Many-body critical phase: extended and nonthermal

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The transition between ergodic phase and many-body localization (MBL) phase lies at the heart in understanding quantum thermalization of many-body systems. Here we predict a many-body critical phase in the one-dimensional extended Aubry-André-Harper-Hubbard model, which is different from both the ergodic phase and MBL phase, implying that the quantum system hosts three different fundamental phases in the thermodynamic limit. It is shown that the level statistics in the many-body critical phase are well characterized by the so-called critical statistics, and the wave functions generally exhibit a multifractal behavior. We further study the half-chain entanglement entropy (EE) and thermalization properties by exact diagonalization, which show that the EE in this critical phase manifest a volume law EE scaling while the many-body states violate the eigenstate thermalization hypothesis. This work unveils a novel many-body phase which is extended but nonthermal.

Introduction.—In the past decade, the eigenstate thermalization hypothesis (ETH) [1–4] has become an essential theoretical underpinning for understanding thermalization in quantum systems. Eigenstates in an ergodic phase obey the ETH, while including disorder in some interacting systems can lead to the many-body localization (MBL) if the disorder strength is strong enough. The MBL phase violates the ETH, i.e., the states in MBL cannot thermalize. The existence of the MBL phase has been well established in one-dimensional interacting systems with random disorder [1–5] or incommensurate potential [12–21], and has also been observed in interacting ultracold atomic gases trapped in incommensurate optical lattices [25–29].

The nature of the transition from an ergodic phase to MBL remains an active area of research. The entanglement entropy (EE) of eigenstates in an ergodic phase follows the volume law, but in MBL obeys an area law [30–32]. In transition between such two phases, the EE changes in a singular way, rendering an eigenstate phase transition. On the other hand, in view of the long-time dynamics in the ergodic phase and MBL phase [33–38], a dynamical phase transition is manifested between the two phases. The critical features of the transition are also examined in the recent works [21] [30–31] [39] to reveal the nature of the critical points. In particular, an outstanding question is, whether there exists some sort of critical phase, other than the critical point in the phase transition? The results based on the finite-size analyses showed that a quantum critical region in the finite-size interacting system vanishes in the thermodynamic limit [21] [33] [40] [52], for which the emergence of a many-body critical phase is yet illusive. To confirm the existence of such critical phase which is different from both the ergodic and MBL phases is undoubtedly important in understanding the quantum thermalization physics of many body disorder systems.

In this work, we show that a many-body critical phase can exist in thermodynamic limit in an extended Harper model [53] [56] with Hubbard interaction. Employing exact diagonalization (ED) to obtain various diagnostics such as level statistics, multifractal, EE and thermalization properties, we confirm the existence of this new phase of quantum matter which is different from the ergodic phase and MBL phase. The level statistics is shown to follow the so-called critical statistics [57] [62] and the wave functions exhibit generally multifractal behaviors. Moreover, the eigenstates in this critical phase violate the ETH but their EE follow a volume law, so this many-body critical phase is an extended nonthermal phase.

Model and phase diagram.—We start with the extended Harper model from the Hamiltonian [53] [56]

\[ H_0 = \sum_j \{(1 + \mu \cos[2\pi(j + 1/2)\alpha + \delta])c_j^\dagger c_{j+1} + H.c. \}

+ V \cos(2\pi j \alpha + \delta) c_j^\dagger c_j \}

(1)

where \( c_j \) (\( c_j^\dagger \)) is the fermion annihilation (creation) operator at the site \( j \), \( \mu \) represents the amplitude of the modulation in the off-diagonal hopping, \( V \) is the strength of the on-site incommensurate potential and \( \delta \) is an arbitrary phase shift. We take \( \alpha = \frac{\sqrt{5} - 1}{2} \), then both the on-site potential and the hopping amplitude between the nearest-neighbor (NN) lattice sites are incommensurate. Fig. 1(a) shows the phase diagram of this system [55] [56], where the region I, II and III correspond to the single-particle extended, critical, and localized phases respectively. When \( \mu = 0 \), this model is reduced to the Aubry-André-Harper model [63] and \( V = 2 \) is the transition point from the extended to the localized phases.
We then add the NN repulsive Hubbard interaction and the total Hamiltonian is described by
\[ H = H_0 + U n_j n_{j+1}, \] (2)
where \( n_j = c_j^\dagger c_j \) is the fermion number operator and \( U \) represents the interaction strength. We consider the half-filling case, with the numbers of fermions \( N \) and the lattice sites \( L \) being fixed to \( N/L = 1/2 \). Since the sample-to-sample fluctuations in quasiperiodic models are much weaker than those in random disorder models, the number of samples used ranges from 500 (\( L = 10 \)) to 50 (\( L = 16 \)) for our study, where a sample is specified by choosing an initial phase \( \delta \). We take open boundary conditions unless otherwise stated and use ED for calculation. The main results are shown in Fig. 1(b), where the interaction is taken \( U = 1 \) as an example, and three fundamental phases, i.e., the ergodic phase (I), the MBL (II) and the many-body critical phase (III) which constitutes the most important prediction in this work, are uncovered. In the following we confirm the phase diagram and further explore the fundamental properties of the many body critical phase.

Energy level statistics and multifractal analysis. — The many-body critical phase in the region II in Fig. 1(b) can be identified from the energy level statistics and multifractal behavior. Defining the energy spacing as \( \delta_n = E_{n+1} - E_n \), where the eigenvalues \( E_n \) have been listed in ascending order. Then we can obtain the ratio of adjacent gaps as \( r_n = \frac{\min(b_n, b_{n+1})}{\max(b_n, b_{n+1})} \) and average it over all gaps and samples. For the system in the ergodic phase, its level statistics follow Gaussian-orthogonal ensemble (GOE): \( P_G = \frac{2}{\pi} \frac{\delta}{\langle \delta \rangle} \exp\left(-\frac{\delta^2}{4\langle \delta \rangle^2}\right) \), where \( \langle \delta \rangle \) is the mean spacing and \( \langle \delta \rangle \) converges to 0.529. In the MBL phase, the level statistics are Poisson: \( P_P = \frac{1}{\langle \delta \rangle} \exp\left(-\frac{\delta}{\langle \delta \rangle}\right) \) and \( \langle \delta \rangle \approx 0.137 \) [6, 7]. One can see \( P_P \) is maximum at \( \delta = 0 \) while in ergodic phase \( P_G(\delta = 0) = 0 \). The latter corresponds to the level repulsion of spectra in the ergodic phase. The larger \( \langle \delta \rangle \) in the ergodic phase tells that the spectrum of the ergodic phase is more uniform than that in the MBL phase. However, the value of \( \langle \delta \rangle \) in the region II of Fig. 1(b) is neither 0.387 nor 0.529 (see more details in Supplemental Materials [64]), implying that the level statistics are neither GOE nor Poisson.

To characterize the statistical properties of the energy spectra in the region II, we consider the level number variance \( \langle \delta^2 \rangle \), which is given by: \( \langle \delta^2 \rangle = \langle M^2(\epsilon) \rangle - \langle M(\epsilon) \rangle^2 \), with \( \langle M(\epsilon) \rangle \) counting the number of levels in a strip of width \( \epsilon \) on the unfolded scale [17, 65, 66]. The unfolding procedure is using a smooth function to fit the staircase function \( \eta(E) = \sum_m \Theta(E - E_m) \), which counts the number of eigenvalues less than and equal to \( E \) (see Supplemental Material [64]). The angular bracket denotes the average over different regions of the spectrum and different samples. In the MBL phase, the spectrum has no correlations, and therefore the number variance is exactly linear with slope one, i.e., \( \langle \delta^2 \rangle = \langle \delta \rangle^2 \). The spectrum in the ergodic phase, as mentioned above, is more uniform due to the level repulsion, so the number variance displays a slow logarithmic growth: \( \langle \delta^2 \rangle \approx \frac{2}{\pi} \ln(2\pi M) \) (red solid curve in Fig. 2). When \( M \) is bigger, the number variance in the ergodic phase shows a much faster
power-law growth, which is considered to be a evidence of the existence of the Thouless energy [17, 18]. The number variance in the critical phase is qualitatively different, which is linear $\Sigma^2(M) \sim \chi M$ but with a slope less than one $\chi < 1$, as given in Fig. 2. In order to see it clearly, we re-display the number variance of the critical regime in the inset and confirm that the number variances are asymptotically linear with slopes $1/2 < \chi < 1$, which is a signature of critical statistics. The critical statistics can be used to describe the energy spectrum of a high-dimensional, noninteracting disordered system near the extended-Anderson localization transition point. The number variance in the critical phase is intermediate between the ergodic phase and the MBL phase, which means that the uniformity and the strength of level repulsions of the spectrum are also in between.

We further confirm the existence of the many body critical phase from the multifractal analysis, which can be performed by examining the participation entropies (PE) [68, 69], defined for each eigenstate $|n\rangle$ as $S^p_{\mu}(|n\rangle) = \frac{1}{\lambda_i^{2p}} \ln(\sum_j \psi_j^{2p})$, where $\psi_j$ is the wave function coefficient of the eigenstate $|n\rangle$ in the computational basis $\{|j\rangle\}$, given by $\psi_j = \langle j |n\rangle$, and $D = \langle \lambda_i^2 \rangle$ is the Hilbert space size. We focus on $S^p_{\mu} = -\ln(\sum_j |\psi_j|^4)$ of the $q = 2$ case, where $\sum_j |\psi_j|^4$ is the usual inverse participation ratio [70]. The fractal dimension $a$ of a wave function can be defined as $a = \frac{S^p_{\mu}}{\ln D}$ with $D \to \infty$, and a multifractal behavior corresponds to $0 < a < 1$. Fig. 3 (a) show the fractal dimensions $a$, which is obtained by using $\langle S^p_{\mu} \rangle = a \ln D + c$ to fit $\langle S^p_{\mu} \rangle$ and $\ln D$ with different sizes, where $c$ is a constant. We see that $a \approx 1$ in the ergodic phase and $a \ll 1$, or $a = 0$ within error bars [46] in the MBL phase. By contrast, in the critical phase, the fractal dimension $a$ is less than 1 but not close to 0, indicating a deep multifractal behavior. Further, more careful analysis shows a logarithmic subleading correction to the scaling of $\langle S^p_{\mu} \rangle$, i.e., $\langle S^p_{\mu} \rangle = a S_0 + b \ln(S_0) + o(S_0)$ with $S_0 = \ln D$, and $b \neq 0$ indicates a nontrivial multifractal behavior. We find that $b$ is negative, negative, and positive for the ergodic, critical, and MBL phase respectively as showed in Fig. 3 (b). We summarize the coefficients $a$ and $b$ in different phases in the Table 1.

| Coefficients | Phases            |
|--------------|--------------------|
| $a \approx 1$, $b < 0$ | ergodic phase     |
| $0 < a < 1$, $b < 0$ | critical phase    |
| $a \ll 1$ or $a = 0$, $b > 0$ | MBL phase         |

Table 1: A comparison of the coefficients $a$ and $b$ in different phases.

![Figure 3](image3.png)

Figure 3: (a) Fractal dimension $a$ numerically obtained by averaging over the mid one-third states and samples with system sizes $L \in \{10, 12, 14, 16\}$ as a function of $V$ and $\mu$. (b) The averaged PE $\langle S^p_{\mu} \rangle$ as a function of $S^p_{\mu}$ for system sizes $L$ (to reduce errors, here we also consider the case of $L = 18$). The fitting coefficients are $a = 1.03 \pm 0.07$, $b = 1.62 \pm 0.39$ for $V = 2$, $\mu = 0.5$ (in the ergodic phase), $a = 0.11 \pm 0.07$, $b = 0.3 \pm 0.23$ for $V = 4$, $\mu = 0.5$ (in the MBL phase), the slope of $a$ will include $0$ when increasing the system size [68] and $a = 0.66 \pm 0.15$, $b = -0.35 \pm 0.63$ for $V = 1$, $\mu = 1.2$ (in the many-body critical phase).

![Figure 4](image4.png)

Figure 4: (a) The sample averaged EE $\langle S \rangle$ as a function of $V$ with fixed $\mu = 0.5$ and (b) $\langle S \rangle$ as a function of $\mu$ with fixed $V = 1$. Finite-size scaling analysis of $\langle S \rangle$ as a function of $(V-V_c)L^{1/\nu}$ with fixed $\mu = 0.5$ in (c) and $(\mu-\mu_c)L^{1/\nu}$ with fixed $V = 1$ in (d). Rescaled $\langle S/\langle S \rangle \rangle$, so that all curves for different sizes converge to a single curve.
independent of $L$ in the deep MBL phase, which fulfills an area law. Unlike the MBL, we show in Fig. 4(b) that the average EE follows a volume law when the system is the critical phase. For a more precise study, we perform a finite-size scaling analysis for the EE which is rescaled by the Page value $S_T = 0.5 L \ln 2 - 1$ [71, 72], as shown in Fig. 4(c) and (d), where the results are fit to $(S/S_T) = f[(V - V_c) L^{1/\nu}]$ with fixed $\mu = 0.5$, and $(S/S_T) = g[(\mu - \mu_c) L^{1/\nu}]$ with fixed $V = 1$, respectively. Here $V_c$ and $\mu_c$ denote the transition point from the ergodic phase to MBL phase and many body critical phase, respectively, and $\nu$ is the associated critical exponent. The results of the best fit are (c) $V_c = 3.09 \pm 0.05$ and $\nu = 0.71 \pm 0.06$ and (d) $\mu_c = 1.03 \pm 0.02$ and $\nu = 0.32 \pm 0.03$ (see details in Supplemental Material [64]). The critical exponent $\nu$ can be determined similarly for generic parameters $V$ and $\mu$ near the phase boundaries, and exhibits a variation range in the whole phase boundaries with different $V_c$ and $\mu_c$, which is typical for disordered systems. The approximate ranges are

$$\nu = \begin{cases} 
0.6(0) \sim 0.8(2), & \text{for } I \to III, \\
0.3(5) \sim 0.4(7), & \text{for } I \to II, \\
0.6(5) \sim 0.8(4), & \text{for } II \to III,
\end{cases}$$

(3)

where I, II and III are the ergodic, many body critical and MBL phases respectively. The scaling exponents for the transitions from ergodic to MBL phases and from critical to MBL phases are close. This is consistent with the result that the many-body critical phase is extended, with EE satisfying volume law and being more similar to the ergodic phase.

Note that a many-body critical regime was found in a finite-size system near the transition point between the ergodic and MBL phases [99,103]. However, even an ergodic (or MBL) system can display critical behavior if the system size is smaller than the correlation length (or localization length). The critical regime vanishes in the thermodynamic limit after performing finite-size analyses [103]. From the finite-size scaling analysis, we confirm that the many-body critical phase predicted here exists in the thermodynamic limit.

**Thermalization properties.**— We finally study the thermalization properties of the many-body critical phase. For this we consider the average deviation of the half-chain particle-number distribution from the half-number $(N/2)$ of particles, which can characterize the thermalization of the system and is defined by $T = \{1/D\} \sum_{m=1}^{D} (O(E_m) - N/2)^2\}^{1/2}$, with many-body eigenstates of eigenvalue $E_m$ being summed over. Here the observable $O(E) = \sum_{j=1}^{L/2}\langle \psi_E | n_j | \psi_E \rangle$ quantifies the number of particles distributed in the half chain of the lattice for the many-body eigenstate $| \psi_E \rangle$ with energy $E$ [14]. The large fluctuation of $O(E)$ among nearby eigenstates signifies the violation of the ETH. In the ergodic phase, the fluctuations of $O$ are small and the ETH is satisfied [Fig. 5(a)], while in the MBL phase, the fluctuations are obviously larger and the ETH is violated [Fig. 5(b)]. For parameters in the critical regime, as shown in Fig. 5 (c), the fluctuations of $O$ are also large, which implies that the ETH is violated and the eigenstates are non-thermal. This feature can be even clearer by examining the qualitative behaviors of $T$ which we define above. Note that as the system size $L$ increases, the value $O(E)$ tends to $N/2$ in the ergodic phase but keeps fluctuating for non-thermal phase. We expect that $T$ decreases for the ergodic phase and increases to finite value with increasing $L$. The numerical results of the sample averaged $(T)$ are presented in Fig. 5 (d) and (e), as a function of $V$ and $\mu$. With the increasing of system size $L$, we see that $(T)$ decreases if the phase is ergodic, but enlarges in the MBL and critical phases. Therefore, the critical phase is non-thermal, similar to the MBL phase. Together with the preceding discussion on the EE, we conclude that the many-body critical phase is an extended non-thermal phase.

**Conclusions.**— We have predicted that a many-body critical phase exists in the thermodynamic limit in the 1D extended Aubry-André-Harper-Hubbard model. The many-body critical phase shows basic properties that are distinct from the ergodic and MBL phases. First, by analysing number variance, we found that the level statistics in the critical phase are neither GOE for ergodic regime nor Poisson for MBL, but are well described by critical statistics. Further, from a multifractal analysis we showed that the many-body states in the critical phase exhibit the multifractal behavior which is different from the ergodic and MBL phases. Finally, we unveiled that the predicted critical phase violates the ETH but their EE exhibits a volume law, implying that this exotic crit-
ical phase is delocalized but non-ergodic and not thermalizable. Being a new critical interacting phase, many interesting issues, such as the dynamical properties, deserve further investigation in the next works. Our work opens a door to study of quantum thermalization physics in the many-body critical phases.

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SUPPLEMENTAL MATERIAL

In the Supplementary Materials, we first show the average ratio \( \langle r \rangle \). Then, we provide the details in unfolding procedure and scaling analysis.

I. Energy level statistics

When we research the energy level statistics in the main text, we define the ratio of adjacent gaps as

\[
    r_n = \frac{\min(\delta_n, \delta_{n+1})}{\max(\delta_n, \delta_{n+1})},
\]

where \( \delta_n = E_{n+1} - E_n \), where the eigenvalues \( E_n \) have been listed in ascending order. Fig. S1 (a) displays the ratio \( \langle r \rangle \) averaged over all gaps and samples as a function of \( V \) with fixed \( \mu = 0.5 \) and we see that \( \langle r \rangle \) changes from 0.529 to 0.387 when \( V \) is increased from the ergodic phase to the many body localization (MBL) phase. Fig. S1 (b) shows \( \langle r \rangle \) as a function of \( \mu \) with \( V = 1 \) and one can see that \( \langle r \rangle \) in the critical phase is neither equal to 0.387 nor equal to 0.529, which indicates that the level statistics are neither Gaussian-orthogonal ensemble nor Poisson. As discussed in the main text, the value of \( \langle r \rangle \) presents uniform degree of the spectrum. We see that \( \langle r \rangle \) in the critical phase is intermediate between the ergodic and MBL phases, which means that the uniformity of the spectrum are also between the two, which is agreement with the conclusion obtained by using the level number variance.

II. Unfolding procedure

We have studied the level statistics in the main text by unfolding the spectrum. We describe the unfolding procedure here. For an ordered sequence of eigenvalues \( \{E_1, E_2, \cdots, E_M\} \), we can define a stick spectral function:

\[
    D(E) = \sum_{m=1}^{M} \delta(E - E_m). \tag{S1}
\]

To unfold this spectrum, we further define a cumulative spectral function \( \eta \),

\[
    \eta(E) = \int_{-\infty}^{E} D(E')dE' = \sum_{m=1}^{M} \Theta(E - E_m). \tag{S2}
\]

This function counts the number of eigenvalues less than and equal to \( E \) and it is a staircase function, which can be decomposed into a smooth part \( \bar{\eta}(E) \) and a fluctuating part \( \delta\eta(E) \): \( \eta(E) = \bar{\eta}(E) + \delta\eta(E) \).

Unfolding procedure corresponds to mapping the eigenvalues to the smooth part, \( E_n \rightarrow \bar{\eta}(E_n) \), i.e., mapping the sequence \( \{E_1, E_2, \cdots, E_M\} \) to the numbers \( \{\bar{\eta}(E_1), \bar{\eta}(E_2), \cdots, \bar{\eta}(E_M)\} \). As required, the density of state of the unfolded spectrum, i.e. the derivative of the smooth part, is unity.

In practice, the separation of a spectrum into a smooth part and a fluctuating part is not a trivial task. There are many ways of unfolding [1, 2]. After making some tests and comparisons, the method we used is using a polynomial regression of degree 3 to fit the staircase function in the range we considered [1, 3], which can keep the correlations at both short-range and long-range.

![Figure S1](image-url)  
Figure S1: The average ratio \( \langle r \rangle \) of adjacent energy gaps versus (a) \( V \) with fixed \( \mu = 0.5 \) and (b) \( \mu \) with fixed \( V = 1 \).
III. Scaling analysis

The transition points $V_c$, $\mu_c$ and the corresponding critical exponents $\nu$ can be estimated using a scaling analysis on the average half-chain entanglement entropy $\langle S \rangle$. The angular bracket denotes the average over the mid one-third states and over different samples. $\langle S/S_T \rangle$, where $S_T$ is the Page value $S_T = 0.5[L \ln 2 - 1]$, can be written as a function of $\langle P - P_c \rangle L^{1/\nu}$, where $P$ denotes $V$ or $\mu$ and $P_c$ denotes $V_c$ or $\mu_c$. $P_c$ and the corresponding $\nu$ can be determined by minimizing the quantity $[1]$

$$Sr(P_c, \nu) = \frac{1}{P_{\text{max}} - P_{\text{min}}} \int_{P_{\text{min}}}^{P_{\text{max}}} \text{Var}_L \{ \langle S/S_T \rangle \langle (P - P_c) L^{1/\nu} \rangle \} dP,$$

where $\text{Var}_L$ is the variance over different sizes $L$. For different sizes, we need use the equal $\langle P - P_c \rangle L^{1/\nu}$ to calculate the variance for each $P$, so the corresponding $\langle S/S_T \rangle$ need to be interpolated using cubic splines. Where $P_{\text{min}}$, $P_{\text{max}}$, $P_c$ and $\nu$ can be extracted in an appropriate range to obtain $P_c$, $\nu$ and their errors. For the case of fixing $\mu = 0.5$, as shown in Fig.2(a) in the main text, $V_{\text{min}}$ is extracted from a box distribution between 2 and 2.5, and $V_{\text{max}}$ between 3.5 and 4. $V_c$ and $\nu$ are taken from a box distribution $[2.5, 3.5]$ and $[0.1, 1]$ respectively. For the selected $V_{\text{min}}$, $V_{\text{max}}$, $V_c$ and $\nu$, $Sr$ can be obtained. When fixing $V_{\text{min}}$ and $V_{\text{max}}$, the minimum value of $Sr$ provide the workable $V_c$ and $\nu$. Choosing different $V_{\text{min}}$ and $V_{\text{max}}$ from the corresponding box distributions, one can obtain different workable $V_c$ and $\nu$, which can provide the errors in $V_c$ and $\nu$. For other parameters, one can obtain the $V_c$, $\mu_c$, the corresponding $\nu$ and their errors by using the same method.

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