Improved Upper Bounds on the Spreads of Some Large Sporadic Groups

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Let $G$ be a group. We say that $G$ has spread $r$ if for any set of distinct elements $\{x_1, \ldots, x_r\} \subset G$ there exists an element $y \in G$ with the property that $\langle x_i, y \rangle = G$ for every $1 \leq i \leq r$. Few bounds on the spread of finite simple groups are known. In this paper we present improved upper bounds for the spread of many of the sporadic simple groups, in some cases improving on the best known upper bound by several orders of magnitude.

1 Introduction

Recall that a group is said to be 2-generated if it can be generated by a just two of its elements. It is well known that every finite simple group is 2-generated [1] and many authors have considered the question of how easily a pair of elements generating a simple group may be obtained. One quantity measuring this introduced by Brenner and Wiegold in [5] is the concept of the spread of a group.

Let $G$ be a group. We say that $G$ has spread $r$ if for any set of distinct elements $X := \{x_1, \ldots, x_r\} \subset G$ there exists an element $y \in G$ with the property that $\langle x_i, y \rangle = G$ for every $1 \leq i \leq r$. We say that this element $y$ is a mate of $X$. We further say that $G$ has exact spread $r$ if $G$ has spread $r$ but not $r + 1$. We write $s(G)$ for the exact spread of $G$.

In [9] Guralnick and Kantor used probabilistic methods to show that if $G$ is a finite simple group then $s(G) \geq 2$. They went on to classify the finite simple groups for which the spread is exactly 2 showing that if $G$ is a finite simple group then $s(G) \geq 3$ unless $G$ is one of $S_{2m}(2)$ ($m \geq 3$), $A_5$, $A_6$ or $\Omega^+_{3}(2)$. Various asymptotic results about spreads are proved by Guralnick and Shalev in [10]. In [2] Binder proved a variety of results regarding the spread of the alternating and symmetric groups and in particular suggested that the exact spread of the alternating group $A_{2n+1}$ tended to infinity with $n$. In [5] Brenner and Wiegold proved that for $q$ a prime-power ($q \geq 11$, if $q$ is odd and $q \geq 4$, otherwise) we have

$$s(L_2(q)) = \begin{cases} 
q - 1 & \text{if } q \equiv 1(4); \\
q - 4 & \text{if } q \equiv 3(4); \\
q - 2 & \text{if } 2 \text{ divides } q.
\end{cases}$$

Much less is known in the case of the sporadic groups, indeed there is only one sporadic group for which we know the precise value of the exact spread: $s(M_{11}) = 3$. This was proved by Woldar in [12] and independently using more geometric methods by Bradley and Holmes in [3]. In [8] Ganif and Moori obtained lower bounds for the spreads of all the sporadic groups and more recently these were improved upon by Breuer, Guralnick and Kantor in [6]. In [4] Bradley and Moori obtained upper bounds for the spreads of all the sporadic
groups, though for many of the smaller sporadic groups these bounds were later improved upon by Bradley and Holmes in [3]. Their methods were heavily computational and as such were unable to tackle the larger sporadic groups. Here we improve upon the upper bounds obtained by Bradley and Moori for the spreads of all the larger sporadic groups, with the exception of the Baby Monster, that were not already improved upon by Bradley and Holmes. More specifically we prove the following theorem.

**Theorem 1** Let $G$ be a sporadic group. Then $s(G)$ is at most the bound given in Table 1.

For comparison we also include in Table 2 the previously best known upper bounds as computed by Bradley and Moori in [4] for the groups we consider here. Whilst not every bound given in Table 1 is new, we include the full list for completeness. The bounds appearing in Table 1 are, as far as the author is aware, the best known, accepting that Bradley and Holmes claim that for the groups they considered ‘better results were obtained for some of the groups in trial runs, but our table gives only the results that were given by known seeds’ [3, p.138].

Whilst our methods are not entirely computer-free, for the most part they solely depend on data given in the Atlas [7] only relying on machine calculations in the most extreme cases.

Throughout we shall use the standard Atlas notation for finite groups, as defined in [7].

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This article is organised as follows. In Section 2 we introduce some concepts that will be of use later on. In Section 3 we prove some preliminary structural results for some of the groups under consideration and finally in Section 3 we pull together these results to prove Theorem 1.

## 2 Preliminaries

In this section we shall describe some concepts that will be useful in proving Theorem 1.

### 2.1 Support

Let $G$ be a group and let $x \in G^\#$ where $G^\# := G \setminus \{e\}$. Following Woldar [12], we define the support of $x$ to be the set

$$supp(x) := \bigcup\{y \in H^\# | H < G, x \in H\}.$$

In other words $y \in supp(x)$ means that $y$ is an element of $G^\#$ that lies in a proper subgroup containing $x$ and so $y$ cannot be a mate for the set $\{x\}$. We extend this to subsets $X \subset G^\#$ as follows:

$$supp(X) := \bigcup\{y \in supp(x) | x \in X\}.$$
Table 1: Upper bounds for the spreads of the sporadic groups.

| $G$   | upper bound on $s(G)$ |
|-------|------------------------|
| $M_{11}$ | 3                      |
| $M_{12}$ | 9                      |
| $J_1$   | 179                    |
| $M_{22}$ | 26                     |
| $J_2$   | 24                     |
| $M_{23}$ | 41020                  |
| HS      | 33                     |
| $J_3$   | 597                    |
| $M_{24}$ | 56                     |
| McL     | 308                    |
| He      | 1223                   |
| Ru      | 1252799                |
| Suz     | 956                    |
| O’N     | 2857238                |
| $Co_3$  | 1839                   |
| $Co_2$  | 1024649                |
| $Fi_{22}$ | 186                  |
| HN      | 75603374               |
| $Ly$    | 1296826874             |
| Th      | 976841774              |
| $Fi_{23}$ | 31670                |
| $Co_1$  | 46621574               |
| $J_4$   | 47766599363            |
| $Fi_{24}'$ | 7819305288794       |
| $B$     | 3843675651630431666542962843030 |
| $M$     | 5791748068511982636944259374 |

Table 2: Best previous upper bounds for the groups considered here.

| $G$   | upper bound on $s(G)$ |
|-------|------------------------|
| Ru    | 12990752                |
| O’N   | 5960127                 |
| $Co_2$ | 5240865                |
| HN    | 229665984               |
| $Ly$  | 112845651178977         |
| Th    | 103613642531            |
| $Fi_{23}$ | 8853365473          |
| $Co_1$ | 58021747714            |
| $J_4$  | 251012689269463297      |
| $Fi_{24}'$ | 309163967798745777216   |
| $M$   | 1458780427083962616126802411518683420794682668030 |
If $Y \subset \text{supp}(X)$ we say that $X$ supports $Y$. In particular, elements of $\text{supp}(X)$ cannot be a mate to $X$. An easy lemma proved in [12, lemma 1.1] is the following.

**Lemma 2** Let $G$ be a group and let $X \subset G^\#$ be a set which supports $G^\#$. Then $s(G) \leq |X| - 1$.

It is this observation that is key to the proof of Theorem 1. For each of the bounds that we improve upon here, our improved bound is obtained by showing that some small conjugacy class of involutions of $G$ supports $G^\#$ in the cases we consider here.

### 2.2 The ‘Even Order Trick’

Let $g \in G$ and suppose that $h \in G$ has the properties that $g = h^i$ for some $i$ and $o(h) = 2k$ for some integer $k$. We have that $g \in C_G(h^k)$ and so $h^k$ supports $g$. Suppose the center of a Sylow 2-subgroup of $G$ contains elements from class $2X$. Then elements from class $2X$ must also lie in $C_G(h^k)$ since the involution $h^k$ must lie in some Sylow 2-subgroup. In particular we have that $g$ supports some element of class $2X$ and so $g$ cannot be a mate to set of all elements in class $2X$.

For example consider the alternating group $A_8$. The center of a Sylow 2-subgroup is generated by an element conjugate to the involution $(12)(34)(56)(78)$. The element $g = (123)$ cannot be a mate to the class of these elements since it is the square of the element $h = (56)(78)(132)$ and thus belongs to the centralizer of $(56)(78)$ which also contains the element $(12)(34)(56)(78)$.

Note in particular if $G$ contains only two classes of involutions then the even order trick immediately implies that any element of even order (and all of their powers) are contained in a proper subgroup containing both classes of involutions (namely involution centralizers).

This simple observation will repeatedly enable us to immediately eliminate most elements of a group from being a mate to the whole of a conjugacy class of involutions. We call this the even order trick.

### 3 Some Preliminary Lemmata

In this section we prove a series of lemmata giving structural properties of several of the sporadic groups.

**Lemma 3** Every element of $Ru$ is contained in a proper subgroup that contains elements of class $2B$.

*Proof.* Since $Ru$ contains only two classes of involutions the even order trick tells us that every element is contained in some proper subgroup containing $2B$ elements except possibly elements in classes 15A and 29A. Elements of class of order 15 are contained in the full $2B$ centralizer which is contained in the maximal subgroup $(2^2 \times Sz(8)) : 3$. The only maximal subgroup containing elements of order 29 have structure $L_2(29)$ and these must contain $2B$ elements since they also contain elements of order 14 and all elements of order 14 in $Ru$ power up to elements in class $2B$. □
Note that we cannot replace class 2B with class 2A (and thus improve upon the above result using our approach) since no proper subgroups contain elements from both classes 29A and 2A.

**Lemma 4** Every maximal subgroup of the Conway group $\text{Co}_2$ contains an element from class 2B.

**Proof.** There are 11 classes of maximal subgroup in $\text{Co}_2$ [ATLAS,p.154]. Three of these contain 2B elements by Sylow’s theorem. A further class of maximal subgroups is the full centralizer of a 2B element. The permutation character for the subgroup $U_6(2):2$ is listed in the ATLAS [7]. The remaining classes are easily verified by Magma. □

Note we cannot replace 2B by another class of involutions since no element from any other class of involutions is contained in a proper subgroup also containing elements of class 23A.

**Lemma 5** Every element of $\text{HN}$ is contained in a proper subgroup containing an element of class 2B.

**Proof.** Since $\text{HN}$ contains only two classes of involutions the even order trick tells us that every element is contained in some proper subgroup containing 2B elements except possibly elements in classes 9A, 19A/B, 25A/B, 35A/B. Each of these remaining cases may be tackled as follows.

- Using the permutation character for the maximal subgroup isomorphic to $A_{12}$ must contain elements of classes 35A/B, 9A and 2B. Since this permutation character is listed in the ATLAS we explicitly give this below.

$$1a+133ab+760a+3344a+8910a+16929a+35112ab+267520a+365750a+406296a$$

- The maximal subgroup with structure $5^{1+4} : 2^{1+4} 5 4$ contains the full centralizer of an element of class 5B. Since elements of class 10C power up to elements in class 2B and 5B this subgroup must also contain elements in class 2B. Elements in classes 25A/B each power up to elements of class 5A so these must also be contained in this subgroup.

- The only maximal subgroups containing elements of order 19 have structure $U_3(8):3$ involutions in these subgroup may be verified to be in class 2B using Magma. □

Note that we cannot replace 2B with 2A since no proper subgroups contain elements of both class 2A and 19A.

**Lemma 6** Every element of $\text{Fi}_{23}$ is contained in a proper subgroup containing an element of class 2A.

**Proof.** Since the center of the Sylow 2-subgroup of $\text{Fi}_{23}$ contains elements from all three classes of involutions, the even order trick tells us that every element is contained in some proper subgroup containing 2A elements except possibly elements in classes 9A, 17A, 23A/B, 27A, 35A or 39A/B. Each of these remaining cases may be tackled as follows.
• Using the permutation character given in the Atlas for the maximal subgroups with structure $O_8^+(3):S_3$ we see that this subgroup contains elements from classes 39A/B, 27A and 9A and 2A.

• The maximal subgroup isomorphic to $S_{12}$ must contain elements in class 2A since only the 2A centralizer is large enough to contain the subgroup $S_{10} \times 2$. This subgroup also contains elements from the class 35A.

• The maximal subgroup isomorphic to $2^{11}.M_{23}$ contains elements from class 2A by Sylow’s theorem and also contains elements from classes 23A/B.

• The maximal subgroup isomorphic to $S_8(2)$ contains elements from both class 2A and from class 17A. This may be verified using Magma.

Note that maximal subgroups of $Fi_{23}$ isomorphic to $L_2(23)$ contain only involutions of class 2C, so a result analogous to Lemma 4 cannot be proved.

**Lemma 7** Every element of $Co_1$ is contained in a proper subgroup containing an element of class 2A.

**Proof.** Since elements from class 2A are contained in the center of the Sylow 2-subgroup, the even order trick tells us that every element is contained in some proper subgroup containing 2A elements except possibly elements in classes 21B, 21C, 23A/B, 33A, 35A and 39A/B. Each of these remaining cases may be tackled as follows.

• Using the permutation character for the maximal subgroup $3\cdot Suz:2$ we find that this subgroups elements from classes 39A/B, 33A, 21B and 2A. Since this permutation character is not listed in the Atlas we explicitly give this below.

\[
1a + 1771a + 27300a + 644644a + 871884a
\]

• The subgroup $(A_5 \times HJ):2$ must contain elements of class 2A since HJ contains elements or order 8 and all elements of order 8 in $Co_1$ power up to elements in class 2A. This subgroup also contains elements from class 35A.

• The maximal subgroup $2^{11}.M_{24}$ contains elements from class 2A by Sylow’s theorem and clearly also contains elements of classes 23A/B.

• Using the permutation character for the maximal subgroup $Co_3$ we find that this subgroups elements from classes 21C and 2A. Since this permutation character is not listed in the Atlas we explicitly give this below.

\[
1a + 299a + 17250a + 80730a + 376740a + 2417415a + 5494125a
\]

**Lemma 8** Every element of $J_4$ is contained in a proper subgroup containing an element of class 2B.

**Proof.** Since the group $J_4$ only has two classes of involutions, the even order trick tells us that every element is contained in some proper subgroup containing 2B elements except possibly elements in classes 23A, 29A, 31A/B/C, 35A/B, 37A/B/C, 43A/B/C. Each of these remaining cases may be tackled as follows.
The only maximal subgroups containing elements of orders 43 or 37 are Frobenious groups. In each case it is easily verifies using Magma that these subgroups contain elements from class 2B.

Elements of order 35 are all contained in some maximal subgroup of structure $2^{3+21} : (S_5 \times L_3(2))$. By Sylow’s theorem this subgroup also contains elements of class 2B.

Elements of order 31 are all contained in some copy of the maximal subgroup with structure $2^{10} : L_5(2)$. From the Atlas specification this subgroup must also contain elements of class 2B.

The only maximal subgroups containing elements of orders 29 are Frobenious groups with structure $2^{9} : 28$. Since all elements of $J_4$ of order 28 power up to 2B elements, the subgroup must contain elements of class 2B.

Elements of order 23 are all contained in the subgroup of structure $2^{11} : M_{23}$. By Sylow’s theorem this subgroup also contains elements of class 2B.

Note that we cannot replace 2B by 2A since no proper subgroup contains elements from both classes 29A and 2A.

Lemma 9 Every element of $Fi'_{24}$ is contained in some proper subgroup containing an element of class 2B.

Proof. Since $Fi'_{24}$ contains only two classes of involutions the even order trick tells us that every element is contained in some proper subgroup containing 2B elements except possibly elements in classes 9D, 15B, 17A, 21B, 27B, 27C, 29A/B, 33A/B, 39A/B, 39C/D, 45A/B. Each of these remaining cases may be tackled as follows.

- The centralizer of an element of class 3B is contained in the maximal subgroup of structure $3^{1+10} : U_5(2) : 2$. Since elements in each of the classes 45A/B, 33A/B, 27B, 27C, 15B and 9D power-up to an element in class 3B these must also be contained in this subgroup. Furthermore $3^{1+10} : U_5(2) : 2$ contains elements of order 16 and all elements of order 16 in $Fi'_{24}$ power up to elements in class 2B.

- Elements of classes 39C/D and 21B are each power up to elements in class 3E in $Fi'_{24}$. Elements of class 6K also power up elements of class 3E as well as elements of class 2B. The centralizer of an element of class 3E will therefore contain elements from each of the classes 3E, 21B and 39C/D.

- Using the permutation character list in the Atlas we find that elements in classes 39A/B, 17A and 2B are contained in copies of the maximal subgroup with structure $Fi_{23}$.

- The only maximal subgroup containing elements of order 29 has structure $29:14$ and involutions in this subgroup may be verified to be in class 2B using Magma.

Note we cannot replace 2B with 2A since no proper subgroup contains elements from classes 29A and 2A.
Lemma 10  Every element of $M$ is contained in a proper subgroup that also contains elements of class 2B.

Proof. Since $M$ contains only two classes of involutions the even order trick tells us that every element is contained in some proper subgroup containing 2B elements except possibly elements in classes 27B, 29A, 39B, 41A, 45A, 51A, 57A, 59A/B, 69A/B, 71A/B, 87A/B, 93A/B, 95A/B, 105A and 119A/B. Recalling the well known fact that $\text{If } g, h \in M \text{ are in class 2A then } o(hg) \leq 6 \ (\dagger)$ each of these remaining classes may be tackled as follows.

- Elements of order 119 are all contained in the maximal subgroup with structure $(7:3 \times \text{He}):2$ since He contains elements of order 17. Since the Held group He contains elements of order 8 this subgroup also contains elements of order 56 and all elements of order 56 in the $M$ power up to elements in class 2B.

- The maximal subgroup with structure $3^+ \text{Fi}_{24}$ contains the full centralizer of a class 3A element. Since elements in each of the classes 105A, 87A/B, 51A, 39B power-up to an element of class 3A these must also be contained in this subgroup. Furthermore $3^+ \text{Fi}_{24}$ contains elements of order 16 and all elements of order 16 in the $M$ power up to elements in class 2B.

- The maximal subgroup with structure $(D_{10} \times \text{HN})^2$ contains the full centralizer of an element of class 5A. Since elements of class 10B power up to elements in class 2B and 5A this subgroup must also contain elements in class 2B. Since elements in classes 95A/B and 45A each power up to elements of class 5A these must also be contained in this subgroup.

- Elements of order 93 are easily seen to be contained in the maximal subgroup with structure $2^{5+10+20}(S_3 \times L_5(2))$ which by Sylow’s theorem also contains elements of class 2B.

- It is shown by Norton and Wilson in [11] that $M$ contains copies of $L_2(q)$ for $q = 29, 59$ or 71 and that these must contain elements of class 2B.

- Elements of order 69 are easily seen to be contained in the subgroup with structure $2^{2+11+22}(M_{24} \times S_3)$ which by Sylow’s theorem also contains elements of class 2B.

- The maximal subgroups of structure $S_3 \times \text{Th}$ contains elements of order 57 since Th contains elements of order 19. Furthermore Th contains involutions whose product has order greater than 6, so by $(\dagger)$ these involutions must be of class 2B of $M$.

- Elements of order 41 are all contained in maximal subgroups with structure 41:40. This subgroup contains only one class of involutions and the product of any two of these has order 41, so by observation $(\dagger)$ they must be in class 2B.
Since the group Suz contains elements of order 16 and elements of order 16 in \( M \) all power up to elements in class 2B the maximal subgroup \( 3^{1+12} \cdot 2 \text{Suz}.2 \) must contain elements of class 2B and by Sylow’s theorem must also contain elements from class 27B.

\( \square \)

Note that we cannot replace 2B by 2A here since this would provide an ‘upper bound’ that was less than the lower bound proved by Bradley and Moori in [4].

4 Proof of Theorem 1

In this section we use the lemmata of the previous section to prove Theorem 1.

The upper bounds given in Table 1 for the groups \( M_{11}, M_{12}, J_1, M_{22}, J_2, M_{23}, HS, J_3, M_{24}, McL, He, Suz, Co_3, Fi_{22} \) are precisely the bounds given by Bradley and Holmes in [3].

The upper bound given in Table 1 for the Baby Monster \( B \) is the bound given by Bradley and Moori [4].

We consider each of the remaining sporadic groups in turn, using the lemmata of the previous section to find in each case a set \( X \subset G \) that supports \( G^# \). By Lemma 2 this proves the bounds given in Table 1.

4.1 The Rudvalis group Ru

By Lemma 3 every element of Ru is contained in a proper subgroup that also contains elements of class 2B. We therefore take \( X \) to be the set of elements of class 2B, proving the bound given.

4.2 The O’Nan group O’N and the Lyons group Ly

Every maximal subgroup of the O’Nan group and the Lyons group Ly have even order. Since each of these groups only have one class of involutions in each case we can take \( X \) to be the set of all involutions of the group, proving the bounds given.

4.3 The Conway group Co_2

By Lemma 4 every element of Co_2 is contained in a proper subgroup that also contains elements of class 2B. We therefore take \( X \) to be the whole conjugacy class of 2B element of the group, proving the bound given.

4.4 The Harada Norton group HN

By Lemma 5 every element of HN is contained in a proper subgroup that also contains elements of class 2B. We therefore take \( X \) to be the whole conjugacy class of 2B element of the group, proving the bound given.
4.5 The Thompson group $\text{Th}$

The Thompson group $\text{Th}$ has a maximal subgroup with structure $31:15=H$.
This clearly only contains elements that are either of order 31 or some power of an element of class 15A. In both cases these elements are contained in copies of the Dempwolff group $2^5\cdot L_5(2)$. We therefore take $X$ to be the whole of the conjugacy class of 2A elements of the group, proving the bound given.

4.6 The Fisher Group $\text{Fi}_{23}$

By lemma 6 every element of $\text{Fi}_{23}$ is contained in some proper subgroup that also contains elements of class 2A. We therefore take $X$ to be the elements of class 2A, proving the bound given.

4.7 The Conway group $\text{Co}_1$

By Lemma 7 every element of $\text{Co}_1$ is contained in some proper subgroup that also contains elements of class 2A. We therefore take $X$ to be the elements of class 2A, proving the bound given.

4.8 The Janko group $\text{J}_4$

By Lemma 8 every element of $\text{J}_4$ is contained in some proper subgroup that also contains elements of class 2B. We therefore take $X$ to be the elements of class 2B, proving the bound given.

4.9 The Fischer Group $\text{Fi}'_{24}$

By Lemma 9 every element of $\text{Fi}'_{24}$ is contained in some proper subgroup that also contains elements of class 2B. We therefore take $X$ to be the elements of class 2B, proving the bound given.

4.10 The Monster group $\text{M}$

By Lemma 10 every element of $\text{M}$ is contained in some proper subgroup that also contains elements of class 2B. We therefore take $X$ to be the elements of class 2B, proving the bound given.

This completes the proof of Theorem 1.

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