Second order pressure estimates for the Crank-Nicolson discretization of the incompressible Navier-Stokes Equations

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We provide optimal order pressure error estimates for the Crank-Nicolson semidiscretization of the incompressible Navier-Stokes equations. Second order estimates for the velocity error are long known, we prove that the pressure error is of the same order if considered at interval midpoints, confirming previous numerical evidence. For simplicity we first give a proof under high regularity assumptions that include nonlocal compatibility conditions for the initial data, then use smoothing techniques for a proof under reduced assumptions based on standard local conditions only.

Keywords: incompressible Navier-Stokes equations, Crank-Nicolson, error estimates

1 Introduction

We consider the Crank-Nicolson timestepping scheme for the Stokes and Navier-Stokes equations as described in the seminal paper by Heywood and Rannacher [8], where optimal velocity error estimates were proven under weak regularity assumptions. Due to its implicit occurrence, the pressure error is only linear in time at the timesteps. Numerically it is well-known that quadratic convergence can be recovered using the midpoint values of the pressure, see for example Reusken and Esser [13], Rank [11] or Hussain, Schieweck and Turek [9] who also consider higher order schemes.

The main result of this paper is a proof of this result: If \( p_k \) denotes the semidiscrete and \( p \) the continuous pressure, we prove in section 2 the temporal \( L^2 \)-estimate

\[
\|a_k p - p_k\|_{L^2(0,T;H^1(\Omega))} \leq Ck^2,
\]

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where $a_k p$ is an average over $p$, and the $L^\infty$-estimate

$$\|m_k p - p_k\|_{L^\infty(0,T;H^1(\Omega))} \leq C k^2$$

where $m_k p$ denotes the midpoint values of $p$, for details on the notation see below. These results hold for solutions of both the Stokes and Navier-Stokes equations. Comparable are temporal $L^2$-estimates by De Frutos, García-Archilla, John and Novo \cite{5} for the Oseen problem in the fully discrete setting. In \cite{6} these results are extended to the Navier-Stokes equations including optimal order estimates for the pressure in the interval midpoints. The evaluation of the adjoint variables in the midpoints by Meidner and Vexler \cite{10} for parabolic optimal control problems is of similar spirit. Rang \cite{11} shows second order pressure convergence for a variant of the Crank-Nicolson timestepping scheme applied to the Navier-Stokes equations, where the pressure is treated like the velocity and split into explicit and implicit part.

The previous results assume highly regular solutions which only exist if the initial data satisfies nonlocal compatibility conditions. Similar to Heywood and Rannacher \cite{8} we reduce the assumptions on the data in section 3 and use the smoothing properties of the solution operator: Replacing the first timestep by an implicit Euler step, we have the $L^2$-estimate

$$\|\tau_k^2 (a_k p - p_k)\|_{L^2(t_1,T;H^1(\Omega))} \leq C k^2$$

where $\tau_k$ is a discretization of the continuous smoothing function $\tau(t) := \min(1, t)$, and replacing the first two steps by implicit Euler steps we have the $L^\infty$-estimate

$$\|\tau_k^2 (m_k p - p_k)\|_{L^\infty(t_2,T;H^1(\Omega))} \leq C k^2.$$ 

These results are derived using energy techniques and technically involved, but based on the simple observation that the discrete smoothing creates jump terms which require a cascade of estimates due to the weak bounds available for the Crank-Nicolson scheme.

In section 4 we present a numerical study illustrating the optimality of the error estimates and the necessity to consider both a weighted norm and initial Euler steps, if the initial data does not satisfy the compatibility conditions.

We interpret the Crank-Nicolson scheme as a discontinuous Petrov-Galerkin in time method, extending a construction by Aziz and Monk \cite{1}: Velocities are approximated using continuous, piecewise-linear functions, the pressure with discontinuous, piecewise-constant functions. The test functions for both velocity and pressure are discontinuous, piecewise-constant in time. This mismatch between velocity test and ansatz spaces is central to the a priori analysis of the problem. Further analysis of Petrov-Galerkin in time methods can be found in Schieweck \cite{15} for parabolic equations, of related interest are also results by Chrysafinos and Karatzas \cite{4} for discontinuous Galerkin methods applied to the Stokes equations, where best approximation results for the velocity error are derived.

### 1.1 General Notation

In the following let $\Omega \subseteq \mathbb{R}^d$ with $d \in \{2,3\}$ be a bounded domain with regularity described later on. Let $I := [0,T)$ with $T > 0$ denote the finite time interval on which
a solution is sought. We write $(\cdot, \cdot)$ for the scalar product on $L^2(\Omega)$ but also for the duality product on a generic Banach space if no confusion is possible. For spatial norms we omit the domain $\Omega$ and write e.g. $\| \cdot \|_{L^p} := \| \cdot \|_{L^p(\Omega)}$. Functions in $L^2_0(\Omega)$ have zero average and those in $L^2_0(\Omega)$ are solenoidal, i.e. have zero (weak) divergence. We write $P$ for the Helmholtz projection.

We use the Bochner spaces $L^p(I, X)$, $W^{m,p}(I, X)$ and $H^m(0, T; X)$ for a Banach space $X$, $m \in \mathbb{N}$ and $1 \leq p \leq \infty$. For $J \subset I$ we write $[\langle \cdot, \cdot \rangle]_J$ for the scalar product on $L^2(J, L^2(\Omega))$, but also for the duality product on $L^2(J, X)$ if no confusion is possible. We write $\| \cdot \|_J$ for the norm on $L^2(J, L^2(\Omega))$ and omit $J$ in both norm and scalar product if $J = I$. For general Bochner spaces we abbreviate $\| \cdot \|_{L^p(I, X)}$ for the norm on $L^p(I, X)$ and those in $L^2(\Omega)$ but also for the duality product on $L^2(I, X)$ and again omit $J$ if $J = I$.

The natural spaces of velocity regularity are induced by the Stokes operator $-P\Delta$, see [16] for details: Since $\Omega$ will have at least a $C^2$-boundary, the Stokes operator $-P\Delta: \mathcal{D}(-P\Delta) \subset L^2_0(\Omega) \rightarrow L^2_0(\Omega)$ has domain $\mathcal{D}(-P\Delta) = L^2_0(\Omega) \cap H^1_0(\Omega) \cap H^2(\Omega)$. Furthermore, $-P\Delta$ is positive, selfadjoint, compact and has a bounded inverse. In particular we may define $V^s := \mathcal{D}(-P\Delta)\bar{\gamma}$ for $s \geq 0$ with graph norm $\| \cdot \|_{V^s}$ and $V^{-s} := (V^s)'$. Then $V^{0} \cong L^2_0(\Omega)$ and $V^s \cong L^2_0(\Omega) \cap H^1_0(\Omega) \cap H^s(\Omega)$ for $s \in \mathbb{N}$. For $u, v \in V^1$ there holds $(\nabla u, \nabla v) = ((-P\Delta)\bar{\gamma}u, (-P\Delta)\bar{\gamma}v)$. For brevity we use the notation $Q^s := L^2_0(\Omega) \cap H^s(\Omega)$ for the pressure regularity spaces with the $H^s(\Omega)$-norm.

We denote the identity operator by $\text{Id}$ and the indicator function on $I$ by $1_I$. For a real number $a \in \mathbb{R}$ we write $(a)^+ := \max\{0, a\}$ and $(a)^- := \min\{0, a\}$. We denote by $C > 0$ a generic constant independent of the timestep size which may change with each occurrence.

### 1.2 Discrete Spaces and Operators

Let $X$ be a real Hilbert space.

**Definition 1.** Let $0 = t_0 < t_1 < \cdots < t_N = T$. We call $\mathbb{I}_k = \{I^1, \ldots, I^N\}$ with $I^n := (t_{n-1}, t_n]$ for $n = 1, \ldots, N$ a discretization of $I$ with nodes $(t_n)_{n=0}^N$. Let $k_n := |I^n|$, $k := \max_{n=1,\ldots,N} k_n$ and denote the midpoints by $t_{n-\frac{1}{2}} := \frac{1}{2}(t_{n-1} + t_n)$ for $n = 1, \ldots, N$.

For the remainder of the paper $\mathbb{I}_k$ is fixed and satisfies, for the Stokes equations,

**Assumption 2.** There exists $\kappa > 0$ such that for $n = 1, \ldots, N - 1$ there holds $\kappa^{-1} \leq \frac{k_{n+1}}{k_n} \leq \kappa$. The generic constants $C > 0$ may depend on $\kappa$.

...and for the Navier-Stokes equations the stronger...

**Assumption 3.** There exists $\rho \geq 1$ such that $\max_{n=1,\ldots,N} k_n \leq \rho \min_{n=1,\ldots,N} k_n$. The generic constants $C > 0$ may depend on $\rho$.

**Definition 4.** We define the continuous and discontinuous Galerkin spaces on $X$ of degree $r \in \mathbb{N}_0$ as

- $cG^r(\mathbb{I}_k, X) := \{ f_k \in C^0([0, T], X) \mid f_k|_{I^n} \in P^r(I^n, X) \forall I^n \in \mathbb{I}_k \}$,
- $dG^r(\mathbb{I}_k, X) := \{ f_k : (0, T) \rightarrow X \mid f_k|_{I^n} \in P^r(I^n, X) \forall I^n \in \mathbb{I}_k \}$,

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where \( P^r(I^n, X) \) is the space of polynomials of degree \( r \) with values in \( X \) on \( I^n \). We equip \( dG^0(I_k, X) \) with the \( L^2(0, T; X) \)-topology.

**Definition 5.** For \( u : I \rightarrow X \) sufficiently regular, we define

- the nodal interpolation operator
  \[
  i_k u \in cG^1(I_k, X), \quad (i_k u)(t_n) := u(t_n) \quad \text{for } n = 0, \ldots, N,
  \]
- the \( L^2 \)-projection onto \( dG^0(I_k, X) \), which corresponds to averaging
  \[
  a_k u \in dG^0(I_k, X), \quad (a_k u)(t_n) := \frac{1}{k_n} \int_{I^n} u(t) \, dt, \quad \text{for } n = 1, \ldots, N,
  \]
- the constant continuation of midpoint values
  \[
  m_k u \in dG^0(I_k, X), \quad (m_k u)(t_n) := u(t_n - \frac{1}{2}) \quad \text{for } n = 1, \ldots, N.
  \]

**Remark 6.** The operator \( i_k \) is used for the velocity and the operator \( m_k \) for the pressure error estimate. The averaging operator \( a_k \) occurs naturally in a-priori estimates for the Crank-Nicolson scheme, see remark 12.

**Lemma 7.** Let \( 2 \leq p \leq \infty \) and \( m \in \{1, 2\} \). For \( u \in W^{m,p}(0, T; X) \) there holds

\[
\| u - i_k u \|_{L^p X} \leq CK^m \| \partial_t^m u \|_{L^p X}.
\]

**Proof.** Follows by reference transformation techniques. \(\square\)

**Lemma 8.** For \( u \in W^{2,\infty}(0, T; X) \) there holds

\[
\| a_k u - m_k u \|_{L^\infty X} \leq CK^2 \| \partial_t u \|_{L^\infty X}.
\]

**Proof.** For each \( I^n = (t_{n-1}, t_n) \in I_k \) we have, by Taylor expansion around \( t_n - \frac{1}{2} \):

\[
\int_{I^n} u(t) - u(t_n - \frac{1}{2}) \, dt = \partial_t u(t_n - \frac{1}{2}) \int_{I^n} t - t_n - \frac{1}{2} \, dt + \int_{I^n} \int_{t_n - \frac{1}{2}}^t (t - s) \partial_t u(s) \, ds \, dt.
\]

The first term on the right vanishes and we can hence conclude

\[
\max_{I^n \in I_k} \| (a_k u - m_k u)(I^n) \|_X \leq CK^2 \| \partial_t u \|_{L^\infty X}.
\]

**Lemma 9.** For \( 2 \leq p \leq \infty \) and \( u \in W^{1,p}(0, T; X) \) there holds:

\[
\| u - a_k i_k u \|_{L^p X} \leq CK \| \partial_t u \|_{L^p X}.
\]

**Proof.** Follows from the reference transformation technique and the Bramble-Hilbert lemma, since \( u - a_k i_k u \) vanishes for piecewise constant functions. \(\square\)

**Remark 10.** In the second part of this paper we apply the estimates of this section to \( L^p \)-spaces equipped with a discrete weight function \( \tau_k^\alpha \in dG^0(I_k, \mathbb{R}_+) \). Since \( \tau_k^\alpha |_{I^n} \equiv \text{const} \) the weighting does not affect the error estimates, such that e.g. lemma 7 reads

\[
\| \tau_k^\alpha (u - i_k u) \|_{L^p X} \leq CK^m \| \tau_k^\alpha \partial_t^m u \|_{L^p X}.
\]
2 Estimates with High Regularity

2.1 Stokes Equations

Let $u$ and $p$ denote a solution of the Stokes equations

$$\partial_t u - \Delta u + \nabla p = f, \quad \text{div } u = 0 \quad \text{in } \Omega,$$

with $u(0) = u^0$, $u|_{\partial \Omega} = 0$ and $\int_\Omega p = 0$. In this section, we require solutions of high regularity and hence the following assumptions on the problem data:

**Assumption 11.** We assume that $\Omega$ has a $C^5$-boundary, the right hand side has regularity $f \in C^0(I, H^3(\Omega)) \cap C^1(I, H^1(\Omega)) \cap H^2(\Omega, L^2(\Omega))$ and that the initial data $u^0 \in V^5$ satisfies the compatibility conditions that $p^0$ and $q^0$ with

$$\Delta p^0 = \nabla \cdot f(0), \quad \Delta q^0 = \nabla \cdot \partial_t f(0) \quad \text{in } \Omega$$

(2)
can be chosen such that

$$\nabla p^0 = f(0) + \Delta u^0, \quad \nabla q^0 = \partial_t f(0) + \Delta f(0) + \Delta^2 u^0 - \Delta \nabla p^0 \quad \text{on } \partial \Omega.$$  

(3)

We write

$$C_{A11} := C \left( \|u^0\|_{V^5} + \|f\|_{C^0H^3} + \|f\|_{C^1H^1} + \|f\|_{H^2L^2} \right)$$

where $C > 0$ is independent of the data and, by abuse of notation, may change with each occurrence of $C_{A11}$.

In [17, theorem 2.1 and theorem 2.2] it is shown that exactly under these conditions there exists a solution to the Stokes equations that satisfies the bound

$$\|u\|_{C^0V^5} + \|u\|_{C^1V^3} + \|u\|_{C^2V^1} + \|u\|_{H^2V^2} + \|p\|_{H^2Q^1} \leq C_{A11}.$$  

(4)

The equations (2) and (3) require $p^0$ and $q^0$ to solve Poisson problems with overdetermined boundary conditions. This makes assumption 11 not only hard to check for given $u^0$ and $f$ but unlikely to hold. Nevertheless we will assume its validity throughout this section. A more technical analysis without compatibility assumptions will be carried out in section 3.

To formulate the timestepping scheme in dual spaces of $dG^0$-functions we note:

**Remark 12.** Let $X$ be a Hilbert space. Since $dG^0(\mathbb{I}_k, X') \subset L^2(I, X')$, we have both

$L^2(I, X') \cong L^2(I, X')'$ and $dG^0(\mathbb{I}_k, X') \cong dG^0(\mathbb{I}_k, X)'$ and the diagram

$$dG^0(\mathbb{I}_k, X') \xrightarrow{\{\phi_k \mapsto \{\cdot, \phi_k\}\}} dG^0(\mathbb{I}_k, X')'$$

$$\cong \xleftarrow{\{\phi_k \mapsto \{\cdot, \phi_k\}\}}$$

$$L^2(I, X') \xrightarrow{\{\phi \mapsto \{\cdot, \phi\}\}} L^2(I, X)'$$
commutes. We emphasize the occurrence of \( a_k \) in this diagram. In particular, any \( f \in L^2(I, X') \) can be understood in \( dG^0(I, X) \) through

\[
\langle f, \phi_k \rangle := \int_I (f(t), \phi_k(t)) \, dt = \int_I (a_k f(t), \phi_k(t)) \, dt = \langle a_k f, \phi_k \rangle
\]

for arbitrary \( \phi_k \in dG^0(I, X) \). For the dual norm we have

\[
\|f\|_{dG^0(I, X)'} := \sup_{\phi_k \in dG^0(I, X)} \frac{\int_I (f(t), \phi_k(t)) \, dt}{\|\phi_k\|_{L^2(X)}} = \|a_k f\|_{L^2(X')}.
\]

We can now give an abstract formulation of the Crank-Nicolson scheme as solution \( u_k \in cG^1(I, V^1) \) and \( p_k \in dG^0(I, Q^0) \) of

\[
\partial_t u_k - \Delta u_k + \nabla p_k = f \quad \text{in } dG^0(I, H^1_0(\Omega))'
\]

with \( u_k(0) = u^0 \). The equation must be understood with left- and right-hand side interpreted as elements of \( dG^0(I, H^1_0(\Omega))' \) as in (5). Using \( a_k \) we have more explicitly:

\[
\partial_t u_k - \Delta a_k u_k + \nabla p_k = a_k f,
\]

now as an equality of \( dG^0 \)-functions. With \( u^n_k := u_k(t_n) \) and \( p^n_k := p_k|_{t^n} \), the equation (6) / (7) can be written as a classical timestepping scheme:

\[
u^n_k - u^{n-1}_k - \frac{k_n}{2} \Delta (u^n_k + u^{n-1}_k) + k_n \nabla p^n_k = \int_{t^{n-1}}^{t^n} f \, dt
\]

for \( n = 1, \ldots, N \) and \( u^0_k := u^0 \). This is the usual Crank-Nicolson scheme with exact integration of the right-hand side. Equation (6) emphasizes the relation to the continuous problem with replaced time-discrete trial and test spaces and can be thought of as integral version of the pointwise-in-time scheme (8). Our error analysis is based on (7) rather than (8), trading notational simplicity for, in our opinion, more transparent proofs since algebraic manipulations and choices of test functions can be identified as standard integration techniques, such as partial integration, and use of temporal projections.

**Remark 13.** We emphasize the purely implicit occurrence of the pressure in (8). The character of the analysis changes significantly if \( \frac{k_n}{2} \nabla (p^n_k + p^{n-1}_k) \) is used as pressure term, which corresponds to (6) with \( p_k \in cG^1(I, Q^0) \) and leads more naturally to second-order pressure estimates. This approach is investigated by Rang [11] and requires the construction of an initial pressure \( p^0 \).

The following stability estimate is formulated in a generality only needed for section 3. In this section only the cases \( s = 1 \) and \( s = 2 \) are used.

**Lemma 14** (Discrete stability). Let \( s \in \mathbb{Z} \). Let \( u_k \in cG^1(I, V^{s+1}) \) and \( r_k \in dG^0(I, V^{s-1}) \) be such that

\[
\partial_t v_k - \Delta v_k = r_k \quad \text{in } dG^0(I, V^{1-s})' \quad \iff \quad \partial_t v_k - \Delta a_k v_k = r_k.
\]

Then we have the stability estimate

\[
\|v_k\|_{L^\infty(V^s)} + \|\partial_t v_k\|_{L^2(V^{s-1})} + \|a_k v_k\|_{L^2(V^{s+1})} \leq C \left( \|v^0_k\|_{V^s} + \|r_k\|_{L^2(V^{s-1})} \right).
\]
Proof. By assumption \((-P\Delta)^sv_k(t) \in V^{1-s}\) for \(t \in I\). Hence (9), as element of \(L^2(I, V^{1-s})'\), can be tested with \((-P\Delta)^sv_k 1_J\) where \(J := (0, t_n]\) for \(n = 1, \ldots, N\):

\[
(\partial_t v_k, (-P\Delta)^sv_k)_J + \langle -P\Delta a_k v_k, (-P\Delta)^sv_k \rangle_J = (r_k, (-P\Delta)^sv_k)_J.
\] (11)

Partial integration in time of the continuous function \(v_k\) implies

\[
(\partial_t v_k, (-P\Delta)^sv_k)_J = \frac{1}{2} \left( \|v_k^0\|_{V^s}^2 - \|v_k^N\|_{V^s}^2 \right).
\]

For the other term on the left of (11) we use the projection property to insert \(a_k\):

\[
(-P\Delta a_k v_k, (-P\Delta)^sv_k)_J = (-P\Delta a_k v_k, (-P\Delta)^sa_k v_k)_J = \|a_k v_k\|_{L^2 V^{s+1}}^2.
\]

For the right-hand side of (11) we again insert \(a_k\) and estimate with Young’s inequality

\[
(r_k, (-P\Delta)^sv_k)_J = (r_k, (-P\Delta)^sa_k v_k)_J \leq \frac{1}{2} \|r_k\|_{L^2 V^{s-1}}^2 + \frac{1}{2} \|a_k v_k\|_{L^2 V^{s+1}}^2
\]

and combination of the three previous estimates implies for (11):

\[
\|v_k^0\|_{V^s}^2 + \|a_k v_k\|_{L^2 V^{s+1}}^2 \leq \|v_k^N\|_{V^s}^2 + \|r_k\|_{L^2 V^{s-1}}^2.
\]

Taking the maximum over \(n = 1, \ldots, N\) we arrive at the estimates for \(\|v_k\|_{L^\infty V^s}\) and \(\|a_k v_k\|_{L^2 V^{s+1}}\). For the remaining estimate of \(\|\partial_t v_k\|_{L^2 V^{s-1}}\) we have by (9)

\[
\|\partial_t v_k\|_{L^2 V^{s-1}} = \|r_k + P\Delta a_k v_k\|_{L^2 V^{s-1}} \leq \|r_k\|_{L^2 V^{s-1}} + \|P\Delta a_k v_k\|_{L^2 V^{s-1}}
\]

and (10) follows by our estimate for \(\|a_k v_k\|_{L^2 V^{s+1}}\). \(\square\)

Remark 15. We emphasize that in (10) only \(\|a_k v_k\|_{L^2 V^{s+1}}\) can be controlled, i.e. the integral of the averages \(\frac{1}{2}(v_k^{n-1} + v_k^n)\) for \(n = 1, \ldots, N\), and not \(\|v_k\|_{L^2 V^{s+1}}\). This will be a central problem in section 3.

Remark 16. Given a sufficiently regular continuous solution \(v\) of

\[
\partial_t v - P\Delta v = r \quad \text{in } L^2(I, V^{1-s})'
\]

we can derive stability estimates by testing with \(\phi = (-P\Delta)^sv 1_{(0,t]}\) for \(t \in I\) a.e. For our time-discrete solution \(v_k\) of

\[
\partial_t v_k - P\Delta v_k = r_k \quad \text{in } dG^0(I_k, V^{1-s})',
\]

we derived stability estimates by testing the equivalent formulation

\[
\partial_t v_k - P\Delta a_k v_k = r_k,
\]

with \(\phi_k = (-P\Delta)^sv_k 1_{(0, t_n]}\) for \(n = 1, \ldots, N\). We note the similarity of the approaches and the fact that \(\phi_k\) is not a valid test function in the first formulation of the time-discrete problem, but in the second one since \(dG^0(I_k, V^{s-1}) \subset L^2(I, V^{s-1}) \cong L^2(I, V^{1-s})'\).
With these preparations we can prove second order error estimates for both velocity and pressure based on the following error identities:

**Lemma 17.** For the solutions \((u, p)\) of the continuous and \((u_k, p_k)\) of the time-discrete Stokes problem we have the velocity error identity

\[
\partial_t (u_k - i_k u) - P \Delta a_k (u_k - i_k u) = -P \Delta a_k (u - i_k u)
\]

and the pressure error identity

\[
\nabla (p_k - a_k p) = (P - \text{Id}) \Delta a_k (u - u_k).
\]

**Proof.** By definition of \((u, p)\) and \((u_k, p_k)\) we have Galerkin orthogonality:

\[
\partial_t (u_k - u) - \Delta (u_k - u) + \nabla (p_k - p) = 0 \quad \text{in} \ dG^0(\mathbf{I}_k, L^2(\Omega))'
\]

and hence in particular

\[
a_k \partial_t (u_k - u) - P \Delta a_k (u_k - u) = 0.
\]

Adding and subtracting \(i_k u\) we arrive at (12) if we can show that \(a_k \partial_t (u - i_k u) = 0\), noting that \(a_k \partial_t (u_k - i_k u) = \partial_t (u_k - i_k u)\). The identity \(a_k \partial_t (u - i_k u) = 0\) follows from

\[
a_k \partial_t (u - i_k u) |_{I^n} = k_n^{-1} \int_{I^n} \partial_t (u - i_k u) \, dt = k_n^{-1} ((u - i_k u)(t_n) - (u - i_k u)(t_{n-1})) = 0
\]

for \(I^n \in \mathbf{I}_k\), since \((u - i_k u)(t_j) = 0\) for all \(j = 0, \ldots, N\). To prove the pressure identity, (13), the Galerkin orthogonality (14) yields that

\[
\nabla (p_k - a_k p) = -\partial_t (u_k - i_k u) - \Delta a_k (u - u_k)
\]

and by (12) we have \(\partial_t (u_k - i_k u) = -P \Delta a_k (u - u_k)\) which implies (13).

**Theorem 18.** Let assumption 11 hold. Then, the Crank-Nicolson time discretization (6) of the Stokes equations (1) satisfies the a priori error estimate

\[
\|u - u_k\|_{L^\infty V^1} + \|a_k (u - u_k)\|_{L^2 V^2} + \|a_k p - p_k\|_{L^2 Q^1} \leq C_{A11} k^2.
\]

**Proof.** We split the velocity error as follows:

\[
\|u - u_k\|_{L^\infty V^1} + \|a_k (u - u_k)\|_{L^2 V^2} \leq \|u - i_k u\|_{L^\infty V^1} + \|a_k (u - i_k u)\|_{L^2 V^2} + \|u_k - i_k u\|_{L^\infty V^1} + \|a_k (u_k - i_k u)\|_{L^2 V^2}.
\]

We apply the discrete stability estimate (10) with \(s = 1\) to the error identity (12) with \(a_k := u_k - i_k u\) and \(r_k := -P \Delta a_k (u - i_k u)\). Let us note that, here and in the following, it is easy to check that the discrete solution has enough regularity to apply the stability
estimate using the regularity theory for the stationary Stokes equations. For the last two terms above this yields
\[
\|u_k - i_k u\|_{L^\infty V^1} + \|a_k(u_k - i_k u)\|_{L^2 V^2} \\
\leq C\|\Delta a_k(u - i_k u)\| = C\|a_k(u - i_k u)\|_{L^2 V^2}.
\]
Combining this estimate with the previous one, using the stability of \(a_k\) and the interpolation error estimate from lemma 7 we arrive at
\[
\|u - u_k\|_{L^\infty V^1} + \|a_k(u - u_k)\|_{L^2 V^2} \leq \|u - i_k u\|_{L^\infty V^1} + C\|a_k(u - i_k u)\|_{L^2 V^2} \\
\leq Ck^2 (\|\partial_t u\|_{L^\infty V^1} + \|\partial_t u\|_{L^2 V^2}) \leq CA_{11} k^2.
\]
For the pressure error we use (13) and the validity of Poincaré’s inequality on \(Q^1\):
\[
\|p_k - a_k p\|_{L^2 Q^1} \leq C\|\nabla (p_k - a_k p)\| = C\|\Delta a_k(u - u_k)\| \\
\leq C\|a_k(u - u_k)\|_{L^2 V^2} \leq CA_{11} k^2. \quad \square
\]
The previous theorem considered the pressure error in an integral sense. For pointwise-in-time pressure errors we need to increase the regularity assumptions:

**Assumption 19.** We assume \(\Omega\) has a \(C^6\)-boundary, the right-hand side has regularity \(f \in C^0(I, H^4(\Omega))\cap C^1(I, H^2(\Omega))\cap C^2(I, L^2(\Omega))\cap H^3(I, V^{-1})\) and the initial data \(u_0 \in V^6\) satisfies the compatibility conditions (2) and (3). We write
\[
CA_{19} := C (\|u_0\|_{V^6} + \|f\|_{C^0 H^4} + \|f\|_{C^1 H^2} + \|f\|_{C^2 L^2} + \|f\|_{H^3 V^{-1}})
\]
where \(C > 0\) is independent of the data and, by abuse of notation, may change with each occurrence of \(CA_{19}\).

It is shown in [17, theorem 2.1 and theorem 2.2] that under these conditions the solution satisfies
\[
\|u\|_{C^0 V^6} + \|u\|_{C^1 V^4} + \|u\|_{C^2 V^2} + \|u\|_{H^2 V^3} + \|p\|_{C^2 Q^1} \leq CA_{19}. \quad (15)
\]

**Theorem 20.** Let assumption 19 hold. Then, the Crank-Nicolson time discretization (6) of the Stokes equations (1) satisfies the a priori error estimate
\[
\|u - u_k\|_{L^\infty V^2} + \|a_k(u - u_k)\|_{L^2 V^3} + \|m_k p - p_k\|_{L^\infty Q^1} \leq CA_{19} k^2.
\]
In particular, the discrete pressure \(p_k\) must be compared to the continuous pressure \(p\) evaluated at interval midpoints for second order convergence.

**Proof.** For the velocity error we can repeat the arguments from theorem 18 with one order of regularity higher, yielding
\[
\|u - u_k\|_{L^\infty V^2} + \|a_k(u - u_k)\|_{L^2 V^3} \leq \|u - i_k u\|_{L^\infty V^2} + C\|a_k(u - i_k u)\|_{L^2 V^3} \\
\leq Ck^2 (\|\partial_t u\|_{L^\infty V^2} + \|\partial_t u\|_{L^2 V^3}) \leq CA_{19} k^2.
\]
For the pressure error we use (13) to get
\[ \|a_k p - p_k\|_{L^\infty Q^1} \leq C\|(P - \text{Id})\Delta a_k (u - u_k)\|_{L^\infty L^2} \leq C\|u - u_k\|_{L^\infty V^2}. \]

Using the already established estimate for \(u - u_k\) and lemma 8 we conclude that
\[ \|m_k p - p_k\|_{L^\infty Q^1} \leq \|a_k p - m_k p\|_{L^\infty Q^1} + \|a_k p - p_k\|_{L^\infty Q^1} \]
\[ \leq C k^2 \|\partial_t u\|_{L^\infty Q^1} + C\|u - u_k\|_{L^\infty V^2} \leq C_{A19} k^2. \]

Remark 21. The proof of theorem 20 shows that \(m_k p\) in the pressure estimate could be replaced with \(a_k p\), the latter in fact being more natural in view of (12). We formulate all \(L^\infty\)-in-time pressure estimates with \(m_k p\) due to its simpler evaluation.

2.2 Navier-Stokes Equations

In this section let \(u\) and \(p\) denote a solution to the Navier-Stokes equations
\[ \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = f, \quad \text{div} u = 0 \quad \text{in } \Omega, \tag{16} \]
with \(u(0) = u^0, u|_{\partial \Omega} = 0\) and \(\int_{\Omega} p = 0\). If \(n = 3\) we require \(T\) or the data to be sufficiently small to guarantee existence of weak solutions. As in the last section we will assume that the data is sufficiently regular to allow the minimal required regularity for quadratic pressure estimates without smoothing techniques. Throughout this section, the problem data has to satisfy the following assumption:

Assumption 22. We assume that \(\Omega\) has a \(C^5\)-boundary, the right hand side is given in \(f \in C^0(I, H^3(\Omega)) \cap C^1(I, H^3(\Omega)) \cap H^2(I, L^2(\Omega))\) and that the initial data is given in \(u^0 \in V^5\) and satisfies the compatibility condition that \(p^0\) and \(q^0\) with
\[ \begin{cases} \Delta p^0 = \nabla \cdot (f(0) - u^0 \cdot \nabla u^0), \\ \Delta q^0 = \nabla \cdot \partial_t f(0) - 2 \text{tr} \left( (\nabla f(0) + \nabla \Delta u^0 - \nabla (u^0 \cdot \nabla u^0) - \nabla^2 p^0) \nabla u^0 \right) \end{cases} \tag{17} \]
in \(\Omega\) can be chosen such that
\[ \begin{cases} \nabla p^0 = f(0) + \Delta u^0, \\ \nabla q^0 = \partial_t f(0) + \Delta f(0) + \Delta^2 u^0 - 2\Delta u^0 \cdot \nabla u^0 - f \cdot \nabla u^0 \\ - \Delta \nabla p^0 + \nabla p^0 \cdot \nabla u^0 \end{cases} \tag{18} \]
on \(\partial \Omega\). We write
\[ C_{A22} := C(\|u^0\|_{V^5}, \|f\|_{C^0 H^3}, \|f\|_{C^1 H^1}, \|f\|_{H^2 L^2}) \]
where \(C(\cdot, \cdot, \cdot, \cdot, \cdot, \cdot)\) is some function independent of the data and, by abuse of notation, may change with each occurrence of \(C_{A22}\).
In [17, theorem 3.1 and theorem 3.2] it is shown that exactly under these conditions there exists a solution to the Navier-Stokes equations that satisfies the bound

$$\|u\|_{C^1V^5} + \|u\|_{C^2V^3} + \|u\|_{C^2V^1} + \|u\|_{H^2V^2} + \|p\|_{H^2Q^1} \leq C A_{22}. \quad (19)$$

Like in the case of the Stokes equations the compatibility condition requires the solution of an overdetermined problem. Again we assume the validity of assumption 22 throughout this section. In section 3.2 we will introduce a smoothing version of all estimates that will allow us to derive optimal order error bounds without relying on such non-local compatibility conditions.

We call functions $u_k \in cG^1(\mathcal{I}, V^1)$ and $p_k \in dG^0(\mathcal{I}, Q^0)$ a solution to the time-discrete Navier-Stokes equations if there holds

$$\partial_t u_k - \Delta u_k + (a_k u_k) \cdot \nabla (a_k u_k) + \nabla p_k = f \quad \text{in } dG^0(\mathcal{I}, H^1_0(\Omega))^t \quad (20)$$

with $u_k(0) = u^0$. As a classical timestepping scheme this is equivalent to

$$u^n_k - u^{n-1}_k - \frac{k_n}{2}(u_n^n + u_{n-1}^n) + \frac{k_n}{2}(u_n^n + u_{n-1}^n) \cdot \nabla (u_n^n + u_{n-1}^n) + k_n \nabla p^n_k = \int_{t^n} f(t) \, dt,$$

for $n = 1, \ldots, N$ with $u^0_k := u^0$.

**Remark 23.** The quadratic form of the nonlinearity makes the approximation $(a_k u_k) \cdot \nabla (a_k u_k)$ instead of $u_k \cdot \nabla u_k$ feasible. This not only simplifies numerical quadrature, but also the analysis since some terms will cancel.

Before we prove a priori estimates we collect some technical results, sometimes in greater generality than needed for this section.

**Lemma 24** (Gronwall’s inequality). Let $\alpha_n > 0$, $\beta_n \geq 0$, $\gamma_n \geq 0$, $\delta \geq 0$ and $x_n \geq 0$ for $n = 0, \ldots, N$. If $x_n$ satisfies

$$\alpha_n x_n + \beta_n \leq \sum_{k=0}^{n-1} \gamma_k x_k + \delta \quad \text{for } n = 0, \ldots, N$$

then there holds

$$\alpha_n x_n + \beta_n \leq \delta c \sum_{k=0}^{n-1} \frac{\gamma_k}{\alpha_k} \quad \text{for } n = 0, \ldots, N.$$

**Lemma 25.** Let $u, v \in V^1$ with additional regularity if necessary.

1. For $s = -3, \ldots, 0$ there holds

$$\|u \cdot \nabla v\|_{H^s} \leq C \|u\|_{L^2} \|v\|_{V^{s+3}}, \quad (21a)$$

$$\|u \cdot \nabla v\|_{H^s} \leq C \|u\|_{V^2} \|v\|_{V^{s+1}}, \quad (21b)$$

$$\|u \cdot \nabla v\|_{H^s} \leq C \|u\|_{V^{s+1}} \|v\|_{V^2}, \quad (21c)$$

where (21b) and (21c) can be sharpened for $s = 0$ to

$$\|u \cdot \nabla v\|_{L^2} \leq C \|\nabla u\|_{L^2} \|\Delta u\|_{L^2} \|v\|_{V^1}, \quad (22a)$$

$$\|u \cdot \nabla v\|_{L^2} \leq C \|u\|_{V^1} \|\nabla v\|_{L^2} \|\Delta v\|_{L^2}. \quad (22b)$$

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2. For $s \geq 1$ we have
\[
\|\nabla^s (u \cdot \nabla v)\|_{L^2} \leq C \sum_{i=1}^{s} \|u\|_{V^{i+1}} \|v\|_{V^{i-1}} = C \sum_{i=1}^{s} \|v\|_{V^{i+1}} \|u\|_{V^{i-1}}. \tag{23}
\]

Proof. We use the embeddings $H^1(\Omega) \hookrightarrow L^6(\Omega) \hookrightarrow L^3(\Omega)$ and $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$.

$s = 0$: With the Gagliardo-Nirenberg inequality $\|u\|_{L^3} \leq C \|u\|_{L^2}^{2/3} \|
abla u\|_{L^2}^{1/3}$ for $u \in H^1(\Omega)$ and Agmon’s inequality $\|u\|_{L^\infty} \leq C \|\nabla u\|_{L^2}^{1/2} \|\Delta u\|_{L^2}^{1/2}$ for $u \in V^2$ we get (21a), (22a) and (22b):
\[
\|u \cdot \nabla v\|_{L^2} \leq C \|u\|_{L^2} \|\nabla v\|_{L^\infty} \leq C \|u\|_{L^2} \|v\|_{V^3},
\]
\[
\|u \cdot \nabla v\|_{L^2} \leq C \|u\|_{L^\infty} \|\nabla v\|_{L^2} \leq C \|\nabla u\|_{L^2}^{1/2} \|\Delta u\|_{L^2}^{1/2} \|v\|_{V^1},
\]
\[
\|u \cdot \nabla v\|_{L^2} \leq C \|u\|_{L^\infty} \|\nabla v\|_{L^3} \leq C \|u\|_{V^1} \|\nabla v\|_{L^2} \|\Delta v\|_{L^2}^{1/2}.
\]

$s = -1$: Let $w \in H_0^1(\Omega)$. Then $(u \cdot \nabla v, w) = - (u \otimes v, \nabla w)$ and thus (21b) since $\|u \cdot \nabla v\|_{H^{-1}} \leq C \|u \otimes v\|_{L^2} \leq C \|u\|_{L^\infty} \|v\|_{L^2} \leq C \|u\|_{V^2} \|v\|_{L^2}$.

(21c) follows by using $\|u \otimes v\|_{L^2} \leq C \|u\|_{L^2} \|v\|_{L^\infty}$ instead. (21c) implies (21a).

$s = -2$: Let $w \in H_0^1(\Omega) \cap H^2(\Omega)$. Then (21a) follows from
\[
(u \cdot \nabla v, w) \leq \|u \cdot \nabla v\|_{L^1} \|w\|_{L^\infty} \leq \|u\|_{L^2} \|\nabla v\|_{L^2} \|w\|_{H^2}.
\]

For (21b) we estimate
\[
(u \cdot \nabla v, w) = -(u \cdot \nabla w, v) \leq C \|u \cdot \nabla w\|_{H^1} \|v\|_{V^{-1}}, \tag{24}
\]
and (21b) follows from $\|u \cdot \nabla w\|_{L^2} \leq C \|u\|_{L^\infty} \|\nabla w\|_{L^2}$ together with
\[
\|\nabla (u \cdot \nabla w)\|_{L^2} \leq C \left( \|\nabla u\|_{L^3} \|\nabla w\|_{L^6} + \|u\|_{L^\infty} \|\nabla^2 w\|_{L^2} \right) \leq C \|u\|_{V^2} \|w\|_{H^2}.
\]

If in (24) we use instead $(u \cdot \nabla w, v) = (u, \nabla wv) \leq \|u\|_{V^{-1}} \|\nabla wv\|_{H^1}$, similar arguments yield (21c).

$s = -3$: Let $w \in H_0^1(\Omega) \cap H^3(\Omega)$. Then (21a) follows from
\[
(u \cdot \nabla v, w) = -(u \otimes v, \nabla w) \leq \|u \otimes v\|_{L^1} \|\nabla w\|_{L^\infty} \leq C \|u\|_{L^2} \|v\|_{L^2} \|w\|_{H^3}.
\]

For (21b) we have $(u \cdot \nabla v, w) \leq C \|u \cdot \nabla w\|_{H^2} \|v\|_{V^{-2}}$ similarly as in (24). Using the estimates for $\|u \cdot \nabla w\|_{L^2}$ and $\|\nabla (u \cdot \nabla w)\|_{L^2}$ from $s = -2$ and
\[
\|\nabla^2 (u \cdot \nabla w)\|_{L^2} \leq C \left( \|\nabla^2 u\|_{L^2} \|\nabla w\|_{L^\infty} + \|\nabla u\|_{L^2} \|\nabla^2 w\|_{L^2} + \|u\|_{L^\infty} \|\nabla^3 w\|_{L^2} \right)
\]
\[
\leq C \|u\|_{V^2} \|w\|_{H^3}
\]
we arrive at (21b). The estimate (21c) follows by a similar argument.
\[ s \geq 1: \text{We have by the product rule} \]
\[
\|\nabla^s(u \cdot \nabla v)\| \leq C \sum_{i=0}^{s} \|\nabla^i u\|_{L^{p_i}} \|\nabla^{s-i+1} v\|_{L^{q_i}}
\]
with \(\frac{1}{p_i} + \frac{1}{q_i} = \frac{1}{2}\) for \(i = 0, \ldots, s\). For \(i > 0\) we set \(p_i = q_i = 4\) and use \(H^1(\Omega) \hookrightarrow L^4(\Omega)\), for \(i = 0\) we set \(p_0 = \infty, q_0 = 2\) and use \(H^2(\Omega) \hookrightarrow L^\infty(\Omega)\). This yields
\[
\|\nabla^s(u \cdot \nabla v)\| \leq C \sum_{i=1}^{s} \|u\|_{V^{s-1}} \|v\|_{V^{s-1}+2} + C \|u\|_{V^2} \|v\|_{V^{s+1}}
\]
and since the last term corresponds to \(i = 1\), (23) follows.

**Lemma 26.** With the operator
\[ N(u, v) := (u - v) \cdot \nabla (u - v) + (u - v) \cdot \nabla v + v \cdot \nabla (u - v). \]
we have the velocity error identity
\[
\partial_t (u_k - i_k u) - P\Delta a_k(u_k - i_k u) = -P\Delta a_k(u - i_k u) + a_k PN(u, a_k i_k u) - PN(a_k u_k, a_k i_k u)
\]
and the pressure error identity
\[
\nabla(p_k - a_k p) = (P - \text{Id})(\Delta a_k(u - u_k) - a_k N(u, a_k i_k u) + N(a_k u_k, a_k i_k u)).
\]

**Proof.** We proceed just as in the linear case, lemma 17. The only exception is the nonlinear term in the Galerkin orthogonality, for which we use the identity
\[ u \cdot \nabla u - a_k u_k \cdot \nabla a_k u_k = N(u, a_k i_k u) - N(a_k u_k, a_k i_k u) \]
which follows by elementary calculations. \(\square\)

**Lemma 27.** Under assumption 22 there holds
\[ \|a_k N(u, a_k i_k u)\|_{L^2 H^{-1}} \leq C_{A22} k^2, \quad \|a_k N(u, a_k i_k u)\| \leq C_{A22} k^2. \]

**Proof.** By definition of \(N\) and the properties of \(a_k\) we have that
\[
a_k N(u, a_k i_k u) = a_k ((u - a_k i_k u) \cdot \nabla (u - a_k i_k u)) + a_k (u - i_k u) \cdot \nabla a_k i_k u + a_k i_k u \cdot \nabla a_k (u - i_k u).
\]
Together with (21b), (21c) and the results from lemmata 7 and 9 this yields
\[
\|a_k N(u, a_k i_k u)\|_{L^2 H^s} \leq C (\|u - a_k i_k u\|_{L^\infty V^{s+1}} \|u - a_k i_k u\|_{L^2 V^2} + \|u\|_{C^0 V^2} \|a_k (u - i_k u)\|_{L^2 V^{s+1}}) \leq C k^2 (\|\partial_t u\|_{L^\infty V^{s+1}} \|\partial_t u\|_{L^2 V^2} + \|u\|_{C^0 V^2} \|\partial_t u\|_{L^2 V^{s+1}})
\]
if \(s \in \{-1, 0\}\). The claim follows since by (19) the right side is bounded by \(C_{A22} k^2\). \(\square\)
Lemma 28. Under assumption 22 there exists $C_0 > 0$ such that there holds

$$\|i_ku - u_k\|_{L^\infty L^2} + \|a_k(i_ku - u_k)\|_{L^2 V^1} \leq C_{A22}k^2$$

if the stepsize condition $k < C_0 \|\nabla u\|_{L^\infty L^2}^{-1} \|\Delta u\|_{L^\infty L^2}^{-1}$ is satisfied.

Proof. Just as in the proof of linear stability, lemma 14 for $s = 0$, we test (25) with $v_k \mathbb{I}_J$ where $v_k := i_k u - u_k$ and $J := [0, t_n]$ for $n = 1, \ldots, N$. This implies

$$\frac{1}{2}\|v_k''\|_{L^2}^2 + \|a_k v_k\|_{L^2 J^V1}^2$$

$$= -\langle(\nabla a_k(u - i_k u), \nabla a_k v_k)\rangle_J + \langle a_k PN(u, a_k i_k u), v_k\rangle_J - \langle PN(a_k u_k, a_k i_k u), v_k\rangle_J$$

$$\leq C \left(\|u - i_k u\|_{L^2 J^V1} + \|a_k PN(u, a_k i_k u)\|_{L^2 J^V1}^{-1}\right) \|a_k v_k\|_{L^2 J^V1}$$

$$- \langle PN(a_k u_k, a_k i_k u), v_k\rangle_J.$$ 

Lemma 7 for the interpolation error and lemma 27 for $N(u, a_k i_k u)$ imply

$$\|P\Delta a_k(u - i_k u)\|_{L^2 J^V1} \leq C_{A22}k^2, \quad \|a_k PN(u, a_k i_k u)\|_{L^2 J^V1} \leq C_{A22}k^2.$$ 

and hence by Young’s inequality

$$\frac{1}{2}\|v_k''\|_{L^2}^2 + \|a_k v_k\|_{L^2 J^V1}^2 \leq C_{A22}k^2 - \langle PN(a_k u_k, a_k i_k u), v_k\rangle_J. \quad (28)$$

For the last term we use the antisymmetry of the nonlinearity, hence implicitly its time discretization, and (21b) with $s = 0$ to estimate

$$\|\langle PN(a_k u_k, a_k i_k u), v_k\rangle_J\| = \|\langle a_k v_k \cdot \nabla a_k i_k u, a_k v_k\rangle_J\|$$

$$\leq C \|\nabla u\|_{L^\infty L^2}^{\frac{3}{2}} \|\Delta u\|_{L^\infty L^2} \|a_k v_k\|_{L^2 J^V1} \|a_k v_k\|_{L^2 J^L^2}$$

$$\leq C \|\nabla u\|_{L^\infty L^2} \|\Delta u\|_{L^\infty L^2} \|a_k v_k\|_J + \frac{1}{4} \|a_k v_k\|_{L^2 J^V1}^2.$$ 

The last term will be moved to the left of (28). Since

$$\|a_k v_k\|_J^2 = \frac{1}{4} \sum_{j=1}^{n} k_j \|v_k^j\|_{L^2} + \|v_k^{j-1}\|_{L^2}^2 \leq C \sum_{j=1}^{n} k_j \|v_k^j\|_{L^2}^2$$

we can use (29) in (28) and conclude the proof with Gronwall’s inequality from lemma 24 if $k < C_0 \|\nabla u\|_{L^\infty L^2}^{-1} \|\Delta u\|_{L^\infty L^2}^{-1}$ for some $C_0 > 0$ independent of the data. \qed

Lemma 29. Under assumption 22 there exists $C_0 > 0$ such that there holds

$$\|i_k u - u_k\|_{L^\infty V^1} + \|a_k(i_k u - u_k)\|_{L^2 V^2} \leq C_{A22}k^2$$ 

if the stepsize condition $k < C_0 \|\nabla u\|_{L^\infty L^2}^{-1} \|\Delta u\|_{L^\infty L^2}^{-1}$ is satisfied.
Proof. We use the stability estimate (10) with \( s = 1 \) for the error equation (25):
\[
\|i_k u - u_k\|_{L^\infty V^1} + \|\xi_k (i_k u - u_k)\|_{L^2 V^2} \\
\leq C \left( \|P \Delta a_k (u - i_k u)\| + \|a_k P N(u, a_k i_k u)\| + \|P N(a_k u_k, a_k i_k u)\| \right)
\]
(31)
For the first and second term on the right we have by lemmata 7 and 27:
\[
\|P \Delta a_k (u - i_k u)\| \leq C_{A22} k^2, \quad \|a_k P N(u, a_k i_k u)\| \leq C_{A22} k^2.
\]
(32)
For the third term on the right we abbreviate \( v_k := i_k u - u_k \) and have by (22a), (22b):
\[
\|N(a_k u_k, a_k i_k u)\| \leq C \left( \|a_k v_k\|_{L^\infty V^1} \|a_k v_k\|_{L^2 V^2}^{\frac{3}{2}} \|a_k v_k\|_{L^2 V^1}^{\frac{1}{2}} + \|u\|_{C^0 V^2} \|a_k v_k\|_{L^2 V^1} \right)
\]
\[
\leq C(\delta) \|a_k v_k\|_{L^2 V^1} \left( \|a_k v_k\|_{L^\infty V^1} + \|u\|_{C^0 V^2} \right) + \delta \|a_k v_k\|_{L^2 V^2}
\]
for arbitrary \( \delta > 0 \). By lemma 28 we have \( \|a_k v_k\|_{L^2 V^1} \leq C_{A22} k^2 \) and with the inverse inequality also \( \|a_k v_k\|_{L^\infty V^1} \leq C k^{-\frac{1}{2}} \|a_k v_k\|_{L^2 V^1} \leq C_{A22} \), hence
\[
\|N(a_k u_k, a_k i_k u)\| \leq C_{A22}(\delta) k^2 + \delta \|a_k v_k\|_{L^2 V^2}.
\]
(33)
Using (32) and (33) in the stability estimate we arrive at (30) after choosing \( \delta \) small enough such that \( \|a_k v_k\|_{L^2 V^2} \) in (33) can be moved to the left-hand side of (31).

\[\square\]

Theorem 30. Let assumption 22 hold. Then there exists \( C_0 > 0 \) such that the Crank-Nicolson time discretization (20) of the Navier-Stokes equations (16) satisfies the a priori bound
\[
\|u - u_k\|_{L^\infty V^1} + \|a_k (u - u_k)\|_{L^2 V^2} + \|a_k p - p_k\|_{L^2 Q^1} \leq C_{A22} k^2
\]
if the stepsize condition \( k < C_0 \|\nabla u\|_{L^\infty L^2}^{-1} \|\Delta u\|_{L^\infty L^2}^{-1} \) is satisfied.

Proof. With \( u - u_k = (u - i_k u) + (i_k u - u_k) \) the estimates for the velocity follow by lemma 29 and the interpolation error estimate from lemma 7. To estimate the pressure error we use (26) and Poincaré’s inequality to get
\[
\|a_k p - p_k\|_{L^2 Q^1} \leq C \left( \|a_k (u - u_k)\|_{L^2 V^2} + \|a_k N(u, a_k i_k u)\| + \|N(a_k u_k, a_k i_k u)\| \right).
\]
The \( k^2 \)-bound follows for the first term using the established velocity estimate and for the second by lemma 27. For the third term we use (33) from the proof of lemma 29:
\[
\|N(a_k u_k, a_k i_k u)\| \leq C_{A22} k^2 + C \|a_k (u_k - i_k u)\|_{L^2 V^2} \leq C_{A22} k^2.
\]
(34)
This concludes the proof that \( \|a_k p - p_k\|_{L^2 Q^1} \leq C_{A22} k^2 \).
\[\square\]

We extend the assumptions on the data to obtain \( L^\infty \)-estimates for the pressure.

Assumption 31. We assume \( \Omega \) has a \( C^6 \)-boundary, the right-hand side has regularity \( f \in C^0(I, H^4(\Omega)) \cap C^1(I, H^2(\Omega)) \cap C^2(I, L^2(\Omega)) \cap H^3(I, V^{-1}) \) and the initial data \( u^0 \in V^6 \) satisfies the compatibility conditions (17) and (18). We write
\[
C_{A31} := C(\|u^0\|_{V^6}, \|f\|_{C^0 H^4}, \|f\|_{C^1 H^2}, \|f\|_{C^2 L^2}, \|f\|_{H^3 V^{-1}})
\]
where \( C(\cdots) \geq 0 \) is a function independent of the data and, by abuse of notation, may change with each occurrence of \( C_{A31} \).

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It is shown in [17, theorem 3.1 and theorem 3.2] that under exactly these conditions the solution satisfies

$$\|u\|_{C^0 V^6} + \|u\|_{C^1 V^4} + \|u\|_{C^2 V^2} + \|u\|_{H^2 V^3} + \|p\|_{C^2 Q^4} + \|p\|_{H^2 Q^2} \leq C_{A31}. \quad (35)$$

**Theorem 32.** Let assumption 31 hold. Then there exists $C_0 > 0$ such that the Crank-Nicolson time discretization (20) of the Navier-Stokes equations (16) satisfies the a priori error estimate

$$\|u - u_k\|_{L^\infty V^2} + \|a_k (u - u_k)\|_{L^2 V^3} + \|m_k p - p_k\|_{L^\infty Q^1} \leq C_{A31} k^2,$$

if the stepsize condition $k < C_0 \|\nabla u\|_{L^\infty L^2}^{-1} \|\Delta u\|_{L^\infty L^2}^{-1}$ is satisfied. In particular, the discrete pressure $p_k$ must be compared to the continuous pressure $p$ evaluated at interval midpoints for second order convergence.

**Proof.** We again split $u - u_k = (u - i_k u) + (i_k u - u_k)$. By lemma 7 there holds

$$\|u - i_k u\|_{L^\infty V^2} + \|a_k (u - i_k u)\|_{L^2 V^3} \leq C k^2 \left( \|\partial_t u\|_{L^\infty V^2} + \|\partial_t u\|_{L^2 V^3} \right) \leq C_{A31} k^2.$$

For $i_k u - u_k$ we use the stability estimate (10) from lemma 14 with $s = 2$ for (25). This implies, abbreviating $v_k := i_k u - u_k$:

$$\|v_k\|_{L^\infty V^2} + \|a_k v_k\|_{L^2 V^3} \leq C \left( \|a_k (u - i_k u)\|_{L^2 V^3} + \|a_k N(u, a_k i_k u)\|_{L^2 H^1} + \|N(a_k u_k, a_k i_k u)\|_{L^2 H^1} \right). \quad (36)$$

The first term on the right can be estimated as above. For the nonlinear terms we only have to derive estimates for the gradients due to lemma 27 and (34). Combining (23) and the representation (27) of $a_k N(u, a_k i_k u)$ we get

$$\|\nabla a_k N(u, a_k i_k u)\|
\leq C \left( \|u - a_k i_k u\|_{L^\infty V^2} \|u - a_k i_k u\|_{L^2 V^2} + \|a_k i_k u\|_{L^\infty V^2} \|a_k (u - i_k u)\|_{L^2 V^2} \right)
\leq C k^2 \left( \|\partial_t u\|_{L^\infty V^2} \|\partial_t u\|_{L^2 V^2} + \|u\|_{C^0 V^2} \|\partial_t u\|_{L^2 V^2} \right) \leq C_{A31} k^2.$$

For the second nonlinear term in (36) we proceed similarly, yielding

$$\|\nabla N(a_k u_k, a_k i_k u)\| \leq C \|a_k v_k\|_{L^2 V^3} \left( \|a_k v_k\|_{L^\infty V^2} + \|u\|_{C^0 V^2} \right) \leq C_{A31} k^2$$
using that $\|a_k v_k\|_{L^2 V^2} \leq C_{A31} k^2$ by lemma 29 and, by the inverse inequality, also $\|a_k v_k\|_{L^\infty V^2} \leq C_{A31}$. Inserting the previous estimates into (36) we arrive at

$$\|v_k\|_{L^\infty V^2} + \|a_k v_k\|_{L^2 V^3} \leq C_{A31} k^2$$
which together with the estimates for $u - i_k u$ concludes the proof for the velocity errors. For the pressure error we get from the error identity (26) that

$$\|a_k p - p_k\|_{L^\infty Q^1}
\leq C \left( \|u - u_k\|_{L^\infty V^2} + \|N(u, a_k i_k u)\|_{L^\infty L^2} + \|N(a_k u_k, a_k i_k u)\|_{L^\infty L^2} \right). \quad (37)$$
For the first term on the right we use the established velocity estimate. For the second term on the right of (37) we repeat the arguments from the proof of lemma 27 with $L^\infty$-in-time estimates:

$$\|N(u, a_k i_k u)\|_{L^\infty L^2} \leq C k^2 (\|\partial_t u\|_{L^\infty V^1} \|\partial_t u\|_{L^\infty V^2} + \|\partial_{tt} u\|_{L^\infty V^1} \|u\|_{C^0 V^2}).$$

Using lemma 25 we get for the third term on the right of (37):

$$\|N(a_k u_k, a_k i_k u)\|_{L^\infty L^2} \leq C \|a_k v_k\|_{L^\infty V^1} (\|a_k v_k\|_{L^\infty V^2} + \|u\|_{C^0 V^2}) \leq C A_{31} k^2$$

using $\|v_k\|_{L^\infty V^1} \leq C A_{31} k^2$ by theorem 30 and $\|a_k v_k\|_{L^\infty V^2} \leq C A_{31}$ by the inverse inequality. Combining these estimates for the right-hand side of (37) with lemma 8 we can hence conclude the proof since

$$\|m_k p - p_k\|_{L^\infty Q^1} \leq \|a_k p - m_k p\|_{L^\infty Q^1} + \|a_k p - p_k\|_{L^\infty Q^1} \leq C A_{31} k^2.$$ 

\[\Box\]

3 Estimates with Low Regularity

The previous estimates require a priori bounds which can only be obtained by strong assumptions on the initial data, in particular nonlocal compatibility conditions (2), (3) for the Stokes and (17), (18) for the Navier Stokes equations. These conditions are hard to verify and do not hold in general.

To avoid this requirement we derive error estimates exploiting the smoothing of the solution operator, quantified through the smoothing function $\tau: I \to \mathbb{R}$ and its discrete counterpart $\tau_k \in dG^0(I_k, \mathbb{R})$ defined as

$$\tau := \min\{t, 1\}, \quad \tau_k|_{I_n} := \min\{t_{n-1}, 1\} \quad \text{for} \quad n = 1, \ldots, N.$$

Note that in particular $\tau_k|_{I_1} \equiv 0$ and $\tau_k \leq \tau$. We remind of remark 10: The presence of the smoothing function does not affect the error estimates for the operators $a_k, i_k$ and $m_k$, this will be used without mention in the following. By weighting norms with powers of the smoothing function and by introducing some few initial steps with the backward Euler scheme we will derive optimal order error estimates that do not require nonlocal compatibility conditions.

3.1 Stokes Equations

Throughout this section we make... 

Assumption 33. In addition to assumption 2 of comparable timestep sizes there also exists $\rho \geq 1$ such that $k \leq \rho k_1$. The generic constants $C > 0$ may depend on $\rho$.

For reasons which will become clear later we first perform $n_0 \in \mathbb{N}$ implicit Euler steps with $n_0$ only depending on the regularity assumptions. We start by formulating this timestepping scheme: Let $u_k^n \in V^1$ and $p_k^n \in Q^0$ solve

$$u_k^n - u_k^{n-1} - k_n \Delta u_k^n + k_n \nabla p_k^n = \int_{I_n} f(t) \, dt$$

(38a)
for \( n = 1, \ldots, n_0 \) with \( u_k^0 \) denoting the time discretization \( I_k \) starting at \( t_{n_0} \). Let \( \Gamma_k^{n_0} \) denote the time discretization \( I_k \) starting at \( t_{n_0} \). Then \( u_k \in cG^1(\Gamma_k^{n_0}, V^1) \) and \( p_k \in dG^0(\Gamma_k^{n_0}, Q^0) \) must satisfy
\[
\partial_t u_k - \Delta u_k + \nabla p_k = f \quad \text{in } dG^0(\Gamma_k^{n_0}, H^1_0(\Omega))' \tag{38b}
\]
with initial value \( u_k(t_{n_0}) = u_k^{n_0} \) from the last implicit Euler step. The results for the Crank-Nicolson scheme from section 2 carry over to this time-shifted variant. We omit a formulation of this scheme in \( dG \)-spaces since we are only interested in pointwise-in-time results.

In contrast to section 2 we combine the assumptions required for \( L^2 \)- and \( L^\infty \)-in-time pressure a priori estimates into one single assumption:

**Assumption 34.** We assume that \( u^0 \in V^r \) with \( r \in \{0, 1, 2\} \). Let \( s_0 \in \{1, 2\} \) encode the regularity of the a priori estimates, in the sense that \( s_0 = 1 \) will yield \( L^2 \)-in-time and \( s_0 = 2 \) will yield \( L^\infty \)-in-time pressure estimates. With
\[ L := 4 + s_0 - r \in \mathbb{N}_0 \tag{39} \]
we can formulate the remaining assumptions: We assume that \( \Omega \) has a \( C^{s_0+L+1} \)-boundary and the right-hand side has the regularity \( f \in C^2(\overline{I}, H^{s_0+L-1}(\Omega)) \). We write
\[ C_{A34} := C \left( \| u^0 \|_{V^r} + \| f \|_{C^2 H^{s_0+L-1}} \right) \]
where \( C > 0 \) is independent of the data and, by abuse of notation, may change with each occurrence of \( C_{A34} \). If this assumption is made, all generic constants \( C > 0 \) may depend on \( r \) and \( s_0 \).

Under assumption 34 for \( s \in \mathbb{Z} \) with \( s_0 - L \leq s \leq s_0 + L \) we define \( \alpha := \frac{(2m+s-r)^+}{2} \) for \( m \in \{0, 1, 2\} \). The solution to the Stokes equations then satisfies the bounds
\[ \| \tau^\alpha \partial_t^m u \|_{C^0 V^s} + \| \tau^\alpha \partial_t^m u \|_{L^2 V^{s+1}} < C_{A34}. \tag{40} \]
Given \( s \geq 2 \) it further holds
\[ \| \tau^\alpha \partial_t^m p \|_{C^0 Q^{-1}} + \| \tau^\alpha \partial_t^m p \|_{L^2 Q^s} < C_{A34}. \tag{41} \]
This follows by similar arguments as in theorems 2.3–2.5 from [7]. By limiting the required regularity of the initial data to \( r \leq 2 \), we avoid nonlocal compatibility conditions. While this, most critical, assumption is removed, assumption 34 is not strictly weaker than those from section 2 due to the stronger assumptions on \( f \) and possibly \( \Omega \), although some of these can be weakened with a more involved analysis.

To use these results for the discrete error we need discrete stability estimates with smoothing:

**Lemma 35** (Discrete stability with smoothing). Let \( s \in \mathbb{Z}, n_0 \in \mathbb{N} \) and \( J := (t_{n_0}, T] \). Let \( v_k \in cG^1(\Gamma_k^{n_0}, V^{s+1}) \) and \( r_k \in dG^0(\Gamma_k^{n_0}, V^{s-1}) \) be such that
\[
\partial_t v_k - P \Delta a_k v_k = r_k. \tag{42} \]
Then we have for \( \ell \in \mathbb{N} \) the stability estimate

\[
\| \tau^f_k v_k \|_{L^\infty T^\ell V^s} + \| \tau^f_k a_k v_k \|_{L^2 T^\ell V^{s+1}} + \| \tau^f_k \partial_t v_k \|_{L^2 T^\ell V^{s-1}} \\
\leq C \left( k^\frac{f}{2} \| v_k \|_{V^s} + \| \tau^f_k r_k \|_{L^2 T^\ell V^{s+1}} + \| \tau^f_k a_k v_k \|_{L^2 T^\ell V^s} + k \| \tau^f_k \partial_t v_k \|_{L^2 T^\ell V^s} \right).
\]

We will refer to (43) as an estimate of regularity (level) \( s \) and smoothing (level) \( \ell \).

**Proof.** Let \( J' := (t_{n_0}, t_n) \) for \( n = n_0 + 1, \ldots, N \). We test (42) with \( \tau^f_k (-P \Delta)^s v_k \mathbb{1}_{J'} \) and proceed as in the proof of lemma 14, yielding

\[
\langle \partial_t v_k, \tau^f_k (-P \Delta)^s v_k \rangle_{J'} + \frac{1}{2} \| \tau^f_k a_k v_k \|^2_{L^2 T^\ell V^{s+1}} \leq \frac{1}{2} \| \tau^f_k r_k \|^2_{L^2 T^\ell V^{s-1}}.
\]

(44)

For the first term on the left we use the piecewise constantness of \( \tau_k \) to get, with a factor 2 for convenience

\[
2 \langle \partial_t v_k, \tau^f_k (-P \Delta)^s v_k \rangle_{J'} = \sum_{j=n_0+1}^n \int_{j}^{j+1} \frac{d}{dt} \left( \tau^f_k \int_{j}^{j+1} \| v_k \|^2_{V^s} \right) dt
\]

\[
= \sum_{j=n_0+1}^n \tau^f_k \left( \| v_k \|^2_{V^s} - \| v_k \|^2_{V^s} \right)
\]

\[
= \tau^f_k \left( \| v_k \|^2_{V^s} - \| v_k \|^2_{V^s} \right) + \sum_{j=n_0+1}^{n-1} \left( \tau^f_k \int_{j}^{j+1} \| v_k \|^2_{V^s} \right).
\]

Moving all but the first term to the right-hand side of (44) and using \( \tau^f_k \big|_{I^{n_0+1}} \leq C k \), we arrive at

\[
\tau^f_k \left( \| v_k \|^2_{V^s} - \| v_k \|^2_{V^s} \right) + \tau^f_k \left( \| v_k \|^2_{V^s} - \| v_k \|^2_{V^s} \right)
\]

\[
\leq C k^\frac{f}{2} \| v_k \|^2_{V^s} + \tau^f_k \left( \| v_k \|^2_{V^s} - \| v_k \|^2_{V^s} \right) + k \sum_{j=n_0+1}^{n-1} \left( \tau^f_k \int_{j}^{j+1} \| v_k \|^2_{V^s} \right).
\]

(45)

We now want to estimate the last term on the right of (45). In the Crank-Nicolson scheme we have no control over \( \| v_k \|^2_{V^s} \) itself, forcing us to use

\[
\| v_k \|^2_{V^s} \leq \| a_k v_k \|_{I_{j}^{}} + \frac{\delta}{2} \| \partial_t v_k \|_{I_{j}^{}}.
\]

(46)

For the smoothing function jump, we have by elementary calculations that

\[
\tau^f_k \big|_{j+1} - \tau^f_k \big|_{j} = \tau^f \left( t_j \right) - \tau^f \left( t_{j-1} \right) \leq \left| t_j - t_{j-1} \right| \leq \left| \tau^f \right|_{\ell^{\infty}_{j}} \leq C k^\frac{f}{2} \tau^f \left( t_{j-1} \right) \big|_{I_{j}^{}}
\]

(47)

for \( t_{j-1} \leq 1 \), but the finale estimate also holds otherwise. From (46) and (47) we get

\[
\sum_{j=n_0+1}^{n-1} \left( \tau^f_k \big|_{j+1} - \tau^f_k \big|_{j} \right) \| v_k \|^2_{V^s} \leq C \sum_{j=n_0+1}^{n-1} k \tau^f \left( t_{j-1} \right) \left( \| a_k v_k \|_{I_{j}^{}} \| v_k \|^2_{V^s} + k^2 \| \partial_t v_k \|_{I_{j}^{}} \| v_k \|^2_{V^s} \right)
\]

\[
\leq C \left( \| a_k v_k \|^2_{L^2 T^{\ell-1} V^s} + k^2 \| \partial_t v_k \|^2_{L^2 T^{\ell-1} V^s} \right)
\]

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and hence we can conclude for (45) that
\[
\tau_k^\ell \left| \int_{I_n} v_k^n \right|^2_{V^s} + \tau_k^\ell \| \partial_t v_k \|_{L^\infty_{\ell+1}}^2 + \tau_k^\ell \| \partial_t v_k \|_{L^\infty_{\ell+1}}^2 \leq C k^\ell \| v_k^n \|_{V^s}^2 + \| \tau_k^\ell r_k \|_{L^\infty_{\ell+1}}^2 + C \| \tau_k^\ell \|_{L^\infty_{\ell+1}}^2 \| \partial_t v_k \|_{L^\infty_{\ell+1}}^2.
\]
Taking the maximum over \( n = n_0 + 1, \ldots, N \) and using
\[
\| \tau_k^\ell v_k \|_{L^\infty_{\ell+1}}^2 \leq C \max_{n = n_0, \ldots, N} \| \tau_k^\ell \|_{I_n} \| v_k^n \|_{V^s}^2
\]
we can conclude the claimed estimate for \( \| \tau_k^\ell v_k \|_{L^\infty_{\ell+1}}^2 \) and \( \| \tau_k^\ell a_k v_k \|_{L^\infty_{\ell+1}}^2 \). Since
\[
\tau_k^\ell \partial_t v_k = \tau_k^\ell r_k + \tau_k^\ell P \Delta a_k v_k
\]
the final estimate for \( \| \tau_k^\ell \partial_t v_k \|_{L^\infty_{\ell+1}}^2 \) follows from the already established one. \(\square\)

**Remark 36.** We emphasize the occurrence of \( a_k v_k \) and \( \partial_t v_k \) on the right-hand side of (43). These terms can be estimated again by (43) with smoothing level \( \ell - 1 \) and regularity level \( s - 1 \) (for \( a_k v_k \)) and \( s + 1 \) (for \( \partial_t v_k \)). The unfavorable increase in regularity to estimate \( \partial_t v_k \) is balanced by the occurrence of the factor \( k \) in (43). The procedure must be repeated until \( \ell = 0 \), where (10) from lemma 14 can be used. The whole process is depicted as a cascade of estimates in figure 1. The origin of the cascade is the lack of an a priori estimate for \( \| v_k \|_{L^\infty_{\ell+1}}^2 \), in the Crank-Nicolson scheme, circumvented by (46).

Let us now turn to the necessity of the implicit Euler steps. If we apply lemma 35 to the discrete error identity (12) we must control \( \| u(t_{n_0}) - u_k^{n_0} \|_{V^s} \) in (43). If \( s > r \) this is impossible in the Crank-Nicolson scheme since the regularity of the initial value limits the regularity of the discrete solution due to the occurrence of \( \Delta u_k^{n-1} \) in (8). The implicit Euler scheme has, in contrast, a smoothing property: From
\[
-k_n P \Delta u_k^n = \int_{I_n} P f(t) \, dt - u_k^n + u_k^{n-1}
\]
we can deduce that \( u_k^n \in V^{r+2n} \) for sufficiently regular \( f \) and \( \Omega \). The number of implicit Euler steps is consequently chosen such that \( u_k^{n_0} \in V^s \) for the largest \( s \) in the cascade from figure 1, which is the bottom-right node with \( s = s_0 + L \). Therefore \( n_0 \) is such that
\[
r + 2n_0 = s_0 + L = 4 + 2s_0 - r, \text{ i.e.}
\]
\[
n_0 := 2 + s_0 - r.
\]

The following lemma quantifies the control over \( \| u(t_{n_0}) - u_k^{n_0} \|_{V^s} \) for the regularity levels \( s \) which will appear in the smoothing cascade, cf. figure 1. For sufficiently small \( s \) we have the usual second order convergence for the local error in the implicit Euler scheme. For larger \( s \) we loose convergence and ultimately stability, the loss of stability of \( \| u_k^n \|_{V^s} \) for large \( s \) can already be seen through the factor \( k_n \) in (48).
Figure 1: Illustration of the cascade of estimates required to reduce (43) from lemma 35 with regularity level $s_0$ and smoothing level $L$ (bold node) to an estimate with only the data on the right-hand side, see remark 36. Each node represents an application of estimate (43) if $\ell > 0$ or (10) if $\ell = 0$. From top-to-bottom the smoothing level decreases while the regularity decreases (estimates for $a_kv_k$) or increases with factor $k$ (estimates for $\partial_t v_k$).
Lemma 37. Let assumption 34 hold. Then for \( s = s_0 - L, \ldots, s_0 + L \) we have

\[
\|u(t_{n_0}) - u^{m}_{k}\|_{V^s} \leq C_{A34}k^{-s/2}.
\]

Proof. We first consider the case \( s \geq r \), which yields non-positive powers of \( k \). We have by assumption 33 that \( t_{n_0} \geq Ck \). The smoothing estimate of \( u \) from assumption 34 hence implies

\[
\|u(t_{n_0})\|_{V^s} = (\tau(t_{n_0}))^{s/2} \|u(t_{n_0})\|_{V^s} \leq C_{A34}(t_{n_0})^{s/2} \leq C_{A34}k^{s/2}.
\]

To estimate \( u^{m}_{k} \) we test, for \( n = 1, \ldots, n_0 \),

\[
u^m_k - k_nP\Delta u^m_k = \int_{t_n} P\phi(t)\,dt + u^{n-1}_k
\]

with \((-P\Delta)^\sigma u^m_k\) where \( r \leq \sigma \leq s - 1 \) is chosen later. Then

\[
\frac{1}{2}\|u^m_k\|_{V^\sigma} + k_n\|u^m_k\|_{V^{\sigma+1}} \leq \int_{t_n} \|P\phi(t)\|_{V^\sigma}\,dt + \|u^{n-1}_k\|_{V^\sigma}^2
\]

and hence, again using assumptions 34 and 33,

\[
\|u^m_k\|_{V^\sigma} \leq C_{A34} + C\|u^{n-1}_k\|_{V^\sigma}, \quad \|u^m_k\|_{V^\sigma+1} \leq C_{A34}k^{-\frac{1}{2}} + Ck^{-\frac{1}{2}}\|u^{n-1}_k\|_{V^\sigma}.
\]

By regularity of the Stokes operator we also conclude from (51) that for \( r \leq \sigma \leq s - 2 \):

\[
\|u^m_k\|_{V^{\sigma+2}} \leq Ck^{-1}\left(\int_{t_n} \|P\phi\|_{V^\sigma}\,dt + \|u^{n-1}_k\|_{V^\sigma} + \|u^m_k\|_{V^\sigma}\right) \leq C_{A34}k^{-1} + Ck^{-1}\|u^{n-1}_k\|_{V^\sigma}.
\]

Let now \( s = r + 2m \) with \( m \geq 0 \). Then \( s \leq s_0 + L \) implies \( m \leq n_0 \). By induction over \( m = 0, \ldots, n_0 \), application of (53) yields

\[
\|u^m_k\|_{V^s} = \|u^m_k\|_{V^{r+2m}} \leq C_{A34}k^{-m} + Ck^{-m}\|u^{m-1}_k\|_{V^{r+2(m-1)}} \leq \ldots \leq C_{A34}k^{-m} + Ck^{-m}\|u^0_k\|_{V^r} \leq C_{A34}k^{-m}
\]

with constants depending on \( n_0 \). For \( m < n_0 \) the estimate (52) implies

\[
\|u^m_k\|_{V^s} \leq C_{A34} + C\|u^{m-1}_k\|_{V^s} \leq \ldots \leq C_{A34} + C\|u^m_k\|_{V^s} \leq C_{A34}k^{-m}
\]

which yields for \( s = r + 2m \) that

\[
\|u^m_k\|_{V^s} \leq C_{A34}k^{-m} = C_{A34}k^{-\frac{s}{2}}.
\]

For \( s = r + 2m - 1 \) we have \( m \leq n_0 \) and use the second estimate in (52) once to get

\[
\|u^m_k\|_{V^s} = \|u^m_k\|_{V^{r+2(m-1)+1}} \leq C_{A34}k^{-\frac{1}{2}} + Ck^{-\frac{1}{2}}\|u^{m-1}_k\|_{V^{r+2(m-1)}}.
\]

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Applying the already established estimate for \( u_k^{m-1} \) and using the stability in \( V^s \) from (52) if \( m < n_0 \) we arrive for \( s = r + 2m - 1 \) at
\[
\| u_k^{m_0} \|_{V^s} \leq C A_{k}^{34} k^r \frac{r-s}{2} = C A_{k}^{34} k^{r-s}.
\]
Combining (54) and (55) yields \( \| u_k^n \|_{V^s} \leq C A_{k}^{34} k^{r-s} \) for any \( s = r, \ldots, s_0 + L \). Together with the estimate (50) for \( u \) this concludes the proof for \( s \geq r \).

For \( s < r \) we define \( v_k^n := u(t_n) - u_k^n \) and derive from (38a) the error identity
\[
v_k^n - v_k^{n-1} - k_n P \Delta v_k^n = P \int_{I^h} \Delta u(t) - \Delta u^n \ dt.
\]
Testing this with \( (-P\Delta)^s v_k^n \), yields for \( n = 1, \ldots, n_0 \) that
\[
\| v_k^n \|_{V^s}^2 + k_n \| v_k^n \|_{V^{s+1}}^2 \leq C \| v_k^{n-1} \|_{V^s}^2 + C \left( \int_{I^n} \| u(t) - u^n \|_{V^{s+2}} \ dt \right)^2
\]
\[
\leq C \| v_k^{n-1} \|_{V^s}^2 + C \left( \int_{I^n} \int_{I^n} \| \partial_t u(\tilde{t}) \|_{V^{s+2}} \ dt \ dt \right)^2.
\]
By assumption 34 we have \( \| \partial_t u(t) \|_{V^{s+2}}^2 \leq C A_{k}^{34} t^{-4+s-r} \), thus
\[
\left( \int_{I^n} \int_{I^n} \| \partial_t u(\tilde{t}) \|_{V^{s+2}} \ dt \ dt \right)^2 \leq C A_{k}^{34} k^{r-s}
\]
and by induction of (56) over \( n = 1, \ldots, n_0 \) the claim for \( s < r \) follows.

For simplicity we always assume in the following that \( t_{n_0} \leq 1 \) such that \( \tau(t_n) = t_n \) for all \( n = 0, \ldots, n_0 \).

**Theorem 38.** Let assumptions 34 hold. If \( s_0 = 1 \) the Crank-Nicolson time discretization with \( n_0 = 3 - r \) initial implicit Euler steps (38) of the Stokes equations (1) satisfies on \( J := (t_{n_0}, T] \) the a priori error estimate
\[
\| \tau_k^{s_0} (u - u_k) \|_{L^\infty V} + \| \tau_k^{s_0} a_k(u - u_k) \|_{L^2 V^2} + \| \tau_k^{s_0} (a_k p - p_k) \|_{L^2 Q^1} \leq C A_{k}^{34} k^2.
\]
If \( s_0 = 2 \) we require \( n_0 = 4 - r \) implicit Euler steps and there holds
\[
\| \tau_k^{s_0} (u - u_k) \|_{L^\infty V} + \| \tau_k^{s_0} a_k(u - u_k) \|_{L^2 V^3} + \| \tau_k^{s_0} (m_k p - p_k) \|_{L^2 Q^1} \leq C A_{k}^{34} k^2.
\]

**Proof.** We note that \( n_0 \) satisfies (49) and \( \tau_k^{s_0} \), with \( L = 4 + s_0 - r \) from (39), matches the smoothing in (57) and (58). We first prove, with \( v_k := u_k - u_k \), that
\[
\| \tau_k^{s_0} v_k \|_{L^\infty V} + \| \tau_k^{s_0} a_k v_k \|_{L^2 V^2} + \| \tau_k^{s_0} \partial_t v_k \|_{L^2 V^3} \leq C A_{k}^{34} k^2.
\]
Specifically, we prove by induction over the levels \( \ell = 0, \ldots, L \) of the cascade that
\[
\| \tau_k^{s_0} v_k \|_{L^\infty V} + \| \tau_k^{s_0} a_k v_k \|_{L^2 V^2} + \| \tau_k^{s_0} \partial_t v_k \|_{L^2 V^3} \leq C A_{k}^{34} k^{2^{\ell}}
\]
for \( s = s_0 - L + \ell + 2i \) with \( i = 0, \ldots, L - \ell \). Then (59) corresponds to (60) with \( \ell = L \) (and \( i = 0 \)). We remark that the order \( k^{2-i} \) may be negative but is sufficient in (60) since \( i \) implicitly counts the number of \( \partial_k v_k \)-estimates in the cascade, each of which yields a factor \( k \), see figure 1. Before we proceed to the induction we prove that

\[
k^\ell \| v_k(t_{no}) \|_{V^s} \leq C A_{34}^k k^{2-i} \quad \text{and} \quad \| \tau_k^{\ell} (u - i_k u) \|_{L^2_j V^{s+1}} \leq C A_{34}^k k^{2-i}
\]

(61)

for all nodes in the cascade. We first note that by definition of \( s \) there holds

\[
\ell + r - s = 4 - 2i.
\]

The first estimate in (61) follows then from lemma 37:

\[
k^\ell \| v_k(t_{no}) \|_{V^s} \leq C A_{34}^k k^{\ell-i} = C A_{34}^k k^{2-i}.
\]

To prove the second inequality in (61) we use lemma 7 to estimate

\[
\| \tau_k^{\ell} (u - i_k u) \|_{L^2_j V^{s+1}} \leq C k^2 \| \tau_k^{\ell} \partial_t u \|_{L^2_j V^{s+1}}.
\]

(63)

Using \( \tau_k \leq \tau \), (62) and \( \tau(t_{no}) = t_{no} \geq C k \) we get

\[
\| \tau_k^{\ell} \partial_t u \|_{L^2_j V^{s+1}} \leq \| \tau_k^{\ell} \partial_t u \|_{L^2_j V^{s+1}} \leq \max_{t \in J} (\tau(t))^{-i} \| \tau_k^{\ell} \partial_t u \|_{L^2_j V^{s+1}} \leq C A_{34}^k k^{-i}
\]

which combined with (63) concludes the proof of (61). The case \( \ell = 0 \) in (60) now follows from the non-smoothing estimate (10) from lemma 14 and (61):

\[
\| v_k \|_{L^\infty_j V^s} + \| a_k v_k \|_{L^2_j V^{s+1}} + \| \partial_t v_k \|_{L^2_j V^{s+1}} \leq C (\| v_k^{no} \|_{V^s} + \| u - i_k u \|_{L^2_j V^{s+1}})
\leq C A_{34}^k k^{2-i}.
\]

For \( \ell = 1, \ldots, L \) and \( i = 0, \ldots, L - \ell \) we use the smoothing estimate (43) from lemma 35 and (61):

\[
\| \tau_k^{\ell} v_k \|_{L^\infty_j V^s} + \| \tau_k^{\ell} a_k v_k \|_{L^2_j V^{s+1}} + \| \tau_k^{\ell} \partial_t v_k \|_{L^2_j V^{s+1}}
\leq C \left( k^\ell \| v_k^{no} \|_{V^s} + \| \tau_k^{\ell} (u - i_k u) \|_{L^2_j V^{s+1}} + \| \tau_k^{\ell} a_k v_k \|_{L^2_j V^s} + k \| \tau_k^{\ell} \partial_t v_k \|_{L^2_j V^s} \right)
\leq C A_{34}^k k^{2-i} + C \left( \| \tau_k^{\ell} a_k v_k \|_{L^2_j V^s} + k \| \tau_k^{\ell} \partial_t v_k \|_{L^2_j V^s} \right).
\]

For the remaining terms on the right-hand side we have by induction hypothesis

\[
\| \tau_k^{\ell} a_k v_k \|_{L^2_j V^s} \leq C A_{34}^k k^{2-i} \quad \text{and} \quad \| \tau_k^{\ell} \partial_t v_k \|_{L^2_j V^s} \leq C A_{34}^k k^{2-(i+1)}
\]

which concludes our induction and hence the proof of (59).
Splitting $u - u_k = (u - i_ku) + (i_ku - u_k)$ we arrive at the velocity error estimates in (57) and (58) using (59) and, for the norms of $i_ku - u_k$, the interpolation estimate for $i_k$ and assumption 34 which imply
\[
\| \tau_k^\ell (u - i_ku) \|_{L_\infty^2 V_\infty^0} \leq C k^2 \| \tau_k^\ell \partial_t u \|_{L_\infty^2 V_\infty^0} \leq C k^2 \| \tau_k^\ell \partial_t u \|_{L_\infty^2 V_\infty^0} \leq C A_{34} k^2.
\]
and similarly $\| \tau_k^\ell a_k(u - i_ku) \|_{L_\infty^2 V_\infty^{n+1}} \leq C A_{34} k^2$.

For the pressure we use the error identity (13), which implies
\[
\tau_k^\ell \nabla (p - a_k p) = (P - \text{Id}) \tau_k^\ell a_k \Delta (u - u_k)
\]
and hence for $s_0 = 1$, by the already established velocity estimate from (57),
\[
\| \tau_k^\ell (a_k p - p) \|_{L_\infty^2 Q^1} \leq C \| \tau_k^\ell a_k (u - u_k) \|_{L_\infty^2 V^2} \leq C A_{34} k^2
\]
and this is the pressure estimate in (57). Similarly for $s_0 = 2$ we have
\[
\| \tau_k^\ell (a_k p - p) \|_{L_\infty^2 Q^1} \leq C \| \tau_k^\ell (u - u_k) \|_{L_\infty^2 V^2} \leq C A_{34} k^2
\]
which implies the pressure estimate in (58) using lemma 8.

**Remark 39.** The techniques used in theorem 38 can easily be generalized to other regularity levels $s_0 \in \mathbb{N}$ with appropriately modified assumptions on the data. The case $s_0 = 0$ and $r = 0$ yields $\| \tau_k^\ell (u - u_k) \|_{L_\infty^2 L^2} \leq C k^2$ after $n_0 = 2$ implicit Euler steps, in agreement with the results in [12] for the fully discrete problem.

### 3.2 Navier-Stokes Equations

We now consider the Navier-Stokes equations with smoothing and restrict ourselves to the situation that $u^0 \in V^2$, which corresponds to $r = 2$ in our linear theory.

**Assumption 40.** Let assumption 34 hold for $r = 2$ and $s_0 \in \{1, 2\}$, in particular $L := 2 + s_0$. In addition, let $\| \nabla u \|_{L_\infty^2 L^2} < \infty$, which is satisfied for $d = 2$ or for sufficiently small data if $d = 3$. We write
\[
C_{A40} := C(\| u^0 \|_{V^r}, \| f \|_{C^2 H^{s_0+l-1}}, \| \nabla u \|_{L_\infty^2 L^2})
\]
where $C(\cdots) > 0$ is some function independent of the data and, by abuse of notation, may change with each occurrence of $C_{A40}$.

Under these assumptions for $s \in \mathbb{Z}$ with $-2 \leq s \leq s_0 + L$ we define $\alpha := \frac{(2m + s - 2)\ell}{2}$ for $m \in \{0, 1, 2\}$. The solution to the Navier-Stokes equations then satisfies the bounds
\[
\| \tau^\alpha \partial_t^m u \|_{C_\ell^s V^s} + \| \tau^\alpha \partial_t^m u \|_{L_\infty^2 V^{s+1}} \leq C_{A40}. \tag{64}
\]
Given $s \geq 2$ it further holds
\[ \|\tau^\alpha \partial_t^m p\|_{C^0 Q_{s-1}} + \|\tau^\alpha \partial_t^m p\|_{L^2 Q_s} \leq C A_{40}. \tag{65} \]
Again, this follows by similar arguments as in theorems 2.3–2.5 from [7]. As in the linear case the required regularity is essentially reduced since we do not ask for nonlocal compatibility conditions to hold.

In the following we apply the linear smoothing techniques to $v_k := u_k - i_k u$ using the error identity (25). To bound the nonlinearity further error estimates are needed, resulting in a double-cascade which is sketched for $s_0 = 1$ in figure 3.2. The derivation of the necessary estimates for all nodes of this double-cascade will be the goal for much of this section.

As in the linear case, we start with $n_0 = 2 + s_0 - r = s_0$ implicit Euler steps. The scheme has the form: Find $u_k^n \in V^1$ and $q_k^n \in Q^0$ such that
\[ u_k^n - u_k^{n-1} - k_n \Delta u_k^n + k_n (u_k^n \cdot \nabla u_k^n) - k_n \nabla p_k^n = \int_{I^n} f(t) \, dt \tag{66a} \]
for $n = 1, \ldots, n_0$ with $u_k^0 := u^0$. Afterwards we use the Crank-Nicolson scheme from section 2, i.e. $u_k \in cG^1(\Gamma_{k_0}^n, V^1)$ and $p_k \in dG^0(\Gamma_{k_0}^n, Q^0)$ must satisfy
\[ \partial_t u_k - \Delta u_k + (a_k u_k) \cdot \nabla (a_k u_k) + \nabla p_k = f \quad \text{in} \ dG^0(\Gamma_{k_0}^n, H_0^1(\Omega))' \tag{66b} \]
with initial value $u_k(t_{n_0}) = u_k^{n_0}$ from the last implicit Euler step.
Lemma 41. Let assumption 40 hold. With $J := (t_{n_0}, T]$, we have for each $s = -2, \ldots, s_0 + L$:

$$
\| \tau_k^{s+2} a_k N(u, a_k i_k u) \|_{L^2 H^{s-1}} \leq C A_{40} k^2. \tag{67}
$$

Proof. Combining the approximation estimates from lemmata 7 and 9 and the regularity results for our continuous solution, we have for $-2 \leq s \leq s_0 + L$:

$$
\left\{ \begin{array}{l}
\| \tau_k^{(s-2)+} u \|_{L^\infty V^s} \leq C A_{40}, \quad \| \tau_k^{(s)+} (u - a_k i_k u) \|_{L^2 V^{s+1}} \leq C A_{40} k, \\
\| \tau_k^{(s)+} (u - a_k i_k u) \|_{L^2 V^s} \leq C A_{40} k, \quad \| \tau_k^{(s)+} (u - i_k u) \|_{L^2 V^{s+1}} \leq C A_{40} k^2.
\end{array} \right. \tag{68}
$$

Combining the estimates for the nonlinearity from (21) with (68) yields (67) for $s \leq 1$:

$$
\| \tau_k^{s+2} N(u, a_k i_k u) \|_{L^2 H^{s-1}}
\leq C \left( \| u - a_k i_k u \|_{L^\infty L^2} \| \tau_k^{(s+1)+} (u - a_k i_k u) \|_{L^2 V^{s+2}} \\
+ \| a_k i_k u \|_{L^\infty V^2} \| \tau_k^{s+2} (u - i_k u) \|_{L^2 V^{s+1}} \right)
\leq C k^2 \left( \| \partial_t u \|_{L^\infty L^2} \| \tau_k^{(s+1)+} \partial_t u \|_{L^2 V^{s+2}} + \| u \|_{C^2 V^2} \| \tau_k^{s+2} \partial_t u \|_{L^2 V^{s+1}} \right)
\leq C A_{40} k^2.
$$

For $s \geq 2$ we use (23) and (68) to get

$$
\| \tau_k^{s+2} a_k \nabla^{s-1} N(u, a_k i_k u) \|_J
\leq C \sum_{i=1}^{s-1} \left( \| \tau_k^{i+2} (u - a_k i_k u) \|_{L^\infty V^{i+1}} \| \tau_k^{i+2} (u - a_k i_k u) \|_{L^2 V^{i+1}} \\
+ \| \tau_k^{i+2} (u - i_k u) \|_{L^2 V^{i+1}} \| \tau_k^{i+1} u \|_{L^\infty V^{i+1}} \right)
\leq C A_{40} k^2
$$

noting in particular that the smoothing on the left-hand side is sufficient for the smoothing used on the right. Combining these estimates and (67) with $s = 1$ for the $L^2$-term, we conclude that (67) is true for the full $H^{s-1}$-norm.

Lemma 42. Let assumption 40 hold. Then there exists $C_0 > 0$ such that for each $s = -2, \ldots, s_0 + L$ we have

$$
\| u(t_{n_0}) - u_{n_k}^{n_0} \|_{V^s} \leq C A_{40} k^{2+s}. \tag{69}
$$

if the stepsize condition $k < C_0 \| \nabla u \|_{L^\infty L^2}^{-1} \| \Delta u \|_{L^\infty L^2}^{-1}$ is satisfied.
Proof. We first prove (69) for $s \leq 2$. With $v^n_k := u^n - u^n_k$ we get from (66a) by elementary calculations the error identity
\[
v^n_k - v^{n-1}_k - k_n P \Delta v^n_k = P \int_n \Delta (u(t) - u^n) - u(t) \cdot \nabla u(t) + u^n_k \cdot \nabla u^n_k \, dt
\]
\[
= P \int_n \Delta (u(t) - u^n) - N(u(t), u^n) + N(u^n_k, u^n) \, dt.
\]
Testing this identity with $(-P\Delta)^s v^n_k$ for $s \leq 1$ yields the stability estimate
\[
\frac{1}{2} \| v^n_k \|^2_{\mathcal{V}*} + k_n \| v^n_k \|^2_{V^{s+1}} \leq \frac{1}{2} \| v^{n-1}_k \|^2_{\mathcal{V}*} + (r^n_{k,1} - r^n_{k,2} + r^n_{k,3}, (-P\Delta)^s v^n_k)
\]
where
\[
\begin{align*}
r^n_{k,1} &= \int_n P \Delta (u(t) - u^n) \, dt, \quad r^n_{k,2} := \int_n N(u(t), u^n) \, dt, \quad r^n_{k,3} := \int_n N(u^n_k, u^n) \, dt.
\end{align*}
\]
To estimate the $r^n_{k,1}$-term we use arguments as in the linear case, see (56) from lemma 37. For $\delta > 0$ there thus holds
\[
(r^n_{k,1}, (-P\Delta)^s v^n_k) \leq C(\delta) \left( \int_n \| u(t) - u^n \|^2_{V^{s+2}} \, dt \right)^2 + \delta \| v^n_k \|^2_{\mathcal{V}*}
\]
\[
\leq C \cdot A_{40} \cdot k^{2-s} + \delta \| v^n_k \|^2_{\mathcal{V}*}.
\]
Using this estimate in (70) and moving the $\delta$-term to the left-hand side, we get
\[
\frac{1}{2} \| v^n_k \|^2_{\mathcal{V}*} + k_n \| v^n_k \|^2_{V^{s+1}} \leq \frac{1}{2} \| v^{n-1}_k \|^2_{\mathcal{V}*} + C \cdot A_{40} k^{2-s} + (r^n_{k,2} + r^n_{k,3}, (-P\Delta)^s v^n_k).
\]
To estimate the $r^n_{k,2}$-term in (71) we use two different splittings of the scalar product:
\[
\begin{align*}
(r^n_{k,2}, (-P\Delta)^s v^n_k) &\leq \begin{cases} C(\delta) \| r^n_{k,2} \|^2_{\mathcal{V}*} + \delta \| v^n_k \|^2_{\mathcal{V}*} & \text{if } s = -2, 0, \\ C(\delta) k^{-1} \| r^n_{k,2} \|^2_{V^{s-1}} + \delta k_n \| v^n_k \|^2_{V^{s+1}} & \text{if } s = -1, 1. \end{cases}
\end{align*}
\]
Note that $\| r^n_{k,2} \|_{\mathcal{V}*}$ now only appears with $s = 0$ and $s = -2$. With $v(t) := u(t) - u^n$, (21b) and (21c) we get for $s = 0$:
\[
\int_n \| N(u(t), u^n) \|_{L^2} \, dt \leq C \int_n \| v(t) \|_{V^1} (\| v(t) \|_{V^2} + \| u^n \|_{V^2}) \, dt \leq C \cdot A_{40} k
\]
since $\| v \|_{C^0 V^2} \leq 2 \| u \|_{C^0 V^2} \leq C \cdot A_{40}$. For $s = -2$ we have
\[
\int_n \| N(u(t), u^n) \|_{H^{-2}} \, dt \leq C \int_n \| v(t) \|_{V^{-1}} (\| v \|_{V^2} + \| u^n \|_{V^2}) \, dt \leq C \cdot A_{40} k^2,
\]
since $\| v(t) \|_{V^{-1}} \leq \int_n \| \partial_t v(t) \|_{V^{-1}} \, dt \leq C \cdot A_{40} k$. Combining these estimates with (72) we get for (70), after choosing a suitable $\delta$, that
\[
\| v^n_k \|^2_{\mathcal{V}*} + k_n \| v^n_k \|^2_{V^{s+1}} \leq C \| v^{n-1}_k \|^2_{\mathcal{V}*} + C \cdot A_{40} k^{2-s} + C k_n (r^n_{k,3}, (-P\Delta)^s v^n_k).
\]
(73)
It remains to estimate the $r_{k,3}^n$-term. For $s = 0$ we use the cancellation property of the nonlinearity and (22a) to get

$$k_n(N(u_k^n, u^n), v_k^n) = k_n(v_k^n \cdot \nabla u^n, v_k^n) \leq C k_n \|\nabla u^n\|_{L^2}^2 \|\Delta u^n\|_{L^2}^2 \|v_k^n\|_{V^1}^2 \leq C(\delta) k_n \|\nabla u^n\|_{L^2} \|\Delta u^n\|_{L^2} \|v_k^n\|_{V^2}^2 + \delta k_n \|v_k^n\|_{V^1}^2.$$

With suitable chosen $\delta > 0$ and using the stepsize condition we may move both terms to the left-hand side of (73), implying that

$$\|v_k^n\|_{V^2}^2 + k_n \|v_k^n\|_{V^1}^2 \leq C \|v_k^{n-1}\|_{V^2}^2 + C_{A40} k^2.$$  

Applying this estimate iteratively for $n = 1, \ldots, n_0$ we arrive at (69) for $s = 0$. Actually this also yields (69) for $s = 1$, but we still consider $s = 1$ in (73) to prove (69) for $s = 2$:

We estimate

$$k_n(N(u_k^n, u^n), (-P\Delta)v_k^n) \leq C(\delta) k_n \|N(u_k^n, u^n)\|_{L^2}^2 + \delta k_n \|v_k^n\|_{V^2}^2$$

and using the estimate for $s = 0$, $\|v_k^n\|_{V^1} \leq C_{A40}$, we get with (22) and (21):

$$\|N(u_k^n, u^n)\|_{L^2} \leq C \left(\|v_k^n\|_{V^1}^2 + \|v_k^n\|_{V^2} \|u^n\|_{V^2} \right) \leq C_{A40} \left(\|v_k^n\|_{V^2}^2 + 1\right).$$

This implies

$$k_n(N(u_k^n, u^n), (-P\Delta)v_k^n) \leq C_{A40}(\delta) k_n \left(\|v_k^n\|_{V^2} + 1\right) + \delta k_n \|v_k^n\|_{V^2}^2 \leq C_{A40}(\delta) k + 2\delta k \|v_k^n\|_{V^2}^2$$

which inserted into (73), moving the $\delta$-term to the left-hand side, yields

$$\|v_k^n\|_{V^1}^2 + k_n \|v_k^n\|_{V^2}^2 \leq C \|v_k^{n-1}\|_{V^1}^2 + C_{A40} k$$

and (69) for $s = 2$ follows. For $s = -1, -2$ we estimate the $r_{k,3}^n$-term in (73) by

$$k_n(N(u_k^n, u^n), (-P\Delta)^sv_k^n) \leq \begin{cases} C(\delta) k_n \|N(u_k^n, u^n)\|_{H^{-2}}^2 + \delta k_n \|v_k^n\|_{V^0}^2 & \text{if } s = -1, \\
C(\delta) k_n^2 \|N(u_k^n, u^n)\|_{H^{-2}}^2 + \delta \|v_k^n\|_{V^{-2}}^2 & \text{if } s = -2 \end{cases}$$

and combined with (73) this proves (69) for $s = -1, -2$ since we have

$$\|N(u_k^n, u^n)\|_{H^{-2}} \leq C \left(\|v_k^n\|_{L^2} \|u^n\|_{V^1} + \|u^n\|_{V^2} \|v_k^n\|_{V^{-1}} \right) \leq C_{A40} k$$

which follows from (21) and the estimate (69) for $s = 0$ and $s = 1$. For $s > 2$ the continuous smoothing result (64) implies

$$\|u(t_m)\|_{V^s} = (\tau(t_m))^{\frac{s}{2}} (\tau(t_m))^{\frac{s}{2}} \|u(t_m)\|_{V^s} \leq C_{A40} (t_m)^{\frac{s}{2}} \leq C_{A40} k^{\frac{s}{2}}.$$
For the corresponding bound for \( u_k \) we prove that for \( m = 1, \ldots, n_0 \) there holds
\[
\|u_k^n\|_{V^{2m+1}} \leq CA_{40}^{k^{-m} + \frac{1}{2}}, \quad \|u_k^n\|_{V^{2m+2}} \leq CA_{40}^{k^{-m}} \tag{74}
\]
for all \( n = m, \ldots, n_0 \). It is easy to see that (74) implies \( \|u_k^n\|_{V^s} \leq CA_{40}^{k^{2-s}} \) for \( 3 \leq s \leq s_0 + L \), e.g. for \( s = s_0 + L = 2 + 2s_0 \) we have by (74) for \( m := n_0 = s_0 \) that
\[
\|u_k^{n_0}\|_{V^{s_0+L}} \leq \|u_k^{n_0}\|_{V^{2m+2}} \leq CA_{40}^{k^{-m}} = CA_{40}^{k^{2-s}}.
\]
in particular the number of implicit Euler steps \( n_0 \) is sufficient. We prove (74) by induction over \( m \), thus let \( m = 1, \ldots, n_0 \) and (74) be valid for smaller \( m \) if \( m > 1 \). For the first estimate in (74) we use the a stability estimate, following from standard arguments:
\[
\|u_k^n\|_{V^{2m}} + k_n\|u_k^n\|_{V^{2m+1}} \leq 2 \int_{I_n} \|Pf(t)\|_{V^{2m}} \, dt + 2\|u_k^{n-1}\|_{V^{2m}} + k_n\|u_k^n\| \cdot \|\nabla u_k^n\|_{H^{2m-1}}^{2}
\]
for \( n = m, \ldots, n_0 \). Either by induction or, for \( m = 1 \), by the already proven estimate (69) with \( s = 2 \), we can bound \( \|u_k^{n-1}\|_{V^{2m}} \) and trivially the \( f \)-term, yielding
\[
\|u_k^n\|_{V^{2m}} + k_n\|u_k^n\|_{V^{2m+1}} \leq CA_{40}^{k^{2(1-m)}} + k_n\|u_k^n\| \cdot \|\nabla u_k^n\|_{H^{2m-1}}^{2} \tag{75}
\]
For the nonlinearity we combine (23) for all derivatives in the \( H^{2m-1} \)-norm and (21b) for the \( L^2 \)-term, to get
\[
\|u_k^n \cdot \nabla u_k^n\|_{H^{2m-1}} \leq C\|u_k^n\|_{V^1} \|u_k^n\|_{V^2} + C \sum_{i=1}^{2m-1} \|u_k^n\|_{V^{i+1}} \|u_k^n\|_{V^{2m-i+1}} \tag{76}
\]
Noting that \( 2 \leq i + 1 \leq 2m \) and \( 2 \leq 2m - i + 1 \leq 2m \) for \( i = 1, \ldots, 2m - 1 \), we can bound all terms on the right-hand side either by induction or using (69) for \( s = 1, 2 \):
\[
\|u_k^n \cdot \nabla u_k^n\|_{H^{2m-1}} \leq CA_{40}^{m} + CA_{40} \sum_{i=1}^{2m-1} k \cdot 2^{-(i+1)} \cdot k^{2(2m+1-i)} \leq CA_{40}^{k^{2-2m}} = CA_{40}^{k^{1-m}}
\]
Combining this inequality with the stability estimate (75) we arrive at
\[
k_n\|u_k^n\|_{V^{2m+1}}^2 \leq \|u_k^n\|_{V^{2m}}^2 + k_n\|u_k^n\|_{V^{2m+1}}^2 \leq CA_{40}^{k^{2(1-m)}}
\]
which yields the first result in (74). For the second estimate we employ the regularity results for the Stokes equations to get
\[
\|u_k^n\|_{V^{2m+2}} \leq Ck^{-1} \left( \int_{I_n} \|Pf(t)\|_{V^{2m}} \, dt + \|u_k^n\|_{V^{2m}} + \|u_k^{n-1}\|_{V^{2m}} \right) + C\|u_k^n\| \cdot \|\nabla u_k^n\|_{V^{2m}}.
\]
Estimating the terms in brackets using (74) by induction, or (69) with \( s = 2 \), yields:
\[
\|u_k^n\|_{V^{2m+2}} \leq CA_{40}^{k^{-m}} + C\|u_k^n\| \cdot \|\nabla u_k^n\|_{V^{2m}} \tag{77}
\]
For the nonlinear term we proceed just as in (76), yielding
\[\|u_k^n \cdot \nabla u_k^n\|_{H^{2m}} \leq C\|u_k^n\|_{V^1}\|u_k^n\|_{V^2} + C \sum_{i=1}^{2m} \|u_k^n\|_{V^{i+1}}\|u_k^n\|_{V^{2m-i+2}}.\]

The norms on the right are at most of order 2 in \(m\). Hence using (74) by induction, the already proven first estimate in (74) for \(m\), and estimates for \(s = 1, 2\) we arrive at
\[\|u_k^n \cdot \nabla u_k^n\|_{H^{2m}} \leq C_{A40} + C_{A40} \sum_{i=1}^{2m} k^{2-(i+1)/2} k^{2-(2m-i+2)/2}\]
\[\leq C_{A40} k^{1-2m} \leq C_{A40} k^{-m}.\]

Combining this result with (77) implies \(\|u_k^n\|_{V^{2m+2}} \leq C_{A40} k^{-m}\), i.e. the second estimate in (74), completing the induction. \(\square\)

**Theorem 43.** Let assumption 40 hold. If \(s_0 = 1\) there exists \(C_0 > 0\) such that the Crank-Nicolson time discretization with \(n_0 = 1\) implicit Euler steps (66) of the Navier-Stokes equations (16) satisfies on \(J := (t_1, T)\) the a priori error estimate
\[\|\tau_k^3(u - u_k)\|_{L_{T}^\infty V^1} + \|\tau_k^3 a_k(u - u_k)\|_{L_{T}^2 V^2} + \|\tau_k^3 (a_k p - p_k)\|_{L_{T}^2 Q^1} \leq C_{A40} k^2\] (78)
if the stepsize condition \(k < C_0 \|u\|_{L_{T}^\infty V^2}^{-2}\) is satisfied. If \(s = 2\) there exists another \(C_0 > 0\) such that after \(n_0 = 2\) implicit Euler steps there holds with \(J := (t_2, T)\):
\[\|\tau_k^2(u - u_k)\|_{L_{T}^\infty V^2} + \|\tau_k^2 a_k(u - u_k)\|_{L_{T}^2 V^3} + \|\tau_k^2 (m_k p - p_k)\|_{L_{T}^2 Q^1} \leq C_{A40} k^2\] (79)
if \(k < C_0 \|u\|_{L_{T}^\infty V^2}^{-2}\) is satisfied.

**Proof.** We note that \(n_0 = s_0\) and the smoothing level in (78) and (79) is \(L = 2 + s_0\). The procedure is just as in the linear case, i.e. theorem 38: For both \(s_0 = 1\) and \(s_0 = 2\) we first bound \(v_k := i_k u - u_k\) by induction over the smoothing cascade, use this to prove the velocity error estimates and then arrive at pressure estimates using (26). We first prove, under the stepsize restriction not mentioned furthermore, that
\[\|\tau_k^\frac{\ell}{k} v_k\|_{L_{T}^\infty V^{s_0}} + \|\tau_k^\frac{\ell}{k} a_k v_k\|_{L_{T}^2 V^{s_0+1}} + \|\tau_k^\frac{\ell}{k} \partial_t v_k\|_{L_{T}^2 V^{s_0-1}} \leq C_{A40} k^2.\] (80)

We proceed by induction over the levels and nodes of the (double) smoothing cascade: For \(\ell = 0, \ldots, L\) and \(s = \ell - 2, \ldots, s_0 + L - \ell\) we prove that
\[\|\tau_k^{\frac{\ell}{k}} v_k\|_{L_{T}^\infty V^s} + \|\tau_k^{\frac{\ell}{k}} a_k v_k\|_{L_{T}^2 V^{s+1}} + \|\tau_k^{\frac{\ell}{k}} \partial_t v_k\|_{L_{T}^2 V^{s-1}} \leq C_{A40} k^{2+\ell-s}.\] (81)

Then (80) corresponds to \(\ell = L\) and \(s = s_0\). We prepare some estimates first: Just as in the linear case, or as in the proof of lemma 41, we get
\[\|\tau_k^{\frac{\ell}{k}} (u - i_k u)\|_{L_{T}^2 V^{s+1}} \leq Ck^{\frac{2+\ell-s}{2}}\|\tau_k^{\frac{2+\ell}{2}} \partial_t u\|_{L_{T}^2 V^{s+1}} \leq C_{A40} k^{\frac{2+\ell-s}{2}}.\] (82)
and from lemma 42 that
\[ k^{2} \| \psi_k(t_{m0}) \|_{V'} \leq C_{A40} k^{2+\frac{t-s}{2}}. \] (83)

Furthermore, \( t_{m0} \geq Ck \) and lemma 41 implies, with \( \ell - s - s \leq 0 \), that
\[ \| \frac{\tau_k^{\ell}}{\| \tau_k \|_{L^2}} k_{a_k} N(u, a_k i_k u) \|_{L^2 V} \leq C \max_{\ell \leq \ell} (\tau_k(t)) \frac{\tau_{k+1}^{2-s}}{\tau_k} \| k_{a_k} N(u, a_k i_k u) \|_{L^2 V} \]
\[ \leq C_{A40} \tau_k^{2-s} \| \|_{L^2 V} \leq C_{A40} k^{\frac{2-s}{2}}. \] (84)

The main difficulty in the proof of (81) is the estimation of \( N(a_k u_k, a_k i_k u) \) in the error identity (25). To resolve the dependencies we prove (81) for \( \ell = 0 \) in the order \( s = 0, -1, 2, 1, \ldots, s_0 + L, \) for \( \ell = 1 \) in the order \( s = 0, -1, 1, \ldots, s_0 + L - 1 \) and for \( \ell \geq 2 \) in the order \( s = -2 + \ell, \ldots, s_0 + L - \ell \). If we consider some \( \ell = 0, \ldots, L \) we implicitly assume in the following that (81) was proven for smaller \( \ell \) if \( \ell > 0 \).

If \( \ell = 1, 2 \) then \( s = 0 \) is a node in the smoothing cascade at level \( \ell \). Proceeding for the error identity (25) just as in the smoothing stability estimate from lemma 35 we have for \( \ell > 0 \) and \( n = n_0 + 1, \ldots, N \) with \( J' := (t_{m0}, t_n) \) that
\[ \| \frac{\tau_k^{\ell}}{\| \tau_k \|_{L^2}} |J| v_k(t_n) \|_{L^2} + \| \frac{\tau_k^{\ell}}{\| \tau_k \|_{L^2}} k_{a_k} v_k \|_{L^2 V} \]
\[ \leq C (k|v_k(t_{m0})|_{L^2} + \| \frac{\tau_k^{\ell}}{\| \tau_k \|_{L^2}} (u - i_k u) \|_{L^2 V} + \| \frac{\tau_k^{\ell}}{\| \tau_k \|_{L^2}} k_{a_k} N(u, a_k i_k u) \|_{L^2 V} \]
\[ + \| \tau_k^{\ell} a_k v_k \|_{L^2} + k^2 \| \tau_k^{\ell} \|_{L^2} \| \partial_t v_k \|_{L^2 V} \) \].

For the first three terms on the right-hand side we use (82), (83) and (84) with \( s = 0 \). For the last two terms we use the validity of (81) for \( \ell = 1 \), just as in the linear case. This yields
\[ \| \frac{\tau_k^{\ell}}{\| \tau_k \|_{L^2}} |J| v_k(t_n) \|_{L^2} + \| \frac{\tau_k^{\ell}}{\| \tau_k \|_{L^2}} k_{a_k} v_k \|_{L^2 V} \]
\[ \leq C_{A40} k^{2+\ell} + C \| \tau_k^{\ell} a_k v_k \|_{L^2} \| \tau_k^{\ell} \|_{L^2 V} \) \].

If \( \ell = 0 \) we can proceed similarly, without the terms \( a_k v_k \) and \( \partial_t v_k \) on the right-hand side, and also arrive at (85). For the nonlinear term in (85) we use the cancellation property to get
\[ \| \tau_k^{\ell} a_k v_k \|_{L^2 V} \| \tau_k^{\ell} a_k v_k \|_{L^2 V} \]
\[ \leq C \| u \|_{C_{0}^{s} V} \| \tau_k^{\ell} a_k v_k \|_{L^2 V} \| \tau_k^{\ell} a_k v_k \|_{L^2 V} \] .

Using Young’s inequality to move the last term to the left-hand side we get
\[ \| \tau_k^{\ell} |J| v_k(t_{m0}) \|_{L^2} + \| \tau_k^{\ell} a_k v_k \|_{L^2 V} \]
\[ \leq C_{A40} k^{2+\ell} + C \| u \|_{C_{0}^{s} V} \| \tau_k^{\ell} a_k v_k \|_{L^2 V} \] .

With the elementary estimate
\[ \| \tau_k^{\ell} a_k v_k \|_{L^2} \]
\[ = \frac{1}{2} \sum_{j=n_0+1}^{n} k_j \| \tau_k^{\ell} \|_{L^2} \| v_k^{j} + v_k^{j-1} \|_{L^2} \]
\[ \leq C \sum_{j=n_0}^{n} k_j \left( \tau_k^{\ell} \| v_k^{j} \|_{L^2} \right)^{2} \] (87)
we can employ Gronwall’s inequality in (86) and conclude that
\[ \| \tau_k^v v_k \|_{L^2}^2 + \| \tau_k^a a_k v_k \|_{L^2}^2 \leq C A 40 k^{2 + \ell} \] (88)
holds under the stepsize condition. This is (81) for \( s = 0 \) except for the \( \partial_t v_k \)-term on the left-hand side, the proof of which must be postponed until \( s = 1 \) has been considered. Any level containing \( s = 0 \) also contains \( s = 1 \) and the \( \partial_t v_k \)-estimate is only used in level \( \ell + 1 \), so the arguments are not disturbed by this postponement.

For \( s = -1 \) and \( \ell = 0, 1 \) or \( s = -2 \) and \( \ell = 0 \) we use arguments similar to those leading to (85), but now also norm estimates for the remaining nonlinear term, to get
\[ \| \tau_k^v v_k(t_n) \|_{L^2}^2 + \| \tau_k^a a_k v_k \|_{L^2}^2 \leq C A 40 k^{2 + \ell - s} + C \| \tau_k^v N(a_k u_k, a_k i_k u) \|_{L^2}^2. \] (89)
Using (21) we can estimate
\[ \| \tau_k^v N(a_k u_k, a_k i_k u) \|_{L^2}^2 \leq C \left( \| a_k v_k \|_{L^2}^2 \tau_k^v a_k v_k \|_{L^2}^2 + \| u \|_{L^2}^2 \| \tau_k^v a_k v_k \|_{L^2}^2 \right). \] (90)
By (88), for the current \( \ell \) and \( \ell = 0 \), we have \( \| a_k v_k \|_{L^2}^2 \tau_k^v a_k v_k \|_{L^2}^2 \leq C A 40 k^{2 + \ell} \) from which we can conclude for \( s = -1 \) that
\[ \| \tau_k^v N(a_k u_k, a_k i_k u) \|_{L^2}^2 \leq C A 40 k^{2 + \ell - s} + C \| u \|_{L^2}^2 \| \tau_k^v a_k v_k \|_{L^2}^2. \] (91)
Using this estimate in (89) and proceeding just as in (87) we can apply Gronwall’s inequality to arrive at
\[ \| \tau_k^v v_k \|_{L^2}^2 + \| \tau_k^v a_k v_k \|_{L^2}^2 \leq C A 40 k^{2 + \ell - s} \] for \( s = -1 \). From (91) we get \( \| \tau_k^v N(a_k u_k, a_k i_k u) \|_{L^2}^2 \leq C A 40 k^{2 + \ell - s} \). This allows us to apply the linear stability estimates from lemma 14 for \( \ell = 0 \) and lemma 35 for \( \ell = 1 \) to arrive at the estimates for \( \partial_t v_k \) as well, concluding the proof of (81) for \( s = -1 \). For \( s = -2 \) we start at (89) and now use \( \| a_k v_k \|_{L^2}^2 \| a_k v_k \|_{L^2}^2 \leq C A 40 k^3 \) in (90) from the established results for \( s = 0 \) and \( s = -1 \). Proceeding just as for \( s = -1 \) we arrive at (81) for \( s = -2 \).

For \( s \geq 1 \) we directly apply the linear stability estimates, from lemma 14 if \( \ell = 0 \) and lemma 35 if \( \ell > 0 \), to the error identity (25). Estimating the initial error, the linear residual \( P \Delta (u - i_k u) \) and using the induction hypothesis for \( \ell > 0 \) just as for \( s \leq 0 \), we arrive at
\[ \| \tau_k^v v_k \|_{L^2}^2 + \| \tau_k^v a_k v_k \|_{L^2}^2 + \| \tau_k^v \partial_t v_k \|_{L^2}^2 \leq C A 40 k^{2 + \ell - s} + C \| \tau_k^v N(a_k u_k, a_k i_k u) \|_{L^2}^2. \] (92)
For $s = 1$ and hence $0 \leq \ell \leq 3$ we have, using \((22a)\) and \((22b)\), that
\[
\|\tau_k^{\ell} N(a_k u_k, a_k i_k u)\|_{L^2_F \ell^2}^2
\leq C \int \tau_k^{\ell} \|a_k v_k\|_{V_1}^2 \|a_k v_k\|_{V_2} + \tau_k^{\ell} \|a_k v_k\|_{V_1}^2 \|a_k i_k u\|_{V_2}^2 \, dt
\leq C(\delta)(\|a_k v_k\|_{L^2_F V_1}^4 + \|u\|_{L^2_F V_1}^2) \|\tau_k^{\ell} a_k v_k\|_{L^2 F V_1}^2 + \delta \|\tau_k^{\ell} a_k v_k\|_{L^2 F V_2}^2
\]
with $\delta > 0$ to be chosen later. Estimate \((88)\) on the level $0 \leq (\ell - 1)^+ \leq 2$ yields
\[
\|\tau_k^{\ell} a_k v_k\|_{L^2_F V_1}^2 \leq \|\tau_k^{\ell - 1} a_k v_k\|_{L^2_F V_1} \leq C A_{40} k^{2 + (\ell - 1)^+} \leq C A_{40} k^{1 + \ell}.
\]
The inverse inequality implies that $\|a_k v_k\|_{L^2_F V_1} \leq C k^{-\frac{\ell}{2}} \|a_k v_k\|_{L^2_F V_1} \leq C A_{40}$ and thus
\[
\|\tau_k^{\ell} N(a_k u_k, a_k i_k u)\|_{L^2_F \ell^2}^2 \leq C A_{40}(\delta) k^{1 + \ell} + \delta \|\tau_k^{\ell} a_k v_k\|_{L^2 F V_2}^2.
\]
Combining this with \((92)\) and moving the last term to the left-hand side for $\delta$ small enough, we arrive at \((81)\) for $s = 1$. For $s \geq 2$ we use \((23)\) to get
\[
\|\tau_k^{\ell} N(a_k u_k, a_k i_k u)\|_{L^2_F H^{s-1}}^2
\leq C \|\tau_k^{\ell} N(a_k u_k, a_k i_k u)\|_{L^2_F \ell^2}^2 + C \sum_{i=1}^{s-1} \|\tau_k^{\ell} a_k v_k\|_{L^2_F V_i+1}^2 \left(\|a_k v_k\|_{L^2_F V_{s-i}}^2 + \|a_k i_k u\|_{L^2_F V_{s-i}}^2\right).
\]
By similar arguments as for the case $s = 1$ we have
\[
\|\tau_k^{\ell} N(a_k u_k, a_k i_k u)\|_{L^2_F \ell^2}^2 \leq C A_{40} \|\tau_k^{\ell} a_k v_k\|_{L^2_F V_1}^2 + C k^{2 + (\ell - 1)^+} \|\tau_k^{\ell} a_k v_k\|_{L^2 F V_2}^2.
\]
Using \((88)\) for the first term we get, since $0 \leq (\ell - s)^+ \leq 2$, that
\[
\|\tau_k^{\ell} a_k v_k\|_{L^2_F V_1}^2 \leq \|\tau_k^{\ell - 1} a_k v_k\|_{L^2_F V_1} \leq C A_{40} k^{2 + (\ell - s)^+} \leq C A_{40} k^{2 + \ell - s}
\]
and using \((81)\) for $s = 1$, since $0 \leq (\ell - s + 1)^+ \leq 3$, that
\[
\|\tau_k^{\ell} a_k v_k\|_{L^2 F V_2}^2 \leq \|\tau_k^{\ell - 1} a_k v_k\|_{L^2 F V_2} \leq C A_{40} k^{2 + (\ell - s + 1)^+ - 1} \leq C A_{40} k^{2 + \ell - s}.
\]
From \((94)\) we hence conclude the $L^2$-estimate for the nonlinear term:
\[
\|\tau_k^{\ell} N(a_k u_k, a_k i_k u)\|_{L^2_F \ell^2}^2 \leq C A_{40} k^{2 + \ell - s}.
\]

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For the remaining terms in (93) we want to prove that, for $i = 1, \ldots, s - 1$,

$$\|\frac{\partial}{\partial t_k} a_k v_k\|_{L^2 V^{\ell + 1}}^2 \| a_k w_k\|_{L^\infty V^{s-i+1}}^2 \leq C A_4^0 k^{2+\ell-s}$$  \hspace{1cm} (96)

with either $w_k = v_k$ or $w_k = i_k u$. Combined with (95) this would imply

$$\|\frac{\partial}{\partial t_k} N(a_k u_k, a_k i_k u)\|_{L^2 H^{s-1}}^2 \leq C A_4^0 k^{2+\ell-s}$$  \hspace{1cm} (97)

which together with (92) would conclude our proof of (81) for $s \geq 2$. For the first term in (96) we use that

$$\|\frac{\partial}{\partial t_k} a_k v_k\|_{L^2 V^{\ell+1}}^2 \leq C A_4^0 k^{2+(\ell-1)^{-i}}$$  \hspace{1cm} (98)

for $i = 1, \ldots, s - 1$ by (81). These $i$ belong to the cascade at level $(\ell-1)^+$ since $s_0 \leq 2$ implies $(\ell-1)^+ - 2 \leq L - 3 \leq s_0 - 1 \leq i$ and from $s \leq s_0 + L - \ell$ it follows that $i \leq s - 1 \leq s_0 + L - (\ell-1) \leq s_0 + L - (\ell-1)^+$. For the second factor in (96) we combine in case $w_k = v_k$ the inverse inequality with (81) for level 0:

$$\|a_k v_k\|_{L^\infty V^{s-i+1}} \leq C k^{-1} \|a_k v_k\|_{L^2 V^{s-i+1}} \leq C A_4^0 k^{1-2(s-i)} \leq C A_4^0 k^{2-(s-i+1)}.$$  

For $w_k = i_k u$ we use the smoothing estimate of $u$ to get

$$\|a_k i_k u\|_{L^\infty V^{s-i+1}}^2 \leq C k^{-(s-i-1)^+} \|\frac{\partial}{\partial t_k} u\|_{C_2 V^{s-i+1}}^2 \leq C A_4^0 k^{-(s-i-1)^+}.$$  

Combining these two estimates for $w_k$ with (98) we get for (96) that

$$\|\frac{\partial}{\partial t_k} a_k v_k\|_{L^2 V^{\ell+1}}^2 \| a_k w_k\|_{L^\infty V^{s-i+1}}^2 \leq C A_4^0 k^{2+(\ell-1)^+ - (s-i)^+}$$

and a simple, but tedious, examination of all cases yields that (96) indeed holds, implying (97) and hence the claimed (81) for $s \geq 2$.

To finish the proof of (81) we must show $\|\frac{\partial}{\partial t_k} v_k\|_{L^2 V^{-1}}^2 \leq C A_4^0 k^{2+\ell}$ from the case $s = 0$. A procedure similar to e.g. $s = -1$, i.e. using the linear stability, yields

$$\|\frac{\partial}{\partial t_k} v_k\|_{L^2 V^{-1}}^2 \leq C A_4^0 k^{2+\ell} + \|\frac{\partial}{\partial t_k} N(a_k u_k, a_k i_k u)\|_{L^2 H^{-1}}^2.$$  \hspace{1cm} (99)

From (90) with $s = 0$ we get

$$\|\frac{\partial}{\partial t_k} N(a_k u_k, a_k i_k u)\|_{L^2 H^{-1}}^2 \leq C \|a_k v_k\|_{L^\infty H^{-1}}^2 \|\frac{\partial}{\partial t_k} a_k v_k\|_{L^2 V^{-1}}^2 + C A_4^0 \|\frac{\partial}{\partial t_k} a_k v_k\|_{L^2 L^2}^2.$$  

Using the estimate (88) together with (81) for $s = -1$ and $s = 1$ we get

$$\|\frac{\partial}{\partial t_k} N(a_k u_k, a_k i_k u)\|_{L^2 H^{-1}}^2 \leq C A_4^0 (k^2 k^{2+\ell-1} + k^{2+\ell+1}) \leq C A_4^0 k^{2+\ell}$$

which, inserted into (99), concludes the case $s = 0$ and hence the proof of (81). Splitting $u - u_k = (u - i_k u) + (i_k u - u_k)$ and using (81) and

$$\|\frac{\partial}{\partial t_k} (u - i_k u)\|_{L^\infty V^{\tau_0}} \leq C k^2 \|\frac{\partial}{\partial t_k} u\|_{L^\infty V^{\tau_0}} \leq C k^2 \|\frac{\partial}{\partial t_k} u\|_{L^\infty V^{\tau_0}} \leq C A_4^0 k^2.$$
and similarly $\| \tau_k^\frac{2}{3} a_k(u-u_k) \|_{L^2_t V^{\alpha_0+1}} \leq C A 40 k^2$, we arrive at the velocity error estimates in (78) and (79):

$$
\| \tau_k^\frac{2}{3} (u - u_k) \|_{L^\infty_t V^{\alpha_0}} \leq C A 40 k^2.
$$

(100)

For the pressure error we use the error identity (26) and Poincaré’s inequality. For $s_0 = 1$ we get

$$
\| \tau_k^\frac{2}{3} (a_k p - p_k) \|_{L^2_t Q^1} \leq C \left( \| \tau_k^\frac{2}{3} a_k (u - u_k) \|_{L^2_t V^2} + \| \tau_k^\frac{2}{3} a_k N(u, a_k u_k) \|_{L^2_t L^2} + \| \tau_k^\frac{2}{3} N(a_k u_k, a_k u_k) \|_{L^2_t L^2} \right).
$$

For the terms on the right we use the velocity estimate (100), lemma 41 and (95) for $s = s_0$ and $\ell = L$. This yields as claimed

$$
\| \tau_k^\frac{2}{3} (a_k p - p_k) \|_{L^2_t Q^1} \leq C A 40 k^2
$$

which finishes the proof of (78). For $s_0 = 2$ we use lemma 8 to estimate

$$
\| \tau_k^2 (m_k p - p_k) \|_{L^\infty_t Q^1} \leq \| \tau_k^2 (m_k p - a_k p) \|_{L^\infty_t Q^1} + \| \tau_k^2 (a_k p - p_k) \|_{L^\infty_t Q^1} \leq C k^2 \| \tau^2 \partial_t p \|_{L^\infty_t Q^1} + \| \tau_k^2 (a_k p - p_k) \|_{L^\infty_t Q^1}
$$

and by (65) there holds $\| \tau^2 \partial_t p \|_{L^\infty_t Q^1} \leq C A 40$. It hence remains to estimate $a_k p - p_k$. The pressure error identity implies

$$
\| \tau_k^2 (a_k p - p_k) \|_{L^\infty_t Q^1} \leq C \left( \| \tau_k^2 a_k (u - u_k) \|_{L^\infty_t V^2} + \| \tau_k^2 a_k N(u, a_k u_k) \|_{L^\infty_t L^2} + \| \tau_k^2 N(a_k u_k, a_k u_k) \|_{L^\infty_t L^2} \right).
$$

(101)

For the first term on the right we again use the velocity estimate (100). For the nonlinear terms no temporal $L^\infty$-estimates have been derived so far. Modifying the proof leading to (67) for $s = 1$ in lemma 41 we get

$$
\| \tau_k^2 N(u, a_k u_k) \|_{L^\infty_t L^2} \leq C k^2 (\| \partial_t u \|_{L^\infty_t L^2} \| \tau_k^2 \partial_t u \|_{L^\infty_t V^3} + \| u \|_{C^0_t V^2} \| \tau_k^2 \partial_t u \|_{L^\infty_t V^2}).
$$

Using (64) this implies that

$$
\| \tau_k^2 N(u, a_k u_k) \|_{L^\infty_t L^2} \leq C A 40 k^2.
$$

(102)

For the second nonlinear term we can proceed as above for $s = 1$, after (92), to get

$$
\| \tau_k^2 N(a_k u_k, a_k u_k) \|_{L^\infty_t L^2} \leq C \left( \| v_k \|_{L^\infty_t V^1} \| \tau_k^2 v_k \|_{L^\infty_t V^1} \| \tau_k^2 v_k \|_{L^\infty_t V^2} + \| \tau_k^2 v_k \|_{L^\infty_t V^1} \| u \|_{C^0_t V^2} \right).
$$

(103)
By (81) for \( \ell = 0 \) and \( s = 1 \) we get \( \|v_k\|_{L^\infty V^1} \leq CA_{40}k^\frac{1}{2} \), for \( \ell = 3 \) and \( s = 1 \) we get \( \|\tau_k^2v_k\|_{L^\infty V^1} \leq CA_{40}k^2 \) and for \( \ell = 4 \) and \( s = 2 \) we get \( \|\tau_k^2v_k\|_{L^\infty V^2} \leq CA_{40}k^2 \). Using these estimates in (103) implies

\[
\|\tau_k^2N(a_ku_k, a_ki_ku_k)\|_{L^\infty L^2} \leq CA_{40}k^2.
\]

Using this estimate, the velocity estimate (100) and (102) in (101) we conclude that

\[
\|\tau_k^2(a_kp - p_k)\|_{L^\infty Q^1} \leq CA_{40}k^2
\]

which finishes the proof of (79).

**Remark 44.** The stepsize condition in theorem 43 is stronger than in section 2 without smoothing. This is due to the application of Gronwall’s inequality in (80) for \( s = -2 \) where sharper estimates for the nonlinearity, like (22), could not be proven.

### 4 Numerical Study

We present a numerical study illustrating the optimality of the error estimates and the necessity to consider both a weighted norm and initial Euler steps, if the initial data does not satisfy the compatibility conditions. On the unit disk \( \Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \) and the temporal interval \( I = [0, 2] \) we study the Navier-Stokes equations with homogeneous Dirichlet data

\[
\partial_t u - 0.01\Delta u + u \cdot \nabla u + \nabla p = f, \quad \text{div } u = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega.
\]

Two different configurations are considered. First, \((i)\), we prescribe homogeneous initial data \( u^0 = 0 \) and the smooth right hand side

\[
f(x, y, t) = 0.2t^2 \exp(-t)( -\sin(4x + y)y, \cos(x - 4y)x).
\]

It holds \( f(\cdot, 0) = 0 \) and \( \partial_t f(\cdot, 0) = 0 \) such that the data satisfies the compatibility conditions. Second, \((ii)\), we consider the homogeneous right hand side \( f = 0 \) and determine the initial condition \( u^0 \) as solution to the stationary Navier-Stokes problem

\[-0.01\Delta u^0 + u^0 \cdot \nabla u^0 + \nabla p^0 = 0.2 \text{sgn}(x) \text{sgn}(y)( -\sin(4x + y)y, \cos(x - 4y)x) \text{ in } \Omega \]

with \( u^0 = 0 \) on the boundary. Since the domain is regular and the initial right hand side is just in \( L^2(\Omega) \), it holds \( u^0 \in V^2 \). The compatibility condition is not satisfied with the right hand side \( f = 0 \).

Spatial discretization is accomplished with quadratic finite elements for velocity and pressure on a mesh with mesh size \( h \approx 0.0025 \). To cope with the missing inf-sup stability we employ the local projection stabilization, see [2]. For temporal discretization we use the Crank-Nicolson scheme as discussed in this paper. To avoid superconvergence effects by symmetry, we consider a base step size \( k \in \{0.02, 0.01, 0.005, 0.0025\} \) and add an alternating variation \( 0.8k, 1.2k, 0.8k, 1.2k, \ldots \) in both test cases. Without this
Figure 3: Results for configuration (i) satisfying the compatibility condition. We observe optimal second order convergence in both norms. No initial Euler steps are required.

modification, no reduction in convergence could be observed in case (ii), even if no initial Euler steps were performed. The nonlinear problems are approximated with a Newton scheme, the resulting linear systems are solved with a geometric multigrid solver. For details on the implementation in GASCOIGNE 3D [3] see [14, chapters 4, 7]. A reference solution \( p_{k_0, h} \) is computed on a uniform time mesh with \( M_0 = \frac{2}{k_0} \) steps of size \( k_0 = 0.0005 \) on the same spatial mesh. Pressure errors are evaluated in the \( L^2 \)- and the \( L^\infty \)-norm, approximated by the midpoint rule on the reference subdivision with stepsize \( k_0 \) in time and by the Euclidean \( l^2 \)-norm on the fixed discretization in space. With \( t_m = (m - \frac{1}{2})k_0 \)

\[
\left\| p_{k, h} - p_{k_0, h} \right\|_{L^2} := \left( \sum_{m=1}^{M_0} k_0 \sum_{i=1}^{N} \left| p_{k, h}(t_m, x_i) - p_{k_0, h}(t_m, x_i) \right|^2 \right)^{\frac{1}{2}},
\]

\[
\left\| p_{k, h} - p_{k_0, h} \right\|_{L^\infty} := \max_{m=1,\ldots,M_0} \left( \sum_{i=1}^{N} \left| p_{k, h}(t_m, x_i) - p_{k_0, h}(t_m, x_i) \right|^2 \right)^{\frac{1}{2}},
\]

where we denote by \( x_i \) for \( i = 1, \ldots, N \) the nodes of the spatial mesh.

The resulting convergence behavior is shown in figure 3 for case (i) and figure 4 for configuration (ii). In configuration (i) we observe optimal second order convergence without any weighting of the norms and without adding Euler steps. Configuration (ii) shows the expected loss of optimality, as the problem regularity is not sufficient. Optimal order convergence is recovered if we add weights to the norms and if we start the procedure with implicit Euler steps according to theorem 43. Adding a proper amount of Euler steps increases the convergence from first to second order. Without
Figure 4: Results for configuration (ii). Without initial Euler steps, \( n_0 = 0 \), only first order convergence is observed for the pressure in both norms. By adding \( n_0 = 1 \) Euler step, we recover second order convergence in the \( L^2 \) norm and by starting with \( n_0 = 2 \) Euler steps, we obtain second order in the \( L^\infty \) norm.

weighting the norms, convergence rates drop to approximately \( \sqrt{k} \).

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