Output feedback sliding mode control based on adaptive sliding mode disturbance observer

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Abstract
In this paper, an adaptive sliding mode disturbance observer is designed to counteract the disturbance actively. By designing the adaptive laws, the assumptions on the disturbance are relaxed in the proposed observer, its first derivative upper bound is considered to be unknown. Based on the proposed disturbance observer, an output feedback sliding mode controller is constructed for the continuous-time linear systems with unknown external disturbance. The proposed controller incorporates only the system output information and has less chattering of the control input. The feasibility of the proposed strategy is shown by numerical simulations.

Keywords
Output feedback control, sliding mode disturbance observer, adaptive control, dynamic sliding surface

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Sliding mode control (SMC) has attracted the attention of many researchers because of its robustness to disturbances and design simplicity.¹⁻⁴ In the area of SMC systems, many of the theoretical developments assume that the system state vector is available.⁵⁻⁷ In most practical situations, it is unrealistic or prohibitively expensive to measure the full state information, and only the output information can be physically measured.⁸⁻¹⁰ Generally, there are static and dynamic output feedback SMC strategies to work around this limitation.¹¹ The static output feedback SMC design problem is studied for a delay system.¹² Song et al.¹³ address the static output-feedback sliding mode control (SMC) problem for a class of uncertain control systems. In order to improve the control performance of static output feedback SMC, the dynamic output feedback SMC can be designed with a compensator. The additional integral term is introduced in the sliding surface, which can provide one more degree of freedom.¹⁴ An asynchronous output feedback sliding controller is proposed for a class of Markovian jump systems.¹⁵ In the aforementioned results for output feedback SMC, the disturbance rejection problem is ignored, which leads to the chattering phenomenon in the system.

Recently, the observer-based method is utilized to eliminate disturbance and reduce chattering.¹⁶,¹⁷ Su et al.¹⁸ present a disturbance observer where the bounds on disturbance derivative are assumed to be known. A new form of the combined observer-controller is designed to provide estimated data of unknown disturbance and unmeasured states in the control law.¹⁹ Lee²⁰ proposed a composite control technique by combining a nonlinear disturbance observer. A fixed-time observer has been put forward, which possessed a better approximation to external disturbances.²¹ However, in the above works the disturbance observer was considered with the assumption of the known maximum upper bound of disturbance. In some practical applications, it is difficult to acquire the disturbance upper bound directly, especially combined with the sliding mode algorithm, too large upper bound will aggravate the chattering phenomenon of the system, and too small upper bound will lead to the instability of the system.

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The sliding mode disturbance observer with adaptive control is proposed to relax the restriction on disturbance. The adaptive control is introduced in the sliding mode disturbance observer that use the adaptive method to ensure the control gain is as small as possible whereas sufficient to eliminate the disturbances or uncertainties.\textsuperscript{22} Negrete-Chávez and Moreno\textsuperscript{23} develop an adaptive second-order SMC to interference disturbances. An adaptive scheme based on equivalent control input,\textsuperscript{24} inverted pendulum on a cart model. The proposed control approaches is illustrated by an example.

When matched uncertainty alone is present, it is sufficient to consider the nominal linear system representation when designing the switching function. The sliding motion depends on the choice of the sliding surface, the precise effect is not readily apparent. Therefore, we need to transform the system into a suitable regular form.

Assume $\text{rank}(CB) = \text{rank}(B) = m$, thus, there exists a non-singular matrix $T \in \mathbb{R}^{n \times n}$ such that $TB = [0 \ B_2]^T$, $CT^{-1} = [0_{q \times (n-q)} \ C_2]$ ($C_2 \in \mathbb{R}^{q \times q}$). Partition $T$ as follows

$$
T = \begin{bmatrix}
T_1 \\
T_2
\end{bmatrix} \begin{bmatrix}
\begin{array}{c}
m1 \\
\vdots \\
m3
\end{array}
\end{bmatrix}, \quad T_1 = \begin{bmatrix}
T_{11} \\
T_{12} \\
T_{13}
\end{bmatrix} \begin{bmatrix}
m1 \\
m2 \\
m3
\end{bmatrix},
$$

where $m_1 \geq 0$, $m_2 \geq 0$, $m_3 \geq 0$, and $m_1 + m_2 + m_3 = n - m$.

Under the coordinate transformation $\bar{x}(t) = Tx(t)$, system (1) is transformed into the following regular form

$$
\dot{\bar{x}}(t) = A\bar{x}(t) + B[u(t) + d(t)] \quad (2)
$$

$$
y(t) = C\bar{x}(t) \quad (3)
$$

where $\bar{x}(t) = \begin{bmatrix}
\bar{x}_1(t) \\
\bar{x}_2(t)
\end{bmatrix}$, $A = TAT^{-1} = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}$, $B = TB = [0 \ B_2]^T$, $C = CT^{-1} = [0_{q \times (n-q)} \ C_2]$, $\bar{x}_1(t) \in \mathbb{R}^{n-m}$, $\bar{x}_2(t) \in \mathbb{R}^{m}$, $B_2 \in \mathbb{R}^{m \times m}$ and $C_2 \in \mathbb{R}^{q \times q}$. Let $C_1 = [0_{(q-m) \times (n-q)} \ I_{q-m}]$, substituting $A$, $B$ and $C$ into (2) and (3), then system (1) is further written as

$$
\dot{\bar{x}}_1(t) = A_{11}\bar{x}_1(t) + A_{12}\bar{x}_2(t) \quad (4)
$$

$$
\dot{\bar{x}}_2(t) = A_{21}\bar{x}_1(t) + A_{22}\bar{x}_2(t) + B_2[u(t) + d(t)] \quad (5)
$$

$$
y(t) = C_2 \begin{bmatrix}
C_1\bar{x}_1(t) \\
\bar{x}_2(t)
\end{bmatrix} \quad (6)
$$

\textbf{Proposed method}

\textbf{Design of dynamic sliding surface}

The dynamic sliding surface is designed as

$$
s(t) = [-w_1, I_m]C_2^{-1}y(t) + w_3\nu(t) \quad (7)
$$

An additional integral term $\nu(t)$ is introduced in the sliding surface, which can provide one more degree of freedom. $\nu(t)$ is defined as follows

$$
\nu(t) = \int_0^t e^{\nu(t-s)}[w_2, \Phi]\Phi^{-1}y(s)ds \quad (8)
$$

and $w_1 \in \mathbb{R}^{n \times (q-m)}$, $w_2 \in \mathbb{R}^{(n-m) \times (q-m)}$, $w_3 \in \mathbb{R}^{(n-m) \times (n-m)}$ are the sliding surface

\textbf{Preliminaries}

Consider the following linear system with matched external disturbance

$$
\dot{x}(t) = Ax(t) + Bu(t) + d(t)
$$

$$
y(t) =Cx(t) \quad (1)
$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^q$ is the output vector, and $m \leq q < n$. $d(t)$ is the unknown disturbance. $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{q \times n}$ are given constant matrices.
The derivative with respect to (8)

\[ \dot{v}(t) = w_3 \int_0^{w_3} e^{w_3(x-x)} [\Phi] \dot{C}_2^{-1} \dot{y}(t) dx + [w_2, \Phi] \dot{C}_2^{-1} \dot{y}(t) \]

where \( I \) is identity matrix. Hence, we have \( F_2 = G_x^{-1}(I - G_1 F_1) \). Define \( F_3 = -F_2 G_x^{-1} G_1 = F_2 (I - G_1^{-1})^{-1} F_2^T \). Applying Schur complement on (16) obtain \( F_1 - G_1^{-1} > 0 \), then

\[
F = H_2 H_1^{-1} = 
\begin{bmatrix}
F_1 & I \\
F_2 & 0
\end{bmatrix}
\begin{bmatrix}
I & G_1 \\
0 & G_2^T
\end{bmatrix}
\]

\[
= 
\begin{bmatrix}
F_1 & (I - G_1) G_2^T \\
F_2 & -F_2 G_x^{-1} G_1
\end{bmatrix}
\]

\[
= 
\begin{bmatrix}
F_1 & F_2^T \\
F_2 & F_3
\end{bmatrix}
> 0
\]

hence exist a positive definite symmetric matrix \( F > 0 \) such that

\[
H_1^T Q H_1^{-1} = A^T H_2 H_1^{-1} + H_1^T H_2 A
\]

\[
= A^T F + FA
\]

\[
< 0
\]

The new sliding motion (14) is asymptotically stable. □

**Design of sliding mode controller**

In this subsection, the control method based on output feedback sliding mode with power term is proposed.

Define the generalized inverse matrix of \( C_1 \) is \( C_1^* \) and \( \zeta_1 = \text{null}(C_1) \) satisfying that \( \zeta_1 C_1^* = 0 \), generalized inverse matrix of \( \zeta_1 \) is \( \zeta_1^* \), then equation \( C_1^* C_1 + \zeta_1^* \zeta_1 = I_{n-m} \) holds. Since \( A \) is Hurwitz matrix, then exist \( \xi > 0 \) and \( P > 0 \) subject to

\[
A^T P + PA + 2\xi P < 0
\]

Denote \( \xi \) as the largest positive scalar such that (17) is feasible. Utilize Cholesky factorization for \( P \) to get

\[
P = P_1 \Theta^T = 
\begin{bmatrix}
P_{11} & 0 \\
P_{21} & P_{22}
\end{bmatrix}
\begin{bmatrix}
P_{11} & 0 \\
P_{21} & P_{22}
\end{bmatrix}^T.
\]

**Remark 1.** Inequality (17) holds which can be explained as follows:

Consider the Lyapunov function for the new sliding motion (14): \( V = x^T P x \)

\[
\dot{V} = x^T P Ax + x^T A^T P x
\]

\[
= x^T (PA + A^T P) x
\]

\[
\leq (PA + A^T P) x^2
\]

Since \( A \) is Hurwitz matrix and \( P > 0, \xi > 0 \), hence \( \dot{V} < 0, PA + A^T P < 0 \). There exists \( \xi > 0 \) for \( (PA + A^T P) + 2\xi P < 0 \) holds.
Proposition 2. Define \( \tau > 0, \varrho > 0, k_1 + k_2 > 1 \)

\[ \text{comma h}_1 > 1, 1 > h_2 > 0, \] then the following output feedback sliding mode controller is proposed:

\[
\dot{u}(t) = -B_2^{-1}[\varpi(t) + \varphi_1(t)(k_1, \text{sgn}^n)(s(t)) + k_2 \text{sgn}^n(s(t))] + w_4w_3v(t) - (A_{23} - w_1C_1A_{11})C_1^T \varphi_2(t) + w_4\Phi \hat{x}_2(t) - \hat{d} \\
+ (A_{23} - w_1C_1A_{11})C_1^T \varphi_2(t) + w_4\Phi \hat{x}_2(t) - \hat{d} \\
+ (A_{22} - w_1C_1A_{12} + w_4\Phi) \hat{x}_2(t) - \hat{d}
\]

(18)

where \( \varphi_1(t) = \psi_0(\dot{\vartheta}_1(t) + q) \) and \( \psi_0 = \|A_{23} - w_1C_1A_{11})C_1^T \| \). \( \dot{\vartheta}_1(t) \) can be obtained through \( \dot{\vartheta}_1(t) = -\xi \dot{\vartheta}_1(t) + e_1 \|s(t)\| \), coefficient \( e_1 = \|P_1^T\Phi\| \) and \( \Phi = [A_{12} \; \Phi]^T \). \( \hat{d} \) is the estimation of the disturbance. Then, the system (2) – (3) can reach the sliding surface.

Proof. The first derivative of the sliding mode surface is given as

\[
\dot{s}(t) = \dot{\hat{x}}_2(t) - w_1C_1\hat{x}_1(t) + w_4\varphi(t) \\
= A_{23}\hat{x}_1(t) + A_{22}\hat{x}_2(t) + B_2[u(t) + d(t)] + w_1C_1[A_{11}\hat{x}_1(t) + A_{12}\hat{x}_2(t)] + w_4[\varphi(t) + W_2C_1\hat{x}_1(t)] \\
[(A_{23} - w_1C_1A_{11})\hat{x}_1(t) + w_4w_2C_1\hat{x}_1(t)] \\
(A_{22} - w_1C_1A_{12} + w_4\Phi) \hat{x}_2(t) + w_4w_3v(t) + B_2[u(t) + d(t)]
\]

(19)

In order to prove that \( s(t) \) can converge to 0, define the Lyapunov function \( V_s(t) = \frac{1}{2}s^T(t)s(t) \), its derivative can be determined and substituted (19) into \( \dot{V}_s(t) \) as

\[
\dot{V}_s(t) = \dot{s}^T(t)[(A_{23} - w_1C_1A_{11})\hat{x}_1(t) + w_4w_2C_1\hat{x}_1(t)] \\
(A_{22} - w_1C_1A_{12} + w_4\Phi) \hat{x}_2(t) + w_4w_3v(t) + B_2[u(t) + d(t)] \\
= s^T(t)[(A_{23} - w_1C_1A_{11})C_1^T \varphi_2(t) + w_4w_2C_1\hat{x}_1(t)] \\
(A_{22} - w_1C_1A_{12} + w_4\Phi) \hat{x}_2(t) + w_4w_3v(t) + B_2[u(t) + d(t)] \\
= -\tau \|s(t)\|^2 + \|\varphi(t)\|^2 - (k_1 + k_2)\psi(t) \\
+ s^T(t)B_2(d(t) - k_2 \text{sgn}(B_2^T s(t))) \\
\leq -\tau \|s(t)\|^2 + \chi(t) \|s(t)\| + (d(t) - \hat{d}) \|B_2^T s(t)\|
\]

(20)

Substituting (18) into (20), one obtains

\[
\dot{V}_s(t) \leq -\tau \|s(t)\|^2 - \psi_0((k_1 + k_2)\|s(t)\| \\
+ \psi_0(s(t))\|P_{11}^T \hat{x}_1(t) + P_{21}^T \varphi(t)\| \\
+ s^T(t)B_2(d(t) - k_2 \text{sgn}(B_2^T s(t))) \\
\leq -\tau \|s(t)\|^2 + \chi(t) \|s(t)\| + (d(t) - \hat{d}) \|B_2^T s(t)\|
\]

(21)

where \( \chi(t) = \psi_0 \|P_{11}^T \hat{x}_1(t) + P_{21}^T \varphi(t)\| - (k_1 + k_2)\psi(t) \). Obviously, \(-\tau \|s(t)\|^2 < 0, \) and assume \( d - \hat{d} = 0 \) holds. In order to ensure that \( V_s(t) < 0 \) holds, we need to discuss the positive and negative of \( \chi(t) \).

Substituting (10) into (4) and (9) yields \( \dot{V}(t) = AX(t) + \Phi_s(t) \). Define new Lyapunov function \( V_s(t) = \sqrt{X^T(t)PX(t)}, \) and derivative of \( V_s \) as follows

\[
\dot{V}_s(t) = \frac{\sqrt{X^T(t)AX(t) + 2X^T(t)PX(t)}}{2X^T(t)PX(t)} \\
\]

(22)

Substituting \( P = P_1^T \), \( V_s(t) = \sqrt{X^T(t)PX(t)} = \|P_{11}^T X(t)\|, \|P_{21}^T X(t)\| = \|P_{11}^T\| \) and (17) into (22), then

\[
\dot{V}_s(t) \leq -\frac{2\xi X^T(t)PX(t) + 2 \|P_{11}^T X(t)\| \|P_{21}^T \Phi\| \|s(t)\|}{2X^T(t)PX(t)} \\
\leq -\xi V_s(t) + e_1 \|s(t)\|
\]

(23)

Define \( e(t) = V_s(t) - \dot{\vartheta}_1(t) \) and \( e(0) \) as the initial value of \( e(t) \). Combining \( \dot{\vartheta}_1(t) = -\xi \dot{\vartheta}_1(t) + e_1 \|s(t)\| \) with (23), we have

\[
e(t) \leq -\xi e(t)
\]

(24)

Integrating the formula (24) yields \( e(t) \leq e^{-\xi t}e(0) \). Combining \( e(t) = V_s(t) - \dot{\vartheta}_1(t) \), one obtains \( V_s(t) \leq \dot{\vartheta}_1(t) + e^{-\xi t}e(0) \). Substitute \( P_1^T \) and \( X(t) \) into the equation \( V_s(t) = \|P_{11}^T X(t)\| \), then

\[
V_s(t) = \sqrt{\|P_{11}^T \hat{x}_1(t) + P_{21}^T \varphi(t)\|^2 + \|P_{21}^T \varphi(t)\|^2}
\]

(25)

Hence \( (k_1 + k_2) \|P_{11}^T \hat{x}_1(t) + P_{21}^T \varphi(t)\| \leq (k_1 + k_2) V_s(t) \). Combining \( k_1 + k_2 > 1 \) with \( \chi(t) = \psi_0 \|P_{11}^T \hat{x}_1(t) + P_{21}^T \varphi(t)\| - (k_1 + k_2)\psi(t) \) one obtains

\[
\chi(t) \leq \psi_0(k_1 + k_2) \leq \psi_0(k_1 + k_2) V_s(t) - (k_1 + k_2)\psi(t) \\
\leq \psi_0(k_1 + k_2) \leq \psi_0(k_1 + k_2) \leq \psi_0(k_1 + k_2) \leq \psi_0(k_1 + k_2) \leq \psi_0(k_1 + k_2)
\]

(26)

Since \( V_s(t) \leq \dot{\vartheta}_1(t) + e^{-\xi t}e(0) \) and \( \varphi_1(t) = \psi_0(\dot{\vartheta}_1(t) + q) \) are known, then

\[
\chi(t) \leq \psi_0(k_1 + k_2) e^{-\xi t}e(0) - q
\]

(27)

In the following, both the cases \( e(0) > 0 \) and \( e(0) \leq 0 \) will be discussed.

If \( e(0) > 0 \), since \( e(t) \leq e^{-\xi t}e(0) \) at \( t = 0 \), thus \( e(t) \leq e(0) \) and \( \chi(t) \leq \psi_0(e(0) - q) \). Assume \( e(0) > q \), there exists \( t > 0 \) such that \( e(t) \geq q \) in \( t \in [0, \; T_1] \) and \( e(t) \leq q \) in \( t > T_1 \). When \( t \in [0, \; T_1] \), then
The coefficients $c_{1d}, c_{2d}, \alpha_{1d}, \alpha_{2d}$ are positive constants with $\alpha_{1d} < 1, \alpha_{2d} > 1$. $v_2$ will be introduced later.

**Proposition 3.** Consider the following sliding surface $s_d$

$$s_d(t) = \dot{e}_a(t) + c_{1d}dsgn(e_a(t))|e_a(t)|^{\alpha_{1d}} + c_{2d}dsgn(e_a(t))|e_a(t)|^{\alpha_{2d}}$$  \hspace{1cm} (34)

If $v_2$ satisfies the following equation

$$v_2(t) = -(k_d(t) + \eta_d)s_d(s_d(t)), v_2(0) = 0$$  \hspace{1cm} (35)

where $\eta_d$ is a positive constant. $k_d(t) > d_1$, $d_1 > |B_2\dot{d}|$ and $d_0 > |B_2d|$. $d$ and $\dot{d}$ are disturbance and its derivatives, $d_1$ and $d_0$ are the upper bounds of $|B_2\dot{d}|$ and $|B_2d|$, respectively. $e_a$ converges to origin in finite time and the estimation of the disturbance as follows

$$v_2(t) = B_2\dot{d}$$  \hspace{1cm} (36)

**Proof.** Consider the derivative of $e_a$ in equation (32) as follows

$$\dot{e}_a(t) = v_2(t) - c_{1d}dsgn(e_a(t))|e_a(t)|^{\alpha_{1d}} - c_{2d}dsgn(e_a(t))|e_a(t)|^{\alpha_{2d}} - B_2\dot{d}(t)$$  \hspace{1cm} (37)

Substitute (37) into (34) one obtains

$$s_d(t) = v_2(t) - B_2\dot{d}(t)$$  \hspace{1cm} (38)

Differentiating the above equation (38) and combining with equation (35) yields

$$\dot{s}_d(t) = \dot{v}_2(t) - B_2\dot{\dot{d}}(t)$$

$$\dot{s}_d(t) = - (k_d(t) + \eta_d)s_d(s_d(t)) - B_2\dot{d}(t)$$  \hspace{1cm} (39)

Consider the Lyapunov function $V_d(t) = \frac{1}{2}s_d^2(t)$, then $V_d(t)$ derivative is as follows

$$\dot{V}_d(t) = s_d(t)\dot{s}_d(t)$$

$$\dot{V}_d(t) = -(k_d(t)s_d(t) - \eta_d|s_d(t)| - B_2\dot{d}(t)s_d(t)$$  \hspace{1cm} (40)

Since $\eta_d$ is a positive constant and $k_d(t) > d_1 > |B_2\dot{d}(t)|$, then (40) can be written as follows

$$\dot{V}_d(t) = s_d(t)\dot{s}_d(t) \leq - \eta_d|s_d(t)|$$  \hspace{1cm} (41)

The $s_d(t)$ will converges to zero in finite time, by utilizing equation (34) one gets

$$\dot{v}_2(t) = -(k_d(t)s_d(t) - \eta_d|s_d(t)| - c_{1d}dsgn(e_a(t))|e_a(t)|^{\alpha_{1d}} - c_{2d}dsgn(e_a(t))|e_a(t)|^{\alpha_{2d}}$$  \hspace{1cm} (42)

By considering Lemma 1, the equilibrium of (42) as $e_a(t) = 0$ is finite time stable.
Consider the estimation error of disturbance as follows
\[ e_d(t) = B_2d(t) - B_2\hat{d} \]  \hfill (43)

Both (36) and (38) are satisfied, \( e_d(t) \) will converge to origin in finite time.

This implies that if \( k_d(t) > d_1 \) is satisfied, the proposed disturbance observer can estimate the disturbance \( d(t) \) in finite time.

**Design adaptation structure**

In this subsection, two adaptation structures are designed, and the assumptions on the disturbance are relaxed in the proposed observer. As is capable of seeing from the proof in the above subsection, the adaptive gain \( k_d(t) \) must satisfy \( k_d(t) > d_1 \) to ensure the system sliding takes place, in which case reachability condition is achieved and sliding motion takes place on \( s_d = \hat{s}_d = 0 \). In other words, unknown disturbance or uncertainty should be completely eliminated, that is \( |u_{eq}(t)| = |d_1| \).

In the first adaptive structure, similar to general adaptive control, it is assumed that \( d_1 \) is known and \( d_0 \) is unknown. The second adaptive strategy is designed which assumes that both \( d_1 \) and \( d_0 \) are unknown, the assumptions on the disturbance are more relaxed.

Consider utilizing a low pass filter to filter the switching signal to obtain a close approximation. Here, the \( \text{sgn}(s_d) \) should take on the average value. Then if \( \hat{u}_{eq}(t) \) satisfies
\[ \dot{\hat{u}}_{eq}(t) = \frac{1}{l} (-k_d(t) + \eta \text{sgn}(s_d) - \hat{u}_{eq}(t)) \]  \hfill (44)

where \( l > 0 \) is a time constant, then \( \hat{u}_{eq}(t) \) almost completely approximates \( u_{eq}(t) \). In order to eliminate the influence of the initial conditions of the filter, assume exist \( 0 < v_1 < 1 \) and \( v_0 > 0 \) such that
\[ ||\hat{u}_{eq}(t)|| - |u_{eq}(t)| < v_1|u_{eq}(t)| + v_0 \]  \hfill (45)

holds. The adaptive algorithm of the control gain is driven by using the equivalent control. Through the following inequality, we introduce the concept of ‘safety margin’
\[ k_d(t) > \frac{1}{\mu} |\hat{u}_{eq}(t)| + v \]  \hfill (46)

where \( 0 < \mu < 1 \) and \( v > 0 \) are design parameters such that
\[ \frac{1}{\mu} |\hat{u}_{eq}(t)| + \frac{v}{2} > |u_{eq}(t)| \]  \hfill (47)

holds. Define error variable \( e_z(t) \) as follows
\[ e_z(t) = k_d(t) - \frac{1}{\mu} |\hat{u}_{eq}(t)| - v \]  \hfill (48)

which shows that if \( e_z(t) = 0 \), then \( k_d(t) > d_1 \), that is, the system will maintain sliding motion. Define adaptive scheme
\[ \dot{k}_d(t) = -g(t)\text{sgn}(s_d) \]  \hfill (49)

where \( g(t) \) is auxiliary scalar, it has the following form
\[ g(t) = f_0 + f(t) \]  \hfill (50)

where \( f_0 \) is a positive scalar and \( f(t) \) is elaborated later.

In this paper, according to whether \( d_1 \) is known, \( f(t) \) will execute different choices. In the next work, we will discuss two situations: \( d_1 \) is known and \( d_1 \) is unknown.

**Case 1:**

Assuming \( d_1 \) is known but \( d_0 \) is unknown. Define
\[ e_z(t) = \frac{\alpha d_1}{\mu} - f(t) \]  \hfill (51)

\[ \dot{f}(t) = \beta|e_z(t)| + f_0 \sqrt{\beta}\text{sgn}(e_z(t)) \]  \hfill (52)

where \( \beta > 0 \), and \( \alpha > 1 \) is designed to ensure \( |\hat{u}_{eq}(t)| < \alpha d_1 \).

**Proposition 4.** As described above, the problem of maintaining sliding is converted to how to yield \( e_z(t) \rightarrow 0 \) in finite time; how to allow \( g(t) \) and \( k_d(t) \) to be bounded. In the following, we will verify the situation where \( d_1 \) is known.

**Proof.** Consider following Lyapunov function
\[ V(t) = \frac{1}{2} e_z^2(t) + \frac{1}{2\beta} e_z^2(t) \]  \hfill (53)

It follows from (53) that
\[ e_z(t)|e_z(t)| \leq -f_0|e_z(t)| + \left( e_z(t) - \frac{\alpha d_1}{\mu} \right)|e_z(t)| \]
\[ + \frac{\alpha d_1}{\mu}|e_z(t)| \leq -f_0|e_z(t)| + e_z(t)|e_z(t)| \]  \hfill (54)

\[ e_z(t)|e_z(t)| \beta = -e_z(t)|e_z(t)| - \frac{f_0|e_z(t)|}{\sqrt{\beta}} \]  \hfill (55)

The derivative with respect to (53) and integrate with (54) and (55), one obtains
\[
\dot{V}(t) = e_k \dot{\epsilon}_k + \frac{e_k \dot{\epsilon}_k}{\beta} \\
\leq -f_0 |\epsilon_k(t)| - f_0 |\epsilon_t(t)| \sqrt{\beta} \\
\leq -\sqrt{2} f_0 \left( \frac{|\epsilon_k(t)|}{\sqrt{2}} + \frac{|\epsilon_t(t)|}{\sqrt{2}\beta} \right) \\
\leq -f_0 \sqrt{2V(t)}
\]

Then \( \epsilon_k(t) = \epsilon_k(0) e^{-\frac{t}{\mu}} \) is uniformly bounded, consequently \( \dot{\epsilon}_k(t) \) is established, (56) can be rewritten as

\[
\dot{V}(t) = e_k \dot{\epsilon}_k(t) + \frac{e_k \dot{\epsilon}_k(t)}{\beta} \\
\leq -f_0 |\epsilon_k(t)| + e_t(t)|\epsilon_k(t)| - \frac{e_k \dot{\epsilon}_k(t)}{\beta} \\
\leq -f_0 |\epsilon_k(t)| + e_t(t)|\epsilon_k(t)| - e_k(t)|\epsilon_k(t)| \\
\leq -f_0 |\epsilon_k(t)|
\]

Define \( V_0 \) to be the initial value of \( V(t) \) and \( t_0 \) to be the time taken for \( V(t) \) to converge to zero, and hence integrating both sides yields \( t_0 < \sqrt{\frac{\alpha \beta}{\mu}} \), which shows that \( \epsilon_k(t) \) and \( \epsilon_t(t) \) will converge to origin in finite time. Then \( k_A(t) > d_l \) will be guaranteed and the reachability condition is satisfied. Obviously \( \epsilon_t(t) \) is bounded, hence \( f(t) \) and \( s(t) \) is bounded.

**Case 2:**
Assuming that both \( d_l \) and \( d_0 \) are unknown. Define

\[
\dot{j}(t) = \begin{cases} 
\beta |\epsilon_k(t)| & \text{if } |\epsilon_k(t)| > \epsilon_0 \\
0 & \text{otherwise}
\end{cases}
\]

(57)

where \( \epsilon_0 \) is a design constant which to eliminate the noise signal in the system.

**Proposition 5.** Both \( d_l \) and \( d_0 \) are unknown, if \( v \) is chosen to satisfy

\[
4\epsilon_0^2 + \frac{4}{v^2} + \frac{\alpha d_l}{\mu}\right)^2 < 1
\]

(58)

then \( |\epsilon_k(t)| < \frac{\epsilon_0}{\sqrt{2}} \) is realized in finite time and the sliding motion is guaranteed.

**Proof.** Consider the Lyapunov function from (53)

\[
e_k \dot{\epsilon}_k \leq -f_0 |\epsilon_k(t)| + e_t(t)|\epsilon_k(t)|
\]

(59)

Suppose \( f(0) = 0 \), from (57) one obtains \( \dot{j}(t) \geq 0 \), then \( f(t) \geq 0 \) holds. Since \( e_t(t) = \frac{\alpha d_l}{\mu} - j(t) \), then \( e_t(t) \leq \frac{\alpha d_l}{\mu} \) is satisfied. In (57), if \( |\epsilon_k(t)| > \epsilon_0 \) with (55), \( \dot{\epsilon}_t(t) = -j(t) = -\beta |\epsilon_k(t)| \) is established, (56) can be rewritten as

\[
\dot{V}(t) = e_k \dot{\epsilon}_k(t) + \frac{e_k \dot{\epsilon}_k(t)}{\beta} \\
\leq -f_0 |\epsilon_k(t)| + e_t(t)|\epsilon_k(t)| - \frac{e_k \dot{\epsilon}_k(t)}{\beta} \\
\leq -f_0 |\epsilon_k(t)| + e_t(t)|\epsilon_k(t)| - e_k(t)|\epsilon_k(t)| \\
\leq -f_0 |\epsilon_k(t)|
\]

If \( |\epsilon_k(t)| < \epsilon_0 \) with \( e_t(t) < 0 \), then \( \dot{V}(t) \leq -f_0 |\epsilon_k(t)| \). Since \( e_t(t) \leq \frac{\alpha d_l}{\mu} \), then outside of the interval

\[
\mathbb{E} = \left\{ (\epsilon_k, e_t) : |\epsilon_k| < \epsilon_0, 0 \leq e_t < \frac{\alpha d_l}{\mu} \right\}
\]

and in the solution domain subject to \( \dot{V} \leq -f_0 |\epsilon_k(t)| \). Draw the smallest ellipse \( F \) with the origin as the center to enclose the \( \mathbb{E} \), then \( F = \{ (\epsilon_k, e_t) : V(\epsilon_k, e_t) < r \} \), where \( r > 0 \) represents the ‘radius’ of the ellipse, and \( r = \frac{\epsilon_0}{\sqrt{2}} + \frac{1}{2\epsilon_0} \left( \frac{\alpha d_l}{\mu} \right)^2 \). Choose the appropriate \( v \) to satisfy (58), the sliding motion will be guaranteed.

Since \( \mathbb{E} \subset F \) and outside of \( F \) in the solution domain, \( \dot{V}(t) \leq 0 \) holds, hence \( F \) is an invariant set. If the \( V(\epsilon_k, e_t) \) enters \( F \), then \( V(\epsilon_k, e_t) \) will not be able to leave \( F \) and from (58), \( |\epsilon_k(t)| < \frac{\epsilon_0}{\sqrt{2}} \) is satisfied. If the \( V(\epsilon_k, e_t) \) does not enter \( F \), then from the above discussion \( \dot{V} \leq -f_0 |\epsilon_k(t)| \) and \( \int_0^T f_0 |\epsilon_k(t)| dt \leq V(0) \) are satisfied. Since \( V(\epsilon_k, e_t) \) is bounded, then \( \epsilon_k(t), e_t(t) \) and \( \dot{\epsilon}_k(t), \dot{\epsilon}_t(t) \) are bounded, hence \( \epsilon_k(t) \) is uniformly continuous, that is, \( \epsilon_k(t) \rightarrow 0 \) when \( t \rightarrow \infty \). It shows that there exists \( t_0 \) such that \( |\epsilon_k(t)| < \frac{\epsilon_0}{\sqrt{2}} \) for \( t > t_0 \). Hence, \( |\epsilon_k(t)| < \frac{\epsilon_0}{\sqrt{2}} \) has always been established in finite time.

Since the \( |\epsilon_k(t)| < \frac{\epsilon_0}{\sqrt{2}} \) holds, therefore \( |\epsilon_k(t)| = |k_A(t) - \frac{1}{\mu} |\dot{u}_{eq}(t)| - v| < \frac{\epsilon_0}{\sqrt{2}} \) from (48), that is \( k_A(t) - \frac{1}{\mu} |\dot{u}_{eq}(t)| - v > -\frac{\epsilon_0}{\sqrt{2}} \), and from (47), one obtains

\[
k_A(t) > \frac{1}{\mu} |\dot{u}_{eq}(t)| + \frac{v}{\sqrt{2}} > |\dot{u}_{eq}(t)| > d_l
\]

(61)

Hence reachability condition is satisfied and the system will maintain sliding motion. Since \( \epsilon_k(t) \) and \( \epsilon_t(t) \) are bounded, consequently \( j(t) \) and \( g(t) \) remains bounded. From (45)

\[
|k_A(t)| < |\epsilon_k(t)| + \frac{1}{\mu} |\dot{u}_{eq}(t)| + v
\]

(62)

obviously \( k_A(t) \) is bounded, and the proof is complete.

**Remark 2.** The method proposed in this paper can be improved by some new fuzzy systems, at the same time, the ASMDO does not require knowledge of the disturbance and its first derivative, hence it can better estimate the disturbances in these systems.

**Remark 3.** The control strategy of this paper can be used to deal with the problem of energy/voltage management in photovoltaic (PV)/battery systems and interval type-3 fuzzy logic systems. The sliding mode disturbance observer can be used to eliminate the influence of the variation of temperature, radiation, and output load on the system, and effectively guarantee better disturbance rejection performance of the controller.

**Simulation example**
A practical example is provided in this section to illustrate the efficiency of the obtained result. Consider an inverted pendulum on a cart model given in Edwards and Spurgeon and the linearization model is as follows.
The non-singular matrix $T$ can be found as

$$ T = \begin{bmatrix} 0 & 0 & 3.1498 & 1.0000 \\ -3.0000 & 0 & 0 & 0 \\ 0 & 2.0000 & 0 & 0 \\ 0 & 0 & -2.0000 & 0 \end{bmatrix} $$

then we can obtain the following transformed regular form

$$ A = \begin{bmatrix} -0.1454 & 0 & 15.4438 & -0.2284 \\ 0 & 0 & 1.5000 & 3.1498 \\ 2.0000 & 0 & 0 & 1.9333 \\ -0.0182 & 0 & 1.9333 & -2.0159 \end{bmatrix} $$

$$ \dot{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.6410 & 0 & 0 & 0 \end{bmatrix} $$

Choose $\Phi$ and $G_2$ as follows

$$ \Phi = \begin{bmatrix} -0.005 & -0.005 \\ 0.2 & 0 \end{bmatrix} $$

$$ G_2 = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 10 \end{bmatrix} $$

Use the LMI toolbox to obtain these parameter matrices $w_1, w_2, w_3, w_4$

$$ w_1 = \begin{bmatrix} 0 & 0.0779 \end{bmatrix} $$

$$ w_2 = \begin{bmatrix} 0.0294 & -1.9900 \\ 0.0913 & -0.9198 \\ -0.1462 & 2.3990 \end{bmatrix} $$

$$ w_3 = \begin{bmatrix} -0.1988 & 1.5439 & 1.3524 \\ 0.3063 & -0.1739 & 0.8509 \\ 19.5966 & -2.0623 & -3.6779 \end{bmatrix} $$

$$ w_4 = \begin{bmatrix} -16.2153 & 1.6530 & 0.4266 \end{bmatrix} $$

The controller (18) parameters are selected as $\tau = 1.8, k_1 = 2, k_2 = 2, h_1 = 3, h_2 = 0.2, \xi = 0.028, \varphi = 0.48$. The initial condition is $\bar{x}(0) = [0.1 \ 0.01 \ 0.2 \ 0.15]$. This system is simulated with disturbance: $\delta(t) = 2 \sin(2t) + 2 \cos(2t)$. The parameters of the sliding mode disturbance observer are selected as $c_{id} = c_{id} = 2, \alpha_{id} = 0.5, \alpha_{id} = 3, \eta_{id} = 0.01$. The disturbance is set to $B_2 \delta(t) = [2 \sin(\pi t) + 0.15 \cos(2\pi t)]$. The parameters of adaptive schemes are chosen as $d_1 = 1, \mu = 0.99, f_0 = 0.6, e_0 = 0.01, \nu = 0.18, l = 0.01$, and starting with zero initial conditions.

Figures 1 and 2 are the control input of the system obtained from Zhang et al.\textsuperscript{11} and (18), respectively. Obviously, the proposed controller possesses less chattering. The fluctuation of the control input from 0 to 6 s is caused by the system states $\dot{x}_1(t)$ and $\dot{x}_2(t)$ in the controller (18). It can be seen from Figure 4 that when the system state tends to be stable, the input of the controller will also tend to stable. The fluctuation of control input after 6 s is caused by $k_1 \text{sgn}^h(\delta(t)) + k_2 \text{sgn}^{l^2}(\delta(t))$ in the controller (18). Figure 3 shows that after the sliding surface converges to 0, there is a slight fluctuation around zero, which causes the $\text{sgn}(\delta)$ in the controller to switch between $-1$ and 1, thus generating the fluctuations in Figure 2. Figure 4 depicts that system state by Zhang et al.\textsuperscript{11} and (18), respectively. Obviously, the system states obtained with the proposed controller have less overshoot and faster convergence performance.
The coefficient of additional term $k_1 + k_2 > 1$ located in denominator, although the item is omitted in $t_e$, actually, it must enable the system to converge faster.

Figure 5 is evolutions of disturbance and its estimation in controller. Compared with Hwang et al., the estimation of disturbance can faster track its true value in finite time under the proposed observer (33). Figures 6 and 7 shown that the adaptive gain $k_d(t)$ still closely follows $B_2\tilde{d}(t)$. As in (48), $k_d(t)$ converges to a safety margin which depends on the parameters $\mu$ and $v$. $k_d(t)$ is always above $B_2\tilde{d}(t)$, the sliding motion will be maintained and the conditions on disturbance bounds are relaxed, the proposed ASMDO does not require information about the bound on the disturbances and their derivatives.

**Conclusion**

This paper has addressed a disturbance observer-based control method for continuous-time linear systems with unknown external disturbance. Based on a novel adaptive sliding mode disturbance observer, the output feedback sliding mode controller has been designed, which guarantees that reachability condition holds strictly. In the absence of upper bound information of the disturbance and its first derivative, the restrictive restraints on disturbance have been relaxed by designing the adaptive laws. It is worth mentioning that the estimation of disturbance can track its true value in finite time under the proposed observer. An inverted pendulum on a cart model has been exploited to illustrate the effectiveness of the proposed controller and observer. Simulation results show that the convergence performance of the controller can be further improved, and the finite time control observer-based method will be studied in future.
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