THE MINIMAL LOG DISCREPANCY

FLORIN AMBRO

Abstract. We survey the known and expected properties of the minimal log discrepancy, the local invariant of a log variety.

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Introduction

The minimal log discrepancy is a fundamental invariant of the singularities that appear in the birational classification of algebraic varieties. Introduced by Shokurov [52] in connection to the termination of a sequence of flips, it appears in the local context of the classification of singularities, or the global context of Fujita’s conjecture on adjoint linear systems. The minimal log discrepancy measures stable vanishing orders of sections of canonical graded rings, or the rate of growth of certain subspaces in the spaces of jets of a singularity. It has an arithmetic flavour, being related to the first minimum of Minkowski in the geometry of numbers.

In this note we introduce the minimal log discrepancy, present some basic open problems, and illustrate them with toric examples. We hope to reinforce the original connection of Reid [50] between discrepancies of singularities on the one hand, and stable vanishing orders of sections of canonical graded rings on the other hand.

The minimal log discrepancy is the local invariant of a log variety. We recall in §1 the construction of canonical models and discrepancies, and their logarithmic version. This is the natural motivation for log varieties with log canonical singularities, which locally are just open subsets of log canonical models. We give the rigorous definition of log varieties and minimal log discrepancies in §2, and present explicit combinatorial formulas in the toric case. We present problems on minimal log discrepancies in §3, and discuss their toric case and some methods, old and new. In §4, we discuss the log canonical threshold and its applications to the problem of constructing flat log structures.

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1-A. Zariski decomposition. The origin of Zariski decomposition is the problem of computing the Hilbert function \( \varphi(n) = \dim \mathbb{C} R(X, D)_n \) of the graded ring

\[
R(X, D) = \bigoplus_{n=0}^{\infty} H^0(X, nD),
\]

for a given divisor \( D \) on a complex manifold \( X \) [61]. We recall the solution to this problem in the case when \( R(X, D) \) is finitely generated and \( D \) is big. We use Zariski’s notation

\[
H^0(X, D) = \{ a \in \mathbb{C}(X)^*; (a) + D \geq 0 \} \cup \{ 0 \},
\]

which makes sense even if \( D \) does not have integer coefficients.

First, if \( D \) is an ample divisor, there exists a polynomial \( P \in \mathbb{Q}[T] \) of degree \( d = \dim(X) \), and a positive integer \( n_0 \), such that \( \varphi(n) = P(n) \) for \( n \geq n_0 \). In general, finite generation means that there exists a positive integer \( r \) such that the natural map \( S^r H^0(X, rD) \rightarrow H^0(X, lrD) \) is surjective for every \( l \geq 1 \). By Hironaka’s resolution of the base locus of a linear system, there exists a birational modification \( f_r : X_r \rightarrow X \) such that if \( F_r \) is the fixed divisor of \( |f_r^*(rD)| \), the mobile part \( M_r = f_r^*(rD) - F_r \) defines a linear system without base points. After taking the Stein factorization, \( |M_r| \) defines a morphism with connected fibers \( g_r : X_r \rightarrow Y_r \). The variety \( Y_r \) has normal singularities and \( M_r = g_r^*(A_r) \), for a normally generated ample Cartier divisor \( A_r \) on \( Y_r \).

We have \( R(X, D) = R(X_r, f_r^*(D)) \), the natural inclusion \( R(X_r, \frac{1}{r} M_r) \subseteq R(X_r, f_r^* D) \) becomes an identity, and \( R(X_r, \frac{1}{r} M_r) = R(Y_r, \frac{n}{r} A_r) \). In particular, \( Y_r = \text{Proj} R(X, D) \) and

\[
\varphi(n) = \dim \mathbb{C} H^0(Y_r, \frac{n}{r} A_r).
\]

There are polynomials \( P_0, \ldots, P_{r-1} \in \mathbb{Q}[T] \), of degree \( d \), and a positive integer \( n_0 \), such that \( \varphi(n) = P_{n-r\lfloor \frac{n}{r} \rfloor}(n) \) for \( n \geq n_0 \). The growth rate of \( \varphi \) is

\[
\lim_{n \rightarrow \infty} \frac{\varphi(n)}{n^d/d!} = \frac{(A_r^d)}{r^d} \in \mathbb{Q}^\mathbb{N}.
\]

The decomposition \( f_r^*(D) = g_r^*(\frac{1}{r} A) + \frac{1}{r} F_r \) is the prototype of a Zariski decomposition. In general, we say that \( D = P + F \) is a Zariski decomposition on a normal variety \( X \), if \( F \) is an effective \( \mathbb{R} \)-divisor and \( P \) is a nef \( \mathbb{R} \)-divisor such that the induced inclusion \( R(X, P) \subseteq R(X, D) \) is surjective. The Zariski decomposition is unique if \( D \) is big, and in this case \( F = \lim_{n \rightarrow \infty} \frac{1}{n} F_n \), where \( F_n \) is the fixed part of \( |nD| \). There are several notions of Zariski decomposition at present, but they all coincide for big divisors (see [61, 29, 22, 15, 16]).

Two useful examples of Zariski decomposition are as follows. For a nef \( \mathbb{R} \)-divisor \( P \), the decomposition \( P = P + 0 \) is a Zariski decomposition. If \( D = P + F \) is a Zariski decomposition on \( X \) and \( F' \) is an effective \( \mathbb{R} \)-divisor supported by the exceptional locus of a birational map \( \mu : X' \rightarrow X \), then \( \mu^* D + E = \mu^* P + (\mu^* F + F') \) is a Zariski decomposition.
1-B. **Canonical models, discrepancies.** Let $X$ be a complex projective manifold of general type, with *canonical divisor* $K_X$. The *canonical ring* $R(X, K_X)$ is expected to be finitely generated, and if it is, we would obtain a natural birational map

$$
\Phi: X \longrightarrow Y := \text{Proj} R(X, K_X).
$$

The birational model $Y$ is called the *canonical model* of $X$. It depends only on the birational class of $X$ and it has a canonical polarization, but it has singularities in general. For example, $Y$ may have some Du Val singularities in dimension two. The singularities that may appear on $Y$ were coined *canonical singularities* by Reid [50].

To get to the formal definition of canonical singularities, let us take a closer look at what $\Phi$ does for $K_X$. By Hironaka’s resolution of singularities, there is a *Hironaka hut*.

![Hironaka hut diagram](image)

that is $X'$ is a projective manifold, $f, g$ are birational morphisms and $\Phi = g \circ f^{-1}$. By definition, $K_X$ is the divisor $(\omega)$ of zeros and poles of a non-zero top rational differential form $\omega \in \wedge^{\dim(X)} \Omega^1_X \otimes_{\mathbb{C}} \mathbb{C}(X)$. Denote $K_{X'} = (f^* \omega)$ and $K_Y = (g_* f^* \omega)$. The latter is a well defined Weil divisor, since $Y$ is normal. Since $X$ has no singularities, the divisor $A_f = K_{X'} - f^*(K_X)$ is effective and supported by the exceptional locus of $f$. Equivalently, the natural map $f_*: R(X', K_{X'}) \rightarrow R(X, K_X)$ is an isomorphism. In particular, $g: X' \rightarrow Y$ is the canonical model of $X'$. Since $g$ is a morphism and $K_{X'}$ is a big divisor, it follows that there exists $r \geq 1$ such that $rK_Y$ defines a projectively normal embedding, and $A_g = \frac{1}{r}(rK_{X'} - g^*(rK_Y))$ is effective and supported by the exceptional locus of $g$. In particular, $g_*: R(X', K_{X'}) \rightarrow R(Y, K_Y)$ is also an isomorphism:

$$
\begin{array}{ccc}
R(X', K_{X'}) & \cong & R(Y, K_Y) \\
\pi & & \pi \\
R(X, K_X) & \cong & R(Y, K_Y)
\end{array}
$$

Reid [50] called a normal germ $P \in Y$ a canonical singularity if $A_g$ is well defined and effective, for a resolution of singularities $g: X' \rightarrow Y$. The coefficients of the $\mathbb{Q}$-divisor $A_g$ are called *discrepancies*. To understand discrepancies in terms of the manifolds that we started with, we go back to our global setting and note that

$$
K_{X'} = g^*(K_Y) + A_g
$$

is a Zariski decomposition of $K_{X'}$, with positive part $g^*(K_Y)$ and fixed part $A_g$. Since $|rK_Y|$ defines a linear system free of base points, $rA_g$ is the fixed part of the linear system $|rK_{X'}|$. Finally, it turns out that $f^*(K_X) = g^*(K_Y) + (A_g - A_f)$ is a Zariski decomposition.

1-C. **Log canonical models of open manifolds.** Let $U$ be a complex quasi-projective manifold of general type, in the sense of Iitaka [25]. By Hironaka’s resolution of singularities, there exists an open embedding $U \subset X$ such that $X$ is a proper manifold, and the complement $X \setminus U = \sum_i E_i$ is a divisor with simple normal crossings. The general type assumption means that the *log canonical divisor* $K_X + \sum_i E_i$ is big. The *log canonical ring* $R(X, K_X + \sum_i E_i)$ is independent of the choice of compactification, and in fact depends
only on the (proper) birational class of $U$. It is expected to be finitely generated, and if it is, we would obtain a natural birational map

$$\Phi: X \to Y := \text{Proj } R(X, K + \sum_i E_i).$$

As before, we can find a Hironaka hut with the extra property that $\text{Exc}(f) \cup f^{-1}(\sum_i E_i)$ is a simple normal crossings divisor $\sum_i E_i$. Denote $B_Y = g_*(\sum_i E_i)$. We imitate the arguments in the compact case, and obtain isomorphisms

$$\begin{align*}
R(X', K_{X'} + \sum_i E_i') & \xrightarrow{\sim} R(Y, K_Y + B_Y) \\
R(X, K_X + \sum_i E_i) & \xrightarrow{\sim} R(Y, K_Y + B_Y)
\end{align*}$$

Again, $|r(K_Y + B_Y)|$ defines a normally generated embedding for some $r \geq 1$, and we have Zariski decompositions $K_{X'} + \sum_i E_i' = g^*(K_Y + B_Y) + A_g$ and $f^*(K_X + \sum_i E_i) = g^*(K_Y + B_Y) + (A_g - A_f)$. One can see that $\Phi^{-1}$ contracts no divisors of $Y$, and $\Phi_*(\sum_i E_i) = B_Y$. The pair $(Y, B_Y)$ is log canonically polarized, and its singularities are log canonical, as we will see shortly. The pair $(Y, B_Y)$ is called the log canonical model of $U$.

For example, let $U \to M_g$ be a resolution of singularities of the moduli space of smooth curves of genus $g \geq 2$. Then the log canonical model of $U$ is $(\overline{M}_g, \delta)$, where $\overline{M}_g$ is the moduli space of stable curves of genus $g$, and $\delta = \overline{M}_g \setminus M_g$ [4].

1-D. Log canonical models of log manifolds, log discrepancies. Log manifolds are the bridge between open and compact manifolds. They are pairs $(X, \sum_i b_i E_i)$, where $X$ is nonsingular, the $E_i$’s are nonsingular divisors intersecting transversely, and $b_i \in [0, 1] \cap \mathbb{Q}$ for all $i$. We call $\sum_i b_i E_i$ the boundary of the log manifold, and denote it by $B$. Suppose moreover that $(X, B)$ is of general type, that is the log canonical divisor $K_X + B$ is big. The log canonical ring $R(X, K_X + B)$ is expected to be finitely generated, and if it is, we would obtain a birational map

$$\Phi: X \to Y := \text{Proj } R(X, K_X + B).$$

Again, we construct a Hironaka hut with the extra property that $\text{Exc}(f) \cup f^{-1}(\sum_i E_i)$ is a simple normal crossings divisor. Let $\cup_j F_j$ be the exceptional locus of $f$ and denote $B_Y = g_*(f^{-1}B + \sum_j F_j)$. We imitate the previous argument, and obtain isomorphisms

$$\begin{align*}
R(X', K_{X'} + f^{-1}B + \sum_j F_j) & \xrightarrow{\sim} R(Y, K_Y + B_Y) \\
R(X, K_X + B) & \xrightarrow{\sim} R(Y, K_Y + B_Y)
\end{align*}$$

There exists $r \in \mathbb{Z}_{\geq 1}$ such that $|r(K_Y + B_Y)|$ defines a normally generated embedding, and we have Zariski decompositions

$$K_{X'} + f^{-1}B + \sum_j F_j = g^*(K_Y + B_Y) + A_g$$

$$f^*(K_X + B) = g^*(K_Y + B_Y) + (A_g - A_f).$$

One can see that $\Phi^{-1}$ contracts no divisors of $Y$, and $\Phi_*(B) = B_Y$. The birational model $\Phi: (X, B) \to (Y, B_Y)$ is called the log canonical model of $(X, B)$. It is polarized by the
log canonical divisor $K_Y + B_Y$ (a $\mathbb{Q}$-divisor), and its singularities are called log canonical singularities. The coefficients of the $\mathbb{Q}$-divisor $A_g$ are called log discrepancies.

2. LOG VARIETIES, MINIMAL LOG DISCREPANCIES

Log varieties with log canonical singularities are locally open subsets of log canonical models. For technical purposes, it is better to work in a slightly more general context, such as non-rational boundaries (to take convex hulls and limits of divisors), or even non-log canonical singularities (to construct a flat log structure at a prescribed point of a polarized manifold, cf §4-A).

Definition 2.1. A log variety $(X, B)$ is a complex normal variety $X$ endowed with an effective $\mathbb{R}$-Weil divisor $B = \sum_i b_i E_i$ such that $K_X + B$ is $\mathbb{R}$-Cartier.

Recall that the canonical divisor $K_X = (\omega)$ is the Weil divisor of zeros and poles of a non-zero top rational differential form $\omega$ (it depends on the choice of $\omega$, but only up to linear equivalence). The $E_i$'s are prime divisors and the $b_i$'s are non-negative real numbers. The $\mathbb{R}$-Cartier assumption means that locally on $X$, $K_X + B$ equals a finite sum $\sum_i r_i(\varphi_i)$, where $r_i \in \mathbb{R}$ and $\varphi_i \in \mathbb{C}(X)^\times$.

Let now $\mu: X' \to X$ be birational morphism, and $E \subset X'$ a prime divisor. We use the same form to define the canonical class of $X'$, that is $K_{X'} = (f^*\omega)$. The log discrepancy of $(X, B)$ at $E$ is defined as

$$a(E; X, B) = \text{mult}_E(K_{X'} + E - \mu^*(K_X + B)) \in \mathbb{R}.$$  

The log discrepancy depends only on the valuation that $E$ induces on $\mathbb{C}(X)$. We call such valuations geometric, and denote $c_X(E) = \mu(E)$. For example, if $E$ is a prime divisor in $X$, then $a(E; X, B) = 1 - \text{mult}_E(B)$.

Definition 2.2. The minimal log discrepancy of a log variety $(X, B)$ at a Grothendieck point $\eta \in X$ is defined as

$$a(\eta; X, B) = \inf\{a(E; X, B); c_X(E) = \eta\} \in \{-\infty\} \cup \mathbb{R}_{\geq 0}.$$  

The global minimal log discrepancy is $a(X, B) = \inf_{\eta \in X} a(\eta; X, B)$.

Example 2.3. $a(0; \mathbb{C}^d) = d$.

The reader may check that $a(\eta; X, B) < 0$ implies $a(\eta; X, B) = -\infty$. Otherwise, $a(\eta; X, B)$ is a non-negative real number, and the infimum is a minimum. If $\eta$ has codimension one, then $a(\eta; X, B) = 1 - \text{mult}_\eta(B)$. Otherwise, construct a log resolution $\mu: X' \to X$ such that $\mu^{-1}(\eta)$ is a divisor, and $\mu^{-1}(\eta), \mu_*^{-1}B$ and Exc$(\mu) = \cup_j F_j$ are all supported by a simple normal crossings divisor. Then $a(\eta; X, B) = \min_{\mu(F_j) = \eta} a(F_j; X, B)$.

The log pullback formula

$$\mu^*(K_X + B) = K_{X'} + \mu_*^{-1}B + \sum_j (1 - a(F_j; X, B))F_j$$

shows that $a(\eta; X, B) \in \frac{1}{r}\mathbb{Z}$ if $r(K_X + B)$ is a Cartier divisor near $\eta$.

Definition 2.4. A log variety $(X, B)$ has log canonical singularities at $\eta$ if $a(\eta; X, B) \geq 0$.

We say that $(X, B)$ has log canonical singularities if $a(X, B) \geq 0$.

Example 2.5. Consider the log variety $(X, \sum_i b_i E_i)$, where $X$ is a manifold, $\sum_i E_i$ a simple normal crossings divisor and $b_i \in [0, 1]$ for all $i$. Then $a(X, \sum_i b_i E_i) = \min_i (1 - b_i)$.  

Example 2.6. Let $X$ be a toric variety and $X \setminus T = \bigcup_i E_i$ the complement of the torus. Then $(X, \sum_i E_i)$ is a log variety with $a(X, \sum_i E_i) = 0$ and $K_X + \sum_i E_i = 0$.

If $a(X, B) \geq 0$, the infimum in its definition is also a minimum. With the notation above, $a(X, B) = \min(\min_i a(E_i; X, B), \min_j a(F_j; X, B))$. Moreover, the formula

$$K_{X'} + \mu_*^{-1} B + \sum_j F_j = \mu^*(K_X + B) + \sum_j a(F_j; X, B)F_j$$

becomes a Zariski decomposition of the relative log manifold of general type $(X', (\mu^{-1})^* B + \sum_j F_j) \rightarrow X$. Its log canonical model in the sense of §1 is $(X/X, B)$, where $X/X$ is the identity morphism.

Definition 2.7. Let $(X/S, B)$ be a log variety $(X, B)$ with $a(X, B) \geq 0$, endowed with a projective morphism $\pi: X \rightarrow S$. We say that $(X/S, B)$ is a

- log Fano if $-(K_X + B)$ is $\pi$-ample.
- log Calabi-Yau if $K_X + B$ is $\pi$-numerically trivial.
- log canonical model if $K_X + B$ is $\pi$-ample.

It is useful to observe that a log variety $(X, B)$ has log canonical singularities at $\eta$ if and only if $(X/X, B)$ is a log canonical model in a neighborhood of $\eta$ (cf. Example 4.6). Note however that the three geometric types above coincide if $\pi$ is the identity map.

Definition 2.8. A germ of log variety $P \in (X, B)$ is called flat if $a(P; X, B) = 0$.

The typical example of flat germ is $0 \in (\mathbb{C}^d, \sum_{i=1}^d H_i)$, where $H_i$ are the coordinate hyperplanes. Our terminology is inspired by an analogy between germs and projective manifolds, where if the minimal log discrepancy corresponds to the Kodaira dimension, flat germs correspond to manifolds with Kodaira dimension zero (an elliptic curve, for example). Also, note that $0 \in (\mathbb{C}, 1 \cdot 0)$ is the only flat log germ in dimension one.

2-A. Examples of minimal log discrepancies. Minimal log discrepancies can be easily computed for log varieties $(X, B)$ such that $X$ is a toric variety and $B$ is supported by the complement of the torus (see [47] for standard terminology on toric varieties). We only consider here $\mathbb{Q}$-factorial, log canonical toric germs of log varieties

$$P \in (X, B) = (T_N \text{emb}(\sigma), \sum_{i=1}^d b_i H_i).$$

They are in one-to-one correspondence with the following data:

- $\sigma = \{x \in \mathbb{R}^d; x_1, \ldots, x_d \geq 0\}$.
- $N \subset \mathbb{R}^d$ is a lattice, containing $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ as primitive vectors.
- $(b_1, \ldots, b_d) \in [0, 1]^d$.

The following basic facts provide lots of examples of minimal log discrepancies:

(a) $a(\eta H_i; X, B) = 1 - b_i$.
(b) Let $x \in N^{prim} \cap \sigma$ be a primitive vector. Then $x$ defines a barycentric subdivision $\Delta_x$ of $\sigma$, and the exceptional locus of the induced birational map $T_N \text{emb}(\Delta_x) \rightarrow T_N \text{emb}(\sigma)$ is a prime divisor $E_x$. Then $a(E_x; X, B) = \sum_{i=1}^d (1 - b_i)x_i$.
(c) Log resolutions exists in the toric category. Therefore minimal log discrepancies can be computed using only valuations $E_x$ as in (b).
Example 2.10. Suppose planes. For the cycle $C$ Suppose Example 2.9.

Example 2.11. For each point $x \in \mathbb{Q}^d \cap (0, 1]^d$, we construct a $d$-dimensional germ of toric log variety $P_x \in (X_x, B_x)$, with minimal log discrepancy

$$a(P_x; X, B_x) = \min_{n \geq 0} \sum_{i=1}^{d} (1 + nx_i - \lfloor nx_i \rfloor).$$

Choose a positive integer $q$ such that $qx \in \mathbb{Z}^d$, and define integers

$$n_j = \frac{\gcd(q, qx_1, \ldots, qx_j)}{\gcd(q, qx_1, \ldots, qx_d)} \quad (1 \leq j \leq d).$$

These integers are independent of the choice of $q$. Let $\sigma \subset \mathbb{R}^d$ be the standard positive cone, and $P_x \in X_x := T_{\mathbb{Z}^d + \mathbb{Z}_x} \text{emb}(\sigma)$ the unique point fixed by the torus. The primitive
vectors of the lattice $\mathbb{Z}^d + \mathbb{Z} \cdot x$ along the rays of $\sigma$ are
\[
\frac{1}{n_1}(1, 0, \ldots, 0), \ldots, \frac{1}{n_d}(0, \ldots, 0, 1).
\]
They define invariant prime divisors $H_1, \ldots, H_d$ in $X$. Set $B_x = \sum_{i=1}^d (1 - \frac{1}{n_i}) H_i$.

It is natural to “compactify” $\mathbb{Q}^d \cap (0, 1]^d$ to $\mathbb{Q}^d \cap [0, 1]^d \setminus 0$, in order to study the limiting behaviour of $a(P_x; X, B_x)$ [9, 5].

The minimal log discrepancy of a toric germ is the “local” version of Minkowski’s first minimum of a convex body about the origin (see [13] for the definition and basic properties of the latter invariant). In our setting, if we denote by $\Delta$ the convex set
\[
\{ x \in \mathbb{R}^d; x_1, \ldots, x_d \geq 0, \sum_{i=1}^d (1 - b_i)x_i \leq 1 \},
\]
then
\[
a(P; X, B) = \sup \{ t \geq 0; N \cap \text{int}(t\Delta) = \emptyset \}.
\]
See [11] for more on minimal log discrepancies versus lattice-point-free convex bodies.

3. PROBLEMS ON MINIMAL LOG DISCREPANCIES

Minimal log discrepancies originate in the problem of termination of log flips: starting with a given log variety, can we perform log flips infinitely many times? Log flips are surgery operations which preserve codimension one cycles, and improve the singularities of higher codimensional cycles. As a measure of this improvement, log discrepancies may only increase after a log flip, and some of them increase strictly [51]. This is the heuristic behind the termination of a sequence of log flips, and it lead Shokurov [51, 52, 55] to question the existence of an infinite increasing sequence of minimal log discrepancies.

First, we fix a log variety $(X, B)$, and investigate the set of minimal log discrepancies of all prime cycles of $X$ [4]. The formula $a(\eta_C; X, B) = a(P; X, B) - \dim(C)$, for a general closed point $P$ of a given prime cycle $C \subset X$, shows that closed points contain the essential information. Consider now the minimal log discrepancy $a(P; X, B)$ as a function on the set of the closed points $P \in X$. This function has a finite image, and in particular the set of minimal log discrepancies of all prime cycles of $X$ is finite. Moreover, the level sets $\{ P \in X; a(P; X, B) \leq t \}$ ($t \geq 0$) are constructible. Simple examples, such as a Du Val singularity $P \in X$, with $a(x; X) = 2$ for $x \neq P$, and $a(P; X) = 1$, suggest that these level sets are in fact closed.

**Conjecture 3.1** ([3]). The minimal log discrepancy $a(P; X, B)$ is lower semi-continuous as a function on the closed points $P$ of $X$.

This behaviour is confirmed in several cases: a) $\dim(X) \leq 3$ [3, 4]; b) $(X, B)$ is a toric log variety [4]; c) $X$ is a local complete intersection [19, 13]. Also, it is equivalent to the inequality $a(P; X, B) \leq a(\eta_C; X, B) + 1$, for every closed point on a curve in $X$ [3].

Now consider the general case, when log flips change the log variety $(X, B)$ in codimension at least two. The coefficients of the boundary are preserved, so we may assume that they belong to a given finite set. More generally, let $\mathcal{B} \subset [0, 1]$ be a set satisfying the \textit{descending chain condition} ($\mathcal{B} = \{ 1 - \frac{1}{n}; n \geq 1 \} \cup \{ 1 \}$ is the typical example), and define
\[
\text{Mld}(d, \mathcal{B}) = \{ a(P; X, B); \dim(X) = d, \text{ coefficients of } B \text{ belong to } \mathcal{B} \}.
\]
The set $\text{Mld}(1, \mathcal{B}) = \{ 1 - b; b \in \mathcal{B} \}$ clearly satisfies the ascending chain condition.

**Conjecture 3.2** (Shokurov [52, 53]). The following properties hold:
Then, if \( a(P; X, B) = 1 \) (Shokurov, unpublished). An important special case of this problem is the classification of terminal 3-fold singularities, \( \text{Mld}(3) \leq 2 \). See also [26] for a higher dimensional reduction to a global problem on adjoint linear systems or the boundedness of log Fano varieties. This was confirmed for surfaces [53, 1, and toric log varieties [9, 5]. By the classification of terminal 3-fold singularities, \( \text{Mld}(3, \{0\}) \cap (1, \infty) = \{1 + \frac{1}{q}, q \geq 1\} \cup \{3\} \). Also, (2) holds if \( X \) is a local complete intersection \([19, 18]\). Shokurov [37] reduced the global problem of the termination of flips to the two local problems 3.1 and 3.2. An interesting problem posed by Shokurov is to relate the minimal log discrepancy and the Cartier index of a singularity. Suppose \( X \) is the germ of a \( d \)-fold with \( a(P; X) \geq 1 \). If \( nK_X \sim 0 \) and Conjecture 3.2 (2) holds, then the minimal log discrepancy can take at most finitely many values: \( a(P; X) \in \{0, \frac{1}{n}, \ldots, \frac{n}{d}\} \). Conversely, is there an integer \( n \), depending only on \( d \) and \( a(P; X) \), such that \( nK_X \sim 0 \)? The answer is positive if \( d = 2 \) (Shokurov, unpublished). An important special case of this problem is \( a(P; X) = 0 \). If \( d = 2 \), then \( n \in \{1, 2, 3, 4, 6\} \). If \( d = 3 \), then \( \varphi(n) \leq 20 \) and \( n \neq 60 \), where \( \varphi \) is the Euler number [26]. See also [21] for a higher dimensional reduction to a global problem on log Calabi-Yau varieties in one dimension less. The boldest conjecture here is the following.

**Conjecture 3.3 (Shokurov [56]).** Let \( P \in (X, B) \) be a log germ such that \( a(P; X, B) = 0 \) and \( B \) has coefficients in a set \( B \subset [0, 1] \cap \mathbb{Q} \) satisfying the descending chain condition. Then \( n(K + B) \sim 0 \) for some positive integer \( n \), depending only on \( \dim(X) \) and \( B \).

A useful formula in inductive arguments in the log category is a comparison of minimal log discrepancies under adjunction, called precise inversion of adjunction.

**Conjecture 3.4 (Shokurov [51, Kollár [53]).** Let \( P \in S \subset (X, B) \) be the germ of a log variety and a normal prime divisor \( S \) with \( \text{mult}_S(B) = 1 \). By adjunction, we have \( (K_S + B)|_S = K_S + B_S \). Then \( a(P; X, B) = a(P; S, B_S) \).

It follows from the Log Minimal Model Program if \( a(P; X, B) \leq 1 \) [33], and it holds if \( X \) is a local complete intersection [19, 18].

Minimal log discrepancies appear naturally in global contexts, such as Fujita’s Conjecture on adjoint linear systems or the boundedness of log Fano varieties.

**Conjecture 3.5 (A. and L. Borisov [8–Alexeev [2]).** Let \( \epsilon \in (0, 1] \) and \( d > 1 \). Then log varieties \( X \) with \( -K_X \) ample, \( a(X) \geq \epsilon \) and \( \dim(X) = d \), form a bounded family.

This is known in several cases: a) \( X \) is toric [8]; b) \( X \) nonsingular [34]; c) \( d = 2 \) [2]; d) \( d = 3, \epsilon = 1 \) [31, 38]; e) \( d = 3, \) and the index of \( K_X \) is fixed [12].

3-A. Toric case. In the assumptions and notations of § 2-A we illustrate some of the local problems on minimal log discrepancies. For lower semi-continuity, it is enough to see that \( a(P; X, B) \leq a(\eta_C; X, B) + 1 \) for a torus-invariant curve \( P \in C \). Suppose \( C \) corresponds to the face \( \tau = \sigma \cap (x_d = 0) \). There exists \( (x', 0) \in N^{prim} \cap \text{relint}(\sigma) \) such that \( a(\eta_C; X, B) = \sum_{i=1}^{d-1} (1 - b_i)x_i \). Then \( (x', 1) \in N \cap \text{int}(\sigma) \) and there exists \( x \in N^{prim} \).
and a positive integer $l$ with $lx = (x', 1)$. We have

$$a(E_x; X, B) \leq la(E_x; X, B) = \sum_{i=1}^{d-1} (1 - b_i)x'_i + 1 - b_d \leq a(\eta_C; X, B) + 1.$$ 

Therefore $a(P; X, B) \leq a(\eta_C; X, B) + 1$.

For precise inversion of adjunction, suppose $B = \sum_{i=1}^{d-1} b_iH_i + H_d$. Then $S = H_d$ is the toric variety $T_{N_d}\text{emb}(\sigma_d)$, where $\sigma_d = \{x \in \mathbb{R}^{d-1}; x_1, \ldots, x_{d-1} \geq 0\}$ and $N_d = \{x \in \mathbb{R}^{d-1}; t \in \mathbb{R}, (x, t) \in N\}$. To bring this to the normal form in §2-A, note that there are positive integers $n_1, \ldots, n_{d-1}$ such that $\frac{1}{n_i}(0, \ldots, 1, \ldots, 0)$ are primitive vectors of $N_d^{prim}$. Then $S = T_{N'}\text{emb}(\sigma')$, where $N' = \{x' \in \mathbb{R}^{d-1}; (n_1x'_1, \ldots, n_{d-1}x'_{d-1}) \in N_d\}$ and $\sigma'$ is the usual positive cone. Let $H'_1, \ldots, H'_{d-1}$ the torus invariant prime divisors of $S$. The key observation is that the log canonical divisor $K_X + B = \sum_{i=1}^{d-1} -(1 - b_i)H_i$ is independent of $H_d$. The boundary induced on $S$ by adjunction is $B_S = \sum_{i=1}^{d-1} (1 - \frac{1-b_i}{n_i})H'_i$, and the equality $a(P; X, B) = a(P; S, B_S)$ is clear.

Finally, for the ascending chain condition, assume by contradiction that we have a strictly increasing sequence $a^1 < a^2 < a^3 < \cdots$, where $a^n = a(P^n; T_{N'}\text{emb}(\sigma))$ for $n \geq 1$. For simplicity, assume that the boundary is zero, so only the lattice changes. We may find $x^n \in (0, 1]^d \cap N^n$ such that $a^n = \sum_{i=1}^{d} x^n_i$. In particular, $a^n \leq d$ for all $n$. Consider now the strictly increasing sequence of open sets $U^n = \{x \in (0, +\infty)^d; \sum_{i=1}^{d} x_i < a^n\}$. By [40], $G^n = \{x \in \mathbb{R}^d; U^n \cap (\mathbb{Z}^d + \mathbb{Z}x) = \emptyset\}$ is the union of finitely many closed subgroups containing $\mathbb{Z}^d$ (the Flatness Theorem of Khinchin [28] gives an alternative proof). We have $G^n \supseteq G^{n+1}$ since $a^n < a^{n+1}$ and $x^n \in G^n \setminus G^{n+1}$, so we obtain a strictly decreasing sequence of finite unions of closed subgroups containing $\mathbb{Z}^d$. This is impossible, since the latter set satisfies the descending chain condition.

3-B. Methods. The toric case (see [13, 10]) suggests that behind the ascending chain condition of minimal log discrepancies lies a deeper fact, the boundedness of singularities with minimal log discrepancy bounded away from zero. Some log canonical singularities are classified in low dimension, but in general we could only expect general structure theorems and boundedness results in terms of minimal log discrepancies. For example, Du Val singularities are classified as follows: $A_n, D_n, E_6, E_7, E_8$. From the above point of view, Du Val singularities are nothing but surface singularities having minimal log discrepancy at least 1, and they come in two types: a 1-dimensional series with two components $(A$ and $D$), and a 0-dimensional series $(E)$.

The known method for bounding germs $P \in X$ is to study the singularities at $P$ of the linear systems $| - mK_X|$ $(m \geq 1)$, and reduce this local problem to the global problem of bounding Fano or log Calabi-Yau varieties in one dimension less [50, 48]. Given that §1 suggests that minimal log discrepancies are invariants of objects of general type, it also natural to investigate the singularities at $P$ of the linear systems $|mK_X|$ $(m \geq 1)$ and $|m_{P, X}|$ $(m \geq 1)$, and relate germs with log canonical models in one dimension less.

Also, it is possible that minimal log discrepancies can be understood from several points of view: analytic, birational, motivic or p-adic. The motivic interpretation of minimal log discrepancies is known in the case when the canonical divisor is Q-Cartier [44, 60]. As for the analytic side, the description of log discrepancies as the coefficients of a Zariski decomposition suggests an interpretation of minimal log discrepancies in terms of Lelong
numbers. For example, the upper bound of Conjecture 3.2 (2) is essentially equivalent to the following problem: let \((X, B)\) be a log manifold of general type having a Zariski decomposition \(K_X + B = P + F\), such that \(F\) is supported by the components of \(B\) with coefficient one. Then some coefficient of \(F\) is at most \(\dim X\).

4. The log canonical threshold

**Definition 4.1.** Let \((X, B)\) be a log variety, \(\eta \in X\) a Grothendieck point such that \(a(\eta; X, B) \geq 0\), and \(D\) an effective \(\mathbb{R}\)-Cartier \(\mathbb{R}\)-divisor passing through \(\eta\). The *log canonical threshold* of \(D\) with respect to \(\eta \in (X, B)\) is defined as

\[
\lct(\eta \in (X, B); D) = \sup \{ t \geq 0; a(\eta; X, B + tD) \geq 0 \}.
\]

The *global log canonical threshold* is \(\lct(X, B; D) = \min_{\eta \in X} \lct(\eta \in (X, B); D)\).

Note that \(a(\eta; X, B) \geq 0\) if and only if \((X, B)\) has log canonical singularities at \(\eta\). Since the support of \(B + tD\) is a fixed divisor \(S\), we may compute the log canonical threshold on a log resolution of \(X\) and \(S\). In particular, \(\lct(\eta \in (X, B); D) \in \mathbb{Q}\) if \(B\) and \(D\) are rational. The inequality \(\lct(P \in (X, B); D) \leq \lct(\eta_C \in (X, B); D)\) holds for a closed point of a prime cycle \(P \in C \subset X\), with equality if \(P\) is general in \(C\).

**Example 4.2.** Let \(\dim X = 1\) and \(P \in X\). Then \(\lct(P \in (X, B); D) = \frac{a(P; X, B)}{\mult P(D)}\).

**Example 4.3** ([16, 23]). Let \(0 \in H: (f = 0) \subset \mathbb{C}^d\) be a hypersurface. Then

\[
\lct(0 \in \mathbb{C}^d; H) = \sup \{ t \geq 0; |f|^{-t} \text{ is } L^2 \text{ near } 0 \}.
\]

The reciprocal number \(\mu(0; H) = 1/\lct(0 \in \mathbb{C}^d; H)\), called Arnold multiplicity, satisfies the inequalities \(\mu(0; H) \leq \mult(0; H) \leq d \cdot \mu(0; H)\).

**Example 4.4.** Let \(H_1, \ldots, H_d\) be the coordinate hyperplanes in \(\mathbb{C}^d, b_1, \ldots, b_d \in [0, 1]\) and \(n_1, \ldots, n_d \in \mathbb{Z}_{\geq 1}\). Then

(i) \(\lct(0 \in (\mathbb{C}^d, \sum_{i=1}^d b_i H_i); (x_1^{n_1} \cdots x_d^{n_d})) = \min_{i=1}^d 1 - \frac{b_i}{n_i}\),

(ii) \(\lct(0 \in (\mathbb{C}^d, \sum_{i=1}^d b_i H_i); (x_1^{n_1} + \cdots + x_d^{n_d})) = \min(1, \sum_{i=1}^d 1 - \frac{b_i}{n_i})\).

**Example 4.5.** Log canonical thresholds of non-degenerate hypersurfaces in toric varieties have a combinatorial description. Let \(P \in (X, B) = (T_N \emb(\sigma), \sum_{i=1}^d b_i H_i)\) be a toric germ as in §2-A, and

\[
f = \sum_{\alpha} \lambda_\alpha x_\alpha^{m_\alpha} \in m_{P,X} \setminus 0.
\]

Here \(m_\alpha\) are finitely many points in \(M \cap \sigma^\vee\), where \(M\) is the lattice dual to \(N\) and \(\sigma \subset M_{\mathbb{R}}\) is the cone dual to \(\sigma\). Note that \((1 - b_1, \ldots, 1 - b_d) \in \sigma^\vee\). Let \(\square\) be the convex hull of the \(\cup_{\alpha}(m_\alpha + \sigma^\vee)\). The ray \(\mathbb{R}_{\geq 0}(1 - b_1, \ldots, 1 - b_d)\) intersects \(\square\) for the first time in a point \(\mu(1 - b_1, \ldots, 1 - b_d)\). We have the following inequality

\[
\lct(P \in (X, B); (f = 0)) \leq \min(1, \frac{1}{\mu}),
\]

and equality holds if the coefficients \(\lambda_\alpha\) are sufficiently general. The generality condition can be made explicit, as in [6].
Example 4.6. The log canonical threshold \( \text{lct}(P \in (X, B); D) \) is the largest \( t \geq 0 \) such that \((X/X, B + tD)\) is a relative log canonical model near \( P \). For example, consider the germ of the cuspidal curve \( 0 \in C : (x^2 - y^3 = 0) \subset \mathbb{C}^2 \), with log canonical threshold \( \frac{5}{6} \). The cusp can be resolved by a composition of three blow-ups \( \mu : X \to \mathbb{C}^2 \). Let \( E_1, E_2, E_3 \) be the proper transforms on \( X \) of the exceptional divisors, in the order of their appearance. We obtain a singular model \( X \to Y \to \mathbb{C}^2 \) by contracting \( E_1 \) and \( E_2 \), and let \( C' \) and \( E'_3 \) be the proper transforms of \( C \) and \( E_3 \) on \( Y \). The relative log variety \((X, E_1 + E_2 + E_3 + t\mu^{-1}(C)) \to \mathbb{C}^2\) is of general type for \( t \in [0, 1] \cap \mathbb{Q} \), and its log canonical model is \((\mathbb{C}^2, tC') \to \mathbb{C}^2\) for \( t \leq \frac{5}{6} \), and \((Y, E'_3 + tC') \to \mathbb{C}^2\) for \( t > \frac{5}{6} \). On the other hand, \((\mathbb{C}^2, tC) \to \mathbb{C}^2\) is a log canonical model on \( \mathbb{C}^2 \setminus 0 \), for every \( t \in [0, 1] \cap \mathbb{Q} \).

4-A. Problems on log canonical thresholds. One can classify a singularity by constructing a flat log structure on it, in an effective way. Log canonical thresholds appear naturally in this construction, as the coefficients of the boundary of the flat log structure.

For example, consider a log germ \( P \in (X, B) \) with \( a(P; X, B) \geq 0 \). If \( a(P; X, B) = 0 \), the log germ is flat. Otherwise, for a general hypersurface \( D_1 \in |m_{P,X}| \), the log canonical threshold \( \gamma_1 = \text{lct}(P \in (X, B); D_1) \) is defined in such a way so that \( a(P; X, B + \gamma_1 D_1) \geq 0 \), and there exists a minimal cycle \( C \supset P \) such that \( a(\eta_C; X, B + \gamma D) \geq 0 \). If \( C = \{P\} \), then \( P \in (X, B + \gamma D) \) is flat. Otherwise, we repeat this process (at most \( \dim(C) \) times) until we obtain a flat log structure \( 0 \in (X, B + \sum \gamma_i D_i) \). Flat germs with certain restrictions on their boundaries are expected to be bounded in a certain sense (cf. §3), so for effective results we should also control the \( \gamma_i \)'s. Shokurov observed that these coefficients satisfy the ascending chain condition in dimension two, and used this to construct 3-fold log flips.

Conjecture 4.7 (Shokurov [54], Kollár [37]). For a positive integer \( d \) and a set \( 0 \in B \subset [0, 1] \) satisfying the descending chain condition, define

\[
\text{Lct}(d, B) = \{ \text{lct}(P \in (X, B); D); \dim(X) = d, \text{coefficients of } B \text{ belong to } B, D \in |m_{P,X}| \}.
\]

The following properties hold:

1. The set \( \text{Lct}(d, B) \) satisfies the ascending chain condition.

2. Assume that \( B \cap [1, 1 - \frac{1}{k}] \) is a finite set for every \( k \geq 1 \). Then the set of accumulation points of \( \text{Lct}(d, B) \) is \( \text{Lct}(d - 1, B') \setminus \{1\} \), for a suitable set \( B' \subset [0, 1] \).

This limiting behaviour is known in several cases: a) \( d = 2 \) [54]; b) \( d = 3 \) [2, 42]; c) \( P \in X \) is toric, \( B \) is invariant and \( D \in |m_{P,X}| \) is a general hypersurface [27]. The set \( \text{Lct}_{d} := \text{Lct}(d, \{0\}) \) has an explicit description in some cases: a) \( \text{Lct}_1 = \{ \frac{1}{n}; n \in \mathbb{Z}_{\geq 1} \} \); b) \( \text{Lct}_2 \cap [\frac{1}{2}, 1] = \{ \frac{1}{2} + \frac{1}{n}; n \geq 3 \} \cup \{1\} \) [39]; c) \( \text{Lct}_3 \cap [\frac{41}{12}, 1] = \{ \frac{41}{12}, 1 \} \) [35]. Conjecture 4.7 would imply that the number \( 1 - \epsilon_d = \max \text{Lct}_{d} \cap (0, 1) \) is well defined, and Kollár [35] suggests that \( \epsilon_d \) is the minimal degree of a log canonical model \( (Y, B_Y) \) of dimension \( d - 1 \), with \( B_Y \) having coefficients in \( \{1 - \frac{1}{n}; n \geq 1\} \cup \{1\} \). For example, \( \epsilon_1 = \frac{1}{2}, \epsilon_2 = \frac{1}{6}, \epsilon_3 = \frac{1}{12} \).

For the relationship between Conjectures 3.2, 3.5 and 4.7, see [7].

Flat log structures have more general applications as well. Given a closed point \( P \) on a polarized manifold \((X, H)\), does there exists \( n \geq 1 \) and \( D \in |nH| \) with \( a(P; X, \frac{1}{n}D) = 0 \)? If such a flat structure exists, then \( P \) is not in the base locus of any divisor \( L \) such that \( L - K_X - H \) ample. This is a powerful technique to study adjoint line bundles, parallel to the \( L^2 \)-methods for singular hermitian metrics in complex geometry (see [30, 17, 37, 3] and [58, 16] for the algebraic and analytic side of the story, respectively). As above, log
canonical thresholds appear naturally. For a log variety \((X, B)\), an ample \(\mathbb{Q}\)-divisor \(H\) and a Grothendieck point \(\eta \in X\), define
\[
\gamma(\eta \in (X, B); H) = \inf \{ \text{lct}(\eta \in (X, B); D) ; \ D \in |nH|, n \geq 1 \}
\]
and \(\gamma(X, B; H) = \inf_{\eta \in X} \gamma(\eta \in (X, B); H)\) (cf. [59]).

**Example 4.8.** Suppose \(\dim(X) = 1\). For every closed point \(P \in X\) we have
\[
\gamma(\eta \in (X, B); H) = \frac{a(P, X, B)}{\deg(H)} \leq \frac{1}{\deg(H)}
\]
and \(\gamma(X, B; H) = \frac{a(X, B)}{\deg(H)}\).

**Lemma 4.9 ([31, 37]).** \(\gamma(P \in (X, B); H) \sqrt{H^d} \leq d\) for every closed point \(P \in X^d\).

For Fujita’s Conjecture on adjoint linear systems, \(\gamma(P \in (X, B); H)\) has to be small. This is achieved by Lemma 4.9 if \(\sqrt{H^d}\) is large. On the other hand, \(\gamma(P \in (X, B); H)\) cannot be too small in a bounded context, and this can be used to bound \(\sqrt{H^d}\) from above. Some known cases are:

a) \(\gamma(\eta_X \in X; -K_X) \geq \frac{1}{d+1}\) if \(X\) is a Fano manifold of dimension \(d\) [23];
b) \(\gamma(G; H) \geq 1\) if \(\mathcal{C}_G(H)\) is the Plücker line bundle on the Grassmanian [24];
c) \(\gamma(A; \Theta) \geq 1\) for a principally polarized abelian variety [30]. Pukhlikov [49] suggests that a Fano variety \(X\) with large \(\gamma(X; -K_X)\) is birationally rigid. Hwang [23] suggests that for a log canonical model \(X\), \(\gamma(\eta_X \in X; K_X) \sqrt{K_X^d}\) is also bounded away from zero, only in terms of dimension. The level sets and critical points of the functional \(\gamma(\eta \in (X, B); H)\), and an extension of Faltings’ product theorem [20, 45, 23], should play a key role in constructing flat log structures.

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RIMS, Kyoto University, Kyoto 606-8502, Japan.
E-mail address: ambro@kurims.kyoto-u.ac.jp