MOMENT INTERMITTENCY IN THE PAM WITH ASYMPTOTICALLY SINGULAR NOISE

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ABSTRACT. Let $\xi$ be a singular Gaussian noise on $\mathbb{R}^d$ that is either white, fractional, or with the Riesz covariance kernel; in particular, there exists a scaling parameter $\omega > 0$ such that $c^{\omega/2} \xi(c \cdot) \Rightarrow \xi$ for all $c > 0$. Let $(\xi_\varepsilon)_{\varepsilon > 0}$ be a sequence of smooth mollifications such that $\xi_\varepsilon \Rightarrow \xi$ as $\varepsilon \to 0$. We study the asymptotics of the moments of the parabolic Anderson model (PAM) with noise $\xi_\varepsilon$ as $\varepsilon \to 0$, both for large (i.e., $t \to \infty$) and fixed times $t$. This approach makes it possible to study the moments of the PAM with regular and singular noises in a unified fashion, as well as interpolate between the two settings. As corollaries of our main results, we obtain the following:

(1) When $\xi$ is subcritical (i.e., $0 < \omega < 2$), our results extend the known large-time moment and tail asymptotics for the Stratonovich PAM with noise $\xi$. Our method of proof clarifies the role of the maximizers of the variational problems (known as Hartree ground states) that appear in these moment asymptotics in describing the geometry of intermittency. We take this opportunity to prove the existence and study the properties of the Hartree ground state with a fractional kernel, which we believe is of independent interest.

(2) When $\xi$ is critical or supercritical (i.e., $\omega = 2$ or $\omega > 2$), our results provide a new interpretation of the moment blowup phenomenon observed in the Stratonovich PAM with noise $\xi$. That is, we uncover that the latter is related to an intermittency effect that occurs in the PAM with noise $\xi_\varepsilon$ as $\varepsilon \to 0$ for fixed finite times $t > 0$.

1. Introduction

In this paper, we are interested in the continuous parabolic Anderson model (PAM) with a time-independent Gaussian noise. That is, the partial differential equation with random coefficients

$$
\begin{aligned}
\partial_t u(t, x) &= (\kappa \Delta + \xi(x)) u(t, x), \\
u(0, x) &= 1,
\end{aligned}
$$

where $d \in \mathbb{N}$ and $\kappa > 0$ are fixed, $\Delta$ is the Laplacian operator, and $\xi : \mathbb{R}^d \to \mathbb{R}$ is a centered stationary Gaussian process. A fundamental problem involving the PAM consists of understanding the occurrence of intermittency at large times. Informally, the latter refers to the observation that the random field $x \mapsto u(t, x)$ tends to develop tall and narrow peak-like formations as $t \to \infty$. We refer to [30, Chapter 1] and references therein for a recent overview.

As explained in [39], arguably the most elementary rigorous indication that intermittency occurs in (1.1) is a growth property of its moments: More specifically, suppose that there exists a scale function $A$ such that $A(t) \to \infty$ as $t \to \infty$ and such that the limits

$$
\ell_p := \lim_{t \to \infty} \frac{\log(u(t, x)^p)}{A(t)},
$$

which are called Lyapunov exponents, exist and are finite for some values of $p > 0$. 

Remark 1.1. Here, as is common in a subset of the PAM literature, we use $\langle \cdot \rangle$ to denote the expectation with respect to $\xi$. Also, since $x \mapsto \xi(x)$ is stationary, so is $x \mapsto u(t, x)$; hence $\ell_p$ does not depend on the choice of $x$.

Definition 1.2. We say that (1.1) is intermittent if there exists a subset of $(0, \infty)$—which contains at least two points—on which the function $p \mapsto \ell_p/p$ is strictly increasing.

We refer to [39, Pages 210–212] for an explanation of the geometric significance of this condition. The classical theory of the continuous PAM concerns the case where $\xi$ is a regular random process (e.g., Hölder continuous). In this setting, the intermittency phenomenon is very well understood, as shown in the works of Carmona, Gärtner, König, and Molchanov [3, 20, 21]. In contrast to that, in recent years there has been a growing interest in extending this theory to singular noises. In this paper, our aim is to contribute to the understanding of intermittency in the PAM with a singular Gaussian noise—where the product $\xi \cdot u(t, \cdot)$ is interpreted in the Stratonovich sense—via a study of the moments. We discuss our approach and state our main results in Section 2.

In the remainder of this section, we introduce the singular noises that we are interested in (Section 1.1), we briefly recall the current state of the art concerning the moments of the regular and singular PAM (Section 1.2), we provide some motivation for our investigations and a brief overview of our results (Section 1.3), we discuss open questions that arise from our results (Section 1.4), and we give an outline of the organization of this paper (Section 1.5).

1.1. Stratonovich PAM with Singular Noise. We make the following assumption on $\xi$:

Assumption 1.3. Informally, we view $\xi$ as a centered Gaussian process on $\mathbb{R}^d$ with covariance
\[
\langle \xi(x)\xi(y) \rangle = \gamma(x - y), \quad x, y \in \mathbb{R}^d,
\]
where $\gamma$ is one of the following three kernels: Letting $\sigma > 0$ be a fixed constant,

1. **(White)** $\gamma(x) = \sigma^2 \delta_0(x)$, where $\delta_0$ denotes the Dirac delta distribution.
2. **(Riesz)** $\gamma(x) = \sigma^2 |x|^{-\omega}$ for some $0 < \omega < d$.
3. **(Fractional)** $\gamma(x) = \sigma^2 \prod_{i=1}^d |x_i|^{-\omega_i}$ for some $\omega_i \in (0, 1)$.

Since $\gamma$ is either a Schwartz distribution or has a singularity at zero, $\xi$ must be defined rigorously as a random Schwartz distribution. More specifically, $\xi$ is a centered Gaussian process on $C_0^\infty(\mathbb{R}^d)$ (the smooth and compactly supported functions) with covariance
\[
\langle \xi(f)\xi(g) \rangle = \int_{[\mathbb{R}^d]^2} f(x)\gamma(x - y)g(y) \, dx\, dy, \quad f, g \in C_0^\infty(\mathbb{R}^d).
\]

Remark 1.4. The three covariance kernels listed in Assumption 1.3 all satisfy
\[
\gamma(cx) = c^{-\omega}\gamma(x), \quad c > 0, \quad x \in \mathbb{R}^d
\]
for some scaling exponent $\omega > 0$ ($\omega = \sum_{i=1}^d \omega_i$ in the case of fractional noise and $\omega = d$ in the case of white noise). $\omega$ can be viewed as a parameter that quantifies the extent to which $\xi$ is singular, i.e., a larger $\omega$ corresponds to a more singular covariance/noise.

The main difficulty in dealing with the PAM with singular noise is that $\xi$ is not actually a function that can be evaluated pointwise. Thus, extending the results on the intermittency of the PAM with regular noise to the singular setting poses serious technical challenges. In fact, the very definition of the singular PAM is nontrivial, as the pointwise product $\xi(x)u(t, x)$ in (1.1) requires a careful justification. One way to get around this issue, which leads to the Stratonovich solution, is to proceed as follows:
Definition 1.5. For every \( \varepsilon > 0 \), denote the Gaussian kernel with variance \( \varepsilon^2 \) as
\[
p_\varepsilon(x) := \frac{e^{-|x|^2/2\varepsilon^2}}{(2\pi\varepsilon^2)^{d/2}}, \quad x \in \mathbb{R}^d,
\]
noting that \( p_\varepsilon(x) = \varepsilon^{-d}p_1(x/\varepsilon) \). We define the mollified noise \( \xi_\varepsilon := \xi * p_\varepsilon/\sqrt{\varepsilon} \), where * denotes the convolution. In particular, \( \xi_\varepsilon \) is a centered Gaussian process on \( \mathbb{R}^d \) with covariance
\[
\langle \xi_\varepsilon(x)\xi_\varepsilon(y) \rangle = \gamma_\varepsilon(x-y), \quad x, y \in \mathbb{R}^d,
\]
where \( \gamma_\varepsilon := \gamma * p_\varepsilon \).

With this in hand, for every \( \varepsilon > 0 \), we may now consider the smoothed PAM
\[
\begin{align*}
\partial_t u_\varepsilon(t, x) &= (\kappa\Delta + \xi_\varepsilon(x)) u_\varepsilon(t, x) \\
u_\varepsilon(0, x) &= 1
\end{align*}
\tag{1.2}
\]
Since \( \xi_\varepsilon \) has smooth sample paths, the solution of (1.2) is classically well defined. Then, one hopes that the solution of (1.1) can be constructed by taking the limit
\[
u(t, x) := \lim_{\varepsilon \to 0} u_\varepsilon(t, x).
\tag{1.3}
\]

On the one hand, if the noise is not too singular—that is, if the condition \( 0 < \omega < 2 \) holds—then the limit (1.3) is nontrivial and gives rise to a meaningful notion of solution (e.g., [29, Section 5]). On the other hand, when \( \omega \geq 2 \), the limit (1.3) blows up. In some such cases, the divergence of \( u_\varepsilon(t, \cdot) \) can be compensated by subtracting diverging renormalization constants from the noise, thus allowing to salvage a meaningful notion of nontrivial solution [10, 14, 24, 26, 27, 28]. However, in some cases, the noise is so singular that it is not clear that any meaningful notion of Stratonovich solution can be salvaged (e.g., the white noise with \( \omega = d \geq 4 \); see [26, Section 1.1, (PAM)]).

Remark 1.6. The procedure described in the above paragraph works for more general mollifiers than the Gaussian kernel used in Definition 1.5. For simplicity and definiteness, in this paper we use the Gaussian kernel.

Remark 1.7. There are alternate ways of defining a solution of the singular PAM, such as the Skorokhod solution (e.g., [8, 11] and [29, Section 3]). The present paper deals exclusively with the Stratonovich solution.

1.2. Moments of the PAM - Known Results. In order to motivate our investigations, we review the known results regarding the moments of the continuous PAM with Gaussian noise.

1.2.1. Regular PAM. Consider \( u_1(t, \cdot) \) (i.e., (1.2) in the case \( \varepsilon = 1 \)), which is the PAM with smooth Gaussian noise \( \xi_1 \). In this case, [21, Theorem 1 and Section 4.1] states that
\[
\log\langle u_1(t, x)^p \rangle = \left( \frac{p^2\gamma_1(0)}{2} \right) t^2 - p^{3/2}\chi^{3/2}(1 + o(1)) \quad \text{as } t \to \infty
\tag{1.4}
\]
for every \( x \in \mathbb{R}^d \) and \( p \geq 1 \). In (1.4), we define
\[
\chi := \inf_{\|f\|_2 = 1} \left( \kappa \int_{\mathbb{R}^d} \|
abla f(x)\|^2 \, dx + \frac{1}{4} \int_{[\mathbb{R}^d]^2} f(x)^2 ((x - y)^\top \Sigma(x - y)) f(y)^2 \, dx \, dy \right),
\tag{1.5}
\]
where \( H^1(\mathbb{R}^d) \) denotes the order 1 Sobolev Hilbert space, \( \| \cdot \|_p \) denotes the Lebesgue \( L^p \) norm, and \( -\Sigma \) is the Hessian matrix of the covariance kernel \( \gamma_1(x) \) evaluated at \( x = 0 \). In particular,
intermittency in the sense of Definition 1.2 occurs with scale function \( A(t) = t^2 \) and Lyapunov exponents \( \ell_p = \frac{2^2 n_1(0)}{2} \) for all \( p \geq 1 \).

The first order in the asymptotic (1.4) (i.e., the term \( \ell_p t^2 \)) was proved by Carmona and Molchanov [3] for all \( p \in \mathbb{N} \). Later, Gärtner and König [21] extended the result to every \( p \geq 1 \) and also provided the second-order asymptotic \( p^{3/2} \chi t^{3/2} \). As explained in [21, Section 0.4.3], the derivation of this second order—and more specifically the minimizers of \( \chi \)—provide a precise description of the geometry of intermittent peaks in \( u_1(t, \cdot) \) for large \( t \).

**Remark 1.8.** The setting considered in [3, 21] is more general than stationary Gaussian noises with a covariance of the form \( \gamma_1 = \gamma * p_1 \) for a \( \gamma \) satisfying Assumption 1.3. Since the latter is the setting that is of interest to us in this paper, we do not state the moment asymptotics of the PAM with regular noise on \( \mathbb{R}^d \) in its full generality.

### 1.2.2. Singular PAM - Subcritical Regime. If \( \xi \) is as in Assumption 1.3 with scaling parameter \( 0 < \omega < 2 \), then we say that it is subcritical. In this regime, it is known that intermittency in the sense of Definition 1.2 occurs. More specifically, define the variational constant

\[
M := \sup_{f \in H^1(\mathbb{R}^d), \|f\|_1=1} \left( \frac{1}{2} \iint_{\mathbb{R}^d} f(x)^2 \gamma(x - y) f(y)^2 \, dx \, dy - \kappa \int_{\mathbb{R}^d} |\nabla f(x)|^2 \, dx \right).
\]

Arguing as in [13, Lemma A.2 with \( \alpha_0 = 0 \)], it can be shown that \( M \in (0, \infty) \). In the case of Riesz and Fractional noise, [12, Theorem 1.1, \( \beta_0 = \rho = 0 \) and \( \alpha = 2 \)] states that

\[
\lim_{t \to \infty} \frac{\log \langle u(t, x)^p \rangle}{t^{(4-\omega)/(2-\omega)}} = p^{(4-\omega)/(2-\omega)} M = \ell_p
\]

for all \( x \in \mathbb{R}^d \) and \( p \in \{1\} \cup [2, \infty) \). For white noise with \( d = \omega = 1 \), [37, (2)] proves (1.7) for \( p = 1 \), and [29, Theorem 6.9 (ii)] shows that there exists constants \( 0 < C < C' \) such that

\[
C(p t)^{(4-\omega)/(2-\omega)} \leq \log \langle u(t, x)^p \rangle \leq C'(p t)^{(4-\omega)/(2-\omega)}
\]

for \( p \in \mathbb{N} \) and \( t > 0 \).

**Remark 1.9.** While we did not find such a statement in the literature, we expect that the general methodology developed in [12, 13] could be suitably adapted to prove (1.7) for \( p \geq 2 \) in the case of one-dimensional white noise. That being said, if \( p \in (0, 1) \cup (1, 2) \), then (1.7) appears to be out of reach of the currently-known techniques for all three noises considered in Assumption 1.3; see [12, (6.2)].

### 1.2.3. Singular PAM - Critical Regime. If \( \xi \) is as in Assumption 1.3 with \( \omega = 2 \), then we say that it is critical. In this regime, it is expected that there exist moment blowup thresholds \( t_0(p) \in (0, \infty) \) such that

\[
\langle u(t, x)^p \rangle \begin{cases} < \infty & \text{if } t < t_0(p) \\ = \infty & \text{if } t > t_0(p) \end{cases}
\]

for all \( p > 0 \); see, e.g., [1, Theorem 1.7], [11, 23], [14, (1.26)], and [38, Corollary 1.2]. In particular, if there is intermittency in this regime, then it does not follow from Definition 1.2; the Lyapunov exponents do not exist when the moments blow up in finite time.

We refer to [11, Contribution 1 and Remarks 1.1 and 1.2] for a related result concerning the Skorokhod PAM when \( p \in \{1\} \cup [2, \infty) \), and to [11, Page 5] for the mention of a conjecture
regarding the value of \( t_0(p) \) in the Stratonovich setting considered in this paper. In both cases, it is known/conjectured that \( t_0(p) \) is related to the best constant

\[
G := \inf \left\{ C > 0 : \int \int (\mathbb{R}^d)^2 f(x)^2 \gamma(x-y)f(y)^2 \, dx \, dy \leq C \int_{\mathbb{R}^d} |\nabla f(x)|^2 \, dx \quad \forall f \in H^1(\mathbb{R}^d) \text{ s.t. } ||f||_2 = 1 \right\};
\]

see [11, (1.9), (3.21), Notation 3.12, and Theorem 3.14].

**Remark 1.10.** The fact that \( G \) is finite for the covariance kernels in Assumption 1.3 when \( \omega = 2 \) is established in [4, (C.1) for \( p = d = 2 \)] for white noise, Proposition 7.1 below for Riesz noise, and [11, Remark 3.11] for fractional noise.

1.2.4. **Singular PAM - Supercritical Regime.** If \( \xi \) is as in Assumption 1.3 with \( \omega > 2 \), then we say that it is supercritical. In this regime, it is expected that \( \langle u(t, x)^p \rangle = \infty \) for all \( p, t > 0 \); see, e.g., [23, Remark 1.7] and [32, Theorem 2 for \( d = 3 \)]. Thus, intermittency cannot be established by studying Lyapunov exponents in this regime either.

1.3. **Motivational Problems and Outline of Results.** The results discussed above raise a number of interesting problems. For instance:

**Problems 1.11.** In the subcritical regime:

- Can the maximizers of the variational problem \( M \) in (1.6) be argued to describe the geometry of intermittent peaks in the singular PAM, in similar fashion to the minimizers of \( \chi \) in (1.4) for the smooth PAM (e.g., the heuristic derivation in [21, Section 0.4.3])?
- If the answer to the above is affirmative, then how/why does the contribution of the geometry of peaks (i.e., the variational constant) undergo a transition from the second-to the first-order asymptotic when going from a regular to a singular noise?
- Can the exact asymptotic (1.7) be established for every \( p > 0 \) and every noise in Assumption 1.3 (in particular, including \( p \in (0, 1) \cup (1, 2) \))? 

**Problems 1.12.** In the critical and supercritical regimes:

- What is the geometric significance (if any) of the blowup of the moments (either in finite time in the critical regime or for all times in the supercritical regime)?
- What is the geometric significance (if any) of the variational constant \( G \) in (1.10) that is expected to characterize the moment blowup thresholds \( t_0(p) \) in (1.9)?

Our main results, which we state in full in Section 2, shed new light on all of these problems. More specifically, our results fit into three main categories:

Firstly, the main theorems proved in this paper are **Theorems 2.5, 2.17, and 2.22**. The latter provide moment asymptotics for the PAM with a noise that becomes increasing singular (i.e., \( \xi_\varepsilon \) as \( \varepsilon \to 0 \)) in the subcritical, critical, and supercritical regimes respectively. The setting that we consider in this paper, which is explained in details in Section 2.1, allows us to treat the PAM with smooth and singular noises in a unified way, as well as interpolate between the two settings. In particular, our method of proof is mostly inspired by the methodology developed in [21] for the smooth setting, and thus provides the same geometric interpretation of intermittency (see Sections 3.2 and 3.3 for the details).

Secondly, we have three corollaries of our main theorems, which improve known results and provide new insights on the behavior of the moments of the PAM with singular noise:
• As a corollary of Theorem 2.5, in Theorem 2.6 we extend (1.7) and (1.8) by establishing precise moment asymptotics in the subcritical regime for all \( p > 0 \), which also allow us to obtain precise tail asymptotics for the solution of the PAM. Combining this with the fact that our moment asymptotics in Theorem 2.5 interpolate between the classical and singular asymptotics in (1.4) and (1.7)/(1.8), as well as the geometric interpretation of our asymptotics in Sections 3.2 and 3.3, this provides a complete answer to Problems 1.11 (see Section 2.2 for more details).

• As corollaries of Theorems 2.17 and 2.22, in Theorems 2.19 and 2.23 we make progress on Problems 1.12 by uncovering that the moment blowup phenomenon discussed in Sections 1.2.3 and 1.2.4 is related to a finite time intermittency effect, which we can detect in the moment asymptotics of \( u_\varepsilon(t, \cdot) \) as \( \varepsilon \to 0 \) with \( t > 0 \) fixed. Moreover, we explain how the best constant \( G \) in (1.10) characterizes the occurrence of this intermittency phenomenon in the critical regime. To the best of our knowledge, these are the first results that establish the existence of an intermittency phenomenon in the PAM with an asymptotically singular noise and a fixed finite time.

Thirdly, in Theorem 2.11, we prove the existence and study the properties of the maximizers of the variational problem \( M \) in (1.6) in the case of the fractional kernel \( \gamma(x) = \sigma^2 \prod_{i=1}^d |x_i|^{-\omega_i} \); the relevance of these maximizers to intermittency is discussed in Section 3.3. Maximizers of variational problems of the form \( M \) for general \( \gamma \) are known in the mathematical physics literature as Hartree ground states and are important objects in quantum mechanics; see, e.g., [19] and references therein. (In that interpretation, the kernel \( \gamma \) represents an interaction potential between distinct particles.) Thus, the existence and properties of these maximizers have been studied for various choices of \( \gamma \), including the white and Riesz noises (see Theorems 2.7 and 2.9). To the best of our knowledge, the standard results in this theory concern the case where \( \gamma \) is symmetric decreasing and has integrable singularities (i.e., the measure of level sets \( \{x \in \mathbb{R}^d : \gamma(x) > t\} \) is finite); see, e.g., [19, Theorem 2]. Neither of these properties hold for the fractional covariance kernel. Thus, we believe that Theorem 2.11 may be of independent interest, as it studies the Hartree ground state problem in a situation where there is less symmetry and integrability than the usual setting.

1.4. Discussion.

1.4.1. Extensions. Many of the moment asymptotics stated in Sections 1.2.2–1.2.4 admit a generalization if

- \( \xi \) is replaced by a possibly time-dependent noise of the form
  \[
  \langle \xi(s, x) \xi(t, y) \rangle = |s - t|^{\omega_0} \gamma(x - y)
  \]
  for some \( \omega_0 \in [0, 1) \); and/or

- the term \( \xi(x) \cdot u(t, x) \) or \( \xi(t, x) \cdot u(t, x) \) in the definition of the PAM is interpreted as a Wick product, which leads to the Skorokhod solution; and/or

- the Laplacian operator \( \Delta \) is replaced by a fractional Laplacian of the form \( -(-\Delta)^s \) for some \( 0 < s < 1 \).

See, for instance, [6, 7, 9, 11, 12, 14]. It is thus natural to ask the following:

Problems 1.13. Is it possible to prove analogues of Theorems 2.5, 2.17, and 2.22 for the PAM with a time-independent noise and/or Skorokhod solution and/or fractional Laplacian?

Among other things, an affirmative answer to these problems could improve the known precise moment asymptotics in [12, Theorem 1.1] to all \( p > 0 \) in the subcritical case, and it
could show that the moment blowup phenomenon proved in [11, Contribution 1] for the critical Skorokhod/time-dependent case is also related to an intermittency effect that occurs in the mollified PAM when $\varepsilon \to 0$ for fixed times.

However, many specifics of the techniques employed in this paper cannot be directly extended to the time-independent noise, Skorokhod, or fractional Laplacian settings. Most notably, our frequent reliance on the spectral expansion (3.3), which is exclusive to the Stratonovich and time-independent settings. Thus, we expect that solving Problems 1.13 would require a number of new ideas (such as replacing spectral expansions with the hypercontractivity trick developed in [33]), which we leave open for future works.

1.4.2. Intermittency in the Critical and Supercritical Regimes. While Theorems 2.19 and 2.23 shed new light on the relationship between intermittency and the moment blowup phenomenon in the critical and supercritical regimes, they do not appear to say anything definitive about the occurrence (or not) of intermittency in the usual large-time setting. At most, these results suggest that if intermittency occurs in the critical or supercritical PAM at large times, then it is likely explained by a different mechanism than the PAM with smooth or subcritical noise.

More specifically, the phase transition in Theorem 2.5 shows that the large-time intermittency in the subcritical PAM comes from the same mechanism as the PAM with smooth noise: As illustrated in the transition from Figure 1 to Figure 3, as we make $\varepsilon$ increasingly small compared to $t$, the geometry of peaks in the noise and the PAM at large times undergoes a transition from the classical asymptotics, and eventually stabilizes to a size and shape independent of $\varepsilon$.

In contrast, Theorems 2.19 and 2.23 show that the intermittency in the PAM with smooth noise cannot transform into an intermittency effect that occurs at large times in the critical or supercritical PAM. Indeed, if we define the renormalized subcritical/critical PAM as

$$u(t, x) = \lim_{\varepsilon \to 0} u_\varepsilon(t, x)e^{-tc_\varepsilon}$$

(for an appropriate choice of diverging renormalization constants $c_\varepsilon$), then the tall and narrow peaks that generate the moment asymptotics in Theorem 2.19 and 2.23 disappear once we take the $\varepsilon \to 0$ limit, and thus have no contribution in the geometry of $u(t, x)$ as $t \to \infty$.

As a final remark, we note the work of König, Perkowski, and Van Zuijlen [31], who studied the almost-sure asymptotics of the critical Stratonovich PAM in the case of white noise. In particular, [31, Theorem 1.1 (b)] states that, from the point of view of almost-sure logarithmic asymptotics of the PAM, an intermittency effect cannot be observed (at least on boxes near the origin of side-length $\sim t^a$ for $0 < a < 1$). This is in sharp contrast to the case of smooth noise, as explained in [20, Section 1.6]. Therefore, it appears that the occurrence of an intermittency phenomenon at large times (as well as its exact nature if it does occur) in the critical and supercritical PAM remains open.

1.5. Organization of this Paper. The remainder of this paper is organized as follows: In Section 2, we explain the approach developed in this paper to tackle Problems 1.11 and 1.12, and then we state our main results. In Section 3, we introduce some notations, present a heuristic derivation of our main theorems, and discuss the geometric significance of the latter. Finally, in Sections 4–7, we prove our main results.

2. Main Results

Throughout this section (and for the remainder of this paper), we assume that $\xi$ is a singular noise that satisfies Assumption 1.3. We also recall that $\langle \cdot \rangle$ denotes the expectation with respect
to $\xi$, that $\omega$ is the scaling parameter introduced in Remark 1.4, and that $\xi_{\varepsilon}$ and $u_{\varepsilon}(t,x)$ ($\varepsilon > 0$) are the mollified noise and PAM introduced in Definition 1.5 and (1.2).

This section is organized as follows. In Section 2.1, we discuss the general philosophy behind the approach used in this paper. In Sections 2.2–2.4, we provide precise statements of the results outlined in Section 1.3. Then, in Section 2.5, we provide an index of where the proofs of each of our main results are located.

2.1. Our Approach: Asymptotically Singular Noise. Instead of studying $u_1(t,x)$ as $t \to \infty$ (i.e., (1.4)) or studying $u_{\varepsilon}(t,x)$ by first sending $\varepsilon \to 0$ and then $t \to 0$ (i.e., Sections 1.2.2–1.2.4), we consider the setting where both $\varepsilon$ and $t$ can vary simultaneously:

Definition 2.1. We introduce the master parameter

$$m \in [0, \infty).$$

We assume that $e = e(m)$ and $t = t(m)$ are positive functions of this parameter such that

$$\sup_{m \geq 0} e \leq 1 \quad \text{and} \quad \inf_{m \geq 0} t > 0. \tag{2.1}$$

The function $e$ represents the dependence of $\varepsilon$ on the master parameter $m$, and $t$ represents the dependence of $t$ on $m$.

Remark 2.2. As we have done in (2.1), in order to improve readability, throughout this paper we mostly keep the dependence of $e$ and $t$ (and various other functions) on the master parameter implicit. For instance, if we state an asymptotic of the form

$$\log \langle u_{e(t,x)^p} \rangle = F(e,t) + o(1) \quad \text{as } m \to \infty$$

for some functional $F$, then it should be understood as

$$\log \langle u_{e(m)}(t(m),x)^p \rangle = F(e(m),t(m)) + o(1) \quad \text{as } m \to \infty.$$

Limits of the form $\lim_{m \to \infty} F(e,t) = l$ should be interpreted in the same way.

We are mainly interested in taking $e \to 0$ as $m \to \infty$. In doing so, our hope is twofold:

1. Since $u_{e(t,\cdot)}$ has a smooth noise for every fixed $\varepsilon > 0$, some of the techniques used for the case $\varepsilon = 1$ in [21] that are not applicable to the singular case remain available.

2. If $\varepsilon$ is small, then $u_{e(t,\cdot)}$’s geometry is similar to that of $u(t,\cdot)$ (possibly up to a renormalization). Thus, if $e \to 0$ at a fast enough rate as $m \to \infty$, then the asymptotics

$$u_{e(t,\cdot)} = u_{e(m)}(t(m),\cdot)$$

might shed new light on the intermittency of the PAM with singular noise.

Since intermittency is a phenomenon that typically occurs at large times, we will in many cases consider the large-$m$ asymptotics of the moments $\langle u_{e(t,x)^p} \rangle$ under the assumption that both $e \to 0$ and $t \to \infty$ as $m \to \infty$. However, as we explain in the coming sections, one of the main insights of this paper is that it is also interesting to consider the case where $t$ remains bounded (see Theorems 2.19 and 2.23).

Remark 2.3. A similar approach was implemented in [22], wherein the main focus was to establish a phase transition in the almost-sure asymptotics of the PAM with asymptotically singular noise (see Theorems 1.7 and 1.11 therein). The moments in the case of asymptotically singular white noise were studied in [22, Theorem 1.17]. However, the latter is weaker and less general than the results in this paper, it is proved using different techniques, and it does not provide any insight on Problems 1.11 and 1.12 (which are the main focus of this paper).

We now proceed to an exposition of our results.
2.2. Main Results Part 1 - Subcritical Regime. Consider the subcritical regime where $0 < \omega < 2$, which we henceforth abbreviate as Sub.

**Assumption 2.4.** In the Sub regime, in addition to (2.1), we always assume that

$$\lim_{m \to \infty} t = \infty.$$  

We consider three cases of the Sub regime, depending on the behavior of $e$ relative to $t$:

- **Case Sub-1:** $\lim_{m \to \infty} e^{t^{1/(2-\omega)}} = \infty$.
- **Case Sub-2:** $\lim_{m \to \infty} e^{t^{1/(2-\omega)}} = c$ for some constant $c \in (0, \infty)$.
- **Case Sub-3:** $\lim_{m \to \infty} e^{t^{1/(2-\omega)}} = 0$.

Our first main result is as follows:

**Theorem 2.5.** Suppose that Assumption 2.4 holds. Let $p > 0$ and $x \in \mathbb{R}^d$ be fixed.

- **In case Sub-1,** as $m \to \infty$, one has

  $$\log \langle u_e(t, x)^p \rangle = \left(\frac{p^2 \gamma_2(0)}{2}\right) e^{-\omega t^2} - p^{3/2} c^{1/2} e^{-(2+\omega)/2} t^{3/2} (1 + o(1)),\tag{2.2}$$

  where we recall that $\chi$ is defined in (1.5).

- **In case Sub-2,**

  $$\lim_{m \to \infty} \frac{\log \langle u_e(t, x)^p \rangle}{t^{(4-\omega)/(2-\omega)}} = p^{(4-\omega)/(2-\omega)} M_{c,p},\tag{2.3}$$

  where we define the variational constant 

  $$M_{c,p} := \sup_{f \in H^1(\mathbb{R}^d)} \left( \frac{1}{2} \int_{\mathbb{R}^d} f(x)^2 \gamma_{\rho_1/(2-\omega)}(x-y)^2 dy - \kappa \int_{\mathbb{R}^d} |\nabla f(x)|^2 dx \right). \tag{2.4}$$

- **In case Sub-3,**

  $$\lim_{m \to \infty} \frac{\log \langle u_e(t, x)^p \rangle}{t^{(4-\omega)/(2-\omega)}} = p^{(4-\omega)/(2-\omega)} M,\tag{2.5}$$

  where we recall that $M$ is defined in (1.6).

2.2.1. Interpretation of Theorem 2.5. In case Sub-1, $t$ blows up quickly relative to $e$’s size. Thus, it is natural to expect that $u_e(t, \cdot)$ should behave similarly to $u_1(t, \cdot)$ as $m \to \infty$. The asymptotic (2.2) confirms this: Apart from the additional factors of $e^{-\omega}$ and $e^{-(2+\omega)/2}$ in the first- and second-order terms respectively, this is the same as the classical result (1.4). (In fact, in the case where $e = 1$ is constant, then we recover exactly (1.4), with the addition that we also prove moment asymptotics for $0 < p < 1$.)

Next, by carefully examining (2.2), it is to be expected that a transition occurs when $e$ is so small that the first- and second-order terms are the same size. This occurs when $e \approx t^{-1/(2-\omega)}$, which implies that $e^{-t^2}$ and $e^{-(2+\omega)/2} t^{3/2}$ are both on the order of $t^{(4-\omega)/(2-\omega)}$. In cases Sub-2 and Sub-3, we show that this transition does indeed occur, and eventually gives rise to the subcritical asymptotics stated earlier in (1.7).

Referring back to Problems 1.11, we note that Theorem 2.5 provides an affirmative answer to the first two items raised therein: Since our proof of Theorem 2.5 mostly follows the general methodology developed in [21], we can infer that the variational problems that appear in (2.3) and (2.5) (and their maximizers) have the same geometric interpretation as the constant $\chi$. 


that appears in the second-order term in (1.4). We refer to Sections 3.2 and 3.3 for a detailed geometric interpretation of the following from the point of view of intermittency:

(1) The maximizers of the variational problems in (2.3) and (2.5); and
(2) the coalescence phenomenon whereby the first- and second-order terms in (2.2) merge into a single term when \( \varepsilon \) is small enough compared to \( t \).

2.2.2. Precise Asymptotics for all \( p > 0 \). Regarding the third item raised in Problems 1.11, unlike the results stated in Section 1.2.2, the asymptotic (2.5) holds for all \( p > 0 \). Thus, as a corollary of (2.5), we obtain the following improvement of [12, Theorem 1.1 with \( \beta_0 = \rho = 0 \) and \( \alpha = 2 \)], [29, Theorem 6.9 (ii)], and [37, (2)], extending the precise asymptotics to all \( p > 0 \):

**Theorem 2.6.** Let \( \xi \) be one of the noises in Assumption 1.3, assuming that \( 0 < \omega < 2 \). For every \( p > 0 \) and \( x \in \mathbb{R}^d \), it holds that
\[
\lim_{t \to \infty} \frac{\log \langle u(t, x)^p \rangle}{t^{(4-\omega)/(2-\omega)}} = p^{(4-\omega)/(2-\omega)} M.
\]

In particular, for every \( \theta > 0 \) and \( x \in \mathbb{R}^d \), we have the tail asymptotic
\[
\lim_{t \to \infty} t^{-(4-\omega)/(2-\omega)} \log \mathbb{P} \left[ u(t, x) \geq \exp \left( \theta t^{(4-\omega)/(2-\omega)} \right) \right] = -\sup_{p>0} \left( \theta p - p^{(4-\omega)/(2-\omega)} M \right) = -2\theta^{(4-\omega)/2}(4 - \omega)^{-1}(4-\omega)/2 \left( \frac{M}{2 - \omega} \right)^{-\omega/2}.
\]

The moment asymptotics in Theorem 2.6 are an immediate consequence of (2.5) and (2.6)
\[
\lim_{\varepsilon \to 0} \frac{\log \langle u_{\varepsilon}(t, x)^p \rangle}{t^{(4-\omega)/(2-\omega)}} = \frac{\log \langle u(t, x)^p \rangle}{t^{(4-\omega)/(2-\omega)}} + o(1) \quad \text{as } m \to \infty.
\]

Once the moment asymptotics are established, the tail asymptotics follow from standard large deviations theory (i.e., the Gärtner-Ellis theorem; see, e.g., item (i) on page 4 of [12]).

2.2.3. Geometry of Maximizers. The significance of the variational constants \( M_{\varepsilon, p} \) and \( M \) in describing the geometry of intermittency (as explained in Sections 3.2 and 3.3) motivates studying the existence and properties of its maximizers.

For the white noise, \( 0 < \omega < 2 \) implies that \( d = 1 \). In this case, we have the following well-known result (e.g., [18, Lemma 3.1] and [37, (3)]):

**Theorem 2.7** ([18, 37]). Suppose that \( d = 1 \) and \( \gamma = \sigma^2 \delta_0 \). The set of maximizers of \( M \) consists of all functions \( f_* \) of the form
\[
f_*(x) = \pm \frac{\sigma}{2^{3/2}\kappa^{1/2}} \sech \left( \frac{\sigma^2}{4\kappa}(x - z) \right),
\]
where \( z \in \mathbb{R} \) can be any real number.

**Remark 2.8.** The importance of the function \( \sech = 1/cosh \) in the description of the geometry of intermittency of the PAM with one-dimensional white noise was also pointed out in [16], in the context of the localization of the associated Schrödinger operator on a large interval.

For the Riesz noise, the minimizers are not explicitly known, but their existence and basic properties are nevertheless well-understood and classical (e.g., [19, Theorem 2]):
Theorem 2.9 ([19]). Suppose that \( \gamma(x) = \sigma^2|x|^{-\omega} \) for some \( 0 < \omega < \min\{2,d\} \). There exists maximizers of \( M \). Suppose that \( f_\ast : \mathbb{R}^d \to \mathbb{R} \) is such a maximizer.

1. \( f_\ast \) is either positive or negative.
2. \( f_\ast \) is smooth and decays exponentially at infinity; that is, there exists some constants \( a, b > 0 \) such that \( |f_\ast(x)| \leq ae^{-b|x|} \) for all \( x \in \mathbb{R}^d \).
3. \( |f_\ast| \) is symmetric decreasing along every axis about a point; that is, there exists some \( z \in \mathbb{R}^d \) and a nonincreasing function \( \rho : [0,\infty) \to [0,\infty) \) such that \( |f_\ast(x - z)| = \rho(|x|) \) for all \( x \in \mathbb{R}^d \).

Remark 2.10. To the best of our knowledge, the uniqueness of the maximizer \( f_\ast \) in Theorem 2.9 (up to a \( \pm \) sign and/or a shift \( f_\ast(\cdot - z) \)) is only known when \( d = 3 \) and \( \omega = 1 \) [34]. Otherwise, the uniqueness appears to be a hard open problem (e.g. [19, Problem 3]).

Next, the existence and basic properties of the maximizers of \( M \) in the case of the fractional noise do not appear to be known. One of the contributions of this paper is to prove the existence and study the geometry of such maximizers:

Theorem 2.11. Let \( \gamma(x) = \sigma^2 \prod_{i=1}^d |x_i|^{-\omega_i} \), where \( \omega_i \in (0,1) \) and \( \omega = \sum_{i=1}^d \omega_i \in (0,2) \). There exists maximizers of \( M \). Suppose that \( f_\ast : \mathbb{R}^d \to \mathbb{R} \) is such a maximizer.

1. \( f_\ast \) is either positive or negative.
2. \( f_\ast \) is smooth and decays exponentially at infinity.
3. \( |f_\ast| \) is symmetric decreasing along every axis about a point; that is, there exists some \( z \in \mathbb{R}^d \) such that for every fixed \( 1 \leq i \leq d \) and \( x_j \in \mathbb{R} \) (for all \( j \neq i \)), there exists a nonincreasing function \( \rho : [0,\infty) \to [0,\infty) \) such that
\[
|f_\ast((x_1,\ldots,x_{i-1},r,x_{i+1},\ldots,x_d) - z)| = \rho(|r|) \quad \text{for every } r \in \mathbb{R}.
\]

Remark 2.12. The fractional covariance kernel \( \gamma(x) = \sigma^2 \prod_{i=1}^d |x_i|^{-\omega_i} \) is symmetric decreasing along every axis about the point \( z = 0 \). Thus, much like the white and Riesz noises, the maximizers of \( M \) for fractional noise inherit some of the same symmetries (up to a translation) as the associated covariance kernel.

Remark 2.13. We do not attempt to settle the question of the uniqueness (up to change of sign or translations) of the maximizers in Theorem 2.11, which we also expect to be a hard problem (see Remark 2.10).

Finally, in Section 3.3 we showcase how the maximizers of \( M_{c,p} \) for \( c \in (0,\infty) \) describe the transition of the geometry of intermittency from the smooth asymptotics in (2.2) to the singular ones in (2.5). This can be made more formal with the following statement:

Proposition 2.14. Let \( 0 < \omega < 2 \). For every \( p, c > 0 \), the constant \( M_{c,p} \) is finite and positive, and it has maximizers. For every fixed \( p > 0 \), we have the limit
\[
(2.7) \quad \lim_{c \to 0} M_{c,p} = M.
\]
Moreover, if \( (f_\ast^{(c)})_{c > 0} \) is a sequence such that \( f_\ast^{(c)} \) is a maximizer of \( M_{c,p} \) for all \( c > 0 \), then every vanishing sequence of \( c \)'s has a subsequence \( (c_n)_{n \in \mathbb{N}} \) along which
\[
(2.8) \quad \lim_{n \to \infty} \|f_\ast^{(c_n)}(\cdot - z_n) - f_\ast\|_{H^1(\mathbb{R}^d)} = 0
\]
for some \( z_n \in \mathbb{R}^d \) and \( f_\ast \) that is a maximizer of \( M \).

Remark 2.15. The limit \( f_\ast \) in (2.8) will in general depend on the subsequence \( c_n \).
2.3. Main Results Part 2 - Critical Regime. Consider the critical regime \( \omega = 2 \), which we henceforth abbreviate as Crt.

**Assumption 2.16.** For Crt, in addition to (2.1), we assume that one of the following two cases holds:

- Case Crt-1: \( \lim_{m \to \infty} t = \infty \).
- Case Crt-2: \( \lim_{m \to \infty} e = 0 \) and \( \lim_{m \to \infty} t = t \) for some fixed time \( t \in (0, \infty) \).

In particular, in this regime \( t \) does not necessarily blow up as \( m \to \infty \).

Our second main result is as follows:

**Theorem 2.17.** Suppose that Assumption 2.16 holds. Let \( p > 0 \) and \( x \in \mathbb{R}^d \) be fixed.

- In case Crt-1, as \( m \to \infty \), one has
  \[
  \lim_{m \to \infty} \frac{\log \langle u_e(t,x)^p \rangle}{e^{-2}} = pt \, M_{t,p}^{\text{crt}},
  \]
  where we define the variational constant

  \[
  M_{t,p}^{\text{crt}} := \sup_{f \in H^2(\mathbb{R}^d) \atop \|f\|_2 = 1} \left( \frac{pt}{2} \int_{(\mathbb{R}^d)^2} f(x)^2 \gamma_1(x-y) f(y)^2 \, dy \, dx - \kappa \int_{\mathbb{R}^d} |\nabla f(x)|^2 \, dx \right).
  \]

2.3.1. Interpretation of Theorem 2.17. In (2.9), the first order term \( e^{-2}t^2 \) always dominates the second order term \( e^{-2}t^{3/2} \) when \( t \to \infty \). Thus, no matter how quickly \( e \) vanishes compared to \( t \)'s size, we do not observe a phase transition in the large-time asymptotics similar to Theorem 2.5 in the Crt regime.

Instead, in order for the the first- and second-order terms in (2.9) to be of the same size and coalesce, \( t \) must remain bounded, in which case both terms grow on the order of \( e^{-2} \). Case Crt-2 describes one aspect of this situation—when \( t \) converges to a constant time \( t \)—which yields the asymptotic (2.10). As it turns out, the latter result provides unexpected insights into Problems 1.12. In order to understand why that is, we state some properties of the variational constant therein:

**Proposition 2.18.** The constant \( M_{t,p}^{\text{crt}} \) is finite for every \( p > 0 \) and \( t > 0 \). Moreover, letting \( G \) be defined as in (1.10), we have that

1. if \( p < \frac{2k}{\ell_0} \) (equivalently \( t < \frac{2k}{\ell_0} \)), then \( M_{t,p}^{\text{crt}} = 0 \); and
2. if \( p > \frac{2k}{\ell_0} \) (equivalently \( t > \frac{2k}{\ell_0} \)) then \( M_{t,p}^{\text{crt}} \) is positive and strictly increasing in \( p \).

In particular, (2.10) and Proposition 2.18 (2) imply the following surprising result:

**Theorem 2.19.** Let \( \xi \) be one of the noises in Assumption 1.3 in the critical regime \( \omega = 2 \). For every fixed \( t > 0 \), the random field \( x \mapsto \xi(t,x) \) is intermittent as \( \varepsilon \to 0 \) with rate function \( A(\varepsilon) = \varepsilon^{-2} \), in the sense that the Lyapunov exponents

\[
\ell_p = ptM_{t,p}^{\text{crt}} = \lim_{\varepsilon \to 0} \frac{\log \langle \xi(t,x)^p \rangle}{\varepsilon^{-2}}
\]

satisfy Definition 1.2 for \( p \in \left( \frac{2k}{\ell_0}, \infty \right) \).
Given the first main result of [11] (see Contribution 1 therein) regarding the moment blowup threshold for the critical Skorokhod PAM, and the blowup threshold [38, Corollary 1.2] for the first moment of the Stratonovich solution with white noise on $\mathbb{R}^2$, it is natural to conjecture the following:

**Conjecture 2.20.** $t_0(p)$ in (1.9) is equal to $\frac{2\kappa}{p^2}$ for all $p > 0$.

If true, then combining this conjecture with Theorem 2.19 answers parts of Problems 1.12 in the following surprising fashion: In the critical Stratonovich PAM, for every $p > 0$, the moment blowup threshold $t_0(p)$ corresponds to the smallest fixed time $t > 0$ for which we observe an intermittency phenomenon as $\varepsilon \to 0$ in the moments $\langle u_\varepsilon(t, x)^q \rangle$ of order $q > p$.

### 2.4. Main Results Part 3 - Supercritical Regime.

Consider the critical regime $\omega > 2$, which we henceforth abbreviate as Sup.

**Assumption 2.21.** In the Sup regime, in addition to (2.1), we assume that at least one of the following holds:

$$
\lim_{m \to \infty} e = 0 \quad \text{or} \quad \lim_{m \to \infty} t = \infty.
$$

**Theorem 2.22.** Suppose that Assumption 2.21 holds. Let $p > 0$ and $x \in \mathbb{R}^d$ be fixed. As $m \to \infty$, one has

$$
\log \langle u_\varepsilon(t, x)^p \rangle = \left( \frac{p^2\gamma(0)}{2} \right) e^{-\omega t^2} - p^{3/2} \chi e^{-(2+\omega)/2t^{3/2}}(1 + o(1)),
$$

where we recall that $\chi$ is defined in (1.5).

**2.4.1. Interpretation of Theorem 2.22.** Our proposed interpretation of Theorem 2.22 is a continuation of Theorems 2.17 and 2.19, and has similar implications for Problems 1.12. More specifically, Theorem 2.22 implies the following:

**Theorem 2.23.** Let $\xi$ be one of the noises in Assumption 1.3 in the supercritical regime $\omega > 2$. For every fixed $t > 0$, the random field $x \mapsto u_\varepsilon(t, x)$ is intermittent as $\varepsilon \to 0$ with scale function $A(\varepsilon) = e^{-\omega}$ and Lyapunov exponents $\ell_p = \frac{p^2\gamma(0)}{2}$.

Moreover, the second-order term in (2.12) suggests that the local geometry of $u_\varepsilon(t, \cdot)$’s intermittent peaks as $\varepsilon \to 0$ is determined by the same mechanism as the PAM with smooth Gaussian noise. This reinforces the idea that the moment blowup phenomenon in the singular PAM can be explained by the occurrence of intermittency in smooth approximations $u_\varepsilon(t, \cdot)$ for fixed $t$ and small $\varepsilon$ (with the difference that, unlike in the Crt regime, in the Sup regime there is no limitation on the moments for which intermittency holds).

**Remark 2.24.** Looking at the specific form of (2.12), it is natural to suspect that a phase transition in the asymptotics occurs when $t$ goes to zero at the same rate or faster than $e^{\omega - 2}$. We do not pursue this direction in this paper, as it is not clear to us that small-time asymptotics have an interesting interpretation in the PAM.

### 2.5. Proof Index.

Theorems 2.5, 2.17, and 2.22 are proved in Sections 3–6. More specifically:

- in Section 3 we provide some notations, a heuristic derivation, and a geometric interpretation of our moment asymptotics; and
- in Sections 4–6, we provide a rigorous proof of every step of the heuristic derivation.
Proposition 2.18 is proved in Section 7.2, and Theorem 2.11 and Proposition 2.14 are proved together in Sections 7.3–7.8. Finally, as Theorems 2.6, 2.19, and 2.23 are immediate consequences of Theorems 2.5, 2.17, and 2.22 and Proposition 2.18, they are not further discussed in the paper.

3. Notations and Heuristics for Moment Asymptotics

In this section, our main purpose is to provide a heuristic derivation and geometric interpretation of Theorems 2.5, 2.17, and 2.22. We take this opportunity to set up some notations that will be used throughout the paper. Among other things, this allows us to formulate a statement that combines Theorems 2.5, 2.17, and 2.22 into a single result, namely, Theorem 3.4. We then formally prove Theorem 3.4 in Sections 4–6.

Remark 3.1. The main source of inspiration for the method developed in this paper is the work of Gärtner and König [21]. Given that some technical estimates used in this paper are cited directly from [21], most of our notations are meant to mirror that of [21] for sake of convenience. However, the difference between the asymptotically singular setting considered herein and that of [21] means that some notations have to be suitably modified. Whenever there are significant differences between the notations, we address it in a remark.

3.1. Semigroup Theory Notations. The main tool used in the proof of our moment asymptotics is the Feynman-Kac formula. For this, we use the following:

- Given any \( r > 0 \), we denote the centered open box with side-length \( 2r \) as \( Q_r := (-r, r)^d \).
- We use \( 1 \) to denote the function equal to one everywhere, and given a set \( K \), we use \( 1_K \) to denote the indicator function of \( K \).
- We use \( (W_t)_{t \geq 0} \) to denote a Brownian motion with generator \( \kappa \Delta \). We use \( E_x \) and \( P_x \) to denote the expectation and probability law of \( W \) conditioned on the starting point \( W_0 = x \). (Remark. We always assume that \( W \) is independent of the noise \( \xi \).)
- We use \( (L_t)_{t \geq 0} \) to denote \( W \)'s normalized occupation measures; that is, \( L_t(K) := \frac{1}{t} \int_0^t 1_{\{W_s \in K\}} \, ds \) for any Borel set \( K \subset \mathbb{R} \).
- For every closed set \( K \subset \mathbb{R}^d \), we use \( \mathcal{P}(K) \) to denote the set of probability measures supported in \( K \).
- We use \( (\cdot, \cdot) \) to denote both the \( L^2 \) inner product and integration against a measure. That is, for two functions \( f, g \) and a measure \( \mu \), one has \( (f, g) := \int_{\mathbb{R}^d} f(x)g(x) \, dx \) and \( (f, \mu) := \int_{\mathbb{R}^d} f(x) \, d\mu(x) \).
- We denote the Dirichlet form of \( \kappa \Delta \) as
  \[
  \mathcal{S}(f) := \kappa \int_{\mathbb{R}^d} |\nabla f(x)|^2 \, dx, \quad f \in H^1(\mathbb{R}^d).
  \]

Henceforth, given a smooth function \( V \) (possibly random), we use \( u^V(t, x) \) to denote the solution of

\[
\begin{cases}
\partial_t u^V(t, x) = (\kappa \Delta + V(x))u^V(t, x), \\
u^V(0, \cdot) = 1
\end{cases}, \quad t \geq 0, \ x \in \mathbb{R}^d.
\]

In particular, with the notation introduced in previous sections, we have that \( u_\varepsilon(t, x) = u^{\varepsilon}(t, x) \).
For every smooth function $V$ and $r > 0$ we use $u^V_r(t, x)$ to denote the solution of
\[
\begin{align*}
\partial_t u^V_r(t, x) &= (\kappa \Delta + V(x)) u^V_r(t, x) \\
u^V_r(0, \cdot) &= 1_{Q_r} \\
u^V_r(t, x)1_{\mathbb{R}^d \setminus Q_r}(x) &= 0
\end{align*}
\]
, \quad t \geq 0, \ x \in \mathbb{R}^d.

By the Feynman-Kac formula,
\begin{equation}
(3.2) \quad u^V_r(t, x) = \mathbb{E}_x \left[ e^{t(V,L_t)} \right] \quad \text{and} \quad u^V_r(t, x) = \mathbb{E}_x \left[ e^{t(V,L_t)} 1_{\{L_t \in \mathcal{P}(Q_r)\}} \right].
\end{equation}

We let $\lambda^V_k(Q_r)$ ($k \in \mathbb{N}$) denote the eigenvalues of the operator $\kappa \Delta + V$ on $Q_r$ with Dirichlet boundary conditions in decreasing order, and we let $e^V_k(Q_r)$ denote the corresponding orthonormal eigenfunctions. In particular, we have the spectral expansion
\begin{equation}
(3.3) \quad u^V_r(t, \cdot) = \sum_{k=1}^{\infty} e^{t \lambda^V_k(Q_r)} (e^V_k(Q_r), 1) e^V_k(Q_r).
\end{equation}

Though the eigenfunctions $e^V_k(Q_r)$ are defined on $Q_r$, we extend their domain to all of $\mathbb{R}^d$ by setting their value to be zero outside of $Q_r$.

### 3.2. Heuristic Derivation of Theorems 2.5, 2.17, and 2.22

If intermittency occurs in $u^\xi(t, \cdot)$, then we expect that there exists sparse localized regions of $\mathbb{R}^d$ inside which $u^\xi(t, \cdot)$ takes unusually large values. In particular, the main contribution to $\langle u^\xi(t, x)^p \rangle$ comes from the event where a large peak in the noise is present near $x$, thus inducing a large peak in $u^\xi(t, \cdot)$ near $x$ as well. Since $u^\xi(t, \cdot)$ is stationary, there is no loss of generality in studying this phenomenon at $x = 0$. In this case, we expect that we can approximate
\begin{equation}
(3.4) \quad \langle u^\xi(t, 0)^p \rangle \approx \langle u^\xi_{R\alpha\varepsilon}(pt, t, 0)^p \rangle = \left( \mathbb{E}_0 \left[ e^{t((\varepsilon, L_t))} 1_{\{L_t \in \mathcal{P}(Q_{R\alpha\varepsilon}(pt))\}} \right] \right)^p,
\end{equation}

where $R > 0$ is a large constant and $\alpha\varepsilon(pt) > 0$ is a scaling function (which depends on $p, \varepsilon$, and $t$) that describes the diameter of $u^\xi(t, 0)$’s large peak near the origin that contributes the most to the $p$th moment. Once we restrict the support of the occupation measure $L_t$ to the box $Q_{R\alpha\varepsilon(pt)}$, we can use the spectral expansion (3.3). If we apply this to (3.4), neglecting all terms but the leading eigenvalue (which, asymptotically, should dominate all other terms), then we are led to
\begin{equation}
(3.5) \quad \langle u^\xi(t, 0)^p \rangle \approx \langle e^{t \lambda^\xi_1(Q_{R\alpha\varepsilon(pt)})} \rangle.
\end{equation}

Thus, we now wish to understand how the geometry of large peaks in $\xi \varepsilon$ determines the behavior of the leading eigenvalue’s moment generating function.

In order to study the asymptotic behavior of (3.5), for every $p, \varepsilon, t > 0$, we introduce a rescaling of the noise of the form
\begin{equation}
(3.6) \quad \Xi_{\varepsilon,t}^\varepsilon(x) := \alpha\varepsilon(pt)^2 \left( \xi\varepsilon(\alpha\varepsilon(pt)x) - \frac{H\varepsilon(pt)}{pt} \right), \quad x \in \mathbb{R}^d
\end{equation}
for an appropriate choice of scaling function $H\varepsilon$. By a rescaling of the eigenvalue, this yields
\begin{equation}
(3.7) \quad \langle e^{t \lambda^\xi_1(Q_{R\alpha\varepsilon(pt)})} \rangle \approx e^{H\varepsilon(pt)} \langle e^{\beta\varepsilon(pt) \lambda^\xi_1(Q_{R\alpha\varepsilon(pt)})} \rangle,
\end{equation}

where we define $\beta\varepsilon(pt) := pt/\alpha^2\varepsilon(pt)$. Suppose that we can prove a large deviation principle for the rescaled noise $\Xi_{\varepsilon,t}^\varepsilon$ as $m \to \infty$, which, informally, takes the following form:
Proposition 3.2 (Informal). For any “shape” function \( V \geq 0 \) supported on \( Q_R \),

\[
\begin{align*}
(3.8) \quad \text{Prob} \left[ \xi_t \text{ has a peak with shape } \frac{H_0(\rho t)}{\rho t} + V(\cdot/\alpha_0(\rho t)) \text{ in } Q_{R_0(\rho t)} \right] \\
= \text{Prob} \left[ \mathbb{E}_{\rho t}^T \text{ has a peak with shape } V \text{ in } Q_R \right] \approx e^{-\beta_0(\rho t)J(\rho t)^2} \\
as m \to \infty \text{ for some rate function } J^p.
\end{align*}
\]

(3.9) \( e^{H_0(\rho t)} \left( e^{\beta_0(\rho t)\lambda_{1/2}^-(\rho t)^2}}(Q_R) \right) \approx e^{H_0(\rho t)} \int e^{\beta_0(\rho t)(\lambda_{1/2}^+(\rho t)^2 - J(\rho t)^2))} \, dV \approx e^{H_0(\rho t) + \beta_0(\rho t) \sup V (\lambda_{1/2}^+(\rho t)^2 - J(\rho t)^2))}.

At this point, if we take \( R \to \infty \) in (3.9) and combine the result with (3.5) and (3.7), then we are led to the asymptotic

\( \langle u^\xi(t,0) \rangle = H_0(\rho t) - \beta_0(\rho t)\chi^p(1 + o(1)) \) as \( m \to \infty \),

where

\( \chi^p = -\sup_{V} \left( \lambda_{1/2}^+(\rho t)^2 - J^p(V) \right) \)

\( = -\sup_{V} \left( \sup_{f} \{ (V, f^2 - \mathcal{J}(f)) - J^p(V) \} = \inf_{f} \left( \mathcal{J}(f) + J^p(f^2) \right) \right) \)

and \(-J^p\) denotes \( J^p\)'s Fenchel-Legendre transform.

With this in hand, the statements of Theorems 2.5, 2.17, and 2.22 are obtained by noting that a formal version of something like Proposition 3.2 (see Propositions 4.4 and 4.5) can be proved using the following dual rate function \( J^p \) and scaling functions \( \alpha_\varepsilon, \beta_\varepsilon \), and \( H_\varepsilon \):

**Definition 3.3.** For every \( \varepsilon, t > 0 \), we define \( \alpha_\varepsilon(t), \beta_\varepsilon(t), \text{ and } H_\varepsilon(t) \) as follows:

| Case | \( \alpha_\varepsilon(t) \) | \( \beta_\varepsilon(t) \) | \( H_\varepsilon(t) \) |
|------|-----------------|-----------------|-----------------|
| Sub-1 | \( \varepsilon^{(2+\omega)/4} t^{-1/4} \) | \( \varepsilon^{-(2+\omega)/2} t^{3/2} \) | \( \varepsilon^{-\omega t^2 \gamma_1(0)/2} \) |
| Sub-2 | \( \varepsilon^{-1/(2-\omega)} t^{(4-\omega)/(2-\omega)} \) | \( \varepsilon^{-1/(2-\omega)} t^{(4-\omega)/(2-\omega)} \) | 0 |
| Sub-3 | \( \varepsilon^{-1/(2-\omega)} t^{(4-\omega)/(2-\omega)} \) | \( \varepsilon^{-1/(2-\omega)} t^{(4-\omega)/(2-\omega)} \) | 0 |
| Crt-1 | \( \varepsilon^{t^{-1/4}} \) | \( \varepsilon^{-\omega t^{2 \gamma_1(0)/2}} \) | \( \varepsilon^{-\omega t^{2 \gamma_1(0)/2}} \) |
| Crt-2 | \( \varepsilon \) | \( \varepsilon^{-2 t^{3/2}} \) | 0 |
| Sup | \( \varepsilon^{(2+\omega)/4} t^{-1/4} \) | \( \varepsilon^{-(2+\omega)/2} t^{3/2} \) | \( \varepsilon^{-\omega t^{2 \gamma_1(0)/2}} \) |

Next, define

\[
J_\infty(\mu) := \frac{1}{4} \int_{\mathbb{R}^d} (x - y) ^\top \sum (x - y) \, d\mu(x) d\mu(y), \quad \mu \in \mathcal{P}(\mathbb{R}^d),
\]

recalling that \(-\sum\) is the Hessian matrix of \( \gamma_1 \) at \( x = 0 \). For every \( c \in [0, \infty) \), let

\[
J_c(\mu) := \frac{1}{2} \int_{\mathbb{R}^d} \gamma_c(x - y) \, d\mu(x) d\mu(y), \quad \mu \in \mathcal{P}(\mathbb{R}^d).
\]

In both cases above (i.e., for \( c \in [0, \infty) \)), if \( f \in H^1(\mathbb{R}^d) \) is such that \( \|f\|_2 = 1 \), then we use the convention that \( J_c(f^2) = J_c(f^2(x)dx) \). For every \( p > 0 \), we define the dual rate function \( J^p \) as

| Case | Sub-1 | Sub-2 | Sub-3 | Crt-1 | Crt-2 | Sup |
|------|-------|-------|-------|-------|-------|-----|
| \( J^p \) | \( J_\infty \) | \( J_{\mu^{1/(2-\omega)} t} \) | \( J_0 \) | \( J_\infty \) | \( ptJ_1 \) | \( J_\infty \) |
Finally, for every $p > 1$, we let

$$\chi^p := \inf_{f \in H^1(\mathbb{R}^d), \|f\|_2=1} \left( \mathcal{J}(f) + J^p(f^2) \right) = \begin{cases} \chi & \text{ (Sub-1, Crt-1, Sup)} \\ -M_{\chi,p} & \text{ (Sub-2)} \\ -M & \text{ (Sub-3)} \\ -M_{\chi,p}^{\text{crit}} & \text{ (Crt-2)} \end{cases}.$$ 

In particular, Theorems 2.5, 2.17, and 2.22 can be summarized as follows:

**Theorem 3.4.** Suppose that one of Assumptions 2.4, 2.16, or 2.21 holds. For every $p > 0$ and $x \in \mathbb{R}^d$, one has

$$\lim_{m \to \infty} \log \frac{\langle u_\varepsilon(t, x)^p \rangle - H_\varepsilon(pt)}{\beta_\varepsilon(pt)} = -\chi^p.$$ 

**Remark 3.5.** In [21], the rescaled noise in (3.6) is instead denoted $\xi_t$. This is because in [21], the scaling parameters $\alpha, \beta$ and $H$ only depend on time (in contrast to depending on both time and the parameter $\varepsilon$ in our setting). Given that $\xi_\varepsilon$ already has an index that keeps track of its dependence on $\varepsilon$, we use the different notation $\Xi_{\varepsilon, t}$.

**Remark 3.6.** In [21], the scaling function denoted by $H$ corresponds to the cumulant generating function of the noise (i.e., $H(t) = \log \langle e^{\varepsilon(0)} \rangle$). In our paper this is still true in cases Sub-1, Crt-1, and Sup (i.e., $H_\varepsilon(t) = \log \langle e^{\varepsilon(0)} \rangle$), but not in cases Sub-2, Sub-3, and Crt-2. Our main purpose for defining $H_\varepsilon(t) = 0$ in those latter three cases is to set up a convenient unified notation, which simplifies some statements. That said, the fact that the leading-order asymptotics of the PAM’s moments are not determined by the cumulant generating function in case Sub-3 has important implications for the geometry of intermittency; see Section 3.3.

**Remark 3.7.** $J^p$ and $\chi^p$ only depend on $p$ in cases Sub-2 and Crt-2. We nevertheless use the superscript in all cases to avoid different notations in different cases. We note that $J^p$ and $\chi^p$ actually depend on another parameter in those two cases, respectively $c$ and $t$. Since these parameters form an integral part of the very definition of cases Sub-2 and Crt-2 (see Assumptions 2.4 and 2.16) and are otherwise fixed, we leave this dependence implicit.

### 3.3. Geometric Interpretation.

Let $x \in \mathbb{R}^d$ and $p > 0$ be fixed. The large deviations heuristics provided in equations (3.4)–(3.9) and Proposition 3.2 suggests that the event that contributes the most to the asymptotics for $\langle u_\varepsilon(t, x)^p \rangle$ (stated in Theorem 3.4) is the following:

1. A large peak in the noise $\xi_\varepsilon$ close to $x$ of the form

$$\xi_\varepsilon(y) \approx \frac{H_\varepsilon(pt)}{pt} + \frac{V_*(y/\alpha_\varepsilon(pt))}{\alpha_\varepsilon(pt)^2} \quad \text{for } y \approx x,$$

where

$$V_* := \arg \sup_V \left( \sup_f \{ (V, f^2) - \mathcal{J}(f) \} - P(V) \right).$$

2. Equivalently, a large peak in $u_\varepsilon(t, \cdot)^p$ close to $x$ of the form

$$u_\varepsilon(t, y)^p \approx e^{H_\varepsilon(pt) - \beta_\varepsilon(pt)\chi^p(1+o(1))} f_* \left( \frac{y}{\alpha_\varepsilon(pt)} \right)^p \quad \text{for } y \approx x,$$

where

$$f_* = \arg \inf_g \left( \mathcal{J}(g) + J^p(g^2) \right).$$
In Cases Sub-1, Crt-2, and Sup, this geometry is similar to the one that gives rise to the classical asymptotics (1.4) uncovered in [21]. That is, as illustrated in Figure 1, the geometry of intermittent peaks requires two distinct orders for its description: On the one hand, the first-order scaling function $H_\varepsilon^p$ (which, in those cases, is the cumulant generating function of the noise; see Remark 3.6) describes the height of the intermittent peaks. On the other hand, the second-order scaling functions $\alpha_\varepsilon$ and $\beta_\varepsilon$ and the minimizers $V^*$ and $f^*$ describe the height, the width, and the shape of those peaks close to their summits.

As a continuation of Sections 2.2.1 and 2.3.1, we note that if $e$ and $t$ are such that $\beta_\varepsilon^p$ is of the same order as the cumulant generating function of $\xi_e$, then the geometry of intermittent peaks undergoes a transition. This gives rise to cases Sub-2 and Crt-2, which are illustrated in Figure 2. Therein, we note that we only need $\alpha_\varepsilon$ and $\beta_\varepsilon$ to describe the magnitude of the peaks, and $V^*$ and $f^*$ describe the shapes of the entire peaks, rather than just the summits.

**Remark 3.8.** In case Crt-2, if $t \to t$, then we expect that Figure 2 only describes the geometry of intermittency when $p > \frac{2\kappa}{|G|}$, since this is the situation when $M^\text{crt}_{t,p} \neq 0$; see Proposition 2.18.
Finally, if we take $c \to 0$ in case Sub-2, then the geometry undergoes yet another transition, giving rise to case Sub-3. The latter is illustrated in Figure 3. Similarly to cases Sub-2 and Crt-2, we only need $\alpha_\epsilon$ and $\beta_\epsilon$ to describe the magnitude of intermittent peaks, and the optimizers of the variational problems describe the shape of the entire peaks. However, there is one meaningful difference between Figures 2 and 3: In case Sub-3, the approximate equality (3.10) should not be interpreted literally in the pointwise sense, but instead in the sense of integration against the noise:

$$\left(\xi_\epsilon, f\right) \approx \left(\xi_\epsilon, f\right) \left(V_{\ast}((pt)^{1/(2-\omega)}), f\right), \quad f \in C_0^\infty(\mathbb{R}^d).$$

Indeed, in case Sub-3 the parameter $\epsilon$ can vanish arbitrarily quickly, meaning that the variance of $\xi_\epsilon(m)$ for large $m$ can be arbitrarily large. In particular, the random field $x \mapsto \xi_\epsilon(m)(x)$ becomes very singular as $m \to \infty$ (as illustrated by the sharp oscillations in the noise on the left-hand side of Figure 3, represented by green “spikes”), and the cumulant generating function of $\xi_\epsilon$ does not appear to describe any aspect of the geometry of intermittency. In particular, the geometry of peaks in this case is independent of the parameter $\epsilon$.

4. PROOF OF THEOREM 3.4 PART 1 - LARGE DEVIATIONS

4.1. Outline. Our main purpose in this section, which consists of the first step in the proof Theorem 3.4, is to formalize the informal large deviation principle stated in Proposition 3.2. This is done in Propositions 4.4 and 4.5 below. Much like in [21], the main difference between the informal statement and Propositions 4.4 and 4.5 is that the latter are proved in the dual setting, using the Varadhan lemma directly. The immediate culmination of these results are the following two lemmas, which are preliminary versions of Theorem 3.4:

**Definition 4.1.** For every $p, R > 0$, define

$$X_R^p := \inf_{f \in H^1(\mathbb{R}^d), \|f\|_2 = 1} \left(\kappa_\epsilon(f) + J^p(f^2)\right).$$

**Lemma 4.2.** Let $p > 0$ be fixed, and let $d = d(m) \geq 0$ be such that $0 \leq d < t$ and

$$\lim_{m \to \infty} \frac{\alpha_\epsilon(p(t - d))}{\alpha_\epsilon(pt)} = 1, \quad \lim_{m \to \infty} \frac{\beta_\epsilon(p(t - d))}{\beta_\epsilon(pt)} = 1, \quad \lim_{m \to \infty} \frac{H_\epsilon(p(t - d)) - H_\epsilon(pt)}{\beta_\epsilon(pt)} = 0.$$
For every $R > 0$, one has

$$
\liminf_{m \to \infty} \frac{\log \left( \left( u_{R}^{e}(pt)(p(t - d), \cdot), 1 \right) \right) - H_{e}(pt)}{\beta_{e}(pt)} \geq -\chi_{R}^{p}.
$$

(4.3)

**Lemma 4.3.** Let $p > 0$ be fixed, and let $R = R(m) > 0$ be such that $\alpha_{e}(pt) = o(R)$ and $R = e^{o(\beta_{e}(pt))}$ as $m \to \infty$. Then, it holds that

$$
\limsup_{m \to \infty} \frac{\log \left( \left( u_{R}^{e}(pt, \cdot), 1 \right) \right) - H_{e}(pt)}{\beta_{e}(pt)} \leq -\chi^{p}
$$

and

$$
\limsup_{m \to \infty} \frac{1}{\beta_{e}(pt)} \left( \log \left( \sum_{k=1}^{\infty} e^{\rho(k)(Q_{R})} \right) - H_{e}(pt) \right) \leq -\chi^{p}.
$$

(4.4)

(4.5)

4.1.1. **Three Remarks.** Before proving Lemmas 4.2 and 4.3, we comment on the differences between the latter and the statement of Theorem 3.4: Indeed, instead of concerning the asymptotics of the moments $\langle u_{R}^{e}(t, x)^{p} \rangle$, Lemmas 4.2 and 4.3 concern the asymptotics of

$$
\left( \langle u_{R}^{e}(p(t - d), \cdot), 1 \rangle \right) \quad \text{and} \quad \left( \sum_{k=1}^{\infty} e^{\rho(k)(Q_{R})} \right)
$$

for an appropriate choice of functions $R, d > 0$. The reasons for this are threefold:

Firstly, since Propositions 4.4 and 4.5 rely on the weak large deviation principle for Brownian occupation measures (e.g., [15, Section 4.2]), it is necessary to restrict our asymptotics to a compact domain. In Sections 5 and 6, we rigorously establish that up to an error of order $e^{o(\beta_{e}(pt))}$, we can replace $\langle u_{R}^{e}(t, x)^{p} \rangle$ by $\langle u_{R}^{e}(t, x)^{p} \rangle$, provided $R$ is not too small.

Secondly, while Propositions 4.4 and 4.5 can easily be used to obtain asymptotics for the first moment $\langle u_{R}^{e}(t, x) \rangle$ (for an appropriate choice of $R$), the same is not true for $\langle u_{R}^{e}(t, x)^{p} \rangle$ when $p \neq 1$. In short, this is because introducing a $p^{th}$ power on the solution of the PAM makes it more difficult to integrate the noise in the Feynman-Kac formula and apply the large deviation principle, as done informally in (3.9). In order to get around this issue, a crucial aspect of the strategy developed in [21] is to show that, up to an error of order $e^{o(\beta_{e}(pt))}$, we can replace $\langle u_{R}^{e}(t, x)^{p} \rangle$ by $\langle u_{R}^{e}(pt, x)^{p} \rangle$. Heuristically, this can be justified by using a spectral expansion and only keeping the leading eigenvalue, as in (3.5). However, given that the integral of the eigenfunctions $\langle e_{R}^{V}(Q_{r}), 1 \rangle$ is more convenient to control than its value $e_{R}^{v}(Q_{r})(x)$ at a single point $x \in \mathbb{R}^{d}$, a formal version of this heuristic is easier to achieve if we instead look at the integral of the solution $\langle u_{R}^{e}(t, \cdot), 1 \rangle^{p}$. In Sections 5 and 6, we rigorously prove that up to an error of order $e^{o(\beta_{e}(pt))}$, one has

$$
\langle u_{R}^{e}(t, x)^{p} \rangle \approx \langle (u_{R}^{e}(t, \cdot), 1)^{p} \rangle \approx \langle (u_{R}^{e}(pt, \cdot), 1) \rangle.
$$

(4.6)

Thirdly, the replacement of $t$ by a truncated time $t - d$ in Lemma 4.2 amounts to a technical concern. More specifically, in order to establish (4.6), it is in many cases necessary to “integrate out” the contribution of the PAM’s solution on the time interval $[0, d]$; we refer to Sections 5 and 6 for the details.

The remainder of Section 4 is now devoted to the proof of Lemmas 4.2 and 4.3.
4.2. Proof of Lemma 4.2. In addition to (4.3), in this proof we also establish
\begin{equation}
\limsup_{m \to \infty} \frac{\log \langle u^{\Xi_{e}}_{R_{\alpha e}(p t)}(p(t-d), \cdot), 1 \rangle - H_{e}(p t)}{\beta_{e}(p t)} \leq -\chi^{p}
\end{equation}
and
\begin{equation}
\limsup_{m \to \infty} \frac{1}{\beta_{e}(p t)} \left( \log \left( \sum_{k=1}^{\infty} e^{p(t-d)} \lambda^{\Xi_{e}}_{k}(Q_{R_{\alpha e}(p t)}) \right) - H_{e}(p t) \right) \leq -\chi^{p}.
\end{equation}

While these two results are not directly related to Lemma 4.2, they are needed in the proof of Lemma 4.3 and use the same ideas as the proof of (4.3).

By Brownian scaling (see [21, (1.3)]) one has
\[ u^{\Xi_{e}}_{R_{\alpha e}(p t)}(p(t-d), \cdot) = e^{H_{e}(p(t-d))} u^{\Xi^{\alpha}_{p,t-d}}_{R_{\alpha e}(p(t-d))} \left( \frac{p(t-d)}{\alpha_{e}(p(t-d))} \cdot \frac{\alpha_{e}(p(t-d))}{\alpha_{e}(p(t-d))} \right) \]
for every \( R > 0 \) and \( p > 0 \), where we recall that \( \Xi^{\alpha}_{p,t} \) is defined as in (3.6). For any \( \theta > 0 \), if \( m \) is large enough, then the first limit in (4.2) implies that
\[ R - \theta < R_{\alpha e}(p(t)/\alpha_{e}(p(t-d))) < R + \theta. \]
Consequently (recalling that \( \beta_{e}(p t) = p t/\alpha_{e}(p t)^{2} \)), for every \( \theta > 0 \), one has
\begin{equation}
e^{H_{e}(p(t-d))} u^{\Xi^{\alpha}_{p,t-d}}_{R_{\alpha e}(p t)} \left( \beta_{e}(p(t-d)), \frac{\alpha_{e}(p(t-d))}{\alpha_{e}(p(t-d))} \right) \leq e^{H_{e}(p(t-d))} u^{\Xi^{\alpha}_{p,t-d}}_{R_{\alpha e}(p t)} \left( \beta_{e}(p(t-d)), \frac{\alpha_{e}(p(t-d))}{\alpha_{e}(p(t-d))} \right),
\end{equation}
provided \( m \) is large enough. Moreover, by the Feynman-Kac formula and Tonelli’s theorem,
\begin{equation}
\langle u^{\Xi_{e}}_{R_{\alpha e}(p t)}(p(t-d), \cdot)/\alpha_{e}(p(t-d)), 1 \rangle = \alpha_{e}(p(t-d))^{d} \int_{Q_{R_{\alpha e}(p t)}} \mathbb{E}_{x} \left[ \langle e^{\beta_{e}(p(t-d))(\Xi^{\alpha}_{p,t-d},L_{\beta_{e}(p(t-d)))} \rangle 1_{L_{\beta_{e}(p(t-d))) \in \mathcal{P}(Q_{R_{\alpha e}(p t)})} \rangle \ dx, \right.
\end{equation}
where the term \( \alpha_{e}(p(t-d))^{d} \) comes from a change of variables in the \( dx \) integral. In order to control the exponential moment of \( \beta_{e}(p(t-d))(\Xi^{\alpha}_{p,t-d},L_{\beta_{e}(p(t-d)))} \), we use the formal version of the large deviation principle stated in Proposition 3.2, which amounts to the following:

**Proposition 4.4.** Let \( d \) satisfy (4.2), let \( R > 0 \) and \( p > 0 \), and suppose that we are in any case except Sub-3. It holds that
\begin{equation}
\limsup_{m \to \infty} \sup_{\mu \in \mathcal{P}(Q_{R})} \frac{1}{\beta_{e}(p(t-d))} \log \langle e^{\beta_{e}(p(t-d))(\Xi^{\alpha}_{p,t-d},\mu)} \rangle + J_{p}(\mu) = 0.
\end{equation}
Moreover, the map
\[ \mu \mapsto J_{p}(\mu) \]
is continuous with respect to the topology of weak convergence on \( \mathcal{P}(Q_{R}) \).

**Proposition 4.5.** Let \( d \) satisfy (4.2). Suppose that we are in case Sub-3, and let \( \varepsilon > 0 \) be arbitrary. For \( m \) large enough we have that
\begin{equation}
e^{-\beta_{e}(p(t-d))J_{e}(L_{\beta_{e}(p(t-d)))}} \leq \langle e^{\beta_{e}(p(t-d))(\Xi^{\alpha}_{p,t-d},L_{\beta_{e}(p(t-d)))}} \rangle \leq e^{-\beta_{e}(p(t-d))J_{0}(L_{\beta_{e}(p(t-d)))}}.
\end{equation}
Moreover, for every $R \in (0, \infty)$ (using the convention that $Q_\infty = \mathbb{R}^d$),

\[
\lim_{\varepsilon \to 0} \frac{1}{\beta_e(p(t-d))} \log \int_{Q_{R+\theta}} \mathbb{E}_x \left( \left< e^{\beta_e(p(t-d))} \mathbb{E}_{t-d}^{p}(L_{\beta_e(p(t-d))}) \right> \mathbb{I}(L_{\beta_e(p(t-d))} \in \mathcal{P}(Q_{R+\theta})) \right) \, dx \to -\chi^p_{R+\theta},
\]

as $m \to \infty$. Indeed, we note that $-\chi^p_{R+\theta} \leq -\chi^p_R$, and that $\chi^p_{R-\theta} \to \chi^p_R$ as $\theta \to 0$ by Lemma 5.3. By (4.11), this is equivalent to

\[
\frac{1}{\beta_e(p(t-d))} \log \int_{Q_{R+\theta}} \mathbb{E}_x \left( \left< e^{\beta_e(p(t-d))} \mathbb{E}_{t-d}^{p}(L_{\beta_e(p(t-d))}) \right> \mathbb{I}(L_{\beta_e(p(t-d))} \in \mathcal{P}(Q_{R-\theta})) \right) \, dx = -\chi^p_{R-\theta},
\]

which is a consequence of Varadhan’s lemma and the weak large deviation principle for Brownian local time with a uniform starting point on $Q_{R+\theta}$ (e.g., [15, Section 4.2]).

Consider now case Sub-3. On the one hand, by the lower bound in (4.12), for every $\varepsilon > 0$, the following holds as $m \to \infty$:

\[
\int_{Q_{R-\theta}} \mathbb{E}_x \left( \left< e^{\beta_e(p(t-d))} \mathbb{E}_{t-d}^{p}(L_{\beta_e(p(t-d))}) \right> \mathbb{I}(L_{\beta_e(p(t-d))} \in \mathcal{P}(Q_{R-\theta})) \right) \, dx \geq \int_{Q_{R-\theta}} \mathbb{E}_x \left( e^{-\beta_e(p(t-d))} J_\varepsilon(L_{\beta_e(p(t-d))}) \mathbb{I}(L_{\beta_e(p(t-d))} \in \mathcal{P}(Q_{R-\theta})) \right) \, dx = \exp \left( \beta_e(p(t-d)) \inf_{\mathcal{P}(Q_{R-\theta})} \left( \kappa \mathcal{J}(f) + J_\varepsilon(f^2) \right) + o(\beta_e(p(t-d))) \right),
\]

where the third line follows from the same application of Varadhan’s lemma as in the previous paragraph. Taking $\varepsilon \to 0$ using (4.13) then yields

\[
\liminf_{m \to \infty} \frac{1}{\beta_e(p(t-d))} \log \int_{Q_{R-\theta}} \mathbb{E}_x \left( \left< e^{\beta_e(p(t-d))} \mathbb{E}_{t-d}^{p}(L_{\beta_e(p(t-d))}) \right> \mathbb{I}(L_{\beta_e(p(t-d))} \in \mathcal{P}(Q_{R-\theta})) \right) \, dx \geq -\chi^p_{R-\theta}.
\]
This concludes the proof of (4.3) for Sub-3 by taking \( \theta \to 0 \) using Lemma 5.3. On the other hand, by the upper bound in (4.12) and removing the indicator of \( L_{\beta_{k}(p(t-d))} \in \mathcal{P}(Q_{R+\theta}) \),

\[
\int_{Q_{R+\theta}} \left\langle \mathbb{E}_{x} \left[ e^{\beta_{k}(p(t-d))}(\Xi_{k}^{p}(y-t, L_{\beta_{k}(y-t)})) \mathbb{I}_{\{L_{\beta_{k}(y-t)} \in \mathcal{P}(Q_{R+\theta})\}} \right] \right\rangle \, dx \\
\leq \int_{Q_{R+\theta}} \mathbb{E}_{x} \left[ e^{-\beta_{k}(p(t-d))}J_{0}(L_{\beta_{k}(p(t-d))}) \right] \, dx = (2(R + \theta))^{d} \mathbb{E}_{0} \left[ e^{-\beta_{k}(p(t-d))}J_{0}(L_{\beta_{k}(p(t-d))}) \right]
\]

for large enough \( m \), where the last equality follows from the fact that \( J_{0} \) is translation invariant. (4.7) for Sub-3 then follows from (1.7) with \( p = 1 \), which implies that

\[
\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}_{0} \left[ e^{-tJ_{0}(L_{t})} \right] = -\chi^{p}.
\]

Indeed, by a Brownian scaling,

\[
-tJ_{0}(L_{t}) = \frac{t^{2}}{t} \int \int_{\mathbb{R}^{2}} \gamma(x-y) \, dL_{t}(x)dL_{t}(y) = \frac{1}{t} \int \int_{[0,t]^{2}} \gamma(W_{r} - W_{s}) \, drds \\
\text{distr.} \\
\int \int_{[0,t/(2-\omega)/(4-\omega)]^{2}} \gamma(W_{r} - W_{s}) \, drds,
\]

and

\[
\left\langle u(t^{(2-\omega)/(4-\omega)},0) \right\rangle = \mathbb{E}_{0} \left[ \exp \left( \int \int_{[0,t/(2-\omega)/(4-\omega)]^{2}} \gamma(W_{r} - W_{s}) \, drds \right) \right].
\]

Next, we consider (4.8). By combining (21, (2.7)–(2.10)) with the argument used to get to (4.10), we know that for every \( \delta > 0 \), if \( m \) is large enough (it suffices that \( \beta_{k}(p(t-d)) > \delta \), which is always possible since \( \beta_{k}(pt) \to \infty \) as \( m \to \infty \) in every case), then

\[
\sum_{k=1}^{\infty} (-1)^{k} \left( e^{\beta_{k}(p(t-d))} \right) \leq (4\pi\delta)^{-d/2} e^{H_{k}(p(t-d))} \\
\cdot \alpha_{k}(p(t-d))^{d} \int_{Q_{R+\theta}} \left\langle \mathbb{E}_{x} \left[ e^{(\beta_{k}(p(t-d)) - \delta)}(\Xi_{k}^{p}(y-t, L_{\beta_{k}(y-t)})) \mathbb{I}_{\{L_{\beta_{k}(y-t)} \in \mathcal{P}(Q_{R+\theta})\}} \right] \right\rangle \, dx.
\]

As subtracting \( \delta \) from \( \beta_{k}(p(t-d)) \) has no effect on the asymptotics, (4.8) follows from (4.7).

We now conclude the proof of Lemma 4.2 by establishing Propositions 4.4 and 4.5.

**Proof of Proposition 4.4.** We begin with the continuity of \( J^{p} \). Since we are not considering case Sub-3, we need only prove the continuity of \( J \) for \( c \in (0, \infty) \). The continuity of \( J_{\infty} \) is established in [21, (8.0) and the following paragraph; see also (4.4)]. Now suppose that \( c \in (0, \infty) \). If \( \mu_{n} \) converges weakly to \( \mu \), then \( \mu_{n} \otimes \mu_{n} \) converges weakly to \( \mu \otimes \mu \). The continuity of \( J_{\infty} \) then follows from the fact that the two-dimensional function \( (x,y) \mapsto \gamma \) is bounded and continuous when \( c \in (0, \infty) \).

We now prove the asymptotic (4.11). Recall the definition of \( \Xi_{\varepsilon,t}^{p} \) in (3.6), and the fact that \( \xi_{\varepsilon} \) has covariance \( \gamma_{\varepsilon} = \varepsilon^{-1} \gamma_{1}(\cdot/\varepsilon) \) for \( \varepsilon > 0 \). With this, it is easy to see by a Gaussian moment
generating function calculation that
\[
\left\langle e^{\beta_c(pt)(\Xi_{t}^{p},\mu)} \right\rangle = e^{-\beta_c(pt)\alpha_c(pt)^2 H_c(pt)/pt} \left\langle e^{\beta_c(pt)\alpha_c(pt)^2 (\xi_c(\alpha_c(pt)\cdot)\mu)} \right\rangle
\]
(4.15)
\[
= \exp\left(\beta_c(pt)\left(-\frac{\alpha_c(pt)^2 H_c(pt)}{pt}\right) + \frac{\beta_c(pt)\alpha_c(pt)^4 e^{-\omega}}{2} \int_{(\mathbb{R}^d)^2} \gamma_1 \left(\frac{\alpha_c(pt)(x-y)}{\varepsilon}\right) d\mu(x)d\mu(y)\right).
\]

From this point on, we argue on a case-by-case basis:

**Case Sub-1:** For sake of readability, let us denote
\[
h = h(m) := e^{(\omega-2)/4}(pt)^{-1/4} = \left(e^{(pt)^{1/(2-\omega)}}\right)^{-(2-\omega)/4}.
\]
By definition of case Sub-1 (recall Assumption 2.4), we note that \(h \to 0\) as \(m \to \infty\). By definition of \(\alpha_c, \beta_c,\) and \(H_c\) in case Sub-1 (recall Definition 3.3), we also have that
\[
\frac{\alpha_c(pt)^2 H_c(pt)}{pt} = \frac{h^{-2} \gamma_1(0)}{2}, \quad \frac{\beta_c(pt)\alpha_c(pt)^4 e^{-\omega}}{2} = \frac{h^{-2}}{2}, \quad \frac{\alpha_c(pt)}{e} = h.
\]
Applying this to (4.15), and using (4.2) to replace \(F_\alpha(p(t-d)) = F_\alpha(pt)(1 + o(1))\) for \(F = \alpha, \beta, H\), we get that for Sub-1,
\[
\frac{1}{\beta_c(p(t-d))} \log \left\langle e^{\beta_c(pt-d)(\Xi_{t-d}^{p},\mu)} \right\rangle
\]
\[
= \frac{h^{-2}}{2} \left(1 + o(1)\right) \left(-\gamma_1(0) + \int_{(\mathbb{R}^d)^2} \gamma_1 \left(h(1 + o(1)) (x-y)\right) d\mu(x)d\mu(y)\right).
\]

Since \(\gamma_1(x)\) is smooth and has a strict maximum at zero, we can approximate its values for small \(x\) using a Taylor expansion of order 2 (e.g., \([25, \text{Theorem 1.23}]\)), which yields
\[
\gamma_1(hx) = \gamma_1(0) - h^2 x^T \Sigma x \frac{1}{2} + h^3 O(|x|^3), \quad x \in \mathbb{R}^d, \ h > 0.
\]
Therefore, given that
\[
\sup_{\mu \in \mathcal{P}(Q_R)} \int_{(\mathbb{R}^d)^2} O(|x|^3) d\mu(x)d\mu(y) < \infty
\]
for every \(R > 0\), we conclude that (4.11) holds for case Sub-1.

**Case Sub-2:** Let us denote
\[
c = c(m) := e^{(pt)^{1/(2-\omega)}}.
\]
Since we are in case Sub-2, we note that \(c \to p^{1/(2-\omega)}c\) as \(m \to \infty\); moreover, according to Definition 3.3 in that same case, we have that
\[
\frac{\alpha_c(pt)^2 H_c(pt)}{pt} = 0, \quad \frac{\beta_c(pt)\alpha_c(pt)^4 e^{-\omega}}{2} = \frac{c^{-\omega}}{2}, \quad \text{and} \quad \frac{\alpha_c(pt)}{e} = \frac{1}{c}.
\]
Thus, by (4.2) and (4.15), we have
\[
\frac{1}{\beta_e(p(t-d))} \log \left( e^{\beta_e(p(t-d))(\Xi_{e,t-d}^p,\mu)} \right) = \frac{1 + o(1)}{2} \int_{(\mathbb{R}^d)^2} \gamma_{c(1+o(1))}(x-y) \, d\mu(x) d\mu(y)
\]
(4.16) \[= -J_{p^{1/(2-\omega)}}(\mu) + o(1) + \frac{1}{2} \int_{(\mathbb{R}^d)^2} \left( \gamma_{c(1+o(1))}(x-y) - \gamma_{p^{1/(2-\omega)}}(x-y) \right) \, d\mu(x) d\mu(y).
\]
As \(\gamma_1\) is smooth and bounded and \(\gamma_c(x) = e^{-\omega}\gamma_1(x/c)\), one has \(\gamma_{c(1+o(1))} \to \gamma_{p^{1/(2-\omega)}}\) uniformly on compacts as \(m \to \infty\). Thus, the integral in (4.16) is uniformly small over \(\mu \in \mathcal{P}(Q_R)\) as \(m \to \infty\), concluding the proof of (4.11) in this case.

**Case Crt-1:** In this case, according to Definition 3.3, we have that
\[
\frac{\alpha_e(pt)^2 H_e(pt)}{pt} = (pt)^{1/2} \frac{\gamma_1(0)}{2}, \quad \frac{\beta_e(pt)\alpha_e(pt)^4 e^{-\omega}}{2} = \frac{(pt)^{1/2}}{2}, \quad \frac{\alpha_e(pt)}{e} = (pt)^{-1/4}.
\]
The proof then follows from the same argument as for case Sub-1, with the only difference that we replace the function denoted \(h\) therein by \((pt)^{-1/4}\).

**Case Crt-2:** In this case, according to Definition 3.3, we have that
\[
\frac{\alpha_e(pt)^2 H_e(pt)}{pt} = 0, \quad \frac{\beta_e(pt)\alpha_e(pt)^4 e^{-\omega}}{2} = \frac{pt}{2}, \quad \text{and} \quad \frac{\alpha_e(pt)}{e} = 1.
\]
Thus, by (4.2) and (4.15), we have
\[
\frac{1}{\beta_e(p(t-d))} \log \left( e^{\beta_e(p(t-d))(\Xi_{e,t-d}^p,\mu)} \right) = \frac{-pt}{2} J_{1+o(1)}(\mu)(1+o(1)).
\]
The result then follows from the facts that \(t \to t\) as \(m \to \infty\) (by definition of case Crt-2; see Assumption 2.16) and, arguing as (4.16),
\[
\lim_{m \to \infty} \sup_{\mu \in \mathcal{P}(Q_R)} \left| J_{1+o(1)}(\mu) - J_1(\mu) \right| = 0.
\]

**Case Sup:** This follows from the same argument as Sub-1 with
\[
h = h(m) := e^{(\omega-2)/4}(pt)^{-1/4} \left( e^{(pt)^{-1/(\omega-2)}} (\omega-2)/4 \right),
\]
which vanishes as \(m \to \infty\) thanks to Assumption 2.21. With this case in hand, the proof of (4.11), and thus Proposition 4.4, is now complete. \(\square\)

**Proof of Proposition 4.5.** In Case Sub-3, the same calculation as in (4.16) implies that
\[
\frac{1}{\beta_e(p(t-d))} \log \left( e^{\beta_e(p(t-d))(\Xi_{e,t-d}^p,\mu)} \right) = \frac{1}{2} \int_{(\mathbb{R}^d)^2} \gamma_{p(t-d)}^{1/(2-\omega)}(x-y) \, dL_{\beta_e(p(t-d))}(x) dL_{\beta_e(p(t-d))}(y)
\]
\[
= -J_{e(p(t-d))^{1/(2-\omega)}}(L_{\beta_e(p(t-d))})/2.
\]
By definition of case Sub-3 and (4.2), \(e(p(t-d))^{1/(2-\omega)} \to 0\) as \(m \to \infty\); thus the proof of (4.12) is simply a matter of noting that for every \(\varepsilon > \varepsilon > 0\) and \(t > 0\), one has
\[
(4.17) \quad -J_{\varepsilon}(L_t) \leq -J_{\varepsilon'}(L_t) \leq -J_0(L_t).
\]
Letting $\hat{\cdot}$ denote the Fourier transform, we have by definition of $L_t$, the Parseval formula, and the convolution theorem that

$$-J_\varepsilon(L_t) = \int_{\mathbb{R}^d} \left| \frac{1}{t} \int_0^t e^{-2\pi i (s,x)} \, ds \right|^2 \hat{\gamma}(x) \hat{\rho}_\varepsilon(x) \, dx. \tag{4.18}$$

Since $\gamma$ is a covariance function, $\hat{\gamma} \geq 0$. Computing explicitly that

$$\hat{\rho}_\varepsilon(x) = \begin{cases} e^{-2\pi^2 |x|^2\varepsilon^2} & \varepsilon > 0 \\ 1 & \varepsilon = 0, \end{cases}$$

and noting that this function is decreasing in $\varepsilon$ for all $x$, we therefore conclude (4.17).

We now prove (4.13). Once again using Fourier transforms, we note that

$$-J_\varepsilon(f^2) = \int_{\mathbb{R}^d} |\hat{f}^2(x)|^2 \hat{\gamma}(x) \hat{\rho}_\varepsilon(x) \, dx. \tag{4.19}$$

Consequently, $J_\varepsilon(f^2) \geq J_0(f^2)$ for every function $f \in H^1(\mathbb{R}^d)$ such that $\|f\|_2 = 1$ and $\varepsilon > 0$. With this in hand, in order to prove (4.13), it suffices to show that for every such $f$, one has

$$\lim_{\varepsilon \to 0} J_\varepsilon(f^2) = J_0(f^2).$$

This follows from (4.19) by monotone convergence. \qed

**4.3. Proof of Lemma 4.3.** We begin with (4.4). Let us define $\tilde{R} = \tilde{R}(m) := R/\alpha_\varepsilon(pt)$. By applying the same scaling identities used to obtain (4.10), we get

$$\left\langle \left( u_{\tilde{R}}^{\varepsilon t}(pt, \cdot) \right), 1 \right\rangle = e^{H_\varepsilon(pt) + o(\beta_\varepsilon(pt))} \left\langle \left( u_{\tilde{R}}^{\varepsilon t}(pt, \cdot) \right), 1 \right\rangle.$$

Thanks to [21, (3.29), (3.31) and (3.32)], there exists some constant $K > 0$ (independent of $m$) such that for every fixed $r \geq 2$, one has

$$e^{H_\varepsilon(pt)} \left\langle \left( u_{\tilde{R}}^{\varepsilon t}(pt, \cdot) \right), 1 \right\rangle \leq O(\tilde{R}^d) e^{K\beta_\varepsilon(pt)/r} \left( \sum_{k=1}^\infty e^{pt\lambda_k^{\varepsilon t}(Q_{r+1})}\alpha_\varepsilon(pt) \right)$$

as $m \to \infty$. Since $\tilde{R}^d = e^{o(\beta_\varepsilon(pt))}$ (as $\alpha_\varepsilon(pt), R = e^{o(\beta_\varepsilon(pt))}$), an application of (4.8) yields

$$\limsup_{m \to \infty} \frac{\log \left\langle \left( u_{\beta_\varepsilon(pt)}^{\varepsilon t}(pt, \cdot) \right), 1 \right\rangle - H_\varepsilon(pt)}{\beta_\varepsilon(pt)} \leq \frac{K}{r} - \chi^p.$$

Since $r \geq 2$ was arbitrary, we can take $r \to \infty$, which then yields (4.4). We then obtain (4.5) from (4.4) by using the same argument as in (4.14), thus concluding the proof of Lemma 4.3.

**5. Proof of Theorem 3.4 Part 2 - Lower Bound**

In this section, we begin implementing the program outlined in Section 4.1.1, which allows us to turn Lemmas 4.2 and 4.3 into Theorem 3.4. More specifically, in this section we provide the following lower bound for Theorem 3.4: Suppose that one of Assumptions 2.4, 2.16, or 2.21 holds. For every $p > 0$ and $x \in \mathbb{R}^d$, one has

$$\lim_{m \to \infty} \inf \frac{\log \langle u^{\varepsilon}(t, x)^p \rangle - H_\varepsilon(pt)}{\beta_\varepsilon(pt)} \geq -\chi^p. \tag{5.1}$$

The remainder of this section is structured as follows: In Sections 5.2 and 5.1, we provide an outline of the proof of (5.1) for $p \geq 1$ and $0 < p < 1$ respectively. This outline relies on five
technical results, namely, Lemmas 5.1–5.5. We then end the section by proving these technical lemmas, in Sections 5.3–5.7.

5.1. Outline for $p \geq 1$. The proof of (5.1) for $p \geq 1$ relies on the following:

**Lemma 5.1.** For every $x \in \mathbb{R}^d$, $p \geq 1$, and $R > 0$,

$$\langle u_{\varepsilon}^{e}(t, x)^p \rangle \geq e^{o(\beta_e(pt))} \left( \langle u_{R_0 e(pt)}^{e}(t, \cdot)^p \rangle \right)$$

as $m \to \infty$.

**Lemma 5.2.** For every $p \geq 1$ and $R > 0$,

$$\liminf_{m \to \infty} \frac{\log \langle u_{R_0 e(pt)}^{e}(t, \cdot)^p \rangle - H_e(pt)}{\beta_e(pt)} \geq p(\chi^p - \chi_R^p) + \liminf_{m \to \infty} \frac{\log \langle u_{R_0 e(pt)}^{e}(pt, \cdot)^p \rangle - H_e(pt)}{\beta_e(pt)},$$

where we recall the definition of $\chi_R^p$ in (4.1).

With these results in hand, (5.1) for $p \geq 1$ is proved as follows: If we combine Lemma 5.1, Lemma 5.2, and (4.3), then we obtain that

$$\liminf_{m \to \infty} \frac{\log \langle u_{\varepsilon}^{e}(t, x)^p \rangle - H_e(pt)}{\beta_e(pt)} \geq p(\chi^p - \chi_R^p) - \chi_R^p$$

for every $x \in \mathbb{R}^d$, $p \geq 1$ and $R > 0$. We then obtain (5.1) for all $p \geq 1$ by taking $R \to \infty$ using the following:

**Lemma 5.3.** For every $p > 0$ and $R' \in (0, \infty]$, one has

$$\lim_{R \to R'} \chi^p_R = \chi^p,$$

where we use the convention that $\chi^p_\infty = \chi^p$.

5.2. Outline for $0 < p < 1$. The proof of (5.1) for $0 < p < 1$ relies on the following:

**Lemma 5.4.** Let $x \in \mathbb{R}^d$, $0 < p < 1$, and $R > 0$ be arbitrary. On the one hand,

$$\langle u_{\varepsilon}^{e}(t, x)^p \rangle \geq e^{o(\beta_e(pt))} \left( \langle u_{R_0 e(pt)}^{e}(pt - \beta_e(pt)^{-1}, \cdot)^p \rangle - e^{\beta_e(t)}e^{-\omega t^2 e^{-3/4t^{1/4}}}) \right)$$

as $m \to \infty$ in Cases Sub-1, Crt-1, and Sup. On the other hand, in Cases Sub-2, Sub-3, and Crt-2 there exists a function $d = d(m)$ satisfying (4.2) such that for every $\theta > 1$, one has

$$\langle u_{\varepsilon}^{e}(t, x)^p \rangle \geq e^{o(\beta_e(pt))} \left( \langle u_{R_0 e(pt)}^{e}(p(t - d), \cdot)^p \rangle \right)^\theta$$

as $m \to \infty$.

We may now prove (5.1) for all $0 < p < 1$: Consider first Cases Sub-1, Crt-1, and Sup. Since $\beta_e(pt)^{-1} = o(1)$, it is clear that $d = \beta_e(pt)^{-1}$ satisfies the first two limits in (4.2). As for the third, if we combine (2.1), Definition 3.3, and the fact that we always have $e \to 0$ or $t \to \infty$,

$$\frac{H_e(p(t - \beta_e(pt)^{-1})) - H_e(pt)}{\beta_e(pt)} = \frac{\gamma_1(t)e^{\omega t^{3/2} e^{-3/4t^{1/4}}}}{2p^{2/3} t^{3/2}} - \frac{\gamma_1(t)e^2}{pt^2} = o(1).$$

Thus, if we apply (4.3) with $d = \beta_e(pt)^{-1}$ to (5.2), then we obtain the lower bound

$$\langle u_{\varepsilon}^{e}(t, x)^p \rangle \geq e^{o(\beta_e(pt))} \left( e^{H_e(pt) - \chi_R^p \beta_e(pt)(1 + o(1))} - e^{\beta_e(t)e^{-\omega t^2 e^{-3/4t^{1/4}}}) \right)$$

$$= e^{H_e(pt) - \chi_R^p \beta_e(pt)(1 + o(1))} \left( 1 - e^{\beta_e(t) - H_e(pt)} + \chi_R^p \beta_e(pt)(1 + o(1)) - e^{\omega t^2 e^{-3/4t^{1/4}}}) \right)$$
for large $m$. In Cases Sub-1, Crt-1, and Sup, for every $q > 0$, it is the case that
\begin{equation}
\lim_{m \to \infty} \frac{\int_{Q_{R^2}} u^{\varepsilon}(t,x)^p}{\beta_{\varepsilon}(qt)} = 0 \quad \text{and} \quad \lim_{m \to \infty} \frac{\int_{Q_{R^2}} u^{\varepsilon}(t,x)^p}{\beta_{\varepsilon}(qt)} = 0;
\end{equation}
we therefore obtain (5.1) for all $0 < p < 1$ from (5.4), as well as Lemma 5.3 to take the limit $R \to \infty$ in the variational constant $\chi^p_R$.

We now conclude the proof of (5.1) for $0 < p < 1$ by dealing with cases Sub-2, Sub-3, and Crt-2. In these cases, we apply (4.3) (except that we replace the noise $\xi$ by $\theta^{-1} \xi$) to (5.3). This yields that for every $\theta > 1$, one has
\begin{equation}
\liminf_{m \to \infty} \log \frac{\int_{Q_{R^2}} u^{\varepsilon}(t,x)^p}{\beta_{\varepsilon}(qt)} = 0 \quad \text{and} \quad \liminf_{m \to \infty} \log \frac{\int_{Q_{R^2}} u^{\varepsilon}(t,x)^p}{\beta_{\varepsilon}(qt)} = 0;
\end{equation}
where for any constant $c > 0$, we define
\begin{equation}
\chi^p_R(c) := \inf_{f \in H^1(Q_R), \|f\|_2 = 1} \left( \kappa \mathcal{J}(f) + c^2 \mathcal{J}(f^2) \right).
\end{equation}
Recalling that $H_\varepsilon = 0$ in cases Sub-2, Sub-3, and Crt-2, we then obtain (5.1) by applying the following lemma, together with the limits $R \to \infty$ and $\theta \searrow 1$ (taken in that order):

**Lemma 5.5.** In cases Sub-2, Sub-3, and Crt-2, for every $p > 0$, we have that
\begin{equation}
\lim_{c \to 1} \lim_{R \to \infty} \chi^p_R(c) = \chi^p.
\end{equation}

### 5.3. Proof of Lemma 5.1.

Since $u^{\varepsilon}(t, \cdot)$ is stationary for every $\varepsilon, t > 0$, we can write
\begin{equation}
\langle u^{\varepsilon}(t,x)^p \rangle = \langle u^{\varepsilon}(t,0)^p \rangle = \left( 2R_{\alpha}(pt) \right)^{-d} \int_{Q_{R_{\alpha}(pt)}} u^{\varepsilon}(t,y)^p \, dy.
\end{equation}
If $p \geq 1$, then Jensen’s inequality yields
\begin{equation}
\left( 2R_{\alpha}(pt) \right)^{-d} \int_{Q_{R_{\alpha}(pt)}} u^{\varepsilon}(t,y)^p \, dy \geq \left( 2R_{\alpha}(pt) \right)^{-d} \int_{Q_{R_{\alpha}(pt)}} u^{\varepsilon}(t,y)^p \, dy.
\end{equation}
We then get Lemma 1 by noting that $u^{\varepsilon}(t, \cdot) \geq u^{\varepsilon}_{R_{\alpha}(pt)}(t, \cdot)$ and $\alpha_{\varepsilon}(pt) = e^{o(\beta_{\varepsilon}(pt))}$.

### 5.4. Proof of Lemma 5.2.

By [21, (2.11) and (2.12)] , if $p \geq 1$, then we have that
\begin{equation}
\left( u^{\varepsilon}_{R_{\alpha}(pt)}(t, \cdot) , 1 \right)^p \geq \left( 2R_{\alpha}(pt) \right)^{-d} \left( \sum_{k=1}^{\infty} c^{p} \chi^p_{R_k}(Q_{R_{\alpha}(pt)}) \right)^p.
\end{equation}
We then obtain Lemma 5.2 by an application of (4.3) and (4.5), and $\alpha_{\varepsilon}(pt) = e^{o(\beta_{\varepsilon}(pt))}$.

### 5.5. Proof of Lemma 5.3.

Note that $\chi^p_R \geq \chi^p_{R'}$ whenever $0 < R < R'$. Thus, it suffices to show that for every $f \in H^1(Q_{R'})$ such that $\|f\|_2 = 1$, supp$(f) \subset Q_{R'}$ (using the convention that $Q_{\infty} = \mathbb{R}^d$), and $\kappa \mathcal{J}(f) + \mathcal{J}(f^2)$ is finite, there exists a sequence $f_R$ of functions such that for each $R$, one has supp$(f) \subset Q_R$ and $\|f_R\|_2 = 1$, and moreover
\begin{equation}
\lim_{R \to R'} \kappa \mathcal{J}(f_R) + \mathcal{J}(f_R^2) = \kappa \mathcal{J}(f) + \mathcal{J}(f^2).
\end{equation}
This follows from the standard proof that smooth and compactly supported functions are dense in $H^1(Q_{R'})$ using smooth cutoff functions, together with the monotone convergence theorem for the convergence $\mathcal{J}(f_R^2) \to \mathcal{J}(f^2)$. 
5.6. Proof of Lemma 5.4. We begin with a general proposition:

**Proposition 5.6.** Let $0 < p < 1$, $R > 0$, and $\theta \geq 1$. For any function $d = d(m)$ such that $0 \leq d < t$, one has

$$\langle (u_{R\alpha(pt)}^{\theta-1}\xi)(t-d, \cdot), 1 \rangle^p \geq e^{o(\beta_k(pt))}\langle (u_{R\alpha(pt)}^{\theta-1}\xi)(p(t-d), \cdot), 1 \rangle \quad \text{as } m \to \infty.$$

**Proof.** For any $\theta \geq 1$, the function

$$f(k) := \frac{(e_k^{\theta-1}\xi(Q_{R\alpha(pt)}), 1)^2}{(2R\alpha(pt))^{2d}}, \quad k \in \mathbb{N}$$

is a probability measure on $\mathbb{N}$. If we use a spectral expansion and then apply Jensen’s inequality to this probability measure (since $0 < p < 1$), we are led to

$$(u_{R\alpha(pt)}^{\theta-1}\xi)(t-d, \cdot, 1)^p = \left( \sum_{k=1}^{\infty} e(t-d)\lambda_k^{\theta-1}\xi(Q_{R\alpha(pt)})(e_k^{\theta-1}\xi(Q_{R\alpha(pt)}), 1)^2 \right)^p \geq (2R\alpha(pt))^{2d(p-1)} \sum_{k=1}^{\infty} e(t-d)\lambda_k^{\theta-1}\xi(Q_{R\alpha(pt)})(e_k^{\theta-1}\xi(Q_{R\alpha(pt)}), 1)^2 \geq e^{o(\beta_k(pt))}(u_{R\alpha(pt)}^{\theta-1}\xi)(p(t-d), \cdot, 1),$$

concluding the proof. \hfill \square

For every $0 \leq d < t$ and $r > 0$, we denote the event

$$(5.8) \quad A_{d,t}(Q_r) := \{ W_s \in Q_r \text{ for every } s \in [d,t] \}.$$ 

By stationarity and the Feynman-Kac formula, we have that

$$\langle u^x(t, x)^p \rangle = \langle u^x(t, 0)^p \rangle \geq \mathbb{E}_0 \left[ \exp \left( \int_0^t \xi_e(W_s) \ ds \right) \mathbb{1}_{A_{d,t}(Q_{R\alpha(pt)})} \right].$$

From this point on, we split the proof into two steps.

5.6.1. Proof of (5.2). Suppose that we are in one of cases Sub-1, Crt-1, or Sup. For simplicity, throughout the proof of (5.2), we denote the functions

$$(5.9) \quad d = \beta_e(pt)^{-1}, \quad c = e^{1/4}t^{-1/4}\beta_e(pt) = p^{3/2}e^{-(3+2\omega)/4}t^{3/4}, \quad \text{and } h = \frac{e^{-\omega/2}t}{p^{3/2}}.$$

Then,

$$\mathbb{E}_0 \left[ \exp \left( \int_0^t \xi_e(W_s) \ ds \right) \mathbb{1}_{A_{d,t}(Q_{R\alpha(pt)})} \right] \geq \mathbb{E}_0 \left[ \exp \left( \int_0^t \xi_e(W_s) \ ds \right) \mathbb{1}_{A_{d,t}(Q_{R\alpha(pt)})} \mathbb{1}_{\{ \int_0^t \xi_e(W_s) \ ds \geq -c \}} \right] \geq e^{-c}\mathbb{E}_0 \left[ \exp \left( \int_0^t \xi_e(W_s) \ ds \right) \mathbb{1}_{A_{d,t}(Q_{R\alpha(pt)})} \mathbb{1}_{\{ \int_0^t \xi_e(W_s) \ ds \geq -c \}} \right] = e^{-c}\mathbb{E}_0 \left[ \exp \left( \int_0^t \xi_e(W_s) \ ds \right) \mathbb{1}_{A_{d,t}(Q_{R\alpha(pt)})} \right] - e^{-c}\mathbb{E}_0 \left[ \exp \left( \int_0^t \xi_e(W_s) \ ds \right) \mathbb{1}_{A_{d,t}(Q_{R\alpha(pt)})} \mathbb{1}_{\{ \int_0^t \xi_e(W_s) \ ds \leq -c \}} \right].$$
Since it is always the case that $e \rightarrow 0$ or $t \rightarrow \infty$, one has $c = o(\beta_e(pt))$. Therefore, given Proposition 5.6 and the fact that $(x - y)^p \geq x^p - y^p$ for all $0 < y < x$ when $0 < p < 1$, in order to prove (5.2) it suffices to show that as $m \rightarrow \infty$, one has

$$
\left(5.10\right) \quad \mathbb{E}_0 \left[ \exp \left( \int_d^t \xi_e(W_s) \, ds \right) 1_{\mathcal{A}_{d,t}(Q_{Roate})} \right]^{p} \geq e^{o(\beta_e(pt))} \left( u_{Roate}^\ell(t - d, \cdot) \right)^{p} \tag{5.10}
$$

and

$$
\left(5.11\right) \quad \mathbb{E}_0 \left[ \exp \left( \int_d^t \xi_e(W_s) \, ds \right) 1_{\mathcal{A}_{d,t}(Q_{Roate})} 1_{\{ f_0^d \xi_e(W_s) \, ds < -c \}} \right]^{p} \leq e^{-phc + pH_e(t)} \tag{5.11}
$$

(Indeed, $phc = e^{-\omega t^2} \cdot e^{-3/4 t^{1/4}}$.)

We begin with (5.10). By the strong Markov property and definition of $A_{d,t}(Q_{Roate})$, we have

$$
\mathbb{E}_0 \left[ \exp \left( \int_d^t \xi_e(W_s) \, ds \right) 1_{\mathcal{A}_{d,t}(Q_{Roate})} \right] = \int_{Roate} \Pi_d(x) u_{Roate}^\ell(t - d, x) \, dx,
$$

where $(\Pi_c)_{c > 0}$ is $W$’s transition density, that is,

$$
\Pi_c(x) := \frac{e^{-|x|^2/4\kappa c}}{(4\kappa c\pi)^{d/2}}, \quad c > 0, \quad x \in \mathbb{R}^d.
$$

We then obtain (5.10) by noting that, since $d = e^{o(\beta_e(pt))}$ and $\alpha_e(pt)^2/d = o(\beta_e(pt))$, one has

$$
\inf_{x \in Q_{Roate}} \Pi_d(x) \geq e^{o(\beta_e(pt))}.
$$

We now prove (5.11). Given that

$$
\lim_{m \rightarrow \infty} (h - 1)d = \lim_{m \rightarrow \infty} \left( p^{-4}e^{-t/2} - e^{(2+\omega)/2(\beta_e(pt)^{-3/2})} \right) = 0,
$$

for large enough $m$ we have that $(h - 1)d \leq t$ (recall that $\inf_{m \geq 0} t > 0$). Therefore,

$$
\mathbb{E}_0 \left[ \exp \left( \int_d^t \xi_e(W_s) \, ds \right) 1_{\mathcal{A}_{d,t}(Q_{Roate})} 1_{\{ f_0^d \xi_e(W_s) \, ds < -c \}} \right] \leq e^{-hc} \mathbb{E}_0 \left[ \exp \left( \int_0^t (1 - (h - 1) 1_{\{0 \leq s \leq d\}}) \xi_e(W_s) \, ds \right) \right].
$$

If we apply Jensen’s inequality to the probability measure with density function

$$
s \mapsto \frac{1 - (h - 1) 1_{\{0 \leq s \leq d\}}}{(t - (h - 1)d)}, \quad s \in [0, t],
$$

then we are led to

$$
\left(5.12\right) \quad \mathbb{E}_0 \left[ \exp \left( \int_d^t \xi_e(W_s) \, ds \right) 1_{\mathcal{A}_{d,t}(Q_{Roate})} 1_{\{ f_0^d \xi_e(W_s) \, ds < -c \}} \right] \leq e^{-hc} \int_0^t \frac{(1 - (h - 1) 1_{\{0 \leq s \leq d\}})}{(t - (h - 1)d)} \mathbb{E}_0 \left[ e^{\ell(t - (h - 1)d) \xi_e(W_s)} \right] \, ds.
$$
At this point, we apply Jensen’s inequality to the expectation $\langle (\cdot)^p \rangle \leq \langle \cdot \rangle^p$ (as $0 < p < 1$) together with the fact that $\xi_e$ is stationary with cumulant generating function $H_e$ to obtain

$$\left\langle E_0 \left[ \exp \left( \int_0^t \xi_e(W_s) \, ds \right) \mathbb{1}_{A_{d,t}(Q_{R\alpha}(pt))} \mathbb{1}_{\{f_{Q_e}^d \xi_e(W_s) \, ds < c\}} \right]^p \right\rangle \leq e^{-\beta_1 c + pH_e(t-(h-1)d)}.$$  

The inequality (5.11) then follows from $H_e(t-(h-1)d) \leq H_e(t)$.

5.6.2. Proof of (5.3). By the reverse Hölder inequality, for every $\theta > 1$ and $0 < p < 1$, one has

$$\langle E_0 \left[ \exp \left( \int_0^d \theta^\prime \xi_e(W_s) \, ds \right) \right] \rangle^p \leq \langle E_0 \left[ \exp \left( \int_0^d \theta^{-1} \xi_e(W_s) \, ds \right) \right] \rangle^\theta \langle E_0 \left[ \exp \left( \int_0^d \theta^1 \xi_e(W_s) \, ds \right) \right] \rangle^{\theta p},$$

where $\theta^\prime = \frac{1}{\theta - 1}$ and the last inequality follows from an application of Jensen’s inequality to the expectation $\langle (\cdot)^p \rangle \leq \langle \cdot \rangle^p$. Thus, by arguing as in (5.10) (except that $\xi_e$ is replaced by $\theta^{-1} \xi_e$), so long as we choose $d$ in such a way that $d = e^{o(\beta_e(pt))}$ and $\alpha_e(pt)^2/d = o(\beta_e(pt))$ in addition to (4.2), it suffices to show that

$$\langle E_0 \left[ \exp \left( \int_0^\delta \theta^\prime \xi_e(W_s) \, ds \right) \right] \rangle \leq e^{o(\beta_e(pt))}.$$  

On the one hand, in cases Sub-2 and Sub-3, it suffices to take any small enough constant $d := \delta$ (so that $\inf_{m \geq 0} t > \delta$): Since $t \to \infty$ and $H_e = 0$ in cases Sub-2 and Sub-3, it is easily checked using Definition 3.3 that (4.2) is met for that choice. Moreover, $\alpha_e(pt)^2/\delta = o(1) = o(\beta_e(pt))$. Finally, in the subcritical regime, (2.6) implies that

$$\lim_{m \to \infty} \left\langle E_0 \left[ \exp \left( \int_0^\delta \theta^\prime \xi_e(W_s) \, ds \right) \right] \right\rangle = \left\langle E_0 \left[ \exp \left( \int_0^\delta \theta^\prime \xi_e(W_s) \, ds \right) \right] \right\rangle < \infty;$$

hence (5.13) holds.

On the other hand, in case Crt-2, we note by Fubini’s theorem that

$$\left\langle E_0 \left[ \exp \left( \int_0^d \theta^\prime \xi_e(W_s) \, ds \right) \right] \right\rangle = E_0 \left[ \exp \left( \frac{(\theta^\prime e)^2}{2} \int_{[0,d]^2} \gamma_1((W_u-W_v)/e) \, dudv \right) \right] \leq e^{(\theta^\prime)^2d^2e^{-2}\|\gamma_1\|_\infty/2};$$

with this inequality in hand, we can take, for example, $d = e^{3/2} = e^{o(\beta_e(pt))}$. Indeed, since $e^{3/2} = o(1)$ in that case, it is clear by Definition 3.3 that (4.2) holds and that $\alpha_e(pt)^2/d = e^{1/2} = o(1) = o(\beta_e(pt))$. This concludes the proof of (5.3), and thus also of Lemma 5.4.
5.7. **Proof of Lemma 5.5.** First, Lemma 5.3 (where we replace \( \xi_n \) by \( c \xi_n \)) implies that

\[
\lim_{c \to 1} \lim_{K \to \infty} \chi_R^P(c) = \lim_{c \to 1} \chi^P(c),
\]

where we define

\[
\chi^P(c) := \inf_{f \in H^1([\mathbb{R}^d]), \|f\|_1 = 1} \left( \kappa \mathcal{J}(f) + c^2 J^P(f^2) \right).
\]

Thus, it now suffices to prove that

\[
(5.14) \quad \lim_{c \to 1} \chi^P(c) = \chi^P
\]

for all \( p > 0 \) in cases Sub-2, Sub-3, and Crt-2.

We begin by proving (5.14) in cases Sub-2 and Crt-2, in which case we have that \( J^p = \zeta(p) J_\varepsilon \) for some \( 0 < \varepsilon < \infty \) (recall Definition 3.3) and constant \( \zeta(p) > 0 \). Since \( J_\varepsilon \leq 0 \) for all \( 0 < \varepsilon < \infty \), we have in those cases that \( \chi^P(c) \leq \chi^P \) when \( c \geq 1 \) and \( \chi^P(c) \geq \chi^P \) when \( c \leq 1 \).

Given that \( \gamma_\varepsilon \) is bounded

\[
J_\varepsilon(f^2) = -\frac{1}{2} \int_{[\mathbb{R}^d]^2} f(x)^2 \gamma_\varepsilon(x-y) f(y)^2 \, dx \, dy \geq -\frac{\|\gamma_\varepsilon\|_\infty \|f\|_2^4}{2} = -\frac{\|\gamma_\varepsilon\|_\infty}{2}
\]

for every \( f \) such that \( \|f\|_2 = 1 \). Thus, if \( c \geq 1 \), then

\[
\chi^P \geq \chi^P(c) = \inf_{f \in H^1([\mathbb{R}^d]), \|f\|_2 = 1} \left( \kappa \mathcal{J}(f) + J^P(f^2) + (c^2 - 1) J^P(f^2) \right) \geq \chi^P - (c^2 - 1) \|\gamma_\varepsilon\|_\infty ;
\]

similarly, if \( c \leq 1 \), then we have that

\[
\chi^P \leq \chi^P(c) \leq \chi^P + (1 - c^2) \|\gamma_\varepsilon\|_\infty.
\]

This implies (5.14).

We now consider case Sub-3, wherein we recall that

\[
J^P(f^2) = -\frac{1}{2} \int_{\mathbb{R}^d} f(x)^2 \gamma(x-y) f(y)^2 \, dx \, dy.
\]

By using the scaling property \( \gamma(cx) = c^{-\omega} \gamma(x) \) for all \( c > 0 \), it can be shown (e.g., [12, (1.13)]) that \( \chi^P(c) = c^{4/(2-\omega)} \chi^P \) for every \( c > 0 \), from which (5.14) immediately follows.

### 6. Proof of Theorem 3.4 Part 3 - Upper Bound

In this section, we prove the following upper bound, which, together with (5.1), concludes the proof of Theorem 3.4: Suppose that one of Assumptions 2.4, 2.16, or 2.21 holds. For every \( p > 0 \) and \( x \in \mathbb{R}^d \), one has

\[
(6.1) \quad \limsup_{m \to \infty} \frac{\log \langle u^{\xi}(t,x) \rangle - H_\varepsilon(pt)}{\beta_\varepsilon(pt)} \leq -\chi^P.
\]

The remainder of this section is structured as follows: In Section 6.1, we provide an outline of the proof of (6.1). This outline relies on two technical results, namely, Lemmas 6.1 and 6.2. These are proved in Sections 6.2 and 6.3, respectively.
6.1. Outline. The proof of (6.1) relies on the following two lemmas:

Lemma 6.1. Let \( x \in \mathbb{R}^d \) and \( p > 0 \). There exists a function \( R \) that satisfies \( \alpha_e(pt) = o(R) \) and \( R = e^{o(\beta_e(pt))} \) such that
\[
\langle u^e(x,t,x)^p \rangle \leq (1 + o(1)) \langle u^e_R(t,x)^p \rangle \quad \text{as} \quad m \to \infty. \tag{6.2}
\]

Lemma 6.2. Let \( x \in \mathbb{R}^d, p > 0, \) and \( R = e^{o(\beta_e(pt))} \). On the one hand, as \( m \to \infty \),
\[
\langle u^e_R(t,x)^p \rangle \leq \begin{cases} 
 e^{o(\beta_e(pt))} \left( \left( \langle u^e_R(pt, \cdot), 1 \rangle \right) + 1 \right) + e^{AH_e(pt)} e^{-\theta t^2 e^{-3/4 t^{1/4}}} & \text{if } p \geq 1 \\
 e^{o(\beta_e(pt))} \left( \sum_{k=1}^{\infty} e^{pt \lambda^e_k(Q_k)} + 1 \right) + e^{4pH_e(t)} e^{-\theta t^2 e^{-3/4 t^{1/4}}} & \text{if } 0 < p < 1 
\end{cases}
\]
in cases Sub-1, Crt-1, and Sup. On the other hand, for every \( \theta > 1 \), it holds as \( m \to \infty \) that
\[
\langle u^e_R(t,x)^p \rangle \leq \begin{cases} 
 e^{o(\beta_e(pt))} \left( \left( \langle u^e_R(pt, \cdot), 1 \rangle \right) + 1 \right)^{1/\theta} & \text{if } p \geq 1 \\
 e^{o(\beta_e(pt))} \left( \sum_{k=1}^{\infty} e^{pt \lambda^e_k(Q_k)} + 1 \right)^{1/\theta} & \text{if } 0 < p < 1 
\end{cases}
\]
in cases Sub-2, Sub-3, and Crt-2.

We may now prove (6.1). On the one hand, in cases Sub-1, Crt-1, and Sup, a combination of (4.4)/(4.5), (5.5), (6.2), and (6.3) implies (6.1). On the other hand, in cases Sub-2, Sub-3, and Crt-2 a combination of (4.4)/(4.5) (where we replace \( \xi_e \) by \( \theta \xi_e \)), (6.2), and (6.4) implies that for every \( \theta > 1 \),
\[
\limsup_{m \to \infty} \frac{\langle u^e(\xi, t, x)^p \rangle}{\beta_e(pt)} \leq \frac{\chi_p(\theta)}{\theta},
\]
where we recall that
\[
\chi_p(\theta) := \inf_{f \in H^1(\mathbb{R}^d), \|f\|_2 = 1} \left( \kappa J(f) + \theta^2 J_p(f^2) \right).
\]
We then obtain (6.1) by taking the limit \( \theta \to 1 \) using (5.14).

6.2. Proof of Lemma 6.1. We claim that it suffices to show that
\[
\lim_{m \to \infty} \frac{\langle (u^e(t, 0) - u^e_R(t, 0))^p \rangle}{\langle u^e(t, 0)^p \rangle} = 0. \tag{6.5}
\]
On the one hand, for \( p \geq 1 \), as remarked in [21, (3.16)], combining the triangle inequality with (6.5) and the fact that \( u^e(x, t, x) = u^e_R(t, 0) \geq u^e_R(t, 0) \geq 0 \) implies that
\[
0 \leq \langle u^e(t, 0)^p \rangle^{1/p} - \langle u^e_R(t, 0)^p \rangle^{1/p} \leq \langle (u^e(t, 0) - u^e_R(t, 0))^p \rangle^{1/p} = o\langle (u^e(t, 0)^p)^{1/p} \rangle.
\]
If we divide all terms in the above inequality by \( \langle u^e(t, 0)^p \rangle^{1/p} \), then we obtain (6.2). On the other hand, for \( 0 < p < 1 \) we have that \( (x + y)^p \leq x^p + y^p \) for all \( x, y \geq 0 \), hence
\[
0 \leq \langle u^e(t, x)^p \rangle - \langle u^e_R(t, 0)^p \rangle \leq \langle (u^e(t, 0) - u^e_R(t, 0))^p \rangle = o\langle (u^e(t, 0)^p) \rangle;
\]
this also implies (6.2).

In order to establish (6.5), we use two tools: Let \( \tau_r := \inf\{t \geq 0 : W_t \not\in Q_r\} \) denote the first hitting time of \( Q_r \)'s boundary by \( W \). By the Feynman-Kac formula, one has
\[
u^e(t, 0) - u^e_R(t, 0) = \mathbb{E}_0 \left[ \exp \left( \int_0^t \xi_e(W_s) \, ds \right) 1_{\{\tau_r \leq t\}} \right], \tag{6.6}
\]
and by the reflection principle,
\begin{equation}
\mathbb{P}_0[\tau_R \leq t] \leq e^{-CR^2/t}
\end{equation}
for some constant $C > 0$ independent of $m$. The details of the proof of (6.5) using these depend on which case we are considering, and thus we split the remainder of the proof into two steps.

6.2.1. Sub-1, Crt-1, and Sup. Suppose first that $p \geq 1$. In that case, Thanks to (6.6) and [21, (3.17)] (note that, in the notation of [21], $R(pt)$ corresponds to $R$ in our setting), we have that for $p \geq 1$ and large $m$,
\begin{equation}
\langle (u^{\xi}(t, 0) - u^{\xi}_R(t, 0))^p \rangle \leq e^{H_e(pt) - CR^2/t}.
\end{equation}

If we combine this with the fact that
\begin{equation}
\langle u^{\xi}(t, 0)^p \rangle \geq e^{-H_e(pt) + \beta_e(pt)\chi_p(1+o(1))}
\end{equation}
as $m \to \infty$ thanks to (5.1), we have that
\[ \frac{\langle (u^{\xi}(t, 0) - u^{\xi}_R(t, 0))^p \rangle}{\langle u^{\xi}(t, 0)^p \rangle} \leq e^{\beta_e(pt)\chi_p(1+o(1)) - CR^2/t} \quad \text{as } m \to \infty. \]

When $0 < p < 1$, we use Jensen’s inequality and (6.8) in the case $p = 1$ to get
\[ \langle (u^{\xi}(t, 0) - u^{\xi}_R(t, 0))^p \rangle \leq \langle (u^{\xi}(t, 0) - u^{\xi}_R(t, 0))^p \rangle \leq e^{H_e(t) - pCR^2/t}. \]

Once again combining this with (5.1) yields
\[ \frac{\langle (u^{\xi}(t, 0) - u^{\xi}_R(t, 0))^p \rangle}{\langle u^{\xi}(t, 0)^p \rangle} \leq e^{H_e(t) - H_e(pt) + \beta_e(pt)\chi_p(1+o(1)) - pCR^2/t} \quad \text{as } m \to \infty. \]

Since $\beta_e(pt) = o(H_e(pt)) = o(H_e(t))$ in cases Sub-1, Crt-1, and Sup, in order to prove (6.5) it suffices to find a function $R$ that satisfies $\alpha_e(pt) = o(R)$, $R = e^{o(\beta_e(pt))}$, and $H_e(pt) = o(R^2/t)$. It is easily seen to be the case that $R = H_e(pt)$ satisfies all three of these conditions.

6.2.2. Sub-2, Sub-3, and Crt-2. If $p \geq 1$, then by Jensen’s inequality and (6.6), we have that
\begin{equation}
\langle (u^{\xi}(t, 0) - u^{\xi}_R(t, 0))^p \rangle \leq \mathbb{E}_0 \left[ \exp \left( \int_0^t p\xi_e(W_s) \, ds \right) \mathbf{1}_{\{\tau_R \leq t\}} \right].
\end{equation}

If we then apply Hölder’s inequality and (6.7), we get that
\[ \langle (u^{\xi}(t, 0) - u^{\xi}_R(t, 0))^p \rangle \leq \mathbb{E}_0 \left[ \exp \left( \int_0^t 2\xi_e(W_s) \, ds \right) \right]^{1/2} e^{-CR^2/2t} \]
as $m \to \infty$. Similarly, when $0 < p < 1$ we have that
\[ \langle (u^{\xi}(t, 0) - u^{\xi}_R(t, 0))^p \rangle \leq \mathbb{E}_0 \left[ \exp \left( \int_0^t 2\xi_e(W_s) \, ds \right) \right]^{p/2} e^{-pCR^2/2t}. \]

In cases Sub-2, Sub-3, and Crt-2, it is always the case that $\alpha_e(pt) = o(\beta_e(pt))$, $\beta_e(pt) = e^{o(\beta_e(pt))}$, and $\beta_e(pt) = o(\beta_e(pt)^2/t)$. Moreover, since $H_e = 0$ in those cases, we know from (5.1) that \( \langle u^{\xi}(t, 0)^p \rangle^{1-p} = e^{O(\beta_e(pt))} \). Thus, by choosing $R = \beta_e(pt)$, in order to prove (6.5), it is enough to show that
\[ \mathbb{E}_0 \left[ \exp \left( \int_0^t \theta\xi_e(W_s) \, ds \right) \right] \leq e^{O(\beta_e(pt))} \]
for every \( \theta > 0 \). On the one hand, in cases Sub-1 and Sub-2, this follows from the fact that
\[
\left\langle \mathbb{E}_0 \left[ \exp \left( \int_0^t \theta \xi_s(W_s) \, ds \right) \right] \right\rangle \leq \left\langle \mathbb{E}_0 \left[ \exp \left( \int_0^t \theta \xi(W_s) \, ds \right) \right] \right\rangle \leq e^{O(\beta_k(pt))},
\]
where the first inequality follows from the same argument as in the proof of (4.17), and the second inequality follows from (1.7) (except that we replace \( \gamma \) by \( \theta^2 \gamma \) in the variational problem \( M \)). On the other hand, in case Crt-2, we simply note that
\[
\left\langle \mathbb{E}_0 \left[ \exp \left( \int_0^t \theta \xi_s(W_s) \, ds \right) \right] \right\rangle = \mathbb{E}_0 \left[ \exp \left( \int_{[0,t]^2} \theta^2 \gamma_s(W_u - W_v) \, dudv \right) \right] \leq e^{\theta^2 t^2 e^{-2\|\gamma\|_\infty}},
\]
recalling that \( e^{-2} = O(\beta_k(pt)) \) and that \( t \) is bounded in that case. With this, the proof of Lemma 6.1 is now complete.

6.3. Proof of Lemma 6.2. The proof of this result is very similar to that of Lemma 5.4. We begin with the following general bound:

**Proposition 6.3.** Let \( p > 0, \theta \geq 1 \) and \( R = e^{O(\beta_k(pt))} \). For any function \( d \) such that \( 0 \leq d < t \),
\[
\left\langle \left( \sum_{k=1}^{\infty} e^{\lambda_k \theta_s(Q_R)} \left( e_k^{\theta_s}(Q_R), 1 \right)^2 + (2R)^{2d} \right)^p \right\rangle.
\]

**Proof.** By arguing as in [21, (3.37)], for every \( p > 0 \), we have that
\[
(\sum_{k=1}^{\infty} e^{\lambda_k \theta_s(Q_R)} \left( e_k^{\theta_s}(Q_R), 1 \right)^2 + (2R)^{2d})^p.
\]
Suppose first that \( p \geq 1 \). Since
\[
f(k) := \frac{\left( \sum_{k=1}^{\infty} e^{\lambda_k \theta_s(Q_R)} \left( e_k^{\theta_s}(Q_R), 1 \right)^2 + (2R)^{2d} \right)^p}{(2R)^{2dp}}, \quad k \in \mathbb{N}
\]
is a probability measure on \( \mathbb{N} \), an application of Jensen’s inequality yields
\[
\left\langle \left( \sum_{k=1}^{\infty} e^{\lambda_k \theta_s(Q_R)} \left( e_k^{\theta_s}(Q_R), 1 \right)^2 + (2R)^{2d} \right)^p \right\rangle \leq (2R)^{2(dp-1)} \sum_{k=1}^{\infty} e^{\lambda_k \theta_s(Q_R)} \left( e_k^{\theta_s}(Q_R), 1 \right)^2 = (2R)^{2(dp-1)} (u_R^{\theta_s} (pt, \cdot), 1).
\]
If we combine this with \( R = e^{O(\beta_k(pt))} \), the inequality \( (x + y)^p \leq 2^p (x^p + y^p) \) for all \( x, y > 0 \) and \( p \geq 1 \), and (6.10), then we obtain the claimed result for \( p \geq 1 \).

Now suppose that \( 0 < p < 1 \). Since \( \sum_k x_k^p \leq \sum_k x_k^p \) for every \( 0 < p < 1 \) and \( x_k \geq 0 \), we get from (6.10) that
\[
(\sum_{k=1}^{\infty} e^{\lambda_k \theta_s(Q_R)} \left( e_k^{\theta_s}(Q_R), 1 \right)^2 + (2R)^{2dp})^p \leq \sum_{k=1}^{\infty} e^{\lambda_k \theta_s(Q_R)} \left( e_k^{\theta_s}(Q_R), 1 \right)^{2p} + (2R)^{2dp}.
\]
Given that \( R = e^{O(\beta_k(pt))} \) and \( \left( e_k^{\theta_s}(Q_R), 1 \right) \leq (2R)^d \), the result follows.

With this in hand, we may now prove (6.3) and (6.4):
6.3.1. Proof of (6.3). Let \( A_{d,t}(Q_r) \) be as in (5.8), and let \( d, c, \) and \( h \) be as in (5.9). By the Feynman-Kac formula,
\[
u^c_R(t,0) \leq E_0 \left[ \exp \left( \int_0^t \xi_e(W_s) \, ds \right) \mathbb{1}_{\{A_{d,t}(Q_r)\}} \mathbb{1}_{\{f_0^d \xi_e(W_s) \, ds \leq c\}} + \mathbb{1}_{\{f_0^d \xi_e(W_s) \, ds > c\}} \right]
\leq e^{c}E_0 \left[ \exp \left( \int_0^t \xi_e(W_s) \, ds \right) \mathbb{1}_{\{A_{d,t}(Q_r)\}} \right] + E_0 \left[ \exp \left( \int_0^t \xi_e(W_s) \, ds \right) \mathbb{1}_{\{f_0^d \xi_e(W_s) \, ds > c\}} \right].
\]
Since \( c = o(\beta_e(pt)) \) and \( (x+y)^p \leq \zeta(p)(x^p + y^p) \) for some constant \( \zeta(p) > 0 \) whenever \( x, y \geq 0 \), thanks to Proposition 6.3, (6.3) will be proved if we show that, for large enough \( m \), one has
\[
E_0 \left[ \exp \left( \int_0^t \xi_e(W_s) \, ds \right) \mathbb{1}_{\{A_{d,t}(Q_r)\}} \right] \leq e^{o(\beta_e(pt))} \langle \langle u^c_R(t - d, \cdot), 1 \rangle \rangle^p
\]
and
\[
E_0 \left[ \exp \left( \int_0^t \xi_e(W_s) \, ds \right) \mathbb{1}_{\{f_0^d \xi_e(W_s) \, ds > c\}} \right]^p \leq \begin{cases} e^{-pc+4H_e(pt)} & \text{if } p \geq 1 \\ e^{-pc+4H_e(t)} & \text{if } 0 < p < 1 \end{cases},
\]
(6.11) can be proved using essentially the same argument as (5.10), except that this time we use the upper bound \( \sup_{x \in \mathbb{R}^d} \Pi_d(x) = O(d^{-d/2}) = e^{o(\beta_e(pt))} \). As for (6.12), essentially the same argument used to arrive at (5.12) together with Jensen’s inequality yields
\[
E_0 \left[ \exp \left( \int_0^t \xi_e(W_s) \, ds \right) \mathbb{1}_{\{f_0^d \xi_e(W_s) \, ds > c\}} \right]^p \leq e^{-pc+4H_e(pt)} \langle \langle e^{(t+hd)\xi_e(W_s)} \mathbb{1}_{\{A_{d,t}(Q_r)\}} \rangle \rangle^p = \begin{cases} e^{-pc+4H_e(p(t+hd))} & \text{if } p \geq 1 \\ e^{-pc+4H_e(t)} & \text{if } 0 < p < 1 \end{cases}.
\]

Then, we obtain (6.12) by noting that since \( hd \leq t \) for large enough \( m \), we have the inequality \((t+hd)^2 \leq 2(t^2 + (hd)^2) \leq 4t^2 \) for large enough \( m \); hence \( H_e(q(t+hd)) \leq 4H_e(qt) \) for all \( q > 0 \).

6.3.2. Proof of (6.4). Let \( \theta > 1 \) be arbitrary, and let \( \theta' > 1 \) be such that \( 1/\theta' + 1/\theta = 1 \). For every function \( d = d(m) \) such that \( 0 \leq d < t \), the Feynman-Kac formula followed by two applications of Hölder’s inequality yields
\[
\langle u^c_R(t, x) \rangle^p = \langle E_0 \left[ \exp \left( \int_0^t \xi_e(W_s) \, ds \right) \mathbb{1}_{\{A_{d,t}(Q_r)\}} \right]^p \rangle^{1/\theta^p} \leq \begin{cases} \langle E_0 \left[ \exp \left( \int_0^t \theta^p \xi_e(W_s) \, ds \right) \mathbb{1}_{\{A_{d,t}(Q_r)\}} \right]^p \rangle^{1/\theta^p} \end{cases}.
\]
By the Markov property,
\[
E_0 \left[ \exp \left( \int_0^t \theta^p \xi_e(W_s) \, ds \right) \mathbb{1}_{\{A_{d,t}(Q_r)\}} \right] \leq O(d^{-d/2})(u^c_R(t - d, \cdot), 1)^p.
\]
Therefore, by Proposition 6.3, it suffices to make sure that it is possible to choose \( d \) such that \( d = e^{o(\beta_e(pt))} \) and
\[
\langle E_0 \left[ \exp \left( \int_0^t \theta^p \xi_e(W_s) \, ds \right) \mathbb{1}_{\{A_{d,t}(Q_r)\}} \right]^p \rangle \leq e^{o(\beta_e(pt))}.
\]
This can be proved using the same argument as in (5.13) (up to applying Jensen’s inequality to move $p$ inside of $E_0$ or outside of $\langle \cdot \rangle$, depending on whether $p \geq 1$ or $0 < p < 1$).

With this, the proof of Lemma 6.2—and therefore also (6.1)—is now complete. In particular, combining this with (5.1), this concludes the proof of Theorem 3.4.

### 7. Variational Problems

In this section, we argue that the constant $G$ in (1.10) is finite for the Riesz noise (see Section 7.1), we prove Proposition 2.18 (see Section 7.2), and we prove Theorem 2.11 and Proposition 2.14 (see Sections 7.3–7.8). Since some of these proofs involve a large number of cumbersome integrals we use two shorthands throughout this section: On the one hand, the notation (3.1) for the Dirichlet form, and on the other hand, the form

$$J_c(f) := -J_c(f^2) = \frac{1}{2} \int \int f(x)^2 \gamma_c(x - y) f(y)^2 \, dx \, dy$$

for $c \in [0, \infty)$, recalling that $\gamma_0 = \gamma$. Moreover, whenever we omit the domain of integration in some integral, it can be assumed that the integral is over all of $\mathbb{R}^d$.

#### 7.1. The Constant $G$ for Riesz Noise

We begin with the following statement:

**Proposition 7.1.** Let $d > 2$ and $\sigma > 0$, and take $\gamma(x) = \sigma^2 |x|^{-d}$. There exists a finite constant $C > 0$ such that for every $f \in H^1(\mathbb{R}^d)$ with $\|f\|_2 = 1$, one has $J_0(f) \leq C \mathcal{I}(f)$.

While we did not find a proof for this exact statement in the literature, it follows from standard scaling arguments (e.g., [5, Lemma A.4]) and the fact that

$$\sup_{f \in H^1(\mathbb{R}^d), \|f\|_2 = 1} \left( \frac{J_0(f)^{1/2}}{\mathcal{I}(f)} \right) < \infty$$

(e.g., apply a trivial change of variables to $[2, (1.19)]$).

#### 7.2. Proof of Proposition 2.18

The fact that $M^r_{t,p} < \infty$ follows from its definition in (2.11), and noting that if $\|f\|_2 = 1$, then

$$J_1(f) \leq \frac{\|\gamma_1\|_\infty \|f\|_2^2}{2} = \frac{\|\gamma_1\|_\infty}{2} \gamma_1 < \infty.$$  

Next, we show that $M^r_{t,p}$ is positive if $p > \frac{2\sigma}{|\omega|}$. The proof is based on scaling arguments. Given $f \in H^1(\mathbb{R}^d)$ with $\|f\|_2 = 1$, we denote $f_\epsilon(x) := \epsilon^{d/2} f(\epsilon x)$. By definition of $\gamma_1$, and since $\omega = 2$, we have

$$\gamma_1(\epsilon^{-1} x) = \gamma \ast p_1(\epsilon^{-1} x) = \epsilon^2 \gamma_1(x).$$

Consequently, for any $\epsilon > 0$ and $f \in H^1(\mathbb{R}^d)$ with $\|f\|_2 = 1$, one has (by a change of variables)

$$pt J_1(f_\epsilon) - \mathcal{I}(f_\epsilon) = pt \epsilon^{2d} J_1(f(\epsilon \cdot)) - \epsilon^2 \mathcal{I}(f) = \epsilon^2 (pt J_1(f) - \mathcal{I}(f)).$$

Thus, an application of monotone convergence (using Fourier transforms as in (4.19)) yields

$$\lim_{\epsilon \to 0} \epsilon^{-2} (pt J_1(f_\epsilon) - \mathcal{I}(f_\epsilon)) = pt J_0(f) - \mathcal{I}(f).$$

This relates the critical variational problem (2.11) to the following new variational problem:

$$M^r_{t,p} := \sup_{f \in H^1(\mathbb{R}^d), \|f\|_2 = 1} \left( pt J_0(f) - \mathcal{I}(f) \right).$$
Indeed, if \( p > \frac{2\omega}{\gamma_p} \), then by definition of \( \mathcal{G} \) we have that \( M^\text{crit}_{t,p} > 0 \) (in fact, we actually have that \( M^\text{crit}_{t,p} = \infty \) by a similar scaling argument as above). Therefore there exists \( f \in H^1(\mathbb{R}^d) \) with \( \|f\|_2 = 1 \) such that
\[
ptJ_0(f) - \mathcal{J}(f) > 0.
\]

Hence by (7.3) we have for \( \varepsilon > 0 \) small enough
\[
ptJ_1(f_\varepsilon) - \mathcal{J}(f_\varepsilon) > 0,
\]
which implies that \( M^\text{crit}_{t,p} \) is positive since \( f_\varepsilon \in H^1(\mathbb{R}^d) \) with \( \|f_\varepsilon\|_2 = 1 \) for every \( f \in H^1(\mathbb{R}^d) \) with \( \|f\|_2 = 1 \).

We now argue that if \( p < \frac{2\omega}{\gamma_p} \), then \( M^\text{crit}_{t,p} = 0 \). In this case by definition of \( \mathcal{G} \), for any \( f \in H^1(\mathbb{R}^d) \) with \( \|f\|_2 = 1 \) we have \( ptJ_0(f) - \mathcal{J}(f) \leq 0 \). Since \( J_1(f) \leq J_0(f) \) (this is easily checked using a Fourier transform as in (4.19)), this implies that \( M^\text{crit}_{t,p} \leq 0 \). The fact that \( M^\text{crit}_{t,p} = 0 \) then immediately follows from (7.3).

We now finish the proof of Proposition 2.18 by showing that \( M^\text{crit}_{t,p} \) is strictly increasing in \( p \) whenever \( M^\text{crit}_{t,p} > 0 \). Take \( p > q > \frac{2\omega}{\gamma_p} \). For every \( \delta > 0 \), we can find some \( g \in H^1(\mathbb{R}^d) \) such that \( \|g\|_2 = 1 \) and
\[
qtJ_1(g) - \mathcal{J}(g) \geq M^\text{crit}_{t,q} - \delta.
\]
Since \( \mathcal{J}(g) \geq 0 \), this also implies that
\[
qtJ_1(g) \geq M^\text{crit}_{t,q} - \delta.
\]

In particular,
\[
M^\text{crit}_{t,p} - M^\text{crit}_{t,q} \geq qtJ_1(g) - \mathcal{J}(g) - (qtJ_1(g) - \mathcal{J}(g) + \delta)
= (p-q)tJ_1(g) - \delta \geq \frac{(p-q)(M^\text{crit}_{t,q} - \delta)}{q} - \delta.
\]

If we take \( \delta \to 0 \) in the above, this yields
\[
M^\text{crit}_{t,p} - M^\text{crit}_{t,q} \geq \frac{p-q}{q}M^\text{crit}_{t,q} > 0,
\]

concluding the proof.

7.3. Outline of Proof of Theorem 2.11 and Proposition 2.14. The remainder of Section 7 is devoted to the proof of Theorem 2.11 and Proposition 2.14. In Section 7.3, we provide an outline of the proof, which relies on a number of technical lemmas (i.e., Lemmas 7.2–7.6). Then, in Sections 7.4–7.8, we prove these technical lemmas.

7.3.1. Step 1. Finiteness, Existence, and Convergence. We begin by proving that \( M \) and \( M^\alpha_{c,p} \) are positive, finite, and have maximizers, as well as the two limits (2.7) and (2.8). The fact that \( M \) is positive and finite can be proved by using the same argument as in [13, Lemma A.2 with \( \alpha_0 = 0 \)]. As for \( M^\alpha_{c,p} \), we have the following:

**Lemma 7.2.** Let Assumption 1.3 hold with \( 0 < \omega < 2 \). \( M^\alpha_{c,p} \in (0, \infty) \) for every \( c, p > 0 \).

The proof of this lemma, which we provide in Section 7.4, uses a scaling argument similar to (7.2) and (7.3). Next, since \( \gamma_p^{(2-\omega)/\omega} \in L^\infty(\mathbb{R}^d) \) for every \( c, p \in (0, \infty) \), a standard application of Lions’ concentration-compactness principle (e.g., combine Lemma 7.2 with [36, Theorem III.2]) implies that maximizers of \( M^\alpha_{c,p} \) exist. Regarding the existence of maximizers of \( M \), we need the following two lemmas:
Lemma 7.3. Let Assumption 1.3 hold with $0 < \omega < 2$. For every $R > 0$, the map $f \mapsto J_0(f)$ is continuous with respect to the $L^2$ norm on the set $\{f \in L^2(\mathbb{R}^d) : \|f\|_{H^1(\mathbb{R}^d)} < R\}$.

Lemma 7.4. Let Assumption 1.3 hold with $0 < \omega < 2$. Let $(f_n)_{n \in \mathbb{N}} \subset H^1(\mathbb{R}^d)$ be a sequence of functions such that $\|f_n\|_2 = 1$ for all $n \in \mathbb{N}$ and
\begin{equation}
\lim_{n \to \infty} \left( J_0(f_n) - J(f_n) \right) = M.
\end{equation}
There exists a subsequence $(n_k)_{k \in \mathbb{N}}$, a function $f_* \in H^1(\mathbb{R}^d)$, and a sequence $(z_k)_{k \in \mathbb{N}} \subset \mathbb{R}^d$ such that
\begin{equation}
\lim_{k \to \infty} \|f_{n_k}(\cdot - z_k) - f_*\|_{H^1(\mathbb{R}^d)} = 0.
\end{equation}

Lemma 7.3, which we prove in Section 7.5, is straightforward. The statement of Lemma 7.4 is only new in the case of fractional noise; in the case of white and Riesz noises, see [36, Theorem III.2] (note that [36, Theorem III.2] does not apply to fractional noise because of condition [36, (24)]). Thus, our proof of Lemma 7.4, which we provide in Section 7.6, only covers the fractional noise. As explained in Section 7.6, the proof relies on the version of Lions’ concentration-compactness principle stated in [36, Lemma III.1].

The existence of Maximizers of $M$ can now be argued as follows: Let $f_{n_k}$ and $f_*$ be as in the statement of Lemma 7.4. Since $M$ is positive and $\omega < 2$, it is clear by (7.5) and (7.9) that there exists some $R > 0$ large enough such that $f_{n_k} \in \{f \in L^2(\mathbb{R}^d) : \|f\|_{H^1(\mathbb{R}^d)} < R\}$ for all $k \in \mathbb{N}$. In particular, by combining Lemma 7.3 with (7.5) and (7.6), we get that
\begin{equation*}
M = \lim_{k \to \infty} \left( J_0(f_{n_k}) - J(f_{n_k}) \right) = \left( J_0(f_*) - J(f_*) \right);
\end{equation*}
hence $f_*$ is a maximizer of $M$.

Finally, (2.7) was proved earlier in (4.13), and the limit (2.8) follows from a combination of Lemmas 7.3 and 7.4 with the fact that for every $\epsilon, p > 0$, one has
\begin{equation*}
M \geq \left( J_0(f^{(\epsilon)}_*) - J(f^{(\epsilon)}_*) \right) \geq \left( J_{p/(2-\omega)\epsilon}(f^{(\epsilon)}_*) - J(f^{(\epsilon)}_*) \right) = M_{\epsilon, p},
\end{equation*}
the latter of which implies by (2.7) that the sequence $f^{(\epsilon)}_*$ satisfies (7.5) as $\epsilon \to 0$.

Step 2. Geometric Properties. Following-up on Section 7.3.1, in order to prove Theorem 2.11 and Proposition 2.14, it only remains to establish properties (1)–(3) in the statement of Theorem 2.11.

We begin with the proof of properties (1) and (2), which follows the outline provided in [19, Theorem 2]. Let $f_*$ be a maximizer of $M$ with $\gamma(x) = \sigma^2 \prod_{i=1}^d |x_i|^{-\omega_i}$. Define the function $W_* := \gamma f_*^2$. By definition of $M$, the function $f_*$ is the ground state (i.e., the eigenvector of the smallest eigenvalue) of the Schrödinger operator $-\kappa \Delta - W_*$, with eigenvalue $-M < 0$. Since $\gamma$ is locally integrable and $\|f_*\|_2 = 1$, $W_*$ is locally integrable. Therefore, by [40, Theorem XIII.48 (a)], $f_*$ is either positive or negative; and by [41, Theorem C.3.3], $f_*$ has exponential decay. Since we can write $f_* = e^{\mathcal{M}t} e^{-(t^2 - \kappa \Delta - W_*)} f_*$, [41, Theorem B.3.2] implies that $f_*$ is continuous. Finally, since $f_*$ is continuous and decays exponentially, the fact that it is smooth follows from a standard elliptic bootstrapping argument, such as [34, Theorem 8 (iii)] (more precisely, we can replace $V_\phi$ in [34] by $W_*$, and replace $Y_\phi$ in [34] by the Green function of $(\kappa \Delta - \mathcal{M})^{-1}$, and then the same argument used to prove [34, Theorem 8 (iii)] can be applied).

It now only remains to prove Theorem 2.11 (3). We recall some standard definitions and terminology: Given a measurable set $A \subset \mathbb{R}^d$ and a unit vector $u \in \mathbb{R}^d$, we use $S_u(A)$ to denote the Steiner symmetrization of $A$ with respect to the plane perpendicular to $u$ (see, e.g., [17,
We recall from [35, Theorem 1.13] that any nonnegative measurable function $f : \mathbb{R}^d \to [0, \infty)$ can be written using the layer cake representation

$$f(x) = \int_0^\infty 1_{\{y \in \mathbb{R}^d : f(y) > \ell\}}(x) \, d\ell,$$

and that the Steiner symmetrization of $f$ along any unit vector $u$ is defined as

$$S_u(f)(x) := \int_0^\infty 1_{S_{\ell u}(\{y \in \mathbb{R}^d : f(y) > \ell\})}(x) \, d\ell$$

(e.g., [35, Page 87]). Finally, let $e_1, e_2, \ldots, e_d$ denote the standard basis vectors in $\mathbb{R}^d$. Given a nonnegative measurable $f$, we denote $f_{\text{coord}} := S_{e_d}(S_{e_{d-1}}(\cdots(S_{e_1}(f)))).$ We have the following two results regarding $f_{\text{coord}}$, which are proved in Sections 7.7 and 7.8, respectively:

**Lemma 7.5.** Let $\gamma(x) = \sigma^2 \prod_{i=1}^d |x_i|^{-\omega_i}$ with $0 < \omega < 2$. Let $f \in H^1(\mathbb{R}^d)$ be nonnegative, smooth and such that $\|f\|_2 = 1$. If there is no $z \in \mathbb{R}^d$ such that $f = f_{\text{coord}}(\cdot - z)$, then

$$\mathcal{J}_0(f) - \mathcal{J}(f) < \mathcal{J}_0(f_{\text{coord}}) - \mathcal{J}(f_{\text{coord}}).$$

**Lemma 7.6.** Let $f : \mathbb{R}^d \to [0, \infty)$ be measurable. For every fixed $1 \leq i \leq d$ and $x_j \in \mathbb{R}$ (for all $j \neq i$), there exists a nonincreasing function $\rho : [0, \infty) \to [0, \infty)$ such that

$$f_{\text{coord}}(x_1, \ldots, x_{i-1}, r, x_{i+1}, \ldots, x_d) = \rho(|r|) \quad \text{for every } r \in \mathbb{R}.$$

We are now in a position to prove Theorem 2.11 (3): By Lemma 7.6, it suffices to prove that if $f_*$ is a maximizer of $M$ for the Riesz noise, then $f_* = (f_*)_{\text{coord}}(\cdot - z)$ for some $z \in \mathbb{R}^d$. This follows from Lemma 7.5. With this, we have now completed the proof of Theorem 2.11 and Proposition 2.14 (up to proving Lemmas 7.2–7.6, which we now carry out).

### 7.4. Proof of Lemma 7.2

The finiteness of $M_{\epsilon,p}$ follows from the same argument as in (7.1). As for positivity, let $f$ be such that $\|f\|_2 = 1$, and let $f_\epsilon(x) := \epsilon^{d/2} f(\epsilon x)$. Arguing as in (7.2) and (7.3), except that in the subcritical case one has

$$\gamma_{\epsilon p^{1/(2-\omega)}}(\epsilon^{-1} x) = \epsilon^{\omega} \gamma_{\epsilon p^{1/(2-\omega)}}(\epsilon^{-1} x),$$

we see that as $\epsilon \to 0$,

$$\mathcal{J}_1(f_\epsilon) - \mathcal{J}(f_\epsilon) = \epsilon^2 \left( \epsilon^{\omega-2} \mathcal{J}_{\epsilon p^{1/(2-\omega)}}(\epsilon f) - \mathcal{J}(f) \right) = \epsilon^2 \left( \epsilon^{\omega-2} (1 + o(1)) \mathcal{J}_0(f) - \mathcal{J}(f) \right).$$

Since $\epsilon^{\omega-2} \to \infty$ when $\epsilon \to 0$, the positivity of $M_{\epsilon,p}$ follows from the existence of a function $f \in H^1(\mathbb{R}^d)$ such that $\mathcal{J}_0(f) > 0$, which is trivial.

### 7.5. Proof of Lemma 7.3

Since $\gamma$ is a covariance function, the bilinear map

$$\langle f, g \rangle_\gamma := \frac{1}{2} \iint f(x) \gamma(x - y) g(y) \, dx \, dy$$

is a semi-inner product. Thus, its induced semi-norm satisfies the (reverse) triangle inequality, whence for every functions $f_1$ and $f_2$ (noting that $\mathcal{J}_0(f) = \langle f^2, f^2 \rangle_\gamma$), one has

$$\left| \sqrt{\mathcal{J}_0(f_1)} - \sqrt{\mathcal{J}_0(f_2)} \right| \leq \sqrt{\langle f_1^2 - f_2^2, f_1^2 - f_2^2 \rangle_\gamma} \leq \sqrt{2} \sum_{i=1}^{\infty} |\langle f_i^2 - f_i^2, f_i^2 \rangle_\gamma|.$$
The claimed result then follows from the fact that in the subcritical regime, there exists a constant $C > 0$ such that for every $f \in H^1(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, one has

$$
(7.9) \quad \int \gamma(x-y)f(y)^2 \, dy \leq C\|f\|_2^{2-\omega}\|\nabla f\|_2^\omega.
$$

Indeed, if we combine (7.9) with (7.8), we get that (7.8) is bounded above by

$$
(7.10) \quad \sqrt{C \left( \int |f_1(x)^2 - f_2(x)^2| \, dx \right)^2} \sum_{i=1}^2 \|f_i\|_2^{2-\omega}\|\nabla f_i\|_2^\omega
= \sqrt{C \left( \int |f_1(x) - f_2(x)| |f_1(x) + f_2(x)| \, dx \right)^2} \sum_{i=1}^2 \|f_i\|_2^{2-\omega}\|\nabla f_i\|_2^\omega,
$$

If we now apply Cauchy-Schwarz in the $dx$ integral in (7.10), then we get that (7.8) is bounded above by

$$
(7.11) \quad \sqrt{C\|f_1 - f_2\|_2\|f_1 + f_2\|_2 \sum_{i=1}^2 \|f_i\|_2^{2-\omega}\|\nabla f_i\|_2^\omega}
$$

which immediately implies the result. To conclude, we note that the inequality (7.9) in the case of Riesz and fractional noise is proved in [5, (A.17) and (A.29)]. In the case of one-dimensional white noise, we remark that every constant $C > 0$ such that for every $x \in \mathbb{R}^d$, we have

$$
\int \delta_0(x-y)f(y)^2 \, dy = f(x)^2 = -\int_x^\infty (f(y)^2)\, dy = -2\int_x^\infty f(y) f'(y) \, dy \leq 2\|f\|_2\|f'\|_2.
$$

7.6. Proof of Lemma 7.4. As mentioned in Section 7.3.1, we only need to prove the result in the case of subcritical fractional noise; we thus henceforth assume that $\gamma(x) = \sigma^2 \prod_{i=1}^d |x_i|^{-\omega_i}$ with $0 < \omega < 2$. Our proof relies on the version of the concentration-compactness principle stated in [36, Lemma III.1], which is as follows:

**Lemma 7.7** ([36]). Let $(f_n)_{n \in \mathbb{N}}$ be such that $f_n \geq 0$ and $\|f_n\|_2 = 1$ for every $n$. There exists a subsequence $(n_k)_{k \in \mathbb{N}}$ that satisfies one of the following three possibilities:

1. **(Compactness)** There exists a sequence $(z_k)_{k \in \mathbb{N}} \subset \mathbb{R}^d$ such that for every $\varepsilon > 0$, there exists $R > 0$ such that

$$
(7.12) \quad \int_{\{x < R\}} f_{n_k}(x-z)^2 \, dx \geq 1 - \varepsilon, \quad k \in \mathbb{N}.
$$

2. **(Vanishing)** For every $R > 0$,

$$
(7.13) \quad \lim_{k \to \infty} \sup_{z \in \mathbb{R}^d} \int_{\{x < R\}} f_{n_k}(x-z)^2 \, dx = 0.
$$

3. **(Dichotomy)** There exists $0 < a < 1$ such that for every $\varepsilon > 0$, there is some $k_0 \in \mathbb{N}$ and nonnegative $f_k^{(1)}$, $f_k^{(2)} \in L^2(\mathbb{R}^d)$ such that for every $k \geq k_0$:

- $\|f_{n_k} - (f_k^{(1)} + f_k^{(2)})\|_p \leq \delta_p(\varepsilon)$ for $2 \leq p \leq 6$, where $\delta_p(\varepsilon) \to 0$ as $\varepsilon \to 0$; and
- $\|f_k^{(1)}\|_2^2 - a \leq \varepsilon$ and $\|f_k^{(2)}\|_2^2 - (1 - a) \leq \varepsilon$.

Moreover,
We now prove that (7.15) with the function

\[ f(y)^2 = f_{n_k}(x+y)^2 g_\theta(y) \leq f_{n_k}(x+y)^2 1_{\{|y|<R_\theta\}} \]

for every \( \theta > 0 \), we can find a smooth and compactly supported function \( g_\theta \) and a large enough \( R_\theta > 0 \) such that

\[ 1_{\{|y|\leq \theta \ \forall 1 \leq i \leq d\}} \leq g_\theta(y) \leq 1_{\{|y|<R_\theta\}}, \quad y \in \mathbb{R}^d. \]

In particular, an application of (7.9) with the function

\[ f(y)^2 = f_{n_k}(x+y)^2 g_\theta(y) \]

Our objective is to show that any sequence \( (f_n)_{n \in \mathbb{N}} \) that satisfies (7.5) must also satisfy the compactness condition stated in (7.12). Indeed, if such is the case, then a direct application of the argument contained in the two paragraphs following [36, Remark III.3] yields (7.6). Thanks to Lemma 7.7, for this it suffices to prove that any sequence satisfying (7.5) cannot also satisfy the vanishing or dichotomy conditions stated in Lemma 7.7.

7.6.1. Vanishing. We begin by proving that vanishing does not occur. Suppose that the sequence \( (f_n)_{n \in \mathbb{N}} \) satisfies (7.5), and that we can find a subsequence such that (7.13) occurs. We claim that this implies that \( M \leq 0 \), which is a contradiction. Thanks to (7.5), in order to prove \( M \leq 0 \), it is enough to show that the vanishing condition (7.13) implies that

\[ \limsup_{k \to \infty} \mathcal{J}_0(f_{n_k}) = 0. \]

For this purpose, for every \( \theta > 0 \), this limsup is bounded above by

\[
\begin{align*}
\limsup_{k \to \infty} \frac{\sigma^2}{2} & \iint_{\{|x-y| \leq \theta \ \forall 1 \leq i \leq d\}} f_{n_k}(x)^2 \prod_{i=1}^d |x_i - y_i|^{-\omega_i} f_{n_k}(y)^2 \, dx \, dy \\
& \quad \quad + \limsup_{k \to \infty} \sum_{j=1}^d \frac{\sigma^2}{2} \iint_{\{|x_j-y_j| > \theta\}} f_{n_k}(x)^2 \prod_{i=1}^d |x_i - y_i|^{-\omega_i} f_{n_k}(y)^2 \, dx \, dy.
\end{align*}
\]

We now prove that (7.14) vanishes for all \( \theta > 0 \), and that (7.15) can be made arbitrarily small by taking \( \theta \to \infty \).

We begin with the claim regarding (7.14). By a straightforward change of variables and the fact that \( \|f_{n_k}\|_2 = 1 \), we have that

\[
\begin{align*}
\iint_{\{|x-y| \leq \theta \ \forall 1 \leq i \leq d\}} f_{n_k}(x)^2 \prod_{i=1}^d |x_i - y_i|^{-\omega_i} f_{n_k}(y)^2 \, dx \, dy \\
& \leq \sup_{x \in \mathbb{R}^d} \int_{\{|y| \leq \theta \ \forall 1 \leq i \leq d\}} \prod_{i=1}^d |y_i|^{-\omega_i} f_{n_k}(x+y)^2 \, dy.
\end{align*}
\]

For every \( \theta > 0 \), we can find a smooth and compactly supported function \( g_\theta \) and a large enough \( R_\theta > 0 \) such that

\[ 1_{\{|y|\leq \theta \ \forall 1 \leq i \leq d\}} \leq g_\theta(y) \leq 1_{\{|y|<R_\theta\}}, \quad y \in \mathbb{R}^d. \]
yields

\[
(7.16) \quad \sup_{x \in \mathbb{R}^d} \int_{\{|y| \leq \theta \forall 1 \leq i \leq d\}} \prod_{i=1}^d |y_i|^{-\omega_i} f_{n_k}(x + y)^2 \, dy \\
\leq C \sup_{x \in \mathbb{R}^d} \|f_{n_k}(x + \cdot)1_{\{|x| < R_\theta\}}\|_2^{2-\omega} \cdot \sup_{x \in \mathbb{R}^d} \left\|\nabla (f_{n_k}(x + \cdot)\sqrt{g_\theta})\right\|_2^{\omega}.
\]

The first supremum on the right-hand side of (7.16) goes to zero as \( k \to \infty \) for every \( R_\theta > 0 \) thanks to the vanishing condition (7.13). Thus, in order to show that the lim sup in (7.14) is zero, it suffices to show that the second supremum on the right-hand side of (7.16) remains bounded as \( k \to \infty \). For this purpose, we apply the product rule and \((x + y)^2 \leq 2(x^2 + y^2)\), which yields

\[
\sup_{k \geq 0} \left\|\nabla (f_{n_k}(x + \cdot)\sqrt{g_\theta})\right\|_2^2 \\
\leq \sup_{k \geq 0} 2 \int \sum_{i=1}^d \left( \frac{\partial}{\partial y_i} f_{n_k}(x + y) \right)^2 g_\theta(y) + \sum_{i=1}^d \left( \frac{\partial}{\partial y_i} \sqrt{g_\theta(y)} \right)^2 f_{n_k}(x + y)^2 \, dy \\
\leq \sup_{k \geq 0} 2 \|\nabla f_{n_k}\|_2^2 + 2 d \max_{1 \leq i \leq d} \left\|\left( \frac{\partial}{\partial y_i} \sqrt{g_\theta} \right) \right\|_\infty^2.
\]

By combining (7.9) with \( M > 0 \) and \( \omega < 2 \), the fact that \((f_n)_{n \in \mathbb{N}}\) satisfies (7.5) implies that

\[
(7.17) \quad \sup_n \|\nabla f_n\|_2 < \infty,
\]

as desired.

We now conclude the proof that vanishing does not occur by showing that (7.15) can be made arbitrarily small by taking \( \theta \to \infty \). Note that, for any \( \theta > 0 \), one has

\[
(7.18) \quad \int \int \{ |x_1 - y_i| > \theta \} f_{n_k}(x)^2 \prod_{i=1}^d |x_i - y_i|^{-\omega_i} f_{n_k}(y)^2 \, dx \, dy \\
\leq \theta^{-\omega_1} \int \int f_{n_k}(x)^2 \prod_{i=2}^d |x_i - y_i|^{-\omega_i} f_{n_k}(y)^2 \, dx \, dy.
\]

Let us denote \( \tilde{x} := (x_2, \ldots, x_d) \) and similarly for \( \tilde{y} \), as well as \( \tilde{\omega} := \sum_{i=2}^d \omega_i \). If we apply (7.9) with \( \gamma(x) = \prod_{i=2}^d |x_i|^{-\omega_i} \), then we get that for every fixed \( x_1 \in \mathbb{R} \)

\[
(7.19) \quad \sup_{\tilde{y} \in \mathbb{R}^{d-1}} \int f_{n_k}(x_1, \tilde{x})^2 \prod_{i=2}^d |x_i - y_i|^{-\omega_i} \, d\tilde{x} \leq C \|f_{n_k}(x_1, \cdot)\|_2^{\frac{2-\tilde{\omega}}{\tilde{\omega}}} \|\nabla f_{n_k}(x_1, \cdot)\|_2^{\tilde{\omega}}.
\]

for some constant \( C > 0 \) independent of \( x_1, k, \) and \( \theta \). In particular, the right-hand side of (7.18) is bounded above by

\[
C \theta^{-\omega_1} \int \int \|f_{n_k}(x_1, \cdot)\|_2^{\frac{2-\tilde{\omega}}{\tilde{\omega}}} \|\nabla f_{n_k}(x_1, \cdot)\|_2^{\tilde{\omega}} f_{n_k}(y)^2 \, dx_1 \, dy.
\]

Now integrate out \( dy \) from the above display, which yields

\[
C \theta^{-\omega_1} \int \|f_{n_k}(x_1, \cdot)\|_2^{\frac{2-\tilde{\omega}}{\tilde{\omega}}} \|\nabla f_{n_k}(x_1, \cdot)\|_2^{\tilde{\omega}} \, dx_1.
\]
Next, applying Hölder’s inequality with \( p = \frac{1}{1-\bar{\omega}/2} \) and \( q = \frac{1}{\bar{\omega}/2} \) gives the upper bound
\[
C\theta^{-\omega_1} \|f_{n_k}(x_1, \cdot)\|_2^2 \|\nabla f_{n_k}(x_1, \cdot)\|_2^\bar{\omega} \leq C\theta^{-\omega_1} \|\nabla f_{n_k}\|_2^\bar{\omega}.
\]
Combining this with (7.17) and (7.18), we conclude that
\[
\limsup_{k \to \infty} \int \int_{\{|x_1-y_1|>\theta\}} f_{n_k}(x)^2 \prod_{i=1}^d |x_i - y_i|^{-\omega_i} f_{n_k}(y)^2 \, dx \, dy = O(\theta^{-\omega_1}).
\]
Using the same argument (but interchanging the roles of \( x_1 \) and \( y_1 \) with \( x_j \) and \( y_j \)), one has
\[
\limsup_{k \to \infty} \int \int_{\{|x_j-y_j|>\theta\}} f_{n_k}(x)^2 \prod_{i=1}^d |x_i - y_i|^{-\omega_i} f_{n_k}(y)^2 \, dx \, dy = O(\theta^{-\omega_j})
\]
for all \( 2 \leq j \leq d \). With this, we obtain that
\[
\lim \limsup_{\theta \searrow 0} \limsup_{k \to \infty} \sum_{j=1}^d \int \int_{\{|x_j-y_j|>\theta\}} f_{n_k}(x)^2 \prod_{i=1}^d |x_i - y_i|^{-\omega_i} f_{n_k}(y)^2 \, dx \, dy = 0,
\]
thus concluding the proof that vanishing cannot occur at the same time as (7.5).

7.6.2. **Dichotomy.** We now conclude the proof of Lemma 7.4 by showing that the dichotomy condition stated in Lemma 7.7 cannot occur at the same time as (7.5). Suppose then that \((f_n)_{n \in \mathbb{N}}\) satisfies both (7.5) and the dichotomy condition. The contradiction that we obtain from this assumption is based on the following quantities: For every \( 0 < a < 1 \), we define
\[
M(a) := \sup_{f \in H^1(\mathbb{R}^d), \|f\|_2 = a} \left( J_0(f) - S(f) \right).
\]
By the straightforward change of variables \( f \mapsto f/\sqrt{a} \), we note that for any \( a < 1 \), one has
\[
M(a) := \sup_{f \in H^1(\mathbb{R}^d), \|f\|_2 = 1} a(aJ_0(f) - S(f)) < aM.
\]
Therefore,
\[
M(a) + M(1-a) < aM + (1-a)M = M \quad \text{for every } 0 < a < 1.
\]
We now prove that the assumption that \((f_n)_{n \in \mathbb{N}}\) satisfies both (7.5) and the dichotomy condition contradicts (7.21).

Let \( 0 < a < 1 \) be as in the dichotomy statement in Lemma 7.7, and for every \( \varepsilon > 0 \), recall the definitions of \( f_k^{(i)} \) and \( \delta_\rho(\varepsilon) \) in the same statement. By (7.5) and the definition of the dichotomy condition, we can write
\[
M = \limsup_{k \to \infty} \left( J_0(f_{n_k}) - S(f_{n_k}) \right)
\]
\[
= \lim \limsup_{\varepsilon \searrow 0} \limsup_{k \to \infty} \left( 2 \sum_{i=1}^2 \left( J_0(f_k^{(i)}) - S(f_k^{(i)}) \right) + \left( S(f_k^{(1)}) + S(f_k^{(2)}) - S(f_{n_k}) \right) \right.
\]
\[
+ \left. \left( J_0(f_{n_k}) - J_0(f_k^{(1)}) - J_0(f_k^{(2)}) \right) \right)
\]
\[
\leq M(a) + M(1-a) + \lim \limsup_{\varepsilon \searrow 0} \limsup_{k \to \infty} \left( J_0(f_{n_k}) - J_0(f_k^{(1)}) - J_0(f_k^{(2)}) \right).
\]
Thus, to get a contradiction, it suffices to prove that the remaining limit in (7.22) is zero.
Recall the bilinear map notation introduced in (7.7). For any \( \varepsilon > 0 \) and \( k \geq 0 \), we can write
\[
(7.23) \quad J_0(f_{nk}^i) - J_0(f_{k}^{(1)}) - J_0(f_{k}^{(2)}) = \left( J_0(f_{nk}^i) - \left( (f_k^{(1)} + f_k^{(2)})^2, (f_k^{(1)} + f_k^{(2)})^2 \right)_\gamma \right)
+ \left( \left( (f_k^{(1)} + f_k^{(2)})^2, (f_k^{(1)} + f_k^{(2)})^2 \right)_\gamma - J_0(f_{k}^{(1)}) - J_0(f_{k}^{(2)}) \right).
\]

We begin by controlling the first term on the right-hand side of (7.23). If we combine (7.8) and (7.11) with the fact that \( \|f_{nk}\|_{H^1(\mathbb{R}^d)} \) and \( \|f_{k}^{(i)}\|_{H^1(\mathbb{R}^d)} \) are uniformly bounded in \( \varepsilon \in (0, 1) \) and \( k \geq 0 \), then we get that
\[
\left| \sqrt{J_0(f_{nk}^i)} - \sqrt{\left( (f_k^{(1)} + f_k^{(2)})^2, (f_k^{(1)} + f_k^{(2)})^2 \right)_\gamma} \right| \leq C\|f_{nk}^i - (f_k^{(1)} + f_k^{(2)})\|_2
\]
for every \( \varepsilon \in (0, 1) \) and \( k \geq 0 \), where the constant \( C > 0 \) is independent of \( k \) and \( \varepsilon \). By definition of the dichotomy condition, for every \( \varepsilon > 0 \), there is some \( k_0 \geq 0 \) such that
\[
\|f_{nk}^i - (f_k^{(1)} + f_k^{(2)})\|_2 \leq \delta_2(\varepsilon)
\]
whenever \( k \geq k_0 \). Recalling that \( \delta_2(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \), we therefore conclude that
\[
(7.24) \quad \lim_{\varepsilon \to 0} \lim_{k \to \infty} \sup \left( J_0(f_{nk}^i) - \left( (f_k^{(1)} + f_k^{(2)})^2, (f_k^{(1)} + f_k^{(2)})^2 \right)_\gamma \right) = 0.
\]

It now only remains to control the second term on the right-hand side of (7.23). For every \( \varepsilon > 0 \), we have that
\[
(7.25) \quad \lim_{k \to \infty} \inf \left\{ |x - y| : x \in \text{supp}(f_k^{(1)}) \text{ and } y \in \text{supp}(f_k^{(2)}) \right\} = \infty.
\]

In particular, if \( k \) is large enough, then \( f_k^{(1)} f_k^{(2)} = 0 \); hence \( (f_k^{(1)} + f_k^{(2)})^2 = [f_k^{(1)}]^2 + [f_k^{(2)}]^2 \). Therefore, since \( J_0(f) = \langle f^2, f^2 \rangle_\gamma \), we can expand
\[
\left( (f_k^{(1)} + f_k^{(2)})^2, (f_k^{(1)} + f_k^{(2)})^2 \right)_\gamma - J_0(f_k^{(1)}) - J_0(f_k^{(2)}) = 2 \left( [f_k^{(1)}]^2, [f_k^{(1)}]^2 \right)_\gamma.
\]

Since the \( \ell_2 \) and \( \ell_\infty \) norms are equivalent on \( \mathbb{R}^d \), (7.25) implies that for every \( \varepsilon > 0 \), there exists a sequence of indices \( 1 \leq i(k) \leq d \) and numbers \( d_k > 0 \) such that
\[
\bullet \quad |x_{i(k)} - y_{i(k)}| > d_k \text{ for every } x \in \text{supp}(f_k^{(1)}) \text{ and } y \in \text{supp}(f_k^{(2)}) \text{; and}
\bullet \quad d_k \to \infty \text{ as } k \to \infty.
\]

In particular, we can write
\[
\left( [f_k^{(1)}]^2, [f_k^{(1)}]^2 \right)_\gamma \leq \frac{\sigma^2}{2} \int \int \left\{ |x_{i(k)} - y_{i(k)}| > d_k \right\} [f_k^{(1)}(x)]^2 \prod_{i=1}^d |x_i - y_i|^{-\omega} [f_k^{(2)}(y)]^2 \, dx \, dy.
\]

As \( \sup_{k \in \mathbb{N}} \|f_k^{(i)}\|_{H^1(\mathbb{R}^d)} < \infty \) for \( i = 1, 2 \), by replicating the argument leading up to (7.20), we conclude that \( \left( [f_k^{(1)}]^2, [f_k^{(1)}]^2 \right)_\gamma \leq C d_k^{\omega(\delta k)} \) for some constant \( C > 0 \) independent of \( k \). Therefore, for every \( \varepsilon > 0 \),
\[
\limsup_{k \to \infty} \left( [f_k^{(1)}]^2, [f_k^{(1)}]^2 \right)_\gamma = 0.
\]

If we combine this with (7.22), (7.23), and (7.24), then we finally obtain that dichotomy cannot occur simultaneously with (7.5), thus concluding the proof of Lemma 7.4.
7.7. Proof of Lemma 7.5. For every fixed \(x_2, \ldots, x_d \in \mathbb{R}\) and nonnegative measurable function \(g\), we note that the one-dimensional function \(x_1 \mapsto S_{e_1}(g)(x_1, x_2, \ldots, x_d)\) is the symmetric decreasing rearrangement of the function \(x_1 \mapsto g(x_1, x_2, \ldots, x_d)\) (see, e.g., [35, Page 80]). Therefore, by the Pólya-Szegő inequality (e.g., [35, Lemma 7.17]),

\[
\int \left( \frac{\partial f(x)}{\partial x_1} \right)^2 \, dx \geq \int \left( \frac{\partial S_{e_1}(f)(x)}{\partial x_1} \right)^2 \, dx,
\]

and since the symmetric decreasing rearrangement preserves the \(L^p\)-norm ([35, Page 81]),

\[
\int \sum_{i=2}^d \left( \frac{\partial f(x)}{\partial x_i} \right)^2 \, dx = \int \sum_{i=2}^d \left( \frac{\partial S_{e_1}(f)(x)}{\partial x_i} \right)^2 \, dx.
\]

In particular,

\[-\mathcal{J}(f) \leq -\kappa \int |\nabla S_{e_1}(f)(x)|^2 \, dx.
\]

Next, by Fubini’s theorem, we can write

\[
\int\int f(x)^2 \prod_{i=1}^d |x_i - y_i|^{-\omega_i} f(y)^2 \, dx \, dy
\]

\[
= \int\int \prod_{i=2}^d |x_i - y_i|^{-\omega_i} \left( \int f(x_1, \tilde{x})^2 |x_1 - y_1|^{-\omega_1} f(y_1, \tilde{y})^2 \, dx_1 \, dy_1 \right) \, dx \, dy,
\]

where we denote \(\tilde{x} := (x_2, \ldots, x_d)\) and similarly for \(\tilde{y}\). Let \(\tilde{x}, \tilde{y} \in \mathbb{R}^{d-1}\) be fixed. By the one-dimensional Riesz rearrangement inequality (e.g., [35, Lemma 3.6]), we have that

\[
\int f(x_1, \tilde{x})^2 |x_1 - y_1|^{-\omega_1} f(y_1, \tilde{y})^2 \, dx_1 \, dy_1
\]

\[
\leq \int [S_{e_1}(f)(x_1 - z_1, \tilde{x})]^2 |x_1 - y_1|^{-\omega_1} [S_{e_1}(f)(y_1 - z_1, \tilde{y})]^2 \, dx_1 \, dy_1.
\]

for any \(z_1 \in \mathbb{R}\). Moreover, since \(x_1 \mapsto |x_1|^{-\omega_1}\) is strictly decreasing in \(|x_1|\), it follows from [35, Theorem 3.9] that the above inequality is an equality only if there exists \(z_1 \in \mathbb{R}\) such that

\[f(x_1, \tilde{x}) = S_{e_1}(f)(x_1 - z_1, \tilde{x}), \quad f(y_1, \tilde{y}) = S_{e_1}(f)(y_1 - z_1, \tilde{y})\]

for almost every \(x_1, y_1 \in \mathbb{R}\). If we choose the same \(z_1\) for all \(\tilde{x}\) and \(\tilde{y}\), this implies that

\[
\int\int f(x)^2 \prod_{i=1}^d |x_i - y_i|^{-\omega_i} f(y)^2 \, dx \, dy
\]

\[
\leq \int\int [S_{e_1}(f)(x_1 - z_1, \tilde{x})]^2 \prod_{i=1}^d |x_i - y_i|^{-\omega_i} [S_{e_2}(f)(y_1 - z_1, \tilde{y})]^2 \, dx \, dy,
\]

for any \(z_1 \in \mathbb{R}\), and if there is no \(z_1 \in \mathbb{R}\) such that \(f = S_{e_1}(f)(\cdot -(z_1, 0, \ldots, 0))\) almost everywhere, then the inequality is strict.

If we then iterate the above argument with the Steiner symmetrizations with respect to the axes \(e_2, \ldots, e_d\), we conclude the proof of Lemma 7.5.
7.8. Proof of Lemma 7.6. By definition,

$$S_{\text{coord}}(f)(x) := \int_0^\infty \mathbf{1}_{S_d(S_{d-1}(...(S_1(\{y \in \mathbb{R}^d: f(y) > \ell\}))...(S_1(A))))}(x) \, d\ell.$$ 

Thus, the lemma follows from the fact that for any measurable set $A \subset \mathbb{R}$, the set

$$S_d(S_{d-1}(...(S_1(A))))$$

is symmetric with respect to every coordinate axis (e.g., [17, Claim #1 in Theorem 2.4]).

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REFERENCES

[1] Romain Allez and Khalil Chouk. The continuous Anderson Hamiltonian in dimension two. Preprint, arXiv:1511.02718, 2015.
[2] Richard Bass, Xia Chen, and Jay Rosen. Large deviations for Riesz potentials of additive processes. Ann. Inst. Henri Poincaré Probab. Stat., 45(3):626–666, 2009.
[3] R. A. Carmona and S. A. Molchanov. Stationary parabolic Anderson model and intermittency. Probab. Theory Related Fields, 102(4):433–453, 1995.
[4] X. Chen. Random walk intersections, volume 157 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2010. Large deviations and related topics.
[5] Xia Chen. Quenched asymptotics for Brownian motion in generalized Gaussian potential. Ann. Probab., 42(2):576–622, 2014.
[6] Xia Chen. Precise intermittency for the parabolic Anderson equation with an (1+1)-dimensional time-space white noise. Ann. Inst. Henri Poincaré Probab. Stat., 51(4):1486–1499, 2015.
[7] Xia Chen. Moment asymptotics for parabolic Anderson equation with fractional time-space noise: in Skorokhod regime. Ann. Inst. Henri Poincaré Probab. Stat., 53(2):819–841, 2017.
[8] Xia Chen. Parabolic Anderson model with rough or critical Gaussian noise. Ann. Inst. Henri Poincaré Probab. Stat., 55(2):941–976, 2019.
[9] Xia Chen. Parabolic Anderson model with rough or critical Gaussian noise. Ann. Inst. Henri Poincaré Probab. Stat., 55(2):941–976, 2019.
[10] Xia Chen, Aurélien Deya, Cheng Ouyang, and Samy Tindel. A K-rough path above the space-time fractional Brownian motion. Stoch. Partial Differ. Equ. Anal. Comput., 9(4):819–866, 2021.
[11] Xia Chen, Aurélien Deya, Cheng Ouyang, and Samy Tindel. Moment estimates for some renormalized parabolic Anderson models. Ann. Probab., 49(5):2599–2636, 2021.
[12] Xia Chen, Yaozhong Hu, Jian Song, and Xiaoming Song. Temporal asymptotics for fractional parabolic Anderson model. Electron. J. Probab., 23:Paper No. 14, 39, 2018.
[13] Xia Chen, Yaozhong Hu, Jian Song, and Fei Xing. Exponential asymptotics for time-space Hamiltonians. Ann. Inst. Henri Poincaré Probab. Stat., 51(4):1529–1561, 2015.
[14] Xia Chen and Jay Rosen. Large deviations and renormalization for Riesz potentials of stable intersection measures. Stochastic Process. Appl., 120(9):1837–1878, 2010.
[15] Jean-Dominique Deuschel and Daniel W. Strook. Large deviations, volume 137 of Pure and Applied Mathematics. Academic Press, Inc., Boston, MA, 1989.
[16] Laure Dumaz and Cyril Labbé. Localization of the continuous Anderson Hamiltonian in 1-D. Probab. Theory Related Fields, 176(1-2):353–419, 2020.
[17] Lawrence C. Evans and Ronald F. Gariepy. Measure theory and fine properties of functions. Textbooks in Mathematics. CRC Press, Boca Raton, FL, revised edition, 2015.

[18] Rupert L. Frank and Leander Geisinger. The ground state energy of a polaron in a strong magnetic field. Comm. Math. Phys., 338(1):1–29, 2015.

[19] Jürg Fröhlich and Enno Lenzmann. Mean-field limit of quantum Bose gases and nonlinear Hartree equation. In Séminaire: Équations aux Dérivées Partielles. 2003–2004, Sémin. Équ. Dériv. Partielles, pages Exp. No. XIX, 26. École Polytech., Palaiseau, 2004.

[20] J. Gärtner, W. König, and S. A. Molchanov. Almost sure asymptotics for the continuous parabolic Anderson model. Probab. Theory Related Fields, 118(4):547–573, 2000.

[21] Jürgen Gärtner and Wolfgang König. Moment asymptotics for the continuous parabolic Anderson model. Ann. Appl. Probab., 10(1):192–217, 2000.

[22] Pierre Yves Gaudreau Lamarre. Phase transitions in asymptotically singular Anderson Hamiltonian and parabolic model. Stoch. Partial Differ. Equ. Anal. Comput., 10(4):1451–1499, 2022.

[23] Yu Gu and Weijun Xu. Moments of 2D parabolic Anderson model. Asymptot. Anal., 108(3):151–161, 2018.

[24] Massimiliano Gubinelli, Peter Imkeller, and Nicolas Perkowski. Paracontrolled distributions and singular PDEs. Forum Math. Pi, 3:e6, 75, 2015.

[25] Osman Güler. Foundations of optimization, volume 258 of Graduate Texts in Mathematics. Springer, New York, 2010.

[26] M. Hairer. A theory of regularity structures. Invent. Math., 198(2):269–504, 2014.

[27] Martin Hairer and Cyril Labbé. A simple construction of the continuum parabolic Anderson model on $\mathbb{R}^2$. Electron. Commun. Probab., 20:no. 43, 11, 2015.

[28] Martin Hairer and Cyril Labbé. Multiplicative stochastic heat equations on the whole space. J. Eur. Math. Soc. (JEMS), 20(4):1005–1054, 2018.

[29] Yaozhong Hu, Jingyu Huang, David Nualart, and Samy Tindel. Stochastic heat equations with general multiplicative Gaussian noises: Hölder continuity and intermittency. Electron. J. Probab., 20:no. 55, 50, 2015.

[30] Wolfgang König. The parabolic Anderson model. Pathways in Mathematics. Birkhäuser/Springer, [Cham], 2016. Random walk in random potential.

[31] Wolfgang König, Nicolas Perkowski, and Willem van Zuijlen. Longtime asymptotics of the two-dimensional parabolic Anderson model with white-noise potential. Ann. Inst. Henri Poincaré Probab. Stat., 58(3):1351–1384, 2022.

[32] Cyril Labbé. The continuous Anderson Hamiltonian in $d \leq 3$. J. Funct. Anal., 277(9):3187–3235, 2019.

[33] Khoa Lê. A remark on a result of Xia Chen. Statist. Probab. Lett., 118:124–126, 2016.

[34] Elliott H. Lieb. Existence and uniqueness of the minimizing solution of Choquard’s nonlinear equation. Studies in Appl. Math., 57(2):93–105, 1976/77.

[35] Elliott H. Lieb and Michael Loss. Analysis, volume 14 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, second edition, 2001.

[36] P.-L. Lions. The concentration-compactness principle in the calculus of variations. The locally compact case. I. Ann. Inst. H. Poincaré Anal. Non Linéaire, 1(2):109–145, 1984.

[37] Ulrich Mansmann. The free energy of the Dirac polaron, an explicit solution. Stochastics Stochastics Rep., 34(1-2):93–125, 1991.

[38] Toyomu Matsuda. Integrated density of states of the Anderson Hamiltonian with two-dimensional white noise. Preprint, arXiv:2011.09180v3, 2022.

[39] Stanislav A. Molchanov. Ideas in the theory of random media. Acta Appl. Math., 22(2-3):139–282, 1991.

[40] M. Reed and B. Simon. Methods of modern mathematical physics. IV. Analysis of operators. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978.

[41] B. Simon. Schrödinger semigroups. Bull. Amer. Math. Soc. (N.S.), 7(3):447–526, 1982.