Fordy-Kulish model and spinor Bose-Einstein condensate

V. A. Atanasov 1,2, V. S. Gerdjikov 1, G. G. Grahovski 1,3, N. A. Kostov 1

1 Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, 72 Tsarigradsko chaussee, 1784 Sofia, Bulgaria
3 Laboratoire de Physique Théorique et Modélisation, Université de Cergy-Pontoise, 2 avenue A. Chauvin, F-95302 Cergy-Pontoise Cedex, France
3 School of Electronic Engineering, Dublin City University, Glasnevin, Dublin 9, Ireland

E-mail: victor@inrne.bas.bg, gerjikov@inrne.bas.bg, grah@inrne.bas.bg, nakostov@inrne.bas.bg

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Abstract

A three-component nonlinear Schrodinger-type model which describes spinor Bose-Einstein condensate (BEC) is considered. This model is integrable by the inverse scattering method and using Zakharov-Shabat dressing method we obtain three types of soliton solutions. The multi-component nonlinear Schrödinger type models related to symmetric spaces $\mathbb{C}.I \simeq \text{Sp}(4)/\text{U}(2)$ is studied.

1 Introduction

The dynamics of spinor BEC is described by a three-component Gross-Pitaevskii (GP) system of equations. In the one-dimensional approximation the GP system goes into the following nonlinear Schrödinger (MNLS) equation in (1D) $x$-space [1]:

$$
\begin{align*}
&i \partial_t \Phi_1 + \partial_x^2 \Phi_1 + 2(|\Phi_1|^2 + 2|\Phi_0|^2)\Phi_1 + 2\Phi_1^* \Phi_0^2 = 0, \\
&i \partial_t \Phi_0 + \partial_x^2 \Phi_0 + 2(|\Phi_{-1}|^2 + |\Phi_0|^2 + |\Phi_1|^2)\Phi_0 + 2\Phi_0^* \Phi_1 \Phi_{-1} = 0, \\
&i \partial_t \Phi_{-1} + \partial_x^2 \Phi_{-1} + 2(|\Phi_{-1}|^2 + 2|\Phi_0|^2)\Phi_{-1} + 2\Phi_{-1}^* \Phi_0^2 = 0.
\end{align*}
$$

(1)

We consider BEC of alkaline atoms in the $F = 1$ hyperfine state, elongated in $x$ direction and confined in the transverse directions $y, z$ by purely optical means. Thus the assembly of atoms in the $F = 1$ hyperfine state can be described by a normalized spinor wave vector

$$
\Phi(x,t) = (\Phi_1(x,t), \Phi_0(x,t), \Phi_{-1}(x,t))^T
$$

(2)

whose components are labelled by the values of $m_F = 1, 0, -1$. The above model is integrable by means of inverse scattering transform method [1]. It also allows an exact description of the dynamics and interaction of bright solitons with spin degrees of freedom. Matter-wave solitons are expected to be useful in atom laser, atom interferometry and coherent atom transport. It could contribute to the realization of quantum information processing or computation, as a part of new field of atom optics. Lax pairs and geometric interpretation of the model [11] are given in [4]. Darboux transformation for this special integrable model is developed in [5]. The aim of present paper is to show that the system [11] is related to symmetric space $\mathbb{C}.I \simeq \text{Sp}(4)/\text{U}(2)$ (in the Cartan classification [8]) with canonical $\mathbb{Z}_2$-reduction and has natural Lie algebraic interpretation. The model allows also a special class of soliton solutions. We will show that they can be obtained by a suitable modification of the generalization of the so-called “dressing method”, proposed in [9].
The model (1) belongs to the class of multicomponent NLS equations that can be solved by the inverse scattering method [7, 6]. It is a particular case of the MNLS related to the C.I type symmetric space Sp(4)/U(2) [4]. These MNLS systems allow Lax representation with the generalized Zakharov–Shabat system as the Lax operator:

\[ L \psi(x, t, \lambda) \equiv i \frac{d \psi}{dx} + (Q(x, t) - \lambda J) \psi(x, t, \lambda) = 0. \]  

where \( J \) and \( Q(x, t) \) are \( 4 \times 4 \) matrices: \( J = \text{diag}(1, 1, -1, -1) \) and \( Q(x, t) \) is a block-off-diagonal matrix:

\[
Q(x, t) = \begin{pmatrix} 0 & q(x, t) \\ p(x, t) & 0 \end{pmatrix}, \quad q(x, t) = \begin{pmatrix} \Phi_0(x, t) & -\Phi_1(x, t) \\ \Phi_1(x, t) - \Phi_0(x, t) \end{pmatrix},
\]

\[
p(x, t) = \begin{pmatrix} \Phi_0^+(x, t) & \Phi_1^*(x, t) \\ -\Phi_1^*(x, t) & -\Phi_0^+(x, t) \end{pmatrix}.
\]  

Solving the direct and the inverse scattering problem for \( L \) uses the Jost solutions \( \phi = (\phi^+, \phi^-) \) and \( \psi = (\psi^-, \psi^+) \) of (3) which are defined by, see [14] and the references therein:

\[
\lim_{x \to -\infty} \phi(x, t, \lambda) e^{i\lambda Jx} = 1, \quad \lim_{x \to \infty} \psi(x, t, \lambda) e^{i\lambda Jx} = 1 \]  

These definitions are compatible with the class of smooth potentials \( Q(x, t) \) vanishing sufficiently rapidly at \( x \to \pm \infty \). It can be shown that \( \phi^+ \) and \( \psi^+ \) (resp. \( \phi^- \) and \( \psi^- \)) composed by 4 rows and 2 columns are analytic in the upper (resp. lower) half plane of \( \lambda \). The scattering matrix associated to (3) is defined as

\[
T(t, \lambda) = (\psi(x, t, \lambda))^{-1} \phi(x, t, \lambda) = \begin{pmatrix} a^+(t, \lambda) & -b^-(t, \lambda) \\ b^+(t, \lambda) & a^-(t, \lambda) \end{pmatrix},
\]

\[
(T(t, \lambda))^{-1} = \begin{pmatrix} c^-(t, \lambda) & d^-(t, \lambda) \\ -d^+(t, \lambda) & c^+(t, \lambda) \end{pmatrix},
\]  

where \( a^\pm(t, \lambda) \) and \( b^\pm(t, \lambda) \) are \( 2 \times 2 \) block matrices. The blocks \( a^\pm, b^\pm, c^\pm \) and \( d^\pm \) satisfy a number of relations [11, 12]; for example

\[
a^+(\lambda)c^-(\lambda) + b^-(\lambda)d^+(\lambda) = 1, \quad a^+(\lambda)d^-(\lambda) - b^-(\lambda)c^+(\lambda) = 0,
\]  

etc. The fundamental analytic solutions (FAS) \( \chi^\pm(x, t, \lambda) \) of \( L(\lambda) \) are analytic functions of \( \lambda \) for \( \text{Im} \lambda \geq 0 \) and are related to the Jost solutions by:

\[
\chi^\pm(x, t, \lambda) = \phi(x, t, \lambda) S^\pm_J(t, \lambda) = \psi(x, t, \lambda) T^\mp_J(t, \lambda).
\]  

Here \( S^\pm_J, T^\pm_J \) upper- and lower- block-triangular matrices:

\[
S^+_{J}(t, \lambda) = \begin{pmatrix} 1 & d^-(t, \lambda) \\ 0 & c^+(t, \lambda) \end{pmatrix}, \quad S^-_{J}(t, \lambda) = \begin{pmatrix} c^-(t, \lambda) & 0 \\ -d^+(t, \lambda) & 1 \end{pmatrix},
\]

\[
T^+_{J}(t, \lambda) = \begin{pmatrix} 1 & -b^-(t, \lambda) \\ 0 & a^-(t, \lambda) \end{pmatrix}, \quad T^-_{J}(t, \lambda) = \begin{pmatrix} a^+(t, \lambda) & 0 \\ b^+(t, \lambda) & 1 \end{pmatrix},
\]
satisfying $T^\pm_j(t, \lambda)S^\pm_j(t, \lambda) = T(t, \lambda)$ and can be viewed as the factors of a generalized Gauss decompositions of $T(t, \lambda)$ \[13\]. If $Q(x, t)$ evolves according to (1) then the scattering matrix and its elements satisfy the following linear evolution equations:

$$
\frac{i}{dt} db^\pm(t, \lambda) + 2\lambda^2 b^\pm(t, \lambda) = 0, \quad \frac{i}{dt} da^\pm = 0, \quad (9)
$$

so the block-matrices $a^\pm(\lambda)$ can be considered as generating functionals of the integrals of motion. The fact that all 4 matrix elements of $a^\pm(\lambda)$ for $\lambda \in \mathbb{C}_+$ (resp. of $a^-\lambda(\lambda)$ for $\lambda \in \mathbb{C}_-$) generate integrals of motion reflect the superintegrability of the model and are due to the degeneracy of the dispersion law of (1).

The system (1) can be written in a Hamiltonian form by introducing the Poisson brackets:

$$
\{q_j(x), p_k(y)\} = 2i\delta_{kj}\delta(x - y), \quad \{q_{12}(x), p_{12}(y)\} = i\delta(x - y), \quad (10)
$$

and the Hamiltonian $H = H_{\text{kin}} + H_{\text{int}}$:

$$
H_{\text{kin}} = \int_{-\infty}^{\infty} dx \left( \frac{\partial \Phi_0}{\partial x}\frac{\partial \Phi^*_0}{\partial x} + \frac{1}{2} \left( \frac{\partial \Phi_1}{\partial x}\frac{\partial \Phi^*_1}{\partial x} + \frac{\partial \Phi_{-1}}{\partial x}\frac{\partial \Phi^*_{-1}}{\partial x} \right) \right), \quad (11)
$$

$$
H_{\text{int}} = -\int_{-\infty}^{\infty} dx \left( (|\Phi_0|^2 + |\Phi_1|^2) + (|\Phi_0|^2 + |\Phi_{-1}|^2) \right)
$$

$$
-\int_{-\infty}^{\infty} dx \left( |\Phi_0\Phi^*_{-1} + \Phi_1\Phi^*_0|^2 \right).
$$

As mentioned above, one can use any of the matrix elements of $a^\pm(\lambda)$ as generating functional of integrals of motion of our model. Generically such integrals would have non-local densities and will not be in involution.

The classical $R$-matrix approach \[6, 4\] is an effective method to determine the generating functionals of local integrals of motion which are in involution. From it there follows that such integrals are generated by expanding $\ln m_k^\pm(\lambda)$ over the inverse powers of $\lambda$, see \[13\]. Here $m_k^\pm(\lambda)$ are the principal minors of $T(\lambda)$; in our case

$$
m_1^+(\lambda) = a_1^+(\lambda), \quad m_2^+(\lambda) = \det a^+(\lambda),
$$

$$
m_1^-(\lambda) = a_2^-(\lambda), \quad m_2^-\lambda(\lambda) = \det a^-\lambda(\lambda). \quad (12)
$$

If we consider

$$
\ln m_k^\pm(\lambda) = \sum_{s=1}^{\infty} \lambda^{-k} I^{(k)}_s,
$$

then one can prove that the densities of $I^{(k)}_s$ are local in $Q(x, t)$. The fact that \[13\] :

$$
\{m_k^\pm(\lambda), m_j^\pm(\mu)\} = 0, \quad \text{for } k, j = 1, 2,
$$

and for all $\lambda, \mu \in \mathbb{C}_\pm$ allow one to conclude that $\{I^{(k)}_s, I^{(j)}_p\} = 0$ for all $k, j = 1, 2$ and $s, p \geq 1$.

In particular, the Hamiltonian of our model is proportional to $I^{(2)}_3$, i.e. $H = 8iI^{(2)}_3$. 

3 Soliton solutions for the spinor BEC: The \( so(5) \) connection

The soliton solutions of the \( \mathfrak{sp}(4) \) MNLS \( (1) \) were derived by using the dressing method \( (3) \). They can be considered as particular cases of the soliton solutions of the generic MNLS eqs., derived through the matrix version of the Gel’fand-Levitan-Marchenko equation, see \( (2, 1, 3) \). Here we extend further these results and combining the ideas of \( (3, 15) \) we specify three types of solitons for the model \( (1) \).

We start our analysis with the well-known isomorphism between the algebras \( \mathfrak{sp}(4, \mathbb{C}) \) and \( \mathfrak{so}(5, \mathbb{C}) \) \( (3) \). Since the Lax representation is of pure algebraical nature it is natural to expect that our model \( (1) \) can be treated also by an equivalent Lax operator \( L' \) whose potential \( Q'(x, t) \) and \( J' \) take values in \( so(5) \). A consequence of the above-mentioned isomorphism is that the typical representation of \( sp(4) \) used above is equivalent to the spinor representation of \( so(5) \).

So we first remind some of our results in \( (10, 11, 12) \), where we have constructed the fundamental analytic solutions, the dressing factors, the soliton solutions etc. for a class of Lax operators (including \( L' \)), related to the simple Lie algebra \( \mathfrak{g} \), in the typical representation of \( \mathfrak{g} \). So we first have to specify (if necessary) \( \mathfrak{g} \simeq so(5) \) and then reformulate the corresponding results for the spinor representation of \( so(5) \).

The main goal of the dressing method is, starting from a solution \( \chi_0^+(x, t, \lambda) \) of \( L_0(\lambda) \) with potential \( Q_{(0)}(x, t) \) to construct a new singular solution \( \chi_1^+(x, t, \lambda) \) with singularities located at prescribed positions \( \lambda_i^\pm \); the reduction \( p = q^1 \) used in eq. \( (4) \) ensures that \( \lambda_1^- = (\lambda_1^+)^* \). The new solutions \( \chi_1^\pm(x, t, \lambda) \) will correspond to a potential \( Q_{(1)}(x, t) \) of \( L(\lambda) \) \( (3) \) with two discrete eigenvalues \( \lambda_i^\pm \). It is related to the regular one by a dressing factor \( u(x, \lambda) \)

\[
\chi_1^\pm(x, t, \lambda) = u(x, \lambda)\chi_0^+(x, t, \lambda)u_*^{-1}(\lambda). \quad u_-(\lambda) = \lim_{x \to -\infty} u(x, \lambda)
\]

Note that \( u_-(\lambda) \) is a diagonal matrix. The dressing factor \( u(x, \lambda) \) must satisfy the equation

\[
i\frac{du}{dx} + Q_{(1)}(x)u - uQ_{(0)}(x) - \lambda[J, u(x, \lambda)] = 0, \quad (14)
\]

and the normalization condition \( \lim_{\lambda \to \infty} u(x, \lambda) = 1 \). Besides \( \chi_i^\pm(x, \lambda), i = 0, 1 \) and \( u(x, \lambda) \) must belong to the corresponding Lie group \( Sp(4, \mathbb{C}) \); in addition \( u(x, \lambda) \) by construction has poles and/or zeroes at \( \lambda_i^\pm \).

The construction of \( u(x, \lambda) \) is based on an appropriate anzats specifying explicitly the form of its \( \lambda \)-dependence.

\[
u(x, \lambda) = 1 + (c_1(\lambda) - 1) P_1(x, t) + \left( \frac{1}{c_1(\lambda)} - 1 \right) \overline{P}_1(x, t),
\]

\[
c_1(\lambda) = \frac{\lambda - \lambda_1^+}{\lambda - \lambda_1^-}, \quad (15)
\]

where the projectors \( P_1(x, t) \) and \( \overline{P}_1(x, t) \) are of rank 1 and are related by \( \overline{P}_1(x) = SPF_1(x)S^{-1} \). They must satisfy \( \overline{P}_1(x, t)P_1(x, t) = P_1(x, t)\overline{P}_1(x, t) = 0 \). By \( S \) we have denoted the special matrix which enters in the definition of the orthogonal algebra, i.e. \( X \in so(5) \) if \( X + SX^T S^{-1} = 0 \). In the typical representation of \( so(5) \) we have \( S = \sum_{k=1}^{5} (-1)^{k+1} E_{k,6-k} \) where \( (E_{ij})_{km} = \delta_{ik}\delta_{jm} \). The construction of \( P_1(x, t) \) and \( \overline{P}_1(x, t) \)
using the ‘polarization’ vectors is outlined in [11] and we skip it. The new potential is obtained from
\[ Q_{(1)}(x,t) - Q_{(0)}(x,t) = (\lambda_1^+ - \lambda_1^-)[J, P_1(x,t) - \overline{P}_1(x,t)]. \]

Here we show that the \( \lambda \)-dependence of \( u(x,\lambda) \) may depend [10] on the choice of the representation of \( so(5,\mathbb{C}) \simeq sp(4,\mathbb{C}) \). For \( so(5) \) it was shown [10, 11] that there are three types of solitons:

- the first type of soliton solutions are generated by dressing factors of the form [15]. For generic choice of the polarization vectors \( P_1(x,t) - \overline{P}_1(x,t) \in so(5) \).

- the second type of soliton solutions are generated analogously with dressing factor [15], but due to a specific choice of the polarization vectors \( P_1(x,t) - \overline{P}_1(x,t) \in so(3) \subset so(5) \).

- the third type of soliton solutions are generated again by [15] but now the corresponding projectors \( P_1(x,t) \) and \( \overline{P}_1(x,t) \) have rank 2.

Each of these types of soliton solutions have their counterpart relevant to our model on \( sp(4) \). To the first type of soliton solutions there correspond dressing factor and potential \( Q_{(1)}(x,t) \) of the form [3]:
\[
\tilde{u}(x,\lambda) = \sqrt{c_1(\lambda)} \pi_1(x) + \frac{1}{\sqrt{c_1(\lambda)}} \overline{\pi}_1(x),
\]
\[
Q_{(1)}(x,t) - Q_{(0)}(x,t) = \frac{1}{2}[J, \pi_1(x,t) - \overline{\pi}_1(x,t)],
\]
where \( \pi_1(x) \) and \( \overline{\pi}_1(x) \) are rank 2 projectors, such that
\[
\overline{\pi}_1(x)\pi_1(x) = \pi_1(x), \quad \overline{\pi}_1(x) = 0, \quad \overline{\pi}_1(x) + \pi_1(x) = 1.
\]

This last property ensures the non-degeneracy of \( u(x,\lambda) \). Note that now the dressing factor is not a rational function of \( \lambda \) but for the dressed FAS \( \chi(x,\lambda) \) eq. (13) we get:
\[
\chi_1^{\pm}(x,t,\lambda) = \left( \pi_1(x,t) + \frac{1}{c_1(\lambda)} \overline{\pi}_1(x,t) \right) \chi_0^{\pm}(x,t,\lambda) \left( \pi_1^- + c_1(\lambda)\overline{\pi}_1^- \right),
\]
\[
\pi_1^- = \lim_{x \to -\infty} \pi_1(x,t),
\]

i.e., the fractional powers of \( c_1(\lambda) \) disappear.

The second type of solitons with rank 2 projector \( P_1(x) \) after recalculating to the spinor representation formally keeps the same form [15] with \( P_1(x) \) replaced by \( A_1(x) \) which has rank 1 but generically is not a projector, see [3].

The third type of solitons is similar to the second one but with additional constraints on the factor \( A_1(x) \) so that \( A_1(x) - \overline{A}_1(x) \in sp(2) \subset sp(4) \).

Consider the purely solitonic case when \( Q_{(0)} = 0 \). From now on we introduce the following notations \( \lambda_1^{\pm} = \mu_1 \pm i\nu_1 \) and
\[
A = -2i((\lambda_1^+)^2 - (\lambda_1^-)^2)t - i(\lambda_1^+ - \lambda_1^-)x,
\]
\[
B = -2((\lambda_1^+)^2 + (\lambda_1^-)^2)t - (\lambda_1^+ + \lambda_1^-)x.
\]
Here \( A(x, t) \) and \( B(x, t) \) are \( x \) and \( t \) dependent real valued functions. Making use of the explicit form of the projectors \( P_{\pm 1}(x) \) valid for the typical representations of \( B_2 \) we obtain\[11\]:

\[
\Phi_{(1)}(x, t) = \frac{4(\lambda^+_1 - \lambda^-_1)}{\langle m|n \rangle} (n_{0,1} m_{0,2} e^A + n_{0,2} m_{0,1} e^{-A}) e^{iB} 
\]

\[
\Phi_{(0)}(x, t) = \frac{2\sqrt{2}(\lambda^+_1 - \lambda^-_1)}{\langle m|n \rangle} (n_{0,1} m_{0,3} e^A - n_{0,3} m_{0,1} e^{-A}) e^{iB} 
\]

\[
\Phi_{(-1)}(x, t) = -\frac{4(\lambda^+_1 - \lambda^-_1)}{\langle m|n \rangle} (n_{0,1} m_{0,2} e^A + n_{0,2} m_{0,1} e^{-A}) e^{iB} 
\]

where the denominator in the above formula is given by:

\[
\langle m|n \rangle = m_{0,1} n_{0,1} (e^{2A}) + m_{0,\bar{1}} n_{0,\bar{1}} (e^{-2A}) 
+ m_{0,2} n_{0,2} + m_{0,3} n_{0,3}.
\]

and \( m_{0,k}, n_{0,k} \) are the components of the polarization vectors.

Choosing appropriately the elements of the polarization vectors \( |n_0 \rangle \) and \( |m_0 \rangle \), one can show that the conjecture that the Zakharov-Shabat dressing procedure and the Gel’fand-Levitan Marchenko formalism lead to comparable soliton solutions is true. It is not a problem to multiply the polarization vectors \( |n_0 \rangle \) and \( |m_0 \rangle \) by an appropriate scalar and thus to adjust the two solutions. Such a multiplication easily goes through the whole scheme outlined above. The involution \( Q^{(1)} = Q(1) \) that the potential of the Lax operator \[3\] associated with the system \[11\] is subject to results in the following relations between the elements of the ”polarization” vectors \( |n_0 \rangle \) and \( \langle m_0 | \rangle \), namely \( n_{0,k} = m^\ast_{0,k} \). Utilizing the above and a proper change of field components, we can relate the solution

\[
\Phi(x, t) = 4\nu_1 C^\dagger e^A + \sigma_2 C^\dagger \sigma_2 \det\{C^\dagger\} e^{-A} e^{iB} + W + |\det\{C\}|^2 e^{-2A} e^{iB},
\]

where \( W = (2|c_{12}|^2 + |c_1|^2 + |c_2|^2) \) and the ”polarization” matrix can be cast into the form

\[
C = \begin{pmatrix} c_{12} & c_1 \\ c_2 & -c_{12} \end{pmatrix}
\]

(25)

In the special case when \( W = 1 \) and \( \det\{C\} = 0 \) we obtain

\[
\Phi(x, t) = \frac{2\nu_1 e^{iB}}{\cosh A} C^\dagger
\]

(26)

Thus we confirm the result obtained in \[1\], acquired with the help of GLM formalism and the solution \[20\], derived within the generalized Zakharov-Shabat dressing procedure, provided we make sure that the extra condition on the vector \( |m \rangle \):

\[
-2m_{0,1} n_{0,\bar{1}} + 2m_{0,2} n_{0,\bar{2}} = (m_{0,3})^2,
\]

and analogous one for \( |n \rangle \) holds true. Setting

\[
m_{0,1} = 1, \quad m_{0,\bar{1}} = -(c_{12}^\ast)^2 - c_1^\ast c_2^\ast,
\]

\[
m_{0,2} = ic_1^\ast, \quad m_{0,\bar{2}} = ic_2^\ast, \quad m_{0,3} = m_{0,\bar{3}} = -\sqrt{2} c_{12}^\ast
\]

we establish the equivalence between the two solutions.
4 Conclusions

We have derived the soliton solutions of the three-component system of NLS type on the symmetric space $\text{Sp}(4)/\text{U}(2)$ which is related to spinor Bose-Einstein condensate model (with $F = 1$). Furthermore, we have described briefly the Hamiltonian properties of the model and the integrals of motion. Using the classical $r$-matrix approach, we showed that the integrals of motion, that belong to the principal series are in involution.

The reduction of the multi-component nonlinear Schrödinger (NLS) equations on symmetric space $\mathbb{C}I \simeq \text{Sp}(2p)/\text{U}(p)$ for $p = 2$ is related to spinor model of Bose-Einstein condensate. Other interesting reductions of MNLS type equations were reported in [11] and a systematic study of the problem is on the way. Recently the authors of [16] develop a perturbation theory for bright solitons of the $F = 1$ integrable spinor BEC model. Both rank-one and rank-two soliton solutions are obtained using Riemann-Hilbert method and are compared with known results.

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