A 3 × 3 matrix spectral problem for AKNS hierarchy and its binary Nonlinearization

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Abstract

A three-by-three matrix spectral problem for AKNS soliton hierarchy is proposed and the corresponding Bargmann symmetry constraint involved in Lax pairs and adjoint Lax pairs is discussed. The resulting nonlinearized Lax systems possess classical Hamiltonian structures, in which the nonlinearized spatial system is intimately related to stationary AKNS flows. These nonlinearized Lax systems also lead to a sort of involutive solutions to each AKNS soliton equation.

1 Introduction

Symmetry constraints have aroused an increasing interest in recent few years due to the important roles they play in soliton theory. Such a kind of very successful symmetry constraint method is the nonlinearization technique for Lax pairs of soliton hierarchies, including mono-nonlinearization proposed by Cao and Geng [7] [8] and further binary nonlinearization [18] [17] [14].

In general, one considers the complicated nonlinear problems to be solved in such a way to break nonlinear problems into several linear or smaller ones and then to solve these resulting problems. It is following this idea that one has introduced the method of Lax pair to study nonlinear soliton equations. The Lax pairs are always linear with respect to their eigenfunctions. Nevertheless, the nonlinearization technique puts this original object, the Lax pair, into a nonlinear and more complicated object, the nonlinearized Lax system. It seems to be not reasonable enough, but in fact, it provides an effective way, different from the usual one, to solve soliton equations. The main reason why the nonlinearization technique takes effect is that kind of specific symmetry constraints expressed through the variational derivative of the potential.

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A similar symmetry constraint procedure for bi-Hamiltonian soliton hierarchies is presented by Antonowicz and Wojciechowski et al \[2\] \[3\] \[22\] and bi-Hamiltonian structures for the resulting classical integrable systems can also be worked out through a Miura map \[3\] \[6\]. A connection between these systems and stationary flows \[5\] is also given by Tondo \[24\] for the case of KdV hierarchy. Because stationary flows may be interpreted as finite dimensional Hamiltonian systems \[1\] based upon the so-called Jacobi-Ostrogradsky coordinates \[16\], a natural generalization of nonlinearization technique to higher order symmetry constraints is made by Zeng \[27\] \[28\] for the KdV and Kaup-Newell hierarchies etc. There have also been some algebraic geometric tricks, proposed by Flaschka et al \[11\] \[1\] \[23\], to deal with similar nonlinearized Lax pairs called Neumann systems.

The study of the nonlinearization theory leads to a large class of interesting finite dimensional Liouville integrable Hamiltonian systems which are connected with soliton hierarchies (for example, see \[8\] \[15\]). However in the literature, most results are presented for the cases of 2 \(\times\) 2 matrix spectral problems. The present paper is devoted to the symmetry constraints in binary nonlinearization for a case of 3 \(\times\) 3 matrix spectral problems. We successfully propose a 3 \(\times\) 3 matrix spectral problem for AKNS soliton hierarchy, motivated by a representation of 3 \(\times\) 3 matrices for the Lie algebra sl(2). Then in Section 3, we consider the Bargmann symmetry constraint for the proposed new Lax pairs and adjoint Lax pairs of AKNS soliton hierarchy. In Section 4, we analyze the nonlinearized Lax systems, especially the nonlinearized temporal systems, and establish a sort of involutive solutions to AKNS soliton equations. Finally in Section 5, some remarks are given.

### 2 New Lax pairs for AKNS equations

We introduce a three-by-three matrix spectral problem

\[
\phi_x = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}_x = U(u, \lambda) \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} -2\lambda & \sqrt{2}q & 0 \\ \sqrt{2}r & 0 & \sqrt{2}q \\ 0 & \sqrt{2}r & 2\lambda \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \tag{2.1}
\]

where the potential \(u = (q, r)^T\). Its adjoint spectral problem reads as

\[
\psi_x = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}_x = -U^T(u, \lambda) \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} 2\lambda & -\sqrt{2}r & 0 \\ -\sqrt{2}q & 0 & -\sqrt{2}r \\ 0 & -\sqrt{2}q & -2\lambda \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \tag{2.2}
\]

Here \(T\) means the transposition of the matrix. Our purpose is to generate AKNS hierarchy of soliton equations from the above specific spectral problem \(2.1\). To this end, we first solve the adjoint representation equation \(V_x = [U, V]\). Take

\[
V = \begin{pmatrix} 2a & \sqrt{2}b & 0 \\ \sqrt{2}c & 0 & \sqrt{2}b \\ 0 & \sqrt{2}c & -2a \end{pmatrix} = \sum_{i=0}^{\infty} \begin{pmatrix} 2a_i & \sqrt{2}b_i & 0 \\ \sqrt{2}c_i & 0 & \sqrt{2}b_i \\ 0 & \sqrt{2}c_i & -2a_i \end{pmatrix} \lambda^{-i} \tag{2.3}
\]
and then we have
\[
[U, V] = \begin{pmatrix}
2(qc - rb) & -2\sqrt{2}(\lambda b + qa) & 0 \\
2\sqrt{2}(ra + \lambda c) & 0 & -2\sqrt{2}(\lambda b + qa) \\
0 & 2\sqrt{2}(ra + \lambda c) & -2(qc - rb)
\end{pmatrix}.
\]
Therefore we easily find that the adjoint representation equation \( V_x = [U, V] \) be-
comes

\[
a_x = qc - rb, \ b_x = -2\lambda b - 2qa, \ c_x = 2\lambda c + 2ra,
\]
which is equivalent to

\[
a_{ix} = qc_i - rb_i, \ b_{ix} = -2b_{i+1} - 2qa_i, \ c_{ix} = 2c_{i+1} + 2ra_i, \ i \geq 0. \quad (2.4)
\]
We fix the initial values

\[
a_0 = -1, \ b_0 = c_0 = 0 \quad (2.5)
\]
and require that

\[
a_{i|u=0} = b_{i|u=0} = c_{i|u=0} = 0, \ i \geq 1, \quad (2.6)
\]
which equivalently select constants of integration to be zero. On the other hand, the above equality (2.4) gives rise to the recursion relation for determining \( a_i, b_i, c_i \):

\[
\begin{align*}
& a_{i+1} = \frac{1}{2}\partial^{-1} (qc_{ix} + rb_{ix}), \\
& b_{i+1} = -\frac{1}{2}b_{ix} - qa_i, \\
& c_{i+1} = \frac{1}{2}c_{ix} - ra_i,
\end{align*} \quad i \geq 0. \quad (2.7)
\]
This recursion relation uniquely determines infinitely many sets of polynomials \( a_i, b_i, c_i, \ i \geq 1 \), in \( u, u_x, \cdots \) under the requirement (2.6). The first two sets are as follows

\[
a_1 = 0, \ b_1 = q, \ c_1 = r; \ a_2 = \frac{1}{2}(qr), \ b_2 = -\frac{1}{2}qx, \ c_2 = \frac{1}{2}rx.
\]
In addition, we have

\[
a^2 + bc = \left( \sum_{i=0}^{\infty} a_i \lambda^{-i} \right)^2 + \left( \sum_{i=0}^{\infty} b_i \lambda^{-i} \right) \left( \sum_{i=0}^{\infty} c_i \lambda^{-i} \right) = 1,
\]
because \( (a^2 + bc)_x = \frac{1}{8} \text{tr}(V^2)_x = \frac{1}{8} \text{tr}[U, V^2] = 0 \) and \( (a^2 + bc)|_{u=0} = 1 \). It follows that \( a_i, b_i, c_i, \ i \geq 1, \) are local.

A direct computation may show that the compatibility conditions of the Lax pairs

\[
\phi_x = U\phi, \ \phi_{tn} = V^{(n)}\phi, \ V^{(n)} = V^{(n)}(u, \lambda) = (\lambda^n V)_+, \ n \geq 0, \quad (2.8)
\]
or the adjoint Lax pairs

\[
\psi_x = -U^T \psi, \ \psi_{tn} = -(V^{(n)})^T \psi, \ n \geq 0, \quad (2.9)
\]
where the symbol + denotes the choice of non-negative power of $\lambda$, engenders a hierarchy of AKNS soliton equations

$$ u_{t_n} = \begin{pmatrix} q \\ r \end{pmatrix}, \quad K_n = \begin{pmatrix} -2b_{n+1} \\ 2c_{n+1} \end{pmatrix} = JL_n \begin{pmatrix} r \\ q \end{pmatrix}, \quad n \geq 0, \quad (2.10) $$

where the Hamiltonian operator $J$ and the recursion operator $L$ read as

$$ J = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} \frac{1}{2}\partial - r\partial^{-1}q & r\partial^{-1}r \\ -q\partial^{-1}q & -\frac{1}{2}\partial + q\partial^{-1}r \end{pmatrix}. \quad (2.11) $$

This AKNS hierarchy is exactly the same as one in Ref. [18], which also shows that the same soliton hierarchy may possess different Lax pairs, even different order spectral matrices. Here the operator $L^*$ is a hereditary operator [13], and $J$ and $JL$ constitute a Hamiltonian pair.

Finally, we would like to elucidate the other two properties on AKNS hierarchy (2.10). First by Corollary 2.1 of Ref. [18], we can obtain

$$ V_{t_n} = [V^{(n)}], \quad n \geq 0 \quad (2.12) $$

when $u_{t_n} = K_n$, i.e. $U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0, \quad n \geq 0$. Second, we can get the Hamiltonian structure of AKNS hierarchy

$$ u_{t_n} = K_n = J \begin{pmatrix} c_{n+1} \\ b_{n+1} \end{pmatrix} = J\delta H_n \delta u, \quad H_n = \frac{2}{n+1}a_{n+2}, \quad n \geq 0, \quad (2.13) $$

by applying the trace identity [25] [26].

3 Binary nonlinearization related to new spectral problem

In order to impose the Bargmann symmetry constraint in binary nonlinearization, we first need to compute the variational derivative of the spectral parameter $\lambda$ with to the potential $u$, which is shown in the following Lemma [12] [18].

**Lemma 3.1** Let $U(u, \lambda)$ be a matrix of order $s$ depending on $u, u_x, \cdots$ and a parameter $\lambda$. Suppose that $\phi = (\phi_1, \phi_2, \cdots, \phi_s)^T, \quad \psi = (\psi_1, \psi_2, \cdots, \psi_s)^T$ satisfy the spectral problem and the adjoint spectral problem

$$ \phi_x = U(u, \lambda)\phi, \quad \psi_x = -U^T(u, \lambda)\psi, $$

and set the matrix $\tilde{V} = \phi\psi^T = (\phi_k\psi_l)_{s \times s}$, then we have the following two results:

(i) the variational derivative of the spectral parameter $\lambda$ with respect to the potential $u$ may be expressed by

$$ \frac{\delta \lambda}{\delta u} = \frac{\text{tr}(\tilde{V}\frac{\partial \tilde{V}}{\partial u})}{-\int_{-\infty}^{\infty} \text{tr}(\tilde{V}\frac{\partial \tilde{V}}{\partial x})dx}, \quad (3.1) $$


(ii) the matrix $\tilde{V}$ is a solution to the adjoint representation equation $V_x = [U, V]$, i.e. $\tilde{V}_x = [U, V]$.

Following (3.1), we have the variational derivative of the spectral parameter for the spectral problem (2.1) and the adjoint spectral problem (2.2)

$$\frac{\delta \lambda}{\delta q} = \frac{\sqrt{2}}{E} (\phi_2 \psi_1 + \phi_3 \psi_2), \quad \frac{\delta \lambda}{\delta r} = \frac{\sqrt{2}}{E} (\phi_1 \psi_2 + \phi_2 \psi_3),$$

where $E = 2 \int_{-\infty}^{\infty} (\phi_1 \psi_1 - \phi_3 \psi_3) dx$.

Let us introduce $N$ ($N \geq 1$) distinct eigenvalues $\lambda_j$, $1 \leq j \leq N$, and denote by

$$\phi^{(j)} = (\phi_{1j}, \phi_{2j}, \phi_{3j})^T, \quad \psi^{(j)} = (\psi_{1j}, \psi_{2j}, \psi_{3j})^T, \quad 1 \leq j \leq N,$$

the eigenfunctions of (2.8) and the adjoint eigenfunctions of (2.9), i.e.

$$\phi^{(j)}_x = U(u, \lambda_j) \phi^{(j)}, \quad \psi^{(j)}_x = -U^T(u, \lambda_j) \psi^{(j)}, \quad 1 \leq j \leq N,$$

$$\phi^{(j)}_t = V^{(n)}(u, \lambda_j) \phi^{(j)}; \quad \psi^{(j)}_t = -V^{(n)}(u, \lambda_j) \psi^{(j)}, \quad 1 \leq j \leq N.$$ (3.4)

Now we make the Bargmann symmetry constraint

$$K_0 = J \frac{\delta H_0}{\delta u} = J \sum_{j=1}^{N} \mu_j E_j \frac{\delta \lambda_j}{\delta u},$$

where $E_j = 2 \int_{-\infty}^{\infty} (\phi_{1j} \psi_{1j} - \phi_{3j} \psi_{3j}) dx$, $1 \leq j \leq N$, and $\mu_j, 1 \leq j \leq N$, are any nonzero constants. By (3.2), this symmetry constraint becomes

$$K_0 = J \sum_{j=1}^{N} \mu_j \left( \frac{\sqrt{2} (\phi_{2j} \psi_{1j} + \phi_{3j} \psi_{2j})}{\sqrt{2} (\phi_{1j} \psi_{2j} + \phi_{2j} \psi_{3j})} \right),$$

from which we get the following explicit expression for the potential $u$

$$u = f(P_1, P_2, P_3; Q_1, Q_2, Q_3) = \sqrt{2} \left( <P_1, BQ_2> + <P_2, BQ_3> - <P_2, BQ_1> + <P_3, BQ_2> \right).$$

Here and hereafter, $<\cdot, \cdot>$ denotes the standard inner product of $\mathbb{R}^N$ and

$$B = \text{diag}(\mu_1, \ldots, \mu_N), \quad \left( \begin{array}{c} P_i \\ Q_i \end{array} \right) = \left( \begin{array}{c} \phi_{1i}, \phi_{2i}, \cdots, \phi_{Ni} \\ \psi_{1i}, \psi_{2i}, \cdots, \psi_{Ni} \end{array} \right)^T, \quad i = 1, 2, 3.$$ (3.7)

The substitution of (3.6) into the spatial system (3.3) and the temporal systems (3.4) for $n \geq 0$ yields the nonlinearized spatial system:

$$\begin{cases}
\left( \begin{array}{c} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{array} \right)_x = U(f, \lambda_j) \left( \begin{array}{c} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{array} \right), & j = 1, 2, \cdots, N, \\
\left( \begin{array}{c} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{array} \right)_x = -U^T(f, \lambda_j) \left( \begin{array}{c} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{array} \right), & j = 1, 2, \cdots, N;
\end{cases}$$ (3.8)
and the nonlinearized temporal systems for \( n \geq 0 \):

\[
\begin{align*}
\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}_{t_n} &= V^{(n)}(f, \lambda_j) \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}, \quad j = 1, 2, \ldots, N, \\
\begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{pmatrix}_{t_n} &= -(V^{(n)})^T(f, \lambda_j) \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{pmatrix}, \quad j = 1, 2, \ldots, N.
\end{align*}
\tag{3.9}
\]

It is obvious that (3.8) is a system of ordinary differential equations and (3.9) is a hierarchy of partial differential equations.

Suppose that \( Z \) is an expression depending on \( u \) and its differentials. From now on we use \( \tilde{Z} \) to denote the expression of \( Z \) depending on \( P_i, Q_i, \quad 1 \leq i \leq 3 \), and their differentials after substituting (3.8) into \( Z \), and use \( \text{Or}(\tilde{Z}) \) to denote the expression of \( \tilde{Z} \) only depending on \( P_i, Q_i, \quad 1 \leq i \leq 3 \), themselves after substituting (3.8) into \( \tilde{Z} \) sufficiently many times. Therefore (3.9) may be transformed into the following systems for \( n \geq 0 \):

\[
\begin{align*}
\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}_{t_n} &= \text{Or}(V^{(n)}(f, \lambda_j)) \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}, \quad j = 1, 2, \ldots, N, \\
\begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{pmatrix}_{t_n} &= -(\text{Or}(V^{(n)}(f, \lambda_j)))^T \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{pmatrix}, \quad j = 1, 2, \ldots, N,
\end{align*}
\tag{3.10}
\]

which are all ordinary differential equations with an independent variable \( t_n \) because the matrices \( \text{Or}(V^{(n)}(f, \lambda_j)) \), \( n \geq 0 \), \( 1 \leq j \leq N \), only depend on \( P_i, Q_i, \quad 1 \leq i \leq 3 \).

We would like to discuss the integrability on the nonlinearized spatial system (3.8) and the nonlinearized temporal systems (3.10) for \( n \geq 0 \) in the Liouville sense (2.1). We shall utilize the symplectic structure \( \omega^2 \) on \( \mathbb{R}^{6N} \)

\[
\omega^2 = \sum_{i=0}^{3} \sum_{j=0}^{N} \mu_j d\phi_{ij} \wedge d\psi_{ij} = \sum_{i=0}^{3} (BdP_i) \wedge dQ_i,
\]

by which one can define the corresponding Poisson bracket for two functions \( F, G \) defined over the phase space \( \mathbb{R}^{6N} \)

\[
\{F, G\} = \omega^2(IdG, IdF) = \omega^2(X_G, X_F) = \sum_{i=1}^{3} \sum_{j=1}^{N} \mu_j^{-1} \left( \frac{\partial F}{\partial \psi_{ij}} \frac{\partial G}{\partial \phi_{ij}} - \frac{\partial F}{\partial \phi_{ij}} \frac{\partial G}{\partial \psi_{ij}} \right) = \sum_{i=1}^{3} \left( \frac{\partial F}{\partial Q_i} B^{-1} \frac{\partial G}{\partial P_i} - \frac{\partial F}{\partial P_i} B^{-1} \frac{\partial G}{\partial Q_i} \right), \tag{3.11}
\]
where $IdH = X_H$ denotes the Hamiltonian vector field with energy $H$ determined by
\[ \omega^2(X, IdH) = \omega^2(X, X_H) = dH(X), \quad X \in T(\mathbb{R}^N), \]
and the corresponding Hamiltonian system with the Hamiltonian function $H$
\[ \dot{x} = IdH(x) = dx(IdH) = \omega^2\{IdH, Idx\} = \{x, H\}, \quad x \in \mathbb{R}^N, \quad (3.12) \]
which possesses an explicit formulation
\[ \dot{P}_i = -B^{-1} \frac{\partial H}{\partial Q_i}, \quad \dot{Q}_i = B^{-1} \frac{\partial H}{\partial P_i}, \quad i = 1, 2, 3. \quad (3.13) \]

Note that there are some authors who use the other Poisson bracket $\{F, G\} = \omega^2(X_F, X_G)$. As remarked by Carroll [9], it doesn’t matter of course but each type has many proponents and hence one must be careful of minus signs in reading various sources. The notation we accept here is the Arnold’s one [4].

**Theorem 3.1** The following functions
\[ \bar{F}_j = \sum_{i=1}^{3} \phi_{ij} \psi_{ij}, \quad 1 \leq j \leq N, \quad (3.14) \]
are all integrals of motion for the nonlinearized spatial system (3.8). Moreover they are in involution under the Poisson bracket (3.11) and independent over the region
\[ \Omega = \{ \mathbb{R}^N | \phi_{ij}, \psi_{ij} \in \mathbb{R}, \sum_{i=1}^{3} (\phi_{ij}^2 + \psi_{ij}^2) \neq 0, \quad 1 \leq j \leq N \}. \]

**Proof:** Let
\[ \bar{V}(\lambda_j) = (\phi_{kj} \psi_{lj})_{k,l=1,2,3}, \quad 1 \leq j \leq N. \]
We can first find that
\[ \bar{F}_j = \text{tr}(\bar{V}(\lambda_j)). \]
On the other hand, by Lemma 3.1 we know that $\bar{V}(\lambda_j)$ satisfies
\[ \bar{V}(\lambda_j)_x = [U(\bar{u}, \lambda_j), \bar{V}(\lambda_j)] \]
when (3.8) holds, and thus
\[ \bar{F}_j = (\text{tr}(\bar{V}(\lambda_j)))_x = \text{tr}(\bar{V}(\lambda_j))_x = \text{tr}[U(\bar{u}, \lambda_j), \bar{V}(\lambda_j)] = 0, \]
which shows that $\bar{F}_j, 1 \leq j \leq N$, are all integrals of motion for the nonlinearized spatial system (3.8). In addition, it is very easy to prove that
\[ \{\bar{F}_k, \bar{F}_l\} = 0, \quad 1 \leq k, l \leq N, \]
which means $\bar{F}_j$, $1 \leq j \leq N$, are in involution. It is also obvious that $\text{grad} \bar{F}_j$, $1 \leq j \leq N$, are everywhere linear independent over $\Omega$ by observing that

\[
\left( \frac{\partial \bar{F}_k}{\partial \phi_{il}} \right)_{k,l=1,\ldots,N} = \begin{pmatrix}
\psi_{i1} & 0 \\
\psi_{i2} & \ddots \\
0 & \ddots & \psi_{iN}
\end{pmatrix}, \quad i = 1, 2, 3,
\]

\[
\left( \frac{\partial \bar{F}_k}{\partial \psi_{il}} \right)_{k,l=1,\ldots,N} = \begin{pmatrix}
\phi_{i1} & 0 \\
\phi_{i2} & \ddots \\
0 & \ddots & \phi_{iN}
\end{pmatrix}, \quad i = 1, 2, 3.
\]

The proof is completed. $\blacksquare$

Throughout our paper, we assume that

\[
A = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N). \tag{3.15}
\]

If all elements $Z_{ij}$, $1 \leq i, j \leq s$ of a given matrix $Z = (Z_{ij})_{s \times s}$ are polynomials in $\lambda$, i.e. $Z_{ij} = \sum_{k=0}^{m} Z_{ik} \lambda^k$, for convenience of presentation, we define a new matrix called $M_A(Z)$ as follows

\[
M_A(Z) = (\sum_{k=0}^{m} Z_{ik} A^k)_{sN \times sN}. \tag{3.16}
\]

Moreover we often accept compact forms, for example,

\[
\frac{\partial}{\partial P_i} = (\frac{\partial}{\partial \phi_{i1}}, \ldots, \frac{\partial}{\partial \phi_{iN}})^T, \quad \{P_i, H\} = (\{\phi_{i1}, H\}, \ldots, \{\phi_{iN}, H\})^T, \quad i = 1, 2, 3.
\]

**Theorem 3.2** We have the explicit integrals of motion for the nonlinearized spatial system (3.8):

\[
\begin{align*}
F_1 &= -8(<P_1, BQ_1> - <P_3, BQ_3>), \\
F_m &= 4 \sum_{i=1}^{m-1} [(<A^{i-1} P_1, BQ_1> - <A^{i-1} P_3, BQ_3>) \times \\
&\quad (<A^{m-i-1} P_1, BQ_1> - <A^{m-i-1} P_3, BQ_3>) \\
&\quad + 2(<A^{i-1} P_1, BQ_2> + <A^{i-1} P_2, BQ_3>) \times \\
&\quad (<A^{m-i-1} P_2, BQ_1> + <A^{m-i-1} P_3, BQ_2>) ] \\
&\quad - 8(<A^{m-1} P_1, BQ_1> - <A^{m-1} P_3, BQ_3>), \quad m \geq 2,
\end{align*}
\]

where $P_i$, $Q_i$, $B$ are defined by (3.7). Moreover they constitute an involutive system together with $\bar{F}_j$, $1 \leq j \leq N$, under the Poisson bracket (3.11), i.e.

\[
\{F_k, F_l\} = \{F_m, \bar{F}_j\} = 0, \quad m, k, l \geq 1, \quad 1 \leq j \leq N.
\]
Further we choose that
\[ \hat{q} = \sqrt{2}(P_1, BQ_2) + (P_2, BQ_3), \quad \hat{r} = \sqrt{2}(P_2, BQ_1) + (P_3, BQ_2); \]
\[ \hat{a}_0 = -1, \quad \hat{b}_0 = \hat{c}_0 = 0; \]
\[ \begin{aligned} \hat{a}_{i+1} &= \langle A^i P_1, BQ_1 \rangle - \langle A^i P_3, BQ_3 \rangle, \quad i \geq 0, \\
\hat{b}_{i+1} &= \sqrt{2}(P_1, BQ_1) + \langle A^i P_2, BQ_3 \rangle, \quad i \geq 0, \\
\hat{c}_{i+1} &= \sqrt{2}(P_2, BQ_1) + \langle A^i P_3, BQ_2 \rangle, \quad i \geq 0. \end{aligned} \]  

\[ \text{(3.18)} \]

Further we choose that
\[ \hat{U} = \begin{pmatrix} -2\lambda & \sqrt{2}\hat{q} & 0 \\ \sqrt{2}\hat{r} & 0 & \sqrt{2}\hat{r} \\ 0 & \sqrt{2}\hat{r} & 2\lambda \end{pmatrix}, \quad \hat{V} = \begin{pmatrix} 2\hat{a} & \sqrt{2}\hat{b} & 0 \\ \sqrt{2}\hat{c} & 0 & \sqrt{2}\hat{b} \\ 0 & \sqrt{2}\hat{c} & -2\hat{a} \end{pmatrix}, \]

where \( \hat{a}, \hat{b} \) and \( \hat{c} \) are defined by
\[ \hat{a} = \sum_{i=0}^{\infty} \hat{a}_i \lambda^{-i}, \quad \hat{b} = \sum_{i=0}^{\infty} \hat{b}_i \lambda^{-i}, \quad \hat{c} = \sum_{i=0}^{\infty} \hat{c}_i \lambda^{-i}. \]

It may be shown that when the nonlinearized spatial system (3.8) holds, we have
\[ \hat{V}_x = [\hat{U}, \hat{V}], \quad \text{i.e.} \quad \hat{a}_x = \hat{q} \hat{c} - \hat{r} \hat{b}, \quad \hat{b}_x = -2\lambda \hat{b} - 2\hat{q} \hat{a}, \quad \hat{c}_x = 2\lambda \hat{c} + 2\hat{r} \hat{a}. \]

Therefore we can compute that
\[ \hat{F}_x := \left( \frac{1}{2} \text{tr}(\hat{V}^2) \right)_x = \frac{1}{2} \text{tr}(\hat{V}^2)_x = \frac{1}{2} \text{tr}[\hat{U}, \hat{V}^2] = 0. \]

On the other hand, we have
\[ \hat{F} = 4(\hat{a}^2 + \hat{b} \hat{c}) = \sum_{m=0}^{\infty} F_m \lambda^{-m}, \quad F_0 = 4, \quad F_m = 4 \sum_{i=0}^{m} (\hat{a}_i \hat{a}_{m-i} + \hat{b}_i \hat{c}_{m-i}), \quad m \geq 1. \]

Hence \( F_m, \ m \geq 1, \) are all integrals of motion for the nonlinearized spatial system (3.8).

Now we turn to the involutivity of integrals of motion. We take
\[ \hat{V}^{(n)}(\lambda) = (\lambda^n \hat{V})_+ \]
and construct a temporal system for \( n \geq 0 \)
\[ \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}_{t_n} = M_A(\hat{V}^{(n)}) \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}_{t_n}, \quad \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}_{t_n} = -(M_A(\hat{V}^{(n)}))^T \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}, \]  

\[ \text{(3.19)} \]

where \( M_A(\hat{V}^{(n)}), \ n \geq 0, \) are determined in the way of (3.16). We can first prove that when this system (3.19) holds, we have
\[ (\hat{V}(\lambda))_{t_n} = [\hat{V}^{(n)}(\lambda), \hat{V}(\lambda)]. \]
Therefore $F_m$, $m \geq 1$, are also integrals of motion for the system \((3.19)\). Secondly, we can verify that
\[
\begin{pmatrix}
B^{-1} \frac{\partial F}{\partial P_3} \\
B^{-1} \frac{\partial F}{\partial P_2} \\
B^{-1} \frac{\partial F}{\partial P_1}
\end{pmatrix} = \begin{pmatrix}
B^{-1} \text{tr}(\dot{V} \frac{\partial}{\partial P_3} \dot{V}) \\
B^{-1} \text{tr}(\dot{V} \frac{\partial}{\partial P_2} \dot{V}) \\
B^{-1} \text{tr}(\dot{V} \frac{\partial}{\partial P_1} \dot{V})
\end{pmatrix} = \sum_{m=0}^{\infty} (M_A(\dot{V}^{(m)}))^T \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} \lambda^{-m-1},
\]
These two equalities show that the system \((3.19)\) for $n \geq 0$ are all Hamiltonian systems with Hamiltonian functions $-F_{n+1}$. Therefore
\[
\{F_{m+1}, -F_n\} = \frac{d}{dt} F_{m+1} = 0, \ m, n \geq 0,
\]
which shows the involutivity of $F_m$, $m \geq 1$. In addition, it is easy to get that
\[
\{F_k, \bar{F}_l\} = \sum_{i=0}^{3} \mu_i^{-1} \left( \frac{\partial F_k}{\partial \psi_{il}} \psi_{il} - \frac{\partial F_k}{\partial \phi_{il}} \phi_{il} \right) = 0, \ k \geq 1, \ 1 \leq l \leq N,
\]
noting the particular form of $F_k$, $k \geq 1$. The proof is finished. 

Because we have $\bar{V}^2(\lambda_j) = \text{tr}(\bar{V}(\lambda_j))\bar{V}(\lambda_j), 1 \leq j \leq N$, $\dot{V}^3 = \frac{1}{2} \text{tr}(\dot{V}^2)\dot{V}$, we cannot obtain new integrals of motion of the nonlinearized spatial system \((3.8)\) from the trace of other power of $\bar{V}(\lambda_j)$ and $\dot{V}$. In addition to this, it is interesting to observe that the determinants of the matrices $\bar{V}(\lambda_j), 1 \leq j \leq N$, and $\dot{V}$ are all zero.

The nonlinearized spatial system is easily rewritten as an Hamiltonian system
\[
P_{ix} = \{P_i, H\} = -B^{-1} \frac{\partial H}{\partial Q_i}, \ Q_{ix} = \{Q_i, H\} = B^{-1} \frac{\partial H}{\partial P_i}, \ i = 1, 2, 3
\]
with the Hamiltonian function
\[
H = 2(<AP_1, BQ_1> - <AP_3, BQ_3>) \\
-2(<P_1, BQ_2> + <P_2, BQ_3>) <P_2, BQ_1> + <P_3, BQ_2>) = -\frac{1}{4} F_2 + \frac{1}{64} F_1^2.
\]
Thus it possesses the following $3N$ involutive integrals of motion
\[
\bar{F}_j, \ 1 \leq j \leq N, \ F_m, \ 1 \leq m \leq 2N.
\]
In some special cases, they may be shown to be independent at least on certain region.
Theorem 3.3 When $N = 1, 2$, the determinant of the matrix

$$D(N) = \begin{pmatrix}
(\text{grad}_{P_1} F_1)^T & (\text{grad}_{P_2} F_1)^T & (\text{grad}_{P_3} F_1)^T \\
\vdots & \vdots & \vdots \\
(\text{grad}_{P_1} F_N)^T & (\text{grad}_{P_2} F_N)^T & (\text{grad}_{P_3} F_N)^T \\
(\text{grad}_{P_1} F_1)^T & (\text{grad}_{P_2} F_1)^T & (\text{grad}_{P_3} F_1)^T \\
\vdots & \vdots & \vdots \\
(\text{grad}_{P_1} F_{2N})^T & (\text{grad}_{P_2} F_{2N})^T & (\text{grad}_{P_3} F_{2N})^T
\end{pmatrix}$$

with $\text{grad}_{P_i} G = (\frac{\partial G}{\partial \phi_{i1}}, \ldots, \frac{\partial G}{\partial \phi_{iN}})^T$, $1 \leq i \leq 3$, is not always zero. Thus the integrals of motion $\bar{F}_j$, $1 \leq j \leq N$, $F_m$, $1 \leq m \leq 2N$, are independent at least on some region.

Proof: We have

$$D(1) = (16\phi_{11}\psi_{11}^3 + 16\phi_{31}\psi_{21}\psi_{31} - 32\phi_{11}\psi_{11}^2\psi_{21}\psi_{31} + 32\phi_{21}\psi_{11}\psi_{21}^2\psi_{31} - 32\phi_{31}\psi_{11}\psi_{21}\psi_{31}^2 - 64\phi_{21}\psi_{11}^2\psi_{21}^2\psi_{31}^2)\mu_1^3$$

and

$$D(2) = (-256\lambda_1^3 + 256\lambda_2^3 - 768\lambda_1\lambda_2^2 + 768\lambda_1^2\lambda_2)\mu_1^3\mu_2^4,$$

when we choose

$$\phi_{11} = 1, \phi_{12} = 0, \phi_{21} = 0, \phi_{22} = -1, \phi_{31} = 0, \phi_{32} = 0, \psi_{11} = 0, \psi_{12} = -1, \psi_{21} = 1, \psi_{22} = 0, \psi_{31} = 0, \psi_{32} = -1.$$ 

These are consequences of computation by the computer algebra system MuPAD. But they may also be shown by some direct computation. The proof is completed.

According to the above theorem, the nonlinearized spatial system (3.8) is Liouville integrable on some region of the phase space, when $N = 1, 2$.

4 Involutive solutions

The aim of this section is to discuss some properties on the nonlinearized spatial system (3.8) and the nonlinearized spatial system (3.10) and to establish a kind of involutive solutions with separated variables for AKNS soliton equations.

Lemma 4.1 When $\phi = (\phi_1, \phi_2, \phi_3)^T$ and $\psi = (\psi_1, \psi_2, \psi_3)^T$ satisfy the spectral problem (2.7) and the adjoint spectral problem (2.3), we have

$$L\begin{pmatrix}
\phi_2\psi_1 + \phi_3\psi_2 \\
\phi_1\psi_2 + \phi_2\psi_3
\end{pmatrix} = \lambda\begin{pmatrix}
\phi_2\psi_1 + \phi_3\psi_2 \\
\phi_1\psi_2 + \phi_2\psi_3
\end{pmatrix} + I\begin{pmatrix}
r \\
q
\end{pmatrix}, \quad (4.1)$$

where $I$ is an integral of motion for (2.7) and (2.3).
Proof: From the spectral problem \( (2.1) \) and the adjoint spectral problem \( (2.2) \), we can find that
\[
\partial(\phi_1\psi_1 - \phi_3\psi_3) = \sqrt{2}q(\phi_2\psi_1 + \phi_3\psi_2) - \sqrt{2}r(\phi_1\psi_2 + \phi_2\psi_3).
\]
This yields
\[
\partial^{-1}[-q(\phi_2\psi_1 + \phi_3\psi_2) + r(\phi_1\psi_2 + \phi_2\psi_3)] = -\frac{1}{\sqrt{2}}(\phi_1\psi_1 - \phi_3\psi_3) + I,
\]
where \( I \) is an integrals of motion for \( (2.1) \) and \( (2.2) \). The relation \( (4.1) \) follows from the above equality.

We recall that \( \tilde{Z} \) denotes the expression of \( Z \) depending on \( P_i, Q_i, 1 \leq i \leq 3 \), and their differentials after the substitution of \( (3.6) \) and into \( Z \), and that \( \text{Or}(\tilde{Z}) \) denotes the expression of \( \tilde{Z} \) only depending on \( P_i, Q_i, 1 \leq i \leq 3 \), themselves after the substitution of \( (3.8) \) into \( \tilde{Z} \) sufficiently many times. A general result on \( \tilde{a}_i, \tilde{b}_i, \tilde{c}_i, i \geq 1 \), is given in the following theorem.

**Theorem 4.1** We have the explicit expressions for \( \tilde{a}_m, \tilde{b}_m, \tilde{c}_m, m \geq 1 \):
\[
\tilde{a}_{m+1} = \sum_{i=0}^{m} I_i(<A^{m-i}P_1, BQ_1> - <A^{m-i}P_3, BQ_3>) - I_{m+1}, m \geq 0, \quad (4.2)
\]
\[
\tilde{b}_{m+1} = \sqrt{2} \sum_{i=0}^{m} I_i(<A^{m-i}P_1, BQ_2> + <A^{m-i}P_2, BQ_3>), m \geq 0, \quad (4.3)
\]
\[
\tilde{c}_{m+1} = \sqrt{2} \sum_{i=0}^{m} I_i(<A^{m-i}P_2, BQ_1> + <A^{m-i}P_3, BQ_2>), m \geq 0, \quad (4.4)
\]
provided that the nonlinearized spatial system \( (3.8) \) is satisfied. Here \( I_m, m \geq 0 \), are defined by
\[
I_0 = 1, \quad I_m = \sum_{n=1}^{m} d_n \sum_{i_1 + \cdots + i_n = m} F_{i_1} \cdots F_{i_n}, m \geq 1, \quad (4.5)
\]
where the constants \( d_n, n \geq 0 \), are determined recursively by
\[
\begin{cases}
\quad d_1 = -\frac{1}{8}, \quad d_2 = \frac{3}{128}, \\
\quad d_n = -\frac{1}{2} \sum_{i=1}^{n-1} d_id_{n-i} - \frac{1}{4} d_{n-1} - \frac{1}{8} \sum_{i=1}^{n-2} d_id_{n-i-1}, \quad n \geq 3,
\end{cases} \quad (4.6)
\]
and the functions \( F_m, m \geq 1 \), are given by \( (3.17) \).

Proof: By the recursion relation \( (2.7) \) and Lemma \( (4.1) \), we can obtain that
\[
\begin{pmatrix}
\tilde{c}_{m+1} \\
\tilde{b}_{m+1}
\end{pmatrix} = \tilde{L}^m \begin{pmatrix}
\tilde{r} \\
\tilde{q}
\end{pmatrix} = \tilde{L}^m \begin{pmatrix}
\sqrt{2}(<P_2, BQ_1> + <P_3, BQ_2>) \\
\sqrt{2}(<P_1, BQ_2> + <P_2, BQ_3>)
\end{pmatrix} = \sum_{i=0}^{m} I_i \begin{pmatrix}
\sqrt{2}(<A^{m-i}P_2, BQ_1> + <A^{m-i}P_3, BQ_2>) \\
\sqrt{2}(<A^{m-i}P_1, BQ_2> + <A^{m-i}P_2, BQ_3>)
\end{pmatrix}, m \geq 0,
\]
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where \( I_0 = 1 \) and \( I_i, 1 \leq i \leq m, \) are integrals of motion for the nonlinear spatial system (3.8). Now we compute \( \bar{a}_{m+1}, m \geq 0, \) by \( \bar{b}_{mx} = -2\bar{b}_{m+1} - 2\bar{q}\bar{a}_m, m \geq 0. \) Noting (3.8), we have for \( m \geq 0 \)

\[
\bar{b}_{mx} = \sum_{i=0}^{m-1} \sqrt{2}I_i(< A^{m-i-1}P_1, BQ_2 > + < A^{m-i-1}P_1, BQ_2 >) + < A^{m-i-1}P_2, BQ_3 > + < A^{m-i-1}P_2, BQ_3 >)
\]

\[
= \sum_{i=0}^{m-1} \sqrt{2}I_i(< -2A^{-1}P_1, BQ_2 > + < A^{m-1}P_1, -\sqrt{2}\bar{q}BQ_1 > + < \sqrt{2}\bar{q}A^{m-1}P_2, BQ_3 > + < A^{m-1}P_2, -2ABQ_3 >)
\]

\[
= -2\bar{b}_{m+1} - 2\bar{q}\left( \sum_{i=0}^{m-1} I_i(< A^{m-i-1}P_1, BQ_1 > - < A^{m-i-1}P_3, BQ_3 >) - I_m \right),
\]

from which (3.2) follows.

In the following we determine the integrals of motion \( I_m, m \geq 0, \) by a relation

\[
\bar{a}^2 + \bar{b}\bar{c} = 1, \quad \bar{a} = \sum_{i=0}^{\infty} \bar{a}_i\lambda^{-i}, \quad \bar{b} = \sum_{i=0}^{\infty} \bar{b}_i\lambda^{-i}, \quad \bar{c} = \sum_{i=0}^{\infty} \bar{c}_i\lambda^{-i},
\]

which gives rise to

\[
2\bar{a}_m = \sum_{i=1}^{m-1} (\bar{a}_i\bar{a}_{m-i} + \bar{b}_i\bar{c}_{m-i}), \quad m \geq 2.
\]

(4.7)

First from \( \bar{a}_1 = 0 \) we have

\[
I_1 = < P_1, BQ_1 > - < P_3, BQ_3 > = -\frac{1}{8}F_1,
\]

which shows \( d_1 = -\frac{1}{8}. \) Now we suppose \( m \geq 2. \) At this moment, we have by (4.7)

\[
2 \sum_{i=0}^{m-1} I_i\bar{a}_{m-i} - 2I_m
\]

\[
= \sum_{i=1}^{m-1} \sum_{k=0}^{i-1} (\sum_{l=0}^{m-i-1} I_{k}\bar{a}_{i-k} - I_i)(\sum_{l=0}^{m-i-1} I_{l}\bar{a}_{m-i-l} - I_{m-i})
\]

\[
+ \sum_{i=1}^{m-1} \sum_{k=0}^{m-i-1} I_{k}\hat{b}_{i-k} \sum_{l=0}^{m-i-1} I_{l}\hat{c}_{m-i-l}, \quad m \geq 2,
\]

where \( \bar{a}_i, \hat{b}_i, \hat{c}_i, i \geq 1, \) are given by (3.18). After interchanging the summing in the above equality, i.e.

\[
\sum_{i=1}^{m-1} \sum_{k=0}^{m-i-1} \sum_{l=0}^{m-i-1} = \sum_{i=1}^{m-1} \sum_{k=0}^{m-i-1} \sum_{l=0}^{m-i-1} = \sum_{i=1}^{m-1} \sum_{k=0}^{m-i-1} \sum_{l=0}^{m-i-1} = \sum_{i=1}^{m-1} \sum_{k=0}^{m-i-1} \sum_{l=0}^{m-i-1} = \sum_{i=1}^{m-1} \sum_{k=0}^{m-i-1} \sum_{l=0}^{m-i-1},
\]

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we may arrive at

\[-2I_m = \sum_{i=1}^{m-1} I_i I_{m-i} + \sum_{k=0}^{m-2} \sum_{l=0}^{(m-2)-k} \sum_{i=k+1}^{m-1} I_k I_l (\hat{a}_{i-k} \hat{a}_{m-i-l} + \hat{b}_{i-k} \hat{c}_{m-i-l})
- \sum_{k=0}^{m-2} \sum_{i=k+1}^{m-1} I_k I_m \hat{a}_{i-k} - \sum_{l=0}^{m-l-1} \sum_{i=1}^{m-2} I_l I_i \hat{a}_{m-i-l} - 2 \sum_{i=0}^{m-1} I_i \hat{a}_{m-i} := B_1 + B_2 + B_3 + B_4 + B_5, \quad m \geq 2. \quad (4.8)\]

Further we have

\[B_3 = -\sum_{k=0}^{m-2} \left( \sum_{i=k+2}^{m} - \sum_{i=k+1}^{m} - \sum_{i=m} \right) I_k I_{m-i} \hat{a}_{i-k}
= -\sum_{k=0}^{m-2} \sum_{i=k+2}^{m} I_k I_{m-i} \hat{a}_{i-k} - \sum_{k=0}^{m-2} \sum_{i=k+2}^{m} I_k I_{m-i-1} \hat{a}_1 + \sum_{k=0}^{m-2} I_k \hat{a}_{m-k},\]

\[B_4 = -\sum_{l=0}^{m-l-2} \left( \sum_{i=0}^{m-l-1} - \sum_{i=m-l} - \sum_{i=m-l-1} \right) I_l I_m \hat{a}_{m-l}
= -\sum_{k=0}^{m-2} \sum_{l=0}^{(m-2)-k} I_k I_l \hat{a}_{m-(k+l)} + \sum_{l=0}^{m-2} I_l \hat{a}_{m-l} - \sum_{l=0}^{m-2} I_l I_{m-l-1} \hat{a}_1.\]

Therefore the latter three terms in the right hand side of (4.8) becomes

\[B_3 + B_4 + B_5 = -2 \sum_{k=0}^{m-2} \sum_{l=0}^{(m-2)-k} I_k I_l \hat{a}_{m-(k+l)} + \frac{1}{4} \sum_{k,l=m-1}^{k+l=m-1} I_k I_l F_1.\]

In this way, from (4.8) we obtain

\[I_m = -\frac{1}{2} \sum_{i=1}^{m-1} I_i I_{m-i} - \frac{1}{8} \sum_{k=0}^{m-2} \sum_{l=0}^{(m-2)-k} I_k I_l F_{m-(k+l)} - \frac{1}{8} \sum_{k,l=m-1}^{k+l=m-1} I_k I_l F_1
= -\frac{1}{2} \sum_{i=1}^{m-1} I_i I_{m-i} - \frac{1}{8} \sum_{k+l=m-1}^{k+l \leq m-1} I_k I_l F_{m-(k+l)}, \quad m \geq 2, \quad (4.9)\]

by which we can determine any \(I_m, \quad m \geq 2,\) starting with \(I_1 = -\frac{1}{8} F_1.\) It is not difficult to find a homogeneous property among the terms of (4.9). Thus we may
assume that

\[ I_m = \sum_{n=1}^{m} d_n \sum_{i_1 + \cdots + i_n = m} F_{i_1} \cdots F_{i_n}, \quad m \geq 2. \]

In general, the coefficients \( d_n \), \( 1 \leq n \leq m \), should depend on \( m \). But the following deduction implies that this assumption is possible. First from (4.9) we easily have

\[ I_2 = \frac{3}{128} F_1^2 - \frac{1}{8} F_2, \]

which leads to \( d_2 = \frac{3}{128} \). When \( m \geq 3 \), (4.9) becomes

\[
I_m = -\frac{1}{2} \sum_{i=1}^{m-1} I_i I_{m-i} - \frac{1}{8} F_m - \frac{1}{4} \sum_{k=1}^{m-1} I_k I_{m-k} - \frac{1}{8} \sum_{k+l \leq m-1} I_k I_l I_{m-(k+l)}, \quad (4.10)
\]

in which the coefficients of the \( F_1^m \) yields the recursion relation (4.6). In what follows, we want to prove that \( I_m, \ m \geq 0 \), determined above satisfy the relation (4.10), indeed. This may be shown by combining the following three equalities. First we have

\[
\begin{align*}
\sum_{i=1}^{m-1} I_i I_{m-i} &= \sum_{i=1}^{m-1} \sum_{k=1}^{m-i} d_k \sum_{i_1 + \cdots + i_k = i} F_{i_1} \cdots F_{i_k} \sum_{l=1}^{m-i} d_l \sum_{j_1 + \cdots + j_l = m-i} F_{j_1} \cdots F_{j_l} \\
&= \sum_{k=1}^{m-1} \sum_{i=k}^{m-i} \sum_{l=1}^{m-i} d_k d_l \sum_{i_1 + \cdots + i_k = i} F_{i_1} \cdots F_{i_k} \sum_{j_1 + \cdots + j_l = m-i} F_{j_1} \cdots F_{j_l} \\
&= \sum_{k=1}^{m-1} \sum_{i=k}^{m-i} \sum_{l=1}^{m-i} d_k d_l \sum_{i_1 + \cdots + i_k = i} F_{i_1} \cdots F_{i_k} \sum_{j_1 + \cdots + j_l = m-i} F_{j_1} \cdots F_{j_l} \\
&= \sum_{n=2}^{m} \sum_{k+l=n}^{m} d_k d_l \sum_{i_1 + \cdots + i_n = m} F_{i_1} \cdots F_{i_n}.
\end{align*}
\]

Similarly we can get the other two equalities

\[
\begin{align*}
\sum_{k=1}^{m-1} I_k I_{m-k} &= \sum_{n=2}^{m} d_{n-1} \sum_{i_1 + \cdots + i_n = m} F_{i_1} \cdots F_{i_n}, \\
\sum_{k+l \leq m-1}^{m-1} I_k I_l I_{m-(k+l)} &= \sum_{n=2}^{m-1} \sum_{i+j=n}^{m-1} d_i d_j \sum_{i_1 + \cdots + i_{n+1} = m} F_{i_1} \cdots F_{i_{n+1}}.
\end{align*}
\]

Therefore the proof is finished. \( \square \)
Theorem 4.2 If \( P_i \) and \( Q_i \), \( 1 \leq i \leq 3 \), solve the nonlinearized spatial system (3.8), then there exist \( N \) integrals of motion \( \alpha_i \), \( 0 \leq i \leq N - 1 \), of (3.8) such that

\[
q = \sqrt{2}(< P_1, BQ_2 > + < P_2, BQ_3 >), \quad r = \sqrt{2}(< P_2, BQ_1 > + < P_3, BQ_2 >)
\]
solve the following \( N \)-th order stationary AKNS equation

\[
K_N + \sum_{i=0}^{N-1} \alpha_i K_i = 0,
\]

where \( K_i \), \( 0 \leq i \leq N \), are defined by (2.10).

Proof: Noting that the expressions (4.3) and (4.4) of \( \bar{b}_{i+1}, c_{i+1}, i \geq 0 \), we can compute that

\[
\sum_{i=0}^{N} \alpha_i \bar{K}_i = \sum_{i=0}^{N} \alpha_i J \left( \frac{\bar{c}_{i+1}}{\bar{b}_{i+1}} \right)
\]

\[
= J \sum_{i=0}^{N} \alpha_i \sum_{j=0}^{1} I_j \left( \sqrt{2}(< A^{i-j}P_2, BQ_1 > + < A^{i-j}P_3, BQ_2 >) \right)
\]

\[
= \sqrt{2}J \sum_{i=0}^{N} \alpha_i \sum_{j=0}^{i} I_j \sum_{l=0}^{N} \lambda_{i}^{l} \mu_{k} \left( \phi_{2k}\psi_{1k} + \phi_{3k}\psi_{2k} \right)
\]

\[
\phi_{1k}\psi_{2k} + \phi_{2k}\psi_{3k}
\]

\[
= \sqrt{2}J \sum_{k=1}^{N} \mu_{k} \left( \sum_{l=0}^{N} \lambda_{i}^{l} \right) \left( \phi_{2k}\psi_{1k} + \phi_{3k}\psi_{2k} \right)
\]

\[
\phi_{1k}\psi_{2k} + \phi_{2k}\psi_{3k}
\]

Secondly we set

\[
G(\lambda) = \prod_{i=1}^{N} (\lambda - \lambda_{i}) = \sum_{i=0}^{N} \beta_{i}\lambda^{i} = \lambda^{N} + \sum_{i=0}^{N-1} \beta_{i}\lambda^{i}
\]

Let us now choose

\[
\sum_{i=l}^{N} \alpha_{i} I_{i-l} = \beta_{l}, \quad 0 \leq l \leq N,
\]

which determines recursively

\[
\alpha_{N} = \beta_{N} = 1, \quad \alpha_{l} = \beta_{l} - \sum_{i=l+1}^{N} \alpha_{i} I_{i-l}, \quad 0 \leq l \leq N - 1,
\]

due to \( I_0 = 1 \). The \( \alpha_{i} \), \( 0 \leq i \leq N - 1 \), are all integrals of motion of (3.8) since they are functions of \( I_{i} \), \( 0 \leq i \leq N \). Further by \( G(\lambda_{k}) = 0, \ 1 \leq k \leq N \), we see that \( \sum_{k=0}^{N} \alpha_{i} \bar{K}_{i} = 0 \), which completes the proof.

The above theorem also implies that the potential determined by the Bargmann symmetry constraint (3.3) is a finite gap potential of the spectral problem (2.1).
Theorem 4.3  Under the control of the nonlinearized spatial system \((3.8)\), the nonlinearized temporal systems \((3.9)\) for \(n \geq 0\) can also be rewritten as the Hamiltonian systems

\[
P_{tn} = \{P_1, H_n\} = -B^{-1} \partial H_n / \partial Q_i, \quad Q_{tn} = \{Q_i, H_n\} = B^{-1} \partial H_n / \partial P_i, \quad i = 1, 2, 3 \quad (4.11)
\]

with the Hamiltonian functions

\[
H_n = -\frac{1}{4} \sum_{m=0}^{n} \frac{d_m}{m+1} \sum_{i_1, \ldots, i_{m+1} = n+1 \atop i_1, \ldots, i_{m+1} \geq 1} F_{i_1} \cdots F_{i_{m+1}},
\]

where \(d_0 = 1\) and \(F_m, m \geq 1\), are defined by \((3.17)\).

Proof: We only prove the former equality of \((4.11)\). We know that under the control of the nonlinearized spatial system \((3.8)\), the results in Theorem 1.1 holds. Hence we have

\[
P_{tn} = 2 \sum_{i=0}^{n} \tilde{a}_i A^{n-i} P_1 + \sqrt{2} \sum_{i=0}^{n} \tilde{b}_i A^{n-i} P_2
\]

\[
= -2A^n P_1 + 2 \sum_{i=1}^{n-1} (I_k \hat{a}_{i-k} - I_i) A^{n-i} P_1 + 2 \sum_{k=0}^{n-1} I_k \hat{b}_{i-k} A^{n-i} P_2
\]

\[
= -2n I_k A^{n-k} P_1 + \sum_{k=0}^{n-1} I_k \sum_{i=k+1}^{n} 2\hat{a}_{i-k} A^{n-i} P_1 + \sum_{k=0}^{n-1} I_k \sum_{i=k+1}^{n} \sqrt{2}\hat{b}_{i-k} A^{n-i} P_2
\]

\[
= \frac{1}{4} \sum_{k=0}^{n} I_k B^{-1} \frac{\partial P_1}{\partial Q_1} + \frac{1}{4} \sum_{k=0}^{n} I_k B^{-1} \frac{\partial F_{n-k+1}}{\partial Q_1} = \frac{1}{4} \sum_{k=0}^{n} I_k B^{-1} \frac{\partial F_{n-k+1}}{\partial Q_1},
\]

where \(\hat{a}_i, \hat{b}_i, i \geq 1\), are given by \((3.18)\). We further note the expression of \(I_m, m \geq 0\), defined by \((4.3)\) and then we may make the following performance

\[
P_{tn} = \frac{1}{4} I_0 B^{-1} \frac{\partial P_{n+1}}{\partial Q_1} + \frac{1}{4} \sum_{m=1}^{n} \sum_{k=m}^{n} d_m \sum_{i_1, \ldots, i_m = k \atop i_1, \ldots, i_m \geq 1} F_{i_1} \cdots F_{i_m} B^{-1} \frac{\partial F_{n-k+1}}{\partial Q_1}
\]

\[
= \frac{1}{4} I_0 B^{-1} \frac{\partial P_{n+1}}{\partial Q_1} + \frac{1}{4} \sum_{m=1}^{n} \sum_{k=m}^{n} d_m \sum_{i_1, \ldots, i_m = k \atop i_1, \ldots, i_m \geq 1} F_{i_1} \cdots F_{i_m} B^{-1} \frac{\partial F_{n-k+1}}{\partial Q_1}
\]

\[
= \frac{1}{4} I_0 B^{-1} \frac{\partial P_{n+1}}{\partial Q_1} + \frac{1}{4} \sum_{m=1}^{n} \frac{d_m}{m+1} B^{-1} \frac{\partial}{\partial Q_1} \sum_{i_1, \ldots, i_{m+1} = n+1 \atop i_1, \ldots, i_{m+1} \geq 1} F_{i_1} \cdots F_{i_{m+1}}
\]

\[
= \frac{1}{4} B^{-1} \frac{\partial}{\partial Q_1} \sum_{m=0}^{n} \frac{d_m}{m+1} \sum_{i_1, \ldots, i_{m+1} = n+1 \atop i_1, \ldots, i_{m+1} \geq 1} F_{i_1} \cdots F_{i_{m+1}} = -B^{-1} \frac{\partial H_n}{\partial Q_1},
\]
where we have accepted $d_0 = 1$. The above manipulation is fulfilled for the case of $n \geq 1$. The case of $n = 1$ needs only a simple calculation. Thus the former equality of (4.11) is true for $n \geq 0$. The latter equality of (4.11) may be proved similarly. The proof is completed.

The above theorem allows us to establish a sort of involutive solutions to AKNS soliton equations, which exhibits a kind of separation of variables for AKNS soliton equations. This is the following result.

**Theorem 4.4** The $n$-th AKNS soliton equation $u_{t_n} = K_n$ has the involutive solution with separated variables

\[
\begin{aligned}
q &= \sqrt{2}(<g^x_H g^t_{Hn} P_1(0,0), B g^x_H g^t_{Hn} Q_2(0,0)>) \\
&\quad + <g^x_H g^t_{Hn} P_2(0,0), B g^x_H g^t_{Hn} Q_3(0,0)>)
\end{aligned}
\]

\[
\begin{aligned}
r &= \sqrt{2}(<g^x_H g^t_{Hn} P_2(0,0), B g^x_H g^t_{Hn} Q_1(0,0)>) \\
&\quad + <g^x_H g^t_{Hn} P_3(0,0), B g^x_H g^t_{Hn} Q_2(0,0)>)
\end{aligned}
\]

(4.12)

where $g^y_G$ denotes the Hamiltonian phase flow of $G$ with a variable $y$ and $P_i(0,0)$ and $Q_i(0,0)$, $1 \leq i \leq 3$, may be arbitrary initial value vectors.

**Proof:** Let

\[
P_i(x, t_n) = g^x_H g^t_{Hn} P_i(0,0), \quad Q_i(x, t_n) = g^x_H g^t_{Hn} Q_i(0,0), \quad 1 \leq i \leq 3.
\]

Then $P_i(x, t_n)$ and $Q_i(x, t_n)$, $1 \leq i \leq 3$, solve the nonlinearized spatial system (3.8) and the Hamiltonian system (4.11). However under the control of (3.8), (4.11) is equivalent to the nonlinearized temporal system (3.9). This shows that $P_i(x, t_n)$ and $Q_i(x, t_n)$ also solve (3.8) and (3.9), simultaneously. Therefore the compatibility condition of (3.8) and (3.9) is satisfied, i.e. (4.12) determines a solution to $u_{t_n} = K_n$.

In addition, since $\{H, H_n\} = 0$, the Hamiltonian phase flows $g^x_H$, $g^t_{Hn}$ may commute with each other. It follows that the resulting solution (4.12) is involutive. The proof is finished.

## 5 Conclusions and remarks

We have introduced a three-by-three matrix spectral problem for the usual AKNS soliton hierarchy and proposed the corresponding Bargmann symmetry constraint on this AKNS hierarchy. Moreover we have exhibited an explicit Poisson algebra

\[
\{\bar{F}_j, 1 \leq j \leq N, F_m, m \geq 1\}
\]

(5.1)

on the symplectic manifold $(\mathbb{R}^{6N}, \omega^2)$ and further a binary nonlinearization procedure is manipulated along with a sort of involutive solutions to AKNS soliton equations. When $N = 1, 2$, we have proved the nonlinearized spatial system (3.8)
is integrable in the Liouville sense, indeed. But we don’t know if we can take out enough independent integrals of motion among the Poisson algebra (5.1) for a general integer $N$. We hope that this Poisson algebra suffices for proving complete integrability of the nonlinearized Lax systems.

It should be pointed out that the Neumann symmetry constraint and the higher order symmetry constraints

$$K_{-1} = J \sum_{j=1}^{N} E_j \delta \lambda_j, \quad K_m = JG_m = J \sum_{j=1}^{N} E_j \delta \lambda_j, \quad (m \geq 1), \quad (5.2)$$

may also be considered. These sorts of symmetry constraints are somewhat different from the Bargmann symmetry constraints because $K_{-1}$ is a constant vector and the conserved covariants $G_m, m \geq 1$, involve the differential of the potential $u$ with respect to the space variable $x$. In order to discuss them, we are required to introduce a new symplectic submanifold of the Euclidean spaces in the case of the Neumann constraint and new dependent variables, i.e. the so-called Jacobi-Ostrogradsky coordinates [14], in the case of higher order constraints. Similarly, we can consider the corresponding $\tau$-symmetry (time first order dependent symmetry) constraints or more generally, time polynomial dependent symmetry constraints. Note that the similar Bargmann symmetry constraints have also been carefully analyzed for KP hierarchy [20] and the symmetries in the right hand side of the Bargmann symmetry constraints are sometimes called additional symmetries [10] and may be taken as sources of soliton equations [13].

We remark that the finite dimensional Hamiltonian systems generated by nonlinearization technique depend on the starting spectral problems. Therefore the same soliton equation may relate to different finite dimensional Hamiltonian systems once it possesses different Lax representations. AKNS soliton equations are exactly such examples. But we don’t know if there exists an interrelation among the different finite dimensional Hamiltonian systems generated from the same soliton equation. In the binary nonlinearization procedure itself, there also exist some intriguing open problems. For example, why do the nonlinearized spatial system and the nonlinearized temporal systems for $n \geq 0$ under the control of the nonlinearized spatial system always possess Hamiltonian structures? We don’t know whether or not the nonlinearized temporal systems for $n \geq 0$ are themselves integrable soliton equations without the control of the nonlinearized spatial system. These problems are worth studying in order to enrich integrable structures of soliton equations.

Acknowledgments: One of the authors (W. X. Ma) would like to thank the Alexander von Humboldt Foundation for a research fellow award and the National Natural Science Foundation of China and the Shanghai Science and Technology Commission of China for financial support. He is also grateful to Drs. P. Zimmermann and G. Oevel for their helpful and stimulating discussions about MuPAD.

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