THE ASYMPTOTIC LIMITS OF SOLUTIONS TO THE Riemann Problem for the Scaled Leroux System

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Abstract. The Riemann problem for the scaled Leroux system is considered. It is proven rigorously that the Riemann solutions for the scaled Leroux system converge to the corresponding ones for a non-strictly hyperbolic system of conservation laws when the perturbation parameter tends to zero. In addition, some interesting phenomena are displayed in the limiting process, such as the formation of delta shock wave and a rarefaction (or shock) wave degenerates to be a contact discontinuity.

1. Introduction. In this paper, we are interested in the Leroux system [17] which is expressed as

\[
\begin{align*}
    u_t + (u^2 + \rho)_x &= 0, \\
    \rho_t + (\rho u)_x &= 0,
\end{align*}
\]

where \( \rho \) and \( u \) represent the density and velocity, respectively. The system (1.1) has been derived in [9] as a hydrodynamic limit under Eulerian scaling for a two-component lattice gas, even beyond the appearance of shock waves. The system (1.1) appears in many applications, such as it can be used to describe a deposition/domain growth in the biological chemotaxis-mechanism model [34] and the stochastic dynamics in the stochastic particle system where the viscosity terms are added [24].

The Leroux system (1.1) is strictly hyperbolic except at the origin \((0, 0)\) and both characteristic fields are genuinely nonlinear. It is remarkable that the Leroux system (1.1) belongs to Temple class [2, 33]. In other words, the shock curves coincide with...
the rarefaction curves in the phase plane due to the special form of (1.1). Thanks to
the above feature, well-posed results for systems of Temple class are available for a
much larger class of initial data compared with general systems of conservation laws
and the Riemann problem can be solved explicitly in the large and wave interactions
have more simplified structures. The global existence of weak solutions for the
general \(2 \times 2\) hyperbolic systems of conservation laws attributed to the Temple
class with any bounded variation initial data was obtained in [25, 26] by using
standard numerical schemes and parabolic regularization. The global existence of
weak solutions to the Riemann problem for the Leroux system (1.1) was established
in [20, 21] and the global boundedness of entropy solution was achieved in [5]. In
addition, the entropy solutions for the Leroux system (1.1) was investigated in [10]
when the Dirac delta function was appeared in the initial data. Recently, the global
solutions to the particular Cauchy problem for the Leroux system (1.1) have been
constructed in [13] when the initial data are taken to be the three piecewise constant
states.

The scaled Leroux system can be expressed as
\[
\begin{align*}
    u_t + (u^2 + \varepsilon \rho)_x &= 0, \\
    \rho_t + (\rho u)_x &= 0,
\end{align*}
\] (1.2)
which can be obtained directly by using \(\varepsilon \rho\) to take the place of \(\rho\) in (1.1). Thus
this scaling or perturbation does not alter the analytical properties of the Leroux
system (1.1). If we take \(\varepsilon \to 0\) in (1.2), then the formal limit would be the following
non-strictly hyperbolic system of conservation laws [32]:
\[
\begin{align*}
    u_t + (u^2)_x &= 0, \\
    \rho_t + (\rho u)_x &= 0.
\end{align*}
\] (1.3)
Actually, (1.3) can also be derived from Euler equations of gas dynamics by letting
density and pressure to be constants in the momentum conservation equation. The
system (1.3) is well known in literature for hyperbolic conservation laws for the
reason that the singular measure-valued solution may appear. In 1994, Tan, Zhang
and Zheng [32] discovered that the form of standard Dirac delta function supported
on a shock wave was used as a part in the Riemann solutions for certain initial data.
It is worthwhile to notice that both the systems (1.1) and (1.3) are derived from
some simplifications of the gas dynamics system. Therefore, the limiting \(\varepsilon \to 0\)
process from (1.2) to (1.3) means the vanishing pressure process since \(\varepsilon \rho\) stands for
the pressure term in the system (1.2).

In this paper, we are mainly concerned with the Riemann problems to (1.2) and
(1.3) with the following Riemann initial data
\[
(u, \rho)(x, 0) = (u_\pm, \rho_\pm), \quad \pm x > 0,
\] (1.4)
where \((u_\pm, \rho_\pm)\) are constant states satisfying \(\rho_\pm \geq 0\). The main purpose of this
paper is to prove rigorously that the limits of Riemann solutions to the scaled
Leroux system (1.2) are in accordance with the corresponding ones to the non-
strictly hyperbolic system (1.3) with the same Riemann initial data. To be more
precise, the main task of the present paper is to prove the following interesting
theorem.
Theorem 1.1. For given Riemann initial data (1.4) satisfying $\rho \pm \geq 0$, the limits $\varepsilon \to 0$ of the Riemann solutions for the scaled Leroux system (1.2) with (1.4) converge to the corresponding Riemann solutions for the non-strictly hyperbolic system (1.3) with the same Riemann initial data.

It is interesting to discover that the limiting behaviors as $\varepsilon \to 0$ of Riemann solutions to the scaled Leroux system (1.2) depend mainly on the limiting behaviors as $\varepsilon \to 0$ of the first and second family wave curves of (1.2) in the $(u, \rho)$ phase plane. Moreover, we point out that there exists the large difference of limit behaviors when $u_-$ or $u_+$ is greater than zero or not. For example, two shock wave solutions for the scaled Leroux system (1.2) may converge to a shock wave plus a contact discontinuity when $u_+ < u_- < 0$, a contact discontinuity plus a shock wave when $0 < u_+ < u_-$ and a delta shock wave when $u_+ < 0 < u_-$. Since the scaled system (1.2) is strictly hyperbolic and while the limiting system (1.3) is non-strictly hyperbolic, the concentration phenomenon in the scaling limit may be regarded as the formation process of resonance between the two characteristic fields. Physically, the reasonable perturbation of flux function is often used to investigate some dynamical behaviors of fluids. However, the limit of flux function may be very singular if we perturb the flux function for some strictly hyperbolic systems. Thus, we should consider it in the space of Radon measures instead of the space of bounded variation functions or the space of Lebesgue integrable functions if we want to study the Cauchy problem for the limiting system. Our result may provide an intuitive example for the related studies.

For the related works, the formation of delta shock waves and vacuum states for the system of pressureless gas dynamics (or called the transport equations)

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2)_x &= 0,
\end{align*}
\]

has been extensively studied. In 2001, Li [18] investigated the limit behavior of the Euler equations

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2 + p(\rho))_x &= 0,
\end{align*}
\]

of isothermal fluids $p(\rho) = T\rho$ as the temperature $T$ drops to zero. It was shown in [18] that the limit solutions, which are the solutions of the Riemann problem to the transport equations, involve delta shock wave and vacuum. In 2003, Chen and Liu [3] considered the Euler equations of isentropic fluids with the perturbed pressure term where $p(\rho) = \varepsilon \rho^\gamma/\gamma$ with $\gamma > 1$. They analyzed and identified the phenomena of concentration and cavitation and the formation of delta shock wave and vacuum state as the pressure vanishes. In [4], they made a step further to study the nonisentropic fluids. In 2007, Mitrovic and Nedeljkov [22] extended the results to the generalized pressureless gas dynamics model. Also see [29, 36, 38] for the related study. About the singular measure-valued solutions for hyperbolic systems of conservation laws, extensive studies have been carried out since 1994, see [1, 7, 8, 11, 12, 14, 15, 16, 19, 23, 27, 28, 30, 31, 37] for instance.

The vanishing pressure limit can be regarded as a singular flux function limit for hyperbolic systems of conservation laws. Compared with the extensive studies on the formation of delta shock wave for the system of pressureless gas dynamics (1.5), to our knowledge, there is no related study on the formation of delta shock wave for the non-strictly hyperbolic system of conservation laws (1.3) by using a singular flux function limit. Moreover, there exist six kinds of Riemann solutions to (1.3)
and (1.4) in contrast with that there only exist two kinds of Riemann solutions to (1.5) and (1.4). Thus, it is more complicated to consider the Riemann solutions to (1.3) and (1.4) by using the method of singular flux function limit.

The rest of the paper is organized as follows. In section 2, we first consider the Riemann problem for the scaled Leroux system (1.2) in details and then simply depict the Riemann solutions for the non-strictly hyperbolic system (1.3) in order to self-contained. In section 3, we rigorously prove that the Riemann solutions for the scaled Leroux system (1.2) converge to the corresponding ones for the non-strictly hyperbolic system (1.3) when the perturbation parameter tends to zero. In fact, we divide our discussions into six situations to deal with separately.

2. The Riemann problem. In this section, we shall first consider the Riemann problem for the scaled Leroux system (1.2) with the Riemann initial data (1.4) in details and examine the dependence of the Riemann solutions on the perturbation parameter \( \varepsilon > 0 \). Consequently, we briefly depict the Riemann solutions to the non-strictly hyperbolic system (1.3) with the Riemann initial data (1.4) which has been well studied in [32].

2.1. The Riemann problem for the scaled Leroux system (1.2). The eigenvalues of the scaled Leroux system (1.2) are

\[
\lambda_1 = \frac{3u - \sqrt{u^2 + 4\varepsilon \rho}}{2}, \quad \lambda_2 = \frac{3u + \sqrt{u^2 + 4\varepsilon \rho}}{2},
\]

so it is strictly hyperbolic in the upper-half plane \( \rho \geq 0 \) in the \((u, \rho)\) phase space except marginal degeneracy at the origin \((0, 0)\) where the two characteristic speeds coincide with each other. The corresponding right eigenvectors are

\[
\vec{r}_1 = \left( \varepsilon, -\frac{u + \sqrt{u^2 + 4\varepsilon \rho}}{2} \right)^T, \quad \vec{r}_2 = \left( \varepsilon, \frac{\sqrt{u^2 + 4\varepsilon \rho} - u}{2} \right)^T.
\]

If \( \varepsilon > 0 \), then we have \( \nabla \lambda_i \cdot \vec{r}_i = 2\varepsilon \neq 0 \) \( (i = 1, 2) \), in which \( \nabla = \left( \frac{\partial}{\partial u}, \frac{\partial}{\partial \rho} \right) \). In other words, both the characteristic fields are genuinely nonlinear when \( \varepsilon > 0 \). Therefore, the associated waves are either rarefaction waves or shock waves for the first and second families. The Riemann invariants along the characteristic fields are

\[
w^\varepsilon = -\frac{u - \sqrt{u^2 + 4\varepsilon \rho}}{2\varepsilon}, \quad z^\varepsilon = -\frac{u + \sqrt{u^2 + 4\varepsilon \rho}}{2\varepsilon}.
\]

Similar notations will be used and not explained any more without confusion.

Since (1.2) and (1.4) are invariant under uniform stretching of coordinates: \((x, t) \to (\alpha x, \alpha t)\) where \( \alpha (> 0) \) is constant, we look for self-similar solutions in the form

\[
(u, \rho)(x, t) = (u, \rho)(\xi), \quad \xi = x/t.
\]

Then, the Riemann problem (1.2) and (1.4) is reduced to the ordinary differential equations

\[
\begin{cases}
-\xi u_\xi + (u^2 + \varepsilon \rho)\xi = 0, \\
-\xi \rho_\xi + (\rho u)\xi = 0,
\end{cases}
\]

with the boundary value \((u, \rho)(\pm \infty) = (u_\pm, \rho_\pm)\). For smooth solutions, (2.5) is rewritten as

\[
\begin{pmatrix}
2u - \xi & \varepsilon \\
\rho & u - \xi
\end{pmatrix}
\begin{pmatrix}
u \\
\rho
\end{pmatrix}_\xi =
\begin{pmatrix}0 \\
0
\end{pmatrix}.
\]
If we substitute $\xi = \lambda_1$ into (2.6), then the 1-rarefaction wave curve is the integral curve of
\[ \frac{d\rho}{du} = -\frac{u + \sqrt{u^2 + 4\varepsilon \rho}}{2\varepsilon}. \] (2.7)
Then, for given $(u_-, \rho_-)$, the 1-rarefaction wave curve $R_1(u_-, \rho_-)$ in the phase plane, which is the set of states that can be connected on the right by the 1-rarefaction wave, is as follows:
\[ \xi = \lambda_1 = \frac{3u - \sqrt{u^2 + 4\varepsilon \rho}}{2}, \] (2.8)
\[ u + \sqrt{u^2 + 4\varepsilon \rho} = u_- + \sqrt{u_-^2 + 4\varepsilon \rho_-}. \] (2.9)
For the 1-rarefaction wave curve, it follows from (2.7) together with (2.9) that
\[ \frac{d\rho}{du} = -\frac{u_- + \sqrt{u_-^2 + 4\varepsilon \rho_-}}{2\varepsilon} = w^\varepsilon. \] (2.10)
Thus (2.9) can also be rewritten as
\[ \rho - \rho_- = w^\varepsilon (u - u_-), \] (2.11)
which implies that the 1-rarefaction wave curve is a straight line in the phase plane. It is easy to get $u_\xi = \frac{1}{2} > 0$ from (2.9) and (2.10) and $\rho_u = w^\varepsilon < 0$ from (2.10). Thus, the set $(u, \rho)$ which can be joined to $(u_-, \rho_-)$ by the 1-rarefaction wave is made up of the half-branch of $R_1(u_-, \rho_-)$ with $u \geq u_-$ and $\rho \leq \rho_-$ when $\varepsilon > 0$.

Similarly, if we substitute $\xi = \lambda_2$ into (2.6), then the 2-rarefaction wave curve is the integral curve of
\[ \frac{d\rho}{du} = \frac{\sqrt{u^2 + 4\varepsilon \rho} - u}{2\varepsilon}. \] (2.12)
Thus, the 2-rarefaction wave curve $R_2(u_-, \rho_-)$ may be expressed as
\[ \xi = \lambda_2 = \frac{3u + \sqrt{u^2 + 4\varepsilon \rho}}{2}, \] (2.13)
\[ \sqrt{u^2 + 4\varepsilon \rho} - u = \sqrt{u_-^2 + 4\varepsilon \rho_- - u_-}. \] (2.14)
For the 2-rarefaction wave curve, it also follows from (2.12) and (2.14) that
\[ \frac{d\rho}{du} = \frac{\sqrt{u_-^2 + 4\varepsilon \rho_- - u_-}}{2\varepsilon} = z^\varepsilon. \] (2.15)
Thus (2.14) can also be rewritten as
\[ \rho - \rho_- = z^\varepsilon (u - u_-), \] (2.16)
which implies that the 2-rarefaction wave curve is still a straight line in the phase plane. We can also obtain $u_\xi = \frac{1}{2} > 0$ from (2.14) and (2.15) and $\rho_u = z^\varepsilon > 0$ from (2.15). Thus, the set $(u, \rho)$ which can be joined to $(u_-, \rho_-)$ by the 2-rarefaction wave is made up of the half-branch of $R_2(u_-, \rho_-)$ with $u \geq u_-$ and $\rho \geq \rho_-$ when $\varepsilon > 0$.

On the other hand, for a bounded discontinuity at $x = x(t)$, the Rankine-Hugoniot condition holds:
\[ \begin{cases} 
-\sigma[u] + [u^2 + \varepsilon \rho] = 0, \\
-\sigma[\rho] + [\rho u] = 0, 
\end{cases} \] (2.17)
where $\sigma = \frac{dx}{dt}$ and $[u] = u_r - u_l$ with $u_l = u(x(t) - 0, t)$ and $u_r = u(x(t) + 0, t)$, etc.

Eliminating $\sigma$ from (2.17) yields

$$[\rho][u^2 + \varepsilon \rho] - [u][\rho u] = 0. \quad (2.18)$$

By simplifying (2.18), we have

$$\varepsilon \left( \frac{\rho_r - \rho_l}{u_r - u_l} \right)^2 + u_l \left( \frac{\rho_r - \rho_l}{u_r - u_l} - \rho_l \right) = 0, \quad (2.19)$$

which implies that

$$\frac{\rho_r - \rho_l}{u_r - u_l} = -\frac{u_l \pm \sqrt{u_l^2 + 4\varepsilon \rho_l}}{2\varepsilon}. \quad (2.20)$$

Through the above analysis, for a given left state $(u_l, \rho_l) = (u_-, \rho_-)$, the 1-shock wave curve $S_1(u_-, \rho_-)$ is shown as

$$\sigma_1 = u + \frac{u_- - \sqrt{u_-^2 + 4\varepsilon \rho_-}}{2}. \quad (2.21)$$

$$\rho - \rho_- = \left( -\frac{u_- - \sqrt{u_-^2 + 4\varepsilon \rho_-}}{2\varepsilon} \right) (u - u_-) = w^-(u - u_-). \quad (2.22)$$

It is easy to obtain that the Lax entropy condition for 1-shock wave is $\rho > \rho_-$ and $u < u_-$. Analogously, we get the 2-shock wave curve $S_2(u_-, \rho_-)$

$$\sigma_2 = u + \frac{u_- + \sqrt{u_-^2 + 4\varepsilon \rho_-}}{2}, \quad (2.23)$$

$$\rho - \rho_- = \left( \frac{\sqrt{u_-^2 + 4\varepsilon \rho_-} - u_-}{2\varepsilon} \right) (u - u_-) = z^-(u - u_-). \quad (2.24)$$

Clearly, the Lax entropy condition for 2-shock wave is $\rho < \rho_-$ and $u < u_-$. Here the shock curves coincide with the rarefaction curves in the phase plane, which is the so-called Temple class [33]. Let us use $W_1(u_-, \rho_-) = S_1(u_-, \rho_-) \cup R_1(u_-, \rho_-)$ and $W_2(u_-, \rho_-) = S_2(u_-, \rho_-) \cup R_2(u_-, \rho_-)$ to denote the first and second family wave curves. It is worthwhile to notice that the lines of $W_1(u_-, \rho_-)$ and $W_2(u_-, \rho_-)$ in the phase plane are exactly tangent to the parabola $\Gamma : u^2 + 4\varepsilon \rho = 0$ and the Riemann invariants (2.3) are the slopes of the two tangent lines passing through $(u_-, \rho_-)$ to the parabola $\Gamma$.

To summarize, the set of states connected on the right consists of $R_1(u_-, \rho_-)$, $S_1(u_-, \rho_-)$, $R_2(u_-, \rho_-)$ and $S_2(u_-, \rho_-)$ for the given left state $(u_-, \rho_-)$. These wave curves divide the phase plane ($\rho \geq 0$) into four parts I, II, III, and IV. According to the right state $(u_+, \rho_+)$ in the different parts, one may obtain the unique global Riemann solution connecting $(u_-, \rho_-)$ and $(u_+, \rho_+)$. We now collect all the wave curves together in the phase plane ($\rho \geq 0$), which may be shown in Figure 1 for $u_- < 0$ and Figure 2 for $u_- > 0$ below. It can be found that the structure of Riemann solution is $S_1 + S_2$ when $(u_+, \rho_+) \in I$, $S_1 + R_2$ when $(u_+, \rho_+) \in II$, $R_1 + S_2$ when $(u_+, \rho_+) \in III$ and $R_1 + R_2$ when $(u_+, \rho_+) \in IV$.

2.2. The Riemann problem for the non-strictly hyperbolic system (1.3).

We are now in a position to simply describe some results on the Riemann solutions to (1.3) and (1.4) which have already been obtained in [32]. The eigenvalues of the system (1.3) are $\lambda_1 = u$ and $\lambda_2 = 2u$. It is worthwhile to notice that $\lambda_1 < \lambda_2$ for $u > 0$ and $\lambda_1 > \lambda_2$ for $u < 0$, so (1.3) is a non-strictly hyperbolic system. The corresponding right eigenvectors are $\vec{v}_1 = (0, 1)^T$ and $\vec{v}_2 = (1, \rho/u)^T$, respectively.
Thus we have \( \nabla \lambda_1 \cdot \mathbf{r}_1 = 0 \) and \( \nabla \lambda_2 \cdot \mathbf{r}_2 \neq 0 \), which means that the characteristic field for \( \lambda_1 \) is linearly degenerate and that for \( \lambda_2 \) is genuinely nonlinear.

Besides the constant state, it is shown that the associated waves \((u, \rho)(\xi) \ (\xi = x/t)\) for \( \lambda_1 \) are contact discontinuities
\[
J : \xi = u_l = u_r,
\]
and those for \( \lambda_2 \) are rarefaction waves
\[
R : \xi = 2u, \quad \frac{\rho}{u} = \frac{\rho_r}{u_r}, \quad \text{for } u_r > u_l > 0 \text{ or } u_l < u_r < 0,
\]
or shock waves
\[
S : \xi = u_l + u_r, \quad \frac{\rho}{u} = \frac{\rho_l}{u_l}, \quad \text{for } u_l > u_r > 0 \text{ or } u_r < u_l < 0.
\]

For the case \( u_- \geq 0 \geq u_+ \), it follows from the result in [32] that the Riemann solution to (1.3) and (1.4) is a delta shock wave \( \delta S \) connecting the two constant states \((u_\pm, \rho_\pm)\). In order to deal with it, we recall the definition of delta shock wave in [32] below.

**Definition 2.1.** Let \( \Gamma = \{(x(s), t(s)) : a < s < b\} \) be a parameterized smooth curve, then a two-dimensional weighted Dirac delta function \( \beta(s) \delta_{\Gamma} \) supported on \( \Gamma \) is defined by
\[
\langle \beta(s) \delta_{\Gamma}, \psi(x,t) \rangle = \int_a^b \beta(s) \psi(x(s), t(s)) ds,
\]
for any test function \( \psi(x,t) \in C^\infty_0(\mathbb{R} \times \mathbb{R}^+) \).

The problem for the multiplication of distributions, such as the product between a discontinuous function with a Dirac delta distribution, can be solved by employing the definition of Volpert’s averaged superposition [35] and the nonconservative products [6]. Furthermore, one can also refer to the exact definition of generalized delta shock wave solution in the framework of weak asymptotic methods which was proposed by Danilov and Shelkovich [8] and developed by Kalisch and Mitrovic [14, 15]. By virtue of the above definitions, one can construct the Riemann solution to (1.3) and (1.4) for the case \( u_- \geq 0 \geq u_+ \) in the following theorem.

**Theorem 2.2** (see [32]). In the case \( u_- \geq 0 \geq u_+ \), the Riemann problem (1.3) and (1.4) has a piecewise smooth solution
\[
(u, \rho)(x,t) = \begin{cases} 
(u_-, \rho_-), & x < \sigma t, \\
u, \beta(t) \delta(x - \sigma t), & x = \sigma t, \\
u_+, \rho_+), & x > \sigma t,
\end{cases} \tag{2.26}
\]
where
\[
u = \sigma = u_- + u_+, \quad \beta(t) = (u_+ \rho_+ - u_- \rho_-) t. \tag{2.27}
\]
The measure-valued solution (2.26) associated with (2.27) satisfies the generalized Rankine-Hugoniot condition as follows:
\[
\frac{dx}{dt} = \sigma, \quad \sigma[u] = [u^2], \quad \frac{d\beta(t)}{dt} = \sigma[\rho] - [\rho u]. \tag{2.28}
\]
To ensure the uniqueness, the \( \delta \)-entropy condition \( \lambda_{2r} \leq \lambda_1 \leq \sigma \leq \lambda_{1l} \leq \lambda_{2l} \) is proposed, which means that all characteristics on both sides of the \( \delta \)-shock wave curve are incoming.
The above-constructed $\delta$-shock wave solution (2.26) associated with (2.27) should obey
\[
\begin{cases}
\langle u, \psi \rangle + \langle u^2, \psi_x \rangle = 0, \\
\langle \rho, \psi \rangle + \langle \rho u, \psi_x \rangle = 0,
\end{cases}
\]  
for any $\psi(x, t) \in C^\infty(R \times R_+)$. It follows from Definition 2.1 that we have
\[
\langle \rho, \psi \rangle = \int_0^\infty \int_{-\infty}^\infty \rho_0 \psi dx dt + \langle \beta(t) \delta_\Gamma, \psi \rangle,
\]  
\[
\langle \rho u, \psi \rangle = \int_0^\infty \int_{-\infty}^\infty \rho_0 u_0 \psi dx dt + \langle \beta(t) u_3 \delta_\Gamma, \psi \rangle,
\]
in which $\Gamma = \{(x, t)|x = \sigma t\}$, $\rho_0 = \rho_+ + [\rho]H(x - \sigma t)$ and $\rho_0 u_0 = \rho_- u_- + [\rho u]H(x - \sigma t)$.

Under the entropy conditions of $S$ and $\delta S$, there are exactly six configurations of Riemann solutions to (1.3) and (1.4) as follows:
\[
\begin{align*}
1. \: & S_1^J(u_+ < u_- < 0), & 2. \: & S_2^J(u_- < u_+ \leq 0), & 3. \: & S_1^R(u_- < 0 < u_+), \\
4. \: & J_1 \rightarrow S_3(u_+ > 0 > u_-), & 5. \: & J_2 \rightarrow R_3(0 \leq u_- < u_+), & 6. \: & \delta S(u_- \geq 0 \geq u_+).
\end{align*}
\]

3. The limits of Riemann solutions for the scaled Leroux system (1.2). In this section, we are dedicated to studying the limit $\varepsilon \to 0$ behaviors of Riemann solutions to the scaled Leroux system (1.2). In view of the discussion on the Riemann solutions to the scaled Leroux system (1.2) in Section 2, let us assume that the intermediate state is $(u^\varepsilon, \rho^\varepsilon)$ in the sense that $(u_-, \rho_-)$ and $(u^\varepsilon, \rho^\varepsilon)$ are connected by 1-wave and $(u^\varepsilon, \rho^\varepsilon)$ and $(u_+, \rho_+)$ are connected by 2-wave, then the Riemann solution to (1.2) and (1.4) is just the combination of the 1-wave and 2-wave as follows:

\[
R_1 : \begin{cases}
\lambda_1 = \xi = \frac{3u - \sqrt{u^2 + 4\varepsilon \rho}}{2}, \\
\rho - \rho_- = w^\varepsilon(u_+ - u_-), \\
u_- \leq u \leq u^\varepsilon, \: \rho^\varepsilon \leq \rho \leq \rho_-.
\end{cases}
\]  

\[
S_1 : \begin{cases}
\sigma_1 = u^\varepsilon + \frac{u_- - \sqrt{u_-^2 + 4\varepsilon \rho_-}}{2}, \\
\rho^\varepsilon - \rho_- = w^\varepsilon(u^\varepsilon - u_-), \\
u^\varepsilon < u_- \leq u^\varepsilon, \: \rho^\varepsilon > \rho_-.
\end{cases}
\]

and

\[
R_2 : \begin{cases}
\lambda_2 = \xi = \frac{3u + \sqrt{u^2 + 4\varepsilon \rho}}{2}, \\
\rho_+ - \rho = z^\varepsilon(u_+ - u), \\
u^\varepsilon \leq u \leq u_+, \: \rho^\varepsilon \leq \rho \leq \rho_+.
\end{cases}
\]  

\[
S_2 : \begin{cases}
\sigma_2 = u_+ + \frac{u^\varepsilon + \sqrt{(u^\varepsilon)^2 + 4\varepsilon \rho_+^2}}{2}, \\
\rho_+ - \rho^\varepsilon = z^\varepsilon(u_+ - u^\varepsilon), \\
u^\varepsilon > u_+, \: \rho^\varepsilon > \rho_+.
\end{cases}
\]

First of all, due to a very particular property of the scaled Leroux system (1.2), we have the following result.
Lemma 3.1. For the intermediate state \((u_\varepsilon^*, \rho_\varepsilon^*)\), we have the following two relations

\[
\begin{align*}
  z_\varepsilon^* &= -u_\varepsilon^* + \sqrt{(u_\varepsilon^*)^2 + 4\varepsilon \rho_\varepsilon^*} = -u_\varepsilon + \sqrt{u_\varepsilon^2 + 4\varepsilon \rho_\varepsilon} = z_+^*, \quad (3.5) \\
  w_-^* &= -u_- - \sqrt{u_-^2 + 4\varepsilon \rho_-} = -u_-^* - \sqrt{(u_-^*)^2 + 4\varepsilon \rho_-^*} = w_-^*. \quad (3.6)
\end{align*}
\]

Proof. It follows from (3.3) (or (3.4)) that

\[
\begin{align*}
  z_\varepsilon^* &= \rho_+ - \rho_- \varepsilon u_\varepsilon^* = \frac{2\rho_+}{u_+ - u_-} = \frac{2\rho_+}{u_+ - \sqrt{u_+^2 + 4\varepsilon \rho_+}} = \frac{\rho_+}{u_+ + \varepsilon z_\varepsilon^*}, \quad (3.7)
\end{align*}
\]

which implies that

\[
\varepsilon (z_\varepsilon^*)^2 + u_+ z_\varepsilon^* - \rho_+ = 0. \quad (3.8)
\]

With \(z_\varepsilon^* > 0\) in mind, we have

\[
\begin{align*}
  z_\varepsilon^* &= -u_\varepsilon + \sqrt{u_\varepsilon^2 + 4\varepsilon \rho_\varepsilon} = z_+^*. \quad (3.9)
\end{align*}
\]

On the other hand, with the similar calculation, (3.6) can also be achieved from (3.1) (or (3.2)). Thus, the conclusion is drawn.

Remark 1. Lemma 3.1 can be explained in the following geometrical sense. The Riemann invariant \(z_\varepsilon^*\) is the slope of the tangent line passing through \((u_\varepsilon^*, \rho_\varepsilon^*)\) to the left-hand side of the parabola \(\Gamma\). We notice that the tangent line also passes through \((u_+, \rho_+\)\). In other words, the tangent line passing through \((u_\varepsilon^*, \rho_\varepsilon^*)\) to the left-hand side of \(\Gamma\) is the same as that passing through \((u_+, \rho_+)\). Thus, they have the same slope. Similarly, the Riemann invariant \(w_-^*\) (or \(w_-^*\)) is the slope of the tangent line passing through \((u_-^*, \rho_-^*)\) (or \((u_-, \rho_-)\)) to the right-hand side of the parabola \(\Gamma\).

It follows from Lemma 3.1 that the intermediate state \((u_\varepsilon^*, \rho_\varepsilon^*)\) can always be expressed as

\[
\begin{align*}
  u_\varepsilon^* &= \frac{w_-^* u_- - z_+^* u_+ + \rho_+ - \rho_-}{w_-^* - z_+^*}, \\
  \rho_\varepsilon^* &= \rho_- + \frac{w_-^* (z_+^* (u_- - u_+) + \rho_+ - \rho_-)}{w_-^* - z_+^*}, \quad (3.10)
\end{align*}
\]

no matter what the Riemann solution is. In order to cover all the situations completely, our discussion should be divided into two parts according to \(u_- < 0\) or not.
3.1. **Part A**: $u_- < 0$. When $u_- < 0$, if we take the limit $\varepsilon \to 0$, then the state variables $(u, \rho)$ on the first family wave curve $W_1(u_-, \rho_-)$ should satisfy

$$
\lim_{\varepsilon \to 0} \rho \frac{\rho - \rho_-}{u - u_-} = -\lim_{\varepsilon \to 0} u_- + \frac{\sqrt{u_-^2 + 4\varepsilon \rho_-}}{2\varepsilon} = -\lim_{\varepsilon \to 0} \frac{2\rho_-}{\sqrt{u_-^2 + 4\varepsilon \rho_-} - u_-} = \frac{\rho_-}{u_-},
$$

namely $\lim_{\varepsilon \to 0} \frac{\rho}{u} = \frac{\rho_-}{u_-}$, which implies that the first family wave curve $W_1(u_-, \rho_-)$ is the line passing through the two states $(0, 0)$ and $(u_-, \rho_-)$ in the limit situation which is in accordance with the second family wave curve for the system (1.3).

On the other hand, when $u_- < 0$, if we take the limit $\varepsilon \to 0$, then the state variables $(u, \rho)$ on the second family wave curve $W_2(u_-, \rho_-)$ should satisfy

$$
\lim_{\varepsilon \to 0} \rho \frac{\rho - \rho_-}{u - u_-} = \lim_{\varepsilon \to 0} \frac{\sqrt{u_-^2 + 4\varepsilon \rho_-} - u_-}{2\varepsilon} = +\infty,
$$

namely $\lim_{\varepsilon \to 0} u = u_-$, which implies that the second family wave curve $W_2(u_-, \rho_-)$ becomes the line $u = u_-$ in the limit situation which is exactly the first family wave curve for the system (1.3).

Let us draw Figure 1 to explain the limit $\varepsilon \to 0$ behaviors of the curves $W_1(u_-, \rho_-)$ and $W_2(u_-, \rho_-)$ when $u_- < 0$. As $\varepsilon \to 0$, the line $W_1(u_-, \rho_-)$ takes $(u_-, \rho_-)$ as the fixed point and rotates clockwise until it arrives at the line $\frac{\rho}{u} = \frac{\rho_-}{u_-}$ in the limit situation. In contrast, $W_2(u_-, \rho_-)$ also takes $(u_-, \rho_-)$ as the fixed point but rotates counterclockwise until it arrives at the line $u = u_-$ in the limit situation.

According to the right state $(u_+, \rho_+)$ in the different positions, our discussion should be divided into the following three cases. Let us first consider the case $u_+ < u_- < 0$. 

![Figure 1](image-url)
Lemma 3.2. For given \((u_+, \rho_+), \) if \(u_+ < u_- < 0, \) then there exists an \(\varepsilon_0 > 0\) such that \((u_+, \rho_+) \in I(u_-, \rho_-)\) when \(\rho_+ < \frac{\rho_+ - u_+}{u_-} \) and \((u_+, \rho_+) \in II(u_-, \rho_-)\) when \(\rho_+ \geq \frac{\rho_+ - u_+}{u_-}\) for \(0 < \varepsilon < \varepsilon_0, \) in which \(\varepsilon_0\) only depends on the initial data.

**Proof.** It follows from (2.22) and (2.24) that if \(\varepsilon\) is small enough to satisfy
\[
\rho_+ + \varepsilon(u_+ - u_-) < \rho_+ < \rho_- + \varepsilon(u_+ - u_-),
\]
then we have \((u_+, \rho_+) \in I(u_-, \rho_-).\) It can be derived from (3.13) that
\[
\frac{-u_- + \sqrt{u_-^2 + 4\varepsilon\rho_-}}{2\varepsilon} > \frac{\rho_+ - \rho_-}{u_+ - u_-} > \frac{-u_- - \sqrt{u_-^2 + 4\varepsilon\rho_-}}{2\varepsilon}.
\]
Moreover, we have
\[
\left(\frac{2\varepsilon(\rho_+ - \rho_-)}{u_+ - u_-} + u_-\right)^2 < u_-^2 + 4\varepsilon\rho_-.
\]
Thus, when \(\rho_+ < \frac{\rho_+ - u_+}{u_-},\) the conclusion can be drawn by taking
\[
\varepsilon_0 = \frac{(u_+ - u_-)(\rho_- - u_+ - \rho_+ u_-)}{(\rho_+ - \rho_-)^2}.
\]
On the other hand, it follows from (2.22) that if \(\varepsilon\) is small enough to satisfy
\[
\rho_+ > \rho_- + \varepsilon(u_+ - u_-),
\]
then we have \((u_+, \rho_+) \in II(u_-, \rho_-).\) For any \(\varepsilon > 0,\) one can see that the inequality (3.17) always holds when \(u_+ < u_- < 0\) and \(\rho_+ \geq \frac{\rho_+ - u_+}{u_-}.\)

If \(u_+ < u_- < 0\) is satisfied, then one can see from Lemma 3.2 that the Riemann solution for the scaled Leroux system (1.2) is \(S_1 + S_2\) if \(\rho_+ < \frac{\rho_+ - u_+}{u_-}\) and otherwise is \(S_1 + R_2\) if \(\rho_+ \geq \frac{\rho_+ - u_+}{u_-}\) provided that \(\varepsilon\) is sufficiently small.

**Theorem 3.3.** In the case \(u_+ < u_- < 0,\) the limit \(\varepsilon \to 0\) of the Riemann solution for the scaled Leroux system (1.2) with the Riemann initial data (1.4) is \(\hat{S} + J\) as follows:

\[
(u, \rho)(\xi) = \begin{cases} 
(u_-, \rho_-), & \xi < u_- + u_+, \\
(u_+, \frac{\rho_+ - u_+}{u_-}), & u_- + u_+ < \xi < u_+, \\
(u_+, \rho_+), & \xi > u_.
\end{cases}
\]

**Proof.** In the case \(u_+ < u_- < 0,\) when \(\rho_+ < \frac{\rho_+ - u_+}{u_-},\) it follows from Lemma 3.2 that the Riemann solution to the scaled Leroux system (1.2) with (1.4) consists of two shock waves \(S_1\) and \(S_2\) for \(0 < \varepsilon < \varepsilon_0\) as follows:

\[
(u^\varepsilon, \rho^\varepsilon)(\xi) = \begin{cases} 
(u_-, \rho_-), & -\infty < \xi < \sigma_1, \\
(u_\xi, \rho_\xi), & \sigma_1 < \xi < \sigma_2, \\
(u_+, \rho_+), & \sigma_2 < \xi < +\infty,
\end{cases}
\]

which satisfies (3.2) and (3.4) together. With \(u_+ < u_- < 0\) in mind, it is shown that
\[
\lim_{\varepsilon \to 0} u^\varepsilon_+ = \frac{\rho_-}{u_-}, \quad \lim_{\varepsilon \to 0} \rho^\varepsilon_+ = +\infty.
\]
In view of (3.20), taking the limit \(\varepsilon \to 0\) in (3.10) leads to
\[
\lim_{\varepsilon \to 0} u^\varepsilon_+ = u_+, \quad \lim_{\varepsilon \to 0} \rho^\varepsilon_+ = \frac{\rho_- u_+}{u_-}.
\]
Moreover, it follows from (3.2) and (3.4) that
\[
\lim_{\varepsilon \to 0} \sigma_1 = u_- + u_+ \quad \text{and} \quad \lim_{\varepsilon \to 0} \sigma_2 = u_+.
\] (3.22)
It is deduced from (3.21) and (3.22) that the shock wave \( S_2 \) will degenerate to be a contact discontinuity \( J \) in the limit situation.

On the other hand, if \( \rho_+ > \frac{\rho_0 - u_+}{u_-} \), then it follows from Lemma 3.2 that the Riemann solution to the scaled Leroux system (1.2) is \( S_1 + R_2 \) as follows:
\[
(u^\varepsilon, \rho^\varepsilon)(\xi) = \begin{cases} 
(u_-, \rho_-), & -\infty < \xi < \sigma_1, \\
(u^\varepsilon_+, \rho_+^\varepsilon), & \sigma_1 < \xi < \lambda_2(u^\varepsilon_+, \rho^\varepsilon_+), \\
R_2, & \lambda_2(u^\varepsilon_+, \rho^\varepsilon_+) \leq \xi \leq \lambda_2(u_+, \rho_+), \\
(u_+, \rho_+), & \lambda_2(u_+, \rho_+) < \xi < \infty,
\end{cases}
\] (3.23)
which satisfies (3.2) and (3.3) together. With the same computation as before, we can also get the formulae (3.20), (3.21) and \( \lim_{\varepsilon \to 0} \sigma_1 = u_- + u_+ \). But here we have
\[
\lim_{\varepsilon \to 0} \lambda_2(u^\varepsilon_+, \rho^\varepsilon_+) = \lim_{\varepsilon \to 0} \lambda_2(u_+, \rho_+) = u_+,
\] (3.24)
in which (3.21) has been used. This is to say that the rarefaction wave \( R_2 \) will also degenerate to be a contact discontinuity \( J \) in the limit situation.

In particular, if \( \rho_+ = \frac{\rho_0 - u_+}{u_-} \), then the Riemann solution to the scaled Leroux system (1.2) can also be expressed as (3.23). But we now get \( \lim_{\varepsilon \to 0}(u^\varepsilon_+, \rho^\varepsilon_+) = (u_+, \rho_+) \) from (3.21), which implies that \( (u_{\pm}, \rho_{\pm}) \) are connected directly by 1-shock and the intermediate state disappears in the limit situation. \( \square \)

Now we return to the case \( u_- < u_+ \leq 0 \) and have the following theorem to depict the limit situation.

**Theorem 3.4.** In the case \( u_- < u_+ \leq 0 \), the limit \( \varepsilon \to 0 \) of the Riemann solution for the scaled Leroux system (1.2) with the Riemann initial data (1.4) is \( \bar{R} + J \) as follows:
\[
(u, \rho)(\xi) = \begin{cases} 
(u_-, \rho_-), & \xi < 2u_- , \\
R_1, & 2u_- \leq \xi \leq 2u_+ , \\
(u_+, \frac{u_+ - \rho_0}{u_-}), & 2u_+ < \xi < u_+ , \\
(u_+, \rho_+), & \xi > u_+ ,
\end{cases}
\] (3.25)
in which the state variables \((u, \rho)\) in \( R_1 \) satisfy \( \frac{\bar{u}}{u_-} = \frac{\rho_0 - u_+}{u_-} \).

**Proof.** Similar to that in Lemma 3.2, in the case \( u_- < u_+ \leq 0 \), we also have \((u_+, \rho_+) \in III(u_-, \rho_-)\) when \( \rho_+ \leq \frac{\rho_0 - u_+}{u_-} \) and \((u_+, \rho_+) \in IV(u_-, \rho_-)\) when \( \rho_+ > \frac{\rho_0 - u_+}{u_-} \) provided that \( \varepsilon \) is sufficiently small.

When \( \rho_+ > \frac{\rho_0 - u_+}{u_-} \), for \( \varepsilon \) small enough, the Riemann solution to the scaled Leroux system (1.2) with (1.4) consists of two rarefaction waves \( R_1 \) and \( R_2 \) as follows:
\[
(u^\varepsilon, \rho^\varepsilon)(\xi) = \begin{cases} 
(u_-, \rho_-), & -\infty < \xi < \lambda_1(u_-, \rho_-), \\
R_1, & \lambda_1(u_-, \rho_-) \leq \xi \leq \lambda_1(u^\varepsilon_+, \rho^\varepsilon_+), \\
(u^\varepsilon_+, \rho_+^\varepsilon), & \lambda_1(u^\varepsilon_+, \rho_+^\varepsilon) < \xi < \lambda_2(u^\varepsilon_+, \rho^\varepsilon_+), \\
R_2, & \lambda_2(u^\varepsilon_+, \rho^\varepsilon_+) \leq \xi \leq \lambda_2(u_+, \rho_+), \\
(u_+, \rho_+), & \lambda_2(u_+, \rho_+) < \xi < \infty,
\end{cases}
\] (3.26)
which satisfies (3.1) and (3.3) together. With the same computation as before, the formulæ (3.20) and (3.21) can also be achieved. Moreover, it follows from (3.1) and (3.3) that
\[
\lim_{\varepsilon \to 0} \lambda_1(u_-, \rho_-) = 2u_-, \quad \lim_{\varepsilon \to 0} \lambda_1(u_*, \rho_*) = 2u_+, \tag{3.27}
\]
\[
\lim_{\varepsilon \to 0} \lambda_2(u_*, \rho_*) = u_+, \quad \lim_{\varepsilon \to 0} \lambda_2(u_+, \rho_+) = u_+, \tag{3.28}
\]

namely the rarefaction wave \( R_2 \) becomes a contact discontinuity \( J \) in the limit situation.

On the other hand, when \( \rho_+ < \frac{\rho_- u_-}{u_+} \), the Riemann solution to the scaled Leroux system (1.2) consists of \( R_1 \) and \( S_2 \) as follows:
\[
(u^*, \rho^*)(\xi) = \begin{cases}
(u_-, \rho_-), & -\infty < \xi < \lambda_1(u_-, \rho_-), \\
R_1, & \lambda_1(u_-, \rho_-) \leq \xi < \lambda_1(u_*, \rho_*), \\
(u_*, \rho_*), & \lambda_1(u_*, \rho_*) < \xi < \sigma_2, \\
(u_+, \rho_+), & \sigma_2 < \xi < \infty,
\end{cases}	ag{3.29}
\]

which should satisfy (3.1) and (3.4) together. Analogously, the formulæ (3.20), (3.21) and (3.27) can also be achieved. So we have \( \lim_{\varepsilon \to 0} \sigma_2 = u_+ \) and then the shock wave \( S_1 \) becomes a contact discontinuity \( J \) for \( \lim_{\varepsilon \to 0} u_* = u_+ \).

If \( \rho_+ = \frac{\rho_- u_-}{u_+} \), then the Riemann solution to (1.2) and (1.4) is still given by (3.29). We also have \( \lim_{\varepsilon \to 0} (u^*, \rho^*) = (u_+, \rho_+) \) from (3.21), namely \( (u_\pm, \rho_\pm) \) are connected directly by 1-rarefaction wave and the intermediate state disappears in the limit situation.

Let us draw our attentions on the case \( u_- < 0 < u_+ \) and the limit situation can be fully depicted in the theorem below.

**Theorem 3.5.** *In the case \( u_- < 0 < u_+ \), the limit \( \varepsilon \to 0 \) of the Riemann solution for the scaled Leroux system (1.2) with the Riemann initial data (1.4) is \( \bar{R} + \bar{R} \) as follows:
\[
(u, \rho)(\xi) = \begin{cases}
(u_-, \rho_-), & \xi < 2u_-, \\
R_1, & 2u_- \leq \xi < 0, \\
R_2, & 0 \leq \xi \leq 2u_+, \\
(u_+, \rho_+), & \xi > 2u_+, 
\end{cases}
\]

in which the state variables \( (u, \rho) \) satisfy \( \xi = \frac{\rho_-}{u_-} \) in \( R_1 \) and \( \xi = \frac{\rho_+}{u_+} \) in \( R_2 \) respectively.*

**Proof.** With the same reason as Lemma 3.2, for given \( (u_+, \rho_+) \), if \( u_- < 0 < u_+ \), then there also exists an \( \varepsilon_0 > 0 \) such that \( (u_+, \rho_+) \in IV(u_-, \rho_-) \) for \( 0 < \varepsilon < \varepsilon_0 \). In other words, the Riemann solution for the scaled Leroux system (1.2) is \( R_1 + R_2 \) provided that \( \varepsilon \) is sufficiently small. More precisely, the Riemann solution to (1.2) and (1.4) can also be expressed by (3.26) which satisfies both (3.1) and (3.3). With \( u_- < 0 < u_+ \) in mind, we have
\[
\lim_{\varepsilon \to 0} w_\varepsilon^- = \frac{\rho_-}{u_-}, \quad \lim_{\varepsilon \to 0} z_\varepsilon^- = \frac{\rho_+}{u_+}, \tag{3.31}
\]
which enables us to derive from (3.10) that
\[
\lim_{\varepsilon \to 0} u_\varepsilon^- = 0, \quad \lim_{\varepsilon \to 0} \rho_\varepsilon^- = 0. \tag{3.32}
\]
Moreover, it follows from (3.1) and (3.3) that
\[
\lim_{\varepsilon \to 0} \lambda_1(u_-, \rho_-) = 2u_-, \quad \lim_{\varepsilon \to 0} \lambda_1(u_+^\varepsilon, \rho_+^\varepsilon) = 0, \quad (3.33)
\]
\[
\lim_{\varepsilon \to 0} \lambda_2(u_+^\varepsilon, \rho_+^\varepsilon) = 0, \quad \lim_{\varepsilon \to 0} \lambda_2(u_+, \rho_+) = 2u_+, \quad (3.34)
\]
which implies that \( R_1 \) and \( R_2 \) are directly connected by the line \( x = 0 \) in the \((x,t)\) plane. In addition, it can be derived from (3.1) and (3.3) together with (3.31) that \((u, \rho)\) in \( R_1 \) and in \( R_2 \) satisfy \( \frac{\varepsilon}{u} = \frac{\rho_1}{\rho} \) and \( \frac{\varepsilon}{u} = \frac{\rho_2}{\rho} \) respectively in the limit situation.

**3.2. Part B:** \( u_- > 0 \). In Part B, we continue to study the limit \( \varepsilon \to 0 \) behaviors of Riemann solutions to the scaled Leroux system (1.2) when \( u_- > 0 \). If we take the limit \( \varepsilon \to 0 \), then the state variables \((u, \rho)\) on the first family curve \( W_1(u_-, \rho_-) \) should satisfy
\[
\lim_{\varepsilon \to 0} \frac{\rho - \rho_-}{u - u_-} = -\lim_{\varepsilon \to 0} \frac{u_- + \sqrt{u_-^2 + 4\varepsilon \rho_-}}{2\varepsilon} = -\infty, \quad (3.35)
\]
which implies that the first family wave curve \( W_1(u_-, \rho_-) \) becomes the line \( u = u_- \), which is identical with the second family wave curve for the system (1.3).

On the other hand, if we take the limit \( \varepsilon \to 0 \), then the state variables \((u, \rho)\) on the second family wave curve \( W_2(u_-, \rho_-) \) should satisfy
\[
\lim_{\varepsilon \to 0} \frac{\rho - \rho_-}{u - u_-} = \lim_{\varepsilon \to 0} \frac{\sqrt{u_-^2 + 4\varepsilon \rho_-} - u_-}{2\varepsilon} = \lim_{\varepsilon \to 0} \frac{2\rho_-}{\sqrt{u_-^2 + 4\varepsilon \rho_-} + u_-} = \frac{\rho_-}{u_-}. \quad (3.36)
\]
\[(u, \rho)(\xi) = \begin{cases} 
(u_-, \rho_-), & \xi < u_-, \\
(u_-, \frac{\rho_+ u_-}{u}), & u_- < \xi < 2u_-, \\
R_2, & 2u_- \leq \xi \leq 2u_+, \\
(u_+, \rho_+), & \xi > 2u_+, 
\end{cases} \quad (3.37)
\]in which the state variables \((u, \rho)\) in \( R_2 \) satisfy \( \frac{\varepsilon}{u} = \frac{\rho_+}{u_+} \).
Figure 2. The phase plane for the scaled Leroux system (1.2) when \( u_- > 0 \), left for \( \varepsilon > 0 \) and right for the limit \( \varepsilon \to 0 \) situation.

**Proof.** With the similar derivation in Lemma 3.2, we can see that if \( 0 < u_- < u_+ \) then we also have \((u_+, \rho_+) \in I(u_-, \rho_-)\) when \( \rho_+ \geq \frac{\rho_- - u_+}{u_-} \) and \((u_+, \rho_+) \in IV(u_-, \rho_-)\) when \( \rho_+ < \frac{\rho_- - u_+}{u_-} \) for \( \varepsilon \) sufficiently small. In other words, the Riemann solution for the scaled Leroux system (1.2) is \( S_1 + R_2 \) if \( \rho_+ \geq \frac{\rho_- - u_+}{u_-} \) and otherwise is \( R_1 + R_2 \) if \( \rho_+ < \frac{\rho_- - u_+}{u_-} \) provided that \( \varepsilon \) is sufficiently small.

If \( \rho_+ > \frac{\rho_- - u_+}{u_-} \), then the Riemann solution to the scaled Leroux system (1.2) consists of \( S_1 \) and \( R_2 \) which can also be expressed as (3.23). In view of \( 0 < u_- < u_+ \), we have

\[
\lim_{\varepsilon \to 0} w_\varepsilon^- = -\infty, \quad \lim_{\varepsilon \to 0} z_\varepsilon^+ = \frac{\rho_+}{u_+}.
\] (3.38)

Together with (3.38), passing to the limit \( \varepsilon \to 0 \) in (3.10) again yields

\[
\lim_{\varepsilon \to 0} u_\varepsilon^+ = u_-, \quad \lim_{\varepsilon \to 0} \rho_\varepsilon^+ = \frac{u_- \rho_+}{u_+}.
\] (3.39)

Then we obtain \( \lim_{\varepsilon \to 0} \sigma_1 = u_- \) from (3.1). Thus, the shock wave \( S_1 \) becomes a contact discontinuity \( J \) in the limit situation. Moreover, it follows from (3.3) and (3.39) that

\[
\lim_{\varepsilon \to 0} \lambda_2(u_\varepsilon^+, \rho_\varepsilon^+) = 2u_-, \quad \lim_{\varepsilon \to 0} \lambda_2(u_+, \rho_+) = 2u_+.
\] (3.40)

If \( \rho_+ = \frac{\rho_- - u_+}{u_-} \), then the Riemann solution to (1.2) and (1.4) is also given by (3.23). One can get \( \lim_{\varepsilon \to 0} (u_\varepsilon^+, \rho_\varepsilon^+) = (u_-, \rho_-) \) from (3.39), namely \((u_+, \rho_+)\) are connected directly by 2-rarefaction wave and the intermediate state disappears in the limit situation.

Otherwise, if \( \rho_+ < \frac{\rho_- - u_+}{u_-} \), then the Riemann solution to the scaled Leroux system (1.2) is \( R_1 \) and \( R_2 \) given by (3.26) for \( \varepsilon \) sufficiently small. Analogously, (3.38)-(3.40) can be obtained by taking the limit \( \varepsilon \to 0 \). In addition, it follows from (3.1) and (3.39) that

\[
\lim_{\varepsilon \to 0} \lambda_1(u_-, \rho_-) = u_-, \quad \lim_{\varepsilon \to 0} \lambda_1(u_\varepsilon^-, \rho_\varepsilon^-) = u_-.
\] (3.41)
which implies that \( R_1 \) also becomes \( J \) in the limit situation.

Let us consider the case \( 0 < u_+ < u_- \) and the limit situation can be fully depicted in the following theorem below.

**Theorem 3.7.** In the case \( 0 < u_+ < u_- \), the limit \( \varepsilon \to 0 \) of the Riemann solution for the scaled Leroux system (1.2) with the Riemann initial data (1.4) is \( J + \overline{S} \) as follows:

\[
(u, \rho)(\xi) = \begin{cases} 
(u_-, \rho_-), & \text{if } \xi < u_-, \\
(u_-, \frac{\rho_- u_-}{u_+}), & \text{if } u_- < \xi < u_- + u_+, \\
(u_+, \rho_+), & \text{if } \xi > u_- + u_+.
\end{cases}
\]

**Proof.** Like as before, if \( 0 < u_+ < u_- \), then we have \((u_+, \rho_+) \in I(u_-, \rho_-)\) when \( \rho_+ > \frac{\rho_- u_+}{u_-} \) and \((u_+, \rho_+) \in III(u_-, \rho_-)\) when \( \rho_+ \leq \frac{\rho_- u_+}{u_-} \) for \( \varepsilon \) sufficiently small.

In other words, the Riemann solution for the scaled Leroux system (1.2) is \( S_1 + S_2 \) if \( \rho_+ > \frac{\rho_- u_+}{u_-} \) and otherwise is \( R_1 + S_2 \) if \( \rho_+ \leq \frac{\rho_- u_+}{u_-} \) when \( \varepsilon \) is sufficiently small.

When \( \rho_+ > \frac{\rho_- u_+}{u_-} \), the Riemann solution to the scaled Leroux system (1.2) consists of \( S_1 \) and \( S_2 \) given by (3.19). Similarly, one can arrives at (3.38) and (3.39). Moreover, it follows from (3.2) and (3.4) that

\[
\lim_{\varepsilon \to 0} \sigma_1 = u_-, \quad \lim_{\varepsilon \to 0} \sigma_2 = u_- + u_+.
\]

Thus, the shock wave \( S_1 \) turns into a contact discontinuity \( J \) in the limit situation.

When \( \rho_+ < \frac{\rho_- u_+}{u_-} \), the Riemann solution to the scaled Leroux system (1.2) consists of \( R_1 \) and \( S_2 \) given by (3.29). Analogously, (3.38), (3.39), (3.41) and \( \lim_{\varepsilon \to 0} \sigma_2 = u_- + u_+ \) can be achieved by taking the limit \( \varepsilon \to 0 \). Here the rarefaction wave \( R_1 \) is transformed into a contact discontinuity \( J \) in the limit situation.

In particular, if \( \rho_+ = \frac{\rho_- u_+}{u_-} \), then the Riemann solution to (1.2) and (1.4) is also given by (3.29). But one now gets \( \lim_{\varepsilon \to 0} (u_+^*, \rho_+^*) = (u_-, \rho_-) \) from (3.39), which implies that \((u_+, \rho_+)\) are connected directly by 2-shock wave in the limit situation.

In the end, we consider the most interesting case \( u_+ \leq 0 < u_- \) in which the delta shock wave can be obtained if we take the limit \( \varepsilon \to 0 \) in the Riemann solution to the scaled Leroux system (1.2). More precisely, we have the following result to depict it.

**Lemma 3.8.** Let us denote \( \sigma = u_- + u_+ \). If \( u_+ \leq 0 < u_- \), then we have the following relations:

\[
\lim_{\varepsilon \to 0} u_+^\varepsilon = \lim_{\varepsilon \to 0} \sigma_1 = \lim_{\varepsilon \to 0} \sigma_2 = \sigma,
\]

\[
\lim_{\varepsilon \to 0} \varepsilon \rho_+^\varepsilon = -u_- u_+, \quad \lim_{\varepsilon \to 0} \int_{\sigma_{1t}}^{\sigma_{2t}} \rho_+^\varepsilon dx = (u_- \rho_+ - u_+ \rho_-) t.
\]

**Proof.** With the same reason as Lemma 3.2, if \( u_+ \leq 0 < u_- \), then we have \((u_+, \rho_+) \in I(u_-, \rho_-)\) provided that \( \varepsilon \) is sufficiently small. In other words, the Riemann solution for the scaled Leroux system (1.2) is \( S_1 + S_2 \) for \( \varepsilon \) sufficiently small.
More precisely, the Riemann solution for the scaled Leroux system (1.2) with the Riemann initial data (1.4) can be expressed by (3.19) which satisfies both (3.2) and (3.4). Taking into account \( u_+ \leq 0 < u_- \), it can be derived from (3.5) and (3.6) that
\[
\lim_{\varepsilon \to 0} w^\varepsilon_+ = -\infty, \quad \lim_{\varepsilon \to 0} z^\varepsilon_+ = +\infty, \quad \lim_{\varepsilon \to 0} \frac{z^\varepsilon_+}{w^\varepsilon_-} = \frac{u_+}{u_-}. \tag{3.47}
\]
Thus, in view of (3.47), it follows from (3.10) that
\[
\lim_{\varepsilon \to 0} u^\varepsilon_+ = u_- + u_+ = \sigma, \quad \lim_{\varepsilon \to 0} \rho^\varepsilon_+ = +\infty, \quad \lim_{\varepsilon \to 0} \varepsilon \rho^\varepsilon_+ = -u_- u_+. \tag{3.48}
\]
Moreover, from (3.2) and (3.4), we deduce that
\[
\lim_{\varepsilon \to 0} \sigma_1 = \lim_{\varepsilon \to 0} \sigma_2 = u_- + u_+ = \sigma. \tag{3.49}
\]
From the above results, one can conclude that the two shock waves coincide as \( \varepsilon \to 0 \). The propagation speed \( \sigma \) is identical with that of the \( \delta \)-shock wave to the system (1.3) with the same Riemann initial data \((u_\pm, \rho_\pm)\).

The Rankine-Hugoniot condition (2.17) for both shock waves \( S_1 \) and \( S_2 \) implies that
\[
\left\{\begin{array}{l}
\sigma_1 (\rho^\varepsilon_- - \rho_-) = \rho^\varepsilon_- u^\varepsilon_- - \rho_- u_-,
\sigma_2 (\rho_+ - \rho^\varepsilon_+) = \rho_+ u_+ - \rho^\varepsilon_+ u^\varepsilon_+.
\end{array}\right. \tag{3.50}
\]
Thus, we have
\[
(\sigma_1 - \sigma_2) \rho^\varepsilon_+ = \rho_+ u_+ - \rho_- u_- + \sigma_1 \rho_- - \sigma_2 \rho_+. \tag{3.51}
\]
Taking the limit \( \varepsilon \to 0 \) in (3.51) yields
\[
\lim_{\varepsilon \to 0} (\sigma_2 - \sigma_1) \rho^\varepsilon_+ = u_- \rho_+ - u_+ \rho_- . \tag{3.52}
\]
Hence, (3.46) can be derived directly from (3.52). It is shown that the state variable \( \rho \) becomes a singular measure on the line \( x = \sigma t \) as \( \varepsilon \to 0 \), which is a linear function of \( t \) and is also consistent with that of the \( \delta \)-shock wave to the system (1.3) with the same Riemann initial data \((u_\pm, \rho_\pm)\). Therefore, the solution is not self-similar any more even though the system (1.3) and the initial data (1.4) are invariant under the self-similar transform.

We are now in a position to show the following theorem characterizing the limit \( \varepsilon \to 0 \) of the Riemann solution for the scaled Leroux system (1.2) with the Riemann initial data (1.4) in the case \( u_+ \leq 0 < u_- \).

**Theorem 3.9.** In the case \( u_+ \leq 0 < u_- \), the limit \( \varepsilon \to 0 \) of the Riemann solution for the scaled Leroux system (1.2) with the Riemann initial data (1.4) is a delta shock wave \( \delta S \) as follows:
\[
(u, \rho)(x, t) = \left\{\begin{array}{ll}
(u_-, \rho_-), & x < \sigma t, \\
(u_\delta, \beta(t) \delta(x - \sigma t)), & x = \sigma t, \\
(u_+, \rho_+), & x > \sigma t,
\end{array}\right. \tag{3.53}
\]
where \( u_\delta = \sigma = u_- + u_+ \) and \( \beta(t) = (u_- \rho_+ - u_+ \rho_-) t \).

**Proof.** In the case \( u_\pm \leq 0 < u_- \), if \( \varepsilon \) is small enough, then the Riemann solution to (1.2) and (1.4) is also given by (3.19). Then, the following weak equalities
\[
\left\langle -\xi (\rho^\varepsilon(x)) \right\rangle + \left( (u^\varepsilon(x))^2 + \varepsilon \rho^\varepsilon(x) \right) \psi(x) = 0, \tag{3.54}
\]
\[
\left\langle -\xi (\rho^\varepsilon(x)) \right\rangle + \left( \rho^\varepsilon(x) u^\varepsilon(x) \right) \psi(x) = 0, \tag{3.55}
\]
hold for any $\psi(\xi) \in C_0^\infty(R)$, which can also be reformulated in the equivalent integral formulations as
\[
\int_{-\infty}^{\infty} (\xi u^*(\xi) - (u^*(\xi))^2 - \varepsilon \rho^*(\xi))\psi'(\xi)d\xi + \int_{-\infty}^{\infty} u^*(\xi)\psi(\xi)d\xi = 0, \tag{3.56}
\]
\[
\int_{-\infty}^{\infty} (\xi - u^*(\xi))\rho^*(\xi)\psi'(\xi)d\xi + \int_{-\infty}^{\infty} \rho^*(\xi)\psi(\xi)d\xi = 0. \tag{3.57}
\]

Let us consider the first integral in (3.56). In view of (3.19), the first integral in (3.56) can be divided into three parts
\[
\int_{-\infty}^{\sigma_1} + \int_{\sigma_1}^{\sigma_2} + \int_{\sigma_2}^{\infty} (\xi u^*(\xi) - (u^*(\xi))^2 - \varepsilon \rho^*(\xi))\psi'(\xi)d\xi. \tag{3.58}
\]
Noticing that $(u^*(\xi), \rho^*(\xi)) = (u_-, \rho_-)$ for $\xi < \sigma_1$, we have
\[
\int_{-\infty}^{\sigma_1} (u_- - u_2 - \varepsilon \rho_-)\psi'(\xi)d\xi = (u_-\sigma_1 - u^2 - \varepsilon \rho_-)\psi(\sigma_1) - \int_{-\infty}^{\sigma_1} u_-\psi(\xi)d\xi. \tag{3.59}
\]
Similarly, taking into account $(u^*(\xi), \rho^*(\xi)) = (u_+, \rho_+)$ for $\xi > \sigma_2$, we also have
\[
\int_{\sigma_2}^{\infty} (u_+ - u_2 - \varepsilon \rho_+)\psi'(\xi)d\xi = (-u_+\sigma_2 + u^2 + \varepsilon \rho_+)\psi(\sigma_2) - \int_{\sigma_2}^{\infty} u_+\psi(\xi)d\xi. \tag{3.60}
\]
Thus, in view of (3.44), the limit $\varepsilon \to 0$ of the sum of the first and last terms in (3.58) is
\[
\lim_{\varepsilon \to 0} \left\{ \int_{-\infty}^{\sigma_1} + \int_{\sigma_1}^{\sigma_2} \right\} (\xi u^*(\xi) - (u^*(\xi))^2 - \varepsilon \rho^*(\xi))\psi'(\xi)d\xi = -\int_{-\infty}^{\infty} u_0(\xi - \sigma)\psi(\xi)d\xi, \tag{3.61}
\]
in which $u_0(\xi) = u_- + (u_+ - u_-)H(\xi)$.

Now, let us turn back to (3.58) and consider the integrand interval $(\sigma_1, \sigma_2)$ where $(u^*(\xi), \rho^*(\xi)) = (u_*, \rho_*)$. Thus, one can obtain that
\[
\int_{\sigma_1}^{\sigma_2} (u_*^2\xi - (u_*^2)^2 - \varepsilon \rho_*^2)\psi'(\xi)d\xi
\]
\[
= u_*^2(\sigma_2\psi(\sigma_2) - (u_*^2)^2) - \varepsilon \rho_*^2)(\psi(\sigma_1)) - \int_{\sigma_1}^{\sigma_2} u_*^2\psi(\xi)d\xi, \tag{3.62}
\]
which converges to zero as $\varepsilon \to 0$ by taking into account the fact that $\varepsilon \rho_*^2$ is bounded, where (3.44) and (3.45) have been used. Thus, for any $\psi(\xi) \in C_0^\infty(R)$, we have
\[
\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} (u^*(\xi) - u_0(\xi - \sigma))\psi(\xi)d\xi = 0. \tag{3.63}
\]

From now on, we are in a position to consider the first integral in (3.57). As before, we also divide the integrand interval $(-\infty, +\infty)$ into three parts. Similarly, we have
\[
\int_{-\infty}^{\sigma_1} (\xi - u_-)\rho_-\psi'(\xi)d\xi = (\sigma_1\rho_- - u_-\rho_-)\psi(\sigma_1) - \int_{-\infty}^{\sigma_1} \rho_-\psi(\xi)d\xi, \tag{3.64}
\]
\[
\int_{\sigma_2}^{\infty} (\xi - u_+)(u_+ - \sigma_2\rho_+)\psi(\sigma_2) - \int_{\sigma_2}^{\infty} \rho_+\psi(\xi)d\xi. \tag{3.65}
\]
By taking the limit \( \varepsilon \to 0 \) in (3.64) and (3.65), in view of (3.44), we achieve

\[
\lim_{\varepsilon \to 0} \left\{ \int_{-\infty}^{\sigma_1} + \int_{\sigma_2}^{\infty} \right\} (\xi - u_\varepsilon(\xi))\rho_\varepsilon(\xi)\psi'(\xi) d\xi = (u_+\rho_- - u_-\rho_+) \psi(\sigma) - \int_{-\infty}^{\infty} \rho_0(\xi - \sigma) \psi(\xi) d\xi.
\]

(3.66)

in which \( \rho_0(\xi) = \rho_- + (\rho_+ - \rho_-)H(\xi) \). Similarly, we also have

\[
\int_{\sigma_1}^{\sigma_2} (\xi - u_\varepsilon^\ast) \rho_\varepsilon^\ast(\xi) d\xi = \rho_\varepsilon^\ast(\sigma_2 \psi(\sigma_2) - \sigma_1 \psi(\sigma_1)) - \rho_\varepsilon^\ast(\psi(\sigma_2) - \psi(\sigma_1)) - \int_{\sigma_1}^{\sigma_2} \rho_\varepsilon^\ast(\xi) d\xi.
\]

(3.67)

Noticing that \( \lim_{\varepsilon \to 0} \rho_\varepsilon^\ast = +\infty \), in order to get the limit \( \varepsilon \to 0 \) in (3.67), one needs to reformulate it as

\[
\int_{\sigma_1}^{\sigma_2} (\xi - u_\varepsilon^\ast) \rho_\varepsilon^\ast(\xi) d\xi = \rho_\varepsilon^\ast(\sigma_2 \psi(\sigma_2) - \sigma_1 \psi(\sigma_1)) - \rho_\varepsilon^\ast(\psi(\sigma_2) - \psi(\sigma_1)) - \int_{\sigma_1}^{\sigma_2} \rho_\varepsilon^\ast(\xi) d\xi.
\]

(3.68)

Taking into account \( \psi(\xi) \in C^\infty_0(R) \) and \( \lim_{\varepsilon \to 0} \sigma_1 = \lim_{\varepsilon \to 0} \sigma_2 = \sigma = u_- + u_+ \), one has

\[
\lim_{\varepsilon \to 0} \frac{\psi(\sigma_2) - \psi(\sigma_1)}{\sigma_2 - \sigma_1} = \psi'(\sigma), \quad \lim_{\varepsilon \to 0} \frac{\sigma_2 \psi(\sigma_2) - \sigma_1 \psi(\sigma_1)}{\sigma_2 - \sigma_1} = \lim_{\varepsilon \to 0} \left( \frac{\sigma_2 \psi(\sigma_2) - \psi(\sigma_1)}{\sigma_2 - \sigma_1} + \psi(\sigma_1) \right) = \sigma \psi'(\sigma) + \psi(\sigma).
\]

(3.69)

(3.70)

Based on (3.45), (3.69) and (3.70), taking the limit \( \varepsilon \to 0 \) in (3.68) leads to

\[
\lim_{\varepsilon \to 0} \int_{\sigma_1}^{\sigma_2} (\xi - u_\varepsilon^\ast) \rho_\varepsilon^\ast(\xi) d\xi = 0,
\]

(3.71)

in which (3.44) and (3.52) have been used. In view of (3.66) and (3.71), it follows from (3.57) that

\[
\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} (\rho_\varepsilon(\xi) - \rho_0(\xi - \sigma)) \psi(\xi) d\xi = (u_-\rho_+ - u_+\rho_-) \psi(\sigma).
\]

(3.72)

In the end, let us consider the limit of \( \rho_\varepsilon \). We need to look for the solution depending on the time \( t \) for it is no longer self-similar in the limit situation. Let \( \phi(x,t) \in C^\infty_0(R \times R^+) \), then we have

\[
\lim_{\varepsilon \to 0} \int_0^\infty \int_{-\infty}^{\infty} \rho_\varepsilon^{(x,t)}(\xi) \phi(x,t) d\xi dt = \lim_{\varepsilon \to 0} \int_0^\infty t \left( \int_{-\infty}^{\infty} \rho_\varepsilon^{(x,t)}(\xi) \phi(\xi,t) d\xi \right) dt.
\]

(3.73)

If \( t \) is taken as a parameter, then it follows from (3.72) that

\[
\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \rho_\varepsilon(\xi) \phi(\xi,t) d\xi = \int_{-\infty}^{\infty} \rho_0(\xi - \sigma) \phi(\xi,t) d\xi + (u_-\rho_+ - u_+\rho_-) \phi(\sigma,t). \]

(3.74)

Substituting \( \xi = \frac{\xi}{t} \) into (3.74) and noting \( \rho_0(\frac{\xi}{t} - \sigma) = \rho_0(x - \sigma t) \), we rewrite (3.74) as

\[
\lim_{\varepsilon \to 0} \int_0^\infty \int_{-\infty}^{\infty} \left( \rho_\varepsilon^{(x,t)} - \rho_0(x - \sigma t) \right) \phi(x,t) d\xi dt = \int_0^\infty (u_-\rho_+ - u_+\rho_-) t \phi(\sigma,t) dt.
\]

(3.75)

Thus, the strength of the delta shock wave is \( \beta(t) = (u_-\rho_+ - u_+\rho_-) t \).
Remark 2. In particular, if \( u_- = 0 \), then both the wave curves \( W_1(u_-, \rho_-) \) and \( W_2(u_-, \rho_-) \) become the line \( u = 0 \) in the limit situation. When \( u_+ < 0 \), the Riemann solution for the scaled Leroux system (1.2) is \( S_1 + S_2 \) given by (3.19) for \( \varepsilon \) sufficiently small. The limit \( \varepsilon \to 0 \) of the Riemann solution to (1.2) and (1.4) is a delta shock wave \( \delta S \) given by (3.53). When \( u_+ > 0 \), the Riemann solution for the scaled Leroux system (1.2) is \( R_1 + R_2 \) given by (3.26) for \( \varepsilon \) sufficiently small. The limit \( \varepsilon \to 0 \) of the Riemann solution to (1.2) and (1.4) is \( J + \overrightarrow{R} \) given by (3.37).

So far, we have finished the discussion for the limit \( \varepsilon \to 0 \) behaviors of Riemann solutions to the scaled Leroux system (1.2) in all kinds of situations. It is clear to see that Theorem 1.1 has been established by gathering Theorems 3.3-3.7 and 3.9 together.

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