifying the existing scheme. It is in line with the meaning of abstraction by Gray and Tall. He writes that “abstraction is the process of drawing from the situation the thinkable concept (the abstraction) under construction.” [4].

In other terms, Gray and Tall use the term compression to describe a mechanism that stores information that depends on a phenomenon who is to name as a word or phrase. Furthermore, some of those phenomena were called as a thinkable concept so that they could be discussed and thought [4]. When the children are forming new concepts based on experience (abstraction), they are actually making a connection between thinkable concept.

Piaget introduces two kinds of abstraction in mathematics and science learning, namely empirical abstraction and reflective abstraction [2, 7]. Skemp’s conception about abstraction consists of the recognition of similarities and followed by realizing these similarities into new mental object [8]. It implies that Skemp’s conception based on experience. Therefore, it is also called as empirical abstraction [8]. Thus, empirical abstraction is a person’s activity when forming a new concept based on experience and observation. This view relates to Herschkowitz, Schwarz, and Dreyfus [9] statement’s who claim that abstraction depends on personal experience.

Mitchelmore and White argue that abstraction in mathematics is a very significant topic in creating effective mathematical learning, including triangles [8]. Research about abstraction in geometry has been carried out in Indonesia by using several approaches and context, such as Nurhasanah et.al [10], Dewi et.al [11], Annas et.al [12], Fitriani et.al [13], and the others. This study will describe students abstraction of low-achievement in the concept and the area of triangles.

By notice that mathematical abstraction will always talk about the mental activities of the students when they are forming new concepts or knowledge through their experiences, this study seeks answers to the question “how is the mathematical abstraction of low-achievement students in the concept and the area of triangles? and what are the abstraction phenomena that arise when students construct their understanding of the concept and the area of triangles?”.

2. Method
This study attempted to describe mathematical abstraction of students who related to the concept and the area of triangles. To address this aim, we conducted a descriptive study in which an individual written test on triangles, followed by student interviews. The subject of this study was 23 students of grade 7 (12-13 years) in Bandung City who had low-achievement. We obtained the low-achievement students from the average of the mark of the last three mathematics test (i.e. lines and angles, triangle, and quadrilaterals) of 98 students grade 7. First, the marks transformed to z-score and then the low-achievement students were the students who have z-score average less than −1 (as 1 was its deviation standard).

Then, twenty three students were given the written test and 4 of them followed by an interview. The interviewee was identified as S2, S3, S4, and S5 who obtained purposively. We analyzed data from the test and the interview qualitatively to find out the mathematical abstraction students with low-achievement in the concept of triangles and the area of triangles.
3. Result and Discussion

3.1. Abstraction in the concept of triangle

When low-achievement students asked to choose a figure which is a triangle, along with the reason, between the two figures in Figure 1, we found that all low-achievement students can choose the correct figure that was a triangle. Based on this information, low-achievement students use the relevant attribute visually, such as angles and straight sides of triangles. Tall stated that visual can help students build mental representation and students engagement in abstraction [14]. However, this study shows that low-achievement students need help, even though they can choose triangle visually, from the teacher to bring students to a more advanced level.

Then, nineteen students write that the sides of the triangle must a straight line, three students write that the triangle must have three angles and the sides are straight lines. Therefore, it can also be revealed that low-achievement students can represent mathematical object with incomplete language. As noted by Horne and Watson, “the language the children used about their reasons…showing that in spite of aspects of their knowledge their attention was still governed by limited visual images.” [15].

![Figure 1. S2 answer’s on the question 1](image)

Through interviews, we confirm the student’s abstraction of the concept of triangle as follows,

| P | …in the number 1, you choose that figure 2 is not triangles, is it? |
| S4 | Which one Mr.? |
| P | This one (pointing S4 answer on the number 1) |
| S4 | Yes, It is not straight lines. |
| P | Ok, It means that the figure 1 is triangles, Isn’t it? |
| S4 | Yes. Because the line is straight. |
| P | So, the triangles is only the line or triangle is the lines within the interior area? |
| S4 | Within the interior area |
| P | Ohh..Now, look at this paper (showing a piece of paper which is treangle shape. Is this a triangles? |
| S4 | Yes. Because the edge is straight. |
| P | (Continue the discussion to inform the concept of triangle) |

The same thing was expressed by the other subject, that initially, low-achievement students tend to interpret the triangle as an area (interior area) which bounded by a straight line. In fact, according to Mulyana, triangle consists of three different line segments where the endpoints of the line segment coincide with the base point of another line segment [16]. Although mathematical language expressed by low-achievement students has a word line as expressed by Mulyana, we presume that low-achievement students can’t be said to have the right definition of the triangle even for themselves. This is in accordance with the result of Annas’s et.al study which states that low-group students tend to define triangles incompletely and they still use routine attributes. It means that low-group students define the concept of triangles visually.
Thus, there are three kinds of abstraction low-achievement students about the concept of triangles, namely the sides of triangles are straight lines, triangles must have three angles and sides, and triangles consist of the area (interior area) which bounded by straight lines and the lines themselves. Then, low-achievement students only can create relationships between the process that have been experienced by students to form a new understanding of triangles through intervention from the teacher. As in this study, even though the subject was able to communicate that the triangles have straight lines, they were unable to use it into new situations, as Mitchelmore and White noted that apply the concept in new situations is one of the teachings for abstraction [8]. Based on this information, we notice that low-achievement students interpret the triangles as an area (interior area) bounded by the straight line.

3.2. Abstraction in the area of triangles

Previously, students with low-achievement tend to understand triangles consisting of the interior area of triangles, not consisting of lines. But, their abstraction does not relevant to the abstraction in the area of triangles. From the interview, low-achievement students tend to differentiate the area of a triangle with the perimeter based on the formula. Students with low-achievement tend not to understand that the area of triangles as the area (interior area) bounded by triangles, but as a half of base times height. It means that they need help with creating relationships between processes to form a new understanding of the area of triangles.

When low-achievement students ask for determining the area of triangles, they did not have an abstraction that the characteristics of the relevant changes in the area are base and height. Of the 23 students who were asked to solve a simple question (calculating the area of the right-angled triangle), only one student who couldn’t solve this question because he doesn’t remember the formula. Meanwhile, the rest of the students can solve the question correctly. Through the interviews with four students, only S4 and S2 can produce that the height and the base of the triangle are perpendicular. As for S3 and S5, they need a little intervention to create a connection between the process that is being experienced to become a new understanding of the base and height.

Figure 2. S3 answer’s on the question 3

Figure 2 shows that S3 identified the height of $\triangle DBC$ incorrectly. When interviewed, S3 stated that AD is the height of $\triangle DBC$ because it is a transversal line for AB and DC lines. Based on this information, we presume that S3 can not bring the abstraction that already they have about the height and base when solving problems because S3 states that the height of triangles is always in the interior area or one of the sides of the triangles. We agree with Dewi et.al which states that students can’t carry out the abstraction process when solving problems [11].

Figure 2 represents one of the types of failures of low-achievement students solving problems in the area of triangles. During the study, as mentioned before, low-achievement students can state that the height and base are perpendicular but they tend not to be able to use their knowledge when faced a complex figure. For example, when S2 completes the problem in Figure 2, S2 draws a line from D to
line BC and claims the distance is 12. From the interview, it was found that S2 could not identify which lines were perpendicular. In other words, low-achievement students can’t separate which lines are needed and not to determine the area of \(\triangle DBC\). As stated by Verschaffel et.al, that weaker student has difficulty with abstraction into sub-problem where they are not able to sort out relevant and irrelevant information [17].

Figure 3 shows that S5 also cannot bring the abstraction that already she has about height and base. In this question, students have to use 24 or 10 as a base or height, but S5 did not think about it. Then, it can be stated that S5 has also not succeeded in creating a relationship between her understanding into a new understanding of the height and base of triangles. The other students (S2, S3, and S4) also do the same thing. Besides that, there is the data was found that low-achievement students use numbers 90 or 180 to replace an unknown variable in the triangles when they are solving the problems. It is due to the tendency of the students to assume that the base is a horizontal line on the triangles.

In this study, even though low-achievement students have been able to know or communicate that the height and the base are perpendiculars, they have difficulty making a new scheme by modifying their existing scheme. It can be seen from the of the students to use a horizontal line as a base firstly. So, S5 and S2 claim the value of KN as 90; S4 claims it to be 8, and S3 claimed to be \(\sqrt{96}\).

Overall, we presume that low-achievement students have not been able to make connections between thinkable concept (for the students) because they have not compressed their ideas become a new understanding or another thinkable concept with the higher level of abstraction. Then, we also presume that there are indications that students with low-achievement need individual guidance from the teacher so that students are able to recognize, organize, and construct the knowledge that they already have into new knowledge. We propose that the need itself are to distinguish triangles and area (interior area) of triangles; draw the height of triangles with the different bases; ensure students to have knowledge about perpendicular lines so that they can choose which segment lines are perpendicular easily; and ensure students to pay attention that the first condition needed to be fulfilled about the base and the height is perpendicular.

4. Conclusion
This study has presented several aspects of the abstraction of low-achievement students in the concept and the area of triangles. Students with low-achievement tend to understand the triangles which consist of the interior area bounded by three straight lines and the lines themselves. They also tend to understand the area of triangles as the result of the pattern of a base times height divided into two. Then, there are indications that low-achievement students need help from teachers so that they can achieve a higher level of abstraction in the aspect of creating relationships between the process to form a new understanding and removing irrelevant attributes from an object. Besides, we also suggest that teachers
have to ensure students understanding of lines and angles is well-owned so that students do the abstraction in the concept and the area of triangles.

5. References

[1] Hazzan O and Zazkis R 2005 Reducing abstraction: the case of school mathematics Educ. Stud. Math. 58 1 pp. 101–119
[2] Dreyfus T 2014 Abstraction in mathematics education Encyclopedia of Mathematics Education ed S Lerman London: Springer pp. 5–8
[3] Mitchelmore M C and White P 2012 Encyclopedia of the Science Learning, ed N M Seel London: Springer pp. 31–33
[4] Gray E and Tall D 2007 Abstraction as a natural process of mental compression Math. Edu. Re. J. 19 2 pp. 23–40
[5] Tall D 2002 Advanced mathematical thinking, ed D Tall London: Kluwer Academic Publishers Group pp 11
[6] Freudenthal H 2002 Revisiting mathematical education London: Kluwer Academic Publisher 9 pp. 41–42
[7] Scheiner T 2016 New light on old horizon: constructing mathematical concepts, underlying abstraction processes, and sense making strategies Educ. Stud. Math. 91 2 pp. 165 – 183
[8] Mitchelmore M and White P 2007 Abstraction in mathematics learning Math. Educ. Res. J. 19 pp. 1–9
[9] Hershkowitz R, Schwarz B B and Dreyfus T 2001 Abstraction in context: epistemic actions J. Res. Math. Edu. pp. 195–222
[10] Nurhasanah F, Kusumah Y S, and Sabandar J 2017 Concept of triangle: examples of mathematical abstraction in two different context Int. J. Emer. Math. Edu. 1 1 pp. 53–70
[11] I Dewi, N Siregar, and A Andriani 2018 The analysis of junior high school students’ mathematical abstraction ability based on cultural wisdom J. Phys.: Conf. Ser. 1088 012076
[12] Annas S, Djadir, and Hasma S M 2018 The Abstraction ability in constructing relation within triangles by the seventh grade students of junior high school J. Phys.: Conf. Ser. 954 012029
[13] Fitriani N, Suryadi D, and Darhim D 2018 Analysis of mathematical abstraction on concept of a three dimensional figure with curved surfaces of junior high school students J. Phys.: Conf. Ser. 1132 012037
[14] Yilmaz R and Argun Z 2018 The role of visualization in mathematical abstraction: the case congruence concept Int. J. Edu. Math. Sci. Tech. 6 1 pp. 41–57
[15] Horne M and Watson K 2008 Developing understanding of triangle, International Group for the Psychology of Mathematics Education 3 2 113–120
[16] Mulyana E 2016 Geometri: untuk siswa dan guru Bandung: Rizqi Press
[17] Verhoeof N C and Broekman H G B 2008 Handbook of Mathematics Teaching Research: Teaching Experiment–A tool for Teacher-Researchers, ed B Czarnocha.

Acknowledgments
Praise the Lord who blessed the process of this study so that the study runs well and we also would like to thank the teachers and the students for their help while the data were collected. We hope that the result will make a contribution in mathematics classrooms.