PROPERTIES OF HIGHER ORDER PREINVEX FUNCTIONS

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Abstract. In this paper, we define and introduce some new concepts of the higher order strongly preinvex functions and higher order strongly monotone operators involving an arbitrary bifunction. Some new relationships among various concepts of higher order strongly preinvex functions have been established. We have shown that the optimality conditions for the preinvex functions can be characterized by class of higher order strongly variational-like inequalities, which appears to be new ones. As a novel applications of the higher order strongly preinvex functions, we have obtained the parallelogram-like laws for the uniformly Banach spaces. As special cases, one can obtain various new and known results from our results. Results obtained in this paper can be viewed as refinement and improvement of previously known results.

1. Introduction. Several extensions and generalizations of the classical convexity have been considered. Hanson [7] introduced the concept of invex function for the differentiable functions, which played significant part in the mathematical programming. Ben-Israel and Mond [2] introduced the concept of invex set and preinvex functions. It is known that the differentiable preinvex function are invex functions. The converse also holds under certain conditions, see [11]. Noor [15] proved that the minimum of the differentiable preinvex functions on the invex set can be characterized by a class of variational inequalities, which is known as the variational-like inequality. For the recent developments in variational-like inequalities and invex equilibrium problems, see [15, 16] and the references therein. These results have inspired a great deal of subsequent work, which has expanded the role and applications of the invexity in nonlinear optimization and engineering sciences. Noor and Noor [18] investigated the properties of the strongly preinvex functions and their variant forms.

Strongly convex functions were introduced and studied by Polyak [20], which played an important part in the optimization theory and related areas. For the applications of strongly convex functions in various fields of pure and applied sciences, for example, see [1, 8, 9, 10, 12, 13, 15, 17, 20, 21, 25] and the references therein. Lin and Fukushima [10] introduced the concept of higher order strongly convex functions and used it in the study of mathematical program with equilibrium constraints. Higher order strongly generalized convex functions play an important role in many fields such as engineering design, economic equilibrium and multilevel game. Noor and Noor [17, 19], Mohsen et al [12] and Alabdali et al [1] introduced

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some new classes of higher order strongly convex functions and studied their characterizations.

Inspired by the research work going in this field, we introduce and consider another class of nonconvex functions with respect to an arbitrary bifunction. This class of nonconvex functions is called the higher order strongly preinvex functions. Several new concepts of monotonicity are introduced. We establish the relationship between these classes and derive some new results under some mild conditions. The optimality conditions of the differentiable higher order strongly preinvex functions can be characterized by a class of higher order variational-like inequalities, which appears to be a new ones. Our results represent a significant refinement and improvement of the results of Alabdali et al [1] and Lin and Fukushima [10] and include the results of Mohsen et al [12] as special cases. As novel and innovative applications of these higher order strongly preinvex functions, we have obtained the parallelogram-like laws for uniformly Banach spaces. As special cases, on can obtain various new and refined versions of known results. It is expected that the ideas and techniques of this paper may stimulate further research in this field.

2. Preliminary Results. Let $K$ be a nonempty closed set in a real Hilbert space $H$. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ be the inner product and norm, respectively. Let $F : K_{\eta} \to R$ be a continuous function and let $\eta(\cdot, \cdot) : K_{\eta} \times K_{\eta} \to R$ be an arbitrary continuous bifunction.

Definition 2.1. [2] The set $K_{\eta}$ in $H$ is said to be invex set with respect to an arbitrary bifunction $\eta(\cdot, \cdot)$, if

$$u + \lambda \eta(v, u) \in K_{\eta}, \quad \forall u, v \in K_{\eta}, \lambda \in [0, 1].$$

The invex set $K_{\eta}$ is also called $\eta$-connected set. Note that the invex set with $\eta(v, u) = v - u$ is a convex set $K$, but the converse is not true. For example, the set $K_{\eta} = R - (-1, 1)$ is an invex set with respect to $\eta$, where

$$\eta(v, u) = \begin{cases} v - u, & \text{for } v > 0, u > 0 \text{ or } v < 0, u < 0 \\ u - v, & \text{for } v < 0, u > 0 \text{ or } v > 0, u < 0. \end{cases}$$

It is clear that $K_{\eta}$ is not a convex set.

From now onward $K_{\eta}$ is a nonempty closed invex set in $H$ with respect to the bifunction $\eta(\cdot, \cdot)$, unless otherwise specified.

We now introduce some new concepts of strongly preinvex functions and their variants forms, which is the main motivation of this paper.

Definition 2.2. The function $F$ on the invex set $K_{\eta}$ is said to be higher order strongly preinvex with respect to the bifunction $\eta(\cdot, \cdot)$, if there exists a constant $\mu > 0$, such that

$$F(u + \lambda \eta(v, u)) \leq (1 - \lambda)F(u) + \lambda F(v) - \mu \{\lambda^p (1 - \lambda) + \lambda (1 - \lambda)^p\} \|\eta(v, u)\|^p, \quad (1)$$

$$\forall u, v \in K_{\eta}, \lambda \in [0, 1], p \geq 1.$$

The function $F$ is said to be higher order strongly preconcave, if and only if, $-F$ is higher order strongly preinvex function. Consequently, we have a new concept.

Definition 2.3. A function $F$ is said to be higher order strongly affine preinvex involving an arbitrary bifunction $\eta(\cdot, \cdot)$, if there exists a constant $\mu > 0$, such that

$$F(u + \lambda \eta(v, u)) = (1 - \lambda)F(u) + \lambda F(v) - \mu \{\lambda^p (1 - \lambda) + \lambda (1 - \lambda)^p\} \|\eta(v, u)\|^p, \quad (2)$$

$$\forall u, v \in K_{\eta}, \lambda \in [0, 1], p \geq 1.$$
Note that every higher order strongly convex function is a higher order strongly preinvex, but the converse is not true. For the applications of the higher order strongly preinvex functions in semi-infinite mathematical programming problems with equilibrium constraints, see [8]. If \( \eta(v, u) = v - u \), then the higher order strongly generalized preinvex function becomes higher order strongly convex functions, that is,

**Definition 2.4.** A function \( F \) is a said to be a higher order strongly convex function, if there exists a constant \( \mu > 0 \), such that
\[
F(u + \lambda(v - u)) \leq (1 - \lambda)F(u) + \lambda F(v) - \mu \lambda\eta(1 - \lambda)\|v - u\|^p, \\
\forall u, v \in K_\eta, \lambda \in [0, 1].
\]

For the properties of the higher order strongly convex functions in variational inequalities and equilibrium problems, see Noor [15, 16, 19].

If \( p = 2 \), then Definition 2.2 becomes:

**Definition 2.5.** A function \( F \) is said to strongly preinvex functions, if
\[
F(u + \lambda\eta(v, u)) \leq (1 - \lambda)F(u) + \lambda F(v) - \mu \lambda(1 - \lambda)\|\eta(v, u)\|^2, \\
\forall u, v \in K_\eta, \lambda \in [0, 1].
\]

For applications in variational inequalities and equilibrium problems, see [15, 16] and the references therein.

**Definition 2.6.** The function \( F \) on the invex set \( K \) is said to be higher order strongly quasi-preinvex with respect to the bifunction \( \eta(\cdot, \cdot) \), if there exists a constant \( \mu > 0 \), such that
\[
F(u + \lambda\eta(v, u)) \leq \max\{F(u), F(v)\} - \mu \lambda\eta(1 - \lambda)\|\eta(v, u)\|^p, \\
\forall u, v \in K_\eta, \lambda \in [0, 1], p \geq 1.
\]

**Definition 2.7.** The function \( F \) on the invex set \( K \) is said to be higher order strongly log-preinvex with respect to the bifunction \( \eta(\cdot, \cdot) \), if there exists a constant \( \mu > 0 \), such that
\[
F(u + \lambda\eta(v, u)) \leq (F(u))^{1-\lambda}(F(v))^\lambda - \mu \lambda\eta(1 - \lambda)\|\eta(v, u)\|^p, \\
\forall u, v \in K_\eta, \lambda \in [0, 1], p \geq 1,
\]
where \( F(\cdot) > 0 \).

From the above definitions, we have
\[
F(u + \lambda\eta(v, u)) \leq (F(u))^{1-\lambda}, (F(v))^\lambda - \mu \lambda\eta(1 - \lambda)\|\eta(v, u)\|^p \\
\leq (1 - \lambda)F(u) + \lambda F(v) - \mu \lambda\eta(1 - \lambda)\|\eta(v, u)\|^p \\
\leq \max\{F(u), F(v)\} - \mu \lambda\eta(1 - \lambda)\|\eta(v, u)\|^p.
\]

This shows that every higher order strongly log-preinvex function is higher order strongly preinvex function and every higher order strongly preinvex function is a higher order strongly quasi-preinvex function. However, the converse is not true.

For \( \lambda = 1 \), Definitions 2.2 and 2.7 reduce to the following condition, which is mainly due to Noor and Noor [15, 16].

**Condition A.**
\[
F(u + \eta(v, u)) \leq F(v), \quad \forall v \in K_\eta.
\]
Definition 2.8. The differentiable function $F$ on the invex set $K_\eta$ is said to be higher order strongly invex function with respect to the bifunction $\eta(\cdot, \cdot)$, if there exists a constant $\mu > 0$ such that
\[
F(v) - F(u) \geq \langle F'(u), \eta(v, u) \rangle + \mu \|\eta(v, u)\|^p, \quad \forall u, v \in K_\eta, p \geq 1,
\]
where $F'(u)$ is the differential of $F$ at $u$.

It is noted that, if $\mu = 0$, then the Definition 2.8 reduces to the definition of the invex function as introduced by Hanson [7]. It is well known that the concepts of preinvex and invex functions play a significant role in the mathematical programming and optimization theory, see [2, 7, 8, 11, 15, 16, 18, 19, 23, 24] and the references therein.

Definition 2.9. An operator $T : K_\eta \to H$ is said to be:

1. higher order strongly $\eta$-monotone, iff, there exists a constant $\alpha > 0$ such that
\[
\langle Tu, \eta(v, u) \rangle + \langle Tv, \eta(u, v) \rangle \leq -\alpha \{\|\eta(v, u)\|^p + \|\eta(u, v)\|^p\}, \quad u, v \in K_\eta.
\]
2. $\eta$-monotone, iff,
\[
\langle Tu, \eta(v, u) \rangle + \langle Tv, \eta(u, v) \rangle \leq 0, \quad \forall u, v \in K_\eta.
\]
3. higher order strongly $\eta$-pseudomonotone, iff, there exists a constant $\nu > 0$ such that
\[
\langle Tu, \eta(v, u) \rangle + \nu \|\eta(v, u)\|^p \geq 0 \Rightarrow -\langle Tv, \eta(u, v) \rangle \geq 0, \quad \forall u, v \in K_\eta.
\]
4. higher order strongly relaxed $\eta$-pseudomonotone, iff, there exists a constant $\mu > 0$ such that
\[
\langle Tu, \eta(v, u) \rangle \geq 0 \Rightarrow -\langle Tv, \eta(u, v) \rangle + \mu \|\eta(u, v)\|^p \geq 0, \quad \forall u, v \in K_\eta.
\]
5. strictly $\eta$-monotone, iff,
\[
\langle Tu, \eta(v, u) \rangle + \langle Tv, \eta(u, v) \rangle < 0, \quad \forall u, v \in K_\eta.
\]
6. $\eta$-pseudomonotone, iff,
\[
\langle Tu, \eta(v, u) \rangle \geq 0 \Rightarrow \langle Tv, \eta(u, v) \rangle \leq 0, \quad \forall u, v \in K_\eta.
\]
7. quasi $\eta$-monotone, iff,
\[
\langle Tu, \eta(v, u) \rangle > 0 \Rightarrow \langle Tv, \eta(u, v) \rangle \leq 0, \quad \forall u, v \in K_\eta.
\]
8. strictly $\eta$-pseudomonotone, iff,
\[
\langle Tu, \eta(v, u) \rangle \geq 0 \Rightarrow \langle Tv, \eta(u, v) \rangle < 0, \quad \forall u, v \in K_\eta.
\]

Definition 2.10. A differentiable function $F$ on the invex set $K_\eta$ is said to be higher order strongly pseudo $\eta$-invex function, iff, if there exists a constant $\mu > 0$ such that
\[
\langle F'(u), \eta(v, u) \rangle + \mu \|\eta(v, u)\|^p \geq 0 \Rightarrow F(v) - F(u) \geq 0, \quad \forall u, v \in K_\eta, p \geq 1.
\]

Definition 2.11. A differentiable function $F$ on $K_\eta$ is said to be higher order strongly quasi-invex function, iff, if there exists a constant $\mu > 0$ such that
\[
F(v) \leq F(u) \Rightarrow \langle F'(u), \eta(v, u) \rangle + \mu \|\eta(u, v)\|^p \leq 0, \quad \forall u, v \in K_\eta, p \geq 1.
\]

Definition 2.12. The function $F$ on the set $K_\eta$ is said to be pseudo-invex, iff,
\[
\langle F'(u), \eta(v, u) \rangle \geq 0 \Rightarrow F(v) \geq F(u), \quad \forall u, v \in K_\eta.
\]
Definition 2.13. The differentiable function $F$ on the $K_\eta$ is said to be quasi-invex, iff,
\[ F(v) \leq F(u) \Rightarrow \langle F'(u), \eta(v, u) \rangle \leq 0, \quad \forall u, v \in K_\eta. \]

If $\eta(v, u) = -\eta(v, u), \forall u, v \in K_\eta$, that is, the function $\eta(\cdot, \cdot)$ is skew-symmetric, then Definitions 2.9-2.13 reduce to the ones in [18, 23, 24]. This shows that the concepts introduced in this paper represent significant improvement of the previously known ones. All these new concepts may play important and fundamental part in the mathematical programming and optimization.

We also need the following assumption regarding the bifunction $\eta(\cdot, \cdot)$.

Condition C[11]. Let $\eta(\cdot, \cdot) : K_\eta \times K_\eta \to H$ satisfy assumptions
\[ \eta(u, u + \lambda \eta(v, u)) = -\lambda \eta(v, u) \]
\[ \eta(v, u + \lambda \eta(v, u)) = (1 - \lambda) \eta(v, u), \quad \forall u, v \in K_\eta, \lambda \in [0, 1]. \]

Clearly for $\lambda = 0$, we have $\eta(u, v) = 0$, if and only if $u = v, \forall u, v \in K_\eta$. One can easily show [23] that $\eta(u + \lambda \eta(v, u), u) = \lambda \eta(v, u), \forall u, v \in K_\eta$.

3. Main Results. In this section, we consider some basic properties of higher order strongly preinvex functions on the invex set $K_\eta$.

Theorem 3.1. Let $F$ be a differentiable function on the invex set $K_\eta$ in $H$ and let the condition C hold. Then the function $F$ is higher order strongly preinvex function, if and only if, $F$ is a higher order strongly invex function.

Proof. Let $F$ be a higher strongly preinvex function on the invex set $K_\eta$. Then
\[ F(u + \lambda \eta(v, u)) \leq (1 - \lambda)F(u) + \lambda F(v) - \mu \{\lambda^p(1 - \lambda) + (1 - \lambda)^p\} \|\eta(v, u)\|^p, \]
\[ \forall u, v \in K_\eta, \lambda \in [0, 1], p \geq 1. \]

which can be written as
\[ F(v) - F(u) \geq \\frac{\langle F'(u + \lambda \eta(v, u)), \eta(v, u) \rangle - \langle F'(u), \eta(v, u) \rangle}{\lambda} + \mu \{\lambda^{p-1}(1 - \lambda) + (1 - \lambda)^p\} \|\eta(v, u)\|^p. \]

Taking the limit in the above inequality as $\lambda \to 0$, we have
\[ F(v) - F(u) \geq \langle F'(u), \eta(v, u) \rangle + \mu \|\eta(v, u)\|^p. \]

This shows that $F$ is a higher order strongly invex function.

Conversely, let $F$ be a higher order strongly invex function on the invex set $K_\eta$. Then, $\forall u, v \in K_\eta, \lambda \in [0, 1], v_\lambda = u + \lambda \eta(v, u) \in K_\eta$ and using the condition C, we have

\[ F(v) - F(u + \lambda \eta(v, u)) \]
\[ \geq \langle F'(u + \lambda \eta(v, u)), \eta(v, u + \lambda \eta(v, u)) \rangle + \mu \|\eta(v, u + \lambda \eta(v, u))\|^p \]
\[ = (1 - \lambda)F'(u + \lambda \eta(v, u)), \eta(v, u)) + \mu(1 - \lambda)^p \|\eta(v, u)\|^p. \quad (4) \]

In a similar way, we have

\[ F(u) - F(u + \lambda \eta(v, u)) \]
\[ \geq \langle F'(u + \lambda \eta(v, u)), \eta(u + \lambda \eta(v, u)) + \mu \|\eta(u + \lambda \eta(v, u))\|^p \]
\[ = -\lambda F'(u + \lambda \eta(v, u)), \eta(v, u)) + \mu \lambda^p \|\eta(v, u)\|^p. \quad (5) \]
Multiplying (4) by \( \lambda \) and (5) by \((1 - \lambda)\) and adding the resultant, we have
\[
F(u + \lambda \eta(v, u)) \leq (1 - \lambda)F(u) + \lambda F(v) - \{\lambda^p(1 - \lambda) + \lambda(1 - \lambda)^p\}\|\eta(v, u)\|^p,
\]
showing that \( F \) is a higher order strongly preinvex function.

**Theorem 3.3.** Let \( F \) be differentiable higher order strongly preinvex function on the invex set \( K \). If \( F \) is a higher order strongly invex function, then
\[
\langle F'(u), \eta(v, u) \rangle + \langle F'(v), \eta(u, v) \rangle \leq -\mu\{\|\eta(v, u)\|^p + \|\eta(u, v)\|^p\}, \forall u, v \in K.
\]

**Proof.** Let \( F \) be a higher order strongly invex function on the invex set \( K \). Then
\[
F(v) - F(u) \geq \langle F'(u), \eta(v, u) \rangle + \mu\|\eta(v, u)\|^p, \quad \forall u, v \in K.
\]
Changing the role of \( u \) and \( v \) in (7), we have
\[
F(u) - F(v) \geq \langle F'(v), \eta(u, v) \rangle + \mu\|\eta(u, v)\|^p, \quad \forall u, v \in K.
\]
Adding (7) and (8), we have
\[
\langle F'(u), \eta(v, u) \rangle + \langle F'(v), \eta(u, v) \rangle \leq -\mu\{\|\eta(v, u)\|^p + \|\eta(u, v)\|^p\}, \forall u, v \in K.
\]
which shows that \( F'(\cdot) \) is a higher order strongly \( \eta \)-monotone operator.

We note that the converse of Theorem 3.2 is true only for \( p = 2 \). However, we have:

**Theorem 3.3.** If the differential \( F'(\cdot) \) is a higher order strongly \( \eta \)-monotone, then
\[
F(v) - F(u) \geq \langle F'(u), \eta(v, u) \rangle + \frac{2}{p}\mu\|\eta(v, u)\|^p.
\]

**Proof.** Let \( F'(\cdot) \) be higher order strongly \( \eta \)-monotone. From (9), we have
\[
\langle F'(v), \eta(u, v) \rangle \geq \langle F'(u), \eta(u, v) \rangle - \mu\{\|\eta(v, u)\|^p + \|\eta(u, v)\|^p\}.
\]
Since \( K \) is an invex set, \( \forall u, v \in K, \lambda \in [0,1] v_\lambda = u + \lambda \eta(v, u) \in K \). Taking \( v = v_\lambda \) in (10) and using Condition C, we have
\[
\langle F'(v_\lambda), \eta(u, u + \lambda \eta(v, u)) \rangle \leq \langle F'(u), \eta(u + \lambda \eta(v, u), u) \rangle
- \mu\{\|\eta(u + \lambda \eta(v, u), u)\|^p + \|\eta(u, u + \lambda \eta(v, u))\|^p\}
= -\lambda\langle F'(u), \eta(v, u) \rangle - 2\lambda^p\mu\|\eta(v, u)\|^p,
\]
which implies that
\[
\langle F'(v_\lambda), \eta(v, u) \rangle \leq \langle F'(u), \eta(v, u) \rangle + 2\lambda^p\mu\|\eta(v, u)\|^p.
\]
If \( g(\lambda) = F(u + \lambda \eta(v, u)) \). Then, from (11), we have
\[
\xi' (\lambda) = \langle F'(u + \lambda \eta(v, u)), \eta(v, u) \rangle \geq \langle F'(u), \eta(v, u) \rangle + 2\lambda^p\mu\|\eta(v, u)\|^p.
\]
Integrating (12) between 0 and 1, we have
\[
\xi(1) - \xi(0) \geq \langle F'(u), \eta(v, u) \rangle + \frac{2}{p}\mu\|\eta(v, u)\|^p,
\]
that is,
\[
F(u + \lambda \eta(v, u)) - F(u) \geq \langle F'(u), \eta(v, u) \rangle + \frac{2}{p}\mu\|\eta(v, u)\|^p.
\]
By using Condition A, we have
\[
F(v) - F(u) \geq \langle F'(u), \eta(v, u) \rangle + \frac{2}{p}\mu\|\eta(v, u)\|^p.
\]
the required result.

We now give a necessary condition for higher order strongly \( \eta \)-pseudo-invex function.

**Theorem 3.4.** Let \( F'(\cdot) \) be a higher order strongly relaxed \( \eta \)-pseudomonotone operator and Condition A and C hold. Then \( F \) is a higher order strongly \( \eta \)-pseudo-invex function.

*Proof.* Let \( F' \) be a higher order strongly relaxed \( \eta \)-pseudomonotone. Then, \( \forall u, v \in K_{\eta}, \)

\[
\langle F'(u), \eta(u, v) \rangle \geq 0,
\]
implies that

\[
-\langle F'(v), \eta(u, v) \rangle \geq \alpha \|\eta(u, v)\|^p. \tag{13}
\]
Since \( K \) is an invex set, \( \forall u, v \in K_{\eta}, \lambda \in [0, 1], v_\lambda = u + \lambda \eta(v, u) \in K_{\eta}. \) Taking \( v = v_\lambda \) in (13) and using condition Condition C, we have

\[
-\langle F'(u + \lambda \eta(v, u)), \eta(u, v) \rangle \geq \lambda \alpha \|\eta(v, u)\|^p. \tag{14}
\]
Let

\[
\xi(\lambda) = F(u + \lambda \eta(v, u)), \quad \forall u, v \in K_{\eta}, \lambda \in [0, 1].
\]
Then, using (14), we have

\[
\xi'(\lambda) = \langle F'(u + \lambda \eta(v, u)), \eta(u, v) \rangle \geq \lambda \alpha \|\eta(v, u)\|^p.
\]
Integrating the above relation between 0 to 1, we have

\[
\xi(1) - \xi(0) \geq \frac{\alpha}{2} \|\eta(v, u)\|^p,
\]
that is,

\[
F(u + \lambda \eta(v, u)) - F(u) \geq \frac{\alpha}{2} \|\eta(v, u)\|^p,
\]
which implies, using Condition A,

\[
F(v) - F(u) \geq \frac{\alpha}{2} \|\eta(v, u)\|^p,
\]
showing that \( F \) is a higher order strongly \( \eta \)-pseudo-invex function.

As special cases of Theorem 3.4, we have the following:

**Theorem 3.5.** Let the differentiable \( F'(u) \) of a function \( F(u) \) on the invex set \( K_{\eta} \) be higher order strongly \( \eta \)-pseudomonotone operator. If Conditions A and C hold, then \( F \) is higher order strongly pseudo \( \eta \)-invex function.

**Theorem 3.6.** Let the differential \( F'(u) \) of a function \( F(u) \) on the invex set \( K_{\eta} \) be higher order strongly \( \eta \)-pseudomonotone. If Conditions A and C hold, then \( F \) is higher order strongly pseudo \( \eta \)-invex function.

**Theorem 3.7.** Let the differential \( F'(u) \) of a function \( F(u) \) on the invex set \( K_{\eta} \) be higher order strongly \( \eta \)-pseudomonotone. If Conditions A and C hold, then \( F \) is higher order strongly pseudo \( \eta \)-invex function.

**Theorem 3.8.** Let the differential \( F'(u) \) of a function \( F(u) \) on the invex set \( K_{\eta} \) be \( \eta \)-pseudomonotone. If Conditions A and C hold, then \( F \) is higher order strongly pseudo \( \eta \)-invex function.
Theorem 3.9. Let the differential $F'(u)$ of a differentiable preinvex function $F(u)$ be Lipschitz continuous on the invex set $K_\eta$ with a constant $\beta > 0$. Then

$$F(u + \eta(v, u)) - F(u) \leq \langle F'(u), \eta(v, u) \rangle + \frac{\beta}{2} \|\eta(v, u)\|^2, \quad u, v \in K_\eta.$$ 

Proof. Its proof follows from Noor and Noor [22].

Definition 3.10. The function $F$ is said to be sharply higher order strongly pseudo-preinvex, if there exists a constant $\mu > 0$ such that

$$\langle F'(u), \eta(v, u) \rangle \geq 0 \Rightarrow F(v) \geq F(v + \lambda \eta(v, u)) + \mu \{\lambda^p (1 - \lambda) + \lambda (1 - \lambda)^p\} \|\eta(v, u)\|^p, \quad \forall u, v \in K_\eta, \lambda \in [0, 1].$$

Theorem 3.11. Let $F$ be a higher order strongly sharply pseudo-preinvex function on $K_\eta$ with a constant $\mu > 0$. Then

$$-\langle F'(v), \eta(v, u) \rangle \geq \mu \|\eta(v, u)\|^p, \quad \forall u, v \in K_\eta.$$ 

Proof. Let $F$ be a higher strongly sharply pseudo preinvex function on $K_\eta$. Then

$$F(v) \geq F(v + \lambda \eta(v, u)) + \mu \{\lambda^p (1 - \lambda) + \lambda (1 - \lambda)^p\} \|\eta(v, u)\|^p, \quad \forall u, v \in K_\eta, \lambda \in [0, 1],$$

from which we have

$$\frac{F(v + \lambda \eta(v, u)) - F(v)}{\lambda} + \mu \{\lambda^{p-1} (1 - \lambda) + (1 - \lambda)^p\} \|\eta(v, u)\|^p \leq 0.$$ 

Taking limit in the above-mentioned inequality, as $\lambda \to 0$, we have

$$-\langle F'(v), \eta(v, u) \rangle \geq \mu \|\eta(v, u)\|^p,$$

the required result.

Definition 3.12. A function $F$ is said to be a pseudo-preinvex function with respect to strictly positive bivariance $W(., .)$, if

$$F(v) < F(u) \Rightarrow F(u + \lambda \eta(v, u)) < F(u) + \lambda (\lambda - 1) W(v, u), \forall u, v \in K_\eta, \lambda \in [0, 1].$$

Theorem 3.13. If the function $F$ is higher order strongly preinvex function such that $F(v) < F(u)$, then the function $F$ is higher order strongly pseudo-preinvex.

Proof. Since $F(v) < F(u)$ and $F$ is higher order strongly preinvex function, then \( \forall u, v \in K_\eta, \lambda \in [0, 1], \) we have

$$F(u + \lambda \eta(v, u)) \leq \frac{F(u) + \lambda (F(v) - F(u))}{\lambda} - \mu \{\lambda^p (1 - \lambda) + \lambda (1 - \lambda)^p\} \|\eta(v, u)\|^p < F(u) + \lambda (1 - \lambda) (F(v) - F(u)) - \mu \{\lambda^p (1 - \lambda) + \lambda (1 - \lambda)^p\} \|\eta(v, u)\|^p = F(u) + \lambda (\lambda - 1) (F(u) - F(v)) - \mu \{\lambda^p (1 - \lambda) + \lambda (1 - \lambda)^p\} \|\eta(v, u)\|^p < F(u) + \lambda (\lambda - 1) W(u, v) - \mu \{\lambda^p (1 - \lambda) + \lambda (1 - \lambda)^p\} \|\eta(v, u)\|^p, \forall u, v \in K_\eta,$$

where $W(u, v) = F(u) - F(v) > 0$. This shows that the function $F$ is a higher order strongly pseudo-preinvex.
We now discuss the optimality for the differentiable higher order strongly preinvex functions.

**Theorem 3.14.** Let $F$ be a differentiable higher order strongly preinvex function with modulus $\mu > 0$. If $u \in K_\eta$ is the minimum of the function $F$, then

$$F(v) - F(u) \geq \mu \|\eta(v, u)\|^p, \quad \forall u, v \in K_\eta. \quad (15)$$

**Proof.** Let $u \in K_\eta$ be a minimum of the function $F$. Then

$$F(u) \leq F(v), \forall v \in K_\eta. \quad (16)$$

Since $K_\eta$ is an invex set, so, $\forall u, v \in K_\eta$, $\lambda \in [0, 1]$,

$$v_\lambda = u + \lambda \eta(v, u) \in K_\eta.$$

Taking $v = v_\lambda$ in (16), we have

$$0 \leq \lim_{\lambda \to 0} \left\{ \frac{F(u + \lambda \eta(v, u)) - F(u)}{\lambda} \right\} = \langle F'(u), \eta(v, u) \rangle. \quad (17)$$

Since $F$ is differentiable higher order strongly preinvex function, so

$$F(u + \lambda \eta(v, u)) \leq F(u) + \lambda(F(v) - F(u)) - \mu \{\lambda^p(1 - \lambda) + \lambda(1 - \lambda)^p\} \|\eta(v, u)\|^p, \forall u, v \in K_\eta,$$

from which, using (17), we have

$$F(v) - F(u) \geq \lim_{\lambda \to 0} \left\{ \frac{F(u + \lambda \eta(v, u)) - F(u)}{\lambda} \right\} + \mu \{\lambda^{p-1}(1 - \lambda) + (1 - \lambda)^p\} \|\eta(v, u)\|^p$$

$$= \langle F'(u), \eta(v, u) \rangle + \mu \|\eta(v, u)\|^p,$$

the required result (15). \qed

**Remark:** We would like to mention that, if $u \in K_\eta$ satisfies the inequality

$$\langle F'(u), \eta(v, u) \rangle + \mu \|\eta(v, u)\|^p \geq 0, \quad \forall u, v \in K_\eta, \quad (18)$$

then $u \in K_\eta$ is the minimum of the function $F$. The inequality of the type (18) is called the strongly variational-like inequality and appears to new one.

It is worth mentioning that inequalities of the type (18) may not arise as the minimization of the preinvex functions. This motivated us to consider a more general variational-like inequality of which (18) is a special case. To be more precise, for given operator $T$, bifunction $\eta(., .)$ and a constant $\mu > 0$, consider the problem of finding $u \in K_\eta$, such that

$$\langle Tu, \eta(v, u) \rangle + \mu \|\eta(v, u)\|^p \geq 0, \forall v \in K_\eta, p \geq 1, \quad (19)$$

which is called higher order strongly variational-like inequality. See also [15, 16, 19] for more details regarding applications of variational-like inequalities and related optimization problems. It is an interesting problem from both analytically and numerically point of view.
4. Applications. In this section, we derive new parallelogram laws for uniformly Banach spaces as a novel application of higher order strongly affine preinvex functions.

Taking $F(u) = \|u\|^p$ in Definition 2.3, we have

$$\|u + \lambda \eta(v, u)\|^p = (1 - \lambda)\|u\|^p + \lambda \|v\|^p - \mu \{\lambda^p(1 - \lambda) + \lambda(1 - \lambda)^p\}\|\eta(v, u)\|^p,$$

$$\forall u, v \in K_\eta, \lambda \in [0, 1], p \geq 1. \quad (20)$$

Taking $\lambda = \frac{1}{2}$ in (20), we have

$$\frac{1}{2}\|u + \eta(v, u)\|^p + \frac{1}{2}\|\eta(v, u)\|^p = \frac{1}{2}\|u\|^p + \frac{1}{2}\|v\|^p, \forall u, v \in K_\eta, \quad (21)$$

which implies that

$$\|2u + \eta(v, u)\|^p + \mu\|\eta(v, u)\|^p = 2^{p-1}\{|\|u\|^p + \|v\|^p\}, \forall u, v \in K_\eta, p \geq 1, \quad (22)$$

which is known as the parallelogram-like laws for the Banach spaces involving bi-function $\eta(v, u)$.

If $\eta(v, u) = v - u$, then (22) reduces to the parallelogram-like law as:

$$\|v + u\|^p + \mu\|v - u\|^p = 2^{p-1}\{|\|u\|^p + \|v\|^p\}, \forall u, v \in K,$$

which are known as the parallelogram-like law for the uniform Banach spaces involving the preinvex functions. Xi [22] obtained these characterizations of $p$-uniform convexity and $q$-uniform smoothness of a Banach space via the functionals $\|\|^{p}$ and $\|\|^q$, respectively. Bynum [3] and Chen et al [4, 6, 8] have studied the properties and applications of the parallelogram laws for the Banach spaces. For the applications of the parallelogram laws in Banach spaces in prediction theory and applied sciences, see [4, 5, 6, 8] and the references therein.

Conclusion. In this paper, we have introduced and studied some new classes of higher order strongly preinvex functions. These concepts are more general and unifying ones. Several new properties of these strongly preinvex functions are discussed and their relations with previously known results are highlighted. It is shown that the optimality conditions of the differentiable higher order strongly preinvex functions can be characterised by a class of variational-like inequalities. This result is used to introduce a more general class of higher order strongly variational-like inequalities (19). It is itself an interesting problem to develop some efficient numerical methods for solving higher-order variational inequalities along with applications in pure and applied sciences. New parallelogram laws for uniformly Banach spaces are obtained as novel and innovative applications of the higher order strongly preinvex functions. To investigate the applications of these parallelogram-like laws in prediction theory and related fields is an open problem. These functions can have important applications in area like machine learning. Despite the current activities, much clearly remains to be done in these fields. It is expected that the ideas and techniques of this paper may be starting point for future research activities.

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