Economics of disagreement

financial intuition for the Rényi divergence*

Andrei N. Soklakov†

Disagreement is an essential element of science and life in general. Currently, the amount of disagreement between two probability distributions is quantified by highly abstract entropic measures such as the Rényi divergence. Despite their widespread use in science and engineering, such quantities lack numerical intuition and their axiomatic definitions contain no practical insight as to how the disagreement can be resolved. An economic approach addresses both of these problems by transforming disagreements into tangible investment opportunities. The Rényi divergence appears connected to the optimized performance of such investments. Optimization around individual opinions provides a social mechanism by which funds flow naturally to support a more accurate view. Such social mechanisms can help to resolve difficult disagreements (e.g., financial arguments concerning the global climate).

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†Strategic Development, Deutsche Bank.

The views expressed herein should not be considered as investment advice or promotion. They represent personal research of the author and do not necessarily reflect the view of his employers, or their associates or affiliates. Andrei.Soklakov@db.com, gmail.com.
Lack of accurate intuition is often cited as a scientific challenge, especially when interpreting probabilistic and statistical research. A popular technique for developing statistical intuition involves imagining a game of chance with well-defined financial outcomes. In 1956 Kelly used this technique to propose an intuitive interpretation of relative entropy [1]. He considered a growth-optimizing investor in a game with mutually exclusive outcomes (a “horse race”) and showed that the rate of return expected by such an investor is equal to the relative entropy which measures disagreement between the investor’s believed probabilities and the official odds. Effectively, Kelly showed that a growth-optimizing investor would expect on average to convert 1 bit of additional information into a 100% return on investment. Notwithstanding the mathematics, this interpretation has been vigorously criticized by Samuelson as dangerous on the grounds that most people are not growth-optimizing [2, 3]. Here we show that the connection between information and expected returns is in fact much deeper than previously thought. Relative to the Kelly information benchmark, variation in people’s risk aversion does cause a drop in expected returns. However, the amount of the drop is also information-driven: it is proportional to the absolute difference between the relative entropy and its celebrated generalization – the Rényi divergence [4]. Given the widespread use of the Rényi divergence in science and engineering, and because financial returns are easier to imagine, we expect our intuition to be useful in a broad range of fields. Financial intuition might also be useful in its own right as an instrument for raising financial support and accelerating the adoption of sound scientific developments.

A game of chance is defined by a random variable, \( X \), and a payoff function, \( F \). The value \( F(x) \) states the amount of reward associated with a particular outcome, \( x \), of the variable. This technical definition covers many practical situations. For instance, \( F(x) \) might denote the payout for medical insurance in the event of a diagnosis \( x \).

Consider an investor who is interested in finding the optimal shape of \( F \) so as to maximize their benefit. Following von Neumann and Morgenstern [5] we understand the benefit as the expected utility of the payoff

\[
E_b U[F] \overset{\text{def}}{=} \int b(x) U(F(x)) \, dx ,
\]

where \( U \) is the utility function and \( b \) is the investor-believed probability distribution for the variable \( X \). It is important to remember that both the probability distribution \( b \) and the utility function \( U \) reflect the investor’s personal understanding of the game.

When maximizing \( E_b U \) the investor is subject to a budget constraint. To state this constraint mathematically we need the ability to price the game. Assuming a very general setting which is used in the financial industry we write the fair price of \( F \) as the average

\[
\text{Price}[F] \overset{\text{def}}{=} \int F(x) \, m(x) \, dx ,
\]

where the positive-valued function \( m \) summarizes the relative prices of ensuring against every outcome \( x \) of the variable \( X \). In game-theoretic illustrations this is often stated in terms of “odds”; economists would recognize this as Arrow-Debreu prices. For our purposes, the important point to remember is that \( m \) is a given property of the market.
and the investor has no choice but to take it into account when understanding their budget. For simplicity we assume that \( m \) is a probability distribution (“fair odds”) and that \( \text{Price}[F] = 1 \).

The payoff \( F \) which achieves the maximum of \( E_b U[F] \) under the constraint \( \text{Price}[F] = 1 \) satisfies the payoff elasticity equation \[ \frac{d \ln F}{d \ln f} = \frac{1}{R}, \]

where \( f = b/m \) and \( R = -FU''_{FF}/U'_F \) is the Arrow-Pratt risk aversion. This equation is used in finance to produce information derivatives – financial instruments which are derived from all relevant information (including market-implied \( m \) and investor-believed \( b \) and \( R \)).

Kelly’s intuition concerns the important special case of \( R = 1 \). This is the famous case of the growth-optimizing investor which was first introduced by Bernoulli in 1738 \[7\]. Indeed, when \( R = 1 \) the utility function \( U(F) \propto \ln(F/\text{Price}[F]) + \text{const} \), so the investor is optimizing the expected logarithmic rate of return:

\[
E_b \text{Rate}[F] = \int b(x) \text{Rate}(F(x)) \, dx, \quad \text{Rate}(F(x)) \overset{\text{def}}{=} \ln \left( \frac{F(x)}{\text{Price}[F]} \right).
\]

Kelly studied the growth-optimizing investor in detail and showed that under some natural assumptions the investor’s expectation for the logarithmic rate of return is exactly the relative entropy \[1\]. The reader can verify this result independently by substituting \( R = 1 \) into Eq. (3) and computing

\[
\max_F E_b \text{Rate}[F] = E_b \text{Rate}[f] = \int b(x) \ln \frac{b(x)}{m(x)} \, dx.
\]

This is the mathematical essence of Kelly’s game-theoretic (financial) interpretation for the information rate.

Kelly’s interpretation depends on the investor to be growth-optimizing. According to Samuelson, this is a major weakness which limits (if not prevents) any use of Kelly’s interpretation in practice \[2, 3\].

In an entirely separate argument, Rényi considered the core mathematical properties of information and recognized relative entropy (the last expression in Eq. (5)) as a special case of a much larger class of information measures \[4\]. This class is spanned by linear combinations of the quantities

\[
D_\alpha(b||m) \overset{\text{def}}{=} \frac{1}{\alpha - 1} \ln \int b(x) \left( \frac{b(x)}{m(x)} \right)^{\alpha - 1} \, dx
\]

with different values of \( \alpha \). An individual \( D_\alpha(b||m) \) is called Rényi’s divergence of order \( \alpha \) of a probability distribution \( b \) from another distribution \( m \). The relative entropy is included in this definition as the limiting case \[4\]

\[
D_1(b||m) \overset{\text{def}}{=} \lim_{\alpha \to 1} D_\alpha(b||m) = \int b(x) \ln \frac{b(x)}{m(x)} \, dx.
\]
In what follows we show that Samuelson’s demands for a more general economic setup and Rényi’s generalization of the relative entropy are in fact a statement and a solution of the same economic problem which translates information content (captured as disagreement between distributions) into financial returns.

Let us consider an investor with an arbitrary constant relative risk aversion, \( R = \text{const} \). Unless \( R = 1 \), i.e. unless the investor is growth-optimizing, the growth rate expected by the investor will be smaller than the Kelly benchmark \( \text{(5)} \). The natural question to ask is how much smaller. Using Eq. \( \text{(3)} \) we derive the optimal payoff

\[
F(x) = \frac{f^{1/R}(x)}{\text{Price}[f^{1/R}]}.
\]  

(8)

By direct substitution into Eq \( \text{(4)} \) we compute

\[
E_b \text{Rate}[F] = \frac{1}{R} E_b \text{Rate}[f] + \frac{R-1}{R} D_{1/R}(b||m). \tag{9}
\]

As a side observation, the reader might be interested to note that the structure of this expression is rather general. In particular, by replacing the investor-believed distribution \( b \) with any other distribution \( p \) one can prove a much more general law with the exact same structure. Expanding \( E_p \text{Rate}[f] = D_1(p||m) - D_1(p||b) \) one can write this as

\[
E_p \text{Rate}[F] = \frac{1}{R} \left( D_1(p||m) - D_1(p||b) \right) + \frac{R-1}{R} D_{1/R}(b||m). \tag{10}
\]

In other words, one can talk about the investor-expected returns (\( p = b \)) or look at the actual realized returns (in which case \( p \) coincides with the actual distribution for \( X \)) or we can take the perspective of a totally independent observer who computes expectations using a very different distribution \( p \) and, in all of these cases, the effect of risk aversion on the financial performance of the growth-optimizing investor would follow the same universal law \( \text{(10)} \).

Coming back to the investor-expected returns, let us investigate the drop in the expected return of \( F \) relative to the growth-optimizing \( f \). Using the Kelly result, \( E_b \text{Rate}[f] = D_1(b||m), \) we compute from Eq. \( \text{(9)} \)

\[
E_b \text{Rate}[f] - E_b \text{Rate}[F] = \frac{R-1}{R} \left( D_1(b||m) - D_{1/R}(b||m) \right)
\]

\[
= \frac{|R-1|}{R} \left| D_1 - D_{1/R} \right|, \tag{11}
\]

where in the last equality we used the fact that \( D_\alpha \) is a nondecreasing function in \( \alpha \).

Equations \( \text{(10)} \) and \( \text{(11)} \) convert the abstract axiomatically-motivated measure of Rényi divergence into a much more intuitive measure of financial returns. This is the point where pure and simple mathematics turns into science about the real world so we need to be extra careful: we need to understand the limitations of the resulting economic intuition before it can be used in practice.

Using Samuelson’s critique as an inspiration, we demand, as a matter of principle, that economic intuition can only be based on realistic investors. Failing that, the researcher
might not be able to relate to the hypothetical investor’s experience correctly, so the resulting economic intuition might be flawed.

Examining the above derivations we see that, besides the information-theoretic definitions, Eq. (11) is a direct consequence of the payoff elasticity equation (3). By the principle of realistic investor, we require Eq. (3) to be tested against the objective records for its ability to explain the observed financial returns.

It turns out that the exact same usage of Eq. (3) as we employed in this paper does indeed explain the observed equity returns [8]. This includes both the forecasted and the realized returns.

Understanding equity returns gives us experimentally confirmed boundaries for risk aversion within which our arguments can be confidently applied. The values of $R$ corresponding to the forecasted returns were mostly in the range between 1 and 2.5 (see Fig. 2 of [8]). The range of $R$ explaining the realized returns was slightly wider: between 0.5 and 3 (see Fig. 4 of [8]). Here we should note that, unlike forecasted returns, realized returns are expected to overestimate the range for $R$. This is because real investors do make some mistakes in their forecasts. For example, we can find historical periods with negative realized returns. Rational investors would accept negative returns only if they had very low levels of risk aversion. In reality, however, the investors can enter a negative period unawares so their actual $R$ might not be as low as suggested by the data. By trying to explain all of the data (as was done in [8]) we assume that the investor foresaw the realized distribution of returns and invested in full knowledge of that distribution. This overestimates the range of $R$ that is necessary to explain the observations.

For completeness we note that the above ranges correspond to equity investors as observed over finite time periods (daily observations from 17 May 2000 to 27 April 2015 for the realized returns and monthly observations from September 2008 to March 2015 for the forecasted returns). Future investigations using Eq. (3) for different types of investors (not necessarily equity) and considering broader time periods will deepen our understanding of the possible ranges for $R$ and this, in turn, will provide better intuition over the Rényi parameter $\alpha$.

We promised the reader an intuitive economic intuition for the Rényi divergence. To see how this works, let us take $D_\alpha(b||m)$ as a function of $\alpha$ for some pair of probability distributions $b$ and $m$ (see, for example, Fig. 1A). The distributions themselves can be very difficult to illustrate (e.g. very high dimensional or even unknown). Our task is to understand intuitively the amount of disagreement between the distributions using nothing but the data in Fig. 1A. To this end we use Eq. (11) and transform the Rényi profile of Fig. 1A into the expected rate of return $E_b[\text{Rate}[F]]$ as a function of risk aversion (see Fig. 1B). The following story illustrates the benefits of this transformation.

Imagine an argument between two scientists, Maggie and Bob, about the distribution of outcomes for some experiment $X$. Maggie believes $m$ is the correct distribution while Bob thinks the real distribution is $b$. Even though the Rényi divergence was specifically designed as a measure of disagreement between distributions, Fig. 1A provides the two scientists with little practical intuition on the extent of their disagreement (not to mention any idea on how it can be resolved).
Maggie happens to be very rich so she decides to prove her point by offering a fair price \( \text{2} \) for any game regarding the experiment. This gives Bob a chance to defend his view by demonstrating a sizeable profit.

By transforming the data in Fig. 1A into financial returns (Fig. 1B) Bob understands the extent of his disagreement with Maggie. Specifically, he expects to make a return in the range between 4.8%-6.9% per run of the experiment\(^1\) (without taking uncomfortable risks, i.e. staying within normal bounds for risk aversion). These returns are very high. Given a statistically significant set of 100 runs, for example, Bob would expect his capital to grow between 120 to 986 times. The enormity of these returns should be obvious to anyone who has ever had a bank account.

Throughout history, mankind’s ability to plan and optimize was key to our survival. We should not be too surprised that this ability can produce strong statistical intuition. The real question is how we can expand and use this intuition in practice.

Regarding the expansion of the intuition beyond the Rényi divergence, the only suggestion we have at the moment is to explore the rich class of investors with non-constant relative risk aversion.

\(^1\)The actual returns are random but, in the long run, Bob can hope to achieve the expected growth of Fig. 1B almost surely by simple repetition, i.e. by reinvesting the actual returns from one run of the experiment into the next. This is, of course, assuming that Bob’s belief \( b \) turns out to be correct. In general, the long-term realized returns will follow an instance of Eq. \((10)\) where \( p \) is chosen to coincide with the correct (i.e. the realized) distribution.
Regarding practical applications, we have shown that framing statistical disagreements in terms of financial optimization quantifies the amount of disagreement in a scientifically sound way. We should also add that this approach can be useful for closing disagreements in a cooperative manner. It would be very interesting, for instance, to see what financial returns modern climatology can demonstrate in a game against the sceptics. We can set up such a game just as we did above for Maggie and Bob. All we would need are concrete examples of some relevant probability distributions (preferably with short time horizons to facilitate convergence via repetition).

In regards to resolving real disagreements, we should note that in the above example Maggie took on the extra responsibility of “making the market”. This was useful to simplify the arguments. In the most general case we might be dealing with multiple players each with their own believed distribution, risk aversion and a limited budget, and there might not be anybody among the players brave enough to make the market. In this case the market should form spontaneously (assuming, of course, everyone can see the opportunity). The corresponding distribution \( m \) would be computed as explained in Appendix V of Ref. [8]. For example, a group of growth-optimizing investors with views \( \{ b_i(x) \} \) and budgets \( \{ w_i \} \) will form a market with \( m(x) = \sum_i w_i b_i(x) / \sum_k w_k \). Whether the market forms spontaneously or is made by one of the players, equations (9) and (10) hold for every individual investor.

The financial attractiveness of the proposed optimized investments is directly related to the amount of genuine disagreement between participants. The optimization ensures the disagreement is utilized to its full potential (from the point of view of each individual participant). Following the optimization, failure to trade would imply that either the disagreement is not large enough (relative to other investment opportunities) or the disagreement itself is not genuine.

**References**

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