A note on the non-diagonal K-matrices for the trigonometric $A_{n-1}^{(1)}$ vertex model

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Abstract

This note presents explicit matrix expressions of a class of recently-discovered non-diagonal K-matrices for the trigonometric $A_{n-1}^{(1)}$ vertex model. From these explicit expressions, it is easily seen that in addition to a discrete (positive integer) parameter $l$, $1 \leq l \leq n$, the K-matrices contain $n+1$ (or $n$) continuous free boundary parameters.

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Inspired from the observation [1] that the generic non-diagonal solutions (or K-matrices) [2, 3] of the reflection equation for the spin-$\frac{1}{2}$ XXZ model are decomposed into the product of intertwiner-matrices and diagonal face-type K-matrix, in [4] an intertwiner-matrix approach was developed and used to construct a class of non-diagonal solutions of the reflection equation for the trigonometric $A_{n-1}^{(1)}$ vertex model. There the K-matrices were expressed in terms of the intertwiner-matrix and a diagonal matrix. To fully realize the application of the solutions obtained in [4], it may be useful to write them in explicit and familiar matrix form. The purpose of this note is to provide such explicit expressions. From these expressions it is easily seen that in addition to a discrete (positive integer) parameter $l$, $1 \leq l \leq n$, the
solutions we constructed in [4] contain $n+1$ (or $n$) continuous free boundary parameters and have $3n-2$ (or $2n-1$) non-vanishing matrix elements.

Our starting point in [4] is the trigonometric R-matrix associated with the $n$-dimensional representation of $A^{(1)}_{n-1}$ given in [5,6]:

$$R(u) = \sum_{\alpha=1}^{n} R^{\alpha}_{\alpha}(u) E_{\alpha} \otimes E_{\alpha} + \sum_{\alpha \neq \beta} \left\{ R^{\alpha\beta}_{\alpha\beta}(u) E_{\alpha} \otimes E_{\beta} + R^{\beta\alpha}_{\alpha\beta}(u) E_{\beta} \otimes E_{\alpha} \right\}, \quad (1)$$

where $E_{ij}$ is the matrix with elements $(E_{ij})^{ij}_k = \delta_{jk}\delta_{il}$. The coefficient functions are

$$R^{\alpha\beta}_{\alpha\beta}(u) = \left\{ \begin{array}{ll}
\frac{\sin(u) e^{-iu}}{\sin(u + \eta)}, & \alpha > \beta, \\
1, & \alpha = \beta, \\
\frac{\sin(u) e^{iu}}{\sin(u + \eta)}, & \alpha < \beta,
\end{array} \right. \quad (2)$$

$$R^{\beta\alpha}_{\alpha\beta}(u) = \left\{ \begin{array}{ll}
\frac{\sin(\eta) e^{iu}}{\sin(u + \eta)}, & \alpha > \beta, \\
1, & \alpha = \beta, \\
\frac{\sin(\eta) e^{-iu}}{\sin(u + \eta)}, & \alpha < \beta.
\end{array} \right. \quad (3)$$

Here $\eta$ is the so-called crossing parameter. In addition to the quantum Yang-Baxter equation, the R-matrix satisfies the following unitarity, crossing-unitarity and quasi-classical relations:

Unitarity : \quad $R_{12}(u) R_{21}(-u) = id$, \quad (4)

Crossing-unitarity : \quad $R_{12}^{t}(u) M_2^{-1} R_{21}^{t}(-u - n\eta) M_2 = \frac{\sin(u) \sin(u + n\eta)}{\sin(u + \eta) \sin(u + n\eta - \eta)} id$, \quad (5)

Quasi-classical property : \quad $R_{12}(u)|_{\eta \to 0} = id$. \quad (6)

Here $R_{21}(u) = P_{12} R_{12}(u) P_{12}$ with $P_{12}$ being the usual permutation operator and $t_i$ denotes the transposition in the $i$-th space. The crossing matrix $M$ is a diagonal $n \times n$ matrix with elements

$$M_{\alpha\beta} = M_\alpha \delta_{\alpha\beta}, \quad M_\alpha = e^{-2i\alpha \eta}, \quad \alpha = 1, \ldots, n. \quad (7)$$

Boundary K-matrices $K^{-}(u)$ and $K^{+}(u)$, which give rise to integrable boundary conditions of an open chain on the right and left boundaries, respectively, satisfy the reflection and dual reflection equations [7,8]:

$$R_{12}(u_1 - u_2) K^{-}_{21}(u_1) R_{21}(u_1 + u_2) K^{-}_{21}(u_2) = K^{-}_{21}(u_2) R_{12}(u_1 + u_2) K^{-}_{21}(u_1) R_{21}(u_1 - u_2), \quad (8)$$

$$R_{12}(u_2 - u_1) K^{+}_{12}(u_1) M^{-1}_{12} R_{21}(-u_1 - u_2 - n\eta) M_{12} K^{+}_{21}(u_2) = M_{12} K^{+}_{21}(u_2) R_{12}(-u_1 - u_2 - n\eta) M^{-1}_{12} K^{+}_{12}(u_1) R_{21}(u_2 - u_1). \quad (9)$$
Different integrable boundary conditions are described by different solutions $K^-(u) \ (K^+(u))$ to the (dual) reflection equation \cite{7-8}.

Let us briefly recall some of the results in \cite{4}. Let $\{\epsilon_i \ | \ i = 1, 2, \ldots, n\}$ be the orthonormal basis of the vector space $\mathbb{C}^n$ such that $\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij}$. For a generic vector $\lambda \in \mathbb{C}^n$, define

$$\lambda_i = \langle \lambda, \epsilon_i \rangle, \ \ |\lambda| = \sum_{k=1}^{n} \lambda_k, \ \ i = 1, \ldots, n.$$  \hspace{1cm} (10)

Let us introduce an $n \times n$ matrix $\Phi(u; \lambda)$ which depends on the spectrum parameter $u$ and $\lambda$. The non-vanishing matrix elements of $\Phi(u; \lambda)$ are given by

$$\begin{pmatrix}
e^{i\eta f_1(\lambda)} & e^{i\eta f_2(\lambda)} & \cdots & e^{i\eta f_n(\lambda)} \\
e^{i\eta F_1(\lambda)} & e^{i\eta F_2(\lambda)} & \cdots & e^{i\eta F_n(\lambda)} \\
\vdots & \vdots & \ddots & \vdots \\
e^{i\eta F_n(\lambda)} & e^{i\eta F_{n+1}(\lambda)} & \cdots & e^{i\eta F_{n-1}(\lambda)}
\end{pmatrix}.$$ \hspace{1cm} (11)

Here $\rho$ is a complex constant with regard to $u$ and $\lambda$, and $\{f_i(\lambda) | i = 1, \ldots, n\}$ and $\{F_i(\lambda) | i = 1, \ldots, n\}$ are linear functions of $\lambda$:

$$f_i(\lambda) = \sum_{k=1}^{i-1} \lambda_k - \lambda_i - \frac{1}{2} |\lambda|, \ \ i = 1, \ldots, n.$$ \hspace{1cm} (12)

$$F_i(\lambda) = \sum_{k=1}^{i} \lambda_k - \frac{1}{2} |\lambda|, \ \ i = 1, \ldots, n-1.$$ \hspace{1cm} (13)

$$F_n(\lambda) = -\frac{3}{2} |\lambda|.$$ \hspace{1cm} (14)

The determinant of $\Phi(u; \lambda)$ is \cite{4}

$$\text{Det} (\Phi(u; \lambda)) = e^{i\eta \sum_{k=1}^{n} \frac{n-2(k+1)}{2} \lambda_k} \ (1 - (-1)^n e^{2iu+\rho}).$$ \hspace{1cm} (15)

For a generic $\rho \in \mathbb{C}$ this determinant is not vanishing and thus the inverse of $\Phi(u; \lambda)$ exists. Associated to a positive integer $l \ (1 \leq l \leq n)$, let us introduce a diagonal matrix

$$D^{(l)}(u) = \text{Diag}(k_1^{(l)}(u), \ldots, k_n^{(l)}(u)),$$ \hspace{1cm} (16)

where $\{k_i^{(l)}(u) | i = 1, \ldots, n\}$ are

$$k_j^{(l)}(u) = \begin{cases} 1, & 1 \leq j \leq l, \\
\frac{\sin(\xi-u)}{\sin(\xi+u)} e^{-2iu}, & l+1 \leq j \leq n.
\end{cases}$$  \hspace{1cm} (17)
Here $\xi$ is free complex parameter. Then one can define the non-diagonal K-matrices $\{K^{(l)}(u)\} | l = 1, \ldots, n \}$ associated with $\{D^{(l)}(u)\} | l = 1, \ldots, n \}$ and $\Phi(u; \lambda)$ as follows:\cite{1}:

$$K^{(l)}(u) = \Phi(u; \lambda) D^{(l)}(u) \{ \Phi(-u; \lambda) \}^{-1}, \quad l = 1, \ldots, n. \quad (18)$$

It has been shown in \cite{4} that the matrix $\Phi(u; \lambda)$ given by \cite{11} is the intertwiner-matrix which intertwines two trigonometric R-matrices, and thus the non-diagonal K-matrices $\{K^{(l)}(u)\}$ given by \cite{18} solve the reflection equation \cite{3} for the trigonometric $A_{n-1}^{(1)}$ vertex model. Moreover, \cite{18} implies that the K-matrices satisfy the regular condition $\{K^{(l)}(0) = id, \quad l = 1, \ldots, n, \}$ and boundary unitarity relation $\{K^{(l)}(u) K^{(l)}(-u) = id, \quad l = 1, \ldots, n.\}$

Through a tedious calculation for $n = 2, 3, 4, 5$ with the help of Mathematica program, we reconfirm the following properties for the non-diagonal K-matrices \cite{18}: $\{K^{(l)}(u)\} | l = 1, \ldots, n-1 \}$ depend on $n + 1$ continuous free parameters $\xi, \{\lambda_i | i = 1, \ldots, n-1 \}$ and $\rho$, and have $3n-2$ non-vanishing matrix elements (c.f. \cite{9,10}); $\{K^{(n)}(u)\}$ depends on $n$ continuous free parameters $\{\lambda_i | i = 1, \ldots, n-1 \}$ and $\rho$, and has $2n-1$ non-vanishing matrix elements (c.f. \cite{9,10}). The dependence on $\lambda_n$ disappears in the final expressions of the K-matrices although it appears in the expression of $\Phi(u; \lambda)$. The above properties are expected to hold for generic $n$. In the rational limit, the trigonometric K-matrices \cite{18} reduce to those corresponding to the rational $A_{n-1}^{(1)}$ vertex model \cite{11,12} with a special choice of the spectral-independent similarity transformation matrix.

In the following, we give the explicit matrix expressions of the K-matrices \cite{18} for the cases $n = 3, 4$.

The $A_{2}^{(1)}$ case:

There are three types of K-matrices for the trigonometric $A_{2}^{(1)}$ model.

- For the K-matrix $K^{(1)}(u)$, the 7 non-vanishing matrix elements $K(u)^k_j$ are given by:

$$K(u)^{1}_1 = e^{2iu(e^{2iu} + e^{\rho})} \left( 1 - e^{\rho} \frac{\sin(u - \xi)}{\sin(u + \xi)} \right), \quad K(u)^{1}_2 = e^{-2i\eta_{1}^{-1} + \rho}(e^{2iu} + e^{\rho}) \left( 1 + e^{2iu} \frac{\sin(u - \xi)}{\sin(u + \xi)} \right),$$

$$K(u)^{1}_3 = -e^{-2i\eta_{1}^{-1} + \rho}(e^{2iu} + e^{\rho}) \left( 1 + e^{2iu} \frac{\sin(u - \xi)}{\sin(u + \xi)} \right),$$

$$K(u)^{2}_1 = e^{2i\eta_{1}^{-1} + \rho}(e^{2iu} + e^{\rho}) \sin(2u) \frac{\sin(u + \xi)}{\sin(u + \xi)}, \quad K(u)^{2}_2 = e^{2iu} \frac{\cos(\eta_{1}^{-1})}{\sin(u + \xi)} \left( e^{\rho} - \frac{\sin(u - \xi)}{\sin(u + \xi)} \right),$$

$$K(u)^{2}_3 = -e^{-2i\eta_{1}^{-1} + \rho}(e^{2iu} + e^{\rho}) \sin(2u) \frac{\sin(u + \xi)}{\sin(u + \xi)}, \quad K(u)^{3}_1 = -e^{-2i\eta_{1}^{-1} + \rho}(e^{2iu} + e^{\rho}) \sin(2u) \frac{\sin(u + \xi)}{\sin(u + \xi)}, \quad K(u)^{3}_2 = e^{-2iu} \frac{\sin(u - \xi)}{\sin(u + \xi)}, \quad K(u)^{3}_3 = e^{-2iu} \frac{\sin(u - \xi)}{\sin(u + \xi)}. \quad (19)$$
• For the K-matrix $K^{(2)}(u)$, the 7 non-vanishing matrix elements $K(u)^j_k$ are given by:

$$
K(u)^1 = \frac{e^{2iu}}{e^{2iu} + \rho} \left( 1 - e^\rho \frac{\sin(u - \xi)}{\sin(u + \xi)} \right), \quad K(u)^2 = \frac{e^{-2\eta \lambda_1 + \rho}}{e^{2iu} + \rho} \left( 1 + e^{2iu} \frac{\sin(u - \xi)}{\sin(u + \xi)} \right),
$$

$$
K(u)^3 = \frac{-e^{-2\eta \lambda_1 + \rho + i(u + \xi)}}{e^{2iu} + \rho} \left( 1 + e^{2iu} \frac{\sin(u - \xi)}{\sin(u + \xi)} \right),
$$

$$
K(u)^2_1 = 1,
$$

$$
K(u)^3_1 = \frac{e^{2iu} \sin(2u)}{(e^{2iu} + \rho) \sin(u + \xi)}, \quad K(u)^3_2 = \frac{e^{2iu + i(u + \xi)} \sin 2u}{(e^{2iu} + \rho) \sin(u + \xi)},
$$

$$
K(u)^3_3 = \frac{1}{e^{2iu} + \rho} \left( \rho - \frac{\sin(u - \xi)}{\sin(u + \xi)} \right).
$$

(20)

• For the K-matrix $K^{(3)}(u)$, the 5 non-vanishing matrix elements $K(u)^j_k$ are given by:

$$
K(u)^1 = \frac{e^{2iu} + e^{4iu} + \rho}{e^{2iu} + \rho}, \quad K(u)^1_2 = -\frac{e^{-2\eta \lambda_1 + \rho} (e^{4iu} - 1)}{e^{2iu} + \rho},
$$

$$
K(u)^1_3 = \frac{e^{-2\eta \lambda_1 + \rho + 2iu} (e^{4iu} - 1)}{e^{2iu} + \rho}, \quad K(u)^2_1 = K(u)^3_3 = 1.
$$

(21)

The $A_3^{(1)}$ case:

There are four types of K-matrices for the trigonometric $A_3^{(1)}$ model.

• For the K-matrix $K^{(1)}(u)$, the 10 non-vanishing matrix elements $K(u)^j_k$ are given by:

$$
K(u)^1 = \frac{e^{2iu}}{e^{2iu} - \rho} \left( 1 + e^\rho \frac{\sin(u - \xi)}{\sin(u + \xi)} \right), \quad K(u)^2 = \frac{e^{-2\eta \lambda_1 + \rho}}{e^{2iu} - \rho} \left( 1 + e^{2iu} \frac{\sin(u - \xi)}{\sin(u + \xi)} \right),
$$

$$
K(u)^3 = \frac{-e^{-2\eta \lambda_1 + \rho + i(u - \xi)}}{e^{2iu} - \rho} \left( 1 + e^{2iu} \frac{\sin(u - \xi)}{\sin(u + \xi)} \right),
$$

$$
K(u)^4 = \frac{1}{e^{2iu} - \rho} \left( \rho + \frac{\sin(u - \xi)}{\sin(u + \xi)} \right), \quad K(u)^2_1 = \frac{e^{2iu} \sin(2u)}{(e^{2iu} - \rho) \sin(u + \xi)},
$$

$$
K(u)^3_2 = \frac{e^{-2\eta \lambda_2 - 2i(u - \xi) + \rho}}{(e^{2iu} - \rho) \sin(u + \xi)}, \quad K(u)^3_1 = K(u)^4_3 = K(u)^4 = \frac{e^{-2iu} \sin(\xi - u)}{\sin(\xi + u)}.
$$

(22)

• For the K-matrix $K^{(2)}(u)$, the 10 non-vanishing matrix elements $K(u)^j_k$ are given by:
matrices obtained in [4] for the trigonometric easily seen that the K-matrices

\[ K(u) = \frac{e^{2iu}}{e^{2iu} - e^{\rho}} \left( 1 + e^{\rho}\frac{\sin(u - \xi)}{\sin(u + \xi)} \right), \quad K(u) = -\frac{e^{-2i\eta\lambda_1 + \rho}}{e^{2iu} - e^{\rho}} \left( 1 + e^{2iu}\frac{\sin(u - \xi)}{\sin(u + \xi)} \right), \]

\[ K(u)_3 = \frac{e^{-2i(H_1 + H_2) + \rho}}{e^{2iu} - e^{\rho}} \left( 1 + e^{2iu}\frac{\sin(u - \xi)}{\sin(u + \xi)} \right), \]

\[ K(u)_4 = -\frac{1}{e^{2iu} - e^{\rho}} \left( e^{\rho} + \frac{\sin(u - \xi)}{\sin(u + \xi)} \right), \quad K(u)_4 = -\frac{1}{e^{2iu} - e^{\rho}} \left( e^{\rho} + \frac{\sin(u - \xi)}{\sin(u + \xi)} \right). \]  

(23)

- For the K-matrix \( K^{(3)}(u) \), the 10 non-vanishing matrix elements \( K(u)_j \) are given by:

\[ K(u)_1 = \frac{e^{2iu}}{e^{2iu} - e^{\rho}} \left( 1 + e^{\rho}\frac{\sin(u - \xi)}{\sin(u + \xi)} \right), \quad K(u)_2 = -\frac{e^{-2i\eta\lambda_1 + \rho}}{e^{2iu} - e^{\rho}} \left( 1 + e^{2iu}\frac{\sin(u - \xi)}{\sin(u + \xi)} \right), \]

\[ K(u)_3 = \frac{e^{-2i(H_1 + H_2) + \rho}}{e^{2iu} - e^{\rho}} \left( 1 + e^{2iu}\frac{\sin(u - \xi)}{\sin(u + \xi)} \right), \]

\[ K(u)_4 = -\frac{1}{e^{2iu} - e^{\rho}} \left( e^{\rho} + \frac{\sin(u - \xi)}{\sin(u + \xi)} \right), \quad K(u)_4 = -\frac{1}{e^{2iu} - e^{\rho}} \left( e^{\rho} + \frac{\sin(u - \xi)}{\sin(u + \xi)} \right). \]  

(24)

- For the K-matrix \( K^{(4)}(u) \), the 7 non-vanishing matrix elements \( K(u)_j \) are given by:

\[ K(u)_1 = \frac{e^{2iu} - e^{4iu + \rho}}{e^{2iu} - e^{\rho}}, \quad K(u)_2 = \frac{e^{-2i\eta\lambda_1 + \rho}(e^{4iu} - 1)}{e^{2iu} - e^{\rho}}, \]

\[ K(u)_3 = -\frac{e^{-2i(H_1 + H_2) + \rho}(e^{4iu} - 1)}{e^{2iu} - e^{\rho}}, \quad K(u)_4 = \frac{e^{2iu}\lambda_1 + \rho(e^{4iu} - 1)}{e^{2iu} - e^{\rho}}, \]

\[ K(u)_2 = K(u)_3 = K(u)_4 = 1. \]  

(25)

In summary, we have presented the explicit matrix expressions of the non-diagonal K-matrices obtained in [4] for the trigonometric \( A^{(1)}_{n-1} \) vertex model. From these results, it is easily seen that the K-matrices \( K^{(l)}(u) \) (\( l = 1, \ldots, n - 1 \)) depend on \( n + 1 \) continuous free
parameters and have $3n - 2$ non-vanishing matrix elements, and that the K-matrix $K^{(n)}(u)$ depends on $n$ continuous free parameters and has $2n - 1$ non-vanishing matrix elements.

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