Phase diagram of a Bose gas near a wide Feshbach resonance

Lan Yin

School of Physics, Peking University, Beijing 100871, China

Abstract

In this paper, we study the phase diagram of a homogeneous Bose gas with a repulsive interaction near a wide Feshbach resonance at zero temperature. The Bose-Einstein-condensation (BEC) state of atoms is a metastable state. When the scattering length $a$ exceeds a critical value depending on the atom density $n$, $na^3 > 0.035$, the molecular excitation energy is imaginary and the atomic BEC state is dynamically unstable against molecule formation. The BEC state of diatomic molecules has lower energy, where the atomic excitation is gapped and the molecular excitation is gapless. However when the scattering length is above another critical value, $na^3 > 0.0164$, the molecular BEC state becomes a unstable coherent mixture of atoms and molecules. In both BEC states, the binding energy of diatomic molecules is reduced due to the many-body effect.

*Electronic address: yinlan@pku.edu.cn
I. INTRODUCTION

Bose gases with Feshbach resonances have shown very interesting properties. The particle loss rate due to the three-body recombination increases enormously near a Feshbach resonance \[1\]. A sudden change in the magnetic field generates oscillations between atoms and diatomic molecules \[2\]. Diatomic molecules can be produced when the magnetic field is either tuned through the resonance \[3\] or oscillates with a frequency corresponding to the molecular binding energy \[4\]. Recently observed Efimov effect \[5\] may help to control the particle loss for more extensive studies near a Feshbach resonance.

The atomic BEC state with strong interactions was explored in a variational approach \[6\]. At resonance the energy per atom was found to be proportional to \( n^{2/3} \) \[6\]. A transition between atomic and molecular BEC states at the resonance was proposed \[7, 8\]. However, these states were found unstable in some regions due to negative compressibility \[9\]. Recently, a transition between the molecular BEC state and a coherent mixture state of atoms and molecules was proposed \[10\].

In a Bose gas with a Feshbach resonance, scattering states in the open channel are coupled to bound states in the closed channel. Eigenstates of diatomic molecules are superpositions of both open and closed channel states. Near a wide resonance where the effective range of the interaction is very small, the magnitude of the close-channel component is much less than that of the open-channel component, which can be seen in the renormalization of the molecular propagator \[11\]. Thus an effective model of open-channel atoms should be able to describe the system with a wide resonance. A single-channel model has been widely used to describe both normal and BEC states of Bose gases \[13\], given by

\[
H = -\frac{\hbar^2}{2m} \nabla^2 \psi + \frac{g}{2} \psi^\dagger \psi^\dagger \psi \psi, \tag{1}
\]

where \( \psi \) is atomic field operator, \( g \) is the coupling constant, \( g \equiv 4\pi \hbar^2 a/m \), and \( a \) is the scattering length. When the scattering length is negative, \( a < 0 \), the system is unstable and subject to mechanical collapse at low temperatures \[14, 15, 16\]. Therefore in the following we focus on the region with repulsive interactions, \( a > 0 \).

A crucial question about the single-channel model is whether or not it is capable of describing diatomic molecules near the resonance. The answer of this question is positive, because the Hamiltonian given by Eq. (1) has bound eigenstates of two atoms at \( a > 0 \).
FIG. 1: The phase diagram of a Bose gas near a wide Feshbach resonance at zero temperature.

The atomic BEC state (ABEC) is metastable with $0 < \mu < 0.52E_a$. The molecular BEC (MBEC) state is with $-0.5E_a < \mu < -0.21E_a$.

These bound states are the eigenstates of diatomic molecules, given by

$$|\phi_p\rangle = \frac{1}{\sqrt{V}} \sum_q \frac{N}{E_a + 2\epsilon_q} \psi_q^\dagger \psi_{q+p/2}^\dagger |0\rangle,$$

where $V$ is the volume, $N$ is a normalization constant, $\epsilon_q \equiv \hbar^2 q^2 / (2m)$, and the molecular binding energy in vacuum is given by $E_a = \hbar^2 / (ma^2)$. The energy eigenvalue of the molecule is given by $-E_a + \epsilon_a/2$, consistent with the energy of shallow bound states in the quantum scattering theory \[17\]. It is important to note that the single-channel model is an effective field-theory model and ultraviolet divergences in calculations need to be removed by the correct renormalization scheme \[18\].

In this paper, the phase diagram of a homogeneous Bose gas near a wide Feshbach resonance is obtained at zero temperature, as shown in Fig. 1. The atomic BEC state is a metastable state when the chemical potential $\mu$ is between 0 and $0.52E_a$, corresponding to $0 < na^3 < 0.035$. The atomic BEC state is unstable against molecule formation when $na^3 > 0.035$. The molecular BEC state is stable when $-0.21E_a > \mu > -0.5E_a$, or $0.0164 > na^3 > 0$. There is no solution when $-0.21E_a < \mu < 0$. The coherent mixture state of atoms and molecules is subject to mechanical collapse, although $\mu$ can be smaller than $-0.5E_a$ at very high densities. In both BEC states, the molecular binding energy is smaller than $E_a$ due to interaction. This phase diagram is valid when $a$ is much larger than the effective range.
II. THE ATOMIC BEC STATE

Bose atoms condense below the BEC transition temperature. The atomic BEC state in the dilute case is well described by Bogoliubov’s theory in which the atomic field operator has a finite expectation value, \( \langle \psi \rangle = \psi_0 \). In the grand canonical ensemble, the grand thermodynamic potential \( F \equiv H - \mu \psi^\dagger \psi \) has a constant part \( F_0 = gn_0^2/2 - \mu n_0 \), where \( \mu \) is the chemical potential and \( n_0 \equiv |\psi_0|^2 \). The saddle-point condition \( \delta F_0 / \delta \psi_0 \) yields \( \mu = gn_0 \).

For convenience \( \psi_0 \) can be chosen to be positive. The quadratic part of the grand potential describes the gaussian fluctuation in the atomic field \( \delta \psi \equiv \psi - \psi_0 \),

\[
F_2 = -\frac{\hbar^2}{2m} \delta \psi^\dagger \nabla^2 \delta \psi + gn_0 \delta \psi^\dagger \delta \psi + \frac{1}{2} gn_0 (\delta \psi^\dagger \delta \psi^\dagger + \text{h.c.}).
\]  

(3)

By Bogoliubov transformation, the quadratic grand potential can be diagonalized,

\[
F_2 = C + \sum_k E_k c_k^\dagger c_k,
\]

where the quasi-particle energy is given by \( E_k = \sqrt{\epsilon_k (\epsilon_k + 2 gn_0)} \), \( \epsilon_k \equiv \hbar^2 k^2 / (2m) \), the quasi-particle operator is given by \( c_k \equiv u_k \psi_k - v_k \psi_k^\dagger \), \( u_k^2 = [1 + (\epsilon_k + gn_0)/E_k]/2 \), and \( v_k^2 = u_k^2 - 1 \). The quantum depletion, the vacuum of quasi-particles, contributes to the ground-state energy by \( C = 8 gn_0 (mgn_0/\hbar^2)^{3/2} / (15 \pi^2) \).

The higher order terms in the grand potential,

\[
g \delta \psi^\dagger \delta \psi[(\psi_0 \delta \psi + \text{h.c.}) + \delta \psi^\dagger \delta \psi/2],
\]

are neglected in Bogoliubov’s approximation. In Popov’s approximation [12], the mean-field terms of non-condensed atoms are added in the particle-hole channel, and the chemical potential is shifted, \( \mu = g(n_0 + 2 \delta n) \), where \( \delta n = 8n_0 \sqrt{n_0 a^3 / \pi} / 3 \) is the atom density of the quantum depletion. In the diagrammatic approach in the traditional theory of a dilute Bose gas [13], thermodynamical properties can be calculated by perturbation in the order of \( \sqrt{na^3} \) beyond Bogoliubov’s approximation.

The traditional theory is accurate in the dilute case, where the perturbation can be stopped at a sufficient order of \( \sqrt{na^3} \) without losing much accuracy. However near the resonance, there are strong fluctuations around the condensate. In addition to single-atom excitations, collective excitations must be considered, such as excitations of diatomic molecules. The dispersion of molecular excitations can be obtained from the poles of the two-particle correlation function of non-condensed atoms. In the atomic BEC state, the particle-particle
channel and the particle-hole channel are coupled, and the correlation function is a $3 \times 3$ matrix given by
\[
\chi_{\alpha\beta}(r - r', t - t') = -\frac{i}{\hbar} \langle T [b_{\alpha}(r, t)b_{\beta}^\dagger(r', t')] \rangle,
\] (5)
where $b_1 = \psi^2$, $b_2 = b_1^\dagger$, $b_3 = 2\psi^\dagger\psi$, and $T$ is the time-ordering operator.

Due to the strong interaction near the resonance, any perturbation theory truncated at any finite order of $\sqrt{na^3}$ will fail. In the following, we adopt the Random Phase Approximation (RPA) which focuses on the renormalization to the two-body interaction. In RPA, there are Feynman diagrams from all orders of perturbation and the result can often be applied to the strong-interaction region with correct qualitative features \[13\].

In RPA, the correlation function is given by
\[
\chi(k, \omega) = [1 - g\chi^{(0)}(k, \omega)]^{-1}\chi^{(0)}(k, \omega).
\] (6)
The function $\chi^{(0)}(k, \omega)$ is the correlation function calculated in Bogoliubov's approximation, given by
\[
\begin{align*}
\chi^{(0)}_{11}(k, \omega) &= \int \frac{d^3k'}{(2\pi)^3} \left( \frac{u_{k'}^2 v_{|k-k'|}^2}{A(k, k', \omega)} + \frac{v_{k'}^2 v_{|k-k'|}^2}{A(k, k', -\omega)} \right), \\
\chi^{(0)}_{12}(k, \omega) &= 2\int \frac{d^3k'}{(2\pi)^3} \frac{u_{k'} v_{|k-k'|} v_{|k-k'|}}{B(k, k', \omega)}, \\
\chi^{(0)}_{13}(k, \omega) &= 2\int \frac{d^3k'}{(2\pi)^3} \left( \frac{u_{k'} v_{|k-k'|} v_{|k-k'|}}{A(k, k', \omega)} + \frac{u_{k'} v_{|k-k'|} v_{|k-k'|}}{A(k, k', -\omega)} \right), \\
\chi^{(0)}_{33}(k, \omega) &= 2\int \frac{d^3k'}{(2\pi)^3} \left( \frac{u_{k'} v_{|k-k'|} + v_{k'} u_{|k-k'|}}{B(k, k', \omega)} \right)^2,
\end{align*}
\] (7)
\[
\chi^{(0)}_{22}(k, \omega) = \chi^{(0)}_{11}(-k, -\omega), \quad \chi^{(0)}_{23}(k, \omega) = \chi^{(0)}_{13}(-k, -\omega), \quad \text{and} \quad \chi^{(0)}_{ij}(k, \omega) = \chi^{(0)}_{ji}(k, \omega) \text{ for } j \neq i.
\]
where $A(k, k', \omega) \equiv \hbar\omega - E_k - E_{|k-k'|} + i\delta$ and $1/B(k, k', \omega) \equiv 1/A(k, k', \omega) + 1/A(k, k', -\omega)$.

The dispersion of diatomic molecules can be obtained from the pole of the correlation function given in Eq. (6),
\[
\det |I - g\chi^{(0)}(k, \omega)| = 0,
\] (8)
where $I$ is the identity matrix. The real part of the molecular excitation energy at $k = 0$ is the negative of the binding energy which is plotted in Fig. 2. When the interaction is weak, $na^3 \ll 1$, the molecular binding energy is approximately given by $E_a - 2gn_0$, which recovers the vacuum result in the dilute limit. The size of the bound state is of the order of the scattering length $a$. When the wavevector $k$ is of the order of $1/a$ or larger,
The atomic-excitation energy is approximately given by $\epsilon_k + gn_0$. This energy shift $gn_0$ in atomic excitation energy at high momentum causes the reduction in the molecular binding energy by $-2gn_0$.

When the interaction is stronger, the imaginary part of the molecular excitation energy becomes noneligible. At the critical value of $na^3 = 0.035$ or $\mu = 0.52E_a$, the real part of the molecular energy vanishes and only the imaginary part is finite, implying that beyond this point the system is totally unstable against molecule formation. This instability is a many-body effect, due to the process described in RPA in which two atoms from two pairs in the quantum depletion form a molecule leaving the rest two atoms in the two pairs excited. Such process is absent in the normal state where there is no off-diagonal correlation. In contrast, the particle loss due to the three-body recombination is a few-body effect which is present in both BEC and normal states.

III. THE MOLECULAR BEC STATE

When the scattering length is positive, $a > 0$, diatomic molecules have lower energy than atoms, and can condense into a molecular BEC state at low temperatures. In the single-channel model, the molecular condensation can be described by the off-diagonal long range
order $\Delta \equiv g\langle \psi \psi \rangle \neq 0$. For simplicity, we assume $\Delta > 0$. The mean-field grand potential describing the molecular BEC state is given by

$$F_m = F_m^{(0)} - \frac{\hbar^2}{2m} \psi \nabla^2 \psi + (2gn - \mu)\psi\psi + \frac{\Delta}{2} (\psi\psi + \psi\psi),$$

(9)

where $F_m^{(0)} = -(gn^2 + \Delta^2/2g)$. This grand potential $F_m$ can be diagonalized by Bogoliubov transformation,

$$F_m = F_m^{(0)} + \sum_k E_k c_k^{\dagger} c_k,$$

(10)

where $E_k = \sqrt{(\epsilon_k + 2gn - \mu)^2 - \Delta^2}$, $c_k = u_k \psi_k - v_k \psi_k^{\dagger}$, $u_k^2 = [1 + (\epsilon_k + 2gn - \mu)/E_k]/2$, $v_k^2 = u_k^2 - 1$, and the constant $F_m^{(0)}$ is the ground state energy

$$F_m^{(0)} = \sum_k \frac{1}{2} [E_k - (\epsilon_k + 2gn - \mu) + \frac{\Delta^2}{2E_k}] - gn.$$

The parameter $\Delta$ can be determined self-consistently,

$$\Delta = g \int u_k v_k \frac{d^3k}{(2\pi)^3},$$

or

$$\frac{1}{g} = \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1}{2E_k} + \frac{1}{2\epsilon_k} \right],$$

(11)

where the last term in the integrand $1/(2\epsilon_k)$ is a counter term. The chemical potential $\mu$ and parameter $\Delta$ can be solved from Eq. (11) and the equation for the density $n$,

$$n = \int \frac{d^3k}{(2\pi)^3} v_k^2 = \int \frac{d^3k}{(2\pi)^3} \frac{\epsilon_k + 2gn - \mu}{2E_k} - 1.$$  

(12)

There is a gap in the atomic excitation given by $E_0 = \sqrt{(2gn - \mu)^2 - \Delta^2}$, consistent with the fact that atoms have higher energy than molecules. The molecular binding energy in the molecular BEC state is given by $2E_0$, equal to the energy difference between two atomic excitations and a molecular excitation at $k = 0$. The binding energy is plotted in Fig. 2. In the dilute limit where $na^3 \ll 1$, the gap $E_0$ is approximately equal to $E_a/2$. However at the critical value of $na^3 = 0.0164$, the gap vanishes. When $na^3 > 0.0164$, there is no solution, indicating that the molecular BEC state no longer exists.

Similar to the case in the atomic BEC state, the dispersion of molecular excitations in the molecular BEC state can be obtained from the poles of the two-particle correlation function $\chi$. In RPA, the pole is given by Eq. (8), and the mean-field correlation function $\chi^{(0)}$ is given by Eq. (7), with the coefficients, $u_k$ and $v_k$, and the excitation energy $E_k$ replaced with
the values in the molecular BEC state. When \( k = 0 \) and \( \omega = 0 \), Eq. (8) is automatically satisfied following Eq. (7, 11), showing that the molecular excitation is gapless. At small \( k \) and \( \omega \), to the leading order of \( k \), the molecular excitation frequency is linearly dispersed, \( \omega_k \approx v_m k \), where \( v_m \) is the molecule velocity.

The molecule velocity is plotted in Fig. 3. In the dilute limit when \( na^3 \ll 1 \), the molecule velocity is approximately given by

\[
v_0 = \frac{\hbar}{m} \sqrt{3\pi na}.
\]  

(13)

If Eq. (13) is compared to the phonon velocity \( v_p = \hbar\sqrt{4\pi na/m} \) in the atomic BEC state, the naive estimation of the molecule-molecule scattering length is \( 6a \), larger than the result from solving the four-body problem [19]. This reason for this discrepancy might be that in the dilute limit the mean-field density given by Eq. (12) is smaller than the true density due to fluctuations, similar to that in the molecular BEC state of the Fermi gas [20]. When the interaction is stronger, at the critical value of \( na^3 = 0.0164 \) or \( \mu = -0.21E_a \), the molecule velocity \( v_m \) drops to zero, which is very similar to softening of phonon modes in solids near structural phase transitions. The vanish of both the atomic excitation gap \( E_0 \) and molecule velocity \( v_m \) at \( na^3 = 0.0164 \) indicates that beyond this critical point the molecular BEC state does no exist.
IV. THE COHERENT MIXTURE OF ATOMS AND MOLECULES

In both BEC states, the molecular binding energy is reduced by the many-body interaction. In the region beyond these BEC states, the system may contain both atoms and molecules. Here we consider a coherent mixture state of the atomic condensation with $\psi_0 \equiv \langle \psi \rangle$, and molecular condensation with $\Delta \equiv g\langle \delta \psi \delta \psi \rangle$, where $\delta \psi \equiv \psi - \psi_0$. In this mixture state, the two parameters $\psi_0$ and $\Delta$ are in principle independent, whereas in the atomic BEC state the parameter $\Delta$ is a function of $\psi_0$.

In the mean-field approximation, the grand potential is given by

$$ F_{\text{mx}} = F_{\text{mx}}^{(0)} - \frac{\hbar^2}{2m} \delta \psi^\dagger \nabla^2 \delta \psi + (2gn - \mu)\delta \psi^\dagger \delta \psi + \frac{1}{2}[(\Delta + g\psi_0^2)\delta \psi^\dagger \delta \psi + \text{h.c}], $$

where $F_{\text{mx}}^{(0)} = -[g\delta n^2 + n_0(\mu - gn_0/2) + |\Delta|^2/(2g)]$, $n_0 \equiv |\psi_0|^2$, $\delta n \equiv \langle \delta \psi^\dagger \delta \psi \rangle$, and $n = n_0 + \delta n$.

Its mean-field expectation value, $\bar{F}_{\text{mx}} \equiv \langle F_{\text{mx}} \rangle$, is given by

$$ \bar{F}_{\text{mx}} = \frac{g}{2} |\psi_0^2 + \Delta| \frac{2}{g} + 2gn_0\delta n + \frac{g}{2} \delta n^2 - \mu(\delta n + n_0). $$

The parameter $\psi_0$ should minimize the grand potential, $\partial F_{\text{mx}}/\partial \psi_0^* = 0$, which yields the saddle-point equation

$$ \mu = g(n_0 + 2\delta n) + \frac{\Delta \psi_0^*}{\psi_0}. $$

Apparently, the product $\Delta \psi_0^2$ is real. For simplicity, we choose $\Delta$ to be positive, which also means that $\psi_0^2$ is real.

The mean-field grand potential can be diagonalized by Bogoliubov transformation, where the field operator of the quasi-particles are given by $c_k = u_k \psi_k - v_k \psi_k^\dagger$, the coefficients are given by $u_k^2 = [1 + (\epsilon_k + 2gn - \mu)/E_k]/2$ and $v_k^2 = u_k^2 - 1$, and the quasi-particle energy is given by $E_k = \sqrt{(\epsilon_k + 2gn - \mu)^2 - (\Delta + g\psi_0^2)^2}$. The parameter $\Delta$ can be determined from the self-consistency equation,

$$ \Delta = g \int \frac{d^3k}{(2\pi)^3} u_k v_k, $$

i. e.

$$ \frac{1}{\Delta + g\psi_0^2} = \int \frac{d^3k}{(2\pi)^3} \left[ -\frac{1}{2E_k} + \frac{1}{2\epsilon_k} \right]. $$

The expression of the total density is now given by

$$ n = n_0 + \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} (\frac{\epsilon_k + 2gn - \mu}{E_k} - 1). $$
FIG. 4: The chemical potential $\mu$ versus $n^{1/3}a$. The solid line is $\mu$ in the atomic BEC state in Popov’s approximation \[12\] for $na^3 < 0.035$. The dashed line is $\mu$ in the molecular BEC state which exists at $na^3 < 0.0164$. The dotted line is $\mu$ in the mixture state.

For a fixed total density $n$, both $\Delta$ and $\psi_0$ can be solved from Eq. \[16, 17, 18\]. In the solution, $\Delta$ and $\psi_0^2$ have opposite signs, whereas in the atomic BEC state $\Delta$ and $\psi_0^2$ have the same sign. At the transition point to the molecular BEC state, the atomic condensation density $n_0$ is zero. The solution shows a reentrant behavior in a tiny region with $0.0156 < na^3 < 0.0164$. When $na^3 > 0.0164$, the solution is unique. However, in this region as shown in Fig. 4, the chemical potential $\mu$ decreases with the increase in density $n$, which means that the compressibility $\partial \mu / \partial n$ is negative and the mixture state is subject to mechanical collapse. This instability is very similar to the instability of Bose gases with attractive interactions \[14, 15, 16\]. In a trap, the mixture state may be stabilized by the finite-size effect under certain conditions, as discovered in trapped Bose gases with attractive interactions \[21\].

**Conclusion and Discussions.** The phase diagram of a homogeneous Bose gas near a wide Feshbach resonance is studied at zero temperature. In the atomic BEC state, the real part of the molecular excitation energy vanish at $na^3 = 0.035$, implying the instability against molecular formation for stronger interactions. In the molecular BEC state, the atomic excitation energy is gapped and the molecular excitation energy is linearly dispersed. Both the gap and molecular velocity vanish at $na^3 = 0.0164$, above which the molecular BEC state no longer exists. In both BEC states molecular binding energies are reduced by the many-body interaction. In the coherent mixture state of atoms and molecules, at $na^3 > 0.0164$, the compressibility is found to be negative, indicating that the mixture is
subject to mechanical collapse, similar to Bose gases with attractive interactions.

It is an open question whether or not there are more exotic states in regions beyond the BEC states. So far three-body interactions are ignored in the BEC states. Although they are crucial to the dynamical properties such as the particle-loss rate and the stability time, the quasi-equilibrium properties is unlikely affected. However triatomic Effimov molecules may form near the resonance due to three-body interactions. Whether or not a gas of Effimov triatomic molecules can exist is at present unknown.

Acknowledgement. We would like to thank D. J. Thouless, T.-L. Ho, and H. Zhai for helpful discussions. This work is supported by NSFC under Grant No. 90303008 and 10674007, and by Chinese MOST under grant number 2006CB921401.

[1] J. Stenger et al., Phys. Rev. Lett. 82, 2422 (1999).
[2] E. A. Donley et al., Nature 417, 529 (2002).
[3] K. Xu et al., Phys. Rev. Lett. 91, 210402 (2003).
[4] S. T. Thompson, E. Hodby, and C. E. Wieman, Phys. Rev. Lett. 95, 190404 (2005).
[5] T. Kraemer et al., Nature 440, 315 (2006).
[6] S. Cowell et al., Phys. Rev. Lett. 88, 210403 (2002).
[7] L. Radzihovsky, J. Park, and P. B. Weichman, Phys. Rev. Lett. 92, 160402 (2004).
[8] M. W. J. Romans, R. A. Duine, S. Sachdev, and H. T. C. Stoof, Phys. Rev. Lett. 93, 020405 (2004).
[9] S. Basu and E. J. Mueller, cond-mat/0507460 (2005).
[10] E. Braaten and D. Zhang, cond-mat/0703308 (2007).
[11] L. Yin and Z.-H. Ning, Phys. Rev. A 68, 033608 (2003); Y. Zhang and L. Yin, Phys. Rev. A 72, 043607 (2005).
[12] V. N. Popov, *Functional integrals and collective excitations* (Cambridge University, Cambridge, 1987).
[13] For example, A. L. Fetter and J. D. Walecka, *Quantum theory of many-particle systems* (McGraw-Hill, San Francisco, 1971); D. Pines, *The many-body problem; a lecture note and reprint volume* (New York, W.A. Benjamin, 1961).
[14] H. T. C. Stoof, Phys. Rev. A 49, 3824 (1994).
[15] E. J. Mueller and G. Baym, Phys. Rev. A 62, 053605 (2000).
[16] G. S. Jeon, L. Yin, S. W. Rhee, and D. J. Thouless, Phys. Rev. A 66, 011603(R) (2002).
[17] J. J. Sakurai and S. F. Tuan, Modern quantum mechanics (Reading, Addison-Wesley, 1994.), chapter 7.
[18] E. Braaten and A. Nieto, Phys. Rev. B 55, 8090 (1997).
[19] D. S. Petrov, C. Salomon, and G. V. Shlyapnikov, Phys. Rev. Lett. 93, 090404 (2004).
[20] P. Nozières and S. Schmitt-Rink, J. Low Temp. Phys. 59, 195 (1985); C. A. R. Sá de Melo, M. Randeria, and J. R. Engelbrecht, Phys. Rev. Lett. 71, 3202 (1993).
[21] P. A. Ruprecht, M. J. Holland, K. Burnett, and M. Edwards, Phys. Rev. A 51, 4704 (1995).