Extended Conformal Symmetry
and
Recursion Formulae for Nekrasov Partition Function

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Abstract

We derive an infinite set of recursion formulae for Nekrasov instanton partition function for linear quiver $U(N)$ supersymmetric gauge theories in 4D. They have the structure of a deformed version of $W_{1+\infty}$ algebra which is called \textit{SH}\textsuperscript{c} algebra (or degenerate double affine Hecke algebra) in the literature. The algebra contains $W_N$ algebra with general central charge defined by a parameter $\beta$, which gives the $\Omega$ background in Nekrasov’s analysis. Some parts of the formulae are identified with the conformal Ward identity for the conformal block function of Toda field theory.

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1 Introduction

Nekrasov partition function [1] is an exact formula for the partition function of four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory, including non-perturbative instanton effects. It is calculated in a deformed four-dimensional Euclidean space, called $\Omega$-background, which is parameterized by two parameters $\epsilon_1, \epsilon_2$. At the same time, it was recognized that the partition function can be identified with the correlation function of two-dimensional conformal field theory. In a recent paper [2], the explicit form of such correspondence was proposed between $N = 2$ gauge theories and Liouville (Toda) conformal blocks (AGT conjecture). In AGT proposal, the instanton part of Nekrasov partition function is identified with the conformal block of $W$-algebra [3,4].

This article is in the line of this development. The instanton partition function for linear quiver gauge theories is decomposed into matrix like product with a factor $Z_{\vec{Y}, \vec{W}}$ which depends on two sets of Young diagrams (eq.2). Here the Young diagrams $\vec{Y} = (Y_1, \cdots, Y_N)$ represent the fixed points of $U(N)$ instanton moduli space under localization. $Z_{\vec{Y}, \vec{W}}$ consists of contributions from one bifundamental hypermultiplet and vectormultiplets. We find that the building block $Z_{\vec{Y}, \vec{W}}$ satisfies an infinite series of recursion relations,

$$\delta_{\pm 1, n} Z_{\vec{Y}, \vec{W}} - U_{\pm 1, n} Z_{\vec{Y}, \vec{W}} = 0,$$

where $\delta_{\pm 1, n} Z_{\vec{Y}, \vec{W}}$ represents a sum of the Nekrasov partition function with instanton number larger or less than $Z_{\vec{Y}, \vec{W}}$ by one with appropriate coefficients and $U_{\pm 1, n}$ are polynomials of parameters such as the mass of bifundamental matter or the VEV of gauge multiletis. The subscript $n$ takes any non-negative integer values. The detailed form of the recursion formula and its derivation are given in the first half of this paper. The recursion formula is derived by a complicated but straightforward calculation from the definition of the factor $Z_{\vec{Y}, \vec{W}}$. We note that a classical limit of such relations was recently explored in [5].

In the latter half of this article, we give an interpretation of (1). We show that the variation in (1) can be understood as an action of an infinite-dimensional extended conformal algebra. It is defined in [6] and called $\mathfrak{SH}^c$ algebra. For this purpose, we construct an explicit representation where the basis of the Hilbert space is labeled by sets of $N$ Young diagrams. Physically, it can be understood that these states correspond to instantons characterized by the same set of Young diagrams. In our previous paper [7], we showed a similar form of recursion formula under self-dual $\Omega$-background ($\epsilon_1 + \epsilon_2 = 0$) and discussed that it can be interpreted in terms of $\mathcal{W}_{1+\infty}$ algebra. The analysis here is a natural generalization to arbitrary $\Omega$-deformation. $\mathfrak{SH}^c$ algebra contains a parameter $\beta$, which is related to $\Omega$-deformation parameters by $\beta = -\epsilon_1/\epsilon_2$. When we take $\beta = 1$, (1) reduces to that in [7] and the action of $\mathfrak{SH}^c$ algebra can be identified with the $\mathcal{W}_{1+\infty}$ algebra. We will also see $\mathfrak{SH}^c$ algebra contains Heisenberg$\times$Virasoro subalgebra and its central charge is the same as that of $\text{Heisenberg} \times W_N$ algebra with background charge $Q = \sqrt{\beta} - 1/\sqrt{\beta}$. The combination of Heisenberg algebra with $W_N$ appears in [8–10], where the authors formally construct a basis of Hilbert space of $\text{Heisenberg} \times W_N$ algebra which reproduces the factorized form of Nekrasov partition function. Such observation implies that one may regard the formula (1) as the conformal Ward identities which characterize the conformal block function.

We mention that there is another one parameter deformation of $\mathcal{W}_{1+\infty}$ algebra [11], $W_\infty[\mu]$ in the context of higher spin supergravity. $\mathfrak{SH}^c$ and $W_\infty[\mu]$ share a property that they are generated by infinite higher spin generators and contain $W_N$ algebra with general $\beta$ as their reduction. Here we use $\mathfrak{SH}^c$ since its action on a basis parametrized by sets of Young diagram is already known. It is natural to expect that these two algebras are identical although they appear to be very different. It should also be noted that the introduction of further deformation parameter is possible [12–14] and was applied to a generalization of AGT conjecture [15].

As we will see later, we expect that the recursion relation from $\mathfrak{SH}^c$ algebra should be regarded as the extended

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1 This name of the algebra appears only in [6]. Degenerate double affine Hecke algebra, or DDAHA in short, may be more appropriate. We thank Y. Tachikawa for informing us of the relevance of [6].
conformal Ward identities and fully reproduce the conformal block function. Because of a technical difficulty to
characterize the vertex operator in $SH^c$, explicit demonstration of the relation is limited to the Heisenberg and
Virasoro subalgebra. For these cases, the recursion relations for $n = 0,1$ can be indeed interpreted as Ward identities.
The algebra $SH^c$ was introduced in [6] to prove the AGT conjecture for pure super Yang-Mills theory. Our analysis
shows that it may be applied to linear quiver gauge theories as well. For the recent development toward such
direction, see also [16].

The rest of this article is organized as follows. In section 2, we describe Nekrasov partition function for linear
quiver gauge theories. In section 3, we derive the recursion formula for Nekrasov partition function. In section 4, we
give the definition of $SH^c$ algebra and a representation of it. The relation to $W_{1+\infty}$ algebra at $\beta = 1$ is also discussed.
In section 5, we show $SH^c$ algebra contains Heisenberg$\times$Virasoro subalgebra and its central charge is equal to that of
Heisenberg$\times W_N$ algebra. In section 6, we discuss that Nekrasov partition function can be interpreted as a correlator
of $SH^c$ algebra. Especially, we explain the recursion formulae for $n = 0,1$ represent the $U(1)$ and Virasoro constraint
for Nekrasov partition function respectively. The vertex operator in the correlator should be chosen to be special
ones which have maximal number of null states at level 1. Since the calculations in this article are very lengthy
but straightforward, most of the detail are not presented. We nevertheless keep some outline of the computation in
Appendix for readers who are interested in the detail.

2 Nekrasov partition function

In this article, we consider four-dimensional $N = 2$ superconformal linear quiver gauge theory with $U(N) \times U(N) \times
\cdots \times U(N)$ gauge group. The instanton partition function of $N = 2$ gauge theories have been developed in [1,17–19].
In this case, it can be written in the following form

$$Z^{\text{Nek}} = \sum_{\vec{Y}(1), \ldots, \vec{Y}(n)} q_i^{\vec{Y}(i)} \vec{V}_{\vec{Y}(1)} \cdot Z_{\vec{Y}(1)} \vec{V}_{\vec{Y}(2)} \cdots Z_{\vec{Y}(n-1)} \vec{V}_{\vec{Y}(n)} \cdot V_{\vec{Y}(n)}. \quad (2)$$

where $q_i = \exp(2\pi i \tau_i)$ represents the complexified coupling constant $\tau_i$ of $i$-th $U(N)$ gauge group, and $\vec{Y}(i)$ is a set
of $N$ Young diagrams characterizing fixed points of localization in the instanton moduli space of the $i$-th $U(N)$. $\vec{a}(i)$
is the VEV for an adjoint scalar field in the vector multiplet of $i$-th $U(N)$ and $\mu(i)$ is the mass parameter for the
bifundamental matter field which interpolates $i$th and $i+1$th gauge groups. We write $\vec{b}$ to represent a set of null
Young diagrams $(\emptyset, \cdots, \emptyset)$.

The building block reads,

$$Z(\vec{a}, \vec{Y}; \vec{b}, W; \mu) = \frac{2\text{bf}}{\text{vect}} = \frac{1}{\left(\prod_{p,q=1}^N g_{Y_p W_q} (a_p - b_q - \mu) \right)^{1/2}}$$

where the numerator ($2\text{bf}$) comes from the contriibution of the bifundamental multiplet and the denominator ($\text{vect}$)
is the contribution from the vector multiplet to which the bifundamental multiplet couples to. The function $g_{Y,W}$ is

$$g_{Y,W}(x) = \prod_{(i,j) \in Y} (x + \beta(Y_j' - i + 1) + W_i - j) \prod_{(i,j) \in W} (-x + \beta(W_j' - i) + Y_i - j + 1). \quad (7)$$

The decomposition of the form (2) seems to be natural if we recall the pants decomposition of multi-point function
on a sphere and the dictionary of AGT relation; A bifundamental and a vector multiplet correspond to a vertex
operator insertion and an internal line respectively (see fig.1).
3 Recursion formula for Nekrasov partition function

In this section, we present the accurate form of the formula (1) and then derive it from the definition (6). For this purpose, we need to introduce some notations. We decompose $Y, W$ into rectangles $Y = (r_1, \ldots, r_f; s_1, \ldots, s_f)$ (with $0 < r_1 < \cdots < r_f$, $s_1 > \cdots > s_f > 0$, see Figure 2 for the parametrization). We use $f_p$ (resp. $\bar{f}_p$) to represent the number of rectangles of $Y_p$ (resp $W_p$). Furthermore, we write (with $r_0 = s_{k+1} = 0$):

\begin{align}
A_k(Y) &= \beta r_{k-1} - s_k - \xi, \quad (k = 1, \cdots, f+1), \quad (8) \\
B_k(Y) &= \beta r_k - s_k, \quad (k = 1, \cdots, f), \quad (9)
\end{align}

where $\xi := 1 - \beta$. $A_k(Y)$ (resp. $B_k(Y)$) represents the $k$th location where a box may be added to (resp. deleted from) the Young diagram $Y$ (Figure 3) composed with a map from location to \( C \).

We denote $Y^{(k,+)}$ (resp. $Y^{(k,-)}$) as the Young diagram obtained from $Y$ by adding (resp. deleting) a box at $(r_{k-1} + 1, s_k + 1)$ (resp. $(r_k, s_k)$). Similarly we use the notation $Y^{(k, \pm)}_p = (Y_1, \cdots, Y^{(k, \pm)}_p, \cdots, Y_N)$ to represent the variation of one Young diagram in a set of Young tables $\tilde{Y}$.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{Locations of boxes}
\end{figure}
One can write the schematic relation (1) more explicitly. We define,

\[
\delta_{-1,n} Z(\vec{a}, \vec{Y}; \vec{b}, \vec{W}; \mu) = \sum_{p=1}^{N} \left( \sum_{k=1}^{f_{p}} (a_{p} + \nu + A_{k}(Y_{p}) + B_{\ell}(Y_{q}) + \xi)^{n} A_{p}^{(k,+)}(\vec{a}, \vec{Y}) Z(\vec{a}, \vec{Y}; \vec{b}, \vec{W}; \mu) \right),
\]

\[
\delta_{1,n} Z(\vec{a}, \vec{Y}; \vec{b}, \vec{W}; \mu) = \sum_{p=1}^{N} \left( \sum_{k=1}^{f_{p}} (a_{p} + \nu + A_{k}(Y_{p}) + B_{\ell}(Y_{q}) + \xi)^{n} A_{p}^{(k,-)}(\vec{a}, \vec{Y}) Z(\vec{a}, \vec{Y}; \vec{b}, \vec{W}; \mu) \right),
\]

where we introduced coefficients \( \Lambda \):

\[
\Lambda_{p}^{(k,+)}(\vec{a}, \vec{Y}) = \left( \prod_{q=1}^{N} \prod_{t=1}^{f_{q}} a_{p} - a_{q} + A_{k}(Y_{p}) - B_{t}(Y_{q}) + \xi \prod_{t=1}^{f_{q}+1} a_{p} - a_{q} + A_{k}(Y_{p}) - A_{t}(Y_{q}) \right) \right) \frac{1}{2},
\]

\[
\Lambda_{p}^{(k,-)}(\vec{a}, \vec{Y}) = \left( \prod_{q=1}^{N} \prod_{t=1}^{f_{q}} a_{p} - a_{q} + B_{k}(Y_{p}) - A_{t}(Y_{q}) + \xi \prod_{t=1}^{f_{q}} a_{p} - a_{q} + B_{k}(Y_{p}) - B_{t}(Y_{q}) \right) \right) \frac{1}{2}.
\]

Prime in the product symbol \( (\prod') \) represents that \((\ell, q) = (k, p)\) is excluded in the product. The parameter \( \nu \) is arbitrary.

In order to define the polynomial \( U_{\pm 1,n} \), we introduce a generating function for multi-variables, \( x_{1}, \cdots, x_{N}, y_{1}, \cdots, y_{N}, \) (the expansion around \( \zeta = \infty \)),

\[
\prod_{l=1}^{N} \frac{\zeta - y_{l}}{\zeta - x_{l}} = 1 + \sum_{n=1}^{\infty} q_{n}(x, y) \zeta^{-n}.
\]

which gives the order \( n \) polynomial \( q_{n} \) in variables \( x_{l} \) and \( y_{l} \). \( U_{\pm 1,n} \) is written in terms of \( q_{n} \) as

\[
U_{\pm 1,n} = \beta^{-1/2} q_{n+1}(x, y),
\]

where we need to make replacements of variables:

\[
x_{l} \rightarrow \{ \nu + A_{k}(Y_{p}) + \nu + B_{\ell}(W_{p}) \}, \quad y_{l} \rightarrow \{ \nu + \mu + A_{k}(W_{p}) + \xi, \nu + B_{\ell}(Y_{p}) - \xi \} \quad \text{for} \quad U_{-1,n},
\]

\[
x_{l} \rightarrow \{ \nu + \mu + A_{k}(W_{p}) + \xi, \nu + B_{\ell}(Y_{p}) \}, \quad y_{l} \rightarrow \{ \nu + A_{k}(Y_{p}) + \xi, \nu + \mu + B_{\ell}(W_{p}) \} \quad \text{for} \quad U_{1,n}.
\]

Here \( k, p \) run over all possible values and the number of variables is \( N = N + \sum_{p=1}^{N} (f_{p} + \bar{f}_{p}) \).

We note that the right hand side of (14) is written as

\[
\exp \left( \sum_{n=1}^{N} \frac{\zeta^{-n}}{n} p_{n}(x, y) \right), \quad p_{n}(x, y) := \sum_{l=1}^{N} (x_{l}^{n} - y_{l}^{n}).
\]

In terms of \( p_{n} \), the function \( q_{n} \) is written as,

\[
q_{1} = p_{1}, \quad q_{2} = \frac{1}{2}(p_{2} + p_{1}^{2}), \cdots
\]

and so on. In general it takes the form of Schur polynomial for single row Young diagram \( (n) \) written in terms of power sum polynomial.
Let us give a proof of the recursion relation (1). It is based on a direct evaluation of the variations of Nekrasov partition function which is given in the appendix A.

By the formulae (117–124), the left hand sides of (10,11) are written in the form,

\[
\zeta \beta^{-1/2} \sum_{l=1}^{N} (x_l)^n \prod_{j=1}^{N} (x_l - y_j) / \prod_{j}(x_l - x_j)
\]

with the replacements (16,17). We rewrite this expression in the form of the generating functional,

\[
\sum_{l=1}^{N} \left( \sum_{n=0}^{\infty} \frac{x^n}{\zeta^{n+1}} \right) \prod_{j=1}^{N} (x_l - y_j) / \prod_{j}(x_l - x_j) = \sum_{l=1}^{N} \frac{1}{\zeta - x_l} \prod_{j}(x_l - x_j) = \frac{\prod_{l=1}^{N} \zeta - y_l}{\zeta - x_l} - 1.
\]

From the second to the third term, we need to use a nontrivial identity [7] which can be proved by comparing the locations of poles and the residue on both hand sides. The third term takes the form of the left hand side of (14). Comparing the coefficients of \(\zeta^{-(n+1)}\), we arrive at the recursion formula (1).

4 Symmetry algebra SH\(^c\)

In this section, we show that the structure of the one box variations in (1) has a nonlinear algebra which is denoted as SH\(^c\) in the paper [6]. It has generators \(D_{r,s}\) with \(r \in \mathbb{Z}\) and \(s \in \mathbb{Z}_{\geq 0}\). We call the first index \(r\) as degree and the second index \(s\) as order of generator. The commutation relations for degree \(\pm 1, 0\) generators are defined by,

\[
[D_{0,l}, D_{1,k}] = D_{1,l+k-1}, \quad l \geq 1,
\]

\[
[D_{0,l}, D_{-1,k}] = -D_{-1,l+k-1}, \quad l \geq 1,
\]

\[
[D_{-1,k}, D_{1,l}] = E_{k+l}, \quad l,k \geq 1,
\]

\[
[D_{0,l}, D_{0,k}] = 0, \quad k,l \geq 0,
\]

where \(E_k\) is a nonlinear combination of \(D_{0,k}\) determined in the form of a generating function,

\[
1 + (1 - \beta) \sum_{l \geq 0} E_l s^{l+1} = \exp(\sum_{l \geq 0} (-1)^{l+1} c_l \pi_l(s)) \exp(\sum_{l \geq 0} D_{0,l+1} \omega_l(s)),
\]

with

\[
\pi_l(s) = s^l G_l(1 + (1 - \beta)s),
\]

\[
\omega_l(s) = \sum_{q=1, -\beta, -\beta - 1} s^l (G_l(1 - qs) - G_l(1 + qs)),
\]

\[
G_0(s) = -\log(s), \quad G_l(s) = (s^{l-1} - 1)/l \quad l \geq 1.
\]

The parameters \(c_l\) \((l \geq 0)\) are central charges. The first few \(E_l\) can be computed more explicitly as,

\[
E_0 = c_0,
\]

\[
E_1 = -c_1 + c_0(c_0 - 1)\xi/2,
\]

\[
E_2 = c_2 + c_1(1 - c_0)\xi + c_0(c_0 - 1)(c_0 - 2)\xi^2/6 + 2\beta D_{0,1},
\]

\[
E_3 = 6\beta D_{0,2} + 2c_0\beta\xi D_{0,1} + \cdots,
\]

\[
E_4 = 12\beta D_{0,3} + 6c_0\beta\xi D_{0,2} + (-c_0\beta^2 + c_0^2\beta^2 - 2c_1\beta\xi + 2 - 4\xi + 4\xi^2 - 2\xi^3) D_{0,1} + \cdots.
\]

where \(\cdots\) are terms which does not contain \(D_{0,l}\).
Other generators are defined recursively by,
\[
D_{l+1,0} = \frac{1}{l} [D_{1,1}, D_{l,0}], \quad D_{-l-1,0} = \frac{1}{l} [D_{-l,0}, D_{-1,1}], \quad (35)
\]
\[
D_{r,t} = [D_{0,t+1}, D_{r,0}], \quad D_{-r,t} = [D_{-r,0}, D_{0,t+1}], \quad (36)
\]
for \( l \geq 0, r > 0 \).

Some of the basic properties of \( SH^c \) [6] are listed as follows:

- The algebra has a natural action on the fixed points of localization in the moduli space of \( SU(N) \) instantons.
- It can be derived as a singular limit of double affine Hecke algebra (DAHA) [12].
- When \( \beta \to 1 \), the algebra reduces to the much simpler algebra \( W_{1+\infty} \).
- For general \( \beta \), the algebra contains \( W_N \) algebra when the representation is constructed out of \( N \) Young diagrams.
- It is closely related to the recursion relations among Jack polynomials.

To see the relation with (1), we introduce a Hilbert space \( \mathcal{H}_{\vec{a}} \) spanned by an basis \( |\vec{a}, \vec{Y}\rangle \) where \( \vec{a} \in \mathbb{C}^N \) and \( \vec{Y} = (Y_1, \cdots, Y_N) \) is a set of \( N \) Young tables. The dual basis \( \langle \vec{a}, \vec{Y} | \) is defined such that
\[
\langle \vec{a}, \vec{Y} | \vec{b}, \vec{W} \rangle = \delta_{\vec{Y},\vec{W}} \delta(\vec{a} - \vec{b}). \quad (37)
\]
We define the actions of \( D_{\pm 1,1}, D_{0,t} \) on the ket and bra basis as,
\[
D_{-1,1}|\vec{b}, \vec{W} \rangle > = (-1)^l \sum_{q=1}^{N} \sum_{t=1}^{j_q} (b_q + B(t(W_q)))^t \Lambda^{(t,-)}(\vec{W}) |\vec{b}, \vec{W}^{(t,-)}\rangle >, \quad (38)
\]
\[
D_{1,1}|\vec{b}, \vec{W} \rangle > = (-1)^l \sum_{q=1}^{N} \sum_{t=1}^{j_q+1} (b_q + A(t(W_q)))^t \Lambda^{(t,+)}(\vec{W}) |\vec{b}, \vec{W}^{(t,+)}\rangle >, \quad (39)
\]
\[
D_{0,t+1}|\vec{b}, \vec{W} \rangle > = (-1)^l \sum_{q=1}^{N} \sum_{\mu \in W_q} (b_q + c(\mu))^t |\vec{b}, \vec{W} \rangle >, \quad (40)
\]
\[
\langle \vec{a}, \vec{Y} | D_{-1,1} \rangle = (-1)^l \sum_{p=1}^{f+1} \sum_{t=1}^{f_p} (a_p + A(Y_p))^t \Lambda^{(t,+)}(\vec{Y}) \langle \vec{a}, \vec{Y}^{(t,+)} |, \quad (41)
\]
\[
\langle \vec{a}, \vec{Y} | D_{1,1} \rangle = (-1)^l \sum_{p=1}^{f} \sum_{t=1}^{f_p} (a_p + B(Y_p))^t \Lambda^{(t,-)}(\vec{Y}) \langle \vec{a}, \vec{Y}^{(t,-)} |, \quad (42)
\]
\[
\langle \vec{a}, \vec{Y} | D_{0,t+1} \rangle = (-1)^l \sum_{p=1}^{f} \sum_{\mu \in Y_p} (a_p + c(\mu))^t \langle \vec{a}, \vec{Y} \rangle, \quad (43)
\]
where \( c(\mu) = \beta i - j \) for \( \mu = (i, j) \).

With such definitions, we claim that the action of \( D_{r,t} \) on the ket and bra basis satisfies \( SH^c \) algebra with central charges
\[
c_l = \begin{cases} 
\sum_{q=1}^{N} (b_q - \xi)^t & \text{(for ket)} \\
\sum_{p=1}^{N} (a_q - \xi)^t & \text{(for bra)}
\end{cases} \quad (44)
\]
We note that the “central charges” depend on the label \( \vec{a}, \vec{b} \) in bra and ket state in general except for \( c_0 = N \). Of course, when the inner product between them becomes nonvanishing \( \langle \vec{a} = \vec{b} \rangle \), they coincide.
Up to overall signs and shift of parameters $a_p \to a_p + \nu$ and $b_p \to b_p + \mu + \nu + \xi$, the coefficients which define $D_{\pm,1,l}$ are identical to the variations $\delta_{\pm,1,l}$ in (10,11). This observation suggests that the partition function may be written as an inner product of the basis $|\bar{a} + \nu \vec{e}, \bar{Y}|$ and $|\bar{b} + (\nu + \mu + \xi)\vec{e}, \bar{W}|$ ($\vec{e} := (1, \cdots, 1)$) with some operator insertions, and the recursion formula should be regarded as the Ward identity for the symmetry algebra $SH^c$. We will pursue this idea in the following.

Actually there exists a small mismatch in the above observation. The coefficient appearing in (11) is shifted from the coefficient in (38) by $\xi$. As we see later, this factor will be canceled by slightly modifying the vertex operator inserted between two basis. With such change, the vertex operator is no more the primary field for the $U(1)$ factor.

We need to perform a lengthy computation to confirm that the action of $D_{\pm,1,l}$ indeed gives a representation of $SH^c$. See the appendix B for some detail.

### 4.1 Comparison with $W_{1+\infty}$

For general value of $\beta$, $SH^c$ is a complicated nonlinear algebra. Simplification occurs when we choose $\beta = 1$. In this case, the nonlinear algebra reduces to a linear algebra $W_{1+\infty}$. It is an algebra of higher order differential operator $z^nD^m$ ($n \in \mathbb{Z}$, $m = 0, 1, 2, \cdots, D = z\partial_z$). Then a quantum generator $W(z^nD^m)$ is assigned to each differential operator (say $z^nD^m$) and satisfies the algebra with a central extension,

$$[W(z^n\,e^{x\bar{D}}), W(z^m\,e^{y\bar{D}})] = (e^{nx} - e^{ny})W(z^{n+m}\,e^{(x+y)\bar{D}}) - C \frac{e^{mx} - e^{ny}}{e^{x+y} - 1} \delta_{n+m,0}. \quad (45)$$

The connection between $SH^c$ and $W_{1+\infty}$ was already explained in appendix F in [6]. In our previous paper [7], we use the explicit action of $W_{1+\infty}$ generators on the free fermion Fock space and have shown that Nekrasov partition function satisfies a recursion formula associated with the symmetry.

Here we make a direct comparison of the action of $W_{1+\infty}$ algebra on the free fermion Fock space in [7] with the corresponding action of $SH^c$ (38–40). For simplicity, we consider the $N = 1$ case.

$$W(zD^i)a, Y\rangle = (-1)^i \sum_{i=1}^{f} (a + B_i(Y) - 1) |a, Y^{(i,-)}\rangle, \quad (46)$$

$$W(z^{-1}D^i)a, Y\rangle = (-1)^i \sum_{i=1}^{f+1} (a + A_i(Y)) |a, Y^{(i,+)}\rangle. \quad (47)$$

We need rewrite $\lambda$ in [7] with $-a$ here. This implies the correspondence in the $\beta \to 1$ limit:

$$D_{-1,l} \leftrightarrow W(z(D + 1)^l) = W(D^l z), \quad (48)$$

$$D_{1,l} \leftrightarrow W(z^{-1}D^l). \quad (49)$$

One may proceed to see the correspondence between the generators in $W_{1+\infty}$ and those in $SH^c$. The recursion formulae and the Ward identity obtained in [7] can be derived from the corresponding formulae in this paper by taking the limit $\beta \to 1$.

### 5 Heisenberg and Virasoro algebra in $SH^c$

In the following, we focus on the important subalgebra in $SH^c$, namely the Heisenberg (or $U(1)$ current) and Virasoro algebras. They are important because we can make the explicit evaluation of Ward identity, while the higher generators in general have nonlinear commutation relation with the vertex operator.
Generators of Heisenberg ($J_l$) and Virasoro algebras ($L_l$) are embedded in $\text{SH}^c$ as [6],

\begin{align}
J_l &= (-\sqrt{\beta})^{-1}D_{l,0}, \quad J_{-l} = (-\sqrt{\beta})^{-1}D_{l,0}, \quad J_0 = E_l/\beta, \\
L_l &= (-\sqrt{\beta})^{-1}D_{l,1}/l + (1-l)c_0\xi J_l/2, \\
L_{-l} &= (-\sqrt{\beta})^{-1}D_{l,1}/l + (1-1)c_0\xi J_{-l}/2, \\
L_0 &= [L_1, L_{-1}]/2 = D_{0,1} + \frac{1}{2\beta} \left( c_2 + c_1(1-c_0)\xi + \frac{\xi^2}{6}c_0(\phi - 1)(\phi - 2) \right).
\end{align}

The commutation relations among these generators are the standard ones,

\begin{align}
[J_n, J_m] &= \frac{nN}{\beta}\delta_{n+m,0}, \\
[L_n, J_m] &= -mJ_{n+m}, \\
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0}.
\end{align}

The derivations of these simple formulae from $\text{SH}^c$ commutator are nontrivial since in the commutation relation of $\text{SH}$, we have generators with degree $\pm 1, 0$ while $J_n$, $L_n$ have degree $n$. Proof of the first line is given in [6]. We need derive the commutation relation among them recursively. The confirmation of Virasoro algebra is much more tedious but we give the explicit computation of $[L_2, L_{-2}]$ in appendix C. This particular commutation relation is important since it implies the central charge of Virasoro algebra is related to those in $\text{SH}$ as,

\begin{equation}
c = \frac{1}{\beta} (-c_0^2 + c_0 - c_0\xi + c_0\xi^2) = 1 + (N-1)(1 - Q^2(N^2 + N)), \quad Q := \sqrt{\beta} - \sqrt{\beta}^{-1} = -\beta^{-1/2}\xi.
\end{equation}

This is the central charge for a combined system of $W_N$ algebra and a free scalar field. It motivate us to propose a free field representation,

\begin{align}
J(z) &= \sum_n J_n z^{-n-1} = \beta^{-1/2}\sum_{i=1}^N \partial_z \varphi^{(i)}(z), \\
T(z) &= \sum_n L_n z^{-n-2} = \sum_{i=1}^N \left( \frac{1}{2} (\partial \varphi^{(i)}(z))^2 - Q \rho_i \partial^2 \varphi^{(i)}(z) \right),
\end{align}

with

\begin{align}
&\varphi^{(i)}(z) = q^{(i)} + \alpha_0^{(i)} \log z - \sum_{n\neq 0} \frac{\alpha_n^{(i)}}{n} z^{-n}, \\
&[\alpha_n^{(i)}, \alpha_m^{(j)}] = n\delta_{n+m,0}\delta_{ij}, \quad [\alpha_n^{(i)}, q^{(j)}] = \delta_{m,0}\delta_{ij}, \\
&\rho_i = \frac{N+1}{2} - i, \quad i, j = 1, \cdots, N.
\end{align}

Eqs.(50, 51) imply

\begin{align}
J_0|\vec{a}, \vec{Y}\rangle &= \frac{1}{\beta} \left( -\sum_i (a_i - \xi) + \frac{\xi N(N-1)}{2} \right) |\vec{a}, \vec{Y}\rangle, \\
L_0|\vec{a}, \vec{Y}\rangle &= \left( |\vec{Y}| + \frac{1}{2\beta} \sum_i (a_i - \xi)^2 + (1-N)\xi \sum_i (a_i - \xi) + \frac{\xi^2}{6}N(N-1)(N-2) \right) |\vec{a}, \vec{Y}\rangle.
\end{align}

We assign the eigenvalue of $\alpha_0^{(i)}$ on the state $|\vec{a}, \vec{Y}\rangle$ as

\begin{equation}
\alpha_0^{(i)} |\vec{a}, \vec{Y}\rangle = p_i |\vec{a}, \vec{Y}\rangle, \quad p_i := -\alpha_i / \sqrt{\beta} = Q i, \quad i = 1, \cdots, N.
\end{equation}

With such assignments, we can rewrite (61, 62) in the more familiar form,

\begin{align}
J_0|\vec{a}, \vec{Y}\rangle &= \frac{1}{\sqrt{\beta}} (\vec{\rho} \cdot \vec{c}) |\vec{a}, \vec{Y}\rangle, \quad L_0|\vec{a}, \vec{Y}\rangle = \left( |\vec{Y}| + \Delta(\vec{\rho}) \right) |\vec{a}, \vec{Y}\rangle, \quad \Delta(\vec{\rho}) := \frac{\vec{\rho} \cdot (\vec{\rho} - 2Q)}{2}.
\end{align}

$\Delta(\vec{\rho})$ is the conformal dimension of a vertex operator $e^{\vec{\rho} \cdot \vec{z}}$ for (57).
6 Nekrasov partition function as a correlator and Heisenberg-Virasoro constraints

In the previous sections, we have seen that the recursion formulae for Nekrasov partition function takes a form of the representation of $\text{SH}^c$ algebra in terms of the orthonormal basis. We have also seen that $\text{SH}^c$ algebra contains Heisenberg and Virasoro algebras as its subalgebras.

We observe that AGT conjecture can be proved once we prove the relation

$$Z(\vec{a}, \vec{Y}; \vec{b}, \vec{W}; \mu) = \langle \vec{a} + \nu \vec{e}, \vec{Y} | V(1) | \vec{b} + (\xi + \nu + \mu) \vec{e}, \vec{W} \rangle,$$  

(65)

with the orthonormal basis $|\vec{a}, \vec{Y}\rangle$ defined in previous sections and a vertex operator $V$. Existence of such basis was formally proved in [9]. The vertex operator is factorized as $V = \tilde{V}^H V^W$ where $V^W$ is the vertex operator for $W_N$ algebra and $\tilde{V}^H$ describes the contribution of $U(1)$ factor. Furthermore it is known that the correlator of Toda theory is calculable only for the special momenta.

$$\vec{p} = -\kappa \vec{e}_1 \text{ or } \vec{p} = -\kappa \vec{e}_N, \quad \vec{e}_1 = (1, 0, \cdots, 0), \quad \vec{e}_N = (0, \cdots, 0, 1).$$

(66)

The new parameter $\kappa$ is to be determined later. For the convenience of the computation, we take the latter choice. $\tilde{V}^H$ and $V^W$ in the decomposition should be written as,

$$\tilde{V}^H = e^{-\frac{\kappa}{2} \vec{e} \cdot \vec{\varphi}}, \quad V^W = e^{-\kappa(\vec{e}_N - \vec{e}) \cdot \vec{\varphi}},$$

(67)

for $\vec{p}$ taking the second value in (66). This form of $W_N$ vertex operator is also important in the context of AGT conjecture. $V^W$ is a vertex operator corresponding to the so-called simple puncture. As we see, we need modify $\tilde{V}^H$ to meet the behavior of $U(1)$ factor in AGT conjecture.

The relation (65) can be established once one proves that the partition function $Z$ satisfies the recursion relation which defines the right hand side [7]. Namely,

$$0 = (\langle \vec{a} + \nu \vec{e}, \vec{Y} | D_{n,m} V(1) | \vec{b} + (\xi + \nu + \mu) \vec{e}, \vec{W} \rangle - \langle \vec{a} + \nu \vec{e}, \vec{Y} | [D_{n,m}, V(1)] | \vec{b} + (\xi + \nu + \mu) \vec{e}, \vec{W} \rangle - \langle \vec{a} + \nu \vec{e}, \vec{Y} | V(1) | D_{n,m} | \vec{b} + (\xi + \nu + \mu) \vec{e}, \vec{W} \rangle).$$  

(68)

The right hand side gives the Ward identity for the conformal block. One may translate such relation into a recursion relation which $Z$ should satisfy if we use the relation (65). It may sound strange to use the relation to be proved. Here we use it as the assumption in the inductive method. It is obvious that the relation (65) holds for the trivial case $\vec{Y} = \vec{W} = \vec{\emptyset}$ with a proper definition of the inner product. General relation (65) will be obtained through the Ward identities by induction.

As we have seen, the recursion relation for $Z$ exists for $n = \pm 1$ and arbitrary $m \geq 0$. Other relations should be derived from them. On the right hand side of (68), we have already defined the action of $D_{n,m}$ on the basis. A problem is that the commutation relation with the vertex operator cannot be written in the closed form except for Heisenberg and Virasoro generators. Thus we focus on these cases in the following though it is not sufficient to complete the inductive proof.

6.1 Modified vertex operator for $U(1)$ factor

While the definition of the vertex operator for $W_N$ algebra is well-known, those for $U(1)$ factor $V^H$ is somewhat tricky [2,9,21]. We give a brief account on the construction.

---

2We thank V. Pasquier to point out this important fact.
The free boson field which describes the $U(1)$ part is given by the operators $J_n$ defined in the previous section. With

$$\alpha_n = \sqrt{\beta/N} J_n,$$  

we define a free boson field as,

$$\phi(z) = q + \alpha_0 \log z - \sum_{n \neq 0} \frac{\alpha_n}{n} z^{-n} = \frac{\vec{z} \cdot \vec{\phi}}{\sqrt{N}},$$

(70)

We modify the vertex operator $\bar{V}^H$ for the $U(1)$ factor as,

$$V^H_\kappa(z) = e^{\frac{1}{\sqrt{N}} (NQ - \kappa) \phi} e^{\frac{1}{\sqrt{N}} \kappa \phi_+},$$

(71)

$$\phi_+ = \alpha_0 \log z - \sum_{n=1}^{\infty} \frac{\alpha_n}{n} z^{-n}, \quad \phi_- = q + \sum_{n=1}^{\infty} \frac{\alpha_n}{n} z^n.$$

Such definition of modified vertex operator is needed to reproduce the contribution of $U(1)$ factor in the correlator $^3$,

$$\langle V^H_\kappa(z_1) \cdots V^H_\kappa(z_n) \rangle = \prod_{i<j} (z_i - z_j)^{-s_i (NQ - s_j) \kappa}.$$  

(73)

Due to the modification, the commutation relation with $U(1)$ current (Heisenberg generator) becomes asymmetric,

$$[\alpha_m, V^H_\kappa(z)] = \frac{1}{\sqrt{N}} (NQ - \kappa) z^m V^H_\kappa(z), \quad [\alpha_{-n}, V^H_\kappa(z)] = -\frac{1}{\sqrt{N}} \kappa z^{-n} V^H_\kappa(z),$$

(74)

for $m \geq 0$, $n > 0$.

Unlike the standard definition of the vertex operator $V = e^{\kappa \phi}$, the conformal property of the modified vertex becomes rather complicated. It is, however, helpful to understand the recursion relations (1) which has some anomaly as well. We define the Virasoro generator for the $U(1)$ factor as,

$$L^H_n = \frac{1}{2} \sum_m \alpha_{n-m} \alpha_m;,$$

(75)

which has $c = 1$. The commutator of the total Virasoro genrators $L_n = L^H_n + L^W_n$ with the vertex $V_\kappa(z) = V^H_\kappa(z) V^W_\kappa(z)$ becomes,

$$[L_n, V_\kappa(z)] = z^{n+1} \partial_z V_\kappa(z) + \frac{(NQ - \kappa)^2}{2N} (n+1) z^n V_\kappa(z) + \sqrt{NQ} \sum_{m=0}^{n} z^{-m} V_\kappa(z) \alpha_m + (n+1) z^n \Delta_W V_\kappa(z), \quad n \geq 0,$$

(76)

$$[L_n, V_\kappa(z)] = z^{n+1} \partial_z V_\kappa(z) + \frac{\kappa^2}{2N} (n+1) z^n V_\kappa(z) - \sqrt{NQ} \sum_{m=1}^{n} z^{n+m} \alpha_{-m} V_\kappa(z) + (n+1) z^n \Delta_W V_\kappa(z), \quad n < 0,$$

(77)

where $\Delta_W = \frac{\kappa (\kappa - Q(N-1)) - \kappa^2}{2N}$ is the conformal dimension of $W_N$ vertex operator $V^W_\kappa$ with Toda momenta $\vec{\beta} = -\kappa (\vec{e}_N - \frac{\vec{e}}{N})$ as in (67). The anomaly due to the modification of $U(1)$ vertex manifests itself through the third term on the right hand side. We write the commutator for the special cases $n = \pm 1, 0$ for the convenience of later calculation.

$$[L_1, V_\kappa(z)] = z^2 \partial_z V_\kappa(z) + \frac{(NQ - \kappa)^2}{N} z V_\kappa(z) + \sqrt{NQ} \partial_z V_\kappa(z) \alpha_0 + \sqrt{NQ} V_\kappa(z) \alpha_1 + 2z \Delta_W V_\kappa(z),$$

(78)

$$[L_0, V_\kappa(z)] = z \partial_z V_\kappa(z) + \frac{(NQ - \kappa)^2}{2N} V_\kappa(z) + \sqrt{NQ} V_\kappa(z) \alpha_0 + \Delta_W V_\kappa(z),$$

(79)

$$[L_{-1}, V_\kappa(z)] = \partial_z V_\kappa(z).$$

(80)

In the following, we examine the relation (68) for Heisenberg ($U(1)$) and Virasoro generators for $D_{n,m}$.

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$^3$Compared with the reference [9], we included the zero mode to modify the commutator with the Virasoro generator.
6.2 Ward identities for $U(1)$ currents

We start from examining the case $n = 0$ which can be interpreted as the Ward identity for $J_{\pm 1}$,

\[ ((\bar{a} + \nu e, \bar{Y}|J_{\pm 1})V(1)(\bar{b} + (\xi + \nu + \mu)e, \bar{W}) - (\bar{a} + \nu e, \bar{Y}|V(1)(J_{\pm 1})(\bar{b} + (\xi + \nu + \mu)e, \bar{W})) = (\bar{a} + \nu e, \bar{Y}|J_{\pm 1}, V(1))|\bar{b} + (\xi + \nu + \mu)e, \bar{W}) \]  

(81)

By the definition of the representation of $SH^c$ algebra (50, 41, 38) and the vertex operator (74), the action of $J_1$ on the bra and ket basis and the commutator with the vertex operator are given as,

\[ (\bar{a} + \nu e, \bar{Y}|J_1) = (-\sqrt{\beta})^{-1} \sum_{p=1}^{N} \sum_{k=1}^{f_p+1} (\bar{a} + \nu e, \bar{Y}^{(k,+)}p|A_p^{(k,+)}(Y), \]  

(82)

\[ J_1|\bar{b} + (\xi + \nu + \mu)e, \bar{W}) = (-\sqrt{\beta})^{-1} \sum_{p=1}^{N} \sum_{k=1}^{f_p} \Lambda_p^{(k,-)}(W)|\bar{b} + (\xi + \nu + \mu)e, \bar{W}^{(k,-)}q, \]  

(83)

\[ [J_1, V_\kappa(1)] = \frac{1}{\sqrt{\beta}} (NQ - \kappa)V_\kappa(1). \]  

(84)

Plugging them into (81) gives,

\[ \left(-\sqrt{\beta}\right)^{-1} \sum_{p=1}^{N} \sum_{k=1}^{f_p+1} A_p^{(k,+)}(Y) (\bar{a} + \nu e, \bar{Y}^{(k,+)}p|V(1)|\bar{b} + (\xi + \nu + \mu)e, \bar{W}) \]  

\[ -\left(-\sqrt{\beta}\right)^{-1} \sum_{q=1}^{N} \sum_{k=1}^{f_q} \Lambda_q^{(k,-)}(W)(\bar{a} + \nu e, \bar{Y}|V(1)|\bar{b} + (\xi + \nu + \mu)e, \bar{W}^{(k,-)q} ) \]  

(85)

\[ = \frac{1}{\sqrt{\beta}} (NQ - \kappa)(\bar{a} + \nu e, \bar{Y}|V(1)|\bar{b} + (\xi + \nu + \mu)e, \bar{W}). \]

Using the assumption (65), the left hand side of (85) becomes

\[ \sqrt{\beta}^{-1} \delta_{-1,0} Z(\bar{a}, \bar{Y}; \bar{b}, \bar{W}; \mu). \]  

(86)

On the other hand, taking account of $U(1)$ charge conservation condition, which is derived from the action of $J_0$,

\[ \kappa = -\beta^{-1/2} \sum_{p=1}^{N} (a_p - b_p - \mu), \]  

(87)

the right hand side of (85) becomes

\[ \frac{1}{\sqrt{\beta}} (NQ - \kappa) Z(\bar{a}, \bar{Y}; \bar{b}, \bar{W}; \mu) = \beta^{-1} N (a_p - b_p - \mu - \xi) Z(\bar{a}, \bar{Y}; \bar{b}, \bar{W}; \mu) = \sqrt{\beta}^{-1} U_{-1,0} Z(\bar{a}, \bar{Y}; \bar{b}, \bar{W}; \mu). \]  

(88)

Thus the Ward identity for $J_1$ is proved since it is identified with the recursion formula $\delta_{-1,0} Z_{\bar{Y}, \bar{W}} - U_{-1,0} Z_{\bar{Y}, \bar{W}} = 0$.

Derivation of the identity for $J_{-1}$ can be performed similarly. The actions of $J_{-1}$ are given by

\[ (\bar{a} + \nu e, \bar{Y}|J_{-1}) = (-\sqrt{\beta})^{-1} \sum_{p=1}^{N} \sum_{k=1}^{f_p} (\bar{a} + \nu e, \bar{Y}^{(k,-)}p|A_p^{(k,-)}(Y), \]  

(89)

\[ J_{-1}|\bar{b} + (\xi + \nu + \mu)e, \bar{W}) = (-\sqrt{\beta})^{-1} \sum_{q=1}^{N} \sum_{k=1}^{f_q+1} \Lambda_q^{(k,+)}(W)|\bar{b} + (\xi + \nu + \mu)e, \bar{W}^{(k,+)q}, \]  

(90)

\[ [J_{-1}, V_\kappa(1)] = -\frac{1}{\sqrt{\beta}} \kappa V_\kappa(1). \]  

(91)
By the assumption (65), we have
\begin{align*}
\langle \vec{a} + \nu \vec{e}, \vec{Y} | J_{-1} V_\kappa(1) | \vec{b} + (\xi + \nu + \mu) \vec{e}, \vec{W} \rangle &= \langle \vec{a} + \nu \vec{e}, \vec{Y} | V_\kappa(1) J_{-1} | \vec{b} + (\xi + \nu + \mu) \vec{e}, \vec{W} \rangle \\
&= -\sqrt{\beta}^{-1} \delta_{1,0} Z(\vec{a}, \vec{Y}; \vec{b}, \vec{W}; \mu),
\end{align*}
(92)
\begin{align*}
\langle \vec{a} + \nu \vec{e}, \vec{Y} | [J_{-1}, V_\kappa(1)] | \vec{b} + (\xi + \nu + \mu) \vec{e}, \vec{W} \rangle &= -\beta^{-1/2} \kappa Z(\vec{a}, \vec{Y}; \vec{b}, \vec{W}; \mu).
\end{align*}
(93)

In the last equality in (93), we use $U(1)$ charge conservation (87). It shows the equivalence between the recursion formula $\delta_{1,0} Z_{\vec{Y},\vec{W}} - U_{1,0} Z_{\vec{Y},\vec{W}} = 0$ and the Ward identity for $J_{-1}$. We note that the modification of the vertex operator is necessary to produce the Ward identities for $U(1)$ currents.

### 6.3 Ward identities for Virasoro generators

We proceed to examine the equivalence of the Ward identity for Virasoro generators and the recursion formula. The actions of $L_1$ on the basis and the vertex operator are evaluated by (51, 38–43, 76),

\begin{align*}
\langle \vec{a} + \nu \vec{e}, \vec{Y} | L_1 = \sqrt{\beta}^{-1} \sum_{p=1}^{N} \sum_{k=1}^{f_p} \langle \vec{a} + \nu \vec{e}, \vec{Y} | (a_p + \nu + A_k(Y_p)) L^{(k,+,p)}(\vec{Y}),
\end{align*}
\begin{align*}
L_1 | \vec{b} + (\xi + \nu + \mu) \vec{e}, \vec{W} \rangle &= \sqrt{\beta}^{-1} \sum_{p=1}^{N} \sum_{k=1}^{f_p} \delta^{(k,+,)}(\vec{W}) (b_q + \nu + \mu + B_t(W_q) + \xi)| \vec{b} + (\xi + \nu + \mu) \vec{e}, \vec{W}^{(\xi,+,q)}.
\end{align*}
\begin{align*}
[L_1, V_\kappa(1)] &= \partial V_\kappa(1) + \frac{(NQ - \kappa)^2}{N} V_\kappa(1) + \sqrt{\kappa} Q N V_\kappa(1) \alpha_0 + \sqrt{\kappa} Q N V_\kappa(1) \alpha_1 + 2 \Delta W V_\kappa(1).
\end{align*}

As we see from the derivative term in the commutator, in order to evaluate the Virasoro Ward identities, we need to evaluate\(\langle \vec{a} + \nu \vec{e}, \vec{Y} | \partial V(1) | \vec{b} + (\nu + \mu + \xi) \vec{e}, \vec{W} \rangle.\) Since the modified vertex operator is not a primary operator, the correlator does not have the standard dependence on the position of the vertex operator. We can, however, derive it through the Ward identity of $L_0$.

According to the actions on the basis (62), we have
\begin{align*}
\langle \vec{a} + \nu \vec{e}, \vec{Y} | L_0 V_\kappa(1) | \vec{b} + (\xi + \nu + \mu) \vec{e}, \vec{W} \rangle - \langle \vec{a} + \nu \vec{e}, \vec{Y} | V_\kappa(1) L_0 | \vec{b} + (\xi + \nu + \mu) \vec{e}, \vec{W} \rangle = \Delta \left( -\frac{\vec{a} + \nu \vec{e}}{\sqrt{\beta}} - Q \vec{b} + Q \frac{N + 1}{2} \vec{e} \right) + |\vec{Y}| - \Delta \left( -\frac{\vec{b} + (\nu + \mu) \vec{e}}{\sqrt{\beta}} - Q \vec{b} + Q \frac{N + 1}{2} \vec{e} \right) - |\vec{W}|.
\end{align*}
(94)

On the other hand, from the commutator between $L_0$ and vertex operator (79), we obtain
\begin{align*}
\langle \vec{a} + \nu \vec{e}, \vec{Y} | [L_0, V_\kappa(1)] | \vec{b} + (\xi + \nu + \mu) \vec{e}, \vec{W} \rangle &= \langle \vec{a} + \nu \vec{e}, \vec{Y} | z \partial V_\kappa(1) | \vec{b} + (\xi + \nu + \mu) \vec{e}, \vec{W} \rangle |_{z=1} - \langle \vec{a} + \nu \vec{e}, \vec{Y} | V_\kappa(1) | \vec{b} + (\xi + \nu + \mu) \vec{e}, \vec{W} \rangle |_{z=1} \frac{(NQ - \kappa)^2}{2N} - \Delta W.
\end{align*}
(95)

Since (94) is identical with (95) by the Ward identity for $L_0$, the derivative term can be evaluated as follows,
\begin{align*}
\langle \vec{a} + \nu \vec{e}, \vec{Y} | \partial V_\kappa(1) | \vec{b} + (\xi + \nu + \mu) \vec{e}, \vec{W} \rangle &= \Delta \left( -\frac{\vec{a} + \nu \vec{e}}{\sqrt{\beta}} - Q \vec{b} + Q \frac{N + 1}{2} \vec{e} \right) + |\vec{Y}| - \Delta \left( -\frac{\vec{b} + (\nu + \mu) \vec{e}}{\sqrt{\beta}} - Q \vec{b} + Q \frac{N + 1}{2} \vec{e} \right) - |\vec{W}| - \frac{\xi}{\beta} \left( -\sum_{p=1}^{N} (b_p + \nu + \mu) + N(N - 1) \xi / 2 \right) - \frac{(NQ - \kappa)^2}{2N} - \frac{\kappa (\kappa - Q(N - 1))}{2} + \frac{\kappa^2}{2N}.
\end{align*}
Now we are ready to check the recursion relation for Virasoro generators. Applying (65), we obtain
\[
\langle \bar{a} + \nu \bar{\epsilon}, \bar{V}[L_1, V_\kappa(1)]|\bar{b} + (\xi + \nu + \mu)\bar{\epsilon}, \bar{W}\rangle - \langle \bar{a} + \nu \bar{\epsilon}, \bar{V}|V_\kappa(1)L_1|\bar{b} + (\xi + \nu + \mu)\bar{\epsilon}, \bar{W}\rangle
\]
\[
= \sqrt{\beta}^{-1} \delta_{-1,1} Z(\bar{a}, \bar{V}; \bar{b}, \bar{W}; \mu) - N \sum_{q=1}^{f_p} \sum_{\ell=1}^{N} \Lambda(\ell, -)q(\bar{W})Z(\bar{a}, \bar{V}; \bar{b}, \bar{W}(\ell, -)q, \mu) .
\]  
(96)

Unlike in the $J_1$ case, an additional term appears because the action of $SHc$ algebra on the ket space is slightly different from the action of $\delta_{-1,1}$ on $Z(\bar{a}, \bar{V}; \bar{b}, \bar{W}; \mu)$ as we have explained previously. The commutator part becomes
\[
\langle \bar{a} + \nu \bar{\epsilon}, \bar{V}|[L_1, V_\kappa(1)]|\bar{b} + (\xi + \nu + \mu)\bar{\epsilon}, \bar{W}\rangle
\]
\[
= N \sum_{q=1}^{f_p} \sum_{\ell=1}^{N} \Lambda(\ell, -)q(\bar{W})Z(\bar{a}, \bar{V}; \bar{b}, \bar{W}(\ell, -)q, \mu) .
\]  
(97)

In the last equality we use (87). This also have an anomalous term since the modified vertex is not primary operator and its commutator with $L_1$ has the $V_\kappa J_1$ term. However, the anomalies in (96) and (97) are identical and the Ward identity for $L_1$ is reduced to the recursion relation $\delta_{-1,1}Z_{\bar{V}, \bar{W}} - U_{-1,1}Z_{\bar{V}, \bar{W}} = 0$ which is already proved. We note that the identity holds only when we have the special value for the vertex momentum (66).

In the same way, for $L_{-1}$, we have
\[
\langle \bar{a} + \nu \bar{\epsilon}, \bar{V}|[L_{-1}, V_\kappa(1)]|\bar{b} + (\xi + \nu + \mu)\bar{\epsilon}, \bar{W}\rangle - \langle \bar{a} + \nu \bar{\epsilon}, \bar{V}|V_\kappa(1)L_{-1}|\bar{b} + (\xi + \nu + \mu)\bar{\epsilon}, \bar{W}\rangle
\]
\[
= \sqrt{\beta}^{-1} \delta_{-1,1} Z(\bar{a}, \bar{V}; \bar{b}, \bar{W}; \mu) ,
\]  
(98)

\[
\langle \bar{a} + \nu \bar{\epsilon}, \bar{V}|[L_{-1}, V_\kappa(1)]|\bar{b} + (\xi + \nu + \mu)\bar{\epsilon}, \bar{W}\rangle
\]
\[
= \left( \Delta \left( \frac{-\bar{\delta} + \nu \bar{\epsilon}}{\sqrt{\beta}} - Q\bar{\epsilon} + \frac{N + 1}{2} \bar{\epsilon} \right) + |\bar{Y}| - \Delta \left( \frac{-\bar{\delta} + (\nu + \mu)\bar{\epsilon}}{\sqrt{\beta}} - Q\bar{\epsilon} + \frac{N + 1}{2} \bar{\epsilon} \right) - |\bar{W}| \right)
\]
\[
- \frac{\kappa}{\beta} \left( - \sum_{p=1}^{N} (b_p + \nu + \mu) + N(N - 1)\xi/2 \right) - \left( \frac{N(\kappa - \kappa^2)}{2N} - \frac{\kappa^2}{2N} \right) Z(\bar{a}, \bar{V}; \bar{b}, \bar{W}; \mu) .
\]  
(99)

Again, we use (87) to derive the last equality in (99). Thus, the recursion formula $\delta_{1,1}Z_{\bar{V}, \bar{W}} - U_{1,1}Z_{\bar{V}, \bar{W}} = 0$ can be identified with the Ward identity. These two consistency conditions are highly nontrivial and strongly suggest that the identity (1) are a part of the Ward identities for the extended conformal symmetry.

7 Conclusion

As a generalization of our last work on $\beta = 1$ case [7], we establish the recursion relations for arbitrary $\beta$, which characterizes the Nekrasov partition function and gives a partial proof of AGT conjecture. This project is much more complicated than before and we have to introduce many new ideas to solve the issues caused by the arbitrary $\beta$. For example, we have to modify the vertex operator to cancel the anomalous terms in the recursion formulae. We also need the help of $SHc$ algebra to define the basis.
Now we have derived the conformal Ward identities for $J_{\pm 1}$ and $L_{\pm 1}$. The derivation of the similar formulae for the general Virasoro and Heisenberg generators ($J_n$, $L_n$) will not be so difficult along the line of [7] while the computation may be tedious and lengthy. What remains to do is to confirm the Ward identities for $L_{\pm 2}$. The identities for other generators can be derived from them. It is supposed to give a proof of AGT conjecture for $SU(2)$ linear quiver gauge theories. For the further generalization to $SU(N)$, we expect that the existence of the recursion formulas for arbitrary $n$ in eq.(1) implies that the Ward identities which completely characterize the conformal block may be reduced to eq.(1) in the end after the proper definition of the vertex operator in $SH^c$.

We also note that there are some important progress in terms of AGT relation [15] for the two parameter extension of $\mathcal{W}_{1+\infty}$ [12]. It is, however, nontrivial to derive AGT from the results of DAHA since the degeneration limit is singular. We hope to come back to this issue in our future work.

We would like to mention some recent papers which are relevant to this work. In [22], large $N$ limit ($N$ is the size of Young tableaux) is taken to relate AGT conjecture to matrix model. There should be a similar limit in our recursion formula where the computation becomes much simpler and the relation with Nekrasov-Shatashvili limit [23] will be clearer. In [24], the correlator of primary fields is defined in terms of null state condition of $W_N$ algebra which in term relates to Calogero-Sutherland system. Since the symmetry of Jack polynomial is identified with $SH^c$, there should be an interesting connection with the current work. In [25], an M-theoretic approach to AGT relation was explored. Furthermore, $SH^c$ seems to have interesting applications to quantum Hall effects or higher spin theories [26]. These may also be interesting directions.

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A Variations of Nekrasov formula

A.1 A useful formula

In order to evaluate the variation of Nekrasov partition function by adding or removing a box, the following formula is essential,

$$g_{Y,W}(x) = \prod_{(i,j) \in Y} (x + \beta(Y_j' - i + 1) + W_i - j) \prod_{(i,j) \in W} (-x + \beta(W_j' - i) + Y_i - j + 1)$$

$$= P_1P_2P_3Q ,$$

$$P_1 = \prod_{(i,j) \in Y} (x + \beta(i - N_2) - j) , \quad P_2 = \prod_{(i,j) \in W} (-x + \beta(i - 1) + N_1 - j + 1) ,$$

$$P_3 = \prod_{i=1}^{N_2} \prod_{j=1}^{N_1} (x + \beta(1 - i) - j)^{-1} , \quad Q = \prod_{i=1}^{N_2} \prod_{j=1}^{N_1} (x + \beta(Y_j' - i + 1) + W_i - j)$$

where $N_1$ and $N_2$ are arbitrary positive integers which should be larger than the size of Young diagrams $Y, W$. This equation is a generalization of Lemma 4 in [20]. Since the proof is exactly parallel, we give outline of proof in the
following.

We rewrite (100) in the following form and give a proof,

$$\prod_{i=1}^{N_2} \prod_{j=1}^{N_1} \frac{x + \beta(-i + 1) - j}{x + \beta(Y_j' - i + 1) + W_i - j} = \prod_{(i,j) \in Y} \frac{x + \beta(i - N_2) - j}{x + \beta(Y_j' - i + 1) + W_i - j} \prod_{(i,j) \in W} \frac{x + \beta(1 - i) - N_1 + j - 1}{x - \beta(W_j' - i) - Y_i + j - 1}. \quad (103)$$

**Proof:**

**Step 1:** Proof for $W = \emptyset$ is straightforward to do so we omit the details.

**Step 2:** We use the induction in the following. Suppose (103) is valid for a Young diagram $W$. Let us construct a Young diagram $Z$ which has only one box difference from $W$: $Z_m = W_m + 1$, $Z_{W_m+1} = m - 1$, $Z_{W_m+1} = m$, with $m$ the length of $W$. (Notice that the special case $W_m = 0$ means $Z_m$ starts from a new column, thus we can build any diagram from null Young diagram). So we just need to prove that

$$\prod_{i=1}^{N_2} \prod_{j=1}^{N_1} \frac{x + \beta(-i + 1) - j}{x + \beta(Y_j' - i + 1) + Z_i - j} \prod_{j=1}^{N_1} \frac{x + \beta(Y_j' - m + 1) + W_m - j}{x + \beta(Y_j' - m + 1) + W_m + 1 - j}. \quad (104)$$

The left hand side of (104) is

$$L = \prod_{i=1}^{N_2} \prod_{j=1}^{N_1} \frac{x + \beta(-i + 1) - j}{x + \beta(Y_j' - i + 1) + W_i - j} \prod_{j=1}^{N_1} \frac{x + \beta(Y_j' - m + 1) + W_m - j}{x + \beta(Y_j' - m + 1) + W_m + 1 - j}. \quad (105)$$

The first factor on the right hand side of (104) is

$$R_1 = \prod_{(i,j) \in Y} \frac{x + \beta(i - N_2) - j}{x + \beta(Y_j' - i + 1) + W_i - j} \prod_{j=1}^{N_1} \frac{x + \beta(Y_j' - m + 1) + W_m - j}{x + \beta(Y_j' - m + 1) + W_m + 1 - j}. \quad (106)$$

The second factor becomes

$$R_2 = \frac{x + \beta(1 - m) - N_1 + W_m}{x - Y_m + W_m} \times \prod_{(i,j) \in W} \frac{x + \beta(1 - i) - N_1 + j - 1}{x - \beta(W_j' - i) - Y_i + j - 1} \prod_{i=1}^{m-1} \frac{x - \beta(m - 1 - i) - Y_i + W_m}{x - \beta(m - i) - Y_i + W_m}. \quad (107)$$

Since we have assumed the equation (103) is correct for $W$, we only need to prove

$$\prod_{j=Y_m+1}^{N_1} \frac{x + \beta(Y_j' - m + 1) + W_m - j}{x + \beta(Y_j' - m + 1) + W_m + 1 - j} = \frac{x + \beta(1 - m) - N_1 + W_m}{x - Y_m + W_m} \times \prod_{i=1}^{m-1} \frac{x - \beta(m - 1 - i) - Y_i + W_m}{x - \beta(m - i) - Y_i + W_m}. \quad (108)$$

The left hand side of the above transforms to

$$L' = \frac{x + \beta(-m + 1) + W_m - N_1}{x + \beta(-m + 1) + W_m - h} \prod_{j=Y_m+1}^{h} \frac{x + \beta(Y_j' - m + 1) + W_m - j}{x + \beta(Y_j' - m + 1) + W_m + 1 - j}. \quad (109)$$

Here $h$ is again the height of $Y$. We name the second term of the last line as $L'_1$.

$$L'_1 = \prod_{j=Y_m+1}^{h} \frac{x + \beta(-m + 1) + W_m - j}{x + \beta(-m + 1) + W_m - h} \times \prod_{j=Y_m+1}^{h} \prod_{i=1}^{Y_j'} \left( \frac{x + \beta(i - m + 1) + W_m - j}{x + \beta(i - m) + W_m - j} \frac{x + \beta(i - m) + W_m + 1 - j}{x + \beta(i - m) + W_m + 1 - j} \right). \quad (110)$$

We call the last term of the last line as $L_3$. The second term on the right hand side of (108) has the form

$$R'_2 = \prod_{i=1}^{m-1} \frac{x - \beta(m - 1 - i) + W_m}{x - \beta(m - i) + W_m} \times \prod_{i=1}^{Y_1} \prod_{j=1}^{m-1} \left( \frac{x + \beta(i - m + 1) + W_m - j}{x + \beta(i - m) + W_m - j} \frac{x + \beta(i - m) + W_m + 1 - j}{x + \beta(i - m) + W_m + 1 - j} \right). \quad (111)$$
so we find
\[
R'_i \frac{L_3}{r} = \prod_{i=1}^{m-1} \frac{x + \beta(i - m) + W_m - Y_m}{x + \beta(i - m) + W_m - Y_m} = \frac{x + W_m - Y_m}{x + \beta(1 - m) + W_m - Y_m}.
\] (112)

Combine (109), (110) and (112), it is straightforward to find that (108) is tenable, thus complete the proof.

### A.2 Variations of Nekrasov formula

We decompose \( Y, W \) into rectangles \( Y = (r_i, \ldots, r_f; s_1, \ldots, s_f) \) and \( W = (t_1, \ldots, t_j; u_1, \ldots, u_j) \). Also we use the same notation such as \( Y^{(k, \pm)} \) and \( W^{(k, \pm)} \). For the variation of \( Y \) (resp. \( W \)), \( P_2, P_3 \) (resp. \( P_1, P_3 \)) remain the same. Variation of \( P_1 \) (resp. \( P_2 \)) produces a term which cancel the \( N_2 \) (resp. \( N_1 \)) dependent term in the variation of \( Q \).

We also uses a notation \( r_0 = s_{f+1} = t_0 = u_{f+1} = 0 \). After some computation, we obtain,

\[
g_{Y^{(k,+)}, W}(x) = \prod_{\ell=1}^{f+1} (x + \beta(r_{k-1} - t_{\ell-1}) + u_\ell - s_k - 1),
\]
(113)

\[
g_{Y^{(k,-)}, W}(x) = \prod_{\ell=1}^{f+1} (x + \beta(r_k - t_{\ell-1}) + u_\ell - s_k),
\]
(114)

\[
g_{Y, W^{(k,+)}}(x) = \prod_{\ell=1}^{f+1} (-x + \beta(t_{\ell-1} - r_k) - u_\ell + s_{k+1}),
\]
(115)

\[
g_{Y, W^{(k,-)}}(x) = \prod_{\ell=1}^{f+1} (-x + \beta(t_{\ell-1} - r_k) - u_\ell + s_k + 1).
\]
(116)

These expressions becomes more compact by the use of the notation \( A_k(Y_p), B_k(Y_p) \) in (8,9),

\[
g_{Y^{(k,+)}, W}(a_p - b_q - \mu) = \prod_{\ell=1}^{f+1} (a_p - b_q - \mu + A_k(Y_p) - A_t(W_q) - \xi),
\]
(117)

\[
g_{Y^{(k,-)}, W}(a_p - b_q - \mu) = \prod_{\ell=1}^{f+1} (a_p - b_q - \mu + B_k(Y_p) - B_t(W_q)),
\]
(118)

\[
g_{Y, W^{(k,+)}}(a_p - b_q - \mu) = \prod_{k=1}^{f+1} (b_q - a_p + \mu + A_t(W_q) - A_k(Y_p) + \xi),
\]
(119)

\[
g_{Y, W^{(k,-)}}(a_p - b_q - \mu) = \prod_{k=1}^{f+1} (b_q - a_p + \mu + B_t(W_q) - B_k(Y_p) + \xi).
\]
(120)

These are sufficient to calculate variation of \( z_{\text{bf}} \) in (6).

To derive the variation of \( z_{\text{rect}} \) for \( p \neq q \), we need the following formulæ which are obtained by putting \( W_q \rightarrow Y_q \),

\[
g_{Y^{(k,+)}, Y_q}(a_p - a_q) g_{Y^{(k,+)}, Y_q}(a_q - a_p) = \prod_{\ell=1}^{f+1} (a_p - a_q + A_k(Y_p) - B_t(Y_q))(a_p - a_q + A_k(Y_p) - B_t(Y_q) + \xi),
\]
(121)

\[
g_{Y^{(k,-)}, Y_q}(a_p - a_q) g_{Y^{(k,-)}, Y_q}(a_q - a_p) = \prod_{\ell=1}^{f+1} (a_p - a_q + B_k(Y_p) - A_t(Y_q))(a_p - a_q + B_k(Y_p) - A_t(Y_q) + \xi).
\]
(122)
For the case \( p = q \), we obtain,

\[
\frac{g_{\gamma_p, \gamma_p}(0)}{g_{Y_p(k+), Y_p(k+)}(0)} = \frac{1}{\beta} \prod_{\ell=1}^{f_p+1} (A_k(Y_p) - B_\ell(Y_p) - \xi), \quad \text{and} \quad \frac{g_{\gamma_p, \gamma_p}(0)}{g_{Y_p(k-), Y_p(k-)}(0)} = \frac{1}{\beta} \prod_{\ell=1}^{f_p+1} (B_k(Y_p) - A_\ell(Y_p) - \xi). \tag{123, 124}
\]

These formulae are sufficient to derive the recursion relation (1).

## B Derivation of commutation relations of \( \text{SH}^c \) algebra

First we notice that

\[
[D_{-1,k}, D_{1,l}][\vec{b}, \vec{W}] > = (-1)^{k+l} \sum_{q=1}^{N} \left\{ \sum_{t=1}^{\hat{q}+1} (b_q + A_t(W_q))^k + (A_q^{(t,+)}(\vec{b}, \vec{W}))^2 \right. \\
- \left. \sum_{i=1}^{\hat{f}} (b_q + B_t(W_q))^k + (A_q^{(t,-)}(\vec{b}, \vec{W}))^2 \right\} [\vec{b}, \vec{W}] > . \tag{125}
\]

We have to be careful that the off-diagonal terms, where the two generator modifies different Young diagrams or two different boxes in the same Young diagram, cancels with each other. This can be checked as below.

### B.1 Cancellation of off-diagonal terms

First, for a Young diagram with one box removed \( W(k,-) \) (or added), we find the relation between \( A_t(W(k,-)) \), \( B_t(W(k,-)) \) and their counterparts of the original yong diagram \( W \).

\[
A_t(W(k,-)) = \begin{cases} 
A_t(W), & 1 \leq t \leq k \\
B_k(W), & t = k + 1 \\
A_{t-1}(W), & k + 2 \leq t \leq \hat{f} + 2
\end{cases}, \quad B_t(W(k,-)) = \begin{cases} 
B_t(W), & 1 \leq t \leq k - 1 \\
B_k(W) - \beta, & t = k \\
B_k(W) + 1, & t = k + 1 \\
B_{t-1}(W), & k + 2 \leq t \leq \hat{f} + 1
\end{cases}
\]

\[
A_s(W(k,+)) = \begin{cases} 
A_s(W), & 1 \leq s \leq k - 1 \\
A_k(W) - 1, & s = k \\
A_k(W) + \beta, & s = k + 1 \\
A_{s-1}(W), & k + 2 \leq s \leq \hat{f} + 2
\end{cases}, \quad B_s(W(k,+)) = \begin{cases} 
B_s(W), & 1 \leq s \leq k - 1 \\
A_k(W), & s = k \\
B_{s-1}(W), & k + 1 \leq t \leq \hat{f} + 1
\end{cases}
\]

With the above relations, we obtain that\(^4\),

\[
D_{-1,k}D_{1,l} [\vec{b}, \vec{W}] > = \sum_{q=1}^{N} \sum_{\gamma=1}^{f_{q}^{(u,+)+1}} (B_t(W_q))^{k} A_q^{(t,+)}(\vec{b}, \vec{W}) > . \tag{126}
\]

\[
D_{1,l}D_{-1,k} [\vec{b}, \vec{W}] > = \sum_{\gamma=1}^{N} \sum_{u=1}^{f_{q}^{(t,-)+1}} (A_u(W_q))^{k} A_q^{(t,-)}(\vec{b}, \vec{W}) > . \tag{127}
\]

\(^4\)For simplicity, in the following we do not write \( b_q \) explicitly, which always comes together with \( A_t(W_q) \) and \( B_t(W_q) \). They may be included by the redefinition of these symbols.
For \( q = \gamma, t \geq u, \)

\[
(A_u(\overline{W}_\gamma^{(t,-),q}))^j \left( - \prod_{\delta=1}^{N} \left( \prod_{v=1}^{j} \frac{A_u(\overline{W}_\gamma^{(t,-),q}) - B_v(\overline{W}_\delta^{(t,-),q}) + \xi \prod_{\delta=1}^{j} A_u(\overline{W}_\gamma^{(t,-),q}) - A_v(\overline{W}_\delta^{(t,-),q}) - \xi}{A_u(\overline{W}_\gamma^{(t,-),q}) - A_v(\overline{W}_\delta^{(t,-),q})} \right) \right)^{1/2}
\]

= common terms \times \frac{A_u(W_\gamma) - (B_t(W_\gamma) - 1)}{A_u(W_\gamma) - (B_t(W_\gamma) - \beta)} \times \frac{A_u(W_\gamma) - (B_t(W_\gamma) + \beta)}{A_u(W_\gamma) - (B_t(W_\gamma) + 1)} \times \frac{A_u(W_\gamma) - (B_t(W_\gamma) + \xi)}{A_u(W_\gamma) - (B_t(W_\gamma) + \xi)}.

(128)

\[
(B_{t+1}(\overline{W}_q^{(u+,\gamma)}))^k \left( - \prod_{p=1}^{N} \left( \prod_{s=1}^{p} \frac{B_{t+1}(\overline{W}_q^{(u+,\gamma)}) - A_u(\overline{W}_p^{(u+,\gamma)}) - \xi \prod_{s=1}^{p} B_{t+1}(\overline{W}_q^{(u+,\gamma)}) - B_v(\overline{W}_p^{(u+,\gamma)}) + \xi}{B_{t+1}(\overline{W}_q^{(u+,\gamma)}) - B_v(\overline{W}_p^{(u+,\gamma)})} \right) \right)^{1/2}
\]

= common terms \times \frac{B_t(W_q) - (A_u(W_q) - \beta)}{B_t(W_q) - (A_u(W_q) - 1)} \times \frac{B_t(W_q) - (A_u(W_q) + 1)}{B_t(W_q) - (A_u(W_q) + \beta)} \times \frac{B_t(W_q) - (A_u(W_q) + \xi)}{B_t(W_q) - (A_u(W_q) + \xi)}.

(129)

Thus we find that \( \sum_{t=0}^{u-1} \gamma \) cancels with \( \sum_{t=0}^{u+1} \gamma \). For \( q = \gamma, t \leq u - 2 \), we have the same result. For \( q = \gamma, t = u - 1 \), we have the direct sum. For \( q \neq \gamma \), using a similar method, we also find that \( \sum_q \sum_{t=0}^{u} \gamma \) cancels with \( \sum_q \sum_{t=0}^{u} \gamma \).

In total we show that all the off-diagonal terms are gone.

### B.2 Evaluation of diagonal terms

Since the right hand side of (125) only depends on \( k + l \), we have \([D_{-1,k}, D_{1,l}] = [D_{-1,0}, D_{1,l+k}]\). We need to define the action of \( D_{0,l} \). For this purpose, we consider

\[
X(s) = \overline{d} \overline{W} \sum_{l \geq 0} [D_{-1,0}, D_{1,l}] s^l |\overline{d}, \overline{W} >.
\]

(130)

Then from the definition of algebra, we obtain

\[
s\xi X(s) = \sum_{q=0}^{N} \left\{ \sum_{t=1}^{f+1} \frac{s\xi}{1 + s(b_q + A_t(W_q))} (A_t^{(q)}(\overline{d}, \overline{W}))^2 - \sum_{t=1}^{f} \frac{s\xi}{1 + s(b_q + B_t(W_q))} (A_t^{(q)}(\overline{d}, \overline{W}))^2 \right\}
\]

\[
= \sum_{q=0}^{N} \left\{ \sum_{t=1}^{f+1} \frac{s\xi}{1 + s(b_q + A_t(W_q))} \prod_{p=1}^{f+1} \frac{b_q - b_p + A_t(W_q) - B_k(W_p) + \xi \prod_{k \neq t}^{f+1} b_q - b_p + A_t(W_q) - B_k(W_p)}{b_q - b_p + A_t(W_q) - B_k(W_p)} \prod_{k \neq t}^{f+1} b_q - b_p + A_t(W_q) - A_k(W_p) \right\}
\]

\[
- \sum_{t=1}^{f} \frac{s\xi}{1 + s(b_q + B_t(W_q))} \prod_{p=1}^{f} \frac{b_q - b_p + B_t(W_q) - B_k(W_p) + \xi \prod_{k \neq t}^{f} b_q - b_p + B_t(W_q) - A_k(W_p)}{b_q - b_p + B_t(W_q) - A_k(W_p)} \right\}
\]

\[
= -1 + \sum_{q=1}^{N} \sum_{l=1}^{f} \frac{1 + s(b_q + B_t(W_q) - \xi \prod_{t=1}^{f+1} 1 + s(b_q + A_t(W_q) + \xi)}{1 + s(b_q + A_t(W_q))}
\]

(131)

The last equality holds because the both sides (i) are degree 0 rational function in \( s \), (ii) have the same simple poles and residues at \( s = -1/(b_q + B_t(W_q)) \), \(-1/(b_q + A_t(W_q))\) and (iii) vanish at \( s = 0 \). We can rewrite (131) as

\[
1 + s\xi X(s) = \prod_{q=1}^{N} \prod_{l=1}^{f} \frac{1 + s(b_q + B_t(W_q) - \xi \prod_{t=1}^{f+1} 1 + s(b_q + A_t(W_q) + \xi)}{1 + s(b_q + A_t(W_q))}
\]

\[
= \exp \left\{ \sum_{q=1}^{N} \sum_{l=1}^{f} \frac{(-1)^{l+1}}{l} \left( \sum_{t=1}^{f} (p_t(b_q + B_t(W_q)) - p_l(b_q + B_t(W_q) - \xi) + \sum_{t=1}^{f+1} (p_l(b_q + A_t(W_q)) - p_l(b_q + A_t(W_q) + \xi)) \right) \right\}
\]

(132)

where \( p_t(x_t) = \sum_t x_t^t \).
We define \( H_l(W_q) := \sum_{i=1}^{l} (p_i(b_q + B_l(W_q)) - p_i(b_q + B_l(W_q) - \xi) + \sum_{i=1}^{l+1} (p_i(b_q + A_l(W_q)) - p_i(b_q + A_l(W_q) + \xi)) \).

Then we use a formula,

\[
H_l(W_q) = (b_q - \xi)^l - (b_q)^l - \sum_{\mu \in W_q} \sigma_l(c_q(\mu)) ,
\]

where \( \sigma_l(x) = (x + 1)^l - (x - 1)^l + (x - \beta)^l - (x + \beta)^l + (x + \beta - 1)^l - (x + 1 - \beta)^l \) and \( c_q(\mu) = b_q + \beta i - j \) for \( \mu = (i, j) \). It was proved in appendix B of [6]. Thus we can proceed as

\[
1 + \xi sX(s) = \exp \left\{ \sum_{q=1}^{\infty} - \frac{s}{q} \sum_{l=1}^{\infty} ((b_q - \xi)^l - (b_q)^l) - \sum_{q=1}^{\infty} \sum_{\mu \in W_q} (1 + \xi s) \sigma_l(c_q(\mu)) \right\}
\]

\[
= \exp \left\{ \sum_{q=1}^{\infty} - \frac{s}{q} \left( \sum_{l=0}^{\infty} (-1)^l \sigma_l(c_q(\mu)) \right) \omega_l(s) \right\} .
\]

In the last equality of (134), we use the following formula

\[
\sum_{l=0}^{\infty} (-1)^l \frac{s^l}{l!} g_l(a+b) = \sum_{l=0}^{\infty} (-1)^l a^l s^l G_l(1+bs) ,
\]

which can be proved using the taylor expansion of \( \log(1+s(a+b)) \). Comparing (134) with (26), the algebra (24) is proved once we set (40, 44). The proof of the algebra for the action on the bra state is similar.

## C Derivation of Virasoro algebra from \( SH^c \)

Here we give a sample computation to give the Virasoro algebra from the definition of \( SH^c \) (22–25) and (51). We focus on the relation

\[
[L_2, L_{-2}] = 4L_0 + \frac{c}{2}
\]

since it gives the simplest commutator to give the Virasoro central charge.

The definition of generators gives

\[
[L_2, L_{-2}] = \frac{1}{4\beta^2} \{ [D_{-2,1}, D_{2,1}] - c_6[2D_{-2,1}, D_{2,0}] - c_6[2D_{-2,0}, D_{2,1}] + (c_6)^2[D_{-2,0}, D_{2,0}] \} .
\]

We express degree 2 generator as the commutator of degree 1 generator

\[
D_{2,0} = [D_{1,1}, D_{1,0}] , \quad D_{-2,0} = [D_{-1,0}, D_{-1,1}] , \\
D_{2,1} = [D_{1,2}, D_{1,0}] , \quad D_{-2,1} = [D_{-1,0}, D_{-1,2}] .
\]

The commutation relation between degree two operators is reduced to those for degree one operator. After some computation we arrive at

\[
[D_{-2,1}, D_{2,1}] = 8\beta E_2 + 6c_6\beta E_1 - c_0^2\beta^2 + c_0^2\beta^2 - 2c_0c_1\beta\xi + 2c_0\beta - 2c_0\beta\xi + 2c_0\beta^2 ,
\]

\[
[D_{-2,0}, D_{2,1}] = -4c_1\beta + 4c_0^2\beta\xi - 2c_0\beta\xi ,
\]

\[
[D_{-2,1}, D_{2,0}] = -4c_1\beta + 4c_0^2\beta\xi - 2c_0\beta\xi ,
\]

\[
[D_{-2,0}, D_{2,0}] = 2c_0\beta .
\]

It gives

\[
[L_2, L_{-2}] = \frac{4}{2\beta} E_2 + \frac{1}{2\beta} \{ -c_6^2\xi^2 + c_0 - c_0\xi + c_0\xi^2 \} .
\]

After identifying \( L_0 = \frac{1}{2\beta} E_2 \), we can identify the Virasoro central charge (55).
References

[1] N. A. Nekrasov, “Seiberg-Witten prepotential from instanton counting,” Adv. Theor. Math. Phys. 7, 831 (2004) [hep-th/0206161]; arXiv:hep-th/0306211;
N. Nekrasov and A. Okounkov, “Seiberg-Witten theory and random partitions,” arXiv:hep-th/0306238.

[2] L. F. Alday, D. Gaiotto and Y. Tachikawa, “Liouville Correlation Functions from Four-dimensional Gauge Theories,” arXiv:0906.3219 [hep-th].

[3] N. Wyllard, “A(N-1) conformal Toda field theory correlation functions from conformal N = 2 SU(N) quiver gauge theories,” JHEP 0911, 002 (2009) [arXiv:0907.2189 [hep-th]].

[4] A. Mironov and A. Morozov, “On AGT relation in the case of U(3),” Nucl. Phys. B 825, 1 (2010) [arXiv:0908.2569 [hep-th]].

[5] N. Nekrasov and V. Pestun, “Seiberg-Witten geometry of four dimensional N=2 quiver gauge theories,” arXiv:1211.2240 [hep-th].

[6] O. Schiffmann and E. Vasserot, “Cherednik algebras, W algebras and the equivariant cohomology of the moduli space of instantons on A2”, arXiv:1202.2756.

[7] S. Kanno, Y. Matsuo and H. Zhang, “Virasoro constraint for Nekrasov instanton partition function,” JHEP 1210, 097 (2012) [arXiv:1207.5658 [hep-th]].

[8] V. A. Alba, V. A. Fateev, A. V. Litvinov and G. M. Tarnopolskiy, “On combinatorial expansion of the conformal blocks arising from AGT conjecture,” Lett. Math. Phys. 98, 33 (2011) [arXiv:1012.1312 [hep-th]].

[9] V. A. Fateev and A. V. Litvinov, “Integrable structure, W-symmetry and AGT relation,” JHEP 1201, 051 (2012) [arXiv:1109.4042 [hep-th]].

[10] A. Belavin and V. Belavin, “AGT conjecture and Integrable structure of Conformal field theory for c=1,” Nucl. Phys. B 850, 199 (2011) [arXiv:1102.0343 [hep-th]].

[11] M. R. Gaberdiel and R. Gopakumar, “Triality in Minimal Model Holography,” JHEP 1207, 127 (2012) [arXiv:1205.2472 [hep-th]].

[12] I. Cherednik, “Double affine Hecke algebras”, London Mathematical Society Lecture Note Series, 319, Cambridge University Press, (Cambridge 2005);
D. Bernard, M. Gaudin, F. D. M. Haldane and V. Pasquier, “Yang-Baxter equation in long range interacting system,” J. Phys. A 26, 5219 (1993).

[13] J. Ding, K. Iohara, ”Generalization of Drinfeld quantum affine algebras”, Lett. Math. Phys. 41 (1997) 181–193.

[14] Kei Miki, “A (q,γ) analog of the W_{1+∞} algebra”, J. Math. Phys. 48 123520 (2007).

[15] see for example, H. Awata, B. Feigin, A. Hoshino, M. Kanai, J. Shiraishi and S. Yanagida, “Notes on Ding-Iohara algebra and AGT conjecture,” arXiv:1106.4088 [math-ph].

[16] D. Maulik and A. Okounkov, “Quantum Groups and Quantum Cohomology,” arXiv:1211.1287 [math.AG].

[17] Hiraku Nakajima and Kota Yoshioka, “Instanton counting on blowup. I. 4-dimensional pure gauge theory”, Invent. Math. 162, 313-355 (2005).
[18] A. Losev, N. Nekrasov and S. L. Shatashvili, “Issues in topological gauge theory,” Nucl. Phys. B 534, 549 (1998) [hep-th/9711108];
G. W. Moore, N. Nekrasov and S. Shatashvili, “Integrating over Higgs branches,” Commun. Math. Phys. 209, 97 (2000) [hep-th/9712241]; A. Lossev, N. Nekrasov and S. L. Shatashvili, “Testing Seiberg-Witten solution,” In *Cargese 1997, Strings, branes and dualities* 359-372 [hep-th/9801061].

[19] R. Flume and R. Poghossian, “An Algorithm for the microscopic evaluation of the coefficients of the Seiberg-Witten prepotential,” Int. J. Mod. Phys. A 18, 2541 (2003) [hep-th/0208176];
U. Bruzzo, F. Fucito, J. F. Morales and A. Tanzini, “Multiinstanton calculus and equivariant cohomology,” JHEP 0305, 054 (2003) [hep-th/0211108];
F. Fucito, J. F. Morales and R. Poghossian, “Instantons on quivers and orientifolds,” JHEP 0410, 037 (2004) [hep-th/0408090].

[20] H. Zhang and Y. Matsuo, “Selberg Integral and SU(N) AGT Conjecture,” JHEP 1112, 106 (2011) [arXiv:1110.5255 [hep-th]].

[21] Erik Carlsson and Andrei Okounkov, “Exts and vertex operators”, Duke Math. J. Volume 161, Number 9 (2012), 1797-1815, [arXiv:0801.2565 [math.AG]].

[22] R. Poghossian, “Deforming SW curve,” JHEP 1104, 033 (2011) [arXiv:1006.4822 [hep-th]]; F. Fucito, J. F. Morales, D. R. Pacifici and R. Poghossian, “Gauge theories on Ω-backgrounds from non commutative Seiberg-Witten curves,” JHEP 1105, 098 (2011) [arXiv:1103.4495 [hep-th]]; F. Ferrari and M. Piatek, “On a singular Fredholm-type integral equation arising in N=2 super Yang-Mills theories,” Phys. Lett. B 718, 1142 (2013) [arXiv:1202.5135 [hep-th]]; J. -E. Bourgine, “Large N limit of beta-ensembles and deformed Seiberg-Witten relations,” JHEP 1208, 046 (2012) [arXiv:1206.1696 [hep-th]]; J. -E. Bourgine, “Large N techniques for Nekrasov partition functions and AGT conjecture,” JHEP 1305, 047 (2013) [arXiv:1212.4972 [hep-th]].

[23] N. A. Nekrasov and S. L. Shatashvili, “Quantization of Integrable Systems and Four Dimensional Gauge Theories,” arXiv:0908.4052 [hep-th].

[24] B. Estienne, V. Pasquier, R. Santachiara and D. Serban, “Conformal blocks in Virasoro and W theories: Duality and the Calogero-Sutherland model,” Nucl. Phys. B 860, 377 (2012) [arXiv:1110.1101 [hep-th]].

[25] Tan, Meng-Chwan ”M-Theoretic Derivations of 4d-2d Dualities: From a Geometric Langlands Duality for Surfaces, to the AGT Correspondence, to Integrable Systems“ arXiv:1301.1977[hep-th].

[26] For example,
S. Iso, D. Karabali and B. Sakita, “Fermions in the lowest Landau level: Bosonization, W infinity algebra, droplets, chiral bosons,” Phys. Lett. B 296, 143 (1992) [hep-th/9209003];
A. Cappelli, C. A. Trugenberger and G. R. Zemba, “Stable hierarchical quantum hall fluids as W(1+infinity) minimal models,” Nucl. Phys. B 448, 470 (1995) [hep-th/9502021]; M. Henneaux and S. -J. Rey, “Nonlinear W_{infinity} as Asymptotic Symmetry of Three-Dimensional Higher Spin Anti-de Sitter Gravity,” JHEP 1012, 007 (2010) [arXiv:1008.4579 [hep-th]]; M. R. Gaberdiel, R. Gopakumar and A. Saha, “Quantum W-symmetry in AdS_{5},” JHEP 1102, 004 (2011) [arXiv:1009.6087 [hep-th]].