On the \( p \)-Laplacian Lichnerowicz equation on compact Riemannian manifolds

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Received January 20, 2020; accepted April 17, 2020; published online July 18, 2020

Abstract In this paper, we deal with a singular quasilinear critical elliptic equation of Lichnerowicz type involving the \( p \)-Laplacian operator. With the help of the subcritical approach from the variational method, we obtain the non-existence, existence, and multiplicity results under some given assumptions.

Keywords \( p \)-Laplacian, critical exponent, negative exponent, variational methods, compact Riemannian manifold

MSC(2020) 58J05, 35J20

Citation: Chen NB, Liu XC. On the \( p \)-Laplacian Lichnerowicz equation on compact Riemannian manifolds. Sci China Math, 2021, 64: 2249–2274, https://doi.org/10.1007/s11425-020-1679-5

1 Introduction and main results

Let \((M, g)\) be a smooth compact Riemannian manifold of dimension \( n \geq 3 \) without boundary. We investigate the following \( p \)-Laplacian Lichnerowicz type equation in \( M \):

\[
\begin{aligned}
\Delta_{p,g} u + hu^{p-1} &= f(x)u^{p^*-1} + a(x)u^{-p^*-1}, \\
u > 0,
\end{aligned}
\]

(1.1)

where \( 1 < p < n \), \( f(x) \) and \( a(x) \) are smooth functions on \( M \), and \( h \) is a negative constant. Here, \( p^* = \frac{np}{n-p} \) is the critical Sobolev exponent for the embedding of \( H^1_0(M) \) into Lebesgue spaces, and \( \Delta_{p,g} := -\text{div}_g(|\nabla_g u|^{p-2}\nabla_g u) \) is the \( p \)-Laplace-Beltrami operator associated with the metric \( g \) on \( M \), which is defined in local coordinates by the expression

\[
\Delta_{p,g} u = -\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left( \sqrt{|g|} g^{ij} |\nabla_g u|^p g^{ij} \frac{\partial u}{\partial x^j} \right),
\]

where \((g^{ij})\) is the inverse of the metric matrix \((g_{ij})\) and \(|g| := \det(g_{ij})\) is the determinant of the metric tensor.

Such type of equations arises from the Hamiltonian constraint equation for the Einstein-scalar field system in general relativity. See, for example, \([7,8,23]\) and the references therein. In the semilinear case

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$p = 2$, with the help of the conformal method, one is led to a simple scalar equation, which is named as the Einstein-scalar field Lichnerowicz equation (the Lichnerowicz equation, in short). Such equations have been the subject of extensive study in recent years due to the nature of their origin.

In the case $p = 2$, there are interesting papers written by Ngô and Xu [24–26]. In particular, in [24], they obtained the non-existence, existence, and multiplicity results for positive solutions for the following Lichnerowicz equation:

$$\Delta_g u + hu = f(x)u^{2^*-1} + \frac{a(x)}{u^{2^*+1}}, \quad u > 0 \text{ in } M,$$

where $h < 0$ is a constant, $f(x)$ and $a(x) \geq 0$ are smooth functions. We should mention that the method they used is based on the method of Rauzy [28] for the prescribed scalar curvature problem on a compact Riemannian manifold with a negative conformal invariant. In [20], Ma and Wei studied the stability and multiplicity of solutions to the Lichnerowicz equation (1.2) for $h > 0$, $f(x) > 0$ and $a(x) \geq 0$. In [14], Hebey et al. established some non-existence and existence results for positive solutions of (1.2) for the case $h > 0$. In the papers [8,15], more advanced existence results were obtained via the sub- and super-solution method for elliptic equations. Some further interesting results on the Lichnerowicz equation have been obtained in [7,11,18,19,21,27,30,34].

For the case $p > 1$, the $p$-Laplacian Lichnerowicz equation is a special case of the following so-called generalised scalar curvature type equation:

$$\Delta_{g,p} u + h(x)u^{p-1} = f(x)u^{p^*-1} + g(x,u), \quad u > 0 \text{ in } M.$$

Such an equation is nonlinear, of degenerate elliptic type, and of critical Sobolev growth, which arises quite naturally in many branches of mathematics. For example, in differential geometry it is an extension of the equation of prescribed scalar curvature.

In the case $p = 2$, the Yamabe problem (see, e.g., [2,31]). The approach is variational and based on the so-called subcritical approach which is widely known for solving the Yamabe problem (see, e.g., [2,31]). The major difficulties come from the following three aspects: critical Sobolev embedding exponent, a sign-changing potential $f(x)$, and a negative power nonlinearity. To overcome these difficulties, we generalize the analysis techniques developed in [24] for $p = 2$ (see also [28]).

Now, our first existence result reads as follows, where the function $f$ involved in the nonlinearity is of changing sign.

**Theorem 1.1.** Let $(M,g)$ be a smooth compact Riemannian manifold of dimension $n$ ($n \geq 3$) without boundary. Let $h < 0$ be a constant, $f$ and $a \geq 0$ be smooth functions on $M$ with $\int_M adv_g > 0$, $\int_M fdv_g < 0$, sup$_M f > 0$ and $|h| < \lambda_f$ where $\lambda_f$ is given in (2.2). Moreover, suppose that the integral of $a$ satisfies

$$\int_M adv_g < \frac{p}{2(n-p)} \left( \frac{2n-p}{2(n-p)} \right) \frac{2n-1}{2} \left( \int_M |h| dv_g \right) \frac{2n}{p} \int_M |f^-| dv_g,$$

where $f^-$ is the negative part of $f$. Then there exists a number $C > 0$ such that, if

$$\sup_M f \left( \int_M |f^-| dv_g \right)^{-1} \leq C,$$
the problem (1.1) admits at least two positive \(C^{1,\alpha}(M)\) solutions with \(\alpha \in (0,1)\).

As a remark, by straightforward calculus, a necessary condition for (1.1) to admit a positive solution is \(\int_M f dv_g < 0\). As another remark, applying Picone’s identity for \(p\)-Laplacian, we also have \(|h| \leq \lambda_f\) if (1.1) admits a positive solution (see, e.g., [22, 24]).

If we assume that \(f\) does not change sign in the sense that \(f \leq 0\) but not strictly negative in \(M\), or \(\sup_M f < 0\), we then obtain the following result.

**Theorem 1.2.** Let \((M, g)\) be a smooth compact Riemannian manifold of dimension \(n \geq 3\) without boundary. Let \(h < 0\) be a constant, \(f\) and \(a\) be smooth functions on \(M\) with \(a \geq 0\) in \(M\) and \(|h| < \lambda_f\).

Moreover, we assume one of the following conditions holds:

1. \(f \leq 0\) but not strictly negative;
2. \(\sup_M f < 0\).

Then the problem (1.1) possesses a positive solution \(u \in C^{1,\alpha}(M)\) for some \(\alpha \in (0,1)\).

Let us point out that in Theorem 1.2 for the case \(p = 2\), \(|h| < \lambda_f\) is a necessary and sufficient solvability condition such that (1.1) admits a positive solution (see [24] for more details). However, due to the quasilinear case \(1 < p < n\), it is difficult to obtain the same necessary and sufficient condition \(|h| < \lambda_f\) as in the case \(p = 2\). Instead of \(|h| < \lambda_f\), we can only get \(|h| \leq \lambda_f\) as the necessary condition in Theorem 1.2. We will focus on this problem in the future work. Now, we describe the proof of our results briefly. Due to the presence of a term with critical exponent and a term with a negative power, we first investigate the following \(\varepsilon\)-approximate subcritical equation:

\[
\Delta_{p, g} u + h|u|^{p-2} u = f(x)|u|^{q-2} u + \frac{a(x)u}{(u^2 + \varepsilon)^{\frac{q}{2} + 1}},
\]

where \(\varepsilon > 0\) is small and \(q \in (p, p^*)\) is sufficiently close to \(p^*\). Based on the mountain pass lemma and the minimization method, we obtain the existence results for (1.6). With the aid of the subcritical approach from the variational method, we will show that the solutions of (1.1) exist as first \(\varepsilon\)-approximate subcritical equation: (1.6) under some given assumptions. We should also mention that though our method is partly similar to the arguments of Ngô and Xu [24], some technical difficulties are completely different in the quasilinear setting. Moreover, compared with the results for \(p = 2\), our study on the \(p\)-Laplacian Lichnerowicz equation is generally harder.

The rest of this paper is organized as follows. In Section 2, we give some notations and prove some basic properties of solutions, including regularity and a non-existence result. In Section 3, we perform analysis for the energy functional associated with (1.6). In Section 4, we prove Theorem 1.1 and finally in Section 5, we complete the proof of Theorem 1.2.

## 2 Preliminaries

Let \((M, g)\) be a smooth compact Riemannian manifold of dimension \(n \geq 3\). For simplicity, we assume that the manifold \(M\) has **unit volume**, i.e., \(\text{Vol}_g(M) = 1\). Let \(L^p(M)\) for \(1 < p < n\) be the usual Lebesgue space on \((M, g)\). For simplicity, we denote by \(\| \cdot \|_p\) the \(L^p\)-norm, i.e., \(\|u\|_p = (\int_M |u|^p dv_g)^{1/p}\) for any \(u \in L^p(M)\). The Sobolev space \(H^1_p(M)\) is defined as the completion of \(C^\infty(M)\) with respect to the Sobolev norm

\[
\|u\|_p = \left( \int_M |\nabla_g u|^p dv_g + \int_M |u|^p dv_g \right)^{1/p}.
\]

By the well-known Sobolev inequality, we know that for any \(\varepsilon > 0\), there exists a constant \(A = A(p, \varepsilon)\) such that for any \(u \in H^1_p(M)\),

\[
\|u\|_{p,\varepsilon}^p \leq (K(n, p)^p + \varepsilon)\|\nabla_g u\|_{p,\varepsilon}^p + A\|u\|_p^p,
\]

where \(K(n, p)\) is the best constant for the embedding of \(H^1_p(\mathbb{R}^n)\) into \(L^{p,\varepsilon}(\mathbb{R}^n)\), i.e.,

\[
K(n, p)^{-1} = \inf_{u \in C^\infty(\mathbb{R}^n) \setminus \{0\}} \frac{\|\nabla u\|_p}{\|u\|_p}.
\]
Proof. (i) We first rewrite (1.6) as

$$\Delta f = \frac{\nabla_\mu u}{|\nabla_\mu u|^2}$$

We always assume that \( f \) is degenerate at the points where \( |\nabla_\mu u| = 0 \). Let \( f(x) \) be a smooth function on \( M \), \( f^- = \min\{f, 0\} \) and \( f^+ = \max\{f, 0\} \). We define the following two numbers:

$$\lambda_f = \begin{cases} \inf_{u \in A} \frac{\int_M |\nabla_g u|^p dv_g}{\int_M |u|^p dv_g}, & \text{if } A \neq \emptyset, \\ +\infty, & \text{if } A = \emptyset \end{cases} \quad (2.2)$$

with

$$A = \left\{ u \in H^p(M) : u \equiv 0, u \neq 0, \int_M |f^-| |u|^{p-1} dv_g = 0 \right\}. \quad (2.3)$$

For \( q \in (p^*, p^*) \) and \( \eta > 0 \), we define

$$\lambda_{f,q} = \inf_{u \in A_{q,\eta}} \frac{\int_M |\nabla_g u|^p dv_g}{\int_M |u|^p dv_g} \quad (2.4)$$

with

$$A_{q,\eta} = \left\{ u \in H^p(M) : \|u\|_q = 1, \int_M |f^-| |u|^q dv_g = \eta, \int_M |f^-| dv_g \right\}. \quad (2.5)$$

Obviously, both \( \lambda_f \) and \( \lambda_{f,q} \) are non-negative. Moreover, the elements in \( A \) are regarded as the functions that vanish on the support of \( f^- \). Similar to the case of \( p = 2 \), the number \( \lambda_f \) will play an important role in solving (1.1). We will approximate \( \lambda_f \) by \( \lambda_{f,q} \) as proposed in [24, 28] in Subsection 3.2 below.

For \( p = 2 \), it is well known that the solutions to (1.1) is \( C^\infty(M) \). For \( p \neq 2 \), since the \( p \)-Laplacian is degenerate at the points where \( \nabla_g u = 0 \), the regularity of the weak solutions to (1.6) is in general of \( C^{1,\alpha}(M) \) for some \( \alpha \in (0, 1) \) but not of \( C^2(M) \) (see, for example, [12, 33]). Inspired by the ideas from [10, 24], we have the following regularity result.

Lemma 2.1. Let \( p < q \leq p^*, \varepsilon > 0 \) be fixed and \( u \in H^p(M) \) be a weak solution of (1.6). Then we have

(i) if \( \varepsilon > 0 \), then \( u \in C^{1,\alpha}(M) \) for some \( \alpha \in (0, 1) \);

(ii) if \( \varepsilon = 0 \) and \( u^{-1} \in L^r(M) \) for all \( r \geq 1 \), then \( u \in C^{1,\alpha}(M) \) for some \( \alpha \in (0, 1) \).

Proof. (i) We first rewrite (1.6) as

$$\Delta_{p,O} u + \tilde{h}(x,u) = 0,$$

where

$$\tilde{h}(x,u) = h|u|^{p-2}u - f(x)|u|^{q-2}u - \frac{a(x)u}{(u^2 + \varepsilon)^{q-1}}.$$

Notice that \( p < q \leq p^* \), and then we have \( |\tilde{h}(x,u)| \leq C_1|u|^{p-1} + C_2 \) for some positive constants \( C_1 \) and \( C_2 \). Accordingly, by the regularity result [10, Theorem 2.3], we get that \( u \in C^{1,\alpha}(M) \) for some \( \alpha \in (0, 1) \).

(ii) We rewrite (1.6) as

$$\Delta_{p,O} u + K(x)|u|^{p-2}u = f(x)$$

with

$$K(x) = h - f(x)|u|^{q-p} \quad \text{and} \quad f(x) = \frac{a(x)}{|u|^{q-p}}.$$

By the Sobolev embedding and the fact that \( \frac{q}{q-p} > \frac{n}{p} \) we have

$$|u|^{q-p} \in L^{\frac{q}{q-p}}(M) \quad \text{and} \quad K(x) \in L^{\frac{n}{p}}(M).$$

Due to the assumptions in the case (ii), we have of course \( f(x) \in L^{\frac{n}{p}}(M) \). Accordingly, by regularity results [10], we get that \( u \in C^{1,\alpha}(M) \) for some \( \alpha \in (0, 1) \).

In order to avoid the lack of regularity, let us first consider the following nondegenerate equation:

$$-\text{div}_g ((\eta + |\nabla_g u|^2)^{\frac{q-2}{2}} \nabla_g u) = g_x \quad \text{in} \quad M \quad (2.6)$$
for a parameter \( \eta > 0 \), where \( g_c = -h|u|^{p-2}u + f(x)|u|^{q-2}u + \frac{a(x)u}{(u^2 + \varepsilon)^{\frac{\delta}{2} + 1}} \) with the same assumptions on \( f \), \( h \) and \( a \) as before. Then (1.6) corresponds to the degenerate case \( \eta = 0 \). Since (2.6) is uniformly elliptic without singularities and the right-hand side is \( C^1 \)-continuous, the solutions \( u_\eta \) are in \( C^2,\delta(M) \) for some \( \delta \in (0,1) \) and the existence of \( u_\eta \) is also ensured by the classical theory (see [16]). With the information in hand we have the following result which plays an important role in the proof of the main results.

**Lemma 2.2.** Let \( u \in C^{1,\alpha}(M) \) be a positive solution of (1.6) with \( h < 0 \). Then, it holds that

\[
\min_M u \geq \min \left\{ \left( \frac{h}{\inf_M f} \right)^{\frac{q}{p-\alpha}}, 1 \right\} > 0
\]

for any \( q \in (p^*,p^*) \) and any \( \varepsilon > 0 \).

**Proof.** Let \( u_\eta \) be a positive classical solution in \( C^{2,\delta}(M) \) to (2.6). From [12,33], \( u_\eta \) is bounded in \( C^{1,\omega}(M) \) independently of \( \eta \in (0,1] \) and thus, up to subsequences, \( u_\eta \) converges to \( u \) in \( C^{1,\omega}(M) \) as \( \eta \to 0 \) for any \( 0 < \omega < \alpha \). Let us assume that \( u_\eta \) achieves its minimum value at \( x_\eta \). Notice that \( u_\eta(x_\eta) > 0 \) since \( u_\eta(x) \) is a positive solution. We then have \( \nabla u_\eta x_\eta = 0 \) and \( \Delta u_\eta x_\eta \leq 0 \). In particular, we have

\[
h(u_\eta(x_\eta))^{p-1} \geq f(x_\eta)(u_\eta(x_\eta))^{q-1} + \frac{a(x_\eta)u_\eta(x_\eta)}{(u_\eta(x_\eta))^2 + \varepsilon)^{\frac{\delta}{2} + 1}} \geq f(x_\eta)(u_\eta(x_\eta))^{q-1}.
\]

Consequently, we get \( f(x_\eta) < 0 \) and thus \( 0 < \frac{h}{f(x_\eta)} \leq (u_\eta(x_\eta))^{q-p} \) which immediately implies

\[
\min_M u_\eta \geq \left( \frac{h}{\inf_M f} \right)^{\frac{q}{p-\alpha}} \geq \min \left\{ \left( \frac{h}{\inf_M f} \right)^{\frac{q}{p-\alpha}}, 1 \right\}
\]

for any \( q \in (p^*,p^*) \). Now, taking \( \eta \to 0 \), we get the desired result.

In the rest of this section, we derive a necessary condition for \( a(x) \) such that the \( p \)-Laplacian Lichnerowicz equation (1.1) admits no solution with finite \( H^p \)-norm. Similar results can be found in [14,24,26] for the case \( p = 2 \).

**Proposition 2.3.** Let \( (M,g) \) be a smooth compact Riemannian manifold of dimension \( n \geq 3 \) without boundary. Let \( a(x) \) and \( f(x) \) be smooth functions on \( M \) with \( a(x) \geq 0 \) in \( M \) and \( h \) a negative constant. If

\[
\int_M a^{\frac{n-p}{2p-n+2}}dv_g > (K(n,p) + 1 + A)^{\frac{2p-n}{2p-n+2}} \Lambda^{\frac{2p-n}{2p-n+2}} \int_M |f^+|^p dv_g \frac{n-p}{2p-n+2}
\]

for some \( \Lambda > 0 \), then the \( p \)-Laplacian Lichnerowicz equation (1.1) has no positive solution \( u \) with the energy \( \|u\| \leq \Lambda \).

**Proof.** Let \( u \) be a positive solution of (1.1). By integrating (1.1) over \( M \) and applying the divergence theorem, we have

\[
\int_M hu^{p-1}dv_g = \int_M f(x)u^{p-1}dv_g + \int_M a(x)u^{p-1}dv_g.
\]

Let

\[
\beta = \frac{p^*}{2p^* + 1}.
\]

Using Hölder’s inequality, we obtain

\[
\int_M a^{2p-n}dv_g \leq \left( \int_M a^{\frac{n}{p^*+1}}dv_g \right)^{1-\beta} \left( \int_M u^{p^*}dv_g \right)^{1-\beta}.
\]

(2.8)

For the second term of the right-hand side of (2.7), notice that \( h < 0 \) and we get

\[
\int_M \frac{a}{u^{p^*+1}}dv_g = \int_M hu^{p-1}dv_g - \int_M f u^{p-1}dv_g \leq \int_M |f||u|^{p-1}dv_g,
\]

(2.9)
while for the first term, we obtain immediately, by Hölder’s inequality,
\[
\int_M |f|^{-p^*} u^{-1} dv_g \leq \left( \int_M |f|^{-p} dv_g \right)^{1 - \frac{1}{p}} \left( \int_M u^p dv_g \right)^{\frac{1}{p}}.
\] (2.10)
Combining (2.7)–(2.10), we finally have
\[
\int_M a^\beta dv_g \leq \left( \int_M |f|^{-p^*} dv_g \right)^{\beta} \left( \int_M u^p dv_g \right)^{1 - \frac{\beta}{p^*}}.
\] (2.11)
Now, suppose that \( \|u\| \leq \Lambda \). By the Sobolev inequality (2.1) with \( \varepsilon = 1 \) and the fact that \( 1 - \frac{\beta}{p^*} = \frac{2p^*}{2p^* + 1} \), we deduce that
\[
\left( \int_M u^p dv_g \right)^{1 - \frac{\beta}{p^*}} \leq (K(n,p) + 1 + A) \frac{2p^* + 1}{p^*} \|u\|^{\frac{2p^* + 1}{p^*}}.
\]
This together with (2.11) implies
\[
\int_M a^\beta dv_g \leq (K(n,p) + 1 + A) \frac{2p^* + 1}{p^*} \Lambda \frac{2p^* + 1}{p^*} \left( \int_M |f|^{-p^*} dv_g \right)^{\frac{p^*}{2p^* + 1}},
\]
which is a contradiction to our assumption. This completes the proof.

**Remark 2.4.** Proposition 2.3 shows that it is reasonable and necessary to have some control on the integral \( \int_M a dv_g \) as we did in Theorem 1.1. Moreover, concerning (2.11) and Proposition 2.3, as in [26], one can estimate the integral \( \int_M a|f|^{-\beta} dv_g \) from above in terms of \( \|u\| \), where \( \alpha \) and \( \beta \) are two positive constants. This also enables us to establish a sufficient condition to guarantee the nonexistence of positive solutions of (1.1).

# 3 The analysis of the energy functionals

Throughout this section, we always assume that \( \sup_M f > 0 \). For each \( q \in (p,p^*) \) and \( k > 0 \), we introduce \( B_{k,q} \), a hypersurface of \( H^p_k(M) \), which is defined as \( B_{k,q} = \{ u \in H^p_k(M) : \|u\|_q^k = k \} \). Clearly, the set \( B_{k,q} \) is non-empty for any \( k > 0 \). Now we construct the approximate energy functional associated with the subcritical problem (1.6). For each \( \varepsilon > 0 \), we define the functional \( I^\varepsilon_q : H^p_k(M) \rightarrow \mathbb{R} \) as
\[
I^\varepsilon_q(u) = \frac{1}{p} \int_M |\nabla u|^p dv_g + \frac{h}{p} \int_M |u|^p dv_g - \frac{1}{q} \int_M f|u|^q dv_g + \frac{1}{q} \int_M \frac{a}{(u^2 + \varepsilon)^{\frac{q}{2}}} dv_g.
\]
By a standard argument, we have \( I^\varepsilon_q \in C^1(H^p_k(M),\mathbb{R}) \). Let \( \delta I^\varepsilon_q \) be the first variation of \( I^\varepsilon_q \), namely,
\[
\delta I^\varepsilon_q(u)(\varphi) = \int_M |\nabla u|^p \varphi - 2g(\nabla u, \nabla \varphi) dv_g + \frac{h}{p} \int_M |u|^p \varphi dv_g + \frac{1}{q} \int_M f|u|^q \varphi dv_g
\] - \( \int_M \frac{a \varphi}{(u^2 + \varepsilon)^{\frac{q}{2}}} dv_g \)
for all \( \varphi \in H^p_k(M) \).
Therefore, the weak solutions of (1.6) correspond to the critical points of \( I^\varepsilon_q \). Set \( \mu^\varepsilon_{k,q} = \inf_{u \in B_{k,q}} I^\varepsilon_q(u) \).
By Hölder’s inequality and the fact that \( \text{Vol}_n(M) = 1 \), it holds that \( I^\varepsilon_q(u) \geq \frac{h}{p} k^{-\frac{p}{2}} - \frac{k}{q} \sup_M f \) for any \( u \in B_{k,q} \). From this we know that \( \mu^\varepsilon_{k,q} > -\infty \) provided that \( k \) is finite. On the other hand, using the test function \( u = k^{\frac{p}{2}} \), we obtain
\[
\mu^\varepsilon_{k,q} \leq \frac{h}{p} k^{\frac{p}{2}} - k \int_M f dv_g + \frac{1}{q} \int_M \frac{a}{(k^2 + \varepsilon)^{\frac{q}{2}}} dv_g,
\] (3.1)
which implies that \( \mu^\varepsilon_{k,q} < +\infty \).
3.1 The asymptotic behavior of $\mu_{k,q}^\varepsilon$

In this subsection, we first show that if $k$, $q$ and $\varepsilon$ are fixed, then $\mu_{k,q}^\varepsilon$ is achieved by some positive function, say $\bar{u}$.

**Lemma 3.1.** $\mu_{k,q}^\varepsilon$ is attained by a positive function $\bar{u} \in C^{1,\alpha}(M)$ for $q < p^*$.

**Proof.** Indeed, let $\{u_j\}_j$ be a minimizing sequence for $\mu_{k,q}^\varepsilon$, i.e.,

$$u_j \in B_{k,q} \text{ and } I_q^\varepsilon(u_j) \rightarrow \mu_{k,q}^\varepsilon.$$

Since $I_q^\varepsilon(u_j) = I_q^\varepsilon(|u_j|)$, we may assume that $u_j \geq 0$ for all $j \geq 1$. By Hölder’s inequality, one has $\|u_j\|_p \leq k^{\frac{q}{p}} \varepsilon$. Since $I_q^\varepsilon(u_j) \leq \mu_{k,q}^\varepsilon + 1$ for sufficiently large $j$, it holds that

$$\frac{1}{p} \int_M |\nabla u_j|^p dv_g \leq \mu_{k,q}^\varepsilon - \frac{h}{p} k^\frac{q}{p} + \frac{k q}{p} \sup M f + 1.$$

Hence, the sequence $\{u_j\}_j \subset H^p(M)$ is bounded and, up to subsequences,

$$u_j \rightharpoonup \bar{u} \text{ weakly in } H^p(M), \quad u_j \rightarrow \bar{u} \text{ strongly in } L^q(M), \quad \text{ and}$$

$$u_j(x) \rightarrow \bar{u}(x) \text{ a.e. in } M \text{ as } j \rightarrow +\infty.$$

This shows that $\bar{u}(x) \geq 0$ a.e. on $M$ and $\|\bar{u}\|_q = k^{\frac{1}{q}}$. In particular, we have $\bar{u} \in B_{k,q}$. Now noticing that $a \varepsilon^{-\frac{2}{p}} \in L^1(M)$, we obtain by Lebesgue’s dominated convergence theorem that

$$\int_M \frac{a}{(u_j^2 + \varepsilon)^\frac{q}{2}} dv_g \rightarrow \int_M \frac{a}{(u^2 + \varepsilon)^\frac{q}{2}} dv_g \text{ as } j \rightarrow +\infty.$$

Hence, from the weak lower semi-continuity of the integral functionals, we get

$$\mu_{k,q}^\varepsilon = \lim_{j \rightarrow +\infty} I_q^\varepsilon(u_j) \geq I_q^\varepsilon(\bar{u}).$$

This and the fact that $\bar{u} \in B_{k,q}$ immediately give us $\mu_{k,q}^\varepsilon = I_q^\varepsilon(\bar{u})$.

Next, we will show the regularity and positivity of $\bar{u}$. Invoked by the Lagrange multiplier rule, we can find $\lambda \in \mathbb{R}$, such that $\bar{u}$ solves

$$\Delta_{p,q} \bar{u} + h|\bar{u}|^{p-2} \bar{u} = (f(x) + \lambda)|\bar{u}|^{q-2} \bar{u} + \frac{a(x) \bar{u}}{(u^2 + \varepsilon)^\frac{q}{2} + 1}$$

in the weak sense. It follows from Lemma 2.1 that $\bar{u} \in C^{1,\alpha}(M)$ for some $\alpha \in (0,1)$ and $\bar{u} \geq 0$ in $M$. Furthermore, applying the strong maximum principle (see [10, Theorem 2.6]) and noticing that $\int_M (\bar{u})^q dv_g = k \neq 0$, we conclude that $\bar{u} > 0$. Thus $\bar{u}$ is a positive solution of (3.2).

The following interesting property of $\mu_{k,q}^\varepsilon$ will also be used in the proofs of our main results.

**Proposition 3.2.** For $\varepsilon > 0$ fixed, $\mu_{k,q}^\varepsilon$ is continuous with respect to $k$.

**Proof.** First, we know that $\mu_{k,q}^\varepsilon$ is well defined for any $k \in (0, +\infty)$. We have to verify that for each $k$ fixed and for any sequence $k_j \rightarrow k$ it holds that $\mu_{k_j,q}^\varepsilon \rightarrow \mu_{k,q}^\varepsilon$ as $j \rightarrow +\infty$. This is equivalent to showing that there exists a subsequence of $\{k_j\}_j$, still denoted by $\{k_j\}_j$, such that $\mu_{k_j,q}^\varepsilon \rightarrow \mu_{k,q}^\varepsilon$ as $j \rightarrow +\infty$. We suppose that $\mu_{k_j,q}^\varepsilon$ and $\mu_{k,q}^\varepsilon$ are achieved by $u_j \in B_{k_j,q}$ and $u \in B_{k,q}$, respectively. From Lemma 3.1, $u_j$ and $u$ are positive functions on $M$. We need to prove the boundedness of $u_j$ in $H^p(M)$. It then suffices to control $\|\nabla u\|_p$. In fact, as in (3.1), one has

$$\int_M |\nabla u_j|^p dv_g < p \left( \mu_{k_j,q}^\varepsilon - \frac{h}{p} k_j^\frac{q}{p} + \frac{k_j q}{p} \sup M f + 1 \right).$$

By the homogeneity we can find a sequence of positive numbers $\{t_j\}_j$ such that $t_j u \in B_{k_j,q}$. Since $k_j \rightarrow k$ as $j \rightarrow +\infty$ and $k_j^{1/q} = \|t_j u\|_q = t_j k^{1/q}$, we immediately see that $t_j \rightarrow 1$ as $j \rightarrow +\infty$. Now, by substituting $u$ with $t_j u$ in $I_q^\varepsilon(u)$, it holds that

$$\mu_{k_j,q}^\varepsilon \leq t_j^p \left( \frac{1}{p} \int_M |\nabla u|^p dv_g + \frac{h}{p} \int_M |u|^p dv_g \right).$$
Notice that $u$ is fixed and $t_j$ belongs to a neighborhood of 1 for large $j$. Thus, $(\mu_{k_j,q})_j$ is bounded which also implies by (3.3) that $\{\|\nabla u_j\|_{p_j}\}_j$ is bounded. Hence, $(u_j)_j$ is bounded in $H^1(M)$. Consequently, there exists a $\bar{u} \in H^1(M)$ such that, up to subsequences, $u_j \to \bar{u}$ strongly in $L^r(M)$ for any $r \in [1, p^*)$, and $\lim_{j \to +\infty} \|u_j\| = \|ar{u}\| = k^\frac{q}{p}$, i.e., $\bar{u} \in B_{k,q}$. Thus, we have $\mathcal{I}_q^\varepsilon(u) = \mu_{k,q}^\varepsilon \leq \mathcal{I}_q^\varepsilon(\bar{u})$. By the weak lower semi-continuity property of $\mathcal{I}_q^\varepsilon$, we deduce that

$$\mathcal{I}_q^\varepsilon(u) \leq \mathcal{I}_q^\varepsilon(\bar{u}) \leq \liminf_{j \to +\infty} \mathcal{I}_q^\varepsilon(u_j).$$

On the other hand, by (3.4) and Lebesgue’s dominated convergence theorem, it holds $\limsup_{j \to +\infty} \mu_{k_j,q}^\varepsilon \leq \mathcal{I}_q^\varepsilon(u)$ since $t_j \to 1$ as $j \to +\infty$. Therefore, $\lim_{j \to +\infty} \mu_{k_j,q}^\varepsilon = \mu_{k,q}^\varepsilon$. This completes the proof. □

The following lemma describes the asymptotic behavior of $\mu_{k,q}^\varepsilon$ as $k$ varies (see Figure 1). We omit its proof since the proof is similar to that for the case $p = 2$ in [24].

**Lemma 3.3.** The following asymptotic behavior holds:

(i) $\mu_{k,q}^\varepsilon \to +\infty$ as $k \to 0^+$. In particular, there exists $k_* \geq 0$ such that $\mu_{k_*}^\varepsilon > 0$ for any $\varepsilon \leq k_*^\varepsilon$.

(ii) $\mu_{k,q}^\varepsilon \to -\infty$ as $k \to +\infty$ provided $\sup_M f > 0$.

(iii) There exists $k_0 = \left(\frac{p+q}{2p}\right)^\frac{q}{2p} \left(\frac{|h|}{\int_M |f^-|dv_g}\right)^{\frac{q}{2p}}$ such that $\mu_{k_0,q}^\varepsilon \leq 0$ for any $\varepsilon > 0$ provided

$$\int_M adv_g \leq \left(\frac{p+q}{2p}\right)^\frac{q}{2p} \left(\frac{|h|}{\int_M |f^-|dv_g}\right)^{\frac{q}{2p}}.$$

In particular, $k_0 > k_*^\varepsilon$.

(iv) Assume that (1.4) holds. Then there exists some constant $\mu$ independent of $q$ and $\varepsilon$ such that $\mu_{k,q}^\varepsilon \leq \mu$ for any $\varepsilon > 0$, $q \in (p^*, p^*)$ and $k \geq k_0$.

(v) There is some $k_{**} \geq k_*$ sufficiently large and independent of both $q$ and $\varepsilon$ such that $\mu_{k,q}^\varepsilon < 0$ for any $k \geq k_{**}$.

The following remarks will be needed in the future argument.

**Remark 3.4.** A simple calculation shows that the following function:

$$\phi(q) := \left(\frac{p+q}{2p}\right)^\frac{q}{2p} \left(\frac{|h|}{\int_M |f^-|dv_g}\right)^{\frac{q}{2p}}$$

is monotone increasing in $(p, p^*)$ provided that $\frac{p}{p^*} |h| \leq \int_M |f^-|dv_g$. Thus, the term on the right-hand side of (1.4) equals $\lim_{q \to p^*} \phi(q)$. This suggests us that a good condition for $\int_M adv_g$ could be (1.4).

![Figure 1](image-url) The asymptotic behavior of $\mu_{k,q}^\varepsilon$ when $\sup_M f > 0$.\p

\[
- \frac{\alpha^q}{2} \int_M f|u|^q dv_g + \frac{1}{q} \int_M \frac{\alpha}{(t_j u)^2 + \varepsilon} dv_g.
\] (3.4)
Remark 3.5. It follows from $q \in (p^*, p^*)$ that

$$\min \left\{ \left( \frac{\int_M |f^{-1}|dv_g}{p} \right)^{\frac{2n-p}{p}}, 1 \right\} \leq k_0,$$

where $k_0$ is given in Lemma 3.3(iii), since $\frac{p+2}{2p} > 1$ and the function $\frac{q}{q-p}$ is monotone decreasing. Thus, as in [24], one can easily control the growth of $\mu_{k_0,q}$ as follows:

$$\mu_{k_0,q} \leq -\frac{1}{p^*} \min \left\{ \left( \frac{|h|}{\int_M |f^{-1}|dv_g} \right)^{\frac{2n-p}{p}}, 1 \right\} \int_M f^+dv_g$$

(3.6)

for any $\varepsilon \geq 0$. The importance of (3.6) is that the right-hand side is strictly negative and independent of both $q$ and $\varepsilon$ provided that $\sup_M f > 0$, which is always the assumption in this subsection.

The next subsection is originally due to Rauzy [28, Subsection IV.3] for prescribing the scalar curvature on a compact Riemannian manifold with a negative conformal invariant. For the sake of clarity, we follow the argument in [28] to re-prove [28, Subsection IV.3] for our quasilinear setting.

3.2 The study of $\lambda_{f,\eta,q}$

The proof of our main result depends on $\lambda_{f,\eta,q}$ (see (2.4)). This quantity was first introduced by Rauzy [28]. Let us first define $\mathcal{A}_{\eta,q} = \inf_{u \in \mathcal{A}_{\eta,q}} \|\nabla_\eta u\|^p_p/\|u\|^p_p$, where

$$\mathcal{A}_{\eta,q} = \left\{ u \in H^p_p(M) : \|u\|_q = 1, \text{ and } \int_M |f^{-1}|u^qdv_g \leq \eta \int_M |f^{-1}|dv_g \right\}.$$ 

For $q$ and $\eta > 0$, the set $\mathcal{A}_{\eta,q}$ is non-empty since it includes the set of functions $u \in H^p_p(M)$ such that $\|u\|_q = 1$ and $\text{supp}(u) \subset \{ x \in M : f(x) > 0 \}$, and thus $\lambda_{f,\eta,q}$ is finite. Moreover, it is easy to check that $\lambda_{f,\eta,q}$ is monotone decreasing with respect to $\eta$. We are now going to prove $\lambda_{f,\eta,q} = \lambda_{f,\eta,q}$. To this end, it suffices to prove $\lambda_{f,\eta,q} \geq \lambda_{f,\eta,q}$ since the reverse is automatically true.

Lemma 3.6. As a function of $\eta > 0$, $\lambda_{f,\eta,q}$ is monotone decreasing.

Proof. We first prove that $\lambda_{f,\eta,q}$ can be achieved. Let $\{v_j\}_j \subset \mathcal{A}_{\eta,q}$ be a minimizing sequence for $\lambda_{f,\eta,q}$. Obviously, the sequence $\{\|v_j\|_j\}_j$ is still a minimizing sequence in $\mathcal{A}_{\eta,q}$, and then we can assume that $v_j \rightharpoonup v$ weakly in $H^p_p(M)$ such that $v_j \rightarrow v$ strongly in $L^q(M)$ for all $r \in [1, p^*)$, and $v_j(x) \rightarrow v(x)$ for almost every $x \in M$ as $j \rightarrow +\infty$.

Consequently, $v(x) \geq 0$ a.e. on $M$ and $\|v\|_q = 1$. Moreover, Lebesgue’s dominated convergence theorem allows us to conclude that $\int_M |f^{-1}|v^qdv_g \leq \eta \int_M |f^{-1}|dv_g$. Hence, $v \in \mathcal{A}_{\eta,q}$. We notice that

$$\lim_{j \rightarrow +\infty} \|v_j\|^p_p = \|v\|^p_p$$

and

$$\lim_{j \rightarrow +\infty} \|\nabla_\eta v_j\|^p_p \leq \|\nabla_\eta v\|^p_p.$$

It follows that $\|\nabla_\eta v\|^p_p/\|v\|^p_p \leq \lambda_{f,\eta,q}$. Thus, $\lambda_{f,\eta,q}$ is achieved by $v$. Moreover, $\|\nabla_\eta v\|^p_p > 0$ since we can assume that $\lambda_{f,\eta,q} > 0$, otherwise $\lambda_{f,\eta,q} = 0$, and this is trivial. By [2, Proposition 3.49], we may assume $\|v\|_q \geq 0$, otherwise we just replace $v$ by $|v|$. We now claim that $v \in \mathcal{A}_{\eta,q}$, where the definition of $\mathcal{A}_{\eta,q}$ is in (2.5). Indeed, we assume by contradiction that $v \not\in \mathcal{A}_{\eta,q}$, i.e., $\int_M |f^{-1}|v^qdv_g < \eta \int_M |f^{-1}|dv_g$. Then there exists a positive constant $\tau$ such that

$$\int_M |f^{-1}|(v + \tau)qdv_g = \eta \int_M |f^{-1}|dv_g \text{ and } \|v + \tau\|_q \geq 1.$$

It follows that $\frac{v + \tau}{\|v + \tau\|_q} \in \mathcal{A}_{\eta,q}$, and then we deduce

$$\left\|\nabla_\eta \left( \frac{v + \tau}{\|v + \tau\|_q} \right) \right\|^p_p/\left\|\frac{v + \tau}{\|v + \tau\|_q} \right\|^p_p = \frac{\|\nabla_\eta (v + \tau)\|^p_p}{\|v + \tau\|^p_p} < \|\nabla_\eta v\|^p_p = \lambda_{f,\eta,q},$$

as desired.
which gives us a contradiction. Hence, \( v \in A_{\eta,q} \). In particular, we have \( \lambda'_{f,\eta,q} = \lambda_{f,\eta,q} \). Therefore, \( \lambda_{f,\eta,q} \) is a decreasing function of \( \eta \).

Lemma 3.6 says that \( A_{\eta,q} \neq \emptyset \). The following lemma gives us a comparison between \( \lambda_{f,\eta,q} \) (see (2.4)) and \( \lambda_f \) (see (2.2)). Intuitively, \( A \) (see (2.3)) is smaller than \( A_{\eta,q} \).

Lemma 3.7. For each \( q \in (p,p^\ast) \) and \( \eta > 0 \) fixed, it holds that \( \lambda_{f,\eta,q} \leq \lambda_f \).

Proof. Let \( u \in A \). Then it holds that \( \int_M u^q dv_y > 0 \), otherwise, \( u \equiv 0 \). By the definition of \( A \), we have \( \int_M |f^\ast| u^{p^\ast - 1} dv_y = 0 \) which also implies that \( \int_M |f|^\ast u^q dv_y = 0 \). Again, from the definition of \( A \) and the fact that \( \sup_M f > 0 \), we must have \( \int_M u^q dv_y > 0 \). We now choose \( \varepsilon > 0 \) such that \( \int_M (\varepsilon u)^q dv_y = 1 \). Then we have \( \varepsilon u \in A'_{\eta,q} \) and

\[
\lambda'_{f,\eta,q} \leq \frac{\|\nabla g(\varepsilon u)\|_p^p}{\|\varepsilon u\|_p^p} = \frac{\|\nabla u\|_p^p}{\|u\|_p^p}.
\]

Since the preceding inequality holds for any \( u \in A \), we may take the infimum on both sides with respect to \( u \) to arrive at \( \lambda_{f,\eta,q} = \lambda'_{f,\eta,q} \leq \lambda_f \).

The next lemma shows that the number \( \lambda_{f,\eta,q} \) can be close arbitrarily to \( \lambda_f \).

Lemma 3.8. For each \( \delta > 0 \) fixed, there exists \( \eta_0 > 0 \) such that for all \( \eta < \eta_0 \), there is a \( q_0 \in (p^\ast, p) \) so that \( \lambda_{f,\eta,q} \geq \lambda_f - \delta \) for every \( q \in (q_0,p^\ast) \).

Proof. By contradiction, assume that there is some \( \delta_0 > 0 \) such that for every \( \eta_0 > 0 \), there exist \( \eta < \eta_0 \) and a monotone sequence \( \{q_j\} \) converging to \( p^\ast \) so that \( \lambda_{f,\eta,q_j} \leq \lambda_f - \delta_0 \) for every \( j \). We can furthermore assume that \( \lambda_{f,\eta,q_j} \) is achieved by \( v_{\eta,q_j} \in A_{\eta,q_j} \). We then immediately have

\[
\frac{\|\nabla g v_{\eta,q_j}\|_p^p}{\|v_{\eta,q_j}\|_p^p} \leq \lambda_f - \delta_0 \quad (3.7)
\]

for any \( j \). Due to the finiteness of \( \lambda_f \), we can prove the boundedness of \( \{v_{\eta,q_j}\}_j \) in \( H_1^p(M) \), which helps us select a subsequence of \( \{v_{\eta,q_j}\}_j \) so that

\[
\left \{ \begin{array}{l}
 v_{\eta,q_j} \rightharpoonup v_{\eta,p^\ast} \quad \text{weakly in } H_1^p(M), \\
 \nabla g v_{\eta,q_j} \rightharpoonup \nabla g v_{\eta,p^\ast} \quad \text{weakly in } L^p(M), \\
v_{\eta,q_j} \rightarrow v_{\eta,p^\ast} \quad \text{strongly in } L^r(M) \text{ for all } r \in [1,p^\ast), \\
v_{\eta,q_j}(x) \rightarrow v_{\eta,p^\ast}(x) \quad \text{almost everywhere on } M
\end{array} \right.
\]

for some \( v_{\eta,p^\ast} \in H_1^p(M) \) as \( j \rightarrow +\infty \). Taking the limit in (3.7), we have

\[
\frac{\|\nabla g v_{\eta,p^\ast}\|_p^p}{\|v_{\eta,p^\ast}\|_p^p} \leq \lambda_f - \delta_0. \quad (3.8)
\]

Besides, Hölder’s inequality tells us \( 1 \leq \|v_{\eta,q_j}\|_{p^\ast} \) for each \( j \), which implies

\[
1 \leq \left( (K(n,p))^p + 1 \right) \frac{\|\nabla g v_{\eta,q_j}\|_{p^\ast}^p}{\|v_{\eta,q_j}\|_p^p} + A \|v_{\eta,q_j}\|_p^p
\]

\[
\leq \left( (K(n,p))^p + 1 \right) (\lambda_f - \delta_0) + A \|v_{\eta,q_j}\|_p^p.
\]

Then it yields

\[
\frac{1}{(K(n,p))^p + 1} (\lambda_f - \delta_0) + A \leq \|v_{\eta,q_j}\|_p^p
\]

(3.9)

for each \( j \). By passing to the limit as \( j \rightarrow +\infty \) in (3.9), one obtains

\[
\frac{1}{(K(n,p))^p + 1} (\lambda_f - \delta_0) + A \leq \|v_{\eta,p^\ast}\|_p^p = \int_M |v_{\eta,p^\ast}|^p dv_y.
\]
For every \( q_j \geq p \), since \( v_{\eta,q_j} \in A_{\eta,q_j} \), it holds that \( \int_M |v_{\eta,q_j}|^p \, dv_g \leq 1 \) and
\[
\int_M |f^- \|v_{\eta,q_j}\|^p \, dv_g \leq \left( \int_M |f^- \|v_{\eta,q_j}\|^q \, dv_g \right)^{\frac{p}{q}} \leq \eta_j^{\frac{p}{q}} \int_M |f^- \, dv_g .
\]

Here, we use Hölder’s inequality. Taking \( j \to +\infty \), by Fatou’s lemma, we deduce that \( \int_M |v_{\eta,p^*}|^p \, dv_g \leq 1 \) and
\[
\int_M |f^- \|v_{\eta,p^*}\|^p \, dv_g \leq \eta^{\frac{p}{p^*}} \int_M |f^- \, dv_g .
\]

Now we let \( \eta_0 \to 0 \), and then clearly \( \eta \to 0 \). The boundedness of \( v_{\eta,p^*} \) in \( H^p(M) \) follows from the facts that \( v_{\eta,q_j} \to v_{\eta,p^*} \) weakly in \( H^p(M) \) and \( \lambda_f \) is finite. Therefore, there exists \( v \in H^p(M) \) such that, up to subsequences,
\[
\left\{ \begin{array}{c}
v_{\eta,p^*} \to v \text{ weakly in } H^p(M), \\
v_{\eta,p^*} \to v \text{ strongly in } L^r(M) \text{ for all } r \in [1,p^*), \\
v_{\eta,p^*}(x) \to v(x) \text{ almost everywhere on } M, 
\end{array} \right. \tag{3.11}
\]
as \( \eta \to 0 \). Before giving out contradiction, we notice from (3.8) that
\[
\|\nabla_g v\|_p^p \leq (\lambda_f - \delta_0)\|v\|_{p^*}^p . \tag{3.12}
\]

Then it is enough to see
\[
0 \leq \int_M |f^- \|v\|^p \, dv_g \leq \lim_{\eta \to 0} \int_M |f^- \|v_{\eta,p^*}\|^p \, dv_g \\
\leq \lim_{\eta \to 0} \left( \eta^{\frac{p}{p^*}} \int_M |f^- \, dv_g \right) = 0 .
\]

Thus, we have \( \int_M |f^- \|v\|^p \, dv_g = 0 \). In particular, \( \int_M |f^- \|v\|^{p-1} \, dv_g = 0 \). By (3.10) and (3.11), we obtain that
\[
\frac{1}{(K(n,p)+1)\lambda_f + \delta} \leq \int_M |v|^p \, dv_g .
\]

Therefore, \( v \not\equiv 0 \), and thus \( |v| \in A \). By the definition of \( \lambda_f \), we know that
\[
\lambda_f\|v\|_p^p \leq \|\nabla_g v\|_p^p = \|\nabla_g v\|_{p^*}^p .
\]

This contradicts (3.12) and concludes the proof. \( \square \)

Now, we prove that, for any \( \varepsilon > 0 \) and some \( k > k_0, \mu^*_k > 0 \). The similar argument can be found in [28, Proposition 2] for the prescribed scalar curvature equation and [24, Proposition 3.14] for the Lichnerowicz equation.

**Proposition 3.9.** Suppose that sup\( M f > 0 \) and \( |h| < \lambda_f \). Then there exist \( \eta_0 > 0 \) and its corresponding \( q_{\eta_0} \) sufficiently close to \( p^* \) such that
\[
\delta := \frac{\lambda_f - \eta_0 - h}{p} > \frac{1}{2p} (\lambda_f + h) . \tag{3.13}
\]

for any \( q \in (q_{\eta_0}, p^*) \). For such \( \delta \), we denote
\[
C_q = \frac{\eta_0}{4|h|} \min_{m} \left\{ \frac{\delta}{A + (K(n,p)+1)(|h| + p\delta) \frac{(p-1)|h|}{p}} \right\} =: \frac{\eta_0}{4|h|} . \tag{3.14}
\]
Now we set
\[ k = \left(\frac{|h|q}{\eta_0 \int_M |f^-| dv_g} \right)^{\frac{p}{q-p}}. \]  
(3.16)

From Lemma 3.3 and the fact that \( 0 < \eta_0 < 2 \) we deduce that \( k_0 < k_{1,q} \). We assume from now on that \( k \geq k_{1,q} \). Let \( u \in B_{k,q} \). We write
\[ I^e_k(u) = G_q(u) - \frac{1}{q} \int_M f^+ |u|^q dv_g + \frac{1}{q} \int_M \frac{a}{(u^2 + \varepsilon)^\frac{q}{2}} dv_g, \]
where
\[ G_q(u) = \frac{1}{p} \int_M |\nabla g u|^p_v dv_g + \frac{h}{p} \int_M |u|^p dv_g + \frac{1}{q} \int_M |f^-| |u|^q dv_g. \]
(3.17)

We then consider the following two possible cases:

**Case 1.** Assume that
\[ \int_M |f^-||u|^q dv_g \geq \eta_0 k \int_M |f^-| dv_g. \]

In this case, by (3.17), (3.16) and the fact that \( k \geq k_{1,q} \), the term \( G_q \) can be estimated from below, i.e.,
\begin{align*}
G_q(u) & \geq \frac{h}{p} \|u\|^p_v + \frac{\eta_0 k}{q} \int_M |f^-| dv_g = -\frac{|h|}{p} \|u\|^p_v + \frac{\eta_0 k}{q} \int_M |f^-| dv_g \\
& \geq \frac{|h|}{p} k \left( \frac{p \eta_0 k^{1-p}}{q|h|} \int_M |f^-| dv_g \right) \geq \frac{(p-1)|h|}{p} k \frac{p}{k^{1-p}}.
\end{align*}
(3.18)

**Case 2.** Assume that
\[ \int_M |f^-||u|^q dv_g < \eta_0 k \int_M |f^-| dv_g. \]

Then it is clear that \( k^{-1/q} u \in \mathcal{A}_{\eta_0,q} \) which implies \( \|\nabla g u\|^p_v \geq \lambda_{f,\eta_0,q} \|u\|^p_v \) by the definition of \( \lambda_{f,\eta_0,q} \).

Therefore, together with (3.17), the term \( G_q \) can be estimated from below by
\[ G_q(u) \geq \delta \|u\|^p_v + \frac{1}{q} \int_M |f^-||u|^q dv_g, \]
(3.19)

where \( \delta \) is as in (3.13). Using (3.17) we have
\[ \|u\|^p_v = \frac{p}{|h|} \left( \frac{1}{p} \|\nabla g u\|^p_v + \frac{1}{q} \int_M |f^-||u|^q dv_g - G_q(u) \right). \]
(3.20)

Now we set \( \delta \|u\|^p_v = \alpha \|u\|^p_v + \beta \|u\|^p_v \), where \( \alpha = \frac{\beta A}{|h| (|K(n,p)| + 1)} \) and \( \delta = \alpha + \beta \). We then apply (3.19) and (3.20) to get
\begin{align*}
G_q(u) & \geq \alpha \|u\|^p_v + \beta \left( \frac{1}{p} \|\nabla g u\|^p_v + \frac{1}{q} \int_M |f^-||u|^q dv_g - G_q(u) \right) + \frac{1}{q} \int_M |f^-||u|^q dv_g \\
& \geq \alpha \|u\|^p_v + \beta \left( \frac{1}{p} \|\nabla g u\|^p_v - G_q(u) \right).
\end{align*}
which yields
\[ G_q(u) \geq \frac{\beta}{|\mathbf{n}| + \beta |p|} \left( \| \nabla_g u \|_p^p + \frac{\alpha |h|}{\beta} \| u \|_p^p \right). \tag{3.21} \]

By the Sobolev inequality, Hölder’s inequality and the equality \( \frac{A}{K(n,p)q+1} = \frac{\alpha |h|}{\beta} \), we deduce that
\[ \| \nabla_g u \|_p^p + \frac{\alpha |h|}{\beta} \| u \|_p^p \geq \frac{k_4^q}{K(n,p)q+1}. \]

Since \( \beta = \frac{(K(n,p)q+1)|h|}{(K(n,p)q+1)|h|+A} \), (3.21) gives
\[ G_q(u) \geq \frac{\beta}{|\mathbf{n}| + \beta |p|} \cdot \frac{k_4^q}{K(n,p)q+1} = \frac{\delta}{A+(K(n,p)q+1)(|h|+\rho\delta)}k_4^q. \tag{3.22} \]

It now follows from (3.14), (3.18) and (3.22) that \( G_q(u) \geq mk_5^q \), where \( m \) is as in (3.14). Thus, we obtain
\[ T^e_q(u) \geq mk_5^q - \frac{k}{q} \sup_M |f|. \]

Thanks to (3.15) and (3.16), we can choose \( k_1,q \leq k < \left( \frac{m(q-\sup_M f)}{2\sup_M f} \right)^{\frac{4}{3\gamma}} \) such that \( \frac{1}{4}mk_5^q - \frac{k}{q} \sup_M |f| > 0 \), and thus we get \( T^e_q(u) \geq \frac{1}{2}mk_5^q > 0 \). From (3.14) and (3.15), we have
\[ \sup_M f \leq C_q \int_M |f^-|dv_g = \frac{\eta_0}{4|h|} \int_M |f^-|dv_g. \tag{3.23} \]

It then follows from (3.16) and (3.23) that
\[ \left( \frac{mq}{2\sup_M |f|} \right)^{\frac{4}{3\gamma}} \geq \left( \frac{2q|h|}{q_0} \int_M |f^-|dv_g \right)^{\frac{4}{3\gamma}} = 2^{\frac{4}{3\gamma}}k_1,q > 2^{\frac{4}{3\gamma}}k_1,q. \]

Hence, if we set \( k_2,q = 2^{\frac{4}{3\gamma}}k_1,q \), then for any \( k \in [k_1,q,k_2,q] \) we always have \( T^e_q(u) \geq \frac{1}{2}mk_5^q > 0 \). In other words, \( \mu_{k,q} > 0 \) for any \( k \in [k_1,q,k_2,q] \) and any \( \varepsilon > 0 \). This completes the proof. \( \square \)

### 3.3 The Palais-Smale condition

In this subsection, we will prove the Palais-Smale compact condition.

**Proposition 3.10.** Suppose that the conditions (3.13)—(3.15) hold. Then for each \( \varepsilon > 0 \) fixed, the functional \( T^e_q(\cdot) \) satisfies the Palais-Smale condition.

**Proof.** Let \( \varepsilon > 0 \) be fixed. Take \( c \in \mathbb{R} \) and assume that \( \{v_j\}_j \subset H^1_0(M) \) is a Palais-Smale sequence at the level \( c \), namely,
\[ T^e_q(v_j) \to c \quad \text{and} \quad \delta T^e_q(v_j) \to 0 \quad \text{as} \quad j \to +\infty. \]

As the first step, we prove that, up to subsequences, \( \{v_j\}_j \) is bounded in \( H^1_0(M) \). Without loss of generality, we may assume that \( \|v_j\| \geq 1 \) for all \( j \). By means of the Palais-Smale condition, one can derive
\[ \frac{1}{p}\|\nabla_g v_j\|_{p}^{p} + \frac{h}{p}\|v_j\|_{p}^{p} - \frac{1}{q}\int_{M}f|v_j|^{q}dv_g + \frac{1}{q}\int_{M}\frac{a}{(v_{j}^{2} + \varepsilon)^{\frac{q}{2}}}dv_g = c + o(1) \tag{3.24} \]

and
\begin{align*}
\int_{M} |\nabla_g v_j|^{p-2}(\nabla_g v_j, \nabla_g \varphi)dv_g &+ h \int_{M} |v_j|^{p-2}v_j \varphi dv_g \\
- \int_{M} f|v_j|^{q-2}v_j \varphi dv_g &- \int_{M} \frac{a|v_j|}{(v_{j}^{2} + \varepsilon)^{\frac{q}{2}+1}}dv_g = o(1)\|\varphi\| \tag{3.25}
\end{align*}
for any $\varphi \in H^1_p(M)$. Letting $\varphi = v_j$ in (3.25), we get
\[ \|\nabla v_j\|_p^p + h\|v_j\|_p^p - \int_M f|v_j|^q dv_g - \int_M \frac{av_j^2}{(v_j^2 + \varepsilon)^{\frac{q}{2} + 1}} dv_g = o(1)||v_j||. \tag{3.26} \]

For simplicity, let $k_j = \int_M |v_j|^q dv_g$. There are two possible cases according to Proposition 3.9.

**Case 1.** Assume that there exists a subsequence of $\{v_j\}_j$, still denoted by $\{v_j\}_j$, such that
\[ \int_M |f^-| |v_j|^q dv_g \geq \eta_0 k_j \int_M |f^-| dv_g. \]

Using (3.14) and (3.15), we get
\[ \mathcal{I}_q^0(v_j) \geq \frac{h}{p} k_j^\frac{2}{p} + \frac{\eta_0 k_j}{q} \int_M |f^-| dv_g - \frac{1}{q} \int_M f^+ |v_j|^q dv_g \]
\[ \geq \frac{h}{p} k_j^\frac{2}{p} + \frac{\eta_0 k_j}{q} \int_M |f^-| dv_g - \frac{k_j}{q} \sup f \]
\[ \geq \frac{h}{p} k_j^\frac{2}{p} + \frac{\eta_0 k_j}{q} \int_M |f^-| dv_g - \frac{k_j (p - 1) \eta_0}{4p} \int_M |f^-| dv_g \]
\[ = \left( \frac{(3p + 1)\eta_0}{4pq} \int_M |f^-| dv_g \right) k_j - \frac{|h|}{p} k_j^\frac{2}{p}. \]

This estimate and the facts that $0 < \frac{p}{q} < 1$ and $\mathcal{I}_q^0(v_j) \to c$ imply that $\{k_j\}_j$ is bounded. In other words, this means that $\{v_j\}_j$ is bounded in $L^q(M)$. Hence, Hölder’s inequality and (3.24) imply that $\{v_j\}_j$ is also bounded in $H^1_p(M)$.

**Case 2.** In contrast to Case 1, for all $j$ sufficiently large, we assume that
\[ \int_M |f^-| |v_j|^q dv_g < \eta_0 k_j \int_M |f^-| dv_g. \]

Applying (3.24) and (3.26), we get
\[ -\frac{1}{q} \int_M f|v_j|^q dv_g = \frac{p}{(q - p)q} \int_M \frac{a}{(v_j^2 + \varepsilon)^{\frac{q}{2}}} dv_g + \frac{1}{q - p} \int_M \frac{av_j^2}{(v_j^2 + \varepsilon)^{\frac{q}{2} + 1}} dv_g \]
\[ - \frac{pc}{q - p} + o(1) + o(1)||v_j||. \]

Therefore, we may rewrite $\mathcal{I}_q^0$ as follows:
\[ \mathcal{I}_q^0(v_j) \geq \frac{1}{p} \left\| \nabla v_j \right\|_p^p + \frac{h}{p} \left\| v_j \right\|_p^p - \frac{pc}{q - p} + o(1) + o(1)||v_j|| + A_j, \tag{3.27} \]
where
\[ A_j = \frac{1}{q - p} \left( \int_M \frac{a}{(v_j^2 + \varepsilon)^{\frac{q}{2}}} dv_g + \int_M \frac{av_j^2}{(v_j^2 + \varepsilon)^{\frac{q}{2} + 1}} dv_g \right). \]

Dividing (3.27) by $\left\| v_j \right\|_p$ and using the equivalent norm to $\left\| v_j \right\|_{H^1_p} = \left\| \nabla v_j \right\|_p + \left\| v_j \right\|_p$, we obtain
\[ \frac{\mathcal{I}_q^0(v_j)}{\left\| v_j \right\|_p} \geq \frac{1}{p} \left( \left\| \nabla v_j \right\|_p^p + h \right) \left\| v_j \right\|_p^{p - 1} + o(1) \frac{\left\| \nabla v_j \right\|_p}{\left\| v_j \right\|_p} \]
\[ - \frac{pc}{(q - p)\left\| v_j \right\|_p} + o(1) + \frac{A_j}{\left\| v_j \right\|_p}. \tag{3.28} \]

Since $k_j^{-1/q} v_j \in A_{\eta_0 q}$, from the definition of $\lambda_{f,\eta_0 q}$, we have $\left\| \nabla v_j \right\|_p^p \geq \lambda_{f,\eta_0 q} \left\| v_j \right\|_p^p$. Therefore, from (3.28) and for $j$ large enough, one gets
\[ \frac{\mathcal{I}_q^0(v_j)}{\left\| v_j \right\|_p} \geq \frac{\lambda_{f,\eta_0 q} + h}{p} \left\| v_j \right\|_p^{p - 1} + o(1)\lambda_{f,\eta_0 q}^{\frac{1}{q}} \]
Combining (3.30) and (3.31), we then infer that

\begin{equation}
\|v_j\|_p \rightarrow +\infty \quad \text{as} \quad j \rightarrow +\infty,
\end{equation}

then we clearly reach a contradiction by taking the limit in (3.29) since \(\lambda_{j,m,q} + h > 0\) and \(A_j > 0\), but

\[\frac{\mathcal{I}_q(v_j)}{\|v_j\|_p} \rightarrow 0 \quad \text{as} \quad j \rightarrow +\infty.\]

Thus, \(\{v_j\}_j\) is bounded in \(L^p(M)\), which also implies that \(\{\nabla_g v_j\}_j\) is bounded in \(L^p(M)\) because of (3.27).

Consequently, \(\{v_j\}_j\) is bounded in \(H^p_1(M)\). Combining Cases 1 and 2 together, we conclude that there exists a bounded subsequence of \(\{v_j\}_j\) in \(H^p_1(M)\), still denoted by \(\{v_j\}_j\). This completes the first step.

We now prove that \(v_j \rightarrow v\) strongly in \(H^p_1(M)\) for some \(v \in H^p_1(M)\). Since \(\{v_j\}_j\) is bounded in \(H^p_1(M)\), there exists \(v \in H^p_1(M)\) such that, up to subsequences,

\[v_j \rightarrow v \quad \text{weakly in} \quad H^p_1(M), \quad v_j \rightarrow v \quad \text{strongly in} \quad L^p(M) \quad \text{for all} \quad r \in [1,p^*),\]

and \(v_j(x) \rightarrow v(x)\) for almost every \(x \in M\) as \(j \rightarrow +\infty\).

From (3.25) and the above convergence properties, we obtain

\[
\int_M (|\nabla_g v_j|_g^{p-2} \nabla_g v_j - |\nabla_g v_i|_g^{p-2} \nabla_g v_i, \nabla_g v_j - \nabla_g v_i) dv_g
\]

\[= |h| \int_M (|v_j|^{p-2} v_j - |v_i|^{p-2} v_i)(v_j - v_i) dv_g + \int_M f(|v_j|^{q-2} v_j - |v_i|^{q-2} v_i)(v_j - v_i) dv_g
\]

\[+ \int_M g \left( \frac{v_j}{v_j^2 + \varepsilon} \right)^{\frac{p}{2}+1} - \frac{v_i}{v_i^2 + \varepsilon} \right)^{\frac{p}{2}+1} (v_j - v_i) dv_g + o(1)\|v_j - v_i\|
\]

\[= o(1).
\]

On the other hand, by Lindqvist’s formula [17, p. 162], for any \(X, Y \in \mathbb{R}^n\), there exists a constant \(c_p > 0\) such that

\[
\langle |X|^{p-2} X - |Y|^{p-2} Y, X - Y \rangle \geq \begin{cases} c_p |X - Y|^p, & \text{if} \quad p \geq 2, \\ c_p \frac{|X - Y|^2}{(|X| + |Y|)^{2-p}}, & \text{if} \quad 1 < p < 2, \end{cases}
\]

where \(\langle \cdot, \cdot \rangle\) denotes the standard scalar product in \(\mathbb{R}^n\). Hence, if \(p \geq 2\), one gets that

\[
c_p \|\nabla_g v_j - \nabla_g v_i\|_p^p \leq \int_M (|\nabla_g v_j|_g^{p-2} \nabla_g v_j - |\nabla_g v_i|_g^{p-2} \nabla_g v_i, \nabla_g v_j - \nabla_g v_i) dv_g. \tag{3.30}
\]

If \(1 < p < 2\), we obtain by Hölder’s inequality that

\[
c_p \|\nabla_g v_j - \nabla_g v_i\|_p^p \leq c_p \left[ \int_M \frac{|\nabla_g (v_j - v_i)|^2}{|\nabla v_j|_g + |\nabla v_i|_g} \right]^{\frac{p}{2}} \left[ \int_M \frac{|\nabla_g v_j|_g + |\nabla g v_i|_g|^{p'}}{p'} \right]^{\frac{p-2}{2}}
\]

\[\leq C \left[ \int_M (|\nabla_g v_j|_g^{p-2} \nabla_g v_j - |\nabla_g v_i|_g^{p-2} \nabla_g v_i, \nabla g v_j - \nabla g v_i) dv_g \right]^{\frac{p}{2}}. \tag{3.31}
\]

Combining (3.30) and (3.31), we then infer that

\[
\|\nabla_g v_j - \nabla_g v_i\|_p \rightarrow 0 \quad \text{as} \quad i, j \rightarrow +\infty.
\]

This shows that \(\{v_j\}_j\) is a Cauchy sequence which, together with the completeness of \(H^p_1(M)\), proves that \(\|v_j - v\| \rightarrow 0\) as \(j \rightarrow +\infty\). We then get the assertion. \(\square\)

4 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. This can be done through three steps.
4.1 The existence of the first solution

In this subsection, we obtain the existence of the first solution of (1.1). Notice that, we require (4.1) to be held. This restriction will be removed by using a scaling argument in the last step.

Proposition 4.1. Let $(M, q)$ be a smooth compact Riemannian manifold of dimension $n$ ($n \geq 3$) without boundary. Let $h < 0$ be a constant, and $f$ and $a \geq 0$ be smooth functions on $M$ with $\int_M adv_g > 0$, $\int_M f dv_g < 0$ and $\sup_M f > 0$. We further assume that

$$|h| \leq \frac{\eta_0}{p^*} \int_M |f|^{-1}dv_g$$

(4.1) and

$$\int_M adv_g \leq \frac{p}{2(n-p)} \left( \frac{2n-p}{2(n-p)} \right)^{\frac{2p}{p^*}-1} \left( \frac{|h|}{\int_M |f|^{-1}dv_g} \right)^{\frac{2p}{p^*}} \int_M f^p dv_g$$

(4.2) hold, where $\eta_0$ is as in Proposition 3.9. Then there exists a number $C_1$ given by (4.5) below such that if

$$\sup_M f \left( \int_M |f|^{-1}dv_g \right)^{-1} \leq C_1,$$

(4.3) then (1.1) possesses at least two positive solutions.

Proof. We divide the proof into several claims for the sake of clarity.

Claim 1. There exists $q_0 \in (p^*, p^*)$ such that for all $q \in (q_0, p^*)$ and $\varepsilon > 0$ small enough, there will be $k_0$ and $k_*$ with the following properties: $k_0 > k_*$ and $\mu_{k_0, q}^+ \leq 0$ while $\mu_{k_*, q}^+ > 0$.

Proof of Claim 1. By Remark 3.4 and (4.2) there is some $q_0 \in (p^*, p^*)$ such that the condition (3.5) holds for all $q \in (q_0, p^*)$. Hence, by Lemma 3.3(iii), there exists $k_0 > 0$ such that $\mu_{k_0, q}^+ \leq 0$. Notice that $p^* > p$ for any $n \geq 3$. From Lemma 3.3(i), we can choose $k_* \in [k_0, 1]$, such that $\mu_{k_*, q}^+ > 0$ for any $\varepsilon \leq k_*$. Thus Claim 1 is proved.

Claim 2. The following subcritical equation:

$$\Delta_{p,q} u + hu^{p-1} = f(x)u^{q-1} + a(x)u^{-q-1},$$

(4.4) admits two positive solutions, say $u_{1,q}$ and $u_{2,q}$. Note that (4.4) corresponds to the case $\varepsilon = 0$ in (1.6).

Proof of Claim 2. By Proposition 3.9, we have $\eta_0$ and its corresponding $q_0 \in (p^*, p^*)$ such that $\delta = \frac{\lambda_f + h}{p} > \frac{1}{2p}(\lambda_f + h)$ for any $q \in (q_0, p^*)$. By Lemma 3.7, we have that $\frac{1}{2p}(\lambda_f + h) < \frac{1}{p}(\lambda_f + h)$. It then follows immediately from this and (3.13) that $C_q \geq C_1$, where

$$C_q = \frac{\eta_0}{4|h|} \min \left\{ \frac{\lambda_f + h}{2p(A + (K(n,p)p + 1)\lambda_f)} \left[ (p-1)|h| \right]^{\frac{1}{p}} \right\}$$

(4.5) and $C_q$ is as in (3.14). Thus the condition (3.15) holds according to (4.3). Note that $C_1$ is independent of $q$ and thus never vanishes for any $q \in (q_0, p^*)$. By using Proposition 3.9 again, there exists an interval $I_q = [k_1,q, k_2,q]$ such that $\mu_{k_*, q}^+ > 0$ for any $k \in I_q$. We then apply Lemma 3.3(iii) and conclude that $k_* < k_0 < k_1,q$, where $k_*$ is given as in Claim 1. Observe that

$$\lim_{q \to p^*} k_{1,q} = \left( \frac{|h|}{\eta_0 \int_M |f|^{-1}dv_g} \right)^{\frac{p}{2}} =: l \quad \text{and} \quad \lim_{q \to p^*} k_{2,q} = 2^n l.$$
where \( D_{k,q} = \{ u \in H^p(M) : \| u \|_q^q = k \text{ and } q \leq k \leq k_1 \} \). Due to (4.1), one can check that \( k_1 \) is strictly monotone increasing with respect to \( q \), and thus \( \| u \|_q^q < l \) for all \( u \in D_{k,q} \). It follows from Section 3 that \( E_1^{k,q} \) is finite and non-positive. Then we have, with the same argument as in Lemma 3.1, \( E_1^{k,q} \) is achieved by some positive function \( u_{1,q}^k \). In particular, \( E_{1,q}^k \) is the energy of \( u_{1,q}^k \). From Remark 3.5 we know that the energy \( E_{1,q}^k \) is strictly negative. Obviously, \( u_{1,q}^k \) is a solution of (1.6). By the Ekeland variational principle and the Palais-Smale compact condition, there exists a bounded minimizing sequence in \( H^p(M) \) for \( E_{1,q}^k \).

Now the lower semi-continuity of \( H^p \)-norm implies that \( \| u_{1,q}^k \| \) is bounded with the bound independent of \( q \) and \( \varepsilon \). If we set \( \| u_{1,q}^k \|_q^q = k_1 \varepsilon \) we then also have \( k_1 \varepsilon \in (k_1,k_1,q) \).

**Step 2.** The existence of the solution \( u_{1,q} \) with strictly negative energy \( \mu_{k_1,q} \) for (4.4).

Let \( \{ \varepsilon_j \} \) be a sequence of positive numbers such that \( \varepsilon_j \to 0 \) as \( j \to +\infty \). For each \( j \), let \( u_{1,q}^{\varepsilon_j} \) be a positive function satisfying the following subcritical equation:

\[
\Delta_{p,q} u_{1,q}^{\varepsilon_j} + h(u_{1,q}^{\varepsilon_j})^{p-1} = f(x)(u_{1,q}^{\varepsilon_j})^{q-1} + \frac{a(x)u_{1,q}^{\varepsilon_j}}{(u_{1,q}^{\varepsilon_j})^2 + \varepsilon_j} \quad (4.6)
\]

in \( M \). Since the sequence \( \{ u_{1,q}^{\varepsilon_j} \} \) is bounded in \( H^p(M) \), up to subsequences, we can assume

\[
\begin{align*}
&\left\{ \begin{array}{l}
 u_{1,q}^{\varepsilon_j} \rightarrow u_{1,q} \text{ weakly in } H^p(M), \\
 u_{1,q}^{\varepsilon_j} \rightarrow u_{1,q} \text{ strongly in } L^r(M) \text{ for all } r \in [1, p^*), \\
 u_{1,q}^{\varepsilon_j} \rightarrow u_{1,q} \text{ for almost every } x \in M
\end{array} \right. \\
&\{ \varepsilon_j \} \to +\infty.
\end{align*}
\]

(4.7)

for some \( u_{1,q} \in H^p(M) \) as \( j \to +\infty \). Moreover, as \( \{ |\nabla g u_{1,q}^{\varepsilon_j}| \} \) is bounded in \( L^p(M) \), one can also assume that

\[
|\nabla g u_{1,q}^{\varepsilon_j} |^{p-2} \nabla g u_{1,q}^{\varepsilon_j} \rightarrow \Sigma_{1,q} \text{ weakly in } (L^p(M))^\prime.
\]

By Lemma 2.2 and Lebesgue’s dominated convergence theorem, we have that \( \int_M (u_{1,q})^{-r} dv_g \) is finite for all \( r \). Taking \( j \to +\infty \) in (4.6), one then gets that

\[
-\text{div}_{g}(\Sigma_{1,q}) + h(u_{1,q})^{p-1} = f(x)(u_{1,q})^{q-1} + a(x)(u_{1,q})^{-q-1}.
\]

Since

\[
-h(u_{1,q})^{p-1} + f(x)(u_{1,q})^{q-1} + a(x)(u_{1,q})^{-q-1}
\]

is bounded in \( L^1(M) \), one can prove that \( \Sigma_{1,q} = |\nabla g u_{1,q}|^{p-2} \nabla g u_{1,q} \) as in [9] (see also [6]). Hence, \( u_{1,q} \) is a weak solution of (4.4). By Lemma 2.1, \( u_{1,q} \in C^{1,\alpha}(M) \) for some \( 0 < \alpha < 1 \). Let \( \| u_{1,q} \|_q^q = k_1 \); by (4.7), we still have \( k_1 \in (k_1,q) \). Consequently, it holds that \( u_{1,q} \neq 0 \). By Lemma 2.1 and the strong maximum principle, we get \( u_{1,q} > 0 \) in \( M \). Since \( u_{1,q}^{\varepsilon_j} \) has strictly negative energy \( E_{1,q}^{\varepsilon_j} \) passing to the limit as \( j \to +\infty \), we have that \( u_{1,q} \) also has strictly negative energy, which we denote by \( \mu_{k_1,q} \). Thus, we have already shown that \( u_{1,q} \) is a positive solution of (4.4) as claimed.

**Step 3.** The existence of the solution \( u_{2,q}^\varepsilon \) with strictly positive energy \( E_{1,q}^{\varepsilon} \) for (1.6).

Let \( k^* \) be a real number such that

\[
\mu_{k^*,q}^* = \max \{ \mu_{k,q} : k_1 \leq k \leq k_2 \}.
\]

From Proposition 3.9, we have \( \mu_{k^*,q}^* > 0 \). According to Proposition 3.2, one can choose \( \bar{k}_1 \in (k_0,k_1,q) \) and \( k_2 \in (k_2,q,k_*) \) such that \( \mu_{k_1,q}^* = \mu_{k_2,q}^* = 0 \). We have proved that \( \mu_{k_1,q}^* \) and \( \mu_{k_2,q}^* \) can be achieved, say by \( u_{k_1,q} \) and \( u_{k_2,q} \), respectively. We now set

\[
\Gamma = \{ \gamma \in C([0, 1]; H^p(M)) : \gamma(0) = u_{k_1,q}, \gamma(1) = u_{k_2,q} \}.
\]

Consider the functional \( E(v) = \mathcal{I}_q(u_{k_1,q} + v) \) for any non-negative real-valued function \( v \) with

\[
\| v \| = \left( \int_M |u_{k_1,q} + v|^q dv_g \right)^{\frac{1}{q}}.
\]
Clearly, we have \( E(0) = 0 \). Let \( \rho = (k^*)^{1/q} \). If \( \|v\| = \rho \), by setting \( u = u_{k_1,q} + v \), then \( \|u\|_q^\rho = k^* \). Therefore
\[
E(v) = \mathcal{I}_q(u) \geq \mu_{k^*,q} > 0.
\]
Next, we set \( v_1 = u_{k_2,q} - u_{k_1,q} \), and then obtain
\[
E(v_1) = \mathcal{I}_q(u_{k_2,q}) = 0 \quad \text{and} \quad \|v_1\| = \left( \int_M \|u_{k_2,q}\|^q dv \right)^{\frac{1}{q}} = (k_2)^{\frac{1}{q}} > \rho.
\]
Notice that the functional \( E \) satisfies the Palais-Smale condition as we have shown for \( \mathcal{I}_q \). Thus, by [31, Chapter II, Theorem 6.1], the following mountain pass level:
\[
\mathcal{E}^\sharp_{2,q} := \inf_{\gamma \in \Gamma \cap [0,1]} \sup_{\gamma(\cdot)} E(\gamma(t) - u_{k_1,q})
\]
is a critical value of the functional \( E \). It is then obvious that \( \mathcal{E}^\sharp_{2,q} > 0 \). Thus, there exists a Palais-Smale sequence \( \{u_j\}_j \subset H^1_\alpha(M) \) for \( \mathcal{I}_q \) at the level \( \mathcal{E}^\sharp_{2,q} \). Since \( \mathcal{I}_q(u_j) = \mathcal{I}_q(|u_j|) \) for every \( j \), we can assume \( u_j \geq 0 \) for any \( j \). Consequently, Proposition 3.10 implies that, up to subsequences, \( u_j \to u_{2,q} \) strongly in \( H^1_\alpha(M) \) for some \( u_{2,q} \in H^1_\alpha(M) \) as \( j \to +\infty \). Therefore, the function \( u_{2,q} \) with positive energy \( \mathcal{E}^\sharp_{2,q} \) satisfies the following subcritical equation:
\[
\Delta_{p,q} u_{2,q} + h(x) (u_{2,q})^{p-1} = f(x) (u_{2,q})^{q-1} + \frac{a(x) u_{2,q}^q}{((u_{2,q})^2 + \varepsilon)^{\frac{q}{2}+1}}
\]
in the weak sense. The non-negativity of \( \{u_j\}_j \) implies that \( u_{2,q} \geq 0 \) almost everywhere, and thus the regularity result, i.e., Lemma 2.1(i) can be applied to (4.8). It follows that \( u_{2,q} \in C^{1,\alpha}(M) \) for some \( \alpha \in (0, 1) \), which also implies \( u_{2,q} \geq 0 \) in \( M \). We claim that \( u_{2,q} \neq 0 \). Indeed, by Lemma 3.3(iv) we have \( \mathcal{E}^\sharp_{2,q} \leq \mu < 0 \). If \( u_{2,q} \equiv 0 \), then we have
\[
\frac{1}{q+1} \int_M a dv = \mathcal{E}^\sharp_{2,q} \leq \mu < 0,
\]
which is impossible if \( \varepsilon \to 0 \). Thus, \( u_{2,q} \geq 0 \) on \( M \) if \( \varepsilon \) is sufficiently small which we will always assume from now on. Let \( \|u_{2,q}\|_q^q = k_2 \). In view of Lemma 3.3(v), we know that \( k_2 > 0 \) is bounded from above by \( k_* \) independent of both \( \varepsilon \) and \( q \).

**Step 4.** The existence of the solution \( u_{2,q} \) with strictly positive energy \( \mu_{k_2,q} \) for (4.4).

Let \( \{\varepsilon_j\}_j \) be a sequence of positive numbers such that \( \varepsilon_j \to 0 \) as \( j \to +\infty \). For each \( j \), let \( u_{2,q}^j \) be a positive function in \( M \) which satisfies the following subcritical equation:
\[
\Delta_{p,q} u_{2,q}^j + h(x) (u_{2,q}^j)^{p-1} = f(x) (u_{2,q}^j)^{q-1} + \frac{a(x) u_{2,q}^j}{((u_{2,q}^j)^2 + \varepsilon_j)^{\frac{q}{2}+1}}
\]
in \( M \). By the discussion above, the sequence \( \{u_{2,q}^j\}_j \) is bounded in \( H^1_\alpha(M) \). Consequently, there exists \( u_{2,q} \in H^1_\alpha(M) \) such that up to subsequences,
\[
\begin{align*}
&u_{2,q}^j \rightharpoonup u_{2,q} \quad \text{weakly in } H^1_\alpha(M), \\
&u_{2,q}^j \to u_{2,q} \quad \text{strongly in } L^r(M) \quad \text{for all } r \in [1,p^*), \\
&u_{2,q}^j(x) \to u_{2,q}(x) \quad \text{for almost every } x \in M \quad \text{as } j \to +\infty.
\end{align*}
\]
Thus, \( u_{2,q} \geq 0 \) almost everywhere in \( M \). We now let \( \|u_{2,q}\|_q^q = k_2 \). By Lemma 2.2, the sequence \( \{u_{2,q}^j\}_j \) is uniformly bounded from below. Applying Lebesgue’s dominated convergence theorem, we have that \( (u_{2,q}^j)^{-1} \in L^r(M) \) for any \( r > 0 \). Taking \( j \to +\infty \) in (4.9) and by deduction as in Step 2, we get that \( u_{2,q} \) is the second weak solution of the following subcritical equation:
\[
\Delta_{p,q} u_{2,q} + h(x) (u_{2,q})^{p-1} = f(x) (u_{2,q})^{q-1} + a(x) (u_{2,q})^{q-1}.
\]
It follows from Lemma 2.1(ii) that \( u_{2,q} \in C^{1,\alpha}(M) \) for some \( \alpha \in (0, 1) \), and thus \( u_{2,q} > 0 \) in \( M \). Since \( u_{2,q}^j \) has positive energy \( \mathcal{E}^\sharp_{2,q} \), taking the limit as \( j \to +\infty \), we know that the energy of \( u_{2,q} \) is still positive, i.e., \( \mu_{k_2,q} > 0 \). Moreover, according to Step 2, it holds that \( u_{1,q} \neq u_{2,q} \). Note that \( k_2 \) is still bounded from above by \( k_* \) independent of both \( \varepsilon \) and \( q \). This completes the proof of Claim 2. \( \square \)
Claim 3. (1.1) has at least one positive solution.

Proof of Claim 3. By Steps 2 and 4 in the proof of Claim 2 above, we know that there exist two positive functions \( u_{1,q} \) and \( u_{2,q} \) which solve (4.4). Moreover, \( \| u_{i,q} \|_q^q = k_i \) \( (i = 1, 2) \). We now estimate \( \mu_{k_1,q} \) and \( \mu_{k_2,q} \). Recall that \( \mu_{k_1,q} \) is the energy of \( u_{i,q} \) found in Claim 2, i.e.,

\[
\mu_{k_1,q} = \frac{1}{p} \| \nabla u_{i,q} \|_p^p + \frac{h}{p} \| u_{i,q} \|_p^p - \frac{1}{q} \int_M f(u_{i,q})^q dv_g + \frac{1}{q} \int_M \frac{a(x)}{u_{i,q}} dv_g.
\]

We have already seen that \( \mu_{k_1,q} < 0 < \mu_{k_2,q} \leq \mu \). Since \( k_1 \in (k_*, k_{1,q}) \) and \( h < 0 \), we obtain

\[
|\frac{1}{p} \| \nabla u_{1,q} \|_p | \leq \mu_{k_1,q} - \frac{h}{p} k_1^{\frac{p}{q}} + \frac{1}{q} \int_M f(u_{1,q})^q dv_g \leq \frac{k_1}{q} \sup_f - \frac{h}{p} k_1^{\frac{p}{q}},
\]

and thus the sequence \( \{ u_{1,q} \}_q \) is bounded in \( H^p_{r}(M) \). Similarly, from Lemma 3.3(iv) and the following estimate:

\[
\frac{1}{p} \| \nabla u_{2,q} \|_p | \leq \mu_{k_2,q} - \frac{h}{p} k_2^{\frac{p}{q}} + \frac{1}{q} \int_M f(u_{2,q})^q dv_g \leq \mu + \frac{k_2}{q} \sup_f - \frac{h}{p} k_2^{\frac{p}{q}},
\]

we know that the sequence \( \{ u_{2,q} \}_q \) is also bounded in \( H^p_{r}(M) \). Combining these facts, we get that for \( i = 1, 2 \),

\[
\| u_{i,q} \|_p = \| \nabla u_{i,q} \|_p + \| u_{i,q} \|_p | \leq p \mu + \frac{p k_i}{q} \sup_f (1 - h) k_i^{\frac{p}{q}}.
\]

Thanks to \( k_* > 1 \), \( k_i \leq k_{**} \) and \( q > p^* \), if we let

\[
\Lambda = \left( p \mu + \left( \sup_M f \right) k_{**} + (1 + |h|) k_{**}^{\frac{p}{q}} \right)^{\frac{1}{2}}, \tag{4.10}
\]

we then get that \( \| u_{i,q} \|_p \leq \Lambda \) for \( i = 1, 2 \). Thus, as usual, up to subsequences, there exist \( u_i \in H^p_{r}(M) \) and \( \Sigma_i \in L^{p^{**}}(M) \) such that, as \( q \) approaches \( p^* \),

\[
\begin{align*}
&\{ u_{i,q} \} \rightarrow u_i \text{ weakly in } H^p_{r}(M), \\
&\{ u_{i,q} \} \rightarrow u_i \text{ strongly in } L^r(M) \text{ for all } r \in [1, p^*], \\
&\{ u_{i,q} \} \rightarrow u_i \text{ for almost every } x \in M, \\
&|\nabla u_{i,q}|_{p}^{-2} \nabla u_{i,q} \rightharpoonup \Sigma_i \quad \text{weakly in } (L^p(M))'.
\end{align*}
\]

By Lemma 2.2 and Lebesgue’s dominated convergence theorem, it holds that

\[
\lim_{q \to p^*} \int_M \frac{a(x)v}{u_{i,q}^{q+1}} dv_g = \int_M \frac{a(x)v}{u_{i}^{q+1}} dv_g \quad \text{for all } v \in H^p_{r}(M).
\]

Moreover, by [2, Theorem 3.45], one has

\[
\{ (u_{i,q})^{q-1} \}_{q} \rightarrow \{ (u_{i})^{p-1} \} \text{ weakly in } L^{p^{**}}(M) \text{ as } q \to p^*, \tag{4.11}
\]

since \( \{ (u_{i,q})^{q-1} \}_{q} \) is bounded in \( L^{p^{**}}(M) \subset L^{p^{**}}(M) \) and \( u_{i,q} \rightarrow u_i \) almost everywhere in \( M \). We thus have, by the Sobolev inequality and the fact that \( (L^{p^{**}}(M))' = L^{p'}(M) \),

\[
\lim_{q \to p^*} \int_M f(u_{i,q})^{q-1} v dv_g = \int_M f(u_i)^{p-1} v dv_g \quad \text{for all } v \in H^p_{r}(M).
\]

Since \( u_{i,q} \) satisfies the following identity:

\[
\int_M (\nabla g u_{i,q})^{p-2} (\nabla g u_{i,q}, \nabla g v) dv_g + h \int_M (u_{i,q})^{p-1} v dv_g
- \int_M f(u_{i,q})^{q-1} v dv_g - \int_M \frac{a(x)v}{u_{i,q}^{q+1}} dv_g = 0 \tag{4.12}
\]
for all \( v \in H^1_0(M) \), passing to the limit for \( q \to p^* \) in (4.12), we obtain for \( i = 1, 2 \) that
\[
\int_M \langle \Sigma_i, \nabla_v v \rangle_g dv_g + h \int_M (u_i)^{p-1} v dv_g
- \int_M f(u_i)^{p-1} v dv_g - \int_M \frac{a(x)v}{(u_i)^{p^*+1}} dv_g = 0 \quad \text{for all } v \in H^1_0(M).
\]

With the similar argument to that for Claim 2, we have \( \Sigma_i = |\nabla_g u_i|^{p-2} \nabla_g u_i \). This shows that \( u_i \) \((i = 1, 2)\) are the weak solutions to (1.1). By the strong maximum principle and Lemma 2.1(ii), one then gets that \( u_i \) is positive and \( u_i \in C^{1,\alpha}(M) \) for some \( \alpha \in (0, 1) \).

The proof of Proposition 4.1 is completed.

4.2 The existence of the second solution

In the previous subsection, we only show that \( u_i \) \((i = 1, 2)\) are the solutions of (1.1). However, it is not clear whether these functions are distinct. In this subsection, we will show that \( u_i \) \((i = 1, 2)\) are in fact different provided \( \sup_M f \) is sufficiently small. Recall that the energy of \( u_i \) is given as follows:
\[
T^0_{p^*}(u_i) = \frac{1}{p} \int_M |\nabla_g u_i|_g^{p} dv_g + \frac{h}{p} \int_M (u_i)^{p} dv_g - \frac{1}{p^*} \int_M f(u_i)^{p^*} dv_g + \frac{1}{p^*} \int_M \frac{a(x)}{(u_i)^{p^*+1}} dv_g.
\]

As in [24], we need compare \( T^0_{p^*}(u_1) \) and \( T^0_{p^*}(u_2) \), and this could be done once we can show that
\[
\lim_{q \to p^*} T^q_{p^*}(u_{i,q}) = T^0_{p^*}(u_i) \quad \text{for } i = 1, 2.
\]

From the expression of \( T^0_{p^*}(u_{i,q}) \), we know that the only difficult part is to show that
\[
\int_M f(u_{i,q})^q dv_g \to \int_M f(u_i)^{p^*} dv_g \quad \text{as } q \to p^*.
\]

To this end, we make \( \sup_M f \) sufficiently small. Intuitively, such a small \( f \) is equivalent to saying, for example, that \( f(u_{i,q})^q \) behaves exactly the same as \( f(u_{i,q})^{q-1} \).

**Proposition 4.2.** Assume that all the requirements in Proposition 4.1 are fulfilled and \( f \) satisfies
\[
\sup_M f < C_2,
\]
where the number \( C_2 \) is given in (4.21) below. Then (4.13) holds.

**Proof.** In (4.12), we choose \( v = (u_{i,q})^{1+p\delta} \) for some \( \delta > 0 \) to be determined later. Then we have
\[
\frac{1}{1+p\delta} \int_M |\nabla_g w_{i,q}|_g^{p} dv_g
= |h| \int_M (w_{i,q})^{p} dv_g + \int_M f(w_{i,q})^{p} (u_{i,q})^{q-p} dv_g + \int_M \frac{a(x)}{(u_{i,q})^{q-p\delta}} dv_g,
\]
where \( w_{i,q} = (u_{i,q})^{b+1} \). Then, it follows from (4.14) and the Sobolev inequality that
\[
\|w_{i,q}\|_{p^*}^p \leq \left( \frac{(K(n,p)+1)(1+\delta)|h|}{1+p\delta} + A \right) \|w_{i,q}\|_p^p
+ \frac{(K(n,p)+1)(1+\delta)p}{1+p\delta} \left( \int_M f^+(w_{i,q})^{p}(u_{i,q})^{q-p} dv_g + \int_M \frac{a(x)}{(u_{i,q})^{q-p\delta}} dv_g \right).
\]

By Hölder’s inequality, we find
\[
\int_M (w_{i,q})^{p} (u_{i,q})^{q-p} dv_g \leq \left( \int_M (w_{i,q})^{p} dv_g \right)^{p/p^*} \left( \int_M (u_{i,q})^\frac{q-p}{p^*} dv_g \right)^{p^*}. \]
Notice that $\frac{\nu(q-p)}{p^*-p} < q$ as long as $q < p^*$. Again by Hölder’s inequality and the Sobolev inequality, one gets
\[
\int_M (u_{i,q})^{\frac{\nu(q-p)}{p^*-p}} dv_g \leq \left( \int_M (u_{i,q})^{\nu} dv_g \right)^{\frac{p^*-p}{p^*}} \leq (K(n,p)^{p} + 1 + A)^{\frac{\nu(q-p)}{p^*-p}} \|u_{i,q}\|^{\frac{\nu(q-p)}{p^*-p}}.
\] (4.17)

Therefore, from (4.16) and (4.17), we have
\[
\int_M (u_{i,q})^{p}(u_{i,q})^{q-p} dv_g \leq \|w_{i,q}\|^{p}(K(n,p)^{p} + 1 + A)^{\frac{\nu(q-p)}{p^*-p}} \|u_{i,q}\|^{q-p}.
\] (4.18)

Now, combining (4.15) and (4.18), we easily get that
\[
\|w_{i,q}\|^{p} \leq \left( \frac{(K(n,p)^{p} + 1)(1 + \delta)^{p}|h|}{1 + p\delta} + A \right)\|w_{i,q}\|^{p} + \frac{(K(n,p)^{p} + 1)(1 + \delta)^{p}}{1 + p\delta} (\sup_M f) \|w_{i,q}\|^{q-p} \|w_{i,q}\|^{p} 
+ \frac{(K(n,p)^{p} + 1)(1 + \delta)^{p}}{1 + p\delta} \int_M \frac{a(x)}{(u_{i,q})^{q-p}} dv_g.
\] (4.19)

We wish to impose some condition of $\sup_M f$ such that
\[
\kappa \leq \frac{(K(n,p)^{p} + 1)(1 + \delta)^{p}}{1 + p\delta} (\sup_M f) \Lambda^{p^*-p} < \frac{1}{2},
\] (4.20)

where $\Lambda > 1$ is as in (4.10). To do so, we let $\sup_M f < C_{2}$ with
\[
C_{2} = \min \left\{ \frac{1}{2(K(n,p)^{p} + 1)} (K(n,p)^{p} + 1 + A)^{-\frac{\nu(q-p)}{p^*-p}} \left( \mu p + k_{*} + (1 + |h|)k_{*}^{\frac{p}{n,p}} \right)^{-\frac{\nu(q-p)}{p^*-p}}, 1 \right\}.
\] (4.21)

Then, we must have
\[
(K(n,p)^{p} + 1)(K(n,p)^{p} + 1 + A)^{-\frac{\nu(q-p)}{p^*-p}} \left( \sup_M f \right) \Lambda^{p^*-p} < \frac{1}{2}.
\] (4.22)

Now, by (4.22) and the fact that $\frac{(1+\delta)^{p}}{1+p\delta} \to 1$ as $\delta \to 0$, we can choose a small $\delta > 0$ such that (4.20) holds. From now on, we fix this $\delta$ in (4.20) with $p(1 + \delta) < p^*$ and $q - p\delta > 0$. In view of (4.19), it is easy to get
\[
\|w_{i,q}\|^{p} \leq 2 \left( \frac{(K(n,p)^{p} + 1)(1 + \delta)^{p}|h|}{1 + p\delta} \right)\|w_{i,q}\|^{p} + 2 \left( \frac{(K(n,p)^{p} + 1)(1 + \delta)^{p}}{1 + p\delta} \right) \int_M \frac{a(x)}{(u_{i,q})^{q-p}} dv_g.
\] (4.23)

Moreover, by Hölder’s inequality, we have
\[
\|w_{i,q}\|^{p} = \|w_{i,q}\|^{1+\delta} \|_{p} = \|u_{i,q}\|^{1+\delta} \|_{p(1+\delta)} \leq \|w_{i,q}\|^{1+\delta}
\] (4.24)

Then together with the Sobolev inequality it yields that $\|w_{i,q}\|_{p}$ can be controlled by some constant depending on $\Lambda$. By Lemma 2.2 and the fact that $q - p\delta > 0$, we know that $\int_M a(u_{i,q})^{-(q-p\delta)} dv_g$ is bounded independently of $q$. Combining this with (4.23)–(4.24), we have that $\|w_{i,q}\|_{p^*}$ is bounded, i.e., $\|u_{i,q}\|_{p^*(1+\delta)}$ is bounded. Again from Hölder’s inequality, we obtain
\[
\|u_{i,q}\|^{q} \|_{p^*(1+\delta)} \leq \|u_{i,q}\|^{q} \|_{p^*(1+\delta)},
\]

which implies that $(u_{i,q})^{q}$ is bounded in $L^{1+\delta}(M)$. Thus, by [2, Theorem 3.45] and $(u_{i,q})^{q} \to (u_{i})^{p^*}$ almost everywhere in $M$ as $q \to p^*$, one gets that $(u_{i,q})^{q} \to (u_{i})^{p^*}$ weakly in $L^{1+\delta}(M)$ as $q \to p^*$. Hence, by the definition of weak convergence and the fact that $L^{1+\delta}(M)$ is the dual space of $L^{1+\delta}(M)$, it holds that
\[
\int_M f(u_{i,q})^{q} dv_g \to \int_M f(u_{i})^{p^*} dv_g
\]
as $q \to p^*$, since we clearly have $f \in L^{1+\delta}(M)$.\]
With the help of Proposition 4.2, we can easily get the following result.

**Proposition 4.3.** Assume that all the requirements in Proposition 4.2 are fulfilled. Then it holds that

$$\|\nabla_g u_{i,q}\|_p \to \|\nabla_g u_i\|_p \text{ as } q \to p^*.$$  

**Proof.** It suffices to prove that $\nabla_g u_{i,q} \to \nabla_g u_i$ strongly in $L^p(M)$ as $q \to p^*$. The choice of $u_{i,q} - u_i$ as a test function in (4.12) gives us

$$\int_M (|\nabla_g u_{i,q}|_g^{p-2} \nabla_g u_{i,q} - |\nabla_g u_i|_g^{p-2} \nabla_g u_i, \nabla_g u_{i,q} - \nabla_g u_i)_g dv_g$$

$$= -h \int_M (u_{i,q})^{p-1}(u_{i,q} - u_i)dv_g + \int_M f(u_{i,q})^{q-1}(u_{i,q} - u_i)dv_g$$

$$+ \int_M \frac{a(x)}{(u_{i,q})^{q+1}}(u_{i,q} - u_i)dv_g - \int_M |\nabla_g u_{i,q}|_g^{p-2}(\nabla_g u_i, \nabla_g (u_{i,q} - u_i))_g dv_g. \ (4.25)$$

We study the right-hand side of (4.25). It is straightforward to check with the usual arguments that, the limits of the first term, the third term and the forth term vanish as $q \to p^*$. While for the second one, we have, using (4.11) and Proposition 4.2,

$$\int_M f(u_{i,q})^{q}dv_g - \int_M f(u_{i,q})^{q-1}dv_g \to 0 \text{ as } q \to p^*.$$  

Therefore, we deduce

$$\int_M (|\nabla_g u_{i,q}|_g^{p-2} \nabla_g u_{i,q} - |\nabla_g u_i|_g^{p-2} \nabla_g u_i, \nabla_g u_{i,q} - \nabla_g u_i)_g dv_g \to 0 \text{ as } q \to p^*.$$  

Now, by a similar argument to that for Proposition 3.10, one gets $\nabla_g u_{i,q} \to \nabla_g u_i$ strongly in $L^p(M)$ as $q \to p^*$. We can now easily conclude that (1.1) has at least two positive solutions.

**Proposition 4.4.** Assume that all the requirements in Proposition 4.2 are fulfilled. Then (1.1) has at least two positive solutions, while one has strictly negative energy and the other has positive energy.

**Proof.** It suffices to compare the energy of $u_i$ ($i = 1, 2$). By Propositions 4.2 and 4.3, one has $\lim_{q \to p^*} T_q^0(u_{i,q}) = T_p^0(u_i)$ for $i = 1, 2$. According to (3.6), we have $T_p^0(u_1) < 0 < T_p^0(u_2)$. Thus, $u_1$ and $u_2$ have different energy. This completes the proof.

### 4.3 The scaling argument

In this subsection, we use the scaling technique to complete the proof of Theorem 1.1 by removing the condition (4.1) mentioned in Proposition 4.1. We first observe that under the variable change $\tilde{u} = \frac{u}{c}$, where $c$ is a suitable constant to be determined later, (1.1) becomes

$$\Delta_{p,g} u + h(x)u^{p-1} = \tilde{f}(x)u^{\tilde{p}*-1} + \frac{\tilde{a}(x)}{u^{\tilde{p}*-1}} \ (4.26)$$

with

$$\tilde{f} = c^{p^*-p} f, \quad \tilde{a} = \frac{a}{c^{p^*+p}}.$$  

We wish to find a suitable constant $c > 0$ such that our new coefficients $\tilde{f}$ and $\tilde{a}$ verify the conditions in Propositions 4.1 and 4.2. Clearly, if $u$ is a solution of (4.26), then $cu$ will solve (1.1) accordingly. Obviously, the coefficient $h$ remains unchanged after scaling and we also have $\lambda_f = \lambda_{\tilde{f}}$ since $c > 0$. Therefore, the following conditions:

$$|h| < \lambda_{\tilde{f}}, \quad \tilde{a} > 0, \quad \int_M \tilde{f}dv_g < 0, \quad \sup_M \tilde{f}^+ > 0$$
are fulfilled. Besides, it is easy to see that
\[
\sup_M \tilde{f} = \frac{\sup_M f}{\int_M |f^{-}|dv_g}.
\]
We now wish to remove (4.1) but still keep other conditions. In other words, we have to choose a suitable \(c\) such that the following conditions:
\[
|h| \leq \frac{\eta_0}{p^*} \int_M |\tilde{f}^{-}|dv_g, \tag{4.27}
\]
and
\[
\sup_M \tilde{f} < C_2, \tag{4.28}
\]
and
\[
\int_M \tilde{a}dv_g < \frac{p}{2(n-p)} \left( \frac{2n-p}{2(n-p)} \right)^{\frac{2n}{p} - 1} \left( \frac{|h|}{\int_M |f^{-}|dv_g} \right)^{\frac{2n}{p}} \int_M |\tilde{f}^{-}|dv_g \tag{4.29}
\]
hold. Indeed, (4.27) and (4.29) can be rewritten as follows:
\[
|h| \leq \frac{\eta_0 p^* - p}{p^*} \int_M \tilde{f}^{-}dv_g \tag{4.30}
\]
and
\[
\int_M \tilde{a}dv_g < \frac{p}{2(n-p)} \left( \frac{2n-p}{2(n-p)} \right)^{\frac{2n}{p} - 1} \left( \frac{|h|}{\int_M |f^{-}|dv_g} \right)^{\frac{2n}{p}} \int_M |f^{-}|dv_g.
\]
Thus, the condition (1.4) is invariant under the variable change. In view of (4.30), we can choose
\[
c = \left( \frac{p^* |h|}{\eta_0 \int_M |f^{-}|dv_g} \right)^{\frac{1}{p^* - p}}.
\]
It suffices to prove that this particular choice of \(c\) and the condition (1.5) are enough to guarantee (4.28). Notice that
\[
\sup_M \tilde{f} = e^{p^* - p} \sup_M f = \left( \frac{p^* |h|}{\eta_0 \int_M |f^{-}|dv_g} \right) \left( \sup_M f \right) = \frac{|h| p^*}{\eta_0} \frac{\sup_M f}{\int_M |f^{-}|dv_g}.
\]
Therefore, if we assume
\[
\frac{\sup_M f}{\int_M |f^{-}|dv_g} < \frac{\eta_0}{|h| p^*} C_2,
\]
then the condition (4.28) holds. In conclusion, if the constant \(C\) in Theorem 1.1 equals
\[
\min \left\{ C_1, \frac{\eta_0}{|h| p^*} C_2 \right\},
\]
we know that (1.1) has at least two positive solutions. This finishes the proof of Theorem 1.1.

5 Proof of Theorem 1.2

In this section, we prove Theorem 1.2 which provides a sufficient condition for the solvability of (1.1). As in the previous sections, we need to study the asymptotic behavior of \(\mu_{k,q}^\varepsilon\) for small \(k\) and large \(k\), respectively.

We first assume that \(f \leq 0\) but not strictly negative. We consider two possible cases.

**Case I.** \(\sup_M f = 0\) and \(\int_{\{f=0\}} dv_g = 0\). In this case, it holds that \(f < 0\) almost everywhere in \(M\) which implies that \(\mathcal{A} = \emptyset\). Hence, it holds that \(\lambda_f = +\infty\). However, for each \(\eta \neq 0\), \(\lambda_{f,\eta,q}\) is well defined as in (2.4), and is monotone decreasing with respect to \(\eta\) whose proof is exactly the same as the proof of Lemma 3.6. Moreover, we have the following lemma which is an analogous version of Lemma 3.8, and we omit its proof.
Lemma 5.1. There exists $\eta_0$ such that for all $\eta < \eta_0$, there exists $q_\eta \in (p^*, p^*)$ such that $\lambda_{f, q_\eta} > |h|$ for every $q \in (q_\eta, p^*)$.

Case II. $\sup_M f = 0$ and $\int_{\{f = 0\}} 1 dv_g > 0$. Under this case, $\lambda_f$ is well defined and finite. A careful analysis shows that all the results from Subsection 3.2 still hold.

We are now in a position to study the behavior of $\mu_{k, q}^\infty$ for $k \to +\infty$ when $\sup_M f = 0$.

Proposition 5.2. Suppose $\sup_M f = 0$. If

- either $\int_{\{f = 0\}} 1 dv_g = 0$,
- or $\int_{\{f = 0\}} 1 dv_g > 0$ and $\lambda_f > |h|$,

then $\mu_{k, q}^\infty \to +\infty$ as $k \to +\infty$ for any $\varepsilon > 0$ sufficiently small and any $q$ sufficiently close to $p^*$ but all are fixed.

Proof. We begin to prove that there exist some $\eta_0 > 0$ and its corresponding $q_{\eta_0} \in (p^*, p^*)$ such that $\delta_0 = (\lambda_{f, q_{\eta_0}} + h)/p > 0$ for any $q \in (q_{\eta_0}, p^*)$. As previously mentioned, we consider two cases separately.

Case 1. Suppose that $\sup_M f = 0$ and $\int_{\{f = 0\}} 1 dv_g = 0$. In this case, $\lambda_f = +\infty$. Since $h$ is fixed, by Lemma 5.1, there exist $\eta_0$ and its corresponding $q_\eta \in (p^*, p^*)$ such that $\lambda_{f, q_\eta} + h > 0$ for any $q \in (q_{\eta_0}, p^*)$. This proves the positivity of $\delta_0$.

Case 2. Suppose that $\sup_M f = 0$ and $\int_{\{f = 0\}} 1 dv_g > 0$. In this case, $\lambda_f$ is well defined and finite. Notice that $\lambda_f > |h|$. Since all the results in Subsection 3.2 still hold, as in the proof of Proposition 3.9, there exist $0 < \eta_0 < 2$ and its corresponding $q_{\eta_0} \in (p^*, p^*)$ such that

$$0 \leq \lambda_f - \lambda_{f, q_{\eta_0}} < \frac{1}{2}(\lambda_f - |h|)$$

for any $q \in (q_{\eta_0}, p^*)$. Therefore, $\delta_0 > \frac{1}{2p}(\lambda_f + h) > 0$.

Now having the strict positivity of $\delta_0$ we can easily go through the proof of Proposition 3.9 and get $G_q(u) \geq mk^2$ where $m$ is given as in (3.14). This implies that $I_q^\infty(u) \geq mk^2$ since $\sup_M f = 0$. Since $\delta_0$ has a strictly positive lower bound, so does $m$. The proof now follows easily.

Remark 5.3. For small $k$, using the same argument as in Lemma 3.3(i), one has $\mu_{k, q}^{\infty} \to +\infty$ as $k \to 0^+$. Thus, compared with the case $\sup_M f > 0$, the curve $k \mapsto \mu_{k, q}^{\infty}$ takes a shape as shown in Figure 2.

We are now in a position to prove Theorem 1.2(1) which is similar to the proof of Theorem 1.1, and therefore we just sketch the proof and omit some details.

Proof of Theorem 1.2(1). Suppose that $\sup_M f = 0$. Let $q \in (q_{\eta_0}, p^*)$. Since

$$I_q^\infty(k^2) = \frac{h}{p}k^\frac{p}{q} - \frac{k}{q} \int_{M} f dv_g + \frac{1}{q} \int_{M} \frac{a}{(k^\frac{p}{q} + \varepsilon)^2} dv_g,$$

by solving the following equation:

$$\frac{h}{p}k^\frac{p}{q} - \frac{k}{q} \int_{M} f dv_g = 0,$$

Figure 2  The asymptotic behavior of $\mu_{k, q}^{\infty}$ when $\sup_M f < 0$
one can easily get
\[ \mu_{k_0,q}^\varepsilon \leq T_q^\varepsilon (k_0^2) = \frac{1}{q} \int_M \frac{a}{(k_0^2 + \varepsilon)^{\frac{q}{2}}} dv_g < \frac{1}{p k_0} \int_M a dv_g, \]
where
\[ k_0 = \left( \frac{q}{p} \frac{h}{\int_M f dv_g} \right)^{\frac{2}{q-p}}. \]

It is then easy to find \( k_1 \) and \( k_2 \) independent of both \( q \) and \( \varepsilon \) such that \( k_1 < k_0 < k_2 \) with \( k_2 > 1 \). Now according to the asymptotic behavior of \( \mu_{k,q}^\varepsilon \), one can find \( k_* \) and \( k_{**} \) independent of both \( q \) and \( \varepsilon \) with \( k_* < k_1 < k_0 < k_2 < k_{**} \) such that \( \mu_{k_0,q}^\varepsilon < \min\{\mu_{k_*}^{\varepsilon,q}, \mu_{k_{**}}^{\varepsilon,q}\} \). Then we define
\[ E_{\varepsilon,q} = \inf_{u \in D_{k,q}} I_{\varepsilon,q} (u) \]
for each \( \varepsilon \) and \( q \) fixed, where \( D_{k,q} = \{u \in H^1_M : \|u\|_q^2 = k \} \). At this point, one can use exactly the same arguments as in the proof of Proposition 4.1 to obtain a positive solution of (1.1).

Let us now assume that \( \sup_M f < 0 \). It suffices to study the asymptotic behavior of \( \mu_{k,q}^\varepsilon \) for large \( k \). Clearly, for any \( u \in B_{k,q} \), we have
\[ T_q^\varepsilon (u) \geq \frac{h}{p} k^\frac{q}{2} + \frac{k}{p^*} \sup_M |f|. \]
We then immediately see that \( \mu_{k,q}^\varepsilon \to +\infty \) as \( k \to +\infty \) since \( p/q < 1 \) (see Figure 2). With the same idea above, we can conclude that (1.1) admits a positive solution. This is the content of the following result whose proof is straightforward.

**Proposition 5.4.** If \( \sup_M f < 0 \), then (1.1) admits a positive solution.

**Remark 5.5.** One can also prove Theorem 1.2(2) with the help of the classical sub- and super-solution method. See [8] for the case \( p = 2 \) and [10] for the case \( a(x) \equiv 0 \).

**Acknowledgements** This work was supported by National Natural Science Foundation of China (Grant Nos. 11771342 and 11571259), and the Natural Science Foundation of Hubei Province (Grant No. 2019CFA007).

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