Lines in affine simplicial toric varieties

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ABSTRACT. We prove that up to automorphisms \( \mathbb{A}^1_k \) admits a unique embedding into the regular part of \( \mathbb{A}^n_k/G \) where \( k \) is an algebraically closed field of characteristic zero, \( n \geq 4 \), \( G \) is a finite subgroup of \( \text{SL}_n(k) \) acting naturally on \( \mathbb{A}^n_k \) and \( \mathbb{A}^n_k/G \) is smooth in codimension 2.

1 Introduction

Let \( \varphi : C \rightarrow C' \) be an isomorphism of two smooth polynomial curves contained in the regular part \( Y_{\text{reg}} \) of an affine algebraic variety \( Y \) over an algebraically closed field \( k \) of characteristic zero. It may happen that \( \varphi \) extends to an automorphism of \( Y \) and our aim is to describe some affine algebraic varieties for which this extension takes place. This problem was studied in several papers [AMo], [Su], [Cr],[Je], [St],[FS], [Ka20] and [AZ]. It turns out that the answer is positive for some classes of flexible varieties of dimension \( n \geq 4 \) where \( Y \) is flexible if the subgroup \( \text{SAut}(Y) \) of the automorphism group \( \text{Aut}(Y) \) of \( Y \) generated by all one-parameter unipotent subgroups acts transitively on \( Y_{\text{reg}} \). Say, this is so if \( Y = \mathbb{A}^n \) with \( n \geq 4 \) [Cr],[Je]. For \( n = 3 \) the answer is unknown but for \( n = 2 \) the famous Abhyankar-Moh-Suzuki theorem [AMo],[Su] states that an isomorphism of two smooth plane polynomial curves always extends to an automorphism of the plane \( \mathbb{A}^2 \). Perhaps, \( \mathbb{A}^2 \) is the only example of a two-dimensional flexible variety with this property. If \( Y \) is an affine simplicial toric variety \( \mathbb{A}^2/G \) where \( G \) is a finite subgroup of \( \text{SL}_2(k) \) acting naturally on \( \mathbb{A}^2 \), then Arzhantsev and Zaidenberg [AZ] showed that the answer is negative. They actually classified up to automorphisms all smooth polynomial curves in \( Y_{\text{reg}} \) (there are only finite number of isomorphism classes of such curves). In this paper we study the case when \( Y \) is an affine simplicial toric variety of dimension \( n \geq 4 \) (i.e., \( Y = \mathbb{A}^n/G \) where \( G \) is a finite subgroup of \( \text{SL}_n(k) \) acting naturally on \( \mathbb{A}^n \)). We show
that the answer to this extension problem is positive under the assumption of smoothness in codimension 2. Furthermore, recall that given a subvariety \( Z \) of \( Y \) with defining ideal \( I \) in the algebra \( k[Y] \) of regular functions on \( Y \) its \( k \)-infinitesimal neighborhood is the scheme with the defining ideal \( I^k \). In particular, if \( W \) is another subvariety of \( Y \) with defining ideal \( J \), then an isomorphism \( Z \to W \) of \( k \)-infinitesimal neighborhoods of \( Z \) and \( W \) is determined by an isomorphism of algebras \( \frac{k[Y]}{I^k} \to \frac{k[Y]}{J^k} \). There are natural obstacles for extending such isomorphisms to automorphisms of \( Y \). Say, let \( Z = W \) be a strict complete intersection given in \( Y \) by \( u_1 = \ldots = u_k = 0 \). Then an automorphism \( \psi: \frac{k[Y]}{I^k} \to \frac{k[Y]}{I^k} \) over \( k[Y] \) of degree at most \( k-1 \). If \( Y \) does not admit nonconstant invertible functions and \( \psi \) is extendable to an automorphism of \( Y \), then one can see that the Jacobian \( \det \left[ \frac{\partial f_i}{\partial u_j} \right]_{i,j=1}^k \) must be equal to a nonzero constant modulo \( I^{k-1} \) in which case we say that \( \psi \) has a nonzero constant Jacobian.

There is also a notion of a nonzero constant Jacobian of an isomorphism \( Z \to W \) in the case when both \( Z \) and \( W \) are smooth polynomial curve in a normal toric variety \( Y \) contained in \( Y_{\text{reg}} \) (see Definition 5.4). The question when such isomorphisms with nonzero constant Jacobians are extendable to automorphisms of \( Y \) was considered in [KaUd] and [Ud]. In combinations with the results of [KaUd] and [Ud] we get the following (Corollary 5.5).

**Theorem 1.1.** Let \( Y \) be an affine simplicial toric variety smooth in codimension 2 such that \( \dim Y \geq 4 \) and \( \varphi: C_1 \to C_2 \) be an isomorphism of \( k \)-infinitesimal neighborhoods of two smooth polynomial curves contained in \( Y_{\text{reg}} \) such that the Jacobian of \( \varphi \) is a nonzero constant. Then \( \varphi \) extends to an automorphism of \( Y \).

Since the paper is heavily based on the technique developed in [Ka20] which was later clarified in [KaUd] we put some necessary facts from [Ka20] and [KaUd] in Appendix. In particular, one can find there formal definitions of locally nilpotent vector fields and flexible varieties. In Section 2 we introduce notations for toric varieties which are used freely throughout the paper and prove some simple facts about normal affine toric varieties. In Section 3 we give an analogue of Holme’s theorem [Hol] about closed embeddings of affine varieties into \( \mathbb{A}^n \) for embeddings into normal toric varieties without torus factors. In particular, we prove that if an affine curve can be embedded as a closed subvariety into some \( \mathbb{A}^n \) (where \( n \) is always at least 4), then it can be also embedded as a closed subvariety into a normal affine toric variety of dimension \( n \) with no torus factors so that its image is contained in the regular part of the toric variety. In Section 4 we study locally nilpotent vector fields on normal affine toric varieties with no torus factors. The properties
of locally nilpotent vector fields are crucial for us since compositions of elements of the flows of such vector fields produce automorphisms that extend isomorphisms of smooth polynomial curves. In Section 5 we prove our main result.

2 Notations and Preliminaries

We suppose that readers are familiar with toric varieties and all information about toric varieties which is used below can be found in the book of Cox, Little and Schrenck [CLS]). We fix the following notations for the rest of the paper.

- $N \cong \mathbb{Z}^n$ - the standard lattice in $\mathbb{R}^n$;
- $M = \text{Hom}_\mathbb{Z}(N, \mathbb{Z})$ - the lattice dual to $N$;
- $< m, u >$ - pairing of $m \in M$ or $M_\mathbb{R} = M \otimes \mathbb{Z} \mathbb{R}$ with $u \in N$ or $N_\mathbb{R} = N \otimes \mathbb{Z} \mathbb{R}$;
- $\sigma$ - a rational convex polyhedral cone in $N_\mathbb{R}$;
- $\sigma^\vee$ - the dual cone of $\sigma$ in $M_\mathbb{R}$;
- $\gamma$ - the set $\{m \in M_\mathbb{R} | < m, \gamma > = 0\}$ where $\gamma$ is any face of $\sigma$;
- $X_\sigma$ - the toric variety of $\sigma$, i.e., $X_\sigma$ is the spectrum of the group algebra of the semigroup $\sigma^\vee \cap M$;
- $\mathbb{T} = \text{Hom}(M, \mathbb{G}_m)$ - the torus acting on $X_\sigma$;
- $\{\varrho_1, \ldots, \varrho_r\}$ - the set of extremal rays of $\sigma$ (by abusing notations we also denote by $\varrho_i$ the ray generator, i.e., the primitive lattice vector on the corresponding ray);
- $\sigma(k)$ - the set of $k$-dimensional faces of $\sigma$ (e.g., $\varrho_i \in \sigma(1)$);
- $O_i$ - the $\mathbb{T}$-orbit in $X_\sigma$ corresponding to $\varrho_i$ by the Orbit-Cone correspondence [CLS, Theorem 3.2.6];
- $D_i$ - the irreducible $\mathbb{T}$-invariant Weil divisor in $X_\sigma$ containing $O_i$ as an open subset (i.e., $D_i$ is the spectrum of the semigroup algebra of $\tau_i = \varrho_i^\perp \cap \sigma^\vee \cap M$);
- $H_i$ - the $\mathbb{G}_m$-subgroup of $\mathbb{T}$ corresponding to $\varrho_i$, i.e., $H_i$ is a unique $\mathbb{G}_m$-subgroup of $\mathbb{T}$ that acts trivially on $D_i$ and for $t \in H_i$ one has $t.\chi^m = t^{<m,\varrho_i>}\chi^m$.

We would like to remind that since $\sigma^\vee \cap M$ and $\tau_i = \varrho_i^\perp \cap \sigma^\vee \cap M$ are saturated affine semigroups the varieties $X_\sigma$ and $D_i$ are always normal (e.g., see [CLS, Theorem 1.3.5]). Furthermore, we consider only the case when $X_\sigma$ has no torus factors (or, equivalently, every invertible function on $X_\sigma$ is constant).

Let $X_\Sigma$ be the toric variety of a fan $\Sigma$ and $r$ be the cardinality of one-dimensional cones in $\Sigma$ (in particular, if $\Sigma = \sigma$ then $r$ is the number of the
ray generators $\varrho$ of $\sigma$). If torus factors are absent, then by [CLS, Theorem 5.1.10] there exists a subgroup $G$ of $\mathbb{G}_m^r$ (which is a product of a torus and a finite group) and a closed subvariety $Z(\Sigma)$ of $\mathbb{A}^r$ such that $Z(\Sigma)$ is invariant under the natural action of $G$ on $\mathbb{A}^r$ and $X_\Sigma$ is isomorphic to $(\mathbb{A}^r \setminus Z(\Sigma))/G$ (while $\mathbb{G}_m^r/G$ is isomorphic to the torus $T$ acting on $X_\Sigma$). We are dealing with the situation when $\Sigma = \sigma$ and in this case $Z(\sigma)$ is empty by construction (see the definition of $Z(\sigma)$ on [CLS, page 206]). Thus, we have the quotient morphism

$$\pi : \mathbb{A}^r \to \mathbb{A}^r//G \simeq X_\sigma$$

(1)

which is $\mathbb{G}_m^r$-equivariant. In connection with this formula we fix the following notations.

- $x_1, \ldots, x_r$ - a fixed coordinate system on $\mathbb{A}^r$;
- $\tilde{T} = \mathbb{G}_m^r$ - the standard torus (with respect to the coordinate system) acting on $\mathbb{A}^r$;
- $\tilde{D}_i$ - the hyperplane in $\mathbb{A}^r$ given by $x_i = 0$;
- $\tilde{H}_i$ - the $\mathbb{G}_m$-subgroup of $\tilde{T}$ acting trivially on $\tilde{D}_i$, i.e., this action is the flow of the semisimple vector field $x_i \frac{\partial}{\partial x_i}$;
- $U$ - the subset of $X_\sigma$ consisting of all point $u \in X_\sigma$ for which $\pi^{-1}(u)$ is a $G$-orbit (in particular, this orbit is closed in $\mathbb{A}^r$);
- $U_0$ - the subset of $X_\sigma$ consisting of all point $u \in U$ for which $\pi^{-1}(u)$ is naturally isomorphic to $G$.

Let us list some properties of the morphism $\pi : \mathbb{A}^r \to X_\sigma$ and the objects introduced before.

**Lemma 2.1.** (i) The morphism $\pi$ is an almost geometric quotient, i.e., $U$ is an open dense subset of $X_\sigma$ and, consequently, general orbits of $G$ in $\mathbb{A}^r$ are closed and isomorphic to $G$. Furthermore, for every $u \in U$ the fiber $\pi^{-1}(u)$ is isomorphic as a homogeneous $G$-space to $G/F$ where $F$ is a finite subgroup of $G$.

(ii) The group $G \cap \tilde{H}_i$ is trivial.

(iii) Let $\theta \in \sigma(k)$ be regular, i.e., the set $\{\varrho_1, \ldots, \varrho_k\}$ of the generators of the extremal rays of $\theta$ can be extended to a basis of $N$. Then $G$ meets the group $\tilde{F}$ generated by $\tilde{H}_{i1}, \ldots, \tilde{H}_{ik}$ at identity only.

(iv) The homomorphism $\pi^* : k[X_\sigma] \to k[x_1, \ldots, x_r]$ induced by $\pi$ is determined by the formula

$$\pi^*(\chi^m) = \prod_{l=1}^r x_l^{<m, \varrho_l>}.$$  

(2)

(v) The image of $\tilde{H}_i$ in $T = \frac{\tilde{T}}{G}$ coincides with $H_i$ and $\pi(\tilde{D}_i) = D_i$. 

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Proof. For the first statement of (i) see [CLS, Theorem 5.1.10]. This implies that \( \dim G = \dim A^r - \dim X_\sigma = r - n \). For every \( u \in U \) one has \( \dim \pi^{-1}(u) = r - n \) since it cannot be less by Chevalley’s theorem [Ha, Chapter II, Exercise 3.22] and it cannot be larger since \( \pi^{-1}(u) \) is a \( G \)-orbit. Being a homogeneous space \( \pi^{-1}(u) \) is of the form \( G/F \) where \( F \) is a subgroup of \( G \) and the condition on the dimension implies that \( F \) is finite. This concludes (i).

By [CLS, Lemma 5.1.1] we have

\[
G = \{ \bar{t} = (t_1, \ldots, t_r) \in \mathbb{G}_m^r | \prod_{i=1}^r t_i^{<\varphi,m>} = 1 \text{ for all } m \in M \}. \tag{3}
\]

Suppose that \( G \cap \tilde{H}_i \) contains a finite subgroup of \( d \)-roots of unity. Assume that \( d > 1 \) and that \( \varepsilon \) is a primitive \( d \)-root of unity. Then Formula (3) implies that \( \varepsilon^{<\varphi,\eta>} = 0 \) for every \( m \in M \), i.e., \( <m, \eta_i> \) is divisible by \( d \). Hence, \( \frac{\eta_i}{d} \in N \) contrary to the fact that \( \eta_i \) is a primitive vector in the lattice \( N \). This yields (ii).

In (iii) let \( G \cap \tilde{F} \) contains a subgroup isomorphic to the group of \( d \)-roots of unity. Then the similar argument implies that for a collection \( \{l_1, \ldots, l_s\} \) of integers with greatest common divisor 1 the sum \( \sum_{k=1}^s l_k \eta_i \) is divisible by \( d \). However, if \( d > 1 \), then this is contrary to the fact that the set \( \{\eta_1, \ldots, \eta_k\} \) is extendable to a basis of \( N \). Thus, we have (iii).

For (iv) see [CLS, page 209]. Statement (v) follows from the explicit construction of \( \pi \) in [CLS, Proposition 5.1.9] which implies, in particular (iv). Vice versa, (v) can be also illustrated by Formula (2). Indeed, this formula implies that for every \( \chi^m, \eta \in \tau_i = \varphi^\perp \cap \sigma^\vee \cap M \) the function \( \pi^*(\chi^m) \) is independent of \( x_i \), i.e., it is fixed under the \( \tilde{H}_i \)-action. In particular, \( \chi^m \) is fixed under the action of the image \( H'_i \) of \( H_i \) in \( T = \mathbb{T} \) (which is isomorphic to \( \tilde{H}_i \) by (ii)). Hence, the \( H'_i \)-action on \( D_i \) is trivial. Since \( H_i \) is a unique \( \mathbb{G}_m \)-subgroup of \( T \) with this property we see that \( H'_i = H_i \). In particular, the divisor \( \pi^{-1}(D_i) \) must be fixed under the \( H_i \)-action which implies that \( \pi^{-1}(D_i) = \tilde{D}_i \) and we are done.

Recall that given an affine algebraic group \( H \) acting on an affine variety \( Z \), an affine variety \( Y \) with a trivial \( H \)-action and an \( H \)-equivariant morphism \( \varrho : Z \to Y \) we call \( Z \) together with the morphism \( \varrho : Z \to Y \) a principal \( H \)-bundle if for every \( y \in Y \) there exists an étale morphism \( \varphi_y : W_y \to Y \) such that \( y \in \text{Im} \varphi_y \) and \( W_y \times_Y Z \) becomes a trivial principal \( H \)-bundle under the natural \( H \)-action. If each \( \varphi_y \) is injective, then we call \( Z \) a locally trivial principal \( H \)-bundle.
Proposition 2.2. (1) The morphism \( \pi|_{\pi^{-1}(U_0)} : \pi^{-1}(U_0) \to U_0 \) is a principal \( G \)-bundle (in particular, \( \pi \) is smooth over \( U_0 \)). Furthermore, if \( G \) is connected, then this principal \( G \)-bundle is locally trivial.

(2) Let \( E \) be the subset of \( \mathbb{A}^r \) consisting of all \( \bar{x} = (x_1, \ldots, x_r) \in \mathbb{A}^r \) such that at most one coordinate \( x_i \) is equal to zero. Then \( \pi(E) \subset U_0 \).

(3) \( U_0 \) is the regular part of \( X_\sigma \).

Proof. If \( u \in U_0 \) and \( w \in \pi^{-1}(u_0) \), then by the Luna étale slice theorem [Lu] (see also [PV, Theorem 6.4]) there exists a smooth subvariety \( V \) of \( \mathbb{P}^n \) transversal to \( \pi^{-1}(u) \) at \( w \) such that \( \pi|_V : V \to U_0 \) is étale. This implies that \( U_0 \) is contained in the regular part of \( X_\sigma \) and that \( \pi \) is smooth over \( U_0 \). Furthermore, the natural \( G \)-action makes \( V \times U_0 \mathbb{A}^r \) a trivial \( G \)-bundle. Hence, \( \pi^{-1}(U_0) \to U_0 \) is a principal \( G \)-bundle. Recall also that if \( G \) is connected, then \( G \) is a special group in the sense of Serre (see, [Gro58, Def. 2 and page 16]) and for any special group \( K \) every \( K \)-principal bundle is locally trivial [Gro58, Theorem 3]. Thus, we have (1).

For (2) and (3) we need to recall that by Orbit-Cone correspondence [CLS, Theorem 3.2.6] every \( \theta \in \sigma(k) \) corresponds to a \( \mathbb{T} \)-orbit \( O(\theta) \subset X_\sigma \) of dimension \( n - k \) where \( O(\theta) \) is the orbit of a so-called distinguished point. The description of this point [CLS, page 116] implies that \( O(\theta) \) consists of all points \( u \in X_\sigma \) such that \( \chi^m(u) \neq 0 \) if and only if \( m \in \theta^\perp \cap M \). In particular, the ring \( k[R] \) of regular functions on the closure \( R \) of \( O(\theta) \) in \( X_\sigma \) can be viewed as the semigroup algebra of \( \theta^\perp \cap M \). Let \( \theta_{i_1}, \ldots, \theta_{i_k} \) be the extremal rays generating \( \theta \) (i.e., \( \theta^\perp = \theta_{i_1}^\perp \cap \cdots \cap \theta_{i_k}^\perp \)) and \( F \) be the subgroup of \( \mathbb{T} \) generated by \( H_{i_1}, \ldots, H_{i_k} \). Note that the natural inclusion \( k[R] \to k[X_\sigma] \) makes \( k[R] \) the subring of \( F \)-invariants and \( R \) is given in \( X_\sigma \) by the ideal generated by \( \{ \chi^m | m \in (\sigma^\perp \setminus \theta^\perp) \cap M \} \). Hence, \( R \) is the fixed point set of the \( F \)-action since for every \( v \in X_\sigma \setminus R \) one can find \( m \in (\sigma^\perp \setminus \theta^\perp) \cap M \) with \( \chi^m(v) \neq 0 \). The difference between the points of \( O(\theta) \) and \( R \setminus O(\theta) \) is that for \( w \in R \setminus O(\theta) \) there exists \( j \in \{1, \ldots, r\} \setminus \{i_1, \ldots, i_k\} \) such that \( w \) is also fixed under the \( H_j \)-action (because, \( w \) is contained in a \( \mathbb{T} \)-orbit of a smaller dimension corresponding to a cone in \( \sigma \) containing \( \theta \) and some \( \theta_j \)), whereas for any point in \( O(\theta) \) such \( j \) does not exist.

Let \( \bar{x} \in \bar{T} = \mathbb{T} \setminus \bigcup_{i=1}^s D_i \). Since \( \pi \) is \( \mathbb{T} \)-equivariant \( G, \bar{x} \) is a general orbit, i.e., \( \pi(\bar{x}) \in U_0 \) by Lemma 2.1 (i). Since \( E \cap \hat{D}_{i_1} \) is a \( \mathbb{T} \)-orbit dense in \( \hat{D}_{i_1} \) and \( \pi(D_{i_1}) = D_{i_1} \) by Lemma 2.1 (v) we see that \( \pi(E \cap D_{i_1}) = O_{i_1} \). For every \( \bar{x} \in E \cap D_{i_1} \) its \( \mathbb{T} \)-orbit \( Q \) is naturally isomorphic to \( G \) by Lemma 2.1 (ii) and \( \pi(x) = u \in O_{i_1} \). Note that if \( Q \) is not closed, then its closure contain a point with some coordinates \( x_j = 0 \) where \( j \neq i_1 \). However, this implies that \( u \) is a fixed point under both \( H_{i_1} \)-action and \( H_j \)-action contrary to the argument before. Hence, \( Q \) is closed. Furthermore, \( Q \) is a unique closed
Consider $u$ in the smooth part of $X_\sigma$. Then $u$ is contained in some $O(\theta)$ as before where $\theta$ must be regular by [CLS, Theorem 1.3.12 and Example 1.2.20]. Let $\tilde{O}(\theta) \subset \mathbb{A}^r$ be the $\tilde{T}$-orbit consisting of all points $\tilde{x}$ whose zero coordinates are exactly $x_{i_1}, \ldots, x_{i_k}$. Let $\theta'$ be a cone in $\sigma$ properly contained in $\theta$, i.e., $O(\theta)$ is contained in the closure of $O(\theta')$. Let us, say, that $\theta'$ is generated by extremal rays $\varrho_{i_2}, \ldots, \varrho_{i_k}$. Then we can suppose by induction that such $\theta'$ is regular and that $\pi(\tilde{O}(\theta')) = O(\theta') \subset U_0$. In particular, $\pi^{-1}(u)$ belongs to the closure of $\tilde{O}(\theta')$ and $\pi^{-1}(u) \cap \tilde{O}(\theta') = \emptyset$. This implies that every $\tilde{x} \in \pi^{-1}(u)$ cannot have a nonzero coordinate $x_i$. Hence, $\tilde{x}$ must be contained in $\tilde{O}(\theta)$ (indeed, if $\tilde{x}$ has a zero coordinate $x_j$ with $j \notin \{i_1, \ldots, i_k\}$, then $u$ is fixed under the $H_f$-action contrary to the argument before). This implies that the $\tilde{T}$-orbit $Q$ of $\tilde{x}$ is closed since otherwise its closure contains a point with an undesirable zero coordinate. By Lemma 2.1(iii) $Q$ is naturally isomorphic to $G$ and arguing as before we see that $\pi^{-1}(u) = Q$. Hence, $u \in U_0$ which yields (3) and concludes the proof. 

**Corollary 2.3.** Let $Y$ be a open subset of $X_\sigma$ such that $\operatorname{codim}_{X_\sigma} X_\sigma \setminus Y \geq 2$. Then $\pi^{-1}(X_\sigma \setminus Y)$ has codimension at least 2 in $\mathbb{A}^r$.

**Proof.** Note that $\pi^{-1}(Y) \subset \pi^{-1}(U \cap Y) \cup (\mathbb{A}^r \setminus E)$. The definition of $U$ implies that $\pi^{-1}(U \setminus Y)$ has codimension at least 2 in $\mathbb{A}^r$ and the same is true for $\mathbb{A}^r \setminus E$. Hence, we have the desired conclusion. 

**Corollary 2.4.** Let $C$ be a closed curve in $X_\sigma$ contained in $U_0$. Suppose that either

1. $C$ is isomorphic to the line $\mathbb{A}^1$ or
2. $G$ is connected and $C$ is a smooth rational curve.

Then there exists a closed curve $\tilde{C} \subset \mathbb{A}^r$ such that $\pi|_{\tilde{C}} : \tilde{C} \rightarrow C$ is an isomorphism.

**Proof.** By Proposition 2.2 $\pi^{-1}(C)$ is a locally trivial principal $G$-bundle. Statement (1) now follows from [FS, Theorem A.1] which states that for each affine algebraic group $F$ every principal $F$-bundle over a line admits a section.

In (2) by Proposition 2.2(1) we can find an open cover $\{V_i\}$ of $C$ for which $\pi^{-1}(V_i)$ is naturally isomorphic to $V_i \times G$. In particular, one has sections $s_i : V_i \rightarrow \pi^{-1}(V_i)$ and $s_j|_{V_i \cap V_j} = g_{ij} s_i$ where $g_{ij} : V_i \cap V_j \rightarrow G$ is a morphism. Since $G \simeq G_m^{r-n}$ we see that $g_{ij}$ can be presented as a collection
of \( r - n \) sections of \( \mathcal{O}_C^* \) over \( V_i \cap V_j \). Hence, \( H^1(C, G) \) is the direct sum of \( r - n \) samples of \( H^1(C, \mathcal{O}_C^*) \). Since \( C \) is a smooth rational curve we have \( H^1(C, \mathcal{O}_C^*) = \text{Pic} C = 0 \) and, hence, \( H^1(C, G) = 0 \). Thus, we can suppose that every pair of sections \( s_i \) and \( s_j \) agree on \( V_i \cap V_j \). Consequently, we have a global section of \( \pi|_{\pi^{-1}(C)} : \pi^{-1}(C) \to C \) which yields the desired conclusion. \( \square \)

### 3 Embedding Theorem

**Lemma 3.1.** Let \( \pi : \mathbb{A}^r \to X_\sigma \) be as in Formula (1) and \( A \) be the algebra of polynomials on \( \mathbb{A}^r \) invariant under the \( G \)-action (i.e., \( A \) can be viewed as the algebra of regular functions on \( X_\sigma \)), let \( \tilde{0} \) be the origin in \( \mathbb{A}^r \) and \( o = \pi(\tilde{0}) \). Then one can choose a collection of monomials as generators of \( A \) and the set \( V \) of common zeros of this collection is contained in \( \pi^{-1}(o) \). In particular, unless \( X_\sigma = U \) (which is the case of a simplicial \( \sigma \) by [CLS, Theorem 5.1.10]) \( V \subset \pi^{-1}(X_\sigma \setminus U) \).

**Proof.** Since the natural \( \tilde{T} \)-action respects monomials the same is true for the \( G \)-action. Thus, any \( G \)-invariant polynomial is the sum of \( G \)-invariant monomials which yields the first claim. Since \( V \subset \mathbb{A}^r \) is closed and \( G \)-invariant \( Z = \pi(V) \) is closed in \( X_\sigma \) and for every \( z \in Z \) the only closed orbit in \( \pi^{-1}(z) \) is contained also in \( V \). Assume that \( Z \) contains two distinct points \( z_1 \) and \( z_2 \) and \( L_i \) is the closed orbit of \( \pi^{-1}(z_i) \). Note that the restriction of every polynomial from \( A \) to \( V \) is constant. Hence, elements of \( A \) do not separate \( L_1 \) and \( L_2 \) contrary to the fact that the regular functions on \( X_\sigma \) separate \( z_1 \) and \( z_2 \). Thus, \( Z \) is at most a singleton. Since \( \tilde{0} \in V \) and \( o \in Z \) we see that \( V \subset \pi^{-1}(o) \). Let \( \dim \pi^{-1}(z) > r - n \) for some \( z \in X_\sigma \setminus o \). Since \( \pi \) is \( \tilde{T} \)-equivariant the same is true for all points in the \( T \)-orbit \( P \) of \( z \) in \( X_\sigma \). The closure of \( P \) contains a \( T \)-orbit \( Q \) of a smaller dimension and for every \( w \in Q \) one has \( \dim \pi^{-1}(w) > r - n \) by Chevalley’s theorem. Reducing the dimension of such \( T \)-orbits further we see that \( \dim \pi^{-1}(o) > r - n \), i.e., \( o \notin U \). This concludes the proof. \( \square \)

**Notation 3.2.** Let \( Z \) be an affine algebraic variety and \( TZ \) be its Zariski tangent bundle. Then we let \( ED(Z) = \max(2 \dim Z + 1, \dim TZ) \).

By [Hol, Theorem 7.4] (later rediscovered in [Ka91] and [Sr]) for every affine algebraic variety \( Z \) there exists a closed embedding of \( Z \) into \( \mathbb{A}^{ED(Z)} \).

**Theorem 3.3.** Let \( Z \) be an affine algebraic variety and \( X_\sigma \) be a normal affine toric variety without torus factors. Let \( \pi : \mathbb{A}^r \to X_\sigma \) be as in Formula (1). Let \( l = \text{codim}_{\mathbb{A}^r} \pi^{-1}(X_\sigma \setminus U_0) \). Suppose that \( ED(Z) \leq \dim X_\sigma \) and \( \dim Z < l \).
Then there exists a closed embedding \( \iota : Z \hookrightarrow X_\sigma \) such that \( \iota(Z) \) is contained in the regular part \( U_0 \) of \( X_\sigma \). Furthermore, \( l \geq 2 \) and, in particular, for every affine curve \( C \) with \( \text{ED}(C) \leq \dim X_\sigma \) there exists a closed embedding of \( C \) in \( X_\sigma \) with the image in \( U_0 \).

**Proof.** By [Hol, Theorem 7.4] \( Z \) can be treated as a closed subvariety of \( \mathbb{A}^r \). Since \( \dim Z < l \) Theorem 6.2 implies that there exists an algebraic family \( \mathcal{A} \) of automorphisms of \( \mathbb{A}^r \) such that for a general \( \alpha \in \mathcal{A} \) the variety \( \alpha(Z) \) does not meet \( \pi^{-1}(X_\sigma \setminus U_0) \). Thus, we suppose further that \( \pi(Z) \subseteq U_0 \).

Note that \( l \geq 2 \) by Corollary 2.3. In particular, by [FKZ, Theorem 2.6], \( \pi^{-1}(U_0) \subset \mathbb{A}^r \) is a flexible variety. By Proposition 2.2(1) the morphism \( \pi|_{\pi^{-1}(U_0)} : \pi^{-1}(U_0) \to U_0 \) is smooth. Hence, by Theorem 6.6 one can suppose that \( \pi|_{\alpha(Z)} : \alpha(Z) \to U_0 \) is an injective immersion since \( \text{ED}(Z) \leq \dim X_\sigma \).

Furthermore, consider the natural embedding \( \mathbb{A}^r \hookrightarrow \mathbb{P}^r \), \( D = \mathbb{P}^r \setminus \mathbb{A}^r \simeq \mathbb{P}^{r-1} \) and \( H = \text{GL}_n(\mathbb{k}) \). Then we have the natural \( H \)-action on \( \mathbb{P}^r \) such that \( D \) is invariant under it. By Proposition 6.8 (3) we can replace \( \mathcal{A} \) by the family \( H \times \mathcal{A} \). That is, for a general \( h \in H \) and a general \( \alpha \in \mathcal{A} \) the morphism \( \pi|_{h \circ \alpha(Z)} : h \circ \alpha(Z) \to U_0 \) is still an injective immersion.

By Lemma 3.1 we can find generators \( g_1, \ldots, g_s \) of \( k[X_\sigma] \) such that the polynomials \( f_i = g_i \circ \pi \) are monomials and the codimension (in \( \mathbb{A}^r \)) of the variety given by \( f_1 = \ldots = f_s = 0 \) is at least \( l \). Note also that \( f_1, \ldots, f_s \) can be viewed as coordinate functions of \( \pi : \mathbb{A}^r \to X_\sigma \subset \mathbb{A}^s \) and they can be extended to rational functions on \( \mathbb{P}^r \). Since each \( f_i \) is homogeneous with respect to the standard degree function the intersection \( R \) of the indeterminacy sets of these extensions is given by the common zeros of \( f_1, \ldots, f_s \) in \( D \).

In particular, \( R \) has codimension at least \( l \) in \( D \). Let \( P \) be the intersection of \( D \) with the closure of \( h \circ \alpha(Z) \) in \( \mathbb{P}^r \), i.e., \( \dim P \leq \dim Z - 1 < l - 1 \).

Since the restriction of the \( H \)-action to \( D \) is transitive \( P \) does not meet \( R \) for general \( h \in H \) and \( \alpha \in \mathcal{A} \) by Theorem 6.2. Hence, \( \pi|_{h \circ \alpha(Z)} : h \circ \alpha(Z) \to X_\sigma \) is a proper morphism by [Ka20, Corollary 5.4]. Consequently, it is a closed embedding which concludes the proof. \( \square \)

**Remark 3.4.** The proof actually implies that we can suppose that \( \iota(Z) \subset V \) where \( V \) is any dense open subset of \( U_0 \) such that \( \text{codim}_{\mathbb{A}^r} \pi^{-1}(X_\sigma \setminus V) \geq l \).

**Corollary 3.5.** Let \( X_\sigma \) be a simplicial toric variety. Let \( Z \) be an affine algebraic variety such that \( \text{ED}(Z) \leq \dim X_\sigma \) and \( \dim Z \) is less than the codimension of the singularities of \( X_\sigma \) in \( X_\sigma \). Then there is a closed embedding of \( Z \) into \( X_\sigma \) with the image in \( U_0 \).
4 Locally nilpotent vector fields on affine toric varieties

Locally nilpotent vector fields on $X_{\sigma}$ were described by Demazure [De] and later by Liendo [Li]. Recall that a Demazure root associated with some $\varrho_i$ is any element $e$ of $M$ such that $< e, \varrho_i >= -1$ and $< e, \varrho_j >$ is nonnegative for every $j \neq i$. The vector field on $X_{\sigma}$ defined by

$$\partial_{\varrho_i,e}(\chi^m) = < m, \varrho_i > \chi^{m+e}$$  \hspace{1cm} (4)

is locally nilpotent and up to a constant factor every homogeneous locally nilpotent vector field is of this form. For a Demazure root $e \in M$ associated with $\varrho_i$ one has $\tilde{e} := \pi^*(\chi^e) = (\tilde{e}_1, \ldots, \tilde{e}_r)$ where by Formula (2) the $i$-th coordinate $\tilde{e}_i$ is equal to $-1$. Let $\tilde{e}' = (\tilde{e}'_1, \ldots, \tilde{e}'_r)$ where the $i$-th coordinate $\tilde{e}'_i$ is equal to zero, whereas $\tilde{e}'_l = \tilde{e}_l$ for $l \neq i$. Formulas (2) and (4) imply now the following fact which was first discovered in [AKuZ19]).

**Lemma 4.1.** The polynomial $\pi^*(\partial_{\tilde{e},e}(\chi^m))$ coincides with $\tilde{\partial}_{\tilde{e},e}(\pi^*(\chi^m))$ where the locally nilpotent vector field $\partial_{\tilde{e},e}$ on $k^r$ is given by $\tilde{x}^{\tilde{e}} \frac{\partial}{\partial x^i}$ with $\tilde{x}^{\tilde{e}} = \prod_{l=1}^{r} x_{i_l}^{\tilde{e}_{i_l}}$, i.e., the flow of $\tilde{\partial}_{\tilde{e},e}$ is given by

$$\tilde{x} = (x_1, \ldots, x_r) \mapsto (x_1, \ldots, x_{i-1}, x_i + t\tilde{x}^{\tilde{e}_i}, x_{i+1}, \ldots, x_r)$$  \hspace{1cm} (5)

where $t$ is the time parameter.

The algebra $k[D_i]$ of regular functions on $D_i$ can be viewed as the semigroup algebra of $\tau_i = g_i^+ \cap \sigma^v \cap M$. Note that $k[D_i]$ is the kernel of $\partial_{\tilde{e},e}$ viewed as a derivation on $k[X_{\sigma}]$. The natural embedding $k[D_i] \rightarrow k[X_{\sigma}]$ yields a dominant $T$-equivariant morphism $\kappa_i : X_{\sigma} \rightarrow D_i$ that is the categorical quotient of the $G_\sigma$-action associated with $\partial_{\tilde{e},e}$. Note that it is also the categorical quotient of the natural $H_i$-action on $X_{\sigma}$ since $k[D_i]$ is the subring of $H_i$-invariants of $k[X_{\sigma}]$. Furthermore, as we mentioned before in the proof of Proposition 2.2 $D_i$ is the fixed point set of the $H_i$-action on $X_{\sigma}$.

**Notation 4.2.** Similarly, consider a cone $\theta \in \sigma(2)$ containing two extremal rays $\varrho_i$ and $\varrho_j$ and the subgroup $H_{ij}$ of $T$ generated by $H_i$ and $H_j$. The dual cone of $\theta$ meets $M$ along $\tau_i \cap \tau_j$. The semigroup algebra of $\tau_i \cap \tau_j$ can be viewed as the algebra of regular functions on $D_{ij} = D_i \cap D_j$. As before, one can see that $D_{ij} = X_\sigma//H_{ij}$ and $D_{ij}$ is the fixed point set of the $H_{ij}$-action on $X_\sigma$. Since $k[D_{ij}]$ has no zero-divisors and its transcendence degree is $n-2$ one can see that $D_{ij}$ is an irreducible $T$-invariant Weil divisor in $D_i$. In particular, $D_{ij}$ contains a dense $T$-orbit $O(\theta)$ (which is associated with $\theta$ via the Orbit-Cone correspondence).
Lemma 4.3. Let Notation 4.2 hold and θ be regular. Then κ_i is smooth over O(θ) and κ_i^{-1}(u) is isomorphic to A^1 for every u ∈ O(θ).

Proof. Since θ is regular O(θ) is contained in the regular part U_0 of X_σ by [CLS, Theorem 1.3.12 and Example 1.2.20]. Let u ∈ O(θ). Then T_u X_σ is equipped with the induced linear H_i-action. By the Luna slice étale theorem for smooth points (e.g., see [PV, Theorem 6.4]) there exists an H_i-equivariant étale morphism φ : Y → T_u X_σ from a dense open H_i-invariant subset Y of X_σ containing u. Hence, since the map T_u X_σ → T_u X_σ//H_i is smooth so is κ_i|_Y : Y → Y//H_i. For every point w ∈ κ_i^{-1}(u) the closure of its H_i-orbit contains the fixed point u (e.g., [PV, Theorem 4.7 and Corollary]), i.e., this orbit is contained in Y. Hence, κ_i^{-1}(u) ⊂ Y (and, consequently, κ_i^{-1}(u) is isomorphic via φ to a line through the origin in T_u X_σ). This yields the desired conclusion.

Lemma 4.4. Suppose that for some g_i every θ ∈ σ(2) containing g_i is regular. Then there exists an open subset V_i of D_i ∩ U_0 such that codim_{D_i} D_i \ V_i ≥ 2 and for every v ∈ V_i one can find a locally nilpotent vector field δ of the form g\partial_{v_i,e} where g ∈ k[D_i] ⊂ k[X_σ] which does not vanish on κ_i^{-1}(v).

Proof. Let g_j_1, ..., g_j_k be the collection of all extremal rays distinct from g_i such that for every s = 1, ..., k there exists θ ∈ σ(2) containing g_j_s and g_i. Formula (4) implies that ∂_{v_i,e} does not vanish over O_i = D_i \ ∪_{s=1}^k D_j_s. Thus, we have to consider the case when v is a general point of some D_j_s. Choose a rational function f_s on D_i with poles on D_i ∩ D_j_s only such that these poles are simple at general points of D_i ∩ D_j_s. Let l_s be the zero multiplicity of ∂_{v_i,e} at general points of D_j_s. Then the vector field δ = f_s l_s ∂_{v_i,e} is regular, locally nilpotent, tangent to the fibers of κ_i, and it does not vanish at general points of D_j_s. By Lemma 4.3 κ_i^{-1}(v) is isomorphic to a line and since v ∈ D_i ∩ D_j_s one has κ_i^{-1}(v) ⊂ D_j_s. Thus, δ|_{κ_i^{-1}(v)} does not vanish since it is tangent to the line κ_i^{-1}(v) and nonzero at a general point of κ_i^{-1}(v). This yields the desired conclusion.

Proposition 4.5. Let every θ ∈ σ(2) containing g_i be regular and let V_i ⊂ D_i be as in Lemma 4.4. Let Z be a closed subvariety of D_i ⊂ X_σ which is contained in V_i. Let s : Z → A^r be a section of π : A^r → X_σ over Z for which Z = s(Z) is closed in A^r. Then one can find a locally nilpotent vector field δ equivalent to ∂_{v_i,e} \textsuperscript{1} and such that δ does not vanish on κ_i^{-1}(Z).

Proof. Lemma 4.4 implies that for every u ∈ Z one can find a locally nilpotent vector field δ_z of the form g_z δ_{v_i,e}, g_z ∈ k[D_i] ⊂ k[X_σ] which does not

\textsuperscript{1}Two locally nilpotent derivations are equivalent if they have the same kernels.
vanish on $\kappa_i^{-1}(u)$. Recall that $g_\hat{z}$ as an element of $k[X_\sigma]$ is invariant under the $H_t$-action. Hence, by Lemma 4.1 $\delta_z = \pi_\sigma(\hat{\delta}_z)$ where $\hat{\delta}_z$ is of the form $\hat{f}_z \frac{\partial}{\partial x_i}$ with $\hat{f}_z$ being a polynomial independent of $x_i$ since $x_i$ is not invariant under the $H_t$-action. By assumption $\hat{\delta}_z$ and, therefore, $\hat{f}_z$ do not vanish at $\pi^{-1}(u) \cap \hat{Z}$. Hence, by the Nullstellensatz one can find polynomials $\hat{h}_z$ such that only finite number of them are nonzero and $\sum_{z} \hat{h}_z \hat{f}_z|_{\hat{Z}} = 1$. By the assumption every regular function on $\hat{Z}$ is a lift of a regular function on $Z$ which extends to an element $k[D_i] = \text{Ker} \partial_{x_i} \subset k[X_\sigma]$. In particular, one can suppose that $\hat{f}_z = \pi^*f_z$ and $\hat{h}_z = \pi^*h_z$ where $f_z, h_z \in k[D_i]$. Hence, $\delta = \sum_z h_z \delta_z$ (resp. $\hat{\delta} = \sum_z \hat{h}_z \delta_z$) is a locally nilpotent non-vanishing vector field on $\kappa_i^{-1}(Z)$ (resp. $\pi^{-1}(\kappa_i^{-1}(Z))$) which yields the desired conclusion. 

Corollary 4.6. Let the assumptions of Proposition 4.5 hold and $Z$ (and, therefore, $\hat{Z}$) be isomorphic to a line $\mathbb{A}^1$ equipped with a coordinate $t$. Let $\delta$ be the locally nilpotent derivation on $\mathbb{A}^r$ as in the proof of Proposition 4.5, i.e., $\pi_\sigma(\delta) = \delta$. Then for every polynomial $h(t)$ there exists a function $g \in k[D_i] \subset k[X_\sigma]$ such for the flow $\beta_h^t$ of the locally nilpotent vector field $\pi^*(g)\delta$ a time 1 one has $x_i \circ \beta_h^1(t) = h(t)$, $t \in \hat{Z}$.

Proof. Since $\delta$ is equivalent to $\partial_{x_i}$ and does not vanish on $Z$ one can suppose (by Lemma 4.1) that the restriction of $\delta$ to $\hat{Z}$ coincides with $\frac{\partial}{\partial x_i}$. Let $\hat{g}(t) = h(t) - x_i(t)$. Note that $\hat{g}(t)$ (as a function on $Z$) admits an extension to a function $g \in k[D_i] = \text{Ker} \delta$. This extension yields the desired function. 

5 Affine simplicial toric varieties

Recall that an affine toric variety $X_\sigma$ is simplicial if every face of $\sigma$ is simplex, i.e., $n = r$ and $G$ is a finite group. This implies, in particular, that for every $j \neq l$ the extremal rays $\theta_l$ and $\theta_j$ are contained in some $\theta \in \sigma(2)$ and $D_{lj} = D_l \cap D_j$ is always a Weil divisor in $D_l$.

Lemma 5.1. Let $X_\sigma$ a simplicial toric variety of dimension at least 4 which is smooth in codimension 2 and $C$ be a smooth polynomial curve in the regular part of $X_\sigma$. Let $V_i$ be as in Lemma 4.4, $V = \bigcap_{i=1}^r \kappa_i^{-1}(V_i)$, $W_l = \kappa_i^{-1}(V_l) \setminus \bigcap_{j \neq l} \kappa_j^{-1}(V_j)$ and $W_l' = \kappa_l(W_l)$. For every $\theta \in \sigma(2)$ containing extremal rays $\theta_l$ and $\theta_j$ let $\psi_\theta : X_\sigma \to D_{lj}$ be the morphism induced by the homomorphism of the semigroup algebras associated with the natural embedding $\tau_l \cap \tau_j \hookrightarrow \sigma$. Then replacing $C$ with its image under an automorphism of $X_\sigma$ one can suppose that

(i) $C$ is contained in $V$;
(ii) $C_l = \kappa_l(C)$ meets $W_l'$ at a finite set for every $l = 1, \ldots, r$;
(iii) \( \kappa_l|_C : C \to D_l \) is a closed embedding for every \( l = 1, \ldots, r \);

(iv) \( \psi_\theta : C \to D_{ij} \) is a birational morphism for every \( \theta \in \sigma(2) \) containing \( g_i \) and \( g_j \).

**Proof.** Since \( D_{ij} \) is a divisor in \( D_l \) and \( D_l \setminus V_l \) has codimension at least 2 in \( D_l \) we see that \( V_l \) contains an open subset of \( D_{ij} \) and \( \kappa_l^{-1}(V_l) \) contains an open part of \( D_j \). Hence, \( X_\sigma \setminus \kappa_l^{-1}(V_l) \) does not contain Weil divisors in \( X_\sigma \), i.e., it is of codimension at least 2. Consequently, \( \text{codim}_{X_\sigma} X_\sigma \setminus V \geq 2 \). Recall that \( X_\sigma \) is flexible by [AKuZ] and, therefore, \( U_0 \) and \( V \) are flexible by [FKZ, Theorem 2.6]. By Theorem 6.2 for a general \( \alpha \) in any perfect family \( \mathcal{A} \) of automorphisms of \( U_0 \) (which are extendable to automorphisms of \( X_\sigma \) by the Hartogs’ theorem) \( \alpha(C) \) is contained in \( V \) and \( \alpha(C) \) meets every \( \kappa_l^{-1}(W_l') \) at a finite set which yields (i) and (ii). Lemma 4.3 implies that every \( \kappa_l \) is smooth over \( \kappa_l(V) \). Thus, by Theorem 6.6 and Proposition 6.8(5) for a general \( \alpha \in \mathcal{A} \) each morphism \( \kappa_l: \alpha(C) \to D_l \) is a closed embedding and each morphism \( \psi_\theta : \alpha(C) \to D_{ij} \) is birational which yields (iii)-(iv) and the desired conclusion. \( \square \)

**Lemma 5.2.** Let the assumptions of Lemma 5.1 hold and \( C \) satisfy conditions (i)-(iv). Suppose that \( i, \delta, h \) and \( g \) are as in Corollary 4.6 and \( \beta_h^i \) is flow of the locally nilpotent vector field \( g \delta \) a time \( 1 \). Suppose further that \( h(t) = ct + d \) where \( c \) and \( d \) are general constants. Then the curve \( \beta_h^i(C) \) also satisfies conditions (i)-(iv).

**Proof.** Let \( \tilde{V} = \pi^{-1}(V) \). By Corollary 2.4 there exists a curve \( \tilde{C} \subset \tilde{V} \) such \( \pi|_{\tilde{C}} : \tilde{C} \to C \) is an isomorphism. Note that \( \pi \circ \tilde{\beta}_h^i = \beta_h^i \circ \pi \) where \( \beta_h^i \) is as in Corollary 4.6. Hence, besides conditions (i) and (ii) for \( \beta_h^i(C) \) it suffices to prove that

(iii’) for \( \tilde{\kappa}_l = \kappa_l \circ \pi \) the morphism \( \tilde{\kappa}_l|_{\tilde{\beta}_h^i(\tilde{C})} : \tilde{\beta}_h^i(\tilde{C}) \to D_l \) is a closed embedding for every \( l = 1, \ldots, r \);

(iv’) for \( \tilde{\psi}_\theta = \psi_\theta \circ \pi \) the morphism \( \tilde{\psi}_\theta|_{\tilde{\beta}_h^i(\tilde{C})} : \tilde{\beta}_h^i(\tilde{C}) \to D_{ij} \) is birational for every \( \theta \in \sigma(2) \) containing \( g_i \) and \( g_j \).

Let us start with (iv’). One can choose coordinate functions of \( \tilde{\psi}_\theta \) in the form \( \pi^*(\chi^m) \) where \( m \in \tau_l \cap \tau_j \). By Formula(2) \( \pi^*(\chi^m) \) is of the form \( x^{k_{m'}}_i y_m^m \) where \( y_m \) is a monomial independent of \( x_i \). Condition (iv) implies that there exist \( m', m'' \in \tau_l \cap \tau_j \) such that for \( t \in \mathbb{A}^1 \simeq \tilde{C} \) the functions \( x^{k_{m'}}_i(t) y_m^m(t) \) and \( x^{k_{m''}}_i(t) y_m^{m''}(t) \) are not proportional and, in particular, \( y_m^{m''}(t) \) is a nonzero rational function. Hence, for general \( c \) and \( d \) the functions \( (ct + d)^{k_{m'}} y_m^m(t) \) and \( (ct + d)^{k_{m''}} y_m^{m''}(t) \) are not proportional and Corollary 4.6 implies that the morphism \( \tilde{\psi}_\theta|_{\tilde{\beta}_h^i(\tilde{C})} : \tilde{\beta}_h^i(\tilde{C}) \to D_{ij} \) is birational which is (iv’).

Let \( S_{ij} \) be a finite subset of \( \tilde{C} \) for which \( \tilde{\psi}_\theta|_{\tilde{\beta}_h^i(\tilde{C} \setminus S_{ij})} : \tilde{\beta}_h^i(\tilde{C} \setminus S_{ij}) \to D_{ij} \) is an embedding. Then condition (iii) implies that for every \( t_0 \in S_{ij} \) there
exists \( m \in \tau_i \cap \tau_j \) such that \( \frac{d}{dt} x_i^{km} y_m |_{t=0} \) is nonzero. This implies that either \( y_m(t_0) \neq 0 \) or \( \frac{d}{dt} y_m |_{t=t_0} \neq 0 \). Consequently, \( \frac{d}{dt} (ct + d)^{km} y_m |_{t=t_0} \) is nonzero for general \( c \) and \( d \). Hence, we can suppose that \( \tilde{\kappa}_l |_{\tilde{\beta}_h(C)} : \tilde{\beta}_h(C) \to D_l \) is an immersion. Furthermore, for every \( t_0 \neq t_1 \in S_{ij} \) there exists \( m \in \tau_i \cap \tau_j \) such that \( x_i^{km} (t_0) y_m (t_0) \neq x_i^{km} (t_1) y_m (t_1) \). Again for general \( c \) and \( d \) this implies that \( (ct_0 + d)^{km} y_m (t_0) \neq (ct_1 + d)^{km} y_m (t_1) \). Hence, \( \tilde{\kappa}_l |_{\tilde{\beta}_h(C)} : \tilde{\beta}_h(C) \to D_l \) is a closed embedding which is (iii').

By (ii) for every \( l \) the curve \( C_l \) meets \( W_l' \) at a finite set \( Q_l' \) or, equivalently, for a general point \( z \in C \) one has \( \kappa_l (z) \notin W_l' \). The same remains true for a general point \( \beta^*_h (z) \) of the curve \( \beta^*_h(C) \). Indeed, it suffices to show that it is true for some point of \( \beta^*_h(C) \). Let \( t \in \tilde{C} \simeq \mathbb{A}^1 \) be the preimage of \( z \) in \( \tilde{C} \). Note that \( \beta^*_h(C) \) meets \( C \) at the points where \( x_i (t) = ct + d \). Since \( c \) and \( d \) are general the solutions of the latter equation yield general points of \( C \) and, hence, condition (ii) for the curve \( \beta^*_h(C) \).

Recall that \( \beta^*_h \) is the flow at time 1 of a locally nilpotent vector field \( g \delta \) as in Corollary 4.6 which is equivalent to a vector field \( \partial_{\beta^*_h,C} \) and, therefore, which is tangent to the fibers of \( \kappa_i \). In particular, \( \beta^*_h (C) \) is contained in \( \kappa_i^{-1} (C_i) \) where \( C_i = \kappa_i (C) \). By construction \( \kappa_i^{-1} (C_i \setminus Q_i') \subset V \). By Lemma 4.3 every fiber \( L \) of \( \kappa_i |_{\kappa_i^{-1} (Q_i')} : \kappa_i^{-1} (Q_i') \to Q_i' \) is a line and such \( L \) meets \( V \) since \( C \) does at some point \( z_0 \in C \) (where this \( z_0 \) is unique since \( \kappa_i : C \to D_l \) is a closed embedding). Let \( t \) be a coordinate on \( \tilde{C} \simeq C \simeq \mathbb{A}^1 \). Recall that by construction in Corollary 4.6 \( g \subset k[D_i] \) is an extension of the function \( ct + d - x_i(t) \). Hence, \( g \delta = \delta_1 + d \delta \) where the locally nilpotent vector field \( \delta_1 \) commutes with \( \delta \). In particular, the flow of \( g \delta \) at time 1 and the composition of the flows of \( \delta_1 \) at time 1 and \( \delta \) at time \( d \). Since by Proposition 4.5 \( \delta \) does not vanish on \( L \) and \( d \) is general we see that \( \beta^*_h(z_0) \) is a general point of \( L \) and, therefore, it belongs to \( V \). This yields condition (i) for \( \beta^*_h(C) \) and the desired conclusion.

Theorem 5.3. Let \( \varphi : C \to C' \) be an isomorphism of two smooth polynomial curve contained in the regular part of an affine simplicial toric variety \( X_\sigma \) of dimension at least 4 such that \( X_\sigma \) is smooth in codimension 2. Then \( \varphi \) extends to an automorphism of \( X_\sigma \).

Proof. Let \( \tilde{C} \) (resp. \( \tilde{C}' \)) be a curve in \( \mathbb{A}^r \) such that \( \pi |_{\tilde{C}} : \tilde{C} \to C \) (resp. \( \pi |_{\tilde{C}'} : \tilde{C}' \to C' \)) is an isomorphism. Let \( t' \) be a coordinate on \( C' \simeq \tilde{C}' \) and \( t = \varphi^*(t') \) be the coordinate on \( C \simeq \tilde{C} \). Applying consequently automorphisms \( \beta^*_h \) as in Lemma 5.2 with \( i \) running over \( \{ 1, \ldots, r \} \) one can suppose that \( \tilde{C} \) is a curve such that \( x_i(t) = c_i t + d_i \) for every \( i \) where \( (c_1, d_1, \ldots, c_r, d_r) \) is a general point in \( \mathbb{A}^{2r} \). Similarly, one can suppose that \( \tilde{C}' \) is a curve such that \( x_i(t') = c'_i t' + d'_i \) for every \( i \) where \( (c'_1, d'_1, \ldots, c'_r, d'_r) \) is a general point in \( \mathbb{A}^{2r} \). Choosing these two general points equal we get the desired conclusion.
We need to remind the following [KaUd, Definition 8.3].

**Definition 5.4.** Let $C_1$ and $C_2$ be smooth curves in a smooth quasi-affine variety $Y$ with defining ideals $I_1$ and $I_2$ in $k[Y]$. We suppose also that $C_1$ and $C_2$ are closed in an affine variety containing $Y$. Let $Y$ possess a volume form $\omega$ (i.e., $\omega$ is a nonvanishing section of the canonical bundle on $Y$) and let each conormal bundle $\frac{I_j}{I_j}$ of $C_j$ in $Y$ be trivial. By [KaUd, Lemma 6.3] there is a neighborhood $W_j$ of $C_j$ in $Y$ in which $C_j$ is a strict complete intersection given by $u_{1,j} = \ldots = u_{n-1,j} = 0$ where $u_{1,j}, \ldots, u_{n-1,j} \in I_j$ and $n = \dim Y$. That is, for $A_j = \frac{k[Y]}{I_j}$ we have the graded algebra $\frac{k[Y]}{I_j} \simeq A_j \oplus \bigoplus_{l=1}^{k-1} \frac{I_j^l}{I_j^{l+1}}$ which can be viewed as the algebra of polynomials in $u_{1,j}, \ldots, u_{n-1,j}$ over $A_j$ of degree at most $k - 1$. Consider an isomorphism $\varphi : \frac{k[Y]}{I_j} \to \frac{k[Y]}{I_j}$ of these algebras for a natural $k$. Up to the induced isomorphism $A_1 \simeq A_2$ this isomorphism $\varphi$ is determined by its values $\varphi(u_{1,i})$, $i = 1, \ldots, n - 1$. These values can be viewed as polynomials in $u_{1,2}, \ldots, u_{n-1,2}$ over $A_2$, i.e., one has the matrix $\left[ \frac{\partial \varphi(u_{1,1})}{\partial u_{s,2}} \right]_{l,s=1}^{n-1}$. Since the normal bundle $N_{Y}C_j$ is trivial, the existence of $\omega$ implies the existence of a volume form on $C_j$. Fix volume forms $\omega_j$ on $C_j$ such that $\varphi^* \omega_1 = \omega_2$ where the isomorphism $\varphi : C_2 \to C_1$ is induced by $\varphi$. Choose a section $pr_j : TY|_{C_j} \to TC_j$ of the canonical inclusion $TC_j \to TY|_{C_j}$ and consider the section $\tilde{\omega}_j = \omega_j \circ pr_j$ of the dual bundle $(TY|_{C_j})'$ of $TY|_{C_j}$. Then one can require that $\omega|_{C_j}$ coincides with $\tilde{\omega}_j \wedge du_{1,j} \wedge \ldots \wedge du_{n-1,j}$. Under this requirement the determinant of $\left[ \frac{\partial \varphi(u_{1,1})}{\partial u_{s,2}} \right]_{l,s=1}^{n-1}$ is well-defined modulo $I_2^{-1}$ (i.e., it is independent of the choice of coordinates $u_{1,j}, \ldots, u_{n-1,j}$). Hence, we say that $\varphi$ has Jacobian $a \in k \setminus \{0\}$ if the determinant of $\left[ \frac{\partial \varphi(u_{1,1})}{\partial u_{s,2}} \right]_{l,s=1}^{n-1}$ is equal to a modulo $I_2^{-1}$.

Note that Definition 5.4 is applicable in the case when $C_1$ and $C_2$ are smooth polynomial curves in a simplicial toric variety $X_\sigma$ contained in its regular part. Indeed, $U_0$ as the regular part of $X_\sigma$ is flexible [AKuZ]. Recall that $A^r$ admits a volume form invariant under the natural $SL_n(k)$-action. Hence, we can push this volume form down to a volume form $\omega$ on $U_0$ since $\pi|_{\pi^{-1}(U_0)} : \pi^{-1}(U_0) \to U_0$ is an unramified covering by Proposition 2.2. Furthermore, the normal bundles of smooth polynomial curves are always trivial, i.e., we are under the assumptions of Definition 5.4.

**Corollary 5.5.** Let $\varphi : C_1 \to C_2$ be an isomorphism of $k$-th infinitesimal neighborhoods of two smooth polynomial curve $C_1$ and $C_2$ contained in the regular part $U_0$ of an affine simplicial toric variety $X_\sigma$ of dimension at least
which is smooth in codimension 2. Suppose that the Jacobian of \( \varphi \) is a nonzero constant \( a \). Then \( \varphi \) extends to an automorphism of \( X_\sigma \).

**Proof.** Recall that by [KaUd, Lemma 6.2] every automorphism \( \alpha \) of \( U_0 \) has a constant Jacobian where the Jacobian is computed as \( \frac{\alpha^*(\omega)}{\omega} \) [KaUd, Lemma 6.2] and if \( \alpha \) is a composition of elements of flows of locally nilpotent vector fields, then its Jacobian is 1. Let \( \psi : C_2 \to C_1 \) be an isomorphism. By Theorem 5.3 \( \psi \) extends to an automorphism \( \Psi \) of \( X_\sigma \) which in turn induces an automorphism \( C_2 \to C_1 \) also denoted by \( \psi \). By construction \( \Psi \) is a composition of elements of flows of locally nilpotent vector fields. Hence, its Jacobian is 1. Taking a composition of \( \varphi \) with the action of an appropriate element of \( T \) and replacing \( C_2 \) with its image under this action we can suppose that the Jacobian of \( \varphi \) is also 1 modulo \( I^{k-1}_j \) in the sense of Definition 5.4 where \( I_j \) is the defining ideal of \( C_j \) in \( k[X_\sigma] \). Then the automorphism \( \lambda := \psi \circ \varphi : C_1 \to C_1 \) has Jacobian 1 modulo \( I^{k-1}_k \). By [KaUd, Theorem 6.5] \( \lambda \) extends to an automorphism \( \Lambda \) of \( X_\sigma \). It remains to note that \( \Psi^{-1} \circ \Lambda \) is the desired extension of \( \varphi \) and we are done.

6 Appendix

In this section we present some technical tools developed in [Ka20] with later clarifications in [KaUd] which we use in this paper.

**Definition 6.1.** (1) Given an irreducible algebraic variety \( A \) and a map \( \varphi : A \to \text{Aut}(X) \) (where \( \text{Aut}(X) \) is the group of algebraic automorphisms of \( X \)) we say that \( (A, \varphi) \) is an algebraic family of automorphisms of \( X \) if the induced map \( A \times X \to X \), \((\alpha, x) \mapsto \varphi(\alpha).x\) is a morphism (see [Raf]).

(2) If we want to emphasize additionally that \( \varphi(A) \) is contained in a subgroup \( G \) of \( \text{Aut}(X) \), then we say that \( A \) is an algebraic \( G \)-family of automorphisms of \( X \).

(3) In the case when \( A \) is a connected algebraic group and the induced map \( A \times X \to X \) is not only a morphism but also an action of \( A \) on \( X \) we call this family a connected algebraic subgroup of \( \text{Aut}(X) \).

(4) Following [AFKKZ, Definition 1.1] we call a subgroup \( G \) of \( \text{Aut}(X) \) algebraically generated if it is generated as an abstract group by a family \( \mathcal{G} \) of connected algebraic subgroups of \( \text{Aut}(X) \).

We have the following important fact [AFKKZ, Theorem 1.15] (which is the analogue of the Kleiman transversality theorem [Kl] for algebraically generated groups).
Theorem 6.2. (Transversality Theorem) Let a subgroup $G \subseteq \text{Aut}(X)$ be algebraically generated by a system $G$ of connected algebraic subgroups closed under conjugation in $G$. Suppose that $G$ acts with an open orbit $O \subseteq X$.

Then there exist subgroups $H_1, \ldots, H_m \in G$ such that for any locally closed reduced subschemes $Y$ and $Z$ in $O$ one can find a Zariski dense open subset $U = U(Y, Z) \subseteq H_1 \times \ldots \times H_m$ such that every element $(h_1, \ldots, h_m) \in U$ satisfies the following:

(a) The translate $(h_1 \cdot \ldots \cdot h_m).Z_{\text{reg}}$ meets $Y_{\text{reg}}$ transversally.

(b) $\dim(Y \cap (h_1 \cdot \ldots \cdot h_m).Z) \leq \dim Y + \dim Z - \dim X$. \footnote{We put the dimension of empty sets equal to $-\infty$.}

In particular $Y \cap (h_1 \cdot \ldots \cdot h_m).Z = \emptyset$ if $\dim Y + \dim Z < \dim X$.

Definition 6.3. (1) A nonzero derivation $\delta$ on the ring $A$ of regular functions on an affine algebraic variety $X$ is called locally nilpotent if for every $0 \neq a \in A$ there exists a natural $n$ for which $\delta^n(a) = 0$. This derivation can be viewed as a vector field on $X$ which we also call locally nilpotent. The set of all locally nilpotent vector fields on $X$ will be denoted by $\text{LND}(X)$. The flow of $\delta \in \text{LND}(X)$ is an algebraic $\mathbb{G}_a$-action on $X$, i.e., the action of the group $(\mathbb{k}, +)$ which can be viewed as a one-parameter unipotent group $U$ in the group $\text{Aut}(X)$ of all algebraic automorphisms of $X$. In fact, every $\mathbb{G}_a$-action is a flow of a locally nilpotent vector field (e.g., see [Fr, Proposition 1.28]).

(2) If $X$ is a quasi-affine variety, then an algebraic vector field $\delta$ on $X$ is called locally nilpotent if $\delta$ extends to a locally nilpotent vector field $\tilde{\delta}$ on some affine algebraic variety $Y$ containing $X$ such that $\tilde{\delta}$ vanishes on $Y \setminus X$ where $\text{codim}_Y(Y \setminus X) \geq 2$. Note that under this assumption $\delta$ generates a $\mathbb{G}_a$-action on $X$ and we use again the notation $\text{LND}(X)$ for the set of all locally nilpotent vector fields on $X$.

Definition 6.4. (1) For every locally nilpotent vector fields $\delta$ and each function $f \in \text{Ker} \delta$ from its kernel the field $f\delta$ is called a replica of $\delta$. Recall that such replica is automatically locally nilpotent.

(2) Let $\mathcal{N}$ be a set of locally nilpotent vector fields on $X$ and $G_{\mathcal{N}} \subseteq \text{Aut}(X)$ denotes the group generated by all flows of elements of $\mathcal{N}$. We say that $G_{\mathcal{N}}$ is generated by $\mathcal{N}$.

(3) A collection of locally nilpotent vector fields $\mathcal{N}$ is called saturated if $\mathcal{N}$ is closed under conjugation by elements in $G_{\mathcal{N}}$ and for every $\delta \in \mathcal{N}$ each replica of $\delta$ is also contained in $\mathcal{N}$.
Definition 6.5. Let $X$ be a normal quasi-affine algebraic variety of dimension at least 2, $\mathcal{N}$ be a saturated set of locally nilpotent vector fields on $X$ and $G = G_{\mathcal{N}}$ be the group generated by $\mathcal{N}$. Then $X$ is called $G$-flexible if for any point $x$ in the smooth part $X_{\text{reg}}$ of $X$ the vector space $T_xX$ is generated by the values of locally nilpotent vector fields from $\mathcal{N}$ at $x$ (which is equivalent to the fact that $G$ acts transitively on $X_{\text{reg}}$ [FKZ, Theorem 2.12]). In the case of $G = \text{SAut}(X)$ we call $X$ flexible without referring to $\text{SAut}(X)$ (recall that $\text{SAut}(X)$ is the subgroup of $\text{Aut} X$ generated by all one-parameter unipotent subgroups).

The following is a simplified version of [Ka20, Theorem 4.2].

Theorem 6.6. Let $X$ be a smooth algebraic variety and $Q$ be a normal algebraic variety. Let $\varphi : X \to Q$ be a dominant morphism. Suppose that $Q_0$ is a smooth open dense subset of $Q$ such that for $X_0 = \varphi^{-1}(Q_0)$ the morphism $\varphi|_{X_0} : X_0 \to Q_0$ is smooth. Let $G \subset \text{Aut}(X)$ be an algebraically generated group acting 2-transitively on $X$ and $Z$ be a locally closed reduced subvariety in $X$.

(i) Let $\dim Q \geq \dim Z + m$ where $m \geq 1$. Then there exists an algebraic $G$-family $\mathcal{A}$ of automorphisms of $X$ such that for a general element $\alpha \in \mathcal{A}$ one can find a constructible subset $R$ of $\alpha(Z) \cap X_0$ of dimension $\dim R \leq \dim Z - m$ for which $\varphi(R)$ and $\varphi(\alpha(Z) \setminus R)$ are disjoint and the restriction $\varphi|_{\alpha(Z) \cap X_0 \setminus R} : (\alpha(Z) \cap X_0) \setminus R \to Q_0$ of $\varphi$ is injective. In particular, if $\dim Q \geq 2 \dim Z + 1$ and $Z'_{\alpha}$ is the closure of $Z'_{\alpha} = \varphi \circ \alpha(Z)$ in $Q$, then for a general element $\alpha \in \mathcal{A}$ the map $\varphi|_{\alpha(Z) \cap X_0} : \alpha(Z) \cap X_0 \to Z'_{\alpha} \cap Q_0$ is a bijection, while in the case of a pure-dimensional $Z$ and $\dim Q \geq \dim Z + 1$ the morphism $\varphi|_{\alpha(Z) \cap X_0} : \alpha(Z) \cap X_0 \to Z'_{\alpha} \cap Q_0$ is birational.

(ii) Let $G$ be generated by a saturated set $\mathcal{N}$ of locally nilpotent vector fields on $X$ (in particular, $X$ is $G$-flexible). Let $\dim TZ \leq \dim Q$ and $2 \dim Z + 1 \leq \dim Q$. Then there exists an algebraic family $\mathcal{A}$ of $G$-automorphisms of $X$ such that for a general element $\alpha \in \mathcal{A}$, every $z \in \alpha(Z) \cap X_0$ the induced map $\varphi_* : T_z\alpha(Z_0) \to T_{\varphi(z)}Q$ of the tangent spaces is injective.

Let us describe some $G$-families $\mathcal{A}$ satisfying the conclusions of Theorem 6.6.

Definition 6.7. Let $X$ be a smooth algebraic variety and $G$ be a subgroup of $X$. Consider $(X \times X) \setminus \Delta$ (where $\Delta$ is the diagonal), the complement $(TX)^\ast$ to the zero section in the tangent bundle of $X$ and the frame bundle $\text{Fr}(X)$ bundle of $TX$ (i.e., the fiber of $\text{Fr}(X)$ over $x \in X$ consists of all bases of $T_xX$). Projectivization of $TX$ replaces $\text{Fr}(X)$ with a bundle $\text{PFr}(X)$ whose fiber over $x$ consists of all ordered $n$-tuples of points in the projectivization $\mathbb{P}^n$. 

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of $T_x X$ (where $n = \dim X$) that are not contained in the same hyperplane of $\mathbb{P}^n$. Then we have natural $G$-actions on all these objects. Let $Y$ be either $X$, or $(X \times_P X) \setminus \Delta$, or $(TX)^*$, or $\operatorname{PFr}(X)$. Suppose that the $G$-action is transitive on $Y$. Then we say that an algebraic $G$-family $A$ of automorphisms of $X$ is a regular $G$-family for $Y$ if

(i) $A = H_m \times \ldots \times H_1$ where each $H_i$ belongs to $\mathcal{G}$;

(ii) for a suitable open dense subset $U \subseteq H_m \times \ldots \times H_1$, the map

$$\Psi : H_m \times \ldots \times H_1 \times Y \longrightarrow Y \times Y \quad \text{with} \quad (h_m, \ldots, h_1, y) \mapsto ((h_m \cdot \ldots \cdot h_1).y, y)$$

(6)

is smooth on $U \times Y$.

An algebraic algebraic $G$-families $A$ that are regular for all four varieties $X$, $(X \times_P X) \setminus \Delta$, $(TX)^*$ and $\operatorname{PFr}(X)$ will be called a perfect $G$-family for $Y$.

**Proposition 6.8.** Let $X$ be a smooth algebraic variety and $G \subset \operatorname{Aut}(X)$ be a group algebraically generated by a family $\mathcal{G}$ of algebraic connected subgroups of $\operatorname{Aut}(X)$. Suppose that $G$ acts transitively on $X$.

(1) Then there exists a regular $G$-family for $X$ (which is of the form $A = H_1 \times \ldots \times H_m$ where each $H_i$ is an element of $\mathcal{G}$).

(2) Every regular $G$-family for $X$ satisfies the conclusions of Theorem 6.2.

(3) If $A$ is a regular (resp. perfect) $G$-family for $X$ and $H$ is an element of $\mathcal{G}$ then $H \times A$ and $A \times H$ are also regular (resp. perfect) $G$-families for $X$.

(4) In particular, if $X$ is $G$-flexible, then there exists a perfect $G$-family.

(5) Let $X$ be $G$-flexible. Every $G$-family regular for $X$ (resp. for $(TX)^*$) satisfies the conclusion of Theorem 6.6 (i) (resp. (ii)). In particular, every perfect $G$-family satisfies the conclusion Theorem 6.6 (i)-(ii).

**Proof.** Statement (1) is proven in [AFKKZ, Proposition 1.15]. Statements (2) and (3) are proven in [Ka20, Proposition 1.10]. The fourth statement follows from the fact that in the flexible case for every $m > 0$ the group $G$ acts $m$-transitively on $X$ and also transitively on $(TX)^*$ and $\operatorname{PFr}(X)$ [AFKKZ, Theorem 4.11 and Remark 4.16]. Statement (5) is essentially the content of the proof of [Ka20, Theorem 4.2].

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