Entanglement of graph states of spin system with Ising interaction and its quantifying on IBM’s quantum computer

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Abstract

We consider graph states generated by operator of evolution with Ising Hamiltonian. It is found that the geometric measure of entanglement of the graph state is related with degree of vertex in the corresponding graph. The graph states of spin system with Ising interaction are prepared and their entanglement is quantified on the 5-qubit IBM’s quantum computer, IBM Q Valencia.

Key words: geometric measure of entanglement; quantum computer; graph states; Ising interaction

1 Introduction

Entanglement is a critical resource in quantum communications and quantum computing (see, for instance, [1] [2] [3] [4] [5] and references therein). Its calculation plays important role in quantum information. Therefore much attention has been devoted to studies of entanglement of quantum states and its detecting on quantum computers [7] [8] [9] [10] [11] [12] [13] [14] [15] [16] [17] [18] [19].

The geometric measure of entanglement is defined as minimal squared Fubiny-Study distance between an entangled state $|\psi\rangle$ and a set of separable pure states $|\psi_s\rangle E(|\psi\rangle) = \min_{|\psi_s\rangle} (1 - |\langle\psi|\psi_s\rangle|^2)$. This measure of entanglement was proposed by Shimony [17].

In the paper [12] it was found that the geometric measure of entanglement of a spin with a quantum system in pure state

$$|\psi\rangle = a |\uparrow\rangle |\Phi_1\rangle + b |\downarrow\rangle |\Phi_2\rangle,$$

(here $a$, $b$ are real and positive constants, $|\Phi_1\rangle$, $|\Phi_2\rangle$ are arbitrary state vectors of a quantum system with norm equals to one, $\langle\Phi_i|\Phi_i\rangle = 1$, $i = 1, 2$) is related with the mean value of the spin. Namely, the entanglement of a spin-1/2 with a quantum system in pure state [1] reads

$$E(|\psi\rangle) = \frac{1}{2} (1 - |\langle\sigma\rangle|),$$

where $|\langle\sigma\rangle| = \sqrt{\langle\sigma\rangle}$, the components $\sigma^x$, $\sigma^y$, $\sigma^z$ of $\sigma$ are Pauli matrixes. Therefore, measuring the mean value of spin one can detect the geometric measure of entanglement.

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Much attention has been devoted to studies of graph states ([14, 15, 16, 17, 18, 19]). Well studied are graph states generated by 2-qubit controlled-Z operator (see, for instance, [16, 17, 18, 19]). In [16] the graph states that corresponds to rings were prepared. It was shown that 16-qubit IBM’s quantum computer (ibmqx5) can be fully entangled. The authors of paper [17] prepared graph state on the 20-qubit quantum device IBM Q Poughkeepsie and examined its entanglement.

In the present paper we consider graph states of spin systems generated by operator of evolution with Ising Hamiltonian. Such generation of the graph states opens possibility to consider them as states of physical systems.

The paper is organized as follows. In the Section 2 we find analytically the geometric measure of entanglement of a spin with other spins in the graph states generated by operator of evolution with Ising Hamiltonian. The relation of the the geometric measure of entanglement with properties of graph is examined. Section 3 is devoted to studies of the geometric measure of entanglement of the graph states on IBM’s quantum computer. We present the results of quantifying entanglement of a spin with other spins in the graph state on 5-qubit IBM’s quantum computer (IBM Q Valencia). Conclusions are presented in Section 4.

2 Geometric measure of entanglement of graph states

Let us study graph states generated by operator of evolution with Ising Hamiltonian. We consider a system of \( N \) spins which is described by the Hamiltonian

\[
H = \frac{1}{2} \sum_{ij} J_{ij} \sigma_i^x \sigma_j^x,
\]

(3)

here \( \sigma_i^x \) is the Pauli matrix of spin \( i \), \( J_{ij} \) is the interaction coupling, \( i, j = 1..N \).

Starting from the initial state

\[
| \psi_0 \rangle = | 00...0 \rangle,
\]

(4)

(the spin states can be associated with qubit states, the state \( | \uparrow \rangle \) corresponds to \( | 0 \rangle \) and \( | \downarrow \rangle \) corresponds to \( | 1 \rangle \)) in result of evolution one obtains the state

\[
| \psi \rangle = e^{-\frac{\mathbb{i} t}{\hbar} \sum_{ij} J_{ij} \sigma_i^x \sigma_j^x} | \psi_0 \rangle,
\]

(5)

which can be associated with a graph \( G(V, E) \). The vertices in the graph \( V \) represent the spins in the system. The edges between the vertexes \( E \) describe interaction between the spins. We consider \( J_{ij} = J \), and associate \( J_{ij} \) with elements of adjacency matrix of the undirected graph \( A_{ij} \) (\( A_{ij} = 1 \) if the interaction between spin \( i \) and spin \( j \) exists, and \( A_{ij} = 0 \) if spin spin \( i \) and spin \( j \) do not interact, \( A_{ij} = J_{ij}/J \)).

Let us calculate the geometric measure of entanglement of one spin with the rest spins of the system described by (3). According to (2) the geometric measure of entanglement
is related with the mean value of the spin. So, let us consider the spin with index \( l \) and find \( \langle \sigma_l \rangle \) in the state (5). We have

\[
\langle \sigma_l^T \rangle = \langle \psi_0 | e^{\frac{i}{\hbar} \sum_{ij} J_{ij} \sigma_i^x \sigma_j^x} e^{-\frac{i}{\hbar} \sum_{ij} J_{ij} \sigma_i^z \sigma_j^z} | \psi_0 \rangle = \langle \psi_0 | \sigma_l^T | \psi_0 \rangle = 0. \tag{6}
\]

The mean value of \( \sigma_l^y \) in state (5) reads

\[
\langle \sigma_l^y \rangle = \langle \psi_0 | e^{\frac{i}{\hbar} \sum_{ij} J_{ij} \sigma_i^x \sigma_j^x} e^{-\frac{i}{\hbar} \sum_{ij} J_{ij} \sigma_i^y \sigma_j^y} | \psi_0 \rangle = \langle \psi_0 | e^{\frac{i}{\hbar} \sum_{ij} J_{ij} \sigma_i^y \sigma_j^y} | \psi_0 \rangle = 0, \tag{7}
\]

where we use the identity

\[
e^{\frac{i}{\hbar} \sum_{ij} J_{ij} \sigma_i^y \sigma_j^y} e^{-\frac{i}{\hbar} \sum_{ij} J_{ij} \sigma_i^y \sigma_j^y} = e^{\frac{i}{\hbar} \sum_{ij} J_{ij} \sigma_i^y \sigma_j^y}, \tag{8}
\]

which follows from the fact that \( \sigma_l^y \) and \( \sigma_l^z \) anticommute. Similarly for \( \langle \sigma_l^z \rangle \) we have

\[
\langle \sigma_l^z \rangle = \langle \psi_0 | e^{\frac{i}{\hbar} \sum_{ij} J_{ij} \sigma_i^x \sigma_j^x} e^{-\frac{i}{\hbar} \sum_{ij} J_{ij} \sigma_i^z \sigma_j^z} | \psi_0 \rangle = \langle \psi_0 | e^{\frac{i}{\hbar} \sum_{ij} J_{ij} \sigma_i^z \sigma_j^z} | \psi_0 \rangle = \langle \psi_0 | e^{\frac{i}{\hbar} \sum_{ij} J_{ij} \sigma_i^z \sigma_j^z} | \psi_0 \rangle. \tag{9}
\]

Taking into account that \( J_{ij} = J \), and using the following notation

\[
\varphi = \frac{2Jt}{\hbar}, \tag{10}
\]

one obtains

\[
\langle \sigma_l^z \rangle = \cos k_l \varphi, \tag{11}
\]

where \( k_l \) is defined as

\[
k_l = \sum_j \frac{J_{lj}}{J}. \tag{12}
\]

Using (2), the geometric measure of entanglement of spin \( l \) with other spins in the system has the following form

\[
E_l( | \psi \rangle) = \frac{1}{2} (1 - | \cos k_l \varphi |) \tag{13}
\]

Note that \( k_l \) given by (12) is the degree of the vertex \( l \) in the graph. So, the geometric measure of entanglement of spin \( l \) with other spins in the graph state (5) is related with the number of edges that are incident to the vertex that represents the spin.

### 3 Quantifying geometric measure of entanglement of graph states on IBM’s quantum computer

Quantum protocol for preparing two qubits \( q[0] \) and \( q[1] \) in the state

\[
| \psi \rangle = e^{-\frac{i}{\hbar} J \sigma_0^z \sigma_1^z} | 00 \rangle, \tag{14}
\]

is presented on Fig. 1.
Figure 1: Quantum protocol for preparing state $|\psi\rangle$. In the protocol controlled-NOT gate, Hadamard gate ($H$), Phase gate ($P(\phi)$) are used.

In the quantum protocol we take into account that the operator $\exp(-itJ_0^x \sigma_1^x / \hbar)$ can be represented as $CX_{01} H_0 P_0 (2Jt / \hbar) H_0 CX_{01}$ where $CX_{01} = |0\rangle_0 \langle 0| \otimes I_1 + |1\rangle_0 \langle 1| \otimes X_1$ is the controlled-NOT gate that acts on qubit labeled by index 0 ($q[0]$) as "control" and on the qubit labeled by index 1 ($q[1]$) as "target". Gate $H_0$ is the Hadamard gate acting on the $q[0]$, and $P_0 (2Jt / \hbar)$ is the Phase gate which acts on $q[0]$. It does not change the state $|0\rangle$ and applies phase multiplier $\exp(i2Jt / \hbar)$ to the state $|1\rangle$. Using protocol Fig. 1 one can prepare states (5) associated with different graphs.

According to relation (2) for quantifying geometric measure of entanglement of spin $l$ with other spins in state (5) on quantum computer the mean values of Pauli operators $\sigma^x_l$, $\sigma^y_l$, $\sigma^z_l$ in the state (5) have to be measured. The protocol for measuring these mean values was proposed in [13]. In the paper it was shown that the mean value of operators $\sigma^x_l$, $\sigma^y_l$, $\sigma^z_l$ can be represented by probabilities which define the result of measure on basis $|0\rangle$, $|1\rangle$. Namely, these mean values can be represented as $\langle \sigma^x \rangle = \langle \psi | \sigma^x | \psi \rangle = |\langle \tilde{\psi}^x | 0 \rangle|^2 - |\langle \tilde{\psi}^y | 1 \rangle|^2$, $\langle \psi | \sigma^y | \psi \rangle = |\langle \tilde{\psi}^y | 0 \rangle|^2 - |\langle \tilde{\psi}^y | 1 \rangle|^2$, where $|\tilde{\psi}^x \rangle = \exp(-i\pi \sigma^x / 4) |\psi \rangle$, $|\tilde{\psi}^y \rangle = \exp(-i\pi \sigma^y / 4) |\psi \rangle$.

From these relations follows that in order to measure the mean value $\langle \sigma^x_l \rangle$ one has to rotate the state of qubit $l$ around $y$ axis by $\pi/2$, to measure the mean value $\langle \sigma^y_l \rangle$ one has to rotate the state by $\pi/2$ around $x$ axis.

We prepare graph state (5) on IBM’s quantum computer IBM Q Valencia v1.4.3. IBM provides free access to this 5-qubit devise [20]. The 5 qubits interacts as it is shown in Fig. 1. Arrows indicate qubits between which the CNOT gate can be directly applied (see Fig. 1). The calibration parameters of IBM Q Valencia v1.4.3 on 24 November 2020 are presented on Table 3 [20].

Taking into account the structure of IBM Q Valencia device, let us examine the graph state defined as

$$| \psi \rangle = e^{-\frac{iJ}{2\hbar}(\sigma_0^x \sigma_1^x + \sigma_1^y \sigma_2^y + \sigma_2^z \sigma_3^z + \sigma_3^x \sigma_4^x)} | 00000 \rangle, \quad (15)$$

[Diagram of the quantum protocol and structure of ibmq_valencia v1.4.3]
Table 1: The calibration parameters of IBM Q Valencia v1.4.3 on 24 November 2020

|                  | $Q_0$ | $Q_1$ | $Q_2$ | $Q_3$ | $Q_4$ |
|------------------|-------|-------|-------|-------|-------|
| Readout error ($10^{-2}$) | 19.96 | 3.48  | 3.56  | 2.35  | 1.67  |
| Single-qubit U2 error rate ($10^{-4}$) | 45.2  | 3.83  | 4.16  | 5.06  | 2.57  |
| CNOT error rate ($10^{-3}$) | CX0_1  | CX1_0  | CX1_2  | CX1_3  | CX2_1  |
|                  | 22.0  | 22.0  | 10.61 | 17.72 | 10.61 |
|                  | CX3_1  | CX3_4  | CX4_3  |       |       |
|                  | 17.72 | 10.99 | 10.99  |       |       |

which can be associated with graph with structure corresponding to the structure of IBM Q Valencia quantum computer. The quantum protocol for preparing graph state (15) is presented on Fig. 3.

Figure 3: Quantum protocol for preparing graph state (15).

According to result (13) obtained analytically in the previous section the entanglement of spin 1 (the spin corresponds to q[1]) with other spins in the graph state (15) reads

$$E_1(|\psi\rangle) = \frac{1}{2}(1 - |\cos^3 \varphi|)$$  \hspace{1cm} (16)

Note, the degree of the vertex 1 in the graph is 3. So, the result (16) is in agreement with conclusion presented in the previous Section. The degree of vertex 3 in the graph is equal to 2 and for geometric measure of entanglement of spin 3 with other spins in the graph state (15) we have the following expression

$$E_3(|\psi\rangle) = \frac{1}{2}(1 - \cos^2 \varphi).$$  \hspace{1cm} (17)

The results of quantifying the geometric measure of entanglement of spin 1 with other spins and spin 3 with other spins on the IBM Q Valencia device are presented on Fig. 4.

Note that for geometric measure of entanglement of spin 3 with other spins in the state we obtained good agreement of experimental results obtained on quantum computer with theoretical ones. For spin 1 because of redout error and single-qubit U2 error the results obtained on quantum computer are not in so good agreement with theoretical ones.

4 Conclusions

The states of many spin systems generated by operator of evolution with Ising Hamiltonian has been considered (5). The states (5) are graph states. They can be associated with
Figure 4: Results of quantifying geometric measure of entanglement on IBM Q Valencia v1.4.3 device (marked by crosses) and analytical results (line) for spin 1 (a) and spin 3 (b) for different values of $\varphi$. 
graphs with vertexes represented by spins and edges corresponding to interaction between
the spins. The geometric measure of entanglement of a spin with other spins in the state
(5) has been found for graph states (5) associated with graphs with arbitrary adjacency
matrixes. We have obtained that the geometric measure of entanglement of a spin with
other spins in the graph state is related with graph properties. Namely it depends on the
degree of vertex which represents the spin in the graph (2).

Entanglement of a spin with other spins in the state (5) was also quantified on quantum
computer. The protocol for preparing graph state (3) has been realized on 5-qubit IBM’s
quantum computer, IBM Q Valencia. The state associated with graph which structure
corresponds to the structure of IBM Q Valencia v1.4.3 device has been prepared and the
geometric measure of entanglement of a spin with other spins has been measured. The
results obtained on quantum computer are in good agreement with theoretical one (see
Fig. 4).

Acknowledgments

This work was supported by Project 2020.02/0196 (No. 0120U104801) from National
Research Foundation of Ukraine.

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