Online Unit Profit Knapsack with Untrusted Predictions

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Abstract

A variant of the online knapsack problem is considered in the settings of trusted and untrusted predictions. In Unit Profit Knapsack, the items have unit profit, and it is easy to find an optimal solution offline: Pack as many of the smallest items as possible into the knapsack. For Online Unit Profit Knapsack, the competitive ratio is unbounded. In contrast, previous work on online algorithms with untrusted predictions generally studied problems where an online algorithm with a constant competitive ratio is known. The prediction, possibly obtained from a machine learning source, that our algorithm uses is the average size of those smallest items that fit in the knapsack. For the prediction error in this hard online problem, we use the ratio \( r = \frac{\hat{a}}{a} \) where \( a \) is the actual value for this average size and \( \hat{a} \) is the prediction. The algorithm presented achieves a competitive ratio of \( \frac{1}{2} r \) for \( r \geq 1 \) and \( \frac{r}{2} \) for \( r \leq 1 \). Using an adversary technique, we show that this is optimal in some sense, giving a trade-off in the competitive ratio attainable for different values of \( r \). Note that the result for accurate advice, \( r = 1 \), is only \( \frac{1}{2} \), but we show that no algorithm knowing the value \( a \) can achieve a competitive ratio better than \( \frac{1}{\sqrt{e}} \approx 0.6321 \) and present an algorithm with a matching upper bound. We also show that this latter algorithm attains a competitive ratio of \( \frac{r}{e} \) for \( r \leq 1 \) and \( \frac{r}{e} + \frac{1}{e} \) for \( 1 \leq r < e \), and no algorithm can be better for both \( r < 1 \) and \( 1 \leq r < e \).

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1 Introduction

In this paper, we consider the Online Unit Profit Knapsack Problem: The request sequence consists of \( n \) item with sizes in \((0, 1]\). An online algorithm receives them one at a time, with no knowledge of future items, and makes an irrevocable decision for each, either accepting or rejecting the item. It cannot accept any item if its size, plus the sum of the sizes of the already accepted items, is greater than 1. The goal is to accept as many items as possible. The obvious greedy algorithm solves the offline Unit Profit Knapsack Problem, since the set consisting of as many of the smallest items that fit in the knapsack is an optimal solution.

Even for this special case of the Knapsack Problem, no competitive online algorithms can exist. Thus, we study the problem under the assumption that (an approximation of) the average item size, \( a \), in an optimal solution is known to the algorithm. We study the case, where the exact value of \( a \) is given to the algorithm as advice by an oracle, as well as the case where \( a \) is untrusted, e.g., estimated using machine learning. For instance, the characteristics of the input may be different depending on the time of day the input is produced, which source produced the input, etc. This could be learned to some extent and result in a prediction, which could be provided to the algorithm.
When considering machine-learned advice, the concepts of consistency and robustness are often considered, describing the balance between performing well on accurate advice and not doing too poorly when the advice is completely wrong. Our setting is different from most work on online algorithms with machine-learned advice, where there is generally a known online algorithm with a constant competitive ratio for the problem without advice. For this problem, if the advice is completely wrong, the algorithm cannot be competitive, since the problem without advice does not allow for competitive algorithms. Despite this hardness for the standard online version of the problem, we obtain results with untrusted predictions that are surprisingly consistent and robust.

1.1 Previous Work

The Knapsack Problem is well studied and comes in many variants; see Kellerer et al. [25]. Cygan et al. [19] refer to the online version we study, where all items give the same profit, as the unit case. They mention that it is well-known that no online algorithm for this version of the problem is competitive, i.e., has a finite competitive ratio. To verify this result, consider, for instance, the family of input sequences $\sigma_j$ consisting of items of sizes $1/i$, $i = 1, 2, 3, \ldots, j$.

In the General Knapsack Problem, each item comes not only with a size, but also with a profit, and the goal is to accept as much profit as possible given that the total size must be at most 1. The ratio of the profit to the size is the importance of an item. (This is sometimes called value, but we want to avoid confusion with other uses of that word.)

The Online Knapsack Problem was first studied by Marchetti-Spaccamela and Vercellis [35]; they prove that the problem does not allow for competitive online algorithms, even for Relaxed Knapsack (fractions of items may be accepted), where all item sizes are 1. They concentrate on a stochastic version of the problem, where both the profit and size coefficients are random variables.

The Online Unweighted (or Simple) Knapsack Problem with advice was studied in [15]. This is also called the proportional or uniform case. In this version, the importance of each item is equal to 1. They show that 1 bit of advice is sufficient to be $\frac{1}{2}$-competitive, $\Omega(\log n)$ bits are necessary to be better than $\frac{1}{2}$-competitive, and $n - 1$ advice bits are necessary and sufficient to be optimal. (As mentioned later, they also considered the General Knapsack Problem in the advice model.) The fundamental issues and many of the early results on oracle-based advice algorithms, primarily in the direction of advice complexity, can be found in [16], though many newer results for specific problems have been published since.

In [46], a knapsack problem is considered in a setting with machine-learned advice, with results incomparable to ours. In their setting, the General Knapsack Problem is considered, and results depend on upper and lower bounds on the importance of the items. The authors define limited classes of algorithms, based on a parameter, leading to some controlled degradation compared to an optimal competitive ratio. Within the defined classes, focus is then on tuning compared with historical data. Decisions to accept or reject an item are based on a threshold function based on the item’s importance. Though the definition of this function is ad hoc, in the sense that it is not derived from some direct optimality criterion, it is well-motivated, aiming to coincide with the behavior found in optimal algorithms for the standard online algorithms setting.

Recently, in [22], the General Knapsack Problem is revisited, again with upper and lower bounds on the possible importance of items. Machine-learned advice is given for each importance $v$, both an upper and a lower bound for the sum of the sizes of the items with importance $v$. The authors present an algorithm which has some similarities to ours. In particular their budget function has a similar function to our threshold function; both
specify the maximum number of the low importance, large items that need to be accepted to obtain the proven competitive ratios. Their results can be extended to the case where the predictions are off by a small amount, the lower bounds can be divided by $1+\varepsilon$, and the upper bounds can be multiplied by $1+\varepsilon$. This is in contrast to ours, where robustness results are proven for arbitrarily large errors in the predictions, but only $a$ is predicted. Since we have no bounds on the ratio of the largest to smallest size, those values do not enter into our results. Their algorithm obtains what they prove to be the optimal competitive ratio (for the given predictions), up to an additive factor that goes to zero as the size of the largest item goes to zero; this result has some of the flavor of our negative result. The authors also consider two related problems.

The Bin Packing Problem is closely related to the Knapsack Problem. This is especially true for the dual variant where the number of bins is fixed and the objective is to pack as many items as possible [17]; the Unit Price Knapsack Problem is Dual Bin Packing with one bin. The standard Bin Packing Problem was considered with machine learning in [3], considering a model of machine learning where, for a given algorithm, $\text{ALG}$, they consider a pair of values, $(r_{\text{ALG}}, w_{\text{ALG}})$, representing worst case ratios compared to the optimal offline algorithm, $\text{OPT}$. The value $r_{\text{ALG}}$ gives the ratio for the best (trusted) advice and $w_{\text{ALG}}$ gives the ratio for the worst possible (untrusted) advice. They use a parameter $\alpha$ in their algorithm, and show that their algorithm achieves values $(r, f(r))$ with $1.5 < r \leq 1.75$ and $f(r) = \max\{33 - 18r, 7/4\}$.

Bin Packing is also studied in [6] in the standard setting for online algorithms with machine learning, giving a trade-off between consistency and robustness, with the performance degrading as a function of the prediction error. They also have experimental results. Since the problem is so difficult, they have restricted their consideration to integer item sizes.

Much additional work has been done for other online problems, studying variants with predictions (machine-learned advice, for instance), initiated by the work of Lykouris and Vassilvitskii [33, 34] and Purohit et al. [40] in 2018, with further work in the directions of search-like problems [2, 7, 14, 30, 31, 36], scheduling [1, 5, 10, 21, 27, 28, 32, 37], rental problems [20, 26, 43], caching/paging [13, 23, 24, 41, 44], and other problems [6, 8, 9, 12, 38, 42], while some papers attack multiple problems [3, 11, 29, 45]. For a survey, see [39].

## 1.2 Preliminaries

We let $a$ denote the average size of items accepted by the offline, optimal algorithm, $\text{OPT}$, that accepts as many of the smallest items as possible. Moreover, we let $\hat{a}$ denote the “guessed” or predicted value of $a$. In the case of accurate advice (received from an oracle), $\hat{a} = a$. If $\hat{a}$ may not be accurate, possibly determined via machine learning, and therefore not necessarily exactly $a$, we define a ratio $r$ such that $a = r \cdot \hat{a}$. This particular advice is considered as a value that might be available or predictable, and the competitive ratios we present are a function of $r$.

We use the asymptotic competitive ratio throughout this paper. Thus, an algorithm $\text{ALG}$ is $c$-competitive if there exists a constant $b$ such that for all request sequences $\sigma$, $\text{ALG}(\sigma) \geq c \text{OPT}(\sigma) - b$, where $\text{ALG}(\sigma)$ denotes $\text{ALG}$'s profit on $\sigma$. $\text{ALG}$’s competitive ratio is then $\sup\{c \mid \text{ALG is } c\text{-competitive}\}$. Note that this is a maximization problem and all competitive ratios are in the interval $[0, 1]$.

We use the notation $\mathbb{N} = \{0, 1, 2, \ldots\}$. At any given time during the processing of the input sequence, the level of the knapsack denotes the total size of the items accepted.
1.3 Our Results

We consider both the case where the advice $\hat{a}$ is known to be accurate, so $r = 1$, and the case where it might not be accurate. Different algorithms are presented for these two cases, but they have a common form.

For our algorithm \textsc{Adaptive Threshold} (AT) where the advice is accurate and, thus, $\hat{a} = a$, the competitive ratio is $\frac{e - 1}{e}$, and we prove a matching upper bound that applies to any deterministic algorithm knowing $a$. This upper bound limits how well any algorithm using trusted predictions can do; the competitive ratio cannot be better than $\frac{e - 1}{e} \approx 0.6321$ for $r = 1$.

If AT is used for untrusted predictions, it obtains a competitive ratio of $r \frac{e - 1}{e}$ for $1 \leq r \leq e$, and 0 for $r \geq e$. No algorithm can be better than this for both $r < 1$ and $1 < r < e$.

For the results for our algorithm, \textsc{Adaptive Threshold for Untrusted Predictions} (ATUP), there are two cases: for $r \leq 1$ the competitive ratio is $\frac{1}{2}$, and for $r \geq 1$ the competitive ratio is $\frac{1}{2r}$. Thus, for accurate advice, the competitive ratio of ATUP is $\frac{1}{2}$, slightly less good than for the other algorithm. We show a negative result implying that an online algorithm cannot both be $\frac{1}{2r}$-competitive for a range of large $r$-values and better than $\frac{1}{2}$-competitive for $r = 1$.

Exact, oracle-based advice is not our focus point, though it is a crucial step in our work towards an algorithm for untrusted predictions. Thus, we do not emphasize the direction of advice complexity, where the focus is on the number of bits of oracle advice used to obtain given competitive ratios (or optimality), but we include a brief discussion in Section 5. Instead, we focus on advice that may be easy to obtain. It seems believable that the average size of requests in an optimal solution would be information easily obtainable. The average size is probably a crucial component with regards to the profit secured by a process and quite possibly crucial with regards to supplying resources (knapsacks) over time. It is a single number (or two numbers: number of items and total size) to collect and store, as opposed to more detailed information about a distribution. So little storage is required that one could keep multiple copies if, for instance, the expected average changes during the day.

Given the simple optimal algorithm for the offline version of unit price knapsack, it seems obvious to consider another possibility for advice, the maximum size, $s$, for items to accept. However, this is insufficient, as there might be many items of that size, but the optimal solution may contain very few of them. Thus, one also needs further advice, including, for example, the fraction of the knapsack filled by items of size $s$. With these parameters given as advice, there would be two possibilities for the error. An extension of this idea is presented in [15], where the minimum importance is used, instead of the maximum size, for the General Knapsack Problem, giving $k$-bit approximations to the advice.

## 2 The Adaptive Threshold Algorithm

In Algorithm 1, we introduce an algorithm template, which can be used to establish an oracle-based advice algorithm as well as an algorithm for untrusted predictions. The template omits the definition of a threshold function, $T$, since it is different for the two algorithms. In both algorithms, the threshold functions have the property that $T(i) > T(i + 1)$ for $i \geq 1$. We use the notation $n_x$ to denote the number of accepted items strictly larger than $x$.

Intuitively, \textsc{Adaptive Threshold} accepts items that fit as long as it has not accepted too many items larger than the current item. The threshold functions are used to determine how many larger items is too many; no more than $i$ items of size larger than $T(i + 1)$ are
Algorithm 1 Algorithm Adaptive Threshold.

1: \( \hat{a} \leftarrow \) predicted average size of Opt’s accepted items
2: level \( \leftarrow 0 \)
3: for each input item \( x \) do
4: \( i = \max_{j \geq 0} \{ n_T(j+1) = j \} \)
5: if size(\( x \)) \( \leq T(i+1) \) and level + size(\( x \)) \( \leq 1 \) then
6: Accept \( x \)
7: level += size(\( x \))
8: else
9: Reject \( x \)

accepted. For smaller item sizes, this number of larger items is larger, since we need to accept more items if there are many small items.

Note that using \( \max_{j \geq 0} \{ n_T(j+1) \geq j \} \) instead of \( \max_{j \geq 0} \{ n_T(j+1) = j \} \) in Line 4 would result in the same algorithm. Thus, \( i \) is nondecreasing through the processing of the input sequence, and the value of the threshold function, \( T(i) \), is decreasing in \( i \), so larger items cannot be accepted after \( i \) increases.

3 Accurate Predictions

In this section, we give an \( \frac{\epsilon-1}{\epsilon} \)-competitive algorithm which receives \( \hat{a} \), the average size of the items in Opt, as advice and prove that it is optimal among algorithms that get only \( \hat{a} \) as advice.

3.1 Positive Result

To define an advice-based algorithm, we define a threshold function; see Algorithm 2. Throughout this section, we assume that \( \hat{a} = a \), but the algorithm is also be used for untrusted predictions in Subsection 4.1.

Algorithm 2 Adaptive Threshold with advice, AT.

1: Define \( T(i) = \frac{\hat{a}e}{\hat{a}e(i-1) + 1} \) for \( i \geq 1 \)
2: Run Adaptive Threshold, Algorithm 1

We first set out to prove that AT with \( \hat{a} = a \) has competitive ratio at least \( \frac{\epsilon-1}{\epsilon} \approx 0.6321 \). For that, we need two simple lemmas. The first involves an obvious generalization of Harmonic numbers to non-integers.

Define \( \sum_{i=x}^{y} f(k) \) for some function \( f \) and real-valued \( x \) and \( y \) such that \( y - x \in \mathbb{N} \) as \( f(x) + f(x+1) + \cdots + f(y) \). We generalize the Harmonic numbers by defining \( H_k = \sum_{i=1+k-\lfloor k \rfloor}^k \frac{1}{x} \), for any real-valued \( k \geq 1 \).

Lemma 1. If \( k \geq p \geq 1 \) and \( k - p \in \mathbb{N} \), then \( \ln k - \ln(p+1) \leq H_k - H_p \leq \ln k - \ln p \).

Proof. Define \( \Delta_k = H_k - \ln k \). First, we argue that \( \Delta_k > \Delta_{k+1} \).

Observe that
\[
\ln(k+1) - \ln k = \int_k^{k+1} \frac{1}{x} dx > \frac{1}{k+1},
\]
since \( \frac{1}{e^{x+1}} \) is the smallest value we are integrating over. So, \( \frac{1}{e^{x+1}} = \ln(k + 1) < - \ln(k) \).

Using this,

\[
\Delta_{k+1} = H_{k+1} - \ln(k+1) = H_{k} + \frac{1}{k+1} - \ln(k+1) < H_{k} - \ln k = \Delta_{k}.
\]

By the definition of \( \Delta_{k} \),

\[
H_{k} - H_{p} = \ln k + \Delta_{k} - (\ln p + \Delta_{p}) \leq \ln k - \ln p, \text{ since, by induction, } \Delta_{k} \leq \Delta_{p}.
\]

From the integral, it follows similarly that \( \ln(p+1) - \ln p < \frac{1}{p} \). Thus,

\[
\Delta_{k+1} = H_{k+1} - \ln(k+1) = H_{k} + 1 - \ln(k+1) < H_{k} - \ln(k) = \Delta_{k}.
\]

By the definition of \( \Delta_{k} \),

\[
H_{k} - H_{p} = \ln k + \Delta_{k} - (\ln p + \Delta_{p}) \leq \ln k - \ln p, \text{ since, by induction, } \Delta_{k} \leq \Delta_{p}.
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\[
\Delta_{k+1} = H_{k+1} - \ln(k+1) = H_{k} + 1 - \ln(k+1) < H_{k} - \ln(k) = \Delta_{k}.
\]

Now, \( H_{k} - \ln k > H_{p} - \ln(p+1) \) clearly holds for \( p = k \), since \( \ln \) is increasing. So, by induction, using the above in the induction step, it holds for smaller \( p \) as well. Thus,

\[
\ln k - \ln(p+1) \leq H_{k} - H_{p} \text{ for } k \geq p. \quad \blacksquare
\]

The next lemma just establishes a simple analytical bound.

\[\textbf{Lemma 2.} \ \forall a > 0: e^{1-ae} \geq e - e^2 a.\]

\[\textbf{Proof.} \ \text{We prove that } e^{1-ae} \text{ is bounded from above by } e^2.\]

The derivative of the term is

\[
ae^{1-ae} \frac{1}{a} = e^{1-ae}(e^{ea} - 1).
\]

The terms \( a^2 \) and \( e^{1-ae} \) are positive. Consider the remaining term, \( ea - e^{ea} + 1 \). For \( a = 0 \), this term is zero. The derivative of \( ea \) is \( e \) and the derivative of \( e^{ea} \) is \( e^{ea} + 1 \). For any \( a > 0 \), \( e^{ea} + 1 > e \), so \( ea - e^{ea} + 1 \) is negative. Thus, for \( a > 0 \), the derivative of \( e^{1-ae} \) is negative, and the term decreases with increasing \( a \). Thus, the limit for \( a \) going towards zero is an upper bound.

Using L'Hôpital's rule,

\[
\lim_{a \to 0^+} e^{1-ae} \frac{1}{a} = \lim_{a \to 0^+} \frac{e^{1-ae} - 1}{1/a} = e^2. \quad \blacksquare
\]

With these two lemmas, we can now prove the theorem.

\[\textbf{Theorem 3.} \ \forall \hat{a} = a, \ AT, \ as \ defined \ in \ Algorithm 2, \ is \ \frac{e-1}{e} - \text{competitive.}\]

\[\textbf{Proof.} \ \text{If } AT \ never \ rejects \ an \ item, \ it \ performs \ optimally. \ So \ assume \ it \ rejects \ an \ item \ at \ some \ point \ in \ the \ request \ sequence } \sigma. \ \text{Considering the conditional statement in the algorithm, if } AT \ rejects \ an \ item, x, \ then \ either \ size(x) > T(i+1) \ or \ level + size(x) > 1.\]

\[\textbf{Case 1:}\]

This is the case where, at some point, \( AT \) rejects an item, \( x \), because \( \text{level} + \text{size}(x) > 1 \).

The value of \( T(k) \) from Algorithm 1 is an upper bound on the size of the \( k \)th largest item accepted by the algorithm. Thus, the \( k \)th largest accepted item has size at most

\[
T(k) = \frac{ae}{ae(k-1) + 1} = \frac{1}{k-1 + \frac{1}{ae}}.
\]
Using the obvious definitions of sums over non-integer values, as outlined above, this gives an upper bound on the total size of items accepted by AT of

\[
\text{level} \leq \sum_{k=1}^{\frac{\text{AT}(\sigma)}{\frac{1}{ae}}} \frac{1}{k - 1 + \frac{1}{ae}} = \sum_{k=\frac{1}{ae}}^{\frac{\text{AT}(\sigma) + \frac{1}{ae}}{\frac{1}{ae}} - 1} \frac{1}{k} = H_{\frac{\text{AT}(\sigma) + \frac{1}{ae}}{\frac{1}{ae}} - 1} - H_{\frac{1}{ae} - 1}.
\]

Simple calculations (detailed in Lemma 1) give,

\[
H_{\frac{\text{AT}(\sigma) + \frac{1}{ae}}{\frac{1}{ae}} - 1} - H_{\frac{1}{ae} - 1} < \ln \left( \frac{\text{AT}(\sigma) + \frac{1}{ae} - 1}{\frac{1}{ae} - 1} \right) - \ln \left( \frac{\text{AT}(\sigma) + \frac{1}{ae} - 1}{\frac{1}{ae} - 1} \right).
\]

By assumption, level + size(x) > 1, and since level \leq \ln \left( \frac{\text{AT}(\sigma) + \frac{1}{ae} - 1}{\frac{1}{ae} - 1} \right), we have

\[
\ln \left( \frac{\text{AT}(\sigma) + \frac{1}{ae} - 1}{\frac{1}{ae} - 1} \right) > 1 - \text{size(x)}
\]

\[\Downarrow\]

\[
\frac{\text{AT}(\sigma) + \frac{1}{ae} - 1}{\frac{1}{ae} - 1} > e^{1 - \text{size(x)}},
\]

\[\Downarrow\]

\[
\text{AT}(\sigma) > \left( \frac{1}{ae} - 1 \right) e^{1 - \text{size(x)}} - \frac{1}{ae} + 1
\]

\[\Downarrow\]

\[
\text{AT}(\sigma) > e^{1 - \text{size(x)}} - 1 + 1 - e^{1 - \text{size(x)}}.
\]

In the algorithm, i is at least zero, so we cannot accept items larger than \(T(1) = ae\).

\[
\text{AT}(\sigma) > e^{1 - \text{size(x)}} - 1 + 1 - e, \quad \text{since } -e^{1 - \text{size(x)}} > -e
\]

\[
> e^{1 - ae} - 1 + 1 - e, \quad \text{by the observation above}
\]

\[
\geq e - e^2 a - 1 + 1 - e, \quad \text{simple calculations, detailed in Lemma 2}
\]

\[
= e - 1 + 2e - 1
\]

\[
\geq e - 1 - 2e + 1. \quad \text{since } \text{OPT}(\sigma) \leq \frac{1}{a}
\]

So, \(\lim_{\text{OPT} \to \infty} \frac{\text{AT}(\sigma)}{\text{OPT}(\sigma)} \geq \frac{e - 1}{e}.

**Case 2:**

This is the case where AT never rejects any item, x, when size(x) \leq T(i+1). Let \(i_t\) denote the final value of i as the algorithm terminates. Suppose \text{OPT} accepts \ell items larger than \(T(i_t + 1)\) and \(s\) items of size at most \(T(i_t + 1)\). Since \text{OPT} accepts \ell items larger than \(T(i_t + 1)\) and \(\ell + s\) items in total, we have \(a > \ell \cdot T(i_t + 1)/(\ell + s)\), which is equivalent to

\[
s > \left( \frac{T(i_t + 1)}{a} - 1 \right) \ell
\]

(1)
By the definition of $T$, we have that $T(i_t + 1) = \frac{ae}{T(i_t + 1)}$. Solving for $i_t$ on the right-hand side, we get
\[ i_t = \frac{1}{T(i_t + 1)} - \frac{1}{ae}. \] (2)
Thus, AT has accepted at least $i_t = \frac{1}{T(i_t + 1)} - \frac{1}{ae}$ items of size greater than $T(i_t + 1)$.

Further, due to the assumption in this second case, AT has accepted all of the $s$ items no larger than $T(i_t + 1)$. To see this, note that the is of the algorithm can only increase, so at no point has there been a size demand more restrictive than $T(i_t + 1)$.

We split in two subcases, depending on how $T(i_t + 1)$ relates to Opt’s average size, $a$.

**Subcase 2a:** $T(i_t + 1) > a$

In this subcase, the lower bound on $s$ of Ineq. (1) is positive.

\[ \frac{AT(\sigma)}{Opt(\sigma)} \geq \frac{(\frac{1}{T(i_t + 1)} - \frac{1}{ae}) + s}{\ell + s}, \quad \text{by Eq.} \ (2) \]
\[ > \frac{(\frac{1}{T(i_t + 1)} - \frac{1}{ae}) + (\frac{T(i_t + 1)}{a} - 1) \ell}{\ell + (\frac{T(i_t + 1)}{a} - 1) \ell}, \quad \text{by Ineq.} \ (1) \]
\[ = \frac{(\frac{1}{T(i_t + 1)} - \frac{1}{ae}) + (\frac{T(i_t + 1)}{a} - 1) \ell}{T(i_t + 1) \ell}. \]

The second inequality follows since the ratio is smaller than one and $s$ is replaced by a smaller, positive term in the numerator as well as the denominator.

We prove that this is bounded from below by $\frac{e - 1}{e}$:

\[ \frac{(\frac{1}{T(i_t + 1)} - \frac{1}{ae}) + (\frac{T(i_t + 1)}{a} - 1) \ell}{T(i_t + 1) \ell} \geq \frac{e - 1}{e} \]
\[ \Downarrow \]
\[ e - \frac{1}{a} + \frac{e}{a} (T(i_t + 1) - 1) \ell \geq e \frac{T(i_t + 1)}{a} \ell - T(i_t + 1) \ell \]
\[ \Downarrow \]
\[ e \frac{1}{a} \geq \left( e - \frac{T(i_t + 1)}{a} \right) \ell \]
\[ \Downarrow \]
\[ ea - T(i_t + 1) \geq ca - T(i_t + 1) \ell \]
\[ \Downarrow \]
\[ \frac{1}{T(i_t + 1)} \geq \ell \]

For the last biimplication, we must argue that $ea - T(i_t + 1) \geq 0$, but this holds since $T(1) = ca$ and $T$ is decreasing. Finally, the last statement, $\frac{1}{T(i_t + 1)} \geq \ell$ holds regardless of the relationship between $T(i_t + 1)$ and $a$, since the knapsack obviously cannot hold more than $\frac{1}{T(i_t + 1)}$ items of size greater than $T(i_t + 1)$.
Subcase 2b: $T(i_t + 1) \leq a$

\[
\frac{\text{AT}(\sigma)}{\text{OPT}(\sigma)} \geq \frac{\left(\frac{1}{T(i_t+1)} - \frac{1}{ae}\right)}{\ell + s}, \text{ by Eq. (2)}
\]
\[
\geq \frac{\left(\frac{1}{T(i_t+1)} - \frac{1}{ae}\right)}{\ell}, \quad \text{since } s \geq 0 \text{ and } \frac{\text{AT}(\sigma)}{\text{OPT}(\sigma)} \leq 1
\]
\[
> \frac{\left(\frac{1}{T(i_t+1)} - \frac{1}{ae}\right)}{\ell}, \quad \text{since, as above, } \ell \leq \frac{1}{T(i_t+1)}
\]
\[
= 1 - \frac{T(i_t+1)}{ae}
\]
\[
\geq 1 - \frac{a}{ae}, \quad \text{by the subcase we are in}
\]
\[
= \frac{e-1}{e}.
\]
This concludes the second case, and, thus, the proof. ◐

3.2 Negative Result

Now, we show that AT is optimal among online algorithms knowing $a$ and nothing else.

▷ Theorem 4. Any algorithm getting only $a$ as advice has a competitive ratio of at most $\frac{e-1}{e}$.

Proof. Let $\text{Alg}$ denote the online algorithm with advice, and let $\sigma$ be the adversarial sequence defined by Algorithm 3, which explains how the adversary defines its sequence based on $\text{Alg}$’s actions.

\begin{algorithm}
  \caption{Adversarial sequence establishing optimality with advice.}
  \begin{algorithmic}[1]
    \State \Comment{Assume $a < \frac{1}{2e}$ and $\frac{1}{e} \in \mathbb{N}$}
    \State $\varepsilon \leftarrow \frac{a}{2e}$
    \State $k \leftarrow \left\lfloor \frac{1}{a} \right\rfloor n$
    \While{$\text{Alg}$’s level $\leq 1 - \frac{1}{k} - k\varepsilon$}
      \For{$k$ times}
        \State Give an item of size $\frac{1}{k} - \varepsilon$
        \State \If{$\text{Alg}$ accepts}
          \State $k++$
          \Comment{\Comment{ALG did not accept any of the $k$ items of this round.}}
        \EndIf
      \EndFor
      \Comment{\Comment{ALG did not accept any of the $k$ items of this round.}}
      \State Give $\frac{1}{k} - k$ items of size $\frac{k\varepsilon}{1 - k\varepsilon}$
    \EndWhile
    \Comment{\Comment{Case 1}}
    \State \Comment{Case 2}
  \EndAlgorithm
\end{algorithm}

Let $k_t$ be the value of $k$ at the beginning of the last iteration of the while-loop. We perform a case analysis based on how the generation of the adversarial sequence terminates.
Case 1:

OPT accepts the $k_t$ items of size $\frac{1}{k_t} - \varepsilon$ in the last iteration of the while-loop and the $\frac{1}{a} - k_t$ items of size $\frac{k_t a \varepsilon}{1 - k_t^2}$ for a total of $\frac{1}{a}$ items of total size $k_t \left( \frac{1}{k_t} - \varepsilon \right) + \left( \frac{1}{a} - k_t \right) \frac{k_t a \varepsilon}{1 - k_t^2} = 1 - k_t \varepsilon + (1 - k_t^2) \frac{k_t^2 a \varepsilon}{1 - k_t^2} = 1$.

Note that the average size of the items accepted by OPT is $\frac{1}{a}$, consistent with the advice.

Alg accepts one item in each iteration of the while-loop, except the last iteration, and at most $\frac{1}{a} - k_t$ items after that, so no more than

$$k_t - \left\lfloor \frac{1}{ae} \right\rfloor + \frac{1}{a} - k_t < \frac{1}{a} - \frac{1}{ae} + 1 = \frac{e - 1}{e} \cdot \frac{1}{a} + 1.$$

Thus, \( \text{Alg}(\sigma) \leq \frac{e - 1}{e} \cdot \frac{1}{a} + 1 = \frac{e - 1}{e} \text{OPT}(\sigma) + 1 \).

Case 2:

OPT accepts the $\frac{1}{a}$ items of size $a$.

For the analysis of Alg, we start by establishing an upper bound on $k_t$. The following inequality holds since Alg accepts one item per round, and Alg’s level just before the last round is at most $1 - \frac{1}{k_t} - k_t \varepsilon$ before the last item of size $\frac{1}{k_t} - \varepsilon$ is accepted.

$$\sum_{k=\left\lfloor \frac{1}{ae} \right\rfloor}^{k_t} \left( \frac{1}{k} - \varepsilon \right) \leq 1 - (k_t + 1) \varepsilon$$

\[ \Downarrow \]

$$H_{k_t} - H_{\left\lfloor \frac{1}{ae} \right\rfloor} - k_t \varepsilon < 1 - k_t \varepsilon$$

\[ \Downarrow \]

$$H_{k_t} - H_{\left\lfloor \frac{1}{ae} \right\rfloor} < 1$$

\[ \Downarrow \]

$$\ln(k_t) - \ln \left( \frac{1}{ae} \right) < 1, \text{ simple calculations, detailed in Lemma 1}$$

\[ \Downarrow \]

$$k_t < e \left\lfloor \frac{1}{ae} \right\rfloor$$

In the case we are treating, Alg leaves the while-loop because its level is more than $1 - \frac{1}{k_t} - (k_t + 1) \varepsilon$. Now, we give a bound on the amount of space available at that point. For the first inequality, note that by the initialization of $k$ in the algorithm, $k_t \geq \left\lfloor \frac{1}{ae} \right\rfloor$.

$$\frac{1}{k_t} + (k_t + 1) \varepsilon < \frac{1}{\left\lfloor \frac{1}{ae} \right\rfloor + 1} + \left( e \left\lfloor \frac{1}{ae} \right\rfloor + 1 \right) \varepsilon < ae + \left( \frac{1}{a} + 1 \right) \frac{a^2}{10}$$

$$< \left( e + \left( \frac{1 + a}{10} \right) \right) a < 3a$$
Thus, after the while-loop, Alg can accept at most two of the items of size \( a \). Clearly, the number of rounds in the while-loop is \( k_t = \lfloor \frac{1}{ae} \rfloor + 1 \). Using \( k_t < e \lfloor \frac{1}{ae} \rfloor < \frac{1}{a} \), we can now bound Alg’s profit:

\[
\text{Alg}(\sigma) \leq k_t - \lfloor \frac{1}{ae} \rfloor + 1 + 2 < (e - 1) \lfloor \frac{1}{ae} \rfloor + 3 \leq \frac{e - 1}{e} a + 3 = \frac{e - 1}{e} \text{OPT}(\sigma) + 3
\]

This establishes the bound on the competitive ratio of \( \frac{e-1}{e} \).

Finally, to ensure that our proof is valid, we must argue that the number of rounds we count in the algorithm and the sizes of items we give are non-negative. For the remainder of this proof, we go through the terms in the algorithm, thereby establishing this.

The largest value of \( k \) in the algorithm is \( k_t \), and we have established that \( k_t < e \lfloor \frac{1}{ae} \rfloor < \frac{1}{a} \). Additionally, from the start value of \( k \), we know that \( \lfloor \frac{1}{ae} \rfloor \leq k \). Using these facts, together with the assumption from the algorithm that \( a < \frac{1}{2e} \), we get the following bounds on the various terms.

\[
1 - \frac{1}{k} - k \varepsilon > 1 - \frac{1}{\lfloor \frac{1}{ae} \rfloor} - \frac{a^2}{10} > 1 - \frac{1}{\lfloor \frac{1}{2e} \rfloor} - \frac{1}{20e} > 0
\]

Further, \( \frac{1}{k} - \varepsilon \geq \frac{1}{a} - \varepsilon > \frac{1}{2} - a^2/10 > 0 \) and \( \frac{1}{a} - k \geq \frac{1}{a} - k_t > \frac{1}{a} - \frac{1}{a} = 0 \).

For the last relevant value, \( 1 - ka \geq 1 - k_t a > 1 - \frac{1}{a} a = 0 \) and from Case 1, we know that the \( \frac{1}{a} - k_t \) items given in Line 9 of the algorithm sum up to at most one.

4 Untrusted Predictions

For the case where the predictions may be inaccurate, the algorithm AT can be used with \( \hat{a} \) possibly not being \( a \) as long as \( r < e \), see Subsection 4.1. In Subsection 4.2, we give an adaptive threshold algorithm, AT\text{up}, that works for all \( r \).

For \( r < \frac{1}{2} (e + \sqrt{e^2 - 2e}) \approx 2.06 \), AT has a better competitive ratio than AT\text{up}. Thus, if an upper bound on \( r \) of approximately 2 (or lower) is known, AT may be preferred, and if a guarantee for any \( r \) is needed, AT\text{up} should be used.

4.1 Semi-Trusted Predictions

In this section, we consider the algorithm AT with a semi-trusted (being guaranteed that \( r < e \)) prediction, \( \hat{a} \), instead of \( a \).

4.1.1 Positive Result

In this section, we consider the algorithm AT with \( \hat{a} \) instead of \( a \). Note that the lower bound of the theorem below is positive only when \( r < e \). For \( r \geq e \), the algorithm may not accept any items, and, hence, its competitive ratio is 0.

\[ c_{\text{AT}}(r) \begin{cases} \frac{e-1}{r}, & \text{if } r \leq 1 \\ \frac{e}{r}, & \text{if } r \geq 1 \end{cases} \]

Proof. The proof is analogous to the proof of Theorem 3.
In Case 1, replacing $a$ by $\hat{a}$, since the algorithm bases its actions on $\hat{a}$ instead of $a$, and setting $\text{OPT} = \frac{1}{ra}$, results in a ratio of
\[
\frac{AT(\sigma)}{\text{OPT}(\sigma)} \geq \frac{e - 1}{e} \cdot r
\]
instead of $\frac{e - 1}{e}$.

In Case 2, the lower bound on $s$ given in Ineq. (1) depends on the actual average size, $a$, whereas the value of $i_t$ given in Eq. (2) depends on $\hat{a}$, since the algorithm uses $\hat{a}$. The subcase distinction is still based on $a$.

In Subcase 2a, we obtain
\[
\frac{AT(\sigma)}{\text{OPT}(\sigma)} \geq \left( \frac{1}{r(i_t + 1)} - \frac{1}{ae} \right) \ell + s \geq \left( \frac{1}{T(i_t + 1)} \right) term. \]

Going through the same calculations as in the proof of Theorem 3, we get that
\[
\frac{1}{T(i_t + 1)} \geq \ell.
\]

In subcase 2b, we obtain
\[
\frac{AT(\sigma)}{\text{OPT}(\sigma)} \geq \left( \frac{1}{r(i_t + 1)} \right) term. \]

Thus, we obtain a lower bound of $\frac{e - 1}{e} \cdot r$ in Case 1 and a lower bound of $\frac{e - 1}{e} \cdot r$ in Case 2.

For $r \leq 1$, $\frac{e - 1}{e} \cdot r \leq \frac{r - e}{e}$, and for $r \geq 1$, $\frac{e - 1}{e} \cdot r \geq \frac{e - r}{e}$.

### 4.1.2 Negative Result

The following result shows that, for $r < e$, no algorithm can be better than $AT$ for both $r < 1$ and $r > 1$.

> **Theorem 6.** If an algorithm is $\frac{e - r}{e}$-competitive for all $1 \leq r < e$, it cannot be better than $r \cdot \frac{e - 1}{e}$-competitive for any $r \leq 1$. If an algorithm is better than $r \cdot \frac{e - 1}{e}$-competitive for some $r \leq 1$, it cannot be $\frac{e - r}{e}$-competitive for all $1 \leq r < e$.

**Proof.** Consider an algorithm, $\text{ALG}$.

Assume that $\text{ALG}$ is $\frac{e - r}{e}$-competitive for all $1 \leq r < e$. Then there exists a constant, $b$, such that $\text{ALG}(\sigma) \geq \frac{e - r}{e} \cdot \text{OPT}(\sigma) - b$, for any sequence $\sigma$ and any $1 \leq r < e$. This constant $b$ is given as a parameter to Algorithm 4, constructing an adversarial sequence, $\sigma$.

Let $k_t$ be the value of $k$ at the end of the last iteration of the while-loop.

If the adversarial algorithm terminates in Line 9, then $\text{ALG}$ has accepted at most $k_t - \lfloor \frac{1}{ae} \rfloor - b - 1$ items. For termination in Line 9, $\text{ALG}$ has not accepted any of the $k_t$ items in
Algorithm 4 Adversarial sequence for $r < e$. The adversarial algorithm takes two parameters, $r_2 < 1$ and $b \geq 0$.

1. $k \leftarrow \left\lfloor \frac{1}{r_2} \right\rfloor - 1$
2. while ALG’s level $\leq 1 - \left\lfloor \frac{1}{r_2} \right\rfloor -(b+1)\hat{a}$ and $\frac{1}{r_2} \geq \hat{a}$ do
3. $k++$
4. for $k$ times do
5. Give an item of size $\frac{1}{k}$
6. if ALG accepts then
7. continue (* the while-loop *)
8. if ALG has accepted fewer than $k - \left\lfloor \frac{1}{r_2} \right\rfloor - b$ items then
9. terminate
10. Give $\frac{1}{r_2}$ items of size $r_2\hat{a}$

the for-loop immediately preceding this, so $k_t$ items of size $\frac{1}{k_t}$ were given. In this case, $\text{OPT}$ accepts exactly these $k_t$ items from the last iteration of the while-loop, and $a = \frac{1}{k_t}$. Let $r_1 = a/\hat{a}$. Since $\frac{1}{k_t} \geq \hat{a}$, $r_1 \geq 1$. Then,

$$\text{ALG}(\sigma) \leq k_t - \left\lfloor \frac{1}{\hat{a}e} \right\rfloor - b - 1 < \text{OPT} - \frac{1}{\hat{a}e} - b = \text{OPT} - \frac{r_1}{\hat{a}e} - b = \text{OPT} - \frac{r_1}{e} \text{OPT} - b$$

contradicting that $\text{ALG}(\sigma) \geq \frac{e-r_1}{e} \text{OPT}(\sigma) - b$. Thus, the adversarial algorithm cannot terminate in Line 9.

Since the adversarial algorithm does not terminate in Line 9, it must accept its first item no later than in the $(b+2)$nd iteration of the while-loop, and the $i$th item accepted by ALG has size at least $\frac{1}{\frac{1}{r_2}+(b+1)} \geq \frac{1}{\frac{1}{r_2}+b+i}$. Thus, the total size, $S_t$, of the items accepted by ALG in the while-loop is

$$S_t \geq \sum_{k=\frac{1}{r_2}+b+1}^{k_t} \frac{1}{k}$$

$$= \sum_{k=1}^{k_t} \frac{1}{k} - \sum_{k=1}^{\frac{1}{r_2}+b} \frac{1}{k} \geq H_{k_t} - H_{\frac{1}{r_2}+b} - (b+1)\hat{a}e$$

$$\geq \ln(k_t) - \ln \left( \frac{1}{\hat{a}e} \right) - (b+1)\hat{a}e, \text{ by Lemma 1.}$$

Thus, we have

$$S_t > \ln(k_t) - \ln \left( \frac{1}{\hat{a}e} \right) - (b+1)\hat{a}e. \quad (3)$$

By the first condition of the while-loop, and since ALG accepts at most one item per iteration, $S_t \leq 1 - (b+1)\hat{a}e$. By Ineq. (3), this means that $\ln(k_t) - \ln \left( \frac{1}{\hat{a}e} \right) - (b+1)\hat{a}e < 1 - \frac{1}{k_t} - (b+1)\hat{a}e$,
and we get
\[
\ln(k_t) - \ln\left(\frac{1}{\hat{a}e}\right) - (b + 1)\hat{a}e < 1 - (b + 1)\hat{a}e
\]
\[\Downarrow\]
\[
\ln(k_t) - \ln\left(\frac{1}{\hat{a}e}\right) < 1
\]
\[\Downarrow\]
\[
\ln\left(\frac{k_t}{\hat{a}e}\right) < 1
\]
\[\Downarrow\]
\[
\frac{k_t}{\hat{a}e} < e
\]
\[\Downarrow\]
\[
k_t < \frac{1}{\hat{a}}.
\] (4)

Furthermore, by the conditions of the while-loop, we have that
\[
S_t > 1 - \frac{1}{k_t + 1} - (b + 1)\hat{a}e
\]
or
\[
\frac{1}{k_t + 1} < \hat{a}.
\]
If
\[
S_t > 1 - \frac{1}{k_t + 1} - (b + 1)\hat{a}e,
\]
then, using that \(k_t \geq \left\lfloor \frac{1}{\hat{a}e} \right\rfloor\),
\[
S_t > 1 - \frac{1}{\left\lfloor \frac{1}{\hat{a}e} \right\rfloor + 1} - (b + 1)\hat{a}e > 1 - \frac{1}{\hat{a}e} - (b + 1)\hat{a}e = 1 - \hat{a}e - (b + 1)\hat{a}e = 1 - (b + 2)\hat{a}e.
\]
Otherwise, we get
\[
\frac{1}{k_t + 1} < \hat{a}
\]
\[\Downarrow\]
\[
k_t > \frac{1}{\hat{a}} - 1.
\] (5)

Plugging this into Ineq. (3), we get
\[
S_t > \ln\left(\frac{1}{\hat{a}} - 1\right) - \ln\left(\frac{1}{\hat{a}e}\right) - (b + 1)\hat{a}e
\]
\[
= \ln\left(\frac{\hat{a} - 1}{\hat{a}e}\right) - (b + 1)\hat{a}e
\]
\[
= \ln\left(e - \hat{a}e\right) - (b + 1)\hat{a}e
\]
\[
= 1 + \ln(1 - \hat{a}) - (b + 1)\hat{a}e
\]
\[
> 1 - (b + 2)\hat{a}e, \text{ since } \hat{a} < \frac{1}{2}.
\]

Thus, in either case, we get \(S_t > 1 - (b + 2)\hat{a}e\). Therefore, the algorithm can fit at most
\[
\frac{(b+2)\hat{a}e}{r_2\hat{a}} = \frac{(b+2)e}{r_2} \text{ of the items of size } r_2\hat{a} \text{ into its knapsack. Since ALG packs at most one}
item per iteration of the while-loop, this means that

\[
\text{ALG}(\sigma) \leq k_t - \frac{1}{\hat{a}c} + 1 + \frac{(b + 2)e}{r_2} \\
< k_t - \frac{1}{\hat{a}c} + 2 + \frac{(b + 2)e}{r_2} \\
< \frac{1}{\hat{a}} - \frac{1}{\hat{a}c} + 2 + \frac{(b + 2)e}{r_2}, \text{ by Ineq. (4)} \\
= \frac{e - 1}{\hat{a}c} + 2 + \frac{(b + 2)e}{r_2} \\
= r_2 \frac{e - 1}{e} \text{OPT}(\sigma) + 2 + \frac{(b + 2)e}{r_2}.
\]

For any \( r < 1 \), this yields an upper bound on the competitive ratio of \( r \frac{e - 1}{e} \), since for any given \( r, 2 + \frac{(b + 2)e}{r} \) is a constant.

This proves the first part of the theorem. The second part of the theorem is just the contrapositive of the first part.

Combining the positive result from Theorem 5 with the negative result from Theorem 6, we obtain that, if \( r \) is guaranteed to be smaller than \( e \), no algorithm can be better than AT for both \( r < 1 \) and \( r > 1 \).

\begin{theorem}
AT has a competitive ratio of

\[
c_{AT}(r) = \begin{cases} 
    r \cdot \frac{e - 1}{e}, & \text{if } r \leq 1 \\
    \frac{e - r}{e}, & \text{if } 1 \leq r \leq e \\
    0, & \text{if } r \geq e
\end{cases}
\]

\end{theorem}

\textbf{Proof.} The lower bounds follow from Theorem 5.

The upper bound for \( 1 \leq r < e \) follows from Theorem 6. Since AT is \( \frac{e - r}{e} \)-competitive for \( 1 \leq r < e \), the upper bound for \( r < 1 \) also follows from Theorem 6.

For \( r > e \), consider the input sequence consisting of \( \frac{1}{r\hat{a}} \) items of size \( \hat{a} \). \text{OPT} accepts all items and AT accepts none.

\subsection{Untrusted Predictions}

\subsection*{4.2 Positive Result}

When considering the case where the average item size is estimated to be \( \hat{a} \), and the accurate value is \( a = r \cdot \hat{a} \), we consider two cases, \( r > 1 \) and \( r < 1 \). In either case, we have the problem that we do not even know which case we are in, so, when large items arrive, we have to accept some to be competitive. The algorithm we consider when the value of \( r \) is not necessarily one achieves similar competitive ratios in both cases. Algorithm 5, AT\textit{up}, is \textsc{Adaptive Threshold} with a different threshold function than was used for accurate advice (and in AT).

Since we need to accept larger items than in the case of accurate advice, we need a threshold function that decreases faster than the threshold function used in Section 3, in order not to risk filling up the knapsack before the small items arrive. Therefore, it may seem surprising that we are using a threshold function that decreases as \( \frac{1}{\sqrt{i}} \), when the threshold
Algorithm 5: Adaptive Threshold for Untrusted Predictions, ATup.

1: Define $T(i) = \sqrt{\frac{\hat{a}}{2i}}$ for $i \geq 1$
2: Run Adaptive Threshold, Algorithm 1

function of Section 3 decreases as $\frac{1}{T(i)}$. However, the $\frac{1}{T(i)}$ function of the algorithm for accurate advice is essentially offset by $\frac{1}{ar}$. We prove a number of more or less technical results before stating the positive results for $r \leq 1$ (Theorem 14) and $r \geq 1$ (Theorem 13).

Lemma 8. For any $k \geq 1$, the total size of the $k$ largest items accepted by ATup is at most $\sqrt{2k\hat{a}}$.

Proof. By the test in ATup, as soon as $i$ items of size greater than $T(i+1)$ have been accepted, no more items larger than $T(i+1)$ are accepted after that. Thus, for each $i \geq 0$, at most $i$ items of size greater than $\sqrt{\frac{\hat{a}}{2(i+1)}}$ are accepted. This means that the $i$th largest item accepted by ATup has size at most $\sqrt{\frac{\hat{a}}{2i}}$. Thus, the total size of the $k$ largest accepted items is bounded by

$$\sum_{i=1}^{k} \sqrt{\frac{\hat{a}}{2i}} \leq \sqrt{\frac{\hat{a}}{2}} \int_{0}^{k} \frac{1}{\sqrt{i}} di = \sqrt{\frac{\hat{a}}{2}} \cdot 2\sqrt{k} = \sqrt{2k\hat{a}},$$

since $f(i) = \frac{1}{\sqrt{i}}$ is a decreasing function. ▶

Corollary 9. If ATup rejects an item based on the level being too high, it has accepted at least $\lfloor \frac{1}{2\hat{a}} \rfloor$ items.

Proof. If ATup has accepted $k$ items when it receives an item with a size no larger than the current bound, $T(i+1)$, that does not fit in the knapsack, then by Lemma 8, $\sqrt{2(k+1)\hat{a}} > 1$. Now,

$$\sqrt{2(k+1)\hat{a}} > 1 \Leftrightarrow k > \frac{1}{2\hat{a}} - 1 \Rightarrow k \geq \left\lfloor \frac{1}{2\hat{a}} \right\rfloor.$$

The following corollary implies that ATup never rejects an item based on the level being too high if $r > 2$. This is because $r > 2$ means that the items in Opt are relatively large compared to $\hat{a}$. Since Opt accepts the smallest items of the sequence, it means that the sequence contains relatively few small items. Thus, the algorithm reserves space for small items that never arrive.

Corollary 10. If ATup rejects an item based on the level being too high, ATup($\sigma$) > $\frac{r}{2}$ Opt($\sigma$) - 1.

Proof. By Corollary 9,

$$\text{ATup}(\sigma) > \frac{1}{2\hat{a}} - 1 = \frac{r}{2} \cdot \frac{1}{r\hat{a}} - 1 \geq \frac{r}{2} \text{ Opt}(\sigma) - 1.$$
Lemma 11. Assume that \( r, \hat{a}, q > 0, i \geq 0, \) and \( \ell < \sqrt{\frac{2(i+1)}{a}} \). If

\[
2r\sqrt{2\hat{a}(i+1)} - 2r\hat{a}(i+1) + q - 2 \leq 0,
\]

then

\[
\frac{(i+1) + \left(\frac{1}{r\sqrt{2\hat{a}(i+1)}} - 1\right)\ell}{r\sqrt{2\hat{a}(i+1)}} > \frac{q}{2}.
\]

Proof.

\[
2r\sqrt{2\hat{a}(i+1)} - 2r\hat{a}(i+1) + q - 2 \leq 0
\]
\[
\Downarrow
\]
\[
2r\sqrt{2\hat{a}(i+1)} - 2 + q \leq 2r(i+1)\hat{a}
\]
\[
\Downarrow
\]
\[
\sqrt{\frac{2(i+1)}{a}}(2r\sqrt{2\hat{a}(i+1)} - 2 + q) \leq 2r(i+1)\sqrt{2\hat{a}(i+1)}
\]
\[
\Downarrow
\]
\[
\ell(2r\sqrt{2\hat{a}(i+1)} - 2 + q) < 2r(i+1)\sqrt{2\hat{a}(i+1)}, \text{ since } \ell < \sqrt{\frac{2(i+1)}{a}}
\]
\[
\Downarrow
\]
\[
\ell q < 2r(i+1)\sqrt{2\hat{a}(i+1)} + 2\ell - 2\ell r\sqrt{2\hat{a}(i+1)}
\]
\[
\Downarrow
\]
\[
\frac{\ell q}{\sqrt{2\hat{a}(i+1)}} < 2r(i+1) + \frac{2\ell}{\sqrt{2\hat{a}(i+1)}} - 2\ell r
\]
\[
\Downarrow
\]
\[
\frac{\ell q}{\sqrt{2\hat{a}(i+1)}} < 2r(i+1) + \frac{\ell}{r\sqrt{2\hat{a}(i+1)}} - \ell
\]
\[
\Downarrow
\]
\[
\frac{q}{2} < \frac{(i+1) + \left(\frac{1}{r\sqrt{2\hat{a}(i+1)}} - 1\right)\ell}{r\sqrt{2\hat{a}(i+1)}}
\]

Lemma 12. Assume that \( \text{Opt} \) accepts \( \ell \) items larger than \( \sqrt{\frac{a}{2(i+1)}} \) and \( s \) items of size at most \( \sqrt{\frac{a}{2(i+1)}} \), \( i \geq 0 \). Then, the following inequalities hold:

1. \( s > \ell \left(\frac{1}{r\sqrt{2\hat{a}(i+1)}} - 1\right) \)
2. \( \ell < \sqrt{\frac{2(i+1)}{a}} \)

Proof. Since \( \text{Opt} \)'s accepted items have average size \( a \), we have that

\[
r\hat{a} = a > \frac{\ell \cdot \sqrt{\frac{a}{2(i+1)}} + s \cdot 0}{\ell + s},
\]

and, equivalently,

\[
s > \ell \left(\frac{1}{r\sqrt{2\hat{a}(i+1)}} - 1\right).
\]
In addition, since $\text{OPT}$ accepts $\ell$ items larger than $\sqrt{\frac{\hat{a}}{2(i_t+1)}}$, $\ell \sqrt{\frac{\hat{a}}{2(i_t+1)}} < 1$, so

$$\ell < \sqrt{\frac{2(i_t+1)}{\hat{a}}}.$$ 

$\blacktriangleleft$

**Theorem 13.** For all request sequences $\sigma$, such that $r \geq 1$,

$$\text{ATup}(\sigma) \geq \frac{1}{2r} \text{OPT}(\sigma) - 1.$$ 

**Proof.** By Corollary 10, if $\text{ATup}$ rejects an item in $\sigma$ due to the knapsack not having room for the item, $\text{ATup}(\sigma) \geq \frac{r}{2} \text{OPT}(\sigma) - 1 \geq \frac{1}{2} \text{OPT}(\sigma) - 1$ for $r \geq 1$.

Now, suppose that $\text{ATup}$ does not reject any item due to it not fitting in the knapsack. If $\text{ATup}$ is not optimal, it must reject due to the size of the item.

Let $i_t$ denote the final value of $i$ when the algorithm is run. This means that $\text{ATup}$ has accepted $i_t$ items of size greater than $\sqrt{\frac{\hat{a}}{2(i_t+1)}}$. We perform a case analysis based on whether this value is smaller or larger than $r\hat{a}$.

**Case 1:** $r \geq \frac{1}{\sqrt{2\hat{a}(i_t+1)}}$

In this case, $i_t + 1 \geq \frac{1}{2r\hat{a}}$ and $\text{OPT}(\sigma) \leq \frac{1}{r\hat{a}} \leq \sqrt{\frac{2(i_t+1)}{\hat{a}}}$. Thus,

$$\frac{\text{ATup}(\sigma) + 1}{\text{OPT}(\sigma)} \geq \frac{i_t + 1}{\sqrt{\frac{2(i_t+1)}{\hat{a}}}} = \sqrt{\frac{\hat{a}(i_t + 1)}{2}} \geq \sqrt{\frac{\hat{a}}{2r^2\hat{a}}} = \frac{1}{2r},$$

Therefore, $\text{ATup}(\sigma) \geq \frac{1}{2r} \text{OPT}(\sigma) - 1$.

**Case 2:** $r < \frac{1}{\sqrt{2\hat{a}(i_t+1)}}$

Suppose $\text{OPT}$ accepts $\ell$ items larger than $\sqrt{\frac{\hat{a}}{2(i_t+1)}}$ and $s$ items of size at most $\sqrt{\frac{\hat{a}}{2(i_t+1)}}$. Note that $\text{ATup}$ also accepts the $s$ items of size at most $\sqrt{\frac{\hat{a}}{2(i_t+1)}}$, since we are in the case where it does not reject items because of the knapsack being too full.

Given the input sequence $\sigma$, we consider the ratio

$$\frac{\text{ATup}(\sigma) + 1}{\text{OPT}(\sigma)} \geq \frac{(i_t + 1) + s}{\ell + s}.$$ 

The result follows if this ratio is always at least $\frac{1}{2r}$.

**Subcase 2a:** $i_t + 1 \geq \frac{1}{2\hat{a}}$

In this case, $\text{ATup}(\sigma) \geq i_t \geq \frac{1}{2\hat{a}} - 1$, while $\text{OPT}(\sigma) \leq \frac{1}{r\hat{a}}$. Thus, $\text{ATup}(\sigma) \geq \frac{1}{2} \text{OPT}(\sigma) - 1$. 

Subcase 2b: $i_t + 1 < \frac{1}{2^a}$

By Ineq. 1 of Lemma 12, and since $\frac{AT_{\text{up}}(\sigma) + 1}{\text{OPT}(\sigma)} \leq 1,$

$$\frac{AT_{\text{up}}(\sigma) + 1}{\text{OPT}(\sigma)} \geq \frac{(i_t + 1) + s}{\ell + s} \geq \frac{(i_t + 1) + \left(\frac{1}{r\sqrt{2^a(i_t + 1)}} - 1\right)\ell}{r\sqrt{2^a(i_t + 1)}}.$$  

We will show that this is at least $\frac{1}{2^a}.$

From our case conditions, $i_t + 1 < \frac{1}{2^a}$ and $1 \leq r < \frac{1}{\sqrt{2^a(i_t + 1)}},$ we get that $\frac{1}{r^2} > 2^a(i_t + 1)$ and $0 < 2^a(i_t + 1) < 1.$ Consider the function

$$f(r) = 2r\sqrt{2^a(i_t + 1)} - 2r^a(i_t + 1) + \frac{1}{r} - 2.$$  

Taking the derivative with respect to $r$ gives

$$f'(r) = 2\sqrt{2^a(i_t + 1) - 2r^a(i_t + 1) - \frac{1}{r^2}}.$$  

Setting this equal to zero and solving for $r,$ we find

$$r^* = \frac{1}{\sqrt{2^a(i_t + 1) - 2r^a(i_t + 1)}}.$$  

The possible maximum value for $f(r)$ in the range for $r$ is then at $1,$ $r^*,$ or $\frac{1}{\sqrt{2^a(i_t + 1)}}.$ For all three values, $f(r) \leq 0.$ The hardest (but still simple) case is for $r = r^*,$ where

$$f(r^*) = \frac{2\sqrt{v} - v}{\sqrt{2\sqrt{v} - v}} + \sqrt{2\sqrt{v} - v} - 2,$$  

where we let $v$ denote $2^a(i_t + 1).$ Note that due to the subcase we are in, $0 < v < 1.$ Now,

$$\frac{2\sqrt{v} - v}{\sqrt{2\sqrt{v} - v}} + \sqrt{2\sqrt{v} - v} - 2 \leq 0$$

$$\Downarrow$$

$$4\sqrt{v} - 2v - 2 \leq 0$$

$$\Downarrow$$

$$2\sqrt{v} \leq v + 1$$

$$\Downarrow$$

$$4v \leq v^2 + 2v + 1$$

$$\Downarrow$$

$$0 \leq (v - 1)^2$$

which clearly holds.

By Ineq. 2 of Lemma 12, the result now follows from Lemma 11 with $q = \frac{1}{r}.$

\[\text{Theorem 14.} \quad \text{For all request sequences } \sigma, \text{ such that } r < 1, \]

$$AT_{\text{up}}(\sigma) \geq \frac{r}{2} \text{OPT}(\sigma) - 1.$$
Proof. The proof follows that of the previous theorem.

Case 1. $i_t + 1 \geq \frac{1}{2r}$.
Since $\ell \geq i_t + 1$ (otherwise ATup is optimal), ATup has accepted at least $\frac{1}{2r} - 1$ items, while Opt can accept at most $\frac{1}{2r}$. Thus, $\text{ATup}(\sigma) \geq \frac{r}{2} \cdot \frac{1}{2r} - 1 \geq \frac{r}{2} \text{Opt}(\sigma) - 1$.

Case 2. $i_t + 1 < \frac{1}{2r}$.
By Ineq. 1 of Lemma 12 and since $\frac{\text{ATup}(\sigma) + 1}{\text{Opt}(\sigma)} \leq 1$,
\[
\frac{\text{ATup}(\sigma) + 1}{\text{Opt}(\sigma)} \geq \frac{(i_t + 1) + s}{\ell + s} \geq \frac{1}{\ell} \left( 1 - \frac{r}{2\sqrt{2a_i(t+1)}} - 1 \right). 
\]
We will show that this is at least $\frac{r}{2}$. Consider the function
\[
f(r) = 2r\sqrt{2a_i(t+1)} - 2r\hat{a}(i_t + 1) + r - 2. 
\]
Taking the derivative with respect to $r$ gives
\[
f'(r) = 2\sqrt{2a_i(t+1)} - 2\hat{a}(i_t + 1) + 1, 
\]
which is positive, since by the case condition, $0 < 2\hat{a}(i_t + 1) < 1$. Thus, $f(r)$ is an increasing function for the values of $\hat{a}$, $i_t + 1$, and $r$ considered in this case, so the maximum value is at the maximum value of $r$, $r = 1$, giving that
\[
f(r) \leq 2r\sqrt{2a_i(t+1)} - 2r\hat{a}(i_t + 1) + r - 2 < 0. 
\]
By Ineq. 2 of Lemma 12, the result now follows from Lemma 11 with $q = r$. ▶

4.2.2 Negative Result

In Section 3, we showed that, even with accurate advice, no algorithm can be better than $\frac{r}{2r}$-competitive. In this section, we give a trade-off in the competitive ratio attained for different values of $r$.

Theorem 15. Let $0 < z \leq 2$. No algorithm can have a competitive ratio better than $\frac{1}{2r}$ for every $r$ between $\frac{z}{4}$ and $\frac{1}{2r}$. Moreover, any algorithm which is $\frac{1}{2r}$-competitive for all $r$ in this interval has a competitive ratio of at most $\frac{z}{4}$, for any positive $r < \frac{2}{z}$.

Proof. We consider the adversary that gives the input sequence $\sigma_z$ defined by Algorithm 6. We begin with the second part of the theorem. Consider an online algorithm, Alg, and assume that there exists a constant, $b$, such that $\text{Alg}(\sigma) \geq \frac{1}{2r} \text{Opt}(\sigma) - b$, for any sequence $\sigma$ and any $r$ such that $\frac{z}{4} \leq r \leq \frac{1}{\sqrt{2z}}$. Now, consider the adversary that gives the input sequence $\sigma_z$ defined by Algorithm 6.

If the adversarial algorithm terminates in Line 6, then, Alg has accepted at most $k - b - 1$ items. In this case, $a = \sqrt{\frac{a}{2z}}$, and Opt accepts exactly the $\left\lfloor \sqrt{\frac{2k}{z}} \right\rfloor$ items from the last iteration of the while-loop. Since $a = r\hat{a}$, $r = \sqrt{\frac{1}{z \hat{a}}}$, which lies between $\sqrt{\frac{1}{2\hat{a}}} \geq \sqrt{\frac{1}{2\hat{a}}} \cdot \frac{4\hat{a}}{z} = \frac{2}{z}$ and $\frac{1}{\sqrt{2z}}$. Thus,
\[
\text{Alg}(\sigma_z) \leq k - b - 1 \leq \left\lfloor \frac{k - 1}{\sqrt{\frac{2k}{z}}} \right\rfloor \text{Opt}(\sigma_z) - b < \frac{k - 1}{\sqrt{\frac{2k}{z}}} \text{Opt}(\sigma_z) - b
\]
\[
< \frac{k - 1}{\sqrt{\frac{2k}{z}}} \text{Opt}(\sigma_z) - b = \frac{1}{2r} \text{Opt}(\sigma_z) - b,
\]
Algorithm 6: Adversarial sequence establishing trade-off on robustness versus consistency. The adversarial algorithm takes parameters, $z$, $q$, and $b$, such that $0 < z \leq 2$, $0 < q < \frac{1}{\sqrt{z}}$, and $b \geq 0$.

1. $p \leftarrow \left\lfloor \frac{z}{2^b} \right\rfloor$
2. $k \leftarrow 0$
3. while $k \leq p - 1$ do
   4. $k++$
   5. Give $\left\lfloor \frac{2k}{b} \right\rfloor$ items of size $\frac{\hat{a}}{z}$
   6. if $\text{Alg}$ has accepted fewer than $k - b$ items then terminate
   7. Give $\frac{1}{q^a}$ items of size $q^a$

where the second strict inequality holds because 1 is added to the numerator and denominator of a positive fraction less than 1. This contradicts the assumption that for each $r$ between $\frac{2}{z}$ and $\frac{1}{\sqrt{z}}$, $\text{Alg}(\sigma) \geq \frac{1}{z} \text{Opt}(\sigma) - b$, for any sequence $\sigma$, when the adversarial algorithm terminates in Line 6. Thus, the adversarial algorithm does not terminate there.

If the adversarial algorithm does not terminate in Line 6, $r = q$ and $\text{Opt}(\sigma) = \frac{1}{q^a} = \frac{1}{r^a}$. Moreover, for $\text{Alg}$, the $i$th accepted item must have size at least $\frac{\hat{a}}{z(i+b)}$, for $1 \leq i \leq p - b$. Thus, these first $p - b$ items fill the knapsack to at least

$$\sum_{i=b+1}^{p} \frac{\hat{a}}{zi} \geq \frac{\hat{a}}{z} \int_{b+1}^{p+1} \frac{1}{\sqrt{i}} di = \frac{\hat{a}}{z} (2\sqrt{p+1} - 2\sqrt{b+1})$$

where we use that $\frac{1}{\sqrt{i}}$ is a decreasing function.

Since the items of size $r\hat{a}$ are the smallest items of the sequence, this means that

$$\text{Alg}(\sigma) \leq p + \frac{1 - \sqrt{z} (2\sqrt{p+1} - 2\sqrt{b+1})}{r\hat{a}}$$

$$\leq \frac{z}{4\hat{a}} + \frac{1 - \sqrt{\frac{a}{z}} (2\sqrt{\frac{z}{4\hat{a}}} - 2\sqrt{\frac{b+1}{z}})}{r\hat{a}}$$

$$= \frac{z}{4\hat{a}} + \frac{1 - 1 + 2\sqrt{\frac{a(b+1)}{z}}}{r\hat{a}}$$

$$= \frac{1}{r\hat{a}} \left( \frac{zr}{4} + 2\sqrt{\frac{a(b+1)}{z}} \right)$$

$$= \left( \frac{zr}{4} + 2\sqrt{\frac{a(b+1)}{z}} \right) \text{Opt}(\sigma)$$

As a function of $\hat{a}$, the lower bound is $\frac{zr}{4} + 2\sqrt{\frac{a(b+1)}{z}}$, but the second term becomes insignificant as $\hat{a}$ approaches zero.

To show that the algorithm cannot have a competitive ratio better than $\frac{1}{2r}$, for every $r$ between $\frac{2}{z}$ and $\frac{1}{\sqrt{z}}$, we consider Algorithm 6 with the item sizes on Line 5 equal to $\frac{\hat{a}}{z} + \varepsilon$. Following the proof above, in the case where the adversarial algorithm terminates in Line 6, $\text{Alg}$’s competitive ratio is at most $\frac{1}{2r}$, for $r$ in this range. However, for small enough $\hat{a}$, the adversarial algorithm must at some point terminate in Line 6, since otherwise the knapsack
would be over-filled: Similar to the calculations above, the first \( p - b \) items fill the knapsack to at least

\[
\sum_{i=b+1}^{p} \sqrt{\frac{\hat{a}}{z^i}} + \varepsilon \geq \sqrt{\frac{\hat{a}}{z}} \int_{b+1}^{p+1} \frac{1}{\sqrt{i}} di + (p - b)\varepsilon
\]

\[
= \sqrt{\frac{\hat{a}}{z}} \left( 2\sqrt{p+1} - 2\sqrt{b+1} \right) + (p - b)\varepsilon
\]

\[
\geq \sqrt{\frac{\hat{a}}{z}} \left( 2\sqrt{\frac{z}{4\hat{a}} - 1} - 2\sqrt{\frac{b}{4\hat{a}}} + 1 \right) + \left( \frac{z}{4\hat{a}} - 1 - b \right)\varepsilon
\]

\[
= 1 - 2\sqrt{b+1} + \left( \frac{z}{4\hat{a}} - 1 - b \right)\varepsilon.
\]

Since \( \hat{a} \) can be arbitrarily small, the last term can dominate the second term, giving a result larger than 1. The result of terminating in Line 6 is the same as above, \( \text{Alg}(\sigma_z) < \frac{1}{r} \text{Opt}(\sigma_z) - b \), giving a contradiction. ▶

Setting \( z = 2 \) in Theorem 15 demonstrates a Pareto-like trade-off between consistency and robustness for ATup:

▶ Corollary 16. No algorithm can have a competitive ratio better than \( \frac{1}{r} \) for every \( r \) between 1 and \( \frac{1}{\sqrt{2}} \). Moreover, any algorithm which is \( \frac{1}{r} \)-competitive for all \( r \) between 1 and \( \frac{1}{\sqrt{2}} \) has a competitive ratio of at most \( \frac{z}{2} \) for any positive \( r < 1 \).

5 Advice Complexity

In this section, we briefly consider the Online Unit Cost Knapsack Problem in terms of advice complexity, concentrating on upper bounds, following the techniques in [15] and many other articles on advice complexity including [4, 18]. One assumes that a certain number of bits are available to approximate actual values (that might not be small integers).

The advice given in the algorithms AT and ATup is a prediction for the value, \( a \), representing the average size of an item that Opt accepts, and it could have some error. One could use AT in the advice complexity setting, assuming that an oracle gives two values: \( z \), the number of zeros after the binary point in the binary representation of \( a \), followed by \( s \), the next \( k \) bits of \( a \). In this case, the prediction \( \hat{a} \) given for \( a \) is \( \frac{z}{2^{k+1}} \). (The numerator should be thought of as the value, e.g., if \( s \) is the bits 1101, the value is 13.) Since the high order bit of \( s \) is 1, this value is at least \( \frac{z}{2^k} \). The error in the prediction, \( \hat{a} \), is only due to the missing low order bits (assumed, possibly incorrectly, to be zero). The missing bits represent a number less than \( \frac{1}{2^k} \). Thus, the ratio, \( r \), in \( a = r\hat{a} \) is in the range \( 1 \leq r \leq 1 + \frac{1}{2^k} \).

By Theorem 5, we can use the algorithm AT (with the modification that it calculates \( \hat{a} = \frac{s}{2^{k+1}} \) after reading and decoding the advice) and obtain that for all \( \sigma \),

\[
\text{AT}(\sigma) \geq \frac{e - 1 - \frac{1}{2^k}}{e} \text{Opt}(\sigma).
\]

Note that the length of the advice is independent of the length of the request sequence, though dependent on the values in that sequence. The value, \( z \), and the bitstring, \( s \), must be specified using self-delimiting encoding, since we do not know how many bits are used for them. For example, \( \lfloor \log(z+1) \rfloor \) could be written in unary (\( \lfloor \log(z+1) \rfloor \) ones, followed by a zero) before writing \( z \) itself in binary. Treating \( s \) similarly, at most \( 2(k + \lfloor \log(z+1) \rfloor + 1) \) bits are used.
Since OPT can be viewed as accepting a prefix of the sequence of items sorted in non-decreasing order of size, there is another obvious type of advice to give. Let the advice be $k$-bit approximations to both the size of the largest item that OPT accepts, $s$, and the fraction, $t$, of the knapsack not filled with items of size strictly smaller than $s$. The approximation to $s$ can be given using the technique above, specifying the number of leading zeros first and then $k$ significant bits. For $t$, we do not use the count of leading zeros and simply use the $k$ most significant bits. There are two reasons that it is necessary to give the fraction of the knapsack not filled with items of at most this size. One reason is that, even if the exact value of $s$ was given, it is unknown if OPT accepts one or many items of that size, and these “large” items could come before any smaller ones. The other reason is that, since the size of this largest accepted item is rounded down, there may be many items that OPT accepts that are larger than this (though never any item as large as $s + \frac{1}{2^k}$). Thus, it can be necessary to accept many items larger than $s$, and we need to know how much space we can use for this, or if space should be saved for many items much smaller than $s$. The algorithm will accept all items that are smaller than $s$, which is the optimal behavior on those items (so in the worst case for the performance ratio, no such items arrive). Thus, we are only interested in items of size between $\frac{s}{2^k}$ and $\frac{s + 1}{2^k}$ and can calculate a bound on the competitive ratio just from the algorithm’s and OPT’s performance on items in that range. Since the algorithm does not accept all items in the worst case, we may assume that there are enough items in this size range that it rejects some. Under this assumption, the algorithm accepts at least $\left\lfloor \frac{t}{2^k} \right\rfloor$ and OPT accepts at most $\frac{t + 1}{2^k}$. For an asymptotic result, ignoring the rounding down on the algorithm’s performance, this gives a performance ratio of at least

$$\frac{t \cdot s}{(s + 1)(t + 1)} \geq \frac{2^k \cdot 2^{z+k}}{(2^k + 1)(2^{t+k} + 1)} \geq \frac{2^{2k}}{(2^k + 1)^2}.$$  

Since we approximate two values, we need twice as much advice as for the first approach, that is $4(k + \lceil \log(z + 1) \rceil + 1)$ bits of advice. The competitive ratio with this approach is better than that of the first approach, but it also uses more advice.

With respect to optimality, we note that the lower bound of $\log n$ from [15] for the general Knapsack Problem cannot be used directly here, since the items used in their sequences all have size 1, so the weights are very important. In contrast to the upper bounds proven above, we prove that for optimality, the number of advice bits needed is a function of $n$, at least $\log(n/3)$. Consider the set of input sequences defined to have length $n$ for which $j^* \neq j'$. If $j^* > j'$, then ALG can accept only $2(k - j')$ items of size $\frac{1}{2^k}$, completely filling up the knapsack. Intuitively, the advice needs to say how many of the first $k$ items to accept. Since there are $n/3$ sequences in all and fewer than $\log(n/3)$ bits of advice, there are at least two of the sequences $I_j$ and $I_{j'}$ for which ALG receives the same advice. Thus, ALG accepts the same number, say $j^*$, items of size $\frac{1}{2^k}$ on both $I_j$ and $I_{j'}$. Without loss of generality, assume that $j^* \neq j'$. If $j^* > j'$, then ALG can accept only $2(k - j^*)$ items of size $\frac{1}{2^k}$. In all, $\text{ALG}(I_{j'}) \leq j^* + 2(k - j^*) < 2k - j' = \text{OPT}(I_{j'}).$ If $j^* < j'$, then $\text{ALG}(I_{j'}) \leq j^* + 2(k - j') < 2k - j' = \text{OPT}(I_{j'})$. Thus, ALG is not optimal on $I_{j'}$, giving a contradiction.
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