Did the Pseudo-Sphere Universe have a Beginning?

G.Oliveira-Neto †

Department of Physics
University of Newcastle-Upon-Tyne
Newcastle-Upon-Tyne, NE1 7RU
U.K.

Pseudo-Spherical Universes

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ABSTRACT

A calculation of the no-boundary wave-function of the universe is put forward for a spacetime with negative curvature. A semi-classical Robertson-Walker approximation is attempted and two solutions to the field equations, one Lorentzian and the other a tunneling one are found. The regularity of those solutions are analysed explicitly, both in $2+1$ and $3+1$ dimensions and a conical singularity is found at the origin of the time axis, contradicting the no-boundary assumption.

1 - Motivation and Introduction

During the last few years, some effort has been spent on understanding models

† Email address: G.OLIVEIRA-NETO@NEWCASTLE.AC.UK
of the universe with negative spacetime curvature[1]. One can think of at least
two main reasons, if the pleasure derived from the geometrical insights furnished
by those models is not enough: The low energy limit of Supergravity with its
negative energy vacuum state [2] and Wheeler’s idea of a foam like structure for
the universe at the Planck scale[3], with all possible topologies contributing to
spacetime.

Our motivation has to do with the second reason. We are interested in build-
ing up wave-functions for universes with negative curvature in a consistent way,
such that we could in the future use them to describe a more complete picture,
which would take in account contributions coming from general topologies. Many
examples of consistent wave-functions for space-times with positive curvature have
already been examined [4]. Consistency of the wave-function involves the defini-
tion of sensible initial conditions, and so far the most appealing proposal is the
one devised by Hartle & Hawking [5], the so called no-boundary proposal.

What we ultimately shall be doing here is trying to derive the no-boundary
wave-function for a space-time with negative curvature, where we foliate the space-
time in spatial sections which evolve in time [6]. We shall not try to derive general
properties of these wave-functions, rather we shall choose a simple example and
work out the particular wave-function arising from this restricted choice. For
mathematical simplicity, and also to account for the present observational data
we shall choose our model to be homogeneous and isotropic, which leads us into a
minisuperspace approximation. So, our metric ansatz will be Robertson-Walker,
such that the spatial sections will have negative curvature, and our unique source
of mass-energy will be a cosmological constant, negative in this case. Besides that,
we shall be concerned with the semi-classical approximation of the no-boundary
wave-function. In this task we shall have to overcome two basic problems:
(1)It is well know that the Hartle & Hawking proposal demands that all space-
times contributing to the wave-function have compact spatial sections. But one’s intuition does not deal easily with the idea of a compact surface with negative curvature [6].

(2) For spacetimes with negative curvature, the signature of solutions is likely to be Lorentzian, whereas the no-boundary proposal prefers Riemannian signature.

Before we proceed any further we should ask what is known already. A couple of years ago, Gibbons & Hartle [7] devised some general statements about the predictions of the no-boundary wave-function for the present large-scale topology and geometry of the universe. There, they argued that among the complex solutions of the Euclidean Einstein’s equations, important in the semi-classical limit, the purely real tunneling ones play a special role. The simplest example of those solutions occurs in a model like ours, differing by the opposite sign of the cosmological constant. In this model, the real tunneling solution (which corresponds to discontinuous metrics) is given by joining of Riemannian metric with a Lorentzian one along a surface characterized by the vanishing of its second fundamental form.

One important point is that the Riemannian region is related to the early universe, in the sense that the scale factor assumes in this sector its smallest values, and grows up until one well defined value, which is dependent on the unique physical parameter in this problem, the cosmological constant. On the other hand, the Lorentzian region, has the property of starting off with the final scale factor from the foregoing region and increases up to the value assigned by the final boundary condition. The schematic picture below (borrowed from them), gives a better idea of what was said above.

The question is: Can we take this result for granted in the case of negatively
1-Tunneling Solution

curved spacetimes? The authors mentioned above [1], assumed the above result also valid for spacetimes with negative curvature, but it will be argued otherwise here.

In the next section, we shall derive the no-boundary solutions for the simple model. In section 3, with the help of few mathematical results, we shall be able to see which are the compact spatial sections allowed in this case. In section 4, we shall determine if these solutions are regular or not.

2 - Euclidean Negatively Curved Robertson-Walker Spacetime

The general expression for the no-boundary wave-function for a space-time [8] is simplified by the assumptions that it is homogeneous, isotropic and its unique source of mass-energy is a negative cosmological constant. This leads to the Robertson-Walker metric ansatz, with the lapse function \( N \) and the scale factor \( a \) depending only on \( t \) and the spatial sections being three-dimensional surfaces with negative curvature (we could have chosen to foliate our four manifold in a different way, but one’s intuition is led to the choice below):
\[ ds^2 = + N^2(t) \, dt^2 + a^2(t) [d\chi^2 + \sinh^2 \chi(d\theta^2 + \sin^2 \theta \, d\phi^2)] \] (1)

With this expression for the metric, the wave-function reduces in complexity hugely to the more workable form, (in the gauge which \( N \) is constant):

\[ \Psi_{nb}(\tilde{a}) = \int dN \int D[a(t)] e^{-I[a(t), N]} \] (2)

where \( I[a(t), N] \) is the Euclidean action for the various different geometries contributing through the integrals of \( N \) and \( a(t) \) to the wave-function. It is given by:

\[ I[N(t), a(t)] = \frac{1}{16\pi} \int (R - 2\Lambda) \sqrt{+g} \, dx^4 + \frac{1}{8\pi} \int K \sqrt{+h} \, dx^3 \] (3)

in this expression we set the velocity of light and the gravitational constant to unity, \( R \) is the scalar curvature, \( K \) is the the trace of the extrinsic curvature of the spacelike boundary surface, \( \Lambda \) is the cosmological constant, \( g \) is the trace of the metric and \( h \) is the trace of the metric on the three-dimensional spatial sections. From the metric ansatz (1), the action (3), can be straightforwardly computed,

\[ I[N, a(t)] = Z \int [\dot{a}^2(t) - N^2 - a^2(t) N^2 \frac{\Lambda}{3}] a(t) \frac{dt}{N} \] (4)

where \( Z \) is a finite defined number, once one specifies the compact spatial slices which will foliate the whole manifold.
All of the instructions needed in order to write down the semi-classical approximation of the no-boundary wave-function (2) for a given spacetime, of which ours is a simple case, are given by Halliwell & Louko [9]. The first step we shall have to make to get the wave-function in the desired approximation, which is given explicitly below, is to find the solutions to the Euclidean-Einstein’s equations. Then,

$$\Psi_{nb}(a) = \sum_k P_k(a) e^{-I_k(a)}$$

Where, $I_k$ are the euclidean actions of the solutions of the Euclidean-Einstein’s equations for the given metric ansatz. $P_k$ is the prefactor and $k$ labels the various possible classical solutions over which we must sum in order to obtain the correct wave-function and $a$ is the scale factor on the final boundary surface.

The pair of independent equations generated by the above metric ansatz (1), coming from the Euclidean-Einstein equations, with a negative cosmological constant are:

$$2\ddot{a}a + \dot{a}^2 - N^2(1 - \frac{a^2}{|\Lambda|}) = 0 \quad (6.1)$$

$$\dot{a}^2 + N^2(1 - |\Lambda|\frac{a^2}{3}) = 0 \quad (6.2)$$

Now, in possession of the general solutions, we must specify that they satisfy the ‘no-boundary’ boundary conditions. For our case [9]:

(1) The manifold must be regular at every point;

(2) There should be a point where the scale factor vanishes;
(3) We should supply the real valued scale factor of the final spatial slice. The first condition we shall tackle after the other two are implemented. The second one is a particularity of the minisuperspace treatment, and we shall choose the point to be the zero value of the time scale. With this choice we shall be able to picture our universe starting at $t=0$, from this surface of zero volume and evolving until $t=t_1$, where we shall furnish the other required value of the scale factor, say, $a(t_1) = a_1$. Introducing those quantities in the above (6.1) and (6.2) equations, and rescaling the time scale in order to write $t_1$ equal to 1, we get the two solutions given below:

The Lorentzian Solution:

$$N = iN_I, \quad \text{where} \quad N_I = \alpha \frac{\pi}{2} \pm \arccos\left[ \frac{1}{\alpha a_1} \right]$$  \hspace{1cm} (7.1)$$

$$a(t) = \pm \alpha \sin\left[ N_I \frac{t}{\alpha} \right] \quad ; \quad \alpha^2 = \frac{3}{|\Lambda|}$$ \hspace{1cm} (7.2)$$

valid for $a_1 < \alpha$. It is important to notice that among the various solutions for $N$, labeled by an integer $m$, we have chosen as a matter of simplicity, the case $m = 0$ (the same remark holds for the complex solution given below). The resulting metric is:

$$ds^2 = -N_I^2 dt^2 + \alpha^2 \sin\left[ \frac{1}{\alpha} N_I t \right]^2 \left[ d\chi^2 + \sinh^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2) \right]$$ \hspace{1cm} (7.3)$$

The complex solution:

$$N = \alpha (N_R + iN_I), \quad \text{where} \quad N_I = \frac{\pi}{2}$$ and
\[ N_R = \arcsin \beta \quad ; \quad \beta = \mp \sqrt{\left( \frac{a_1}{\alpha} \right)^2 - 1} \] (8.1)

\[ a(t) = a_R(t) + ia_I(t) \quad ; \quad \text{where} \]

\[ a_R(t) = \mp \alpha \sin(N_I t) \cosh(N_R t) \quad \text{and} \]

\[ a_I(t) = \pm \alpha \cos(N_I t) \sinh(N_R t) \] (8.2)

and for this case the metric is a complex one, as written below:

\[ ds^2 = \alpha^2(N_R^2 - N_I^2)dt^2 + (a_R^2(t) - a_I^2(t))\tilde{\Omega}_3 + 2i\alpha^2(N_R N_I dt^2 + a_R(t)a_I(t)\tilde{\Omega}_3) \] (8.3)

where \( \tilde{\Omega}_3 \) stands for the metric of the three-dimensional pseudo-sphere, and this solution holds for \( a_1 > \alpha \).

The natural sequence of our investigation, would be to address the question of regularity, as demanded by the condition (1) above, to the two solutions just found, in order for those to be regarded as truly no-boundary solutions of our model. It is important to notice, that these solutions are well defined for all \( t \), except \( t = 0 \). There, the metrics vanish identically, and our analysis of regularity must be focused on this specific point. But before we begin we will try to write our complex solution in the form of a real tunneling one.

As one can see, from the expression of the semi-classical no-boundary wave-function, the quantities which dominate the wave-function are the classical actions, not the metrics themselves. It may be possible to find another metric for which,
the new action is a composition of two actions: the first derived from a Lorentzian metric (which will give the purely imaginary contribution to the action) and the second from a Riemannian metric (responsible for the real part). Fortunately, for our simple model, it is easy to show that such an interpretation of the complex solution is applicable.

We should start by inspecting our action (4), following the path of Halliwell & Hartle [10]. It is easy to see that (4) can be rewritten as an integral over a complex variable, $T$, defined as the product $t \times N$. In other words,

$$ I[a(T)] = Z \int \left[ \left( \frac{da(T)}{dT} \right)^2 - 1 - a^2(T) \frac{\Lambda}{3} \right] a(T) dT \quad (9) $$

Now, we want to deform the contour of integration on the complex $T$ plane, such that our contour breaks off in two parts (in this simple case): one running along or parallel to the real axis (what would be interpreted as the Riemannian sector), and another running along or parallel to the imaginary $T$ axis (interpreted as the Lorentzian sector). One is left with two choices of contour, which are easily determined from the value of the coordinate $T$ at the final boundary surface. The value of the on this surface is:

$$ \bar{T}_f = \bar{T}_{Rf} + i \bar{T}_{If} \quad \text{where,} $$

$$ \bar{T}_{Rf} = \alpha \arcsin \beta \quad ; \quad \bar{T}_{If} = \frac{\alpha \pi}{2} \quad (10) $$

Then, our two possible contour are given in the diagram below:
The circuit (a) above, was extensively studied in reference [10]. The condition one has to satisfy in order to have a purely imaginary action, when our variable \( T \) runs along the second part of this circuit (ii), is that the first derivative of the scale factor at the point where this part of the circuit starts, \( P_a \) in the diagram above, must vanish (this is the condition that the extrinsic curvature vanishes at this point, for our simple case). From the explicit value of \( a(T) \), in terms of this complex variable:

\[
a(T) = -i\alpha \sinh\left(\frac{T}{\alpha}\right); \quad \text{where the minus sign was chosen by convenience.} \tag{11}
\]

It is not a great deal of work to find that the first derivative at the point \( \bar{T} = \bar{T}_{Rf} \) (10), is not zero and rather has the value:

\[
\left.\frac{dT(T)}{dT}\right|_{\bar{T}_{Rf}} = -i \cosh(\text{arcsinh} \beta) \tag{12}
\]

Therefore it is not possible to have a purely imaginary action coming from the circuit where our complex variable \( T \) has a fixed real component and a variable purely imaginary one. So we are left with the option of the second circuit (circuit (b) diagram above). In the first part of the circuit (i), our variable is purely imaginary and can be rewritten as:

\[
\bar{T} = iT; \quad T \text{ is real and has the following range } 0 < T < \bar{T}_{If} \tag{13}
\]
Which implies that the scale factor must have the value below:

\[ a(T) = \alpha \sin\left(\frac{T}{\alpha}\right) \]  

(14)

This part of the circuit gives rise to a Lorentzian metric, contributing with a purely imaginary action, such that our universe starts off with a zero volume and reaches the joining surface with a scale factor \( \alpha \). In the point \( P_b \) of the diagram above, our variable turns abruptly to be fully complex, with a fixed imaginary component and a variable real one. There, along this straight line (ii), we could introduce the following quantity:

\[ \bar{T} = T + i\bar{T}_{ff} \quad ; \quad T \text{ is real and has the limits} \quad 0 < T < \bar{T}_{Rf} \]  

(15)

And from the expression of the scale factor (11) and with the help of (10), we can compute the value of the scale factor over this sector of our contour:

\[ a(T) = \alpha \cosh\left(\frac{T}{\alpha}\right) \]  

(16)

which is real. Then this sector contributes with a Riemannian metric and a real action. Now in this region, the universe has a scale factor \( \alpha \) on the transition surface, which increases until reaches the final boundary surface. As an authentic solution of our equations, its interpretation is not an immediate process, since it points to a universe (if no other phenomenon takes place) which is at large-scale Riemannian!
Once the first stage of determination of the solutions of the classical Euclidean-Einstein's equations is over, we must turn to the second and more complex one. We must find compact surfaces with negative curvature, which will represent our spatial-slices. But before that it is necessary to point out that we have arrived at a very important result, which will facilitate our analysis of regularity immensely: the singular point in each of the two above solutions (7.3), (8.3) or (14), is found to be in the region where our spacetimes are Lorentzian ones. Then, studying the regularity of one of the solutions in this region will be enough to draw conclusion for both of them. As a matter of simplicity, we shall choose the Lorentzian solution (7.3), which will easier to be pictured as a whole.

3 - Compact Negatively Curved Surfaces in Two and Three Dimensions

The idea of compact surfaces with negative curvature is not a new one [12] but its actual implementation in quantum cosmology is very recent [2]. It is interesting to note that in other branches of physics, e.g. classical and quantum chaotic systems, two dimensional compact surfaces of negative curvature have been studied for some time [13]. We shall need now some mathematics in order to justify the next step, which is the determination of the compact spatial sections of the solutions of the Euclidean-Einstein's equations. But before that, it is important to stress that we shall start working in three spacetime dimensions in order to understand the basic features of those compact spatial sections (in this case two-dimensional), and also the application of our procedure to check whether those solutions are regular or not. Then the natural next step will be the analysis of a more realistic case in which the spacetimes will have four-dimensions (and consequently three-dimensional compact spatial sections).
In two dimensions, it was shown in a survey made by Peter Scott [14] that every closed orientable surface of genus at least two allows a geometric structure modeled on the two-dimensional hyperbolic space \( H^2 \), or in summary admits a hyperbolic structure. This result enables one to select an example of a two-dimensional compact surface. It will be a closed orientable surface of genus two.

Unfortunately, there is not such a neat result for three-dimensional hyperbolic spaces \( H^3 \), but much work was done by W. P. Thurston [15], in this area, and from his work we were able to collect one example of a compact negatively curved surface in three-dimensions and it will be this example, which we shall discuss below, that will represent the spatial sections of our four-dimensional hyperbolic world.

As we have said at the end of the last section, our analysis of regularity will be based on the Lorentzian solution, which is simply anti-DeSitter spacetime. Turning to Hawking & Ellis [16], we learn that the anti-DeSitter spacetime, can be represented as the hyperboloid:

\[
-v^2 - u^2 + x^2 + y^2 + z^2 = -1
\]  

(17)

in the flat five-dimensional space \( R^5 \) with metric:

\[
ds^2 = -dv^2 - du^2 + dx^2 + dy^2 + dz^2
\]

(18)

and our Lorentzian solution, (7.3), is obtained by the following transformation of variables:
\[ v = \alpha \cos(N_I \frac{t}{\alpha}) \]
\[ u = \alpha \sin(N_I \frac{t}{\alpha}) \cosh \chi \]
\[ x = \alpha \sin(N_I \frac{t}{\alpha}) \sinh \chi \cos \theta \]
\[ y = \alpha \sin(N_I \frac{t}{\alpha}) \sinh \chi \sin \theta \cos \phi \]
\[ z = \alpha \sin(N_I \frac{t}{\alpha}) \sinh \chi \sin \theta \sin \phi \]

(19)

It is important to comment at this point, that our solution does not cover the whole anti-DeSitter spacetime and there is other coordinate chart which perform this task in a more complete way [16].

Now, if one wants to take a glance at the spatial sections of that manifold, one has to assign a certain value to the coordinate \( t \), which is the same (through (19)) as specifying a value for \( v \), say \( v_0 \). We can see that, equation (17) is transformed to:

\[ u^2 - x^2 - y^2 - z^2 = 1 - v_0^2 \]

(20)

And, the above equation is easily recognizable as the one for a hyperboloid of two sheets but not in Euclidean space, rather in a Minkowskian space. Each of the two sheets is a representation of the 3-dimensional Hyperbolic space \( H^3 \), but it is very difficult to work with those surfaces in this form. A more profitable procedure would be to project the whole of \( H^3 \) into \( R^3 \), such that the projection is isometric. There is more than one stereographic projection of the \( H^3 \), and each of these is suitable for one certain kind of problem. At this point we shall split our analysis in two, in order to study two and three-dimensional surfaces separately.
3.1 - 2 + 1 Spacetimes.

In two dimensions, we shall use the Poincare Disc model for the $H^2$. The relevant spatial sections to be studied now are the ones derived by (20), through the removal of the $z$ coordinate, or,

$$u^2 - x^2 - y^2 = 1 - v_0^2$$  \hspace{1cm} (21)

This is a two-dimensional hyperboloid with two sheets embedded in a Minkowskian space. The Poincare disc model is build up as the projection of the upper sheet of the hyperboloid (21) above, on the plane $(u, x, 0)$, taking as the base point of the projection, the apex $(0, 0, -\sqrt{1 - v_0^2})$. It is important to notice, that our model differs a little from the ordinary one \[13\], in the fact that the apexes are not situated at $(0, 0, 1)$ and $(0, 0, -1)$. The ordinary case is a limiting case when $v_0(t)$ is zero and the other relevant limit, in fact the one we shall be concerned with is furnished when $v_0(t)$ goes to 1. The disc is fabricated such that its center projects the upper apex of the hyperboloid and the outer limiting circumference of radius $\sqrt{1 - v_0^2}$, represents the points at infinity of $H^2$. There is a simple relationship between the coordinates $(\chi, \theta)$ on the $H^2$, and the polar coordinates $(r, \theta)$ on the P.D.,

$$r = \sqrt{1 - v_0^2} \tanh(\frac{\chi}{2}); \quad \theta \text{ is unchanged}$$  \hspace{1cm} (22)

The conformally isometric mapping produces the metric below,
\[ ds^2 = 4(1 - v_0^2)^2 \frac{(dr^2 + r^2d\theta^2)}{(1 - v_0^2 - r^2)^2} \]  

(23)

The mathematical process by which we shall obtain our compact two-dimensional spatial sections is a tessellation [18], a covering of the P.D. by congruent (isometric) repetition of some basic figure, the fundamental region. The fundamental region is defined as the quotient surface \( H^2/\Gamma \), where \( \Gamma \) is a certain symmetry transformation of the \( H^2 \).

The relevant symmetry transformation \( \Gamma \), for a closed orientable region of genus two, would be the ordinary translation if we were on the Euclidean plane (consider the example of a torus derived from identification of opposite sides of a square [19]), but in the Minkowskian space the translation operations are performed by the Lorentzian boosts can be written in the linear fractional form [13]:

\[ z' = T(z) = \frac{(\alpha z + \beta)}{(\beta^*z + \alpha^*)} \]  

(24)

with \( |\alpha|^2 + |\beta|^2 = 1 \) and \( z \) defined as \( z = re^{i\theta} \)

and \( T \) has also a matrix representation given below,

\[
T = \begin{pmatrix}
\alpha & \beta \\
\beta^* & \alpha^*
\end{pmatrix}
\]  

(25)

Our compact surface will be fabricated then by the identification of the sides, through (24), of a polygon (the relevant fundamental region) on the P.D.. The simplest polygon is the regular octagon, with the opposite sides identified, as shown in the figure below.
3-Identified Octagon

For each of the four relevant directions of the transformation we have a generator which is given in its matrix form by:

\[
T_k = \begin{pmatrix}
\cosh\left(\frac{\chi_0}{2}\right) & e^{ik\frac{\pi}{4}} \sinh\left(\frac{\chi_0}{2}\right) \\
e^{-ik\frac{\pi}{4}} \sinh\left(\frac{\chi_0}{2}\right) & \cosh\left(\frac{\chi_0}{2}\right)
\end{pmatrix}
\]  

where \(k = 0, 1, 2, 3\); and \(\chi_0\) is the rapidity over which we have transformed the point on \(H^2\). That octagonal fundamental region with identified opposite sides, has a representation in a three-dimensional space known as: Sphere with two handles or Double Torus \((T_2)\), drawn below:

4-Double Torus

When one notices that the metric on the P.D. is induced on the double torus, one has arrived to the desired two-dimensional compact negatively curved spatial sections. Then, our three-dimensional spacetime is composed of two disjointed double tori, evolving in a symmetrical way in time. The analysis made for one of them will reveal the behaviour of the other.

3.2 - 3 + 1 Spacetimes.
In this case, the model we shall use to describe the three-dimensional negatively curved spatial sections $H^3$, is called the projective model [15]. The first step in order to implement this projection is to repeat what was done in the previous sub-section, to obtain the Poincare Disc model in this higher dimensional space (P.D.3). The metric of this P.D.3 is given by the same formal expression as (23) but now $r$ stands for the radius of a three-dimensional spherical coordinate system. The three-dimensional Poincare Disc obtained in this way passes through the centre of a three sphere. On this three-sphere we can define a Northern Hemisphere and a Southern Hemisphere. The following step is the stereographic projection of the P.D.3 on the Southern Hemisphere of the $S^3$, taking as the base point the northern pole. Finally, perform a Euclidean orthogonal projection of the Southern Hemisphere back to the P.D.3., and this map of P.D.3 onto itself, along with the induced metric coming from the P.D.3, is the projective model of $H^3$ (Pr. 3). It is important to note that in this model the geodesics are straight lines but unfortunately the projection fails to be conformal, in other words angles are not preserved.

The fabrication of our desired three-dimensional compact negatively curved spatial sections can proceed now on this projective model. Let us start with a regular tetrahedron inscribed in the Pr.3, such that all four vertices are on the two-sphere at infinity. The dihedral angles of this ideal hyperbolic simplex are $\pi/3$ (here we have an astonishing verification of the non-conformality of the projective model). Now, let the sides and faces expand in a homogeneous and isotropic way, until the dihedral angles reach the value $\pi/6$. It can be shown, that there exists for each vertex of our simplex, a two-dimensional surface which is perpendicular to all three faces having one of those vertices in common. Along those planes, truncate the tetrahedron such that all four vertices are deleted and all points of our simplex are inside of the Pr.3. Now, repeat all the above steps in order to get
another copy of the truncated figure \( T \), then glue what was left of the former faces following the pattern given below.

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5-The glueying rules to form \( M \)

The resulting complex \( M \) has one boundary. One can show that this boundary is a two-dimensional surface of genus 2, more specifically a double torus (the one constructed in the previous sub-section). But we want a compact region, without boundary. So, we must take two copies of \( M \) and glue them together by identifying the boundaries in a one-to-one map. In the following figure we shall show the identifications of the boundaries of \( T \) in order to obtain the final compact three-dimensional surface.
6-Identification of the boundaries of two complexes $T$

It is that complex, which we shall call $P$, with the metric induced by the Pr.3 which will represent each disjoint sequence of three-dimensional spatial sections of our negatively curved four-dimensional world. As in the previous case it is enough to analyse the regularity of one of the sequence of spatial sections.

Our universe is now completed, either in three or four-dimensions. We must finally turn to the ultimate goal, the determination of the regularity or not of those manifolds.

4 - Regularity Analysis

The classical solutions for the metric ansatz have been entirely determined, and we have only to answer this question: Are they regular? Regularity conditions have been given for spacetimes other than ours (see reference [6] for a brief con-
sideration of few of those proposals). Our manifolds are examples of a very simple
class of solutions of the Einstein equations: they have constant curvature or more
precisely, a constant scalar of curvature $R = -4|\Lambda|$. The important point to
make here is that we shall not be able to use most of the proposals reported in
reference [6], in order to identify our spacetimes as singular or not. This is because
these proposals use the behaviour of scalar functions, derived from the curvature
over the manifold, to identify singular points. If any of these functions has an
unbounded limit in a specific point of spacetime, we say that this point is singular
or has a singularity. This is not the case for our manifolds, for these functions will
be constant. One would be tempted to say at this stage that when all these scalar
functions are well behaved, it should imply that the manifold is regular! This
would be forgetting a distinct type of singularity, the conical singularity. Thus,
even with well defined functions of the scalar of curvature over all their points, our
manifolds could still have a conical singularity at the point $v_0 = 1$, or $t = 0$.

Our analysis, will be based entirely in the application of the concept of holonomy
[20] to the manifold being studied. The holonomy group of a manifold $M$,
equipped with a metric $g$, $(M, G)$ (therefore with an affine connection $\Gamma$), is defined
as the transformations from the tangent space at the point $p$, $T_p M$, to itself, con-
structed by taking a fiducial vector $v$ and parallel transporting it along all closed
curves starting and finishing at $p$ on $T_p M$. For a manifold, the set of all linear
transformations generated by the parallel transport of $v$ along all possible closed
curves from $p$, is called the holonomy group of $M$ at $p$. In fact, we shall not be
interested in computing the holonomy group of the given manifold but we shall use
this concept in a related way. For a manifold $(M, g)$ and a fiducial vector we shall
evaluate the transformation generated by the parallel transport over closed curves
around the point $s$. Each of these curves will have a certain proper length which
vanishes when $t$ tends to some limit $t_0$, as the points defined by closed curves col-
lapse to the point $s$. Then, if the point $s$ is regular, the limit of the transformation induced by the parallel transport of the fiducial vector around a closed curve, as $t$ goes to $t_0$, must be the identity transformation. In other words, the limit of the initial and final values of the vector at $t_0$ must be the same. If this is not the case, the manifold is non-regular at $s$. Let us write it down in a more precise manner.

Let $(M, g)$ be a given manifold factorized in the usual 3 + 1 decomposition. Let us take a point $s$ on $M$ and a certain foliation of $M$. For $t$ being the ordinary time parameter, we choose a point $p(t_a)$ in the neighbourhood $N(s)$ of $s$. We build up now the closed curve $c_p(t_a)$ such that:

$$c_p(t_a) = [c(\lambda, t_a) / c(\lambda_0, t_a) = p(t_a) = c(\lambda_f, t_a) ; \lambda_0 < \lambda < \lambda_f].$$

We furthermore imply that: for $p(t_b) \in N(s)$, if $t_a \neq t_b$ then, $p(t_b) \not\in c_p(t_a)$. With the curves $c_p(t)$ we define the map $T_p(t_a)$:

$$p(t_a) \mapsto p(t_a) / \text{it is } \nabla_{u(c_p(t_a))}v \text{ around } c_p(t_a).$$

For each $c_p(t)$, we define a function $L$ such that $L(c_p(t))$ gives one possible type of length. Now we introduce $t_0$, which is defined by the limit when $L(c_p(t))$ vanishes and $p(t_a)$ goes to $s$. We require for $M$ to be regular, the limit $\lim_{t \to t_0} T_p(t_a)$ to be the identity. Now that our method of analysis is well established, let us turn to the specific manifolds of interest.

4.1 - $2 + 1$ Spacetimes.

The first solution to be analysed and also the one we shall consider in deepest details will be the one with three spacetime dimensions and two-dimensional spatial sections, given in 3.1. The manifold $M$ is defined for this case, by the metric (7.3) without the component $g_{\phi \phi}$, and the two-dimensional metric of the compact spatial foliation derived in (23) with the aid of the coordinate transformation (22) will be used. We shall also change our coordinates in (23), from polar to Cartesian (which will simplify our study) and finally set $N_I$ and $\alpha$ to one. The starting point is the following metric,
\[ ds^2 = -dt^2 + \frac{4(1 - v_0)^2 (dx^2 + dy^2)}{(1 - v_0^2 - x^2 - y^2)^2} \]  

(27)

with the simplifications we made, \( v_0 \) is reduced to \( \cos t \), from (19). The fact that the manifold under study is a Lorentzian one will be helpful, since for this type of spacetime, no other information than the above mentioned limit of the map \( T_p(t) \) is required in order to establish its regularity.

As pointed out by J. Louko in a private communication, for Riemannian manifolds one can obtain unity limit for the map \( T_p(t) \) even for singular points on the manifold. Then, it would be important to have some extra information about the functions composing the metric.

The next step is to find out the transformation \( T_p(t) \), which will be possible only when we have the parallel transport equations. The easiest way to derive them from (27) is by working on the orthonormal basis, defined by the transformation:

\[
w^t = dt ; \\
\quad w^i = \frac{2(1 - v_0^2)dx_i}{(1 - v_0^2 - x^2 - y^2)} \quad \text{for} \quad x_1 = x \quad \text{and} \quad x_2 = y \quad (28)
\]

The non-vanishing connection coefficients components in this base are:

\[
\Gamma^t_{xx} = \frac{2\dot{v}_0(t)v_0(x^2 + y^2)}{(1 - v_0^2)(1 - v_0^2 - x^2 - y^2)} = \Gamma^x_{tx} \quad (29)
\]

\[
\Gamma^t_{yy} = \frac{2\dot{v}_0(t)v_0(x^2 + y^2)}{(1 - v_0^2)(1 - v_0^2 - x^2 - y^2)} = \Gamma^y_{ty} \quad (30)
\]
\[ \Gamma_{yy}^x = \frac{-x}{1 - v_0^2} = -\Gamma_{xy}^y \]  
\[ \Gamma_{xx}^y = \frac{-y}{1 - v_0^2} = -\Gamma_{yx}^x \]  

In this base, the general expression for the parallel transport equation is [21]:

\[ \frac{dv^\alpha}{d\lambda} + v^\beta \Gamma^\alpha_{\beta\gamma} \frac{dx^\gamma}{d\lambda} = 0 \]  

Two important remarks must be make at this point. The first is that, as we stressed before, the map \( T_p(t) \) is computed for one given value of the coordinate \( t \). It means that at the set of equations (33), we must consider the path or paths for constant \( t \). The other point is related to the fact that we are working with a non-coordinate basis and in reality we are willing to describe our paths in terms of elements of a coordinate basis. In order to do this we shall have to modify our set of parallel transport equations and introduce the relevant transformation matrix.

Then, the new set of equation is written,

\[ \frac{dv^\alpha}{d\lambda} + v^\beta \Gamma^\alpha_{\beta\gamma} \Lambda^\gamma_\delta \frac{dx^\delta}{d\lambda} = 0 \]  

where \( \Lambda \) is the basis one-form transformation matrix, also responsible for the transformation of vector components. We must now concentrate on the following question: what are all possible independent closed curves defined at one moment of time in our compact two-dimensional spatial surface? The answer comes from geometrical considerations on our fundamental region, figure 3. For our closed paths to take in account the fact that our fundamental region is compact, they should use the identification of the opposite sides at least once. In other words,
our paths must exit from one side and reappear in the opposite one, at least one
time. But of course, the paths can do so more than once, which leave us with
an infinity number of closed trajectories to be analysed. Fortunately, that infinity
number is only apparent because they are not all independent paths, more than
that, they are all constructed out of a finite number of paths. The basic paths
are in reality just four (and also the ones derived from those basic ones by means
of allowed symmetry transformations, as parity and rotation under some specific
angles) and they are the only possible ones which exit the fundamental region
at one side reappearing at the opposite side once and just once. So, all other
paths which use the identification of opposite sides more than once are built up
as combinations of those basic ones, which are given below:
Then, in order to demonstrate the regularity of our manifold, we should determine the transformations $T_p(t)$, and its limit as $v_0$ goes to 1 or identically as $t$ goes to 0, over all those four closed curves. On the other hand, if we find out that for one of the four $T_p(t)$ defined in that way, we do not have the desired limit in order for $M$ to be a regular manifold, we shall be able to conclude without any more argument that our manifold is not regular at the spacetime event under consideration. Let us start with the simplest one, (a), at the figure 7 above.

This path, is defined by: $y = 0$ and we can set the coordinate $x$ to be described by the simplest non-constant function of the parameter $\lambda$, which is $x = \lambda$ with the resulting range for $\lambda$: $-x_0(t) < \lambda < x_0(t)$. It is important to note here, that, as $\lambda$ is not a function of $t$, all information about the contraction or expansion
of our fundamental region, as the time passes is contained at the extreme values of $\lambda$, which are given by (22):

$$x = \sqrt{1 - v_0^2 \tanh(\frac{\chi_0}{2})} ; \quad \theta = 0, \pi \quad (35)$$

where $\chi_0$ is one fixed value of $\chi$. Then the parallel transport equations (34), becomes, with the aid of (28) (for the relevant components of the basis one-form transformation $\Lambda$), (29), (31) and (32) (for the necessary connection coefficient components):

$$\frac{dv^y}{d\lambda} = 0 \quad (36)$$

$$\frac{dv^t}{d\eta} + \frac{2\beta \eta^2 v^x}{\alpha(1 - \eta^2)^2} = 0 \quad (37)$$

$$\frac{dv^x}{d\eta} + \frac{2\beta \eta^2 v^t}{\alpha(1 - \eta^2)^2} = 0 \quad (38)$$

where we have defined:

$$\beta = 2\dot{v}_0 v_0 \quad ; \quad \alpha^2 = 1 - v_0^2 \quad (39)$$

and we also have defined the new variable $\eta$, related to $\lambda$ by, $\eta = \lambda/\alpha$. The range of this new variable is given according to its definition by: $-x_0(t)/\alpha < \eta < x_0(t)/\alpha$. But from (35) we see that the limits are simply $\pm \tanh(\chi_0/2)$, which are independent of $t$, then all the dependence of $T_p(t)$ now must come exclusively from the quantities $\alpha$ and $\beta$ defined above.
The solution of the equation (36) is straightforward, and for an initial value of the component $y$ of the fiducial vector to be parallel transported, equal to $v_y^0$ (or $v_y(\eta = -\eta_0) = v_y^0$), we have,

$$v_y = v_y^0$$  \hspace{1cm} (40)

The other two equations form a coupled system of first order differential equations and its solution will require a very particular change of variable as we shall see next. Define a new variable $\rho$ such that the differential variation of it is related to the differential variation of the old one $\lambda$ by: $d\rho = \frac{2\beta\eta^2}{\alpha(1-\eta^2)^2}$. Through explicit integration we can obtain a relationship between the variables themselves:

$$\rho = \frac{\beta}{\alpha} \left[ \frac{\eta}{1-\eta^2} - \frac{1}{2} \log(\frac{1+\eta}{1-\eta}) \right]$$  \hspace{1cm} (41)

and the range of this new variable is given by:

$$-\rho_0 < \rho < \rho_0$$  \hspace{1cm} (42)

where $\rho_0$ is defined such that $\rho(\eta = \eta_0) = \rho_0$ and $\rho(\eta = -\eta_0)$ is easily seen to be $-\rho_0$. The equations (37) and (38) are now expressed in terms of $\rho$ in the following very pure way:

$$\frac{dv^t}{d\rho} + v^x = 0$$  \hspace{1cm} (43)
\[
\frac{dv^x}{d\rho} + v^t = 0 \quad (44)
\]

The general solution is given in terms of two constants, \( A \) and \( \delta_0 \), to be determined by the initial conditions, and that solution is:

\[
v^x(\rho) = A \sinh(\rho + \delta_0) \quad (45)
\]

\[
v^t(\rho) = A \cosh(\rho + \delta_0) \quad (46)
\]

The initial conditions will be given by: \( v^x(\rho = -\rho_0) = v^x_0 \) and \( v^t(\rho = -\rho_0) = v^t_0 \). If one introduces those quantities in the equations (45) and (46), after some algebraic calculations one finds,

\[
v^x(\rho) = \cosh(\rho + \rho_0)v^x_0 + \sinh(\rho + \rho_0)v^t_0 \quad (47)
\]

\[
v^t(\rho) = \sinh(\rho + \rho_0)v^x_0 + \cosh(\rho + \rho_0)v^t_0 \quad (48)
\]

Then, joining together all the solutions of our parallel transport equations, (40), (47) and (48), we can write them in a matrix form, from which it will be straightforward for one to read the \( T_p(t) \) map.

\[
v(\rho) = T_p(t)v_0, \quad \text{where}
\]

\[
T_p(t) = \begin{pmatrix}
\cosh(\rho + \rho_0) & \sinh(\rho + \rho_0) & 0 \\
\sinh(\rho + \rho_0) & \cosh(\rho + \rho_0) & 0 \\
0 & 0 & 1
\end{pmatrix} \quad (49)
\]
For simplicity and relaxing a little the proper use of the concepts, we shall refer to the matrix $T_p(t)$ as ”holonomy matrix”. It is important to notice the general behaviour of a fiducial vector parallel transported along the chosen closed curve, which can be readily seen with the help of the holonomy matrix. If we set $\rho = -\rho_0$, at (49), which means that we are in the initial point, we get the initial value of the fiducial vector, as defined. But if we set $\rho = \rho_0$, in general, our vector does not return to the initial value, although by definition our curve has returned to its initial point. This fact, was expected, and it is just a verification that our manifold has a non-vanishing curvature. The final step in our present analysis, is the evaluation of the holonomy matrix after a complete loop in our closed curve in the limit when $t$ goes to 0.

The relevant components of the holonomy matrix to be analysed when we have performed a complete loop around the path, or in other words at the spacetime point $\rho = \rho_0$ are:

$$\cosh(2\rho_0) \quad \text{and} \quad \sinh(2\rho_0) \quad (50)$$

We should now take the limit of those quantities when $t$ goes to zero. The starting point must be the explicit expressions of (50), in terms of the variable time. This can be achieved when one notice that from the definition of the terms $\alpha$ and $\beta$ (39), and the fact already mentioned that $v_0$ is simply $\cos t$ we obtain the following value for the ratio $\beta/\alpha$:

$$\frac{\beta}{\alpha} = -2 \cos t \quad (51)$$

This ratio is where all dependence on $t$ of the functions (50) is concentrated.
It is readily seen by direct observation of the expression (41), which gives the relationship between the variables \( \rho \) and \( \eta \), when we set \( \rho = \rho_0 \). When one takes the exponential of both sides of this expression evaluated at that particular value of \( \rho \), or better saying \( \eta \), one obtains:

\[
e^{2\rho_0} = \left( \frac{1 + \tanh(\frac{\chi_0}{2})}{1 - \tanh(\frac{\chi_0}{2})} \right)^2 e^{-4 \cos t \sinh(\chi_0)}
\] (52)

That expression and the one derived from it by replacing \( \rho \) by \(-\rho\), are the building blocks of the quantities given in (50). Then, the limit of the desired elements of the holonomy matrix, (50), when \( t \) goes to 0 is:

\[
\lim_{t \to 0} \cosh 2\rho_0 = \cosh[2(\chi_0 - \sinh(\chi_0))] \] (53)

\[
\lim_{t \to 0} \sinh 2\rho_0 = \sinh[2(\chi_0 - \sinh(\chi_0))] \] (54)

It is not difficult to see that the holonomy matrix will go to the unity at that limit, if and only if, the arguments of the hyperbolic functions on the right-hand side of the expressions (53) and (54) vanish. The unique possibility for this to happen, is if the quantity \( \chi_0 \), which gives the initial size of our fundamental region is zero. But it is not possible since our fundamental region starts with a finite, non-vanishing well defined size. We are then forced to conclude that there is a singularity at the point \( t = 0 \), of our manifold, for this choice of closed path. But, if for one of the four basic closed loops of our fundamental region, we found that there is a singularity sat at the point \( t = 0 \), it must not be a particularity of the chosen closed path. Then, irrespective of the closed basic curve we chose we should find the same result. More than that, this result is also independent of the coordinates in which
one decides to describe our spacetime, as long as, one keeps untouched one of the basic features of our anti-DeSitter model, the homogeneity of its spatial sections. It is more clearly appreciated if one realizes that the effect of a basis one-form coordinate transformation of the metric components, given by the S matrix, upon our holonomy matrix is the similarity transformation $ST_p(t)S^{-1}$. But it is easy to see that no transformation of this kind would be able to furnish the desired unity limit to our holonomy matrix, when $t$ goes to 0. So, we must finally answer the question posed by the title of this article: Yes, the 3-dimensional pseudo-sphere universe had a beginning.

4.2 - 3 + 1 Spacetimes.

The results and the various steps followed in the previous section, will form the core of the work planned for the present section. We shall be interested in compute the holonomy matrix for the more physical 4-dimensional version of the anti-DeSitter spacetime given by the Lorentzian solution (7.3), now in its complete form, and the three-dimensional compact spatial sections will be described by the three-dimensional version of (23), in Cartesian coordinates, when the coordinate transformation (22) holds. The explicit form of the metric is then,

$$\text{ds}^2 = -dt^2 + \frac{4(1 - v_0^2)^2(dx^2 + dy^2 + dz^2)}{(1 - v_0^2 - x^2 - y^2 - z^2)^2}$$ (55)

where the same simplifications for $\alpha$ and $N_I$ were made, leading to a $v_0 = \cos t$. The non-vanishing components of the connection coefficients for that metric (55) are:
\[ \Gamma^t_{ii} = \frac{2\dot{v}_0 v_0 (x^2 + y^2 + z^2)}{(1 - v_0^2)(1 - v_0^2 - x^2 - y^2 - z^2)} = \Gamma^i_{ti} \]

\[ \Gamma^i_{jj} = \frac{-i}{1 - v_0^2} = -\Gamma^j_{ij} \quad \text{where} \quad i, j = x, y, z \quad (57) \]

From here onward the analysis will be a repetition of the previous case, except by the choice of the basic closed loops on our three-dimensional fundamental region given now by the complex \( P \), introduced in 3.2. As one can imagine, it will be much more difficult to find in this case the analogue of the four basic closed curves on the identified octagon. Because of this huge complexity and also due to the fact that we have learned how to derive the limit of \( T_p(t) \) out of a very simple closed loop, in the last section, we shall not try to identify all possible basic closed paths on the complex \( P \). Rather, we shall try to find the path similar to the one we used on the section 4.1, on the identified octagon on our present fundamental region. The desired closed curve is not difficult to be found, one has only to inspect two complexes \( T \), figure 6, disposed in such a way that one out of the five one-to-one identification between two of the ten boundaries (five for each complex) is displayed explicitly. This can be seen in the figure below:
8-Basic loop in the fundamental complex $P$

It is easy to see now, that with the Cartesian coordinate axis placed as in the figure above, the curve $C'$ which runs along with the $z$ axis, starting at $-z_0(t)$ and moving up until $z_0(t)$, and of course returning to $-z_0(t)$, since both boundaries at $z_0(t)$ and $-z_0(t)$ are identified, is the analogue of the basic loop we chose in the last section. Then, choosing this closed path to parallel transport our fiducial vector we shall be able to determine like before a holonomy matrix and finally its regularity as $t$ goes to 0. Introducing the equation for the curve $C'$:

\[ x = y = 0 \quad \text{and} \quad z = \lambda; \quad \text{where} \quad -z_0(t) < \lambda < z_0(t) \]  \hspace{1cm} (57)
on the parallel transport equations (34), (and the basis one-form transformation (28) suitably modified to deal with the presence of an extra dimensional coordinate) along with the non-vanishing components of the connection coefficients (56), we get for the initial conditions:

\[
\begin{align*}
  v^t(\lambda = -z_0(t)) &= v_0^t; \\
  v^x(\lambda = -z_0(t)) &= v_0^x; \\
  v^y(\lambda = -z_0(t)) &= v_0^y; \\
  v^z(\lambda = -z_0(t)) &= v_0^z
\end{align*}
\]

The following holonomy matrix in terms of the variable \( \rho \) introduced in the last section:

\[
v(\rho) = T_p(t)v_0 \quad \text{where}
\]

\[
T_p(t) = \begin{pmatrix}
\cosh(\rho + \rho_0) & 0 & 0 & \sinh(\rho + \rho_0) \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\sinh(\rho + \rho_0) & 0 & 0 & \cosh(\rho + \rho_0)
\end{pmatrix}
\]

and the limit of the relevant elements of \( T_p(t) \) given above, when \( t \) goes to 0, is given by (53) and (54). And as before, we conclude this time for the four-dimensional version of the anti-DeSitter space time equipped with compact spatial sections that there is a singularity placed at \( t = 0 \).

\textbf{V - Conclusions}

The results obtained in this article prompt some questions: Do all negatively curved spacetimes decompose, after a deformation of the contour of integration on
the complex $T$ plane (figure 1), into an initial Lorentzian spacetime and a later Riemannian spacetime? Are those space-times always singular, preventing then the construction of no-boundary wave-function? We are working with a simple model and cannot derive such a general result.

Another line of work is suggested by the conclusions we have got here. Supposing it turns out that the non-regularity of negatively curved Lorentzian spacetimes is a general one, and one wants to try to construct a sensible wave-function for those manifolds. It is clear that the initial conditions will have to take in account the presence of the singularity. But how? The first and simplest idea is to add to the empty spacetime a matter distribution which could conceal out the singular behaviour of the given universe. Is there any matter distribution which gives rise to a conical like singularity? In $2 + 1$ dimensions, S. Deser and R. Jackiw [22], have shown that a negatively curved spacetime like ours, where all the curvature comes from the presence of a cosmological constant, develop conical-like singularities around point-like mass-energy distributions. For a $3 + 1$ dimensional universe, the analogy with the $2 + 1$ dimensional case [22], [23] and some results for flat four-dimensional spacetimes [24], imply that the most natural candidate for a mass-energy distribution generating a smooth spacetime is an infinite string. Then, one would be lead to include scalar matter fields as well as gravity.

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**Figure Captions**

Figure 1 - Tunneling Solution
Figure 2 - Deformations of contour on the complex $\bar{T}$ plane
Figure 3 - Identified Octagon
Figure 4 - Double Torus
Figure 5 - The glueying rules to form $M$
Figure 6 - Identification of the boundaries of two complexes $T$
Figure 7 - Basic loops on the identified octagon
Figure 8 - Basic loop in the fundamental complex $P$