RESTRICTIONS OF $SL_3$ MAASS FORMS TO MAXIMAL FLAT SUBSPACES

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Abstract. Let $\psi$ be a Hecke-Maass form on a cubic division algebra over $\mathbb{Q}$. We apply arithmetic amplification to improve the local bound for the $L^2$ norm of $\psi$ restricted to maximal flat subspaces.

1. Introduction

Let $S$ be the globally symmetric space $SL_3(\mathbb{R})/SO(3)$. Let $\Gamma \subset SL_3(\mathbb{R})$ be a cocompact congruence lattice arising from a cubic division algebra over $\mathbb{Q}$, and let $X = \Gamma \backslash S$ (see Section 2 for definitions). Let $\psi$ be a Hecke-Maass form on $X$, that is to say an eigenfunction of the full ring of invariant differential operators and the Hecke operators. We assume that $\|\psi\|_2 = 1$. We also assume that the spectral parameter of $\psi$ is of the form $t\lambda$, where $t \in \mathbb{R}_{>0}$ and $\lambda \in B^*$, and $B^*$ is a fixed compact regular subset of $a^*$.

Let $E$ be a ball of radius 1 inside a maximal flat subspace of $S$. It was proven in [13], Theorem 1.2 that $\|\psi|_E\|_2 \ll t^{3/4}$, and moreover that this bound is sharp on the compact globally symmetric space $SU(3)/SO(3)$ which is dual to $S$. Note that Theorem 3 of [4] and the $L^\infty$ bound of [15] also provide bounds of this form with exponents of 1 and 3/2 respectively. In this paper, we apply arithmetic amplification to improve this exponent further.

Theorem 1.1. There is an absolute constant $\delta > 0$ and $C = C(\Gamma, B^*) > 0$ such that $\|\psi|_E\|_2 \leq Ct^{3/4-\delta}$.

We have chosen this particular restriction problem because it is one of only two cases in which we can restrict to a maximal flat subspace and observe a regime change, or kink point, in the local bound for the $L^p$ norm of the restricted eigenfunction. The other case is a geodesic on a surface, treated in [12]; see Theorem 1.2 of [13] for the proof of this classification. Both of these features simplify the problem, as the flatness prevents us from having to use the nonabelian Fourier transform on the subspace, and the presence of a regime change lies behind the strong bound we are able to prove for the ‘off-diagonal’ oscillatory integrals in Proposition 4.1. It would be interesting to see whether Theorem 1.1 could be used to prove a power saving for the global $L^p$ norms of $\psi$ for small $p$ as in [1, 2, 17].

The restriction to division algebras is not essential, and our methods could be applied on $GL_3$ to prove a result identical to Theorem 1.1 with the exception that it would only be uniform for $E$ in compact sets.

1.1. Outline of the proof. We shall prove Theorem 1.1 in the same way as Theorem 1.1 of [12]. Let $B_\alpha \subset a$ be the ball of radius 1 about the origin with respect to the norm $\| \cdot \|$.

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We have $E = g \exp(B_d)$ for some $g \in SL_3(\mathbb{R})$, and by abuse of notation we also let $E : H \mapsto g \exp(H)$ be a parametrisation of $E$. If we fix a function $b \in C_0^\infty(a)$ that is real-valued and satisfies $\text{supp}(b) \subseteq B_a$, it suffices to estimate the norm of $b(H)\psi(E(H)) \in L^2(a)$, where the measure $dH$ on $a$ is that associated to $\| \cdot \|$. We define the Fourier transform on $L^2(a)$ by

$$\hat{f}(\mu) = \int_a f(H) e^{-i\mu(H)} dH.$$ 

Let $\beta$ be a parameter satisfying $1 \leq \beta \leq t^{1/2}$. For $\mu \in a^*$, define $H(\mu, \beta) \subset L^2(a)$ to be the space of functions whose Fourier support lies in the ball of radius $\beta$ around $\mu$ with respect to the norm dual to $\| \cdot \|$. Let $\Pi(\mu, \beta)$ be the projection operator onto $H(\mu, \beta)$. Define

$$H_\beta = \bigoplus_{w \in W} H(w\lambda, \beta)$$

and let $\Pi_\beta$ be the projection onto $H_\beta$. We shall bound $\Pi(w\lambda, \beta) b\psi$ and $(1 - \Pi_\beta) b\psi$ separately, by applying amplification to the former and a local bound to the latter. The results we obtain are the following.

**Proposition 1.2.** There is an absolute $\delta > 0$ and $C = C(\Gamma, B^*, b) > 0$ such that $\|\Pi(w\lambda, \beta) b\psi\|_2 \leq C t^{3/4 - \delta} \beta^{3/4}$ for all $w \in W$.

**Proposition 1.3.** For any $\epsilon > 0$ there is $C = C(\Gamma, B^*, b, \epsilon) > 0$ such that $\|(1 - \Pi_\beta) b\psi\|_2 \leq C t^{3/4 + \epsilon} \beta^{-1/4}$.

Combining these two results with $\beta = t^\delta$ gives Theorem 1.1.

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2. Notation

Throughout the paper, the notation $A \ll B$ will mean that there is a positive constant $C$ such that $|A| \leq CB$, and $A \sim B$ will mean that there are positive constants $C_1$ and $C_2$ such that $C_1 B \leq A \leq C_2 B$.

2.1. Division algebras and adelic groups. Let $D$ be a cubic division algebra over $\mathbb{Q}$. We denote the reduced norm on $D$ by $nr$, and denote the kernel of $nr$ by $D^1$. We let $D$, $D^\times$, and $D^1$ be the algebraic groups over $\mathbb{Q}$ such that $D(\mathbb{Q}) = D$, $D^\times(\mathbb{Q}) = D^\times$, and $D^1(\mathbb{Q}) = D^1$. We denote the center of $D^\times$ by $Z$. Let $S_f$ be the set of places at which $D$ is ramified, and let $S = S_f \cup \{\infty\}$. We define $D_v = D \otimes_\mathbb{Q} \mathbb{Q}_v$, and choose an isomorphism $\phi_v : D_v \simeq M_3(\mathbb{Q}_v)$ for every $v \notin S_f$. We shall implicitly identify $D_v$ with $GL_3(\mathbb{Q}_p)$ via $\phi_p$ for $p \notin S_f$.

Let $R \subset D$ be a maximal order, and let $R^1 = R \cap D^1$. We define $R_p = R \otimes_\mathbb{Z} \mathbb{Z}_p \subset D_p$ for every prime $p$. $R_p$ is a maximal order in $D_p$ for all $p$, and for $p \notin S_f$ we can choose $\phi_p$ so that $\phi_p(R_p) = M_3(\mathbb{Z}_p)$. We define $K_p = R_p^\times \subset D_p$, so that for $p \notin S_f$ we have $\phi_p(K_p) = GL_3(\mathbb{Z}_p)$. For each $p$, we define $d_{K_p}$ to be the Haar measure on $D_p^\times$ that assigns mass 1 to $K_p$. We define $K_f = \otimes_p K_p$, which is a maximal compact subgroup of $D^\times(\mathbb{A}_f)$. We define $K_\infty = \phi^{-1}_\infty(SO(3))$, and let $K = K_\infty \otimes K_f$.

Define $X = D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A})/K \mathbb{Z}(\mathbb{A})$. Our assumption that $D$ is a division algebra implies that $X$ is compact. $X$ is also connected, by the Hasse-Schilling Norm Theorem (see
Theorem 33.15 of [14], and the fact that $D^1$ is simply connected and hence satisfies strong approximation. If we define $\Gamma \subset SL_3(\mathbb{R})$ to be the image of $\phi_\infty(R^*) \subset GL_3(\mathbb{R})$ under central projection, we have $X = \Gamma \backslash SL_3(\mathbb{R})/SO(3)$.

2.2. Lie groups and algebras. We define $A$ to be the subgroup of $SL_3(\mathbb{R})$ consisting of diagonal matrices with positive entries, and let $Z_A$ be the centraliser of $A$ in $M_3(\mathbb{R})$. We define $N$ to be the subgroup of strictly upper triangular matrices. We denote the Lie algebras of $N$, $A$, and $SO(3)$ by $n$, $a$ and $\mathfrak{k}$ respectively. We denote the roots of $a$ in $\mathfrak{g}$ by $\Delta$, and the set of positive roots corresponding to $n$ by $\Delta^+$. We denote the set of regular and singular points in $a$ and $a^*$ by $a_r$, $a_s$, etc. We write the Iwasawa decomposition as

$$g = n(g) \exp(A(g))k(g) = \exp(N(g)) \exp(A(g))k(g).$$

We let $M$ and $M'$ be the centraliser and normaliser of $a$ in $SO(3)$, and define the Weyl group $W = M'/M$.

We equip $M_3(\mathbb{R})$ with the standard Euclidean norm as a 9-dimensional vector space, which we denote by $\| \cdot \|$. We obtain a positive definite norm on $\mathfrak{g}$ from $\| \cdot \|$ under the natural restriction, as well as a norm on $a^*$ by duality. We shall also denote these norms by $\| \cdot \|$, the particular one we are using will be clear from the context. We let $B_a$, $B_\mathfrak{a}$ and $B_\mathfrak{k}$ be the unit balls in $n$, $a$, and $\mathfrak{k}$ with respect to $\| \cdot \|$. We define a flat disk in $S$ to be any subset of the form $g \exp(B_\mathfrak{a})$ for $g \in SL_3(\mathbb{R})$. The Killing form on $\mathfrak{g}$ will always be denoted by $\langle \cdot, \cdot \rangle$, and we give $S$ the Riemannian structure determined by $\langle \cdot, \cdot \rangle$.

2.3. Hecke algebras. If $p \not\in S_f$, we let $\mathcal{H}_p$ denote the space of functions in $C_0^\infty(GL_3(\mathbb{Q}_p))$ that are bi-invariant under $K_p$. It is an algebra under convolution with respect to the measure $dg_p$. If $(a, b, c) \in \mathbb{Z}^3$, we define $K_p(a, b, c)$ to be the double coset

$$K_p(a, b, c) = K_p \left( \begin{array}{ccc} p^a & & \\ & p^b & \\ & & p^c \end{array} \right) K_p$$

and let $\Phi_p(a, b, c)$ be the characteristic function of $K_p(a, b, c)$.

If $(a, b, c) \in \mathbb{Q}^3$, define $K(a, b, c)$ to be

$$K(a, b, c) = \bigotimes_{p \in S_f} K_p \otimes \bigotimes_{p \not\in S_f} K_p(\text{ord}_p(a), \text{ord}_p(b), \text{ord}_p(c)),$$

and let $\Phi(a, b, c)$ be the characteristic function of $K(a, b, c)$. If $\gamma \in D^* \cap K(a, b, c)$ we have $\text{nr}(\gamma) = \pm abc$, and we define

$$R(a, b, c) = \{ \gamma \in D^* \cap K(a, b, c) | \text{nr}(\gamma) > 0 \},$$

so that $R(a, b, c) \subset R$ and $R(1, 1, 1) = R^1$. It may be seen that $R^1 \backslash R(a, b, c)$ is finite for all $(a, b, c) \in \mathbb{Q}^3$. The map $\overline{\phi}_\infty : R(a, b, c) \to SL_3(\mathbb{R})$ given by composing $\phi_\infty$ with central projection is an injection, and we define the Hecke operator $T(a, b, c)$ on functions on $X$ by

$$T(a, b, c) f(x) = \sum_{\gamma \in R^1 \backslash R(a, b, c)} f(\overline{\gamma}_\infty x).$$
It may be seen that $T(a, b, c)$ is the same as the operator of right convolution by $\Phi(a, b, c)$, and that the adjoint of $T(a, b, c)$ is $T(-a, -b, -c)$. It follows that the operators $T(a, b, c)$ are normal and commute with one another.

2.4. Maass forms. Let $\pi = \otimes_v \pi_v$ be an automorphic representation of $D^\times(A)$ that is unramified everywhere with respect to $K$ and has trivial central character. Let $\psi \in \pi$ be a $K$-fixed vector, so that $\psi$ defines a function on $X$ that is an eigenfunction of the ring of invariant differential operators and of the Hecke operators defined above. We let the $L$-function of $\psi$ be

$$L(s, \psi) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}.$$  

It is known that

(2) $T(p, 1, 1)\psi = pa(p)\psi$

for $p \notin S_f$, see for instance Proposition 7.2 of [5] or Section 6.4 of [9]. The bound

(3) $\sum_{n \leq X} |a(n)|^2 \ll \epsilon X^{1+\epsilon t^\epsilon}$

follows from Corollary 2 of [3] or Theorem 2 of [11].

3. Constructing an amplifier

We now construct the amplifier that we shall apply to our Maass form, using the following relation between the Hecke operators on $GL_3$.

Lemma 3.1. We have

$$\Phi_p(1, 0, 0) * \Phi_p(1, 1, 0) = \Phi_p(2, 1, 0) + (p^2 + p + 1)\Phi_p(1, 1, 1)$$

Proof. As $\Phi_p(1, 0, 0) * \Phi_p(1, 1, 0)$ must be supported on those double cosets $K_p(a, b, c)$ with $a + b + c = 3$ and $a, b, c \geq 0$, we must have

(4) $\Phi_p(1, 0, 0) * \Phi_p(1, 1, 0) = a\Phi(1, 1, 1) + b\Phi_p(2, 1, 0) + c\Phi_p(3, 0, 0)$

for some $a, b$, and $c \in \mathbb{R}$. Taking adjoints gives

$$\Phi_p(-1, 0, 0) * \Phi_p(-1, -1, 0) = a\Phi(-1, -1, -1) + b\Phi_p(-2, -1, 0) + c\Phi_p(-3, 0, 0),$$

and it follows that $c = 0$ as there can be no matrices whose entries have denominator $p^3$ in the support of $\Phi_p(-1, 0, 0) * \Phi_p(-1, -1, 0)$.

If we translate (4) by $p^{-1}I$ we obtain

(5) $\Phi_p(1, 0, 0) * \Phi_p(-1, 0, 0) = a\Phi(0, 0, 0) + b\Phi_p(1, 0, -1),$ 

and it may be easily shown that

\[4\]
\[ \mu(K_p(1,0,0)) = p^2 + p + 1, \quad \mu(K_p(1,0,-1)) = (p^2 + p)(p^2 + p + 1). \]

Evaluating (5) at the identity gives
\[ a = \Phi_p(1,0,0) \ast \Phi_p(-1,0,0)(e) = \|\Phi_p(1,0,0)\|_2^2 = p^2 + p + 1. \]

Integrating (5) over \(GL_3(\mathbb{Q}_p)\) gives
\[ \mu(K_p(1,0,0))^2 = a + b\mu(K_p(1,0,-1)), \]
and it follows that \(b = 1\).

Lemma 3.1 and (2) imply that if we define
\[ T_p = p^{-1}T(p,p,1)a(p) - p^{-2}T(p^2,p,1), \]
then when \(p \notin S_f\) we have
\[ T_p\psi = \frac{p^2 + p + 1}{p^2}\psi. \]

We shall need the following bound for the coefficients in the expansion of \(T_p T^*_p\).

**Lemma 3.2.** Write
\[ T^*_p T_p = \sum_{a \geq b \geq c} \alpha(a,b,c) T(p^a, p^b, p^c). \]
If \(\alpha(a,b,c) \neq 0\) then we have \(|a|, |b|, |c| \leq 2\), and
\[ \alpha(a,b,c) \ll p^{c-a}(1 + |a(p)|^2) \]
where the implied constant is independent of \(p\).

**Proof.** If we define
\[ \Phi_p = p^{-1}\Phi_p(1,1,0)|a(p)| + p^{-2}\Phi_p(2,1,0) \]
and write
\[ \Phi^*_p \ast \Phi_p = \sum_{a \geq b \geq c} \beta(a,b,c) \Phi_p(a,b,c), \]
then we have \(\beta(a,b,c) \geq 0\) and \(|\alpha(a,b,c)| \leq \beta(a,b,c)\). The condition that \(|a|, |b|, |c| \leq 2\) follows by considering denominators as in Lemma 3.1.

To bound \(\beta(a,b,c)\), we use the spherical transform. Define \(A_p : GL_3(\mathbb{Q}_p) \rightarrow \mathbb{Z}^3\) to be the \(p\)-adic Iwasawa \(A\) co-ordinate, and let
\[ \rho : \mathbb{Z}^3 \rightarrow \mathbb{Z} \]
\[ (a,b,c) \mapsto a - c \]
be the half sum of the positive roots. We define the function \( \varphi_0 \) by

\[
\varphi_0(g) = \int_{K_p} p^{\rho(A(kg))} dg_p
\]

so that \( \varphi_0 \) is the spherical function with trivial Satake parameter. It follows from Proposition 7.2 of [5] that

\[
\Phi_p(1, 0, 0) \varphi_0 = \Phi_p(1, 1, 0) \varphi_0 = 3p \varphi_0,
\]

and combining this with Lemma 3.1 gives

\[
(\Phi_p * \Phi_p) \varphi_0 = (3|a(p)| + 8 - p^{-1} - p^{-2})^2 \varphi_0.
\]

Taking only one term in the expansion (9) and evaluating at the identity gives

\[
\beta(a, b, c)(\Phi_p(a, b, c) \varphi_0)(e) \leq (3|a(p)| + 8 - p^{-1} - p^{-2})^2.
\]

We have

\[
(\Phi_p(a, b, c) \varphi_0)(e) = \int_{K_p(a,b,c)} \varphi_0(g) dg
\]

\[
= \int_{K_p(a,b,c)} p^{\rho(A(kg))} dg
\]

\[
\geq p^{a-c},
\]

and (8) now follows.

□

4. Amplification of periods along flats

We now prove Proposition 1.2. We let the spectral parameter of \( \psi \) be \( t \lambda \), where \( t \in \mathbb{R}_{>0} \) and \( \lambda \in B^* \), and \( B^* \) is a fixed compact subset of \( a^* \). Fix a real-valued positive function \( h \in S(a^*) \) of Payley-Wiener type that is \( >1 \) on \( B_a \). Define \( h_t \) by

\[
h_t(\mu) = \sum_{w \in W} h(w \mu + t \lambda),
\]

and let \( k_t^0 \) be the \( K \)-invariant function on \( S \) with Harish-Chandra transform \( h_t \) (see [8] for definitions). The Payley-Wiener theorem of Gangolli [7] implies that \( k_t^0 \) is of compact support that is uniform in \( t \lambda \), and may be chosen to be arbitrarily small. Let \( K_t^0 \) be the point-pair invariant on \( S \) associated to \( k_t^0 \), which satisfies \( K_t^0(x, y) = K_t^0(y, x) \). Let \( A_t^0 \) the operator on \( X \) with integral kernel

\[
A_t^0(x, y) = \sum_{\gamma \in \Gamma} K_t^0(x, \gamma y).
\]

It follows that \( A_t^0 \) is a self-adjoint approximate spectral projector onto the eigenfunctions in \( L^2(X) \) with spectral parameter near \( t \lambda \). Let \( k_t \) be the \( K \)-invariant function on \( S \) with Harish-Chandra transform \( h_t^2 \), and let \( K_t \) and \( A_t \) be associated to \( k_t \) in the same way. It follows that \( A_t = (A_t^0)^2 \).
It suffices to estimate $\langle \psi, b\phi \rangle$ for $\phi \in H_\beta$ with $\|\phi\|_2 = 1$. Let $1 \leq N \leq t$ be an integer to be chosen later, and define $T$ to be the Hecke operator

$$T = \sum_{1 \leq p \leq N} T_p$$

where $T_p$ is as in (6). Equation (7) and our construction of $A^0_t$ imply that

$$TA^0_t \psi = C^0_t \psi$$

with $C^0_t \in \mathbb{R}$ and $C^0_t \gg N^{1-\epsilon}$. We shall estimate $\langle \psi, b\phi \rangle$ by estimating $\langle TA^0_t \psi, b\phi \rangle$. We first take adjoints to obtain $\langle TA^0_t \psi, b\phi \rangle = \langle \psi, T^*A^0_t b\phi \rangle$, where $T^*A^0_t b\phi$ is the function on $X$ given by

$$A^0_t b\phi(y) = \int_a A^0_t(y, E(H))b(H)\phi(H)dH.$$ We then apply Cauchy-Schwarz to obtain

$$|\langle \psi, T^*A^0_t b\phi \rangle| \leq \langle T^*A^0_t b\phi, T^*A^0_t b\phi \rangle^{1/2}$$

$$= \langle b\phi, TT^*A^0_t b\phi \rangle^{1/2}.$$ (11)

We shall estimate (11) with the aid of two propositions. The first says that the only significant contribution to (11) comes from those isometries $\pi^-_{\infty}$ in $TT^*$ for which $\pi^-_{\infty} E$ and $E$ are close. To define the notion of close that we shall use, if $E_1$ and $E_2$ are a pair of flat disks in $S$ we define

$$d(E_1, E_2) = \inf\{d(p, q) | p \in E_1, q \in E_2\},$$

where $d(p, q)$ is the distance function on $S$. If we let $E_i = g_i \exp(B_a)$, then the condition that $d(E_1, E_2) \leq C$ for some $C$ implies that $g_1^{-1}g_2$ lies in a compact subset $D \subset SL_3(\mathbb{R})$. The second distance function we define is

$$n(E_1, E_2) = \inf\{\|g_1^{-1}g_2 - x\| | x \in Z_A\}.$$ In particular, $n(E_1, E_2) = 0$ if $g_1^{-1}g_2 \in MA$, so that the infinite extensions of $E_1$ and $E_2$ coincide and have the same orientations.

If we define

$$I(t, \phi, E_1, E_2) = \int_a b(H_1)b(H_2)\phi(H_1)\overline{\phi(H_2)}K_t(E_1(H_1), E_2(H_2))dH_1dH_2,$$

then the contribution to (11) from an isometry $\pi^-_{\infty}$ is $I(t, \phi, E, \pi^-_{\infty} E)$. We assume that $k_t$ is supported in the ball of radius 1 about the identity, so that $I(t, \phi, E_1, E_2) = 0$ unless $d(E_1, E_2) \leq 1$. We have the following bounds for $I(t, \phi, E_1, E_2)$.

**Proposition 4.1.** Suppose $d(E_1, E_2) \leq 1$, and let $\epsilon > 0$. We have

$$|I(t, \phi, E_1, E_2)| \ll \epsilon t^{3/2+\epsilon} \beta^{3/2}$$

for all $E_1$ and $E_2$, while if $n(E_1, E_2) \geq t^{-1/2+\epsilon} \beta^{1/2}$ we have

$$|I(t, \phi, E_1, E_2)| \ll \epsilon t^{3/2+\epsilon} \beta^{3/2}$$

for all $E_1$ and $E_2$. 




The second proposition bounds the number of isometries that map $E$ close to itself. We define the counting function

$$M(E, n, \kappa) = \sum_{(a,b,c) \in \mathbb{Z}^3 \atop abc = n} |\{ \gamma \in R(a,b,c) \mid d(E, \overline{\gamma} E) \leq 1, n(E, \overline{\gamma} E) \leq \kappa \}|.$$ 

We may prove the following bound for $M(E, n, \kappa)$. Note that we explicate which of our bounds depend on $\Gamma$ in Proposition 4.2 and Lemma 4.3.

**Proposition 4.2.** There is an absolute $C > 0$ and a $C_1 > 0$ depending on $\Gamma$ such that $M(E, n, C_1 n^{-C}) \ll \Gamma, \epsilon, n$, uniformly in $E$.

**Proof.** As $M(E, n, \kappa) = M(\gamma E, n, \kappa)$ for $\gamma \in \Gamma$, we may assume that $E$ lies in a compact subset of $S$, and hence that $E = g \exp(B_a)$ for $g$ lying in a compact subset $K_1 \subset SL_3(\mathbb{R})$ depending on $\Gamma$. Suppose that $\gamma$ contributes to $M(E, n, \kappa)$. The assumption that $d(E, \overline{\gamma} E) \leq 1$ implies that $g^{-1} \overline{\gamma} g$ also lies in a compact subset of $SL_3(\mathbb{R})$, and combining these gives that $\overline{\gamma}$ lies in a compact subset $K_2$. As $n r(\gamma) = n$, we have $\gamma \in n^{1/3} K_2$ and hence $\|\gamma\| \ll \Gamma n^{1/3}$.

The assumption $n(E, \overline{\gamma} E) \leq \kappa$ gives

$$\inf \{ \|\gamma - x\| \mid x \in gZAg^{-1} \} \leq C_n\kappa n^{1/3}$$

for some $C'$ depending only on $K_1$, and hence $\Gamma$. Lemma 4.3 below implies that there is an absolute $C > 0$ and $C_1 > 0$ depending on $\Gamma$ such that if $\kappa \leq C_1 n^{-C}$, then all $\gamma$ satisfying (13) must lie in a cubic subfield $F \subset D$.

As $\gamma \in R \cap F$, $\gamma$ must lie in the ring of integers $O_F$ of $F$, and as the reduced norm $n r$ on $D$ agrees with the norm $N$ on $F$ we must have $N(\gamma) = n$. The condition $\gamma \in n^{1/3} K_2$ implies that the image of $\gamma$ under all archimedean embeddings of $F$ must be $\ll \Gamma n^{1/3}$, and the number of algebraic integers in $F$ satisfying these conditions may easily be seen to be $\ll \Gamma, \epsilon, n'$, uniformly in $F$.

The following lemma is a small modification of Lemma 4.1 of [16]. We thank the authors for permission to reproduce it here.

**Lemma 4.3.** Let $S \subset M_3(\mathbb{R})$ be an $\mathbb{R}$-subalgebra isomorphic to $\mathbb{R}^3$. For $c > 0$ larger than an absolute constant, and $c' > 0$ sufficiently small depending on $D$ and $\phi_{\infty}$, the set of $x \in R$ satisfying

$$\|x\| \leq X, \quad \inf \{ \|x - s\| \mid s \in S \} \leq \epsilon$$

is contained in a subalgebra $F \subset D$ of dimension at most 3 as long as

$$\epsilon X^c < c'.$$
Proof. There is a polynomial function \( P : D^4 \to \mathbb{Q} \), with integral coefficients with respect to \( R \), such that \( P(\alpha_1, \ldots, \alpha_4) = 0 \) exactly when \( \alpha_1, \ldots, \alpha_4 \) span a linear subspace of dimension \( \leq 3 \). For example, one may use the sum of the squares of the minors of a suitable matrix.

Take \( x_1, \ldots, x_4 \) belonging to the set defined by \([13]\). There are \( y_1, \ldots, y_4 \) such that \( \| x_i - y_i \| \leq \epsilon \), and so \( P(x_1, \ldots, x_4) \ll X^{O(1)} \epsilon \). On the other hand, if \( P(x_1, \ldots, x_4) \neq 0 \) then we must have \( |P(x_1, \ldots, x_4)| \geq 1 \) by the condition that \( P \) had integral coefficients with respect to \( R \). It follows that if a condition of the form \([16]\) holds for \( c \) and \( c' \) as stated, then \( x_1, \ldots, x_4 \) span a \( \mathbb{Q} \)-linear space of dimension at most 3.

Now let \( F \) be the \( \mathbb{Q} \)-algebra generated by those \( x \) satisfying \([15]\). It is clear that \( F \) is in fact spanned by monomials in such \( x \) of length at most \( 9 = \dim_{\mathbb{Q}} D \). Each such monomial \( y \) satisfies the bounds

\[
\| y \| \ll X^{O(1)}, \quad \inf\{\| x_\infty - s \| | s \in S \} \leq X^{O(1)} \epsilon.
\]

After increasing \( c \) and decreasing \( c' \) as necessary, it follows that \( F \) also has dimension at most 3.

With these results, we may now estimate \([11]\). We have

\[
TT^* = \sum_{p,q \leq N} T_p T_q^* + \sum_{p \leq N} T_p T_p^*.
\]

The action of \( T_q^* \) on \( X \) is the same as that of

\[
T_q' = q^{-1}a(q)T(q, 1, 1) - q^{-2}T(q^2, q, 1),
\]

and we have

\[
T_p T_q' = (pq)^{-1}a(p)a(q)T(pq, p, 1) - p^{-2}q^{-1}a(q)T(p^2q, p, 1) - p^{-1}q^{-2}a(p)T(pq^2, pq, 1) + (pq)^{-2}T(p^2q^2, pq, 1).
\]

We shall describe how to deal with the first term in this expansion, the remainder being identical. We have

\[
\langle b_\phi, T(pq, p, 1)A_\varphi b_\phi \rangle = \sum_{\gamma \in R(pq, p, 1)} I(t, \phi, E, \gamma_\infty E).
\]

Proposition \([41]\) implies that we only need to consider those \( \gamma \) for which \( d(E, \gamma_\infty E) \leq 1 \) and \( n(E, \gamma_\infty E) \leq t^{-1/2+\epsilon} \beta^{1/2} \). Proposition \([42]\) implies that there is a \( \delta > 0 \) such that if \( N \leq t^\delta \) then

\[
M(E, p^2q, t^{-1/2+\epsilon} \beta^{1/2}) \ll \epsilon N^\epsilon.
\]

We therefore have \( \langle b_\phi, T(pq, p, 1)A_\varphi b_\phi \rangle \ll \epsilon N^\epsilon t^{3/2+\epsilon} \beta^{9/2} \). Bounding the other terms in the same way gives

\[
\langle b_\phi, T_p T_q' A_\varphi b_\phi \rangle \ll \epsilon N^\epsilon (pq)^{-1}(1 + |a(p)|)(1 + |a(q)|)t^{3/2+\epsilon} \beta^{9/2}.
\]

Summing over \( p \) and \( q \) and applying \([3]\) gives
Our assumptions that \( \text{inj rad}(\langle \cdot \rangle) \leq N^{1/2+\varepsilon} \beta^{3/2} \) to the sum, while the contribution from the other terms is smaller. Summing over \( N \) and applying (3) again, we obtain

\[
\sum_{p \leq N} \langle b\phi, T_p T_p^* A_t b\phi \rangle \ll \varepsilon N^{1+\varepsilon} t^{3/2+\varepsilon} \beta^{9/2}.
\]

Combining (17) and (18) gives \( \langle b\phi, T T^* A_t b\phi \rangle \ll \varepsilon N^{1+\varepsilon} t^{3/2+\varepsilon} \beta^{9/2} \), and when we combine this with (11) and (10) we have \( \langle \psi, b\phi \rangle \ll \varepsilon N^{-1/2+\varepsilon} t^{3/4+\varepsilon} \beta^{9/4} \). Choosing \( N = t^\delta \) and \( \beta = t^{5/10} \) completes the proof.

5. Bounds away from the spectrum

We now give the proof of Proposition 1.3. We are free to assume that \( \beta \geq t' \), as otherwise the result follows from the local bound \( \|\psi\|_2 \ll t^{3/4} \). As we will not be using Hecke operators, we are free to replace \( \Gamma \) by a finite index sublattice with \( \text{inj rad}(\langle \cdot \rangle) \geq 10 \). It suffices to estimate \( \langle \psi, b\phi \rangle \) for \( \phi \in H_{\beta}^\perp \) with \( \|\phi\|_2 = 1 \). If \( k_t, K_t \) and \( A_t \) are as before, it again follows that

\[ |\langle A_t^0 \psi, b\phi \rangle| \leq \langle b\phi, A_t b\phi \rangle^{1/2}, \]

where

\[ \langle b\phi, A_t b\phi \rangle = \int \int a_b(H_1) b(H_2) \phi(H_1) \overline{\phi(H_2)} K_t(E(H_1), E(H_2)) dH_1 dH_2. \]

Our assumptions that \( \text{inj rad}(X) \geq 10 \) and \( k_t \) is supported in a ball of radius 1 imply that only the term \( \gamma = e \) makes a contribution to the inner sum, so that

\[ \langle b\phi, A_t b\phi \rangle = \int \int a_b(H_1) b(H_2) \phi(H_1) \overline{\phi(H_2)} K_t(E(H_1), E(H_2)) dH_1 dH_2. \]

We have \( K_t(E(H_1), E(H_2)) = k_t(\exp(H_1 - H_2)) \). Therefore, if we define \( p_t(H) = k_t(\exp(H)) \) and let \( P_t \) be the operator on \( a \) with integral kernel \( P_t(H_1, H_2) = p_t(H_1 - H_2) \), we have \( \langle b\phi, A_t b\phi \rangle = \langle b\phi, P_t b\phi \rangle \).

For \( \nu \in a^* \) and \( C \in \mathbb{R}, \) we define \( B(W \nu, C) \) to be the union of the balls of radius \( C \) around the points in \( W \nu \). Write \( b\phi = \phi_1 + \phi_2 \), where the Fourier transform of \( \phi_2 \) is supported on \( B(W t \lambda, \beta/2) \) and the transform of \( \phi_1 \) is supported on \( a \setminus B(W t \lambda, \beta/2) \). Because \( b \) was a fixed smooth function, we have \( \|\phi_2\|_2 \ll \beta^{-A} \ll_{\varepsilon, A} t^{-A} \). Because the kernel of \( P_t \) is translation invariant, we have

\[ \langle b\phi, P_t b\phi \rangle = \langle \phi_1, P_t \phi_1 \rangle + \langle \phi_2, P_t \phi_2 \rangle \leq \sup_{\mu \in B(W t \lambda, \beta/2)} |\hat{P}_t(\mu)| + O_{\varepsilon, A}(t^{-A}) \|\hat{P}_t\|_\infty. \]
Proposition [L3] now follows from the lemma below.

**Lemma 5.1.** We have \( \|\hat{p}_t\|_\infty \ll t^{3/2} \), and \( \hat{p}_t(\mu) \ll \epsilon^{3/2+\epsilon} \beta^{-1/2} \) for \( \mu \notin B(Wt\lambda, \beta/2) \).

**Proof.** The first assertion is proven in Section 3 of [13], or can be deduced easily from the bound

\[
2 \leq 3 \prod_{\alpha \in \Delta^+} (1 + t|\alpha(H)|)^{-1/2}
\]

proven in Lemma 2.6 of [13]. We shall prove the second assertion with the aid of two asymptotic formulae for \( \varphi_{tw} \) taken from Theorem 1.3 and 1.4 of [13]. To state them, let \( B \) and \( B_0^* \) be compact subsets of \( a \) and \( a^* \) respectively. For \( H \in a \) we define

\[
\sigma_w(H) = \sum_{\alpha \in \Delta^+} \text{sgn}(w\alpha(H)),
\]

and

\[
d(H) = \inf\{\|H - Z\| | Z \in a_s\}.
\]

If \( H \in B \) and \( \nu \in B_0^* \), we have

\[
(19) \quad \varphi_{tw}(\exp(H)) \ll_{B, B_0^*} \prod_{\alpha \in \Delta^+} (1 + t|\alpha(H)|)^{-1/2},
\]

and there are functions \( C_w \in C^\infty(a \times a^*) \) for \( w \in W \) such that

\[
(20) \quad \varphi_{tw}(\exp(H)) = t^{-3/2} \prod_{\alpha \in \Delta^+} |\alpha(H)|^{-1/2} \sum_{w \in W} C_w(H, \nu) \exp(it\nu(wH) + i\pi\sigma_w(H)/4)
\]

\[
+ O_{B, B_0^*}(1)d(H)^{-1}t^{-5/2} \prod_{\alpha \in \Delta^+} |\alpha(H)|^{-1/2}.
\]

Let \( \chi \) be the characteristic function of the set of points in \( a \) that are at distance at most 2 from \( a_s \). Fix a function \( b_0 \in C^\infty_0(a) \) with \( \text{supp}(b_0) \subseteq B_0 \), and let \( b_1 = \chi \ast b_0 \). The function \( b_1 \) then has the property that \( b_1(H) = 1 \) if the distance from \( H \) to \( a_s \) is at most 1. Fix a function \( b_2 \in C^\infty_0(a) \) with \( \text{supp}(b_2) \subseteq 2B_0 \) that is equal to 1 on \( B_0 \). We wish to estimate

\[
\hat{p}_t(\mu) = \int_a b_2(H)k_t(\exp(H))e^{-i\mu(H)}dH.
\]

By inverting the Harish-Chandra transform and applying our assumption that \( \beta \geq t^\epsilon \) as in the proof of Lemma 4.4. of [12] or Lemma 2.6 of [13], it suffices to prove the bound

\[
\int_a b_2(H)\varphi_{tw}(\exp(H))e^{-i\mu(H)}dH \ll \epsilon^{3/2+\epsilon} \beta^{-1/2}
\]

uniformly for \( \mu \notin B(W\lambda, \beta/2t) \) and \( \nu \in B(W\lambda, \beta/4t) \). We decompose the integral as

\[
\int_a b_2(H)b_1(t^{-\epsilon}\beta H)\varphi_{tw}(\exp(H))e^{-i\mu(H)}dH + \int_a b_2(H)(1 - b_1(t^{-\epsilon}\beta H))\varphi_{tw}(\exp(H))e^{-i\mu(H)}dH.
\]
The bound \((19)\) implies that the first integral is \(\ll \epsilon^{-3/2+\epsilon} \beta^{-1/2}\) as in Section 2.5 of \([13]\). We shall estimate the second integral by applying \((20)\) with \(B = 2B_\alpha\) and \(B_0^\ast\) containing an open neighbourhood of \(B^\ast\). It may be shown in the same way that

\[
\int_a b_2(H)(1 - b_1(t^{-\epsilon} \beta H)))d(H)^{-1}t^{-5/2} \prod_{\alpha \in \Delta^+} |\alpha(H)|^{-1/2}dH \ll_B \epsilon^{-5/2+\epsilon} \beta^{1/2},
\]

and our assumption that \(\beta \leq t^{1/2}\) implies that this is \(\ll_B \epsilon^{-3/2} \beta^{-1/2}\). It therefore suffices to bound the contribution from the main terms in \((20)\). As these may all be handled in the same way, we shall deal only with the term corresponding to \(w = e\). We wish to estimate

\[
\int_a b_2(H)(1 - b_1(t^{-\epsilon} \beta H)))C_e(H, \nu) \prod_{\alpha \in \Delta^+} |\alpha(H)|^{-1/2} \exp(it(\nu - \mu)(H) + i\pi \sigma_e(H)/4)dH.
\]

After changing variable from \(H\) to \(t^{-\epsilon} \beta H\) this becomes

\[
t^{\epsilon/2} \beta^{1/2} \int_a b_2(t^{\epsilon} \beta^{-1} H)(1 - b_1(H)))C_e(t^{\epsilon} \beta^{-1} H, \nu) \prod_{\alpha \in \Delta^+} |\alpha(H)|^{-1/2} \exp(it^{1+\epsilon} \beta^{-1}(\nu - \mu)(H) + i\pi \sigma_e(H)/4)dH.
\]

Our assumptions on \(\mu\) and \(\nu\) imply that \(t^{1+\epsilon} \beta^{-1}\|\nu - \mu\| \geq t^{\epsilon}/4\), and the assumption that \(\beta \geq t^{\epsilon}\) implies that all derivatives of the amplitude factors in the above integral are bounded. Integration by parts then implies that the integral is \(\ll_{\epsilon, A} t^{-A}\), which completes the proof.

\[
\square
\]

6. Oscillatory Integrals

We now prove the bounds of Proposition \((4.1)\) for \(I(t, \phi, E_1, E_2)\). In this section, \(t\) and \(\beta\) will be real positive quantities satisfying \(1 \leq \beta \leq t^{1/2}\). \(B^\ast\) will denote a fixed compact subset of \(\alpha^\ast\) that is not necessarily the same as the set \(B^\ast\) used earlier. We set \(K = SO(3)\). We begin by calculating a certain derivative of the function \(A\).

**Lemma 6.1.** Let \(g \in SL_3(\mathbb{R})\) have Iwasawa decomposition \(g = nak\). If \(H_1, H_2 \in \alpha\), then

\[
\frac{\partial}{\partial s} \langle H_1, A(g \exp(sH_2)) \rangle \bigg|_{s=0} = \langle H_1, \text{Ad}(k)H_2 \rangle.
\]

**Proof.** We have

\[
\frac{\partial}{\partial s} \langle H_1, A(g \exp(sH_2)) \rangle \bigg|_{s=0} = \frac{\partial}{\partial s} \langle H_1, A(k \exp(sH_2)) \rangle \bigg|_{s=0} = \frac{\partial}{\partial s} \langle H_1, A(\exp(s\text{Ad}(k)H_2)) \rangle \bigg|_{s=0} = \langle H_1, \text{Ad}(k)H_2 \rangle.
\]

\[
\square
\]
For \( g \in SL_3(\mathbb{R}) \), let \( \Phi_g : K \to K \) be the map sending \( k \) to \( k(kg) \). The smoothness of the Iwasawa decomposition implies that \( \Phi_g \) is a diffeomorphism that depends smoothly on \( g \). When \( g \in NA \) it may be seen that \( \Phi_g(m) = m \) and \( \Phi_g(mk) = m\Phi_g(k) \) for \( m \in M \).

We now estimate two integrals that comprise \( I(t, \phi, E_1, E_2) \).

Proposition 6.2. Fix \( C, \epsilon > 0 \), a compact set \( D_B \subset NA \), and a function \( b \in C_0^{\infty}(a) \) with \( \text{supp}(b) \subseteq B_a \). If \( k \in K \) and \( \lambda, \nu \in B^* \) satisfy

\[
(21) \quad k \notin M \exp(Ct^{-1/2+\epsilon} \beta^{1/2} B_t) \quad \text{and} \quad \| \lambda - \nu \| \leq \beta/t,
\]

and \( g \in D_B \), then we have

\[
(22) \quad \int_a b(H) \exp(it\lambda(H) - it\nu(A(kg \exp(H)))) dH \ll_A t^{-A}.
\]

The implied constant depends on \( B^*, D_B, C, \epsilon \), and the size of the derivatives of \( b \) of order at most \( n \), where \( n \) depends on \( \epsilon \).

Proof. Define

\[
\psi(H, k, g) = \lambda(H) - \nu(A(kg \exp(H)))
\]
to be the phase of the integral (22). Lemma 6.1 implies that

\[
H_\nu \psi(H, k, g) = \langle H_\nu, H_\lambda \rangle - \langle H_\nu, \text{Ad}(\Phi_g \exp(H)(k))H_\nu \rangle
\]

\[
= \langle H_\nu, H_\lambda - H_\nu \rangle + \langle H_\nu, (1 - \text{Ad}(\Phi_g \exp(H)(k)))H_\nu \rangle.
\]

(23)

It is proven in Proposition 5.4 of [6] that the function \( \langle H_\nu, \text{Ad}(k)H_\nu \rangle \) has a critical point at the identity, and it follows from Proposition 6.5 of [6] or Proposition 4.4 of [13] that the Hessian at this critical point is negative definite, uniformly for \( \nu \in B^* \). It follows that there is an open neighbourhood \( 0 \in U_{k1} \subseteq \mathfrak{k} \) such that for \( X \in U_{k1} \) we have

\[
(24) \quad \langle H_\nu, H_\nu \rangle - \langle H_\nu, e^{\text{ad}(X)}H_\nu \rangle \sim \| X \|^2,
\]

and \( \exp \) gives a diffeomorphism \( U_{k1} \to U_1 := \exp(U_{k1}) \). Let \( 0 \in U_\nu \subseteq \mathfrak{k} \) be an open set such that \( \exp \) gives a diffeomorphism \( U_\nu \to U := \exp(U_\nu) \), and

\[
U \subseteq \bigcap_{\Phi_g(B_a)} \Phi_g^{-1}(U_1).
\]

Assume that \( k \notin MU \). Theorem 8.2 of [10] implies that \( \langle H_\nu, \text{Ad}(k)H_\nu \rangle \leq \langle H_\nu, H_\nu \rangle \) with equality iff \( k \in M \), so that we have

\[
\langle H_\nu, H_\nu \rangle - \langle H_\nu, \text{Ad}(k)H_\nu \rangle > \delta > 0.
\]

The condition (21) implies that

\[
|\langle H_\nu, H_\lambda - H_\nu \rangle| \ll \beta/t \ll t^{-1/2},
\]

so that \( H_\nu \psi \geq \delta/2 \) for \( t \) sufficiently large. The bound (22) now follows by integration by parts.
Now assume that $k \in MU$. As $\psi(H, mk, g) = \psi(H, k, g)$ for $m \in M$, we may assume that $k = \exp(X) \in U$ with $X \in U_t$. The assumption \((21)\) implies that

\[(26) \quad \|X\|^2 \geq C^2 t^{-1+2\epsilon} \beta.\]

Define $X(H, k, g) \in U^1_t$ by requiring that $\Phi_{\exp(H)}(k) = \exp(X(H, k, g))$. Because $\Phi_g$ is a smoothly varying family of diffeomorphisms, we have $\|X(H, k, g)\| \sim \|X\|$, and hence \((24)\) gives

\[\langle H_\nu, (1 - \Ad(\Phi_g(\exp(H)))H_\nu) \rangle \gg \|X\|^2.\]

It follows by combining this with \((25)\) and \((26)\) that $H_\nu \psi \gg \|X\|^2$. Combining \((28)\) and \((24)\) gives

\[H_\nu \psi(H, k, g) - \langle H_\nu, H_\lambda - H_\nu \rangle \ll \|X\|^2,\]

which implies that $H_\nu \psi(H, k, g) - \langle H_\nu, H_\lambda - H_\nu \rangle$ vanishes to second order at $k = e$. It follows that

\[H_\nu^n \psi(H, k, g) \ll_n \|X\|^2\]

for $n \geq 2$. As $t\|X\|^2 \geq C^2 t^{2\epsilon} \beta$, the result now follows by integration by parts with respect to $H_\nu$.

\[\square\]

**Proposition 6.3.** Fix $C$, $\epsilon > 0$, a compact set $D_B \subset NA$, and a function $b \in C_0^\infty(a)$ with $\text{supp}(b) \subseteq B_a$. If $g = na \in D_B$ and $\lambda, \nu \in B^*$ satisfy

\[n \notin \exp(C t^{-1/2+\epsilon} \beta^{1/2} B_a) \quad \text{and} \quad \|\lambda - \nu\| \leq \beta/t,\]

then

\[(27) \quad \int_a b(H) e^{it\lambda(H)} \varphi_{-\nu}(g \exp(H)) dH \ll_A t^{-A}.\]

The implied constant depends on $B^*$, $D_B$, $C$, $\epsilon$, and the size of the derivatives of $b$ of order at most $n$, where $n$ depends on $\epsilon$.

**Proof.** If we substitute the formula for $\varphi_{-\nu}$ as an integral of plane waves into \((27)\), it becomes

\[\int_a \int_K b(H) \exp(it\lambda(H) + (\rho - it\nu)(A(kg \exp(H)))) dk dH.\]

Choose a function $f \in C_0^\infty(\mathfrak{t})$ with $\text{supp}(f) \subseteq 2B_\epsilon$ and $f(X) = 1$ for $X \in B_\epsilon$. Let $C_1 > 0$ be a constant to be chosen later, define $b_1 \in C^\infty(K)$ to be the pushforward of $f(C_1^{-1}t^{1/2-\epsilon} \beta^{-1/2}X)$ under exp, and define $b_2$ by $b_2(k) = 1 - \sum_{m \in M} b_1(mk)$ so that

\[
\text{supp}(b_1) \subseteq \exp(2C_1 t^{-1/2+\epsilon} \beta^{1/2} B_\epsilon), \\
\text{supp}(b_2) \subseteq K \setminus M \exp(C_1 t^{-1/2+\epsilon} \beta^{1/2} B_\epsilon).
\]

Proposition 6.2 implies that
\[
\int_{a}^{b} \int_{K} b_2(k)b(H) \exp(it\lambda(H) + (\rho - it\nu)(A(kg \exp(H))))dkdH \ll t^{-A}.
\]

It therefore suffices to estimate

\[
\int_{a}^{b} \int_{K} b_1(mk)b(H) \exp(it\lambda(H) + (\rho - it\nu)(A(kg \exp(H))))dkdH
\]

for \( m \in M \), and we assume without loss of generality that \( m = e \). We shall do this by estimating the integrals

\[
\int_{K} b_1(k) \exp(-it\nu(A(kg)))dk,
\]

where we have absorbed \( \exp(H) \) into \( g \) after enlarging \( D_B \), and absorbed the factor \( \exp(\rho(A(kg \exp(H)))) \) into \( b_1(k) \) and suppressed the dependence on \( H \) and \( g \).

We define the phase functions

\[
\psi(k, g) = \nu(A(kg)), \quad k \in K, g \in NA,
\]

and

\[
\psi_S(x) = \nu(A(x)), \quad x \in S.
\]

If \( X \in \mathfrak{g} \), we let \( X^S \) be the vector field on \( S \) whose value at \( x \in S \) is \( \frac{\partial}{\partial t} \exp(tX)x|_{t=0} \). It may be shown that these vector fields satisfy \([X^S, Y^S] = -[X, Y]^S\), where the first Lie bracket is on \( S \) and the second is in \( \mathfrak{g} \). We choose \( X_\alpha \in \mathfrak{g}_\alpha \) for each \( \alpha \in \Delta \) so that the relations \( X_{-\alpha} = \theta X_\alpha \) and \( \langle X_\alpha, X_{-\alpha} \rangle = -1/2 \) are satisfied, where \( \theta \) is the Cartan involution on \( \mathfrak{g} \). If \( \alpha \in \Delta^+ \), we let \( K_\alpha = X_\alpha + X_{-\alpha} \in \mathfrak{t} \). It may be seen that the fields \( \{X_\alpha^S \mid \alpha \in \Delta^+\} \) form a basis for the normal bundle to \( A \) in \( S \).

Proposition 5.4 of [6] implies that the set of \( x \in S \) where the functions \( \{K_\alpha^S\psi_S \mid \alpha \in \Delta^+\} \) vanish simultaneously is exactly \( A \). The following lemma shows that these functions in fact form a co-ordinate system transversely to \( A \).

**Lemma 6.4.** If \( \alpha, \beta \in \Delta^+ \) and \( x \in A \), we have \( X_\alpha^S K_\beta^S \psi_S(x) = \delta_{\alpha\beta} \langle \alpha, \nu \rangle \psi_S(x)/2 \).

**Proof.** We have

\[
X_\alpha^S K_\beta^S \psi_S = K_\beta^S X_\alpha^S \psi_S + [X_\alpha^S, K_\beta^S] \psi_S.
\]

The first term vanishes, as \( X_\alpha^S \psi_S \equiv 0 \). If \( \alpha = \beta \) then

\[
[X_\alpha^S, K_\alpha^S] = -[X_\alpha, K_\alpha]^S = \frac{1}{2} H_\alpha^S,
\]

and the lemma follows. If \( \alpha \neq \beta \), then along \( A \) we have

\[
[X_\alpha, K_\beta]^S \in \text{span}(X_\gamma^S \mid \gamma \in \Delta^+).
\]

As the fields \( X_\gamma^S \) annihilate \( \psi_S \), the lemma follows. \( \square \)
Fix a compact set $D_A \subset A$ such that $D_B \subset ND_A$, and a second compact set $D'_A \subset A$ that contains an open neighbourhood of $D_A$. Let $S_n$ be the unit sphere in $n$ with respect to $\| \cdot \|$. It follows from Lemma 6.3 that there exist open sets $U'_\alpha \subset S_n$ for each $\alpha \in \Delta^+$, and $\sigma$, $\delta > 0$ such that the following holds: we have $S_n = \bigcup U'_\alpha$, and if $x \in \exp([0, \sigma)U'_\alpha)D_A$ then $|K^n_\alpha \psi_S(x)| \geq \delta \|N(x)\|$ for each $\alpha$, where $N(x)$ is as in (11). For each $\alpha \in \Delta^+$, we choose a second open set $U_\alpha \subset S_n$ such that $\overline{U_\alpha} \subset U'_\alpha$ and we still have $S_n = \bigcup U_\alpha$.

Suppose that $g \in \exp(\sigma B_n/2)D_A$. Let $\alpha \in \Delta^+$ be such that $g \in \exp([0, \sigma/2)U_\alpha)D_A$. As we have assumed that $\|N(g)\| \geq C t^{-1/2+\epsilon} \beta^{1/2}$, it follows that if $t$ is sufficiently large and $C_1$ is sufficiently small, $k \in \exp(2C_1 t^{-1/2+\epsilon} \beta^{-1/2} B_t)$ implies that $kg \in \exp([0, \sigma)U'_\alpha)D_A$ and $\|N(kg)\| \gg t^{-1/2+\epsilon} \beta^{1/2}$. It follows that

$$|K_\alpha \psi(k, g)| = |K^n_\alpha \psi_S(kg)| \geq \delta \|N(kg)\| \gg t^{-1/2+\epsilon} \beta^{1/2}$$

for $k \in \text{supp}(b_1)$. As we have $K^n_\alpha \psi(k, g) \ll_k 1$, the proposition follows by integration by parts after pulling the integral back to one on $t$ and scaling by $t^{-1/2}$.

If $g \in D_B \setminus \exp(\sigma B_n/2)D_A$, it follows that there exists an $\alpha \in \Delta^+$ and $\delta > 0$ such that $|K_\alpha \psi(k, g)| \geq \delta$ for $k \in \text{supp}(b_1)$ and $t$ sufficiently large. The proposition again follows by integration by parts.

We now combine Propositions 6.2 and 6.3 to prove the following, which will imply Proposition 4.4 after inverting the various transforms.

**Proposition 6.5.** Fix $\epsilon > 0$, and functions $b_1, b_2 \in C^\infty(a)$ with $\text{supp}(b_1) \subseteq B_a$. If $g \in SL_3(\mathbb{R})$ and $\nu, \lambda_1, \lambda_2 \in B^*$ satisfy

$$d(\exp(B_a, g, \exp(B_a))) \leq 1 \quad \text{and} \quad \|\lambda_i - \nu\| \leq \beta/t,$$

then we have

$$\int a b_1(H_1) b_2(H_2) e^{it(\lambda_1(H_1) - \lambda_2(H_2))} \varphi_{-\nu}(\exp(H_1), g \exp(H_2))dH_1dH_2 \ll t^{-3/2+\epsilon} \beta^{3/2}. \quad (29)$$

If we also have $n(\exp(B_a), g \exp(B_a)) \geq t^{-1/2+\epsilon} \beta^{1/2}$, then

$$\int a b_1(H_1) b_2(H_2) e^{it(\lambda_1(H_1) - \lambda_2(H_2))} \varphi_{-\nu}(\exp(H_1), g \exp(H_2))dH_1dH_2 \ll t^{-A}. \quad (30)$$

The implied constants depend only on $B^*$, $\epsilon$, and the size of the derivatives of $b_i$ of order at most $n$, where $n$ depends on $\epsilon$.

**Proof.** We begin by expressing $\varphi_{-\nu}$ as an integral of plane waves. For $y, z \in S$, we have

$$\varphi_{-\nu}(y, z) = \int K_z \exp((\rho - it\nu)(A(k_z y) - A(k_z z)))dk_z \quad (31)$$

where $K_z$ is the stabiliser of $z$. If we let $g_z \in NA$ be the element that maps $z$ to $e$, then $K_z = g_z^{-1}Kg_z$. If we let $k_z = g_z^{-1}kg_z$ with $k \in K$, then we have
If in addition we have

\[ A(g_z^{-1}k g_z y) - A(g_z^{-1}k g_z z) = A(k g_z y) - A(k g_z z) \]

\[ = A(\Phi_{g_z}(k) y) - A(\Phi_{g_z}(k) z). \]

Substituting this into (31) gives

\[ \varphi_{-t\nu}(y, z) = \int_K \exp((\rho - it\nu)(A(\Phi_{g_z}(k) y) - A(\Phi_{g_z}(k) z))) dk \]

\[ = \int_K \exp((\rho - it\nu)(A(k y) - A(k z))) D\Phi^{-1}(k, z) dk, \]

where \( D\Phi^{-1}(k, z) \) is the determinant of the Jacobian of \( \Phi^{-1}(k) \). If we substitute this into the LHS of (29), we obtain the integral

\[ \int_a \int_K b_1(H_1)b_2(H_2)e^{it(\lambda_1(H_1) - \lambda_2(H_2))} \exp((\rho - it\nu)(A(k \exp(H_1)) - A(k g \exp(H_2)))) \]

\[ D\Phi^{-1}(k, g \exp(H_2)) dk dH_1 dH_2. \]

Choose a constant \( C > 0 \). If \( k \notin \exp(Ct^{-1/2+\epsilon} \beta^{1/2} B_t) \), then Proposition 6.2 implies that the integral of (32) with respect to \( H_1 \) is \( \ll_A t^{-A} \). We may therefore restrict the integral over \( K \) to \( \exp(Ct^{-1/2+\epsilon} \beta^{1/2} B_t) \), and this gives the bound (29).

We now assume that \( n(\exp(B_a), g \exp(B_a)) \geq t^{-1/2+\epsilon} \beta^{1/2} \), or that

\[ \|g - x\| \geq t^{-1/2+\epsilon} \beta^{1/2}, \quad x \in Z_A. \]

Let \( g = k_1 n_1 a_1 \). Condition (28) implies that \( n_1 a_1 \) is restricted to a compact set \( D_B \subset NA \). As \( k g = kk_1 n_1 a_1 \), the integral of (32) with respect to \( H_2 \) will also be negligible unless \( kk_1 \in \exp(Ct^{-1/2+\epsilon} \beta^{1/2} B_t) \). Combined with \( k \in \exp(Ct^{-1/2+\epsilon} \beta^{1/2} B_t) \), we see that (32) will be \( \ll_A t^{-A} \) unless \( k_1 \in \exp(2Ct^{-1/2+\epsilon} \beta^{1/2} B_t) \).

If \( C \) is chosen sufficiently small, combining \( k_1 \in \exp(2Ct^{-1/2+\epsilon} \beta^{1/2} B_t) \) with (33) gives an absolute constant \( C_1 > 0 \) such that \( \|n_1 - e\| \geq C_1 t^{-1/2+\epsilon} \beta^{1/2} \). Shrinking \( C \) further if necessary, this implies that \( \exp(B_a) \) and \( g \exp(B_a) \) are separated in the sense that there is an absolute constant \( C_2 > 0 \) such that

\[ d(p, g \exp(B_a)) \geq C_2 t^{-1/2+\epsilon} \beta^{1/2} \]

for all \( p \in \exp(B_a) \). The result now follows by applying Proposition 6.3 to the integral of the LHS of (30) over \( H_2 \) for each fixed \( H_1 \).

\[ \square \]

**Corollary 6.6.** Fix \( \epsilon > 0 \), and functions \( b_1, b_2 \in C_0^\infty(a) \) with \( \text{supp}(b_1) \subset B_a \). Let \( \nu \in B^* \), and choose \( \phi \in H(\nu, \beta) \) with \( \|\phi\|_2 = 1 \). If \( g \in SL_3(\mathbb{R}) \) satisfies \( d(\exp(B_a), g \exp(B_a)) \leq 1 \), we have

\[ \int_a \int_K b_1(H_1)b_2(H_2)\phi(H_1)\overline{\phi(H_2)}\varphi_{-t\nu}(E(H_1), gE(H_2)) dH_1 dH_2 \ll t^{-3/2+\epsilon} \beta^{3/2}. \]

If in addition we have \( n(\exp(B_a), g \exp(B_a)) \geq t^{-1/2+\epsilon} \beta^{1/2} \), then
\[
\int a b_1(H_1)b_2(H_2)\phi(H_1)\overline{\phi(H_2)}\varphi_{-\nu}(E(H_1), gE(H_2))dH_1dH_2 \ll_A t^{-A}.
\]

The implied constants depend only on \(B^*, \epsilon\), and the size of the derivatives of \(b_i\) of order at most \(n\), where \(n\) depends on \(\epsilon\).

**Proof.** The follows immediately from Proposition 6.5 after inverting the Fourier transform and noting that \(\|\hat{\phi}\|_1 \leq \|\hat{\phi}\|_2 (4\pi \beta^3/3)^{1/2} \ll \beta^{3/2}\).

\(\square\)

Proposition 4.1 now follows by a standard inversion of the Harish-Chandra transform as in Section 6.3 of [12] or Lemma 2.6 of [13].

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