Important Criteria for Asymptotic Properties of Nonlinear Differential Equations

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Abstract: In this article, we prove some new oscillation theorems for fourth-order differential equations. New oscillation results are established that complement related contributions to the subject. We use the Riccati technique and the integral averaging technique to prove our results. As proof of the effectiveness of the new criteria, we offer more than one practical example.

Keywords: fourth-order differential equations; neutral delay; oscillation

1. Introduction

In this manuscript, we are concerned with the asymptotic behavior of solutions to fourth-order differential equations:

\[
\left( m(z) \Psi_1 \left( \frac{\Psi_2(z)}{\omega(z)} \right) \right) + \omega(z) \Psi_2 \left( \delta(\alpha(z)) \right) = 0,
\]

(1)

where \( \Psi_i \)s, i = 1, 2, \( \omega(z) > 0, 0 \leq \hat{y}(z) < \hat{y}_0 < \infty, \hat{\alpha}, \alpha \in [z_0, \infty), \hat{\alpha}(z) \leq z, \lim_{z \to \infty} \hat{\alpha}(z) = \infty \); and \( r_1 \) and \( r_2 \) are quotients of odd positive integers, under the assumption of the following:

\[
\int_{z_0}^{\infty} \frac{1}{m^{1/r_1}(z)} \, ds = \infty.
\]

(2)

The theory of the oscillation of delay of differential equations is a fertile study area and has attracted the attention of many authors recently. This is due to the existence of many important applications of this theory in neural networks, biology, social sciences, engineering, etc.; see [1,2].

A study of the behavior of solutions to higher order differential equations yields much fewer results than for the least order equations although they are of the utmost importance in a lot of applications, especially neutral delay differential equations.

Currently, there are studies on the oscillation results of differential equations, so many of these studies have been devoted to study the oscillation of different classes of differential equations by using different techniques in order to establish sufficient conditions to ensure the oscillatory behavior of the solutions of (1), see [3–5].
The motivation for studying this article is complemented by the results reported in [6,7]; therefore, we discuss their findings and results below.

Xing et al. [6] presented criteria for oscillation of the equation as follows:

\[
\left( m(z) \left( \zeta^{(n-1)}(z) \right)^{\gamma_1} \right)' + \hat{\omega}(z) \delta^\gamma_1(\alpha(z)) = 0,
\]

under the conditions

\[
\left( \alpha^{-1}(z) \right)' \geq \alpha_0 > 0, \quad \hat{\alpha}'(z) \geq \hat{\alpha}_0 > 0, \quad \hat{\alpha}^{-1}(\alpha(z)) < z
\]

and

\[
\liminf_{z \to \infty} \int^z_{\hat{\alpha}^{-1}(\alpha(z))} \frac{\hat{\omega}(s)}{m(s)} \left( \zeta^{(n-1)}(s)^{\gamma_1} \right) \, ds > \left( \frac{1}{\alpha_0} + \frac{\hat{y}_0^\gamma_1}{\alpha_0 \hat{\alpha}_0} \right) > \frac{(n-1)!}{e} \gamma_1,
\]

where \(0 \leq \hat{y}(z) < \hat{y}_0 < \infty\) and \(\hat{\omega}(z) := \min\{\hat{\omega}(\alpha^{-1}(z)), \hat{\omega}(\alpha^{-1}(\hat{\alpha}(z)))\}\). Moreover, the authors used the comparison method to obtain oscillation conditions for this equation.

Bazighifan et al. [7] presented oscillation results for the following fourth-order equation:

\[
\left( m(z) \left( \zeta'''(z) \right)^{\gamma_1} \right)' + \hat{\omega}(z) \delta^\gamma_1(\alpha(z)) = 0,
\]

under the conditions

\[
\int^{\infty}_{z_0} \frac{1}{m^{1/\gamma_1}(s)} \, ds < \infty
\]

using the Riccati technique.

Zhang et al. [8] established oscillation criteria for the following equation:

\[
\left( m(z) \left( \zeta^{(n-1)}(z) \right)^{\gamma_1} \right)' + \hat{\omega}(z) f(\delta(\alpha(z))) \, ds = 0
\]

and under the condition

\[
\int^{\infty}_{z_0} \left( k \rho(z) E(z) - \frac{1}{4 \lambda} \left( \frac{\rho'(z)}{\rho(z)} \right)^2 \eta(z) \right) \, ds = \infty.
\]

Chatzarakis et al. [9], by using the Riccati technique, established asymptotic behavior for the following neutral equation:

\[
\left( m(z) \left( \zeta'''(z) \right)^{\gamma_1} \right)' + \int_{\alpha}^{\beta} \hat{\omega}(z,s) f(\delta(\alpha(z,s))) \, ds = 0.
\]

The authors in [6,7] used the comparison technique that differs from the one we used in this article. Their approach is based on using these mentioned methods to reduce Equation (1) into a first-order equation, while in our article, we discuss the oscillatory properties of differential equations with a middle term and with a canonical operator of the neutral-type, and we employ a different approach based on using the integral averaging technique and the Riccati technique to reduce the main equation into a first-order inequality to obtain more effective oscillatory properties.

The purpose of this article is to establish new oscillation criteria for (1). The methods used in this paper simplify and extend some of the known results that are reported in the literature [6,7]. The authors in [6,7] used a comparison technique that differs from the one we used in this article.

2. Oscillation Criteria

We next present the lemmas needed for the proof of the original results:
Lemma 1 ([10]). If \( \delta^{(i)}(z) > 0, i = 0, 1, \ldots, n, \) and \( \delta^{(n+1)}(z) < 0, \) then the following holds:

\[
\frac{n! \delta(z)}{z^n} \geq \frac{(n-1)! \delta'(z)}{z^{n-1}}.
\]

Lemma 2 ([11]). Let \( \delta \in C^n([z_0, \infty)), (0, \infty)) \). Assume that \( \delta^{(n)}(z) \) is of fixed sign and not identically zero on \([z_0, \infty)\) and that there exists a \( z_1 \geq z_0 \) such that \( \delta^{(n+1)}(z_1) \delta^{(n)}(z) \leq 0 \) for all \( z \geq z_1 \). If \( \lim_{z \to \infty} \delta(z) \neq 0, \) then for every \( \mu \in (0, 1) \) there exists \( z_\mu \geq z_1 \) such that the following holds:

\[
\delta(z) \geq \frac{\mu}{(n-1)!} z^{n-1} |\delta^{(n-1)}(z)| \text{ for } z \geq z_\mu.
\]

Lemma 3 ([12]). Let \( a \geq 0; \) then, the following holds:

\[
X\delta - Y\delta^{(a+1)/a} \leq a^a (a + 1)^{-a} Y^{-a} X^{a+1},
\]

where \( Y > 0 \) and \( X \) are constants.

Lemma 4 ([13]). Assume that \( \delta(z) \) is an eventually positive solution of Equation (1). Then,

- Case \((N_1)\) : \( \zeta(z) > 0, \zeta'(z) > 0, \zeta'''(z) > 0, \)
- Case \((N_2)\) : \( \zeta(z) > 0, \zeta'(z) > 0, \zeta''''(z) < 0, \zeta''''(z) > 0. \)

Here are the notations used for our study:

\[
E_1(z) = \beta(z)\bar{\omega}(z)(1 - \bar{y}_0)^{r_2} A_1^{\beta - r_1} \left( \frac{a(z)}{z} \right)^{3r_2},
\]

\[
\Phi(z) = (1 - \bar{y}_0)^{r_2/r_1} h(z) A_2^{1/r_1 - 1} \left( \frac{1}{m(u)} \int_u^\infty \bar{\omega}(s) \frac{a^r(s)}{s^{r_1}} ds \right)^{1/r_1} du
\]

and

\[
\Theta(z) = r_1 \mu_1 \frac{z^2}{2m^{1/r_1}(z)\beta^{1/r_1}(z)}.
\]

Lemma 5. Let \( \delta(z) \) is an eventually positive solution of Equation (1), then

\[
\left( m(z)(\zeta''''(z))^{r_1} \right)' \leq -G(z)(\zeta''''(a(z)))^{r_2}, \tag{3}
\]

where

\[
G(z) = \bar{\omega}(z)(1 - \bar{y}_0)^{r_2} \left( \frac{\mu}{6} a(z)^3 \right)^{r_2}.
\]

Proof. Let \( \delta(z) \) is an eventually positive solution of Equation (1). From definition of \( \zeta(z) = \delta(z) + \bar{y}(z)\delta(\bar{\delta}(z)), \) we obtain the following:

\[
\delta(z) \geq \zeta(z) - \bar{y}_0 \delta(\bar{\delta}(z)) \geq \zeta(z) - \bar{y}_0 \zeta(\bar{\delta}(z)) \geq (1 - \bar{y}_0)\zeta(z),
\]

which with (1), results in the following:

\[
\left( m(z)(\zeta''''(z))^{r_1} \right)' + \bar{\omega}(z)(1 - \bar{y}_0)^{r_2} \zeta''''(a(z)) \leq 0. \tag{4}
\]

Using Lemma 2, we see the following:

\[
\zeta(z) \geq \frac{\mu}{6} z^3 \zeta''''(z). \tag{5}
\]
Combining (4) and (5), we find the following:

\[
\left( m(z) (\xi''''(z))^{r_1} \right) + \omega(z) (1 - \gamma_0)^{r_2} \left( \frac{\mu}{6} (\xi(z))^3 \right)^{r_2} (\xi'''(\xi(z)))^{r_2} \leq 0.
\]

Thus, (3) holds. This completes the proof. \( \square \)

**Lemma 6.** Let \( \delta(z) \) is an eventually positive solution of Equation (1) and

\[
B'(z) \leq \frac{\beta'(z)}{\beta(z)} B(z) - E_1(z) - r_1 \mu_1 \frac{z^2}{2^{m^{1/\alpha}(z)} \beta^{1/\alpha}(z) B^{n+1}(z)}, \text{ if } \xi \text{ satisfies } (N_1)
\]

and

\[
A'(z) \leq -\Phi(z) + \frac{h'(z)}{h(z)} A(z) - \frac{1}{h(z)} A^2(z), \text{ if } \xi \text{ satisfies } (N_2),
\]

where

\[
B(z) := \beta(z) \frac{m(z) (\xi'''(z))^{r_1}}{c^{r_1}(z)} > 0
\]

and

\[
A(z) := h(z) \frac{c(z)}{c(z)} / z \geq z_1.
\]

**Proof.** Let \( \delta(z) \) is an eventually positive solution of Equation (1). Let \((N_1)\) holds. From (8) and (4), we find the following:

\[
B'(z) \leq \frac{\beta'(z)}{\beta(z)} B(z) - \beta(z) \omega(z) (1 - \gamma_0)^{r_2} c^{r_2}(a(z)) - r_1 \beta(z) \frac{m(z) (\xi'''(z))^{r_1}}{c^{r_1}(z)} \xi'(z).
\]

Using Lemma 1, we find

\[
\xi(z) \geq z \xi'(z)
\]

and hence,

\[
\frac{\xi(a(z))}{\xi(z)} \geq \frac{a^3(z)}{z^3}.
\]

It follows from Lemma 2 that

\[
\xi'(z) \geq \frac{\mu_1}{2} z^2 \xi'''(z),
\]

for all \( \mu_1 \in (0, 1) \) and every sufficiently large \( z \). Thus, by (10)–(12), we obtain the following:

\[
B'(z) \leq \frac{\beta'(z)}{\beta(z)} B(z) - \beta(z) \omega(z) (1 - \gamma_0)^{r_2} c^{r_2}(a(z)) \left( \frac{a(z)}{z} \right)^{3r_2} - r_1 \mu_1 \frac{z^2}{2^{m^{1/\alpha}(z)} \beta^{1/\alpha}(z) B^{n+1}(z)}.
\]

Since \( \xi'(z) > 0 \), there exist \( z_2 \geq z_1 \) and \( A_1 > 0 \) such that the following holds:

\[
\xi(z) > A_1.
\]

Thus, we obtain the following:

\[
B'(z) \leq \frac{\beta'(z)}{\beta(z)} B(z) - \beta(z) \omega(z) (1 - \gamma_0)^{r_2} A^{3r_2} \left( \frac{a(z)}{z} \right)^{3r_2} - r_1 \mu_1 \frac{z^2}{2^{m^{1/\alpha}(z)} \beta^{1/\alpha}(z) B^{n+1}(z)},
\]
which yields the following:
\[ B'(z) \leq \frac{\beta'(z)}{\beta(z)} B(z) - E_1(z) - r_1 \mu_1 \frac{z^2}{2m^{1/\eta}(z) \beta^{1/\eta}(z)} B^{r_1 + 1}(z). \]

Thus, (6) holds.

Let \((N_2)\) hold. Integrating (4) from \(z\) to \(u\), we find the following:
\[ m(u) \left( \zeta'''(u) \right)^{r_1} - m(z) \left( \zeta'''(z) \right)^{r_1} \leq - \int_z^u \omega(s) (1 - \gamma_0)^2 \zeta'\zeta^2(a(s)) ds. \]  
(14)

From Lemma 1, we obtain the following:
\[ \zeta(z) \geq z \zeta'(z) \]
and hence,
\[ \zeta(a(z)) \geq \frac{a(z)}{z} \zeta(z). \]  
(15)

For (14), letting \(u \to \infty\) and using (15), we obtain the following:
\[ m(z) \left( \zeta'''(z) \right)^{r_1} \geq (1 - \gamma_0)^2 \zeta^{r_2}(z) \int_z^\infty \omega(s) \frac{a^{r_2}(s)}{s^{r_2}} ds. \]  
(16)

Integrating (16) from \(z\) to \(\infty\), we find the following:
\[ \zeta''(z) \leq - (1 - \gamma_0)^{r_2/\eta_1} \zeta^{r_2/\eta_1}(z) \int_z^\infty \left( \frac{1}{m(u)} \int_u^\infty \omega(s) \frac{a^{r_2}(s)}{s^{r_2}} ds \right)^{1/\eta_1} du, \]  
(17)

From the definition of \(A(z)\), we see that \(A(z) > 0\) for \(z \geq z_1\), and using (13) and (17), we find the following:
\[ A'(z) = \frac{h'(z)}{h(z)} A(z) + h(z) \frac{\zeta''(z)}{\zeta(z)} - h(z) \left( \frac{\zeta'(z)}{\zeta(z)} \right)^2 \]
\[ \leq \frac{h'(z)}{h(z)} A(z) - \frac{1}{h(z)} A^2(z) \]
\[ - (1 - \gamma_0)^{r_2/\eta_1} h(z) \zeta^{r_2/\eta_1-1}(z) \int_z^\infty \left( \frac{1}{m(u)} \int_u^\infty \omega(s) \frac{a^{r_2}(s)}{s^{r_2}} ds \right)^{1/\eta_1} du. \]

Since \(\zeta'(z) > 0\), there exist \(z_2 \geq z_1\) and \(A_2 > 0\) such that the following holds:
\[ \zeta(z) > A_2. \]

Thus, we obtain the following:
\[ A'(z) \leq - \Phi(z) + \frac{h'(z)}{h(z)} A(z) - \frac{1}{h(z)} A^2(z), \]

Thus, (7) holds. Proof of the theorem is completed. □

**Definition 1.** Let
\[ D = \{ (z, s) \in \mathbb{R}^2 : z \geq \gamma \geq z_0 \} \text{ and } D_0 = \{ (z, s) \in \mathbb{R}^2 : z > \gamma \geq z_0 \}. \]

The function \(G_i \in C(D, \mathbb{R})\) fulfills the following conditions:
(i) \(G_i(z, s) = 0\) for \(z \geq z_0\), \(G_i(z, s) > 0\), \((z, s) \in D_0\);
(ii) The functions $h, v \in C^1([z_0, \infty), (0, \infty))$ and $g_l \in C(D_0, \mathbb{R})$ such that

$$\frac{\partial}{\partial s} G_1(z, s) + \frac{\beta'(s)}{\beta(s)} G_1(z, s) = g_1(z, s) G_1^{r_1/(r_1+1)}(z, s)$$

and

$$\frac{\partial}{\partial s} G_2(z, s) + \frac{h'(s)}{h(s)} G_2(z, s) = g_2(z, s) \sqrt{G_2(z, s)}.$$  

Now, we present some Philos-type oscillation criteria for (1).

Theorem 1. Let (24) hold. If $\beta, h \in C^1([z_0, \infty), \mathbb{R})$ such that

$$\limsup_{z \to \infty} \frac{1}{G(z, z_1)} \int_{z_1}^{z} G(z, s) E_1(s) - \delta_1^{r_1+1} \frac{G_1^{r_1}(z, s)}{(r_1+1)^{r_1+1}} \frac{2^n m(s) \beta(s)}{(\mu_1 z^2)^{r_1}} \, ds = \infty$$

for all $\mu_2 \in (0, 1)$, and

$$\limsup_{z \to \infty} \frac{1}{G_2(z, z_1)} \int_{z_1}^{z} \left( G_2(z, s) \Phi(s) - \frac{h(s) G_2^2(z, s)}{4} \right) \, ds = \infty,$$

then (1) is oscillatory.

Proof. Let $\delta$ be a non-oscillatory solution of (1), we see that $\delta > 0$. Assume that (N$_1$) holds. Multiplying (6) by $G(z, s)$ and integrating the resulting inequality from $z_1$ to $z$, we obtain the following:

$$\int_{z_1}^{z} G(z, s) E_1(s) \, ds \leq B(z_1) G(z, z_1) + \int_{z_1}^{z} \left( \frac{\partial}{\partial s} G(z, s) + \frac{\beta'(s)}{\beta(s)} G(z, s) \right) B(s) \, ds$$

$$- \int_{z_1}^{z} \Theta(s) G(z, s) B^{r_1+1}(s) \, ds.$$

From (18), we obtain the following:

$$\int_{z_1}^{z} G(z, s) E_1(s) \, ds \leq B(z_1) G(z, z_1) + \int_{z_1}^{z} g_1(z, s) G_1^{r_1/(r_1+1)}(z, s) B(s) \, ds$$

$$- \int_{z_1}^{z} \Theta(s) G(z, s) B^{r_1+1}(s) \, ds. \tag{22}$$

Using Lemma 3 with $V = \Theta(s) G(z, s)$, $U = g_1(z, s) G_1^{r_1/(r_1+1)}(z, s)$ and $\delta = B(s)$, we obtain the following:

$$g_1(z, s) G_1^{r_1/(r_1+1)}(z, s) B(s) - \Theta(s) G(z, s) B^{r_1+1}(s)$$

$$\leq \frac{g_1^{r_1+1}(z, s) G_1^{r_1}(z, s)}{(r_1+1)^{r_1+1}} \frac{2^n m(z) \beta(z)}{(\mu_1 z^2)^{r_1}},$$

which, with (22) gives the following:

$$\int_{z_1}^{z} \left( G(z, s) E_1(s) - \frac{g_1^{r_1+1}(z, s) G_1^{r_1}(z, s)}{(r_1+1)^{r_1+1}} \frac{2^n m(s) \beta(s)}{(\mu_1 z^2)^{r_1}} \right) \, ds \leq B(z_1),$$

which contradicts (20).
Assume that \((N_2)\) holds. Multiplying (7) by \(G_2(z,s)\) and integrating the resulting inequality from \(z_1\) to \(z\), we find the following:

\[
\int_{z_1}^{z} G_2(z,s) \Phi(s) \, ds \leq A(z_1)G_2(z,z_1) + \int_{z_1}^{z} \left( \frac{\partial}{\partial s} G_2(z,s) + \frac{h'(s)}{h(s)} G_2(z,s) \right) A(s) \, ds - \int_{z_1}^{z} \frac{1}{h(s)} G_2(z,s) A^2(s) \, ds.
\]

Thus,

\[
\int_{z_1}^{z} G_2(z,s) \Phi(s) \, ds \leq A(z_1)G_2(z,z_1) + \int_{z_1}^{z} g_2(z,s) \sqrt{G_2(z,s)} A(s) \, ds - \int_{z_1}^{z} \frac{1}{h(s)} G_2(z,s) A^2(s) \, ds
\]

and so

\[
\frac{1}{G_2(z,z_1)} \int_{z_1}^{z} \left( G_2(z,s) \Phi(s) - \frac{h(s)g_2^2(z,s)}{4} \right) \, ds \leq A(z_1),
\]

which contradicts (21). Proof of the theorem is completed. \(\square\)

**Corollary 1.** Let (24) hold. If \(\beta, h \in C^1([z_0, \infty), \mathbb{R})\) such that

\[
\int_{z_0}^{\infty} \left( E_1(s) - \frac{2^{r_1}}{(r_1 + 1)^{r_1 + 1}} \frac{m(s)(\beta'(s))^{r_1 + 1}}{h(s)^2s^{2r_1}\beta'(s)} \right) \, ds = \infty \quad (23)
\]

and

\[
\int_{z_0}^{\infty} \left( \Phi(s) - \frac{(h'(s))^2}{4h(s)} \right) \, ds = \infty, \quad (24)
\]

for some \(\mu_1 \in (0,1)\) and every \(A_1, A_2 > 0\), then (1) is oscillatory.

### 3. Example

This section presents some interesting examples to examine the applicability of theoretical outcomes.

**Example 1.** Consider the following equation:

\[
\left( \delta + \frac{1}{2} \delta \left( \frac{1}{3} z \right) \right)^{(4)} + \frac{\omega_0}{z^4} \delta \left( \frac{1}{2} z \right) = 0, \ z \geq 1, \ \omega_0 > 0. \quad (25)
\]

Let \(r_1 = r_2 = 1, \ m(z) = 1, \ \gamma(z) = 1/2, \ \gamma(z) = z/3, \ a(z) = z/2 \) and \(\tilde{\omega}(z) = \omega_0/z^4\).

Hence, it is easy to see that

\[
\int_{z_0}^{\infty} \frac{1}{m^{1/r_1} (s)} \, ds = \infty, \ E_1(z) = \frac{\omega_0}{16z}
\]

and

\[
\Phi(z) := \frac{\omega_0}{24}.
\]
If we put $\beta(s) = z^3$ and $h(z) = z^2$, then we find the following:

$$\int_{z_0}^{\infty} \left( E_1(s) - \frac{2r_1}{(r_1 + 1)^{r_1 + 1}} \frac{m(s)(\beta'(s))^{r_1 + 1}}{\mu_1} z^{2r_1} \beta^{r_1}(s) \right) ds$$

$$= \int_{z_0}^{\infty} \left( \frac{\omega_0}{16s} - \frac{9}{2s} \frac{1}{\mu_1} \right) ds$$

and

$$\int_{z_0}^{\infty} \left( \Phi(s) - \frac{(h'(s))^2}{4h(s)} \right) ds$$

$$= \int_{z_0}^{\infty} \left( \frac{\omega_0}{24} - 1 \right) ds.$$

Thus,

$$\omega_0 > 72$$

and

$$\omega_0 > 24.$$

From Corollary 1, Equation (25) is oscillatory if $\omega_0 > 72$.

Example 2. Consider the following equation:

$$\left( z(\delta + \bar{\gamma}(\gamma z))^{r_1} \right)' + \frac{\omega_0}{z^3} \delta(\eta z) = 0, \ z \geq 1,$$

(28)

where $\bar{\gamma} \in [0, 1)$, $\gamma, \eta \in (0, 1)$ and $\omega_0 > 0$. Let $r_1 = r_2 = 1$, $m(z) = z$, $\bar{\gamma}(z) = \bar{\gamma}_0$, $\delta(z) = \gamma z$, $\eta(z) = \eta z$ and $\omega(z) = \omega_0/z^3$. Hence, if we set $\beta(s) = z^2$ and $h(z) = z$, then we get

$$E_1(z) = \frac{\omega_0(1 - \bar{\gamma}_0)\eta^3}{z}, \ \Phi(z) = \frac{\omega_0(1 - \bar{\gamma}_0)\eta}{4z}.$$ 

Thus, (23) and (24) become the following:

$$\int_{z_0}^{\infty} \left( E_1(s) - \frac{2r_1}{(r_1 + 1)^{r_1 + 1}} \frac{m(s)(\beta'(s))^{r_1 + 1}}{\mu_1} z^{2r_1} \beta^{r_1}(s) \right) ds$$

$$= \int_{z_0}^{\infty} \left( \frac{\omega_0}{s} - \frac{2}{\mu_1 s} \right) ds$$

and

$$\int_{z_0}^{\infty} \left( \Phi(s) - \frac{(h'(s))^2}{4h(s)} \right) ds$$

$$= \int_{z_0}^{\infty} \left( \frac{\omega_0(1 - \bar{\gamma}_0)\eta}{4s} - \frac{1}{4s} \right) ds.$$

So,

$$\omega_0 > \frac{2}{(1 - \bar{\gamma}_0)\eta^3}$$

(29)

and

$$\omega_0 > \frac{1}{(1 - \bar{\gamma}_0)\eta}.$$ 

From Corollary 1, Equation (28) is oscillatory if (29) holds.
4. Conclusions

In this work, we prove some new oscillation theorems for (1). New oscillation results are established that complement related contributions to the subject. We used the Riccati technique and integral averages technique to obtain some new results to the oscillation of Equation (1) under the condition \( \int_0^\infty \frac{1}{m^{1/r}(s)} \, ds = \infty \). In future work, we will study this type of equation under the following condition:

\[
\int_0^\infty \frac{1}{m^{1/r}(s)} \, ds < \infty,
\]

We also introduce some important oscillation criteria of differential equations of the fourth-order and under the following:

\[
\zeta(z) = \delta(z) + \delta(z) \sum_{i=1}^{j} \delta_i(\hat{\delta}(z)).
\]

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