Explicit solution of the Cauchy problem for cellular automaton rule 172

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Abstract

Cellular automata (CA) are fully discrete alternatives to partial differential equations (PDE). For PDEs, one often considers the Cauchy problem, or initial value problem: find the solution of the PDE satisfying a given initial condition. For many PDEs of the first order in time, it is possible to find explicit formulae for the solution at the time $t > 0$ if the solution is known at $t = 0$. Can something similar be achieved for CA? We demonstrate that this is indeed possible in some cases, using elementary CA rule 172 as an example. We derive an explicit expression for the state of a given cell after $n$ iteration of the rule 172, assuming that states of all cells are known at $n = 0$. We then show that this expression (“solution of the CA”) can be used to obtain an expected value of a given cell after $n$ iterations, provided that the initial condition is drawn from a Bernoulli distribution. This can be done for both finite and infinite lattices, thus providing an interesting test case for investigating finite size effects in CA.

1. Introduction

Cellular automata are often described as fully discrete alternatives to partial differential equations (PDEs). In one dimension, a PDE which is first-order in time can be written as

$$u_t(x, t) = F(u, u_x, u_{xx}, \ldots),$$

where $u(x, t)$ is the unknown function, and the independent variables $t$ and $x$ are commonly interpreted as, respectively, time and position in space. Both variables $t$ and $x$, as well as $u(x, t)$, take values in the set of real numbers.

Cellular automata (CA), on the other hand, are typically written as

$$u(i, n + 1) = f(u(i - r, n), u(i - r + 1, n), \ldots, u(i + r, n)),$$

where $f$ is called a local function and the integer $r$ is called a radius of the cellular automaton. For CA independent variables $n$ (representing time) and $i$ (representing space) are integers, while $u(i, n)$ takes values in a finite set of symbols, usually integers. In the case of binary cellular automata, which are the main focus of this paper, $u(i, n)$ takes values in the set $\{0, 1\}$, so that $f : \{0, 1\}^{2r+1} \rightarrow \{0, 1\}$. 
Comparing eqs. (1) and (2) we conclude that discrete time \( n \) in CA plays the role of \( t \) in PDEs, \( i \) in CA plays the role of \( x \) in PDEs, and \( r \) in CA plays a similar role as the degree of the highest derivative in PDEs. In fact, there are some further analogies, but we will not discuss them here. We will only mention that there exist discretization schemes (such as ultradiscretization) which allow to construct CA from PDE while preserving some features of the dynamics, but they are beyond the scope of this paper. We merely want to indicate here that conceptually, cellular automata are closely related to PDEs, although in contrast to PDEs, all variables in CA are discrete. Moreover, dependent variable \( u \) is bounded in the case of CA – a restriction which is not normally imposed on the dependent variable of a PDEs.

For PDEs, the initial value problem (also called the Cauchy problem) is often considered. It is the problem of finding \( u(x,t) \) for \( t > 0 \) subject to

\[
\begin{align*}
  u_t(x,t) &= F(u,u_x,u_{xx},\ldots), \quad \text{for } x \in \mathbb{R}, \ t > 0, \\
  u(x,0) &= G(x) \quad \text{for } x \in \mathbb{R},
\end{align*}
\]

where the function \( G : \mathbb{R} \to \mathbb{R} \) represents given initial data. A similar problem can be formulated for cellular automata: given

\[
\begin{align*}
  u(x,t + 1) &= f(u(x - r,t),u(x - r + 1,t),\ldots,u(x + r,t)), \\
  u(x,0) &= g(x),
\end{align*}
\]

find \( u(x,t) \) for \( t > 0 \), where the initial data is represented by the function \( g : \mathbb{Z} \to \{0,1\} \).

For the problem (4), it is easy to find the value of \( u(x,t) \) for any \( x \in \mathbb{Z} \) and any \( t \in \mathbb{N} \) by direct iteration of the cellular automaton equation (2). Thus, in the algorithmic sense, problem (4) is always solvable – all one needs to do is to take the initial data \( g(x) \) and perform \( n \) iterations.

In contrast to this, the initial value problem for PDE cannot be solved exactly by direct iteration. In some cases, however, one can obtain exact solution in the sense of a formula for \( u(x,t) \) involving \( G(x) \). To give a concrete example, consider the classical Burgers equation,

\[
u_t = u_{xx} + uu_x.
\]

If \( u(x,0) = G(x) \), one can show that for \( t > 0 \),

\[
u(x,t) = 2 \frac{\partial}{\partial x} \ln \left\{ -\frac{(x - \xi)^2}{4t} \exp \left[ -\frac{(x - \xi)^2}{4t} - \frac{1}{2} \int_0^\xi G(\xi')d\xi' \right] d\xi \right\}.
\]

Can we obtain similar formulae for cellular automata? The answer is affirmative in some cases. These cases usually involve “simple” CA rules. The goal of this paper is to demonstrate how to obtain solution of a CA in one of such “simple” cases, using elementary CA rule 172 as an example. We will also show some applications of the solution. Some ideas presented here appeared in a preliminary form in an earlier conference proceedings paper [2].

2. Basic definitions

Let \( \mathcal{A} = \{0,1\} \) be called a symbol set, and let \( \mathcal{S} = \{0,1\}^\mathbb{Z} \) be the set of all bisequences over \( \mathcal{A} \), to be called a configuration space.

A block or word of length \( n \) is an ordered set \( b_0 b_1 \ldots b_{n-1} \), where \( n \in \mathbb{N} \), \( b_i \in \mathcal{A} \). Let \( n \in \mathbb{N} \) and let \( \mathcal{B}_n \) denote the set of all blocks of length \( n \) over \( \mathcal{A} \) and \( \mathcal{B} \) be the set of all finite blocks over \( \mathcal{A} \).
For $r \in \mathbb{N}$, a mapping $f : \{0, 1\}^{2r+1} \rightarrow \{0, 1\}$ will be called a cellular automaton rule of radius $r$. Alternatively, the function $f$ can be considered as a mapping of $\mathcal{B}_{2r+1}$ into $\mathcal{B}_0 = \mathcal{A} = \{0, 1\}$.

Corresponding to $f$ (also called a local mapping) we define a global mapping $F : \mathcal{S} \rightarrow \mathcal{S}$ such that $(F(s))_i = f(s_{i-r}, \ldots, s_i, \ldots, s_{i+r})$ for any $s \in \mathcal{S}$.

A block evolution operator corresponding to $f$ is a mapping $f : \mathcal{B} \rightarrow \mathcal{B}$ defined as follows. Let $r \in \mathbb{N}$ be the radius of $f$, and let $a = a_0 a_1 \ldots a_{n-1} \in \mathcal{B}_n$ where $n \geq 2r + 1 > 0$. Then

$$f(a) = \{f(a_i, a_{i+1}, \ldots, a_{i+2r})\}_{i=0}^{n-2r-1}.$$  \hspace{1cm} (7)

Note that if $b \in \mathcal{B}_{2r+1}$ then $f(b) = f(b)$. The set of $n$-step preimages of the block $b$ under the rule $f$ is defined as the set $f^{-n}(b) = \{c \in \mathcal{B} : f^n(c) = b\}$. Note that the notion of block preimages has been, in somewhat different context, studied in many earlier works, including [3, 4, 5, 6].

Binary rules of radius 1 are called elementary rules, and they are usually identified by their Wolfram number $W(f)$, defined as

$$W(f) = \sum_{x_1, x_2, x_3=0}^{1} f(x_1, x_2, x_3) 2^{(2^2 x_1 + 2^1 x_2 + 2^0 x_3)}.$$  \hspace{1cm} (8)

We will consider, as an example of a “solvable” CA rule, one of the elementary rules, namely the rule with Wolfram number 172. Its local function $f : \{0, 1\}^3 \rightarrow \{0, 1\}$ is defined as

$$f(x_1, x_2, x_3) = \begin{cases} x_2 & \text{if } x_1 = 0, \\ x_3 & \text{if } x_1 = 1. \end{cases}$$  \hspace{1cm} (9)

It is easy to verify that for the above $f$ we have $W(f) = 172$.

The reason for which rule 172 was selected is that its dynamics is simple enough to render it “solvable”, yet it is not entirely trivial. Further explanation regarding the meaning of “non-trivial” will be given in the conclusion section. We shall also add that many properties of rule have been studied in the past, usually in the context of other elementary CA. Some of these include place of rule 172 in various CA classifications, structure of its Garden of Eden configurations, algebraic properties, and various properties of its global function [8, 9, 10, 11].

In what follows, whenever we use the symbol $f$ it will signify the local function defined in eq. (9), while $f$ and $F$ will denote, respectively, the corresponding block evolution operator and the global function. To familiarize the reader with the concept of the block evolution operator, let us take as an example $b = 1001010$. We can compute $f(b)$ by applying $f$ to all consecutive triples of symbols, that is, $f(b) = f(100)f(001)f(010)f(101)f(010) = 00111$. If we apply $f$ again to 00111, we will obtain $f^2(b) = 011$, and yet another application of $f$ yields $f^3(b) = 1$. It is sometimes convenient to write consecutive images of $b$ under each other, as follows:

$$1001010$$ $$00111$$ $$011$$ $$1$$

The above shows, starting from the top, $b$, $f(b)$, $f^2(b)$, and $f^3(b)$.

We shall also note that there is usually more than one block $c$ such that $f(b) = c$. For example, for rule 172, $f(0010) = f(0011) = f(1101) = 01$. We can, therefore, write $f^{-1}(01) = \{0010, 0011, 1101\}$. Similarly, we can write

$$f^{-2}(101) = \{001101, 0101101, 0111101, 1011101, 1101101, 1111101\},$$  \hspace{1cm} (10)
because all 6 blocks on the right hand side of the above (and only these blocks) have the property that after applying $f$ to them twice, one obtains 101.

Our strategy for constructing the solution of rule 172 will be as follows. First, we will construct $n$-step preimages of 1 (i.e., sets $f^{-n}(1)$) for various $n$. We will then try to find patterns in these sets which would allow us to give a combinatorial description of them, as sets of binary strings satisfying certain conditions. Once this is done, we will construct a Boolean function which is an indicator function of $f^{-n}(1)$. Such function will then be used to construct an explicit expression for $[F^n(x)]_i$ for any $x \in \{0, 1\}^\mathbb{Z}$, $i \in \mathbb{Z}$ and $n \in \mathbb{N}$.

3. Structure of preimage sets

Suppose now that we have a string $b$ of length $2n + 1$ and we want to find out the necessary and sufficient conditions for $f^n(b) = 1$. We will try to “guess” these conditions first, formulate them in a rigorous way, and then prove them.

In order to “guess” the conditions, one can generate sets $f^{-n}(1)$ for various values of $n$ and try to discover obvious patterns in them. From author’s experience, a good way to do this is to build minimal finite state machines generating words of $f^{-n}(1)$. This can be done using AT&T FSM Library [12, 13], and Figure 1 shows an example of a minimal finite state machine (FSM) generating $f^{-7}(1)$ for rule 172. In order to generate a preimage of 1 using this picture, start on the left (at circled zero) and follow the arrows writing down all encountered edge labels until you reach the final state (doubly circled 27). The string of 15 labels obtained this way will be a possible preimage of 1, one of many. Obviously there are as many preimage string as paths joining the initial state and the final state. Note that circled numbers denote internal states of the FSM, and are irrelevant for our purposes.

From Figure 1 it is clear that the first 5 symbols of $f^{-6}(1)$ are arbitrary, and then we have two possibilities:

(i) 001 followed by 6 arbitrary symbols, or

(ii) string of 8 symbols without 00 pair anywhere, followed by 10 or 11 (if it ends with 0) or by 01 or 11 (if it ends by 1).

This observation can be generalized and summarized as the following proposition.

**Proposition 3.1** Block $b$ of length $2n + 1$ belongs to $f^{-n}(1)$ if and only if it has the structure

$$b = \underbrace{\star \ldots \star} _{n-2} 001 \underbrace{\star \ldots \star} _n,$$  \hspace{1cm} (11)

or

$$b = \underbrace{\star \ldots \star} _{n-2} a_1 a_2 \ldots a_{n+1} c_1 c_2,$$ \hspace{1cm} (12)
Figure 2: Example of a spatiotemporal pattern produced by rule 172.

where \( a_1a_2 \ldots a_n \) is a binary string which does not contain any pair of adjacent zeros, and

\[
    c_1c_2 = \begin{cases} 
    1\star, & \text{if } a_{n+1} = 0, \\
    \star1, & \text{otherwise.} 
    \end{cases}
\]  

We will sketch the proof of the above, leaving out some tedious details. It will be helpful to inspect spatiotemporal pattern generated by rule 172 first, as shown in Figure 2. Careful inspection of this pattern reveals three facts, each of them easily provable in a rigorous way:

(F1) A cluster of two or more zeros keeps its right boundary in the same place for ever.

(F2) A cluster of two or more zeros extends its left boundary to the left one unit per time step as long as the left boundary is preceded by two or more ones. If the left boundary of the cluster of zeros is 01, the cluster does not grow.

(F3) Isolated zero moves to the left one step at a time as long as it has at least two ones on the left. If an isolated zero is preceded by 01, it disappears in the next time step.

Suppose now that we have a string \( b \) of length \( 2n + 1 \) and we want to find out the necessary and sufficient conditions for \( f^n(b) = 1 \). From (F1) it is clear that the word 001 will remain in the same position forever, which means that if

\[
b = \star \ldots \star 001 \star \ldots \star, \]

then \( f^n(b) = 1 \). What are the other possibilities for \( b \) which would result in \( f^n(b) = 1 \)?

From (F2) we deduce that if there is no cluster of two or more more zeros somewhere in the last \( n + 3 \) bits of \( b \), then there is no possibility of the growth of cluster of zeros producing \( f^n(b) = 0 \). The only way to get zero after \( n \) iterations of \( f \) in such a case would be having zero at the end of \( b \) preceded by 11. This means that in order to avoid this scenario, the last 3 bits of \( b \) must be 010, 011, 101 or 111, or, in other words, the last three bits must be 01\( \star \) or 1\( \star \)1, as in eq. (13). □

4. Solving rule 172

We are now almost ready to construct the solution of rule 172. Suppose that \( x \in \{0,1\}^\mathbb{Z} \) is an initial configuration, and that we iterate rule 172 \( n \) times. What is the value of the central site
after \( n \) iterations of the rule, that is, the value of \([F^n(x)]_0\)? Obviously it can be either 0 or 1, so let us suppose that it is 1, which means that \( x_{-n}x_{-n+1} \ldots x_n \in \mathcal{F}^{-n}(1) \). By the virtue of Proposition 3.1, \( x_{-n}x_{-n+1} \ldots x_n \) must take one of the two forms, the first of them being

\[
x_{-n}x_{-n+1} \ldots x_n = \underbrace{* \ldots *}_{n-2} 001 \underbrace{* \ldots *}_{n}.
\]

The above means that \( x_{-2}x_{-1}x_0 = 001 \), and it will be true if and only if

\[
(1 - x_{-2})(1 - x_{-1})x_0 = 1.
\]

The second possibility, according to Proposition 3.1, is

\[
x_{-n}x_{-n+1} \ldots x_n = \underbrace{* \ldots *}_{n-2} a_1a_2 \ldots a_{n+1}c_1c_2,
\]

where \( a_ia_{i+1} \neq 0 \) for \( i = 1, 2, \ldots, n \) and \( c_1, c_2 \) satisfy condition of eq. (13). This means that \( a_1a_2 \ldots a_{n+1} = c_2c_{-1} \ldots c_{t-2} \), and therefore \( x_i = x_{i+1} \neq 0 \) for \( i = -2, -1, \ldots, t-3 \), as well as

\[
x_{t-1} = \begin{cases} 1*, & \text{if } x_{t-2} = 0, \\ 1, & \text{otherwise}. \end{cases}
\]

The second possibility will be realized if and only if

\[
\left( \prod_{i=-2}^{t-3} (1 - \bar{x}_i) \right) \left( \bar{x}_{t-2}x_{t-1} + x_{t-2}x_t \right) = 1,
\]

where we used notation \( \bar{x}_i = 1 - x_i \). Combining eqs. (16) and (19) we obtain

\[
[F^n(x)]_0 = \bar{x}_{-2}\bar{x}_{-1}x_0 + \left( \prod_{i=-2}^{n-3} (1 - \bar{x}_i) \right) \left( \bar{x}_{n-2}x_{n-1} + x_{n-2}x_n \right).
\]

This is the desired solution expressing the value of the central site after \( n \) iterations of rule \( F \) starting from an initial configuration \( x \). Of course, we can now obtain analogous expression for any other site \([F^n(x)]_j\), by simply translating the above formula from \( j = 0 \) to an arbitrary position \( j \).

**Proposition 4.1** Let \( F \) be the global function of elementary CA rule 172 and \( x \in \{0, 1\}^\mathbb{Z} \). Then, after \( n \in \mathbb{N} \) iterations of \( F \), for any \( j \in \mathbb{Z} \),

\[
[F^n(x)]_j = \bar{x}_{j-2}\bar{x}_{j-1}x_j + \left( \prod_{i=j-2}^{j+n-3} (1 - \bar{x}_i) \right) \left( \bar{x}_{j+n-2}x_{j+n-1} + x_{j+n-2}x_{j+n} \right).
\]

The above could be called a solution of rule 172. It is an explicit solution of the Cauchy problem for this rule, expressing the state of a site at position \( j \) after \( n \) iterations in terms of initial site values. The formula for the solution is very simple, utilizing only addition, subtraction, and multiplication of site values. As we will see in subsequent sections, it can be very useful in practice, for example for constructing probabilistic solutions of the CA rule and investigating finite size effects.
5. Probabilistic solution for infinite configurations

Let us now assume that the initial configuration $x$ is drawn from a Bernoulli distribution. More precisely, let $x_j = X_j$ for $j \in \mathbb{Z}$, where $X_j$ are independent and identically distributed random variables such that $Pr(X_j = 1) = q$, $Pr(X_j = 0) = 1 - q$, where $q \in [0, 1]$. What is the expected value of $[F^n(x)]_j$ in such circumstances? Denoting the expected value by $\langle \cdot \rangle$, let us first note that due to translational invariance, $\langle [F^n(x)]_j \rangle = \langle [F^n(x)]_0 \rangle$. In order to compute $\langle [F^n(x)]_0 \rangle$, we need to calculate the expected value of the right hand side of eq. (20). The following lemma will be useful for this purpose.

**Lemma 5.1** Let $q \in (0, 1)$ and let $X_i$ be independent and identically distributed Bernoulli random variables for $i \in \{0, 1, \ldots, n\}$ such that $Pr(X_i = 1) = q$, $Pr(X_i = 0) = 1 - q$. Then

$$\langle \prod_{i=1}^{n-1} (1 - \bar{X}_i \bar{X}_{i+1}) \rangle = \frac{q}{\lambda_2 - \lambda_1} (\alpha_1 \lambda_2^{n-2} + \alpha_2 \lambda_2^{-2}), \quad (22)$$

where

$$\lambda_{1, 2} = \frac{1}{2} q \pm \frac{1}{2} \sqrt{q(4 - 3q)}, \quad (23)$$

$$\alpha_{1, 2} = \left( \frac{q}{2} - 1 \right) \sqrt{q(4 - 3q)} \pm \left( \frac{q}{2} - 1 \right). \quad (24)$$

To prove it, let us first define

$$U_n = \prod_{i=1}^{n-1} (1 - \bar{X}_i \bar{X}_{i+1}) \quad \text{and} \quad V_n = \bar{X}_n \prod_{i=1}^{n-1} (1 - \bar{X}_i \bar{X}_{i+1}). \quad (25)$$

We observe that

$$U_n = U_{n-1}(1 - \bar{X}_{n-1}\bar{X}_n) = U_{n-1} - \bar{X}_n V_{n-1}, \quad (26)$$

and

$$V_n = \bar{X}_n U_n = \bar{X}_n U_{n-1}(1 - \bar{X}_{n-1}\bar{X}_n) = \bar{X}_n U_{n-1} - \bar{X}_n V_{n-1}, \quad (27)$$

where we used the fact $X_n$ is a Boolean variable, thus $\bar{X}_n^2 = \bar{X}_n$. This yields the system of recurrence equations for $U_n$ and $V_n$,

$$U_n = U_{n-1} - \bar{X}_n V_{n-1}, \quad (28)$$

$$V_n = \bar{X}_n U_{n-1} - \bar{X}_n V_{n-1}. \quad (29)$$

Since $\bar{X}_n$ is independent of both $U_{n-1}$ and $V_{n-1}$, we can write

$$\langle U_n \rangle = \langle U_{n-1} \rangle - \langle \bar{X}_n \rangle \langle V_{n-1} \rangle, \quad (29)$$

$$\langle V_n \rangle = \langle \bar{X}_n \rangle \langle U_{n-1} \rangle - \langle \bar{X}_n \rangle \langle V_{n-1} \rangle. \quad (30)$$

Now, taking into account that $\langle \bar{X}_n \rangle = 1 - q$, we obtain

$$\begin{bmatrix} \langle U_n \rangle \\ \langle V_n \rangle \end{bmatrix} = M \begin{bmatrix} \langle U_{n-1} \rangle \\ \langle V_{n-1} \rangle \end{bmatrix}, \quad (31)$$

where

$$M = \begin{bmatrix} 1 & q - 1 \\ 1 - q & q - 1 \end{bmatrix}. \quad (32)$$
This recurrence equation is easy to solve,

\[
\begin{bmatrix}
\langle U_n \rangle \\
\langle V_n \rangle
\end{bmatrix} = M^{n-2} \begin{bmatrix}
\langle U_2 \rangle \\
\langle V_2 \rangle
\end{bmatrix}.
\]

(33)

Since \( \langle U_2 \rangle \) and \( \langle V_2 \rangle \) can be directly computed,

\[
\langle U_2 \rangle = (1 - \bar{X}_1 \bar{X}_2) = 1 - (1 - q)^2 = 2q - q^2,
\]

(34)

\[
\langle V_2 \rangle = \langle \bar{X}_2(1 - \bar{X}_1 \bar{X}_2) \rangle = \langle \bar{X}_2 - \bar{X}_1 \bar{X}_2 \rangle = 1 - q - (1 - q)^2 = q - q^2,
\]

(35)

we obtain

\[
\begin{bmatrix}
\langle U_n \rangle \\
\langle V_n \rangle
\end{bmatrix} = M^{n-2} \begin{bmatrix}
2q - q^2 \\
q - q^2
\end{bmatrix}.
\]

(36)

The only thing left is to compute is \( M^{n-2} \). This can be done by diagonalizing \( M \),

\[
M^{n-2} = P \begin{bmatrix}
\lambda_1^{n-2} & 0 \\
0 & \lambda_2^{n-2}
\end{bmatrix} P^{-1},
\]

(37)

where \( \lambda_{1,2} \) are eigenvalues of \( M \), as defined in eq. (23), and \( P \) is the matrix of eigenvectors of \( P \),

\[
P = \begin{bmatrix}
\frac{1-q}{1-\lambda_1} & \frac{1-q}{1-\lambda_2} \\
1 & 1
\end{bmatrix}, \quad P^{-1} = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix}
\frac{(\lambda_1 - 1)(\lambda_2 - 1)}{-1+q} & \lambda_1 - 1 \\
\frac{(\lambda_1 - 1)(\lambda_2 - 1)}{-1+q} & -\lambda_2 + 1
\end{bmatrix}.
\]

(38)

The final formula for \( U_n \) and \( V_n \) is

\[
\begin{bmatrix}
\langle U_n \rangle \\
\langle V_n \rangle
\end{bmatrix} = P \begin{bmatrix}
\lambda_1^{n-2} & 0 \\
0 & \lambda_2^{n-2}
\end{bmatrix} P^{-1} \begin{bmatrix}
2q - q^2 \\
q - q^2
\end{bmatrix}.
\]

(39)

By carrying out the multiplications of matrices on the right hand side, after some algebra, we obtain

\[
\langle U_n \rangle = \frac{q}{\lambda_2 - \lambda_1} \left( (-1 - q + q^2 + 2 \lambda_2 - q \lambda_2) \lambda_1^{n-2} + (1 + q - q^2 - 2 \lambda_1 + q \lambda_1) \lambda_2^{n-2} \right),
\]

(40)

in agreement with eq. (22). □

Before we take the expected value of both sides of eq. (20), let us first rewrite the last factor,

\[
\bar{x}_{n-2}x_{n-1} + x_{n-2}x_n = (1 - x_{n-2})x_{n-1} + x_{n-2}x_n = x_{n-1} + x_{n-2}(x_n - x_{n-1}).
\]

(41)

Using the above, we obtain from eq. (20),

\[
\langle [F^n(x)]_0 \rangle = \langle \bar{x}_{n-2}x_{n-1}x_0 \rangle + \left\langle \prod_{i=-2}^{n-3} (1 - \bar{x}_i \bar{x}_{i+1}) \right\rangle x_{n-1} + \left\langle \prod_{i=-2}^{n-3} (1 - \bar{x}_i \bar{x}_{i+1}) \right\rangle x_{n-2}(x_n - x_{n-1})
\]

(42)
Using the fact that the expected value of the product of independent random variables is equal to the product of their expected values, this yields

\[
\langle [F_n(x)]_0 \rangle = \langle \bar{x}_{-2} \rangle \langle x_0 \rangle + \left\langle \prod_{i=-2}^{n-3} (1 - \bar{x}_i \bar{x}_{i+1}) \right\rangle \langle x_{n-1} \rangle + \left\langle \prod_{i=-2}^{n-3} (1 - \bar{x}_i \bar{x}_{i+1}) \right\rangle \langle x_{n-2} \rangle \langle x_{n} - x_{n-1} \rangle.
\] (43)

Because \( \langle x_n - x_{n-1} \rangle = 0 \), the last term vanishes, and, using the fact that \( \langle x_i \rangle = q \) and \( \langle \bar{x}_i \rangle = 1 - q \), we obtain

\[
\langle [F_n(x)]_0 \rangle = (1 - q)^2 q + q \left\langle \prod_{i=-2}^{n-3} (1 - \bar{x}_i \bar{x}_{i+1}) \right\rangle.
\] (44)

Using Lemma 5.1 and remembering that the expected value must be the same for any index \( j \), the final result is thus

\[
\langle [F_n(x)]_j \rangle = (1 - q)^2 q + \frac{q^2}{\lambda_2 - \lambda_1} \left( \alpha_1 \lambda_1^{n-1} + \alpha_2 \lambda_2^{n-1} \right).
\] (45)

The above could be called a probabilistic solution of CA rule 172 for infinite Bernoulli initial configuration. Note that since \( |\lambda_{1,2}| < 1 \), we have

\[
\lim_{n \to \infty} \langle [F_n(x)]_j \rangle = (1 - q)^2 q.
\] (46)

When \( q = 1/2 \), eq. (45) becomes, after simplification,

\[
\langle [F_n(x)]_j \rangle = \frac{1}{8} + \left( \frac{1}{4} + \frac{\sqrt{5}}{10} \right) \left( \frac{1}{4} + \frac{\sqrt{5}}{4} \right)^n + \left( \frac{1}{4} - \frac{\sqrt{5}}{10} \right) \left( \frac{1}{4} - \frac{\sqrt{5}}{4} \right)^n.
\] (47)

Since \( \frac{1}{4} + \frac{\sqrt{5}}{4} \) is half of ratio divina (the golden ratio), one recognizes a link to Fibonacci numbers in the above. Indeed, it is easy to show that for \( q = 1/2 \),

\[
\langle [F_n(x)]_j \rangle = \frac{1}{8} + \frac{\mathcal{F}_{n+3}}{2^{n+2}},
\] (48)

where \( \mathcal{F}_n \) is the \( n \)-th Fibonacci number.

6. Probabilistic solution for periodic configuration

Suppose now that the initial condition is periodic with period \( k \), that is, \( x_i = x_{i+k} \) for all \( i \in \mathbb{Z} \). Although one could of course consider all finite configurations of a given length and determine their attractors, it will nevertheless be useful to construct a general formula valid for arbitrary \( k \). This could be, for example, useful if one wants to study the dependence of the speed of convergence to the steady state as a function of \( k \).

We will take \( i = 0, \ldots, k - 1 \) as the principal period. The solution will be given by the same formula as before, except that all indices are to be taken modulo \( k \). Let us further assume that, as before, \( x_j = X_j \) for \( j \in \{0, 1, \ldots, k - 1\} \), where \( X_j \) are independent and identically distributed random variables such that \( Pr(X_j = 1) = q, Pr(X_j = 0) = 1 - q \), where \( q \in [0, 1] \).
In order to compute the expected value of a site after \(n\) iterations of rule 172, we take expected value of both sides of eq. (21) (remembering that indices are now modulo \(k\)), obtaining

\[
\langle [F^n(x)]_j \rangle = (1 - q)^2 q + \left( \prod_{i=j-2}^{j+n-3} (1 - \bar{x}_i \bar{x}_{i+1}) \right) \left( \bar{x}_{j+n-2} x_{j+n-1} + x_{j+n-2} x_{j+n} \right) .
\]  

(49)

Observe that when \(j = -2\) (remember that the expected value must be \(j\)-independent, so the choice of \(j\) does not matter), the only indices of \(x\) occurring on the right hand side of the above will be in the range from 0 to \(n + 2\). This means that for \(n \leq k - 3\), we will never actually need to use modulo \(k\) operation to bring the index to the principal period range. For this reason, eq. (45) remains valid in the periodic case as long as \(n \leq k - 3\).

Let us now suppose that \(n \geq k\). In this case,

\[
\prod_{i=j-2}^{j+n-3} (1 - \bar{x}_i \bar{x}_{i+1}) = \prod_{i=0}^{k-1} (1 - \bar{x}_i \bar{x}_{i+1}),
\]  

(50)

because in the product on the left hand side there are only \(k\) different factors, and \((1 - \bar{x}_i \bar{x}_{i+1})^m = (1 - \bar{x}_i \bar{x}_{i+1})\) for any positive integer \(m\). We will once again take advantage of the translational symmetry. Since \(\langle [F^n(x)]_j \rangle\) should be the same for all \(j\), we will take \(j = k - n\),

\[
\langle [F^n(x)]_j \rangle = \langle [F^n(x)]_{k-n} \rangle = (1 - q)^2 q + \left( \prod_{i=0}^{k-1} (1 - \bar{x}_i \bar{x}_{i+1}) \right) \left( \bar{x}_{k-2} x_{k-1} + x_{k-2} x_0 \right) .
\]  

(51)

Before we proceed further, let us note that

\[
\bar{x}_{k-2} x_{k-1} + x_{k-2} x_0 = \bar{x}_{k-2} (1 - \bar{x}_{k-1}) + (1 - \bar{x}_{k-2}) (1 - \bar{x}_0) = 1 - \bar{x}_{k-2} \bar{x}_{k-1} - \bar{x}_0 + \bar{x}_0 \bar{x}_{k-2}.
\]  

(52)

Since \((1 - \bar{x}_{k-2} \bar{x}_{k-1}) \prod_{i=0}^{k-1} (1 - \bar{x}_i \bar{x}_{i+1}) = \prod_{i=0}^{k-1} (1 - \bar{x}_i \bar{x}_{i+1})\), we obtain

\[
\langle [F^n(x)]_j \rangle = (1 - q)^2 q + \left( \prod_{i=0}^{k-1} (1 - \bar{x}_i \bar{x}_{i+1}) \right) + \left( \bar{x}_0 (\bar{x}_{k-2} - 1) \prod_{i=0}^{k-1} (1 - \bar{x}_i \bar{x}_{i+1}) \right) .
\]  

(53)

We will deal with the two expected values on the right hand side separately. Let us start from the
first one.

\[
\langle k-1 \prod_{i=0}^{k-1} (1 - \bar{x}_i \bar{x}_{i+1}) \rangle = \langle (1 - \bar{x}_0 \bar{x}_1) \prod_{i=1}^{k-2} (1 - \bar{x}_i \bar{x}_{i+1})(1 - \bar{x}_{k-1} \bar{x}_0) \rangle
\]

\[
= \langle (1 - \bar{x}_0 \bar{x}_1)(1 - \bar{x}_{k-1} \bar{x}_0) \prod_{i=1}^{k-2} (1 - \bar{x}_i \bar{x}_{i+1}) \rangle
\]

\[
= \langle (1 - \bar{x}_0 \bar{x}_{k-1} - \bar{x}_0 \bar{x}_1 + \bar{x}_0 \bar{x}_1 \bar{x}_{k-1}) \prod_{i=1}^{k-2} (1 - \bar{x}_i \bar{x}_{i+1}) \rangle
\]

\[
= \prod_{i=1}^{k-2} (1 - \bar{x}_i \bar{x}_{i+1}) - \langle \bar{x}_0 \rangle \prod_{i=1}^{k-2} (1 - \bar{x}_i \bar{x}_{i+1})
\]

\[
- \langle \bar{x}_0 \rangle \prod_{i=1}^{k-2} (1 - \bar{x}_i \bar{x}_{i+1}) + \langle \bar{x}_0 \rangle \bar{x}_1 \prod_{i=1}^{k-2} (1 - \bar{x}_i \bar{x}_{i+1})
\]

Recall that

\[
U_n = \prod_{i=1}^{n-1} (1 - \bar{X}_i \bar{X}_{i+1}) \quad \text{and} \quad V_n = \bar{X}_n \prod_{i=1}^{n-1} (1 - \bar{X}_i \bar{X}_{i+1}),
\]

(54)

and define

\[
U'_n = \bar{X}_1 \prod_{i=1}^{n-1} (1 - \bar{X}_i \bar{X}_{i+1}) \quad \text{and} \quad V'_n = \bar{X}_1 \bar{X}_n \prod_{i=1}^{n-1} (1 - \bar{X}_i \bar{X}_{i+1}).
\]

(55)

Now we have

\[
\langle k-1 \prod_{i=0}^{k-1} (1 - \bar{x}_i \bar{x}_{i+1}) \rangle = \langle U_{k-1} \rangle - (1 - q)\langle V_{k-1} \rangle - (1 - q)\langle U'_{k-1} \rangle
\]

\[
+ (1 - q)\langle V'_{k-1} \rangle = \langle U_k \rangle + q\langle U'_{k-1} \rangle - \langle U'_k \rangle + (1 - q)\langle V'_k \rangle
\]

\[
= \langle U_k \rangle + q\langle U'_{k-1} \rangle - \langle U'_k \rangle,
\]

(56)

where we used the fact that \(\langle U'_n \rangle, \langle V'_n \rangle\) satisfy the same recurrence equations as \(\langle U_n \rangle, \langle V_n \rangle\).
The second expected value in eq. (53) can be calculated as follows.

\[
\left\langle x_0(x_{k-2} - 1) \prod_{i=0}^{k-3} (1 - \bar{x}_i x_{i+1}) \right\rangle 
= \left\langle x_0(x_{k-2} - 1)(1 - \bar{x}_{k-2} \bar{x}_{k-1})(1 - \bar{x}_{k-1} x_0) \prod_{i=0}^{k-3} (1 - \bar{x}_i x_{i+1}) \right\rangle 
= \left\langle x_0(x_{k-2} - \bar{x}_{k-2} \bar{x}_{k-1} - 1 + \bar{x}_{k-1}) \prod_{i=0}^{k-3} (1 - \bar{x}_i x_{i+1}) \right\rangle 

- 2 \left\langle x_0 \prod_{i=0}^{k-3} (1 - \bar{x}_i x_{i+1}) \right\rangle 
+ \left\langle x_0 \bar{x}_{k-1} \prod_{i=0}^{k-3} (1 - \bar{x}_i x_{i+1}) \right\rangle = \langle V'_{k-1} \rangle + \langle U'_{k} \rangle - 2\langle U'_{k-1} \rangle + (1 - q)\langle U'_{k-1} \rangle.
\]

Combining both expected values computed above we get

\[
\langle [F^n(x)]_j \rangle = (1 - q)^2 q + \langle U_k \rangle + q\langle U'_{k-1} \rangle - \langle U'_{k} \rangle + \langle V'_{k-1} \rangle + \langle U'_{k} \rangle 
- 2\langle U'_{k-1} \rangle + (1 - q)\langle U'_{k-1} \rangle = (1 - q)^2 q + \langle U_k \rangle - \langle U'_{k-1} \rangle + \langle V'_{k-1} \rangle.
\]

Since

\[
\langle V'_{k} \rangle = (1 - q)\langle U'_{k-1} \rangle - (1 - q)\langle V'_{k-1} \rangle,
\]

we obtain

\[
\langle U'_{k-1} \rangle - \langle V'_{k-1} \rangle = \frac{1}{1 - q} \langle V'_{k} \rangle,
\]

and therefore

\[
\langle [F^n(x)]_j \rangle = (1 - q)^2 q + \langle U_k \rangle - \frac{1}{1 - q} \langle V'_{k} \rangle.
\]

Now we need to find \( \langle V'_{k} \rangle \). As noted before, the recurrence equations for \( \langle U'_{n} \rangle, \langle V'_{n} \rangle \) are the same as for \( \langle U_{n} \rangle, \langle V_{n} \rangle \), that is, as in eq. (28). Only initial conditions are different,

\[
\langle U'_{2} \rangle = \langle \bar{X}_1 (1 - \bar{X}_1 \bar{X}_2) \rangle = \langle \bar{X}_1 - \bar{X}_1 \bar{X}_2 \rangle = 1 - q - (1 - q)^2 = q - q^2,
\]

\[
\langle V'_{2} \rangle = \langle X_1 X_2 (1 - X_1 X_2) \rangle = \langle X_1 X_2 - X_1 X_2 \rangle = 0.
\]

The formula (39) thus becomes

\[
\begin{bmatrix} \langle U'_{n} \rangle \\ \langle V'_{n} \rangle \end{bmatrix} = P \begin{bmatrix} \lambda_1^{n-2} & 0 \\ 0 & \lambda_2^{n-2} \end{bmatrix} P^{-1} \begin{bmatrix} q - q^2 \\ 0 \end{bmatrix},
\]

where \( P \) and \( P^{-1} \) are defined in eq. (38). After carrying out matrix multiplication and simplification this yields

\[
\langle V'_{n} \rangle = \frac{q(1 - q)^2}{\lambda_1 - \lambda_2} \left( \lambda_1^{n-2} - \lambda_2^{n-2} \right).
\]
The final result is thus
\[
\langle [F^n(x)]_j \rangle = (1 - q)^2 q + \frac{q}{\lambda_2 - \lambda_1} \left( \alpha_1 \lambda_1^{k-2} + \alpha_2 \lambda_2^{k-2} \right) - \frac{q(1 - q)}{\lambda_1 - \lambda_2} \left( \lambda_2^{k-2} - \lambda_1^{k-2} \right),
\]
which simplifies to
\[
\langle [F^n(x)]_j \rangle = (1 - q)^2 q + \frac{q}{\lambda_2 - \lambda_1} \left( \alpha_1 + 1 - q \right) \lambda_1^{k-2} + (\alpha_2 - 1 + q) \lambda_2^{k-2},
\]
where \(\lambda_{1,2}\) and \(\alpha_{1,2}\) are defined in eq. (23). Note that the above expression does not depend on \(n\), which means that \(\langle [F^n(x)]_j \rangle\) becomes constant when \(n \geq k\).

In fact, one can show that when \(n = k - 1\) and \(n = k - 2\), eq. (65) remains valid. We will omit details, but the reasoning is very similar as in the case of \(n \geq k\). The final result for the periodic case can thus be summarized as follows.
\[
\langle [F^n(x)]_j \rangle = (1 - q)^2 q + \left\{ \begin{array}{ll}
\frac{q^2}{\lambda_2 - \lambda_1} \left( \alpha_1 \lambda_1^{n-1} + \alpha_2 \lambda_2^{n-1} \right) & \text{if } n \leq k - 3, \\
\frac{q}{\lambda_2 - \lambda_1} \left( \alpha_1 + 1 - q \right) \lambda_1^{k-2} + (\alpha_2 - 1 + q) \lambda_2^{k-2} & \text{if } n \geq k - 2.
\end{array} \right.
\]

7. Finite vs. infinite configurations

The case of the periodic initial configuration is often interpreted as a finite configuration of \(k\) sites with periodic boundary conditions. We can say, therefore, that we have obtained probabilistic solutions for both infinite (eq. 45) and finite (eq. 66) configurations. Let us briefly describe differences between them. To simplify notation, we will define \(c_n = \langle [F^n(x)]_j \rangle\), and we will call \(c_n\) the density of ones after \(n\) iterations of the rule.

Figure 3 shows plots of the density \(c_n\) versus \(n\) for both finite \((k = 20)\) and infinite configurations with \(q = 0.5\). One can see that initially they are identical, and at \(n = k - 2 = 18\) they split. The steady state density is clearly higher in the finite case, albeit not too much. The difference between finite and infinite configurations is much more dramatic when the initial density \(q\) becomes closer to 1. This is demonstrated Figure 4 which shows the graph of \(c_\infty = \lim_{n \to \infty} c_n\) as a function of \(q\).

One can see that the difference between steady states of finite and infinite configurations grows rapidly when \(q\) approaches 1. In fact, in the vicinity of \(q = 1\), finite configurations tend to density approaching zero, while infinite configurations tend to density approaching 1. This shows the danger of using finite lattices with periodic boundaries as somewhat “resembling” infinite ones, as it is sometimes done in cellular automata simulations and models. We can conclude from Figure 4 that finite size effects can be very significant in CA, even leading to outcomes completely opposite than those expected for an infinite system.

8. Conclusions

We have demonstrated that a simple CA rule, namely rule 172, can be explicitly solved, meaning that it is possible to obtain a closed form formula for the state of a given cell after \(n\) iterations of the rule, as in eq. (21). Such formula is useful for further analysis of properties of the rule. Using
Figure 3: Plot of $c_n$ as a function of $n$ for $q = 1/2$ for infinite initial configuration (solid line) and periodic initial configuration with period $k = 20$ (dashed line). Dots represent average value of the state of site $i = 0$ after $n$ iterations of rule 172, for finite periodic configuration of 20 sites, obtained by direct numerical iteration of randomly generated initial configuration repeated $10^6$ times.

Figure 4: Plot of $c_\infty$ as a function of $q$ for infinite initial configuration (solid line) and periodic initial configuration with period $k = 20$ (dashed line).
it, we obtained “probabilistic” solutions, that is, the expected value of a cell after \( n \) iterations for both infinite and finite configurations, assuming that the initial state is drawn from a Bernoulli distribution. This, in turn, allowed us to investigate the role and significance of finite size effects for rule 172.

It should be stressed that probabilistic solution obtained here is exact, and thus it should not be confused with approximate methods such as the mean-field theory \[14\] or local structure theory \[15, 16\].

Although the method presented here is, obviously, not a general one, it can be used for other rules providing that their dynamics is not overly complicated. While it is difficult to pinpoint what “not overly complicated” means precisely, some empirical observations can be made. First of all, let us notice that in rule 172 if a pair of zeros occurs somewhere in the initial condition, it stays in the same place throughout iterations of the rule. The word 00 is thus a blocking word \[17\] for precise definition. It is known that all CA rules possessing a blocking word are almost equicontinuous \[17\], thus rule 172 has that property too. Existence of a blocking word severely limits propagation of information between sites, thus making the rule “simple”, and a good candidate for solving using the method outlined in this paper. One should add, however, that the rule 172 is only almost equicontinuous, but not equicontinuous. Furthermore, its entropy is positive, as shown in the appendix, thus its dynamics is not entirely trivial. The fact that it can nevertheless be solved is encouraging, and it seems highly probable that other almost-equicontinuous rules with positive entropy could be solved in a similar fashion.

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Appendix: entropy of rule 172
Let us recall that \( \mathcal{A} = \{0, 1\} \) and that the local function of rule 172 is defined as

\[
f(x_1, x_2, x_3) = \begin{cases} 
  x_2 & \text{if } x_1 = 0, \\
  x_3 & \text{if } x_1 = 1.
\end{cases}
\]  

(67)

Corresponding to \( f \), we define a global mapping \( F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}} \) such that \( (F(x))_i = f(x_{i-1}, x_i, x_{i+1}) \) for any \( x \in \mathcal{A}^{\mathbb{Z}} \). We will be interested in the entropy of the dynamical system \( (\mathcal{A}^{\mathbb{Z}}, F) \), to be denoted by \( h(\mathcal{A}^{\mathbb{Z}}, F) \).

Proposition 8.1 Entropy of \( (\mathcal{A}^{\mathbb{Z}}, F) \), where \( F \) is the global function of CA rule 172, is positive.

Let \( \Sigma_{\{00,010\}} \) be the set of all elements of \( \mathcal{A}^{\mathbb{Z}} \) in which words 00 and 010 do not occur. \( (\Sigma_{\{00,010\}}, \sigma) \) is then a subshift of finite type, where \( \sigma \) is the usual shift map, defined as \( \sigma(x)_i = x_{i+1} \). We will first show that rule 172 restricted to \( \Sigma_{\{00,010\}} \) is equivalent to shift map,

\[
F|_{\Sigma_{\{00,010\}}} = \sigma.
\]  

(68)

Indeed, consider the value of \( f(x_{i-1}, x_i, x_{i+1}) \) for \( x \in \Sigma_{\{00,010\}} \). If \( x_{i-1} = 0 \), we must have \( x_i = 1 \) (because double zeros are forbidden), and \( x_{i+1} = 1 \) (because isolated ones are forbidden), therefore \( f(0, x_i, x_{i+1}) = f(0, 1, 1) = 1 = x_{i+1} \). If, on the other hand, \( x_{i-1} = 1 \), then by the definition of \( f \)
for rule 172 shown in eq. (9), \(f(1, x_i, x_{i+1}) = x_{i+1}\). This means that \(f(x_{i-1}, x_i, x_{i+1}) = x_{i+1}\) for all \(x \in \Sigma_{\{00, 010\}}\), as required.

Let us now compute the entropy of \((\Sigma_{\{00, 010\}}, \sigma)\) using the method outlined in [18]. First, we need to find a Markov shift conjugate to \((\Sigma_{\{00, 010\}}, \sigma)\). This can be done by defining new symbol set \(B = \{011, 101, 111\} := \{a, b, c\}\). If \(\Sigma_{\{ba\}}\) denotes the set of points of \(B^\mathbb{Z}\) in which the word \(ba\) does not occur, then it is easy to show that the subshift \((\Sigma_{\{00, 010\}}, \sigma)\) is conjugate to \((\Sigma_{\{ba\}}, \sigma)\), which is a Markov subshift. Adjacency matrix \(3 \times 3\) for \((\Sigma_{\{ba\}}, \sigma)\) is defined by \(M_{i,j} = 1\) if \((i, j) \neq (b, a)\) and \(M_{i,j} = 0\) if \((i, j) = (b, a)\). Spectral radius of this matrix is \((3 + \sqrt{5})/2\), therefore

\[
h(\Sigma_{\{ba\}}, \sigma) = h(\Sigma_{\{00, 010\}}, \sigma) = h(\Sigma_{\{00, 010\}}, F) = \ln \frac{3 + \sqrt{5}}{2}, \quad (69)
\]

where we used the fact that the entropy is invariant with respect to conjugacy. This shows that \(h(\Sigma_{\{00, 010\}}, F) > 0\). Since \(\Sigma_{\{00, 010\}} \subset A^\mathbb{Z}\), we conclude that \(h(A^\mathbb{Z}, F) > 0\).

References

[1] T. Tokihiro, *Discrete Integrable Systems*, ch. Ultradiscrete Systems (Cellular Automata), pp. 383–424. Springer, Berlin, Heidelberg, 2004.

[2] H. Fukš, “Probabilistic initial value problem for cellular automaton rule 172,” *DMTCS proc. AL* (2010) 31–44.

[3] E. Jen, “Table of preimage formulae for elementary rules,” Report LA-UR-88-3359, Los Alamos National Laboratory, 1988.

[4] E. Jen, “Enumeration of preimages in cellular automata,” *Complex Systems* 3 (1989) 421–456.

[5] B. H. Voorhees, *Computational analysis of one-dimensional cellular automata*. World Scientific, Singapore, 1996.

[6] H. V. McIntosh, *One Dimensional Cellular Automata*. Luniver Press, 2009.

[7] S. Wolfram, *Cellular Automata and Complexity: Collected Papers*. Addison-Wesley, Reading, Mass., 1994.

[8] G. J. Martinez, “A note on elementary cellular automata classification,” *Journal of Cellular Automata* 8 (2013) 233–259.

[9] V. Bulitko, B. Voorhees, and V. Bulitko, “Discrete baker transformations for linear cellular automata analysis,” *Journal of Cellular Automata* 1 (2006) 41–70.

[10] B. Voorhees, “Commutation of cellular automata rules,” *Complex Systems* 7 (1993) 309–325.

[11] H. Nishio, “Automorphissm classification of cellular automata,” *Fundamenta Informaticae* 104 (2010), no. 1–2, 125–140.

[12] S. Skiena, “AT&T Finite-State Machine Library, version 4.0.”

http://www3.cs.stonybrook.edu/~algorith/implement/fsm/implement.shtm

[13] S. Skiena, *The Algorithm Design Manual*. Springer, 2008.
[14] H. Gutowitz and C. Langton, “Mean field theory of the edge of chaos,” in in Proceedings of ECAL3, pp. 52–64. Springer, 1995.

[15] H. A. Gutowitz, J. D. Victor, and B. W. Knight, “Local structure theory for cellular automata,” Physica D 28 (1987) 18–48.

[16] H. Fukš, “Construction of local structure maps for cellular automata,” J. of Cellular Automata 7 (2013) 455–488.

[17] P. Kürka, “Topological dynamics of cellular automata,” in Encyclopedia of Complexity and System Science, R. A. Meyers, ed. Springer, 2009.

[18] D. Lind and B. Marcus, Symbolic dynamics and coding. Cambridge UP, Cambridge, 1995.