Perspectives on the formation of peakons in the stochastic Camassa–Holm equation

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A famous feature of the Camassa–Holm equation is its admission of peaked soliton solutions known as peakons. We investigate this equation under the influence of stochastic transport. Noting that peakons are weak solutions of the equation, we present a finite-element discretization for it, which we use to explore the formation of peakons. Our simulations using this discretization reveal that peakons can still form in the presence of stochastic perturbations. Peakons can emerge both through wave breaking, as the slope turns vertical, and without wave breaking as the inflection points of the velocity profile rise to reach the summit.

1. Introduction

(a) Overview

We are dealing with stochastic perturbations of partial differential equations (PDEs) for fluid dynamics, whose deterministic solutions develop singularities by producing a vertical derivative, which is commonly known as wave breaking. A famous example is the Burgers equation, which has been studied with a stochastic component several times, for instance by [1,2]. In the latter case, the stochastic perturbations take the form of multiplicative noise, which will be our focus here. A striking feature is that the addition of stochastic perturbations has been found to affect the development of singularities; examples include [3,4].

Here, we consider the effect of stochastic perturbations on a well-known equation that can also exhibit wave breaking and singularity formation: the Camassa–Holm (CH) equation of [5], which has recently been...
studied in the stochastic case by [6,7]. This stochastic equation can be expressed as

\[ dm(x, t) + (\partial_x m + m \partial_x)(u \, dt + \mathcal{E}(x) \circ dW) = 0, \quad \text{with } m = u - \alpha^2 u_{xx}, \]  

(1.1)

for the evolution in time \( t \in \mathbb{R} \) of the fluid momentum density \( m(x, t) \) and the velocity \( u(x, t) \) on the real line \( x \in \mathbb{R} \). The constant \( \alpha \) has dimensions of length. In equation (1.1), the time differential \( d \) is short notation for a stochastic integral; the spatial partial derivative \( \partial_x \) acts rightwards on all products; the symbol \( \circ \) denotes a Stratonovich stochastic process; \( dW \) is a Brownian motion with spatially modulated amplitude \( \mathcal{E}(x) \); and one interprets the effect of the stochasticity as a noisy perturbation of the transport velocity. See [4] for more discussion of such stochastic transport. In the deterministic case, with \( \mathcal{E}(x) = 0 \), it is well known that (1.1) admits weak singular solutions known as peakons, which refers to their shape with a vertical derivative at their peak.

The paper begins by presenting a finite-element discretization to a form of (1.1), inspired by the acknowledgement that peakons are weak solutions. We then use this discretization to study the formation of peakons in deterministic and stochastic cases. We look at the formation of peakons both with and without wave breaking. In this second case, rather than the slope turning vertical the inflection points of the profile rise until they reach the summit. With the addition of stochastic noise, our simulations suggest that peakon formation (with and without wave breaking) persists.

(b) A stochastic variational principle

Stochastic perturbations are of current interest in the modelling of geophysical fluids, where they can express the uncertainty in the effects of the unresolved processes upon the resolved flow. In this context, a new class of stochastic fluid equations was introduced by [8], which crucially preserves many of the circulation properties of the respective deterministic equations. For instance in the stochastic quasi-geostrophic equations studied by [9,10], the potential vorticity is still preserved by stochastic material transport. These equations are derived from a stochastically constrained variational principle \( \delta S = 0 \), where the action integral \( S \) is given by

\[ S = (u, p, q) = \int \left[ \ell(u, q) \, dt + \langle p, dq + \mathcal{L}_{dx} q \rangle \right], \]

(1.2)

where \( u \) is the velocity advecting quantity \( q \) and \( \ell(u, q) \) is the Lagrangian of the deterministic fluid. The Lagrange multiplier \( p \) enforces the stochastic transport of \( q \) by the Lie derivative \( \mathcal{L}_{dx} \) along a vector field \( dx_t \), which is given by

\[ dx_t = u(x, t) \, dt - \sum_j \mathcal{E}^j(x) \circ dW^j(t), \]

(1.3)

with the sum over multiple Wiener processes, which are modulated in space by a series of functions \( \mathcal{E}^j(x) \), indexed by superscript \( j \). The angled brackets denote the spatial integral over the domain \( \Omega \) of the pairing of \( p \) and \( q \),

\[ \langle p, q \rangle = \int_\Omega pq \, dx. \]

(1.4)

A thorough description of this formulation can be found in [8].

(c) The deterministic and stochastic Camassa–Holm equations

The deterministic CH equation of [5] can be expressed as

\[ u_t = -\partial_x \left\{ \frac{1}{2} u^2 + K \ast \left[ u^2 + \frac{\alpha^2}{2} (u_x)^2 \right] \right\}, \]

(1.5)

which is sometimes known as the advective or hydrodynamic form of the equation. In contrast to (1.1), this does not contain double spatial derivatives of \( u \). For more discussion of this form, see
for example [7,11]. The operator $K$ is the Green’s function for the Helmholtz operator $(1 - \alpha^2 \partial_{xx})$ that relates the momentum density $m$ and the velocity $u$, so that

$$K \ast m = K \ast \left( u - \alpha^2 u_{xx} \right) = u.$$  \hspace{1cm} (1.6)

The CH equation (1.5) was derived at one order beyond the celebrated Korteweg-de Vries equation in the asymptotic expansion of the Euler fluid equations for nonlinear shallow water waves on a free surface propagating under the restoring force of gravity [12–14]. It is integrable [5,15] and has a bi-Hamiltonian structure [16,17].

The stochastic Camassa–Holm (SCH) equation (1.1) was derived by [6] from the stochastic variational principle of [8] discussed in §1b. As seen in [7], (1.1) also possesses a hydrodynamic form

$$d u = -\partial_x \left\{ \frac{1}{2} u^2 + K \ast \left[ u^2 + \frac{\alpha^2}{2} (u_x)^2 \right] \right\} dt - \sum_j \left\{ u_x \xi^j + K \ast \left[ 2u \xi^j_x + \alpha^2 u_x \xi^j_{xx} \right] \right\} \circ dW^j.$$ \hspace{1cm} (1.7)

Unlike (1.5), the right-hand side of (1.7) cannot be written purely as a gradient, but it also contains no double derivatives of $u$. Other SCH equations have also been studied, including that of [18], which used additive noise, and that of [19], which used a type of multiplicative noise that differs from that derived from the variational principle of [8]. Equation (1.7) will be the focus for the rest of this paper.

The deterministic CH equation (1.5) admits solutions with singularities known as peakons, which possess a sharp peak at the apex of their velocity profile. In fact, due to the singularity in the spatial derivative of the peakon solutions, they should be interpreted as weak solutions to the CH equation (1.5), as discussed by [11,20,21]. The shape of the peakon profile turns out to be the Green’s function $K$. This can be explained by recalling that the peakon solutions for the momentum density in the deterministic form of (1.1) may be expressed by the singular momentum map [22],

$$m(x,t) = \sum_k p_k(t) \delta \left( \frac{x - q_k(t)}{\alpha} \right) \quad \text{so} \quad u(x,t) = \sum_k p_k(t) K \ast \delta \left( \frac{x - q_k(t)}{\alpha} \right),$$ \hspace{1cm} (1.8)

in which $K(x - q_k(t)) = \frac{1}{2} \exp(-|x - q_k(t)|/\alpha)$ is the Green’s function on the real line with homogeneous boundary conditions. Remarkably, the purely discrete spectrum of the isospectral eigenvalue equation, which demonstrates that the CH equation is a completely integrable Hamiltonian soliton system, also implies that only the singular solutions in (1.8) will persist.

The continuation of peakon solutions beyond their emergence has been investigated by studies including [23,24]. For such a sum of peakons, the canonical coordinates giving the peakon momentum $p_k$ and position $q_k$ obey the following coupled ordinary differential equations:

$$\frac{dp_k}{dt} = -p_k \frac{\partial u(q_k)}{\partial q_k} \quad \text{and} \quad \frac{dq_k}{dt} = u(q_k).$$ \hspace{1cm} (1.9)

The stochastic equation (1.7) also admits weak peakon solutions via (1.8), as mentioned by [6]. Holm & Tyranowski [6] derived the stochastic (ordinary) differential equations obeyed by these peakons from the stochastic Hamiltonian, giving

$$dp_k = -p_k(t) \frac{\partial x(q_k(t))}{\partial q_k} \quad \text{and} \quad dq_k = dx(q_k(t)),$$ \hspace{1cm} (1.10)

where

$$dx(q_k(t)) = u(q_k(t)) \, dt + \sum_j \xi^j(q_k(t)) \circ dW^j.$$ \hspace{1cm} (1.11)
Evaluating (1.10) gives
\[ dp_k = \frac{p_k}{2\alpha} \sum_l p_l \text{sign}(q_k - q_l) e^{-|q_k - q_l|/\alpha} dt - p_k \sum_j \mathcal{Z}_k^j(q_k) \circ dW^j \] (1.12a)
and
\[ dq_k = \frac{1}{2} \sum_l p_l e^{-|q_k - q_l|/\alpha} dt + \sum_j \mathcal{Z}_k^j(q_k) \circ dW^j, \] (1.12b)
where we take sign(0) = 0. For a discussion of the isospectral problem and integrability of the stochastic equation (1.7), see [7]. Holm & Tyranowski [6] discretized (1.12) using a variational integrator and used it to investigate the interaction of peakons. This was followed by [25,26], which further explored stochastic Hamiltonian systems. This paper continues in the same spirit as [6] of using numerical methods to understand the equation, as we present a finite-element discretization for (1.7) and using it to investigate the formation of peakons in the SCH equation.

A final property to mention concerning peakons is the steepening lemma of [5], which is repeated for instance in [7]. This demonstrates that peakons emerge from a class of smooth, confined initial velocity profiles. Such profiles have an inflection point with a negative slope s at x = \tilde{x} to the right of their maxima. The steepening lemma of [5], as well as the argument of [20], shows that if s is sufficiently negative then the slope will become vertical within finite time, and thus a peakon emerges through wave breaking. Considering a velocity profile \( u \) with momentum \( m = u - \alpha^2 u_{xx} \), [27] argues that wave breaking will occur if there is some negative m at points \( x_- \) that lie to the right of points \( x_+ \) at which m is positive. In a periodic domain this must be the case, unless \( m > 0 \) everywhere.

Crisan & Holm [7] then investigated the analogous steepening lemma for the SCH equation, again looking at only a class of initially smooth, confined solutions whose negative slope s at \( \tilde{x} \) was sufficiently negative. Focusing on the case of a single stochastic basis function, \( \mathcal{Z}(x) = \xi \) (a constant), the authors found that the expectation of the slope at the inflection point also blows up in finite time. Despite this, for individual realizations of the noise it was unclear whether wave breaking would always occur, as [7] found that the slope becomes vertical with a positive probability, which may not necessarily be unity. This was interpreted as the probability of peakon formation.

As we discuss in §5, peakons can still form from smooth, confined initial conditions, even if s is not sufficiently negative for wave breaking to occur. Instead, the inflection point rises up the velocity profile, with the peakon emerging as the inflection point reaches the profile’s maximum. This emphasizes that the stochastic steepening lemma of [7] describes the probability of wave breaking (for initial profiles with sufficiently negative s) and not the probability of peakon formation.

Some outstanding questions therefore remain. While the introduction of stochastic transport into the SCH equation cannot be expected to preserve the complete integrability of the unperturbed CH equation, one may ask whether peakons still form from smooth, confined initial conditions. We look first at the case of \( \mathcal{Z}(x) = \xi \) (a constant) considered by [6,7]. In the situation of [7], with sufficiently negative s, is the probability of wave breaking actually unity or merely non-zero? What about the probability of peakon formation? When the initial condition has a shallower slope, do peakons still form under stochastic perturbation, and can wave breaking now occur even when it would not in the deterministic case? The purpose of this paper is investigate these questions numerically via a finite-element discretization for (1.7).

Here is the paper’s structure. In §2, we verify that peakons do indeed satisfy the SCH equation by writing it in hydrodynamic form. In §3, we present a finite-element discretization for the SCH equation, showing that it numerically converges to the stochastic (ordinary) differential equations for peakons in §4. In §5, we numerically investigate the formation of peakons using our discretization.
2. Peakon solutions to the SCH equation

In this section, we verify that peakons are also solutions to the SCH equation. These peakons take the form

$$u = \sum_k p_k(t) K \ast \delta \left( \frac{x - q_k(t)}{\alpha} \right), \quad (2.1)$$

in which $q_k(t)$ represents the position of the peakons and $p_k(t)$ represents their momentum, and together they form pairs of canonical coordinates satisfying a Hamiltonian system. Although this was described in [7], here we emphasize that peakons are weak solutions to (1.7), since the derivative $u_x$ is not defined at the peak. In other words, peakons are solutions to

$$\int_{\Omega} \phi \, du \, dx = \int_{\Omega} \phi_x \left\{ \frac{1}{2} u^2 + K \ast \left[ u^2 + \frac{\alpha^2}{2} (u_x)^2 \right] \right\} \, dt \, dx$$

$$- \sum_j \int_{\Omega} \phi \left\{ u_x \Sigma^j + K \ast \left[ 2u \Sigma^j_x + \alpha^2 u_x \Sigma^j_{xx} \right] \right\} \circ dW^j \, dx, \quad (2.2)$$

for all $\phi \in H^1(\Omega)$. In later sections, we numerically solve the SCH equation in the periodic interval $x \in [0, L]$ so we take this here for our domain $\Omega$. In this case, the operator $K(x)$ is given by

$$K(x) = \frac{1}{2(1 - e^{-L/\alpha})} \left[ e^{-\text{mod}(x,L)/\alpha} + e^{-L/\alpha} e^{\text{mod}(x,L)/\alpha} \right]. \quad (2.3)$$

On the periodic interval, the velocity for series of peakons with momenta $p_k$ centred at positions $q_k$ is given by

$$u = \sum_k u^k = \frac{1}{2} \sum_k \frac{p_k(t)}{1 - e^{-L/\alpha}} \left[ e^{-\text{mod}(x-q_k(t),L)/\alpha} + e^{-L/\alpha} e^{\text{mod}(x-q_k(t),L)/\alpha} \right], \quad (2.4)$$

where we define $u^k$ to be the contribution from a single peakon. Other works using periodized peakons include [28–30]. This reduces to the usual set of peakons on $\mathbb{R}$ when $L \to \infty$

$$u = \frac{1}{2} \sum_k p_k(t) e^{-|x-q_k(t)|/\alpha}. \quad (2.5)$$

In the periodic case, the stochastic ODEs for the evolution of the momenta $p_k$ and positions $q_k$ are given by

$$dp_k = \frac{p_k}{2\alpha(1 - e^{-L/\alpha})} \sum_l p_l \text{sign}(q_k - q_l) \left[ e^{-\text{mod}(q_k-q_l,L)/\alpha} - e^{-L/\alpha} e^{\text{mod}(q_k-q_l,L)/\alpha} \right] \, dt$$

$$- p_k \sum_j \Sigma^j_k(q_k) \circ dW^j \quad (2.6a)$$

and

$$dq_k = \frac{1}{2(1 - e^{-L/\alpha})} \sum_l p_l \left[ e^{-\text{mod}(q_k-q_l,L)/\alpha} + e^{-L/\alpha} e^{\text{mod}(q_k-q_l,L)/\alpha} \right] \, dt + \sum_j \Sigma^j_k(q_k) \circ dW^j, \quad (2.6b)$$

which is the periodic equivalent of (1.12). Taking the differential of (2.4) and substituting in equation (1.10) yields

$$du = \sum_k \left[ \frac{\partial u^k}{\partial p_k} \, dp_k + \frac{\partial u^k}{\partial q_k} \, dq_k \right], \quad (2.7a)$$

$$= - \sum_k \left[ u^k \frac{\partial u(q_k)}{\partial q_k} + u^k_x u(q_k) \right] \, dt + \sum_j \left( u^k \Sigma^j_k(q_k) + u^k_x \Sigma^j_k(q_k) \right) \circ dW^j. \quad (2.7b)$$
In order to verify that (2.4) satisfies (2.2), we will substitute it and (2.7b) into (2.2) and show that the deterministic and stochastic components of the equation will both vanish separately. First, substituting just (2.7b) into the left-hand side of (2.2) and grouping terms, with the deterministic parts on the left-hand side and stochastic terms on the right-hand side, gives

\[ -\int_\Omega \phi \sum_k \left( u^k \frac{\partial u(q_k)}{\partial q_k} + u_x^k u(q_k) \right) \, dt \, dx \]

\[ = \int_\Omega \phi \left\{ \sum_k \left[ \left( u^k \mathcal{S}_x^j(q_k) + u_x^k \mathcal{S}_x^j(q_k) \right) - u_x^k \mathcal{S}_x^j - K \ast \left( 2u \mathcal{S}_x^j + \alpha^2 u_x \mathcal{S}_x^j \right) \right] \right\} \circ dW^j \, dx. \quad (2.8) \]

Now we consider the multi-peakon solution for \( u \) from (2.4). Although we do not give explicit proof here (for more details, see [21, 28] for the periodic case), this peakon is a weak solution to (2.8) cancel almost everywhere, so that the integral on the right-hand side vanishes.

Since \( u \) is a linear sum of \( u^k \) for each term on the right-hand side of (2.8), we only need to consider a single peakon solution \( u^k \) and a single \( \mathcal{S}^j \), taking

\[ \mathcal{S}^j = \xi \left[ C \cos \left( \frac{2\pi j x}{L} \right) + D \sin \left( \frac{2\pi j x}{L} \right) \right], \quad (2.9) \]

as in a periodic domain, any choice of \( \mathcal{S} \) will be a linear combination of such functions. With this choice, careful computation of the convolution term in the stochastic part of (2.8) reveals that for \( x \neq q_k \)

\[ K \ast \left( 2u^k \mathcal{S}^j + \alpha^2 u_x^k \mathcal{S}^j_{xx} \right) \]

\[ = \frac{1}{2\alpha(1 - e^{-L/\alpha})} \left[ e^{-\mod(x-y,L)/\alpha} + e^{-L/\alpha} e^{\mod(x-y,L)/\alpha} \right] \]

\[ \times \left[ 2u^k(y) \mathcal{S}^j_y(y) + \alpha^2 u_x^k(y) \mathcal{S}^j_{yy}(y) \right] \, dy \]

\[ = \frac{\xi p_k}{2(1 - e^{-L/\alpha})} \left\{ \frac{2\pi j}{L} \left( e^{-\mod(x-q_k,L)/\alpha} + e^{-L/\alpha} e^{\mod(x-q_k,L)/\alpha} \right) \right\} \]

\[ \times \left[ -C \sin \left( \frac{2\pi j x}{L} \right) + D \cos \left( \frac{2\pi j x}{L} \right) \right] \]

\[ - \frac{\pi \alpha}{\xi} \left( e^{-\mod(x-q_k,L)/\alpha} - e^{-L/\alpha} e^{\mod(x-q_k,L)/\alpha} \right) \]

\[ \times \left[ -C \cos \left( \frac{2\pi j x}{L} \right) + D \sin \left( \frac{2\pi j x}{L} \right) \right] \} \]

\[ = u^k \mathcal{S}_x^j(q_k) + u_x^k \mathcal{S}_x^j(q_k) - u_x^k \mathcal{S}_x^j(x). \quad (2.10) \]

Consequently, the term inside the curly brackets of the stochastic of (2.8) vanishes almost everywhere, and the integral on the right-hand side of (2.8) is zero. Since both the deterministic and stochastic parts of (2.8) vanish when the velocity profile is given by (2.4), this periodic peakon is indeed a weak solution of the SCH equation (2.2), with the momenta \( p_k \) and \( q_k \) obeying (2.6).

Alternatively, taking a constant \( \mathcal{S}^j = \xi \) in (2.8) then the convolution term vanishes as \( \mathcal{S}_x^j = 0 \) and \( \mathcal{S}_x^j_{xx} = 0 \). Since for constant \( \xi \), \( \mathcal{S}^j(x) = \mathcal{S}^j(q_k) \), the remaining terms on the right-hand side of (2.6) cancel and again we see that (2.4) is a solution of (2.2).
3. A finite-element discretization for the SCH equation

As our aim is to numerically investigate the formation of peakons within the SCH equation, we look for weak solutions of the hydrodynamic form (1.7), acknowledging that peakons are weak solutions of this equation. Numerical solutions will then be valid before and after the moment of peakon formation, as we avoid direct evaluation of any spatial derivatives of second order or higher. While looking for weak solutions, it is natural to discretize the equation using a finite-element method.

The CH equation has been studied numerically using a range of methods, including finite difference, e.g. [31], finite volume, e.g. [32] and particle methods, e.g. [33]. Previous examples of finite-element methods to discretize the CH equation include discontinuous Galerkin methods, such as those of [29,34,35], and the continuous methods of [36,37]. In contrast to those examples, we discretize the CH equation in its hydrodynamic form.

We now present a new finite-element discretization of the SCH equation that we will use in the remainder of the paper.

To obtain the discretization, we introduce the variables $F$ and $G^j$ and write (1.7) as

\[
\frac{du}{dt} = -\partial_x \left\{ \frac{1}{2} u^2 + F \right\} dt - \sum_j \left\{ u_x \mathcal{E}^j + G^j \right\} \circ dW^j, \tag{3.1a}
\]

\[
F = K \ast \left[ u^2 + \frac{\alpha^2}{2} (u_x)^2 \right], \tag{3.1b}
\]

and

\[
G^j = K \ast \left[ 2 u \mathcal{E}_x^j + \alpha^2 u_x \mathcal{E}_x^j \right]. \tag{3.1c}
\]

Using that $K$ is the Green’s function for the Helmholtz operator, this becomes

\[
\frac{du}{dt} = -\partial_x \left\{ \frac{1}{2} u^2 + F \right\} dt - \sum_j \left\{ u_x \mathcal{E}^j + G^j \right\} \circ dW^j, \tag{3.2a}
\]

\[
F - \alpha^2 F_{xx} = u^2 + \frac{\alpha^2}{2} (u_x)^2, \tag{3.2b}
\]

and

\[
G^j - \alpha^2 G^j_{xx} = 2 u \mathcal{E}_x^j + \alpha^2 u_x \mathcal{E}_x^j. \tag{3.2c}
\]

We restrict ourselves to considering a periodic one-dimensional domain $\Omega$ of length $L_d$. Now we introduce our finite-element space $V$, which is the space of continuous piecewise-linear functions over some partition of $\Omega$ (often this is known as the continuous Galerkin space $CG_1$). We multiply each of the equations in (3.2) by the test functions $\psi$, $\phi$ and $\zeta$, all in $V$. After integration by parts, we obtain

\[
\int_\Omega \psi \frac{du}{dx} dx + \int_\Omega \psi u_x \left( \frac{du}{dt} + \sum_j \mathcal{E}^j \circ dW^j \right) dx
- \int_\Omega \psi_x F \, dt + \sum_j \int_\Omega \psi G^j \circ dW^j \, dx = 0, \tag{3.3a}
\]

\[
\int_\Omega \phi F \, dx + \int_\Omega \alpha^2 \phi_x F_x \, dx - \int_\Omega \phi \left[ u^2 + \frac{\alpha^2}{2} (u_x)^2 \right] \, dx = 0 \tag{3.3b}
\]

and

\[
\int_\Omega \zeta G^j \, dx + \int_\Omega \alpha^2 \zeta_x G^j_x \, dx - \int_\Omega \zeta \left[ 2 u \mathcal{E}_x^j + \alpha^2 u_x \mathcal{E}_x^j \right] \, dx = 0. \tag{3.3c}
\]

To discretize in time, we use the implicit midpoint rule, defining

\[
u^{(n+1)} = \frac{1}{2} (u^{(n+1)} + u^{(n)}), \tag{3.4}
\]
where \( u^{(n)} \) is the value of \( u \) at the \( n \)th time level. This is the natural choice for discretizing a Stratonovich process and is unconditionally stable for Stratonovich stochastic differential equations (SDEs) (see [38]). This gives us our mixed finite-element discretization, in which we search for the simultaneous fields

\[
\left( u^{(n+1)}, \Delta F^{(h)}, \Delta G^{(h)} \right) \in (V, V, V),
\]  

(3.5)

which satisfy, for all \((\psi, \phi, \zeta) \in (V, V, V)\), the equations

\[
\begin{align*}
\int_{\Omega} \psi \left( u^{(n+1)} - u^{(n)} \right) \, dx + \int_{\Omega} \psi u_{x}^{(h)} \Delta u^{(h)} \, dx - \int_{\Omega} \psi \Delta F^{(h)} \, dx + \int_{\Omega} \psi \Delta G^{(h)} \, dx &= 0, \\
\int_{\Omega} \phi \Delta F^{(h)} \, dx + \alpha^{2} \int_{\Omega} \phi \Delta G^{(h)} \, dx - \int_{\Omega} \phi u_{x}^{(h)} \Delta t \, dx - \frac{\alpha^{2}}{2} \int_{\Omega} \phi u_{xx}^{(h)} \Delta t \, dx &= 0
\end{align*}
\]  

(3.6a)

and

\[
\begin{align*}
\int_{\Omega} \zeta \Delta G^{(h)} \, dx + \alpha^{2} \int_{\Omega} \zeta \Delta F^{(h)} \, dx - \sum_{j} \int_{\Omega} \left( 2 \zeta u^{(h)} \varepsilon_{x}^{j} + \alpha^{2} \zeta u_{x}^{(h)} \varepsilon_{xx}^{j} \right) \Delta W^{j} \, dx &= 0,
\end{align*}
\]  

(3.6b)

where \( \Delta t \) is the time step and \( \Delta W^{j} \) is a normally distributed number with zero mean and variance \( \Delta t \). The stochastic velocity \( \Delta u^{(h)} \) is given by

\[
\Delta u^{(h)} = u^{(h)} \Delta t + \sum_{j} \varepsilon^{j} \Delta W^{j}.
\]  

(3.7)

We have also absorbed the time step \( \Delta t \) and the increments \( \Delta W \) into \( F \) and \( G^{j} \), writing these as \( \Delta F \) and \( \Delta G \) to emphasize that they contain increments in time.

If the system is deterministic, there are no \( \varepsilon^{j} \) functions that are non-zero and the equations reduce so that \( \Delta G = 0 \) and \( \Delta u^{(h)} = u^{(h)} \Delta t \).

To implement this scheme, we used the Firedrake software described in [39].

4. Convergence of the discretization to peakon solutions

We showed in §2 that the peakon (2.4) satisfying (2.6) is a weak solution to (2.2). In this section, we demonstrate that, when describing a single peakon, our discretization of the PDE (1.7) also converges to the SDEs (2.6) for the evolution of the canonical coordinates \( p \) and \( q \), in the limit that the grid spacing \( \Delta x \to 0 \). We also demonstrate that the discretization of the PDE has strong (pathwise) convergence to the peakon SDEs as the time step \( \Delta t \to 0 \).

To do this, we first re-cast equations (2.6) for a single peakon in Itô form

\[
dp = \frac{p}{2} \sum_{j} \left[ \left( \varepsilon_{x}^{j}(q) \right)^{2} - \varepsilon^{j}(q) \varepsilon_{xx}^{j}(q) \right] \, dt - p \sum_{j} \varepsilon_{x}^{j}(q) \, dW^{j}
\]  

(4.1a)

and

\[
\dq = \frac{1}{2} \left[ \frac{p(1 + e^{-L/\alpha})}{1 - e^{-L/\alpha}} + \sum_{j} \varepsilon^{j}(q) \varepsilon_{x}^{j}(q) \right] \, dt + \sum_{j} \varepsilon^{j}(q) \, dW_{j}.
\]  

(4.1b)

We solve both (3.6) and (4.1) in a periodic one-dimensional domain of length \( L = 40 \), and take \( \alpha = 1 \). For this section, for the stochastic basis functions we used only a single constant function

\[
\varepsilon^{0} = 1.
\]  

(4.2)

Our strategy is to establish a numerical solution of the SDEs (4.1) that is well resolved temporally, by solving it using Milstein’s method for discretizing in time; see for instance [40] for more details of this scheme. We used an initial condition of \( p = 1 \) and \( q = 20 \) and with \( \Delta t = 10^{-6} \). Using the resulting \( p \) and \( q \), we can construct the velocity field \( u \) corresponding to this peakon via (2.4). This can be compared with the velocity field \( u \) that has been computed by solving the stochastic PDE (1.7) using the discretization presented in §3.
Figure 1. Plots showing the convergence (a) as $\Delta x \to 0$ and (b) as $\Delta t \to 0$ of the discretization from §3 to the equations (4.1) that describe the evolution of peakon solutions. The field $u_{\text{PDE}}$ was computed by solving the discretization from an initial condition resembling a peakon. The equations (4.1) were solved using Milstein’s method. The resulting values of $p$ and $q$ were used to reconstruct the field $u_{\text{SDE}}$. We then computed the error between $u_{\text{PDE}}$ and $u_{\text{SDE}}$ using the $L^2$ norm. Crosses mark the data points, with a solid best fit line overlaid.

Our initial condition for the PDE, corresponding to $p = 1$ and $q = 20$, was

$$u = \begin{cases} 
\frac{1}{2(1 - e^{-L/\alpha})} \left[ e^{(x-L/2)/\alpha} + e^{-L/\alpha} e^{-(x-L/2)/\alpha} \right], & x < L/2, \\
\frac{1}{2(1 - e^{-L/\alpha})} \left[ e^{-L/2} e^{-(x-L/2)/\alpha} + e^{-L/\alpha} e^{-(x-L/2)/\alpha} \right], & x \geq L/2.
\end{cases}$$

(4.3)

Taking different choices of $\Delta x$ and $\Delta t$, we computed the state $u_{\text{PDE}}$ at $t = 0.1$ for various spatial resolutions. Then we measured the error in the $L^2$-norm between $u_{\text{PDE}}$ and the corresponding $u_{\text{SDE}}$ that was reconstructed from the $p$ and $q$ computed at $t = 0.1$ from the numerical solution to the SDE.

We did two separate investigations into the convergence of this error: first by reducing $\Delta x$ with $\Delta t = 10^{-6}$ kept constant, and second by reducing $\Delta t$ with $\Delta x = 0.002$ kept constant. In the second case, as $\Delta t$ was changed it was important to ensure that the stochastic realization still corresponded to that of the ‘true’ solution computed from the SDEs. Let \{ $\Delta \tilde{W}_1, \ldots, \Delta \tilde{W}_n$ \} be the set of randomly generated increments used for the ‘true’ solution, with $\Delta \tilde{t} = 10^{-6}$. Then, when taking a larger time step, such that $\Delta t = M\Delta \tilde{t}$, we took

$$\Delta W_i = \frac{1}{\sqrt{M}} \sum_{j=1}^{M} \Delta \tilde{W}_{(i-1)M+j},$$

(4.4)

where the square root is necessary to keep the variance of $\Delta W$ equal to $\Delta t$. A description of this approach can be found in [40].

The errors are plotted as functions of $\Delta x$ and $\Delta t$ in figure 1. This demonstrates that when our discretization presented in §3 describes a peakon, it does indeed converge to the coupled SDEs (2.6) describing the evolution of a peakon. Since the peakon profile does not have a continuous derivative, we do not expect to obtain a convergence rate approaching second order as $\Delta x \to 0$. Although a discretization using the implicit midpoint rule should have first-order strong (pathwise) convergence as $\Delta t \to 0$ (see [41] or [38]), again, due to the lack of smoothness in the solution, we expect the reduced convergence that we have observed.
Figure 2. The emergence of a peakon via wave breaking from the initial condition (5.1). Each panel shows the velocity profile $u$ in a periodic domain (although only part of the domain is shown) at a different moment in time. This initial condition has a momentum $m$ with negative parts and satisfies the assumptions of the steepening lemma of [5] and we see that the slope approaches vertical and a peakon forms through wave breaking. In fact, we also see the formation of anti-peakons (peakons with negative velocity values) to the left of the primary peakon.

5. Numerical investigations

(a) Deterministic formation of peakons via wave breaking

In this section, we use the finite-element discretization to investigate the formation of peakons. For all of §5, we take $\alpha = 1$ and use a periodic domain of length $L = 40$.

First, we will demonstrate the formation of a peakon via wave breaking in the deterministic case from a smooth, confined initial condition. The steepening lemma of [5] considers profiles whose initial negative slope $s$ at the inflection point and the integral $H_1 = \frac{1}{2} \int_\Omega (u^2 + \alpha^2 u_x^2) \, dx$ satisfy $s < -\sqrt{2H_1/\alpha^3}$. Here, we choose an initial condition satisfying this

$$u = \frac{1}{2} \sech \left( \frac{x - L/4}{L/240} \right), \quad (5.1)$$

so that $s = -1.5$, $H_1 \approx 0.542$ and $\sqrt{2H_1/\alpha^3} \approx 1.041$. Computing $m = u - \alpha^2 u_{xx}$ for this profile reveals $m$ to have a negative section to the right of a positive section, and as shown by [27] and discussed by [42] this is a condition for wave breaking. The deterministic case is achieved in our discretization by simply taking $\mathcal{E} = 0$. Figure 2 shows the velocity profile found using the discretization of §3 with $\Delta x = 0.002$ and $\Delta t = 0.001$ at a range of times. We see that the peak leans to the right and the negative slope steepens until it becomes vertical. Then the wave breaks and a peakon emerges to propagate to the right.
Figure 3. The emergence of a peakon without wave breaking. The initial condition used was (5.2), whose slope is shallower than that assumed for the steepening lemma of [5] and whose momentum $m > 0$. As the profile moves rightwards, although we see it tilt the slope never becomes vertical. Instead, the inflection points (marked with dashed lines) rise until they meet at the top of the profile. At this moment, the peakon emerges.

(b) Deterministic formation of peakons without wave breaking

Now we consider an initial condition with a shallower slope

$$u = \frac{1}{2} \text{sech} \left( \frac{x - L/4}{L/40} \right).$$

(5.2)

This does not satisfy the assumptions of the steepening lemma of [5], as $s = -0.25$, $H_1 \simeq 0.333$ and $\sqrt{2H_1/\alpha^3} \simeq 0.816$. In contrast to the steep initial condition in the previous section, computing $m = u - \alpha^2 u_{xx}$ at each point gives a momentum that is entirely positive. This is a condition for no wave breaking discussed in [27].

Figure 3 illustrates peakon formation from this initial condition. It shows the velocity profile at a series of times, with the inflection points highlighted with dashed lines. As the velocity is represented by continuous, piecewise linear function in our discretization, the gradient $u_x$ of the field is a series of discontinuous piecewise constants. The positions of the inflection points are then identified as the centre of the cells containing the maxima and minima in $u_x$. As we see in figure 3, the inflection points rise up the profile until they eventually meet at the maximum, which is the moment of peakon formation. At no point does the slope turn vertical, and thus we see that peakon formation can occur without wave breaking.

(c) Diagnostics for peakon formation and wave breaking

Before we investigate the formation of peakons in the stochastic case, we first define two diagnostics that allow us to identify whether a velocity profile is a peakon, and whether wave breaking has taken place.
Our first diagnostic, which we denote by $v$, describes the jump in $u_x$ around the peak and is given by

$$v := \frac{\max_x(u_x) - \min_x(u_x)}{x_{\text{min}} - x_{\text{max}}},$$

(5.3)

where $x_{\text{max}}$ and $x_{\text{min}}$ are the locations of the maximum and minimum in $u_x$. For a peakon, as the grid spacing $\Delta x \to 0$, $v \to \infty$, whereas it will tend to some finite value when not representing a peakon.

Figure 4 shows the evolution with time of $v$ for three values of $\Delta x$, using the shallow set-up outlined in §5b. The time step used was $\Delta t = 0.0005$, although when this was repeated with smaller values of $\Delta t$ there was no significant difference in the values. We see that as the slope on the leading edge of the velocity profile steepens, the value of $v$ rises, but there is agreement as the resolution is refined and $v$ tends to some finite value. Then the values of $v$ jump sharply, and as expected it appears that as $\Delta x \to 0$, $v \to \infty$. The subsequent band of oscillating values corresponds to the changing value of $(x_{\text{max}} - x_{\text{min}})$ as the peakon moves through a cell. To detect the presence of a peakon, at each point in time, we inspect the values of $dv/d(\Delta x)$ from $t \in [15, 20]$ s. These times are shaded in grey on the central figure. (c) The values of $dv/d(\Delta x)$ at these times are plotted as a histogram, with the mean value shown with the dashed line, giving the value $\Pi$ of equation (5.4).

$$\Pi := \langle \frac{dv}{d(\Delta x)} \rangle,$$

(5.4)

with the average denoted by $\langle \cdot \rangle$ taken over $t \in [15, 20]$ s. Thus the presence of a peakon is signified by a large negative value of $\Pi$. The lack of a peakon would be represented by a value of $\Pi$ close to zero.
To detect wave breaking, we look at $\min(x)(u_x)$ as a function of time. If wave breaking is taking place and the slope turns vertical, then as $\Delta x \to 0$ we will see $\min(x)(u_x) \to -\infty$. Again for each point in time, we fit a linear curve to $\min(x)(u_x)$ as a function of $\Delta x$ and inspect the gradient, with strongly positive values indicating wave breaking. Figure 5 shows this for the set-ups of §5a,b, clearly showing wave breaking occurring with the steeper initial condition of equation (5.1) but not with that of equation (5.2). As before, it is helpful to define a single value to determine whether wave breaking has taken place. We therefore look at

$$\omega := \max_t \left[ \frac{d\min(x)(u_x)}{d(\Delta x)} \right],$$

(5.5)

with large positive values of $\omega$ representing wave breaking and values of $\omega$ close to zero showing a lack of wave breaking.

(d) Stochastic formation of peakons

In [7], the authors investigated the formation of peakons with the case that $S(x) = \xi$, a constant. They did this by considering a stochastic form of the steepening lemma of [5], applied to initial conditions satisfying the same condition of $s < -\sqrt{2H_1/\alpha^3}$. The authors found that wave breaking
did occur with a positive probability, but it was unclear whether this probability was necessarily unity. Although this was interpreted as the probability of peakon formation, the results from §5a–c indicate that instead this was the probability of the wave breaking process occurring.

In this section, we use the discretization of §3 to investigate both wave breaking and the formation of peakons in the SCH equation. First, we aim to determine qualitatively whether wave breaking and peakon formation may still occur following the introduction of stochastic transport. Finding that they do, we may ask whether the probabilities of these events is merely non-zero or in fact unity. To answer these questions, we use the diagnostics $\Pi$ and $\omega$ and the methodologies that were laid out in §5c. Large negative values of $\Pi$ indicate the formation of a peakon, while large positive values of $\omega$ indicate that wave breaking has taken place.

For both the steep and shallow initial profiles laid out in §5a,b, we now add a stochastic component to the transport with $\Sigma = \xi$ (a constant), for a range of $\xi$ values. Simulations were all computed up to $t = 20$ s. To investigate the probabilities of peakon formation and wave breaking, we solved the SCH equation under 100 different realizations of the noise, for each value of $\xi$ and both the steep and shallow initial profiles. In each case, we computed values of the diagnostics $\Pi$ and $\omega$, which are plotted as histograms for each $\xi$ in figures 6 and 7.
For each realization of the noise, we performed the simulation at five different resolutions, in each case dividing our domain uniformly into 1000, 1500, 2000, 2500 and 3000 cells. The values of \( \nu \) and \( \min_x (u_x) \) at each of these resolutions were used to determine the diagnostics \( \Pi \) and \( \omega \). We used a time step of \( \Delta t = 0.0005 \) s. As shown in §4, for a stochastic peakon the convergence as \( \Delta t \to 0 \) is slow. Thus we found that refining \( \Delta t \) further did improve the results, but limits on computational resources then restricted the number of realizations of noise that it was feasible to use.

We look first at figure 6, which uses the steep initial condition from §5a. The values of \( \Pi \) are negative in every single case, indicating that peakons can indeed form. Two positive values of \( \Pi \) were recorded, though further inspection of the fields from these simulations suggested that these were still peakons. Similarly, the values of \( \omega \) are positive in every case, particularly strong when the noise is weak, suggesting that wave breaking can occur from the initial condition of equation (5.1).

Figure 7 plots the diagnostics \( \Pi \) and \( \omega \) for the shallow initial condition of §5b. As with the steeper initial condition, the results suggest that peakon formation does take place. However, we
did not see a clear example of wave breaking, although it is possible that for large values of $\xi$ that wave breaking may take place.

Both figures 6 and 7 show that as the strength of the noise is increased, the signals of both peakon formation and wave breaking get weaker, as the diagnostics of $v$ and $\min_x(u_x)$ become more noisy. As the simulations had not converged with respect to $\Delta t$, we did see improvements in these signals by reducing $\Delta t$ further, but as mentioned above this results in a trade-off with the number of realizations of the noise. However, even for the strongest values of $\xi$ we do see evidence of peakon formation.

It is also worth noting that our $\Pi$ diagnostic could show false-positive results for velocities profiles that are not peakons, but whose inflection points are separated by less than a distance of the order of the smallest $\Delta x$ that we used. However, we have demonstrated in §2 that peakons are weak solutions to the SCH equation and in §4 that our discretization will converge to such solutions.

The conclusions that we draw must come with caveats. There are infinitely many initial conditions that we have not considered, with infinitely many choices of $\Xi$ and infinitely many realizations of the noise. It may be that peakons might not form under larger values of $\xi$, although the physical motivation of including stochastic transport to represent unresolved processes suggests that values of $\xi$ should be small. Despite this, we did not see any examples in which a peakon did not appear to form. In this section, we have therefore seen that under the addition of stochastic perturbations both peakon formation and wave breaking can still occur.

6. Conclusion

We have considered the CH equation of [5] with the addition of the stochastic transport of [8]. Taking the advective form of this SCH equation, we showed that peakons are weak solutions to this equation in §2. Inspired by looking for weak solutions, we presented a finite-element discretization to the advective form of the equation in §3 and demonstrated that it converges to the peakon solutions in §4. The strength of using such a finite-element discretization is that it can be used to describe solutions of the equation through the process of peakon formation.

The simulations presented in §5 demonstrate the formation of peakons through two mechanisms. First through wave breaking, as described by the steepening lemma of [5] when the slope at the inflection point to the right of the initial peak is sufficiently negative. For shallower initial profiles, peakons can still form without wave breaking taking place. Instead, the inflection points rise up the profile without the slope ever turning vertical and the peakon emerges as the inflection points reach the summit.

We then investigated these processes with stochastic perturbations using a constant $\Xi$, testing the findings of [7]. Our results suggest that peakon formation does still occur, and that wave breaking can still occur when the initial slope is steep. We did not see any examples when peakons did not form in these situations.

A clear avenue for further work is to investigate whether spatially varying $\Xi$ may prevent the formation of peakons. The (impossibly unlikely) choice of $\sum_j \Xi^j \circ dW^j = -u \, dt$ in equation (1.1) yields $du = 0$, suggesting that this may be possible. We have also not discussed the effect that the noise has upon the time of the peakon formation.

Data accessibility. The data used in the numerical investigations can be reproduced from the experiments kept in the Github repository that can be found at https://github.com/tommbendall/peakondrake.

Authors’ contributions. The numerical experiments were performed by T.M.B., who wrote the corresponding code and drafted the manuscript. The calculation of §2 was performed by C.J.C. and T.M.B. D.D.H. and C.J.C. guided the work, conceived of the experiments and revised the manuscript.

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