Hidden Variables, Non Contextuality and Einstein-Localit y in Quantum Mechanics

Virendra Singh
Tata Institute of Fundamental Research, Mumbai 400 005, India

1. The Formalism of Quantum Mechanics:
   (a) States, (b) Physical Observables, (c) Dynamics, (d) Statistical Postulate.

2. Why Hidden Variables in Quantum Mechanics
   2.1 Determinism
   2.2 The Problem of Measurement and Incompleteness
       2.2.1 Bohr : Quantum System-Classical Apparatus
       2.2.2 Von-Neumann : Wave function Collapse
       2.2.3 Non Local Correlations

3. Von-Neumann’s Proof of the Impossibility of Hidden Variable Theories
   3.1 Proof
   3.2 Homogeneous Ensembles
   3.3 Reactions to Von-Neumann’s Theorem

4. Bells’ Hidden Variable Model for a Spin One-half Particle
   4.1 The Model
   4.2 Analysis of Von-Neumann Theorem
   4.3 Jauch-Piron Impossibility Proof.

5. Non Contextuality and Quantum Mechanics
   5.1 Gleason’s Theorem and Bell’s Proof of it
   5.2 Non Contextuality
   5.3 Kochen-Specker Theorem
   5.4 Mermin’s Examples
   5.5 Some Other Aspects
       5.5.1 Stochastic Noncontextual Hidden Variable Theories
5.5.2 Finite Precision Measurements and Kochen-Specker Theorem

6. Einstein-Locality and Quantum Mechanics
   6.1 The Background
   6.2 Einstein-Podolsky-Rosen Theorem
   6.3 Einstein Locality
   6.4 Bohm’s Version of the EPR Analysis
   6.5 Bell’s Inequalities and Bell’s Theorem
   6.6 Clauser-Home-Shimony-Hoft (CHSH) form of Bell’s Inequalities
   6.7 Wigner’s Proof of Bell-CHSH Inequalities
   6.8 Experimental Tests of Bell Inequalities
   6.9 Bell’s Theorem without Inequalities
      6.9.1 Greenberger-Horne-Zeilinger (GHZ) Proof
      6.9.2 Hardy’s Version of EPR Correlations
   6.10 Superluminal Signalling
   6.11 More Bell’s Inequalities

7. Envoi

8. Bibliographical Notes
1 The Formalism of Quantum Mechanics

There is general agreement on the basic formalism of Quantum Mechanics which one uses in its physical applications. Briefly we can recapitulate it as follows:

(a) The States:

The state of a quantum system is specified by a state vector $|\psi >$. Further

(i) The state vectors $|\psi >$ and $|\psi' > = \exp(i\theta)|\psi >$, where $\theta$ is a real constant, represent the same physical state,

(ii) Let $|\psi_1 >$ and $|\psi_2 >$ specify two different possible states of a system $S$, then the linear combination,

$$a|\psi_1 > + b|\psi_2 >,$$

where $a$ and $b$ are complex coefficients, also represents a possible state of the system $S$.

(iii) The possible state vectors of a system $S$ belong to a Hilbert space $\mathcal{H}_S$ equipped with a complex scalar product. The scalar product of a state vector $|\psi >$ with the state vector $|\phi >$ is denoted by $<\phi|\psi>$ and satisfies the relation

$$<\phi|\psi>^* = <\psi|\phi>.$$

The norm of a state $|\psi >$, $||\psi||$, is given by

$$||\psi||^2 = <\psi|\psi>.$$

We shall normally use normalized $|\psi >$’s ie we take $||\psi|| = 1$.

(b) Physical Observables:

Any physical observable $A$, of a system $S$, is represented by a unique linear self-adjoint operator $A$ acting on $|\psi > \subset D(A) \subset \mathcal{H}_S$, where $D(A)$ is called the domain of the operator $A$.

The spectrum of the operator $A$, $\text{spec} (A)$, is the set of all those complex numbers $z$ for which the operator $(A - zI)$ cannot be inverted. For self-adjoint operators in Hilbert space it is a set of real numbers $a_j \in \text{spec} (A)$. Depending on $A$, $\text{spec} (A)$ can be a discrete set, or a continuous set or a mixture of both. Any $A$ measurement of a physical observable $A$, always results in one of real numbers $a_j$ occuring in $\text{spec} (A)$. Let

$$A|a_j >= a_j|a_j >, \quad a_j \epsilon \text{spec}(A).$$

For the sake of brevity we will refers to $a_j$’s as eigenvalues of $A$ and the $|a_j >$’s as the corresponding eigenvectors belonging to the eigenvalue $a_j$ of $A$. All the eigenvectors of $A$ together form a complete set and we can choose them to be an orthonormal set ie

$$<a_j|a_k > = \delta_{(a_j,a_k)}$$

where $\delta_{(a_j,a_k)}$ is Kronecker delta symbol (for discrete eigenvalues) or appropriate Dirac delta distribution (for continuous part of the spectrum). The completeness relation can be stated as

$$\sum_{a_j} |a_j > < a_j| = 1$$
where the sigma symbol over \(a_j\) stands for summation over discrete part of the spectrum \(a_j \in \text{spec}(A)\) and for an integral, with proper measure, for the continuous part of the spectrum.

(c) Dynamics:
The dynamical evolution of the state vector \(|\psi(t)\rangle\), of a system, with time \(t\), is given by the Schrödinger equation
\[
    i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H|\psi(t)\rangle
\]
\[
    \text{i.e. } \psi(t) = U(t-t')\psi(t'),
\]
with \(U(t) = \exp(-iHt/\hbar)\)

where \(H\) is the Hamiltonian operator for the system \(S\), corresponding to it’s physical energy observable. This evolution is valid for a free evolution of the system between it’s preparation and subsequent measurement.

(d) Statistical Postulate:
If we measure a physical observable \(A\), on a system \(S\), in the state \(|\psi\rangle\), then the probability of measurement resulting in the value \(a_j \in \text{spec}(A)\) is given
\[
    \text{prob}(A, a_j, |\psi\rangle) = |<a_j|\psi>|^2.
\]
Using this expression the expectation value \(E_{\psi}(A)\), of \(A\) in the state \(|\psi\rangle\) is given by
\[
    E_{|\psi\rangle}(A) = \sum_{a_j} a_j |<a_j|\psi>|^2 = <\psi|A|\psi>.
\]

If we define the projection operator \(P(|\phi\rangle)\) corresponding to the state \(|\phi\rangle\) by
\[
    P(|\phi\rangle) = |\phi><\phi| = P_\phi
\]
then we can write
\[
    E_{|\psi\rangle}(A) = \text{Tr}(A|\psi><\psi|) = \text{Tr}(AP_\psi),
\]
and
\[
    \text{prob}(A, a_j, |\psi\rangle) = \text{Tr}(|a_j><a_j|\psi><\psi|) = \text{Tr}(P_{a_j}P_\psi).
\]
We shall refer to the projection operator \(|\psi><\psi|\) when the system is in the state \(|\psi\rangle\) as it’s density operator \(\rho\) ie
\[
    \rho = |\psi><\psi|.
\]
If, however, the system is not in a pure state but we know that it is in mixture of states \(|\psi_k\rangle\) with probability \(p_k\), then it has to be represented by the density matrix
\[
    \rho = \sum_k p_k |\psi_k><\psi_k|
\]
with \(\sum_k p_k = 1, p_k \geq 0\). We now have
\[
    \text{prob}(A, a_k; |\psi_k\rangle, p_k) = \text{Tr}(P_{a_k}\rho)
\]
\[
    E_{||\psi_k\rangle, p_k}(A) = \text{Tr}(A\rho).
\]
2 Why Hidden Variables in Quantum Mechanics?

2.1 Determinism

In the formalism of the quantum mechanics the state vector $|\psi>$ specifies the state of the system completely. If we now measure a physical observable $A$ for the system in the state $|\psi>$ we obtain a value $a_k$, which is one of the possible eigenvalues of $A$. The statistical postulate tells us about the probability with which a particular value $a_k$, belonging to the spec ($A$), will occur. We are not able to predict as to which of the possible eigenvalues $a_k$ of $A$ will occur. It is as if we had not specified our state of the system completely by a knowledge of the state vector $|\psi>$. The quantum mechanical state vector $|\psi>$ of a system seems to encode only the statistical information.

Before the discovery of quantum mechanics, the statistical description had been used in classical physics. For example, in classical statistical mechanics we use such a description with great success, to describe thermodynamic states of a classical system. These states are a complicated average over microscopic states of the system which are well defined in terms of positions and momentum coordinates of the particles composing the system. In thermodynamics we are simply not interested in such a detailed description of the system. Besides such a detailed description, since it deals with extremely large number of variables, is not always feasible.

Another example of random motion in classical physics is provided by the Brownian motion of pollen grains in a fluid. We know here that the randomness of the motion of pollen grains is due to their being constantly buffeted by the large number of molecules of the fluid. The molecules of the fluid remain hidden to our view under the microscope used to observe pollen grain. If we include these molecules in the dynamics then the motion is deterministic and not random. Incidentally the Einstein’s expression for the root mean square displacement of a Brown particle, divided by the time duration, has the same structure as Heisenberg’s uncertainty principle except that it involves properties of fluid, such as diffusion coefficient, instead of Planck’s constant, in it.

In both these examples the systems for which we gave statistical description in classical physics were not inherently statistical but were deterministic. The statistical or random element arose because of our averaging over a large number of variables which we ignored ie which remained hidden. Could it be that the statistical description given by quantum mechanics could also be arising from the same cause? We could then try to supplement the quantum mechanical specification of the state of a system by it’s state vector $|\psi>$ by adding extra number of variables, referred to as “hidden variables”, to provide a more complete deterministic description of the system.

2.2 The Problem of Measurement and Incompleteness

2.2.1 Bohr : Quantum System-Classical Apparatus Split

According to Niels Bohr the language of classical physics is the only language available to us to describe experimental results. Classical physics is thus needed to describe the results of experiments even on the quantum systems which are usually microscopic. The quantum system obeys laws of quantum physics. In contrast the measuring apparatus,
which are generally macroscopic, obey laws of classical physics. There is thus a split of
the world into system, which obeys quantum laws, and apparatus, which obeys classical
physics laws.

There is a problem of principle here. how much of the ‘system plus apparatus’ is to
be put on quantum side ie into ‘system’ and how much of it is to put on classical side ie
into ‘apparatus’. This is not a severe problem in practice in as much as we put enough of
‘system + apparatus’ on the ‘system’ side so that we achieve sufficient degree of accuracy
of description. The convergence is supposed to be ensured by the folkloric “correspon-
dence principle” according to which quantum mechanics, in “appropriate” limits, goes
into classical physics. But in principle the split remains.

This split of the world into a microscopic quantum system and a mesoscopic classical
apparatus has been found deeply disturbing by many persons. Is there a definite boundary
between quantum and classical at some mesoscopic level, which we will discover. This
possibility looks unlikely in view of the existence of macroscopic quantum systems like
squids. It is more likely that there is no boundary at all. As John Bell says “It is hard
for me to envisage intelligible discourse about a world with no classical part – no base of
given events, be they only mental events in a single consciousness, to be correlated. On
the other hand, it is easy to imagine that the classical domain could be extended to cover
the whole. The wave function would prove to be a provisional or incomplete description
of the quantum mechanical put, of which an objective account would be come possible.
It is this possibility of a homogeneous account of the world, which is for me the chief
motivation of the study of the so called ‘hidden variable’ possibility”.

2.2.2 Von-Neumann : Wavefunction Collapse

It would be more satisfactory to describe both the system and apparatus by the same
quantum laws especially if quantum mechanics has pretensions to be the fundamental
theory of physics. John Von-Neumann regarded both the system and apparatus to be so
describable by quantum mechanics. He postulates a measurement interaction such that
the energy-coupling between the system, in a pure state \( |\psi_n \rangle \) of an observable \( \Omega \) (ie
\( \Omega |\psi_n \rangle = \omega_n |\psi_n \rangle \)) with eigenvalue \( \omega_n \), and apparatus, for measuring the observable \( \Omega \),
in a pure state \( |f(a)\rangle \) where \( a \) is the pointer reading, causes the system-apparatus to go
from the initial state \( |\psi_n \rangle |f(a)\rangle \) to the final state \( |\psi \rangle |f(a_n)\rangle \). The pointer-reading
\( a_n \) thus indicates that the system was having the eigenvalue \( \omega_n \) of the observable \( \Omega \). We
shall further restrict ourselves to measurement interaction of the type such that

\[
|\psi_n \rangle |f(a)\rangle \rightarrow |\psi_n \rangle |f(a_n)\rangle .
\]

If the system is not initially in a pure state of the observable \( \Omega \) by is in a linear
combination of such states ie the initial state of the system \( |\psi_{initial} \rangle \) is given by

\[
|\psi_{initial} \rangle = \sum_n c_n |\psi_n \rangle ,
\]

the linearity of quantum mechanics leads to the system apparatus initial state

\[
|\psi_{initial} \rangle |f(a)\rangle
\]
to evolve into the final state
\[ \sum_n c_n|\psi_n > |f(a_n) >. \]
Thus the apparatus is in a superposition of states corresponding to various pointer-readings and the probability of opening a pointer reading \( a_N \) given by \( |c_N|^2 \).

How does it happen that apparatus gives us, at the end of measurement, a definite pointer reading say \( a_N \) ie what causes the collapse of the wave function
\[ \sum_n c_n|\psi_N > |f(a_N) > to |\psi > |f(a_N) > \]
with probability equal to \( |c_N|^2 \). According to Von-Neumann this is caused by a casual and discontinuous process, called “process of the first kind” by him, which take place only when a measurement is completed. These are quite different by what he terms as “processes of the second kind” which are causal and continuous and obey Schrödinger equation, and by which a quantum system evolves in time between its preparation and a measurement on it. Incidentally the preparatus of a system in a definite state can also be regarded as a special kind of measurement.

It is abundantly clear that even here quantum mechanics is not a closed system since it has to be supplemented by “processes of the first kind” which cause the collapse of a wavefunction at the end of a measurement. It could be that we have to add extra dynamical hidden variables to explicate these processes.

2.2.3 Non-local Correlations

The Einstein-Podolsky-Rozen (EPR) correlations are quite nonlocal. Let two particles, each having spin \( \frac{1}{2} \hbar \), in a simplest state separate, one moving to the left and the second moving to the right. Then a measurement of any component \( \sigma \cdot \hat{a} \) of one of them allows us to know what \( \sigma \cdot \hat{a} \) would be if measured even if these two particles are extremely for apart. It is conceivable, in principle, that this nonlocal correlation comes about through “local causal” correlations through the involvement of hidden variables. If it turns out to be possible then one would not have to invoke non-local causal mechanics. We will be discussing EPR correlations in detail later on.

3 Von-Neumann’s Proof of the Impossibility of Hidden-Variable Theories

3.1 Proof

John Von-Neumann published, in 1932, a result which seemed to say that it is impossible to have a hidden variable completion of the quantum mechanics. This result, given the immense prestige of Von-Neumann as a mathematician, effectively blocked any search for such theories for a long time. We therefore look at this proof at this stage before proceeding any further.

The assumptions that go in the proof of Von-Neumann’s theorem are as follows. We use Gothic letters to denote physical observables and the corresponding Latin symbol for
the Hermitian operator associated with it in quantum mechanics e.g. for the physical observable $R$ the Hermitian operator associated is denoted by $R$. The expectation value of $R$ is denoted by $<R>$ in the physical ensemble.

(0) To every physical observable $A$ there corresponds only one Hermitian operator $A$. We use the notation $A \mapsto A$ for this correspondence.

(I) If a physical observable $R$ is non-negative then its expectation value $<R>$ is also non-negative.

(II) If $R_1 + R_2$ are two physical observables then the physical observable associated with their sum $R_1 + R_2$ is associated with the Hermitian operator $R_1 + R_2$.

(III) If $R_1, R_2, \cdots$ are physical observables and $a_1, a_2, \cdots$ real numbers then

$$< (a_1 R_1 + a_2 R_2 + \cdots) > = a_1 < R_1 > + a_2 < R_2 > + \cdots$$

(IV) If a physical observable $R$ is associated with a Hermitian operator $R$ then a function $f(R)$ is associated with the Hermitian operator $f(R)$.

Von-Neumann first proves, using only assumptions (0)–(III), the following theorem. **Theorem:** There exists a density operator $\rho$ such that

$$< R > = \text{Tr}(\rho R)$$

for all $R$. Here the density operator depends only on the ensemble of physical states but is independent of the physical observable $R$. Further

$$\text{Tr} \rho = 1.$$

**Proof:** Given a complete orthonormal set $|n>$, ie

$$< n|m > = \delta(n, m) \quad \text{and} \quad \sum_n |n><n| = 1.$$

We can express any Hermitian operator $R$, as

$$R = \sum_{n,m} |n><m| R_{nm}$$

where $R_{nm} = < n|R|m >$. Further we have

$$R_{nm}^* = R_{nm} = < m|R|n >^*.$$

We can rearrange various terms to rewrite

$$R = \sum_n U_{nn} R_{nn} + \sum_{n>m} V_{nm} \text{Re} R_{nm} + \sum_{n>m} W_{nm} \text{Im} R_{nm}$$
where
\[
U_{nn} = |n><n|,
V_{nm} = |n><m| + |m><n|,
W_{nm} = i(|n><m| - |m><n|)
\]

The operators $U_{nn}, V_{nm}$ and $W_{nm}$ are all hermitian and will be taken to represent the observables according to the correspondence, (assumption (0)),
\[
U_{nn} \mapsto U_{nn},
V_{nm} \mapsto V_{nm},
W_{nm} \mapsto W_{nm}.
\]

By assumption (0) and (II), we have an observable $\Re$ given by
\[
\Re = \sum_n U_{nn} R_{nn} + \sum_{n>m} \Re_{nm} R_{nm} + \sum_{n>m} W_{nm} \Im R_{nm}
\]

We thus have the correspondence
\[
\Re \mapsto R.
\]

By assumption III, the expectation value $<\Re>$ of $\Re$ in the ensemble corresponding to a quantum state is given by
\[
<\Re> = \sum_n <U_{nn} > R_{nn} + \sum_{n>m} <V_{nm} > R_{nm} + \sum_{n>m} <W_{nm} > \Im R_{nm}.
\]

We now introduce the notation,
\[
\rho_{nn} = <U_{nn} >,
\rho_{nm} = \frac{1}{2} [<V_{nm} > + i <W_{nm} >] \quad \text{for } n > m,
\rho_{mn} = \frac{1}{2} [<V_{nm} > - i <W_{nm} >] \quad \text{for } n > m.
\]

In terms of the expressions $\rho$, we can express
\[
<\Re> = \sum_{n,m} \rho_{mn} R_{nm}.
\]

We now define a hermitian density operator $\rho$ by
\[
<m|\rho|n> = \rho_{mn}
\]

Using the density operator $\rho$, we can finally express
\[
<\Re> = \text{Tr}(\rho R).
\]

Note that $\rho$ is independent of $\Re$ but only depends on the ensemble.
For the constant unit observable $R = 1 \mapsto R = 1$, we get

$$\text{Tr}\rho = 1.$$  

Von-Neumann next uses projection operators to show that the density operator is positive semidefinite. Consider the correspondence

$$P_\phi \mapsto |\phi><\phi|$$

By (IV), we get

$$(P_\phi)^2 \mapsto (|\phi><\phi|)^2 = |\phi><\phi|$$

we thus get

$$<P_\phi^2> = <P_\phi >= \text{Tr}(\rho P_\phi) = \text{Tr}(\rho|\phi><\phi|)$$

By (1), it follows that

$$<P_\phi^2> \geq 0$$

and therefore

$$<\phi|\rho|\phi> \geq 0.$$  

**Theorem**: The density operator $\rho$ is hermitian and positive semidefinite.

We now proceed to prove the Von-Neumann theorem on the absence of hidden variable in quantum mechanics using these results and assumption (IV). We are going to show that the dispersion-free states do not exist. If they do than for them, we must have

$$<R^2> = <R>^2.$$  

By assumption (IV) if

$$R \mapsto R$$

then

$$R^2 \mapsto R^2.$$  

The dispersion free condition then becomes

$$\text{Tr}(\rho R^2) = [\text{Tr}(\rho R)]^2.$$  

In particular for $R = |\phi><\phi| = P_\phi$

$$\text{Tr}(\rho P_\phi^2) = [\text{Tr}(\rho P_\phi)]^2$$

using $P_\phi^2 = P_\phi$ we get

$$\text{Tr}(\rho P_\phi)[1 - \text{Tr}(\rho P_\phi)] = 0$$

ie

$$\text{Tr}(\rho P_\phi) = <\phi|\rho|\phi>$$
can be either 0 or 1. If we take

\[ \phi = \cos \theta \phi_1 + \sin \theta \phi_2 \]

then \( < \phi|\rho|\phi > \) varies continuously with \( \theta \). Thus we have only the following two possibilities left (i) \( < \phi|\rho|\phi > = 0 \) for all \( \phi \) or (ii) \( < \phi|\rho|\phi > = 1 \) for all \( \phi \). Thus either \( \rho = 0 \) or \( \rho = 1 \). Both these possibilities are unacceptable physically. Hence no dispersion-free states can exist.

As John Von-Neumann concluded "It is therefore not, as is often assumed a question of reinterpretation of quantum-mechanics, the present system of quantum-mechanics would have to be objectively false, in order that another description of the elementary process than the statistical one be possible”.

### 3.2 Homogeneous Ensembles

An ensemble characterized by a density matrix \( \rho \) is called homogeneous, if whenever we write

\[ \rho = \rho_1 + \rho_2 \]

and where

\[ \rho_1 = \rho_1^{\dagger}, \quad \rho_1 \geq 0, \]

and

\[ \rho_2 = \rho_2^{\dagger}, \quad \rho_2 \geq 0 \]

then

\[ \rho_1 = c_1 \rho, \quad \rho_2 = c_2 \rho \]

where

\[ c_1 \geq 0, \quad c_2 \geq 0, \quad c_1 + c_2 = 1. \]

Von-Neumann showed that \( \rho \) corresponding to a homogeneous ensemble has to be some projection operator ie \( \rho = P_\phi \) for some \( \phi \). We thus have for homogeneous ensembles

\[ < R > = \text{Tr}(RP) = < \phi | R | \phi > \]

which is the usual quantum-mechanical expectation value.

### 3.3 Reactions to Von-Neumann’s Theorem

After its publication in 1932, Von-Neumann’s theorem soon acquired the status of a dogmatic pronouncement against hidden variable theories. At a conference on “New Theories of Physics” at Warsaw in 1938, Niels Bohr, the guiding spirit behind the long-dominant Copenhagen interpretation of Quantum mechanics, publicly endorsed it after a presentation of it by Von-Neumann. Bohr expressed his admiration for the result. He also mentioned that the conclusion one of his own papers was essentially the same even though he arrived at it by using more elementary ways.

Max Born, in his book “Natural Philosophy of Cause and Chance”, first published in 1949, said “The result is that the formalism of quantum mechanics is uniquely determined
by these axioms; in particular no concealed parameters can be introduced with the help of which the indeterministic description could be transformed into a deterministic one. Hence if a future theory should be deterministic, it cannot be a modification of the present one but must be essentially different. How this should be possible without sacrificing a whole treasure of well-established results I leave to the determinist to worry about”. This quotation is from an English translation of Born’s 1949 book which was published as late as in 1964.

Even more telling are the recollection of Paul Feyeraband, the well known philosopher of science, of a public lecture and seminar by Bohr in Askov (sometime during 1949-1952). He says “In Askov, I also met Niels Bohr. He came for a public lecture and conducted a seminar, both in Danish … At the end of the lecture he left, and the discussion proceeded without him. Some speakers attacked his qualitative arguments — there seemed to be lots of loopholes. The Bohrians did not clarify the arguments; they mentioned an alleged proof by Von Neumann, and that settled the matter. Now I very much doubt that those who mentioned the proof, with the possible exception of one or two of them, could have explained it. I am also sure their opponents had no idea of it’s details. yet, like magic, the mere name “Von Neumann’ and the me word “proof” silenced the objectors. I found this very strange but was relieved to remember that Bohr himself had never used such tricks”.

Despite the fact that in 1952 David Bohm actually published a hidden variable model of quantum mechanics, the spell of von-Neumann’s theorem remained unbroken. The “hidden-variables” in Bohm’s model were particle trajectories. It was a resurrection of earlier de-Broglie pilot wave theory model of 1927 which had been abandoned by its’ originator under criticism by Pauli and Einstein. Bohm was however able to take care of various criticisms of Pauli, Einstein and of de Broglie, besides supplementing it with a theory of measurements. Bohm however did not analyze Von-Neumann’s proof to pinpoint as to how he was able to circumvent Von-Neumann’s theorem. He must have clearly violated some assumption involved in it. Such as analysis was first carried out by John Bell, in the context of his hidden variable model for a spin one-half particle, in 1966. The spell of Von-Neumann’s theorem was at last broken as a result of Bell’s work.

4 Bell’s Hidden Variable Model for a Spin One-half Particle

4.1 The Model

We consider, following Bell, a quantum mechanical spin one-half particle and ignore its translational degrees of freedom. It’s description requires a two dimensional Hilbert space. The most general observable $M$ is specified by four real numbers, $\alpha$ and $\vec{\beta}$, and can be taken to be

$$M(\alpha, \vec{\beta}) = \alpha \mathbf{1} + \vec{\beta} \cdot \vec{\sigma}$$

where $\sigma$’s are usual three Pauli-matrices and $\mathbf{1}$ is a unit matrix. The eigenvalues of $M(\alpha, \beta)$ are given by $(\alpha + |\vec{\beta}|)$ and $(\alpha - |\vec{\beta}|)$. The physical observable $M$ can take only
these two eigenvalue as it’s values.

The state of this system is in quantum-mechanics described by a state vector \( |\psi> \) which we take to be normalised ie \(<\psi|\psi>=1\). We now add a hidden variable \( \lambda \), which is a real variable having the range \( \frac{1}{2} \geq \lambda \geq -\frac{1}{2} \), to the quantum state vector \( |\psi> \) to complete the specification of dispersion free states.

For the dispersion free state \((|\psi>, \lambda)\) we associate the value for the observable \( M(\alpha, \vec{\beta}) \) given by

\[
M(\alpha, \vec{\beta}, |\psi>, \lambda) = \alpha + |\vec{\beta}| \text{sgn}[<\psi|\vec{\beta}\cdot\vec{\sigma}|\psi>] \times \text{sgn}\{\lambda|\vec{\beta}| + \frac{1}{2} |<\psi|\vec{\beta}\cdot\vec{\sigma}|\psi>|\}.
\]

Clearly \( M(\alpha, \vec{\beta}, |\psi>, \lambda) \) only taken the values \( \alpha \pm |\vec{\beta}| \) in the dispersion free states \((|\psi>, \lambda)\) as is required.

The ensemble of dispersion free states is taken be one in which the hidden value \( \lambda \) is distributed with equal probability over it’s range \( \frac{1}{2} \geq \lambda \geq -\frac{1}{2} \). We shall now check whether the ensemble average \(<M>\) over this ensemble reproduces the quantum mechanical expectation value of \( M(\alpha, \vec{\beta}) \) which is given by

\[
<\psi|M(\alpha, \vec{\beta}, |\psi>)>.
\]

We therefore integrate \( M(\alpha, \vec{\beta}, |\psi>, \lambda) \) over \( \lambda \) ie

\[
<M> = \int_{-\frac{1}{2}}^{\frac{1}{2}} d\lambda M(\alpha, \vec{\beta}, |\psi>, \lambda).
\]

An easy calculation leads to

\[
<M> = \alpha + <\psi|\vec{\beta}\cdot\vec{\sigma}|\psi> = <\psi|M(\alpha, \vec{\beta})|\psi>.
\]

It is thus equal to the quantum mechanical expectation value of \( M(\alpha, \vec{\beta}) \). Bell thus succeeded in providing a hidden variable model of a quantum translationless spin \( \frac{1}{2} \) particle.

4.2 Analysis of Von-Neumann Theorem

Now that we have an explicit example of the hidden variable model we can use it to analyse the Von-Neumann’s impossibility proof. The guilty assumption turns out to the innocuous looking one III

\[
a_1 <A_1> + a_2 <A_2> = <a_1 A_1 + a_2 A_2>.
\]

even when \( A_1 \) and \( A_2 \) are noncommuting Hermian operators. While it is true of quantum mechanical states but is not true for dispersion free states. After all the eigenvalues \( \pm \sqrt{2} \) of the operator \( \sigma_x + \sigma_y \) are not sum of the eigenvalues \( \pm 1 \) of \( \sigma_2 \) and eigenvalues \( \pm 1 \) of \( \sigma_y \). A physical explanation of this phenomenon is that we require three different orientation of Stern-Gerlach magnets to measure \( \sigma_x, \sigma_y \) and \( (\sigma_x + \sigma_y) \).
4.3 Jauch-Piron Impossibility Proof

Bell also analysed a new proof of impossibility of hidden variables in quantum mechanics which was given by Jauch and Piron in 1963. They deal with expectation values of only projection operators. For a projection operator $P$, since $P^2 = P$ an eigenvalue can be only 0 or 1. The expectation value of $< P >$ thus gives us the probability that we observe the eigenvalue 1 for it.

Jauch and Piron assume

$$< A > + < B > = < (A + B) >$$

only for commuting projection operators $A$ and $B$. To this extent it is an improvement of Von-Neumann’s work where this assumption was made for all operators $A$ and $B$ whether they were commuting or noncommuting. Given two projection operators $A$ and $B$ they also define another projection operator $(A \cap B)$ which projects out the intersection of the subspaces of $A$ and $B$. They further assume that if, in some state,

$$< A >= < B >= 1$$

then $< A \cap B >= 1$ for that state. This assumption is by analogy to logic of propositions in where the value 1 corresponds to ‘truth’ and value 0 to ‘falsehood’. It corresponds to logical proposition that if $A$ is ‘true’ and $B$ is ‘true’ then $(A \ ‘and’ \ B)$ is also ‘true’.

Consider a two dimension subspace and the projection operators $\frac{1}{2}(1 + \vec{a} \cdot \vec{\sigma})$ and $\frac{1}{2}(1 - \vec{a} \cdot \vec{\sigma})$ where $\vec{a}$ is a unit vectors. Since they commute and add upto 1, we have

$$< \frac{1}{2}(1 + \vec{a} \cdot \vec{\sigma}) > + < \frac{1}{2}(1 - \vec{a} \cdot \vec{\sigma}) > = 1.$$

Thus, for dispersion free states, either $< \frac{1}{2}(1 + \vec{a} \cdot \vec{\sigma}) >= 1$ or $< \frac{1}{2}(1 - \vec{a} \cdot \vec{\sigma}) >= 1$. Similarly either

$$< \frac{1}{2}(1 + \vec{\beta} \cdot \vec{\sigma}) >= 1 \ or \ < \frac{1}{2}(1 - \vec{\beta} \cdot \vec{\sigma}) >= 1$$

for another unit vector $\vec{\beta}$. Thus by choosing $A$ to be either $\frac{1}{2}(1 + \vec{a} \cdot \vec{\sigma})$ or $\frac{1}{2}(1 - \vec{a} \cdot \vec{\sigma})$, and $B$ to be either $\frac{1}{2}(1 + \vec{\beta} \cdot \vec{\sigma})$ or $\frac{1}{2}(1 - \vec{\beta} \cdot \vec{\sigma})$ appropriately we can arrange

$$< A > = < B >= 1$$

But now $A \cap B = 0$, so that $< A \cap B >= 0$ which contradicts the assumption made earlier. Hence dispersion free states cannot exist.

The objection to Jauch-Piron proof, as Bell pointed out, is again that we are dealing with measurements and not with logical propositions. Their second assumption, while obeyed by quantum mechanical states is not necessarily obeyed by the hidden variable states.
5 Non Contextuality and Quantum Mechanics

5.1 Gleason’s Theorem and Bell’s Proof of it

We saw that a major ingredient of Von-Neumann’s impossibility proof of hidden variables was his proof of existence of a Hermitian positive semidefinite density matrix. Von-Neumann had to assume that expectation value of a sum of given observables is a sum of expectation values of these observable even if they are a noncommuting set. Gleason (1957) was able to prove the existence of such a density matrix by making the assumption of additivities of expectation values only for commuting observables provided the dimension of the Hilbert space for the system is greater than or equal to three. This would allow us again to prove the impossibility of hidden variables in quantum mechanics for systems with \( \dim \mathcal{H}_s \geq 3 \). Bell’s counter exempts of spin one-half particle, since it’s Hilbert space is two dimensional, has no relevance for this case.

We shall now proceed to reproduce Bell’s proof (1966) of Gleason result. Let a set \((\phi_1, \phi_2, \phi_3, \cdots)\) be orthogonal and complete. The projection operators

\[ P(\phi_i) = |\phi_i><\phi_i| \]

commute with other ie

\[ [P(\phi_i), P(\phi_i)] = 0. \]

Further

\[ \sum_i P(\phi_i) = 1. \]

Using the additivity assumption for the expectation value of commuting observables only we get

\[ \langle \sum_i P(\phi_i) \rangle = \sum_i \langle P(\phi_i) \rangle = 1. \] (1)

We also note, since any eigenvalue of a projection operator is either 0 or 1,

\[ \langle P(\phi_i) \rangle \geq 0. \] (2)

We thus have two corollaries (A) and (B):

(A) If in a given state, we have \( \langle P(\phi)\rangle = 1 \) then \( \langle P(\phi')\rangle = 0 \) for \( \phi' \) orthogonal to \( \phi \) ie \( \langle \phi' | \phi \rangle = 0 \).

This corollary is obvious since we can always have a complete orthonormal set containing \( |\phi\rangle, |\phi'\rangle \) and using (1) and (2).

(B) If, in a given state, we have

\[ \langle P(\phi_1) \rangle = \langle P(\phi_2) \rangle = 0 \quad \text{for some} \quad \langle \phi_1 | \phi_2 \rangle = 0 \]

then \( \langle P(\alpha \phi_1 + \beta \phi_2) \rangle = 0 \) for all \( \alpha, \beta \).

We note here

\[ 0 = \langle P(\phi_1) \rangle + P(\phi_2) = 1 - \sum_{i \neq 1,2} \langle P(\phi_i) \rangle \]

\[ = \langle P(\psi_1) \rangle + \langle P(\psi_2) \rangle \]
where

\[\psi_1 = \alpha \phi_1 + \beta \phi_2\]
\[\psi_2 = -\beta \phi_1 + \alpha \phi_2\]

since \((\psi_1/|\psi_1|, \psi_2/|\psi_2|, \phi_3, \phi_4, \ldots)\) is also a complete set if \((\phi_1, \phi_2, \phi_3, \phi_4, \ldots)\) is the corollary (B) follows from the positivity of \(\langle P(\psi_1) \rangle\) and \(\langle P(\psi_2) \rangle\).

The strategy now would be to first prove that if \(\langle P(\psi) \rangle = 1\) and \(\langle P(\phi) \rangle = 0\) then \(|\psi\rangle\) and \(|\phi\rangle\) can not arbitrarily close to each other or more precisely

\[||(\psi - \phi)|| \geq k ||\psi||\] with \(k > 0\). \hfill (3)

Note that the constant \(k\) is strictly positive in (3). In fact we will show that \(k\) can be chosen to be \(\sqrt{(2 - 4/\sqrt{5})}\). This result will then be used to prove the absence of dispersion free states. To prove the result in eqn. (3) we shall use repeatedly corollaries (A) and (B). We now give the proof. Let

\[\phi = \sqrt{1 - \epsilon^2} \psi + \epsilon \psi'\]

where \(\langle \psi' | \psi \rangle = 0\). Let \(\psi''\) be a state vector such that

\[\langle \psi'' | \psi \rangle = \langle \psi'' | \psi' \rangle = 0, \quad \langle \psi'' | \psi'' \rangle = 1.\]

It follow \(\langle \phi | \psi'' \rangle = 0\).

Using (A) we get

\[\langle P(\psi'') \rangle = 0\] since \(\langle \psi'' | \psi \rangle = 0\),

and \[\langle P(\psi') \rangle = 0\] since \(\langle \psi' | \psi \rangle = 0\).

Now, using (B) we have

\[\langle P(\phi + \frac{\epsilon}{\gamma} \psi'') \rangle = 0\]

since \(\langle P(\phi) \rangle = \langle P(\psi'') \rangle = 0\). Here \(\gamma\) is a constant to be chosen later. We also have, using (B),

\[\langle P(-\epsilon \psi' + \gamma \epsilon \psi'') \rangle = 0\] since \(\langle P(\psi') \rangle = \langle P(\psi'') \rangle = 0\)

We now note that

\[\langle -\epsilon \psi' + \gamma \epsilon \psi'' | \phi + \frac{\epsilon}{\gamma} \psi'' \rangle = -\epsilon \langle \psi' | \phi \rangle + \epsilon^2 = 0\]

We can, using (B), see

\[\langle P(\phi + \frac{\epsilon}{\gamma} \psi'') + \langle P(-\epsilon \psi' + \gamma \epsilon \psi'') \rangle \]
\[= \langle P(\sqrt{1 - \epsilon^2} \psi + \epsilon (\gamma + \frac{1}{\gamma}) \psi'') \rangle = 0.\]
We can now choose, if $\epsilon^2 \leq \frac{1}{5}$, $\gamma$ such that

$$(\gamma + \frac{1}{\gamma})\epsilon = \pm \sqrt{1 - \epsilon^2}.$$ 

We thus get

$$\langle P(\psi \pm \psi'') \rangle = 0.$$ 

Using (B) once again we get

$$\langle P(\psi) \rangle = 0.$$ 

Since $< P(\psi) > = 1$ we have a contradiction. We must therefore have $\epsilon^2 \geq \frac{1}{5}$. This leads to

$$|\psi - \phi| = \sqrt{2 - 2\sqrt{(1 - \epsilon^2)}|\psi| \geq (2 - \frac{4}{\sqrt{5}})^{1/2}|\psi| > .$$

The proof needs at least three linearly independent vectors $\psi, \psi', \psi''$ and thus the Hilbert space must dimension $\geq 3$.

If there are dispersion freestates then each projector has expectation value either 0 or 1. Consider the expectation value projector

$$P(\theta) = \langle P(\cos \theta \psi + \sin \theta \phi) \rangle.$$ 

We have $P(\theta) = 0$ or 1, $P(0) = 1, P(\frac{\pi}{2}) = 0$. There must thus be at least one $\theta$ value, say $\theta_0$, at which $P(\theta)$ jumps from the value 1 to value 0.

We thus have two states which are arbitrarily close such that for one of them projector has expectation value equal to 1 and for the other one it is zero. This contradicts the result derived above. We thus have proved the impossibility of hidden variable theories assuming only the innocuous looking assumption about additivity of expectation values of only commuting physical observables.

How did this happen? To trace the culprit, we note that we can have a set of orthonormal states

$$\{\phi_1, \phi_2, \phi_3, \cdots\}$$

and another set of orthonormal states, also containing $\phi_1$,

$$\{\phi_1', \phi_2', \phi_3', \cdots\}.$$ 

We would then have

$$1 = P(\phi_1) + P(\phi_2) + P(\phi_3) + \cdots$$

with $[P(\phi_j), P(\phi_k)] = 0$ for $j \geq 1, k \geq 1$, and

$$1 = P(\phi_1') + P(\phi_2') + P(\phi_3') + \cdots$$

with

$$[P(\phi_1), P(\phi_1')] = 0 \text{ for } k \geq 2;$$

and

$$[P(\phi_j'), P(\phi_k')] = 0 \text{ for } j \geq 2, k \geq 2.$$
What is assumed implicitly in Gleason’s proof is that measuring $P(\phi_1)$ is independent of whether one is measuring $P(\phi_2), P(\phi_3), \cdots$ along with it or one is measuring $P(\phi'_2), P(\phi'_3), \cdots$ along with it. However in general, since

$$[P(\phi_j), P(\phi'_k)] \neq 0 \text{ for } j \geq 2, k \geq 2.$$  

We cannot measure $P(\phi_j)$ and $P(\phi'_k)(j \geq 2, k \geq 2)$ together. We thus can raise physical objections against the Gleason’s proof as against Von-Neumann’s proof.

### 5.2 Non Contextuality

We now summarise the moral learned from Gleason-Bell proof. Let a physical observable $A$ commute with $B_1, B_2 \cdots$ which mutually commute with each other i.e.

$$[A, B_j] = 0, \quad [B_i, B_j] = 0.$$  

We can measure $A$ along with $B_j$’s together. The set $[A, B_1, B_2, \cdots]$ will be said to form a context. Let there be another set $(A, C_1, C_2, \cdots)$ of mutually commuting observables and it will form another context. In general however $B_j$’s will not commute with $C_k$’s i.e

$$[B_j, C_k] \neq 0$$  

and are thus not simultaneously observable. The measurement of $A$ thus should be expected to depend on the context. The culprit in Bell-Gleason proof of nonexistence of hidden variable theories was implicit assumption of Non-contextuality. As Bell said “It was tacitly assumed that measurement of an observable must yield the same value independently of what other measurements may be made simultaneously”.

### 5.3 Kochen-Specker Theorem

Bell’s proof involves a consideration of a continuum of projection operators to prove “non existence” of hidden variable theories using noncontextuality and additivity of expectation values of only commuting projection operators. Philosophers of science refer to it as a “continuum proof”. They prefer another version which uses only a finite number of projection operators which was given by S. Kochen and E. Specker in 1967. This proof is given for a spin-one quantum particle which needs a Hilbert space of three dimensions. As one can always embed a 3-dimensional Hilbert space in a higher dimensional one, the result obtained remains valid for any system with Hilbert space of three or more dimensions.

Consider the angular-momentum vector $\vec{J} = (J_x, J_y, J_z)$ for a spin-one particle. Using the standard commutation rules for angular momentum operators, it is easy to see that the three operators $J_x^2, J_y^2, J_z^2$ commute with other and further

$$J_x^2 + J_y^2 + J_z^2 = \hbar^2.$$  

We will set $\hbar = 1$ from now on. The eigenvalues of $J_x^2, J_y^2, J_z^2$ are 0 and 1. It follows that, in a dispersion free state, out of these three operators two of them must take the value
1 and the third one the value zero. Otherwise they cannot sum up to two. We can also recast this example in terms of three commuting projection operators \( P_x, P_y, P_z \) which sum to unity, and whose eigenvalues are 0 and 1. Then, in a dispersion free state, one of the three projection operator would take the value 1, while the other two would take the value 0. As it looks more physical we will continue to take in terms of squares of the components of the angular momentum. Similarly if \((n_1, n_2, n_3)\) is triad of unit vectors which are perpendicular to each other, than again \(J^2_{n_1}, J^2_{n_2}, J^2_{n_3}\) commute with other and add upto 2. Further, in a dispersion free state two of them must take the value 1. while the third one should be zero.

More playfully the problem is same as the following Kochen-Specker colouring problem. Take a number of orthogonal triads and start coloring them green if \(S^2_n = 0\) or red if \(S^2_n = 1\). In each triad \((n_1, n_2, n_3)\) two of vectors will be coloured red and the third one green. Is it always possible to carry out this colouring job consistently without ever coming to impasse that the same vector has to coloured both red and green? Kochen and Specker considered a set of 117 directions and showed that such is not the case. Their construction is highly intricate geometrically. A Peres found, in 1991, a choice of particularly elegant choice of 33 directions which also allow one to prove the same result. These are all the rays obtained by taking the squares of three direction cosines to be \((0, 0, 1)\), \((0, \frac{1}{2}, \frac{1}{2})\), \((0, \frac{1}{3}, \frac{2}{3})\) and \((\frac{1}{4}, \frac{1}{2}, \frac{1}{2})\). A ray with direction cosines \((a, b, c)\) is regarded equivalent to one with \((-a, -b, -c)\). A figure containing exactly these 33 directions occurs in one of Escher’s drawing of an impossible waterfalls on the top of one of the towers. The record of the minimum directions needed for proof is 31 and is due to Conway and Kochen.

We thus see that it not possible for a non contextuatic hidden variable theory of reproduce quantum mechanical results.

### 5.4 Mermin’s Examples

The proofs of Kochen-Specker theorem involve more intricate geometry than is to the taste of most physicists. Mermin considered a set of nine physical observables for a system of two spin one-half particles. For this system, with a Hilbert space having dimension equal to 4, it is possible to prove Bell-Kochen-Specker result in a very simple algebraic fashion using only the properties of Pauli-matrices.

Let \(A, B, C, \cdots\) be a commuting set of observables. Let eigenvalues of \(A, B, C, \cdots\) be denoted by \(a, b, c, \cdots\). Let there exist a functional relationship between these commuting observables is given by

\[
f(A, B, C, \cdots) = 0.
\]

Let \(v(A)\) be the value the observable \(A\) in the dispersion free state. We shall take \(v(A)\) to be one of the eigenvalues \(a\) of \(A\). Similarly for \(v(B)\) etc. These values \(v(A), v(B), \cdots\) must also satisfy

\[
f(v(A), v(B), v(C), \cdots) = 0.
\]
The set of nine observables are arranged in a $3 \times 3$ array given below:

\[
\begin{array}{ccc}
V1 & V2 & V3 \\
H1: & \sigma_x^{(1)} & \sigma_x^{(2)} & \sigma_x^{(1)}\sigma_x^{(2)} \\
H2: & \sigma_y^{(2)} & \sigma_y^{(1)} & \sigma_y^{(1)}\sigma_y^{(2)} \\
H3: & \sigma_x^{(1)}\sigma_y^{(2)} & \sigma_x^{(2)}\sigma_y^{(1)} & \sigma_x^{(1)}\sigma_z^{(2)}
\end{array}
\]

where $\sigma_x^{(\alpha)}, \sigma_y^{(\alpha)}, \sigma_z^{(\alpha)}$ are the Pauli-matrices for the particle $\alpha, (\alpha = 1, 2)$. All the three operators in either a row (denoted by $H1$, $H2$, $H3$) or a column (denoted by $V1$, $V2$, $V3$) commute with each other. Further the product of all the three operators in either a row or in column $V1$ and $V2$ is equal to 1, while the product of the three operators in column $V3$ equals $(-1)$. Further the eigenvalues of all the nine operators are $+1$ or $-1$.

It follows that the product of the values $v(A)$ of all the observables in horizontal row and first and second vertical rows must be equal to 1, and for the third vertical column it should be equal to $-1$. This is however impossible since the product all the nine values of operators, as calculated from the product of values of operators in the horizontal rows comes out to 1, while the same product, as calculated from the product of values of operators in the vertical rows comes out to be $-1$. We have thus shown that such a noncontextual value assignment to the physical observables in not possible, thus proving Bell-Kochen-Specker theorem. This is a far simpler proof of it, albeit for only system with a dimension of Hilbert space larger equal to 4, while the original Bell-Kochen-Specker applied for systems with Hilbert space dimension greater or equal to 3.

We may mention here that, if we consider more generalised measurements using “Positive Operator Valued Measures” (POVM’s), instead of only Von-Neumann measurements using only “Projectors”, we can extend both Gleasons and Kocher-Specker Theorems to Hilbert space of a qubits with dimension 2.

### 5.5 Some Further Aspects

#### 5.5.1 Stochastic Noncontextual Hidden Variable Theories

As we have seen deterministic noncontextual hidden variable theories are ruled out by Gleason-Bell and Kochen-Specker theorems. In quantum theory however we have a kind of “Statistical Noncontextuality” in that the expectation values of any observable is unchanged by a simultaneous or previous measurement of observable which commute with it. Roy and Singh therefore investigated whether a “statistically noncontextual” hidden variable theory could reproduce this feature of quantum mechanics. They showed that it is not possible to do so.

#### 5.5.2 Finite Precision Measurements and Kochen-Specker Theorem

We saw that it was not possible to carry out Kochen-Specker colouring of the surface of a unit phase. Note that we are specifying each direction precisely with no latitude for error.
Meyer, in 1999, followed by Kent, raised the issue whether a noncontextual hiddle variable model will allow us to reproduce the predictions of quantum theory if we allow finite fixed precision in experiments. Clifton and Kent later presented K-S colouring which claim to do so over any dense subset. These colourings are extremely discontinuous. There has been a lot of criticism of these claims and an extended discussion of what they mean? Since any neighbourhood, no matter how small, contains points coloured both red and green, it follows that these models do not satisfy the “faithful measurement condition”. If an observable has a value, it is not guaranteed that in an experiment, no matter how precise, we will obtain a value close to it’s value with a high probability.

6 Einstein- Locality and Quantum Mechanics

6.1 The Background
The early phase of quantum theory beginning with Planck’s work in 1900 on black-body radiation and ending with the discovery of quantum mechanics by W. Heisenberg (1925), Paul Dirac (1925) and E. Schrödinger (1926) is referred to as old quantum theory. In this period Einstein beginning with his paper on “light quantum hypothesis” in 1905, and Niels Bohr, beginning with his papers on atomic structure and spectra in 1913, had been the main guiding lights of quantum theory. After the discovery of the mathematical formation of quantum mechanics in 1925-26, Niels Bohr, together with Heisenberg and others, hammered out the “Copenhagen Interpretation of quantum mechanics” which was to remain the dominant way to think about quantum mechanics for a long time. Einstein, with his deep commitment to realism, did not subscribe to this interpretation. His initial efforts were to see whether Heisenberg’s uncertainty principle can be circumvented and thereby showing the provisional nature of quantum mechanics. These discussions have been summarised by Bohr in his account of famous Einstein-Bohr dialogues. Einstein here conceded to Bohr that this was not possible. Einstein however raised a much more substantial argument, against the prevailing interpretation, which depended on some unusual non-local aspects of two-particle systems in quantum mechanics.

6.2 Einstein-Podolsky-Rosen Theorem
In 1935, A. Einstein, B. Podolsky and N. Rosen (EPR) published a paper “Can Quantum Mechanical Description of Reality be Considered Complete?” in Physical Review. It had a rather unusual title for a paper for this journal. In view of this they provided the following two definitions at the beginning of the paper: (1) A necessary condition for the completeness of a theory is that every element of the physical reality must have a counterpart in the physical theory. (2) A sufficient condition to identify an element of reality: “If, without in any way disturbing a system, we can predict with certainty (ie with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity”.

We now illustrate the use of these definitions for a single-particle system. Let the position and momentum observable of the particle be denoted by $Q$ and $P$ respectively.
Since in an eigenstate of \( Q \), we can predict with certainty the value of \( Q \), which is given by its eigenvalue in that eigenstate, it follows that the position \( Q \) of the particle is an element of physical reality (e.p.r.). Similarly the momentum \( P \) is also an e.p.r. The position \( Q \) and the momentum \( P \) however are not simultaneous e.p.r. So at the single particle level there is no problem with quantum mechanics, as far as these definitions of ‘completeness’ and ‘elements of reality’ are concerned.

The interesting new things are however encountered when a two particle system is considered. Let the momenta and position of the two particles be denoted respectively by \( P_1 \) and \( Q_1 \) for the first particle and by \( P_2 \) and \( Q_2 \) for the second particle. Consider now the two-particle system in the eigenstate of the relative-position operator, \( Q_2 - Q_1 \) with eigenvalue \( q_0 \). The relative position \( Q_2 - Q_1 \) can be predicted to have a value \( q_0 \) with probability one in this state and thus qualifies to be an e.p.r. We can also consider an eigenstate of the total momentum operator, \( P_1 + P_2 \), with an eigenvalue \( p_0 \). The total momentum can be predicted to have a value \( p_0 \) with probability one and thus also qualifies to be an e.p.r. Furthermore relative position operator, \( Q_2 - Q_1 \), and total momentum operator, \( P_1 + P_2 \), commute with each other and thus can have a common eigenstate, and thus qualify to be simultaneous elements of physical reality.

We consider the two-particle system in which two particles are flying apart from each other having momenta in opposite directions and are thus having a large spatial separation. The separation will be taken so that no physical signal can reach between them. Let a measurement of position be made on the first particle in the region \( R_1 \) and let the result be \( q_1 \). It follows from standard quantum mechanics that instantaneously the particle 2, which is a spatially far away region \( R_2 \), would be in an eigenstate \( q_0 + q_1 \) of \( Q_2 \). The \( Q_2 \) is thus an e.p.r. the position of second particle gets fixed to the value \( q_0 + q_1 \) despite the fact that no signal can reach from region \( R_1 \) to \( R_2 \) where the second particle is A “spooky action at a distance” indeed. On the other hand a measurement of the momentum \( P_1 \) of the first particle, in the region \( R_1 \) can be carried out and let it result in a measured value \( p_1 \). It then follows from the standard quantum mechanics, that the particle 2, in the region \( R_2 \) would be in an eigenstate of its momentum \( P_2 \) with and eigenvalue \( p_0 - p_1 \). The \( P_2 \) is thus also an e.p.r. This however leads to a contradiction since \( Q_2 \) and \( P_2 \) can not be a simultaneous e.p.r. as they do not commute. We quote the resulting conclusion following from this argument as given by Einstein in 1949,

**EPR Theorem**: The following two assertions are not compatible with each other

“(1) the description by means of the \( \psi \)-function is complete

(2) the real states of spatially separated objects are independent of each other”.

The predilection of Einstein was that the second postulate, now referred to as “Einstein locality” postulate, was true and thus EPR theorem establishes the incompleteness of quantum mechanics.

Einstein, Podolsky and Rosen were aware of a way out of the above theorem but they rejected it as unreasonable. As they said “Indeed one would not arrive at our conclusion if one insisted that two or more quantities can be regarded as simultaneous elements of reality only when they can be simulatenously measured or predicted. On this point of view, either one or the other, but not both simultaneously, of the quantities \( P \) and \( Q \) can be predicted, they are not simultaneously real. This makes the reality of \( P \) and \( Q \) depend upon the process of measurement carried out on the first system, which does not disturb
the second system in any way. No reasonable definition of reality could be expected to permit this”.

6.3 Einstein Locality

We present here a selection of Einstein quotations on this important concept.

(i) From his “autobiographical notes” (1949)

“But on one supposition we should, in my opinion, absolutely hold fast: the real factual situation of the system \( S_2 \) is independent of what is done, with system \( S_1 \), which is spatially separated from the formed”.

(ii) From Einstein-Born correspondence (March 1948)

“That which really exists in B should \( \cdots \) not depend on what kind of measurement is carried out in part of space A; it should also be independent of whether or not any measurement at all is carried out in space A. If one adheres to this program, one can hardly consider the quantum theoretical description as a complete representation of the physically real. If one tries to do so in spite of this, one has to assume that the physically real in B suffers a sudden change as a result of a measurement in A. My instinct for physics bristles at this”.

(iii) From Einstein-Born Correspondence (March 1947)

“I can not seriously believe in (the quantum theory) because it can not be reconciled with the idea that physics should represent a reality in time and space free from spooky actions at a distance (Spukhafte Fernwirkungen)”.

6.4 Bohm’s Version of the EPR Analysis

Einstein-Podolsky-Rosen discussion suffers from some inessential mathematical technicalities, which were noted by a number of authors. Some of these are, (i) the use of non normalisable plane-wave eigenfunctions, (ii) neglect of time dependence of wave functions, and (iii) non-self-adjointness of momentum operator over half-space. Bohm, in order to present a clear account of the essential features of the EPR analysis, reformulated it in terms of two spin-one-half particles. This formalism has also been influential in later work of John Bell and others on the foundations of quantum mechanics. We therefore present it now.

Consider a spin one-half particle with spin angular momentum \( \vec{S} = \frac{1}{2}\vec{\sigma}\hbar \). Here \( \vec{\sigma} \) are the usual Pauli matrices. From now we will often set \( \hbar = 1 \). The component of spin along the direction unit-vector \( \hat{n} \) is given by \( \vec{S} \cdot \hat{n} \). Let us denote it’s eigenvectors by \( \chi_m(\hat{n}) \) with \( m \) specifying the eigenvalue ie

\[
\vec{S} \cdot \hat{n} \chi_m(\hat{n}) = m \chi_m(\hat{n}).
\]

Here \( m \) can take only two values given by \( m = \frac{1}{2} \), corresponding to spin pointing up in \( \hat{n} \) direction and \( m = -\frac{1}{2} \), corresponding to spin pointing down in \( \hat{n} \) direction. We also denote \( S \cdot \hat{e} \) by \( S_z \) and \( \vec{S} \cdot \hat{x} \) by \( S_x \). In an eigenstate of \( S_z \), we can predict with certainty
it’s value and thus it is an e.p.r. Similarly one can see that $S_x$ is also an e.p.r. They are however not simultaneous e.p.r. as they do not commute.

We now consider, a two spin one half particle system in a singlet state ie which has total spin-angular-momentum, $S^{(1)} + S^{(2)}$, equal to zero. We will use superscripts 1 and 2 to denote the quantities referring the particle 1 and 2 respectively. We will also ignore the inessential space degree of freedom. The eigenfunction of the singlet state is given by

$$\psi(\text{singlet}) = \frac{1}{\sqrt{2}} \left[ \chi^{(1)}_{1/2}(\hat{z})\chi^{(2)}_{-1/2}(\hat{z}) - \chi^{(1)}_{-1/2}(\hat{z})\chi^{(2)}_{1/2}(\hat{z}) \right].$$

We now imagine this singlet state two-particle state, at rest, decays into it’s two constituents, which fly apart with opposite momenta. Let the particle 1 and 2 be respectively in regions $R_1$ and $R_2$ which are far apart. If we now measure $S_x^{(1)}$ for particle 1 in the region $R_1$ we will obtain a definite value which is $+\frac{1}{2}$ or $-\frac{1}{2}$. We can therefore assert with certainty that $S_x^{(2)}$ has a definite value, which is negative of the value found for $S_x^{(1)}$, in the far away region $R_2$.

We could however have decided to measure $S_x^{(1)}$ instead of $S_x^{(1)}$. The singlet eigenfunction can be also reexpressed as

$$\psi(\text{singlet}) = \frac{1}{\sqrt{2}} \left[ \chi^{(1)}_{1/2}(\hat{x})\chi^{(2)}_{-1/2}(\hat{x}) - \chi^{(1)}_{-1/2}(\hat{x})\chi^{(2)}_{1/2}(\hat{x}) \right].$$

A measurement of $S_x^{(1)}$ in the region $R_1$ would lead to a definite value for $S_x^{(1)}$ which is either $+\frac{1}{2}$ or $-\frac{1}{2}$. We can therefore assert with certainty that the value of $S_x^{(2)}$ also has a definite value, which is negative of $S_x^{(1)}$ value found in $R_1$, in the far away region $R_2$. However the particle 2 can not have a definite value for both $S_x^{(2)}$ and $S_x^{(2)}$ since these do not commute. The EPR theorem, therefore, follows. Surprising element here is not that the value of $S_x^{(2)}$ (or $S_x^{(1)}$) is negative to that of $S_x^{(1)}$ (or $S_x^{(1)}$), but rather that the far away particle 2 is in an eigenstate of $S_x^{(2)}$ or $S_x^{(2)}$ depending on what we decide to measure in region $R_1$. As Greenberger and Yasin said “Reality should be made of sterner stuff”.

### 6.5 Bell’s Inequalities and Bell’s Theorem

As a consequence of E.P.R. work we see Einstein nonlocality is a feature of quantum mechanics. On the other hand Einstein-locality looks a very nice and desirable feature to have in a basic theory of nature. Can we retain Einstein-locality if we admit of a hidden variable substratum to quantum mechanics? John Bell investigated that possibility by giving concept of Einstein-locality a precise mathematics formulation.

Let the set of hidden variables be denoted by $\lambda$. We take then to be distributed probability distribution $\rho(\lambda)$ which is positive semidefinite and is normalised to unity. We use Bohm formulation of EPR work and we shall be measuring correlations of spin components $\sigma^{(1)} \cdot \hat{a}$ and $\sigma^{(2)} \cdot \hat{b}$ for the two spin $\frac{1}{2}$ particles. Let the values fixed by hidden variables $\lambda$ for $\sigma^{(1)} \cdot \hat{a}$ be denoted by $A(\hat{a}, \lambda; b)$ while for $\sigma^{(2)} \cdot \hat{b}$ be denoted by $B(\hat{b}, \lambda; a)$. since the observable values of $\sigma^{(1)} \cdot \hat{a}$ and $\sigma^{(2)} \cdot \hat{b}$ are their eigenvalues, which are $+1$ or $-1$, we have to take allowed value of $A(\hat{a}, \lambda; b)$ and $B(\hat{b}, \lambda; a)$ to also be $+1$ or $-1$. All the quantities $\rho(\lambda), A(\hat{a}, \lambda; b)$ and $B(\hat{b}, \lambda; a)$ will have dependence on the wavefunction
ψ of the system as well which we donot explicitly indicate. In the hidden variable theory
$P(\hat{a}, \hat{b})$, ie the expectation value of the product of $\sigma^{(1)} \cdot \hat{a}$ and $\sigma^{(2)} \cdot \hat{b}$, would be given by

$$P_{h.v}(\hat{a}, \hat{b}) = \int d\lambda \rho(\lambda) A(\hat{a}, \lambda; \hat{b}) B(\hat{b}, \lambda; \hat{a}).$$

We have so far not used the concept of Einstein locality. In it’s spirit we assume, following Bell, that the hidden variable value $A(\hat{a}, \lambda; \hat{b})$ of the observable $\sigma^{(1)} \cdot \hat{a}$ should not depend on the setting $\hat{b}$ of the Stern-Gerlach magnet used to measure $\sigma^{(2)} \cdot \hat{b}$ in the far away region ie

$$A(\hat{a}, \lambda; \hat{b}) = A(\hat{a}, \lambda).$$

Similarly

$$B(\hat{b}, \lambda; \hat{a}) = B(\hat{b}, \lambda).$$

So, in an Einstein-locality obeying hidden variable theory

$$P_{h.v}(\hat{a}, \hat{b}) = \int d\lambda \rho(\lambda) A(\hat{a}, \lambda) B(\hat{b}, \lambda)$$

where

$$|A(\hat{a}, \lambda)| = 1$$

$$|B(\hat{b}, \lambda)| = 1$$

and

$$\rho(\lambda) \geq 0, \quad \int d\lambda \rho(\lambda) = 1.$$

The correlation coefficient $P(\hat{a}, \hat{b})$ is given in quantum mechanics by

$$P_{QM}(\hat{a}, \hat{b}) = \langle \psi | \sigma^{(1)} \cdot \hat{a} \sigma^{(2)} \cdot \hat{b} | \psi \rangle$$

$$= -\hat{a} \cdot \hat{b} \quad \text{for} \ \psi = \psi(\text{singlet}).$$

Note that in quantum mechanics we have perfect anticorrelation given by

$$P_{QM}(\hat{a}, \hat{a}) = -1 \quad \text{for} \ \psi = \psi(\text{singlet}).$$

If we specialise to the case of singlet wavefunction and demand perfect anticorrelation, an extra assumption, for the correlation function for hidden variable theories it

$$P_{h.v}(\hat{a}, \hat{a}) = -1$$

we must have

$$B(\hat{a}, \lambda) = -A(\hat{a}, \lambda).$$

We then have the symmetric expression

$$P_{h.v}(\hat{a}, \hat{b}) = -\int d\lambda \rho(\lambda) A(\hat{a}, \lambda) A(\hat{b}, \lambda).$$

It is easy to see that

$$|P_{h.v}(\hat{a}, \hat{b}) - P_{h.v}(\hat{a}, \hat{c})| \leq 1 + P_{h.v}(\hat{b}, \hat{c})$$

25
Proof:

\[ P_{h.v}(\hat{a}, \hat{b}) - P_{h.v}(\hat{a}, \hat{c}) = -\int d\lambda \rho(\lambda) A(\hat{a}, \lambda) [A(\hat{b}, \lambda) - A(\hat{c}, \lambda)] , \]

\[ |P_{h.v}(\hat{a}, \hat{b}) - P_{h.v}(\hat{a}, \hat{c})| \leq \int d\lambda \rho(\lambda) |A(\hat{a}, \lambda)A(\hat{b}, \lambda)| [1 - A(\hat{b}, \lambda)A(\hat{c}, \lambda)] \]

\[ \leq \int d\lambda \rho(\lambda) [1 - A(\hat{b}, \lambda)A(\hat{c}, \lambda)] = 1 + P_{h.v}(\hat{b}, \hat{c}) . \]

We can, in fact, derive four inequalities in a similar way,

\[ 1 \geq \eta_a \eta_b P_{h.v}(\hat{a}, \hat{b}) + \eta_a \eta_c P_{h.v}(\hat{a}, \hat{c}) + \eta_b \eta_c P_{h.v}(\hat{b}, \hat{c}) \]

where \((\eta_a)^2 = (\eta_b)^2 = (\eta_c)^2 = 1\). These were the first inequalities, following from Einstein-locality for the correlation coefficients, which were derived by John Bell. All such inequalities on correlation coefficients are now known, generically, as Bell’s inequalities.

Bell made the remarkable discovery that these inequalities are not always consistent with the prediction of quantum-mechanics, \( P_{Q.M}(\hat{a}, \hat{b}) = -\hat{a} \cdot \hat{b} \) given above. For example let \( \hat{a} = (1, 0, 0), \hat{b} = (-\frac{1}{2}, \sqrt{3}/2, 0), \hat{c} = (-\frac{1}{2}, -\sqrt{3}/2, 0) \) and choose \( \eta_a = \eta_b = \eta_c = 1 \). We then have \( \hat{a} \cdot \hat{b} = \hat{a} \cdot \hat{c} = \hat{b} \cdot \hat{c} = -\frac{1}{2} \) and thus Bell’s inequality, given above becomes

\[ 1 \geq \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2} \]

which is clearly not satisfied. It follows that,

Bell’s Theorem:
Quantum mechanics is inconsistent with a Einstein-locality obeying hidden variable theories.

Stapp has called it the most profound discovery of twentieth century physics.

The above discussion can be generalised to include additional hidden variables for the measuring instruments as well provided the distribution of these hidden variables does not depend on the setting of the far away measuring instrument. Denoting by \( \bar{A}(\hat{a}, \lambda) \) and \( \bar{B}(\hat{b}, \lambda) \), the averaging of \( A(\hat{a}, \lambda) \) and \( B(\hat{b}, \lambda) \) respectively, we have

\[ |\bar{A}(\hat{a}, \lambda)| \leq 1, \quad |\bar{B}(\hat{b}, \lambda)| \leq 1 \]

and we have to use these averaged quantities \( \bar{A}(\hat{a}, \lambda) \) and \( \bar{B}(\hat{b}, \lambda) \), instead of \( A(\hat{a}, \lambda) \) and \( B(\hat{b}, \lambda) \) respectively in the expression for \( P_{h.v}(\hat{a}, \hat{b}) \) given earlier. The same Bell’s inequalities still follow.

6.6 Clauser-Horne-Shimony-Holt (CHSH) Form of Bell’s Inequalities

The assumption of perfect correlation \( P(\hat{a}, \hat{a}) = -1 \) for a singlet two spin one-half particles may be hard to meet in practice as it would require ideal (and may be identical) detectors on both sides. The CHSH formulation does not assume this. Using

\[ P_{h.v}(\hat{a}, \hat{b}) = \int d\lambda \rho(\lambda) \bar{A}(\hat{a}, \lambda) \bar{B}(\hat{b}, \lambda) \]
with
\[|A(\hat{a}, \lambda)| \leq 1, \quad |B(\hat{b}, \lambda)| \leq 1,\]
\[\rho(\lambda) \geq 0, \quad \int d\lambda \rho(\lambda) = 1,\]
it is possible to derive the following Bell-CHSH equalities
\[S = |P_{h.v}(\hat{a}, \hat{b}) - P_{h.v}(\hat{a}', \hat{b}')| + |P_{h.v}(\hat{a}', \hat{b}) + P_{h.v}(\hat{a}, \hat{b}')| \leq 2.\]
In fact we have the same inequalities valid even if
\[P(\hat{a}_i, \hat{b}_j) \to Q(\hat{a}_i, \hat{b}_j) = \eta_i \eta_j P(\hat{a}_i, \hat{b}_j)\]
where
\[(\hat{a}_1, \hat{a}_2, \hat{b}_1, \hat{b}_2) = (\hat{a}, \hat{a}', \hat{b}, \hat{b}')\quad \eta_i^2 = \eta_j^2 = 1.\]
The proof is similar to Bell’s inequalities given earlier.
If these inequalities are consistent with Quantum-mechanics we should have
\[|\hat{a} \cdot \hat{b} - \hat{a} \cdot \hat{b}'| + |\hat{a}' \cdot \hat{b} + \hat{a} \cdot \hat{b}'| \leq 2\]
for all possible choices of unit vectors. Choosing
\[\hat{a} = (0, 1, 0), \hat{b} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), \hat{a}' = (1, 0, 0), \hat{b}' = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)\]
we see that lefthand side is equal to $2\sqrt{2}$ while the right hand side of the inequality is equal to 2. We thus see that quantum mechanics is in general not consistent with hidden variable Einstein-local theories. The violation, which is maximum possible, is a factor of $\sqrt{2}$ here.

6.7 Wigner’s Proof of Bell-CHSH Inequalities

We should also refer to a proof of Bell’s inequalities by Wigner. His assumption are (i) physical realism: spin components of the two particles have definite preassigned values even for non-commuting observables, and (ii) Einstein-locality: a measurement on a spin component on one side does not modify the preassigned value of the far away spin component. In Wigner’s proof probabilities appear for the first time. Let $w(s, s', t, t')$ be the fraction of population of those particles for which preassigned values are given by
\[A(\hat{a}) = s, \quad A(\hat{a}') = s', \quad B(\hat{b}) = t \quad \text{and} \quad B(\hat{b}') = t'.\]
Here $s, s', t, t'$ are all equal to $\pm 1$. We then have
\[\sum w(s, s', t, t') = 1, \quad w(s, s', t, t') \geq 0,\]
\[P(\hat{a}, \hat{b}) = \sum w(s, s', t, t')st,\]
\[P(\hat{a}, \hat{b}') = \sum w(s, s', t, t')st',\]
\[P(\hat{a}', \hat{b}) = \sum w(s, s', t, t')s't,\]
\[P(\hat{a}', \hat{b}') = \sum w(s, s', t, t')s't'.\]
Again it is easy to derive Bell-CHSH inequality using these expression. Note that probability concept is used here in the same way as in classical statistical physics.
6.8 Experimental Tests of Bell’s Inequalities

The Bell-CHSH inequalities, published in 1969, were amenable to experimental tests and showed the possibility of experimental tests for discriminating between hidden variable theories obeying Einstein locality and quantum mechanics. The first generation experiments were carried in nineteen seventies. Better second generations experiments were carried out by Alain Aspét at Orsay in 1980-82. For the settings, where the conflict is maximu, it is found that

\[ S_{\text{exp}} = 2.697 \pm 0.015 \]

The quantum mechanical prediction is

\[ S_{\text{QM}} = 2.70 \pm 0.05. \]

The uncertainties in \( S_{\text{QM}} \) refer to slight lack of symmetry, about \( \pm 1\% \), of two channels of polarisers used in the experiment (ideally \( S_{\text{QM}} = 2\sqrt{2} \)). The local hidden variable theory has, as follows for CHSH inequality

\[ |S_{h.v}| \leq 2.0 \]

It would seem that experiments are in favour of quantum mechanics and Einstein nonlocality is a feature of nature.

More precise third generation experiments started in late ninetten eighties and still are in progress. They would take care of various loopholes in earlier experiments such as (i) low detection efficiencies of photon detectors, (ii) possibility of light signalling between the detectors by having polarisers which can be randomly oriented in timing faster than the time taken by light to reach from one polariser to another.

6.9 Bell’s Theorem without Inequalities

6.9.1 Greenberger-Horne-Zeilinger (GHZ) Proof

It was realised by Greenberger, Horne and Zeilinger that for a three spin-\( \frac{1}{2} \) particle system, it is possible to demonstrate a conflict between local realism and quantum mechanics more directly without using Bell’s inequalities.

Consider the three spin-\( \frac{1}{2} \) particle system in the state

\[ \psi = \frac{1}{\sqrt{2}}[|↑↑↑⟩ − |↓↓↓⟩] \]

where \( |↑↑↑⟩ (|↓↓↓⟩ \) denotes the state with all the three particles have \( z \) component of the spin \( S_z = \frac{1}{2} \sigma_z \) equal to \( +\frac{1}{2} (−\frac{1}{2}) \). It is easy to verify that

\[ \sigma_z^{(1)} \sigma_z^{(2)} \sigma_z^{(3)} |\psi⟩ = |ψ⟩ \]
\[ \sigma_y^{(1)} \sigma_x^{(2)} \sigma_y^{(3)} |\psi⟩ = |ψ⟩ \]
\[ \sigma_y^{(1)} \sigma_y^{(2)} \sigma_x^{(3)} |\psi⟩ = |ψ⟩. \]
Here superscripts will refer to the particles. Let the three particle separate and be in a far away regions. We can measure $\sigma_y^{(2)}$ with a result $m_y^{(2)}$, and $\sigma_y^{(3)}$ with a result $m_y^{(3)}$, we can predict the value $m_x^{(1)}$ of $\sigma_x^{(1)}$ with certainty. It would be given by

$$m_x^{(1)} m_y^{(2)} m_y^{(3)} = 1.$$  

Thus $\sigma_x^{(1)}$ would be an e.p.r. Similarly the value $m_y^{(1)}$ of $\sigma_y^{(1)}$ can be predicted with certainty by measuring the value $m_x^{(2)}$ of $\sigma_x^{(2)}$ and the value $m_y^{(3)}$ of $\sigma_y^{(3)}$ in the far away regions. It would be given by

$$m_y^{(1)} m_x^{(2)} m_y^{(3)} = 1.$$  

Thus $\sigma_y^{(1)}$ is also an e.p.r. Similarly $\sigma_x^{(2)}$, $\sigma_y^{(2)}$, $\sigma_x^{(3)}$ and $\sigma_y^{(3)}$ are also e.p.r. We have one more relation

$$m_y^{(1)} m_x^{(2)} m_y^{(3)} = 1.$$  

By multiplying these three relations it follows that

$$m_x^{(1)} m_x^{(2)} m_x^{(3)} = 1.$$  

Since $(m_y^{(i)})^2 = 1$ for $i = 1, 2, 3$. But this is not consistent with quantum mechanics, since we also have

$$\sigma_x^{(1)} \sigma_x^{(2)} \sigma_x^{(3)} |\psi\rangle = (-1) |\psi\rangle$$  

which would imply in a local realist theory

$$m_x^{(1)} m_x^{(2)} m_x^{(3)} = -1.$$  

This argument demonstrate the conflict between deterministic local-hidden variable theories. Of course Bell’s argument is generalisable to some forms of Stochastic local-hidden variable theories as well.

### 6.9.2 Hardy’s Version of EPR Correlations

L. Hardy found that, surprisingly there exist two spin one-half particle states for which it is possible to demonstrate the conflict between local realism and quantum mechanics. We first present it in a paraphrase by Stapp.

Consider again two spin one half particles moving away from each other towards region $A$ and $B$ which are far from each other.

We denote the two physical observable, to be measured in each region $A$ and $B$ by “size” and “colour”. The “size” can take two values viz “large” and “small”. The “colour” can be “black” or “white”. The constructed state has the following perfect corelations built in

(i) If “size” was measured in region $A$ and was found to have the value “large”, then if “colour” was measured in region $B$ it would be found to be “white” with probability equal to 1 ie

$[\text{region } A: \text{“size” = “large”}] \rightarrow [\text{region } B: \text{“colour” = “white”}]$,

(ii) $[\text{region } B: \text{“colour” = “white”}] \rightarrow [\text{region } A: \text{“colour” = “black”}]$,
Would it now be correct to conclude that the following correlation should also be perfect, i.e., probability equal to 1,

\[ \text{[region } A: \text{“size”} = \text{“large”]} \rightarrow \text{[region } B: \text{“size”} = \text{“small”}], \]

as it seems to be implied by classical physics and common sense. Hardy showed that, in a quantum world, this is not necessarily so but one can find, with a probability of up to about 9%, the correlation

\[ \text{[region } A: \text{“size”} = \text{“large”]} \rightarrow \text{[region } B: \text{“size”} = \text{“large”}]. \]

We now present briefly some mathematical details. Consider the observables \( U^{(1)} \) (and \( U^{(2)} \)) for the particle 1 (and 2) given by

\[ U^{(i)} = \left| u^{(i)} \right\rangle \langle u^{(i)} | \]

where \( |u^{(i)}\rangle \) and \( |v^{(i)}\rangle \) forms an orthonormal basis states for particle \( i (= 1, 2) \). Consider also another observables \( U^{(i)'} (i = 1, 2) \) for the two particles given by

\[ U^{(i)'} = \left| u^{(i)'} \right\rangle \langle u^{(i)'} | \]

where \( |u^{(i)'}\rangle \) and \( |v^{(i)'}\rangle \) forms another orthonormal basis states of the particle \( i (= 1, 2) \). All these observables are projection operators with eigenvalues equal to 0 or 1.

Hardy discovered that two spin-\( \frac{1}{2} \) particles states \( |\psi\rangle \) exists which satisfy, (with \( V^{(i)'} = 1 - U^{(i)} \))

(i) \( U^{(1)} U^{(2)} |\psi\rangle = 0 \)

(ii) \( (1 - U^{(1)}) V^{(2)'} |\psi\rangle = 0 \) i.e. \( U^{(2)'} = 0 \implies U^{(1)} = 1 \)

(iii) \( V^{(1)'} (1 - U^{(2)}) |\psi\rangle = 0 \) i.e. \( U^{(2)} = 0 \implies U^{(1)'} = 1 \)

(iv) there is a non-zero probability \( p \) for finding \( U^{(1)'} = U^{(2)'} = 0 \).

More explicitly, we have

\[ \sqrt{1-p_1 p_2} |\psi\rangle = \sqrt{(1 - p_1)(1 - p_2)} |v^1, v^2\rangle - \sqrt{p_1 (1 - p_2)} |u^1, v^2\rangle - \sqrt{p_2 (1 - p_1)} |v^1, u^2\rangle \]

\[ p = \frac{p_1 (1 - p_1)(p_2)(1 - p_2)}{1 - p_1 p_2} \]

where \( p_1, p_2 \) are real numbers lying between 0 and 1.

The maximum value of \( p \) is obtained by choosing \( p_1 = p_2 = \frac{1}{\tau} \) where \( \tau = \frac{\sqrt{5} + 1}{2} \) is the golden ratio. We thus have

\[ 0 \leq p \leq \frac{1}{\tau} \approx 0.09017 \cdots \]

By using Hardy ladders, i.e., by a consideration of \( N \) observables, each two valued, in each region \( A \) and \( B \), the probability of violation of local realism \( p_N \) can be made as large as \( \frac{1}{2} \) by letting \( N \rightarrow \infty \).
6.10 Superluminal Signalling

Quantum mechanics satisfies a signal locality, i.e. no faster than light signalling, requirement for statistical averages. Let $B$ denote a physical observable being measured in a space like region separated from the region to which a variable $A$ refers. On measuring $B$ the density matrix $\rho$ changes to $\rho'$, i.e.

$$\rho \rightarrow \rho' = \sum_{\beta} P_{\beta} \rho P_{\beta}$$

where $P_{\beta}$'s are projectors to different eigenstates of the observable $B$. So if $A$ is measured after $B$ is measured first, we get the expectation value $\langle A \rangle$ of $A$,

$$\langle A \rangle = \text{Tr}(\rho' A) = \text{Tr}(\sum_{\beta} P_{\beta} \rho P_{\beta} A)$$

$$= \text{Tr} \sum_{\beta} (P_{\beta} \rho AP_{\beta})$$

since $[A, P_{\beta}] = 0$ as $[A, B] = 0$. Using cyclicity of trace and general properties of projectors

$$P_{\beta}^2 = P_{\beta}, \quad \sum_{\beta} P_{\beta} = 1,$$

we finally obtain

$$\langle A \rangle = \text{Tr}(\rho A).$$

This however is also the expectation value of $A$ even if $B$ had not been measured earlier. We can not use the earlier measurement of $B$ to do any signalling as long as only expectation values are measured.

It was shown that requirement of signal locality can also be formulated for local hidden variable theories and in general leads to testable inequalities.

It might appear as if EPR correlations might lead to superluminal signalling. This is not so, since the sequence of measurements of a spin component in a region produces a random sequence, which not having any structure, can not contain information.

6.11 More Bell’s Inequalities

The context of Einstein locality is not exhausted, even for two spin-$\frac{1}{2}$ particle system, by the Bell’s inequalities given by Bell himself or by CHSH discussed earlier. A large number of these have written down for two spin-$\frac{1}{2}$ particle system, multiple particle system and higher spin systems. They have also been written for phase space.

7 Envoi

The subject of “foundations of quantum mechanics”, which includes EPR correlation, was somewhat philosophical and generally hard-boiled physicist would turn their noses at it. John Bell’s work in early sixties showed that these philosophical discussion can be subject to precise experimentation. If the experiments had been found not to violate
Bell’s inequalities, quantum mechanics would have been in serious trouble. The Einstein nonlocal nature of quantum mechanics, which was exposed through EPR correlations, have been found in last two decades to be, far from being an embarrassment, a resource in many technical engineering applications. These applications include newly emerging areas of quantum cryptography and quantum teleportation. The whole area of quantum information and computing is intensely active. One has come a long way from philosophy to technology.
8 Bibliographical Notes

1. For the basic formalism of Quantum mechanics, see,
   Dirac, P.A.M., *The Principles of Quantum Mechanics*, (fourth edition), Oxford 1958;
   Von-Neumann, J., *Mathematical Foundations of Quantum Mechanics* (in German),
   Berlin, 1932, (trans. by R.T. Beyer) Princeton, 1955;
   Bohm, A., *The Rigged Hilbert Space and Quantum Mechanics*, Springer lecture notes
   in Physics, ol. 78, 1978.

2. For a historical account of Quantum Mechanics, see,
   Jammer, M., *The Conceptual Development of Quantum Mechanics*, New York, 1966;
   and *The Philosophy of Quantum Mechanics*, New York, 1974;
   Mehra, J. and Rechenberg, H., *The Historical Development of Quantum Mechanics*,
   Springer, 1982-...;
   Whittaker, E.T., *A Theory of Aether and Electricity*, Vol. 2, *Modern
   Theories (1900-1926)*, Harper Torchbacks, 1960;
   Beller, M., *Quantum dialogue*, Chicago, 1999.

3. (i) A number of important basic sources on fundamental aspects of Quantum Me-
   chanics are collected in,
   Wheeler, J.A. and Zurek, W.H., *Quantum Theory and Measurement*, Princeton,
   1983;
   (ii) The writings of John Bell on quantum mechanics and these are a must for any-
   one interested in this area, are available in the collection,
   Bell, J.S., *Speakable and unspeakable in quantum mechanics*, Cambridge, 1987.

4. Some popular level books on quantum mechanics are
   Pagels, H.R., *The Cosmic Code: Quantum Physics as the Language of Nature*, Penguin
   Books, 1984;
   Polkinghorn, J.C., *The Quantum World*, Longmans, London, 1984;
   Rae, A., *Quantum Physics: Illusion or Reality*, Cambridge, 1986;
   deEspagnat, B., *Conceptual Foundations of Quantum Mechanics*, Benjamin, 1976.

5. For a survey of hidden variable theories we refer to,
   Belinfante, F.J., *A Survey of Hidden Variable Theories*, Pergamon, Oxford, 1973;
   One should of course read the masterly treatment in,
   Bell, J.S., *On the Problem of Hidden variables in Quantum Mechanics*, Rev. Mod.
   Phys. **38**, 447 (1966); (to be referred to as Bell (1966)).

6. For Niels Bohr’s views see
   Bohr, N., *Atomic Theory and the Description of Nature*, Cambridge, 1934;
   Bohr, N., *Atomic Physics and Human Knowledge*, Wiley, New York, 1958;
   Bohr, N., *Essays 1958/1962 on Atomic Physics and Human Knowledge*, Wiley, New
   York, 1963.

7. Von-Neumann’s theory of measurement is contained in his book referred to earlier.
   See also
London F., and Bauer, E., *La Theorie de L’ observation en Mecanique Quantique*, Paris, 1939.  
An english translation is available in Wheeler and Zurek collection referred to earlier.

8. The original E.P.R. paper appeared in  
Einstein, A., Podolsky, B. and Rosen, N., Phys. Rev. 57, 777 (1935);  
Einstein’s restatements occur in  
Schilpp, P.A., *Albert Einstein: Philosopher-Scientist*, Library of Living Philosophers, Evanston III, 1949.

9. Von Neumann’s impossibility proof of hidden variable theories also occurs in his book cited earlier. See also,  
Albertson, J., Am. J. Phys. 29, 478 (1961).  
The reaction of Bohr is quoted in  
Selleri, F., *Quantum paradoxes and Physical Reality*, Kluwer, 1990;  
The comments of Feyerabend are from his autobiography,  
Feyerabend, P., *Killing Time*, Chicago, 1995.

10. The original papers on deBroglie-Bohm theory are,  
deBroglie, L., J. Physique, 6th series, 8, 225 (1927);  
Bohm, D., Phys. Rev. 85, 166 (1952);  
Bohm, D., Phys. Rev. 89, 458 (1953).

11. John Bell’s spin one half hidden variable theory and his analysis of von-Neuman and other proofs is given in Bell (1966).  
For Jauch-Piron work, see  
Jauch, J.M. and Piron, C., Helv. Phys. Acta 36, 827 (1963).

12. For Gleason’s proof of his theorem, see  
Gleason, A.M., J. Math & Mech. 6, 885 (1957);  
For Bell’s proof of Gleason’s theorem see Bell (1966).

13. (i) For Kochen-Specker theorem see  
Kochen, S. and Specker, E., Jour. of Math. and Mechanics, 17, 59-87 (1967);  
Redhead, M., *Incompletenes, Nonlocality and Realism*, Clarendon, Oxford, 1987.  
For 33 ray proof by A. Peres see  
Peres, A., J. Phys. A24, L 175 (1991);  
Mermin, N., Rev. Mod. Phys. 65, 803 (1993).  
The extension of Kochen-Specker theorem using POVM is given in  
Cabello, A., Phys. Rev. Lett. 90, 190401 (2003).  
For the extention of Gleason’s theorem using POVM, see Bush, P., arXive: quant-ph/9909073  
(ii) Stochastic noncontextual hidden variable theories are discussed in  
Roy, S.M. and Singh, V., Phys. Rev. A48, 3379 (1993).  
(iii) For the implications of finite precision measurements for $K−S$ theorem see  
Meyer, D.A., Phys. Rev. Lett. 83, 3751 (1999);  
Kent, A., Phys. Rev. Lett. 83, 3755 (1999);  
Clifton, R. and Kent, A., Proc. Roy. Soc. (London) A456, 2101 (2000);
14. (i) Bohm’s version appears in
Bohm, D., Quantum Theory, Prentice Hall, 1951.
(ii) For E.P.R. Paradox and Bell’s theorem, see
Redhead, M., Incompleteness, Nonlocality and Realism, Clarendon, Oxford, 1987;
Selleri, F., Quantum Mechanics versus Local Realism: The
Einstein-Podolsky-Rosen Paradox, Plenum, 1988;
Cushing, J.T. and Mc.Mullin, E., Philosophical Consequences of Quantum
Theory, Reflections on Bell’s Theorem, Notre Dame, 1789;
Selleri, F., Quantum Paradoxes and Physical Reality, Kluwer, 1990;
Home, D., Conceptual Foundations of Quantum Physics, Planum, New York, 1997;
Shimony, A., New aspects of Bell’s theorem, in “Quantum Reflections” edited by J.
Ellis and D. Amati, Cambridge (2000).

15. The first Bell’s inequalities appeared in
Bell, J., Physics 1, 195 (1964);
The CHSH inequalities appeared in
Clauser, J.F., Horne, M.A., Shimony, A. and Hoet, R.A., Phys. Rev. Lett. 26, 880
(1969);
For Wigner’s proof, see
Wigner, E.P., Am. J. Phys. 38, 1005 (1970).
Stapp’s comment on Bell’s theorem occurs in
Stapp, H.P., Nuovo Cimento B40, 191 (1977).

16. For experimental tests of Bell’s inequalities, see
Aspect, A. and Grangier, P., Proc. Int. Symp. Foundations of Quantum Mechanics,
Tokyo, p. 214-224 (1983);
Tittel, W., Brendel, J., Zbinden, H. and Gisin, N., Phys. Rev. Lett. 81, 3563 (1998);
Aspect, A. in “Quantum [un] speakables – From Bell to Quantum Information” edited
by R.A. Bertlmann and A. Zeilinger, Springer (2002) = arXive: quant-ph/0402001.
See also,
Zeilinger, A., Rev. Mod. Phys. 71, 5288 (1999);
Whitaker, M.A.B., Progress in Quantum Electronics, 29, 1 (2000).

17. On Bell’s theorem without inequalities, see
(i) for G.H.Z. proof,
Greenberger, D.M., Horne, M. and Zeilinger, A., Going beyond Bell’s theorem, in
“Bell’s theorem, Quantum Theory and Conception of the Universe” edited by M. Kafatos,
Kluwer, 1989;
Greenberger, D.M., Horne, M.A., Shimony, A. and Zeilinger, A., Am. J. Phys. 58,
1131 (1990);
Arvind, P.K., Found. of Phys. Lett. 15, 297 (2002).
(ii) for Hardy’s work,
Hardy, L., Phys. Rev. Lett. 68, 2981 (1992);
Hardy, L., Phys. Rev. Lett. **71**, 1665 (1993);
for Hardy Ladders,
Boschi, D., Branca, S., De Martini, F. and Hardy, L., Phys. Rev. Lett. **79**, 2755 (1997).
for Stapp’s paraphrase,
Stapp, H.P., *Mind, Matter and Quantum Mechanics*, p. 5-7, Springer, 1993.

18. (i) For signal locality in quantum mechanics, see
Ghirardi, G.C., Rimini, A. and Weber, T., Lett. Nuovo Cimento **27**, 293 (1980).
(ii) For experimental tests of signal locality, see
Roy, S.M. and Singh, V., Phys. Letters A**139**, 437 (1989), and Phys. Rev. Lett. **67**, 2761 (1991);
Singh, V. and Roy, S.M., Theories with Signal locality, in *M.A.B. Bég Memorial Volume*
edited by A. Ali and P. Hoodbhoy, World Scientific, 1991.
(iii) For discussions on possibility of EPR correlations being used for superluminal signalling, see
Kennedy, J.B., Philosophy of Science **62**, 543 (1995),
Berkovitz, J., Stud. Hist. Phil. Mod. Phys. **29**, 183 (1998) and **29**, 509 (1998).

19. A selection of some of the paper dealing with more Bell’s inequalities, see
(i) For a two particle system:
Roy, S.M. and Singh, V., Jour. Phys. A**11**, L 167 (1978) and Jour. Phys. A**12**, 1003 (1979);
Garuccio, A. and Selleri, F., Found. Phys. **10**, 209 (1980);
Garg, A. and Mermin, N.D., Phys. Rev. Lett. **49**, 901, 1220 (1987);
Gisin, N., Phys. Lett. A**260**, 1 (1990).
Pitowsky, I. and Svozil, K., Phys. Rev. A**64**, 014102 (2001).
Collins, D., Gisin, N., Linden, N., Massar, S. and Popescu, S., Phys. Rev. Lett. **88**, 040404 (2002).
(ii) For multiparticle systems see
Mermin, N.D., Phys. Rev. Lett. **65**, 1838 (1990);
Roy, S.M. and Singh, V., Phys. Rev. Lett. 67, 2761 (1991);
Home, D. and Majumdar, A.S., Phys. Rev. A**52**, 4959 (1995).
(iii) for phase space inequalities, see
Auberson, G., Mahoux, G., Roy S.M. and Singh, V., Phys. Lett. A**300**, 327 (2002),
J. Math. Phys. **44**, 2729 (2003) and J. Math. Phys. **45**, 4832 (2004).

20. For recent applications to Quantum information theory and quantum computation,
see the text book
Nielsen, M. and Chuang, I.L., *Quantum Computation and Quantum Information*,
Cambridge, 2000.