Comments on noncommutative quantum mechanical systems associated with Lie algebras

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Abstract

We consider quantum mechanics on the noncommutative spaces characterized by the commutation relations

$$[x_a, x_b] = i\theta f_{abc} x_c,$$

where $f_{abc}$ are the structure constants of a Lie algebra. We note that this problem can be reformulated as an ordinary quantum problem in a commuting momentum space. The coordinates are then represented as linear differential operators $\hat{x}_a = -i\hat{D}_a = -iE_{ab}(p)\partial/\partial p_b$. Generically, the matrix $E_{ab}(p)$ represents a certain infinite series over the deformation parameter $\theta$: $E_{ab} = \delta_{ab} + \ldots$. The deformed Hamiltonian, $\hat{H} = -\frac{1}{2}\hat{D}_a^2$, describes the motion along the corresponding group manifolds with the characteristic size of order $\theta^{-1}$. Their metrics are also expressed into certain infinite series in $\theta$, with $E_{ab}$ having the meaning of vielbeins.

For the algebras $su(2)$ and $u(N)$, it has been possible to represent the operators $\hat{x}_a$ in a simple finite form. A byproduct of our study are new nonstandard formulas for the metrics on all the spheres $S^n$, on the corresponding projective spaces $RP^n$ and on $U(2)$. 
1 Introduction

The quantum mechanical systems defined on noncommutative spaces \[ [1] \] have recently attracted a considerable interest \[ [2] \]. The simplest such system involves only two dynamic variables \( x, y \) with a nontrivial commutator

\[
[x, y] = i\theta. \tag{1.1}
\]

Probably, the best way to deal with such a system \[ [4] \] is to go over to the momentum space, \((p_x, p_y) \equiv (X, Y)\), in which case \( X \) and \( Y \) commute, while the original noncommuting coordinates \( x, y \) can be interpreted as differential operators, which are quantum counterparts of the velocity components of the moving particle of unit mass.

\[
x \equiv \hat{v}_X = -i\frac{\partial}{\partial X} + \frac{\theta}{2} Y, \\
y \equiv \hat{v}_Y = -i\frac{\partial}{\partial Y} - \frac{\theta}{2} X. \tag{1.2}
\]

A naturally defined Hamiltonian,

\[
H = \frac{1}{2}(\hat{v}_X^2 + \hat{v}_Y^2),
\]

describes then the motion on the plane \((X, Y)\) with a homogeneous orthogonal magnetic field \( \theta \) of a particle of unit mass. The canonical momenta are \( \hat{P}_X = \hat{v}_X - \theta Y/2 \) and \( \hat{P}_Y = \hat{v}_Y + \theta X/2 \).

2 Noncommutative systems involving the Lie algebra commutators

The R.H.S. of the commutator \( [1.1] \) is constant. In this case, \( \partial_x [x, y] = \partial_y [x, y] = 0 \), which is true if the derivative operator satisfies the ordinary Leibnitz rules. More general, for a space of arbitrary dimension with the constant commutators \( [x_a, x_b] = i\theta f_{abc} x_c \), \( f_{abc} \) are structure constants of an arbitrary Lie algebra. Such spaces were considered in \[ [5] \] where it was shown that, to satisfy the Jacobi identities, one has to postulate the following nontrivial commutators:

\[
[\partial_a, x_b] = \delta_{ab}, \quad [\partial_a, \partial_b] = 0
\]

and observe that the Jacobi identities still hold.

But it is not generically true for more complicated noncommutative spaces when \( [x_a, x_b] \) exhibit a nontrivial coordinate dependence. In this paper, we consider a particular class of such spaces characterised by the commutators

\[
[x_a, x_b] = i\theta f_{abc} x_c, \tag{2.1}
\]

where \( f_{abc} \) are structure constants of an arbitrary Lie algebra. Such spaces were considered in \[ [5] \] where it was shown that, to satisfy the Jacobi identities, one has to postulate the following nontrivial commutators:

\[
[\tilde{\partial}_a, x_b] = \left[ \frac{i\theta F(\tilde{\partial})}{e^{i\theta F(\tilde{\partial})} - 1} \right]_{ab} = \delta_{ab} + \sum_{n=1}^{\infty} (-i\theta)^n \frac{B_n^+}{n!} [F^n(\tilde{\partial})]_{ab}, \quad [\tilde{\partial}_a, \tilde{\partial}_b] = 0, \tag{2.2}
\]

\[ ^1 \text{Also noncommutative field theories have been intensely studied [2], but they are not the subject of our present discussion.} \]
where
\[ F_{ab}(\tilde{\partial}) = f_{acb} \tilde{\partial}_c \] (2.3)

and \( B^+_n \) are the Bernoulli numbers\(^2\).

The Jacobi identities for the algebra (2.1), (2.2) are satisfied, as one can be convinced quite directly in each order of the expansion in \( \theta \), using the recursive relations for the Bernoulli numbers. In the limit \( \theta \rightarrow 0 \), the algebra elements \( \tilde{\partial}_a \) acquire the meaning of the ordinary partial derivatives \( \partial_a \) with respect to the commuting variables \( x_a \).

In the spirit of our remark in the Introduction, we perform the quantum canonical transformation,
\[
\hat{p}_a = -i\tilde{\partial}_a \rightarrow -X_a, \quad x_a \rightarrow \hat{P}_a, \tag{2.4}
\]
so that the elements \( \tilde{\partial}_a \) are traded for the commuting coordinates \( X_a \), whereas the original coordinates \( x_a \) are realized as the linear differential operators,
\[
x_a \rightarrow -i\hat{D}_a = -i \left[ \frac{F(\theta X)}{1 - e^{-F(\theta X)}} \right]_{ab} \frac{\partial}{\partial X_b}. \tag{2.5}
\]

The operators \( \hat{D}_a \) satisfy the relations
\[
[\hat{D}_a, \hat{D}_b] = -\theta f_{abc} \hat{D}_c. \tag{2.6}
\]

In other words, we have constructed a representation of the generators of an arbitrary Lie group not in the matrix form, as is usually done, but in the form of linear differential operators acting on the space of the dimension coinciding with the dimension of the group. The operators \( \hat{D}_a \) have a transparent geometric meaning. They are none other than the generators of the right group rotations. Indeed, one can show that
\[
e^{i\theta X_a t_a} e^{i\theta \epsilon_a t_a} = e^{i\theta t_a(X_a + \delta X_a)}, \tag{2.7}
\]
where\(^3\)
\[
\delta X_a = \sum_{n=0}^{\infty} B^+_n \frac{[F^n(\theta X)]_{ab}}{n!} \epsilon_b + o(\epsilon). \tag{2.8}
\]

And this is, of course, the reason why \( \hat{D}_a \) satisfy the Lie algebra commutation relations.

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\(^2\)The Bernoulli numbers are defined from the Taylor expansion,
\[
\frac{t}{1 - e^{-t}} = \sum_{n=0}^{\infty} B^+_n \frac{t^n}{n!}.
\]
The first few Bernoulli numbers are \( B^+_0 = 1, B^+_1 = 1/2, B^+_2 = 1/6, B^+_3 = 0 \) and \( B^+_4 = -1/30 \). Note that \( B^+_{2m+1} \) all vanish for \( m > 0 \).

\(^3\)A simple derivation of this relation (which represents a particular case of the general Baker-Campbell-Hausdorff formula) is given in the appendix.
This paper is devoted to studying the properties of the Hamiltonian
\[ \hat{H} = -\frac{1}{2}\hat{D}_a^2. \]  
(2.9)

The classical Hamiltonian corresponding to the quantum Hamiltonian (2.9) reads
\[ H^{\text{cl}} = \frac{1}{2}g^{jk}P_j P_k, \]  
(2.10)

where
\[ g^{jk} = E_a^j E_a^k \]  
(2.11)

with
\[ E_a^j = \left[ \frac{F(\theta X)}{1 - e^{-F(\theta X)}} \right]_a^j. \]  
(2.12)

This Hamiltonian describes the motion over the manifold with the inverse metric \( g^{jk} \). The objects \( E_a^j \) have the meaning of the vielbeins. The classical Lagrangian corresponding to the Hamiltonian (2.10) depends on the covariant metric tensor and reads
\[ L = \frac{1}{2}g_{jk}\dot{X}^j\dot{X}^k. \]  
(2.13)

**Theorem 1.** The metric \( g_{jk} \) in (2.13) coincides with the invariant metric on the group manifold.

**Proof.** The canonical metric on a group manifold, which is invariant under the left and right group rotations, is
\[ g_{jk} = \frac{1}{h\theta^2} \text{Tr} \left\{ \partial_j \omega^{-1} \partial_k \omega \right\}, \]  
(2.14)

where
\[ \omega(X) = \exp\{i\theta X^j t_j\} \]  
(2.15)

and \( t_j \) are the generators in a given representation with the Dynkin index \( h \). Then \( g_{jk}(X = 0) = \delta_{jk} \).

Consider the distance between two close points \( \omega(X + \delta X) \) and \( \omega(X) \). For the invariant metric, this distance is the same as the distance between \( \omega^{-1}(X)\omega(X + \delta X) = \omega(\epsilon) \) and the group unity. The latter is \( ds^2 = \epsilon_a \epsilon_a \). To find the metric at the vicinity of \( X \), we have only

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4 Having said that, we went to the middle of the alphabet to denote the world tensor indices, while keeping its beginning for the flat tangent space indices.

5 It is familiar to physicists, appearing e.g. in the effective chiral Lagrangian in QCD, \[ \mathcal{L} \propto \text{Tr} \left\{ \partial_\mu \omega^{-1} \partial^\mu \omega \right\} \] with \( \omega \in SU(2) \) or \( SU(3) \).
express $\epsilon_a$ in terms of $\delta X^j$. To do so, we use the formula \[7\] (its derivation is explained in the appendix)
\[
\frac{1}{\theta} \omega^{-1}(-i\partial_\omega) = \int_0^1 d\tau e^{-\tau R} t_j e^{\tau R} = t_j - \frac{1}{2}[R, t_j] + \frac{1}{6}[R, [R, t_j]] - \ldots = E_j^a t_a \tag{2.16}
\]
where $R = i\theta t_j X^j$ and
\[
E_j^a = \left[\frac{1 - e^{-F(\theta X)}}{F'(\theta X)}\right]_j^a = \sum_{n=0}^{\infty} \frac{[F^n(-\theta X)]_j^a}{(n+1)!}
\tag{2.17}
\]
is the inverse vielbein. Then $\epsilon^a = E_j^a \delta X^j$ and the metric is
\[
g_{jk} = E_j^a E_k^a, \tag{2.18}
\]
which matches (2.11).

Thus, the classical Hamiltonian (2.10) describes the motion of a particle along the group manifold.

For $SU(2) \equiv S^3$, the sums for the vielbeins in (2.12) and (2.17) can be done:
\[
E_a^j = \delta_{aj} + \frac{\theta}{2} \varepsilon_{apj} X_p + \left(\delta_{ja} - \frac{X_a X_j}{X^2}\right) \left[\frac{\theta X/2}{\tan(\theta X/2)} - 1\right]
\]
\[
E_j^a = \delta_{ja} - 2 \sin^2(\theta X/2) \varepsilon_{jpa} X_p + \left(\delta_{ja} - \frac{X_a X_j}{X^2}\right) \left[\frac{\sin(\theta X)}{\theta X} - 1\right], \tag{2.19}
\]
where $X = \sqrt{X_p X_p}$ and the positions of the indices in the right-hand sides do not have a tensorial meaning and are irrelevant.

The metric (2.11), (2.18) reads
\[
g^{jk} = A^{-1}(X) \left(\delta_{jk} - \frac{X_j X_k}{X^2}\right) + \frac{X_j X_k}{X^2},
\]
\[
g_{jk} = A(X) \left(\delta_{jk} - \frac{X_j X_k}{X^2}\right) + \frac{X_j X_k}{X^2}, \tag{2.20}
\]
where
\[
A(X) = \frac{4 \sin^2(\theta X/2)}{\theta^2 X^2}.
\]
At the vicinity of the origin,
\[
g_{jk} = \delta_{jk} + \frac{\theta^2}{12} (X_j X_k - X^2 \delta_{jk}) + O(X^4). \tag{2.21}
\]
The corresponding scalar curvature is $R = 3\theta^2/2$. Bearing in mind that $R = 6/\rho^2$ where $\rho$ is the radius of $S^3$, we derive
\[
\rho = \frac{2}{\theta}. \tag{2.22}
\]
\[6\]This formula coincides with Eq. (4.80) in recent [8], though its interpretation there was completely different.
The metric (2.18) looks quite different from the familiar conformally flat metric on $S^3$,

$$g_{jk} = \frac{\delta_{jk}}{[1 + Z^2/(4\rho^2)]^2}.$$  \hfill (2.23)

But these two metrics describe the same geometry, which means that the only difference between them is the choice of coordinates $Z \to X$ [see Eqs. (2.35), (2.36) below].

Consider now the quantum Hamiltonian (2.9) implying a particular way of ordering of the coordinates and canonical momenta.

**Theorem 2.** The Hamiltonian (2.9) coincides up to the factor $-1/2$ with the Laplace-Beltrami operator on the group manifold,

$$\triangle = \frac{1}{\sqrt{g}} \partial_j (\sqrt{g} g^{jk}) \partial_k,$$ \hfill (2.24)

with the invariant metric (2.18).

**Proof.** Clearly the operator $(\hat{D}_a)^2$ commutes with $\hat{D}_a$. This means that the Hamiltonian (2.9) is invariant under right group rotations. But it must also be invariant under left group rotations,

$$e^{i\theta t} x^j \to e^{i\theta t} e^{i\theta t} X^j.$$ \hfill (2.25)

Indeed, right and left rotations commute and their generators $\hat{D}_a$ and $\hat{D}'_a$ must also commute. Both of them can be represented as infinite sums

$$\hat{D}_a = \sum_{n=0}^{\infty} \frac{B^+_n \theta^n [F^n(X)]_a^p}{n!} \partial_p,$$

$$\hat{D}'_a = \sum_{n=0}^{\infty} \frac{B^-_n \theta^n [F^n(X)]_a^p}{n!} \partial_p.$$ \hfill (2.26)

The only difference is that the sum for $\hat{D}_a$ involves the Bernoulli numbers $B^+_n$, while the sum for $\hat{D}'_a$ involves the Bernoulli numbers $B^-_n$. The latter coincide with $B^+_n$ for all $n$ except $n = 1$ where $B^-_1 = B^+_1 = -1/2$.

Left and right rotations commute as well as their generators. The vanishing of the commutator $[\hat{D}'_a, \hat{D}_b] = 0$ entails the vanishing of $[\hat{D}'_a, H^{au}]$. The only second order differential operator that is invariant under both left and right group rotations is, up to a constant, the Laplace-Beltrami operator.

Note that the operators $\hat{D}'_a$ satisfy the commutation relations

$$[\hat{D}'_a, \hat{D}'_b] = \theta f_{abc} \hat{D}'_c$$ \hfill (2.27)

with the opposite sign, compared to (2.6). The origin of this extra sign is quite clear bearing in mind (2.7) and (2.25). The composition of two right rotations $g_1$ and $g_2$ is $g_1 g_2$, while for left rotations it is $g_2 g_1$.  

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Another remark is the following. The Hamiltonian (2.24) seems to be not Hermitian. But it is nothing but an “optical illusion”. It is Hermitian in the Hilbert space where the wave functions are normalized with the covariant measure,

$$\int d^dX \sqrt{g} |\Psi(X)|^2 = 1,$$

(2.28)

g = \det |g_{jk}|. In the Hilbert space with the flat measure $d^dX$, the Hamiltonian would include extra wrapping factors: $H_{\text{flat}} = g^{1/4} H_{\text{flat}}^{1/4}$.

The same concerns the operator $\hat{D}_a$ defined in (2.5): it is anti-Hermitian in the Hilbert space including the factor $\sqrt{g}$ in the measure.

We hasten to comment, however, that the representation (2.5) for $\hat{D}_a$ is not unique. For example, for $su(2)$, the following nice representation is possible:

$$\hat{D}_a = \frac{\partial}{\partial Y_a} - \frac{\theta}{2} \varepsilon_{abc} Y_b \frac{\partial}{\partial Y_c} + \frac{\theta^2}{4} Y_a Y_b \frac{\partial}{\partial Y_b}. $$

(2.29)

The algebra (2.6) still holds.

The operator $(\hat{D}_a)^2$ describes then the motion of a particle over the 3-dimensional manifold with the metric

$$g^{jk} = \delta^{jk} + \kappa(Y^2 \delta^{jk} + Y^j Y^k) + \kappa^2 Y^j Y^k,$$

$$g_{jk} = \frac{\delta_{jk}}{1 + \kappa Y^2} - \frac{\kappa Y_j Y_k}{(1 + \kappa Y^2)^2},$$

(2.30)

where $\kappa = \theta^2/4$.

The corresponding Christoffel symbols are

$$\Gamma^p_{kl} = -\frac{\kappa}{1 + \kappa Y^2} (Y_k \delta_{pk} + Y_l \delta_{pk}).$$

(2.31)

The Riemann tensor reads

$$R^i_{kpl} = \partial_p \Gamma^i_{kl} - \partial_l \Gamma^i_{kp} + \Gamma^i_{mp} \Gamma^m_{kl} - \Gamma^i_{nl} \Gamma^m_{kp}$$

$$= \frac{\kappa}{1 + \kappa Y^2} (\delta_{kl} \delta_{ip} - \delta_{kp} \delta_{il}) + \frac{\kappa^2 Y_k (Y_p \delta_{il} - Y_l \delta_{ip})}{(1 + \kappa Y^2)^2} = \kappa (\delta_{kl} g_{ip} - \delta_{kp} g_{il}).$$

(2.32)

Hence $R_{ikpl} = \kappa (g_{ip} g_{kl} - g_{il} g_{kp})$.

It follows that all the curvature invariants are constant:

$$R = 6 \kappa = \frac{3 \theta^2}{2}, \quad R_{ikl} R^{kl} = 12 \kappa^2 = \frac{3 \theta^4}{4}, \quad R_{ikp} R^{pm} R^{n_k} = 24 \kappa^3 = \frac{3 \theta^6}{8},$$

$$R_{ikpl} R^{ikpl} = 12 \kappa^2 = \frac{3 \theta^4}{4}, \ldots$$

(2.33)

One is then tempted to say that our manifold represents the “round” 3-sphere of radius (2.22), but it is not exactly so. The determinant of the metric is

$$g = \frac{1}{(1 + \kappa Y^2)^4}.$$

\footnote{Cf. Eq. (37) in Ref. [9].}
The volume of the manifold,

\[ V = \int \int \int_{-\infty}^{\infty} \sqrt{g} \, d^3x = \pi^2 \rho^3, \] (2.34)

is two times smaller than the volume of \( S^3 \), which means that we are dealing with the projective space \( RP^3 = S^3/Z_2 \equiv SO(3) \). In fact, one can confirm it, writing an explicit coordinate change [10], that brings the metric (2.30) in the form (2.20):

\[ Y^p = X^p \frac{2 \tan \left( \frac{\theta X}{2} \right)}{\theta X}. \] (2.35)

Then \( Y^p = \infty \) corresponds to \( X = \sqrt{X^p X^p} = \pi/\theta \). And the latter value corresponds to the equator on \( S^3 \) in the exponential parameterization (2.15).

Another variable change,

\[ Y^p = Z^p \frac{1}{1 - \kappa Z^2/4}, \] (2.36)

brings the metric (2.30) in the habitual conformally flat form.

### 2.1 Higher spheres

The expressions (2.20) and (2.30) describe the metric of \( S^n \) in any dimension. Indeed, the stereographic projection leading to the metric (2.23) can be performed in any dimensions, and the variable changes (2.36) (2.35) bring it in the forms (2.20), (2.30) in any dimension.

However, the manifolds \( S^3 \) and \( S^7 \) are special. These spheres are parallelizable and admit quaternion (resp. octonion) algebraic structure. The case of \( S^3 \) we already discussed. It is just a group manifold and the Hamiltonian describing its motion is related to the Lie algebra (2.6).

But the motion over \( S^7 \) can also be associated with a not so simple nonlinear algebra.

Define the operators \( \hat{D}_A = E_A^J \partial_J \) with the vielbeins

\[ E_A^J = \delta_{AJ} + \frac{\theta}{2} \eta_{APJ} X_P + \left( \delta_{AJ} - \frac{X_A X_J}{X^2} \right) \left[ \frac{\theta X/2}{\tan(\theta X/2)} - 1 \right]. \] (2.37)

They have the same form as in (2.19), but \( X_J \) are now the coordinates on \( S^7 \) and the antisymmetric tensor \( \eta_{ABC} \) defines the rules of octonion multiplication\(^8\)

\[ e_A e_B = -\delta_{AB} + \eta_{ABC} e_C. \] Using the identity

\[ \eta_{ABC} \eta_{DEC} = \delta_{AD} \delta_{BE} - \delta_{AE} \delta_{BD} - \eta_{ABDE} \] (2.38)

with

\[ \eta_{ABDE} = \frac{1}{6} \epsilon_{ABDEFGH} \eta_{FGH}. \] (2.39)

\(^8\)Different choices for \( \eta_{ABC} \) are possible. One of them is

\[ \eta_{123} = \eta_{456} = \eta_{147} = \eta_{657} = 1, \]

the other nonzero components being restored by antisymmetry.
one can be convinced that the convolution of the vielbeins (2.37) gives the metric on $S^7$, which, as we have mentioned, has the same form as in (2.20), but with seven coordinates.

The difference is that the operators $\hat{D}_A$ do not form a Lie algebra as in (2.6), but their commutators have a more complicated form with the coordinate dependence in the right hand side:

$$[\hat{D}_A, \hat{D}_B] = -\theta \eta_{ABC} \hat{D}_C + \frac{\theta \sin(\theta X)}{X} \eta_{ABCP} X_P \hat{D}_C + 2 \frac{\theta \sin^2(\theta X/2)}{X^2} \eta_{ABCP} \eta_{CDQ} X_P X_Q \hat{D}_D. \quad (2.40)$$

### 3 Gurevich-Saponov model.

#### 3.1 $u(2)$ model.

In Refs. [12, 13], a noncommutative quantum mechanical system was studied with the following nontrivial commutation relations:

$$[x_a, x_b] = i \theta \epsilon_{abc} x_c, \quad [x_a, \tau] = 0, \quad (3.1)$$

supplemented by

$$[\tilde{\partial}_a, x_b] = \delta_{ab} \left(1 + \frac{i \theta}{2} \tilde{\partial}_\tau\right) + \frac{i \theta}{2} \epsilon_{abc} \tilde{\partial}_c, \quad [\tilde{\partial}_a, x_a] = -\frac{i \theta}{2} \tilde{\partial}_a,$$

$$[\tilde{\partial}_a, \tau] = 1 + \frac{i \theta}{2} \tilde{\partial}_\tau, \quad [\tilde{\partial}_a, \tilde{\partial}_b] = [\tilde{\partial}_a, \tilde{\partial}_\tau] = 0. \quad (3.2)$$

It is easy to check that the Jacobi identities hold. This model belongs to the large class of the models (2.1), but the algebra here is $u(2)$, which is not simple. Note that the R.H.S. of the commutators in (3.2) does not represent an infinite series as in (2.2), but has a simple form involving only the first generalized derivatives [cf. Eq. (2.29), which, in contrast to (2.5), involves only the linear and quadratic terms in $\tilde{\partial}_a \to -i Y_a$].

Proceeding in the same way as before, we may trade the coordinates for momenta and vice versa to arrive at the ordinary QM system with four commuting dynamic variables $X_a, T$ and former coordinates and now momenta being realized as the differential operators:

$$P_a = -i \hat{D}_a = -i \left[\left(1 + \frac{\theta T}{2}\right) \frac{\partial}{\partial X_a} - \theta X_a \frac{\partial}{\partial T} + \frac{\theta \epsilon_{abc} X_b}{2} \frac{\partial}{\partial X_c} \right];$$

$$P_0 = -i \hat{D}_0 = -i \left[\left(1 + \frac{\theta T}{2}\right) \frac{\partial}{\partial T} + \theta X_a \frac{\partial}{\partial X_a} + \theta \right]. \quad (3.3)$$

The algebra

$$[\hat{D}_a, \hat{D}_b] = -\theta \epsilon_{abc} \hat{D}_c, \quad [\hat{D}_a, \hat{D}_0] = 0$$

\footnote{In different terms, this result was derived in [11] using (2.38) and the identity

$$\eta_{ABC} \eta_{AFDE} = \delta_{CF} \eta_{BDE} + \delta_{CD} \eta_{BEF} + \delta_{CE} \eta_{BFD} - (B \leftrightarrow C),$$

but we translated it in our notation.}

\footnote{We added a constant in $P_0$ to make it Hermitian with the flat measure $d\mu = d^3 X DT$.}
holds.

Consider now the operator

$$\hat{H} = -\frac{1}{2}(\hat{D}_0 \hat{D}_0 + \hat{D}_a \hat{D}_a).$$

(3.4)

Its classical counterpart is

$$H^{cl} = \frac{P_0^2 + P_a^2}{2} \left[ 1 + \theta T + \frac{\theta^2}{4}(T^2 + X_a^2) \right].$$

(3.5)

This Hamiltonian describes the motion of a particle over a 4-dimensional manifold with the conformally flat metric

$$g_{\mu\nu} = \frac{\delta_{\mu\nu}}{F(T, X_a)} = \frac{\delta_{\mu\nu}}{1 + \theta T + \theta^2(T^2 + X_a^2)/4}.$$ 

(3.6)

The Ricci tensor has the following components:

$$R_{00} = \frac{\theta^4}{8F^2}, \quad R_{0a} = -\frac{\theta^3(1 + \theta T/2)X_a}{4F^2}, \quad R_{ab} = \frac{\theta^2}{2F} \left( \delta_{ab} - \frac{\theta^2 X_a X_b}{4F} \right).$$

(3.7)

An explicit calculation shows that all the curvature invariants made from the Ricci tensor are constants:

$$R = R_{\mu\nu} g^{\mu\nu} = \frac{3\theta^2}{2}, \quad R_{\mu\nu} R^{\mu\nu} = \frac{3\theta^4}{4}, \quad R_{\mu\nu} R^{\nu\rho} R_{\rho} = \frac{3\theta^6}{8}, \text{ etc.}$$

(3.8)

They are the same as for a 3-dimensional “round” sphere of radius (2.22) [see Eq. (2.33)].

In other words, our manifold is \(S^3 \times \mathbb{R}\) or maybe \(SO(3) \times \mathbb{R}\) — it is difficult to say based solely on the form (3.6) of the metric where the coordinates \(X_a\) and \(T\) are intertwined. This result is quite natural as the Gurevich-Saponov model involves the nontrivial commutators (3.1) of the \(u(2)\) algebra. After interchanging the coordinates and momenta, we are arriving at the system that describes the motion along \(U(2)\) in a non-compact realization.

### 3.2 Higher \(N\)

One can also consider \[13\] the \(u(N)\) generalization of the algebra (3.1), (3.2). Introduce \(N^2 + N^2\) elements \(l^m_j, \tilde{\partial}^m_j\) and postulate the following commutators:

$$[l^m_j, l^n_k] = \theta(\delta^m_j l^n_k - \delta^n_k l^m_j), \quad [\tilde{\partial}^m_j, l^n_k] = \delta^m_j \delta^n_k + \theta \delta^m_j \tilde{\partial}^n_k.$$ 

(3.9)

The first relation is the habitual \(u(N)\) algebra. It is supplemented by the commutation laws for the quasiderivatives \(\tilde{\partial}^m_i\). It is not difficult to check that the Jacobi identities are satisfied.

\[11\] It is not \(S^3 \times S^1\) or \(SO(3) \times S^1\) because the integral

$$\int d^3X dT \sqrt{g} = \int d^3X dT \frac{F}{F^2}$$

diverges and the manifold is not compact.
To make contact with (3.1), (3.2), we introduce the notation

\[ x_A = (t_A)_a^i l^a_j, \quad \tilde{\partial}_A = 2(t_A)_a^i \tilde{l}_j^a, \]  

where \( A = (a, 0) \) and \( t_A \) are the generators of \( U(N) \) in the fundamental representation — the Hermitian \( N \times N \) matrices normalized in a habitual way

\[ \text{Tr}\{t_A t_B\} = \frac{1}{2} \delta_{AB}. \]

In particular, \( t_0 = 1/\sqrt{2N} \), so that \( x_0 \equiv t = l^i_j / \sqrt{2N} \) and \( \tilde{\partial}_t = \sqrt{2/\!N} \tilde{l}_j^i \). It is also convenient to introduce \( \tau = it, \tilde{\partial}_\tau = -i \tilde{l}_j^i \). In these terms, the algebra (3.9) reads

\[ [x_a, x_b] = i\theta f_{abc} x_c, \quad [x_a, \tau] = 0, \]
\[ [\tilde{\partial}_a, x_b] = \delta_{ab} \left( 1 + \frac{i\theta}{\sqrt{2N}} \tilde{l}_j^i \right) + \frac{i\theta}{2} (f_{abc} - i d_{abc}) \tilde{l}_j^a, \]
\[ [\tilde{\partial}_a, \tau] = -[\tilde{\partial}_\tau, x_a] = \frac{i\theta}{\sqrt{2N}} \tilde{l}_j^a, \quad [\tilde{\partial}_\tau, \tau] = 1 + \frac{i\theta}{\sqrt{2N}} \tilde{l}_j^\tau, \]  

(3.11)

where \( f_{abc} \) are the \( su(N) \) structure constants and \( d_{abc} \) is a symmetric tensor entering the relation

\[ \text{Tr}\{t_a t_b t_c\} = \frac{1}{4} (i f_{abc} + d_{abc}). \]

For \( N = 2 \), \( d_{abc} \) vanishes and (3.11) coincides with (3.1), (3.2).

We perform now our quantum canonical transformation to represent this algebra in terms of the commuting coordinates (former momenta) \( X_a, T \) and the differential operators

\[ x_a \rightarrow P_a = -i \hat{D}_a = -i \left[ \left( 1 + \frac{\theta T}{\sqrt{2N}} \right) \frac{\partial}{\partial X_a} - \frac{\theta}{\sqrt{2N}} X_a \frac{\partial}{\partial T} + \frac{\theta}{2} (f_{abc} + i d_{abc}) X_b \frac{\partial}{\partial X_c} \right], \]
\[ \tau \rightarrow P_0 = -i \hat{D}_0 = -i \left[ \left( 1 + \frac{\theta T}{\sqrt{2N}} \right) \frac{\partial}{\partial T} + \frac{\theta}{\sqrt{2N}} X_a \frac{\partial}{\partial X_a} \right]. \]  

(3.12)

The algebra

\[ [\hat{D}_a, \hat{D}_b] = -\theta f_{abc} \hat{D}_c, \quad [\hat{D}_a, \hat{D}_0] = 0 \]

holds.\(^\text{12}\)

Note now the intrinsic complexity \( \sim f_{abc} + i d_{abc} \) in the expression for \( \hat{D}_a \). That means that the would-be vielbein

\[ E_a^i = \left( 1 + \frac{\theta T}{\sqrt{2N}} \right) \delta_{aj} + \frac{\theta}{2} (f_{abj} + i d_{abj}) X_b \]  

(3.14)

\(^\text{12}\)It can be checked explicitly using the identities (see e.g. Appendix A of Ref. [14])

\[ f_{abc} f_{cde} + f_{cae} f_{bde} + f_{bce} f_{ade} = 0, \]
\[ f_{abc} d_{cde} + f_{ace} d_{bde} + f_{bce} d_{ade} = 0, \]
\[ f_{abc} f_{cde} = \frac{2}{N} (\delta_{ac} \delta_{bd} - \delta_{bc} \delta_{ad}) + d_{ace} d_{bde} - d_{bce} d_{ade}. \]  

(3.13)
is complex, and the same concerns the metric $g^{jk} = E^j_A E^k_A$. The determinant of the latter is also complex.

Complex metric and complex volume have little geometric sense. Also the canonical momenta $\hat{D}_a$ and the Hamiltonian (3.4) are not Hermitian for $N > 2$. There are some Hamiltonians that do not seem to be Hermitian, but are in fact Hermitian in disguise and have a real spectrum [15]. We do not think that it is so in our case, but a special study of this question would be interesting.

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Appendix

To make the paper self-contained, we will explain here how the identities (2.16) and (2.8) are derived.

The formula (2.16) has a nice physical interpretation [7]. Consider the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V}$ and the corresponding evolution operator

$$U = e^{i(\hat{H}_0 + \hat{V})t} = \lim_{N \to \infty} \left[ 1 + \frac{it}{N} (\hat{H}_0 + \hat{V}) \right]^N. \quad (A.1)$$

Suppose that the perturbation $\hat{V}$ is small and keep only the linear in $\hat{V}$ terms. The perturbation can be inserted at any time moment $t'$ between 0 and $t$, and we obtain

$$U = e^{i\hat{H}_0 t} + i \int_0^t dt' e^{i(t-t')\hat{H}_0} \hat{V} e^{it'\hat{H}_0} + \ldots . \quad (A.2)$$

Choosing $\hat{H}_0 = \theta X^j t_j$, $\hat{V} = \theta \delta X^j t_j$, $t = 1$, renaming $t' \to \tau$ and recalling the notation $\omega = e^R = e^{i\theta X^j t_j}$, we derive

$$\omega(X + \delta X) = \omega(X) + \delta X^j \partial_j \omega(X) + \ldots = \omega(X) \left[ 1 + i \theta \int_0^1 d\tau e^{-\tau R} \delta X^j t_j e^{\tau R} + \ldots \right], \quad (A.3)$$

from which (2.16) follows.

As a consequence, we derive for the product in (2.7) that $\epsilon^a$ is equal to $E^a_j \delta X^j$, with the vielbein $E^a_j$ having the form (2.17). Then $\delta X^j$ is expressed via $\epsilon^a$ with the vielbein $E^i_a$ written in (2.8).

References

[1] M. Chaichian, M.M. Sheikh-Jabbari and A. Tureanu, Hydrogen atom spectrum and the Lamb shift in noncommutative QED, Phys. Rev. Lett. 86 (2001) 2716,
This problem has many facets, see e.g.

L.G. Aldrovandi and F.A. Schaposnik, *Quantum mechanics in non(anti)commutative superspace*, JHEP **08** (2006) 081, arXiv:hep-th/0604197. B. Bagchi and A. Fring, *Minimal length in Quantum Mechanics and non-Hermitian Hamiltonian systems*, Phys. Lett. A **373** (2009) 4307, arXiv:0907.5354 [hep-th]; V.G. Kupriyanov, *A hydrogen atom in curved noncommutative space* J. Phys. A **46** (2013), 245303, arXiv:1209.6105 [math-ph]; O. Bertolami and P. Leal, *Aspects of phase-space noncommutative quantum mechanics*, Phys. Lett. B **750** (2015) 6, arXiv:1507.07722 [gr-qc]; T. Kanazawa, G. Lambiase, G. Vilas, and A. Yoshioka, *Noncommutative Schwarzschild geometry and generalized uncertainty principle*, Eur. Phys. J. C **79** (2019) 95.

M.R. Douglas and N.A. Nekrasov, *Noncommutative field theory*, Rev. Mod. Phys. **73** (2001) 977, arXiv:hep-th/0106048.

F. Delduc, Q. Duret, F. Gieres and M. Lefrançois, *Magnetic fields in noncommutative quantum mechanics*, J. Phys. Conf. Ser. **103** (2008) 012020, arXiv:0710.2239 [quant-ph].

N. Durov, S. Meljanac, A. Samsarov, Z. Škoda, *A universal formula for representing Lie algebra generators as formal power series with coefficients in the Weyl algebra*, J. Algebra **309** (2007) 318, arXiv:math/0604096.

See e.g. H. Leutwyler, *Chiral perturbation theory*, Scholarpedia **7** (2012) 8708.

R. Karplus and J. Schwinger, *A note on saturation in microwave spectroscopy*, Phys. Rev. **73** (1948) 1020; R.P. Feynman, *An operator calculus having applications in quantum electrodynamics*, Phys. Rev. **84** (1951) 108.

V. G. Kupriyanov, *Poisson gauge theory*, JHEP **09** (2021) 016, arXiv:2105.14965 [hep-th].

G. Gubitosi, F. Lizzi, J.J. Relancio, and P. Vitale, *Double quantization*, arXiv:2112.11401 [hep-th].

V.G. Kupriyanov, M.A. Kurkov and P. Vitale, *Poisson gauge models and Seiberg-Witten map*, in preparation.

V. G. Kupriyanov, *Recurrence relations for symplectic realization of (quasi)-Poisson structures*, J. Phys. A **52** (2019), 225204, arXiv:1805.12040 [math-ph].

D. Gurevich and P. Saponov, *Noncommutative geometry and dynamical models on $U(u(2))$ background*, J. Gen. Lie Theory Appl. **91** (2015), arXiv:1311.6231 [math.QA].

D. Gurevich and P. Saponov, *Noncommutative geometry on central extension of $U(u(2))$*, arXiv:2009.05807 [math.QA].

V.S. Fadin and R. Fiore, *Non-forward NLO BFKL kernel*, Phys. Rev. D **72**, 014018 (2005), arXiv:hep-ph/0502045.

C. Bender, *PT symmetry in quantum and classical physics*, World Scientific, 2019.