ON THE MAXIMUM PRINCIPLE FOR THE RIESZ TRANSFORM

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Abstract. Let \( \mu \) be a measure in \( \mathbb{R}^d \) with compact support and continuous density, and let
\[
R^s \mu(x) = \int \frac{y - x}{|y - x|^{s+1}} \, d\mu(y), \quad x, y \in \mathbb{R}^d, \quad 0 < s < d.
\]
We consider the following conjecture:
\[
\sup_{x \in \mathbb{R}^d} |R^s \mu(x)| \leq C \sup_{x \in \text{supp} \mu} |R^s \mu(x)|, \quad C = C(d, s).
\]
This relation was known for \( d - 1 \leq s < d \), and is still an open problem in the general case. We prove the maximum principle for \( 0 < s < 1 \), and also for \( 0 < s < d \) in the case of radial measure. Moreover, we show that this conjecture is incorrect for non-positive measures.

1. Introduction

Let \( \mu \) be a non-negative finite Borel measure with compact support in \( \mathbb{R}^d \), and let \( 0 < s < d \). The truncated Riesz operator \( R^s_{\mu, \varepsilon} \) is defined by the equality
\[
R^s_{\mu, \varepsilon} f(x) = \int_{|y - x| > \varepsilon} \frac{y - x}{|y - x|^{s+1}} f(y) \, d\mu(y), \quad x, y \in \mathbb{R}^d, \quad f \in L^2(\mu), \quad \varepsilon > 0.
\]
For every \( \varepsilon > 0 \) the operator \( R^s_{\mu, \varepsilon} \) is bounded on \( L^2(\mu) \). By \( R^s_{\mu} \) we denote a linear operator on \( L^2(\mu) \) such that
\[
R^s_{\mu} f(x) = \int \frac{y - x}{|y - x|^{s+1}} f(y) \, d\mu(y),
\]
whenever the integral exists in the sense of the principal value. We say that \( R^s_{\mu} \) is bounded on \( L^2(\mu) \) if
\[
\|R^s_{\mu}\| := \sup_{\varepsilon > 0} \|R^s_{\mu, \varepsilon}\|_{L^2(\mu) \to L^2(\mu)} < \infty.
\]
In the case \( f \equiv 1 \) the function \( R^s_{\mu} 1(x) \) is said to be the \( s \)-Riesz transform (potential) of \( \mu \) and is denoted by \( R^s \mu(x) \). If \( \mu \) has continuous density with respect to the Lebesgue measure \( m_d \) in \( \mathbb{R}^d \), that is if \( d\mu(x) = \rho(x) \, dm_d(x) \) with \( \rho(x) \in C(\mathbb{R}^d) \), then \( R^s \mu(x) \) exists for every \( x \in \mathbb{R}^d \).

By \( C, c \), possibly with indexes, we denote various constants which may depend only on \( d \) and \( s \).

We consider the following well-known conjecture.

Conjecture 1.1. Let \( \mu \) be a nonnegative finite Borel measure with compact support and continuous density with respect to the Lebesgue measure in \( \mathbb{R}^d \). There is a constant \( C \) such that
\[
\sup_{x \in \mathbb{R}^d} |R^s \mu(x)| \leq C \sup_{x \in \text{supp} \mu} |R^s \mu(x)|. \tag{1.1}
\]
For \( s = d - 1 \) the proof is simple. Obviously,
\[
R^s \mu(x) = \nabla U^s_\mu(x), \tag{1.2}
\]
where
\[ U^s_\mu(x) = \frac{1}{s-1} \int \frac{d\mu(y)}{|y-x|^{s-1}}, \quad s \neq 1, \quad U^1_\mu(x) = - \int \log|y-x| d\mu(y). \]

Thus each component of the vector function \( R^s_\mu(x), \ s = d - 1, \) is harmonic in \( \mathbb{R}^d \setminus \text{supp} \mu. \)

Applying the maximum principle for harmonic functions we get (1.1).

For \( d - 1 < s < d, \) the relation (1.1) was established in [2] under stronger assumption that \( \rho \in C^\infty(\mathbb{R}^d). \) In fact it was proved that (1.1) holds for each component of \( R^s_\mu \) with \( C = 1 \) as in the case \( s = d - 1. \) The proof is based on the formula which recovers a density \( \rho \) from \( U^s_\mu. \) But this method does not work for \( s < d - 1. \)

The problem under consideration has a very strong motivation and also is of independent interest. In [2] it is an important ingredient of the proof of the following theorem. By \( \mathcal{H}^s \) we denote the \( s \)-dimensional Hausdorff measure.

**Theorem 1.2** ([2]). Let \( d - 1 < s < d, \) and let \( \mu \) be a positive finite Borel measure such that \( \mathcal{H}^s(\text{supp} \mu) < \infty. \) Then \( \| R^s_\mu \|_{L^\infty(\mu)} = \infty \) (equivalently, \( \| R^s_\mu \| = \infty). \)

If \( s \) is integer, the conclusion of Theorem 1.2 is incorrect. For \( 0 < s < 1 \) Theorem 1.2 was proved by Prat [10] using different approach. The obstacle for extension of this result to all noninteger \( s \) between 1 and \( d - 1 \) is the lack of the maximum principle. The same issue concerns the quantitative version of Theorem 1.2 obtained by Jaye, Nazarov, and Volberg [3].

The maximum principle is important for other problems on the connection between geometric properties of a measure and boundedness of the operator \( R^s_\mu \) on \( L^2(\mu) \) – see for example [3], [5], [6], [7]. All these results are established for \( d - 1 < s < d \) or \( s = d - 1. \)

The problem of the lower estimate for \( \| R^s_\mu \| \) in terms of the Wolff energy (a far going development of Theorem 1.2) which is considered in [3], [5], was known for \( 0 < s < 1. \) And the results in [6], [7] are \( (d - 1) \)-dimensional analogs of classical facts known for \( s = 1 \) (in particular, [7] contains the proof of the analog of the famous Vitushkin conjecture in higher dimensions). For \( 0 < s \leq 1, \) the proofs essentially use the Melnikov curvature techniques and do not require the maximum principle. But this tool is absent for \( s > 1. \)

At the same time the validity of the maximum principle itself remained open even for \( 0 < s < 1. \) It is especially interesting because the analog of (1.1) does not hold for each component of \( R^s_\mu \) when \( 0 < s < d - 1 \) unlike the case \( d - 1 \leq s < d \) – see Proposition 2.1 below.

We prove Conjecture 1.1 for \( 0 < s < 1 \) in Section 2 (Theorem 2.3). The proof is completely different from the proof in the case \( d - 1 \leq s < d. \) In Section 3 we prove Conjecture 1.1 in the special case of radial density of \( \mu \) (that is when \( d\mu = h(|x|) \, dm_d(x) \)), but for all \( s \in (0, d). \) Section 4 contains an example showing that Conjecture 1.1 is incorrect for non-positive measures, even for radial measures with \( C^\infty \)-density (note that in [14, Conjecture 7.3] Conjecture 1.1 was formulated for all finite signed measures with compact support and \( C^\infty \)-density).

## 2. The case \( 0 < s < 1 \)

We start with a statement showing that the maximum principle fails for every component of \( R^s_\mu \) if \( 0 < s < d - 1. \)
Proposition 2.1. For any $d \geq 2$, $0 < s < d - 1$, and any $M > 0$, there is a positive measure $\mu$ in $\mathbb{R}^d$ with $C^\infty$-density such that

$$\sup_{x \in \mathbb{R}^d} |R^s_1 \mu(x)| > M \sup_{x \in \text{supp } \mu} |R^s_1 \mu(x)|,$$

where $R^s_1 \mu$ is the first component of $R^s \mu$.

Proof. Let $E = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_1 = 0, x_2^2 + \cdots + x_d^2 \leq 1\}$, and let $E_\delta$, $\delta > 0$, be a $\delta$-neighborhood of $E$ in $\mathbb{R}^d$. Let $\mu = \mu_\delta$ be a positive measure supported on $E_\delta$ with $\mu(E_\delta) = 1$ and with $C^\infty$-density $\rho(x)$ such that $\rho(x) < 2/\text{vol}(E_\delta) \leq C_d \delta$. Then

$$|R^s_1 \mu(x')| > A_\delta,$$

where $x' = (1,0,\ldots,0)$, $0 < \delta < 1/2$.

On the other hand, for $x \in \text{supp } \mu$ integration by parts yields

$$|R^s_1 \mu(x)| < \int_{|y-x| < \delta} \frac{1}{|y-x|^s} \, d\mu(y) + \int_{|y-x| \geq \delta} \frac{\delta}{|y-x|^{s+1}} \, d\mu(y)$$

$$= \frac{\mu(B(x, \delta))}{\delta^s} + s \int_0^{\mu(B(x, r))} dr + \delta(s + 1) \int_\delta^\infty \frac{\mu(B(x, r))}{r^{s+2}} dr$$

$$< C \frac{C_d \delta^d}{\delta^s} + C \frac{s}{\delta} \int_0^{\delta} \frac{r^d}{r^{s+1}} dr + C \delta(s + 1) \int_\delta^2 \frac{r^{d-1}}{r^{s+2}} dr + C \delta.$$

Here by $C$ we denote different constants depending only on $d$, and $B(x, r) := \{y \in \mathbb{R}^d : |y-x| < r\}$. We have

$$\delta \int_\delta^2 \frac{r^{d-1}}{r^{s+2}} dr = \begin{cases} \delta \ln \frac{2}{\delta}, & s = d - 2, \\ \frac{1}{d - s - 2}(2^{d-s-2} - \delta^{d-s-1}), & s \neq d - 2. \end{cases}$$

Thus, all terms in the right-hand side of the estimate for $|R^s_1 \mu(x)|$ tend to 0 as $\delta \to 0$, and we may choose $\delta$ and a corresponding measure $\mu$ satisfying (2.1). \qed

We need the following lemma. The notation $A \approx B$ means that $cA < B < CB$ with constants $c, C$ which may depend only on $d$ and $s$.

Lemma 2.2. Let $\mu$ be a non-negative measure in $\mathbb{R}^d$ with continuous density and compact support. Let $0 < s < d - 1$. Then for every ball $B = B(x_0, r)$,

$$\left| \int_{\partial B} (R^s \mu \cdot \mathbf{n}) \, d\sigma \right| \approx r^{d-s-1} \mu(B) + r^d \int_r^\infty \frac{d\mu(B(x_0, t))}{ts^{s+1}},$$

where $\mathbf{n}$ is the outer normal vector to $B$ and $\sigma$ is the surface measure on $\partial B$.

Proof. We will use the Ostrogradsky-Gauss Theorem and differentiation under the integral sign. To justify these operations and make an integrand sufficiently smooth, we approximate $K(x) = x/|x|^{s+1}$ by the smooth kernel $K_\varepsilon$ in the following standard way. Let $\phi(t)$, $t \geq 0$, be a $C^\infty$-function such that $\phi(t) = 0$ as $0 \leq t \leq 1$, $\phi(t) = 1$ as $t \geq 2$, and $0 \leq \phi'(t) \leq 2$, $t > 0$. Let $\phi_\varepsilon(t) := \phi(t/\varepsilon)$, $K_\varepsilon(x) := \phi_\varepsilon(|x|)K(x)$, and $\tilde{R}^s_\varepsilon \mu := K_\varepsilon * \mu$. We have

$$\int_{\partial B} (\tilde{R}^s_\varepsilon \mu \cdot \mathbf{n}) \, d\sigma = \int_B \nabla \cdot \tilde{R}^s_\varepsilon \mu(x) \, dm_d(x) = \int_B \left[ \int_{\mathbb{R}^d} \nabla \cdot \phi_\varepsilon(|y-x|) \frac{y-x}{|y-x|^{s+1}} \, d\mu(y) \right] dm_d(x).$$

The inner integral is equal to

$$\int_{|y-x| \leq 2\varepsilon} \nabla \cdot \phi_\varepsilon(|y-x|) \frac{y-x}{|y-x|^{s+1}} \, d\mu(y) + \int_{|y-x| > 2\varepsilon} \nabla \cdot \frac{y-x}{|y-x|^{s+1}} \, d\mu(y) =: I_1(x) + I_2(x).$$
One can easily see that
\[
\left| \frac{\partial}{\partial x_i} \phi(x)(y) \right| \frac{y - x}{|y - x|^{s+1}} \leq C \left( \frac{1}{1 + |x|} + \frac{1}{1 + |y|} \right) < \frac{C}{|y - x|^{s+1}} , \quad |y - x| \leq 2 \varepsilon.
\]
Hence,
\[
|I_1(x)| \leq C \int_{|y-x| \leq 2 \varepsilon} \frac{1}{|y - x|^{s+1}} d\mu(y) < C \int_0^{2 \varepsilon} \frac{1}{t^{s+1}} d\mu(B(x,t)) \approx \frac{\mu(B(x,2 \varepsilon))}{(2 \varepsilon)^{s+1}} + \int_0^{2 \varepsilon} \frac{\mu(B(x,t))}{t^{s+2}} dt.
\]
Since \( \mu \) has a continuous density with respect to \( m_d \), we have \( \mu(B(x,t)) \leq A_{\mu,B} t^d \) as \( t \leq 2 \varepsilon < 1 \), \( x \in B \). Taking into account that \( s < d - 1 \), we obtain the relation \( \int_B I_1(x) \, dm_d(x) \to 0 \) as \( \varepsilon \to 0 \).

To estimate the integral of \( I_2(x) \) we use the equality \( \nabla \cdot \frac{x}{|x|^{s+1}} = \frac{d-s-1}{|x|^{s+1}} \). Thus,
\[
\left| \int_B I_2(x) \, dm_d(x) \right| = C \int_B \left[ \int_{|y-x| > 2 \varepsilon} \frac{d\mu(y)}{|y - x|^{s+1}} \right] dm_d(x)
\]
\[
= C \left( \int_{B(x_0,r+2 \varepsilon)} \left[ \int_{B \cap \{|y-x| > 2 \varepsilon\}} \frac{dm_d(x)}{|y - x|^{s+1}} \right] d\mu(y) \right.
\]
\[
+ \int_{\mathbb{R}^d \setminus B(x_0,r+2 \varepsilon)} \left[ \int_B \frac{dm_d(x)}{|y - x|^{s+1}} \right] d\mu(y) \bigg) =: C(J_1 + J_2).
\]
Obviously,
\[
\int_B \frac{dm_d(x)}{|y - x|^{s+1}} \approx \left\{ \begin{array}{ll}
\int_0^r \frac{t^{d-1}}{t^{s+1}} dt & \approx r^{d-s-1} , |y - x_0| \leq r , \\
\int_r^{+\infty} \frac{r^d}{(|y-x_0|^{s+1}} & |y - x_0| > r .
\end{array} \right.
\]

In order to estimate \( J_1 \) we note that for sufficiently small \( \varepsilon \),
\[
\int_{B \cap \{|y-x| > 2 \varepsilon\}} \frac{dm_d(x)}{|y - x|^{s+1}} \approx \int_B \frac{dm_d(x)}{|y - x|^{s+1}} \approx r^{d-s-1} , \quad y \in B(x_0,r+2 \varepsilon).
\]
Hence, \( J_1 \approx r^{d-s-1} \mu(B(x_0,r+2 \varepsilon)) \). Moreover,
\[
J_2 \approx \int_{\mathbb{R}^d \setminus B(x_0,r+2 \varepsilon)} \frac{r^d}{|y - x_0|^{s+1}} d\mu(y) = r^d \int_{r+2 \varepsilon}^{+\infty} \frac{d\mu(B(x_0,t))}{t^{s+1}}.
\]
Passing to the limit as \( \varepsilon \to 0 \), we get (2.2) \( \Box \).

Now we are ready to prove our main result.

**Theorem 2.3.** Let \( \mu \) be a non-negative measure in \( \mathbb{R}^d \) with continuous density and compact support. Let \( 0 < s < 1 \). Then (1.1) holds with a constant \( C \) depending only on \( d \) and \( s \).

**Proof.** Let us sketch the idea of proof. Let a measure \( \mu \) be such that \( \mu(B(y,t)) \leq C t^s, \quad y \in \mathbb{R}^d, \quad t > 0 \). For Lipschitz continuous compactly supported functions \( \varphi, \psi \), define the form \( \langle R^s(\psi \mu), \varphi \rangle \) by the equality
\[
\langle R^s(\psi \mu), \varphi \rangle = \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{y - x}{|y - x|^{s+1}} (\psi(y) \varphi(x) - \psi(x) \varphi(y)) \, d\mu(y) \, d\mu(x);
\]
the double integral exists since \( |\psi(y) \varphi(x) - \psi(x) \varphi(y)| \leq C_{\psi, \varphi} |x-y| \). If we assume in addition that \( \int \psi \, d\mu = 0 \), we may define \( \langle R^s(\psi \mu), \varphi \rangle \) for any (not necessarily compactly supported)
bounded Lipschitz continuous function $\varphi$ on $\mathbb{R}^d$; here we follow [4]. Let $\text{supp } \psi \in B(0, R)$. For $|x| > 2R$ we have

$$\left| \int_{\mathbb{R}^d} \frac{y-x}{|y-x|^{s+1}} \psi(y) \, d\mu(y) \right| = \left| \int_{\mathbb{R}^d} \left[ \frac{y-x}{|y-x|^{s+1}} + \frac{x}{|x|^{s+1}} \right] \psi(y) \, d\mu(y) \right| \leq \frac{C}{|x|^{s+1}} \int_{\mathbb{R}^d} |y\psi(y)| \, d\mu(y) = \frac{C_{\psi}}{|x|^{s+1}}.$$ 

Choose a Lipschitz continuous compactly supported function $\xi$ which is identically 1 on $B(0, 2R)$. Then we may define the form $\langle R^s(\psi \mu), \varphi \rangle_\mu$ as

$$\langle R^s(\psi \mu), \varphi \rangle_\mu = \langle R^s(\psi \mu), \xi \varphi \rangle_\mu + \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}^d} \frac{y-x}{|y-x|^{s+1}} \psi(y) \, d\mu(y) \right] (1 - \xi(x)) \varphi(x) \, d\mu(x).$$

The repeated integral is well defined because

$$\int_{|x| > 2R} \frac{d\mu(x)}{|x|^{s+1}} \leq C \int_{2R}^\infty \frac{\mu(B(0, t))}{t^{s+2}} \, dt \leq C \int_{2R}^\infty \frac{1}{t^2} \, dt.$$

Assuming that Theorem 2.3 is incorrect and using the Cotlar inequality we establish the existence of a positive measure $\nu$ such that $\nu$ has no point masses, the operator $R_n^s$ is bounded on $L^2(\nu)$, and $\langle R^s(\psi \nu), 1 \rangle_\nu = 0$ for every Lipschitz continuous function $\psi$ with $\int \psi \, d\mu = 0$. It means that $\nu$ is a reflectionless measure, that is a measure without point masses with the following properties: $R_n^s$ is bounded on $L^2(\nu)$, and $\langle R^s(\psi \nu), 1 \rangle_\nu = 0$ for every Lipschitz continuous compactly supported function $\psi$ such that $\int \psi \, d\mu = 0$. But according to the recent result by Prat and Tolsa [11] such measures do not exist for $0 < s < 1$. We remark that the proof of this result contains estimates of an analog of the Melnikov’s curvature of a measure. This is the obstacle to extent the result to $s \geq 1$. We now turn to the details.

Suppose that $C$ satisfying (1.1) does not exists. Then for every $n \geq 1$ there is a positive measure $\mu_n$ such that

$$\sup_{x \in \mathbb{R}^d} |R^s \mu_n(x)| = 1, \quad \sup_{x \in \text{supp } \mu_n} |R^s \mu_n(x)| \leq \frac{1}{n}.$$

Let

$$\theta_\mu(x, r) := \frac{\mu(B(x, r))}{r^s}, \quad \theta_\mu := \sup_{x, r} \theta_\mu(x, r).$$

We prove that

$$0 < c < \theta_\mu < C. \quad (2.3)$$

The estimate from above is a direct consequence of Lemma 2.2. Indeed, for any ball $B(x, r)$ (2.2) implies the estimate

$$c_d r^{d-1} \geq \left| \int_{\partial B} (R^s \mu_n \cdot n) \, d\sigma \right| \geq C r^{d-s-1} \mu_n(B),$$

which implies the desired inequality.

The estimate from below follows immediately from a Cotlar-type inequality

$$\sup_{x \in \mathbb{R}^d} |R^s \mu_n(x)| \leq C \left[ \sup_{x \in \text{supp } \mu_n} |R^s \mu_n(x)| + \theta_\mu \right]$$

(see [8, Theorem 7.1] for a more general result).
Let $B(x_n, r_n)$ be a ball such that $\theta_{\mu_n}(x_n, r_n) > c = c(s, d)$, and let $\nu_n(\cdot) = r_n^{-s}\mu_n(x_n + r_n\cdot)$. Then
\[
R_s\mu_n(x) = R_s\nu_n\left(\frac{x - x_n}{r_n}\right), \quad \theta_{\nu_n}(y, t) = \theta_{\mu_n}(r_ny + x_n, r_nt).
\]
In particular, $\nu_n(B(0, 1)) = \theta_{\mu_n}(x_n, r_n) > c$. Choosing a weakly converging subsequence of $\{\nu_n\}$, we obtain a positive measure $\nu$. If we prove that
(a) $\nu(B(y, t)) \leq Ct^s$,
(b) $\langle R^s\nu, \psi \rangle_\nu = 0$ for every Lipschitz continuous compactly supported function $\psi$ with $\int \psi d\nu = 0$,
(c) the operator $R_s^\nu$ is bounded on $L^2(\nu)$,
then $\nu$ is reflectionless, and we come to contradiction with Theorem 1.1 in [11] mentioned above. Thus, the proof would be completed.

The property (a) follows directly from (2.3). For weakly converging measures $\nu_n$ with $\theta_{\mu_n} < C$, we may apply Lemma 8.4 in [4] which yields (b). To establish (c) we use the inequality
\[
R_{s, \varepsilon}^* \mu(x) := \sup_{\varepsilon > 0} |R_{\mu, \varepsilon}^* 1(x)| \leq \|R_s^\nu\|_{L^\infty(m_\rho)} + C, \quad x \in \mathbb{R}^d, \quad C = C(s),
\]
for any positive Borel measure $\mu$ such that $\mu(B(x, r)) \leq r^s$, $x \in \mathbb{R}^d$, $r > 0$, – see [12, Lemma 2] or [13, p. 47], [1, Lemma 5.1] for a more general setting. Thus, $R_{\nu_n}^n(x) := R_{s, \varepsilon}^* \mu_n(x) \leq C$ for every $\varepsilon > 0$. Hence, $R_{s, \varepsilon}^* \mu_n(x) \leq C$ for $\varepsilon > 0$, $x \in \mathbb{R}^d$, and the non-homogeneous $T1$-theorem [9] implies the boundedness of $R_{s, \varepsilon}^* \nu_n$ on $L^2(\nu)$.

3. The case of radial density

Lemma 2.2 allows us to prove the maximum principle for all $s \in (0, d)$ in the special case of radial density.

**Proposition 3.1.** Let $d\mu(x) = h(|x|) \, dm_\rho(x)$, where $h(t)$ is a continuous function on $[0, \infty)$, and let $s \in (0, d - 1)$. Then (1.1) holds with a constant $C$ depending only on $d$ and $s$.

We remind that for $s \in [d - 1, d)$ Conjecture 1.1 is proved in [2] for any compactly supported measure with $C^\infty$ density. Thus, for compactly supported radial measures with $C^\infty$ density (1.1) holds for all $s \in (0, d)$.

**Proof.** Because $\mu$ is radial, by (2.2) we have
\[
c_{d}r^{d-1}|R_s^\mu(x)| = \left| \int_{\partial B(0, r)} (R_s^\mu \cdot \mathbf{n}) \, d\sigma \right|
\approx r^{d-s-1}\mu(B(0, r)) + r^d \int_r^\infty \frac{d\mu(B(0, t))}{ts^{s+1}}, \quad r = |x|.
\]
Thus,
\[
|R_s^\mu(x)| \approx \frac{\mu(B(0, r))}{r^s} + r \int_r^\infty \frac{d\mu(B(0, t))}{ts^{s+1}}.
\]
Fix $w \notin \text{supp} \mu$, and let $r = |w|$. If
\[
\frac{\mu(B(0, r))}{r^s} \geq r \int_r^\infty \frac{d\mu(B(0, t))}{ts^{s+1}},
\]
then there is \( r_1 \in (0, r) \) such that \( \{ y : |y| = r_1 \} \subset \text{supp } \mu \) and \( \mu(B(0, r)) = \mu(B(0, r_1)) \). Hence,

\[
|R^s \mu(w)| \approx \frac{\mu(B(0, r))}{r^s} < \frac{\mu(B(0, r_1))}{r_1^s} \leq C|R^s \mu(x_1)|, \quad |x_1| = r_1.
\]

If

\[
\frac{\mu(B(0, r))}{r^s} < r \int_r^\infty \frac{d\mu(B(0, t))}{t^{s+1}},
\]

then there is \( r_2 > r \) such that \( \{ y : |y| = r_2 \} \subset \text{supp } \mu \) and \( \mu(B(0, r)) = \mu(B(0, r_2)) \). Hence,

\[
\frac{\mu(B(0, r_2))}{r_2^s} < \frac{\mu(B(0, r))}{r^s} < r \int_{r_2}^\infty \frac{d\mu(B(0, t))}{t^{s+1}},
\]

and we have

\[
|R^s \mu(w)| \approx r \int_{r_2}^\infty \frac{d\mu(B(0, t))}{t^{s+1}} \leq C|R^s \mu(x_2)|, \quad |x_2| = r_2.
\]

\[
\Box
\]

4. Counterexample

Given \( \varepsilon > 0 \), we construct a signed measure \( \nu = \nu(\varepsilon) \) in \( \mathbb{R}^5 \) with the following properties:

(a) \( \nu \) is a radial signed measure with \( C^\infty \)-density;

(b) \( \text{supp } \nu \subset D_\varepsilon := \{ 1 - \varepsilon \leq |x| \leq 1 + \varepsilon \} \);

(c) \( |R^2 \nu(x)| < \varepsilon \) for \( x \in \text{supp } \nu \); \( |R^2 \nu(x)| > a > 0 \) for \( |x| = 2 \), where \( a \) is an absolute constant. Here \( R^2 \nu \) means \( R^s \nu \) with \( s = 2 \).

Let \( \Delta^2 := \Delta \circ \Delta \), and let

\[
u(x) = \begin{cases} 
2/3, & |x| \leq 1, \\
\frac{1}{|x|} - \frac{1}{3|x|^3}, & |x| > 1.
\end{cases}
\]

Note that \( \Delta(\frac{1}{|x|}) = 0 \) and \( \Delta^2(\frac{1}{|x|}) = 0 \) in \( \mathbb{R}^5 \setminus \{0\} \). Hence, \( \Delta^2 u(x) = 0, |x| \neq 1 \). Moreover, \( \nabla u \) is continuous in \( \mathbb{R}^5 \) and \( \nabla u(x) = 0, |x| = 1 \).

For \( \delta \in (0, \varepsilon) \), let \( \varphi_\delta(x) \) be a \( C^\infty \)-function in \( \mathbb{R}^5 \) such that \( \varphi_\delta > 0 \), \( \text{supp } \varphi_\delta = \{ x \in \mathbb{R}^5 : |x| \leq \delta \} \), and \( \int \varphi_\delta(x) \, dm_5(x) = 1 \) (for example, a bell-like function on \( [0, \delta] \)). Let \( U_\delta := u \ast \varphi_\delta \). Then \( \nabla \Delta U_\delta(x) = 0 \) as \( x \notin D_\delta \). Also, \( \Delta U_\delta(x) \to 0 \) as \( |x| \to \infty \). Hence, the function \( \Delta U_\delta \) can be represented in the form \( \Delta U_\delta = c(\frac{1}{|x|} \ast \Delta^2 U_\delta) \) (here and in the sequel by \( c \) we denote various absolute constants). Set \( d\nu_\delta = \Delta^2 U_\delta \, dm_5 \). Then \( \text{supp } \nu_\delta \subset D_\delta \), and (b) is satisfied. Since \( \Delta(\frac{1}{|x|}) = \frac{\delta}{|x|^2} \), we have \( \Delta U_\delta = c(\frac{1}{|x|} \ast \Delta^2 U_\delta) \), that is \( U_\delta = c(\frac{1}{|x|} \ast \Delta^2 U_\delta) + h \), where \( h \) is a harmonic function in \( \mathbb{R}^5 \). Since both \( U_\delta \) and \( \frac{1}{|x|} \ast \Delta^2 U_\delta \) tend to 0 as \( x \to \infty \), we have

\[
U_\delta = c\left( \frac{1}{|x|} \ast \Delta^2 U_\delta \right).
\]

Thus,

\[
R^2 \nu(x) = c \int \frac{y-x}{|y-x|^3} \, d\nu_\delta(y) = c \nabla U_\delta(x).
\]

Obviously, \( \nabla U_\delta = \nabla (u \ast \varphi_\delta) = (\nabla u) \ast \varphi_\delta \), and hence \( \max_{x \in D_\delta} |\nabla U_\delta(x)| \to 0 \) as \( \delta \to 0 \). On the other hand, for fixed \( x \) with \( |x| > 1 \) (say, \( |x| = 2 \)) we have \( \lim_{\delta \to 0} (\nabla u \ast \varphi_\delta) = |\nabla u(x)| > 0 \). Thus, (c) is satisfied if \( \delta \) is chosen sufficiently small.
Remark. It is well-known that the maximum principle (with a constant $C$) holds for potentials $\int K(|x-y|)d\mu(y)$ with non-negative kernels $K(t)$ decreasing on $(0, \infty)$, and non-negative finite Borel measures $\mu$. Our arguments show that for non-positive measures the analog of (1.1) fails even for potentials with positive Riesz kernels. In fact we have proved that for every $\varepsilon > 0$, there exists a signed measure $\eta = \eta(\varepsilon)$ in $\mathbb{R}^2$ with $C^\infty$-density and such that $\sup \eta \in D_\varepsilon := \{1-\varepsilon \leq |x| \leq 1+\varepsilon\}$, $|u_\eta(x)| < \varepsilon$ for $x \in \sup \eta$, but $|u_\eta((2,0,\ldots,0))| > b > 0$, where $u_\eta(x) := \int \frac{d\eta(y)}{|y-x|}$, and $b$ is an absolute constant.

Indeed, for the first component $R^2_1 \nu$ of $R^2 \nu$ we have

$$R^2_1 \nu = c \frac{\partial}{\partial x_1} U_\delta = c \left( \frac{1}{|x|} \ast \frac{\partial}{\partial x_1} (\Delta^2 U_\delta) \right) = cu_\eta, \text{ where } d\eta = \frac{\partial}{\partial x_1} (\Delta^2 U_\delta) d\nu.$$

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