Dispersive derivation of the trace anomaly

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Abstract

We present a simple derivation of the one-loop trace anomaly in spinor QED through dispersion relations, avoiding completely any ultraviolet regularization. The anomaly can be expressed as a convergent sum rule for the imaginary part of a relevant formfactor. In the massless limit, the imaginary part produces a delta-function singularity at zero external momentum squared. Such a treatment reveals an "infrared face" of the trace anomaly, in striking similarity with the well-known case of the axial anomaly.
1 Introduction

Quantum anomalies constitute one of the fundamental chapters in modern field theory and play a remarkably ubiquitous role in particle physics [1]. A thorough understanding of their various aspects therefore represents an important goal of the theory. In the archetypal example - the axial anomaly [2] - one observes a peculiar feature: The relevant quantity can be calculated in several clearly distinct ways, which in turn reveal diverse faces of the anomaly. To put it in more explicit terms, let us recall that the most familiar calculational methods refer to the ultraviolet properties of the basic triangle graph (exemplified by the Pauli-Villars or dimensional regularization). On the other hand, the axial anomaly can be computed in a radically different way, namely by using (convergent) dispersion relations and some remarkable infrared properties of the relevant imaginary part [3, 4]. Thus, one may avoid completely a regularization of the ultraviolet divergences associated with the triangle graph. This "infrared face" of the axial anomaly is relatively rarely mentioned in the literature, in comparison with the ultraviolet approaches discussed amply in any modern field theory textbook (see, however, [1, 5]). Nevertheless, it represents a conceptually independent, complementary view of the anomaly phenomenon.

In this context, another potentially interesting example would be the well-known trace anomaly [6, 7, 8], related to the broken dilatation (scale) invariance. The trace anomaly (which manifests itself as an additional quantum contribution to the trace of the energy-momentum tensor) has been first calculated by using an explicit ultraviolet regularization of the relevant correlation functions, and interpreted in terms of the corresponding high-momentum asymptotic behaviour [6, 7]. Subsequently it has been related to the Gell-Mann-Low beta function associated with coupling constant renormalization [8], and this "ultraviolet face" of the trace anomaly has become a common wisdom in modern particle theory. It would be interesting to know whether the trace anomaly could also be recovered through dispersion relations (and related to infrared properties of the imaginary part of an appropriate correlation function), in analogy with what has been done before in the case of the axial anomaly [3, 4]. To our knowledge, there has been no detailed discussion of this point in the available literature, although the trace anomaly already has a rather long history. The purpose of the present paper is to examine this problem explicitly, employing the familiar frame-
work of canonical Ward identities (zero-energy theorems) for broken scale invariance. To fix a reference point for the subsequent discussion and for a later comparison, in the next section we summarize briefly some basic results concerning the dispersive approach to the axial anomaly (cf. [1, 3, 4, 5]). The dispersive derivation of the trace anomaly within spinor QED is given in Section 3, which in fact represents the central part of the paper. Some other aspects of this approach are discussed in Section 4 and the main results are summarized briefly in Section 5.

2 Paradigm of the axial anomaly

Consider the familiar \( VVA \) triangle fermion loop and denote the corresponding (properly normalized) amplitude as \( T_{\alpha\mu\nu}(k, p) \), with \( k, p \) being the external momenta outgoing from vector vertices (labelled by \( \mu, \nu \)); for simplicity, we take \( k^2 = p^2 = 0 \). One may introduce formfactors via an appropriate tensor decomposition as

\[
T_{\alpha\mu\nu}(k, p) = F_1(q^2)q_\alpha\varepsilon_{\mu\nu\rho\sigma}k^\rho p^\sigma + F_2(q^2)(p_\nu\varepsilon_{\alpha\mu\rho\sigma} - k_\mu\varepsilon_{\alpha\nu\rho\sigma})k^\rho p^\sigma
\]  

Note that in the last expression we have already imposed the vector Ward identities (i.e. vector current conservation). The associated \( VVP \) amplitude can be written as

\[
T_{\mu\nu}(k, p) = G(q^2)\varepsilon_{\mu\nu\rho\sigma}k^\rho p^\sigma
\]  

and the (anomalous) axial Ward identity then reads

\[
q^2 F_1 = 2mG + \frac{1}{2\pi^2}
\]  

where \( m \) denotes the fermion mass; the constant \( 1/2\pi^2 \) is the celebrated axial anomaly [2] (we have suppressed here any possible coupling constants). If one wants to reproduce the anomaly through the corresponding imaginary parts (following [3,4] ), one may define the formfactors in (1) and (2) by means of dispersion relations (which turn out to converge without subtractions). The imaginary parts are well-defined finite quantities and must therefore satisfy the canonical (i.e. non-anomalous) Ward identities, i.e.

\[
t \text{Im} F_1(t; m^2) = 2m \text{Im} G(t; m^2)
\]  

2
where \( t \) stands for the kinematical variable \( q^2 \) (we have marked explicitly also the parametric dependence on the fermion mass). Using unsubtracted dispersion relations for the \( F_1 \) and \( G \) and employing the identity (4), one gets readily

\[
q^2 F_1 = 2mG - \frac{1}{\pi} \int_{4m^2}^{\infty} \text{Im} F_1(t; m^2) dt \quad (5)
\]

An explicit calculation gives

\[
\text{Im} F_1(t; m^2) = -\frac{1}{\pi} \frac{m^2}{t^2} \ln \frac{1 + \sqrt{1 - \frac{4m^2}{t}}}{1 - \sqrt{1 - \frac{4m^2}{t}}} \quad (6)
\]

for \( t > 4m^2 \), and taking the integral one obtains

\[
\int_{4m^2}^{\infty} \text{Im} F_1(t; m^2) dt = -\frac{1}{2\pi} \quad (7)
\]

Using (5) and (7), the value of the axial anomaly in (3) is recovered. In such a derivation one thus obviously avoids any ultraviolet regularization and the anomaly emerges as a sum rule for the imaginary part of a relevant formfactor (which is simply related to the classical symmetry-breaking term - cf. (4) ).

In the massless limit, the \( \text{Im} F_1(t; m^2) \) is seen to vanish pointwise while the integral remains constant, independent of \( m \). It means that, in fact

\[
\lim_{m \to 0} \text{Im} F_1(q^2; m^2) = -\frac{1}{2\pi} \delta(q^2) \quad (8)
\]

In the approach outlined above, the fermion mass \( m \) serves as an infrared cut-off and the anomaly, being the net effect persisting in the massless limit, is a consequence of a delta-function singularity of the \( \text{Im} F_1 \) at \( q^2 = 0 \) (in the full formfactor \( F_1 \) it is manifested as an "anomaly pole" at \( q^2 = 0 \)). These facts, in a nutshell, describe the infrared (or low-energy) face of the axial anomaly.

3 Trace anomaly in spinor QED through dispersion relations

Let us now examine the trace anomaly, which shows up as an extra term in canonical Ward identities describing formally the (softly broken) scale
invariance of a quantum field theory model. The formal theory of broken scale invariance has been a classic theme in field theory since the early 1970’s, so we may perhaps start the subsequent discussion by writing down immediately a typical Ward identity, referring for a general background to the standard literature (see e.g. [6, 9, 10, 11]). For the sake of technical simplicity, we will stay within the framework of spinor QED, at the level of one-loop Feynman graphs.

Let us consider a simple Ward identity involving the correlation function of the trace of energy-momentum tensor and two electromagnetic currents (cf. [7]), which reads, in its naive canonical form

\[
\left(2 - p \cdot \frac{\partial}{\partial p}\right) \Pi_{\mu\nu}(p) = \Delta_{\mu\nu}(p)
\]  

(9)

Here the \(\Pi_{\mu\nu}\) stands for the vacuum polarization tensor

\[
\Pi_{\mu\nu}(p) = i \int d^4x e^{ipx} \langle 0 | T(J_\mu(x)J_\nu(0)) | 0 \rangle
\]

(10)

and the \(\Delta_{\mu\nu}\) represents the three point vertex function

\[
\Delta_{\mu\nu}(p) = \int d^4x d^4y e^{ipx} \langle 0 | T(\theta_{\alpha}(x)J_\mu(y)J_\nu(0)) | 0 \rangle
\]

(11)

where \(\theta_{\alpha}\) means the trace of the (“improved”) energy-momentum tensor, equal to the divergence of the dilatation current [9]. In spinor QED one has, in the lowest order

\[
\theta_{\alpha}(x) = m \bar{\psi}(x)\psi(x)
\]

(12)

The relation (9) is a “zero-energy theorem”, since the divergence of dilatation current (and hence the mass insertion in (11)) is taken at zero external momentum. It can be derived formally by means of the naive canonical manipulations with the correlation function involving the dilatation current and reflects thus the scale invariance of QED, softly broken by the fermion mass term (12). Let us recall that the expression on the left-hand side of (9) comes from the commutators of currents with the dilatation charge, while the right-hand side corresponds to the classical symmetry-breaking term. It is also useful to remember that the mass insertion at zero momentum can formally be produced by means of a differentiation with respect to mass, so that

\[
m \frac{\partial}{\partial m} \Pi_{\mu\nu}(p; m) = \Delta_{\mu\nu}(p; m)
\]

(13)
The current conservation implies transversality of the $\Pi_{\mu\nu}$ and $\Delta_{\mu\nu}$, so one may define the corresponding formfactors by

$$
\Pi_{\mu\nu}(p) = \Pi(p^2)(p_\mu p_\nu - p^2 g_{\mu\nu})
$$

$$
\Delta_{\mu\nu}(p) = \Delta(p^2)(p_\mu p_\nu - p^2 g_{\mu\nu})
$$

(14)

(notice that the $\Pi$ and $\Delta$ are dimensionless). The naive Ward identity (9) may then be recast as

$$
-2p^2 \frac{\partial}{\partial p^2} \Pi(p^2; m^2) = \Delta(p^2; m^2)
$$

(15)

Of course, in arriving at (15) one ignores any problems connected with ultraviolet divergences of the underlying Feynman diagrams. Although (15) represents a relation between two finite quantities, these should be defined properly, e.g. by means of an appropriate gauge invariant ultraviolet regularization. When this is done, the naive identity picks up an additional term - the trace anomaly [6, 7]. At one-loop level one gets

$$
-2p^2 \frac{\partial}{\partial p^2} \Pi(p^2; m^2) = \Delta(p^2; m^2) + \frac{1}{6\pi^2} e^2
$$

(16)

with $e$ being the electromagnetic coupling constant. The identity (16) exemplifies the widely known connection of the trace anomaly with the charge renormalization beta function [8] (let us recall that such a connection becomes rather transparent when considering the limit $p^2 \to \infty$ in (16) and taking into account the dominance of logarithmic asymptotics of the $\Pi(p^2)$ over the $\Delta(p^2)$).

We will now show that the anomalous term in (16) can be recovered through dispersion relations, and expressed in terms of the imaginary part of the formfactor $\Delta(p^2)$ (obviating thus the problems of ultraviolet nature), in close analogy with the case of axial anomaly described in the preceding section. To this end, it is natural to define the derivative of the $\Pi(p^2)$ by means of a differentiated dispersion relation

$$
\frac{\partial}{\partial p^2} \Pi(p^2) = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{\partial}{\partial p^2} \frac{\text{Im} \Pi(t)}{t - p^2} dt
$$

(17)

It is easy to see that (17) also follows readily from the usual (once subtracted) dispersion relation for the $\Pi(p^2)$

$$
\frac{1}{p^2} \Pi(p^2) = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{\text{Im} \Pi(t)}{t - p^2} \frac{dt}{t}
$$

(18)
Similarly, the $\Delta(p^2)$ can be defined by an unsubtracted dispersion relation

$$\Delta(p^2) = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{\text{Im} \Delta(t)}{t - p^2} \, dt$$  \hspace{1cm} (19)

In writing (17) and (19) we have simply assumed the necessary convergence properties of the dispersion integrals, i.e. a proper asymptotic behaviour of the imaginary parts; an explicit illustration at the one-loop level is given below. The imaginary parts are well-defined finite quantities and should therefore satisfy the canonical (non-anomalous) Ward identity

$$-2t \frac{\partial}{\partial t} \text{Im} \Pi(t; m^2) = \text{Im} \Delta(t; m^2)$$  \hspace{1cm} (20)

To have some explicit one-loop expressions at hand, one may remember the familiar formula

$$\text{Im} \Pi(t; m^2) = \frac{e^2}{12\pi} \left( 1 + \frac{2m^2}{t} \right) \sqrt{\frac{1}{1 - 4m^2}} t$$  \hspace{1cm} (21)

for $t > 4m^2$ (see e.g. [11]). Using (20), one immediately gets also

$$\text{Im} \Delta(t; m^2) = -\frac{2e^2}{\pi} \frac{m^4}{t^2} \sqrt{1 - \frac{4m^2}{t}}$$  \hspace{1cm} (22)

(the last result can of course be checked independently by a Feynman graph calculation). Let us now calculate the quantity on the right-hand side of (16). Using the definition (17) and integrating by parts, one obtains first

$$-2p^2 \frac{\partial}{\partial p^2} \Pi(p^2) = -2p^2 \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{1}{(t - p^2)^2} \text{Im} \Pi(t) \, dt$$

$$= -2p^2 \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{1}{t - p^2} \frac{\partial}{\partial t} \text{Im} \Pi(t) \, dt$$  \hspace{1cm} (23)

In arriving at the last expression we have dropped the corresponding surface term; this is justified in view of the boundary values of the $\text{Im} \Pi(t)$ (cf. (21)). In (23) one may now employ the identity (20) to get, after a simple manipulation

$$-2p^2 \frac{\partial}{\partial p^2} \Pi(p^2) = p^2 \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{\text{Im} \Delta(t)}{t(t - p^2)} \, dt$$

$$= \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{\text{Im} \Delta(t)}{t - p^2} \, dt - \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{\text{Im} \Delta(t)}{t} \, dt$$  \hspace{1cm} (24)
Taking into account the definition (19) one may thus write finally

\[- 2p^2 \frac{\partial}{\partial p^2} \Pi(p^2) = \Delta(p^2) - \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{\text{Im} \Delta(t)}{t} \, dt \]  

(25)

Having reproduced the form of the anomalous Ward identity (16), one should check that the correct value of the trace anomaly is indeed recovered in (25). Using the expression (22) and performing the integral one obtains

\[ \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{\text{Im} \Delta(t)}{t} \, dt = - \frac{1}{6\pi} e^2 \]  

(26)

so that the anticipated result (16) is confirmed. From (22) and (26) it is also clear that

\[ \lim_{m \to 0} \frac{1}{t} \text{Im} \Delta(t; m^2) = - \frac{e^2}{6\pi} \delta(t) \]  

(27)

Thus, within the dispersion relation approach the trace anomaly is tantamount to the sum rule (26) for the imaginary part of the classical symmetry-breaking term \( \Delta(t; m^2) \), and in the massless limit it is manifested through the delta-function singularity of the relevant imaginary part at \( t = 0 \). A remarkable feature of such an approach is that one obtains a quantity intimately related to the charge-renormalization beta function (which is inherently of ultraviolet nature) without ever mentioning an ultraviolet regularization. There is obviously a striking similarity between the results found here and those described in the preceding section for the well-known axial anomaly. In particular, it is worth emphasizing that in both cases the anomaly can actually be expressed in the same way, namely through the imaginary part of the classical symmetry-breaking term: Indeed, in the case of axial anomaly (cf. (5)) one has \( \text{Im} F_1(t) = 2m \text{Im} G(t)/t \) (see (4)), and the term \( 2mG(t) \) is clearly a natural counterpart of the \( \Delta(t) \).

4 Further remarks on the imaginary parts

Let us now add some further observations concerning the imaginary parts of the considered correlation functions in the context of the trace anomaly problem.

First, it is easy to arrive at an alternative representation of the trace anomaly in terms of \( \text{Im} \Pi(p^2) \). Denoting the anomalous term embodied in
(25) by \( A \), i.e.

\[
A = -\frac{1}{\pi} \int_{4m^2}^{\infty} \text{Im} \Delta(t; m^2) \frac{dt}{t}
\]

(28)

then employing the identity (20) one gets immediately

\[
A = \frac{2}{\pi} \int_{4m^2}^{\infty} \frac{\partial}{\partial t} \text{Im} \Pi(t) dt
\]

(29)

and therefore

\[
A = \lim_{p^2 \to \infty} \frac{2}{\pi} \text{Im} \Pi(p^2; m^2)
\]

(30)

Note that a relation of this type was in fact considered earlier (cf. [7], where it was deduced in a somewhat indirect way). Within our dispersive approach, the result (30) emerges as a straightforward consequence of the basic integral representation (28).

Another remarkable technical feature of the anomaly (28) is that it may be recast as

\[
A = -\Delta(0; m^2)
\]

(31)

The relation (31) has also been discussed in the early days of the anomaly theory (see [7]) by means of different (short-distance) methods. Within our approach it becomes obvious immediately, by comparing (28) with the definition (19). In fact, (31) is easily understood if one takes into account that the \( \Pi(p^2; m^2) \) has no singularity at \( p^2 = 0 \) for \( m \neq 0 \) (note, however, that in the massless limit the \( \Pi(p^2; 0) \) does have a logarithmic infrared singularity, which reproduces precisely the trace anomaly in (16)). In explicit terms, (31) means that the trace anomaly is determined by the value of the three point function formfactor \( \Delta \) with all external momenta set to zero. In our calculation, the external momentum attached to the mass insertion is set to zero from the start and then the limit \( p^2 \to 0 \) can be taken. As an additional check of our approach, one would like to see whether a limiting procedure taken in reverse order would produce the same result. To clarify this point, let us consider the relevant three point function (denoted here as \( \tilde{\Delta}_{\mu\nu} \) to distinguish it from the previous kinematical configuration) with light-like momenta \( k, p \) attached to the currents; the corresponding formfactor is then a function of the kinematical variable \( q^2 \), with \( q \) being the four-momentum attached to the mass-insertion vertex (i.e. \( q^2 = (k + p)^2 \)). Again, one may
write an unsubtracted dispersion relation for the $\tilde{\Delta}(q^2; m^2)$, i.e.

$$\tilde{\Delta}(q^2; m^2) = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{\text{Im} \tilde{\Delta}(s)}{s - q^2} ds$$  \hspace{1cm} (32)$$

A straightforward calculation now yields

$$\text{Im} \tilde{\Delta}(s; m^2) = -\frac{e^2}{8\pi} \frac{4m^2}{s} \left(1 - \frac{4m^2}{s}\right) \ln \frac{1 + \sqrt{1 - \frac{4m^2}{s}}}{1 - \sqrt{1 - \frac{4m^2}{s}}}$$  \hspace{1cm} (33)$$

The quantity

$$\tilde{\Delta}(0) = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{\text{Im} \Delta(s; m^2)}{s} ds$$  \hspace{1cm} (34)$$

is then expected to be equal to the $\Delta(0)$ calculated before (we have suppressed the dependence on $m^2$ as it is obviously trivial for zero external virtualities). The relevant integral occurring in (34) may be easily evaluated as

$$\int_{1}^{\infty} \frac{1}{y} \left(1 - \frac{1}{y}\right) \ln \frac{1 + \sqrt{1 - \frac{1}{y}}}{1 - \sqrt{1 - \frac{1}{y}}} dy = \frac{4}{3}$$  \hspace{1cm} (35)$$

which then obviously reproduces the anticipated result

$$\tilde{\Delta}(0) = \Delta(0)$$  \hspace{1cm} (36)$$

(cf.(26)) and an equivalence of the $t$-channel and $s$-channel dispersive calculations of the trace anomaly is thus verified.

The last remark concerns a possible generalization of the preceding discussion to other field theory models. The previous results can be extended to the scalar electrodynamics without any major changes. In a previous paper \cite{12} we have discussed briefly the problem of a dispersive derivation of the trace anomaly due to the $W$ boson loops within the standard model of electroweak interactions, employing the $s$-channel dispersion relations (in the sense indicated above) in connection with the Higgs boson decay into two photons. The corresponding argument becomes somewhat obscured by technical complications due to the delicate nature of massive vector bosons in spontaneously broken gauge theories and a corresponding $t$-channel calculation (which would be analogous to the main line of the present paper) also requires special care. In general, the problem of dispersive derivation of the trace anomaly in non-Abelian gauge theory models would deserve a separate treatment.
5 Summary

Let us now summarize briefly the main results obtained in this paper. Invoking the paradigm of the well-known axial anomaly, we have discussed a simple derivation of the trace anomaly in spinor QED by means of dispersion relations. The trace anomaly, defined as an extra term in a canonical Ward identity (zero-energy theorem) for the broken scale invariance, has been expressed as a convergent sum rule for the imaginary part of the classical symmetry-breaking term (i.e. as an integral along the cut associated with the non-zero imaginary part of a relevant formfactor). In the massless limit, one gets a delta-function singularity at zero external momentum squared, which reveals an alternative "infrared face" of the trace anomaly. We have checked explicitly that the same result for the anomaly is recovered when employing dispersion relations in different kinematical variables ($t$- and $s$-channel resp.). The results obtained here exhibit a striking similarity with the case of the axial anomaly, for which the dispersive infrared approach was pioneered by Dolgov and Zakharov many years ago [3].

In general, the infrared (low-energy) aspects of quantum anomalies are not discussed frequently in the literature, and for various reasons the ultraviolet (short-distance) nature of these phenomena is usually emphasized. For the trace anomaly, its remarkable connection with the coupling-constant renormalization beta function is certainly the best known (ultraviolet) aspect. Nevertheless, it may be useful to know its dispersive derivation and the associated infrared face as well, in analogy with the archetypal example of the axial anomaly.

Finally, let us remark that some of the points emphasized in the present paper were also mentioned briefly, in a slightly different context, in the recent paper [13] by Deser. Another independent investigation in this direction (for a scalar field model) has recently been communicated to us by O. Teryaev [14].

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