ANALYSIS OF OBSERVABLES IN
CHERN-SIMONS PERTURBATION THEORY

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ABSTRACT

Chern-Simons Theory with gauge group $SU(N)$ is analyzed from a perturbation theory point of view. The vacuum expectation value of the unknot is computed up to order $g^6$ and it is shown that agreement with the exact result by Witten implies no quantum correction at two loops for the two-point function. In addition, it is shown from a perturbation theory point of view that the framing dependence of the vacuum expectation value of an arbitrary knot factorizes in the form predicted by Witten.

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1. Introduction

Chern-Simons gauge theory was solved exactly by Witten [1] using non-perturbative methods. This solution has been obtained subsequently by other groups using both, the point of view of canonical quantization [2-10], and of current algebra [11,5] as originally proposed in [1]. The exact result for the vacuum expectation value of the observables of the theory is analytic in the inverse of the Chern-Simons parameter $k$. Defining the Chern-Simons coupling constant as $g = \sqrt{4\pi/k}$ the exact result suggests that the small coupling constant perturbation expansion should reproduce the exact result. One does not expect any non-perturbative effect in Chern-Simons gauge theory. Perturbative approaches to the theory under consideration have been carried out during the last two years [12-25]. The main question which have been addressed in these works is which one is the renormalization scheme which leads to the exact result obtained by Witten. In Chern-Simons gauge theory there are two problems which must be taken into account. On the one hand, the loop expansion possesses divergences already at one loop for two-point functions which must be regularized. On the other hand, some of the observables of the theory, the Wilson lines, possess products of operators at coincident points in their integration regions. The loop expansion divergences must be regulated in perturbation theory to obtain a finite answer to be compared to the exact result. The ambiguities present when considering products of operators at coincident points forces to make a choice in defining the observables of the theory. The main goal of this paper is to give a regularization procedure and a choice to solve the problem of the ambiguity when considering products of operators at coincident points whose perturbative expansion coincides with the exact result [1]. The second aspect of the problem was solved successfully in [12] and we will follow here their approach.

To handle the ambiguity associated to products of operators at coincident points one must consider framed links instead of links [26,1,12], in other words one must introduce a band instead of a knot and the corresponding integer number
$n$ which indicates the number of times that the band is twisted. In the non-perturbative approach leading to the exact result [1] the origin of the dependence on the framing comes about because one must construct the observables on the surface of a Riemann surface which then must be glued to another Riemann surface to build a three-dimensional manifold. The same knot can be obtained using that procedure in a variety of ways leading to different quantities for observables which, however, differ by a factor which is associated to the framing. As first found out in [1] this factor is just $\exp(2\pi i n h)$ where $h$ is the conformal weight of the representation (for $SU(N)$ in the fundamental representation, $h = c_2(R)/(k + N)$, $c_2(R) = (N^2 - 1)/2N$) carried out by the Wilson line.

The presence of ultraviolet divergences in the loop expansion of Chern-Simons theory forces to regularize the theory and consequently to choose a renormalization scheme. Certainly, from a perturbation theory point of view all schemes are physically equivalent since they differ by a finite renormalization which can be accomplished by adding finite counterterms to the action. However, one would like to know if there exist a scheme which leads naturally to the exact result obtained by Witten. By naturalness we understand a scheme in which the intermediate regularized action leads after taking the limit in which the cutoff is removed to the exact result obtained by Witten where the constant $k$ we started with (bare $k$) is the same constant $k$ as the one appearing in the exact result. Certainly, this concept of naturalness has only meaning in a theory like Chern-Simons theory in which the beta function as well as the anomalous dimensions of the elementary fields vanish at any order in perturbation theory [20,21,24,27]. This was the point of view taken in [13] where a scheme based on Pauli-Villars regularization seemed to be natural in the sense discussed above. Indeed, the results obtained in [13] showed that a choice of scheme of that type seems to lead to the shift $k \rightarrow k + N$ which appears in many of the equations corresponding to the exact result. The fact that the origin of the shift is a quantum effect was first pointed out by Witten [1] who showed its appearance using a gauge invariant regularization based on the eta function. There are other schemes which lead to results in agreement with
[13,1] as the one used in [18]. All these schemes which seem to provide at one loop an explanation of the origin of the shift share the common feature that the intermediate regularized action is gauge invariant.

So far all calculations involving quantum corrections using gauge invariant regularized actions have concentrated on the effective action. The fact that the quantum correction leads to an effective action whose constant $k$ has been shifted indicates that one would observe such effect when computing observables using those schemes. However, at present, only indirect calculations of observables have been carried out taking into account this quantum correction [25,12]. In this paper we are going to present the computation of the Wilson line corresponding to the unknot in the fundamental representation of $SU(N)$ up to order $g^6$. This calculation involves diagrams at two-loops for the two-point function whose calculation in some scheme whose regularized action is gauge invariant has to be carried out. We do not perform in this paper such a two-loop calculation but we will show that agreement with the exact result implies that there is no correction a two-loops for the two-point function. The computation of the two-point function at two-loops using the Pauli-Villars regularization plus higher derivatives proposed in [13] is being carried out [28]. So far we have been able to prove that there is no need to introduce higher derivative terms to regulate the theory at two loops and that a single generation of Pauli-Villars is sufficient to render the two-loop graphs finite. However, we have not finished the calculation of the finite part that according to the results which we present in this work should be zero to have agreement with the exact result, and, therefore, to be able to consider the scheme based on Pauli-Villars as natural.

To end with this introduction we will reproduce here the result obtained by Witten in [1] for the framed unknot in the fundamental representation of $SU(N)$ lying on $S^3$ with framing $n$. The corresponding vacuum expectation value is,

$$\langle W \rangle = \frac{q^{\frac{N}{2}} - q^{-\frac{N}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} q^{\frac{N^2-1}{2}} ,$$

(1.1)
where,

\[ q = \exp\left( \frac{2\pi i}{k + N} \right). \]  

(1.2)

Expanding (1.1) in terms of \( 1/k \) up to terms of order \( 1/k^3 \) one finds,

\[
\langle W \rangle = N + \frac{1}{k} i \pi n (N^2 - 1) \\
+ \frac{1}{k^2} \left[ -\frac{\pi^2}{6} N (N^2 - 1) - \frac{\pi^2 n^2}{2N} (N^2 - 1)^2 - i \pi n N (N^2 - 1) \right] \\
+ \frac{1}{k^3} \left[ -\frac{i \pi^3}{6} n (N^2 - 1)^2 - \frac{i \pi^3}{6N^2} n^3 (N^2 - 1)^3 + \pi^2 n^2 (N^2 - 1)^2 \\
+ \frac{\pi^2}{3} N^2 (N^2 - 1) + i \pi n N^2 (N^2 - 1) \right] \\
+ O\left( \frac{1}{k^4} \right). 
\]  

(1.3)

Notice that in this expansion all terms containing a power of \( \pi \) different that the power of \( 1/k \) are originated by the fact that \( k \) appears shifted into \( k + N \) in (1.2).

In our analysis we will show that those terms do indeed correspond to diagrams which contain one-loop quantum corrections. Notice also that in the standard framing (\( n = 0 \)) the series expansion has a simpler form. As a consequence of our analysis we will be able to identify very simply all diagrams which provide the framing dependence. Actually, we will derive from a perturbation theory point of view the form of the framing dependence of the vacuum expectation value of an arbitrary knot. If, on the other hand, it turns to be correct the picture in which there are only one-loop corrections (which just account for the shift \( k \to k + N \)) one could extract all the effects due to framing and therefore one would be left with a series of diagrams which constitute the building blocks of the knot invariant.

These building blocks lead to topological invariants which after considering them as the coefficients of a power series build the knot invariants leading to the Jones polynomials [29] and its cousins [30,31]. We will discuss in more detail this picture of the perturbation theory series expansion in our concluding remarks.

In this paper we will consider the three dimensional manifold as \( R^3 \) which allows us to identify the corresponding observables to the ones in \( S^3 \). We will not discuss
the effect of the framing of the three-dimensional manifold from a perturbation theory point of view. A good discussion of this point can be found in [25].

The paper is organized as follows. In sect. 2 we define the regularized Chern-Simons gauge theory using Pauli-Villars fields which we claim to correspond to a natural scheme in the sense discussed above. In sect. 3 we compute (1.3) in perturbation theory the vacuum expectation value of the unknot carrying the fundamental representation of $SU(N)$ up to order $g^6$. In sect. 4 we will identify all the framing dependence of the vacuum expectation value of a knot and we will show its factorization in the form predicted by Witten. Finally, in sect. 5 we state our conclusions and make some final remarks. Several appendices deal with our conventions and with the proof of some results which are used in sects. 3 and 4.
2. Perturbative Chern-Simons gauge theory

In this section we will define Chern-Simons gauge theory from a perturbation theory point of view. This is carried out in two steps. First a gauge fixing is performed. Second, after analyzing the ultraviolet behavior of the theory a regularized action using Pauli-Villars fields is provided. Let us consider an $SU(N)$ gauge connection $A_\mu$ on a boundaryless three-dimensional manifold $M$ and the following Chern-Simons action,

$$S(A_\mu) = \frac{k}{4\pi} \int_M \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A),$$

(2.1)

where $k$ is an arbitrary positive integer and “Tr” denotes the trace in the fundamental representation of $SU(N)$ (normalized in such a way that $\text{Tr}(T^a T^b) = -\frac{1}{2} \delta^{ab}$). A summary of our group-theoretical conventions is contained in Appendix A. In defining the theory from a perturbation theory point of view we must give a meaning to vacuum expectation values of operators, i.e., to quantities of the form,

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int [DA_\mu] \mathcal{O} \exp \left( iS(A_\mu) \right),$$

(2.2)

where $Z$ is the partition function,

$$Z = \int [DA_\mu] \exp \left( iS(A_\mu) \right).$$

(2.3)

The operators entering (2.2) are gauge invariant operators which do not depend on the three-dimensional metric. These operators are knots, links and graphs [1,32]. The first issue in defining (2.2) is to take care of the gauge fixing. Indeed, the

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* A negative $k$ will change the $\epsilon$ prescription in the perturbative series expansion leading to a shift of $k$ at one loop with the opposite sign. With a negative $k$ one makes connection with the exact result (1.1) after replacing $q \rightarrow q^*$. 
exponential in (2.2) is invariant under gauge transformations of the form,

\[ A_\mu \rightarrow h^{-1} A_\mu h + h^{-1} \partial_\mu h, \]  

(2.4)

where \( h \) is an arbitrary continuous map \( h : M \rightarrow SU(N) \). Before carrying out the gauge fixing let us redefine the constant \( k \) and the field \( A_\mu \) in such a way that the action (2.1) becomes standard from a perturbation theory point of view. We define,

\[ g = \sqrt{\frac{4\pi}{k}}. \]  

(2.5)

Then, after rescaling the gauge connection,

\[ A_\mu \rightarrow g A_\mu, \]  

(2.6)

one obtains the following Chern-Simons action,

\[ S'(A_\mu) = \int_M \text{Tr}(A \wedge dA + \frac{2}{3} g A \wedge A \wedge A). \]  

(2.7)

This form of the action contains the standard 1/2 factor for the kinetic term after using (A3). From now on we will restrict ourselves to the case in which the three-dimensional manifold \( M \) is \( \mathbb{R}^3 \) which is the simplest case to treat from a perturbation theory point of view. Though (2.7) is metric independent, we will be forced to introduce a metric in carrying out the gauge fixing. We will assume that this metric has signature \((1, -1, -1)\).

Our gauge choice will be the same as the one taken in [13]. The Lorentz-like gauge condition \( \partial^\mu A_\mu = 0 \) is imposed using the standard Fadeev-Popov construction which leads to the following action to be added to (2.7),

\[ S_{gf}(A_\mu, c, \bar{c}, \phi) = \int \text{Tr} \left( 2\bar{c} \partial_\mu D^\mu c - 2\phi \partial_\mu A^\mu - \lambda \phi^2 \right), \]  

(2.8)

where \( \phi \) is the Lagrange multiplier which imposes the gauge condition, \( c \) and \( \bar{c} \) are the Fadeev-Popov ghost, and \( \lambda \) is a gauge fixing parameter. In (2.8), \( D_\mu \) is
the covariant derivative, $D_\mu c = \partial_\mu c + g[A_\mu, c]$. The action (2.7) as well as the gauge-fixing action (2.8) are invariant under the following BRST transformations,

$$sA_\mu = D_\mu c, \quad sc = -cc, \quad s\bar{c} = \phi, \quad s\phi = 0. \quad (2.9)$$

The field $\phi$ can be integrated out easily providing the following functional integral for vacuum expectation values as the ones in (2.2):

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int [DA_\mu DcD\bar{c}] \mathcal{O} \exp \left( iI(A_\mu, c, \bar{c}) \right), \quad (2.10)$$

where,

$$I(A_\mu, c, \bar{c}) = \int \text{Tr} \left( \epsilon^{\mu\nu\rho}(A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho) - \lambda^{-1} A_\mu \partial^\mu \partial^\nu A_\nu + 2 \bar{c} \partial_\mu D_\mu c \right). \quad (2.11)$$

Of course, $Z$ in (2.10) is appropriately defined taking into account the gauge fixing. The quantities obtained in (2.10) are independent of the value of $\lambda$. In order to avoid the presence of infrared divergences we will work in the Landau gauge in which $\lambda = 0$.

The perturbative series expansion which one obtains from (2.10) and (2.11) possesses some divergences which need to be regularized. The analysis of the nature of these divergences was carried out in [13] by performing the corresponding power counting. There are many ways to regularize these divergences giving physically equivalent results. In this work we will follow the regularization procedure introduced in [13], i.e., we will use a gauge invariant regularization based on the introduction of Pauli-Villars fields and, if needed, higher-derivative terms. This seems to provide a scheme which is natural in the sense explained in sect. 1. Further work have shown [28] that there is no need to introduce higher-derivative terms. The Pauli-Villars fields which one introduces to regulate at one loop seem to be sufficient to render the theory finite to any order. Of course, after the gauge fixing has been performed, when talking about a gauge invariant regularization we mean a regularization which preserves the BRST symmetry (2.9).
Following [13] we introduce Pauli-Villars fields $A^{(j)}_{\mu}, \phi^{(i)}$ and $\bar{\phi}^{(i)}, j = 1, ..., J$ and $i = 1, ..., I$. The regularized functional integral takes the form,

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int [DA_{\mu}DcD\bar{c}] \mathcal{O} \exp \left( iI(A_{\mu}, c, \bar{c}) \right) \left( \prod_{j=1}^{J} \det^{-b_{j}/2} A_{j} \right) \left( \prod_{i=1}^{I} \det^c C_{i} \right),$$

(2.12)

where, of course, the same type of regularization is used for $Z$, and all the dependence on the Pauli-Villars fields is contained in the determinants,

$$\det^{-1/2} A_{j} = \int [DA^{(j)}_{\mu}] \exp \left( i \int \text{Tr}(\epsilon^{\mu\nu\rho} A_{\mu}^{(j)} D_{\nu} A_{\rho}^{(j)} + M_{j} A_{\mu}^{(j)} A^{(j)\mu}) \right)$$

$$\det C_{i} = \int [Dc^{(i)}D\bar{c}^{(i)}] \exp \left( 2i \int \text{Tr}(\bar{c}^{(i)} D_{\mu} D^{\mu} c^{(i)} - m_{j}^{2} c^{(i)} c^{(i)}) \right).$$

(2.13)

The masses entering into the determinants in (2.13) as well as the integers $b_{j}, j = 1, ..., J$ and $c_{i}, i = 1, ..., I$ are the regulating parameters. The relative values of these masses and these integers are fixed to make the theory finite in the limit in which the common scale of the masses $\Lambda$ becomes large. In [13] was shown that the following choice makes the theory finite at one loop,

$$\sum_{j=1}^{J} b_{j} = 1, \quad \sum_{j=1}^{J} \frac{b_{j}}{M_{j}} = 0, \quad \sum_{j=1}^{J} \frac{b_{j}}{M_{j}^{2}} = 0,$$

$$I = J, \quad c_{j} = \frac{1}{2} b_{j}, \quad m_{j} = M_{j}.$$

(2.14)

We conjecture that the limit $\Lambda \to \infty$ of (2.12) with the choice (2.14) generates the same values for the observables of the theory (once the ambiguities originated at coincidence points of products of operators are handled as shown in the next section) as the ones in the exact result obtained by Witten [1]. The results presented in this paper and in [28] provide certain evidence towards the validity of this conjecture.

As shown in [13] the regularized action entering (2.12) is BRST invariant. Indeed, defining the BRST transformations of the Pauli-Villars fields as just gauge
transformations of fields transforming in the adjoint representation whose gauge parameter is the ghost field \( c \),

\[
sA^{(j)}_\mu = [A^{(j)}_\mu, c], \quad sc^{(i)} = \{\bar{c}^{(i)}, c\}, \quad sc^{(i)} = \{c^{(i)}, c\},
\]

it is simple to prove that the determinants entering (2.12) are BRST invariant.

To end this section let us summarize the Feynman rules of the theory as well as the one-loop results obtained in [13]. They will become very useful in the next section where the Wilson line corresponding to the unknot will be computed to order \( g^6 \). We will work in space-time space. The two basic Feynman rules entering our calculations are summarized in Fig. 1. In particular, the propagator associated to the gauge field takes the form,

\[
\Sigma^{\mu\nu}_{ab}(x, y) = \frac{i}{4\pi} \delta_{ab} \epsilon^{\mu\rho\nu} \frac{(x - y)_\rho}{|x - y|^3}.
\]

We do not give the Feynman rules corresponding to ghost and Pauli-Villars fields since these fields only enter in loops and we will take the results obtained in [13] for one-loop Green functions. These results are summarized in Fig. 2.
3. Unknot to order $g^6$.

In this section we will compute the vacuum expectation value of the Wilson line corresponding to the unknot in the fundamental representation of $SU(N)$ using the functional integral defined in (2.12). This calculation will provide the tools and methods to analyze general features of the perturbative series expansion of vacuum expectation values of knots as the one considered in the next section. Taking into account the rescaling (2.6), the operator $O$ entering in (2.12) has the form,

$$W = \text{Tr} \left( P e^{g \oint A} \right),$$

(3.1)

where the trace is taken over the fundamental representation and $P$ denotes path-ordered product. This choice of sign in the exponential leads to the convention (A1). The contour integral in (3.1) corresponds to any path diffeomorphic to the unknot. To compute the vacuum expectation value of this operator in perturbation theory we have to consider all diagrams which are not vacuum diagrams since, as shown in (2.12), we consider normalized vacuum expectation values, i.e., in (2.12), the functional integration where the operator is inserted is divided by the partition function $Z$. The expansion of the path-ordered exponential in (3.1) reduces the calculation to certain integrals of $n$-point functions. These $n$-point functions need to be computed perturbatively up to certain order. We will use the standard Feynman diagrams to denote these $n$-point functions. To denote the contour integral we will attach their $n$-points by a circle. For convenience, let us express the perturbative series corresponding to the vacuum expectation value of (3.1) as,

$$\langle W \rangle = \sum_{i=0}^{\infty} w_{2i} g^{2i}.$$  

(3.2)

Clearly, to order $g^0$ the computation of the vacuum expectation value of (3.1) reduces to the trace of the unit operator in the fundamental representation which
is just \( N \),

\[
    w_0 = N, \quad (3.3)
\]

in agreement with (1.3). Higher orders up to \( g^6 \) will be computed in the following subsections.

### 3.1. ORDER \( g^2 \)

To this order, since there is a factor \( g \) in the exponential (3.1) there is only one diagram which just involves the propagator (2.16). This diagram is shown in Fig. 3. Its contribution to the perturbative series (3.2) is just,

\[
    w_2 = \text{Tr}(T^a T^b) \oint dx^\mu \int dy^\nu \frac{i}{4\pi} \delta_{ab\rho\sigma} \frac{(x - y)\rho}{|x - y|^3} = \frac{1}{2} \text{Tr}(T^a T^b) \oint dx^\mu \oint dy^\nu \frac{i}{4\pi} \delta_{ab\rho\sigma} \frac{(x - y)\rho}{|x - y|^3}. \quad (3.4)
\]

To perform the step carried out in obtaining the second expression for \( w_2 \) one must first realize that the integration is well defined and finite, and symmetric under the interchange \( x^\mu \leftrightarrow y^\nu \). Notice that although it seems that there are singularities at coincident points, a careful analysis of the integral shows that this is not the case [26,12]. However, from a quantum field theory point of view the quantity entering (3.4) is not well defined. The reason is that among the points of integration there are points where one is using quantities like \( \langle A_\mu(x) A_\nu(x) \rangle \) which are not well defined from a field theory point of view. One could add a finite part at those coincident points making the integration ambiguous. As shown in [12] there is way to solve this ambiguity providing a procedure which is metric independent as it would be desirable from the point of view of topological field theory. The idea is to introduce an unit vector \( n^\mu \) normal to the path of integration and consider the path corresponding to \( y^\nu \) as the one constructed by \( y^\nu = x^\nu + \varepsilon n^\nu \). The resulting integral depends on the choice of \( n^\mu \) and it corresponds to the the Gauss integral.
which can be normalized such that its value is an integer \( n \),

\[
n = \frac{1}{4\pi} \oint dx^\mu \oint' dy^\nu \epsilon^{\mu\nu\rho} \frac{(x - y)_\rho}{|x - y|^3}.
\] (3.5)

In this equation the prime denotes that the second path is slightly separated from the first path as dictated by the unit vector \( n^\mu \). Often we will refer to this situation as saying that \( x \) runs over the knot and \( y \) over its frame. The integer value \( n \) is the linking number of the two non-intersecting paths. In general, the perturbative expansion of the Wilson line will possess terms containing the ambiguity discussed here. From a field theory point of view, one may detect the presence of this ambiguity just observing if in the integrations of products of operators one is integrating over coincident points. Fortunately, this seems to happen only when the two end points of a propagator may get together ("collapsible" propagator). It turns out that the three-point function possesses milder singularities than the propagator at coincident points and it does not introduce any ambiguity. We will discuss in more detail this feature in the next section.

Using (3.5) and (A3) one finds for \( w_2 \) in (3.4),

\[
w_2 = \frac{i}{4} n (N^2 - 1),
\] (3.6)

which is in agreement with (1.3) after taking into account that \( g^2 = 4\pi/k \).

3.2. Order \( g^4 \)

The diagrams contributing to this order are depicted in Fig. 4. It is at this order where the first appearance of a diagram involving quantum corrections is present. Namely, diagram \( a \) of Fig. 4 contains the full two-point function at one loop. This two-point function was computed in [13] in the scheme adopted in this paper. The result obtained there has been summarized in Fig. 2. Taking into account that result and the previous calculation leading to \( w_2 \) we can write very
simply the contribution of diagram \(a\) of Fig. 4 to this order,

\[
w_4^{(a)} = - \frac{N}{4\pi} w_2 = - \frac{i}{16\pi} nN(N^2 - 1).
\] (3.7)

Notice that this contribution corresponds to the last one at order \(1/k^2\) in (1.3). This term in (1.3) is such that the power of \(\pi\) and the power of \(1/k\) are different and therefore corresponds to the type of terms which are in the expansion of \(\langle W \rangle\) because \(k\) appears shifted into \(k + N\) in the exact result (1.1). This is the first case in which we will observe that a diagram present because of the existence of quantum corrections gives a contribution which corresponds to the one originated by the shift present in the exact result.

The contribution of diagrams \(b\), and \(c_1, c_2\) and \(c_3\) of Fig. 4 has been analyzed in detail in [12,25]. We will use here their results and we will make a series of remarks which will be useful in computations at higher order. The contribution from \(b\) is,

\[
w_4^{(b)} = \text{Tr}(T^a T^b T^c) \oint dx^\mu \int dy^\nu \int dz^\rho \int d^3\omega \left( -i f^{abc} \epsilon_{\nu_1 \nu_2 \nu_3} \right)
\]

\[
i \frac{4\pi}{\rho_1} \frac{\epsilon_{\mu \rho \nu_1} (x-w)^\rho_1 \epsilon_{\nu_2 \nu_3} (y-w)^\rho_2 \epsilon_{\rho \nu_3} (z-w)^\rho_3}{|x-w|^3 |y-w|^3 |z-w|^3} \rho_1.
\] (3.8)

where we have used (A1) and (A3) and,

\[
\rho_1(C') = \frac{1}{32\pi^3} \oint dx^\mu \int dy^\nu \int dz^\rho \int d^3\omega \left( \epsilon_{\mu \nu_2 \nu_1} \epsilon_{\nu_2 \nu_3} \epsilon_{\rho \nu_3} \epsilon_{\nu_1 \nu_2 \nu_3} \right)
\]

\[
\frac{(x-w)\rho_1(y-w)\rho_2(z-w)\rho_3}{|x-w|^3 |y-w|^3 |z-w|^3}.
\] (3.9)

This quantity has a special significance which we will discuss after analyzing the contribution from the rest of diagrams at this order. The argument of \(\rho_1(C)\), \(C\), is the integration path. Notice that the integration entering \(\rho_1(C)\) does not possess
any ambiguity due to the presence of products of operators at coincident points and therefore it is framing independent. The reason why ambiguities are not present is that coincident points occur pairwise, i.e., the three endpoints never get together in the integration, and singularities associated to this case are too mild to introduce ambiguities. Of course, this assertion needs a careful proof which indeed has been carry out indirectly in [12,25]. Form a quantum field theory point of view, it seems plausible and we will think about it as a general feature of the perturbative series expansion. For the unknot the quantity $\rho_1(C)$ was computed in [12] obtaining the result,

$$\rho_1|_{\text{unknot}} = -\frac{1}{12}. \quad (3.10)$$

Taking into account this value, the contribution from diagram $b$ of Fig. 4 has the form,

$$w_4^{(b)} = -\frac{1}{96} N(N^2 - 1), \quad (3.11)$$

which is just the first term of order $1/k^2$ in (1.3) after taking into consideration that $g^2 = 4\pi/k$.

We are left with the contributions from diagrams $c_1$, $c_2$ and $c_3$ of Fig. 4. Diagrams $c_1$ and $c_2$ give the same contribution. However, diagram $c_3$ has an entirely different nature. On the one hand, notice that diagram $c_3$ does not possess ambiguities. The endpoints of a propagator never get together since they always enclose an endpoint of another propagator. This means in particular that the contribution from such a diagram is framing independent. In addition, the group factor from this diagram is different than the one from the other two diagrams. Non-planar diagrams as $c_3$ possess different group factors than the corresponding planar ones. In general, using (A1) the group factor of a non-planar diagram can be decomposed in a part containing the same structure as the planar one plus another contribution. Namely using (A1) one finds,

$$\text{Tr}(T^aT^bT^aT^b) = \text{Tr}(T^aT^aT^bT^b) + f^{abc}\text{Tr}(T^aT^cT^b)$$

$$= \frac{(N^2 - 1)^2}{4N} - \frac{1}{4}N(N^2 - 1). \quad (3.12)$$
the first group factor has the same form as the group factors of diagrams $c_1$ and $c_2$ and we will consider all three contributions together. Actually it is simple to realize that the resulting expression once the three contributions are taken into account possesses an integrand that is symmetric. This allows to enlarge the integration region symmetrically and divide by a factor $4!$. On the other hand the contribution due to the second group factor in (3.12) is proportional to,

$$\rho_2(C) = \frac{1}{8\pi^2} \int dx^\mu \int dy^\nu \int dz^\rho \int d\omega^\tau \epsilon^{\mu\nu\sigma_1\rho} \epsilon^{\tau\sigma_2\omega} (x-z)_{\sigma_1} (y-w)_{\sigma_2} \left( \frac{\epsilon^{\mu\nu}}{4\pi} \frac{(x-y)_{\sigma_1}}{|x-y|^3} \frac{\epsilon^{\tau\omega}}{4\pi} \frac{(z-w)_{\sigma_2}}{|z-w|^3} \right), \quad (3.13)$$

which vanishes for the case in which the contour $C$ can be contained in a plane as it is the case for the unknot. Therefore, the contribution from diagrams $c_1$, $c_2$ and $c_3$ of Fig. 4 takes the form,

$$w^{(c)}_4 = -\frac{(N^2 - 1)^2}{4N} \frac{3}{4!} \int dx^\mu \int dy^\nu \int dz^\rho \int d\omega^\tau \left( \frac{\epsilon^{\mu\nu}}{4\pi} \frac{(x-y)}{|x-y|^3} \frac{\epsilon^{\tau}}{4\pi} \frac{(z-w)}{|z-w|^3} \right)$$

$$= -\frac{n^2(N^2 - 1)^2}{32N}, \quad (3.14)$$

where in the last step we have used (3.5). This contribution is just the remaining one at order $1/k^2$ in the expansion (1.3). Therefore, to this order we have full agreement between the exact result and the perturbative calculation. Notice that to achieve this we have defined products of operators at coincident points in a very precise manner. We have argued that the ambiguity in those products only produces a relevant effect when the points of coincidence are joined by a propagator. Coincidence of end-points which belong to a connected part of an $n$-point function with $n > 2$ does not introduce any ambiguity. One may verify that the singularities appearing when $n > 2$ are milder than in the case $n = 2$ to justify in certain sense that assertion. However, a complete proof of it would be desirable. For the case of $\rho_1(C)$ and $\rho_2(C)$, it has been shown [12,25] that both are framing independent, in agreement with our statement. Their sum must therefore correspond to a knot
invariant. In fact, it was shown in [12] that

\[ \rho(C) = \rho_1(C) + \rho_2(C) \] (3.15)

can be identified with the second coefficient of the Alexander-Conway polynomial. In general, the picture that emerges from the perturbative calculation is that the connected \( n \)-point functions, \( n > 2 \), constitute the main building blocks of the knot invariant (1.1). This building blocks are knot invariants and build the perturbative series leading to (1.1). The two-point function takes care of the framing (planar contribution) and of some corrections to the connected \( n \)-point functions, \( n > 2 \), as \( \rho_2(C) \) above (non-planar contribution). We will see how these facts are realized at next order in perturbation theory. Their general features will be discussed in sect. 4.

3.3. Order \( g^6 \)

This is the first order where a two-loop diagram takes place. The diagrams contributing to this order are represented in Fig. 5 and Fig. 6. Diagram \( a_1 \) involves the full two-loop one particle irreducible two-point function. This quantity has not been computed yet in the regularization scheme considered in this paper. One of the aims of this work is to demonstrate that it must vanish in order to have agreement with the exact result (1.1). We will compute in this section all other contributions at this order and we will prove that they generate all the terms at order \( 1/k^3 \) in (1.3).

The contribution from diagram \( a_2 \) is straightforward after using the expression in Fig. 2. It turns out,

\[ w_6^{(a_2)} = \frac{i}{64\pi^2} nN^2(N^2 - 1). \] (3.16)

This contribution corresponds to the last term at order \( 1/k^3 \) in (1.3). Notice that this is one of the terms where the power of \( \pi \) is different than the power of \( 1/k \) and
therefore is shift related. The other diagrams containing one-loop corrections are \( b, c_1, c_2 \) and \( c_3 \), and \( d_1, ..., d_6 \) of Fig. 5. The contribution from these diagrams are simple to compute using the form of the one-particle irreducible diagrams in Fig. 2, and the results of the previous order. From diagram \( b \) one finds,

\[
w_6^{(b)} = \frac{1}{32\pi} N^2(N^2 - 1)\rho_1
\]

while, similarly, from diagrams \( c_1, c_2 \) and \( c_3 \), which all give the same contribution,

\[
w_6^{(c)} = -\frac{3}{32\pi} N^2(N^2 - 1)\rho_1.
\]

Finally, after rearranging the group factors as in (3.12), the contribution from diagrams \( d_1, ..., d_6 \) is,

\[
w_6^{(d)} = \frac{1}{64\pi} n^2(N^2 - 1)^2 - \frac{1}{16\pi} N^2(N^2 - 1)\rho_2.
\]

Collecting all the contributions and using (3.10) and the fact that for the unknot \( \rho_2 = 0 \) one finds,

\[
w_6^{(b)} + w_6^{(c)} + w_6^{(d)} = -\frac{1}{16\pi} N^2(N^2 - 1)(\rho_1 + \rho_2) + \frac{1}{64\pi} n^2(N^2 - 1)^2
\]

\[
= -\frac{1}{192\pi} N^2(N^2 - 1) + \frac{1}{64\pi} n^2(N^2 - 1)^2,
\]

which correspond to the other two contributions in (1.3) (all except the last one) whose power of \( \pi \) does not coincide with the power of \( 1/k \).

The rest of the diagrams contributing at this order do not contain loop corrections and are depicted in Fig. 6. Diagrams \( e_1, e_2 \) and \( e_3 \) of Fig. 6 involve the tree-level four-point function. Clearly, the first two diagrams are planar and identical while the third one is non-planar. This third diagram, \( e_3 \), possesses the group factor,

\[
\text{Tr}(T^a T^b T^c T^d) f^{ace} f^{ebd},
\]

which, as shown in Appendix A, vanishes (see equation (A7)). The other two diagrams, \( e_1 \) and \( e_2 \) of Fig. 6, which are the same, vanish for the case of the unknot as it is shown in Appendix B.
Let us compute the contribution form the ten diagrams \( f_1, ..., f_{10} \) of Fig. 6. These diagrams can be divided in planar and non-planar ones. As in previous cases, non-planar diagrams possess group factors which decompose into the group factors of the planar ones plus an additional contribution. Indeed, from a diagram like \( f_6 \) the group factor is,
\[
\text{Tr}(T^a T^a T^b T^c T^d) f^{bcd} = -\frac{1}{8} (N^2 - 1)^2,
\]
while from a diagram like \( f_2 \) the group factor is,
\[
\text{Tr}(T^a T^b T^c T^d T^b) f^{acd} = -\frac{1}{8} (N^2 - 1)^2 + \frac{1}{8} N^2 (N^2 - 1).
\]

Non-planar diagrams of this type do not contribute for the case in which the Wilson line corresponds to the unknot. This can be shown writing explicitly the integration involved or using the lemma below. The main idea behind the argument based on that lemma is that non-planar diagrams of the type under consideration are framing independent so one can choose any framing to compute it. For the unknot it is simple to realize that choosing a framing which is contained in the same plane as the unknot the integrand vanishes trivially. Before stating and proving this lemma let us define “free” propagators as the ones that have both endpoints on the knot.

**Lemma.** Every framing independent diagram of the unknot containing a free propagator is zero.

**Proof.** Let us place the unknot \( C \) in a plane. Being the diagram framing independent, choose a frame \( C_f \) coplanar to it. The diagram contains the part corresponding to the free propagator,
\[
\int \cdots dx^\mu dy^\nu \epsilon_{\mu \rho \nu} (x - y)^\rho |x - y|^3 \cdots
\]
where \( dx^\mu \in C \) and \( dy^\nu \in C_f \). Due to the coplanarity, the previous term is a 3 \( \times \) 3 determinant whose rows are linearly dependent. Then, it is zero and the
lemma is proved. This result is very powerful once all the framing independent diagrams of the perturbative series expansion are identified. The theorem stated in the next section allows to characterize very simply all those diagrams. As we will discuss there, it turns out that those diagrams are the ones not containing collapsible propagators. Thus, using the lemma above, we conclude that the only non-vanishing diagrams contributing to the perturbative series expansion of the unknot are the ones with no free propagators.

We are left with planar diagrams of type $f$ in Fig. 5. Actually, it will be much more convenient to consider the whole set of the ten diagrams all with the same group factor (3.22). The reason for this is that then one can show the factorization of the contribution into a product of contributions of the type appearing in Fig. 3 times contributions of the type $b$ in Fig. 4. This phenomena of factorization is general for diagrams with disconnected one-particle irreducible subdiagrams. Indeed, in Appendix C we show the general form of the factorization theorem. The result of applying this theorem for diagrams $f_1, ..., f_{10}$ of Fig. 6 is explained as an example in Appendix C. It turns out that it can be written as the following product:

$$w_6^{(f)} = \frac{-i}{8}(N^2 - 1)^2 g^6 \frac{1}{4\pi^2} \int dx_1^{a_1} \int dx_2^{a_2} \epsilon^{a_1, a_2, a_3, a_4} \frac{1}{|x_1 - x_2|^3} 64\pi^3 \times$$

$$\int dx_3^{a_3} \int dx_4^{a_4} \int dx_5^{a_5} \int d^3 z \epsilon^{a_3, a_6, a_7, a_8} \epsilon^{a_4, a_5, a_6, a_7, a_8} \frac{(z - x_3) \gamma (z - x_4) \delta (z - x_5) \beta}{|z - x_3|^3 |z - x_4|^3 |z - x_5|^3},$$

i.e., a product of a linking number times $\rho_1$,

$$w_6^{(f)} = \frac{i}{32} n(N^2 - 1)^2 \rho_1 = -\frac{i}{384} n(N^2 - 1)^2.$$  (3.26)

In obtaining (3.26) we have used (3.6), (3.8) and (3.10). This contribution is just the first one at order $1/k^3$ in (1.3) after using the fact that $g^2 = 4\pi/k$. This procedure of using the lemma plus the factorization theorem of Appendix C is a general feature of the unknot. In general, for an arbitrary knot, the factorization
theorem would force us to overcount diagrams giving additional contributions. However, for the unknot all those contributions vanish.

To complete the perturbative computation at order $g^6$ we are left with diagrams $g_1, ..., g_{15}$ of Fig. 6. Again these diagrams can be divided into planar and non-planar ones. However, now the non-planar ones can be divided in three groups depending on the number crossings. The group factor decomposes differently in each group. A given diagram produces an additional group factor for each uncrossing needed to make it planar. If the group factor of the planar diagrams, $g_1$ to $g_5$ is

$$\text{Tr}(T^a T^a T^b T^b T^c T^c) = -\frac{1}{8N^2} (N^2 - 1)^3,$$  \hspace{1cm} (3.27)

the group factor of diagrams $g_6, ..., g_{11}$, which are of the first type, takes the form,

$$\text{Tr}(T^a T^a T^b T^c T^b T^c) = -\frac{1}{8N^2} (N^2 - 1)^3 + \frac{1}{8}(N^2 - 1)^2,$$  \hspace{1cm} (3.28)

where we have used simply (A1). The diagrams of the second type are $g_{12}, g_{13}$ and $g_{14}$, which similarly generate the following group factor,

$$\text{Tr}(T^a T^b T^c T^a T^b T^c) = -\frac{1}{8N^2} (N^2 - 1)^3 + \frac{1}{8}(N^2 - 1)^2 + \frac{1}{8}(N^2 - 1)^2,$$  \hspace{1cm} (3.29)

Finally, diagram $g_{15}$ generates,

$$\text{Tr}(T^a T^b T^c T^a T^b T^c) = -\frac{1}{8N^2} (N^2 - 1)^3 + \frac{1}{8}(N^2 - 1)^2 + \frac{1}{4}(N^2 - 1)$$  \hspace{1cm} (3.30)

Of all three types of group factors only the first one contributes in the case of the unknot. To the second group factor, $\frac{1}{8}(N^2 - 1)^2$, there are contributions from the last 10 diagrams. Using the factorization theorem of Appendix C one finds that this contribution is proportional to $\rho_2$ (diagram $c_3$ in Fig. 4) and therefore vanishes. We have rearranged the group factors in order to get the right weights which make explicit the factorization of $\rho_2$. To the third group factor, $\frac{1}{8}(N^2 - 1)$, there are
contributions from the last 4 diagrams which can be shown explicitly to vanish for the case of the unknot. We are left with the first group factor, $-\frac{1}{8N^2}(N^2 - 1)^3$. There are contributions from all diagrams. One can use the factorization theorem of Appendix C to write this contribution as a product of contributions of the type shown in Fig. 3. Using (3.6) one then finds,

$$w_6^{(g)} = -\frac{i}{384N^2}n^3(N^2 - 1)^3,$$

which indeed corresponds to the second contribution at order $1/k^3$ in (1.3). The calculation is described in some detail as an example in Appendix C. This was the only contribution left to be obtained from the perturbative series expansion. The agreement found between the two results shows that the contribution from diagram $a_1$ of Fig. 5 must be zero. This implies that the one-particle irreducible diagram corresponding to the two-point function must vanish at two loops.
4. Factorization of the framing dependence

In this section we state a theorem about the framing independence of diagrams which do not contain one-particle irreducible subdiagrams corresponding to two-point functions whose endpoints could get together. This theorem refers to any kind of knot. Before stating it, some remarks are in order. Let us consider an arbitrary diagram whose $n$ legs are attached to $n$ points on the knot. The resulting integral runs over these points on the knot in a given order, $i_1 < i_2 < \cdots < i_n$. Suppose that our diagram has a propagator with endpoints attaching two consecutive points, say $i_1$ and $i_2$. Remember that the path ordered integration will make $i_1 \rightarrow i_2$, and that the propagator is singular in that case. Albeit this singularity exists, the integral is finite but shape-dependent, as is well-known. The results for a circumference and for an ellipse are different and then it is not a topological invariant. The way out of this difficulty is the introduction of framings [26,1,12]. When the propagator connects the knot and the frame, the resulting integral is the linking number of the frame around the knot, and this is a topological invariant. This suggests that the framing is relevant only when there are collapsible (free propagators whose endpoints may get together upon integration) propagators. This is the idea behind this theorem. Its statement is:

**Theorem.** A diagram gives a framing dependent contribution to the perturbative expansion of the knot if and only if it contains at least one collapsible propagator. Moreover, the order of $n$ in its contribution, the linking number, equals the number of collapsible propagators.

Diagrams $b$ and $c_3$ of Fig. 4, $e_1, e_2$ and $e_3$, $f_1$ to $f_5$, and $g_{12}$ to $g_{15}$ of Fig. 6 are examples of framing independent diagrams. Diagrams $a$, $c_1$ and $c_2$ of Fig. 4, and $f_6$ to $f_{10}$, and $g_1$ to $g_{11}$ of Fig. 6 are examples of framing dependent ones.

Although we have no rigorous proof of this theorem, we do have results that suggest its validity. Two of them are the framing independence of $\rho_1(C)$ and $\rho_2(C)$.
separately. The framing independence of these objects has been rigorously proven [12,25]. As argued in the previous section, from a quantum field theory point of view one would expect that the ambiguity present in $n$-point functions at coincident points would play a role when all $n$ points get together. Since a Wilson line consists of a path-ordered integration, such coincident points may occur only for the case of two-point functions ($n = 2$), in particular when they are collapsible. By no means this argument provides a proof of the theorem but it makes its validity plausible. In rigorous terms one should think of the theorem above as a conjecture. In the rest of this section we will find further evidence regarding its validity. Assuming that the theorem holds the following corollary follows.

**Corollary.** If all the contribution to the self-energy comes from one loop diagrams, then $\langle W(C) \rangle = F(C; N) q^{n(N^2-1)/2N}$ where $F(C; N)$ is framing independent but knot dependent, and the exponential is manifestly framing dependent but knot independent.

**Proof.** Let us prove this corollary first forgetting about the shift $k \rightarrow k + N$, or in other words, not including loops. Let us recall that free propagators are the ones with both endpoints on the knot. For example, diagram $b$ of Fig. 4 does not contain free propagators while diagrams $c_2$ and $c_3$ of Fig. 4 contain two free propagators. In diagram $c_2$ of Fig. 4 these two free propagators are collapsible. To prove the corollary we will organize the perturbative series expansion of the Wilson line in the following way. First, select all the diagrams which do not contain free propagators. Let us denote by $\mathcal{M}$ the set of these diagrams. The simplest diagram of this set is the one with no internal line at all, which is the zeroth order diagram. Diagram $b$ of Fig. 4 is the order $g^4$ diagram in the set $\mathcal{M}$. In virtue of the theorem above the contribution of each of the diagrams in this set is framing independent. Now take each of the diagrams of this set and dress it with free propagators in all possible ways. Certainly, this organization exhausts the perturbative series. The proof will consist in demonstrating that the effect of dressing by free propagators each diagram in $\mathcal{M}$ is such that the contribution to the perturbative series expansion factorizes as stated in the corollary. To be more
specific, we will show that the form of the contribution of a diagram in $\mathcal{M}$ plus all the diagrams resulting of its dressing by free propagators factorizes in a part containing all the framing dependence which has the form $q^n(N^2-1)/2N$ times a part which is framing independent.

Let us consider a diagram $A \in \mathcal{M}$ and let us denote by $\{D^p_A\}$ the set of diagrams resulting after dressing the diagram $A$ with $p$ free propagators. This set of diagrams in $\{D^p_A\}$ has been schematically drawn in Fig. 7. Given a diagram in $\{D^p_A\}$, one can work out its group trace and notice that after commuting appropriately the generators of $SU(N)$ entering into this trace in such a way that generators with the same index get together one generates a series of terms, being the last of them the group factor of the diagram in $\{D^p_A\}$ with $p$ collapsible propagators. For example, the representative of $\{D^p_A\}$ shown in Fig. 7 would provide a group structure whose last (or leading) term is as the one of the diagram pictured in Fig. 8. This procedure is the one which we have followed, for example, in the derivation of the group factors (3.28), (3.29) and (3.30). To gain a better understanding about the types of group factors which appear we will consider several subsets of $\{D^p_A\}$. At first sight one could think that diagrams with $p$ free propagators with the same number of crossings lead to the same group structure. This is not entirely true. It holds for diagrams without three-vertices with one crossing of free propagators but it is not true in general. Given a diagram in $\{D^p_A\}$ with $c$ crossings one finds different group factors. For example one can check explicitly that the group factor of the diagram in Fig. 9 is different than the one of diagram $g_{12}$ of Fig. 6 with one more (collapsible) propagator. Let us denote by $\{D^{p,c,j}_A\}$ the set of diagrams in $\{D^p_A\}$ with $c$ crossings and group factor $j$. Certainly,

$$\{D^{p,c,j}_A\} \subset \{D^p_A\}. \quad (4.1)$$

Given a diagram $D^{p,c,j}_A$ one finds after working out the group factor that it always contains one which corresponds to the power of order $p$ of the quadratic Casimir, $[(N^2 - 1)/2N]^p$ times the group factor corresponding to diagram $A$. In
the process one finds other group structures with lower powers. Let us concentrate first on the group structure with the highest power. Certainly, all diagrams in $D_{p,c,j}^{p,c,j}$ for a fixed value of $p$ contribute to this group structure. To apply the factorization theorem of Appendix C we need to have as many diagrams as domains. As shown at the end of Appendix C, if diagram $A$ is connected the difference between the number of domains and the number of diagrams comes about because while diagrams with $n_i$ identical subdiagrams count as one, from the point of view of domains they should count as $n_i!$ to have the adequate relabelings to be in the hypothesis of the factorization theorem. Thus, for the case in which $A$ is connected one just has to repeat $p!$ times the diagrams and make the adequate relabelings to be in hypothesis of the factorization theorem. Of course, this implies that one must divide the result of the theorem by $p!$. For $A$ connected the contribution corresponding to the group structure $[(N^2 - 1)/2N]^p$ from all diagrams in $\{D_{A}^{p,c,j}\}$ is,

$$1 \frac{n^p}{p!} 2^p (ig^2)^p \left( \frac{N^2 - 1}{2N} \right)^p D_A = 1 \frac{n^p}{p!} \left( i \frac{2\pi N^2 - 1}{k} \right)^p D_A, \quad (4.2)$$

where the factor $2^p$ appears after enlarging the integral of each propagator to the whole knot (which provides the factor $n^p$, where $n$ is the winding number (3.5)). Notice that in (4.2) $D_A$ represents the contribution from diagram $A$, which is framing independent, and that we have used (2.5). Notice also that after summing in $p$ (4.2) gives the form stated for the Wilson line in the corollary. However, this is not the only framing dependent contribution. One certainly has more contributions with other group structures. Also, one has to discuss the situation in which $A$ is not connected. We will consider that situation later.

To the next group structure (next to leading) not all the diagrams in $D_{A}^{p,c,j}$ contribute. Indeed, only diagrams with $c > 0$ do. There are, however, diagrams which contribute and are framing dependent so we have to work out this dependence. For example, diagrams $g_6$ to $g_{11}$ of Fig. 6 are framing dependent. One would like to have enough diagrams to be able to use the factorization theorem of Appendix C and factorize the contribution as the one from diagram $A$ with one
crossing of free propagators, which is framing independent, times the contribution due to \( p - 2 \) collapsible propagators which is framing dependent and proportional to \( n^{p-2} \). To explain how one must arrange the perturbative series to extract the effect of the framing we will consider first in some detail the case corresponding to diagrams \( g_6 \) to \( g_{11} \) of Fig. 6. This is a particular case in which \( p = 3 \) and \( A \) is trivial but it possesses the essential features in which we will focus our attention in the general proof.

Diagrams \( g_1 \) to \( g_{15} \) of Fig. 6 contribute to the group structures worked out in (3.27), (3.28), (3.29) and (3.30). All fifteen diagrams \( g_1 \) to \( g_{15} \) shown in Fig. 6 contribute to the group structure \(- (N^2 - 1)^3/8N^2\). It corresponds to the general case which led to (4.2). We will describe it in this example for completeness. The contribution to this group structure can be written as follows:

\[
I_3 = \oint_{i<j<k<l<m<n} \left[ f(ij, kl, mn) + f(ij, km, ln) + f(ij, kn, lm) + f(ik, jl, mn) \\
+ f(ik, jm, ln) + f(ik, jn, lm) + f(il, jk, mn) + f(il, jm, kn) \\
+ f(il, jn, km) + f(im, jk, ln) + f(im, jl, kn) + f(im, jn, kl) \\
+ f(in, jk, lm) + f(in, jl, km) + f(in, jm, kl) \right]
\]

(4.3)

We have use a notation in which \( i, j, k, l, m, n \) represent the six points attached to the knot and \( f(ij, kl, mn) \) the corresponding integrand. As explained, not all possible domains are represented in (4.3). One possesses 90 domains while there are only 15 diagrams. The ratio between these two numbers is just \( 3! \), the number of possible orderings of the three free propagators which from the point of view of domains should be different. Introducing all the relabelings needed to apply the factorization theorem of Appendix C one must therefore divide by \( 3! \). The result is,

\[
I_3 = \frac{1}{6} \oint_{i<j<k<l<m<n} \oint_{i<j} p(i,j) p(k,l) p(m,n),
\]

(4.4)

where \( p(i,j) \) represents the integrand corresponding to a free propagators with its
endpoints $i$ and $j$ attached to the knot. The linking number $n$ appears when we let the endpoints of each propagator go freely over the knot and the frame, and multiply by $1/2$ per propagator. This provides the additional factor $(1/2)^3 = 1/8$. The result, in agreement with (4.2) is,

$$I_3 = \frac{n^3}{3!2^3}.$$ (4.5)

The diagrams contributing to the next group structure, are $g_6$ to $g_{15}$ of Fig. 6, and in order to apply the factorization theorem we need to have 15 diagrams since that is the number of domains. Therefore, we have to overcount some of them. Notice that now one of the subdiagrams is a free propagator while the other is $c_3$ of Fig. 4, which is not connected. We can not apply the simple strategy described at the end of Appendix C. The number of domains is different than the number of diagrams because in diagrams like $g_{12}$, $g_{13}$ and $g_{14}$ one has two possible choices of domains while in diagrams $g_{15}$ one has three. Thus let us add what we need, i.e., let us consider two diagrams of types $g_{12}$, $g_{13}$ and $g_{14}$ and make the aptoipate relabelings in one of each pair, and 3 diagrams of type $g_{15}$ and make relabelings in two of them. Certainly, we must subtract what we have added. Now one can not just simply divide by a factor. We need therefore to subtract once diagrams $g_{12}$, $g_{13}$ and $g_{14}$ and twice diagram $g_{15}$. These diagrams are framing independent and therefore they do not contribute to the framing dependent part. In the general proof, at this stage, the diagrams which one must subtract contribute to a lower power of $n$. Therefore, we may use the algorithm safely power by power. Using our previous notation, the contribution to the next group structure, $\frac{1}{8}(N^2 - 1)^2$ is,
\[ I_1 = \oint_{i<j<k<l<m<n} \left[ f(ij, km, ln) + f(jk, ln, mi) + f(kl, jm, ln) + f(lm, ik, jn) \\
+ f(mn, ik, jl) + f(ni, jl, km) + f(ik, jm, ln) + f(im, jl, kn) \\
+ f(il, jn, km) + f(il, jm, kn) \right] \\
= \left( \oint_{i<j<k<l<m<n} + \oint_{k<i<j<l<m<n} + \oint_{k<l<i<j<m<n} + \oint_{k<l<m<i<j<n} \\
+ \oint_{k<l<m<i<j} + \oint_{i<k<l<m<n} + \oint_{i<k<l<m<n} \right) f(ij, km, ln), \]
\]

which, after adding single replicas for the integrands corresponding to \( g_{12}, g_{13} \) and \( g_{14} \) and double ones for \( g_{15} \), using statement 1 of Appendix C, and making the relabelings,

\[ \sigma_7 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 1 & 6 & 2 \end{pmatrix}, \quad \sigma_8 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 2 & 5 & 6 \end{pmatrix}, \quad \sigma_9 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 5 & 2 & 6 \end{pmatrix} \]

\[ \sigma_{10} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 2 & 6 \end{pmatrix}, \quad \sigma'_{10} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 5 & 6 & 2 \end{pmatrix} \]

one finds,

\[ I_1 = \tilde{I}_1 - \hat{I}_1, \]

where,
\[ I_1 = \left( \oint_{i<j<k<l<m<n} + \oint_{i<k<j<l<m<n} + \oint_{i<k<l<j<m<n} + \oint_{i<k<l<m<j<n} + \oint_{i<k<l<m<n<j} + \oint_{k<i<j<l<m<n} + \oint_{k<i<l<j<m<n} + \oint_{k<i<l<m<j<n} + \oint_{k<i<l<m<n<j} + \oint_{k<l<i<j<m<n} + \oint_{k<l<i<m<j<n} + \oint_{k<l<i<m<n<j} \right) f(ij, km, ln) \]

\[ = \oint_{i<j} p(i, j) \oint_{k<l<i<m<n} f_2(kl, mn). \] 

\[ \] 

and,

\[ I_1 = \left( \oint_{k<l<i<m<j<n} + \oint_{k<l<i<m<n<j} + \oint_{k<l<i<m<j<n} + \oint_{k<l<m<i<j<n} + \oint_{k<l<m<i<n<j} \right) f(ij, km, ln). \] 

In (4.8), the quantity \( f_2 \) is the integrand corresponding to diagram \( c_3 \) of Fig. 4. After integration, it results the quantity \( \rho_2 \) in (3.13). The contributions remaining to the next group structures plus the left over represented by (4.9) are framing independent. This means that we have extracted all the framing dependence from the set of diagrams \( g_1 \) to \( g_{15} \) of Fig. 6. Using (4.5) and (4.8) we can write such a contribution as

\[ -\frac{1}{8N^2} (N^2 - 1)^3 \frac{n^3}{3!2^3} + \frac{1}{8} (N^2 - 1) \frac{3n^2}{2}, \]

\[ = -\frac{N}{8} \frac{1}{3!} \left( \frac{(N^2 - 1)n}{2N} \right)^3 + \frac{N}{8} (N^2 - 1) \frac{1}{1!} \left( \frac{N^2 - 1}{2N} \right)^1 \rho_2 \] 

In the second line of this equation we have rewritten the contribution to show the general structure which we will find out now.
Let us discuss which one is the algorithm to arrange the perturbative series expansion to extract the framing dependence for the group structure next to the leading one. In the example considered we have seen that diagrams contributing to this type of group structure were \( g_6 \) to \( g_{15} \) of Fig. 6. Certainly, these were not all the 15 diagrams which were needed to use the factorization theorem and factorize the contribution to the group factor \((N^2 - 1)\) as a product of a diagram like the one in Fig. 3 times the one originating \( \rho_2 \) (diagram \( c_3 \) of Fig. 4). However, we were able to add pieces of diagrams with the adequate group structure with two or more crossings (which one must subtract when considering the situation leading to a factorization of diagrams with two crossings) to have the 15 needed and apply the factorization theorem of Appendix C. Clearly, this is the procedure which one must carry out when considering the next to the leading group structure. One adds pieces of diagrams to complete the set in such a way that the factorization theorem of Appendix C can be applied, and, on the other hand one subtracts them. The important point is that all diagrams which are involved in this operation contain a lower power of \( n \), the linking number, and therefore, in the process one extracts all the framing dependence for the next to leading group structure. The corresponding contribution from all the diagrams in \( D^{p,c,j}_A \) for a fixed value of \( p \) is,

\[
\frac{1}{(p-2)!} \frac{n^{(p-2)}}{2^{(p-2)}} \left( \frac{N^2 - 1}{2N} \right)^{(p-2)} \left( \frac{2\pi N^2 - 1}{2N} \right)^{(p-2)} \frac{1}{D^{(1,i)}_A} = \frac{1}{(p-2)!} \left( \frac{2\pi N^2 - 1}{2N} \right)^{(p-2)} \frac{1}{D^{(1,i)}_A},
\]

(4.11)

where the origin of each factor is similar to the case of (4.2) and \( D^{(1,i)}_A \) is one of the possible types of diagrams resulting after dressing diagram \( A \) with two free propagators with one crossing. In Fig. 10 a particular situation of (4.11) has been depicted.

One has now to analyze the next group structure. In this case one has to take into consideration all the left-overs from the previous one. Certainly, this is going to change the numerical factor in front of this contribution but once this is taken into account one may proceed similarly as the previous case. It is clear now that one may proceed performing this construction for a fixed value of \( p \) with all types
which appear at each value of \(c\). The result that one obtains in this way is,

\[
\frac{1}{p!} \left( \frac{2\pi N^2 - 1}{2N} \right)^p \sum_{c=1}^{p} \sum_{i=1}^{n_c} \frac{1}{(p - c - 1)!} \left( \frac{2\pi N^2 - 1}{2N} \right)^{(p-c-1)} D_A^{(c,i)},
\]

(4.12)

where \(n_c\) is the number of types which one finds at \(c\) crossings, and we have used the fact that for the case of \(p\) free propagators the maximum number of possible crossings is \(\binom{p}{2}\). It is important to remark that the coefficients \(D_A, D_A^{(c,i)}\), which appear in (4.12) are framing independent. In (4.12) we have singled out all the framing dependence at a given order of free propagators for a diagram \(A \in \mathcal{M}\). Summing over \(p\) one obtains the anticipated exponential behavior:

\[
(D_A + \sum_{c=1}^{\infty} \sum_{i=1}^{n_c} D_A^{(c,i)}) \exp \left( \frac{1}{2} \frac{\pi g N^2 - 1}{2N} \right) = (D_A + \sum_{c=1}^{\infty} \sum_{i=1}^{n_c} D_A^{(c,i)}) q^{n \frac{N^2 - 1}{2N}},
\]

(4.13)

where we have used (2.5) and (1.2). Therefore the corollary is proven.

As this exponential is a common factor, the sum of the perturbative series factorizes as the sum of all diagrams without collapsible propagators (framing independent, but knot dependent) multiplied by some adequate coefficients times the preceding exponential (manifestly framing dependent). Note that this factor does not depend on the kind of knot because the Gauss integral only sees the linking number. Also is interesting to note that the framing independent and knot dependent factor is intrinsic to the knot and then includes all the diagrams that give the building blocks of its topological invariants. This is schematically represented in Fig. 11.

All along our discussion regarding the proof of the corollary we have assumed that \(A\) was a connected diagram. Let us remove now that fact. If \(A\) is not connected it is clear that one can use the technic of adding and removing pieces of diagrams to apply the factorization theorem at each stage of the proof described for the case in which \(A\) was connected. This will introduce some numerical factors in (4.13) which are important in what regards the building blocks of the knot, but are irrelevant.
for the framing dependence since the exponential behavior has been shown for each term.

Finally we have to include the shift $k \rightarrow k + N$. Assuming that diagrams with more than one loop do not contribute to the self-energy, we have to factorize propagators with and without self-energy insertions, and then sum the resulting series. Remember again that the rest of the diagrams add up to a framing independent and knot dependent factor.

This series is hard to manage because there can be any number of self-energy insertions at each propagator, and any number of propagators. The best organization is as follows: call $\{D^q_p\}$ the set of diagrams with $p$ propagators in which we have inserted $q$ self-energies in all the possible ways, and $f(n)$, the sum of all of them, which is the framing dependent and knot independent factor in $\langle W(C) \rangle$. Notice that,

$$f(n) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \{D^q_p\}.$$  

(4.14)

The important point to use here is that the distribution of self-energies is such that they are indistinguishable. The number of possible distributions of $q$ identical insertions in $p$ lines is a Bose-Einstein combinatorial factor. The lines in which we insert self-energies are also indistinguishable. For example, insertions done in sets of diagrams as the ones in Fig. 7 introduces a factor $1/p!$. Therefore, the prefactor of $\{D^q_p\}$ is

$$\frac{1}{p!} \binom{p + q - 1}{q}.$$  

(4.15)

Now, each insertion amounts to a factor $-N/k$ ($k = 4\pi/g^2$), and each line to a factor $x/k$ ($x = i n 2\pi (N^2 - 1)/2N$). Hence,

$$f(n) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{1}{p!} \binom{p + q - 1}{q} \left( \frac{x}{k} \right)^p \left( \frac{-N}{k} \right)^q.$$  

(4.16)
This series in \( q \) is the expansion of a factor that provides the shift:

\[
\left(1 + \frac{N}{k}\right)^{-p} = \sum_{q=0}^{\infty} \left(\frac{p+q-1}{k}\right) \left(\frac{-N}{k}\right)^q,
\]

and therefore, the final result is

\[
f(n) = \sum_{p=0}^{\infty} \frac{1}{p!} \left(\frac{x}{k}\right)^p \left(1 + \frac{N}{k}\right)^{-p} = e^{\frac{\alpha x(N^2-1)}{2N}} \frac{1}{k+N} = q^{n \frac{N^2-1}{2N}},
\]

where we have used (1.2). This proof also works in the other way around. Suppose that two-loop diagrams also contribute. These are identical among themselves, but distinguishable from the one loop insertions, and so there must appear two “bosonic” combinatory factors. The sum of this series has to be different from the shifted exponential, because expanding the exponential with shift we find just one bosonic combinatory factor. Then, the exact result implies that the only quantum corrections relevant to the framing dependent part are the one-loop self-energies.

This corollary shows from a perturbation theory point of view that all dependence on the framing in the vacuum expectation value of a knot factorizes in the form predicted by Witten [1]. Notice also that assuming that there are only one-loop quantum corrections, we have found for an arbitrary knot in the fundamental representation of \( SU(N) \) the shift in the framing dependent factor of the Wilson Line through a purely perturbative approach.

We have proved that the effect of one-loop contributions on the framing dependent factorized part of the vacuum expectation value is just a shift in \( k \). Certainly, this is going to hold also for the rest. This, together with the factorization of the framing dependence means that we can write from (1.1) the full contribution from the building blocks or diagrams in \( \mathcal{M} \) with no loop insertions. One just has to set \( n = 0 \) in (1.1) and remove the shift.
In this paper we have shown that agreement between the exact result found by Witten [1] for the vacuum expectation value of the unknot and the Chern-Simons perturbative series expansion implies that the two-loop contribution to the one-particle irreducible two-point function must vanishes. We have worked within a renormalization scheme which is gauge invariant and which provides a one loop correction to the two and three-point functions which, as shown here, is responsible for the shift of $k$ into $k + N$ observed in [1]. Consistency with the exact result implies that the two-loop contribution in renormalization schemes providing quantum corrections at one loop must vanish. Work is under completion regarding this issue [28] for the renormalization scheme proposed in sect. 2 of this paper.

Our analysis of the structure of the series expansion which appears in the perturbative calculation of the vacuum expectation value of Wilson lines shows that in general one can disentangle the framing dependence from the rest of the contribution. We have shown that under the assumption that the theorem stated in sect. 4 holds, the framing dependence of the vacuum expectation value of Wilson lines factorizes in the form predicted in [1]. We have shown this for a Wilson line carrying the fundamental representation of $SU(N)$ but it is clear from the proof that similar arguments hold for any other representation. Although a rigorous proof of the theorem (conjecture) stated in sect. 4 would be very valuable to make our discussion on the factorization of the framing dependence complete, the arguments based on physical grounds utilized in sect. 3 make the validity of this conjecture rather plausible.

It is important to remark that the factorization theorem proved in Appendix C has played an essential role in the factorization of the framing dependence achieved in sect. 4, as well as in the explicit calculation of the vacuum expectation value of the unknot in the fundamental representation at order $g^6$. In general, this theorem decreases the number of integrations needed at a given order by one, reducing the computation to just the building blocks of the perturbative expansion.
The study carried out in sect. 4 to extract the framing dependence out of the perturbative series expansions of the vacuum expectation value of a Wilson line has also provided information about the building blocks of the perturbative series expansion. Presumably, these building blocks generate a whole series of topological invariants whose integral form is easy to write down using the Feynman rules of the theory. Certainly, these building blocks are framing independent since, according to the corollary of sect. 4, all framing dependence has been factorized out. To prove their topological invariance is much harder and it may well happen that at a given order in $g$ not all the building blocks by themselves are topological invariants but adequate combinations of them. It would be desirable to have some general result in this respect.

In this paper we have carried out an explicit calculation of the vacuum expectation value of the unknot in the fundamental representation of $SU(N)$ up to order $g^6$. It is straightforward to generalize this calculation to any other representation. One would like, however, to analyze the case of a non-trivial knot to verify if the same conclusion holds and to compute some of its building blocks. Chern-Simons theory provides a whole series of topological invariants whose integral form is simple (but tedious) to write down, which would be interesting to classify and characterize. For example, one would like to know if the degree of complexity of a knot is related to the number of building blocks which are different from zero. For the case of the unknot we have that the lemma of sect. 3 plus the theorem of sect. 4 imply that its building blocks are diagrams which do not contain free propagators. The quantity $\rho_1$ defined in (3.10) is the first non-trivial building block. It is represented by diagram $b$ of Fig. 4. The building blocks of the unknot at next order are represented by diagrams $e_1$, $e_2$ and $e_3$ of Fig. 6. As shown in sect. 3 together with Appendix B the contribution from these diagrams vanishes. Therefore, the next possibly non-vanishing building blocks for the unknot corresponds to diagrams containing two three-vertices with all their legs attached to the Wilson line (a representative is diagram $a$ of Fig. 12) and diagrams containing three three-vertices (a representative is diagram $b$ of Fig. 12). The contribution from
these building blocks should be computed and compared to the exact result. As argued at the end of Appendix B, all building blocks of the unknot corresponding to connected tree-level diagrams with an even number of vertices vanish. This is in agreement with the full result (1.1). From (1.1), as explained at the end of sect. 4, it is rather simple to obtain the contribution from the building blocks. One has just to set $n = 0$ and remove the shift. The remaining series is clearly even in $1/k$ which implies that only terms at order $g^{4m}$ are different from zero, in agreement with the observation made at the end of Appendix B.

In this work we have shown how to extract from the perturbative series expansion of knots in Chern-Simons theory their framing dependence as well as the effect of quantum corrections. This leaves the series with the essential ingredients which we have called building blocks and contain all the topological information. Further work is needed to study the general features and the classification of these building blocks.
APPENDIX A

In this appendix we present a summary of our group-theoretical conventions. We choose the generators of $SU(N)$, $T^a$, $a = 1, \ldots N^2 - 1$, to be antihermitian such that

$$[T^a, T^b] = -f^{abc}T^c,$$  \hspace{1cm} (A.1)

and $f^{abc}$ are completely antisymmetric, satisfying,

$$f^{acd}f^{bcd} = N\delta^{ab}. \hspace{1cm} (A.2)$$

The convention chosen in (A.1) seems unusual but it is the right one when the Wilson line is defined as in (3.1). If we had chosen $if^{abc}$ instead of $-f^{abc}$, the exponential of the Wilson line would have had $ig$ instead of $g$. Our convention also introduces a $-1$ in the vertex (see Fig. 1). The fundamental representation of $SU(N)$ is normalized in such a way that,

$$\text{Tr}(T^aT^b) = -\frac{1}{2}\delta^{ab}. \hspace{1cm} (A.3)$$

The quadratic Casimir in the fundamental representation has the form,

$$\sum_{a=1}^{N^2-1} T^aT^a = -\frac{N^2-1}{2N}. \hspace{1cm} (A.4)$$

One of the group factors which appear in subsect. 3.3 of the paper is the following,

$$\text{Tr}(T^aT^bT^cT^d)f_{ace}f_{ebd}, \hspace{1cm} (A.5)$$

which can be shown to be zero. In fact, using the invariance of the trace under cyclic permutations one finds, after relabeling,

$$\text{Tr}(T^aT^bT^cT^d)f_{ace}f_{ebd} = \text{Tr}(T^bT^cT^dT^a)f_{ace}f_{ebd} = \text{Tr}(T^aT^bT^cT^d)f_{dbec}f_{eac}, \hspace{1cm} (A.6)$$

which is just the same as (A.5) but with the opposite sign. Therefore,

$$\text{Tr}(T^aT^bT^cT^d)f_{ace}f_{ebd} = 0. \hspace{1cm} (A.7)$$
In this appendix we show that the contribution from diagrams \( e_1 \) or \( e_2 \) of Fig. 6 vanishes for the case of the unknot. These are the integrals that appear in the four-point \( g^6 \) contribution to the unknot, represented in diagrams \( e_1, e_2 \) and \( e_3 \) of Fig. 6. The idea of the calculation is as follows. According to the framing independence theorem of sect. 4, each diagram is framing independent, so we can think that the four points are all in the unknot. Also we assume that it corresponds to a topological invariant and therefore we choose the unknot to be a circumference on the \( x_0 = 0 \) plane, centered at the origin. Call \( p \) and \( q \) the points of integration over \( R^3 \otimes R^3 \). Now observe that the integrand contains an odd number of \( \epsilon_{\alpha\beta\gamma} \) contracted in such a way that it is a pseudoscalar. Its sign is different in the \( x_0 > 0 \) and \( x_0 < 0 \) regions. In other words, for each \( (p, q) \in R^3 \otimes R^3 \) there are \( (p', q') \in R^3 \otimes R^3 \) such that \( p'_0 = -p_0, \quad q'_0 = -q_0 \) and all other components unchanged, for which the integrands are equal in magnitude but different in sign. Then, in the \( p_0, q_0 \) plane we have and odd integrand and so the integral vanishes. Let us verify this explicitly.

The integrations entering this contribution are of the type,

\[
\int d^3 p \, d^3 q \, \epsilon_{\mu\rho_1\nu_1} dx^\mu \frac{(x - p)^\rho_1}{|x - p|^3} \epsilon_{\mu\rho_2\tau_2} dy^\nu \frac{(y - p)^\rho_2}{|y - p|^3} \epsilon_{\rho_3\nu_2\tau_1} dx^\rho \frac{(z - q)^\rho_3}{|z - q|^3} \\
\epsilon_{\rho_4\nu_4} dw^\tau \frac{(w - q)^\rho_4}{|w - q|^3} \epsilon_{\nu_3\nu_2\tau_2} \epsilon_{\tau_1\tau_2} \frac{(p - q)^\rho_5}{|p - q|^3},
\]

(B.1)

where \( x, y, z, w \) lie on the knot and hence \( x_0 = y_0 = z_0 = w_0 = 0 \). The denominator of the integrand of this expression is invariant under \( p_0, q_0 \to -p_0, -q_0 \). The structure of the numerator is more complicated and we will consider its form separately.

The first three factors of the numerator of (B.1) become,

\[
\epsilon_{1\rho_1\nu_1} dx^1 (x - p)^\rho_1 + \epsilon_{2\rho_1\nu_1} dx^2 (x - p)^\rho_1 \\
= \epsilon_{1\nu_1} dx^1 (x - p)^0 + \epsilon_{12\nu_1} dx^1 (x - p)^2 + \epsilon_{20\nu_1} dx^2 (x - p)^0 + \epsilon_{21\nu_1} dx^2 (x - p)^1.
\]

(B.2)
The following three factors give an analogous contribution,

$$
\epsilon_{1\rho_1\nu_2}dy^1(y-p)^\rho_1 + \epsilon_{2\rho_1\nu_2}dy^2(y-p)^\rho_1
= \epsilon_{10\nu_2}dy^1(y-p)^0 + \epsilon_{12\nu_2}dy^1(y-p)^2 + \epsilon_{20\nu_2}dy^2(y-p)^0 + \epsilon_{21\nu_2}dy^2(y-p)^1,
$$

(B.3)

which becomes (B.2) after changing $y \to x$ and $\nu_2 \to \nu_1$. Contracting (B.2) and (B.3) with $\epsilon^{\nu_1 \nu_2 \tau}$ one obtains,

$$
\epsilon_{20\tau_1}[dx^1p^0dy^1(y-p)^2 - dx^1p^0dy^2(y-p)^1 - dx^1(x-p)^2dy^1p^0 + dx^2(x-p)^1dy^1p^0]
- \epsilon_{01\tau_1}[dx^1p^0dy^2(x-p)^2 - dx^2p^0dy^2(y-p)^2 + dx^2(y-p)^1dy^2p^0 - dx^2(x-p)^1dy^2p^0]
+ \epsilon_{21\tau_1}[-dx^1(p^0)^2dy^2 + dx^2(p^0)^2dy^1].
$$

(B.4)

The rest of the factors in (B.1) except the last two are treated similarly, obtaining an expression similar to (B.4) with $x \to z$, $y \to w$, $p \to q$, and $\tau_1 \to \tau_2$. Finally, multiplying (B.4) by the remaining factor of (B.1), $\epsilon_{\tau_1 \tau_2 \rho_5}(p-q)\rho_5$, one gets,

$$
-p_0q_0^2(p-q)^2[\cdots] + p_0q_0(p-q)[\cdots] - q_0p_0^2(p-q)_2[\cdots] - q_0p_0^2(p-q)\nu_1[\cdots]
+ p_0q_0(p-q)[\cdots] - p_0q_0^2(p-q)\nu_1[\cdots],
$$

(B.5)

where by $[\cdots]$ it is meant a part that does not depend on $p_0$ or $q_0$. As argued above, this expression is odd under $p_0, q_0 \to -p_0, -q_0$ and therefore the integration over $p_0, q_0$ in (B.1) vanishes. This way of showing the vanishing of integrations as (B.1) suggests that this property is a general feature of tree level connected diagrams with an even (and non zero) number of $R^3$ points of integration. As we discuss in sect. 5, this assertion is substantiated by the full result (1.1).
APPENDIX C

In this appendix we state and prove the factorization theorem. First, let us introduce some notation. We will be considering diagrams corresponding to a given order $g^{2m}$ in the perturbative expansion of a knot, and to a given number of points running over it, namely $n$. Note that $n$ and $m$ fix the number of vertices, $n_v$, and propagators, $n_p$, that are in each diagram: $n_v = 2m - n$ and $n_p = 3m - n$. We will denote by $\{i_1, i_2, \ldots, i_n\}$ a domain of integration where the order of integration is $i_1 < i_2 < \ldots < i_n$, being $i_1, i_2, \ldots, i_n$ the points on the knot (notice the condensed notation) where the internal lines of the diagram are attached. The integrand corresponding to that diagram will be denoted as $f(i_1, i_2, \ldots, i_n)$. Diagrams are in general composed of subdiagrams, which may be connected or non-connected. For a given diagram we will make specific choices of subdiagrams depending on the type of factorization which is intended to achieve. For example, for a diagram like $g_6$ of Fig. 6 one may choose as subdiagrams the three free propagators, or one may choose a subdiagram to be the collapsible propagator and other subdiagram to be the one built by two crossed free propagators. We will consider a set of diagrams $\mathcal{N}$ corresponding to a given order $g^{2m}$, to given number of points attached to the knot, $n$, and to a given kind. By kind we mean all diagrams containing $n_i$ subdiagrams of type $i$, $i = 1, \ldots, T$. By $p_i$ we will denote the number of points which a subdiagram of type $i$ has attached to the knot. For example, if one considers diagrams at order $g^6$ with $n = 6$ points attached to the knot, with three subdiagrams which are just free propagators, this set is made out of diagrams $g_1$ to $g_{15}$ of Fig. 6. However, if one considers diagrams at order $g^6$ with $n = 6$ with a subdiagram consisting of a free propagator and another subdiagram of the type $c_3$ of Fig. 4, this set is made out of diagrams $g_6$ to $g_{15}$. The contribution from all diagrams in $\mathcal{N}$ can be written as the following sum:

$$\sum_{\sigma \in \Pi_n} \int f(i_{\sigma(1)}, i_{\sigma(2)}, \ldots, i_{\sigma(n)})$$ (C.1)

where $\sigma \in \Pi_n$, being $\Pi_n \subset P_n$ a subset of the symmetric group of $n$ elements.
Notice that $\Pi_n$ reflects the different shapes of the diagrams in $\mathcal{N}$. In (C.1) the integration region has been left fixed for all the diagrams and one has introduced different integrands. One could have taken the opposite choice, namely, one could have left fixed the integrand and sum over the different domains associated to $\mathcal{N}$. The first statement regarding the factorization theorem just refers to these two possible choices. Let us define the domain resulting of permuting $\{i_1, i_2, \ldots, i_n\}$ by an element $\sigma$ of the symmetric group $P_n$ by

$$d_\sigma = \{ i_{\sigma(1)}, i_{\sigma(2)}, \ldots, i_{\sigma(n)} \},$$  \hspace{1cm} (C.2)$$

then the following result immediately follows.

**Statement 1.** The contribution to the Wilson line of the sum of diagrams whose integrands are of the form:

$$f(i_{\sigma(1)}, i_{\sigma(2)}, \ldots, i_{\sigma(n)}),$$  \hspace{1cm} (C.3)$$

where $\sigma$ runs over a given subset $\Pi_n \in P_n$ with a common domain of integration is equal to the sum of the integral of $f(i_1, i_2, \ldots, i_n)$ over $d_\sigma$ where $\sigma \in \Pi_n^{-1}$:

$$\oint_{i_1,i_2,\ldots,i_n} \sum_{\sigma \in \Pi_n} f(i_{\sigma(1)}, i_{\sigma(2)}, \ldots, i_{\sigma(n)}) = \sum_{\sigma \in \Pi_n^{-1}} \oint_{d_\sigma} f(i_1, i_2, \ldots, i_n).$$  \hspace{1cm} (C.4)$$

The idea behind the factorization theorem is to organize the diagrams in $\mathcal{N}$ in such a way that one is summing over all possible permutations of domains. Summing over all domains implies that one can consider the integration over the points corresponding to each subdiagram as independent and therefore one can factorize the contribution into a product given by the integrations of each subdiagram independently. Our aim in the rest of this appendix will be to prove the following statement.
Statement 2. (Factorization theorem) Let $\Pi'_n$ be the set of all possible permutations of the domains of integration of diagrams containing subdiagrams of types $i = 1, ..., T$. If $\Pi^{-1}_n = \Pi'_n$, the sum of integrals over $d\sigma$, $\sigma \in \Pi^{-1}_n$, is the product of the integrals of the subdiagrams over the knot, being the domains all independent,

$$\sum_{\sigma \in \Pi^{-1}_n} \oint f(i_1, \ldots, i_n) = \prod_{i=1}^{T} \left( \oint f(i_1, \ldots, i_{p_i}) \right)^{n_i},$$  \hspace{1cm} (C.5)

In (C.5) $n_i$ denotes the number of subdiagram of type $i$ and $p_i$ its number of points attached to the knot.

The proof of this statement is trivial since having all possible domains it is clear that one can write the integration considering subdiagram by subdiagram, the result being the product of all the partial integrations over subdiagrams.

In sect. 4 we considered situations where we were forced to add and subtract pieces of diagrams in such a way that the theorem above was utilized. For completeness, we will show now that for the case in which all subdiagrams are connected the overcounting needed to apply the theorem is very simple and that it just amount to divide by an adequate combinatorial factor. Let us discuss first an example to understand the strategy leading to the general situation.

Let us consider the four point $g^4$ contribution or, better to say, its part with $(N^2 - 1)^2/4N$ as $SU(N)$ factor. The diagrams are $c_1, c_2$ and $c_3$ of Fig. 4, whose contribution can be written according to Statement 1, is,

$$\oint \left[ f(i_1, i_2, i_3, i_4) + f(i_1, i_3, i_2, i_4) + f(i_1, i_4, i_2, i_3) \right]$$

$$= \left( \oint + \oint + \oint \right) p(i_1, i_2) p(i_3, i_4),$$  \hspace{1cm} (C.6)

where $p(i_1, i_2)$ represents a free propagator attached to the knot at points $i_1$ and $i_2$. There are no more than three diagrams, and the number of domains is six.
These are:

\[ i_1 < i_2 < i_3 < i_4 \quad i_3 < i_1 < i_2 < i_4 \]
\[ i_1 < i_3 < i_2 < i_4 \quad i_3 < i_1 < i_4 < i_2 \]
\[ i_1 < i_3 < i_4 < i_2 \quad i_3 < i_4 < i_1 < i_2 \]  \hspace{1cm} (C.7)

so we need three more diagrams. Now the overcounting consists of rewriting the diagrams after a relabeling. In the course of the relabeling we will use the fact that \( p(i_1, i_2) = p(i_2, i_1) \). The relabelings needed in the overcounting are the following:

\[ \sigma_1 = \left( \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{array} \right) \quad \sigma_2 = \left( \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{array} \right) \quad \sigma_3 = \left( \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{array} \right) \]  \hspace{1cm} (C.8)

Here the subindex of each \( \sigma \) indicates the integrand over which it acts. Note also that each relabeling is in fact a repetition of the diagram, due to the symmetry of the propagator pointed out above. Therefore, all we have to do is to multiply the sum of the six terms by \( 1/2 \). The new integrands are:

\[ \sigma_1[f(i_1, i_2, i_3, i_4)] = f(i_3, i_4, i_1, i_2) = f(i_1, i_2, i_3, i_4), \]
\[ \sigma_2[f(i_1, i_3, i_2, i_4)] = f(i_3, i_4, i_1, i_2) = f(i_1, i_2, i_3, i_4), \quad \text{(C.9)} \]
\[ \sigma_3[f(i_1, i_4, i_2, i_3)] = f(i_3, i_4, i_1, i_2) = f(i_1, i_2, i_3, i_4), \]

and the result is:

\[
\oint_{i_1 < \ldots < i_4} \left[ f(i_1, i_2, i_3, i_4) + f(i_1, i_3, i_2, i_4) + f(i_1, i_4, i_2, i_3) \right] \\
\quad = \frac{1}{2} \left( \oint_{i_1 < i_2 < i_3 < i_4} + \oint_{i_1 < i_3 < i_2 < i_4} + \oint_{i_1 < i_4 < i_2 < i_3} + \oint_{i_3 < i_1 < i_2 < i_4} + \oint_{i_3 < i_2 < i_1 < i_4} + \oint_{i_3 < i_4 < i_1 < i_2} \right) p(i_1, i_2) p(i_3, i_4). \]  \hspace{1cm} (C.10)

Now, in the language of the theorem, \( \Pi_4^{-1} = \Pi'_4 \), i.e., each propagator runs freely
over the knot and therefore we obtain,

\[ \oint p(i_1, i_2) p(i_3, i_4) \]

\( i_1 < ... < i_4 \)

\[ \oint [f(i_1, i_2, i_3, i_4) + f(i_1, i_3, i_2, i_4) + f(i_1, i_4, i_2, i_3)] = \frac{1}{2} \oint \oint p(i_1, i_2) p(i_3, i_4). \]

\( i_1 < i_2 \quad i_3 < i_4 \)

(C.11)

The linking number of the frame and the knot arises as an integral of a propagator with one of its endpoints running over the knot and the other over the frame, without any ordering. This is easily achieved in the integrals we have by simply leaving \( i_1 \) and \( i_2 \) free, and multiplying by \( 1/2 \). The same should be done for the pair \( i_3 \) and \( i_4 \). This is again an example of factorization. The final \( 1/8 \) is the factor \( 3/4! \) that appears for \( w_4^{(c)} \) in (3.14).

The example suggests the idea of a general proof. We should count the number of domains and the number of diagrams and observe their relation, as well as the origin of their difference. We are able to provide formulae for the number of domains for an arbitrary diagram. However, our formula for the number of diagrams in terms of the features of their subdiagrams only holds when all subdiagrams are connected. Using these formulae we will show that we can make equivalent the overcounting and the original contribution simply introducing a combinatorial factor.

First, let us compute the number of domains, \( d' \), corresponding to a general set of diagrams at a given order in the perturbative expansion of the knot, to the same number of points on the knot, and to the same types of subdiagrams. Suppose that we construct the diagram adding its subdiagrams in a given order. The first one can put its \( p_i \) points in \( n \) places. The second one has to distribute its points in the remaining \( n - p_i \), and so on. For example, if there are only \( a \) points attached by propagators and \( b \) points attached by three-vertices (and so \( n = 2a + 3b \)),

\[ d' = \binom{n}{2} \binom{n-2}{2} \binom{n-4}{2} \ldots \binom{n-2a+2}{2} \binom{n-2a}{3} \binom{n-2a-3}{3} \ldots \]

\[ \ldots \binom{n-2a-3b+3}{3} = \frac{n!}{(2!)^a (3!)^b}. \]

(C.12)

In general the denominator will include the product of every \( p_{i!} \), where \( p_i \) denotes
the number of points attached to the knot corresponding to a subdiagram of type $i$.
If we denote by $d$ the number of diagrams of the set under consideration, it is clear
that $d \leq d'$ due to the possible identity of some subdiagrams. In those cases we have
to divide $d'$ by the number of permutations of all the identical subdiagrams. If some
of the subdiagrams were not connected we would have to consider additional factors
which would imply to introduce more data about each subdiagram. Therefore, let
us restrict ourselves to the case of connected subdiagrams. The final form of the
formulae is:

$$d' = \frac{n!}{\prod_{i=1}^{k} (p_i!)^{n_i}}; \quad d = \frac{n!}{\prod_{i=1}^{k} n_i!(p_i!)^{n_i}}.$$  \hspace{1cm} (C.13)

The relation between $d$ and $d'$ is always an integer:

$$\frac{d'}{d} = \prod_{i=1}^{k} n_i!.$$  \hspace{1cm} (C.14)

This is the combinatory factor we were searching for. Therefore, when the subdi-
agrams are connected one just has to overcount evenly each diagram to have as
many diagrams as domains and divide by (C.14). For non-connected subdiagrams
the previous formula for $d$ fails. An example is the factorization of $\rho_2(C)$ in the
$\frac{1}{8}(N^2 - 1)^2$ part of diagrams $g_6, \ldots, g_{15}$ of Fig. 6. The subdiagram that we would
factorize is the corresponding to diagram $c_3$ of Fig. 4, which is not connected. The
previous formula gives $d = d' = 15$, exactly the same result as if the subdiagram
were $e_1$ or $e_2$ of Fig. 6, but there are just 10 diagrams. The number of domains,
however, is correct.
FIGURE CAPTIONS

1) Basic Feynman rules of the theory.

2) Two-point function and three-point function at one loop.

3) Diagrams corresponding to $g^2$. Thick lines represent the Wilson line while thin lines refer to Feynman diagrams.

4) Diagrams corresponding to $g^4$.

5) Part of the diagrams corresponding to $g^6$.

6) Rest of diagrams corresponding to $g^6$.

7) A general set of diagrams with $p$ propagators in the knot

8) The less crossed diagram of Fig. 7.

9) A framing independent diagram.

10) Factorization of $\rho_2$

11) Factorization of the framing dependence.

12) Diagrams at order $g^8$ corresponding to building blocks
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