New Developments in Mean Curvature Flow of Arbitrary Codimension Inspired By Yau Rigidity Theory

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Abstract. In this survey, we will focus on the mean curvature flow theory with sphere theorems, and discuss the recent developments on the convergence theorems for the mean curvature flow of arbitrary codimension inspired by the Yau rigidity theory of submanifolds. Several new differentiable sphere theorems for submanifolds are obtained as consequences of the convergence theorems for the mean curvature flow. It should be emphasized that Theorem 4.1 is an optimal convergence theorem for the mean curvature flow of arbitrary codimension, which implies the first optimal differentiable sphere theorem for submanifolds with positive Ricci curvature. Finally, we present a list of unsolved problems in this area.

1. Huisken’s classical theorems for MCF of hypersurfaces

Let \((M,g)\) be a closed Riemannian \(n\)-manifold, and let \(F_t : M^n \rightarrow N^{n+q}\) be a one-parameter family of smooth submanifolds immersed in an \((n+q)\)-dimensional Riemannian manifold \((N,h)\). We say that \(M_t = F_t(M)\) is a solution of the mean curvature flow if \(F_t\) satisfies

\begin{equation}
\begin{cases}
\frac{\partial}{\partial t} F(x,t) = H(x,t), \\
F(x,0) = F_0(x),
\end{cases}
\end{equation}

where \(F(x,t) = F_t(x)\), \(H(x,t)\) is the mean curvature vector, and \(F_0(M)\) is the initial submanifold immersed in \(N\).

In 1984, Huisken first studied the mean curvature flow for compact hypersurfaces in the Euclidean and proved the following convergence theorem [19].

**Theorem 1.1.** Let \(M_0\) be an \(n\)-dimensional \((n \geq 2)\) closed hypersurface in \(\mathbb{R}^{n+1}\). Assume that \(M_0\) is uniformly convex. Then the mean curvature flow with initial value \(M_0\) has a smooth solution on a finite time interval \(0 \leq t < T\), and converge to a single round point as \(t \rightarrow T\).

Afterwards, Huisken [20] generalized Theorem 1.1 to the case of the mean curvature flow of compact hypersurfaces in a Riemannian manifold \(N^{n+1}\) with bounded geometry, which is stated as follows.

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Theorem 1.2. Let \( n \geq 2 \) and \( N^{n+1} \) be a complete Riemannian manifold which satisfies uniform bounds

\[-K_1 \leq \bar{K}_N \leq K_2, \quad |\bar{\nabla} \bar{R}| \leq L, \quad \text{inj}(N) \geq i_N,\]

for nonnegative constants \( K_1, K_2, L \) and positive constant \( i_N \). Let \( M_0 \) be a closed hypersurface immersed in \( N \), and suppose that \( M_0 \) satisfies the pinching condition

\[|H|h_{ij} > nK_1g_{ij} + \frac{n^2}{|H|}Lg_{ij}.\]

Then the mean curvature flow with initial value \( M_0 \) has a smooth solution on a finite time interval \( 0 \leq t < T \), and converge to a single round point as \( t \to T \).

Motivated by the rigidity theorem for hypersurfaces with constant mean curvature in a sphere due to Okumura [40], Huisken [21] proved the following convergence theorem for the mean curvature flow in a sphere.

Theorem 1.3. Let \( n \geq 2 \) and \( F^{n+1}(c) \) be a spherical space form with positive constant curvature \( c \). Let \( M_0 \) be a closed hypersurface immersed in \( F^{n+1}(c) \), and suppose that \( M_0 \) satisfies the pinching condition

\[
|A|^2 < \begin{cases} 
\frac{2}{n} |H|^2 + \frac{2}{3}c, & n = 2, \\
\frac{1}{n-1} |H|^2 + 2c, & n \geq 3.
\end{cases}
\]

Then one of the following holds:

(i) The mean curvature flow with initial value \( M_0 \) has a smooth solution \( M_t \) on a finite time interval \( 0 \leq t < T \) and the \( M_t \)'s converge uniformly to a single round point as \( t \to T \).

(ii) The mean curvature flow with initial value \( M_0 \) has a smooth solution \( M_t \) for all \( 0 \leq t < \infty \) and the \( M_t \)'s converge in the \( C^\infty \)-topology to a smooth totally geodesic hypersurface \( M_\infty \).

During the past three decades, there are many progresses on the theory of mean curvature flows. Most of the results focus on the mean curvature flow of hypersurfaces. On the other hand, fruitful results on the mean curvature flow of submanifolds of higher codimension were obtained by several geometers. Mean curvature flow of surfaces in 4-manifolds, Lagrangian mean curvature flow of higher codimension, graphic mean curvature flow, mean curvature flow with convex Gauss image were investigated by Chen, Li, Smoczyk, Wang, Xin and others [8, 50, 51, 52, 53, 55, 56].

In the present article, we will introduce the recent developments on the convergence theorems for the mean curvature flow of arbitrary codimension inspired by the Yau rigidity theory of submanifolds with parallel mean curvature. Several new differentiable sphere theorems for submanifolds are obtained as consequences of the convergence theorems for the mean curvature flow. Finally, we present a list of conjectures in this area.

2. Yau rigidity theory of submanifolds and its developments

More than forty years ago, Okumura [39, 40] investigated compact submanifolds with parallel mean curvature and flat normal bundle in \( S^{n+1} \) whose squared norm of the second fundamental form satisfies \( |A|^2 < 2 + \frac{|H|^2}{n-1} \). Meanwhile, Chen-Okumura [7] proved a rigidity theorem for compact submanifolds with parallel mean curvature.
curvature in Euclidean spaces under pinching condition $|A|^2 < \frac{\mu n^2}{n^2 - 1}$. In 1975, Yau [65] established the rigidity theory of submanifolds with parallel mean curvature, which includes the following important results.

**Theorem 2.1.** Let $M$ be an $n$-dimensional compact submanifold with parallel mean curvature in $\mathbb{S}^{n+q}$ with $q > 1$. If $|A|^2 < n|3 + n^2 - (q - 1)|^{-1}$, then $M$ lies in a totally geodesic sphere $\mathbb{S}^{n+1}$.

**Theorem 2.2.** Let $M$ be an $n$-dimensional oriented compact minimal submanifold in $\mathbb{S}^{n+q}$. If $K_M \geq \frac{q-1}{2q-1}$, then either $M$ is the totally geodesic sphere, the standard immersion of the product of two spheres, or the Veronese surface in $\mathbb{S}^4$.

Notice that Yau's pinching constant $\frac{q-1}{2q-1}$ is better than the pinching constant $\frac{1}{2}$ given by Simon [48]. Moreover, Yau's pinching constant above is the best possible in the case where $q = 1$, or $n = 2$ and $q = 2$. Inspired by the Yau rigidity theorem, Ejiri [11] obtained a rigidity theorem for compact and simply connected minimal submanifolds in a sphere with $Ric_M \geq n - 2$. The Yau parameter method was first introduced in the proof of the Yau rigidity theorem, which has many applications in the study of the Chern conjecture and other rigidity problems.

Many other important results were obtained by Yau [65]. For example, he obtained the geometric structure theorem for non-negatively curved submanifolds with parallel mean curvature in spaces forms, the classification theorem for surfaces with parallel mean curvature in spaces forms, and the codimension reduction theorem for submanifolds in a conformally flat manifold.

The Yau rigidity theory of submanifolds plays a very important role in the study of geometry, topology and curvature flows of submanifolds. During the past four decades, the Yau rigidity theory has been developed by several geometers [17]. In 1984, Cheng, Li, and Yau [9] proved the following volume gap theorem for minimal submanifolds.

**Theorem 2.3.** Let $M$ be an $n$-dimensional compact minimal submanifold in $\mathbb{S}^{n+q}$. Then there exists an explicit positive constant $\varepsilon_n$ depending only on $n$ such that if $Vol(M) < Vol(\mathbb{S}^n) + \varepsilon_n$, then $M$ is congruent to the great sphere $\mathbb{S}^n$.

The possible generalization of the Cheng-Li-Yau theorem will give a proof of the higher dimensional version of the Willmore conjecture on the total mean curvature for closed submanifolds in Euclidean spaces.

In 1990, Xu [57] proved the generalized Simons-Lawson-Chern-do Carmo-Kobayashi theorem for compact submanifolds with parallel mean curvature in a sphere.

**Theorem 2.4.** Let $M$ be an $n$-dimensional oriented compact submanifold with parallel mean curvature in $\mathbb{S}^{n+q}$. If $|A|^2 \leq C_0(n,q,|H|)$, then $M$ is either a totally umbilic sphere $\mathbb{S}^n(\sqrt{|H|^2 + n^2})$, a Clifford hypersurface in an $(n+1)$-sphere, or the Veronese surface in $\mathbb{S}^4(2/\sqrt{|H|^2 + 4})$. Here the constant $C_0(n,q,|H|)$ is defined by

$$C_0(n,q,|H|) := \begin{cases} \alpha(n,|H|), & q = 1, \text{or } q = 2 \text{ and } |H| \neq 0, \\ \frac{n}{2(n-1)}, & q \geq 2 \text{ and } |H| = 0, \\ \min \left\{ \alpha(n,|H|), \frac{n^2 + |H|^2}{(2(n-1)n)^2} + \frac{|H|^2}{n} \right\}, & q \geq 3 \text{ and } |H| \neq 0, \end{cases}$$

$$\alpha(n,|H|) = n + \frac{n}{2(n-1)}|H|^2 - \frac{n-2}{2(n-1)}\sqrt{|H|^4 + 4(n-1)|H|^2}.$$
In [29, 30], Li and Li improved Simons’ pinching constant for \( n \)-dimensional compact minimal submanifolds in \( S^{n+q} \) to max\( \{ \frac{n}{n+q}, \frac{2}{3n} \} \). Using Li-Li’s matrix inequality, Xu [58] improved the pinching constant \( C_0(n, q, |H|) \) in Theorem 2.4 to

\[
C(n, q, |H|) = \begin{cases} 
\alpha(n, |H|), & q = 1, \text{ or } q = 2 \text{ and } |H| \neq 0, \\
\min \left\{ \alpha(n, |H|), \frac{2n}{n+q} + \frac{n}{3n}|H|^2 \right\}, & \text{otherwise.}
\end{cases}
\]

This is the best pinching constant for \( n \)-dimensional compact submanifolds with parallel mean curvature in \( S^{n+q} \) up to date. In fact, Theorem 2.4 is stronger than the rigidity results in [1, 3, 42]. In [59], Xu obtained an optimal rigidity theorem for complete submanifolds with parallel mean curvature in hyperbolic spaces.

Let \( F^{n+q}(c) \) be an \((n+q)\)-dimensional simply connected space form with constant curvature \( c \). Put

\[
\alpha(n, |H|, c) = nc + \frac{n}{2(n-1)|H|^2} - \frac{n-2}{2(n-1)} \sqrt{4(n-1)c^2 - |H|^4 + 4(n-1)c|H|^2}.
\]

If \( c > 0 \), we have \( \min_{|H|} \alpha(n, |H|, c) = 2\sqrt{n-1}c \).

Using the techniques in stable currents and algebraic topology, Shiohama and Xu [46] proved the following optimal topological sphere theorem.

**Theorem 2.5.** Let \( M^n(n \geq 4) \) be an oriented complete submanifold in \( F^{n+q}(c) \) with \( c \geq 0 \). Suppose that \( \sup_M (|A|^2 - \alpha(n, |H|, c)) < 0 \). Then \( M \) is homeomorphic to a sphere.

After the work by Shen [43] and Lin-Xia [32], Xu [57] proved the following theorem.

**Theorem 2.6.** Let \( M \) be an \( n \)-dimensional compact submanifold with parallel mean curvature in the space form \( F^{n+q}(c) \) with \( c \geq 0 \). Denote by \( A \) the traceless second fundamental form of \( M \). If

\[
\int_M |\tilde{A}|^n dM < C_1(n),
\]

where \( C_1(n) \) is an explicit positive constant depending only on \( n \), then \( M \) is a totally umbilic sphere.

Applying the Morse theory of submanifolds, Shiohama and Xu [47] proved the following topological sphere theorem for compact submanifolds.

**Theorem 2.7.** Let \( M \) be an \( n \)-dimensional compact submanifold in the space form \( F^{n+q}(c) \) with \( c \geq 0 \). There exists a positive constant \( C_2(n) \) depending only on \( n \) such that if

\[
\int_M |\tilde{A}|^n dM < C_2(n),
\]

then \( M \) is homeomorphic to \( S^n \).

After the work due to Ejiri [11], Shen [45], Li [31] and Xu-Tian [63], Xu-Gu [61] proved the following pinching theorem for submanifolds with pinched Ricci curvatures.
Theorem 2.8. Let $M$ be an $n$-dimensional oriented compact submanifold with parallel mean curvature in the space form $\mathbb{F}^{n+q}(c)$. If

$$Ric_M \geq (n-2)(c + \frac{|H|^2}{n^2}),$$

where $n^2c + |H|^2 > 0$, then $M$ is either a totally umbilic sphere, a Clifford hypersurface $\mathbb{S}^m(r) \times \mathbb{S}^n(r)$ in an $(n+1)$-sphere with $n = 2m$, or $\mathbb{C}P^2(\frac{4}{3}c + \frac{1}{12}|H|^2)$ in $\mathbb{S}^7(\frac{4}{\sqrt{16c+|H|^2}})$.

Remark 2.1. When $n$ is even, the pinching condition in Theorem 2.8 is optimal. Furthermore, Gu-Tian-Xu [61, 63] obtained the sharp topological and differentiable sphere theorems for submanifolds with pinched Ricci curvatures.

Using the DDVV inequality verified by Ge-Tang [12] and Lu [37] and the Yau parameter method, Gu-Xu [14] proved the following rigidity theorem for minimal submanifolds in spheres.

Theorem 2.9. Let $M$ be an $n$-dimensional oriented compact minimal submanifold in the unit sphere $\mathbb{S}^{n+q}$. If

$$K_M \geq \frac{q \cdot \text{sgn}(q-1)}{2(q+1)},$$

then $M$ is either a totally geodesic sphere, the standard immersion of the product of two spheres, or the Veronese surface in $\mathbb{S}^4$. Here $\text{sgn}(\cdot)$ is the standard sign function.

Remark 2.2. When $2 < q < n$, the pinching constant in Theorem 2.9 is better than ones given by Yau [65] and Itoh [22].

Combing Theorems 2.2, 2.9, and rigidity results in [14, 22, 44], we present a general version of the Yau rigidity theorem for submanifolds with parallel mean curvature in spaces forms.

Generalized Yau rigidity theorem. Let $M$ be an $n$-dimensional oriented compact submanifold with parallel mean curvature in $\mathbb{F}^{n+q}(c)$, where $n^2c + |H|^2 > 0$. Set $k(m, n) = \min\{m \cdot \text{sgn}(m-1), n\}$. Then we have

(i) if $|H| = 0$ and

$$K_M \geq \frac{k(q, n)c}{2[k(q, n) + 1]},$$

then $M$ is either a totally geodesic sphere, the standard immersion of the product of two spheres, or the Veronese submanifold in $\mathbb{F}^{n+d}(c)$, where $d = \frac{1}{2}n(n+1) - 1$;

(ii) if $|H| \neq 0$ and

$$K_M \geq \frac{k(q-1, n)(c + \frac{1}{n^2}|H|^2)}{2[k(q-1, n) + 1]},$$

then $M$ is either a totally umbilical sphere, a Clifford hypersurface in an $(n+1)$-sphere, a product of three spheres in an $(n+2)$-sphere, or the Veronese submanifold in $\mathbb{F}^{n+d}(c + \frac{1}{n^2}|H|^2)$, where $d = \frac{1}{2}n(n+1) - 1$. 
Remark 2.3. Motivated by the generalized Yau rigidity theorem, Gu and Xu [16] verified a sharp differentiable sphere theorem for submanifolds with positive sectional curvature.

### 3. MCF meets Ricci flow: positive sectional curvature

Using the Ricci flow and stable currents [5, 6, 18, 24], Xu and Zhao [64] initiated the study of differentiable pinching problems for submanifolds, and proved the following differentiable sphere theorem.

**Theorem 3.1.** Let $M$ be an $n$-dimensional oriented complete submanifold in the unit sphere $S^{n+q}$. Then

(i) if $n = 4, 5, 6$ and $\sup_M(|A|^2 - \alpha(n, |H|)) < 0$, then $M$ is diffeomorphic to $S^n$;

(ii) if $n \geq 7$ and $|A|^2 < 2\sqrt{2}$, then $M$ is diffeomorphic to $S^n$.

In [60], Xu and Gu proved an optimal differentiable sphere theorem for submanifolds in the space form $F^{n+q}(c)$ with constant curvature $c \geq 0$.

**Theorem 3.2.** Let $M$ be an $n$-dimensional oriented complete submanifold in $F^{n+q}(c)$ with $c \geq 0$. If $\sup_M \left( |A|^2 - \frac{|H|^2}{n-1} - 2c \right) < 0$, then $M$ is diffeomorphic to $S^n$.

Meanwhile, Andrews-Baker [2] proved the following convergence theorem for the mean curvature flow of arbitrary codimension in Euclidean space, which implies a differentiable sphere theorem for submanifolds in Euclidean spaces.

**Theorem 3.3.** Let $M_0$ be an $n$-dimensional compact submanifold immersed in $R^{n+q}$. If $M_0$ has $|H| \neq 0$ everywhere and satisfies

$$|A|^2 \leq \begin{cases} \frac{4}{3n}|H|^2, & n = 2, 3, \\ \frac{1}{n-1}|H|^2, & n \geq 4, \end{cases}$$

then the mean curvature flow has a smooth solution $F : M \times [0, T) \to R^{n+q}$ on a finite maximal time interval, and $M_t$ converges uniformly to a point $q \in R^{n+q}$ as $t \to T$. The rescaled maps $\tilde{F}_t = \frac{F_t - x_0}{\sqrt{2n(t-t_0)}}$ converge in $C^\infty$ to a limiting embedding $\tilde{F}(M)$ such that $\tilde{F}(M)$ is the unit $n$-sphere in some $(n + 1)$-dimensional subspace of $R^{n+q}$.

In [15], Gu and Xu obtained a refined version of Theorem 3.2. Later, Baker [4] proved a sharp convergence theorem for the mean curvature flow of submanifolds in spheres.

**Theorem 3.4.** Let $M_0$ be an $n$-dimensional compact submanifold immersed in $S^{n+q}$. If $M_0$ satisfies

$$|A|^2 \leq \begin{cases} \frac{4}{3n}|H|^2 + \frac{2(n-1)}{3}, & n = 2, 3, \\ \frac{1}{n-1}|H|^2 + 2, & n \geq 4, \end{cases}$$

then either $M_t$ shrinks uniformly to a round point $p \in S^{n+q}$ as $t \to T < \infty$, or $M_t$ converges to a totally geodesic sphere in $S^{n+q}$ as $t$ tends to infinity.

In [34], Liu-Xu-Ye-Zhao proved the following sharp convergence theorem for the mean curvature flow of submanifolds in hyperbolic spaces.
**Theorem 3.5.** Let $F_0 : M^n \to \mathbb{F}^{n+q}(c)$ be an $n$-dimensional closed submanifold in a hyperbolic space with constant curvature $c < 0$. Assume $F_0$ satisfies

$$|A|^2 \leq \begin{cases} \frac{1}{n} |H|^2 + \frac{3}{4} c, & n = 2, 3, \\ \frac{1}{n-1} |H|^2 + 2c, & n \geq 4. \end{cases}$$

Then the mean curvature flow with $F_0$ as initial value converges to a round point in finite time. In particular, $M$ is diffeomorphic to $S^n$.

More generally, Xu and Gu [60] proved the following differentiable sphere theorem, which generalized the Brendle-Schoen [6] differentiable sphere theorem for manifolds with strictly $1/4$-pinched curvatures in the pointwise sense to the cases of submanifolds in a Riemannian manifold with codimension $q(\geq 0)$.

**Theorem 3.6.** Let $M^n$ be an $n$-dimensional complete submanifold in an $(n+q)$-dimensional Riemannian manifold $N^{n+q}$. If

$$|A|^2 < \frac{8}{3} \left( K_{\min} - \frac{1}{4} K_{\max} \right) + \frac{|H|^2}{n-1},$$

then $M$ is diffeomorphic to a space form. In particular, if $M$ is simply connected, then $M$ is diffeomorphic to $S^n$ or $\mathbb{R}^n$.

Let $N^{n+q}$ be an $(n + q)$-dimensional complete Riemannian manifold. Suppose that $N$ satisfies

1. the sectional curvature $-K_1 \leq K_N \leq K_2$ for $K_1, K_2 \geq 0$;
2. $|\nabla \bar{R}| \leq L$ for $L \geq 0$;
3. the injectivity radius $\text{inj}(N) \geq i(N) > 0$.

$N$ is called a Riemannian manifold with bounded geometry. We call $K_1, K_2, L$, and $i_N$ the bound constants.

In 2012, Liu-Xu-Zhao [35] proved the following convergence theorem.

**Theorem 3.7.** Let $F_0 : M^n \to N^{n+q}$ be an $n$-dimensional closed submanifold in an $(n+q)$-dimensional complete Riemannian manifold with bounded geometry with bound constants $K_1, K_2, L$, and $i_N$. There is an explicit constant $b_0$ depending on $n$, $q$, $K_1$, $K_2$ and $L$ such that if $F_0$ satisfies

$$|A|^2 < \begin{cases} \frac{1}{n} |H|^2 - b_0, & n = 2, 3, \\ \frac{1}{n-1} |H|^2 - b_0, & n \geq 4, \end{cases}$$

then the mean curvature flow with $F_0$ as initial value contracts to a round point in finite time. In particular, $M$ is diffeomorphic to $S^n$.

In [33], Liu-Xu-Ye-Zhao obtained some convergence theorems for the mean curvature flow of closed submanifolds in Euclidean spaces under suitable integral curvature pinching conditions.

### 4. MCF with positive Ricci curvature

There are very few optimal differentiable spheres in curvature and topology of manifolds. In 1995, Grove and Wilhelm [13] first verified an optimal differentiable sphere theorem for a class of Riemannian manifolds with positive sectional curvature. Using the Ricci flow and stable currents, Xu and Gu [60] proved an optimal
differentiable sphere theorem for certain submanifolds with positive sectional curvature. A challenging problem is: is it possible to prove an optimal differentiable sphere theorem for certain manifolds with positive Ricci curvature?

In 2015, the authors [25] first proved the following optimal convergence theorem for the mean curvature flow of arbitrary codimension in hyperbolic spaces whose initial submanifold has positive Ricci curvature.

**Theorem 4.1.** Let \( F_0 : M \to \mathbb{H}^{n+q}(c) \) be an \( n \)-dimensional \((n \geq 6)\) complete submanifold in a hyperbolic space with negative constant curvature \( c \). If \( M \) satisfies
\[
\sup_{M}(|A|^2 - \alpha(n, |H|, c)) < 0, \quad \text{where} \quad n^2c + |H|^2 > 0,
\]
then the mean curvature flow with initial value \( F_0 \) has smooth solution \( F_t(\cdot) \), and \( F_t(\cdot) \) converges to a round point in finite time. In particular, \( M \) is diffeomorphic to \( S^n \).

**Remark 4.1.** Since \( \alpha(n, |H|, c) > \frac{1}{n-2} |H|^2 + 2c \), our theorem above improves Theorem 3.5 for \( n \geq 6 \). Note that almost all initial submanifolds in the convergence results possess positive curvature. The pinching condition in Theorem 4.1 implies that the Ricci curvature of the initial submanifold is positive, but does not imply positivity of the sectional curvature. The following example shows that the pinching condition in Theorem 4.1 is optimal for arbitrary \( n \geq 6 \). Therefore, Theorem 4.1 is an optimal convergence theorem for the mean curvature flow of arbitrary codimension, which implies the first optimal differentiable sphere theorem for submanifolds with positive Ricci curvature.

**Example.** Let \( \lambda, \mu \) be positive constants satisfying \( \lambda \mu = -c \) and \( \lambda > \sqrt{-c} \), where \( c < 0 \). For \( n \geq 3 \), we consider the submanifold \( M = \mathbb{F}^{n-1}(c + \lambda^2) \times \mathbb{F}^1(c + \mu^2) \subset \mathbb{H}^{n+q}(c) \). Then \( M \) is a complete submanifold with parallel mean curvature, which satisfies \( |H| \equiv (n-1)\lambda + \mu > n\sqrt{-c} \) and \( |A|^2 \equiv (n-1)\lambda^2 + \mu^2 = \alpha(n, |H|, c) \).

The key ingredient of the proof of Theorem 4.1 is to establish the elaborate estimates for the pinching quantity \( \hat{\alpha} = \alpha(n, |H|, c) - \frac{1}{n} |H|^2 \), because our pinching condition is sharper than that in Theorem 3.5. Using the properties of \( \hat{\alpha} \) and the evolution equations, we first derive that \( |A|^2 < \hat{\alpha} \) is preserved along the mean curvature flow. Applying a new auxiliary function \( f_\sigma = |\hat{A}|^2 / \hat{\alpha}^{1-\sigma} \), we deduce that \( |\hat{A}|^2 \leq C_0 |H|^{2(1-\sigma)} \) via the De Giorgi iteration. We then obtain an estimate for \( |\nabla H| \). Finally, using estimates for \( |\nabla H| \) and the Ricci curvature, we show that \( \text{diam}(M_t) \to 0 \) and \( |H|_{\text{min}} / |H|_{\text{max}} \to 1 \) as \( t \to T \). This implies the flow shrinks to a round point.

**Proof of Theorem 4.1.** We split the proof into several steps.

**Step 1.** We first show that the pinching condition in Theorem 4.1 is preserved under the mean curvature flow.

Suppose that \( M_0 \) is an \((n \geq 6)\)-dimensional complete submanifold satisfying
\[
\sup_{M}(|A|^2 - \alpha(n, |H|, c)) < 0, \quad |H|^2 + n^2c > 0.
\]
From a theorem due to Shiohama-Xu [46], we see that \( M_0 \) is actually a compact submanifold.
For a positive integer $n$ and a negative constant $c$, we define a function $\hat{\alpha} : (-n^2c, +\infty) \to \mathbb{R}$ by

$$\hat{\alpha}(x) := nc + \frac{n^2 - 2n + 2}{2(n - 1)n} x - \frac{n - 2}{2(n - 1)} \sqrt{x^2 + 4(n - 1)cx}.$$  

Then $|\mathcal{A}|^2 < \alpha(n, |H|, c)$ is equivalent to $|\mathcal{A}|^2 < \hat{\alpha}(|H|^2)$.

Note that $\hat{\alpha}(|H|^2) \to 0$ as $|H|^2 \to -n^2c$. This implies that if $|\mathcal{A}|^2 < \hat{\alpha}(|H|^2)$ remains true along the flow, then $|H|^2 + n^2c > 0$ also remains true.

We consider $U = |\mathcal{A}|^2 - \hat{\alpha}(|H|^2)$. Along the mean curvature flow, $U$ satisfies

$$(\partial_t - \Delta) U$$

$$= (\partial_t - \Delta) |\mathcal{A}|^2 - \hat{\alpha}'(|H|^2) \cdot (\partial_t - \Delta) |H|^2$$

$$+ \hat{\alpha}''(|H|^2) \nabla |H|^2]^2$$

$$\leq - 2|\nabla \mathcal{A}|^2 + 2 \left( \frac{1}{n} + \hat{\alpha}'(|H|^2) \right) |\nabla H|^2 + \hat{\alpha}''(|H|^2) \cdot |\nabla |H|^2|^2$$

$$+ 2 |\mathcal{A}|^2 (|\mathcal{A}|^2 - nc) - 2 \hat{\alpha}'(|H|^2) |H|^2 (|\mathcal{A}|^2 + nc).$$

For convenience, we denote $\hat{\alpha}(|H|^2)$, $\hat{\alpha}'(|H|^2)$ and $\hat{\alpha}''(|H|^2)$ by $\hat{\alpha}$, $\hat{\alpha}'$ and $\hat{\alpha}''$, respectively.

Using the inequality $|\nabla \mathcal{A}|^2 \leq \frac{1}{n+2} |\nabla H|^2$ on submanifolds of space forms, we obtain

$$(\partial_t - \Delta) U$$

$$\leq 2U \left( 2\hat{\alpha} + \frac{1}{n} |H|^2 - nc - \hat{\alpha}' \cdot |H|^2 + U \right)$$

$$+ 2 \left[ \frac{2(n-1)}{n(n+2)} + \hat{\alpha}' + 2 |H|^2 \hat{\alpha}'' \right] |\nabla H|^2$$

$$+ 2 \left[ \hat{\alpha} \cdot \left( \hat{\alpha} + \frac{1}{n} |H|^2 - nc \right) - \hat{\alpha}' \cdot |H|^2 \cdot \left( \hat{\alpha} + \frac{1}{n} |H|^2 + nc \right) \right].$$

To apply the maximum principle for the parabolic partial differential equation, we need to prove the two expressions in the square brackets of the above formula are non-positive.

By some computations, we prove the following.

**Lemma 4.2.** If $x > -n^2c$, then $\hat{\alpha}$ has the following properties.

(i) $2\hat{\alpha}''(x) + \hat{\alpha}'(x) < \frac{2(n-1)}{n(n+2)}$.

(ii) $x\hat{\alpha}'(x) \cdot \left( \hat{\alpha}(x) + \frac{1}{n} x + nc \right) \equiv \hat{\alpha}(x) \cdot \left( \hat{\alpha}(x) + \frac{1}{n} x - nc \right).$

It follows from Lemma 4.2 that $(\partial_t - \Delta) U \leq 0$ at the point where $U = 0$. Hence, if $U$ is initially negative, then $U < 0$ is preserved along the mean curvature flow. Therefore, we obtain the following.

**Lemma 4.3.** If $M_0$ satisfies $|\mathcal{A}|^2 < \hat{\alpha}$ and $|H|^2 + n^2c > 0$, then there exist small positive constants $\varepsilon$ and $\delta$, such that for all time $t \in [0, T_{max})$, we have

$$|\mathcal{A}|^2 < \hat{\alpha} - \varepsilon |H|^2 \quad \text{and} \quad |H|^2 + n^2c > \delta.$$

**Step 2.** We next prove a pinching estimate for the traceless second fundamental form, which guarantees that $M_t$ becomes spherical along the mean curvature flow.

**Lemma 4.4.** There are positive constants $C_0$ and $\sigma$ independent of $t$ such that

$$|\mathcal{A}|^2 \leq C_0 |H|^{2(1-\sigma)}$$
holds along the mean curvature flow.

To prove the above lemma, we introduce an auxiliary function:

\[ f_\sigma = \frac{|A|^2}{\alpha^{1-\sigma}}, \]

where \( \sigma \in (0, 1) \). We need to prove that \( f_\sigma \) is bounded.

Computing the evolution equation of \( f_\sigma \), we get

\[
\frac{\partial}{\partial t} f_\sigma \leq \Delta f_\sigma + \frac{2}{|A|} |\nabla f_\sigma||\nabla H| - 12\varepsilon \frac{f_\sigma}{|A|^2} |\nabla H|^2 - 4cf_\sigma + \sigma |H|^2 f_\sigma.
\]

Because of the existence of term \( \sigma |H|^2 f_\sigma \) in the inequality above, the maximum principle doesn’t work here. To deal with this term, we need the following estimate.

Lemma 4.5. If a submanifold in \( \mathbb{H}^{n+q}(c) \) satisfies \( |\hat{\alpha}|^2 < \hat{\alpha} - \varepsilon |H|^2 \) and \( |H|^2 + n^2 c > 0 \), then we have

\[
\Delta |\hat{\alpha}|^2 \geq 2 \left( \hat{\alpha}, \nabla^2 H \right) + \frac{\varepsilon}{2} |H|^2 |\hat{\alpha}|^2.
\]

From Lemma 4.5, we obtain the following inequality.

\[
\Delta f_\sigma \geq \frac{2f_\sigma}{|A|^2} \left( \hat{\alpha}, \nabla^2 H \right) + \frac{\varepsilon}{2} |H|^2 f_\sigma - (1 - \sigma \frac{f_\sigma}{\alpha}) \Delta \hat{\alpha} - \frac{2}{|A|} |\nabla f_\sigma||\nabla H|.
\]

Integrating both sides of the inequality above and making use of the divergence theorem, we get

\[
\frac{\varepsilon}{2} \int_{M_t} |H|^2 f_\sigma^p \, d\mu_t \leq \int_{M_t} \left( \frac{3pf_\sigma^{p-1}}{|A|} |\nabla f_\sigma||\nabla H| + \frac{5f_\sigma^p}{|A|^2} |\nabla H|^2 \right) \, d\mu_t.
\]

Then we get the following estimate for the time derivative of the integral of \( f_\sigma \).

\[
\frac{d}{dt} \int_{M_t} f_\sigma^p \, d\mu_t = p \int_{M_t} f_\sigma^{p-1} \frac{\partial f_\sigma}{\partial t} \, d\mu_t - \int_{M_t} f_\sigma^p |H|^2 \, d\mu_t \leq p \int_{M_t} f_\sigma^{p-2} \left[ -(p-1) |\nabla f_\sigma|^2 + \left( 2 + \frac{6sp}{\varepsilon} \right) \frac{f_\sigma}{|A|} |\nabla f_\sigma||\nabla H| \right.
\]

\[
- \left( 12\varepsilon - \frac{10\sigma}{\varepsilon} \right) \frac{f_\sigma}{|A|^2} |\nabla H|^2 \right] \, d\mu_t - 4cp \int_{M_t} f_\sigma^p \, d\mu_t.
\]

Note that the expression in the square bracket of the above formula is a quadratic polynomial. By choosing suitable constants \( \sigma \) and \( p \), we make this quadratic polynomial non-positive.

Hence we get

\[
\frac{d}{dt} \int_{M_t} f_\sigma^p \, d\mu_t \leq -4cp \int_{M_t} f_\sigma^p \, d\mu_t.
\]

It can be proved that the maximal existence time of the mean curvature flow is finite. Therefore, we obtain
Lemma 4.6. There exist a constant $C$ independent of $t$, such that

$$\left( \int_{M_t} f_\sigma^p \, d\mu_t \right)^{\frac{1}{p}} \leq C.$$ 

Then we show that $f_\sigma$ is bounded along the mean curvature flow via a Stampacchia iteration procedure.

Step 3. We establish a gradient estimate for the mean curvature flow, which will be used to compare the mean curvature at different points of the submanifold.

Lemma 4.7. For every $\eta > 0$, there exists a constant $C_\eta$ independent of $t$ such that for all $t \in [0, T_{\text{max}})$, the following inequality holds

$$|\nabla H|^2 \leq \eta |H|^4 + C_\eta.$$ 

Proof. Consider the function

$$f = |\nabla H|^2 + (N_1 + N_2 |A|^2) |\dot{A}|^2 - \eta |H|^4,$$

where $N_1$ and $N_2$ are sufficiently large positive constants independent of $t$.

By the evolution equations of $|H|^4$ and $|\nabla H|^2$, we obtain

$$\frac{\partial}{\partial t} f \leq \Delta f + C_5.$$ 

The assertion follows from the maximum principle and the definition of $f$. □

Step 4. Now we finish the proof of Theorem 4.1.

By an estimate for Ricci curvature due to Shiohama-Xu [46], we have

$$\text{Ric}_{M_t} \geq \frac{n-1}{n} \left( nc + \frac{2}{n} |H|^2 - |A|^2 - \frac{n-2}{\sqrt{n(n-1)}} |H||\dot{A}| \right).$$

By the preserved pinching condition $|\dot{A}|^2 < \dot{\alpha} - \epsilon |H|^2$ and the following identity

$$\frac{n-2}{\sqrt{n(n-1)}} \sqrt{x\dot{\alpha}(x)} \equiv \frac{1}{n} x - \dot{\alpha}(x) + nc,$$

we see that there exists a positive constant $\epsilon_0$ independent of $t$ such that $\text{Ric}_{M_t} \geq \epsilon_0 |H|^2$.

Since $T_{\text{max}}$ is finite, $\max_{M_t} |A|^2 \to \infty$ as $t \to T_{\text{max}}$. At a time $t$ close to $T_{\text{max}}$, let $x$ be a point on $M_t$ where $|H|$ achieves its maximum. From the gradient estimate of $H$, along all geodesics of length $l = (2\eta |H|_{\text{max}})^{-1}$ starting from $x$, we have $|H| > (1 - \eta) |H|_{\text{max}}$. With $\eta$ small enough, the Ricci curvature of $M_t$ satisfies $\text{Ric}_{M_t} > (n-1)^2 / l^2$ on these geodesics. Then from Myers’ theorem, these geodesics can reach any point of $M_t$.

Thus we have $|H|_{\text{min}} > (1 - \eta) |H|_{\text{max}}$ and $\text{diam}(M_t) \leq (2\eta |H|_{\text{max}})^{-1}$. So, $\frac{\max_{M_t} |H|}{\min_{M_t} |H|} \to 1$ and $\text{diam}(M_t) \to 0$ as $t \to T_{\text{max}}$. Hence $M_t$’s converge to a single point $o$ as $t \to T_{\text{max}}$.

Take a rescaling around $o$ such that the total area of the expanded submanifolds are fixed. Then the rescaled immersions converge to a totally umbilical immersion as $t \to T_{\text{max}}$.

Recently, Lei-Xu [26] proved the following sharp convergence theorem for the mean curvature flow of hypersurfaces in spheres.
Theorem 4.8. Let $F_0 : M^n \to \mathbb{F}^{n+1}(c)$ be an $n$-dimensional ($n \geq 3$) closed hypersurface immersed in a spherical space form. If $F_0$ satisfies

$$|A|^2 < \gamma_1(n, |H|, c),$$

then the mean curvature flow with initial value $F_0$ has a unique smooth solution $F : M \times [0, T) \to \mathbb{F}^{n+1}(c)$, and $F_t$ converges to a round point in finite time, or converges to a totally geodesic hypersurface as $t \to \infty$. Here $\gamma_1(n, H, c)$ is an explicit positive scalar defined by

$$\gamma_1(n, H, c) = \min\{\alpha(H^2), \beta(H^2)\},$$

where

$$\alpha(x) = nc + \frac{n}{2(n-1)} x - \frac{n-2}{2(n-1)} \sqrt{x^2 + 4(n-1)cx},$$

$$\beta(x) = \alpha(x_0) + \alpha'(x_0)(x - x_0) + \frac{1}{2} \alpha''(x_0)(x - x_0)^2,$$

$$x_0 = y_n c, \quad y_n = 4(1 - n) + \frac{2(n^2 - 4)}{\sqrt{2n - 5}} \cos \left( \frac{1}{3} \arctan \frac{n^2 - 4n + 6}{2(n-1)^2} \right).$$

Remark 4.2. The scalar $\gamma_1(n, |H|, c)$ satisfies (i) $\gamma_1(n, H, c) > \frac{1}{n+1} H^2 + 2c$; (ii) $\gamma_1(n, |H|, c) > \frac{2}{n+2} \sqrt{n-1} c$; (iii) $\gamma_1(n, |H|, c) = \alpha(n, |H|, c)$, for $|H|^2 \geq y_n c$. Thus Theorem 4.8 substantially improves the famous convergence theorem due to Huisken [21].

Now we state the idea of the proof of Theorem 4.8.

Step 1. We first prove that the pinching condition in Theorem 4.8 is preserved under the mean curvature flow.

Suppose that $M_0$ is an $n$-dimensional ($n \geq 3$) closed hypersurface satisfying $|A|^2 < \gamma_1(|H|^2)$. Here $\gamma_1$ is an explicit function defined by

$$\gamma_1(x) = \begin{cases} \alpha(x), & x \geq x_0, \\ \beta(x), & 0 \leq x < x_0. \end{cases}$$

Consider $U = |A|^2 - \gamma_1(H^2)$. Computing the evolution equation of $U$, we get

$$\left( \frac{\partial}{\partial t} - \Delta \right) U \leq 2U \left( 2\gamma_1(H^2) - nc - \gamma_1'(H^2) H^2 + U \right) + 2 \left[ \frac{3}{n+2} + \gamma_1'(H^2) + 2H^2 \gamma_1''(H^2) \right] |\nabla H|^2 + 2 2cH^2 + \gamma_1(H^2)^2 - nc\gamma_1(H^2) - (\gamma_1(H^2) + nc) H^2 \gamma_1'(H^2) \right].$$

To apply the maximum principle, we need to prove the two expressions in the square brackets of the above formula are non-positive.

By some complicated calculations, we get the following lemma.

Lemma 4.9. For $x \geq 0$, the function $\gamma_1$ has the following properties.

(i) $2x\gamma_1''(x) + \gamma_1'(x) \leq \frac{1}{n+2}$.

(ii) $(\gamma_1(x) + nc)\gamma_1'(x) \geq 2cx + \gamma_1(x)^2 - nc\gamma_1(x).$
It follows from Lemma 4.9 that \((\partial_t - \Delta)U \leq 0\) at the point where \(U = 0\). Applying the maximum principle, we prove that if \(U\) is initially negative, then \(U < 0\) is preserved along the mean curvature flow.

**Step 2.** We prove a pinching estimate for the traceless second fundamental form, which guarantees that \(M_t\) becomes spherical along the mean curvature flow.

**Lemma 4.10.** There are positive constants \(C_0\) and \(\sigma\) independent of \(t\) such that

\[
|\bar{A}|^2 \leq C_0 (H^2 + c)^{1-\sigma} e^{-2\sigma ct}
\]

holds along the mean curvature flow.

To prove the lemma above, we introduce an auxiliary function:

\[
f_{\sigma} = |\bar{A}|^2 / \gamma_1(H^2)^{1-\sigma},
\]

where \(\sigma \in (0, 1)\). We need to show that \(f_{\sigma}\) decays exponentially.

Let \(p\) be a sufficiently large number. Computing the evolution equation of \(f_{\sigma}^p\), we get

\[
\frac{d}{dt} \int_{M_t} f_{\sigma}^p d\mu_t \leq -3\sigma cp \int_{M_t} f_{\sigma}^p d\mu_t.
\]

Thus we obtain

**Lemma 4.11.** There exist a constant \(C\) independent of \(t\), such that

\[
\left(\int_{M_t} f_{\sigma}^p d\mu_t\right)^\frac{1}{p} \leq C e^{-3\sigma ct}.
\]

Let \(g_{\sigma} = f_{\sigma} e^{2\sigma ct}\). Then we show that \(g_{\sigma}\) is bounded along the mean curvature flow via a Stampacchia iteration procedure. This proves the lemma.

**Step 3.** We establish a gradient estimate for the mean curvature flow, which will be used to compare the mean curvature at different points of the hypersurface.

**Lemma 4.12.** For every \(\eta > 0\), there exists a constant \(C_\eta\) independent of \(t\) such that for all \(t \in [0, T_{\text{max}}]\), there holds

\[
|\nabla H|^2 < (\langle H \rangle^4 + C_\eta^2) e^{-\sigma ct}.
\]

**Proof.** Consider the function

\[
f = \left(\left|\nabla H\right|^2 + (N_1 + N_2 |A|^2)|\bar{A}|^2\right) e^{\sigma ct} - \eta |H|^4,
\]

where \(N_1\) and \(N_2\) are sufficiently large positive constants independent of \(t\). By the evolution equations of \(|H|^4\) and \(|\nabla H|^2\), we obtain

\[
\frac{\partial}{\partial t} f \leq \Delta f + C_5 e^{-\sigma ct}.
\]

The assertion follows from the maximum principle and the definition of \(f\). \(\Box\)

**Step 4.** Now we finish the proof of Theorem 4.8. We have two cases as follows.

Case (i). Suppose that \(T_{\text{max}}\) is finite. By an estimate for Ricci curvature due to Shiohama-Xu [40], we obtain \(\text{Ric}_{M_t} \geq \varepsilon_0 (H^2 + c)\). Since \(T_{\text{max}}\) is finite, \(\max_{M_t} |A|^2 \to \infty\) as \(t \to T_{\text{max}}\). Using the gradient estimate for \(H\) and Myers’ theorem, we obtain \(\frac{\max_{M_t} |H|}{\min_{M_t} |H|} \to 1\) and \(\text{diam}(M_t) \to 0\) as \(t \to T_{\text{max}}\). Hence \(M_t\)’s converge to a single point \(o\) as \(t \to T_{\text{max}}\).
We take a rescaling around \( o \). Then the rescaled immersions converge to a totally umbilical immersion as \( t \to T_{\text{max}} \).

Case (ii). Suppose that \( T_{\text{max}} = \infty \). It follows from the gradient estimate of \( H \) that \( H^2 < C e^{-\alpha t} \). Noting that \( |A|^2 \leq C_0 H^{2(1-\sigma)} \cdot e^{-2\sigma t} \), we get \( |h|^2 \leq C e^{-\alpha t} \).

Since \( |h| \to 0 \) as \( t \to \infty \), \( M_t \) converges to a totally geodesic hypersurface.

This completes the proof of Theorem 4.8.

More recently, Lei-Xu \cite{27} proved a sharp convergence theorem for mean curvature flow (\( n \geq 6 \)) of arbitrary codimension in spheres, which substantially improves the convergence theorem for mean curvature flow of arbitrary codimension in spheres due to Baker \cite{4}.

**Theorem 4.13.** Let \( F_0 : M^n \to \mathbb{F}^{n+q}(c) \) be an \( n \)-dimensional (\( n \geq 6 \)) closed submanifold immersed in a spherical space form. If \( F_0 \) satisfies \( |A|^2 < \gamma(n, |H|, c) \),

then the mean curvature flow with initial value \( F_0 \) has a unique smooth solution \( F : M \times [0, T) \to \mathbb{F}^{n+q}(c) \), and \( F_t \) converges to a round point in finite time, or converges to a totally geodesic sphere as \( t \to \infty \). Here \( \gamma(n, |H|, c) \) is an explicit positive scalar defined by

\[
\gamma(n, |H|, c) = \min \{ \alpha(|H|^2), \beta(|H|^2) \},
\]

where

\[
\alpha(x) = nc + \frac{n}{2(n-1)} x - \frac{n-2}{2(n-1)} x^2 + (n-1)cx,
\]

\[
\beta(x) = \alpha(x_0) + \alpha'(x_0)(x - x_0) + \frac{1}{2} \alpha''(x_0)(x - x_0)^2,
\]

\[
x_0 = y_n c, \quad y_n = \frac{2n + 2}{n-4} \sqrt{n-1} \left( \sqrt{n-1} - \frac{n-4}{2n+2} \right)^2.
\]

**Remark 4.3.** The scalar \( \gamma(n, |H|, c) \) satisfies the following: (i) \( \gamma(n, H, c) > \frac{1}{4n-2} H^2 + 2c \); (ii) \( \gamma(n, |H|, c) > \frac{1}{4n-1} c \); (iii) \( \gamma(n, |H|, c) = \alpha(n, |H|, c) \), for \( |H|^2 \geq 2c \). Hence Theorem 4.13 substantially improves Theorem 3.4.

In \cite{21}, Pipoli and Sinestrari obtained the following convergence theorem for mean curvature flow of small codimension in the complex projective space.

**Theorem 4.14.** Let \( F_0 : M^n \to \mathbb{CP}^{n+q} \) be a closed submanifold of dimension \( n \) and codimension \( q \) in the complex projective space with Fubini-Study metric. Let \( F : M^n \times [0, T) \to \mathbb{CP}^{n+q} \) be the mean curvature flow with initial value \( F_0 \). If \( F_0 \) satisfies

\[
|h|^2 < B_0(n, q, |H|) := \left\{ \begin{array}{ll}
\frac{1}{n-1} |H|^2 + 2c, & q = 1 \text{ and } n \geq 5, \\
\frac{1}{n-1} |H|^2 + \frac{n-3-4q}{n}, & 2 \leq q < \frac{n-3}{4},
\end{array} \right.
\]

then \( F_t \) converges to a round point in finite time, or converges to a totally geodesic submanifold as \( t \to \infty \). In particular, \( M \) is diffeomorphic to either \( S^n \) or \( \mathbb{CP}^{n/2} \).

Most recently, Lei-Xu \cite{28} proved a new convergence theorem for mean curvature flow of arbitrary codimension in complex projective spaces, which substantially improves the convergence theorem for mean curvature flow of small codimension
in complex projective spaces due to Pipoli-Sinestrari [41]. Precisely, we prove the following theorem.

**Theorem 4.15.** Let $F_0 : M^n \to \mathbb{C}P^{n+q}$ be an $n$-dimensional closed submanifold in $\mathbb{C}P^{n+q}$. Let $F : M^n \times [0, T) \to \mathbb{C}P^{n+q}$ be the mean curvature flow with initial value $F_0$. If $F_0$ satisfies

$$\|h\|^2 < B(n, q, |H|) := \begin{cases}
\frac{1}{n} |H|^2, & q = 1 \text{ and } n \geq 3, \\
\frac{1}{n} |H|^2 + 2 - \frac{3}{n}, & 2 \leq q < n - 4, \\
\psi(|H|^2), & q \leq n - 4 \geq 2,
\end{cases}$$

then $F_t$ converges to a round point in finite time, or converges to a totally geodesic submanifold as $t \to \infty$. In particular, $M$ is diffeomorphic to $\mathbb{S}^n$ or $\mathbb{C}P^{n/2}$.

Here $\varphi(|H|^2)$ and $\psi(|H|^2)$ are given by

$$\varphi(|H|^2) = 2 + a_n + \left(b_n + \frac{1}{n-1}\right) |H|^2 - \sqrt{\frac{2}{n} |H|^4 + 2a_n b_n |H|^2},$$

$$\psi(|H|^2) = \frac{9}{n^2 - 3n - 3} + \frac{n^2 - 3n}{n^2 + 3} |H|^2 - \frac{3(n-1)(n^2-3)|H|^2+9(n-1)^2}{n^2-4n^2+3},$$

where $a_n = 2\sqrt{n^2 - 4n + 3} b_n$, $b_n = \min \left\{ \frac{n-3}{4n-4}, \frac{2n-5}{n^2+n-2} \right\}$.

**Remark 4.4.** In particular, (i) if $q = 1$ and $n \geq 5$, we have $B(n, q, |H|) > B_0(n, q, |H|)$ and $B(n, q, |H|) > \sqrt{2(n-3)}$: (ii) if $2 \leq q < \frac{4n-5}{n-1}$, we have $B(n, q, |H|) - B_0(n, q, |H|) = 1 + \frac{42}{n}$. Therefore, Theorem 4.15 substantially improves Theorem 4.14.

5. Problem section

In this section, we give several unsolved problems on the convergence theorems for the mean curvature flow and sphere theorems for submanifolds.

Denote by $\mathbb{F}^{n+q}(c)$ the $(n + q)$-dimensional complete simply connected space form of constant sectional curvature $c$. Let $M$ be an $n$-dimensional oriented compact submanifold in $\mathbb{F}^{n+q}(c)$. Based on the discussions in [14, 17, 26, 27, 33, 36, 38, 46, 47, 61, 62], we present the following conjectures.

**Conjecture 5.1 (Liu-Xu-Ye-Zhao [33]).** Let $F_0 : M \to \mathbb{F}^{n+q}(c)$ be an $n$-dimensional closed submanifold in $\mathbb{F}^{n+q}(c)$ with $c > 0$. Suppose $F_0$ satisfies

$$|A|^2 < o(n, |H|, c),$$

then the mean curvature flow with $F_0$ as initial value has a unique solution $F : M \times [0, T) \to \mathbb{F}^{n+q}(c)$ on a maximal time interval, and either (i) $T < \infty$ and $M_t$ converges to a round point as $t \to T$; or (ii) $T = \infty$ and $M_t$ converges to a totally geodesic submanifold in $\mathbb{F}^{n+q}(c)$ as $t \to \infty$.

In particular, $M$ is diffeomorphic to the standard $n$-sphere.

**Conjecture 5.2 (see also [33]).** Let $F_0 : M \to \mathbb{F}^{n+q}(c)$ be an $n$-dimensional closed submanifold in $\mathbb{F}^{n+q}(c)$. Then there exists a positive constant $C_n$ depending only on $n$, such that if $M$ satisfies

$$\int_M |A|^p dM < C_n,$$
then the mean curvature flow with $F_0$ as initial value has a unique solution $F : M \times [0, T) \to \mathbb{F}^{n+q}(c)$ on a maximal time interval, and either

(i) $T < \infty$ and $M_t$ converges to a round point as $t \to T$; or

(ii) $c > 0$, $T = \infty$ and $M_t$ converges to a totally geodesic submanifold in $\mathbb{F}^{n+q}(c)$ as $t \to \infty$.

In particular, $M$ is diffeomorphic to the standard $n$-sphere.

**Conjecture 5.3.** Let $F_0 : M \to \mathbb{R}^{n+q}$ be an $n$-dimensional closed submanifold in $\mathbb{R}^{n+q}$. There exists a positive constant $C_n$ depending only on $n$, such that if $M$ satisfies

$$\int_M |H|^n dM < n^n Vol(S^n) + C_n,$$

then the mean curvature flow with $F_0$ as initial value has a unique solution $F : M \times [0, T) \to \mathbb{R}^{n+q}$ on a finite maximal time interval, and $M_t$ converges to a round point as $t \to T$. In particular, $M$ is diffeomorphic to the standard $n$-sphere.

A challenging problem is: what is the best pinching constant in Conjecture 5.3? In particular, we have the following stronger version of the Willmore theorem verified by Marques and Neves [38].

**Conjecture 5.4.** Let $F_0 : M^2 \to \mathbb{R}^3$ be a closed surface in $\mathbb{R}^3$. If $M$ satisfies

$$\int_M |H|^2 dM < 8\pi^2,$$

then the mean curvature flow with $F_0$ as initial value has a unique solution $F : M \times [0, T) \to \mathbb{R}^3$ on a finite maximal time interval, and $M_t$ converges to a round point as $t \to T$. In particular, $M$ is diffeomorphic to $S^2$.

**Conjecture 5.5 (Xu-Gu [61]).** Let $F_0 : M \to \mathbb{F}^{n+q}(c)$ be an $n(\geq 3)$-dimensional closed submanifold in an $(n + q)$-dimensional space form $\mathbb{F}^{n+q}(c)$ with $n^2c + |H|^2 > 0$. If the Ricci curvature of $M$ satisfies

$$Ric_M > (n - 2)(c + \frac{1}{n^2}|H|^2),$$

then the mean curvature flow with $F_0$ as initial value has a unique solution $F_t(\cdot)$ on a maximal time interval, and $M_t$ converges to a round point in finite time, or $c > 0$ and $M_t$ converges to a totally geodesic sphere as $t \to \infty$. In particular, $M$ is diffeomorphic to $S^n$.

**Conjecture 5.6 (Gu-Xu [14]).** Let $F_0 : M \to \mathbb{F}^{n+q}(c)$ be an $n$-dimensional closed submanifold in an $(n + q)$-dimensional space form $\mathbb{F}^{n+q}(c)$ with $n^2c + |H|^2 > 0$. Set $k(q, n) = \min\{q \cdot \text{sgn}(q - 1), n\}$. If the sectional curvature of $M$ satisfies

$$K_M > \frac{k(q, n)(c + \frac{1}{n^2}|H|^2)}{2[k(q, n) + 1]},$$

then the mean curvature flow with $F_0$ as initial value has a unique solution $F_t(\cdot)$ on a maximal time interval, and $M_t$ converges to a round point in finite time, or $c > 0$ and $M_t$ converges to a totally geodesic sphere as $t \to \infty$. In particular, $M$ is diffeomorphic to $S^n$.

Even for $q = 1$ and $c = 1$, the above problem is still open.
Conjecture 5.7. Let $F_0 : M \rightarrow \mathbb{S}^{n+q}$ be an $n$-dimensional closed submanifold in an $(n + q)$-dimensional unit sphere. If $|h(u, u)|^2 < \frac{1}{3}$ for any $u \in UM$, then the mean curvature flow with $F_0$ as initial value has a unique solution $F_t(\cdot)$ on a maximal time interval, and $M_t$ converges to a round point in finite time, or converges to a totally geodesic sphere as $t \rightarrow \infty$. In particular, $M$ is diffeomorphic to $\mathbb{S}^n$.

Conjecture 5.8 (Xu-Huang-Zhao [62]). Let $F_0 : M \rightarrow \mathbb{S}^{n+q}$ be an $n$-dimensional closed submanifold in an $(n + q)$-dimensional unit sphere. Set

$$
\tau(x) = \max_{u,v \in U_x M, u \perp v} |h(u, u) - h(v, v)|^2.
$$

If $\tau(x) < \frac{4}{3}$ for all $x \in M$, then the mean curvature flow with $F_0$ as initial value has a unique solution $F_t(\cdot)$ on a maximal time interval, and $M_t$ converges to a round point in finite time, or converges to a totally geodesic sphere as $t \rightarrow \infty$. In particular, $M$ is diffeomorphic to $\mathbb{S}^n$.

Notice that if $|h(u, u)|^2 < \frac{1}{3}$ for any $u \in U_x M$, then $\tau(x) < \frac{4}{3}$. This implies that Conjecture 5.8 is stronger than Conjecture 5.7.

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