On the non-physical concavity of the quark potentials within the thick center vortex model

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Abstract: Lattice gauge theory (LGT) results denote to the confinement for the quark potential in various Yang–Mills theories. This property can also be obtained for the $G(2)$ gauge theory. LGT calculations show that quark potential should have the downward concavity behavior. Confinement properties can be explained using the thick center vortex model. However using this model, an upward concavity is seen in some intervals of the quark potential. Considering the reason of this concavity, it is shown that the non-physical upward concavity (convexity) can be reduced by taking an arbitrary symmetric vortex flux in the space–time plane of the lattice.

Keywords: Quantum chromodynamics; Non-perturbative methods; Lattice gauge theory; Quark confinement; Center vortex model; Thick center vortex model; Wilson loop; Quark potential

1. Introduction

The problem of the quantum chromodynamics (QCD), relating to vacuum properties, especially confinement, is an interesting subject in the particle physics [1–5]. There is not yet an analytic proof of color confinement in any non-Abelian gauge theory. Nonlinear properties of the confinement make it hard to study using the usual perturbation quantum field theory such as Feynman diagrams techniques. So, non-perturbation methods are used to study this phenomenon. Lattice gauge theory is a useful method to explain such nonlinear phenomena[6–8]. The confining phase is usually defined by the behavior of the action of the Wilson loop. Wilson loop is simply the path in space–time traced out by a quark antiquark pair created at one point and annihilated at another point [9]. In a non-confining theory, the action of such a loop is proportional to its perimeter. However, in a confining theory, the action of the loop instead of its perimeter is proportional to its area. Since the area is proportional to the separation of the quark antiquark pair, free quarks are suppressed. Mesons are allowed in such a picture, since a loop-containing another loop with the opposite orientation has only a small area between the two loops. Topological properties of non-Abelian theories seem interesting for studying the quark confinement. Phenomenological models are used to study the confinement with different approaches. In these models, the QCD vacuum is filled with some topological configurations which are confining colored objects. The most popular candidates among these topological fields are monopoles and vortices [10–22]. Other candidates include merons, calorons, etc. There are very strong correlations between these various objects, though.

The center vortex model was initially introduced by ’t Hooft [23–27]. According to the numerical simulation, center vortices have been identified as physical objects in the continuum limit of lattice gauge theory. Vortex vacuum model reproduces almost the full string tension as well as the correct scaling behavior, if the continuum limit is approached. Center vortex density has been shown an independent relation to the temperature of lattice simulation, and this indicates that density of vortices is a renormalization group invariant and therefore is a physical quantity. A density of two center vortices in one fermi distance has been discussed and considered in [28]. This shows that vortices are not lattice artifacts but physical objects. The vortex picture successfully describes the finite temperature deconfinement transition. The deconfinement phase transition in the vortex picture can thus be understood as a transition from a percolating to a non-percolating phase [29]. The relation between center vortices and dynamical breaking of chiral symmetry using the quark
propagator under the removal of center vortices has been investigated before. It has been shown that dynamical mass generation disappears if those vortices are removed, and surprisingly, much of it resides in the vortex-only part \[30\]. Also, it has been shown that the constrained cooling results in Yang–Mills streamline configurations which not only give rise to spontaneous chiral symmetry, but also to quark confinement. The structure, however, differs from a gas or liquid of instantons and resemble vortex or string-like structure \[31\]. Center vortex model is able to explain the confinement of quark pairs at the asymptotic region, but it is not able to explain the confinement at intermediate distances especially for the higher representations. The model then is modified to the thick center vortex model \[32–35\]. Within this model, one can obtain the Casimir scaling and \(N\)-ality behavior of the gauge theory. This is done before for the \(SU(2)\), \(SU(3)\) and \(SU(4)\) \[36–40\]. Using other modifications of the model, it is possible to describe the properties of gauge theory without nontrivial center elements such as the \(G(2)\) gauge theory \[41–44\].

One of the properties of the quark potentials is downward concavity. Due to this condition, it is not possible to observe an upward concavity in the quark potential. However using the thick center vortex model, such behavior is observed. In this article after studying a proof for this condition, the reason of the presence of such nonphysical properties is understood more. Due to this study, considering an arbitrary symmetric vortex flux in the space–time is suggested to reduce the upward concavity (convexity). In the next section, the thick center vortex is introduced. In Sect. 3, a proof for the concavity of the quark potential using the lattice gauge theory is introduced. In Sect. 4, it is shown that an arbitrary symmetric vortex flux is needed to avoid the upward concavity of the potential. Also in this section, a method to avoiding the quark upward concavity is introduced and then applied to the thick center vortex model. Elimination of the upward concavity is shown for the \(SU(2)\) and \(SU(3)\) in different representations in the Casimir region, and a whole reduction is observed. Section 5 is devoted to the Casimir scaling properties of the Casimir using this method. Finally, our conclusion is given in Sect. 6.

2. Thick center vortex model

To obtain a quark confinement picture, it is common to generalize the mechanisms which prevent spontaneous symmetry breaking in ferromagnets with a continuous global symmetry group \(G\) to Euclidean gauge field theories. Natural analog of Bloch walls in ferromagnets exists in Yang–Mills theories with a gauge group \(G\) that has a nontrivial center element. These structures are center vortex. In ferromagnets, spontaneous magnetization breaks down when Bloch walls of large extension become sufficiently abundant. The absence of spontaneous magnetization leads to falloff of the two-point spin correlation function with distance. The simplest Bloch walls appear in two-dimensional Ising ferromagnets. Changes of spin direction from \(+1\) to \(−1\) lead to Bloch walls. These Bloch walls have a thickness of only one lattice spacing. However, in ferromagnets with continuous symmetry group, thick Bloch walls can also appear in which the spin direction rotates very gently as one crosses from one side to the other. These thick Bloch walls can be made responsible for the absence of spontaneous symmetry breakdown in two-dimensional ferromagnets.

In the reference \[13\], Mack and Petkova showed analytically that considering such thick center vortex volumes in \(D = 3\) and \(4\) dimension is sufficient to obtain a linear potential (confinement). In non-Abelian gauge theories, one hopes for an approximately linear rise of \(V(R)\) with \(R\). Vortices can (in principle) produce a falloff of \((TrU[C])\) with \(R\) much as Bloch walls can produce a falloff of spin correlation functions in ferromagnets. Mack and Petkova found after some calculation without any approximation that the quark potential obey the following inequality:

\[
V(R) > \text{const.} \times R \ln(R)^{-2},
\]

in which \(V(R)\) rises approximately linearly. They showed that this sufficient condition for confinement of static quarks is applicable for any arbitrary compact gauge group \(G\) with nontrivial center \(Z\), in particular for \(G = SU(N)\), with center subgroup \(Z_N\), \(N = 2, 3, \ldots\) (Among the simply connected compact simple Lie groups, only \(G_2, F_4, E_6\) have trivial center.). Following their calculation, they showed an estimation for thin vortices in an Abelian theory. They found a lower bound as follows:

\[
V(R) > c(\beta) \ln R.
\]

This is the confining Coulomb potential without linear behavior (due to the absence of nontrivial center element). Based on such idea, thick center vortex model appeared as a theory of confinement.

The confinement mechanism can be related to the Abelian sector especially center elements of a gauge group \[45\]. This is done through the center vortex idea \[46, 49–51\]. Vortices here are typically \(1 + 1\)-dimensional soliton-like solution which is embedded in \(1 + 3\) dimension. They form closed surfaces which can be linked to Wilson loop. In the center vortex picture, the presence of vortices
in the vacuum is due to the center elements and their fluctuation in the number of center vortices linked to the loop. It is leading to an area law for Wilson loop from which a linear potential or string-like behavior is appearing. Wilson loops are gauge-invariant observable obtained from the holonomy of the gauge connection around given loops. Confinement is obtained from random fluctuations in the linking number. A vortex piercing a Wilson loop contributes to a center element somewhere between the group elements \( U \) of the gauge group. The Wilson loop is represented by

\[
W(C) = Tr[UUU...U] \rightarrow Tr[UU...U(Z)U]
\]  
(3)

Center elements commute with all elements of the group, so the location of \( Z \) in Eq. (3) can be changed by changing the place of discontinuity which lead to a vortex formation. In the SU(2) group, for example, the string tension \( \sigma \) can be obtained considering the vacuum expectation value of the Wilson loop such that:

\[
\langle W(C) \rangle = \prod \{(1 - f) + f(-1)\} \langle W_0(C) \rangle
= \exp[-\sigma(C)A]\langle W_0(C) \rangle.
\]  
(4)

\( f \) here is the probability of piercing a plaquette with a thin vortex somewhere on a Wilson loop, and \( W_0(C) \) is the Wilson loop with no linking to a vortex. \( A \) is the area of the Wilson loop and is equal to \( R \times T \). \( R \) is for the space side, and \( T \) is for time side of the Wilson loop. String tension can be obtained as follows:

\[
\sigma = -\frac{1}{A} \ln(1 - 2f).
\]  
(5)

Center vortex scenario can explain the asymptotic string behavior in different representations of the gauge group, but cannot explain the intermediate behavior of the quark potential. In this scenario, the vortices are considered thin. However, the results of lattice gauge theory (LGT) show that these vortex structures have a comparable thickness. There is evidence in the SU(2) theory that the vortices defined by center gauging and center projection indeed are localize thick vortices as defined by their center element contributions to linked Wilson loops [47, 48]. Having an explanation for these properties a thickness must be considered for the vortices. This is done by considering the parameter \( G \) instead of the center element \( Z \) such that

\[
G(x, s) = S \exp(i \alpha(x) \vec{n} \vec{L}) S^{-\dagger}
\]  
(6)

in above equation \( L_i \)'s are the generators of the group in the representation \( j \), \( n \) is a unit vector and \( S \) is an element of the group SU(N) in the representation \( j \). Also in Eq.(6), \( \alpha(x) \) gives the profile of the vortex, and it depends on that fraction of the vortex which is pierced by the loop. In fact, it depends on the shape of the loop \( C \) and the position of the center of the vortex relative to the perimeter of the loop. The Wilson loop expectation value considering the thickness is obtained as follows:

\[
V(R) = \sum_x \ln \{1 - \sum_{n=1}^{N-1} f_n(1 - \text{Reg}_R[\alpha^n_C(x)])\},
\]  
(7)

In Eq.(6), \( g_r[\alpha^n_C(x)] \) is obtained by averaging over group space direction as follows:

\[
g_r[\alpha] = \frac{1}{d_r} \text{Tr}(\exp[i \vec{n} \vec{H}])
\]  
(8)

In this equation, \( H_s \) are the diagonal generators of the representation of the group. For the vortex flux, the following ansatz is considered

\[
\alpha^n_C(x) = N^n [1 - \tan h(ay(x) + \frac{b}{R})]
\]  
(9)

\( a \) and \( b \) are the parameters of the model, and \( y(x) \) is given by

\[
y(x) = \begin{cases} 
-x & |R - x| > |x| \\
-x - R & |R - x| \leq |x|
\end{cases}
\]  
(10)

\( y(x) \) is the nearest distance of the vortex center \( x \), from time-like side of the Wilson loop. The normalization constant \( N^n \) is obtained from the maximum flux condition, where the loop contains the vortex completely, such as

\[
\exp(i \vec{x} \vec{H}) = z_n I
\]  
(11)

where \( z_n \) is presented by

\[
z_n = \exp \left( \frac{2 \pi i n}{N} \right) \in Z_N.
\]  
(12)

\( I \) is the unit element of the group. \( z_n \)'s are the center elements of the group. Using this model, the quark potential in different gauge groups can be obtained. Here, the model is applied to the simplest non-Abelian group SU(2) and the QCD color symmetry SU(3). To obtain the quark potential considering this model, the diagonal generators are needed, and the normalization constant should be obtained. The fundamental representation generators are proportional to the Pauli matrices for the SU(2) group. The free parameters of the model are considered as follows:

\[
a = 0.05, b = 4, f = 0.1.
\]

The quark potentials behavior using this model is shown in Fig. 1. For the fundamental representation and also the adjoint and also four representations, the potentials are obeying the 2-ality in the final quark potential behavior. An upward concavity is observed in the quark potential at distances 10–30 which is not
According to the quark potential condition, it is not possible for the quark potential to have an upward concavity.

In the $SU(3)$ gauge group, the generators of the fundamental representation are Gell–Mann matrices. In $SU(3)$, there are two diagonal matrices which are used in this model. The normalization conditions are applied for each
representation. Also diagonal generators of higher representations are obtained using the tensor method. Using the thick center vortex model, the quark potential behavior is shown in different representations of the SU(3) group in Fig. 2. Again, the true 3-ality is observed in different representations of the group. For the adjoint representation with 0-ality, a screening asymptotic behavior is observed. For the other representations, asymptotic linear behavior is observed. Again, an upward concavity is observed in the quark potential with respect to the distance between quark and antiquark. The force between quark and antiquark should be attractive. A proof for the concavity of quark potential is length, and its unit is lattice unit. This non-physical behavior should be removed from the model. Previously, some methods are used to remove this non-physical behavior of the quark potential using this model in the SU(3) gauge theory can be written as follows:

\[ V(R) = \lim_{T \to \infty} \left\{ -\frac{1}{T} \ln(g \langle trU(W) \rangle) + \text{const.} \right\} \tag{16} \]

Where expectation value of a quantity can be obtained using Feynman path integral method as follows:

\[ \langle trU(W) \rangle = \frac{\int \prod_p [dU(P)] e^{-\frac{1}{g^2} S}}{\int \prod_p [dU(P)] e^{-\frac{1}{g^2} S}} \tag{17} \]

In this equation, \( dU(P) \) is the group measure. We will use the notation \( QZ \) as \( QZ = \int \prod_p [dU(P)] e^{-S} \) in our further calculation. The action has reflection positivity which is due to a positive metric Hilbert space. To achieve this reality, it is better to consider a three-dimensional hyperplane normal to the primary axis of the lattice. For example, consider \( p^1 = 0 \) hyperplane and denote the sites of plaquettes with minimal area that lie on the right and left sides of the border with \( L_R, L_L, L_D \), respectively. Taking the border place is arbitrary and due to the \( p^1 \) in lattice site, the reflection operator \( \Theta \) can be considered as follows:

\[ \Theta F(U(P)) = F^*(\Theta U(P)) = F^*(U(\Theta P)) \tag{18} \]

\[ \Theta p = \Theta(p^1, p^2, p^3, p^4) = (-p^1, p^2, p^3, p^4), \tag{19} \]

\[ \Theta P = \Theta(p, p') = (\Theta p, \Theta p') \tag{20} \]

\( p \) is abbreviation of lattice site coordinate \( p^u = (p^1, p^2, p^3, p^4) \). For the calculation of the plaquettes, the presence of the minimal area of the plaquettes on the border or at right side or left side of the border is important. Using \( \Theta \) operator, one can introduce the Osterwalder–Schrader positivity property or reflective positivity [55, 56] which is the main ingredient for establishing the existence of a positive semi definite self-adjoint Hamiltonian. Using this operator, it is possible to write all functions \( F \) of link variable on \( L_R, L_D \) as

\[ S = \frac{1}{g^2} \sum_{\Delta \Delta} \Re Tr \prod_{t \in \square} U_t(P) \tag{15} \]
\[\langle F_1 \hat{\Theta} F_2 \rangle = Q_2^{-1} \int \prod_{P \in L_L} [dU(P)] \exp\left(-\frac{1}{g} \sum_{q \in L_L} \text{ReTr} U_q(P)\right) \]
\[\times \prod_{P \in L_R} [dU(P)] F(U(P)) \exp\left(-\frac{1}{g} \sum_{q \in L_R} \text{ReTr} U_q(P)\right)\]
\[\times \prod_{P \in L_L} [dU(P)] F'(U'(\hat{\Theta})) \exp\left(-\frac{1}{g} \sum_{q \in L_L} \text{ReTr} U_q(P)\right) \]
\[\times \prod_{P \in L_R} [dU(P)] F(U(P)) \exp\left(-\frac{1}{g} \sum_{q \in L_R} \text{ReTr} U_q(P)\right)\]
\[= Q_2^{-1} \prod_{P \in L_L} [dU(P)] \exp\left(-\frac{1}{g} \sum_{q \in L_L} \text{ReTr} U_q(P)\right)\]
\[\times \prod_{P \in L_R} [dU(P)] F(U(P)) \exp\left(-\frac{1}{g} \sum_{q \in L_R} \text{ReTr} U_q(P)\right)\]
\[\quad \geq 0\] (21)

In above equation, Schwarz-type inequality 
\[\langle F_1 \hat{\Theta} F_2 \rangle^2 \leq \langle F_1 \hat{\Theta} F_1 \rangle \langle F_2 \hat{\Theta} F_2 \rangle\]
is used. The former part of the above equation is only a separation of the functional integral into three parts relative to the border. In the latter part, we have used the Schwarz inequality to convert the \(L_L\) to a \(L_R\) integral. We have three types plaquettes in 4D relative to border which are the ones only on the hyperplane or border. These plaquettes minimal area are exactly on the hyperplane. No link variable is outside this surface. In fact, these plaquettes make the border. There are plaquettes which their minimal area is not on the border. On the other words, the minimal area of the plaquettes is important to consider them at the \(L_R\) and \(L_L\), not their interaction link in the hyperplane of the border. In addition to the plaquettes with minimal area outside the border at the right side with link variables that contain lattice sites on the border, there is also plaquette with minimal area outside the border at the left side with link variables that contain lattice sites on the border. Consider this inequality for the hyperplane parallel to the time axis and normal to the Wilson loop in Fig. 3. Then using the reflection properties, it leads to:
\[\langle trU(W) \rangle = \sum_{ij} \langle U(W_1)_{ij} \hat{\Theta} U(W_2)_{ij} \rangle\]
\[\langle trU(W) \rangle = \sum_{ij} \langle U(W_1)_{ij} \hat{\Theta} U(W_1)_{ij} \rangle^\dagger \langle U(W_2)_{ij} \hat{\Theta} U(W_2)_{ij} \rangle^\dagger\]
\[\langle trU(W) \rangle = \langle tr[U(W_1) U(-\hat{\Theta} W_1)] \rangle^\dagger \langle tr[U(W_2) U(-\hat{\Theta} W_2)] \rangle^\dagger\] (22)

Using the definition for the quark potential, this inequality means
\[V(R) \geq \frac{1}{2} V(R - r) + \frac{1}{2} V(R + r)\] (23)

This means a concave quark potential. So, it seems the reflection symmetry for the functions, and variable is the building block of concavity criteria, and if in a model, such reflection properties are not considered, it leads to the violation of the concavity condition. In the thick center vortex idea, a vortex flux is considered which is not reflective relative to the an arbitrary line. So, to obey the concavity in the model, it seems such arbitrary symmetry is essential. In the next section, it is shown that such symmetry is not considered for the thick center vortex flux in plane and try to consider reflective symmetric vortex fluxes relative to an arbitrary line and study its effects on the quark potential.

4. Elimination of the upward concavity using an arbitrary symmetric vortex flux in the space–time hyperplane space–time

In the previous section, a proof for the concavity is explained. A symmetry in the space–time hyperplane (Fig. 3) for the Wilson loop is essential to obtain this criterion. However, considering the presence of the thick center vortices in the plane breaks this symmetry. Then for
such asymmetry, the formulas (21,22,23) cannot lead to the concavity criterion. The question here is how we can use the concavity proof again in the presence of the vortices in the plane? Fig. 4.i shows the situation of the presence of the vortices in the plane. As it is clear, the vortex flux in the plane breaks the arbitrary symmetry of the plane. To investigate this, we take a look at the center vortex mechanism and try to apply the changes to the thick center vortex model related to the concavity proof.

In the center vortex mechanism, the vortices effects on a Wilson loop are considered through the center elements of the group $Z$. No thickness is considered for the vortices in this model. This is clear in Eq. 3. However, to consider the concavity proof for the center vortex model, the symmetry of piercing of the vortices should be accounted because of the presence of the $Z$ in the upper or lower plane. This can be done through considering two vortex piercing, one in the lower plane and one in the upper plane. The additional vortex leads to a symmetry for the Wilson loop in the upper and lower plane. A $Z^+$ is considered for the presence of a symmetric vortex with respect to the $Z$ taking the reflection operator $\hat{\Theta}$. This is the simplest ansatz one can consider. Due to this, we can write:

$$
\langle F \Theta F \rangle = Q_0^2 \int \prod_{\mathcal{I}_d} |dU(P)| \exp \left( -\frac{1}{g^2} \sum_{\mathcal{I}_d} \text{Re} \text{Tr} \prod_{\mathcal{I}_d} U_i(P) \right) \int \prod_{\mathcal{I}_d} |dU(P)| |U_1 U_2 Z \ldots U_1| \exp \left( -\frac{1}{g^2} \sum_{\mathcal{I}_d} \text{Re} \text{Tr} \prod_{\mathcal{I}_d} U_i(P) \right) \\
\times \int \prod_{\mathcal{I}_d} |dU(P)| |U_1 U_2 Z \ldots U_1| \exp \left( -\frac{1}{g^2} \sum_{\mathcal{I}_d} \text{Re} \text{Tr} \prod_{\mathcal{I}_d} U_i(P) \right) \\
= Q_0^2 \int \prod_{\mathcal{I}_d} |dU(P)| \exp \left( -\frac{1}{g^2} \sum_{\mathcal{I}_d} \text{Re} \text{Tr} \prod_{\mathcal{I}_d} U_i(P) \right) \\
\int \prod_{\mathcal{I}_d} |dU(P)| |U_1 U_2 Z \ldots U_1| \exp \left( -\frac{1}{g^2} \sum_{\mathcal{I}_d} \text{Re} \text{Tr} \prod_{\mathcal{I}_d} U_i(P) \right) \geq 0
$$

Again the Schwarz inequality is applied. By omitting $ZZ^* = |Z|^2$, we arrive at:

$$
|Z|^4 \langle F_1 \hat{\Theta} F_2 \rangle^2 \leq |Z|^2 \langle F_1 \hat{\Theta} F_1 \rangle |Z|^2 \langle F_2 \hat{\Theta} F_2 \rangle
$$

Again, all of the previous proof of the concavity is getting validation. So, in the center vortex model, considering a symmetric ansatz for the piercing of the vortex in the Wilson loop, the center vortex model can explain the performed situation.

For the thick center vortex, $W(C) = \text{Tr}[UUU...U] \rightarrow \text{Tr}[UU...(G(x,s))U]$ is used for the piercing of the vortex where $G(x, s)$ is introduced in Eq. 6. Now, the validity of the concavity could be examined. Here, a vortex ansatz with a symmetric $G$ ansatz is considered. Due to the vortex thickness, a $G_1$ portion is considered for the presence of the vortex in the lower plane, and a $G'$ portion is considered for the piercing of the vortex in the upper plane and also
consider $G_1(x,s)G_1^*(\hat{\Theta}x,s) = G$. So, it seems that relative to the border, the ansatz divides the vortex flux into the two symmetric parts; one in the upper plane and one in the lower plane. Consequently, we can write:

$$\tilde{z}_1(x) = N^n[1 - \tanh(ay(x + R) + \frac{b}{R})]$$

$$\tilde{z}_2(x) = N^n[1 - \tanh(ay(x - R) + \frac{b}{R})]$$

$$\tilde{z}(x) = \tilde{z}_1(x) + \tilde{z}_2(x)$$

(27)

$$\langle F \tilde{\Theta} F \rangle = Q^{-1}_Z \int \prod_{P \in L_a} [dU(P)] \exp(-\frac{1}{g^2} \sum_{i \in L_a} \text{Re} \text{Tr} \prod_{U_i(P)}(\prod_{U_i(P)}) \int \prod_{P \in L_r} [dU(P)] U_1U_2G_1...U_l \exp(-\frac{1}{g^2} \sum_{i \in L_r} \text{Re} \text{Tr} \prod_{U_i(P)})$$

$$\times \prod_{P \in L_r} \left[ dU(P) \right] U_1U_2U_3...U_l \exp\left(-\frac{1}{g^2} \sum_{i \in L_r} \text{Re} \text{Tr} \prod_{U_i(P)} \right)$$

$$= Q^{-1}_Z \int \prod_{P \in L_a} [dU(P)] \exp(-\frac{1}{g^2} \sum_{i \in L_a} \text{Re} \text{Tr} \prod_{U_i(P)}(\prod_{U_i(P)}) \int \prod_{P \in L_r} [dU(P)] U_1U_2U_3...U_l \exp\left(-\frac{1}{g^2} \sum_{i \in L_r} \text{Re} \text{Tr} \prod_{U_i(P)} \right)$$

(26)

Again, the previous proof can be true by considering an ansatz with symmetric $G$ relative to the border of the upper and lower plane. So, if the symmetry restored, the concavity proof would be applicable again. A symmetric ansatz with the equal portion relative to the border of the upper and lower plane leads again to the validation of concavity formulas. Figure 4.i shows this situation in which a symmetric vortex profile with equal distances to the intersection is considered. To consider such situation, two vortex fluxes are introduced as follows:

$$\tilde{z}_1(x) = N^n[1 - \tanh(ay(x + R) + \frac{b}{R})]$$

$$\tilde{z}_2(x) = N^n[1 - \tanh(ay(x - R) + \frac{b}{R})]$$

$$\tilde{z}(x) = \tilde{z}_1(x) + \tilde{z}_2(x)$$

In this equation, $R$ is the space side of the Wilson loop. Using such ansatz, the symmetry is restored, and the concavity proof becomes valid. For the symmetric flux in the lower plane, a vortex with overlapping with the dual Wilson loop ($W_2 \cup \hat{\Theta}W_1$) is considered. Figures 5 and 6 show the behavior of the vortex flux relative to the center of the vortices. Figure 7 shows the quark potentials for the different representations of the $SU(2)$ group using the symmetric vortex flux. Figure 8 shows the quark potentials for the representation of the $SU(3)$ group. As it is seen, the

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**Fig. 5** Two symmetric vortex fluxes are considered to restore the arbitrary symmetry of the space–time hyperplane for the intervals of x+R and x-R in the $SU(2)$ gauge group
Fig. 6 Two symmetric vortex fluxes are considered to restore the arbitrary symmetry of the space–time hyperplane for the intervals of $x+R$ and $x-R$ in the $SU(3)$ gauge group.

Fig. 7 Quark potentials behavior for the $SU(2)$ gauge group and its various representations are plotted with the introduced arbitrary symmetric flux. By comparing this figure and Fig. 1, it is well seen that the non-physical upward concavity especially for $4_s$ representation is reduced.

non-physical upward concavity present in Figs. 1 and 2 is removed in Figs. 7 and 8 in the Casimir region due to the restoration of the arbitrary symmetry in the plane. So to obtain the true quark potentials using the thick center vortex model, a symmetric vortex flux can be introduced in the plane. The Casimir region of the potential is the part of the potential which is due to the vortex thickness. Considering symmetric vortex thickness, the upward concavity from this region is removed, and an overall reduction in the upward concavity occurs.
Note that the line for considering such symmetry is arbitrary line which can be considered in the vacuum. There is no forced place for the presence of such border in the vacuum. Especially the line is considered somewhere with no symmetry relative to the Wilson loop to show such arbitrariness of the position of the line or border between upper and lower hyperplanes. The line can be considered randomly, or for every vortex in the QCD vacuum, we can find at least one vortex with similar flux present in the QCD vacuum. The line in two dimensions or hyperplane in four dimensions can be considered with equal distance between these two vortices. For every Wilson loop with the splitting line, the proof is correct. For example, if we consider another Wilson loop in the vacuum space again, we can consider such arbitrary line which divide the space of the vacuum into the upper and lower space. This arbitrariness of the position of the line excludes any new symmetry for the QCD vacuum. But relative to any splitting line, we consider a symmetric vortex flux. Due to the fluctuation of the vortex fluxes, this consideration is not very far reaching. Instead of considering any fluctuation of the vortex fluxes, a symmetric vortex fluxes are considered for the vortices in the vacuum relative to this arbitrary line. Also, we do not consider the position of vortices to be fixed in the space–time and considering a movement of the vortices by enlarging the Wilson loop. This is done by considering the center of vortex fluxes at $x - R$ and $x + R$ in which $R$ is the space side of the Wilson loop. For example, by expanding the Wilson loop from $R = 10$ to $R = 100$, the position of these vortex fluxes will change. The movement of the vortex fluxes with such function can be a simple model for the fluctuation of the vortices within the QCD vacuum for this model. So, the symmetry applied in the article is not a real QCD vacuum symmetry but an accidental symmetry in the QCD vacuum due to the fluctuation and movement of vortices in the QCD vacuum.

Also the domain structure idea for the QCD vacuum can be examined [32–35, 41, 42]. In the domain structure picture of the QCD vacuum, trivial center element of the group also can affect the behavior of the linear potential. So in this model which is a modification of the thick center vortex model, the trivial center element portion is considered in the model through the quark potential equation as follows:

$$V(R) = \sum_x Ln\{1 - \sum_{n=0}^{N-1} f_n (1 - Reg_{\text{trivial}}[\mathcal{Z}_C(x)])\}.$$  \hspace{1cm} (28)

$\mathcal{Z}_C(x)$ is the center element portion, and its flux is normalized to the trivial center element $I$. In Eq. (28), $f_n$ with $n = 0$, $f_0$, is the probability of trivial domain piercing the Wilson loop minimal area. A $f_0 = 0.025$ is considered for the $SU(2)$, and $f_0 = 0.05$ is considered for the $SU(3)$ group to obtain the quark potential. Figures 9 and 11 show the quark potential for the $SU(2)$ and $SU(3)$ before considering the symmetric profile ansatz. Again, the upward concavity is present at distances $R = 10–40$. Figures 10 and 12 show the quark potential for the $SU(2)$ and $SU(3)$ considering the symmetric ansatz using the domain structure model. As it
can be seen, again the upward concavity is removed from the Casimir region and even an overall better reduction of the upward concavity is observed even with respect to the thick center vortex model.

5. Casimir scaling considering the symmetric vortex fluxes

One of the characteristic properties of the quark potentials is Casimir scaling. According to this property, the quark linear potentials at the intermediate region should be
proportional to the Casimir eigen value of the representation of the group. Consider \( C_r \) as the Casimir eigen value of the representation \( r \) and the quark potentials slope as \( k_r \) then according to the Casimir rule:

\[
k_r = \frac{C_r}{C_F} k_F
\]  

In which \( F \) is the index to denote for the fundamental representation. Figure 13 shows a close look at the potentials in this region for the representations of the \( SU(2) \) with
non-symmetric vortex fluxes. Figure 14 shows the potentials behavior at intermediate region with two vortex fluxes. The Casimir ratios using the model can be obtained by dividing the potential values to the potential values of the fundamental representation for each distance at the intermediate region. Figure 15 shows the Casimir ratios for the $SU(2)$ by considering one vortex flux in the plane. Figure 16 shows Casimir ratios using two vortex fluxes with
symmetric fluxes. The Casimir ratios are $\frac{C_3}{C_2} = \frac{8}{3}$, $\frac{C_4}{C_2} = 5$.

Figure 17 shows the quark potentials for the $SU(3)$ representations at the intermediate region with non-symmetric vortex fluxes. Figure 18 shows the quark potentials for the $SU(3)$ representations at the intermediate region with symmetric vortex fluxes. Figure 19 shows the Casimir ratios for the $SU(3)$ group with one vortex flux. The Casimir ratios are $\frac{C_6}{C_3} = 2.5$, $\frac{C_8}{C_3} = 2.25$. Figure 20 shows
Casimir ratios when two symmetric vortex are present in the plane. For these groups, the effects of two symmetric vortex fluxes in the plane on the Casimir scaling are not very much, and the Casimir scaling behavior is present using these symmetric vortex fluxes. So, by considering symmetric vortex fluxes, the upward concavity is removed with an acceptable Casimir scaling.
6. Conclusions

Thick center vortex model and the domain structure picture of the QCD vacuum are the successful methods to explain the confinement problem. One of the important properties of the quark potentials is the downward concavity. As it is seen, this property is violated using the thick center vortex model. According to the reference [53], downward concavity should be valid for every gauge group. The Bachas proof does not relate to any gauge group. Even in an Abelian theory according to the reference [15], sufficient condition is proportional to the $\ln(R)$ potential which is
concave but is not linear due to the absence of nontrivial center in this gauge group. Here, it is shown that this property can be obtained using an arbitrary symmetric vortex flux. By considering such situation, the upward concavity is removed from the quark potentials at the intermediate region and also an overall reduction of the upward concavity is observed for the quark potential. Applying the domain structure idea and considering a portion for the trivial center element in the quark potential behavior leads to even better reduction of the upward concavity. The vortex thickness is related to the slope of the intermediate region of the quark potentials which are obtained from the thick center vortex model. The symmetric vortex flux leads to the elimination of the upward concavity within the intermediate region. By adding the trivial domain portion to the model and reduction of the upward concavity, it seems that the remaining upward concavity is causing by a reason other than the symmetric vortex profile. It may be related to the free parameters of the models such as $f$, $a$ and $b$. The presence of two vortices in the plane can lead to interesting properties of the interaction between vortices. This can be studied as a further research task by considering the interaction between vortices [57].

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