Vibration analysis of a rotating axially functionally graded tapered beam with hollow circular cross-section

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Abstract. In this paper, the free vibrations of a rotating axially functionally graded (FG) tapered cantilever beam with hollow circular cross-section attached to a rigid hub are studied. To capture the additional dynamic stiffening terms, the longitudinal shrinkage of the beam is considered. The dynamic governing equations are derived via employing the assumed modes method and Lagrange’s equations. Through ignoring the higher-order quantities, the first order approximate coupling (FOAC) dynamic model can be acquired. Based on the FOAC dynamic model, influences of the angular speed, the taper ratio, the hub radius, the slenderness ratio, and the functionally gradient index on natural frequencies are studied. Frequency veering and mode shape interaction of the system are discussed when the bending-stretching (B-S) mode coupling effect of the beam is considered.

1. Introduction
Dynamics of flexible attachments of multibody systems with large overall motions are very complicated due to the coupling effect between the rigid motion and elastic deformation of the flexible body. Rotating hub-beam systems usually can be used to do modeling and dynamic analysis of space manipulators or slender rotor blades, which are usually made of advanced composite materials. For such a rigid-flexible coupled system, there are also complicated coupling effect between different vibrations modes of the flexible beam, and the vibration characteristics of the flexible beam structures should be well examined. Functionally graded materials (FGM) are well received in many engineering fields because they can integrate the excellent properties of two or more materials. Beam structures with large overall motions such as turbine blades, helicopter rotor blades, and manipulators can be designed to be made of FGM so that they can possess excellent dynamic characteristics and work well under some extreme conditions.

In recent years, with the vigorous development of aerospace, robotics and other engineering fields, the dynamics of composite beam structures made of FGM has drawn more and more attention by researchers. Kien [1] investigated the response of axially FG tapered cantilever beams. Li et al. [2] investigated vibration problems of rotating FG beams, discussed effects of functional gradient index. Li and Zhang [3] also used the B-spline method to research the response and vibration of axially FG tapered beams. In Refs. [2] and [3], the coupling effect of longitudinal displacement and transverse displacement of FG beams was included. Since the FGM beam was modeled in a planar coordinate system, the flapwise bending vibration of the beam was ignored. Mazanoglu et al. [4] first used Rayleigh-Ritz method to discuss vibrations of rotating axially FG tapered beams, both flapwise
bending and chordwise bending vibrations were studied, and the effect of centrifugal rigidity was considered. Huang et al. [5] studies free vibrations of axially FG Timoshenko beams, where the beam rotating about its neutral axis. Gao et al. [6] employed the asymptotic development method (ADM) to investigate vibrations of axially FG beams. Rezaiee-Pajand et al. [7] proposed an analytical and numerical method which can be used to study vibration problems of double-axially FG beams. Chen et al. [8] first used isogeometric analysis method to investigate vibration characteristics of three-dimensional axially FG beams, where the thickness of the beam was assumed to be variable. Sari et al. [9] proposed a new Timoshenko beam model with nonlocal residuals to solve the vibration problem of axially FG non-uniform beams. In Refs. [10-12], axially FG beams are also studied.

In this paper, the free vibrations of a planar rotating axially FG tapered beam with hollow circular cross-section are investigated, where rotary inertia is included. The FG beam material characteristics are supposed to change along the axis $x$. As the B-S vibration coupling effect is taken into account, frequency veering and mode shape interaction of the system are observed and discussed.

2. Derivation of differential equations of motion

2.1. Physical model

Figure 1 shows the schematic of the hub-beam system. A planar body-fixed coordinate system $OXY$ with origin $O$ fixed on the center of the hub is established. To describe the displacement field of any point $P_0$ on the beam, a floating coordinate system $oxy$, with its origin $o$ locates on the connection point between the beam and the rigid hub, is also defined. $\theta$ is the rotating angle of the hub.

![Figure 1. The displacement field of any point on the beam axis.](image)

Figure 2 shows geometry of the FG tapered beam with hollow circular cross-section. The length and density of the beam are $L$ and $\rho(x)$, respectively, the modulus of elasticity is $E(x)$, the outer diameter of the beam is linear, and the inner diameter is kept constant along the $x$ axis. The starting radius is $R_1$, the end radius is $R_2$, and the wall thickness $e(x)$ changes with $x$. $R(x)$ is the radius of the middle line of the wall thickness of any section, and $d$ is the hollow diameter.
The FG beam material characteristics are written as follows [9]:

\[ E(n) = (E_r - E_e) \left( \frac{x}{L} \right)^n + E_e \]  \hspace{1cm} (1)

\[ \rho(x) = (\rho_r - \rho_e) \left( \frac{x}{L} \right)^n + \rho_e \]  \hspace{1cm} (2)

2.2. The description of the deformation field

According to Figure 1, the position vector \( \mathbf{r}_p \) can be expressed as

\[ \mathbf{r}_p = (a + x + u_x) \mathbf{i} + u_y \mathbf{j} \]  \hspace{1cm} (3)

where \( u_x = w_1 + w_c + w_d, \) \( u_y = w_2 \)  \hspace{1cm} (4)

in which \( w_1 \) is the axial displacement, \( w_c = -(1/2) \int_0^x (\partial w_2 / \partial \xi) \, d\xi \) is the axial shrinkage of the beam caused by the transverse displacement, \( w_d = -\gamma w' \) is the axial displacement caused by the rotation of the cross section.

The velocity vector of \( \mathbf{r}_p \) is

\[ \mathbf{r}_p = (\dot{u}_x - u_x \dot{\theta}) \mathbf{i} + \left[ (a + x + u_x) \dot{\theta} + \dot{u}_y \right] \mathbf{j} \]  \hspace{1cm} (5)

2.3. Kinetic energy of system

The kinetic energy of the system can be written as

\[ T = \frac{1}{2} \int_{\text{dom}} \int_r \rho(x) \left( (u_x - u_x \dot{\theta})^2 + \left[ (a + x + u_x) \dot{\theta} + \dot{u}_y \right]^2 \right) \, dV \]

\[ = \frac{1}{2} \int_{\text{dom}} \int_{2\pi} \rho(x) S(x) \left( (w_1 + w_c + w_d - w_2 \dot{\theta})^2 + \left[ (a + x + w_1 + w_c) \dot{\theta} + \dot{w}_2 \right]^2 \right) \, dx \]

\[ + \frac{1}{2} \int_{\text{dom}} \rho(x) I(x) \left( \dot{\theta}^2 w_{z_1}^2 + w_{z_2}^2 \right) \, dx \]  \hspace{1cm} (6)

where \( S(x) = 2\pi R(x)e(x) \), \( I(x) = \pi R^4(x)e(x) \left( 1 + \frac{e^2(x)}{4R^2(x)} \right) \)
2.4. Potential energy of the system
Neglecting torsional effect and deformation energy caused by shear deformation, the potential energy can be expressed as

\[ U = \frac{1}{2} \int \int \sigma \varepsilon \, dV = \frac{1}{2} \int \int E(x) \varepsilon^2 \, dV \]  

(7)

The normal strain of any point can be written as

\[ \varepsilon_x = \frac{\partial w_1}{\partial x} - \gamma \frac{\partial^2 w_2}{\partial x^2} \]  

(8)

thus, the potential energy of the rotating axially FG tapered beam with hollow circular cross-section can be acquired as

\[ U = \frac{1}{2} \int \int \left( \frac{\partial w_1}{\partial x} \right)^2 - \gamma \int \int \left( \frac{\partial^2 w_2}{\partial x^2} \right)^2 \, dV \]  

(9)

2.5. Assumed modes discretization
Employing the assumed modes method to approximate variables, the axial deformation \( w_1 \), transverse deformation \( w_2 \) can be expressed respectively as follows:

\[ \begin{cases} w_1 = \Phi_1(x) A(t) \\ w_2 = \Phi_2(x) B(t) \end{cases} \]  

(10)

Substituting Eq. (10) into Eq. (6) and Eq. (9), in order to study vibration characteristics more conveniently, the higher-order quantities are ignored. Take \( q = (\theta^T \quad A^T \quad B^T)^T \) as the generalized coordinate vector, and employing Lagrange’s equations of the second kind:

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}} \right) - \frac{\partial T}{\partial q} = \frac{\partial U}{\partial q} + F_q \]  

(11)

Acquire the coupling equations of the system as follow:

\[ \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{A} \\ \dot{B} \end{bmatrix} = \begin{bmatrix} Q_\theta \\ \dot{Q}_A \\ \dot{Q}_B \end{bmatrix} \]  

(12)

3. Analysis of frequencies

3.1 Without the mode coupling effect
The rotating angular speed is assumed to be a constant, \( \dot{\theta} = \Omega \), hence \( \ddot{\theta} = 0 \), the transverse bending vibration equation can be obtained from Eq. (12) as

\[ M_{33} \ddot{B} + K_{33} B = 0 \]  

(13)

In order to acquire the natural frequencies, some dimensionless variables are adopted:

\[ \psi = t/T \quad , \quad \xi = x/L \quad , \quad \kappa_2 = B/L \quad , \quad \gamma = T\dot{\theta} \quad , \quad \delta = a/L \quad , \quad r = \frac{I_1}{S L^2} \]  

(14)
where \( T = \sqrt{\frac{\rho S L^4}{E I}} \), \( S_i = \pi R_i^4 / 4 \), \( \gamma \) is called the dimensionless angular speed, \( \delta \) is called the ratios of hub radius to beam length and \( r \) is called the slenderness ratio.

Thus, the dimensionless expression of equation (13) can be written as

\[
\ddot{\bar{\kappa}}_i + \bar{\kappa} = 0
\]

(15)

Chordwise bending natural frequencies of rotating solid Euler-Bernoulli beams when \( \lambda = 1, \delta = 0, \beta = 0, r = 0, N = 0 \) are shown in Table 1, where \( \lambda = R_1 / R_0 \) is called the taper ratio, \( \beta = d / (2R_0) \) is called the ratio of hollow radius to the starting radius. By selecting the above values of the dimensionless parameters, the present model can be degraded into a model for traditional homogeneous uniform beams. The results from the present model are consistent with those from Ref. [13], so the correctness can be guaranteed. In order to neglect torsional effect, the wall thickness should not be too small. Table 2 shows chordwise bending natural frequencies of rotating FG tapered beams for different gradient index when \( \lambda = 0.75, \delta = 0, \beta = 0.3, r = 1 / 50 \). As it is shown in Tables 2, as the gradient index increases, the first order natural frequency increases, while the second and the third decrease. And some special situation can also be found at some angular speed.

Table 1 The first three frequencies with \( \lambda = 1, \delta = 0, \beta = 0, r = 0, N = 0 \).

| \( \gamma \) | First | Ref [13] | Second | Ref [13] | Third | Ref [13] |
|---|---|---|---|---|---|---|
| 0 | 3.51602 | 3.51602 | 22.0345 | 22.0345 | 61.6972 | 61.6972 |
| 2 | 3.6218 | 3.62182 | 22.5263 | 22.5264 | 62.2411 | 61.8337 |
| 5 | 4.07403 | 4.07471 | 24.9504 | 24.9520 | 65.014 | 65.0195 |
| 10 | 5.05185 | 5.06384 | 32.1247 | 32.1408 | 73.9888 | 74.0626 |

Table 2 The first three frequencies for different gradient index with \( \lambda = 0.75, \delta = 0, \beta = 0.3, r = 1 / 50 \).

| \( N \) | \( \gamma = 5 \) | \( \gamma = 25 \) | \( \gamma = 50 \) | \( \gamma = 5 \) | \( \gamma = 25 \) | \( \gamma = 50 \) | \( \gamma = 5 \) | \( \gamma = 25 \) | \( \gamma = 50 \) |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 3.58542 | 7.07773 | 10.9724 | 22.1908 | 61.8248 | 118.207 | 56.0363 | 113.576 | 204.5 |
| 5 | 3.72593 | 7.33778 | 11.1986 | 21.6083 | 61.6427 | 118.037 | 53.8176 | 112.055 | 203.581 |
| 10 | 3.89257 | 7.57291 | 11.4489 | 21.1558 | 59.588 | 113.986 | 52.2764 | 109.031 | 199.113 |
| 20 | 4.04168 | 7.7719 | 11.6758 | 21.0564 | 57.8016 | 110.323 | 51.5763 | 105.809 | 193.004 |
| 50 | 4.17248 | 7.93964 | 11.8731 | 21.2945 | 56.8521 | 108.174 | 51.8137 | 103.371 | 187.255 |
| 100 | 4.22644 | 8.00713 | 11.9534 | 21.4958 | 56.7137 | 107.745 | 52.2481 | 102.862 | 185.588 |

Figure 3 shows variations of chordwise bending natural frequencies for \( \lambda = 1, 0.75, 0.5 \) when \( \delta = 0, \beta = 0.3, N = 5 \), and \( r = 1 / 30 \). It is found from Figure 3 that the taper ratio variation of the beam has little impact on the first dimensionless natural frequencies, but has observable impact on the second as well as the third frequencies. In addition, the first frequencies increase while the second and third frequencies decrease with the taper ratio, respectively. Figure 4 shows variations of chordwise bending natural frequencies for \( r = 1 / 30, 1 / 50, 1 / 100 \) when \( \lambda = 0.75, \delta = 0, \beta = 0.3 \), and \( N = 2 \). As it is shown in Figure 4, the impact of the slenderness ratio on chordwise bending natural frequencies is slight (see the third dimensionless natural frequency loci), the first two natural frequencies have low sensitive to this parameter. The difference between the third frequencies of the beams with small slenderness ratios (\( r = 1 / 100 \) and \( r = 1 / 50 \)) is quite small.
Figure 3. Variations of chordwise bending natural frequencies for \( \lambda = 1, 0.75, 0.5 \); \( \delta = 0 \), \( \beta = 0.3 \), \( N = 5 \), and \( r = 1/30 \).

Figure 4. Variations of chordwise bending natural frequencies for \( r = 1/30, 1/50, 1/100 \); \( \lambda = 0.75 \), \( \delta = 0 \), \( \beta = 0.3 \), and \( N = 2 \).

3.2 With the coupling effect

Following the basic definitions in the above section, the transverse and longitudinal vibration mode coupled free vibration equations of flexible beams can be written as:

\[
\begin{bmatrix}
M_{22} & 0 \\
0 & M_{33}
\end{bmatrix} \ddot{A} + \begin{bmatrix}
G_{23} \\
0
\end{bmatrix} \dot{B} = \begin{bmatrix}
K_{22} & 0 \\
0 & K_{33}
\end{bmatrix} \begin{bmatrix}
A \\
B
\end{bmatrix}
\]

Define another dimensionless variable \( \kappa = A/L \), the dimensionless expression of equation (16) is obtained as

\[
\begin{bmatrix}
\bar{M}_{22} & 0 \\
0 & \bar{M}_{33}
\end{bmatrix} \ddot{\kappa}_{1} + \begin{bmatrix}
\bar{G}_{23} \\
0
\end{bmatrix} \dot{\kappa}_{2} = \begin{bmatrix}
\bar{K}_{22} & 0 \\
0 & \bar{K}_{33}
\end{bmatrix} \begin{bmatrix}
\kappa_{1} \\
\kappa_{2}
\end{bmatrix}
\]

Table 3 shows the first two frequencies include the coupling effect and without the coupling effect, it also shows the difference ratio. As it is shown in Table 3, when the rotating angular speed increases, through observing the difference ratio, one can get that the bending-stretching coupling effect becomes obvious, and the effect of the coupling on the first frequency is larger than the second; when the angular speed is high, in order to get more accurate results, the coupling effect should be considered. And as the ratios of hub radius to beam length increases, the effect of coupling also increases.
Table 3 The coupling effect on the first two natural frequencies; $\lambda = 0.75$, $\beta = 0.3$, $r = 1/50$, $N = 1$.

| $\delta$ | $\gamma$ | First frequency | Second frequency |
|----------|----------|-----------------|-----------------|
|         |          | Neglect B-S     | Consider B-S    | Difference(%)  | Neglect B-S     | Consider B-S    | Difference(%)  |
| 0       | 10       | 4.60284         | 4.4364          | 3.6            | 30.071           | 29.9048        | 0.55            |
|         | 20       | 6.27552         | 5.4388          | 13.3           | 49.7633          | 48.7203        | 2.1             |
|         | 50       | 10.9848         | 5.0133          | 54.4           | 115.067          | 98.0341        | 14.8            |
| 1       | 10       | 13.0867         | 12.6017         | 3.7            | 41.4926          | 41.2388        | 0.6             |
|         | 20       | 25.2281         | 21.6145         | 14.3           | 75.1318          | 71.8936        | 4.3             |
|         | 50       | 62.1433         | 25.4411         | 59             | 180.695          | 120.472        | 33.3            |
| 5       | 10       | 27.6748         | 26.5287         | 4.1            | 69.8572          | 68.5643        | 1.85            |
|         | 20       | 54.8344         | 44.0023         | 19.8           | 134.157          | 84.7652        | 36.8            |
|         | 50       | 136.662         | 40.6456         | 70.3           | 330.54           | 166.927        | 49.5            |

Figure 5 shows chordwise bending natural frequency variations when $\lambda = 0.75$, $\beta = 0.3$, $N = 1$, $r = 1/50$, and $\delta = 0$, where the coupling effect is considered. The fourth frequency is expressed by $S_1$, the seventh frequency is expressed by $S_2$, because they correspond to the first and the second order frequencies of stretching vibration when $\gamma = 0$, respectively; the others are expressed by $B_1$-B7, respectively.

In order to observe in detail, the first sharpest veering areas are enlarged in Figure 6, as it is shown in Figure 6, as the dimensionless angular speed varies, modes of $S_1$ and $B_3$ also change. $\gamma=13.1$ is a turn angular speed related to $B_3$ and $S_1$, the $B_3$ mode transforms from transverse bending vibration dominance to longitudinal stretching vibration dominance, while the $S_1$ mode is contrary. Figure 7 shows the bending and stretching mode shape vibration along $B_3$ and $S_1$. In order to observe easily, all the amplitudes are normalized.

Figure 5. Chordwise bending natural frequency variations with the B-S included.
Figure 6. The enlarged drawing of the first frequency veering region.

Figure 7. (a) chordwise bending mode shape variation in B3 when $\gamma = 10$ and S1 when $\gamma = 18$, respectively; (b) stretching mode shape variation in B3 when $\gamma = 18$ and S1 when $\gamma = 10$, respectively.

4. Conclusion
In this paper, free vibration characteristics of a rotating axially FG tapered beam with hollow circular cross-section are studied. The obtained results show that the variations of hub radius, functionally gradient index, the taper ratio, and the slenderness ratio have different degrees of influence on natural frequencies of each order. Influences of these parameter factors on beam natural frequencies of each mode increase with the rotating angular speed of the hub. The longitudinal vibration has considerable impact on transverse bending vibration. And the coupling effect cannot be ignored if the angular speed is large. With such mode coupling effect, frequency veering and mode shape interaction can be observed. Research in this paper are helpful to understand the dynamic characteristics of hollow circular cross-section beams which are frequently used in practical engineering fields and may provide theoretical guidance for vibration prediction of such flexible structures.

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