FACTORIZATION IN TOPOLOGICAL MONOIDS

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Abstract. The aim of this paper is sketch a theory of divisibility and factorisation in topological monoids, where finite products are replaced by convergent products. The algebraic case can then be viewed as the special case of discretely topologized topological monoids.

In particular, we define the topological factorisation monoid, a generalisation of the factorisation monoid for algebraic monoids, and show that it is always topologically factorial: any element can be uniquely written as a convergent product of irreducible elements.

1. A PRIMER ON FACTORIZATION IN DISCRETE MONOIDS

In this section, we give some basic definitions on the divisibility and factorisation theory of algebraic monoids. For additional information we refer to [3, 4].

An (algebraic) monoid \( H \) is a semi-group with a neutral element. In this paper, \( H \) is assumed to be abelian and cancellative. Unless otherwise stated, we write \( H \) multiplicatively and denote by \( 1 \in H \) its neutral element. However, the monoid \( \mathbb{N} = (\mathbb{N}, +) \) will be written additively, with \( 0 \in \mathbb{N} \) the neutral element. We will also write the monoid \( \mathbb{N}^X \) (with \( X \) a set) additively, along with its sub-monoids.

We denote the set of units in \( H \) by \( H^\times \), and say that \( H \) is reduced if \( H^\times = \{1\} \). Since \( H^\times \) is a subgroup of \( H \), we can form the factor monoid \( H_{\text{red}} = H/H^\times \), which is reduced. More explicitly, \( H_{\text{red}} = H/\sim \) where \( a \sim b \) iff \( a = eb \) for some \( e \in H^\times \). By passing to \( H_{\text{red}} \) if necessary, we will in what follows assume that \( H \) is reduced. We denote the set of non-units by \( H^* \), i.e. \( H^* = H \setminus \{1\} \). Since \( H \) is assumed to be both cancellative and reduced, it follows that it is also torsion-free.

If \( a, b \in H \) then we say that \( a \) divides \( b \), and write \( a \mid b \), if there exists a (necessarily unique) element \( c \in H \) such that \( ac = b \). An element \( p \in H^* \) is said said to be irreducible if \( p = a_1 \cdots a_r \) with \( a_j \in H \) implies that \( p = a_j \) for some \( j \). The irreducible elements in \( H \) are called atoms; we denote by \( \mathcal{A}(H) \) the set of all atoms in \( H \).

We say that \( p \in H^* \) is prime if whenever \( p \) divides \( a_1 \cdots a_r \), it divides some \( a_i \). Note that a prime element is always irreducible.

A monoid \( H \) is atomic if \( p \in H \) can be written as a finite product of atoms, and factorial if this factorisation is unique. In a factorial monoid, irreducible elements are prime. Hence unique factorisation into atoms implies unique factorisation into primes. A factorisation into

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primes is always unique, so if every element in \( H \) can be written as a product of primes, then \( H \) is factorial.

2. Infinite Products in Topological Monoids

We assume standard notions of topology and topological semigroups, as used for instance in [3, 6, 2].

In what follows, \( H \) will be a topological monoid, that is, a monoid with a topology on its underlying set such that the multiplication map \( H \times H \to H \) is continuous. Following [3], we assume that the topology on \( H \) is Hausdorff. Note that any algebraic monoid is a topological monoid when endowed with the discrete topology. We also assume that \( H \) is abelian, cancellative and reduced.

To be able to talk about infinite products, we make the following definition (this is the same definition as is used in [2], but differently phrased):

**Definition 2.1.** A (possibly infinite) product

\[
g = \prod_{j \in J} f_j, \quad f_j \in H, \quad J \text{ any set}
\]

is convergent iff the net \((\phi, \Delta)\) converges to \( g \), where \( \Delta \) is the directed set of all finite subsets of \( J \), and for \( F \in \Delta \), \( \phi(F) = \prod_{j \in F} f_j \).

In detail: for every neighbourhood \( U \) of \( g \), there is a finite subset \( S \subset J \), such that for all finite subsets \( S \subset T \subset J \) we have that \( \prod_{j \in T} f_j \in U \).

**Definition 2.2.** We say that \( H \) allows arbitrary decimation if whenever \( b = \prod_{s \in S} e_s \) is convergent, and \( T \subset S \), then \( \prod_{s \in T} e_s \) is convergent. We say that \( H \) allows finite decimation if \( \prod_{s \in H} e_s \) is convergent whenever \( S \setminus T \) is finite.

For instance, if \( H \) is complete, then \( H \) allows arbitrary decimation; this follows as in [3, III, §5.3, Proposition 3] (which treats the case of complete groups).

**Example 2.3.** Let \( \mathbb{R}^+ \) denote the monoid of non-negative real numbers, with the usual topology, and addition as the operation (which we’ll write additively). Let \( \mathbb{Q}^+ \) denote the sub-monoid of non-negative rational numbers. Then \( \mathbb{Q}^+ \) is reduced, cancellative, Hausdorff, but not complete, and it does not allow arbitrary decimation. To see this, consider the sum

\[
\sum_{k=1}^{\infty} 2^{-k} = 1,
\]

which (in \( \mathbb{R}^+ \)) can be decimated to yield any real number \( x \in [0, 1] \).

Henceforth, we assume that \( H \) allows arbitrary decimation

We can regard any infinite product of the form

\[
U = \prod_{i \in I} g_i^{\alpha(i)}, \quad \alpha(i) \in \mathbb{N}
\]

as an instance of (1) by taking \( J = I \) and \( f_j = g_i^{\alpha(i)} \), or by replacing \( g^{\alpha(i)} \) with \( \alpha(i) \) copies of \( g \) and enlarging the index set accordingly.

**Lemma 2.4.** Suppose that \( x = \prod_{i \in I} e_i^{\alpha(i)} \) is a convergent product in \( H \). Then if \( \alpha(i) \geq \beta(i) \) for all \( i \in I \) then the element \( y = \prod_{i \in I} e_i^{\beta(i)} \) is a divisor of \( x \).
Lemma 2.5. If $x \in H^*$ then the sequence $(x^n)_{n=0}^\infty$ diverges.

Proof. Suppose that $x^n \to a \in H$. Then $x \cdot x^n \to xa$, but we also have that $x \cdot x^n \to a$, hence $xa = a$. This is impossible since $H$ is cancellative and $x \neq 1$. \qed

Lemma 2.6. Suppose that $x = \prod_{i \in I} e_i^{\alpha(i)}$ is a convergent product in $H$. Then all $\alpha(i) < \infty$, and there is no infinite subset $J \subset I$ such that $k, \ell \in J \implies e_k = e_\ell$.

Proof. If $\prod_{i \in I} e_i^{\alpha(i)}$ is convergent, then so is all its sub-products. By Lemma 2.5, this means that no element can occur infinitely many times in the product. \qed

Theorem 2.7. Any convergent product $A = \prod_{j \in J} e_j$ can be expressed as

$$A = \prod_{h \in H^*} h^{m(h)}, \quad m(h) = \# \{ j \in J | e_j = h \} < \infty.$$  \hfill (4)

The study of infinite products in $H$ is thus reduced to the study of certain multisets on $H$. In the next section, we shall exploit a variant of this, when we consider multisets on the set of irreducible elements.

Definition 2.8. For $M \subset H$ we denote by $[M]^*$ the sub-monoid generated by all convergent products of elements in $M$.

Lemma 2.9. If $M \subset H$, then

$$M \subset [M] \subset [M]^* \subset \overline{[M]} \subset H$$ \hfill (5)

where $\overline{[M]}$ is the topological closure of $[M]$.

Proof. Note that the subspace $[M]$ is a sub-monoid of $H$, since the closure of a sub-monoid is a sub-monoid.

The only non-trivial inclusion is $[M]^* \subset \overline{[M]}$. Let, as in (1), $U = \prod_{j \in J} f_j$ be a convergent product, with $J$ any set; let $\Delta$ be the directed set of all finite subsets of $J$, let for $F \in \Delta$, $\phi(F) = \prod_{j \in F} f_j$, and let $(\phi, \Delta)$ be the corresponding net. By definition of convergent products, the net converges to $U \in H$. Since $\phi(F) = \prod_{j \in F} f_j \in [M]$ for all $F \in \Delta$ it follows that $U \in \overline{[M]}$. \qed

Example 2.10. Let

$$X = \{0\} \cup \{ 1/n | n \in \mathbb{N}^+ \} \subset [0, 1] \subset \mathbb{R}$$

be given the subspace topology, and let $H$ be the following abelian topological monoid. As an algebraic monoid, $H$ is the free abelian monoid on $X$; we denote by $e_0$ the basis vector corresponding to 0, and by $e_i$ the basis vector corresponding to $1/i$. There is a surjection

$$\phi : H \to \mathbb{R}$$

$$e_0 \mapsto 0$$

$$e_i \mapsto 1/i \quad \text{for } i > 0$$

We give $H$ the smallest topology such that $\phi$ is continuous, that is, a sequence $f_n \to f$ in $H$ iff $\phi(f_n) \to \phi(f)$ in $\mathbb{R}$. We will write the commutative monoid operation on $H$ additively.
Let $M = \{ e_n \mid n \in \mathbb{N}^+ \} \subset H$. Since $1/n \to 0$ in $\mathbb{R}$, $e_n \to e_0$ in $H$, hence $e_0 \in [M]^*$. We claim that $e_0 \not\in [M]^*$. To see this, let $g : X \to \mathbb{R}$ be the natural inclusion, which is of course continuous and closed. Suppose that $e_0 = \sum_{i \in I} e_{c_i}$ with $c_i \in \mathbb{N}^+$ is a convergent sum, then by continuity of $g$ we get that $0 = g(e_0) = g(\sum_{i \in I} e_{c_i}) = \sum_{i \in I} g(e_{c_i})$. But $g(e_{c_i}) > 0$ in the natural total order on $\mathbb{R}$, hence $\sum_{i \in I} g(e_{c_i}) > 0$, a contradiction.

**Definition 2.11.** We say that the topological monoid $H$ is *almost discrete* if all convergent products in $H$ are finite, that is, for all $M \subset H$, we have that $[M] = [M]^*$.

**Example 2.12.** Let $M$ be the multiplicative monoid of $K[[x_1, \ldots, x_n]]$, with the $\mathfrak{m}$-adic topology, and let $H = \{1\} \cup \{ f \in M \mid f(0) = 0 \}$. Then $H$ is almost discrete (any convergent infinite product of elements in $K[[x_1, \ldots, x_n]]$ will converge to $0 \not\in H$, but not discrete (every non-polynomial in $H$ is the limit of polynomials).

3. Factorisation: atoms, primes, and their topological counterparts

**Definition 3.1.** We say that $p \in H$ is *topologically prime* if whenever $p$ divides a convergent product, it divides some factor. We denote by $B(H)$ the set of topologically prime elements in $H$.

**Lemma 3.2.** Call $a \in H^*$ topologically irreducible if it can not be written as a convergent product of elements, all different from $a$. Then $a$ is topologically irreducible if and only if it is irreducible.

**Proof.** Since $H$ is reduced, $a$ is irreducible if and only if it can not be written as a finite product of non-units, all different from $a$. Thus if $a$ is topologically irreducible, it is irreducible.

For the converse, we need to use that $H$ allows finite decimation. Suppose that $a$ is irreducible, and that $a = \prod_{i \in I} e_{i}$. Let $j \in I$. Then $a = e_{j} b$, where $b = \prod_{i \in I \setminus \{j\}} e_{i}$. Since $a$ is irreducible, either $a = e_{j}$ or $a = b$. If $a = e_{j}$, we are done. If $a = b$, then $a = e_{j} b = b$, which is impossible since $H$ is cancellative. \qed

**Definition 3.3.** Suppose that

$$f = \prod_{i \in I} a_{i} = \prod_{j \in J} b_{j}$$

are two convergent factorisations of $f$ into non-units. We say that these factorisations are equivalent if

$$\forall h \in H^* : \quad \# \{ i \in I \mid a_{i} = h \} = \# \{ i \in I \mid a_{i} = h \}$$

**Definition 3.4.** $H$ is *topologically [atomic, prime atomic]* if every non-unit $h$ can be written as a convergent product of [atoms, topologically prime elements]. It is *topologically [factorial, prime factorial]* if any two such factorisations of $h$ are equivalent.

**Proposition 3.5.** Suppose that $H$ is topologically prime atomic, and let $x \in H^*$. Then the following are equivalent:

(i) $x$ is topologically prime,
(ii) $x$ is prime,
(iii) $x$ is irreducible.
Proof. It suffices to show that an irreducible element is topologically prime, so suppose that $x$ is irreducible. Since $H$ is topologically prime atomic, $x$ may be written as a convergent product of topologically prime elements. We claim that this product must have only one factor. Hence, $x$ is topologically prime.

To establish the claim, we argue by contradiction, and write $x = \prod_{i \in I} e_i$ with $e_i$ topologically prime, and where $I$ is not a singleton. Choose an $j \in I$ and put $y = e_j$ and $z = \prod_{i \in I \setminus \{j\}} e_i$. Since $H$ allows finite decimation, the latter product is convergent. Hence $x = yz$, in contradiction to the fact that $x$ is irreducible. \hfill \Box$

**Proposition 3.6.** Suppose that $H$ is topologically factorial. Then atoms in $H$ are prime.

**Proof.** Suppose that $x$ is an atom in $H$, and that $x | ab$, with $a, b \in H$. Then there exists $y \in H$ with $xy = ab$. Since $H$ is topologically factorial, we can uniquely factor $a, b, y$ into atoms:

$$a = \prod_{i \in I} u_i, \quad b = \prod_{j \in J} v_j, \quad y = \prod_{k \in K} w_k.$$ 

We can assume that $I, J, K$ are pair-wise disjoint. Applying [2, III, §5.3, Proposition 3], we have that

$$x \prod_{k \in K} w_k = \prod_{i \in I} u_i \times \prod_{k \in K} w_k = \prod_{\ell \in I \cup J} z_\ell, \quad z_\ell = \begin{cases} u_i & \text{if } \ell = i \in I, \\ v_j & \text{if } \ell = j \in J. \end{cases}$$

Since $H$ is topologically factorial, factorisation into atoms is unique, hence $x = z_\ell$ for some $\ell \in I \cup J$. Without loss of generality, assume that $\ell = i \in I$, so that $x = u_i$. Then the fact that $H$ allows finite decimation implies that $x | a$. \hfill \Box

**Definition 3.7.** $H$ allows dissociation if whenever $b = \prod_{s \in S} e_s$ is convergent, and for each $s \in S$, $e_s = \prod_{t \in G_s} f_t$ is convergent, then $b = \prod_{t \in G} f_t$ is convergent, where $G$ is the disjoint union of the $G_s$’s.

**Example 3.8.** There are lots of non-reduced, but cancellative and even complete monoids which do not allow expansion. For instance, in the additive group of the reals, if we put $e_k = k + (-k) = 0$ we have that $0 = \sum_{k \in \mathbb{Z}^+} (k + (-k))$, but $\sum_{k \in \mathbb{Z}} k$ is not convergent.

**Example 3.9.** Consider the monoid $\mathbb{Q}^+$ of Example 2.3. We claim that this monoid allows dissociation, but does not allow us to perform the reordering $\sum_{i \in I} \sum_{j \in J} e_{ij} = \sum_{j \in J} \sum_{i \in I} e_{ij}$. To establish the first claim, we note that in $\mathbb{R}^+$ all summable nets are countable, thus we need only to consider sequences. Furthermore, since everything is positive, all convergent sums in $\mathbb{R}^+$ are absolutely convergent. Thus, if $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = A$, then $\sum_{(i,j)} a_{ij} = A$. If the first sum has all summands in $\mathbb{Q}^+$ and converges to a rational value, then so does the second sum.

However, the “column sums” $\sum_{i=1}^{\infty} a_{ij}$ need no be rational.

We believe that our assumptions on $H$ (cancellative, reduced) are not enough to guarantee that $H$ allows dissociation, thus we postulate this in the next proposition.

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1That Proposition deals with topological groups, but the proof works verbatim for topological monoids.
**Proposition 3.10.** Suppose that $H$ is topologically factorial and allows dissociation. Then atoms in $H$ are topologically prime.

*Proof.* Suppose that $x$ is an atom in $H$. By the previous proposition, $x$ is prime. Suppose that $x \mid \prod_{i \in I} f_i$, with $f_i \in H$, and write each $f_i$ as a convergent product $f_i = \prod_{a \in \mathcal{A}(H)} a^{g_i(a)}$.

$$\prod_{i \in I} f_i = \prod_{i \in I} \prod_{a \in \mathcal{A}(H)} a^{g_i(a)},$$

but on the other hand, $\prod_{i \in I} f_i = xb$ for some $b \in H$. Write $b = \prod_{a \in \mathcal{A}(H)} a^{h(a)}$, then

$$\prod_{i \in I} \prod_{a \in \mathcal{A}(H)} a^{g_i(a)} = \prod_{(i,a) \in I \times \mathcal{A}(H)} a^{h(a)},$$

and since $H$ is topologically factorial, factorisation into atoms is unique, hence $x$ occurs in the right hand side. \hfill \Box

**Proposition 3.11.** If $b = \prod_{s \in S} e_s^{\alpha_s}$ is a convergent product in $H$ of topologically prime elements, then $\alpha_s$ is the maximal integer $r \geq 0$ such that $e_s^r \mid b$. Hence, the factorisation of an element into a convergent product of topologically prime elements, if it exists, is unique.

*Proof.* For any topologically prime element $p \in M$, we have that $p \mid b$ iff $p = e_s$ for some $s \in S$. To see this, first note that if $p = e_s$ then $b = e_s(e_s^{\alpha_s-1} \prod_{t \in S \setminus \{s\}} e_t^{\alpha_t})$. The right hand side is a product of $e_s$ and a convergent since $H$ allows finite decimation.

Conversely, if $p \mid b$ then by definition of topologically primeness there is an $s \in S$ such that $p \mid e_s^{\alpha_s}$. Applying the property of topologically primeness again, we get that $p \mid e_s$. Clearly, different topologically prime elements do not divide each other, hence $p \mid e_s$ iff $p = e_s$. It follows that $\alpha_s$ is the maximal integer $r$ such that $e_s^r \mid b$. \hfill \Box

**Corollary 3.12.** A topologically prime atomic monoid is topologically prime factorial.

So, we have the following implications:

topologically prime atomic $\iff$ topologically prime factorial $\implies$ topologically factorial.

The last implication can be reversed if $H$ allows dissociation.

4. **The Topological Factorisation Homomorphism**

Recall that the free abelian monoid $\mathcal{F}(\mathcal{A}(H))$ is called the factorisation monoid of $H$, and the canonical homomorphism

$$\pi_H : \mathcal{F}(\mathcal{A}(H)) \to H \quad (8)$$

is called the factorisation homomorphism. If $p \in H$, then the elements in $\pi_H^{-1}(p)$ are called the factorisations of $p$. This homomorphism gives a lot of information about the factorisation properties of $H$: we have that $H$ is atomic iff $\pi_H$ is surjective, and factorial iff $\pi_H$ is bijective.

We now make a construction which captures also the infinite factorisations. First, we introduce some notation for the topological monoid $\mathbb{N}^X$, the set of all functions $X \to \mathbb{N}$,
where \( X \) is any set. This is a topological monoid with the operation of point-wise addition (we'll write the operation additively), and the topology of point-wise convergence. It is also a partially ordered set with point-wise comparison.

**Definition 4.1.** For \( x \in X \), we define

\[
\chi_x : X \to \mathbb{N} \\
\chi_x(y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}
\]

Thus any \( f : X \to \mathbb{N} \) may be written as a convergent sum \( f = \sum_{x \in X} f(x)\chi_x \).

**Definition 4.2.** The partially defined map

\[
\pi_H : \mathbb{N}^{A(H)} \to H \\
\sum_{a \in A(H)} v(a)\chi_a \mapsto \prod_{a \in A(H)} a^{v(a)}
\]

is defined whenever the right-hand side of (9) is a convergent product. Denote by \( Z(H) \subset \mathbb{N}^{A(H)} \) the domain of definition of \( \pi_H \). We call \( Z(H) \) **topological factorisation monoid** of \( H \); note that it contains the factorisation monoid of \( H \), since the latter corresponds to the finitely supported maps \( A(H) \to \mathbb{N} \). In what follows, we regard \( \pi_H \) as a map \( Z(H) \to H \) and call it the **topological factorisation homomorphism**.

If \( p \in H \), then \( \pi_H^{-1}(p) \) is the set of (topological) factorisations of \( p \).

Clearly, \( H \) is topologically atomic iff \( \pi_H \) is surjective, and topologically factorial iff \( \pi_H \) is bijective.

For each \( a \in A(H) \), the projection map \( \text{pr}_a : Z(H) \to \mathbb{N} \) is defined by \( (e_a)_{v \in A(H)} \mapsto e_a \). We topologize \( Z(H) \) by giving it the initial topology with respect to \( \pi_H \) and all the projection maps \( \text{pr}_a \), where \( \mathbb{N} \) is discretely topologized.

Thus, \( Z(H) \) has the weakest topology such that \( \pi_H \) and all the projections \( \text{pr}_a \) are continuous, and a net \( f_i \) converges to \( f \) in \( Z(H) \) iff \( \pi_H(f_i) \to \pi_H(f) \) and \( \forall a \in A : \text{pr}_a(f_i) \to \text{pr}_a(f) \).

It is easy to see that this topology is Hausdorff.

**Lemma 4.3.** With respect to the component-wise partial order on \( \mathbb{N}^{A(H)} \), \( Z(H) \) is an order ideal, i.e. if \( c, d : A(H) \to \mathbb{N} \), \( c \in Z(H) \), and \( d(a) \leq c(a) \) for all \( a \in A \), then \( d \in Z(H) \).

**Proof.** This is a direct consequence of our assumption that \( H \) allows arbitrary decimation. \( \square \)

**Proposition 4.4.** \( Z(H) \) is a topological monoid, and \( \pi_H \) is an homomorphism of topological monoids.

**Proof.** To show that \( Z(H) \) is a algebraic monoid, we must show that if \( f, g \in Z(H) \) then \( h = f + g \in Z(H) \) is in fact in \( Z(H) \). So, we must show that \( \prod_{a \in A(H)} a^{h(a)} \) is convergent.

Let \( W \) be a neighbourhood of \( \pi_H(h) \in H \). Since multiplication in \( H \) is continuous, there is a neighbourhood \( U \) of \( \pi_H(f) \) and a neighbourhood \( V \) of \( \pi_H(g) \) such that \( UV \subset W \). Since \( \prod_{a \in A(H)} a^{f(a)} \) and \( \prod_{a \in A(H)} a^{g(a)} \) are convergent, there is a finite \( S \subset A(H) \) such that for any finite subset \( S \subset T \subset A(H) \) we have that \( \prod_{a \in T} a^{f(a)} \in U \) and \( \prod_{a \in T} a^{g(a)} \in V \), hence \( \prod_{a \in T} a^{h(a)} \in UV \subset W \). Thus, \( \prod_{a \in A(H)} a^{h(a)} \) is convergent, showing that \( Z(H) \) is an algebraic
monoid. The product obviously converges to $\pi_H(h)$, so $\pi_H$ is a homomorphism of algebraic monoids. By definition, $\pi_H$ is continuous.

It remains to see that addition in $Z(H)$ is continuous. Let $f_i \to f$, $g_i \to g$ in $Z(H)$, then by definition $\pi_H(f_i) \to \pi_H(f)$ and $pr_a(f_i) \to pr_a(f)$, and likewise for $g$. Since $H$ is a topological monoid, $\pi_H(f_i + g_i) = \pi_H(f_i)\pi_H(g_i) \to \pi_H(f)\pi_H(g) = \pi_H(f + g)$, and similarly $pr_a(f_i + g_i) \to pr_a(f + g)$. Thus $f_i + g_i \to f + g$. \qed

**Lemma 4.5.** $Z(H)$ is reduced, cancellative, and allows arbitrary decimation.

*Proof.* Since $\mathbb{N}^X$ is reduced and cancellative for all $X$, we have that $Z(H)$ is a submonoid of a reduced, cancellative monoid, and hence it is reduced and cancellative.

Let $\sum_{i \in I} g_i = h$ be a convergent sum in $Z(H)$, and let $J \subset I$. We want to show that $\sum_{i \in J} g_i$ is convergent, i.e. that the following two conditions hold:

1. $\prod_{i \in I} \pi_H(g_i)$ converges,
2. for all $a \in AH$, $\sum_{i \in J} g_i(a) < \infty$.

The first property follows since we know that $\prod_{i \in I} \pi_H(g_i)$ converges, and that $H$ allows arbitrary decimation. The second property follows since we know that $\sum_{i \in I} g_i(a) < \infty$. \qed

We now show that this definition generalises the discrete one.

**Theorem 4.6.** If $H$ is discrete, then so is $Z(H)$.

*Proof.* The fact that $H$ is discrete means that $Z(H)$ consists precisely of the finitely supported maps $A(H) \to H$. Suppose that $c_i \to c \in Z(H)$. We put $x = \pi_H(c)$, and note that since $\pi_H(c_i) \to \pi_H(c) = x$, and since $H$ is discrete, there is an $N_1$ such that $\pi(c_i) = x$ for all $i > N_1$.

We have that $c$ is supported on a finite set $A \subset A(H)$, and for each $a \in A$, there is an $M_a$ such that $pr_a(c_i) = pr_a(c)$ whenever $i > M_a$. Thus there is an $N_2$ such that $pr_a(c_i) = pr_a(c)$ whenever $i > N_2$ and $a \in A$.

Suppose now that $i > N_1, N_2$. Then

$$\pi_H(c_i) = \prod_{a \in A} a^{pr_a(c_i)} \prod_{b \in A, \neq A} b^{pr_b(c_i)} = \pi_H(c) = \prod_{a \in A} a^{pr_a(c)} \prod_{b \in A, \neq A} b^{pr_b(c)} = \prod_{a \in A} a^{pr_a(c)} ,$$

so

$$\prod_{b \in A, \neq A} b^{pr_b(c_i)} = 1,$$

which means that $pr_b(c_i) = 0$ for all $b \notin A$. Hence $c_i = c$ when $i > N_1, N_2$, so all convergent nets are stationary after a finite number of steps, which means that $Z(H)$ has the discrete topology. \qed

**Lemma 4.7.** The atoms (and the topologically prime elements) of $Z(H)$ are precisely the elements $\{\chi_a | a \in A(H)\}$

*Proof.* If $a \in A(H)$ then $\chi_a \in Z(H) \subset \mathbb{N}^{A(H)}$. Since $\chi_a$ is irreducible in $\mathbb{N}^{A(H)}$, it is irreducible in $Z(H)$. If $f \in Z(H)$, then $f = \sum_{a \in A(H)} f(a)\chi_a$, so if $f(a), f(b) \neq 0$ for $a \neq b$, then $f$ is not topologically irreducible, hence (Lemma 3.3) not irreducible. We have thus shown that $A(Z(H)) = \{\chi_a | a \in A(H)\}$.

By Lemma 4.3 we have that all $\chi_a$ are topologically prime. A topologically prime element is prime, hence irreducible, hence of the form $\chi_a$. \qed
Lemma 4.8. If $H$ is a topologically factorial, and if $H \ni f = \prod_{a \in \mathcal{A}(H)} a^{v(a)}$, $H \ni g = \prod_{a \in \mathcal{A}(H)} a^{w(a)}$, then $f \mid g$ iff $\forall a \in \mathcal{A}(H) : v(a) \leq w(a)$.

Proof. The underlying algebraic monoid of $H$ is isomorphic to $Z(H) \subseteq \mathbb{N}^{\mathcal{A}(H)}$, and $Z(H)$ is an order ideal.

Example 4.9. For a topologically factorial monoid the topological factorisation homomorphism is an isomorphism of algebraic monoids, but it need not be an isomorphism of topological monoids. As an example, if $H$ is the topological monoid of Example 2.10 then the factorisation homomorphism $\pi_H$ maps $a_0$ to $e_0$ and $a_n$ to $e_n$. Now, consider the sequence $f_i = a_i$ for $i > 1$. We have that $\pi_H(f_i) \to e_0 = \pi_H(f_0)$. However, $pr_{a_0}(f_i) = 0$ for all $i > 0$, so $pr_{a_0}(f_i) \to 0$, whereas $pr_{a_0}(f_0) = 1$. Thus, the inverse is not continuous.

Recall that for a discretely topologized monoid, the factorisation monoid $\mathcal{F}(\mathcal{A}(H))$ is free abelian. Similarly:

Theorem 4.10. $Z(H)$ is topologically prime factorial.

Proof. Each element $f \in Z(H)$ can be written uniquely as $f = \sum_{a \in \mathcal{A}(H)} \chi_a f(a)$; this sum is convergent with respect to the topology of point-wise convergence, and by construction, its image under $\pi_H$ is also convergent.

Moreover:

Theorem 4.11. $Z(Z(H)) \simeq Z(H)$ as topological monoids.

Proof. We define

$$\Xi : Z(H) \to \mathbb{N}^{\mathcal{A}(Z(H))}$$

$$\sum_{a \in \mathcal{A}(H)} f(a)a \mapsto \sum_{a \in \mathcal{A}(H)} f(a)\chi_a$$

where we have used the fact that $\mathcal{A}(Z(H)) = \{ \chi_a \mid a \in \mathcal{A}(H) \}$ (Lemma 4.7). This map is obviously an injective homomorphism of algebraic monoids. Since

$$\sum_{a \in \mathcal{A}(H)} f(a)\chi_a \in Z(Z(H))$$

if and only if

$$\sum_{a \in \mathcal{A}(H)} f(a)a \text{ converges in } Z(H)$$

if and only if

$$\prod_{a \in \mathcal{A}(H)} a^{f(a)} \text{ converges in } H$$

if and only if

$$\sum_{a \in \mathcal{A}(H)} f(a)a \in Z(H)$$

we get that the image of $\Xi$ is exactly $Z(Z(H))$, thus that $Z(H) \simeq Z(Z(H))$ as algebraic monoids. Henceforth, we regard $\Xi$ as a map $\Xi : Z(H) \to Z(Z(H))$. A simple calculation
shows that it is a section to the topological factorisation homomorphism \( \pi_Z : Z(Z(H)) \to Z(H) \), i.e. that \( \pi_Z \circ \Xi : Z(H) \to Z(H) \) is the identity.

It remains to show that \( \Xi \) is continuous, with continuous inverse. Let \( (g_v)_{v \in V} \) be a net in \( Z(H) \), and let \( g \in Z(H) \), with

\[
g = \sum_{a \in A(H)} r(a)a, \quad g_v = \sum_{a \in A(H)} r_v(a)a
\]

Then

\[
g_v \to g \text{ in } Z(H)
\]

if and only if

\[
\pi_H(g_v) \to \pi_H(g) \text{ in } H \quad \text{and} \quad \forall b \in A(H) : pr_b(g_v) \to pr_b(g)
\]

if and only if

\[
\pi_H \left( \sum_{a \in A(H)} r_v(a)a \right) \to \pi_H \left( \sum_{a \in A(H)} r(a)a \right) \quad \text{in } H
\]

and

\[
\forall b \in A(H) : pr_b \left( \sum_{a \in A(H)} r_v(a)a \right) \to pr_b \left( \sum_{a \in A(H)} r(a)a \right)
\]

which holds if and only if

\[
\prod_{a \in A(H)} a^{r_v(a)} \to \prod_{a \in A(H)} a^{r(a)} \quad \text{and} \quad \forall b \in A(H) : r_v(b) \to r(b).
\]

On the other hand,

\[
\Xi(g_v) \to \Xi(g) \text{ in } Z(Z(H))
\]

if and only if

\[
\pi_Z(H)(\Xi(g_v)) \to \pi_Z(H)(\Xi(g)) \text{ in } Z(H) \quad \text{and} \quad \forall b \in A(Z(H)) : pr_b(\Xi(g_v)) \to pr_b(\Xi(g)),
\]

if and only if

\[
g_v \to g \text{ in } Z(H) \quad \text{and} \quad \forall b \in A(H) : r_v(b) \to r(b)
\]

Hence,

\[
g_v \to g \iff \Xi(g_v) \to \Xi(g),
\]

so \( \Xi \) is continuous, with continuous inverse. \( \square \)
5. The case of restricted decimation — a counterexample

If $H$ does not allows arbitrary decimation, then few of the previous results holds. We’ll explore this in an example.

Consider the submonoid $H \subset \mathbb{N}^\mathbb{N}$ consisting of those functions $\mathbb{N} \to \mathbb{N}$ which are either finitely supported, or $\geq 1$ everywhere. Then, the the functions $\chi_j$, with $j \in \mathbb{N}$, are topologically prime, and topologically irreducible. Now consider the function $f$ which is constantly 1. Note that $f$ is irreducible, since it can not be decomposed as a non-trivial finite product. In fact, $f$ is prime: if $f$ divides $\prod_{j=1}^n g_j$ then at least one $g_j$ must be $\geq 1$ everywhere, and then $f$ divides that $g_j$. However, $f$ can be written as a convergent (infinite) product of atoms, so it is neither topologically irreducible, nor topologically prime.

Note that $f = \prod_{i=0}^\infty \chi_i$, yet $\chi_0$ is not a divisor of $f$, since

$$\frac{f}{\chi_0} = \prod_{i=1}^\infty \chi_i \notin H.$$  

So $Z(H)$ is not an order ideal. This is true even if we instead define the topological factorisation monoid as a set of maps from the topologically irreducible elements to $\mathbb{N}$.

References

[1] Daniel D. Anderson, editor. *Factorization in integral domains*, volume 189 of *Lecture notes in pure and applied mathematics*. Marcel Dekker, Inc, 1997.

[2] Nicolas Bourbaki. *General Topology*. Springer Verlag, 1989.

[3] J. H. Carruth, J. A. Hildebrant, and R. J. Koch. *The theory of topological semigroups*. Pure and Applied Mathematics. Marcel Dekker, Inc, 1983.

[4] Scott Chapman and Alfred Geroldinger. Krull Domains and Monoids, Their Sets of Lengths, and Associated Combinatorial Problems. In Anderson [1], pages 73–112.

[5] Franz Halter-Koch. Finitely Generated Monoids, Finitely Primary Monoids, and Factorization Properties of Integral Domains. In Anderson [1], pages 31–72.

[6] J. Nagata. *Modern general topology*, volume 7 of *Series bibilotheca mathematica*. North-Holland publishing company, 1968.

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