LINEAR EIGENVALUE STATISTICS OF RANDOM MATRICES WITH A VARIANCE PROFILE

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Abstract. We give an upper bound on the total variation distance between the linear eigenvalue statistic, properly scaled and centered, of a random matrix with a variance profile and the standard Gaussian random variable. The second order Poincaré inequality type result introduced in [12] is used to establish the bound. Using this bound we prove Central limit theorem for linear eigenvalue statistics of random matrices with different kind of variance profiles. We re-establish some existing results on fluctuations of linear eigenvalue statistics of some well known random matrix ensembles by choosing appropriate variance profiles.

Keywords: Linear eigenvalue statistics, random matrix, variance profile, Central limit theorem, Band matrix, sparse matrix, Erdős-Rényi random graph, patterned random matrix.

1. Introduction

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of an $n \times n$ matrix $M$ with real or complex entries. The linear eigenvalue statistic of $M$ is a function of the form

$$\sum_{k=1}^{n} f(\lambda_k)$$

where $f$ is a fixed function. The function $f$ is known as the test function. In this article we study the fluctuations of linear eigenvalue statistics of random matrices of the following form

(1) $Y = A \circ X,$

where $\circ$ is the matrix Hadamard product, $A$ is an $n \times n$ deterministic matrix with non-negative entries $a_{ij}$ and $X$ is an $n \times n$ random matrix. In literature $A$ is referred as the standard deviation profile, and $A \circ A$ is referred as the variance profile. We call $Y$ as a random matrix with a variance profile $A \circ A.$ If $X$ is symmetric (Hermitian) and $A$ is symmetric, then $Y$ is symmetric (Hermitian) with a variance profile $A \circ A.$

Random matrices and random matrices with variance profiles have been used in different areas of sciences, for instance, in ecology to study the stability of an ecological system with different species, and in neuroscience to model networks. For an overview, we refer to [28, 9, 5, 13, 38, 3, 4] and [32].

In recent years there have been increasing interest to study random matrices with a variance profile. For results on Hermitian matrices with a variance profile,
Recently Cook et. al. [14] have studied the limiting spectral distribution of non-Hermitian random matrices with different types of variance profile matrices. In this article, our goal is to study the fluctuations of linear eigenvalue statistics of random matrices with different type of variance profiles for polynomial test functions. We investigate the convergence of the fluctuations of linear eigenvalue statistics in total variation norm.

The literature on linear eigenvalue statistics is quite large. To the best of our knowledge, the fluctuations of linear eigenvalue statistics was first considered by Arharov [8] in 1971 for sample covariance matrices. In 1982, Jonsson [23] proved the Central limit theorem (CLT) type results of linear eigenvalue statistics for Wishart matrices using method of moments. In last three decades the fluctuations of linear eigenvalue statistics have become one of the popular field of research in random matrix theory. There are several results on the fluctuations of linear eigenvalue statistics of random matrix ensembles of different type. To get an overview on the results on Wigner and sample covariance matrices, we refer the readers to [22], [37], [9], [27], [34], [39] and the reference there in. For band and sparse symmetric random matrices, see [7], [20], [25], [35]. For Toeplitz, Hankel and circulant matrices, see [12], [26], [1] and [2], and for non-Hermitian matrices, see [33], [30] and [31].

In this article, we derive a simple bound on the total variation distance between the linear fluctuations, for the polynomial test functions, of \( A \odot X \) and the standard Gaussian variable. The bound is given in terms of the entries of \( A \), the degree of the polynomial and the dimension of \( X \). see Theorem 1. Using Theorem 1 we establish the Central limit theorem for the linear eigenvalue statistics of \( A \odot X \) with different kind of variance profile matrices. For instance, in Corollary 9 we studied the fluctuation of \( A \odot X \) when \( A \) is an adjacency matrix of an Erdős-Rényi random graph. We re-establish some existing results on the fluctuations of linear eigenvalue statistics of random matrices very easily with the appropriate choices of \( A \). For example, in Corollary 7 we showed that the fluctuation of linear eigenvalue statistics of sample covariance matrix is Gaussian. In Corollary 8, we established that the fluctuation of linear eigenvalue statistics of finite product of independent copies of i.i.d. matrices is Gaussian. We also derived the fluctuations of linear eigenvalue statistics for diagonal, anti-diagonal and sparse random matrices using Theorem 1, see Corollary 6, Corollary 12 and Remark 14.

Now, we briefly outline the rest of the article. In Section 1.1 we introduce the notations frequently used in this article. In Section 2, we introduce the assumptions on the test functions, on the entries of \( X \), and state our main result, Theorem 1. In Section 3, we derive some new results and re-establish some existing results on fluctuations of linear eigenvalue statistics from Theorem 1. In Section 4, we give the proof of Theorem 1 and in Section 4.1, we collect the results needed to prove Theorem 1. In Section 5, we give definitions of some variance profiles to make this article self contained.

1.1. Notations. Here we introduce some basic definitions and notations used in this article.

(i) Let \( B(\mathbb{R}) \) be the Borel sigma algebra on \( \mathbb{R} \). Then we define \( d_{TV}(\mu, \nu) := \sup_{B \in B(\mathbb{R})} |\mu(B) - \nu(B)| \) the total variation distance between two probability measures \( \mu \) and \( \nu \) on the real line.

(ii) Unless otherwise specified, all matrices are \( n \times n \) square matrices with growing \( n \). We suppress the subscript \( n \) to avoid notational complexity.
(iii) Let \( A = (a_{ij})_{n \times n} \) and \( B = (b_{ij})_{n \times n} \) be two matrices of same dimension. Then \( A \circ B \) denotes the standard Hadamard product, that is, \((A \circ B)_{ij} = a_{ij}b_{ij}\).

(iv) \( \{e_1, e_2, \ldots, e_n\} \) is the canonical basis of \( \mathbb{R}^n \).

(v) \( [n] = \{1, 2, \ldots, n\} \).

(vi) \( I_k = \{(i_1, \ldots, i_k) \in [n]^k : i_p \neq i_q \text{, for all } p \neq q\} \).

(vii) Let \( \{a_n\}_n \) and \( \{b_n\}_n \) be two sequence of non-negative real numbers. We write \( a_n \lesssim b_n \) if there exists \( c > 0 \) such that \( a_n \leq cb_n \) for all \( n \). We write \( a_n \sim b_n \) if \( \lim_{n \to \infty} (a_n/b_n) = 1 \).

2. Main result

Let \( P_k(x) = \sum_{i=0}^{k} c_i x^i \), where \( c_k \neq 0 \) and \( c_i \in \mathbb{R} \cap [-\tau, \tau] \), for all \( 1 \leq i \leq k \), for some \( k \)-independent constant \( \tau > 0 \). In other words, the coefficients of the polynomial remain bounded even when the degree of the polynomial goes to infinity. In this article, we shall consider this type of polynomial test functions only. Now define

\[
Z_k(A \circ X) = \frac{\text{Tr}(P_k(A \circ X)) - \mathbb{E}[\text{Tr}(P_k(A \circ X))]}{\sqrt{\text{Var}(P_k(A \circ X))}},
\]

where \( A = (a_{ij}) \) is an \( n \times n \) deterministic matrix with non-negative \( a_{ij} \) and \( X = (X_{ij}) \) is an \( n \times n \) random matrix.

In our results, \( X \) is either an i.i.d. random matrix or a symmetric random matrix. By i.i.d. random matrix we mean \( \{X_{ij} : i, j \geq 1\} \) are i.i.d. random variables, and by symmetric random matrix we mean \( X_{ji} = X_{ij} \) and \( \{X_{ij} : 1 \leq i \leq j\} \) are i.i.d. random variables.

We show that \( Z_k(A \circ X) \) converges to the standard Gaussian distribution in total variation norm when \( X \) is an i.i.d. or a symmetric random matrix and \( X_{ij} \) belongs to a specific class of distributions, namely, \( \mathcal{L}(c_1, c_2) \) for some \( c_1, c_2 > 0 \).

For each \( c_1, c_2 > 0 \), \( \mathcal{L}(c_1, c_2) \) is a class of probability measures on \( \mathbb{R} \) that arise as laws of random variables like \( u(Z) \), where \( Z \) is a standard Gaussian random variable and \( u \) is a twice continuously differentiable function such that for all \( x \in \mathbb{R} \)

\[
|u'(x)| \leq c_1 \text{ and } |u''(x)| \leq c_2.
\]

For example, the standard Gaussian random variable is in \( \mathcal{L}(1, 0) \). The uniform distributed random variable on \([0, 1]\) is in \( \mathcal{L}((2\pi)^{-1/2}, (2\pi)^{-1/2}) \). To the best of our knowledge, the linear eigenvalue statistics with this class of random variables was first considered in [12].

**Theorem 1.** Let \( X = (X_{ij}) \) be an \( n \times n \) i.i.d. (respectively, symmetric) random matrix, where \( X_{ij} \) are symmetric random variables with variance one and \( X_{ij} \in \mathcal{L}(c_1, c_2) \) for some \( c_1, c_2 > 0 \). Let \( A = (a_{ij}) \) be an \( n \times n \) deterministic (respectively, symmetric) matrix with non-negative entries and

\[
b_n = \max_{i,j} \left\{ \sum_{k=1}^{n} a_{ik}^2, \sum_{k=1}^{n} a_{kj}^2, \log n \right\}.
\]

Then

\[
d_{TV}(Z_k(A \circ X), Z) \lesssim \left( \frac{\max\{|a_{ij}|\}^{2k} \sqrt{m_k} - 1}{\sum_{I_k} a_{i_1 i_2}^2 a_{i_2 i_3}^2 \cdots a_{i_k i_1}^2} \right),
\]

where \( Z \) is a standard Gaussian random variable and \( I_k = \{(i_1, \ldots, i_k) \in [n]^k : i_p \neq i_q, \text{ for all } p \neq q\} \).
Observe that, when the degree of the polynomial $k$ is fixed, the factor $k^5$ in (2) can be absorbed in the implying constant of ‘$\leq$’. We kept $k^5$ in the expression as we will change the degree of the polynomial with $n$.

To prove the theorem we use second order Poincaré inequality type result introduced in [12, Theorem 3.1]. Suppose $X = (X_1, X_2, \ldots, X_n)$ is a vector of independent standard Gaussian random variables and $g : \mathbb{R}^n \to \mathbb{R}$ is smooth function. Then Gaussian Poincaré inequality says that $\text{Var}(g(X)) \leq E\| \nabla g(X) \|$, that is, if $\| \nabla g(X) \|$ is small then $g(X)$ has small fluctuation. Now the second order Poincaré inequality says if the second order derivatives of $g$ have good behaviour then $g(X)$ is close to Gaussian. We use this idea to prove the bound in (2). For that we need to estimate a lower bound of the variance of $Z_k(A \circ X)$ and a non-asymptotic upper bound for the norm of $A \circ X$. In Lemma 15 we show that the variance of $Z_k(A \circ X)$ is bounded below by $\sum I_k a_{i_1 j_1}^2 a_{i_2 j_2}^2 \cdots a_{i_k j_k}^2$. In Lemma 19 we show that the norm of $\frac{1}{\sqrt{b_n}} A \circ X$ is bounded almost surely. The main ingredients of the proof of Lemma 19 are the concentration type inequalities and a sharp non-asymptotic bound on the expected norm of a symmetric random matrix derived in [10, Corollary 3.5]. The restriction on $b_n$ in Theorem 1 that is, $b_n \geq \log n$ is required to establish the norm bound of $\frac{1}{\sqrt{b_n}} A \circ X$.

In the next section, using Theorem 1 we show that the fluctuations of the linear eigenvalue statistics of $A \circ X$ for various types of variance profile matrices are asymptotically Gaussian.

3. Applications of Theorem 1

There are various kind of variance profile matrices that have appeared in different branches of sciences. For a good overview on variance profile matrices, we refer the readers to a recent article by Cook et. al. [14]. In the following corollaries, we show that the linear eigenvalue statistics of $A \circ X$ with different type of variance profile matrices converge to Gaussian distribution in total variation norm. We start with a basic variance profile matrix, namely, separable variance profile.

**Corollary 2** (Separable variance profile). Let $c > 0$, and $v, w \in \{c, 1\}^n$ be two deterministic vectors for each $n \geq 1$. Consider $A \circ A = vw^T$.

Suppose $P_k$ is a polynomial of degree $k$ where $k = o(\log n)$ and $X$ is same as in Theorem 1, then $d_{TV}(Z_k(A \circ X), Z) \to 0$ as $n \to \infty$.

**Proof.** Note that $a_{ij}^2 = v_i w_j$, for $1 \leq i, j \leq n$. Therefore we have

$$\sum_{k=1}^{n} a_{ik}^2 = \sum_{k=1}^{n} v_i w_k \leq n \quad \text{and} \quad \sum_{k=1}^{n} a_{kj}^2 \leq n.$$

Now, using Stirling’s approximation we have

$$\frac{1}{n^k} |I_k| = \frac{n!}{n^k (n-k)!} \sim e^{-k} \left(1 - \frac{k}{n}\right)^{-n+k} \sim 1,$$

where the last asymptotic equality follows whenever $k = o(\sqrt{n})$. Therefore,

$$\sum_{I_k} a_{i_1 j_1}^2 a_{i_2 j_2}^2 \cdots a_{i_k j_k}^2 \geq c^2 k |I_k| = c^2 k \Omega(n^k).$$
Now from Theorem 1 for $k = o(\log n)$, we have
\[
d_{TV}(Z_k(A \circ X), Z) \lesssim \frac{k^5n^{k-1}{\sqrt n}}{c^{2k}n^c} = \frac{k^5}{c^{2k}} \to 0, \text{ as } n \to \infty.
\]
Hence the result. ■

In the following corollary, the variance profile matrix is constructed from a continuous positive real valued function $f(\cdot, \cdot)$ defined on $[0, 1]^2$. This type of variance profile is known as Sampled variance profile.

If $f(x, y) = g(x)h(y)$ where $g, h : [0, 1] \to (0, \infty)$ are two continuous functions, then the corresponding variance profile is known as Separate and sampled variance profile (see [14]).

**Corollary 3** (Sampled variance profile). Consider $A = (a_{ij})$ with $a_{ij}^2 = f(i/n, j/n)$, where $f(\cdot, \cdot)$ is a positive continuous on $[0, 1]^2$ with $\int_0^1 f(x, y)dy = 1$ and $\int_0^1 f(x, y)dx = 1$. Suppose $P_k$ is a polynomial of degree $k$ where $k = o(\log n)$ and $X$ is same as in Theorem 1 then $d_{TV}(Z_k(A \circ X), Z) \to 0$ as $n \to \infty$.

**Proof.** Since $f$ is a continuous function on $[0, 1]^2$, there exists a positive constant $C$ such that $f \leq C$. Therefore, for $1 \leq i, j \leq n$, we have
\[
\sum_{k=1}^n a_{ik}^2 \leq Cn \quad \text{and} \quad \sum_{k=1}^n a_{kj}^2 \leq Cn.
\]
Again, using (35) we get
\[
\lim_{n \to \infty} \frac{1}{n^k} \sum_{l_k} a_{1i_1}^2 a_{i_2j}^2 \cdots a_{ik}^2 = \lim_{n \to \infty} \frac{1}{n^k} \sum_{l_k} f\left(\frac{i_1}{n}, \frac{i_2}{n}\right) \cdots f\left(\frac{i_k}{n}, \frac{i_1}{n}\right) \\
= \int_0^1 \cdots \int_0^1 f(x_1, x_2)f(x_2, x_3)\cdots f(x_k, x_1)dx_1 \cdots dx_k \\
= M \ (\text{say})
\]
Therefore from Theorem 1 for $k = o(\log n)$ and $\epsilon > 0$ such that $M - \epsilon > 0$, we have
\[
d_{TV}(Z_k(A \circ X), Z) \lesssim \frac{k^5C^k n^{k-1}{\sqrt n}}{n^k(M - \epsilon)} = \frac{k^5C^k}{n(M - \epsilon)} \to 0, \text{ as } n \to \infty.
\]
Hence the result. ■

**Remark 4.** Anderson and Zeitouni [7] considered $n \times n$ symmetric random matrix $Y$ with on or above diagonal terms are of the form $\frac{1}{\sqrt n} f(i/n, j/n)X_{ij}$ where $X_{ij}$ are zero mean unit variance i.i.d. random variables with all moments bounded and $f$ is a continuous function on $[0, 1]^2$ such that $\int_0^1 f^2(x, y)dy = 1$. They established CLT for linear eigenvalue statistics of $Y$ with polynomial test functions. They used moment method, and nice combinatorial arguments inspired from [10]. If the test function is continuously differentiable with polynomial growth, then they established the CLT for the random variables $X_{ij}$ which satisfy a Poincaré inequality with common constant $c$.

In the previous corollary, the variance profile was constructed from a continuous function. In particular, $a_{ij}$ were allowed to take the zero values. However if $\{a_{ij}\}$ do not originate from a continuous function, we need to assume that $\{a_{ij} : 1 \leq i, j \leq n\}$
band matrices are well studied in literature. Li and Soshnikov [25] considered symmetric periodic linear eigenvalue statistics of symmetric band random matrices. In the following corollary we show that the linear eigenvalue statistics of band random matrices converge to Gaussian distribution in total variation norm. This is shown in the following corollary.

**Corollary 5.** Let $A$ be an $n \times n$ matrix such that $\frac{1}{n^\alpha} \lesssim a_{ij} \leq 1$ with $\alpha < \frac{1}{2}$ and $cd_n \leq \sum_{k=1}^{n} a_{ik}^2, \sum_{k=1}^{n} a_{kj}^2 \leq Cd_n$, for some positive constants $c$ and $C$. If $d_n \geq \log n$, $P_k$ is a polynomial of degree $k$ with $k = o(n^{1/2 - \frac{\alpha}{2}})$ and $X$ is as in Theorem 1, then $d_{TV}(Z_k(A \circ X), Z) \to 0$ as $n \to \infty$.

**Proof.** Note that we have

$$b_n = \max_{i,j} \left\{ \sum_{k=1}^{n} a_{ik}^2, \sum_{k=1}^{n} a_{kj}^2, \log n \right\} \leq Cd_n.$$ 

For $1 \leq i_1, i_3 \leq n$, we have

$$\sum_{i_2} a_{i_1i_2}^2 a_{i_2i_3}^2 \gtrsim \frac{1}{n^{2\alpha}} \sum_{i_2} a_{i_1i_2}^2 \gtrsim \frac{cd_n}{n^{2\alpha}}.$$ 

Using the last inequality and summing over the rest of the indices one by one, we have

$$\sum_{k \in 1 \backslash \{i_2\}} a_{i_1i_2}^2 \cdots a_{i_{k-1}i_k} \gtrsim \frac{cd_n}{n^{2\alpha}} \sum_{i_2} a_{i_1i_2}^2 \cdots a_{i_{k-1}i_k} \gtrsim \frac{cd_n}{n^{2\alpha}} n(c_d)^{k-2} = \frac{(c_d)^{k-1}}{n^{2\alpha-1}}.$$ 

Therefore from (2), for $\alpha < \frac{1}{4}$ and $k = o(\log n)$, we get

$$d_{TV}(Z_k(A \circ X), Z) \lesssim \frac{k^5(c_d)^{k-1} \sqrt{m^{2\alpha-1}}}{(c_d)^{k-1}} = k^5(c/c)^{k-1}n^{2\alpha-\frac{1}{2}} \to 0,$$

as $n \to \infty$. Hence the result.

Now we shall consider the linear eigenvalue statistics of band random matrices. The fluctuations of linear eigenvalue statistics of symmetric band random matrices are well studied in literature. Li and Soshnikov [29] considered symmetric periodic band matrices $Y = (Y_{ij})$ where

$$Y_{ij} = Y_{ji} = \begin{cases} \frac{1}{\sqrt{b_n}} X_{ij} & \text{if } \min\{|i-j|, n-|i-j|\} \leq b_n, \\ 0 & \text{otherwise.} \end{cases}$$

and $\{X_{ij}; i \leq j\}$ are i.i.d. random variables. They established CLT for linear eigenvalue statistics of $Y$ when $X_{ij}$ satisfies Poincaré inequality with some constant $c$ and the test function $g$ has continuous bounded derivative, and $\sqrt{n} << b_n << n$ ($a_n << b_n$ means $a_n/b_n \to 0$ as $n \to \infty$). Later the conditions on the band width $b_n$ and on the test function were improved in [21] and [35].

Now observe that the periodic (or non-periodic) symmetric band random matrix can be seen as a symmetric random matrix with a variance profile. For example, let $A$ be a periodic (or non-periodic) band matrix with band length $b_n$, that is,

$$a_{ij} = \begin{cases} 1 & \text{if } \min\{|i-j|, n-|i-j|\} \leq b_n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $A \circ X$ is a periodic (or non-periodic) symmetric band random matrix if $X$ is a symmetric random matrix. In the following corollary we show that the linear eigenvalue statistics of band random matrices, symmetric and non-symmetric both, converge to Gaussian distribution in total variation norm.
Corollary 6. Let $A$ be a periodic (or non-periodic) band matrix with band length $b_n$ as defined in [1]. Suppose $b_n \geq \log n$, $P_k$ is a polynomial of degree $k$ where $k = o(n^{1/10})$ and $X$ is same as in Theorem [1] then

$$d_{TV}(Z_k(A \circ X), Z) \to 0 \quad \text{as} \quad n \to \infty.$$ 

Proof. If $A$ be a band matrix with band length $b_n$, so is $A \circ A$. Now it is easy to see that

$$\sum_{l_k} a_{i_1i_2}^2 a_{i_2i_3}^2 \cdots a_{i_ki_1}^2 \gtrsim n b_n^{k-1}.$$ 

The result follows from (2). 

In the next corollary we obtain the fluctuation of linear eigenvalue statistics for sample covariance matrix $XX^t$ using Theorem [1]

Corollary 7. Let $X = (X_{ij})$ be an $n \times m$ random matrix with i.i.d. entries, where $X_{ij}$ are as in Theorem [1]. If $\frac{m}{n} \to c$ as $n \to \infty$ for some $c > 0$. Then, for $k = o(\log n)$,

$$d_{TV}(Z_k(XX^t), Z) \to 0, \quad \text{as} \quad n \to \infty,$$

where $P_k$ as in Theorem [1] and $X^t$ denotes the transpose of $X$.

Proof. Without loss of generality we assume that $n \leq m$. Let $Y$ be a $(n+m) \times (n+m)$ symmetric matrix as in Theorem [1]. Observed that

$$(A \circ Y)^2 = \begin{bmatrix} XX^t & 0 \\ 0 & XX^t \end{bmatrix}.$$ 

Let $Q_{2k}(x) = P_k(x^2)$. Then $\text{Tr}(Q_{2k}(A \circ Y)) = 2\text{Tr}(P_k(XX^t)) + (m-n)c_0$, where $c_0$ is the constant term in the polynomial $P_k$. Also observe that

$$Z_k(XX^t) = \frac{\text{Tr}(P_k(XX^t)) - \mathbb{E}[\text{Tr}(P_k(XX^t))]}{\sqrt{\text{Var} \text{Tr}(P_k(XX^t))}}$$

$$= \frac{\text{Tr}(Q_{2k}(A \circ Y)) - \mathbb{E}[\text{Tr}(Q_{2k}(A \circ Y))]}{\sqrt{\text{Var} \text{Tr}(Q_{2k}(A \circ Y))}} =: Z_{2k}^Q(A \circ Y) \text{ (say).}$$

Note that in $A \circ Y$, $b_{n+m} = m$ and we have

$$\sum_{l_{2k}} a_{i_1i_2}^2 \cdots a_{i_{2k}i_1}^2 = \sum_{i_1 \neq i_2 \neq \cdots \neq i_{2k-1} = 1 \atop i_2 \neq i_4 \neq \cdots \neq i_{2k} = n+1} \sum_{i_2}^{n+m} a_{i_1i_2}^2 \cdots a_{i_{2k}i_1}^2 \sim (nm)^k.$$ 

Therefore from Theorem [1] for $k = o(\log n)$, we get

$$d_{TV}(Z_{2k}^Q(A \circ Y), Z) \lesssim \frac{(2k)^5 \sqrt{n + m} m^{2k-1}}{(nm)^k} \lesssim \frac{k^5}{\sqrt{n}} \frac{m}{n} \left(\frac{m}{n}\right)^{k-1} \to 0, \quad \text{as} \quad n \to \infty.$$ 

Hence the result.
In the next corollary we derive the fluctuations of linear statistics for the product of independent copies of i.i.d random matrices. The fluctuation of linear eigenvalue statistics of a single i.i.d. matrix was considered in [33]. Coston and O’Rourke have considered the fluctuations of linear eigenvalue statistics for product of \( m \) independent copies of i.i.d. matrices with analytic test functions. Here we derive that result with polynomial test functions when the entries of the i.i.d. matrices are in \( L(c_1, c_2) \).

**Corollary 8.** Let \( P_k \) and \( X \) be as defined in Theorem 1. Let \( X_1, \ldots, X_m \) be \( m \) independent copies of \( X \). Then, for fixed \( m \) and \( k = o(n^{1/10}) \),

\[
d_{TV}(Z_k(X_1 \cdots X_m), Z) \to 0, \quad \text{as } n \to \infty.
\]

**Proof.** For \( m = 1 \), the proof is straight forward. For \( m = 2 \), the idea of the proof is similar to the proof of Corollary 1. The only difference is that here we start with a \( 2n \times 2n \) i.i.d. random matrix \( Y \) instead of a \( 2n \times 2n \) symmetric random matrix. Then

\[
A \circ Y = \begin{bmatrix} 0 & X_1 \\ X_2 & 0 \end{bmatrix}, \quad \text{where } A = \begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix}
\]

and \( A_{12}, A_{21} \) are two \( n \times n \) matrices with all entries 1. Observe that \( X_1, X_2 \) are two independent \( n \times n \) i.i.d. matrices. Now the rest of the proof is similar to Corollary 1. We skip the detail.

Now we prove the result for \( m = 3 \). For \( m > 3 \), it can be proved in a similar way. Let \( Y \) be a \( 3n \times 3n \) dimensional i.i.d. matrix as in Theorem 1. Then

\[
A \circ Y = \begin{bmatrix} 0 & X_1 & 0 \\ 0 & 0 & X_2 \\ X_3 & 0 & 0 \end{bmatrix}, \quad \text{where } A = \begin{bmatrix} 0 & A_{12} & 0 \\ 0 & 0 & A_{23} \\ A_{31} & 0 & 0 \end{bmatrix}
\]

and \( A_{12}, A_{23}, A_{31} \) are three \( n \times n \) matrices with all entries 1. Observe that \( X_1, X_2, X_3 \) are three independent \( n \times n \) i.i.d. random matrices and

\[
(A \circ Y)^3 = \begin{bmatrix} X_1X_2X_3 & 0 & 0 \\ 0 & X_2X_3X_1 & 0 \\ 0 & 0 & X_3X_2X_1 \end{bmatrix}.
\]

Now, let \( Q_{3k}(x) = P_k(x^3) \). Then \( \text{Tr}(Q_{3k}(A \circ Y)) = 3\text{Tr}(P_k(X_1X_2X_3)) \) and

\[
Z_k(X_1X_2X_3) = \frac{\text{Tr}(P_k(X_1X_2X_3)) - \mathbb{E}[\text{Tr}(P_k(X_1X_2X_3))]}{\sqrt{\text{Var}(\text{Tr}(P_k(X_1X_2X_3)))}}
\]

\[
= \frac{\text{Tr}(Q_{3k}(A \circ Y)) - \mathbb{E}[\text{Tr}(Q_{3k}(A \circ Y))]}{\sqrt{\text{Var}(Q_{2k}(A \circ Y))}} =: Z_{3k}^Q(A \circ Y) \quad \text{(say)}.
\]

Note that in \( A \circ Y \), \( b_{3n} = n \) and we have

\[
\sum_{i_1 \neq i_2 \ldots \neq i_{3k-1} \neq i_{3k} = 1}^n \sum_{i_2 \neq i_3 \ldots \neq i_{3k-2} = 1}^{2n} \ldots \sum_{i_{3k-1} \neq i_{3k} = 2n+1}^{3n} \sum_{i_1 \neq i_2 \ldots \neq i_{3k} = 2n+1}^{2n+1} a_{i_1i_2} a_{i_2i_3} \ldots a_{i_{3k}i_1} \sim n^{3k}.
\]

Therefore from Theorem 1 for \( k = o(n^{1/10}) \), we get

\[
d_{TV}(Z_{3k}^Q(A \circ Y), Z) \lesssim \frac{(3k)^5 \sqrt{3n^{3k-1}}}{n^{3k}} \lesssim \frac{k^5}{\sqrt{n}} \to 0, \quad \text{as } n \to \infty.
\]
Hence the result.

In the last corollary we considered finite product of independent copies of i.i.d. random matrices. However, using the same technique one can study the fluctuation of linear eigenvalue statistics for product of rectangular random matrices with i.i.d. entries. Suppose $X_1, X_2, \ldots, X_m$ are independent rectangular random matrices with i.i.d. entries of dimensions $n_1 \times n_2, n_2 \times n_3, \ldots, n_m \times n_1$, respectively. With out loss of generality we assume that $n_1 = \min_{1 \leq k \leq m} n_k$. Now following the idea of the proofs of Lemma 7 and Lemma 8, it is easy to show that the fluctuation of linear eigenvalue statistics of $X_1 X_2 \cdots X_m$ is Gaussian if $\max_{1 \leq k \leq m} n_k/n_1 \to c < \infty$ as $n_1 \to \infty$.

In the next corollary we consider $A$ as an adjacency matrix of an Erdős-Rényi random graph $G(n, p_n)$.

**Corollary 9.** Let $X$ be a random matrix as defined in Theorem 1, and $A$ be the adjacency matrix of a random Erdős-Rényi random graph $G(n, p_n)$ independent of $X$, where $p_n \geq n^{-\gamma}$ for some $\gamma \in [0, 1/2)$. Let $P_k$ be a polynomial of degree $k$, where $k = o(\log n / \log(2/p_n))$. Then

$$d_{TV}(Z_k(A \circ X), Z) \to 0, \text{ almost surely as } n \to \infty.$$ 

Here almost surely is with respect to the probability measure on $G(n, p_n)$.

The following lemma will be used in the proof of Corollary 9.

**Lemma 10.** Let $A$ be the matrix as defined in Corollary 9 with $p_n \geq n^{-\gamma}$ for some fixed $\gamma \in [0, 1/2)$. Then

(i) $\frac{1}{(np_n)^k} \sum_{I_k} a_{i_1 i_2}^2 \cdots a_{i_k i_1}^2 \to 1$, almost surely, as $n \to \infty$,

(ii) $\max_i \sum_{j=1}^n a_{ij}^2 \leq (1 + \epsilon_n)np_n$, almost surely, as $n \to \infty$,

for $k = o(\log n / \log(2/p_n))$ and $\epsilon_n = O(n^{-\alpha} p_n^{-1})$ with $\alpha \in (\gamma, 1/2)$.

The following inequality will be used in the proof of Lemma 10.

**Result 11** (Bounded difference inequality). [29, Lemma 1.2] Let $Z = f(X_1, X_2, \ldots, X_n)$ be a function of independent random variables $X_1, \ldots, X_n$. Let $X'_i$ be an independent copy of $X_i, i = 1, \ldots, n$. Suppose $c_1, \ldots, c_n$ are constants such that for each $i$,

$$|f(X_1, \ldots, X_{i-1}, X'_i, X_{i+1}, \ldots, X_n) - f(X_1, \ldots, X_n)| \leq c_i \text{ almost surely.}$$

Then for any $t \geq 0$ we have

$$\mathbb{P}(Z - \mathbb{E}[Z] \geq t) \leq \exp \left\{ - \frac{2t^2}{\sum_{i=1}^n c_i^2} \right\}.$$ 

**Proof of Lemma 10.** The entries of $A$ can be thought as a vector of length $n(n + 1)/2$, use dictionary order. Define

$$f(A) = \sum_{I_k} a_{i_1 i_2}^2 \cdots a_{i_k i_1}^2.$$
Let $a'_{ij}$ be an independent copy of $a_{ij}$, for $1 \leq i \leq j \leq n$. Then

$$|f(A) - f(A - a_{12}e_1e_2^t + a'_{12}e_1e_2^t)| = k|a_{12}^2 - a'_{12}^2| \sum_{l=2}^k a_{2i_1i_2}^2 \cdots a_{ik-1}^2 \leq 2k|I'_{k-2}| \leq 2kn^{k-2}. $$

Similarly, for $1 \leq i, j \leq n$, we have

$$|f(A) - f(A - a_{ij}e_ie_j^t + a'_{ij}e_ie_j^t)| \leq 2kn^{k-2}. $$

Note that only $(i, j)$-th element $a_{ij}$ is replaced by $a'_{ij}$. Since $a_{i_1i_2}, \ldots, a_{ik-1}$, for $(i_1, \ldots, i_k) \in I_k$, are independent random Bernoulli random variables with parameter $p_n$, we have

$$\mathbf{E}\left[\sum_{I_k} a_{i_1}^2 \cdots a_{i_k}^2\right] = |I_k|p_n^k. $$

Using Stirling’s approximation, we have

$$\frac{1}{n^k}|I_k| = \frac{n!}{n^k(n-k)!} \sim e^{-k} \left(1 - \frac{k}{n}\right)^{-n+k} \sim 1,$$

where the last asymptotic equality follows from $k = o(\sqrt{n})$. Now, using the Result [11] for $\delta > 0$, we get

$$\mathbf{P}\left(\left|\frac{\sum I_k a_{i_1i_2}^2 \cdots a_{i_k}^2 - \mathbf{E}[\sum I_k a_{i_1i_2}^2 \cdots a_{i_k}^2]}{(np_n)^k}\right| > \delta\right) \leq \exp\left\{-\frac{\delta^2n^2p_n^{2k}}{2k^2}\right\}. $$

According to the choices of $k$ and $p_n$, we have $np_n^k/k \geq \sqrt{n}/\log n$. Hence the result (i) follows from the Borel-Cantelli lemma.

To prove (ii), let us define

$$\Omega_n = \{A \in G(n, p_n): \sum_{i=1}^n a_{ij}^2 \leq (1 + \epsilon_n)np_n, \text{ for all } i\},$$

where $\epsilon_n = p_n^{-1}n^{-\alpha}$. Using union bound and result [11]

$$\mathbf{P}(\Omega_n^c) \leq n\mathbf{P}\left(\sum_{i=1}^n a_{ij} > (1 + \epsilon_n)np_n\right) \leq n\exp\{-\epsilon_n^2np_n^2/2\}. $$

According the choice of $\epsilon_n$, and $p_n$, we have $\epsilon_n^2np_n^2 \geq n^{1-2\alpha}$. Consequently, $\sum_{n=1}^\infty n\exp\{-\epsilon_n^2np_n^2\} < \infty$. Hence the result (ii) follows from the Borel-Cantelli lemma.

**Proof of Corollary** [9] By Theorem [11] Lemma [11] almost surely for each $A$, we have

$$d_{TV}(Z_k(A \circ X), Z) \leq \frac{k^5 \sqrt{n}n^{k-1}}{p_n \sqrt{n} \sum I_k a_{i_1i_2}^2 \cdots a_{i_k}^2} \leq \frac{k^5 \sqrt{n}(1 + \epsilon_n)^{k-1}(np_n)^{k-1}}{p_n \sqrt{n} \sum I_k a_{i_1i_2}^2 \cdots a_{i_k}^2} = \frac{k^5(1 + \epsilon_n)^{k-1}(np_n)^k}{p_n \sqrt{n} \sum I_k a_{i_1i_2}^2 \cdots a_{i_k}^2}. $$

If we choose $k = o(\log n/\log(2/p_n))$, then the above converges to zero as $n \to \infty$. Hence the result.
In the following corollary we consider the variance profiles which have an $m \times m$ block of ones along the diagonal, where $m$ is of the order of $n$. So this corollary captures the fluctuation of linear eigenvalue statistics of a specific type of (symmetric or non-symmetric) sparse random matrices which have a block of non-zero entries of dimension $cn \times cn$ along the diagonal for some $c \in (0, 1)$. For more results on fluctuations of symmetric sparse random matrices, we refer to [35].

**Corollary 12.** Let $P_k, X, A$ be as in Theorem 1. Assume that there exists $J \subseteq [n]$ such that $a_{ij} = 1$ for all $i, j \in J$, where $|J| \geq cn$ for some $0 < c < 1$. Then for any $k = o(\log n)$,

$$d_{TV}(Z_k(A \circ X), Z) \to 0 \quad \text{as} \quad n \to \infty.$$ 

**Proof.** We observe that $b_n \leq n$ and from the assumption, we have

$$\sum_{I_k} a_{i_1 i_2}^2 \cdots a_{i_k i_1}^2 \geq \sum_{J_k} a_{i_1 i_2}^2 \cdots a_{i_k i_1}^2 = c_k n^k.$$ 

Now using the above estimates in Theorem 1 we have the result. 

**Remark 13.** At first glance the assumption on $A$ in Corollary 12 looks quite similar to super regularity or broad connectivity condition (see [14]). For readers’ convenience we give the definitions of super regularity and broad connectivity in Section 5. But a close check reveals that the assumption in Corollary 12 is weaker than super regularity, and it is not comparable with broad connectivity. To see the difference from broad connectivity consider the following examples:

(i) Consider an $n \times n$ matrix $A$ of the form

$$A = \frac{1}{\sqrt{n}} \begin{bmatrix} 0 & A_{12} & A_{13} & 0 & 0 \\ 0 & 0 & A_{23} & A_{24} & 0 \\ A_{41} & 0 & 0 & 0 & A_{45} \\ A_{51} & A_{52} & 0 & 0 & 0 \end{bmatrix},$$

where each $A_{ij}$ is $\lfloor \frac{n}{5} \rfloor \times \lfloor \frac{n}{5} \rfloor$ matrices with all 1. The above matrix does not satisfy the assumption of Corollary 12 since there does not exist any index set $J \subseteq [n]$ such that $a_{ij} = 1$ for all $i, j \in J$. However, $A$ is a $(\delta, \nu)$-broadly connected matrix. Now we see that $Z_k(A) \equiv 0$ for all $n$ and $k = 1, 2$, that is, $Z_k(A) \not\to Z$.

(ii) Now consider an $n \times n$ matrix $A$ of the form

$$A = \frac{1}{\sqrt{n}} \begin{bmatrix} 0 & A_{12} & 0 & 0 & 0 \\ 0 & 0 & A_{23} & 0 & 0 \\ 0 & 0 & 0 & A_{34} & 0 \\ 0 & 0 & 0 & 0 & A_{45} \\ A_{51} & 0 & 0 & 0 & 0 \end{bmatrix},$$

where each $A_{ij}$ is $\lfloor \frac{n}{5} \rfloor \times \lfloor \frac{n}{5} \rfloor$ matrices with all 1. In this case, it is easy to see that $Z_k(A \circ X) \equiv 0$ if $k$ is not a multiple of 5. In other cases, $Z_k(A \circ X) \to Z$ as $n \to \infty$. Here also, $A$ is a $(\delta, \nu)$-broadly connected matrix but it does not satisfy the assumptions of Corollary 12.

(iii) Consider the following $n \times n$ matrix:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & O \end{bmatrix}$$
where $A_{11}$ is $n/4 \times n/4$ matrix will all entries 1, $A_{12}$ is $n/4 \times 3n/4$ matrix will all entries 1, $A_{21}$ is $3n/4 \times n/4$ matrix will all entries 1 and $O$ is $3n/4 \times 3n/4$ matrix will all entries 0. The matrix $A$ satisfies the assumptions of Corollary 12 but it is not $(\bar{\delta}, \nu)$-broadly connected. Observe that $\delta \leq 1/4$ and $A$ does not satisfy condition (iii) of the definition of broad connectivity (see Section 3). For example, take $J = \{n/4 + 1, \ldots , n\}$. Then $N_{A\tau}(J) = \{i \in [n] : |N_{A}(i) \cap J| \geq \frac{2n}{\nu} \} = \{1, 2, \ldots , n/4\}$. So $|N_{A\tau}(J)| = n/4 \ngeq \min\{n, (1+\nu)\frac{2n}{\nu}\}$. However, $Z_{k}(A \circ X)$ will converge to $Z$ in total variation norm for $k = o(\sqrt{n})$.

In Remark 13 we showed that the fluctuations of linear eigenvalue statistics are not Gaussian in general for some specially structured variance profile matrices. But such specially structured matrix can arise in an Erdős-Rényi random graph. So an obvious question arise, whether our observation in Remark 13 contradicts Corollary 8 or not? It does not contradict, since in Erdős-Rényi random graphs the probability that any special structure emerges in the graph (and hence in the adjacency matrix) is almost zero.

**Remark 14.** From Corollary 12 it also follows that the linear eigenvalue statistics of (symmetric and non-symmetric) anti-diagonal band random matrices converge to Gaussian distribution in total variation norm when the band length is of the order of $n$. By anti-diagonal random band matrix, we mean the matrices of the form $A \circ X$ with the following type of $A$:

$$
A = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{bmatrix}
$$

or

$$
A = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}
$$

4. Proof of Theorem 1

In this section we give the proof of Theorem 1. The following lemmata will be used in the proof of Theorem 1.

4.1. Preliminary lemmas.

**Lemma 15.** Suppose $A$ and $X$ are as in Theorem 1. Then

$$
\text{Var}(\text{Tr}((A \circ X)^k)) \geq \sum_{I_k} a_{i_1i_2}^2 \cdots a_{i_ki_1}^2,
$$

where $I_k = \{(i_1, \ldots , i_k) \in [n]^k : i_p \neq i_q, \text{ for all } p \neq q\}$.

**Proof.** For a positive integer $k$, we have

$$
\text{Var}(\text{Tr}(A \circ X)^k) = E[(\text{Tr}(A \circ X)^k)^2] - E[(\text{Tr}(A \circ X)^k)]^2
$$

$$
= \sum_{I_k, j_k} (E[X_{i_1j_1} \cdots X_{i_kj_k}X_{j_1j_2} \cdots X_{j_kj_1}])a_{i_1j_1} \cdots a_{i_kj_k}a_{j_1j_2} \cdots a_{j_kj_1},
$$
where $I'_k, J'_k \in [n]^k$. Note that, we have

$$E[X_{i_1j_1} \cdots X_{i_kj_k}] - E[X_{i_1j_2} \cdots X_{i_kj_1}] = \prod_{i,j} E[X_{ij}^{\alpha_{ij} + \beta_{ij}}] - \prod_{i,j} E[X_{ij}^{\alpha_{ij}}]E[X_{ij}^{\beta_{ij}}] \geq 0,$$

(7) for some $\alpha_{ij}, \beta_{ij} \in \{0, 1, 2, \ldots\}$ and $i, j \in \{1, 2, \ldots, n\}$. The last inequality follows from the fact that $E[X_{ij}^{\alpha_{ij} + \beta_{ij}}] \geq E[X_{ij}^{\alpha_{ij}}]E[X_{ij}^{\beta_{ij}}]$, as $X_{ij}$ are symmetric random variables. Indeed, if $\alpha_{ij} + \beta_{ij}$ is odd then $E[X_{ij}^{\alpha_{ij} + \beta_{ij}}] = 0$ and one of $E[X_{ij}^{\alpha_{ij}}]$ and $E[X_{ij}^{\beta_{ij}}]$ is zero, as a symmetric random variable has odd moments zero. Suppose $\alpha_{ij} + \beta_{ij}$ is even and both $\alpha_{ij}, \beta_{ij}$ are odd, then $E[X_{ij}^{\alpha_{ij}}] = E[X_{ij}^{\beta_{ij}}] = 0$. Finally if both $\alpha_{ij}, \beta_{ij}$ are even, then by Hölder inequality we have

$$E[X_{ij}^{\alpha_{ij}}] \leq (E[X_{ij}^{\alpha_{ij} + \beta_{ij}}])^{\alpha_{ij}/2 + \beta_{ij}/2} \text{ and } E[X_{ij}^{\beta_{ij}}] \leq (E[X_{ij}^{\alpha_{ij} + \beta_{ij}}])^{\beta_{ij}/2 + \alpha_{ij}/2}.$$

Therefore $E[X_{ij}^{\alpha_{ij} + \beta_{ij}}] \geq E[X_{ij}^{\alpha_{ij}}]E[X_{ij}^{\beta_{ij}}]$. Since $a_{ij}$ are non-negative, from (6) and (7) we get

$$\text{Var}(\text{Tr}(A \circ X)^k) \geq \sum_{I_k} E[X_{i_1j_2}^2 \cdots X_{i_kj_1}^2] a_{i_1j_2}^2 \cdots a_{i_kj_1}^2 = \sum_{I_k} a_{i_1j_2}^2 \cdots a_{i_kj_1}^2.$$

In the last line we used the fact that $E[X_{i_1j_2} \cdots X_{i_kj_1}] = 0$ and $E[X_{i_1j_2}^2 \cdots X_{i_kj_1}^2] = 1$ for $(i_1, \ldots, i_k) \in I_k$, as $X_{ij}$ are i.i.d. random variables with mean zero and variance one. This completes the proof of the lemma.

\begin{result}
\textbf{Theorem 1.2} Let $\mu$ be a probability measure on $\mathbb{R}^n$ such that $\mu(dx) \propto \exp\{\sum_{i=1}^n V(x_i)\}$. Assume that $V: \mathbb{R} \to \mathbb{R}$ is a $\kappa$-convex function i.e., $V(x) - \kappa x^2/2$ is a convex function for some fixed $\kappa > 0$. Then for any $g \in H^2(\mu)$,

$$\text{Ent}(g^2) \leq \frac{2}{\kappa} \int_{\mathbb{R}^n} |\nabla g|^2 \, d\mu,$$

where $\text{Ent}(g)$ is defined by

$$\text{Ent}(g) := \int_{\mathbb{R}^n} g \log g \, d\mu - \left( \int_{\mathbb{R}^n} g \, d\mu \right) \log \left( \int_{\mathbb{R}^n} g \, d\mu \right)$$

for any $g > 0$.

\begin{result}
\textbf{Proposition 2.3} Let $\{X_1, X_2, \ldots, X_k\}$ be a collection of random variables with $E[e^{X_i}] < \infty$ for all $\lambda \in \mathbb{R}$ and

$$\text{Ent}(e^{X_i}) \leq c^2 E[e^{X_i}] \text{ for } 1 \leq i \leq k.$$Suppose $F: \mathbb{R}^k \to \mathbb{R}$ be a Lipschitz function with $\|F\|_{\text{Lip}} \leq 1$. Then

$$P (F(X_1, \ldots, X_k) \geq E[F(X_1, \ldots, X_k)] + t) \leq e^{-t^2/c^2},$$

for every $t \geq 0$.

\begin{result}
\textbf{Corollary 3.5} Let $X$ be an $n \times n$ symmetric matrix with $X_{ij} = \xi_{ij} b_{ij}$, where $\{\xi_{ij} : i \geq j\}$ are independent centered random variables and $\{b_{ij} : i \geq j\}$ are given scalars. If $E[|\xi_{ij}|^{2p}] \leq Kp^\beta$ for some $K, \beta > 0$ and all $p, i, j$ then

$$E\|X\| \lesssim \max_i \sqrt{\sum_j b_{ij}^2} + \max_{ij} |b_{ij}|(\log n)^{\max(\beta, 1)/2}.$$
The universal constant in the above inequality depends on $\beta$ only.

**Lemma 19.** Let $X = (X_{ij})_{n \times n}$ be a random matrix, where \{(X_{ij}, X_{ji}) : 1 \leq i, j \leq n\} be a collection of independent centered random vectors with $E[X_{ij}^2] = E[X_{ji}^2] = 1$, $E[X_{ij}X_{ji}] = \rho \in \{0, 1\}$ and $X_{ij} \in \mathcal{L}(c_1, c_2)$, for some fixed $c_1 > 0, c_2 > 0, \forall i, j$. Let $A = (a_{ij})_{n \times n}$ be a fixed deterministic matrix such that $a_{ij} \in [0, 1]$, and in addition, $a_{ij} = a_{ji}$ for all $i, j$ if $\rho = 1$. Then

$$
(8) \quad P \left( \|A \circ X\| > K \max_{i,j} \left( \sqrt{\sum_k a_{ik}^2} + \sqrt{\sum_k a_{kj}^2} + |a_{ij}| \sqrt{\log n} \right) + t \right) \leq e^{-t^2/c^2},
$$

where $K > 0$ is an universal constant. In particular, with probability one

$$
\|A \circ X\| \leq K \max_i \sqrt{\sum_j a_{ij}^2} + K \max_j \sqrt{\sum_i a_{ij}^2} + (K \max_{ij} |a_{ij}| + \sqrt{2c_1}) \sqrt{\log n}
$$

for all but finitely many $n$.

**Proof.** This proof is based on [10] Corollary 3.5. First of all if $\rho = 1$, then both $A$ and $X$ are symmetric matrices. In that case, the result[10] gives bound on $E\|A \circ X\|$. If $\rho = 0$, let us write the matrix $A \circ X$ in the following way.

$$
A \circ X = \frac{1}{2} [(A \circ X) + (A \circ X)^\dagger] + \frac{1}{2} [(A \circ X) - (A \circ X)^\dagger]
$$

$$
= (B_n^+ \circ X^+) + (B_n^- \circ X^-),
$$

where

$$(B_n^+)_{ij} := \sqrt{a_{ij}^2 + a_{ji}^2}, \quad (B_n^-)_{ij} := \sqrt{a_{ij}^2 + a_{ji}^2},
$$

$$(X^+)_{ij} := (a_{ij}X_{ij} + a_{ji}X_{ji})/(2B_n^+)_{ij}, \quad (X^-)_{ij} := (a_{ij}X_{ij} - a_{ji}X_{ji})/(2B_n^-)_{ij}.
$$

In the above definitions, we use the convention that $(X^+)_{ij} = 0 = (X^-)_{ij}$ if both $a_{ij} = 0 = a_{ji}$. Since $X_{ij} \in \mathcal{L}(c_1, c_2)$, we may write

$$
X_{ij} = u(0) + u'(\theta)_{ij} Z_{ij},
$$

where $Z_{ij} \sim N(0, 1)$, and $\theta_{ij}$ lies in between 0 and $Z_{ij}$. Since $X_{ij} \in \mathcal{L}(c_1, c_2)$, we have $|u'(\theta Z)| \leq c_1$. Therefore for any $p \in \mathbb{N},$

$$
E[|X_{ij}^{2p}|^{1/2p} \leq |u(0)| + c_1 E|Z_{ij}^{2p}|^{1/2p} \leq |u(0)| + c_1 \sqrt{p} \leq C \sqrt{p},
$$

where $C = |u(0)| + c_1$. Consequently by Minkowski’s inequality, we have

$$
E[|X_{ij}^{\nu}|^{2p}]^{1/2p} \leq \frac{C}{2(B_{ij}^\nu)} \sqrt{[|a_{ij}| + |a_{ji}|]} \sqrt{p} \leq C \sqrt{p}, \quad \nu \in \{+, -\},
$$

where the last inequality follows from the fact that $|a_{ij}| + |a_{ji}| \leq 2(B_{ij}^\nu)$ for $\nu \in \{+, -\}$. Hence using result[10], we conclude that for $\nu \in \{+, -\},$

$$
E\|B_n^\nu \circ X^\nu\| \leq K \left[ \max_i \sqrt{\sum_j (B_{ij}^\nu)^2} + \max_{ij} (B_{ij}^\nu) \sqrt{\log n} \right].
$$
\[ \leq K \max_{i} \sqrt{\sum_{j} a_{ij}^2} + K \max_{j} \sqrt{\sum_{i} a_{ij}^2} + K \max_{ij} |a_{ij}| \sqrt{\log n}, \]

where \( K > 0 \) is a universal constant.

To prove the almost sure bound, we invoke the results\([16]\) and\([17]\). If \( Z \sim N(0, 1) \) and \( u : \mathbb{R} \to \mathbb{R} \) is a differentiable function with \( |u'| \leq c_1 \) uniformly, then taking \( \kappa = 1/2 \) in result\([16]\)

\[ \text{Ent} \left( e^{u(Z)} \right) \leq 4E \left[ \left| \frac{u'(Z)}{2} \right|^2 e^{u(Z)} \right] \leq c_1^2 E \left[ e^{u(Z)} \right]. \]

Now since \( \|M\| \leq \sqrt{\sum_{i,j} M_{ij}^2} \), \( M \mapsto \|M\| \) is a Lipschitz function with Lipschitz constant equal to 1. Therefore, the estimate\([3]\) follows from the result\([17]\) \( \blacksquare \).

4.2. Proof of the Theorem\([1]\). We first introduce some notations which will be used in the proof of Theorem\([1]\).

Let \( J \) be a finite index set. Define

\[ R = \{ \alpha \in \mathbb{C}^J : \sum_{\alpha \in J} |\alpha_u|^2 = 1 \}, \quad S = \{ \beta \in \mathbb{C}^{n \times n} : \sum_{i,j=1}^n |\beta_{i,j}|^2 = 1 \}. \]

Let \( H(x) = (h_{ij}) \) be an \( n \times n \) matrix, where \( h_{ij} : \mathbb{R}^J \to \mathbb{C} \) are \( C^2 \) maps for \( 1 \leq i,j \leq n \). Define three functions \( \gamma_0, \gamma_1 \) and \( \gamma_2 \) on \( \mathbb{R}^J \) as follows

\[ \gamma_0(x) = \sup_{u \in J, \|Bu\| = 1} \left| \text{Tr} \left( B \frac{\partial H}{\partial x_u} \right) \right|, \]
\[ \gamma_1(x) = \sup_{\alpha \in R, \beta \in S} \left| \sum_{u \in J} \sum_{i,j} \alpha_u \beta_{i,j} \frac{\partial h_{ij}}{\partial x_u} \right|, \]
\[ \gamma_2(x) = \sup_{\alpha, \alpha' \in R, \beta \in S} \left| \sum_{u, v \in J} \sum_{i,j} \alpha_u \alpha'_{i,j} \frac{\partial^2 h_{ij}}{\partial x_u \partial x_v} \right|. \]

Let \( \lambda(x) = \|H(x)\| \), \( r(x) = \text{rank}(H(x)) \). Define a few more functions

\[ \eta_0(x) = \gamma_0(x) f_1(\lambda(x)), \quad \eta_1(x) = \gamma_1(x) f_1(\lambda(x)) \sqrt{r(x)}, \]
\[ \eta_2(x) = \gamma_2(x) f_1(\lambda(x)) \sqrt{(r(x))} + \gamma_1(x)^2 f_2(\lambda(x)), \]
\[ \kappa_0 = \left( \mathbb{E} \eta_0(x)^2 \right)^{\frac{1}{2}}, \quad \kappa_1 = \left( \mathbb{E} \eta_1(x)^4 \right)^{\frac{1}{4}}, \quad \kappa_2 = \left( \mathbb{E} \eta_2(x)^4 \right)^{\frac{1}{4}}. \]

The following result from\([12]\) is the main ingredient of our proof of Theorem\([1]\).
Result 20. [12] Theorem 3.1] Let all notation be as above. Suppose $W = \text{Re} \text{Tr} f(H(x))$ has finite fourth moment and let $\sigma^2 = \text{Var}(W)$. Let $Z$ be a random variable with the same mean and variance as $W$. Then

$$d_{TV}(W, Z) \leq \frac{2\sqrt{5}(c_1c_2\kappa_0 + c_3\kappa_1\kappa_2)}{\sigma^2}.$$  

In our setting, $H_n = A \circ X$ and $J = \{(i, j) : 1 \leq i, j \leq n\}$ when $X$ is an iid matrix. We will use Result 20 to prove the theorem and for that we first estimate $\kappa_0, \kappa_1$ and $\kappa_2$ for $A \circ X$. Note

$$\gamma_0(x) = \sup_{u \in J, \|u\| = 1} \left| \text{Tr} \left( B \frac{\partial(A \circ X)}{\partial x_u} \right) \right| \leq \max |a_{ij}|,$$

$$\gamma_1(x) = \sup_{\alpha, \beta \in \mathbb{S}} \left| \sum_{i \in J} \sum_{j} \alpha_i \beta_j \frac{\partial(a_{ij}x_{ij})}{\partial x_u} \right| \leq \max |a_{ij}|,$$

$$\gamma_2(x) = \sup_{\alpha, \alpha' \in \mathbb{R}, \beta \in \mathbb{S}} \left| \sum_{i \in J} \sum_{j} \alpha_i \alpha'_j \frac{\partial^2(a_{ij}x_{ij})}{\partial x_u \partial x_v} \right| = 0.$$  

Let $f(z) = P_k(z) = \sum_{i=0}^n c_i z^i$, where $c_0, \ldots, c_k \in \mathbb{R}$ with $c_k \neq 0$. Then, for $\lambda \geq 1$,

$$f(\lambda) = \lambda^k \sum_{i=0}^k c_i \lambda^{i-k} \leq k \lambda^k \max \{|c_0|, \ldots, |c_k|\} \lesssim k \lambda^k,$$

$$f_1(\lambda) = k \lambda^{k-1} \sum_{i=1}^k |c_i| \frac{\lambda^{i-k}}{k} \lesssim k^2 \lambda^{k-1},$$

$$f_2(\lambda) = k(k-1) \lambda^{k-2} \sum_{i=2}^k |c_i| \frac{i(i-1)}{k(k-1)} \lambda^{i-k} \lesssim k^3 \lambda^{k-2}.$$  

Then we have

$$\eta_0(x) = \gamma_0(x)f_1(\lambda_n(x)) \lesssim \max |a_{ij}| k^2 \lambda_n(x)^{k-1},$$

$$\eta_1(x) = \gamma_1(x)f_1(\lambda_n(x)) \sqrt{r_n(x)} \lesssim \max |a_{ij}| k^2 \lambda_n(x)^{k-1} \sqrt{r_n(x)},$$

$$\eta_2(x) = \gamma_2(x)f_1(\lambda_n(x)) \sqrt{(r_n(x))} + \gamma_1(x)f_2(\lambda(x)) \lesssim (\max |a_{ij}|)^2 k^3 \lambda_n(x)^{k-2},$$

where $\lambda_n = \|A \circ X\|$. Again, note that, Lemma 19 implies that $\lambda_n \leq \sqrt{b_n}$. Since $\text{rank}(A \circ X) = r_n \leq n$ almost surely

$$\kappa_0 = (\mathbb{E} \eta_0(x)^2 \eta_1(x)^2)^{\frac{1}{2}} \lesssim (\max |a_{ij}|)^2 k^4 b_n^{k-1} \sqrt{n}, \text{ almost surely},$$

$$\kappa_1 = (\mathbb{E} \eta_1(x)^4)^{\frac{1}{2}} \lesssim (\max |a_{ij}|)^2 k^2 b_n^{k-1} \sqrt{n}, \text{ almost surely},$$

$$\kappa_2 = (\mathbb{E} \eta_2(x)^4)^{\frac{1}{2}} \lesssim (\max |a_{ij}|)^2 k^3 b_n^{\frac{k-2}{2}} \sqrt{n}, \text{ almost surely}.$$  

Therefore from Result 20 and Lemma 15, we have

$$d_{TV}(W_n, Z_n) \lesssim \frac{(\max |a_{ij}|)^2 k^5 b_n^{k-1} \sqrt{n}}{\sigma^2_n}.$$  

This completes proof of the Theorem 1 for non-symmetric matrix.

For the symmetric case, that is, when both $X$ and $A$ are symmetric, proof goes in the similar line. In this case also $\|A \circ X\| \lesssim b_n$ a.s. and it follows immediately from [10] Corollary 3.12. Here we skip the details.
5. Appendix

Here we give the definitions of broadly connected and super regular variance profiles mentioned in Remark [13]. For further information and results on these variance profiles, we refer to [14]. Let $A$ be an $n \times m$ deterministic matrix with non-negative entries $a_{ij}$ and define

\[
N_A(i) = \{ j \in [m] : a_{ij} > 0 \},
\]

\[
e_A(I, J) = |\{(i, j) \in [n] \times [m] : a_{ij} > 0\}|.
\]

**Definition 21** (Broad connectivity). Let $A = (a_{ij})$ be an $n \times m$ matrix with non-negative entries. For $I \subset [n]$ and $\delta \in (0, 1)$, define the set $\delta$-broadly connected neighbours of $I$ as

\[
N_{A^\delta}(I) = \{ j \in [m] : |N_{A^\delta}(j) \cap I| \geq \delta|I| \}.
\]

For $\delta, \nu \in (0, 1)$, we say that $A$ is $(\delta, \nu)$-broadly connected if

(i) $|N_A(i)| \geq \delta m$, for all $i \in [n]$,

(ii) $|N_{A^\delta}(j)| \geq \delta n$ for all $j \in [m]$,

(iii) $|N_{A^\delta}(J)| \geq \min(n, (1 + \nu)|J|)$ for $|J| \subset [m]$.

**Definition 22** (Super regularity). Let $A$ be an $n \times m$ matrix with non-negative entries, for $\delta, \epsilon \in (0, 1)$, we say that $A$ is $(\delta, \epsilon)$-super regular if the following hold:

(i) $|N_A(i)| \geq \delta m$ for all $i \in [n]$,

(ii) $|N_{A^\delta}(j)| \geq \delta n$ for all $j \in [m]$,

(iii) $e_A(I, J) \geq \delta |I||J|$ for all $I \subset [n], J \subset [m]$ such that $|I| \geq \epsilon n$ and $|J| \geq \epsilon m$.

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