Tight Analysis of Priority Queuing Policy for Egress Traffic

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Abstract

Recently, the problems of evaluating performances of switches and routers has been formulated as online problems, and a great amount of results have been presented. In this paper, we focus on managing outgoing packets (called egress traffic) on QoS (Quality of Service) switches, and analyze the performance of one of the most fundamental scheduling policies Priority Queuing (PQ) using competitive analysis. We formulate the problem of managing egress queues as follows: An output interface is equipped with \( m \) queues, each of which has a buffer of size \( B \). The size of a packet is unit, and each buffer can store up to \( B \) packets simultaneously. Each packet is associated with one of \( m \) priority values \( \alpha_j \) \( (1 \leq j \leq m) \), where \( \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_m \), \( \alpha_1 = 1 \), and \( \alpha_m = \alpha \) and the task of an online algorithm is to select one of \( m \) queues at each scheduling step. The purpose of this problem is to maximize the sum of the values of the scheduled packets.

For any \( B \) and any \( m \), we show that the competitive ratio of PQ is exactly \( 1 + \frac{\sum_{j=1}^{m'} \alpha_j}{\sum_{j=1}^m \alpha_j} \), where \( m' \in \arg \max_{x \in \{1, m-1\}} \{ \frac{\sum_{j=1}^x \alpha_j}{\sum_{j=1}^{x+1} \alpha_j} \} \). Moreover, we show that no deterministic online algorithm can have a competitive ratio smaller than \( 1 + \frac{\alpha^4 + \alpha^2 + \alpha}{\alpha^4 + 4\alpha^2 + 3\alpha + 4\alpha + 1} \).

1 Introduction

In recent years, the Internet provides a rich variety of applications, such as teleconferencing, video streaming, IP telephone, mainly thanks to the rapid growth of the broadband technology. To enjoy such services, the demand for the Quality of Service (QoS) guarantee is crucial. For example, usually there is little requirement for downloading programs or picture images, whereas real-time services, such as distance meeting, require constant-rate packet transmission. One possible way of supporting QoS is differentiated services (DiffServ) [15]. In DiffServ, a traffic descriptor assigns a value to each packet according to the importance of the packet. Then, QoS switches decide the order of packets to be processed, based on the value of packets. In such a mechanism, one of the main issues in designing algorithms is how to treat packets depending on the priority in buffering or scheduling. This kind of problems was recently modeled as an online problem, and the competitive analysis of algorithms has been done.

Aiello et al. [1] was the first to attempt this study, in which they considered a model with only one First In First Out (FIFO) queue. This model mainly focuses on the buffer management issue of the input port of QoS switches: There is one FIFO queue of size \( B \), meaning that it can store...
up to $B$ packets. An input is a sequence of events. An event is either an arrival event, at which a packet with a specified priority value arrives, or a scheduling event, at which the packet at the head of the queue will be transmitted. The task of an online (buffer management) algorithm is to decide, when a packet arrives at an arrival event, whether to accept or to reject it (in order to keep a room for future packets with higher priority). The purpose of the problem is to maximize the sum of the values of the transmitted packets. Aiello et al. analyzed the competitiveness of the Greedy Policy, the Round Robin Policy, the Fixed Partition Policy, etc.

After the publication of this seminal paper, more and more complicated models have been introduced and studied, some of which are as follows: Azar et al. [9] considered the multi-queue switch model, which formulates the buffering problem of one input port of the switch. In this problem, an input port has $N$ input buffers connected to a common output buffer. The task of an online algorithm is now not only buffer management but also scheduling. At each scheduling event, an algorithm selects one of $N$ input buffers, and the packet at the head of the selected buffer is transmitted to the inside of the switch through the output buffer. There are some formulations that models not only one port but the entire switch. For example, Kesselman et al. [27] introduced the Combined Input and Output Queue (CIOQ) switch model. In this model, a switch consists of $N$ input ports and $N$ output ports, where each port has a buffer. At an arrival phase, a packet (with the specified destination output port) arrives at an input port. The task of an online algorithm is buffer management as mentioned before. At a transmission phase, all the packets at the top of the nonempty buffers of output ports are transmitted. Hence, there is no task of an online algorithm. At a scheduling phase, packets at the top of the buffers of input ports are transmitted to the buffers of the output ports. Here, an online algorithm computes a matching between input ports and output ports. According to this matching, the packets in the input ports will be transmitted to the corresponding output ports. Kesselman et al. [30] considered the crossbar switch model, which models the scheduling phase of the CIOQ switch model more in detail. In this model, there is also a buffer for each pair of an input port and an output port. Thus, there arises another buffer management problem at scheduling phases.

In some real implementation, additional buffers are equipped with each output port of the switch for supporting QoS to control the outgoing packets (called egress traffic). Suppose that there are $m$ priority values of packets $\alpha_1, \alpha_2, \ldots, \alpha_m$ such that $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_m$. Then, $m$ FIFO queues $Q^{(1)}, Q^{(2)}, \ldots, Q^{(m)}$ are introduced for each output port, and a packet with the value $\alpha_i$ arriving at this output port is stored in the queue $Q^{(i)}$. Usually, this buffering policy is greedy, namely, when a packet arrives, it is rejected if the corresponding queue is full, and accepted otherwise. The task of an algorithm is to decide which queue to transmit a packet at each scheduling event.

Several practical algorithms, such as Priority Queuing ($PQ$), Weighted Round-Robin ($WRR$) [23], and Weighted Fair Queuing ($WFQ$) [19], are currently implemented in network switches. $PQ$ is the most fundamental algorithm, which selects the highest priority non-empty queue. This policy is implemented in many switches by default. In the $WRR$ algorithm, queues are selected according to the round robin policy based on the weight of packets corresponding to queues, i.e., the rate of selecting $Q^{(i)}$ in one round is proportional to $\alpha_i$ for each $i$. This algorithm is implemented in Cisco’s Catalyst 2950 [17] and so on. In the $WFQ$ algorithm, length of packets, as well as the priority values, are taken into consideration so that shorter packets are more likely to be scheduled. This algorithm is implemented in Cisco’s Catalyst 6500 [18] and so on.

In spite of intensive studies on online buffer management and scheduling algorithms, to the best of our knowledge, there have been no research on the egress traffic control, which we focus on in this paper.
Our Results. In this paper, we formulate this problem as an online problem, and perform competitive analysis for \( PQ \). \( PQ \) is trivial to implement, and has a lower computational load than the other policies, such as \( WRR \) and \( WFQ \). Hence, it is meaningful to analyze the performance of \( PQ \). For any \( B \), we show that the competitive ratio of \( PQ \) is exactly \( 1 + \frac{\sum_{j=1}^{m} \alpha_{j}}{\sum_{j=1}^{m+1} \alpha_{j}} \), where \( m' \in \arg\max_{x \in [1,m-1]} \{ \frac{\sum_{j=1}^{x} \alpha_{j}}{\sum_{j=1}^{x+1} \alpha_{j}} \} \). Moreover, we present a lower bound of \( 1 + \frac{\alpha_{3}^{2} \alpha_{2}^{2} + \alpha_{2} \alpha_{3}}{\alpha_{4} + 4 \alpha_{3} + 3 \alpha_{2} + \alpha_{1} + 1} \) on the competitive ratio of any deterministic algorithm.

Independently of our work, Al-Bawani and Souza [2] and Itoh and Yoshimoto [22] have recently considered much the same model. (\( PQ \) is called the greedy algorithm in their papers. They consider the case where \( 0 < \alpha_{1} < \alpha_{2} < \cdots < \alpha_{m} \). Also, they suppose that for any \( j \in [1,m] \), the \( j \)-th queue can store at most \( B_{j} \in [1,B] \) packets at a time. In practical switches, the sizes of any two egress queues attached to the same output port are generally equivalent. Thus, we assume that the size of each queue is \( B \). However, if we consider the setting where for any \( j \), the size of the \( j \)-th queue is \( B_{j} \), we can show that the competitive ratio of \( PQ \) is exactly \( 1 + \frac{\sum_{j=1}^{m'} \alpha_{j} B_{j+1}}{\sum_{j=1}^{m+1} \alpha_{j} B_{j}} \), where \( m' \in \arg\max_{x \in [1,m-1]} \{ \frac{\sum_{j=1}^{x} \alpha_{j} B_{j+1}}{\sum_{j=1}^{x+1} \alpha_{j} B_{j}} \} \). Our analysis in this paper does not depend on the numbers of packets stored in buffers. Instead it depends on whether buffers are full of packets.) Al-Bawani and Souza [2] showed that the competitive ratio of \( PQ \) is at most \( 2 \) for any \( m \) and \( B \). When \( m = 2 \), they presented an upper bound of \( 1 + \frac{1}{\alpha} \) on the competitive ratio of \( PQ \) for any \( B \). In addition, when \( B = 1 \), they established a lower bound of \( 2 - \frac{\alpha}{m} \) on the competitive ratio of any deterministic algorithm for any \( m \). Itoh and Yoshimoto [22] showed that the competitive ratio of \( PQ \) is at most \( 1 + \max_{i \in [1,m-1]} \frac{\alpha_{i} + \sum_{j=1}^{i} 2^{j-1} \alpha_{i-j}}{\alpha_{i+1} + \sum_{j=1}^{i} 2^{j-1} \alpha_{i-j}} \) for any \( m \) and \( B \). Moreover, using the technique in [9], it is easy to show that the competitive ratio of any greedy algorithm is at most \( 2 \). Compared with our upper bound, \( 2 > 1 + \max_{i \in [1,m-1]} \frac{\alpha_{i} + \sum_{j=1}^{i} 2^{j-1} \alpha_{i-j}}{\alpha_{i+1} + \sum_{j=1}^{i} 2^{j-1} \alpha_{i-j}} > 1 + \frac{\sum_{j=1}^{m'} \alpha_{j}}{\sum_{j=1}^{m+1} \alpha_{j}} \) by the inequality in page 17 of [22].

Related Work. As mentioned earlier, there are a lot of studies concentrating on evaluating performances of functions of switches and routers, such as queue management and packet scheduling. The most basic one is the model consisting of single FIFO queue by Aiello et al. [1] mentioned above. In their model, each packet can take one of two values \( 1 \) or \( \alpha (> 1) \). Andelman et al. [7] generalized the values of packets to any value between \( 1 \) and \( \alpha \). Another generalization is to allow preemption, namely, one may drop a packet that is already stored in a queue. Results of the competitiveness on this model are given in [1, 24, 37, 26, 7, 6, 5, 20].

The multi-queue switch model [9, 11, 34] consists of \( m \) FIFO queues. In this model, the task of an algorithm is to manage its buffers and to schedule packets. The problem of designing only a scheduling algorithm in multi-queue switches is considered in [4, 8, 13, 33, 14]. Moreover, Albers and Jacobs [3] performed an experimental study for the first time on several online scheduling algorithms for this model. Also, the overall performance of several switches, such as shared-memory switches [21, 25, 32], CIOQ switches [27, 10, 31, 28], and crossbar switches [29, 30], are extensively studied.

Fleischer and Koga [35] and Bar-Noy et al. [12] studied the online problem of minimizing the length of the longest queue in a switch, in which the size of each queue is unbounded. In [35] and [12], they showed that the competitive ratio of any online algorithm is at least \( \Omega(\log m) \), where \( m \) is the number of queues in a switch. Fleischer and Koga [35] presented a lower bound of \( \Omega(m) \) for the Round Robin policy. In addition, in [35] and [12], the competitive ratio of a greedy algorithm called Longest Queue First is at most \( O(\log m) \).
2 Model Description

In this section, we formally define the problem studied in this paper. Our model consists of $m$ queues, each with a buffer of size $B$. The size of a packet is unit, which means that each buffer can store up to $B$ packets simultaneously. Each packet is associated with one of $m$ values $\alpha_i$ ($1 \leq i \leq m$), which represents the priority of this packet where a packet with larger value is of higher priority. Without loss of generality, we assume that $\alpha_1 = 1$, $\alpha_m = \alpha$, and $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_m$. The $i$th queue is denoted $Q^{(i)}$ and is also associated with its priority value $\alpha_i$. An arriving packet with the value $\alpha_i$ is stored in $Q^{(i)}$. We assign index numbers 1 through $B$ to each position of a queue from the head to the tail in increasing order. The $j$th position of $Q^{(i)}$ is called the $j$th cell.

An input for this model is a sequence of events. Each event is an arrival event or a scheduling event. At an arrival event, a packet $p$ arrives at one of $m$ queues, and the packet is accepted to the buffer when the corresponding queue has free space. Otherwise, it is rejected. If a packet is accepted, it is stored at the tail of the corresponding queue. At a scheduling event, an online algorithm selects one non-empty queue and transmits the packet at the head of the selected queue.

The gain of an algorithm is the sum of the values of transmitted packets. Our goal is to maximize it. The gain of an algorithm $\text{ALG}$ for an input $\sigma$ is denoted by $V_{\text{ALG}}(\sigma)$. If $V_{\text{ALG}}(\sigma) \geq V_{\text{OPT}}(\sigma)/c$ for an arbitrary input $\sigma$, we say that $\text{ALG}$ is $c$-competitive, where $\text{OPT}$ is an optimal offline algorithm for $\sigma$.

We make some assumptions that are well-known to have no effect on the analysis of the competitive ratio. For a lower bound analysis, we consider only algorithms which transmit a packet at a scheduling event whenever their buffers are not empty. (Such algorithms are called work-conserving. See e.g. [9]) For ease of presentation, an event time denotes a time when an event happens, and a non-event time denotes a time when an event does not happen.

3 Analysis of Priority Queuing

3.1 Priority Queuing

$\text{PQ}$ is a greedy algorithm. At a scheduling event, $\text{PQ}$ selects the non-empty queue with the largest index. For analysis, we assume that $\text{OPT}$ does not reject an arriving packet. This assumption does not affect the analysis of the competitive ratio. (See Lemma A.1 in Appendix A.)

3.2 Overview of the Analysis

We define an extra packet as a packet which is accepted by $\text{OPT}$ but rejected by $\text{PQ}$. In the following analysis, we evaluate the sum of the values of extra packets to obtain the competitive ratio of $\text{PQ}$. We introduce some notation for our analysis. For any input $\sigma$, $k_{j}(\sigma)$ denotes the number of extra packets arriving at $Q^{(j)}$ when treating $\sigma$. We call a queue at which at least one extra packet arrives a good queue when treating $\sigma$. $n(\sigma)$ denotes the number of good queues for $\sigma$. Moreover, for any input $\sigma$ and any $i(\in [1, n(\sigma)])$, $q_{i}(\sigma)$ denotes the good queue with the $i$th minimum index. That is, $1 \leq q_{1}(\sigma) < q_{2}(\sigma) < \cdots < q_{n(\sigma)}(\sigma) \leq m$. Also, we define $q_{n(\sigma)+1}(\sigma) = m$. In addition, for any input $\sigma$, $s_{j}(\sigma)$ denotes the number of packets which $\text{PQ}$ transmits from $Q^{(j)}$. We drop the input $\sigma$ from the notation when it is clear. Then, $V_{\text{PQ}}(\sigma) = \sum_{j=1}^{m} \alpha_{j}s_{j}$, and $V_{\text{OPT}}(\sigma) = V_{\text{PQ}}(\sigma) + \sum_{i=1}^{m} \alpha_{q_{i}}k_{q_{i}}$.

First, we show that $k_{n} = 0$ in Lemma 3.2. That is, $q_{n} + 1 \leq m$ holds. Furthermore, we define the set $S^{*}$ of the inputs $\sigma'$ satisfying the following five conditions: (i) for any $i(\in [1, n(\sigma') - 1])$,
\( q_i(\sigma') + 1 = q_{i+1}(\sigma') \), (ii) for any \( i \in [1, n(\sigma')] \), \( k_{q_i(\sigma')(\sigma')} = B \), (iii) for any \( j \in [q_1(\sigma'), q_n(\sigma')(\sigma') + 1] \), \( s_{j}(\sigma') = B \), (iv) for any \( j \in [1, q_1(\sigma') - 1] \), \( s_{j}(\sigma') = 0 \) if \( q_1(\sigma') - 1 \geq 1 \), and (v) for any \( j \in [q_n(\sigma')(\sigma') + 2, m] \), \( s_{j}(\sigma') = 0 \) if \( q_n(\sigma')(\sigma') + 2 \leq m \). Then, in Lemma 3.10, we show that there exists an input \( \sigma' \in S^* \) such that \( \max_{\sigma'} \left\{ \frac{V_{OPT}(\sigma')}{V_{PQ}(\sigma')} \right\} = \frac{V_{OPT}(\sigma')}{V_{PQ}(\sigma')} \). For ease of presentation, we write \( s_i(\sigma^*), n(\sigma^*), q_i(\sigma^*) \) and \( k_i(\sigma^*) \) as \( s_i^*, n^*, q_i^* \) and \( k_i^* \), respectively. Thus, \( \frac{V_{OPT}(\sigma^*)}{V_{PQ}(\sigma^*)} = \frac{V_{PQ}(\sigma^*) + \sum_{j=1}^{m} \alpha_j q_j^*}{V_{PQ}(\sigma^*)} = 1 + \frac{B \sum_{j=1}^{m} \alpha_j}{B \sum_{j=1}^{m+1} \alpha_j} \). Therefore, we have the following theorem:

**Theorem 3.1** The competitive ratio of \( PQ \) is exactly \( 1 + \frac{\sum_{j=1}^{m} \alpha_j}{\sum_{j=1}^{m+1} \alpha_j} \), where \( m' \in \arg\max_{\sigma^* \in [1, m]} \left\{ \frac{\sum_{j=1}^{x} \alpha_j}{\sum_{j=1}^{x+1} \alpha_j} \right\} \).

### 3.3 Analysis

We give some definitions. For any non-event time \( t \), suppose that the \( j \)th cell in \( Q^{(i)} \) of \( PQ \) holds a packet at \( t \) but the \( j \)th cell in \( Q^{(i)} \) of \( OPT \) does not at \( t \). Then, we call \( c \) a free cell at \( t \). For any non-event time \( t \), let \( h^{(j)}_{ALG}(t) \) denote the number of packets which an algorithm \( ALG \) stores in \( Q^{(j)} \), Note that any extra packet is accepted at a free cell. We first prove the following lemma.

**Lemma 3.2** \( k_m = 0 \).

**Proof.** By the definition of \( PQ \), \( PQ \) selects the non-empty queue with the highest priority. Thus, \( h^{(m)}_{PQ}(t) \leq h^{(m)}_{OPT}(t) \) holds at any non-event time \( t \). Therefore, there is no free cell in \( Q^{(m)} \) of \( OPT \) at any time. Since any extra packet is accepted to a free cell, \( k_m = 0 \).

We give some definitions. For any event time \( t \), \( t^- \) denotes the non-event time before \( t \) and after the previous event time. Also, \( t+ \) denotes the non-event time after \( t \) and before the next event time. For any non-event time \( t \), let \( f^{(j)}(t) \) denote the number of free cells in \( Q^{(j)} \), namely, \( f^{(j)}(t) = \max\{h^{(j)}_{PQ}(t) - h^{(j)}_{OPT}(t), 0\} \). We first verify the change in the number of free cells at each event. Note that \( OPT \) does not reject any packet by our assumption (Lemma A.1). Thus, for any non-event time \( t \), \( \sum_{j=1}^{m} h^{(j)}_{OPT}(t) > 0 \) if \( \sum_{j=1}^{m} h^{(j)}_{PQ}(t) > 0 \).

**Arrival event:** Let \( p \) be a packet arriving at \( Q^{(x)} \) at an event time \( t \).

**Case A1:** Both \( PQ \) and \( OPT \) accept \( p \), and \( h^{(x)}_{PQ}(t^-) - h^{(x)}_{OPT}(t^-) > 0 \): Since \( h^{(x)}_{PQ}(t^+) = h^{(x)}_{PQ}(t^-) + 1 \) and \( h^{(x)}_{OPT}(t^+) = h^{(x)}_{OPT}(t^-) + 1 \), \( h^{(x)}_{PQ}(t^+) - h^{(x)}_{OPT}(t^+) > 0 \). Thus, the \( (h^{(x)}_{PQ}(t^+)+1) \)st cell of \( Q^{(x)} \) becomes free in place of the \( (h^{(x)}_{OPT}(t^-)+1) \)st cell of \( Q^{(x)} \). Hence \( f^{(x)}(t^+) = f^{(x)}(t^-) \).

**Case A2:** Both \( PQ \) and \( OPT \) accept \( p \), and \( h^{(x)}_{PQ}(t^-) - h^{(x)}_{OPT}(t^-) \leq 0 \): Since \( h^{(x)}_{PQ}(t^+) = h^{(x)}_{PQ}(t^-) + 1 \) and \( h^{(x)}_{OPT}(t^+) = h^{(x)}_{OPT}(t^-) + 1 \), \( h^{(x)}_{PQ}(t^+) - h^{(x)}_{OPT}(t^+) \leq 0 \). Since the states of all the free cells do not change before and after \( t \), \( f^{(x)}(t^+) = f^{(x)}(t^-) \).

**Case A3:** \( PQ \) rejects \( p \), but \( OPT \) accepts \( p \): \( p \) is an extra packet since only \( OPT \) accepts \( p \). \( p \) is accepted into the \( (h^{(x)}_{OPT}(t^-)+1) \)st cell, which is free at \( t^- \), of \( Q^{(x)} \). \( h^{(x)}_{PQ}(t^+) = h^{(x)}_{PQ}(t^-) = B \), and \( h^{(x)}_{OPT}(t^+) = h^{(x)}_{OPT}(t^-) + 1 \), which means that \( f^{(x)}(t^+) = f^{(x)}(t^-) - 1 \).

**Scheduling event:**

If \( PQ \) (\( OPT \), respectively) has at least one non-empty queue, suppose that \( PQ \) (\( OPT \), respectively) transmits a packet from \( Q^{(y)} \) (\( Q^{(z)} \), respectively) at \( t \).
Case S: $\sum_{j=1}^{m} h^{(j)}_{PQ}(t) > 0$ and $\sum_{j=1}^{m} h^{(j)}_{OPT}(t) > 0$:

Case S1: $y = z$:

Case S1.1: $h^{(y)}_{PQ}(t) - h^{(y)}_{OPT}(t) > 0$:

Since $h^{(y)}_{PQ}(t) = h^{(y)}_{OPT}(t) - 1$ and $h^{(y)}_{OPT}(t) = h^{(y)}_{OPT}(t) - 1$, $h^{(y)}_{PQ}(t) - h^{(y)}_{OPT}(t) > 0$ holds.

Thus, the $h^{(y)}_{OPT}(t)$-th cell of $Q^{(y)}$ becomes free in place of the $h^{(y)}_{PQ}(t)$-th cell of $Q^{(y)}$. Hence $f^{(y)}(t) = f^{(y)}(t)$.

Case S1.2: $h^{(y)}_{PQ}(t) - h^{(y)}_{OPT}(t) \leq 0$:

Since $h^{(y)}_{PQ}(t) = h^{(y)}_{PQ}(t) - 1$ and $h^{(y)}_{OPT}(t) = h^{(y)}_{OPT}(t) - 1$ hold, $h^{(y)}_{PQ}(t) - h^{(y)}_{OPT}(t) \leq 0$.

Thus, the states of all the free cells do not change before and after $t$.

Case S2: $y > z$:

Case S2.1: $h^{(z)}_{PQ}(t) - h^{(z)}_{OPT}(t) < 0$:

Since $h^{(z)}_{PQ}(t) = h^{(z)}_{PQ}(t)$ and $h^{(z)}_{OPT}(t) = h^{(z)}_{OPT}(t)$ hold, $h^{(z)}_{PQ}(t) \leq h^{(z)}_{OPT}(t)$. Thus, the states of all the free cells of $Q^{(z)}$ do not change before and after $t$.

Case S2.1.1: $h^{(y)}_{PQ}(t) - h^{(y)}_{OPT}(t) > 0$:

Since $h^{(y)}_{PQ}(t) = h^{(y)}_{PQ}(t) - 1$ and $h^{(y)}_{OPT}(t) = h^{(y)}_{OPT}(t)$, $f^{(y)}(t) = f^{(y)}(t) - 1$ holds.

Case S2.1.2: $h^{(y)}_{PQ}(t) - h^{(y)}_{OPT}(t) \leq 0$:

Since $h^{(y)}_{PQ}(t) = h^{(y)}_{PQ}(t) - 1$, $h^{(y)}_{OPT}(t) = h^{(y)}_{OPT}(t)$ and $h^{(y)}_{PQ}(t) < h^{(y)}_{OPT}(t)$. Hence, the states of all the free cells of $Q^{(y)}$ do not change before and after $t$.

Case S2.2: $h^{(z)}_{PQ}(t) - h^{(z)}_{OPT}(t) \geq 0$:

$h^{(z)}_{PQ}(t) = h^{(z)}_{PQ}(t)$ and $h^{(z)}_{OPT}(t) = h^{(z)}_{OPT}(t)$ - 1. Thus, the $h^{(z)}_{OPT}(t)$-th cell of $Q^{(z)}$ becomes free, which means that $f^{(z)}(t) = f^{(z)}(t) + 1$ holds.

Case S2.2.1: $h^{(y)}_{PQ}(t) - h^{(y)}_{OPT}(t) > 0$:

Since $h^{(y)}_{PQ}(t) = h^{(y)}_{PQ}(t) - 1$ and $h^{(y)}_{OPT}(t) = h^{(y)}_{OPT}(t)$, $f^{(y)}(t) = f^{(y)}(t) - 1$.

Case S2.2.2: $h^{(y)}_{PQ}(t) - h^{(y)}_{OPT}(t) \leq 0$:

Since $h^{(y)}_{PQ}(t) = h^{(y)}_{PQ}(t) - 1$ and $h^{(y)}_{OPT}(t) = h^{(y)}_{OPT}(t)$, $h^{(y)}_{PQ}(t) < h^{(y)}_{OPT}(t)$, which means that the states of all the free cells of $Q^{(y)}$ do not change before and after $t$.

Case S3: $y < z$:

Since $h^{(z)}_{PQ}(t) = h^{(z)}_{PQ}(t) = 0$ by the definition of $PQ$, no new free cell arises in $Q^{(z)}$.

Case S3.1: $h^{(y)}_{PQ}(t) - h^{(y)}_{OPT}(t) > 0$:

Since $h^{(y)}_{PQ}(t) = h^{(y)}_{PQ}(t) - 1$ and $h^{(y)}_{OPT}(t) = h^{(y)}_{OPT}(t)$, $f^{(y)}(t) = f^{(y)}(t) - 1$ holds.

Case S3.2: $h^{(y)}_{PQ}(t) - h^{(y)}_{OPT}(t) \leq 0$:

Since $h^{(y)}_{PQ}(t) = h^{(y)}_{PQ}(t) - 1$, $h^{(y)}_{OPT}(t) = h^{(y)}_{OPT}(t)$, $h^{(y)}_{PQ}(t) < h^{(y)}_{OPT}(t)$ holds. Hence, the states of all the free cells of $Q^{(y)}$ do not change before and after $t$.

Case $\bar{S}$: $\sum_{j=1}^{m} h^{(j)}_{PQ}(t) = 0$ and $\sum_{j=1}^{m} h^{(j)}_{OPT}(t) > 0$:

Since the buffer of $PQ$ is empty, there does not exist any free cell in it.

Based on the change in the state of free cells, we “match” a packet transmitted by $PQ$ with each extra packet according to the matching routine in Table 1. Using this matching, we can bound
the number of extra packets arriving at each queue.

Table 1: Matching routine

| Matching routine: | Let \( t \) be an event time. |
|-------------------|-------------------------------|
| Arrival event:    | Suppose that a packet \( p \) arrives at \( Q(x) \) at \( t \). Execute one of the following three cases at \( t \). |
| Case A1: Both \( PQ \) and \( OPT \) accept \( p \), and \( h_{PQ}^{(x)}(t-) - h_{OPT}^{(x)}(t-) > 0 \): |
|                   | Let \( c \) be \( OPT \)'s \((h_{OPT}^{(x)}(t-)+1)\)st cell of \( Q(x) \), which is free at \( t- \) but is not at \( t+ \). Let \( c' \) be \( OPT \)'s \((h_{PQ}^{(x)}(t-)+1)\)st cell which is not free at \( t- \) but is free at \( t+ \). There exists the packet \( q \) matched with \( c \) at \( t- \). (Lemma 3.3.) Change the matching partner of \( q \) from \( c \) to \( c' \). |
| Case A2: Both \( PQ \) and \( OPT \) accept \( p \), and \( h_{PQ}^{(x)}(t-) - h_{OPT}^{(x)}(t-) \leq 0 \): |
|                   | Do nothing. |
| Case A3: \( PQ \) rejects \( p \), but \( OPT \) accepts \( p \): |
|                   | Let \( c \) be \( OPT \)'s \((h_{OPT}^{(x)}(t-)+1)\)st cell of \( Q(x) \), that is, the cell to which the extra packet \( p \) is now stored. Note that \( c \) is free at \( t- \) but is not at \( t+ \). There exists the packet \( q \) matched with \( c \) at \( t- \). (Lemma 3.3.) Change the partner of \( q \) from \( c \) to \( p \). |
| Scheduling event: | If \( PQ \) (\( OPT \), respectively) has at least one non-empty queue at \( t- \), suppose that \( PQ \) (\( OPT \), respectively) transmits a packet from \( Q(y) \) (\( Q(z) \), respectively) at \( t \). Execute one of the following three cases at \( t \). |
| Case S1.1: \( \sum_{j=1}^{m} h_{PQ}^{(y)}(t-) > 0, \sum_{j=1}^{m} h_{OPT}^{(y)}(t-) > 0, y = z, \) and \( h_{PQ}^{(y)}(t-) - h_{OPT}^{(y)}(t-) > 0 \): |
|                   | Let \( c \) be \( OPT \)'s \((h_{OPT}^{(y)}(t-)+1)\)th cell of \( Q(y) \), which is free at \( t- \) but is not free at \( t+ \). Let \( c' \) be \( OPT \)'s \((h_{PQ}^{(y)}(t-)+1)\)th cell of \( Q(y) \), which is not free at \( t- \) but is free at \( t+ \). There exists the packet \( q \) matched with \( c \) at \( t- \). (Lemma 3.3.) Change the matching partner of \( q \) from \( c \) to \( c' \). |
| Case S2.2: \( \sum_{j=1}^{m} h_{PQ}^{(y)}(t-) > 0, \sum_{j=1}^{m} h_{OPT}^{(y)}(t-) > 0, y > z, \) and \( h_{PQ}^{(y)}(t-) - h_{OPT}^{(y)}(t-) \geq 0 \): |
|                   | Let \( c \) be \( OPT \)'s \((h_{OPT}^{(y)}(t-)+1)\)th cell of \( Q(y) \), which becomes free at \( t+ \). Since the packet \( p \) transmitted from \( Q(y) \) by \( PQ \) is not matched with anything (Lemma 3.3), match \( p \) with \( c \). |
| Otherwise:       | Do nothing. |

We give some definitions. For any non-event time \( t \) and any packet \( p \) which arrives before \( t \), \( g(p) \) denotes the index of the queue at which \( p \) arrives. Also, for any cell \( c \), \( g(c) \) denotes the index of the queue including \( c \). We now show the feasibility of the routine.

Lemma 3.3 For any non-event time \( t' \), any extra packet \( p \) which arrives before \( t' \) and some packet \( p' \) which \( PQ \) transmits before \( t' \) such that \( g(p) < g(p') \), \( p \) is matched with \( p' \) at \( t' \). Moreover, for any free cell \( c \) at \( t' \), and some packet \( p'' \) which \( PQ \) transmits before \( t' \) such that \( g(c) < g(p'') \), \( c \) is matched with \( p'' \) at \( t' \).

Proof. The proof is by induction on the event time. The base case is clear. Let \( t \) be any event time. We assume that the statement is true at \( t- \), and prove that it is true at \( t+ \).

First, we discuss the case where the routine executes Case A1 or Case S1.1 at \( t \). Let \( c \) be the cell which becomes free at \( t \). Also, let \( c' \) be the cell which is free at \( t- \) and not free at \( t+ \). By the induction hypothesis, a packet \( p \) which is transmitted by \( PQ \) before \( t- \) is matched with \( c' \) at \( t- \). Then, the routine unmatches \( p \), and matches \( p \) with \( c \) by the definitions of Cases A1 and S1.1.
\(g(c) = g(c')\) clearly holds. Also, since \(g(c') < g(p)\) by the induction hypothesis, the statement is true at \(t^+\).

Next, we consider the case where the routine executes Case A3 at \(t\). Let \(p'\) be the extra packet accepted by \(OPT\) at \(t\). Also, let \(c\) be the free cell into which \(OPT\) accepts \(p'\) at \(t\). By the induction hypothesis, a packet \(p\) which is transmitted by \(PQ\) before \(t^--\) is matched with \(c\) at \(t^--\). Then, by the definition of Case A3, the routine unmatches \(p\), and matches \(p\) with \(p'\). \(g(c) = g(p')\) holds by definition. In addition, \(g(c) < g(p)\) by the induction hypothesis. Thus, \(g(p') < g(p)\), which means that the statement holds at \(t^+\).

Third, we investigate the case where the routine executes Case S2.2 at \(t\). Suppose that \(PQ\) transmits a packet \(p\) at \(t\), and a new free cell \(c\) arises at \(t\). By the induction hypothesis, any \(PQ\)’s packet which is matched with a free cell or an extra packet is transmitted before \(t\). Hence, \(p\) is not matched with anything at \(t^--\). Thus, the routine can match \(p\) with \(c\) at \(t\). Moreover, \(g(c) < g(p)\) by the induction hypothesis, the statement is true at \(t^+\).

In the other cases, a new matching does not arise. Thus, the statement is clear by the induction hypothesis.

Next, we sequentially restrict the set of inputs to be analyzed in the following lemmas, which ultimately yields \(S^*\). At the same time, we evaluate the number of extra packets arriving at each good queue.

**Lemma 3.4** Let \(\sigma\) be an input such that for some \(u(\in [1, m])\), \(s_u(\sigma) > B\). Then, there exists an input \(\hat{\sigma}\) such that for each \(j(\in [1, m])\), \(s_j(\hat{\sigma}) \leq B\), and \(\frac{V_{OPT}(\sigma)}{V_{PQ}(\sigma)} < \frac{V_{OPT}(\hat{\sigma})}{V_{PQ}(\sigma)}\).

**Proof.** Let \(z\) be the minimum index such that \(s_z(\sigma) > B\). Then, there exist the three event times \(t_1, t_2(> t_1)\) and \(t_3(> t_2)\) satisfying the following five conditions: (i) \(t_2\) is the arrival event time when the \((B + 1)st\) packet which \(PQ\) accepts at \(Q(z)\) arrives, (ii) \(t_1\) is an event time when \(OPT\) transmits a packet from \(Q(z)\), (Since \(OPT\) accepts any arriving packet by our assumption, \(OPT\) certainly transmits at least one packet from \(Q(z)\) before \(t_2\).) (iii) \(OPT\) does not transmit any packet from \(Q(z)\) during time \((t_1, t_2)\), (iv) \(t_3\) is an event time when \(PQ\) transmits a packet from \(Q(z)\), and (v) \(PQ\) does not transmit any packet from \(Q(z)\) during time \((t_2, t_3)\). We construct \(\sigma'\) by removing the events at \(t_1\) and \(t_2\) from \(\sigma\). Suppose that \(\frac{V_{OPT}(\sigma)}{V_{PQ}(\sigma)} < \frac{V_{OPT}(\sigma')}{V_{PQ}(\sigma')}\). If we remove some events corresponding to \(Q(j)\) in ascending order of index \(j\) in \(\{x | s_x(\sigma) > B\}\), then we can construct an input \(\hat{\sigma}\) such that for each \(j(\in [1, m])\), \(s_j(\hat{\sigma}) \leq B\), and \(\frac{V_{OPT}(\sigma)}{V_{PQ}(\sigma)} < \frac{V_{OPT}(\hat{\sigma})}{V_{PQ}(\sigma)}\), which completes the proof. Hence, we next show that \(\frac{V_{OPT}(\sigma)}{V_{PQ}(\sigma)} < \frac{V_{OPT}(\sigma')}{V_{PQ}(\sigma')}\).

First, we discuss the value gained by \(OPT\) for \(\sigma'\). Let \(ALG\) be the offline algorithm for \(\sigma'\) such that for each scheduling event \(e\) in \(\sigma'\), \(ALG\) selects the queue which \(OPT\) selects at \(e\) in \(\sigma\). We consider the number of packets in \(ALG\)’s buffer during time \((t_1, t_3)\) for \(\sigma'\). For any non-event time \(t(\in (t_1, t_3))\), and any \(y(\neq z)\), \(h^{(y)}_{ALG}(t) = h^{(y)}_{OPT}(t)\). For any non-event time \(t(\in (t_1, t_2))\), \(h^{(z)}_{ALG}(t) = h^{(z)}_{OPT}(t) + 1\). Also, for any non-event time \(t(\in (t_2, t_3))\), \(h^{(z)}_{ALG}(t) = h^{(z)}_{OPT}(t)\). By the above argument, \(V_{OPT}(\sigma') \geq V_{ALG}(\sigma') = V_{OPT}(\sigma) - \alpha_z\).

Next, we evaluate the value gained by \(PQ\) for \(\sigma'\). For notational simplicity, we describe \(PQ\) for \(\sigma'\) as \(PQ'\). First, we consider the case where there does not exist any packet which \(PQ\) accepts but \(PQ'\) rejects during time \((t_1, t_3)\). \(V_{PQ}(\sigma') = V_{PQ}(\sigma) - \alpha_z\) clearly holds. Now, we discuss the numbers of packets which \(PQ\) and \(PQ'\) store in their buffers after \(t_1\). For any non-event time \(t(\in (t_1, t_2))\), \(\sum_{j=1}^{m} h^{(j)}_{PQ'}(t) = \sum_{j=1}^{m} h^{(j)}_{PQ}(t) + 1\). For any non-event time \(t\), we define
For any \( w(t) = \arg \max \{ j \mid h_{PQ}^{(j)}(t) > 0 \} \). Specifically, \( h_{PQ'}^{(w(t))}(t) = h_{PQ}^{(w(t))}(t) + 1 \). (We call this fact the property (a).) Moreover, for any non-event time \( t(\in (t_2, t_3)) \), \( \sum_{j=1}^{m} h_{PQ}^{(j)}(t) = \sum_{j=1}^{m} h_{PQ'}^{(j)}(t) \). However, if \( w(t) > z \), then \( h_{PQ}^{(w(t))}(t) = h_{PQ}^{(w(t))}(t) + 1 \). Also, \( h_{PQ}^{(z)}(t) = h_{PQ}^{(z)}(t) - 1 \). If \( w(t) = z \), then for any \( j(\in [1, m]) \), \( h_{PQ}^{(j)}(t) = h_{PQ}^{(j)}(t) \). For any non-event time \( t(> t_3) \) and any \( j(\in [1, m]) \), \( h_{PQ}^{(j)}(t) = h_{PQ}^{(j)}(t) \).

Secondly, we consider the case where there exists at least one packet which \( PQ \) accepts but \( PQ' \) rejects arrives. Let \( t' \) be the first event time when the packet \( p \) which \( PQ \) accepts but \( PQ' \) rejects arrvies. Then, suppose that \( t' \in (t_1, t_2) \). By the definition of \( z \), \( p \) arrives at \( Q(z') \) such that \( z' \geq z \). By the property (a), for \( j(\in [1, m]) \), \( h_{PQ'}^{(j)}(t') = h_{PQ}^{(j)}(t') \). Thus, packets accepted by \( PQ \) during time \( t'(t_2, t_3) \) can be accepted by \( PQ' \). Only \( PQ \) accepts the packet arriving at \( Q(z) \) at \( t_2 \) by the definition of \( \sigma' \). Hence, \( h_{PQ}^{(j)}(t_2) = h_{PQ}^{(j)}(t_2) + 1 \), and for any \( j(\in [1, m]) \) such that \( j \neq z \), \( h_{PQ}^{(j)}(t_2) = h_{PQ}^{(j)}(t_2) \). (We call this fact the property (b).) If all the packets which \( PQ \) accepts after \( t_2 \) are the same as those accepted by \( PQ' \) after \( t_2 \), \( V_{PQ}^{(\sigma')} = V_{PQ}^{(\sigma)} - \alpha_z - \alpha_{z'} \). Then, we consider the case where there exists at least one packet \( p' \) which \( PQ \) rejects but \( PQ' \) accepts after \( t_2 \). By the same argument, \( V_{PQ}^{(\sigma')} = V_{PQ}^{(\sigma)} - \alpha_z - \alpha_z \leq V_{PQ}^{(\sigma)} - \alpha_z \).

Finally, we consider the case where \( t' \in (t_2, t_3) \). By the same argument as the case where \( t' \in (t_1, t_2) \), we can prove this case. Specifically, the number of packets which \( PQ \) rejects but \( PQ' \) accepts after \( t' \) is exactly one. This packet arrives at \( Q(z'') \), where some \( z'' \leq z \). Therefore, \( V_{PQ}^{(\sigma')} = V_{PQ}^{(\sigma)} - \alpha_z - \alpha_z \leq V_{PQ}^{(\sigma)} - \alpha_z \).

By the above argument, \( \frac{V_{OP}^{(\sigma')}}{V_{PQ}^{(\sigma)}} \geq \frac{V_{OP}^{(\sigma)}}{V_{PQ}^{(\sigma)}} \geq \frac{V_{OP}^{(\sigma)} - \alpha_z}{V_{PQ}^{(\sigma)} - \alpha_z} > \frac{V_{OP}^{(\sigma)}}{V_{PQ}^{(\sigma)}} \).

We introduce the notation. \( S \) denotes the set of inputs \( \sigma \) such that for any \( j(\in [1, m]) \), \( s_j(\sigma) \leq B \). In what follows, we analyze only inputs in \( S \) by Lemma 3.4.

**Lemma 3.5** For any \( x(\in [1, n]) \), \( \sum_{i=x}^{n} k_i \leq \sum_{j=q_x+1}^{m} s_j \).

*Proof.* By Lemma 3.3, each extra packet \( p \) is matched with a packet \( p' \) which is transmitted by \( PQ \) by the end of the input. In addition, for any non-event time \( t \), \( g(p) < g(p') \) if an extra packet \( p \) is matched with a packet \( p' \) of \( PQ \) at \( t \). Thus, \( k_{q_x} \leq \sum_{j=q_x+1}^{m} s_j \), \( k_{q_n-1} \leq (\sum_{j=q_n-1}^{m} s_j) - k_{q_n} \), \( \cdots \), and \( k_{q_1} \leq (\sum_{j=q_1+1}^{m} s_j) - \sum_{j=2}^{n} k_j \). Therefore, for any \( x(\in [1, n]) \), \( \sum_{i=x}^{n} k_i \leq \sum_{j=q_x+1}^{m} s_j \). 

Throughout the proofs in all the following lemmas, we drop \( \sigma \) from \( s_j(\sigma), n(\sigma), q_i(\sigma) \) and \( k_j(\sigma) \).

**Lemma 3.6** If any input \( \sigma \in S \), there exists an input \( \hat{\sigma}(\in S) \) such that (i) for any \( \hat{i}(\in [1, n(\hat{\sigma})) \), \( k_{q_i}(\hat{\sigma}) = \sum_{j=q_i(\hat{\sigma})+1}^{n(\hat{\sigma})} s_j(\hat{\sigma}) \), (ii) for any \( j(\in [1, q_i(\hat{\sigma}) - 1]) \), \( s_j(\hat{\sigma}) = 0 \) if \( q_i(\hat{\sigma}) - 1 \geq 1 \), and (iii) \( \frac{V_{OP}^{(\sigma)}}{V_{PQ}^{(\sigma)}} \leq \frac{V_{OP}^{(\hat{\sigma})}}{V_{PQ}^{(\hat{\sigma})}} \).

*Proof.* For any input \( \sigma \in S \), we construct \( \sigma' \) from \( \sigma \) according to the following steps. First, for each \( j(\in [q_i, m]) \), \( s_j \) events at which \( s_j \) packets at \( Q^{(j)} \) arrive occur during time \((0, 1) \). Since
\( s_j \leq B \) by the definition of \( S \), \( PQ \) accepts all the packets which arrive at these events. \( \sum_{i=1}^{n} k_{q_i} \) packets arrive after time 1, and \( PQ \) cannot accept them. Specifically, for any \( i \in [1, n] \), we define \( a_i = \sum_{j=q_{i-1}+1}^{q_{i+1}-1} s_j \) and \( a_0 = 0 \). Then, for each \( x \in [0, n-1] \), a scheduling event occurs at each integer time \( t = (\sum_{j=0}^{x} a_j) + 1, \ldots, \sum_{j=0}^{x+1} a_j \), and an arrival event where a packet arrives at \( Q^{(q_{n-1})} \) occurs at each time \( t + \frac{1}{2} \). After time \( (\sum_{j=0}^{n} a_j) + 1 \), sufficient scheduling events to transmit all the arriving packets occur.

For these scheduling events, \( PQ \) transmits a packet from \( Q^{(j)} \) at \( t \) such that \( j \in [q_{n-1} + 1, q_{n-1} + n] \). Also, let \( ALG \) be an offline algorithm. \( ALG \) transmits a packet from \( Q^{(q_{n-1})} \) at \( t \). Since for any \( i \in [1, n] \), at least one extra packet arrives at \( Q^{(q_{i})} \), \( s_{q_i} = B \) holds. Hence, since for any \( i \in [1, n] \), \( h_{PQ}^{(q_{i})}(1-) = B \), \( PQ \) cannot accept the packet which arrives at each \( t + \frac{1}{2} \). However, \( ALG \) can accept all these packets, which means that \( ALG \) is an optimal offline algorithm. Then, \( n(\sigma') = n \), and for any \( i \in [1, n] \), \( q_i(\sigma') = q_i \).

By the above argument, \( V_{PQ}(\sigma') = V_{PQ}(\sigma) - \sum_{j=1}^{q_{n-1}} \alpha_j s_j \). Furthermore, for each \( i \in [1, n] \), \( k_{q_i}(\sigma') = \sum_{j=q_i+1}^{q_{i+1}} s_j \). By these equalities, \( V_{ALG}(\sigma') = V_{PQ}(\sigma') + \sum_{i=1}^{n} \alpha_i k_{q_i}(\sigma') = V_{PQ}(\sigma) + \sum_{i=1}^{n} \alpha_i k_{q_i} \geq V_{PQ}(\sigma) + \sum_{i=1}^{n} \alpha_i k_{q_i} = V_{OPT}(\sigma) - \sum_{j=1}^{q_{n-1}} \alpha_j s_j \).

Therefore, \( \frac{V_{OPT}(\sigma)}{V_{PQ}(\sigma)} \geq \frac{V_{OPT}(\sigma) - \sum_{j=1}^{q_{n-1}} \alpha_j s_j}{V_{PQ}(\sigma) - \sum_{j=1}^{n} \alpha_j s_j} \geq \frac{V_{OPT}(\sigma)}{V_{PQ}(\sigma)} \). Moreover, by the definition of \( \sigma' \), \( \sigma' \) satisfies the condition (ii) in the statement, and is included in \( S \).

We introduce the notation. \( S_2 \) denotes the set of inputs \( \sigma \in (S) \) satisfying the following conditions: (i) for any \( i \in [1, n] \), \( k_{q_i} = \sum_{j=q_i+1}^{q_{i+1}} s_j \), (ii) for any \( j \in [q_1, m] \), \( s_j \leq B \), and (iii) for any \( i \in [1, q_1 - 1] \), \( s_j = 0 \) if \( q_i - 1 \geq 1 \).

**Lemma 3.7** Let \( \sigma \in (S_2) \) be an input such that for some \( z \leq n(\sigma) - 1 \), \( q_z(\sigma) + 1 < q_{z+1}(\sigma) \). Then, there exists an input \( \sigma' \in (S_2) \) such that (i) for each \( i \in [1, n(\sigma) - 1] \), \( q_i(\sigma') = q_i(\sigma) + 1 \) and \( k_{q_i(\sigma')}(\sigma') = B \), and (ii) \( \frac{V_{OPT}(\sigma)}{V_{PQ}(\sigma)} \leq \frac{V_{OPT}(\sigma')}{V_{PQ}(\sigma')} \).

**Proof.** For any \( j \in [1, m] \) such that \( j \neq q_{z+1} - 1 \), we define \( s_{j'} = s_j \). Also, we define \( s_{q_{z+1} - 1} = B \).

We construct \( \sigma' \) from \( \sigma \) in the following way. This approach is similar to those in the proof of Lemma 3.6. First, for each \( j \in [q_1, m] \), \( s_{j'} \) events at which \( s_{j'} \) packets at \( Q^{(j)} \) arrive occur during time \( (0, 1) \). Since \( s_{j'} \leq B \) by definition, \( PQ \) accepts all these packets. In addition, for any \( i \in [1, z] \), we define \( q_{i'} = q_i \). We define \( q_{z+1} = q_{z+1} - 1 \). For any \( i \in [1, n + 1] \), we define \( q_i = q_i(\sigma) + 1 \). Moreover, for any \( i \in [1, n + 1] \), we define \( a_i = \sum_{j=q_i+1}^{q_{i+1}} s_j \) and \( a_0 = 0 \). For any \( x \in [0, n] \), a scheduling event occurs at each integer time \( t = (\sum_{j=0}^{x} a_j) + 1, \ldots, \sum_{j=0}^{x+1} a_j \). Also, an arrival event where a packet arrives at \( Q^{(q_{n-1}+1)} \) occurs at each time \( t + \frac{1}{2} \). After time \( (\sum_{j=0}^{n+1} a_j) + 1 \), sufficient scheduling events to transmit all the arriving packets occur.

Then, \( PQ \) transmits a packet from \( Q^{(j)} \) at \( t \) such that \( j \in [q_{n-1}+1, q_{n-1}+2] \). Let \( ALG \) be an offline algorithm which transmits a packet from \( Q^{(q_{n-1}+1)} \) at \( t \). By the definition of \( q_{i'} \), for any \( i \in [1, n+1] \), \( h_{PQ}^{(q_i)}(1-) = B \). Thus, \( PQ \) cannot accept any packet arriving at \( t + \frac{1}{2} \), but \( ALG \) can accept all the arriving packets. That is to say, \( ALG \) is optimal.

By the above argument, \( V_{PQ}(\sigma') = V_{PQ}(\sigma) + \alpha_{q_{n-1}+1}(B - s_{q_{n-1}+1}) \). Furthermore, for any \( i \in [1, z-1] \), \( k_{q_i}(\sigma') = k_{q_i} \). Also, \( k_{q_{n-1}}(\sigma') = k_{q_{n-1}} - s_{q_{n-1}+1}, k_{q_{n-1}+1}(\sigma') = B \), and for any \( i \in [z+1, n] \),
Lemma 3.6 and 3.7. First, for each integer time \( t \leq 0 \), \( q_1(t') = q'_1 \). Moreover, \( V_{OPT}(\sigma') = V_{ALG}(\sigma') = V_{PQ}(\sigma') + \sum_{i=1}^{n(\sigma')} \alpha_{q_i(\sigma')}k_{q_i(\sigma')}(\sigma') \).

By the above equalities, \( \sum_{i=1}^{n(\sigma')} \alpha_{q_i(\sigma')}k_{q_i(\sigma')}(\sigma') = (\sum_{i=1}^{n(\sigma')} \alpha_{q_i(\sigma')}k_{q_i(\sigma')}) + \alpha_{q_{i+1}-1}B \geq (\sum_{i=1}^{n(\sigma')} \alpha_{q_i(\sigma')}k_{q_i(\sigma')}) + \alpha_{q_{i+1}-1}(B-s_{q_{i+1}-1}) \). Hence, \( \sum_{i=1}^{n(\sigma')} \alpha_{q_i(\sigma')}k_{q_i(\sigma')}(\sigma') \geq (\sum_{i=1}^{n(\sigma')} \alpha_{q_i(\sigma')}k_{q_i(\sigma')}) + \alpha_{q_{i+1}-1}(B-s_{q_{i+1}-1}) \).

By the definition of \( \sigma' \), \( \sigma' \in S_2 \) holds. By the above argument, for any \( \sigma' \) such that \( q_{i+1} < q_{i+1} \), we recursively construct an input in the above way, and then we can obtain an input satisfying the lemma.

We define the set \( S_3 \) of inputs. \( S_3 \) denotes the set of inputs \( \sigma \in S_2 \) such that (i) for each \( i \in [1, n-1] \), \( q_i + 1 = q_{i+1} \), (ii) for each \( i \in [1, n-1] \), \( k_{q_i} = B \), (iii) for each \( j \in [q_1, q_n] \), \( s_j = B \), and (iv) for any \( j \in [1, q_1-1] \), \( s_j = 0 \) if \( q_1 - 1 \geq 1 \), (By Lemma 3.2, \( q_n + 1 \leq m \)) and (v) for each \( j \in [q_n + 1, m] \), \( s_j \leq B \).

Lemma 3.8 For any input \( \sigma \in S_3 \), there exists an input \( \sigma' \in S_3 \) such that (i) we define \( u = \frac{\sum_{j=q_n(\sigma)+1}^{m} s_j(\sigma)}{B} \), \( q_n(\sigma)+u-1(\sigma') = (\sum_{j=q_n(\sigma)+1}^{m} s_j(\sigma)) - uB \), and for any \( j \in [q_n(\sigma), q_n(\sigma)+u] \), \( s_j(\sigma') = B \), and (ii) \( \frac{V_{OPT}(\sigma')}{V_{PQ}(\sigma')} \leq \frac{V_{OPT}(\sigma)}{V_{PQ}(\sigma)} \).

Proof. For any \( j \in [1, q_n] \), we define \( s_j'' = s_j \). Furthermore, for each \( j \in [q_n + 1, q_n + u] \), we define \( s_j'' = B \), and \( s_j'' + 1 = (\sum_{j=q_n}^{m} s_j) - uB \).

We construct \( \sigma' \) from \( \sigma \) in the following way. This approach is similar to those in the proof of Lemmas 3.6 and 3.7. First, for each \( j \in [q_1, m] \), \( s_j'' \) events at which \( s_j \) packets at \( Q^{(j)} \) arrive occur during time \( (0, 1) \). Since \( s_j'' \leq B \) by definition, \( PQ \) accepts all these packets. Then, for any \( i \in [1, n] \), we define \( a_i = \sum_{j=q_{i+1}}^{m} s_j + a_0 = 0 \). For any \( x \in [0, n-1] \), a scheduling event occurs at each integer time \( t = (\sum_{j=x}^{a_i} + 1), \ldots, \sum_{j=x}^{a_i} \). Also, at each time \( t + \frac{1}{2} \), an arrival event where a packet arrives at \( Q^{(q_{n-x})} \) occurs. After time \( \sum_{j=x}^{a_i} \), sufficient scheduling events to transmit all the arriving packets occur. It is easy to see that \( V_{PQ}(\sigma') \leq V_{PQ}(\sigma) \) and \( V_{OPT}(\sigma') = V_{OPT}(\sigma) \). Moreover, by the definition of \( \sigma', \sigma' \in S_3 \) holds, and \( \sigma' \) satisfies the condition (i) in the statement.

Lemma 3.9 Let \( \sigma \in S_4 \) be an input such that \( q_n(\sigma) + 2 \leq m \), \( s_{q_n(\sigma)+1}(\sigma) = B \), and \( \sum_{j=q_n(\sigma)+2}^{m} s_j(\sigma) > 0 \).

Then, there exists an input \( \sigma \in S_4 \) such that (i) \( n(\sigma) = n(\sigma) + 1 \), (ii) for each \( i \in [1, n(\sigma)-1] \), \( q_i(\sigma) = q_i(\sigma) \), and \( q_n(\sigma)(\sigma) = q_n(\sigma)(\sigma) + 1 \), and (iii) \( \frac{V_{OPT}(\sigma)}{V_{PQ}(\sigma)} \leq \frac{V_{OPT}(\sigma)}{V_{PQ}(\sigma)} \).

Proof. We construct \( \sigma' \) from \( \sigma \) as follows: First, for each \( j \in [q_1, m] \), \( s_j \) events at which \( s_j \) packets at \( Q^{(j)} \) arrive occur during time \( (0, 1) \). Since \( s_j \leq B \) by the definition of \( S_4 \), \( PQ \) accepts all these packets.
arriving packets. For any \(i(\in [1, n])\), we define \(q'_i = q_i, q'_i + 1 = q_n + 1\) and \(q'_n + 2 = m\). Moreover, for any \(i(\in [1, n + 1])\), we define \(a_i = \sum_{j=q'_i+3-1}^{q'_n+3-1} s_j\) and \(a_0 = 0\). Then, for any \(x(\in [0, n])\), a scheduling event occurs at each integer time \(t = (\sum_{j=0}^{x-1} a_j) + 1, \ldots, \sum_{j=0}^{x+1} a_j\). In addition, for any \(x(\in [0, n])\), an arrival event where a packet arrives at \(Q(q'_n+1-x)\) occurs at each time \(t + \frac{1}{2}\). After time \((\sum_{j=0}^{n+1} a_j) + 1\), sufficient scheduling events to transmit all the arriving packets occur.

Then, the packets which \(PQ\) transmits at each scheduling event for \(\sigma'\) are equivalent to those for \(\sigma\). Consider an offline algorithm \(ALG\) which transmits a packet from \(Q(q'_n+1-x)\) at \(t\). By the definition of \(q'_i\), since for any \(i(\in [1, n + 1])\), \(h(q'_i)(1-) = B\), \(PQ\) cannot accept any packet which arrives at each time \(t + \frac{1}{2}\), but \(ALG\) can accept all the packets, which means that \(ALG\) is optimal. Hence, \(n(\sigma') = n + 1\) and for any \(i(\in [1, n + 1])\), \(q_i(\sigma') = q'_i\).

Since for any \(j(\in [1, m])\), \(s_j(\sigma') = s_j\), \(V_{PQ}(\sigma') = V_{PQ}(\sigma)\). Moreover, for any \(i(\in [1, n + 1])\), \(k_{q_i}(\sigma') = k_{q_i}, k_{q_i}(\sigma') = s_{q_i+1},\) and \(k_{q_i+1}(\sigma') = \sum_{j=q_i+2}^{s_{q_i}+1} s_j\). Therefore, \(\sigma' \in S_4\) holds, \(\sigma'\) satisfies the conditions (i) and (ii) in the statements. Also, \(V_{OPT}(\sigma') = V_{ALG}(\sigma') = V_{OPT}(\sigma) + (\alpha_{q_i+1} - \alpha_{q_i}) \sum_{j=q_i+2}^{s_{q_i}+1} s_j \geq V_{OPT}(\sigma)\).

We define the set \(S_5\) of inputs. \(S_5\) denotes the set of inputs \(\sigma(\in S_4)\) satisfying the following five conditions: (i) for each \(i(\in [1, n - 1])\), \(q_i + 1 = q_{i+1}\), (ii) for each \(i(\in [1, n - 1])\), \(k_{q_i} = B\), (iii) for each \(j(\in [q_1, q_n])\), \(s_j = B\), (iv) for any \(j(\in [1, q_1 - 1])\), \(s_j = 0\) holds if \(q_1 - 1 \geq 1\), (v) \(k_{q_i} = s_{q_i+1}\). (By Lemma 3.2, \(q_n + 1 \leq m\).) \(1 \leq s_{q_n+1} \leq B\), and (vi) for any \(j(\in [q_n + 2, m])\), \(s_j = 0\) holds if \(q_n + 2 \leq m\).

**Lemma 3.10** For any input \(\sigma(\in S_5)\), there exists an input \(\hat{\sigma}(\in S_5)\) such that (i) \(s_{q_n(\sigma)}(\hat{\sigma})+1 = B\), and (ii) \(\frac{V_{OPT}(\sigma)}{V_{PQ}(\sigma)} \leq \frac{V_{OPT}(\hat{\sigma})}{V_{PQ}(\sigma)}\).

That is, there exists an input \(\sigma^*(\in S^*)\) such that \(\max_{\sigma(\in S_4)}\{\frac{V_{OPT}(\sigma)}{V_{PQ}(\sigma)}\} = \frac{V_{OPT}(\sigma^*)}{V_{PQ}(\sigma^*)}\).

**Proof.** Since \(\sigma(\in S_5)\) holds, \(\frac{V_{OPT}(\sigma)}{V_{PQ}(\sigma)} = \frac{V_{PQ}(\sigma) + \sum_{j=q_1}^{q_n+1} s_j k_{q_j}}{V_{PQ}(\sigma)} \leq 1 + \frac{B(s_{q_n+1})+a_n s_{n+1}}{\sum_{j=q_1}^{q_n+1} s_j k_{q_j}} \leq 1 + \frac{B(s_{q_n+1})+a_n s_{n+1}}{B(s_{q_n+1})+a_n s_{n+1}},\) which we define as \(x(s_{q_n+1})\).

Let \(\sigma_1, \sigma_2 \in S_5\) be any inputs such that (i) \(n = n(\sigma_2) = n(\sigma_1) + 1\), (ii) for any \(i(\in [1, n - 1])\), \(q_i = q_i(\sigma_1) = q_i(\sigma_2)\), (iii) \(q_n = q_n(\sigma_2)\), and (iv) \(s_{q_n+1}(\sigma_1) = B\) and \(s_{q_n+1}(\sigma_2) = B\). Then, since \(x(s_{q_n+1})\) is monotone (increasing or decreasing) as \(s_{q_n+1}\) increases, \(\frac{V_{OPT}(\sigma_2)}{V_{PQ}(\sigma_2)} \leq \max\{\frac{V_{OPT}(\sigma_1)}{V_{PQ}(\sigma_1)}, \frac{V_{OPT}(\sigma_2)}{V_{PQ}(\sigma_2)}\}\).

Therefore, let \(\hat{\sigma}\) be the input such that \(\hat{\sigma} \in \arg\max\{\frac{V_{OPT}(\sigma_1)}{V_{PQ}(\sigma_1)}, \frac{V_{OPT}(\sigma_2)}{V_{PQ}(\sigma_2)}\}\), which means that the statement is true.

## 4 Lower Bound

In this section, we show a lower bound for any deterministic algorithm.

**Theorem 4.1** No deterministic online algorithm can achieve a competitive ratio smaller than \(1 + \frac{\alpha^3 + \alpha^2 + 3\alpha}{\alpha^4 + 4\alpha^3 + 3\alpha^2 + 4\alpha + 1}\).

**Proof.** Fix an online algorithm \(ON\). Our adversary constructs the following input \(\sigma\). Let \(\sigma(t)\) denote the prefix of the input \(\sigma\) up to time \(t\). OPT can accept and transmit all arriving packets.
in this input. 2B arrival events occur during time (0, 1), and B packets arrive at Q(1) and Q(m), respectively. In addition, B scheduling events occur during time (1, 2). For σ(2), suppose that ON transmits B(1 − x) packets and Bx ones from Q(1) and Q(m), respectively. 2B arrival events occur during time (0, 1), and B packets arrive at Q(1) and Q(m), respectively. After time 2, our adversary selects one queue from Q(1) and Q(m), and makes some packets arrive at the queue.

**Case 1:** If αx ≥ 1 − x: B arrival events occur during time (2, 3), and B packets arrive at Q(1). Then, the total value of packets which ON accepts by time 3 is (α + 1 + 1 − x)B. Moreover, B scheduling events occur during time (3, 4). For σ(4), suppose that ON transmits B(1 − y) packets and By packets from Q(1) and Q(m), respectively. After time 4, in the same way as time 2, our adversary selects one queue from Q(1) and Q(m), and makes some packets arrive at the queue.

**Case 1.1:** If α(x + y) ≥ 1 − y: B arrival events occur during time (4, 5), and B packets arrive at Q(1). Furthermore, 2B scheduling events occur during time (5, 6).

For this input, VON(σ) = (α + 1 + 1 − x + α(x + y))B, and VOPT(σ) = (α + 1 + 1 + 1)B.

**Case 1.2:** If α(x + y) < 1 − y: B arrival events occur during time (4, 5), and B packets arrive at Q(m). Moreover, 2B scheduling events occur during time (5, 6).

For this input, VON(σ) = (α + 1 + 1 − x + α(x + y))B, and VOPT(σ) = (α + 1 + 1 + α)B.

**Case 2:** If αx < 1 − x: B arrival events occur during time (2, 3), and B packets arrive at Q(m). Then, the total value of packets which ON accepts by time 3 is (α + 1 + αx)B. Moreover, B scheduling events occur during time (3, 4). For σ(4), ON transmits B(1 − z) packets and Bz ones from Q(1) and Q(m), respectively. After time 4, in the same way as the above case, ON selects one queue from Q(1) and Q(m), and causes some packets to arrive at the queue.

**Case 2.1:** If αz ≥ 1 − x + 1 − z: B arrival events occur during time (4, 5), and B packets arrive at Q(1). Also, 2B scheduling events occur during time (5, 6).

For this input, VON(σ) = (α + 1 + αx + 1 − x + 1 − z)B, and VOPT(σ) = (α + 1 + α + 1)B.

**Case 2.2:** If αz < 1 − x + 1 − z: B arrival events occur during time (4, 5), and B packets arrive at Q(m). In addition, 2B scheduling events occur during time (5, 6).

For this input, VON(σ) = (α + 1 + αx + αz)B, and VOPT(σ) = (α + 1 + α + α)B.

By the above argument, we define c1(x) = \min y \{\frac{\alpha+1+1+1}{\alpha+1+1-1-x+y} \frac{\alpha+1+1+1}{\alpha+1+1-x+y}\} and c2(x) = \min z \{c_1(x), c_2(x)\}.

Thus, \( c_1(x) = \frac{\alpha+1+1+1}{\alpha+1+1-x+1-y} = \frac{\alpha+1+1+1}{\alpha+1+1-x+1-y} \). Then, \( y = \frac{\alpha(a+3)+(-a^2-4a+1)x}{a^2+5a+2} \).

Thus, \( c_2(x) = \frac{\alpha+1+1+1}{\alpha+1+1-x+1-z} = \frac{\alpha+1+1+1}{\alpha+1+1-x+1-z} \). Then, \( z = \frac{\alpha^2+6a+1+(a^2-4a-1)x}{2a^2+5a+1} \).

Finally, \( \min x \{c_1(x), c_2(x)\} \) is minimized when \( c_1(x) = c_2(x) \), that is \( \frac{\alpha^2+5a+2}{a^2+4a+2-\alpha} = \frac{\alpha^2+5a+2}{a^2+4a+2-\alpha} \).

Therefore, since \( x = a^2+4a+2+\alpha \), \( \min x \{c_1(x), c_2(x)\} \geq \frac{\alpha^2+5a+2+4a+2+\alpha a^2}{\alpha^2+4a^2+3a^2+4a+1} = 1+\frac{\alpha^2+4a^2+3a^2+4a+1}{\alpha^2+4a^2+3a^2+4a+1} \).

5 Concluding Remarks

A lot of packets used by multimedia applications arrive in a QoS switch at a burst, and managing queues to store outgoing packets (egress traffic) can become a bottleneck. In this paper, we
have formulated the problem of controlling egress traffic, and analyzed Priority Queuing policies (PQ) using competitive analysis. We have shown that the competitive ratio of PQ is exactly
\[
1 + \frac{\sum_{j=1}^{m'} \alpha_j}{\sum_{j=1}^{m-1} \alpha_j},
\]
where \(m' \in \arg \max_{x \in [1,m-1]} \left\{ \frac{\sum_{j=1}^{x} \alpha_j}{\sum_{j=1}^{x+1} \alpha_j} \right\} \). Moreover, we have shown that there is no competitive deterministic algorithm.

We present some open questions as follows: (i) What is the competitive ratio of other practical policies, such as WRR? (ii) We consider the case where the size of each packet is one, namely fixed. In the setting where packets with variable sizes arrive, what is the competitive ratio of PQ or other policies? (iii) An obvious open question is to close the gap between the competitive ratio of PQ and our lower bound for any deterministic algorithm.

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A Restriction of Input

Lemma A.1 Let σ be an input such that OPT rejects at least one packet at an arrival event. Then, there exists an input σ′ such that

$$\frac{V_{OPT}(\sigma)}{V_{PQ(\sigma)}} \leq \frac{V_{OPT}(\sigma')}{V_{PQ(\sigma')}}$$

and OPT accepts all arriving packets.

Proof. Let e be the first arrival event where OPT rejects a packet happens, let p be an arriving packet at e, and let t be the event time when e happens. We construct a new input σ′′ by removing e from a given input σ. Then, PQ for σ′′ might accept a packet q which is not accepted for σ after t. Suppose that PQ handles priorities to packets in its buffers, and transmits the packet with the highest priority at each scheduling event. Let Q(i) be a queue at which p arrives at e. Then, at a scheduling event after t, a priority which PQ handles to a packet in Q(j) (j ≤ i) for σ′′ is higher than that for σ. However, a priority which PQ handles to a packet in Q(j) (j > i) for σ′′ is equal to that for σ. Thus, a time when a packet is transmitted from Q(j) (j > i) in σ′′ is the same as that in σ. Also, the number of packets which PQ stores in Q(i) (j > i) in σ′′ is equivalent to that in σ. For the value αk of q, i ≥ k holds. Hence, $V_{PQ(\sigma'')} \leq V_{PQ(\sigma)}$. On the other hand, $V_{OPT(\sigma'')} = V_{OPT(\sigma)}$. According to the inequality and the equality, $\frac{V_{OPT(\sigma)}}{V_{PQ(\sigma)}} \leq \frac{V_{OPT(\sigma'')}}{V_{PQ(\sigma')}}$. As a result, we construct a new input σ′ by removing all arrival events at which OPT rejects a packet from σ. Then, $\frac{V_{OPT(\sigma)}}{V_{PQ(\sigma)}} \leq \frac{V_{OPT(\sigma')}}{V_{PQ(\sigma')}}$. \qed