BLACK HOLES IN EINSTEIN-LOVELOCK GRAVITY*

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Abstract

Static, spherically symmetric solutions of the field equations for a particular dimensional continuation of general relativity with negative cosmological constant are found. In even dimensions the solution has many similarities with the Schwarzschild metric. In odd dimensions, the equations of motion are explicitly anti de-Sitter invariant, and the solution is alike in many ways to the 2+1 black hole.

This talk will be devoted to the study of lower and higher dimensional black holes in the Einstein-Lovelock theory of gravity. The results presented here have been developed in collaboration with C. Teitelboim and J. Zanelli [1,2]. I thank them for their great encouragement and guidance while this work was in preparation. Of course, errors and omissions in this report are my responsibility.

In dimensions greater than four the usual Einstein equations are not the most general equations that give rise to second order tensorial equations for the metric. In contrast, a large class of equations, parametrized by a number of independent dimensionful parameters, can be considered [3]. This equations are known as Einstein-Lovelock equations. In this work black-hole solutions for a particular class of these equations will be found.

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The construction of the most general set of equations for gravity in an arbitrary spacetime dimension $D$ is more easily done in the first order formalism. In this framework, the canonical variables are the matrix $e^a_{\mu}$ and the spin connection $w^{ab}_{\mu}$. These two fields are related with the metric and the Christoffel symbols by the change of coordinates

$$g_{\mu\nu} = e^a_{\mu} \eta_{ab} e^b_{\nu}$$

and

$$\Gamma^a_{\mu\nu} = e^a_{\rho} w^{ab}_{\nu} e^b_{\mu} + e^a_{\nu} e^a_{\mu,\nu}$$

where $e^a_{\mu}$ acts as the matrix of change of basis between the coordinate and orthonormal basis.

Consider the pair $\{ e^a_{\mu} d^\mu, w^{ab}_{\mu} d^\mu \}$ as a connection 1-form for a gauge theory for the anti-de Sitter group. This group has generators $J_a$ and $J_{ab}$ satisfying the commutation relations

$$[J_a, J_b] = l^{-2} J_{ab}$$

$$[J_{ab}, J_c] = J_a \eta_{bc} - J_b \eta_{ac}$$

$$[J_{ab}, J_{cd}] = -J_{ac} \eta_{bd} + J_{ad} \eta_{bc} + J_{bc} \eta_{ad} - J_{bd} \eta_{ac}$$

where $l$ is a parameter with dimensions of a length.

Define the 1-form gauge field $A$ as

$$A \equiv e^a_{\mu} J_a + \frac{1}{2} w^{ab}_{\mu} J_{ab}$$

and the 2-form Yang-Mills curvature as

$$F = dA + A \wedge A$$

$$\equiv F^a_{\mu} J_a + \frac{1}{2} F^{ab}_{\mu} J_{ab}$$

Using the commutation relations (3) it is simple to find $F^a_{\mu}$ and $F^{ab}_{\mu}$ in terms of $e^a_{\mu}$ and $w^{ab}_{\mu}$,

$$F^a_{\mu} = de^a_{\mu} + w^{a}_{\nu} e^b_{\mu}$$

$$F^{ab}_{\mu} = R^{ab}_{\mu} + l^{-2} e^a_{\nu} e^b_{\mu}$$
where $R^{ab} = dw^{ab} + w^a_c w^{cb}$ is the spacetime curvature tensor.

We will use the anti-de Sitter 2-form curvature $F$ to write the equations of motion. Consider first the three dimensional case. As the lagrangian is a 3-form, and the gauge field $A$ is a 1-form, the equations of motion must be 2-forms. By inspection it is easy to check that, if no other fields are included, the only anti-de Sitter invariant equations that one can write are

$$F = 0.$$ (8)

If Eq. (8) is written in terms of the metric using (2), one can easily prove that it is equivalent to the usual Einstein equation in three dimensions. Black-hole solutions for (8) exists [1]. The metric takes the form

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (N^\phi dt + d\phi)^2$$ (9)

where the squared lapse $N^2(r)$ and the angular shift $N^\phi(r)$ are given by

$$N^2(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}$$ (10)
$$N^\phi(r) = -\frac{J}{2r^2}$$ (11)

with $-\infty < t < \infty$, $0 < r < \infty$ and $0 \leq \phi \leq 2\pi$. The parameters $M$ and $J$ are, respectively, the mass and the angular momentum of the solution.

An interesting property of three dimensions is that a general solution for the equations (8) can be given. In fact, the expression

$$A = U^{-1} dU.$$ (12)

where $U$ is an element of the gauge group is the most general solution for (8). The group element $U$ that reproduces the black-hole metric (9) was found by Cangemi et al. [5]. Of course, $U$ is not trivial in the sense that it cannot be continuously deformed to the identity. (If this can be done, $A$ would not represent a physically interesting solution since it can be set equal to zero by a gauge transformation.)
From the point of view of the metric, the non-trivial topology of the manifold is seen from the fact that the metric (1) can be obtained from anti-de Sitter space by means of an identification with a discrete subgroup of the symmetry group. This identification changes the topology of anti-de-Sitter space and non-contractible loops appear [4].

Let us now study the higher odd-dimensional cases. In all odd dimensions the equations of motion must be a \((2n - 2)\)-form where \(2n - 1\) is the corresponding spacetime dimension. As the only tensor in the theory (without including external fields), is the curvature 2-form \(F\) it is not difficult to check that the only possible equations are

\[
\begin{align*}
F \wedge F &= 0 & D &= 5 \\
F \wedge F \wedge F &= 0 & D &= 7 \\
&\vdots \\
F \wedge \ldots \wedge F &= 0 & D &= 2n - 1
\end{align*}
\] (13)

It is remarkable that, although the equations (13) are much more complicated than the case \(F = 0\), it is still possible to solve them in closed form for a spherically symmetric static metric. For any odd dimension \(D = 2n - 1\) the black hole metric

\[
ds^2 = -\left[1 - \left(M + 1\right)^{\frac{1}{n-1}} + (r/l)^2\right]dt^2 + \frac{dr^2}{1 - \left(M + 1\right)^{\frac{1}{n-1}} + (r/l)^2} + r^2d\Omega^2
\] (14)

is an exact solution of Eqs. (13).

This geometry represents a natural extension of the three dimensional case (without angular momentum). A horizon exists only for positive masses and, in the special case \(M = -1\), the solution reduces to anti-de Sitter space. An important difference with the three dimensional case is that the curvature tensor for \(D > 3\) is no longer constant. Moreover, the curvature is singular at the origin although its effects are not observed from outside due to the presence of the horizon.

In the even dimensional case it is not possible to write a set of equations invariant under de anti-de Sitter group. This is simply observed from the fact that in even dimensions, the equations must be forms of odd degree and there are no odd tensor forms in the theory. It is necessary to break the symmetry under the full group. The best alternative is to consider
a set of equations invariant only under the Lorentz group as it is a subgroup of the anti-de Sitter group. (That can be seen from the fact that the $J_{ab}$ form a subalgebra in (3).) Thus, besides the curvature 2-form $F$, one has the 1-form $e^a$ that transforms as a vector under Lorentz transformations. In four dimensions the simplest equation that one can write is

$$F \wedge e = 0$$  \hspace{1cm} (15)

and it is a simple matter to check that (15) are equivalent to the usual Einstein equations with cosmological constant. In higher dimensions, a natural choice for the remaining equations is, in view of (13),

$$F \wedge F \wedge e = 0 \hspace{1cm} D = 6$$

$$F \wedge F \wedge F \wedge e = 0 \hspace{1cm} D = 8$$

$$\vdots$$

$$F \wedge F \wedge \ldots \wedge F \wedge e = 0 \hspace{1cm} D = 2n$$  \hspace{1cm} (16)

[Note that the equations (16) are the dimensionally continued version of the equations (13).]

Again, these equations can be solved in the spherically symmetric case obtaining the black hole geometry

$$ds^2 = - \left[ 1 - (2M/r)^{1/n-1} + (r/l)^2 \right] dt^2 + \frac{dr^2}{1 - (2M/r)^{1/n-1} + (r/l)^2} + r^2 d\Omega^2.$$  \hspace{1cm} (17)

This metric represents a natural extension for the Schwarzschild solution. There exists a horizon only for positive masses. Anti-de Sitter space, in this case, has zero mass. The curvature tensor is singular at the origin for non-zero mass but this singularity is hidden by the horizon.

It should be stressed here that the equations of motion written in (16) are not the most general equations that can be considered without breaking the Lorentz symmetry. For example, in six dimensions one can consider

$$F \wedge F \wedge e + a F \wedge e \wedge e \wedge e + b e \wedge e \wedge e \wedge e \wedge e = 0$$  \hspace{1cm} (18)
where $a$ and $b$ are arbitrary parameters. In this more general case the solution can still be found, but it cannot be written explicitly as the metric components are the roots of algebraic equations that cannot be solved in a unique way. Our case, which corresponds to take

$$a = b = 0,$$  \hspace{1cm} (19)

can be thought of as the dimensionally continued version of equations (13) for any even dimension. It is interesting to note that in the odd dimensional case, if one break down the symmetry to the Lorentz group, then the same arbitrariness in the coefficients appear. However, in odd dimensions there exist the privileged case with a larger symmetry that fixes the coefficients.

Note that the equations (13) and (16) do not coincide with the usual dimensionally continued Einstein equations which are linear in the curvature tensor,

$$F^\wedge e_\wedge \cdots \wedge e_{D-3} = 0.$$ \hspace{1cm} (20)

It is only in the special cases of $D = 3, 4$ when (20) coincides with (8) and (15).

The black hole solutions (14) and (17) admit electric charge. We will not write this solution here but only mention that a new curvature singularity develops, although it is also hidden by the horizon [2]. These black holes have thermodynamical properties similar to those in three and four dimensions. The temperature goes to infinity as the black hole dissapears in all even dimensions while in the odd dimensional case the temperature goes to zero. The entropy, on the other hand, is no longer proportional to the area of the black hole but it is still an increasing function of the black hole radius and mass [2].

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REFERENCES

[1] M. Bañados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett., 69(1992)1849.

[2] M. Bañados, C. Teitelboim and J. Zanelli, CECS/IAS preprint (1993), gr-qc/9307033,
   to appear in Phys. Rev. D.

[3] D. Lovelock, J. of Math. Phys. 12(1971)498.

[4] M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, CECS/IAS-preprint (1992),
   gr-qc/9302012, to appear in Phys. Rev. D.

[5] D. Cangemi, M. Leblanc and R.B. Mann, CTP#2162 preprint, to appear in Phys. Rev. D.