Factorizations, Riemann-Hilbert problems and the corona theorem

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Abstract. The solvability of the Riemann-Hilbert boundary value problem on the real line is described in the case when its matrix coefficient admits a Wiener-Hopf type factorization with bounded outer factors but rather general diagonal elements of its middle factor. This covers, in particular, the almost periodic setting, when the factorization multiples belong to the algebra generated by the functions $e^{i\lambda x}$, $\lambda \in \mathbb{R}$. Connections with the corona problem are discussed. Based on those, constructive factorization criteria are derived for several types of triangular $2 \times 2$ matrices with diagonal entries $e_{\pm \lambda}$ and non-zero off diagonal entry of the form $a_- e^{-\beta} + a_+ e\nu$ with $\nu, \beta \geq 0$, $\nu + \beta > 0$ and $a_{\pm}$ analytic and bounded in the upper/lower half plane.

1. Introduction

The (vector) Riemann-Hilbert boundary value problem on the real line $\mathbb{R}$ consists in finding two vector functions $\phi_{\pm}$, analytic in the upper and lower half plane $\mathbb{C}^\pm = \{ z \in \mathbb{C} : \pm \text{Im } z > 0 \}$ respectively, satisfying the condition

$$\phi_- = G\phi_+ + g,$$

imposed on their boundary values on $\mathbb{R}$. Here $g$ is a given vector function and $G$ is a given matrix function defined on $\mathbb{R}$, of appropriate sizes. It is well known that

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various properties of (1.1) can be described in terms of the (right) factorization of its matrix coefficient $G$, that is, a representation of $G$ as a product

$$G = G_+ D G_+^{-1},$$

(1.2)

where $G_{\pm}$ and their inverses are analytic in $\mathbb{C}^\pm$ and $D$ is a diagonal matrix function with diagonal entries $d_j$ of a certain prescribed structure. An exact definition of the factorization (1.2) is correlated with the setting of the problem (1.1), that is, the requirements on the boundary behavior of $\phi_{\pm}$. To introduce a specific example, denote by $H^\pm$ the Hardy classes in $\mathbb{C}^\pm$ and by $L^p$ the Lebesgue space on $\mathbb{R}$, with $p \in (0, \infty]$. Let us also agree, for any set $X$, to denote by $X^n (X^{n\times n})$ the set of all $n$-vectors (respectively, $n \times n$ matrices) with entries in $X$.

With this notation at hand, recall that the $L^p$ setting of (1.1) is the one for which $g \in L^n$ and $\phi_{\pm} \in (H^\pm)^n$. An appropriate representation (1.2), in this setting with $p > 1$, is the so called $L^p$ factorization of $G$: the representation (1.2) in which

$$\lambda_\pm^{-1} G_{\pm} \in (H^\pm)^{n\times n}, \quad \lambda_\pm^{-1} G_{\pm}^{-1} \in (H^\pm)^{n\times n}$$

and $d_j = (\lambda_- / \lambda_+)^{\kappa_j}$.

(1.3)

Here

$$1 < p < \infty, \quad q = \frac{p}{p-1}, \quad \lambda_{\pm}(z) = z \pm i,$$

and the integers $\kappa_j$ are called the (right) partial indices of $G$.

A full solvability picture for the problem (1.1) with $L^p$ factorable $G$ can be extracted from [12, Chapter 3], see also [10]. The central result in this direction is (the real line version of) the Simonenko’s theorem, according to which (1.1) has a unique solution for every right hand side $g$ — equivalently, the associated Toeplitz operator $T_G := P_+ G | (H^+_p)^n$ is invertible — if and only if $G$ admits an $L^p$ factorization (1.2) with $D = I$, subject to the additional condition

$$G_- P_+ G_-^{-1}$$

(1.4)

is a densely defined bounded operator on $L^n$. Here $P_+$ is the projection operator of $L^p$ onto $H^+_p$ along $H^-_p$, defined on vector (or matrix) functions entrywise.

In this paper, we take particular interest in bounded factorizations for which in (1.2), by definition,

$$G^\pm \in (H^\pm)^{n\times n}, \quad G^\pm \in (H^-_\infty)^{n\times n}.$$  

(1.5)

Of course, with $d_j$ as in (1.3) a bounded factorization of $G$ is its $L^p$ factorization simultaneously for all $p \in (1, \infty)$, and the additional condition (1.4) is satisfied. However, some meaningful conclusions regarding the problem (1.1) can be drawn from the relation (1.2) satisfying (1.5) even without any additional information about the diagonal entries of $D$. This idea for $L^p$ factorization on closed curves was first discussed in [13]; in Section 2 we give a detailed account of the bounded factorization version. That includes in particular the interplay between the factorization problem and the corona theorem.
Section 3 deals with the almost periodic (AP for short) setting, in which the elements of the matrix function involved belong to the algebra AP generated by the functions
\[ e_\lambda(x) = e^{i\lambda x}, \quad \lambda \in \mathbb{R}, \quad (1.6) \]
the diagonal elements \( d_j \) being chosen among its generators \( e_\lambda \). In this case not only we consider the solvability of (1.1) when \( G \) admits an AP factorization, but also address the converse question: what information on the existence and the properties of that factorization can be obtained from a solution to a homogeneous problem
\[ G\phi_+ = \phi_-, \quad \phi_\pm \in (H_p^\pm)^n \quad (1.7) \]
with \( p = \infty \).

In Sections 4, 5 we consider classes of matrix functions \( G \) for which (1.1) is closely related with a convolution equation on an interval of finite length. By determining a solution to the homogeneous Riemann-Hilbert problem (1.7) in \( H_\infty^\pm \) and applying the results of the previous sections, we study the factorability of \( G \) and the properties of the related Toeplitz operator \( T_G \). In particular, invertibility conditions for this operator are obtained and a subclass of matrix functions is identified for which invertibility of \( T_G \) is (somewhat surprisingly) equivalent to its semi-Fredholmness.

2. Riemann-Hilbert problems and factorization

We start with the description of the solutions to (1.1), in terms of a bounded factorization (1.2).

**Theorem 2.1.** Let \( G \) admit a bounded factorization (1.2). Then all solutions of the problem (1.1) satisfying \( \phi_\pm \in (H_p^\pm)^n \) for some \( p \in [1, \infty] \) are given by
\[ \phi_+ = \sum_j \psi_j g^+_j, \quad \phi_- = \sum_j d_j \psi_j g^-_j + g. \quad (2.1) \]
Here \( g^\pm_j \) stands for the \( j \)-th column of \( G^\pm \):
\[ G_- = \begin{bmatrix} g^-_1 & g^-_2 & \cdots & g^-_n \end{bmatrix}, \quad G_+ = \begin{bmatrix} g^+_1 & g^+_2 & \cdots & g^+_n \end{bmatrix}, \quad (2.2) \]
and \( \psi_j \) is an arbitrary function satisfying
\[ \psi_j \in H_p^+, \quad d_j \psi_j + (G^{-1} g)_j \in H_p^- \quad (2.3) \]
In other words, the Riemann-Hilbert problem (1.1) with a matrix coefficient \( G \) admitting a bounded factorization can be untangled into \( n \) scalar Riemann-Hilbert problems, in the same \( L_p \) setting.

The proof of Theorem 2.1 is standard in the factorization theory, based on a simple change of unknowns \( \phi_\pm = G^\pm \psi_\pm \). We include it here for completeness.
Proof. If \((\phi_+, \phi_-)\) is a solution to (1.1), then defining \(\psi := (\psi_j)_{j=1, \ldots, n} = G^{-1}_{\phi_+} \phi_+\), we get \(\phi_+ = G_+ \psi, \phi_- = G_- D \psi + g\), which is equivalent to (2.4), and (2.5) is satisfied. Conversely, if (2.5) holds for all \(j = 1, \ldots, n\), then \(\phi_+ = G_+ \psi \in (H_p^+)^n\), \(\phi_- = G_- D \psi + g \in (H_p^-)^n\), and (1.1) holds. \(\square\)

We will say that a function \(f\), defined a.e. on \(\mathbb{R}\), is of non-negative type if
\[
f \in H^+_{\infty} \text{ or } f^{-1} \in H^-_{\infty}. \tag{2.4}\]
The type is non-positive if
\[
f \in H^-_{\infty} \text{ or } f^{-1} \in H^+_{\infty}, \tag{2.5}\]
(strictly) positive if (2.4) holds while (2.5) does not, and neutral if both (2.4), (2.5) hold.

Lemma 2.2. For \(d_j\) of positive type, there is at most one function \(\psi_j\) satisfying (2.3).

Proof. It suffices to show that the only function \(\psi \in H^+_p\) satisfying \(d_j \psi \in H^-_p\) is zero.

If the first condition in (2.4) holds for \(f = d_j\), then \(d_j \psi \in H^+_p\) simultaneously with \(\psi\) itself. From here and \(d_j \psi \in H^-_p\) it follows that \(d_j \psi\) is a constant. If this constant is non-zero (which is only possible if \(p = \infty\)), then \(d_j\) is invertible in \(H^+_\infty\), which contradicts the strict positivity of its type. On the other hand, the product \(d_j \psi\) of two analytic functions may be identically zero only if one of them is. It cannot be \(d_j\) (once again, since otherwise the first condition in (2.5) would hold); thus, \(\psi = 0\).

The second case of (2.4) can be treated in a similar way. \(\square\)

As an immediate consequence we have:

Corollary 2.3. If \(G\) admits a bounded factorization with all \(d_j\) of positive type, then the homogeneous Riemann-Hilbert problem (1.7) has only the trivial solution \(\phi_+ = \phi_- = 0\) for any \(p \in [1, \infty]\).

If \(d_j\) is of neutral type, then by definition it is either invertible in \(H^+_\infty\), or in \(H^-_{\infty}\), or is equal to zero. Disallowing the latter case, and absorbing \(d_j\) in the column \(g_j^\pm\) in the former, we may without loss of generality suppose that all such \(d_j\) are actually equal 1. With this convention in mind, the following result holds.

Corollary 2.4. Let \(G\) admit a bounded factorization with all \(d_j\) of non-negative type, \(d_j \neq 0\). Then the homogeneous problem (1.7) for \(1 \leq p < \infty\) has only the trivial solution, and for \(p = \infty\) all its solutions are given by
\[
\phi_+ = \sum_{j \in J} c_j g_j^+, \quad \phi_- = \sum_{j \in J} c_j g_j^- \tag{2.6}\]
Here \(g_j^\pm\) are as in (2.2), \(c_j \in \mathbb{C}\), and \(j \in J\) if and only if \(d_j\) is of neutral type.
Proof. From (2.1) and from Lemma 2.2 we have
\[ \phi_+ = \sum_{j \in J} \psi_j g_j^+, \quad \phi_- = \sum_{j \in J} d_j \psi_j g_j^-, \]
while our convention regarding the neutral type allows us to drop the functions \(d_j\) in the expression for \(\phi_-\). Finally, (2.3) with \(d_j\) of neutral type and \(g = 0\) means that \(\psi_j \in H^+_p \cap H^-_p\), and thus \(\psi_j\) is a constant \((= 0\) if \(p < \infty\)). □

Recall that the factorization (1.2) is canonical if the middle factor \(D\) of it is the identity matrix, and can therefore be dropped:
\[ G = G_- G_+^{-1}. \] (2.7)
The following criterion for bounded canonical factorability is easy to establish, and actually well known. We state it here, with proof, for the sake of completeness and ease of references.

Lemma 2.5. \(G\) admits a bounded canonical factorization (2.7) if and only if problem (1.7) with \(p = \infty\) has solutions \(\phi^+_1, \ldots, \phi^+_{n}\), \(\phi^-_1, \ldots, \phi^-_{n}\) such that
\[ \det[\phi^+_1 \ldots \phi^+_n] \] is invertible in \(H^\pm_\infty\). (2.8)
If this is the case, then one of the factorizations is given by
\[ G_\pm = [\phi^+_1 \ldots \phi^+_n], \] (2.9)
and all solutions to (1.7) in \(H^\pm_\infty\) are linear combinations of \(\phi^\pm_j\).

Proof. If (2.7) holds with \(G_\pm\) satisfying (1.6), then one may choose \(\phi^\pm_j\) as the \(j\)-th column of \(G_\pm\). Conversely, if \(\phi^\pm_j\) satisfy (1.7) and (2.8), then \(G_\pm\) given by (2.7) satisfy \(GG_+ = G_-\) and (1.5). Therefore, (2.7) holds and delivers a bounded canonical factorization of \(G\).

The last statement now follows from Corollary 2.4. □

Observe that for \(G\) with constant non-zero determinant, the determinants of matrix functions \(G_\pm\) given by (2.9) also are necessarily constant. So, (2.8) holds if and only if the vector functions \(\phi^+_1(z), \ldots, \phi^+_n(z)\) (or \(\phi^-_1(z), \ldots, \phi^-_n(z)\)) are linearly independent for at least one value of \(z \in \mathbb{C}^+\) (resp., \(\mathbb{C}^-\)).

As it happens, if \(G\) admits a bounded canonical factorization, all its bounded factorizations (with no a priori conditions on \(d_j\)) are forced to be “almost” canonical. The precise statement is as follows.

Theorem 2.6. Let \(G\) have a bounded canonical factorization \(G = G_- G_+^{-1}\). Then all its bounded factorizations are given by (1.2), where each \(d_j\) has a bounded canonical factorization
\[ d_j = d_j - d_j^{-1}, \quad j = 1, \ldots, n, \] (2.10)
\[ G_\pm = \tilde{G}_\pm Z D_\pm^{-1}, \quad D_\pm = \text{diag}[d_1, \ldots, d_n], \] (2.11)
and \(Z\) is an arbitrary invertible matrix in \(\mathbb{C}^{n \times n}\).
Proof. Equating two factorizations \( \tilde{G} - \tilde{G}^{-1} \) and \( G - DG^{-1} \) yields
\[
D = G^{-1} \tilde{G} - \tilde{G}^{-1} G = F - F^{-1},
\]
where \( F, F^{-1} \in (H^\pm_{\infty})^{n \times n} \). Consequently, \( D \) admits a bounded canonical factorization, and therefore the Toeplitz operator \( T_D \) is invertible on \((H^+_\infty)^n\) for \( p \in (1, \infty) \). Being the direct sum of \( n \) scalar Toeplitz operators \( T_{d_j} \), this implies that each of the latter also is invertible, on \( H^+_p \). Thus, each of the scalar functions \( d_j \) admits a canonical \( L^p \) factorization. Let (2.10) be such a factorization, corresponding to \( p = 2 \). Then, according to (2.12) the elements \( f_{ij} \pm \) of the matrix functions \( F \pm \) are related as \( f_{ij}^- = d_j f_{ij}^+ \). Due to the invertibility of \( F \pm \), for each \( j \) the functions \( f_{ij}^\pm \) are non-zero for at least one value of \( i \). Choosing such \( i \) arbitrarily, and abbreviating the respective \( f_{ij}^\pm \) simply to \( f_j^\pm \), we have
\[
f_j^- - d_j^- = f_j^+ - d_j^+.
\]
The left and right hand side of the latter equality is a function in \( \lambda - H^-_\infty \) and \( \lambda + H^+_{\infty} \), respectively. Hence, each of them is just a scalar (non-zero, due to our choice of \( i \)). So, \( d_{j,\pm} \in H^\pm_{\infty} \).

Letting \( d_{\pm} = \prod_{j=1}^n d_{j,\pm} \), from here we obtain that \( \det D = d_- d_+^{-1} \), with \( d_{\pm} \in H^\pm_{\infty} \). But (2.12) implies also that \( \det D \) admits the bounded analytic factorization \( \det F_- / \det F_+ \). Thus,
\[
d_+ / \det F_- = d_- / \det F_+,
\]
with left/right hand side lying in \( H^\pm_{\infty} \), respectively. Hence, \( d_{\pm} \) differs from \( \det F_{\pm} \) only by a (clearly, non-zero) scalar multiple, and therefore is invertible in \( H^\pm_{\infty} \). This implies the invertibility of each multiple \( d_{j,\pm} \) in \( H^\pm_{\infty} \), \( j = 1, \ldots, n \), so that each representation (2.10) is in fact a bounded canonical factorization.

With the notation \( D_{\pm} \) as in (2.11), the first equality in (2.12) can be rewritten as
\[
\tilde{G} - \tilde{G}^{-1} G = \tilde{G}^{-1} G D_{\pm}.
\]
Since the left/right hand side is invertible in \( (H^\pm_{\infty})^{n \times n} \), each of them is in fact an invertible constant matrix \( Z \). This implies the first formula in (2.11). □

According to (2.11) with \( D = I \), two bounded canonical factorizations of \( G \) are related as
\[
G_{\pm} = \tilde{G}_{\pm} Z, \quad \text{where } Z \in \mathbb{C}^{n \times n}, \det Z \neq 0,
\]
— a well-known fact.

When \( n = 2 \), the results proved above simplify in a natural way. We will state only one such simplification, once again, for convenience of references.

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1. The interpolation property of factorization \cite{12} Theorem 3.9] implies that in our setting the canonical \( L^p \) factorization of \( d_j \) is the same for all \( p \in (1, \infty) \) but this fact has no impact on the reasoning.
Theorem 2.7. Let $G$ be a $2 \times 2$ matrix function admitting a bounded factorization \((1.2)\) with one of the diagonal entries (say $d_2$) of positive type. Then the problem \((1.7)\) has non-trivial solutions in $H^+_p$ for some $p \in [1, \infty]$ if and only if $d_1$ admits a representation $d_1 = d_1^- d_1^+$ with $d_1^\pm \in H^\pm_p$. If this condition holds, then all the solutions of \((1.7)\) are given by

$$\phi_+ = \psi g_1^+, \quad \phi_- = d_1 \psi g_1^-,$$

where $g_1^+$ is the first column of $G_\pm$ in the factorization \((1.2)\) and $\psi \in H^+_p$ is an arbitrary function satisfying $d_1 \psi \in H^-_p$.

Proof. Sufficiency. If $d_1 = d_1^- d_1^+$ with $d_1^\pm \in H^\pm_p$, then obviously $d_1^+ \neq 0$ and

$$\phi_+ = d_1^- g_1^+, \quad \phi_- = d_1^- g_1^-$$

is a non-trivial solution to \((1.7)\).

Necessity. By Lemma 2.2 and Theorem 2.1 the solution must be of the form $\phi_+ = \psi g_1^+, \phi_- = d_1 \psi g_1^-$ with $\psi \in H^+_p \setminus \{0\}, d_1 \psi \in H^-_p$. It remains to set $d_1^+ = \psi, d_1^- = d_1 \psi$. \hfill \Box

More interestingly, there is a close relation between factorization and corona problems.

Recall that a vector function $\omega$ with entries $\omega_1, \ldots, \omega_n \in H^+_\infty$ satisfies the corona condition in $\mathbb{C}^+$ (notation: $\omega \in CP^+$) if and only if

$$\inf_{z \in \mathbb{C}^+} (|\omega_1(z)| + \cdots + |\omega_n(z)|) > 0.$$

The corona condition in $\mathbb{C}^-$ for a vector function $\omega \in (H^-_\infty)^n$ and the notation $\omega \in CP^-$ are introduced analogously.

By the corona theorem, $\omega \in CP^+$ if and only if there exists $\omega^* = (\omega^*_1, \ldots, \omega^*_n) \in (H^+_{\infty})^n$ such that

$$\omega_1 \omega^*_1 + \cdots + \omega_n \omega^*_n = 1.$$

Theorem 2.8. If an $n \times n$ matrix function $G$ admits a bounded canonical factorization, then any non-trivial solution of problem \((1.7)\) in $(H^\pm_\infty)^n$ actually lies in $CP^\pm$.

Proof. Let $G$ admit a bounded canonical factorization \((2.7)\). By Corollary 2.4 every non-trivial solution $\phi_\pm$ of \((1.7)\) is a nontrivial linear combination of the columns $g_j^+, j = 1, \ldots, n$. According to \((2.13)\), any such combination, in turn, can be used as a column of some (perhaps, different) bounded canonical factorization of $G$. Being a column of an invertible element of $(H^\pm_\infty)^{n \times n}$, it must lie in $CP^\pm$. \hfill \Box

The following result is a somewhat technical generalization of Theorem 2.8 which will be used later on.

Theorem 2.9. Let $G$ be an $n \times n$ matrix function admitting a bounded factorization \((1.2)\) in which for all $k = 2, \ldots, n$ either $d_k = d_1 \neq 0$ or $d_1^{-1} d_k$ is a function
of positive type. Then for any pair of non-zero vector functions \( \phi \pm \in (H^\pm_\infty)^n \) satisfying \( G\phi_+ = \phi_- \), \( d_1 \phi_+ \in (H^+_\infty)^n \), in fact stronger conditions

\[
d_1 \phi_+ \in CP^+, \quad \phi_- \in CP^-
\]

hold. In order for such pairs to exist, \( d_1 \) has to be of non-positive type.

Proof. Let \( \tilde{G} = d_1^{-1} G \). Then, due to (1.2),

\[
\tilde{G} = G_- \tilde{D} G_+^{-1} \quad \text{with} \quad \tilde{D} = \text{diag}[1, d_1^{-1} d_2, \ldots, d_1^{-1} d_n],
\]

which of course is a bounded factorization of \( \tilde{G} \).

Condition \( \phi_- = G\phi_+ \) implies that \( \phi_- = \tilde{G} d_1 \phi_+ \), so that the pair \( d_1 \phi_+, \phi_- \) is a non-trivial solution of the homogeneous Riemann-Hilbert problem with the coefficient \( \tilde{G} \).

If \( d_1 = d_2 = \cdots = d_n \), then (2.15) delivers a bounded canonical factorization of \( \tilde{G} \), so that the desired result follows from Theorem 2.8. If, on the other hand, \( d_1^{-1} d_2, \ldots, d_1^{-1} d_n \) are all of positive type, then \( d_1 \phi_+ \) and \( \phi_- \) differ only by a (non-zero) constant scalar multiple from the first column of \( G_+ \) and \( G_- \) respectively, according to Corollary 2.4. This again implies (2.14).

Finally, from \( d_1 \phi_+ \in CP^+ \) and \( \phi_- \in (H^+_\infty)^n \) it follows that \( d_1^{-1} \in H^+_\infty \), that is, \( d_1 \) is of non-positive type. \( \square \)

The exact converse of Theorem 2.8 is not true. However, a slightly more subtle result holds.

**Theorem 2.10.** Let \( G \in L^2_{\infty}^{2 \times 2} \) be such that there exists a solution of problem (1.7) in \( CP^\pm \). Then the Toeplitz operators \( T_G \) on \((H^+_\infty)^2\) and \( T_{\det G} \) on \( H^+_\infty \) are Fredholm only simultaneously, and their defect numbers coincide.

Proof. The existence of the above mentioned solutions implies (see, e.g., computations in [3, Section 22.1]) that

\[
G = X_- \begin{bmatrix} \det G & 0 \\ * & 1 \end{bmatrix} X_+,
\]

where \( X_\pm \) is an invertible element of \((H^\pm_\infty)^{2 \times 2}\). From here and elementary properties of block triangular operators it follows that the respective defect numbers (and thus the Fredholm behavior) of \( T_G \) and \( T_{\det G} \) are the same. \( \square \)

According to Theorem 2.10, in the particular case when \( \det G \) admits a canonical factorization, the operator \( T_G \) is invertible provided that (1.7) has a solution in \( CP^\pm \). For \( \det G \equiv 1 \) the latter result was essentially established in [1]. An alternative, and more detailed, proof of Theorem 2.10 can be found in [3], Theorems 4.1 and 4.4.

Let now \( \mathcal{B} \) be a subalgebra of \( L_\infty \) (not necessarily closed in \( L_\infty \) norm) such that, for any \( n \), a matrix function \( G \in \mathcal{B}^{n \times n} \) admits a bounded canonical factorization if and only if the operator \( T_G \) is invertible in \((H^+_p)^n\) for at least one (and therefore all) \( p \in (1, \infty) \). There are many classes satisfying this property, e.g.,
decomposable algebras of continuous functions (see [8, 12]) or the algebra $APW$ considered below.

**Theorem 2.11.** Let $G \in B^{2 \times 2}$ with $\det G$ admitting a bounded canonical factorization, and let $\phi_\pm \in (H^\pm_\infty)^2$ be a non-zero solution to (1.7). Then $G$ has a bounded canonical factorization if and only if $\phi_\pm \in CP^\pm$.

**Proof.** Necessity follows from Theorem 2.8 and sufficiency from Theorem 2.10. The latter can also be deduced from [1, Theorem 3.4] formulated there for $G$ with constant determinant but remaining valid if $\det G$ merely admits a bounded canonical factorization. □

### 3. $AP$ factorization

We will now recast the results of the previous section in the framework of $AP$ factorization. To this end, recall that $AP$ is the uniform closure of all linear combinations $\sum c_\lambda e_\lambda$, with $c_\lambda \in \mathbb{C}$, defined by (1.6), while these linear combinations themselves form the set $APP$ of all almost periodic polynomials. Properties of $AP$ functions are discussed in detail in [9, 11], see also [3, Chapter 1]. In particular, for every $f \in AP$ there exists its mean value

$$M(f) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(t) \, dt.$$ 

This yields the existence of $\hat{f}(\lambda) := M(e^{-\lambda}f)$, the Bohr-Fourier coefficients of $f$.

For any given $f \in AP$, the set

$$\Omega(f) = \{ \lambda \in \mathbb{R} : \hat{f}(\lambda) \neq 0 \}$$

is at most countable, and is called the Bohr-Fourier spectrum of $f$. The formal Bohr-Fourier series $\sum_{\lambda \in \Omega(f)} \hat{f}(\lambda)e_\lambda$ may or may not converge; we will write $f \in APW$ if it does converge absolutely. The algebras $AP$ and $APW$ are inverse closed in $L^\infty$; moreover, for an invertible $f \in AP$ there exists an (obviously, unique) $\lambda \in \mathbb{R}$ such that a continuous branch of $\log(e^{-\lambda}f) \in AP$. This value of $\lambda$ is called the mean motion of $f$; we will denote it $\kappa(f)$.

Finally, let

$$AP^\pm = \{ f \in AP : \Omega(f) \subset \mathbb{R}_\pm \},$$

where of course $\mathbb{R}_\pm = \{ x \in \mathbb{R} : \pm x \geq 0 \}$. Denote also

$$APW^\pm = AP^\pm \cap APW, \quad APP^\pm = AP^\pm \cap APP.$$ 

Clearly,

$$APP^\pm \subset APW^\pm \subset AP^\pm \subset H_\infty^\pm.$$ 

An $AP$ factorization of $G$, by definition, is a representation (1.2) in which $G_\pm$ are subject to the conditions

$$G_\pm \in (AP^\pm)^{n \times n}, \quad G_\pm \in (AP^\pm)^{n \times n}, \quad (3.1)$$
more restrictive than \((1.5)\), and the diagonal entries of \(D\) are of the form \(d_j = e^{\delta_j}\), \(j = 1 \ldots , n\). The real numbers \(\delta_j\) are called the (right) partial \(AP\) indices of \(G\), and by an obvious column permutation in \(G_\pm\) we may assume that they are arranged in a non-decreasing order: \(\delta_1 \le \delta_2 \le \cdots \le \delta_n\).

A particular case of \(AP\) factorization occurs when conditions \((3.1)\) are changed to more restrictive ones:

\[
G_+^{\pm 1} \in (APW^+)^{n \times n}, \quad G_-^{\pm 1} \in (APW^-)^{n \times n},
\]
or even

\[
G_+^{\pm 1} \in (APP^+)^{n \times n}, \quad G_-^{\pm 1} \in (APP^-)^{n \times n}.
\]

These are naturally called \(APW\) and \(APP\) factorization of \(G\), respectively. Of course, \(G\) has to be an invertible element of \((APW^+)^{n \times n}\), \((APW^-)^{n \times n}\), \((APP^+)^{n \times n}\), or even \((APP^-)^{n \times n}\), provided that \(G\) is respectively \(APW\)- or \(APP\)-factorable).

Corollary \(2.4\) for example, applies to \(AP\)-factorable matrix functions \(G\) with non-negative partial \(AP\) indices. Formulas \((2.6)\) imply then that all solutions of \((1.7)\) in \((H_\pm^\infty)^n\) are automatically in \((AP^\pm)^n\) (and even \((APW^\pm)^n\) or \((APP^\pm)^n\)), provided that \(G\) is respectively \(APW\)- or \(APP\)-factorable).

Lemma \(2.5\) takes the following form.

**Theorem 3.1.** An \(n \times n\) matrix function \(G\) admits a canonical \(AP\) \((APW)\) factorization if and only if there exist \(n\) solutions \((\psi^+_1, \psi^-_1, \ldots , \psi^+_n, \psi^-_n)\) to \((1.7)\) in \((AP^\pm)^n\) (resp., \((APW^\pm)^n\)), such that \(\det[\psi^+_1 \cdots \psi^+_n, \psi^-_1 \cdots \psi^-_n]\) are bounded from zero in \(C^\pm\).

The respective criterion for \(APP\) factorization is slightly different, because \((APP^\pm)^n\) is not inverse closed in \(H_\pm^\infty\). Moreover, the only invertible elements of \((APP^\pm)^n\) are non-zero constants. Therefore, we arrive at

**Corollary 3.2.** An \(n \times n\) matrix function \(G\) admits a canonical \(APP\) factorization if and only if there exist \(n\) solutions \((\psi^+_1, \psi^-_1, \ldots , \psi^+_n, \psi^-_n)\) to \((1.7)\) in \((APP^\pm)^n\) with constant non-zero \(\det[\psi^+_1 \cdots \psi^+_n, \psi^-_1 \cdots \psi^-_n]\).

Similarly to the case in Section \(2\) for matrix functions \(G\) with constant determinant the condition on \(\det[\psi^+_1 \cdots \psi^+_n, \psi^-_1 \cdots \psi^-_n]\) holds whenever at least one of them is non-zero at just one point of \(C^\pm \cup \mathbb{R}\). All non-trivial solutions to \((1.7)\) are actually in \(CP^\pm\), as guaranteed by Theorem \(2.8\).
Theorem 2.1 of course remains valid when $G$ admits an $AP$ factorization; the only change needed is that $d_j$ in formulas \ref{eq:2.1}, \ref{eq:2.3} should be substituted by $e_{\delta_j}$.

For the homogeneous problem \eqref{eq:1.7} this yields the following.

**Theorem 3.3.** Let $G$ admit an $AP$ factorization \eqref{eq:1.2}. Then the general solution of problem \eqref{eq:1.7} in $(H^\pm_\infty)^n$ is given by
\[
\phi_+ = \sum_j \psi_j g_j^+, \quad \phi_- = \sum_j e_{\delta_j} \psi_j g_j^-,
\]
where the summation is with respect to those $j$ for which $\delta_j \leq 0$, $\psi_j$ are constant whenever $\delta_j = 0$ and satisfy
\[
\psi_j \in H^+_{\infty} \cap e^{-\delta_j} H^-_{\infty} \text{ whenever } \delta_j < 0.\]

Observe that $\phi_\pm$ given by \eqref{eq:3.3} belong to $AP^n$ if and only if condition \eqref{eq:3.4} is replaced by a more restrictive
\[
\psi_j \in AP, \quad \Omega(\psi_j) \subset [0, -\delta_j]
\]
(\text{where by convention } \psi_j = 0 \text{ if } \delta_j > 0), since
\[
\psi := (\psi_j) = G_+^{-1}\phi_+ = D_+^{-1}G_+^{-1}\phi_-.
\]

Moreover, if in fact $G$ is $APW$ factorable, then the functions \eqref{eq:3.3} are in $APW^n$ if and only if
\[
\psi_j \in APW, \quad \Omega(\psi_j) \subset [0, -\delta_j].
\]

Solutions of \eqref{eq:1.7} in $(H^\pm_\infty)^n$ are automatically in $AP$ ($APW$) if $G$ is $AP$- (resp., $APW$-) factorable with non-negative partial $AP$ indices, since in this case $D_+^{-1} \in APP^{-}$ and \eqref{eq:3.3} implies that $\psi \in C^n$. On the other hand, if $G$ is $APW$ factorable with at least one negative partial $AP$ index, then all three classes are distinct. Indeed, for any $j$ corresponding to $\delta_j < 0$ there is a plethora of functions $\psi_j$ satisfying \eqref{eq:3.4} not lying in $AP$, as well as functions in $AP \setminus APW$ with the Bohr-Fourier spectrum in $[0, -\delta_j]$.

The case of exactly one non-positive partial $AP$ index is of special interest.

**Corollary 3.4.** Let $G$ admit an $AP$ factorization with the partial $AP$ indices $\delta_1 \leq 0 < \delta_2 \leq \cdots$. Then all solutions to \eqref{eq:1.7} in $(H^\pm_\infty)^n$ ($AP^n$, $APW^n$) are given by
\[
\phi_+ = fg_1^+, \quad \phi_- = e_{\delta_1} fg_1^-,
\]
where $f$ is an arbitrary $H^+_{\infty}$ function such that $e_{\delta_1} f \in H^-_{\infty}$ (resp., $f \in AP$ or $f \in APW$ and $\Omega(f) \subset [0, -\delta_1]$).

For $n = 2$ the reasoning of Theorem 2.9 suggests an appropriate modification of \eqref{eq:1.7} for which some solutions are forced to lie in $AP$. Recall our convention $\delta_1 \leq \delta_2$ according to which the condition on $d_1, d_2$ in Theorem 2.9 holds automatically.
Theorem 3.5. Let $G$ be a $2 \times 2$ AP factorable matrix function with partial indices $\delta_1, \delta_2$ ($\delta_1 \leq \delta_2$). Then any non-zero pair $(\phi_+, \phi_-)$ with $\phi_+ \in (H^+_{\infty})^2 \cap e_{-\delta_1}(H^+_{\infty})^2$, $\phi_- = G\phi_+ \in (H^-_{\infty})^2$ satisfies

\[ \phi_+ \in (\text{AP}^\pm)^2, \quad e_{\delta_1} \phi_+ \in \text{CP}^+, \quad \phi_- \in \text{CP}^-, \]

and in order for such pairs to exist it is necessary and sufficient that $\delta_1 \leq 0$. If $\delta_2 > \delta_1$, all those solutions have the form

\[ \phi_+ = c e_{-\delta_1} g_1^+ \cdot \phi_- = c g_1^- \quad \text{with} \quad c \in \mathbb{C} \setminus \{0\}. \]

For $\delta_2 = \delta_1$, $\phi_+$ and $\phi_-$ are the same non-trivial linear combinations of the columns of $e_{-\delta_1} G_+$ and $G_-$. Of course, Theorem 3.5 holds with AP changed to APW or APP everywhere in its statement.

Recall that a Toeplitz operator with scalar AP symbol $f$ is Fredholm on $H^p$ for some (equivalently: all) $p \in (1, \infty)$ if and only if it is invertible if and only if $f$ is invertible in AP with mean motion zero. Therefore, Theorem 2.10 implies

Lemma 3.6. Let $G \in \text{AP}^{2 \times 2}$ be such that there exists a solution of (1.7) in CP$^\pm$. Then the Toeplitz operator $T_G$ is invertible on $(H^p_{\infty})^2$, $1 < p < \infty$, if and only if $\kappa(\det G) = 0$.

Passing to the APW setting, we invoke the result according to which $T_G$ with $G \in \text{APW}^{n \times n}$ is invertible if and only if $G$ admits a canonical AP (or APW) factorization. Lemma 3.6 then implies (compare with Theorem 2.11):

Theorem 3.7. Let $G \in \text{APW}^{2 \times 2}$. Then $G$ admits a canonical AP factorization if and only if $\kappa(\det G) = 0$ and problem (1.7) has a solution in CP$^\pm$. If this is the case, then every non-zero solution of (1.7) is in $(\text{APW}^\pm)^2 \cap \text{CP}^\pm$.

The first part of Theorem 3.7 for $G$ with $\det G \equiv 1$ (so that $\kappa(\det G) = 0$ automatically) is in [3] (see Theorem 23.1 there). Essentially, it was proved in [1], with sufficiency following from Theorems 3.4, 6.1 and necessity from Theorem 3.5 there.

Our next goal is the APW factorization criterion in the not necessarily canonical case.

Theorem 3.8. Let $G$ be a $2 \times 2$ invertible APW matrix function. Denote $\delta = \kappa(\det G)$. Then $G$ admits an APW factorization if and only if the Riemann-Hilbert problem

\[ e_{-\delta} G \psi_+ = \psi_- \quad \psi_+ \in (\text{APW}^\pm)^2 \quad (3.7) \]

admits a solution $(\psi_+, \psi_-)$ such that

\[ \tilde{\psi}_+ := e_{-\tilde{\delta}} \psi_+ \in \text{CP}^+ \text{ for some } \tilde{\delta} \geq 0 \text{ and } \psi_- \in \text{CP}^- \quad (3.8) \]

If this is the case, then the partial AP indices of $G$ are $\delta_1 = -\tilde{\delta} + \frac{\delta}{2}$, $\delta_2 = \tilde{\delta} + \frac{\delta}{2}$ and the factors $G_\pm$ can be chosen in such a way that $\psi_+$ is the first column of $G_+$ and $\psi_-$ is the first column of $G_-$. 

Proof. If $G$ admits an APW factorization, then $\delta = \delta_1 + \delta_2$ due to (3.2). In its turn, $\psi_+ = e^{\frac{\delta}{2} - \delta}g_1^+$, $\psi_- = g_1^-$ is a solution of (3.7) if $\frac{\delta}{2} - \delta \geq 0$. It remains to set $\tilde{\delta} = \delta - \delta_1$ in order to satisfy (3.8) by analogy with Theorem 3.5. Formulas $\delta_1 = \frac{\delta}{2} - \tilde{\delta}$, $\delta_2 = \frac{\delta}{2} + \tilde{\delta}$ for the partial $AP$ indices then also hold.

Suppose now that (3.7) has a solution for which (3.8) holds. From the corona theorem in the APW setting (see [3, Chapter 12]), there exist $h_\pm = (h_{1\pm}, h_{2\pm}) \in (APW^\pm)^2$ such that

$$
\psi_1 - h_1 - \psi_2 - h_2 = 1, \quad e^{-\tilde{\delta}}(\psi_1 + h_1 + \psi_2 + h_2) = 1.
$$

(3.9)

In other words, the matrix functions

$$
H_+ = \begin{bmatrix}
  e^{\frac{\delta}{2} - \tilde{\delta}} & -h_{2+} \\
  e^{\frac{\delta}{2} - \tilde{\delta}} & h_{1+}
\end{bmatrix} \quad \text{and} \quad H_- = \begin{bmatrix}
  \psi_1 & -h_{2-} \\
  \psi_2 & h_{1-}
\end{bmatrix}
$$

(3.10)

have determinants equal to 1 and are therefore invertible in $(APW^+)^{2 \times 2}$ and $(APW^-)^{2 \times 2}$ respectively. Thus the matrix functions $G_1 = H_+^{-1}GH_+$ and $G$ are only simultaneously APW factorable, and their partial $AP$ indices coincide.

For the first column of $G_1$, taking (3.9) into account, we have

$$
e^{-\tilde{\delta}}H_+^{-1}G_+ = e^{\frac{\delta}{2} - \tilde{\delta}}H_-^{-1}\psi_- = \begin{bmatrix}
  e^{\frac{\delta}{2} - \tilde{\delta}} \\
  0
\end{bmatrix}.
$$

Thus the second diagonal entry in $G_1$ must be equal to

$$
e^{-\frac{\delta}{2}} \det G = \gamma_+ e^{-\frac{\delta}{2} + \tilde{\delta}} \gamma_+^{-1},
$$

where

$$
\det G = \gamma_+ e^{-\frac{\delta}{2} + \tilde{\delta}} \gamma_+^{-1}
$$

is a factorization of the scalar APW function $\det G$. Consequently,

$$
G_1 = \begin{bmatrix}
  1 & 0 \\
  0 & \gamma_-
\end{bmatrix} e^{\frac{\delta}{2} - \tilde{\delta}} \begin{bmatrix}
  g & 0 \\
  0 & \gamma_+
\end{bmatrix} \begin{bmatrix}
  1 & 0 \\
  0 & \gamma_+
\end{bmatrix}^{-1}
$$

(3.11)

with $g \in APW$ given by $g = [1 \ 0 | G_1 | 0 \ \gamma_+]^T$. Finally, the middle factor in the right-hand side of (3.11) is APW factorable with the partial indices $\frac{\delta}{2} - \tilde{\delta}$, $\frac{\delta}{2} + \tilde{\delta}$ equal to the mean motions of its diagonal entries:

$$
\begin{bmatrix}
  e^{\frac{\delta}{2} - \tilde{\delta}} & g \\
  0 & e^{\frac{\delta}{2} + \tilde{\delta}}
\end{bmatrix} = \begin{bmatrix}
  1 & g \\
  0 & 1
\end{bmatrix} \begin{bmatrix}
  e^{\frac{\delta}{2} - \tilde{\delta}} & 0 \\
  0 & e^{\frac{\delta}{2} + \tilde{\delta}}
\end{bmatrix} \begin{bmatrix}
  1 & -g_+ \\
  0 & 1
\end{bmatrix}^{-1}.
$$

(3.12)

The only condition on $g_\pm \in APW^\pm$ is

$$
ge^{\frac{\delta}{2} - \tilde{\delta}} = g_+ + g_- e^{2\tilde{\delta}},
$$

(3.13)

and it can be satisfied since $\tilde{\delta} \geq 0$. Clearly, making use of (2.13) we can always choose $G_\pm$ in such a way that $\psi_+$ is the first column of $G_+$ and $\psi_-$ is the first column of $G_-$. \qed
The proof of the preceding theorem provides, via (3.10), (3.11)–(3.13), formulas for an APW factorization of \( G = H_1^{-1}G_1H_+ \), in terms of the solutions to (3.7) and the corona problems (3.9).

4. Applications to a class of matrices with a spectral gap near zero

We consider now the factorability problem for a class of triangular matrix functions, closely related to the study of convolution equations on an interval of finite length \( \lambda \) (see, e.g., [3, Section 1.7] and references therein), of the form

\[
G = \begin{bmatrix}
  e^{-\lambda} & 0 \\
  g & e^\lambda
\end{bmatrix}.
\]

(4.1)

Throughout this section we assume that

\[
g = a_- e^{-\beta} + a_+ e^\nu \quad \text{for some } a_\pm \in H^\pm_\infty \text{ and } 0 \leq \nu, \beta \leq \lambda, \nu + \beta > 0.
\]

(4.2)

Representation (4.2), when it exists, is not unique. In particular, it can be rewritten as

\[
g = \tilde{a}_- e^{-\tilde{\beta}} + \tilde{a}_+ e^{\tilde{\nu}}
\]

with

\[
\tilde{\nu} \in [0, \nu], \quad \tilde{\beta} \in [0, \beta], \quad \tilde{a}_+ = a_+ e^{\nu - \tilde{\nu}}, \quad \tilde{a}_- = a_- e^{\tilde{\beta} - \beta}.
\]

(4.3)

Among all the representations (4.2) choose those with the smallest possible value of

\[
N = \left\lceil \frac{\lambda}{\nu + \beta} \right\rceil,
\]

(4.4)

where as usual \( \lceil x \rceil \) denotes the smallest integer which is greater or equal to \( x \in \mathbb{R} \). Of course, \( N \geq 1 \) due to the positivity of \( \frac{\lambda}{\nu + \beta} \).

Formula (4.4) means that

\[
N - 1 < \frac{\lambda}{\nu + \beta} \leq N.
\]

Decreasing \( \beta, \nu \) as described in (4.3), we may turn the last inequality into an equality. In other words, without loss of generality we may (and will) suppose that

\[
\frac{\lambda}{\nu + \beta} = N
\]

(4.5)

is an integer.

We remark that even under condition (4.5) representation (4.2) may not be defined uniquely.

Given \( N \geq 1 \), we denote by \( S_{\lambda,N} \) the class of functions \( g \) satisfying (4.2), (4.5) for which

\[
b_+ := e^{-\beta} a_- \in H^+_\infty, \quad b_- := e^{-\nu} a_+ \in H^-_\infty \quad \text{if } N > 1.
\]

(4.6)

By \( \mathcal{G}_{\lambda,N} \) we denote the class of \( 2 \times 2 \) matrix functions \( G \) of the form (4.1) with \( g \in S_{\lambda,N} \).
Remark 4.1. If \( g \in S_{\lambda,N} \) with \( N > 1 \), then necessarily in (4.2) \( \beta, \nu > 0 \). Indeed, if say \( \nu = 0 \), then (4.6) implies that \( a_+ \) is a constant. Consequently, \( g \in H^{-}_{\infty} \), and setting \( a_- = g, a_+ = 0, \beta = 0, \nu = \lambda \) in (4.2) would yield \( N = 1 \) — a contradiction with our convention to choose the smallest possible value of \( N \). Note also that, due to (4.6), \( a_{\pm} \) are entire functions when \( N > 1 \).

We start by determining a solution to (1.7) for \( G \) in \( S_{\lambda,N} \).

Theorem 4.2. Let \( G \in S_{\lambda,N} \), with \( g \) given by (4.2). Then

\[
\phi_1^+ = e_{\lambda - \nu} \sum_{j=0}^{N-1} \left( (-1)^j a_+^{N-1-j} b_+^j \right) , \quad \phi_2^+ = -a_+^N ,
\]

(4.7)

\[
\phi_1^- = e_{-\lambda} \phi_1^+ , \quad \phi_2^- = (-1)^{N-1} a_+^N
\]

(4.8)

deliver a solution \( \phi_{\pm} = (\phi_{1\pm}, \phi_{2\pm}) \) to the Riemann-Hilbert problem (1.7). 

Proof. A direct computation based on the equality

\[
x^N + (-1)^{N-1} y^N = (x+y) \sum_{j=0}^{N-1} ((-1)^j x^{N-1-j} y^j)
\]

shows that \( G\phi_+ = \phi_- \). Obviously, \( \phi_{2\pm} \in H_{\pm}^{\infty} \). So, it remains to prove only that \( \phi_{1\pm} \in H_{\pm}^{\infty} \). For \( N = 1 \), this is true because the definition of \( \phi_1^+ \) from (4.7) collapses to \( \phi_1^+ = e_\beta \). The case \( N > 1 \) is slightly more involved.

Namely, for \( N > 1 \) from (4.6) it follows that

\[
e_{\frac{a_+}{N-1}} a_- = b_+ \in H_\infty^+ ,
\]

so that

\[
\phi_1^+ = \sum_{j=0}^{N-1} \left( (-1)^j a_+^{N-1-j} b_+^j e_{\beta - j \frac{a_-}{N-1}} e_{\lambda - (j+1) \frac{a_-}{N}} \right) \in H_\infty^+ .
\]

(4.9)

Analogously, from

\[
e_{-\frac{a_-}{N-1}} a_+ = b_- \in H_\infty^-
\]

we have

\[
\phi_1^- = \sum_{j=0}^{N-1} \left( (-1)^j b_-^{N-1-j} a_-^j e_{-j \frac{a_+}{N-1}} e_{\lambda - (j+1) \frac{a_+}{N}} \right) \in H_\infty^- .
\]

(4.10)

□

This theorem, along with Theorem 2.10, allows to establish sufficient conditions, which in some cases are also necessary, for invertibility in \( (H_\infty^+)^2, p > 1 \), of Toeplitz operators with symbol \( G \in S_{\lambda,N} \). To invoke Theorem 2.10 however, we need to be able to check when the pairs \( (\phi_{1\pm}, \phi_{2\pm}) \) defined by (4.7), (4.8) belong to \( CP^+ \) or \( CP^- \). The following result from [3] (see Theorem 2.3 there) will simplify this task.
Theorem 4.3. Let a $2 \times 2$ matrix function $G$ and its inverse $G^{-1}$ be analytic and bounded in a strip
\[ S = \{ \xi \in \mathbb{C} : -\varepsilon_2 < \text{Im} \xi < \varepsilon_1 \} \quad \text{with} \quad \varepsilon_1, \varepsilon_2 \in [0, +\infty[ , \quad (4.11) \]
and let $\phi_{\pm} \in (H^2_\infty)^2$ satisfy (4.7). Then $\phi_+ \in CP^+$ (resp. $\phi_- \in CP^-$) if and only if
\[ \inf_{C^+ + i\varepsilon_1} (|\phi_{1+}| + |\phi_{2+}|) > 0 \quad \text{(resp.,} \quad \inf_{C^- - i\varepsilon_2} (|\phi_{1-}| + |\phi_{2-}|) > 0) \quad (4.12) \]
and one of the following (equivalent) conditions is satisfied:
\[ \inf_S (|\phi_{1+}| + |\phi_{2+}|) > 0 , \quad (4.13) \]
\[ \inf_S (|\phi_{1-}| + |\phi_{2-}|) > 0 . \quad (4.14) \]

Here and in what follows, we identify the functions $\phi_{1+}, \phi_{2+}$ (resp., $\phi_{1-}, \phi_{2-}$) with their analytic extensions to $\mathbb{C}^+ - i\varepsilon_2$ (resp. $\mathbb{C}^+ + i\varepsilon_1$) and, for any real-valued function $\phi$ defined on $S$, abbreviate $\inf_{\xi \in S} \phi(\xi)$ to $\inf_S \phi$.

We will see that for $G \in \mathfrak{G}_{\lambda, N}$, $N \geq 1$, the behavior of the solutions “at infinity”, that is, condition (4.12) for sufficiently big $\varepsilon_1, \varepsilon_2 > 0$, is not difficult to study. Therefore, due to Theorem 4.3, we will be left with studying the behavior of $\phi_+$ or $\phi_-$ in a strip of the complex plane. According to the next result this, in turn, can be done in term of the functions $a_\pm$ from (4.12) or, equivalently, of $g_\pm$

\[ g_+ = e_\nu a_+ , \quad g_- = e_\nu - a_- \]

It should be noted that, for $N > 1$, $a_\pm$ and $g_\pm$ are entire functions. Moreover, even if the behavior of $a_+$ and $a_-$ in a strip $S$ may be difficult to study, it is clear from (4.7) and (4.8) that this is in general a much simpler task than that of checking whether (4.12) is satisfied using the expressions for $\phi_{1\pm}, \phi_{2\pm}$.

Lemma 4.4. Let $G \in \mathfrak{G}_{\lambda, N}$ for some $N > 1$, and let $\phi_{\pm}$ be given by (4.7), (4.8).
Then for any strip (4.11) we have
\[ \inf_S (|\phi_{1+}| + |\phi_{2+}|) > 0 \iff \inf_S (|\phi_{1-}| + |\phi_{2-}|) > 0 \iff \inf_S (|a_+| + |a_-|) > 0 . \quad (4.15) \]

Proof. Since the last two conditions in (4.15) are obviously equivalent, and (4.13) is equivalent to (4.14) due to Theorem 4.3 we need to prove only that
\[ \inf_S (|\phi_{1+}| + |\phi_{2+}|) > 0 \iff \inf_S (|a_+| + |a_-|) > 0 . \]

Suppose first that
\[ \inf_{\xi \in S} (|a_+(\xi)| + |a_-(\xi)|) = 0 . \]

Then there is a sequence $\{\xi_n\}_{n \in \mathbb{N}}$ with $\xi_n \in S$ such that $a_+(\xi_n) \to 0$ and $a_-(\xi_n) \to 0$. Taking into account the expressions for $\phi_{1+}, \phi_{2+}$ given by (4.7), we must have $\phi_{1+}(\xi_n) \to 0$ and $\phi_{2+}(\xi_n) \to 0$. Therefore,
\[ \inf_{\xi \in S} (|\phi_{1+}(\xi)| + |\phi_{2+}(\xi)|) = 0 . \]
Conversely, if
\[ \inf_{\xi \in S} (|\phi_{1+}(\xi)| + |\phi_{2+}(\xi)|) = 0, \]
then for some sequence \( \{\xi_n\} \) with \( \xi_n \in S \) for all \( n \in \mathbb{N} \), we have \( \phi_{1+}(\xi_n) \to 0 \) and \( \phi_{2+}(\xi_n) \to 0 \). Thus, from the expression for \( \phi_{2+} \) given by (1.7), it follows that \( a_+(\xi_n) \to 0 \). From the expression for \( \phi_{1+} \) in (1.7), we then conclude
\[ a_+^{-1} = (-1)^{N-1} e_{\nu-} \phi_{1+} + (-1)^N e^{\frac{i\pi}{2}N} \sum_{j=0}^{N-2} (-1)^j a_+^{N-1-j} a_-^j e^{-j \frac{\pi}{2}}. \]
Since \( \phi_{1+}(\xi_n) \to 0 \) and \( a_+(\xi_n) \to 0 \), then also \( a_-(\xi_n) \to 0 \) and therefore
\[ \inf_{\xi \in S} (|a_+(\xi)| + |a_-(\xi)|) = 0. \]

\[ \square \]

We can now state the following.

**Theorem 4.5.** Let \( G \in \mathfrak{S}_{\lambda,N} \) for some \( N \in \mathbb{N} \), and let \( \phi_{\pm} \) be the solutions to (1.7) given by (1.7), (1.8). Then:

(i): For \( N = 1 \), \( \phi_{\pm} \in CP^\pm \) if and only if
\[ \inf_{\mathbb{C}^+ \ni \varepsilon_1} |a_+| > 0, \quad \inf_{\mathbb{C}^- \ni \varepsilon_2} |a_-| > 0 \quad \text{for some } \varepsilon_1, \varepsilon_2 > 0. \] (4.16)

(ii): For \( N > 1 \), \( \phi_{\pm} \in CP^\pm \) if and only if, with \( b_+, b_- \) defined by (4.6),
\[ \inf_{\mathbb{C}^+ \ni \varepsilon_1} (|b_+| + |a_+|) > 0, \quad \inf_{\mathbb{C}^- \ni \varepsilon_2} (|b_-| + |a_-|) > 0 \quad \text{for some } \varepsilon_1, \varepsilon_2 > 0 \] (4.17)
and, for any \( S \) of the form (4.11),
\[ \inf_S (|a_+| + |a_-|) > 0. \] (4.18)

**Proof.** Part (i) follows immediately from the explicit formulas
\[ \phi_+ = (e^{\beta}, -a_+), \quad \phi_- = (e^{-\beta}, a_-). \] (4.19)

(iii) For \( N > 1 \) we have, from (4.7)–(4.10),
\[ \phi_{1+} = (-1)^{N-1} b_{1+}^{-1} + \sum_{j=0}^{N-2} (-1)^j a_+^{N-1-j} b_+^j e_{(N-1-j) \frac{\pi}{2}} \quad \text{and} \quad \phi_{2+} = (-1)^N a_N^{-1} \] (4.20)
\[ \phi_{1-} = b_{1-}^{-1} + \sum_{j=1}^{N-1} (-1)^j a_-^j b_{-1}^{N-1-j} e_{-j \frac{\pi}{2}}, \quad \phi_{2-} = (-1)^N a_N. \] (4.21)

Since \( \nu, \beta < \lambda \) when \( N > 1 \), we see that for any sequence \( \{\xi_n\} \) with \( \xi_n \in \mathbb{C}^+ \) and \( \text{Im}(\xi_n) \to +\infty \),
\[ |\phi_{1+} - (-1)^{N-1} b_{1+}^{N-1}|_{(\xi_n)} \to 0, \] (4.22)
and, for any sequence \( \{\xi_n\} \) with \( \xi_n \in \mathbb{C}^- \) and \( \text{Im}(\xi_n) \to -\infty \),
\[ |\phi_{1-} - b_{1-}^{N-1}|_{(\xi_n)} \to 0. \] (4.23)
It follows from (4.20)–(4.23) that there exist \( \varepsilon_1, \varepsilon_2 > 0 \) such that (4.17) holds if and only if there exist \( \varepsilon_1, \varepsilon_2 > 0 \) such that (4.12) holds. Moreover, by Lemma 4.4, (4.18) is equivalent to (4.13), thus the result follows from Theorem 4.3. \( \Box \)

Note that \( \det G \equiv 1 \) for all matrix functions of the form (4.1). Therefore, Theorems 2.10, 2.11 and 4.5 combined imply the following.

**Corollary 4.6.** Let the assumptions of Theorem 4.5 hold. Then condition (4.16) (for \( N = 1 \)) and (4.17), (4.18) (for \( N > 1 \)) imply the invertibility of \( T_G \). The converse is also true (and, moreover, \( G \) admits a bounded canonical factorization) provided that \( G \in B^{2 \times 2} \).

For \( N = 1 \), this result was proved (assuming \( \lambda = 1 \), which amounts to a simple change of variable) in [6], Theorem 4.1 and Corollary 4.5.

For the particular case when \( a_- \) (or \( a_+ \)) is just a single exponential function, condition (4.18) is always satisfied and we can go deeper in the study of the properties of \( T_G \). Before proceeding in this direction, however, it is useful to establish a more explicit characterization of the classes \( S_{\lambda,N} \) under the circumstances. Without loss of generality, let us concentrate on the case when \( a_- \) is an exponential.

**Lemma 4.7.** Given \( \lambda > 0 \), let

\[
g = e^{-\sigma} + g_+, \tag{4.24}
\]

where \( g_+ \in H^+_{\infty} \) is not identically zero, and \( 0 < \sigma < \lambda \). Then \( g \in S_{\lambda,N} \) for some \( N \in \mathbb{N} \) if and only if

\[
e^{-\nu} g_+ \in H^+_{\infty}, \quad e^{-\frac{\lambda}{N} - \sigma} g_+ \in H^-_{\infty} \tag{4.25}
\]

for some

\[
\nu \in \left[ \frac{\lambda}{N} - \sigma, \frac{\lambda}{N} - \frac{N - 1}{N} \sigma \right] \tag{4.26}
\]

(of course, the second condition in (4.26) applies only for \( N > 1 \)).

Note that conditions (4.25), (4.26) imply

\[
e^{-\frac{\lambda}{N} - \sigma} g_+ \in H^+_{\infty}, \quad e^{-\frac{\lambda}{N} - \sigma} g_+ \in H^-_{\infty},
\]

and therefore may hold for at most one value of \( N \).

**Proof.** Necessity. Suppose \( g \in S_{\lambda,N} \). Comparing (4.2) and (4.24) we see that

\[
a_- = e^{\beta - \sigma} \in H^+_{\infty}, \quad a_+ = e^{-\nu} g_+ \in H^-_{\infty}. \tag{4.27}
\]

On the other hand, (4.6) takes the form

\[
e^{-\frac{\lambda}{N} - \sigma} g_+ \in H^+_{\infty}, \quad e^{-\frac{\lambda}{N} - \sigma} g_+ \in H^-_{\infty} \tag{4.28}
\]

The first containments in (4.27), (4.28) are equivalent to

\[
\frac{N - 1}{N} \sigma \leq \beta \leq \sigma,
\]
which along with (4.15) yields that $\nu = \frac{\lambda}{N} - \beta$ satisfies (4.20). The second containments in (4.24), (4.25) then imply (4.26).

Sufficiency. Given (4.20), (4.26), let $\beta = \frac{\lambda}{N} - \nu$, and define $a_\pm$ by (4.25). Then (4.2), (4.5) and (4.6) hold (the latter for $N > 1$). □

**Theorem 4.8.** Let $G$ be given by (4.1) with $g$ of the form

$$g = e^{-\sigma} + e_\mu a_+, \quad \mu, \sigma > 0, \quad a_+ \in H^1_\infty,$$

(4.29)

where $\mu + \sigma \geq \lambda$. Then the Toeplitz operator $T_G$ is invertible if (and only if, provided that $G \in B^{2 \times 2}$)

$$\mu + \sigma = \lambda \quad \text{and} \quad \inf_{C^+ + i\varepsilon} |a_+| > 0 \text{ for some } \varepsilon > 0,$$

(4.30)

and $T_G$ is not semi-Fredholm if $\mu + \sigma > \lambda$.

**Proof.** Condition (4.29) implies that $g \in S_{\lambda, 1}$ with $\beta = \sigma, \nu = \lambda - \sigma$, and a solution to (1.7) is given by

$$\phi_+ = (e_\sigma, -e_{\mu + \sigma - \lambda} a_+), \quad \phi_- = (e_{\sigma - \lambda}, 1).$$

Clearly, $\phi_- \in CP^-$, while $\phi_+ \in CP^+$ if and only if (4.30) holds. The part of the statement pertinent to the case $\lambda = \sigma + \mu$ now follows from Theorems 2.10, 2.11.

For $\mu + \sigma > \lambda$, following the proof of [5, Theorem 5.3] observe that

$$1 - e^{-\gamma(z)}$$

(4.31)

deliver a solution to (1.7) in $L^p$, for any $\gamma$ between 0 and $\min\{\sigma, \mu + \sigma - \lambda\}$; thus, the operator $T_G$ has an infinite dimensional kernel in $(H^1_\mu)^2$ for any $p \in (1, \infty)$.

Denote by $G^{-T}$ the transposed of $G^{-1}$. A direct computation shows that for the matrix under consideration, due to its algebraic structure,

$$G^{-T} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} G \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

(4.32)

Therefore, the operator $T_G^{-\tau}$ also has an infinite dimensional kernel. But this means (see, e.g., [12, Section 3.1]) that the cokernel of $T_G$ is infinite dimensional. Therefore, the operator $T_G$ is not even semi-Fredholm on $(H^1_\mu)^2$, $1 < p < \infty$. □

**Theorem 4.9.** Let, as in Theorem 4.8, (4.11) and (4.20) hold, but now with $g$ of the form

$$g = e^{-\sigma} + e_\mu a_+, \quad \mu, \sigma > 0, \quad a_+ \in H^-_\infty$$

(4.33)

for some integer $N > 1$. Then $T_G$ is invertible if (and only if, for $G \in B^{2 \times 2}$) for some $\varepsilon > 0$ one of the following three conditions holds:

$$\sigma + \mu = \frac{\lambda}{N} \quad \text{and} \quad \inf_{C^+ + i\varepsilon} |a_+| > 0,$$

(4.34)

or

$$\frac{N - 1}{N} \sigma + \mu = \frac{\lambda}{N} \quad \text{and} \quad \inf_{C^- - i\varepsilon} |e^{-\frac{\mu}{\sigma - \lambda}} a_+| > 0,$$

(4.35)

or

$$\inf_{C^+ + i\varepsilon} |a_+| > 0, \quad \text{and} \quad \inf_{C^- - i\varepsilon} |e^{-\frac{\mu}{\sigma - \lambda}} a_+| > 0.$$
If, on the other hand,
\[
\sigma + \mu > \frac{\lambda}{N} \quad \text{and} \quad e^{-\frac{\mu}{N}} a_+ \in H_n^-
\]
(4.34)

or
\[
\frac{N-1}{N} \sigma + \mu < \frac{\lambda}{N} \quad \text{and} \quad e^{-\delta a_+} \in H_n^+
\]
(4.35)

for some \( \delta > 0 \), then \( T_G \) is not even semi-Fredholm.

Proof. According to Lemma 4.7, \( G \in S_{\lambda,N} \). Moreover, one can choose in (4.2) \( \nu = \mu \), \( \beta = \frac{\lambda}{N} - \mu \) and \( a_- = e_{\frac{\mu}{N}-\mu} \). Then formulas (4.7), (4.8) yield the following solution to (1.7):

\[
\phi_1^+ = e^{\lambda - N(\mu + \sigma)} \sum_{j=0}^{N-1} (-1)^j a_j^{N-1-j} e^{(N-1-j)(\mu + \sigma)},
\]
\[
\phi_2^+ = -a_+^N,
\]
\[
\phi_1^- = \sum_{j=0}^{N-1} (-1)^j (e^{-\mu} a_+)^{N-1-j} e^{-j(\frac{\mu}{N} + \sigma)},
\]
\[
\phi_2^- = (-1)^{N-1} e^{\lambda - N(\sigma + \mu)}.
\]

Clearly, \( (\phi_1^-, \phi_2^-) \in CP^- \) if and only if the first condition in (4.32) or the second condition in (4.33) holds. Similarly, \( (\phi_1^+, \phi_2^+) \in CP^+ \) is equivalent to the first condition in (4.33) or the second condition in (4.32). Since the first conditions in (4.32), (4.33) cannot hold simultaneously, the statement regarding the invertibility of \( T_G \) now follows from Theorems 2.10, 2.11.

If (4.34) or (4.35) holds, then \( \phi_- = e^{-\delta} \tilde{\phi}_- \) or \( \phi_+ = e^{\delta} \tilde{\phi}_+ \) with \( \delta > 0 \), \( \tilde{\phi}_\pm \in (H_\infty)\), respectively. It follows that the kernel of \( T_G \) is infinite dimensional, as in the proof of Theorem 4.8. Using (4.31), we in the same manner derive that the cokernel of \( T_G \) also is infinite dimensional. So, \( T_G \) is not semi-Fredholm.

\[\square\]

5. AP matrix functions with a spectral gap around zero

The results of the previous section take a particular and, in some sense, more explicit form when considered in the almost periodic setting. The first natural question is, which functions \( g \in AP \) belong to \( S_{\lambda,N} \) for some \( N \in \mathbb{N} \), with \( a_\pm \in AP_\pm \) in (4.2).

According to Remark 4.1 we may have \( 0 \in \Omega(g) \) only if \( N = 1 \) and, in addition, \( g = a_- + a_+ e_\lambda \) with \( 0 \in \Omega(a_-) \) or \( g = a_- e^{-\lambda} + a_+ \) with \( 0 \in \Omega(a_+) \). In either case the operator \( T_G \) is invertible, as can be deduced from the so called one sided case, see [3, Section 14.1]. The easiest way to see that directly, however, is by observing that problem (1.7) has a solution on \( CP^\pm \): \( \phi_+ = (1, -a_+) \), \( \phi_- = (e^{-\lambda}, a_-) \) in the first case, \( \phi_+ = (e_\lambda, a_+) \), \( \phi_- = (1, a_-) \) in the second.
Therefore, in what follows we restrict ourselves to the case \(0 \notin \Omega(g)\). Then
\[
g = g_- + g_+ \quad \text{with} \quad g_\pm \in \mathcal{AP}^\pm, \quad 0 \notin \Omega(g_\pm)
\] (5.1)
with \(g_\pm\) uniquely defined by \(g\). Comparing with (4.2), we have
\[
g_+ = a_+ e_\nu, \quad g_- = a_- e^{-\beta}.
\] (5.2)

Let
\[
\eta_1^- = -\sup \Omega(g_-), \quad \eta_2^- = -\inf \Omega(g_-),
\] (5.3)
\[
\eta_1^+ = \inf \Omega(g_+), \quad \eta_2^+ = \sup \Omega(g_+).
\] (5.4)

Here \(\Omega(g_+), -\Omega(g_-)\) are thought of as subsets of \(\mathbb{R}_+\) (possibly empty), so that \(\eta_1^\pm, \eta_2^\pm \in [0, +\infty] \cup \{-\infty\}\).

**Theorem 5.1.** Let \(g\) be given by (5.1). Then
(i) \(g \in S_{\lambda,1}\) if and only if \(\eta_1^++\eta_1^- \geq \lambda\); 
(ii) \(g \in S_{\lambda,N}\) with \(N > 1\) if and only if
\[
N = \left\lceil \frac{\lambda}{\eta_1^- + \eta_1^+} \right\rceil,
\] (5.5)
while
\[
\eta_1^- \geq \frac{N - 1}{N} \eta_2^-, \quad \eta_1^+ \geq \frac{N - 1}{N} \eta_2^+, \quad \eta_2^+ + \eta_2^- \leq \frac{\lambda}{N-1}.
\] (5.6)

Under these conditions, any \(\nu\) satisfying
\[
M := \max \left\{ \frac{\lambda}{N} - \eta_1^-, \frac{N - 1}{N} \eta_2^- \right\} \leq \nu \leq \min \left\{ \eta_1^+, \frac{\lambda - (N-1)\eta_2^-}{N} \right\} =: m
\] (5.7)
and
\[
a_+ = g_+ e^{-\nu}, \quad \beta = \frac{\lambda}{N} - \nu, \quad a_- = g_- e^{\lambda - \eta_1^-}
\] (5.8)
deliver a representation (4.2).

**Proof.** (i) If \(g \in S_{\lambda,1}\), then from (5.2) with \(\nu + \beta = \lambda\) it follows that \(\eta_1^++\eta_1^- \geq \lambda\). Conversely, setting \(a_\pm = 0\) if \(g_\pm = 0\), \(a_+ = g_+ e^{-\eta_1^+}, \quad a_- = g_- e^{\lambda - \eta_1^+}\) if \(g_+ \neq 0\), and \(a_+ = g_+ e^{-\lambda + \eta_1^-}, \quad a_- = g_- e^{\eta_1^-}\) if \(g_- \neq 0\), we can write \(g\) as in (4.2) with \(\nu + \beta = \lambda\), so that \(g \in S_{\lambda,1}\).

(ii) **Necessity.** Formulas for \(a_\pm\) in (5.8) follow from the uniqueness of \(g_\pm\) in the representation (5.1). The condition \(a_\pm \in H_\infty^\pm\) is therefore equivalent to
\[
\beta \leq \eta_1^-, \quad \nu \leq \eta_1^+.
\] (5.9)

Conditions (5.10), in their turn, are equivalent to
\[
\beta \geq \frac{N - 1}{N} \eta_2^-, \quad \nu \geq \frac{N - 1}{N} \eta_2^+.
\] (5.10)
Comparing the respective inequalities in (5.9) and (5.10) shows the necessity of the first two conditions in (5.6). To obtain the third condition there, just add the two inequalities in (5.10):

\[ \beta + \nu \geq \frac{N - 1}{N} (\eta_2 + \eta_2 -), \]

and compare the result with (4.5).

On the other hand, adding the inequalities in (5.9) yields, once again with the use of (4.5),

\[ \lambda N = \beta + \nu \leq \eta_1 + \eta_1 - \]

So,

\[ \frac{\lambda}{\eta_1 + \eta_1 -} \leq N \leq 1 + \frac{\lambda}{\eta_2 + \eta_2 -}. \]  

(5.11)

If at least one of the inequalities \( \eta_2 \pm \eta_1 \pm \) holds, the difference between the right- and left-hand sides of the inequalities (5.11) is strictly less than 1, and therefore an integer \( N \) is defined by (5.11) uniquely, in accordance with (4.4). Otherwise, \( \eta_1 \pm = \eta_2 \pm \), which means that \( g = c_1 e_{\eta_1 -} + c_2 e_{\eta_1 +} \) with \( c_1, c_2 \in \mathbb{C} \setminus \{0\} \). Since by definition \( N \) is the smallest possible number satisfying (4.4) with \( \nu, \beta \) such that (4.2) holds, we arrive again at (5.5).

**Sufficiency.** Let (5.6) hold for \( N \) defined by (5.5). Then \( m, M \) defined in (5.7) satisfy \( M \leq m \), so that \( \nu \) may indeed be chosen as in (5.7). With such \( \nu \), and \( a \pm \) defined by (5.8), we have (4.2), (4.5), and (4.6). □

The results of Theorem 4.5 and Corollary 4.6 combined with Theorem 5.1 yield the following.

**Theorem 5.2.** Let \( g \in S_{\lambda, N} \) be written as (5.1), and let \( \eta_j \pm (j = 1, 2) \) be defined by (5.3)–(5.4). Then the Toeplitz operator \( T_G \) with symbol \( G \) given by (4.1) is invertible if (and, for \( g \in APW \), only if) one of the following conditions holds:

(i) \( N = 1 \) and

\[ \eta_1 + \in \Omega(g_+), \ -\eta_1 - \in \Omega(g_-), \ \eta_1 + \eta_1 - = \lambda; \]  

(5.12)

(ii) \( N > 1 \) and

\[ \eta_1 + \in \Omega(g_+), \ -\eta_1 - \in \Omega(g_-), \ \eta_1 + \eta_1 - = \frac{\lambda}{N}; \]  

(5.13)

(iii) \( N > 1 \) and

\[ \eta_1 +, \eta_2 + \in \Omega(g_+), \ \eta_2 + = \frac{N}{N - 1} \eta_1 +; \]  

(5.14)

(iv) \( N > 1 \) and

\[ -\eta_1 -, -\eta_2 - \in \Omega(g_-), \ \eta_2 - = \frac{N}{N - 1} \eta_1 -; \]  

(5.15)
(v): \( N > 1 \) and
\[
\eta_2^+ \in \Omega(g_+), \quad -\eta_2^- \in \Omega(g_-), \quad \eta_2^+ + \eta_2^- = \frac{\lambda}{N-1}; \quad (5.16)
\]
and, whenever \( N > 1 \),
\[
\inf_S \{|g_+| + |g_-|\} > 0 \quad \text{for any strip } S \text{ of the form } (4.11). \quad (5.17)
\]

**Proof.** For \( N = 1 \), (5.12) is equivalent to (4.16).

For \( N > 1 \), setting
\[
a_- = e^{\lambda - \nu} g_- \quad \text{and} \quad a_+ = e^\nu g_+ \quad (5.18)
\]
where \( \beta = \frac{\lambda}{N} - \nu \), we deduce from (4.6) that
\[
b_- = e^{-N \nu \frac{N-1}{N}} g_+ \quad \text{and} \quad b_+ = e^{-N \nu \frac{N-1}{N}} g_- \quad (5.19)
\]
Hence
\[
M(a_+) \neq 0 \quad \text{if and only if} \quad \eta_1^+ = \nu \in \Omega(g_+),
\]
\[
M(b_+) \neq 0 \quad \text{if and only if} \quad -\eta_2^- = -\frac{\lambda - N\nu}{N-1} \in \Omega(g_-),
\]
\[
M(a_-) \neq 0 \quad \text{if and only if} \quad -\eta_1^- = \nu - \frac{\lambda}{N} \in \Omega(g_-),
\]
\[
M(b_-) \neq 0 \quad \text{if and only if} \quad \eta_2^+ = -\frac{N\nu}{N-1} \in \Omega(g_+).
\]

Thus, the first inequality in (4.17) holds if and only if either \( \eta_1^+ = \nu \in \Omega(g_+) \) or \(-\eta_2^- = -\frac{\lambda - N\nu}{N-1} \in \Omega(g_-) \), and the second inequality in (4.17) holds if and only if either \(-\eta_1^- = \nu - \frac{\lambda}{N} \in \Omega(g_-) \) or \( \eta_2^+ = -\frac{N\nu}{N-1} \in \Omega(g_+) \).

Taking now \( \eta_1^+ = \nu \in \Omega(g_+) \) and \(-\eta_2^- = -\frac{\lambda - N\nu}{N-1} \in \Omega(g_-) \), we get the equivalence of (4.17) and (5.13); taking \( \eta_1^+ = \nu \in \Omega(g_+) \) and \( \eta_2^+ = -\frac{N\nu}{N-1} \in \Omega(g_+) \), we get the equivalence of (4.17) and (5.14); taking \( -\eta_1^- = \nu - \frac{\lambda}{N} \in \Omega(g_-) \) and \( \eta_2^+ = -\frac{N\nu}{N-1} \in \Omega(g_+) \), we get the equivalence of (4.17) and (5.15). Thus, we see that (4.17) is equivalent to one of the conditions (ii)–(v) of the theorem being satisfied.

The result now follows from Theorem 4.5 and Corollary 4.6 and the second equivalence in (4.15). \( \square \)

From (5.7) it follows that in the case (ii) we have \( \nu = \frac{\lambda}{N} - \eta_1^- = \eta_1^+ \) and therefore
\[
\lambda \geq \max \{ N\eta_1^- + (N-1)\eta_2^+, N\eta_1^+ + (N-1)\eta_2^- \},
\]
in the case (iii) we have \( \nu = \frac{N-1}{N} \eta_2^+ = \eta_1^+ \) so that
\[
N\eta_1^+ + (N-1)\eta_2^- \leq \lambda \leq N\eta_1^- + (N-1)\eta_2^+,
\]
in the case (iv) we have \( \nu = \frac{1}{N} - \eta_1 - \frac{N-1}{N} \eta_2 \) and therefore
\[ N\eta_1 + (N-1)\eta_2 \leq \lambda \leq N\eta_1 + (N-1)\eta_2, \]
in the case (v) we have \( \nu = \frac{N-1}{N} \eta_2 = \frac{1}{N} - \frac{N-1}{N} \eta_2 \) so that
\[ \lambda \leq \min \{ N\eta_1 + (N-1)\eta_2, N\eta_1 + (N-1)\eta_2 \}. \]

We also note that if \( \lambda = N\eta_1 - (N-1)\eta_2 \), then condition (5.14) is equivalent to
\[ \eta_1 + \eta_2 \in \Omega(g_+), \quad \eta_2 + \eta_1 - \frac{\lambda}{N}; \quad (5.20) \]
while condition (5.15) is equivalent to
\[ -\eta_1, -\eta_2 \in \Omega(g_-), \quad \eta_2 + \eta_1 - \frac{\lambda}{N-1}; \quad (5.21) \]
If \( \lambda = N\eta_1 + (N-1)\eta_2 \), then condition (5.14) is equivalent to (5.21), while condition (5.15) is equivalent to (5.20).

Observe that necessity of conditions (5.12)–(5.16) persists for \( g \in AP \) without an additional restriction \( g \in APW \). To see that, suppose that \( T_G \) is invertible in one of the cases (i)–(v) while the respective condition (5.12)–(5.16) fails. Approximate \( g \) by a function in \( APW \) with the same \( \eta_{j\pm} \) and so close to \( g \) in the uniform norm that the respective Toeplitz operator is still invertible. This contradicts the necessity of (5.12)–(5.16) in the \( APW \) case.

It is not clear, however, whether the condition (5.17) remains necessary in the \( AP \) setting.

Remark 5.3. Part (i) of Theorem 5.2 means that, for \( T_G \) to be invertible in the case when the length of the spectral gap of \( g \) around zero is at least \( \lambda \), it in fact must equal \( \lambda \) and, moreover, both endpoints of the spectral gap must belong to \( \Omega(g) \). In contrast with this, for \( N > 1 \) according to parts (ii)–(v) \( T_G \) can be invertible when one (or both) of the endpoints of the spectral gap around zero is missing from \( \Omega(g) \), and the length of this spectral gap can be greater than \( \lambda/N \).

For \( g \in APW \) Theorem 5.2 delivers the invertibility criterion of \( T_G \), and thus a necessary and sufficient condition for \( G \) to admit a canonical \( APW \) factorization. Using Theorem 3.8, however, will allow us to tackle the non-canonical \( AP \) factorability of \( G \) as well.

We assume from now on that \( g \in APW \) is given by (5.1), so that in fact \( g_{\pm} \in APW_{\pm} \), and that \( g \in S_{\lambda,N} \) as described by Theorem 5.1.

In the notation of this theorem, for \( N = 1 \) we have \( \eta_1 + \eta_1 - \lambda \) — the so called big gap case, — and a solution to (1.7) is given by
\[ \phi_+ = (e_{\lambda-\nu}, e_{\nu}-e_{1+\eta_1}g_+), \quad (5.22) \]
\[ \phi_- = (e_{-\nu}, e_{\nu-\eta_1}g_-), \quad (5.23) \]
where
\[ \hat{g}_+ = e_{-\eta_1}g_+ \quad (0 = \inf \Omega(\hat{g}_+)), \quad (5.24) \]
\[ \hat{g}_- = e_{\eta_1}g_- \quad (0 = \sup \Omega(\hat{g}_-)), \quad (5.25) \]
Knowing these solutions and using Theorem 3.4 with \( f \in APW^+ \) as in (3.6), we will be able to complete the consideration of \( AP \) factorability in the big gap case.

It was shown earlier (see [3, Chapter 14], [7, Theorem 2.2]) that \( G \) is \( APW \) factorable if, in addition to the big gap requirement \( \eta_1^+ + \eta_1^- \geq \lambda \), also
\[
\eta_1^+ \in \Omega(g_+) \text{ or } \eta_1^+ \geq \lambda, \quad -\eta_1^- \in \Omega(g_-) \text{ or } \eta_1^- \geq \lambda.
\] (5.27)

However, the \( AP \) factorability of \( G \) if \( \lambda > \eta_1^+ \) or \( \lambda > \eta_1^- \) remained unsettled. As the next theorem shows, in these cases \( G \) does not have an \( AP \) factorization.

**Theorem 5.4.** Let \( g \in APW \) be given by (5.1), with \( \eta_1^\pm \) defined by (5.3), (5.4) and satisfying \( \eta_1^+ + \eta_1^- \geq \lambda \). Then the matrix function (4.1) is \( AP \) factorable if and only if (5.27) holds. In this case \( G \) actually admits an \( APW \) factorization and its partial indices are \( \pm \mu \) with
\[
\mu = \min\{\lambda, \eta_1^+, \eta_1^-, \eta_1^+ + \eta_1^- - \lambda\}.
\] (5.28)

In particular, the factorization is canonical if and only if \( \eta_1^+ = 0 \) or \( \eta_1^- = 0 \) or \( \eta_1^+ + \eta_1^- = \lambda \).

**Proof.** **Sufficiency.** Although it was established earlier, we give here a (much) shorter and self-contained proof, based on the results of Theorem 3.8. Namely, if (5.27) is satisfied, then (5.22)–(5.26) hold with \( 0 = \min \Omega(\tilde{g}_+) = \max \Omega(\tilde{g}_-) \).

Writing
\[
\phi_+ = e^{\mu_1 \tilde{\psi}_+} \quad \text{with} \quad \mu_1 = \min\{\lambda - \nu, -\nu + \eta_1^+\},
\]
\[
\phi_- = e^{-\mu_2 \tilde{\psi}_-} \quad \text{with} \quad \mu_2 = \min\{\nu, \eta_1^- + \nu - \lambda\},
\]
we see that \( \tilde{\psi}_\pm \in APW^\pm \cap CP^\pm \) and
\[
Ge^{\mu_1+\mu_2 \tilde{\psi}_+} = \tilde{\psi}_-,
\]
so that, according to Theorem 3.8 \( G \) admits an \( APW \) factorization with partial indices \( \pm \mu \) where
\[
\mu = \mu_1 + \mu_2 = \min\{\lambda, \eta_1^+, \eta_1^-, \eta_1^+ + \eta_1^- - \lambda\}
\]
(as can be checked straightforwardly).

**Necessity.** Suppose that \( \Omega(g_+) \not\supset \eta_1^+ < \lambda \); the case \( -\Omega(g_+) \not\supset \eta_1^- < \lambda \) can be treated analogously. Then a solution to (1.7) with \( \phi_\pm \in (APW^\pm)^2 \) is given by (5.22)–(5.26).

It follows from these formulas that \( \phi_2^+ = -e_{-\nu + \eta_1^+} \tilde{g}_+ \), where \( -\nu + \eta_1^+ \geq 0 \) due to (5.25). On the other hand, \( 0 \notin \Omega(\tilde{g}_+) \) because \( \eta_1^+ \not\in \Omega(g_+) \). Therefore, for any \( \varepsilon > 0 \) and \( \nu = \eta_1^+ \) there is \( y_\varepsilon \in \mathbb{R}^+ \) such that
\[
\inf_{C^+ + iy_\varepsilon} |\phi_2^+| = \inf_{C^+ + iy_\varepsilon} |e_{-\nu + \eta_1^+} \tilde{g}_+| < \varepsilon
\]
\[
\inf_{C^+ + iy_+} |\phi_{1+}| = \inf_{C^+ + iy_+} |e_{\lambda - \nu}| < \varepsilon.
\]
Thus \(\phi_+ = (\phi_{1+}, \phi_{2+}) \notin CP^+\) and we conclude from Theorem 5.7 that \(G\) cannot have a canonical AP factorization.

Now, if \(G\) admits a non-canonical factorization, which must have partial AP indices \(\pm \mu\) with \(\mu > 0\), then according to Corollary 3.4 we have \(f \in AP^+\), \(\Omega(f) \subset [0, \mu]\). Denoting \(g_+^\pm = (g_{1+}^\pm, g_{2+}^\pm)\), and considering in particular the first component of \(\phi_+\), we thus have from (5.22):
\[
e_{\lambda - \nu} = fg_+^1.
\]
(5.29)

In addition, from the factorization it follows directly that
\[
e_{-\lambda + \mu}g_{11}^+ = g_{11}^-.
\]
Consequently, the Bohr-Fourier spectrum of \(g_{11}^+\) also is bounded, and \(g_{11}^+\) therefore holds everywhere in \(C\). In particular, \(f\) and \(g_{11}^+\) do not vanish in \(C\). But then (see [9] Lemma 3.2 or [11] p. 371) \(\Omega(f), \Omega(g_{11}^+)\) must each contain the maximum and the minimum element, which implies that
\[
\max \Omega(f) + \max \Omega(g_{11}^+), \min \Omega(f) + \min \Omega(g_{11}^+) \in \Omega(fg_{11}^+) = \{\lambda - \nu\}.
\]
We conclude that \(\min \Omega(f) = \max \Omega(f)\) and thus \(f = e_\gamma\) for some \(\gamma \in [0, \mu]\).

But then, from (5.22) and (5.6),
\[(g_{11}^+, g_{21}^+) = (e_{\lambda - \nu - \gamma}, e_{-\nu + \eta_{1+} - \gamma}) \in CP^+,
\]
which is impossible when \(\Omega(g_{1+}) \notin \eta_{1+} < \lambda\). Indeed, in this case \(\lambda - \nu - \gamma > \eta_{1+} - \nu - \gamma \geq 0\) and \(0 \notin \Omega(\tilde{g}_{1+})\).

Finally, the criterion for the AP factorization of \(G\) to be canonical, when it exists, follows immediately from formulas (5.28).

\textbf{Remark 5.5.} The last statement of Theorem 3.8 implies that the construction in the proof of Theorem 5.4 delivers not only the partial AP indices but also a first column of \(G_+\) and \(G_-\). Namely, they may be chosen equal to \(\tilde{\psi}_+\) and \(\tilde{\psi}_-\), respectively.

Now we move to the case \(N > 1\).

Knowing a solution (4.20), (4.21) of (1.7) and using Theorem 3.8 (with \(\det G \equiv 1\), and therefore \(\delta = 0\)), we can obtain sufficient conditions for AP factorability of \(G \in \mathbb{S}_{\lambda,N}, N > 1\).

\textbf{Theorem 5.6.} Let \(g \in APW\) be such that \(g \in S_{\lambda,N}, N > 1\), as described in Theorem 5.4 with (5.17) satisfied. Then \(G\) admits an APW factorization with partial AP indices \(\pm \mu\) where:
(i) \(\mu = N(\eta_{1+} + \eta_- - \lambda)\) if
\[
\eta_{1+} \in \Omega(g_+), \ -\eta_- \in \Omega(g_-)
\]
and
\[
\lambda \geq \max \{N\eta_{1+} + (N - 1)\eta_-, N\eta_- + (N - 1)\eta_{2+}\};
\]
(ii): $\mu = N\eta_{1+} - (N-1)\eta_{2+}$ if

$$\eta_{1+}, \eta_{2+} \in \Omega(g_+)$$

and

$$N\eta_{1+} + (N-1)\eta_{2-} \leq \lambda \leq N\eta_{1-} + (N-1)\eta_{2+};$$

(iii): $\mu = N\eta_{1-} - (N-1)\eta_{2-}$ if

$$-\eta_{1-}, -\eta_{2-} \in \Omega(g_-)$$

and

$$N\eta_{1-} + (N-1)\eta_{2+} \leq \lambda \leq N\eta_{1+} + (N-1)\eta_{2-};$$

(iv): $\mu = \lambda - (N-1)(\eta_{2+} + \eta_{2-})$ if

$$\eta_{2+} \in \Omega(g_+), -\eta_{2-} \in \Omega(g_-)$$

and

$$\lambda \leq \min \{N\eta_{1+} + (N-1)\eta_{2-}, N\eta_{1-} + (N-1)\eta_{2+}\}.$$  

**Proof.** Consider the solution to (1.7) given by (4.20), (4.21), with $a_\pm, b_\pm$ as in (5.18), (5.19). Then we obtain

$$\phi_{1+} = e_{\lambda - N\nu} \sum_{j=0}^{N-1} \left((-1)^j g_-^j g_+^{N-1-j}\right)$$

$$= \sum_{j=0}^{N-1} \left((-1)^j e_{\lambda - N\nu - j\eta_{2-} + (N-1-j)\eta_{1+}} (e_{\eta_{2+}, g_-}^j (e_{-\eta_{1+}, g_+})^{N-1-j}\right)$$

$$= e_{\lambda - N\nu - (N-1)\eta_{2-}} \phi_{1+},$$

with $\phi_{1+} \in APW^+$ where $\lambda - N\nu - (N-1)\eta_{2+} \geq 0$ due to (5.7) and $0 = \inf \Omega(\phi_{1+})$ (= $\min \Omega(\phi_{1+})$ if $-\eta_{2+} \in \Omega(g_-)$);

$$\phi_{2+} = -e_{-N\nu}^{N} = -e_{-N\nu + N\eta_{1+}} (e_{-\eta_{1+}, g_+})^{N} = e_{-N\nu + N\eta_{1+}} \phi_{2+},$$

with $\phi_{2+} \in APW^+$ where $-N\nu + N\eta_{1+} \geq 0$ due to (5.7) and $0 = \inf \Omega(\phi_{2+})$ (= $\min \Omega(\phi_{2+})$ if $\eta_{1+} \in \Omega(g_+)$);

$$\phi_{1-} = e_{-N\nu} \sum_{j=0}^{N-1} \left((-1)^j g_-^j g_+^{N-1-j}\right)$$

$$= \sum_{j=0}^{N-1} \left((-1)^j e_{-N\nu - j\eta_{2+} + (N-1-j)\eta_{1-}} (e_{\eta_{1-}, g_-}^j (e_{-\eta_{2+}, g_+})^{N-1-j}\right)$$

$$= e_{-N\nu + (N-1)\eta_{2+}} \phi_{1-},$$

with $\phi_{1-} \in APW^-$ where $-N\nu + (N-1)\eta_{2+} \leq 0$ due to (5.7) and $0 = \sup \Omega(\phi_{1-})$ (= $\max \Omega(\phi_{1-})$ if $\eta_{2+} \in \Omega(g_-)$);

$$\phi_{2-} = (-1)^{N-1} e_{\lambda - N\nu} g_-^{N} = (-1)^{N-1} e_{\lambda - N\nu - N\eta_{1-}} (e_{\eta_{1-}, g_-})^{N} = e_{\lambda - N\nu - N\eta_{1-}} \phi_{2-},$$
with \( \tilde{\phi}_{2-} \in APW^- \) where \( \lambda - N\nu + N\eta_{1-} \geq 0 \) due to (5.31) and \( 0 = \sup \Omega(\tilde{\phi}_{2-}) \) (= max \( \Omega(\tilde{\phi}_{2-}) \)) if \( -\eta_{1-} \in \Omega(g_-) \).

Hence,

\[
G \left[ e_{\lambda - N\nu + (N-1)\eta_{2-}} \tilde{\phi}_{1+} \mid e_{-N\nu + N\eta_{1+}} \tilde{\phi}_{2+} \right] = \left[ e_{-N\nu + (N-1)\eta_{2+}} \tilde{\phi}_{1-} \mid e_{\lambda - N\nu - N\eta_{1-}} \tilde{\phi}_{2-} \right].
\]

(5.38)

Setting now \( \phi_+ = e_{\mu_1} \tilde{\psi}_+ \) and \( \phi_- = e_{-\mu_2} \tilde{\psi}_- \) where

\[
\mu_1 = -N\nu + \min \{ \lambda - (N-1)\eta_{2-} , N\eta_{1+} \} \geq 0,
\]

\[
\mu_2 = N\nu + \min \{ - (N-1)\eta_{2+} , N\eta_{1-} - \lambda \} \geq 0,
\]

we infer from (5.38) that \( G\tilde{\psi}_+ = \tilde{\psi}_- \), with \( \psi_+ = e_{\mu} \tilde{\psi}_+ \) and

\[
\mu = \mu_1 + \mu_2 = \min \{ \lambda - (N-1)\eta_{2-} , N\eta_{1+} \} + \min \{ - (N-1)\eta_{2+} , N\eta_{1-} - \lambda \}
\]

\[
= \min \left\{ \begin{array}{l}
N(\eta_{1+} + \eta_{1-}) - \lambda, N\eta_{1+} - (N-1)\eta_{2+}, \\
N\eta_{1-} - (N-1)\eta_{2-}, \lambda - (N-1)(\eta_{2+} + \eta_{2-}) \end{array} \right\}.
\]

(5.39)

We consider separately the cases (i)-(iv).

(i) If (5.30) and (5.31) hold, then \( \mu = N(\eta_{1+} + \eta_{1-}) - \lambda \) due to (5.39) and

\[
\tilde{\psi}_+ = \left[ e_{-N\nu + (N-1)\eta_{2-}} \tilde{\phi}_{1+} \mid e_{\lambda - N\nu - N\eta_{1-}} \tilde{\phi}_{2-} \right], \quad \tilde{\psi}_- = \left[ e_{-N\nu + N\eta_{1+}} \tilde{\phi}_{2+} \mid e_{\lambda - N\nu + (N-1)\eta_{2+}} \tilde{\phi}_{1-} \right]
\]

where \( M(\tilde{\phi}_{2+}) \) \( \neq 0 \) if and only \( \eta_{1+} \in \Omega(g_+) \), and \( M(\tilde{\phi}_{2-}) \) \( \neq 0 \) if and only \( -\eta_{1-} \in \Omega(g_-) \). Hence, by (5.30), \( \psi_+ = e_{-\mu} \tilde{\psi}_+ \in CP^+ \) and \( \tilde{\psi}_- \in CP^- \). The result now follows from Theorem 3.8.

(ii) If (5.32) and (5.33) hold, then \( \mu = N\eta_{1+} - (N-1)\eta_{2+} \) due to (5.39) and

\[
\tilde{\psi}_+ = \left[ e_{\lambda - N\nu + (N-1)\eta_{2-}} \tilde{\phi}_{1+} \mid e_{\lambda - N\nu - (N-1)\eta_{1-}} \tilde{\phi}_{2-} \right], \quad \tilde{\psi}_- = \left[ e_{-N\nu + N\eta_{1+}} \tilde{\phi}_{2+} \mid e_{-N\nu + (N-1)\eta_{2+}} \tilde{\phi}_{1-} \right]
\]

where \( M(\tilde{\phi}_{2+}) \) \( \neq 0 \) if and only \( \eta_{1+} \in \Omega(g_+) \), and \( M(\tilde{\phi}_{2-}) \) \( \neq 0 \) if and only \( \eta_{2+} \in \Omega(g_+) \). Hence, by (5.32), \( \psi_+ = e_{-\mu} \tilde{\psi}_+ \in CP^+ \) and \( \tilde{\psi}_- \in CP^- \). The result now follows from Theorem 3.8.

(iii) If (5.31) and (5.35) hold, then \( \mu = N\eta_{1-} - (N-1)\eta_{2-} \) due to (5.39) and

\[
\tilde{\psi}_+ = \left[ e_{-\mu} \tilde{\phi}_{1+} \mid e_{-N\nu + (N-1)\eta_{2-}} \tilde{\phi}_{2+} \right], \quad \tilde{\psi}_- = \left[ e_{-N\nu + N\eta_{1+}} \tilde{\phi}_{2+} \mid e_{-N\nu + (N-1)\eta_{2+}} \tilde{\phi}_{1-} \right]
\]

where \( M(\tilde{\phi}_{1+}) \) \( \neq 0 \) if and only \( -\eta_{2-} \in \Omega(g_-) \), and \( M(\tilde{\phi}_{2-}) \) \( \neq 0 \) if and only \( -\eta_{1-} \in \Omega(g_-) \). Hence, by (5.31), \( \psi_+ = e_{-\mu} \tilde{\psi}_+ \in CP^+ \) and \( \tilde{\psi}_- \in CP^- \). The result now follows from Theorem 3.8.
(iv) If (5.36) and (5.37) hold, then \( \mu = \lambda - (N - 1)(\eta_{2+} + \eta_{2-}) \) due to (5.39) and
\[
\tilde{\psi}_+ = \begin{bmatrix} \tilde{\phi}_1^+ \\ e^{-\lambda + N\eta_{1+} + (N-1)\eta_{2-}} \phi_{2+} \end{bmatrix}, \quad \tilde{\psi}_- = \begin{bmatrix} \tilde{\phi}_1^- \\ e^{\lambda - N\eta_{1-} - (N-1)\eta_{2+}} \phi_{2-} \end{bmatrix}
\]
where \( M(\tilde{\phi}_1^+) \neq 0 \) if and only \( -\eta_{2-} \in \Omega(g_-) \), and \( M(\tilde{\phi}_1^-) \neq 0 \) if and only \( \eta_{2+} \in \Omega(g_+). \) Hence, by (5.39), \( \tilde{\psi}_+ = e_{-\mu}\tilde{\psi}_+ \in CP^+ \) and \( \tilde{\psi}_- \in CP^- \). The result again follows from Theorem 3.8.

**Remark 5.7.** If \( \lambda = N\eta_{1+} + (N-1)\eta_{2-} = N\eta_{1-} + (N-1)\eta_{2+} \), then all the numbers
\[
N(\eta_{1+} + \eta_{1-}) - \lambda, \quad N\eta_{1+} - (N-1)\eta_{2+},
\]
\[
N\eta_{1-} - (N-1)\eta_{2-}, \quad \lambda - (N-1)(\eta_{2+} + \eta_{2-})
\]
coincide, and therefore \( \mu \) in Theorem 5.6 is equal to their common value. Analogously, if \( \lambda = N\eta_{1+} + (N-1)\eta_{2-}, \) then
\[
N(\eta_{1+} + \eta_{1-}) - \lambda = N\eta_{1-} - (N-1)\eta_{2-},
\]
\[
\lambda - (N-1)(\eta_{2+} + \eta_{2-}) = N\eta_{1+} - (N-1)\eta_{2+},
\]
and if \( \lambda = N\eta_{1-} + (N-1)\eta_{2+}, \) then
\[
N(\eta_{1+} + \eta_{1-}) - \lambda = N\eta_{1+} - (N-1)\eta_{2+},
\]
\[
\lambda - (N-1)(\eta_{2+} + \eta_{2-}) = N\eta_{1-} - (N-1)\eta_{2-}.
\]
Hence, in the latter two cases \( \mu = \min \{ N(\eta_{1+} + \eta_{1-}) - \lambda, \lambda - (N-1)(\eta_{2+} + \eta_{2-}) \}. \)

**Remark 5.8.** The main difficulty in applying Theorem 5.6 lies in verifying whether or not condition (5.17) holds. In this regard, Theorems 3.1 and 3.4 of [4] may be helpful. Also, as was mentioned before, (5.17) holds if \( a_+ \) or \( a_- \) is a single exponential. A class of matrix functions with such \( a_+ \) was studied in [7], where the APW factorization of \( G \) was explicitly obtained. Naturally, conclusions of [7] match those that can be obtained by applying Theorem 5.6 to the same class. Furthermore, combining Corollary 5.3 and Theorem 5.5 of the present paper with the APW factorization obtained in [7], it is possible to characterize completely the solutions of (1.7) in that case.

Below we give examples of two cases in which condition (5.17) is also not hard to verify.

**Example 5.9.** Let the off-diagonal entry \( g \in S_{\lambda, N} \) of the matrix (4.1) be given by
\[
g = c_+ e^{-\eta_{2-}} + c_- e^{-\eta_{1-}} + g_+
\]
with \( c_-, c_- \in \mathbb{C}, 0 \leq \eta_{1-} < \eta_{2-} \) and \( g_+ \in APW^+ \) with Bohr-Fourier spectrum containing its maximum and minimum points \( \eta_{j+}, j = 1, 2. \)

If \( N = 1 \), which happens in particular if \( c_- = c_+ = 0 \), then \( G \) is APW factorable with partial AP indices given by Theorem 5.3.
If \( N > 1 \), then it follows from Theorem 5.6 that \( G \) admits an APW factorization with partial AP indices \( \pm \mu \), where
\[
\mu = \begin{cases} 
N(\eta_1 + \eta_{-1}) - \lambda & \text{if } \lambda \geq \max \{N\eta_1 + (N-1)\eta_{-1}, N\eta_{-1} + (N-1)\eta_2\}, \\
N\eta_1 - (N-1)\eta_{2+} & \text{if } N\eta_1 + (N-1)\eta_{-1} \leq \lambda \leq N\eta_{-1} + (N-1)\eta_2, \\
N\eta_{-1} - (N-1)\eta_{2-} & \text{if } N\eta_{-1} + (N-1)\eta_{2+} \leq \lambda \leq N\eta_{1} + (N-1)\eta_{2-}, \\
\lambda - (N-1)(\eta_{2+} + \eta_{2-}) & \text{if } \lambda \leq \min\{N\eta_1 + (N-1)\eta_{2-}, N\eta_{-1} + (N-1)\eta_{2+}\},
\end{cases}
\]
whenever (5.17) holds. Moreover, the expressions given in the proof of Theorem 5.6 for \( \phi_{1\pm}, \phi_{2\pm} \) in each case also provide, by using Theorem 3.8, one column for the factors \( G_{\pm} \) in an APW factorization of \( G \).

In its turn, condition (5.17) is satisfied if and only if one of the coefficients \( c_{-1}, c_{-2} \) is zero or (if \( c_{-1}c_{-2} \neq 0 \))
\[
\inf_{k \in \mathbb{Z}} |g_+(z_k)| > 0 \quad (5.40)
\]
where \( z_k, k \in \mathbb{Z} \), are the zeros of \( g_+ = c_{-2}e^{-\eta_{-2}} + c_{-1}e^{-\eta_{-1}}, \) i.e.,
\[
z_k = \frac{1}{\eta_{2-} - \eta_{1-}} \left( \arg \left( -\frac{c_{-2}}{c_{-1}} \right) + 2k\pi - i \log \left| \frac{c_{-2}}{c_{-1}} \right| \right).
\]
If, in particular, \( g_+ \) also is a binomial, i.e.,
\[
g_+ = c_1e_{\eta_{1+}} + c_2e_{\eta_{2+}} \quad (c_1, c_2 \in \mathbb{C}, 0 \leq \eta_{1+} < \eta_{2+})
\]
then (5.40) is satisfied whenever one of the coefficients \( c_1, c_2 \) is zero. On the other hand, for \( c_1, c_2 \neq 0 \) condition (5.40) is equivalent to (cf. Lemma 3.3 in [2])
\[
\frac{c_1}{c_2} \left| \frac{\eta_{2-} - \eta_{1-}}{\eta_{2+} - \eta_{1+}} \right| \neq \left| \frac{c_{-2}}{c_{-1}} \right| \quad \text{if } \eta_{2+} - \eta_{1+} \in \mathbb{R}\backslash\mathbb{Q};
\]
and to
\[
\left( \frac{-c_1}{c_2} \right)^q \neq \left( \frac{-c_{-2}}{c_{-1}} \right)^p \quad \text{if } \frac{\eta_{2+} - \eta_{1+}}{\eta_{2-} - \eta_{1-}} = \frac{p}{q}, \quad \text{with } p, q \in \mathbb{N} \text{ relatively prime.}
\]

Example 5.10. Let \( G \in \mathfrak{S}_{\lambda,N}, N > 1 \), with the off-diagonal entry \( g \in \text{APW} \) of the form \( g_+ = g_- + g_+ \) where
\[
g_+ = c_\alpha e_\alpha g_- + c_\mu e_\mu,
\]
\( \alpha, \mu > 0, c_\alpha, c_\mu \in \mathbb{C}, c_\mu \neq 0 \) and \( \eta_{1\pm}, \eta_{2\pm} \in \Omega(g_+) \) (see (5.3), (5.4)). It is easy to see that (5.17) holds. Therefore, Theorem 5.6 implies therefore that \( G \) admits an APW factorization with partial AP indices as indicated in that theorem.
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