BORNOLOGIES AND FILTERS IN SELECTION PRINCIPLES ON FUNCTION SPACES

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ABSTRACT. We extend known results of selection principles in $C_p$-theory to the context of spaces of the form $C_B(X)$, where $B$ is a bornology on $X$. Particularly, by using the filter approach of Jordan to $C_p$-theory, we show that $\gamma$-productive spaces are productive with a larger class of $\gamma$-like spaces.

1. Introduction

The framework of selection principles, introduced by Scheepers in [19], provides a uniform manner to deal with diagonalization processes that appears in several mathematical contexts since the 1920’s. Detailed surveys on this subject are provided in [22][24]. Here we present a brief introduction, in order to fix notations.

Given an infinite set $S$, let $A$ and $C$ be families of nonempty subsets of $S$. We consider the following classic selection principles:

• $S_1(A, C)$: for each sequence $(A_n : n \in \omega)$ of elements of $A$ there is a sequence $(C_n : n \in \omega)$ such that $C_n \in A_n$ for all $n$ and $\{C_n : n \in \omega\} \in C$;

• $S_{\text{fin}}(A, C)$: for each sequence $(A_n : n \in \omega)$ of elements of $A$ there is a sequence $(C_n : n \in \omega)$ such that $C_n \in [A_n]^{<\omega}$ for all $n$ and $\bigcup_{n \in \omega} C_n \in C$.

There are natural infinite games of perfect information associated with these selection principles. In the same setting of the above paragraph, a play of the game $G_1(A, C)$ is defined as follows: for every inning $n < \omega$, Player I chooses an element $A_n \in A$, and then Player II picks a $C_n \in A_n$; Player II wins the play if $\{C_n : n \in \omega\} \in C$. The game $G_{\text{fin}}(A, C)$ is defined in a similar way.

For $J \in \{I, II\}$, we denote the sentence “Player $J$ has a winning strategy in the game $G$” by $J \uparrow G$, while its negation is denoted by $J \not\uparrow G$. The interest about these games lies on finding winning strategies for some of the players and, in the topological context, asking how the topological properties of a space determine these strategies for particular instances of families $A$ and $C$.

In the next diagram, the straight arrows summarize the general implications between these principles.

$$\begin{align*}
\text{II} \uparrow G_1(A, C) & \implies \text{I} \not\uparrow G_1(A, C) \implies S_1(A, C) \\
\text{II} \uparrow G_{\text{fin}}(A, C) & \implies \text{I} \not\uparrow G_{\text{fin}}(A, C) \implies S_{\text{fin}}(A, C)
\end{align*}$$

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The dashed arrows above mark implications that are not necessarily true in general. An important situation, in which these converses hold, occurs when one takes \( \mathcal{A} = \mathcal{C} = \mathcal{O}(X) \), where \( \mathcal{O}(X) \) denotes the family of all open coverings of a topological space \( X \).

**Theorem 1.1** (Hurewicz, 1925). *For a topological space \( X \), \( S_{\text{fin}}(\mathcal{O}(X), \mathcal{O}(X)) \) is equivalent to \( \not\uparrow \mathcal{G}_{\text{fin}}(\mathcal{O}(X), \mathcal{O}(X)) \).*

**Theorem 1.2** (Pawlikowski, 1994). *For a topological space \( X \), \( S_1(\mathcal{O}(X), \mathcal{O}(X)) \) is equivalent to \( \not\uparrow \mathcal{G}_1(\mathcal{O}(X), \mathcal{O}(X)) \).*

We introduce in Section 2 a variation of the principles defined above, allowing us to treat simultaneously of several selection principles, based on the ideas presented in [2, 6]. We shall use these principles in connection with function spaces. In this context, many dualities are known between selective local properties of \( C_p(X) \) and selective covering properties of \( X \), where \( X \) is a Tychonoff space and \( C_p(X) \) denotes the space of the continuous real functions on \( X \) with the topology of the pointwise convergence.

Particularly, we are interested in the dualities summarized in the next diagram, where \( \Omega \) stands for the collection of \( \omega \)-coverings of \( X \) – those open coverings \( U \) such that each finite subset \( F \) of \( X \) is contained in some element of \( U \), \( \Omega_o \) denotes the family \( \{ A \subset C_p(X) : o \in A \} \) and \( o \) is the constant zero function.

The work of Caserta et al. [4], generalizing the bottom horizontal equivalences of the diagram above, motivated us to investigate the other equivalences in a broader context, for spaces of the form \( C_B(X) \), where \( B \) is a bornology with a compact base. In [3] we presented generalizations for the horizontal equivalences, but at that time we were not able to solve the vertical ones, originally proved by Scheepers [20] in the context of \( C_p \)-theory.

In Section 3 we settle these remaining equivalences, by using the upper semi-finite topology [14] on the family \( B \), and we analyze its consequences accordingly to the framework presented in Section 2. Back to the \( C_p \)-theory context, Jordan [9] obtains general dualities by using filters, and Sections 4 and 5 are dedicated to extend his results for spaces of the form \( C_B(X) \). Particularly, in the last section we show that the class of \( \gamma \)-productive spaces is productive with a class (formally) larger than the class of \( \gamma \)-spaces – both definitions are recalled there.

### 2. In between \( S_1/G_1 \) and \( S_{\text{fin}}/G_{\text{fin}} \)

In this section we fix an infinite set \( S \) and families \( \mathcal{A} \) and \( \mathcal{C} \) of subsets of \( S \). We shall denote by \([2, \aleph_0] \) (resp. \([2, \aleph_0) \)) the set of all cardinals \( \alpha \) such that \( 2 \leq \alpha \leq \aleph_0 \).
(resp. \(2 \leq \alpha < \aleph_0\)) and for \(n \geq 1\), let \(\mu : \omega \to [2, \aleph_0]\) be the constant function given by \(m \mapsto n + 1\) for all \(m \in \omega\).

For a function \(\varphi : \omega \to [2, \aleph_0]\), we consider the following selection principle:

- \(S_\varphi(A, \mathcal{C})\): for each sequence \((A_n : n \in \omega)\) of elements of \(A\) there is a sequence \((C_n : n \in \omega)\) such that \(C_n \in \{A_n\}^{< \varphi(n)}\) for all \(n\) and \(\bigcup_{n \in \omega} C_n \in \mathcal{C}\).

Note that for \(\varphi \equiv \aleph_0\), one gets the definition of the selection principle \(S_{\text{fin}}\). On the other hand, \(S_1(A, \mathcal{C}) \Rightarrow S_{\text{fin}}(A, \mathcal{C})\) and it is formally stronger. However, since \(S_1(A, \mathcal{C}) = S_1(A, \mathcal{C})\) holds for all pairs \((A, \mathcal{C})\) considered along this work, we shall not worry about this, and for simplicity we assume this equality as an additional hypothesis for the general case. Hence, for each \(n \geq 1\) it makes sense to denote the selection principle \(S_n(A, \mathcal{C})\) as \(S_n(A, \mathcal{C})\).

The original prototype of the above selection principle was defined in [6], where the authors concerned about variations of tightness by taking

\[
\mathcal{A} = \mathcal{C} = \Omega_x := \{A \subset X : x \in \overline{A}\}.
\]

In [2], the natural adaptation of the principle \(S_\varphi\) to the context of games was analyzed for the same pair \((\Omega_x, \Omega_x)\). This motivates our next definition.

For \(A, \mathcal{C}\) and \(\varphi\) as before, let \(G_\varphi(A, \mathcal{C})\) be the infinite game of perfect information between Player I and Player II, defined as follows:

- for every inning \(n < \omega\), Player I chooses an element \(A_n \in A\), and then Player II picks a \(C_n \in \{A_n\}^{< \varphi(n)}\);
- Player II wins if \(\bigcup_{n \in \omega} C_n \in \mathcal{C}\).

Again, the constant function \(\varphi \equiv \omega\) yields the game \(G_{\text{fin}}(A, \mathcal{C})\), and since we assume \(G_1(A, \mathcal{C}) = G_1(A, \mathcal{C})\), we may denote the game \(G_n(A, \mathcal{C})\) as \(G_n(A, \mathcal{C})\).

The general relationship between these principles is stated in the following

**Proposition 2.1.** Let \(\varphi\) and \(\psi\) be functions of the form \(\omega \to [2, \aleph_0]\) such that \(\psi \leq \varphi\). Then

\[
\begin{align*}
\text{II} \uparrow G_\psi(A, \mathcal{C}) & \rightarrow \text{I} \uparrow G_\psi(A, \mathcal{C}) \rightarrow S_\psi(A, \mathcal{C}) \\
\text{II} \uparrow G_\varphi(A, \mathcal{C}) & \rightarrow \text{I} \uparrow G_\varphi(A, \mathcal{C}) \rightarrow S_\varphi(A, \mathcal{C})
\end{align*}
\]

Particularly, note that for \(\psi = 1\) and \(\varphi \equiv \aleph_0\), the above diagram yields the first one presented in Introduction. Also, for \(A = \mathcal{C} = \mathcal{O}(X)\), one has the following

**Theorem 2.2** (Garca-Ferreira and Tamariz-Mascara [6]). \(S_f(\mathcal{O}(X), \mathcal{O}(X))\) and \(S_1(\mathcal{O}(X), \mathcal{O}(X))\) are equivalent for any space \(X\) and any function \(f : \omega \to [2, \aleph_0]\).

Thus, in this context, all of the following statements are equivalent,

- (i) \(I \uparrow G_1(\mathcal{O}, \mathcal{O})\)
- (ii) \(I \uparrow G_f(\mathcal{O}, \mathcal{O})\)
- (iii) \(S_f(\mathcal{O}, \mathcal{O})\)
- (iv) \(S_1(\mathcal{O}, \mathcal{O})\),

because \(S_1(\mathcal{O}, \mathcal{O}) \Rightarrow I \uparrow G_1(\mathcal{O}, \mathcal{O})\) by Theorem 1.2.
Remark 1. The natural question then is whether the games $G_f(\mathcal{O}(X), \mathcal{O}(X))$ and $G_{k-1}(\mathcal{O}(X), \mathcal{O}(X))$ are equivalent or not. Nathaniel Hiers, in a joint work with Logan Crone, Lior Fishman, and Stephen Jackson, recently\footnote{At the Conference Frontiers of Selection Principles, that took place on Warsaw during the two last weeks of August, 2017.} presented an affirmative answer concerning the game $G_2$, for any Hausdorff space $X$.

Although their solution can possibly be extended for any function $f: \omega \rightarrow [2, \aleph_0)$, we mention that in this general case, an affirmative answer can also be obtained when $X$ is a $T_1$ second countable space or a Hausdorff space with $G_3$-points.

However, the above equivalences do not hold in the tightness context. Denoting as $\text{Id}: \omega \rightarrow [2, \aleph_0)$ the function given by $\text{Id}(n) = n + 2$ for all $n \in \omega$, one has the following

**Theorem 2.3** (García-Ferreira and Tamariz-Mascarúa, [6]). Let $Y$ be a topological space, $y \in Y$ and let $f: \omega \rightarrow [2, \aleph_0)$ be a function.

1. If $f$ is bounded, then $S_f(\Omega_y, \Omega_y)$ is equivalent to $S_1(\Omega_y, \Omega_y)$.
2. If $f$ is unbounded, then $S_f(\Omega_y, \Omega_y)$ is equivalent to $S_{\text{Id}}(\Omega_y, \Omega_y)$.

Examples 3.7 and 3.8 in [6] show that in general the implications

$$S_1(\Omega_y, \Omega_y) \equiv S_{\text{Id}}(\Omega_y, \Omega_y) \Rightarrow S_\text{fin}(\Omega_y, \Omega_y)$$

are not reversible. Still, the authors also show that for spaces of the form $C_p(X)$, where $X$ is a Tychonoff space, the converse of $(\#)$ holds. We prove this in the next section in a more general context. We finish this section with the counterpart of Theorem 2.3 for games, that will be useful later.

**Theorem 2.4** (Aurichi, Bella and Dias [2]). Let $Y$ be a topological space, $y \in Y$ and let $f: \omega \rightarrow [2, \aleph_0)$ be a function.

1. If $f$ is bounded, then the games $G_f(\Omega_y, \Omega_y)$ and $G_{k-1}(\Omega_y, \Omega_y)$ are equivalent, where $k = \limsup_{n \in \omega} f(n)$.
2. If $f$ is unbounded, then $G_f(\Omega_y, \Omega_y)$ and $G_{\text{Id}}(\Omega_y, \Omega_y)$ are equivalent.

3. Bornologies as hyperspaces

We recall the basic definitions from [3]. A bornology $\mathcal{B}$ on a topological space $X$ is an ideal of subsets of $X$ that covers the space. A subset $\mathcal{B}'$ of $\mathcal{B}$ is called a compact base for the bornology $\mathcal{B}$ if $\mathcal{B}'$ cofinal in $\mathcal{B}$ with respect to inclusion and all its elements are compact subspaces of $X$.

For a topological space $X$ and a bornology $\mathcal{B}$ on $X$, we call the topology of the uniform convergence on $\mathcal{B}$, denoted by $\mathcal{T}_{\mathcal{B}}$, as the topology on $C(X)$ having as a neighborhood base at each $f \in C(X)$ the sets of the form

$$\langle B, \varepsilon \rangle[f] := \{g \in C(X) : \forall x \in B(|f(x) - g(x)| < \varepsilon)\},$$

for $B \in \mathcal{B}$ and $\varepsilon > 0$. By $C_{\mathcal{B}}(X)$ we mean the space $(C(X), \mathcal{T}_{\mathcal{B}})$.

It can be showed that $\mathcal{T}_{\mathcal{B}}$ is obtained from a separating uniformity over $C(X)$, from which it follows that $C_{\mathcal{B}}(X)$ is a Tychonoff space (see McCoy and Ntantu [13]). It is also worth to mention that $C_{\mathcal{B}}(X)$ is a homogeneous space, so there is no loss of generality in fixing an appropriate point from $C_{\mathcal{B}}(X)$ in order to analyze its closure properties – in this case, we fix the zero function $0: X \rightarrow \mathbb{R}$.
A collection \( C \) of open sets of \( X \) is a \( B \)-covering for \( X \) if for every \( B \in \mathcal{B} \) there is a \( C \in C \) such that \( B \subset C \). Following the notation of Caserta et al. \[4\], we denote by \( \mathcal{Q}_B \) the collection of all open \( B \)-coverings for \( X \). When \( \mathcal{U} \in \mathcal{Q}_B \) is such that \( X \not\in \mathcal{U} \), the \( B \)-covering \( \mathcal{U} \) is said to be nontrivial. An important fact about nontrivial \( B \)-coverings is that any cofinite subset of it is also a \( B \)-covering of \( X \).

The main examples of bornologies with a compact base on a topological space \( X \) are the bornologies \( \mathcal{F} = [X]^{<\aleph_0} \) and \( \mathcal{K} = \{ A \subset X : \exists K \subset X \text{ compact and } A \subset K \} \) — if \( X \) is a Hausdorff space, then \( \mathcal{K} = \{ A \subset X : \overline{A} \text{ is compact} \} \). For \( B = \mathcal{F} \), one has \( C_\mathcal{F}(X) = C_p(X) \) and the \( \mathcal{F} \)-coverings turns out to be the \( \omega \)-coverings of \( X \). Also, if \( X \) is Hausdorff, it follows that \( C_\mathcal{K}(X) = C_k(X) \), where \( C_k(X) \) denotes the set \( C(X) \) with the compact-open topology. One readily sees that \( \mathcal{O}_K = \mathcal{K} \), where \( \mathcal{K} \) denotes the set of the so called \( K \)-coverings of \( X \).

Now, recall we want to generalize the following theorem for \( B \)-coverings.

**Theorem 3.1** (Scheepers \[20\]). Let \( X \) be a Tychonoff space.

1. \( S_{\text{fin}}(\Omega, \Omega) \) is equivalent to \( 1 \not\uparrow G_{\text{fin}}(\Omega, \Omega) \).
2. \( S_1(\Omega, \Omega) \) is equivalent to \( 1 \not\uparrow G_1(\Omega, \Omega) \).

Although the requirement of a compact base is necessary to settle the dualities between local properties of \( C_{\mathcal{B}}(X) \) and covering properties of \( X \), the generalization of the above theorem for \( B \)-covering does not need any requirement on the bornology \( \mathcal{B} \): in fact, it holds for an arbitrary family \( \mathcal{B} \) of subsets of \( X \).

**Remark 2** (Scheepers’ key idea). It may be enlightening to review Scheepers’ original proof. The key idea in his arguments for proving Theorem \[5,1\] consists in finding an appropriate hyperspace \( Y = Y(X) \) such that \( S_*(\mathcal{O}(Y), \mathcal{O}(Y)) \) in \( X \) translates as \( S_*(\mathcal{O}(Y), \mathcal{O}(Y)) \) in \( Y \). This is done in such a way that he can carry back and forth strategies and plays from the game \( G_*(\mathcal{O}(Y), \mathcal{O}(Y)) \) in \( X \) to the game \( G_*(\mathcal{O}(Y), \mathcal{O}(Y)) \). This allows him to reduce the problem to a scenario where Theorems \[1.1\] and \[1.2\] are available.

The difficulty in following the above sketch when trying to generalize it to \( B \)-coverings consists in finding an appropriate hyperspace \( Y(X) \). Scheepers originally used \( Y(X) = \sum_{n \in \omega} X^n \), but we were not able to relate this construction to the bornology \([X]^{<\aleph_0}\). The way we found to solve this problem was to consider \( Y(X) \) as the bornology itself, with an appropriate topology.

More generally, given a family \( \mathcal{B} \) of nonempty subsets of a topological space \( X \), we consider the topology on \( \mathcal{B} \) whose basic open neighborhoods are sets of the form
\[
\langle U \rangle := \{ B \in \mathcal{B} : B \subset U \},
\]
for \( U \subset X \) open. This type of hyperspace has been studied already in the literature\[3\] in \[14\]. Michael considers over \( \mathcal{A}(X) := \{ A \subset X : A \neq \emptyset \} \) the topology generated by sets of the form
\[
U^+ := \{ A \in \mathcal{A}(X) : A \subset U \},
\]
with \( U \) ranging over the open sets of \( X \), and he calls it as the upper semi-finite topology on \( \mathcal{A}(X) \); by restricting this construction to the family of all nonempty closed subsets of \( X \), one obtains the so called upper Vietoris topology \[8\].

\footnote{We would like to thank Valentin Gutev for pointing this out.}
Since the topology on \( \mathcal{B} \) generated by the family \( \{ (U) : U \subset X \text{ is open} \} \) is the topology of \( \mathcal{B} \) as a subspace of \( \mathcal{A}(X) \), we shall write \( \mathcal{B}^+ \) to denote the family \( \mathcal{B} \) endowed with this topology.

The main problem with the hyperspace \( \mathcal{B}^+ \) concerns its poor separation properties: one readily sees that if there are \( A, B \in \mathcal{B} \) such that \( A \subset B \), then they cannot be separated as points of \( \mathcal{B}^+ \), showing that \( \mathcal{B}^+ \) is not \( T_1 \). However, this lack of separation properties will be harmless in our context.

**Lemma 3.2.** Let \( X \) be a topological space and let \( \mathcal{B} \) be a family of subsets of \( X \).

1. If \( \mathcal{U} \) is a \( \mathcal{B} \)-covering for \( X \), then \( \langle \mathcal{U} \rangle := \{ \langle U \rangle : U \in \mathcal{U} \} \) is an open covering for \( \mathcal{B}^+ \).
2. If \( \mathcal{W} \) is an open covering for \( \mathcal{B}^+ \) consisting of basic open sets, then the family \( \mathcal{W} := \{ \{ U \} : \{ U \} \in \mathcal{W} \} \) is a \( \mathcal{B} \)-covering for \( X \).

Then, let \( \varphi: \omega \to [2, \aleph_0] \) be a function.

3. \( S_\varphi(\mathcal{O}_B, \mathcal{O}_B) \) holds in \( X \) if and only if \( S_\varphi(\mathcal{O}(\mathcal{B}^+), \mathcal{O}(\mathcal{B}^+)) \) holds.
4. The games \( G_\varphi(\mathcal{O}_B, \mathcal{O}_B) \) in \( X \) and \( G_\varphi(\mathcal{O}(\mathcal{B}^+), \mathcal{O}(\mathcal{B}^+)) \) are equivalent.

**Proof.** The items (1) and (2) follow from the definition of \( \mathcal{B}^+ \). The other items hold because one can replace arbitrary open coverings in \( \mathcal{B}^+ \) with open coverings consisting of basic open sets, what enables one to use the previous items. \( \square \)

**Theorem 3.3.** Let \( X \) be a topological space and let \( \mathcal{B} \) be a family of subsets of \( X \).

1. If \( S_1(\mathcal{O}_B, \mathcal{O}_B) \) holds, then \( \exists \mathcal{G}_1(\mathcal{O}_B, \mathcal{O}_B) \) also holds.
2. If \( S_{\text{fin}}(\mathcal{O}_B, \mathcal{O}_B) \) holds, then \( \exists \mathcal{G}_{\text{fin}}(\mathcal{O}_B, \mathcal{O}_B) \) also holds.

**Proof.** Repeat the steps in Remark 2 with \( Y(X) = \mathcal{B}^+ \). \( \square \)

**Corollary 3.4.** Let \( X \) be a topological space and let \( \mathcal{B} \) be a family of subsets of \( X \).

For a function \( f: \omega \to [2, \aleph_0] \), the following are equivalent:

1. \( \exists \mathcal{G}_1(\mathcal{O}_B, \mathcal{O}_B) \); 
2. \( \exists \mathcal{G}_f(\mathcal{O}_B, \mathcal{O}_B) \);
3. \( S_f(\mathcal{O}_B, \mathcal{O}_B) \);
4. \( S_1(\mathcal{O}_B, \mathcal{O}_B) \).

**Proof.** These equivalences hold for the pair \( (\mathcal{O}(\mathcal{B}^+), \mathcal{O}(\mathcal{B}^+)) \). Apply Lemma 3.2 to finish. \( \square \)

By replacing \( \mathcal{B} \) with \( [X]^{<\aleph_0} \) in the above corollary results in a strengthening of Theorem 3.1 while taking \( \mathcal{B} \) as the family of all compact subsets of \( X \) yields new results about \( K \)-coverings. Well, almost new results, as we explain below.

**Remark 3.** Following the announcement of this work, Boaz Tsaban brought to our attention that a result similar to Theorem 3.3 also appears in the (thus far, unpublished) MSc thesis of his student Nadav Samet [18].

Instead of considering a topology over a family of subsets of \( X \), they take a family \( \mathcal{P} \) of filters of open sets and observe that sets of the form \( O_U := \{ p \in \mathcal{P} : U \in p \} \) define a base for a topology over \( \mathcal{P} \) when \( U \) ranges over the open sets of \( X \).

In connection with spaces of the form \( C_B(X) \), we first state a generalization of some of our results in [3].

**Proposition 3.5.** Let \( X \) be a Tychonoff space and let \( \mathcal{B} \) be a bornology on \( X \) with a compact base. Consider a function \( \varphi: \omega \to [2, \aleph_0] \).
(1) $S_\varphi(O_B,O_B) \text{ holds in } X \text{ if and only if } S_\varphi(\Omega_\omega,\Omega_\omega) \text{ holds in } C_B(X)$.

(2) If $\varphi$ is non-decreasing, then the games $G_{\varphi}(O_B,O_B)$ in $X$ and $G_{\varphi}(\Omega_\omega,\Omega_\omega)$ in $C_B(X)$ are equivalent.

The proof is essentially an adaptation of the arguments presented in [3], and it follows with appropriate applications of the following lemma, adapted from [4].

**Lemma 3.6.** Let $X$ be a Tychonoff space and let $B$ be a bornology with a compact base on $X$.

1. If $U$ is a collection of open sets of $X$ such that $X \notin U$, then $U \in O_B$ if and only if $A(U) = \{ f \in C_B(X) : \exists U \in U \text{ with } f \upharpoonright (X \setminus U) \equiv 1 \} \in \Omega_\omega$.
2. Let $A \subseteq C_B(X)$, $n \in \omega$ and set $U_n(A) = \left\{ f^{-1}\left(\left(-\frac{1}{n+1},\frac{1}{n+1}\right)\right) : f \in A \right\}$. If $o \in \overline{A}$, then $U_n(A) \in O_B$.
3. If $(A_n)_{n \in \omega}$ is a sequence of finite subsets of $C_B(X)$ such that $\bigcup_{n \in \omega} U_n(A_n)$ is a nontrivial $B$-covering, then $\bigcup_{n \in \omega} A_n \in \Omega_\omega$.
4. If $(A_n)_{n \in \omega}$ is a sequence of finite subsets of $C_B(X)$ such that $U_{n \in \omega} A_n \in \Omega_\omega$ and for each $n \in \omega$ and each $g \in A_n$ there is a proper open set $U_g \subset X$ such that $g \upharpoonright (X \setminus U_g) \equiv 1$, then $\bigcup_{n \in \omega} \{U_g : g \in A_n\} \in O_B$.

**Proof.**

1. If $U \in O_B$ and $(B,\varepsilon)[o]$ is a neighborhood of $o$, then we obtain a function $f \in A(U) \cap (B,\varepsilon)[o]$, because $B$ is compact and $X$ is a Tychonoff space (see [3] Theorem 3.1.7). Conversely, for a $B \in B$ we take an $f \in A(U) \cap (B,1)[o]$, from which we obtain an open set $U \in U$ such that $B \subset U$.

2. It follows because for a function $f \in C_B(X)$, $f \in (B,\frac{1}{n+1})[o]$ if and only if $B \subset f^{-1}\left(\left(-\frac{1}{n+1},\frac{1}{n+1}\right)\right)$.

3. In addition to the previous observation, we use the fact that if $U \in O_B$ is nontrivial, then $U \setminus F \in O_B$ for any finite subset $F \subset U$.

4. For a $B \in B$, we take a $g \in \bigcup_{n \in \omega} A_n \cap (B,1)[o]$, from which it follows that $B \subset U_g$, because $g \upharpoonright (X \setminus U_g) \equiv 1$.

Particularly, the monotonicity hypothesis in Proposition [3.5] can be dropped if the function $\varphi$ is of the form $\omega \to [2,\aleph_0]$. In fact, this follows from Theorem [2.3] and its counterpart for $\mathbf{B}$-coverings, which we state below.

**Proposition 3.7.** Let $X$ be a topological space with a bornology $B$, and consider a function $f : \omega \to [2,\aleph_0]$.

1. If $f$ is bounded, then the games $G_f(O_B,O_B)$ and $G_f(O_B,O_B)$ are equivalent, where $k = \limsup_{n \in \omega} f(n)$;

2. If $f$ is unbounded, then $G_f(O_B,O_B)$ and $G_{\mathbf{B}}(O_B,O_B)$ are equivalent.

**Proof.** In face of Corollary [3.4] we just need to worry about Player \textsc{II}. For the first case where $f$ is bounded, note that there exists an $m_0 \in \omega$ such that $f(n) \leq k$ for all $n \geq m_0$. Thus, if $\mu$ is a winning strategy for Player \textsc{II} in $G_f$, then $\mu$ induces a winning strategy on $G_k$ simply by ignoring the $m_0$ first innings – here we also use the fact that $U \setminus F \in O_B$ whenever $U \in O_B$ is nontrivial and $F \in \mathcal{U}_<\omega$. The converse holds because the set $N = \{n \in \omega : f(n) = k\}$ is infinite.

\^3Particularly, everything still works if $X$ is a normal space and $B$ is a bornology with a closed base.
Now, if \( f \) and \( g \) are unbounded, by symmetry it is enough to show that
\[
\Pi \uparrow G_f(\mathcal{O}_B, \mathcal{O}_B) \Rightarrow \Pi \uparrow G_g(\mathcal{O}_B, \mathcal{O}_B).
\]
Indeed, if \( \mu \) is a strategy for Player \( \Pi \) in \( G_f \), we fix a sequence \((n_i)_{i \in \omega}\) of natural numbers such that \( g(n_i) \geq f(i) \) for all \( i < \omega \) and then we induce a winning strategy for Player \( \Pi \) in the game \( G_g \) by using \( \mu \) only in the innings \( n \in \{ n_i : i < \omega \} \). \( \square \)

Now, we can translate Corollary 3.4 for function spaces automatically.

**Corollary 3.8.** Let \( X \) be a Tychonoff space and let \( \mathcal{B} \) be a bornology on \( X \) with a compact base. For a function \( f : \omega \to [2, \aleph_0) \), the following are equivalent:

1. \( \Pi \notin G_1(\Omega_o, \Omega_o) \);
2. \( \Pi \notin G_f(\Omega_o, \Omega_o) \);
3. \( S_f(\Omega_o, \Omega_o) \);
4. \( S_1(\Omega_o, \Omega_o) \).

**Remark 4.** We still do not know if the games \( G_f(\mathcal{O}, \mathcal{O}) \) and \( G_1(\mathcal{O}, \mathcal{O}) \) are equivalent for arbitrary topological spaces. If it become to be true, at least for \( T_\eta \)-spaces, then an analogous result can be derived for function spaces with the tools we presented above.

### 4. Bornologies and filters

Recall that a filter \( \mathcal{F} \) on a set \( C \) is a family of subsets of \( C \) closed upwards and closed for taking finite intersections – it is called a proper filter if \( \emptyset \notin \mathcal{F} \). For a topological space \( Y \) and a point \( y \in Y \), we consider the neighborhood filter of \( y \),

\[
\mathcal{N}_y := \{ N \subset Y : \exists V \subset Y, V \text{ is open and } y \in V \subset N \}.
\]

In this section we intend to generalize Theorem 3 in [9], but first we make the necessary definitions, adapted from [9][11].

For a fixed set \( C \), we denote by \( \mathcal{P}(C) \) the family of the proper filters of \( C \), and for a cardinal \( \kappa \geq \aleph_0 \) we let \( \mathcal{F}_{\kappa}(C) \) be the family of those proper filters of \( C \) of the form

\[
\mathcal{G} \uparrow := \{ F \subset Y : \exists G \in \mathcal{G} \ (G \subset F) \},
\]

where \( \mathcal{G} \in [\mathcal{P}(C)]^{\leq \kappa} \) – particularly, we call the elements in \( \mathcal{F}_1(C) \) and \( \mathcal{F}_{\aleph_0}(C) \) as principal filters and countable based filters, respectively. If \( R \subset C \times D \) is a binary relation on the sets \( C \) and \( D \) and if \( \mathcal{F} \) is a collection of subsets of \( C \), we set \( R(\mathcal{F}) := \{ R[F] : F \in \mathcal{F} \}^\uparrow \). It is worth to mention that the correspondence

\[
\varphi(\varphi(C)) \ni \mathcal{G} \mapsto \mathcal{G} \uparrow \in \mathcal{P}(\varphi(C)),
\]

does not determine a function \( \varphi(\varphi(C)) \rightarrow \mathcal{P}(C) \). In fact, \( \mathcal{G} \uparrow \) is a proper filter if and only if \( \emptyset \notin \mathcal{G} \) and for all \( A, B \in \mathcal{G} \) there is a \( G \in \mathcal{G} \) such that \( G \subset A \cap B \).

By a class of filters \( \mathcal{K} \) we mean a property about filters, and we write \( \mathcal{F} \in \mathcal{K} \) to indicate that the filter \( \mathcal{F} \) has property \( \mathcal{K} \). We also say that a topological space \( Y \) is a \( \mathcal{K} \)-space if \( \mathcal{N}_y \in \mathcal{K} \) for all \( y \in Y \). For instance, the class of \( \mathcal{F}_{\aleph_0} \)-spaces is the class of spaces with countable character.

We say that a class of filters \( \mathcal{K} \) is \( \mathcal{F}_1 \)-composable if for any sets \( C, D \) and any relation \( R \subset C \times D \), the following holds:

\[
\mathcal{F} \in \mathcal{F}(C) \cap \mathcal{K} \text{ and } R(\mathcal{F}) \in \mathcal{F}(D) \Rightarrow R(\mathcal{F}) \in \mathcal{K}.
\]
Note that the condition “\( R(F) \in \mathbb{F}(D) \)” in the left hand of the above implication is necessary, because in general there is no guarantee that \( R(F) \) is a proper filter on \( D \). If \( R \) is just a relation in \( C \times D \), it may happen that \( R[F] = \emptyset \) for some \( F \in F \), and in this case \( R(F) \) is not a proper filter. However, the situation becomes simpler if \( R \) is a function.

**Lemma 4.1.** Let \( C \) and \( D \) be sets and let \( f: C \to D \) be a function.

1. If \( F \in \mathbb{F}(C) \), then \( f(F) \in \mathbb{F}(D) \).
2. If \( G \in \mathbb{F}(D) \), then \( f^{-1}(G) \in \mathbb{F}(C) \) if and only if \( G \cap f[C] \neq \emptyset \) for all \( G \in \mathcal{G} \).

Finally, a class of filters \( \mathcal{K} \) is called \( \mathbb{F}_\omega \)-steady if for any set \( C \) and each pair of filters \( F \in \mathbb{F}(C) \cap \mathcal{K} \) and \( G \in \mathbb{F}_\omega(C) \) such that \( F \cap G \neq \emptyset \) for all \( (F,G) \in F \times G \), the following holds

\[
F \vee G := \left\{ F \cap G : (F,G) \in F \times G \right\}^\uparrow \in \mathcal{K}.
\]

Given a Tychonoff space \((X, \tau)\) and a bornology \( \mathcal{B} \) on \( X \), we denote by \( \Gamma_\mathcal{B}(X) \) the filter on \( \tau \) generated by the sets

\[
V(B) := \{ U \in \tau : B \subset U \},
\]

i.e., \( \Gamma_\mathcal{B}(X) := \{ V(B) : B \in \mathcal{B} \}^\uparrow \). Note that whenever \( \mathcal{B}' \subset \mathcal{B} \) is a base for \( \mathcal{B} \), then \( \{ V(B) : B \in \mathcal{B}' \}^\uparrow = \Gamma_\mathcal{B}(X) \).

In [9], Jordan considers the filter \( \Gamma(X) \) on \( \tau \), which coincides with \( \Gamma_{|X|<\omega}(X) \) according to our previous definition. Jordan shows that if a class of filters \( \mathcal{K} \) is \( \mathbb{F}_1 \)-composable and \( \mathbb{F}_\omega \)-steady, then \( \Gamma(X) \in \mathcal{K} \) if and only if \( C_\mathcal{B}(X) \) is a \( \mathcal{K} \)-space. Our next results aim to extend this conclusion to \( \Gamma_\mathcal{B}(X) \) and \( C_\mathcal{B}(X) \). Their proofs are natural adaptations from [9].

**Proposition 4.2.** Let \((X, \tau)\) be a Tychonoff space and let \( \mathcal{B} \) be a bornology with a compact base on \( X \). Suppose that \( \mathcal{K} \) is an \( \mathbb{F}_1 \)-composable class of filters. If \( C_\mathcal{B}(X) \) is a \( \mathcal{K} \)-space, then \( \Gamma_\mathcal{B}(X) \in \mathcal{K} \).

**Proof.** It is enough to present a neighborhood filter \( \mathcal{F} \) in \( C_\mathcal{B}(X) \), a set \( Y \) with functions \( \pi: Y \to \tau \) and \( \Phi: Y \to C_\mathcal{B}(X) \) such that \( \Phi^{-1}(F) \in \mathbb{F}(Y) \) and \( \Gamma_\mathcal{B}(X) = \pi(\Phi^{-1}(F)) \). In fact, since \( \mathcal{F} \in \mathcal{K} \) and \( \mathcal{K} \) is \( \mathbb{F}_1 \)-composable, it follows by the previous lemma that \( \Phi^{-1}(F) \in \mathcal{K} \) and \( \pi(\Phi^{-1}(F)) \in \mathcal{K} \).

Let \( \mathcal{B}_0 \) be a compact base for \( \mathcal{B} \). Let \( Y := \{ (B, U) \in \mathcal{B}_0 \times \tau : B \subset U \} \) and define \( \pi: Y \to \tau \) by \( \pi(B, U) = U \). Since \( X \) is a Tychonoff space, for each \((B, U) \in Y \) there exists some \( f = f_{(B, U)} \in C_\mathcal{B}(X) \) such that \( f \upharpoonright B \equiv 0 \) and \( f \upharpoonright X \setminus U \equiv 1 \), so we may set \( \Phi: Y \to C_\mathcal{B}(X) \) by \( \Phi(B, U) = f_{(B, U)} \) for each \((B, U) \in Y \).

Now, we take \( \mathcal{F} := \mathcal{N}_{0,C_\mathcal{B}(X)} = \{ (B, [_{n \in \omega}^n f]) : B \in \mathcal{B}_0, n \in \omega \} \). In order to prove \( \pi(\Phi^{-1}(F)) = \Gamma_\mathcal{B}(X) \), it is enough to note that both filters are generated by the same base. Indeed, for any \( B \in \mathcal{B}_0 \) and \( \varepsilon \in (0,1) \) one has

\[
V(B) = \pi(\Phi^{-1}(\mathcal{N}_{0,C_\mathcal{B}(X)})),
\]

from which the desired equality follows. \qed

**Proposition 4.3.** Let \( \mathcal{K} \) be a class of filters that is \( \mathbb{F}_1 \)-composable and \( \mathbb{F}_\omega \)-steady. For a topological space \((X, \tau)\) and a bornology \( \mathcal{B} \) on \( X \), \( \Gamma_\mathcal{B}(X) \in \mathcal{K} \) implies that \( C_\mathcal{B}(X) \) is a \( \mathcal{K} \)-space.
Proof. We use a similar strategy as in the proof of the previous proposition. We define a set \( Y \) with a proper countably based filter \( \mathcal{H} \in \mathcal{F}_\omega(Y) \) and functions \( \pi: Y \to C_B(X) \) and \( \Phi: Y \to \tau \) such that \( \Phi^{-1}(\Gamma_B(X)) \) and \( \Phi^{-1}(\Gamma_B(X)) \vee \mathcal{H} \) are proper filters in \( Y \), and \( \pi(\Phi^{-1}(\Gamma_B(X)) \vee \mathcal{H}) = \mathcal{N}_{o,C_B(X)} \). Again, the hypotheses over \( K \) guarantee that \( \pi(\Phi^{-1}(\Gamma_B(X)) \vee \mathcal{H}) \in K \), which is enough to finish the proof, since \( C_B(X) \) is a homogeneous space.

For brevity, we call \( I_n := (-\frac{1}{n+1}, \frac{1}{n+1}) \subset \mathbb{R} \) for each \( n \in \omega \).

Let \( Y := \{(f, B, n) \in C_B(X) \times B \times f[B] \subset I_n \} \) and \( \mathcal{H} := \{M_n : n \in \omega\}^\dagger \), where \( M_n := \{(f, B, m) \in Y : m \geq n\} \) for each \( n \in \omega \). Now, let \( \pi: Y \to C_B(X) \) be defined by \( \pi(f, B, n) = f \) and \( \Phi: Y \to \tau \) defined as \( \Phi(f, B, n) = f^{-1}[I_n] \). Since \( \Gamma_B(X) \in K \) and \( K \) is \( \mathcal{F}_1 \)-composible, it follows that \( \Phi^{-1}(\Gamma_B(X)) \in K \). Also, since \( \Phi^{-1}([V(B)] \cap M_n \neq \emptyset) \) for all \( n \in \omega \), the \( \mathcal{F}_\omega \)-steadiness of \( K \) gives \( \Phi^{-1}(\Gamma_B(X)) \vee \mathcal{H} \in K \) and, again by the \( \mathcal{F}_1 \)-composability of \( K \), \( \pi(\Phi^{-1}(\Gamma_B(X)) \vee \mathcal{H}) \in K \). So, in order to finish the proof we must show that \( \mathcal{N}_{o,C_B(X)} = \pi(\Phi^{-1}(\Gamma_B(X)) \vee \mathcal{H}) \). The desired equality follows because

\[
\left( B, \frac{1}{n+1}\right) \cdot [o] = \pi [\Phi^{-1} [V(B)] \cap M_n] \]

holds for any \( B \in \mathcal{B} \) and \( n \in \omega \). \( \square \)

Altogether, the propositions above yields the following

**Theorem 4.4.** Let \( X \) be a Tychonoff space and let \( \mathcal{B} \) be a bornology with a compact base on \( X \). If \( K \) is a class of filters that is \( \mathcal{F}_1 \)-composible and \( \mathcal{F}_\omega \)-steady, then \( \Gamma_B(X) \in K \) if and only if \( C_B(X) \) is a \( K \)-space.

**Example 1.** For a cardinal \( \kappa \geq \aleph_0 \), we say that a filter \( \mathcal{F} \) on \( C \) belongs to \( T_\kappa \) if for all \( A \subset C \) such that \( A \cap C \neq \emptyset \) holds for every element of \( \mathcal{F} \), there is a \( B \in [A]^{\leq \kappa} \) such that \( B \cap F \neq \emptyset \) for all \( F \in \mathcal{F} \). It can be shown that \( T_\kappa \) is a class of filters \( \mathcal{F}_1 \)-composible and \( \mathcal{F}_\omega \)-steady. So, under the assumptions of the previous corollary, \( C_B(X) \) has tightness less than or equal to \( \kappa \) if and only if \( \Gamma_B(X) \in T_\kappa \), i.e., any \( \mathcal{B} \)-covering of \( X \) has a \( \mathcal{B} \)-subcovering of cardinality \( \leq \kappa \), a result originally due to McCoy and Ntantu [13].

**Example 2.** Let \( \mathcal{F} \) be a proper filter on a set \( C \), and consider the family

\[
\mathcal{M} := \{ A \subset C : \forall F \in \mathcal{F} (A \cap F \neq \emptyset) \}.
\]

Note that for any function \( \varphi: \omega \to [2, \aleph_0] \), both classes of filters \( S_\varphi(M, M) \) and \( \mathcal{I} \uparrow G_\varphi(M, M) \) are \( \mathcal{F}_\omega \)-composible. Thus, the directions “property in \( C_B(X) \) implies property in \( X \)” of Proposition [12, Corollary concerning both \( S_\varphi \) and Player \( \mathcal{I} \) follow from Proposition [12]. On the other hand, the converses follow from Proposition [4, 3] whenever \( \varphi \) is a constant function, because in this case it can be proved that \( S_\varphi(M, M) \) and \( \mathcal{I} \uparrow G_\varphi(M, M) \) are also \( \mathcal{F}_\omega \)-steady classes of filters. However, we were not able to prove similar statements regarding Player \( \mathcal{I} \) in the game \( G_\varphi(M, M) \).

5. \( \gamma \)-PRODUCTIVE SPACES

Gerlits and Nagy [7] had introduced the concept of point-cofinite open coverings\(^4\) in their analysis of Fréchet property in \( C_p(X) \). Recall that a topological space \( Y \)

\(^4\) Usually called \( \gamma \)-coverings in the literature.
is Fréchet (resp. strictly Fréchet) if for each \( y \in Y \) and for each \( A \in \Omega_y \) there is a subset \( B \subset A \) such that \( B \in \Gamma_y \), where
\[
\Gamma_y := \{ A \subset Y : |A \setminus V| < \aleph_0 \text{ for all } V \in \mathcal{N}_{y,Y} \}.
\]
(resp. if \( S_1(\Omega_y, \Gamma_y) \) holds). We say that an infinite collection \( \mathcal{U} \) of proper open sets of \( X \) is called a point-cofinite covering if for all \( x \in X \) the set \( \{ U \in \mathcal{U} : x \notin U \} \) is finite, and we denote by \( \Gamma \) the collection of all point-cofinite coverings of \( X \) - particularly, note that \( \Gamma \subset \Omega \). A space \( X \) is called a \( \gamma \)-space if any nontrivial \( \omega \)-covering has a (countable) \( \gamma \)-subcovering. Then we have the following

**Theorem 5.1** (Gerlits and Nagy [7]). For a Tychonoff space \( X \), the following are equivalent:

1. \( X \) is a \( \gamma \)-space;
2. \( S_1(\Omega, \Gamma) \) holds;
3. \( C_p(X) \) is a strictly Fréchet space;
4. \( C_p(X) \) is a Fréchet space.

In [9], Jordan obtain a variation of the above theorem as a corollary of Theorem 4.4, with appropriate definitions for Fréchet filters and strongly \( F \)-Fréchet filters, which turns out to be \( F_1 \)-composable and \( F_{\omega \cdot} \)-steady classes of filters.

However, we shall pay more attention to the following characterization.

**Theorem 5.2** (Jordan and Mynard [10]). A topological space \( Y \) is productively Fréchet if and only if \( Y \times Z \) is a Fréchet space for any strongly Fréchet space \( Z \).

In the above theorem, the sentence “\( Y \) is productively Fréchet” has a very precise meaning – it is a space such that all of its neighborhood filters belongs to the class of productively Fréchet filters:
\[
\mathcal{F} \in \mathcal{F}(C) \text{ is productively Fréchet if for each strongly Fréchet filter } \mathcal{H} \in \mathcal{F}(C) \text{ such that } \mathcal{F} \cap \mathcal{H} \neq \emptyset \text{ for all } (\mathcal{F}, \mathcal{H}) \in \mathcal{F} \times \mathcal{H} \text{ there is a filter } \mathcal{G} \in \mathcal{F}_{\omega \cdot}(C) \text{ refining } \mathcal{F} \lor \mathcal{H}.
\]

The considerations above justify the following definition of Jordan [9]: a Tychonoff space \( X \) is called \( \gamma \)-productive if the filter \( \Gamma(X) \) is productively Fréchet. Then, by using the fact that productively Fréchet filters are \( F_1 \)-composable and \( F_{\omega \cdot} \)-steady, Jordan proves the following.

**Proposition 5.3** (Jordan [9]). If \( X \) is \( \gamma \)-productive, then \( X \times Y \) is a \( \gamma \)-space for all \( \gamma \)-space \( Y \).

By extending this definition to the filter \( \Gamma_B(X) \), we will prove that the productivity of \( \gamma \)-productive spaces is far more strong. In order to do this, we will follow the indirect approach presented by Miller, Tsaban and Zdomskyy in [16] to derive Proposition 5.3.

Let \( X \) be a Tychonoff space and let \( \mathcal{B} \) be a bornology on \( X \). We say that an infinite collection \( \mathcal{U} \) of proper open sets of \( X \) is a \( \mathcal{B} \)-cofinite covering if for all \( B \in \mathcal{B} \) the set \( \{ U \in \mathcal{U} : B \not\subset U \} \) is finite, and we denote by \( \Gamma_B \) the collection of all \( \mathcal{B} \)-cofinite coverings of \( X \). Naturally, we say that \( X \) is a \( \gamma_B \)-space if any nontrivial \( \mathcal{B} \)-cofinite covering has a (countable) \( \mathcal{B} \)-cofinite subcovering.

\[\text{Strictly Fréchet spaces and strongly Fréchet spaces are not formally the same: the later were independently introduced by Michael [15] and Siwiec [23]. Although the definition is similar, in the strong case there is the additional requirement for the sequence } (A_n)_{n \in \omega} \text{ with } y \in \bigcap_{n \in \omega} A_n \text{ to be decreasing.}\]
McCoy and Ntantu [13] have introduced $\mathcal{B}$-cofinite open coverings in their generalization of Theorem 5.1, calling them as $\mathcal{B}$-sequences there. Although they just stated items (1) and (4) of the theorem below, their arguments, which are adapted from Gerlits and Nagy [7], can be used to prove the following.

**Theorem 5.4** (McCoy and Ntantu [13]). Let $X$ be a Tychonoff space and let $\mathcal{B}$ be a bornology with a compact base on $X$. The following are equivalent:

1. $X$ is a $\gamma_\mathcal{B}$-space;
2. $\mathcal{S}_1(\mathcal{O}_\mathcal{B}, \Gamma_\mathcal{B})$ holds;
3. $\mathcal{C}_\mathcal{B}(X)$ is strictly Fréchet;
4. $\mathcal{C}_\mathcal{B}(X)$ is Fréchet.

**Remark 5.** In particular, all of the above conditions are equivalent to require that $\mathcal{C}_\mathcal{B}(X)$ is strongly Fréchet.

Now, we will say that a Tychonoff space $X$ with a bornology $\mathcal{B}$ is $\gamma_\mathcal{B}$-productive if the filter $\Gamma_\mathcal{B}(X)$ is productively Fréchet – note that for $\mathcal{B} = [X]^{<\aleph_0}$, one obtains the original definition of $\gamma$-productive spaces.

Since we will work with the product of spaces endowed with different bornologies, we need to describe their behavior under products.

**Proposition 5.5.** Given a family $\{X_t: t \in T\}$ of pairwise disjoint topological spaces, consider for each $t \in T$ a set $\mathcal{B}_t \subset \mathcal{P}(X_t)$. Let $\mathcal{B}_0 = \{\prod_{t \in T} \mathcal{B}_t: \forall t (\mathcal{B}_t \in \mathcal{B}_t)\}$ and $\mathcal{B}_1 = \{\bigsqcup_{t \in T} \mathcal{B}_t: \mathcal{B}_t \in \mathcal{B}_t$ for finitely many $t \in T$, $\mathcal{B}_t = \emptyset$ otherwise\}. If $\mathcal{B}_t$ is a (compact) base for each $t \in T$, then $\mathcal{B}_0$ and $\mathcal{B}_1$ are (compact) bases for bornologies on $\prod_{t \in T} X_t$ and $\sum_{t \in T} X_t$, respectively.

We denote by $\bigotimes_{t \in T} \mathcal{B}_t$ and $\bigoplus_{t \in T} \mathcal{B}_t$ the bornologies generated by the bases $\mathcal{B}_0$ and $\mathcal{B}_1$ in the above proposition, respectively.

**Proposition 5.6.** Let $\{X_t: t \in T\}$ be a family of topological spaces, and for each $t \in T$ let $\mathcal{B}_t$ be a bornology on $X_t$. Then $C_{\bigotimes_{t \in T} \mathcal{B}_t}(\sum_{t \in T} X_t)$ is homeomorphic to $C_{\bigoplus_{t \in T} \mathcal{B}_t}(\sum_{t \in T} X_t)$.

**Proof.** Note that the map

$$C_{\bigotimes_{t \in T} \mathcal{B}_t}(\sum_{t \in T} X_t) \ni f \mapsto (f \upharpoonright X_t)_{t \in T} \in \prod_{t \in T} C_{\mathcal{B}_t}(X_t)$$

is continuous and it has a continuous inverse. \qed

**Remark 6.** Particularly, it follows from the previous proposition that

(8) $C_p(\sum_{t \in T} X_t)$ is homeomorphic to $\prod_{t \in T} C_p(X_t)$,

and, if each $X_t$ is a Hausdorff space, then

(9) $C_k(\sum_{t \in T} X_t)$ is homeomorphic to $\prod_{t \in T} C_k(X_t)$.

For brevity, if $X_t = X$ and $\mathcal{B}_t = \mathcal{B}$ for all $t \in T$, we will write $\mathcal{B}^{[T]}$ instead of $\bigotimes_{t \in T} \mathcal{B}$.

The next lemma will be very useful later.
Lemma 5.7 (Miller, Tsaban and Zdomskyy [10]). Let \( \mathfrak{P} \) be a topological property hereditary for closed subspaces and preserved under finite power. Then for any pair of topological spaces \( X \) and \( Y \), \( X \times Y \) has the property \( \mathfrak{P} \) provided that \( X + Y \) has the property \( \mathfrak{P} \).

The first three items in the next proposition states that the property “having a bornology \( \mathcal{B} \) with a compact base such that it is a \( \gamma_{\mathcal{B}} \)-space” satisfies the conditions of the previous lemma.

Proposition 5.8. Let \( X \) be a topological space and let \( \mathcal{B} \) be a bornology on \( X \).
(a) If \( Y \subset X \), then \( \mathcal{B}_Y := \{ B \cap Y : B \in \mathcal{B} \} \) is a bornology on \( Y \). If \( Y \) is closed and \( \mathcal{B} \) has a compact base on \( X \), then \( \mathcal{B}_Y \) has a compact base on \( Y \).
(b) If \( X \) is a \( \gamma_{\mathcal{B}} \)-space and \( Y \subset X \) is closed, then \( Y \) is a \( \gamma_{\mathcal{B}_Y} \)-space.
(c) If \( X \) is a \( \gamma_{\mathcal{B}} \)-space and \( \mathcal{B} \) has a compact base, then \( X^n \) is a \( \gamma_{\mathcal{B}^n} \)-space for any \( n \in \omega \).
(d) If \( X \) is a \( \gamma_{\mathcal{B}} \)-space and \( Y \) is a \( \gamma_{\mathcal{L}} \)-space for a bornology \( \mathcal{L} \) in \( Y \) such that \( X \times Y \) is a \( \gamma_{\mathcal{B} \otimes \mathcal{L}} \)-space, then \( X \sqcup Y \) is a \( \gamma_{\mathcal{B} \oplus \mathcal{L}} \)-space.

Corollary 5.9. Let \( X \) and \( Y \) be topological spaces with bornologies \( \mathcal{B} \) and \( \mathcal{L} \), respectively, both of them with compact bases. Then \( X \times Y \) is a \( \gamma_{\mathcal{B} \otimes \mathcal{L}} \)-space if and only if \( X + Y \) is a \( \gamma_{\mathcal{B} \oplus \mathcal{L}} \)-space.

We finally can state and prove the desired extension of Proposition 5.3.

Corollary 5.10. Let \( X \) be a Tychonoff space and let \( \mathcal{B} \) be a bornology with a compact base on \( X \). If \( X \) is \( \gamma_{\mathcal{B}} \)-productive, then \( X \times Y \) is a \( \gamma_{\mathcal{B} \otimes \mathcal{L}} \)-space for any Tychonoff space \( Y \) endowed with a bornology \( \mathcal{L} \) with a compact base such that \( \mathcal{L} \) is a \( \gamma_{\mathcal{L}} \)-space.

Proof. If \( X \) is \( \gamma_{\mathcal{B}} \)-productive then \( C_{\mathcal{B}}(X) \) is productively Fréchet. Since \( Y \) is a \( \gamma_{\mathcal{L}} \)-space, it follows that \( C_{\mathcal{L}}(Y) \) is Fréchet, hence \( C_{\mathcal{B}}(X) \times C_{\mathcal{L}}(Y) \) is (strongly) Fréchet. But \( C_{\mathcal{B}}(X) \times C_{\mathcal{L}}(Y) \) is homeomorphic to \( C_{\mathcal{B} \otimes \mathcal{L}}(X + Y) \), thus \( X + Y \) is a \( \gamma_{\mathcal{B} \otimes \mathcal{L}} \)-space, and the conclusion follows from the last corollary.

Particularly, if a Tychonoff space \( X \) is \( \gamma \)-productive, then \( X \times Y \) is a \( \gamma_{\mathcal{L}} \)-space whenever \( Y \) is a Tychonoff \( \gamma_{\mathcal{L}} \)-space, where \( \mathcal{L} \) is a bornology with a compact base on \( Y \). This suggests that the converse of Proposition 5.3 may be false.

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