Higher derivative corrections to Lifshitz backgrounds

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Abstract: We explore the effect of curvature-square corrections on Lifshitz solutions to the Einstein-Maxwell-dilaton system. After exhibiting the renormalized Lifshitz scaling solution to the system with parameterized $R^2$ corrections, we turn to a toy model with coupling $g(\phi)C^2_{\mu\nu\rho\sigma}$ and demonstrate that such a term can both stabilize the dilaton and resolve the Lifshitz horizon to $\text{AdS}_2 \times \mathbb{R}^2$. As an example, we construct numerical flows from $\text{AdS}_4$ in the UV to an intermediate Lifshitz region and then to $\text{AdS}_2 \times \mathbb{R}^2$ in the deep IR.
1 Introduction

The application of holographic techniques to condensed matter systems has led to the study of non-relativistic fixed points invariant under the non-relativistic scaling

\[ t \rightarrow \lambda^z t, \quad \vec{x} \rightarrow \lambda \vec{x}, \quad (1.1) \]

where \( z \) is the dynamical exponent. Holographically, this scaling may be realized by taking a bulk metric of the form \([1]\)

\[ ds^2_{d+1} = -e^{2\gamma r/L} dt^2 + e^{2\gamma r/L} d\vec{x}^2 + dr^2, \quad (1.2) \]

where \( r \) is a radial coordinate, and \( L \) sets the length dimension in the bulk. The scaling (1.1) is then accompanied by the transformation

\[ r \rightarrow r - L \log \lambda. \quad (1.3) \]

This background is generally referred to as a Lifshitz spacetime, and it has been the subject of much recent interest.

\[ ^{1}\text{There are, of course, several equivalent ways of writing this metric, and we will make use of this freedom below when investigating the Lifshitz flows in section 3.} \]
Exact Lifshitz geometries were constructed in [2] based on a simple model of a massive vector field coupled to Einstein gravity. Turning on the time component of the vector breaks $d$-dimensional Lorentz symmetry and gives rise to a family of backgrounds with $z \geq 1$. Alternatively, Lifshitz backgrounds may be obtained in the near horizon region of dilatonic branes. A simple realization is to take an Einstein-Maxwell-dilaton system of the form [3–8]

$$e^{-1} \mathcal{L} = R - \frac{1}{2}(\partial \phi)^2 - f(\phi) F_{\mu \nu} F^{\mu \nu} - V(\phi).$$

(1.4)

Lifshitz scaling is obtained by taking a single exponential for the gauge kinetic function along with a constant potential\(^2\)

$$f(\phi) = e^{\lambda_1 \phi}, \quad V(\phi) = -\Lambda.$$  

(1.5)

The scaling solution has a running dilaton and a dynamical exponent given by the relation

$$\lambda_1^2 = \frac{2(d - 1)}{z - 1}.$$  

(1.6)

In addition, full solutions may be constructed that interpolate between AdS\(_{d+1}\) in the UV and Lifshitz in the IR.

As a consequence of the running dilaton, the Lifshitz solution runs into strong coupling either in the UV for the electrically charged solution or the IR for the magnetic solution (in the case $d = 3$). For the magnetic case, the possibility of quantum corrections was investigated in [9] by constructing a toy model where the gauge kinetic function picks up an expansion in the effective coupling $g \equiv e^{-\frac{1}{2} \lambda_1 \phi}$

$$f(\phi) = \frac{1}{g^2} + \xi_1 + \xi_2 g^2 + \cdots.$$  

(1.7)

Under appropriate conditions, these loop corrections will stabilize the dilaton and lead to the emergence of an AdS\(_2 \times \mathbb{R}^2\) geometry in the deep IR. The emergence of this AdS\(_2 \times \mathbb{R}^2\) region also has the benefit of resolving the Lifshitz horizon, which would otherwise lead to tidal singularities [1, 10, 11].

In contrast with the magnetic solution, the electric solution ought not to pick up quantum corrections in the IR, as the dilaton runs to weak coupling. In this case, the Lifshitz horizon would not get resolved by the same mechanism. However, in a stringy context (or in that of any UV complete theory of gravity), there is another potential source of corrections that arise from higher curvature terms. Although Riemann invariants remain finite at the tidal singularity, this singularity is nevertheless felt by strings [11]. Hence the Lifshitz horizon would presumably be resolved in a consistent manner in a stringy realization.

In this paper, we investigate the possibility that higher curvature terms can resolve the Lifshitz horizon into an AdS\(_2\) region in the deep IR. In particular, we add $R^2$ terms to the

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\(^2\)Backgrounds dual to systems exhibiting hyperscaling violation may be obtained by instead taking an exponential potential.

\(^3\)Note that the dilaton can also be stabilized in the dyonic case [3, 8], as well as in models with multiple Maxwell fields [12, 13].
Einstein-Maxwell-dilaton system (1.4) and seek electrically charged brane solutions that flow from AdS_{d+1} in the UV to Lifshitz and then to AdS_2 \times \mathbb{R}^{d-1} in the deep IR. As demonstrated in [14], higher curvature terms do not necessarily destroy the Lifshitz scaling solution, but simply renormalize the dynamical exponent z. Thus we expect that brane solutions with a large intermediate Lifshitz region do exist. However, whether such solutions will flow smoothly into AdS_2 \times \mathbb{R}^{d-1} will depend on the parameters of the model. We investigate the d = 3 case in some detail below, and in particular we confirm numerically that smooth flows do exist that interpolate from AdS_4 to Lifshitz to AdS_2 \times \mathbb{R}^2.

This paper is organized as follows. In section 2, we extend the Einstein-Maxwell-dilaton system by adding parameterized R^2 corrections and construct the resulting renormalized Lifshitz solutions. Since these solutions involve a running dilaton, we demonstrate in section 3 that the dilaton can be stabilized by introducing a dilatonic coupling to R^2. The resulting geometry then takes the form AdS_2 \times \mathbb{R}^{d-1} in the deep IR. Finally, in section 4, we conclude with a discussion on some open issues.

2 Lifshitz solutions in higher derivative gravity

Lifshitz solutions in the presence of higher curvature terms were previously investigated in [15–22]. Here, we focus on the Einstein-Maxwell-dilaton system, (1.4), and take the potential to be a constant, V(φ) = −Λ, so that Lifshitz scaling may be obtained at the two-derivative level. The first set of corrections occurs at the four-derivative level, and in the gravitational sector may be parameterized by three constants, \( \alpha_1, \alpha_2 \) and \( \alpha_3 \), where the action is given by

\[
S = \int d^{d+1}x \sqrt{-g} \left( R + \Lambda - \frac{1}{2} (\partial \phi)^2 - f(\phi) F_{\mu \nu} F^{\mu \nu} + \alpha_1 R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma} + \alpha_2 R_{\mu \nu} R^{\mu \nu} + \alpha_3 R^2 \right).
\]

(2.1)

Using Bianchi identities, we may write Einstein’s equations as:

\[
T_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R + 2 \alpha_1 R_{\mu \rho \lambda \sigma} R^{\rho \lambda \sigma} + (4 \alpha_1 + 2 \alpha_2) R_{\mu \nu \rho \lambda} R^{\rho \lambda} - 4 \alpha_1 R_{\mu \nu} R_{\rho} - \frac{1}{2} g_{\mu \nu} \left[ \alpha_1 R_{\rho \lambda \sigma \kappa} R^{\rho \lambda \sigma \kappa} + \alpha_2 R_{\mu \nu} R^{\mu \nu} + \alpha_3 R^2 - (\alpha_2 + 4 \alpha_3) \Box R \right],
\]

(2.2)

where the energy momentum tensor is given by

\[
T_{\mu \nu} = \frac{1}{2} \partial_{\mu} \phi \partial_{\nu} \phi + 2 f(\phi) (F_{\mu \rho} F^{\rho \nu} - \frac{1}{4} g_{\mu \nu} F_{\rho \sigma} F^{\rho \sigma}) + \frac{1}{2} g_{\mu \nu} (\Lambda - \frac{1}{2} \partial^{\rho} \phi \partial_{\rho} \phi).
\]

(2.3)

These equations need to be supplemented with the equations of motion of \( F_{\mu \nu} \) and \( \phi \):

\[
\nabla_{\mu} (f(\phi) F^{\mu \nu}) = 0,
\]

(2.4)

\[
\Box \phi - f'(\phi) F_{\mu \nu} F^{\mu \nu} = 0,
\]

(2.5)

where \( f'(\phi) \) is the derivative of \( f(\phi) \) with respect to \( \phi \).
Our goal is to find a matter field background that supports the Lifshitz metric, (1.2). From here on, we set \( L = 1 \) without loss of generality. Thus we have
\[
ds^2_{d+1} = -e^{2zr}dt^2 + e^{2r}d\vec{x}^2 + dr^2,
\]
and we want to determine the form of \( F_{\mu\nu} \) and \( \phi \). We first note that Maxwell’s equations, (2.4), can be integrated to obtain an electric solution:
\[
F = \frac{Q}{f(\phi)} e^{(z-(d-1))r} dr \wedge dt,
\]
where \( Q \) is an integration constant (the electric charge). Note that we allow \( \phi \) to depend on \( r \) only. The components of the energy momentum tensor are then given by
\[
T_{00} = g_{00} \left( -\frac{Q^2}{f(\phi)} e^{-2(d-1)r} - \frac{1}{4} (\phi')^2 + \frac{\Lambda}{2} \right),
\]
\[
T_{rr} = g_{rr} \left( -\frac{Q^2}{f(\phi)} e^{-2(d-1)r} + \frac{1}{4} (\phi')^2 + \frac{\Lambda}{2} \right),
\]
\[
T_{ij} = g_{ij} \left( -\frac{Q^2}{f(\phi)} e^{-2(d-1)r} - \frac{1}{4} (\phi')^2 + \frac{\Lambda}{2} \right).
\]
Invariance of \( T_{\mu\nu} \) under Lifshitz scaling requires \( \phi \propto r \) and \( f^{-1} \propto e^{2(d-1)r} \). More explicitly, we may rewrite Einstein’s equations, (2.2), as:
\[
(\phi')^2 = 2(e^{-2zr} RHS_{00} + RHS_{rr}),
\]
\[
\Lambda = RHS_{rr} + e^{-2r} RHS_{ii},
\]
\[
\frac{Q^2}{f(\phi)} e^{-2(d-1)r} = \frac{1}{2} (e^{-2zr} RHS_{00} + RHS_{ii}).
\]
The right hand side of each equation is a fourth order polynomial in \( z \) and does not depend on \( r \). (The curvatures are computed in Appendix A.) After integrating out the electric field, the dilaton equation of motion reads
\[
\phi''(r) + (d-1)\phi'(r) + 2 \frac{f'(\phi)}{f(\phi)} \frac{Q^2}{f(\phi)} e^{-2(d-1)r} = 0.
\]
Plugging in \( f \propto e^{-2(d-1)r} \) and recalling that \( \phi \) is linear in \( r \), we now find that the gauge kinetic function has to be a single exponential \( f(\phi) = e^{\lambda_1 \phi} \).

Before we write down the final solution, let us change to a more convenient basis of higher derivative terms by writing the corresponding Lagrangian as
\[
\mathcal{L}_{\text{hid}} = \alpha_w C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \alpha_G G + \alpha_R R^2,
\]
with the Weyl tensor
\[
C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{d-1} \left( g_{[\mu|\rho} R_{\sigma]\nu} - g_{[\nu|\rho} R_{\sigma]\mu} \right) + \frac{1}{d(d-1)} g_{[\mu|\rho} g_{\sigma]\sigma} R_{\mu\nu},
\]
and the Gauss-Bonnet combination

\[ G = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2. \]  

(2.15)

The Gauss-Bonnet term is topological in four dimensions and vanishes in fewer than four dimensions. Hence we expect the equations of motion to be independent of \( \alpha_{GB} \) for \( d \leq 3 \). The coefficients in (2.13) and (2.1) are related via

\[
\begin{align*}
\alpha_1 &= \alpha_{GB} + \alpha_W, \\
\alpha_2 &= -4\alpha_{GB} - \frac{4}{d-1}\alpha_W, \\
\alpha_3 &= \alpha_{GB} + \frac{2}{d(d-1)}\alpha_W + \alpha_R.
\end{align*}
\]

(2.16)

In this new basis, the final solution is given by the Lifshitz metric (2.6), the Maxwell field (2.7), and the dilaton

\[ \phi = -\frac{2(d-1)}{\lambda_1^2} r + C, \]

(2.17)

where \( C \) is a constant of integration. The gauge kinetic function is

\[ f(\phi) = e^{\lambda_1 \phi}, \]

(2.18)

The electric charge \( Q \) and the cosmological constant \( \Lambda \) are given in terms of \( z \) according to

\[
\begin{align*}
Q^2e^{-\lambda_1 C} &= \frac{1}{2}(z-1)(z+d-1) \left[ 1 - \frac{4(d-2)}{d}\alpha_W z(z-d-1) - 2(d-2)(d-3)\alpha_{GB} \right] \\
&\quad - 2\alpha_R \left[ z^4 + 2(d - \frac{3}{2})z^3 + \frac{3}{2}(d-1)z^2 + \frac{1}{2}(d-1)(d^2 - 4d + 2)z - \frac{1}{2}d(d-1)^2 \right], \\
\Lambda &= (z+d-1)(z+d-2) - \frac{4(d-2)^2}{d}\alpha_W z(z-1) \left( z - 2 \frac{d-1}{d-2} \right) \\
&\quad - 2(d-2)(d-3)\alpha_{GB} \left[ z^2 + 2(d - \frac{3}{2})z + \frac{1}{2}(d-1)(d-4) \right] \\
&\quad + 4\alpha_R \left[ z^3 - \frac{3}{2}(d-1)(d-\frac{8}{3})z^2 - (d-1)(d^2 - \frac{7}{2}d + 2)z - \frac{1}{4}d(d-1)^2(d-4) \right] .
\end{align*}
\]

(2.19)

As expected, for \( d = 2 \) and \( 3 \), the Gauss-Bonnet combination does not contribute to the equations of motion. Notice also that due to the shift symmetry \( \phi \mapsto \phi + C, F_{\mu\nu} \mapsto F_{\mu\nu}e^{-\frac{1}{2}\lambda_1 C} \), only the combination \( Q^2e^{-\lambda_1 C} \) is fixed.

We see that the higher derivative action (2.1) admits Lifshitz solutions with an electric background gauge potential and \( \phi \propto r \). The “running” of the dilaton has physical conse-
quences: The effective gauge coupling $f^{-\frac{1}{2}}$ runs from weak coupling in the IR ($r \to -\infty$) to strong coupling in the UV ($r \to \infty$).4

The above solution is the straightforward generalization of the previously known Maxwell-dilaton background to the case of four-derivative gravity. The effect of the higher derivative corrections is to renormalize the cosmological constant and electric charge by inducing corrections of order $z^4$. We will demonstrate below that this leads to some nontrivial features of the solution.

2.1 Lifshitz solutions in Einstein-Weyl gravity

Let us now focus on the special case of Einstein-Weyl gravity. This theory will be of particular interest to us in the following section, where we will construct smooth flows from AdS$_4$ to Lifshitz to AdS$_2 \times \mathbb{R}^2$. Lifshitz solutions in pure Einstein-Weyl gravity without additional matter fields have also been studied in [22].

Setting $\alpha_{GB} = \alpha_R = 0$, the solution simplifies to

$$Q^2 e^{-\lambda_1 C} = \frac{1}{2} (z - 1)(z + d - 1) \left( 1 - \frac{4(d - 2)}{d} \alpha_W z(z - d - 1) \right),$$

$$\Lambda = (z + d - 1)(z + d - 2) - \frac{4(d - 2)^2}{d} \alpha_W z(z - 1) \left( z - 2 \frac{d - 1}{d - 2} \right).$$

This solution has some interesting features. For $d = 2$, the Weyl-tensor vanishes identically and so there are no higher derivative corrections. Next, notice that if $Q^2 \to 0$, $\lambda_1 \to \infty$, $\phi \to \text{const.}$, the matter fields decouple and we recover a purely gravitational solution. There are two distinct ways to achieve this: The first one is the case $z = 1$, corresponding to pure AdS$_{d+1}$ without matter fields. Note that because the Weyl tensor vanishes in AdS, the cosmological constant is not renormalized.

As a second possibility, we may choose

$$\alpha_W = \frac{d}{4(d - 2)z(z - d - 1)},$$

In this case we recover purely gravitational Lifshitz solutions, with

$$Q^2 e^{-\lambda_1 C} = 0,$$

$$\phi = \text{const.},$$

$$\Lambda = (z + d - 1)(z + d - 2) - (d - 2)(z - 1) \frac{z - 2 \frac{d - 1}{d - 2}}{z - d - 1}.$$  

It is interesting to consider the limit of conformal gravity, where $\alpha_W \to \infty$. From (2.23), we expect the scaling parameter to take two possible values, $z = 0$, or $z = d + 1$. However, in the latter case, $\Lambda$ blows up for general $d$. It is only in the case $d = 3$ that the second solution with $z = 4$ is well behaved. Finally, notice also that for any given $\alpha$ and $\lambda_1$, there may be multiple solutions for $z$ (see also Appendix B).

4In four dimensions, we can use electric-magnetic duality to obtain a magnetic solution, $\tilde{F} \equiv f(\phi) \star F = Q_m dx \wedge dy$, with magnetic charge $Q_m$. Since the duality transformation also requires $f \mapsto f^{-1}$, the dilaton now runs towards strong coupling in the IR.
3 Smoothing out the singularity

The Lifshitz solutions of the previous section have a physical singularity in the infrared. For \( z \neq 1 \), an infalling extended object, such as a string, experiences infinitely strong tidal forces as \( r \to -\infty \) \[11\]. Hence pure Lifshitz solutions are ‘IR incomplete’. However, one might argue that this kind of pathological behavior is simply a signal that our solutions should not be trusted in this particular regime and the singularity would presumably be resolved in a more complete string theory picture. Some compelling evidence supporting this point of view has been presented in \[9, 23, 24\].

The analysis of the previous section suggests a straightforward way of resolving the Lifshitz singularity: In general, a nonzero coupling of the dilaton to higher derivative terms will generate corrections to its effective potential. In this section, we will use a simple toy model in four dimensions to show that by choosing such a coupling appropriately, the dilaton can be stabilized at some finite value \( \phi_0 \). As a result, the geometry flows smoothly from Lifshitz to \( \text{AdS}_2 \times \mathbb{R}^2 \) in the deep IR, which is free of physical singularities.

In order to imitate the effect of generic higher derivative corrections from string theory, we consider the following theory:

\[
S = \int d^4 x \sqrt{-g} \left( R + \Lambda - \frac{1}{2} (\partial \phi)^2 - f(\phi) F_{\mu\nu} F^{\mu\nu} + g(\phi) C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right). \tag{3.1}
\]

Since the Weyl tensor vanishes in \( \text{AdS}_4 \), the higher derivative terms do not source the dilaton in the UV. We therefore expect a smooth flow from \( \text{AdS}_4 \) to Lifshitz, much like the domain-wall solutions found in \[3, 7\]. As we flow further towards the IR, the Weyl-squared term ought to become more important, and the dilaton-Weyl coupling, \( g(\phi) \), may then stabilize the dilaton. To be concrete, we choose \( g(\phi) \) to be

\[
g(\phi) = \frac{3}{4} (\alpha + \beta e^{\lambda_2 \phi}). \tag{3.2}
\]

For \( \beta e^{\lambda_2 \phi} \ll \alpha \), \( g(\phi) \) is approximately constant and we expect to find Lifshitz scaling solutions of the form described in the previous section. With an appropriate choice of parameters, the exponential becomes more and more important as \( \phi \) runs towards weak coupling and it eventually stabilizes the dilaton in the deep IR.

Since we have introduced a Weyl-squared correction, it is convenient to choose the following parametrization of the metric\(^5\):

\[
ds^2 = a^2(\ell) \left( -dt^2 + dr^2 + b^2(r)(dx^2 + dy^2) \right), \tag{3.3}
\]

With this choice, the Weyl-invariance of the higher derivative Lagrangian is manifest as a rescaling of \( a(\ell) \). In practice, this means that only \( b(r) \) will receive higher derivative corrections in the equations of motion. Fixing \( \Lambda = 1 \), the \( \text{AdS}_4 \) solution is given by

\[
a = \frac{\sqrt{6}}{r}, \quad b = \text{const.}, \tag{3.4}
\]

\(^5\)We will work in units where \( L = 1 \) in what follows.
while the Lifshitz solution takes the form
\[ a \propto \frac{1}{r}, \quad b \propto r^{\tilde{z}}. \] (3.5)

For this metric, the scaling symmetry (1.1) and (1.3) becomes
\[ t \to \lambda t, \quad x \to \lambda^{1-\tilde{z}} x, \quad r \to \lambda r. \] (3.6)

In changing from the more common form of the metric, (1.2), to the Weyl form, (3.3), we need to make the following identifications:
\[ z = \frac{1}{1 - \tilde{z}}, \]
\[ L \to \frac{L}{1 - \tilde{z}}, \]
\[ \alpha \to (1 - \tilde{z})^2 \alpha. \] (3.7)

As before, we choose a background electric charge:
\[ F = \frac{Q}{b^2 f(\phi)} dr \wedge dt. \] (3.8)

Einstein’s equations are:
\[
T_{00} = -2 \left( \frac{a''}{a} + \frac{b''}{b} \right) + \left( \frac{a'}{a} \right)^2 - \left( \frac{b'}{b} \right)^2 - 4 \frac{a' b'}{ab}
\]
\[ - \frac{4}{3} \frac{g(\phi)}{a^2} \left[ \frac{b^{(4)}}{b} + \frac{b^{(3)} b'}{b^2} - 2 \frac{b'' (b')^2}{b^3} - \frac{1}{2} \left( \frac{b''}{b} \right)^2 + \frac{1}{2} \left( \frac{b'}{b} \right)^4 + \left( \frac{b''}{b} - \left( \frac{b'}{b} \right)^2 \right) \frac{g'}{g} \phi'' \right.
\]
\[ + \left( 2 \frac{b^{(3)}}{b} - b'' (b')^2 - \left( \frac{b'}{b} \right)^3 \right) \frac{g'}{g} \phi' + \left( \frac{b''}{b} - \left( \frac{b'}{b} \right)^2 \right) \frac{g''}{g} \left( \phi' \right)^2 \left[ \right], \] (3.9)

\[
T_{rr} = 3 \left( \frac{a'}{a} \right)^2 + \left( \frac{b'}{b} \right)^2 + 4 \frac{a' b'}{ab}
\]
\[ + \frac{4}{3} \frac{g(\phi)}{a^2} \left[ - \frac{b^{(3)} b'}{b^2} + \frac{1}{2} \left( \frac{b'}{b} \right)^4 + \frac{1}{2} \left( \frac{b''}{b} \right)^2 + \left( \frac{b'}{b} \right)^2 - \frac{b'' b'}{b^2} \right] \frac{g'}{g} \phi' \right], \] (3.10)

\[
T_{ii} = 2 \frac{a''}{a} + \frac{b''}{b} - \left( \frac{a'}{a} \right)^2 + 2 \frac{a' b'}{ab}
\]
\[ - \frac{4}{3} \frac{g(\phi)}{a^2} \left[ \frac{1}{2} \left( \frac{b^{(4)}}{b} \right)^2 + \frac{1}{2} \left( \frac{b''}{b} \right)^4 - \frac{(b')^2 b''}{b^3} + \frac{1}{2} \left( \frac{b'}{b} \right)^2 - \left( \frac{b'}{b} \right)^2 \right] \left( \frac{g'}{g} \phi'' + \frac{g''}{g} \left( \phi' \right)^2 \right)
\]
\[ + \left( \frac{b^{(3)}}{b} - \frac{b'' b'}{b^2} \right) \frac{g'}{g} \phi' \right], \] (3.11)
with

\[ T_{00} = \frac{Q^2}{a^2 b^4 f(\phi)} + \frac{1}{4} \left( \frac{\partial}{\partial r} \right)^2 - \frac{a^2 \Lambda}{2}, \]  
(3.12)

\[ T_{rr} = -\frac{Q^2}{a^2 b^4 f(\phi)} + \frac{1}{4} \left( \frac{\partial}{\partial r} \right)^2 + \frac{a^2 \Lambda}{2}, \]  
(3.13)

\[ T_{ij} = b^2 \delta_{ij} \left( \frac{Q^2}{a^2 b^4 f(\phi)} - \frac{1}{4} \left( \frac{\partial}{\partial r} \right)^2 + \frac{a^2 \Lambda}{2} \right). \]  
(3.14)

If we demand that \( \phi \) depends only on \( r \), the dilaton equation of motion simplifies to

\[ \phi'' + 2 \left( \frac{d'}{a} + \frac{b'}{b} \right) \phi' + a^2 V'_{\text{eff}}(\phi) = 0, \]  
(3.15)

where

\[ V'_{\text{eff}}(\phi) \equiv \frac{2Q^2 f'(\phi)}{a^4 b^4 f^2(\phi)} + \frac{4}{3a^4} \left( \frac{d^2 \log(b)}{dr^2} \right)^2 g'(\phi). \]  
(3.16)

Hence the effect of the higher derivative terms is to generate a correction to the effective dilaton potential.

We would like to find out which choices of \( g(\phi) \) allow for an emerging AdS\(_2 \times \mathbb{R}^2 \) geometry in the deep IR. Corresponding to AdS\(_2 \times \mathbb{R}^2 \), we make the ansatz

\[ a(r) = \frac{1}{r}, \]

\[ b(r) = b_0 r, \]

\[ \phi(r) = \phi_0. \]  
(3.17)

Solving (3.9)-(3.11), we find

\[ \Lambda = 1, \]

\[ \frac{Q^2}{b_0^4} = \frac{f(\phi_0)}{2} \left( 1 - \frac{4}{3} g(\phi_0) \right). \]  
(3.18)

Since only the ratio \( Q/b_0^4 \) is fixed, we are free to set \( b_0 \equiv 1 \) in what follows. Equation (3.15) gives us the condition

\[ V'_{\text{eff}}(\phi_0) = \frac{f'(\phi_0)}{f(\phi_0)} \left( 1 - \frac{4}{3} g(\phi_0) \right) + \frac{4}{3} g'(\phi_0) = 0. \]  
(3.19)

Let us now specialize to the case \( f(\phi) = e^{\lambda_1 \phi} \). Since the dilaton runs towards weak coupling as \( r \to \infty \), this ansatz is valid even in the deep IR. With our choice of \( g(\phi) \), the solution to (3.18) and (3.19) is given by

\[ Q^2 = \frac{(\alpha - 1) \lambda_2}{\lambda_1 - \lambda_2}, \]

\[ \phi_0 = \frac{1}{\lambda_2} \log \left( \frac{\lambda_1 - \lambda_2}{\lambda_1 - \alpha} \right). \]  
(3.20)

Clearly this solution only makes sense for a certain choice of \( \lambda_i, \alpha, \beta \). We will discuss the constraints on these parameters at the end of the next section.
3.1 Perturbations around $\text{AdS}_2 \times \mathbb{R}^2$

We would like to find numerical solutions that smoothly interpolate between $\text{AdS}_4$ and $\text{AdS}_2 \times \mathbb{R}^2$, with some intermediate Lifshitz regime. This is most easily accomplished numerically by using the “shooting” technique, starting in the deep IR ($r \to \infty$). The initial conditions have to be chosen such that we follow perturbations that are irrelevant in the IR. These are perturbations that fall off faster than the background solution as $r \to \infty$. In other words, they allow a smooth flow away from $\text{AdS}_2 \times \mathbb{R}^2$ as $r$ decreases. Requiring the existence of such perturbations will introduce nontrivial constraints on the parameters of our model.

We start by perturbing the $\text{AdS}_2 \times \mathbb{R}^2$ solution, (3.17), in the following way

$$a(r) = \frac{1}{r} + \delta a(r), \quad b(r) = r + \delta b(r), \quad \phi(r) = \phi_0 + \delta \phi(r). \quad (3.21)$$

Using the conditions (3.18) and (3.19) repeatedly, the linearized equations of motion may be written as

$$\frac{3}{2g_0} \left( r^3 \delta a' \right)' + \left( r^2 \delta b' \right)' - 2 \frac{f_0 g_0}{f'_0 g_0} \left( \frac{\delta b'}{r} \right)' - \frac{g'_0}{g_0} \frac{(r \delta \phi')'}{r} + \frac{g'_0}{g_0} \frac{\delta \phi}{r^2} = 0, \quad (3.22)$$

$$\frac{3}{2g_0} \left( r^2 \delta a' \right)' + r^2 \delta b^{(3)} - 2 \frac{f_0 g'_0}{f'_0 g_0} \delta b' - \frac{g'_0}{g_0} (r \delta \phi)' = 0, \quad (3.23)$$

$$-3 \left( r^2 \delta a' \right)' + r^2 g_0 \delta b^{(4)} - 2 \left( g_0 + \frac{3}{4} \right) r^2 \left( \frac{\delta b'}{r^2} \right)' - 6 \frac{\delta b}{r^2} - g'_0 r \left( \delta \phi'' - 2 \frac{\delta \phi}{r^2} \right) = 0, \quad (3.24)$$

$$\delta \phi'' + V_{\text{eff}}'(\phi_0) \frac{\delta \phi}{r^2} - \frac{8}{3} \frac{g'_0}{g_0} r \left( \frac{\delta b'}{r^2} \right)' = 0, \quad (3.25)$$

where $f_0 \equiv f(\phi_0)$, etc. The presence of the $\delta b$ term in the last equation emphasizes the fact that the higher derivative corrections generate a gravitational effective potential for the dilaton. This is different from the case of a quantum-corrected $f(\phi)$, and will in general lead to a nontrivial mixing of $\phi$ perturbations with gravitational perturbations. Since the first three equations are related via a Bianchi identity, it is possible to eliminate the $\delta b^{(4)}$ terms and reduce the system to a third order coupled ODE. Hence there are only seven independent solutions:

$$\delta a = -r, \quad \delta b = r^3, \quad \delta \phi = 0; \quad (3.26)$$

$$\delta a = - \frac{3}{4} \left( \log(r) \right), \quad \delta b = \log r, \quad \delta \phi = \frac{\xi}{r}; \quad (3.27)$$

$$\delta a = - \frac{1}{r^2}, \quad \delta b = 1, \quad \delta \phi = 0; \quad (3.28)$$

$$\delta a = A_0 r^{\nu-1}, \quad \delta b = B_0 r^{\nu+1}, \quad \delta \phi = P_0 r^{\nu}. \quad (3.29)$$

Here

$$\xi = \frac{6 \lambda_1 \lambda_2 (1 - \alpha)}{\lambda_1 \lambda_2 (\lambda_1 + \lambda_2) (\alpha - 1) + 2 (\lambda_2 - \lambda_1)}, \quad (3.30)$$
and the constants $A_0$, $B_0$ and $P_0$ in (3.29) are related by

$$A_0 = \frac{2g_0}{3} \left( \frac{g_0'}{g_0} \left( P_0 + 2\frac{f_0}{f_0'} \right) - \nu (\nu - 1) \right) B_0,$$

$$P_0 = \frac{8}{3} \frac{g_0'}{V''(\phi_0)} + \nu (\nu - 1) B_0.$$  \hspace{1cm} (3.31)

There are four solutions for the exponent in (3.29):

$$\nu \equiv \frac{1}{2} + \tilde{\nu},$$

$$2\tilde{\nu}^2 = - \left( V_{\text{eff}}''(\phi_0) - \frac{1}{4} - \frac{8}{3} \frac{(g_0')^2}{g_0} + x \right) \pm \left[ \left( V_{\text{eff}}''(\phi_0) - \frac{1}{4} - \frac{8}{3} \frac{(g_0')^2}{g_0} - x \right)^2 - \frac{16}{3} \left( \frac{g_0'}{g_0} \right)^2 \right]^{\frac{1}{2}},$$

$$x \equiv \frac{5}{12} - \frac{1}{2g_0} - \frac{4}{3f_0g_0}.$$  \hspace{1cm} (3.32)

For our choice of $g(\phi)$, given by (3.2), we get

$$\tilde{\nu} = \pm \frac{1}{2} \left[ 1 - \frac{(1 - \alpha) \lambda_2}{\lambda_1} \left[ 2\lambda_1 - \frac{4}{3\lambda_1} + 2\alpha \lambda_2 \right] \right. \pm \frac{2}{\lambda_1} \left( \lambda_1^4 + 2\alpha \lambda_2 \lambda_1^3 + \left( \alpha^2 - 4 \right) \lambda_1^2 + \frac{4}{3} \alpha \lambda_1 \lambda_2 + \frac{4}{9} \right)^{\frac{1}{2}} \right].$$  \hspace{1cm} (3.33)

Regardless of the form of the effective potential, there always exist two irrelevant perturbations, (3.27) and (3.28). Whether or not the solutions (3.29) are irrelevant depends on the choice of parameters. Although in general there is a mixing of $\phi$ with $a$ and $b$ due to the dilaton coupling to $C^2_{\mu\nu\rho\sigma}$, one can check that for $g(\phi) \equiv 0$ the ansatz (3.29) reproduces the purely dilatonic perturbations of the two-derivative theory [9]. Although not technically correct, we will therefore still refer to those perturbations as “dilaton perturbations” in what follows.

To find the desired numerical solutions, we impose the following set of conditions:

1. $\lambda_2/\lambda_1 > 0$: This ensures that $g(\phi) \approx \text{const.}$ during the Lifshitz scaling stage and in the deep UV. Thus $g'(\phi)$ only becomes important in the IR, where it stabilizes the dilaton. Since (3.1) is invariant under $\phi \mapsto -\phi$, $\lambda_1 \mapsto -\lambda_1$, we shall assume without loss of generality that $\lambda_1 > 0$ and $\lambda_2 > 0$.

2. $V_{\text{eff}}'(\phi_0) = 0$ for some $\phi_0$ (see (3.19)): The effective potential stabilizes the dilaton and admits an AdS$_2 \times \mathbb{R}^2$ solution.

3. We focus on the case $g(\phi_0) > 0$. For negative $g(\phi)$, we numerically find either singular solutions or solutions with $\phi' \ll 0$ as we approach AdS$_4$. It is unclear whether the sign of higher derivative terms has a physical interpretation in terms of unitarity or causality or a generalized null energy condition.
4. $Q^2 > 0$, i.e. the vector potential is real-valued.

5. Our numerical analysis, as well as the analysis performed in [9, 23] strongly suggest that we need at least one of the dilaton perturbations to be irrelevant in order to “kick” $\phi$ out of its local minimum in the IR and roll towards large negative values in the UV. We therefore demand that $\nu < 0$ for at least one of the dilaton perturbations. Notice that there can be at most two solutions that satisfy this condition.

6. We require $\nu$ to be real-valued; that is, we exclude oscillating perturbations. We take the existence of complex eigenvalues as an indication of a dynamical instability. However, due to the higher-derivative nature of our theory, a more detailed analysis of the time-dependent perturbations would be needed in order to determine whether the theory is truly unstable for complex exponents.

Let us now find out what these conditions imply for our parameters $\alpha, \beta, \lambda_1, \lambda_2$. Conditions 3 and 4 allow for two possible choices:

1) $\alpha \geq 1, \quad \alpha \frac{\lambda_2}{\lambda_1} \leq 1$;

2) $0 < \alpha < 1, \quad \alpha \frac{\lambda_2}{\lambda_1} > 1$.  \hfill (3.34)

In both cases, condition 2 then requires that $\beta < 0$. Recall that in the electric case $\phi \leq \phi_0$, so choosing the sign of $g(\phi)$ in the IR determines the sign everywhere\(^6\). Finally, we would like the dilaton perturbations to be non-oscillating (condition 5) and demand that at least one of them should be irrelevant (condition 6). The details of the corresponding calculations can be found in Appendix C. Our results are summarized in Figures 1 and 2. In the green region, all of our conditions are satisfied. The gray region is inconsistent with conditions 1-4, while in the red region $g(\phi) < 0$. The yellow region has $g(\phi) > 0$, but has either no irrelevant dilaton perturbations, or oscillating modes.

For $\alpha < 1$, we find either one or no irrelevant dilaton perturbations, while for $\alpha > 1$ we find either two or none. This result seems to be related to the fact that in the $\alpha > 1$ case, $\lambda_1^2(\tilde{z})$ is not injective, i.e. there exist two possible scaling parameters $\tilde{z}_1, \tilde{z}_2$ for any given $\lambda_1$ (see Appendix B). We will address this issue further at the end of the following section. Notice also that while for $\alpha < 1$, all values of $\lambda_1$ and $\tilde{z}$ are allowed, for $\alpha > 1$ there is a lower bound on $\lambda_1$ and an upper bound on $\tilde{z}$. These bounds stems from the condition that $\lambda_1^2 > 0$ and equation (B.5). We conclude that for a given choice of $\alpha$, there exists a large region in parameter space that is consistent with our conditions and hence admits the desired $\text{AdS}_4 \rightarrow \text{Lif}_1^2 \rightarrow \text{AdS}_2 \times \mathbb{R}^2$ solutions.

\(^6\)Although we will not consider the case of negative $g(\phi)$, let us point out that in this case we would also have to take $\beta < 0$ to satisfy condition 2, so this is a universal result.
Figure 1. Plot of different regions in parameter space, characterized by the number of irrelevant dilaton perturbations ($\alpha = 0.9$). The regions are bounded by the curves (C.1), (C.2), (3.34) and $\lambda_2 = \lambda_1$. They are colored as follows: $g(\phi) < 0$ (red), $g(\phi) > 0$ but no irrelevant perturbations or oscillating modes (gold), $g(\phi) > 0$ and at least one irrelevant perturbation (green). In the gray area, at least one of the conditions 1-4 is violated.

Figure 2. The same plot for $\alpha = 3$. Now the green region has 2 irrelevant perturbations. There is a lower bound on $\lambda_1$ and an upper bound on $\tilde{z}$, as indicated by vertical lines.
3.2 Numerical results

In order to find numerical solutions to our equations, we proceed as follows: We set initial conditions at large $r$ by adding irrelevant perturbations to the exact $\text{AdS}_2 \times \mathbb{R}^2$ solution:

$$a(r) = \frac{1}{r} + \sum_{i=1}^{3(4)} D_i \delta a_i,$$

$$b(r) = r + \sum_{i=1}^{3(4)} D_i \delta b_i,$$

$$\phi(r) = \phi_0 + \sum_{i=1}^{3(4)} D_i \delta \phi_i. \quad (3.35)$$

We focus here on the case $\alpha < 1$, for which there are three irrelevant perturbations. The case $\alpha > 1$ is discussed briefly at the end of this section. The amplitudes $D_i$ have to be tuned in order to find a solution that has both an intermediate Lifshitz regime and a smooth flow to $\text{AdS}_4$ in the UV. In practice, it is however easier to choose a different basis for the perturbations in (3.35), which allows us to specify $a', b', \phi'$ directly. The role of the three initial conditions is then roughly the following: The value of $\phi'$ determines how long the solution stays approximately $\text{AdS}_2 \times \mathbb{R}^2$. There is a minimum value $\phi'_{\text{min}}$ that is required to “kick” $\phi$ out of its local minimum and run logarithmically during the Lifshitz stage. The transition stage from $\text{AdS}_2 \times \mathbb{R}^2$ to Lifshitz is shifted towards the IR as we increase $\phi'$. The value of $a'$ determines the duration of the scaling stage: We find that in the space of initial conditions, there exists a two-dimensional submanifold $(a'_{\text{crit}}, b', \phi')$ with attractor-like behavior. As we approach this critical plane, we observe the emergence of an intermediate Lifshitz stage, which gets wider and wider as $a'$ approaches $a'_{\text{crit}}$ from below, while for $a' > a'_{\text{crit}}$ the solution becomes singular. Therefore, by tuning $a'$ we can in principle make the Lifshitz stage arbitrarily long. Finally, the value of $b'$ needs to be tuned in order to achieve a smooth flow to $\text{AdS}_4$ in the UV.

The parameters and initial conditions of our numerical solutions are summarized in Table 1. Figure 3 shows the evolution of the metric components $g_{00}$ (black) and $g_{ii}$ (blue) for solution #1. (The individual metric functions $a(r)$ and $b(r)$ as well as the dilaton $\phi(r)$ are plotted in Figures 4 and 5.) We chose to plot $d \log g_{\mu\nu}/d \log r$ versus $\log r$ so that power-law relations are clearly visible as horizontal lines. The solution is asymptotically $\text{AdS}_2 \times \mathbb{R}^2$ with $g_{00} \propto r^{-2}$, $g_{ii} \propto r^0$ for large $r$. At $r \approx 10^{-6}$, the solution approaches an approximate Lifshitz scaling stage with $g_{00} \propto r^{-2}$, $g_{ii} \propto r^{2(\tilde{z}-1)}$, where it remains for several decades. This stage is characterized by an effective scaling parameter $\tilde{z}_{\text{eff}} \approx 0.73$ (or $z \approx 3.7$). Notice that $\tilde{z}_{\text{eff}}$ decreases slowly towards the UV, as indicated by the slightly positive slope of $d \log g_{ii}/d \log r$. This is due to the fact that $e^{\lambda_2 \phi}$ is small but nonzero: Effectively, the coupling constant $\alpha$ is reduced, which in turn increases $\tilde{z}_{\text{eff}}$ (see Figure 8). We expect that as we approach the attractor, the solution will take the exact form (3.5) with the predicted value of $\tilde{z} \simeq 0.71$ for
Table 1. Parameters, initial conditions and fit parameters for numerical solutions.

| #  | 1     | 2     |
|----|-------|-------|
| λ₁ | 1.2   | 1.1   |
| λ₂ | 2     | 0.24  |
| α  | 0.9   | 3     |
| β  | -1    | -1    |
| Q  | 0.20  | 4.55  |
| φ₀ | -0.95 | 3.91  |
| a' | -4.9 · 10⁻⁹ | -1.5 · 10⁻⁷ |
| b' | -5.8 · 10⁻⁶ | 2.0 · 10⁻⁴ |
| φ' | 8 · 10⁻⁷  | 10⁻⁵  |
| b''| -     | 10⁻⁵  |
| ε  | 0.73  | 0.78  |
| K  | 0.91  | 0.82  |
| φ_{UV} | -19.8 | -7.6 |
| b_{UV} | 1.5 · 10⁻¹¹ | 2.9 · 10⁻⁸ |

Figure 3. Plot of the metric components $g_{00}$ (black) and $g_{ii}$ (blue) for solution #1 (see Table 1). The figure on the right is a magnified view of the Lifshitz region for $g_{ii}$. Constant values of $d \log g_{\mu \nu} / d \log r$ indicate a power-law relation. One can clearly see the emergence of an intermediate Lifshitz geometry with $g_{00} \propto r^{-2}$ and $g_{ii} \propto r^{2(\tilde{\varepsilon} - 1)}$. The dotted lines indicate the exact AdS$_2 \times \mathbb{R}^2$ solution with $g_{ii} \propto r^0$ in the IR and AdS$_4$ with $g_{ii} \propto r^{-2}$ in the UV.

$r \to 0$. Finally, it is worth mentioning that both $g_{ii}$ and $g_{00}$ initially overshoot slightly before flowing to Lifshitz.

The dilaton starts out at some large negative value $\phi_{UV}$ for small $r$ and runs towards weak coupling during the scaling stage. In this intermediate regime, $e^{\lambda_2 \phi} \ll 1$ and $\phi$ grows approximately logarithmically, as in (B.2). As $\phi$ increases, the $e^{\lambda_2 \phi}$-term becomes more
Figure 4. Plot of $a \cdot r$ for solution #1. The figure on the right is a magnified view of the Lifshitz region. The dotted lines represent the exact $\text{AdS}_2 \times \mathbb{R}^2$ solution with $a \cdot r = 1$ in the IR and $\text{AdS}_4$ with $a \cdot r = \sqrt{6}$ in the UV.

Figure 5. Plot of the metric function $b$ (left) and the dilaton $\phi$ (right) for solution #1. The dotted line represents the asymptotic value $\phi_0$ given by (3.20).

and more important until at large $r$, the higher derivative corrections eventually modify the effective potential and stabilize the dilaton at $\phi_0$.

In the case of $\alpha \geq 1$, there are two possible dynamical exponents $\tilde{z}_1 < \tilde{z}_2$ (see Appendix B). There is one additional dilaton perturbation, which we can use to fix the value of $b''$ in the IR. Numerically, we were only able to find flows from $\text{AdS}_2 \times \mathbb{R}^2$ to $\text{Lif}_{\tilde{z}_2}$, with $\tilde{z}_2 \approx 0.78$ ($z_2 \approx 4.4$). The metric components for this solution are shown in Figure 6. The corresponding values for the exact analytical solution are $\tilde{z}_1 \approx 0.38$ and $\tilde{z}_2 \approx 0.73$. Although a simple counting of dilaton perturbations would suggest that there is one irrelevant deformation leading to each of the two Lifshitz solutions, we were not able to numerically shoot to $\text{Lif}_{\tilde{z}_1}$. It therefore remains unclear whether flows to $\text{Lif}_{\tilde{z}_1}$ exist.
3.3 Flow to AdS$_4$ in the UV

Our numerical analysis suggests that the solutions exhibit some interesting behavior as they approach asymptotic AdS$_4$ for $r \to 0$. It is worthwhile to analyze this asymptotic behavior analytically. To lowest order, the solution to the linearized equations of motion is given by

$$a(r) = \frac{\sqrt{6}}{r} \left(1 + a_+ r^{\nu_+} + a_- r^{\nu_-} + a_3 r^3 + a_4 r^4 + \cdots\right),$$
$$b(r) = b_{UV} + b_+ r^{\nu_+} + b_- r^{\nu_-} + b_3 r^3 + b_4 r^4 + \cdots,$$
$$\phi(r) = \phi_{UV} + \phi_3 r^3 + \phi_4 r^4 + \cdots,$$  \hspace{1cm} (3.36)

where $a_3, \phi_3, a_\pm$ are free constants and

$$a_4 = \frac{1}{180} \frac{Q^2 (9 + 2 g(\phi_{UV}))}{b_{UV}^4 f'(\phi_{UV}) (1 + 2 g(\phi_{UV}))},$$
$$b_3 = -2 b_{UV} a_3,$$
$$b_4 = -\frac{1}{12} \frac{Q^2}{(1 + 2 g(\phi_{UV})) f'(\phi_{UV}) b_{UV}^3},$$
$$b_\pm = -\frac{3 (\nu + 1)}{2\nu} b_{UV} a_\pm,$$
$$\phi_4 = -\frac{1}{12} \frac{Q^2 f'(\phi_{UV})}{b_{UV}^4 f(\phi_{UV})^2},$$
$$\nu_\pm = \frac{3}{2} \pm \frac{1}{2} \sqrt{1 - \frac{16}{g(\phi_{UV})}}.$$  \hspace{1cm} (3.37)

The leading order perturbations $r^{\nu}$ are purely gravitational. They survive in the limit of pure Einstein-Weyl gravity (i.e. $Q \to 0$) [22]. For $g(\phi_{UV}) < 16$, $\nu$ becomes complex and the
perturbations oscillate as

\[
\begin{align*}
    a &\sim r^{\Re(\nu)-1} \cos (\Im(\nu) \log (r) + \varphi_a), \\
    b &\sim r^{\Re(\nu)} \cos (\Im(\nu) \log (r) + \varphi_b),
\end{align*}
\]

where \(\varphi_{a/b}\) are constant phases. Notice that for \(\alpha \to 0\), the imaginary part of \(\nu\) blows up, so these perturbations do not decouple in the two-derivative limit. Figure 7 shows the asymptotic behavior of one of our numerical solutions (parameter set \#1). One can clearly see that \(a\) and \(b\) oscillate according to (3.38), while \(\phi\) simply decreases monotonically.

As it turns out, the oscillating nature of our solutions makes it necessary to switch to a “stiff” method when trying to find exact numerical solutions in the UV. In addition, the attractor-mechanism of the Lifshitz stage tends to “wipe out” initial conditions, which makes it more and more difficult to exactly hit AdS\(_4\) numerically as the scaling stage gets wider. We therefore content ourselves with presenting the asymptotic behavior for a solution with a relatively narrow scaling stage. A more efficient way of studying the UV-asymptotics would

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**Figure 7.** Flow to asymptotic AdS\(_4\) in the deep UV. The first graph shows the metric components \(g_{00}\) (black) and \(g_{ii}\) (blue). While the metric functions \(a\) and \(b\) oscillate according to (3.38), the dilaton decreases monotonically.
be to directly shoot from the UV.

4 Discussion

We first showed that Lifshitz backgrounds are renormalized in the presence of higher derivative corrections, and in particular Weyl-squared corrections, according to (2.18)-(2.20). However, the exceptions to this are solutions of conformal gravity, which may be obtained in the formal limit $\alpha_w \to \infty$. The variation of $C^2_{\mu \nu \rho \sigma}$ is proportional to the Bach-tensor

$$B_{\mu \nu} = \left( \nabla^{\rho \sigma} + \frac{1}{2} R^{\rho \sigma} \right) C_{\mu \rho \nu \sigma}. \tag{4.1}$$

Notice that the Bach-tensor vanishes identically for Einstein metrics. Moreover, since $B_{\mu \nu}$ is a conformal tensor, it also vanishes on spacetimes that are conformally Einstein. Hence, these backgrounds are not renormalized. For the solutions of the form (1.2), the cases $z = 0$, $z = 1$ (AdS) and, in four dimensions, $z = 4$ are conformally related to Einstein metrics\textsuperscript{7} via

$$g_{\mu \nu} \mapsto e^{-2(2+z^2)/(2+z)} g_{\mu \nu} \text{ (for } d = 3)$$

and are therefore protected against renormalization.

We then demonstrated in a toy model that higher curvature corrections, such as those that arise from the string $\alpha'$ expansion, may resolve the Lifshitz horizon into $\text{AdS}_2 \times \mathbb{R}^2$. In particular, we have constructed numerical flows from $\text{AdS}_4$ to an intermediate Lifshitz region and finally to $\text{AdS}_2 \times \mathbb{R}^2$ in the deep IR in the Einstein-Maxwell-dilaton system with a four-derivative correction of the form

$$\delta L = \frac{3}{4} (\alpha + \beta e^{\lambda_2 \phi}) C^2_{\mu \nu \rho \sigma}. \tag{4.2}$$

The dilaton coupling $\beta$ is introduced to stabilize the dilaton, so that an emergent $\text{AdS}_2 \times \mathbb{R}^2$ may appear in the IR.

The existence of flows to $\text{AdS}_2 \times \mathbb{R}^2$ is not universal, but depends on the parameters $\alpha$, $\beta$, $\lambda_1$ and $\lambda_2$. For $\alpha < 1$, there is at most one irrelevant dilaton perturbation that can induce a flow from $\text{AdS}_2 \times \mathbb{R}^2$ in the deep IR to an intermediate Lifshitz region. We have presented a numerical example of such a flow for $\alpha = 0.9$. On the other hand, for $\alpha \geq 1$, if any irrelevant dilaton perturbations exist, then they necessarily come as a pair. Furthermore, in this case there are two possible dynamical exponents, $\tilde{z}_1$ and $\tilde{z}_2$ (where we take $\tilde{z}_1 < \tilde{z}_2$), allowed in the Lifshitz region. We have constructed a numerical example for $\alpha = 3$ that flows from $\text{AdS}_2 \times \mathbb{R}^2$ in the deep IR to an intermediate $\text{Lif}^{\tilde{z}_2}_4$. However, we were unable to find numerical flows to $\text{Lif}^{\tilde{z}_1}_4$. It remains unclear whether such flows are possible. From a simple counting of irrelevant perturbations, we expect that these flows should indeed exist. In any case, the natural question that arises is whether or not the additional irrelevant perturbation that appears for $\alpha \geq 1$ leads to an interesting geometry. To make a definitive statement about the flows that are allowed, a study of perturbations around the different Lifshitz backgrounds would be required. A similar analysis was carried out for the massive vector case in [25, 26].

\textsuperscript{7}They are, in fact, conformally Ricci flat.
It would also be interesting to see if one can find numerical solutions that interpolate between the two Lifshitz solutions.

It is also worth noting that, for a certain choice of parameters, we found irrelevant perturbations that oscillate around \( \text{AdS}_2 \times \mathbb{R}^2 \). We have discarded such cases, as our intuition from two-derivative theories suggests that this should be taken as a sign of a dynamical instability. However, we would have to perform a more detailed analysis to see whether the existence of such oscillatory perturbations actually leads to an instability. If in these cases the IR geometry is truly unstable, this raises the question of what the geometry would decay into and consequently what the true ground state of the theory is. Another source of instability that we did not consider here is the formation of striped phases [27–29].

Of course, it would be desirable to explore whether a realistic string model would lead to either \( \alpha' \) or string loop corrections of the form needed to resolve the Lifshitz horizon. The \( \alpha' \) corrections extend beyond the gravitational sector, and for example may include \( RF^2 \) terms at the four derivative level. Even in the gravitational sector, one would expect to have a more general form of the four-derivative corrections, similar to (2.13), but also with possible dilaton couplings. We expect that the mechanism to resolve the Lifshitz singularity in the IR will also work in the more general case with \( \alpha_R \neq 0 \) and \( \alpha_{GB} \neq 0 \). However, the smooth flow to \( \text{AdS}_4 \) in the UV observed here relies on the fact that the Weyl tensor vanishes quickly enough so that it does not source the dilaton for small \( r \). It is unclear whether the UV asymptotics would remain unchanged for generic higher derivative corrections.

In the case of an electrically charged brane considered here, there is another source of corrections that might modify the UV dynamics: Since the dilaton runs towards strong coupling, we expect quantum corrections to the gauge kinetic function \( f(\phi) \) to become important and modify the effective potential in this regime. In addition, there is a priori no reason why magnetic solutions should not be equally sensitive to \( \alpha' \)-corrections. We therefore expect our mechanism to be relevant also in the magnetic case. Since in this case the dilaton runs towards strong coupling in the IR, a consistent approach would be to consider both \( \alpha' \) and quantum corrections at the same time.

We expect that our analysis can be easily extended to geometries with hyperscaling violation. These backgrounds can be parametrized by a metric of the form

\[
d s_{d+1}^2 = e^{2\gamma r} \left( -e^{2z/L} dt^2 + e^{2r/L} d\vec{x}^2 + dr^2 \right).
\]

(4.3)

For \( \gamma \neq 0 \) this metric is invariant under the scale transformation (1.1) and (1.3) only up to a rescaling of \( ds \). One may construct solutions of this type by choosing an exponential potential for the dilaton, which as a result runs linearly with \( r \). Flows to \( \text{AdS}_2 \times \mathbb{R}^2 \) were constructed using a quantum corrected gauge kinetic function \( f(\phi) \) in [23].

Finally, although an emergent \( \text{AdS}_2 \times \mathbb{R}^2 \) geometry provides a non-singular resolution of the Lifshitz scaling solution, the presence of a non-contracting transverse \( \mathbb{R}^2 \) leads to non-zero entropy at zero temperature in the dual non-relativistic system. A more realistic situation where the entropy vanishes at zero temperature may potentially be obtained by flowing into \( \text{AdS}_4 \) in the deep IR. Thus one may imagine constructing flows from \( \text{AdS}_4 \) to Lifshitz to
AdS$_4$. This would be a special case of an AdS to AdS domain wall solution, in which case the holographic c-theorem would apply. It would be interesting to see whether such flows may be constructed in a toy model admitting AdS$_4$ solutions with two distinct AdS radii.

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A Metric ansatz and curvature

Here we provide the curvature components used in the derivation of the Lifshitz solution in section 2. Although not needed for the Lifshitz case, we consider slightly more general metrics of the form

\[ ds^2 = -e^{2b_0(r)} dt^2 + dr^2 + \sum_{i=1}^{d-1} e^{2b_i(r)} (dx^i)^2. \]  

(A.1)

The nonvanishing curvature terms are

\[ R^\rho\sigma_{\mu\nu} = \delta^\rho_{\nu} \eta_{\mu\sigma} b'_\rho b'_\sigma, \]  

(A.2)

\[ R^r_{\mu r\nu} = -\eta_{\mu\nu} e^{2b_\nu} \left(b''_\nu + (b'_\nu)^2\right) \]  

(A.3)

\[ R_{rr} = -\sum_\lambda (b''_\lambda + (b'_\lambda)^2), \]  

(A.4)

\[ R_{\mu\nu} = -\eta_{\mu\nu} e^{2b_\nu} \left(b''_\nu + b'_\nu \sum_\lambda b'_\lambda\right), \]  

(A.5)

\[ R = -\sum_\lambda \left(2b''_\lambda + (b'_\lambda)^2\right) - \left(\sum_\lambda b'_\lambda\right)^2, \]  

(A.6)

where $\mu, \nu = 0, \ldots, d-1$, and repeated indices are not summed over unless explicitly stated. The Lifshitz solution is given by

\[ b_0(r) = zr, \quad b_i(r) = r, \quad z > 1, \]  

(A.7)
and the case \(z = 1\) corresponds to \(AdS_{d+1}\). For this class of solutions, we have

\[
\begin{align*}
R_{00}^r &= z^2 e^{2z}, \\
R_{ij}^r &= -\delta_{ij} e^{2z}, \\
R_{00}^0 &= -z^2 + d - 1, \\
R_{ij}^0 &= -\delta_{ij} (z + d - 1) e^{2z}, \\
R &= -(z^2 + d - 1 + (z + d - 1)^2). \\
\end{align*}
\]

(A.8)

B Lifshitz solutions in alternative gauge

In our numerical analysis, we chose the parametrization (3.3) for the metric, which is different from (A.1). In this gauge, the Lifshitz metric of section 2 takes the form:

\[
ds^2 = \frac{1}{r^2} (-dt^2 + dr^2 + r^{2\tilde{z}} (dx^2 + dy^2)).
\]

(B.1)

The scaling parameters are related via \(z = (1 - \tilde{z})^{-1}\). Furthermore,

\[
\phi = \frac{4}{\lambda_1} (1 - \tilde{z}) \log r + C,
\]

(B.2)

\[
Q^2 e^{-\lambda_1 C} = \left(\frac{3}{2} - \tilde{z}\right) \tilde{z} \left(1 - 4\alpha \left(\tilde{z} - \frac{3}{4}\right)\right),
\]

(B.3)

\[
\Lambda = 2 \left(\frac{3}{2} - \tilde{z}\right) \tilde{z} \left(2 - \tilde{z}\right) + 4\alpha \tilde{z} \left(1 - \tilde{z}\right) \left(\frac{3}{4} - \tilde{z}\right),
\]

(B.4)

\[
\lambda_1^2 = \frac{4}{Q^2 e^{-\lambda_1 C}} \left(\frac{3}{2} - \tilde{z}\right) \left(1 - \tilde{z}\right) \left(1 - \frac{1 - \tilde{z}}{\tilde{z} \left(1 - \alpha \left(\tilde{z} - \frac{3}{4}\right)\right)}\right).
\]

(B.5)

It is straightforward to show that \(\lambda_1^2 (\tilde{z})\) has a local minimum at

\[
\tilde{z}_\pm = 1 \pm \frac{1}{2} \sqrt{1 - \frac{1}{\alpha}},
\]

(B.6)

provided that \(\alpha \geq 1\). In this case there are two different scaling parameters \(\tilde{z}_1 < \tilde{z}_2\) for any given \(\lambda_1^2\) (away from the minimum) (see Figure 8). Notice also that \(\lambda_1^2\) blows up for \(\tilde{z}_* = 3/4 + 1/(4\alpha)\), which is within the range of physical solutions for \(\alpha \geq 1\) only. To summarize, the possible ranges for the parameters are:

\[
\begin{align*}
\alpha < 1 : & \quad 0 \leq \lambda_1 < \infty, \quad 0 < \tilde{z} \leq 1, \\
\alpha \geq 1 : & \quad \lambda_{\text{min}} \leq \lambda_1 < \infty, \quad 0 < \tilde{z} < \tilde{z}_*,
\end{align*}
\]

(B.7)

where \(\lambda_{\text{min}} \equiv \lambda_1 (\tilde{z}_-).\)
C Irrelevant Perturbations

There are two ways in which the exponent of the dilaton perturbations, $\nu$ may become complex:

1. The smaller square-root in (3.33) becomes imaginary. This happens when

$$\lambda_2 = \frac{1}{\alpha \lambda_1} \left( - \left( \lambda_1 - \frac{2}{\sqrt{3}} \right)^2 + \frac{2}{3} \right). \tag{C.1}$$

2. Even if the small root is real-valued, $\tilde{\nu}^2$ may still cross zero, which happens at

$$\lambda_2 = \frac{1}{\lambda_1 (11 \alpha^2 - 19 \alpha + 8)} \left[ \frac{4}{3} (1 - \alpha) \lambda_1^2 + \frac{11}{8} \alpha - 1 \right.$$ 

$$\left. \pm \frac{3}{2} \left( (\alpha - 1)^2 \lambda_1^4 - \frac{1}{2} (11 \alpha^2 - 19 \alpha + 8) \lambda_1^2 + \left( \frac{11}{12} \alpha - \frac{2}{3} \right)^2 \right)^{\frac{1}{2}} \right]. \tag{C.2}$$

To find out when the dilaton perturbations are irrelevant, i.e. $\tilde{\nu}^2 = 1/4$, notice that $\tilde{\nu}^2 - 1/4$ can only change its sign as we go from case 1) to case 2) in (3.34). As a consequence, irrelevant perturbations will stay irrelevant as long as $\alpha \lambda_2 / \lambda_1 \geq 1$. In practice, it is therefore easiest to plot the curves (C.1)/(C.2) and determine the number of irrelevant perturbations numerically, making use of continuity arguments (see Figures 1 and 2).

References

[1] S. Kachru, X. Liu and M. Mulligan, \textit{Gravity Duals of Lifshitz-like Fixed Points}, Phys. Rev. D \textbf{78}, 106005 (2008) [arXiv:0808.1725 [hep-th]].
[2] M. Taylor, *Non-relativistic holography*, arXiv:0812.0530 [hep-th].

[3] K. Goldstein, S. Kachru, S. Prakash and S. P. Trivedi, *Holography of Charged Dilaton Black Holes*, JHEP **1008** (2010) 078 [arXiv:0911.3586 [hep-th]].

[4] M. Cadoni, G. D’Appollonio and P. Pani, *Phase Transitions Between Reissner-Nordstrom and Dilatonic Black Holes in 4D AdS Spacetime*, JHEP **1003** (2010) 100 [arXiv:0912.3520 [hep-th]].

[5] C.-M. Chen and D.-W. Pang, *Holography of Charged Dilaton Black Holes in General Dimensions*, JHEP **1006** (2010) 093 [arXiv:1003.5064 [hep-th]].

[6] C. Charmousis, B. Gouteraux, B. S. Kim, E. Kiritsis and R. Meyer, *Effective Holographic Theories for Low-Temperature Condensed Matter Systems*, JHEP **1011** (2010) 151 [arXiv:1005.4690 [hep-th]].

[7] E. Perlmutter, *Domain Wall Holography for Finite Temperature Scaling Solutions*, JHEP **1102** (2011) 013 [arXiv:1006.2124 [hep-th]].

[8] K. Goldstein, N. Iizuka, S. Kachru, S. Prakash and A. Westphal, *Holography of Dyonic Dilaton Black Branes*, JHEP **1010** (2010) 027 [arXiv:1007.2490 [hep-th]].

[9] S. Harrison, S. Kachru and H. Wang, *Resolving Lifshitz Horizons*, arXiv:1202.6635 [hep-th].

[10] K. Copsey and R. Mann, *Pathologies in Asymptotically Lifshitz Spacetimes*, JHEP **1103** (2011) 039 [arXiv:1011.3502 [hep-th]].

[11] G. T. Horowitz and B. Way, *Lifshitz Singularities*, Phys. Rev. D **85** (2012) 046008 [arXiv:1111.1243 [hep-th]].

[12] J. Tarrio and S. Vandoren, *Black Holes and Black Branes in Lifshitz Spacetimes*, JHEP **1109** (2011) 017 [arXiv:1105.6335 [hep-th]].

[13] M. R. Mohammadi Mozaffar and A. Mollabashi, *Holographic Quantum Critical Points in Lifshitz Space-Time*, JHEP **1304** (2013) 081 [arXiv:1212.6635 [hep-th]].

[14] A. Adams, A. Maloney, A. Sinha and S. E. Vazquez, *1/N Effects in Non-Relativistic Gauge-Gravity Duality*, JHEP **0903** (2009) 097 [arXiv:0812.0166 [hep-th]].

[15] E. Ayon-Beato, A. Garbarz, G. Giribet and M. Hassaine, *Lifshitz Black Hole in Three Dimensions*, Phys. Rev. D **80** (2009) 104029 [arXiv:0909.1347 [hep-th]].

[16] R.-G. Cai, Y. Liu and Y.-W. Sun, *A Lifshitz Black Hole in Four Dimensional $R^2$ Gravity*, JHEP **0910** (2009) 080 [arXiv:0909.2807 [hep-th]].

[17] D.-W. Pang, *On Charged Lifshitz Black Holes*, JHEP **1001** (2010) 116 [arXiv:0911.2777 [hep-th]].

[18] E. Ayon-Beato, A. Garbarz, G. Giribet and M. Hassaine, *Analytic Lifshitz Black Holes in Higher Dimensions*, JHEP **1004** (2010) 030 [arXiv:1001.2361 [hep-th]].

[19] M. H. Dehghani and R. B. Mann, *Lovelock-Lifshitz Black Holes*, JHEP **1007** (2010) 019 [arXiv:1004.4397 [hep-th]].

[20] M. H. Dehghani and R. B. Mann, *Thermodynamics of Lovelock-Lifshitz Black Branes*, Phys. Rev. D **82** (2010) 064019 [arXiv:1006.3510 [hep-th]].

[21] W. G. Brenna, M. H. Dehghani and R. B. Mann, *Quasi-Topological Lifshitz Black Holes*, Phys. Rev. D **84** (2011) 024012 [arXiv:1101.3476 [hep-th]].
[22] H. Lu, Y. Pang, C. N. Pope and J. F. Vazquez-Poritz, AdS and Lifshitz Black Holes in Conformal and Einstein-Weyl Gravities, Phys. Rev. D 86, 044011 (2012) [arXiv:1204.1062 [hep-th]].

[23] J. Bhattacharya, S. Cremonini and A. Sinkovics, On the IR completion of geometries with hyperscaling violation, JHEP 1302, 147 (2013) [arXiv:1208.1752 [hep-th]].

[24] N. Bao, X. Dong, S. Harrison and E. Silverstein, The Benefits of Stress: Resolution of the Lifshitz Singularity, Phys. Rev. D 86, 106008 (2012) [arXiv:1207.0171 [hep-th]].

[25] H. Braviner, R. Gregory and S. F. Ross, Flows involving Lifshitz solutions, Class. Quant. Grav. 28, 225028 (2011) [arXiv:1108.3067 [hep-th]].

[26] J. T. Liu and Z. Zhao, Holographic Lifshitz Flows and the Null Energy Condition, arXiv:1206.1047 [hep-th].

[27] A. Donos, J. P. Gauntlett and C. Pantelidou, Spatially Modulated Instabilities of Magnetic Black Branes, JHEP 1201 (2012) 061 [arXiv:1109.0471 [hep-th]].

[28] A. Donos, J. P. Gauntlett and C. Pantelidou, Magnetic and Electric AdS Solutions in String- and M-theory, Class. Quant. Grav. 29 (2012) 194006 [arXiv:1112.4195 [hep-th]].

[29] S. Cremonini and A. Sinkovics, Spatially Modulated Instabilities of Geometries with Hyperscaling Violation, arXiv:1212.4172 [hep-th].