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SUBCONVEXITY FOR MODULAR FORM $L$-FUNCTIONS IN THE $t$ ASPECT

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Abstract. Modifying a method of Jutila, we prove a $t$-aspect subconvexity estimate for $L$-functions associated to primitive holomorphic cusp forms of arbitrary level that is of comparable strength to Good’s bound for the full modular group, thus improving on a 36-year-old result. A key innovation in our proof is a general form of Voronoi summation that applies to all fractions, even when the level is not squarefree.

1. Introduction

Let $f \in S_k(\Gamma_0(N), \xi)$ be a primitive holomorphic cusp form of weight $k$, level $N$, and nebentypus character $\xi$. (Here and throughout the paper, “primitive” means that $f$ is a normalized Hecke eigenform in the new space.) Let

$$f(z) = \sum_{n \geq 1} \lambda_f(n) n^{(k-1)/2} e^{2\pi i n z} \quad \text{for } \Im(z) > 0$$

be the normalized Fourier expansion of $f$ at the cusp $\infty$. The $L$-function associated to $f$ is defined by

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_{p \text{ prime}} \left( 1 - \frac{\lambda_f(p)}{p^s} + \frac{\xi(p)}{p^{2s}} \right)^{-1}$$

for $\Re(s) > 1$, and by analytic continuation on the rest of $\mathbb{C}$.

The analogue of the Lindelöf hypothesis for $L(s, f)$, in the $t$ aspect, is the conjecture that

$$|L(\frac{1}{2} + it, f)| \ll_{\varepsilon} (1 + |t|)^{\varepsilon}$$

for any $\varepsilon > 0$.

A standard application of the Phragmén–Lindelöf principle shows that

$$|L(\frac{1}{2} + it, f)| \ll_{\varepsilon} (1 + |t|)^{\frac{1}{2} + \varepsilon}.$$ 

This is called the convexity estimate for $L(s, f)$ (in the $t$ aspect), and any improvement on the size of the exponent on the right-hand side of the inequality is referred to as a subconvexity estimate.

For $N = 1$, Good [8] showed that

$$|L(\frac{1}{2} + it, f)| \ll |t|^\frac{3}{4} (\log |t|)^{\frac{1}{4}} \quad \text{for } |t| \geq 2,$$

using the spectral theory of the Laplacian for the modular group to estimate so-called shifted convolution sums. Good’s approach implicitly relies on the fact that the Selberg eigenvalue conjecture holds for level 1 [13, Theorem 11.4]. To generalize it to arbitrary level, one would...
have to consider the possibility of exceptional eigenvalues, which could potentially lead to a weaker estimate. There are situations where this numerical weakening can be circumvented; for instance, Lau, Liu, and Ye [17] showed, for a related problem, that the contribution from exceptional eigenvalues can be controlled and causes no harm to their final result. It is possible that a similar analysis could be carried out in the present context.

In this paper, we instead consider a subsequent, more elementary, approach developed by Jutila. Using only Voronoi summation, Farey fractions, and estimates for exponential integrals, Jutila proved (again for $N = 1$) that

$$|L(\frac{1}{2} + it, f)| \ll \varepsilon |t|^\frac{1}{3} + \varepsilon$$

for $|t| \geq 2$.

We generalize Huxley’s treatment [12] of Jutila’s method and obtain a result for arbitrary level that is essentially as strong as Good’s:

**Theorem 1.1.** Let $f \in S_k(\Gamma_1(N))$ be a primitive cusp form. Then

$$|L(\frac{1}{2} + it, f)| \ll |t|^\frac{1}{2} \log |t|$$

for $|t| \geq 2$,

with an implied constant that is polynomial in $k$ and $N$.

**Remark.** Combining the resolution of the Sato–Tate conjecture [3] with general estimates for sums of multiplicative functions due to Shiu [21], we can marginally improve the inequality in Theorem 1.1 to

$$|L(\frac{1}{2} + it, f)| \ll |t|^{\frac{1}{2}} (\log |t|)^{\frac{5}{8} + \frac{\pi}{27}}.$$  

However, the implied constant need no longer be polynomial in $k$ and $N$; see Lemma 2.2. One could specify the dependence on $k$ and $N$ more precisely with additional work, but it seems unlikely to be competitive with recent hybrid subconvexity bounds for most ranges of the parameters.

Munshi [19] has recently improved Good’s bound for the full modular group, proving that

$$|L(\frac{1}{2} + it, f)| \ll \varepsilon |t|^\frac{1}{2} - \frac{1}{16m} + \varepsilon$$

for any $\varepsilon > 0$ when $N = 1$. For $N > 1$, prior to this paper, it was known that

$$|L(\frac{1}{2} + it, f)| \ll \varepsilon |t|^\frac{1}{2} - \delta + \varepsilon$$

for any $\varepsilon > 0$ with $\delta = \frac{1-2\theta}{8}$ for any primitive $f \in S_k(\Gamma_1(N))$ by the work of Wu [22] and with $\delta = \frac{1-2\theta}{2(3-2\theta)}$ for $k \geq 4$ by the work of Kuan [16]. Here $\theta$ is any exponent toward the Ramanujan-Petersson conjecture. Using $\theta = \frac{7}{64}$, we note that $\frac{1-2\theta}{8} = \frac{25}{256}$ and $\frac{1-2\theta}{2(3-2\theta)} = \frac{25}{178}$.

Our main theorem is an instance of what is commonly referred to as a Weyl-type subconvexity estimate which, in the $t$ aspect, states that $|L(\frac{1}{2} + it)| \ll \varepsilon |t|^\frac{1}{2} + \varepsilon$ for an $L$-function, $L(s)$, of degree $m$. Classically such estimates are known for the Riemann zeta-function and Dirichlet $L$-functions. For degree 2, Good [8] and Meurman [18] proved results of this strength for the $L$-functions associated to holomorphic modular forms and Maass forms on the full modular group. Theorem 1.1 extends Good’s work to arbitrary level (while the analogous extension for Maass forms remains an open problem). For primitive $L$-functions of higher degree, obtaining a subconvexity estimate in the $t$ aspect of Weyl strength remains elusive. Recently Blomer and Milićević [6] have developed a $p$-adic analogue of Jutila’s argument to prove a subconvexity estimate for $L(s, f \times \chi)$ in the character aspect, for a level
Lemma 2.1. Let \( g : (0, \infty) \to \mathbb{R} \) be a smooth function with functional equation \( g(x) + g(1/x) = 1 \) and derivatives decaying faster than any negative power of \( x \) as \( x \to \infty \). Then
\[
L(\frac{1}{2} + it, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^{\frac{1}{2} + it}} g\left(\frac{n}{\sqrt{C}}\right) + \epsilon_f \frac{\Gamma_{\mathbb{C}}(\frac{k}{2} + it)}{\Gamma_{\mathbb{C}}(\frac{k}{2})} \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^{\frac{1}{2} + it}} g\left(\frac{n}{\sqrt{C}}\right) + O_{\epsilon, g}\left(N^\frac{k-3}{2} C^{-\frac{1}{2} + \epsilon}\right)
\]
for any \( \epsilon > 0 \), where \( C = C(f, t) \) is the analytic conductor, defined by
\[
C := \frac{N}{\pi^2} \left|\frac{k+1}{2} + it\right| \left|\frac{k+3}{2} + it\right|.
\]

Proof. This is a special case of a result of Harcos [11, Theorem 2.5]. \( \square \)

Next, we need upper estimates for \( |\lambda_f(n)| \) on average:
Lemma 2.2. Let \( \delta \in \{0, 1 - \frac{8}{3\pi}\} \), \( \alpha \geq 0 \), \( x \geq \frac{3}{2} \), and \( h \geq 1 \). Then

(i) \( \sum_{x < n \leq x + h} |\lambda_f(n)|^2 \ll_{k,N} \max(h, x^{\frac{3}{2}}) \);

(ii) \( \sum_{x < n \leq x + h} |\lambda_f(n)| \ll_{\delta,k,N} \max(h, x^{\frac{3}{2}})(\log x)^{-\delta} \);

(iii) for \( \alpha > 1 \), \( \sum_{n > x} \frac{|\lambda_f(n)|}{n^\alpha} \ll_{\delta,\alpha,k,N} x^{1-\alpha}(\log x)^{-\delta} \);

(iv) for \( \alpha < 1 \), \( \sum_{n \leq x} \frac{|\lambda_f(n)|}{n^{-\alpha}} \ll_{\delta,\alpha,k,N} x^{1-\alpha}(\log x)^{-\delta} \).

Moreover, when \( \delta = 0 \), the implied constants are polynomial in \( k \) and \( N \).

Proof. In his work introducing the Rankin–Selberg method, Rankin [20] proved the estimate

\[
\sum_{n \leq x} |\lambda_f(n)|^2 = A_f x + O_{k,N}(x^{\frac{3}{2}}),
\]

for a certain explicit \( A_f > 0 \). One can see that both \( A_f \) and the implied constant above grow at most polynomially in \( k \) and \( N \), and (i) follows.

As for (ii), when \( \delta = 0 \), Cauchy’s inequality and (i) imply that

\[
\sum_{x < n \leq x + h} |\lambda_f(n)| \ll_{\delta,k,N} \sqrt{h \max(h, x^{\frac{3}{2}})} \leq \max(h, x^{\frac{3}{2}}),
\]

again with a polynomial implied constant. For \( \delta = 1 - \frac{8}{3\pi} \), it follows from the resolution of the Sato–Tate conjecture [3] that

\[
\sum_{p \leq x} \frac{|\lambda_f(p)|}{p} \leq (1 - \delta + o_{k,N}(1)) \log \log x \quad \text{as} \quad x \to \infty.
\]

Inserting this into Shiu’s estimate [21, Theorem 1], for any fixed \( \varepsilon > 0 \) we derive that

\[
\sum_{x < n \leq x + h} |\lambda_f(n)| \ll_{\varepsilon,k,N} h(\log x)^{-\delta} \quad \text{uniformly for} \quad h \geq x^{\varepsilon},
\]

which is clearly sufficient for (ii).

Note that (ii) implies \( \sum_{n \leq x} |\lambda_f(n)| \ll_{\delta,k,N} x(\log x)^{-\delta} \). Using this, a simple exercise in partial summation implies (iii) and (iv). \( \square \)

Finally, we require a form of Voronoi/Wilton summation. As this name is usually understood, such a formula exists for every fraction \( \alpha \in \mathbb{Q} \) only when the level is squarefree. Since we do not want to impose such a restriction on \( f \) in our hypotheses, we prove a generalization, the basic idea of which is to replace additive characters by multiplicative characters at finitely many bad primes. To this end, for any Dirichlet character \( \chi \pmod{q} \), let \( f^\chi \) denote the unique primitive cusp form whose Fourier coefficients \( \lambda_{f^\chi}(n) \) satisfy \( \lambda_{f^\chi}(n) = \lambda_f(n)\chi(n) \) for all \( n \) coprime to \( q \); this is guaranteed to exist by [2, Theorem 3.2] and has level dividing \( Nq^2 \).

\( ^1\)Equality holds in this estimate when \( k > 1 \) and \( f \) does not have CM. The remaining cases must be handled separately, but are easier to prove and lead to slightly improved estimates. Specifically, \( \frac{8}{3\pi} \) can be replaced by \( \frac{2}{\pi} \) for CM forms, at most \( \frac{2}{3} \) for dihedral forms, \( \frac{5}{6} \) for tetrahedral forms, \( \frac{5 + 3\sqrt{2}}{12} \) for octahedral forms, and \( \frac{11 + 6\sqrt{5}}{30} \) for icosahedral forms.
Lemma 2.3. Let \( \alpha = a/q \in \mathbb{Q} \), and define \( q^* = \prod_{p \mid q} p^{1+\text{ord}_p q} \). Then

\[
\sum_{n=1}^{\infty} \frac{\lambda_f(n)e(\alpha n)}{n^s} = \sum_{\chi \pmod{q}} \sum_{m \mid \text{gcd}(N, q^2)} \frac{C(f, \alpha, m, \chi)}{m^s} L(s, f^\chi),
\]

for some numbers \( C(f, \alpha, m, \chi) \in \mathbb{C} \) satisfying \( C(f, \alpha, m, \chi) \lesssim_q 1 \).

Proof. Let us first assume that \( q = p^e \) is a prime power and \( r \) is a positive integer coprime to \( p \). Then the additive twist of \( r^{-s}L(s, f) \) by \( \alpha = a/q \) equals

\[
\sum_{n=1}^{\infty} \frac{\lambda_f(n)e(\alpha n)}{n^s} = \sum_{k=0}^{e-1} \frac{\lambda_f(p^k)}{(rp^k)^s} \sum_{(n,p)=1} \frac{\lambda_f(n)e(\alpha n)}{n^s} + \sum_{k=e}^{\infty} \frac{\lambda_f(p^k)}{(rp^k)^s} \sum_{(n,p)=1} \frac{\lambda_f(n)}{n^s}
\]

\[
= \sum_{k=0}^{e-1} \frac{\lambda_f(p^k)}{(rp^k)^s} \sum_{\chi \pmod{p^{e-k}}} \frac{\tau(\chi)\chi(\alpha)\varphi(p^{e-k})}{\varphi(p^{e-k})} \sum_{(n,p)=1} \frac{\lambda_f(n)\chi(n)}{n^s} + \sum_{k=e}^{\infty} \frac{\lambda_f(p^k)}{(rp^k)^s} \sum_{(n,p)=1} \frac{\lambda_f(n)}{n^s}
\]

\[
= \sum_{k=0}^{e-1} \frac{\lambda_f(p^k)}{(rp^k)^s} \sum_{\chi \pmod{p^{e-k}}} \frac{\tau(\chi)\chi(\alpha)\varphi(p^{e-k})}{\varphi(p^{e-k})} E_{f^\chi,p}(p^{-s})L(s, f^\chi)
\]

\[
+ \left[r^{-s} - E_{f^\chi,p}(p^{-s}) \sum_{k=0}^{e-1} \frac{\lambda_f(p^k)}{(rp^k)^s}\right] L(s, f),
\]

where \( E_{f,p} \) and \( E_{f^\chi,p} \) denote the Euler factor polynomials of \( f \) and \( f^\chi \) at \( p \), respectively. Note that this is a linear combination of terms of the form \( (rp^j)^{-s}L(s, f^\chi) \) for characters \( \chi \pmod{p^e} \).

In the general case, by partial fractions, we may express \( \alpha \) as a sum of fractions of the form \( a/p^e \), and applying the prime power case inductively yields a decomposition of the required type. The estimate \( C(f, \alpha, m, \chi) \lesssim_q 1 \) follows from the fact that the coefficients in the above polynomials depend only on local data of \( f^\chi \), together with Deligne’s bound. It remains only to be seen that the values of \( m \) that occur must divide \( \text{lcm}(N, q^2)^s, q^* \), which reduces to the following two assertions in the prime power case:

(2.2) \[ k + \deg E_{f^\chi,p} \leq e + 1 \]

and

(2.3) \[ k + \deg E_{f^\chi,p} + \text{ord}_p \text{cond}(f^\chi) \leq e + \max(e, \text{ord}_p N). \]

Since \( k \leq e - 1 \) and \( \deg E_{f^\chi,p} \leq 2 \), the assertion in (2.2) is clear. As for (2.3), since \( \chi \) is a character \( \pmod{p^{e-k}} \), it follows from [2, Theorem 3.1] that

\[ \text{ord}_p \text{cond}(f^\chi) \leq e - k + \max(e - k, \text{ord}_p N), \]

so (2.3) holds when \( \deg E_{f^\chi,p} = 0 \). In particular, this is the case when the local constituent \( \pi_p \) of the automorphic representation associated to \( f \) is supercuspidal. If \( \pi_p \) is special then we might have \( \deg E_{f^\chi,p} = 1 \), but this happens only when \( \text{ord}_p \text{cond}(f^\chi) = 1 \), in which case the left-hand side of (2.3) is at most \( k + 2 \leq e + 1 \leq 2e \). Finally, suppose that \( \pi_p \) is a principal series, say \( \pi_p = \pi(\chi_1, \chi_2) \). If \( \deg E_{f^\chi,p} = 2 \) then \( \text{ord}_p \text{cond}(f^\chi) = 0 \), so again we get
the upper bound $k + 2 \leq 2e$. If $\deg E_{f,x,p} = 1$ then we may assume that $\chi \chi_1$ is unramified, so that
\[
\ord_p \cond(f^\chi) = \ord_p \cond(\chi \chi_2) \leq \ord_p \cond(\chi_1) + \ord_p \cond(\chi_2) = \ord_p N,
\]
and the left-hand side of (2.3) is at most $e + \ord_p N$. \hfill \square

**Lemma 2.4.** Let $a/q \in \mathbb{Q}$, and let $F : (0, \infty) \to \mathbb{C}$ be a $C^2$ function of compact support. Define
\[
N_1 = (N,q), \quad N_2 = \frac{N}{N_1}, \quad q_2 = (N_2^\infty, q), \quad q_1 = \frac{q}{q_2},
\]
and write
\[
\frac{a}{q} = \frac{a_1}{q_1} + \frac{a_2}{q_2},
\]
with the fractions on the right-hand side in lowest terms. Then
\[
\sum_{n=1}^{\infty} \lambda_f(n) e\left(\frac{an}{q}\right) F(n) = \sum_{\chi \mod N_1} \sum_{r|N_2q_2^2} c(f,a/q,r,\chi) \sum_{n=1}^{\infty} \lambda_{f^x}(n) e\left(-\frac{a_1 r n}{q_1}\right) 
\]
\[
\cdot 2\pi i^k \int_0^\infty F(q_1 x) J_{k-1}\left(4\pi \sqrt{\frac{n x}{q_1 r}}\right) dx,
\]
where $\overline{a_1 r F}$ denotes a multiplicative inverse of $a_1 r (\mod q_1)$, and $c(f,a/q,r,\chi) \ll q_2 1$.

**Remark.** Note that $q_2 | N_1$, so that both $r$ and the coefficients $c(f,a/q,r,\chi)$ are $O_N(1)$, independent of $q$.

**Proof.** Since $q_2 | N_1$, we have $\lcm(Nq_2,q_2^2) = Nq_2$, so applying Lemma 2.3 with $\alpha = a_2/q_2$, we get
\[
\lambda_f(n) e\left(\frac{a_2 n}{q_2}\right) = \sum_{\chi \mod q_2} \sum_{m | \frac{Nq_2}{\cond(\chi) n}} C(f,a_2/q_2,m,\chi) \lambda_{f^x}\left(\frac{n}{m}\right),
\]
for some numbers $C(f,a_2/q_2,m,\chi) \in \mathbb{C}$.

Next we compute the sum
\[
\sum_{n \geq 1 \atop m \in n} \lambda_{f^x}\left(\frac{n}{m}\right) e\left(\frac{a_1 n}{q_1}\right) F(n) = \sum_{n=1}^{\infty} \lambda_{f^x}(n) e\left(\frac{a_1 mn}{q_1}\right) F(mn)
\]
by applying the Voronoi summation formula [15, Theorem A.4]. Let us first suppose that $F$ is smooth, which is a hypothesis of loc. cit. Put $g = f^\chi$ and $D = \cond(g)$, so that $g \in S_k(\Gamma_0(D),\psi)$, where $\psi = \xi \chi^2$. Set $D_1 = (D,q_1) = N_1/q_2$, $D_2 = D/D_1$, and split the character $\psi$ as a product $\psi_{D_1} \psi_{D_2}$ of characters modulo $D_1$ and $D_2$, respectively. Then [15, Theorem A.4] yields
\[
\sum_{n=1}^{\infty} \lambda_g(n) e\left(\frac{a_1 mn}{q_1}\right) F(mn)
\]
\[
= \psi_{D_1}(a_1 m) \psi_{D_2}(-q_1) \eta_g(D_2) \sum_{n=1}^{\infty} \lambda_g(\overline{D_2} n) e\left(-\frac{a_1 m D_2}{q_1}\right) \int_0^\infty F(m q_1 x) J_{k-1}\left(4\pi \sqrt{\frac{n x}{q_1 D_2}}\right) dx,
\]

where \( \eta_g(D_2) \) is a constant of modulus 1 and \( \overline{a_1 m D_2} \) is an inverse of \( a_1 m D_2 \) (mod \( q_1 \)). Note that \( g^{\overline{D_2}} = \overline{g}^{D_1} \), where \( \overline{g} \in S_k(\Gamma_0(D), \overline{\psi}) \) is the dual of \( g \). Since \( D_1 \) is coprime to the modulus of \( \chi \), we further have \( \psi_{D_1} = \xi_{D_1} = \xi_{N_1/q_2} \), so that \( \overline{g}^{D_1} = \overline{f}_{\mathbf{R}N_1/q_2} \). Since \( m \) is restricted to the divisors of \( Nq_2/D_1 \), we see that \( mD_2 \) divides \( Nq_2/D_1 = 2q_2^2 \). Writing \( r = mD_2 \) and making the change of variables \( x \mapsto x/m \), the last line becomes

\[
\frac{\psi_{D_1}(a_1 m) \psi_{D_2}(-q_1) \eta_g(D_2)}{m \sqrt{D_2}} \sum_{n=1}^{\infty} \lambda_f \overline{\chi_{N_1/q_2}} (n) e \left( -\frac{a_1 r n}{q_1} \right) \int_0^\infty F(q_1 x) J_{k-1} \left( 4\pi \sqrt{\frac{nx}{q_1 r}} \right) dx.
\]

From the estimate provided by Lemma 2.3, we have

\[
\frac{\psi_{D_1}(a_1 m) \psi_{D_2}(-q_1) \eta_g(D_2)}{m \sqrt{D_2}} C(f, a_2/q_2, m, \chi) \ll q_2.
\]

Making the change of variables \( \chi \mapsto \xi_{N_1/q_2} \) yields (2.4).

It remains only to see that (2.4) is valid if \( F \) is merely \( C^2 \) and not necessarily smooth. Making the substitution \( x = q_1 r \left( \frac{u}{4\pi} \right)^2 \), we have

\[
\int_0^\infty F(q_1 x) J_{k-1} \left( 4\pi \sqrt{\frac{nx}{q_1 r}} \right) dx = \frac{q_1 r}{8\pi^2} \int_0^\infty u^{-k} F \left( \frac{q_1^2 ru^2}{16\pi^2} \right) u^k J_{k-1}(\sqrt{nu}) du.
\]

Applying integration by parts twice and using the estimates

\[
\frac{d}{dx} \{ x^k J_k(x) \} = x^k J_{k-1}(x), \quad \frac{d}{dx} \{ x^{k+1} J_{k+1}(x) \} = x^{k+1} J_k(x), \quad \text{and} \quad J_{k+1}(x) \ll_k \frac{1}{\sqrt{x}},
\]

we see that this integral is \( O(n^{-\frac{k}{2}}) \). Thus, the sum over \( n \) on the right-hand side of (2.4) is absolutely convergent, and the lemma follows by a standard argument using smooth approximations of \( F \).

\[\square\]

3. Proof of Theorem 1.1

3.1. Initial reduction. Let \( f \) be as in the statement of Theorem 1.1, and let \( t \in \mathbb{R} \). By replacing \( f \) with \( \overline{f} \) if necessary, we may assume without loss of generality that \( t \geq 0 \). We may further assume that

\[
(3.1) \quad t \geq \max \{ k^{\frac{3}{2}} \log k, N^{\frac{3}{2}}, t_0 \}
\]

for a large constant \( t_0 \), as otherwise the convexity bound implies Theorem 1.1.

Let \( C \) denote the analytic conductor defined in (2.1), and fix, for the remainder of the paper, a choice of \( \delta \in \{ 0, 1 - \frac{\delta}{3\pi} \} \). With \( \delta = 0 \), all implied constants depend at most polynomially on \( k \) and \( N \). In this section, we prove that

\[
(3.2) \quad |L(\frac{1}{2} + it, f)| \ll \sum_M \frac{1}{\sqrt{M}} \max_{M_1 \in [M, 2M]} \left| \sum_{M_1 \leq n \leq 2M} \lambda_f(n) n^{-it} \right| + O_{k,N} \left( \sqrt{M_0} (\log M_0)^{-\delta} \right)
\]

for any integer \( M_0 \in [2, \sqrt{C}] \), where \( M \) runs through numbers of the form \( 2^K M_0 \) for integers \( K \in [0, \log_2 \frac{\sqrt{C}}{M_0} + 1] \). Therefore, in order to prove Theorem 1.1, it suffices to estimate exponential sums of the form

\[
(3.3) \quad \sum_{M_1 \leq n \leq M_2} \lambda_f(n) n^{-it}
\]
where $M_1 \leq M_2 \leq 2M_1$.

Our starting point for the proof of (3.2) is the approximate functional equation for $L(\frac{1}{2} + it, f)$ in the form of Lemma 2.1. We remark that, without loss of generality, we may suppose that the test function $g$ appearing there is decreasing and supported on the interval $[0, 2]$. For example, the function

$$g(x) = \begin{cases} 
1, & \text{if } x < \frac{1}{2}, \\
\alpha \int_{\log_2 x}^{1} e^{-\frac{1}{2-x^2}} \, dt, & \text{if } \frac{1}{2} \leq x \leq 2, \\
0, & \text{if } x > 2,
\end{cases}$$

where $\alpha = e^{\frac{1}{2}}/(K_1(\frac{1}{2}) - K_0(\frac{1}{2})) = 2.25228\ldots$ is chosen so that $g(\frac{1}{2}) = 1$, has these properties ($K_n(z)$ denotes the usual $K$-Bessel function). Since $|\epsilon_f \Gamma C(\frac{k}{2} - it)/\Gamma C(\frac{k}{2} + it)| = 1$, by (3.1) we have

$$|L(\frac{1}{2} + it, f)| \leq 2 \left| \sum_{n=1}^{\infty} \lambda_f(n) g\left(\frac{n}{\sqrt{C}}\right) \right| + O(1).$$

By the triangle inequality, since $0 \leq g(x) \leq 1$, we have

$$\sum_{n=1}^{\infty} \lambda_f(n) g\left(\frac{n}{\sqrt{C}}\right) \leq \sum_{n \leq M_0} |\lambda_f(n)| \sum_{n=M_0}^{\infty} |\lambda_f(n) g\left(\frac{n}{\sqrt{C}}\right)|,$$

with $M_0 \leq \sqrt{C}$ and $M = 2^k M_0$ as above. Note that only finitely many values of $M$ are relevant, since $\text{supp}(g) \subseteq [0, 2]$. We will choose $M_0$ (depending on $t$) at the end of the proof, but to fix ideas, we note that

$$\left(\frac{t}{\log t}\right) \ll M_0 \ll t^\frac{2}{3}.$$  

Applying Lemma 2.2(iv), we have

$$\sum_{n \leq M_0} |\lambda_f(n)| \ll_{k,N} M_0 (\log M_0)^{-\delta}.$$

We now simplify the second sum on the right-hand side of (3.4) using [12, Lemma 5.1.1]. Defining $G(x) = \frac{1}{\sqrt{x}} g\left(\frac{x}{\sqrt{C}}\right)$, we may assume that $G$ is decreasing, and hence

$$\sum_{M < n \leq 2M} |\lambda_f(n)| g\left(\frac{n}{\sqrt{C}}\right) \ll \frac{1}{\sqrt{M}} \max_{M_1 \in [M, 2M]} \left| \sum_{M_1 \leq n \leq 2M} \lambda_f(n) n^{-it} \right|.$$

Combining estimates yields (3.2).

3.2. Farey fractions and stationary phase. We now turn our attention to estimating the sum in (3.3). Let $M$ be a size parameter and suppose that $M_0 \leq M \ll M_1 \leq M_2 \leq 2M_1 \ll M$. Define

$$R = \sqrt{\frac{M}{M_0}} \quad \text{and} \quad H := \left\lfloor \frac{M^2}{R^2 t} \right\rfloor,$$

and let

$$\mathcal{F}(R) = \left\{ \frac{u}{v} : u, v \in \mathbb{Z}, (u, v) = 1, 0 < v \leq R \right\}.$$
denote the set of extended Farey fractions with denominator less than or equal to $R$. Consider the interval $$\left[ -\frac{t}{2\pi(M_1+2H)}, -\frac{t}{2\pi(M_2-2H)} \right]$$ and denote the elements of $\mathcal{F}(R)$ in this interval by $\alpha_j$, $j = 1, \ldots, J$, in increasing order. (If $M_2 - M_1 < 4H$ or if this interval contains no elements of $\mathcal{F}(R)$ then (3.3) is trivially bounded by the error term in (3.9), below. Hence we may assume that this is not the case.)

We make the labeling $\alpha_j = -\frac{u_j}{v_j}$, where $u_j, v_j \in \mathbb{Z}_{>0}$, $(u_j, v_j) = 1$, and $v_j \leq R$. For consecutive Farey fractions $\alpha_j = -\frac{u_j}{v_j}$ and $\alpha_{j+1} = -\frac{u_{j+1}}{v_{j+1}}$ the mediant, denoted $\rho_j$, is given by $\rho_j = -\frac{u_j + u_{j+1}}{v_j + v_{j+1}}$. Note that (3.6) $|\rho_j - \alpha_j| = \frac{1}{v_j(v_j + v_{j+1})} \approx \frac{1}{v_j R}$

and similarly $|\rho_j - \alpha_{j+1}| \approx \frac{1}{v_j R}$. Define the function $h(y) = -\frac{t}{2\pi y}$ and integers $N_0 = M_1 + 2H$, $N_j = M_2 - 2H$, and $N_j = \lfloor h(\rho_j) + \frac{1}{2} \rfloor$ for $j = 1, \ldots, J - 1$. Evidently,

$$M_1 < N_0 < N_1 < \cdots < N_{J-1} < N_J < M_2.$$ 

Using the above and assuming that $t_0$ is sufficiently large, we have

$$N_j - h(\alpha_j) = h(\rho_j) - h(\alpha_j) + O(1) = \frac{t}{2\pi} \frac{\rho_j - \alpha_j}{\alpha_j \rho_j} + O(1)$$

$$\approx \left( \frac{t}{M} \right)^2 \frac{2\pi}{v_j R} \approx \frac{HR}{v_j R}.$$ 

By a similar argument we see that $h(\alpha_j) - N_j - 1 \approx \frac{HR}{v_j}$, and thus

(3.7) $$N_j - N_{j-1} \approx \frac{HR}{v_j}.$$ 

Next let

(3.8) $$\omega_j(x) = \omega(x - N_{j-1}) - \omega(x - N_j),$$

where

$$\omega(x) = \begin{cases} 
  1, & \text{for } x \geq H, \\
  \frac{1}{2} \left( 1 + \sin^s \left( \frac{\pi x}{2H} \right) \right), & \text{for } |x| \leq H, \\
  0, & \text{for } x \leq -H,
\end{cases}$$

for an integer $s \geq 2$. These functions provide a $C^s$ partition of unity on the interval $[M_1 + 2H, M_2 - 2H]$. In particular,

$$\sum_{j=1}^{J} \omega_j(n) = \begin{cases} 
  0, & \text{for } x \leq M_1, \\
  1, & \text{for } M_1 + 2H \leq x \leq M_2 - 2H, \\
  0, & \text{for } x \geq M_2.
\end{cases}$$
From this we observe that
\[
\sum_{M_1 \leq n \leq M_2} \frac{\lambda_f(n)}{n^H} - \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \lambda_f(n)n^{-it} \omega_j(n) \leq \sum_{m=M_1}^{M_1 + 2H} |\lambda_f(m)| + \sum_{m=M_2 - 2H}^{M_2} |\lambda_f(m)|.
\]
(3.9)
\[\ll k,N \max(H, M^{\frac{3}{2}})(\log M)^{-\delta},\]
by Lemma 2.2(ii). Hence
\[
\sum_{M_1 \leq n \leq M_2} \frac{\lambda_f(n)}{n^H} = \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \lambda_f(n)n^{-it} \omega_j(n) + O_{k,N}(\max(H, M^{\frac{3}{2}})(\log M)^{-\delta})
\]
(3.10)
\[= \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \lambda_f(n)e(\alpha_j n)F_j(n) + O_{k,N}(\max(H, M^{\frac{3}{2}})(\log M)^{-\delta}),\]
where \(F_j(n) = n^{-it}e(-\alpha_j n)\omega_j(n)\).

We would now want to apply Lemma 2.4 to the sum involving \(F_j(n)\). If the level \(N\) is not squarefree, then we do not have a Voronoi formula for every Farey fraction \(\alpha_j\). To circumvent this issue, we decompose each fraction into a ‘good part’ and a ‘bad part’ where the bad parts range over a finite set, the additive twists involving the good part of the fraction can be handled by Voronoi summation, and the additive characters involving the bad part can be handled by decomposing into multiplicative characters. To that end, define
\[N^b = N \prod_{p|N} p^{-1} \quad \text{and} \quad \mathcal{B}(N^b) = \left\{ \frac{b}{N^b} : 0 \leq b < N^b \right\}\]
and write
\[
\alpha_j = -\frac{u_j}{v_j} = -\frac{a_j}{q_j} + \beta_j = -\frac{a_j}{q_j} + \frac{c_j}{d_j},
\]
where \(a_j, q_j, c_j, d_j \in \mathbb{Z}_{\geq 0}, (a_j, q_j) = (c_j, d_j) = (q_j, d_j) = 1, \beta_j \in \mathcal{B}(N^b),\) and for every prime \(p | q_j\) we have \(\operatorname{ord}_p(q_j) \geq \operatorname{ord}_p(N)\). Such a decomposition always exists and is uniquely determined; concretely,
\[d_j = \prod_{p|d_j} p^{\operatorname{ord}_p(v_j)}, \quad q_j = \frac{v_j}{d_j},\]
and \(c_j\) is the unique integer in \([0, d_j)\) satisfying \(q_j c_j \equiv -u_j \pmod{d_j}\). Since \(v_j = d_j q_j\), this congruence is equivalent to
\[d_j u_j \equiv -c_j v_j \pmod{d_j^2}\]

Next we apply Voronoi summation in the form of Lemma 2.4 to see that
\[
\sum_{n=1}^{\infty} \lambda_f(n)e(\alpha_j n)F_j(n) = \sum_{\beta \in \mathcal{B}(N^b)} \sum_{p|N^b} \sum_{\chi \pmod{N}} c(f, r, \chi; j) \sum_{\ell=1}^{\infty} \lambda_{\chi \ell}(\ell) e\left(\frac{\ell r \ell^j}{q_j}\right) 2\pi i^k \int_{0}^{\infty} F_j(q_j x) J_{k-1} \left(4\pi \sqrt{\frac{x}{q_j}}\right) dx
\]
(3.11)
for some complex numbers \(c(f, r, \chi; j)\) satisfying \(c(f, r, \chi; j) \ll_{N^s} 1\). Applying stationary phase to the integral on the right-hand side of this equation we derive the following proposition, deferring the proof until Section 4.

**Proposition 3.1.** Given \(\beta = \frac{t}{\ell} \in B(N^s)\) and \(r \mid NN^s\), let

\[
J(\beta, r) = \{ j \in \{1, \ldots, J\} : \beta_j = \beta, (q_j, r) = 1 \}.
\]

If \(s \geq 6\), then

\[
\sum_{j \in J(\beta, r)} c(f, r, \chi; j) \sum_{\ell = 1}^{\infty} \lambda_{f\chi}(\ell) e\left(\frac{\ell x_j}{q_j}\right) 2\pi i^k \int_{0}^{\infty} F_j(q_j, x) J_{k-1} \left(4\pi \sqrt{\frac{\ell x_j}{q_j}}\right) dx
\]

\[
= \sum_{j \in J(\beta, r)} c(f, r, \chi; j) \sum_{\ell \leq K_1 r d^{-2}} \lambda_{f\chi}(\ell) e\left(\frac{\ell x_j}{q_j}\right) \omega_j(x_j^\pm(\ell)) h_j^\pm(\ell) e\left(g_j^\pm(\ell)\right)
\]

\[
+ O_{K, N, s} \left(\sqrt{M} \left(\frac{M}{R^2}\right)^{\frac{1}{2(\ell-1)}} + \frac{M^{\frac{1}{2}} R^2}{H^3}\right)(\log M_0)^{-\delta},
\]

where \(x_j^\pm(\ell)\) are stationary points defined by

\[
x_j^\pm(\ell) = \left(\frac{d}{2 u_j}\right)^2 \left(\sqrt{\ell + \frac{2 t u_j q_j}{\pi d} \pm \sqrt{\ell}}\right)^2,
\]

\[
g_j^\pm(\ell) = -\frac{t}{2\pi} \log x_j^\pm(\ell) + \frac{u_j}{u_j} x_j^\pm(\ell) \mp \frac{2}{q_j} \sqrt{\ell x_j^\pm(\ell)} + \frac{1}{8} \mp \frac{1}{8},
\]

\[
h_j^\pm(\ell) = \left(\frac{q_j t \sqrt{\ell}}{\pi x_j^\pm(\ell)^{\frac{3}{2}} \pm \ell x_j^\pm(\ell)}\right)^{-\frac{1}{2}},
\]

and \(K_1 \approx M/R^2 = M_0\).

Combining (3.10), (3.11), and Proposition 3.1, we have

\[
\sum_{M_1 \leq n \leq M_2} \frac{\lambda_f(n)}{n^{it}} =
\sum_{\beta \in B(N^s) \cap [NN^s]} \sum_{\chi \mod N} \sum_{j \in J(\beta, r)} c(f, r, \chi; j) \sum_{\ell \leq K_1 r d^{-2}} \lambda_{f\chi}(\ell) e\left(\frac{\ell x_j}{q_j}\right) \omega_j(x_j^\pm(\ell)) h_j^\pm(\ell) e\left(g_j^\pm(\ell)\right)
\]

\[
+ O_{K, N, s} \left(\sqrt{M} \left(\frac{M}{R^2}\right)^{\frac{1}{2(\ell-1)}} + \frac{M^{\frac{1}{2}} R^2}{H^3} + \max(H, M^{\frac{1}{3}})\right)(\log M_0)^{-\delta}.
\]

Since \(R = \sqrt{M/M_0} \geq 1\), \(M \ll \sqrt{N} t\), \(M_0 \ll t^\frac{3}{2}\), and \(|t| \geq N^\frac{3}{4}\), we see that

\[
\max(H, M^{\frac{1}{3}}) \ll \frac{M^{\frac{1}{2}} R^2}{H^3} \quad \text{and} \quad \frac{M}{R} \ll \frac{R^8 t^3}{M^2}.
\]

Therefore, assuming \(s \geq 2\) and using the definition of \(H\), we find that the error term in (3.16) can be replaced by \(O_{K, N}(R^8 t^3 M^{-\frac{1}{2}})(\log M_0)^{-\delta}\).
The next step is to split the sum in (3.16) so that the integers \( \ell \) lie in dyadic intervals and the sum over \( j \) is reorganized so that the \( v_j \) lie in dyadic intervals. Thus we have

\[
\sum_{M_1 \leq n \leq M_2} \frac{\lambda_f(n)}{n^{it}} = \sum_{\beta \in B(N^\beta)} \sum_{r | N N^\beta} \chi(n) \pm \sum_{L, U, V} (\mp 1)^k \sum_{\ell = L_1}^{L_2} \lambda_f(x(\ell)) \sum_{\ell \in J(\beta, r)} c(f, r, \chi; j) e(\frac{\ell}{q_j}) \omega_j(x_j^\pm(\ell)) \omega_j(g_j^\pm(\ell)) \]

\[
\cdot \sum_{U \leq u_j \leq U_2} \sum_{V \leq v_j \leq V_2} \sum_{\substack{u_j \equiv -cv \pmod{d^2}, \\ d \text{ odd}}} \sum_{u_j \equiv -cv \pmod{d^2}} \sum_{V \leq v_j \leq V_2} \sum_{u_j \equiv -cv \pmod{d^2}} + O_{k, N}(R^8 t^2 M^{-2}(\log M_0)^{-\delta})
\]

where \( L_1, L_2, U_1, U_2, V_1, \) and \( V_2 \) are positive integers satisfying \( L \ll L_1 \leq L_2 \leq L, U \ll U_1 \leq U_2 \leq U, \) and \( V \ll V_1 \leq V_2 \leq V \) and \( L, U, V \) run through powers of 2 satisfying

\[
L \ll \frac{r M}{(dR)^2}, \quad V \ll R, \quad \text{and} \quad U \asymp \frac{tV}{M}.
\]

Observe that the third condition follows from the fact that if \( v_j \asymp V \), then \( u_j \asymp \frac{V}{M} \) since \( \frac{u_j}{v_j} \asymp \frac{t}{M} \). The next step is to apply a large sieve inequality for Farey fractions to the inner double sum in (3.17). In fact, we found a more general sum with Proposition 3.2, below.

### 3.3. The large sieve and conclusion of the proof

To estimate the main term on the right-hand side of (3.16), we employ the following large sieve inequality, deferring the proof until Section 5.

**Proposition 3.2.** Let notation be as above and fix \( \beta = \frac{\xi}{d} \in B(N^\beta) \) and \( r | N N^\beta \). Let \( L_1, L_2, U_1, U_2, V_1, \) and \( V_2 \) be positive integers satisfying \( L \ll L_1 \leq L_2 \leq L, U \ll U_1 \leq U_2 \leq U, \) and \( V \ll V_1 \leq V_2 \leq V \) where \( L, U, \) and \( V \) are size parameters satisfying

\[
L \ll \frac{r M}{(dR)^2}, \quad V \ll R, \quad \text{and} \quad U \asymp \frac{tV}{M}.
\]

Define

\[
\mathcal{R} = \left\{ (u, v) \in \mathbb{Z}^2 : U_1 \leq u \leq U_2, V_1 \leq v \leq V_2, (u, v) = 1, du \equiv -cv \pmod{d^2} \right\}
\]

and

\[
\eta = \sqrt{\frac{d^2 t}{r L U V}}, \quad X_0 = \sqrt{L \max(\eta, 1)}.
\]

Then, given any complex numbers \( \nu(j), \lambda(\ell) \) for \( j \in J(\beta, r) \) and \( \ell \in \mathbb{Z} \cap [L_1, L_2] \), we have

\[
\left| \sum_{\ell = L_1}^{L_2} \lambda(\ell) \sum_{j \in J(\beta, r)} \nu(j) h_j^\pm(\ell/r) \omega_j(x_j^\pm(\ell/r)) e(g_j^\pm(\ell/r)) \right|^2 \ll \max_{j \in J(\beta, r)} |\nu(j)|^2 \sum_{\ell = L_1}^{L_2} |\lambda(\ell)|^2 \cdot \frac{\eta r V}{U} \left\{ X_0 (\# \mathcal{R})^2 + \int_{X_0}^L B(\Delta_1(X), \Delta_2(X)) \, dX \right\},
\]

in (3.18).
where the implied constant is absolute, $\Delta_1$ and $\Delta_2$ are functions satisfying
\begin{equation}
\Delta_1(X) \ll \frac{L}{X^2} \quad \text{and} \quad \Delta_2(X) \ll \frac{L}{\eta X^2},
\end{equation}
and $B(\Delta_1, \Delta_2)$ is the number of pairs $(i, j) \in J(\beta, r)^2$ such that $(u_i, v_i), (u_j, v_j) \in \mathcal{R}$,
\begin{equation}
\left\| \frac{a_i r}{q_i} - \frac{a_j r}{q_j} \right\| \leq \Delta_1, \quad \text{and} \quad |u_iv_i - u_jv_j| \leq UV\Delta_2.
\end{equation}

For fixed $\beta = \frac{c}{r}$ and $r$, define $\varphi(y) = \sqrt{y - \beta r}$ and recall that $\frac{u_i}{v_i} = \frac{a_i}{q_i} - \beta$ for $j \in J(\beta, r)$. This, together with the condition $|u_iv_i - u_jv_j| \leq UV\Delta_2$, implies that
\begin{equation}
\frac{d}{\sqrt{r}} \left| q_i \varphi(ra_i/q_i) - q_j \varphi(ra_j/q_j) \right| = \left| \sqrt{u_iv_i} - \sqrt{u_jv_j} \right| \ll \sqrt{UV}\Delta_2.
\end{equation}

We want to count pairs of fractions $(\frac{ra_i}{q_i}, \frac{ra_j}{q_j})$ in the interval $\left[ \frac{rU_1}{V_1} + \beta r, \frac{rU_2}{V_1} + \beta r \right]$ satisfying the inequalities in (3.20). On this interval, we have $\varphi(y) \asymp \Phi := \sqrt{\frac{rU}{V}}$, and hence
\begin{equation}
\left| q_i \varphi(ra_i/q_i) - q_j \varphi(ra_j/q_j) \right| \ll \frac{\Phi V}{d} \Delta_2.
\end{equation}

Note that under the conditions of Proposition 3.2, we have
\begin{equation}
\frac{V_1}{d} \leq q_i \leq \frac{V_2}{d} \quad \text{and} \quad \frac{r}{d}(U_1 + \beta V_1) \leq ra_j \leq \frac{r}{d}(U_2 + \beta V_2).
\end{equation}

Assuming that $U_2 \leq 2U_1$ and $V_2 \leq 2V_1$, we thus see that $ra_j$ and $q_j$ lie in dyadic intervals. This allows us to apply the following estimate of Graham and Kolesnik [10] with parameters $A = \frac{r}{d}(U_1 + \beta V_1)$ and $C = \frac{V_1}{d}$.

**Lemma 3.1.** Suppose $A$ and $C$ are positive integers, and that $\Delta_1$ and $\Delta_2$ are positive real numbers not exceeding 1. Suppose that $\varphi(x)$ is a real positive continuously differentiable function defined on a subinterval $I$ of $[A/(2C), 2A/C]$. Suppose that there is a constant $C_0$ and a parameter $\Phi$ such that
\begin{equation}
\frac{\Phi}{C_0} \leq \varphi(x) \leq C_0\Phi, \quad \frac{\Phi}{C_0} \leq |x\varphi'(x)| \leq C_0\Phi, \quad \text{and} \quad \frac{\Phi}{C_0} \leq |\varphi(x) - x\varphi'(x)| \leq C_0\Phi
\end{equation}
whenever $x$ is an element of $I$. Let $B$ be the number of solutions of the inequalities
\begin{equation}
\left\| \frac{r}{q} - \frac{r_1}{q_1} \right\| \leq \Delta_1 \quad \text{and} \quad \left| q\varphi(r/q) - q_1\varphi(r_1/q_1) \right| \leq C\Phi\Delta_2
\end{equation}
when $(r, q) = 1, (r_1, q_1) = 1$, $A < r, r_1 \leq 2A$, and $C < q, q_1 \leq 2C$ where $r\overline{r} \equiv 1 \pmod{q}$ and $r_1\overline{r_1} \equiv 1 \pmod{q_1}$. Then
\begin{equation}
B \ll \Delta_1\Delta_2 A^2C^2 + \Delta_1^2 A^2C^2 + AC + \Delta_2 A^2 + \Delta_2 C^2
\end{equation}
where the implied constant depends only on $C_0$.

**Proof.** This is [10, Lemma 7.18] and is a variation of a counting problem first considered by Bombieri and Iwaniec [5] in connection to subconvexity estimates for the Riemann zeta-function. \hfill \square
Since $V \ll R$, $U \ll \frac{tR}{M}$, and $L \ll rM/(dR)^2$, we have $LUV \ll rt/d^2$. Hence

(3.21) \[ \eta \approx_N \sqrt{\frac{t}{LUV}} \approx_N 1 \quad \text{and} \quad X_0 \approx_N \sqrt{\eta L} \approx_N \left( \frac{tL}{UV} \right)^{\frac{1}{4}}. \]

Therefore, in the notation of Proposition 3.2, we have

\[
X_0(\#\mathcal{R})^2 + \int_{X_0}^L B(\Delta_1(X), \Delta_2(X)) \, dX \\
\ll X_0(\#\mathcal{R})^2 + \int_{X_0}^L \left\{ \frac{A^2C^2}{X^4} + \frac{L^2}{X^4}(1+\eta^{-1}) + AC + \frac{L}{\eta X^2}(A^2 + C^2) \right\} \, dX \\
\ll X_0(\#\mathcal{R})^2 + \frac{A^2C^2L^2}{X^3} \left( 1 + \eta^{-1} \right) + ACL + \frac{L}{\eta X_0}(A^2 + C^2) \\
\ll_N X_0U^2V^2 + \frac{U^2V^2L^2}{X^3} + LUV + \frac{U^2L}{\eta X^2}.
\]

The inequality $\eta \gg_N 1$ implies that $X_0 \gg_N \sqrt{L}$ which, in turn, implies that the first term on the right-hand side of the above expression dominates the second and fourth terms. Thus, using the estimates in (3.21) and that $U \approx \frac{V}{M}$, we have

\[
\frac{\eta V}{U} \left\{ X_0(\#\mathcal{R})^2 + \int_{X_0}^L B(\Delta_1(X), \Delta_2(X)) \, dX \right\} \ll_N \left( LMV^2 \right)^{\frac{1}{4}} + \left( L^{-1}M^3V^2 \right)^{\frac{1}{4}}.
\]

We apply the large sieve with $\lambda(\ell) = \lambda_{f,j}(\ell)$ and $\nu(j) = c(f, r, \chi; j)$. By Lemma 2.2(i), we have $\sum_{1 \leq \ell \leq L} |\lambda(\ell)|^2 \ll_{k,N} L$, and thus the left-hand side of the large sieve inequality is $\ll_{k,N} \left( L^3MV^2 \right)^{\frac{1}{4}} + \left( L^3M^3V^2 \right)^{\frac{1}{4}}$. Estimating the sums over $\beta$, $r$, and $\chi$ trivially in (3.16), we deduce that

\[
\sum_{M_1 \leq n \leq M_2} \frac{\lambda_f(n)}{n^{it}} \ll_{k,N} \sum_{L, U, V} \left\{ \left( L^3MV^2 \right)^{\frac{1}{4}} + \left( L^3M^3V^2 \right)^{\frac{1}{4}} \right\} + \frac{R^8t^3}{M^\frac{1}{2}} (\log M_0)^{-\delta},
\]

where $L$, $U$, and $V$ run over powers of 2 satisfying $L \ll rM/(dR)^2$, $V \ll R$, and $U \approx \frac{V}{M}$. Note that there are boundedly many values of $U$ corresponding to each $V$. Hence, we derive that

\[
\sum_{M_1 \leq n \leq M_2} \frac{\lambda_f(n)}{n^{it}} \ll_{k,N} \frac{M}{R} + \frac{R^8t^3}{M^\frac{1}{2}} (\log M_0)^{-\delta} \ll \sqrt{M} \left\{ \sqrt{M_0} + M_0^{-4}t^3(\log t)^{-\delta} \right\}.
\]

Therefore (3.2) becomes

\[
|L(\frac{1}{2} + it, f)| \ll_{k,N} \left\{ \sqrt{M_0} + M_0^{-4}t^3(\log t)^{-\delta} \right\} \log C + \sqrt{M_0}(\log M_0)^{-\delta} \\
\ll_{k,N} \sqrt{M_0} \log t + M_0^{-4}t^3(\log t)^{1-\delta}.
\]

Choosing $M_0 = \lfloor t^2(\log t)^{-\frac{24}{7}} \rfloor$ balances the two terms on the right-hand side and proves Theorem 1.1.
4. Proof of Proposition 3.1

4.1. Preliminary lemmas. Before proving Proposition 3.1, we state three lemmas. For $k$ a positive integer, let $C^k([\alpha, \beta])$ denote the space of $k$ times continuously differentiable real-valued functions on the interval $[\alpha, \beta]$. The next two lemmas on exponential integrals are Lemma 5.5.5 and Lemma 5.5.6 of [12].

**Lemma 4.1.** Let $F \in C^3([\alpha, \beta])$ and let $G \in C^2([\alpha, \beta])$. Suppose there exist positive parameters $M, H, t, U$, with $M \geq \beta - \alpha$, and positive constants $C_{r_1}, C_{r_2}$ such that, for $x \in [\alpha, \beta]$, we have

$$|F^{(r_1)}(x)| \leq C_{r_1} t/M^{r_1} \quad \text{and} \quad |G^{(r_2)}(x)| \leq C_{r_2} U/H^{r_2}$$

for $r_1 \in \{2, 3\}$, and $r_2 \in \{0, 1, 2, 3\}$. If $F'$ and $F''$ do not change sign on $[\alpha, \beta]$, then

$$I = \int_\alpha^\beta \frac{G(x)e(F(x))}{\sqrt{F''(x)}} \, dx = \frac{G(\beta)e(F(\beta))}{2\pi i F'(\beta)} - \frac{G(\alpha)e(F(\alpha))}{2\pi i F'(\alpha)} + O\left(\frac{U}{M^2} \left(1 + \frac{M}{H} \frac{M^2}{H^2} \min \left|\frac{F'(x)}{x}\right|\right) \min \left|\frac{F''(x)}{x}\right|^3\right).$$

**Lemma 4.2.** Let $F \in C^4([\alpha, \beta])$ and let $G \in C^3([\alpha, \beta])$. Suppose there exist positive parameters $M, H, T, U$, with $M \geq \beta - \alpha$, $H \geq M/\sqrt{t}$, and positive constants $C_{r_1}, C_{r_2}$ such that, for $x \in [\alpha, \beta]$, we have

$$|F^{(r_1)}(x)| \leq C_{r_1} t/M^{r_1} \quad \text{and} \quad |G^{(r_2)}(x)| \leq C_{r_2} U/H^{r_2}$$

for $r_1 \in \{2, 3, 4\}$ and $r_2 \in \{0, 1, 2, 3\}$, and a positive constant $\tilde{C}$ such that

$$F^{(2)}(x) \geq t/\tilde{C} M^2.$$

Suppose also that $F'(x)$ changes sign from negative to positive at a point $x = \gamma$ with $\alpha < \gamma < \beta$. If $t$ is sufficiently large with respect to the constants $C_{r_1}, C_{r_2}, \tilde{C}$, then

$$\int_\alpha^\beta \frac{G(x)e(F(x))}{\sqrt{F''(x)}} \, dx = \frac{G(\gamma)e(F(\gamma) + \frac{1}{8})}{\sqrt{F''(\beta)}} + \frac{G(\beta)e(F(\beta))}{2\pi i F'(\beta)} - \frac{G(\alpha)e(F(\alpha))}{2\pi i F'(\alpha)}$$

$$+ O\left(\frac{M^4 U}{t^2} \left(1 + \frac{M}{H} \frac{M^2}{H^2} \frac{1}{(\alpha - \gamma)^3} + \frac{1}{(\beta - \gamma)^3}\right)\right) + O\left(\frac{M U}{t^2} \left(1 + \frac{M}{H}\right)^2\right).$$

The third lemma provides bounds for derivatives of $F_j$.  

**Lemma 4.3.** For any integer $s \geq 0$, we have

$$\left| \frac{d^s}{dx^s} \left\{ F_j(x) x^{-\frac{k+1}{2}} \right\} \right| \ll_s (v_j R)^{-s} x^{-\frac{k+1}{2}}.$$

**Proof.** By continuity, it suffices to consider $x = x_0 \in (N_{j-\frac{1}{2}} - H, N_j + H) \setminus \{N_{j-\frac{1}{2}} + H, N_j - H\}$. For any fixed $x_0$, the function $F_j(z) z^{-\frac{k+1}{2}}$ agrees with an analytic function $g(z)$ for $z$ in a neighborhood of $x_0$. We estimate $g^{(s)}(x_0) = \left. \frac{d^s}{dz^s} (F_j(z) x^{-\frac{k+1}{2}}) \right|_{z=x_0}$ via the Cauchy integral formula. Since $v_j R \leq R^2 \ll M$, we may fix a small constant $c > 0$ such that $Y = cv_j R \leq \frac{1}{2} M_1$, and integrate over the circle $C_Y(x_0)$ of radius $Y$ around $x_0$:

$$|g^{(s)}(x_0)| = \left| \frac{s!}{2\pi i} \int_{C_Y(x_0)} \frac{g(z)}{(z-x_0)^{s+1}} \, dz \right| \leq s! Y^{-s} \sup_{z \in C_Y(x_0)} |g(z)| \ll_s (v_j R)^{-s} \sup_{z \in C_Y(x_0)} |g(z)|.$$
Hence, it suffices to show that $g(z) \ll_s x_0^{-k+1}$ for $z \in C_Y(x_0)$.

Let $z = x + iy \in C_Y(x_0)$, so that $x = x_0 + O(v_jR) \times M$ and $y \ll v_jR$. We have $\frac{z - x_0}{x_0^2} \ll \frac{v_jR}{x_0^2}$ so, by (3.1) and (3.5), $z^{-\frac{k-1}{2}} = x_0^{-\frac{k-1}{2}} e^{O((k-1)/M)} \ll x_0^{-\frac{k-1}{2}}$. Next, observe that

$$|z^{-it} e(-\alpha_j z)| = e^t \arctan(y/x) + 2\pi \alpha_j y.$$ 

The exponent here is bounded since

$$t \arctan\left(\frac{y}{x}\right) - \frac{ty}{x} \ll \frac{t(v_jR)^3}{M^3} \ll \frac{t}{M_0^3} \ll 1.$$ 

Moreover, by the estimates in Section 3.2, we have $x_0 = h(\alpha_j) + O(HR/v_j)$. Since $M_0^2 \gg t$, we also have $v_jR \ll HR/v_j$, so that $x = h(\alpha_j) + O(HR/v_j)$. Thus

$$\frac{ty}{x} + 2\pi \alpha_j y = \frac{2\pi y \alpha_j}{x} (x - h(\alpha_j)) \ll v_jR \cdot \frac{t}{M^2} \cdot \frac{HR}{v_j} \ll 1.$$ 

Finally, for $z$ near $x_0$, $\omega_j(z)$ is a linear combination of 1, $\sin^{*+1}(\frac{\pi}{2H} (z-N_{j-1}))$, and $\sin^{*+1}(\frac{\pi}{2H} (z-N_j))$. As above, we have $\frac{\pi y}{2H} \ll \frac{v_jR}{HR} \ll 1$, so that

$$\sin^{*+1}\left(\frac{\pi}{2H} (z-N_i)\right) \ll_s 1 \quad \text{for} \quad N \in \{N_{j-1}, N_j\},$$ 

as desired. □

4.2. Outline of the proof. Since the proof of Proposition 3.1 is long, we give an outline. The left-hand side of equation (3.12) is written as $S = \sum_{j \in J(\beta, r)} c(f, r, \chi; j) S(j)$ where

$$S(j) = 2\pi i^k \sum_{\ell=1}^{\infty} \lambda_j f_\chi(\ell) e\left(\frac{\pi n^2}{q_j}\right) I_j(\ell)$$

and

$$I_j(\ell) = \frac{1}{d_j} \int_0^\infty F_j(x) J_{k-1}\left(\frac{4\pi \ell^2}{q_j \sqrt{x}}\right) dx.$$ 

Our goal is to develop an approximate formula for $S$. This is done in five steps. In the first four steps we determine an approximate formula for $S(j)$, and in the final step these approximations are summed over $j$ to obtain our formula for $S$. We choose real parameters $K_1$ and $K$ such that $rK_1/d^2 \leq K$ and we decompose $S(j) = S_{[1,K]}(j) + S_{(K,\infty]}(j)$, where for an interval $I \subset \mathbb{R}$, $S_I(j) := 2\pi i^k \sum_{\ell \in I} \lambda_j f_\chi(\ell) e\left(\frac{\pi n^2}{q_j}\right) I_j(\ell)$. Our bounds for the sums $S_I(j)$ will depend on bounds for $\sum_{\ell \leq x} |\lambda_j f_\chi(\ell)|$. These steps are now described in more precise detail.

Step 1. We first bound $S_{(K,\infty]}(j)$. For $\ell > K$, the integral $I_j(\ell)$ is estimated by integration by parts, making use of the smoothness of $F_j$ and bounds for Bessel functions.

Step 2. Next we insert the asymptotic formula [9, §8.451, Eqn. 1]

$$J_{\nu}(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi \nu}{2} - \frac{\pi}{4}\right) + O(x^{-\frac{3}{2}}),$$ 

which holds for $\nu \in \mathbb{Z}_{\geq 0}$ as $x \to \infty$, and estimate the corresponding error terms for $\ell \leq K$ to deduce that $S_{[1,K]}(j)$ equals $\tilde{S}_{[1,K]}(j)$ plus an error term, where $\tilde{S}_{[1,K]}(j)$ is a simplified sum. We choose $K$ as a function of $v_j$, $M$, and $R$ to balance the error terms in steps 1 and 2.
Step 3. We are left with sums of the shape \( \tilde{S}_{[1,K]}(j) = \sum_{\ell \leq K} \alpha_{\ell,j,r} \lambda_{\ell}(x) e\left(\frac{2\pi i}{\ell q_j} \right) I_j^\pm(\ell) \) where \( \alpha_{\ell,j,r} \in \mathbb{C}, \ I_j^\pm(\ell) = \frac{1}{q_j} \int_0^\infty F_j(x)e(\phi_\pm(x)) \, dx \), and \( \phi_\pm(x) \) is a function depending on parameters \( \ell, j, t, r \). We then choose \( K_1 \) so that \( \phi_\pm(x) \) does not change sign for \( x \in \text{supp}(F_j) \). For those \( \ell \) with \( rK_1d^{-2} \leq \ell \leq K \), the integrals \( I_j^\pm(\ell) \) are estimated using a weighted first derivative estimate (Lemma 4.1).

Step 4. Next, we treat the sum \( \sum_{\ell \leq K} \alpha_{\ell,j,r} \lambda_{\ell}(x) e\left(\frac{2\pi i}{\ell q_j} \right) I_j^\pm(\ell) \). In this range of \( \ell \) the integrals \( I_j^\pm(\ell) \) possess stationary points \( x_j^\pm(\ell) \). Each integral is treated with Lemma 4.2, leading to an expression \( S(j) = \mathcal{M}(j) + O(\tilde{E}_1(j) + \tilde{E}_2(j) + \tilde{E}_3(j)) \) where \( \mathcal{M}(j) \) is a main term and the \( \tilde{E}_i(j) \) are error terms.

Step 5. Finally, using \( c(f, r, \chi; j) \ll N \), we are left with the sum

\[
S = \sum_{j \in \mathcal{J}(\beta, r)} c(f, r, \chi; j) \mathcal{M}(j) + O\left( \sum_{j=1}^J (\tilde{E}_1(j) + \tilde{E}_2(j) + \tilde{E}_3(j)) \right).
\]

In this last step, the error terms \( \tilde{E}_i(j) \) are bounded as \( j \) ranges over all Farey fractions.

### 4.3. Proof of Proposition 3.1

We now commence with the proof.

**Step 1.** By repeated application of the identity [9, §8.472, Eqn. 3]

\[
\frac{d}{dx}(x^n J_n(x)) = x^n J_{n-1}(x),
\]

we have

\[
(2 \frac{s}{A})^s \frac{d^s}{dx^s}\left(x^{\frac{k+s-1}{2}} J_{k+s-1}(A \sqrt{x})\right) = x^{\frac{k-1}{2}} J_{k-1}(A \sqrt{x}) \text{ for } s \geq 0 \text{ and } A \neq 0.
\]

Setting \( A = \frac{4\pi \sqrt{q_j}}{\ell q_j} \), we have \( I_j(\ell) = \frac{1}{q_j} \int_0^\infty F_j(x)x^{\frac{k-1}{2}} (x^{\frac{k-1}{2}} J_{k-1}(A \sqrt{x})) \, dx \). Integrating by parts \( s \) times and using (4.3), it follows that

\[
I_j(\ell) = \left(\frac{-1}{q_j}\right)^s (2 \frac{s}{A})^s \int_0^\infty \frac{d^s}{dx^s}(F_j(x)x^{-\frac{k-1}{2}})(x^{\frac{k+s-1}{2}} J_{k+s-1}(A \sqrt{x})) \, dx.
\]

The asymptotic formula in (4.2) gives \( J_{k+s-1}(A \sqrt{x}) \ll_{k,s} A^{-\frac{1}{2}} x^{-\frac{k}{4}} \). This estimate, along with Lemma 4.3, implies that

\[
|I_j(\ell)| \ll_{k,s} \frac{1}{q_j A^{s+\frac{1}{2}}(v_j R)^s} \int_{\text{supp}(\omega_j)} x^{\frac{2s-1}{4}} \, dx.
\]

Using this bound for \( \ell > K \) and \( A \asymp \ell^2 q_j^{-1} r^{-\frac{1}{2}} \), we deduce that

\[
S_{(K, \infty)}(j) \ll_{k,s} r^{\frac{s}{2} + \frac{1}{4}} q_j^{s-\frac{1}{2}} M^{\frac{2s-1}{4}} \left(\frac{1}{v_j R}\right)^s \sum_{\ell > K} \frac{|\lambda_{\ell}(x)|}{(\ell^2 + 1)^{\frac{s}{4}}} |\text{supp}(\omega_j)|.
\]

Note that this last sum only converges for \( s \geq 2 \). By Lemma 2.2(iii), we have

\[
S_{(K, \infty)}(j) \ll_{k,N,s} r^{\frac{s}{2} + \frac{1}{4}} q_j^{s-\frac{1}{2}} M^{\frac{2s-1}{4}} \left(\frac{1}{v_j R}\right)^s |\text{supp}(\omega_j)| K^{\frac{3}{4} - \frac{s}{2}} (\log K)^{-\delta}.
\]
Step 2. The asymptotic estimate in (4.2) implies that

\[ J_{k-1}(\frac{\pi}{q}(\frac{\ell x}{r})) = \frac{r}{2\sqrt{2\pi}} \frac{q_j^{\frac{1}{2}}}{(\ell x)^{\frac{1}{2}}} \left( e(-\frac{k}{q})e(2(\frac{\ell x}{r}q_j)^{\frac{1}{2}} + \frac{1}{8}) + e(\frac{k}{q})e(-2(\frac{\ell x}{r}q_j)^{\frac{1}{2}} - \frac{1}{8}) \right) + O_k \left( \frac{r^3 q_j^3}{x^{\frac{3}{2}} \ell^{\frac{3}{2}}} \right). \]

Inserting this expression into (4.1) for each \( \ell \leq K \) and estimating the error terms, we have

\[ S_{[1,K]}(j) = \frac{i^k r^\frac{1}{2}}{\sqrt{2} q_j^\frac{1}{2}} \sum_{\ell \leq K} \frac{\lambda_{jx}(\ell) e(\frac{\ell q_j}{q})}{\ell^\frac{1}{2}} \sum_{\pm} e(\pm \frac{k}{q}) \int_0^\infty F_j(x) x^{-\frac{1}{2}} e(\mp 2(\frac{\ell x}{r}q_j)^{\frac{1}{2}} \mp \frac{1}{8}) dx + E_0(j), \]

where

\[ E_0(j) \ll \sum_{\ell \leq K} \frac{|\lambda_{jx}(\ell)|}{\ell^\frac{1}{2}} \frac{1}{q_j} \int_{-\infty}^\infty |F_j(x)| \frac{r^3 q_j^3}{x^3} dx \ll_k, N \frac{r^3 q_j^3}{M^4} |\text{supp}(\omega_j)| K^\frac{1}{2} (\log K)^{-\delta} \]

by Lemma 2.2(iv) since \( x \approx M \). Choosing \( K := \left( \frac{M}{v_j r} \right)^{\frac{3}{8}} M_0 \) and recalling that \( |\text{supp}(\omega_j)| \ll \frac{HR}{v_j} \), it follows that

\[ (4.4) \quad E_1(j) := S_{(K,\infty)}(j) + E_0(j) \ll_{k, N, s} \frac{HM^{\frac{2-s}{(s-1)}} R^{\frac{s-2}{(s-1)}}}{v_j^{\frac{4-s}{(s-1)}}} (\log M_0)^{-\delta}. \]

Therefore \( S(j) = \tilde{S}_{[1,K]}(j) + E_1(j) \), where

\[ (4.5) \quad \tilde{S}_{[1,K]}(j) = i^k \sum_{\ell \leq K} \frac{\lambda_{jx}(\ell) e(\frac{\ell q_j}{q})}{\ell^\frac{1}{2}} \left( e(\frac{k}{q}) I_j^+(\ell) + e(-\frac{k}{q}) I_j^-(\ell) \right), \]

\[ (4.6) \quad I_j^+(\ell) = \int_0^\infty g_j(x) e(\phi_+(x)) dx, \]

\[ (4.7) \quad g_j(x) = x^{-\frac{1}{2}} \omega_j(x), \]

and

\[ (4.8) \quad \phi_+(x) := -\frac{t}{2\pi} \log x - \alpha_j x \mp 2(\frac{\ell x}{r}q_j)^{\frac{1}{2}} \mp \frac{1}{8}. \]

Step 3. Let \( K_1 = \left[ \frac{C M}{R^2} \right] \), where \( C \) is a sufficiently large positive constant. For \( \ell \) satisfying \( rK_1d^{-2} \leq \ell \leq K \), we bound the integral in (4.6) by showing that \( |\phi_+(x)| \gg \frac{1}{q_j} \left( \frac{\ell}{rM} \right)^{\frac{1}{2}} \) and using a weighted first derivative estimate (Lemma 4.1). It is convenient to write

\[ \phi_+(x) = f(x) + \frac{u_j}{v_j} x - \frac{2}{q_j} \left( \frac{\ell x}{r} \right)^{\frac{1}{2}} - \frac{1}{8}, \]

where \( f(x) = -\frac{t}{2\pi} \log x \). We shall make use of

\[ (4.9) \quad f^{(j)}(x) = \frac{(j-1)!(-1)^j t}{2\pi x^j} \gtrsim \frac{t}{M^j} \quad \text{for} \quad x \in [M_1, M_2] \]

for \( j \geq 1 \) and the identity

\[ (4.10) \quad y = f'(h(y)). \]
For \( x \in \text{supp}(\omega_j) \), the mean value theorem implies there exists \( \xi \in \text{supp}(\omega_j) \) such that

\[
f'(x) = f'(h(\rho_j)) + f''(\xi)(x - h(\rho_j)) = \rho_j + O\left(\frac{t}{M^2} |\text{supp}(\omega_j)|\right),
\]
based on (4.9) and (4.10). By (3.6) and (3.7) this is

\[
f'(x) = \alpha_j + O\left(\frac{1}{v_j R} + \frac{t}{M^2} \frac{HR}{v_j}\right).
\]

Let \( c_2 \) be such that \( M_2 \leq c_2 M \). By the previous equation there exists \( c_0 > 0 \) such that

\[
|f'(x) - \alpha_j| \leq c_0 \left(\frac{1}{v_j R} + \frac{t}{M^2} \frac{HR}{v_j}\right) \leq \frac{2c_0}{q_j dR} \leq \frac{1}{2q_j} \left(\frac{\ell}{rc_2 M}\right)^{\frac{1}{2}},
\]
as long as \( \ell \gg \frac{M}{dR^2} \). Since \( x \leq M_2 \leq c_2 M \), we obtain \( |f'(x) - \alpha_j| \leq \frac{1}{2q_j} \left(\frac{\ell}{rM}\right)^{\frac{1}{2}} \). It follows from (4.8) and (4.11) that for \( \ell \geq K_1 d^{-2} \),

\[
|\phi'_+(x)| = |f'(x) - \alpha_j - \frac{1}{q_j} (\frac{\ell}{rM})^{\frac{1}{2}}| \geq \frac{1}{2q_j} (\frac{\ell}{rM})^{\frac{1}{2}} \gg \frac{1}{2q_j} (\frac{\ell}{rM})^{\frac{1}{2}}.
\]

We now compute the derivatives of \( F(x) = \phi_+(x) \) and \( G(x) = g_j(x) \) given by (4.7). We have

\[
F^{(r_1)}(x) = f^{(r_1)}(x) - \frac{2}{q_j} \sqrt{\frac{\ell}{r}} \frac{d^{r_1}}{dx_1} x^{\frac{1}{2}} \omega_j^{(i_2)}(x) \ll \frac{t}{M^{r_1}} + \frac{2}{q_j} \sqrt{\frac{\ell}{r}} \frac{1}{M^{r_1} - \frac{1}{2}} \ll \frac{t}{M^{r_1}},
\]

which follows (after some calculation) using the facts that \( \ell \leq K \) and \( s \geq 6 \). Also we have

\[
G^{(r_2)}(x) = \sum_{i_1 + i_2 = r_2} \left(\frac{r_2}{i_1}\right) \frac{d^{i_1}}{dx_1} x^{-i_2} \omega_j^{(i_2)}(x) \ll \sum_{i_1 + i_2 = r_2} M^{-\frac{1}{2} - 2i_1} H^{-i_2} \ll M^{-\frac{1}{4}} H^{-r_2}
\]

where we used \( H = \frac{M^2}{R^2} \leq M \). We now invoke Lemma 4.1 with \( \alpha = N_{j-1} - H \), \( \beta = N_j + H \), and \( U = M^{-\frac{1}{4}} \) and make use of the lower bound \( F'(x) \gg \frac{1}{q_j} (\frac{\ell}{rM})^{\frac{1}{2}} \) for \( x \in [\alpha, \beta] \). With these choices the condition \( M \geq \beta - \alpha \) is satisfied and \( G(\alpha) = G(\beta) = 0 \). Thus, for \( rK_1 d^{-2} \leq \ell \leq K \), this lemma gives

\[
I^\pm_j(\ell) \ll M^{-\frac{1}{4}} \frac{t}{M^2} \left(1 + \frac{M}{H} + \frac{M^2}{H^2} \frac{1}{q_j} (\frac{\ell}{rM})^{\frac{1}{2}} \right) \left(\frac{q_j^2 r M}{\ell}\right)^{\frac{3}{2}}.
\]

Since \( H \leq M \), we also have

\[
r^\pm q_j^\pm \ell^{-\frac{1}{2}} I^\pm_j(\ell) \ll r^\pm (q_j^2 \ell M)^{-\frac{1}{4}} \frac{t}{M^2} \left(\frac{M}{H} + \frac{M^3}{H^2} \frac{1}{q_j} (\frac{\ell}{rM})^{\frac{1}{2}} \right) \left(\frac{q_j^2 r^3 M^3}{\ell^2}\right)^{\frac{3}{2}} \]

\[
\ll \frac{r^\pm}{\ell^2} \left(\frac{2}{q_j^2} \left(\frac{M}{\ell}\right)^{\frac{3}{4}} \frac{M}{H^2 R^2} + q_j^2 \left(\frac{M}{\ell}\right)^{\frac{3}{4}} \frac{M}{H^2} \sqrt{r}\right),
\]

as long as \( rK_1 d^{-2} \leq \ell \leq K \).
where we used the definition of $H$ in the last line. It follows that the contribution of the range $\ell \in (rK_1d^{-2}, K]$ to (4.5) is

$$\mathcal{E}_2(j) \ll r^{\frac{7}{2}} \sum_{rK_1d^{-2} < \ell \leq K} \left( \frac{\lambda_{J_4}(\ell)}{\ell} \right) \left( \frac{M}{\ell} \right)^{\frac{3}{2}} \left( \frac{M}{H^2 R^2} + q_j \right) \left( \frac{M}{\ell} \right)^{\frac{3}{2}} \left( \frac{M}{H^2 \sqrt{r}} \right)$$

$$\ll r^{\frac{7}{2}} \left( \sum_{\ell > rK_1d^{-2}} \frac{\left| \lambda_{J_4}(\ell) \right|}{\ell^{\frac{3}{2}}} \right) + \sum_{\ell > rK_1d^{-2}} \left| \lambda_{J_4}(\ell) \right|.$$

Using Lemma 2.2(iii) and the estimate $K_1 \approx M/R^2$, we have

$$\mathcal{E}_2(j) \ll_{k,N} r^\frac{7}{2} \left( \frac{q_j^2 M^\frac{5}{2}}{H^2 R^2} \left( \frac{rM}{R^2 d^2} \right)^{-\frac{3}{2}} + \frac{q_j^2 M^\frac{5}{2}}{H^2 \sqrt{r}} \left( \frac{rM}{R^2 d^2} \right)^{-\frac{3}{2}} \right) \right) \left( \log(2 + rK_1d^{-2}) \right)^{-\frac{1}{2}}$$

$$(4.14) \quad \ll_{k,N} \left( \frac{u_j}{R^2} + R^{\frac{1}{2}} \right) \left( \log M_0 \right)^{-\frac{1}{2}} \ll \left( \frac{u_j}{R} \right)^{\frac{3}{2}} \frac{M^\frac{5}{2} R^2}{H^2} \left( \log M_0 \right)^{-\frac{1}{2}},$$

since $v_j \leq R$.

**Step 4.** We have shown that $\mathcal{S}(j) = \tilde{\mathcal{S}}_{[1,rK_1d^{-2}]}(j) + \mathcal{E}_1(j) + \mathcal{E}_2(j)$ where

$$\tilde{\mathcal{S}}_{[1,rK_1d^{-2}]}(j) = i^k \sum_{\ell \leq rK_1d^{-2}} \frac{\lambda_{J_4}(\ell) e\left(\frac{\ell x}{q_j}\right)}{\sqrt{2\left(\frac{\ell}{\ell} \frac{3}{2} q_j \right)}} \left( e\left(\frac{\xi}{q_j} I_j^+(\ell) \right) + e\left(-\frac{\xi}{q_j} I_j^-(\ell) \right) \right),$$

and $\mathcal{E}_1(j)$ and $\mathcal{E}_2(j)$ are estimated by (4.4) and (4.14), respectively. For $\ell \leq rK_1d^{-2}$, we extract the stationary phase terms of the integrals $I_j^\pm(\ell)$ given by (4.6). Let $x_j^\pm(\ell)$ be the roots of

$$\frac{d}{dx} \left( -\frac{t}{2\pi} \log x - \alpha_j x \mp 2\left(\frac{\ell x}{q_j} \right)^{\frac{1}{2}} \mp \frac{1}{2} \right) = 0.$$

Notice that the numbers $x_j^\pm(\xi_j)$ are the stationary points satisfying $\phi'_\pm(x_j^\pm(\xi_j)) = 0$. It follows that the $x_j^\pm(\ell)$ are the positive roots of

$$\frac{t}{2\pi x} - \frac{u_j}{v_j} x + \left( \frac{\ell}{x} \right)^{\frac{1}{2}} \frac{1}{q_j} = 0,$$

since $\alpha_j = -\frac{u_j}{v_j}$. Multiplying by $x$ this becomes

$$\frac{u_j}{v_j} x^2 + \frac{\ell}{q_j} x^{\frac{1}{2}} x - \frac{t}{2\pi} = 0 \quad \text{so that} \quad x = \frac{\pm \frac{\ell}{q_j} \pm \sqrt{\frac{\ell}{q_j} + 4\frac{u_j}{v_j} \frac{t}{\pi d^2}}}{2\frac{u_j}{v_j}}.$$

We discard those solutions corresponding to the second $-$ sign since $\sqrt{x}$ is necessarily positive. With a little calculation, it follows that

$$x_j^\pm(\ell) = \left( \frac{\pm \frac{\ell}{q_j} + \sqrt{\frac{\ell}{q_j} + 2\frac{u_j}{v_j} \frac{t}{\pi d^2}}}{2\frac{u_j}{v_j}} \right)^2 = \left( \frac{d}{2u_j} \right)^2 \left( \sqrt{\ell + \frac{2u_j q_j t}{\pi d} \pm \sqrt{\ell}} \right)^2,$$
since $v_j = dq_j$. Finally, we apply the stationary phase lemma (Lemma 4.2) to those $I_j^\pm(\ell)$ with $\ell \leq rK_1d^{-2}$. We choose $F(x) = \phi_+(x)$, $G(x) = g_j(x)$, $\alpha = x_j^+(\xi) - M/4$, $\beta = x_j^+(\xi) + M/4$, and $\gamma = x_j^+(\xi)$. We also have the parameters $t$, $M$, $H$, and $U = M^{-\frac{1}{4}}$ which correspond to those of Lemma 4.2 and we have the derivative bounds (4.12) and (4.13). Observe that the conditions $M \geq \beta - \alpha$ and $H \geq M/\sqrt{r}$ are both met. With these choices we now demonstrate that supp$(\omega_j) = [\mathcal{N}_{j-1} - H, \mathcal{N}_j + H] \subseteq [\alpha, \beta]$ so that $G(\alpha) = G(\beta) = 0$. We aim to show
\[
\alpha = x_j^+(\xi) - M/4 \leq \mathcal{N}_{j-1} - H.
\]
By the mean value theorem there exists $\xi \in \text{supp}(\omega_j)$ such that
\[
(4.17) \quad |f'(\mathcal{N}_{j-1}) - f'(x_j^+(\xi))| = |f''(\xi)||\mathcal{N}_{j-1} - x_j^+(\xi)| \gg \frac{t}{M^2} |\mathcal{N}_{j-1} - x_j^+(\xi)|.
\]
Similarly there exists $\xi' \in \text{supp}(\omega_j)$ such that
\[
f'(\mathcal{N}_{j-1}) = f'(h(\rho_{j-1})) + O(f''(\xi')) = \rho_{j-1} + O\left(\frac{t}{M^2}\right),
\]
by (4.10). By (4.16) $f'(x_j^+(\xi)) = -\frac{t}{2\pi x_j^+(\xi)} = \alpha_j \pm \left(\frac{\ell/r}{x_j^+(\xi)}\right)^\frac{1}{2}$. Using (3.6) it follows that
\[
(4.18) \quad |f'(\mathcal{N}_{j-1}) - f'(x_j^+(\xi))| \ll \frac{1}{v_j} + \frac{t}{M^2} + \left(\frac{\ell/r}{M}\right)^\frac{1}{2} \frac{1}{q_j} \ll \frac{1}{v_j} + \frac{t}{M^2},
\]
since $\frac{\ell}{r} \leq \frac{CM}{d tR^2}$. Combining (4.17) and (4.18) yields $|\mathcal{N}_{j-1} - x_j^+(\xi)| \ll \frac{M^2}{tR} + 1$ and thus
\[
x_j^+(\xi) - \mathcal{N}_{j-1} + H \leq O\left(\frac{M^2}{tR} + 1\right) + \frac{M^2}{tR^2} \leq \frac{M}{4},
\]
assuming that $t_0$ is sufficiently large. Hence $\alpha \leq \mathcal{N}_{j-1} - H$, and an analogous argument establishes that $\mathcal{N}_j + H \leq \beta$. The stationary point of $G = \phi_+$ is $x_j^+(\xi)$. Hence the main term in Lemma 4.2 is
\[
\frac{x_j^+(\xi) - \frac{1}{2} \omega_j(x_j^+(\xi))e(\phi_+(x_j^+(\xi)) + \frac{1}{8})}{\sqrt{\phi_+'(x_j^+(\xi))}}
\]
and since $\gamma - \alpha = \beta - \gamma = M/4$, the error term is
\[
\ll \frac{M^4 M^{-\frac{1}{4}}}{t^2} \left(1 + \frac{M}{H}\right)^2 M^{-3} + \frac{MM^{-\frac{1}{4}}}{t^2} \left(1 + \frac{M}{H}\right)^2 \ll M^{-\frac{1}{4}} \frac{M^3}{t^2 H^2},
\]
as the second error term dominates the first and $H \leq M$. A similar argument establishes the analogous result for $I_j^-(\ell)$. Thus,
\[
I_j^\pm(\ell) = \frac{x_j^\pm(\xi) - \frac{1}{2} \omega_j(x_j^\pm(\xi))e(\phi_\pm(x_j^\pm(\xi)) + \frac{1}{8})}{\sqrt{\phi_\pm'(x_j^\pm(\xi))}} + O\left(\frac{M^2}{t^2 H^2}\right) \text{ for } \ell \leq rK_1d^{-2}.
\]
The error term, when inserted into (4.15), becomes
\[
\mathcal{E}_3(j) \ll \frac{r^\frac{1}{2}}{q_j^\frac{1}{2}} \sum_{\ell \leq rK_1d^{-2}} |\lambda_j f_\ell(\ell)| M^\frac{1}{4} \ll_{k,N} \frac{r^\frac{1}{2} M^\frac{1}{4}}{q_j^\frac{1}{2} H^2 t^\frac{3}{2}} \left(\frac{r M}{R^2 d^2}\right)^\frac{3}{4} (\log M_0)^{-\delta},
\]
where $M_0$ is the mean value of $M$. This completes the proof of Proposition 1.2.
expression in (4.19) simplifies to

\[ E_3(j) \ll_{k,N} M^{3/2} (\log M_0)^{-\delta} q_j^2 H^{3/2} \ll_{k,N} M^{3/2} R_2^{3/2} (\log M_0)^{-\delta}, \]

since \( H \geq \frac{M^2}{H_{ij}} \) and \( v_j = dq_j \). Hence we have established

\[ S(j) = M(j) + O_{k,N,s} \left( (\tilde{E}_1(j) + \tilde{E}_2(j) + \tilde{E}_3(j)) (\log M_0)^{-\delta} \right), \]

where

\[ M(j) = i^k e\left( \frac{\ell}{q_j} \right) \sum_{\ell \leq \tau K_1 d^{-2}} \frac{\lambda_{f,x}(\ell)e\left( \frac{r \phi_j}{q_j} \right) \phi_j(x_j^+(\ell)) e(\phi_+(x_j^+(\ell)) + \frac{1}{8})}{\sqrt{2(\ell)^{3/2} q_j x_j^+(\ell))^{1/2} \sqrt{\phi_+^2(x_j^+(\ell))}} \]

(4.19)

\[ + i^k e\left( -\frac{\ell}{q_j} \right) \sum_{\ell \leq \tau K_1 d^{-2}} \frac{\lambda_{f,x}(\ell)e\left( \frac{r \phi_j}{q_j} \right) \phi_j(x_j^-(\ell)) e(\phi_-(x_j^-(\ell)) + \frac{1}{8})}{\sqrt{2(\ell)^{3/2} q_j x_j^-(\ell))^{1/2} \sqrt{\phi_-^2(x_j^-(\ell))}} \]

\( \tilde{E}_1(j) = \frac{H M^{2/3} R_2^{2/3}}{v_j^{2/3}}, \)

\( \tilde{E}_2(j) = \left( \frac{v_j}{R} \right)^{3/2} \frac{M^{3/2} R_2^{3/2}}{H^{2/3}}, \)

and

\( \tilde{E}_3(j) = \frac{M^{3/2} R_2^{3/2}}{H^{1/3} v_j}. \)

We now simplify the expression for \( M(j) \). By (4.8), it follows that

\[ x^{1/2} (\phi_+^2(x))^{1/2} = \left( \frac{t}{2 \pi x^2} \right)^{1/2} \left( \frac{1}{2 q_j} \sqrt{\frac{t}{r} x^{-3/2}} \right)^{1/2} = \left( \frac{t}{2 \pi x^2} \right)^{1/2} \left( \frac{1}{2 q_j} \sqrt{\frac{t}{r} x^{-3/2}} \right)^{1/2}, \]

and thus

\[ \sqrt{2(\ell)^{3/2} q_j x_j^+(\ell))^{1/2} (\phi_+^2(x_j^+(\ell)))^{1/2} = \left( \frac{\sqrt{r} q_j^{1/2}}{\pi x_j^+(\ell)^{1/2}} \right)^{1/2} \left( \frac{1}{x_j^+(\ell)^{1/2}} \right)^{1/2}. \]

Since \( i^k e(\pm \frac{k}{4}) = (\pm 1)^k \), the expression in (4.19) simplifies to

\[ M(j) = \sum_{\pm} (\pm 1)^k \sum_{\ell = 1}^{\tau K_1 d^{-2}} \lambda_{f,x}(\ell) e\left( \frac{r \phi_j}{q_j} \right) \phi_j(x_j^+(\ell)) h_j^+(\ell) e(g_j^+(\ell)), \]

where \( x_j^+(\ell), h_j^+(\ell), \) and \( g_j^+(\ell) \) are given by (3.13), (3.14), and (3.15), respectively. Therefore,

(4.21)

\[ S = \sum_{j \in J(\beta,r)} c(f,r,\chi;j) M(j) + O_{k,N,s} \left( (\log M_0)^{-\delta} \sum_{j = 1}^{J} (\tilde{E}_1(j) + \tilde{E}_2(j) + \tilde{E}_3(j)) \right), \]

since \( \sum_{j \in J(\beta,r)} \tilde{E}_i(j) \leq \sum_{j = 1}^{J} \tilde{E}_i(j) \).

**Step 5.** In this final step, we bound the error terms in (4.21). First, we divide the sum over \( j \) into subsums where the \( v_j \) lie in dyadic intervals \([Q,2Q]\) where \( Q = 2^i, i \geq 0, \) and \( Q \leq R \). We require a bound for the number of \( v_j \) in \([Q,2Q]\). Observe that the Farey fractions \( -\frac{u}{v_j} \)

lie in the interval \( I = [-\frac{t}{2\pi(M_1+2H)}, -\frac{t}{2\pi(M_2-2H)}] \) of length \( |I| \approx \frac{t}{M} \approx \frac{M}{H_{ij}}, \) since \( H \geq \frac{M^2}{H_{ij}} \).

Now if \( F(Q) \) denotes the extended Farey fractions with denominator less than or equal to \( Q \), then [12, Lemma 1.2.3] gives

\[ \sum_{\alpha \in F(Q) \cap I} 1 \leq Q^2 + 1, \]

\[ 22 \]
where $I$ is an interval of length $\Delta$. Applying this estimate with $\Delta = \frac{M}{HR^2}$, we have

$$\sum_{Q \leq v_j \leq 2Q} 1 \leq \sum_{\alpha \in F(2Q) \cap I} 1 \ll \frac{MQ^2}{HR^2} + 1.$$  

(4.22)

Consequently,

$$\sum_{j=1}^{J} \tilde{E}_1(j) \ll H \sum_{Q \leq R} M^{\frac{2-s}{2(s-1)}} R^{\frac{s-2}{2(s-1)}} \left( \sum_{Q \leq v_j \leq 2Q} v_j^{\frac{1}{2(s-1)}} \right)$$

$$\ll H \sum_{Q \leq R} \frac{M^{\frac{2-s}{2(s-1)}} R^{\frac{s-2}{2(s-1)}}}{Q^{\frac{1}{2(s-1)}}} \left( \frac{MQ^2}{HR^2} + 1 \right)$$

$$= H \sqrt{\frac{R}{M}} \left( \frac{M}{R} \right)^{\frac{1}{2(s-1)}} \left( \frac{M}{HR^2} \sum_{Q \leq R} Q^{\frac{2}{2(s-1)}} + \sum_{Q \leq R} Q^{-\frac{1}{2(s-1)}} \right).$$

Using the elementary estimate

$$\sum_{Q \leq R} Q^{c_1} \ll_c \begin{cases} R^{c_1}, & \text{for } c_1 > 0, \\ 1, & \text{for } c_1 < 0, \end{cases}$$

with $c_1 = \frac{3}{2} - \frac{s}{2(s-1)}$ and $c_1 = -\frac{1}{2} - \frac{s}{2(s-1)}$ (for $s \geq 2$) it follows that

$$\sum_{j=1}^{J} \tilde{E}_1(j) \ll H \sqrt{\frac{R}{M}} \left( \frac{M}{R} \right)^{\frac{1}{2(s-1)}} \left( \frac{M}{HR^2} R^{\frac{3}{2}} - \frac{1}{2(s-1)} + 1 \right)$$

$$\ll H \sqrt{\frac{R}{M}} \left( \frac{M}{R} \right)^{\frac{1}{2(s-1)}} \left( \frac{M}{HR^2} R^{\frac{3}{2}} - \frac{1}{2(s-1)} + H \right).$$

Observe that $H \ll \frac{M^2}{HR^2} \ll \frac{M}{R}$. Since $s \geq 6$, the second term in the brackets is bounded by the first, and

$$\sum_{j=1}^{J} \tilde{E}_1(j) \ll \sqrt{\frac{R}{M}} \left( \frac{M}{R} \right)^{\frac{1}{2(s-1)}} \frac{M}{HR^2} R^{\frac{3}{2}} - \frac{1}{2(s-1)} = \sqrt{M} \left( \frac{M}{R^2} \right)^{\frac{1}{2(s-1)}}.$$  

(4.24)

Turning to the second error term in (4.21), we have

$$\sum_{j=1}^{J} \tilde{E}_2(j) \ll \frac{M^2 R^2}{H^2} \sum_{Q \leq R} \sum_{Q \leq v_j \leq 2Q} \left( \frac{v_j}{R} \right)^{\frac{1}{2}} \ll \frac{M^2 R^2}{H^2} \sum_{Q \leq R} \left( \frac{Q}{R} \right)^{\frac{1}{2}} \left( \frac{MQ^2}{HR^2} + 1 \right).$$

Again applying (4.23) with $c_1 = \frac{7}{2}$ and $c_1 = \frac{3}{2}$, we find that

$$\sum_{j=1}^{J} \tilde{E}_2(j) \ll \frac{M^2 R^2}{H^2} \left( \frac{M}{H} \right) \sum_{Q \leq R} \left( \frac{Q}{R} \right)^{\frac{7}{2}} + \sum_{Q \leq R} \left( \frac{Q}{R} \right)^{\frac{1}{2}} \ll \frac{M^2 R^2}{H^2} \left( \frac{M}{H} + 1 \right) \ll \frac{M^2 R^2}{H^3},$$  

(4.25)
since $H \ll \frac{M}{R} \ll M$. The third error term in (4.21) is
\[
\sum_{j=1}^{J} \bar{\xi}_3(j) \ll \frac{M^{\frac{1}{2}} R^2}{\sqrt{H}} \sum_{Q \leq R} \sum_{Q \leq v_j \leq 2Q} v_j^{-\frac{1}{2}} \ll \frac{M^{\frac{1}{2}} R^2}{\sqrt{H}} \sum_{Q \leq R} Q^{-\frac{1}{2}} \left( \frac{M^2 Q^2}{HR^2} + 1 \right).
\]
by (4.22). By (4.23) with $c_1 = \frac{3}{2}$ and $c_2 = -\frac{1}{2}$, we have
\[
(4.26) \quad \sum_{j=1}^{J} \bar{\xi}_3(j) \ll \frac{M^{\frac{1}{2}} R^2}{\sqrt{H}} \left( \frac{M}{HR^2} R^2 + 1 \right) = \frac{M^2 R}{H^2} \left( 1 + \frac{R H}{M} \right) \ll \frac{M^2 R}{H^2}.
\]
since $\frac{R^2}{M} \ll R^{-\frac{1}{2}} \ll 1$. Collecting the estimates in (4.24), (4.25), and (4.26), we find that
\[
(4.27) \quad \sum_{j=1}^{J} (\bar{\xi}_1(j) + \bar{\xi}_2(j) + \bar{\xi}_3(j)) \ll \sqrt{M} \left( \frac{M}{R^2} \right)^{\frac{1}{2(r+1)}} + \frac{M^2 R^2}{H^2} + \frac{M^2 R}{H^2}.
\]
Note that the third error term is dominated by the first two. To see this, note that if $MR \geq H^\frac{3}{2}$, then $\frac{M^2 R}{H^2} \leq \frac{M^2 R^2}{H^2}$; while if $MR \leq H^\frac{3}{2}$, then $\frac{M^2 R}{H^2} = \sqrt{M} \frac{MR}{H^\frac{3}{2}} \leq \sqrt{M} \ll \sqrt{M} (\frac{M}{R^2})^{\frac{1}{2(r+1)}}$.
Therefore the right-hand side of (4.27) is $O((\sqrt{M} (\frac{M}{R^2})^{\frac{1}{2(r+1)}} + \frac{M^2 R^2}{H^2}) (\log M_0)^{-\frac{1}{2}})$. Proposition 3.1 now follows by combining (4.20), (4.21), and (4.27).

5. Proof of Proposition 3.2

Define the functions
\[
g_{ijr}^{\pm}(\ell) = \frac{g_{ijr}^{\pm}(\ell/r)}{q_j}, \quad h_{ijr}^{\pm}(\ell) = \frac{\ell}{r} h_{ijr}^{\pm}(\ell/r),
\]
\[
H_{ijr}^{\pm}(\ell) = h_{ijr}^{\pm}(\ell) h_{ijr}^{\pm}(\ell), \quad \text{and} \quad W_{ijr}^{\pm}(\ell) = \omega_i(x_{ijr}^{\pm}(\ell/r)) \omega_j(x_{ijr}^{\pm}(\ell/r)),
\]
where we recall that $g_{ijr}^{\pm}, h_{ijr}^{\pm}, x_{ijr}^{\pm},$ and $\omega_j$ are given by (3.14), (3.15), (3.13), and (3.8), respectively. Note that $|W_{ijr}^{\pm}(\ell)| \leq 1$ and, since $x_{ijr}^{\pm}(\ell)$ is monotonic, $W_{ijr}^{\pm}(\ell)$ has bounded variation (over all of $\mathbb{R}$). Applying Cauchy’s inequality in the $\ell$ variable in (3.18) and then expanding out the resulting square, we have
\[
\left| \sum_{\ell=L_1}^{L_2} r \lambda(\ell) \frac{\nu(j) e(g_{ijr}^{\pm}(\ell)))}{q_j} h_{ijr}^{\pm}(\ell) \omega_j(x_{ijr}^{\pm}(\ell/r)) \right|^2 
\leq r^2 \sum_{\ell=L_1}^{L_2} \frac{\lambda(\ell)^2}{\ell^2} \sum_{i,j \in J(\beta, r)} \nu(i) \nu(j) \sum_{\ell=L_1}^{L_2} W_{ijr}^{\pm}(\ell) H_{ijr}^{\pm}(\ell) e(g_{ijr}^{\pm}(\ell) - g_{ijr}^{\pm}(\ell))
\leq \frac{r^2}{L^2} \sum_{\ell=L_1}^{L_2} \max_{j \in J(\beta, r)} |\nu(j)|^2 \cdot \sum_{i,j \in J(\beta, r)} \sum_{\ell=L_1}^{L_2} W_{ijr}^{\pm}(\ell) H_{ijr}^{\pm}(\ell) e(g_{ijr}^{\pm}(\ell) - g_{ijr}^{\pm}(\ell))
\cdot \left| \sum_{\ell=L_1}^{L_2} W_{ijr}^{\pm}(\ell) H_{ijr}^{\pm}(\ell) e(g_{ijr}^{\pm}(\ell) - g_{ijr}^{\pm}(\ell)) \right|.
\]
Let
\[
Y_{ijr}(L_1, L_2) = |W_{ijr}(L_1) H_{ijr}(L_1)| + \int_{L_1}^{L_2} \left| \frac{d}{dx} W_{ijr}(x) H_{ijr}(x) \right| dx.
\]
We now apply [12, Lemma 5.1.1] to see that
\[ \left| \sum_{\ell=L_1}^{L_2} W_{ijr}^\pm(\ell) H_{ijr}^\pm(\ell) e\left( g_{ir}^\pm(\ell) - g_{jr}^\pm(\ell) \right) \right| \leq Y_{ijr}^\pm(L_1, L_2) \max_{L'_1 \in [L_1, L_2]} \left| \sum_{\ell=L'_1}^{L_2} e\left( g_{ir}^\pm(\ell) - g_{jr}^\pm(\ell) \right) \right|. \]

We first estimate \( Y_{ijr}^\pm(L_1, L_2) \). For any function \( W \) of bounded variation, we have
\[
\int \left| \frac{d}{dx}(WH) \right| \, dx \ll \max |H| + \int |H'(x)| \, dx.
\]
Therefore
\[
Y_{ijr}^\pm(L_1, L_2) \ll \max_{\ell \in [L_1, L_2]} H_{ijr}^\pm(\ell) + \int_{L_1}^{L_2} \left| \frac{d}{dx} H_{ijr}^\pm(x) \right| \, dx,
\]
since \( H_{ijr}^\pm(\ell) \) is positive. We now estimate the right-hand side of this inequality. Observe that
\[
h_{jr}^\pm(\ell) = q_j \sqrt{\frac{\ell}{\eta L^2}} \left( \sqrt{\frac{\ell y_j}{\pi}} \pm \frac{\ell y_j}{\pi} \right) \pm 1 \right]^{-\frac{1}{2}}.
\]
Let \( y_j = \frac{d}{2r^2r_jq_j} \). Then
\[
\sqrt{\frac{\ell}{\eta L^2}} x_j^\pm(\ell/r) = \sqrt{(\ell y_j)^2 + \frac{\ell \ell y_j}{\pi} \pm \ell y_j} = \frac{\ell y_j}{\pi} \sqrt{y_j^2 + \frac{\ell y_j}{\pi \pm y_j}}.
\]
From the third formula, we see that this is an increasing function of \( \ell \) for \( \ell > 0 \) and thus \( h_{jr}^\pm(\ell) \) is an increasing function of \( \ell \) for \( \ell > 0 \), as well. Therefore \( Y_{ijr}^\pm(L_1, L_2) \ll H_{ijr}^\pm(L_2) \).

Since \( R = \sqrt{M/M_0} \) and \( M \ll \sqrt{C} \), by (3.1) and (3.5) we have \( M \ll tR \). This implies that
\[
\frac{d^2 L}{R^2} \ll \frac{rUVM^2}{tR^2} \ll rtUV
\]
so that \( \frac{\ell}{\pi \ell y_j} \gg 1 \) for all \( j \). Therefore
\[
\sqrt{\frac{\ell}{\eta L^2}} x_j^\pm(\ell/r) = \sqrt{(\ell y_j)^2 + \frac{\ell \ell y_j}{\pi} \pm \ell y_j} \approx \eta L
\]
and thus
\[
h_{jr}^\pm(\ell) \approx \frac{V}{d} \sqrt{\frac{(\eta L)^3}{t}}.
\]
This means that
\[
Y_{ijr}^\pm(L_1, L_2) \ll \frac{V^2}{d^2} (\eta L)^3 = \frac{\eta V L^2}{rU}.
\]
We now estimate
\[
\Sigma_{ijr}^\pm(L_1, L_2) := \max_{L_1 \leq L'_1 \leq L_2} \left| S_{ijr}^\pm(L'_1, L_2) \right|
\]
where
\[
S_{ijr}^\pm(L'_1, L_2) := \sum_{\ell=L'_1}^{L_2} e\left( g_{ir}^\pm(\ell/r) - g_{jr}^\pm(\ell/r) \right).
\]
In order to estimate the exponential sum $S_{ijr}(L_1, L_2)$, we use van der Corput first and second derivative estimates in the form of [12, Lemmas 5.1.2 and 5.1.3]. In particular, we need to study the derivatives of the functions $g_{jr}^\pm(\ell)$.

5.1. **First derivative estimate.** Note that

\[
\frac{d}{d\ell} g_{jr}^\pm(\ell) = \frac{1}{r} (g_{jr}^\pm)'(\ell/r) + \frac{ra_j}{q_j}.
\]

For the stationary point $x_j^\pm(\ell)$, we have

\[-\frac{t}{2\pi x_j^\pm(\ell)} + \frac{u_j}{v_j} + \frac{1}{q_j} \sqrt{\frac{\ell}{x_j^\pm(\ell)}} = 0,
\]

so that

\[
\frac{d}{d\ell} g_{jr}^\pm(\ell) = \left\{ -\frac{t}{2\pi x_j^\pm(\ell)} + \frac{u_j}{v_j} + \frac{1}{q_j} \sqrt{\frac{\ell}{x_j^\pm(\ell)}} \right\} \frac{dx_j^\pm(\ell)}{d\ell} + \frac{1}{q_j} \sqrt{\frac{x_j^\pm(\ell)}{\ell}} = +\frac{1}{q_j} \sqrt{\frac{x_j^\pm(\ell)}{\ell}}
\]

and

\[
(5.1) \quad \frac{d}{d\ell} g_{jr}^\pm(\ell) = \mp \frac{1}{rq_j} \sqrt{\frac{x_j^\pm(\ell/r)}{\ell/r}} + \frac{ra_j}{q_j}. \tag{5.1}
\]

5.2. **Second derivative estimate.** We have

\[
\frac{d^2}{d\ell^2} g_{jr}^\pm(\ell) = \mp \frac{1}{rq_j} \frac{d}{d\ell} \sqrt{\frac{x_j^\pm(\ell/r)}{\ell/r}}.
\]

Again writing $y_j = \frac{d}{2ru_jq_j}$, we have

\[
\frac{1}{rq_j} \sqrt{\frac{x_j^\pm(\ell/r)}{\ell/r}} = \frac{d}{2ru_jq_j} \left( \sqrt{1 + \frac{2trujq_j}{\pi d\ell}} \pm 1 \right) = y_j \left( \sqrt{1 + \frac{t}{\pi y_j \ell}} \pm 1 \right),
\]

and so

\[
\frac{d^2}{d\ell^2} g_{jr}^\pm(\ell) = \mp \frac{1}{rq_j} \frac{d}{d\ell} \sqrt{\frac{x_j^\pm(\ell/r)}{\ell/r}} = \pm \frac{t}{2\pi \ell^2} \left( 1 + \frac{t}{\pi ly_j} \right)^{-\frac{1}{2}}.
\]
This is clearly a monotonic function of \( y_j \). Therefore \((g^+_i - g^-_j)^n(\ell)\) is either identically zero, or it is never zero, so \((g^+_i - g^-_j)'(\ell)\) is monotone in \( \ell \). Hence
\[
\frac{d^2}{d\ell^2} [g^+_i(\ell) - g^-_j(\ell)] = \pm \frac{t}{2\pi \ell^2} \left\{ \frac{1}{\sqrt{1 + \frac{t}{\pi \ell y_i}} - \frac{1}{\sqrt{1 + \frac{t}{\pi \ell y_j}}} \right\}
\]
\[
= \pm \frac{t}{2\pi \ell^2} \left\{ \frac{t}{\pi \ell y_i} - \frac{t}{\pi \ell y_j} \right\} \left(1 + \frac{t}{\pi \ell y_i} + 1 + \frac{t}{\pi \ell y_j}\right)
\]
\[
= \pm \frac{rt^2}{\pi^2 \ell^3} \left\{ \frac{u_j q_j - u_i q_i}{1 + \frac{t}{\pi \ell y_i} + 1 + \frac{t}{\pi \ell y_j}} \right\}.
\]
As shown above, we have \( \frac{t}{\pi \ell y_j} \gg 1 \), so that
\[
\left| \frac{d^2}{d\ell^2} [g^+_i(\ell) - g^-_j(\ell)] \right| \leq \frac{rt^2}{L^3} \frac{|u_i q_i - u_j q_j|}{(rt/UV \ell)^{3/2}} \ll \eta \frac{|u_i v_i - u_j v_j|}{L UV}.
\]

5.3. Applying van der Corput estimates. Our strategy is to use both van der Corput first and second derivative estimates. If neither of these estimates is small, then this implies constraints on the sizes of \( |u_i v_i - u_j v_j| \) and \( \|rt/v_i - rt/v_j\| \). This leads to the counting problem given in Proposition 3.2. To this end, define
\[
N(X) = \# \{(i, j) \in J(\beta, r)^2 : (u_i, v_i), (u_j, v_j) \in \mathcal{R} \text{ and } \Sigma^\pm_{ijr}(L_1, L_2) \geq X\}.
\]
Trivially \( \Sigma^\pm_{ijr}(L_1, L_2) \leq L \), so we have
\[
\sum_{(i, j) \in J(\beta, r)^2} \Sigma^\pm_{ijr}(L_1, L_2) = - \int_0^L X \, dN(X) = \int_0^L N(X) \, dX
\]
\[
\leq X_1 (\# \mathcal{R})^2 + \int_{X_1}^L N(X) \, dX,
\]
where we take \( X_1 := A \sqrt{L} \max(\sqrt{\eta}, 1) \) for a sufficiently large constant \( A \).

Suppose that \( \Sigma^\pm_{ijr}(L_1, L_2) \geq X \geq X_1 \). If \( u_i v_i - u_j v_j \neq 0 \) then we get a bound for \( \Sigma^\pm_{ijr} \) by the second derivative test. In particular, by [12, Lemma 5.1.3], we have
\[
X \leq \Sigma^\pm_{ijr}(L_1, L_2) \ll L \sqrt{\lambda_2} + \frac{1}{\sqrt{\lambda_2}} \ll \sqrt{\eta L} + \frac{1}{\sqrt{\lambda_2}} \leq \frac{X}{A} + \frac{1}{\sqrt{\lambda_2}},
\]
where \( \lambda_2 = \frac{\eta |u_i v_i - u_j v_j|}{UV} \). If \( A \) is large enough, then this implies that \( 1/\sqrt{\lambda_2} \gg X \) so therefore \( \lambda_2 \ll X^{-2} \). Thus, we have
\[
|u_i v_i - u_j v_j| \ll UV \frac{L}{\eta X^2},
\]
and obviously this holds also when \( u_i v_i - u_j v_j = 0 \).
Next we apply a first derivative estimate. Let \( z_{ijr} = \frac{\alpha r}{q_r} - \frac{\alpha r}{q_r} \). Then, with \( y_j \) as above, from (5.1) we derive that

\[
\left| \frac{d}{d\ell} \left[ g_{ir}^+(\ell) - g_{jr}^-(\ell) \right] - z_{ijr} \right| = \left| \left( \sqrt{y_i^2 + \frac{t y_i}{\pi \ell}} \pm y_i \right) - \left( \sqrt{y_j^2 + \frac{t y_j}{\pi \ell}} \pm y_j \right) \right|
\]

(5.4)

\[
= |y_i - y_j| \left| \frac{y_i + y_j + \frac{t}{\pi \ell} \pm \sqrt{y_i^2 + \frac{t y_i}{\pi \ell}} + \sqrt{y_j^2 + \frac{t y_j}{\pi \ell}}} \pm 1 \right|
\]

\[
\leq |y_i - y_j| \sqrt{\frac{rUVT}{d^2 L}} \leq \eta \frac{|u_i v_i - u_j v_j|}{UV}.
\]

Therefore, using the upper bound for \( |u_i v_i - u_j v_j| \) in (5.3), we have

\[
\frac{d}{d\ell} \left[ g_{ir}^+(\ell) - g_{jr}^-(\ell) \right] - z_{ijr} \ll \frac{L}{X^2} \leq \frac{L}{X_1^2} \ll 1.
\]

By (possibly) increasing the size of the constant \( A \) in the definition of \( X_1 \), we see that there exist numbers \( \mu \) and \( \nu \) such that \( \nu - \mu < \frac{1}{2} \) and \( \mu \leq \frac{d}{d\ell} \left[ g_{ir}^+(\ell) - g_{jr}^-(\ell) \right] \leq \nu \). Now we use the truncated Poisson summation formula [12, Lemma 5.4.3], which states that

\[
\sum_{\ell = L_1}^{L_2} e\left( g_{ir}^+(\ell) - g_{jr}^-(\ell) \right) = \sum_{\mu - \frac{1}{4} \leq n \leq \nu + \frac{1}{4}} \int_{L_1}^{L_2} e\left( g_{ir}^+(x) - g_{jr}^-(x) - nx \right) dx + O(1).
\]

Note that the sum on the right-hand side contains at most one term. Hence

\[
\Sigma_{ijr}^\pm(L_1, L_2) \leq O(1) + \max_{L_1 \in [L_1, L_2]} \left| \int_{L_1}^{L_2} e\left( g_{ir}^+(x) - g_{jr}^-(x) - nx \right) dx \right|
\]

\[
= O(1) + \max_{L_1 \in [L_1, L_2]} \left| \int_{L_1}^{L_2} e\left( g_{ir}^+(x) - g_{jr}^-(x) - nx \right) dx \right|.
\]

Since \( z_{ijr} \) is only defined modulo 1, we are free to shift \( g_{ir}^+(\ell) - g_{jr}^-(\ell) \) by any integer multiple of \( \ell \). Thus, if there is an integer \( n \in [\mu - \frac{1}{4}, \nu + \frac{1}{4}] \), we may assume that \( n = 0 \). Therefore,

\[
\Sigma_{ijr}^\pm(L_1, L_2) \leq \max_{L_1 \in [L_1, L_2]} \left| \int_{L_1}^{L_2} e\left( g_{ir}^+(x) - g_{jr}^-(x) \right) dx \right| + O(1).
\]

Define

\[
\lambda_1 = \min_{\ell \in [L_1, L_2]} \left| \frac{d}{d\ell} \left[ g_{ir}^+(\ell) - g_{jr}^-(\ell) \right] \right|.
\]

Then by [12, Lemma 5.1.2] we have

\[
\max_{L_1 \in [L_1, L_2]} \left| \int_{L_1}^{L_2} e\left( g_{ir}^+(x) - g_{jr}^-(x) \right) dx \right| \ll \frac{1}{\lambda_1}.
\]
This implies that \( X \leq \Sigma_{ijr}^\pm(L_1,L_2) \ll \frac{1}{X^2} + 1 \), so that \( \lambda_1 \ll \frac{1}{X} \) if the constant \( A \) is sufficiently large. Thus, by (5.4), we find that
\[
\|z_{ijr}\| \leq |z_{ijr}| = \lambda_1 + O\left( \frac{\eta|u_iv_i - u_jv_j|}{UV} \right) \ll \frac{1}{X} + \frac{L}{X^2} \ll \frac{L}{X^2},
\]
since \( X \leq L \).

In summary, we have found that \( \Sigma_{ijr}^\pm(L_1,L_2) \geq X \geq X_1 \) implies the inequalities (5.3) and (5.5). In other words, \( N(X) \leq B(\Delta_1(X),\Delta_2(X)) \) for certain functions \( \Delta_1(X) \) and \( \Delta_2(X) \) satisfying the conditions in (3.19). From (5.2), we derive that
\[
\sum_{(i,j) \in J(\beta,r)^2} \Sigma_{ijr}^\pm(L_1,L_2) \leq X_1(\#\mathcal{R})^2 + \int_{X_1}^L B(\Delta_1(X),\Delta_2(X)) \, dX.
\]
By extending the definitions of \( \Delta_1(X) \) and \( \Delta_2(X) \) to be zero for \( X < X_1 \), we note that the right-hand side of this expression is
\[
\ll X_0(\#\mathcal{R})^2 + \int_{X_0}^L B(\Delta_1(X),\Delta_2(X)) \, dX
\]
for \( X_0 \) defined in Proposition 3.2 so long as \( A \geq 1 \). The proposition now follows.

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