The sequence of ideas in a re-discovery of
the Colombeau algebras

Andre Gsponer
Independent Scientific Research Institute
Oxford, OX4 4YS, England

SEQUEN.10 October 6, 2008

Abstract

This is a gentle introduction to Colombeau nonlinear generalized func-
tions, a generalization of the concept of distributions such that distributions
can freely be multiplied. It is intended to physicists and applied mathemati-
cians who prefer a ‘step-by-step approach’ to a ‘top-down indoctrination.’

No particular prerequisite knowledge is necessary — and in less than one hour you should know everything you need to know and were afraid to ask about Colombeau algebras and their applications in physics...

A selected bibliography is appended, giving examples of applications to partial differential and wave equations, electrodynamics, hydrodynamics, general relativity, and quantum field theory.

The goal of this tutorial is to lead the reader to rediscover by himself the key ideas which led Colombeau to define the proper generalization of the concept of distributions such that multiplication is always possible and meaningful.

The emphasis is on concepts and methods, and the intent is to convince the reader that working with Colombeau nonlinear generalized functions (in short, \( \mathcal{G} \)-functions), which can be differentiated and multiplied freely, is not more comp-
plicated than working with the familiar \( C^\infty \)-functions.

Since everything is self-contained and kept simple there are only few references in the text. On the other hand, a selected bibliography with references to major publications on the subject is appended at the end.

While Colombeau’s seminal books \([4, 5]\) are still highly valuable, the most recent comprehensive textbook is \([11]\). Short summaries of the main features of the Colombeau theory are included in most publications cited in the bibliography. Furthermore, an alternative primer on Colombeau algebras is given in \([12]\).

\(^1\)The adjective ‘nonlinear’ emphasizes that Colombeau generalized functions form an algebra.
1 Regular and irregular distributions

The discovery of the Colombeau algebras is certainly one of the great events of the Twentieth Century history of mathematics. To understand how it came about, let us start from another great invention, that of the theory of distributions by Bochner, Sobolev, Mikusinski, and Schwartz. Indeed, whereas a regular distribution is a functional \( \phi(T) \) having the representation

\[
\phi(T) := \int dx \, \phi(x) T(x), \quad \forall T(x) \in \mathcal{D},
\]

(1.1)

where \( \phi(x) \) is a locally integrable function\(^2\) and \( T(x) \) a ‘test’ function\(^3\) there is no such representation for the Dirac ‘function’ \( \delta(x) \) which is defined by the functional

\[
\delta(T) := T(0), \quad \forall T(x) \in \mathcal{D}.
\]

(1.2)

Thus, before the theory of such singular distributions was invented, the only thing that could be done was to write, symbolically,

\[
\delta(T) = \int dx \, \delta(x) T(x), \quad \forall T(x) \in \mathcal{D},
\]

(1.3)

and to refer to the definition (1.2) for the interpretation of (1.3).

2 The abstract and sequential views of distributions

Schwartz showed that \( \delta(x) \) can be interpreted as an element of the space \( \mathcal{D}' \) of continuous linear functionals on \( \mathcal{D} \), and its derivatives defined as the derivatives of these functionals. That is, if \( \gamma \in \mathcal{D}' \) is any distribution, its derivatives in the ‘distributional sense’ are such that, \( \forall T \in \mathcal{D} \) and \( D^n = \partial^n \partial x^n \),

\[
\int dx \, (D^n \gamma)(x) \, T(x) = (-1)^n \int dx \, \gamma(x) \, (D^n T)(x).
\]

(2.1)

\(^2\)In simple words, a function is locally integrable if it is integrable on every compact set.

\(^3\)\( \mathcal{D} \) is the space \( \mathcal{D}(\Omega) \) of \( \mathcal{C}^\infty \) functions with compact support on an open subset \( \Omega \subset \mathbb{R} \). For simplicity of notation we write \( \mathcal{C} \), \( \mathcal{C}^m \), and \( \mathcal{C}^\infty \) for the continuous, \( m \)-times continuously differentiable, and respectively smooth functions with compact support on \( \Omega \). We similarly write \( \mathcal{C}_p \) for the piece-wise continuous functions. Then \( \mathcal{C}^\infty \subset \mathcal{C}^m \subset \mathcal{C}^0 = \mathcal{C} \subset \mathcal{C}_p \). We also tacitly assume that all integrations are over \( \mathbb{R} \), and that all functions are extended to \( \mathbb{R} \) by setting them equal to zero outside of \( \Omega \). Finally, we set \( \mathbb{N}_0 = \{0, \mathbb{N}\} \).
Alternatively, following Mikusinski, (1.3) can be written as the weak limit of a sequence of $C^\infty$ functions $\delta_\epsilon$, that is,

$$
\delta(T) := \lim_{\epsilon \to 0} \int dx \, \delta_\epsilon(x) T(x) = T(0), \quad \forall T(x) \in D. \quad (2.2)
$$

Indeed, if $\delta_\epsilon(x)$ is any family of functions

$$
\delta_\epsilon(x) = \rho_\epsilon(x) := \frac{1}{\epsilon} \rho\left(\frac{x}{\epsilon}\right), \quad (2.3)
$$

where $\epsilon \in ]0, 1[$ is a parameter, and $\rho$ taken in the set

$$
A_0 := \left\{ \rho(x) \in S, \quad \text{and} \quad \int dx \, \rho(x) = 1 \right\}, \quad (2.4)
$$

making the change of variable $x = \epsilon y$ and taking the limit, it comes

$$
\int dx \, \delta(x) T(x) := \lim_{\epsilon \to 0} \int dy \, \rho(y) T_\epsilon(y) = T(0). \quad (2.5)
$$

Returning to equation (2.3) one can observe that the sequence $\delta_\epsilon$ representing the $\delta$-distribution can actually be written as the convolution

$$
\delta_\epsilon(x) = \int dy \, \rho_\epsilon(x - y) \delta(y) = \rho_\epsilon(x), \quad (2.6)
$$

provided the symbol $\delta$ inside the integral is interpreted according to its functional definition (1.2).

This method is general: It can be proved that convoluting a regular or singular distribution with any $\rho_\epsilon$ provides a representative sequence

$$
\gamma_\epsilon(x) := \rho_\epsilon(x) * \gamma(x) = \int dy \, \rho_\epsilon(x - y) \gamma(y), \quad (2.7)
$$

of that distribution. As $\gamma_\epsilon(x) \in C^\infty$ this process of generating a smooth representative of $\gamma(x) \in D'$ is called a regularization, and the regularizing functions defined by (2.4) are termed regularizers or mollifiers. Thus, if $\gamma$ is any regular or singular distribution, (1.1) can be written

$$
\gamma(T) := \lim_{\epsilon \to 0} \int dx \, \gamma_\epsilon(x) T(x), \quad \forall T(x) \in D, \quad (2.8)
$$

$^{4}$S is the space of $C^\infty$ functions with steep descent, i.e., such that $f(x) \in S$ and its derivatives decrease more rapidly than any power of $1/|x|$ as $x$ tends to infinity. In distribution theory one usually takes $\rho \in D \subset S$ because this enables to deal with distributions with non-compact support unrestrictedly.
and \( \gamma \) is then interpreted as the equivalence class of the weakly convergent sequences of the smooth functions \( \gamma_\epsilon \) modulo weak zero-sequencies.

In comparison to Schwartz’s abstract theory, the advantages of Mikusinski’s sequential view are that it provides explicit representations for the distributions, and that their derivatives are simply obtained by differentiating the representative sequencies.

3 Schwartz’s local structure theorem

A particularly important contribution of Laurent Schwartz is the formulation of his local structure theorem stating that “any distribution is locally a partial derivative of a continuous function” [1, Theorems XXI and XXVI]:

**Theorem 1** Let \( \mathcal{D}'(\Omega) \) be the space of distributions on the compact set \( \Omega \). Then every \( \gamma \in \mathcal{D}' \) is of the form

\[
\gamma(x) = \sum_n D^n g_n(x),
\]

(3.1)

where \( n \in \mathbb{N}_0 \), and the support of each \( g_n \in \mathcal{C}(\Omega) \) is contained in an arbitrary compact neighborhood \( K \subset \Omega \).

For example, Dirac’s function \( \delta(x) \) is generated by the second distributional derivative of the absolute value \( |x| \in \mathcal{C}^0 \), i.e., \( \delta(x) = 1/2 \ D^2 |x| \).

Differentiation induces therefore the following remarkable cascade of relationships: continuously differentiable functions \( \rightarrow \) continuous functions \( \rightarrow \) distributions. This gives a unique position to Schwartz distributions because they constitute the smallest space in which all continuous functions can be differentiated any number of times. For this reason it is best to reserve the term ‘distribution’ to them, and to use the expression ‘generalized function’ for any of their generalizations. On the other hand, classical generalizations of the concept of function such as piece-wise continuous functions, measures, Cauchy and Hadamard finite-parts of integrals, etc., are all distributions.

For application in physics Schwartz’s structure theorem is of great significance because it asserts that singular distributions do not come ‘from nowhere,’ but derive from a generating function \( g(x) \in \mathcal{C}^0 \). For example, the classical electron charge distribution originates from the absolute value in the definition of the
Coulomb potential, i.e., $\phi = e/|\vec{r}|$, and due consideration to this fact leads to a distributionally consistent introduction of point charges and dipoles in classical electrodynamics [25, 27, 28, 29, 30].

4 Schwartz’s multiplication impossibility theorem

Distributions generalize ordinary functions, which can be regarded as trivial cases of distributions. They enjoy most of the properties of $C^\infty$ functions (e.g., they can be differentiated any number of time) with the notable exception of multiplication.

For example, if the product of distributions is defined in the most natural way, i.e., by multiplying representative sequences, the square of the $\delta$-function corresponds to $(\delta^2)\epsilon(x) = (\delta\epsilon)^2(x) = \rho^2(x)$. Then, when evaluated on a test function $T \in \mathcal{D}$ according to (2.8), we get,

$$\int dx \, \delta^2(x)T(x) := \lim_{\epsilon \to 0} \int dx \, \rho^2_\epsilon(x)T(x) = \lim_{\epsilon \to 0} \frac{T(0)}{\epsilon} \int dy \, \rho^2(y) = \infty, \quad (4.1)$$

which implies that $\delta^2$ is not a distribution. Many mathematicians have of course tried to define a consistent product of distributions. But these efforts only confirmed that there is no multiplication on all of $\mathcal{D}'$ which still has values in $\mathcal{D}'$, unless some essential properties are given up. For instance, in any formulation such that the usual relations $x \cdot 1/x = 1$ and $x \cdot \delta(x) = 0$ hold, associativity leads to the contradiction

$$(x \cdot \frac{1}{x}) \cdot \delta(x) = \delta(x), \quad \text{whereas} \quad \frac{1}{x} \cdot (x \cdot \delta(x)) = 0. \quad (4.2)$$

The goal therefore shifted towards finding an algebra $\mathcal{G}$ of generalized functions containing the distributions and preserving most of the desirable properties of ordinary functions. But even that less ambitious goal turned out to be quite difficult. In particular, there are many options and it is not possible to know a priori which ‘essential properties’ should be preserved. For instance, possibly inspired by the cardinal position of continuous functions in his structure theorem (3.1), Laurent Schwartz was particularly attached to the idea that these functions should have a similar position in $\mathcal{G}$. He therefore postulated a set of minimum requirements which can be phrased as follows

$\mathcal{G}$ The differential algebra $\mathcal{G}$ is associative and commutative. Its elements are written $[u]$ when it is useful to emphasize that $u \in \mathcal{G}$.

$^5$See for example [11] p.6.]
1. The space of distributions $\mathcal{D}'$ is linearly embedded into $\mathcal{G}$, and the function $f(x) \equiv 1$ is the unit element for their product ‘$\circ$’ in $\mathcal{G}$, i.e., $\forall \gamma \in \mathcal{D}'$, there is an embedding $\mathcal{D}' \rightarrow \mathcal{G}$, $\gamma \mapsto [\gamma]$, and $[1] \circ [\gamma] = [\gamma]$;

2. There exists a derivation operator $D : \mathcal{G} \rightarrow \mathcal{G}$ that is linear and satisfies the Leibniz rule, i.e., $\forall u, v \in \mathcal{G}$, $D(u \circ v) = (Du) \circ v + u \circ (Dv)$;

3. $D$ restricted to $[\mathcal{D}']$ is the usual partial derivative consistent with the integration by parts formula (2.1);

4. The product of two continuous functions embedded in $\mathcal{G}$ coincides with the usual pointwise product ‘.’ in $\mathcal{C}$, i.e., $\forall f, g \in \mathcal{C}$, $[f] \circ [g] = f \cdot g$.

Unfortunately, on the basis of simple counter-examples, it is easy to show that there is no associative and commutative differential algebra $\mathcal{G}$ satisfying the requirements 1–4. For example, they lead to the conclusion $D^2 |x| = 0$, whereas, as recalled above, $D^2 |x| = 2\delta(x)$ in distribution theory. This is the famous Schwartz impossibility theorem of 1954.

5. Colombeau’s breakthrough

It was only in 1983 that Jean-François Colombeau was able to show in a truly satisfactory manner that it is actually possible to construct associative and commutative algebras satisfying 1–3, provided 4 is replaced by

4’. The product of two $C^\infty$ functions embedded in $\mathcal{G}$ coincides with the usual pointwise product ‘.’ in $C^\infty$, i.e., $\forall f, g \in C^\infty$, $[f] \circ [g] = f \cdot g$.

Therefore, since $C^\infty \subset C$, it was by relaxing the requirement 4 that it became possible to move forward: As $C^\infty$ functions have much more powerful properties than continuous functions in general, e.g., Taylor’s theorem with remainder, the problem became manageable.

Original discoveries of Colombeau were made in a different context, and arose from more abstract considerations. But their success can be traced to the emphasis given to $C^\infty$ rather than to continuous functions in general, an emphasis which may have a deep physical significance.
6 The embedding space

Colombeau’s axiom [4'] combined with axioms [1] – [3] implies that $G$ contains $C^\infty$ as a differential subalgebra: This opens the way to the possibility that $G$ could be similarly contained in a larger differential algebra $E$ such that its elements would be $C^\infty$ in the variable $x$. Since the mollified sequences $\gamma_\epsilon(x)$ representing the distributions are precisely $C^\infty$ in the variable $x$, this suggests to define $E$, the embedding space, as the set of maps,
\[
E := \left\{ (f_\epsilon) : \mathcal{A}_q \times \Omega \to \mathbb{R}, \quad (\eta, x) \mapsto (f_\epsilon)(\eta, x) \right\}, \tag{6.1}
\]
which are $C^\infty$ functions in the variable $x \in \Omega$ for any given Colombeau mollifier $\eta \in \mathcal{A}_q$, where $\mathcal{A}_q \subset \mathcal{A}_0$ remains to be specified, and which depend on the parameter $\epsilon \in [0, 1]$ through the scaled mollifier
\[
\eta_\epsilon(x) := \frac{1}{\epsilon} \eta \left( \frac{x}{\epsilon} \right). \tag{6.2}
\]
Obviously, $E$ is an associative and commutative differential algebra with unit $(\eta, x) \mapsto 1$ with respect to pointwise multiplication. It contains $C^\infty$ as the subset of the maps (6.1) which do not depend on $\eta$, i.e., $(f_\epsilon)(x) \equiv f(x)$.

The distributions $f \in \mathcal{D}'$ are then embedded in $E$ as the convolutions
\[
(f_\epsilon)(x) := \eta_\epsilon(-x) * f(x) = \int \frac{dy}{\epsilon} \eta \left( \frac{y-x}{\epsilon} \right) f(y) = \int dz \; \eta(z) \; f(x+\epsilon z), \tag{6.3}
\]
where, in order to define $G \subset E$, the Colombeau mollifiers $\eta \in \mathcal{A}_q$ may need to have specific properties in addition to those implied by (2.4). In particular, since $C^\infty \subset \mathcal{D}'$ there are two distinct embeddings of $C^\infty$ in $E$: Its embedding by (6.3) as a subset of $\mathcal{D}'$, and its direct inclusion by (6.1) according to the maps $(f_\epsilon)(x) \equiv f(x)$. To be consistent with axiom [4'], the mollifiers $\eta \in \mathcal{A}_q$ have thus to be such that $[(f_\epsilon)_\epsilon](x) = [f](x) = f(x)$ for all $f \in C^\infty$.

\[\text{The notation } (f_\epsilon)_\epsilon \text{ where } 0 < \epsilon < 1, \text{ which will be later abbreviated as } f_\epsilon, \text{ emphasizes that } (f_\epsilon)_\epsilon \text{ is an element of } E \text{ rather than a usual representative sequence (2.7).}\]

\[\text{This definition due to Colombeau differs by a sign from the usual definition (2.7) of regularization.}\]


7 Embedding of \( C^\infty \) functions

To find these additional properties we begin by studying the embeddings and products of \( C^\infty \) functions. We therefore calculate (6.3) for \( f \in C^\infty \) and apply Taylor’s theorem to obtain at once

\[
(f_\epsilon)_\epsilon(x) = f(x) \int dz \, \eta(z) + \ldots 
\]

(7.1)

\[
+ \frac{\epsilon^n}{n!} f^{(n)}(x) \int dz \, z^n \eta(z) + \ldots
\]

(7.2)

\[
+ \frac{\epsilon^{(q+1)}}{(q+1)!} \int dz \, z^{q+1} \eta(z) \, f^{(q+1)}(x + \vartheta \epsilon z),
\]

(7.3)

where \( f^{(n)}(x) \) is the \( n \)-th derivative of \( f(x) \), and \( \vartheta \in ]0, 1[ \). Then, since \( \eta \in S \) and \( f \) has a compact support, the integral in (7.3) is bounded so that the remainder is of order \( O_x(\epsilon^{q+1}) \) at any fixed point \( x \).

Moreover, if following Colombeau the mollifier \( \eta \) is chosen in the set

\[
A_q := \{ \eta(x) \in A_0, \quad \text{and} \quad \int dz \, z^n \eta(z) = 0, \quad \forall n = 1, \ldots, q \},
\]

(7.4)

all the terms in (7.2) with \( n \in [1, q] \) are zero and we are left with

\[
\forall f \in C^\infty, \quad (f_\epsilon)_\epsilon(x) = f(x) + O_x(\epsilon^{q+1}).
\]

(7.5)

Therefore, provided the set \( A_q \) is not empty and \( q \) can take any value in \( \mathbb{N} \), it is possible to make the difference \( (f_\epsilon)_\epsilon(x) - f(x) \) as small as we please even if \( \epsilon \in ]0, 1[ \) is kept finite. If we now consider a product of two \( C^\infty \) functions, it is easily seen that equation (7.5) immediately leads to

\[
\forall u, v \in C^\infty, \quad (u_\epsilon)_\epsilon(x) \cdot (v_\epsilon)_\epsilon(x) = u(x) \cdot v(x) + O_x(\epsilon^{q+1}),
\]

(7.6)

where the remainder \( O_x(\epsilon^{q+1}) \) is still as small as we please for any \( \epsilon \in ]0, 1[ \) if \( q \) is large enough.

8 Colombeau mollifiers and Fourier transformation

Colombeau proved that \( A_q \) is not empty and provided a recursive algorithm for constructing the corresponding mollifiers for all \( q \in \mathbb{N} \). He also showed [5, p.7], [5, p.113], [33, p.169] that due to the Fourier transformation identities

\[
\int dx \, \eta(x) = \hat{\eta}(0), \quad \text{and} \quad \int dx \, x^n \eta(x) = (-i)^n \frac{d^n \hat{\eta}}{dp^n}(0),
\]

(8.1)
which are valid $\forall \eta \in \mathcal{S}$, the conditions (7.4) on the moments of $\eta(x)$ can be replaced by equivalent conditions on the derivatives of its Fourier transform $\hat{\eta}(p)$. Thus by taking for $\eta(x)$ any real functions such that $\hat{\eta}(p) \equiv 1$ in a finite neighborhood of $p = 0$, one automatically satisfies the conditions (7.4) for any $n \in \mathbb{N}$, that is for $q$ as large as we please. For this reason the set of mollifiers

$$\mathcal{A}_\infty := \left\{ \eta(x) \in \mathcal{S}, \text{ such that } \hat{\eta}(0) \equiv 1 \right\}. \quad (8.2)$$

is written $\mathcal{A}_\infty$. In this paper all Colombeau mollifiers will be taken in that set.

For example, with $\eta \in \mathcal{A}_\infty$ the Colombeau embeddings of any two polynomials, and the products of these embeddings, are identical to these polynomials and to their ordinary products. That is, axiom 4' is identically satisfied for polynomials. But for the other $C^\infty$ functions there will still be a remainder to be taken care of, even if it is infinitesimal.

### 9 Embedding of continuous functions

Let us now consider the embedding of continuous functions assuming that the only things that are known is that they are continuous and compactly supported. Then, a priori, there is little more that can be done than writing

$$\forall f \in \mathcal{C}, \quad (f_\epsilon)(x) = \int dz \, \eta(z) f(x + \epsilon z), \quad (9.1)$$

because neither Taylor’s formula nor the mean-value theorem can be applied to transform the right-hand side into a more useful expression. In fact, the only fully general expression comparable to (7.5) is

$$\forall f \in \mathcal{C}, \quad (f_\epsilon)(x) = f(x) + o_{x, \epsilon}(1), \quad (9.2)$$

which simply means that $(f_\epsilon)(x)$ converges uniformly to $f(x)$ as $\epsilon \to 0$ because $f$ has compact support. Any more precise statement requires that the continuous function is further specified.

For example, if $f$ is $m$-times continuously differentiable we can write

$$\forall f \in \mathcal{C}^m, \quad (f_\epsilon)(x) = f(x) + O_x(\epsilon^m), \quad (9.3)$$

where, in contrast to (7.5), $m \geq 1$ is a fixed integer.
Embedding of distributions

We now turn to distributions. By Schwartz’s local structure theorem (3.1) they can be written
\[ \gamma(x) = D^n g(x) \]
where \( g \in C^\infty \) if we restrict ourselves to a single generating function with support in a compact set \( K \). Then for \( x \in K \) their embeddings (6.3) are, using the integration by parts formula (2.1),
\[
(\gamma_\epsilon)_\epsilon(x) = \int dz \, \eta(z) \, D^n g(x + \epsilon z), \tag{10.1}
\]
\[
= \frac{1}{\epsilon^n} \int dz \, \eta(z) \, D^n g(x + \epsilon z), \tag{10.2}
\]
\[
= \left( \frac{-1}{\epsilon} \right)^n \int dz \, (D^n \eta)(z) \, g(x + \epsilon z). \tag{10.3}
\]

Since \( \eta \in S \), and \( g \in C^\infty \) is compactly supported, the last integral is bounded and we get
\[
\forall \gamma \in \mathcal{D}', \quad (\gamma_\epsilon)_\epsilon(x) = O_x(1/\epsilon^n), \quad \text{as } \epsilon \to 0. \tag{10.4}
\]

This bound is compatible with the bounds (7.5, 9.2, and 9.3) because \( \mathcal{D}' \) contains all continuous functions. To illustrate its significance for non-trivial distributions we need to consider generating functions \( g \in C^0 \).

For example, we know that \( \delta(x) = D^2 g(x) \) with \( g(x) = |x|/2 \). On the other hand, the Colombeau embedding (6.3) of \( \delta(x) \) is
\[
(\delta_\epsilon)_\epsilon(x) = \frac{1}{\epsilon} \eta\left( -\frac{x}{\epsilon} \right) = O_x(1/\epsilon), \tag{10.5}
\]
which has an \( \epsilon \)-dependent bound consistent with the bound (10.4), i.e., 1/\( \epsilon \) \( < \) 1/\( \epsilon^2 \), although their exponents disagree by one unit. This is because (10.4) is fully general and thus does not take the particular properties of \( g(x) \) into account. In the present case it is easy to calculate \((|x|_\epsilon)_\epsilon(x)\) with (6.3) and to verify that \( 2(\delta_\epsilon)_\epsilon(x) = D^2 (|x|_\epsilon)_\epsilon(x) = O_x(1/\epsilon) \) rather than \( O_x(1/\epsilon^2) \). For the same reason the embedding of the Heaviside function \( H(x) = D|x|/2 \)
\[
(H_\epsilon)_\epsilon(x) = \int_{-\infty}^{x/\epsilon} dz \, \eta(-z) = O_x(1). \tag{10.6}
\]
which, rather than \( O_x(1/\epsilon) \), has the \( \epsilon \)-dependence \( O_x(1) \) characteristic of a piece-wise continuous function because of its jump at \( x = 0 \).

To give another example, the singular distributions generated by the derivatives of the \( C^0 \) function equal to 0 for \( x \leq 0 \) and to \( x^r \) for \( x > 0 \), with \( r \in ]0, 1[ \), have an \( \epsilon \)-dependent bound \( O_x(\epsilon^{r-n}) \) where \( r - n < 0 \) in \( \mathbb{R} \).
Local structure of embeddings and of their differences in $E$

| $f$ | $(f_\epsilon)_\epsilon(x)$ |
|-----|--------------------------|
| $\mathcal{C}_\infty$ | $f(x)$ | Directly included smooth function |
| $\mathcal{C}_\infty$ | $f(x) + O_x(\epsilon^q)$ | Smooth function, $q > \forall p \in \mathbb{N}$ |
| $\mathcal{C}$ | $f(x) + o_{x,\epsilon}(1)$ | Continuous function |
| $\mathcal{C}_p$ | $O_x(1)$ | Piece-wise continuous function |
| $\mathcal{D}'$ | $O_x(\epsilon^{-N})$ | Singular distribution, $N \in \mathbb{N}$ |
| $\mathcal{N}$ | $O_x(\epsilon^q)$ | Negligible function, $q > \forall p \in \mathbb{N}$ |

Table 1: The differential algebra $E$ contains the smooth functions as direct embeddings $f \in \mathcal{C}_\infty \subset E$, and also as mollified embeddings $(f_\epsilon)_\epsilon \in (\mathcal{C}_\infty)_\epsilon \subset E$. The embeddings $(f_\epsilon)_\epsilon$ of the continuous functions and of the distributions are sorted in terms of the behavior of the bound on their $\epsilon$-dependent part as $\epsilon \to 0$. $f(x)$ is the point-value of the continuous functions at $\epsilon = 0$. The negligible functions are infinitesimally small elements such as the differences between the direct inclusions and the Colombeau-mollified embeddings of the $\mathcal{C}_\infty$ functions.

In summary, (10.4) provides a conservative bound for the $\epsilon$-dependence of all Schwartz distributions. In the case of singular distributions, the exponent in the bound (10.4) can be any integer $n \in \mathbb{N}$.

11 Linear operations and negligible functions

In Table 1 the usual functions and the distributions are classified according to the structure of their embeddings in $E$. Referring to this table it is easy to predict the structure of the result of binary algebraic operations in $E$, and thus to get clues on how to define the algebra $\mathcal{G}$.

For instance, in the last line of Table 1 the set denoted by $\mathcal{N}$ consists of functions which are not the direct result of embeddings: It is the algebra of the so-called negligible functions,\footnote{A proper definition of negligible functions will be given shortly.} which arise in particular from subtracting the two different inclusions of the $\mathcal{C}_\infty$ functions, i.e.,

$$\forall f \in \mathcal{C}_\infty, \quad \forall q \in \mathbb{N}, \quad (f_\epsilon)_\epsilon(x) - f(x) = O_x(\epsilon^q) \in \mathcal{N}. \quad (11.1)$$
Then for all linear operations (i.e., addition/subtraction, multiplication by a scalar, and differentiation) it is clear that the results will always be in one of the sets listed in the first column of Table 1, which is therefore a suitable classification of the embeddings of the usual functions and distributions with regards to linear operations in $\mathcal{E}$.

Of course, the negligible functions of the type $(11.1)$ are precisely the differences that are to be taken care of in order to satisfy axiom $4'$. In particular, they will remain ‘negligible’ as long as they are not multiplied by ‘very large’ functions. This is why we have now to look at the nonlinear operations in $\mathcal{E}$.

## 12 Nonlinear operations and moderate functions

We know that the product of two distributions (or of a continuous function and a distribution) will, in general, not be a distribution. For example, the $n$-th power of the Dirac $\delta$-function can be defined by the $n$-th power of the embedding $(10.5)$, i.e.,

$$
(\delta^n_\epsilon)(x) = \frac{1}{\epsilon^n} \eta^n \left(-\frac{x}{\epsilon}\right) = O_x(\epsilon^{-n}).
$$

(12.1)

But, despite that $(\delta^n_\epsilon)(x)$ has a $O_x(\epsilon^{-n})$ dependence similar to that of a ‘distribution,’ it is not a distribution in the sense of Schwartz and Mikusinski — rather, it is an element of a larger set of ‘generalized functions’ containing the distributions as a subspace.

This led Colombeau to define the set $\mathcal{E}_M$, which he called *moderate functions*[^10]

$$
\forall (g_\epsilon) \in \mathcal{E}_M : \exists N \in \mathbb{N}_0, \text{ such that } (g_\epsilon)(x) = O_x(\epsilon^{-N}).
$$

(12.2)

It is evident that $\mathcal{N} \subset (C^\infty)_\epsilon \subset (C)_\epsilon \subset (D')_\epsilon \subset \mathcal{E}_M$, and a matter of elementary calculations to verify that $\mathcal{E}_M$ and $\mathcal{N}$ are algebras for the usual pointwise operations in $\mathcal{E}$, and that $\mathcal{N}$ is an ideal of $\mathcal{E}_M$. Indeed, the product of two moderate functions is still moderate — they are *multipliable* — and as $q$ in $(11.1)$ is as large as we please, and $N$ in $(12.2)$ a fixed integer, the product of a negligible function by a moderate one will always be a negligible function. Moreover, $\mathcal{E}_M$ is a differential algebra satisfying axioms $[1-3]$, and it is not difficult to show that $\mathcal{E}_M$ is the largest differential subalgebra (i.e., stable under partial differentiation) of $\mathcal{E}$ in which $\mathcal{N}$ is a differential ideal.

[^10]: A proper definition of moderate functions will be given shortly.
### Table 2: The elements of \( \mathcal{E} \) remain in their respective subalgebras \( \mathcal{N} \), \( \mathcal{E}_M \), or \( \mathcal{E}_M \setminus \mathcal{E} \) when multiplied by directly included \( C^\infty \) functions. The subalgebra \( \mathcal{N} \subset \mathcal{E}_M \) is an ideal of \( \mathcal{E}_M \). The products of negligible and moderate elements with elements in the complement of \( \mathcal{E}_M \) in \( \mathcal{E} \) are in general undefined.

| Multiplication in \( \mathcal{E} \) | \( C^\infty \) | \( \mathcal{N} \) | \( \mathcal{E}_M \) | \( \mathcal{E}_M \setminus \mathcal{E} \) |
|----------------------------------|----------|----------|----------|----------------|
| \( C^\infty \)                  | \( C^\infty \) | \( \mathcal{N} \) | \( \mathcal{E}_M \) | \( \mathcal{E}_M \setminus \mathcal{E} \) |
| \( \mathcal{N} \)               | \( \mathcal{N} \) | \( \mathcal{N} \) | \( \mathcal{N} \) | \( \mathcal{E} \) |
| \( \mathcal{E}_M \)             | \( \mathcal{E}_M \) | \( \mathcal{N} \) | \( \mathcal{E}_M \) | \( \mathcal{E} \) |
| \( \mathcal{E}_M \setminus \mathcal{E} \) | \( \mathcal{E}_M \setminus \mathcal{E} \) | \( \mathcal{E} \) | \( \mathcal{E} \) | \( \mathcal{E}_M \setminus \mathcal{E} \) |

Furthermore, one can also consider infinite sums of products of moderate functions and take their limits in \( \mathcal{E} \). It is then easily verified that, for example, \( \sqrt{\delta^n}_\epsilon(x) \) and \( \sin(\delta^n)_\epsilon(x) \) are elements of \( \mathcal{E}_M \). On the other hand

\[ \exp(|\delta|)_\epsilon(0) = O_x(\exp(1/\epsilon)) \not\in \mathcal{E}_M, \]

so that \( \exp(\delta)_\epsilon(x) \) is a non-moderate function, and thus an element of the complement of \( \mathcal{E}_M \) in \( \mathcal{E} \). Conversely, \( \exp(-|\delta|)_\epsilon(x) \) is a negligible function, so that \( \mathcal{N} \) contains elements of exponentially fast decrease.

Consequently, when operating in full generality in \( \mathcal{E} \), that is when including multiplication and limiting processes, one is led to consider its elements as in Table 2, i.e., as members of the differential algebras \( C^\infty, \mathcal{N}, \) and \( \mathcal{E}_M \), rather than as members of the embeddings of the classical spaces \( C^\infty, \mathcal{C}, \) and \( \mathcal{D}' \) as in Table 1.

### 13 Discovery of the Colombeau algebra

The fact that \( \mathcal{N} \) is an ideal of \( \mathcal{E}_M \) is the key to defining an algebra containing the distributions and satisfying axiom 4. Indeed, if we conventionally write \( \mathcal{N} \) for any negligible function, then

\[ \forall g_\epsilon, h_\epsilon \in \mathcal{E}_M, \quad (g_\epsilon + \mathcal{N}) \cdot (h_\epsilon + \mathcal{N}) = g_\epsilon \cdot h_\epsilon + \mathcal{N}. \]

\[ \text{[11]From now on we abbreviate } (f_\epsilon)_\epsilon \text{ as } f_\epsilon. \]

13
Similarly, using the same convention, Eq. (7.6) giving the product of the Colombeau embeddings of two $C^\infty$ functions can be written
\[ \forall u, v \in C^\infty, \quad (u + N) \cdot (v + N) = u \cdot v + N, \] (13.2)
whereas axiom $[4']$ demands
\[ \forall u, v \in C^\infty, \quad [u + N] \odot [v + N] = [u \cdot v + N] = u \cdot v. \] (13.3)
Thus, it suffice to define the elements of $G$ as the elements of $E_M$ modulo $N$, e.g., to identify $[g_\epsilon + N]$ and $[g_\epsilon]$, so that $[u + N] = [u]$ because $(C^\infty)_\epsilon \subset E_M$, and axiom $[4']$ will be satisfied.

This immediately leads to the definition of the Colombeau algebra as the quotient
\[ G := \frac{E_M}{N}. \] (13.4)
That is, an element $g \in G$ is an equivalence class $[g] = [g_\epsilon + N]$ of an element $g_\epsilon \in E_M$, which is called a representative of the generalized function $g$. The product $g \odot h$ is defined as the class of $g_\epsilon \cdot h_\epsilon$ where $g_\epsilon$ and $h_\epsilon$ are (arbitrary) representatives of $g$ and $h$; similarly $Dg$ is the class of $Dg_\epsilon$ if $D$ is any partial differentiation operator. Therefore, when working in $G$, all algebraic and differential operations (as well as composition of functions, etc.) are performed component-wise at the level of the representatives $g_\epsilon$.

$G$ is an associative and commutative differential algebra because both $E_M$ and $N$ are such. The two main ingredients which led to its definition are the primacy given to $C^\infty$ functions, and the use of the Colombeau mollifiers for the embeddings.

### 14 Special and general Colombeau algebras

Depending on the precise definitions of the moderate and negligible functions, as well as of any further specification constraining the Colombeau mollifiers, there can be many variants of $G$, even if the domain and range of the generalized functions are simply a subset of $\mathbb{R}$. There are however two general types of Colombeau algebras: The ‘special’ (or ‘simple’) algebras, and the ‘general’ (also called ‘full’ or ‘elementary’) algebras.

For example, let us define a special Colombeau algebra of generalized functions on $\Omega \in \mathbb{R}^n$ with value in $\mathbb{C}$. Then, using the standard multi-index notation
\[ D^\alpha = \frac{\partial^{\vert \alpha \vert}}{(\partial x_1)^{\alpha_1} \cdots (\partial x_n)^{\alpha_n}}, \] (14.1)
where \( \alpha \in \mathbb{N}_0^n \) and \(|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n\), the distributions will be the partial derivatives \( D^\alpha f(\vec{x}) \) of the continuous function \( f(\vec{x}) \in \mathcal{C}(\Omega) \). A possible definition of \( \mathcal{G}^s(\Omega) \), which can easily be adapted to more complicated manifolds, is as follows:

**Definition 1 (Embedding space)** Let \( \Omega \) be an open set in \( \mathbb{R}^n \), let \( \epsilon \in ]0, 1[ \) be a parameter, and let \( \eta \in \mathcal{A}_\infty \) be an arbitrary but fixed Colombeau mollifier. The ‘embedding space’ is the differential algebra

\[
\mathcal{E}^s(\Omega) := \left\{ f_\epsilon : \mathcal{D}^0 f_\epsilon(\vec{x}) \right\}, \quad (\epsilon, \vec{x}) \mapsto f_\epsilon(\vec{x}), \quad (14.2)
\]

where the sequences \( f_\epsilon \) are \( \mathcal{C}^\infty \) functions in the variable \( \vec{x} \in \Omega \). The compactly supported distributions\(^{12}\) are embedded in \( \mathcal{E}^s \) by convolution with the scaled mollifier \( \eta_\epsilon \), i.e.,

\[
f_\epsilon(\vec{x}) := \int d^n \eta(\frac{\vec{y} - \vec{x}}{\epsilon}) f(\vec{y}) = \int d^n \eta(\vec{z}) f(\vec{x} + \epsilon \vec{z}). \quad (14.3)
\]

**Definition 2 (Moderate functions)** The differential subalgebra \( \mathcal{E}^s_M(\Omega) \subset \mathcal{E}^s \) of ‘moderate functions’ is

\[
\mathcal{E}^s_M(\Omega) := \left\{ f_\epsilon : \forall K \text{ compact in } \Omega, \forall \alpha \in \mathbb{N}_0^n, \exists N \in \mathbb{N}_0 \text{ such that,} \right. \\
\sup_{\vec{x} \in K} |D^\alpha f_\epsilon(\vec{x})| = O(\frac{1}{\epsilon^N}) \text{ as } \epsilon \to 0 \left\}. \quad (14.4)
\]

**Definition 3 (Negligible functions)** The differential ideal \( \mathcal{N}^s \subset \mathcal{E}^s_M \) of ‘negligible functions’ is

\[
\mathcal{N}^s(\Omega) := \left\{ f_\epsilon : \forall K \text{ compact in } \Omega, \forall \alpha \in \mathbb{N}_0^n, \forall q \in \mathbb{N}, \sup_{\vec{x} \in K} |D^\alpha f_\epsilon(\vec{x})| = O(\epsilon^q) \text{ as } \epsilon \to 0 \right\}. \quad (14.5)
\]

**Definition 4 (Special algebra)** The special Colombeau algebra is the quotient

\[
\mathcal{G}^s(\Omega) := \frac{\mathcal{E}^s_M(\Omega)}{\mathcal{N}^s(\Omega)} . \quad (14.6)
\]

\(^{12}\)The embedding of all of \( \mathcal{D}' \) is achieved by a more complicated formula based on sheaf theoretic arguments, see [11 Proposition 1.2.13].
The main differences with the ‘naive’ definitions (11.1) and (12.2) are: (i) The resort to the compact subset \( K \subset \Omega \), which is necessary because of the local character of the concept of distribution, as is clearly stipulated by Schwartz’s structure theorem; (ii) the need to consider the supremum over all \( K \subset \Omega \) in order to take into account all possible discontinuities when \( x \) ranges in \( \Omega \); and (iii) the need to consider all possible derivatives of \( f_\varepsilon \) in order that the moderate and negligible functions have the required properties for all their derivatives.

A general Colombeau algebra \( G^\eta \) is an enlargement of \( G^s \), obtained by considering all \( \eta \in A_\infty \) and by replacing (in both \( N^s \) and \( E^s_M \)) the functions \( x \to f_\varepsilon(x) \) by the set of functions \( x \to f_\varepsilon(x, \eta) \) depending on \( \eta \). Since all possible \( \eta \) are considered the arbitrariness characteristic of \( G^s(\Omega) \) disappears, and the embeddings of the distributions and functions with finite differentiability become ‘canonical’ since they do not depend any more on a fixed mollifier. However, while this is conceptually interesting from the mathematical point of view, it is not a necessity since the particular mollifier (or set of mollifiers) defining a special Colombeau algebra \( G^s \subset G^\eta \) may have a physical interpretation. For this reason the dependence of the embeddings on the mollifiers is not a defect, but rather a positive feature in many applications of the special Colombeau algebras.

### 15 Interpretation of distributions within \( G \)

To construct the Colombeau algebra we have been led to embed the distributions as the representative sequences \((\gamma_\varepsilon)_\varepsilon \in \mathcal{E}\) defined by (6.3) where \( \eta_\varepsilon \in A_\infty \) is a Colombeau mollifier, that is not as the usual representative sequences defined by (2.7) where \( \rho_\varepsilon \in A_0 \). However, since \( A_\infty \subset A_0 \), we can still recover any distribution \( \gamma \) from \( \gamma_\varepsilon = (\gamma_\varepsilon)_\varepsilon \) by means of (2.8), i.e., as the equivalence class

\[
\gamma(T) := \lim_{\varepsilon \to 0} \int dx \, \gamma_\varepsilon(x) \, T(x), \quad \forall T(x) \in \mathcal{D},
\]

where \( \gamma_\varepsilon \) can be any representative of the class \([\gamma] = [\gamma_\varepsilon + \mathcal{N}]\) because negligible elements are zero in the limit \( \varepsilon \to 0 \).

Of course, as we work in \( G \) and its elements get algebraically combined with other elements, there can be generalized functions \([g_\varepsilon]\) different from the class \([\gamma_\varepsilon]\) of an embedded distribution which nevertheless correspond to the same distribution \( \gamma \). This leads to the concept of association: We say that two generalized functions
\[ g \text{ and } h \text{ are associated, and we write } g \asymp h \text{ iff } \lim_{\varepsilon \to 0} \int dx \left( g_\varepsilon(x) - h_\varepsilon(x) \right) T(x) = 0, \quad \forall T(x) \in \mathcal{D}. \quad (15.2) \]

Thus, if \( \gamma \) is a distribution and \( g \) some generalized function, the relation \( g \asymp \gamma \) implies that \( g \) admits \( \gamma \) as ‘associated distribution,’ and \( \gamma \) is called the ‘distributional shadow’ (or ‘distributional projection’) of \( g \) because the mapping \( \gamma_\varepsilon \mapsto \gamma \) defined by (15.1) is then a projection \( \mathcal{G} \to \mathcal{D}' \) for all \( g_\varepsilon \) associated to \( \gamma_\varepsilon \).

13 In the literature the symbol \( \approx \) is generally used for association. We prefer to use \( \asymp \) because association is not some kind of an ‘approximate’ relationship, but rather the precise statement that a generalized function corresponds to a distribution.

16 Multiplication of distributions in \( \mathcal{G} \)

The continuous functions and their derivatives, i.e., the distributions, are not subalgebras of \( \mathcal{G} \): Only the smooth functions have that property. Thus we do not normally expect that their products in \( \mathcal{G} \) will be associated to some continuous functions or distributions: In general these products will be genuine generalized functions, i.e., new mathematical objects — which constitute one of the main attractions of \( \mathcal{G} \).

For example, the \( n \)-th power of Dirac’s \( \delta \)-function in \( \mathcal{G} \), Eq. (12.1), has no associated distribution. But \( \delta^n \) is a moderate function and thus makes perfectly sense in \( \mathcal{G} \). Moreover, its point-value at zero, \( \eta^n(0)/\varepsilon^n \) can be considered as a ‘generalized number.’

On the other hand, we have elements like the \( n \)-th power of Heaviside’s function, Eq. (10.6), which has an associated distribution but is such that \([H^n](x) \neq [H](x)\) in \( \mathcal{G} \), whereas \( H^n(x) = H(x) \) as a distribution in \( \mathcal{D}' \). Similarly, we have \([x] \circ [\delta](x) \neq 0 \) in \( \mathcal{G} \), whereas \( x\delta(x) = 0 \) in \( \mathcal{D}' \). In both cases everything is consistent: Using (15.2), one easily verifies that indeed \([H^n](x) \asymp [H](x)\) and \([x] \circ [\delta](x) \asymp 0 \).

These differences between products in \( \mathcal{G} \) and in \( \mathcal{D}' \) stem from the fact that distributions embedded and multiplied in \( \mathcal{G} \) carry along with them infinitesimal.
information on their ‘microscopic structure.’ That information is necessary in order that the products and their derivatives are well defined in \( \mathcal{G} \), and is lost when the factors are identified with their distributional projection in \( \mathcal{D}' \).

Let us illustrate this essential point with a concrete example. In physics the Heaviside function \( H(x) \) represents a function whose values jump from 0 to 1 in a tiny interval of width \( \epsilon \) around \( x = 0 \). Thus it is obvious that \( \int (H^2(x) - H(x)) T(x) \, dx \) tends to 0 when \( \epsilon \to 0^+ \) if \( T \) is a bounded function, i.e., \( H^2 \asymp H \). But since \( H' \) is unbounded one has \( \int (H^2(x) - H(x)) \cdot H'(x) \, dx = 1/3 - 1/2 = -1/6 \), as obvious from elementary calculations. This shows that one is not allowed to state \( H^2 \asymp H \) in a context where the function \( H^2 - H \) could be multiplied by a function taking infinite values such as the Dirac function \( \delta = H' \).

Therefore, the distinction between \( \mathcal{G} \)-functions that are ‘infinitesimally nonzero’ such as \( H^2 - H \) from the genuine zero function insures that multiplication is coherent in \( \mathcal{G} \), because ‘infinitesimally nonzero quantities,’ when multiplied by ‘infinitely large quantities,’ can give significant nonzero results. At the same time, this distinction insures that all calculations are consistent with those in \( \mathcal{D}' \). In particular, if at any point it is desirable to look at the intermediate results of a calculation from the point of view of distribution theory, one can always use the concept of association to retrieve their distributional content. In fact, this is facilitated by a few simple formulas which easily derive from the definition (15.2). For instance,

\[
\forall f_1, \forall f_2 \in \mathcal{C} \quad \Rightarrow \quad [f_1] \circ [f_2] \asymp [f_1 \cdot f_2], \quad (16.1)
\]

and,

\[
\forall f \in \mathcal{C}^\infty, \forall \gamma \in \mathcal{D}' \quad \Rightarrow \quad [f] \circ [\gamma] \asymp [f \cdot \gamma], \quad (16.2)
\]

but, in general,

\[
\forall \gamma_1, \forall \gamma_2 \in \mathcal{D}' \quad \Rightarrow \quad [\gamma_1] \circ [\gamma_2] \not\asymp [\gamma_1 \cdot \gamma_2], \quad (16.3)
\]

whereas,

\[
\forall g_1, \forall g_2 \in \mathcal{G}, \quad g_1 \asymp g_2 \quad \Rightarrow \quad D^\alpha g_1 \asymp D^\alpha g_2. \quad (16.4)
\]

For example, applying the last equation to \( [H^2](x) \asymp [H](x) \) one proves the often used distributional identity \( 2[\delta](x)[H](x) \asymp [\delta](x) \).

In summary, one calculates in \( \mathcal{G} \) as in \( \mathcal{C}^\infty \) by operating on the representatives \( g_\epsilon \in \mathcal{E} \) with the usual operations \( \{+, -, \times, d/dx\} \). The distributional aspects, if required, can be retrieved at all stages by means of association.
17 Working with distributions versus working with $G$-functions

An interesting feature of Colombeau algebras is that they enable, in many cases, to set aside the concept of distributions and to replace it by the more general and flexible one of $G$-functions.

Indeed, a distribution cannot be the end result of a calculation in any physical theory: It is a functional which has to be integrated over its argument to yield a quantity comparable to experiment. Similarly, a measurement is always some kind of an average over a continuous distribution of matter supported by bodies of finite extension. Thus, if one takes the sequential view, one is often led in physics to consider integrals of the type (2.8), i.e.,

$$g(S) = \lim_{\epsilon \to 0} \int dx \, g_\epsilon(x) \, S(x), \quad (17.1)$$

where $g(x)$ may be any regular or singular distribution corresponding to a basic physical quantity (e.g., an energy density), and $S(x) \in D$ a smooth function (e.g., $S(x)dx$ could be a volume element).

There are then two options:

- In conventional ‘distribution theory’ the distributional aspect is emphasized throughout the calculation and all intermediate results are interpreted as distributions. In particular, when working according to the sequential view, limits similar to that in (17.1) are taken at all stages so that information that could be relevant to nonlinear operations is discarded. (In the language of generalized functions, one systematically works with the distributional shadows rather than with the generalized functions themselves.) The method is therefore restricted to linear theories, and if the limit $\epsilon \to 0$ is undefined the end result will in general be meaningless even if $\epsilon$ is kept finite. (Because infinitesimal information that could have been significant before passing to the limit may have been discarded.)

Example: If the electrostatic Coulomb potential is defined as a distribution, the $G$-embedded Coulomb field has the form [24, 27, 30]

$$\vec{E}_\epsilon(\vec{r}) = e \lim_{a \to 0} \left( \frac{\vec{r}}{r^3} H(r - a) \right)_\epsilon - e \lim_{a \to 0} \left( \frac{\vec{r}}{r^2} \delta(r - a) \right)_\epsilon$$

(17.2)

$$\approx e \lim_{a \to 0} \frac{\vec{r}}{r^3} H(r - a). \quad (17.3)$$
In distribution theory, the second part of (17.2), which contains the $\delta$-function, is discarded [24, p. 144]. Expression (17.3) cannot therefore be used in non-linear calculations such as the self-energy of an electron.

- In ‘$G$-function theory’ all non-smooth functions $f$ are represented by their Colombeau mollified sequence $f_\epsilon$, and there is a unique parameter $\epsilon$ which is kept finite until the end of the calculation. (There is also possibly a unique common mollifier $\eta$ if one works in a special Colombeau algebra.) All operations are then performed on these representatives, and at any stage one can verify the validity of the calculations by checking that the intermediate results are moderate functions. It is possible to consistently manipulate singular distributions in nonlinear calculations, and the end results are obtained by taking the limit $\epsilon \to 0$ as in (17.1). If the theory is linear, these results are identical to those of the conventional ‘distribution theory’ option. In linear or nonlinear theories which lead to divergent quantities as $\epsilon \to 0$ the parameter $\epsilon$ can be left finite, and the end result can be interpreted as a ‘generalized number.’ This generalized number may then be renormalized to some finite quantity, which implies that any dependence on $\epsilon$ and on the arbitrary mollifier $\eta$ is removed at this final stage.

Example: Calculating the self-energy of an electron involves integrating the square of its electric field. In distribution theory, where this square is undefined, the result using (17.3) is the well-known expression

$$U_{\text{self}}(D') = e^2 \lim_{a \to 0} \frac{1}{2a},$$

(17.4)

which diverges as $a \to 0$, and which corresponds to the square of the first term in (17.2), i.e., to the energy of the field surrounding the electron. On the other hand, when calculated in $G$ using (17.2) the self-energy is [27, 30]

$$U_{\text{self}}(G) = e^2 \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{-\infty}^{+\infty} dx \eta^2(-x).$$

(17.5)

which is independent of the ‘cut-off’ $a$, and which corresponds to the square of the second term in (17.2), i.e., to the square of a $\delta$-function. Thus, when calculated in $G$, the self-energy is entirely located at the position of the electron, i.e., precisely where its ‘inertia’ as a point-mass is supposed to reside. It remains therefore to renormalize $U_{\text{self}}(G)$ to the measured mass of the electron, and everything makes mathematically and physically sense.

That discussion permits to conclude this paper by an analogy: The relations of the usual functions and distributions to the $G$-functions are somewhat analogous
to those of the real to the complex numbers. If one looks at $\epsilon$ as an analog of $i$, then taking the real part of a complex number corresponds to restricting a generalized function to its associated function or distribution by taking the limit $\epsilon \to 0$. Working in $\mathbb{R}$ or $\mathcal{D}'$ is therefore less general and flexible than working in $\mathbb{C}$ or $\mathcal{G}$.

**Acknowledgments**

The author is very much indebted to Drs. G. Falquet and J.-P. Hurni for stimulating discussions and comments and for a critical reading of the manuscript.
Appendix: Special algebra of tempered $G$-functions

In many mathematical and physical applications the restriction of the test functions to the space $\mathcal{D}$ of $C^\infty$ functions with compact support is too restrictive [8]. This is the case of the Fourier transform which, in its simplest form, has as a kernel $\cos(px)$ which is not integrable over the whole space $\Omega = \mathbb{R}$. Thus, just like in Schwartz distribution theory, an extension of the Colombeau theory to ‘tempered $G$-functions,’ see, e.g., [11, p. 15 and p. 65], is essential when dealing with functions that are integrated over the whole space $\Omega = \mathbb{R}^n$. Moreover, the algebra $G^t$ of tempered $G$-functions has a property that is important from a practical point of view: In $G^t$, componentwise composition is a well defined operation generalizing composition of $C^\infty$ functions.

To use this extension it is necessary to be careful about the definitions of the pertinent function spaces. We therefore recall [11, p. 15]:

**Definition 5 (Algebras $S$, $O_C$, and $O_M$)** Let $\Omega^t \subset \mathbb{R}^n$ be a $n$-dimensional box, i.e., a subset of the form $\omega_1 \times \cdots \times \omega_n$ where $\omega_i$ is a finite or infinite open interval in $\mathbb{R}$. Then,

$$S := \left\{ f \in C^\infty(\Omega^t) : \forall m \in \mathbb{N}, \forall \alpha \in \mathbb{N}_0^n, \sup_{\vec{x} \in \Omega^t} (1 + |\vec{x}|)^{-m}\left|D^\alpha f(\vec{x})\right| < \infty \right\}, \quad (18.1)$$

$$O_C := \left\{ f \in C^\infty(\Omega^t) : \exists m \in \mathbb{N} such that, \forall \alpha \in \mathbb{N}_0^n, \sup_{\vec{x} \in \Omega^t} (1 + |\vec{x}|)^{-m}\left|D^\alpha f(\vec{x})\right| < \infty \right\}, \quad (18.2)$$

$$O_M := \left\{ f \in C^\infty(\Omega^t) : \forall \alpha \in \mathbb{N}_0^n, \exists m \in \mathbb{N}, \sup_{\vec{x} \in \Omega^t} (1 + |\vec{x}|)^{-m}\left|D^\alpha f(\vec{x})\right| < \infty \right\}. \quad (18.3)$$

$O_C$ and $O_M$ correspond to two closely related definitions of functions with polynomial growth as $|\vec{x}| \to \infty$. But, while $O_M$ corresponds to the usual definition, it is the algebra $O_C$ which in the $G$-context provides the proper ‘tempered’ extension of the notion of $C^\infty$ functions with compact support.

**Definition 6 (Embedding space of temperate distributions)** Let $\Omega^t \subset \mathbb{R}^n$ be a $n$-dimensional box, let $\epsilon \in ]0, 1[$ be a parameter, and let $\eta \in A_\infty \subset S$ be an
arbitrary but fixed Colombeau mollifier. The ‘embedding space’ is the differential algebra

\[ \mathcal{E}^t(\Omega^t) := \left\{ \epsilon, \vec{x} \mapsto f_\epsilon(\vec{x}) \right\}, \]  

(18.4)

where the sequences \( f_\epsilon \) are \( C^\infty \) functions in the variable \( \vec{x} \in \Omega^t \). The tempered distributions \( g \in \mathcal{S}' \) are embedded in \( \mathcal{E}^t \) by convolution with the scaled mollifier \( \eta_\epsilon \), i.e.,

\[ g_\epsilon(\vec{x}) := \int \frac{dy^n}{\epsilon^n} \eta_\epsilon \left( \frac{\vec{y} - \vec{x}}{\epsilon} \right) g(\vec{y}) = \int \frac{dz^n}{\epsilon^n} \eta(\vec{z}) g(\vec{x} + \epsilon \vec{z}). \]  

(18.5)

The functions \( h \in \mathcal{O}_C \) are directly embedded, i.e.,

\[ \forall h \in \mathcal{O}_C, \quad (h_\epsilon)(x) = h(x), \]  

(18.6)

so that \( \mathcal{O}_C \) is a subalgebra of \( \mathcal{E}^t(\Omega^t) \).

**Definition 7 (Temperate moderate functions)** The differential subalgebra \( \mathcal{E}^t_M \subset \mathcal{E}^t \) of ‘moderate functions’ is

\[ \mathcal{E}^t_M(\Omega^t) := \left\{ f_\epsilon : \forall K \text{ compact in } \Omega^t, \forall \alpha \in \mathbb{N}_0^n, \exists N \in \mathbb{N}_0 \text{ such that}, \right. \]

\[ \left. \sup_{\vec{x} \in K} (1 + |\vec{x}|)^{-N}|D^\alpha f_\epsilon(\vec{x})| = O\left(\frac{1}{\epsilon N}\right) \text{ as } \epsilon \to 0 \right\}. \]  

(18.7)

**Definition 8 (Temperate negligible functions)** The differential ideal \( \mathcal{N}^t \subset \mathcal{E}^t_M \) of ‘negligible functions’ is

\[ \mathcal{N}^t(\Omega^t) := \left\{ f_\epsilon : \forall K \text{ compact in } \Omega^t, \forall \alpha \in \mathbb{N}_0^n, \exists m \in \mathbb{N} \text{ such that}, \forall q \in \mathbb{N}, \right. \]

\[ \left. \sup_{\vec{x} \in K} (1 + |\vec{x}|)^{-m}|D^\alpha f_\epsilon(\vec{x})| = O(\epsilon^q) \text{ as } \epsilon \to 0 \right\}. \]  

(18.8)

**Definition 9 (Special algebra of tempered \( G \)-functions)** The special Colombeau algebra is the quotient

\[ \mathcal{G}^t(\Omega^t) := \frac{\mathcal{E}^t_M(\Omega^t)}{\mathcal{N}^t(\Omega^t)}. \]  

(18.9)
References

Distributions

[1] L. Schwartz, Théorie des Distributions (new edition, Hermann, Paris 1966) 420 pp.

[2] Y. Choquet-Bruhat, C. DeWitt-Morette, and M. Dillard-Bleick, Analysis, Manifolds and Physics (North-Holland, Amsterdam, 1982) 630 pp.

[3] T. Schücker, Distributions, Fourier transforms, and Some of Their Applications to Physics (World Scientific, Singapore, 1991) 167 pp.

Colombeau generalized functions

[4] J.F. Colombeau, New Generalized Functions and Multiplication of Distributions, North-Holland Math. Studies 84 (North-Holland, Amsterdam, 1984) 375 pp.

[5] J.F. Colombeau, Elementary Introduction to New Generalized Functions, North-Holland Math. Studies 113 (North Holland, Amsterdam, 1985) 281 pp.

[6] T.D. Todorov, Sequential approach to Colombeau’s theory of generalized functions, Report IC/87/126 (International Center for Theoretical Physics, Trieste, July 1987) 53 pp.

[7] H.A. Biagioni, A Nonlinear Theory of Generalized Functions, Lecture Notes in Mathematics 1421 (Springer Verlag, 1990) 215 pp.

[8] J. Schmeelk, A guided tour of new tempered distributions, Found. Phys. Lett. 3 (1990) 403–423.

[9] J.F. Colombeau, Multiplication of distributions, Bull. Am. Math. Soc. 23 (1990) 251–268.

[10] M. Nedeljkov, S. Pilipović, and D. Scarpalézos, The Linear Theory of Colombeau Generalized Functions, Pitman Research Notes in Mathematics 385 (Longman, Harlow, 1998) 156 pp.

[11] M. Grosser, M. Kunzinger, M. Oberguggenberger, and R. Steinbauer, Geometric Theory of Generalized Functions with Applications to General Relativity, Mathematics and its Applications 537 (Kluwer Acad. Publ., Dordrecht-Boston-New York, 2001) 505 pp.
Partial differential equations and waves equations

[15] M. Oberguggenberger, Multiplication of Distributions and Applications to Partial Differential Equations, Pitman Research Notes in Mathematics 259 (Wiley & Sons, New York, 1992) 312 pp.

[16] H. A. Biagioni. M. Oberguggenberger, Generalized solutions to the Korteweg - de Vries and the regularized long-wave equations, SIAM J. Math. Anal. 23 (1992) 923–940.

[17] M. Oberguggenberger, Y.-G. Wang, Reflection of delta-waves for nonlinear wave equations in one space variable, Nonlinear Analysis 22 (1994) 983–992.

[18] A. Antonevitch, The Schrödinger equation with point interaction in an algebra of new generalized functions, in M. Grosser, et al., Eds., Nonlinear Theory of Generalized Functions, Research Notes in Math. 401 (Chapman and Hall/CRC, London) 23–34.

[19] M. Oberguggenberger, Generalized solutions to nonlinear wave equations, Mat. Contemp. 27 (2004) 169–187.

[20] M. Oberguggenberger, Colombeau solutions to nonlinear wave equations (15 December 2006) 14 pp. e-print [arXiv:math/0612445]

[21] E. Mayerhofer, The Wave Equation on Singular Space-times, Doctoral dissertation (University of Vienna, 1 August 2006) 102 pp. e-print [arXiv:0802.1616]

General physics and mechanics
[22] S. Konjik, M. Kunzinger, and M. Oberguggenberger, *Foundations of the calculus of variations in generalized function algebras* (12 July 2007) 27 pp. e-print [arXiv:0707.1842](https://arxiv.org/abs/0707.1842).

[23] G. Hörmann and L. Oparnica, *Distributional solution concepts for the Euler-Bernoulli beam equation with discontinuous coefficients* (12 April 2007) 13 pp. e-print [arXiv:math/0606058](https://arxiv.org/abs/math/0606058).

**Electrodynamics**

[24] G. Temple, *Theories and applications of generalized functions*, J. Lond. Math. Soc. **28** (1953) 134–148.

[25] F.R. Tangherlini, *General relativistic approach to the Poincaré compensating stresses for the classical point electron*, Nuovo Cim. **26** (1962) 497–524.

[26] G. Hörmann and M. Kunzinger, *Nonlinearity and self-interaction in physical field theories with singularities*, Integral Transf. Special Funct. **6** (1998) 205–214.

[27] A. Gsponer, *A concise introduction to Colombeau generalized functions and their applications to classical electrodynamics* (2006) 19 pp. To be published in Eur. J. Phys. e-print [arXiv:math-ph/0611069](https://arxiv.org/abs/math-ph/0611069).

[28] A. Gsponer, *Distributions in spherical coordinates with applications to classical electrodynamics*, Eur. J. Phys. **28** (2007) 267–275; Corrigendum Eur. J. Phys. **28** (2007) 1241. e-print [arXiv:physics/0405133](https://arxiv.org/abs/physics/0405133).

[29] A. Gsponer, *On the electromagnetic momentum of static charge and steady current distributions*, Eur. J. Phys. **28** (2007) 1021–1042. e-print [arXiv:physics/0702016](https://arxiv.org/abs/physics/0702016).

[30] A. Gsponer, *The classical point-electron in Colombeau’s theory of generalized functions*, J. Math. Phys. **49** (2008) 102901 (22 pages). e-print [arXiv:0806.4682](https://arxiv.org/abs/0806.4682).

**Hydrodynamics**

[31] J.F. Colombeau and A.Y. Le Roux, *Multiplication of distributions in elasticity and hydrodynamics*, J. Math Phys. **29** (1988) 315–319.

[32] J.F. Colombeau, *The elastoplastic shock problem as an example of the resolution of ambiguities in the multiplication of distributions*, J. Math. Phys. **30** (1989) 2273–2279.
[33] J.F. Colombeau, Multiplication of Distributions — A tool in Mathematics, Numerical Engineering and Theoretical Physics, Lect. Notes in Math. 1532 (Springer-Verlag, Berlin, 1992) 184 pp.

[34] J. Hu, The Riemann problem for pressureless fluid dynamics with distribution solution in Colombeau’s sense, Comm. Math. Phys. 194 (1998) 191–205.

[35] S. Bernard, J.-F. Colombeau, A. Meril, and L. Remaki, Conservation laws with discontinuous coefficients, J. Math. Anal. Appl. 258 (2001) 63–86.

[36] J.F. Colombeau, Nonlinear generalized functions and nonlinear numerical simulations in fluid and solid continuum mechanics (5 February 2007) 17 pp. e-print arXiv:math-ph/0702014.

[37] R. S. Baty, F. Farassat, and D. H. Tucker, Nonstandard analysis and jump conditions for converging shock waves, J. Math. Phys. 49 (2008) 063101 (18 pages).

**General Relativity**

[38] R. Steinbauer, The ultrarelativistic Reissner-Nordstrøm field in the Colombeau algebra, J. Math. Phys. 38 (1997) 1614–1622. e-print arXiv:gr-qc/9606059

[39] H. Balasin, Geodesics for impulsive gravitational waves and the multiplication of distributions, Class. Quant. Grav. 14 (1997) 455–462.

[40] R. Steinbauer, Geodesics and geodesic deviation for impulsive gravitational waves, J. Math. Phys. 39 (1998) 2201–2212.

[41] M. Kunzinger and R. Steinbauer, A rigorous solution concept for geodesic and geodesic deviation equations in impulsive gravitational waves, J. Math. Phys. 40 (1999) 1479–1489.

[42] M. Grosser, M. Kunzinger, M. Oberguggenberger, and R. Steinbauer, Geometric Theory of Generalized Functions with Applications to General Relativity, Mathematics and its Applications 537 (Kluwer Acad. Publ., Dordrecht-Boston-New York, 2001) 505 pp.

[43] A. Gsponer, On the physical interpretation of singularities in Lanczos-Newman electrodynamics, Report ISRI-04-04 (2004) 22 pp. e-print arXiv:gr-qc/0405046
[44] M. Kunzinger, R. Steinbauer, and J.A. Vickers, Generalised connections and curvature, Math. Proc. Cambridge Philos. Soc., 139 (2005) 497–521.

[45] R. Steinbauer and J.A. Vickers, The use of generalized functions and distributions in general relativity, Class. Quant. Grav. 23 (2006) R91–114. e-print arXiv:gr-qc/0603078.

[46] C. Castro, The Euclidian gravitational action as black hole entropy, singularities, and spacetime voids, J. Math. Phys. 49 (2008) 042501 (30 pages).

Quantum field theory

[47] J. Schmeelk, Fourier transforms in generalized Fock spaces, Internat. J. Math. and Math. Sci. 13 (1990) 431–442.

[48] E. Charpentier, Sur l’´elimination des “infinis” en th´eorie quantique des champs : la r´egularisation zeta`a l’´epreuve de l’interpr´etation de Colombeau et vice versa, Diss. Math. 383 (Polish Academy of Sciences, Warszawa, 1999) 56 pp.

[49] H. Grosse, M. Oberguggenberger, and I.T. Todorov, Generalized functions for quantum fields obeying quadratic exchange relations, Proc. Steklov Inst. Math. 228 (2000) 81–91. e-print arXiv:math-ph/9902008.

[50] J.F. Colombeau, A. Gsponer, and B. Perrot, Nonlinear generalized functions and the Heisenberg-Pauli foundations of quantum field theory (2007) 20 pp. e-print arXiv:0705.2396.

[51] J.F. Colombeau and A. Gsponer, The Heisenberg-Pauli canonical formalism of quantum field theory in the rigorous setting of nonlinear generalized functions (Part I) (2008) 101 pp. e-print arXiv:0807.0289.