A second countable locally compact transitive groupoid
without open range map

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Abstract. Dana P. Williams raised in [Proc. Am. Math. Soc., Ser. B, 2016] the following question: Must a second countable, locally compact, transitive groupoid have open range map? This paper gives a negative answer to that question. Although a second countable, locally compact transitive groupoid $G$ may fail to have open range map, we prove that we can replace its topology with a topology which is also second countable, locally compact, and with respect to which $G$ is a topological groupoid whose range map is open. Moreover, the two topologies generate the same Borel structure and coincide on the fibres of $G$.

1. Introduction

In order to construct convolution algebras associated to a locally compact topological groupoid one needs an analogue of the Haar measure on a locally compact group. Starting with the work of Jean Renault [7], this analogue is a system of measures, called Haar system, subject to suitable support, invariance and continuity conditions. According to a result of Anthony Seda [8], the continuity assumption is essential for the Renault’s construction [7] of the $C^*$-algebra associated to a locally compact groupoid. This continuity assumption entails that the range map is open ([9] Proposition I.4 or [8] p. 118]). As Williams pointed out [10 Question 3.5], while there certainly exist groupoids that fail to have open range maps, most of these examples are group bundles which are as far as from being principal groupoids or transitive groupoids as possible. This led him to the next question [10 Question 3.5]: must a second countable, locally compact, transitive groupoid have open range map? In this paper we construct a second countable, locally compact, transitive and principal groupoid that fails to have open range map (and hence open domain map).

In addition, we prove that for every second countable, locally compact topology $\tau$ on a transitive groupoid $G$ making $G$ a topological groupoid, there is a topology $\tilde{\tau}$ which is also second countable, locally compact, and with respect to which $G$ is a topological groupoid with open range map. Moreover, $\tau$ and $\tilde{\tau}$ generate the same Borel structure and coincide on the fibres of $G$.

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We shall use the definition of a topological groupoid given by Jean Renault in [7]. For a groupoid $G$, $G^{(2)}$ will denote the set of the composable pairs and $G^{(0)}$ its unit space. As usual the inverse map will be written $x \mapsto x^{-1} : G \to G$ and the product map will be written $(x, y) \mapsto xy : G^{(2)} \to G$. For each $x \in G$, $r(x) = xx^{-1}$, respectively, $d(x) = x^{-1}x$ will denote the range, respectively the domain (source) of $x$ in $G^{(0)}$ (thus $r : G \to G^{(0)}$, respectively $d : G \to G^{(0)}$ will be the range map, respectively the domain/source map). For each $u \in G^{(0)}$, the fibre of the range, respectively domain map over $u$ is denoted $G^u = r^{-1}\{u\}$, respectively, $G_u = d^{-1}\{\{u\}\}.

A groupoid $G$ is said to be transitive if for every $u, v \in G^{(0)}$, there is $x \in G$ such that $r(x) = u$ and $d(x) = v$. A groupoid is called principal if the map $(r, d) : G \to G^{(0)} \times G^{(0)}$, defined by $(r, d)(x) = (r(x), d(x))$ for all $x \in G$, is injective.

If $A, B \subseteq G$ and $x \in G$, one may form the following subsets of $G$:

\[
A^{-1} = \{x \in G : x^{-1} \in A\} \\
AB = \{xy : (x, y) \in G^{(2)} \cap (A \times B)\} \\
xA = \{x\}A \text{ and } Ax = A\{x\}.
\]

A topological groupoid consists of a groupoid $G$ and a topology compatible with the groupoid structure. This means that the inverse map $x \mapsto x^{-1} : G \to G$ is continuous, as well as the product map $(x, y) \mapsto xy : G^{(2)} \to G$ is continuous, where $G^{(2)}$ has the induced topology induced from $G \times G$. By a locally compact groupoid we mean a topological groupoid whose topology is locally compact (Hausdorff)

2. A second countable locally compact transitive groupoid without open range map

Let us modify the usual topology of the space of real numbers $\mathbb{R}$ in the points of the form $\frac{3}{2^{n+2}}$ and $\frac{5}{2^{n+2}}$ with $n \in \mathbb{N}$. For every $x \in \mathbb{R}$, let

\[
B_x = \begin{cases} \\
\left\{ \left[ \frac{3}{2^{n+2}}, \frac{3}{2^{n+2}} + \varepsilon \right], \varepsilon > 0 \right\}, & \text{if } x = \frac{3}{2^{n+2}} (n \in \mathbb{N}) \\
\left\{ \left[ \frac{5}{2^{n+2}}, \frac{5}{2^{n+2}} - \varepsilon \right], \varepsilon > 0 \right\}, & \text{if } x = \frac{5}{2^{n+2}} (n \in \mathbb{N}) \\
\left\{ (x - \varepsilon, x + \varepsilon), \varepsilon > 0 \right\}, & \text{if } x \in \mathbb{R} \setminus \bigcup_{n=0}^{\infty} \left\{ \frac{3}{2^{n+2}}, \frac{5}{2^{n+2}} \right\}.
\end{cases}
\]

and

\[
\mathcal{F}_x = \{ V \subseteq \mathbb{R} : \text{there is } U \in B_x \text{ such that } U \subseteq V \}\]

Let

\[
\tau_0 = \{ O \subseteq \mathbb{R} : \text{if } x \in O, \text{ then } O \in \mathcal{F}_x \}
\]

be the unique topology on $X = \mathbb{R}$ with the property that for every $x \in X$, $\mathcal{F}_x$ is the family of neighborhoods of $x$. Then $(X, \tau_0)$ is a Hausdorff second countable topological space. Moreover if $x \in \mathbb{R} \setminus \left( \bigcup_{n=0}^{\infty} \left\{ \frac{3}{2^{n+2}}, \frac{5}{2^{n+2}} \right\} \cup \{0\} \right)$ and $\varepsilon > 0$ is small enough such that $[x - \varepsilon, x + \varepsilon] \cap \bigcup_{n=0}^{\infty} \left\{ \frac{3}{2^{n+2}}, \frac{5}{2^{n+2}} \right\} = \emptyset$, then $[x - \varepsilon, x + \varepsilon]$ is a compact neighborhood of $x$ with respect to the topology $\tau_0$. Also if $n \in \mathbb{N}$ and $0 < \varepsilon < \frac{1}{2^n}$, then $\left[ \frac{3}{2^{n+2}}, \frac{3}{2^{n+2}} + \varepsilon \right]$ is a compact neighborhood of $\frac{3}{2^{n+2}}$ and $\left[ \frac{5}{2^{n+2}} - \varepsilon, \frac{5}{2^{n+2}} \right]$ is a compact neighborhood of $\frac{5}{2^{n+2}}$. 
Let \( G = X \times X = \mathbb{R} \times \mathbb{R} \) be the pair groupoid (product: \( (x,y)(y,z) = (x,z) \)), inverse: \( (x,y)^{-1} = (y,x) \)). For every \( (x,y) \in G \) let

\[
\mathcal{B}_{(x,y)} = \begin{cases} 
\{ (x,y) \in A \times B : A,B \in \mathcal{B}_x \}, & \text{if } x \neq 0 \text{ and } y \neq 0 \\
\{ (0,y) \in B : y \in B, B \in \mathcal{B}_0 \}, & \text{if } x = 0 \text{ and } y \neq 0 \\
\{ (x,0) \in A : x \in A, A \in \mathcal{B}_0 \}, & \text{if } x \neq 0 \text{ and } y = 0 \\
\{ (0,0) \} \cup U_n : n \in \mathbb{N} \}, & \text{if } (x,y) = (0,0),
\end{cases}
\]

where \( U_n = \bigcup_{k=n}^{\infty} \left[ \frac{3}{2k+2}, \frac{5}{2k+2} \right] \times \left[ \frac{3}{2k+2}, \frac{5}{2k+2} \right] \) for all \( n \in \mathbb{N} \).

and

\[
\mathcal{F}_{(x,y)} = \{ V \subset \mathbb{R} \times \mathbb{R} : \text{there is } U \in \mathcal{B}_{(x,y)} \text{ such that } U \subset V \}.
\]

Let us endow \( G = \mathbb{R} \times \mathbb{R} \) with the unique topology \( \tau_1 \) with the property that for every \( (x,y) \in G, \mathcal{F}_{(x,y)} \) is the family of neighborhoods of \( (x,y) \). It is easy to see that the inverse map is continuous with respect to \( \tau_1 \). Since for all subsets \( A,B,C \subset X = \mathbb{R}, \)

\[(A \times C) (C \times B) \subset A \times B \]

it follows that the product map is continuous in the points of the form \( ((x,y), (y,z)) \in G^2 \) with \( x \neq 0 \) and \( z \neq 0 \). For every \( n \in \mathbb{N} \), let \( U_n = \bigcup_{k=n}^{\infty} \left[ \frac{3}{2k+2}, \frac{5}{2k+2} \right] \times \left[ \frac{3}{2k+2}, \frac{5}{2k+2} \right] \).

The continuity of the product map in \( ((0,0), (0,0)) \) is the consequence of the fact that

\[
\{ (0,0) \} \cup U_n \cup \{ (0,0) \} \cup U_n = \{ (0,0) \} \cup U_n.
\]

Since for all subsets \( B \subset X = \mathbb{R}, \)

\[
\{ (0) \times B \} (B \times \{0\}) = \{ (0,0) \} \subset \{ (0,0) \} \cup U_n
\]

it follows that the product map is continuous in the points of the form \( ((0,y), (y,0)) \) such that \( y \neq 0 \). The fact that for all subsets \( B \subset X = \mathbb{R}, \)

\[
\{ (0,0) \} \cup U_n \cup \{ (0) \times B \} = \{ (0,0) \} \cup \{ (0) \times B \}
\]

implies that the product map is continuous in the points of the form \( ((0,0), (0,y)) \) such that \( y \neq 0 \). Similarly, the product map is continuous in the points of the form \( ((y,0), (0,0)) \) such that \( y \neq 0 \). Therefore \((G, \tau_1)\) is a topological groupoid.

For every \( x \in \mathbb{R} \setminus \{0\} \), let \( K_x \) be a compact neighborhood of \( x \) with respect to \( \tau_0 \). Then \( K_x \times K_y \) is a compact neighborhood of \( (x,y) \) with respect to the topology \( \tau_1 \). For every \( x \in \mathbb{R} \setminus \{0\} \), \{0\} \times K_x, respectively \( K_x \times \{0\} \) is a compact neighborhood of \( (0,x) \), respectively \( (x,0) \) with respect to \( \tau_1 \). Let us prove that for every \( m \in \mathbb{N}, \)

\[
K_m = \{ (0,0) \} \bigcup \left( \bigcup_{k=m}^{\infty} \left[ \frac{3}{2k+2}, \frac{5}{2k+2} \right] \times \left[ \frac{3}{2k+2}, \frac{5}{2k+2} \right] \right)
\]

is a compact neighborhood of \( (0,0) \). Let \( ((x_n, y_n))_n \in K_m \). If

\[
\{ n \in \mathbb{N} : (x_n, y_n) = (0,0) \}
\]

is infinite, then \( ((x_n, y_n))_n \) has a subsequence converging to \( (0,0) \in K_m \). If \( \{ n \in \mathbb{N} : (x_n, y_n) = (0,0) \} \) is finite, then there is an integer \( n_0 \geq m \) such that \( (x_n, y_n) \in \bigcup_{k=n}^{\infty} \left[ \frac{3}{2k+2}, \frac{5}{2k+2} \right] \times \left[ \frac{3}{2k+2}, \frac{5}{2k+2} \right] \) for all \( n \geq n_0 \). In this case for every
\( n \geq n_0 \), there are \( u_n, v_n \in [3, 5] \) and \( k_n \in \mathbb{N} \), such that \( x_n = \frac{u_n}{2^{n+2}} \) and \( y_n = \frac{v_n}{2^{n+2}} \). If \( (k_n)_n \) is unbounded, then \( (k_n)_n \) has a subsequence that diverges to \( \infty \) and hence \( (x_n)_n \) and \( (y_n)_n \) have subsequences which converge to 0 in the usual topology on \( \mathbb{R} \). Since for all \( n \),

\[
K_m \cap \left( \left( -\frac{5}{2^{n+2}}, \frac{5}{2^{n+2}} \right) \times \left( -\frac{5}{2^{n+2}}, \frac{5}{2^{n+2}} \right) \right) \subseteq \mathbb{Q}
\]

it follows that \( ((x_n, y_n))_n \) has a subsequence converging to \( (0, 0) \) with respect to \( \tau_1 \).

If \( (k_n)_n \) is bounded, then it has a convergent subsequence in the usual topology on \( \mathbb{R} \), or equivalently a stationary subsequence. Also \( (u_n)_n \) and \( (v_n)_n \) have convergent subsequences in the usual topology on \( \mathbb{R} \). Thus \( (x_n)_n \) (respectively, \( (y_n)_n \)) has a subsequence converging to \( \frac{u}{2^{n+2}} \) (respectively, \( \frac{v}{2^{n+2}} \)) in the usual topology on \( \mathbb{R} \).

Therefore \( ((x_n, y_n))_n \) has a subsequence converging to \( \left( \frac{u}{2^{n+2}}, \frac{v}{2^{n+2}} \right) \) in the topology \( \tau_1 \) of \( G = X \times X = \mathbb{R} \times \mathbb{R} \). Thus \( K_m \) is a compact neighborhood of \( (0, 0) \).

Therefore \( (G, \tau_1) \) is a second countable locally compact groupoid. Since for any \( B \in \tau_0 \),

\[
r\{(0, 0) \times B\} = \{(0, 0)\}
\]

is not open in \( G^{(0)} = \{(x, x) : x \in X\} \), it follows that the range map is not open (and hence the domain map \( d \) is not open).

3. Replacing the topology of a locally compact transitive groupoid with a locally transitive topology

We prove that for every second countable, locally compact, transitive groupoid \( G \) there is a second countable, locally compact topology \( \tilde{\tau} \) making \( G \) a topological groupoid with open range map. Moreover, the original topology on \( G \) and \( \tilde{\tau} \) generate the same Borel structure and coincide on the \( r \)-fibres and \( d \)-fibres of \( G \). The topology \( \tilde{\tau} \) is in fact the topology \( \tau_W \) introduced in [2] Definition 3.1], where \( W \) is a suitable \( G \)-uniformity. Let us recall that a \( G \)-uniformity (in the sense of [2] Definition 2.1]) is a collection \( \{W\}_{W \in \mathcal{W}} \) of subsets of a groupoid \( G \) satisfying the following conditions:

(1) \( G^{(0)} \subseteq W \subseteq G \) for all \( W \in \mathcal{W} \).

(2) If \( W_1, W_2 \in \mathcal{W} \), then there is \( W_3 \subseteq W_1 \cap W_2 \) such that \( W_3 \in \mathcal{W} \).

(3) For every \( W_1 \in \mathcal{W} \) there is \( W_2 \in \mathcal{W} \) such that \( W_2 W_2 \subseteq W_1 \).

(4) \( W = W^{-1} \) for all \( W \in \mathcal{W} \).

If \( G \) is groupoid endowed with a topology, then a \( G \)-uniformity \( \mathcal{W} \) is said to be compatible with the topology of the \( r \)-fibres (in the sense of [2] Definition 3.4]) if for every \( u \in G^{(0)} \) and every open neighborhood \( U \) of \( u \), there is \( W \in \mathcal{W} \) such that \( W \cap G^u \subseteq U \cap G^u \) and \( u \) is in the interior of \( W \cap G^u \) with respect to the topology on \( G^u \) coming from \( (G, \tau) \).

A subset \( K \) of \( G \) is diagonally compact (in the sense of [5] p. 10]) if \( K \cap r^{-1} (L) \) and \( K \cap d^{-1} (L) \) are compact whenever \( L \) is a compact subset of \( G^{(0)} \).
Proposition 1. If $G$ is a second countable, locally compact Hausdorff groupoid, then $G$ admits a countable $G$-uniformity $\{W_n\}_{n\in\mathbb{N}}$ compatible with the topology of the $r$-fibres such that for every $n \in \mathbb{N}$, $W_n$ is a diagonally compact neighborhood of $G^{(0)}$.

Proof. Since $G$ is a second countable, locally compact Hausdorff space, it follows that $G$ is metrizable. Let us denote the metric by $d$. Also since $G$ is a second countable, locally compact Hausdorff space, it follows that $G$ as well as $G^{(0)}$ are paracompact spaces. Thus $G$ has a fundamental system of diagonally compact neighborhoods of $G^{(0)}$ (by [5] Lemma 2.10/p. 10 or the proof of [7] Proposition 1.9]). For each $n \in \mathbb{N} \setminus \{0\}$ let us write

$$D_n = \left\{ x \in G : d(x, r(x)) < \frac{1}{n+1} \right\}.$$  

Let $W_0$ be a diagonally compact symmetric neighborhood of $G^{(0)}$ such that $W_0 \subset D_0$. Inductively we construct a $G$-uniformity $\{W_n\}_{n\in\mathbb{N}}$ consisting in diagonally compact symmetric neighborhoods of $G^{(0)}$ such that $W_n \subset D_n$ for all $n \in \mathbb{N}$. Suppose a symmetric neighborhood $W_n$ of $G^{(0)}$ has already been built. Let $V_n$ be a diagonally compact neighborhood of $G^{(0)}$ such that $V_n \subset D_{n+1} \cap W_n$. Since $G$ is paracompact, according to [6] p. 361-362], there is a neighborhood $U_n$ of $G^{(0)}$ such that $U_nU_n \subset V_n$. Let $W_{n+1}$ be a diagonally compact neighborhood of $G^{(0)}$ such that $W_{n+1} \subset U_n$. Replacing $W_{n+1}$ with $W_{n+1} \cap W_{n+1}^{-1}$, we may assume that $W_{n+1} = W_{n+1}^{-1}$. Thus we obtain a diagonally compact symmetric neighborhood of $G^{(0)}$ such that

$$W_{n+1} \subset W_{n+1}W_{n+1}^{-1} \subset U_nU_n \subset V_n \subset D_{n+1} \cap W_n.$$  

Let us remark that for every $u \in G^{(0)}$ we have

$$W_n \cap G^u \subset D_n \cap G^u = \left\{ x \in G^u : d(x, u) < \frac{1}{n+1} \right\}$$

for all $n \in \mathbb{N}$. Consequently, $\{W_n\}_{n\in\mathbb{N}}$ is compatible with the topology of the $r$-fibres. \hfill \Box

Let us recall that $A \subset G$ is open with respect to the topology $\tau_W$ [2] Definition 3.1] (respectively, $\tau_W$ [1] p. 59]) associated to a $G$-uniformity $W$ [2] Definition 2.1] if and only if for every $x \in A$ there is $W_x \in W$ such that $W_x xW_x \subset A$ (respectively, $xW_x \subset A$). If $G$ is a topological groupoid and if $W$ is a $G$-uniformity compatible with the topology of the $r$-fibres, then by [2] Proposition 3.6] for every $W_1 \in W$ and $x \in G$ there is $W_2 \in W$ such that $W_2 \cap G^{\delta_{d(x)}}_d \subset x^{-1}W_1x$ and by [2] Proposition 3.7], $G$ endowed with $\tau_W$ is a topological groupoid. Moreover the topologies induced by $\tau_W$ and $\tau_W$ on $r$-fibres coincide. According [2] Proposition 3.8], the compatibility of the $G$-uniformity $W$ with the topology of $G$ ensures that the topologies induced by $\tau_W$ and the original topology of $G$ on the $r$-fibres $G^u$ coincide. Thus for each $u \in G^{(0)}$ and each $x \in G^u$, $\{xW\}_{W \in W}$ is a neighborhood basis (local basis) for $x$ with respect to the topology induced from $G$ on $G^u$.

Proposition 2. Let $G$ be a topological transitive groupoid endowed with a $G$-uniformity $W$ compatible with the topology of the $r$-fibres and let $u \in G^{(0)}$. If $S \subset G^u$ is a dense subset of $G^u$ with respect to the topology induced from $G$, then $S^{-1}S$ is a dense subset of $G$ with respect to the topology $\tau_W$. 

A topological groupoid is said to be locally transitive ([8], p. 119) (or groupoïde microtransitif [4]) if for every \( u \in G(0) \) the map \( r_u \) is open, where \( r_u : G_u \to G(0) \) is defined by \( r_u(x) = r(x) \) for all \( x \in G_u \) and \( G_u \) is endowed with the topology coming from \( G \). Hence the maps \( d_u \) are open, where \( d_u : G^u \to G(0) \), \( d_u(x) = d(x) \) for all \( x \in G^u \). Obviously, every locally transitive groupoid has open range and domain maps. Conversely, according ([4], Theorem 2.2 A amd Theorem 2.2 N] or [6] Theorem 3.2], if \( G \) is a second countable, locally compact, transitive groupoid with open range map, then \( G \) is locally transitive. The example constructed in Section 2 demonstrates that there are second countable, locally compact, transitive groupoids which are not locally transitive. The following theorem shows that if \( G \) is a second countable, locally compact, transitive groupoid, then we can eventually replace the topology of \( G \) with a second countable, locally compact topology \( \tilde{\tau} \) making \( G \) a locally transitive groupoid. In addition, the original topology on \( G \) and \( \tilde{\tau} \) generate the same Borel structure and coincide on the \( r \)-fibres (hence on \( d \)-fibres) of \( G \).

**Theorem 1.** Let \( G \) be a transitive groupoid endowed with second countable locally compact Hausdorff topology \( \tau \) making \( G \) a topological groupoid. Then the topology \( \tau \) of \( G \) can be replaced with a topology \( \tilde{\tau} \) such that:

1. \( G \) is a (topological) locally transitive groupoid with respect to the topology \( \tilde{\tau} \) (hence \( G \) has open range map with respect to \( \tilde{\tau} \)).
2. The topology \( \tilde{\tau} \) is in general finer than \( \tau \). However \( \tau \) and \( \tilde{\tau} \) coincide if \((G, \tau)\) is locally transitive (i.e. for every \( u \in G(0), r|G_u \) is open with respect to the topology induced by \( \tau \) on \( G_u \)).
3. The topologies induced by \( \tau \) and \( \tilde{\tau} \) on \( r \)-fibres (respectively, on \( d \)-fibres) of \( G \) coincide.
4. The topology \( \tilde{\tau} \) is second countable and locally compact Hausdorff.
5. The topologies \( \tau \) and \( \tilde{\tau} \) generate the same Borel structure on \( G \) (the Borel sets of a topological space are taken to be the \( \sigma \)-algebra generated by the open sets).

**Proof.** According Proposition 1 G admits a countable \( G \)-uniformity \( W = \{W_n\}_{n \in \mathbb{N}} \) compatible with the topology of the \( r \)-fibres such that for every \( n \in \mathbb{N} \), \( W_n \) is a diagonally compact neighborhood of \( G(0) \). Let \( \tilde{\tau} = \tau_W \).

Then [2] Proposition 3.6 and [2] Proposition 3.7 show 1 and [2] Proposition 3.8] implies 2 and 3.

Let us fix \( u \in G(0) \) and let \( \{x_n, n \in \mathbb{N} \} \) be a dense subset of \( G^u \).

4. For each \( n \), let \( \tilde{W_n} \) denote the interior of \( W_n \) with respect to \( \tau \). We claim \( \{\tilde{W_k}x_m^{-1}x_n\tilde{W_k} : m, n, k \in \mathbb{N} \} \) is a countable base for \( \tilde{\tau} = \tau_W \). Let us prove that for every \( n, m, k \in \mathbb{N} \) and \( x \in G \), \( \tilde{W_k}x\tilde{W_k} \) is an open set with respect to \( \tau_W \). Let \( s \in \tilde{W_k} \cap G_{r(x)} \) and \( t \in \tilde{W_k} \cap G^{d(x)} \). Since \( r(s) \) \( s \in \tilde{W_k} \) and \((G, \tau)\) is a
topological groupoid, there is an open neighborhood $V$ of $r\left(s\right)$ (with respect to $\tau$) such that $Vs \subset W_k$. The fact that $\mathcal{W} = \{W_n\}_{n \in \mathbb{N}}$ is compatible with the topology of the $r$-fibres implies that there is $p \in \mathbb{N}$ such that $W_p \cap G^r(s) \subset V^{-1} \cap G^r(s)$ and consequently, $W_p \cap G_r(s) \subset V \cap G_r(s)$. Hence

$$W_{p,s} \subset \{W_p \cap G_r(s)\} s \subset V \subset W_k.$$ 

Similarly, there $q \in \mathbb{N}$ such that $tW_q \subset W_k$. If $r \in \mathbb{N}$ is such that $W_r \subset W_p \cap W_q$, then

$$sx \subset \{W_r \cap G_s(s)\} \subset W_r \subset W_k.$$ 

Thus $W_kxW_k$ is an open set with respect to $\tau_W$.

Let us prove that $\{\hat{W}_k x^{-1} W_k : m, n, k \in \mathbb{N}\}$ is a base for $\hat{\tau} = \tau_W$. Indeed, let $x \in G$ and let $A$ be an open subset of $G$ with respect to $\tau_W$ such that $x \in A$. Then there is $k \in \mathbb{N}$ such that $x \in W_k x W_k \subset A$. Let $r \in \mathbb{N}$ such that $W_r W_r \subset W_k$. Proposition 2 implies that $\{x^{-1} W_n : m, n \in \mathbb{N}\}$ is dense in $G$ with respect to $\tau_W$. Hence there are $m, n \in \mathbb{N}$ such that $x^{-1} W_n \subset W_r x W_r$, or equivalently, $x \in W_r x^{-1} W_n W_r$. Therefore

$$x \in W_r x^{-1} W_n W_r \subset W_r x^{-1} W_n W_r \subset W_r x W_r W_r \subset W_k x W_k \subset A.$$ 

5. Since the topology $\hat{\tau}$ is finer than $\tau$, it suffices to prove that each open set with respect to $\hat{\tau}$ belong to the Borel structure generated by $\tau$. But as we have proved every open set with respect to $\hat{\tau}$ is a countable union of sets of the form $W_r x^{-1} W_n W_r$ which are compact with respect to $\tau$ (because $W_r$ is diagonally compact).

\[\Box\]

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