1. Introduction

At the heart of the axiomatic formulation of 1+1-dimensional topological field theory is the set of all surfaces with boundary assembled into a category. This category of surfaces has compact 1-manifolds as objects and smooth oriented cobordisms as morphisms. Taking disjoint unions gives a monoidal structure and a 1+1-dimensional topological field theory can be defined to be a monoidal functor \( E : \mathcal{S} \to \text{Vect}_\mathbb{C} \) normalised so that \( E(\text{cylinder}) = \text{identity} \). Here \( \text{Vect}_\mathbb{C} \) is the category of finite dimensional complex vector spaces and linear maps, with monoidal structure the usual tensor product. It is well known that to specify a 1+1-dimensional topological field theory is the same thing as specifying a finite dimensional commutative Frobenius algebra.

There is a natural generalisation of the category of surfaces, where one introduces a background space \( X \) and considers maps of surfaces to \( X \). Once again a monoidal category emerges. In this paper we explain how for a simply connected background space, monoidal functors from this category to \( \text{Vect}_\mathbb{C} \) can be interpreted in terms of Frobenius algebras with additional structure. The main result is Theorem 4.1. Although we use \( \mathbb{C} \) throughout, any other field will give the same results.

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2. Surface categories

There are a number of candidates for the title of surface category of a space, all requiring some care in their formulation. Firstly recall the definition of the category of surfaces which we denote by \( \mathcal{S} \) (see for example [1], [5]). Let \( C \) be a parametrised circle. There is an object \( S_n \) of \( \mathcal{S} \) for each natural number \( n \) consisting of an ordered set of \( n \)-copies of \( C \). Morphisms from \( S_m \) to \( S_n \) are pairs \( (\Sigma, \alpha) \) where \( \Sigma \) is a smooth oriented surface equipped with an orientation preserving homeomorphism \( \alpha : \partial \Sigma \to S_m^{\text{op}} \sqcup S_n \), where \( S_m^{\text{op}} \) means \( S_m \) with its opposite orientation. Two pairs are identified if there is a diffeomorphism \( T : \Sigma_1 \to \Sigma_2 \) satisfying \( \alpha_2 \circ T|_{\partial \Sigma} = \alpha_1 \). Composition of morphisms is by gluing surfaces via the maps \( \alpha \) (in fact because of the identifications of morphisms this is determined by a labelling of the boundary components and their orientations). Also \( \mathcal{S} \) has a monoidal structure corresponding to juxtaposition of objects and surfaces.

It is a common procedure (e.g. in topological sigma models, string theory etc.) to introduce a background manifold \( X \) and no longer consider abstract
surfaces but maps (of a suitable kind) from surfaces to \( X \). That one can assemble a category in this way is an idea of Segal and we refer to [3] for an early appearance of this idea. Of the possible variants available we require our surface categories to have a minimum of structure, in particular we can deform surfaces in \( X \) by homotopies relative to the boundary.

**Definition 2.1.** The homotopy surface category of \( X \), denoted by \( \mathcal{S}X \), is the category defined as follows:

- **Objects are pairs** \((S, s)\) where \( S \) is an object of \( \mathcal{S} \) and \( s: S \to X \) is a continuous function.

- **Morphisms from** \((S, s)\) **to** \((S', s')\) **are triples** \((\Sigma, \alpha, \sigma)\) where
  1. \( \Sigma \) is a smooth oriented surface
  2. \( \alpha: \partial\Sigma \to S \) is an orientation preserving homeomorphism and
  3. \( \sigma: \Sigma \to X \) is a continuous function such that \( \sigma|\partial\Sigma \circ \alpha^{-1} = s \sqcup s' \).

We identify \((\Sigma_1, \alpha_1, \sigma_1)\) with \((\Sigma_2, \alpha_2, \sigma_2)\) if there exists a diffeomorphism \( T: \Sigma_1 \to \Sigma_2 \) satisfying \( \alpha_2 \circ T|\partial\Sigma = \alpha_1 \) and \( \sigma_2 \circ T \simeq_{\partial\Sigma} \sigma_1 \) (here \( \simeq_{\partial\Sigma} \) means homotopy relative to the boundary).

It is sometimes useful to think of this as the loop world equivalent of the fundamental groupoid. Composition is by gluing using the maps \( \alpha \) for identification. \( \mathcal{S}X \) has a natural monoidal structure: define \((S \otimes S', s \otimes s')\) and \((\Sigma_1 \otimes \Sigma_2, \alpha_1 \otimes \alpha_2, \sigma_1 \otimes \sigma_2)\) by taking the maps \( s \otimes s' \) and \( \sigma_1 \otimes \sigma_2 \) to be the disjoint union of the two original maps on the juxtaposed components.

**Remark 2.2.** When \( X = pt \) there is a unique map \( \Sigma \to X \) for any surface \( \Sigma \) and we have \( \mathcal{S}X = \mathcal{S} \).

Let \( \text{Vect}_C \) be the category of finite dimensional complex vector spaces and linear maps, with monoidal structure the usual tensor product and recall:

**Definition 2.3.** A monoidal representation of a monoidal category \( \mathcal{C} \) is a functor \( F: \mathcal{C} \to \text{Vect}_C \) respecting the monoidal structure i.e. \( F(0) = C \) and \( F(A \otimes B) = F(A) \otimes_C F(B) \).

We will write \( \text{MonRep}_C(\mathcal{S}X) \) for the set of monoidal representations of \( \mathcal{S}X \). This is in fact a category where a morphism between \( F_1 \) and \( F_2 \) is a natural transformation of monoidal functors. It is this category for a simply connected space which this paper investigates.

Segal suggests that just as representations of path categories give rise to the geometry of vector bundles, so should the representations of surface categories give rise to interesting geometry involving the free loop space. For the simply connected case and monoidal representations we will write about this elsewhere.

### 3. \( A \)-Frobenius Algebras

There is a “folk” theorem (made precise by Abrams [4]) stating that there is a 1-1 correspondence between 1+1-dimensional topological field theories and finite dimensional commutative Frobenius algebras over \( \mathbb{C} \). Recall that a Frobenius algebra over \( \mathbb{C} \) is an algebra \( V \) over \( \mathbb{C} \) with unit for which there exists an isomorphism of \( V \)-modules \( V \cong V^* \). Equivalently there exists a linear form \( \theta: V \to \mathbb{C} \) whose kernel contains no non-trivial ideals. The
correspondence assigns the vector space $V = E(\text{circle})$ to a 1+1-dimensional topological field theory $E : S \to \text{Vect}_\mathbb{C}$ and the genus zero surface with 2+1 boundary circles (the pair of pants) gives the algebra multiplication and the cap gives the trace map $\theta$.

In order to generalise this to monoidal representations of $S^X$ for simply connected $X$ we need the notion of an $A$-Frobenius algebra, where $A$ is a finitely generated abelian group.

**Definition 3.1.** Let $A$ be a finitely generated abelian group. An $A$-Frobenius algebra over $\mathbb{C}$ is a commutative $\mathbb{C}[A]$-algebra $V$ with unit, for which there exists an isomorphism of $V$-modules $V \cong V^*$, where $V^* = \text{Hom}_\mathbb{C}(V, \mathbb{C})$.

**Remark 3.2.** Alternatively one can think of an $A$-Frobenius algebra over $\mathbb{C}$ as a commutative Frobenius algebra $V$ together with a representation $i : A \to GL(V)$ such that

$$i(a)(x \cdot y) = i(a)x \cdot y = x \cdot i(a)y$$

where $a \in A$, $x, y \in V$ and $x \cdot y$ is the algebra product.

$A$-Frobenius algebras form a category which we will denote $\text{FrobAlg}_{\mathbb{C}[A]}$. Morphisms are taken to be $\mathbb{C}[A]$-algebra isomorphisms preserving the form $\theta$.

**Example 3.3.** One of the first examples of a Frobenius algebra is the group ring $\mathbb{C}[H]$ for a finite abelian group $H$ where the Frobenius map is given by $\theta = [1]^*$. Here $\{[h]\}_{h \in H}$ is a basis for the vector space $\mathbb{C}[H]$, and $\{[[h]]^*\}$ the dual basis of $\mathbb{C}[H]^*$. For details see Abrams [4]. Now let $A$ be a subgroup of $H$ then $\mathbb{C}[H]$ is an $A$-Frobenius algebra under the natural action of $A$ i.e. for $a \in A$

$$i(a)(\sum_{h \in H} \alpha_h [h]) = \sum_{h \in H} \alpha_h [ah]$$

**Example 3.4.** Another source of Frobenius algebras is the cohomology ring $H^*(M)$ of a compact manifold where Poincaré duality ensures the Frobenius structure. Quantum cohomology also provides Frobenius algebras. Allowing $V$ to be infinite dimensional it is known that $QH^*(M)$ of a symplectic manifold is a Frobenius algebra, but it is also naturally a $\Gamma$-Frobenius algebra, where $\Gamma$ is the image of the Hurewicz homomorphism into integral homology modulo torsion. Here we are taking $QH^*(M) \cong H^*(M) \otimes \Lambda$ where $\Lambda$ is the Novikov ring associated to the homomorphism $\omega : \Gamma \to \mathbb{Z}$ induced by the symplectic form (see McDuff and Salamon [2] for details). Using the notation of [2] an element $x \in QH^*(M)$ may be written in the form

$$x = \sum_{b \in \Gamma} x_b e^{2\pi ib}$$

and the $\Gamma$ action is given by

$$i(a)x = \sum_{b \in \Gamma} x_b e^{2\pi i(b+a)}$$
4. An equivalence of categories

For the rest of the paper $X$ will be simply connected. The main theorem of the paper is:

**Theorem 4.1.** Let $X$ be a simply connected topological space. There is an equivalence of categories

$$\text{MonRep}_\mathbb{C}(\mathcal{S}X) \leftrightarrow \text{FrobAlg}_\mathbb{C}[\pi_2X]$$

In order to prove this result we need to first cut down the size of $\mathcal{S}X$ by considering its skeleton, that is we identify all isomorphic objects of the category to obtain a new equivalent category.

**Lemma 4.2.** For $X$ simply connected, $(S, s) \cong (S', s')$ if and only if $S = S'$.

**Proof.** Suppose that $S = S'$, then since $X$ is simply connected $s \simeq s'$, that is we have a homotopy $S \times I \to X$ agreeing with $s$ and $s'$ on the boundary. Thus taking $\Sigma = S \times I$ with the obvious identifications with $S$ at the boundary we see we have a morphism from $(S, s)$ to $(S', s')$. It is clear this morphism is invertible and hence $(S, s) \cong (S', s')$.

Conversely, suppose that $(S, s) \cong (S', s')$ via a morphism $(\Sigma, \alpha, \sigma)$ with inverse $(\Sigma^*, \alpha^*, \sigma^*)$. Then the composition $(\Sigma, \alpha, \sigma)(\Sigma^*, \alpha^*, \sigma^*)$ must be identified with the identity morphism on $(S, s)$. In particular the glued surface $\Sigma \Sigma^*$ must be diffeomorphic to $S \times I$ in which case $\Sigma$ and $\Sigma^*$ must be made up of cylinders and so $S = S'$.

We chose a representative of each isomorphism class as follows. Define $c_n: S_n = \coprod_n S^1 \to X$ to be the unique collapse map to a chosen basepoint of $X$. Define $\mathcal{T}X$ to be the full sub category of $\mathcal{S}X$ with set of objects $\{(S_n, c_n)\}$. By the above lemma $\mathcal{T}X$ is the skeleton of $\mathcal{S}X$.

Observe that $\mathcal{T}X$ has a natural monoidal structure induced by that on $\mathcal{S}X$. So we have:

**Lemma 4.3.** There is an equivalence of monoidal categories $\mathcal{S}X \leftrightarrow \mathcal{T}X$.

For convenience we will denote the object $((S_n, c_n))$ simply as $n$. Note that using the monoidal structure $n = 1 \otimes 1 \otimes \cdots \otimes 1$.

**Lemma 4.4.** Let $\Sigma$ be an arbitrary surface of genus $l \geq 0$ consisting of $d$ connected components (closed or with boundary). Pick a base point $*$ on $X$. If $X$ is simply connected then there is a one-to-one correspondence

$$[\Sigma, \partial \Sigma; X, *] \leftrightarrow (\pi_2X)^d$$

**Proof.** First assume $\Sigma$ is closed, connected of genus $g$. There is a cofibration

$$S^1 \xrightarrow{\chi} \bigvee_{2g} S^1 \xrightarrow{} \Sigma$$

where $\chi$ is the “commutator” map. Applying $\text{Map}_*(-, X)$ we get a cofibration

$$\text{Map}_*(\Sigma, X) \xrightarrow{} (\Omega X)^{2g} \xrightarrow{\chi} \Omega X$$
The long exact homotopy sequence gives

\[ \cdots \to (\pi_1 \Omega X)^{2g} \xrightarrow{\chi_*} \pi_1 \Omega X \to [\Sigma, X]_* \to \pi_0(\Omega X)^{2g} \to \cdots \]

\(\chi_*\) is trivial since \(\Omega X : \Omega(\Omega X)^{2g} \to \Omega^2 X\) is null by homotopy commutativity. Since \(\pi_1 X = \pi_0 \Omega X\) is trivial it follows that the middle map is an equivalence. Moreover for simply connected \(X\) we have \([\Sigma, X]_* = [\Sigma, X]\) so \(\pi_2 X = [\Sigma, X] = [\Sigma, \partial \Sigma; X, \ast]\).

Next assume that \(\Sigma\) is connected, genus \(g\) and \(\partial \Sigma \neq \emptyset\) with \(k\) boundary components. Let \(\tilde{\Sigma}\) be the surface obtained from \(\Sigma\) by sewing in \(k\) discs. Let \(p_i\) be a point in the \(i\)'th disc. There is a fibration

\[ \text{Map}(\tilde{\Sigma}, \{p_1, \ldots, p_k\}; X, \ast) \to \text{Map}(\tilde{\Sigma}, X) \to X^k \]

whence an exact sequence

\[ (\pi_1 X)^k \to [\Sigma, \{p_1, \ldots, p_k\}; X, \ast] \to [\tilde{\Sigma}, X] \to (\pi_0 X)^k \]

The extremities are trivial so \([\tilde{\Sigma}, \{p_1, \ldots, p_k\}; X, \ast] = [\tilde{\Sigma}, X]\). Thus

\[ [\Sigma, \partial \Sigma; X, \ast] = [\tilde{\Sigma}, \{p_1, \ldots, p_k\}; X, \ast] = [\tilde{\Sigma}, X] = \pi_2 X. \]

Finally for a general surface each of its \(d\) connected components (closed or not) corresponds to an element of \(\pi_2 X\) by the above. \(\qed\)

In view of this discussion we will denote a morphism \((\Sigma, \alpha, \sigma)\) of \(T X\) by \(\Sigma_g\) where \(g = (g_1, \ldots, g_d) \in (\pi_2 X)^d\) is the sequence of elements of \(\pi_2 X\) given by the above Lemma. The maps \(\alpha\) will be suppressed. The unit of \(\pi_2 X\) will be denoted by \(1\) and sequence of \(1\)'s by \(1 \in (\pi_2 X)^d\).

**Remark 4.5.** In other words morphisms of \(T X\) are morphisms of \(S\) with a labelling of each component from \(\pi_2 X\). In light of this it is easy to see that if \(X\) is 2-connected then there is an equivalence of categories \(\mathfrak{S}X \leftrightarrow S\).

**Remark 4.6.** It is tempting to think there might be a monoidal equivalence between \(T X\) and \(S \times \pi_2 X\). This cannot be the case in general: any such equivalence takes \(c_0\) to \((S_0, 1)\) and would induce an equivalence between the monoids of endomorphisms of these objects, however these can be seen to be different.

**Lemma 4.7.** Let \(\Sigma\) and \(\Sigma'\) be surfaces. Then

\[ [\Sigma']_g [\Sigma]_h = ([\Sigma']_f )_{f_i} \]

where \(f_i\) is the product of all \(g_j\)'s and \(h_k\)'s forming the connected component \(f_i\) of \(\Sigma' \Sigma\).

**Proof.** Gluing of two connected components corresponds to multiplication in \(\pi_2 X\), and for the \(i\)'th component of \(\Sigma' \Sigma\) we multiply all components of \(\Sigma'\) and \(\Sigma\) forming that component.
In particular for connected morphisms $\Sigma_{g_1}$ and $\Sigma_{g_2}$ (where $g_i \in \pi_2X$),

$$\Sigma'_{g_1}\Sigma_{g_2} = (\Sigma'\Sigma)_{g_1g_2}$$

Now let

$$E : \mathcal{T}X \to \text{Vect}_\mathbb{C}$$

be a monoidal representation of $\mathcal{T}X$. $E(n)$ is determined by $V := E(1)$ since

$$E(n) = E(1 \otimes \cdots \otimes 1) = E(1) \otimes \cdots \otimes e(1) = V^\otimes n$$

To each morphism $\Sigma \in \mathcal{T}X(m, n)$, $E$ assigns a linear map

$$E(\Sigma) : V^\otimes m \to V^\otimes n$$

**Lemma 4.8.** A monoidal representation $E : \mathcal{T}X \to \text{Vect}_\mathbb{C}$ induces a Frobenius algebra structure on $E(1)$.

**Proof.** Let $\mathcal{T}^1X$ be the subcategory of $\mathcal{T}X$ with objects those of $\mathcal{T}X$ and morphisms

$$\{\Sigma_{\bot} : \Sigma \in \mathcal{S}\} \subset \text{morphisms of } \mathcal{T}X$$

Note that the morphisms of $\mathcal{T}^1X$ are closed under composition by Lemma 4.7. The correspondence $\Sigma \leftrightarrow \Sigma_{\bot}$ then induces an equivalence of categories $\mathcal{S} \leftrightarrow \mathcal{T}^1X$. The composition $\mathcal{S} \leftrightarrow \mathcal{T}^1X \to \mathcal{T}X \to \text{Vect}_\mathbb{C}$ gives a Frobenius algebra structure on $E(1)$ by Proposition 13 in [1].

The remainder of this section is devoted to the proof of Theorem 4.1.

First we see that a monoidal representation $E : \mathcal{T}X \to \text{Vect}_\mathbb{C}$ gives rise to a $\pi_2X$-Frobenius algebra. Let $V = E(1)$ which courtesy of Lemma 4.8 is a Frobenius algebra. Denote by $I_g \in \mathcal{T}X(1, 1)$ the cylinder labelled with $g \in \pi_2X$. Define

$$i : \pi_2X \to GL(V)$$

by

$$i(g) = E(I_g)$$

Writing $i_g$ for $i(g)$, note that $i_g$ is indeed invertible since

$$E(I_g)E(I_{g^{-1}}) = E(I_gI_{g^{-1}}) = E(I_1) = \text{id}_V.$$
The property
\[(1) \quad i_g(x \cdot y) = i_g x \cdot y = x \cdot i_g y.\]
follows from recalling that the pair-of-pants induces the algebra structure on \(V\).

Moreover a natural transformation of monoidal functors induces a morphism of \(\pi_2\) Frobenius algebras. Consider a monoidal natural transformation \(\phi\) between \(E_1\) and \(E_2\) which by definition assigns to object \(n\) a map \(\phi_n : E_1(1)^\otimes n \to E_2(1)^\otimes n\) such that every morphism \(\Sigma_g\) gives a commutative diagram

\[
\begin{array}{ccc}
E_1(1)^\otimes m & \xrightarrow{\phi_m} & E_2(1)^\otimes m \\
\downarrow & & \downarrow \\
E_1(\Sigma_g) & \xrightarrow{\phi_n} & E_2(\Sigma_g)
\end{array}
\]

By the definition of a monoidal functor we have \(\phi_n = \phi_1^\otimes n\) and it follows from [1] that \(\phi_1\) is a Frobenius algebra isomorphism. \(\pi_2\) linearity follows from the commutativity of

\[
\begin{array}{ccc}
E_1(1) & \xrightarrow{\phi_1} & E_2(1) \\
E_1(I_g) & \xrightarrow{\phi_1} & E_2(I_g)
\end{array}
\]

We have thus defined a functor
\[F : \text{MonRep}_\mathbb{C}(TX) \to \text{FrobAlg}_\mathbb{C}[\pi_2 X].\]

Now consider a Frobenius algebra \(V\) together with a representation \(i\) of \(\pi_2\) on \(V\) satisfying [1]. To such a Frobenius algebra we wish to assign a monoidal representation \(E\) of \(TX\). We will set \(E(1) = V\) and define \(E(\text{pair-of-pants}) : V \otimes V \to V\) to be the multiplication in \(V\). Letting \(I_g^m = I_{g_1} \otimes \cdots \otimes I_{g_m}\) be a morphism consisting of \(m\) straight cylinders such that \(\prod g_i = g\) \((g, g_i \in \pi_2 X)\), define \(E(I_g^m) = i_{g_1} \otimes \cdots \otimes i_{g_m}\), recalling that \(i_{g_k} \in GL(V)\). To define \(E\) for an arbitrary surface we first show that the entire category \(TX\) can be recovered from \(T^1 X\) and \(\{I_h\}_{h \in \pi_2 X}\) (by composition and monoidal structure).
First assume $\Sigma_g \in TX(m,n)$ is a connected surface and not a morphism to the empty manifold (that is $n > 0$). Then the following diagram commutes

![Diagram]

If $m > 0$ we have similarly

![Diagram]

For a closed connected surface $\Sigma_g$ ($n = m = 0$), capping the cylinder $I^1_g$ suffices; let $D, E$ be the discs to and from the empty manifold respectively and $\overline{\Sigma}_g$ be the morphism in $TX(1,1)$ obtained from $\Sigma_g$ by removing a disc at each end. We have

![Diagram]

It follows from this discussion that any surface $\Sigma_g \in TX(m,n)$ can be decomposed (non-uniquely) into surfaces in $TX$ and $\{I^m_g\}_{g \in \pi_2 X}$. Using such a decomposition we can define a map

$$E(\Sigma_g) : V^{\otimes m} \to V^{\otimes n}$$

A priori this depends on the decomposition, but as we now see it is in fact independent of that choice. The hard work is contained in the proof that $\text{MonRep}_C(S) \leftrightarrow \text{FrobAlg}_C$ (see [1]) where it is shown that any decomposition of a surface into the five basic surfaces is independent of the slicing. We need only in addition to know that the map $E(\Sigma_g)$ above is independent of the labellings of the components by $\pi_2 X$. This follows from the fact that $i_g(x \cdot y) = i_g x \cdot y = x \cdot i_g y$.

Furthermore, a morphism $\phi$ of $\pi_2 X$-Frobenius algebras induces a monoidal natural transformation by setting $\phi_n = \phi^{\otimes n}$. The discussion above and $\pi_2 X$-linearity ensure we get a monoidal natural transformation.

So we have a functor

$$G : \text{FrobAlg}_{C[\pi_2 X]} \to \text{MonRep}_C(TX)$$

It is easy to see that $F \circ G = 1$ and $G \circ F = 1$ so there is an isomorphism of categories

$$\text{MonRep}_C(TX) \cong \text{FrobAlg}_{C[\pi_2 X]}$$

Finally using Lemma 4.3 we get the desired equivalence

$$\text{MonRep}_C(\mathcal{S}X) \leftrightarrow \text{FrobAlg}_{C[\pi_2 X]}$$

completing the proof of the main theorem.
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