ENERGY DISTRIBUTION OF A STATIONARY BEAM OF LIGHT

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ABSTRACT. Aguirregabiria et al showed that Einstein, Landau and Lifshitz, Papapetrou, and Weinberg energy-momentum complexes coincide for all Kerr-Schild metric. Bringley used their general expression of the Kerr-Schild class and found energy and momentum densities for the Bonnor metric. We obtain these results without using Aguirregabiria et al results and verify that Bringley’s results are correct. This also supports Aguirregabiria et al results as well as Cooperstock hypothesis. Further, we obtain the energy distribution of the space-time under consideration.

1. INTRODUCTION

One of the most interesting problems which remains unsolved since Einstein proposal of general theory of relativity, is the energy-momentum localization. After Einstein [1] obtained an expression for the energy-momentum complexes many physicists, such as Landau and Lifshitz [2], Tolman [3], Papapetrou [4], and Weinberg [5] had given different definitions for the energy-momentum complex. These definitions were restricted to evaluate energy distribution in quasi-cartesian coordinates. This motivated Møller [6] and many other, like Komar [7] and Penrose [8], to construct coordinate independent definitions. Each of these has its own drawbacks (see [9]).

In the past decade much work has been produced showing the Einstein, Tolman, Landau and Lifshitz, Papapetrou and Weinberg (later ETLLPW) complexes give meaningful results for many well known space-times (see [10]-[22]). Aguirregabiria et al [23] showed that the energy-momentum complexes of ETLLPW give the same result for any Kerr-Schild (KS) metric if the computations are performed in Kerr-Schild cartesian coordinates. The energy and momentum densities are calculated for a stationary beam of light using the definitions of ELLPW. It is shown that the four energy-momentum complexes of ELLPW coincide if the calculations are carried out in Kerr-Shild cartesian coordinates.

In the present paper we calculate the five energy-momentum complexes of ETLLPW for the Bonnor metric if the calculations are performed in cartesian coordinates.

We use the convention that Latin indices take values from 0 to 3 and Greek indices take values from 1 to 3, and take units where $G = 1$ and $c = 1$. 
The Bonnor metric is given by

\[ ds^2 = (1 + m)dt^2 - 2m dt dz - dx^2 - dy^2 - (1 - m) dz^2, \]

where \( m \) is a function of \( x \) and \( y \) only.

The non-zero components of the energy momentum tensor \( T^a_b \)

\[ -T^3_3 = -T^0_3 = T^3_0 = T^0_0 = \rho, \]

where \( \rho \) is the energy density, and

\[ \nabla^2 m = 16\pi \rho. \]

\( m \) must be such \( \rho \) is non-negative. The line element (2.1) describing a stationary beam of light flowing in the \( z \)-direction.

The determinant and the non-zero components of the contravariant metric tensor are

\[
\begin{align*}
g &= -1 \\
g^{00} &= 1 - m, \\
g^{03} &= -m, \\
g^{11} &= -1, \\
g^{22} &= -1, \\
g^{33} &= -(1 + m).
\end{align*}
\]

Now, we can require the following list of non-vanishing components of the Christoffel symbol

\[
\begin{align*}
\Gamma^0_{01} &= \frac{1}{2} m_x, & \Gamma^0_{02} &= \frac{1}{2} m_y, \\
\Gamma^0_{13} &= -\frac{1}{2} m_x, & \Gamma^0_{23} &= -\frac{1}{2} m_y, \\
\Gamma^1_{00} &= \frac{1}{2} m_x, & \Gamma^1_{03} &= -\frac{1}{2} m_x, \\
\Gamma^1_{33} &= \frac{1}{2} m_x, & \Gamma^2_{00} &= \frac{1}{2} m_y, \\
\Gamma^2_{03} &= -\frac{1}{2} m_y, & \Gamma^2_{33} &= \frac{1}{2} m_y, \\
\Gamma^3_{01} &= \frac{1}{2} m_x, & \Gamma^3_{02} &= \frac{1}{2} m_y, \\
\Gamma^3_{13} &= -\frac{1}{2} m_x, & \Gamma^3_{23} &= \frac{1}{2} m_y,
\end{align*}
\]

where subscripts denote partial differentiation.
3. Energy-momentum Complex as Defined by Einstein

The energy-momentum complex as defined by Einstein is given by

\[ \theta^k_i = \frac{1}{16\pi} H^{kl}_{i,l}, \]

where

\[ H^{kl}_{i} = -H^{lk}_{i} = \frac{g_{in}}{\sqrt{-g}} \left[ -g(g^{kn}g^{lm} - g^{ln}g^{km}) \right]_{,m}. \]

\( \theta^0_0 \) and \( \theta^0_\alpha \) are the energy and momentum density components, respectively. The energy-momentum complex \( \theta^k_i \) satisfies the local conservation law

\[ \frac{\partial \theta^k_i}{\partial x^k} = 0. \]

The energy and momentum in Einstein’s prescription are given by

\[ P_i = \int \int \int \theta^0_i dx^1 dx^2 dx^3. \]

Using the Gauss’s theorem, we get

\[ P_i = \frac{1}{16\pi} \int \int H^{0\alpha}_{i} n_\alpha ds, \]

where \( n_\beta \) is the outward unit normal vector over an infinitesimal surface element \( ds \). \( P_0 \) and \( P_\alpha \) are the energy and momentum components.

In order to evaluate the energy and momentum densities in Einstein’s prescription associated with the Bonnor space-time, we evaluate the non-zero components of \( H^{kl}_{i} \)

\[ H^{01}_{0} = m_x, \]
\[ H^{02}_{0} = m_y, \]
\[ H^{01}_{3} = -m_x, \]
\[ H^{02}_{3} = -m_y. \]

Using these components and equation (2.2) in equation (3.1), we get the energy and momentum densities as following

\[ \theta^0_0 = \rho. \]

\[ \theta^{03} = \eta^{a3} \theta^0_3 = \rho. \]

Using (3.6) in equation (3.3), we get the expression of the energy in the form

\[ E_{Ein} = M. \]
4. Energy Distribution in Tolman’s Prescription

The energy-momentum complex of Tolman is

\[ \mathcal{I}^i_k = \frac{1}{8\pi} U^i_{k,j}, \]

where

\[ U^i_j = \sqrt{-g} \left[ g^{pi} \left( -\Gamma^j_{kp} + \frac{1}{2} \delta^j_k \Gamma^a_{ap} + \frac{1}{2} \delta^j_p \Gamma^a_{ak} \right) \right] + \frac{1}{2} \delta^j_k g^{pm} \left( -\Gamma^j_{pm} + \frac{1}{2} \delta^j_p \Gamma^a_{am} + \frac{1}{2} \delta^j_m \Gamma^a_{ap} \right), \]

\( \mathcal{I}^0_0 \) is the energy density, \( \mathcal{I}^\alpha_0 \) are the components of the energy current density, and \( \mathcal{I}^0_\alpha \) are the momentum density components.

The energy-momentum complex \( \mathcal{I}^i_k \) satisfies the local conservation law

\[ \frac{\partial \mathcal{I}^i_k}{\partial x^i} = 0. \]

The energy distribution in the Tolman’ definition \( E_{Tol} \) is given by

\[ E_{Tol} = \int \int \int \mathcal{I}^0_0 \, dx \, dy \, dz. \]

Using the Gauss theorem (noting that the space-time under consideration is static), one has

\[ E_{Tol} = \frac{1}{8\pi} \int \int U^0_\beta n^\beta(\alpha) \, ds(\alpha), \]

where \( n^\beta \) is the unit vector over an infinitesimal surface element, \( ds \).

Using (2.3) and the components of Christoffel symbol in (1.2), after straightforward but rather lengthy calculations, we get

\[ U^0_1 = \frac{1}{2} m_x, \]
\[ U^0_2 = \frac{1}{2} m_y, \]
\[ U^0_3 = -\frac{1}{2} m_x, \]
\[ U^0_3 = -\frac{1}{2} m_y, \]

Using (1.3) and the condition (2.2) in (1.1), we obtain the energy density and momentum density in the form

\[ \mathcal{I}^0_0 = \rho \]
\[ \mathcal{I}^{03} = \eta^{33} \mathcal{I}^0_3 = \rho. \]
Using (4.6) in equation (4.3), then the energy distribution associated with the Bonnor space-time is given by

\[ E_{\text{Tol}} = M. \]

5. The Landau and Lifshitz Energy-momentum Complex

Landau and Lifshitz’s energy-momentum complex is given by

\[ L^{ij} = \frac{1}{16\pi} S^{ijkl}_{,kl}, \]

where

\[ S^{ijkl} = -g^{ij}g^{kl} - g^{il}g^{kj}. \]

\( L^{ij} \) is symmetric in its indices, \( L^{00} \) is the energy density and \( L^{0\alpha} \) are the momentum (energy current) density components. \( S^{ijkl} \) has symmetries of the Riemann curvature tensor.

The expression

\[ P^i = \int \int \int L^{i0} dx^1 dx^2 dx^3 \]

gives the energy \( P_0 \) and the momentum \( P_\alpha \) components. Further Gauss’s theorem furnishes the energy \( E_{LL} \) given by

\[ E_{LL} = \frac{1}{16\pi} \int \int S^{0\alpha\beta}_\alpha n_\beta ds, \]

where \( n_\beta \) is the outward unit normal vector over an infinitesimal surface element \( ds \).

The non-zero components of \( S^{ijkl} \) for the line element (2.1) are

\[ S^{0101} = -(1 - m), \]
\[ S^{0202} = -(1 - m), \]
\[ S^{0311} = m, \]
\[ S^{0322} = m. \]

Now, substituting (5.5) and using the condition (2.2), we obtain the energy and momentum densities of the line element (2.1) in the sense of Landau and Lifshitz in the following form

\[ L^{00} = \rho. \]

\[ L^{03} = \rho \]

Using (5.6) in equation (5.3), we get the energy

\[ E_{LL} = M. \]
6. The Energy-Momentum Complex of Papapetrou

The symmetric energy-momentum complex of Papapetrou is given by

\[ \Omega^{ij} = \frac{1}{16\pi} \gamma^{ijkl}_{,kl}, \]  

where

\[ \gamma^{ijkl} = \sqrt{-g} \left( g^{ij} \eta^{kl} - g^{ik} \eta^{jl} + g^{jl} \eta^{ij} - g^{ij} \eta^{jl} \right), \]

and \( \eta^{ik} \) is the Minkowski metric with signature \(-2\). \( \Omega^{00} \) and \( \Omega^{0\alpha} \) are the energy and momentum density components. The energy and momentum components are given by

\[ P^i = \int \int \int \Omega^{0dx^1dx^2dx^3}. \]

Using the Gauss theorem, the energy \( E_P \) takes the following form

\[ E_P = \frac{1}{16\pi} \int \int \gamma^{0\alpha\beta}_{,\beta} h_{\alpha} ds. \]

To find the energy and momentum densities of the space-time under consideration, we require the following non-zero components of \( \gamma^{ijkl} \)

\[ \begin{align*}
\gamma^{0011} &= m - 2, \\
\gamma^{0022} &= m - 2, \\
\gamma^{0311} &= m, \\
\gamma^{0322} &= m.
\end{align*} \]

Using these components in (6.1), we get the energy and momentum densities in the following form

\[ \begin{align*}
\Omega^{00} &= \rho, \\
\Omega^{03} &= \rho.
\end{align*} \]

The energy \( E_P \) of the Bonnor space-time is obtained by using (6.6) in (6.3)

\[ E_P = M. \]

7. The Weinberg Energy-Momentum Complex

The symmetric energy-momentum complex of Weinberg is given by

\[ W^{ij} = \frac{1}{16\pi} \Delta^{ijk}_{,k}, \]

where

\[ \Delta^{ijk} = \frac{\partial h_a^i}{\partial x^1} \eta^{jk} - \frac{\partial h_a^j}{\partial x^1} \eta^{ik} - \frac{h^{ai}}{\partial x^A} \eta^{jk} + \frac{h^{aj}}{\partial x^A} \eta^{ik} \]

\[ + \frac{\partial h^{jk}}{\partial x^i} - \frac{\partial h^{ij}}{\partial x^k}, \]
and
\[ h_{ij} = g_{ij} - \eta_{ij}. \]

\( \eta_{ij} \) is the Minkowski metric with signature \(-2\). The indices on \( h_{ij} \) or \( \frac{\partial}{\partial x^i} \)
are raised or lowered with the help of \( \eta \)'s. The Weinberg energy-momentum complex \( W^{ij} \) satisfies the local conservation laws
\[ \frac{\partial W^{ij}}{\partial x^j} = 0. \]

\( W^{00} \) and \( W^{\alpha 0} \) are the energy and momentum density components. The energy and momentum components are given by
\[ (7.3) \quad P^i = \int \int W^{i0} dx^1 dx^2 dx^3. \]

The only required components of \( \triangle^{ijk} \) in the calculation of the energy and momentum densities are the following
\[ \begin{align*}
\triangle^{001} &= m_x, \\
\triangle^{002} &= m_y, \\
\triangle^{031} &= m_x, \\
\triangle^{032} &= m_y.
\end{align*} \]

Using these results in (7.4) and using the condition (2.2), we obtain the energy density and momentum density of the space-time (2.1)
\[ (7.5) \quad W^{00} = \rho. \]
\[ (7.6) \quad W^{03} = \rho. \]

Using (7.5) in (7.3) the energy \( E_W \) is
\[ (7.7) \quad E_W = M. \]

8. Summary

It is well-known that the subject of the energy-momentum localization is associated with much debate. Misner et al. [25] argued that the energy is localizable only for spherical systems. Cooperstock and Saracino [26] gave their viewpoint that if the energy is localizable in spherical systems then it is localizable for all systems. Bondi [27] gave his opinion that a nonlocalizable form of energy is not admissible in relativity so its location can in principle be found.

Some interesting results which have been found recently (see for example, [10], [15], [16]) lend support to the idea that the several energy-momentum complexes can give the same and acceptable result for a given space-time. Virbhadra [28] emphasized that though that the energy-momentum complexes are non-tensors under general coordinate transformations, the local conservation laws with them hold in all coordinate system. Aguirregabiria
et al. [23] showed that different energy-momentum complexes yield the same energy distribution for any Kerr-Schild class metric.

In this paper we obtained the energy distribution associated with the Bonnor metric describing a stationary beam of light. We used the energy-momentum complexes of Einstein, Tolman, Landau and Lifshitz, Papapetrou and Weinberg. All these definitions give the same results for the energy density, momentum density, and energy distribution. Our calculations are performed in Cartesian coordinates.

The results for the energy and momentum densities obtained here are the same as the results obtained by Bringley [21] using the energy-momentum complexes of ELLPW in Kerr-Schild cartesian coordinates.

Our results sustain the Aguirregabiria et al. [23] results as well as Cooperstock hypothesis [29] (which essentially states that the energy and momentum in a curved space-time are confined to the regions of non-vanishing energy-momentum tensor $T_{ij}$ of the matter and all non-gravitational fields).

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