Subsequential tightness of the maximum of two dimensional Ginzburg-Landau fields

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Abstract

We prove the subsequential tightness of centered maxima of two-dimensional Ginzburg-Landau fields with bounded elliptic contrast.

1 Introduction

Let \(V \in C^2(\mathbb{R})\) satisfy

\[
V(x) = V(-x), \\
0 < c_- \leq V''(x) \leq c_+ < \infty,
\]

where \(c_-, c_+\) are positive constants. The ratio \(\kappa = c_+ / c_-\) is called the elliptic contrast of \(V\). We assume (1) and (2) throughout this note without further mentioning it.

We treat \(V\) as a nearest neighbor potential for a two-dimensional Ginzburg-Landau gradient field. Explicitly, let \(D_N := [-N, N]^2 \cap \mathbb{Z}^2\) and let the boundary \(\partial D_N\) consist of the vertices in \(D_N\) that are connected to \(\mathbb{Z}^2 \setminus D_N\) by some edge. The Ginzburg-Landau field on \(D_N\) with zero boundary condition is a random field denoted by \(\phi^{D_N,0}\), whose distribution is given by the Gibbs measure

\[
d\mu_N = Z_N^{-1} \exp \left[ - \sum_{v \in D_N} \sum_{i=1}^2 V(\nabla_i \phi(v)) \right] \prod_{v \in D_N \setminus \partial D_N} d\phi(v) \prod_{v \in \partial D_N} \delta_0(\phi(v)),
\]

where \(\nabla_i \phi(v) = \phi(v + e_i) - \phi(v), e_1 = (1,0)\) and \(e_2 = (0,1)\), and \(Z_N\) is the normalizing constant ensuring that \(\mu_N\) is a probability measure, i.e. \(\mu_N(\mathbb{R}^{D_N}) = 1\).

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1. We denote expectation with respect to \( \mu_N \) by \( \mathbb{E}_N \), or simply by \( \mathbb{E} \) when no confusion can occur.

Ginzburg-Landau fields with convex potential, which are natural generalizations of the standard lattice Gaussian free field corresponding to quadratic \( V \) (DGFF), have been extensively studied since the seminal works [FS97, HS94, NS97]. Of particular relevance to this paper is Miller’s coupling, described in Section 2.2 below, which shows that certain multi-scale decompositions that hold for the Gaussian case continue to hold, approximately, for the Ginzburg-Landau model.

In this paper, we study the maximum of Ginzburg-Landau fields. Given \( U \subset D_N \), let

\[
M_U := \max_{x \in U} \phi^{D_N,0}(x),
\]

and set \( M_N = M_{D_N} \). For the Gaussian case, we write \( M^G_N \) for \( M_N \). Much is known about \( M^G_N \), following a long succession of papers starting with [Bra83]. In particular, see [BDZ16] and [BL16], \( M^G_N - m^G_N \) converges in distribution to a randomly shifted Gumbel, with \( m^G_N = c_1 \log N - c_2 \log \log N \) and explicit constants \( c_1, c_2 \).

Much less is known concerning the extrema in the Ginzburg-Landau setup, even though linear statistics of such fields converge to their Gaussian counterparts [NS97]. A first step toward the study of the maximum was undertaken in [BW16], where the following law of large numbers is proved:

\[
\frac{M_{D_N}}{\log N} \to 2\sqrt{7} \text{ in } L^2, \quad \text{for some } g = g(c_+, c_-). \tag{4}
\]

In this note we prove that the fluctuations of \( M_{D_N} \) around its mean are tight, at least along some (deterministic) subsequence.

**Theorem 1** There is a deterministic sequence \( \{n_k\} \) with \( n_k \to k \to \infty \infty \) such that the sequence of random variables \( \{M_{D_{n_k}} - \mathbb{E}M_{D_{n_k}}\} \) is tight.

As will be clear from the proof, the sequence \( \{n_k\} \) can be chosen with density arbitrarily close to 1. Theorem 1 is the counterpart of an analogous result for the Gaussian case proved in [BDZ11], building on a technique introduced by Dekking and Host [DH91]. The Dekking-Host technique is also instrumental in the proof of Theorem 1. However, due to the fact that the Ginzburg-Landau field does not possess good decoupling properties near the boundary, significant changes need to be made. Additional crucial ingredients in the proof are Miller’s coupling and a decomposition in differences of harmonic functions introduced in [BW16].

### 2 Preliminaries

#### 2.1 The Brascamp-Lieb inequality

One can bound the variances and exponential moments with respect to the Ginzburg-Landau measure by those with respect to the Gaussian measure, us-
ing the following Brascamp-Lieb inequality. Let \( \phi \) be sampled from the Gibbs measure (3). Given \( \eta \in \mathbb{R}^{DN} \), set 
\[
\langle \phi, \eta \rangle := \sum_{v \in DN} \phi_v \eta(v).
\]

Lemma 2 (Brascamp-Lieb inequalities [BL76]) Assume that \( V \in C^2(\mathbb{R}) \) satisfies \( \inf_{x \in \mathbb{R}} V''(x) \geq c_\ast > 0 \). Let \( \mathbb{E}_{GFF} \) and \( \text{Var}_{GFF} \) denote the expectation and variance with respect to the DGFF measure (that is, (3) with \( V(x) = x^2/2 \)). Then for any \( \eta \in \mathbb{R}^{DN} \),
\[
\text{Var}(\phi, \eta) \leq c_\ast^{-1} \text{Var}_{GFF}(\phi, \eta),
\]
(5)
\[
\mathbb{E}[\exp(\langle \phi, \eta \rangle - \mathbb{E}(\phi, \eta))] \leq \exp\left(\frac{1}{2}c_\ast^{-1}\text{Var}_{GFF}(\phi, \eta)\right).
\]
(6)

2.2 Approximate harmonic coupling

By their definition, the Ginzburg-Landau measures satisfy the domain Markov property: conditioned on the values on the boundary of a domain, the field inside the domain is again a gradient field with boundary condition given by the conditioned values. For the discrete GFF, there is in addition a nice orthogonal decomposition. More precisely, the conditioned field inside the domain is the discrete harmonic extension of the boundary value to the whole domain plus an independent copy of a zero boundary discrete GFF.

While this exact decomposition does not carry over to general Ginzburg-Landau measures, the next result due to Jason Miller, see [Mii11], provides an approximate version.

Theorem 3 ([Mii11]) Let \( D \subset \mathbb{Z}^2 \) be a simply connected domain of diameter \( R \), and denote \( D' = \{x \in D : \text{dist}(x, \partial D) > r\} \). Let \( \Lambda \) be such that \( f : \partial D \to \mathbb{R} \) satisfies \( \max_{x \in \partial D} |f(x)| \leq 2\Lambda |\log R|^{-\Lambda} \). Let \( \phi \) be sampled from the Ginzburg-Landau measure (3) on \( D \) with zero boundary condition, and \( \phi_f \) be sampled from Ginzburg-Landau measure on \( D \) with boundary condition \( f \). Then there exist constants \( c, \gamma, \delta' \in (0, 1) \), that only depend on \( V \), so that if \( r > cR^{1/2} \) then the following holds. There exists a coupling \( (\phi, \phi_f) \), such that if \( \hat{\phi} : D' \to \mathbb{R} \) is discrete harmonic with \( \hat{\phi}|_{\partial D'} = \phi_f - \phi|_{\partial D'} \), then
\[
\mathbb{P}(\phi_f = \phi + \hat{\phi} \text{ in } D') \geq 1 - c(\Lambda) R^{-\delta'}.
\]

Here and in the sequel of the paper, for a set \( A \subset \mathbb{Z}^2 \) and a point \( x \in \mathbb{Z}^2 \), we use \( \text{dist}(x, A) \) to denote the (lattice) distance from \( x \) to \( A \).

2.3 Pointwise tail bound

We also recall the pointwise tail bound for the Ginzburg-Landau field (3), proved in [BW16].
Theorem 4 Let $g$ be the constant as in (4). For all $u > 0$ large enough and all $v \in D_N$ we have

$$P(\phi_v \geq u) \leq \exp\left(-\frac{u^2}{2g \log \text{dist}(v, \partial D_N)} + o(u)\right).$$

(7)

This allows us to conclude that the maximum of $\phi^{D_N,0}$ does not occur within a thin layer near the boundary.

Lemma 5 Given $\delta < 1$, there exists $\delta' > 0$ such that

$$P\left(M_{A_{N,N^\delta}} > (2\sqrt{g} - \delta') \log N\right) \leq N^{\frac{-1}{2}}.$$

where

$$A_{N,N^\delta} := \{v \in D_N : \text{dist}(x, \partial D_N) < N^\delta\}.$$

Proof. Let $\Delta = \text{dist}(x, \partial D_N)$. For $\delta'$ small enough, applying Theorem 4 with $u = (2\sqrt{g} - \delta') \log N$ yields

$$P(\phi_v \geq (2\sqrt{g} - \delta') \log N) \leq \exp\left(-\frac{2 \log N^2}{\log \Delta} + \frac{2\delta' \log N^2}{\sqrt{g} \log \Delta} + o(\log N)\right) \leq N^{-2+2\delta'/\sqrt{g}+o(1)},$$

for all $v \in A_{N,N^\delta}$. Therefore a union bound yields

$$P\left(M_{A_{N,N^\delta}} \geq (2\sqrt{g} - \delta') \log N\right) \leq N^{\delta-1+2\delta'/\sqrt{g}+o(1)}.$$

It suffices to take $\delta'$ such that $2\delta'/\sqrt{g} < \frac{1-\delta}{2}$. $\blacksquare$

3 The recursion and proof of Theorem 1

We prove Theorem 1 by establishing a recursion for some random variable $M_{Y_N}$, where $Y_N \subset D_N$ is a specific subset defined below. Denote by $T_N = [-N, N] \times \{N\} \subset D_N$ the top boundary of $D_N$. For fixed $\varepsilon > 0$, define

$$Y_N = \{v \in D_N : \text{dist}(v, \partial D_N) \geq \varepsilon N\} \cup \{v \in D_N : \text{dist}(v, \partial D_N) = \text{dist}(v, T_N)\}.$$

For $\delta \in (0,1)$, we also define $Y_{N,\delta} \subset Y_N$ as

$$Y_{N,\delta} = \{v \in Y_N : \text{dist}(v, T_N) > N^{1-\delta}\},$$

see Figure II.

Lemma 6 For the constant $g = g(c_+, c_-)$ in (4), we have

$$\frac{M_{Y_{N,\delta}}}{\log N} \to 2\sqrt{g} \text{ in } L^2.$$

(8)
εN

\[ N - \delta \]

\[ 2N \]

\[ Y_{N,\delta} \]

Figure 1: The domain \( Y_{N,\delta} \).

**Proof.** Let \( D^\varepsilon_N := \{ v \in D_N : \text{dist}(v, \partial D_N) \geq \varepsilon N \} \). Since

\[
\frac{M_{D^\varepsilon_N}}{\log N} \leq \frac{M_{Y_{N,\delta}}}{\log N} \leq \frac{M_{D_N}}{\log N},
\]

the claim (8) follows from [BW16], since the upper control on \( M_{D_N}/\log N \) follows from \( (4) \) while the lower control on \( M_{D^\varepsilon_N}/\log N \) follows from the display below (5.19) in [BW16].

We now switch to dyadic scales. For \( n \in \mathbb{N} \), set \( N = 2^n \) and \( m_n := M_{Y_{2^n,\delta}} \).

We set up a recursion for \( m_n \). Clearly,

\[ \mathbb{E}m_{n+2} = \mathbb{E}M_{Y_{4N,\delta}} \geq \mathbb{E} \max \left\{ \max_{v \in Y_{N,\delta}^{(1)}} \phi_{v}^{D_{4N,0}}, \max_{v \in Y_{N,\delta}^{(2)}} \phi_{v}^{D_{4N,0}} \right\}, \]

where \( Y_{N,\delta}^{(i)} \) are the translations of \( Y_{N,\delta} \), defined by \( Y_{N,\delta}^{(1)} = Y_{N,\delta} + (-1.1N, 3N) \), \( Y_{N,\delta}^{(2)} = Y_{N,\delta} + (1.1N, 3N) \), see Figure 2.

The next two lemmas will allow us to control the difference between \( \phi_{D_{4N,0}} \) and \( \phi_{D_{N,0}} \) (and as a consequence, between \( m_{n+2} \) and \( m_n \)).

**Lemma 7** There exist \( \delta', 1 > \delta > \gamma > 0 \), such that the following statement holds. Set \( D_{N}^{(1)} = D_{N} + (-1.1N, 3N) \), \( D_{N}^{(2)} = D_{N} + (1.1N, 3N) \). Let \( D_{N}^{\gamma, (i)} := \{ v \in D_{N} : \text{dist}(v, \partial D_{N}^{(i)}) \geq N^{\gamma} \} \). Then there exists a coupling \( \mathbb{P} \) of

\[ (\phi_{D_{4N,0}}, \phi_{D_{N}^{(1),0}}, \phi_{D_{N}^{(2),0}}) \]

and an event \( \mathcal{G} \) with \( \mathbb{P}(\mathcal{G}^c) \leq N^{-\delta'} \), such that with
Figure 2: The domains \( Y_{N,\delta}^{(i)} \), with the boundary pieces \( R, Q \).

\( h_v^{(i)} \) being harmonic functions in \( D_N^{(i)} \) with boundary conditions \( \phi^{D_N,0} - \phi^{D_N,0} \), on the event \( \mathcal{G} \), we have

\[
\phi_v^{D_N,0} = \phi_v^{D_N,0} + h_v^{(i)}, \text{ for all } v \in Y_{N,\gamma}^{(i)}, \text{ for } i = 1, 2.
\]

Moreover, there is a constant \( C_0 = C_0(\delta) \), such that, for any \( 1 > \delta > \gamma \),

\[
\max_{i=1,2} \max_{v \in Y_{N,\delta}^{(i)}} \text{Var}(h_v^{(i)}) \leq C_0(\delta).
\]

**Lemma 8** With notation as in Lemma 7, there exists a constant \( C_1 < \infty \), such that

\[
\mathbb{E} \min_i \min_{v \in Y_{N,\delta}^{(i)}} h_v^{(i)} = -\mathbb{E} \max_i \max_{v \in Y_{N,\delta}^{(i)}} h_v^{(i)} \geq -C_1.
\]

The proof of Lemmas 7 and 8 are postponed to Section 4. In the rest of this section, we bring the proof of Theorem 1.

**Proof of Theorem 1** Denote by \( m_n^* \) an independent copy of \( m_n \). We
combine Lemmas 7 and 8 to conclude

\[ E_{m+n} \geq \mathbb{E} \left[ 1_{G_{m+n}^{(i)}} \right] \geq \mathbb{E} \max \{ m, m^* \} + 2 \mathbb{E} \min \min i_{v \in Y_{N,i}^{(i)}} h_v^{(i)} - 2 \mathbb{E} \left[ 1_{G^c \cdot m_n} \right]. \]

We apply (4) to conclude that

\[ \mathbb{E} \left[ 1_{G^c \cdot m_n} \right] \leq \mathbb{P} \left( G_{m_n} \right) \leq \frac{C \log N}{N^{8/2}}, \]

Thus for all large \( n \), we can apply Lemma 8 to get

\[ E_{m+n+2} \geq \mathbb{E} \max \{ m, m^* \} - 3C_1. \]

Using \( \max \{ a, b \} = \frac{1}{2} (a + b + |a - b|) \) and Jensen’s inequality, we obtain

\[ E_{m+n+2} - E_m \geq \frac{1}{2} \mathbb{E} |m_n - m_n^*| - 3C_1 \geq \frac{1}{2} \mathbb{E} |m_n - \mathbb{E} m_n^*| - 3C_1. \quad (9) \]

We need the following lemma.

**Lemma 9** There exists a sequence \( \{ n_k \} \) and a constant \( K < \infty \) such that

\[ \mathbb{E} m_{n_k+2} \leq \mathbb{E} m_{n_k} + K. \]

**Proof of Lemma 9** Let \( I_{n,K} = \{ j \in \{ n, n + 2, \ldots, 2n \} : \mathbb{E} m_{j+2} \geq \mathbb{E} m_j + K \} \). Then Lemma 8 implies that \( \mathbb{E} m_n/n \to 2 \sqrt{g} \), while (4) gives \( \mathbb{E} m_{n+2} \geq \mathbb{E} m_n - 3C_1 \). Therefore, for any fixed \( \eta > 0 \) and all large \( n \),

\[ K |I_{n,K}| - 3C_1 \left( \frac{n}{2} - |I_{n,K}| \right) \leq 2 \sqrt{g} n (1 + \eta) \leq 4 \sqrt{g} n, \]

giving that for all large \( n \),

\[ |I_{n,K}| \leq \frac{n - 4 \sqrt{g} + \frac{3}{K + 3C_1}}{K + 3C_1}. \]

Choosing \( K > 16 \sqrt{g} + 3C_1 \) gives that for all \( n \) large, \( |I_{n,K}| \leq n/4 \). It follows that for all large \( n \), there exists \( n' \in [n, 2n] \), such that

\[ \mathbb{E} m_{n'+2} \leq \mathbb{E} m_{n'} + K. \]

This completes the proof of Lemma 9.

We continue with the proof of Theorem 1. Using the subsequence \( \{ n_k \} \) from Lemma 9 we have from (9) that

\[ \mathbb{E} |m_{n_k} - \mathbb{E} m_{n_k}^*| \leq 2K + 6C_1, \]
which implies, using Jensen’s inequality, that \( \{m_{n_k} - E m^*_n\} \) is tight. This implies that the sequence of random variables

\[
\bar{M}_{D^\delta N_k} := \max \left\{ \phi^D_{N_k,0} : v \in D_{N_k}, \text{dist}(v, \partial D_{N_k}) \geq N_k^{1-\delta} \right\}
\]

is tight around its mean because \( \bar{M}_{D^\delta N_k} \) is the maximum of 4 rotated copies of \( m_{n_k} \).

Finally, combining (4) and Lemma 5 we obtain

\[
P \left( M_{D N_k} > \bar{M}_{D^\delta N_k} \right) \leq 2^{n_k(\delta - 1)/2},
\]

and

\[
E M_{D N_k} - E \bar{M}_{D^\delta N_k} \leq E M_{D N_k} \mathbb{1}\{ M_{D N_k} > \bar{M}_{D^\delta N_k} \} \leq P \left( M_{D N_k} > \bar{M}_{D^\delta N_k} \right)^{1/2} \left( E M_{D N_k}^2 \right)^{1/2} \leq 2^{n_k(\delta - 1)/2} O (\log N_k) \to 0.
\]

We conclude that the sequence \( \{M_{D N_k} - E M_{D N_k}\} \) is tight.

4 Proof of Lemma 7 and 8

Proof of Lemma 7. The existence of the harmonic decomposition is implied by the Markov property and Theorem 3 (with \( \delta', \gamma \) taken as the constants in Theorem 3). It thus suffices to obtain an upper bound for \( \text{Var} \left( h_v^{(1)} \right) \). Write \( h_v^{(i)} = \hat{h}_v^{(i)} - \tilde{h}_v^{(i)} \), where \( \hat{h}_v^{(i)} \) is the harmonic function in \( D^\gamma N,0 \) with boundary value \( \phi^D_{N,0} \), and \( \tilde{h}_v^{(i)} \) is the harmonic function in \( D^\gamma N,0 \) with boundary value \( \phi^D_{N,0} \). Without loss of generality we set \( i = 1 \). Applying the Brascamp-Lieb inequality (5) we get

\[
\text{Var} \left( h_v^{(1)} \right) \leq c^{-1} \text{Var}_{GFF} \left( h_v^{(1)} \right).
\]

The orthogonal decomposition for GFF implies

\[
\text{Var}_{GFF} \left( h_v^{(1)} \right) = \text{Var}_{GFF} \left( E_{GFF} \left[ \phi^D_{N,0} | F \partial D^{\gamma (i)}_{N} \right] \right) = \text{Var}_{GFF} \left[ \phi^D_{N,0} - \text{Var}_{GFF} \left[ \phi^D_{N,0} \right] \right]
\]

and

\[
\text{Var}_{GFF} \left( \hat{h}_v^{(1)} \right) = \text{Var}_{GFF} \left[ \phi^{D^{(i)}(1)}_{N,0} \right] - \text{Var}_{GFF} \left[ \phi^{D^{(i)}(1),0}_{N,0} \right].
\]
We now estimate the last two expressions for different regions of \( v \in Y_{N,\delta}^{(1)} \). First of all, it suffices to control \( h_v^{(1)} \) for \( v \in \partial Y_{N,\delta}^{(1)} \). Let

\[
Q : = \left\{ v \in \partial Y_{N,\delta}^{(1)} : \text{dist}(v, \partial D_N) = \text{dist}(v, T) \right\}, \\
R : = \left\{ v \in \partial Y_{N,\delta}^{(1)} : \text{dist}(v, \partial D_N) = \varepsilon N \right\}.
\]

We first show that

\[
\max_{v \in R} \text{Var}_{\text{GFF}}(\hat{h}_v^{(1)}) \leq C(\varepsilon), \\
\max_{v \in Q \cup R} \text{Var}_{\text{GFF}}(\tilde{h}_v^{(1)}) \leq C_0 N^{\gamma - \delta}.
\]

Indeed, standard asymptotics for the lattice Green’s function (following e.g. from [Law96, Proposition 1.6.3]) give, for some constant \( g_0 \),

\[
\text{Var}_{\text{GFF}}[\phi_v^{D_{4N},0}] - \text{Var}_{\text{GFF}}[\phi_v^{D_N^{(1)},0}] = g_0 \left( \log \text{dist}(v, \partial D_N) - \log \text{dist}(v, \partial D_N^{(1)}) \right) + o_N(1) \\
\leq g_0 \log \frac{4N}{\varepsilon N - N^\gamma} + o_N(1) \leq C(\varepsilon),
\]

and similarly,

\[
\text{Var}_{\text{GFF}}[\phi_v^{D_N^{(1)},0}] - \text{Var}_{\text{GFF}}[\phi_v^{D_N^{(1)},0}] = g_0 \left( \log \text{dist}(v, \partial D_N^{(1)}) - \log \text{dist}(v, \partial D_N^{(1)}) \right) + O(N^{-1}) \\
\leq g_0 \log \frac{N^\delta}{N^\delta - N^\gamma} + O(N^{-1}) \leq C_0 N^{\gamma - \delta}.
\]

To conclude the proof, we also claim for \( \delta \in (\gamma, 1) \)

\[
\max_{v \in Q} \text{Var}_{\text{GFF}}(\tilde{h}_v^{(1)}) \leq C N^{\gamma - \delta}.
\]

Indeed, denote by \( T_\gamma \) the top boundary of \( D_N^{\gamma} \), we apply asymptotics for lattice Green’s function to obtain

\[
\text{Var}_{\text{GFF}}[\phi_v^{D_{4N},0}] - \text{Var}_{\text{GFF}}[\phi_v^{D_N^{(1)},0}] = g_0 \left( \log \text{dist}(v, \partial D_N) - \log \text{dist}(v, D_N^{(1)}) \right) + O(N^{-1}) \\
= g_0 \left( \log \text{dist}(v, T) - \log \text{dist}(v, T_\gamma) \right) + O(N^{-1}).
\]

Since

\[
\log \frac{\text{dist}(v, T)}{\text{dist}(v, T_\gamma)} \leq \log \frac{N^\delta}{N^\delta - N^\gamma} \leq C N^{\gamma - \delta},
\]

[9]
we obtain (11). □

Proof of Lemma 8. Recall that \( h^{(i)} = \hat{h}^{(i)} - \tilde{h}^{(i)} \). We will prove that there exist \( C_0 < \infty \) and \( \alpha > 0 \), such that for all \( C_1 > C_0 \),

\[
P \left( \max_{v \in Q} \hat{h}^{(1)} > C_1 \right) \leq e^{-\alpha C_1}, \tag{12}
\]

\[
P \left( \max_{v \in R} \hat{h}^{(1)} > C_1 \right) \leq e^{-\alpha C_1}, \tag{13}
\]

\[
P \left( \min_{v \in Q \cup R} \tilde{h}^{(1)} < -C_1 \right) \leq e^{-\alpha C_1}. \tag{14}
\]

Indeed, (12) follows from (11) and the exponential Brascamp-Lieb inequality (6):

\[
P \left( \max_{v \in Q} \hat{h}^{(1)} > C_1 \right) \leq |Q| \max_{v \in Q} P \left( \hat{h}^{(1)} > C_1 \right) \leq C_3 N \exp \left( -\frac{C_2^2}{C_2 \text{Var}_{GFF} (\hat{h}^{(1)})} \right) \leq C_3 N \exp \left( -\frac{C_2^2}{C_2 N^{d-\gamma}} \right),
\]

where \( C_2, C_3 \) are some fixed constants. The same argument using (10) gives (14).

We now prove (13) using chaining. Omitting the superscripts (1) in \( \hat{h}^{(1)} \) and \( \tilde{h}^{(1)} \), we claim that there exists \( K < \infty \), such that for \( u, v \in R \),

\[
\text{Var}_{GFF} \left[ \hat{h}_u - \hat{h}_v \right] \leq K \frac{|u - v|}{\varepsilon N}. \tag{15}
\]

Applying the orthogonal decomposition of the DGFF we obtain

\[
\phi^{D_{\gamma}^{(1)},0}_u - \phi^{D_{\gamma}^{(1)},0}_v = \phi^{D_N^{(1)},0}_u - \phi^{D_N^{(1)},0}_v + \hat{h}_u - \hat{h}_v,
\]

and therefore, by the independence of \( \phi^{D_N^{(1)},0}_u - \phi^{D_N^{(1)},0}_v \) and \( \hat{h}_u - \hat{h}_v \) under the DGFF measure,

\[
\text{Var}_{GFF} \left[ \hat{h}_u - \hat{h}_v \right] = \text{Var}_{GFF} \left[ \phi^{D_{\gamma}^{(1)},0}_u - \phi^{D_{\gamma}^{(1)},0}_v \right] \text{Var}_{GFF} \left[ \phi^{D_N^{(1)},0}_u - \phi^{D_N^{(1)},0}_v \right]. \tag{16}
\]

We now apply the representation of the lattice Green’s function, see, e.g., [Law90 Proposition 1.6.3],

\[
G_D^{D_N}(u, v) = \sum_{y \in \partial D_N} H_{\partial D_N}(u, y)a(y - v) - a(u - v),
\]
where \( H_{\partial D_N}(u, \cdot) \) is the harmonic measure of \( D_N \) seen at \( u \) and \( a \) is the potential kernel on \( \mathbb{Z}^2 \) which satisfies the asymptotics

\[
a(x) = \frac{2}{\pi} \log |x| + D_0 + O \left( |x|^{-2} \right),
\]

where \( D_0 \) is an explicit constant (see e.g. [Law96, Page 39] for a slightly weaker result which nevertheless is sufficient for our needs). Substituting into (16), we see that

\[
\text{Var}_{GFF}[\phi^{D_N,0}_{D_N} - \phi^{D_N,0}_{D_N}] - \text{Var}_{GFF}[\phi^{D_{\gamma},0}_{D_{\gamma}} - \phi^{D_{\gamma},0}_{D_{\gamma}}]
\]

\[
= G^{D_N}(u, u) + G^{D_N}(v, v) - 2G^{D_N}(u, v) - \left( G^{D_N}(u, u) + G^{D_N}(v, v) - 2G^{D_N}(u, v) \right)
\]

\[
= \sum_{z \in \partial D_N} H_{\partial D_N}(u, z) a(u - z) + \sum_{z \in \partial D_N} H_{\partial D_N}(v, z) a(v - z)
\]

\[
- 2 \sum_{z \in \partial D_N} H_{\partial D_N}(u, z) a(v - z)
\]

\[
- \sum_{z \in \partial D_N} H_{\partial D_{\gamma}}(u, z) a(u - z) - \sum_{z \in \partial D_{\gamma}} H_{\partial D_{\gamma}}(v, z) a(v - z)
\]

\[
+ 2 \sum_{z \in \partial D_{\gamma}} H_{\partial D_{\gamma}}(u, z) a(v - z)
\]

\[
= A_{D_N} - A_{D_{\gamma}}
\]

We now apply the Harnack inequality, see [Law96 Theorem 1.7.1],

\[
|H_{\partial D_N}(u, z) - H_{\partial D_N}(v, z)| \leq \frac{|u - v|}{4N}
\]

to obtain

\[
A_{D_N} = \sum_{z \in \partial D_N} H_{\partial D_N}(u, z) (a(u - z) - a(v - z))
\]

\[
+ \sum_{z \in \partial D_N} (H_{\partial D_N}(v, z) - H_{\partial D_N}(u, z)) a(v - z)
\]

\[
\leq \frac{|u - v|}{4N} \sum_{z \in \partial D_N} H_{\partial D_N}(u, z)
\]

\[
+ \sum_{z \in \partial D_N} \left( H_{\partial D_N}(v, z) - H_{\partial D_N}(u, z) \right) \left( a(v - z) - \frac{2}{\pi} \log N - D_0 \right)
\]

\[
\leq K \frac{|u - v|}{N}, \quad \text{for some } K < \infty.
\]

The same argument gives \( \left| A_{D_{\gamma}} \right| \leq K \frac{|u - v|}{\varepsilon N} \), thus (15) is proved.
Now fix a large $k_0$. For $k \geq k_0$ let $P_k$ be subsets of $R$ that plays the role of dyadic approximations: $P_k$ contains $O\left(2^k\right)$ vertices that are equally spaced and the graph distance between adjacent points is $\varepsilon N 2^{-k}$. For $v \in R$, denote by $P_k(v)$ the $k^{th}$ dyadic approximation of $v$, namely the vertex in $P_k$ that is closest to $v$. Then for $v \in R$,

$$\hat{h}_v = P_{k_0}(v) + \sum_{k \geq k_0} \hat{h}_{P_{k+1}(v)} - \hat{h}_{P_k(v)}.$$ 

We now apply the exponential Brascamp-Lieb inequality (6), (15), and a union bound to obtain

$$\mathbb{P}\left(\max_{v \in R} \left[\hat{h}_{P_{k+1}(v)} - \hat{h}_{P_k(v)}\right] > \sqrt{K \left(\frac{3}{2}\right)^{-k}}\right) \leq C_3 2^k \exp\left(-K \left(\frac{3}{2}\right)^{-k} \cdot \frac{C_4}{2 \cdot 2^{-k}}\right)$$

$$\leq C_3 2^k \exp\left(-C_4 \left(\frac{4}{3}\right)^k \frac{K}{2}\right),$$

for some constant $C_4$. Since both $\sqrt{K \left(\frac{3}{2}\right)^{-k}}$ and the tail probability are summable in $k$, we conclude that (13) holds.

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