On Balanced Games with Infinitely Many Players:
Revisiting Schmeidler’s Result

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Abstract

We consider transferable utility cooperative games with infinitely many players and the core understood in the space of bounded additive set functions. We show that, if a game is bounded below, then its core is non-empty if and only if the game is balanced.

This finding is a generalization of Schmeidler’s (1967) original result “On Balanced Games with Infinitely Many Players”, where the game is assumed to be non-negative. We furthermore demonstrate that, if a game is not bounded below, then its core might be empty even though the game is balanced; that is, our result is tight.

We also generalize Schmeidler’s (1967) result to the case of restricted cooperation too.

Keywords
TU games with infinitely many players, core, balancedness, TU games with restricted cooperation, signed TU games, bounded additive set functions

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1 Introduction

The core (Shapley, 1953; Gillies, 1959) is one of the most important solution concepts of cooperative game theory. It is important not only from the theory viewpoint, but for its simple and easy to understand nature, it also helps to solve various problems arising in practice.

In the transferable utility setting (henceforth TU games) the Bondareva-Shapley Theorem (Bondareva, 1963; Shapley, 1967; Faigle, 1989) provides a necessary and sufficient condition for the non-emptiness of the core of a finite TU game; it states that the core of a finite TU game with our without restricted cooperation is not empty if and only if the TU game is balanced. The textbook proof of the Bondareva-Shapley Theorem goes by the strong duality theorem of linear programs, see e.g. Peleg and Sudhölter (2007).

Schmeidler (1967), Kannai (1969, 1992), and Pintér (2011), among others, considered TU games with infinitely many players. All these papers studied the case when the core consists of bounded additive set functions. Schmeidler (1967) and Kannai (1968) showed respectively that the core of a non-negative
TU game with infinitely and countably infinitely many players is not empty if and only if the TU game is balanced.

In this paper we consider infinite signed TU games (sign unrestricted TU games with infinite many players) with and without restricted cooperation. Particularly, we follow Schmeidler (1967) and assume that the allocations are bounded additive set functions.

Applications of infinite signed TU games go back in times at least as early as Shapley and Shubik (1969a) (economic systems with externalities), which generalize market games (Shapley and Shubik, 1969a). Further applications are (semi-) infinite transportation games (Sanchez-Soriano et al, 2002; Timmer and Llorca, 2002), infinite sequencing games (Fragnelli et al, 2010), and somehow less directly the line of literature represented by e.g. Montrucchio and Semeraro (2008) among others.

While we can analyze the non-emptiness of the core in the finite setting by using the aforementioned Bondareva-Shapley Theorem (Bondareva, 1963; Shapley, 1967; Faigle, 1989), we have been missing an appropriate tool for such TU games with infinitely many players.

Our contribution is an extension of Schmeidler’s (1967) result saying a non-negative infinite TU game without restricted cooperation has a non-empty core if and only if it is balanced, to the general case saying a bounded below infinite TU game with or without restricted cooperation has a non-empty core if and only if it is balanced (Theorems 4 and 8).

It is worth mentioning that neither Schmeidler’s (1967) nor Kannai’s (1969; 1992) approach (proof) can be applied to achieve our generalization (Theorems 4 and 8). Our approach is different from the previous ones.

The set-up of this paper is as follows. In Sections 2 and 3, we introduce basic notions of TU games with infinitely many players, including the core and balancedness, and we present our main result (Theorem 4). In Sections 4 and 5, we recall some useful concepts pertaining functional spaces, topology and compactness, and we prove our main result. We additionally give examples to show the tightness of our main result and we also mention an interesting “limiting” property of the core. Finally, in Section 6, we discuss the case of restricted cooperation and give our second main result (Theorem 8).

2 Preliminaries of infinite TU games

We consider transferable utility cooperative games with a finite or infinite set \( N \) of players. A coalition is a subset \( S \subseteq N \), so the power set \( P(N) = \{ S : S \subseteq N \} \) is the collection of all coalitions that can be considered. Let \( \mathcal{A} \subseteq P(N) \) be the collection of all feasible coalitions, which are those that can potentially emerge. In the case of no restricted cooperation, we assume that \( \mathcal{A} \) is a field of sets over \( N \); that is, the collection \( \mathcal{A} \) is such that \( \emptyset \in \mathcal{A} \) and, if \( S, T \in \mathcal{A} \), then \( N \setminus S \in \mathcal{A} \) and also \( S \cup T \in \mathcal{A} \). In the case of restricted cooperation, we assume only that \( \emptyset, N \in \mathcal{A} \).

Then a transferable utility cooperative game (henceforth game for short) is represented by its coalition function, which is a mapping \( v : \mathcal{A} \to \mathbb{R} \) such that \( v(\emptyset) = 0 \). For any coalition \( S \in \mathcal{A} \), the value \( v(S) \) is understood as the payoff that the coalition \( S \) receives if it is formed.

Assume that the players form the grand coalition \( N \in \mathcal{A} \). Then \( v(N) \in \mathbb{R} \) is
the value of the grand coalition \( N \), and the issue is to allocate this value among the players. Following Schmeidler (1967), we define the allocations as bounded additive set functions \( \mu : A \rightarrow \mathbb{R} \); that is, a function such that \( |\mu(S)| \leq C \) for all \( S \in A \) for some constant \( C \in \mathbb{R} \) and \( \mu(S \cup T) = \mu(S) + \mu(T) \) for any disjoint \( S, T \in A \). Let \( ba(A) = \{ \mu : A \rightarrow \mathbb{R} : \mu \) is a bounded additive set function \} denote the space of all bounded additive set functions on \( A \). Then the core of the game by \( v \) is the set

\[
\text{ba-core}(v) = \{ \mu \in \text{ba}(A) : \mu(N) = v(N), \mu(S) \geq v(S) \quad \text{for all} \quad S \in A \setminus \{N\} \}.
\]

In words, the core consists of all the allocations of the value \( v(N) \) among the players (efficiency) such that any coalition \( S \in A \setminus \{N\} \) that could potentially emerge gets by the proposed allocations at least as much as the value \( v(S) \) (coalitional rationality), see Shapley (1955), Gillies (1959), Kannai (1992), and Zhao (2018).

It is worth noticing that calling any game where the class of feasible coalitions is not a field even if the field is not the power set of the player set misleading because any additive set function defined on a subfield can be extended to the power set. Therefore, an allocation from the core of a game without restricted cooperation gives rise (typically in a non-unique way) to an allocation defined on the power set of the player set.

The case when the class of feasible coalitions is not a field, however, leads to the very same features of the core as restricted cooperation leads in the finite setting (see Faigle (1989)), explaining why we call this case restricted cooperation.

The key question is whether the core is non-empty. An answer is provided by the Bondareva-Shapley Theorem.

### 3 The Bondareva-Shapley Theorem

Consider a game having finitely many players without restricted cooperation. In this case, we have \( N = \{1, 2, \ldots, n\} \) for some natural number \( n \) and \( A = \mathcal{P}(N) \). Moreover, in this setting, the allocations of the value \( v(N) \) among the players are given by payoff vectors, any of them is an \( n \)-tuple \( a = (a_i)_{i=1}^{n} \in \mathbb{R}^n \) of real numbers; number \( a_i \) means the payoff allocated to player \( i \) for \( i = 1, 2, \ldots, n \). Then the core of this game is defined to be the set

\[
\text{core}(v) = \{ a \in \mathbb{R}^N : \sum_{i \in N} a_i = v(N), \sum_{i \in S} a_i \geq v(S) \quad \text{for all} \quad S \in \mathcal{P}(N) \setminus \{N\} \}.
\]

The intuitive meaning of the core \( v \) is the same as that of the ba-core \( v \), see above. Clearly, given a payoff vector \( a \in \mathbb{R}^N \), we can define the corresponding additive set function \( \mu : \mathcal{P}(N) \rightarrow \mathbb{R} \) by \( \mu(S) = \sum_{i \in S} a_i \) for any \( S \in \mathcal{P}(N) \). Conversely, given an additive set function \( \mu : \mathcal{P}(N) \rightarrow \mathbb{R} \), we can define the corresponding payoff vector \( a \in \mathbb{R}^N \) by \( a_i = \mu(\{i\}) \) for \( i = 1, 2, \ldots, n \). Here any additive set function \( \mu : \mathcal{P}(N) \rightarrow \mathbb{R} \) is bounded as the number of the players is finite. We thus have a one-to-one correspondence between the ba-core \( v \) and the core \( v \). Hence, the notion of bounded additive function \( \mu \in \text{ba}(A) \) naturally
extends the concept of the payoff vector \( a \in \mathbb{R}^N \) when the set \( N \) of the players is infinite.

Regarding the question whether the core(\( v \)) is non-empty, for any coalition \( S \subseteq N \), define its characteristic vector to be the row vector \( \chi_S = (\chi_S(1) \quad \chi_S(2) \quad \ldots \quad \chi_S(n)) \) with \( \chi_S(i) = 1 \) if \( i \in S \), and with \( \chi_S(i) = 0 \) if \( i \notin S \), for \( i = 1, 2, \ldots, n \). We say that a collection \( S = \{S_1, S_2, \ldots, S_r\} \subseteq \mathcal{P}(N) \) of coalitions is balanced if there exist non-negative real numbers \( \lambda_1, \lambda_2, \ldots, \lambda_r \), called balancing weights, such that

\[
\sum_{p=1}^{r} \lambda_p \chi_{S_p} = \chi_N.
\]  

Moreover, we say that a game \( v \) is balanced if

\[
\sum_{p=1}^{r} \lambda_p v(S_p) \leq v(N)
\]

for every balanced collection \( \{S_1, S_2, \ldots, S_r\} \subseteq \mathcal{P}(N) \) of coalitions. The following result due to Bondareva (1963) and Shapley (1967), later extended by Faigle (1989) to the restricted cooperation case, has become classical:

**Theorem 1 (Bondareva-Shapley Theorem).** Consider a game with finitely many players, with or without restricted cooperation, represented by a coalition function \( v: \mathcal{P}(N) \to \mathbb{R} \). Then the core(\( v \)) is non-empty if and only if the game is balanced.

Consider now a general game without restricted cooperation; that is, the set \( N \) of the players can be finite or infinite and the class of feasible coalitions \( \mathcal{A} \subseteq \mathcal{P}(N) \) is a field of sets over \( N \). Concerning the question whether \( \text{ba-core}(v) \) is non-empty we follow Schmeidler (1967), who proceeds analogously as in the classical case; that is:

For any subset \( S \subseteq N \), define its characteristic function \( \chi_S: N \to \{0,1\} \) by letting \( \chi_S(i) = 1 \) if \( i \in S \), and \( \chi_S(i) = 0 \) if \( i \notin S \), for every \( i \in N \). We say that a collection \( S = \{S_1, S_2, \ldots, S_r\} \subseteq \mathcal{A} \) of coalitions is balanced if there exist non-negative real numbers \( \lambda_1, \lambda_2, \ldots, \lambda_r \), called balancing weights, such that

\[
\sum_{p=1}^{r} \lambda_p \chi_{S_p} = \chi_N.
\]

Furthermore, we say that a game \( v \) is balanced if

\[
\sum_{p=1}^{r} \lambda_p v(S_p) \leq v(N)
\]

for every balanced collection \( \{S_1, S_2, \ldots, S_r\} \subseteq \mathcal{A} \) of coalitions.

**Remark 2.** Schmeidler (1967) actually defines balancedness in a slightly different way: “A game is balanced if sup \( \sum a_i v(A_i) \leq v(S) \) when the sup is taken over all finite sequences of \( a_i \) and \( A_i \), where the \( a_i \) are non-negative numbers, the \( A_i \) are in \( \Sigma \), and \( \sum a_i \chi A_i \leq \chi S \).” Considering non-negative games, Schmeidler explains that his definition is different from the “definition with equality” only
in its form: “It is easy to verify that this sup does not change even if it is constrained by \( \sum a_i \chi_{A_i} = \chi_S \) (instead of the inequality); also, for balanced games, the sup equals \( v(S) \).” — See Schmeidler (1967, p. 1). In the case of non-negative games Schmeidler’s definition of balancedness is equivalent with the “definition with equality”; however, in the general, signed case, those are different.

Then Schmeidler (1967) proves the following result, see Kannai (1969) for another proof:

**Theorem 3** (Bondareva-Shapley Theorem, Schmeidler (1967)). Given a finite or infinite set \( N \) of the players and a field of sets \( A \subseteq \mathcal{P}(N) \) over \( N \), consider a game represented by a coalition function \( v: A \rightarrow \mathbb{R} \). If the game is non-negative; that is,

\[
\forall S \in A: \quad v(S) \geq 0,
\]

then the \( \text{ba-core}(v) \) is non-empty if and only if the game is balanced.

It is easy to see that Theorem 3 is a generalization of Theorem 1 if the game is non-negative. Our goal, nonetheless, is to establish the following result:

**Theorem 4** (Bondareva-Shapley Theorem, a generalization). Given a finite or infinite set \( N \) of the players and a field of sets \( A \subseteq \mathcal{P}(N) \) over \( N \), consider a game represented by a coalition function \( v: A \rightarrow \mathbb{R} \). If the game is bounded below; that is,

\[
\exists L \in \mathbb{R} \quad \forall S \in A: \quad v(S) \geq L,
\]

then the \( \text{ba-core}(v) \) is non-empty if and only if the game is balanced.

Notice that Theorem 4 directly generalizes both Theorems 1 and 3 because a game with finitely many players is always bounded below.

Before we present our proof of Theorem 4, we find it appropriate to introduce and recall several notions and concepts.

4 Several notions and concepts

Let \( N \) be a set and let \( A \subseteq \mathcal{P}(N) \) be a field of sets over \( N \). Then the pair \((N, A)\) is called a chargeable space. Recall that, for any \( S \subseteq N \), the symbol \( \chi_S \) denotes the characteristic function \( \chi_S: N \rightarrow \{0, 1\} \) of the set \( S \). Given a function \( f: N \rightarrow \mathbb{R} \), we say it is a simple function if \( f = \lambda_1 \chi_{S_1} + \lambda_2 \chi_{S_2} + \cdots + \lambda_r \chi_{S_r} \) for some natural number \( r \), for some real numbers \( \lambda_1, \lambda_2, \ldots, \lambda_r \), and for some sets \( S_1, S_2, \ldots, S_r \in A \). Let \( \Lambda(A) = \{ f: N \rightarrow \mathbb{R} : \text{f is a simple function} \} \) denote the vector (i.e. linear) space of all simple functions defined over \((N, A)\), where the sum of two functions and the multiplication of a function by a constant are both defined in the usual way, i.e. pointwise. For a simple function \( f \in \Lambda(A) \), define its norm to be

\[
\|f\| = \sup_{i \in N} |f(i)|,
\]

so \( \Lambda(A) \) is a normed linear space.

Likewise, notice that the space \( \text{ba}(A) \) of all bounded additive set functions on \( A \) is also a vector space; for a \( \mu \in \text{ba}(A) \), define its norm to be

\[
\|\mu\| = \sup_{S_1, S_2, \ldots, S_r \in A, S_1 \cup S_2 \cup \cdots \cup S_r = N} \{ |\mu(S_1)| + |\mu(S_2)| + \cdots + |\mu(S_r)| \}.
\]
It is well-known that the topological dual \((\Lambda(A))^*\) of the vector space \(\Lambda(A)\): that is, the space of all continuous linear functionals on \(\Lambda(A)\), is isometrically isomorphic to the space \(ba(A)\) (see, e.g., Dunford and Schwartz\(^2\) (1958), Theorem IV.5.1, p. 258). Indeed, a continuous linear functional \(\mu' \in (\Lambda(A))^*\) induces a bounded additive set function \(\mu \in ba(A)\) by letting \(\mu(S) = \mu'(\chi_S)\) for \(S \in A\), and, conversely, a bounded additive set function \(\mu \in ba(A)\) induces a continuous linear functional \(\mu' \in (\Lambda(A))^*\) by letting

\[
\mu'(f) = \lambda_1\mu(S_1) + \lambda_2\mu(S_2) + \cdots + \lambda_r\mu(S_r)
\]

for any simple function \(f = \lambda_1\chi_{S_1} + \lambda_2\chi_{S_2} + \cdots + \lambda_r\chi_{S_r} \in \Lambda(A)\). This is the reason why, for simplicity, we shall identify the space \((\Lambda(A))^*\) with \(ba(A)\).

Consider now a game represented by a coalition function \(v : A \to \mathbb{R}\), and let the game be bounded below: that is, there exists a constant \(L \in \mathbb{R}\) such that \(v(S) \geq L\) for all \(S \in A\). Assume that a \(\mu \in ba-core(v)\). Let \(S_1, S_2, \ldots, S_r \in A\) be pairwise disjoint and such that \(N = S_1 \cup S_2 \cup \cdots \cup S_r\). Then

\[
\sum_{p=1}^{r} |\mu(S_p)| = \sum_{\mu(S_p) \geq 0} \mu(S_p) - \sum_{\mu(S_p) < 0} \mu(S_p) = \mu\left(\bigcup_{\mu(S_p) \geq 0} S_p\right) - \mu\left(\bigcup_{\mu(S_p) < 0} S_p\right) = \mu(N) - \mu\left(\bigcup_{\mu(S_p) < 0} S_p\right) \leq \mu(N) - 2L = v(N) - 2L.
\]

By taking the definition \(1\) of the norm into account, it follows the ba-core\((v)\) is contained in the closed ball \(B_R = \{ \mu \in ba(A) : \|\mu\| \leq v(N) - 2L\}\) of radius \(R = v(N) - 2L\). (Notice that \(v(N) - 2L \geq 0\), for if we had \(v(N) < 2L\), then the ba-core\((v)\) would obviously be empty, contradicting the assumption that \(\mu \in ba-core(v)\).)

We endow the space \(ba(A)\) with the weak* topology with respect to \(\Lambda(A)\). The topology will be introduced if we describe all the neighborhoods of a point. A set \(U \subseteq ba(A)\) is a weak* neighborhood of a \(\mu_0 \in ba(A)\) if there exist a natural number \(r\) and functions \(f_1, f_2, \ldots, f_r \in \Lambda(A)\) such that \(\bigcap_{p=1}^{r} \{ \mu \in ba(A) : |\mu'(f_p) - \mu_0'(f_p)| < 1 \} \subseteq U\), where \(\mu'\) and \(\mu_0'\) is the continuous linear functional induced by \(\mu\) and \(\mu_0\), respectively, see \(1\). By Alaoglu’s Theorem (see, e.g., Aliprantis and Border\(^2\) (2006), Theorem 6.21, p. 235), the closed ball \(B_R\) is compact in the weak* topology. That is, if \(G_i \subseteq ba(A)\) are weakly* open sets for \(i \in I\), where \(I\) is an index set, such that \(\bigcup_{i \in I} G_i \supseteq B_R\), then \(\bigcup_{i=1}^{n} G_{i_j} \supseteq B_R\) for some natural number \(n\) and for some \(j_1, j_2, \ldots, j_n \in I\).

Let \(F_i \subseteq B_R\) be weakly* closed sets for \(i \in I\), where \(I\) is an index set. We say the collection \(\{F_i\}_{i \in I}\) is a centered system of sets if \(\bigcap_{i=1}^{n} F_{i_j} \neq \emptyset\) for any
natural number \( n \) and for any \( i_1, i_2, \ldots, i_n \in I \). By considering the complements \((G_i = ba(A) \setminus F_i)\), it follows \( \bigcap_{i \in I} F_i \neq \emptyset \).

In our proof of Theorem 4 we consider the weakly* closed sets
\[
F_S = \left\{ \mu \in ba(A) : \mu(N) = v(N) \text{ and } \mu(S) \geq v(S) \text{ and } \|\mu\| \leq R \right\}
\]
for \( S \in A \). The main idea is to show that, if the game \( v \) is balanced, then the system \( \{F_S\}_{S \in A} \) is centered. Noticing that ba-core(\( v \)) = \( \bigcap_{S \in A} F_S \neq \emptyset \), the proof will be done.

We are now ready to present our proof of Theorem 4.

5 Proof of Theorem 4

Below we give our proof of Theorem 4. The aforegiven notions and concepts are utilized in the proof, and it will be seen that its main ingredience is the use of compactness.

**Proof of Theorem 4.** Assume that the given coalition function \( v : A \rightarrow \mathbb{R} \) is bounded below by \( L \). We are to show that ba-core(\( v \)) \( \neq \emptyset \) if and only if the given game is balanced. The "only if" part is obvious. Assume that a \( \mu \in ba(v) \) and let \( S = \{S_1, S_2, \ldots, S_r\} \subseteq A \) be a balanced collection of coalitions, so that (3) holds for some non-negative balancing weights \( \lambda_1, \lambda_2, \ldots, \lambda_r \). Then \( \sum_{p=1}^{r} \lambda_p v(S_p) \leq \sum_{p=1}^{r} \lambda_p \mu(S_p) = \mu(N) = v(N) \), so (4) is satisfied, and the game is balanced. It remains to prove the "if" part.

Pick up any sets \( S_0, S_1, \ldots, S_n \in A \). Our purpose is to show that \( \bigcap_{j=0}^{n} F_{S_j} \neq \emptyset \). We can assume w.l.o.g. that the sets \( S_0, \ldots, S_n \) are distinct with \( S_0 = \emptyset \) and \( S_n = N \), and that the collection \( \{S_0, \ldots, S_n\} \subseteq A \) is a field of sets. (Roughly speaking, the more sets we pick up, the smaller the intersection \( \bigcap_{j=0}^{n} F_{S_j} \) is. Having to show the intersection is non-empty anyway, we can include the empty and the grand coalition among the sets. Moreover, we can add further sets from \( A \) so that the collection \( \{S_0, \ldots, S_n\} \) becomes a finite field of sets.

We can also assume w.l.o.g. that \( S_1, \ldots, S_n \) are all the atoms of the field; that is, they are all the minimal elements in the collection \( \{S_1, \ldots, S_n\} \). Obviously, the atoms \( S_1, \ldots, S_n \) are pairwise disjoint, and it holds \( n = 2^n - 1 \).

Now, the sets \( \emptyset, S_0, S_1, \ldots, S_n \) being fixed, we apply balancedness to the sets \( S_1, \ldots, S_n \):

\[
\forall \lambda_1, \ldots, \lambda_n \geq 0: \quad \lambda_1 \chi_{S_1} + \cdots + \lambda_n \chi_{S_n} = \chi_N \implies \lambda_1 v(S_1) + \cdots + \lambda_n v(S_n) \leq v(N). \quad (7)
\]

It shall follow hence that the system of relations
\[
\mu(N) = v(N), \quad \mu(S_1) \geq v(S_1), \quad \mu(S_n) \geq v(S_n) \quad (8)
\]
has a solution \( \mu \in ba(A) \) such that \( \|\mu\| \leq R \). To see that, we apply the Bondareva-Shapley Theorem for finite games (Theorem 1).

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Consider a new finite game $v': \mathcal{P}(N') \to \mathbb{R}$ with the set of the players $N' = \{1, \ldots, n'\}$. Define the game as follows. Recall first that the collection \( \{S_0, \ldots, S_n\} \) is a field of sets and that $S_1, \ldots, S_{n'}$ are all its atoms, which are pairwise disjoint. Now, for an $S' \subseteq N'$, let $S = \bigcup_{i \in S'} S_i$, notice $S \in \{S_0, \ldots, S_n\}$, and put $v'(S') = v(S)$. The new finite game $v'$ has been defined thus.

Now, condition (7) equivalently says that the new game $v'$ is balanced. By the Bondareva-Shapley Theorem (Theorem 1), its core is non-empty: there exist thus.

The atoms $S_1, \ldots, S_{n'}$ being non-empty sets, there exist elements $x_i \in S_i$ for $i' = 1, \ldots, n'$. Consider the measure

$$\mu = a_1 \delta_{x_1} + \cdots + a_{n'} \delta_{x_{n'}},$$

where $\delta_{x_i}$ is the Dirac measure concentrated at $x_i$. We have $\mu(N) = \mu(S_1 \cup \cdots \cup S_{n'}) = a_1 + \cdots + a_{n'} = v(N)$. For any $j = 1, \ldots, n$, let $S_j' = \{i' \in N': S_i \subseteq S_j\}$. Then $S_j = \bigcup_{i' \in S_j'} S_i$, and $\mu(S_j) = \sum_{i' \in S_j'} a_{i'} \geq v'(S_j') = v(S_i)$. We have shown thus that the $\mu$ is a solution to the system of inequalities (8).

Finally, let us calculate the norm $\|\mu\|$ of the solution, see (9). For a $T \in \mathcal{A}$, we observe that

$$\mu(T) = \sum_{i' \in T} a_{i'}.$$ 

Given pairwise disjoint sets $T_1, \ldots, T_s \in \mathcal{A}$ such that $N = T_1 \cup \cdots \cup T_s$, and recalling $\sum_{i' = 1}^{n'} a_{i'} = v(N)$, we have

$$\sum_{q = 1}^{s} \left| \sum_{i' \in T_q} a_{i'} \right| \leq \sum_{q = 1}^{s} n' \sum_{i' \in T_q} |a_{i'}|$$

$$= \sum_{i' = 1}^{n'} |a_{i'}| = \sum_{i' = 1}^{n'} a_{i'} - \sum_{i' = 1}^{n'} a_{i'}$$

$$= v(N) - 2 \sum_{i' = 1}^{n'} a_{i'} \leq v(N) - 2 v\left( \bigcup_{i' = 1}^{n'} S_i \right)$$

$$\leq v(N) - 2L = R.$$

It follows that $\|\mu\| \leq R$. To conclude, we have a $\mu \in \text{ba}(\mathcal{A})$ such that it is a solution to (8) and $\|\mu\| \leq R$, which means $\mu \in \bigcap_{j=1}^{n} F_{S_j}$. Since $F_{S_j} \subseteq F_\emptyset$ for $j = 1, \ldots, n$, it holds $\mu \in \bigcap_{j=0}^{n} F_{S_j}$. We have shown thus that the system $\{F_S\}_{S \in \mathcal{A}}$ is centered. As the closed $R$-ball $B_R$ is weakly* compact, we have $\text{ba-core}(v) = \bigcap_{S \in \mathcal{A}} F_S \neq \emptyset$.

The following example demonstrates that Theorem 3 cannot be generalized further. It presents an unbounded below game that is balanced, but its core is empty.

**Example 5.** Let the player set be $N = \mathbb{N}$, and let $\mathcal{A} = \{S \subseteq N : S \text{ is finite or } N \setminus S \text{ is finite}\}$. Consider the game represented by the coalition function
v: \mathcal{A} \to \mathbb{R} defined as follows: for any S \in \mathcal{A}, let

\[v(S) = \begin{cases} 
1 & \text{if } S = \{1\}, \\
1 + \frac{1}{n} & \text{if } S = \{1, n\} \text{ for } n = 2, 3, \ldots , \\
-\sum_{n \in T} \frac{1}{n} & \text{if } S = N \setminus T \text{ for a finite } T \in \mathcal{A}, \\
0 & \text{otherwise.}
\]

It is easy to see that this game is balanced. Assuming that a \(\mu \in \text{ba-core}(v)\), then \(\mu(\{1\}) \geq v(\{1\}) = 1\) and \(\mu(N \setminus \{1\}) \geq v(N \setminus \{1\}) = -1\). Since \(\mu(\{1\}) + \mu(N \setminus \{1\}) = \mu(N) = v(N) = 0\), we have \(\mu(\{1\}) = 1\). As \(\mu(\{1, n\}) \geq v(\{1, n\}) = 1 + 1/n\) for all \(n = 2, 3, \ldots \), it follows \(\mu(\{n\}) \geq 1/n\) for all \(n = 2, 3, \ldots \). Summing up, we have \(\mu(\{1, \ldots , n\}) \geq \ln(n+1)\), so \(\mu \notin \text{ba}(\mathcal{A})\) because \(\mu\) is not bounded. It follows \(\text{ba-core}(v) = \emptyset\).

The following “limiting” property of the ba-core is interesting. It is obtained as a corollary of Theorem 4 by considering the balancedness condition \(\mathbb{3}\) and \(\mathbb{4}\).

Corollary 6. Given a finite or infinite set \(N\) of the players and a field of sets \(\mathcal{A} \subseteq \mathcal{P}(N)\) over \(N\), let the game represented by a coalition function \(v: \mathcal{A} \to \mathbb{R}\) be bounded below. For any \(\varepsilon > 0\), define the coalition function \(v_\varepsilon: \mathcal{A} \to \mathbb{R}\) as follows: let \(v_\varepsilon(N) = v(N) + \varepsilon\) and \(v_\varepsilon(S) = v(S)\) for all \(S \in \mathcal{A} \setminus \{N\}\). Then: if \(\text{ba-core}(v_\varepsilon) \neq \emptyset\) for all \(\varepsilon > 0\), then \(\text{ba-core}(v) \neq \emptyset\).

Under the assumptions of Corollary 6, the converse statement is clear: if \(\text{ba-core}(v) \neq \emptyset\), then \(\text{ba-core}(v_\varepsilon) \neq \emptyset\) for all \(\varepsilon > 0\). We thus conclude that a game represented by the coalition function \(v\) is balanced if and only if \(\text{ba-core}(v_\varepsilon) \neq \emptyset\) for all \(\varepsilon > 0\).

6 Games with restricted cooperation

We now pay attention to games with restricted cooperation. In general, the cooperation is restricted whenever the collection \(\mathcal{A} \subseteq \mathcal{P}(N)\) of coalitions that can potentially emerge is a proper subset of \(\mathcal{P}(N)\). In this sense, Theorem 4 covers the case of restricted cooperation, under the additional assumption that \(\mathcal{A} \subseteq \mathcal{P}(N)\) is a field of sets over \(N\), too. Now, let \(\mathcal{A} \subseteq \mathcal{P}(N)\) be the collection of all coalitions that can potentially emerge; the collection \(\mathcal{A}'\) need not be a field of sets now. Assume only \(\emptyset, N \in \mathcal{A}'\). Then any coalition function \(v': \mathcal{A}' \to \mathbb{R}\), such that \(v'(\emptyset) = 0\), represents a game with restricted cooperation.

To introduce the concept of core of this game with restricted cooperation, let \(\mathcal{A} = \text{field}(\mathcal{A}')\) be the field hull of \(\mathcal{A}'\); that is, the minimal collection \(\mathcal{A} \supseteq \mathcal{A}'\) that is a field of sets over \(N\). Then the core of a game \(v'\) with restricted cooperation is the set

\[\text{ba-core}(v') = \{ \mu \in \text{ba}(\text{field}(\mathcal{A}')) : \mu(N) = v'(N), \mu(S) \geq v'(S) \text{ for all } S \in \mathcal{A}' \setminus \{N\} \}.\]

We again ask whether \(\text{ba-core}(v') \neq \emptyset\).

The following example presents a non-negative game with restricted cooperation which is balanced as by \(\mathbb{3}\) and \(\mathbb{4}\), but only the feasible coalitions are considered, but its core is empty.
Example 7. Let the player set be $N = \mathbb{N}$ and let $A' = \{\emptyset\} \cup \{(1,i) : i = 1, 2, 3, \ldots\} \cup \{N \setminus \{1\}\} \cup \{N\}$. Consider the game represented by the coalition function $v' : A' \to \mathbb{R}$ defined as follows: for any $S \in A$, let

$$
v'(S) = \begin{cases} 
2 & \text{if } S = \{1\}, \\
2 + \frac{1}{n} & \text{if } S = \{1,n\} \text{ for } n = 2, 3, \ldots, \\
0 & \text{if } S = N \setminus \{1\} \text{ or } S = \emptyset, \\
1 & \text{if } S = N.
\end{cases}
$$

Notice that this game is analogous to that presented in Example 5. The field hull of $A'$ is $A = \{S \subseteq N : S \text{ is finite or } N \setminus S \text{ is finite}\}$. The fact that this game is balanced as by (3) and (4), but replace $A$ and $v$ with $A'$ and $v'$, respectively, is clear. To show that $\text{ba-core}(v') = \emptyset$, it is enough to follow the arguments presented in Example 5.

Due to Example 7 we have to introduce a new notion of balancedness in the case of restricted cooperation. A game represented by a coalition function $v'$ defined on the class of feasible coalitions $A'$ is bounded-balanced if there exists a bounded below balanced game $v$ defined on $A$, where $A$ is the field hull of $A'$ such that for every $S \in A'$ it holds that $v(S) = v'(S)$. It is clear that if $A'$ is a field and the game is bounded below (as in Theorem 4), then we get back to the notion of balancedness applied in Theorem 4. Moreover, notice that for finite games bounded-balancedness and balancedness by Faigle (1989) are equivalent.

The game in Example 7 above has empty core because even if it is non-negative, none of its bounded below “extensions” onto $A$ is balanced and non of its balanced “extensions” onto $A$ is bounded below.

Then the following theorem extends Theorem 4 to the class of games with restricted cooperation, hence it extends Faigle (1989).

**Theorem 8.** Consider a coalition function $v' : A' \to \mathbb{R}$, where $\emptyset, N \in A' \subseteq \mathcal{P}(N)$ and $N$ is a finite or infinite set of the players. If $v'$ is bounded below, then the $\text{ba-core}(v') \neq \emptyset$ if and only if $v'$ is bounded-balanced.

**Proof.** If $A'$ is a field then we are back at Theorem 4, hence nothing to do.

Suppose that $A'$ is not a field. The game $v'$ is bounded below, take any game $v$ which makes $v'$ be bounded below. Then by Theorem 4 $\text{ba-core}(v) \neq \emptyset$ if and only $v$ is balanced. Since if $v$ is balanced then $v'$ is bounded-balanced by $v$, we get if $v'$ be bounded below, then the $\text{ba-core} v' \neq \emptyset$ if and only if $v'$ is bounded-balanced.

Notice that the game $v'$ of Example 7 is not bounded-balanced, hence $\text{ba-core}(v') = \emptyset$.

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