Violation of the Ikeda Sum Rule and the Self-consistency in the Renormalized Quasiparticle Random Phase Approximation and the Nuclear Double-beta Decay

F. Krmpotić, T.T.S. Kuo, A. Mariano, E.J.V. de Passos and A.F.R. de Toledo Piza

Abstract

The effect of the inclusion of ground state correlations into the QRPA equation of motion for the two-neutrino double beta ($\beta\beta_{2\nu}$) decay is carefully analyzed. The resulting model, called renormalized QRPA (RQRPA), does not collapse near the physical value of the nuclear force strength in the particle-particle channel, as happens with the ordinary QRPA. Still, the $\beta\beta_{2\nu}$ transition amplitude is only slightly less sensitive on this parameter in the RQRPA than that in the plain QRPA. It is argued that this fact reveals once more that the characteristic behaviour of the $\beta\beta_{2\nu}$ transition amplitude within the QRPA is not an artifact of the model, but a consequence of the partial restoration of the spin-isospin $SU(4)$ symmetry. It is shown that the price paid for bypassing the collapse in the RQRPA is the violation of the Ikeda sum rule.

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1 Introduction

The quasiparticle random phase approximation (QRPA) is the most frequently used nuclear structure method for evaluating $\beta\beta$ rates both for the two-neutrino decay mode ($\beta\beta^{2\nu}$) and for the neutrinoless mode ($\beta\beta^{0\nu}$). The general feature of this method is that the resulting nuclear matrix elements $M_{2\nu}$ and $M_{0\nu}$ turn out to be highly sensitive to the particle-particle force in the $S = 1, T = 0$ channel \cite{1, 2, 3, 4, 5, 6}. Furthermore, the QRPA collapses very close to the physical value for this force. One thus may suspect that this method yields relatively small values of $M_{2\nu}$ simply because the approximation breaks up. In other words, the smallness of $M_{2\nu}$ in the QRPA could be just an artifact of the model.

Several modifications of the QRPA have been proposed in order to amend the above behavior in a qualitative way, including higher order RPA corrections \cite{7}, nuclear deformation \cite{8}, single-particle self-energy BCS terms \cite{9}, particle number projection \cite{10} and the proton-neutron pairing \cite{11}. Yet, none of these inhibits the collapse, which is just the famous ”phase transition” where the RPA develops zero or imaginary frequency solutions. One should remember in this connection that in the derivation of the ordinary RPA it is assumed that we can replace, when evaluating the equations of motion, the RPA (correlated) ground state by the Hartree-Fock ground state. Discrepancies due to this replacement will obviously get more serious the more significant are the ground state correlations. Attempts to correct this have been made by embodying the effect of GSC in the RPA equations of motion \cite{12, 13, 14, 15}. The corresponding formalism, named self-consistent or renormalized RPA (RRPA), generally yields better results in the sense that the instability develops at a larger interaction strength or is fully taken away.

Recently, the renormalized QRPA (RQRPA) has been applied to the $\beta\beta^{2\nu}$ decays by Toivanen and Suhonen \cite{16} (in $^{100}Mo$) and by Schwieger, Šimkovic and Faessler \cite{17} (in $^{76}Ge$, $^{82}Se$, $^{128}Te$ and $^{130}Te$ nuclei). We have also discussed the $^{100}Mo$ $2\nu$-decay but only in the framework of a schematic model \cite{18}. In this work we present a detailed study of several $\beta\beta$ decaying nuclei within the RQRPA. Particular attention is given to the non-conservation of the non energy weighted sum rule (called Ikeda sum rule for $J^\pi = 1^+$) within this approximation. The self-consistency between the residual interaction and the mean field \cite{5} is also addressed.
2 Formalism

The RQRPA formalism has been presented in several articles [15, 16, 17, 18] and we will therefore only sketch the main formulae needed in the discussion.

2.1 RQRPA Equation and the Transition Moments

The excited states $|\lambda JM\rangle$ are built by the action of the charge-exchange operators

$$\Omega^\dagger(\lambda JM) = \sum_{pn} \left[ X_{pn}(\lambda J)A_{pn}(JM) - Y_{pn}(\lambda J)A_{pn}(\overline{JM}) \right],$$

(1)

on the QRPA correlated ground state $|\tilde{0}\rangle$. Here $A_{pn}(JM) = [\alpha^+_p\alpha^+_n]^{JM}$, $A_{pn}(\overline{JM}) = (-1)^{J+M}A_{pn}(J - M)$, and $\alpha^+_t$ are quasiparticle creation operators for protons ($t = p$) and neutrons ($t = n$). The amplitudes $X$ and $Y$ and the eigenvalues $\omega_\lambda$ satisfy the RQRPA equations

$$\begin{pmatrix} A(J) & B(J) \\ B^*(J) & A^*(J) \end{pmatrix} \begin{pmatrix} X(\lambda J) \\ Y(\lambda J) \end{pmatrix} = \omega_\lambda \begin{pmatrix} X(\lambda J) \\ Y(\lambda J) \end{pmatrix},$$

(2)

where

$$X_{pn}(\lambda J) \equiv X_{pn}(\lambda J)D_{pn}^{1/2} \quad \text{and} \quad Y_{pn}(\lambda J) \equiv Y_{pn}(\lambda J)D_{pn}^{1/2}$$

(3)

are the renormalized amplitudes,

$$D_{pn} = 1 - N_p - N_n$$

(4)

and

$$\mathcal{N}_t = \tilde{j}_t^{-1}\langle \tilde{0}|[\alpha_t^\dagger\alpha_t^0]|\tilde{0}\rangle$$

(5)

are the quasiparticle occupations ($\tilde{J} \equiv \sqrt{2J + 1}$). Here we are assuming that the single quasi-particle density is diagonal, $\langle \tilde{0}|[\alpha_t^\dagger\alpha_t^0]|\tilde{0}\rangle = \delta_{tt'}\langle \tilde{0}|[\alpha_t^\dagger\alpha_t^0]|\tilde{0}\rangle$. The submatrices $A(J)$ and $B(J)$ are
\[
A_{pn,p'n'}(J) = (\epsilon_p + \epsilon_n)\delta_{pp'}\delta_{nn'} + D_{pn}^{1/2} [F(pn, p'n', J)(u_p v_n u_{p'} v_{n'} + v_p u_n v_{p'} u_{n'}) + G(pn, p'n', J)(u_p u_n v_{p'} v_{n'} + v_p v_n u_{p'} u_{n'})] D_{p'n'}^{1/2},
\]

\[
B_{pn,p'n'}(J) = D_{pn}^{1/2} [F(pn, p'n', J)(v_p u_n u_{p'} v_{n'} + u_p v_n v_{p'} u_{n'}) - G(pn, p'n', J)(u_p v_n v_{p'} u_{n'} + v_p u_n u_{p'} u_{n'})] D_{p'n'}^{1/2},
\]

(6)

where \(F\) and \(G\) are the usual particle-hole (ph) and particle-particle (pp) coupled two-body matrix elements.

The QRPA ground state is determined by the condition

\[
\Omega(\lambda J M) |\tilde{0}\rangle = 0.
\]

(7)

Writing the ground state in the form

\[
|\tilde{0}\rangle = N_0 e^S |BCS\rangle,
\]

(8)

using equation (7) and making the quasi-boson approximation, we find

\[
S = \frac{1}{2} \sum_{pn p'n'} \hat{j}^{-1} D_{pn}^{-1/2} C_{pn p'n'}(J) D_{p'n'}^{-1/2} [A_{pn}^\dagger(J) A_{p'n'}^\dagger(J)]^0,
\]

(9)

where the matrix \(C\) is the solution of

\[
\sum_{pn} X_{pn}^*(\lambda J) C_{pn p'n'}(J) = Y_{p'n'}^*(\lambda J), \text{ for all } \lambda J.
\]

(10)

From these equations one gets that

\[
\mathcal{N}_p = \sum_{\lambda J n'} \hat{j}^2 j_p^{-2} D_{pn} |Y_{pn'}(\lambda J)|^2,
\]

\[
\mathcal{N}_n = \sum_{\lambda J p'} \hat{j}^2 j_n^{-2} D_{p'n} |Y_{p'n}(\lambda J)|^2.
\]

(11)

To evaluate the transition matrix elements

\[
\langle \lambda J | \mathcal{O}_\pm(J) | \tilde{0}\rangle = \langle 0 | \left[ \Omega(\lambda J), \mathcal{O}_\pm(J) \right]^0 | \tilde{0}\rangle,
\]

(12)

the \(\beta^\pm\) decay operators
\[ \mathcal{O}_\pm (JM) = \sum_i O(JM; i) t_\pm (i); \quad \text{(with } t_-|n\rangle = |p\rangle) \] (13)

are expressed in the form

\[ \mathcal{O}_\pm (JM) = \mathcal{O}^{20}_\pm (JM) + \mathcal{O}^{11}_\pm (JM), \] (14)

where

\[ \mathcal{O}^{20}_\pm (JM) = \sum_{pn} \left[ \Lambda^\pm_{pn} (J) A^\dagger_{pn}(JM) + (-1)^J \Lambda^{\mp*}_{pn}(J) A_{pn}(JM) \right], \]

\[ \mathcal{O}^{11}_\pm (JM) = \sum_{pn} \left[ \Delta^\pm_{pn} (J) B^\dagger_{pn}(JM) + (-1)^J \Delta^{\mp*}_{pn}(J) B_{pn}(JM) \right], \] (15)

with

\[ \Lambda^+_{pn} (J) = -(-1)^J \hat{j}^{-1} u_p u_n \langle p||O(J)||n\rangle^*, \quad \Lambda^-_{pn} (J) = -\hat{j}^{-1} u_p v_n \langle p||O(J)||n\rangle, \]

\[ \Delta^+_{pn} (J) = -(-1)^J \hat{j}^{-1} v_p u_n \langle p||O(J)||n\rangle^*, \quad \Delta^-_{pn} (J) = \hat{j}^{-1} u_p u_n \langle p||O(J)||n\rangle, \] (16)

\[ B^\dagger_{pn}(JM) = [\alpha^\dagger_p \alpha_n]^J_M \] and \[ B_{pn}(JM) = (-1)^{J+M} B_{pn}(J - M). \] Notice that, whereas the operator \( \mathcal{O}^{20}_\pm (JM) \) creates and destroys a \( np \) quasi-particle pair, the scattering operator \( \mathcal{O}^{11}_\pm (JM) \) conserves the number of quasi-particles.

The evaluation of (12) goes now via the following relationships:

\[ \hat{j}^{-1} \langle \bar{0} | \left[ A_{pn}(\bar{J}), A^\dagger_{p'n'}(J) \right] | \bar{0}\rangle = \delta_{pp'} \delta_{nn'} L_{pn}, \]

\[ \hat{j}^{-1} \langle \bar{0} | \left[ B_{pn}(\bar{J}), B^\dagger_{p'n'}(J) \right] | \bar{0}\rangle = \delta_{pp'} \delta_{nn'} (N_n - N_p), \]

\[ \hat{j}^{-1} \langle \bar{0} | \left[ A^\dagger_{pn}(J), B^\dagger_{p'n'}(J) \right] | \bar{0}\rangle = -\delta_{nn'} \hat{J}_p^{-1} \langle \bar{0} | \left[ \alpha^\dagger_p \alpha^\dagger_{n'} \right] | \bar{0}\rangle \equiv 0, \]

\[ \hat{j}^{-1} \langle \bar{0} | \left[ A^\dagger_{pn}(J), B_{p'n'}(J) \right] | \bar{0}\rangle = -\delta_{pp'} \hat{J}_n^{-1} \langle \bar{0} | \left[ \alpha^\dagger_n \alpha^\dagger_{n'} \right] | \bar{0}\rangle \equiv 0. \] (17)

The expectation values \( \langle \bar{0} | \left[ \alpha^\dagger_p \alpha^\dagger_{n'} \right] | \bar{0}\rangle \) are identically zero since \( |\bar{0}\rangle \) is a superposition of states with equal number of neutron and proton quasi-particles (see eqs. (8) and (9)).
Thus, only the first term in eq. (14) contributes to the transition matrix element and one gets

\[ \langle \lambda J | O_{\pm}(J) | \tilde{0} \rangle \equiv \langle \tilde{0} | \left[ \Omega(\lambda \tilde{J}), O_{\pm}^{20}(J) \right]^{0} | \tilde{0} \rangle = \hat{j} \sum_{pn} \left[ \Lambda_{pn}^{\pm}(J)X_{pn}^{*}(\lambda J) + (-1)^{J}\Lambda_{pn}^{\pm*}(J)Y_{pn}^{*}(\lambda J) \right] D_{pn}^{1/2} \]  

(18)

The corresponding RQRPA total strengths are

\[ \tilde{S}_{\pm}(J) = \hat{j}^{-2} \sum_{\lambda} |\langle \lambda J | O_{\pm}(J) | \tilde{0} \rangle|^{2}, \]  

(19)

and from (18) we obtain that

\[ \tilde{S}_{-}(J) - \tilde{S}_{+}(J) = \sum_{pn}(|\Lambda_{pn}^{-}(J)|^{2} - |\Lambda_{pn}^{+}(J)|^{2})D_{pn} \]
\[ = \hat{j}^{-2} \sum_{pn}(v_{n}^{2} - v_{p}^{2})D_{pn} |\langle \nu | O(J) | n \rangle|^{2}. \]  

(20)

For the evaluation of the \( \beta \beta_{2\nu} \) matrix elements, instead of the two-vacua QRPA method introduced in ref. [19], we will here simply use the expression

\[ \mathcal{M}_{2\nu}(J) = \sum_{\lambda} \frac{\langle \tilde{0} | O_{-}(J) | \lambda J \rangle \langle \lambda J | O_{-}(J) | \tilde{0} \rangle}{\omega_{\lambda J}}, \]  

(21)

but solving the gap equations for the intermediate nucleus [20]. We have tested numerically that both methods yield almost identical results.

### 2.2 Sum Rule

It is well known that the total \( \beta^{\pm} \) strengths \( S_{\pm}(J) \)

\[ S_{\pm}(J) = \hat{j}^{-2} \sum_{\nu} |\langle \nu J | O_{\pm}(J) | 0 \rangle|^{2} \]  

(22)

can be expressed in the form

\[ S_{\pm}(J) = (-)^{J}\hat{j}^{-1}\langle 0 | [O_{\pm}(J)O_{\pm}(J)]^{0} | 0 \rangle, \]  

(23)
when \(|\nu J\rangle\) is the complete set of excited states that can be reached by operating with \(O_\pm(J)\) on the initial state \(|0\rangle\). It follows at once that

\[
S_-(J) - S_+(J) = (-)^J \hat{J}^{-1} \langle 0 | [O_+(J), O_-(J)]^0 |0\rangle,
\]

i.e., the difference between the exact total \(S_-(J)\) and \(S_+(J)\) strengths equals the expectation value of the operator \((-)^J \hat{J}^{-1} [O_+(J), O_-(J)]^0\) in the ground state \(|0\rangle\). In particular, for the observables of interest here, i.e., the Fermi (F) and Gamow-Teller (GT) operators \(t_\pm\) and \(t_\pm \sigma\), one gets [21, 22]

\[
S_-(J^\pi = 0^+, 1^+) - S_+(J^\pi = 0^+, 1^+) = \langle 0 | \hat{N} - \hat{Z} |0\rangle.
\]

Moreover, if the ground state is chosen so that

\[
\langle 0 | \hat{N} - \hat{Z} |0\rangle = N - Z,
\]

one obtains the well known result

\[
S_-(J^\pi = 0^+, 1^+) - S_+(J^\pi = 0^+, 1^+) = N - Z.
\]

It can now be easily shown that the relations (23) and (24) are not valid within the RQRPA, and this leads to the violation of the Ikeda sum rule. In fact, using eqs. (17) the right hand side of eq. (24) evaluates in the RQRPA to

\[
(-)^J \hat{J}^{-1} \langle \tilde{0} | [O_+(J), O_-(J)]^0 |\tilde{0}\rangle = (-)^J \hat{J}^{-1} \langle \tilde{0} | [O_+^{20}(J), O_-^{20}(J)]^0 |\tilde{0}\rangle
\]

\[+ (-)^J \hat{J}^{-1} \langle \tilde{0} | [O_+^{11}(J), O_-^{11}(J)]^0 |\tilde{0}\rangle.
\]

Calculating the first term on the right hand side of eq. (28) one finds that it reproduces the difference between the total RQRPA strengths \(\tilde{S}_-(J)\) and \(\tilde{S}_+(J)\) given in eq. (20), that is

\[
(-)^J \hat{J}^{-1} \langle \tilde{0} | [O_+^{20}(J), O_-^{20}(J)]^0 |\tilde{0}\rangle = \hat{J}^{-2} \sum_{pn} (v_n^2 - v_p^2) D_{pn} |\langle p | O(J) |n\rangle|^2 = \tilde{S}_-(J) - \tilde{S}_+(J).
\]
As for the second term on the right hand side of eq. (28) one obtains

\[-J J^* - \sum_{pn} |\langle p||O(J)||n \rangle|^2 (u_n^2 - v_p^2) (N_n - N_p)\].

This quantity is different from zero because the states $O_{\pm}^{11}(JM)|\tilde{0}\rangle$ are both non-null and orthogonal to the RQRPA model space (spanned by the states $\Omega^\dagger(\lambda JM)|\tilde{0}\rangle$).

Using the above results we can express $\tilde{S}_-(J) - \tilde{S}_+(J)$ for F and GT transitions as

\[
\tilde{S}_-(J^\pi = 0^+, 1^+) - \tilde{S}_+(J^\pi = 0^+, 1^+) = \langle \tilde{0} | \hat{N} - \hat{Z} | \tilde{0} \rangle - \sum_{pn} |\langle p||O(J^\pi = 0^+, 1^+)||n \rangle|^2 (u_n^2 - v_p^2) (N_n - N_p),
\]

and the violation of the corresponding sum rules within the RQRPA is associated with a nonvanishing value of the quantity

\[
\Delta \tilde{S}(J^\pi = 0^+, 1^+) = (N - Z) - [\tilde{S}_-(J^\pi = 0^+, 1^+) - \tilde{S}_+(J^\pi = 0^+, 1^+)].
\]

As discussed in Sect. 3 below, this depends quantitatively on the way the gap equations are solved. In fact, the usual constraints lead to results different from those of ref. \[18\]

\[
N = \langle BCS|\hat{N}|BCS \rangle = \sum_n j_n^2 v_n, \\
Z = \langle BCS|\hat{Z}|BCS \rangle = \sum_p j_p^2 v_p,
\]

lead to results different from those of ref. \[18\]

\[
N = \langle \tilde{0} | \hat{N} | \tilde{0} \rangle = \sum_n j_n^2 [v_n^2 + (u_n^2 - v_n^2) N_n], \\
Z = \langle \tilde{0} | \hat{Z} | \tilde{0} \rangle = \sum_p j_p^2 [v_p^2 + (u_p^2 - v_p^2) N_p].
\]

In ordinary QRPA, where $|\tilde{0}\rangle \rightarrow |BCS\rangle$, the states $O_{\pm}^{11}(JM)|BCS\rangle$ are null-vectors and the F and GT sum rules are fulfilled, i.e., the quantity corresponding in this approximation to eq. (32) vanishes.
2.3 Self-consistency between the Residual Interaction and the Mean Field

It is well known that in the limit of exact isospin symmetry all the $S_-(J^\pi = 0^+)$ strength is concentrated in the isobaric analog state (IAS), there is no $\beta^+$ strength and the $\beta\beta_{2\nu}$ decay is forbidden, i.e.,

$$S_{IAS} \equiv S_-(J^\pi = 0^+), \quad S_+(J^\pi = 0^+) \equiv 0, \quad \mathcal{M}_{2\nu}(J^\pi = 0^+) \equiv 0. \quad (35)$$

The question of which conditions have to be fulfilled in order for these relations to hold in the QRPA has been discussed in ref. [5]. Within the RQRPA they read

$$X_{pn}^{IAS} = D^{1/2}_{pn} \Lambda_+^{pn}(J^\pi = 0^+)/\sqrt{N - Z} = \sqrt{N - Z} D^{1/2}_{pn} \hat{J}_p u_p v_n,$$
$$Y_{pn}^{IAS} = -D^{1/2}_{pn} \Lambda_+^{pn}(J^\pi = 0^+)/\sqrt{N - Z} = -\sqrt{N - Z} D^{1/2}_{pn} \hat{J}_p u_n v_p, \quad (36)$$

as can be checked from eqs. (16) and (18). Putting these expressions into the RQRPA equations (2) we get

$$\epsilon_p + \epsilon_n - \omega_{IAS} = -U_{jp=jn}^{IAS} + \frac{u_p}{v_p} \Delta_+^{IAS} + \frac{u_n}{v_n} \Delta_+^{IAS},$$
$$\epsilon_p + \epsilon_n + \omega_{IAS} = U_{jp=jn}^{IAS} + \frac{u_p}{v_p} \Delta_+^{IAS} + \frac{v_n}{u_n} \Delta_+^{IAS}, \quad (37)$$

where

$$\Delta_+^{IAS} = -\frac{1}{2} \sum_{p'} \hat{J}_{p'} \hat{J}_p^{-1} G(pp', p'p', J^\pi = 0^+) u_p v_{p'} D_{p'p'},$$
$$\Delta_+^{IAS} = -\frac{1}{2} \sum_{n'} \hat{J}_{n'} \hat{J}_n^{-1} G(nn', n'n', J^\pi = 0^+) u_n v_{n'} D_{n'n'}, \quad (38)$$
$$U_{jp=jn}^{IAS} = \sum_{j_{p'}=j_{n'}} \hat{J}_{p'} \hat{J}_p^{-1} F(pp', p'n', J^\pi = 0^+) (v_{n'}^2 - v_{p'}^2) D_{p'n'}. $$

In deriving the eqs. (37) the relation $2G(nn', p'n', J^\pi = 0^+) = G(pp, p'p', J^\pi = 0^+) = G(nn, n'n', J^\pi = 0^+)$ has been used. We have also assumed that $J^\pi = 0^+$ is the only
important multipole affecting the solution of eq. (11). This leads to the identity \( D_{pn} = D_{pp} = D_{nn} \).

Summing and subtracting the two equations in (37) we also obtain

\[
\epsilon_p + \epsilon_n = \frac{\Delta_{p\text{AS}}}{2v_p u_p} + \frac{\Delta_{n\text{AS}}}{2v_n u_n},
\]

\[
\omega_{\text{IAS}} = U_{j_p = j_n}^{\text{IAS}} + (\epsilon_p - \lambda_p) \frac{\Delta_{p\text{AS}}}{\Delta_p} - (\epsilon_n - \lambda_n) \frac{\Delta_{n\text{AS}}}{\Delta_n}.
\] (39)

As \( \epsilon_t = \Delta_t/2v_t u_t \), the first relation in (39) will be an identity only if \( \Delta_t = \Delta_{t\text{AS}} \). This means that in solving the gap equations the substitution

\[
G(tt, t't', J^\pi = 0^+) \rightarrow G(tt, t't', J^\pi = 0^+) D_{t't'}
\] (40)

has to be done.

Also from the last relation in (39) we obtain

\[
\omega_{\text{IAS}} + \lambda_p - \lambda_n = \epsilon_p - \epsilon_n + U_{j_p = j_n}^{\text{IAS}}.
\] (41)

The left hand side in this equation is just the excitation energy of the IAS relative to the ground state of the initial nucleus, i.e., \( E_{\text{IAS}} - E_i \). Further \( \epsilon_p - \epsilon_n = \Delta_C - U_{j_p = j_n}^{\text{sym}} \), where \( \Delta_C \) and \( U_{j_p = j_n}^{\text{sym}} \) are, respectively, the Coulomb displacement energy and the symmetry energy. Thus, \( E_{\text{IAS}} - E_i \) will be equal to \( \Delta_C \) only if \( U_{j_p = j_n}^{\text{sym}} = U_{j_p = j_n}^{\text{IAS}} \). But, in the BCS approximation, the symmetry energy reads

\[
U_{j_p = j_n}^{\text{sym}} = \sum_{j_p' = j_n'} \hat{j}_p' \hat{j}_n' - 1 F(pn, p'n'; 0^+)(v_{n'}^2 - v_p^2)
\]
\[
= \frac{1}{2} \sum_{j_p' = j_n'} \sum_{JT} (-)^{T+1} j^2 j^2 - 2 G(jj'jj'; JT)(v_{j_n'}^2 - v_{j_p'}^2),
\] (42)

and is equal to

\[\text{It should be remembered that in the BCS approximation the energy difference between the ground states of an even-even (N, Z) nucleus and an odd-odd (N - 1, Z + 1) nucleus is } E_{N-1,Z+1} - E_{N,Z} = \Delta_p + \Delta_n + \lambda_p - \lambda_n; \text{ also } E_{N-2,Z+2} - E_{N,Z} = 2(\lambda_p - \lambda_n).\]
where

\[ \mu_{jn} = \sum_{n'} \hat{j}_n^{1-1} j_{n'} v_{n'}^2 F(nn, n'n'; 0^+) + \sum_{p'} \hat{j}_n^{1-1} j_{p'} v_{p'}^2 F(nn, p'p'; 0^+) \]

\[ = \frac{1}{2} \sum_{J_0 = J_0'} \sum_J J^2 J^{-2} [G(jj'jj'; JT = 1) (2 v_{j_0}^2 + v_{j_0}^2) + G(jj'jj'; JT = 0)] v_{j_0}^2, \]

\[ \mu_{jp} = \sum_{p'} \hat{j}_p^{1-1} j_{p'} v_{p'}^2 F(pp, p'p'; 0^+) + \sum_{n'} \hat{j}_n^{1-1} j_{n'} v_{n'}^2 F(pp, n'n'; 0^+) \]

\[ = \frac{1}{2} \sum_{J_0 = J_0'} \sum_J J^2 J^{-2} [G(jj'jj'; JT = 1) (2 v_{j_0}^2 + v_{j_0}^2) + G(jj'jj'; JT = 0)] v_{j_0}^2, \] (44)

are, respectively, the neutron and proton self-energies. This suggests that, when the self-energies are included in a RQRPA calculation, the substitution

\[ F(tt, t't', J^\pi = 0^+) \rightarrow F(tt, t't', J^\pi = 0^+) D_{t't'} \] (45)

is also pertinent.

\section{Results and Discussion}

The numerical calculations reported below are performed with two-body delta-force and the single particle energies used previously \[5\]. We define the ratios

\[ s = \frac{v_{st}}{v_{spair}}, \quad t = \frac{v_{tt}}{v_{t'spair}}, \]

between the \(T = 1, S = 0\) and \(T = 0, S = 1\) coupling constants in the \(pp\) channels and the pairing force constant, respectively. Their physical values are \(s \cong 1.0\) and \(t \cong 1.5\). An eleven dimensional model space, including all the single-particle orbitals from oscillator shells \(3\hbar\omega\) and \(4\hbar\omega\) plus \(0\hbar_{9/2}\) and \(0\hbar_{11/2}\) from the \(5\hbar\omega\) oscillator shell, is used.

The results for two different RQRPA calculations will be discussed, namely:

1. \textit{Approximation I (AI)}: the BCS equations are solved in the usual way with the constraint (33).
2. Approximation II (AII): the pairing matrix elements are renormalized as indicated in (II) and the constraint (III) is used.

As done in ref. [16], we analyzed the difference between the double iteration (DI) procedure, when the proton and neutron occupations are given by (II), and the simple iteration (SI) procedure, when the coefficients $D_{pn}$ on the right hand side of eq. (II) are put equal to unity. Besides, the influence of different number of multipolarities $J^\pi$ in the summations involved in eqs. (II) was studied by performing calculations either with only the $J^\pi = 1^+$ state (calculations denoted $SI1$ and $DI1$) or with the states $J^\pi = 1^+, 2^-, 3^+$ and $4^-$ (calculations denoted $SI4$ and $DI4$). The quantities that are of interest for the discussion are: the amount of ground state correlations, defined as

$$ C = \sum_p N_p + \sum_n N_n, $$

the normalized GT strengths

$$ s_- = \tilde{S}_-(J^\pi = 1^+)/\left(N - Z\right); \quad s_+ = \tilde{S}_+(J^\pi = 1^+)/\left(N - Z\right), $$

the normalized GT sum rule,

$$ S_R = s_- - s_+, $$

the lowest energy eigenvalue $\omega_{J^\pi=1^+}$ and the $\beta\beta_{2\nu}$ matrix element given by (21).

As an example, we first discuss the calculations of these observables in $^{76}Ge$ nucleus. They are shown in Fig. 1 for different AIFs ($SI1$, $DI1$, $SI4$ and $DI4$), as a function of the parameter t. The QRPA results, up to the collapse of this approximation, are also presented in the same figure. One can immediately see that:

a) At variance with the QRPA, none of the RQRPA’s collapses for physically meaningful values of t, and all four AIFs yield quite similar results for $\omega_{J^\pi=1^+}$.

b) The difference between the $SI$ and $DI$ procedures is always rather small.

c) The GSC generated by the states with multipolarities $J^\pi \neq 1^+$ are not so significant as one would perhaps expect.

d) The violation of the Ikeda sum rule is more pronounced when more GSC are taken into account. As the $\beta^+$-strengths within the RQRPA and the QRPA are practically the
same, the deviation from the QRPA value $S_R = 1$, mainly arises from the decrease of the $\beta^-$-strength.

e) The different $AI$'s yield very similar results for the $\beta\beta_{2\nu}$ matrix element $M_{2\nu}(J^\pi = 1^+)$, and it is difficult to say which one is "better" and which one is "worse". As a function of $t$, they all seem to behave in the same way as the corresponding QRPA matrix element. Still, the RQRPA results are qualitatively different in the sense that they cannot be fitted by the $(1,1)$-Padé approximant

$$M_{2\nu} \cong M_{2\nu}(t = 0) \frac{1 - t/t_0}{1 - t/t_1}, \quad (49)$$

as happens in the QRPA case \[3\]. ($t_0$ and $t_1$ denote, respectively, the zero and the pole of $M_{2\nu}(J^\pi = 1^+)$. ) This is a direct consequence of the absence of the collapse within the RQRPA for the physical values of $t$.

All the above comments are also valid for the nuclei $^{82}Se$, $^{100}Mo$, $^{128}Te$ and $^{130}Te$. From now on we discuss the differences between the $AI$ and $AII$ results. In doing this we will recur only to the $SI$ method. In Fig. 2 are compared the behaviors of the Ikeda reduced sum rule $S_R$ for all five nuclei.

One immediately sees that in all the cases, except for $^{100}Mo$, the violation of this sum rule is quite more pronounced in the $AI$ than in the $AII$. Fig. 3 focuses the crossings of the calculated $M_{2\nu}(J^\pi = 1^+)$ matrix elements and the corresponding matrix elements deduced from the experimental half-lives (including the experimental errors) \[23\]. Again, as we do not know which value of the parameter $t$ should be used in each case, it is hard to say that the RQRPA is a better model than the QRPA. What is definitively clear is that the sensitivity of $M_{2\nu}(J^\pi = 1^+)$ on this parameter is unavoidable and that it is not a consequence of the collapse of the QRPA. It has been pointed out more than once \[19, 23\] that the $M_{2\nu}(J^\pi = 1^+)$ amplitude goes to zero within the QRPA because of the partial restoration of the Wigner spin-isospin SU(4) symmetry. This is a result of two antagonistic

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Footnotes:

1. It is worth noting that, when the $SI$ procedure is used, $N_n = N_p$ and therefore $S_-(J^\pi = 0^+) = \bar{S}_+(J^\pi = 0^+) = \langle 0|\bar{N} - \bar{Z}|0 \rangle$. Thus from (22) we see that the $F$ sum rule is exactly conserved within $AII$. Contrarily, in the $AI$ it is broken by $\lesssim 1\%$ for $s = 1$ and by $\gtrsim 10\%$ for $s = 2$. The $F$ matrix elements $M_{2\nu}(J^\pi = 0^+)$ behave in the same way as the GT matrix elements and pass through zero at $s \cong 1$.

2. From the theoretical point of view, the $M_{2\nu}(J^\pi = 1^+)$ matrix element in $^{100}Mo$ is in same sense peculiar, because of the strong predominance of the $[0g_7/2(n)0g_9/2(p); J^\pi = 1^+]$ configuration.
effects: the spin-orbit term in the mean-field potential that destroys the SU(4) symmetry (since it singles out one spin direction over the other) and the $pn$ residual interaction that favors the $LS$ coupling over the $j − j$ coupling scheme \[^{24}\]. A physical criterion for fixing the triplet $pp$ coupling strength $t$, based on the maximal restoration of these symmetries, has also been suggested ($t_{sym}$). \[^{4}\] In table 1 we compare the values of $t_{sym}$, obtained using the recipe just mentioned, with those that are necessary to reproduce the experimental $M_{2\nu}(J^\pi = 1^+)$ amplitudes ($t_{exp}$). The differences are of the order of 10% what is quite auspicious since the values of the nuclear parameter (both for schematic and realistic interactions) are not known with such a precision. We also remind that some additional degrees of freedom, not considered in the present calculations, such as the quadrupole and octupole charge-conserving vibrations can play an important role for all the nuclei discussed here. Besides, the contributions of odd-parity nuclear operators, arising from the $p$-wave Coulomb corrections to the electron wave functions and the recoil corrections to the nuclear current, are also significant for the $\beta\beta_{2\nu}$-decays of $^{128}Te$ and $^{130}Te$ nuclei \[^{25}\].

In summary, when the GSC are taken into account the collapse of the QRPA does not develop in the physical region of the $pp$-strength parameter \[^{16, 17, 18}\]. Yet, the GSC only slightly mitigate the strong dependence of the $\beta\beta_{2\nu}$ transition amplitude on this parameter, which is set on by the partial restoration of the SU(4) symmetry. The price that is paid to avoid the collapse within the QRPA is the non-conservation of the Ikeda sum rule within the RQRPA. This violation cannot be eluded and comes from the fact that the states generated by the action of the scattering part of the GT operator on the RQRPA ground state is not contained in the model space.

\[^{4}\]By maximal restoration of the SU(4) symmetry we mean that for $t = t_{sym}$ the maximal concentration of the $\beta^-$-strength within the GT resonance takes place and the $\beta^+$-strength is minimal. In no way it connotes that the $J^\pi = 1^+$ states belong to a single SU(4) multiplet.
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Figure Captions

Figure 1: Behavior of several relevant quantities (defined in the text) within different RQRPA-AI approximations in the case the $^{76}\text{Ge}$ nucleus, as a function of $t$. The QRPA results, up to the collapse of this approximation, are also exhibited. The matrix elements $M_{2\nu}$ are given in units of $[\text{MeV}]^{-1}$.

Figure 2: Ikeda sum rule within the approximations I (upper panel) and II (lower panel), when the simple iteration method is used with only one intermediate state.

Figure 3: Calculated matrix elements $M_{2\nu}$ (in units of $[\text{MeV}]^{-1}$) for several nuclei, as a function of $t$. Dashed and dotted curves correspond to the approximations I and II, respectively, and the solid lines are the QRPA results. The fill patterns indicate the experimental results.

Table 1: Comparison between the values of the parameter $t$ necessary to reproduce the experimental $M_{2\nu}(J^\pi = 1^+)$ amplitudes ($t_{\text{exp}}$) and those that lead to maximal restoration of the SU(4) symmetry ($t_{\text{sym}}$).

|       | $^{76}\text{Ge}$ | $^{82}\text{Se}$ | $^{90}\text{Mo}$ | $^{128}\text{Te}$ | $^{130}\text{Te}$ |
|-------|------------------|------------------|------------------|------------------|------------------|
| QRPA |                  |                  |                  |                  |                  |
| $t_{\text{exp}}$ | 1.37             | 1.35             | 1.52             | 1.32             | 1.30             |
| $t_{\text{sym}}$ | 1.23             | 1.27             | 1.47             | 1.40             | 1.40             |
| AI   |                  |                  |                  |                  |                  |
| $t_{\text{exp}}$ | 1.44             | 1.40             | 1.67             | 1.33             | 1.32             |
| $t_{\text{sym}}$ | 1.25             | 1.32             | 1.55             | 1.42             | 1.42             |
| AII  |                  |                  |                  |                  |                  |
| $t_{\text{exp}}$ | 1.42             | 1.37             | 1.65             | 1.32             | 1.30             |
| $t_{\text{sym}}$ | 1.25             | 1.30             | 1.55             | 1.40             | 1.40             |
