Universality and Ultraviolet Regularizations of Chern-Simons Theory

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Abstract

The universality of radiative corrections to the gauge coupling constant $k$ of Chern-Simons theory is studied in a very general regularization scheme. We show that the effective coupling constant $k$ induced by radiative corrections depends crucially on the balance between the ultraviolet behavior of scalar and pseudoscalar terms in the regularized action. There are three different regimes. When the ultraviolet leading term is scalar the coupling $k$ is shifted to $k + h^\vee$. However, if the leading term is pseudoscalar the shift is $k + sh^\vee$ with $s = 0$ or $s = 2$ depending on the sign of such a term. In the borderline case when the scalar and pseudoscalar terms have the same ultraviolet behavior the shift of $k$ becomes arbitrary (even non-integer) and depends on the parameters of the regularization. We also show that the coefficient of the induced gravitational Chern-Simons term is different for the three regimes and has the same universality properties than the effective coupling constant $k$. The results open the possibility of a connection with non-rational two-dimensional conformal theories in the borderline regime.
1. Introduction

The invariance of Chern-Simons theory under gauge transformations imposes a quantization condition on its coupling constant $k \in \mathbb{Z}$ when the gauge group is compact. This constraint arises in the covariant formalism as a consistency condition for the definition of the euclidean functional integral due to the special transformation properties of the Chern-Simons action under large gauge transformations [1]. In the canonical formalism it appears as a necessary condition for the integration of Gauss law constraint on the physical states [2]. Both interpretations are based on non-infinitesimal symmetries and therefore the quantization condition can not be derived from perturbative arguments. However, unexpectedly the perturbative contributions of quantum fluctuations do not seem to change the integer nature of the Chern-Simons coupling constant in the renormalization schemes considered so far. Regularization methods which do preserve the pseudoscalar character of Chern-Simons interaction do not yield any renormalization of the coupling constant [3] [4] and the regularizations which introduce scalar Yang-Mills like selfinteractions generate by one loop corrections an integer shift of the coupling constant of the form $k \rightarrow k + h^\nu$ [5]–[9]. The behavior of the effective coupling constant $k$ with these two types of regularizations seems to indicate the existence of an universal property of $k$ perhaps related to the topological nature of the Chern-Simons theory [1]. However from a pure quantum field theory point of view this behavior is unusual because in absence of perturbative symmetry constraints [8] [11] there must always exist a regularization scheme where the effective values of coupling constants are arbitrary. In a recent paper [4] we have found by a suitable introduction of pseudoscalar interactions with Pauli-Villars ghosts that the effective value of $k$ is not constrained to be only $k$ or $k + h^\nu$ but $k + nh^\nu$, where $n$ is an arbitrary integer which depends on the number of ghosts fields interacting with gauge fields by means of pseudoscalar couplings [6]. However, even within this general scheme the integer character of the shift is preserved and this property requires a physical explanation.

In this paper we find a physical interpretation of this phenomenon by considering the quantization of Chern-Simons theory in a more general regularization scheme. The regularization involves pseudoscalar and scalar terms with arbitrary number of covariant derivatives in such a way that gauge invariance is preserved. We observe an interplay

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1 A similar phenomenon occurs for non-compact gauge groups [10]

2 This result is reminiscent of that obtained by Coste-Lüsher for the coefficient of the Chern-Simons term in the effective action generated by Dirac fermions in $2 + 1$ space-time dimensions
between the ultraviolet behaviors of scalar and pseudoscalar terms, which generates three different regimes for the radiative corrections generated by loops of gluons. In the first regime the leading ultraviolet selfinteractions of gluons are scalar. The effective Chern-Simons coupling constant gets shifted in this case by the dual Coxeter number $h^\vee$ of $G$, i.e. $k \to k + h^\vee$, due to one loop gluonic radiative corrections. The second regime is characterized by an ultraviolet behavior dominated by pseudoscalar selfinteractions and the absence of radiative corrections to $k$ if the sign of those leading terms match that of the Chern-Simons term. If the signs of those terms are opposite, then, the radiative corrections shift $k$ to $k + 2h^\vee$. In the borderline regime scalar and pseudoscalar terms have the same ultraviolet behavior. In such a case the quantum corrections to $k$ can take any real value and do depend on the relative coefficients of the leading terms of scalar and pseudoscalar interactions.

The universality properties of the above three regimes are not only present in the renormalization of the coupling constant $k$. They also appear in other observables of the quantum theory, e.g. in the form of the induced gravitational action. The one-loop contribution to coefficient $\kappa$ of the induced gravitational Chern-Simons term is $\dim G/24$ in the regime dominated by scalar interactions, vanishes or is equal to $\dim G/12$ in the axial regime and depends on the relative coefficients of the leading terms of scalar and axial interactions like the shift of $k$ in the borderline regime. This features indicate that the three regimes correspond in fact to different physical behaviors, and point out the physical relevance of the balance of the ultraviolet behaviors of the axial and scalar regulating terms.

The structure of the paper is as follows. In section 2 we introduce the most general regularization within the framework of geometric regularizations which will be used throughout the paper. The consistency of the regularization is analyzed in section 3. This imposes some restrictions on the nature of ghosts interactions. In particular, finiteness and gauge invariance can only be achieved, for regularizations with leading axial terms in the ultraviolet regime, by means of pseudo-differential non-local operators in the interaction of ghosts with gluons. In section 4 we carry out the calculation of one-loop corrections to the effective action. We observe the three regimes already mentioned leading to different quantum corrections. In most cases there is a non-vanishing renormalization of the gauge field, first pointed out in [8]. In section 5 we analyse the nature of the induced gravitational action and in particular we evaluate the coefficient of the gravitational Chern-Simons term.
which is supposed to be proportional to the central charge of the associated Wess-Zumino-Witten model. Finally in section 6 we summarize the conclusions and implications of the present study. Some technical aspects concerning perturbative calculations are summarized in the two appendices.

2. Ultraviolet Regularization

Because of the pseudoscalar character of the Chern-Simons action, standard perturbative regularization methods can not be applied. This fact, has recently stimulated the interest on the application of different perturbative regularization prescriptions to Chern-Simons theories [3]–[12]. We will consider an ultraviolet regularization based on the geometric regularization scheme introduced by two of us for gauge theories in ref. [13]. The essential features of this type of regularization are based on the observation that the relevant space for covariant quantization is the space of gauge fields modulo gauge transformations, i.e. the space of gauge orbits $\mathcal{M}$, endowed with a Riemannian volume element [14]. Since $\mathcal{M}$ is a curved $\infty$-dimensional (Riemannian) manifold the regularization of a functional integral defined over $\mathcal{M}$ does not simply requires a regularization of the action, as in ordinary field theories with flat configuration spaces, but also a non-trivial regularization of the functional (Riemannian) volume element [15]. In this way it is possible to obtain a regularization which overcomes the Gribov problem and is free of overlapping divergences usually associated to the regularization by means of higher covariant derivatives and Pauli-Villars ghosts, whereas it preserves all topological properties of continuum approaches and has a non-perturbative interpretation (see Refs. [13]).

The geometric regularization method proceeds by three steps.

1. Regularization of the classical action by higher covariant derivatives. There are two classes of regulating terms which preserve gauge invariance: scalar (real) terms of Yang-Mills type [3], e.g.

$$S_{\Lambda}^{s} = \frac{k}{8\pi\Lambda} (F(A), (I + \Delta_{A}/\Lambda^{2})^{m} F(A))$$

with $\Delta_{A} = d_{A}^{*} d_{A} + d_{A} d_{A}^{*}$ and pseudoscalar (axial, imaginary) terms, e.g.

$$S_{\Lambda}^{a} = -\frac{ik}{8\pi\Lambda^{2}} (* F(A), (I + \Delta_{A}/\Lambda^{2})^{n} * d_{A} (I + \Delta_{A}/\Lambda^{2})^{n} * F(A))$$

\[^{3}\text{A pure Yang-Mills term is not sufficient to regulate 2-loops divergences [7].}\]
which were first introduced in ref. \[8\] for different purposes. We shall consider the most general gauge invariant structure of the regularized action

\[ S_{\Lambda} = S^{cs} + \lambda S^{s}_{\Lambda} + \lambda' S^{a}_{\Lambda} \]  

(2.3)

involving both types of regulating terms with relative (real) weights \( \lambda, \lambda' \) in addition to the Chern-Simons action

\[ S^{cs} = \frac{ik}{4\pi} \int_{T^3} tr (A \wedge dA + \frac{2}{3} A \wedge A \wedge A). \]  

(2.4)

We assume that \( \lambda \geq 0 \) to have a damping contribution to the functional integral and use the compact notation introduced in ref. \[13\] for covariant gauge differential calculus.

2. **Regularization of the volume of gauge orbits**,

\[ \det^{1/2} \Delta^{0}_{\Lambda} = \det^{1/2} d^{*}_{A} d_{A}, \]  

by Pauli-Villars method

\[ \det_{A} \Delta^{0}_{A} = \det^{-1} (I + \Delta^{0}_{A}/\Lambda^{2})^{2m_{0}} \det^{-1} (I + \Delta^{0}_{A}/\Lambda^{2})^{2(m_{2}-m_{1})} \]  

\[ \det \Delta^{0}_{A} \left( [(I + \lambda'(I + \Delta^{0}_{A}/\Lambda^{2})^{2n} \Delta^{0}_{A}/\Lambda^{2})]^{2} + \lambda^{2} (I + \Delta^{0}_{A}/\Lambda^{2})^{2m} \Delta^{0}_{A}/\Lambda^{2} \right) \]  

(2.5)

and the crucial step

3. **Regularization of the volume element of the gauge orbit space** \( \mathcal{M} = \mathcal{A}/\mathcal{G} \). In the case of scalar fields the configuration space has a Hilbert space structure \( \mathcal{H} \) and a Gaussian measure in \( \mathcal{H} \) is defined by means of the nuclear structure associated to a trace class operator (Minlos’ theorem). The generalization of this structure for \( \infty \)-dimensional curved spaces is also necessary for the construction of functional measures on Hilbert Lie groups \[16\] and arbitrary \( \infty \)-dimensional Hilbert manifolds including gauge orbits spaces \[15\]. Geometric regularization is based on the implementation of this construction for the covariant formalism \[13\]. This is achieved by means of a binuclear Riemannian structure \( (g^{0}_{A}, G^{1}_{A}, G^{2}_{A}) \) of \( \mathcal{M} \), which consists of a Riemannian metric \( g^{0}_{A} \) and two families of selfadjoint trace class operators \( G^{1}_{A}, G^{2}_{A} \) acting on the tangent spaces of \( \mathcal{M} \). In our case the binuclear Riemannian structure can be defined by means of the three Riemannian metrics \( g^{i}_{A}, i = 0, 1, 2 \) of \( \mathcal{M} \) given by

\[ g^{i}_{A}(\tau, \eta) = (P_{A} \tilde{\tau}, (I + \Delta_{A}/\Lambda^{2})^{m_{i}} P_{A} \tilde{\eta}) \quad i = 0, 1, 2, \]  

(2.6)

for any tangent vectors \( \tilde{\tau}, \tilde{\eta} \) of \( T_{A} \mathcal{A} \) whose projections on \( \mathcal{M} \) are \( \tau \) and \( \eta \), respectively. \( P_{A} \) denotes the projection operator

\[ P_{A} = 1 - d_{A} (d^{*}_{A} d_{A})^{-1} d^{*}_{A} \]  

(2.7)
onto the subspace of tangent vectors $\xi \in T_A A$ which are transverse to the gauge orbit of $A$, $d^*_A \xi = 0$. The operators $G^i_\Lambda : T_{[A]} M \rightarrow T_{[A]} M$ $(i = 1, 2)$ defined by

$$g^i_\Lambda(\tau, \eta) = g^0_\Lambda(\tau, (G^i_\Lambda)^{-1} \eta)$$

(2.8)

are selfadjoint and trace class with respect to $g^0_\Lambda$ for $m_i \geq m_0 + 2$, and define a binuclear Riemannian structure in the orbit space $M$.

In the local coordinates of $M$ defined by the gauge condition

$$d^* A_c = 0$$

(2.9)

in a neighborhood $C_0$ of the orbit of $A = 0$, the regularized functional integral reads

$$\int_{C_0} \delta A_c \, det^{1/2}_c g^0_\Lambda(A_c) \, det^{1/2}_c (G^1_\Lambda)^{-1} \, det^{1/2}_c (G^2_\Lambda)^{-1} \, det^1_c \, \Delta^0_A \, e^{-S_\Lambda(A)},$$

(2.10)

where the functional formal measure

$$\delta \mu_{g^0_\Lambda G^1_\Lambda G^2_\Lambda}(A_c) = \delta A_c \, det^{1/2}_c g^0_\Lambda(A_c) \, det^{1/2}_c (G^1_\Lambda)^{-1} \, det^{1/2}_c (G^2_\Lambda)^{-1} \, G^1_\Lambda.$$ 

(2.11)

can be considered as a regularization of the volume element associated to the binuclear Riemannian structure $(g^0_\Lambda, G^1_\Lambda, G^2_\Lambda)$.

We remark that the definition of the regularization is based on pure geometrical terms and makes sense beyond perturbation theory. Therefore the regularized functional integral

$$\int_M \delta \mu_{g^0_\Lambda G^1_\Lambda G^2_\Lambda}([A]) \, det^{1/2}_c \, \Delta^0_A \, e^{-S_\Lambda(A)},$$

(2.12)

has a global meaning on the orbit space $M$ because volume element (2.11) under changes of local coordinates is similar to that of a Riemannian measure in a finite dimensional manifold. Notice that $\delta \mu_{g^0_\Lambda} = \delta A_c \, det^{1/2}_c g_\Lambda(A_c)$ is the (formal) volume element of the Riemannian metric $g^0_\Lambda$, and the remaining factors associated to the nuclear structures $G^1_\Lambda$ and $G^2_\Lambda$ are gauge invariant.
3. Finiteness and Gauge Invariance

Let us analyze the finite character of the above regularization in perturbation theory. Because of the presence of a large enough number of higher covariant derivatives in the regulators of the functional measure (2.10), the degree of ultraviolet divergences in diagrams involving more than one loops is negative by power counting. However, one loop divergences can not removed by this method and have to be cancelled by an appropriate choice of the exponents of the regulating terms. The leading divergences arise in the 2-point gluonic function. They depend on the relative value of the leading exponents in the scalar and axial terms $m$ and $n$ (see Appendix A for Feynman rules).

In order to properly handle one-loop divergences it is convenient to introduce an auxiliary regularization because the finite radiative corrections might depend on the prescription used to cancel the divergences of each diagram. A very natural prescription for the auxiliary ultraviolet regularization is the introduction of a momentum cut-off $|p| \leq \Omega$ for all propagating modes. The only problem with this prescription is that it breaks gauge invariance and is therefore necessary to impose some subsidiary conditions to ensure that it is restored when the pre-cutoff $\Omega$ is removed [17].

The divergent contributions of gluonic loops come from the diagrams (1) and (2) of fig. 1 at zero external momentum, and are given by

\[
\frac{2h^\vee}{3\pi^2} (m + 1)^2 \Omega \delta^{ab} \delta_{\nu\mu} \quad \text{if } m \geq 2n + 1/2 \quad (3.1a)
\]

\[
\frac{2h^\vee}{3\pi^2} (2n + \frac{3}{2})^2 \Omega \delta^{ab} \delta_{\nu\mu} \quad \text{if } m \leq 2n + 1/2 \quad (3.1b)
\]

and

\[
-\frac{2h^\vee}{3\pi^2} ((m + 1)^2 + \frac{m}{2}) \Omega \delta^{ab} \delta_{\nu\mu} \quad \text{if } m \geq 2n + 1/2 \quad (3.2a)
\]

\[
-\frac{2h^\vee}{3\pi^2} ((2n + \frac{3}{2})^2 + n + \frac{1}{4}) \Omega \delta^{ab} \delta_{\nu\mu} \quad \text{if } m \leq 2n + 1/2 \quad (3.2b)
\]
respectively (see appendix B). $h^\vee$ denotes the dual Coxeter number of $G$, e.g. $h^\vee = N$ for $G = SU(N)$. Therefore the total divergence associated to gluonic loop reads

\[- \frac{h^\vee}{3\pi^2} m \Omega \delta^{ab} \delta_{\nu\mu} \quad \text{if} \quad m \geq 2n + 1/2. \quad (3.3a)\]

\[- \frac{h^\vee}{6\pi^2} (4n + 1) \Omega \delta^{ab} \delta_{\nu\mu} \quad \text{if} \quad m \leq 2n + 1/2 \quad (3.3b).\]

This divergent contribution to the vacuum polarization tensor can be cancelled by the contributions of one-loop diagrams of metric and nuclear ghosts. The divergent contributions generated by one loop radiative corrections of metric ghosts (diagrams (1) and (2) of fig. 2) are

This divergent contribution to the vacuum polarization tensor can be cancelled by the contributions of one-loop diagrams of metric and nuclear ghosts. The divergent contributions generated by one loop radiative corrections of metric ghosts (diagrams (1) and (2) of fig. 2) are
\[-\frac{2h^\vee}{3\pi^2} \Omega \delta^{ab}\delta_{\mu\nu} m_0^2 \] (3.4)

and

\[ \frac{h^\vee}{3\pi^2} (2m_0^2 + m_0 - 1) \Omega \delta^{ab}\delta_{\mu\nu}, \] (3.5)

respectively.

In a similar way, the corresponding diagrams of nuclear ghost loops yield

\[-\frac{2h^\vee}{3\pi^2} [(m_1 - m_0)^2 + (m_2 - m_1)^2] \Omega \delta^{ab}\delta_{\mu\nu} \] (3.6)

and

\[ \frac{h^\vee}{3\pi^2} (2(m_1 - m_0)^2 + 2(m_2 - m_1)^2 + m_2 - m_0) \Omega \delta^{ab}\delta_{\mu\nu}, \] (3.7)

respectively. Therefore the total divergent contribution of metric and nuclear ghosts is

\[ \frac{h^\vee}{3\pi^2} (m_2 - 1) \Omega \delta^{ab}\delta_{\mu\nu}. \] (3.8)

The sum of linear divergences (3.3a) (3.3b) and (3.8) vanishes provided the finiteness condition

\[ m_2 = \max\{m + 1, 2n + \frac{3}{2}\} \] (3.9)

is satisfied. The absence of divergences in the radiative corrections to the ghost propagators is guaranteed by the choice of regulating exponents satisfying the following inequalities

\[ \max\{m, 2n + \frac{1}{2}\} > \{1, m_0, m_1 - m_0, m_2 - m_1\}, \] (3.10)

Those conditions are closely related to the binuclear character of the orbit space \( \mathcal{M} \) in geometric regularization and allow us to get rid of the problem of overlapping divergences usually associated with regularizations by higher covariant derivatives \([18]\). The importance of nuclear structures can be understood from a perturbative point of view as a necessary condition to compensate the ultraviolet behavior of the ghost propagators with the singularities associated to ghost-gluon interactions. The divergences generated by the scalar ghosts involved in the regularization of the volume element of the gauge orbits also cancel out under the conditions (3.3) (3.10). In fact, those divergent contributions are given by

\[ \frac{4h^\vee}{3\pi^2} [m_0^2 + (m_1 - m_0)^2 + (m_2 - m_1)^2 - (m + 1)^2] \Omega \delta^{ab}\delta_{\mu\nu} \]
if \( m \geq 2n + 1/2 \) or

\[
\frac{4h^\vee}{3\pi^2} \left[ m_0^2 + (m_1 - m_0)^2 + (m_2 - m_1)^2 - (2n + \frac{3}{2})^2 \right] \Omega \delta^{ab} \delta_{\mu\nu}
\]

if \( m \leq 2n + 1/2 \) for diagrams of type (1), and

\[
-\frac{2h^\vee}{6\pi^2} \left( 4m_0^2 + 4(m_1 - m_0)^2 + 4(m_2 - m_1)^2 + m_2 - 4(m + 1)^2 - m - 1 \right) \Omega \delta^{ab} \delta_{\mu\nu}
\]

if \( m \geq 2n + 1/2 \) or

\[
-\frac{2h^\vee}{6\pi^2} \left( 4m_0^2 + 4(m_1 - m_0)^2 + 4(m_2 - m_1)^2 + m_2 - 4(2n + \frac{3}{2})^2 - 2n - \frac{3}{2} \right) \Omega \delta^{ab} \delta_{\mu\nu}
\]

if \( m \leq 2n + 1/2 \) for diagrams of type (2). They cancel each other out under the condition (3.9). The splitting of the cancellation of one loop divergences into two parts, one corresponding to vector fields and another to scalar fields, is one of the characteristics of geometric regularization. Divergences generated by one gluonic loop diagrams are cancelled by the diagrams associated to the volume element of the binuclear structure of the orbit space and the divergences corresponding to the volume of the gauge fibers are regularized by standard Pauli-Villars methods.

On the other hand there are not divergences associated to three-point functions. In this case the potentially logarithmic divergent terms cancel out by algebraic reasons and the final contribution always remains finite.

Although the regularization is formally gauge invariant the introduction of the auxiliary momentum cut-off \( \Omega \) for the analysis of the cancellation of one loop divergences breaks explicitly gauge invariance. This invariance might be recovered after removing \( \Omega \), but in general the restoration of the symmetry requires to impose some constraints on the exponents of the regularization \[17\]. The conditions which guarantee the vanishing of anomalous terms in Slavnov-Taylor identities are

\[
\max\{m + 1, 2n + \frac{3}{2}\}^2 = m_0^2 + (m_1 - m_0)^2 + (m_2 - m_1)^2.
\] (3.11)

The constraints arise from the analysis of possible sources of violation of those identities. The diagrammatic derivation of the identities involves a comparison of the contributions of diagrams of type (1) and (2) in two points functions. Now, the domain of integration in diagrams of type (1) is \(|p| \leq \Omega, |p + q| \leq \Omega\) whereas in diagrams of type (2) it is just \(|p| \leq \Omega\), and such a difference of domains generates an anomalous contribution because both
diagrams were originally divergent. Three and higher point functions, being logarithmically
divergent or finite, do not generate in the limit of $\Omega \to \infty$ any anomalous contribution
associated to the differences of domains of integration in diagrams with different topologies.

A way of overcoming the problems with the choice of domains in Feynman integrals
is to impose finiteness on the sum of diagrams with the same topology separately, then
the sum of the integrals associated to diagrams of type (1) is finite when $\Omega \to \infty$ and it is
independent of which of the two domains of integration we consider.

It can be shown that it is precisely the condition (3.11) which together with (3.9)
implies the vanishing of the sum of divergences of diagrams with the same topology. In
particular, the divergent contributions of diagrams of type (1) of figs. 1 and 2, (3.16) (3.17),
(3.4) and (3.6) cancel out if and only if (3.11) is satisfied. The cancellation of the contributions of diagrams of type (2) follows from (3.9) and (3.11).

If conditions (3.9) and (3.11) are satisfied we can completely forget about the auxiliary
cut-off $\Omega$ and use the standard Pauli-Villars prescription where the internal momenta
of all divergent diagrams with the same topology are parametrized in an identical way.
However the use of the auxiliary cut-off $\Omega$ is very convenient because it provides a non-
perturbative approach to the functional integral and gives an unambiguous prescription for
the parametrization of individual diagrams in Pauli-Villars regularizations which involve
different fields with different gauge interactions.

After this short digression regarding the relevance of (3.9) and (3.11) to guarantee
finiteness and gauge invariance in the regularization, we shall consider the solutions of
those conditions. When the scalar part of the action dominates the ultraviolet behavior
the general solution of conditions (3.9) and (3.11) with integer exponents is [17]

$$m + 1 = m_2 = c (a^2 + b^2 + ab)$$
$$m_0 = c (a^2 + ab)$$
$$m_1 = c (a^2 + b^2 + 2ab)$$

(3.12)

where $a, b, c$ are any three positive integers with $a, b$ relatively primes and $a < b$. When
the pseudoscalar part of the action is ultraviolet dominant there are not solutions with
integer coefficients. This implies the presence of non-local pseudo-differential operators in
the regulating terms of ghost-gluon interactions. An explicit solution of the conditions in this case can be given by,

\[ 4n + 3 = 2m_2 = c(a^2 + b^2 + ab) \]
\[ 2m_0 = c(a^2 + ab) \]  
\[ 2m_1 = c(a^2 + b^2 + 2ab), \]

as in (3.12). An infinity of values of \(a, b\) and \(c\) satisfy the remaining inequality constraints (3.10).

4. One-loop Radiative Corrections

Once we have shown that the regularization cancels all perturbative divergences we can address the calculation of finite corrections generated by one loop diagrams. The finite scalar contributions to the two-point function generated by one gluon loops are

\[ -\frac{h^\vee}{8\pi^2} (m+1)^2 |q| \left( \delta_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) \delta^{ab} + \mathcal{O}(\Omega^{-1}, \Lambda^{-1}) \]  

if \( m \geq 2n + 1/2 \) and

\[ -\frac{h^\vee}{8\pi^2} (2n + 3/2)^2 |q| \left( \delta_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) \delta^{ab} + \mathcal{O}(\Omega^{-1}, \Lambda^{-1}) \]  

if \( m \leq 2n + 1/2 \) (see Appendix B). These contributions cancel up to terms of order \( \mathcal{O}(\Omega^{-1}, \Lambda^{-1}) \) with those generated by metric and nuclear ghost loops

\[ \frac{h^\vee}{8\pi^2} (m_0)^2 |q| \left( \delta_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) \delta^{ab} + \mathcal{O}(\Omega^{-1}, \Lambda^{-1}) \]
\[ \frac{h^\vee}{8\pi^2} (m_1 - m_0)^2 |q| \left( \delta_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) \delta^{ab} + \mathcal{O}(\Omega^{-1}, \Lambda^{-1}) \]
\[ \frac{h^\vee}{8\pi^2} (m_2 - m_1)^2 |q| \left( \delta_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) \delta^{ab} + \mathcal{O}(\Omega^{-1}, \Lambda^{-1}) \]

by condition (3.11). Similar contributions are generated by ghosts associated to the volume of the gauge orbits (2.5) but cancel out among themselves by the same reason. Non-trivial contributions to the mass term generated by diagrams of type (1) cancel out with those of diagrams of type (2). On the other hand it is easy to see that the anomalous contributions to the Slavnov-Taylor identities involving two and three point functions also cancel out.
when the condition (3.11) is satisfied. The non-analytic dependence of expressions (4.1)
(4.3) come from the constraint imposed by the non-perturbative cut-off Ω on the internal
momenta of diagrams of type (1), \(|p| < \Omega\) and \(|p + q| < \Omega|\).

After the removal of the ultraviolet regulating parameter Λ the only quantum correc-
tions to the two and three point functions are local and pseudoscalar like the two terms
of the Chern-Simons action. The radiative corrections to the two point Green function
generated by gluonic loops (diagrams (1)–(2) of fig. 1) are gi-
gen by
\[
\Gamma_{\mu\nu}^{ab} = \frac{h^\vee}{3\pi^2} \int_0^\infty dp \frac{\Sigma(p)}{\rho(p)} \delta^{ab} \epsilon_{\mu\sigma\nu} q^\sigma,
\]
where
\[
\Sigma(p) = 2S(p) + \frac{3}{2} \left[ R(p) \frac{d(S(p)p)}{dp} - pS(p) \frac{dR(p)}{dp} \right],
\]
\[
\rho(p) = p^2 S(p)^2 + R(p)^2,
\]
and
\[
S(p) = \lambda(1 + p^2)^m \quad R(p) = 1 + \lambda' p^2 (1 + p^2)^{2n}
\]
(see appendix B for more details).

There are not further one loop corrections to the pseudoscalar part of the effective
action because loops of metric and nuclear ghosts do not involve pseudoscalar couplings.

The radiative correction to the quadratic term of the pseudoscalar effective action
contains the renormalization of the gauge field and the renormalization of the coupling
constant. In order to split the two contributions we calculate the renormalization of the
gauge field by analysing the renormalization of the coupling of gauge fields to Faddeev-
Popov ghost\(^4\). The renormalization of the ghost field \(c\) is obtained from the diagram (1)
of fig. 3 whose contribution reads
\[
\Sigma_{ab} = q^2 \frac{4h^\vee}{3k\pi} \int_0^\infty dp \frac{S(p)}{\rho(p)} \delta_{ab}.
\]
This leads to a finite renormalization of Faddeev-Popov ghost fields
\[
c_R = \left[ 1 - \frac{2h^\vee}{3k\pi} \int_0^\infty dp \frac{S(p)}{\rho(p)} \right] c
\]
\(^4\) Although those ghosts do not appear in the geometric formulation of the functional integral
they appear in Slavnov-Taylor identities which are also satisfied by the regularization provided
the condition (3.11) is satisfied.\(^1\)}
One-loop radiative corrections to the gluon-ghost interaction (diagrams (2) (3) of fig. 3) vanish. Therefore the renormalization of the gauge field is given by

\[ A_\mu^R = \left[ 1 + \frac{4h^\vee}{3k\pi} \int_0^\infty \frac{S(p)}{\rho(p)} dp \right] A_\mu. \]  \hspace{1cm} (4.9)

We remark the existence of a non-trivial renormalization of the gauge field \( A \) except in the pure axial case \( \lambda = 0 \).

Plugging the renormalized fields into the two point gluonic function yields a finite renormalization of the Chern-Simons coupling constant of the form

\[ k_R = k + \frac{4h^\vee}{3\pi} \left[ \int_0^\infty \frac{\Sigma'(p)}{\rho(p)} dp \right] \]  \hspace{1cm} (4.10)

where

\[ \Sigma'(p) = \Sigma - 2S(p). \]
If we perform the change of variables \( \phi = \phi(p) = pS(p)/R(p) \) the second term on the right hand side of (4.10) for \( \lambda' > 0 \) becomes

\[
\frac{2h^\vee}{\pi} \int_{\phi(0)}^{\phi(\infty)} \frac{d\phi}{1 + \phi^2},
\]

and can be exactly calculated. However, when \( \lambda' < 0 \) the change of variables has a singular point \( p_\infty \) where \( \phi(p_\infty) = \mp \infty \) (see fig. 4).

\[\Phi 4\] The three regimes of ultraviolet behavior of the \( \phi \) function for \( \lambda' < 0 \). The infinite gap at \( p_\infty \) corresponds to a zero value of the pseudoscalar leading term which appears in all regimes.

Therefore, the domain of integration in \( \phi \) variable splits into two domains

\[
\frac{2h^\vee}{\pi} \left[ \int_{\phi(0)}^{\phi(p_\infty)^-} + \int_{\phi(p_\infty)^+}^{\phi(\infty)} \right] \frac{d\phi}{1 + \phi^2}.
\]

(4.12)

There are three different regimes for the behavior of the effective coupling constant \( k \) (see Figs. 4,5).

\(^5 \) An additional shift by \( 2sh^\vee \) could be obtained in any regime by considering pseudoscalar
Figure 5. The three regimes of ultraviolet behavior of the φ function for \( \lambda' > 0 \). \( p_0 \) is the critical point where the function φ reaches its maximal value in the regime \( m < 2n + \frac{1}{2} \).

i) If \( m > 2n + \frac{1}{2} \) and \( \lambda' > 0 \), since \( \phi(p) \) is a continuous function with \( \phi(0) = 0 \) and \( \phi(\infty) = \infty \) the shift is universal and independent of the parameters of the regularization

\[
k_R = k + h^\vee
\]

The same shift occurs in that case for \( \lambda' < 0 \) because the second integral in (4.12) vanishes. This regime includes all the cases where the ultraviolet divergences are regulated by Yang-Mills like terms [5]–[9]. Most of the regularizations analysed in the literature are of this type.

ii) If \( m < 2n + 1/2 \), \( \phi(0) = 0 \) and \( \phi(\infty) = 0 \). Therefore, if \( \lambda' > 0 \), \( \phi(p) \) is a continuous function and there is no shift of \( k \) for any value of the other parameters of the regularization. This explains why it is possible to find a gauge invariant regularization scheme where the couplings between gluons and ghosts [4]. For simplicity, we shall not consider such a possibility in this paper. We will restrict ourselves to the analysis of the behavior of the corrections generated only by gluon selfinteractions.
effective value of the coupling constant equals its bare value. This universality class of regularizations was first analyzed in a gauge invariant way in ref. However there is another universality class associated to the same regime. If \( \lambda' < 0 \) the function \( \phi(p) \) develops a pole at the only real root \( p_\infty \) of the polynomial \( 1 + \lambda' p^2 (1 + p^2)^{2n} \). Therefore the shift of \( k \) is in this case \( k + 2h^\vee \). This gives rise to a new universality class valid for all regularizations with leading pseudoscalar coupling with opposite sign to the Chern-Simons term.

iii) If \( m = 2n + 1/2 \), since \( \phi(0) = 0 \) but \( \phi(\infty) = \lambda/\lambda' \) the shift is not universal and does depend on the parameters of the regularization

\[
k_R = k + 2h^\vee \pi \arctan \frac{\lambda}{\lambda'},
\]

with \( \arctan \) taking values in \([0, \pi]\). This is a novel regime and its discovery is one of the major issues of this work.

One remarkable aspect of the above results is the existence of universality in the renormalization of \( k \) in the first two regimes and its violation in the borderline case. The physical explanation of this fact can be learned from (4.11) and (4.12) where it is shown that the shift is originated by the balance between ultraviolet and infrared behaviors of the regulating terms of the real and imaginary parts of the regularized action. The infrared behavior is unchanged in the three different cases whereas the ultraviolet regime depends on which terms has more derivatives. Once there exists a dominant scalar or axial part the shift is constant and therefore universal; at most it might depend on the sign of the leading pseudoscalar term \( \lambda' \). However, in the borderline case with a hybrid ultraviolet behavior the result depends on the relative weight of the coefficients of the terms with higher number of derivatives. In such a case any value for the effective coupling constant can be attained.

Notice that the constraint associated to the invariance under large gauge transformations only applies to the bare coupling constant. Therefore nothing prevents the existence of non-integer values for the effective coupling constant. The reason why this possibility has never considered previously is because it required a regularization by means of pseudodifferential operators. Any other choice leads to an integer valued effective coupling constant.

In the case where the effective coupling constant is non-integer the connection with conformal field theory is presumable through non-rational conformal field theories.

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6 Notice that one of the two exponents \( m \) or \( n \) have to be half integer in the borderline case because of the identity \( m = 2n + 1/2 \)

7 In fact, in the three regimes above discussed there is a larger ambiguity in the shift induced
5. Topological Anomaly

In order to explore further consequences of the existence of three renormalization regimes of the coupling constant we investigate the behavior of the radiative corrections to the vacuum energy to elucidate whether the three regimes give rise to different physical effects in this case too.

In particular, since Chern-Simons theory is a topological theory the radiative corrections should preserve the vanishing of the vacuum energy. However due to the existence of a topological anomaly a finite contribution can appear in the form of a gravitational Chern-Simons term

\[ G_{cs} = \frac{i\kappa}{4\pi} \int \text{tr} \left( \omega \wedge d\omega + \frac{2}{3} \omega \wedge \omega \wedge \omega \right). \quad (5.1) \]

defined by the Levi-Civita connection \( \omega \) of \( TM \) and a non-local framing-dependent term (framing anomaly) with a non-analytic dependence on the dreivein field \( e^a_{\mu} \) \([19]\).

If the space time manifold is of the form \( \Sigma \times S^1 \) and the metric splits into a product of metrics of \( \Sigma \) and \( S^1 \), there exists a framing where the gravitational Chern-Simons term vanishes. Therefore the partition function becomes completely metric independent. However, for a general three manifold \( M \) and arbitrary framing of its tangent bundle \( TM \) the gravitational Chern-Simons terms does not vanish (see \([20]\) for a general discussion of metric (in)dependence of the partition function of Chern-Simons theory). Moreover Witten conjectured an universal behavior for the coefficient of this pseudoscalar term of the form \([19]\)

\[ \kappa = \frac{c}{24} \]

which provides a three-dimensional interpretations of the central charge of the corresponding Wess-Zumino-Witten model,

\[ c = \frac{k \dim G}{k + h^\vee}. \quad (5.2) \]

by radiative corrections. Besides the shift by \( 2sh^\vee \) induced by radiative corrections generated by Pauli-Villars ghost fields with pseudoscalar couplings afore mentioned, there is another source of ambiguity of the same type purely generated by gluon radiative corrections. If instead of regulating the Chern-Simons action by polynomials on the covariant laplacian \( \Delta_A \) of the form \((1 + \Delta_A)^n\) we consider a more general polynomial in \( \Delta_A \) the \( \phi(p) \) function might develop a richer critical structure with more zeros and poles. In particular, if it has \( s \) poles with \( s - 1 \) zeros intercalated between those poles the integrals \((4.11)\) and \((4.12)\) would give rise to an additional shift of \( k \) by \( 2sh^\vee \) for any of the three regimes discussed above.
Since \( c \) is an analytic function of \( h^\vee / k \)

\[
c = \dim G \sum_{n=0}^{\infty} \left( \frac{-h^\vee}{k} \right)^n.
\] (5.3)

it can be computed in perturbation theory.

We shall calculate the first corrections to this coefficient in the regularization scheme discussed in the previous sections to see if the existence of three renormalization regimes has some consequences on Witten’s conjecture.

The remaining non-local contributions can also be indirectly calculated in this approximation and its presence is reflected in the existence of a framing anomaly. It follows from the observation that since the regularization method is independent of any framing of the tangent bundle. Then the effective action must be framing independent. Now the gravitational Chern-Simons term is frame dependent, therefore, it must exist a non-analytic locally constant dependence on the dreivein field \( e_i^\alpha \) whose variation under large frame transformations counterbalance the variation of the gravitational Chern-Simons term \( \mathcal{E} \). We can eliminate the contribution of the Chern-Simons gravitational term by a local counterterm but the contribution of the non-local term cannot be cancelled in such a way and this fact implies the existence of a framing anomaly in such a case \( \mathcal{F} \).

A general feature of the geometric regularization method is that once the ultraviolet divergences are cancelled in a flat space-time they do not reappear when the theory is defined on curved background metric \( \mathcal{24} \). However, when the regulator is removed some divergences might appear unless we take special care of the Pauli-Villars regularization of the scalar ghost fields associated to the volume of the orbits of the gauge group. The special form of this regularization chosen in \( \mathcal{24} \) guarantees that not only the theory is completely regularized in our case, but there is not need of an infinite renormalization in curved spacetimes backgrounds, at least at one loop level. A general discussion of finiteness and metric (in)dependence of the partition function on arbitrary curved space-times can be found in \( \mathcal{24}-\mathcal{26} \).

---

8 Those non-analytic terms also appear in the spectral asymmetry function \( \eta \) which is associated \( \mathcal{19} \) to the elliptic operator \( \ast d + d\ast \) acting on \( \text{Lie–}G \) valued one-forms and are connected with its spectral flow and the index of its four dimensional extension.

9 However, it is always possible to choose a framing of the double-tangent bundle \( T^2M \) of \( M \) (the canonical framing) where this non-local framing dependent term vanishes \( \mathcal{22} \).
In this section we are only interested on the calculation of local (finite) radiative corrections of pseudoscalar type in order to verify Witten’s conjecture.

Since ghosts fields have only scalar interactions the lowest order pseudoscalar radiative corrections can only appear from one gluon loop corrections to the vacuum energy,

\[ S_{\text{eff}} = -\frac{1}{2} \log \det((\lambda (I + \Delta / \Lambda^2)^m d^* d/\Lambda - i * d (I + \lambda' (I + \Delta / \Lambda^2)^2n d^* d/\Lambda^2)). \] (5.4)

Due to the independence on the gauge field \( A \), the calculation of the determinants simplifies considerably in operator formalism. In particular the transverse projectors necessary to restrict the differential operators to transverse modes can be suppressed because all the operators are already transverse, i.e.

\[ S_{\text{eff}} = -\frac{1}{2} \text{tr}_c \log((\lambda (I + d^* d/\Lambda^2)^m d^* d/\Lambda - i * d (I + \lambda' (I + d^* d/\Lambda^2)^2n d^* d/\Lambda^2)). \] (5.5)

The pseudoscalar terms come from the imaginary part of the effective action

\[ \frac{i}{2} \text{tr}_c \frac{\lambda' (I + d^* d/\Lambda^2)^2n d^* d/\Lambda}{\lambda (I + d^* d/\Lambda^2)^m d^* d/\Lambda}. \] (5.6)

Now, since the gravitational Chern-Simons term is analytic in metric variables \( g_{\mu \nu} \) we can calculate its coefficient by a weak metric expansion. If we consider a background metric \( g_{\mu \nu} = \delta_{\mu \nu} + h_{\mu \nu} \) in \( \mathbb{R}^3 \) which is a slight perturbation of the Euclidean metric \( \delta_{\mu \nu} \), the first contribution to the Chern-Simons gravitational term is quadratic in the metric perturbation \( h \) and cubic in space-time derivatives

\[ G_{\text{cs}} = \frac{iK}{8\pi} \int d^3 x \epsilon^{\mu \nu \sigma} (\partial_{\nu'} h_{\mu \nu} - \partial_{\mu'} h_{\mu \nu}) \partial_{\sigma} (\partial_{\mu'} h_{\nu \nu} - \partial_{\nu'} h_{\nu \nu}) + O(h^3). \] (5.7)

We remark that the expression (5.6) is a functional of the single operator \( *d \). Therefore, its second variation with respect to the metric perturbation yields,

\[ \frac{i}{2} \frac{\delta^2}{\delta g_{\mu \nu} \delta g_{\mu' \nu'}} \text{tr}_c \frac{\lambda' (I + d^* d/\Lambda^2)^2n d^* d/\Lambda}{\lambda (I + d^* d/\Lambda^2)^m d^* d/\Lambda} \]

\[ = \frac{i}{2} \sum_{j=1}^{\infty} a_j \frac{\delta^2}{\delta g_{\mu \nu} \delta g_{\mu' \nu'}} \text{tr}_c (*d)^{2j+1} \] (5.8)

where \( a_j \) are the coefficients of the Taylor expansion of the analytic function

\[ \Xi(t) = \frac{1 + \lambda' (1 + t^2 / \Lambda^2)^2n t^2 / \Lambda^2}{\lambda (1 + t^2 / \Lambda^2)^m t / \Lambda} = \sum_{j=0}^{\infty} a_j t^{2j+1} \] (5.9)
We only need to recall the expression of the differential operator \( *d \) in local coordinates,

\[
(*d)_{\mu}^{\nu} = \frac{1}{\sqrt{g}} g_{\mu\gamma} \epsilon^{\gamma\sigma\nu} \partial_{\sigma}.
\]  

(5.10)

Hence,

\[
\frac{\delta(*d)_{\mu}^{\nu}}{\delta g_{\alpha\beta}} \bigg|_{\hat{g}} = -\frac{1}{2} \frac{1}{\sqrt{\hat{g}}} \hat{g}_{\mu\gamma} \epsilon^{\gamma\alpha\beta} \epsilon^{\gamma\sigma\nu} \partial_{\sigma} + \frac{1}{\sqrt{\hat{g}}} \delta_{\mu}^{\alpha} \delta_{\nu}^{\beta} \epsilon^{\gamma\sigma\nu} \partial_{\sigma}
\]  

(5.11)

which in our particular case of flat background metric \( \hat{g}_{\mu\nu} = \delta_{\mu\nu} \) reads,

\[
\frac{\delta(*d)_{\mu}^{\nu}}{\delta g_{\alpha\beta}} = -\frac{1}{2} \delta_{\mu\nu} \epsilon^{\alpha\beta} \epsilon^{\gamma\sigma\nu} \partial_{\sigma} + \delta_{\mu}^{\alpha} \delta_{\nu}^{\beta} \epsilon^{\gamma\sigma\nu} \partial_{\sigma}.
\]  

(5.12)

Now, terms which involve traces of \( h_{\alpha\beta} \) do not appear in the gravitational Chern-Simons action \( G_{cs} \). Therefore, the first term of (5.11) do not contribute to \( G_{cs} \). The second term is not dependent on \( \hat{g} \), which implies that the relevant dependence of \( *d \) on the metric \( \hat{g} \) is linear. Hence the non-trivial contribution of (5.8) to \( G_{cs} \) is

\[
\frac{i}{2} \sum_{j=1}^{\infty} \sum_{l=0}^{2j-1} (2j + 1)a_j \text{tr} \frac{\delta(*d)^{l}}{\delta g_{\mu\nu}} \frac{\delta(*d)^{l}}{\delta g_{\mu'\nu'}} (*d)^{2j-l-1}.
\]  

(5.13)

Since we are perturbing the Euclidean metric of \( \mathbb{R}^3 \), the trace

\[
\text{tr} \frac{\delta(*d)^{l}}{\delta g_{\mu\nu}} \frac{\delta(*d)^{l}}{\delta g_{\mu'\nu'}} (*d)^{2j-l-1}
\]  

(5.14)

can be evaluated in momentum space. The contribution of the one loop diagrams to the quadratic graviton-graviton pseudoscalar term (5.8) reads (see fig. 6),

\[
\text{Diagram (1)} \quad \text{Diagram (2)}
\]

Figure 6. One loop contributions to the graviton 2-point function involving gluon loops.
where $q$ denotes the external momentum. In order to get the local terms with cubic dependence on the external momentum we have to expand the function

$$
\Theta(p, q) = \sum_{j=1}^{\infty} (2j + 1) a_j \sum_{s=0}^{j-1} (p + q)^{2s} p^{2j-2s-2}
$$

up to third order in the power Taylor expansion. Plugging the corresponding terms

$$
\Theta(p, q) = \sum_{j=1}^{\infty} (2j + 1) a_j \sum_{s=0}^{j-1} (p + q)^{2s} p^{2j-2s-2} = \frac{d \Xi(p+q)}{dp} - \frac{d \Xi(p)}{dp}
$$

into (5.15), using symmetry properties of metric tensors and internal momentum integrals and integrating by parts we get

$$
-\frac{1}{48 \pi^2} \dim G \epsilon^{\mu \sigma \nu} q_\sigma (q_\nu q_\nu' - q_\nu q_\nu') \int_0^\infty dp \frac{d \Xi(p)}{dp}.
$$

In fact, only the second and third terms of the $q$–expansion (5.17) give a non trivial contribution because the other terms cancel out. The $q$–independent and $q$–linear contributions which are potentially linear and logarithmic divergent in the borderline case also vanish.

This implies that the quantum corrections to the vacuum energy generate a gravitational Chern-Simons term $G_{cs}$ with coefficient

$$
\kappa = -\frac{\dim G}{12 \pi} \int_0^\infty dp \frac{d \Xi(p)}{dp}.
$$

In this case one must use the Feynman rules for pseudodifferential operators. We follow the standard prescription in analogy to the Feynman rules for gauge field couplings described in Appendix A.
The value of this coefficient is different in the three regimes of ultraviolet regularization. In the regime dominated by scalar terms \(\Xi(\infty) - \Xi(0) = -\pi/2\) and the coefficient takes values \(\kappa = \dim G/24\) which are in agreement with Witten’s conjecture (see also [27] [28].

In the axial regime if \(\lambda' > 0\), \(\Xi\) is a positive function and \(\Xi(\infty) - \Xi(0) = 0\). Therefore, there is not gravitational Chern-Simons term in the induced action. The result is easy to understand in the case where the regularized action is purely axial (i.e. \(\lambda = 0\)), because the radiative corrections to the gravitational Chern-Simons term vanish in such a case for all odd orders of perturbation theory. This leads to conjecture that the same result holds for \(\lambda \neq 0\) but \(m \leq 2n + 1/2\). In fact the most natural result in this regime would be \(\kappa = 0\) for all orders of perturbation theory. However, Axelrod-Singer have calculated the second order correction in this regime and found a non-vanishing contribution [25]. This is a very intriguing result. A two-loop calculation in geometric regularization scheme would be very interesting to verify whether the above conjecture holds beyond one loop approximation. In any case since all odd powers of \(\kappa\) in the expansion (5.3) vanish Witten’s conjecture can never be satisfied in this regime. If \(\lambda' < 0\) then \(\Xi(0) = 0\) and also \(\Xi(\infty) = 0\), but there exist a singular value of \(p\) where \(\Xi\) jumps from \(\infty\) to \(-\infty\). In such a case

\[
\int_0^\infty dp \frac{d\Xi(p)}{dp} = \pi,
\]

and the coefficient \(\kappa\) gets a non-null value

\[
\kappa = \frac{\dim G}{12\pi} \arctan \frac{\lambda}{\lambda'}.
\]

It would be very interesting to calculate the higher order corrections to this coefficient and see if the generalization of Witten’s conjecture also holds for this new universality class.

In the borderline regime \(m = 2n + 1/2\) the coefficient \(\kappa\) depends on \(\lambda\) and \(\lambda'\)

\[
\kappa = \frac{\dim G}{12\pi} \arctan \frac{\lambda}{\lambda'} \tag{5.20}
\]

in a similar way to the shift of \(k\).

Therefore, if \(\kappa\) is still related to the central charge of a two dimensional conformal theory it has to be a non-rational one.

In this case the non-local term which counterbalances the framing dependence of gravitational Chern-Simons term \(G_{cs}\) of the effective action can not be simply associated to the standard regularization of the spectral asymmetry function of the operator \(*d + d*\). The simplest interpretation of this fact is that the gravitational induced effective action can
not be associated in this case to the gravitational part of standard Atiyah-Patodi-Singer \( \eta \)-function \[29\]. There is an additional multiplicative fractional factor which is regularization dependent. Whether the same factor appears in higher order corrections is an interesting conjecture which requires to be tested beyond the weak metric approximation and deserves further investigation.

On the other hand, the dependence on the gauge field of the effective action or spectral asymmetry in the above regularizations, gets also multiplied by the same factor, except in the borderline regime, which makes very interesting to study the spectral asymmetry in this regime.

6. Discussion and Conclusions

The results analyzed in previous sections show that the three regimes of ultraviolet regularization of the Chern-Simons action correspond in fact to three different physical behaviors. In particular, if the gravitational Chern-Simons term is eliminated by the introduction of a local counterterm in order to get a metric independent effective action a different framing anomaly is induced in each regime. In the axial regime there is no framing anomaly if \( \lambda' > 0 \) or is a multiple of \( \text{dim } G/12\pi \) if \( \lambda' < 0 \). In the Yang-Mills regime the coefficient of this anomaly is universal and does not depend, for instance, on the parameters of the regulators. This behavior was first pointed out by Witten \[30\]. In the borderline regime such a coefficient depends on the weights \( \lambda \) and \( \lambda' \) of the axial and scalar regulators and does not correspond to any previously expected behavior. This property indicates a possible connection with non-rational conformal field theories.

The same analysis holds for the gauge field effective action as shown in section 4. The effective coupling constant behaves in a different way for each regime. It is remarkable the correspondence between the shift of \( k \) and the value of \( \kappa \). Such a relationship is more evident in the borderline case where both coefficients depend on the parameters \( \lambda \) and \( \lambda' \). However, such a connection does not hold if we had chosen the scalar laplacian \( \Delta^0_A \) instead of the Hodge Laplace-Beltrami operator

\[
\Delta_A = d_A^* d_A + d_A d_A^* = \Delta^0_A + \hat{R}(g) + [F(A), \cdot]. \quad (6.1)
\]

for the regularization of the gluonic action. The difference between both operators being the Ricci operator \( \hat{R}(g) \) and the curvature operator \([F(A), \cdot] \). In such a case the results for the first two regimes are unchanged but in the borderline case the shift of \( k \) becomes

\[
k_R = k + \frac{2h^\vee}{\pi} \left[ \arctan \frac{\lambda}{\lambda'} + \frac{\lambda\lambda'}{3(\lambda^2 + \lambda'^2)} \right], \quad (6.2)
\]

23
whereas the coefficient $\kappa$ of the induced Chern-Simons gravitational term reads

$$\kappa = \frac{\dim G}{12\pi} \left[ \arctan \frac{\lambda}{\lambda'} + \frac{16n\lambda\lambda'}{5(\lambda^2 + \lambda'^2)} \right].$$

(6.3)

The new terms $\lambda\lambda'/(\lambda^2 + \lambda'^2)$ come from the boundary values of integration by parts in the diagrams of the type (1) in figures 1 and 4 and give different weights for the radiative corrections to $k$ and $\kappa$. This fact stresses the very deep nature of the connection between both quantities in the natural regularizations formulated purely in terms of Hodge-Laplace-Beltrami operators $\Delta_A$. It is only for those regularizations where the ratio between both corrections agrees with Witten’s conjecture. It would be very interesting to investigate if this property also holds beyond one loop approximation.

On the other hand, there is an additional non-analytic contribution to the effective gauge action in the borderline regime. It appears in one loop approximation to the effective action to compensate the anomalous transformation law of Chern-Simons terms under large gauge transformations. The contribution is similar to the one which appears in the spectral asymmetry $\eta$–function of the operator $*d_A + d_A*$ induced by the changes of signs in the spectral flow [31] [32]. The coefficient of such a term also depends on the ultraviolet behavior of the regularized action, which suggests the existence of a prefactor in the dependence of the effective action on the standard $\eta$-function and the spectral flow. This means that those regularizations which provide different results correspond in fact to different regularizations of the spectral asymmetry [32]. The existence of higher order corrections to this coefficient remains unclear and requires further study. A more detailed account of this phenomenon will be carried out elsewhere [32].

Although the value of the effective coupling constant can always be modified by a different choice of renormalization scheme, we remark that the borderline quantization can not be reduced to the other two regimes by renormalization. The behavior of Chern-Simons term under large gauge transformations implies that the functional integral (2.12) is ill defined unless the bare coupling constant $k$ is an integer number. Such a constraint is based on a non-perturbative effect, because large gauge transformations map small fields into large gauge fields and, therefore, they are genuine non-perturbative symmetries. In consequence, although in perturbation theory any local BRST invariant counterterm is valid, only counterterms which preserve the non-perturbative consistency condition can be added to the bare action. This condition imposes a very restrictive constraint on counterterms which have to preserve the integer valued character of the bare coupling
constant $k$. In particular, if the effective value of $k_R$ is not an integer, we cannot reduce the physical behavior of the system to the standard integer valued case by a consistent renormalization. Therefore, whereas theories defined by axial or scalar regularizations might be physically equivalent, because it suffices to consider different values for the bare coupling $k$ for each case, the borderline regime yields a new different theory.

Axial and scalar regularizations define quantum theories in the covariant formalism which agree with the ones obtained by canonical quantization \[3\] \[7\]. They have a finite (although different for the same bare coupling constant $k$) number of physical states in a two-dimensional space with the topology of a torus; and those states are in one-to-one correspondence with the primary fields of the corresponding conformal field theories \[19\]. If the borderline regularization really defines a new type of theory, it must have a different number of states. In this way it will not be related to rational conformal field theories. Therefore in order to elucidate this possibility it is very interesting to calculate the number of states of the theory on a two dimensional torus. It can be obtained in the covariant formalism by the analysis of the expectation value of an unknotted Wilson loop \( \langle W_J^\mu(A) \rangle \) in the spin $J$ representation of $SU(2)$. Following the methods and renormalization prescriptions used in Refs. \[33\] it is easy to show that with our regularization we get

\[
\langle W_J^\mu(A) \rangle = 2 - \frac{\pi^2}{k^2} + \xi \frac{2\pi^2}{k^3} + O\left(\frac{1}{k^4}\right),
\]

(6.4)

where $\xi = k_R - k$ denotes the shift of the effective coupling constant $k_R$. The absence of frame dependent terms is due to the fact that in the regularized theory there are not divergences in the radiative corrections to the Wilson loop. Logarithmic divergences, however, reappear in the limit $\Lambda \to \infty$. They can be removed by appropriate counterterms which in our case have been chosen in such a way that the whole contribution of perturbative diagrams with an isolated propagator vanishes. Such a prescription also removes the (writhe) metric dependent part of the Wilson loop. The different terms of (6.4) agree with the first terms of the perturbative expansion of the formula

\[
\langle W_J^\mu(A) \rangle = \frac{\sin[(2J + 1)\pi/(k + \xi)]}{\sin[\pi/(k + \xi)]},
\]

(6.5)

which generalizes Witten’s conjecture \[19\].

In axial ($\xi = 0$) and scalar regimes ($\xi = 2$) the vacuum expectation value of the unknotted Wilson loop \( \langle W_J^\mu(A) \rangle \) is periodic function in $J$, which indicates that the number of genuine primary fields of the corresponding $SU(2)$ affine algebra is finite, $k - 1$ and $k + 1$, respectively.
respectively. Moreover, there is in each case a null representation, \( J = \frac{1}{2}(k \mp 1) \) where the value of the Wilson loop vanishes. It corresponds to a null state in the module of primary fields of the corresponding rational conformal field theory. However, in the borderline regime (6.5) is no longer periodic in \( J \), because \( \xi \) is not an integer (4.10), and there are not null states which could be associated to null vectors in integrable representations of the \( \widehat{SU}(2) \) affine algebra. Since it is not possible to mod out by null vectors, the number of Chern-Simons states when the physical space is a torus is infinite, which is a way of interpolating between the dimensions of the Hilbert spaces of the theory in the axial \((k-1)\) and scalar regimes \((k+1)\).

On the other hand the underlying \( SU(2)_q \) quantum group symmetry, gets a deformation parameter \( q = \exp[2\pi i/(k + \xi)] \) which in that regime is not a root of unity. Such a behavior points out again that such a theory cannot be related to rational conformal field theories.

The fact that different regularizations of the theory give rise to different quantum theories is quite surprising. Usually, the effect of the irrelevant terms introduced by the different regularizations can be absorbed into the renormalization of coupling constants. However, in the present case this is not possible because of global constraints. This feature can be associated to the topological character of the theory. In topological theories, expectation values of topological observables are usually quantized, therefore, there exists the possibility of having irrelevant perturbations which break diffeomorphism invariance and modify the expectation values in such a way that the original values cannot be recovered by means of a consistent renormalization of coupling constants. For instance, in topological quantum mechanics on a Riemann surface \( \Sigma \) of genus \( h \) in the presence of a magnetic field \( A \) with magnetic charge \( k \), the dimension of the space of quantum states is given by the Riemann-Roch theorem: \( \dim \mathcal{H}_k^0 = 1 - h + k \), for \( k > h - 1 \). The quantum hamiltonian is trivial \( (H = 0) \) as corresponds to a topological theory. However, if we regularize the theory by means of a metric dependent kinetic term,

\[
L(x, \dot{x}) = \frac{1}{2\Lambda} g_{ij} \dot{x}^i \dot{x}^j + A_i \dot{x}^i,
\]

the Hamiltonian becomes \( H_\Lambda = \frac{\Delta}{2} \Delta \), and the topological limit \( \Lambda \to \infty \) is governed by the ground states of \( H_\Lambda \). The quantum Hilbert space of the topological field theory obtained by this method can have a dimension lower than \( 1 - h + k \), depending on the symmetries of the background metric \( g \) of \( \Sigma \) [34]. In particular, this is the case when the metric \( g \) breaks the
degeneracy of the ground state of the covariant Laplacian $\Delta^g_A$. This simple example shows how in topological theories it is possible to obtain different physical theories by means of different choices of regularization. In some sense, only regularizations which preserve certain symmetries of the theory belong to the same universality class. For instance, in the previous example only the metrics which are compatible with the magnetic field $B = dA$, in the sense that they define a Kähler structure on $\Sigma$, lead to the same quantum system.

In Chern-Simons theory the only regime which can be characterized by some extra symmetries of the ultraviolet regularization is the axial regime when $\lambda = 0$. Only in such a case, the odd character of the bare action under parity transformations is preserved. The effect of this remnant symmetry is to preserve the gauge fields unrenormalized, and consequently the supersymmetry associated to Chern-Simons in Landau gauge is recovered in the limit $\Lambda \to \infty$. Such a regime is defined by a special universality class of regularizations, which give rise to the standard Chern-Simons theory defined by canonical quantization after a finite renormalization of the coupling constant. The other regimes introduce regularized actions with an hybrid behavior under parity symmetry which is reflected in the renormalization gauge fields and the breaking of the special supersymmetry associated to Landau gauge. Nevertheless, once such a renormalization of the gauge field wave function is taken into account, the theory defined by the scalar regime agrees with the standard theory defined by canonical quantization.

However, the borderline regime provides a non-standard quantization prescription for Chern-Simons theory in the covariant formalism which is presumably related to non-rational conformal field theories. The analysis of such a connection in the canonical formalism is a challenging open problem.

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Appendix A.

The propagator of gluons is obtained from the restriction of the inverse of the operator

$$\Gamma_2 = -\frac{k}{4\pi} \left\{ -\lambda \frac{\Delta}{\Lambda^2} \left( 1 + \frac{\Delta}{\Lambda^2} \right)^m + i * d \left[ 1 + \lambda' \frac{\Delta}{\Lambda^2} \left( 1 + \frac{\Delta}{\Lambda^2} \right)^{2n} \right] \right\}$$

to the subspace of transverse fields $A \{d^*A = 0\}$:

$$\Pi = (1 - d(d^*d)^{-1}d^*)\Gamma_2^{-1}(1 - d(d^*d)^{-1}d^*)$$

It is given by

$$\Pi^{ab}_{\mu\nu}(p) = -\frac{4\pi}{k} \frac{\delta^{ab}}{\Lambda} \left[ -\lambda(1 + p^2/\Lambda^2)^m(p^2\delta_{\nu\mu} - p_{\nu}p_{\mu})/\Lambda + \epsilon_{\mu\nu\rho}\epsilon_{\rho\beta}(1 + \lambda'(1 + p^2/\Lambda^2)^{2n}p^2/\Lambda^2) \right] / \left( p^2[\lambda^2(1 + p^2/\Lambda^2)^{2m}p^2/\Lambda^2 + (1 + \lambda'(1 + p^2/\Lambda^2)^{2n}p^2/\Lambda^2)^2] \right)$$

The simplest gluon selfinteraction is given by the three points vertex function, which in generic regimes with integer exponents $n$ and $m$ reads

$$\Gamma^{abc}_{\mu\nu\rho}(p, q, r) = \frac{ik}{24\pi} f^{abc} \epsilon_{\mu\nu\rho} - \frac{ik\lambda}{48\pi\Lambda} f^{abc} \left\{ \epsilon_{\alpha\beta\mu}\epsilon_{\gamma\nu\rho}p_{\beta} \left( 1 + \frac{p^2}{\Lambda^2} \right)^m \right. - \frac{1}{\Lambda^2} \epsilon_{\sigma\beta\mu}\epsilon_{\alpha\gamma\rho}p_{\beta}r_{\gamma}(2r_{\nu}\delta_{\sigma\alpha} + q_{\sigma}\delta_{\alpha\nu} - q_{\alpha}\delta_{\sigma\nu}) \sum_{j=0}^{m-1} \left( 1 + \frac{p^2}{\Lambda^2} \right)^j \left( 1 + \frac{r^2}{\Lambda^2} \right)^{m-j-1} \}$$

$$- \frac{1}{\Lambda^2} \epsilon_{\alpha\beta\mu}\epsilon_{\gamma\nu\rho}\epsilon_{\gamma\sigma\rho}p_{\beta}r_{\sigma} \left( 1 + \frac{p^2}{\Lambda^2} \right)^n \left( 1 + \frac{r^2}{\Lambda^2} \right)^n$$

$$+ \frac{1}{\Lambda^2} r_{\sigma}^2 \epsilon^{\mu\alpha\rho} \left( q_{\gamma}\epsilon_{\gamma\alpha\mu}\delta_{\sigma\nu} - q_{\sigma}\epsilon_{\nu\alpha\mu} \right) (r_{\sigma}r_{\rho} - r^2\delta_{\sigma\rho})$$

$$+ 2r_{\nu}r_{\rho}^2 \epsilon^{\mu\alpha\rho} \sum_{j=0}^{n-1} \left( 1 + \frac{p^2}{\Lambda^2} \right)^j \left( 1 + \frac{r^2}{\Lambda^2} \right)^{2n-j-1}$$

$$- \frac{1}{\Lambda^2} \left[ 2\epsilon^{\mu\alpha\rho}r_{\alpha}r_{\nu}p^2 + r_{\alpha}\epsilon_{\gamma\alpha\rho}(q_{\gamma}\delta_{\sigma\nu} - q_{\sigma}\delta_{\gamma\nu}) (p_{\sigma}p_{\mu} - p^2\delta_{\mu\nu}) \right]$$

$$\sum_{j=0}^{n-1} \left( 1 + \frac{p^2}{\Lambda^2} \right)^{2n-j-1} \left( 1 + \frac{r^2}{\Lambda^2} \right)^j \} + \text{perms}[(p, \nu, a), (q, \mu, b), (r, \rho, c)]$$

The four point interaction has a similar expression although considerably much longer. In order to keep the discussion in reasonable terms we omit the explicit expression which on the other hand can be derived by standard methods.

There is a subtle point in the analysis of Feynman rules for the intermediate case. There, since $m = 2n + 1/2$ the regularization involve pseudodifferential operators in the
regulators of gauge fields. If we assume \( n \) to be an integer then the pseudodifferential operators appear in the Yang-Mills like part of the regularized action

\[
(F(A), (I + \Delta A/\Lambda^2)^m F(A))
\]

We assume the standard definition of pseudodifferential operators and we proceed by expanding the corresponding perturbative expression in a momentum basis. If we denote by \( F(A) \) the pseudodifferential operator \((I + \Delta A/\Lambda^2)^m\) the corresponding Feynman rules can be derived from the following perturbative expansion:

\[
F(A)^{ac}_{\mu\nu}(p, -p - q) = \delta^{ac}\delta_{\mu\nu}\delta(q)F_0(p) + \frac{1}{\Lambda^2} \left\{ i f^{abc} A^b_\nu(q) [(2p + q)_\nu \delta_{\mu\rho} - q_\mu \delta_{\nu\rho} + q_\rho \delta_{\nu\mu}] 
+ \int \frac{d^3q'}{(2\pi)^3} A^b_\nu(q - q') A^{b'}_\nu(q')(f^{bb'c} f^{acc'} \delta_{\mu\nu'} \delta_{\rho\nu} - f^{abc} f^{eb'c} \delta_{\mu\rho} \delta_{\nu'} \delta_{\nu\sigma} - (q_\mu - q'_\mu) \delta_{\nu\sigma}) 
+ (q_\sigma - q'_\sigma) \delta_{\nu\mu} [((2p + q - q')_\nu \delta_{\sigma\rho} - q'_{\sigma} \delta_{\nu\rho} + q_{\rho} \delta_{\nu\sigma}] \frac{1}{p^2 - (p + q)^2} 
\frac{F_0(p + q - q') - F_0(p)}{(p + q - q')^2 - p^2} - \frac{F_0(p + q - q') - F_0(p + q)}{(p + q - q')^2 - (p + q)^2} \right\} + O(A^3),
\]

where \( F_0(p) = (1 + p^2/\Lambda)^m \).

Feynman rules for the interaction of nuclear and metric ghosts can be found in [13] (see also [36]).

**Appendix B.**

Using the Feynman rules of Appendix A it is possible to calculate the radiative corrections to the two-point function generated by gluonic loops (diagrams (1) and (2) of Fig. 1). The scalar parts of those contributions are

\[
^{(1)} \Gamma^{ab}_{\mu\nu}(q) = \frac{2h^\nu}{3\pi^2} (m + 1)^2 \Omega \delta^{ab} \delta_{\mu\nu} 
+ \frac{h}{3\pi^2} \Lambda I(n, m) \delta^{ab} \delta_{\mu\nu} 
- \frac{h^\nu}{8\pi^2} (m + 1)^2 \delta^{ab} |q| \left( \delta_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) 
+ O(\Omega^{-1}, \Lambda^{-1})
\]
and

\begin{equation}
\Gamma^{ab}_{\mu\nu}(q) = -\frac{h^\gamma}{3\pi^2}(2m^2 + 5m + 2)\,\Omega\,\delta^{ab}\delta_{\mu\nu}
\end{equation}

\begin{equation}
-\frac{h^\gamma}{3\pi^2}\Lambda\,I(n, m)\,\delta^{ab}\delta_{\mu\nu}
+ O(\Omega^{-1}, \Lambda^{-1}),
\end{equation}

if \( m \geq 2n + 1/2 \)

\begin{equation}
\Gamma^{ab}_{\mu\nu}(q) = \frac{2h^\gamma}{3\pi^2}(2n + 3/2)^2\,\Omega\,\delta^{ab}\delta_{\mu\nu}
\end{equation}

\begin{equation}
+ \frac{h^\gamma}{3\pi^2}\Lambda\,I(n, m)\,\delta^{ab}\delta_{\mu\nu}
- \frac{h^\gamma}{32\pi^2}(4n + 3)^2\,\delta^{ab}\delta_{\mu\nu}\left(\,\Omega\,\delta^{ab}\delta_{\mu\nu} + \frac{q\mu q\nu}{q^2}\right)
+ O(\Omega^{-1}, \Lambda^{-1}),
\end{equation}

and

\begin{equation}
\Gamma^{ab}_{\mu\nu}(q) = -\frac{h^\gamma}{3\pi^2}(8n^2 + 14n + 5)\,\Omega\,\delta^{ab}\delta_{\mu\nu}
\end{equation}

\begin{equation}
- \frac{h^\gamma}{3\pi^2}\Lambda\,I(n, m)\,\delta^{ab}\delta_{\mu\nu}
+ O(\Omega^{-1}, \Lambda^{-1}),
\end{equation}

when \( m \leq 2n + 1/2 \), with

\begin{equation}
I(n, m) = \int_0^\infty \frac{\tau(p)}{\rho(p)}dp
\end{equation}

\begin{equation}
\rho(p) = \lambda^2 p^2 (1 + p^2)^{2m} + [1 + \lambda p^2 (1 + p^2)^{2n}]^2,
\end{equation}

and

\begin{align}
\tau(p) = & 2\lambda'[1 + \lambda' p^2 (1 + p^2)^{2n}][5 + 2 p^2 (5 + 9n)] p^2 (1 + p^2)^{2n-2} - \\
& - \lambda^2 [m p^2 (1 + p^2)^{2m-2}+[2m + 5] + (4m + 3)p^2] + (2m^2 + 5m + 2)]
\end{align}

The pseudoscalar contributions to \( \Gamma^{ab}_{\mu\nu} \) generated by gluonic loops (diagrams (1)–(2) of fig. 1) are given by

\begin{equation}
\star \Gamma^{ab}_{\mu\nu}(q) = \frac{h^\gamma}{3\pi^2} \int_0^\infty \frac{A(p)}{\rho(p)^2}dp \,\delta^{ab} \epsilon_{\mu\sigma\nu} q^\sigma,
\end{equation}

and

\begin{equation}
\star \Gamma^{ab}_{\mu\nu}(q) = \frac{h^\gamma}{3\pi^2} \int_0^\infty \frac{A(p)}{\rho(p)^2}dp \,\delta^{ab} \epsilon_{\mu\sigma\nu} q^\sigma,
\end{equation}
respectively, where

\[ A(p) = B(p) \rho(p) + C(p) \frac{d \rho(p)}{dp} + D(p) \frac{d \phi(p)}{dp}, \quad (B.8) \]

\[ \tilde{A}(p) = 2 \lambda R(p) + 3 S(p) R(p) - 5 S(p) - m \lambda [7 p^2 + (5 + 2m) p^4] (1 + p^2)^{-m-2} R(p) \]

\[ + S(p) 2n \lambda' p^4 [9 + (7 + 4n) p^2] (1 + p^2)^{-2n-2}, \quad (B.9) \]

and

\[ B(p) = - \lambda R(p) - S(p) R(p) + 3 S(p) + \frac{d (S(p)p)}{dp} + 3 p \lambda \frac{d R(p)}{dp} \]

\[ + (6m - 8n) \lambda' p^4 (1 + p^2)^{-2n-1} S(p) \]

\[ C(p) = - 2 p \lambda R(p) + \frac{1}{2} p \frac{d (S(p)p)}{dp} + p S(p) \left[ \frac{3}{2} + m \lambda' p^4 (1 + p^2)^{-2n-1} - \frac{1}{2} p \frac{d R(p)}{dp} \right] \]

\[ D(p) = \lambda p^2 S(p) R^{-2}(p) [2 + (1 + p^2)^m], \]

\[ S(p) = \lambda (1 + p^2)^m \quad R(p) = 1 + \lambda' p^2 (1 + p^2)^{2n} \quad \phi(p) = p \frac{S(p)}{R(p)}. \]

In the first two terms of (B.8) into (B.6) one \( \rho(p) \)-factor can be removed from the denominator by integration by parts. Adding the result to the contribution (B.7) of the diagram (2) we obtain

\[ \left[ \int_0^\infty \frac{\theta(p)}{\rho(p)} dp + \chi_1 \right] \delta^{ab} \epsilon_{\mu \sigma \nu} q^\sigma \frac{h^\nu}{3\pi^2}, \quad (B.10) \]

with

\[ \theta(p) = - \lambda R(p) + \lambda p \frac{d R(p)}{dp} + 2 S(p) + R(p) \frac{d (S(p)p)}{dp} - p S(p) \frac{d R(p)}{dp} \]

and

\[ \chi_1 = \left[ \lim_{p \to \infty} - \lim_{p \to 0^+} \right] \frac{1}{\rho(p)} \left[ \frac{1}{2} p^2 S(p) \frac{d (R(p))}{dp} + \frac{1}{2} p S(p) R(p) - 2 p S(p) \right. \]

\[- \frac{1}{2} p R(p) \frac{d (S(p)p)}{dp} + 2 \lambda p R(p) \left. \right] \]

The remaining term of (B.8) after integration by parts yields

\[ \frac{h^\nu}{6\pi^2} \left[ \int_0^\infty \frac{\xi(p)}{\rho(p)} dp - \chi_2 \right] \delta^{ab} \epsilon_{\mu \sigma \nu} q^\sigma, \quad (B.11) \]

where

\[ \xi(p) = 2 \lambda R(p) - 2 \lambda p \frac{d R(p)}{dp} + R(p) \frac{d (S(p)p)}{dp} - p S(p) \frac{d R(p)}{dp} \]

and

\[ \chi_2 = \left[ \lim_{p \to \infty} - \lim_{p \to 0^+} \right] \frac{\lambda p [2 + (1 + p^2)^m][1 + \lambda' p^2 (1 + p^2)^{2n}]}{\rho(p)}. \quad (B.12) \]
The global contribution of gluonic loops to the two point function is obtained by summing up the contributions (B.10) and (B.11),

\[ \Pi_{\mu\nu}^{ab}(q) = \frac{h^\nu}{3\pi^2} \int_0^\infty dp \frac{\Sigma(p)}{\rho(p)} \delta^{ab} \epsilon_{\mu\sigma\nu} q^\sigma \]  

(B.13)

where

\[ \Sigma(p) = 2S(p) + 3 \left[ R(p) \frac{d(S(p)p)}{dp} - pS(p) \frac{dR(p)}{dp} \right]. \]  

(B.14)

Notice that the calculation in the borderline case which involves pseudodifferential operators reduces to the analytic continuation in the regulating parameters \( n \) and \( m \) of the other two cases.

Finally, the radiative corrections to the vacuum polarization tensor generated by loops of metric and nuclear ghosts loops are purely scalar because their interactions do not involve pseudoscalar couplings. They read

\[ \Pi'_{\mu\nu}^{ab}(q) = + \frac{h^\nu}{3\pi^2} (m_2 - 1) \Omega \delta^{ab} \delta_{\mu\nu} \]

\[ + \frac{h^\nu}{8\pi^2} [(m_0)^2 + (m_2 - m_1)^2 + (m_1 - m_0)^2] \delta^{ab} |q| \left( \delta_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) \]  

(B.15)

\[ + O(\Omega^{-1}, \Lambda^{-1}). \]
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