Analysis of a free boundary problem modeling the growth of necrotic tumors*

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Abstract

In this paper we make rigorous mathematical analysis to a free boundary problem modeling the growth of necrotic tumors. A remarkable feature of this free boundary problem is that it contains two different-type free surfaces: One is the tumor surface whose evolution is governed by an evolution equation and the other is the interface between the living shell of the tumor and the necrotic core which is an obstacle-type free surface, i.e., its evolution is not governed by an evolution equation but instead is determined by some stationary-type equation. In mathematics, the inner free surface is induced by discontinuity of the nonlinear reaction functions in this model, which causes the main difficulty of analysis of this free boundary problem. Previous work on this model studies spherically symmetric situation which is in essence an one-dimension free boundary problem. The purpose of this paper is to make rigorous analysis in general spherically asymmetric situation. By applying the Nash-Moser implicit function theorem, we prove that the inner free surface is smooth and depends on the outer free surface smoothly when it is a small perturbation of the surface of a sphere. By applying this result and some abstract results for parabolic differential equations in Banach manifolds we prove that the unique radial stationary solution of this free boundary problem is asymptotically stable under small non-radial perturbations.

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1 Introduction

The study of mathematical theory of tumor growth has lasted for over eighty years. It is motivated by a basic observation that under constant conditions, an evolutionary tumor undergoes three typical periods of growth: Firstly it goes through a nearly exponential growth period, next it experiences a nearly linear growth period, and finally it evolves into a stationary or dormant state [1, 6, 26, 27, 30]. In the dormant state, the tumor usually contains an inner necrotic core made by dead cells, an outer proliferating shell occupied by proliferating cells, and an intermediate region occupied by quiescent living cells [1, 4]. During 1970’s, Greenspan proposed the first mathematical model in the form of free boundary problem of reaction diffusion equations to explain this phenomenon [22, 23]. His model was very well improved by Byrne and

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 Chaplain during 1990’s [3, 4]. Since then many different tumor models have been established by different groups of researchers, cf. the reviewing articles [2, 17] and references cited therein. Rigorous mathematical analysis of such free boundary problems has attracted much attention during the past twenty years and many interesting results have been obtained, cf. [7, 8, 9, 12, 13, 14, 18, 19, 20, 21, 31, 32] and references cited therein.

In this paper we study the following free boundary problem modeling the growth of necrotic tumors:

\[
\begin{align*}
\Delta \sigma &= f(\sigma) & \text{in } \Omega(t), \quad t > 0, \\
-\Delta p &= g(\sigma) & \text{in } \Omega(t), \quad t > 0, \\
\sigma &= \bar{\sigma} & \text{on } \partial \Omega(t), \quad t > 0, \\
p &= \gamma \kappa & \text{on } \partial \Omega(t), \quad t > 0, \\
V_n &= -\partial_n p & \text{on } \partial \Omega(t), \quad t > 0, \\
\Omega(0) &= \Omega_0.
\end{align*}
\tag{1.1}
\]

Here \(\Omega(t)\) is the domain in \(\mathbb{R}^3\) occupied by the tumor at time \(t\), \(\sigma = \sigma(x, t)\) and \(p = p(x, t)\) are the nutrient concentration in the tumor region and the pressure between tumor cells, respectively, \(\partial_n\) represents the derivative in the direction of the outward normal \(n\) of the tumor surface \(\partial \Omega(t)\), \(\kappa\) is the mean curvature of the tumor surface \(\partial \Omega(t)\) whose sign is designated by the convention that for the sphere it is positive, \(\bar{\sigma}\) is a positive constant reflecting the surface tension of the tumor surface and is usually referred as surface tension coefficient, \(V_n\) is the normal velocity of the tumor surface movement, \(\Omega_0\) is the domain that the tumor initially occupies, and \(f, g\) are given functions respectively having the following forms:

\[
f(\sigma) = \lambda \sigma H(\sigma - \bar{\sigma}), \quad g(\sigma) = \mu(\sigma - \bar{\sigma})H(\sigma - \bar{\sigma}) - \nu,
\tag{1.2}
\]

where \(H\) is the Heaviside function: \(H(s) = 1\) for \(s > 0\) and \(H(s) = 0\) for \(s \leq 0\), and \(\lambda, \mu, \nu, \bar{\sigma}\) and \(\bar{\sigma}\) are positive constants, with \(\lambda\) being the consumption rate coefficient of nutrient by tumor cells, \(\mu\) the proliferation rate coefficient of tumor cells (= the birth rate of tumor cells that a unit amount of nutrient can sustain), \(\nu\) the dissolution rate of dead cells, \(\bar{\sigma}\) a threshold value of nutrient concentration to sustain tumor cells alive and proliferating, i.e., only in the region where \(\sigma > \bar{\sigma}\) tumor cells are alive and proliferating, and \(\bar{\sigma} = \sigma - (\nu/\mu)\). We assume that \(0 < \bar{\sigma} < \bar{\sigma}, \nu < \mu \bar{\sigma}\) (so that \(0 < \bar{\sigma} < \bar{\sigma}\)) and, for simplicity of notations \(\lambda = \bar{\sigma} = 1\), which can always be achieved through rescaling. For more information concerning the above model, we refer the reader to see the reference [4]. Note that in the above model it is assumed that all living cells are proliferating and the state of quiescence of cells is neglected. This is merely for the purpose to make the model simpler.

If instead of (1.2) the functions \(f, g\) are smooth monotone increasing functions in \([0, \infty)\) and satisfy the properties \(f(0) = 0, g(0) < 0\) and \(g(\infty) > 0\), the problem (1.1) models the growth of nonnecrotic tumors, cf. [3], which has been intensively studied, cf. [7, 12, 13, 14, 18, 19, 20] and references therein. It was proved that there exists a threshold value \(\gamma^* > 0\) for the surface tension coefficient \(\gamma\), such that if \(\gamma > \gamma^*\) then the unique radial stationary solution is asymptotically stable modulo translations, whereas if \(\gamma < \gamma^*\) then it is unstable. In the necrotic case, analysis of the model is much harder: In addition to the outer free boundary \(\partial \Omega(t)\) whose evolution is governed by the equation \(V_n = -\partial_n p\), discontinuity of the functions \(f, g\) at \(\sigma = \bar{\sigma}\) produces an inner free boundary or interface \(\Gamma(t)\) dividing the domain \(\Omega(t)\) into two disjoint regions, with the outer region

\[
\Omega_{\text{liv}}(t) = \{x \in \Omega(t) : \sigma(x, t) > \bar{\sigma}\}
\tag{1.3}
\]

being the living shell and the inner region

\[
\Omega_{\text{dec}}(t) = \text{int}\{x \in \Omega(t) : \sigma(x, t) = \bar{\sigma}\}
\tag{1.4}
\]
the necrotic core of the tumor. $\Gamma(t)$ is therefore the common boundary of these two parts:

$$\Gamma(t) = \partial\Omega_{\text{nec}}(t) \cap \partial\Omega_{\text{liv}}(t).$$  \hfill (1.5)

Main difficulty of analysis of the model (1.1) is caused by existence of $\Gamma(t)$, which is implicitly contained in the problem (1.1) and there is not an obvious evolution equation to govern its movement.

In [8] the spherically symmetric version of the problem (1.1) was studied, improving some earlier results of [14]. It was proved that this problem has a unique radial stationary solution which is asymptotically stable under spherically symmetric perturbations. However, whether this stationary solution is asymptotically stable under spherically asymmetric perturbations has been kept unknown for more than ten years. Recently some numerical results on this problem were obtained by Hao et al [25]. In this paper we aim at making a rigorous analysis to the problem (1.1) and establishing a similar result as that obtained in [7], [13], [18] for the nonnecrotic case, i.e., we want to prove that there exists a threshold value $\gamma^*$ for the surface tension coefficient $\gamma$ such that if $\gamma > \gamma^*$ then the unique radial stationary solution ensured by the reference [8] is asymptotically stable module translations under spherically asymmetric perturbations, whereas if $\gamma < \gamma^*$ then it is unstable under spherically asymmetric perturbations. To get this result, the crucial step is to prove that the inner free boundary $\Gamma(t)$ is smooth and depends on the outer free boundary $\partial\Omega(t)$ smoothly at least when the outer free boundary $\partial\Omega(t)$ is sufficiently close to the surface of a sphere. This will be proved by applying the Nash-Moser implicit function theorem. Note that this latter result has clearly its own independent significance.

To state the main results of this paper, let us first recall some results obtained in [8]. Consider the following elliptic boundary value problem:

$$\begin{cases}
\Delta \sigma = f(\sigma) & \text{in } \Omega, \\
\sigma = 1 & \text{on } \partial\Omega,
\end{cases}$$  \hfill (1.6)

where $\Omega$ is a given bounded domain in $\mathbb{R}^3$ with a $C^2$ boundary and $f$ is as in (1.2) (with $\lambda = 1$). It is not hard to prove that this problem has a unique solution $\sigma \in \cap_{1 < q < \infty} W^{2,q}(\Omega)$ which satisfies $\hat{\sigma} \leq \sigma \leq 1$ (see Lemma 2.2 in Section 2). Besides, it is not very difficult to prove that (see Lemmas 3.1, 3.2 of [8]) if $\Omega = B(0,R)$ for some $R > 0$, then the unique solution of the above problem is a radial function, given by $\sigma(x) = U(|x|,R)$, where $U(r,R)$ is defined as follows: Let $R^*$ be the unique positive number solving

$$\frac{\sinh R^*}{R^*} = \frac{1}{\hat{\sigma}}. \quad (1.7)$$

Then $U(r,R) = \frac{R \sinh r}{r \sinh R}$ for $0 < R \leq R^*$ and

$$U(r,R) = \begin{cases} 
\hat{\sigma}[\sinh(r-K) + K \cosh(r-K)]/r & \text{for } K \leq r \leq R \\
\hat{\sigma} & \text{for } r < K
\end{cases} \quad (1.8)$$

for $R > R^*$, where for $R > R^*$, $K = K(R)$ is the unique solution of the following equation in the interval $(0,R)$:

$$\sinh(R-K) + K \cosh(R-K) = \frac{R}{\hat{\sigma}}. \quad (1.9)$$
Now let

\[
F(R) = \frac{1}{4\pi R^3} \int_{B(0,R)} g(U(|x|, R))dx
\]

\[
= \frac{1}{R^3} \left\{ \mu \int_{U(r, R)} (U(r, R) - \hat{\sigma}) r^2 dr - \frac{1}{3} \nu R^3 \right\}
\]

\[
= \left\{ \begin{array}{ll}
\mu & 0 < R < R^*, \\
\mu G(R) - \frac{1}{3} \nu \left( \frac{K}{R} \right)^3 \mu \hat{\sigma} & R > R^*,
\end{array} \right.
\]

(1.10)

where \( K = K(R) \) is as before, and

\[
G(R) = \frac{1}{R^3} \left\{ (R - K) \cosh(R - K) + (RK - 1) \sinh(R - K) + \frac{1}{3} K^3 \right\}, \quad R > R^*.
\]

In case \( \Omega(t) = B(0, R(t)) \), the problem (1.1) reduces into the following initial value problem for a first-order differential equation:

\[
\begin{cases}
R'(t) = R(t) F(R(t)), \quad t > 0, \\
R(0) = R_0,
\end{cases}
\]

where \( R_0 > 0 \) is the number such that \( \Omega_0 = B(0, R_0) \). It was proved (see Lemma 4.2 of [3]) that \( F \) is continuously differentiable in \((0, \infty), F'(R) < 0 \) for all \( R > 0 \), and

\[
\lim_{R \to 0^+} F(R) = \mu \left( \frac{1}{3} (1 - \hat{\sigma}) \right) > 0, \quad \lim_{R \to +\infty} F(R) = -\frac{\nu}{3} < 0.
\]

Hence the function \( F \) has a unique positive root \( R_s \), and \( F(R) > 0 \) for \( 0 < R < R_s \), \( F(R) < 0 \) for \( R > R_s \).

It follows that for any \( R_0 > 0 \), the solution of the above problem exists for all \( t \geq 0 \), and

\[
\lim_{t \to \infty} R(t) = R_s.
\]

Moreover, since the function \( R \mapsto \frac{\sinh R}{R} \cdot \frac{R \coth R - 1}{(R)^2} \) is strictly monotone increasing, converges to \( \frac{1}{3} \) as \( R \to 0^+ \) and tends to \( +\infty \) as \( R \to +\infty \), it follows that the following relation is true:

\[
\frac{R^* \coth R^* - 1}{(R^*)^2} > \frac{1}{3} \hat{\sigma}.
\]

From this relation it follows that \( R_s > R^* \), which implies that the dormant or stationary tumor must have a necrotic core with radius \( K_s = K(R_s) \). It was also proved in [3] that the necrotic core is formed at finite time.

Given \( R > R^* \) and \( \rho, \eta \in C^{2+\mu}(\mathbb{S}^2) \) \( (0 < \mu < 1) \) with \( \|\rho\|_{C^{2+\mu}(\mathbb{S}^2)} \) and \( \|\eta\|_{C^{2+\mu}(\mathbb{S}^2)} \) sufficiently small, we denote

\[
\Omega_\rho = \{ x \in \mathbb{R}^3 : r < R[1 + \rho(\omega)] \}, \quad D_{\rho,\eta} = \{ x \in \mathbb{R}^3 : K[1 + \eta(\omega)] < r < R[1 + \rho(\omega)] \},
\]

\[
S_\rho = \{ x \in \mathbb{R}^3 : r = R[1 + \rho(\omega)] \}, \quad \text{and} \quad \Gamma_\eta = \{ x \in \mathbb{R}^3 : r = K[1 + \eta(\omega)] \}.
\]

It is not hard to prove (see Lemma 2.3 in Section 2) that given \( \rho \in C^{2+\mu}(\mathbb{S}^2) \) \( (0 < \mu < 1) \) with \( \|\rho\|_{C^{2+\mu}(\mathbb{S}^2)} \) sufficiently small and letting \( \Omega = \Omega_\rho \), solving the problem (1.6) is equivalent to seeking \((\sigma, \eta)\) such that it solves the following problem:

\[
\begin{cases}
\Delta \sigma = \sigma & \text{in} \ D_{\rho,\eta}, \\
\sigma = 1 & \text{on} \ S_\rho, \\
\sigma = \hat{\sigma} & \text{on} \ \Gamma_\eta, \\
\partial_r \sigma = 0 & \text{on} \ \Gamma_\eta,
\end{cases}
\]

(1.11)
where \( \partial_r \) denotes the derivative in radial direction. We note that if \( \| \eta \|_{C^1(S^2)} \) is sufficiently small then at any point in \( \Gamma_\eta \), the radial direction is not tangent to \( \Gamma_\eta \), so that the boundary value condition (1.1)_4 is regular. Later on we shall also use the following abbreviations:

\[
D = D_{0,0} = B(0, R) \setminus B(0, K), \quad S_0 = \partial B(0, R), \quad \text{and} \quad \Gamma_0 = \partial B(0, K),
\]

where \( K = K(R) \).

As we mentioned earlier, the problem (1.6) has a unique solution \( \sigma \in \cap_{1 \leq q < \infty} W^{2,q}(\Omega) \). Hence if \( R > R^* \) then the problem (1.11) has a unique solution. What we are concerned with is regularity of the free boundary \( \Gamma_\eta \), or equivalently the regularity of the function \( \eta \), and, more importantly, the regularity of the mapping \( \rho \mapsto \eta \). This leads to the following result:

**Theorem 1.1** Let \( R > R^* \) be given. Let the integer \( m \geq 2 \) and the number \( 0 < \mu < 1 \) be fixed. Then there exists a constant \( \delta > 0 \) such that for any \( \rho \in C^{m+\mu}(S^2) \) with \( \| \rho \|_{C^{m+\mu}(S^2)} < \delta \), the problem (1.11) has a unique solution \((\sigma, \eta)\) with \( \eta \in C^\infty(S^2) \) and \( \sigma \in C^{m+\mu}(\overline{D}_{\rho,\eta}) \cap C^\infty(D_{\rho,\eta} \setminus S_\rho) \), and the mapping \( \rho \mapsto \eta \) from the open set \( \| \rho \|_{C^{m+\mu}(S^2)} < \delta \) in \( C^{m+\mu}(S^2) \) to the Frechét space \( C^\infty(S^2) \) is smooth.

The above result will be proved by using the Nash-Moser implicit function theorem; see section 2.

In order to state our result on asymptotic stability of the radial stationary solution of the problem (1.1), we need to introduce some more notations. Let \( m \) and \( \mu \) be as above. Recall that a bounded domain \( \Omega \subseteq \mathbb{R}^3 \) is called a \( C^{m+\mu} \)-domain if it is \( C^{m+\mu} \)-diffeomorphic to the unit sphere \( B(0,1) \subseteq \mathbb{R}^3 \), and \( \Omega \) is called a \( \dot{C}^{m+\mu} \)-domain if it is \( C^{m+\mu} \)-diffeomorphic to the unit sphere \( B(0,1) \), where \( C^{m+\mu} \) refers to \( m+\mu \)-th order little Hölder continuous class. We use the notation \( \mathcal{D}^{m+\mu}(\mathbb{R}^3) \) to denote the Banach manifold of all \( C^{m+\mu} \)-domains in \( \mathbb{R}^3 \); cf. [10] for details. We denote

\[
\mathcal{M} := \mathcal{D}^{m+\mu}(\mathbb{R}^3), \quad \mathcal{M}_0 := \mathcal{D}^{m+\mu+3}(\mathbb{R}^3).
\]

From [10] we know that \( \mathcal{M}_0 \) is a \( C^3 \)-embedded Banach submanifold of \( \mathcal{M} \), so that every point in \( \mathcal{M}_0 \) is \( C^3 \)-differentiable as a point of \( \mathcal{M} \). Given \( \Omega \in \mathcal{M}_0 \), the equations (1.1)_1–(1.1)_4 with \( \Omega(t) \) replaced by \( \Omega \) has a unique solution \((\sigma, p)\) satisfying the following properties:

\[
\begin{aligned}
\left\{ \begin{array}{l}
\sigma, p \in W^{2,q}(\Omega) \ (\forall q \in [1, \infty)), \\
\sigma \in \dot{C}^\infty(\Omega_{\text{div}} \cup \Omega_{\text{nge}}), \\
p \in C^{m+1+\mu}(\Omega_{\text{div}} \cup \partial \Omega),
\end{array} \right.
\end{aligned}
\]  

where \( \Omega_{\text{div}} \) and \( \Omega_{\text{nge}} \) are defined similarly as in (1.3) and (1.4), respectively. We define a vector field \( \mathcal{G} \) in \( \mathcal{M} \) with domain \( \mathcal{M}_0 \) (i.e., \( \mathcal{G} : \mathcal{M}_0 \to \mathcal{T}_{\mathcal{M}_0}(\mathcal{M}) \)) by letting

\[
\mathcal{G}(\Omega) = -\partial_{\sigma} p|_{\partial \Omega}, \quad \forall \Omega \in \mathcal{M}_0.
\]  

Then the problem (1.1) reduces into the following differential equation in the Banach manifold \( \mathcal{M} \):

\[
\begin{aligned}
\left\{ \begin{array}{l}
\Omega'(t) = \mathcal{G}(\Omega(t)), \\
\Omega(0) = \Omega_0.
\end{array} \right.
\end{aligned}
\]  

We note that when the domain \( \Omega(t) \) is determined, the other two components \( \sigma, p \) of the solution of the problem (1.1) will then follow from solving the elliptic boundary value problem (1.1)_1–(1.1)_4, and properties of \( \sigma, p \) are fully determined by those of the domain \( \Omega(t) \). Hence, to avoid a very complex statement, we only give the precise statement of our result on the initial value problem (1.14), which reads as follows:
Theorem 1.2. Let $\mathcal{M}_c$ be the 3-dimensional submanifold of $\mathcal{W}_0$ consisting of all surface spheres in $\mathbb{R}^3$ of radius $R_s$. There exists a constant $\gamma^* > 0$ such that if $\gamma > \gamma^*$ then the following assertions hold:

1. There is a neighborhood $\mathcal{O}$ of $\mathcal{M}_c$ in $\mathcal{W}_0$ such that for any $\Omega_0 \in \mathcal{O}$, the initial value problem (1.14) has a unique solution $\Omega \in C((0, \infty), \mathcal{M}_c) \cap C^1((0, \infty), \mathcal{W}_0)$.

2. There exists a submanifold $\mathcal{M}_a$ of $\mathcal{W}_0$ of codimension 3 passing $\Omega_s = B(0, R_s)$ such that for any $\Omega_0 \in \mathcal{M}_a$, the solution of the problem (1.14) satisfies $\lim_{t \to \infty} \Omega(t) = \Omega_s$ and, conversely, if the solution of (1.14) satisfies this property then $\Omega_0 \in \mathcal{M}_a$.

3. For any $\Omega_0 \in \mathcal{O}$ there exist unique $x_0 \in \mathbb{R}^3$ and $\Omega'_0 \in \mathcal{M}_a$ such that $\Omega_0 = x_0 + \Omega'_0$ and, for the solution $\Omega = \Omega(t)$ of (1.14), we have

$$\lim_{t \to \infty} \Omega(t) = B(x_0, R_s),$$

Moreover, convergence rate of the above limit relations is of the form $Ce^{-\nu t}$ for some positive constants $C$ and $\nu$ depending on $\gamma$. If on the contrary $0 < \gamma < \gamma^*$ then the radial stationary solution of the problem (1.14) is unstable. \(\square\)

The above result will be proved in Section 3 by using some abstract result for parabolic differential equations in Banach manifold recently established in [11]. A crucial step is to show that the representation of the vector field $\mathcal{G}$ in certain local chart of $\mathcal{M}$ is smooth, which is ensured by Theorem 1.1.

Remark 1. If $R_s > R^*$ then from (1.8) we see that the stationary tumor has a necrotic core, i.e., the set $\Omega^*_{\text{nec}} = \text{int}\{x \in \Omega_s : \sigma(x) = \sigma^*\}$ is nonempty. By Theorem 1.1 and Lemma 2.6 in Section 2 we see that if $\Omega_0$ is sufficiently close to $\Omega_s$ then for any $t > 0$ the tumor also has a necrotic core, i.e., the set $\Omega_{\text{nec}}(t) = \text{int}\{x \in \Omega(t) : \sigma(x, t) = \sigma^*\}$ is nonempty. From the assertion (3) and the proof of Theorem 1.1 it easily follows that the following relation holds:

$$\lim_{t \to \infty} \Omega_{\text{nec}}(t) = \Omega^*_{\text{nec}}.$$

If $R_s < R^*$ then it is also not hard to prove that the tumor does not have a necrotic core for all $t > 0$ and the results of [11] apply to this situation.

Remark 2. As we pointed earlier, properties of the other two components $\sigma, p$ of the solution of the problem (1.1) are fully determined by those of the domain $\Omega(t)$; they can be deduced from Theorem 1.2, Theorem 1.1 and Lemma 2.6. We omit this discussion here.

The organization of the rest part is as follows. In Section 2 we study the problem (1.6) and the equations (1.1)−(1.4) for $\gamma = 0$ and fixed $t > 0$. This section is divided into three subsections. In Subsection 2.1 we make a short review to the Nash-Moser implicit function theorem. In Subsection 2.2 we use the Nash-Moser implicit function theorem to prove that the interface $\Gamma$ of the problem (1.6) caused by discontinuity of the nonlinear reaction function $f(\sigma)$ is smooth and depends on $\partial \Omega$ smoothly, provided $\Omega$ is a small perturbation of a sphere. In Subsection 2.3 we use this result to prove that the surface tension free part $\pi_o$ of the solution of the equations (1.1)2 and (1.1)4 has the property that the map $\Omega \mapsto \partial_n \pi_o|_{\partial \Omega}$ is smooth. In Section 3 we give the proof of Theorem 1.2.

2 The Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1. We shall use Nash-Moser implicit function theorem to prove this theorem. Since Nash-Moser implicit function theorem might be not very familiar to the
reader, we shall first make a short review to this theorem. Hence this section is divided into three subsections: In Subsection 2.1 we make a short review to Nash-Moser implicit function theorem. In Subsection 2.2 we give the proof of Theorem 1.1. In Subsection 2.3 we use Theorem 1.1 to prove that the representation of the vector field $\mathcal{G}$ in (1.13) in certain local chart of the manifold $\mathfrak{S}$ is smooth.

### 2.1 Review of Nash-Moser implicit function theorem

Recall that implicit function theorem in Banach space says that for three Banach spaces $X, Y, Z$ and a $C^1$-mapping $F : U \subseteq X \times Y \to Z$, where $U$ is an open subset of $X \times Y$, if $F(x_0, y_0) = 0$ for some $(x_0, y_0) \in U$ and $\partial_y F(x_0, y_0) : Y \to Z$ is an isomorphism of Banach spaces, where $\partial_y F(x, y)$ denotes the partial Fréchet derivative of $F(x, y)$ in the variable $y$, then there exist $\varepsilon, \delta > 0$ with $B(x_0, \varepsilon) \times B(y_0, \delta) \subseteq U$ and a unique $C^1$-mapping $f : B(x_0, \varepsilon) \to B(y_0, \delta)$ such that

$$F(x, f(x)) = 0 \quad (2.1)$$

for all $x \in B(x_0, \varepsilon)$. This is a fundamental result in the field of mathematical analysis. Naturally one may ask whether this result can be extended to general topological vector spaces or more specifically to general Fréchet spaces. Unfortunately, this is false; cf. counterexamples in Section 5.5 of Part I of [24]. However, if we make suitable restrictions to the spaces $X, Y, Z$ as well as the mapping $F$, and suitably strengthen the conditions on $\partial_y F(x, y)$, then a generalization named Nash-Moser implicit function theorem holds. Basic idea of this theorem was first discovered by Nash in [29] and later fashioned by Moser in [28] into an abstract theorem in functional analysis of wide applicability. An excellent exposition of this theorem was given in [24]. In what follows we make a brief review to this theorem in the spirit of [24]. Let us start by introducing some basic concepts.

A graded Fréchet space is a topological vector space $X$ whose topology is defined by an increasing sequence of seminorms $\{\| \cdot \|_n\}_{n=0}^{\infty}$, i.e.,

$$\| x \|_0 \leq \| x \|_1 \leq \| x \|_2 \leq \cdots \leq \| x \|_n \leq \cdots, \quad \forall x \in X.$$ 

A linear map $L : X \to Y$ between two graded Fréchet spaces $X, Y$ is called a tame linear map if there exist nonnegative integers $n_0, r$ and for each integer $n \geq n_0$ a corresponding constant $C_n > 0$ such that for any integer $n \geq n_0$ the following estimate holds:

$$\| Lx \|_n \leq C_n \| x \|_{n+r}, \quad \forall x \in X.$$ 

A nonlinear map $F : U \subseteq X \to Y$ between two graded Fréchet spaces $X, Y$, where $U$ is an open subset of $X$, is called a tame nonlinear map if $F$ is continuous and for any point $x_0 \in U$ there exist a neighborhood $U_{x_0} \subseteq U$ of $x_0$, nonnegative integers $n_0 = n_0(x_0)$, $r = r(x_0)$ and for each integer $n \geq n_0$ a corresponding constant $C_n(x_0) > 0$ such that for any integer $n \geq n_0$ the following estimate holds:

$$\| F(x) \|_n \leq C_n(x_0) (1 + \| x \|_{n+r}), \quad \forall x \in U_{x_0}.$$ 

In this case we often simply say that $F$ is tame. If $F : U \subseteq X \to Y$ is smooth and not only itself is tame, but also all its derivatives $D^k F$ ($k = 1, 2, \cdots$) are tame, then we call $F$ a smooth tame map.

A graded Fréchet space $X$ is called a tame direct summand of another graded Fréchet space $Y$ if there exist tame linear maps $F : X \to Y$ and $G : Y \to X$ such that $G(F(x)) = x$ for all $x \in X$. For a Banach space $B$, the notation $\Sigma(B)$ denotes the space of all sequences $\{x_k\}$ of elements in $B$ such that

$$\| \{ x_k \} \|_n = \sum_{k=1}^{\infty} e^{nk} \| x_k \|_B < \infty, \quad n = 0, 1, 2, \cdots.$$ 

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Concerning tame maps and smooth tame maps, we have the following assertions:

Smooth sections of $V$ map from a finite dimensional space to a graded Fréchet space is tame.

Fortunately, we have the following basic results:

1. All Banach spaces are tame spaces.
2. If $X$ is a compact manifold then $C^\infty(X)$ is a tame space.
3. If $X$ is a compact manifold with boundary then both $C^\infty(X)$ and $C^\infty_0(X)$ are tame spaces.
4. If $X$ is a compact manifold and $V$ is a vector bundle over $X$ then the space $C^\infty(X, V)$ of all smooth sections of $V$ over $X$ is a tame space.
5. A tame direct summand of a tame space is tame.
6. A cartesian product of two tame spaces is tame.

Concerning tame maps and smooth tame maps, we have the following assertions:

7. Any continuous map from a graded Fréchet space to a Banach space is tame. Any continuous map from a finite dimensional space to a graded Fréchet space is tame.
8. A composition of tame maps is tame.
9. Let $X$ be a compact manifold and $V, W$ be vector bundles over $X$. Let $U$ be an open subset of $V$ and $p : U \subseteq V \to W$ be a smooth map of $U$ into $W$ which takes fibres into fibres. Let $U \subseteq C^\infty(X, V)$ be the set of smooth sections of $V$ over $X$ whose image lies in $U$. Then $U$ is an open subset of $C^\infty(X, V)$ and the map $P : U \subseteq C^\infty(X, V) \to C^\infty(X, W)$ defined by $Pf(x) = p(f(x))$ (for $f \in U$), called nonlinear vector bundle operator, is tame.
10. Let $X, V, W$ be as above, $m$ a positive integer and $U$ an open subset of $C^\infty(X, V)$. A smooth nonlinear partial differential operator $P$ of order $m$ from $V$ to $W$ is a map $P : U \subseteq C^\infty(X, V) \to C^\infty(X, W)$ such that for any $f \in U$ and $x \in X$, $Pf(x)$ is a smooth function of $f(x)$ and partial derivatives of $f$ at $x$ of degree at most $m$ in any local charts. A smooth nonlinear partial differential operator is a smooth tame map.

Nash-Moser implicit function theorem reads as follows:

**Theorem 2.1** Let $X, Y, Z$ be tame Fréchet spaces and let $F : U \subseteq X \times Y \to Z$ be a smooth tame map, where $U$ is an open subset of $X \times Y$. Let $(x_0, y_0) \in U$ be such that $F(x_0, y_0) = 0$. Assume that there exists a smooth tame map $A : (U \subseteq X \times Y) \times Z \to Y$ of the form $A(x, y, z) = L(x, y)z$, where for each $(x, y) \in U$, $L(x, y)$ is a linear map from $Z$ to $Y$, such that

$$L(x, y)\partial_y F(x, y)u = u \quad \text{and} \quad \partial_y F(x, y)L(x, y)z = z$$

(2.2)

for all $(x, y) \in U$, $u \in Y$ and $z \in Z$. Then there exist neighborhoods $B_1, B_2$ of $x_0$ and $y_0$, respectively, with the property that $B_1 \times B_2 \subseteq U$, and a smooth tame map $f : B_1 \to B_2$, such that

$$f(x_0) = y_0 \quad \text{and} \quad F(x, f(x)) = 0$$

for all $x \in B_1$. Moreover, for any $x \in B_1$, $y = f(x)$ is the unique solution of the equation $F(x, y) = 0$ in $B_2$.

For the proof of the above theorem, we refer the reader to see Theorems 3.3.1 and 3.3.3 in Part III.
of [24]. We note that in those theorems some quadratic error terms are included in the right-hand sides of the two equations in (2.2). Here we take such terms to be identically vanishing.

Remark. Comparing Nash-Moser implicit function theorem with the implicit function theorem in Banach space, we see that a significant difference between these theorems is that in the Fréchet space case, the partial derivative \( \partial_y F(x, y) \) of \( F(x, y) \) should be invertible not merely at the single point \((x_0, y_0)\) as in the Banach space case, but at all points in a neighborhood of this point. Partial reason for this difference to occur is due to the fact that for two Banach spaces \(X\) and \(Y\), the set of invertible continuous linear maps from \(X\) to \(Y\) is an open subset of \(L(X, Y)\), whereas if \(X\) and \(Y\) are Fréchet spaces, this is not the case, even if they are tame; cf. the counterexample 5.3.3 in Part I of [24].

\[ 2.2 \text{ The Proof of Theorem 1.1} \]

In this subsection we give the proof of Theorem 1.1.

Let \( R > R^* \) be given and set \( K = K(R) \). Let \( m, \mu \) be as in Theorem 1.1, i.e., \( m \) is an integer not less than 2 and \( 0 < \mu < 1 \). We know that \( C^\infty(S^2) \) with the family of seminorms \( \{ \| \cdot \|_{C^k(S^2)} \} \cup \{ \| \cdot \|_{C^{k+m}(S^2)} \} \) for \( k \) is a tame Frechét space. We also regard the Banach space \( C^{m+\mu}(S^2) \) as a tame Frechét space. For sufficiently small \( \delta, \delta' > 0 \) we denote

\[ O_\delta = \{ \rho \in C^{m+\mu}(S^2) : \| \rho \|_{C^{m+\mu}(S^2)} < \delta \}, \quad O'_{\delta'} = \{ \eta \in C^\infty(S^2) : \| \eta \|_{C^{m+\mu}(S^2)} < \delta' \}; \]

they are open subsets of \( C^{m+\mu}(S^2) \) and \( C^\infty(S^2) \), respectively. We define a map \( A : O_\delta \times O'_{\delta'} \subseteq C^{m+\mu}(S^2) \times C^\infty(S^2) \rightarrow C^\infty(S^2) \) as follows: Given \( \rho \in O_\delta \) and \( \eta \in O'_{\delta'} \), let \( \sigma = \sigma(\rho, \eta, \omega) \) be the unique solution of the equations (1.11) and (1.11), and define

\[ A(\rho, \eta) = [\omega \mapsto \sigma(K[1 + \eta(\omega)] ; \omega, \rho, \eta) - \delta, \omega \in S^2]. \] (2.3)

Clearly \( A(0, 0) = 0 \). We shall prove that if \( \delta, \delta' \) are sufficiently small then there exists a unique smooth mapping \( \varphi : O_\delta \rightarrow O'_{\delta'} \) such that \( A(\rho, \varphi(\rho)) = 0 \) for all \( \rho \in O_\delta \).

**Lemma 2.2** A is a smooth tame map.

**Proof.** Choose a function \( \phi \in C^\infty[K, R] \) such that it satisfies the following conditions:

\[ 0 \leq \phi \leq 1; \quad \phi(R) = \phi(K) = 1; \quad \phi(t) = 0 \text{ for } \frac{3}{4} K + \frac{1}{4} R \leq t \leq \frac{1}{4} K + \frac{3}{4} R; \]

\[ \phi'(t) \leq 0 \text{ for } K \leq t \leq \frac{3}{4} K + \frac{1}{4} R; \quad \phi'(t) \geq 0 \text{ for } \frac{1}{4} K + \frac{3}{4} R \leq t \leq R. \]

Let \( M_0 = \max_{K \leq t \leq R} |\phi'(t)| \) and assume \( \delta, \delta' \) are small enough such that \( \delta < (1 + M_0 R)^{-1} \), \( \delta' < (1 + M_0 K)^{-1} \) and \( \max\{ \delta, \delta' \} < \frac{1}{3} \frac{R - K}{R + K} \). Consider the variable transformation \( y = \Psi_{\rho, \eta}(x) \) from \( \overline{D}_{\rho, \eta} \) to \( \overline{D} \), where for \( x \in \overline{D}_{\rho, \eta} \),

\[ \Psi_{\rho, \eta}(x) = \begin{cases} x - R \rho(\omega) \phi(\frac{r}{1 + \rho(\omega)}) \omega & \text{if } r \geq \frac{1}{2} (K + R), \\ x - K \eta(\omega) \phi(\frac{r}{1 + \eta(\omega)}) \omega & \text{if } r < \frac{1}{2} (K + R). \end{cases} \] (2.4)

It is easy to see that \( \Psi_{\rho, \eta} \) is a \( C^{m+\mu} \) diffeomorphism from \( \overline{D}_{\rho, \eta} \) onto \( \overline{D} \). Moreover, denoting

\[ E_{\eta} = \{ x \in \mathbb{R}^3 : K[1 + \eta(\omega)] < r < \frac{1}{2} (K + R) \}, \quad E = \{ x \in \mathbb{R}^3 : K < r < \frac{1}{2} (K + R) \}, \]

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we see that the restriction of \( \Psi_{\rho, \eta} \) on \( \overline{E}_\eta \) is a \( C^\infty \)-diffeomorphism from \( \overline{E}_\eta \) onto \( \overline{E} \), due to the facts that \( \eta \in C^\infty (S^2) \) and that this restriction is independent of \( \rho \). Because of the latter property, we re-denote the restriction of \( \Psi_{\rho, \eta} \) on \( \overline{E}_\eta \) as \( \Psi_\eta \), and denote by \( \psi_\eta \) the restriction of \( \Psi_\eta \) to \( \Gamma_\eta \), which is clearly a \( C^\infty \)-diffeomorphism from \( \Gamma_\eta \) onto \( \Gamma_0 \). Now define operators \( \mathcal{A}(\rho, \eta) : C^{m+\mu}(\overline{D}) \cap C^\infty (\overline{E}) \to C^{m-2+\mu}(\overline{D}) \cap C^\infty (\overline{E}) \), \( \mathcal{A}(\eta) : C^\infty (\overline{E}) \to C^\infty (\overline{E}) \) and \( \mathcal{N}(\eta) : C^\infty (\overline{E}) \to C^\infty (\Gamma_0) \) respectively as follows:

\[
\mathcal{A}(\rho, \eta)u = [\Delta (u \circ \Psi_{\rho, \eta})] \circ \Psi_\eta^{-1} \quad \text{for } u \in C^{m+\mu}(\overline{D}) \cap C^\infty (\overline{E}),
\]

\[
\mathcal{A}(\eta)u = [\Delta (u \circ \Psi_\eta)] \circ \Psi_\eta^{-1} \quad \text{for } u \in C^\infty (\overline{E}),
\]

\[
\mathcal{N}(\eta)u = [\partial_i (u \circ \Psi_\eta)|_{\Gamma_\eta}] \circ \psi_\eta^{-1} \quad \text{for } u \in C^\infty (\overline{E}).
\]

Let \( u = \sigma \circ \Psi_\eta^{-1} \). After the variable transformation \( x \to \Psi_{\rho, \eta}(x) \), the problem (1.11)_1, (1.11)_2 and (1.11)_4 transforms into the following problem:

\[
\begin{cases}
\mathcal{A}(\rho, \eta)u = u & \text{in } D, \\
u = 1 & \text{on } S_0, \\
\mathcal{N}(\eta)u = 0 & \text{on } \Gamma_0.
\end{cases}
\]

(2.5)

It is clear that all coefficients of the operator \( \mathcal{A}(\rho, \eta) \) belong to \( C^{m-2+\mu}(\overline{D}) \) and the \( C^{m-2+\mu}(\overline{D}) \)-norms of all these coefficients are bounded with a constant depending only on \( m, \mu, \delta, \delta' \), and similarly all coefficients of the operator \( \mathcal{N}(\eta) \) belong to \( C^{m-1+\mu}(\Gamma_0) \) and the \( C^{m-1+\mu}(\Gamma_0) \)-norms of all these coefficients are also bounded with a constant depending only on \( m, \mu, \delta, \delta' \). Besides, it is also clear that the smallest eigenvalue of the second-order coefficient matrix of the operator \( -\mathcal{A}(\rho, \eta) \) is bounded below by a positive constant depending only on \( m, \mu, \delta, \delta' \). Hence, by a standard result in the theory of elliptic boundary value problems we see that the solution \( u \) satisfies the following estimates:

\[
\|u\|_{C^{m+\mu}(\overline{D})} \leq C(m, \mu, \delta, \delta') < \infty.
\]

(2.6)

Next, choose a number \( \varepsilon > 0 \) sufficiently small and let

\[
\Lambda_i = \left\{ x \in \mathbb{R}^3 : \frac{1}{2} (K + R) - i \varepsilon < r < \frac{1}{2} (K + R) + i \varepsilon \right\}, \quad i = 1, 2, 3.
\]

Note that \( \Lambda_1 \subset \subset \Lambda_2 \subset \subset \Lambda_3 \). Since in \( \Psi_{\rho, \eta}(x) = x \) for \( x \in \Lambda_3 \), from the equation \( \mathcal{A}(\rho, \eta)u = u \) we have

\[
\Delta u = u \quad \text{in } \Lambda_3.
\]

From this fact and the first estimate in (2.6), by using some standard estimates for elliptic equations we get:

\[
\|u\|_{C^{k+\mu}(\overline{\Delta}_j)} \leq C(k), \quad k = m, m + 1, m + 2, \cdots.
\]

(2.7)

In \( E \) we have

\[
\begin{cases}
\mathcal{A}(\eta)u = u & \text{in } E, \\
\mathcal{N}(\eta)u = 0 & \text{on } \Gamma_0.
\end{cases}
\]

The operators \( \mathcal{A}(\eta) \) and \( \mathcal{N}(\eta) \) respectively have the following forms:

\[
\mathcal{A}(\eta)u = \sum_{i,j=1}^{3} a_{ij}(x, \eta, \nabla \eta) \partial_{ij} u + \sum_{i=1}^{3} b_i(x, \eta, \nabla \eta, \nabla^2 \eta) \partial_i u,
\]

\[
\mathcal{N}(\eta)u = \sum_{i=1}^{3} c_i(x, \eta, \nabla \eta) \partial_i u,
\]

\[
10
\]
where $a_{ij}(x, \eta, \nabla \eta)$'s are quadratic functions in $\nabla \eta$ with coefficients being smooth functions of $x$ and $\eta$, $b_i(x, \eta, \nabla \eta, \nabla^2 \eta)$'s are sums of linear functions in $\nabla^2 \eta$ and quadratic functions in $\nabla \eta$ with coefficients being smooth functions of $x$ and $\eta$, and $c_i(x, \eta, \nabla \eta)$'s are linear functions in $\nabla \eta$ with coefficients being smooth functions of $x$ and $\eta$. Using these facts, the estimates in (2.6) and some standard arguments as in the proofs of higher-order interior and boundary regularity estimates for elliptic boundary value problems, we see that the following estimates hold:

$$
\|u\|_{C^{k+\mu}(\mathcal{D},A)} \leq C(k, \mu, \delta', (1 + \|\eta\|_{C^{\nu}(S^2)})(1 + \|\eta\|_{C^{\nu}(S^2)}), \quad k = m + 1, m + 2, \ldots. \tag{2.8}
$$

Combining the estimates (2.6), (2.7) and (2.8), we get the following estimates:

$$
\|u\|_{C^{m+\mu}(\mathcal{D})} + \|u\|_{C^{k+\mu}(\mathcal{D})} \leq C(1 + \|\eta\|_{C^{\nu}(S^2)}), \quad k = m, m + 1, m + 2, \ldots. \tag{2.9}
$$

Next, for any positive integer $p$, if we denote by $u_p$ any one of the partial Fréchet derivatives of order $p$ of $u$ in $\rho, \eta$ (recall that $u_p$ is a continuous $p$-linear operator in $C^{\infty}(S^2)$ with value in $C^{m+\mu}(\mathcal{D}) \cap C^{\infty}(\mathcal{D})$), then from (2.5) we see easily that for any $\xi_1, \xi_2, \ldots, \xi_p \in C^{\infty}(S^2)$, $u_p(\xi_1, \xi_2, \ldots, \xi_p)$ satisfies an elliptic boundary value problem of the following form:

$$
\left\{
\begin{array}{ll}
\mathcal{A}(p, \eta)u_p(\xi_1, \xi_2, \ldots, \xi_p) = u_p(\xi_1, \xi_2, \ldots, \xi_p) + \mathcal{F}_p(\rho, \eta, u, Du_1, \ldots, D^{p-1}u, \xi_1, \xi_2, \ldots, \xi_p) & \text{in } D, \\
u_p(\xi_1, \xi_2, \ldots, \xi_p) = 0 & \text{on } \mathcal{S}_0, \\
\mathcal{G}_p(\rho, \eta, u, Du_1, \ldots, D^{p-1}u, \xi_1, \xi_2, \ldots, \xi_p) & \text{on } \Gamma_0,
\end{array}
\right.
$$

where $D^i u$ denotes the set of partial Fréchet derivatives of $u$ in $\rho, \eta$ of order $i$, $i = 1, 2, \ldots, p - 1$, $\mathcal{F}_p(\rho, \eta, u, Du_1, \ldots, D^{p-1}u, \xi_1, \xi_2, \ldots, \xi_p)$ and $\mathcal{G}_p(\rho, \eta, u, Du_1, \ldots, D^{p-1}u, \xi_1, \xi_2, \ldots, \xi_p)$ denotes functionals in $\rho, \eta, u, Du_1, \ldots, D^{p-1}u, \xi_1, \xi_2, \ldots, \xi_p$, both linear in $(u, Du_1, \ldots, D^{p-1}u)$ and $p$-linear in $(\xi_1, \xi_2, \ldots, \xi_p)$. Using these facts, and a similar argument as above and induction in $p$, we see that similar estimates as in (2.9) also hold for $u_p$ for any $p \geq 1$, i.e., for any $(\rho, \eta) \in O_\delta \times O'_\delta$, and any $\xi_1, \xi_2, \ldots, \xi_p \in C^{\infty}(S^2)$,

$$
\|u_p(\xi_1, \xi_2, \ldots, \xi_p)\|_{C^{m+\mu}(\mathcal{D})} + \|u_p(\xi_1, \xi_2, \ldots, \xi_p)\|_{C^{k+\mu}(\mathcal{D})} \leq C_{p,k}(1 + \|\eta\|_{C^{\nu}(S^2)}) \sum_{i=1}^{p} \|\xi_1\|_{C^{\nu}(S^2)} \cdots \|\xi_i-1\|_{C^{\nu}(S^2)} \|\xi_i\|_{C^{\nu}(S^2)} \|\xi_{i+1}\|_{C^{\nu}(S^2)} \cdots \|\xi_p\|_{C^{\nu}(S^2)},
$$

$$
\|\xi_1\|_{C^{\nu}(S^2)} \cdots \|\xi_1\|_{C^{\nu}(S^2)} \|\xi_i\|_{C^{\nu}(S^2)} \|\xi_{i+1}\|_{C^{\nu}(S^2)} \cdots \|\xi_p\|_{C^{\nu}(S^2)}, \quad k = m, m + 1, m + 2, \ldots. \tag{2.10}
$$

From (2.9) and (2.10) we see that the solution map $(\rho, \eta) \mapsto u$ of the problem (2.8) is a smooth tame map. Since

$$
A(\rho, \eta) = \left[ \omega \mapsto u(K\omega) - \sigma, \omega \in S^2 \right],
$$

the desired assertion immediately follows. □

**Lemma 2.3** There exist $\delta, \delta' > 0$ sufficiently small such that for all $\rho \in O_\delta$ and $\eta \in O'_\delta$, $\partial_\eta A(\rho, \eta)$ is invertible, and the map $(\rho, \eta, \xi) \mapsto \partial_\eta A(\rho, \eta)^{-1} \xi$ is a smooth tame map.

**Proof.** We note that from the definition of $A(\rho, \eta)$ (see (2.3)), we see that if as in (2.3) we denote by $\sigma = \sigma(r, \omega; \rho, \eta)$ the solution of the problem (1.11)$_1$, (1.11)$_2$ and (1.11)$_4$, then for any $\xi \in C^{\infty}(S^2)$ we have

$$
\partial_\eta A(\rho, \eta) \xi = \partial_\eta \sigma(K[1 + \eta(\omega)], \omega; \rho, \eta) \xi + \partial_\sigma(K[1 + \eta(\omega)], \omega; \rho, \eta) K \xi
$$

$$
= \partial_\eta \sigma(K[1 + \eta(\omega)], \omega; \rho, \eta) \xi \quad \text{(by (1.11)$_4$)}.
$$
Let \( v(r, \omega) = \partial_\rho \sigma(r, \omega; \rho, \eta) \xi \). Then \( \partial_\eta A(\rho, \eta) \xi = [\omega \mapsto v(K[1 + \eta(\omega)], \omega), \omega \in S^2] \). A simple computation shows that \( v \) is the solution of the following problem:
\[
\Delta v = v \text{ in } D_{\rho, \eta}, \quad v = 0 \text{ on } S_\rho, \quad \partial_\nu v = -K \partial_\nu^2 \sigma|_{\Gamma, \xi} \text{ on } \Gamma_{\eta}.
\]
If \( \rho = 0 \) and \( \eta = 0 \) then \( \sigma = U(\rho, R) \), so that \( \partial_\rho^2 \sigma(\cdot; 0, 0)|_{\Gamma, \eta} = \partial_\rho^2 U(K, R) = U(K, R) = \sigma \). We now choose \( \delta, \delta' > 0 \) sufficiently small such that for all \( \rho \in O_\delta \) and \( \eta \in O_{\delta'} \) there hold \((1/2)\delta \leq \partial_\rho^2 \sigma(\cdot; \rho, \eta)|_{\Gamma, \eta} \leq 2\delta\). Then for any \( \rho \in O_\delta \) and \( \eta \in O_{\delta'} \) the operator \( \partial_\eta A(\rho, \eta) \) is invertible, with
\[
[\partial_\eta A(\rho, \eta)]^{-1} = \frac{\partial_\rho w(\cdot; \rho, \eta, \zeta)}{K \partial_\rho^2 \sigma(\cdot; \rho, \eta)}|_{\Gamma, \eta}, \quad \forall \zeta \in C^\infty(S^{n-1}),
\]
where \( w = w(\cdot; \rho, \eta, \zeta) \) is the solution of the following problem:
\[
\Delta w = w \text{ in } D_{\rho, \eta}, \quad w = 0 \text{ on } S_\rho, \quad w = \zeta \text{ on } \Gamma_{\eta}.
\]

**Proof of Theorem 1.1.** Having proved Lemmas 2.2 and 2.3, by using the Nash-Moser implicit function theorem we conclude that by choosing \( \delta, \delta' > 0 \) smaller when necessary, it follows that for any \( \rho \in O_\delta \) there exists \( \eta \in O_{\delta'} \) such that it is the unique solution of the equation \( A(\rho, \eta) = 0 \) in \( O_{\delta'} \), and the map \( \rho \mapsto \eta \) from \( O_\delta \subseteq C^{m+\mu}(S^2) \) to \( O_{\delta'} \subseteq C^{\infty}(S^2) \) is a smooth tame map. This shows that for any \( \rho \in C^{m+\mu}(S^2) \) with \( \|\rho\|_{C^{m+\mu}(S^2)} < \delta \), the free boundary \( \Gamma_{\eta} \) is smooth and the mapping \( \rho \mapsto \eta \) is also smooth. Having proved smoothness of the free boundary \( \Gamma_{\eta} \), the assertion \( \sigma \in C^{m+\mu}(\overline{\Omega}_{\rho, \eta}) \cap C^{\infty}(\overline{\Omega}_{\rho, \eta} \setminus S_\rho) \) follows immediately. This proves Theorem 1.1. \( \square \)

**Remark.** The reader might argue why we don’t use the implicit function theorem in Banach space to prove theorem 1.1, as in the proof of Lemma 5.1 of \([9]\). The reason is that the method used in the proof of Lemma 5.1 of \([9]\) does not work to the present problem, due to the fact that \( \sigma'(K) = 0 \), where \( \sigma(r) = U(r, R) \).

### 2.3 Smoothness of a map related to the free boundary problem (1.1)

In this subsection we prove that for the solution \((\sigma, \pi_0)\) of the boundary value problem
\[
\begin{align*}
\Delta \sigma &= f(\sigma) & & \text{in } \Omega, \\
-\Delta \pi_0 &= g(\sigma) & & \text{in } \Omega, \\
\sigma &= 1 & & \text{on } \partial \Omega, \\
\pi_0 &= 0 & & \text{on } \partial \Omega,
\end{align*}
\]
(2.11)
where \( f, g \) are the discontinuous functions given in (1.2) (with \( \lambda = 1 \)), the mapping \( \Omega \mapsto \partial_\eta \sigma|_{\partial \Omega} \) from a neighborhood of a sphere in \( \mathfrak{M}_0 := D^{m+3+\mu}(R^3) \subseteq \mathfrak{M} := D^{m+\mu}(R^3) \) to \( \mathfrak{T}_{\mathfrak{M}_0}(\mathfrak{M}) \) is smooth, i.e., representation of this mapping in some regular local chart of \( \mathfrak{M} \) at every sphere is smooth, where \( \partial_\eta \) denotes the derivative in the outward normal direction \( n \) of \( \partial \Omega \). This result is crucial in the study of the free boundary problem (1.1) to be given in the next section. We note that since the functions \( f, g \) are discontinuous, such a result apparently looks unbelievable.
We point out that although here we only consider the three dimension case, a similar discussion also works for general dimension \( n \geq 2 \) case; in order to do so the discussion in [8] must be first extended, which is not hard.

We begin by proving that the problem (1.6) is equivalent the problem (1.11), i.e., we have the following preliminary result:

**Lemma 2.4** For \( \rho, \eta \in C^{2+\mu}(S^2) \) \((0 < \mu < 1)\) with \( \|\rho\|_{C^{2+\mu}(S^2)} \) and \( \|\eta\|_{C^{2+\mu}(S^2)} \) sufficiently small, if \( \Omega = \Omega_\rho \) and \( \Gamma = \Gamma_\eta \) then the problem (1.6) is equivalent to the problem (1.11).

**Proof.** Let \( \nu \) be the outward unit normal field of the inner part boundary \( \Gamma_\eta \) of \( D_{\rho,\eta} \). It is easy to see that under the assumption \( \rho, \eta \in C^{2+\mu}(S^2) \), \( \Omega = \Omega_\rho \) and \( \Gamma = \Gamma_\eta \), the problem (1.6) is equivalent to the following problem:

\[
\begin{cases}
\Delta \sigma = \sigma & \text{in } D_{\rho,\eta}, \\
\sigma = 1 & \text{on } S_\rho, \\
\sigma = \hat{\sigma} & \text{on } \Gamma_\eta, \\
\partial_\nu \sigma = 0 & \text{on } \Gamma_\eta.
\end{cases}
\tag{2.12}
\]

That is, if \( \sigma \) is a solution of (1.6) for \( \Omega = \Omega_\rho \) and its free boundary \( \Gamma \) has the form \( \Gamma = \Gamma_\eta \) with \( \eta \in C^{2+\mu}(S^{n-1}) \), then the restriction of \( \sigma \) to \( \bar{D}_{\rho,\eta} \) is a solution of (2.12); conversely, if \( \sigma \) is a solution of (2.12) then by extending it into the whole domain \( \Omega = \Omega_\rho \) such that it identically takes the value \( \hat{\sigma} \) in \( \Omega \setminus \bar{D}_{\rho,\eta} \), then after such extension \( \sigma \) is a solution of (1.6). The condition \( \sigma = \hat{\sigma} \) on \( \Gamma_\eta \) implies \( \nabla_\omega \sigma(\eta(\omega), \omega) = -\partial_\nu \sigma(\eta(\omega), \omega) \nabla_\omega \eta(\omega) \) for \( \omega \in S^{n-1} \). Since \( \nu = [\omega - (1/r)\nabla_\omega \eta(\omega)]/\sqrt{1+1/(1/r^2)\nabla_\omega \eta(\omega)^2} \) and \( \nabla \sigma = (\partial_\nu \sigma)\omega + (1/r)\nabla_\omega \sigma \), so that

\[
\partial_\nu \sigma |_{r=\eta(\omega)} = \frac{\partial_\nu \sigma - (1/r^2)\nabla_\omega \eta(\omega) \cdot \nabla_\omega \sigma}{\sqrt{1+1/(1/r^2)\nabla_\omega \eta(\omega)^2}} |_{r=\eta(\omega)},
\]

we see that the condition \( \sigma = \hat{\sigma} \) on \( \Gamma_\eta \) implies \( \partial_\nu \sigma |_{r=\eta(\omega)} = \sqrt{1+1/(1/r^2)\nabla_\omega \eta(\omega)^2} \partial_\nu \sigma |_{r=\eta(\omega)} \). Hence the problems (1.11) and (2.12) are equivalent and, consequently, the problems (1.6) and (1.11) are equivalent. This proves the desired assertion. \( \square \)

Let us now consider the problem (2.11). Let \( \delta \) and \( O_\delta \) be as in the proof of Theorem 1.1. Given \( \rho \in O_\delta \), we first solve the equation (2.11)_{1} subject to the boundary value condition (2.11)_{2}. By Lemma 2.4 and Theorem 1.1, this problem has a unique solution \( \sigma \). Next we substitute \( \sigma \) into (2.11)_{2} and take (2.11)_{4} into account. Then we obtain the following elliptic boundary value problem:

\[
\begin{cases}
-\Delta \pi_0 = g(\sigma) & \text{in } \Omega, \\
\pi_0 = 0 & \text{on } \partial \Omega.
\end{cases}
\tag{2.13}
\]

By applying the standard theory for elliptic boundary value problems and using the properties of \( \sigma \) given in (1.12), we see the above problem has a unique solution satisfying the following properties:

\[
\pi_0 \in W^{2,q}(\Omega) \forall p \in [1, \infty), \quad \pi_0 \in C^\infty(O_{\text{nec}} \cup \Omega_{\text{nec}}) \quad \pi_0 \in C^{m+\mu}(\Omega_{\text{nec}} \cup \partial \Omega). \tag{2.14}
\]

It follows that \( \partial_\nu \pi_0 |_{\partial \Omega} \in C^{m-1+\mu}(\partial \Omega) \), where \( \nu \) denotes the unit outward normal field of \( \partial \Omega \). In this way we obtain a map \( F_0 : O_{\delta} \subseteq C^{m+\mu}(S^2) \to C^{m-1+\mu}(S^2) \) defined as follows: For any \( \rho \in O_{\delta} \),

\[
F_0(\rho) = [\omega \mapsto \partial_\nu \pi_0(R[1+\rho(\omega)], \omega), \omega \in S^2].
\]

Our next goal of this section is to prove the following result:
Lemma 2.5 \( F_0 \in C^\infty(O_\delta, C^{m-1+\mu}(S^2)) \).

Proof. From the proof of Lemma 2.2 we easily see that not only the map \( \rho \mapsto \eta \) is tame, but also the map \( \rho \mapsto \sigma[\mathbf{E}] \) from \( O_\delta \subseteq C^{m+\mu}(S^2) \) to \( C^\infty(\mathbf{E}) \) is tame. Let \( D_{\rho,\eta}, S_\rho \) and \( \Gamma_\eta \) be as before and set \( B_{\eta} = \{ x \in \mathbb{R}^3 : r < K[1 + \eta(\omega)] \} \), \( B_0 = B(0, K) \).

Then (2.13) can be rewritten as the following equivalent problem:

\[
\begin{cases}
-\Delta \pi_0 = a(\sigma - \bar{\sigma}) - b & \text{in } D_{\rho,\eta}, \\
-\Delta \pi_0 = -b & \text{in } B_{\eta}, \\
\pi_0 = 0 & \text{on } S_\rho, \\
\pi_0, \partial_\nu \pi_0 & \text{are continuous across } \Gamma_\eta,
\end{cases}
\]

(2.15)

where \( \nu \) is as before (note that it is also the inward unit normal field of the boundary \( \Gamma_\eta \) of \( B_{\eta} \)). Let \( \Psi_{\rho,\eta}, \Psi, \psi, \omega \) in Lemma 2.2 and define \( \mathcal{B}(\eta)u = [\partial_\nu (u \circ \Psi_{\eta})|_{\Gamma_\eta}] \circ \psi^{-1} \) for \( u \in C^\infty(\mathbf{E}) \).

Choose another smooth function \( \phi_1 \in C^\infty[0, K] \) such that it satisfies the following conditions:

\[ 0 \leq \phi_1 \leq 1; \quad \phi_1' > 0; \quad \phi_1(t) = 0 \text{ for } 0 \leq t \leq \frac{1}{2}K; \quad \phi_1(K) = 1. \]

Let \( M_1 = \max_{0 \leq t \leq K} |\phi_1'(t)| \) and assume \( \delta' > 0 \) is small enough such that in addition to the conditions appearing in the proof of Lemma 2.2 we have also \( \delta' < (1 + M_1 K)^{-1} \). Let \( \Psi_{\eta}^1 : B_{\eta} \to B_0 \) be as follows:

\[
\Psi_{\eta}^1(x) = x - K\eta(\omega)\phi_1\left(\frac{r}{1 + \eta(\omega)}\right)\omega \quad \text{for } x \in B_{\eta}.
\]

(2.16)

Define \( \omega_{\eta} : C^\infty(B_0) \to C^\infty(B_0) \) and \( \omega_{\eta} : C^\infty(B_0) \to C^\infty(\Gamma_0) \) respectively as follows:

\[
\omega_{\eta}u = [\Delta (u \circ \Psi_{\eta}^1)] \circ (\Psi_{\eta}^1)^{-1} \quad \text{for } u \in C^\infty(B_0),
\]

\[
\omega_{\eta}u = [\partial_\nu (u \circ \Psi_{\eta}^1)|_{\Gamma_\eta}] \circ (v_{\eta}^{-1})_\eta \quad \text{for } u \in C^\infty(B_0),
\]

(2.17)

where \( v_{\eta} = \Psi_{\eta}^1|_{\Gamma_\eta} \). Let \( v = \pi_0|_{B_{\rho,\eta}} \circ \Psi_{\rho,\eta}^{-1} \) and \( v_1 = \pi_0|_{B_{\eta}} \circ (\Psi_{\eta}^1)^{-1} \). After the variable transformation \( x \mapsto \Psi_{\rho,\eta}(x) \) (for \( x \in B_{\rho,\eta} \)) and \( x \mapsto \Psi_{\eta}^1(x) \) (for \( x \in B_{\eta} \)), the problem (2.15) transforms into the following problem:

\[
\begin{cases}
-\omega\rho,\eta)v = a(u - \bar{\sigma}) - b & \text{in } D, \\
\omega_\eta v_1 = b & \text{in } B_0, \\
v = 0 & \text{on } S_\rho, \\
v = v_1 & \text{on } \Gamma_\eta, \\
\mathcal{B}(\eta)v = \omega_\eta v_1 & \text{on } \Gamma_\eta.
\end{cases}
\]

Lemma 2.5 is an immediate consequence of Theorem 1.1 and the following preliminary result:

Lemma 2.6 Let \( \delta, O_{\delta}' \) be as in the proof of Lemma 2.2 and \( \delta' \) as stated above. Let \( O_{\delta}' = \{ \eta \in C^{m+\mu}(S^2) : \|\eta\|_{C^{m+\mu}(S^2)} < \delta' \} \). Given \( (\rho, \eta, u) \in O_{\delta}' \times O_{\delta}' \times C^{m+\mu}(\mathbf{D}) \), the problem (2.17) has a unique solution \( (v, v_1) \in C^{m+\mu}(\mathbf{D}) \times C^{m+\mu}(\mathbf{B}_0) \), and the solution map \( (\rho, \eta, u) \mapsto (v, v_1) \) from \( O_{\delta}' \times O_{\delta}' \times C^{m+\mu}(\mathbf{D}) \subseteq C^{m+\mu}(S^2) \times C^{m+\mu}(S^2) \times C^{m+\mu}(\mathbf{D}) \) to \( C^{m+\mu}(\mathbf{D}) \times C^{m+\mu}(\mathbf{B}_0) \) is smooth.
We introduce a mapping \( G(\xi) \) (cf. Lemmas 3.1 and 3.2 of [8]). It is clear that \( G(\xi) \) are constant functions such that the relations \( V(K^+, R) = V(K^-, R) \) hold; in particular,

\[
\begin{align*}
D = \frac{1}{3} \tilde{\sigma} K^3 - a \int_K U(\eta, R) \eta^2 d\eta.
\end{align*}
\]

(c.f. Lemmas 3.1 and 3.2 of [8]).

Given \( (\rho, \eta, u, \xi) \in O_0' \times O_0'' \times C^{m+\mu}(\overline{D}) \times C^{m+\mu}(\Gamma_0), \) we consider the following two elliptic boundary value problems:

\[
\begin{cases}
-\mathcal{A}(\rho, \eta) v = a(u - \tilde{\sigma}) - b & \text{in } D, \\
v = 0 & \text{on } S_0, \\
v = \xi & \text{on } \Gamma_0,
\end{cases}
\quad
\begin{cases}
\mathcal{A}_1(\eta) v_1 = b & \text{in } B_0, \\
v_1 = \xi & \text{on } \Gamma_0.
\end{cases}
\]

(2.18)

Clearly, these problems have unique solutions \( v \in C^{m+\mu}(\overline{D}) \) and \( v_1 \in C^{m+\mu}(\overline{B}_0) \), respectively. Define \( \mathcal{D}(\rho, \eta) : C^{m+\mu}(\Gamma_0) \to C^{m-1+\mu}(\Gamma_0) \) and \( \mathcal{D}_1(\eta) : C^{m+\mu}(\Gamma_0) \to C^{m-1+\mu}(\Gamma_0) \) respectively as follows:

\[
\mathcal{D}(\rho, \eta) \xi = \mathcal{D}(\xi) v, \quad \mathcal{D}_1(\eta) \xi = \mathcal{D}_1(\eta) v_1 \quad \text{for } \xi \in C^{m+\mu}(\Gamma_0).
\]

(2.19)

The problem (2.17) is equivalent to the following problem: Find \( \xi \in C^{m+\mu}(\Gamma_0) \) such that

\[
\mathcal{D}(\rho, \eta) \xi = \mathcal{D}_1(\eta) \xi.
\]

We introduce a mapping \( \mathcal{D} : O_0' \times O_0'' \times C^{m+\mu}(\overline{D}) \times C^{m+\mu}(\Gamma_0) \to C^{m-1+\mu}(\Gamma_0) \) by defining

\[
\mathcal{D}(\rho, \eta, u, \xi) = \mathcal{D}(\rho, \eta) \xi - \mathcal{D}_1(\eta) \xi \quad \text{for } \rho \in O_0', \ \eta \in O_0'', \ u \in C^{m+\mu}(\overline{D}), \ \xi \in C^{m+\mu}(\Gamma_0).
\]

(2.20)

It is clear that \( \mathcal{D} \in C^{\infty}(O_0' \times O_0'' \times C^{m+\mu}(\overline{D}) \times C^{m+\mu}(\Gamma_0), C^{m-1+\mu}(\Gamma_0)) \), and

\[
\mathcal{D}(0, 0, u_0, \xi_0) = 0,
\]

where \( u_0 = U(r, R) \) and \( \xi_0 \) represents the following constant function in \( \Gamma_0 \): \( \xi_0(x) = V(K, R) \) for \( x \in \Gamma_0 \). A simple computation shows that for any \( \zeta \in C^{m+\mu}(\Gamma_0) \), \( L\zeta := \partial_{\nu} \mathcal{D}(0, 0, u_0, \xi_0) \zeta = \partial_{\nu} w|_{\Gamma_0} - \partial_{\nu} w_1|_{\Gamma_0} = \partial_r w_1|_{\Gamma_0} - \partial_r w|_{\Gamma_0} \), where \( w, w_1 \) are respectively the unique solutions of the following problems:

\[
\begin{cases}
\Delta w = 0 & \text{for } K < r < R, \\
w = 0 & \text{for } r = R, \\
w = \xi & \text{for } r = K,
\end{cases}
\quad
\begin{cases}
\Delta w_1 = 0 & \text{for } r < K, \\
w_1 = \xi & \text{for } r = K.
\end{cases}
\]

(2.21)

From this fact it is not hard to see that if \( L\zeta = 0 \) then \( \zeta = 0 \). Indeed, if \( L\zeta = 0 \) then by letting \( z = w \) for \( K \leq r \leq R \) and \( z = w_1 \) for \( r < K \), we get a weak solution of the boundary value problem

\[
\begin{cases}
\Delta z = 0 & \text{in } B(0, R), \\
z = 0 & \text{on } \partial B(0, R),
\end{cases}
\]

(2.22)
which implies, by Green’s second identity, that $\zeta = 0$. This shows that $\text{Ker } L = \{0\}$. Since $L$ is a sum of two Dirichlet-Neumann operators, it is a first-order pseudo-differential operator of elliptic type (cf. [15], [16]), so that standard Schauder estimate for elliptic pseudo-differential operator applies to it: There exists a positive constant $C > 0$ such that

$$\|\zeta\|_{C^{m+\mu}(\Gamma_0)} \leq C(\|L\zeta\|_{C^{m-1+\mu}(\Gamma_0)} + \|\zeta\|_{L^\infty(\Gamma_0)}), \quad \forall \zeta \in C^{m+\mu}(\Gamma_0).$$

Since $\text{Ker } L = \{0\}$, this implies, by a standard argument, the following estimate:

$$\|\zeta\|_{C^{m+\mu}(\Gamma_0)} \leq C\|L\zeta\|_{C^{m-1+\mu}(\Gamma_0)}, \quad \forall \zeta \in C^{m+\mu}(\Gamma_0).$$

(2.21)

For every $k \in \mathbb{Z}_+$ let $\{Y_{kl}(\omega)\}_{l=1}^{2k+1}$ be the normalized orthogonal basis (in $L^2(S^2)$ inner product) of the linear space of $k$-th order spherical harmonics. A simple computation shows that for any $\zeta \in C^\infty(\Gamma_0)$,

$$L\zeta(\omega) = \sum_{k=0}^\infty \sum_{l=1}^{2k+1} \frac{(2k+1)c_{kl}}{[1 - (K/R)^{2k+1}]} Y_{kl}(\omega) \text{ if } \zeta(K\omega) = \sum_{k=0}^\infty \sum_{l=1}^{2k+1} c_{kl} Y_{kl}(\omega).$$

From this expression of $L$ it is easy to see that for any $\eta \in C^\infty(\Gamma_0)$ the equation $L\zeta = \eta$ has a unique solution $\zeta \in C^\infty(\Gamma_0)$ (see the proofs of Lemma 3.3 and Corollary 3.4 in the next section for more details in this argument). It follows, by using the estimate (2.21) and a standard approximation argument, that also for any $\eta \in C^{m-1+\mu}(\Gamma_0)$ the equation $L\zeta = \eta$ has a unique solution $\zeta \in C^{m+\mu}(\Gamma_0)$.$^4$

This shows that $L = \partial_\zeta \mathcal{G}(0, 0, u_0, \xi_0) : C^{m-1+\mu}(\Gamma_0) \to C^{m-1+\mu}(\Gamma_0)$ is a (linear and topological) isomorphism. Hence, by applying the implicit function theorem in Banach spaces, we see that by choosing $\delta, \delta' > 0$ further small when necessary, there exists a smooth mapping $\chi : O'_\delta \times O''_{\delta'} \times C^{m+\mu}(\overline{D}) \subseteq C^{m+\mu}(S^2) \times C^{m+\mu}(S^2) \times C^{m+\mu}(\overline{D}) \to C^{m+\mu}(\Gamma)$ such that $\chi(0, 0, u_0) = \xi_0$ and for any $\rho \in O'_\delta$, $\eta \in O''_{\delta'}$ and $u \in C^{m+\mu}(\overline{D})$, $\xi = \chi(\rho, \eta, u)$ is the unique solution of the equation $\mathcal{G}(\rho, \eta, u, \xi) = 0$ in a small neighborhood of $\xi_0$ in $C^{m+\mu}(\Gamma)$. This proves unique solvability of the equation (2.19) and smoothness of the solution map $\xi = \chi(\rho, \eta, u)$. As a result, the solution map $(\rho, \eta, u) \mapsto (v, \gamma_1)$ (from $O'_\delta \times O''_{\delta'} \times C^{m+\mu}(\overline{D}) \subseteq C^{m+\mu}(S^2) \times C^{m+\mu}(\overline{D})$ to $C^{m+\mu}(\overline{D}) \times C^{m+\mu}(\overline{D})$) of the problem (2.17) is smooth. This proves Lemma 2.6 and also completes the proof of Lemma 2.5. $^\square$

**Remark.** We note that all Hölder spaces appearing in this section can be replaced with corresponding little Hölder spaces. In the next section we shall use this fact without making further explanation.

### 3 The proof of Theorem 1.2

In this section we prove Theorem 1.2.

We first recall that radial stationary solution $(\sigma_s, \pi_s, \Omega_s)$ of the problem (1.1) is given by

$$\sigma_s(r) = U(r, R_s), \quad \pi_s(r) = \frac{\gamma}{R_s} + V(r, R_s), \quad \Omega_s = B(0, R_s),$$

(3.1)

$^4$Since $C^\infty(\Gamma_0)$ is not dense in $C^{m-1+\mu}(\Gamma_0)$, one might argue validity of the approximation argument. It is as follows: For any $\eta \in C^{m-1+\mu}(\Gamma_0)$ choose a sequence $\{\eta_j\}_{j=1}^\infty \subseteq C^\infty(\Gamma_0)$ such that it is bounded in $C^{m-1+\mu}(\Gamma_0)$ and converges to $\eta$ in $C^{m-1+\mu}(\Gamma_0)$ for any $0 < \mu' < \mu$ (e.g., choose a mollification sequence of $\eta$). For every $j$ let $\zeta_j \in C^\infty(\Gamma_0)$ be the unique solution of the equation $L\zeta_j = \eta_j$. The estimate (2.21) ensures the sequence $\{\zeta_j\}_{j=1}^\infty$ is bounded in $C^{m+\mu}(\Gamma_0)$ and converges to a function $\zeta$ in $C^{m+\mu}(\Gamma_0)$ for any $0 < \mu' < \mu$, which implies that $\zeta$ is a solution of the equation $L\zeta = \eta$. Boundedness of the sequence $\{\zeta_j\}_{j=1}^\infty$ in $C^{m+\mu}(\Gamma_0)$ implies $\zeta \in C^{m+\mu}(\Gamma_0)$.
where $U$, $V$ are as in the previous section, and $R_s$ is the root of the equation $\pi_\sigma'(R_s) = 0$ or $\partial_s V(R_s, R_s) = 0$, i.e.,
\[
\frac{1}{3}a\hat{\sigma}K^3(R_s) + a \int_{K(R_s)}^R U(\eta, R_s)\eta^2 d\eta = \frac{1}{3}(a\hat{\sigma} + b)R_s^3.
\]
Since $0 < \hat{\sigma} < 1$, by Lemma 4.2 of [8] we know that this equation has a unique solution $R_s > R^*$, which means the problem (1.1) with $f$, $g$ given in (1.2) has a unique radial stationary solution.

We shall use Theorem 1.1 of [1] to prove Theorem 1.2. To this end, let us first reduce the problem (1.1) into a differential equation in a Banach manifold. Let $m$ be a positive integer $\geq 2$ and $0 < \mu < 1$. Let $\mathcal{M} := \mathcal{D}^{m,\mu}(\mathbb{R}^3)$ and $\mathcal{M}_0 := \mathcal{D}^{m,\mu+3}(\mathbb{R}^3)$. Given $\Omega \in \mathcal{M}_0$, we have seen that the problem (2.11) has a unique solution $(\sigma, \pi_0)$ satisfying the following properties:
\[
\sigma, \pi_0 \in W^{2,q}(\Omega) \quad (\forall q \in [1, \infty)), \quad \sigma, \pi_0 \in \dot{C}^\infty(\Omega_{\text{liv}} \cup \Omega_{\text{fuc}}) \quad \text{and} \quad \sigma, \pi_0 \in \dot{C}^{m+3+\mu}(\Omega_{\text{liv}} \cup \partial \Omega).
\]
It follows that $\partial_\pi \pi_0|_{\partial \Omega} \in \dot{C}^{m+\mu+2}(\partial \Omega)$. We define $\mathcal{F}_0 : \mathcal{M}_0 \to \mathcal{T}_{\mathcal{M}_0}(\mathcal{M})$ by letting
\[
\mathcal{F}_0(\Omega) = -\partial_\pi \pi_0|_{\partial \Omega}, \quad \forall \Omega \in \mathcal{M}_0.
\]
Next, given $\Omega \in \mathcal{M}_0$, let $\pi \in \dot{C}^{m+\mu+1}(\Omega)$ be the unique solution of the following elliptic boundary value problem:
\[
\begin{cases}
-\Delta \pi = 0 & \text{in } \Omega, \\
\pi = \kappa & \text{on } \partial \Omega,
\end{cases}
\]
where $\kappa$ is as explained in Section 1, and define $\mathcal{F} : \mathcal{M}_0 \to \mathcal{T}_{\mathcal{M}_0}(\mathcal{M})$ by letting
\[
\mathcal{F}(\Omega) = -\partial_\pi \pi|_{\partial \Omega}, \quad \forall \Omega \in \mathcal{M}_0.
\]
We now define $\mathcal{G} : \mathcal{M}_0 \to \mathcal{T}_{\mathcal{M}_0}(\mathcal{M})$ as follows:
\[
\mathcal{G}(\Omega) = \gamma \mathcal{F}(\Omega) + \mathcal{F}_0(\Omega), \quad \forall \Omega \in \mathcal{M}_0. \tag{3.2}
\]
Then $\mathcal{F}$ is a vector field in $\mathcal{M}$ with domain $\mathcal{M}_0$, and the problem (1.1) reduces into the following differential equation in the Banach manifold $\mathcal{M}$:
\[
\begin{cases}
\Omega'(t) = \mathcal{G}(\Omega(t)), & t > 0, \\
\Omega(0) = \Omega_0.
\end{cases} \tag{3.3}
\]
The fact that $(\sigma_s, \pi_s, \Omega_s)$ is a stationary solution of the problem (1.1) implies that $\Omega_s$ is a stationary solution of the equation $\Omega' = \mathcal{G}(\Omega)$.

Let $G_{ul} = \mathbb{R}^n$ be the additive group of $n$-vectors. Given $z \in \mathbb{R}^n$ and $\Omega \in \mathcal{M}$, let
\[
p(z, \Omega) = \Omega + z = \{x + z : x \in \Omega\}.
\]
It is clear that $p(z, \Omega) \in \mathcal{M}$, $\forall \Omega \in \mathcal{M}$, $\forall z \in \mathbb{R}^n$. It can be easily seen that $(G_{ul}, p)$ is a Lie group action on $\mathcal{M}$. By Lemma 4.1 of [8] we know that the action $p(z, \Omega)$ is differentiable at every point $\Omega \in \mathcal{M}_0$, and $\text{rank} D_z p(z, \Omega) = n$, $\forall z \in G_{ul}$, $\forall S \in \mathcal{M}_0$.

**Lemma 3.1** The vector field $\mathcal{G}$ is invariant under the group action $(G_{ul}, p)$.

**Proof.** The proof is similar to that of Lemma 5.3 of [1]. We omit it here. \(\square\)
Next we consider representation of the problem \((3.3)\) in a regular local chart \((\mathcal{U}, \varphi)\) of \(\mathfrak{M}\) at the point \(\Omega_s\). We denote
\[
X = \mathcal{C}^{m+\mu}(S^2), \quad X_0 = \mathcal{C}^{m+\mu+3}(S^2), \quad X_1 = \mathcal{C}^{m+\mu+2}(S^2),
\]
and for a sufficiently small number \(\delta > 0\) let
\[
O_\delta = \{\rho \in X, \|\rho\|_X < \delta\}, \quad O'_\delta = \{\rho \in X_0, \|\rho\|_{X_0} < \delta\},
\]
\[
\mathcal{U} = \{\Omega_\rho : \rho \in O_\delta\}, \quad \mathcal{U}' = \{\Omega_\rho : \rho \in O'_\delta\},
\]
where \(\Omega_\rho\) is as before, i.e.,
\[
\Omega_\rho = \{x \in \mathbb{R}^3 : r < R_s[1 + \rho(\omega)]\}.
\]
It is clear that \(\Omega_\rho \in \mathfrak{M}\) for \(\rho \in O_\delta\) and \(\Omega_\rho \in \mathfrak{M}_0\) for \(\rho \in O'_\delta\), so that \(\mathcal{U}\) and \(\mathcal{U}'\) are neighborhoods of \(\Omega_s\) in \(\mathfrak{M}\) and \(\mathfrak{M}_0\), respectively. We define \(\varphi : \mathcal{U} \to X\) by letting \(\varphi(\Omega_\rho) = \rho, \forall \rho \in O_\delta\). Then \((\mathcal{U}, \varphi)\) is a regular local chart of \(\mathfrak{M}\) at the point \(\Omega_s\), with base space \(X\). We denote by \(F, F_0\) and \(G\) the representations of the vector fields \(\mathcal{F}, \mathcal{F}_0\) and \(\mathcal{G}\), respectively, in this local chart, i.e., for any \(\rho \in O'_\delta\),
\[
F(\rho) = \varphi'(\Omega_\rho)\mathcal{F}(\Omega_\rho), \quad F_0(\rho) = \varphi'(\Omega_\rho)\mathcal{F}_0(\Omega_\rho) \quad \text{and} \quad G(\rho) = \varphi'(\Omega_\rho)\mathcal{G}(\Omega_\rho);
\]
see Section 2 of \([11]\) for details of these notions. Then \(G(\rho) = \gamma F(\rho) + F_0(\rho)\), and representation of the problem \((3.3)\) in the local chart \((\mathcal{U}, \varphi)\) is the following initial value problem in the Banach space \(X\):
\[
\begin{cases}
\rho'(t) = G(\rho(t)), & t > 0, \\
\rho(0) = \rho_0,
\end{cases}
\]
(3.4)
where \(\rho_0 \in O'_\delta\) is the function such that \(\Omega_0 = \Omega_{\rho_0}\). It is clear that \(F \in C^\infty(O'_\delta, X)\). By Lemma 2.5 we know that also \(F_0 \in C^\infty(O'_\delta, X_0)\). Hence we have

**Lemma 3.2** \(G \in C^\infty(O'_\delta, X)\). \(\square\)

**Lemma 3.3** The differential equation \((3.3)\) is of parabolic type.

**Proof.** It is well-known that for any \(\rho \in O'_\delta\), \(F'(\rho)\) is a sectorial operator in \(X\) with domain \(X_0\); cf., e.g., \([2]\), \([3]\). Lemma 2.5 ensures that for any \(\rho \in O'_\delta, F_0(\rho) \in L(X_0, X_1)\). Since \(G'(\rho) = \gamma F'(\rho) + F_0'(\rho)\) and \(X_1\) is an intermediate space between \(X_0\) and \(X\), by a well-known perturbation theorem for sectorial operators it follows that \(G'(\rho)\) is also a sectorial operator in \(X\) with domain \(X_0\). Besides, it is easy to check that the graph norm of \(\text{Dom} G'(\rho) = X_0\) is equivalent to the norm of \(X_0\). Hence the desired assertion follows. \(\square\)

Following \([21]\) and \([13]\), we compute \(G'(0)\) as follows: Let
\[
\rho(\omega) = \varepsilon \xi(\omega), \quad \eta(\omega) = \varepsilon \zeta(\omega), \quad \sigma(r, \omega) = \sigma_s(r) + \varepsilon u(r, \omega).
\]
Substituting these expressions into \((2.12)\) and the equation \(\Delta \sigma = 0\) for \(r < K_s\), and comparing coefficients of first-order terms of \(\varepsilon\), we obtain the following equations:
\[
\begin{cases}
\Delta u = u \quad &\text{for } K_s < r < R_s, \\
\Delta u = 0 \quad &\text{for } r < K_s, \\
u = -R_s \sigma_s'(R_s) \xi(\omega) \quad &\text{for } r = R_s, \\
u = 0 \quad &\text{for } r = K_s, \\
\partial^+_r u = -\delta K_s \zeta(\omega) \quad &\text{for } r = K_s.
\end{cases}
\]
(3.5)
Here \( K_s = K(R_s) \) and for \( r = K_s, \frac{\partial^+ v}{\partial r} u = \frac{\partial^+ v}{\partial r}(K_s^+, \omega) \). Similarly, by letting \( \pi(r, \omega) = \pi_s(r) + \varepsilon v(r, \omega) \), we get the following equations for \( v \):

\[
\begin{aligned}
\Delta v &= -au & \text{for } K_s < r < R_s, \\
\Delta v &= 0 & \text{for } r < K_s, \\
v &= -\frac{\gamma}{R_s} \left( \xi(\omega) + \frac{1}{2} \Delta \omega \xi(\omega) \right) & \text{for } r = R_s, \\
v^+ &= v^- & \text{for } r = K_s, \\
\partial_r^+ v - \partial_r^- v &= a(\tilde{\sigma} - \tilde{\sigma}) K_s \xi(\omega) & \text{for } r = K_s.
\end{aligned}
\]

(3.6)

Here for \( r = K_s, v^\pm = v(K_s^\pm, \omega) \) and \( \partial_r^\pm v = \frac{\partial^\pm v}{\partial r}(K_s^\pm, \omega) \). After solving these equations (with \( u, v \) and \( \xi \) being unknown functions) for any given function \( \xi = \xi(\omega) \), we then have

\[
G'(0)\xi(\omega) = -\frac{\partial v}{\partial r}(R_s, \omega) + g(1)R_s \xi(\omega), \quad \forall \xi \in X_0
\]

(3.7)

(cf. (4.4) in [13], but be aware that here the perturbation of the sphere \( r = R_s \) is given by \( r = R_s[1+\varepsilon \xi(\omega)] \), not as in [13] given by \( r = R_s + \varepsilon \xi(\omega) \)).

We now use spherical harmonics expansions of functions in \( S^2 \) to solve the problems (3.5) and (3.6). Hence let \( \{ Y_{kl}(\omega) : k = 0, 1, 2, \cdots, l = 1, 2, \cdots, 2k + 1 \} \) be a normalized orthogonal basis of \( L^2(S^2) \) consisting spherical harmonics on \( S^2 \), where for every \( k \in \mathbb{Z}_+, Y_{kl}(\omega), l = 1, 2, \cdots, 2k + 1 \), are spherical harmonics of degree \( k \), so that

\[
\Delta_\omega Y_{kl}(\omega) = -\lambda_k Y_{kl}(\omega), \quad k = 0, 1, 2, \cdots, l = 1, 2, \cdots, 2k + 1.
\]

where \( \lambda_k = k(k+1) \), \( k = 0, 1, 2, \cdots \). A simple computation shows that if a given function \( \xi \in C^\infty(S^2) \) has a spherical harmonics expansion

\[
\xi(\omega) = \sum_{k=0}^\infty \sum_{l=1}^{2k+1} c_{kl} Y_{kl}(\omega),
\]

then the solution \( (u, \xi) \) of the problem (3.5) is given by

\[
u(r, \omega) = \begin{cases} 
-R_s \sigma_s(R_s) \sum_{k=0}^\infty \sum_{l=1}^{2k+1} \left( \frac{r}{R_s} \right)^k \tilde{u}_k(r)c_{kl} Y_{kl}(\omega) & \text{for } K_s < r \leq R_s \\
0 & \text{for } r \leq K_s
\end{cases}
\]

and \( \xi(\omega) = -\partial_r^+ u / \tilde{\sigma} K_s \), where \( \tilde{u}_k \) is the unique solution of the following boundary value problem:

\[
\begin{pmatrix}
\tilde{u}_k''(r) + \frac{2(k+1)}{r} \tilde{u}_k'(r) = \tilde{u}_k(r) & \text{for } K_s < r \leq R_s, \\
\tilde{u}_k(K_s) = 0, & \tilde{u}_k(R_s) = 1.
\end{pmatrix}
\]

(3.9)

Substituting these expressions of \( u \) and \( \xi \) into (3.6), we easily obtain the following solution of that problem:

\[
v(r, \omega) = \sum_{k=0}^\infty \sum_{l=1}^{2k+1} \left[ \frac{\gamma}{2R_s} (k-1)(k+2) + a R_s \sigma_s(R_s) \tilde{v}_k(r) \right] \left( \frac{r}{R_s} \right)^k c_{kl} Y_{kl}(\omega),
\]

(3.10)
where $\tilde{v}_k$ is the unique solution of the following boundary value problem:

$$\begin{cases}
\tilde{v}'_k(r) + \frac{2(k+1)}{r} \tilde{v}_k(r) = \tilde{u}_k(r) & \text{for } K_s < r < R_s, \\
\tilde{v}'_k(r) + \frac{2(k+1)}{r} \tilde{v}_k(r) = 0 & \text{for } r < K_s, \\
\tilde{v}_k(R_s) = 0, \\
\tilde{v}_k(K_+^s) = \tilde{v}_k(K_-^s), \\
\tilde{v}'_k(K_+^s) - \tilde{v}'_k(K_-^s) = \frac{\sigma - \tilde{\sigma}}{\sigma} \tilde{u}_k(K_+^s).
\end{cases} \quad (3.11)$$

From (3.11) we get the following relation:

$$\tilde{v}'_k(R_s) = \frac{\sigma - \tilde{\sigma}}{\sigma} \tilde{u}_k(K_+^s) \left( \frac{K_+^s}{R_s} \right)^{2(k+1)} + \int_{K_s}^{R_s} \tilde{u}_k(\tau) \left( \frac{\tau}{R_s} \right)^{2(k+1)} d\tau. \quad (3.12)$$

Now let

$$a_k(\gamma) = -\frac{\gamma}{2R_s^2} k(k-1)(k+2) - aR_s \sigma'(R_s) \tilde{v}_k(R_s) + g(1)R_s. \quad (3.13)$$

Then from (3.8), (3.10), (3.11) and (3.12) we obtain the following preliminary result:

**Lemma 3.4** $G'(0)$ is a Fourier multiplier in the sense that if $\xi \in C^\infty(S^2)$ has expansion (3.8), then

$$G'(0)\xi = \sum_{k=0}^{\infty} \sum_{l=1}^{2k+1} a_k(\gamma) c_{kl} Y_{kl}(\omega) \in C^\infty(S^2). \quad \square$$

**Corollary 3.5** $\sigma(G'(0)) = \{a_k(\gamma) : k = 0, 1, 2, \cdots \}$.

**Proof.** Since $G'(0) \in L(X_0, X)$ and it is a sectorial operator in $X$ with domain $X_0$, the inverse mapping theorem implies that for any $\lambda \in \rho(G'(0))$ we have $|\lambda - G'(0)|^{-1} \in L(X, X_0)$. As a consequence, $R(\lambda, G'(0)) = |\lambda - G'(0)|^{-1}$ is a compact linear operator in $X$, so that its spectrum contains only eigenvalues. It follows that $\sigma(G'(0))$ also contains only eigenvalues. Next, for every $s \geq 0$ let $H^s(S^2)$ be the standard Sobolev space on $S^2$ with index $s$. We know that $H^s(S^2)$ has an equivalent norm $\|\xi\|_{H^s(S^2)} = \left[ \sum_{k=0}^{\infty} \sum_{l=1}^{2k+1} (1 + \lambda_k)^s |c_{kl}|^2 \right]^{1/2}$, if $\xi \in H^s(S^2)$ has the expression (3.8) as an element of $L^2(S^2)$.

Since $C^\infty(S^2) = \bigcap H^s(S^2)$ and it is dense in every $H^s(S^2)$, $s \geq 0$, Lemma 3.4 shows that the operator $G'(0) : C^{m+3+\mu}(S^2) \to C^{m+\mu}(S^2)$ can be uniquely extended into a bounded linear operator from $H^3(S^2)$ to $L^2(S^2)$, and after extension we have the relation $\sigma(G'(0)) = \{a_k(\gamma) : k = 0, 1, 2, \cdots \}$. Moreover, the eigenspace corresponding to the eigenvalue $a_k(\gamma)$ is span\{$Y_{kl}(\omega) : l = 1, 2, \cdots, 2k+1$\} $\subseteq C^\infty(S^2)$, i.e., all eigenfunctions are smooth. Hence, since if $\xi \in C^{m+3+\mu}(S^2)$ is an eigenvector of the operator $G'(0) : C^{m+3+\mu}(S^2) \to C^{m+\mu}(S^2)$ then it is also an eigenvector of the operator $G'(0) : H^3(S^2) \to L^2(S^2)$, we obtain the desired assertion. \square

It is clear that $a_0(\gamma)$ and $a_1(\gamma)$ are independent of $\gamma$. Hence we re-denote them as $a_0$ and $a_1$, respectively, i.e.,

$$a_0 = g(1)R_s - aR_s \sigma'(R_s) \tilde{v}_0(R_s), \quad a_1 = g(1)R_s - aR_s \sigma'(R_s) \tilde{v}_1(R_s).$$

For $k \geq 2$ we denote

$$\gamma_k = \frac{2R_s^3}{k(k-1)(k+2)} [g(1) - a\sigma'(R_s) \tilde{v}_k(R_s)]. \quad (3.14)$$
Then from (3.13) we have
\[ a_k(\gamma) = -\frac{1}{2R^2_s}k(k-1)(k+2)(\gamma - \gamma_k), \quad k = 2, 3, \ldots. \] (3.15)

We shall prove \( a_0 < 0, a_1 = 0 \) and \( \gamma_k > 0 \) for \( k \geq 2 \). For this purpose we need the following lemma:

**Lemma 3.6** For the solution of the problem (3.9) we have the following assertions:

1. \( 0 < \bar{u}_k(r) < 1 \) and \( \bar{u}_k'(r) > 0 \) for \( K_s < r < R_s \).
2. If \( k > l \) then \( \bar{u}_k(r) > \bar{u}_l(r) \) for \( K_s < r < R_s \), and \( \bar{u}_k'(K_s) \geq \bar{u}_l'(K_s) \), \( \bar{u}_k'(R_s) \leq \bar{u}_l'(R_s) \).
3. If \( k > l \) then \( \bar{u}_k(r)(r/R_s)^{k+1} < \bar{u}_l(r)(r/R_s)^{k+1} \) for \( K_s < r < R_s \).

**Proof.** The assertion \( 0 < \bar{u}_k(r) < 1 \) for \( K_s < r < R_s \) is an immediate consequence of the maximum principle. Note that this assertion joint with the boundary value conditions \( \bar{u}_k(K_s) = 0 \) and \( \bar{u}_k(R_s) = 1 \) implies that \( \bar{u}_k'(K_s) \geq 0 \) and \( \bar{u}_k'(R_s) \leq 0 \). Next we let \( w_k(r) = \bar{u}_k'(r) \). A simple computation shows that \( w_k \) satisfies the following equation:
\[ w''_k(r) + \frac{2(k+1)}{r}w'_k(r) - \left( \frac{2(k+1)}{r^2} + 1 \right)w_k(r) = 0 \quad \text{for} \quad K_s < r < R_s. \]

Since \( w_k(K_s) \geq 0 \) and \( w_k(R_s) \leq 0 \), again by the maximum principle we see that \( w_k(r) > 0 \) for \( K_s < r < R_s \). This proves the assertion (1). From the property \( \bar{u}_k'(r) > 0 \) for \( K_s < r < R_s \) it follows that if \( k > l \) then
\[ \bar{u}_k''(r) + \frac{2(l+1)}{r}\bar{u}_k'(r) - \bar{u}_k(r) < 0 \quad \text{for} \quad K_s < r < R_s. \]

Hence by the maximum principle we obtain \( \bar{u}_k(r) > \bar{u}_l(r) \) for \( K_s < r < R_s \) and \( k > l \), which easily implies that \( \bar{u}_k'(K_s) \geq \bar{u}_l'(K_s) \) and \( \bar{u}_k'(R_s) \leq \bar{u}_l'(R_s) \) for \( k > l \). This proves the assertion (2). Finally we let \( z_k(r) = \bar{u}_k(r)(r/R_s)^{k+1} \). It can be easily seen that \( z_k(r) \) is a solution of the following problem:
\[
\begin{cases}
  z_k''(r) - \left( \frac{k(k+1)}{r^2} + 1 \right)z_k(r) = 0 & \text{for} \quad K_s < r < R_s, \\
  z_k(K_s) = 0, \quad z_k(R_s) = 1.
\end{cases}
\]

Since \( z_k > 0 \) for \( K_s < r < R_s \), a similar argument as in the proof of the assertion (2) shows that if \( k > l \) then \( z_k(r) < z_l(r) \) for \( K_s < r < R_s \). This proves the assertion (3) and completes the proof of the lemma. \( \square \)

**Lemma 3.7** We have the following assertions:

1. \( a_0 < 0 \) and \( a_1 = 0 \).
2. \( \gamma_k > 0 \) for all \( k = 2, 3, \ldots \), and \( \gamma_k \sim 2R^3_s \gamma(1)k^{-3} \) as \( k \to \infty \).

**Proof.** From (3.9) we have
\[ \bar{u}_k'(R_s) = \bar{u}_k'(K_s) \left( \frac{K_s}{R_s} \right)^{2(k+1)} + \int_{K_s}^{R_s} \bar{u}_k'(\tau) \left( \frac{\tau}{R_s} \right)^{2(k+1)} d\tau. \]

Hence the relation (3.12) can be rewritten as follows:
\[ \bar{u}_k'(R_s) = \frac{\bar{u}_k'(K_s)}{\gamma_k} + \frac{\bar{u}_k'(R_s)}{\gamma_k} \int_{K_s}^{R_s} \bar{u}_k(\tau) \left( \frac{\tau}{R_s} \right)^{2(k+1)} d\tau. \]
Using this relation and the assertions (2) and (3) of Lemma 3.6, we conclude that
\[ \bar{v}_k'(R_s) < \bar{v}_l'(R_s) \quad \text{if} \quad k > l. \]
Hence, all the desired assertions will follow if we prove that \( a_1 = 0 \) and \( \bar{v}_k'(R_s) = O(1/k) \) as \( k \to \infty \). The proof that \( \bar{v}_k'(R_s) = O(1/k) \) as \( k \to \infty \) is easy and is omitted. In what follows we prove \( a_1 = 0 \).

It is easy to check
\[ \bar{u}_1(r) = \frac{R_s \sigma_s'(r)}{r \sigma_s'(R_s)} \quad \text{for} \quad K_s \leq r \leq R_s. \tag{3.16} \]
In what follows we prove
\[ \bar{v}_1(r) = -\frac{R_s \pi_s'(r)}{ar \sigma_s'(R_s)} \quad \text{for} \quad K_s \leq r \leq R_s. \tag{3.17} \]
We use the notation \( q(r) \) to denote the function on the right-hand side of the above relation. To prove the above relation, we only need to show that \( q(r) \) is a solution of the following problem:
\[
\begin{cases}
q''(r) + \frac{4}{r}q'(r) = \bar{u}_1(r) & \text{for} \quad K_s < r < R_s, \\
q(R_s) = 0, \quad q'(K_s^+) = \frac{\sigma - \bar{\sigma}}{\sigma} \bar{u}_1(K_s^+).
\end{cases}
\]
The equation in the first line is easy to check, and the boundary value condition \( q(R_s) = 0 \) is clear. Since
\[ q'(r) = -\frac{R_s}{a \sigma_s'(R_s)} \left[ \frac{\pi_s''(r)}{r} - \frac{\pi_s'(r)}{r^2} \right] = \frac{R_s}{a \sigma_s'(R_s)} \left[ \frac{3 \pi_s'(r)}{r^2} + \frac{g'(\sigma_s(r))}{r} \right], \]
we have
\[ q'(K_s^+) = \frac{R_s}{a \sigma_s'(R_s)} \left[ \frac{3 \pi_s'(K_s)}{K_s^2} + \frac{g'(\bar{\sigma})}{R_s} \right] = 3R_s \pi_s'(K_s). \]
From the equation \( \Delta \pi_s = -b \) (for \( r < K_s \)) we see that \( \pi_s'(K_s) = (1/3)bK_s = (a/3)(\sigma - \bar{\sigma})K_s \), and it is clear that \( \bar{u}_1'(K_s^+) = R_s \sigma_s''(K_s^+)/K_s \sigma_s'(R_s) = \bar{\sigma}R_s/K_s \sigma_s'(R_s) \). Combining these relations, we see that the boundary value condition \( q'(K_s^+) = \frac{3 \bar{\sigma} - \sigma}{\sigma} \bar{u}_1'(K_s^+) \) is also satisfied. Hence (3.19) is true. The assertion \( a_1 = 0 \) is an immediate consequence of the relation (3.19) and the fact that \( \pi_0''(R_s) = -g(1). \]

**Lemma 3.8** Let \( \gamma^* = \max\{\gamma_k : k = 2, 3, \ldots\} \). The following assertions hold:

1. If \( \gamma > \gamma^* \) then \( \text{sup}\{\Re \lambda : \lambda \in \sigma(G'(0)) \setminus \{0\}\} < 0 \) and \( \text{Ker} \ G'(0) = \text{span}\{Y_{12}(\omega), Y_{13}(\omega), Y_{14}(\omega)\} \), so that \( \dim \text{Ker} \ G'(0) = 3 \). If instead \( 0 < \gamma < \gamma^* \) then \( \text{sup}\{\Re \lambda : \lambda \in \sigma(G'(0))\} > 0 \).

2. Let \( \gamma > \gamma^* \). Then \( \text{Range} \ G'(0) \) is closed, and \( X = \text{Ker} \ G'(0) \oplus \text{Range} \ G'(0) \).

**Proof.** Assertions in (1) are immediate consequences of Corollary 3.5, the expression (3.15) and Lemma 3.7. To prove the assertion (2), we note that \( G'(0) = \gamma F'(0) + F'_0(0) \). From [13] we know that \( F'(0) \) is a third-order elliptic pseudo-differential operator on the sphere \( S^2 \), and Lemma 2.5 shows that \( F'_0(0) \) is a lower-order perturbation. It follows that standard \( C^k \)-estimates work for \( G'(0) \) and, consequently, the Fredholm alteration principle applies to it, by a similar argument as in the proof of Lemma 2.5. Hence the assertion (2) follows. \( \square \)

**Proof of Theorem 1.2** From (3.15) and Corollary 3.5 (as well as the fact that \( a_0 < 0 \)) we see that if \( \gamma > \gamma^* \) then \( \text{sup}\{\Re \lambda : \lambda \in \sigma(G'(0))\} < 0 \). This fact and Lemmas 3.1, 3.2, 3.3, 3.7, 3.8 show that Theorem 1.1 of [11] (with \( N = 1 \)) applies to the equation (3.3). Hence, by applying Theorem 1.1 of [11] we see that if \( \gamma > \gamma^* \) then the assertions (1)–(3) of Theorem 1.2 hold. If on the other hand \( \gamma < \gamma^* \) then from (3.15) we see that there exists integer \( k \geq 2 \) such that \( a_k(\gamma) > 0 \), so that \( \text{sup}\{\Re \lambda : \lambda \in \sigma(G'(0))\} > 0 \). It follows from linearized instability criterion for parabolic equations in Banach spaces (i.e. Theorem 9.1.3 of [7]) we obtain the last assertion of Theorem 1.2. This proves Theorem 1.2. \( \square \)
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