An isomorphism between the fusion algebras of $V_L^+$ and type $D^{(1)}$ level 2

by Michael Cuntz and Christopher Goff*

Joint Mathematics Meetings
Washington DC, 2009
Fusion Algebras

• Commutative, associative, with 1
• Has Z-basis and well-behaved involution
• Similar to Grothendieck rings of semisimple tensor categories
• Arise in Representation Theory: of finite groups, f.d.s.s. Lie algebras, etc., including Kac-Moody Lie algebras (KMLAs) and vertex operator algebras (VOAs)
KMLAs

- Generalization of f.d.s.s. Lie algebras
- Useful in Conformal Field Theory
- We looked at type $D^{(1)}$, affinization of type $D$ (orthogonal) Lie algebra
- Kac-Peterson $S$-matrix (from modular group representation) determines the fusion algebra of irreducible representations
Kac-Peterson $S$-matrix

$$S_{\Lambda, \Lambda'} = c \sum_{w \in W^o} \det(w) \exp \left( -\frac{2\pi i \langle \Lambda + \rho | w(\Lambda' + \rho) \rangle}{k + h^\vee} \right)$$

- We explicitly calculated $S$ for type $D^{(1)}$, at level 2.
- (The Weyl group with respect to the appropriate basis consists of permutation matrices with an even number of -1 entries.)
- In previous work of Cuntz, he recognizes $S$ entries as determinants of certain matrices.
VOAs

- Algebraic relative of conformal field theory
- Fusion algebra of $V_L$ is $L^0/L$
- $V_L^+$ refers to the $Z_2$-orbifold
- We looked at rank one even lattices $L$
- Fusion rules worked out (for any positive definite even lattice) by Abe, Dong, and Li [2003]
Our Work

• We calculated the $S$-matrix.
• We showed that the $S$-matrix from the KMLA “was” the $S$-matrix for the VOA. Hence the corresponding fusion algebras are isomorphic.
• Apparently, this was known to physicists:
  – Fuchs, Schellekens, and Schweigert, ‘95
  – Schellekens and Yankielowicz, ‘89
  – Dijkgraaf, Verlinde, and Verlinde, ‘88
• We found a computational proof.