HOMOCLINIC ORBITS AND CHAOS IN THE GENERALIZED LORENZ SYSTEM

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Abstract. This paper investigates the homoclinic orbits and chaos in the generalized Lorenz system. Using center manifold theory and Lyapunov functions, we get non-existence conditions of homoclinic orbits associated with the origin. The existence conditions of the homoclinic orbits are obtained by Fishing Principle. Therefore, sufficient and necessary conditions of existence of homoclinic orbits associated with the origin are given. Furthermore, with the broken of the homoclinic orbits, we show that the chaos is in the sense generalized Shil’nikov homoclinic criterion.

1. Introduction. Chaos, as one of the most significant discoveries in the 20th century, has raised widespread concerns, and also has a great applying potentiality in different fields including nonlinear circuits, secure communications, biological system and other areas [7, 21, 1, 18, 19, 10, 17, 15, 16].

In the investigation of three-dimensional autonomous systems, the Shil’nikov criteria [20, 22] is regarded as a classical method in proving the existence of chaos. With regard to the criteria, the system has to process the homoclinic orbits or heteroclinic cycles at some condition. In Shil’nikov homoclinic criteria, there exist two situations. The one (classical Shil’nikov homoclinic criteria [20]) is that the equilibrium associating the homoclinic orbit is a saddle-focus whose eigenvalues \( \gamma, \alpha \pm i\beta \) satisfy \( \gamma \alpha < 0 \) and \( |\gamma| > |\alpha| > 0 \). Then, the chaos could appear near the homoclinic orbit. The other (generalized Shil’nikov homoclinic criteria [22]) requires that a system has two homoclinic orbits associated with a saddle with real eigenvalues \(-\lambda_2 > \lambda_1 > -\lambda_3 > 0\) and the two homoclinic orbits could been broken by varying a parameter. Then, the chaos would arise after the homoclinic orbits disappear. This is a difference between the latter situation with the former situation where the broken of the homoclinic orbit is not necessary for the appearance of chaos.

In the application of the Shil’nikov criteria, searching for the connection orbits becomes a key. However, for a concrete three-dimensional autonomous system, it is a hard task to obtaining the existence of the homoclinic orbits. Hastings and Tory [9] obtained the existence of a homoclinic orbit for a small set of parameters in Lorenz system by using a shooting argument which is the first pure mathematical

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proof. Chen [6] established the existence of homoclinic orbits for a large set of parameters based on Lyapunov functions and the shooting argument. Leonov [11, 12, 13, 14] proved the existence of homoclinic orbits by Fishing Principle which is firstly proposed in [11]. Bao and Yang [2] provided a new series method to find exact homoclinic orbits. Coomes, Koçak and Palmer [8] gave a general theorem that guarantees the existence of a homoclinic orbit associated with a hyperbolic equilibrium.

In this paper, we consider homoclinic orbits of the generalized Lorenz system [3, 4, 5]

\[
\begin{align*}
\dot{x} &= -s_1 x + s_2 y, \\
\dot{y} &= Rx + dy - xz, \\
\dot{z} &= -qz + xy,
\end{align*}
\]

where \( \dot{\cdot} = \frac{d}{dt} \), \( s_1 > d > 0 \), \( q > 0 \) and the divergence \( \nabla = -s_1 + d - q < 0 \).

First, the stability of the nonhyperbolic equilibrium are analyzed by center manifold theory. Furthermore, when the origin is a saddle, the nonexistence of homoclinic orbits are obtained by constructing Lyapunov functions. Second, the existence of homoclinic orbits are investigated by Fishing Principle. As a result, the sufficient and necessary conditions of existence of homoclinic orbits associated with the origin of the generalized Lorenz system are given. Third, when the homoclinic orbits break, such system shows a chaotic attractor. And the chaos here is in generalized Shil’nikov’s condition.

This paper is organized as follows. In section 2, by using center manifold theory and Lyapunov functions, we give the non-existence conditions of homoclinic orbits. The existence conditions of homoclinic orbits are obtained by Fishing Principle in section 3. Section 4 shows the chaos which is produced by breaking the homoclinic orbits and finds the chaos is in the sense generalized Shil’nikov homoclinic criterion. The final section concludes this paper.

2. Non-existence of homoclinic orbits. For system (1), it is easy to see that the eigenvalues of the origin are

\[
\begin{align*}
\lambda_1 &= \frac{1}{2} \left( -(s_1 - d) + \sqrt{(s_1 - d)^2 + 4(s_1 d + s_2 R)} \right), \\
\lambda_2 &= \frac{1}{2} \left( -(s_1 - d) - \sqrt{(s_1 - d)^2 + 4(s_1 d + s_2 R)} \right), \\
\lambda_3 &= -q.
\end{align*}
\]

**Theorem 2.1.** Let \( s_1 d + s_2 R \leq 0 \), then the origin of system (1) is local asymptotically stable.

*Proof.* It is obvious that the origin is local asymptotically stable if \( s_1 d + s_2 R < 0 \). Next, we consider the stability of the origin if \( s_1 d + s_2 R = 0 \). In this case, the three eigenvalues are

\[
\lambda_1 = 0, \quad \lambda_2 = -(s_1 - d), \quad \lambda_3 = -q,
\]

which means the origin is an nonhyperbolic equilibrium. Make the following transformation

\[ X = AY, \]
where \( X = (x, y, z)^T, Y = (y_1, y_2, y_3)^T \) and
\[
A = \begin{pmatrix}
  s_2 & s_2 & 0 \\
  s_1 & d & 0 \\
  0 & 0 & 1
\end{pmatrix}.
\]
Then system (1) is changed into
\[
\begin{pmatrix}
  \dot{y}_1 \\
  \dot{y}_2 \\
  \dot{y}_3
\end{pmatrix} = \begin{pmatrix}
  0 & 0 & 0 \\
  0 & -(s_1 - d) & 0 \\
  0 & 0 & -q
\end{pmatrix} \begin{pmatrix}
  y_1 \\
  y_2 \\
  y_3
\end{pmatrix} + \begin{pmatrix}
  -\frac{s_2}{s_1 - d}(y_1 + y_2) \nu \\
  \frac{s_2}{s_1 - q}(y_1 + y_2) \nu \\
  s_2(y_1 + y_2)(s_1 y_1 + d y_2)
\end{pmatrix}.
\]
(5)

Based on the center manifold theory, there exists a center manifold which can be expressed locally as follows:
\[
W^c(0) = \{(y_1, y_2, y_3) \in \mathbb{R}^3 | y_2 = h_2(y_1), y_3 = h_3(y_1), h_2(0) = h_3(0) = 0, Dh_2(0) = Dh_3(0) = 0\}.
\]
Assume
\[
h_2(y_1) = a_2 y_1^2 + a_3 y_1^3 + \cdots, \quad (6)
\]
\[
h_3(y_1) = b_2 y_1^2 + b_3 y_1^3 + \cdots. \quad (7)
\]
Then, substituting (6) and (7) into (5), by comparing coefficients, one gets
\[
a_2 = 0, \quad a_3 = \frac{s_1 s_2^2}{q(s_1 - d)^2}, \quad b_2 = \frac{s_1 s_2}{q}. \quad (8)
\]
Finally, substituting (6), (7) and (8) into (5), one obtains the vector field reduced to the center manifold
\[
\dot{y}_1 = -\frac{s_1 s_2^2}{q(s_1 - d)} y_1^2 + O(y_1^4), \quad (9)
\]
which implies \( y_1 = 0 \) is local asymptotically stable. It is worth noting that the other two eigenvalues are \( \lambda_2 = -(s_1 - d), \lambda_3 = -q \). Therefore, the origin \( O(0, 0, 0) \) is local asymptotically stable.

**Remark 1.** When the origin is stable, System (1) has no homoclinic orbits associated with the origin. Moreover, System (1) has no homoclinic orbits associated with the origin if \( s_2 = 0 \). In fact, in this case, \( \dot{x} = -s_1 x, \dot{y} = Rx + dy - xz, \dot{z} = -qz + xy \).

Suppose this system has a homoclinic orbit, it must be on the invariant plane \( x = 0 \) since \( \dot{x} = -s_1 x \). At this time, \( \dot{y} = dy, \dot{z} = -qz \). Consequently, \( y(t) = e^{dt} y(0), z(t) = e^{-qt} z(0) \) on the plane \( x = 0 \) which means there are no homoclinic orbits on the plane \( x = 0 \).

Now, we consider the situation \( s_2 \neq 0, s_1 d + s_2 R > 0 \). In this case, \( \lambda_1 > 0, \lambda_{2,3} < 0 \) which means that there are a two-dimensional stable manifold \( \Pi \) and a one-dimensional unstable manifold \( \gamma^+ \cup O \cup \gamma^- \) where \( \gamma^+ \) and \( \gamma^- \) are symmetric with respect to \( z \) axis. Besides, by calculating the eigenvectors corresponding to \( \lambda_1, \lambda_2 \) and \( \lambda_3 \), one can obtain a asymptotic behavior of \( \gamma^+ \):
\[
\left(x, \frac{\lambda_1 + s_1}{s_2} x + O(x^2), \frac{\lambda_1 + s_1}{s_2(2\lambda_1 + q)} x^2 + O(x^3)\right)
\]
for small positive \( x \).

Let
\[
Q = z - \frac{1}{2s_2} x^2, \quad \eta = -s_1 x + s_2 y, \quad \nu = \frac{2s_1 - q}{2s_2},
\]
then the system (1) is equivalent to the following system

\[
\begin{aligned}
\dot{x} & = \eta, \\
\dot{\eta} & = (d - s_1)\eta + s_2x(R + \frac{s_1d}{s_2} - Q - \frac{x^2}{2s_2}), \\
\dot{Q} & = -qQ + \nu x^2. \\
\end{aligned}
\]  
(11)

**Lemma 2.2.** Let \(d < s_1 \leq \frac{q}{2}\). Then for any \(R\) satisfying \(s_1d + s_2R > 0\), system (1) has no homoclinic orbits associated with any equilibrium.

**Proof.** The proof is divided into two steps: Case I. \(d < s_1 < \frac{q}{2}\), and Case II. \(d < s_1 = \frac{q}{2}\).

**Case I.** \(d < s_1 < \frac{q}{2}\).

Define Lyapunov function

\[
V_1 = \eta^2 - \frac{s_2}{2\nu q} (-qQ + \nu x^2)^2 + \frac{s_1}{2q} \left( x^2 - \frac{s_2}{s_1} (R + \frac{s_1d}{s_2})q \right)^2.
\]

Then we get

\[
\begin{aligned}
\dot{V}_1 = & \eta^2 - \frac{s_2}{2\nu q} (-qQ + \nu x^2)^2 + \frac{s_1}{2q} \left( x^2 - \frac{s_2}{s_1} (R + \frac{s_1d}{s_2})q \right)^2 \\
& + \frac{s_1}{2q} \left( x^2 - 2\frac{s_2}{s_1} (R + \frac{s_1d}{s_2})q + \frac{s_2^2}{s_1^2} (R + \frac{s_1d}{s_2})^2 q^2 \right) \\
= & \eta^2 + s_2x^2 \left( \frac{1}{4s_2} x^2 + Q - (R + \frac{s_1d}{s_2}) \right) - \frac{s_2q}{2\nu} Q^2 + \frac{s_2^2}{2s_1} (R + \frac{s_1d}{s_2})^2.
\end{aligned}
\]

Then, along any trajectory of system (11),

\[
\begin{aligned}
\dot{V}_1 = & 2\eta \dot{\eta} + 2s_2x \dot{x} \left( \frac{1}{4s_2} x^2 + Q - (R + \frac{s_1d}{s_2}) \right) + s_2x^2 \left( \frac{1}{2s_2} x \dot{x} + \dot{Q} - \frac{s_2q}{\nu} Q \dot{Q} \right) \\
= & 2\eta \left( (d - s_1)\eta + s_2x(R + \frac{s_1d}{s_2} - Q - \frac{x^2}{2s_2}) \right) + 2s_2x \dot{\eta} \left( \frac{1}{4s_2} x^2 + Q - (R + \frac{s_1d}{s_2}) \right) \\
& + s_2x^2 \left( \frac{1}{2s_2} x \dot{\eta} + (-qQ + \nu x^2) \right) - \frac{s_2q}{\nu} Q(-qQ + \nu x^2) \\
= & -2(s_1 - d)\eta^2 + \frac{s_2}{\nu} (-qQ + \nu x^2)^2 \\
= & -2(s_1 - d) \dot{x}^2 + \frac{s_2}{\nu} \dot{Q}^2.
\end{aligned}
\]

Since \(\frac{s_2}{\nu} = \frac{2s_2}{2s_1}e^{-q} < 0\), \(\dot{V}_1 \leq 0\) for all \(t \in (-\infty, \infty)\). Besides, \(\dot{V}_1 = 0\) if and only if \(\dot{x} = 0\) and \(\dot{Q} = 0\). However, a homoclinic orbit is the trajectory that has to tend to the same point as \(t \to \infty\) and \(t \to -\infty\). Therefore, from the properties of the Lyapunov function \(V_1\), system (1) has no homoclinic orbits associated with any equilibrium.

**Case II.** \(d < s_1 = \frac{q}{2}\).

In this case, \(\nu = \frac{2s_1}{2s_2} = 0\) which implies that \(\dot{Q} = -qQ\). Consequently, \(Q(t) = Q(0)e^{-qt}\). In particular, \(Q(t) \equiv 0\) along \(s^+\), since when \(t \to -\infty\), \(Q(t) \to 0\) holds only in the condition that \(Q(0) = 0\).

We define

\[
V_2 = \eta^2 + \frac{1}{4} \left( x^2 - 2s_2(R + \frac{s_1d}{s_2}) \right)^2,
\]
then along any trajectory satisfying \( Q(0) = 0 \),
\[
\dot{V}_2 = 2\eta \dot{\eta} + \frac{1}{2} \left( x^2 - 2s_2(R + \frac{s_1 d}{s_2}) \right) 2x \ddot{x}
\]
\[
= 2\eta \left( (d - s_1)\eta + s_2 x(R + \frac{s_1 d}{s_2} - Q - \frac{x^2}{2s_2}) \right) + x\eta \left( x^2 - 2s_2(R + \frac{s_1 d}{s_2}) \right)
\]
\[
= -2(s_1 - d)\eta^2 \leq 0.
\]
From above, we obtain that \( \gamma^+ \) cannot be a homoclinic orbit. For any other trajectories satisfying \( Q(0) \neq 0, Q(t) \to 0 \) as \( t \to +\infty \) and \( |Q(t)| \to \infty \) as \( t \to -\infty \).

As a result, there are no homoclinic orbits associated with any equilibrium. \( \square \)

**Lemma 2.3.** Let \( \frac{2}{3} < s_1 \leq \frac{2q - d}{3} \). Then for any \( R \) satisfying \( s_1 d + s_2 R > 0 \), system (1) has no homoclinic orbits associated with the origin.

**Proof.** If \( s_2 > 0 \), On \( \gamma^+ \), we have that \( z > \frac{1}{2s_2} x^2 \) for small positive \( x \) since
\[
\frac{\lambda_1 + \lambda_1^+}{s_2(2\lambda_1 + \eta)} > \frac{\lambda_1 + \lambda_1^+}{s_2(2\lambda_1 + 2s_2)} = \frac{1}{2s_2}. \]
Obviously, we can choose \( a_0 \) such that \( Q(t) > 0 \) for \( t \in (-\infty, t_0) \). From \( \dot{Q} = -qQ + \nu x^2 \), we get \( \frac{d}{dt}(e^{\nu t}Q) = e^{\nu t}\nu x^2 \geq 0 \) which means \( e^{\nu t}Q(t) \) is a monotone increasing function with \( t \). Consequently, \( Q(t) > 0 \) for all \( t \in (-\infty, \infty) \). Similarly, we can get that \( Q(t) < 0 \) for all \( t \in (-\infty, \infty) \) on \( \gamma^+ \) if \( s_2 < 0 \). Therefore, on \( \gamma^+ \), \( s_2 Q(t) > 0 \) is valid for all \( t \in (-\infty, \infty) \).

Now we define
\[
V(x, \eta, Q) = \eta^2 + s_2 x^2 \left( \frac{x^2}{4s_2} + Q - (R + \frac{s_1 d}{s_2}) \right) + (s_1 - d)x\eta. \quad (12)
\]
Then along \( \gamma^+ \),
\[
\dot{V} = 2\eta \dot{\eta} + 2s_2 x \ddot{x} \left( \frac{x^2}{4s_2} + Q - (R + \frac{s_1 d}{s_2}) \right) + s_2 x^2 (\frac{1}{2s_2} x \ddot{x} + \dot{Q})
\]
\[
+ (s_1 - d)x \dot{\eta} + (s_1 - d)x \dot{\eta}
\]
\[
= \eta (2\eta + (s_1 - d)x)
\]
\[
+ \dot{x} \left( 2s_2 x \left( \frac{x^2}{4s_2} + Q - (R + \frac{s_1 d}{s_2}) \right) + \frac{1}{2} x^3 + (s_1 - d)\eta \right) + Qs_2 x^2
\]
\[
= \left( (d - s_1)\eta + s_2 x(R + \frac{s_1 d}{s_2} - Q - \frac{x^2}{2s_2}) \right) (2\eta + (s_1 - d)x)
\]
\[
+ \eta \left( 2s_2 x \left( \frac{x^2}{4s_2} + Q - (R + \frac{s_1 d}{s_2}) \right) + \frac{1}{2} x^3 + (s_1 - d)\eta \right)
\]
\[
+ (-qQ + \frac{2s_1 - q}{2s_2} x^2)s_2 x^2
\]
\[
= (d - s_1)\eta^2 - (s_1 - d)^2 x\eta + s_2 (s_1 - d)x^2 (R + \frac{s_1 d}{s_2} - Q - \frac{x^2}{2s_2})
\]
\[
- s_2 q x^2 Q + (s_1 - \frac{q}{2})x^4.
\]
So, by simple calculation, we get the following
\[
\dot{V} + (s_1 - d) V = \frac{3}{4} (s_1 - \frac{2q - d}{3}) x^4 - s_2 q x^2 Q. \quad (13)
\]
In this situation, one can conclude \( \dot{V}(t) + (s_1 - d) V(t) \leq 0 \) on \( \gamma^+ \) for all \( t \in (-\infty, \infty) \) since \( s_1 \leq \frac{2q - d}{3} \) and \( s_2 Q > 0 \) on \( \gamma^+ \). It means that \( \frac{d}{dt}(e^{(s_1 - d)t} V(t)) \leq 0 \), so that
$V(t) \leq 0$ for all $t$ with $V(\infty) = 0$. Then $V(t) \leq e^{-(s_1-d)t}V(0)$ for any $t \geq 0$. Consequently, $|V(t)| \geq |V(0)|e^{-(s_1-d)t}$ for any $t \geq 0$.

From system (11), one can obtain $\dot{x} + (s_1 - d)\dot{x} - s_2x(R + \frac{\lambda_1d}{2} - Q - \frac{x^2}{2s_2}) = 0$. If a trajectory is on the stable manifold of the origin and is not the $z$ axis, then for $t$ sufficiently large, $Q(t)$ is sufficiently small. So, $\dot{x} + (s_1 - d)\dot{x} - s_2(R + \frac{\lambda_1d}{2} + o(1))x = 0$. In this case, $x, \dot{x}, \ddot{x} = O(e^{(s_1+d)(1)\gamma})$ as $t \to \infty$. If $\gamma^+ \to 0$ as $t \to \infty$, then $V(t) = O(e^{s(1)\gamma})$. Since $\lambda_2 < -(s_1 - d), |V(t)| < e^{-2(s_1-d)t}$ for sufficiently large $t$, which contradict with $|V(t)| \geq |V(0)|e^{-(s_1-d)t}$ for $t \geq 0$.

As a result, there is no homoclinic orbit associated with the origin. □

Combining Lemma 2.2 with Lemma 2.3, we can conclude the following:

**Theorem 2.4.** Let $d < s_1 \leq \frac{2q-d}{3}$. Then for any $R$ satisfying $s_1d + s_2R > 0$, system (1) has no homoclinic orbits associated with the origin.

3. **Existence of homoclinic orbits.** At the beginning of this section, we firstly recall Fishing Principle [13].

Consider the following differential system

$$\frac{dx}{dt} = f(x, p),$$

where $x \in \mathbb{R}^n$, parameter $p \in \mathbb{R}^m$ and $f(x, p)$ is a smooth vector-function.

Consider the Tricomi problem: Let $\Gamma(s), s \in [0, 1]$ be a smooth path in the space of parameters $\mathbb{R}^m$ of system (14). Does there exist a point $p_0 \in \Gamma(s)$ such that system (14) has a homoclinic orbit?

To answer the problem, we introduce some notations.

Suppose $\Omega$ is a $(n-1)$ dimensional closed bounded region with piecewise smooth edge $\partial\Omega$ and equilibrium $x_0 \in \partial\Omega$. $x(t, s)^+$ is a separatrix of saddle point $x_0$ satisfying $\lim_{t \to -\infty} x(t, s)^+ = x_0$. $x(s)^+$ is a point of the first crossing of separatrix $x(t, s)^+$ with region $\Omega$:

$$x(t, s)^+ \notin \Omega, \ t \in (-\infty, T),$$

$$x(T, s)^+ = x(s)^+ \in \Omega.$$

If such a crossing is nonexistence, $x(s)^+ = \emptyset$.

**Fishing Principle.** Suppose that for the path $\Gamma(s)$ and $(n-1)$ dimensional closed bounded region $\Omega$ with the piecewise smooth edge $\partial\Omega$, the following properties are satisfied:

(1) for any $x \in \Omega \setminus \partial\Omega$ and $s \in [0, 1]$, the vector $f(x, \Gamma(s))$ is transversal to the manifold $\Omega$;

(2) for any $s \in [0, 1], f(x_0, \Gamma) = 0$ and the point $x_0 \in \partial\Omega$ is a saddle of system (14);

(3) the inclusion $x(0)^+ \in \Omega \setminus \partial\Omega$ is satisfied;

(4) the relation $x(1)^+ = \emptyset$ is valid;

(5) for any $s \in [0, 1]$ and $y \in \partial\Omega \setminus x_0$, there exists a neighborhood $U(y, \delta) = \{x | x - y < \delta\}$ such that $x(s)^+ \notin U(y, \delta)$;

(6) for all $s \in [0, 1]$ such that $x(s)^+ \in \Omega$ and for all $t \in (-\infty, T)$ there exists a number $R$ such that $|x(t, s)^+| \leq R$. Here, $x(T, s)^+ = x(s)^+$.

**Lemma 3.1.** If conditions (1) – (6) are satisfied, then there exists $s_0 \in [0, 1]$ such that $x(t, s_0)^+$ is a homoclinic trajectory of the saddle $x_0$. 
We assume that the separatrix $\gamma^+ = (x(t)^+, y(t)^+, z(t)^+)$ of system (1) for convenience. From (10), there exists a constant $T_1$ such that $x(t)^+ > 0$, $y(t)^+ > 0$, $z(t)^+ > 0$ for all $t \in (-\infty, T_1)$ for $s_2 > 0$ or $x(t)^+ > 0$, $y(t)^+ < 0$, $z(t)^+ < 0$ for all $t \in (-\infty, T_1)$ for $s_2 < 0$. Let $\epsilon = \frac{1}{\sqrt{(\frac{s_2}{2s_1} + 1)}}$, under the following transformation

$$\theta = \frac{\epsilon x}{\sqrt{2s_1}} \frac{1}{d}, \eta_1 = \frac{\epsilon^2}{\sqrt{2s_1}} (-s_1 x + s_2 y), \xi = \frac{\epsilon^2}{s_1} \frac{1}{s_1^2} (z - \frac{s_1}{qs_2} x^2), t_1 = \frac{\sqrt{s_1 d}}{\epsilon} t,$$

system (1) is changed into

$$\begin{cases}
\dot{\theta} = \eta_1,
\dot{\eta}_1 = -\mu \eta_1 - \xi \theta + \theta - \frac{2s_1 \theta^3}{q},
\dot{\xi} = -\alpha \xi - \beta \theta \eta_1,
\end{cases} \quad (15)$$

where

$$\mu = \frac{\epsilon (s_1 - d)}{\sqrt{s_1 d}}, \alpha = \frac{\epsilon q}{s_1^2}, \beta = 2(\frac{2s_1}{q} - 1)$$

and dot denotes derivative with respect to time $t_1$.

**Lemma 3.2.** Let $s_1 > \frac{2q - d}{3}$. Then for sufficiently large $R$, there exists a number $T$ such that $x(T)^+ = 0$.

**Proof.** We consider system

$$\begin{cases}
W \frac{dW}{d\theta} = -\mu W - P \theta + \theta - \frac{2s_1 \theta^3}{q},
W \frac{dP}{d\theta} = -\alpha P - \beta \theta W.
\end{cases} \quad (16)$$

Obviously, system (15) is equivalent to system (16) on the set $\{\theta \geq 0, \eta_1 > 0\}$ and $\{\theta \geq 0, \eta_1 < 0\}$.

By ignoring the parameters $\mu$ and $\alpha$, we can get the following system

$$\begin{cases}
W \frac{dW}{d\theta} = -P \theta + \theta - \frac{2s_1 \theta^3}{q},
P \frac{dP}{d\theta} = -\beta \theta W.
\end{cases} \quad (17)$$

Under the initial condition $W(0) = 0$, $P(0) = 0$, system (17) has solution

$$W(\theta) = \pm \theta \sqrt{1 - \frac{\theta^2}{2}}, \ P(\theta) = -\frac{\beta}{2} \theta^2, \ \theta \in [0, \sqrt{2}]. \quad (18)$$

Since $R$ is a large number, $\mu$ and $\alpha$ are small parameters. In this case, solution (18) can be used to obtain the first approximations of solutions $W(\theta)$ and $P(\theta)$ of system (16), which correspond to the separatrix of a saddle:

$$W_1(\theta)^2 = \theta^2 - \frac{\theta^4}{2} - 2\mu \int_0^\theta u \sqrt{1 - \frac{u^2}{2}} du + 2\alpha \beta \int_0^\theta u \sqrt{1 - \frac{u^2}{2} - 1} du,$$

$$P_1(\theta) = -\frac{\beta}{2} \theta^2 - \alpha \beta (1 - \frac{\theta^2}{2} - 1),$$

and

$$W_2(\theta)^2 = \theta^2 - \frac{\theta^4}{2} - \frac{2}{3} \alpha \beta - \frac{3}{4} \mu + 2\mu \int_\sqrt{1}^\theta \sqrt{1 - \frac{u^2}{2}} du.$$
\[-2\alpha\beta \int_0^\theta u(\sqrt{1 + \frac{u^2}{2}} - 1)du,\]
\[P_2(\theta) = -\frac{\beta}{2} \theta^2 + \alpha\beta(\sqrt{1 - \frac{\theta^2}{2}} + 1),\]

where \(W_1(\theta) \geq 0\) and \(W_2(\theta) \leq 0\).

Then, we have
\[W_2(0)^2 = \frac{8\epsilon}{3\sqrt{s_1}d}(3s_1 - 2q + d),\]
\[P_2(0) = \frac{4\epsilon}{\sqrt{s_1}d}(2s_1 - q).\]

This implies that if \(3s_1 - 2q + d > 0\) and \(R\) sufficiently large, then there exists \(T\) such that
\[\theta(T) = 0,\]
\[\eta_1(T) \approx W_2(0) = -\frac{8\epsilon}{3\sqrt{s_1}d}(3s_1 - 2q + d),\]
\[\xi(T) \approx P_2(0) = \frac{4\epsilon}{\sqrt{s_1}d}(2s_1 - q).\]

Consequently, one obtain \(x(T)^+ = 0\).

**Remark 2.** In this paper, \(R\) sufficiently large means that sufficiently large \(R > 0\) if \(s_2 > 0\) or sufficiently large \(-R > 0\) if \(s_2 < 0\). Since \(s_1 > \frac{2q - d}{3}\) implies \(2s_1 > q\), we can get that \(Q(t)^+ \geq 0\) for \(t \in (-\infty, \infty)\) for \(s_2 > 0\) or \(Q(t)^+ \leq 0\) for \(t \in (-\infty, \infty)\) for \(s_2 < 0\) from Lemma 2.3. Moreover, from continuous dependence of solutions on parameters \(s_1, s_2, R, d, q\), the separatrix \(x(t)^+, \eta(t)^+, Q(t)^+\) are uniformly bounded on \((-\infty, T]\) for any compact set of parameters.

**Lemma 3.3.** Let \(s_1 > \frac{2q-d}{3}\), then
\[\eta(t)^+ \leq Mx(t)^+,\] (19)
for \(t \in (-\infty, T_1)\), where
\[M = \frac{-(s_1 - d) + \sqrt{(s_1 - d)^2 + 4(s_1 d + s_2 R)}}{2}.\]

**Proof.** Consider system
\[
\begin{align*}
\dot{x} &= \bar{\eta}, \\
\dot{\eta} &= (d - s_1)\bar{\eta} + (s_1 d + s_2 R)\bar{x}.
\end{align*}
\] (20)

Obviously, \(\bar{\eta}(t)^+ = M\bar{x}(t)^+\) where \(\bar{\eta}(t)^+\), \(\bar{x}(t)^+\) is a separatrix of saddle of system (20).

Compare system (20) with the first two equation of system (11), for any \(t_0 \in (-\infty, T_1)\) and \((x(t_0), \eta(t_0)) = (\bar{x}(t_0), \bar{\eta}(t_0))\), one has
\[\dot{\eta}|_{t_0} \leq \dot{\bar{\eta}}|_{t_0},\]
which means that
\[\eta(t)^+ \leq \bar{\eta}(t)^+, \forall t \in (-\infty, T_1).\]
As a result, \(\eta(t)^+ \leq Mx(t)^+\) for any \(t \in (-\infty, T_1)\). \(\square\)
Lemma 3.4. Let $s_1 > \frac{2q-d}{3}$, then
\[
s_2 Q(t)^+ \geq \frac{2s_1 - q}{2(2M + q)} (x(t)^+)^2, \forall t \in (-\infty, T_1).
\]

Proof. From (19), we get
\[
(s_2 Q(t)^+ - h(x(t)^+)^2)' + q(s_2 Q(t)^+ - h(x(t)^+)^2) = (\frac{2s_1 - q}{2} - hq)(x(t)^+)^2 - 2hx(t)^+ \eta(t)^+
\]
\[
\geq (\frac{2s_1 - q}{2} - hq - 2hM)(x(t)^+)^2 = 0,
\]
where
\[
h = \frac{2s_1 - q}{2(2M + q)}.
\]
This implies that $e^t(s_2 Q(t)^+ - h(x(t)^+)^2)$ is a monotone increasing function with $t$. Since $e^t(s_2 Q(t)^+ - h(x(t)^+)^2) \to 0$ as $t \to -\infty$, we obtain
\[
s_2 Q(t)^+ \geq h(x(t)^+)^2, \forall t \in (-\infty, T_1).
\]

Lemma 3.5. Let $0 < s_1 - \frac{2q-d}{3} < \frac{2q(2s_1-q)}{3(2M+q)}$. Then
\[
x(t)^+ > 0, \forall t \in (-\infty, \infty).
\]

Proof. From (12), (13) and Lemma 3.4, we obtain
\[
\dot{V} + (s_1 - d)V \leq 3 \left( s_1 - \frac{2q-d}{3} - \frac{2q(2s_1-q)}{3(2M+q)} \right) (x(t)^+)^4 < 0, \forall t \in (-\infty, T_1),
\]
which means
\[
V(x(t)^+, \eta(t)^+, Q(t)^+) < 0, \forall t \in (-\infty, T_1).
\]

Obviously,
\[
V(x(T_1)^+, \eta(T_1)^+, Q(T_1)^+) < 0.
\]
It is worth noting that $V(0, \eta(t)^+, Q(t)^+) = (\eta(t)^+)^2 \geq 0$ for any $t \in (-\infty, \infty)$. So, $x(T_1)^+ \neq 0$ which implies $x(t)^+ > 0$ for any $t \in (-\infty, \infty)$. \hfill \Box

Remark 3. If $s_1 d + s_2 R$ is sufficiently small, $M$ is also sufficiently small. In this case, the condition of Lemma 3.5, $0 < s_1 - \frac{2q-d}{3} < \frac{2q(2s_1-q)}{3(2M+q)}$, is always valid for $s_1 > d$.

To apply Fishing Principle to system (1), we define
\[
\Omega = \begin{cases} \{x = 0, y \leq 0, y^2 + z^2 \leq R_0\}, & \text{for } s_2 > 0; \\ \{x = 0, y \geq 0, y^2 + z^2 \leq R_0\}, & \text{for } s_2 < 0; \end{cases}
\]
where $R_0$ is a sufficiently large number.

Obviously, conditions (1) and (2) of Fishing Principle are satisfied.

According to Lemma 3.2, condition (3) of Fishing Principle are satisfied for $s_1 > \frac{2q-d}{3}$ and sufficiently large $R$.

According to Lemma 3.5, condition (4) of Fishing Principle are satisfied for $s_1 > \frac{2q-d}{3}$ and sufficiently small $s_1 d + s_2 R$. 

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System (1) has a solution
\[ x(t) \equiv 0, \quad y(t) \equiv 0, \quad z(t) = e^{-qt}z(0). \]
Then,
\[ \lim_{t \to -\infty} |z(t)| = +\infty. \quad (21) \]
According to continuous dependence of solutions on initial condition and (21), condition (5) of Fishing Principle is satisfied.

According to Remark 2, under condition \( s_1 > 2q - d \), condition (6) of Fishing Principle is satisfied for any compact set of parameters \( s_1, s_2, R, d \) and \( q \).

In this case, all conditions of Fishing Principle are satisfied. Moreover, system (1) is invariant under the change of coordinates \((x, y, z) \to (-x, -y, z)\). According to Lemma 3.1, one obtains the following:

**Theorem 3.6.** Let \( s_1 > 2q - d, \quad s_2 \neq 0 \). Then there exists \( s_1d + s_2R > 0 \) such that system (1) has two symmetrical homoclinic orbits associated with the origin.

From Theorem 2.1, Theorem 2.4 and Theorem 3.6, we conclude the main result as follows:

**Theorem 3.7.** Let \( s_1 > d > 0, \quad q > 0 \). Then system (1) has two symmetrical homoclinic orbits for some \( R \) satisfying \( s_1d + s_2R > 0 \) if and only if \( s_1 > 2q - d \) and \( s_2 \neq 0 \).

To justify the result, we consider a group of parameters \( s_1 = 5, \quad s_2 = 4, \quad d = 1.5, \quad q = 2 \) which satisfies the condition \( s_1 > 2q - d \) for system (1). By numerical method, we obtain the estimate of parameter \( R \) corresponding to homoclinic orbits:
\[ R \in (3.0843, 3.0844) \]
As shown in Fig. 1, two symmetrical homoclinic orbits are obtained.

4. **Chaos in homoclinic bifurcation.** From Theorem 3.7 and (2)-(4), we know that the origin associating with homoclinic orbits is a saddle with a two-dimensional stable manifold and a one-dimensional unstable manifold. Therefore, the homoclinic
orbits do not satisfy the classical Shil’nikov theorem which requires the equilibrium to be a saddle-focus. However, when parameter takes small changes, the homoclinic orbits break and a chaotic attractor arises. Therefore, one could conclude that the chaos is in the sense generalized Shil’nikov homoclinic criteria, which requires that the system has two homoclinic orbits associated with a saddle with real eigenvalues \(-\lambda_2 > \lambda_1 > -\lambda_3 > 0\) and the two homoclinic orbits need been broken by varying a parameter. It is worth noting that, in the generalized Shil’nikov criterion, the breaking of two homoclinic orbits associated with a saddle is an essential condition for the chaos.

Indeed, in Fig. 1, the eigenvalues of the origin satisfy \(-\lambda_2 > \lambda_1 > -\lambda_3 > 0\) and there exists two homoclinic orbits associated with the origin. Also, the system does not show chaotic behaviors at this time. Then, taking \(d = 1.6\) without changing the other parameters, one could find that the homoclinic orbits disappear and a chaotic attractor \(\mathcal{A}\) arises, as shown in Fig. 2. The Lyapunov exponents of \(\mathcal{A}\) are

\[
\chi_1 = 0.2095, \quad \chi_2 = -0.0000, \quad \chi_3 = -5.6094.
\]

![Figure 2. Chaotic attractor \(\mathcal{A}\) of system (1) with \(s_1 = 5, s_2 = 4, d = 1.5, q = 2, R = 3.08435\).](image)

5. Conclusion. This paper investigates homoclinic orbits and chaos in the generalized Lorenz system. The non-existence conditions of homoclinic orbits are given by center manifold theory and Lyapunov functions. Using Fishing Principle, we get the existence conditions of homoclinic orbits. As a result, the sufficient and necessary conditions of existence of homoclinic orbits associated with the origin of the generalized Lorenz system are given. Furthermore, by changing parameters, it is found that the homoclinic orbits disappear and a chaotic attractor arises. The phenomenon could be a validation that the chaos in the generalized Lorenz system is in the sense generalized Shil’nikov homoclinic criteria.

REFERENCES

[1] A. Ashraf and A. Abdulnasser, On the design of chaos-based secure communication systems, *Commun. Nonlin. Sci.*, 16 (2011), 3721–3737.

[2] J. Bao and Q. Yang, A new method to find homoclinic and heteroclinic orbits, *Appl. Math. Comput.*, 217 (2011), 6526–6540.

[3] S. Čelikovský and G. Chen, On a generalized Lorenz canonical form of chaotic systems, *Int. J. Bifurcat. Chaos*, 12 (2002), 1789–1812.
[4] S. Čelikovský and G. Chen, Secure synchronization of a class of chaotic systems from a nonlinear observer approach, *IEEE Trans. Automat. Contr.*, 50 (2005), 76–82.

[5] S. Čelikovský and G. Chen, On the generalized Lorenz canonical form, *Chaos Solitons Fractals*, 26 (2005), 1271–1276.

[6] X. Chen, Lorenz equations part I: Existence and nonexistence of homoclinic orbits, *SIAM J. Math. Anal.*, 27 (1996), 1057–1069.

[7] L. O. Chua, M. Komuro and T. Matsumoto, The double scroll family, *IEEE Trans. Circuits Syst.*, 33 (1986), 1072–1118.

[8] B. A. Coomes, H. Kočak and K. J. Palmer, A Computable Criterion for the Existence of Connecting Orbits in Autonomous Dynamics, *J. Dyn. Differ. Equ.*, 28 (2016), 1081–1114.

[9] S. P. Hastings and W. C. Troy, A proof that the Lorenz equations have a homoclinic orbit, *J. Differ. Equ.*, 113 (1994), 166–188.

[10] M. W. Hirsch, S. Smale and R. L. Devaney, *Differential Equations, Dynamical Systems, and an Introduction to Chaos*, Third edition. Elsevier/Academic Press, Amsterdam, 2013.

[11] G. A. Leonov, Attractors, limit cycles and homoclinic orbits of low dimensional quadratic systems: analytical methods, *Can. Appl. Math. Q.*, 17 (2009), 121–159.

[12] G. A. Leonov, General existence conditions of homoclinic trajectories in dissipative systems, Lorenz, Shimizu-Morioka, Lu and Chen systems, *Phys. Lett. A*, 376 (2012), 3045–3050.

[13] G. A. Leonov, Fishing principle for homoclinic and heteroclinic trajectories, *Nonlinear Dyn.*, 78 (2014), 2751–2758.

[14] G. A. Leonov, Existence Conditions of Homoclinic Trajectories in Tigan System, *Int. J. Bifurcat. Chaos*, 25 (2015), 1550175.

[15] E. N. Lorenz, Deterministic non-periodic flow, *J. Atmos. Sci.*, 20 (1963), 130–141.

[16] P. Namayanja, Chaotic dynamics in a transport equation on a network, *Discrete Contin. Dyn. Syst. Ser. B*, 23 (2018), 3415–3426.

[17] J. Palis, J. P. Júnior and F. Takens, *Hyperbolicity and Sensitive Chaotic Dynamics at Homoclinic Bifurcations: Fractal Dimensions and Infinitely Many Attractors in Dynamics*, Cambridge University Press, Cambridge, 1995.

[18] K. Rajagopal, A. Akgul, S. Jafari and B. Aricioglu, A chaotic memcapacitor oscillator with two unstable equilibriums and its fractional form with engineering applications. *Nonlinear Dynam.*, 91 (2018), 957–974.

[19] L. P. Shil’nikov, A. L. Shil’nikov, D. V. Turaev and L. O. Chua, *Methods of Qualitative Theory in Nonlinear Dynamics*, World Scientific, Singapore, 2001.

[20] C. P. Silva, Shil’nikov’s theorem-a tutorial, *IEEE Trans. Circuits Syst.*, 40 (1993), 675–682.

[21] S. H. Strogatz, *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering*, CRC Press, 2018.

[22] S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, 2nd edition, Springer-Verlag, New York, 2003.

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