ON THE INERTIA CONJECTURE
AND ITS GENERALIZATIONS

BY

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ABSTRACT

We study the two-point branched Galois covers of the projective line given by explicit equations and with prescribed branched data. We also obtain several results using formal patching techniques that are useful in realizing new Galois covers. As a consequence, we prove the Inertia Conjecture for the alternating groups $A_{p+1}$, $A_{p+3}$, $A_{p+4}$ when $p \equiv 2 \pmod{3}$ is a odd prime and for the group $A_{p+5}$ when additionally $4 \nmid (p + 1)$ and $p \geq 17$. We also pose a general question motivated by the Inertia Conjecture and obtain some affirmative results. A special case of this question, which we call the Generalized Purely Wild Inertia Conjecture, is shown to be true for the groups for which the purely wild part of the Inertia Conjecture is already established. We show that if this generalized conjecture is true for the groups $G_1$ and $G_2$ which do not have a common quotient, then the conjecture is also true for the product $G_1 \times G_2$.

1. Introduction

Let $k$ be an algebraically closed field of characteristic $p > 0$ and $U$ be a smooth connected affine $k$-curve. In general, the full structure of the étale fundamental group $\pi_1(U)$ is not known. One interesting problem is to understand the set $\pi_A(U)$ of isomorphism classes of the finite (continuous) group quotients of $\pi_1(U)$, or equivalently the finite groups that occur as the Galois groups of

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Galois étale connected covers of $U$. In 1957, Abhyankar conjectured (known as Abhyankar’s Conjecture on the affine curves, [1], [13, Conjecture 3.2]; now a theorem due to Serre, Raynaud and Harbater) on which groups occur in the set $\pi_A(U)$. In particular, this shows that $\pi_A(\mathbb{A}^1)$ is the set of isomorphism classes of the quasi $p$-groups, i.e., the groups $G$ which are generated by their Sylow $p$-subgroups. Given $G \in \pi_A(U)$, the next natural problem is to study the inertia groups that occur over the points in $X - U$. By [20, Chapter IV, Corollary 4], any inertia group must be of the form $P \rtimes \mathbb{Z}/m$ for a $p$-group $P$ and a coprime to $p$ integer $m$. Studying the branched covers of $\mathbb{P}^1$ using explicit equations, Abhyankar proposed the following conjecture, known as the Inertia Conjecture (IC).

**Conjecture 1.1 (IC, [3, Section 16]):** Let $G$ be a finite quasi $p$-group. Let $I$ be a subgroup of $G$ which is an extension of a $p$-group $P$ by a cyclic group of order prime-to-$p$. Then there is a connected $G$-Galois cover of $\mathbb{P}^1$ étale away from $\infty$ such that $I$ occurs as an inertia group at a point over $\infty$ if and only if the conjugates of $P$ in $G$ generate $G$.

A special case of the above conjecture is when $I$ is a $p$-group. This is known as the Purely Wild Inertia Conjecture (PWIC).

**Conjecture 1.2 (PWIC):** Let $G$ be a finite quasi $p$-group. A $p$-subgroup $P$ of $G$ occurs as the inertia group at a point above $\infty$ in a connected $G$-Galois cover of $\mathbb{P}^1$ branched only at $\infty$ if and only if the conjugates of $P$ in $G$ generate $G$.

If a $G$-Galois cover exists as in Conjecture 1.1, then it is well known that the conjugates of the $p$-subgroup $P$ in $G$ generate $G$. This follows from the fact that the tame fundamental group of the affine line is trivial. The other direction of the conjecture is proved to be true only in a few cases (see [5], [17], [16], [15], [6] and [13] for more details) and even the PWIC remains wide open at this moment. Using a formal patching technique ([8, Theorem 2]), Harbater has shown that the PWIC is true when $P$ is a Sylow $p$-subgroup of $G$. One particular class of groups of interest are the alternating groups. It is known that the IC is true for the $A_p$ when $p \geq 5$ ([5, Theorem 1.2]) and for the $A_{p+2}$ when $p \equiv 2 \pmod 3$ is an odd prime ([16, Theorem 1.2]).

In whatever follows, we assume that $p$ is an odd prime.

One of our main results in this paper is the following.
Theorem 1.3: When \( p \equiv 2 \pmod{3} \) is an odd prime, the IC is true for the groups \( A_{p+1}, A_{p+3} \) and \( A_{p+4} \). When \( p \equiv 2 \pmod{3} \), \( 4 \nmid (p + 1) \) and \( p \geq 17 \), the IC is true for \( A_{p+5} \).

The result for \( A_{p+1} \) is of special interest since till now there was no example of an \( A_{p+1} \)-Galois étale cover of the affine line such that the tame part of the inertia group at a point above \( \infty \) is non-trivial. The strategy of the proofs is to first identify the inertia groups that are needed to be realized over \( \infty \) for the corresponding Galois étale covers of the affine line. Using Abhyankar’s Lemma ([18, XIII, Proposition 5.2]) and Lemma 5.2 we reduce this list of potential inertia groups. Then we construct two-point branched covers of \( \mathbb{P}^1 \) given by some explicit affine equations (Section 3) to resolve these cases.

In this context, Proposition 3.11 is an interesting, useful result that enables us to construct general covers of \( \mathbb{P}^1 \) with prescribed ramification over 0 and \( \infty \) (tamely ramified over 0 and wildly ramified over \( \infty \)). Under an assumption (Assumption 3.8) on the existence of solutions to a system of equations, these covers are étale away from \( \{0, \infty\} \), and except for a few exceptions, \( S_d \) or \( A_d \) occur as the Galois groups.

In the next part of this article, we study the Galois étale covers of an affine \( k \)-curve, in general. We pose some general questions (see Section 6) related to their ramification in this direction, and try to answer them in specific cases (see Section 9). Reserving the details for the later sections, we mention our Conjectures which are directly motivated by the IC or the PWIC. The following conjecture (the Generalized Inertia Conjecture or the GIC) generalizes the IC to the case of multiple branched points.

Conjecture 1.4 (GIC, Conjecture 6.7): Let \( r \geq 1 \) and \( G \) be a finite quasi \( p \)-group. For \( 1 \leq i \leq r \) let \( I_i \) be a subgroup of \( G \) which is an extension of a \( p \)-group \( P_i \) by a cyclic group of order prime-to-\( p \) such that \( G = \langle P_1^G, \ldots, P_r^G \rangle \). Let \( B = \{x_1, \ldots, x_r\} \) be a set of closed points in \( \mathbb{P}^1 \). Then there is a connected \( G \)-Galois cover of \( \mathbb{P}^1 \) étale away from \( B \) such that \( I_i \) occurs as an inertia group above the point \( x_i \) for \( 1 \leq i \leq r \).

When the inertia groups are the \( p \)-groups, we pose the Generalized Purely Wild Inertia Conjecture (GPWIC, Conjecture 6.10). We see that for the groups for which the PWIC is already known to be true, the GPWIC is also true. Namely, we prove the following result.
Theorem 1.5 (Corollary 7.7): Let $G$ be a quasi $p$-group, and $P_1, \ldots, P_r$ be $p$-subgroups of $G$ for some $r \geq 1$ such that

$$G = \langle P_1^G, \ldots, P_r^G \rangle.$$ 

Let $B := \{x_1, \ldots, x_r\}$ be a set of closed points in $\mathbb{P}^1$. There is a connected $G$-Galois cover of $\mathbb{P}^1$ that is étale away from $B$ and $P_i$ occurs as an inertia group above $x_i$ where $G$ is one of the following groups:

1. $G$ is a $p$-group;
2. $G$ has order strictly divisible by $p$;
3. $G = G_1 \times \cdots \times G_u$ where each $G_i$ is either a simple alternating group of degree coprime to $p$ or a $p$-group or a simple non-abelian group of order strictly divisible by $p$.

We also show that the GPWIC holds for certain product of groups if it holds for individual groups. This generalizes [15, Corollary 4.6].

Theorem 1.6 (Theorem 7.5): Let $G_1$ and $G_2$ be two finite quasi $p$-groups such that they have no non-trivial quotient in common. If the GPWIC is true for the groups $G_1$ and $G_2$, then the GPWIC is also true for $G_1 \times G_2$.

The structure of the article is as follows. In Section 3, we introduce covers given by some explicit general affine equations and study their ramification behaviour. These covers will play a vital role in our study for the IC. In Section 4, we introduce and recall some results on constructing covers using formal patching techniques. These important results together with the covers obtained using explicit equations allow us to construct the new classes of covers of curves, whose existence were previously unknown. Section 5 is devoted to the new evidence (Theorem 1.3) towards the IC for the alternating group covers. In Section 6, we introduce some general questions and conjectures that generalize the IC to a great extent. In Sections 7 and 8, we see results in the support of the GPWIC. Finally, Section 9 contains our results towards the GIC and the aforementioned general questions.

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2. Notation and conventions

We fix the following notation throughout this article.

(1) Let $p$ be an odd prime and $k$ be an algebraically closed field of characteristic $p$.

(2) All the $k$-curves considered will be smooth connected curves, unless otherwise specified.

(3) All the Alternating and Symmetric groups considered will be of degree $\geq 5$.

(4) For a finite group $G$, let $p(G)$ denote the subgroup of $G$ generated by all the Sylow $p$-subgroups of $G$.

(5) For a finite group $G$ and a subgroup $H \subset G$, let $H^G$ denote the set of conjugacy classes of $G$ and $\langle H^G \rangle$ be the subgroup of $G$ that is generated by all the conjugates of $H$ in $G$.

Definition 2.1: For a finite group $G$ and a subgroup $I \subset G$, we say that the pair $(G, I)$ is **realizable** if there exists a connected $G$-Galois cover of $\mathbb{P}^1$ branched only at $\infty$ such that $I$ occurs as an inertia group above $\infty$.

In the above definition, $G$ is necessarily a **quasi $p$-group**, i.e., $G = p(G)$.

Definition 2.2: Let $n$ be coprime to $p$. The **$[n]$-Kummer cover** is the unique connected $\mathbb{Z}/n$-Galois cover $\psi: Z \cong \mathbb{P}^1 \to \mathbb{P}^1$ étale away from $\{0, \infty\}$ over which the cover is totally ramified. Let $\phi: Y \to \mathbb{P}^1$ be a connected $G$-Galois cover. Let $W$ be a dominant component in the normalization of $Y \times_{\mathbb{P}^1} Z$. We say that the cover $W \to Z$ is **obtained by a pullback** by the $[n]$-Kummer cover.

3. Constructing covers by explicit equations

A powerful tool to construct finite étale covers of $\mathbb{A}^1$ is via explicit affine equations. This technique is prominent in the works of Abhyankar; see [13, Section 4] for examples of such covers. We are interested in the cases where the Galois groups are Alternating or Symmetric groups of degree $d \geq p$. Such a cover will be constructed as the Galois closure of a finite degree-$d$ cover $\mathbb{P}^1 \to \mathbb{P}^1$, étale away from $\{0, \infty\}$. Any finite cover $\mathbb{P}^1_y \to \mathbb{P}^1_x$ is given by an affine equation of the form $xf(y) - g(y) = 0$ for some polynomials $f(y)$ and $g(y)$ in $k[y]$ such that they do not have any common zero. We impose conditions on these polynomials so that the resulting cover has the desired inertia and Galois groups. The main
results of this section will be implemented in the later parts and in particular,
to prove the Inertia Conjecture for certain alternating groups (cf. Section 5).
We start with the group theoretic results associated to the study of such covers.

Let $d \geq p$ and $\tau$ be a $p$-cycle in $S_d$. By [6, Proposition 2.1], for any transitive
quasi $p$-subgroup $G \subset S_d$ containing $\tau$, the normalizer $N_G(\langle \tau \rangle)$ is given by

$$N_G(\langle \tau \rangle) = (\langle \tau, \theta \rangle \times \text{Sym}(\{1, \ldots, d\} \setminus \text{Supp}(\tau))) \cap G$$

where $\text{Supp}(\tau) \subset \{1, \ldots, d\}$ is the support of $\tau$ and $\theta \in \text{Sym}(\text{Supp}(\tau)) \cap N_{S_d}(\langle \tau \rangle)$
is an element of order $p - 1$ such that the conjugation by $\theta$ is a generator
of $\text{Aut}(\langle \tau \rangle)$. Arguing as in the proof of [13, Proposition 4.16], we obtain the
following result.

**Lemma 3.1:** Let $d \geq p$ and $G$ be a transitive subgroup of $S_d$. Let $\phi: Z \to X$ be a $G$-Galois cover of smooth connected projective $k$-curves and $x \in X$ be a closed point. Let $I$ be an inertia group over $x$. Consider the subgroup $S_{d-1}$ of $S_d$ fixing the element 1. Set

$$H := G \cap S_{d-1}.$$  

The $G$-Galois cover $\phi$ factors as a composition of a connected $H$-Galois cover $Z \to Y$ followed by a degree-$d$ cover $\psi: Y \to X$. Then the set $\psi^{-1}(x) \subset Y$ is in a bijective correspondence with the set of orbits of the action of $I$ on $\{1, \ldots, d\}$. Moreover, for a point $y \in \psi^{-1}(x)$, the ramification index of $y$ over $x$ is given by the length of the corresponding orbit.

**Lemma 3.2:** Under the hypothesis of Lemma 3.1, we have the following.

1. Suppose that $\psi^{-1}(x)$ consists of $s$ points with the ramification indices $n_1, \ldots, n_s$ where each $n_i$ is coprime to $p$ and $\sum_{i=1}^{s} n_i = d$. Then $\phi$ is tamely ramified over $x$. If $\gamma$ is a generator of $I$, a disjoint cycle decomposition of $\gamma$ in $S_d$ consists of $s$ cycles of length $n_1, \ldots, n_s$.

2. If $d = p$, and $\psi^{-1}(x)$ consists of a unique point with the ramification index $p$, then $I$ is of the form

$$I = \langle \tau \rangle \rtimes \langle \theta^i \rangle$$

for a $p$-cycle $\tau$ and $1 \leq i \leq p - 1$. If $d \geq p + 1$ and $\psi^{-1}(x)$ consists of $r + 1$ points with the ramification indices $p, m_1, \ldots, m_r$ where all $m_i$ are coprime to $p$ and $\sum_{i=1}^{r} m_i = d - p$, then $I$ is of the form

$$I = \langle \tau \rangle \rtimes \langle \theta^i \omega \rangle$$
for a $p$-cycle $\tau$, $1 \leq i \leq p - 1$ and an element $\omega \in \text{Sym}(\{p + 1, \ldots, d\})$ having a disjoint cycle decomposition consisting of $r$ cycles of length $m_1, \ldots, m_r$. Here $\theta$ is as in Equation (3.1).

Proof. Under the hypothesis of (1), it is well known (also can be proved using the argument for (2) below) that the cover $\phi$ is tamely ramified over $x$. Then the structure of a generator $\gamma$ of $I$ is a direct consequence of Lemma 3.1.

Now we prove (2). By Lemma 3.1, the points in $\psi^{-1}(x)$ are in a bijective correspondence with the $I$-orbits and the ramification indices at these points are equal to the corresponding orbit lengths. Then our hypotheses imply that there is a unique point $y \in \psi^{-1}(x)$ having ramification index $p$, and let $z \in Z$ be a point lying above $y$ such that $I$ is the inertia group at $z$. We first claim that $p^2 \nmid |I|$. When $d = p$, $G \subset S_p$, and this is immediate. So let $d \geq p + 1$, and assume on the contrary. Let $\tau \in p(I) \cap H$ be an element of order $p$. Let $y' \in \psi^{-1}(x)$, $y' \neq y$. The ramification index at $y'$ over $x$ is coprime to $p$. So for $z' \in \phi^{-1}(x)$ that lies over $y'$, if $I'$ is the inertia group at $z'$ over $x$, we have $\tau \in p(I') \cap H$. As the inertia groups at the points in $\phi^{-1}(x)$ are conjugate to each other, we conclude that $g^{-1}\tau g \in H$ for all $g \in G$. But since $G$ acts transitively on $\{1, \ldots, d\}$, there is a $g \in G$ such that $g^{-1}\tau g$ does not fix 1, a contradiction.

Since $p$ divides $|I|$, $p(I)$ is a $p$-cyclic group generated by an element $\tau_1 \cdots \tau_a$ of order $p$, where the $\tau_i$ are disjoint $p$-cycles in $S_d$. By [6, Proposition 2.1],

$$I = \langle \tau, \theta_i, \sigma \rangle \times \langle \omega \rangle$$

for some $1 \leq i \leq p - 1$, $\omega \in \text{Sym}(\{1, \ldots, d\} - \text{Supp}(\tau))$ and $\sigma \in \text{Sym}($\text{Supp}(\tau)) of order prime-to-$p$ and $\sigma$ acts on $\tau$ via conjugation as follows: for $1 \leq i \leq a$,

$$\sigma \tau_i \sigma^{-1} = \tau_j$$

for some $1 \leq j \leq a$. So by Lemma 3.1, the fibre $\psi^{-1}(x)$ consists of points with the ramification indices $a_1p, \ldots, a_t p$, $u_1, \ldots, u_{t'}$, where the $a_\nu$ and the $u_\eta$ are coprime to $p$.

So if $d = p$ and $\psi^{-1}(x)$ consists of a unique point with the ramification index $p$, then $\tau$ must be a $p$-cycle, and $I$ is of the form $I = \langle \tau \rangle \times \langle \theta_i \rangle$ for some $1 \leq i \leq p - 1$. In the second case, $\tau$ is again a $p$-cycle, $I = \langle \tau \rangle \times \langle \theta_i \omega \rangle$ for some $1 \leq i \leq p - 1$, and the disjoint cycle decomposition of $\omega$ in $\text{Sym}(\{p + 1, \ldots, d\})$ consists of $r$ cycles length $m_1, \ldots, m_r$. □
Using the above results and a technique used in [4, Proposition 1.3] (when \( p \) strictly divides the order of \( G \)), we obtain the following result for a certain type of two-point branched Galois cover of the projective line \( \mathbb{P}^1 \). This result will be very useful in understanding the inertia groups for the covers we will obtain later.

**Proposition 3.3:** Let \( p \) be a prime, \( d \geq p \). Let \( G \) be a transitive subgroup of \( S_d \). Let \( \phi: Z \to \mathbb{P}^1 \) be a \( G \)-Galois cover of smooth projective connected \( k \)-curves. Consider the degree-\( d \) cover \( \psi: Y := Z/(G \cap S_{d-1}) \to \mathbb{P}^1 \) of smooth projective connected \( k \)-curves where \( S_{d-1} \) is the subgroup of elements in \( S_d \) fixing 1. Assume that the following hold.

(i) There are exactly \( s \) points in \( \psi^{-1}(0) \) with ramification indices \( n_1, \ldots, n_s \) such that each \( n_i \) is coprime to \( p \) and \( \sum_{i=1}^{s} n_i = d \);
(ii) when \( d = p \), there is a unique point in \( Y \) lying over \( \infty \) with ramification index \( p \). When \( d > p \), there are exactly \( r + 1 \) points in \( \psi^{-1}(\infty) \) with ramification indices \( p, m_1, \ldots, m_r \) such that each \( m_l \) is coprime to \( p \) and \( \sum_{l=1}^{r} m_l = d - p \).

Then \( \phi \) is tamely ramified over 0, and \( I = \langle (1, \ldots, p) \rangle \rtimes \langle \theta^i \omega \rangle \) occurs as an inertia group over \( \infty \) for some \( 1 \leq i \leq p - 1 \) (\( \theta, \omega \) are as in Equation (3.1)). If \( \gamma \) is a generator of an inertia group over 0, a disjoint cycle decomposition of \( \gamma \) in \( S_d \) consists of \( s \) cycles of length \( n_1, \ldots, n_s \). If \( d = p \), \( \omega \) is the trivial permutation and if \( d \geq p + 1 \), a disjoint cycle decomposition of \( \omega \) in \( \text{Sym} \{p + 1, \ldots, d\} \) consists of \( r \) cycles length \( m_1, \ldots, m_r \).

Moreover, if the cover \( \phi \) is étale away from \( \{0, \infty\} \) and \( g(Y) \) is the genus of \( Y \), the upper jump for any local extension above \( \infty \) is \( \frac{2g(Y) + s + r - 1}{p - 1} \), and we have

\[
\text{ord}(\theta^i) = \frac{p - 1}{(p - 1, 2g(Y) + s + r - 1)}.
\]

**Proof.** The structures of the inertia groups are the consequence of Lemma 3.2. Suppose that the cover \( \phi \) is étale away from \( \{0, \infty\} \). We use the Riemann–Hurwitz formula for the two Galois covers \( \phi \) and \( Z \to Y \) to obtain the upper jump \( \frac{2g(Y) + s + r - 1}{p - 1} \) at \( \infty \) (see [6, Equation (2.1)] for such techniques). By [6, Lemma 2.6, Equation (2.2)], \( \text{ord}(\theta^i) = \frac{p - 1}{(p - 1, 2g(Y) + r + s - 1)} \).

Now we proceed towards the construction of the covers. We will consider the following assumption.
Assumption 3.4: Let \( s \geq 2 \) be an integer. Let \( n_1, \ldots, n_s \) be coprime to \( p \) such that \( \sum_{i=1}^{s} n_i = p \). Let \( \alpha_1 = 0 \). Assume that there exist non-zero distinct elements \( \alpha_2, \ldots, \alpha_s \) in \( k \) so that the polynomial
\[
\sum_{i=1}^{s} n_i \prod_{j \neq i, 1 \leq j \leq s} (y - \alpha_j) \in k[y]
\]
is a non-zero constant in \( k \). In terms of the coefficients of this polynomial, the assumption is equivalent to the existence of non-zero distinct \( \alpha_i \)'s, \( 2 \leq i \leq s \), such that for each \( 1 \leq \nu \leq s - 2 \),
\[
\sum_{2 \leq i_1 < \cdots < i_{s-\nu-1} \leq s} \left( \sum_{i \in \{i_1, \ldots, i_{s-\nu-1}\}} n_i \right) \alpha_{i_1} \cdots \alpha_{i_{s-\nu-1}} = 0.
\]

Remark 3.5: Note that Assumption 3.4 is satisfied when \( s = 2 \) or \( s = 3 \). It is immediate when \( s = 2 \). For \( s = 3 \), the assumption holds for the pair
\[
(\alpha_2, \alpha_3) = \left(1, -\frac{n_2}{n_3}\right).
\]

Under the above assumption, the following result constructs a degree-\( p \) cover of \( \mathbb{P}^1 \) that is étale away from \( \{0, \infty\} \) and with prescribed ramification over the branched points.

Proposition 3.6: Let \( p \geq 5 \) be a prime and \( s \geq 2 \) be an integer. Let \( n_1, \ldots, n_s \) be coprime to \( p \) such that \( \sum_{i=1}^{s} n_i = p \). Let \( \alpha_1, \ldots, \alpha_s \) be distinct elements in \( k \). Let \( \psi : Y \to \mathbb{P}^1 \) be the degree-\( p \) cover given by the affine equation \( \bar{f}(x, y) = 0 \) where
\[
(3.2) \quad \bar{f}(x, y) := \prod_{i=1}^{s} (y - \alpha_i)^{n_i} - x.
\]

Let \( \phi : Z \to \mathbb{P}^1 \) be the Galois closure of \( \psi \) with group \( G \). Then the following hold:

1. \( G \) is a primitive subgroup of \( S_p \).
2. \( \phi \) is tamely ramified with cyclic inertia group of order \( \text{lcm}\{n_1, \ldots, n_s\} \) over 0. If \( \gamma \) is one of its generators, then \( \gamma \) has a disjoint cycle decomposition in \( S_p \) with cycle lengths \( n_1, \ldots, n_s \).
3. Over \( \infty \), the inertia group is of the form \( I = \langle(1, \ldots, p)\rangle \times \langle \theta^i \rangle \) for some \( 1 \leq i \leq p - 1 \) and where \( \theta \) is as in Equation (3.1).
Additionally, if \( \alpha_i \)'s satisfy Assumption 3.4, then the cover \( \phi \) is étale away from \( \{0, \infty\} \). Then we have

\[
\text{ord}(\theta^i) = \frac{p - 1}{(p - 1, s - 1)}, \quad |I| = \frac{p(p - 1)}{(p - 1, s - 1)},
\]

and the upper jump for any \( I \)-Galois local extension over \( \infty \) is given by \( \frac{s - 1}{p - 1} \).

Moreover, if there is a positive integer \( j \) such that \( \gamma^j \) is a non-trivial cycle fixing \( \geq 3 \) points in \( \{1, \ldots, p\} \) or if \( p \neq 11, 23, p \not\in \{q^n - 1 \mid q \text{ prime power, } n \geq 2\} \), and \( \gamma \) is not a conjugate of \( \theta^i \) for any \( 1 \leq i \leq p - 1 \),

\[
G = \begin{cases} 
A_p, & \text{if } \gamma \text{ is an even permutation,} \\
S_p, & \text{if } \gamma \text{ is an odd permutation.}
\end{cases}
\]

**Proof.** The polynomial \( \bar{f}(x, y) \) is linear and monic in \( x \). So it is irreducible in \( k[y][x] \) and hence in \( k(x)[y] \). So \( G \) is a transitive subgroup of \( S_p \), and hence it is a primitive subgroup of \( S_p \). This proves (1).

From the equation \( \bar{f}(x, y) = 0 \) it follows that \( v_{(y - \alpha_i)}(x) = n_i \) for \( 1 \leq i \leq s \), and \( v_{(y - 1)}(x^{-1}) = p \). Since \( \Sigma n_i = p \), we see that the fibre \( \psi^{-1}(0) \) consists of \( s \) points in \( Y \), and the ramification index at the point \( (y = \alpha_i) \) is given by \( n_i \). Also there is a unique point in \( Y \) lying above \( \infty \) at which the ramification index is \( p \). Then (2) and (3) follow from Proposition 3.3.

Now suppose that \( \alpha_i \)'s satisfy Assumption 3.4. The \( y \)-derivative of \( \bar{f}(x, y) \) is given by

\[
\bar{f}_y(x, y) = \prod_{i=1}^{s} (y - \alpha_i)^{n_i - 1} \left( \sum_{i=1}^{s} n_i \prod_{j \neq i, 1 \leq j \leq s} (y - \alpha_j) \right).
\]

Let \((a, b)\) be a common zero of \( \bar{f} \) and \( \bar{f}_y \). Then \( a = 0 \) if \( n_i > 1 \) for some \( i \), and there is no common zero if \( n_i = 1 \) for all \( 1 \leq i \leq s \). So the cover \( \psi \), and hence \( \phi \) is étale away from \( \{0, \infty\} \). By Proposition 3.3, the upper jump is \( \frac{s - 1}{p - 1} \), \nord(\theta^j) = \frac{p - 1}{(p - 1, s - 1)}, \) So \( |I| = \frac{p(p - 1)}{(p - 1, s - 1)} \).

Since \( G \) is a primitive subgroup of \( S_p \) containing a \( p \)-cycle, under the additional hypothesis on \( p \) and \( \gamma \), \( G \) contains \( A_p \) by [14, Theorem 1.2]. So if \( \gamma \) is an odd permutation, \( G = S_p \). Now let \( \gamma \) be an even permutation. Assume that \( G = S_p \). Then the connected \( \mathbb{Z}/2 \)-Galois cover \( \mathbb{Z}/A_p \to \mathbb{P}^1 \) is étale away from \( \infty \) and is tamely ramified above \( \infty \), a contradiction to the fact that \( \pi_1^t(\mathbb{A}^1) \) is the trivial group. So if \( \gamma \) is an even permutation, \( G = A_p \). \( \blacksquare \)
Remark 3.7: In [2, Section 20], Abhyankar introduced the following cover and calculated its Galois group. Consider the degree-$p$ cover of $\mathbb{P}^1$ given by the affine equation $\tilde{f} = 0$ where

$$\tilde{f}(x, y) = y^p - y^t + x.$$ 

Consider its Galois closure $\tilde{Y} \to \mathbb{P}^1$ with group $G$. Abhyankar showed that for $2 \leq t \leq p - 3$, $G = S_p$ if $t$ is even, and $G = A_p$ if $t$ is odd. This is a special case of Proposition 3.6.

Now we construct covers of degree $d \geq p + 1$. Similar to the previous case, we consider the following assumption.

**Assumption 3.8:** Let $p$ be an odd prime, $t \geq 1$ be coprime to $p$ such that $d := p + t \geq 5$. Let $r$ and $s$ be two positive integers. Let $n_1, \ldots, n_s, m_1, \ldots, m_r$ be coprime to $p$ integers such that $\sum_{i=1}^s n_i = p + t$, $\sum_{l=1}^r m_l = t$. Assume that there exist distinct elements $\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_r$ in $k$ such that the polynomial

$$g(y) := \prod_{l=1}^r (y - \beta_l) \left( \sum_{i=1}^s n_i \prod_{j \neq i, 1 \leq j \leq s} (y - \alpha_j) \right)$$

$$- \prod_{i=1}^s (y - \alpha_i) \left( \sum_{l=1}^r m_l \prod_{u \neq l, 1 \leq u \leq r} (y - \beta_u) \right)$$

in $k[y]$ is a non-zero constant in $k$.

Remark 3.9: In particular, if $r = 1$, setting $\beta_1 = 0$, Assumption 3.8 says that there are non-zero distinct elements $\alpha_i$ in $k$, $1 \leq i \leq s$, such that the polynomial

$$y \left( \sum_{i=1}^s n_i \prod_{j \neq i, 1 \leq j \leq s} (y - \alpha_j) \right) - t \sum_{i=1}^s (y - \alpha_i)$$

is a non-zero constant in $k$. In terms of coefficients, we need the $\alpha_i$’s to satisfy the following condition for each $1 \leq \nu \leq s - 1$:

$$\sum_{1 \leq i_1 < \cdots < i_{s-\nu} \leq s} (n_{i_1} + \cdots + n_{i_{s-\nu}}) \alpha_{i_1} \cdots \alpha_{i_{s-\nu}} = 0.$$ 

When $s = 1$, setting $\alpha_1 = 0$, we also get a similar condition on the choice of $\beta_l$’s.

Before proceeding to the construction of the covers, let us see some of the cases where Assumption 3.8 is satisfied.
Lemma 3.10: Assumption 3.8 holds with a choice of distinct $\alpha_i$'s and $\beta_l$'s in the following cases:

1. $s = 1 = r$ with $(\alpha_1, \beta_1) = (1, 0)$;
2. $s = 2, r = 1$ with $(\alpha_1, \alpha_2, \beta_1) = (1, -n_1/n_2, 0)$;
3. $s = 1, r = 2$ with $(\alpha_1, \beta_1, \beta_2) = (0, 1, -m_1/m_2)$;
4. $s = 3, r = 1$ with $(\alpha_1, \alpha_2, \alpha_3, \beta_1) = (t + 2, t + 2, 1, 0)$ where $(p, t + 2) = 1 = (p, t - 2)$ and $n_1 = p - 2, n_2 = 2, n_3 = t$;
5. $r = s = 2$ with $(\alpha_1, \alpha_2, \beta_1, \beta_2) = (1, \frac{n_2 - n_1}{2n_2}, 0, \frac{t}{2n_2})$ where $n_i \equiv m_i \mod p$ and $n_1 \neq n_2$ in $k$.

Proof. It is easy to see that in each of the cases the assigned values of $\alpha_i$'s and $\beta_l$'s are all distinct and they satisfy Assumption 3.8. ■

The following result produces our main example of an $S_d$-Galois or an $A_d$-Galois two-point branched cover of $\mathbb{P}^1$.

Proposition 3.11: Let $p$ be an odd prime such that $d := p + t$. Let $r$ and $s$ be two positive integers. Let $n_1, \ldots, n_s, m_1, \ldots, m_r$ be coprime to $p$ such that

$$\sum_{i=1}^{s} n_i = p + t, \quad \sum_{l=1}^{r} m_l = t.$$

Let $\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_r$ be distinct elements in $k$. Let $\psi: Y \to \mathbb{P}^1$ be the degree-$d$ cover given by the affine equation $f(x, y) = 0$ where

$$f(x, y) = \prod_{i=1}^{s} (y - \alpha_i)^{n_i} - x \prod_{l=1}^{r} (y - \beta_l)^{m_l}. \quad (3.6)$$

Let $\phi: Z \to \mathbb{P}^1$ be its Galois closure with group $G$.

1. Then $G$ is a transitive subgroup of $S_d$.
2. The cover $\phi$ is tamely ramified with cyclic inertia group generated by an element $\gamma \in G$ of order $\text{lcm}\{n_1, \ldots, n_s\}$ over 0, whose disjoint cycle decomposition in $S_d$ consists of $s$ disjoint cycles of length $n_1, \ldots, n_s$.
3. Over $\infty$, the inertia group is of the form $I = \langle (1, \ldots, p) \rangle \rtimes \langle \theta^i \omega \rangle$ for some $1 \leq i \leq p-1$, where $\theta$ is as in Equation (3.1) and $\omega \in \text{Sym}\{p+1, \ldots, d\}$ is a product of $r$ disjoint cycles of length $m_1, \ldots, m_r$. 

Additionally, if \((\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_r)\) satisfies Assumption 3.8, the cover \(\phi\) is \'{e}tale away from \(\{0, \infty\}\). Also,

\[
\text{ord}(\theta^i) = \frac{p-1}{(p-1,r+s-1)}, \quad |I| = p \times \text{lcm}\{\text{ord}(\theta^i), \text{ord}(\omega)\},
\]

and the upper jump for any \(I\)-Galois local extension over \(\infty\) is given by \(\frac{r+s-1}{p-1}\).

Moreover, if \(t < p\), \(G\) is a primitive subgroup of \(S_d\). Furthermore, if either there is a positive integer \(j\) such that \(\gamma^j\) is a non-trivial cycle fixing \(\geq 3\) points in \(\{1, \ldots, d\}\) or if \(3 \leq t \leq p-1\),

\[
G = \begin{cases} 
A_d, & \text{if } \gamma \text{ is an even permutation,} \\
S_d, & \text{if } \gamma \text{ is an odd permutation.}
\end{cases}
\]

**Proof.** Since \(\alpha_i\)'s are distinct from \(\beta_l\)'s by our assumption and the polynomial \(f(x,y)\) is linear in \(x\), it is irreducible in \(k(x)[y]\). So \(G\) is a transitive subgroup of \(S_d\), proving (1).

From the equation \(f(x,y) = 0\) we have \(v_{(y-\alpha_i)}(x) = n_i\) for \(1 \leq i \leq s\), \(v_{(y-\beta_l)}(x^{-1}) = m_l\) for \(1 \leq l \leq r\) and \(v_{(y^{-1})}(x^{-1}) = p\). Since \(\sum n_i = p + t\) the fibre \(\psi^{-1}(0)\) consists of \(s\) points in \(Y\) with the ramification index at the point \((y = \alpha_i)\) given by \(n_i\), and also since \(\sum m_l = t\), there are exactly \(r + 1\) points in \(Y\) lying above \(\infty\) with ramification indices given by \(p, m_1, \ldots, m_r\). Then the description of the inertia groups above 0 and \(\infty\) ((2) and (3)) follow from Proposition 3.3.

Now suppose that Assumption 3.8 holds. The \(y\)-derivative of the polynomial \(f(x,y)\) is given by

\[
f_y(x,y) = \prod_{i=1}^{s} (y - \alpha_i)^{n_i-1} \left( \sum_i n_i \prod_{j \neq i} (y - \alpha_j) \right) - x \prod_{l=1}^{r} (y - \beta_l)^{m_l-1} \left( \sum_l m_l \prod_{u \neq l} (y - \beta_u) \right).
\]

Let \((a,b)\) be a common zero of \(f\) and \(f_y\). Then

\[
0 = f_y(a,b) \prod_{l=1}^{r} (b - \beta_l) = g(b) \prod_{i} (b - \alpha_i)^{n_i-1}
\]

(where \(g\) is the polynomial given by Equation (3.3)). Then \(a = 0\) if \(n_i \geq 2\) for some \(i\) and there is no such common zero otherwise. So the cover \(\psi\), and hence \(\phi\) is \'{e}tale away from \(\{0, \infty\}\). Again by Proposition 3.3, the upper jump is \(\frac{r+s-1}{p-1}\) and \(\text{ord}(\theta^i) = \frac{p-1}{(p-1,r+s-1)}\).
Since $G$ is a transitive subgroup of $S_d$ containing the $p$-cycle $\tau$ which fixes $t$ points in $\{1, \ldots, d\}$ and $t < \frac{p+1}{2}$, by [14, Remark 1.6], $G$ is primitive.

Finally, if some power of $\gamma$ is a non-trivial cycle fixing $\geq 3$ points or if $3 \leq t \leq p-1$, by [14, Theorem 1.2], $G$ contains $A_d$. The rest follows as in Proposition 3.6.

We will frequently obtain covers of $\mathbb{P}^1$ with Galois groups $S_d$ or $A_d$, étale away from two points. To obtain our required covers, we will need to consider a certain Kummer pullback (Definition 2.2) of such covers. In these situations, we will use the following result.

**Lemma 3.12:** Let $G \in \{S_d, A_d\}$, $d \geq p$, $d \geq 5$. Let $\gamma \in G$ be an element of order prime-to-$p$ and $n$ be coprime to $p$. Let $\phi_1 : \mathbb{Z} \to \mathbb{P}^1$ be a $G$-Galois cover of smooth projective connected $k$-curves that is étale away from $\{0, \infty\}$ such that $\langle \gamma \rangle$ occurs as an inertia group above $0$, and $I = P \rtimes \langle \beta \rangle$ occurs as an inertia group above $\infty$ for some $p$-subgroup $P$ and an element $\beta$ of order prime-to-$p$.

Then the $[n]$-Kummer pullback of $\phi_1$ is a connected Galois cover of $\mathbb{P}^1$ with

$$\text{Galois group} = \begin{cases} A_d, & \text{if } \gamma^n \text{ is an even permutation}, \\ S_d, & \text{if } \gamma^n \text{ is an odd permutation}. \end{cases}$$

étale away from $\{0, \infty\}$, such that the group $\langle \gamma^n \rangle$ of order $\frac{\text{ord}(\gamma)}{\text{ord}(\gamma^n)}$ is an inertia group above $0$, and the group $P \rtimes \langle \beta^n \rangle$ of order $\text{ord}(P) \times \frac{\text{ord}(\beta)}{\text{ord}(\beta^n)}$ is an inertia group above $\infty$.

In particular, if $n = \text{ord}(\gamma)$, then the resulting cover is a connected $A_d$-Galois cover of $\mathbb{P}^1$, branched only at $\infty$, and $P \rtimes \langle \beta^n \rangle$ occurs as an inertia group above $\infty$.

**Proof.** Let $\eta : \mathbb{P}^1 \to \mathbb{P}^1$ denote the $\mathbb{Z}/n$-Galois Kummer cover (cf. Definition 2.2). Let $\phi : Z \to \mathbb{P}^1$ be the cover obtained after the $[n]$-Kummer pullback of $\phi_1$ with Galois group $H$. When $\phi_1$ and $\eta$ are linearly disjoint, the statement is the Refined Abhyankar’s Lemma ([16, Lemma 4.1]). Using the same argument, the structure and the orders of the inertia groups follow, and $\phi$ is étale away from $\{0, \infty\}$. By our construction, $H$ is the Galois group of the compositum $k(Z)k(\mathbb{P}^1)/k(\mathbb{P}^1)$ of the function field extensions of $k(\mathbb{P}^1)$, and so $H \subset G \times \mathbb{Z}/n$. Now, as in the proof of Proposition 3.6, using the fact that there is no non-trivial étale cover of $\mathbb{P}^1$, it follows that $H = A_d$ if and only if $\gamma^n$ is an even permutation, and $H = S_d$ otherwise. ■
From the above proof it also follows that if $G = A_d$, then we always have $H = A_d$. As an application of the above lemma and Proposition 3.11, we deduce the following results which will be used later.

**Corollary 3.13**: Let $p$ be an odd prime. Let $3 \leq t \leq p-2$ be an integer such that $(t, p-1) = 1$. Set $d = p + t$. Then there is a connected $A_d$-Galois étale cover of the affine line such that $I := \langle (1, \ldots, p) \rangle \rtimes \langle \theta^2(p+1, \ldots, d) \rangle$ occurs as an inertia group at a point above $\infty$.

**Proof.** Take $s = 2, r = 1, n_1 = p + t - 1, n_2 = 1$ in Proposition 3.11. By Lemma 3.10(2), Assumption 3.8 is satisfied. As $(t, p-1) = 1, t$ is an odd integer. So a $(p + t - 1)$-cycle is an even permutation. By Proposition 3.11, there is a connected $A_d$-Galois cover $\phi_1 : \mathbb{Z}_1 \to \mathbb{P}^1$ branched only at 0 and $\infty$ with the following properties: $\mathbb{Z}/(p + t - 1)$ occurs as an inertia group above 0 with a $(p + t - 1)$-cycle as a generator, and since $(p-1, r + s - 1) = (p-1, 2) = 2$, the inertia groups above $\infty$ are the conjugates of $I = \langle (1, \ldots, p) \rangle \rtimes \langle \theta^2(p+1, \ldots, d) \rangle$.

Consider a $[(p + t - 1)]$-Kummer pullback $\phi: Z \to \mathbb{P}^1$ of the cover $\phi_1$. By Lemma 3.12, the cover $\phi$ is a connected $A_d$-Galois cover, étale away from $\infty$. Since $(p-1, t) = 1$ by our hypothesis, we have $(p + t - 1, t) = 1 = (p + t - 1, p-1)$, and so $(p + t - 1, \text{ord}(\theta^2(p+1, \ldots, d))) = 1$. Thus, $I$ occurs as an inertia group in the cover $\phi$ above $\infty$.

**Corollary 3.14**: Let $p$ be an odd prime, $4 \leq t \leq p-1$ an integer such that $(t+1, p-1) = 1 = (t-1, p+1)$ and $d = p + t$. Then there is a connected $A_d$-Galois étale cover of the affine line such that $I := \langle (1, \ldots, p) \rangle \rtimes \langle \theta^2(p+1, \ldots, d-1) \rangle$ occurs as an inertia group at a point above $\infty$.

**Proof.** Take $s = 1, r = 2, m_1 = t - 1, m_2 = 1$ in Proposition 3.11. By Lemma 3.10(3), Assumption 3.8 holds. Since $(t+1, p-1) = 1$, and $p$ is an odd prime, $t$ must be even. Apply Proposition 3.11 to obtain a connected $A_d$-Galois cover $\phi_1 : \mathbb{Z}_1 \to \mathbb{P}^1$, étale away from $\{0, \infty\}$, such that $\mathbb{Z}/(p + t)$ occurs as an inertia group above 0 with a $(p + t)$-cycle as a generator, and the inertia groups above $\infty$ are the conjugates of

$\langle (1, \ldots, p) \rangle \rtimes \langle \theta^{(p-1, r+s-1)}(p+1, \ldots, d-1) \rangle = \langle (1, \ldots, p) \rangle \rtimes \langle \theta^2(p+1, \ldots, d-1) \rangle$

in $A_d$. Let $\phi$ be the cover obtained from $\phi_1$ after a $[p + t]$-Kummer pullback. Since $(p + t, p-1) = (t+1, p-1) = 1$ and $(p + t, t-1) = (p + t, t-1) = 1$ by our hypothesis, $(p + t, \text{ord}(\theta^2(p+1, \ldots, d-1))) = 1$. From Lemma 3.12 it follows that the cover $\phi$ has the required properties. ■
COROLLARY 3.15: Let $p \equiv 2 \pmod{3}$ be an odd prime. Let $3 \leq t \leq p - 1$ be an integer, $d = p + t$. Then there is a connected $A_d$-Galois étale cover of the affine line such that $I := \langle (1, \ldots, p) \rangle \rtimes \langle \theta(t, p-1)(p+1, \ldots, d-1) \rangle$ occurs as an inertia group at a point above $\infty$.

Proof. Take $s = 2$, $r = 2$, $n_1 = p + t - 1$, $n_2 = 1$, $m_1 = t - 1$, $m_2 = 1$ in Proposition 3.11. By Lemma 3.10(5), Assumption 3.8 is satisfied. A $(p + t - 1)$-cycle is an odd permutation if and only if $t$ is an even integer. By Proposition 3.11, there is a connected $G$-Galois cover of $\mathbb{P}^1$ branched only at 0 and $\infty$, where $G = A_d$ if $t$ is odd and $G = S_d$ if $t$ is even. Over 0, the inertia groups are generated by conjugates of a $(p + t - 1)$-cycle in $G$, and

$$\langle (1, \ldots, p) \rangle \rtimes \langle \theta(p-1, 3)(p+1, \ldots, d-1) \rangle = \langle (1, \ldots, p) \rangle \rtimes \langle \theta(p+1, \ldots, d-1) \rangle$$

occurs (as $p \equiv 2 \pmod{3}$) as an inertia group above $\infty$. Note that

$$(p + t - 1, \text{ord}(\theta(p+1, \ldots, d-1))) = (t, p - 1) = 1.$$  

After a pullback via the $[p + t - 1]$-Kummer cover, we obtain (see Lemma 3.12) a connected $A_d$-Galois cover $\phi: Y \to \mathbb{P}^1$ étale away from $\infty$ such that $I$ occurs as an inertia group above $\infty$.  

4. Constructing covers by formal patching

For the details of the formal patching, we refer to [8]. The results of this section will be used later. Throughout this section, we fix the following notation.

Notation 4.1: Set $R = k[[t]], K = k((t))$. Let $s$ denote the closed point of $S := \text{Spec}(R)$. For any integral $k$-algebra $A$ with $L = QF(A)$ and any $k$-scheme $W$, set

$$W_A := W \times_k A, \quad W_L := W_A \times_A L.$$  

For any closed point $w \in W$, $w_A := w \times_k A, \quad w_L := w_A \times_A L$.

For any integral scheme $X$ and a closed point $x \in X$ let

$$K_{X,x} := QF(\hat{O}_{X,x}).$$

As an application of [11, Proposition 3.11], the following result shows that we can construct a family of Galois covers of $S$ with prescribed two special fibres. This will be helpful in construction of covers of curves using formal patching.
Lemma 4.2: Let $I$ be an extension of a $p$-group by a cyclic group of order $m$, $p \nmid m$. Suppose that $V_1 \to S$ and $V_2 \to S$ are two connected $I$-Galois covers that are totally ramified over $s$. Then there is a connected $I$-Galois cover $V \to S \times \mathbb{A}^1_u$ of integral schemes with branched locus $s \times \mathbb{A}^1_u$ over which it is totally ramified, and we have the isomorphisms
\[
V \times_{S \times \mathbb{A}^1_u} (S \times (u = 1)) \cong V_1,
\]
\[
V \times_{S \times \mathbb{A}^1_u} (S \times (u = -1)) \cong V_2,
\]

as $I$-Galois covers over $S$.

Proof. Let $i \in \{1, 2\}$. Then $V_i = \text{Spec}(A_i)$ for a complete discrete valuation ring $A_i$ with residue field $k$. Let $L_i$ be the field of fractions of $A_i$. After a change of variable, we may assume that
\[
I^{p(I)}_i = K' = k((T))
\]
where the extension $K'/K$ is given by $T^m = t$. Consider the trivial deformation $\text{Spec}(K'[u]) \to \text{Spec}(K[u])$ of integral $K$-curves of the cover $\text{Spec}(K') \to \text{Spec}(K)$. The compositions
\[
\text{Spec}(L_1) \to \text{Spec}(K') \to \text{Spec}(K) = \text{Spec}(K[u]) \times_k (u = 1),
\]
\[
\text{Spec}(L_2) \to \text{Spec}(K') \to \text{Spec}(K) = \text{Spec}(K[u]) \times_k (u = -1),
\]
are connected $I$-Galois étale covers. By applying [11, Proposition 3.11] with $\Gamma = I$, $G = \mathbb{Z}/m$, $X$ as the affine $u$-line over $K$ and $X'$ as the closed subset of $X$ consisting of the points $(u = 1)$ and $(u = -1)$, we obtain a connected $p(I)$-Galois étale cover $V' \to \mathbb{A}^1_{K'}$ of integral curves such that the composition $V' \to \mathbb{A}^1_{K'}$, $\to \mathbb{A}^1_K$ is a connected $I$-Galois cover,
\[
V' \times_{\mathbb{A}^1_{K'}} (u = 1) \cong \text{Spec}(L_1) \quad \text{and} \quad V' \times_{\mathbb{A}^1_{K'}} (u = -1) \cong \text{Spec}(L_2)
\]
as $I$-Galois covers of $\text{Spec}(K)$. Take $V$ to be the normalization of $S \times \mathbb{A}^1_u$ in the function field of $V'$. Then we have the isomorphisms
\[
V \times_{S \times \mathbb{A}^1_u} (S \times (u = 1)) \cong V_1,
\]
\[
V \times_{S \times \mathbb{A}^1_u} (S \times (u = -1)) \cong V_2,
\]
as $I$-Galois covers over $S$. We will show that the cover
\[
f: V \to S \times \mathbb{A}^1_u = \text{Spec}(k[[t]][u])
\]
has the stated ramification theoretic properties.
By construction,
\[ V \times_{S \times \mathbb{A}^1_k} \text{Spec}(K[u]) \cong V' \]
as Galois covers of \( \mathbb{A}^1_K \). So the branched locus of \( f \) is contained in \( s \times \mathbb{A}^1_u \).
The cover \( f \) factors as the composition \( V \to \text{Spec}(k[[T]][u]) \to \text{Spec}(k[[t]][u]) \) (with the relation \( T_m = t \)). Let \( \mathcal{M}_{p(I)}^{\text{loc}} \) denote the coarse moduli space of \( p(I) \)-Galois étale covers of \( \text{Spec}(K') \) (see [7, Proposition 2.1]) and let \( \mathcal{M}'_{p(I)} \subset \mathcal{M}_{p(I)}^{\text{loc}} \) denote the dense open subset parametrizing the covers that are irreducible ([7, Remark 2.5(b)]). So the cover \( V' \to \text{Spec}(K'[u]) \) induces a morphism \( \text{Spec}(K'[u]) \to \mathcal{M}_{p(I)}^{\text{loc}} \) such that the images of \( (u = 1) \) and \( (u = -1) \) lie in \( \mathcal{M}'_{p(I)} \). Since \( \mathcal{M}'_{p(I)} \) is open and dense in \( \mathcal{M}_{p(I)}^{\text{loc}} \), for all points \( (u = a) \) in a dense open subset of \( \text{Spec}(K'[u]) \), the image of \( (u = a) \) lies in \( \mathcal{M}'_{p(I)} \). Equivalently, the inertia group above \( (T = 0, u = a) \) in the fibre
\[ V \times_{\text{Spec}(k[[T]][u])} \left( \text{Spec}(k[[T]]) \times (u = a) \right) \to \text{Spec}(k[[T]]) \times (u = a) \]
is \( p(I) \). By construction, for every \( b \in k \), the inertia group above \( (s, u = b) \) in the fibre \( \text{Spec}(k[[T]][u]) \times_{\text{Spec}(k[[t]][u])} \left( S \times (u = b) \right) \to S \times (u = b) \) has inertia group \( \mathbb{Z}/m \). As the ramification is multiplicative for a composition of covers, we conclude that for all points \( (u = a) \) in a dense open subset of \( \mathbb{A}^1_u \), the fibre \( V \times_{S \times \mathbb{A}^1} \left( S \times (u = a) \right) \to S \) is totally ramified above \( s \).

We obtain the following result using the formal patching results from [9] and [6]. The above lemma provides a compatibility of local extensions. Similar results appeared in the literature for the purely wild ramification (cf. [19, Theorem 2.2.3]).

**Theorem 4.3:** Let \( X \) be a smooth projective connected \( k \)-curve, \( B_1 \subset X \) be a finite set of closed points, \( \xi \in B_1 \). Let \( G \) be a finite group, \( I \subset G \) be an extension of a \( p \)-group by a cyclic group of order \( m \), \( p \nmid m \). Let \( G_1 \) and \( G_2 \) be two subgroups of \( G \). Suppose that \( I_1 \) and \( I_2 \) are subgroups of \( I \) with \( |I_1/p(I_1)| = m = |I_2/p(I_2)| \) such that the following hold:

1. \( f_1: Y_1 \to X \) is a connected \( G_1 \)-Galois cover étale away from \( B_1 \subset X \) such that \( I_1 \) occurs as an inertia group above \( \xi \). For any point \( b_1 \neq \xi \) in \( B_1 \), let \( J_{b_1} \) denote an inertia group above \( b_1 \).
2. \( f_2: Y_2 \to \mathbb{P}^1 \) is a connected \( G_2 \)-Galois cover with branched locus \( B_2 \subset \mathbb{P}^1 \), \( 0 \in B_2 \). Suppose that \( I_2 \) occurs as an inertia group above \( 0 \), and for points \( y \neq 0 \) in \( B_2 \), let \( I'_y \) denote an inertia group above \( y \).
If $G = \langle G_1, G_2, I \rangle$, then there is subset $B_0 \subset X$ disjoint from $B_1$ together with a bijection $\eta: B_0 \rightarrow B_2 - \{0\}$ and a connected $G$-Galois cover of $X$ étale away from $B_1 \sqcup B_0$ such that $I$ occurs as an inertia group above $x$, for all points $b_1 \neq \xi$ in $B_1$, $J_{b_1}$ occurs as an inertia group above $b_1$, and for all $b_0$ in $B_0$, $I'_{\eta(b_0)}$ occurs as an inertia group above $b_0$.

Proof. We first deform the covers $f_1$ and $f_2$ so that we can apply Lemma 4.2. Applying [12, Theorem 3.6] to the $G_1$-Galois cover $f_1$, we obtain a connected $H_1 := \langle G_1, I \rangle$-Galois cover $g_1: C_1 \rightarrow X$ that is étale away from $B_1$, $I$ occurs as an inertia group above $\xi$, and for $b_1 \in B$, $b_1 \neq \xi$, $J_{b_1}$ occurs as an inertia group above $b_1$. Similarly, by the same result applied to the cover $f_2$ produces a connected $H_2 := \langle G_2, I \rangle$-Galois cover $g_2: C_2 \rightarrow \mathbb{P}^1$ with branched locus $B_2$ such that $I$ occurs as an inertia group above 0, and for $y \in B_2$, $y \neq 0$, $I'_y$ occurs as an inertia group above $y$.

Now we further deform these covers such that the resulting local extensions at $\xi$ and 0 in the respective covers become isomorphic. This will allow us to apply formal patching techniques. Let $c_1 \in g_1^{-1}(\xi) \subset C_1$ be a point such that $I$ is the Galois group of the field extension $K_{C_1,c_1}/K_{X,\xi}$. Also let $c_2 \in C_2$ be a point lying above 0 such that $I$ is realized as the Galois group of the field extension $K_{C_2,c_2}/k((x))$. We will identify $\hat{O}_{X,\xi}$ and $k[[x]]$. Then taking

$$V_1 = \text{Spec}(\hat{O}_{C_1,c_1}) \rightarrow \text{Spec}(k[[x]])$$

and

$$V_2 = \text{Spec}(\hat{O}_{C_2,c_2}) \rightarrow \text{Spec}(k[[x]])$$

in Lemma 4.2, we obtain a connected $I$-Galois cover $g: V \rightarrow \text{Spec}(k[[x]][u])$ of integral schemes with branched locus $x \times \mathbb{A}_u^1$ over which it is totally ramified and such that

$$V \times_{\text{Spec}(k[[x]][u])} \text{Spec}(k[[x]]) \times \{u = -1\} \cong V_2$$

as $I$-Galois covers over $\text{Spec}(k[[x]])$. Then the cover $g_1$ and the $I$-Galois cover $g$ satisfy the hypothesis of [6, Lemma 3.2]. By [6, Lemma 3.3], there is an open dense set $\mathcal{W}_1 \subset \mathbb{A}_u^1$ such that for each closed point $(u = \beta)$ in $\mathcal{W}_1$ the following

\[1\] Thus, by abuse of notation, the local coordinates at 0 in $\mathbb{P}^1$ and at $\xi$ in $X$ are both denoted by $x$. 

holds. There is a connected $H_1$-Galois cover $h_1: Z_1 \to X$ étale away from $B_1$ such that there is a point $z_1 \in Z_1$ above $\xi$ for which $\text{Spec}(\hat{O}_{Z_1, z_1})$ is equal to the fibre of the cover $g: V \to \text{Spec}(k[[x]][u])$ over $(u = \beta)$ as $I$-Galois covers over $\text{Spec}(k[[x]])$, and for any point $b_1 \neq \xi$ in $B_1$, $J_{b_1}$ occurs as an inertia group above $b_1$. Similarly, the covers $g_2$ and $g$ satisfy the hypothesis of [6, Lemma 3.3] and by [6, Lemma 3.3], there is an open dense set $W_2 \subset \mathbb{A}^1_u$ such that for each closed point $(u = \alpha)$ in $W_2$ the following holds. There is a connected $H_2$-Galois cover $h_2: Z_2 \to \mathbb{P}^1$ étale away from $B_2$ such that there is a point $z_2 \in Z_2$ above $0$ for which $\text{Spec}(\hat{O}_{Z_2, z_2})$ as an $I$-Galois cover of $\text{Spec}(k[[x]])$ is equal to the fibre of $g$ over $(u = \alpha)$, and for $y \in B_2$, $y \neq 0$, $I_y'$ occurs as an inertia group above $y$.

We fix a closed point $(u = a)$ in $W_1 \cap W_2$ and consider the corresponding covers $h_1: Z_1 \to X$ and $h_2: Z_2 \to \mathbb{P}^1$ as above. Then $\text{Spec}(\hat{O}_{Z_1, z_1})$ and $\text{Spec}(\hat{O}_{Z_2, z_2})$ are isomorphic $I$-Galois covers of $\text{Spec}(k[[x]])$.

Let $T^*$ be a regular irreducible projective $R$-curve with generic fibre $X_K$ together with a cover $T^* \to \mathbb{P}^1_R$ and whose closed fibre $T'$ is the union of two irreducible components $X$ and $\mathbb{P}^1_y$ meeting at a point $\eta$ and such that the complete local ring of $T^*$ at $\eta$ is given by

$$\hat{O}_{T^*, \eta} = k[[x, y]][t]/(t - xy) \cong k[[x, y]].$$

Now consider the trivial deformation $\text{Spec}(\hat{O}_{Z_2, z_2}[y]) \to \text{Spec}(k[[x]][y])$. Let

$$\hat{N}^* := \text{Spec}(\hat{O}_{Z_2, z_2}[y]) \times_{\text{Spec}(k[[x]][y])} \text{Spec}(k[[x, y]]).$$

Let $X - x = \text{Spec}(A)$, $Z_1 - z_1 = \text{Spec}(B_1)$ and $Z_2 - z_2 = \text{Spec}(B_2)$. Then the hypothesis of [9, Proposition 2.3] is satisfied with the covers

$$W_1^* = \text{Spec}(B_1[[t]]) \to X_1^* = \text{Spec}(A[[t]])$$

and

$$W_2^* = \text{Spec}(B_2[[t]]) \to X_2^* = \text{Spec}(k[y^{-1}][[t]])$$

induced by $h_1$ and $h_2$, respectively, and with the isomorphisms

$$\hat{N}^* \times_{\text{Spec}(k[[x, y]])} \text{Spec}(K_{X, x}[[t]]) \cong \text{Spec}(K_{Z_1, z_1})$$

and

$$\hat{N}^* \times_{\text{Spec}(k[[x, y]])} \text{Spec}(k((y))[[t]]) = \text{Spec}(K_{Z_2, z_2}).$$

\footnote{The choice of this closed point is not important in the proof; any such choice produces covers with a compatibility in the appropriate local extensions which enables us to apply a formal patching technique.}
By [9, Proposition 2.3], there is an irreducible normal $G$-Galois cover $h^*: V^* \rightarrow T^*$ such that

$$V^* \times_{T^*} X'_1 \cong \text{Ind}^G_{H_1} W'_1,$$

$$V^* \times_{T^*} X'_2 \cong \text{Ind}^G_{H_2} W'_2$$

and

$$V^* \times_{T^*} \hat{O}_{T^*} \eta \cong \text{Ind}^G_I \hat{N}^*$$

as $G$-Galois covers. Consider the generic fibre $h^0: V^0 \rightarrow X_K$ of the cover $h^*$. Then there is a set $B_0 \subset X$ disjoint from $B_1$ together with a bijection $\eta: B_0 \rightarrow B_2 - \{0\}$ of sets such that $h^0$ is étale away from $\{x'_K| x' \in B_1 \cup B_0\}$, $I$ occurs as an inertia group above $\xi_K$, for $b_1 \neq \xi$ in $B_1$, $J_{b_1}$ occurs as an inertia group above $b_{1,K}$ and for $b_0$ in $B_0$, $I'_{\eta(b_0)}$ occurs as an inertia group above $b_{0,K}$. Since $Z_1 \times_R K$, $Z_2 \times_R K$ and $T \times_R K$ are smooth over $K$, $V_0$ is also smooth over $K$. Also since $T'$ is generically smooth and the cover $V^* \rightarrow T^*$ is generically unramified, the closed fibre $V^* \times_{T^*} T' \rightarrow T'$ of $h^*$ is generically smooth. Now the result follows by [9, Corollary 2.7].

By induction on $n$ and using the above theorem, we obtain the following result which generalizes a patching result by Raynaud ([19, Theorem 2.2.3]). This result allows us to construct Galois covers with control on the inertia groups from covers with smaller Galois groups. As an application (Lemma 5.2) of it, we will see that the study of the IC (Conjecture 1.1) gets a little easier.

**Corollary 4.4:** Let $n \geq 2$ be an integer. Let $G$ be a finite group, $I \subset G$ be an extension of a $p$-group by a cyclic group of order $m$, $p \nmid m$. For $1 \leq i \leq n$, let $G_i$ be subgroups of $G$, $I_i$ be a subgroup of $I$ with $|I_i/p(I_i)| = m$ such that the following hold:

1. $X$ is a smooth projective connected $k$-curve, $f: Y \rightarrow X$ is a connected $G_1$-Galois cover étale away from $B \subset X$. Let $x \in B$ be a closed point and let $I_1$ occur as an inertia group above $x$. For $x' \neq x$ in $B$ let $J_{x'}$ occur as an inertia group above $x'$.

2. For each $2 \leq i \leq n$ the pair $(G_i, I_i)$ is realizable.

If $G = \langle G_1, \ldots, G_n, I \rangle$, then there is a connected $G$-Galois cover $Z \rightarrow X$ étale away from $B$ such that $I$ occurs as an inertia group above $x$ and for $x' \neq x$ in $B$, $J_{x'}$ occurs as an inertia group above $x'$. 
For our next result, let us fix the following notation.

Notation 4.5: Let $G_1, G_2$ be two finite groups, $X$ a smooth projective connected $k$-curve. Let $B \subset X$ be a finite set of closed points. Let $i \in \{1, 2\}$. Suppose that for $x \in B$, $P_{x,i}$ is a $p$-subgroup (possibly trivial) of $G_i$, and there is a connected $G_i$-Galois cover $f_i: Y_i \to X$, étale away from $B$, and $P_{x,i}$ occurs as the inertia group above $x \in B$. For each $x \in B$, let $Q_{x}$ be a $p$-group (possibly trivial), and let $N_{x,i}$ be a normal subgroup of $P_{x,i}$ such that $P_{x,i}/N_{x,i} \cong Q_{x}$.

The following result shows that in the set-up of the above Notation 4.5, a certain kind of field extensions can be realized as local extensions by the Galois covers for both the groups $G_1$ and $G_2$. This will be used (Theorem 7.5) to show that the GPWIC (Conjecture 6.10) is true for a certain product of groups. As in Theorem 4.3, we again use a technique from [6] to realize a common local extension at a point over $X$ for two different covers.

Lemma 4.6: Assume that Notation 4.5 holds. Then for $i = 1, 2$, there is a connected $G_i$-Galois cover $Z_i \to X$ étale away from $B$ such that $P_{x,i}$ occurs as an inertia group above $x \in B$ and such that there is a point $z_i \in Z_i$ over $x$ with $K_{Z_i,z_1,N_{x,1}}/K_{X,x} \cong K_{Z_2,z_2,N_{x,2}}/K_{X,x}$ as $Q_x$-Galois extensions of $K_{X,x}$.

Proof. Let $x \in B$, $i = 1, 2$. Without loss of generality, we may assume that all the groups $P_{x,i}$ and $Q_{x}$ are non-trivial. Let $y_{x,i} \in Y_i$ with $f_i(y_{x,i}) = x$ such that $P_{x,i}$ is the Galois group of the field extension $K_{Y_i,y_{x,i}}/K_{X,x}$. Take $I = Q_{x}$ and

$$V_i = \text{Spec}(\hat{O}_{Y_i,y_{x,i}}^{N_{x,i}})$$

in Lemma 4.2. Then we obtain a connected $Q_{x}$-Galois cover

$$V_x \to \text{Spec}(\hat{O}_{X,x}) \times \mathbb{A}_u^1$$

of integral schemes with branched locus $x \times \mathbb{A}_u^1$ over which it is totally ramified such that

$$V_x \times_{\mathbb{A}_u^1} (u = 1) \cong V_1 \quad \text{and} \quad V_x \times_{\mathbb{A}_u^1} (u = -1) \cong V_2$$

as $Q_{x}$-Galois covers over $\text{Spec}(\hat{O}_{X,x})$. Taking $\Gamma = P_{x,1}$, $G = Q_{x}$, $X$ as the affine $u$-line over $K_{X,x}$ and $X' \subset X$ as the point $(u = 1)$ in [11, Theorem 3.11], there
is a connected $P_{x,1}$-Galois étale cover $W'_{x,1} \to \text{Spec}(K_{X,x}) \times A^1_u$ that dominates the $Q_x$-Galois étale cover

$$V_x \times_{\text{Spec}(\hat{O}_{X,x}) \times A^1_u} (\text{Spec}(K_{X,x}) \times A^1_u)$$

of integral $K_{X,x}$-curves. Taking normalization of $\text{Spec}(\hat{O}_{X,x}) \times A^1_u$ in the function field of $W'_{x,1}$, we obtain a connected $P_{x,1}$-Galois cover

$$g_{x,1} : W_{x,1} \to \text{Spec}(\hat{O}_{X,x}) \times A^1_u$$

dominating $V_x \to \text{Spec}(\hat{O}_{X,x}) \times A^1_u$ such that the fibre of $g_{x,1}$ over $(u = 1)$ is $\text{Spec}(\hat{O}_{Y_1,y,x})$. Similarly, we obtain a connected $P_{x,2}$-Galois cover

$$g_{x,2} : W_{x,2} \to \text{Spec}(\hat{O}_{X,x}) \times A^1_u$$

dominating $V_x \to \text{Spec}(\hat{O}_{X,x}) \times A^1_u$ such that the fibre of $g_{x,2}$ over $(u = -1)$ is $\text{Spec}(\hat{O}_{Y_2,y,x})$. As in the proof of Lemma 4.2, we choose the covers $g_{x,1}$ and $g_{x,2}$ such that the fibres over every point in a dense open subset of $A^1_u$ have inertia group $P_{x,1}$ and $P_{x,2}$, respectively, over the closed point $x$ of $\text{Spec}(\hat{O}_{X,x})$.

By [6, Lemma 3.2, Remark 3.4], there are connected $G_1$-Galois and $G_2$-Galois covers of $X_R$ satisfying the hypothesis of [6, Lemma 3.3, Remark 3.4]. We conclude that for each $x \in B$ and $i = 1, 2$, there is a dense open subset $W_{x,i} \subset A^1_u$ such that for every closed point $(u = \beta_{x,i})$ in $W_{x,i}$, the following hold. There are connected $G_1$-Galois and $G_2$-Galois covers $Z_1 \to X$ and $Z_2 \to X$, respectively, both covers are étale away from $B$; for $x \in B$, $P_{x,1}$ and $P_{x,2}$ occur as the inertia groups at points $z_{x,1} \in Z_1$ and $z_{x,2} \in Z_2$ above $x$ in the respective covers. Moreover, for each $x \in B$ we have the isomorphisms

$$\text{Spec}(K_{Z_1,z_{x,1}}) \cong W'_{x,1} \times_{\text{Spec}(K_{X,x}) \times A^1_u} (\text{Spec}(K_{X,x}) \times (u = \beta_{x,1})),$$

$$\text{Spec}(K_{Z_2,z_{x,2}}) \cong W'_{x,2} \times_{\text{Spec}(K_{X,x}) \times A^1_u} (\text{Spec}(K_{X,x}) \times (u = \beta_{x,2})), $$

as $P_{x,1}$-Galois and $P_{x,2}$-Galois covers of $\text{Spec}(K_{X,x})$, respectively. In particular, for each $x \in B$, taking any point $(u = \beta_x)$ in $W_{x,1} \cap W_{x,2}$, we obtain covers $Z_1 \to X$ and $Z_2 \to X$ with Galois groups $G_1$ and $G_2$, respectively, both the covers are étale away from $B$; for $x \in B$, $P_{x,1}$ and $P_{x,2}$ occur as the inertia groups at points $z_{x,1} \in Z_1$ and $z_{x,2} \in Z_2$ above $x$ in the respective covers with $K_{z_{x,1},z_{x,1}}^N/K_{X,x} \cong K_{z_{x,2},z_{x,2}}^N/K_{X,x}$ as $Q_x$-Galois extensions.

We recall and restate the following theorem due to Harbater which will be used throughout Sections 7–9.
Theorem 4.7 ([10, Corollary to Patching Theorem]): Let \( r \geq 1 \) be an integer. Let \( G \) be a finite group, \( G_1 \) and \( G_2 \) be two subgroups of \( G \) such that \( G = \langle G_1, G_2 \rangle \). Let \( X \) be a smooth projective connected \( k \)-curve. Let \( B \) be a finite set of closed points of \( X \) containing a point \( x_0 \). Let \( B' := \{ \eta_0, \ldots, \eta_r \} \) be a set of distinct points of \( \mathbb{P}^1 \). Let \( a \in G_1 \cap G_2 \) be an element of order prime-to-\( p \). Assume that

1. there is a connected \( G_1 \)-Galois cover \( f: Y \to X \) étale away from \( B \) such that \( I_x \) occurs as an inertia group above \( x \in B \) and \( I_{x_0} = \langle a \rangle \);
2. there is a connected \( G_2 \)-Galois cover \( g: W \to \mathbb{P}^1 \) étale away from \( B' \) such that \( J_i \) occurs as an inertia group above \( \eta_i, 0 \leq i \leq r \), and such that \( J_0 = \langle a^{-1} \rangle \).

Then there is a set \( B'' = \{ x_1, \ldots, x_{r-1} \} \) of closed points of \( X \) disjoint from \( B \) and a connected \( G \)-Galois cover of \( X \) étale away from \( B \cup B'' \) such that \( I_x \) occurs as an inertia group above \( x \in B \setminus \{ x_0 \} \), \( J_r \) occurs as an inertia group above \( x_0 \) and \( J_i \) occurs as an inertia group above \( x_i, 1 \leq i \leq r - 1 \).

A special case of the following result appeared in [12] that solves the split quasi-\( p \) embedding problem.

Theorem 4.8: Let \( G \) be a finite group, \( \psi: Y \to X \) be a smooth connected \( G \)-Galois cover. Let \( x_0 \in X \) be a closed point. Let \( \Gamma \) be a finite group generated by \( G \) and a quasi \( p \)-group \( H \) such that there is a \( p \)-subgroup \( P \) of \( H \) which is normalized by \( G \) and such that the pair \( (H, P) \) is realizable. Then there is a \( \Gamma \)-Galois cover \( \phi: Z \to X \) of smooth projective connected \( k \)-curves dominating the cover \( \psi \) such that the following hold:

1. For a closed point \( x \neq x_0 \), if \( I_x \) occurs as an inertia group at a point over \( x \) for the cover \( \psi \), then \( I_x \) also occurs as an inertia group at a point over \( x \) for the cover \( \phi \);
2. if \( I_0 \) occurs as an inertia group at a point above \( x_0 \) for the cover \( \psi \), \( I_0P \) occurs as an inertia group at a point above \( x_0 \) for the cover \( \phi \);
3. the covers

\[
Z/\langle H^\Gamma \rangle \to X \quad \text{and} \quad Y/(G \cap \langle H^\Gamma \rangle) \to X
\]

are isomorphic as \( \Gamma/\langle H^\Gamma \rangle \)-Galois covers of \( X \).
Proof. By [12, Theorem 2.1, Theorem 4.1] the above conclusion holds when $P$ is replaced by a Sylow $p$-subgroup of $H$ which is normalized by $G$. But the same proof works under our hypothesis with the additional assumption that there is a connected $H$-Galois étale cover of the affine line such that $P$ occurs as an inertia group above $\infty$. ■

5. Inertia Conjecture for the alternating groups

In this section, we prove the Inertia Conjecture (Conjecture 1.1) for some alternating groups. Recall that the IC was proved to be true for $A_p$ ([5, Theorem 1.2]) and when $p \equiv 2 \pmod{3}$ for $A_{p+2}$([16, Theorem 1.2]). We show that when $p \equiv 2 \pmod{3}$ the IC is true for the groups $A_{p+1}$, $A_{p+3}$ and $A_{p+4}$, and with some extra condition on $p$ the IC is also true for $A_{p+5}$. The covers will be constructed using the results and techniques from Section 3 and Corollary 4.4 from the previous section. Throughout this section $\tau$ denotes the $p$-cycle $(1, \ldots, p)$ in $S_p$.

In view of Equation (3.1), to prove the IC for $A_d$, $p < d < 2p$, we need to prove that there is a connected $A_d$-Galois étale cover of the affine line such that $I = \langle \tau \rangle \rtimes \langle \theta^i \omega \rangle$ occurs as an inertia group above $\infty$ (i.e., the pair $(A_d, I)$ is realizable) for every $1 \leq i \leq p-1$ and $\omega \in \text{Sym}(\{p+1, \ldots, d\})$ such that $\theta^i \omega$ is an even permutation. Note that for $\theta^i \omega \in A_d$, $i$ is an even integer if and only if $\omega$ is an even permutation. Now we make some observations which reduce the proof of the IC to the realization of the pair $(A_d, I)$ to a fewer cases.

Remark 5.1: Since $\langle \theta^i \omega \rangle = \langle \theta^{i(p-1)} \omega \rangle$, it is enough to consider the $i$’s dividing $p - 1$. Also using Abhyankar’s Lemma ([18, XIII, Proposition 5.2]) it is enough to prove for the cases when $I$ is a maximal inertia group in the sense of [13, Section 4.9].

In fact, the following result shows that it is enough to consider the more restricted cases when $\omega$ acts on the set $\{p+1, \ldots, d\}$ either with one fixed point or without any fixed point, provided the IC is true for the alternating groups of lower degrees.

Lemma 5.2: Let $p$ be an odd prime, $p + 1 \leq d \leq 2p - 1$. Assume that the pair $(A_d, I)$ is realizable for a subgroup $I \subset A_d$ which fixes $\geq 1$ points in $\{1, \ldots, d\}$. Then for all $d' \geq d$, the pair $(A_{d'}, I)$ is also realizable.
In particular, to prove the IC for $A_d$, $p + 2 \leq d \leq 2p - 1$, it is enough to prove that the IC is true for each $A_u$, $p \leq u \leq d - 2$, and that the pair $(A_d, I)$ is realizable for each $I \subset N_{A_d}(<\tau>)$ such that $I$ fixes 0 or 1 point in the set $\{1, \ldots, d\}$.

Proof. This is a direct consequence of Corollary 4.4 with $G_i = \text{Alt}(\text{Supp}(I) \cup S_i)$ for every set $S_i \subset \{1, \ldots, d'\} \setminus \text{Supp}(I)$ of size $d - |\text{Supp}(I)|$. The second statement follows from the first one. 

We are now ready to prove that IC for certain alternating groups.

**Theorem 5.3:** Let $p \equiv 2 \pmod{3}$ be an odd prime. Then the IC is true for the group $A_{p+1}$.

**Proof.** By Remark 5.1, it is enough to prove that the pair $(A_{p+1}, I)$ is realizable for $I = <\tau> \rtimes <\theta^2>$. Set $s = 3$, $r = 1$, $n_1 = p - 2$, $n_2 = 2$, $n_3 = 1$. By Lemma 3.10(4), Assumption 3.8 holds for the choice $(\alpha_1, \alpha_2, \alpha_3, \beta_1) = (3/4, 1/4, 1, 0)$.

Let $\psi: Y \to \mathbb{P}^1$ be the degree-$(p+1)$ cover given by the affine equation

$$\prod_{i=1}^{3}(y - \alpha_i)^{n_i} - xy = 0.$$  

Let $\phi: Z \to \mathbb{P}^1$ be the Galois closure of the cover $\psi$ with Galois group $G$. Then by Proposition 3.11, $G$ is a primitive subgroup of $S_{p+1}$ and the cover $\phi$ is étale away from $\{0, \infty\}$ such that $<(1, \ldots, p-2)(p-1, p)>$ occurs as an inertia group over 0 and $<\tau> \rtimes <\theta>$ occurs as an inertia group over $\infty$. Since $p - 2$ is odd, $G$ contains the transposition $(p-1, p)$ and since $G$ also contains the $p$-cycle $\tau$, by [21, Lemma 4.4.3], $G = S_{p+1}$. After the pullback of $\phi$ under the $[2(p-2)]$-Kummer cover we obtain a connected $A_{p+1}$-Galois étale cover of the affine line such that $I$ occurs as an inertia group above $\infty$. 

**Theorem 5.4:** Let $p \equiv 2 \pmod{3}$ be an odd prime. Then the IC is true for the group $A_{p+3}$.

**Proof.** When $p \equiv 2 \pmod{3}$, by Abhyankar’s Lemma it is enough to prove that there is a connected $A_{p+3}$-Galois cover of $\mathbb{P}^1$ étale away from $\infty$ such that $I = <\tau> \rtimes <\beta>$ occurs as an inertia group at a point above $\infty$ where $\beta$ is of the form $\beta = \theta^2(p+1, p+2, p+3)$ or $\beta = \theta(p+1, p+2)$. These are immediate from Corollary 3.13 and Corollary 3.15.
Theorem 5.5: Let \( p \equiv 2 \pmod{3} \) be an odd prime. Then the IC is true for the group \( A_{p+4} \).

Proof. Let \( p \equiv 2 \pmod{3} \) be an odd prime. By Abhyankar's Lemma, to prove the IC for \( A_{p+4} \) it is enough to prove that there is a connected \( A_{p+4} \)-Galois cover of \( \mathbb{P}^1 \) étale away from \( \infty \) such that \( I = \langle \tau \rangle \times \langle \beta \rangle \) occurs as an inertia group at a point above \( \infty \) where \( \beta \) is of the form \( \beta = \theta(p+1, p+2) \) or \( \beta = \theta^2(p+1, p+2, p+3) \) or \( \beta = \theta(p+1, p+2, p+3, p+4) \).

By Theorem 5.4, the pair \( (A_{p+3}, \langle \tau \rangle \times \langle \theta(p+1, p+2) \rangle) \) is realizable. So by Lemma 5.2, the pair \( (A_{p+4}, \langle \tau \rangle \times \langle \theta(p+1, p+2) \rangle) \) is realizable.

Now fix an element \( w_3 \in k \) such that \( w_3^2 = 3 \). Then for \( s = 2, r = 2, n_1 = p+2, n_2 = 2, m_1 = 3, m_2 = 1 \), Assumption 3.8 holds for the choice

\[
(\alpha_1, \alpha_2, \beta_1, \beta_2) = \left(1 + \frac{w_3}{4}, 1 - \frac{w_3}{4}, 0, 1\right).
\]

Apply Proposition 3.11 to obtain a connected \( S_{p+4} \)-Galois cover of \( \mathbb{P}^1 \) étale away from \( \{0, \infty\} \) such that \( \langle \tau \rangle \times \langle \theta^2(p+1, p+2, p+3) \rangle \) occurs as an inertia group above \( 0 \) and \( \langle \tau \rangle \times \langle \theta(p+1, p+2, p+3) \rangle \) occurs (since \( (p-1, r+s-1) = (p-1, 3) = 1 \) by our assumption on \( p \)) as an inertia group above \( \infty \). Consider a \( [2(p+2)] \)-Kummer pullback of the above cover to obtain the Galois cover \( \phi: Y \to \mathbb{P}^1 \).

Note that

\[
(2(p+2), p-1) = 2(p+2, p-1) = 2(3, p-1) = 2
\]

and

\[
(2(p+2), 3) = (p+2, 3) = (p-1, 3) = 1.
\]

So by Lemma 3.12, the cover \( \phi \) is a connected \( A_{p+4} \)-Galois étale cover of the affine line such that \( \langle \tau \rangle \times \langle \theta^2(p+1, p+2, p+3) \rangle \) occurs as an inertia group above \( \infty \).

For the last case, fix an element \( w_2 \in k \) such that \( w_2^2 = 2 \). Then for \( s = 3, r = 1, n_1 = p-2, n_2 = n_3 = 3 \), Assumption 3.8 holds for the choice

\[
(\alpha_1, \alpha_2, \alpha_3, \beta_1) = \left(1, \frac{1+w_2}{3}, \frac{1-w_2}{3}, 0\right).
\]

By Proposition 3.11, there is a connected \( A_{p+4} \)-Galois cover of \( \mathbb{P}^1 \) étale away from \( \{0, \infty\} \) such that

\[
\langle (1, \ldots, p-2)(p-1, p, p+1)(p+2, p+3, p+4) \rangle
\]
occurs as an inertia group above 0 and
\[ \langle \tau \rangle \rtimes \langle \theta(p + 1, p + 2, p + 3, p + 4) \rangle \]
occurs (since \((p - 1, r + s - 1) = (p - 1, 3) = 1\)) as an inertia group above \(\infty\). Let \(\phi: Y \to \mathbb{P}^1\) be the Galois cover obtained by a \([\text{lcm}\{3, p - 2\}]\)-Kummer pullback. Since
\[ (3(p - 2), p - 1) = (p - 2, p - 1) = 1 \]
and
\[ (3(p - 2), 4) = (p - 2, 4) = 1, \]
we also have \((\text{lcm}\{3, p - 2\}, p - 1) = 1 = (\text{lcm}\{3, p - 2\}, 4)\). So by Lemma 3.12, \(\phi\) is a connected \(A_{p+4}\)-Galois étale cover of \(\mathbb{A}^1\) and
\[ \langle \tau \rangle \rtimes \langle \theta(p + 1, p + 2, p + 3, p + 4) \rangle \]
occurs as an inertia group over \(\infty\); in other words, the pair
\[ (A_{p+4}, \langle \tau \rangle \rtimes \langle \theta(p + 1, p + 2, p + 3, p + 4) \rangle) \]
is realizable. \(\blacksquare\)

**Lemma 5.6:** When \(p \equiv 2 \pmod{3}\) is a prime \(> 5\), the pair \((A_{p+5}, I_i)\) is realizable for \(2 \leq i \leq 5\), where
\[
I_2 = \langle \tau \rangle \rtimes \langle \theta(p + 1, p + 2) \rangle, \\
I_3 = \langle \tau \rangle \rtimes \langle \theta^2(p + 1, p + 2, p + 3) \rangle, \\
I_4 = \langle \tau \rangle \rtimes \langle \theta(p + 1, p + 2, p + 3, p + 4) \rangle, \\
I_5 = \langle \tau \rangle \rtimes \langle \theta^2(p + 1, p + 2, p + 3, p + 4, p + 5) \rangle. 
\]
Additionally if \(4 \nmid (p + 1)\), the pair
\[ (A_{p+5}, \langle \tau \rangle \rtimes \langle \theta(p + 1, p + 2)(p + 3, p + 4, p + 5) \rangle) \]
is also realizable.

**Proof.** Let \(p \equiv 2 \pmod{3}\) be a prime \(> 5\). Since the IC is true for \(A_{p+3}\) (Theorem 5.4) and for \(A_{p+4}\) (Theorem 5.5), by Lemma 5.2, the first two cases follow.

Now we consider the realization of \(I_4\) as an inertia group for a connected \(A_{p+5}\)-Galois étale cover of \(\mathbb{A}^1\). Fix \(w_{2/3} \in k\) such that \(w_{2/3}^2 = 2/3\). Then
for \( s = 2, r = 2, n_1 = p + 2, n_2 = 3, m_1 = 4, m_2 = 1, \) Assumption 3.8 holds for the choice

\[
(\alpha_1, \alpha_2, \beta_1, \beta_2) = \left( \frac{1 - 3w_2/3}{5}, \frac{1 + 2w_2/3}{5}, 0, 1 \right).
\]

So we can apply Proposition 3.11 to obtain a connected \( A_{p+5} \)-Galois cover of \( \mathbb{P}^1 \) étale away from \( \{0, \infty\} \) such that

\[
\langle (1, \ldots, p + 2)(p + 3, p + 4, p + 5) \rangle
\]

(of order \( \text{lcm}\{3, p + 2\} = \frac{3(p+2)}{3(p+2)} = 3(p+2) \)) occurs as an inertia group above 0 and

\[
\langle \tau \rangle \times \langle \theta(p+1, p+2, p+3, p+4) \rangle
\]
occurs (since \( (p-1, r+s-1) = (p-1, 3) = 1 \)) as an inertia group above \( \infty \). As in the proofs of the other cases, we again consider a \([3(p+2)]\)-Kummer pullback and obtain a Galois cover \( Y \to \mathbb{P}^1 \). As \( (3(p+2), p-1) = 1 = (3(p+2), 4) \), by Lemma 3.12, this cover is a connected \( A_{p+5} \)-Galois cover, étale away from \( \infty \) and \( I_4 \) occurs as an inertia group above \( \infty \). So the pair \( (A_{p+5}, I_4) \) is realizable.

For the next case, when \( (5, p-1) = 1 \), the pair \( (A_{p+5}, I_5) \) is realizable by Corollary 3.13. So let \( (5, p-1) = 5 \). Fix \( w_7 \in k \) such that \( w_7^2 = 7 \). Then for \( s = 3, r = 1, n_1 = p - 2, n_2 = 6, n_3 = 1, m_1 = 5, \) Assumption 3.8 holds for the choice

\[
(\alpha_1, \alpha_2, \alpha_3, \beta_1) = \left( \frac{w_7 + 1}{2}, \frac{w_7 - 1}{6}, 2, 0 \right).
\]

By Proposition 3.11, there is a connected \( S_{p+5} \)-Galois cover of \( \mathbb{P}^1 \) étale away from \( \{0, \infty\} \) such that

\[
\langle (1, \ldots, p - 2)(p - 1, p + 1, p + 2, p + 3, p + 4) \rangle
\]
occurs as an inertia group above 0 (of order \( 2 \times \text{lcm}\{3, p - 2\} \)) and

\[
\langle \tau \rangle \times \langle \theta(p+1, p+2, p+3, p+4, p+5) \rangle
\]
occurs (since \( (p-1, r+s-1) = (p-1, 3) = 1 \)) as an inertia group above \( \infty \). Since \( 5|(p-1) \) by our assumption, we have \( (p - 2, 5) = 1 \). Also, \( (3(p-2), p-1) = 1 \). These imply that

\[
(2 \times \text{lcm}\{3, p - 2\}, p - 1) = 2 \quad \text{and} \quad (2 \times \text{lcm}\{3, p - 2\}, 5) = 1.
\]

So by Lemma 3.12, after a \([\text{lcm}\{6, p - 2\}]\)-Kummer pullback, we obtain a connected \( A_{p+5} \)-Galois étale cover of the affine line such that \( I_5 \) occurs as an inertia group above \( \infty \).
Now we consider the last case with the additional assumption \((p + 1, 4) = 2\) (this is equivalent to \(4 \nmid (p+1)\)). Set \(s = 2 = r, m_1 = 2, m_2 = 3, n_1 = n_2 = \frac{p+5}{2}\). Choose an element \(w_3 \in k\) such that \(w^2 = 3\). Then for
\[
(\alpha_1, \alpha_2, \beta_1, \beta_2) = \left(\frac{3 + 2w_3}{5}, \frac{3 - 2w_3}{5}, 0, 1\right),
\]
Assumption 3.8 holds. So we can apply Proposition 3.11 to obtain a connected \(A_{p+5}\)-Galois cover of \(\mathbb{P}^1\) branched only at 0 and \(\infty\) such that
\[
\langle \left(1, \ldots, \frac{p+5}{2}\right) \left(\frac{p+5}{2} + 1, \ldots, p + 5\right) \rangle
\]
occurs as an inertia groups above 0 and
\[
I = \langle \tau \rangle \rtimes \langle \theta(p + 1, p + 2)(p + 3, p + 4, p + 5) \rangle
\]
occurs (as \((p - 1, r + s - 1) = (p - 1, 3) = 1\)) as an inertia group at a point over \(\infty\). Observe the following congruence relations:
\[
(p + 5, p - 1) = (6, p - 1) = 2, \\
(p + 5, 2) = 2, \\
(p + 5, 3) = 1.
\]
By our assumption, \((p + 1, 4) = 2\), and so \(\frac{p+5}{2}\) is an odd integer. Now the above relations imply that
\[
\left(\frac{p+5}{2}, p - 1\right) = 1 \quad \text{and} \quad \left(\frac{p+5}{2}, 6\right) = 1.
\]
So, after a \([\frac{p+5}{2}]\)-Kummer pullback (by Lemma 3.12), we obtain a connected \(A_{p+5}\)-Galois étale cover of \(A^1\) and \(I_5\) occurs as an inertia group above \(\infty\).

Using the above lemma and Abhyankar’s Lemma ([18, XIII, Proposition 5.2]), we conclude the following result.

**Theorem 5.7:** Let \(p \equiv 2 \pmod{3}\) be a prime \(\geq 17\) such that \(4 \nmid (p+1)\). Then the IC is true for the group \(A_{p+5}\).

### 6. Generalizations of the Inertia Conjecture

In this section, we pose certain questions which generalize the Inertia Conjecture. Although these generalizations make sense for \(p = 2\), we restrict ourselves to \(p > 2\). Consider the following notation for the rest of this section.
Notation 6.1: Let $r \geq 1$, $X$ be a smooth projective connected $k$-curve, $G$ be a finite group. Let $B = \{x_1, \ldots, x_r\} \subset X$ be a set of closed points in $X$. Let $P_1, \ldots, P_r$ be $p$-subgroups of $G$, possibly trivial. Define a subnormal series $\{H_j\}_{j \geq 0}$ of $G$ inductively as follows.

$$H_0 := G, H_{j+1} := \langle P_i \cap H_j | 1 \leq i \leq r \rangle \subset G.$$ 

Then each $H_{j+1}$ is a normal subgroup of $H_j$ containing all the $P_i$’s. Since $G$ is a finite group, there is a minimal non-negative integer $l$ such that $H_j = H_l$ for all $j \geq l$.

Under the above notation, we observe the following. Let $\phi: Z \to X$ be a connected $G$-Galois cover étale away from $B$ and $I_i$ be an inertia group above $x_i$, $1 \leq i \leq r$. Set $Y_0 := X$. Since $H_1$ is a normal subgroup of $G$, the cover $\phi$ factors through a connected $G/H_1$-Galois cover $\psi_1: Y_1 \to X$, étale away from $B$, and $I_i / I_i \cap H_1$ occurs as an inertia group above $x_i$, $1 \leq i \leq r$. Inductively for $1 \leq j \leq l$, the $H_{j-1}$-Galois cover $Z \to Y_{j-1}$ factors through a connected $H_{j-1}/H_j$-Galois cover $\psi_j: Y_j \to Y_{j-1}$ that is étale away from $B$. Moreover, if $y_{i,j}$ is a point of $Y_j$ lying above $x_i$, then $I_i \cap H_{j-1} / I_i \cap H_j$ occurs as an inertia group above $y_{i,j-1}$ in the cover $\psi_j$, $1 \leq i \leq r$. So $\phi$ is the composition of a tower

$$Z \longrightarrow Y_1 \longrightarrow Y_{l-1} \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0 = X,$$

where $\psi_j: Y_j \to Y_{j-1}$ is an $H_{j-1}/H_j$-Galois cover of smooth projective connected $k$-curves for $1 \leq j \leq l$. Also note that $l$ is the minimal non-negative integer such that $H_l = \langle P_i H_j | 1 \leq i \leq r \rangle$.

Note that, when $X = \mathbb{P}^1$ in the above question and $|B| = 1$, the above observation is the necessary condition on the inertia groups from the IC. In other words, the unsolved forward direction of the IC is the converse of the above observation in the case $X = \mathbb{P}^1$ and $|B| = 1$. Motivated by this, we ask the following question.

Question 6.2 ($Q[r, X, B, G]$): Let $r$, $X$, $G$ and $B = \{x_1, \ldots, x_r\} \subset X$ be as in Notation 6.1. For $1 \leq i \leq r$ let $I_i$ be a subgroup of $G$ which is an extension of a $p$-group $P_i$ (possibly trivial) by a cyclic group of order prime-to-$p$ such that there is a tower

$$Y_1 \longrightarrow Y_{l-1} \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0 := X$$
of covers of smooth projective connected $k$-curves with $l$ and $H_j$’s are as in Notation 6.1, $\psi_j : Y_j \to Y_{j-1}$ is an $H_{j-1}/H_j$-Galois cover, and if $y_{i,j}$ is a point of $Y_j$ lying above $x_i$, then $I_i \cap H_{j-1}/I_i \cap H_j$ occurs as an inertia group above $y_{i,j-1}$ in the cover $\psi_j$, $1 \leq i \leq r$. Let $\psi : Y_l \to X$ denote the composite morphism.

Does there exist a connected $G$-Galois cover $\phi$ of $X$ étale away from $B$ dominating the cover $\psi$ such that $I_i$ occurs as an inertia group above the point $x_i$ for $1 \leq i \leq r$?

**Remark 6.3:** We have the following preliminary observations.

1. Since $p(I_i) = P_i \subset H_l$ for all $i$, the cover $\psi$ and each Galois cover $\psi_i$ is tamely ramified.

2. If $B_{j-1} \subset Y_{j-1}$ is the branched locus of the cover $\psi_j$, $1 \leq j \leq l$, then $H_{j-1}/H_j \in \pi_A^{\text{ét}}(Y_{j-1} - B_{j-1})$. But only assuming this, we get a negative answer to the above question, as seen from Example 6.4 below. This is why we put the hypothesis about the existence of the cover $\psi$ with the given ramification behavior.

3. Example 6.5 shows that only assuming the existence of $\psi_j$’s for $j < l$, we get a negative answer to Question 6.2.

4. When all the $p$-groups $P_i$ are trivial, taking $\phi = \psi$ gives the answer to the question.

**Example 6.4:** Take $r = 1$, $X$ as an elliptic curve with origin $0$, $B = \{0\}$, $P_1$ as the trivial group, $G = I_1 = \mathbb{Z}/m$ where $m$ is coprime to $p$. Then $H_1$ is the trivial group. Since there is a connected $\mathbb{Z}/m$-Galois étale cover of $X$, we have $G/H_1 = \mathbb{Z}/m \in \pi_A^{\text{ét}}(X - B)$. From the Riemann–Hurwitz formula, we obtain

$$2g_Y - 2 = m(2 - 2) + \frac{m}{m}(m - 1).$$

So $m$ must be an odd integer. Thus if $m$ is an even integer, there is no $\mathbb{Z}/m$-Galois connected cover $Y \to X$ that is étale away from $\{0\}$ over which it is totally ramified.

**Example 6.5:** Let $H$ be the elementary abelian group

$$H = \langle \tau_1 \rangle \times \cdots \times \langle \tau_p \rangle \cong \mathbb{Z}/p \times \cdots \times \mathbb{Z}/p$$

of $p$-exponent $p$. Then $\langle \sigma \rangle \cong \mathbb{Z}/p$ acts on $H$ via the action of $\sigma$ on the set $\{1, \ldots, p\}$ as an element of $S_p$. Consider the wreath product

$$G = H \rtimes \langle \sigma \rangle \cong \mathbb{Z}/p \wr \mathbb{Z}/p.$$
Take $X$ to be an ordinary elliptic curve $E$ with origin 0. Take $r = 1$, $P = I = \langle \tau_1 \rangle$. Then $H_1 = H$, and $H_2 = P$. There is a unique connected $G/H \cong \mathbb{Z}/p$-Galois étale cover $\psi_1 : Y_1 := E \to E$. Now assume that there is a connected $G$-Galois cover $Y \to E$ dominating $\psi_1$ such that $P$ occurs as an inertia group over 0. As observed before, the connected $H_1$-Galois cover $Y \to Y_1$ factors via an étale Galois cover $Y_2 \to Y_1 = E$ with group $H_1/H_2$ which is an elementary abelian group of order $p^{l_1 - 1}$, a contradiction since the pro-$p$ étale fundamental group $\pi^p_1(E)$ of $E$ is a free profinite group on 1 generator.

In whatever follows, assume that $P_1$ is non-trivial (cf. (4)).

So $H_j$ is a non-trivial normal quasi $p$-subgroup of $H_{j-1}$ for $1 \leq j \leq l$. We make the following observation when $X = \mathbb{P}^1$ and $l \in \{0, \infty\}$.

Remark 6.6: When $l = 0$, we have $G = \langle P_i^G | 1 \leq i \leq r \rangle$, and so each inertia group $I_i$ is contained in $H_0 = G$. Now let $X = \mathbb{P}^1$. Since there is no non-trivial étale cover of $\mathbb{P}^1$, if each $I_i \subset H_l$, then $\psi$ is the identity map $\mathbb{P}^1 \to \mathbb{P}^1$, and so $l = 0$, i.e.,

$$G = H_0 = \langle P_i^G | 1 \leq i \leq r \rangle.$$ 

These equivalent conditions imply that $G$ must be a quasi $p$-group. Also note that if $l = 1$ and $\psi_1$ is a two-point branched Galois cover of $\mathbb{P}^1$ (so in particular, $r = 2$), by the Riemann–Hurwitz formula, $\psi$ is the $\mathbb{Z}/n$-Galois Kummer cover of $\mathbb{P}^1$ branched at $\{x_1, x_2\}$ which is totally ramified above these points.

With the above observation, a special case of Question 6.2 (i.e., $Q[r, \mathbb{P}^1, B, G]$ with each $I_i \subset H_l$ or equivalently $l = 0$) is the following which we pose as the Generalized Inertia Conjecture (GIC).

**Conjecture 6.7 (GIC):** Let $r \geq 1$, $G$ be a finite quasi $p$-group. For $1 \leq i \leq r$ let $I_i$ be a subgroup of $G$ which is an extension of a $p$-group $P_i$ by a cyclic group of order prime-to-$p$ such that $G = \langle P_1^G, \ldots, P_r^G \rangle$. Let $B = \{x_1, \ldots, x_r\}$ be a set of closed points in $\mathbb{P}^1$. Then there is a connected $G$-Galois cover of $\mathbb{P}^1$ étale away from $B$ such that $I_i$ occurs as an inertia group above the point $x_i$ for $1 \leq i \leq r$.

Remark 6.8: As in the case of Question 6.2 we again allow $P_i$’s to be trivial for $2 \leq i \leq r$. As noted earlier, when $r = 1$, this is the unsolved direction of the IC (Conjecture 1.1).
Remark 6.9: Under the notation and the hypothesis of Question 6.2, we can ask the following weaker questions.

(1) \((Q[r, X, B, G]_{\text{weak}})\) Does there exist a connected \(G\)-Galois cover \(\phi\) of \(X\) that is étale away from \(B\) and \(I_i\) occurs as an inertia group above \(x_i\), \(1 \leq i \leq r\)? (Here we drop the condition on \(\phi\) to dominate the cover \(\psi\).)

(2) \((Q[r, X, G])\) Do there exist a set \(B' = \{x'_1, \ldots, x'_r\}\) of closed points in \(X\) and a connected \(G\)-Galois cover \(\phi\) of \(X\) étale away from \(B'\) such that \(I_i\) occurs as an inertia group above \(x'_i\), \(1 \leq i \leq r\)?

Note that when the \(\psi_j\)'s in Question 6.2 are uniquely determined, \(Q[r, X, B, G]_{\text{weak}}\) is equivalent to \(Q[r, X, B, G]\). Later, we will see some partial answers to Question 6.2 for \(l \in \{0, 1\}\) as the application of the formal patching technique and by constructing covers given by the explicit equations. When the inertia groups \(I_i\) are \(p\)-groups \(P_i\), we have the following special case of the GIC which we see as a generalization of the PWIC (Conjecture 1.2).

Conjecture 6.10 (GPWIC or Generalized Purely Wild Inertia Conjecture): Let \(r \geq 1\) and \(G\) be a finite quasi \(p\)-group. Let \(P_1, \ldots, P_r\) be non-trivial \(p\)-subgroups of \(G\) such that \(G = \langle P^G_1, \ldots, P^G_r \rangle\). Let \(B = \{x_1, \ldots, x_r\}\) be a set of closed points in \(\mathbb{P}^1\). Then there is a connected \(G\)-Galois cover of \(\mathbb{P}^1\) étale away from \(B\) such that \(P_i\) occurs as an inertia group above the point \(x_i\) for \(1 \leq i \leq r\).

7. Towards the GPWIC

In this section, we see some affirmative results for the GPWIC (Conjecture 6.10). Let \(G\) be a quasi \(p\)-group, \(P_1, \ldots, P_r\) be non-trivial \(p\)-subgroups of \(G\) for some \(r \geq 1\) such that \(G = \langle P^G_1, \ldots, P^G_r \rangle\). Let \(B = \{x_1, \ldots, x_r\}\) be a set of closed points in \(\mathbb{P}^1\). Then the GPWIC says that there is a connected \(G\)-Galois cover of \(\mathbb{P}^1\) étale away from \(B\) such that \(P_i\) occurs as an inertia group above the point \(x_i\). We use these notations throughout this section and use the results from Section 4 for the proofs. We start with the following group theoretic observation.

Lemma 7.1: Let \(G\) be a \(p\)-group, \(r \geq 1\) be an integer. Let \(P_1, \ldots, P_r\) be \(p\)-subgroups of \(G\) such that \(G = \langle P^G_1, \ldots, P^G_r \rangle\). Then

\[ G = \langle P_1, \ldots, P_r \rangle. \]
Proof. The result holds for an abelian $p$-group $G$. So assume that $G$ is non-abelian. Then the Frattini subgroup $\Phi(G)$ of $G$ is non-trivial. Consider the Frattini quotient $G \twoheadrightarrow G/\Phi(G)$. Under this map, $P_i$ has image

$$P_i/P_i \cap \Phi(G)$$

and

$$G/\Phi(G) = \langle (P_i/P_i \cap \Phi(G))/\Phi(G) \mid 1 \leq i \leq r \rangle.$$ 

Since $G/\Phi(G)$ is elementary abelian,

$$G/\Phi(G) = \langle P_1/P_1 \cap \Phi(G), \ldots, P_r/P_r \cap \Phi(G) \rangle.$$ 

Since $\Phi(G)$ is the set of non-generators of $G$, we obtain

$$G = \langle P_1, \ldots, P_r, \Phi(G) \rangle = \langle P_1, \ldots, P_r \rangle. \quad \blacksquare$$

**Theorem 7.2:** The GPWIC holds for every $p$-group.

Proof. By Lemma 7.1, $G = \langle P_1, \ldots, P_r \rangle$. We proceed via induction on $r$. The pair $(P_r, P_r)$ is realizable. By the induction hypothesis, there is a connected $G_1 := \langle P_1, \ldots, P_{r-1} \rangle$-Galois cover of $\mathbb{P}^1$ étale away from $\{x_1, \ldots, x_{r-1}\}$ such that $P_i$ occurs as an inertia group above the point $x_i$ for $1 \leq i \leq r - 1$. Now the result follows from Theorem 4.7. \thmbox{\blackslug}

**Theorem 7.3:** The GPWIC holds for any quasi $p$-group $G$ whose order is strictly divisible by $p$. In particular, it holds for alternating groups $A_d$ with $p \leq d \leq 2p - 1$ and for $\text{PSL}_2(p)$ when $p$ is an odd prime $\geq 5$.

Proof. Since $p^2 \nmid |G|$, each $P_i$ is a $p$-cyclic Sylow $p$-subgroup of $G$. Since $G$ is a quasi $p$-group, for each $i$,

$$G = \langle P_i^G \rangle.$$ 

We proceed by induction on $r$. By Raynaud’s proof of the Abhyankar’s Conjecture on the affine line ([19, Corollary 2.2.2]), the pair $(G, P_r)$ is realizable. Now we argue as in the proof of Theorem 7.2. \thmbox{\blackslug}

**Theorem 7.4:** Let $u \geq 1$ be an integer. Let $G = G_1 \times \cdots \times G_u$ where each $G_i$ is either a non-abelian simple quasi $p$-group of order strictly divisible by $p$ or a simple alternating group of degree coprime to $p$. Then the GPWIC is true for $G$. 

Proof. For any subset $\Lambda \subseteq \{1, \ldots, u\}$, set
\[ H_\Lambda := \prod_{\lambda \in \Lambda} G_\lambda, \]
and let $\pi_\Lambda : G \to H_\Lambda$ be the projection. For each $1 \leq i \leq r$, consider the set $\alpha_i$ consisting of $j \in \{1, \ldots, u\}$ such that $\pi_j(P_i)$ is non-trivial. Since $G_i$’s are simple non-abelian groups, the conjugates of $\pi_{\alpha_i}(P_i)$ in $H_{\alpha_i}$ generate $H_{\alpha_i}$. We proceed via induction on $r$. If $r = 1$, by [6, Remark 5.8, Corollary 5.7], the pair $(G, P_1)$ is realizable. So let $r \geq 2$. By the induction hypothesis, there is a connected $H_{\bigcup_{i=1}^{r-1} \alpha_i}$-Galois cover of $\mathbb{P}^1$ étale away from $\{x_1, \ldots, x_{r-1}\}$ such that $P_i$ occurs as an inertia group above the point $x_i$ for $1 \leq i \leq r - 1$. If $H_{\bigcup_{i=1}^{r-1} \alpha_i} = G$, the result follows by [8, Theorem 2]. Otherwise, set
\[ S := \{1, \ldots, u\} \setminus \bigcup_{i=1}^{r-1} \alpha_i. \]
Since $\bigcup_{1 \leq i \leq r} \alpha_i = \{1, \ldots, u\}$, we must have $S \subset \alpha_r$. Again by [6, Remark 5.8, Corollary 5.7], the pair $(H_{\alpha_r}, P_r)$ is realizable. Then the result follows by applying Theorem 4.7 with the groups $H_{\bigcup_{i=1}^{r-1} \alpha_i}$ and $H_{\alpha_r}$.

In the following, we show that the GPWIC is true for a certain product of groups if it is true for the individual groups.

**Theorem 7.5:** Let $G_1$ and $G_2$ be two finite quasi $p$-groups such that they have no non-trivial quotient in common. If the GPWIC is true for the groups $G_1$ and $G_2$, then it is also true for $G_1 \times G_2$.

**Proof.** Set $G := G_1 \times G_2$, and let $\pi_j : G \to G_j$ denote the projections for $j \in \{1, 2\}$. Let $r \geq 1$, $P_1, \ldots, P_r$ be non-trivial $p$-subgroups of $G$ such that $G = \langle P_1^G, \ldots, P_r^G \rangle$. Then for each $1 \leq i \leq r$, the two groups $\pi_1(P_i)$ and $\pi_2(P_i)$ cannot be simultaneously trivial. For $1 \leq i \leq r$, by Goursat’s lemma, there is a $p$-group $Q_i$ (possibly trivial) and the normal subgroups $N_{j,i}$ of $\pi_j(P_i)$ such that
\[ \pi_j(P_i)/N_{j,i} \cong Q_i \quad \text{and} \quad P_i \cong \pi_1(P_i) \times_{Q_i} \pi_2(P_i). \]
Let $Q_i'$ be a common quotient of $\pi_1(P_i)$ and $\pi_2(P_i)$ such that $Q_i$ is a quotient of $Q_i'$. Then $\pi_1(P_i) \times_{Q_i'} \pi_2(P_i) \subset \pi_1(P_i) \times_{Q_i} \pi_2(P_i)$, and so by [8, Theorem 2], it is enough to consider $Q_i$ to be a maximal common quotient of $\pi_1(P_i)$ and $\pi_2(P_i)$. Now
\[ G_j = \langle \pi_j(P_1)^{G_j}, \ldots, \pi_j(P_r)^{G_j} \rangle. \]
Let $B = \{x_1, \ldots, x_r\}$ be a set of closed points in $\mathbb{P}^1$. By the hypothesis, for $j = 1, 2$, there is a connected $G_j$-Galois cover $f_j: Y_j \to \mathbb{P}^1$ étale away from $B$ such that $\pi_j(P_i)$ occurs as an inertia group above the point $x_i \in B$. By Lemma 4.6, for $j = 1$ and 2, there is a connected $G_j$-Galois cover $Z_j \to \mathbb{P}^1$ étale away from $B$ such that $\pi_j(P_i)$ occurs as an inertia group above $x_i$, $1 \leq i \leq r$, and such that there is a point $z_{j,i} \in Z_j$ over $x_i$ with

$$K_{Z_1,z_{1,i}}^{N_{1,i}}/K_{\mathbb{P}^1,x_i} \cong K_{Z_2,z_{2,i}}^{N_{2,i}}/K_{\mathbb{P}^1,x_i}$$

as $Q_i$-Galois extensions. Since $Q_i$ is a maximal common quotient of $\pi_1(P_i)$ and $\pi_2(P_i)$, the extensions $K_{Z_1,z_{1,i}}/K_{\mathbb{P}^1,x_i}$ and $K_{Z_2,z_{2,i}}/K_{\mathbb{P}^1,x_i}$ are linearly disjoint over $K_{Z_{1,2},z_{1,i}} \cong K_{Z_{2,2},z_{2,i}}/K_{\mathbb{P}^1,x_i}$. Let $W$ be a dominant connected component of the normalization of $Z_1 \times_{\mathbb{P}^1} Z_2$. Since $G_1$ and $G_2$ have no common non-trivial quotient in common, the cover $\Psi: W \to \mathbb{P}^1$ is a connected $G_1 \times G_2$-Galois cover. For a point $w = (z_{1}', z_{2}')$ in $W$ with $f_j(z_{j}') = x$ the extension $K_{W,w}/K_{\mathbb{P}^1,x}$ is the compositum of the extension $K_{Z_1,z_{1,i}}/K_{\mathbb{P}^1,x}$ with the extension $K_{Z_2,z_{2,i}}/K_{\mathbb{P}^1,x}$. So the cover $\Psi$ is étale away from $B$ and

$$\pi_1(P_i) \times_Q \pi_2(P_2) = P_i$$

occurs as an inertia group above $x_i$.

Remark 7.6: The above theorem generalizes [15, Corollary 4.6] where the result was proved for a perfect group $G_1$ and a $p$-group $G_2$. The assumption on the non-existence of a common non-trivial quotient is for the benefit of the proof and we do not know whether the result is still valid otherwise.

We summarize the results of this section.

Corollary 7.7: The GPWIC (Conjecture 6.10) is true for the following quasi $p$-groups $G$:

1. $G$ is a $p$-group;
2. $G$ has order strictly divisible by $p$;
3. $G = G_1 \times \cdots \times G_u$ where each $G_i$ is either a simple alternating group of degree $d \geq p$, where $d = p$ or $(d,p) = 1$ or $\text{PSL}_2(p)$ or a $p$-group or a simple non-abelian group of order strictly divisible by $p$.  


8. Weaker results to the GPWIC

Although the status of the GPWIC (Conjecture 6.10) is open in general, some weaker statements related to the GPWIC are true. Namely, when we allow the branched locus to be sufficiently large or if we allow bigger inertia groups, there are suitable covers with the prescribed ramification. We also show that when the group is $A_d$ it is enough to add only one more branch point. Using [8, Theorem 2] one can increase the wild part of the inertia groups of a cover. In particular, let $G$ be a quasi $p$-group, $P_1, \ldots, P_r$ be $p$-subgroups of $G$. Let $Q_i$ be a Sylow $p$-subgroup of $G$ containing $P_i$. Let $B = \{x_1, \ldots, x_r\}$ be a set of closed points of $\mathbb{P}^1$. Then there is a connected $G$-Galois cover of $\mathbb{P}^1$ étale away from $B$ such that $Q_i$ occurs as an inertia group above $x_i$ for $1 \leq i \leq r$. In the following we show that the inertia groups can be taken as the Sylow $p$-subgroups of the normal quasi $p$-groups $\langle P_i \rangle$.

**Proposition 8.1:** Under the notation and hypothesis of Conjecture 6.10, for each $i$, there is a $p$-subgroup $Q_i \supset P_i$ in $\langle P_i \rangle$ such that there is a connected $G$-Galois cover of $\mathbb{P}^1$ étale away from $B$ and $Q_i$ occurs as an inertia group above the point $x_i$ for $1 \leq i \leq r$.

**Proof.** We proceed by induction on $r$. When $r = 1$, it is the consequence of [8, Theorem 2]. So let $r \geq 2$. For $1 \leq i \leq r$, let $Q_i$ be a Sylow $p$-subgroup of $H_i = \langle P_i \rangle$ containing $P_i$. Since $H_i$ is a quasi $p$-group, $H_i = \langle Q_i^{H_i} \rangle$ and by the $r = 1$ case, the pair $(H_i, Q_i)$ is realizable. By the induction hypothesis there is a connected $G_1 := \langle H_1, \ldots, H_{r-1} \rangle$-Galois cover of $\mathbb{P}^1$ étale away from $\{x_1, \ldots, x_{r-1}\}$ and $Q_i$ occurs as an inertia group above $x_i$. If $G_1 = G$, apply [8, Theorem 2]. Otherwise the result follows by Theorem 4.7 with $G_2 = H_r$.  

**Proposition 8.2:** Assume that the hypothesis and notation of Conjecture 6.10 hold. For each $1 \leq i \leq r$, let $C_i := \{P_{i,j}\}_{1 \leq j \leq t_i}$ be the set of all conjugates of $P_i$ in $G$. Let $\emptyset \neq A_i \subset C_i$ such that $G = \langle P \mid P \in A_i, 1 \leq i \leq r \rangle$. Let $l := \Sigma_i |A_i|$ and let $B = \{x_{i,j} \mid 1 \leq i \leq r, P_{i,j} \in A_i\}$ be a set of $l$ closed points in $\mathbb{P}^1$ such that $x_{i,1} = x_i$ for each $i$. Then there is a connected $G$-Galois cover of $\mathbb{P}^1$ étale away from $B$ and $P_i$ occurs as an inertia group above the point $x_{i,j} \in B$.

**Proof.** Since the inertia groups above a point in a connected Galois cover are conjugates, it is enough to prove that there is a connected $G$-Galois cover of $\mathbb{P}^1$ étale away from $B$ such that $P_{i,j}$ occurs as an inertia group above the
point \(x_{i,j} \in B\). We proceed via induction on \(l\). If \(l = 1\) then \(G = P_1\) and the result follows. Let \(l \geq 2\). Fix an \(i_0, 1 \leq i_0 \leq r\), and element \(P_{i_0,j_0}\) in \(A_{i_0}\). Then the pair \((P_{i_0,j_0}, P_{i_0,j_0})\) is realizable. By the induction hypothesis, we may assume that there is a connected \(G_1 := \{P_{i,j} | 1 \leq i \leq r, j \in A_i\} - \{P_{i_0,j_0}\}\)-Galois cover of \(\mathbb{P}^1\) étale away from \(B - \{x_{i_0,j_0}\}\) and \(P_{i,j}\) occurs as an inertia group above the point \(x_{i,j} \in B\) for \((i,j) \neq (i_0,j_0)\). Now use Theorem 4.7 with \(G_2 = P_{i_0,j_0}\).

By the above proposition, if we allow enough numbers of branch points we can obtain covers with the desired purely wild ramification. By [6, Corollary 5.5], for \(d = p\) or \(d\) is coprime to \(p\), the GPWIC holds for \(A_d\). So assume that \(a \geq 2\) and \(d = ap\). The following result shows that in this case we only need one extra branched point.

**Proposition 8.3:** Let \(p\) be an odd prime, \(a \geq 2\) be an integer, \(d = ap\). Let \(r \geq 1\) be an integer and \(P_1, \ldots, P_r\) be non-trivial \(p\)-subgroups of \(A_d\). Let \(B = \{x_1, \ldots, x_r\}\) be a set of closed points in \(\mathbb{P}^1\) and let \(x_0 \in \mathbb{P}^1\) be a closed point outside \(B\). Fix \(1 \leq i_0 \leq r\). Then there is a connected \(A_d\)-Galois cover of \(\mathbb{P}^1\) étale away from \(B \sqcup \{x_0\}\) such that \(P_i\) occurs as an inertia group above \(x_i\) and \(P_{i_0}\) occurs as an inertia group above \(x_0\).

**Proof.** We may assume that \(i_0 = 1\). By [8, Theorem 2] it is enough to consider the case \(r = 1\) and when \(P_1 = \langle \tau \rangle\) for an element \(\tau\) of order \(p\). Without loss of generality we may assume that \(\tau = \tau_1 \cdots \tau_v\) where \(\tau_i\) is a the \(p\)-cycle \(((i-1)p+1, \ldots, ip)\), \(1 \leq i \leq v\). By [6, Corollary 5.6] we may assume that \(v = a\).

For \(1 \leq i \leq a\), set \(H_{i1} := \text{Alt}(\{(i-1)p+1, \ldots, ip\})\). For \(1 \leq j \leq a-1\) set \(H_{j2} := \text{Alt}(\{(j-1)p+2, \ldots, jp+1\})\) and let

\[H_{a2} := \text{Alt}(\{(a-1)p+2, \ldots, ap+1\}).\]

For \(i = 1, 2\), set \(G_i := H_{i1} \times \cdots \times H_{ai} \subset A_d\). For \(1 \leq j \leq a\), consider the \(p\)-cycle \(\sigma_j\) given by \(\sigma_j := ((j-1)p+2, \ldots, jp+1)\) for \(1 \leq j \leq a-1\), \(\sigma_a := ((a-1)p+2, \ldots, ap, 1)\). Consider the element \(\sigma := \sigma_1 \cdots \sigma_a\) in \(A_d\) of order \(p\). Now by Theorem 7.4 the pairs \((G_1, \langle (\tau_1, \ldots, \tau_a) \rangle)\) and \((G_2, \langle (\sigma_1, \ldots, \sigma_a) \rangle)\) are realizable. Set \(G := \langle G_1, G_2 \rangle \subset A_d\). Since each \(H_{ij}\) are generated by \(p\) cycles, so is \(G\). Also the 3-cycle \((1, 2, 3) \in H_{11}\) is contained in \(G\). So by [21, Lemma 4.4.4] \(G = A_d\). Since \(\sigma\) is a conjugate of \(\tau\) in \(A_d\), by Theorem 4.7, there is a connected \(A_d\)-Galois cover of \(\mathbb{P}^1\) étale away from \(\{x_0, x_1\}\) such that \(\langle \tau \rangle\) occurs as an inertia group over \(x_0\) and over \(x_1\).
9. Towards the general question

In this final section, we see some evidence for $Q[r,X,B,G]$ (Question 6.2). The following result is a consequence of the formal patching technique (Theorem 4.7).

**Proposition 9.1:** Let $r \geq 1$, $X$ be a smooth projective connected $k$-curve, $G$ be a finite group. Let $B = \{x_1, \ldots, x_r\} \subset X$ be a set of closed points in $X$. For $1 \leq i \leq r$, let $I_i$ be a subgroup of $G$ which is an extension of a $p$-group $P_i$ by a cyclic group of order prime-to-$p$, and set $H := \langle P_i^G | 1 \leq i \leq r \rangle$. Assume that $H$ has a complement $H'$ in $G$. Suppose that the following hold:

1. There is a connected $H'$-Galois étale cover $\psi: Z \to X$.
2. There is a connected $H$-Galois cover of $\mathbb{P}^1$ étale away from a set $\{\eta_1, \ldots, \eta_r\}$ of $r$-distinct points such that $I_i$ occurs as an inertia group above the point $\eta_i$ for $1 \leq i \leq r$.

Then there is a connected $G$-Galois cover of $X$ étale away from a set $B' = \{x'_1, \ldots, x'_r\}$ of closed points such that $I_i$ occurs as an inertia group above the point $x'_i$, $1 \leq i \leq r$. Moreover, we can choose an $i$ such that $x'_i = x_i$. Furthermore, if each $(H, I_i)$ is realizable, we can take $x'_i = x_i$ for all $1 \leq i \leq r$.

**Proof.** By the hypothesis, $G/H$ is a subgroup of $G$ which together with $H$ generates $G$. By Theorem 4.7, there is a connected $G$-Galois cover of $X$, étale away from $B'$, such that $I_i$ occurs as an inertia group above the point $x'_i$ for $1 \leq i \leq r$, and we can choose one $1 \leq i \leq r$ such that $x'_i = x_i$. For the last assertion, we use Theorem 4.7 inductively for $r$. ■

In the above proposition, we assumed the existence of an étale cover $\psi$. Although the étale covers of an arbitrary smooth projective connected $k$-curve are not well understood in general, they are known to exist when the Galois group is either of coprime to $p$ order or is a $p$-group, i.e., if either $(|H'|, p) = 1$ or if $H'$ is a $p$-group of rank $s$ and $X$ has $p$-rank $\geq s$, there is a connected $H$-Galois étale cover $\psi$ as in the proposition.

**Remark 9.2:** Note that Proposition 9.1 can be applied to the Question $Q[r,X,B,G]$ if there exists $0 \leq j \leq l$ such that $H_j$ is normal in $G$ which has a complement in $G$, the composite cover $\psi_j \circ \cdots \circ \psi_1$ is étale and such that each pair $(\langle P_i^G \rangle, I_i)$ is realizable.
Using the IC for the alternating groups proved in Section 5, the above corollary implies the following result.

**Corollary 9.3** \((Q[1, X, \{\ast\}, S_d])\): Let \(p\) be an odd prime and \(X\) be any smooth projective \(k\)-curve of genus \(\geq 1\). Then for \(r = 1\), Question 6.2 has an affirmative answer for the group \(S_p\) and when \(p \equiv 2 \pmod{3}\) for the groups \(S_{p+1}, S_{p+2}, S_{p+3}, S_{p+4}\).

**Proof.** Let \(d = p\) or when \(p \equiv 2 \pmod{3}\), \(d \in \{p + 1, p + 2, p + 3, p + 4\}\). Set \(G = S_d\). Let \(x \in X\) be a closed point, \(I \subset G\) be an extension of a \(p\)-group \(P\) by a cyclic group of order prime-to-\(p\). Then \(\langle P_G \rangle = A_d\). Let \(\psi: Y \to X\) be a connected \(\mathbb{Z}/2\)-Galois cover of \(X\) étale away from \(x\). By the Riemann–Hurwitz formula, the ramification index above \(x\) must be an odd integer. So \(\psi\) is étale everywhere and \(I \subset A_d\). Now the result follows from Proposition 9.1 applied to [5, Theorem 1.2], [16, Theorem 1.2] and Theorem 5.3–Theorem 5.5. \(\blacksquare\)

Using a formal patching technique, we conclude the following result towards the GIC (Conjecture 6.7).

**Proposition 9.4:** Let \(G\) be a finite quasi \(p\)-group, \(r \geq 1\) be an integer. Let \(P_1 \subset G\) be a non-trivial \(p\)-subgroup and \(\beta_1 \in N_G(P_1)\) be an element of order \(m_1, p \nmid m_1\). For \(2 \leq i \leq r\), suppose that \(I_i \subset G\) is of the form

\[
I_i = \begin{cases} 
    P_i \rtimes \langle \beta_i \rangle, & P_i \text{ a non-trivial } p\text{-subgroup of } G, \\
    \langle \gamma_i \rangle, & \beta_i \in N_G(P_i) \text{ of order } m_i, p \nmid m_i \\
    \gamma_i \in G \text{ of coprime top order } m_i 
\end{cases}
\]

such that the following hold:

1. For any \(P_i\) as above, the pair \((G, I_i)\) is realizable.
2. For \(2 \leq i \leq r\), if \(I_i \cong \mathbb{Z}/m_i\), there is a \(2 \leq j \leq r, j \neq i, \text{ with } I_j = I_i\).

Then there exists a connected \(G\)-Galois cover of \(\mathbb{P}^1\) étale away from a set \(B = \{x_1, \ldots, x_r\}\) of closed points in \(\mathbb{P}^1\) such that \(I_i\) occurs as an inertia group above \(x_i, 1 \leq i \leq r\). Moreover, if all the \(I_i\) \((2 \leq i \leq r)\) are equal whenever \(I_i \cong \mathbb{Z}/m_i\), the set \(B\) can be chosen arbitrarily.

**Proof.** Set \(E := \{2 \leq i \leq r \mid I_i \cong \mathbb{Z}/m_i\}\). For \(i \in E\), set \(S_i := \{2 \leq j \leq r \mid I_j = I_i\}\).

By our assumption, the set \(\{1, \ldots, r\} \setminus E\) is non-empty, as well as the sets \(S_i\) for \(i \in E\) (whenever non-empty). For \(i \in E\), consider the connected \(I_{r}\)-Galois cover of \(\mathbb{P}^1\) obtained from the Kummer theory, that is branched at \(|S_i|\)-many
closed points of \( \mathbb{P}^1 \), and totally ramified over each of these branched points. For \( j \in \{1, \ldots, r\} \setminus E \), let \( \phi_j \) be a connected \( G \)-Galois cover of \( \mathbb{P}^1 \) branched only at \( \infty \) such that \( I_j \) occurs as an inertia group above \( \infty \). The statement follows from applying Theorem 4.7 inductively to the above covers \( \phi_j \)'s and \( \psi_i \)'s.  

In view of the cases of the IC proved earlier, we can conclude the following result towards the GIC for alternating groups.

**Corollary 9.5:** Let \( p \geq 5 \) be a prime number. Let \( d= p \) or when \( p \equiv 2 \pmod{3} \), \( d \in \{p+1, p+2, p+3, p+4\} \). Let \( r \geq 1 \) be an integer. Let \( P := \langle (1, \ldots, p) \rangle \subset A_d \). Let \( I_1 := P \rtimes \langle \beta_1 \rangle \), \( \beta_1 \in N_{A_d}(P) \) be of order prime-to-\( p \). For \( 2 \leq i \leq r \), suppose that

\[
I_i = \begin{cases} 
P \rtimes \langle \beta_i \rangle, \quad \beta_i \in N_{A_d}(P) \text{ of order } m_i, p \nmid m_i \\
\langle \gamma_i \rangle, \quad \gamma_i \in A_d \text{ of coprime top order } m_i \end{cases}
\]

such that: if \( I_i \cong \mathbb{Z}/m_i \), there is a \( 2 \leq j \leq r \), \( j \neq i \), with \( I_j = I_i \).

Then there exists a connected \( A_d \)-Galois cover of \( \mathbb{P}^1 \) that is étale away from a set \( B = \{x_1, \ldots, x_r\} \) of closed points in \( \mathbb{P}^1 \) such that \( I_i \) occurs as an inertia group above \( x_i \), \( 1 \leq i \leq r \). Moreover, if all the \( I_i \) are equal whenever \( P_i \) is trivial, the set \( B \) can be chosen arbitrarily.

**Proof.** The statement follows from Proposition 9.4 together with [5, Theorem 1.2], [16, Theorem 1.2] and Theorem 5.3–Theorem 5.5.  

Now onward, we study the general question for \( X = \mathbb{P}^1 \) and \( r = 2 \). We have the following result when \( G = P \rtimes \mathbb{Z}/n \) for a \( p \)-group \( P \) and \( n \) coprime to \( p \).

**Theorem 9.6 (Q[2, \mathbb{P}^1, \{0, \infty\}, P \rtimes \mathbb{Z}/n]):** Let \( G = P \rtimes \mathbb{Z}/n \) for a \( p \)-group \( P \) and \( (p, n) = 1 \). Then Question 6.2 has an affirmative answer for \( G, \mathbb{P}^1 \) and \( r = 2 \).

**Proof.** In view of Theorem 7.2, we may assume \( n \geq 2 \). Let \( P_1, P_2 \) be two \( p \)-subgroups of \( P \) where \( P_1 \) is non-trivial and \( P_2 \) is possible trivial and such that \( \langle P_1^G, P_2^G \rangle = P \). For \( i = 1, 2 \), let \( I_i = P_i \rtimes \mathbb{Z}/m_i \). Let \( \psi : Y \to \mathbb{P}^1 \) be a connected \( \mathbb{Z}/n \)-Galois cover étale away from \( \{0, \infty\} \) such that \( \mathbb{Z}/m_1 \) occurs as an inertia group above 0 and \( \mathbb{Z}/m_2 \) occurs as an inertia group above \( \infty \). By the Riemann-Hurwitz formula, \( m_1 = m_2 = n \), and \( \psi \) is the \( \mathbb{Z}/n \)-Galois Kummer cover totally ramified over 0 and \( \infty \). Since \( I_i \) normalizes \( P_i \), \( \mathbb{Z}/n \) also normalizes \( P_i \) in \( G \). So \( \langle P_1^P, P_2^P \rangle = P \). By Lemma 7.1, \( P = \langle P_1, P_2 \rangle \). Consider the connected \( P_1 \rtimes \mathbb{Z}/n \)-Galois HKG cover \( \psi \) of \( \mathbb{P}^1 \) étale away from \( \{0, \infty\} \).
which is totally ramified above $\infty$ and such that $\mathbb{Z}/n$ occurs as an inertia group above 0 (Theorem 4.8). If $P_2$ is the trivial group, $G = P_1 \rtimes \mathbb{Z}/n$. Otherwise apply [12, Theorem 3.6] to this cover to obtain the result.

In the rest of this article, we study some $S_d$-Galois covers with $X = \mathbb{P}^1$ and $r = 2$. These realization results are the evidence for the Question $Q[2, \mathbb{P}^1, \{0, \infty\}, S_d]$. When $P_2$ is the trivial group, we have seen some of the cases that can occur from studying explicit equations (Section 3). The following result shows the existence of another such cover with $P_2 = \{1\}$ and with the same tame part of the inertia groups over both points as an application of Theorem 4.8 to [6, Corollary 5.5].

**Corollary 9.7:** Let $p$ be an odd prime, $d \geq p$ be an integer such that either $d = p$ or $(d, p) = 1$. Let $I = \langle (1, \ldots, p) \rangle \rtimes \langle \gamma \rangle \subset S_d$ where $\gamma$ is an odd permutation in $S_d$. Then there is a connected $S_d$-Galois cover of $\mathbb{P}^1$ étale away from $\{0, \infty\}$ such that $\langle \gamma \rangle$ occurs as an inertia group at a point over 0 and $I$ occurs as an inertia group at a point over $\infty$.

**Proof.** Set $n := \text{ord}(\gamma)$. Consider the $\mathbb{Z}/n$-Galois Kummer cover $\psi: \mathbb{P}^1 \to \mathbb{P}^1$ totally ramified over $\{0, \infty\}$ and étale everywhere else. By [6, Corollary 5.5], the pair $(A_d, P)$ is realizable. Now the result follows from Theorem 4.8 by taking

$$\Gamma = S_d = \langle A_d, \langle \gamma \rangle \rangle.$$

In whatever follows, assume that $P_1$ and $P_2$ are both non-trivial subgroups of $S_d$, $p \leq d \leq 2p - 1$.

Without loss of generality, we assume that $P_1 = P_2 = \langle \tau \rangle$ where $\tau$ is the $p$-cycle $(1, \ldots, p)$. Question $Q[2, \mathbb{P}^1, \{0, \infty\}, S_d]$ then asks whether the following statement is true.

\begin{center}
\begin{itemize}
  \item \textbf{(*)} For $1 \leq j_1, j_2 \leq p - 1$, $\omega_1, \omega_2 \in \text{Sym}\{p + 1, \ldots, d\}$ such that $\theta^{j_1} \omega_1$ and $\theta^{j_2} \omega_2 \in S_d$ are odd permutations (when $d = p$, $\omega_i$ are the trivial permutations), there is a connected $S_d$-Galois cover of $\mathbb{P}^1$, étale away from $\{0, \infty\}$, such that $\langle \tau \rangle \rtimes \langle \theta^{j_1} \omega_1 \rangle$ occurs as an inertia group above 0, and $\langle \tau \rangle \rtimes \langle \theta^{j_2} \omega_2 \rangle$ occurs as an inertia group above $\infty$.
\end{itemize}
\end{center}

Similar to the case of proving the IC for the alternating groups, we make the following observation that will be used frequently.
Lemma 9.8: Let $p$ be an odd prime and $p \leq d \leq 2p - 1$. Suppose that there exists a $\gamma \in S_d$ so that the following hold. For each $1 \leq i \leq p - 1$ dividing $p - 1$ and $\omega \in \text{Sym}\{p + 1, \ldots, d\}$ with $\theta^i \omega$ being an odd permutation, there is a connected $S_d$-Galois cover of $\mathbb{P}^1$ étale away from $\{0, \infty\}$ such that $\langle \gamma \rangle$ occurs as an inertia group above $0$, and $\langle \tau \rangle \rtimes \langle \theta^i \rangle$ occurs as an inertia group above $\infty$.

Then Statement $(*)$ holds.

Proof. First note that to obtain a connected $S_d$-Galois cover of $\mathbb{P}^1$ that is branched only at $0$ and $\infty$ such that $I_1$ and $I_2$ are the respective inertia groups above $0$ and $\infty$, by Theorem 4.7, it is enough to show the existence of an odd permutation $\gamma \in S_d$ of order prime-to-$p$ and of two connected $S_d$-Galois covers $\phi_1$ and $\phi_2$ of $\mathbb{P}^1$, both of which are branched at $0$ and $\infty$, with the following ramification properties. $\langle \gamma \rangle$ occurs as an inertia group above $0$ for both these covers, and $I_1$ (respectively, $I_2$) occurs as the inertia group above $\infty$ for the cover $\phi_1$ (respectively, for the cover $\phi_2$). Now for Statement $(*)$ to hold, we need to show the above where $I_1$ and $I_2$ vary over all possible inertia groups, namely, each $I_i$ is of the form $\langle \tau \rangle \rtimes \langle \theta^i \rangle$, $1 \leq j_i \leq p - 1$, $\omega_i \in \text{Sym}\{p + 1, \ldots, d\}$ such that $\theta^{j_i} \omega_i$ is an odd permutation. Moreover since $\langle \theta^i \rangle = \langle \theta^{(i,p-1)} \rangle$, we may consider that $j_1$ and $j_2$ both divide $p - 1$; this is precisely our hypothesis.

Theorem 9.9: Let $p \geq 5$ be a prime. Statement $(*)$ is true for $S_p$. In other words, for odd integers $1 \leq i, j \leq p - 1$, $I_1 := \langle \tau \rangle \rtimes \langle \theta^i \rangle$, $I_2 := \langle \tau \rangle \rtimes \langle \theta^j \rangle \subset S_d$, there is a connected $S_p$-Galois cover of $\mathbb{P}^1$, étale away from $\{0, \infty\}$, such that $I_1$ occurs as an inertia group above $0$, and $I_2$ occurs as an inertia group above $\infty$.

Proof. We first show that for each odd divisor $i$ of $p - 1$, there is a connected $S_p$-Galois cover of $\mathbb{P}^1$, étale away from $\{0, \infty\}$, such that $\langle (1,2) \rangle$ occurs as an inertia group above $0$ and $\langle \tau \rangle \rtimes \langle \theta^i \rangle$ occurs as an inertia group above $\infty$.

Consider the degree $p$ cover $\psi: Y \rightarrow \mathbb{P}^1$ given by the affine equation $f(x, y) = 0$ where $f(x, y) = y^p - y^2 - x = 0$.

Let $\phi: \tilde{Y} \rightarrow \mathbb{P}^1$ be its Galois closure. By Remark 3.7, $\phi$ is a connected $S_p$-Galois cover of $\mathbb{P}^1$, étale away from $\{0, \infty\}$, such that $\langle (1,2) \rangle$ is an inertia group over $0$, and $\langle \tau \rangle \rtimes \langle \theta^i \rangle$ occurs as an inertia group above $\infty$. Since $i$ is odd, after the $[i]$-Kummer pullback of $\phi$ (see Definition 2.2), we obtain a connected $S_p$-Galois cover of $\mathbb{P}^1$, étale away from $\{0, \infty\}$, such that $\langle (1,2) \rangle$ occurs as an inertia group over $0$, and $\langle \tau \rangle \rtimes \langle \theta^i \rangle$ occurs as an inertia group above $\infty$.

Now the result follows by applying Lemma 9.8.
Theorem 9.10: Let $p \equiv 2 \pmod{3}$ be an odd prime. Statement $(\ast)$ holds for $S_{p+1}$.

Proof. Consider the $S_{p+1}$-Galois cover $\phi : \tilde{Y} \to \mathbb{P}^1$ which is the Galois closure of the degree-$(p+1)$ cover of $\mathbb{P}^1$ given by the affine equation (5.1). As in the proof of Theorem 5.3, the cover $\phi$ is étale away from $\{0, \infty\}$ such that $\langle (1, \ldots, p-2)(p-1, p) \rangle$ occurs as an inertia group above 0, and $\langle \tau \rangle \times \langle \theta \rangle$ occurs as an inertia group above $\infty$. After a $[p-2]$-Kummer pullback we obtain a connected $S_{p+1}$-Galois cover of $\mathbb{P}^1$ étale away from $\{0, \infty\}$ such that $\langle (p-1, p) \rangle$ is an inertia group above 0 (since the $(p-2)^{th}$ power of the element $(1, \ldots, p-2)(p-1, p)$ is the transposition $(p-1, p)$), and $\langle \tau \rangle \times \langle \theta \rangle$ is an inertia group above $\infty$. Now apply Lemma 9.8. 

Theorem 9.11: Let $p \equiv 2 \pmod{3}$ be a prime $> 5$. Statement $(\ast)$ holds for $S_{p+2}$.

Proof. Let $\gamma$ be the $(p-1)$-cycle $(1, \ldots, p-1)$ in $S_{p+2}$. By Lemma 9.8, it is enough to show the following:

1. For each odd divisor $i$ of $p-1$, there is a connected $S_{p+2}$-Galois cover of $\mathbb{P}^1$, étale away from $\{0, \infty\}$, such that $\langle \gamma \rangle$ is an inertia group above 0, and $\langle \tau \rangle \times \langle \theta^i \rangle$ is an inertia group above $\infty$.

2. For each even divisor $j$ of $p-1$, there is a connected $S_{p+2}$-Galois cover of $\mathbb{P}^1$, étale away from $\{0, \infty\}$, such that $\langle \gamma \rangle$ occurs as an inertia group above 0, and $\langle \tau \rangle \times \langle \theta^j(p+1, p+2) \rangle$ occurs as an inertia group above $\infty$.

Setting $s = 2 = r$, $n_1 = p-1$, $n_2 = 3$, $m_1 = 1 = m_2$, we see that Assumption 3.8 holds with $(\alpha_1, \alpha_2, \beta_1, \beta_2) = (5, 2, 1, 0)$. By Proposition 3.11, we obtain a connected $S_{p+2}$-Galois cover $\phi_1 : Y_1 \to \mathbb{P}^1$ that is étale away from $\{0, \infty\}$ such that $\langle \gamma \rangle$ occurs as an inertia group above 0, and $\langle \tau \rangle \times \langle \theta \rangle$ is an inertia group above $\infty$ (since $p \equiv 2 \pmod{3}$). This is (1).

Now let $s = 2$, $r = 1$, $n_1 = p-1$, $n_2 = 3$. Then by Lemma 3.10(2), Assumption 3.8 holds. Consider the connected $S_{p+2}$-Galois cover $\phi_2$ of $\mathbb{P}^1$ obtained from Proposition 3.11. Now $\phi_2$ is étale away from $\{0, \infty\}$, $\langle \gamma \rangle$ is an inertia group above 0, and $\langle \tau \rangle \times \langle \theta^2(p+1, p+2) \rangle$ occurs as an inertia group above $\infty$ (as $(p-1, r + s - 1) = (p-1, 2) = 2$); this is (2).
THEOREM 9.12: Let $p$ be an odd prime such that $p \equiv 2 \pmod{3}$ and $4 \nmid (p-1)$. Statement (*) holds for $S_{p+3}$.

Proof. Consider the $(p + 3)$-cycle $\gamma := (1, \ldots, p + 3)$ in $S_{p+3}$. In view of Lemma 9.8, it is enough to show that for

$$\beta = \begin{cases} 
\beta_1 = \theta^{i_1}, & i_1 \text{ is an odd divisor of } p - 1, \\
\beta_2 = \theta^{i_2}(p + 1, p + 2), & i_2 \text{ is an even divisor of } p - 1, \\
\beta_3 = \theta^{i_3}(p + 1, p + 2, p + 3), & i_3 \text{ is an odd divisor of } p - 1,
\end{cases}$$

the following holds. There is a connected $S_{p+3}$-Galois cover of $\mathbb{P}^1$ étale away from $\{0, \infty\}$ such that $\langle \gamma \rangle$ is an inertia group above 0, and $I = \langle \tau \rangle \rtimes \langle \beta \rangle$ is an inertia group above $\infty$.

CASE $\beta = \beta_1$: Set $s = 1$, $r = 3$, $m_1 = m_2 = m_3 = 1$. Choose an element $w \in k$ such that $w^2 = p - 3$. With the choice $(\alpha_1, \beta_1, \beta_2, \beta_3) = (0, 1, \frac{w-1}{2}, -\frac{w+1}{2})$, Assumption 3.8 is satisfied. By Proposition 3.11, there is a connected $S_{p+3}$-Galois cover of $\mathbb{P}^1$, étale away from $\{0, \infty\}$, such that $\langle \gamma \rangle$ occurs as an inertia group above 0, and $(p + 1, r + s - 1) = (p - 1, 3) = 1$ $I = \langle \tau \rangle \rtimes \langle \theta \rangle$ occurs as an inertia group above $\infty$. Since $i_1 |(p - 1)$ is odd and $(p - 1, p + 3) = (p - 1, 4) = 2$, after an $[i_1]$-Kummer pullback, we obtain the required cover.

CASE $\beta = \beta_2$: Take $s = 1$, $r = 2$, $m_1 = 2$, $m_2 = 1$. By Lemma 3.10(3), for the choice $(\alpha_1, \beta_1, \beta_2) = (0, 1, -2)$, Assumption 3.8 holds. Apply Proposition 3.11 to obtain a connected $S_{p+3}$-Galois cover of $\mathbb{P}^1$, étale away from $\{0, \infty\}$, such that $\langle \gamma \rangle$ occurs as an inertia group above 0, and $I = \langle \tau \rangle \rtimes \langle \theta^2(p+1, p+2) \rangle$ occurs $((p - 1, r + s - 1) = (p - 1, 2) = 2)$ as an inertia group above $\infty$. Since $4 \nmid (p - 1)$ by our assumption, $(p + 3, i_2/2) = 1$. So after an $[i_2/2]$-Kummer pullback, we obtain the required cover with $\beta = \beta_2 = \theta^{i_2}(p + 1, p + 2)$.

CASE $\beta = \beta_3$: Finally, take $s = 1 = r$. Then by Lemma 3.10(1), Assumption 3.8 holds, and by Proposition 3.11, there is a connected $S_{p+3}$-Galois cover of $\mathbb{P}^1$ that is étale away from $\{0, \infty\}$ such that $\langle \gamma \rangle$ is an inertia group above 0, and $I = \langle \tau \rangle \rtimes \langle \theta(p+1, p+2, p+3) \rangle$ is $((p - 1, r + s - 1) = 1$) an inertia group above $\infty$. Since $i_3$ is an odd divisor of $p - 1$, via an $[i_3]$-Kummer pullback, we obtain the required cover with $\beta = \theta^{i_3}(p + 1, p + 2, p + 3)$. ■
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