Black brane solutions related to non-singular Kac-Moody algebras

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Abstract

A multidimensional gravitational model containing scalar fields and antisymmetric forms is considered. The manifold is chosen in the form $M = M_0 \times M_1 \times \ldots \times M_n$, where $M_i$ are Einstein spaces ($i \geq 1$). The sigma-model approach and exact solutions with intersecting composite branes (e.g., solutions with harmonic functions and black brane ones) with intersection rules related to non-singular Kac-Moody (KM) algebras (e.g. hyperbolic ones) are considered. Some examples of black brane solutions are presented, e.g., those corresponding to hyperbolic KM algebras: $H_2(q,q) (q > 2)$, $HA_2^{(1)} = A_2^{++}$ and to the Lorentzian KM algebra $P_{10}$.
1 Introduction

In this paper, we consider certain classes of black brane solutions related to non-singular Kac-Moody algebras. At present, Kac-Moody (KM) Lie algebras [1, 2] play rather an important role in different areas of mathematical physics (see [3, 4] and references therein). We recall that KM Lie algebra is a Lie algebra generated by the relations [3]

\[ [h_i, h_j] = 0, \quad [e_i, f_j] = \delta_{ij}h_j, \quad (1.1) \]
\[ [h_i, e_j] = A_{ij}e_j, \quad [h_i, f_j] = -A_{ij}f_j, \quad (1.2) \]
\[ (ade_i)^{1-A_{ij}}(e_j) = 0 \quad (i \neq j), \quad (1.3) \]
\[ (adf_i)^{1-A_{ij}}(f_j) = 0 \quad (i \neq j). \quad (1.4) \]

Here \( A = (A_{ij}) \) is a generalized Cartan matrix, \( i, j = 1, \ldots, r \), and \( r \) is the rank of the KM algebra. It means that all \( A_{ii} = 2 \); \( A_{ij} \) for \( i \neq j \) are non-positive integers and \( A_{ij} = 0 \) implies \( A_{ji} = 0 \). In what follows, the matrix \( A \) is restricted to be non-degenerate (i.e. \( \det A \neq 0 \)) and symmetrizable, i.e. \( A = BD \), where \( B \) is a symmetric matrix and \( D \) is an invertible diagonal matrix.

If \( A \) is positive-definite (the Euclidean case) we get ordinary finite-dimensional Lie algebras [3, 4]. For non-Euclidean signatures of \( A \) all KM algebras are infinite-dimensional. Among these the Lorentzian KM algebras with pseudo-Euclidean signatures (\( - \)) of the Cartan matrix \( A \) are of current interest since they contain a subclass of the so-called hyperbolic KM algebras widely used in modern mathematical physics. Hyperbolic KM algebras are by definition Lorentzian Kac-Moody algebras with the property that, removing any node from their Dynkin diagram leaves one with a Dynkin diagram of the affine or finite type. The hyperbolic KM algebras were completely classified in [7, 8]. They have rank \( 2 \leq r \leq 10 \). For \( r \geq 3 \) there is a finite number of hyperbolic algebras. For rank 10, there are four algebras, known as \( E_{10}, BE_{10}, CE_{10} \) and \( DE_{10} \). Hyperbolic KM algebras appeared in ordinary gravity [9] (\( F_3 = AE_3 = H_3 \)), supergravity: [10, 11] (\( E_{10} \)), [12] (\( F_3 \)), strings [13], oscillating behaviour near the singularity [14] (see also [15, 16], etc.

It has been proposed by P. West that the Lorentzian (non-hyperbolic) KM algebra \( E_{11} \) is responsible for a hidden algebraic structure characterizing 11D supergravity [17]. The same very extended algebra occurs in \( IIA \) [17] and \( IIB \) supergravities [18].

Here we briefly consider another possibility of utilizing non-singular (e.g. hyperbolic) KM algebras, suggested in our three papers [19, 20, 21]. This possibility also implicitly assumed in [22, 23, 24, 25, 26]. It is related to certain classes of exact solutions describing intersecting composite branes in a multidimensional gravitational model containing scalar fields and antisymmetric forms defined on (warped) product manifolds \( M = M_0 \times M_1 \times \ldots \times M_n \), where \( M_i \) are Ricci-flat spaces (\( i \geq 1 \)). From a pure mathematical point of view, these solutions may be obtained from sigma-models or Toda chains corresponding to certain non-singular KM algebras. The information about a (hidden) KM algebra is encoded in intersection rules which relate the dimensions of brane intersections with non-diagonal components of the generalized Cartan matrix \( A \) [27]. We deal here with generalized Cartan matrices of the form

\[ A_{ss'} = \frac{2(U_s, U_{s'})}{(U_s, U_s)}. \quad (1.5) \]

\( s, s' \in S \), with \( (U_s, U_s) \neq 0 \), for all \( s \in S \) (\( S \) is a finite set). Here \( U_s \) are the so-called brane (co-)vectors. They are linear functions on \( \mathbb{R}^N \), where \( N = n + l \) and \( l \) is the number of scalar fields. The indefinite scalar product \( (, ,) \) [28] is defined on \( (\mathbb{R}^N)^* \) and has the signature \( (-1, +1, \ldots, +1) \) if all scalar fields have positive kinetic terms, i.e. there are no phantoms (or ghosts). The matrix \( A \) is symmetrizable. \( U^* \)-vectors may be put in one-to-one correspondence with simple roots \( \alpha_s \) of the generalized KM algebra after a suitable normalization.

For \( D = 11 \) supergravity [29] and ten-dimensional \( IIA \), \( IIB \) supergravities all \( (U_s, U_s) = 2 \) [27, 30], and corresponding KM algebras are simply laced. It was shown in our papers [31, 32] that the inequality \( (U_s, U_s) > 0 \) is a necessary condition for the formation of a billiard wall (if one approaches the singularity) by the s-th matter source (e.g., a fluid component or a brane).

The scalar products for brane vectors \( U_s \) were found in [28]

\[ (U_s, U_s') = d_{ss'} + \frac{d_s d_{s'}}{2-D} + \chi_s \chi_{s'} < \lambda_s, \lambda_{s'}>, \quad (1.6) \]
where $d_s$ and $d_{s'}$ are the dimensions of brane world volumes corresponding to branes $s$ and $s'$, respectively, $d_{ss'}$ is the dimension of the intersection of brane world volumes, $D$ is the total space-time dimension, $\chi_s = +1, -1$ for electric or magnetic brane respectively, and $< \lambda_s, \lambda_{s'} >$ is the non-degenerate scalar product of the $l$-dimensional dilatonic coupling vectors $\lambda_a$ and $\lambda_{a'}$ corresponding to branes $s$ and $s'$.

The relations (1.5), (1.6) determine the brane intersection rules [27]:

$$d_{ss'} = d_{s's'}^0 + \frac{1}{2} K_{s's'} A_{ss'},$$  \hspace{1cm} (1.7)

$s \neq s'$, where $K_s = (U^s, U^s)$ and

$$d_{s's'}^0 = \frac{d_s d_{s'}}{D-2} - \chi_s \chi_{s'} < \lambda_s, \lambda_{s'} >$$  \hspace{1cm} (1.8)

is the dimension of the so-called orthogonal (or $(A_1 \oplus A_1)$-) intersection of branes following from the orthogonality condition [28]:

$$(U^s, U^{s'}) = 0,$$  \hspace{1cm} (1.9)

$s \neq s'$.

The relations (1.6) and (1.8) were derived in [28] under rather general assumptions: the branes were composite, the number of scalar fields $l$ was arbitrary as well as the signature of the bilinear form $<.,.>$ (or, equivalently, the signature of the kinetic term for scalar fields), Ricci-flat factor spaces $M_i$ had arbitrary dimensions $d_i$ and signatures. The intersection rules (1.8) appeared earlier in [33, 34] for all $d_i = 1$ ($i > 0$) and $<.,.>$ being positive-definite.

2 The model

2.1 The action

We consider a model governed by the action

$$S = \frac{1}{2 \kappa^2} \int_M d^D z \sqrt{|g|} \{ R[g] - 2 \Lambda - h_{\alpha\beta} g^{MN} \partial_M \varphi^\alpha \partial_N \varphi^\beta - \sum_{a \in \Delta} \frac{\theta_a}{n_a!} \exp[2 \lambda_a(\varphi)] (F^a)^2 \} + S_{GH},$$  \hspace{1cm} (2.1)

where $g = g_{MN} dz^M \otimes dz^N$ is the metric on the manifold $M$, dim $M = D$, $\varphi = (\varphi^a) \in \mathbb{R}^l$ is a vector of dilatonic scalar fields, $(h_{\alpha\beta})$ is a non-degenerate symmetric $l \times l$ matrix ($l \in \mathbb{N}$), $\theta_a \neq 0$,

$$F^a = dA^a = \frac{1}{n_a!} F^a_{M_1 \ldots M_{n_a}} dz^{M_1} \wedge \ldots \wedge dz^{M_{n_a}}$$

is an $n_a$-form ($n_a \geq 2$) on the $D$-dimensional manifold $M$, $\Lambda$ is a cosmological constant and $\lambda_a$ is a 1-form on $\mathbb{R}^l$: $\lambda_a(\varphi) = \lambda_{a\alpha} \varphi^\alpha$, $a \in \Delta$, $\alpha = 1, \ldots, l$. In (2.1) we denote $|g| = |\text{det}(g_{MN})|$, $(F^a)^2 = F^a_{M_1 \ldots M_{n_a}} F^a_{N_1 \ldots N_{n_a}} g^{M_1 N_1} \ldots g^{M_{n_a} N_{n_a}}$, $a \in \Delta$, where $\Delta$ is some finite set (for example, of positive integers), and $S_{GH}$ is the standard Gibbons-Hawking boundary term [35]. In models with one time all $\theta_a = 1$ and the signature of the metric is $(-1, +1, \ldots, +1)$; $\kappa^2$ is the multidimensional gravitational constant.

2.2 The Ansatz for composite branes

Consider the manifold

$$M = M_0 \times M_1 \times \ldots \times M_n,$$  \hspace{1cm} (2.2)

with the metric

$$g = e^{2\gamma(x)} g^0 + \sum_{i=1}^n e^{2\gamma^i(x)} \hat{g}^i,$$  \hspace{1cm} (2.3)
where \( g^0 = g^0_{\nu\mu}(x)dx^\nu \otimes dx^\mu \) is an arbitrary metric with any signature on the manifold \( M_0 \) and \( g^i = g^i_{m,n_i}(y_i)dy^m_i \otimes dy^n_i \) is a metric on \( M_i \) satisfying the equation

\[
R_{m,n_i}[g^i] = \xi_ig^i_{m,n_i},
\]

(2.4)

\( m_i, n_i = 1, \ldots, d_i; \xi_i = \text{const}, i = 1, \ldots, n. \) Here \( \hat{g}^i = p^*_i g^i \) is the pullback of the metric \( g^i \) to the manifold \( M \) by the canonical projection: \( p_i : M \to M_i, i = 0, \ldots, n. \) Thus, \( (M_i, g^i) \) are Einstein spaces, \( i = 1, \ldots, n. \) The functions \( \gamma, \phi^i : M_0 \to \mathbb{R} \) are smooth. We denote \( d_\nu = \dim M_\nu; \nu = 0, \ldots, n; \)

\[
D = \sum_{\nu=0}^n d_\nu.
\]

We assume all manifolds \( M_\nu, \nu = 0, \ldots, n, \) to be oriented and connected. Then the volume \( d_i \)-form

\[
\tau_i \equiv \sqrt{|g^i(y_i)|} dy^1_i \wedge \ldots \wedge dy^d_i,
\]

(2.5)

and signature parameter

\[
\varepsilon(i) \equiv \text{sign}(\det(g^i_{m_i,n_i})) = \pm 1
\]

(2.6)

are correctly defined for all \( i = 1, \ldots, n. \)

Let \( \Omega = \Omega(n) \) be a set of all non-empty subsets of \( \{1, \ldots, n\}. \) The number of elements in \( \Omega \) is

\[
|\Omega| = 2^n - 1.
\]

For any \( I = \{i_1, \ldots, i_k\} \in \Omega, i_1 < \ldots < i_k, \) we denote

\[
\tau(I) \equiv \hat{\tau}_{i_1} \wedge \ldots \wedge \hat{\tau}_{i_k},
\]

(2.7)

\[
\varepsilon(I) \equiv \varepsilon(i_1) \ldots \varepsilon(i_k),
\]

(2.8)

\[
M_I \equiv M_{i_1} \times \ldots \times M_{i_k},
\]

(2.9)

\[
d(I) \equiv \sum_{i \in I} d_i,
\]

(2.10)

Here \( \hat{\tau}_i = p^*_i \hat{\tau}_i \) is the pullback of the form \( \tau_i \) to the manifold \( M \) by the canonical projection: \( p_i : M \to M_i, i = 1, \ldots, n. \) We also put \( \tau(\emptyset) = \varepsilon(\emptyset) = 1 \) and \( d(\emptyset) = 0. \)

For fields of forms we consider the following composite electromagnetic ansatz:

\[
F^a = \sum_{I \in \Omega_{a,e}} F^{(a,e,I)} + \sum_{J \in \Omega_{a,m}} F^{(a,m,J)}
\]

(2.11)

where

\[
F^{(a,e,I)} = d\Phi^{(a,e,I)} \wedge \tau(I),
\]

(2.12)

\[
F^{(a,m,J)} = e^{-2\alpha_0(x)} * (d\Phi^{(a,m,J)} \wedge \tau(J))
\]

(2.13)

are elementary forms of electric and magnetic types, respectively, \( a \in \Delta, I \in \Omega_{a,e}, J \in \Omega_{a,m} \) and \( \Omega_{a,v} \subset \Omega, v = e, m. \) In \( \{2.13\} * = *[g] \) is the Hodge operator on \( (M, g). \)

For scalar functions we put

\[
\varphi^a = \varphi^a(x), \quad \Phi^s = \Phi^s(x),
\]

(2.14)

\( s \in S. \) Thus \( \varphi^a \) and \( \Phi^s \) are functions on \( M_0. \)

Here and below

\[
S = S_e \sqcup S_m, \quad S_v = \sqcup_{a \in \Delta} \{a\} \times \{v\} \times \Omega_{a,v},
\]

(2.15)

\( v = e, m. \) Here and in what follows \( \sqcup \) means the union of non-intersecting sets. The set \( S \) consists of elements \( s = (a_s, v_s, I_s), \) where \( a_s \in \Delta \) is color index, \( v_s = e, m \) is the electro-magnetic index and the set \( I_s \in \Omega_{a,v} \) describes the location of a brane.

Due to \( \{2.12\} \) and \( \{2.13\} \)

\[
d(I) = n_a - 1, \quad d(J) = D - n_a - 1,
\]

(2.16)

for \( I \in \Omega_{a,e} \) and \( J \in \Omega_{a,m} \)(i.e., in the electric and magnetic case, respectively).
2.3 The sigma model

Let \( d_0 \neq 2 \) and

\[
\gamma = \gamma_0(\phi) = \frac{1}{2 - d_0} \sum_{j=1}^{n} d_j \phi^j, \tag{2.17}
\]
i.e., the generalized harmonic gauge (frame) is used.

We put two restrictions on the sets of branes that guarantee the block-diagonal form of the energy-momentum tensor and the existence of the sigma-model representation (without additional constraints):

\[
\textbf{(R1)} \quad d(I \cap J) \leq d(I) - 2, \tag{2.18}
\]
for any \( I, J \in \Omega_{a,v}, \ a \in \Delta, \ v = e, m \) (here \( d(I) = d(J) \)) and

\[
\textbf{(R2)} \quad d(I \cap J) \neq 0 \text{ for } d_0 = 1, \quad d(I \cap J) \neq 1 \text{ for } d_0 = 3. \tag{2.19}
\]

It was proved in \cite{28} that equations of motion for the model \cite{24} and the Bianchi identities:

\[
d^F = 0, \tag{2.20}
\]
s \( \in S_m \), for fields from \((2.3), (2.11)-(2.14)\), under the restrictions \((2.18)\) and \((2.19)\), are equivalent to the equations of motion of the \(\sigma\)-model governed by the action

\[
S_{\sigma 0} = \frac{1}{2\kappa_0^2} \int d^6 x \sqrt{|g^0|} \left\{ R[g^0] - \hat{G}_{AB} g^{0\mu\nu} \partial_{\mu} \sigma^{A} \partial_{\nu} \sigma^{B} - \sum_{s \in S} \varepsilon_s \exp \left( -2U^s_\lambda \sigma^A \right) g^{0\mu\nu} \partial_{\mu} \Phi^s \partial_{\nu} \Phi^s - 2 \Phi^s \right\}, \tag{2.21}
\]
where \((\sigma^A) = (\phi^i, \varphi^\alpha), \ k_0 \neq 0\), the index set \( S \) is defined in \((2.15)\),

\[
V = V(\phi) = \Lambda e^{2\gamma(\phi)} - \frac{1}{2} \sum_{i=1}^{n} \xi_i d_i e^{-2\phi^i + 2\gamma(\phi)} \tag{2.22}
\]
is the potential,

\[
(G_{AB}) = \begin{pmatrix} G_{ij} & 0 \\ 0 & h_{\alpha\beta} \end{pmatrix} \tag{2.23}
\]
is the target space metric with

\[
G_{ij} = d_i \delta_{ij} + \frac{d_i d_j}{d_0 - 2} \tag{2.24}
\]
and co-vectors

\[
U^s_A = U^s_\lambda \sigma^A = \sum_{i \in I_s} d_i \phi^i - \chi_s \lambda_{a_s}(\varphi), \quad (U^s_A) = (d_i \delta_{I_s}, -\chi_s \lambda_{a_s}), \tag{2.25}
\]
s \( = (a_s, v_s, I_s) \). Here \( \chi_e = +1 \) and \( \chi_m = -1; \)

\[
\delta_{il} = \sum_{j \in l} \delta_{ij} \tag{2.26}
\]
is an indicator of \( i \) belonging to \( I \): \( \delta_{il} = 1 \) for \( i \in I \) and \( \delta_{il} = 0 \) otherwise; and

\[
\varepsilon_s = (-\varepsilon[g])^{(1-\chi_e)/2} \varepsilon(I_s) \theta_{a_s}, \tag{2.27}
\]
s \( \in S, \varepsilon[g] \equiv \text{sign det}(g_{MN}) \). More explicitly, \((2.27)\) reads

\[
\varepsilon_s = \varepsilon(I_s) \theta_{a_s} \text{ for } v_s = e; \quad \varepsilon_s = -\varepsilon[g] \varepsilon(I_s) \theta_{a_s} \text{ for } v_s = m. \tag{2.28}
\]

For finite internal space volumes \( V_i \) (e.g. compact \( M_i \)) and electric \( p \)-branes (i.e. all \( \Omega_{a,m} = \emptyset \)) the action \((2.21)\) coincides with the action \((2.4)\) when \( \kappa^2 = \kappa_0^2 \prod_{i=1}^{n} V_i \).
3 Solutions governed by harmonic functions

3.1 Solutions with a block-orthogonal set of $U^s$ and Ricci-flat factor-spaces

Here we consider a special class of solutions to the equations of motion governed by several harmonic functions, where all factor spaces are Ricci-flat and the cosmological constant is zero, i.e., $\xi = \Lambda = 0$, $i = 1, \ldots, n$. In certain situations these solutions describe extremal black branes charged by fields of forms.

The solutions crucially depend upon scalar products of $U^s$-vectors $(U^s, U'^s)$; $s, s' \in S$, where

$$(U, U') = G^{AB}U_AU'_B,$$  \hspace{1cm} (3.1)

for $U = (U_A), U' = (U'_A) \in \mathbb{R}^N$, $N = n + l$ and

$$(\hat{G}^{AB}) = \left( \begin{array}{cc} \delta^{ij} & 0 \\ 0 & h^{\alpha\beta} \end{array} \right)$$  \hspace{1cm} (3.2)

is the inverse matrix to the matrix $G^{ij}$. As in [36], we have

$G^{ij} = 0 \text{ for all } i, j, = 1, \ldots, n$.

The scalar products $\langle U^s, U'^s \rangle$ of the vectors $U^s$ were calculated in [28] and are given by

$$\langle U^s, U'^s \rangle = \frac{d(I_s \cap I_{s'})}{d} + \frac{d(I_s) d(I_{s'})}{2 - D} + \chi_s \chi'_s \lambda_{a,\alpha} \lambda_{a,\beta} h^{\alpha\beta},$$  \hspace{1cm} (3.4)

where $(h^{\alpha\beta}) = (h, -1)^{-1}$, and $s = (a_s, v_s, I_s)$, $s' = (a_{s'}, v_{s'}, I_{s'})$ belong to $S$. This relation is very important one since it encodes brane data (e.g., intersections) via the scalar products of $U$-vectors.

Let

$$S = S_1 \sqcup \ldots \sqcup S_k,$$  \hspace{1cm} (3.5)

$S_i \neq \emptyset$, $i = 1, \ldots, k$, and

$$\langle U^s, U'^s \rangle = 0$$  \hspace{1cm} (3.6)

for all $s \in S_i$, $s' \in S_{j}$, $i \neq j$; $i, j = 1, \ldots, k$. The relation $\langle U^s, U'^s \rangle = 0$ means that the set $S$ is a union of $k$ non-intersecting (non-empty) subsets $S_1, \ldots, S_k$. According to (3.6) the set of vectors $(U^s, s \in S)$ has a block-orthogonal structure with respect to the scalar product (3.1), i.e., it splits into $k$ mutually orthogonal blocks $(U^s, s \in S_i)$, $i = 1, \ldots, k$.

Here we consider exact solutions in the model (2.1) where vectors $(U^s, s \in S)$ obey the block-orthogonal decomposition (3.5), (3.6) with scalar products defined in (3.4) [19]. These solutions were obtained from the corresponding solutions to the $\sigma$-model equations of motion [19].

Proposition 1. Let $(M, g^0)$ be Ricci-flat: $R_{\mu\nu}[g^0] = 0$. Then the field configuration

$$g^0, \quad \sigma^A = \sum_{s \in S} \xi_s U^sA \nu_s^2 \ln H_s, \quad \Phi^s = \frac{\nu_s}{H_s},$$  \hspace{1cm} (3.7)

$s \in S$, satisfies the field equations corresponding to the action (2.27) with $V = 0$ if the real numbers $\nu_s$ obey the relations

$$\sum_{s' \in S} \langle U^s, U'^s \rangle \nu_s \nu_{s'}^2 = -1$$  \hspace{1cm} (3.8)

$s \in S$, the functions $H_s > 0$ are harmonic, i.e. $\Delta [g^0] H_s = 0, s \in S$, and $H_s$ coincide inside the blocks: $H_s = H_{s'}$ for $s, s' \in S_i, i = 1, \ldots, k$.

Using the sigma-model solution from Proposition 1 and the relations for contra-variant components [28].
defines the so-called "orthogonal" intersection rules (28) (see also (33, 34) for
rules (3.18) in this case have a simpler form (27):

\[ s = (a_s, v_s, I_s), \text{ we get (33):} \]

\[
g = \left( \prod_{s \in S} H_s^{2d(I_s)} \right)^{1/(2-D)} \left\{ g^0 + \sum_{s \in S} \left( \prod_{s \in S} H_s^{2\epsilon_s v_s^2} \delta_{I_s} \right) g^1 \right\}, \tag{3.10} \]

\[
\varphi^a = - \sum_{s \in S} \lambda^a_s \epsilon_s v_s^2 \ln H_s, \tag{3.11} \]

\[
F^a = \sum_{s \in S} F^s \delta^a_{\alpha_s}, \tag{3.12} \]

where \( i = 1, \ldots, n, \alpha = 1, \ldots, l, a \in \Delta \)

\[
\mathcal{F}^s = \nu_s dH_s^{-1} \wedge \tau(I_s), \text{ for } v_s = e, \tag{3.13} \]

\[
\mathcal{F}^s = \nu_s (\ast_0 dH_s) \wedge \tau(I_s), \text{ for } v_s = m, \tag{3.14} \]

\( H_s \) are harmonic functions on \( (M_0, g^0) \) which coincide inside the blocks (i.e., \( H_s = H_{s'} \) for \( s, s' \in S_i, \)
\( i = 1, \ldots, k \)) and the relations (3.8) on the parameters \( \nu_s \) are imposed. The matrix \( ((U^s, U^{s'})) \) and the parameters \( \epsilon_s, s \in S, \) are defined in (3.10) and (2.27), respectively; \( \lambda^a_s = \hbar^{\alpha \beta} \lambda_{a, \alpha} \) \( \ast_0 = \ast [g^0] \) is the Hodge operator on \( (M_0, g^0) \) and

\[
\tilde{I} = \{ 1, \ldots, n \} \setminus I \tag{3.15} \]

is the dual set. (In (3.14) we have redefined the sign of the parameter \( \nu_s. \))

3.2 Solutions related to non-singular KM algebras

Now we will study the solutions (3.10)-(3.14) in more detail and show that some of them may be related to non-singular KM algebras. We put

\[
K_s = (U^s, U^{s'}) \neq 0 \quad (3.16) \]

for all \( s \in S \) and introduce the matrix \( A = (A_{ss'}) \):

\[
A_{ss'} = \frac{2(U^s, U^{s'})}{(U^{s'}, U^{s'})}, \tag{3.17} \]

\( s, s' \in S. \) Here some ordering in \( S \) is assumed.

Using this definition and (3.11) we obtain the intersection rules (27)

\[
d(I_s \cap I_{s'}) = \Delta(s, s') + \frac{1}{2} K_{s'} A_{ss'}, \tag{3.18} \]

where \( s \neq s' \), and

\[
\Delta(s, s') = \frac{d(I_s) d(I_{s'})}{D - 2} - \chi_s \chi_{s'} \lambda_{a, \alpha} \lambda_{a', \beta} \hbar^{\alpha \beta} \tag{3.19} \]

defines the so-called “orthogonal” intersection rules (28) (see also (33, 34) for \( d_l = 1 \)).

In \( D = 11 \) and \( D = 10 \) (IIA and IIB ) supergravity models, all \( K_s = 2 \) and hence the intersection rules (3.18) in this case have a simpler form (27):

\[
d(I_s \cap I_{s'}) = \Delta(s, s') + A_{ss'}, \tag{3.20} \]

where \( s \neq s' \), implying \( A_{ss'} = A_{s's} \). The corresponding KM algebra is simply-laced in this case.

For \( \det A \neq 0 \) relation (3.8) may be rewritten in the equivalent form

\[
- \epsilon_s v_s^2 (U^{s}, U^s) = 2 \sum_{s' \in S} A_{ss'} = b_s, \tag{3.21} \]
where \( s \in S \), and \( (A^{ss'}) = A^{-1} \). Thus eq. (3.28) may be resolved in terms of \( \nu_s \) for certain \( \varepsilon_s = \pm 1 \), \( s \in S \). We note that due to (3.6) the matrix \( A \) has a block-diagonal structure and, hence, for any \( i \)-th block the set of parameters \( (\nu_s, s \in S_i) \) depends on the matrix inverse to the matrix \( (A_{ss'}; s, s' \in S_i) \).

Now we consider the one-block case such that the brane intersections are related to some non-singular KM algebras.

**Finite-dimensional Lie algebras** [20]

Let \( A \) be the Cartan matrix of a simple finite-dimensional Lie algebra. In this case \( A_{ss'} \in \{0, -1, -2, -3\} \), \( s \neq s' \). The elements of the inverse matrix \( A^{-1} \) are positive (see Ch. 7 in [4]) and hence we get from (3.21) the same signature relation as in the orthogonal case [28]:

\[
\varepsilon_s(U_s^a, U_s^b) < 0, \quad s \in S.
\]

If all \( (U_s^a, U_s^b) > 0 \), we get \( \varepsilon_s < 0 \), \( s \in S \). Moreover, all \( b_s \) are positive integers:

\[
b_s = n_s \in \mathbb{N}, \quad s \in S.
\]

The integers \( n_s \) coincide with the components of the twice dual Weyl vector in the basis of simple co-roots (see Ch. 3.1.7 in [4]).

**Hyperbolic KM algebras**

Let \( A \) be a generalized Cartan matrix corresponding to a simple hyperbolic KM algebra. For hyperbolic algebras, the following relations are satisfied

\[
\varepsilon_s(U_s^a, U_s^b) > 0, \quad s \in S.
\]

since all \( b_s \) are negative in the hyperbolic case [37]:

\[
b_s < 0, \quad s \in S.
\]

For \( (U_s^a, U_s^b) > 0 \) we get \( \varepsilon_s > 0 \), \( s \in S \). If \( \theta_{a_s} > 0 \) for all \( s \in S \), then

\[
\varepsilon(I_s) = 1 \quad \text{for} \quad v_s = e; \quad \varepsilon(I_s) = -\varepsilon(g) \quad \text{for} \quad v_s = m. \quad (3.26)
\]

For a pseudo-Euclidean metric \( g \) all \( \varepsilon(I_s) = 1 \), and hence all branes are Euclidean or should contain even number of time directions: 2, 4, ... For \( \varepsilon[g] = 1 \) only magnetic branes may be pseudo-Euclidean.

**B\(_D\)-models.** The \( B_D \)-model has the action [27]

\[
S_D = \int d^Dz \sqrt{|g|} \left\{ R[g] + g^{MN} \partial_M \varphi \partial_N \varphi - \sum_{a=4}^{D-7} \frac{1}{a!} \exp[2\varphi_a](F^a)^2 \right\}, \quad (3.27)
\]

where \( \varphi = (\varphi^1, \ldots, \varphi^l) \in \mathbb{R}^l \), \( \tilde{\lambda}_a = (\lambda_{a1}, \ldots, \lambda_{al}) \in \mathbb{R}^l \), \( l = D - 11 \), \( \text{rank}F^a = a \), \( a = 4, \ldots, D - 7 \). Here vectors \( \tilde{\lambda}_a \) satisfy the relations

\[
\tilde{\lambda}_a \tilde{\lambda}_b = N(a, b) - \frac{(a - 1)(b - 1)}{D - 2} = \Lambda_{ab}, \quad (3.28)
\]

\[
N(a, b) = \min(a, b) - 3, \quad (3.29)
\]

\( a, b = 4, \ldots, D - 7 \) and \( \tilde{\lambda}_{D-7} = -2\lambda_4 \). For \( D > 11 \) vectors \( \tilde{\lambda}_4, \ldots, \tilde{\lambda}_{D-8} \) are linearly independent. (It may be verified that the matrix \( (\Lambda_{ab}) \) is positive-definite and hence the set of vectors obeying (3.28) does exist.)

The model (3.27) contains \( l \) scalar fields with a negative kinetic term (i.e., \( h_{a\beta} = -\delta_{a\beta} \) in (2.11)) coupled to several forms (the number of forms is \( (l + 1) \)).

For the brane worldvolumes we have the following dimensions (see (2.16)): \( d(I) = 3, \ldots, D - 8 \) for \( I \in \Omega_{a,c} \) and \( d(I) = D - 5, \ldots, 6 \) for \( I \in \Omega_{a,m} \). Thus there are \( (l + 1) \) electric and \( (l + 1) \) magnetic \( p \)-branes, \( p = d(I) - 1 \). In \( B_D \)-model all \( K_s = 2 \). (For \( B_{12} \)-model see [11].)
Example 1: $H_2(q_1, q_2)$ algebra [19]. Let

$$A = \begin{pmatrix} 2 & -q_1 \\ -q_2 & 2 \end{pmatrix}, \quad q_1 q_2 > 4,$$

$q_1, q_2 \in \mathbb{N}$. This is the Cartan matrix of the hyperbolic KM algebra $H_2(q_1, q_2)$ [3]. From (3.21) we get

$$\varepsilon_s \nu_s^2 (U^s, U^s) (q_1 q_2 - 4) = 2q_s + 4,$$

$s \in \{1, 2\} = S$.

An example of the $H_2(q, q)$-solution for $B_D$-model with two electric $p$-branes ($p = d_1, d_2$), corresponding to $F^a$ and $F^b$ fields and intersecting on a time manifold, is as follows [19]:

$$g = H^{-2/(q-2)} g^0 - H^{2/(q-2)} dt \otimes dt + \hat{g}^1 + \hat{g}^2,$$

$$F^a = \nu_1 dH^{-1} \wedge dt \wedge \hat{r}_1,$$

$$F^b = \nu_2 dH^{-1} \wedge dt \wedge \hat{r}_2,$$

$$\varphi = - (\hat{\lambda}_a + \hat{\lambda}_b)(q - 2)^{-1} \ln H$$

where $d_0 = 3, d_1 = a - 2, a = q + 4, a < b, d_2 = b - 2, D = a + b$, and $\nu_1^2 = \nu_2^2 = (q - 2)^{-1}$. The signature restrictions are: $\varepsilon_1 = \varepsilon_2 = -1$. Thus the space-time $(M, g)$ should contain at least three time directions. The minimal $D$ is 15. For $D = 15$ we get $a = 7, b = 8, d_1 = 5, d_2 = 6$ and $q = 3$.

4 Black brane solutions

In this section we consider spherically symmetric solutions with $d_0 = 1$ and $M_1 = S^{d_1}, g^0 = d\Omega_{d_1}^2$, where $d\Omega_{d_1}^2$ is the canonical metric on a unit sphere $S^{d_1}, d_1 \geq 2$. The manifold $M_0$ corresponds to a radial variable $R$. We put also $M_2 = \mathbb{R}, g^2 = -dt \otimes dt$, i.e., $M_2$ is a time manifold and

$$2 \in I_s, \quad \forall s \in S,$$

i.e., all branes have a common time direction $t$.

In [23], [24], [25], the following solutions with a horizon were obtained:

$$g = \left( \prod_{s \in S} H_s^{2h_s, d(I_s)/(D-2)} \right) \left\{ \left( 1 - \frac{2\mu}{R^d} \right)^{-1} dR \otimes dR + R^2 d\Omega_{d_1}^2 \right\}$$

$$- \left( \prod_{s \in S} H_s^{-2h_s} \right) \left( 1 - \frac{2\mu}{R^d} \right) dt \otimes dt + \sum_{s \in S} \left( \prod_{s \in S} H_s^{-2h_s, d_s, I_s} \hat{g}^i \right),$$

$$\exp(\varphi_0) = \prod_{s \in S} H_s^{h_s, x_s, \lambda_{as}}$$

where $F^a = \sum_{s \in S} \delta^a_{as} F^s$, and

$$F^s = - \frac{Q_s}{R d^s} \left( \prod_{s' \in S} H_{s'}^{-A_{s'}} \right) dR \wedge \tau(I_s),$$

$s \in S_e,$

$$F^s = Q_s \tau(I_s),$$

$s \in S_m.$

Here $Q_s \neq 0, h_s = K_s^{-1}, s \in S$, and the generalized Cartan matrix $(A_{s,s'})$ is non-degenerate.

The functions $H_s > 0$ obey the equations

$$\frac{d}{dz} \left( \frac{1 - 2\mu z}{H_s} \frac{d}{dz} H_s \right) = \tilde{B}_s \prod_{s' \in S} H_s^{-A_{s,s'},}$$

(4.6)
\begin{align}
H_s((2\mu)^{-1} - 0) &= H_{s0} \in (0, +\infty), \\
H_s(+0) &= 1,
\end{align}

\( s \in S \), where \( H_s(z) > 0, \mu > 0, z = R^{-d} \in (0, (2\mu)^{-1}) \), \( d = d_1 - 1 \) and \( \tilde{B}_s = \varepsilon_s K_s Q_s^2 / \tilde{d}^2 \neq 0 \).

There exist solutions to eqs. \((4.6)-(4.7)\) of polynomial type. The simplest example occurs in orthogonal case \([40, 27]\) (for \( d_i = 1 \) see also \([38, 39]\)):

\((U^s, U^{s'}) = 0\), for \( s \neq s', s, s' \in S \). In this case \((A_{s'}) = \text{diag}(2, \ldots, 2)\) is a Cartan matrix of the semisimple Lie algebra \( A_1 \oplus \ldots \oplus A_1 \) and

\[ H_s(z) = 1 + P_s z \]

with \( P_s \neq 0 \), satisfying

\[ P_s(P_s + 2\mu) = -\tilde{B}_s, \]

\( s \in S \).

In \([43, 44]\) this solution was generalized to the block-orthogonal case \([45, 46]\). In this case \((4.9)\) is modified as follows

\[ H_s(z) = (1 + P_s z)^b_s, \]

where \( b_s \) are defined as follows

\[ b_s = 2 \sum_{s' \in S} A^{ss'} \]

and parameters \( P_s \) coincide within blocks, i.e., \( P_s = P_{s'} \) for \( s, s' \in S_i \), \( i = 1, \ldots, k \). The parameters \( P_s \neq 0 \) satisfy the relations \([44, 26]\)

\[ P_s(P_s + 2\mu) = -\tilde{B}_s / b_s, \]

\( s \in S \), and the parameters \( \tilde{B}_s / b_s \) coincide within blocks, i.e., \( \tilde{B}_s / b_s = \tilde{B}_{s'} / b_{s'} \) for \( s, s' \in S_i \), \( i = 1, \ldots, k \).

**Finite-dimensional Lie algebras.**

Let \( (A_{s'}) \) be a Cartan matrix for a finite-dimensional semisimple Lie algebra \( \mathcal{G} \). In this case all powers in \((4.12)\) are positive integers which coincide with the components of twice the dual Weyl vector in the basis of simple co-roots \([4]\), and hence all functions \( H_s \) are polynomials, \( s \in S \).

**Conjecture 1.** Let \( (A_{s'}) \) be a Cartan matrix for a semisimple finite-dimensional Lie algebra \( \mathcal{G} \). Then the solutions to eqs. \((4.6)-(4.8)\) (if any) have a polynomial structure:

\[ H_s(z) = 1 + \sum_{k=1}^{n_s} P_s^{(k)} z^k, \]

where \( P_s^{(k)} \) are constants, \( k = 1, \ldots, n_s \); \( n_s = b_s = 2 \sum_{s' \in S} A^{ss'} \in \mathbb{N} \) and \( P_s^{(n_s)} \neq 0 \), \( s \in S \).

In the extremal case \((\mu = +0)\), an analogue of this conjecture was suggested previously in \([42]\). Conjecture 1 was verified for the \( A_m \) and \( C_{m+1} \) Lie algebras in \([24, 25]\). Explicit expressions for polynomials corresponding to the Lie algebras \( C_2 \) and \( A_3 \) were obtained in \([45]\) and \([46]\), respectively. Recently, a family of black brane type solutions in the model with multicomponent anisotropic fluid were found in \([48]\).

**Hyperbolic KM algebras.** Let \( (A_{s'}) \) be a Cartan matrix for the infinite-dimensional hyperbolic KM algebra \( \mathcal{G} \). In this case, all powers in \((4.12)\) are negative, and hence we have no chance to get a polynomial structure for \( H_s \). Here we are led to an open problem of seeking solutions to the set of “master” equations \((4.6)-(4.7)\). These solutions determine special solutions to the Toda-chain equations corresponding to the hyperbolic KM algebra \( \mathcal{G} \).

**Example 2.** Black hole solutions for the KM algebras \( A_1 \oplus A_1, A_2 \) and \( H_2(q, q) \) \([47]\). Consider a 4-dimensional model governed by the action

\[ S = \int_M d^4z \sqrt{|g|} \{ R[g] - \varepsilon g^{MN} \partial_M \varphi \partial_N \varphi - \frac{1}{2} e^{2\lambda \varphi} (F^1)^2 - \frac{1}{2} e^{-2\lambda \varphi} (F^2)^2 \} \]

\[ (4.15) \]

Here \( F^1 \) and \( F^2 \) are 2-forms, \( \varphi \) is a scalar field and \( \varepsilon = \pm 1 \).
We consider a black brane solution defined on $\mathbb{R}_s \times S^2 \times \mathbb{R}$ with two electric branes $s_1$ and $s_2$, corresponding to the forms $F^1$ and $F^2$, respectively, with the sets $I_1 = I_2 = \{2\}$. Here $\mathbb{R}_s$ is a subset of $\mathbb{R}$, $M_1 = S^2$, $g^1 = d\Omega^2_2$ is the canonical metric on $S^2$, $M_2 = \mathbb{R}$, $g^2 = -dt \otimes dt$ and $\varepsilon_1 = \varepsilon_2 = -1$.

The scalar products of the $U$-vectors are (we identify $U^1 = U^{s_i}$):

$$(U^1, U^1) = (U^2, U^2) = \frac{1}{2} + \varepsilon \lambda^2 \neq 0, \quad (U^1, U^2) = \frac{1}{2} - \varepsilon \lambda^2.$$  

(4.16)

The matrix $A$ from (3.17) is a generalized non-degenerate Cartan matrix if and only if

$$\frac{2(U^1, U^2)}{(U^2, U^2)} = -q,$$

or, equivalently,

$$\varepsilon \lambda^2 = \frac{2 + q}{2(2 - q)},$$

(4.17)  (4.18)

where $q = 0, 1, 3, 4, \ldots$. This takes place if $\varepsilon = +1$ for $q = 0, 1$ and $\varepsilon = -1$ for $q = 3, 4, 5, \ldots$.

The first branch ($\varepsilon = +1$) corresponds to the finite-dimensional Lie algebras $A_1 \oplus A_1$ ($q = 0$), $A_2$ ($q = 1$) and the second one ($\varepsilon = -1$) to the hyperbolic KM algebras $H_2(q, q)$, $q = 3, 4, \ldots$. In the hyperbolic case, the scalar field $\varphi$ is a phantom.

The black brane solution reads (see (4.2)-(4.4))

$$g = (H_1 H_2)^{\delta} \left\{ \left( 1 - \frac{2 \mu}{R} \right)^{-1} dR \otimes dR + R^2 d\Omega^2_2 \right. - (H_1 H_2)^{-2h} \left( 1 - \frac{2 \mu}{R} \right) dt \otimes dt \right\},$$

$$\exp(\varphi) = (H_1 / H_2)^{\varepsilon h},$$

$$F^s = \frac{Q_s}{R^2} H_s^{-2}(H_5)^q dt \wedge dR,$$

(4.19)  (4.20)  (4.21)

$s = 1, 2$. Here $h = (2 - q)/2$ and $\delta = 2, 1$ for $s = 1, 2$, respectively.

The moduli functions $H_s > 0$ obey the equations (see (4.6))

$$\frac{d}{dz} \left( \frac{(1 - 2 \mu z)}{H_s} \frac{d}{dz} H_s \right) = \frac{2Q^2_s}{q - 2} H_s^{-2}(H_5)^q,$$

(4.22)

with the boundary conditions $H_s((2\mu)^{-1} - 0) = H_{s0} \in (0, +\infty)$, $H_s(+0) = 1$, $s = 1, 2$. Here $\mu > 0$, $z = 1/R \in (0, (2\mu)^{-1})$. For $q = 0, 1$ the solutions to eqs. (4.22) with these boundary conditions were given in [23] [24] [25]. They are polynomials of degrees 1 and 2 for $q = 0$ and $q = 1$, respectively. For $q = 3, 4, \ldots$ the exact solutions to eqs. (4.22) are yet unknown.

In the special case $Q^2_1 = Q^2_2$ the metric coincides with that of the Reissner-Nordström solution [17].

**Example 3: Black brane solution corresponding to the KM algebra $HA_2^{(1)} = A_2^{++}$.**

Now we consider the $B_{15}$-model in 15-dimensional pseudo-Euclidean space of signature $(-, +, \ldots, +)$ with the forms $F^4, \ldots, F^8$.

Here we deal with four electric branes $s_1, s_2, s_3, s_4$ corresponding to the 6-form $F^6$. The brane sets are: $I_1 = \{1, 2, 3, 11, 12\}$, $I_2 = \{4, 5, 6, 11, 12\}$, $I_3 = \{7, 8, 9, 11, 12\}$, $I_4 = \{1, 4, 10, 11, 12\}$.

It may be verified that these sets obey the intersection rules corresponding to the hyperbolic KM algebra $HA_2^{(1)}$ with the Cartan matrix

$$A = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix},$$

(4.23)
is in agreement with the fact that the metric (4.24) has a singularity at the boundary conditions

\[ \alpha_{\mu} > 0, \]

\( \text{Example 4: Black brane solution corresponding to the Lorentzian KM algebra} \)

This solution is valid when a special set of charges is considered:

The Hawking temperature in this case is \[ T_H = (1 + P/2\mu)^{22}/(8\pi\mu). \] It diverges as \( \mu \to +0 \). This is in agreement with the fact that the metric (4.24) has a singularity at \( R = +0 \) if \( \mu = +0 \).

**Example 4:** Black brane solution corresponding to the Lorentzian KM algebra \( P_{10} \).
Now we consider another solution for the $B_{15}$-model in 15-dimensional pseudo-Euclidean space of signature $(-, +, ..., +)$ with the non-zero 6-form $F^6$.

Here we deal with ten electric branes $s_1, ..., s_{10}$ corresponding to the 4-form $F^4$. The brane sets are: $I_1 = \{1, 4, 7, 11, 12\}$, $I_2 = \{8, 9, 10, 11, 12\}$, $I_3 = \{2, 5, 7, 11, 12\}$, $I_4 = \{4, 6, 10, 11, 12\}$, $I_5 = \{2, 3, 9, 11, 12\}$, $I_6 = \{1, 2, 8, 11, 12\}$, $I_7 = \{1, 3, 10, 11, 12\}$, $I_8 = \{4, 5, 8, 11, 12\}$, $I_9 = \{3, 6, 7, 11, 12\}$, $I_{10} = \{5, 6, 9, 11, 12\}$.

These sets obey the intersection rules corresponding to the Lorentzian KM algebra $P_{10}$ (we call it the Petersen algebra) with the following Cartan matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 2 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & 0 & 0 \end{pmatrix} \quad (4.32)$$

The Dynkin diagram for this Cartan matrix could be represented by the Petersen graph “a star inside a pentagon”. $P_{10}$ is a Lorentzian KM algebra. It is a subalgebra of $E_{10}$ [3].

Let us present a black brane solution for the configuration of 10 electric branes under consideration. The metric \ref{g} reads

$$g = \left( \prod_{s=1}^{10} H_s \right)^{5/13} \left\{ \left( 1 - \frac{2\mu}{R} \right)^{-1} dR \otimes dR + R^2 d\Omega_2^2 \right\} \quad (4.33)$$

The form field

$$F^6 = -Q_1 R^{-2} H_1^{-2} H_2 H_5 H_{10} dR \wedge dt^1 \wedge dt^2 \wedge dx^1 \wedge dx^4 \wedge dx^7 \quad (4.34)$$

$$-Q_2 R^{-2} H_1 H_2^{-2} H_3 H_9 dR \wedge dt^1 \wedge dt^2 \wedge dx^8 \wedge dx^9 \wedge dx^{10}$$

$$-Q_3 R^{-2} H_3 H_2^{-2} H_7 H_5 dR \wedge dt^1 \wedge dt^2 \wedge dx^2 \wedge dx^5 \wedge dx^7$$

$$-Q_4 R^{-2} H_3 H_4^{-2} H_5 H_9 dR \wedge dt^1 \wedge dt^2 \wedge dx^4 \wedge dx^6 \wedge dx^{10}$$

$$-Q_5 R^{-2} H_1 H_4 H_5^{-2} H_8 dR \wedge dt^1 \wedge dt^2 \wedge dx^2 \wedge dx^3 \wedge dx^9$$

$$-Q_6 R^{-2} H_4 H_1 H_5^{-2} H_9 dR \wedge dt^1 \wedge dt^2 \wedge dx^4 \wedge dx^8$$

$$-Q_7 R^{-2} H_3 H_7^{-2} H_8 H_{10} dR \wedge dt^1 \wedge dt^2 \wedge dx^1 \wedge dx^3 \wedge dx^{10}$$

$$-Q_8 R^{-2} H_5 H_7 H_9^{-2} H_5 dR \wedge dt^1 \wedge dt^2 \wedge dx^4 \wedge dx^5 \wedge dx^8$$

$$-Q_9 R^{-2} H_2 H_5 H_9^{-2} dR \wedge dt^1 \wedge dt^2 \wedge dx^3 \wedge dx^6 \wedge dx^7$$

$$-Q_{10} R^{-2} H_1 H_5 H_7 H_5^{-2} dR \wedge dt^1 \wedge dt^2 \wedge dx^5 \wedge dx^6 \wedge dx^9.$$
where $Q_s \neq 0$, $s = 1, \ldots, 10$.

The scalar fields reads

$$\varphi^\alpha = \frac{1}{2} \lambda_{6\alpha} \ln \left( \prod_{s=1}^{10} H_s \right),$$

(4.35)

$\alpha = 1, 2, 3, 4$. Here $H_s > 0$ obey the equations

$$\frac{d}{dz} \left( \frac{1 - 2\mu z}{H_s} \frac{d}{dz} H_s \right) = 2Q_s^2 \prod_{s'=1}^{10} H_{s'}^{-A_{ss'}},$$

(4.36)

with the boundary conditions $H_s((2\mu)^{-1} - 0) = H_{s0} \in (0, +\infty)$, and $H_s(+0) = 1$, $s = 1, \ldots, 10$. Here $\mu > 0$, $z = R^{-1} \in (0, (2\mu)^{-1})$, and $(A_{ss'})$ is the Cartan matrix (4.23) for the KM algebra $P_{10}$.

**Special 1-block solution.** Now we consider a special 1-block solution. This solution is valid if a special set of charges is considered: $Q_s^2 = 2Q^2$ ($Q \neq 0$) in agreement to (4.28) and

$$b_s = 2 \sum_{s'=1}^{10} A_{ss'} = -2,$$

(4.37)

for $s = 1, \ldots, 10$. In this case the functions $H_s$ are

$$H_s = H^{-2}, \quad H = 1 + P/R,$$

(4.38)

where $P(P+2\mu) = 2Q^2$. As in the previous case, we get a well-defined solution for $P = -\mu + \sqrt{\mu^2 + 2Q^2} > 0$ and $\mu > 0$.

The Hawking temperature in this case has the following form: $T_H = (1 + P/2\mu)^{10}/(8\pi\mu)$. It is smaller than that in the previous example but it also diverges as $\mu \to +0$. It is in agreement with the singularity of the metric (4.33) at $R = +0$ for $\mu = +0$.

5 Conclusions

We have considered several classes of exact solutions in multidimensional gravity with a set of scalar fields and fields of forms related to non-singular (e.g., hyperbolic) KM algebras.

The solutions describe composite electromagnetic branes defined on warped products of Ricci-flat, or sometimes Einstein, spaces of arbitrary dimensions and signatures. The metrics are block-diagonal, and all scale factors, scalar fields and fields of forms depend on points of some manifold $M_0$. The solutions include those depending on harmonic functions, and spherically-symmetric black-brane solutions. Our approach is based on the sigma-model representation obtained in [28] under rather a general assumption on intersections of composite branes (such that the stress-energy tensor has a diagonal structure).

We have also considered a class of black brane configurations with intersection rules [27] governed by an invertible generalized Cartan matrix corresponding to a certain generalized KM Lie algebra $G$. The “master” equations for moduli functions have polynomial solutions in the finite-dimensional case (according to our conjecture [24, 25]), while in the infinite-dimensional case we have only a special family of the so-called block-orthogonal solutions corresponding to semi-simple non-singular KM algebras. Certain examples of black brane solutions are presented, corresponding to the hyperbolic KM algebras: $H_2(q, q)$ ($q > 2$), $A_{2}^{(1)} = A_{2}^{++}$ and the Lorentzian KM algebra $P_{10}$.

The last two solutions (which are new) may be analyzed from the viewpoints (i) of post-Newtonian parameters $\beta$ and $\gamma$ corresponding to the 4-dimensional section of the metric and (ii) of thermodynamic properties of the black branes under consideration. These and some other tasks may be a subject of a separate publication.

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