Elliptic Ruijsenaars difference operators, symmetric polynomials, and Wess–Zumino–Witten fusion rings

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Abstract
The fusion ring for \( \hat{\mathfrak{sl}}(n)_m \) Wess–Zumino–Witten conformal field theories is known to be isomorphic to a factor ring of the ring of symmetric polynomials presented by Schur polynomials. We introduce a deformation of this factor ring associated with eigenpolynomials for the elliptic Ruijsenaars difference operators. The corresponding Littlewood–Richardson coefficients are governed by a Pieri rule stemming from the eigenvalue equation. The orthogonality of the eigenbasis gives rise to an analog of the Verlinde formula. In the trigonometric limit, our construction recovers the refined \( \hat{\mathfrak{sl}}(n)_m \) Wess–Zumino–Witten fusion ring associated with the Macdonald polynomials.

Keywords
Symmetric functions · Elliptic Ruijsenaars system · Macdonald polynomials · Wess–Zumino–Witten fusion ring · Verlinde algebra

Mathematics Subject Classification
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1 Introduction
It is well-known that the structure constants for the ring of symmetric polynomials in \( n \) variables in the basis of Schur polynomials \( s_\lambda(x_1, \ldots, x_n) \), the Littlewood–Richardson coefficients, count the tensor multiplicities in the decomposition of tensor products of irreducible representations for the Lie algebra \( \mathfrak{sl}(n; \mathbb{C}) \) (cf. e.g. [33, 40]). Here the partitions \( \lambda = (\lambda_1, \ldots, \lambda_n) \) label the dominant weights and one divides out by an
ideal generated by the relation $x_1 x_2 \ldots x_n = 1$. If additionally—upon fixing a level $m \in \mathbb{N}$—the ideal generated by the Schur polynomials corresponding to partitions such that $\lambda_1 - \lambda_n = m + 1$ is divided out, then one arrives at a finite-dimensional factor ring that is isomorphic to the fusion ring for $\hat{\mathfrak{sl}}(n)_m$ Wess–Zumino–Witten conformal field theories [11, 15, 17, 19, 25, 30]. The factor ring in question is also often referred to as the Verlinde algebra in the literature and it has a natural basis of Schur classes labeled by partitions $\lambda$ with $\lambda_1 - \lambda_n \leq m$ (which encode the dominant weights of the underlying affine Lie algebra $\hat{\mathfrak{sl}}(n)_m$). The corresponding structure constants, i.e. the affine counterparts of the Littlewood–Richardson coefficients at level $m$, describe the fusion rules for primary fields of the associated conformal field theories. From a mathematical point of view these structure constants compute, cf. e.g. [18], the Littlewood–Richardson coefficients of Hecke algebras at roots of unity [19] and the dimensions of spaces of conformal blocks of three-point functions in Wess–Zumino–Witten conformal field theories [46].

Generalizations of the $\hat{\mathfrak{sl}}(n)_m$ fusion ring have been constructed by means of parameter deformations of the Schur polynomials; important examples arise this way from the Hall–Littlewood polynomials [29] and from the Macdonald polynomials [1, 6, 27, 37]. While the deformed fusion ring stemming from the Hall–Littlewood polynomials has been related to a deformation of the Verlinde algebra appearing in connection with an index formula of Teleman and Woodward [2, 38, 44, 45], in the case of the Macdonald polynomials one deals rather with a Verlinde algebra that is intimately connected to the computation of refined Chern-Simons invariants for torus knots [1, 7, 21, 22, 37]. From the point of view of quantum integrable particle dynamics, the Verlinde algebra for $\hat{\mathfrak{sl}}(n)_m$ Wess–Zumino–Witten conformal field theories can be interpreted as a Hilbert space for the phase model of impenetrable bosons on the one-dimensional periodic lattice [30]. From this perspective, the deformations of the Verlinde algebra associated with the Hall–Littlewood polynomials and with the Macdonald polynomials correspond in turn to Hilbert spaces for the periodic $q$-boson model [29] and for the quantized trigonometric Ruijsenaars–Schneider system on $\mathbb{CP}^{n-1}$ [10, 20] (cf. also [3]), respectively.

The idea of the present work is to introduce a further generalization of the (refined) Verlinde algebra originating from the elliptic Ruijsenaars operators; these are commuting difference operators with coefficients built from products of Weierstrass' sigma functions [42, 43] that reduce to the Macdonald-Ruijsenaars $q$-difference operators [33] in the trigonometric limit. The study of their eigenfunctions points towards the existence of an elliptic counterpart of Macdonald’s theory of symmetric polynomials [12, 31, 32, 36, 41]. Let us recall at this point that in [33, Chapter VI] Macdonald’s symmetric polynomials arise as an eigenbasis for the (trigonometric) commuting Macdonald-Ruijsenaars $q$-difference operators. Upon evaluation on the spectrum Macdonald’s eigenpolynomials exhibit a fundamental duality symmetry which implies that classical identities for the Schur polynomials such as the principal specialization formula, norm formulas, Pieri rules, Cauchy identity, tableaux formulas, …admit beautiful generalizations to the level of the Macdonald polynomials. Important approaches towards the elliptic generalization of Macdonald’s theory were developed in [12] (using quantum affine algebras), [41] (using elliptic Selberg integrals), [31] (using string theory), and [32] (using Ruijsenaars’ elliptic generalization of the Ruijsenaars-Macdonald
To arrive at our elliptic generalization of the (refined) Verlinde algebra, however, we will rather employ a dual picture and study symmetric polynomials generated by an elliptic generalization of the Pieri rules for the Macdonald polynomials in the spirit of [9, 36]. Since the heuristics of the duality symmetries pertaining to Ruijsenaars’ difference operators turns out to be quite intricate at the elliptic level [31], the expected loss of self-duality predicts that any corresponding generalization of Macdonald’s theory for the polynomials under consideration will be necessarily far from straightforward (cf. e.g. Remarks 5.11 and 5.13 below).

Our starting point will be to discretize the Ruijsenaars operators on a lattice of points labeled by partitions. Specifically, the lattice points are given by the $\mathfrak{gl}(n)$ dominant weights shifted by a Weyl vector that is rescaled linearly (as usual) with the multiplicity (or coupling) parameter. The eigenfunctions of these discrete Ruijsenaars operators are subsequently constructed in terms of polynomials determined by a recurrence stemming from the eigenvalue equation. The polynomials at issue turn out to provide a basis for the ring of symmetric polynomials. The associated Littlewood–Richardson coefficients constitute an elliptic deformation of Macdonald’s $(q, t)$-Littlewood–Richardson coefficients [33]. In this setup, the eigenvalue equation for the elliptic Ruijsenaars operator entails an explicit formula for the elliptic deformation of the Pieri rule. Next, we perform the reduction to the (deformed) $\hat{\mathfrak{sl}}(n)$ character ring and construct the corresponding elliptic analog of the $\hat{\mathfrak{sl}}(n)_m$ fusion ring, for which we compute the structure constants in terms of elliptic Littlewood–Richardson coefficients. Now this deformed Verlinde algebra constitutes the Hilbert space for the compact elliptic Ruijsenaars model on $\mathbb{C}P^{n-1}$, whose classical and quantum dynamics was studied in [9, 13], respectively. This particle interpretation gives rise to an orthogonality relation for the eigenpolynomials of the elliptic Ruijsenaars operators [9], from which we derive a Verlinde formula for the structure constants upon identifying the pertinent elliptic deformation of the $\hat{\mathfrak{sl}}(n)_m$ Kac-Peterson modular $S$-matrix [25].

The material is organized as follows. We start by discretizing the elliptic Ruijsenaars operators onto partitions in Sect. 2. Next, in Sect. 3, we construct an eigenbasis for these discrete Ruijsenaars operators; this gives rise to an elliptic deformation of Littlewood–Richardson coefficients generated by explicit Pieri rules. By dividing out the ideal generated by basis polynomials labeled by partitions with $\lambda_1 - \lambda_n = m + 1$, we arrive in Sect. 4 at an elliptic deformation of the $\hat{\mathfrak{sl}}(n)_m$ Wess–Zumino–Witten fusion ring; we compute its structure constants in terms of elliptic Littlewood–Richardson coefficients. In Sect. 5, a Verlinde formula for these structure constants is presented in terms of the relevant elliptic deformation of the Kac-Peterson modular $S$-matrix. We wrap up by detailing how the structure constants for the $\hat{\mathfrak{sl}}(n)_m$ Wess–Zumino–Witten fusion ring and its refined deformation associated with the Macdonald polynomials are recovered from their elliptic counterparts via parameter degenerations.
2 Discrete Ruijsenaars operators on partitions

2.1 Ruijsenaars’ commuting difference operators

The Ruijsenaars difference operators \( D_1, \ldots, D_n \) are of the form

\[
D_r = \sum_{J \subset \{1, \ldots, n\} \atop |J| = r} V_J(x) T_J, \quad V_J(x) = \prod_{j \in J \atop k \notin J} \frac{|x_j - x_k + g|}{|x_j - x_k|} \quad (r = 1, \ldots, n).
\] (2.1)

Here \(|\cdot|\) denotes the cardinality of the set in question and \(T_J\) acts by translation on complex functions \( f(x) = f(x_1, \ldots, x_n) \):

\[
(T_J f)(x) = f(x + \epsilon_J) \quad \text{with} \quad \epsilon_J = \sum_{j \in J} \epsilon_j
\]

(where \(\epsilon_1, \ldots, \epsilon_n\) refer to the standard unit basis of \(\mathbb{C}^n\)). For \(g \in \mathbb{C}\), these difference operators commute if the coefficients are built from a function \([z]\) that factorizes into the product of a Weierstrass sigma function \(\sigma(z)\) and a Gaussian of the form \(\exp(az^2 + bz)\) (for any \(a, b \in \mathbb{C}\)) \([42, 43]\). In Ruijsenaars’ original approach this was shown by deducing that the commutativity is guaranteed provided \([z]\) solves a functional identity that is satisfied by functions of the form \(\sigma(z) \exp(az^2 + bz)\). More algebraic proofs explaining the commutativity of the elliptic Ruijsenaars difference operators within the context of the quantum Yang-Baxter equation can be found in \([5, 14, 23, 28]\). For our purposes it is convenient to pick

\[
[z] = [z; p] = \frac{\vartheta_1(\frac{\alpha}{2} z; p)}{\vartheta_1'(0; p)} \quad (z \in \mathbb{C}, \alpha > 0, 0 < p < 1), \quad (2.2a)
\]

where \(\vartheta_1\) denotes the Jacobi theta function

\[
\vartheta_1(z; p) = 2 \sum_{l \geq 0} (-1)^l p^{(l + \frac{1}{2})^2} \sin(2l + 1)z,
\]

\[
= 2p^{1/4} \sin(z) \prod_{l \geq 1} (1 - p^{2l})(1 - 2p^{2l} \cos(2z) + p^{4l}). \quad (2.2b)
\]

The conversion to the Weierstrass sigma function associated with the period lattice \(\Omega = 2\omega_1 \mathbb{Z} + 2\omega_2 \mathbb{Z}\) is governed by the relation (cf. e.g. \([39, \S 23.6(i)]\)):

\[
[z; p] = \sigma(z) e^{-\frac{\zeta(\omega_1)}{2\omega_1} z^2}
\]

with \(\alpha = \frac{\pi}{\omega_1}, p = e^{i\pi \tau}, \tau = \frac{\omega_2}{\omega_1}, \) and \(\zeta(z) = \frac{\sigma'(z)}{\sigma(z)}\). Below we will often use that \([z; p]\) extends analytically in \(p\) to the interval \(-1 < p < 1\) with \([z; 0] = \frac{2}{\alpha} \sin(\frac{\alpha z}{2})\).
2.2 Discretization on partitions

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ denote a partition of length $\ell(\lambda) \leq n$, i.e. $\lambda$ belongs to

$$\Lambda^{(n)} = \{ \lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \}. \quad (2.3)$$

Unless explicitly stated otherwise, it will be assumed that the value of $g$ is chosen generically in $\mathbb{R}$ such that

$$\frac{g}{j} \not\in \mathbb{Z}_{\leq 0} + \frac{2\pi}{\alpha} \mathbb{Z} \quad \text{for } j = 1, \ldots, n-1 \quad (2.4a)$$

(where $\mathbb{Z}_{\leq 0} = \mathbb{Z} \setminus \mathbb{N} = \{0, -1, -2, -3, \ldots\}$), which ensures in particular that

$$\prod_{1 \leq j < k \leq n} [\lambda_j - \lambda_k + (k-j)g] \neq 0 \quad (\forall \lambda \in \Lambda^{(n)}). \quad (2.4b)$$

The following lemma now allows us to restrict $D_r (2.1)$ to a discrete difference operator acting on functions supported on partitions shifted by

$$\rho_g = ((n-1)g, (n-2)g, \ldots, g, 0). \quad (2.5)$$

**Lemma 2.1** (Boundary Condition) *For any $\lambda \in \Lambda^{(n)}$ and $J \subset \{1, \ldots, n\}$ (with $g$ generic as detailed above), one has that*

$$V_J(\rho_g + \lambda) = 0 \quad \text{if } \lambda + \varepsilon_J \notin \Lambda^{(n)}. \quad (2.6)$$

**Proof** If $\lambda \in \Lambda^{(n)}$ and $\mu = \lambda + \varepsilon_J \notin \Lambda^{(n)}$ then $\mu_j - \mu_{j+1} < 0$ for some $1 \leq j < n$, which implies that $j \notin J$, $j+1 \in J$ and $\lambda_j - \lambda_{j+1} = 0$. Since the denominators do not vanish because of Eq. (2.4b), one then picks up a zero of $V_J(x)$ at $x = \rho_g + \lambda$ from the factor $[x_{j+1} - x_j + g]$. \qed

Specifically, by means of Lemma 2.1 we cast the corresponding action of $D_r (2.1)$ in terms of a discrete difference operator in the space $\mathcal{C}(\Lambda^{(n)})$ of complex lattice functions $\lambda \mapsto f_\lambda$:

$$\left(D_r f\right)_\lambda = \sum_{\lambda \subset v \subset \lambda+1^n \atop |v| = |\lambda| + r} B_{v/\lambda}(\alpha, g; p) f_v \quad (f \in \mathcal{C}(\Lambda^{(n)}), \ \lambda \in \Lambda^{(n)}), \quad (2.7a)$$

with

$$B_{v/\lambda}(\alpha, g; p) = \prod_{1 \leq j < k \leq n} \frac{[\lambda_j - \lambda_k + g(k-j + \theta_j - \theta_k)]}{[\lambda_j - \lambda_k + g(k-j)]} \quad \text{and } \theta = v - \lambda, \quad (2.7b)$$
through the dictionary $f(\rho g + \lambda) = f_\lambda, \lambda + \epsilon_J = \nu$ (which implies that $V_\lambda(\rho g + \lambda) = B_{\nu/\lambda}(\alpha, g; p)$). Here and below we have employed the following standard notational conventions regarding partitions:

$$|\lambda| = \lambda_1 + \cdots + \lambda_n, \quad m^r = (m, \ldots, m, 0, \ldots, 0),$$

and $\forall \lambda, \mu \in \Lambda^{(n)}$: $\lambda \subset \mu$ iff $\lambda_j \leq \mu_j$ for $j = 1, \ldots, n$. Since $D_r$ (2.7a), (2.7b) amounts to a discretization of the elliptic Ruijsenaars operator $D_r$ (2.1), the commutativity is inherited automatically.

**Corollary 2.2 (Commutativity)** The discrete Ruijsenaars operators $D_1, \ldots, D_n$ (2.7a), (2.7b) commute in $C(\Lambda^{(n)})$.

### 3 Eigenpolynomials

#### 3.1 Joint eigenfunctions

We will now construct joint eigenfunctions for the elliptic Ruijsenaars operators in $C(\Lambda^{(n)})$. The values of these eigenfunctions at the lattice points are given by polynomials in the eigenvalues that are defined by means of a recurrence stemming from the corresponding eigenvalue equations. The commutativity of the elliptic Ruijsenaars operators guarantees that the solution of the recurrence in question indeed produces the asserted eigenfunctions.

Specifically, let us define polynomials $P_\mu(e) = P_\mu(e; \alpha, g; p)$, $\mu \in \Lambda^{(n)}$ in the variables $e = (e_1, \ldots, e_n)$ by means of the following recurrence relation

$$P_\mu(e) = e_r P_\lambda(e) - \sum_{\lambda \subset \nu \subset \lambda + 1^n, |\nu| = |\mu|, s.t. \nu \in \Lambda^{(n)} \setminus \{\mu\}} \psi'_{\nu/\lambda}(\alpha, g; p) P_\nu(e) \quad \text{if } \mu \neq 0, \quad (3.1a)$$

and $P_\mu(e) = 1$ if $\mu = 0$. Here $\lambda = \mu - 1^r$, where

$$r = r_\mu = \min\{1 \leq j \leq n \mid \mu_j - \mu_{j+1} > 0\} \quad (3.1b)$$

(with the convention $\mu_{n+1} \equiv 0$) and

$$\psi'_{\nu/\lambda}(\alpha, g; p) = \prod_{1 \leq j < k \leq n} \frac{[\nu_j - \nu_k + g(k-j+1)] [\lambda_j - \lambda_k + g(k-j-1)]}{[\nu_j - \nu_k + g(k-j)] [\lambda_j - \lambda_k + g(k-j)]} \quad \text{with } \theta = \nu - \lambda. \quad (3.1c)$$

Moreover, let

$$d_\mu = \mu_1 - \mu_n \quad (3.2)$$
and let $\preceq$ denote the dominance partial order on $\Lambda^{(n)}$, i.e.

$$\forall \lambda, \mu \in \Lambda^{(n)}: \lambda \preceq \mu \iff |\lambda| = |\mu|$$

and

$$\sum_{1 \leq j \leq r} \lambda_j \leq \sum_{1 \leq j \leq r} \mu_j \text{ for } r = 1, \ldots, n - 1$$

(and $\lambda < \mu$ if $\lambda \preceq \mu$ and $\lambda \neq \mu$).

**Proposition 3.1** (Triangularity) The polynomials $P_{\mu}(e)$, $\mu \in \Lambda^{(n)}$ are uniquely determined by the recurrence (3.1a)–(3.1c) (from the initial condition $P_0(e) = 1$), and their expansion in the monomial basis is unitriangular with respect to the dominance order:

$$P_{\mu}(e) = e_\mu + \sum_{v \in \Lambda^{(n)}, v < \mu} u_{\mu,v} e_v \quad \text{with } u_{\mu,v} = u_{\mu,v}(\alpha, g; p) \in \mathbb{R} \quad (3.3a)$$

and

$$e_\mu = \prod_{1 \leq j \leq n} e_j^{\mu - \mu_{j+1}}. \quad (3.3b)$$

**Proof** The proof is by lexicographical induction in $(d_\mu, r_\mu)$, with $r_\mu$ and $d_\mu$ as in Eqs. (3.1b) and (3.2), respectively.

If $d_\mu = 0$, then either $\mu = 0$ or $r = n$, i.e. $\mu = m^n$ with $m = \mu_1$. When $m = 0$ we have that $\mu = 0$, so $P_{\mu}(e) = e_\mu = 1$ by the initial condition, while for $m > 0$ the recurrence entails that $P_{\mu}(e) = P_{m^n}(e) = e_n P_{(m-1)^n}(e) = e_{\mu}^{(n)} = e_\mu$. Hence, in either case the induction hypothesis guarantees that $P_{\mu}(e)$ is uniquely determined by the recurrence relations through a monomial expansion consisting of $e_\mu$ perturbed by a linear combination of $e_\nu$ with $\nu < \mu$. Moreover, one either has that $d_\nu < d_\mu$ or that $d_\nu = d_\mu$ with $r_\nu < r_\mu$. Hence, the remaining terms on the RHS of the recurrence (3.1a) consist in turn of a $\mathbb{R}$-linear combination of $P_{\nu}(e)$ with $d_\nu \leq d_\mu$ and $\nu < \mu$. Upon combining all these terms appearing on the RHS of the recurrence (3.1a), one confirms that monomial expansion of $P_{\mu}(e)$ is of the form asserted in Eq. (3.3a) with expansion coefficients $u_{\mu,v}(\alpha, g; p) \in \mathbb{R}$ that are determined uniquely by the recurrence relation.

For $e \in \mathbb{C}^n$, we define $p(e) = p(e; \alpha, g; p) \in C(\Lambda^{(n)})$ in terms of the normalized polynomials

$$P_{\mu}(e) = c_\mu P_{\mu}(e) \quad (\mu \in \Lambda^{(n)}), \quad (3.4a)$$

with
\[ c_\mu = c_\mu(\alpha, g; p) = \prod_{1 \leq j < k \leq n} \frac{[(k-j)g]_{\mu_j-\mu_k}}{[(k-j+1)g]_{\mu_j-\mu_k}}, \]  
\( (3.4b) \)

where \([z]_k, k = 0, 1, 2, \ldots\) denotes the elliptic factorial

\[ [z]_k = \prod_{0 \leq l < k} [z + l] \quad \text{with} \quad [z]_0 = 1. \]

Notice that the regularity assumption on the parameter \( g \) in Eq. (2.4a) ensures that both the numerator and the denominator of \( c_\mu \) do not vanish.

**Theorem 3.2** (Joint Eigenfunctions) (i) The functions \( p(e), e \in \mathbb{C}^n \) constitute a family of joint eigenfunctions for the elliptic Ruijsenaars operators \( D_1, \ldots, D_n \) (2.7a), (2.7b) in \( \mathcal{C}(\Lambda^{(n)}) \):

\[ D_r p(e) = e_r p(e) \quad \text{for} \quad r = 1, \ldots, n. \]  
\( (3.5) \)

(ii) The vector of joint eigenvalues \( e = (e_1, \ldots, e_n) \in \mathbb{C}^n \) for the simultaneous eigenvalue problem in Eq. (3.5) is multiplicity free in \( \mathcal{C}(\Lambda^{(n)}) \).

**Proof** When evaluating at \( \mu \in \Lambda^{(n)} \), the \( s \)th eigenvalue equation in Eq. (3.5) reads

\[ \sum_{\mu \subset \nu \subset \mu + 1^n, |\nu| = |\mu| + s} B_{\nu/\mu}(\alpha, g; p) p_\nu(e) = e_s p_\mu(e). \]  
\( (3.6a) \)

The explicit product formulas for \( B_{\mu/\lambda}(\alpha, g; p) \) (2.7b), \( \psi'_{\mu/\lambda}(\alpha, g; p) \) (3.1c) and \( c_\mu \) (3.4b) reveal that for all \( \nu \in \Lambda^{(n)} \) such that \( \mu \subset \nu \subset \mu + 1^n + 1 \):

\[ \psi'_{\nu/\mu}(\alpha, g; p) c_\mu = B_{\nu/\mu}(\alpha, g; p) c_\nu. \]  
\( (3.6b) \)

Hence, Eq. (3.6a) can be rewritten in terms of \( P_\mu(e) \) as follows:

\[ \sum_{\mu \subset \nu \subset \mu + 1^n, |\nu| = |\mu| + s} \psi'_{\nu/\mu}(\alpha, g; p) P_\nu(e) = e_s P_\mu(e). \]  
\( (3.6c) \)

Since \( \psi'_{\mu+1^n/\mu}(\alpha, g; p) = 1 \), the equality in Eq. (3.6c) is immediate from the recurrence for \( P_{\mu+1^n}(e) \) if \( s \leq r_\mu \) with the convention that \( r_0 = n \) (cf. Eqs. (3.1a)–(3.1c)). This settles the proof of Eq. (3.6a) for \( s \leq r_\mu \), but it remains to check that the identity in question also holds if \( r_\mu < s \leq n \). To this end we perform induction with respect to the lexicographical order on \((d_\mu, r_\mu)\) as in the proof of Proposition 3.1:

\[ e_s P_\mu(e) = c_\mu e_s P_\mu(e) \]
\[ \text{Eq. (3.1a)} \]
\[ = c_\mu e_s \left( e_r P_\lambda(e) - \sum_{\lambda \subset \nu \subset \lambda + 1^n, |\nu| = |\lambda|, \text{s.t. } \nu \in \Lambda^{(n)} \setminus \{\mu\}} \psi'_{\nu/\lambda}(\alpha, g; p) P_\nu(e) \right) \]
Elliptic Ruijsenaars difference operators, symmetric... Page 9 of 29

(\text{where } r = r_\mu \text{ and } \lambda = \mu - 1^r)

\begin{align*}
\text{induction} &\rightarrow c_\mu \left( e_s e_r P_\lambda(e) - \sum_{\lambda \subset \nu \subset \lambda + 1^n, |\nu| = |\mu|} \psi_{\nu/\lambda}^r(\alpha, g; p) c_{\nu}^{-1} (D_s p(e))_{\nu} \right) \\
\text{Eq. (3.6b)} &\equiv (D_s p(e))_\mu + c_\mu \left( e_s e_r P_\lambda(e) - (D_r D_s p(e))_\lambda \right) = (D_s p(e))_\mu
\end{align*}

as desired. To verify the cancellation of the underbraced terms it is essential to exploit the commutativity of the elliptic Ruijsenaars operators:

\[(D_r D_s p(e))_\lambda \overset{\text{Cor. 2.2}}{=} (D_s D_r p(e))_\lambda = \sum_{\lambda \subset \nu \subset \lambda + 1^n, |\nu| = |\lambda| + r} B_{\nu/\lambda}(\alpha, g; p) (D_r p(e))_\nu\]

\[(D_s p(e))_\mu \overset{\text{induction}}{=} e_r \sum_{\lambda \subset \nu \subset \lambda + 1^n, |\nu| = |\lambda| + s} B_{\nu/\lambda}(\alpha, g; p) p(e)_\nu = e_r (D_s p(e))_\lambda \]

In step * we used that either \((d_\nu, r_\nu) < (d_\mu, r_\mu)\) in the lexicographical order (in which case the equality \((D_r p(e))_\nu = e_r p(e)_\nu\) stems from the induction hypothesis), or else \(r = r_\mu \leq r_\nu\) (in which case the equality in question is plain from the recurrence relation for \(P_{\nu+1^r}(e)\) in combination with Eq. (3.6b)).

This completes the proof of part (i) of the Theorem. Part (ii) follows in turn from the observation that, upon normalizing such that \(p_0(e) = 1\) (cf. Remark 3.3 below), any joint eigenfunction \(p(e) \in \mathcal{C}(\Lambda^{(n)})\) solving the eigenvalue equations in Eq. (3.5) automatically gives rise to functions \(P_\mu(e) (3.4a), (3.4b)\) satisfying the relations in Eq. (3.6c) for \(s = 1, \ldots, n\). This implies in particular that the functions in question obey the recurrence (3.1a)–(3.1c), so Proposition 3.1 guarantees that the corresponding joint eigenfunction is unique, i.e. the vector of joint eigenvalues \(e = (e_1, \ldots, e_n)\) is multiplicity free in \(\mathcal{C}(\Lambda^{(n)})\).

\textbf{Remark 3.3} Our choice for the unit-normalization at the origin does not contemplate the construction of joint eigenfunctions \(p(e)\) that vanish at \(\mu = 0\). Indeed, such eigenfunctions cannot exist in \(\mathcal{C}(\Lambda^{(n)})\) because from the recurrence in Eqs. (3.1a)–(3.1c) it is manifest that the initial condition \(P_0(e) = 0\) would imply that \(P_\mu(e) = 0\) for all \(\mu \in \Lambda^{(n)}\) (by lexicographical induction in \((d_\mu, r_\mu)\) as in the proof of Proposition 3.1).

\subsection*{3.2 Elliptic Littlewood–Richardson coefficients}

It is clear from Proposition 3.1 that the polynomials \(P_\mu(e), \mu \in \Lambda^{(n)}\) form a basis for the polynomial ring \(\mathbb{R}[e_1, \ldots, e_n]\). The corresponding structure constants give rise to
an elliptic generalization of the Littlewood–Richardson coefficients:

\[ P_\lambda P_\mu = \sum_{\nu \in \Lambda(n)} c^{\nu}_{\lambda, \mu}(\alpha, g; p) P_{\nu} \]  

(3.7)

(where the arguments \( e \) are suppressed). For \( \mu = 1^r \), an explicit product formula for these elliptic Littlewood–Richardson coefficients is immediate from Theorem 3.2.

**Corollary 3.4** (Pieri Rule) For \( \mu \in \Lambda(n) \) and \( 1 \leq r \leq n \), one has that

\[ P_\lambda P_{1^r} = \sum_{\nu \subset \lambda + 1^r} c^{\nu}_{\lambda, 1^r}(\alpha, g; p) P_{\nu}. \]  

(3.8)

**Proof** Since \( P_{1^r}(e) = e_{1^r} = e_r \), the asserted Pieri rules encode the eigenvalue equations of Theorem 3.2 in the reformulation of Eq. (3.6c) (obtained via Eq. (3.6b)). \( \square \)

With the aid of the Pieri rule, it is readily seen that the classical Littlewood–Richardson coefficients \( c^{\nu}_{\lambda, \mu} \) for the Schur polynomials [33, Chapter I.9] and their two-parameter deformation \( f^{\nu}_{\lambda, \mu}(q, t) \) associated with the Macdonald polynomials [33, Chapter VI.7] arise as suitable parameter specializations of the three-parameter elliptic Littlewood–Richardson coefficients \( c^{\nu}_{\lambda, \mu}(\alpha, g; p) \) (3.7).

**Proposition 3.5** (Degenerations) The classical Littlewood–Richardson coefficients \( c^{\nu}_{\lambda, \mu} \) and Macdonald’s/ \( (q, t) \)-Littlewood–Richardson coefficients \( f^{\nu}_{\lambda, \mu}(q, t) \) are recovered from \( c^{\nu}_{\lambda, \mu}(\alpha, g; p) \) (3.7) in the following way:

\[ \lim_{g \to 1} c^{\nu}_{\lambda, \mu}(\alpha, g; p) = c^{\nu}_{\lambda, \mu} \]  

(3.9a)

(provided \( \frac{2\pi}{\alpha} > 0 \) is irrational), and

\[ \lim_{p \to 0} c^{\nu}_{\lambda, \mu}(\alpha, g; p) = f^{\nu}_{\lambda, \mu}(q, q^g) \text{ with } q = e^{i\alpha} \]  

(3.9b)

(provided \( jg \notin \mathbb{Z}_{\leq 0} + \frac{2\pi}{\alpha} \mathbb{Z} \) for \( j = 1, \ldots, n \)).

In particular, for \( \lambda, \nu \in \Lambda(n) \) such that \( \lambda \subset \nu \subset \lambda + 1^n \) one has with these genericity assumptions in place that

\[ \lim_{g \to 1} \psi^{\nu}_{\nu/\lambda}(\alpha, g; p) = 1 \]  

(3.10a)

and

\[ \lim_{p \to 0} \psi^{\nu}_{\nu/\lambda}(\alpha, g; p) = \prod_{1 \leq j < k \leq n \atop \theta_j - \theta_k = -1} \frac{[v_j - v_k + g(k-j+1)]_q}{[v_j - v_k + g(k-j)]_q} \frac{[\lambda_j - \lambda_k + g(k-j)-1]_q}{[\lambda_j - \lambda_k + g(k-j)]_q}, \]  

(3.10b)
where \( \theta = v - \lambda \) and
\[
[z]_q = \frac{\sin\left(\frac{\theta z}{2}\right)}{\sin\left(\frac{z}{2}\right)} = \frac{q^{\frac{z}{2}} - q^{-\frac{z}{2}}}{q^\frac{1}{2} - q^{-\frac{1}{2}}}.
\]

**Proof** By definition, the polynomial ring \( \mathbb{R}[e_1, \ldots, e_n] \) is generated by the monomials \( e_r = P_r(e), r = 1, \ldots, n \). It is therefore sufficient to verify the limits in Eqs. (3.9a), (3.9b) for \( \lambda = 1^r \) (\( r = 1, \ldots, n \)), which—by the Pieri rule of Corollary 3.4—amounts to checking the limits in Eqs. (3.10a), (3.10b). The genericity assumptions on the parameters ensure that none of the denominators vanish.

Specifically, in the product formula for \( \psi'_{v/\lambda} \) (3.1c) one has that \( \lambda_j - \lambda_k = v_j - v_k + 1 \) if \( \theta_j - \theta_k = -1 \), so the limit in Eq. (3.10a) is evident. We thus recover in this manner the (dual) Pieri rules for the Schur polynomials \( s_{\lambda}(x) \) [33, Ch. I, Eq. (5.17)] from Corollary 3.4, which in turn implies the limit in Eq. (3.9a).

Similarly, since \([z; p] = q^2 \sin\left(\frac{z}{2}\right)\), the limit in Eq. (3.10b) is also manifest from Eq. (3.1c). Upon comparing with [33, Ch. VI, Eqs. (6.7'), (6.13)], we see that at \( p = 0 \) the Pieri rules for the Macdonald polynomials \( P_{\mu}(x; q, q^g) \) are recovered from Corollary 3.4, therewith settling the limit in Eq. (3.9b).

The Pieri rule also confirms that some well-known vanishing properties enjoyed by the Littlewood–Richardson coefficients persist at the elliptic level.

**Proposition 3.6** (Vanishing Terms) For \( \lambda, \mu, v \in \Lambda^{(n)} \), the elliptic Littlewood–Richardson coefficient \( c_{\lambda, \mu}(\alpha, g; p) \) vanishes unless \( \lambda \subset v \) and \( \mu \subset v \) with \( |\lambda| + |\mu| = |v| \).

**Proof** It suffices to mimic the proof for the corresponding statement at \( p = 0 \) from [33, Ch. VI, Eq. (7.4)]. Since the monomial expansion of \( P_{\kappa}(e) \) in Proposition 3.1 involves only monomials \( e_{\kappa} \) with \( |\kappa| = |\mu| \) and \( e_{\kappa} e_{\tilde{\kappa}} = e_{\kappa + \tilde{\kappa}} \), it is clear that \( c_{\lambda, \mu}(\alpha, g; p) \) can only be nonzero provided \( |v| = |\lambda| + |\mu| \). Moreover, let \( \mathcal{I}_\lambda \) denote the subspace of \( \mathbb{R}[e_1, \ldots, e_n] \) spanned by the \( P_{\kappa}(e) \) with \( \lambda \subset \kappa \). It is manifest from the Pieri rule for the Schur polynomials \( s_{\lambda}(x) \) that \( e_{\lambda} \mathcal{I}_\lambda \subset \mathcal{I}_\mu \) for \( r = 1, \ldots, n \), so \( \mathcal{I}_\lambda \) is an ideal in \( \mathbb{R}[e_1, \ldots, e_n] \). It thus follows that \( P_{\lambda}(e) P_{\mu}(e) \in \mathcal{I}_\lambda \cap \mathcal{I}_\mu \).

### 3.3 Symmetric polynomials

Let us recall (cf. the proof of Proposition 3.5) that \( s_{\mu}(x) = s_{\mu}(x_1, \ldots, x_n) \) and \( P_{\mu}(x; q, t) = P_{\mu}(x_1, \ldots, x_n; q, t) \) refer to the Schur polynomials [33, Chapter I] and the Macdonald polynomials [33, Chapter VI], respectively. The symmetric polynomials in question are monic in the sense that their leading monomial is given by
\[
m_{\mu}(x) = m_{\mu}(x_1, \ldots, x_n) = \sum_{\nu \in S_n(\mu)} \lambda_1^{\nu_1} \cdots \lambda_n^{\nu_n} \quad (\mu \in \Lambda^{(n)}),
\] (3.11)
where the sum is over all compositions reordering the parts of \( \mu \) (i.e. over the orbit of \( \mu \) with respect to the action of the permutation-group \( S_n \) of permutations.
σ = (1/σ1 2/σ2 ... n/σn) on μ1, μ2, ..., μn). It is instructive to describe the precise relation between our elliptic eigenpolynomials and these two standard bases for the ring \( \mathcal{A}^{(n)} = \mathbb{R}[x_1, \ldots, x_n] S_n \) of symmetric polynomials in the variables \( x_1, \ldots, x_n \).

To this end, let us observe that the polynomials \( P_\mu(e) \), \( \mu \in \Lambda_1^{(n)} \) give rise to a (monic) basis for \( \mathcal{A}^{(n)} \) through the ring isomorphism \( \mathbb{R}[e_1, \ldots, e_n] \cong \mathcal{A}^{(n)} \) determined by the injection \( e_r \rightarrow m_{1r}(x), r = 1, \ldots, n \):

\[
R_\mu(x; \alpha, g; p) = P_\mu(e) \quad \text{with} \quad e = (m_{11}(x), m_{12}(x), \ldots, m_{1n}(x)).
\]  

(3.12)

From the limits in Proposition 3.5, it is then clear that the corresponding degenerations of the recurrence relations (3.1a)–(3.1c) reproduce the Schur polynomials and the Macdonald polynomials respectively (cf. [33, Ch. I, Eq. (5.17)] and [33, Ch. VI, Eqs. (6.7′), (6.13)]).

Corollary 3.7 (The Schur and Macdonald Limits) For any \( \mu \in \Lambda^{(n)} \), the Schur polynomial \( s_\mu(x) \) and the Macdonald polynomial \( P_\mu(x; q, q^g) \) are recovered from \( R_\mu(x; \alpha, g; p) \) (3.12) in the following way:

\[
\lim_{g \to 1} R_\mu(x; \alpha, g; p) = s_\mu(x) \quad \text{(3.13a)}
\]

(provided \( 2\pi/\alpha > 0 \) is irrational), and

\[
\lim_{p \to 0} R_\mu(x; \alpha, g; p) = P_\mu(x; q, q^g) \quad \text{with} \quad q = e^{i\alpha} \quad \text{(3.13b)}
\]

(provided \( jg \notin \mathbb{Z} + 2\pi/\alpha \mathbb{Z} \) for \( j = 1, \ldots, n \)).

4 Generic elliptic deformation of the fusion ring for \( \widehat{\mathfrak{sl}}(n)_m \)

4.1 Character ring for \( \mathfrak{sl}(n) \)

When \( r = n \) the Pieri rule (3.8) simply states that \( P_{\mu+1^n}(e) = P_\mu(e)P_1^n(e) = P_\mu(e)e_n \). Hence, one has more generally that

\[
P_\mu(e) = P_\mu(e)e_n^{\mu_n} \quad \text{with} \quad \underline{\mu} = (\mu_1 - \mu_n, \mu_2 - \mu_n, \ldots, \mu_{n-1} - \mu_n, 0). \quad (4.1)
\]

To divide out this translational symmetry one substitutes \( e_n = 1 \), therewith reducing to the ring

\[
\mathcal{R}_0^{(n)} = \mathbb{R}[e_1, \ldots, e_n]/\langle e_n - 1 \rangle \quad \text{(4.2a)}
\]

for which (the corresponding specialization of) the polynomials

\[
P_\mu(e) \quad \text{with} \quad \mu \in \Lambda_0^{(n)} = \{ \lambda \in \Lambda^{(n)} \mid \lambda_n = 0 \} \quad \text{(4.2b)}
\]
provide a basis. In view of Proposition 3.6, it is immediate from Eq. (4.1) that the corresponding structure constants for $R_0^{(n)}$ can be expressed in terms of elliptic Littlewood–Richardson coefficients $c^v_{\lambda, \mu}(\alpha, g; p)$ (3.7) as follows:

$$P_{\lambda}P_{\mu} = \sum_{v \supset \lambda, v \supset \mu \atop |v| = |\lambda| + |\mu|} c^v_{\lambda, \mu}(\alpha, g; p) P_v \quad (\lambda, \mu \in \Lambda^0_0, v \in \Lambda^{(n)}).$$ (4.3a)

In particular, for $1 \leq r < n$ and $\lambda \in \Lambda^0_0$ one retrieves from Corollary 3.4 that

$$P_{\lambda}P_1 = \sum_{\lambda \subset v \subset \lambda + 1^n \atop |v| = |\lambda| + r} \psi^v_{1/\lambda}(\alpha, g; p) P_v.$$ (4.3b)

Proposition 3.5 and Corollary 3.7 entail that at $p = 0$ Eq. (4.3a) recovers the structure constants for the multiplication in the basis of Macdonald polynomials associated with (the root system of) the complex simple Lie algebra $\mathfrak{sl}(n)$ [34, 35]. Indeed, Eq. (4.3b) degenerates to the Pieri rule for the $\mathfrak{sl}(n)$ Macdonald polynomials in this situation (which is obtained from the Pieri formula for $P_{\mu}(x_1, \ldots, x_n; q, q^g)$ in [33, Ch. VI, Eqs. (6.7′), (6.13)] by dividing out the ideal generated by $m_1(x_1, \ldots, x_n) - 1$). Here partitions $\lambda \in \Lambda^0_0$ are identified with dominant weight vectors for $\mathfrak{sl}(n)$ in the standard way:

$$\lambda \leftrightarrow \sum_{1 \leq r < n} (\lambda_r - \lambda_{r+1}) \sigma_r,$$ (4.4)

where $\sigma_r \leftrightarrow 1^r$, $r = 1, \ldots, n - 1$ refers to the corresponding basis of fundamental weight vectors (labeled in accordance with the plates of [4]).

Similarly, for $g \to 1$ Eq. (4.3a) encodes the structure constants for the character ring of $\mathfrak{sl}(n)$ in the basis of the irreducible characters. For instance, in this limit the Pieri rule (4.3b) counts the tensor multiplicities for tensoring with a fundamental representation, cf. e.g. [40, Chapter 9.10].

### 4.2 Fusion ideal

We now scale $\alpha$ in terms of $g \in \mathbb{R} \setminus \mathbb{Q}$ as follows:

$$\alpha = \frac{2\pi}{m + ng} \quad \text{with } m \in \mathbb{N}. \quad (4.5)$$

The irrationality of $g$ then guarantees that the regularity requirement in Eq. (2.4b) is satisfied, so the polynomials $P_{\mu}(e)$ (3.1a)–(3.1c) are well-defined for this parameter specialization. Let us consider the following ideal in $R_0^{(n)}$ (4.2a):

$$\mathcal{I}^{(n,m)} = \langle P_{\mu}(e) \mid \mu \in \Lambda^0_0 \text{ with } d_\mu = m + 1 \rangle. \quad (4.6)$$
This ideal should be viewed as an elliptic \((g, p)\)-deformation associated with the elliptic Ruijsenaars model of the fusion ideal for \(\mathfrak{sl}(n)_m\) Wess–Zumino–Witten conformal field theories (cf. e.g. [11, 15, 17, 19, 30] and references therein).

With the aid of the Pieri rule (4.3b) and following lemma, one arrives at a convenient basis for \(\mathcal{I}^{(n, m)}\) in terms of the eigenpolynomials associated with the elliptic Ruijsenaars lattice model.

**Lemma 4.1** (Level m Boundary Condition) Let \(\lambda \in \Lambda_0^{(n)}\) with \(d_{\lambda} = m + 1\) and let \(\lambda \subset \nu \subset \lambda + 1^n\). Then one has that

\[
\psi'_{\nu/\lambda} \left( \frac{2\pi}{m + ng} \cdot g; p \right) = 0 \quad \text{if} \quad d_{\nu} \leq m
\]

(assuming \(g \in \mathbb{R} \setminus \mathbb{Q}\)).

**Proof** The conditions imply that \(\nu = \lambda + \theta\) with \(\theta\) a vertical \(r\)-strip \((1 \leq r < n)\), so \(d_{\nu} = d_{\lambda} + 1 - \theta_n\) with \(\theta_1, \theta_n \in \{0, 1\}\). If \(d_{\nu} \leq m\), we must in fact have that \(\theta_1 = 0\), \(\theta_n = 1\) and \(d_{\nu} = m\) as \(d_{\lambda} = m + 1\). In this situation, \(\psi'_{\nu/\lambda} \left( \frac{2\pi}{m + ng} \cdot g; p \right)\) (3.1c) picks up a zero from the factor \([v_j - v_k + (k - j + 1)g]\) in the numerator for \(j = 1\) and \(k = n: [v_1 - v_n + ng] = [m + ng] = \left[\frac{2\pi}{g}\right] = 0\).

**Proposition 4.2** (Basis for \(\mathcal{I}^{(n, m)}\)) For \(\alpha = \frac{2\pi}{m + ng}\) and \(g \in \mathbb{R} \setminus \mathbb{Q}\), the polynomials \(P_{\mu}(e)\) with \(\mu \in \Lambda_0^{(n)}\) such that \(d_{\mu} > m\) constitute a basis for \(\mathcal{I}^{(n, m)}\).

**Proof** By definition, the ideal \(\mathcal{I}^{(n, m)}\) (4.6) consists of polynomials of the form

\[
\sum_{\lambda \in \Lambda_0^{(n)}, \ d_{\lambda} = m + 1} a_{\lambda}(e) P_{\lambda}(e) \quad \text{with} \quad a_{\lambda}(e) \in \mathcal{R}_0^{(n)}. \tag{4.8}
\]

Since the monomials \(e_r = P_r(e)\) \((1 \leq r < n)\) generate \(\mathcal{R}_0^{(n)}\), it is immediate from the Pieri rule (4.3b) and Lemma 4.1 that all products \(a_{\lambda}(e) P_{\lambda}(e)\) in the sum of Eq. (4.8) expand as \(\mathbb{R}\)-linear combinations of basis polynomials \(P_{\mu}(e)\) with \(\mu \in \Lambda_0^{(n)}\) and \(d_{\mu} > m\). It remains to check that the pertinent basis polynomials \(P_{\mu}(e)\) indeed belong to \(\mathcal{I}^{(n, m)}\). If \(d_{\mu} = m + 1\) this is the case by definition, while for \(d_{\mu} > m + 1\) it follows by lexicographical induction in \((d_{\mu}, r_{\mu})\) from the recurrence in Eqs. (3.1a)–(3.1c) with the aid of Lemma 4.1 (using also that \(P_{\nu}(e) = P_{\lambda}(e)\) in \(\mathcal{R}_0^{(n)}\)). Indeed, on the RHS of (3.1a) one has that \(d_{\lambda} = d_{\mu} - 1 > m\), so \(e_r P_{\lambda}(e) \in \mathcal{I}^{(n, m)}\) since \(P_{\lambda}(e) \in \mathcal{I}^{(n, m)}\) by virtue of the induction hypothesis. The remaining terms on the RHS of Eq. (3.1a) involve polynomials \(P_{\nu}(e) = P_{\mu}(e)\) with either \(d_{\nu} = d_{\mu} > m\) and \(r_{\nu} < r_{\mu}\) or (using Lemma 4.1) with \(d_{\mu} > d_{\nu} > m\); the induction hypothesis therefore guarantees again that all of these terms belong to \(\mathcal{I}^{(n, m)}\). We may thus conclude that \(P_{\mu}(e)\) (3.1a) lies in \(\mathcal{I}^{(n, m)}\), therewith completing the induction step. \(\square\)
4.3 Fusion ring

Upon dividing out $\mathcal{I}^{(n,m)}$, one arrives in turn at a corresponding elliptic deformation of the fusion ring for $\widehat{\mathfrak{sl}}(n)_m$ Wess–Zumino–Witten conformal field theories:

$$\mathcal{R}_0^{(n,m)} = \mathcal{R}_0^{(n)} / \mathcal{I}^{(n,m)} \quad (4.9)$$

(cf. again [11, 15, 17, 19, 30] and references therein). For $P \in \mathcal{R}_0^{(n)}$, we will denote its coset $P + \mathcal{I}^{(n,m)}$ in $\mathcal{R}_0^{(n,m)}$ by $[P]$. The cosets of the elliptic eigenpolynomials labeled by bounded partitions in

$$\Lambda_0^{(n,m)} = \{ \lambda \in \Lambda_0^{(n)} | d_\lambda \leq m \} \quad (4.10)$$

provide a basis for our elliptic fusion ring. Notice in this connection that the bijection (4.4) maps the bounded partitions in question to (the nonaffine parts of) the dominant weights of the affine Lie algebra $\widehat{\mathfrak{sl}}(n)_m$ (cf. e.g. [30, Section 2.1]).

**Proposition 4.3** (Basis for $\mathcal{R}_0^{(n,m)}$) For $\alpha = \frac{2\pi}{m+n} g$ and $g \in \mathbb{R} \setminus \mathbb{Q}$, the cosets $[P_\mu(e)]$, $\mu \in \Lambda_0^{(n,m)}$ constitute a basis for $\mathcal{R}_0^{(n,m)}$.

**Proof** Since the polynomials $P_\mu(e)$, $\mu \in \Lambda_0^{(n)}$ form a basis for $\mathcal{R}_0^{(n)}$, the assertion in the proposition is immediate from Proposition 4.2. Indeed, the kernel of the ring homomorphism $P \to [P]$ from $\mathcal{R}_0^{(n)}$ onto $\mathcal{R}_0^{(n,m)}$ is equal to the $\mathbb{R}$-span of the polynomials $P_\mu(e)$, $\mu \in \Lambda_0^{(n)} \setminus \Lambda_0^{(n,m)}$ (by Proposition 4.2), so the cosets $[P_\mu(e)], \mu \in \Lambda_0^{(n,m)}$ constitute a basis for $\mathcal{R}_0^{(n,m)}$. \qed

It is now straightforward to express the structure constants of $\mathcal{R}_0^{(n,m)}$ in the basis $[P_\mu(e)], \mu \in \Lambda_0^{(n,m)}$ in terms of elliptic Littlewood–Richardson coefficients.

**Theorem 4.4** (Structure Constants of $\mathcal{R}_0^{(n,m)}$) For $g \in \mathbb{R} \setminus \mathbb{Q}$, the structure constants of $\mathcal{R}_0^{(n,m)}$ in the basis $[P_\mu(e)], \mu \in \Lambda_0^{(n,m)}$ can be expressed in terms of elliptic Littlewood–Richardson coefficients as follows:

$$[P_\lambda][P_\mu] = \sum_{\nu \supset \lambda, \nu \supset \mu} c_{\lambda, \mu}^{\nu}(\frac{2\pi}{m+n} g; p)[P_\nu] \quad (\lambda, \mu \in \Lambda_0^{(n,m)}, \nu \in \Lambda^{(n)}).$$

(4.11a)

In particular, for $1 \leq r < n$ and $\lambda \in \Lambda_0^{(n,m)}$ one has explicitly that

$$[P_\lambda][P_1^r] = \sum_{\lambda \subset \nu \subset \lambda + m+1^n} \psi_{\nu/\lambda}^{\nu}(\frac{2\pi}{m+n} g; p)[P_\nu].$$

(4.11b)
5 \hat{\mathfrak{sl}}(n)_m Verlinde algebras from elliptic Ruijsenaars systems

5.1 Analyticity

For \( g > 0 \) and \( \mu \in \Lambda_0^{(n)} \) with \( d_\mu \leq m + 1 \), we now employ analytic continuation in the parameters of \( P_\mu(\mathbf{e}; \frac{2\pi}{m+ng}, g; p) \in \mathcal{R}_0^{(n)} \) so as to remove the restriction that \( g \) be irrational.

Lemma 5.1 (Positivity of the Recurrence Coefficients) Let \( \alpha = \frac{2\pi}{m+ng} \), \( g > 0 \), \( -1 < p < 1 \), and \( v \in \Lambda^{(n)} \).

(i) For \( 1 \leq j < k \leq n \), one has that \( [v_j - v_k + (k - j)g; p] > 0 \) if \( v_j - v_k \leq m \).

(ii) For \( \lambda \in \Lambda_0^{(m,n)} \) and \( \lambda. Let \( \nu/\lambda \) be the polynomial in question. For later reference, let us also explicitly check that the pertinent normalization coefficients \( c_\mu(\alpha, g; p) \) from Eq. (3.4b) permit analytic continuation to \( g > 0 \).
Lemma 5.3 (Positivity of the Normalization Constants) For $\mu \in \Lambda_{0}^{(n,m)}$, $g > 0$ and $-1 < p < 1$, the normalization coefficient $c_{\mu}(\frac{2\pi}{m+ng}, g; p)$ (3.4b) is positive.

Proof The arguments of the theta functions in the numerator and the denominator of $c_{\mu}(\frac{2\pi}{m+ng}, g; p)$ are of the form $l + kg$, with $0 \leq l < m$ and $1 \leq k \leq n$, so $0 < l + kg < m + ng = \frac{2\pi}{\alpha}$. The corresponding values of the theta function are thus positive by the product formula (2.2b).

5.2 Spectral variety

In [9] it was shown that for $\alpha$ (4.5) with $g > 0$, the elliptic Ruijsenaars operators truncate to commuting discrete difference operators of the form (cf. Equations (2.7a), (2.7b)):

$$(Dr f)_{\lambda} = \sum_{\lambda \subset \nu \subset \lambda + 1} B_{\nu/\lambda}(\frac{2\pi}{m+ng}, g; p) f_{\nu}, \quad 1 \leq r < n.$$  

These operators turn out to be normal in the Hilbert space $\ell^{2}(\Lambda_{0}^{(n,m)}, \Delta)$ with the inner product

$$\langle f, g \rangle_{\Delta} = \sum_{\lambda \in \Lambda_{0}^{(n,m)}} f_{\lambda} \overline{g_{\lambda}} \Delta_{\lambda} \quad (f, g \in \ell^{2}(\Lambda_{0}^{(n,m)}, \Delta)),$$  

where

$$\Delta_{\lambda} = \Delta_{\lambda}(\alpha, g; p) = \prod_{1 \leq j < k \leq n} \frac{[\lambda_{j} - \lambda_{k} + (k-j)g]}{[k-j]g} \frac{[\lambda_{j} - \lambda_{k}]}{[1+(k-j-1)g]^{\lambda_{j} - \lambda_{k}}}$$  

(cf. [9, Proposition 6]). Their joint spectrum is moreover given by $(n-1+m)_{m}$ multiplicity-free vectors (cf. [9, Corollary 10])

$$e_{\nu} = e_{\nu}(\frac{2\pi}{m+ng}, g; p) = (e_{1,\nu}, \ldots, e_{n-1,\nu}, 1) \in \mathbb{C}^{n} \quad (\nu \in \Lambda_{0}^{(n,m)})$$  

such that

$$(Dr p(e_{\nu})) = e_{r,\nu} p(e_{\nu}) \quad (1 \leq r < n, \: \nu \in \Lambda_{0}^{(n,m)},)$$  

which are analytic in $p \in (-1, 1)$ with

$$\lim_{p \to 0} e_{r,\nu}(\frac{2\pi}{m+ng}, g; p) = q^{-r \left( \frac{1}{m} + \frac{(n-1)g}{2} \right)} m_{1}^{1}(q^{|v_{1}|+(n-1)g}, q^{|v_{2}|+(n-2)g}, \ldots, q^{|v_{n-1}|+g}, 1).$$  

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Here \( p(e) \) is given by Eqs. (3.4a), (3.4b) and \( q = e^{\frac{2\pi i}{m+ng}} \).

Let us now define the spectral variety as the zero locus \( V(\mathcal{I}^{(n,m)}) \) of the fusion ideal \( \mathcal{I}^{(n,m)} \) (4.6):

\[
V(\mathcal{I}^{(n,m)}) = \{ e = (e_1, \ldots, e_{n-1}, 1) \in \mathbb{C}^n \mid P(e) = 0, \forall P \in \mathcal{I}^{(n,m)} \}. \tag{5.4}
\]

In other words, \( V(\mathcal{I}^{(n,m)}) \) arises as the intersection of the zero loci of all polynomials in \( \mathcal{I}^{(n,m)} \).

**Proposition 5.4** (Spectral Variety) For \( \alpha = \frac{2\pi}{m+ng} \) with \( g \in (0, \infty) \), the spectral variety \( V(\mathcal{I}^{(n,m)}) \) (5.4) is given by the joint spectrum \( e_v \left( \frac{2\pi}{m+ng}, g; p \right) \) (5.3a)–(5.3c) of the truncated elliptic Ruijsenaars operators \( D_r \) (5.1):

\[
V(\mathcal{I}^{(n,m)}) = E_0^{(n,m)} = \{ e_v \left( \frac{2\pi}{m+ng}, g; p \right) \mid v \in \Lambda_0^{(n,m)} \}. \tag{5.5}
\]

**Proof** Upon specializing the parameters in the Pieri rule (4.3b) with the aid of Proposition 5.2, one sees that for \( \lambda \in \Lambda_0^{(n,m)}, g \in (0, \infty) \) and \( e \in V(\mathcal{I}^{(n,m)}) \) (5.4):

\[
e_r P_\lambda(e; \frac{2\pi}{m+ng}, g; p) = \sum_{\lambda \subseteq \upsilon \subseteq \lambda + \{g \}, \vert \upsilon \vert = \vert \lambda \vert + r, d_v \leq m} \psi_{\nu/\lambda} \left( \frac{2\pi}{m+ng}, g; p \right) P_{\nu} \left( e; \frac{2\pi}{m+ng}, g; p \right) \text{ for } 1 \leq r < n. \tag{5.6}
\]

As in the proof of Theorem 3.2, we can now employ Eq. (3.6b) and Lemma 5.3 to rewrite Eq. (5.6) in the form \( D_r p(e; \frac{2\pi}{m+ng}, g; p) = e_r p(e; \frac{2\pi}{m+ng}, g; p) \) with \( D_r \) and \( p_\mu(e; \frac{2\pi}{m+ng}, g; p) \) taken from Eq. (5.1) and Eqs. (3.4a), (3.4b), respectively. Since \( p_0(e; \frac{2\pi}{m+ng}, g; p) = 1 \neq 0 \), this implies that \( e \) must belong to the joint spectrum \( E_0^{(n,m)} \) (5.5) of the operators \( D_1, \ldots, D_{n-1} \) in \( \ell^2(\Lambda_0^{(n,m)}), \Delta \). Reversely, if we assume that \( e \in E_0^{(n,m)} \) (5.5) then the eigenvalue equation entails that Eq. (5.6) holds for \( 1 \leq r < n \). The recurrence relations (3.1a)–(3.1c) thus yield in this situation that \( P_\mu(e; \frac{2\pi}{m+ng}, g; p) = 0 \) if \( d_\mu = m + 1 \) (where we use that \( \psi'_{\nu/\lambda} \left( \frac{2\pi}{m+ng}, g; p \right) = 1 \) and that the solution of the recurrence is unique). In other words, the vector of joint eigenvalues \( e \) necessarily lies on the zero locus \( V(\mathcal{I}^{(n,m)}) \) (5.4) of \( \mathcal{I}^{(n,m)} \) (4.6). \( \square \)

**Remark 5.5** One learns from Proposition 5.4 that for the indicated parameter regime the spectral variety \( V(\mathcal{I}^{(n,m)}) \) is a zero-dimensional variety consisting of \( \binom{n-1+m}{m} \) points in \( \mathbb{C}^n \). As a matter of fact, since the commuting operators \( D_r \) (5.1) are normal in \( \ell^2(\Lambda_0^{(n,m)}, \Delta) \) the existence of an orthonormal joint eigenbasis is actually guaranteed a priori by the corresponding spectral theorem in finite dimension (cf. e.g. [16, Chapter IX.15] or [24, Chapter 2.5]). Moreover, the operators in question and the orthogonality measure \( \Delta \) are analytic in \( p \in (-1, 1) \) so the normality ensures additionally that the pertinent eigenvalues also depend analytically on this parameter (cf. e.g. [26, Chapter 2, Theorem 1.10]). At \( p = 0 \) the eigenvalues can be computed explicitly with the aid of the theory of Macdonald’s polynomials [33], which gives rise to Eq. (5.3c)
(cf. [9, Section 5]). In particular, this explicit formula shows that when \( g \) is positive integral the eigenvalues at \( p = 0 \) are given by (homogenizations of) the elementary symmetric polynomials \( m_\nu, 1 \leq r < n \) specialized at roots of unity (cf. Equation (5.15) below). Since the elementary symmetric functions of interest separate the points of the underlying Weyl alcove (cf. [10, Section 4.2]), the explicit values in Eq. (5.3c) confirm that the spectral points of \( E_0^{(n,m)} \) (5.5) are indeed all distinct at \( p = 0 \). More generally, as functions of \( p \in (-1, 1) \) the trajectories of the spectral points are given by \( (n-1+m) \)-analytic curves \( e_v(\frac{2\pi}{m+ng}; g; p), v \in \Lambda_0^{(n,m)} \) in \( \mathbb{C}^n \) that pass through the explicit Macdonald spectrum \( e_v(\frac{2\pi}{m+ng}; g; 0), v \in \Lambda_0^{(n,m)} \) at \( p = 0 \). The uniqueness of the recursive construction of the eigenfunctions in terms of polynomials on the spectrum reveals that the spectral points of \( E_0^{(n,m)} \) (5.5) are in fact multiplicity-free for any \( p \in (-1, 1) \). In other words, the curves \( e_v(\frac{2\pi}{m+ng}; g; p), v \in \Lambda_0^{(n,m)} \) never cross each other when \( p \) varies over the interval \((-1, 1)\) and collisions between the spectral points are thus avoided when \( p \)-deforming away from the Macdonald spectrum.

### 5.3 Verlinde algebra

Let us now define the **Verlinde algebra** associated with the elliptic Ruijsenaars system as the \( (n-1+m) \)-dimensional algebra of complex functions on the joint spectrum \( E_0^{(n,m)} \) (5.5):

\[
\mathcal{F}_0^{(n,m)} = \{ f : E_0^{(n,m)} \to \mathbb{C} \},
\]

and let us write \( \mathcal{R}_0^{(n)} \) and \( \mathcal{R}_0^{(n,m)} \) for the algebras obtained by complexifying \( \mathcal{R}_0^{(n)} \) (4.2a) and \( \mathcal{R}_0^{(n,m)} \) (4.9).

**Proposition 5.6** (Basis for \( \mathcal{F}_0^{(n,m)} \)) For \( \alpha = \frac{2\pi}{m+ng} \) with \( g > 0 \), the restrictions of the polynomials \( P_\mu(e; \frac{2\pi}{m+ng}, g; p), \mu \in \Lambda_0^{(n,m)} \) on the joint spectrum \( E_0^{(n,m)} \) (5.5) constitute a basis for the Verlinde algebra \( \mathcal{F}_0^{(n,m)} \) (5.7).

**Proof** It is immediate from the fact that the eigenfunctions \( p(e_v), v \in \Lambda_0^{(n,m)} \) of the normal operators \( D_1, \ldots, D_{n-1} \) (5.1) provide an orthogonal basis for the Hilbert space \( L^2(\Lambda_0^{(n,m)}, \Delta) \) (cf. [9, Theorem 8]) that the square matrix \( [P_\mu(e_v)]_{\mu, v \in \Lambda_0^{(n,m)}} \) is of full rank. \( \Box \)

It is plain from Propositions 5.4 and 5.6 that for \( g \in (0, \infty) \) the vanishing ideal \( \mathcal{I}(E_0^{(n,m)}) = \{ P \in \mathcal{R}_0^{(n,m)} | P(e) = 0, \forall e \in E_0^{(n,m)} \} \) is equal to \( \mathcal{I}^{(n,m)} \) (4.6). It means that in this situation the Verlinde algebra \( \mathcal{F}_0^{(n,m)} \) and the \( \mathfrak{sl}(n)_m \) elliptic fusion algebra \( \mathcal{R}_0^{(n,m)} \) are isomorphic, as the kernel of the evaluation homomorphism \( P(e) \to P(e_v) \) from \( \mathcal{R}_0^{(n,m)} \) onto \( \mathcal{F}_0^{(n,m)} \) coincides with the complexification of \( \mathcal{I}^{(n,m)} \) (4.6).
Corollary 5.7 \((\mathcal{R}_{0, \mathbb{C}}^{(n,m)} \cong \mathcal{F}_{0}^{(n,m)})\) For \(\alpha = \frac{2\pi}{m + ng}\) with \(g \in (0, \infty)\), the evaluation homomorphism \(P(e) \rightarrow P(e_{\nu})\) from \(\mathcal{R}_{0, \mathbb{C}}^{(n,m)}\) onto \(\mathcal{F}_{0}^{(n,m)}\) induces an algebra isomorphism \([P] \rightarrow P(e_{\nu})\) from \(\mathcal{R}_{0, \mathbb{C}}^{(n,m)}\) onto \(\mathcal{F}_{0}^{(n,m)}\).

Corollary 5.7 makes it trivial to determine further algebraic properties of \(\mathcal{R}_{0, \mathbb{C}}^{(n,m)}\) and \(\mathcal{I}^{(n,m)}\). For instance, it is manifest from the isomorphism that the factor ring \(\mathcal{R}_{0}^{(n,m)}\) does not have nilpotents \(\neq 0\), i.e. it is a reduced ring and \(\mathcal{I}^{(n,m)}\) is a radical ideal. However, clearly the factor ring \(\mathcal{R}_{0}^{(n,m)}\) has zero divisors (stemming from functions in \(\mathcal{F}_{0}^{(n,m)}\) with disjoint support), i.e. it is not an integral domain and the ideal \(\mathcal{I}^{(n,m)}\) is therefore neither maximal nor prime.

With the aid of the Verlinde algebra \(\mathcal{F}_{0}^{(n,m)}\) we are now in the position to extend the elliptic fusion rules in Theorem 4.4 so as to include the case of positive rational values for \(g\).

Theorem 5.8 \((\text{Structure Constants of } \mathcal{F}_{0}^{(n,m)})\) For \(\alpha = \frac{2\pi}{m + ng}\) with \(g > 0\), the structure constants of the Verlinde algebra \(\mathcal{F}_{0}^{(n,m)}\) in the basis \(P_{\mu}(e; \frac{2\pi}{m + ng}, g; p), \mu \in \Lambda_{0}^{(n,m)}\) can be expressed in terms of elliptic Littlewood–Richardson coefficients as follows:

\[
P_{\lambda}(e; \frac{2\pi}{m + ng}, g; p) P_{\mu}(e; \frac{2\pi}{m + ng}, g; p) = \sum_{\nu \supset \lambda, \nu \supset \mu} \left( \lim_{c \in \mathbb{R} \setminus \mathbb{Q}} \psi^{\nu}_{\lambda, \mu}(\frac{2\pi}{m + ng}, c; p) \right) P_{\nu}(e; \frac{2\pi}{m + ng}, g; p)
\]

\((5.8a)\)

(with \(\lambda, \mu \in \Lambda_{0}^{(n,m)}, \nu \in \Lambda^{(n)}\) and \(e \in \mathbb{E}_{0}^{(n,m)}\).

In particular, for \(1 \leq r < n, \lambda \in \Lambda_{0}^{(n,m)}\) and \(e \in \mathbb{E}_{0}^{(n,m)}\) one has explicitly that

\[
P_{\lambda}(e; \frac{2\pi}{m + ng}, g; p) P_{1^{r}}(e; \frac{2\pi}{m + ng}, g; p) = \sum_{\lambda \subset \nu \subset \lambda + 1^{m}} \psi^{1^{r}}_{\nu/\lambda}(\frac{2\pi}{m + ng}, g; p) P_{\nu}(e; \frac{2\pi}{m + ng}, g; p).
\]

\((5.8b)\)

Proof If we pick \(g > 0\) irrational, then it is clear from Theorem 4.4 and Corollary 5.7 that on \(\mathbb{E}_{0}^{(n,m)}\):

\[
P_{\lambda}(e; \frac{2\pi}{m + ng}, g; p) P_{\mu}(e; \frac{2\pi}{m + ng}, g; p) = \sum_{\nu \supset \lambda, \nu \supset \mu} \psi^{\nu}_{\lambda, \mu}(\frac{2\pi}{m + ng}, g; p) P_{\nu}(e; \frac{2\pi}{m + ng}, g; p),
\]

which establishes in particular the validity of the Pieri rule \((5.8b)\) in this situation. Rational values of \(g > 0\) can now be included by analytic continuation, which gives rise to Eq. \((5.8a)\). Indeed, the case of the Pieri rule the analytic continuation is
achieved through the explicit formulas with the aid of Lemma 5.1 and Proposition 5.2 (where one also uses that the spectral points $e_v = e_v(g, p)$ are analytic in $g > 0$ by the normality of $D_r (5.1)$ in $\ell^2(\Lambda_0^{(n,m)}, \Delta)$, cf. e.g. [26, Chapter 2, Theorem 1.10]). Since the monomials $P_{1r}(e; \frac{2\pi}{m+ng}, g; p) = e_r, 1 \leq r < n$ generate $F_0^{(n,m)}$, the analyticity of the Pieri rule is inherited in turn by the structure constants of $F_0^{(n,m)}$ in general. To see this, it suffices to expand one of the two factors in the product on the LHS of Eq. (5.8a) in monomials (cf. Proposition 3.1). Since the corresponding expansion coefficients are analytic by virtue of Proposition 5.2, iterated application of the Pieri rules then leads to the analyticity of the structure constants as claimed.

\[ \sum_{\lambda, \mu \in \Lambda_0^{(n,m)}} \langle P_{\lambda}, P_{\mu} \rangle_{\Delta} = \begin{cases} 1, & \text{if } \lambda = \mu, \\ 0, & \text{if } \lambda \neq \mu, \end{cases} \quad (\lambda, \mu) \in \Lambda_0^{(n,m)} \]

5.4 Verlinde formula

Let us assume that $g > 0$ and denote the squared norms of the eigenbasis $p(e_v), v \in \Lambda_0^{(n,m)}$ for the elliptic Ruijsenaars operators $D_r (5.1)$ in the Hilbert space $\ell^2(\Lambda_0^{(n,m)}, \Delta)$ by (cf. Equations (5.2a), (5.2b)):

\[ \hat{\Delta}_v = 1/\langle p(e_v), p(e_v) \rangle_{\Delta} \quad (v \in \Lambda_0^{(n,m)}). \]  

We now endow the Verlinde algebra $F_0^{(n,m)} (5.7)$ with the following inner product:

\[ \langle f, g \rangle_{\Delta} = \sum_{v \in \Lambda_0^{(n,m)}} f(e_v)g(e_v)\hat{\Delta}_v \quad (f, g \in F_0^{(n,m)}). \]  

The basis of the Verlinde algebra in Proposition 5.6 is then orthogonal with respect to this inner product [9, Corollary 11], viz. one has that

\[ \forall \lambda, \mu \in \Lambda_0^{(n,m)} : \quad \langle P_{\lambda}, P_{\mu} \rangle_{\Delta} = \begin{cases} 1, & \text{if } \lambda = \mu, \\ 0, & \text{if } \lambda \neq \mu, \end{cases} \quad (\lambda, \mu) \in \Lambda_0^{(n,m)} \]

with $c_{\lambda}$ and $\Delta_{\lambda}$ given by Eqs. (3.4b) and (5.2b). With the aid of this dual orthogonality relation one arrives at a Verlinde formula expressing the structure constants

\[ P_{\lambda}(e; \frac{2\pi}{m+ng}, g; p) P_{\mu}(e; \frac{2\pi}{m+ng}, g; p) \]

\[ = \sum_{\kappa \in \Lambda_0^{(n,m)}} N_{\kappa, \lambda}(e; \frac{2\pi}{m+ng}, g; p) P_{\kappa}(e; \frac{2\pi}{m+ng}, g; p) \quad (e \in E_0^{(n,m)}) \]

for $F_0^{(n,m)}$ in terms of a corresponding $S$-matrix stemming from the elliptic Ruijsenaars model.

**Theorem 5.9** (Verlinde Formula) Let $g > 0$. For $p \in (-1,1)$ generic such that $\prod_{\lambda \in \Lambda_0^{(n,m)}} P_{\lambda}(e_0) \neq 0$, the structure constants $N_{\lambda, \mu}^\kappa(e; \frac{2\pi}{m+ng}, g; p)$ (5.12) for the
Verlinde algebra $\mathcal{F}_0^{(n,m)}$ (5.7) are given by

$$N^\kappa_{\lambda,\mu}\left(\frac{2\pi}{m+ng}, g; p\right) = \sum_{v \in \Lambda_0} \frac{S_{\lambda,v} S_{\mu,v} S_{v,\kappa}^{-1}}{S_{0,v}}$$

(5.13a)

with

$$S_{\lambda,v} = S_{\lambda,v}\left(\frac{2\pi}{m+ng}, g; p\right) = P_{\lambda}(e_v; \frac{2\pi}{m+ng}, g; p) P_{\mu}(e_v; \frac{2\pi}{m+ng}, g; p)$$

(5.13b)

and

$$S_{\lambda,v}^{-1} = |P_{\lambda}(e_0)|^{-2} \Delta_{\lambda,\mu}^2 \Delta_v.$$

(5.13c)

**Proof** Let us first observe that it is clear from the orthogonality relation in Eq. (5.11) that the inverse of the $S$-matrix $S_{\lambda,v}$ (5.13b) is given by the matrix $S_{\lambda,v}^{-1}$ (5.13c). Hence, upon pairing both sides of Eq. (5.12) against $P_{\kappa}(e; \frac{2\pi}{m+ng}, g; p), \kappa \in \Lambda_0^{(n,m)}$ with $(\cdot, \cdot)$, the Verlinde formula readily follows via the orthogonality relation (5.11):

$$N^\kappa_{\lambda,\mu}\left(\frac{2\pi}{m+ng}, g; p\right) = \langle P_{\lambda} P_{\mu}, P_{\kappa} \rangle_{\hat{\Delta}} = c^2_{\lambda} \Delta_{\kappa} \sum_{v \in \Lambda_0} P_{\lambda}(e_v) P_{\mu}(e_v) P_{\kappa}(e_v) \hat{\Delta}_v$$

$$= \sum_{v \in \Lambda_0} \frac{S_{\lambda,v} S_{\mu,v} S_{v,\kappa}^{-1}}{S_{0,v}}. \quad \square$$

It is useful to recall though that we already know from Theorem 5.8 that many of the structure constants in Eq. (5.12) in fact vanish:

$$N^\kappa_{\lambda,\mu}\left(\frac{2\pi}{m+ng}, g; p\right)$$

$$= \begin{cases} 
\lim_{c \to g} c^\mu \left(\frac{2\pi}{m+nc}, c; p\right) & \text{if } \kappa = v \text{ with } v \supset \lambda, v \supset \mu, \text{ and } |v| = |\lambda| + |\mu|, \\
0 & \text{otherwise}.
\end{cases}$$

Notice also that it follows from Eq. (5.13c) that the absolute value of the determinant of the $S$-matrix is given by

$$|\det S| = \prod_{\lambda \in \Lambda_0^{(n,m)}} \frac{|P_{\lambda}(e_0)|}{c_{\lambda} \sqrt{\Delta_{\lambda} \hat{\Delta}_{\lambda}}}.$$

(5.14)

**Remark 5.10** Macdonald’s principal specialization formula (cf. Remark 5.11 below) guarantees that the technical condition in Theorem 5.9 that $\prod_{\lambda \in \Lambda_0^{(n,m)}} P_{\lambda}(e_0)$ be nonzero is satisfied in a neighborhood of $p = 0$. Hence—by the analyticity in $p$—one
has \( \forall \epsilon \in (0, 1) \) and \(-\epsilon \leq p \leq \epsilon\) that \( \prod_{\lambda \in \Lambda_0^{(n,m)}} P_{\lambda}(e_0) \neq 0 \) for all but at most a finite number of values of \( p \) within the window \([-\epsilon, \epsilon]\).

### 5.5 Degenerations

To see how the structure constants of the \( \widehat{\mathfrak{sl}}(n)_m \) Wess–Zumino–Witten fusion ring and its refined deformation are recovered from \( \mathcal{F}_0^{(n,m)} \), let us put

\[
q = e^{\frac{2\pi i}{m+n g}} \quad \text{and} \quad e_r = (x_1 \cdots x_n)^{-r/n} m_1(x) \quad 1 \leq r < n. \tag{5.15}
\]

Then for \( g > 0, -1 < p < 1 \) and \( \lambda, \mu \in \Lambda_0^{(n,m)} \) we have that:

\[
\psi_{\nu/\lambda}^{\psi} \left( \frac{2\pi i}{m+n g}, g; p \right) = \begin{cases} 
\prod_{1 \leq j < k \leq n} \frac{[\nu_j - \nu_k + g(k-j+1)]_q}{[\nu_j - \nu_k + g(k-j)]_q} & \text{if } p = 0, \\
1 & \text{if } g = 1
\end{cases} \tag{5.16a}
\]

(for \( \lambda \subset \nu \subset \lambda + 1^n \) with \( \theta = \nu - \lambda \) and \( \nu \in \Lambda_0^{(n,m)} \)),

\[
P_{\mu}(e; \frac{2\pi i}{m+n g}, g; p) = \begin{cases} 
(x_1 \cdots x_n)^{-|\mu|/n} P_{\mu}(x; q, q^{\theta}) & \text{if } p = 0, \\
(x_1 \cdots x_n)^{-|\mu|/n} s_{\mu}(x) & \text{if } g = 1,
\end{cases} \tag{5.16b}
\]

and

\[
e_{r, \nu} \left( \frac{2\pi i}{m+n g}, g; p \right) = \begin{cases} 
q^{-r\left(\frac{|\nu| + (n-1)g}{2}\right)} m_1(\sqrt{q}^{v_1+(n-1)g}, q^{v_2+(n-2)g}, \ldots, q^{v_{n-1}+g}, 1) & \text{if } p = 0, \\
q^{-r\left(\frac{|\nu| + n g}{2}\right)} m_1(\sqrt{q}^{v_1+n-1}, q^{v_2+n-2}, \ldots, q^{v_{n-1}+1}, 1) & \text{if } g = 1
\end{cases} \tag{5.16c}
\]

(for \( \nu \in \Lambda_0^{(n,m)} \)). Indeed, Eqs. (5.16a) and (5.16b) follow from Proposition 3.5 and Corollary 3.7 by Lemma 5.1, while Eq. (5.16c) is immediate from Eq. (5.3c) when \( p = 0 \), and hence also when \( g = 1 \) (since the \( p \)-dependence drops out at \( g = 1 \)).

At \( p = 0 \), Theorem 5.8 therefore computes the structure constants of the refined Verlinde algebra [1, 6, 27, 37] in terms of Macdonald’s \( (q, t) \)-Littlewood–Richardson coefficients (cf. Proposition 3.5):

\[
N_{\lambda, \mu}^{\nu} \left( \frac{2\pi i}{m+n g}, g; 0 \right) = \lim_{e \to 0} \varphi_{\lambda, \mu}(e^{\frac{2\pi i}{m+n c}}, e^{\frac{2\pi i c}{m+n c}}) \quad \text{if } \kappa = \nu \text{ with } \nu \supset \lambda, \nu \supset \mu, \text{ and } |\nu| = |\lambda| + |\mu|, \\
0 \quad \text{otherwise.} \tag{5.17a}
\]
In the case of the Pieri rule this becomes explicitly

$$N^\kappa, \nu \left( \frac{2\pi}{m+n}, g; 0 \right) = \begin{cases} 
\psi'_{/\lambda} \left( \frac{2\pi}{m+n}, g; 0 \right) & \text{if } \kappa = \nu \text{ with } \lambda \subset \nu \subset \lambda + 1^n, \text{ and } |\nu| = |\lambda| + r, \\
0 & \text{otherwise,} 
\end{cases}$$

(5.17b)

At $g = 1$, on the other hand, we see from the Pieri rule in Theorem 5.8 that

$$N^\kappa, \nu \left( \frac{2\pi}{m+n}, 1; p \right) = N^\kappa, \nu \left( \frac{2\pi}{m+n}, 1; 0 \right) = \begin{cases} 
1 & \text{if } \kappa = \nu \text{ with } \lambda \subset \nu \subset \lambda + 1^n, \text{ and } |\nu| = |\lambda| + r, \\
0 & \text{otherwise.} 
\end{cases}$$

(5.18a)

More generally, Theorem 5.8 thus states that

$$N^\kappa, \mu \left( \frac{2\pi}{m+n}, 1; p \right) = N^\kappa, \mu \left( \frac{2\pi}{m+n}, 1; 0 \right) = \begin{cases} 
\lim_{c \to 1} f\nu_{/\lambda, \mu} \left( e^{2\pi i m + n c}, e^{2\pi i c} \right) & \text{if } \kappa = \nu \text{ with } \nu \supset \lambda, \nu \supset \mu, \text{ and } |\nu| = |\lambda| + |\mu|, \\
0 & \text{otherwise,} 
\end{cases}$$

(5.18b)

which retrieves the structure constants of the $\hat{sl}(n)_m$ Wess–Zumino–Witten fusion ring [11, 15] through Macdonald’s $(q, t)$-Littlewood–Richardson coefficients. In other words, $(q, t)$-deformation can be used as a vehicle for computing structure constants in the fusion ring (i.e. modulo the fusion ideal) by degeneration from (deformed) Littlewood–Richardson coefficients in the ring of symmetric polynomials itself, as was previously pointed out in [8].

Finally, we see from Eqs. (5.16b), (5.16c) that the corresponding degenerations of the elliptic $S$-matrix (5.13b) recover, respectively, the $S$-matrix for refined Chern-Simons knot invariants [1, 7, 21, 22, 37] originating from [6, 27]:

$$S_{\lambda, \nu} \left( \frac{2\pi}{m+n}, g; 0 \right) = q^{-\frac{1}{n}|\lambda| - \frac{1}{n-1}(n-1)\left(|\lambda| + |\nu|\right)g} \times P_{\lambda}(q^{v_1+(n-1)g}, q^{v_2+(n-2)g}, \ldots, q^{v_{n-1}+g}, 1; q, q^g) \times P_{\nu}(q^{(n-1)g}, q^{(n-2)g}, \ldots, q^g, 1; q, q^g),$$

(5.19a)

and the celebrated Kac-Peterson modular $S$-matrix for the $\hat{sl}(n)_m$ Wess–Zumino–Witten fusion ring [11, 15, 25]:

$$S_{\lambda, \nu} \left( \frac{2\pi}{m+n}, 1; p \right) = q^{-\frac{1}{n}|\lambda| - \frac{1}{n-1}(n-1)\left(|\lambda| + |\nu|\right)} \times S_{\lambda}(q^{v_1+n-1}, q^{v_2+n-2}, \ldots, q^{v_{n-1}+1}, 1) \times S_{\nu}(q^{n-1}, q^{n-2}, \ldots, q, 1).$$

(5.19b)
Remark 5.11 The principal specialization formula for Macdonald polynomials [33, Chapter VI, (6.11)] states that:

\[
q^{-\frac{1}{2}(n-1)g} P_v(q^{(n-1)g}, q^{(n-2)g}, \ldots, q^g, 1; q, q^g) = \prod_{1 \leq j < k \leq n} \frac{[k-j+1]q, \nu_j - \nu_k}{[(k-j)g]q, \nu_j - \nu_k}
\]

(= 1/c_v(\alpha, g; 0)), where [z]_{q,k} = \prod_{0 \leq l < k} [z + l]_q with [z]_{q,0} = 1. By Eqs. (5.16b), (5.16c), this means that

\[
P_v(e_0; \frac{2\pi}{m+ng}, g; 0) = 1/c_v(\frac{2\pi}{m+ng}, g; 0)
\]

(≠ 0) and

\[
P_v(e_0; \frac{2\pi}{m+n}, 1; p) = 1/c_v(\frac{2\pi}{m+n}, 1; 0)
\]

(≠ 0). Because c_v(\frac{2\pi}{m+ng}, 1; p) ≠ c_v(\frac{2\pi}{m+n}, 1; 0) if p ≠ 0 and v ≠ 0, the discrepancy at g = 1 reveals that the principal specialization formula does not lift to the elliptic level in the naive way:

\[
P_v(e_0; \frac{2\pi}{m+ng}, g; p) ≠ 1/c_v(\frac{2\pi}{m+ng}, g; p) \quad \text{if } p ≠ 0 \text{ and } v ≠ 0 \quad (5.20)
\]

(as an inequality of analytic functions in g > 0).

5.6 Explicit computations for \(\widehat{\mathfrak{sl}}(2)_m\)

For notational convenience let us identify \(\lambda = (\lambda_1, 0)\) and \(e = (e_1, 1)\) with \(\lambda_1\) and \(e_1\), respectively, and subsequently suppress the overall subindex 1 (thus committing a slight abuse of notation). In the \(\mathfrak{sl}(2)\) character ring \(\mathcal{R}_0^{(2)}\) (4.2a) the recurrence for the elliptic basis polynomials (4.2b) stemming from Eqs. (3.1a)–(3.1c) then becomes:

\[
e P_{\lambda}(e) = P_{\lambda+1}(e) + \psi'_{(\lambda),1}/\lambda P_{\lambda-1}(e), \quad \lambda = 0, 1, 2, \ldots, \quad (5.21a)
\]

with \(P_0(e) = 1\), \(P_{-1}(e) = 0\), and

\[
\psi'_{(\lambda),1}/\lambda = \frac{[\lambda-1+2g][\lambda]}{[\lambda-1+g][\lambda+g]}. \quad (5.21b)
\]

Upon comparing the recurrence with the expansion of the following determinant with respect to the last row/column it is clear that

\[
P_{\lambda+1}(e) = \det(J^{(\lambda)} + eI_{\lambda+1}) \quad \text{for } \lambda = 0, 1, 2, \ldots, \quad (5.22a)
\]
where $I_{\lambda+1}$ denotes the identity matrix of dimension $\lambda + 1$ and

$$J^{(\lambda)} = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
\vdots & & \ddots & & \vdots \\
0 & \psi'_{(1,1)/1} & 0 & \cdots & 1 \\
\vdots & & \ddots & & 0 \\
0 & \cdots & 0 & \psi'_{(\lambda,1)/\lambda} & 0 \\
\end{bmatrix}.$$  

(5.22b)

From now on we put $\alpha = \frac{2\pi}{m+2g}$ with $g > 0$. The $\hat{sl}(2)_m$ elliptic Verlinde algebra $\mathcal{F}_0^{(2,m)}$ (5.7) is supported on the spectrum $E_0^{(2,m)}$ (5.5) given by the roots $e_\nu = e_\nu\left(\frac{2\pi}{m+2g}, g; p\right), \nu = 0, 1, \ldots, m$ of $P_{m+1}(e)$ (5.22a), (5.22b). The recurrence (5.21a), (5.21b) implies that for $e \in E_0^{(2,m)} = \{e_0, e_1, \ldots, e_m\}$:

$$eP^{(m)}(e) = J^{(m)}P^{(m)}(e)$$

with $P^{(m)}(e) = \begin{bmatrix}
P_0(e) \\
P_1(e) \\
P_2(e) \\
\vdots \\
P_m(e)
\end{bmatrix}$,

and thus

$$P_\lambda(e)P^{(m)}(e) = P_\lambda(J^{(m)})P^{(m)}(e).$$

The structure constants for $\mathcal{F}_0^{(2,m)}$ in Eq. (5.12) are therefore given by the corresponding matrix elements:

$$N^{\kappa}_{\lambda,\mu} = N^{\kappa}_{\lambda,\mu}\left(\frac{2\pi}{m+2g}, g; p\right) = P_\lambda(J^{(m)})_{\mu,\kappa} \quad (0 \leq \lambda, \mu, \kappa \leq m).$$  

(5.23)

From the matrix recurrence

$$P_\lambda(J^{(m)}) = J^{(m)}P_{\lambda-1}(J^{(m)}) - \psi'_{(\lambda-1,1)/\lambda-1}P_{\lambda-2}(J^{(m)})$$

with the initial conditions $P_0(J^{(m)}) = I_{m+1}$ and $P_1(J^{(m)}) = J^{(m)}$, it is now immediate that the pertinent structure constants can be computed inductively in $\lambda$ by means of the recurrence

$$N^{\kappa}_{\lambda,\mu} = N^{\kappa}_{\lambda-1,\mu+1} + \psi'_{(\mu,1)/\mu}N^{\kappa}_{\lambda-1,\mu-1} - \psi'_{(\lambda-1,1)/\lambda-1}N^{\kappa}_{\lambda-2,\mu}$$  

(5.24a)

with the initial conditions

$$N^{\kappa}_{1,\mu} = \begin{cases} 
1, & \text{if } \mu = \kappa - 1 \\
\psi'_{(\mu,1)/\mu}, & \text{if } \mu = \kappa + 1 \\
0, & \text{otherwise}
\end{cases} \quad \text{and} \quad N^{\kappa}_{0,\mu} = \begin{cases} 
1, & \text{if } \mu = \kappa \\
0, & \text{otherwise}
\end{cases}.$$  

(5.24b)
Remark 5.12 For $n = 2$ it is readily inferred from Theorem 5.8 that the structure constants $N_{\lambda,\mu}^\kappa \left( \frac{2\pi}{m+2g}, g; p \right)$, $0 \leq \lambda, \mu, \kappa \leq m$ of the elliptic Verlinde algebra $\mathcal{F}_{0}^{(2,m)}$ vanish unless

$$|\lambda - \mu| \leq \kappa \leq \min(\lambda + \mu, 2m - \lambda - \mu) \quad \text{with} \quad \lambda + \mu + \kappa \text{ even.}$$

This confirms that the structure constant in question does not vanish at the elliptic level as an analytic function of $g > 0$ if and only if the pertinent $\widehat{sl}(2)_m$ Wess–Zumino–Witten fusion coefficient at $g = 1$ is positive (cf. e.g. [11, Eq. (16.48)]).

Remark 5.13 It is well-known that Macdonald’s duality symmetry [33, Chapter VI, (6,6)] states that the $S$-matrix for refined Chern-Simons knot invariants in Eq. (5.19a) is symmetric:

$$S_{\lambda,\nu} \left( \frac{2\pi}{m+ng}, g; 0 \right) = S_{\nu,\lambda} \left( \frac{2\pi}{m+ng}, g; 0 \right). \quad (5.25)$$

From Eqs. (5.22a), (5.22b) one sees, moreover, that for $n = 2$ the elliptic $S$-matrix in the Verlinde formula of Theorem 5.9 is given explicitly by

$$S_{\lambda,\nu} \left( \frac{2\pi}{m+2g}, g; p \right) = \det(\mathbf{J}^{(\lambda-1)} + e_\nu \mathbf{I}_\lambda) \det(\mathbf{J}^{(\nu-1)} + e_0 \mathbf{I}_\nu)$$

(for $0 \leq \lambda, \nu \leq m$ with the convention that $\det(\mathbf{J}^{(-1)} + e_\nu \mathbf{I}_0) \equiv 1$). Numerical experiments indicate that the (duality) symmetry (5.25) will not be preserved at the elliptic level, i.e. generally

$$S_{\lambda,\nu} \left( \frac{2\pi}{m+2g}, g; p \right) \neq S_{\nu,\lambda} \left( \frac{2\pi}{m+2g}, g; p \right) \quad \text{when} \quad p \neq 0 \text{ and } g \neq 1$$

(provided $0 < \lambda, \nu < m$ with $\lambda \neq \nu$ and $\lambda + \nu \neq m$).

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