Morgan type uncertainty principle and unique continuation properties for abstract Schrödinger equations

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Abstract

In this paper, Morgan type uncertainty principle and unique continuation properties of abstract Schrödinger equations with time dependent potentials in vector-valued $L^2$ classes are obtained. The equation involves a possible linear operators considered in the Hilbert space $H$. So, by choosing the corresponding spaces $H$ and operators we derived unique continuation properties for numerous classes of Schrödinger type equations and its systems which occur in a wide variety of physical systems.

Key Word: Schrödinger equations, Positive operators, Semigroups of operators, Unique continuation, Morgan type uncertainty principle

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1. Introduction, definitions

In this paper, the unique continuation properties of the abstract Schrödinger equations

$$i\partial_t u + \Delta u + Au + V(x,t)u = 0, \quad x \in \mathbb{R}^n, \quad t \in [0,1],$$

(1.1)

are studied, where $A$ is a linear operator, $V(x,t)$ is a given potential operator function in a Hilbert space $H$, subscript $t$ indicates the partial derivative with respect to $t$, $\Delta$ denotes the Laplace operator in $\mathbb{R}^n$ and $u = u(x,t)$ is the $H$-valued unknown function. The goal is to obtain sufficient conditions on the operator $A$, the potential $V$ and the behavior of the solution $u$ at two different times, $t_0 = 0$ and $t_1 = 1$ which guarantee that $u \equiv 0$ in $\mathbb{R}^n \times [0,1]$.

This linear result was then applied to show that two regular solutions $u_1$ and $u_2$ of non-linear Schrödinger equations

$$i\partial_t u + \Delta u + Au = F(u,\bar{u}), \quad x \in \mathbb{R}^n, \quad t \in [0,1],$$

(1.2)

and for general non-linearities $F$, must agree in $\mathbb{R}^n \times [0,1]$, when $u_1 - u_2$ and its gradient decay faster than any quadratic exponential at times 0 and 1.

Unique continuation properties for Schrödinger equations studied e.g in [6-8], [21-23] and the references therein. In contrast to the mentioned above results we will study the unique continuation properties of abstract Schrödinger equations with operator potentials. Abstract differential equations studied e.g. in [1], [5], [10, 11], [13 - 19], [25, 26]. Since the Hilbert space $H$ is arbitrary and $A$ is a possible linear operator, by choosing $H$ and $A$ we can obtain numerous classes
of Schrödinger type equations and its systems which occur in a wide variety of physical systems. If we choose the abstract space $H$ a concrete Hilbert space, for example $H = L^2(G)$, $A = L$, where $G$ is a domain in $R^n$ with sufficiently smooth boundary and $L$ is elliptic operator, then we obtain the unique continuation properties of following Schrödinger equation

$$\partial_t u = i \left[ \Delta u + Lu + V(x,t) u \right], \quad x \in R^n, \quad y \in G, \quad t \in [0,1],$$

(1.3)

where $L$ is an elliptic operator with respect to variable $y \in G \subset R^m$ and $u = u(x,y,t)$.

Moreover, let $H = L^2(0,1)$ and $A$ to be differential operator with generalized Wentzell-Robin boundary condition defined by

$$D(A) = \{ u \in W^{2,2}(0,1), \quad B_j u = Au(j) = 0, \quad j = 0, 1 \},$$

(1.4)

$$Au = au^{(2)} + bu^{(1)},$$

where $a$ is positive and $b$ is a real-valued functions on $(0,1)$. Then, we get the unique continuation properties of the Wentzell-Robin type boundary value problem (BVP) for the following Schrödinger type equation

$$i \partial_t u + \Delta u + a \frac{\partial^2 u}{\partial y^2} + b \frac{\partial u}{\partial y} = F(u, \bar{u}) ,$$

(1.5)

$$u = u(x,y,t), \quad x \in R^n, \quad y \in G, \quad t \in [0,1].$$

(1.6)

Let $E$ be a Banach space and $\gamma = \gamma(x)$ be a positive measurable function on a domain $\Omega \subset R^n$. Here, $L_{p,\gamma}(\Omega; E)$ denotes the space of strongly measurable $E$-valued functions that are defined on $\Omega$ with the norm

$$\|f\|_{p,\gamma} = \|f\|_{L_{p,\gamma}(\Omega; E)} = \left( \int \|f(x)\|^p_E \gamma(x) \, dx \right)^{\frac{1}{p}} , \quad 1 \leq p < \infty, \quad \|f\|_{L_{\infty,\gamma}(\Omega; E)} = \text{ess sup}_{x \in \Omega} \|f(x)\|^p_E \gamma(x) , \quad p = \infty.$$
For $\gamma(x) \equiv 1$ the space $L_{p,\gamma}(\Omega; E)$ will be denoted by $L_p = L_p(\Omega; E)$ for $p \in [1, \infty]$.

Let $\mathbf{p} = (p_1, p_2)$ and $\Omega = \Omega_1 \times \Omega_2$, where $\Omega_k \in \mathbb{R}^{n_k}$. $L_\mathbf{p} = L_\mathbf{p}(\Omega; E)$ will denote the space of all $E$-valued $\mathbf{p}$-summable functions with mixed norm, i.e., the space of all measurable functions $f$ defined on $\Omega$ equipped with norm

$$
\|f\|_{L_{p_1}^{p_1}L_{p_2}^{p_2}(\Omega; E)} = \left( \frac{\int_{\Omega_1} \|f(x,t)\|_E^{p_2} \, dt}{\|f(x)\|_E^{p_1}} \right)^{\frac{1}{p_1}}.
$$

For $p = 2$ and $H$ Hilbert space we get Hilbert space of $H$-valued functions with inner product of two elements $f, g \in L_2(\Omega; H)$:

$$(f, g)_{L^2(\Omega; H)} = \int_{\Omega} (f(x), g(x))_H \, dx.$$  

Let $C(\Omega; E)$ denote the space of $E$-valued, bounded uniformly continuous functions on $\Omega$ with norm

$$
\|u\|_{C(\Omega; E)} = \sup_{x \in \Omega} \|u(x)\|_E.
$$

$C^m(\Omega; E)$ will denote the space of $E$-valued bounded uniformly strongly continuous and $m$-times continuously differentiable functions on $\Omega$ with norm

$$
\|u\|_{C^m(\Omega; E)} = \max_{0 \leq |\alpha| \leq m} \sup_{x \in \Omega} \|D^\alpha u(x)\|_E.
$$

$C_0^\infty(\Omega; E)$—denotes the space of $E$-valued infinity many differentiable finite functions.

Let $O_R = \{x \in \mathbb{R}^n, \ |x| < R\}, \ R > 0$.

Let $\mathbb{N}$ denote the set of all natural numbers, $\mathbb{C}$ denote the set of all complex numbers.

Let $\Omega$ be a domain in $\mathbb{R}^n$ and $m$ be a positive integer. $W^{m,p}(\Omega; E)$ denotes the space of all functions $u \in L^p(\Omega; E)$ that have the generalized derivatives $\frac{\partial^m u}{\partial x_k^m} \in L^p(\Omega; E)$, $1 \leq p \leq \infty$ with the norm

$$
\|u\|_{W^{m,p}(\Omega; E)} = \|u\|_{L^p(\Omega; E)} + \sum_{k=1}^{n} \left( \frac{\partial^m u}{\partial x_k^m} \right)_{L^p(\Omega; E)} < \infty.
$$

Let $E_0$ and $E$ be two Banach spaces and $E_0$ is continuously and densely embedded into $E$. Here, $W^{m,p}(\Omega; E_0, E)$ denote the space $W^{m,p}(\Omega; E) \cap L^p(\Omega; E)$ equipped with norm

$$
\|u\|_{W^{m,p}(\Omega; E_0, E)} = \|u\|_{L^p(\Omega; E_0)} + \sum_{k=1}^{n} \left( \frac{\partial^m u}{\partial x_k^m} \right)_{L^p(\Omega; E)} < \infty.
$$
Let $E_1$ and $E_2$ be two Banach spaces. $L(E_1, E_2)$ will denote the space of all bounded linear operators from $E_1$ to $E_2$. For $E_1 = E_2 = E$ it will be denoted by $L(E)$.

A linear operator $A$ is said to be positive in a Banach space $E$ with bound $M > 0$ if $D(A)$ is dense on $E$ and $\|(A+sI)^{-1}\|_{L(E)} \leq M (1 + |s|)^{-1}$ for any $s \in (-\infty, 0)$, where $I$ is the identity operator in $E$.

Let $[A, B]$ be a commutator operator, i.e.

$$[A, B] = AB - BA$$

for linear operators $A$ and $B$.

Sometimes we use one and the same symbol $C$ without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say $\alpha$, we write $C_\alpha$.

2. Main results for abstract Schrödinger equation

First of all, we generalize the result G. W. Morgan (see e.g. [7]) about Morgan type uncertainty principle for Fourier transform. Let

$$X = L^2(R^n; H), \ Y^k = W^{2,k}(R^n; H), k \in \mathbb{N}. $$

**Lemma 2.1.** Let $f(x) \in L^1(R^n; H) \cap X$ and

$$\int_{R^n} \int_{R^n} \|f(x)\|_H \left\| \hat{f}(\xi) \right\|_H e^{ix\cdot\xi} \, dx \, d\xi < \infty.$$  

Then $f(x) \equiv 0$.

In particular, using Young’s inequality this implies:

**Result 2.1.** Let $f(x) \in L^1(R^n; H) \cap X$, $p \in (1, 2)$, $\frac{1}{p} + \frac{1}{q} = 1, \alpha, \beta > 0$

and

$$\int_{R^n} \|f(x)\|_H e^{\alpha|x|^p} \, dx + \int_{R^n} \left\| \hat{f}(\xi) \right\|_H e^{\beta|\xi|^q} \, d\xi < \infty, \alpha \beta > 1.$$  

Then $f(x) \equiv 0$.

The Morgan type uncertainty principle, in terms of the solution of the free Schrödinger equation will be as:

Let

$$u_0(\cdot) \in L^1(R^n; H) \cap X$$

and for some $t \neq 0$

$$\int_{R^n} \|u_0(x)\|_H e^{\frac{\alpha|x|^p}{p}} \, dx + \int_{R^n} \left\| e^{it(\Delta + A)}u_0(x) \right\|_H e^{\frac{\beta|\xi|^q}{q}} \, d\xi < \infty, \alpha \beta > 1.$$  


Then \( u_0 (x) \equiv 0. \)

**Condition 1.** Assume \( A \) is a positive operator in Hilbert space \( H \) and \( iA \) generates a semigroup \( U(t) = e^{iAt} \). Suppose

\[
\|V\|_{L^\infty (R^n \times (0,1);L(H))} \leq C \tag{2.1}
\]

and

\[
\lim_{R \to \infty} \|V\|_{L^1 L^\infty (L(H))} = 0, \tag{2.2}
\]

where

\[
L^1 L^\infty (L(H)) = L^1 (0, 1; L^\infty (R^n/O_R); L(H)).
\]

Here,

\[
\sigma(t) = \frac{1}{\alpha (1 - t) + \beta t}.
\]

In [17, Theorem 1] we proved the following result:

**Theorem A.** Assume the Condition 1 holds or \( V(x, t) = V_1(x) + V_2(x, t) \), where \( V_1(x) \in L(H) \) for \( x \in R^n \) and

\[
\sup_{t \in [0,1]} \left\| e^{ix^2 \sigma^2(t)} V_2(., t) \right\|_B < \infty.
\]

Suppose \( \alpha, \beta > 0 \) and \( \alpha \beta < 4 \) such that any solution \( u \in C([0,1];X) \) of (1.1) satisfy

\[
\left\| e^{\frac{ix^2}{\alpha}} u(.,0) \right\|_X < \infty, \left\| e^{\frac{ix^2}{\beta}} u(.,1) \right\|_X < \infty.
\]

Then \( u(x, t) \equiv 0. \)

Our main result in this paper is the following:

**Theorem 1.** Assume the Condition 1 holds and there exist constants \( a_0, a_1, a_2 > 0 \) such that for any \( k \in Z^+ \) a solution \( u \in C([0,1];X) \) of (1.1) satisfy

\[
\int_{R^n} \|u(x,0)\|_H^2 e^{2a_0 |x|^p} dx < \infty, \text{ for } p \in (1,2), \tag{2.3}
\]

\[
\int_{R^n} \|u(x,1)\|_H^2 e^{2k |x|^p} dx < a_2 e^{2a_1 k \frac{p}{p+q}}, \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{2.4}
\]

Moreover, there exists \( M_p > 0 \) such that

\[
a_0 a_1^{p-2} > M_p. \tag{2.5}
\]

Then \( u(x, t) \equiv 0. \)

**Corollary 1.** Assume the Condition 1 holds and

\[
\lim_{|R| \to \infty} \int_0^1 \sup_{|x| > R} \|V(x,t)\|_{L(H)} dt = 0.
\]
There exist positive constants $\alpha$, $\beta$ such that a solution $u \in C([0, 1] ; X)$ of (1.1) satisfy
\[
\int_{\mathbb{R}^n} \|u(x, 0)\|_H^2 e^{\frac{2|\alpha x|^p}{p}} dx + \int_{\mathbb{R}^n} \|u(x, 1)\|_H^2 e^{\frac{2|\beta x|^q}{q}} dx < \infty,
\] (2.6)
with
\[
p \in (1, 2), \quad \frac{1}{p} + \frac{1}{q} = 1
\]
and there exists $N_p > 0$ such that
\[
\alpha \beta > N_p.
\] (2.7)

Then $u(x, t) \equiv 0$.

As a direct consequence of Corollary 1 we have the following result regarding the uniqueness of solutions for nonlinear equation (1.2)

**Theorem 2.** Assume the Condition 1 holds and $u_1$, $u_2 \in C([0, 1] ; Y^{2,k})$ strong solutions of (1.2) with $k \in \mathbb{Z}^+$, $k > \frac{n}{p}$. Suppose $F : H \times H \to H$, $F \in C^k$, $F(0) = \partial_u F(0) = \partial_x F(0) = 0$ and there exist positive constants $\alpha$, $\beta$ such that
\[
e^{-\frac{p|\alpha x|^p}{p}} (u_1(., 0) - u_2(., 0)) \in X, \quad e^{-\frac{q|x|^q}{q}} (u_1(., 0) - u_2(., 0)) \in X,
\] (2.8)
with
\[
p \in (1, 2), \quad \frac{1}{p} + \frac{1}{q} = 1
\]
and there exists $N_p > 0$ such that
\[
\alpha \beta > N_p.
\] (2.9)

Then $u_1(x, t) \equiv u_2(x, t)$.

**Corollary 2.** Assume the Condition 1 holds and there exist positive constants $\alpha$ and $\beta$ such that a solution $u \in C([0, 1] ; X)$ of (1.1) satisfy
\[
\int_{\mathbb{R}^n} \|u(x, 0)\|_H^2 e^{\frac{2|\alpha x|^p}{p}} dx + \int_{\mathbb{R}^n} \|u(x, 1)\|_H^2 e^{\frac{2|\beta x|^q}{q}} dx < \infty,
\] (2.10)
for $j = 1, 2, \ldots, n$ and $p \in (1, 2)$, $\frac{1}{p} + \frac{1}{q} = 1$. Moreover, there exists $N_p > 0$ such that
\[
\alpha \beta > N_p.
\] (2.11)
Then $u(x, t) \equiv 0$.

**Remark 2.1.** The Theorem 3 still holds, with different constant $N_p > 0$, if one replaces the hypothesis (2.8) by
\[
e^{\frac{p|\alpha x|^p}{p}} (u_1(., 0) - u_2(., 0)) \in X, \quad e^{\frac{q|x|^q}{q}} (u_1(., 0) - u_2(., 0)) \in X,
\]
for $j = 1, 2, ..., n$.

Next, we shall extend the method used in the proof Theorem 3 to study the blow up phenomenon of solutions of nonlinear Schrödinger equations

$$i\partial_t u + \Delta u + Au + F(u, \bar{u}) u = 0, \quad x \in \mathbb{R}^n, \quad t \in [0, 1],$$  \hspace{1cm} (2.12)

where $A$ is a linear operator in a Hilbert space $H$.

Let $u(x, t)$ be a solution of the equation (2.12). Then it can be shown that the function

$$v(x, t) = U(x, 1 - t) u \left( \frac{x}{1 - t} \cdot \frac{t}{1 - t} \right),$$  \hspace{1cm} (2.13)

is a solution of the focussing $L^2$-critical solution of abstract Schrodinger equation

$$i\partial_t u + \Delta u + Au + \|u\|^{\frac{4}{n}} u = 0, \quad x \in \mathbb{R}^n, \quad t \in [0, 1]$$  \hspace{1cm} (2.14)

which blows up at time $t = 1$, where $U(x, t)$ is a fundamental solution of the Schrödinger equation

$$i\partial_t u + \Delta u + Au = 0, \quad x \in \mathbb{R}^n, \quad t \in [0, 1],$$

i.e.

$$U(x, t) = \frac{1}{t^{n/2}} \exp \left\{ i \left( A + |x|^2 \right) / 4t \right\}.$$

Assume that the function $v$ defined by (2.13) satisfies the following estimate

$$\|v(x, t)\|_H \leq \frac{1}{(1 - t)^{n/2}} Q \left( \frac{|x|}{1 - t} \right), \quad t \in (-1, 1)$$  \hspace{1cm} (2.15)

where

$$Q(x) = b_1^{-\frac{n}{2}} e^{b_2|x|^p}, \quad b_1, b_2 > 0, \quad p > 1$$  \hspace{1cm} (2.16)

The following result will be get:

**Theorem 3.** Assume the Condition 1 holds and there exist positive constants $b_0$ and $\theta$ such that a solution $u \in C([-1, 1]; X)$ of (2.12) satisfied

$$\|F(u, \bar{u})\|_H \leq b_0 \|u\|_H^\theta$$

for $\|u\|_H > 1$.  \hspace{1cm} (2.17)

Suppose

$$\|u(., t)\|_X = \|u(., 0)\|_X = \|u_0\|_X = a, \quad t \in (-1, 1)$$

and that (2.15) holds with $Q(.)$ satisfies (2.16). If $p > p(\theta) = \frac{2(\theta n - 2)}{(\theta n - 1)}$, then $a \equiv 0$.

**3. Some properties of solutions of abstract Schrödinger equations**

Let
\[ \sigma(t) = [\alpha(1 - t) + \beta t]^{-1}, \quad \eta(x, t) = (\alpha - \beta)\|x\|^2 [4i(\alpha(1 - t) + \beta t)]^{-1}, \]

\[ \nu(s) = \left[ \gamma \alpha \beta \sigma^2(s) + \frac{(\alpha - \beta) a}{4(a^2 + b^2)} \sigma(s) \right], \quad \phi(x, t) = \frac{\gamma a |x|^2}{a + 4\gamma(a^2 + b^2)t}. \]

We recall the following lemma (see [17, Lemma 3.1]).

Let

\[ \Phi(A, V) v = a \Re((A + V) v, v)_H - b \Im((A + V) v, v)_H, \]

for \( v = v(x, t) \in H(A). \)

**Lemma A.** Assume \( a > 0, b \in \mathbb{R}, \) \( A \) is a symmetric operator in \( H. \)

Moreover, there is a constant \( C_0 > 0 \) so that

\[ |\Phi(A, V) v(x, t)| \leq C_0 \mu(x, t) \|v(x, t)\|_H^2, \]

for \( x \in \mathbb{R}^n, t \in [0, T], \gamma \geq 0, T \in [0, 1] \) and \( v \in H(A), \) where \( \mu \) is a positive function in \( L^1(0, T; L^\infty(\mathbb{R}^n)). \)

Then the solution \( u \) of (3.0) belonging to \( L^\infty(0, 1; X) \cap L^2(0, 1; Y^1) \) satisfies the following estimate

\[ e^{MT} \left\| e^{\sigma(t)T} u(., T) \right\|_X \leq M_T \left\| e^{\gamma|x|^2} u(., 0) \right\|_X + \sqrt{a^2 + b^2} \left\| e^{\sigma(t)F} \right\|_{L^1(0, T; X)}, \]

where

\[ M_T = ||\mu||_{L^1(0, T; L^\infty(\mathbb{R}^n))}. \]

Let \( u = u(x, s) \) be a solution of the equation

\[ \partial_s u = i \left[ \Delta u + Au + V(y, s) u + F(y, s) \right], \quad y \in \mathbb{R}^n, \quad s \in [0, 1]. \]

and \( a + ib \neq 0, \gamma \in \mathbb{R}, \alpha, \beta \in \mathbb{R}_+. \) Set

\[ \tilde{u}(x, t) = \left( \sqrt{\alpha \beta \sigma(t)} \right)^{\frac{3}{2}} u \left( \sqrt{\alpha \beta x \sigma(t)}, \beta t \sigma(t) \right) e^{\eta}. \quad (3.1) \]

Then, \( \tilde{u}(x, t) \) verifies the equation

\[ \partial_t \tilde{u} = i \left[ \Delta \tilde{u} + A\tilde{u} + \tilde{V}(x, t) \tilde{u} + \tilde{F}(x, t) \right], \quad x \in \mathbb{R}^n, \quad t \in [0, 1] \quad (3.2) \]

with

\[ \tilde{V}(x, t) = \alpha \beta \sigma^2(t) V \left( \sqrt{\alpha \beta x \sigma(t)}, \beta t \sigma(t) \right), \quad (3.3) \]
\[ \tilde{F}(x, t) = \left( \sqrt{\alpha \beta \sigma(t)} \right)^{2} \left( \sqrt{\alpha \beta x \sigma(t)} \right) \cdot \beta t \sigma(t). \] (3.4)

Moreover,

\[ \left\| e^{\gamma|x|^2} \tilde{F}(., t) \right\|_X = \alpha \beta \sigma^{2}(t) e^{\nu|y|^2} \| F(s) \|_X \text{ and } \left\| e^{\gamma|x|^2} \tilde{u}(., t) \right\|_X = e^{\nu|y|^2} \| u(s) \|_X \] (3.5)

when \( s = \beta t \sigma(t) \).

**Remark 3.1.** Let \( \beta = \beta(k) \). By assumption we have

\[ \left\| e^{a_0|x|^p} u(x, 0) \right\|_X = A_0, \]
\[ \left\| e^{a_0|x|^p} u(x, 0) \right\|_X = A_k \leq a_2 e^{2a_1 k^{\frac{p}{2} - p}} = a_2 e^{2a_1 k^{\frac{p}{2}}} . \] (3.6)

Thus, for \( \gamma = \gamma(k) \in [0, \infty) \) to be chosen later, one has

\[ \left\| e^{\gamma|x|^p} \tilde{u}_k(x, 0) \right\|_X = \left\| e^{(\beta \alpha)^{p/2}|x|^p} u_k(x, 0) \right\|_X = B_0, \] (3.7)
\[ \left\| e^{\gamma|x|^p} \tilde{u}_k(x, 1) \right\|_X = \left\| e^{(\beta \alpha)^{p/2}|x|^p} u_k(x, 1) \right\|_X = A_k. \]

Let us take

\[ \gamma \left( \frac{\alpha}{\beta} \right)^{p/2} = a_0 \text{ and } \gamma \left( \frac{\beta}{\alpha} \right)^{p/2} = k, \]

i.e.

\[ \gamma = (k a_0)^{\frac{1}{p}}, \beta = k^{\frac{1}{p}}, \alpha = a_0^{\frac{1}{p}} . \] (3.8)

Let

\[ M = \int_{0}^{1} \| V(., t) \|_{L^\infty(R^n; H)} dt = \int_{0}^{1} \| V(., s) \|_{L^\infty(R^n; H)} ds. \]

From (3.2), using energy estimates it follows

\[ e^{-M} \| u(., 0) \|_X \leq \| u(., t) \|_X = \| \tilde{u}(., s) \|_X \leq e^{M} \| u(., 0) \|_X , t, s \in [0, 1] , \] (3.9)

where

\[ s = \beta t \sigma(t) . \]

Consider the following problem

\[ i \partial_t u + \Delta u + Au = V(x, t) u + F(x, t) , x \in R^n , t \in [0, 1] , \] (3.10)

\[ u(x, 0) = u_0(x) , \]

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where $A$ is a linear operator, $V(x,t)$ is a given potential operator function in a Hilbert space $H$ and $F$ is a $H$-valued function.

Let us define operator valued integral operators in $L^p(Ω; E)$. Let $k: R^n \setminus \{0\} → L(E)$. We say $k(x)$ is a $L(E)$-valued Calderon-Zygmund kernel ($C - Z$ kernel) if $k \in C^∞(R^n \setminus \{0\}, L(E))$, $k$ is homogenous of degree $-n$, $\int_B k(x)\,dσ = 0$.

where

$$B = \{ x ∈ R^n : |x| = 1 \}.$$  

For $f ∈ L^p(Ω; E), p ∈ (1, \infty), a ∈ L^∞(R^n)$, and $x ∈ Ω$ we set the Calderon-Zygmund operator

$$K_ε f = \int_{|x−y|>ε, y∈Ω} k(x,y)f(y)\,dy, \quad K f = \lim_{ε→0} K_ε f$$

and commutator operator

$$[K;a]f = a(x)Kf(x)−K(af)(x) = \lim_{ε→0} \int_{|x−y|>ε, y∈Ω} k(x,y)[a(x)−a(y)]f(y)\,dy.$$  

By using Calderón’s first commutator estimates [4], convolution operators on abstract functions [2] and abstract commutator theorem in [20] we obtain the following result:

**Theorem A.** Assume $k(\cdot)$ is $L(E)$-valued $C - Z$ kernel that have locally integrable first-order derivatives in $|x| > 0$, and

$$\|k(x,y)−k(x',y)\|_{L(E)} ≤ M |x−x'| |x−y|^{−(n+1)} \quad \text{for} \quad |x−y| > 2 |x−x'|.$$  

Let $a(\cdot)$ have first-order derivatives in $L^r(R^n), 1 < r ≤ ∞$. Then for $p, q ∈ (1, ∞), q^{-1} = p^{-1} + r^{-1}$ the following estimates hold

$$\|[K;a]\partial_x f\|_{L^q(R^n; E)} ≤ C \|f\|_{L^p(R^n; E)} ,$$

$$\|\partial_x [K;a] f\|_{L^q(R^n; E)} ≤ C \|f\|_{L^p(R^n; E)} ,$$

for $f ∈ C^∞_0(R^n; E)$, where the constant $C > 0$ is independent of $f$.

Let

$$X_γ = L^2_γ (R^n; H).$$

By following [4, Lemma 2.1] let us show the following

**Lemma 3.1.** Assume the Condition 1 holds and there exists $ε_0 > 0$ such that

$$\|V\|_{L^1(L^∞_x(R^n × [0,1]; L(H)))} < ε_0.$$  

Moreover, suppose $u ∈ C([0,1]; X)$ is a strong solution of (3.10) with

$$u_0, \quad u_1 = u(x,1) ∈ X_γ, \quad F ∈ L^1(0,1; X_γ)$$
for \( \gamma (x) = e^{2\lambda x} \) and for some \( \lambda \in \mathbb{R}^n \). Then there exists a positive constant \( M_0 = M_0 (n, A, H) \) independent of \( \lambda \) such that

\[
\sup_{t \in [0,1]} \| u (., t) \|_{X_\gamma} \leq M_0 \left[ \| u_0 \|_{X_\gamma} + \| u_1 \|_{X_\gamma} + \int_0^1 \| F (., t) \|_{X_\gamma} dt \right]. \tag{3.11}
\]

**Proof.** First, we consider the case, when \( \gamma (x) = \beta (x) = e^{2\beta x_1} \). Without loss of generality we shall assume \( \beta > 0 \). Let \( \varphi_n \in C^\infty (\mathbb{R}) \) such that \( \varphi_n (\tau) = 1, \tau \leq n \) and \( \varphi_n (\tau) = 0 \) for \( \tau \geq 10n \) with \( 0 \leq \varphi_n \leq 1, \left| \varphi_n^{(j)} (\tau) \right| \leq C_j \tau^{-j} \). Let

\[
\theta_n (\tau) = \beta \int_0^\tau \varphi_n^2 (s) ds
\]

so that \( \theta_n \in C^\infty (\mathbb{R}) \) nondecreasing with \( \theta_n (\tau) = \beta \tau \) for \( \tau < n \), \( \theta_n (\tau) = C_n \beta \) for \( \tau > 10n \) and

\[
\theta_n^{(j)} (\tau) = \beta \varphi_n^2 (\tau) \leq \beta, \theta_n^{(j)} (\tau) = \beta C_j \tau^{-j}, j = 1, 2, ..., \tag{3.12}
\]

Let \( \phi_n (\tau) = \exp (2 \theta_n (\tau)) \) so that \( \phi_n (\tau) \leq \exp (2 \beta \tau) \) and \( \phi_n (\tau) \rightarrow \exp (2 \beta \tau) \) for \( n \rightarrow \infty \). Let \( u (x, t) \) be a solution of the equation (3.10), then one gets the equation \( v_n (x, t) = \phi_n (x_1) u (x, t) \) satisfies the following

\[
i \partial_t v_n + \Delta v_n + Av_n = V_n (x, t) v_n + \phi_n (x_1) F (x, t), \tag{3.13}
\]

where

\[
V_n (x, t) v_n = V (x, t) v_n + 4 \beta \varphi_n (x_1) \partial_{x_1} v_n + \left[ 4 \beta \varphi_n (x_1) \varphi'_n (x_1) - 4 \beta^2 v_n^4 \right] v_n.
\]

Now, we consider a new function

\[
w_n (x, t) = e^{\mu} v_n (x, t), \mu = -i 4 \beta^2 \varphi_n^4 (x_1) t. \tag{3.14}
\]

Then from (3.13) we get

\[
i \partial_t w_n + \Delta w_n + Aw_n = \tilde{V}_n (x, t) w_n + \tilde{F}_n (x, t), \tag{3.13}
\]

where

\[
\tilde{V}_n (x, t) w_n = V (x, t) w_n + h (x_1, t) + a^2 (x_1) \partial_{x_1} w_n + it h (x_1) \partial_{x_1} w_n
\]

\[
\tilde{F}_n (x, t) = e^{\mu} \phi_n (x) F (x, t),
\]

when

\[
h (x_1, t) = (i 16 \beta^2 \varphi_n^3 \varphi'_n t)^2 + i 48 \beta^2 \varphi_n^2 (\varphi'_n)^2 t + i 16 \beta^2 \varphi_n^3 \varphi^{(2)}_n t +
\]

\[
+ (i 16 \beta^2 \varphi_n^3 \varphi'^2_n t + i 48 \beta^2 \varphi_n^2 (\varphi'^2_n)^2 t + i 16 \beta^2 \varphi_n^3 \varphi^{(3)}_n t +
\]

\[
+ (i 16 \beta^2 \varphi_n^3 \varphi^{(4)}_n t + ...
\]
4\beta \varphi_n \varphi_n' + i64\beta^2 \varphi_n^3 \varphi_n'^3 t, \ a^2 = 4\beta \varphi_n^2 (x_1), \ b = -32\beta^2 \varphi_n^3 \varphi_n'.

It is clear to see that

\[ \| \partial^j_x h (x_1, t) \|_{L^\infty (R \times [0, 1])} \leq C_J n^{-j+1}, \ j = 1, 2, \ldots, \] (3.14)

\[ a^2 (x_1) \geq 0, \ \| \partial^j_x a (x_1) \|_{L^\infty (R)} \leq C_J n^{-j}, \ j = 1, 2, \ldots, \] (3.15)

\[ \| \partial^j_x b (x_1) \|_{L^\infty (R)} \leq C_J n^{-j}, \ j = 1, 2, \ldots. \]

Consider the smooth function \( \eta \in C^\infty (R^n) \) such that \( \eta (x) = 1 \) for \( |x| \leq \frac{1}{2} \)

and \( \eta (x) = 0 \) for \( |x| \geq 1, \) with \( 0 \leq \eta (x) \leq 1. \) Moreover, let

\[ \chi_{\pm} (\xi_1) = \begin{cases} 1, & \xi_1 > 0 (\xi_1 < 0) \\ 0, & \xi_1 < 0 (\xi_1 > 0) \end{cases}. \]

Define the multipliers operators

\[ P_\varepsilon f = F^{-1} \left( \eta (\varepsilon \xi) \hat{f} (\xi) \right), \ P_{\pm} f = F^{-1} \left( \chi_{\pm} (\varepsilon \xi) \hat{f} (\xi) \right) \text{ for } 0 < \varepsilon \leq 1. \]

Then by applying the equation (3.10) for example to \( P_\varepsilon P_+ w_n \) we get

\[ i \partial_t P_\varepsilon P_+ w_n + \Delta P_\varepsilon P_+ w_n + AP_\varepsilon P_+ w_n = P_\varepsilon (V w_n) + P_\varepsilon (h w_n) + \] (3.16)

\[ P_\varepsilon P_+ (a^2 (x_1) \partial_x w_n) + P_\varepsilon P_+ (ib (x_1) \partial_x w_n) + P_\varepsilon P_+ \left( \hat{F}_n \right). \]

From (3.16) we obtain

\[ i \partial_t \left( P_\varepsilon P_+ w_n, v \right) + \left( \Delta P_\varepsilon P_+ w_n, v \right) + \left( AP_\varepsilon P_+ w_n, v \right) = \]

\[ \left( P_\varepsilon P_+ (V w_n), v \right) + \left( P_\varepsilon P_+ (h w_n), v \right) + \left( P_\varepsilon P_+ (a^2 (x_1) \partial_x w_n), v \right) + \]

\[ \left( P_\varepsilon P_+ (ib (x_1) \partial_x w_n), v \right) + \left( P_\varepsilon P_+ \left( \hat{F}_n \right), v \right), \] (3.17)

for all \( v \in C^\infty_0 (R^n, H) \), where \((u, v)\) denotes scalar product of \( u (x, t) \) and \( v (x) \) in \( H \) for all \( x \in R^n \) and \( t \in [0, 1] \). Taking the complex conjugate of the above, we get the equation

\[ \bar{K} \left( i \partial_t \left( P_\varepsilon P_+ w_n, v \right) \right) + \bar{K} \left( \Delta P_\varepsilon P_+ w_n, v \right) + \bar{K} \left( AP_\varepsilon P_+ w_n, v \right) = \]

\[ \bar{K} \left( P_\varepsilon P_+ (V w_n), v \right) + \bar{K} \left( P_\varepsilon P_+ (h w_n), v \right) + \bar{K} \left( P_\varepsilon P_+ (a^2 (x_1) \partial_x w_n), v \right) + \]

(3.18)
\( \tilde{K} (P_z P_+ (ib (x_1) \partial_{x_1}, w_n), v) + \tilde{K} \left( P_z P_+ \left( \tilde{F}_n \right), v \right) , \)

here \( \tilde{K} (u) \) denotes the complex conjugate of \( u \).

Multiplying (3.17) and (3.18) by \( \tilde{K} ((P_z P_+ w_n, v)) \) and \( -(P_z P_+ w_n, v) \) respectively, and adding the result, we obtain

\[
i \partial_t \| (P_z P_+ w_n, v) \|^2 + (\Delta P_z P_+ w_n, v) \tilde{K} ((P_z P_+ w_n, v)) = 0
\]

By taking the imaginary part in (3.19) we get

\[
2 \text{Im} (\Delta P_z P_+ w_n, v) (\tilde{K} (P_z P_+ w_n, v)) + 2 \text{Im} (AP_z P_+ w_n, v) (\tilde{K} (P_z P_+ w_n, v)) + 2 \text{Im} ((P_z P_+ hw_n), v) (\tilde{K} (P_z P_+ w_n, v))
\]

Since for all \( n \in \mathbb{Z}^+, w_n (.) \in X, \tilde{F}_n (., t) \in X \) and for a.e. \( t \in [0,1] \) by integrating both sides of (3.20) on \( \mathbb{R}^3 \) we get

\[
\int_{\mathbb{R}^3} \text{Im} (\Delta P_z P_+ w_n, v) (\tilde{K} (P_z P_+ w_n, v)) \, dx = 0.
\]
It is clear to see that
\[(u, v)_X = (u (.), v (.)_H)_{L^2(R^n)}, \text{ for } u, v \in X.
\]

Then applying the Cauchy-Schwartz and Holder inequalities for a.e. \( t \in [0, 1] \) we obtain
\[
\int_{R^n} \text{Im} (P_\varepsilon P_+ V w_n, v) (K (P_\varepsilon P_+ w_n, v)) \, dx \leq C \|V\|_B \|w_n\|_X^2 \|v\|_X^2, \tag{3.21}
\]
\[
\int_{R^n} \text{Im} (P_\varepsilon P_+ h w_n, v) K (P_\varepsilon P_+ w_n, v) \, dx \leq C \|h\|_{L^\infty} \|w_n\|_X^2 \|v\|_X^2, \tag{3.22}
\]
\[
\int_{R^n} \text{Im} (P_\varepsilon P_+ \tilde{V} w_n, v) (K (P_\varepsilon P_+ w_n, v)) \, dx \leq C \|\tilde{V}\|_X \|w_n\|_X^2 \|v\|_X^2. \tag{3.23}
\]

Moreover, again applying the Cauchy-Schwartz and Holder inequalities due to symmetricity of the operator \( A \), for a.e. \( t \in [0, 1] \) we get
\[
\int_{R^n} \text{Im} (A P_\varepsilon P_+ w_n, v) (K (P_\varepsilon P_+ w_n, v)) \, dx \leq C \|A w_n\|_X \|w_n\|_X \|v\|_X^2 \tag{3.24}
\]
where, the constant \( C \) in (3.21) – (3.24) is independent of \( v \in C^\infty_0 (R^n, H) \), \( \varepsilon \in (0, 1) \) and \( n \in Z^+ \).

Since \( C^\infty_0 (R^n, H) \) is dense in \( X \), from (3.24)-(3.27) in view of operator theory in Hilbert spaces, we obtain the following
\[
\int_{R^n} \text{Im} (P_\varepsilon P_+ V w_n, KP_\varepsilon P_+ w_n) \, dx \leq C \|V\|_B \|w_n\|_X^2, 
\]
\[
\int_{R^n} \text{Im} (P_\varepsilon P_+ h w_n, KP_\varepsilon P_+ w_n) \, dx \leq C \|h\|_{L^\infty} \|w_n\|_X^2 \leq C \frac{1}{n} \|w_n\|_X^2, 
\]
\[
\int_{R^n} \text{Im} (P_\varepsilon P_+ \tilde{V} w_n, KP_\varepsilon P_+ w_n) \, dx \leq C \|\tilde{V}\|_X \|w_n\|_X^2, \tag{3.25}
\]
\[
\int_{R^n} \text{Im} (A P_\varepsilon P_+ w_n, KP_\varepsilon P_+ w_n) \, dx \leq C \|A w_n\|_X \|w_n\|_X^2.
\]

For bounding the last two terms in (3.20) we will use the abstract version of Calderón’s first commutator estimates [4]. Really by Cauchy-Schwartz inequality and in view of Theorem A_{1} we get
\[
\|([P_{\pm} a] \partial_x f, v)\|_X \leq C \|\partial_x a\|_{L^\infty} \|f\|_X \|v\|_X, \tag{3.26}
\]
\[ \| \partial_x (\mathbb{P}_a f, v) \|_X \leq C \| \partial_x a \| L^\infty \| f \|_X \| v \|_X, \quad (3.27) \]

Also, from the calculus of pseudodifferential operators with operator coefficients (see e.g. [5]) and the inequality (3.15), we have

\[ \| (C_0 \partial_x a f, v) \|_X \leq C_n \| f \|_X \| v \|_X, \quad (3.28) \]

\[ \| \partial_x (C_0 a f, v) \|_X \leq C_n \| f \|_X \| v \|_X, \quad (3.29) \]

where the constant \( C \) in (3.26) – (3.29) is independent of \( \varepsilon \in [0, 1] \) and \( n \).

We remark that estimates (3.26) – (3.29) also hold with \( b(x_1) \) replacing \( a(x_1) \). Since \( C_0^\infty (R^n; H) \) is dense in \( X \), from (3.26)-(3.29) in view of operator theory in Hilbert spaces, we obtain

\[ \| (P_0 a) \partial_x f \|_X \leq C \| \partial_x a \| L^\infty \| f \|_X, \quad (4.1) \]

\[ \| (P_0 a) \partial_x (f, v) \|_X \leq C_n \| f \|_X \| v \|_X, \quad (4.2) \]

and the same estimates (4.30) with \( b(x_1) \) replacing \( a(x_1) \).

By reasoning as in [4, Lemma 2.1] (claim 1 and 2) from (3.30) we obtain

\[ \| \text{Im} \left( P_0^2 (x_1) \partial_x w_n, \mathcal{K} P_0 P_1 w_n \right) \| \leq O \left( n^{-1} \| w_n \|_X \right), \quad (4.3) \]

\[ \| \text{Im} \left( P_0 b(x_1) \partial_x w_n, \mathcal{K} P_0 w_n \right) \| \leq O \left( n^{-1} \| w_n \|_X \right). \]

Now, the estimates (3.25) and (4.3) imply the assertion.

4. Proof of Theorem 1.

We will apply Lemma 3.1 to a solution of the equation (3.2). Since \( 0 < \alpha < \beta = \beta(k) \) for \( k > k_0 \) it follows that \( \alpha \leq \sigma(t) \leq \beta \) for any \( t \in [0, 1] \). Therefore if \( y = \sqrt{\alpha \beta x} \) (t), then from (3.8) we get

\[ \sqrt{\alpha \beta} |x| \leq |y| \sqrt{\alpha^{-1} \beta} |x| = (k a_0^{-1}) \hat{\beta} |x| \quad (4.1) \]

Thus,

\[ \| \alpha \beta \partial_x^2 (t) V \left( \sqrt{\alpha \beta x} \sigma(t), \beta t \sigma(t) \right) \|_{L^2(H)} \leq \alpha^{-1} \beta \| V \|_B \leq \left( k a_0^{-1} \hat{\beta} \right) \| V \|_B \quad (4.2) \]

and so,

\[ \| \hat{V} (., t) \|_{L^2(R^n; H)} \leq \left( k a_0^{-1} \right) \hat{\beta} \| V (., t) \|_{L^2(R^n; H)}, \quad (4.3) \]
Also, for \( s = \beta t \sigma(t) \) it is clear that
\[
\frac{ds}{dt} = \alpha \beta \sigma^2(t), \quad dt = (\alpha \beta)^{-1} \sigma^{-2}(t) \, ds.
\] (4.4)

Therefore,
\[
\int_0^1 \left\| \tilde{V}(.,t) \right\|_{L^\infty(R^n;H)} \, dt = \int_0^1 \left\| V(.,s) \right\|_{L^\infty(R^n;H)} \, ds,
\]
and from (4.1) we get
\[
\int_0^1 \left\| \tilde{V}(.,t) \right\|_{L^\infty(|x| > R;H)} \, dt = \int_0^1 \left\| V(.,s) \right\|_{L^\infty(|y| > \kappa;H)} \, ds,
\] (4.5)

where \( \kappa = (a_0 k^{-1})^{\frac{1}{p}} R \).

So, if
\[
\int_0^1 \left\| V(.,s) \right\|_{L^\infty(|y| > \kappa;H)} \, ds < \varepsilon_0
\]
then,
\[
\int_0^1 \left\| \tilde{V}(.,t) \right\|_{L^\infty(|y| > \kappa;H)} \, ds < \varepsilon_0, \quad \text{for} \quad R = \kappa (ka_0^{-1})^{\frac{1}{p}}
\]
and we can apply Lemma 3.1 to the equation (3.2) with
\[
\tilde{V} = \tilde{V}_{X(|x| > R)} (x,t), \quad \tilde{F} = \tilde{V}_{X(|x| < R)} (x,t) \tilde{u}(x,t)
\]
to get the following estimate
\[
\sup_{t \in [0,1]} \left\| e^{\nu} \tilde{u}(.,t) \right\|_{L^\infty} \leq M_0 \left( \left\| e^{\nu} \tilde{u}(.,0) \right\|_{L^\infty} + \left\| e^{\nu} \tilde{u}(.,1) \right\|_{L^\infty} \right) + M_0 e^{M_0 e^{\nu_0}} \left\| \tilde{V} \right\|_{B} \left\| u(.,0) \right\|_{L^\infty},
\] (4.6)

where \( M \) a positive constant defined in Remark 2.1 and
\[
B = L^\infty(R^n \times [0,1]; L(H)), \quad \nu = (2p)^{\frac{1}{p}} \gamma \frac{1}{p} \lambda \frac{x}{2}, \quad \nu_0 = |\lambda| (2p)^{\frac{1}{p}} \gamma \frac{1}{p} \frac{R}{2}.
\]

From (4.6) we have
\[
\sup_{t \in [0,1]} \int_{R^n} \left\| e^{\nu} \tilde{u}(.,t) \right\|^2_H \, dx \leq M_0 \int_{R^n} e^{\nu} \left( \left\| \tilde{u}(.,0) \right\|^2_H + \left\| \tilde{u}(.,1) \right\|^2_H \right) \, dx +
\]
and multiply the above inequality by $e^{|\lambda|/q |\lambda|^{n(q-2)/2}}$, integrate in $\lambda$ and in $x$, use Fubini theorem and the following formula

$$
 M_0 e^{M} |\lambda|^2 \gamma^\frac{1}{p} \mathbb{E} \|V\|_B \|u(.,0)\|^2_X,
$$

proven in [7, Appendix] to obtain

$$
 \int_{|x|>1} e^{2|\lambda|p} \|\tilde{u}(.,t)\|^2_B \, dx \leq M_0 \int_{R^n} e^{2|\lambda|p} \left(\|\tilde{u}(.,0)\|^2_H + \|\tilde{u}(.,1)\|^2_H\right) \, dx +

M_0 e^{2M} e^{2\gamma R} R^C \mathbb{E} \|V\|_B \|u(.,0)\|^2_X.
$$

Hence, the estimates (3.6), (3.8), (3.9), (4.3) and (4.8) imply

$$
 \sup_{t \in [0,1]} \|e^{|\alpha|p} \tilde{u}(.,t)\|_X \leq M_0 \left( \|e^{|\alpha|p} \tilde{u}(.,0)\|_X + \|e^{|\alpha|p} \tilde{u}(.,1)\|_X \right) +

M_0 e^{M} \|u(.,0)\|_X + M_0 \left( k a_0^{-1} \right) \mathbb{E} \|u(.,0)\|_X \|V\|_B \leq (4.9)

M_0 (A_0 + A_k) + M_0 e^M \|u(.,0)\|_X \left( \gamma + \left( k a_0^{-1} \right) \mathbb{E} \|V\|_B \right) e^{k \gamma p} \leq

M_0 A_k = M_0 e^{a_1 k^{1/2-p}} \text{ for } k > k_0 (M_0) \text{ sufficiently large.}

Next, we shall obtain bounds for the $\nabla \tilde{u}$. Let $\tilde{\gamma} = \frac{\gamma}{2}$ and $\varphi$ be a strictly convex complex valued function on compact sets of $R^n$, radial such that (see [7])

$$
 D^2 \varphi \geq p(p-1)|x|^{(p-2)}, \text{ for } |x| \geq 1,

\varphi \geq 0, \|\partial^\alpha \varphi\|_{L^\infty} \leq C, 2 \leq |\alpha| \leq 4, \|\partial^\alpha \varphi\|_{L^\infty(|x|<2)} \leq C \text{ for } |\alpha| \leq 4,

\varphi(x) = |x|^p + O(|x|), \text{ for } |x| > 1.

Let us consider the equation

$$
 \partial_t v = i (\Delta v + Av + F(x,t)), \ x \in R^n, \ t \in (0,1),
$$

where $F(x,t) = \tilde{V} v$, $A$ is a symmetric operator in $H$ and $\tilde{V}$ is a operator in $H$ defined by (3.3).

Let

$$
 f(x,t) = e^{\tilde{\gamma} \varphi} v(x,t), \ Q(t) = (f(x,t), f(x,t))_H,
$$

where $v$ is a solution of (4.10). Then, by reasoning as in [17, Lemma 3.3] we have

$$
 \partial_t f = S f + K f + i \left[ A + e^{\tilde{\gamma} \varphi} F \right], \ (x,t) \in R^n \times [0,1],
$$

(4.11)
where $S$, $K$ are symmetric and skew-symmetric operator, respectively given by

$$S = -i\hat{\gamma} (2\nabla \varphi \cdot \nabla + \Delta \varphi), \quad K = i \left( \Delta + A + \hat{\gamma}^2 |\nabla \varphi|^2 \right). \quad (4.12)$$

Let

$$[S, K] = SK - KS.$$

A calculation shows that,

$$SK = \hat{\gamma} (2\nabla \varphi \cdot \nabla + \Delta \varphi) \left( \Delta + A + \hat{\gamma}^2 |\nabla \varphi|^2 \right) = \hat{\gamma} (2\nabla \varphi \cdot \nabla + \Delta \varphi) \Delta +$$

$$\hat{\gamma} (2\nabla \varphi \cdot \nabla + \Delta \varphi) A + \hat{\gamma}^3 |\nabla \varphi|^2 (2\nabla \varphi \cdot \nabla + \Delta \varphi),$$

$$KS = \hat{\gamma} \left( \Delta (2\nabla \varphi \cdot \nabla + \Delta \varphi) + A (2\nabla \varphi \cdot \nabla + \Delta \varphi) \right) + \hat{\gamma}^3 |\nabla \varphi|^2 (2\nabla \varphi \cdot \nabla + \Delta \varphi),$$

$$[S, K] = \hat{\gamma} \left( (2\nabla \varphi \cdot \nabla + \Delta \varphi) \Delta - \Delta (2\nabla \varphi \cdot \nabla + \Delta \varphi) \right) + 2\hat{\gamma} (\nabla \varphi \cdot \nabla A - A\nabla \varphi \cdot \nabla).$$

By [17, Lemma 3.2]

$$Q''(t) = 2\partial_t \text{Re} \left( \partial_t f - Sf - Kf, f \right)_X + 2 \left( S_t f + [S, K] f, f \right)_X +$$

$$\| \partial_t f - Sf + Kf \|^2_X - \| \partial_t f - Sf - Kf \|^2_X, \quad (4.14)$$

so,

$$Q''(t) = 2\partial_t \text{Re} \left( \partial_t f - Sf - Kf, f \right)_X + 2 \left( S_t f + [S, K] f, f \right)_X. \quad (4.15)$$

Multiplying (4.15) by $t(1-t)$ and integrating in $t$ we obtain

$$\int_0^1 t(1-t) \left( S_t f + [S, K] f, f \right)_X dt \leq M_0 \left[ \sup_{t \in [0,1]} \| e^{\hat{\gamma} \varphi} v \| X + \sup_{t \in [0,1]} \| e^{\hat{\gamma} \varphi} F \| X \right].$$

This computation can be justified by parabolic regularization using the fact that we already know the decay estimate for the solution of (4.10). Hence, combining (3.8), (4.3) and (4.9) it follows that

$$\hat{\gamma} \int_0^1 \int_{\mathbb{R}^n} t(1-t) D^2 \varphi (x, t) (\nabla f, \nabla f)_H \, dx \, dt + \hat{\gamma}^3 \int_0^1 \int_{\mathbb{R}^n} t(1-t) D^2 \varphi (x, t) (\nabla f, \nabla f)_H \, dx \, dt \leq$$

$$M_0 \left[ \sup_{t \in [0,1]} \| e^{\hat{\gamma} \varphi} v \| X \left( 1 + \| \tilde{V} (., t) \|_{L^\infty (\mathbb{R}^n; L(H))} \right) + \hat{\gamma} \sup_{t \in [0,1]} \| e^{\hat{\gamma} \varphi} v \| X \right] \leq \quad (4.16)$$
\[ M_0 k^C p A_k. \]

It is clear to see that
\[ \nabla f = \tilde{\gamma} e^\tilde{\varphi} \nabla \varphi + e^{\tilde{\varphi}} \nabla v. \]

So, by using the properties of \( \varphi \) we get
\[ \left| e^{2\tilde{\varphi}} D^2 \varphi \right| \leq C \rho e^{3\gamma \varphi^2}. \]

From here, we can conclude that
\[ \gamma \int_0^1 \int_R (1 - t) (1 + |x|)^{p-2} \| \nabla v (x, t) \|_H e^{\gamma |x|^p} dxd + \sup_{t \in [0, 1]} \left\| e^{\frac{\gamma |x|^p}{\alpha}} v (., t) \right\|_X \leq C_0 k^C p A_k^2 = C_0 k^C p e^{a(k, p)} (4.17) \]

for \( k \geq k_0(M_0) \) sufficiently large, where
\[ a(k, p) = C \mu M_0 k^C p e^{2a_1 (2 - p)}. \]

For proving Theorem 1 first, we deduce the following estimate
\[ \int_{|x| < \frac{4}{\nu_1}} \int_R \| \tilde{u} (x, t) \|_H dtdx \geq C_0 e^{-M} \| u (. , 0) \|_X, \quad (4.18) \]

for \( R \) sufficiently large, \( \nu_1, \nu_2 \in (0, 1), \nu_1 < \nu_2 \) and \( \nu = (\nu_1, \nu_2) \). From (3.1) by using the change of variables \( s = \beta \sigma (t) \) and \( y = \sqrt{\alpha \beta x \sigma (t)} \) we get
\[ \int_{|x| < \frac{4}{\nu_1}} \int_R \| \tilde{u} (x, t) \|_H^2 dtdx = (\alpha \beta)^{\frac{3}{2}} \int_{|x| < \frac{4}{\nu_1}} \int_R \| \sigma (t) \|^\alpha \left\| u \left( \sqrt{\alpha \beta x \sigma (t)} , \beta \sigma (t) \right) \right\|_H^2 dtdx \geq \]
\[ M_0 \frac{\beta}{\alpha} \int_{|y| < R_0 s^2} \| u (y, s) \|^\frac{2}{H} dsdy \geq M_0 \frac{\beta}{\alpha} \int_{|y| < R_0 s^2} \| u (y, s) \|^\frac{2}{H} dsdy \quad (4.19) \]

for \( k > M_0, s\nu_1 > \frac{1}{2} \) and \( R_0 = R \left( ka_0^{-1} \right)^{\frac{1}{2}} \). Thus, taking
\[ R > \omega \left( ka_0^{-1} \right)^{\frac{1}{2}} \quad (4.20) \]

with \( \omega = \omega (u) \) a constant to be determined, it follows that
\[ \Phi \geq M_0 \frac{\beta}{\alpha} \int_{|y| < \omega s^2} \| u (y, s) \|^\frac{2}{H} dsdy, \]
where the interval $I = I_k = [s \nu_1, s \nu_2]$ satisfies $I \subset [1/2, 1]$ for $k$ sufficiently large. Moreover, given $\varepsilon > 0$ there exists $k_0(\varepsilon) > 0$ such that for any $k \geq k_0$ one has that $I_k \subset [1 - \varepsilon, 1]$. By hypothesis on $u(x, t)$, i.e. the continuity of $\|u(\cdot, s)\|_X$ at $s = 1$, it follows that there exists $\omega > 1$ and $K_0 = K_0(u)$ such that for any $k \geq K_0$ and for any $s \in I_k$

$$\int_{|y| < \omega} \|u(y, s)\|^2_H \, dy \geq C_\nu e^{-M} \|u(\cdot, 0)\|_X,$$

which yields the desired result. Next, we deduce the following estimate

$$\int \int_{|x| < R} \left( \|\tilde{u}(x, t)\|_H^2 + \|\nabla \tilde{u}(x, t)\|_H^2 + \|A\tilde{u}(x, t)\|_H^2 \right) \, dtdx \leq C_{\nu_1} C_\nu k C_p e^{\alpha(k, p)},$$

(4.21)

for $R$ sufficiently large, $\nu_1, \nu_2 \in (0, 1)$, $\mu_1 = \frac{(\nu_2 - \nu_1)}{8}$, $\mu_2 = 1 - \mu_1$, $\mu_1 < \mu_2$ and $\mu = (\mu_1, \mu_2)$.

Indeed, from (3.9) and (4.17) we obtain

$$\int \int_{|x| < R \mu_1} \|\tilde{u}(x, t)\|_H \, dtdx \leq C_{\mu} e^{2M} \|u(\cdot, 0)\|_X,$$

$$\int \int_{\mu_1 |x| < R} \|\nabla \tilde{u}(x, t)\|_H \, dtdx \leq C_{\nu_1} \int \int_{\mu_1 |x| < R} t(1 - t) \|\nabla u(x, t)\|_H e^{-\gamma|x|^p} \, dtdx \leq$$

$$C_{\mu} \gamma^{-1} R^{2-p} C_\nu k C_p A_k^2 \leq C_{\mu} C_\nu k C_p e^{2\alpha(k, p)^{-1}}.$$

Hence, from (4.22) we get (4.21) for $k \geq k_0(C_\nu)$ sufficiently large.

Let $Y = L^2(R^n \times [0, 1]; H)$. By reasoning as in [6, Lemma 3.1] we obtain

**Lemma 4.1.** Let $A$ be a positive operator in the Hilbert space $H$ and $iA$ generates a semigroup $U(t) = e^{iAt}$. Assume that $R > 0$ and $\varphi : [0, 1] \to \mathbb{R}$ is a smooth function. Then, there exists $C = C(n, \varphi, H, A) > 0$ such that the inequality

$$\frac{x^2}{R^2} \left\| e^{\alpha|\varphi|} g \right\|_Y \leq C \left\| e^{\alpha|\varphi|} i (\partial_t g + \Delta g + Ag) \right\|_Y$$

holds, for $\alpha \geq CR^2$ and $g \in C_0(\mathbb{R}^{n+1}; H)$ with support contained in the set

$$\left\{ x, t : |\varphi(x, t)| = \frac{x}{R} + \varphi(t) e_1 \geq 1 \right\}.$$

**Proof.** Let $f = e^{\alpha|\varphi(x, t)|^2} g$. Then, by acts of Schredinger operator $(i\partial_t + \Delta + A)$ to $f \in X$ we get

$$e^{\alpha|\varphi(x, t)|^2} ((i\partial_t g + \Delta g + Ag)) = S_\alpha f - 4\alpha A_\alpha f,$$

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where

\[ S_\alpha = (i \partial_t + \Delta + A) + \frac{4 \alpha^2}{R^2} |\psi(x,t)|^2, \]

\[ A_\alpha = \frac{1}{R} \psi(x,t) \nabla + \frac{n}{R^2} + \frac{i \varphi'}{2} \left( \frac{x}{R} + \varphi(t) \right). \]

Hence,

\[ (S_\alpha)^* = S_\alpha, \quad (A_\alpha)^* = A_\alpha \]

and

\[ \left\| e^{\alpha |\psi(x,t)|^2} (i \partial_t g + \Delta g + Ag) \right\|_X^2 = (S_\alpha f - 4 \alpha A_\alpha f, S_\alpha f - 4 \alpha A_\alpha f)_X \geq -4 \alpha ((S_\alpha A_\alpha - A_\alpha S_\alpha), f)_X. \]

A calculation shows that

\[ [S_\alpha, A_\alpha] = \frac{2}{R^2} \Delta - \frac{4 \alpha^2}{R^4} \left| \frac{x}{R} + \varphi e_1 \right|^2 - \frac{1}{2} + \frac{2i \varphi'}{R} \partial_{x_1} \]

and

\[ \left\| e^{\alpha |\psi(x,t)|^2} (i \partial_t g + \Delta g + Ag) \right\|_X^2 \geq \frac{16 \alpha^3}{R^4} \int \left| \frac{x}{R} + \varphi e_1 \right|^2 \|f(x,t)\|_H^2 \ dx \ dt + \frac{8 \alpha}{R^2} \int \|\nabla f(x,t)\|_H^2 \ dx \ dt + \]

\[ 2 \alpha \int \left[ \left( \frac{x}{R} + \varphi \right) \varphi'' + (\varphi')^2 \right] \|f(x,t)\|_H^2 \ dx \ dt - \frac{8 \alpha}{R} \int \varphi' \partial_{x_1} f \ f \ dx \ dt. \]

Hence, using the hypothesis on the support on \(g\) and the Cauchy–Schwarz inequality, the absolute value of the last two terms in (4.23) can be bounded by a fraction of the first two terms on the right-hand side of (4.23), when \(\alpha > CR^2\) for some large \(C\) depending on \(\|\varphi\|_\infty + \|\varphi''\|_\infty\). This yields the assertion.

Now, from (3.3) we have

\[ \left\| \hat{V} (x,t) \right\|_H \leq \frac{\alpha}{\beta} \mu_1^{-2} \left\| V \right\|_B \leq \mu_1^{-2} \mu_2^\frac{1}{2} k^{-\frac{1}{2}} \left\| V \right\|_B. \]

Then from (4.20) we get

\[ \left\| \hat{V} \right\|_{L^\infty(\mathbb{R}^n \times [\mu_1, \mu_2]; L(H))} \leq R. \quad (4.24) \]

Define

\[ \delta^2(R) = \int_{\mu_1}^{\mu_2} \int_{R^{1-R} \leq |x| \leq R} \left( \|\hat{u}(x,t)\|_H^2 + \|\nabla \hat{u}(x,t)\|_H^2 + \|A\hat{u}(x,t)\|_H^2 \right) \ dt \ dx. \quad (4.25) \]
Let \( \nu_1, \nu_2 \in (0, 1) \), \( \nu_1 < \nu_2 \) and \( \nu_2 < 2\nu_1 \). We choose \( \varphi \in C^\infty([0,1]) \) and \( \theta, \theta_R \in C^0_{\infty}(\mathbb{R}^n) \) satisfying

\[
0 \leq \varphi(t) \leq 3, \quad \varphi(t) = 3 \text{ for } t \in [\nu_1, \nu_2], \quad \varphi(t) = 0 \text{ for } t \in [0, \nu_2 - \nu_1] \cup \left[ \nu_2 + \frac{\nu_2 - \nu_1}{2}, 1 \right],
\]

\[
\theta_R(x) = 1 \text{ for } |x| < R - 1, \quad \theta_R(x) = 0, \text{ for } |x| > R,
\]

and

\[
\theta(x) = 1 \text{ for } |x| < 1, \quad \theta(x) = 0, \text{ for } |x| \geq 2.
\]

Let \( g(x, t) = \theta_R(x) \theta(\psi(x, t)) \hat{u}(x, t) \), \hspace{1cm} (4.26)

where \( \hat{u}(x, t) \) is a solution of (3.2) when \( \hat{V} = \hat{F} \equiv 0 \). It is clear to see that

\[
|\psi(x, t)| \geq 5 \text{ for } |x| < R_2 \text{ and } t \in [\nu_1, \nu_2].
\]

Hence,

\[
g(x, t) = \hat{u}(x, t) \text{ and } e^{\kappa|\psi(x, t)|^2} \geq e^{\frac{2\kappa}{R}} \text{ for } |x| < \frac{R}{2} \text{ and } t \in [\nu_1, \nu_2].
\]

Moreover, from (4.26) also we get that

\[
g(x, t) = 0 \text{ for } |x| \geq R \text{ or } t \in [0, \nu_2 - \nu_1] \cup \left[ \nu_2 + \frac{\nu_2 - \nu_1}{2}, 1 \right],
\]

so

\[
supp g \subset \{ |x| \leq R \} \times \left[ \nu_2 - \nu_1, \nu_2 + \frac{\nu_2 - \nu_1}{2} \right] \cap \{ |\hat{u}(x, t)| \geq 1 \}.
\]

Then, for \( \xi = \psi(x, t) \) we have

\[
\left( i\partial_t + \Delta + A + \hat{V} \right) g = [\theta(\xi) (2\nabla \theta_R(x) \cdot \hat{u} + \hat{u} \Delta \theta_R(x)) + 2 \nabla \theta(\xi) \cdot \nabla \theta_R \hat{u}] + \theta_R(x) \left[ 2R^{-1} \nabla \theta(\xi) \cdot \nabla \hat{u} + R^{-2} \hat{u} \Delta \theta(\xi) + i\varphi' \partial_x \theta(\xi) w \right] = B_1 + B_2.
\]

Note that,

\[
supp B_1 \subset \{ (x, t) : R - 1 \leq |x| \leq R, \mu_1 \leq t \leq \mu_2 \}
\]

and

\[
supp B_2 \subset \{ (x, t) \in \mathbb{R}^n \times [0,1], 1 \leq |\psi(x, t)| \leq 2 \}.
\]

Now applying Lemma 4.1 choosing \( \kappa = d_n^2 R^2 \), \( d_n^2 \geq \|\varphi'\|_\infty + \|\varphi''\|_\infty \) it follows that

\[
R \left\| e^{\kappa|\psi|^2} g \right\| \leq C \left\| e^{\kappa|\psi|^2} i (\partial_t g + \Delta g + Ag) \right\| \leq \quad (4.27)
\]
\[
C \left[ \| e^{\kappa |\psi|^2} \hat{\psi} \|_Y + \| e^{\kappa |\psi|^2} B_1 \|_Y + \| e^{\kappa |\psi|^2} B_2 \|_Y \right] = D_1 + D_2 + D_3.
\]

Since
\[
\left\| \hat{\psi} \right\|_{L^\infty(\mathbb{R}^n \times [\mu_1, \mu_2]; L(H))} < R,
\]

\(D_1\) can be absorbed in the left hand side of (4.27). Moreover, \(|\psi(x,t)| \leq 4\) on the support of \(B_1\), thus
\[
D_2 \leq C \delta(R) e^{16\kappa}.
\]

Let
\[
R_\mu = \{ (x,t) : |x| \leq R, \mu_1 \leq t \leq \mu_2 \}
\]

Then \(R_\mu \subset \text{supp} \ B_2\), and \(1 \leq |\psi(x,t)| \leq 2\), so
\[
D_3 \leq C e^{4\kappa} \| \tilde{u} + \nabla \tilde{u} \|_{L^2(\mathbb{R}^n \times [\mu_1, \mu_2]; H)}.
\]

By using (4.18) and (4.22) we have
\[
C_\mu e^{-M} e^{\frac{2\kappa}{\nu}} \| u(.,0) \|_X \leq Re^{\frac{2\kappa}{\nu}} \left[ \int_{|x| < \frac{1}{2} \nu_1} \int_{\nu_1}^{\nu_2} \| \tilde{u}(x,t) \|_H \, dt \, dx \right] \leq (4.28)
\]

\[
C_\mu \delta(R) e^{\frac{2\kappa}{\nu}} + C_\mu e^{4\kappa} \| \tilde{u} + \nabla \tilde{u} \|_{L^2(R_\mu; H)} \leq C_\mu \delta (R) e^{16\kappa} + C_\mu C_0 k^2 e^{4\kappa} e^{2a_1 k^{(2-p)^{-1}}}
\]

Putting \(\kappa = d_a R^2 = 2a_1 k^{\frac{1}{2-p}}\) it follows from (4.28) that, if \(\|u(.,0)\|_X \neq 0\) then
\[
\delta(R) \geq C_\mu \| u(.,0) \|_X e^{-(M + 10\kappa)} = C_\mu \| u(.,0) \|_X e^{-(M + 20)a_1 k^{\frac{1}{2-p}}} \quad (4.29)
\]

for \(k \geq k_0(C_\mu)\) sufficiently large.

Now, by (4.22) we get
\[
\delta^2(R) = \int_{\nu_1}^{\nu_2} \int_{R-1 \leq |x| \leq R} \left( \| \tilde{u}(x,t) \|_H^2 + \| \nabla \tilde{u}(x,t) \|_H^2 + \| A \tilde{u}(x,t) \|_H^2 \right) \, dt \, dx \leq
\]
\[
\int_{\nu_1}^{\nu_2} \int_{R-1 \leq |x| \leq R} \left( \| \tilde{u}(x,t) \|_H^2 + \| A \tilde{u}(x,t) \|_H^2 \right) \, dt \, dx +
\]
\[
C_\mu \int_{\nu_1}^{\nu_2} \int_{R-1 \leq |x| \leq R} t (1-t) \| \nabla \tilde{u}(x,t) \|_H^2 \, dt \, dx \leq
\]

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\[ C_{\mu} e^{-\gamma(R-1)^{p}} \sup_{t \in [0,1]} \left| e^{\gamma|x|^{p/2}} \tilde{u}(x,t) \right|_{X}^{2} + C_{\mu} \gamma^{-1} R^{2-p} e^{-\gamma(R-1)^{p}} \times \]

\[ \int_{\mu} \int_{R^{n}} \left[ \frac{t(1-t)}{(1+|x|)^{2/p}} \left\| \nabla \tilde{u}(x,t) \right\|_{H}^{2} + \| A \tilde{u}(x,t) \|_{H}^{2} \right] dt \|_{x} \leq (4.30) \]

The estimates (4.28) – (4.30) imply

\[ C_{\mu} e^{-2M} e^{\frac{2a_{1}}{2-p} \| u(.,0) \|_{X}} \leq C_{0} k C_{p} e^{\omega(p)} + O \left( \frac{k^{1/2(2-p)}}{2a_{1}} \right), \quad \omega(p) = 42a_{1} k^{1/(2-p)} - a_{0} \left( \frac{2a_{1}}{d_{n}} \right)^{2} k^{1/(2-p)}. \]

Hence, if \( 42a_{1} < \sqrt{a_{1}^{2} a_{0} \left( \frac{2a_{1}}{d_{n}} \right)^{2}} \), by letting \( k \) tends to infinity it follows from (4.31) that \( \| u(.,0) \|_{X} = 0 \), which gives \( u(x,t) \equiv 0 \).

**Proof of Corollary 1.** Since

\[ \int_{R^{n}} \left\| u(x,1) \right\|_{H}^{2} e^{2b|x|^{q}} dx < \infty \text{ for } b = \frac{\beta^{q}}{q} \]

one has that

\[ \int_{R^{n}} \left\| u(x,1) \right\|_{H}^{2} e^{2b|x|^{q}} dx \leq \left\| e^{2k|x|^{q} - \beta b|x|^{q}} \right\|_{H} \int_{R^{n}} \left\| u(x,1) \right\|_{H}^{2} e^{2b|x|^{q}} dx. \]

Then, by reasoning as in [7, Corollary 1] we obtain the assertion.

**Proof of Theorem 2.** Indeed, just applying Corollary 1 with

\[ u(x,t) = u_{1}(x,t) - u_{2}(x,t) \]

and

\[ V(x,t) = \frac{F(u_{1}, \bar{u}_{1}) - F(u_{2}, \bar{u}_{2})}{u_{1} - u_{2}} \]

we obtain the assertion of Theorem 2.

**5. Proof of Theorem 3.**

First, we deduce the corresponding upper bounds. Assume

\[ \| u(.,t) \|_{X} = a \neq 0. \]

Fix \( \bar{t} \) near 1, and let

\[ v(x,t) = u(x,t - 1 + \bar{t}), \quad t \in [0,1] \]

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which satisfies the equation (2.12) with
\[
|v(x,0)| \leq \frac{b_1}{(2-\tilde{t})^{n/2}} e^{-\frac{|x|^p}{|x|^p}}, \quad |v(x,1)| \leq \frac{b_1}{(1-\tilde{t})^{n/2}} e^{-\frac{|x|^p}{|x|^p}}
\] (5.1)
where \( A \) is a linear operator, \( V(x,t) \) is a given potential operator function in a Hilbert space \( H \).

From (5.1) we get
\[
\int_{\mathbb{R}^n} \|v(x,0)\|^2_H e^{A_0|x|^q} \, dx = a_0^2, \quad \int_{\mathbb{R}^n} \|v(x,1)\|^2_H e^{A_1|x|^q} \, dx = a_1^2,
\]
where
\[
A_0 = \frac{b_2}{(2-t)^p}, \quad A_1 = \frac{b_2}{(1-t)^p}.
\] (5.2)

For \( V(x,t) = F(u,\bar{u}) \), by hypothesis
\[
\|V(x,t)\|_H \leq C \|u(x,t-1+\tilde{t})\|_H^\theta \leq \frac{C}{(2-t-\tilde{t})^{n/2}} e^{C|x|^p}.
\]

By using Appell transformation if we suppose that \( v(y,s) \) is a solution of
\[
\partial_s v = i (\Delta v + A v + V(y,s) v), \quad y \in \mathbb{R}^n, \quad t \in [0,1],
\]
\( \alpha \) and \( \beta \) are positive, then
\[
\tilde{u}(x,t) = \left( \sqrt{\alpha\beta\sigma(t)} \right)^{\frac{p}{2}} u \left( \sqrt{\alpha\beta\sigma(t)}, \beta t \sigma(t) \right) e^{\eta}.
\] (5.3)

verifies the equation
\[
\partial_t \tilde{u} = i \left[ \Delta \tilde{u} + A \tilde{u} + \tilde{V}(x,t) \tilde{u} + \tilde{\Phi}(x,t) \right], \quad x \in \mathbb{R}^n, \quad t \in [0,1]
\] (5.4)
with \( \tilde{V}(x,t) \), \( \tilde{\Phi}(x,t) \) defined by (3.3), (3.4) and
\[
\left\| e^{\gamma|x|^p} \tilde{u}_k(x,0) \right\|_X = \left\| e^{\gamma\left(\frac{x}{p}\right)^{p/2}|x|^p} v(x,0) \right\|_X = a_0,
\] (5.5)
\[
\left\| e^{\gamma|x|^p} \tilde{u}_k(x,1) \right\|_X = \left\| e^{\gamma\left(\frac{x}{p}\right)^{p/2}|x|^p} v(x,1) \right\|_X = a_1.
\]
It follows from expressions (3.6) and (3.8) that
\[
\gamma \sim \frac{1}{(1-\tilde{t})^{p/2}}, \quad \beta \sim \frac{1}{(1-\tilde{t})^{p/2}}, \quad \alpha \sim 1.
\] (5.6)

Next, we shall estimate
\[
\left\| \tilde{V}(x,t) \right\|_{L^1_t L^\infty_x(L(H),R)},
\]
where 
\[ L^1_1 L^\infty_\infty(L(H), R) = L^1(0, 1; L^\infty(R^n/O_R); L(H)). \]
Thus,
\[ \| \tilde{V}(x, t) \|_{L(H)} \leq \frac{\beta}{\alpha} \| V(y, s) \|_{L(H)} \leq \frac{\beta}{\alpha} \frac{C}{(1 - \bar{t})^{\theta_n/2}} e^{C|y|^p}, \]
with
\[ |y| = \sqrt{\alpha \beta} |x| \sigma(t) \geq \tilde{R} \sqrt{\frac{\alpha}{\beta}} \sim \frac{R}{\sqrt{\beta}} = CR(1 - \bar{t})^{1/2}. \]
Hence,
\[ \| \tilde{V}(., t) \|_{L^\infty(R^n; L(H))} \leq \frac{\beta}{\alpha} \| V(., s) \|_{L^\infty(R^n; L(H))} \leq \frac{C}{(1 - \bar{t})^{1+\theta_n/2}} \]
and
\[ \| \tilde{V}(x, t) \|_{L^1_1 L^\infty_\infty(L(H), R)} \leq \frac{\beta}{\alpha} \| V(y, s) \|_{L^1_1 L^\infty_\infty(R, \beta)} \leq \frac{C}{(1 - \bar{t})^{1+\theta_n/2}} e^{-CR^p(1 - \bar{t})^{1/2}}, \]
where
\[ L^1_1 L^\infty_\infty(R, \beta) = L^1(0, 1; L^\infty(R^n/O_{CR/\sqrt{\pi}}); L(H)). \]
To apply Lemma 3.1 we need
\[ \| \tilde{V}(x, t) \|_{L^1_1 L^\infty_\infty(L(H), R)} \leq \frac{C}{(1 - \bar{t})^{1+\theta_n/2}} e^{-CR^p(1 - \bar{t})^{1/2}} \leq \delta_0. \]
for some \( R \), i.e.,
\[ R \sim \frac{C}{(1 - \bar{t})^{p/2}} \delta(t), \]
where
\[ \delta(t) = \log \frac{1}{\alpha} \phi(t), \phi(t) = \frac{C}{\delta_0 (1 - \bar{t})^{\theta_n/2}}. \]
Let
\[ \mathcal{V} = \tilde{V}_{\chi_{(x > R)}}(x, t), \mathcal{F} = \tilde{V}_{\chi_{(x < R)}}(x, t) \tilde{u}(x, t). \]
By using (5.5) - (5.10), by virtue of Lemma 3.1 and (4.7) we deduced
\[ \sup_{t \in [0, 1]} \| e^{\gamma|x|^p} \tilde{u}(., t) \|_X^2 \leq C \left( \| e^{\gamma|x|^p} \tilde{u}(., 0) \|_X^2 + \| e^{\gamma|x|^p} \tilde{u}(., 1) \|_X^2 \right) + \]
\[ Ca^2 e^{C\gamma R^p} \| \tilde{V}(x, t) \|_X \leq \frac{Ca^2}{(1 - \bar{t})^p} e^{\delta(t)}, \]
where
\[ a = \| u_0 \|_X. \]
Next, using the same argument given in section 4, (4.10) – (4.22), one finds that
\[
\gamma \int_0^1 \int_{\mathbb{R}^n} t(1-t)(1+|x|)^{p-2} \| \nabla \tilde{u}(x,t) \|_H e^{\gamma |x|^p} \, dx \, dt \leq \frac{C a^2}{(1-t)^p} e^{\delta(t)}. 
\]
Now we turn to the lower bounds estimates. Since they are similar to those given in detail in section 3, we obtain that the estimate (4.24) for potential operator function \( \tilde{V}(x,t) \) when \( \theta_n - 1 < \frac{2}{2(2-p)} \), i.e., \( p > \frac{2(\theta_n - 2)}{\theta_n - 1} \).

Finally, we get
\[
e^{C \gamma R^p} \log \phi(t) \leq e^{C\gamma R^p} \leq e^{C (1-t)^{\frac{p}{2}}}, \quad \phi(t) = C (1-t)^{\frac{p}{2}},
\]
for \( p < \frac{\theta}{2} + \frac{p^2}{2(2-p)} \), i.e. \( p > 1 \) that assumed in conditon of Theorem 4, i.e. we obtain the assertion of Theorem 4.

**Remark 5.1.** Let us consider the case \( \theta = 4/n \) in Theorem 4, i.e. \( p > 4/3. \) Then from Theorem 4 we obtain the following result

**Result 5.1.** Assume the conditions of Theorem 4 are satisfied for \( p > 4/3. \) Then \( u(x,t) \equiv 0. \)

**6. Unique continuation properties for the system of Schrödinger equation**

Consider the Cauchy problem for the finite or infinite system of Schrödinger equation

\[
\frac{\partial u_m}{\partial t} = i \left[ \Delta u_m + \sum_{j=1}^{N} a_{mj} u_j + \sum_{j=1}^{N} b_{mj} u_j \right], \quad x \in \mathbb{R}^n, \quad t \in (0,T),
\]

where \( u = (u_1, u_2, \ldots, u_N) \), \( u_j = u_j(x,t) \), \( a_{mj} \) are complex numbers and \( b_{mj} = b_{mj}(x,t) \) are complex valued functions. Let \( l_2 = l_2(N) \) and \( l_2^s = l_2^s(N) \) (see [23, § 1.18]). Let \( A \) be the operator in \( l_2(N) \) defined by

\[
D(A) = \left\{ u = \{u_j\} \in l_2, \quad \|u\|_{l_2(N)} = \left( \sum_{j=1}^{N} 2^{sj} |u_j|^2 \right)^{\frac{1}{2}} < \infty \right\},
\]

\[
A = [a_{mj}], \quad a_{mj} = g_m 2^{sj}, \quad s > 0, \quad m, j = 1, 2, \ldots, N, \quad N \in \mathbb{N}
\]

and

\[
D(V(x,t)) = \{ u = \{u_j\} \in l_2^s \},
\]

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\[ V(x,t) = [b_{mj}(x,t)], \quad b_{mj}(x,t) = g_m(x,t)2^{mj}, \quad m, j = 1, 2, ..., N. \]

Let
\[ X_2 = L^2(R^n; l^2), Y_{s^2} = H^{s^2}(R^n; l^2). \]

From Theorem 1 we obtain the following result

**Theorem 6.1.** Assume there exist the constants \( a_0, a_1, a_2 > 0 \) such that for any \( k \in \mathbb{Z}^+ \) a solution \( u \in C([0,1]; X_2) \) of (6.1) satisfy
\[
\int_{R^n} \|u(x,0)\|_2^2 e^{2a_0|x|^p} dx < \infty, \quad \text{for} \quad p \in (1,2),
\]
\[
\int_{R^n} \|u(x,1)\|_2^2 e^{2k|x|^p} dx < 2e^{2a_1k^{\frac{q}{1+p}}}, \quad \frac{1}{p} + \frac{1}{q} = 1.
\]

Moreover, there exists \( M_p > 0 \) such that
\[
a_0a_1^{p-2} > M_p.
\]

Then \( u(x,t) \equiv 0. \)

**Proof.** It is easy to see that \( A \) is a symmetric operator in \( l^2 \) and other conditions of Theorem 1 are satisfied. Hence, from Theorem 1 we obtain the conclusion.

### 7. Unique continuation properties for nonlinear anisotropic Schrödinger equation

The regularity property of BVP for elliptic equations were studied e.g. in [1, 2]. Let \( \Omega = R^n \times G, \ G \subset R^d, \ d \geq 2 \) is a bounded domain with \((d-1)\)-dimensional boundary \( \partial G \). Let us consider the following problem
\[
i\partial_t u + \Delta_x u + \sum_{|\alpha| \leq 2m} a_\alpha(y) D^\alpha_y u(x,y,t) + F(u, \bar{u}) u = 0, \quad (7.1)
\]
\[
x \in R^n, \ y \in G, \ t \in [0,1],
\]
\[
B_j u = \sum_{|\beta| \leq m_j} b_{j\beta}(y) D^\beta_y u(x,y,t) = 0, \ x \in R^n, \ y \in \partial G, \ j = 1, 2, ..., m, \quad (7.2)
\]
where \( a_\alpha, b_{j\beta} \) are the complex valued functions, \( \alpha = (\alpha_1, \alpha_2, ..., \alpha_n), \ \beta = (\beta_1, \beta_2, ..., \beta_n), \ \mu_i < 2m, \ K = K(x,y,t) \) and
\[
D^k_x = \frac{\partial^k}{\partial x^k}, \quad D_j = -i \frac{\partial}{\partial y_j}, \quad D_y = (D_1, ..., D_n), \quad y = (y_1, ..., y_n).
\]
Let
\[ \xi' = (\xi_1, \xi_2, \ldots, \xi_{n-1}) \in R^{n-1}, \quad \alpha' = (\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) \in Z^n, \]
\[ A \left( y_0, \xi', D_y \right) = \sum_{|\alpha'|+j\leq 2m} a_{\alpha'} \left( y_0 \right) \xi_1^{\alpha_1} \xi_2^{\alpha_2} \ldots \xi_{n-1}^{\alpha_{n-1}} D^j_y \text{ for } y_0 \in \bar{G} \]
\[ B_j \left( y_0, \xi', D_y \right) = \sum_{|\beta'|+j\leq m_j} b_{j\beta'} \left( y_0 \right) \xi_1^{\beta_1} \xi_2^{\beta_2} \ldots \xi_{n-1}^{\beta_{n-1}} D^j_y \text{ for } y_0 \in \partial G \]

**Theorem 7.1.** Let the following conditions be satisfied:

1. \( G \in C^2, \ a_\alpha \in C \left( \bar{G} \right) \) for each \(|\alpha| = 2m\) and \( a_\alpha \in L_\infty \left( G \right) \) for each \(|\alpha| < 2m\);
2. \( b_{j\beta} \in C^{2m-m_j} \left( \partial G \right) \) for each \( j, \beta \) and \( m_j < 2m, \sum_{j=1}^{m} b_{j\beta} \left( y' \right) \sigma_j \neq 0, \) for \(|\beta| = m_j, \ y' \in \partial G, \) where \( \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in R^n \) is a normal to \( \partial G \);
3. for \( y \in \bar{G}, \ \xi \in R^n, \ \lambda \in S \left( \varphi_0 \right) \) for \( 0 \leq \varphi_0 < \pi, \ |\xi| + |\lambda| \neq 0 \) let \( \lambda + \sum_{|\alpha|=2m} a_\alpha \left( y \right) \xi^\alpha \neq 0; \)
4. for each \( y_0 \in \partial G \) local BVP in local coordinates corresponding to \( y_0: \)
\[ \lambda + A \left( y_0, \xi', D_y \right) \theta \left( y \right) = 0, \]
\[ B_j \left( y_0, \xi', D_y \right) \theta \left( 0 \right) = h_j, \ j = 1, 2, \ldots, m \]
has a unique solution \( \theta \in C_0 \left( \mathbb{R}_+ \right) \) for all \( h = (h_1, h_2, \ldots, h_m) \in \mathbb{C}^n \) and for \( \xi' \in R^{n-1}; \)
5. there exist positive constants \( b_0 \) and \( \vartheta \) such that a solution \( u \in C \left( [-1, 1]; X_2 \right) \) of (7.1) − (7.2) satisfied
\[ \| F \left( u, \bar{u} \right) \|_{L^2 \left( \bar{G} \right)} \leq b_0 \| u \|_{L^2 \left( \bar{G} \right)} \theta \text{ for } \| u \|_{L^2 \left( \bar{G} \right)} > 1; \]
6. Suppose
\[ \| u \left( .., t \right) \|_{L^2 \left( \mathbb{R}^n \times \bar{G} \right)} = \| u \left( .., 0 \right) \|_{L^2 \left( \mathbb{R}^n \times \bar{G} \right)} = \| u_0 \|_{L^2 \left( \mathbb{R}^n \times \bar{G} \right)} = a \]
for \( t \in [-1, 1] \) and that (2.15) holds with \( Q \left( .. \right) \) satisfies (2.16) for \( H = L^2 \left( \bar{G} \right) \).

If \( p > p \left( \theta \right) = \frac{2 \left( \theta n - 2 \right)}{\left( \theta n - 1 \right)}, \) then \( a = 0. \)

**Proof.** Let us consider operators \( A \) and \( V \left( x, \xi \right) \) in \( H = L^2 \left( \bar{G} \right) \) that are defined by the equalities
\[ D \left( A \right) = \left\{ u \in W^{2m, 2} \left( \bar{G} \right), \ B_j u = 0, \ j = 1, 2, \ldots, m \right\}, \ A u = \sum_{|\alpha| \leq 2m} a_\alpha \left( y \right) D_y^m u \left( y \right), \]

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Then the problem (7.1)−(7.2) can be rewritten as the problem (2.12), where $u(x) = u(x,.)$, $f(x) = f(x,.)$, $x \in R^n$ are the functions with values in $H = L^2(G)$. By virtue of [1] operator $A + \mu$ is positive in $L^2(G)$ for sufficiently large $\mu > 0$. Moreover, in view of (1)-(6) all conditions of Theorem 3 are hold. Then Theorem 3 implies the assertion.

8. The Wentzell-Robin type mixed problem for Schrödinger equations

Consider the problem (1.5)−(1.6). Let

$$\sigma = R^n \times (0,1), X_2 = L^2(R^n \times (0,1)), Y^{2,k} = H^{2,k}(R^n \times (0,1)).$$

Suppose $\nu = (\nu_1, \nu_2, ..., \nu_n)$ are nonnegative real numbers. In this section, from Theorem 1 we obtain the following result:

**Theorem 8.1.** Suppose the following conditions are satisfied:
(1) $a$ is positive, $b$ is a real-valued functions on $(0,1)$. Moreover, $a(.) \in C(0,1)$ and

$$\exp \left( - \int_{0}^{x} b(t) a^{-1}(t) \, dt \right) \in L^1(0,1);$$

(2) $u_1, u_2 \in C([0,1]; Y^{2,k})$ are strong solutions of (1.2) with $k > \frac{q}{p}$;
(3) $F : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, $F \in C^k$, $F(0) = \partial_\nu F(0) = \partial_\bar{\nu} F(0) = 0$;
(4) there exist positive constants $\alpha$ and $\beta$ such that

$$e^{\frac{|x|\alpha}{p}} (u_1(.,0) - u_2(.,0)) \in X_2, \quad e^{\frac{|x|\beta}{q}} (u_1(.,0) - u_2(.,0)) \in X_2,$$

with

$$p \in (1,2), \frac{1}{p} + \frac{1}{q} = 1;$$

(5) there exists $N_p > 0$ such that

$$\alpha \beta > N_p.$$

Then $u_1(x,t) \equiv u_2(x,t)$.

**Proof.** Let $H = L^2(0,1)$ and $A$ is a operator defined by (1.4). Then the problem (1.5)−(1.6) can be rewritten as the problem (1.2). By virtue of [10, 11] the operator $A$ generates analytic semigroup in $L^2(0,1)$. Hence, by virtue of (1)-(5) all conditions of Theorem 1.2 are satisfied. Then Theorem 1.2 implies the assertion.

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