SCHATTEN CLASS BERGMAN-TYPE AND SZEGÖ-TYPE OPERATORS ON BOUNDED SYMMETRIC DOMAINS

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Abstract. In this paper, we investigate singular integral operators induced by the Bergman kernel and Szegö kernel on the irreducible bounded symmetric domain in its standard Harish-Chandra realization. We completely characterize when Bergman-type operators and Szegö-type operators belong to Schatten class operator ideals by several analytic numerical invariants of the bounded symmetric domain. These results generalize a recent result on the Hilbert unit ball due to the author and his coauthor but also cover all irreducible bounded symmetric domains. Moreover, we obtain two trace formulae and a new integral estimate related to the Forelli-Rudin estimate. The key ingredient of the proofs involves the function theory on the bounded symmetric domain and the spectrum estimate of Bergman-type and and Szegö-type operators.

1. Introduction

Let \( \Omega \) be an irreducible bounded symmetric domain in the standard Harish-Chandra realization, namely \( \Omega \) can be realized as the open unit ball with respect to the so-called spectral norm in a finite dimensional complex vector space \([10, 20]\), thus we can identify \( \Omega \) with the spectral unit ball. Denote the normalized Lebesgue measure on \( \Omega \) by \( dv \) which means that the volume measure of \( \Omega \) is one. It is well known \([10]\) that there exists a unique generic polynomial \( h(z, w) \) in \( z, \bar{w} \) and an analytic numerical invariant \( N \) satisfying the Bergman kernel of \( \Omega \) is given by

\[
K(z, w) = h(z, w)^{-N},
\]

where the invariant \( N \) is also called the genus of domain \( \Omega \).

Bergman-type operators are singular integral operators induced by the (modified) Bergman kernel. In this paper, we will mainly consider the Bergman-type integral operator in the following form. Denote the \( K \)-invariant normalized measure \( dv_\gamma \) on \( \Omega \) by

\[
dv_\gamma(w) = c_\gamma h(w, w)^\gamma dv(w)
\]

for \( \gamma > -1 \), where \( c_\gamma \) is the normalized constant and \( K \) is the isotropic subgroup in the identity connected component of the biholomorphic automorphism group of \( \Omega \).

Definition 1.1. For \( \alpha \in \mathbb{R} \) and \( \gamma > -1 \), the Bergman-type integral operator \( B_{\alpha,\gamma} \) on \( L^1(dv_\gamma) \) is defined by

\[
B_{\alpha,\gamma} f(z) = \int_{\Omega} \frac{f(w)}{h(z, w)^\alpha} dv_\gamma(w), \quad z \in \Omega.
\]
Bergman-type integral operators play an important role in complex analysis and operator theory. Note that $B_{N,0}$ is precisely the standard Bergman projection on $L^2(dv)$. The boundedness of Bergman projection on $L^p(dv)$ for $1 < p < \infty$ is a fundamental and open problem on the bounded symmetric domain, which is only solved in the case of Hilbert unit ball $\mathbb{B}$ that is the unit ball of rank one in the usual complex Euclidean norm. Moreover, Bergman projection has a close relationship with classical integral operators such as Toeplitz operators and Hankel operators on Bergman space over the bounded symmetric domain. In [26, 27], the Bergman-type integral operator was applied to develop the holomorphic Besov function theory on the bounded symmetric domain, which was then used for investigating boundedness, compactness and Schatten membership [19, 29] of Hankel operators. In the case of Hilbert unit ball, the Bergman-type integral operator can be used to study more holomorphic function spaces such as Bergman space and Bloch space, but also frequently applied to operator theory in function space [25, 28, 29].

Recently, the boundedness, compactness and Schatten membership of Bergman-type, Volterra-type and Toeplitz-type operators on the Hilbert unit ball have attracted more attention [4, 6, 7, 8, 14, 17, 18, 24]. These researches more or less base on the Forelli-Rudin asymptotic estimate of Bergman kernel on the Hilbert unit ball [11, 28]. However, in the high rank case, the Forelli-Rudin asymptotic estimate is a long time unsolved problem [9, 10, 23]. The first purpose of this paper is to characterize Schatten class Bergman-type operator $B_{\alpha,\gamma}$ on the bounded symmetric domain. To overcome the main difficulty that there is no complete Forelli-Rudin asymptotic estimate in the high rank case, the author and his coauthor provide an alternative approach based on the spectrum estimate in the case of Hilbert unit ball [7]. In this paper we will systematically develop this approach on the bounded symmetric domain. More precisely, we introduce the notation of prediagonal operator on abstract Hilbert space and reduce the Bergman-type operator $B_{\alpha,\gamma}$ to the prediagonal operator. In our case, the prediagonal operator is normal compact and takes its eigenvalues (counting multiplicities) as its characteristic. Then combing with asymptotic estimates of Gindikin’s Gamma functions [10] and dimension of irreducible polynomial spaces occur in the Peter-Schmid-Weyl decomposition [10, 21, 22] follows our main results (Theorem 3.1 and 3.2). As its application, we obtain two trace formulae (Proposition 4.1), a new integral estimate (Corollary 4.3) related to the Forelli-Rudin estimate on the bounded symmetric domain and a compactness result (Corollary 4.4) of the operator $B_{\alpha,\gamma}$ from $L^\infty(dv_\gamma)$ to $L^1(dv_\gamma)$. Similarly, we also consider the Schatten class Szegö-type operators which are integral operators introduced by the Szegö kernel.

These results generalize our recent result on the Hilbert unit ball [7, Theorem 3], but also covers all irreducible bounded symmetric domains. It is worth mentioning that the condition in our results is necessary and sharp. This work can be also viewed as an attempt to study the boundedness and compactness of classical integral operators without Forelli-Rudin estimate on bounded symmetric domains. Since there exists an one-to-one corresponding between semi-simple Lie algebras of Hermitian type and Hermitian Jordan triple Systems [20, 21, 22], we believe that our work should be reproduced in term of Jordan triple Systems.
The paper is organized as follows. In Section 2, we give some preliminaries of
the function theory on bounded symmetric domains in term of the Harish-Chandra
realization. In section 3, we give and prove our main results on Bergman-type opera-
tors. In section 4, we prove two trace formulae of the Bergman-type operator, as its
application, we give an integral estimate related to the Forelli-Rudin estimate and a
compactness result on the Bergman-type operator $B_{\alpha,\gamma}$. In section 5 we give our results
on Szeg"o-type operators.

2. PRELIMINARIES

In this section, we will recall some basic results about the bounded symmetric domain
in its standard standard Harish-Chandra realization without proofs, and the interested
readers can consult [10, 12, 20] for details. Let $\Omega$ be an irreducible bounded symmet-
ric domain in $\mathbb{C}^d$. Let $G$ be the identity connected component of the biholomorphic
automorphism group of $\Omega$, and its isotropic subgroup $K$ is given by

$$K := \{g \in G : g(o) = o\},$$

for any fixed $o \in \Omega$. Then $\Omega = G \cdot o \approx G/K$. Denote the Lie algebra of $G, K$ by $\mathfrak{g}, \mathfrak{k}$
respectively. Then $\mathfrak{g}$ is semi-simple Lie algebra and $\mathfrak{k}$ has nontrivial center with the
Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$  

Chose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{k}$, and it is also a Cartan subalgebra of $\mathfrak{p}$. Denote by
$\mathfrak{g}^\mathbb{C}, \mathfrak{h}^\mathbb{C}$ the corresponding complexification of $\mathfrak{g}, \mathfrak{h}$ respectively. A nonzero root $\gamma$ of $\mathfrak{g}^\mathbb{C}$
with respect to $\mathfrak{h}^\mathbb{C}$ is called compact if $\gamma \in \mathfrak{k}^\mathbb{C}$ and is called noncompact if $\gamma \in \mathfrak{p}^\mathbb{C}$. Let
$\theta$ be the conjugation with respect to $\mathfrak{k} \oplus i\mathfrak{p}$ and $\{e_\gamma\}$ be a base of root vectors such that

$$\theta e_\gamma = -e_{-\gamma}, [e_\gamma, e_{-\gamma}] = h_\gamma, [h_\gamma, e_{\pm\gamma}] = \pm 2e_{\pm\gamma}.$$  

Set $\mathfrak{p}^\pm = \sum_\gamma \mathbb{C}e_{\pm\gamma}$, thus we have the following canonical decomposition

$$\mathfrak{g}^\mathbb{C} = \mathfrak{h}^\mathbb{C} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-.$$  

Denote by $G^\mathbb{C}$ the adjointgroup of $g^\mathbb{C}$. Let $G, K, K^\mathbb{C}, P^\pm$ be the analytic subgroups
corresponding to $\mathfrak{g}, \mathfrak{k}, \mathfrak{k}^\mathbb{C}, \mathfrak{p}^\pm$, respectively. It follows from a well-known lemma [20,
Lemma 4.2] that there exists a natural (open) embedding:

$$\Omega \approx G/K \approx GK^\mathbb{C}P^+ / K^\mathbb{C}P^- \hookrightarrow P^+ K^\mathbb{C}P^- / K^\mathbb{C}P^- \approx P^+ \approx \mathfrak{p}^+.$$  

Thus the embedding $\varphi : \Omega \to \mathfrak{p}^+$ is given by the following rule:

$$\Omega \to \mathfrak{p}^+$$

$$x = g \cdot o \Leftrightarrow (\exp z)^{-1} g,$$

namely $\varphi$ maps $g$ to the $K^\mathbb{C}P^-$-part of in the direct product decomposition $P^+ K^\mathbb{C}P^-$. 

Due to Harish-Chandra, there exist $r$ linearly independent positive non-compact roots
$\{\gamma_1, \ldots, \gamma_r\}$ (relative to $\mathfrak{h}^\mathbb{C}$) such that $\gamma_1 > \gamma_2 > \cdots > \gamma_r$ and $\gamma_i \pm \gamma_j$ is not a root
whenever $i \neq j$ which are called strongly orthogonal roots, the positive integer $r$ is
called the rank of domain $\Omega$. Let

$$e_j := e_{\gamma_j}, h_j := h_{\gamma_j}.$$  

and

\[ e := \sum_{j=1}^{r} e_j. \]

For any \( z \in p^+ \approx C^d \), there exist \( k \in K \) and \( (\xi_1, \cdots, \xi_r) \in \mathbb{R}^r \) such that

\[ z = Ad(k)(\sum_{j=1}^{r} \xi_j e_j), \tag{2.1} \]

and the \( r \)-tuple \( (\xi_1, \cdots, \xi_r) \) is uniquely determined by \( z \) up to the order and the sign. The expression (2.1) in fact induces the so-called spectral norm \( \| \cdot \|_{sp} \) on \( p^+ \approx C^d \) which is given by

\[ \|z\|_{sp} = \max_{1 \leq j \leq r} |\xi_j|. \]

Indeed, it can be proved that \( \varphi(\Omega) = \{z \in C^d : \|z\|_{sp} < 1\} \). This is the so-called bounded Harish-Chandra realization. Note that any two norm of the finite dimensional complex vector space \( C^d \) are equivalent, thus the topologies induced by the spectral norm and the usual Euclidian norm coincide on \( C^d \). Since the spectral unit ball in \( C^d \) is a bounded balanced convex domain, the one-to-one correspondence between the bounded balanced convex domains and Minkowski norm functions \[13\] on \( C^d \) implies that the spectral norm coincides with the Minkowski function of the spectral unit ball.

Denote by \( c = \exp i(\xi)(e - \theta e) \) the Cayley transform and by \( ^c g \) the Lie algebra of the group \( ^c G = cGc^{-1} \). Let \( \mathfrak{h}^- \) be the real span of the vectors in \( i\mathfrak{h} \), then \( i\mathfrak{h}^- \) is a Cartan subalgebra of the pair \( (^c \mathfrak{g}, ^c \mathfrak{k}) \), and \( \pm 1/2(\gamma_j \pm \gamma_k), \pm \gamma_j, \pm i\gamma_j \) are roots with respective multiplicities \( a, 1, 2b \) of \( ^c g \) with respect to \( i\mathfrak{h}^- \), where \( a, b \) are independent of indexes \( j, k \). Indeed, the nonnegative integer triple \((a, b, r)\) uniquely determines an irreducible bounded symmetric domain up to biholomorphism \[26\]. It follows from the dimension count that

\[ d = r + \frac{a}{2}r(r-1) + br. \]

The Hermitian inner product induced from the Killing form \( B \) and the the conjugation \( \theta \) on \( p^+ \) is defined by

\[ (z, w) = -B(z, \theta w), \]

for \( z, w \in p^+ \). Let \( P(p^+) \) be the set of all holomorphic polynomials on \( p^+ \), endowed with the \( K \)-invariant Fischer-Fock inner product

\[ (p, q) = \int_{p^+} p(z)\overline{q(z)}e^{-\langle z, z \rangle}dz, \tag{2.2} \]

for all \( p, q \in P(p^+) \). By \[10\] \[22\] the natural action of \( K \) on \( P(p^+) \) induces the Peter-Schmid-Weyl decomposition

\[ P(p^+) = \sum_{m \geq 0} P_m(p^+), \]

where \( m = (m_1, \cdots, m_r) \geq 0 \) runs over all integer partitions, namely

\[ m_1 \geq \cdots \geq m_r \geq 0. \]

The decomposition is irreducible under the action of \( K \) and is orthogonal under the Fischer-Fock inner product.
Let $I_\Omega$ be the set of all integer partitions with length $r$. Define
\[
I_\Omega(0) = \{0\},
\]
\[
I_\Omega(j) = \{(m_1, \cdots, m_j, 0, \cdots, 0) : m_1 \geq \cdots \geq m_j > 0\}, 1 \leq j \leq r - 1,
\]
\[
I_\Omega(r) = \{(m_1, \cdots, m_r) : m_1 \geq \cdots \geq m_r > 0\}.
\]

It obvious that
\[
I_\Omega = \bigcup_{j=0}^{r} I_\Omega(j) \quad \text{and} \quad I_\Omega(i) \cap I_\Omega(j) = \emptyset
\] whenever $i \neq j$. Let $a^+$ be the real span of $\{e_1, \cdots, e_r\}$, denote the $K$-invariant polynomial $h$ on the real span $a^+ \subset p^+$ by
\[
h(\sum_{j=1}^{r} t_j e_j) = \prod_{j=1}^{r} (1 - t_j^2),
\]
for real $r$-tuple $t = (t_1, \cdots, t_r)$. Note that $h$ is a real polynomial, thus it can be polarized, from [10], we know that
\[
h(z, w) = \exp \sum_{j} z_j \frac{\partial}{\partial z_j} \exp \sum_{j} \bar{w}_j \frac{\partial}{\partial \bar{w}_j} h(z)
\]
is still a polynomial in $z, \bar{w}$. In particular, for $z$ in the form (2.1), the following identity holds
\[
h(z, z) = \prod_{j=1}^{r} (1 - \xi_j^2)
\] (2.4)

Denote by $A^2(dv_\gamma)$ the weighted Bergman space consisted of square integrable holomorphic functions with respect to the measure $dv_\gamma$ on $\Omega$. The genus $N$ of $\Omega$ is defined by
\[
N := 2 + a(r - 1) + b,
\]
then the Bergman kernel $K_\gamma(z, w)$ of the spectral unit ball is given by
\[
K_\gamma(z, w) = h(z, w)^{-(N+\gamma)},
\]
which is degenerated to the formula (1.1) when $\gamma = 0$. The weighted Bergman space $A^2(dv_\gamma)$ is a reproducing kernel Hilbert function space, since
\[
f(z) = \langle f, K_\gamma z \rangle_\gamma = \int_\Omega f(w) \overline{K_\gamma(z, w)} dv_\gamma(w), \quad z \in \Omega
\] (2.5)
for any $f \in A^2(dv_\gamma)$, where
\[
K_\gamma(z, w) = h(z, w)^{-(N+\gamma)}, \quad z, w \in \Omega;
\]
is also called the reproducing kernel of $A^2(dv_\gamma)$.

For an integer $r$-tuple $s = (s_1, \cdots, s_r) \in \mathbb{Z}^r$, the Gindikin’s Gamma function [10] is given by
\[
\Gamma_\Omega(s) = (2\pi)^{\frac{ar(r-1)}{4}} \prod_{j=1}^{r} \Gamma(s_j - \frac{a}{2}(j - 1)),
\]
of the usual Gamma function $\Gamma$ whenever the right side is well defined. The multi-variable Pochhammer symbol \cite{10, 22} is
\[
(\lambda)_s = \frac{\Gamma_{\Omega}(\lambda + s)}{\Gamma_{\Omega}(s)}.
\]
It can be verified that
\[
(\lambda)_s = \prod_{j=1}^{r}(\lambda + \frac{a}{2}(j - 1))_{s_j}
\]
of the usual Pochhammer symbols $(\mu)_m = \prod_{j=1}^{m}(\mu + j - 1)$.

3. Schatten class Bergman-type operators

In this section, we will state and prove our main results on Bergman-type operators, which completely characterize the Schatten class Bergman-type operator on the irreducible bounded symmetric domain. Before stating our main theoremes, we first introduce some notations involved. We define two sets $\mathcal{F}$ and $\mathcal{B}_\gamma$ of real numbers related to the bounded symmetric domain $\Omega$:
\[
\mathcal{F} := \{f = a_2(l - 1) - k : 1 \leq l \leq r, k \in \mathbb{N}\},
\]
\[
\mathcal{B}_\gamma := \{s < N + \gamma : s \notin \mathcal{F}\},
\]
where $\mathbb{N}$ is the set of all nonnegative integers. Note that
\[
\max \mathcal{F} < \gamma + N.
\]
It is easy to verify that $\alpha \in \mathcal{F}$ if and only if $(\alpha)_m = 0$ for all but finitely many integer partitions $m \geq 0$.

Let $H$ be a separable Hilbert space, denote by $S_p(H)$ the Schatten $p$-class (or ideal) on $H$ where $0 < p < \infty$. As we all known, Schatten classes are more refined classification of compact operators on Hilbert spaces, but also involve global estimates of the spectrum of compact operators. Moreover, $S_p(H)$ can be viewed as a noncommutative generalization of $\ell^p$ space; in particular, $S_p(H)$ will become a Banach space when provided a suitable norm for $1 \leq p < \infty$. We refer the reader to \cite{18, 19, 29} for details about Schatten class on the Hilbert space. Now we can state our main results.

**Theorem 3.1.** If $\alpha \in \mathcal{F}$, then the followings hold.

1. The operator $B_{\alpha, \gamma} \in S_p(L^2(dv_\gamma))$ for any $p > 0$.
2. The operator $B_{\alpha, \gamma} \in S_p(A^2(dv_\gamma))$ for any $p > 0$.

**Theorem 3.2.** Suppose $\alpha \in \mathcal{B}_\gamma$ and $0 < p < \infty$, then the following statements are equivalent.

1. $B_{\alpha, \gamma} \in S_p(L^2(dv_\gamma))$.
2. $B_{\alpha, \gamma} \in S_p(A^2(dv_\gamma))$.
3. $\widetilde{B}_{\alpha, \gamma} \in L^p(d\lambda)$.
4. $p > \frac{N - 1}{N + \gamma - \alpha}$.

Where $\widetilde{B}_{\alpha, \gamma}$ is the Berezin transform of $B_{\alpha, \gamma}$ on Bergman space $A^2(dv_\gamma)$ and $d\lambda$ is the Möbius invariant measure on $\Omega$, the exact definition will be given later. The condition $\alpha \in \mathcal{B}_\gamma$ in Theorem 3.2 is necessary and sharp.
Example 3.3. For a moment, let us make our results more precisely. Due to E. Cartan [5], there only exist six type irreducible bounded symmetric domains up to biholomorphism, and the first four types are also called classical domains. For convenience, we list their standard Harish-Chandra realization as follows [16, Chapter 17].

| Type | $\mathfrak{p}^+$ | $d$ | $a$ | $b$ | $r$ | $N$ |
|------|-----------------|-----|-----|-----|-----|-----|
| I    | $\mathbb{C}^{1\times 1}$ | 1   | 1   | 0   | 1   | 2   |
| I    | $\mathbb{C}^{r\times s}, r \leq s$ | $rs \geq 2$ | 2   | $s-r$ | $r+s$ |
| II   | $\mathbb{C}_{asym}^{(2r+\epsilon)\times (2r+\epsilon)}$ | $r(2r+2\epsilon-1) \geq 5$ | 4   | $2\epsilon$ | $4r+2\epsilon-2$ |
| III  | $\mathbb{C}_{sym}^{r\times r}$ | $r(r+1) \geq 2$ | 1   | 0   | $r+1$ |
| IV   | $\mathbb{C}_{spin}^{s\times s}$ | $s \geq 4$ | $s-2$ | 0   | 2   | $s$ |
| V    | $\mathbb{O}_C^{1\times 2}$ | 16  | 6   | 4   | 2   | 12  |
| VI   | $\mathcal{H}_3^C(1) \otimes \mathbb{C}$ | 27  | 8   | 0   | 3   | 18  |

Where $\epsilon = 0, 1$. Thus, by direct calculation, the above Theorem 3.2 can be exactly restated case-by-case in the following table.

| Type | $\mathcal{F}$ | $\mathcal{B}_\gamma$ | $B_{a,\gamma} \in S_p$|
|------|----------------|----------------------|-----------------|
| I    | $\bigcup_{k=0}^\infty \{r-1-k\}$ | $(\infty, r+s) \setminus \bigcup_{k=0}^\infty \{r-1-k\}$ | $p > \frac{r+s-1}{r+s+\gamma-\alpha}$ |
| II   | $\bigcup_{k=0}^\infty \{2r-2-k\}$ | $(\infty, 4r+2\epsilon-2) \setminus \bigcup_{k=0}^\infty \{2r-2-k\}$ | $p > \frac{4r+2\epsilon-3}{4r+2\epsilon+2\gamma+\alpha}$ |
| III  | $\bigcup_{k=0}^\infty \{r-1-k, r-2-k\}$ | $(\infty, r+1) \setminus \bigcup_{k=0}^\infty \{r-1-k, r-2-k\}$ | $p > \frac{r}{r+1+\gamma-\alpha}$ |
| IV   | $\bigcup_{k=0}^\infty \{s-2-k, -k\}$ | $(\infty, s) \setminus \bigcup_{k=0}^\infty \{s-2-k, -k\}$ | $p > \frac{s-1}{s+\gamma-\alpha}$ |
| V    | $\bigcup_{k=0}^\infty \{3-k\}$ | $(\infty, 12) \setminus \bigcup_{k=0}^\infty \{3-k\}$ | $p > \frac{11}{12+\gamma-\alpha}$ |
| VI   | $\bigcup_{k=0}^\infty \{8-k\}$ | $(\infty, 18) \setminus \bigcup_{k=0}^\infty \{8-k\}$ | $p > \frac{17}{18+\gamma-\alpha}$ |

Now we turn to prove the main theorems. We first establish some lemmas. The following lemma gives the regularity of the image of $B_{a,\gamma}$.

Lemma 3.4. For any $\alpha \in \mathbb{R}$ and $\gamma > -1$, then $B_{a,\gamma}f$ is holomorphic on $\Omega$ for any $f \in L^1(d\nu_\gamma)$.

Proof. It is sufficient to show that $B_{a,\gamma}f$ is holomorphic on every point of $\Omega$. Suppose $z_0$ is an arbitrary point in $\Omega$. Let $\| \cdot \|$ be the spectral norm of $\Omega$. Chose a real $r$ satisfying $\|z_0\| < r < 1$. Then $z_0 \in B_r$, where $B_r = \{ \|z\| < r \}$ is an open ball with
radius \( r > 0 \) in the spectral norm. It follows from [10, Theorem 3.8] that

\[
h(z, w)^{-\alpha} = \sum_{m \geq 0} (\alpha)_m K^m(z, w)
\]

converges uniformly and absolutely on \( B_r \times \overline{\Omega} \), where \( K^m \) is the reproducing kernel of \( P_m(p^+) \) in the Fischer-Fock inner product (2.2). Thus the dominated convergence theorem implies that

\[
B_{\alpha, \gamma} f(z) = \int_{\Omega} f(w) h(z, w) d\gamma(w)
\]

\[
= \int_{\Omega} f(w) \sum_{m \geq 0} (\alpha)_m K^m(z, w) d\gamma(w)
\]

\[
= \sum_{m \geq 0} (\alpha)_m \int_{\Omega} f(w) K^m(z, w) d\gamma(w),
\]

for any \( z \in B_r \). Note that \( \int_{\Omega} f(w) K^m(z, w) d\gamma(w) \) is a holomorphic polynomial in \( z \) for each \( m \geq 0 \). Combining this with (3.1) follows that \( B_{\alpha, \gamma} f \) is holomorphic on the spectral ball \( B_r \). It leads to the desired result. □

Recall that a bounded operator on a Hilbert space is called a finite rank operator if the range of the operator has finite dimension. Obviously, finite rank operators must be compact and belong to every Schatten \( p \)-class with \( 0 < p < \infty \).

**Definition 3.5.** Let \( H \) be a separable Hilbert space, a linear operator \( T : H \to H \) is called prediagonal if there exist an orthogonal basis \( \{e_n\} \) of \( H \) and a complex number sequence \( \{\lambda_n\} \) such that

\[
T(\sum f_n e_n) = \sum \lambda_n f_n e_n
\]

whenever \( \sum f_n e_n \in H \) and \( \sum \lambda_n f_n e_n \in H \).

In particular,

\[
Te_n = \lambda_n e_n
\]

for each \( n \), thus the operator \( T \) is densely defined. In the following, we will frequently say that an operator \( T \) is a prediagonal operator with the characteristic \( \{\lambda_n\} \) if there exist an orthogonal basis \( \{e_n\} \) and a sequence \( \{\lambda_n\} \) satisfying (3.2). Note that a closed operator \( T \) is prediagonal if and only if \( T^* \) is prediagonal, where \( T^* \) is the adjoint of \( T \); moreover, if \( \{\lambda_n\} \) is the characteristic of \( T \) with respect to the orthogonal basis \( \{e_n\} \), then \( \{\bar{\lambda}_n\} \) is the characteristic of \( T^* \) and

\[
T^* e_n = \bar{\lambda}_n e_n
\]

for each \( n \). When \( T \) is a bounded prediagonal operator, it can be proved that \( T \) must be normal and the characteristic of \( T \) is unique up to the order and the characteristic is nothing but the sequence of eigenvalues (counting geometric multiplicities). If \( \dim H < \infty \), then \( H \) can be isometrically embedded an infinite dimensional Hilbert space, and \( T \) will be identified with its natural zero extension on the infinite dimensional Hilbert space. Moreover, in the finite dimensional case, the prediagonal operator is just a matrix in the unitary equivalent class of some diagonal matrix; in the infinite
dimensional case, all normal compact operators are prediagonal as above. The following lemma gives some criteria for boundedness, compactness and Schatten membership of prediagonal operators by using its characteristic.

**Lemma 3.6.** Let $T$ be a prediagonal operator with characteristic $\{\lambda_n\}$ as above, then the following hold.

1. $T$ is bounded if and only if $\{\lambda_n\} \in \ell^\infty$.
2. $T$ is compact if and only if $\{\lambda_n\} \in c_0$.
3. $T \in S_p(H)$ if and only if $\{\lambda_n\} \in \ell^p$ for $0 < p < \infty$.

**Proof.** (1) It is obvious that if $T$ is bounded then $\{\lambda_n\} \in \ell^\infty$. Conversely, for any $f = \sum f_n e_n \in H$, since $\{\lambda_n\} \in \ell^\infty$, Parseval equality implies that

$$\| \sum \lambda_n f_n e_n \| \leq \| \{\lambda_n\} \| \| f \|.$$  

Thus

$$\| T f \| = \| \sum \lambda_n f_n e_n \| \leq \| \{\lambda_n\} \| \| f \|.$$  

Since $f$ is arbitrary, it implies that $T$ is bounded.

(2) Suppose that $T$ is compact. Note that $e_n \to 0$ weakly as $n \to 0$. Combing with the well-known fact that a compact operator maps a weakly convergent sequence into a strongly convergent one, we thus obtain that

$$\lim |\lambda_n| = \lim \| Te_n \| = 0,$$

namely $\{\lambda_n\} \in c_0$. Conversely, suppose $\{\lambda_n\} \in c_0$. By (1) we know that $T$ is bounded. Define a finite rank operator sequence $\{T_k\}$ on $H$ by

$$T_k f = \sum_{j=1}^k \lambda_j \langle f, e_j \rangle e_j,$$

for any $f = \sum f_n e_n \in H$. Since $\{\lambda_n\} \in c_0$, it is easy to see that $T_k \to T$ in the operator norm. Thus $T$ is compact.

(3) It suffices to consider the case that $T$ is a normal compact operator. Denote the point spectrum of $T$ by $\sigma(T)$. Then $\{\lambda_n : n \in \mathbb{N}\} \subset \sigma(T)$ as set. If there exists $\lambda \in \sigma(T)$ but $\lambda \notin \{\lambda_n\}$, then there exists a nonzero $e \in H$ satisfying $Te = \lambda e$ and $e \perp \{e_n\}$. Note that $\{e_n\}$ is an orthogonal basis of $H$, it implies that $e = 0$, a contradiction with $e \neq 0$. Thus $\{\lambda_n\} = \sigma(T)$ as set. Now suppose $\lambda' \in \{\lambda_n\} = \sigma(T)$. Note that $M(\lambda') \leq \text{dim Ker}(\lambda' - T)$, which is the geometric multiplicity of the nonzero eigenvalue $\lambda$. Then the multiplicity $M(\lambda')$ of $\lambda'$ in $\{\lambda_n\}$ must be finite. We turn to prove that $M(\lambda') = \text{dim Ker}(\lambda' - T)$. If $M(\lambda') < \text{dim Ker}(\lambda' - T)$, then there exists a nonzero $e' \in H$ such that $Te' = \lambda' e'$ and $e' \perp \{e_n\}$, this is a contradiction since $\{e_n\}$ is an orthogonal basis. It prove that $M(\lambda') = \text{dim Ker}(\lambda' - T)$. Thus $\{\lambda_n\} = \sigma(T)$ as set and the multiplicity $M(\lambda_n)$ coincides with its geometric multiplicity for any nonzero $\lambda_n$. On the other hand, $T$ is normal compact, it follows from the spectral theorem and functional calculus for compact normal operators that $\{|\lambda_n|\} = \sigma(|T|)$ as set and the multiplicity $M(|\lambda_n|)$ coincides with its geometric multiplicity for any nonzero $|\lambda_n|$. Thus $T \in S_p(H)$ if and only if $\{\lambda_n\} \in \ell^p$ for any $0 < p < \infty$. \qed
Lemma 3.7. The operator $B_{\alpha, \gamma} : L^2(d\nu_\gamma) \rightarrow L^2(d\nu_\gamma)$ is bounded if and only if $\alpha \leq N + \gamma$.

**Proof.** Clearly that $B_{N+\gamma, \gamma} : L^2(d\nu_\gamma) \rightarrow A^2(d\nu_\gamma)$ is the Bergman orthogonal projection and is bounded. In what follows, we show that the operator identity
\[ B_{\alpha, \gamma}B_{N+\gamma, \gamma} = B_{\alpha, \gamma} \] (3.3)
holds on $L^2(d\nu_\gamma)$ for any $\alpha \in \mathbb{R}$. It suffices to show that
\[ B_{\alpha, \gamma}(\text{Id} - B_{N+\gamma, \gamma}) = 0 \]
on $L^2(d\nu_\gamma)$. Since Bergman space $A^2(d\nu_\gamma)$ is a closed subspace of the Hilbert space $L^2(d\nu_\gamma)$, together with the orthogonal projection theorem, it suffices to show that
\[ B_{\alpha, \gamma}|_{(A^2(d\nu_\gamma))^\perp} = 0, \] (3.4)
where $(A^2(d\nu_\gamma))^\perp$ is the orthogonal complement of Bergman space $A^2(d\nu_\gamma)$ in $L^2(d\nu_\gamma)$. Suppose $f \in (A^2(d\nu_\gamma))^\perp$, namely $f \in L^2(d\nu_\gamma)$ and $\langle f, g \rangle = 0$ for any $g \in A^2(d\nu_\gamma)$; in particular,
\[ \langle f, P \rangle = 0 \] (3.5)
for any holomorphic polynomial $P \in \mathcal{P}(\mathfrak{p}^+)$. Now we prove that $B_{\alpha, \gamma}f = 0$ for any $f \in (A^2(d\nu_\gamma))^\perp \subset L^2(d\nu_\gamma)$. It implies from Lemma 3.4 that $B_{\alpha, \gamma}f$ is holomorphic on $\Omega$. By the uniqueness theorem of holomorphic functions, it suffices to show that $B_{\alpha, \gamma}f = 0$ on a spectral ball $B_r$ with $0 < r < 1$. Then (3.1) and (3.5) yield that
\[ B_{\alpha, \gamma}f(z) = \int_{\Omega} \frac{f(w)}{h(z, w)} d\nu_\gamma(w) = \sum_{m \geq 0} (\alpha)_m \int_{\Omega} f(w) K^m(z, w) d\nu_\gamma(w) = \sum_{m \geq 0} (\alpha)_m \langle f, K^m_\gamma \rangle = 0, \] (3.6)
for any $z \in B_r$. The last equality in (3.6) holds, since $K^m_z \in \mathcal{P}(\mathfrak{p}^+)$ for any fixed $z \in B_r$ and $m \geq 0$. Thus the operator identity (3.3) holds. Combing Lemma 3.4 and the operator identity (3.3) yields that $B_{\alpha, \gamma} : L^2(d\nu_\gamma) \rightarrow L^2(d\nu_\gamma)$ is bounded if and only if $B_{\alpha, \gamma} : A^2(d\nu_\gamma) \rightarrow A^2(d\nu_\gamma)$ is bounded. In the following, we prove that $B_{\alpha, \gamma} : A^2(d\nu_\gamma) \rightarrow A^2(d\nu_\gamma)$ is a prediagonal operator. Suppose $f = \sum_{m \geq 0} f_m \in A^2(d\nu_\gamma)$ with the Peter-Schmid-Wely decomposition of $A^2(d\nu_\gamma)$. By Schur’s lemma, the ratio of any two $K$-invariant inner products is a constant on the irreducible space $\mathcal{P}_m(\mathfrak{p}^+)$, in fact, by [10, Corollary 3.7] we know that
\[ \frac{\langle p, q \rangle_\gamma}{\langle p, q \rangle} = \frac{1}{(N + \gamma)_m}, \] (3.7)
for any $p, q \in \mathcal{P}_m(\mathfrak{p}^+)$. Since $K$ is a compact group, it follows that the orbit $K \cdot z \subset \Omega$ is compact for any $z \in \Omega$. Then, combing this with the integral formula (1.11) in [10]
implies that

\[ B_{\alpha,\gamma}f(z) = \sum_{m \geq 0} (\alpha)_m \int_{\Omega} \sum_{t \geq 0} f_t(w) K^m(z, w) dv_\gamma(w) \]

\[ = \sum_{m \geq 0} c_\gamma \cdot (\alpha)_m \int_0^1 \cdots \int_0^1 \prod_{l<k} t_j^{2b+1}(1 - t_j^2)^\gamma \prod_{l<k} |t_l^2 - t_k^2|^a dt_1 \cdots dt_r \]

\[ \times \int_K \sum_{n \geq 0} f_n(g \sum_{j=1}^r t_j e_j) K^m(z, g \sum_{j=1}^r t_j e_j) dg \]

\[ = \sum_{m \geq 0} c_\gamma c \cdot (\alpha)_m \int_0^1 \cdots \int_0^1 \prod_{l<k} t_j^{2b+1}(1 - t_j^2)^\gamma \prod_{l<k} |t_l^2 - t_k^2|^a dt_1 \cdots dt_r \]

\[ \times \sum_{n \geq 0} \int_K f_n(g \sum_{j=1}^r t_j e_j) K^m(z, g \sum_{j=1}^r t_j e_j) dg \]

\[ = \sum_{m \geq 0} (\alpha)_m \langle f_m, K^m_z \rangle \gamma \]

\[ = \sum_{m \geq 0} \frac{(\alpha)_m}{(N+\gamma)_m} f_m(z), \]

where \( c \) is an integral constant. It implies from (3.8) that \( B_{\alpha,\gamma} \) is a prediagonal operator with characteristic \( \{ (\alpha)_m \} \) on \( A^2(\Omega, \gamma) \), where of cause counting multiplicities. Thus by Lemma 3.6, \( B_{\alpha,\gamma} \) is bounded on \( A^2(\Omega, \gamma) \) if and only if \( \{ \frac{(\alpha)_m}{(N+\gamma)_m} \} \in \ell^\infty \). By the definition of the multi-variable Pochhammer symbol, we obtain that

\[ \frac{(\alpha)_m}{(N+\gamma)_m} = \frac{\Gamma_\Omega(\alpha + m)}{\Gamma_\Omega(N + \gamma + m)} \frac{\Gamma_\Omega(N + \gamma)}{\Gamma_\Omega(\alpha)} = \frac{\Gamma_\Omega(N + \gamma)^r}{\Gamma_\Omega(\alpha)} \prod_{j=1}^r \frac{\Gamma(\alpha + m_j - (j - 1)\frac{\gamma}{2})}{\Gamma(N + \gamma + m_j - (j - 1)\frac{\gamma}{2})}. \]

(3.9)

Then Stirling’s formula and (3.9) imply that

\[ \left| \frac{(\alpha)_m}{(N+\gamma)_m} \right| \sim \prod_{j=1}^k \frac{1}{m_j^{N+\gamma-\alpha}}, \]

as \( I_\Omega(k) \geq m \rightarrow \infty \), for \( 1 \leq k \leq r \). The notation \( A(m) \sim B(m) \) as \( I_\Omega(k) \geq m \rightarrow \infty \) means that the ratio \( \frac{A(m)}{B(m)} \) has a positive finite limit as

\[ m_1 \rightarrow \infty, \cdots, m_k \rightarrow \infty. \]
for \(1 \leq k \leq r\). Consequently, \(\{\frac{(\alpha)_{m}}{(N+\gamma)_{m}}\} \in \ell^{\infty}\) if and only if \(\alpha \leq N + \gamma\). It finishes the proof. \(\square\)

**Remark 3.8.** In the case of Hilbert unit ball, the operator identity (3.3) holds on \(L^{p}(dv_{\gamma})\) for any \(1 < p < \infty\), since the Bergman projection \(B_{N, \gamma}\) is bounded on \(L^{p}(dv_{\gamma})\) for any \(1 < p < \infty\); see [4, 8]. However, in the high rank case, the boundedness of Bergman projection \(B_{N+\gamma, \gamma}\) on \(L^{p}(dv_{\gamma})\) for all \(1 < p < \infty\) is not valid; see [2, 3].

We have proved that \(B_{\alpha, \gamma}: A^{2}(dv_{\gamma}) \to A^{2}(dv_{\gamma})\) is a prediagonal operator. Note that \(A^{2}(dv_{\gamma})\) is a closed subspace of the Hilbert space \(L^{2}(dv_{\gamma})\) and \(B_{\alpha, \gamma}(A^{2}(dv_{\gamma}))^{\perp} = 0\), it follows that \(B_{\alpha, \gamma}: L^{2}(dv_{\gamma}) \to L^{2}(dv_{\gamma})\) is prediagonal. In Definition 1.1 \(B_{\alpha, \gamma}\) is only defined on \(L^{1}(dv_{\gamma})\); however, \(B_{\alpha, \gamma}\) can be defined on \(L^{1}(dv_{\beta})\) for any \(\beta > -1\). Denote by \(H_{\lambda} \subset L^{2}(dv_{\beta})\) the solution space of the following linear integral equation

\[
\lambda f(z) - \frac{c_{\gamma}}{c_{\beta}} \int_{\Omega} h(z, z)^{\gamma - \beta} h(z, w)^{-\alpha} f(w) dv_{\beta}(w) = 0,
\]

for parameter \(\lambda \in \mathbb{C}\). It is obvious that \(H_{\lambda} \perp H_{\mu}\) if \(\lambda \neq \mu\), and there only exist countably many \(\lambda \in \mathbb{C}\) satisfying \(H_{\lambda} \neq \emptyset\). Then Fubini’s Theorem and the previous discussion of prediagonality between a closed operator and its adjoint operator imply the following.

**Corollary 3.9.** The operator \(B_{\alpha, \gamma}: L^{2}(dv_{\beta}) \to L^{2}(dv_{\beta})\) is prediagonal if and only if

\[
L^{2}(dv_{\beta}) = \bigoplus_{\lambda \in \mathbb{C}} H_{\lambda}.
\]

Combing the definition of \(\mathcal{F}\) with (3.8) yields the following conclusion.

**Proposition 3.10.** The operator \(B_{\alpha, \gamma}: L^{2}(dv_{\gamma}) \to L^{2}(dv_{\gamma})\) is finite rank operator if and only if \(\alpha \in \mathcal{F}\).

It is easy to see that Theorem 3.1 is a direct corollary of Lemma 3.4 and Proposition 3.10. Denote by \(\sigma(B_{\alpha, \gamma}, A^{2}(dv_{\gamma}))\) the point spectrum (the collections of eigenvalues) of \(B_{\alpha, \gamma}\) on the Bergman space \(A^{2}(dv_{\gamma})\).

**Lemma 3.11.** For \(\alpha \in \mathbb{R}\), then

\[
\sigma(B_{\alpha, \gamma}, A^{2}(dv_{\gamma})) = \left\{\frac{(\alpha)_{m}}{(N+\gamma)_{m}} : m \geq 0\right\}.
\]

**Proof.** Denote \(\mu_{m} = \frac{(\alpha)_{m}}{(N+\gamma)_{m}}\) for all \(m \geq 0\). By formula (3.8), it implies that \(\{\mu_{m} : m \geq 0\} \subset \sigma(B_{\alpha, \gamma}, A^{2})\).

Thus, it suffices to prove that \(\sigma(B_{\alpha, \gamma}, A^{2}(dv_{\gamma})) \subset \{\mu_{m} : m \geq 0\}\).

Suppose \(\mu \in \sigma(B_{\alpha, \gamma}, A^{2}(dv_{\gamma}))\), then there exists a nonzero \(f \in A^{2}(dv_{\gamma})\) such that

\[
B_{\alpha, \gamma}f = \mu f.
\]

Let \(E_{m} = \{e_{\gamma}^{j} : j = 1, \cdots, d_{m}\}\) be an orthogonal basis of \(\mathcal{P}_{m}(p^{\perp})\) under the Bergman inner product \(\langle \cdot, \cdot \rangle_{\gamma}\) for all \(m \geq 0\), where \(d_{m} = \text{dim} \mathcal{P}_{m}(p^{\perp})\). It is easy to see that
$\bigcup_{m \geq 0} E_m$ is an orthogonal basis of the Bergman space $A^2(dv_{\gamma})$. Thus $f$ has the following representation

$$f = \sum_{m \geq 0} \sum_{j=1}^{d_m} \langle f, e_m^{(j)} \rangle e_m^{(j)}. \quad (3.12)$$

Combing this representation with (3.1), (3.7) and (3.11), we obtain that

$$\sum_{m \geq 0} \sum_{j=1}^{d_m} (\mu - \mu_m) \langle f, e_m^{(j)} \rangle e_m^{(j)} e_m^{(j)} = 0 \quad (3.13)$$

as function on the spectral ball $B_r$ with $0 < r < 1$. Note that $B_r$ is biholomorphic to $\Omega = B_1$ for any $0 < r < 1$. This along with formula (1.11) in [10], it is easy to see that

$$\bigcup_{m \geq 0} r^{-(d+|m|)} \sqrt{\frac{(N)_m}{(N+\gamma)_m}} E_m$$

is an orthogonal basis of $A^2(B_r, dv)$. The reproducing property (2.5), Cauchy’s inequality and (2.4) imply that convergences in (3.12) and (3.13) are uniform in the compact subset. Thus we have

$$r^{2(d+|m|)} \frac{(N+\gamma)_m}{(N)_m} (\mu - \mu_{m'}) \langle f, e_{m'}^{(j)} \rangle \gamma e_m^{(j)} e_m^{(j)} dv$$

$$= \int_{B_r} (\mu - \mu_{m'}) \langle f, e_m^{(j)} \rangle e_m^{(j)} e_m^{(j)} e_m^{(j)} dv$$

$$= \sum_{m \geq 0} \sum_{j=1}^{d_m} \int_{B_r} (\mu - \mu_m) \langle f, e_m^{(j)} \rangle e_m^{(j)} e_m^{(j)} e_m^{(j)} dv \quad (3.14)$$

$$= \int_{B_r} \sum_{m \geq 0} \sum_{j=1}^{d_m} (\mu - \mu_m) \langle f, e_m^{(j)} \rangle e_m^{(j)} e_m^{(j)} dv$$

$$= 0,$$

for any $m' \geq 0$. Since $f$ is nonzero, (3.12) implies that there exist $m_0 \geq 0$ and $j_0$ such that

$$\langle f, e_m^{(j_0)} \rangle \neq 0. \quad (3.15)$$

Then (3.14) and (3.15) yield that $\mu = \mu_{m_0}$ and

$$f = \sum_{j=1}^{d_{m_0}} \langle f, e_m^{(j_0)} \rangle e_m^{(j_0)}. \quad (3.16)$$

It completes the proof. \(\square\)

**Remark 3.12.** Indeed, (3.8), (3.11), (3.12) and (3.16) imply that $B_{\alpha, \gamma} : A^2(dv_{\gamma}) \to A^2(dv_{\gamma})$ is the prediagonal operator with point spectrum (counting geometric multiplicities) as its characteristic whether $B_{\alpha, \gamma}$ is bounded or not.
Lemma 3.13. (1) There exists an uniform positive constant $C$ satisfying

$$
\dim \mathcal{P}_m(p^+) \leq C \prod_{j=1}^{k} m_j^{(r-j)a+b}
$$

(3.17)

for any $m \in I_{\Omega}(k), 1 \leq k \leq r$.

(2) Moreover, there exists an uniform positive constant $C$ satisfying

$$
\frac{1}{C} m_1^{(r-1)a+b} \leq \dim \mathcal{P}_m(p^+) \leq C m_1^{(r-1)a+b}
$$

(3.18)

for any $m \in I_{\Omega}(1)$.

Proof. (1) For any $\beta, \gamma \in \mathbb{C}$ and $k \in \mathbb{N}$, we define $(\beta)_{k, \gamma}$ by

$$
(\beta)_{k, \gamma} = \prod_{j=0}^{k-1} (\beta + j \cdot \gamma).
$$

Obviously, $(\beta)_{k, 1} = (\beta)_k$ for any $\beta \in \mathbb{C}$ and $k \in \mathbb{N}$. Now suppose $m \in I_{\Omega}(k), 1 \leq k \leq r$. By [21, Lemma 2.6 and 2.7] it follows that

$$
\dim \mathcal{P}_m(p^+) = \frac{(\rho)_m}{(\rho - b)_m} \prod_{1 \leq i < j \leq r} \frac{m_i - m_j + \frac{a}{2}(j - i)}{\frac{a}{2}(j - i)} \cdot \frac{(m_i - m_j + \frac{a}{2}(j - i - 1))_a}{(1 + \frac{a}{2}(j - i - 1))_a}
$$

$$
\overset{\text{def}}{=} \frac{(\rho)_m}{(\rho - b)_m} I(a, m),
$$

where $\rho = 1 + \frac{a}{2}(r - 1) + b$. By the definition and Stirling’s formula, we then get that

$$
\frac{(\rho)_m}{(\rho - b)_m} = \frac{\Gamma_{\Omega}(\rho + m)}{\Gamma_{\Omega}(\rho - b + m)} \frac{\Gamma_{\Omega}(\rho - b)}{\Gamma_{\Omega}(\rho)}
$$

$$
= \frac{\Gamma_{\Omega}(\rho - b)}{\Gamma_{\Omega}(\rho)} \prod_{j=1}^{r} \frac{\Gamma(\rho + m_j - \frac{a}{2}(j - 1))}{\Gamma(\rho - b + m_j - \frac{a}{2}(j - 1))}
$$

(3.19)

$$
\sim \prod_{j=1}^{k} m_j^b,
$$
as \( I_{\Omega}(k) \ni m \to \infty \). On the other hand,

\[
I(a, m) = \prod_{1 \leq i < j \leq r} \left( \frac{m_i - m_j + \frac{a}{2}(j - i)}{\frac{a}{2}(j - i)} \right) \prod_{1 \leq i < j \leq r} \left( \frac{m_i - m_j + \frac{a}{2}(j - i - 1)}{1 + \frac{a}{2}(j - i - 1)} \right)_{a-1}
\]

\[
= \left( \prod_{j=1}^{k} m_j^{r-j} \prod_{1 \leq i < j \leq k} \left( 1 - \frac{m_i}{m_j} + \frac{\frac{a}{2}(j-i)}{m_j} \right) \right) \prod_{1 \leq i < j \leq r} \left( \frac{1 + \frac{a}{2}(j-i)}{\frac{a}{2}(j-i)} \right)_{a-1}
\]

\[
\times \prod_{1 \leq i < j \leq r} \left( 1 + \frac{\frac{a}{2}(j-i)}{m_j} \right)_{a-1, \frac{1}{m_j}} \prod_{k+1 \leq i < j \leq r} \left( \frac{\frac{a}{2}(j-i-1)}{\frac{a}{2}(j-i-1)} \right)_{a-1}
\]

\[
\leq C_k \prod_{j=1}^{k} m_j^{(r-j)a}. \tag{3.20}
\]

Combing (3.19) with (3.20), we thus obtain

\[
\dim \mathcal{P}_m(p^+) = \frac{(\rho)m}{(\rho - b)m} I(a, m) \leq C \prod_{j=1}^{k} m_j^{(r-j)a+b},
\]

for any \( m \in I_{\Omega}(k), 1 \leq k \leq r \), where \( C = \max_{1 \leq k \leq r} \{ C_k \} \).

(2) Now suppose \( m = (m_1, 0, \cdots, 0) \in I_{\Omega}(1) \). The direct calculation implies that

\[
I(a, m) = m_1^{(r-1)a} \prod_{j=2}^{r} \left( 1 + \frac{\frac{a}{2}(j-2)}{m_1} \right)_{a-1, \frac{1}{m_1}} \prod_{2 \leq i < j \leq r} \left( \frac{\frac{a}{2}(j-i-1)}{\frac{a}{2}(j-i-1)} \right)_{a-1}
\]

This along with (3.19) yields the desired result. \( \square \)

**Remark 3.14.** In the case of Hilbert unit ball, the \( I(\Omega) \) is just the set of nonnegative integers and \( \mathcal{P}_m(p^+) \) is just homogeneous polynomial space with degree \( m \). In this case, the dimension of \( \mathcal{P}_m(p^+) \) is

\[
\dim \mathcal{P}_m(p^+) = \frac{(m + 1)d-1}{(d-1)!}, \quad m \geq 0,
\]

which coincides with (2) of Lemma 3.13.

Now we recall the definition of Berezin transform related to the Bergman space \( A^2(dv_\gamma) \) on the bounded symmetric domain \( \Omega \). Recall that Bergman space \( A^2(dv_\gamma) \) is a reproducing kernel Hilbert function space, whose reproducing kernel is given by

\[
K_{\gamma,w}(z) = K_{\gamma}(z, w) = h(z, w)^{(N+\gamma)}, \quad z, w \in \Omega.
\]

The normalized reproducing kernel of \( A^2(dv_\gamma) \) is

\[
k_{\gamma,w}(z) = \frac{K_{\gamma}(z, w)}{\sqrt{K_{\gamma}(w, w)}} = \frac{h(w, w)^{N+\gamma}}{h(z, w)^{N+\gamma}}, \quad z, w \in \Omega. \tag{3.21}
\]
For a linear operator $T$ on $A^2(dv_\gamma)$, the Berezin transform $\tilde{T}$ of $T$ is given by

$$\tilde{T}(z) = \langle Tk_{\gamma,z}, k_{\gamma,z}\rangle_\gamma, \quad z \in \Omega.$$  \hfill (3.22)

The Möbius invariant measure $d\lambda$ on $\Omega$ is defined by

$$d\lambda(z) = \frac{dv(z)}{h(z, z)^N}.$$  

It is easy to verify that the measure $d\lambda$ is biholomorphic invariant. The Berezin transform is an important tool in the operator theory on the holomorphic function space on the Hilbert ball; see [28, 29] for more details. In what follows, we exactly calculate the Berezin transform of $B_{\alpha,\gamma}$ on $\Omega$.

**Lemma 3.15.** For any $\alpha \in \mathbb{R}$,

$$\tilde{B}_{\alpha,\gamma}(z) = h(z, z)^{N+\gamma-\alpha},$$

on $\Omega$.

**Proof.** For any fixed $z \in \Omega$, [10, Theorem 3.8] implies that $h(w, z)^{-\alpha}$ is a bounded holomorphic function on $\Omega$, in particular, $h(w, z)^{-\alpha} \in A^2(dv_\gamma)$ for any $\alpha \in \mathbb{R}$. Then (2.5) implies that

$$B_{\alpha,\gamma}K_{\gamma,z}(w) = \int_\Omega \frac{1}{h(w, u)^\alpha} \frac{1}{h(u, z)^{N+\gamma}} dv_\gamma(u)$$

$$= \int_\Omega \frac{1}{h(u, w)^\alpha} \frac{1}{h(z, u)^{N+\gamma}} dv_\gamma(u)$$

$$= \langle h^{-\alpha}, K_{\gamma,z}\rangle_\gamma$$

$$= h^{-\alpha}(w, z).$$  

Combining this with (3.21) and (3.22) shows that

$$\tilde{B}_{\alpha,\gamma}(z) = \langle B_{\alpha,\gamma}k_{\gamma,z}, k_{\gamma,z}\rangle_\gamma$$

$$= \frac{1}{K_\gamma(z, z)} \langle B_{\alpha,\gamma}K_{\gamma,z}, K_{\gamma,z}\rangle_\gamma$$

$$= h(z, z)^{N+\gamma-\alpha}.$$  

It completes the proof. \hfill \square

**Proof of Theorem 3.2.** It follows from Lemma 3.11 and Remark 3.12 that $B_{\alpha,\gamma} : A^2(dv_\gamma) \to A^2(dv_\gamma)$ is a prediagonal operator with characteristic $\{(\alpha)_m\}_{(N+\gamma)_m}$, where of course counting multiplicities. Since $A^2(dv_\gamma)$ is a closed subspace of the Hilbert space $L^2(dv_\gamma)$, combing with (3.3) implies that $B_{\alpha,\gamma} : L^2(dv_\gamma) \to L^2(dv_\gamma)$ is a prediagonal operator with characteristic $\{(\alpha)_m\}_{(N+\gamma)_m} \cup \{0\}$. Then Lemma 3.6 yields that $B_{\alpha,\gamma} \in S_p(L^2(dv_\gamma))$ if and only if $B_{\alpha,\gamma} \in S_p(A^2(dv_\gamma))$ if and only if $\{(\alpha)_m\}_{(N+\gamma)_m} \in \ell^p$ for $0 < p < \infty$. Thus (1) is equivalent to (2) which is equivalent to $\{(\alpha)_m\}_{(N+\gamma)_m} \in \ell^p$. It remains to prove $\{(\alpha)_m\}_{(N+\gamma)_m} \in \ell^p$ is equivalent to (4) and (3) is equivalent to (4) in the condition
\( \alpha \in \mathcal{B}_\gamma \). We first prove the equivalence of (3) and (4). From Lemma 3.15 we know that \( \widetilde{B}_{\alpha, \gamma} \in L^p(d\lambda) \) if and only if

\[
\int_{\Omega} |\widetilde{B}_{\alpha, \gamma}|^p d\lambda = \int_{\Omega} h^{p(N+\gamma-\alpha)-N} dv < \infty.
\]

This along with the well-known fact that \( \int_{\Omega} h^t dv < \infty \) for \( t \in \mathbb{R} \) is equivalent to \( t > -1 \) implies that \( p > \frac{1+b+(r-1)a}{N+\gamma-\alpha} = \frac{N-1}{N+\gamma-\alpha} \), which proves the equivalence of (3) and (4).

Now we turn to prove \( \{\frac{\alpha m}{(N+\gamma) m}\} \in \ell^p \) for \( 0 < p < \infty \) is equivalent to \( p > \frac{N-1}{N+\gamma-\alpha} \). Suppose \( p > \frac{N-1}{N+\gamma-\alpha} = \frac{1+b+(r-1)a}{N+\gamma-\alpha} \), namely \( p(N+\gamma-\alpha) - (r-1)a - b > 1 \), so

\[
p(N+\gamma-\alpha) - (r-j)a - b > 1, j = 1, \ldots, r.
\]

Then (2.3), (3.10) and Lemma 3.13 imply that

\[
\sum_{m \in I(\Omega)} \left| \frac{\alpha m}{(N+\gamma) m} \right|^p \dim \mathcal{P}_m(p^+) = \left| \frac{\alpha 0}{(N+\gamma) 0} \right|^p \dim \mathcal{P}_0(Z) + \sum_{k=1}^r \sum_{m \in I_k(\Omega)} \left| \frac{\alpha m}{(N+\gamma) m} \right|^p \dim \mathcal{P}_m(p^+)
\]

\[
\leq \left| \frac{\alpha 0}{(N+\gamma) 0} \right|^p \dim \mathcal{P}_0(Z) + C \sum_{k=1}^r \sum_{m \in I_k(\Omega)} \prod_{j=1}^k \frac{1}{m_j^{p(N+\gamma-\alpha)-(r-j)a-b}}
\]

\[
\leq \left| \frac{\alpha 0}{(N+\gamma) 0} \right|^p \dim \mathcal{P}_0(Z) + C \sum_{k=1}^r \prod_{j=1}^k \left( \sum_{m=1}^{\infty} \frac{1}{m^{p(N+\gamma-\alpha)-(r-j)a-b}} \right)
\]

\[
< \infty,
\]

which means that \( \{\frac{\alpha m}{(N+\gamma) m}\} \in \ell^p \) for \( 0 < p < \infty \).

Suppose \( \{\frac{\alpha m}{(N+\gamma) m}\} \in \ell^p \) for \( 0 < p < \infty \). Then (2.3) and Lemma 3.13 imply that

\[
\infty > \sum_{m \in I(\Omega)} \left| \frac{\alpha m}{(N+\gamma) m} \right|^p \dim \mathcal{P}_m(p^+)
\]

\[
= \left| \frac{\alpha 0}{(N+\gamma) 0} \right|^p \dim \mathcal{P}_0(Z) + \sum_{k=1}^r \sum_{m \in I_k(\Omega)} \left| \frac{\alpha m}{(N+\gamma) m} \right|^p \dim \mathcal{P}_m(p^+)
\]

\[
> \sum_{m \in I_1(\Omega)} \left| \frac{\alpha m}{(N+\gamma) m} \right|^p \dim \mathcal{P}_m(p^+)
\]

\[
\geq \frac{1}{C} \sum_{m=1}^{\infty} \frac{1}{m^{p(N+\gamma-\alpha)-(r-1)a-b}},
\]

which implies that \( p(N+\gamma-\alpha) - (r-1)a - b > 1 \), namely \( p > \frac{1+b+(r-1)a}{N+\gamma-\alpha} = \frac{N-1}{N+\gamma-\alpha} \). \( \square \)

4. Trace formulae for Bergman-type operators

From Theorem 3.13.2 we know that \( B_{\alpha, \gamma} \) belongs to the trace class \( S_1(L^2(dv_\gamma)) \) or \( S_1(A^2(dv_\gamma)) \) if and only if \( \alpha < 1 + \gamma \) or \( \alpha \in \mathcal{F} \). Since Bergman space \( A^2(dv_\gamma) \) is a
closed subspace of $L^2(d\nu_\gamma)$ and \((\overline{3.3})\), it follows that the trace $\text{Tr}(B_{\alpha,\gamma})$ on $L^2(d\nu_\gamma)$ and $A^2(d\nu_\gamma)$ are the same when $\alpha < 1 + \gamma$ or $\alpha \in \mathcal{F}$. Moreover, $B_{\alpha,\gamma}$ belongs to the Hilbert-Schmidt class $S_2(L^2(d\nu_\gamma))$ or $S_2(A^2(d\nu_\gamma))$ if and only if $\alpha < \frac{N+1+2\gamma}{2}$. Similarly, the trace $\text{Tr}(B_{\alpha,\gamma}^*B_{\alpha,\gamma})$ on $L^2(d\nu_\gamma)$ and $A^2(d\nu_\gamma)$ are the same when $\alpha < \frac{N+1+2\gamma}{2}$.

**Proposition 4.1.** (1) The operator $B_{\alpha,\gamma} \in S_1$ if and only if $\alpha < 1 + \gamma$ or $\alpha \in \mathcal{F}$. In this case,

$$\text{Tr}(B_{\alpha,\gamma}) = \int_{\Omega} h(z, z)^{-\alpha}d\nu_\gamma(z).$$

(2) The operator $B_{\alpha,\gamma} \in S_2$ if and only if $\alpha < \frac{N+1+2\gamma}{2}$. In this case,

$$\text{Tr}(B_{\alpha,\gamma}^*B_{\alpha,\gamma}) = \int_{\Omega \times \Omega} |h(w, z)|^{-2\alpha}d\nu_\gamma(w)d\nu_\gamma(z).$$

**Proof.** (1) It suffices to prove the trace formula. It implies from \((\overline{3.8})\) that the operator $B_{\alpha,\gamma} : A^2(d\nu_\gamma) \rightarrow A^2(d\nu_\gamma)$ is a prediagonal operator. Let $\{\lambda_n\}$ be its characteristic and $\{e_n\}$ be an orthogonal base of $A^2(d\nu_\gamma)$ such that $B_{\alpha,\gamma}e_n = \lambda_ne_n$ for each $n$. Since $B_{\alpha,\gamma} \in S_1$, it follows from Lemma \(3.6\) that $\{\lambda_n\} \in l^1$. Then the dominated convergence theorem implies that

$$\text{Tr}(B_{\alpha,\gamma}) = \sum \langle B_{\alpha,\gamma}e_n, e_n \rangle_{\gamma}$$

$$= \int \sum \langle B_{\alpha,\gamma}e_n, K_{\gamma,z} \rangle_{\gamma} \bar{e}_n(z)d\nu_\gamma(z)$$

$$= \int \sum \langle e_n \bar{e}_n(z), B_{\alpha,\gamma}^*K_{\gamma,z} \rangle_{\gamma}d\nu_\gamma(z)$$

$$= \int \langle \sum e_n \bar{e}_n(z), B_{\alpha,\gamma}^*K_{\gamma,z} \rangle_{\gamma}d\nu_\gamma(z)$$

$$= \int \langle B_{\alpha,\gamma}K_{\gamma,z}, K_{\gamma,z} \rangle_{\gamma}d\nu_\gamma(z).$$

This along with and \((3.23)\) shows that

$$\text{Tr}(B_{\alpha,\gamma}) = \int_{\Omega} h(z, z)^{-\alpha}d\nu_\gamma(z).$$

(2) Similarly, we have

$$\text{Tr}(B_{\alpha,\gamma}^*B_{\alpha,\gamma}) = \sum \langle B_{\alpha,\gamma}^*B_{\alpha,\gamma}e_n, e_n \rangle_{\gamma}$$

$$= \int \sum \langle B_{\alpha,\gamma}^*B_{\alpha,\gamma}e_n, K_z \rangle_{\gamma} \bar{e}_n(z)d\nu_\gamma(z)$$

$$= \int \sum \langle e_n \bar{e}_n(z), B_{\alpha,\gamma}^*B_{\alpha,\gamma}K_z \rangle_{\gamma}d\nu_\gamma(z)$$

$$= \int \langle B_{\alpha,\gamma}K_z, B_{\alpha,\gamma}K_z \rangle_{\gamma}d\nu_\gamma(z)$$

$$= \int_{\Omega \times \Omega} |h(w, z)|^{-2\alpha}d\nu_\gamma(w)d\nu_\gamma(z).$$

\(\square\)
Lemma 4.5. A Banach space is compact if and only if the operator maps every bounded subset to its closure in the norm topology is compact. Obviously, an operator between two precompact ones; in particular, the compact operators are all bounded.

Remark 4.2. In fact, the trace \( Tr(B_{\alpha,\gamma}) \) can be exactly calculated as follow.

\[
Tr(B_{\alpha,\gamma}) = \frac{c_{\gamma}}{c_{\gamma-\alpha}} = \frac{\Gamma_{\Omega}(N+\gamma)}{\Gamma_{\Omega}(N+\gamma-\alpha)} \frac{\Gamma_{\Omega}(\frac{\alpha}{2}(r-1) + 1 + \gamma - \alpha)}{\Gamma_{\Omega}(\frac{\alpha}{2}(r-1) + 1 + \gamma)}. 
\]

We first note that \( Tr(B_{\alpha,\gamma}^* B_{\alpha,\gamma}) < \infty \) if and only if \( \alpha < \frac{N+1+2\gamma}{2} \). An application, we establish an integral estimate related to the Forelli-Rudin estimate on the bounded symmetric domain. Denote \( J_{\beta,\gamma} \) on \( \Omega \) by

\[
J_{\beta,\gamma}(z) = \int_{\Omega} \frac{h(w,w)^\gamma}{h(z,w)^{N+\beta+\gamma}} dv(w), \quad z \in \Omega, 
\]

where of course we assume \( \gamma > -1 \). The asymptotic estimation of \( J_{\beta,\gamma} \) for \(|\beta| > \frac{\alpha}{2}(r-1)\) have been obtained in \([10]\); see \([9,23]\) for the gap \(|\beta| \leq \frac{\alpha}{2}(r-1)\) in some spacial cases.

Corollary 4.3. The integral \( \int_{\Omega} J_{\beta,\gamma} dv < \infty \) if and only if \( \beta < 1 \).

Proof. It follows from \([10]\) Corollary 3.7 and Theorem 3.8 that

\[
J_{\beta,\gamma}(z) = \sum_{m \geq 0} \frac{|(N+\beta+\gamma)/2m|^2}{(\gamma+N)m} K_m(z,z) 
\]

On the other hand, Stirling’s formula implies that there exists a positive constant \( C \) such that

\[
\frac{1}{C} \frac{|(N+\beta)/2m|^2}{|(N)m|^2} \leq \frac{|(N+\beta+\gamma)/2m|^2}{|(\gamma+N)m|m} \leq C \frac{|(N+\beta)/2m|^2}{|(N)m|^2}
\]

for any \( m \geq 0 \). Combining this with \((4.1)\) shows that the integral \( \int_{\Omega} J_{\beta,\gamma} dv < \infty \) if and only if the integral \( \int_{\Omega} J_{\beta,0} dv < \infty \). Then Proposition \(4.1\) implies that the integral \( \int_{\Omega} J_{\beta,0} dv < \infty \) if and only if \( \beta < 1 \). It completes the proof.

In the case of Hilbert unit ball, Corollary \(4.3\) is immediate from the Forelli-Rudin estimate \([28\) Theorem 1.12]. To end this section, we give a compactness result on \( B_{\alpha,\gamma} \).

Corollary 4.4. If \( \alpha < N + 1 + 2\gamma \), then \( B_{\alpha,\gamma} : L^\infty(dv_\gamma) \rightarrow L^1(dv_\gamma) \) is compact.

To prove the above corollary, we need the following lemma which provides a criteria of precompactness in \( L^p(dv_\gamma) \) with \( \gamma > -1 \) and \( 1 \leq p < \infty \), whose proof is similar to \([1]\) Theorem 2.33]. Recall that a subset in a Banach space is called precompact if its closure in the norm topology is compact. Obviously, an operator between two Banach spaces is compact if and only if the operator maps every bounded subset to precompact one; in particular, the compact operators are all bounded.

Lemma 4.5. \([1]\) Let \( 1 \leq p < \infty \) and \( E \subset L^p(dv_\gamma) \). Suppose there exists a sequence \( \{\Omega_j\} \) of subdomains of \( \Omega \) having the following properties:

1. \( \Omega_j \subset \Omega_{j+1} \).
(2) The set of restrictions to \( \Omega_j \) of the functions in \( E \) is precompact in \( L^p(\Omega_j, dv_\gamma) \) for each \( j \).

(3) For every \( \varepsilon > 0 \) there exists a \( j \) such that

\[
\int_{\Omega - \Omega_j} |f|^p dv_\gamma < \varepsilon, \quad \forall f \in E.
\]

Then \( E \) is precompact in \( L^p(dv_\gamma) \).

**Proof of Corollary 4.4.** Suppose \( \{f_j\} \) is an arbitrary bounded sequence in \( L^\infty(dv_\gamma) \), without loss of generality, we can suppose that

\[
\|f_j\|_{L^\infty} \leq 1, \quad j = 1, 2, \ldots.
\]

Denote the bounded domain \( B'_j \) by \( B'_j = \{z \in \mathbb{C}^d : |z| \leq 1 - \frac{1}{j}\}, j = 1, 2, \ldots \). Clearly, \( B'_j \) is compactly contained in \( \Omega \) and \( B'_j \subset B'_{j+1} \subset \Omega \), for every \( j \). We first prove that the set of restrictions to \( B'_j \) of the functions in \( \{B_{\alpha,\gamma} f_n\} \) is precompact in \( L^p(B'_j) \) for each \( j \).

In view of Lemma 3.4, we know the functions in \( \{B_{\alpha,\gamma} f_n\} \) are all continuous on \( \Omega \) and uniformly continuous on every \( B'_j \⊂ \Omega \). Combining with the fact that the embedding \( C(B'_j) \subset L^p(B'_j, dv_\gamma) \) is continuous for every \( j \), it is enough to prove \( \{(B_{\alpha,\gamma} f_n)|_{B'_j}\} \) is precompact in \( C(B'_j) \) for every \( j \).

Note that \( h(z, w)^{-\alpha} \) is holomorphic on \( B'_j \times \Omega \subset \Omega \times \Omega \), so \( h(z, w)^{-\alpha} \) is uniformly continuous on \( B'_j \times \Omega \) and there exists \( C_j > 0 \) such that

\[
|h(z, w)^{-\alpha}| \leq C_j
\]
on \( B'_j \times \Omega \) for every \( j \). We then have that

\[
\|(B_{\alpha,\gamma} f_n)|_{B'_j}\|_{L^\infty} = \sup_{z \in B'_j} |B_{\alpha,\gamma} f_n|
\]

\[
= \sup_{z \in B'_j} \left| \int_\Omega f_n(w) dv_\gamma(w) \right| h(z, w)^\alpha \leq C_j \|f_n\|_{L^\infty}
\]

for every \( j \). The estimate (4.2) implies that \( \{(B_{\alpha,\gamma} f_n)|_{B'_j}\} \) are uniformly bounded in \( C(B'_j) \) for every \( j \). Meanwhile, the uniform continuity of the function \( h(z, w)^{-\alpha} \) on \( B'_j \times \Omega \) implies that \( \{(B_{\alpha,\gamma} f_n)|_{B'_j}\} \) is equicontinuous on \( B'_j \) for every \( j \). Then Arzelà-Ascoli theorem implies that \( \{(B_{\alpha,\gamma} f_n)|_{B'_j}\} \) is precompact in \( C(B'_j) \) for every \( j \).

Due to Proposition 4.1 we obtain that, for any fixed \( \alpha < N + 1 + 2 \gamma \) and for any \( \varepsilon > 0 \), there exists a \( J > 0 \) satisfying

\[
\int_{\Omega - B'_j} dv_\gamma(z) |h(z, w)|^{-\alpha} dv_\gamma(w) < \varepsilon, \quad \forall j > J,
\]

(4.3)
since the absolute continuity of the integral and \( \lim_{j \to \infty} v_{\gamma}(\Omega - B'_j) = 0 \). It follows from (4.3) that
\[
\int_{\Omega - B'_j} |B_{\alpha, \gamma} f_n(z)| dv_{\gamma}(z) = \int_{\Omega - B'_j} \left| \int_{\Omega} \frac{f_n(w) dv_{\gamma}(w)}{h(z, w)^\alpha} \right| dv_{\gamma}(z)
\leq \|f_n\|_{L^\infty} \int_{\Omega - B'_j} |h(z, w)|^{-\alpha} dv_{\gamma}(w)
\leq \varepsilon,
\]
for any \( j > J \) and \( n = 1, 2, \ldots \). Combining with Lemma 4.5 it implies that \( \{B_{\alpha, \gamma} f_n\} \) is precompact in \( L^1(dv_{\gamma}) \). Thus \( B_{\alpha, \gamma} : L^\infty(dv_{\gamma}) \to L^1(dv_{\gamma}) \) is compact. It completes the proof. \( \Box \)

5. Schatten class Szeg\"o-type operators

We have characterized the Schatten class Bergman-type operators on the Bergman space \( A^2(dv_{\gamma}) \) so far. The above argument can be used for the characterization of Schatten class of the so-called Szeg\"o-type integral operators on Hardy spaces. Let \( S \) be the Shilov boundary of \( \Omega \) with normalized \( K \)-invariant Haar measure \( d\sigma \), the Hardy space \( H^2(\Omega) \) is consists of all holomorphic functions on \( \Omega \) satisfying
\[
\sup_{R \to 1^-} \int_S |f(Rz)|^2 d\sigma(z) < \infty.
\]
There exists a canonical unitary isomorphism between \( H^2(\Omega) \) and a closed subspace \( H^2(S) \) of \( L^2(S, d\sigma) \); see [15, 21, 22] for details about Hardy spaces. Thus we can identify \( H^2(\Omega) \) with \( H^2(S) \). Szeg\"o kernel of \( \Omega \) is given by
\[
S(z, \xi) = h(z, \xi)^{-\rho}, \quad (z, \xi) \in \Omega \times S,
\]
where \( \rho := 1 + \frac{\alpha}{2} (r - 1) + b \).

**Definition 5.1.** For \( \alpha \in \mathbb{R} \), the Szeg\"o-type integral operator \( H_\alpha \) on \( L^1(S, d\sigma) \) is defined by
\[
H_\alpha f(z) = \int_S \frac{f(\xi)}{h(z, \xi)^\alpha} d\sigma(\xi), \quad z \in \Omega.
\]

Note that \( H_\rho \) is the Szeg\"o projection from \( L^2(S) \) onto \( H^2(\Omega) \). The same argument in Lemma 3.7 shows that \( H_\alpha : L^2(S) \to L^2(S) \) is bounded if and only if \( H_\alpha : H^2(\Omega) \to H^2(\Omega) \) is bounded if and only if \( \alpha \leq \rho \). Denote the set \( \mathcal{F} \) by
\[
\mathcal{F} := \{ s < \rho : s \notin \mathcal{F} \}.
\]

Now we list the main results of Schatten class of Szeg\"o-type operators, the proof is similar to the case of Bergman-type case and omitted.

**Theorem 5.2.** If \( \alpha \in \mathcal{F} \), then the followings hold.

1. The operator \( H_\alpha \in S_p(L^2(S)) \) for any \( p > 0 \).
2. The operator \( H_\alpha \in S_p(H^2(\Omega)) \) for any \( p > 0 \).

**Theorem 5.3.** Suppose \( \alpha \in \mathcal{F} \) and \( 0 < p < \infty \), then the following statements are equivalent.
(1) \( H_\alpha \in S_p(L^2(S)) \).
(2) \( H_\alpha \in S_p(H^2(\Omega)) \).
(3) \( \tilde{H}_\alpha \in L^p(d\lambda) \).
(4) \( p > \frac{N-1}{n-\alpha} \).

Where \( \tilde{H}_\alpha \) is the Berezin transform of \( H_\alpha \) on Hardy space \( H^2(\Omega) \).

References

[1] A. Adama, F. Fournier, Sobolev Spaces, Pure and Applied Mathematics (Amsterdam), Elsevier/Academic Press, Amsterdam (2003)
[2] D. Békollé, A. Bonami, Estimates for the Bergman and Szegő projections in two symmetric domains of \( \mathbb{C}^n \), Colloq. Math. 68, 81-100 (1995)
[3] A. Bonami, G. Garrigós, C. Nana, \( L^p-L^q \) estimates for Bergman projections in bounded symmetric domains of tube type, J. Geom. Anal. 24, 1737-1769 (2014)
[4] G. Cheng, X. Fang, Z. Wang, J. Yu, The hyper-singular cousin of the Bergman projection, Trans. Amer. Math. Soc. 369, 8643-8662 (2017)
[5] E. Cartan, Sur les domaines bornés homogénes de l’espace des variables complexes (French), Abh. Math. Sem. Univ. Hamburg, 11, 116-162 (1935)
[6] G. Cheng, X. Hou, C. Liu, The singular integral operator induced by Drury-Arveson kernel, Complex Anal. Oper. Theory, 12, 917-929 (2018)
[7] L. Ding, J. Fan, On the compactness of Bergman-type integral operators, arXiv:2004.13635, (2020)
[8] L. Ding, K. Wang, The \( L^p-L^q \) problems of Bergman-type operators, arXiv:2003.00479, (2018)
[9] M. Englis, G. Zhang, On the Faraut-Koranyi hypergeometric functions in rank two, Ann. Inst. Fourier (Grenoble), 54, 1855-1875 (2005)
[10] J. Faraut, A. Korányi, Function spaces and reproducing kernels on bounded symmetric domains, J. Funct. Anal., 88, 64-89 (1990)
[11] F. Forelli, W. Rudin, Projections on spaces of holomorphic functions in balls, Indiana Univ. Math. J. 24, 593-602 (1974)
[12] S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces, Academic Press, New York, (1978)
[13] M. Jarnicki, P. Pflug, Invariant distances and metrics in complex analysis, Second extended edition, De Gruyter Expositions in Mathematics, 9, Walter de Gruyter, Berlin (2013)
[14] H. Kaptanoğlu, A. Ureyen, Singular integral operators with Bergman-Besov kernels on the ball, Integral Equations Operator Theory, 91, No. 30 (2019)
[15] A. Korányi, The Poisson integral for generalized half-planes and bounded symmetric domains, Ann. Math. (2), 82, 332-350 (1965)
[16] O. Loos, Jordan Pairs, Lecture Notes in Math., 460, Springer (1975)
[17] S. Mihkinnen, J. Pau; A. Perälä, M. Wang, Volterra type integration operators from Bergman spaces to Hardy spaces, J. Funct. Anal. 279, 32 pp (2020)
[18] J. Pau, Jordi, A. Peralà, A Toeplitz-type operator on Hardy spaces in the unit ball, Trans. Amer. Math. Soc., 373, 3031-3062 (2020)
[19] J. Ringrose, Compact non-self-adjoint operators, Van Nostrand Reinhold Co., London (1971)
[20] I. Satake, Algebraic structures of symmetric domains, Iwanami Shoten, Tokyo, and Princeton Univ. Press, Princeton, NJ, (1980)
[21] H. Upmeier, *Toeplitz operators on bounded symmetric domains*, Trans. Amer. Math. Soc. **280**, 221-237 (1983)

[22] H. Upmeier, K. Wang, *Dixmier trace for Toeplitz operators on symmetric domains*, J. Funct. Anal., **271**, 532-565 (2016)

[23] Z. Yan, *A class of generalized hypergeometric functions in several variables*, Canad. J. Math., **44**, 1317-1338 (1992)

[24] R. Zhao, *Generalization of Schur’s test and its application to a class of integral operators on the unit ball of $\mathbb{C}^n$*, Integral Equations Operator Theory, **82**, 519-532 (2015)

[25] R. Zhao, K. Zhu, *Theory of Bergman Spaces in the Unit Ball of $\mathbb{C}^n$*, Mém. Soc. Math. Fr. **115** (2009)

[26] K. Zhu, *Holomorphic Besov spaces on bounded symmetric domains*, Quart. J. Math. Oxford, **46**, 239-256 (1995)

[27] K. Zhu, *Holomorphic Besov spaces on bounded symmetric domains. II*, Indiana Univ. Math. J., **44**, 1017-1031 (1995)

[28] K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, Graduate Texts in Mathematics, **226**, Springer-Verlag, New York (2005)

[29] K. Zhu, *Operator Theory in Function Spaces. Second Edition*, Mathematical Surveys and Monographs, 138, American Mathematical Society, Providence, RI (2007)

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