Quantum state engineering by nonlinear quantum interference

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Abstract

Multi-photon quantum interference is the underlying principle for optical quantum information processing protocols. Indistinguishability is the key in quantum interference. Therefore, the success of many protocols in optical quantum information processing relies on the availability of photon states with a well-defined spatial and temporal mode. Photons in single spatial mode can be obtained from nonlinear process in a single-mode waveguide. For the temporal mode, the common approach is to engineer the nonlinear processes to achieve the required spectral properties for the generated photons. But this approach is complicated because the spectral properties and the nonlinear interaction are often intertwined through phase matching condition. In this paper, we study a different approach that separates the spectral control from nonlinear interaction, leading to versatile and precise engineering in the production of two-photon states from spontaneous parametric processes. The approach is based on an SU(1,1) nonlinear interferometer with a pulsed pump and a controllable spectral phase shift for precise engineering. We analyze systematically the important figures of merit such as modal purity and heralding efficiency in characterizing a photon state and use this analysis to investigate the feasibility of this interferometric approach based on four-wave mixing process with dispersion-shifted fibers as the nonlinear media and a standard single-mode fiber as the phase control medium. Both modal purity and efficiency are improved simultaneous with this technique. Furthermore, a novel multi-stage nonlinear interferometer is proposed and shown to achieve more precise state engineering for near ideal single-mode operation and near unity efficiency. Our investigation provides a new approach for modifying the spectral property of photon pairs in such a way that they simultaneously possess the properties of high purity, high collection efficiency and high brightness.

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I. INTRODUCTION

Many protocols in quantum information and quantum communication were first demonstrated in optics [1, 2] because of the simplicity in photons and the easiness to implement them with linear optics [3, 4]. This requires high quality single-photon and multi-photon sources with superior modal purity and efficiency. One approach is to produce single photon on demand [5]. Despite of constant improvement of technology that leads to high quality in photon indistinguishability of the single-photon source [6], this type of photon source still lacks the consistency in repeatability, that is, the quality varies from one source to another. This limits the applicability of the source. Another common approach that began from the early stage is the correlated photon pair generation from spontaneous emission of nonlinear parametric processes, which has become a popular multi-photon source ever since its discovery. A single-photon state can be produced by heralding on the detection of one of the photon pair [7]. Because of its simplicity, this type of photon source has been used in a wide range of applications in quantum information processing (QIP).

Because of the way they are generated, the photon pairs from spontaneous parametric emission (SPE) are highly correlated in frequency and time. This, on the one hand, is highly desirable in studying quantum entanglement in frequency and time, on the other hand leads to distinguishability in time due to difference in the production time of the photon pairs and becomes troublesome for quantum interference, in particular, in the QIP protocols involving the quantum interference among multiple sources, such as the generation of multi-photon entanglement [8] and quantum teleportation [9]. To tackle this problem, ultra-short pulses are used to eliminate the time uncertainty and define a proper temporal mode [10, 11]. This effort, however, was hampered by the dispersion in nonlinear optical media due to the ultra-fast process [12, 13] and leads to even more complicated temporal modes. Ironically, to obtain a better temporal mode for the two-photon fields, it is desirable to have no frequency correlation between the photons so that each photon can have a definite temporal mode of their own [14]. This leads to the requirement of factorization of two-photon wave-function or joint spectral function (JSF) [15].

Efforts in acquiring a factorized JSF have been under way for quite some years ever since it was discovered that high visibility in multi-photon quantum interference relies on the factorization of the JSF [15]. In the early days, the factorization of JSF was realized by utilizing
passive filtering \[9–11\]. However, it is well known that this method will result in a reduction of the brightness. Moreover, the collection efficiency of photon pairs, which corresponds to the heralding efficiency of heralded single photons, will be significantly reduced because the filtering process will cut out photons randomly to destroy photon correlation and degrades the quality of the quantum correlated photon pairs. \[12, 13, 16\]. Then came the idea of engineering the source of photon pairs to achieve factorization without filtering. Over the years, many techniques have been deployed in order to directly engineer the JSF into a factorized form. They include the employment of photonic grating for active temporal mode shaping \[17\], special selection of $\chi^{(2)}$-nonlinear crystals with the desired properties \[18\], engineering of the dispersion of nonlinear optical fiber \[19–22\], and engineering of the structure of the nonlinear photonic crystals \[23, 24\].

The common goal in the techniques mentioned above is to engineer the JSF by manipulating the linear spectral properties of the nonlinear media to achieve an un-correlated and near factorized JSF without passive filtering. The key parameters for a successful engineering are the high modal purity and the good collection or heralding efficiency while maintaining a high photon pair production rate. While most have achieved the aforementioned goals to some extent, many are limited to specific wavelengths of operation due to strict requirement on dispersion and are therefore lack of tunability.

Two factors need to be considered in the engineering of the JSF: (1) dispersion of the media for tailoring the spectral shape of JSF and (2) phase matching for achieving efficient nonlinear interaction. Most of the schemes implemented so far for quantum state engineering have the two aspects intertwined: changing one will affect the other and everything has to be just right to achieve the goals. This is why most of the schemes are lack of tunability.

In this paper, we consider a totally different approach in which we separate the nonlinear gain control and dispersion engineering by the method of SU(1,1)-type nonlinear quantum interference \[25\]. The SU(1,1)-type nonlinear interferometer (NLI), first proposed by Yurke et al. \[26\] and recently realized experimentally \[27, 28\], is analogous to a conventional Mach-Zehnder interferometer (MZI) but with the two splitting mirrors being substituted by two nonlinear media. Originally designed to achieve the Heisenberg limit in precision phase measurement, this type of NLI has found applications in quantum interferometry beyond standard quantum limit \[28\], imaging with undetected photons \[29\], and infrared spectroscopy \[30\], and has been realized with atoms in a Bose-Einstein condensate \[31\],
phonons in an opto-mechanical system \[32\], microwaves in low noise RF amplifiers \[33\], and a combined atom-photon system in hybrid atom-light interferometers \[34\]. Here we propose and analyze a new type of reshaping method for the JSF of photon pairs based on the SU(1,1)-type NLI, in which the phase matching of parametric process is controlled by the nonlinear media whereas the spectral shaping is achieved via dispersive phase control of the interferometer \[35\]. With the roles of phase matching and spectral reshaping separated, we are able to achieve fine control of the parameters in engineering the JSF by introducing the dispersive phase control with a programmable optical filter commonly employed in ultra-fast pulse shaping \[36, 37\]. Better control and finer engineering of the JSF can also be achieved with a novel multi-stage nonlinear interference scheme for the production of higher quality two-photon state. The involvement of dispersive media in the interference process leads to active spectral filtering, which, different from passive filtering with regular filters, maintains the original high collection efficiency for good photon heralding efficiency and keeps in the meantime a good modal purity with high brightness, all desirable in many quantum information protocols.

The rest of the paper is organized as follows. We first lay the groundwork for quantum state engineering in Sect. II with a characterization of multi-mode two-photon state from SPE by defining some key parameters such as state purity and heralding efficiency. Then, we introduce the SU(1,1)-type NLI in Sect. III for the engineering of JSF and apply it to an optical fiber system and demonstrate the improvement of the key parameters by the new scheme. To make a better control and finer engineering, we introduce the techniques of programmable optical filtering and multi-stage interference in Sect. IV. Finally, we conclude with a summary and discussion in Sect. V.

II. GENERATION AND CHARACTERIZATION OF TWO-PHOTON STATES AND HERALDED SINGLE-PHOTON STATES BY SPONTANEOUS PARAMETRIC PROCESSES

A. Two-photon states and Schmidt mode decomposition

Two-photon states are usually generated in the signal and idler field through nonlinear interactions of three- or four-wave mixing with one or two strong pump fields. When the
pump power is relatively low, the dominating interaction leads to two-photon generation. If the spatial modes are well-defined, as in optical fiber, we can use one-dimensional description for the generated signal and idler fields and the output quantum state takes the form of

$$|\Psi\rangle \approx |\text{vac}\rangle + G|\Psi_2\rangle$$  \hspace{1cm} (1)

with the two-photon state term

$$|\Psi_2\rangle = \int d\omega_s d\omega_i F(\omega_s, \omega_i) \hat{a}_s^\dagger(\omega_s) \hat{a}_i^\dagger(\omega_i) |\text{vac}\rangle,$$  \hspace{1cm} (2)

where $\hat{a}_s^\dagger(\omega_s)$ and $\hat{a}_i^\dagger(\omega_i)$ are the creation operators of the signal and idler fields at $\omega_s$ and $\omega_i$, respectively. The coefficient $G$ is proportional to $\gamma L P_p (\gamma L \sqrt{P_p})$ for four(three)-wave mixing with $\gamma$ and $L$ respectively denoting the nonlinear coefficient and length of the nonlinear medium and $P_p$ being the peak power of the pump. The JSF $F(\omega_s, \omega_i)$ is normalized as

$$\int d\omega_s d\omega_i |F(\omega_s, \omega_i)|^2 = 1$$  \hspace{1cm} (3)

and can be expressed via singular mode decomposition method as Schmidt mode expansion $^{38, 39}$:

$$F(\omega_s, \omega_i) = \sum_k r_k \psi_k(\omega_s) \phi_k(\omega_i)$$  \hspace{1cm} (4)

with mode expansion coefficients $r_k \geq 0 \ (k = 1, 2, ...), \ \sum_k r_k^2 = 1$, and two sets of orthonormal functions $\{\psi_k(\omega_s), \phi_k(\omega_i)\}$ satisfying

$$\int d\omega_s \psi_k^\star(\omega_s) \psi_{k'}(\omega_s) = \delta_{kk'}, \quad \int d\omega_i \phi_k^\star(\omega_i) \phi_{k'}(\omega_i).$$  \hspace{1cm} (5)

With mode decomposition in Eq.(4), the state in Eq.(1) can be rewritten as

$$|\Psi\rangle \approx |\text{vac}\rangle + G \sum_k r_k \hat{A}_k^\dagger \hat{B}_k^\dagger |\text{vac}\rangle = |\text{vac}\rangle + G \sum_k r_k |1_k\rangle_s |1_k\rangle_i,$$  \hspace{1cm} (6)

where operators

$$\hat{A}_k^\dagger \equiv \int d\omega \psi_k(\omega) \hat{a}_s^\dagger(\omega), \quad \hat{B}_k^\dagger \equiv \int d\omega \phi_k(\omega) \hat{a}_i^\dagger(\omega)$$  \hspace{1cm} (7)

define single temporal modes for the signal and idler fields, respectively. $|1_k\rangle_s \equiv \hat{A}_k^\dagger |\text{vac}\rangle$, $|1_k\rangle_i \equiv \hat{B}_k^\dagger |\text{vac}\rangle$ are the single-photon states in those temporal modes $^{39}$. The way in which $|\Psi\rangle$ is expressed in terms of the temporal modes in Eq.(6) indicates that it is a multi-mode
two-photon state and is in the form of high-dimensional entanglement \([38, 40]\). The Schmidt mode number \(K\) is defined through the coefficients \(r_k\) by

\[
K \equiv 1 \sqrt{\sum_k r_k^4}. \tag{8}
\]

Take, for example, the case of \(M\) modes with equal weight: \(r_k^2 = 1/M (k = 1, 2, ..., M)\) but \(r_k = 0\) for other \(k\). We have from Eq. (8) \(K = 1/(M \times (1/M^2)) = M\), i.e., the number of modes. Hence, the Schmidt number is an approximate measure of the number of modes in the two-photon state \(|\Psi\rangle\) in Eq. (6).

Experimentally, it is hard to measure the JSF and make the decomposition in Eq. (4). Thus, it is impractical to use Eq. (8) to obtain the mode number. On the other hand, it has been shown that the measurable quantity \(g_{s(i)}^{(2)}\), i.e., the normalized intensity correlation of the individual signal (idler) field alone, which comes from the four-photon state term in spontaneous parametric process (see later in Eqs. (29) and (30)), can be expressed in terms of the Schmidt number as \([15, 41]\)

\[
g_{s(i)}^{(2)} \equiv \int dt_1 dt_2 \langle \hat{I}_{s(i)}(t_1) \hat{I}_{s(i)}(t_2) \rangle = 1 + \frac{\mathcal{E}}{\mathcal{A}} = 1 + \sum_k r_k^4 = 1 + \frac{1}{K}, \tag{9}
\]

where \(\hat{I}_{s(i)}(t) = \hat{E}_{s(i)}^\dagger(t) \hat{E}_{s(i)}(t)\) with \(\hat{E}_{s(i)}(t) = \frac{1}{\sqrt{2\pi}} \int d\omega \hat{a}_{s(i)}(\omega) e^{-j\omega t}\) being the electric field operator of the signal (idler) field and

\[
\mathcal{E} \equiv \int d\omega_s d\omega_i d\omega'_s d\omega'_i \left| F(\omega_s, \omega_i) F(\omega'_s, \omega'_i) F^*(\omega_s, \omega'_i) F^*(\omega'_s, \omega_i) \right|^2 = \sum_k r_k^4
\]

\[
\mathcal{A} \equiv \int d\omega_s d\omega_i d\omega'_s d\omega'_i \left| F(\omega_s, \omega_i) \right|^2 \left| F(\omega'_s, \omega'_i) \right|^2 = 1. \tag{10}
\]

Thus, the measurement of \(g_{s(i)}^{(2)}\) will lead to \(K\) or the number of modes of the two-photon fields and \(g_{s(i)}^{(2)} = 2\) or \(K = 1\) will be a good indication for single-mode operation.

The actual function of the JSF \(F(\omega_s, \omega_i)\) depends on the nonlinear processes and can be engineered accordingly for various tasks in quantum information processing. One of the important tasks is to produce a transform-limited single-photon state by heralding on the detection of one of the correlated photon pair, say, the idler. So, before going to the specific form of \(F(\omega_s, \omega_i)\), let us first examine in the following some key parameters such as the state purity and heralding efficiency for the characterization of the heralded single-photon state.
B. Heralded single-photon state and its purity

The heralding process is a quantum projection in the form of a detection of the idler photon at time $t$, leading to the un-normalized heralded state as

$$|\Psi_1(t)\rangle = \hat{E}_i(t)|\Psi\rangle,$$  \hspace{1cm} (11)

Substituting Eq.(1) into the above, we have

$$|\Psi_1(t)\rangle = \frac{G}{\sqrt{2\pi}} \int d\omega_s d\omega_i \hat{a}_s(\omega) e^{-j\omega t} F(\omega_s, \omega_i) \hat{a}_s(\omega_s) \hat{a}_i(\omega_i) |\text{vac}\rangle,$$

$$= \frac{G^2}{\sqrt{2\pi}} \int d\omega_s d\omega_i e^{-j\omega t} F(\omega_s, \omega_i) \hat{a}_s(\omega_s) |\text{vac}\rangle,$$  \hspace{1cm} (12)

where we used the commutation relation $[\hat{a}_i(\omega), \hat{a}_i^\dagger(\omega_i)] = \delta(\omega - \omega_i)$. If the detection process does not have a good time resolution, especially in the case of two-photon states produced by ultra-fast pulses, the heralded state is a mixed state with average over all time:

$$\hat{\rho}_1 = \int dt |\Psi_1(t)\rangle \langle \Psi_1(t)|$$

$$= G^2 \int d\omega_s d\omega_i d\omega'_s \psi_k(\omega_s) \psi_k^*(\omega'_s) \hat{a}_s(\omega_s) |\text{vac}\rangle \langle \text{vac}| a_s(\omega'_s),$$  \hspace{1cm} (13)

where we used the relation $(1/2\pi) \int d\omega e^{j\omega t} = \delta(\omega)$. Notice that the density operator in Eq.(13) is not normalized due to state projection. With decomposition in Eq.(1) and after proper normalization, we obtain

$$\hat{\rho}_1 = \sum_k r_k^2 \int d\omega_s d\omega'_k \psi_k(\omega_s) \psi_k^*(\omega'_k) \hat{a}_s(\omega_s) |\text{vac}\rangle \langle \text{vac}| \hat{A}_k$$

$$= \sum_k r_k^2 \hat{A}_k^\dagger |\text{vac}\rangle \langle \text{vac}| \hat{A}_k$$

$$= \sum_k r_k^2 |1_k\rangle \langle 1_k|,$$  \hspace{1cm} (14)

where we used the orthonormal relation in Eq.(5) for $\phi_k(\omega_i)$, and $|1_k\rangle \equiv \hat{A}_k^\dagger |\text{vac}\rangle$ is a single-photon state in a single temporal mode $k$ defined by $\hat{A}_k^\dagger \equiv \int d\omega_s \psi_k(\omega_s) \hat{a}_s(\omega_s)$. Eq.(14) describes a mixed multi-mode single-photon state with a state purity of

$$\gamma_P \equiv \text{Tr} \hat{\rho}_1^2 = \sum_k r_k^4 = 1 - \sum_k r_k^2(1 - r_k^2) \leq 1,$$  \hspace{1cm} (15)

where we used $\sum_k r_k^2 = 1$ and the equal sign stands only for the single-mode case of $r_1 = 1, r_k = 0 \ (k \neq 1)$. Note that we have $\gamma_P = 1/K$ from Eq.(8). So, the non-unit purity is because of the multi-mode nature of the two-photon state in Eq.(1), as expressed in the
mode decomposition in Eq. (11). The single-mode case of \( r_1 = 1 \) corresponds to a factorized JSF: \( F(\omega_s, \omega_i) = \psi_1(\omega_s)\phi_1(\omega_i) \) and a purity equal to 1. But non-factorized JSFs will lead to a multi-mode situation with \( r_1 < 1 \) and the heralded photon state has a purity less than 1.

C. Effects of passive optical filtering

Almost all experiment involves optical filtering to discriminate against background light. While the use of passive optical filtering is necessary in experiment, its role on the properties of the filtered photon pairs are mixed. On the one hand, it can reshape the JSF to make it more factorized and improve the mode structure. On the other hand, it destroys the photon correlation between the signal and the idler fields by deleting one of the photons and leads to poor collection and heralding efficiencies, as we will see later. So, we next examine the property of the generated signal (idler) field passing through passive optical filters, which can be modeled as frequency-dependent beam splitter with amplitude transmissivity \( f_{s(i)}(\omega_{s(i)}) \) and reflectivity \( r_{s(i)}(\omega_{s(i)}) \) \( ([f_{s(i)}(\omega_{s(i)})]^2 + [r_{s(i)}(\omega_{s(i)})]^2 = 1) \). Then the state in Eq. (11) is changed to

\[
|\tilde{\Psi}\rangle \approx |\text{vac}\rangle + G \int d\omega_s d\omega_i F(\omega_s, \omega_i)[f_s(\omega_s)\hat{a}_s^\dagger(\omega_s) + r_s(\omega_s)\hat{a}_{sv}(\omega_s)]
\]

\[
\times [f_i(\omega_i)\hat{a}_i^\dagger(\omega_i) + r_i(\omega_i)\hat{a}_{iv}^\dagger(\omega_i)]|\text{vac}\rangle,
\]

where \( \hat{a}_{sv} \) and \( \hat{a}_{iv} \) denote the modes that the filters reject and are replaced by vacuum. The un-normalized projected state after heralding is then

\[
|\tilde{\Psi}_1(t)\rangle = \frac{G}{\sqrt{2\pi}} \int d\omega_s d\omega_i e^{-i\omega_i t} F(\omega_s, \omega_i) f_i(\omega_i)[f_s(\omega_s)\hat{a}_s^\dagger(\omega_s) + r_s(\omega_s)\hat{a}_{sv}^\dagger(\omega_s)]|\text{vac}\rangle.
\]

The heralded photon state, after time integral similar to Eq. (13), becomes

\[
\hat{\rho}_1 = G^2 \int d\omega_s d\omega_s' F(\omega_s, \omega_i) F^*(\omega_s', \omega_i) f_i^2(\omega_i)
\]

\[
\times \left[
\left[f_{s}\left(\omega_{s}\right)f_{s}\left(\omega_{s}'\right)|1_s(\omega_{s})\rangle\langle 1_s(\omega_{s}')\right| + r_{s}(\omega_{s})r_{s}(\omega_{s}')|1_{sv}(\omega_{s})\rangle\langle 1_{sv}(\omega_{s}')\right|
\right]
\]

\[
+ f_{s}(\omega_{s})r_{s}(\omega_{s}')|1_s(\omega_{s})\rangle\langle 1_{sv}(\omega_{s}')| + r_{s}(\omega_{s})f_{s}(\omega_{s}')|1_{sv}(\omega_{s})\rangle\langle 1_s(\omega_{s}')\rangle\right].
\]

Normalization requires the evaluation of the trace of the density operator above:

\[
\text{Tr}\hat{\rho}_1 = G^2 \int d\omega_s d\omega_i |F(\omega_s, \omega_i) f_i(\omega_i)|^2 = G^2 \tilde{A}_1^{1/2},
\]

where \( \tilde{A}_1 \equiv \left[ \int d\omega_s d\omega_i |F(\omega_s, \omega_i) f_i(\omega_i)|^2 \right]^2 \) is similar to that in Eq. (10) but with \( F(\omega_s, \omega_i) \) replaced by \( F(\omega_s, \omega_i) f_i(\omega_i) \). After tracing out the filter-rejected states \( |1_{sv}(\omega)\rangle \) in Eq. (18)
and proper normalization, we arrive at

$$\hat{\rho}' = \text{Tr}_{\bar{A}} \hat{\rho}_1 / \text{Tr} \hat{\rho}_1 = T \sum_k \bar{r}_k^2 |\bar{I}_k\rangle \langle \bar{I}_k| + R |\text{vac}\rangle \langle \text{vac}|,$$

(20)

with

$$T \equiv \frac{\int d\omega_s d\omega_i |F(\omega_s, \omega_i)|^2 f_s^2(\omega_s) f_i^2(\omega_i)}{\int d\omega_s d\omega_i |F(\omega_s, \omega_i)|^2 f_i^2(\omega_i)}$$

$$R \equiv \frac{\int d\omega_s d\omega_i |F(\omega_s, \omega_i)|^2 \bar{r}_k^2(\omega_s) f_i^2(\omega_i)}{\int d\omega_s d\omega_i |F(\omega_s, \omega_i)|^2 f_i^2(\omega_i)},$$

(21)

where $\bar{r}_k$ and $|\bar{I}_k\rangle = \hat{\bar{A}}_k^\dagger |\text{vac}\rangle$ are obtained by Schmidt mode expansion of the filtered JSF

$\hat{F}(\omega_s, \omega_i) \equiv F(\omega_s, \omega_i) f_s(\omega_s) f_i(\omega_i)/N_{si}$ with normalization constant

$$N_{si}^2 \equiv \int d\omega_s d\omega_i |F(\omega_s, \omega_i)|^2 f_s^2(\omega_s) f_i^2(\omega_i).$$

(22)

The purity of the single-photon state in Eq.(20) is

$$\gamma'_{\rho'} = \text{Tr}(\hat{\rho}_1' \hat{\rho}_1') = T^2 \sum_k \bar{r}_k^4 + R^2 = 1 - T^2 \sum_k \bar{r}_k^2(1 - \bar{r}_k^2) - 2TR \leq 1,$$

(23)

where we used $T + R = 1$, $T^2 + R^2 = 1 - 2TR \leq 1$ and $\sum_k \bar{r}_k^2 = 1$ with the equal sign stands if $T = 1$, $R = 0$ and $\bar{r}_1 = 1$, $\bar{r}_k = 0$ ($k \neq 1$). The reduction of the state purity comes from two sources: (i) multi-mode nature, similar to Eq.(14), and (ii) rejection of correlated signal photons due to filtering of the modes and thus the introduction of vacuum. The latter can be understood in terms of the quantity of collection efficiency and the heralding efficiency discussed in the following.

Another key parameter in characterizing the quality of the photon pairs is the collection efficiency of photon pairs, which is defined through the single-photon detection probability and two-photon coincidence detection probability for photon pairs. When signal and idler photons are respectively measured by two detectors, the probability of detecting one photon in individual signal (idler) band per pulse is expressed as

$$P_{s(i)} = \eta_{s(i)} G^2 \int dt \langle \hat{E}^\dagger_{s(i)}(t) \hat{E}_{s(i)}(t) \rangle_{\bar{\Psi}}$$

$$= \eta_{s(i)} G^2 \int d\omega_s d\omega_i |F(\omega_s, \omega_i) f_{s(i)}(\omega_{s(i)})|^2,$$

(24)

where $\eta_{s(i)}$ is the total detection efficiency in the signal (idler) band, respectively, and the average is over the filtered two-photon state $|\bar{\Psi}\rangle$ in Eq.(16). The two-photon coincidence
detection probability per pulse of a photon pair, one from the signal and the other from idler field, is

\[ P_c = \eta_s \eta_r G^2 \int dt_1 dt_2 \langle \hat{E}^\dagger_s(t_1) \hat{E}^\dagger_i(t_2) \hat{E}_i(t_2) \hat{E}_s(t_1) \rangle \psi \]

\[ = \eta_s \eta_r G^2 \int d\omega_s d\omega_i |F(\omega_s, \omega_i) f_s(\omega_s) f_i(\omega_i)|^2. \quad (25) \]

Accordingly, for a photon detected in the idler (signal) band, the probability of detecting its twin photon at signal (idler) band, i.e., the collection efficiency is simply the conditional probability

\[ \xi_s(i) \equiv \frac{P_c}{P_s(i)} = \frac{\eta_s \int d\omega_s d\omega_i |F(\omega_s, \omega_i) f_s(\omega_s) f_i(\omega_i)|^2}{\int d\Omega_s d\omega_i |F(\omega_s, \omega_i) f_s(\omega_i)|^2}, \quad (26) \]

and the probability of a photon emerging at signal (idler) band upon the detection of an idler (signal) photon, or the heralding efficiency

\[ h_s(i) = \frac{\xi_s(i)}{\eta_s(i)} \quad (27) \]

Notice from Eq. (21) and the above that \( T = h_s \). We thus relate the reduction of purity in Eq. (23) directly to the heralding efficiency: when \( \eta_s = 1 \) and \( f_s(\omega_s) \equiv 1 \), meaning no loss of photon for the signal field, the collection efficiency \( \xi_s \) is unit, or \( P_c = P_s \), which leads to a pure single-photon state in the signal field when heralded on the idler photon detection if it is in single-mode (\( \hat{r}_1 = 1 \)).

In addition to modal purity and the collection efficiencies, passive spectral filtering also affects the relation between the modal purity \( \gamma_{P'} \) and the value of \( \tilde{g}^{(2)}_{s(i)} \) in the filtered individual signal (idler) band, which can be calculated using Eq. (9) but with filters at the detectors and is expressed as

\[ \tilde{g}^{(2)}_{s(i)} = 1 + \frac{\tilde{E}_{s(i)} \tilde{A}_{s(i)}}{A_{s(i)}} = 1 + \frac{\int d\omega_s d\omega_i |\hat{f}_{s(i)}(\omega_s(\tilde{s})) f_s(\omega_s(\tilde{s})) F^*(\omega_s, \omega_i) f_i(\omega_i) F(\omega_s, \omega_i)|^2}{\int d\omega_s d\omega_i |\hat{f}_{s(i)}(\omega_s(\tilde{s})) f_s(\omega_s(\tilde{s})) F(\omega_s, \omega_i)|^2}, \quad (28) \]

where \( \tilde{E}_{s(i)}, \tilde{A}_{s(i)} \) are given in Eq. (10) but with the original JSF \( F(\omega_s, \omega_i) \) replaced by the one-side-filtered JSF \( f_s(\omega_s(\tilde{s})) F(\omega_s, \omega_i) \). The dependence on only one filter function \( f_s(\omega_s(\tilde{s})) \) is because it is measured on one side only and has nothing to do with the filter on the other side.

On the other hand, while \( \tilde{g}^{(2)}_{s(i)} \) is an experimentally measurable quantity, the Schmidt number is related to the mode coefficients \( \tilde{r}_k \) from the two-side filtered JSF \( f_s(\omega_s) f_i(\omega_i) F(\omega_s, \omega_i) \),
from which the intensity correlation function $\bar{g}^{(2)}$ can be calculated in Eq.(9) with JSF replaced by the two-side filtered JSF. Since $\bar{g}^{(2)}$ takes the maximum value of 2 for factorized JSF and the more filtered two-side filtered JSF tends to be more close to a factorized function than one-side-filtered JSF, we expect $\bar{g}_{s(i)}^{(2)} \leq \bar{g}^{(2)}$. Although we cannot prove this in general, it is true for the special Gaussian shaped JSF and filtering functions [42]. So, the experimentally measurable $\bar{g}_{s(i)}^{(2)}$ sets a lower bound for $\bar{g}^{(2)}$ which is directly related to the filter-modified Schmidt number $\bar{K} \equiv 1/\sum k \bar{r}_k^4$ or the mode property of the filtered photon pairs.

D. Effects of higher order contributions from multi-pair events

From the discussions of last section, it seems that in order to obtain high purity heralded single photons in the signal band, we only need to improve $h_s$, which can be made equal to 1 by removing the filter in the signal field, and $\bar{g}^{(2)}$, which can be made equal to 2 by heavily filtering the idler field. Of course, this strategy will lead to extremely small $h_i$, which does not seem to matter that much if our interest is in the signal field only. However, when high brightness of the sources is required in some of the multi-photon experiments, higher order contributions of multi-pair events are significant and must be included. But as we will show next, low value of $h_i$ will also hamper the purity of the heralded single-photon state due to the higher photon number events such as four-photon state.

The contributions from multi-pair events will become prominent when the pump power in spontaneous parametric processes is high in order to increase the brightness of the source. In this case, the output quantum state in Eq.(11) needs to be modified to include the next order of four-photon state as [15]

$$|\Psi\rangle \approx |vac\rangle + G|\Psi_2\rangle + (G^2/2)|\Psi_4\rangle$$

(29)

with $|\Psi_2\rangle$ given in Eq.(2) and

$$|\Psi_4\rangle = |\Psi_2\rangle \otimes |\Psi_2\rangle = \int d\omega_s d\omega_i d\omega'_s d\omega'_i F(\omega_s, \omega_i) F(\omega'_s, \omega'_i) \hat{a}_s^\dagger(\omega_s) \hat{a}_s^\dagger(\omega'_s) \hat{a}_i^\dagger(\omega_i) \hat{a}_i^\dagger(\omega'_i) |vac\rangle,$$

(30)

corresponding to a four-photon state due to independent two-pair generation. Using the procedure for the heralded state in Eq.(20) but involving more complicated derivations (see
Appendix A), we find the normalized heralded state as

$$\tilde{\rho}'' = N \left[ T \sum_k \tilde{\rho}_k^2 |\tilde{1}_k\rangle \langle \tilde{1}_k| + R |\text{vac}\rangle \langle \text{vac}| + G^2 \tilde{\rho}_2'/4\tilde{A}_i^{1/2} \right]$$

(31)

with the two-pair contribution as a two-photon state:

$$\tilde{\rho}_2' = \int d\omega_s d\omega_i d\omega_i' f_s(\omega_i) f_s(\omega_i') f_i(\omega_s, \omega_i) \times \int d\omega_i d\omega_i' F(\omega_s, \omega_i) F(\omega_i', \omega_i') \left[ F^*(\omega_s, \omega_i) F^*(\omega_i', \omega_i') + F^*(\omega_s, \omega_i') F^*(\omega_i, \omega_i) \right]$$

$$\times \left[ f_i^2(\omega_i) + f_i^2(\omega_i') \right] |1_s(\omega_i)1_s(\omega_i')\rangle \langle 1_s(\omega_i)1_s(\omega_i')|,$$

(32)

where $|1_s(\omega)\rangle \equiv \hat{a}_s^\dagger(\omega)|\text{vac}\rangle$. $N < 1$ is the normalization factor related to $G$. The existence of the two-photon state will reduce the purity for large $G$. But the more damaging consequence is a nonzero heralded auto-intensity correlation function $\tilde{g}_s^{(2)}$, which is defined as

$$\tilde{g}_s^{(2)} \equiv \frac{\int dt_1 dt_2 \tilde{\Gamma}_s^{(2)}(t_1, t_2)}{\left[ \int dt \tilde{\Gamma}_s^{(1)}(t) \right]^2}$$

(33)

with $\tilde{\Gamma}_s^{(2)}(t_1, t_2) \equiv \text{Tr}[\tilde{\rho}'' \hat{I}_s(t_1) \hat{I}_s(t_2)]$, $\tilde{\Gamma}_s^{(1)}(t) \equiv \text{Tr}[\tilde{\rho}'' \hat{I}_s(t)]$. From this definition, it is obvious that $\tilde{g}_s^{(2)}$ is zero for the heralded state in Eq. (20), which is the signature property of a single-photon state. For the state in Eq. (31), $\tilde{g}_s^{(2)}$ can be calculated through a lengthy derivation (see Appendix A) to have the form of

$$\tilde{g}_s^{(2)} = \frac{2P_c}{\hbar_s \hbar_i} \left( 1 + \frac{\bar{E}}{\bar{A}} \right),$$

(34)

where $h_s(\omega)$ is the heralding efficiency given in Eq. (27), $P_c$ is given in Eq. (25) with $\eta_i = 1 = \eta_s$ and $\bar{E}, \bar{A}$ are given in Eq. (10) but with factors $f_s^2(\omega_s)$, $f_s^2(\omega_i')$, and $f_i^2(\omega_i)$ included.

Equation (34) shows that in order to reduce $\tilde{g}_s^{(2)}$ for high quality heralded single-photon state in the signal field, we need to improve the collection efficiencies in both signal and idler fields. From the discussions in Sect.IIB and above, we find that a high quality two-photon state from spontaneous parametric emission process requires high collection efficiencies in both signal and idler fields, and a high modal purity with a factorized JSF for single-mode operation. However, such strict requirements are difficult to meet from a common two-photon source, as we will see next, unless specific attention is paid to engineer the JSF.

E. An example of typical two-photon sources

To see how well the parameters in the previous sections measure up for some common sources, we next consider a specific form of JSF from spontaneous four-wave mixing (SFWM)
process in a single-mode nonlinear optical fiber [41]. For the generation of two photons with well-defined time, ultrafast pulses are usually deployed as the pump field and we have

\[ F(\omega_s, \omega_i) = N_j \alpha(\omega_s, \omega_i) \times \kappa(\omega_s, \omega_i), \]  

(35)

where \( N_j \) is the normalization factor to make the expression of JSF always satisfying Eq. (3),

\[ \alpha(\omega_s, \omega_i) = \exp \left[ - \frac{(\omega_s + \omega_i - 2\omega_p)^2}{4\sigma_p^2} (1 + jC_p) \right], \]  

(36)

describes the pulsed pump field with a Gaussian spectral envelop of width \( \sigma_p \), central frequency \( \omega_p \), and linear chirp of \( C_p \), and

\[ \kappa(\omega_s, \omega_i) = \text{sinc} \left( \frac{\Delta k L}{2} \right) e^{j\frac{\Delta k L}{2}} \]  

(37)

is the phase matching function with

\[ \Delta k = 2k(\omega_p) - k(\omega_s) - k(\omega_i) - 2\gamma P_p \]  

(38)

as the wave vector mismatch and \( L \) denoting the length of the fiber. In the expression of \( \Delta k \), \( k(\omega_l) \) \((l = p, s, i)\) is the wave vector at \( \omega_l \), \( \gamma \) is the nonlinear coefficient, and \( P_p \) is the peak power of pump. Note that we have assumed that photon pairs at \( \omega_s \) and \( \omega_i \) are created through the scattering of two frequency degenerate pump photons at \( \omega_p \), thus we have the energy conservation relation \( 2\omega_p = \omega_s + \omega_i \).

After omitting the second and higher order dispersive terms in \( \Delta k \), the JSF in Eq. (35) can be written as:

\[ F(\Omega_s, \Omega_i) = N_j \exp \left[ - \frac{(\Omega_s + \Omega_i)^2}{4\sigma_p^2} (1 + jC_p) \right] \times \text{sinc} \left( \frac{\Omega_s A + \Omega_i B}{A} \right) e^{j\left( \frac{\Omega_s A}{A} + \frac{\Omega_i B}{B} \right)} \]  

(39)

where \( \Omega_s = \omega_s - \omega_{s0} \) and \( \Omega_i = \omega_i - \omega_{i0} \) are the frequency biases of the signal and idler photons from the perfectly phase matched frequencies of the signal and idler fields, \( \omega_{s0} \) and \( \omega_{i0} \), respectively, and \( A = 2(k_{p0}^{(1)} - k_{s0}^{(1)})^{-1} L^{-1} \), \( B = 2(k_{p0}^{(1)} - k_{i0}^{(1)})^{-1} L^{-1} \) with \( k_l^{(1)} = dk(\omega)/d\omega|_{\omega_l} \) \((l = p0, s0, i0)\) are parameters depending on the linear dispersion and length of the fiber.

Figure 1(a) shows the contour plot of the JSF in Eq. (39) when \( A = 1.2\sigma_p \) and \( B = 1.8\sigma_p \), in which anti-correlation in frequency is exhibited between the signal and idler fields. Note that we actually plot the absolute square of the JSF, \( |F(\Omega_s, \Omega_i)|^2 \), since it is directly related to the intensity of the photon pairs. The quality of the two-photons with this JSF is then characterized by the Schmidt decomposition. The Schmidt mode expansion coefficients \( r_k^2 \)
FIG. 1. (a) Contour plot of the absolute square of the JSF in Eq. (39), $|F(\Omega_s, \Omega_i)|^2$, when $A = 1.2\sigma_p$ and $B = 1.8\sigma_p$. (b) Calculated Schmidt mode expansion coefficients $r_k^2$. (c) Calculate $\bar{g}_s^{(2)}$ and $\xi_s$ as functions of the bandwidth of filters applied to both signal and idler channels.

is presented in Fig. (b), which shows the multi-mode nature with a Schmidt mode number $K = 6.1$. Such a source is usually not useful for any quantum information protocol.

To produce a better quality two-photon state, a common practice is to use optical filters to modify the JSF. Assuming both filters applied to the signal and idler bands are rectangular shaped with a common filter bandwidth $\sigma_f$ (see Eq. (49) in Sect. III C), we plot $\bar{g}_s^{(2)}$ (calculated via Eq. (28) and related to $K$ from Eq. (9)) and the collection efficiency $\xi_s$ (calculated via Eq. (26) with $\eta_s = 1$) as functions of $\sigma_f$ in Fig. (c). Indeed, filtering can greatly improve the mode structure but at a cost of reduced collection efficiency. It can be seen from Fig. (c) that there are opposite effects on $\bar{g}_s^{(2)}$ and $\xi_s$: $\bar{g}_s^{(2)}$ improves at the expense of the drop of $\xi_s$ and vice versa.

Realizing that filtering has some detrimental effects on collection efficiency, attentions were focused on engineering the JSF into a factorized form, which gives automatically the single-mode case without filtering. The general idea underlying these efforts is that we can engineer the dispersion of the nonlinear medium and therefore the phase mismatch $\Delta k$ in Eq. (37). Together with the control of pump bandwidth $\sigma_p$, we can manipulate the JSF $F(\omega_s, \omega_i)$ to achieve engineering of the two-photon quantum state in Eq. (1) [19–22]. However, as can be seen, the pump spectral function leads to a frequency anti-correlation between $\omega_s$ and $\omega_i$, so, to manipulate $F(\omega_s, \omega_i)$ into a factorable form, the dispersion of the nonlinear medium should fulfill a crucial condition $AB \leq 0$ [19]. Even if it is possible, usually it only works at certain wavelengths determined by the aforementioned parameters, and there is basically no tunability here.

In the next three sections, we will discuss a different method of using a nonlinear interfer-
ometer to engineer JSF without changing the parameters mentioned above. So, we will start
with the JSF not directly factorable, i.e., the signal and idler photon pairs directly out of the
optical fiber are anti-correlated in frequency as shown in Fig. I(a). The state engineering is
achieved through a linear dispersive medium that is independent of two-photon generation
processes but induces a frequency-dependent phase shift and therefore alters the outputs of
the interferometer.

III. ENGINEERING QUANTUM STATES BY A TWO-STAGE NONLINEAR INTERFEROMETER (NLI)

A. The two-stage NLI

Our SU(1,1)-type NLI for demonstrating quantum state engineering consists of two identi-
cal single-mode nonlinear fibers (NFs) with one linear dispersive medium (DM) in between,
as shown in Fig. II. When acting alone, each NF with length $L$ functions as a nonlinear
medium for SFWM process, and the wave vector mismatch in the NF is $\Delta k$ (see Eq. (38)).
For a single NF being pumped by a Gaussian shaped pulse, the two-photon state pro-
duced is in the form of Eq. (1) with the JSF given in Eq. (35). When the NFs and DM are
connected as in Fig. II quantum interference occurs between the fields produced in NF$_1$
and NF$_2$ with the phase being modulated by DM. The DM-induced phase shift, $\Delta \phi_{DM}$, is
frequency(wavelength)-dependent and is a key element in quantum state engineering. When
the pump, signal and idler fields co-propagate through the DM, the phase shift between the
three fields is then

$$\Delta \phi_{DM} = 2\phi_{DM}(\omega_p) - \phi_{DM}(\omega_s) - \phi_{DM}(\omega_i) = \Delta k_{DM} L_{DM}, \quad (40)$$

where $\phi_{DM}(\omega_j)$ ($j = p, s, i$) is the phase of the corresponding field after propagation,

$$\Delta k_{DM} = 2k_{DM}(\omega_p) - k_{DM}(\omega_s) - k_{DM}(\omega_i) \quad (41)$$

is the wave vector difference in the DM and $L_{DM}$ is the length of the DM.

Under the assumption of neglecting all transmission losses, the JSF of photon pairs at
the output of the NLI can be calculated as

$$F_{NLI}(\omega_s, \omega_i) = N_j \exp \left[ - \frac{(\omega_s + \omega_i - 2\omega_p)^2}{4\sigma_p^2} \right] (1 + jC_p) \times \left[ \text{sinc} \left( \frac{\Delta k L}{2} \right) e^{j\Delta \phi_{DM}} \right]$$
FIG. 2. Schematic of the two-stage nonlinear interferometer (NLI).

\[
+\text{sinc}\left(\frac{\Delta k L}{2}\right)e^{j(\frac{\Delta k L}{2}+\Delta k L+\Delta \phi_{DM})}
\]

\[
= N_j \exp\left[-\frac{(\omega_s + \omega_i - 2\omega_{p0})^2}{4\sigma_p^2}(1 + jC_p)\right]\text{sinc}\left(\frac{\Delta k L}{2}\right)e^{j(\frac{\Delta k L}{2}+\theta)\cos\theta}, \quad (42)
\]

where

\[
\theta = \frac{\Delta k L}{2} + \frac{\Delta \phi_{DM}}{2}. \quad (43)
\]

\(\cos\theta\) is the interference factor that can be seen as a result of the two-photon quantum interference. The working principle of the NLI can be explained as follows. When pumped, both NF\(_1\) and NF\(_2\) can produce photon pairs. Furthermore, the two photon-pair generation processes will interfere with each other. The phase difference between the two processes is the phase difference between the pump field (responsible for fields generated in NF\(_2\)) and the signal and idler fields generated by NF\(_1\), and is determined by the phase mismatch in both NF\(_1\) and the DM. Therefore, the overall photon-pair production rate depends on the wavelengths and this NLI scheme functions as an active filter for photon pairs. Usually, we have \(\Delta k \to 0\) to guarantee a significant pair production rate from each NF, so, \(\Delta \phi_{DM}\) becomes the main term determining \(\theta\) (see Eq. (43)). This is exactly what we expect: the NFs are responsible for producing photon pairs with a certain spectrum, while the DM modifies the spectra as an active filter.

In the following subsections, we will characterize the output quantum state from the NLI by substituting actual experimental parameters into the expression of JSF and simulating some key parameters discussed in Sect. II. The effect of \(\Delta \phi_{DM}\) on the modification of JSF will then be visualized through the simulations.
B. Modification of the joint spectrum

In our simulation model, we use single-mode dispersion-shifted fiber (DSF) and standard single-mode fiber (SMF) as the NF and DM, respectively. The experimental realization of this configuration is straightforward [20, 43]. The wavelengths of the signal, idler, and pump fields are all in the 1550 nm telecom band. Although our simulations will be performed in the angular frequency space using the equations in Sect. II, for the sake of convenient demonstration, the results will be presented in the wavelength space, e.g., the JSF will be plotted as a function of the signal and idler wavelengths, $\lambda_s$ and $\lambda_i$, and the optical bandwidths will be specified in terms of wavelength. Note that the angular frequency of light is related to wavelength via $\omega_l = \frac{2\pi c}{\lambda_l}$ ($l = p, s, i, p0, s0, i0$) with $c$ denoting the speed of light in vacuum.

In order to calculate the JSF, we first simplify the expressions of the phase shift induced by SMF and the wave vector mismatch in DSF, i.e., $\Delta \phi_{DM}$ and $\Delta k$. After using the Taylor series of $k_{DM}(\omega)$ and omitting the third- and higher-order terms, the phase shift induced by SMF can be written as

$$\Delta \phi_{DM} = \frac{\lambda_{p0}^2 D_{SMF} L_{DM}}{8\pi c} (\omega_s - \omega_i)^2,$$  \hspace{1cm} (44)

where $\lambda_{p0}$ is the central wavelength of pump and $D_{SMF}$ is the group velocity dispersion (GVD) coefficient at $\lambda_{p0}$. For $\Delta k$ in DSF, the higher-order dispersion is more significant, so we omit the fourth- and higher-order terms of the Taylor series and arrive

$$\Delta k = \frac{k_{p0}^{(2)}}{4} (\omega_s - \omega_i)^2 + \frac{k_{p0}^{(3)}}{8} (\omega_s + \omega_i - 2\omega_{p0})(\omega_s - \omega_i)^2 - 2\gamma P_p,$$  \hspace{1cm} (45)

with $k_{p0}^{(2)} = \frac{\lambda_{p0}^2}{2\pi c} D_{slope}(\lambda_{p0} - \lambda_z)$ and $k_{p0}^{(3)} = -\frac{\lambda_{p0}^3}{(2\pi c)^2} D_{slope}$ where $\lambda_z$ is zero GVD wavelength of DSF and $D_{slope}$ is the GVD slope at $\lambda_z$. We list below the detailed parameters in the simulation. The pump is Gaussian-shaped with central wavelength $\lambda_{p0}=1548.5$ nm, linear chirp parameter $C_p=0$, and bandwidth (full width at half maximum, FWHM) $\Delta \lambda_p=1$ nm. The DSFs have a zero GVD wavelength $\lambda_z=1548.2$ nm with GVD slope $D_{slope}=0.075$ ps/(km-nm$^2$), and the nonlinear self phase modulation term $\gamma P_p=1$ km$^{-1}$. The length of each DSF is $L=50$ m. As for the SMF, the GVD coefficient is $D_{SMF}=17$ ps/(km-nm) at $\lambda_{p0}$, and the length is $L_{DM}=7$ m. By substituting the parameters into Eqs. (44) and (45), we can calculate the JSF at the output of the NLI by using Eq. (42).
For the sake of comparison, we first perform calculation for the non-NLI case of a single-piece 100-m-long DSF, which is equivalent to the NLI case but with the SMF being removed and the two DSFs being connected directly. The JSF of the non-NLI case \((L_{DM}=0 \text{ m})\) is shown in Fig. 3(a), which exhibits a strong frequency anti-correlation between the signal and idler bands. In this case, the only quantity we can control is the width of the JSF stripe, by adjusting the pump bandwidth and/or the length of DSF.

In the NLI case \((L_{DM}=7 \text{ m})\) shown in Fig. 3(b), due to the interference factor \(\cos \theta\), the JSF follows a quasi-periodically varying interference profile and exhibits some kind of “islands” pattern. The maxima of the islands correspond to the maximum-amplitude points of \(\cos \theta\) (for \(\theta = n\pi \ (n = 0, 1, 2, \ldots)\)), while the valleys correspond to zero-amplitude points (for \(\theta = \pi/2 \pm n\pi \ (n = 0, 1, 2, \ldots)\)). The central wavelengths and widths (along the symmetric line) of each island are mainly determined by the DM-induced phase shift \(\Delta \phi_{DM}\). The quasi-periodicity is because \(\Delta \phi_{DM}\) quadratically depend on the frequency detuning between signal and idler (see Eq. 44). For convenience of discussion, as shown in Fig. 3(b), we number the island of the JSF starting from the pump as \(m=0\) and the first whole island is the \(m=1\) island. We also denote the central wavelength of \(m\)th island in the signal (idler) band by \(\lambda_{s0(i0)}^{(m)}\). This numbering rule will be adapted in the rest of this paper. For the \(m=1\) island in Fig. 3(b), we find \(\lambda_{s0(i0)}^{(1)}=1556.7 \ (1540.4) \text{ nm}\).

![FIG. 3. The contour plots of JSF for (a) the non-NLI case of a single-piece 100-m-long DSF \((L_{DM}=0 \text{ m})\) and (b) the NLI case with 7-m-long DM placed in between two 50-m-long DSFs \((L_{DM}=7 \text{ m})\). The marginal intensity distributions are plotted next to the corresponding axes.](image-url)
We then examine the interference pattern of individual signal and idler bands by calculating the marginal spectral distribution, $F_{s(i)}(\omega_{s(i)})$, which is the projection of JSF on the signal (idler) axis: $F_{s(i)}(\omega_{s(i)}) = \int d\omega_i(s) F(\omega_s, \omega_i)$. The results are shown by the curves next to the corresponding axes. From Fig. 3(b), one sees the quasi-periodically varying interference profile in the marginal distribution. This type of interference in frequency domain was observed before in phase-sensitive fiber amplifier [43] and inhomogeneous fibers [44]. It is worth noting that unlike the case of using a single frequency continuous-wave laser as the pump [43], the fringe patterns presented here in the signal and idler bands for the pulse-pumped NLI are asymmetrical. This asymmetry originates from the higher order dispersion of the NF (see the term of $\Delta k L / 2$ in Eq. (43)). Moreover, the visibility of the fringe decreases with the increase of the island number $m$. The non-zero values at the minimum is due to the spectral overlap of adjacent islands. More overlap occurs as $m$ increases. The overlap can be seen as a consequence of pulsed pumping.

The results in Fig. 3 clearly demonstrate that the JSF from NLI is modified by the phase shift in the DM (i.e., the SMF). In increase of $m$, the frequency correlation of each island is changed from negatively-correlated, un-correlated, and positively-correlated. Because the signal and idler photon pairs are amplified or de-amplified in pairs, this effect in NLI can be viewed as a multi-channel band-pass filtering. Different from the passive filtering after the photon pair generation (as discussed in Eq. (16)), the filtering effect in NLI is active and will not introduce loss and un-correlated noise photons, which is in a similar way to an optical parametric oscillator far below threshold [45]. As we will see later, this type of active filter can improve the mode purity of the photon pairs, but will not reduce the collection efficiency as does the passive filtering.

C. Mode structure

With the modified JSF in Eq. (42), let us examine the mode structure for the fields from the NLI and compare it to that without the DM. First of all, since the JSF from NLI is divided into islands, which are separated from or orthogonal to each other, each island can be viewed as an individual JSF by filtering so that the state in Eq. (1) can be rewritten as

$$|\Psi\rangle \approx |\text{vac}\rangle + G_m \sum_{m=0} |\Psi_2^{(m)}\rangle,$$

(46)
where $G_m = G N_m$ and
\[
|\Psi_2^{(m)}\rangle = N_m^{-1} \int d\omega_s d\omega_i F(\omega_s, \omega_i) f_s^{(m)}(\omega_s) f_i^{(m)}(\omega_i) \hat{a}_s^\dagger(\omega_s) \hat{a}_i^\dagger(\omega_i) |\text{vac}\rangle
\] (47)
is the two-photon state of the $m$th island with
\[
N_m^2 \equiv \int d\omega_s d\omega_i |F(\omega_s, \omega_i)|^2 |f_s^{(m)}(\omega_s) f_i^{(m)}(\omega_i)|^2
\] (48)
as the normalization factor and $f_s^{(m)}(\omega_s), f_i^{(m)}(\omega_i)$ as the proper filter functions to isolate the $m$th island in JSF. More specifically, we use the rectangular-shaped filter functions
\[
f_s^{(m)}(\omega_s) = \begin{cases} 
1, & \text{if } |\omega_s - \omega_s^{(m)}| \leq \frac{\sigma_s^{(i)}}{2} \\
0, & \text{if } |\omega_s - \omega_s^{(m)}| > \frac{\sigma_s^{(i)}}{2}
\end{cases}
\] (49)
with $\omega_s^{(m)}$ as the central frequency of the $m$th island and $\sigma_s^{(i)}$ as the bandwidth of the filter.

The two-photon state in Eq.(46) is an entangled state of multiple frequency components and can be viewed as in the form of multi-dimensional entangled states with each island representing a component in the high dimensional space. However, this view relies on that each island in the JSF represents a single-mode two-photon state, which is exactly what we would like to achieve with our NLI. To find the mode property of each island, we examine next the modal purity of each island in the JSF from the NLI.

**D. Modal purity and collection efficiencies**

To examine the modal purity of the islands in JSF from the NLI, we now calculate the intensity correlation function $\bar{g}_{s(i)}^{(2)}$ for the filtered individual signal (idler) photons. The one-side-filtered $\bar{g}_{s(i)}^{(2)}$ sets a lower bound for the two-side filtered $\bar{g}^{(2)}$, which is directly related to the Schmidt number $K$ and describes the modal purity of the filtered photon pairs (see Sect. IIC). For the sake of consistency, the simulation is based on the NLI model described in Sect. IIIB. We calculate $\bar{g}_{s(i)}^{(2)}$ for the first three islands in Fig.3(b) by using Eq.(28) with the JSF and rectangular filter function being given in Eqs.(42) and (49), respectively. Both filters in the signal and idler fields are assumed to have the same bandwidth, i.e., $\sigma_s = \sigma_i = \sigma_f$, where $\sigma_f$ denoting the common filter bandwidth. For comparison to the NLI cases, we also calculate $\bar{g}_{s(i)}^{(2)}$ for the non-NLI case. In the non-NLI case, there is no island structure and the
marginal distributions are relatively flat within the plotted range, therefore, without loss of generality, we use the same filters as that for the \( m=1 \) island.

Figures 4(a) and 4(b) respectively present the calculated \( \bar{g}_s^{(2)} \) and \( \bar{g}_i^{(2)} \) as functions of the common filter bandwidth in terms of wavelength, \( \Delta \lambda_f \). Here we have used the approximate relation \( \Delta \lambda_f = \frac{(1550 \text{ nm})^2}{2\pi c} \sigma_f \) for the 1550 nm band filters. The dashed, dotted, and dash-dotted curves are the result for islands with island numbers \( m=1 \), 2, and 3, respectively, while the solid curves are the result for the non-NLI case. Comparing the results of the signal and idler fields, we find the general trends are similar in both cases, except the differences originated from the spectral asymmetry depicted in Fig. 3(b). Therefore, in the following discussion we will focus only on the results of the signal field (i.e., Fig. 4(a)).

![FIG. 4. Calculated one-side-filtered second-order intensity correlation functions and collection efficiencies as functions of the common filter bandwidth \( \Delta \lambda_f \). (a) and (b) show the one-side-filtered \( \bar{g}_s^{(2)} \) and \( \bar{g}_i^{(2)} \), (c) and (d) show the collection efficiencies \( \xi_s \) and \( \xi_i \). The dashed, dotted, and dash-dotted curves are the results for three NLI cases with island numbers \( m=1 \), 2, and 3, respectively, while the solid curves are results for the non-NLI case.](image)

One sees from Fig. 4(a) that \( \bar{g}_s^{(2)} \) of all the four cases are very close to 2 when \( \Delta \lambda_f < 0.2 \) nm, showing the powerful mode-cleaning effect of an extremely narrow bandpass filter. With the increase of \( \Delta \lambda_f \), the advantage of NLI becomes significant. In some certain range of \( \Delta \lambda_f \), \( \bar{g}_s^{(2)} \) of the NLI cases are higher than that of the non-NLI case, which means an improvement...
of the modal purity. Moreover, for each case in Fig. 4(a), $\bar{g}_s^{(2)}$ decreases with the increase of $\Delta \lambda_f$, but the descent rate for each case is different. Particularly, one sees that there exists a plateau before the sharp drop of $\bar{g}_s^{(2)}$ in the NLI cases whereas in the non-NLI case $\bar{g}_s^{(2)}$ decreases with a nearly constant rate. The plateaus can be seen as the results of the island structure of the interference pattern, while the sharp drop after each plateau is because the components of adjacent islands are also collected as the filter bandwidth increases. The turning point of the sharp drop of $\Delta \lambda_f$ is determined by the valley-to-valley width of the specific island. For example, the turning point for the $m=2$ island is approximately at $\Delta \lambda_f = 3$ nm.

As discussed in Sect.IIC, collection efficiency $\xi_{s(i)}$ is another important factor in obtaining high quality heralded single-photon source. It has an opposite trend to $\bar{g}_s^{(2)}$ as the bandwidth of the filters changes. Now let us examine how the collection efficiencies are affected in the selection of a specific island in the JSF from NLI by filtering. Equation (26) will be used for the calculation of $\xi_{s(i)}$ with $\eta_{s(i)} = 1$. Again, we perform the calculation for the three NLI cases (the islands with numbers $m=1, 2, \text{and } 3$) as well as the non-NLI case. Figures 4(c) and 4(d) respectively show the calculated $\xi_s$ and $\xi_i$ as functions of the common filter bandwidth $\Delta \lambda_f$. From the results of the signal field shown in Fig. 4(c), one sees that although the general trends of $\xi_s$ are still opposite to those of $\bar{g}_s^{(2)}$ due to the detrimental effect of passive filtering, $\xi_s$ for the NLI cases are in general significantly higher than that of the non-NLI case. In particular, there exists a maximum value of $\xi_s$ for the NLI cases, corresponding to the plateau turning point of $\bar{g}_s^{(2)}$ in Fig. 4(a). The existence of the maximum of $\xi_s$ is due to the same reason for the tuning point of $\bar{g}_s^{(2)}$ that some uncorrelated photons from the adjacent islands are also collected by the filter. The maxima of $\xi_s$ are about 98%, 97%, and 96% for the islands with $m=1, 2, \text{and } 3$, respectively. Notice this decreasing trend of the maxima of $\xi_{s(i)}$ with $m$ is in accordance with the drop of fringe visibility in Fig. 3(b). Therefore, we believe the improvement of $\xi_s$ by using NLI is originated from the active filtering effect. Higher visibility of interference fringe means less uncorrelated photons being collected by the filter.

Finally, inspecting Figs 4(a)-(d) together, we find there is an optimum value of filter bandwidth $\Delta \lambda_f$ for which both $\bar{g}_s^{(2)}$ and $\xi_{s(i)}$ are relatively high, for example, $m=3$ island giving the best number of $\xi_s = 96\%$ with $\bar{g}_s^{(2)} = 1.91$. Therefore, passive filtering with proper bandwidth for the output of NLI does not harm the collection efficiency as much as the
non-NLI case. The less-than-ideal performance is because the different islands in the JSF of Fig. 3(b) do not separate far enough to have a clean cut for the filters. This leaves us rooms for further improvement. We will discuss next some methods to increase the separation between different islands and obtain better values for both \( \bar{g}^{(2)}_{s(i)} \) and \( \xi_s(i) \).

IV. FURTHER ENGINEERING FOR BETTER CONTROL OF JSF

In the above theoretical analysis, we have demonstrated that the JSF of photon pairs can be engineered by using a DM-based two-stage NLI to have some sort of island pattern due to the quantum interference. We also find that the overlapping of the adjacent islands is detrimental to creating JSF with a spectrally factorable island. In this section we will respectively resort to two different methods, namely, the programmable optical filtering technology and the multi-stage NLI scheme, to realize a more flexible and precise engineering of JSF. Using these methods, we can create JSF with island patterns that are more factorable and sufficiently-isolated, which is desirable in generating multi-dimensional entanglement. Moreover, we will also discuss how to make full use of each island of the JSF to achieve multi-channel outputs.

A. Using programmable optical filter for arbitrary spectral engineering

A programmable optical filter (POF) can introduce arbitrary phase at different frequency (wavelength), which can be described by phase function \( \phi_{POF}(\omega) \). If we replace the DM with a POF in the two-stage NLI, as shown in Fig. 5(a), the DM-induced phase shift \( \Delta \phi_{DM} \) in Eq. (43) will be accordingly replaced with the POF-induced phase shift \( \Delta \phi_{POF} \), then the interference factor in Eq. (43) becomes

\[
\cos \theta = \cos \left( \frac{\Delta k L}{2} + \frac{\Delta \phi_{POF}}{2} \right),
\]

with

\[
\Delta \phi_{POF} = 2\phi_{POF}(\omega_p) - \phi_{POF}(\omega_s) - \phi_{POF}(\omega_i).
\]

In this case, we can tailor the JSF with much more flexibility by arbitrarily controlling the phase function of POF \( \phi_{POF}(\omega) \).
Different from the DM-induced phase function $\phi_{DM}(\omega)$, which is a continuous function, we can define $\phi_{POF}(\omega)$ as a piecewise function to increase the flexibility of spectral control. For example, suppose we are required to create a JSF with two factorable islands and the $m$th ($m=1, 2$) island should center at some arbitrary frequency $\omega_{(m)}^{(i_0)}$ in the signal (idler) band with $\omega_{s0}^{(m)} + \omega_{i0}^{(m)} = 2\omega_{pc}$, we define $\phi_{POF}(\omega)$ as

$$\phi_{POF}(\omega) = \begin{cases} 
\frac{\pi}{2}, & |\omega - \omega_{pc}| \leq 2\sigma_p, \\
\frac{\pi}{2\sigma_c}(\omega_{s0}^{(m)} - \omega) + \frac{[2+(-1)^m]\pi}{2}, & |\omega - \omega_{s0}^{(m)}| \leq \sigma_c, \\
\frac{\pi}{2\sigma_c}(\omega - \omega_{i0}^{(m)}) + \frac{[2+(-1)^m]\pi}{2}, & |\omega - \omega_{i0}^{(m)}| \leq \sigma_c, \\
\pi, & \omega_{s0}^{(2)} + \sigma_c < \omega < \omega_{s0}^{(1)} - \sigma_c \text{ or } \\
0, & \omega_{i0}^{(1)} + \sigma_c < \omega < \omega_{i0}^{(2)} - \sigma_c \\
\end{cases} \quad (52)$$

where $\sigma_p$ is the bandwidth of the Gaussian shaped pump and $\sigma_c$ is the width of the phase-control range around the central frequency of each island. The dashed lines in Fig. 5(b) mark the phase-control range around $\omega_{s0}^{(1)}$, the signal central frequency of the $m=1$ island. It should be pointed out that although the central frequencies of the islands are arbitrarily selected, the separation of adjacent islands should be larger than $2\sigma_c$, i.e., $|\omega_{s0}^{(1)} - \omega_{s0}^{(2)}| \geq 2\sigma_c$.

To explain Eq. (52), we examine the phase-control range of the $m=1$ island. After substituting Eq. (52) into Eq. (51), we find

$$\Delta \phi_{POF} = \frac{\pi}{2\sigma_c}(\omega_s - \omega_i + \omega_{i0}^{(1)} - \omega_{s0}^{(1)}). \quad (53)$$
One sees the center of the phase-control range \((\omega_s = \omega_s^{(1)}, \omega_i = \omega_i^{(1)})\) corresponds to the maximum-amplitude point of \(\cos \theta\), while the two boundaries of the range correspond to the zero-amplitude points (assuming \(\Delta k \to 0\)). Therefore, together with the pump envelop, an island structure can be created, whose width (along the symmetric line) is determined by \(\sigma_c\). In order to shape the islands into a circular pattern for factorization, we let \(\sigma_c = 1.2\sigma_p\) to approximately match the bandwidths of the cosine function and the Gaussian pump envelop.

Now we find the result of POF by simulation. The simulation is based on the parameters given in Sect. IIB where DSF is used as the NF. Without loss of generality, we set the central wavelengths of the \(m=1(2)\) island of the JSF to be 1554 (1558) nm and 1543 (1539.1) nm in the signal and idler bands, respectively. Based on Eq. (52), we plot the phase function of POF for this situation in Fig. 5(b). Then we calculate the JSF output from the POF-based NLI, which is shown in Fig. 5(c). One sees from Fig. 5(c) that both islands are located at the expected wavelengths. The separation of the two islands is sufficient to have a clean cut by using proper rectangular filters (see the yellow dashed lines). However, although the \(m=1\) island is nearly factorable, the \(m=2\) island is distorted and there exist some unwanted noise pattern on the background of JSF. The distortion and noise are mainly due to the dispersion - especially the high-order dispersion - of the NF (see the term of \(\frac{\Delta k L}{2}\) in Eq. (50)), which is also the same reason responsible for the asymmetry of islands with large \(m\) in Fig. 3(b).

The good news is that we can further tailor the phase function of POF to completely compensate for the dispersion of NF. The modified phase function of POF can be expressed as

\[
\phi'_{POF}(\omega) = \phi_{POF}(\omega) + \phi_C(\omega),
\]

where \(\phi_C(\omega)\) is the compensation term and can be straightforwardly defined as

\[
\phi_C(\omega) = \begin{cases} 
-k(\omega)L + \gamma P_p L, & |\omega - \omega_{pc}| \leq 2\sigma_p, \\
-k(\omega)L, & \text{otherwise} 
\end{cases}
\]

We test this modified \(\phi'_{POF}(\omega)\) by applying it in our simulation. According to the simplified expression of \(\Delta k\) in DSF (Eq. (45)), \(\phi_C(\omega)\) can be accordingly written as

\[
\phi_C(\omega) = \begin{cases} 
-\frac{k^{(2)}}{2}(\omega - \omega_{j0})^2 + \frac{k^{(3)}}{6}(\omega - \omega_{j0})^3 \bigg[ L + \gamma P_p L, & |\omega - \omega_{pc}| \leq 2\sigma_p, \\
-\frac{k^{(2)}}{2}(\omega - \omega_{j0})^2 + \frac{k^{(3)}}{6}(\omega - \omega_{j0})^3 \bigg] L, & \text{otherwise} 
\end{cases}
\]
Then, together with Eqs. (52) and (54), we obtain the modified $\phi'_{POF}(\omega)$ and calculate the JSF for this situation. The results are shown in Figs. (a) and (b), respectively. One sees from the JSF in Fig. (b) that both islands are nearly factorable and a nearly ideal visibility is achieved in the marginal intensity distributions, showing the compensation for the dispersion of DSF is effective. Further more, we calculate the one-side-filtered intensity correlation function $\bar{g}^{(2)}_s$ and collection efficiency $\xi_s$ for the two islands shown in Fig. (b). The results for both islands are almost the same, so we only plot the result of the $m=1$ island in Fig. (c). One sees that $\bar{g}^{(2)}_s > 1.95$ and $\xi_s > 99\%$ can be simultaneously achieved when the common filter bandwidth $\Delta \lambda_f$ in the range of 2.5 nm to 3 nm, so that photon pairs with both high modal purity and collection efficiency can be realized.

The above simulations have revealed that JSF with sufficiently-isolated and factorable islands at arbitrarily chosen wavelengths can be realized using a POF-based NLI. It paves the way for developing a multi-channel source of photon pairs with high modal purity, high collection efficiency, and arbitrary output wavelengths. However, one problem is that the currently available POFs have a relatively high insertion loss, which could limit the performance of the POF-based NLI.
B. Multi-stage NLI for more precise engineering

Another method of fine engineering the JSF is to use a multi-stage NLI. As shown in Fig. 7, an N-stage NLI consists of $N$ pieces of NFs and $N-1$ pieces of DM placed in between every two NFs. Assuming all the NFs (DMs) of the multi-stage NLI are identical and the insertion and transmission losses can be neglected, the two-photon state from the NLI with stage number $N$ is in the form of Eq. (1) but with the JSF being modified as

$$F_{NLI}^{(N)}(\omega_s, \omega_i) = \exp\left[-\frac{(\omega_s + \omega_i - 2\omega_p)^2}{4\sigma_p^2}\right] \times \sin\left(\frac{\Delta k L}{2}\right) \times H(\theta), \quad (57)$$

with

$$H(\theta) = 1 + \sum_{n=1}^{N-1} e^{2jn\theta} = \frac{\sin N\theta}{\sin \theta} e^{j(N-1)\theta}, \quad (58)$$

where $H(\theta)$ with $\theta = \Delta k L/2 + \Delta\phi_{DM}/2$, (Eq. (43)) is a modulation function similar to the interference factor of multi-slit interferometer in classical optics.

To demonstrate the performance of the multi-stage NLI, we perform simulations in the same way as that for the two-stage NLI in Sect.III. Again we employ the DSFs (each of length 50 m) and SMFs (each of length 7 m) as the NFs and DMs, respectively. The parameters given in Sect. IIIB are used as well. Using Eq. (57), we plot the JSFs from the two-stage NLI ($N=2$) and the multi-stage NLIs ($N=3, 4,$ and $5$), which are depicted in Figs. 8(a)-(d), respectively. From Fig. 8, we find the following output features of multi-stage NLI. First, with the increase of stage number $N$, the central wavelengths of the primary islands do not vary but the width of each island (along the symmetric line) decreases. The central wavelengths of the $m=1, 2$ and $3$ islands in the signal (idler) band are 1556.7 nm (1540.4 nm), 1560.2 nm (1537.0 nm), and 1562.8 nm (1534.5 nm), respectively. Second, in
the cases of $N \geq 3$, there exists $N - 2$ secondary islands between two adjacent primary islands, and the ratio of the intensities of the primary and secondary islands increases with the increase of $N$. As a result, the fringe visibility of marginal intensity distributions in the signal band accordingly increases with $N$ as well. These results indicate that JSF with sufficiently-isolated islands can be realized using a multi-stage NLI, which is important for improving the collection efficiency of the photon pairs.

FIG. 8. Contour plots of JSF and intensity distributions in the signal band. (a)-(d) are the results of NLIs with stage number $N=2$, 3, 4, and 5, respectively.

On the other hand, the width (along the symmetric line) of a specified island, which is a crucial factor in creating factorability, is related to the island number $m$ as well as the stage number $N$. For an $N$-stage NLI, it is possible to find a specific island with a definite $m$ number, which is the most factorable. This definite $m$ varies for different stage number $N$. For example, from Figs. 8(b)-(d), we find the most factorable islands are: (i) $m = 3$ island for $N = 3$, (ii) $m = 2$ island for $N = 4$, and (iii) $m = 1$ island for $N = 5$. To characterize the three cases, we respectively calculate the one-side-filtered intensity correlation function $\bar{g}_s^{(2)}$ and collection efficiency $\xi_s$ as functions of the common filter bandwidth $\Delta \lambda_f$ for each island. As shown in Fig. 9 in each case, $\xi_s$ rises quickly with $\Delta \lambda_f$ when the bandwidth $\Delta \lambda_f$ is less than 1.5 nm, but approaches to unity for $\Delta \lambda_f$ in the range of 1.5 nm to 3 nm. Meanwhile, although $\bar{g}_s^{(2)}$ decreases with the increase of $\Delta \lambda_f$, the descending rate is relatively low and depends on the stage number $N$. We find that $\bar{g}_s^{(2)} > 1.95$ and $\xi_s > 95\%$ can be simultaneously achieved for all the three cases.

Although for an $N$-stage NLI only one island can be the most factorable, we can make full use of the multiple stages by successively carving out the factorable islands, which
FIG. 9. Calculated one-side-filtered intensity correlation function $g_s^{(2)}$ and collection efficiency $\xi_s$ as function of the common filter bandwidth $\Delta \lambda_f$ for the mostly factorable islands in Figs. (b)-(d). The island and stage numbers for the three cases are: (a) $m = 3$ and $N = 3$, (b) $m = 2$ and $N = 4$, and (c) $m = 1$ and $N = 5$.

can be realized by inserting suitable dual-band band-pass filters (BPFs) having reflection ports. Figure 10 depicts an example of a five-stage NLI, in which the dispersion induced by the BPFs is neglected. Firstly, the $m=3$ island (see Fig. 8(b)) is filtered and selected by BPF1 after DSF$_3$. The two 1.5-nm-width rectangular passbands of BPF1 are centering at 1562.8 nm and 1534.5 nm, respectively, so the signal and idler photons of $m=3$ island can be extracted while the residual photons and pump can be sent to the next stage via the reflection port. The contour plot of the JSF for the output of BPF1 (Output1) is shown in Fig. 10. Similarly, the $m=2$ and $m=1$ islands in Fig. 8(c) and (d) are extracted by BPF2 and BPF3, respectively, which also have rectangular shaped passbands with suitable bandwidths and central wavelengths. The JSFs of the corresponding outputs (Output2, Output3) are also depicted in Fig. 10. It can be seen that all the JSFs are nearly round and factorized. In this way, a multi-channel source of photon pair with high purity and efficiency can be realized, which can be further used to obtain multi-dimensional entanglement or multi-channel single photon source [40]. Moreover, it is worth noting that the multi-stage NLI can benefit from the low insertion loss of the fiber system. Compared with POF-based two-stage NLI, the multi-stage NLI have the advantages of low cost, low insertion loss, and high flexibility of selective output.
FIG. 10. Three-channel source of spectrally factorable photon pairs based on a five-stage NLI. Contour plots of the JSFs for the outputs 1-3 are shown next to the corresponding ports of BPF1-BPF3, respectively. DSF, dispersion shifted fiber; SMF, single-mode fiber, BPF, dual-channel bandpass filter with a reflection port.

V. SUMMARY AND DISCUSSION

In this paper, we propose and analyze a novel interferometric method to engineer the joint spectral function (JSF) of a two-photon state generated from spontaneous parametric emission. We achieve this by employing nonlinear interferometer (NLI) schemes. We successfully modify an original frequency anti-correlated JSF from dispersion shifted fiber to a nearly factorized JSF using a two-stage NLI. We further refine the two-stage NLI with a programmable optical filter and extend our discussion to a multi-stage NLI for finer engineering.

The role played by NLI in JSF engineering is spectral filtering. But different from passive filtering after the production of the photon pairs, which may destroy the correlation between the photons in a pair, the spectral filtering achieved in NLI is an active filtering scheme that selectively produces the photon pair and thus preserves the photon correlation. Because of this, the engineered two-photon states maintain a high collection efficiency while have the spectral properties modified to become frequency uncorrelated for near unity modal purity, which gives rise to a transform-limited two-photon state.

Our investigation provides a new approach for engineering the spectral property of photon
pairs and for obtaining narrow band photon pairs simultaneously possessing the advantages of high purity, high collection efficiency and high brightness, which are an important resource of quantum states for quantum information and communication. Compared with the methods of directly generating factorable photon pairs by engineering the dispersion property of nonlinear medium, our NLI approach is easy to implement and has flexible output. The wavelength of photon pairs can be conveniently tuned for various tasks. Compared with the methods of generating factorable photon pairs by applying narrow band filters in signal and idler bands, our NLI approach does not decrease the brightness of photon pairs and collection efficiency. Moreover, by introducing the multi-stage design into our NLI approach, we are able to develop a source delivering fully factorable photon pairs entangled in multi-frequency channels for high dimensional entanglement.

The interferometric approach discussed here can also be applied to high gain situation for modal purity, which is even more critical in exploring quantum entanglement with continuous variables [39]. However, the case becomes complicated when the pump power is increased in nonlinear fibers for high gain because power-dependent nonlinear phase shift will alter the spectral properties of the generated fields as well. Losses is also a critical factor to consider in quantum entanglement with continuous variables in the design involving a programmable optical filter and multiple stages. Nevertheless, the interferometric approach provides us more flexibility in engineering quantum states.

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Appendix A: Derivation of the heralded auto-intensity correlation function

In this appendix we give the procedure for deriving Eqs. (31) to (34), which finally lead to the expression of the heralded auto-intensity correlation function $\tilde{g}^{(2)}_s$. The state given in Eq. (11) is a low pump power approximation for the quantum state of the light field from spontaneous parametric processes. Higher photon number terms will start to contribute when the pump power is high in order to increase the brightness of the source. The next order modification includes a four-photon state and leads to a state of

$$|\Psi\rangle \approx |vac\rangle + G|\Psi_2\rangle + (G^2/2)|\Psi_4\rangle$$

with $|\Psi_2\rangle$ given in Eq. (2) and the four-photon modification of

$$|\Psi_4\rangle = |\Psi_2\rangle \otimes |\Psi_2\rangle = \int d\omega_s d\omega_i d\omega'_s d\omega'_i F(\omega_s, \omega_i) F(\omega'_s, \omega'_i) \hat{a}_s^\dagger(\omega_s) \hat{a}_i^\dagger(\omega_i) \hat{a}_s^\dagger(\omega'_s) \hat{a}_i^\dagger(\omega'_i) |vac\rangle.$$

The four-photon term corresponds to two-pair production and will lead to a two-photon state in the heralded field. We will calculate this two-photon state from the four-photon modification term in Eq. (A2) in this Appendix.

For completeness of discussion, we will include the passive filters. As in Eq. (16), the passive filters are modeled as beam splitters by replacing $\hat{a}_s(\omega)$ with $\hat{a}_s'(\omega) \equiv f_s(\omega)\hat{a}_s(\omega) + r_s(\omega)\hat{a}_sv(\omega)$ and $\hat{a}_i(\omega)$ with $\hat{a}_i'(\omega) \equiv f_i(\omega)\hat{a}_i(\omega) + r_i(\omega)\hat{a}_iv(\omega)$. The contributions to the heralded state from the first two terms in Eq. (A1) are exactly the same as Eq. (18). So, we will only calculate the contribution from the four-photon term here.

The heralding detection of an idler photon at time $t$ collapses the four-photon state in Eq. (A2) to

$$\hat{E}_i(t) |\Psi_4\rangle = \int d\omega_s d\omega_i d\omega'_s d\omega'_i F(\omega_s, \omega_i) F(\omega'_s, \omega'_i) \hat{a}_s^\dagger(\omega_s) \hat{a}_i^\dagger(\omega_i) \hat{a}_s^\dagger(\omega'_s) \hat{a}_i^\dagger(\omega'_i) \hat{E}_i(t) \hat{a}_i^\dagger(\omega_i) \hat{a}_i^\dagger(\omega'_i) |vac\rangle = \frac{1}{\sqrt{2\pi}} \int d\omega_s d\omega_i d\omega'_s d\omega'_i F(\omega_s, \omega_i) F(\omega'_s, \omega'_i) \hat{a}_s^\dagger(\omega_s) \hat{a}_i^\dagger(\omega_i) \hat{a}_s^\dagger(\omega'_s) \hat{a}_i^\dagger(\omega'_i)$$
where $|\Psi_{4}\rangle$ is the filtered state and $\hat{E}_i(t) = \frac{1}{\sqrt{2\pi}} \int d\omega e^{-j\omega t} \hat{a}_i(\omega)$. With slow heralding detectors unable to time resolve the pulsed field, the projected density operator becomes

$$\hat{\rho}_{\text{proj}} = \frac{G^4}{4} \int dt \hat{E}_i(t) |\Psi_{4}\rangle \langle \Psi_{4} | \hat{E}_i^\dagger(t)$$

$$= \frac{G^4}{4} \int d\omega_i d\omega'_i d\omega_s d\omega'_s |1'_s(\omega_s)1'_s(\omega'_s)\rangle \langle 1'_s(\omega_s)1'_s(\omega'_s)|$$

$$\times \int d\omega_i d\omega'_i d\omega_s d\omega'_s |1'_s(\omega_s)1'_s(\omega'_s)\rangle \langle 1'_s(\omega_s)1'_s(\omega'_s)|$$

$$\times \frac{1}{2\pi} \int dt \left[ e^{-j\omega_{i}t} f_i(\omega_i) |1'_s(\omega'_s)\rangle \langle 1'_s(\omega'_s)| + e^{-j\omega'_{i}t} f'_i(\omega'_i) |1'_s(\omega'_s)\rangle \langle 1'_s(\omega'_s)| \right].$$

(A4)

After carrying out the time integral and trace out the idler photons, we obtain

$$\hat{\rho}_2 = \text{Tr} \hat{\rho}_{\text{proj}}$$

$$= \frac{G^4}{4} \int d\omega_i d\omega'_i d\omega_s d\omega'_s |1'_s(\omega_s)1'_s(\omega'_s)\rangle \langle 1'_s(\omega_s)1'_s(\omega'_s)|$$

$$\times \int d\omega_i d\omega'_i F(\omega_i, \omega'_i)[f_i^2(\omega_i) + f'_i^2(\omega'_i)]$$

$$\times [F^*(\omega_s, \omega_i) F^*(\omega'_s, \omega'_i) + F^*(\omega_s, \omega'_i) F^*(\omega'_s, \omega_i)].$$

(A5)

Proper normalization requires the evaluation of the trace of the density operator, which gives

$$\text{Tr} \hat{\rho}_2 = G^4 \int d\omega_i d\omega'_i \int d\omega_j d\omega'_j f_j^2(\omega_j) F(\omega_j, \omega_i) F(\omega'_j, \omega'_i)$$

$$\times [F^*(\omega_j, \omega_i) F^*(\omega'_j, \omega'_i) + F^*(\omega_j, \omega'_i) F^*(\omega'_j, \omega_i)]$$

$$= G^4 (\hat{A}_i + \hat{E}_i),$$

(A6)

where $\hat{A}_i, \hat{E}_i$ are similar to those in Eq.(13) except the filtering factor $f_i^2(\omega_i)$ is included. Tracing out the vacuum photons in the signal field, we obtain the two-photon part of the heralded state in the signal field upon detection of one idler photon:

$$\hat{\rho}'_2 = \hat{\rho}_0 + \hat{\rho}_1 + \hat{\rho}_2,$$

(A7)

where $\hat{\rho}_0, \hat{\rho}_1$ are the vacuum and one-photon terms whose exact forms are unimportant because they only give higher order corrections to the state in Eq.(13); $\hat{\rho}_2$ are the two-photon term and has the form of

$$\hat{\rho}_2 = \frac{G^4}{4} \int d\omega_i d\omega'_i d\omega_s d\omega'_s f_s(\omega_s) f_s(\omega'_s) f_s(\omega_s) f_s(\omega'_s)$$

$$\times e^{-j\omega_{i}t} \hat{a}_i^\dagger(\omega_i) f_i(\omega_i) + e^{-j\omega'_{i}t} \hat{a}_i^\dagger(\omega'_i) f'_i(\omega'_i)|\text{vac}\rangle,$$

(A3)
\[
\times \int d\omega_1d\omega_1'F(\omega_s,\omega_1)F(\omega_1',\omega_1')\left[F^*(\bar{\omega}_s,\omega_1)F^*(\bar{\omega}_s',\omega_1') + F^*(\bar{\omega}_s,\omega_1')F^*(\bar{\omega}_s',\omega_1)\right]
\times \left[f_i^2(\omega_i) + f_i^2(\omega_i')\right] |1_s(\omega_1)|^2 |1_s(\omega_1')|^2 |1_s(\omega_s)|^2 |1_s(\omega_s')|^2].
\tag{A8}
\]

Combining the above with Eq. (19) which is the contribution from the first two terms of \(\tilde{\Gamma}_s\), we find the normalized heralded state as

\[
\hat{\rho}'' = \mathcal{N} \left[ T \sum_k \hat{\rho}_k^2 |\hat{I}_k\rangle\langle \hat{I}_k| + R|\text{vac}\rangle\langle\text{vac}| + G^2(\hat{\rho}_1' + \hat{\rho}_1 + \hat{\rho}_2')/4 \bar{A}_1^{1/2} \right],
\tag{A9}
\]

where \(\hat{\rho}_l' \equiv 4\hat{\rho}_l/G^4\) for \(l = 0, 1, 2\) with \(\bar{A}_i\) given in Eq. (19). \(\mathcal{N} = [1 + CG^2]^{-1}\) is the normalization factor with \(C\) as some constant related to the JSF \(F(\omega_s,\omega_i)\) and filter functions \(f_s, f_i, r_s, r_i\). \(\mathcal{N} \approx 1\) for \(G \ll 1\).

The heralded auto-intensity correlation function \(\hat{g}_{s}^{(2)}\) is defined as

\[
\hat{g}_{s}^{(2)} \equiv \frac{\int dt_1dt_2 \hat{\Gamma}_s^{(2)}(t_1,t_2)}{\int dt \hat{\Gamma}_s^{(1)}(t)^2},
\tag{A10}
\]

with

\[
\hat{\Gamma}_s^{(2)}(t_1,t_2) \equiv \text{Tr}[\hat{\rho}'' \hat{I}_s(t_1)\hat{I}_s(t_2)], \quad \hat{\Gamma}_s^{(1)}(t) \equiv \text{Tr}[\hat{\rho}'' \hat{I}_s(t)].
\tag{A11}
\]

\(\hat{\rho}_2'\) is the only term in \(\hat{\rho}''\) that will contribute to \(\hat{\Gamma}_s^{(2)}(t_1,t_2)\), which is calculated as

\[
\hat{\Gamma}_s^{(2)}(t_1,t_2) = \text{Tr} \left[ \hat{\rho}'' \hat{I}_s(t_1)\hat{I}_s(t_2) \right]
= \frac{1}{(2\pi)^2} \frac{G^2}{4 \bar{A}_1^{1/2}} \text{Tr} \left[ \hat{\rho}_2' \int d\bar{\omega}_s1d\bar{\omega}_s1'd\bar{\omega}_s2d\bar{\omega}_s2'e^{-i(\bar{\omega}_s1t_1+\bar{\omega}_s2t_2)} \times e^{i(\bar{\omega}_s1t_1+\bar{\omega}_s2t_2)} \hat{a}_{s}^\dagger(\bar{\omega}_s1)\hat{a}_{s}^\dagger(\bar{\omega}_s2)\hat{a}_{s}(\bar{\omega}_s1)\hat{a}_{s}(\bar{\omega}_s2) \right]
= \frac{1}{(2\pi)^2} \frac{G^2}{4 \bar{A}_1^{1/2}} \int d\omega_s d\omega_s' d\omega_s d\omega_s' f_s(\omega_s)f_s(\omega_s')f_s(\omega_s')f_s(\omega_s') \times \int d\omega_1d\omega_1'F(\omega_s,\omega_1)F(\omega_1',\omega_1') \left[f_i^2(\omega_1) + f_i^2(\omega_1')\right]
\times \left[F^*(\bar{\omega}_s,\omega_1)F^*(\bar{\omega}_s,\omega_1') + F^*(\bar{\omega}_s,\omega_1')F^*(\bar{\omega}_s,\omega_1)\right]
\times \left[e^{-i(\omega_s1+\omega_s2)} + e^{-i(\omega_s1+\omega_s2)}\right]
\times \left[e^{i(\omega_s1+\omega_s2)} + e^{i(\omega_s1+\omega_s2)}\right].
\tag{A12}
\]

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Then we carry out the time integral

\[
\int dt_1 dt_2 \tilde{\Gamma}^{(2)}_s(t_1, t_2) = \frac{G^2}{4A_1^{1/2}} \int d\omega s d\omega' s d\omega'_s d\omega'_s f_s(\omega_s) f_s(\omega'_s) f_s(\omega_s) f_s(\omega'_s) \\
\times \int d\omega i d\omega'_i F(\omega_s, \omega_i) F(\omega'_s, \omega'_i) \left[ f^2_i(\omega_i) + f^2_i(\omega'_i) \right] \\
\times \left[ F^*(\omega'_s, \omega_i) F^*(\omega'_s, \omega'_i) + F^*(\omega_s, \omega'_i) F^*(\omega'_s, \omega_i) \right] \\
\times 2 [\delta(\omega_s - \omega'_s) \delta(\omega'_s - \omega'_i) + \delta(\omega'_s - \omega_s) \delta(\omega_s - \omega'_i)]
\]

(A13)

with

\[
\tilde{A} = \int d\omega_s d\omega'_s \int d\omega_i d\omega'_i f^2_s(\omega_s) f^2(\omega'_s) f^2_i(\omega_i) \left| F(\omega_s, \omega_i) F(\omega'_s, \omega'_i) \right|^2
\]

(A14)

and

\[
\tilde{E} = \int d\omega_s d\omega'_s \int d\omega_i d\omega'_i f^2_s(\omega_s) f^2(\omega'_s) f^2_i(\omega_i) \\
\times F(\omega_s, \omega_i) F^*(\omega'_s, \omega_i) F(\omega'_s, \omega'_i) F^*(\omega'_s, \omega_i),
\]

where \( P_c, P_s, P_i \) are given in Eq. (24, 25) with \( \eta_s = 1 = \eta_i \). From Eq. (18), we find

\[
\tilde{\Gamma}^{(1)}_s(t) = \text{Tr} \left[ \tilde{\rho}'' \tilde{I}(t) \right] \\
= \text{Tr} \left[ \tilde{\rho}' \tilde{E}_s(t) \tilde{E}_s(t) \right] \\
= \frac{1}{2\pi} \frac{1}{P_i} \text{Tr} \left[ \int d\omega_s d\omega'_s d\omega_i F(\omega_s, \omega_i) F^*(\omega'_s, \omega_i) f^2_i(\omega_i) f_s(\omega_s) f_s(\omega'_s) e^{-i(\omega_s - \omega'_s)t} \right],
\]

(A16)

then we carry out the time integral

\[
\int dt \tilde{\Gamma}^{(1)}_s(t) = \frac{1}{2\pi} \frac{1}{P_i} \int dt \text{Tr} \left[ \int d\omega_s d\omega'_s d\omega_i F(\omega_s, \omega_i) F^*(\omega'_s, \omega_i) \\
\times f^2_i(\omega_i) f_s(\omega_s) f_s(\omega'_s) e^{-i(\omega_s - \omega'_s)t} \right]
\]

(A17)

\[
= \frac{1}{P_i} \int d\omega_s d\omega_i |F(\omega_s, \omega_i)|^2 f^2_s(\omega_s) f^2_i(\omega_i)
\]

\[
= \frac{P_c}{P_i}.
\]

After substituting Eqs. (A13) and (A17) into Eq. (A10), we arrive the heralded auto-intensity correlation function

\[
\tilde{g}^{(2)}_s = \frac{2P_c}{h_i h_s} (1 + \frac{\tilde{E}}{\tilde{A}}),
\]

(A18)

where \( h_i, h_s \) are the heralding efficiencies given in Eq. (26) with \( \eta_i = 1 = \eta_s \).