Open string amplitudes of closed topological vertex

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Abstract

The closed topological vertex is the simplest ‘off-strip’ case of non-compact toric Calabi–Yau threefolds with acyclic web diagrams. By the diagrammatic method of topological vertex, open string amplitudes of topological string theory therein can be obtained by gluing a single topological vertex to an ‘on-strip’ subdiagram of the tree-like web diagram. If non-trivial partitions are assigned to just two parallel external lines of the web diagram, the amplitudes can be calculated with the aid of techniques borrowed from the melting crystal models. These amplitudes are thereby expressed as matrix elements, modified by simple prefactors, of an operator product on the Fock space of 2D charged free fermions. This fermionic expression can be used to derive \(q\)-difference equations for generating functions of special subsets of the amplitudes. These \(q\)-difference equations may be interpreted as the defining equation of a quantum mirror curve.

Keywords: closed topological vertex, open string amplitude, mirror curve, free fermion, quantum torus, shift symmetry, \(q\)-difference equation

1. Introduction

Topological vertex \cite{1} is a diagrammatic method that captures a-model topological string theory on non-compact toric Calabi–Yau threefolds. In the case of ‘on-strip’ geometry (see appendix A for a precise setup), this method works particularly well to calculate both the closed string partition function and open string amplitudes in an explicit form \cite{2}. Since the
calculation in the on-strip case relies on the linear shape of the toric diagram, it is a technical challenge to extend this result to an ‘off-strip’ case.

The closed topological vertex [3] is one of the simplest examples of ‘off-strip’ geometry. Its web diagram is acyclic (in other words, the threefold has no compact 4-cycle), and the toric diagram has a triangular shape (see figure 1). The closed string partition function in this case is calculated by several methods including topological vertex [3–5]. The final expression of the partition function resembles the on-strip case, but the method of derivation is more subtle. Moreover, Karp et al [4, section 6.4] argued that such a closed expression of the partition function will cease to exist if branches of the tree-like web diagram are prolonged to arbitrary lengths. In this sense, the closed topological vertex is rather special among off-strip geometry without compact 4-cycle.

In this paper, we calculate open string amplitudes of the closed topological vertex in the case where non-trivial boundary conditions of the string world sheet are imposed on two parallel external lines of the web diagram. Although lacking full generality, this is the first attempt in the literature to calculate open string amplitudes of the closed topological vertex explicitly. Moreover, we use this result to derive $q$-difference equations for generating functions of special subsets of these amplitudes. In the perspectives of mirror geometry of topological string theory [6, 7], the $q$-difference equations may be interpreted as the defining equations of a ‘quantum mirror curve’. This quantum mirror curve will be a new example of quantum curves in the topological recursion program [8].

To calculate the open string amplitudes in question, we use techniques that were developed in our previous work on the melting crystal models [9–12]. A clue of these techniques is the notion of ‘shift symmetries’ in a quantum torus algebra. This algebra is realized by operators on the Fock space of 2D charged free fermions\footnote{The same fermionic realization of the quantum torus algebra appears in the work of Okounkov and Pandharipande [13] on the Gromov–Witten invariants of $\mathbb{CP}^1$.}. The shift symmetries act on a set of basis elements $V_{m}^{(k)}$ of this algebra so as to shift the indices $k, m$ in a certain way. This enables us to relate the commutative subalgebra spanned by $V_{m}^{(k)}$ to the $U(1)$ current algebra spanned by $V_{m}^{(0)}$. In our previous work, this algebraic machinery is used to convert the partition functions of the melting crystal models to tau functions of the 2D Toda hierarchy. In this paper, we employ the same method to express the open string amplitudes as matrix elements, modified by simple prefactors, of an operator product on the fermionic Fock space.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{web_toric.pdf}
\caption{Web diagram (left) and toric diagram (right) of closed topological vertex.}
\end{figure}

\footnote{Its role as symmetries in the KP and 2D Toda hierarchies was independently studied by Harnad and Orlov [14].}
Our calculation starts from a cut-and-glue description of the amplitude [5]. Namely, the web diagram is cut into two subdiagrams by removing an internal line, and glued together along this line after calculating the contributions of these two parts. One of them is a single topological vertex, and the other is an on-strip diagram for which the result of Iqbal and Kashani-Poor [2] can be used. To glue these two parts again, we have to calculate an infinite sum with respect to a partition on the internal line. This is the place where the aforementioned techniques are used. The amplitude of the closed topological vertex thereby boils down to a product of simple factors and a matrix element of an operator on the Fock space. Moreover, the matrix element turns out to be the open string amplitude of a new on-strip web diagram.

The final expression of the open string amplitudes enables us to derive $q$-difference equations for the generating functions of special subsets of the amplitudes. The generating functions are the Baker–Akhiezer functions in the context of integrable hierarchies, and play the role of ‘wave functions’ of a probe D-brane [6, 7]. Our result is an extension of known results on the resolved conifold [15–17] and more general on-strip geometry [18]. The structure of the $q$-difference equation is, so to speak, a mixture of the $q$-difference equations of the quantum dilogarithmic functions [19, 20] and the $q$-hypergeometric equations that appear in the resolved conifold and more general on-strip geometry. Our result shows that quantum mirror curves beyond on-strip geometry can have an intricate origin.

This paper is organized as follows. In section 2, the diagrammatic construction of the open string amplitudes are reformulated in a partially summed form. Fermionic tools for the subsequent calculation are also reviewed here. In section 3, the techniques borrowed from the melting crystal models are used to calculate the amplitudes in terms of fermions. In section 4, the fermionic expression of the amplitudes is further converted to a final form. A technical clue therein is the cyclic symmetry among ‘two-leg’ topological vertices. This well known symmetry is translated to a kind of ‘operator-state correspondence’ in the fermionic Fock space, and used to rewrite the fermionic expression of the amplitudes. In section 5, the generating functions of special subsets of the amplitudes are introduced, and shown to satisfy $q$-difference equations. The structure of the $q$-difference equations is examined in the perspectives of mirror geometry. In section 6, these results are shown to be consistent with a flop

![Figure 2. Setup for open string amplitude $Z_{\mathcal{P}_{12}}^{\text{ctv}}$.](image-url)
2. Construction of open string amplitudes

The setup for the open string amplitudes in question is shown in figure 2. \( Q_1, Q_2, Q_3 \) are Kähler parameters on the internal lines, \( \beta_1 \) and \( \beta_2 \) are partitions assigned to the two lower external lines. The other external lines are given the trivial partition \( \emptyset \). Let \( Z^\alpha_1 \beta_1^\beta_2 \) denote the amplitude in this setup. \( Z^\alpha_1 \beta_1^\beta_2 \) is a sum of weights over all possible values of the partitions \( \alpha_1, \alpha_2, \alpha_3 \) on the internal lines. The weight for a given configuration of \( \alpha_1, \alpha_2, \alpha_3 \) is a product of vertex weights and edge weights. These weights depend on the parameter \( q \) in the range \(|q| < 1\).

2.1. Vertex weights and gluing rules

The vertex weight at each vertex is the topological vertex\(^6\)
\[
C_{\lambda \mu \nu} = q^{-\mu/2} s_{\lambda}(q^{-\nu}) \sum_{\eta \in \mathcal{P}} s_{\lambda/\eta}(q^{-\nu-\rho}) s_{\mu/\eta}(q^{-\nu-\rho}),
\]
where the sum with respect to \( \eta \) ranges over the set \( \mathcal{P} \) of all partitions. \( \lambda = (\lambda_i)_{i=1}^\infty \), \( \mu = (\mu_i)_{i=1}^\infty \) and \( \nu = (\nu_i)_{i=1}^\infty \) are the partitions assigned to the three legs of the vertex that are ordered anti-clockwise, and \( ^{\dagger}\nu \) denotes the conjugate (or transposed) partition of \( \nu \). \( \kappa(\mu) \) is the second Casimir invariant
\[
\kappa(\mu) = \sum_{i=1}^\infty \mu_i(\mu_i - 1/2) - (i + 1/2)^2 \cdot
\]
\( s_{\nu}(q^{-\rho}), s_{\lambda/\eta}(q^{-\nu-\rho}) \) and \( s_{\mu/\eta}(q^{-\nu-\rho}) \) are special values of the infinite-variate Schur function \( s_\nu(x) \) and the skew Schur functions \( s_{\lambda/\eta}(x), s_{\mu/\eta}(x), x = (x_1, x_2, \ldots) \), at
\[
q^{-\rho} = (q^{1/2})_{i=1}^\infty, \quad q^{-\nu-\rho} = (q^{-\nu+1/2})_{i=1}^\infty, \quad q^{-\nu-\rho} = (q^{-\nu+1/2})_{i=1}^\infty.
\]
The vertex weight enjoy the cyclic symmetry
\[
C_{\lambda \mu \nu} = C_{\mu \nu \lambda} = C_{\nu \lambda \mu}
\]
that can be deduced from the crystal interpretation of the vertex weight [23].

The vertex weights \( C_{\lambda \mu \nu} \) and \( C_{\nu \lambda \mu} \) at two vertices connecting an internal line are glued together by the following rules:

(i) The partitions on the internal line, say \( \lambda \) and \( \lambda' \), are matched as
\[
\lambda' = ^{\dagger}\lambda.
\]

(ii) The product of the vertex weights is multiplied by the edge weight
\[
(-Q)^{\lambda_1}(-1)^{\mu_1} q^{-\kappa(\lambda)/2},
\]
where \( Q \) is the Kähler parameter of the internal line, and \( n \) is an integer called ‘the framing number’.

\(^6\) We follow a definition commonly used in the recent literature [21, 22]. This definition differs from the earlier one [1, 6] in that \( q \) is replaced by \( q^{-1} \) and an overall factor of the form \( q^{(\lambda_1)/2 + (\mu_1)/2 + (\nu_1)/2} \) is multiplied.
The framing number is defined as
\[ n = v' \wedge v = w' \wedge w, \]  
where \( v, w, v', w' \) are vectors in the web diagram that emanate from the two vertices (see figure 3). The wedge product means the determinant of the \( 2 \times 2 \) matrix formed by the two vectors, i.e.
\[ v' \wedge v = v'_1 v_2 - v'_2 v_1 \]
for \( v' = (v'_1, v'_2) \) and \( v = (v_1, v_2) \). These vectors \( v, w, v', w' \) are chosen along with the third vectors \( u, u', u + u' = 0 \), in such a way that \( u, v, w \) and \( u', v', w' \) are ordered anti-clockwise and satisfy the zero-sum relations
\[ u + v + w = 0, \quad u' + v' + w' = 0. \]
These sets of vectors are uniquely determined as far as the toric diagram is fully triangulated (i.e., the area of each triangle is \( 1/2 \)).

2.2. Reformulation of amplitude

The amplitude \( Z^{ctv}_{\beta_1 \beta_2} \) is given by a sum of the product of these weights over \( \alpha_1, \alpha_2, \alpha_3 \in P \). Following Sulkowski’s formulation [5], we decompose this sum to a partial with sum respect to \( \alpha_1, \alpha_2 \) at the first stage and a sum with respect to \( \alpha_3 \) at the next stage. The full amplitude can be thus reformulated as
\[ Z^{ctv}_{\beta_1 \beta_2} = \sum_{\alpha_3 \in P} Z^{ctv}_{\beta_1 \beta_2 | \alpha_3} (-Q_1)^{[\alpha_3]} C_{\alpha_3 \varnothing \varnothing}. \]

\( Z^{ctv}_{\beta_1 \beta_2 | \alpha_3} \) is the partial sum with respect to \( \alpha_1, \alpha_2 \) and represents the contribution from the lower part of the web diagram. This part is glued with the upper part via the internal line carrying \( \alpha_3 \). \((-Q_1)^{[\alpha_3]}\) is the edge weight of this internal line. Note that the framing number (2.3) in this case is equal to 0. \( C_{\alpha_3 \varnothing \varnothing} \) is the contribution from the upper part of the web diagram. By the cyclic symmetry (2.2), this vertex weight reduces to a special value of the Schur function:
\[ C_{\alpha_3 \varnothing \varnothing} = C_{\varnothing \varnothing \varnothing} = \delta_{\alpha_3}(q^{-r}). \]

The partial sum \( Z^{ctv}_{\beta_1 \beta_2 | \alpha_3} \) itself may be thought of as an open string amplitude of the web diagram (called ‘double-\( \varnothing \)’) shown in figure 4. Since this is a diagram ‘on a strip’, the
associated open string amplitude can be calculated by the well known result [2] (see appendix A):

\[
Z_{\beta_1,\beta_2,\alpha_1} = s_{\beta_1}(q^{-\rho})s_{\beta_2}(q^{-\rho})s_{\alpha_1}(q^{-\rho}) \prod_{i,j=1}^{\infty} \left( 1 - Q_1 Q_2 q^{-\beta_1_i - \beta_2_i + i + j - 1} \right)^{-1} \\
\times \prod_{i,j=1}^{\infty} \left( 1 - Q_1 q^{-\beta_1_i - \alpha_1_i + i + j - 1} \right) \prod_{i,j=1}^{\infty} \left( 1 - Q_2 q^{-\alpha_1_i - \beta_2_i + i + j - 1} \right).
\]

(2.6)

Plugging these building blocks into (2.4), we obtain the following expression of \(Z^{ctv}_{\beta_1,\beta_2,\alpha_1}:

\[
Z^{ctv}_{\beta_1,\beta_2,\alpha_1} = s_{\beta_1}(q^{-\rho})s_{\beta_2}(q^{-\rho}) \prod_{i,j=1}^{\infty} \left( 1 - Q_1 Q_2 q^{-\beta_1_i - \beta_2_i + i + j - 1} \right)^{-1} \\
\times \sum_{\alpha_1 \in \mathbb{P}} s_{\alpha_1}(q^{-\rho}) s_{\alpha_1}(q^{-\rho}) \prod_{i,j=1}^{\infty} \left( 1 - Q_1 q^{-\beta_1_i - \alpha_1_i + i + j - 1} \right) \\
\times \prod_{i,j=1}^{\infty} \left( 1 - Q_2 q^{-\alpha_1_i - \beta_2_i + i + j - 1} \right).
\]

(2.7)

Note here that the sum with respect to \(\alpha_1\) resembles the partition function of the modified melting model [11, 12]; the main part of the Boltzmann weight therein takes the product form \(s_{\alpha_1}(q^{-\rho}) s_{\alpha_1}(q^{-\rho})\), and this weight is deformed by external potentials depending on \(\alpha_1\). To calculate this sum, we use the machinery of 2D charged free fermions.

2.3. Fermionic Fock space and operators

The setup of the fermionic Fock space and operators is the same as used for the melting crystal models [9–12]. Let \(\psi_n, \psi_n^*\) \(n \in \mathbb{Z}\) denote the Fourier modes of the 2D charged free fermion fields \(\psi(z), \psi^*(z)\). They satisfy the anti-commutation relations

\[
\psi_m \psi_n + \psi_n^* \psi_m = \delta_{m+n,0}, \quad \psi_m \psi_n^* + \psi_n^* \psi_m^* = 0, \quad \psi_m^* \psi_n^* + \psi_n \psi_m^* = 0.
\]

The associated Fock space and its dual space are decomposed to the charge-\(s\) sectors for \(s \in \mathbb{Z}\). It is only the charge-0 sector that is relevant to the calculation of (2.7). An orthonormal basis of the charge-0 sector is given the ground states

\[
|0\rangle = (-\infty|\cdots|\psi_{-1}^*|\psi_0^*|\cdots|\psi_{-i+1}^*|\psi_{-i}^*|\cdots|\psi_0^*|\cdots|\psi_{-i}^*|\psi_{-i+1}^*|\psi_{-i+2}^*|\cdots|\infty\rangle
\]

\[
|0\rangle = \psi_0 \psi_{-1}^* \psi_{-2}^* \cdots |\cdots|\psi_{-i}^* |\cdots|\psi_{-i+1}^* |\psi_{-i+2}^* |\cdots |\infty\rangle
\]
and the excited states
\[ \langle \lambda | = \{-\infty | \cdots \psi^{*}_{\lambda_{-i}+1} \cdots \psi^{*}_{\lambda_{-i+1}} \psi^{*}_{\lambda}, \]
\[ | \lambda \rangle = \psi^{*}_{\lambda} \psi^{*}_{\lambda-1} \cdots \psi^{*}_{\lambda-i+1} \cdots \psi^{*}_{\lambda_{-i}} | - \infty \}

labelled by partitions. The normal ordered product \( \psi^{*}_{m} \psi^{*}_{n} \) is defined as
\[ \psi^{*}_{m} \psi^{*}_{n} = 0 \]
\[ \psi^{*}_{m} \psi^{*}_{n} = | 0 \rangle \]
\[ \psi^{*}_{m} \psi^{*}_{n} = 0 \].

The following operators on the Fock space are used as fundamental tools in our calculation.

(i) The zero-modes
\[ L_0 = \sum_{n \in \mathbb{Z}} n \psi^{*}_{-n} \psi^{*}_{n}, \quad W_0 = \sum_{n \in \mathbb{Z}} n^2 \psi^{*}_{-n} \psi^{*}_{n}, \]

of the Virasoro and \( W_3 \) algebras and the Fourier modes
\[ J_m = \sum_{n \in \mathbb{Z}} \psi^{*}_{-n} \psi^{*}_{n+m}, \quad m \in \mathbb{Z}, \]

of the fermionic current \( \psi(z) \psi^{*}(z) \).

(ii) The fermionic realization
\[ K = \sum_{n \in \mathbb{Z}} (n - 1/2)^2 : \psi^{*}_{-n} \psi^{*}_{n} := W_0 - L_0 + J_0/4 \]

of the so called ‘cut-and-join operator’ [24, 25].

(iii) The basis elements
\[ V^{(k)} = q^{-km/2} \sum_{n \in \mathbb{Z}} q^{kn} \psi^{*}_{-n} \psi^{*}_{n}, \quad k, m \in \mathbb{Z}, \]

of a fermionic realization of the quantum torus algebra [9, 13].

(iv) The vertex operators [26, 27]
\[ \Gamma_{\pm}(z) = \exp \left( \sum_{k=1}^{\infty} \frac{z^k}{k} J_{\pm k} \right), \quad \Gamma_{\mp}(z) = \exp \left( -\sum_{k=1}^{\infty} \frac{(-z)^k}{k} J_{\pm k} \right) \]

and the multi-variable extensions
\[ \Gamma_{\pm}(\mathbf{x}) = \prod_{i>1} \Gamma_{\pm}(x_i), \quad \Gamma^{\prime}_{\pm}(\mathbf{x}) = \prod_{i>1} \Gamma^{\prime}_{\pm}(x_i). \]

The matrix elements of these operators are well known. \( J_0, L_0, W_0, K \) are diagonal with respect to the basis \( \{| \lambda \rangle \}_{\lambda \in \mathbb{P}} \) in the charge-0 sector:
\[ \langle \lambda | J_0 | \mu \rangle = 0, \quad \langle \lambda | L_0 | \mu \rangle = \delta_{\mu \lambda} | \lambda \rangle, \]
\[ \langle \lambda | W_0 | \mu \rangle = \delta_{\mu \lambda} (\kappa(| \lambda \rangle) + | \lambda \rangle), \quad \langle \lambda | K | \mu \rangle = \delta_{\mu \lambda} \kappa(| \lambda \rangle). \quad (8.2) \]

The matrix elements of \( \Gamma_{\pm}(\mathbf{x}) \) and \( \Gamma^{\prime}_{\pm}(\mathbf{x}) \) are skew Schur functions [28, 29]:
\[ \langle \lambda | \Gamma_{\pm}(\mathbf{x}) | \mu \rangle = \langle \mu | \Gamma_{\pm}(\mathbf{x}) | \lambda \rangle = s_{\lambda / \mu}(\mathbf{x}), \]
\[ \langle \lambda | \Gamma^{\prime}_{\pm}(\mathbf{x}) | \mu \rangle = \langle \mu | \Gamma^{\prime}_{\pm}(\mathbf{x}) | \lambda \rangle = s_{\lambda^{\prime} / \mu}(\mathbf{x}). \quad (8.9) \]
3. Calculation of sum in (2.7)

Let us proceed to calculation of the sum in (2.7). This comprises two steps. In the first step, we express \( s_\alpha(q^{-\rho}) \) and \( s_\alpha(q^{-\rho'}) \) in a fermionic form, and convert the \( c \)-number factors \( \prod_{i,j=1}^{\infty} (1 - Q_i \cdots) \) and \( \prod_{i,j=1}^{\infty} (1 - Q_2 \cdots) \) to operators inserted in the fermionic expression of the Schur functions. The sum with respect to \( \alpha_3 \) thereby turns into the vacuum expectation value of an operator product on the Fock space. In the second step, we use the ‘shift symmetries’ of the quantum torus algebra \([9–12]\) to rewrite the vacuum expectation value further. This calculation is more or less parallel to the way the partition functions of the various melting crystal models are converted to tau functions of the 2D Toda hierarchy.

3.1. Step 1: translation to fermionic language

The infinite products \( \prod_{i,j=1}^{\infty} (1 - Q_i \cdots) \) and \( \prod_{i,j=1}^{\infty} (1 - Q_2 \cdots) \) can be re-expressed in an exponential form as

\[
\prod_{i,j=1}^{\infty} (1 - Q_i q^{-\beta_3 - \alpha_3 i + j - 1}) = \exp\left(-\sum_{i,k=1}^{\infty} \frac{(Q_i q^{-\beta_3})^k}{k} \sum_{j=1}^{\infty} q^{-k(\alpha_3 - j + 1)}\right)
\]

and

\[
\prod_{i,j=1}^{\infty} (1 - Q_2 q^{-3\alpha_3 - \beta_3 i + j - 1}) = \exp\left(-\sum_{j,k=1}^{\infty} \frac{(Q_2 q^{-\beta_3})^k}{k} \sum_{i=1}^{\infty} q^{-k(3\alpha_3 - i + 1)}\right)
\]

We convert these \( c \)-number factors to operators inserted in the fermionic expression

\[
s_{\alpha_3}(q^{-\rho}) = \langle 0 \vert \Gamma^\alpha_3(q^{-\rho}) \vert \alpha_3 \rangle, \quad s_{\alpha_1}(q^{-\rho}) = \langle \alpha_3 \vert \Gamma^\alpha_3(q^{-\rho}) \vert 0 \rangle
\]

of the special values of the Schur functions.

To this end, let us note that \( \sum_{j=1}^{\infty} q^{-k(\alpha_3 - j + 1)} \) and \( \sum_{j=1}^{\infty} q^{-k(3\alpha_3 - i + 1)} \) are related to eigenvalues of \( V^{(i, k)}_0 \) as shown below.

**Lemma 1.** For any \( k > 0 \) and any \( \lambda \in \mathcal{P} \),

\[
\left(V^{(-k)}_0 + \frac{1}{1 - q^k}\right) |\lambda\rangle = \sum_{i=1}^{\infty} q^{-k(\lambda_i - i + 1)} |\lambda\rangle,
\]

\[
\left(V^{(k)}_0 - \frac{q^k}{1 - q^k}\right) |\lambda\rangle = -q^k \sum_{i=1}^{\infty} q^{-k(\lambda_i - i + 1)} |\lambda\rangle.
\]

**Remark 1.** Equations (3.1) and (3.2) imply the relations

\[
\langle \lambda \vert V^{(-k)}_0 + \frac{1}{1 - q^k}\rangle = \langle \lambda \vert \sum_{i=1}^{\infty} q^{-k(\lambda_i - i + 1)},
\]

\[
\langle \lambda \vert V^{(k)}_0 - \frac{q^k}{1 - q^k}\rangle = -q^k \sum_{i=1}^{\infty} q^{-k(\lambda_i - i + 1)}
\]

in the dual Fock space as well.
Proof. It is straightforward to derive (3.1):

\[ V^{(k)}_0 \| \lambda \rangle = \sum_{j=1}^{\infty} \left( q^{-k}(\lambda_j-1) - q^{-k(j+1)} \right) \| \lambda \rangle \]

\[ = \left( \sum_{j=1}^{\infty} q^{-k(\lambda_j-1)} - \frac{1}{1 - q^k} \right) \| \lambda \rangle. \]

The subtraction term \( q^{-k(j+1)} \) in this calculation originates in the normal ordering

\[ \psi_n \psi_n^* \psi_n^* \psi_n \]

\[ := \begin{cases} \psi_n \psi_n^* & \text{for } n > 0, \\ \psi_n \psi_n^* - 1 & \text{for } n \leq 0. \end{cases} \]

It is not straightforward to derive (3.2). Let \( n \) be an integer greater than or equal to the length of \( \lambda \). Accordingly, \( \lambda_i = i \) for \( i > n \). Since the set of all integers \( i \leq n \) can be divided into two disjoint sets as

\[ \{i \mid i \leq n\} = \{1, \lambda_i - i + 1 | i \geq 1\} \cup \{-\lambda_i + i | 1 \leq i \leq n\}, \]

one obtains the identity

\[ \sum_{i=1}^{\infty} q^{-k(\lambda_i-i+1)} + \sum_{i=1}^{n} q^{-k(-\lambda_i+i)} = \sum_{i=-\infty}^{n} q^{-ki} = \frac{q^{-kn}}{1 - q^k}, \]

which implies that

\[ \sum_{i=1}^{\infty} q^{-k(\lambda_i-i+1)} = -\sum_{i=1}^{n} q^{-k(-\lambda_i+i)} + \frac{q^{-kn}}{1 - q^k} \]

\[ = -\sum_{i=1}^{n} \left( q^{-k(-\lambda_i+i)} - q^{-ki} \right) + \frac{1}{1 - q^k} \]

\[ = -q^{-k} \sum_{i=1}^{n} \left( q^{k(\lambda_i-i+1)} - q^{k(-i+1)} \right) + \frac{1}{1 - q^k}. \]

Consequently

\[ V^{(k)}_0 \| \lambda \rangle = \sum_{i=1}^{n} \left( q^{k(\lambda_i-i+1)} - q^{k(-i+1)} \right) \| \lambda \rangle \]

\[ = \left( -q^k \sum_{i=1}^{\infty} q^{-k(\lambda_i-i+1)} + \frac{q^k}{1 - q^k} \right) \| \lambda \rangle. \]

Equation (3.2) can be thus derived. \[ \square \]

By (3.1) and (3.2), the \( c \)-number factors \( \prod_{j=1}^{\infty} (1 - Q \cdots) \) and \( \prod_{j=1}^{\infty} (1 - Q \cdots) \) can be converted to operators on the Fock space as

\[ \prod_{i,j=1}^{\infty} \left( 1 - Q_i q^{-\beta_{i,j} a_{i,j} + 1} \right) \cdot \mathcal{A}(q^{-\rho}) \]

\[ = (0 \mid \Gamma(q^{-\rho}) \exp \left( -\sum_{i,k=1}^{\infty} \frac{Q_i q^{-\beta_{i,j}+1}}{k} \left( V^{(k)}_0 + \frac{1}{1 - q^k} \right) \right) \mid 0 \rangle \] (3.3)
and
\[
\prod_{i,j=1}^{\infty} \left( 1 - Q_2 q^{-\alpha_{i,j} - \beta_2 + i + j - 1} \right) s_{\alpha_3}(q^{-\rho}) \exp \left( \sum_{j,k=1}^{\infty} \frac{Q_2 q^{-(\beta_2 + j - 1)k}}{k} \left( V_0^{(k)} - \frac{q^k}{1 - q^k} \right) \right) \Gamma_{-}(q^{-\rho}) |0\rangle.
\] (3.4)

Moreover, the factor \(-Q_3)^{\alpha_3}\) can be identified with the diagonal matrix element of \(-Q_3)^{\alpha_3}\). Having derived these building blocks, we can now use the partition of unity
\[
\sum_{\alpha_3 \in P} |\alpha_3\rangle \langle \alpha_3| = 1
\]
in the charge-0 sector to rewrite the sum in (2.7) to the vacuum expectation value of an operator product:
\[
\sum_{\alpha_3 \in P} s_{\alpha_3}(q^{-\rho}) s_{\alpha_3}(q^{-\rho}) \prod_{i,j=1}^{\infty} \left( 1 - Q_1 \right) \prod_{i,j=1}^{\infty} \left( 1 - Q_2 \right)
\]
\[
= \langle 0 | \Gamma_+(q^{-\rho}) \exp \left( - \sum_{i,k=1}^{\infty} \frac{Q_1 q^{-\beta_1 + 1}k}{k} \left( V_0^{(-k)} + \frac{1}{1 - q^k} \right) \right) \Gamma_{-}(q^{-\rho}) |0\rangle.
\] (3.5)

### 3.2. Step 2: use of shift symmetries

Let us recall the following consequence of the shift symmetries of the quantum torus algebra [9–11]. Note that the last one (3.8) is modified from the previous formulation in terms of \(W_0\).

#### Lemma 2.

\[
\Gamma_+(q^{-\rho}) \Gamma_+(q^{-\rho}) V_0^{(-k)} \left( V_0^{(-k)} + \frac{1}{1 - q^k} \right) = V_k^{(-k)} \Gamma_+(q^{-\rho}) \Gamma_+(q^{-\rho}),
\] (3.6)

\[
\Gamma_{-}(q^{-\rho}) \Gamma_{-}(q^{-\rho}) V_0^{(k)} \left( V_0^{(k)} - \frac{q^k}{1 - q^k} \right) = \Gamma_{-}(q^{-\rho}) \Gamma_{-}(q^{-\rho}) (-1)^k V_{-k}^{(k)},
\] (3.7)

\[
V_{-k}^{(-k)} = q^{-k/2} \delta_{k,0} q^{-k/2}, \quad V_{-k}^{(k)} = q^{k/2} \delta_{k,0} q^{k/2}.
\] (3.8)

We use these operator identities to rewrite (3.5) further. Let us first examine the left side of \(-Q_3)^{\alpha_3}\) in (3.5). Upon inserting \(q^{-k/2} \Gamma_{-}^{(q^{-\rho})} \) to the right of \(0\) as
\[
\langle 0 | \Gamma_+(q^{-\rho}) = \langle 0 | q^{-k/2} \Gamma_+(q^{-\rho}) \Gamma_+(q^{-\rho}),
\]

we can use (3.6) and (3.8) to rewrite the left side of \((-Q_3)^L_3\) as

\[
\langle 0 | \Gamma'_+ (q^{-\rho}) \exp \left( - \sum_{i,k=1}^{\infty} \frac{(Q_i q^{-\beta_i} + i)^k}{k} \left( V_0^{(i)} - \frac{1}{1 - q^k} \right) \right) | 0 \rangle
\]

\[
= \langle 0 | q^{-K/2} \exp \left( - \sum_{i,k=1}^{\infty} \frac{(Q_i q^{-\beta_i} + i)^k}{k} V_k^{(i)} \right) \Gamma'_+ (q^{-\rho}) \Gamma'_+ (q^{-\rho}) | 0 \rangle
\]

\[
= \langle 0 | \exp \left( - \sum_{i,k=1}^{\infty} \frac{(Q_i q^{-\beta_i} + i)^k}{k} q^{-k/2} \right) \Gamma'_+ (q^{-\rho}) \Gamma'_+ (q^{-\rho}) | 0 \rangle
\]

The exponential operator in the last line is essentially a vertex operator

\[
\exp \left( - \sum_{i,k=1}^{\infty} \frac{(Q_i q^{-\beta_i} + i)^k}{k} \right) = (Q_i)^{-L_0} \Gamma'_+ (q^{-\beta_i} q^{-\rho}) | 0 \rangle.
\]

In exactly the same manner, using (3.7) and (3.8), we can rewrite the right side of \((-Q_3)^L_0\) as

\[
\exp \left( \sum_{j,k=1}^{\infty} \frac{Q_j q^{-\beta_j} + i)^k}{k} V_k^{(j)} - \frac{q^k}{1 - q^k} \right) \Gamma'_+ (q^{-\rho}) | 0 \rangle
\]

\[
= \Gamma'_+ (q^{-\beta_i} q^{-\rho}) q^{K/2} (Q_3)^{L_0} \Gamma'_+ (q^{-\rho}) | 0 \rangle.
\]

Plugging (3.9) and (3.10) into (3.5) yields the following expression of the sum in (2.7):

\[
\sum_{a \in \mathcal{P}} s_{a_1} (q^{-\rho}) s_{a_1} (q^{-\rho}) (Q_3)^{|a_1|} \prod_{i,j=1}^{\infty} \left( 1 - Q_{i_1} \cdots Q_{i_1} \right) \prod_{i,j=1}^{\infty} \left( 1 - Q_{2_1} \cdots Q_{2_1} \right)
\]

\[
= \langle 0 | \Gamma'_+ (q^{-\beta_i} q^{-\rho}) q^{K/2} (Q_3)^{L_0} \Gamma'_+ (q^{-\rho}) | 0 \rangle.
\]
4. Final expression of open string amplitudes

We have thus derived the following intermediate expression of $Z_{\beta_1\beta_2}$:

$$Z_{\beta_1\beta_2} = s_{\beta_1}(q^{-\beta_1}) s_{\beta_2}(q^{-\beta_2}) \prod_{i,j=1}^{\infty} \left( 1 - Q_1 Q_2 q^{-(\beta_1,\beta_2)_{ij} + i + j - 1} \right)^{-1}$$

$$\times \langle 0 | \Gamma'_+ \left( q^{-\beta_1-\beta_2} \right) \left( -Q_1 \right)^{l_0} q^{-K/2} \Gamma'_- \left( q^{-\beta_1-\beta_2} \right) \Gamma'_+ \left( q^{-\beta_1-\beta_2} \right) \left( -Q_2 \right)^{l_0} \rangle$$

$$\times \Gamma_\lambda (q^{-\beta_1}) \Gamma_\lambda (q^{-\beta_2}) q^{K/2} \left( -Q_1 \right)^{l_0} \Gamma_\lambda \left( q^{-\beta_1-\beta_2} \right) \langle 0 |.$$  \hfill (4.1)

As a final step, we use the following relations in the fermionic Fock space that can be derived from a special case of the cyclic symmetry (2.2). This is a kind of operator-state correspondence that maps vertex operators of the form $\Gamma_\lambda(q^{-\lambda-\rho})$ and $\Gamma'_\pm(q^{-\lambda-\rho})$ to the state vectors $\langle \lambda | \Gamma_\lambda \rangle$ and $\langle \lambda | \Gamma'_\pm \rangle$ in the Fock space.

**Lemma 3.** For any $\lambda \in \mathcal{P}$,

$$s_{\lambda}(q^{-\rho}) \Gamma'_+ \left( q^{-\lambda-\rho} \right) | 0 \rangle = q^{K/2} \Gamma_- \left( q^{-\rho} \right) \Gamma_\lambda \left( q^{-\rho} \right) | \lambda \rangle,$$  \hfill (4.2)

$$s_{\lambda}(q^{-\rho}) \Gamma_- \left( q^{-\lambda-\rho} \right) | 0 \rangle = q^{K/2} \Gamma'_+ \left( q^{-\rho} \right) \Gamma'_- \left( q^{-\rho} \right) \Gamma'_+ \left( q^{-\rho} \right) | \lambda \rangle.$$  \hfill (4.3)

**Remark 2.** There are a number of apparently different, but equivalent forms of these relations. For example, one can use the well known identity [28]

$$s_{\lambda}(q^{-\rho}) = q^{\kappa(\lambda)/2} s_\lambda(q^{-\rho})$$  \hfill (4.4)

to rewrite (4.3) as

$$s_{\lambda}(q^{-\rho}) \Gamma_- \left( q^{-\lambda-\rho} \right) | 0 \rangle = q^{-K/2} \Gamma'_+ \left( q^{-\rho} \right) \Gamma'_- \left( q^{-\rho} \right) | \lambda \rangle.$$  \hfill (4.5)

Equation (4.5), in turn, is equivalent to (4.2) (with $\lambda$ being replaced by $\langle \lambda |$) as one can see from the identities

$$\langle \lambda | \Gamma_- \left( q^{-\lambda-\rho} \right) | 0 \rangle = \langle \mu | \Gamma'_+ \left( q^{-\lambda-\rho} \right) | 0 \rangle,$$

$$\langle \mu | \Gamma'_- \left( q^{-\rho} \right) \Gamma'_+ \left( q^{-\rho} \right) | \lambda \rangle = \langle \mu | \Gamma_- \left( q^{-\rho} \right) \Gamma_\lambda \left( q^{-\rho} \right) | \lambda \rangle$$

and the fact that $\langle \lambda |$ and $| \lambda \rangle$ are eigenvectors of $K$ with eigenvalue $\kappa(\lambda)$. Equation (4.2), (4.3) and (4.5) imply the relations

$$s_{\lambda}(q^{-\rho}) \langle 0 | \Gamma'_+ \left( q^{-\lambda-\rho} \right) = \langle \lambda | \Gamma_- \left( q^{-\rho} \right) \Gamma_\lambda \left( q^{-\rho} \right) q^{K/2},$$

$$s_{\lambda}(q^{-\rho}) \langle 0 | \Gamma_- \left( q^{-\lambda-\rho} \right) = q^{\kappa(\lambda)/2} \langle \lambda | \Gamma'_+ \left( q^{-\rho} \right) \Gamma'_- \left( q^{-\rho} \right) \Gamma'_+ \left( q^{-\rho} \right) q^{-K/2},$$

$$s_{\lambda}(q^{-\rho}) \langle 0 | \Gamma_\lambda \left( q^{-\lambda-\rho} \right) = \langle \lambda | \Gamma'_+ \left( q^{-\rho} \right) \Gamma'_- \left( q^{-\rho} \right) q^{-K/2}$$

in the dual Fock space as well.
Proof. The topological vertex has the fermionic expression
\[ C_{\mu \nu} = q^{\epsilon(\mu)/2} S_\nu \langle q^{-\nu} \rangle \{ \langle 1 \rangle | \Gamma_+ (q^{-\nu - \rho}) \Gamma_+ (q^{-\nu + \rho}) \} | \mu \rangle \]
\[ = q^{\epsilon(\mu)/2} S_\nu \langle q^{-\nu} \rangle \langle \lambda | \Gamma_+ (q^{-\nu - \rho}) \Gamma_+ (q^{-\nu + \rho}) \} | \mu \rangle \right). \quad (4.6) \]
The ‘two-leg’ case \( C_{\mu \nu \lambda} = C_{\mu \nu \sigma} \) of the cyclic symmetry \(^7 \) (2.2) thereby turns into the relation
\[ s_{\lambda \beta} (q^{-\nu}) \{ 0 | \Gamma_+ (q^{-\nu - \rho}) \} | \mu \rangle = \{ \langle 1 \rangle | \Gamma_+ \Gamma_+ (q^{-\nu - \rho}) \} \langle \lambda | \mu \rangle q^{\epsilon(\lambda)/2}, \quad (4.7) \]
Among matrix elements of operators on the Fock space. Since this identity holds for any \( \mu \), one obtains \((4.2)\) in the dual form. Similarly, the symmetry relation \( C_{\mu \nu \lambda} = C_{\mu \lambda \sigma} \) yields the identity
\[ s_{\lambda \beta} (q^{-\nu}) \{ \mu | q^{K/2} \Gamma_+ (q^{-\nu - \rho}) \} | 0 \rangle = \{ \mu | \Gamma_+ \Gamma_+ (q^{-\nu - \rho}) \} q^{\epsilon(\lambda)/2}, \quad (4.8) \]
and this implies \((4.3)\).

We can use the specialization
\[ s_{\lambda \beta} (q^{-\nu}) \{ 0 | \Gamma_+ \Gamma_+ (q^{-\nu - \rho}) \} | \mu \rangle = \{ \langle 1 \rangle | \Gamma_+ \Gamma_+ (q^{-\nu - \rho}) \} \langle \lambda | \mu \rangle q^{\epsilon(\lambda)/2}, \]
of \((4.2)\) and \((4.3)\) to \( \lambda = \beta \) and \( \lambda = \beta_2 \) to rewrite \((4.1)\) as
\[ Z_{\beta_1 \beta_2}^{\text{ctv}} = q^{\epsilon(\beta_1)/2} \sum_{i,j=1}^\infty \left( 1 - Q_1 Q_2 q^{-\beta_1 - i + j - 1} \right)^{-1} \]
\[ \times \{ \langle 1 \rangle | \Gamma_+ \Gamma_+ (q^{-\nu - \rho}) \} q^{K/2} \Gamma_+ (q^{-\nu - \rho}) \Gamma_+ (q^{-\nu + \rho}) \}
\[ \times \langle \lambda | \mu \rangle q^{\epsilon(\lambda)/2}, \quad (4.9) \]
we arrive at the following final expression of \( Z_{\beta_1 \beta_2}^{\text{ctv}} \).

Theorem 1. The open string amplitude \( Z_{\beta_1 \beta_2}^{\text{ctv}} \) can be expressed as
\[ Z_{\beta_1 \beta_2}^{\text{ctv}} = q^{\epsilon(\beta_1)/2} \sum_{i,j=1}^\infty \left( 1 - Q_1 Q_2 q^{-\beta_1 - i + j - 1} \right)^{-1} \]
\[ \times \{ \langle 1 \rangle | \Gamma_+ \Gamma_+ (q^{-\nu - \rho}) \} q^{K/2} \Gamma_+ (q^{-\nu - \rho}) \Gamma_+ (q^{-\nu + \rho}) \}
\[ \times \langle \lambda | \mu \rangle q^{\epsilon(\lambda)/2}, \quad (4.10) \]
Let us note here that the main part \( \{ \langle 1 \rangle | \Gamma_+ \Gamma_+ (q^{-\nu - \rho}) \} \) of this expression coincides with the open string amplitude of the on-strip web diagram shown in figure 5 (see appendix A for general formulae of amplitudes). Thus, speaking schematically, gluing the one-leg vertex \((2.5)\) to the on-strip web diagram of figure 4 generates another on-strip web diagram and its correction

\(^7\) Zhou [30] gave a direct proof of the two-leg cyclic symmetry without relying on the crystal interpretation of Okounkov et al [23]. We present another direct proof in appendix B that employs the same techniques as used in section 3.
This structure of (4.10) is a key to derive \( q \)-difference equations for generating functions.

5. \( q \)-difference equations for generating functions

The foregoing expression (4.10) of the open string amplitudes can be used to derive \( q \)-difference equations for the generating functions

\[
\Psi(x) = \frac{1}{Z_{\text{ctv}}} \sum_{k=0}^{\infty} Z_{ctv}^{(k)} x^k, \quad (5.1)
\]

\[
\tilde{\Psi}(x) = \frac{1}{Z_{\text{ctv}}} \sum_{k=0}^{\infty} Z_{ctv}^{(k)} \tilde{x}^k \quad (5.2)
\]

of special subsets of the normalized amplitudes \( Z_{\text{ctv}}^{(k)} / Z_{\text{ctv}} \). Note that \((l^k)\) \((k\)-copies of 1) and \((k)\) represent Young diagrams with a single column or row. These generating functions are the Baker–Akhiezer functions\(^8\) of an integrable hierarchy, and \( x \) amounts to the spectral variable therein \([18]\). One can derive \( q \)-difference equations for generating functions of \( Z_{\text{ctv}}^{(k^j)} / Z_{\text{ctv}}^{(k)} \) and \( Z_{\text{ctv}}^{(k^j)} / Z_{\text{ctv}}^{(k)} \) as well, though they become slightly more complicated because of the presence of the factor \( q^{(j^2)/2} \).

5.1. Derivation of \( q \)-difference equation

A key towards the derivation of a \( q \)-difference equation is to compare \( \Psi(x) \) and \( \tilde{\Psi}(x) \) with another pair of generating functions

\[
\Phi(x) = \frac{1}{Y_{\text{ctv}}} \sum_{k=0}^{\infty} Y^{(l^k)} x^k, \quad (5.3)
\]

\[
\tilde{\Phi}(x) = \frac{1}{Y_{\text{ctv}}} \sum_{k=0}^{\infty} Y^{(k^j)} \tilde{x}^k \quad (5.4)
\]

\(^8\) Speaking more precisely, it is rather \( \Psi(-x) \) and \( \tilde{\Psi}(x) \) that literally correspond to the dual pair of Baker–Akhiezer functions. Because of this, the \( q \)-difference equations for \( \Psi(x) \) and \( \tilde{\Psi}(x) \) presented below are not fully symmetric. This is also the case for another pair \( \Phi(x) \) and \( \tilde{\Phi}(x) \) of generating functions introduced below.
obtained from the main part

\[
Y_{\beta_1,\beta_2} = \left( \prod_{i,j=1}^{\beta_1} \frac{1 - Q_1 Q_2 q^{i+j-i-j}}{1 - Q_1 Q_2 q^{i+j-i-j}} \right)^{-1} \prod_{i,j=1}^{\beta_2} \frac{1 - Q_1 Q_2 q^{i+j-i-j}}{1 - Q_1 Q_2 q^{i+j-i-j}}
\]

(5.5)

deleted

Let us first note the following relation between the coefficients of \(\Psi(x)\) and \(\Phi(x)\).

**Lemma 4.** The coefficients of the expansion

\[
\Psi(x) = \sum_{k=0}^{\infty} a_k x^k, \quad \Phi(x) = \sum_{k=0}^{\infty} b_k x^k, \quad a_0 = b_0 = 1,
\]

are related as

\[
a_k = b_k \prod_{i=1}^{k} \left( 1 - Q_1 Q_2 q^{i-i-1} \right)^{-1} \text{ for } k \geq 1.
\]

(5.6)

**Proof.** \(a_k/b_k\) is given by the ratio of the values of the prefactor in (4.10) for \(\beta_1 = 1\), \(\beta_2 = \emptyset\) and \(\beta_1 = \beta_2 = \emptyset\):

\[
\frac{a_k}{b_k} = \prod_{i,j=1}^{\beta_1} \frac{1 - Q_1 Q_2 q^{i+i+j-j-1}}{1 - Q_1 Q_2 q^{i+i+j-j-1}} \prod_{i,j=1}^{\beta_2} \frac{1 - Q_1 Q_2 q^{i+i+j-j-1}}{1 - Q_1 Q_2 q^{i+i+j-j-1}}
\]

\[
= \prod_{i=1}^{k} \prod_{j=1}^{\infty} \left( 1 - Q_1 Q_2 q^{i+j-i-j} \right)^{-1} \prod_{i=1}^{\infty} \prod_{j=1}^{k} \left( 1 - Q_1 Q_2 q^{i+j-i-j} \right)^{-1}
\]

\[
= \prod_{i=1}^{k} \left( 1 - Q_1 Q_2 q^{i-i-1} \right)^{-1}.
\]

The next step is to derive a \(q\)-difference equation for \(\Phi(x)\).

**Lemma 5.** \(\Phi(x)\) can be expressed in the infinite-product form

\[
\Phi(x) = \prod_{i=1}^{\infty} \frac{\left( 1 - Q_1 q^{i-i-1/2} \right) \left( 1 - Q_1 Q_2 Q_3 q^{i-i-1/2} \right)}{\left( 1 - q^{i-i-1/2} \right) \left( 1 - Q_1 Q_2 q^{i-i-1/2} \right)},
\]

(5.7)

and satisfies the \(q\)-difference equation

\[
\Phi(qx) = \frac{\left( 1 - q^{i-i+1/2} \right) \left( 1 - Q_1 Q_3 q^{i-i-1/2} \right)}{\left( 1 - Q_1 q^{i-i/2} \right) \left( 1 - Q_1 Q_2 Q_3 q^{i-i/2} \right)} \Phi(x).
\]

(5.8)

**Remark 3.** The infinite product \(\prod_{i=1}^{\infty} \left( 1 - q^{i-i-1/2} \right)^{-1}\) is an expression of the quantum dilogarithmic function [19, 20]. Thus \(\Phi(x)\) is a multiplicative combination of four quantum dilogarithmic functions.
Proof. \( Y_{(1)} \) can be expressed as
\[
Y_{(1)} = \langle \langle k \rangle | \tilde{\\Gamma}_-(q^{-\rho}) \Gamma_+(q^{-\rho}) (-Q_1)^{k_0} \Gamma'_-(q^{-\rho}) \Gamma'_+(q^{-\rho}) (-Q_3)^{k_0} \times \tilde{\\Gamma}_-(q^{-\rho}) \Gamma_+(q^{-\rho}) (-Q_2)^{k_0} \Gamma'_-(q^{-\rho}) \Gamma'_+(q^{-\rho}) | 0 \rangle.
\]

By the fundamental properties of the single-variate vertex operators\([26, 27]\), the generating function of \( Y_{(1)} \)’s can be expressed as
\[
\sum_{k=0}^{\infty} x^k \langle \langle k \rangle | \tilde{\\Gamma}_+(x) \rangle = \langle \langle 0 \rangle | \Gamma_+(x) \rangle,
\]
(5.9)
of the single-variate vertex operators\([26, 27]\), the generating function of \( Y_{(1)} \)’s can be expressed as
\[
\sum_{k=0}^{\infty} Y_{(1)} x^k = \langle \langle 0 \rangle | \Gamma_+(x) \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) (-Q_1)^{k_0} \Gamma'_-(q^{-\rho}) \Gamma'_+(q^{-\rho}) (-Q_3)^{k_0} \times \tilde{\\Gamma}_-(q^{-\rho}) \Gamma_+(q^{-\rho}) (-Q_2)^{k_0} \Gamma'_-(q^{-\rho}) \Gamma'_+(q^{-\rho}) | 0 \rangle.
\]

One can now use the commutation relations\([26, 27]\)
\[
\Gamma_+(x) \Gamma_-(y) = (1 - xy)^{-1} \tilde{\\Gamma}_-(y) \Gamma_+(x),
\]
\[
\Gamma'_+(x) \Gamma'_-(y) = (1 - xy)^{-1} \tilde{\\Gamma}'_+(y) \Gamma'_+(x),
\]
\[
\Gamma_+(x) \Gamma'_-(y) = (1 + xy) \tilde{\\Gamma}'_-(y) \Gamma_+(x),
\]
\[
\Gamma'_+(x) \Gamma_-(y) = (1 + xy) \tilde{\\Gamma}_-(y) \Gamma'_+(x)
\]
(5.10)
of the single-variate vertex operators to move \( \Gamma_+(x) \) to the right until it hits \( | 0 \rangle \) and disappears. This yields the infinite-product expression
\[
\sum_{k=0}^{\infty} Y_{(1)} x^k = \prod_{i=1}^{\infty} \left( 1 - \frac{Q_i Q_2 Q_3 q^{i-1/2x}}{1 - q^{i-1/2x}} \right) Y_{(1)}
\]
of the unnormalized generating function, hence the expression (5.7) of \( \Phi(x) \). The \( q \)-difference equation (5.8) is an immediate consequence of (5.7).

To derive a \( q \)-difference equation for \( \Psi(x) \), let us rewrite (5.8) as
\[
\left( 1 - Q_1 (1 + Q_2 Q_3) q^{1/2x} + Q_1^2 Q_2 Q_3 q^{3/2x} \right) \Phi(qx)
\]
\[
= \left( 1 - (1 + Q_1 Q_3) q^{1/2x} + Q_1 Q_3 q^{3/2x} \right) \Phi(x)
\]
and extract the coefficients of \( x^k \). This yields the recursion relations
\[
q^k b_k = Q_1 (1 + Q_2 Q_3) q^{1/2k-1} b_{k-1} + Q_1^2 Q_2 Q_3 q^{k-2} b_{k-2}
\]
\[
= b_k - (1 + Q_1 Q_3) q^{1/2k-1} b_{k-1} + Q_1 Q_3 q b_{k-2}
\]
(5.11)
(1 - Q_1 Q_2 q^{k-2}) (1 - Q_1 Q_2 q^{k-1}) q^k a_k 
- Q_1 (1 + Q_2 Q_1) q^{1/2} (1 - Q_1 Q_2 q^{k-2}) q^{k-1} a_{k-1} 
+ Q_1^2 Q_2 Q_3 q q^{k-2} a_{k-2}
= (1 - Q_1 Q_2 q^{k-2}) (1 - Q_1 Q_2 q^{k-1}) q^k a_k 
- (1 + Q_1 Q_2) q^{1/2} (1 - Q_1 Q_2 q^{k-2}) a_{k-1} 
+ Q_1 Q_3 q a_{k-2} \tag{5.12}

for a_k’s. Multiplying these equations by x^k and taking the sum over k = 0, 1, ..., we can derive a q-difference equation for \Phi(x).

To state this result in a compact form, let us use the shift operator \( q^{x_0} \), \( \partial_x = \partial / \partial x \), that acts on a function \( f(x) \) of \( x \) as
\[
q^{x_0} f(x) \equiv f(qx).
\]

**Theorem 2.** \( \Psi(x) \) satisfies the q-difference equation
\[
(1 - Q_1 Q_2 q^{-x_0}) (1 - Q_1 Q_2 q^{-x_0}) \Psi(qx) 
- Q_1 (1 + Q_2 Q_1) q^{1/2} (1 - Q_1 Q_2 q^{-x_0}) \Psi(qx) + Q_1^2 Q_2 Q_3 q^2 \Psi(qx)
= (1 - Q_1 Q_2 q^{-x_0}) (1 - Q_1 Q_2 q^{-x_0}) \Psi(x) 
- (1 + Q_1 Q_2) q^{1/2} (1 - Q_1 Q_2 q^{-x_0}) \Psi(x) + Q_1 Q_3 q^2 \Psi(x). \tag{5.13}
\]

A q-difference equation for \( \tilde{\Psi}(x) \) can be derived in the same way from the q-difference equation
\[
\tilde{\Phi}(qx) = \frac{(1 + Q_1 q^{1/2}) (1 + Q_1 Q_2 Q_3 q^{1/2})}{(1 + q^{1/2}) (1 + Q_1 Q_3 q^{1/2})} \tilde{\Phi}(x) \tag{5.14}
\]
for \( \tilde{\Phi}(x) \) and the relation
\[
\tilde{a}_k = \tilde{b}_k \prod_{i=1}^{k} \left(1 - Q_1 Q_2 q^{-i}\right)^{-1} \text{ for } k \geq 1 \tag{5.15}
\]
between the coefficients of the expansion
\[
\Psi(x) = \sum_{k=0}^{\infty} a_k x^k, \quad \Phi(x) = \sum_{k=0}^{\infty} b_k x^k, \quad \tilde{a}_0 = \tilde{b}_0 = 1,
\]
of \( \Psi(x) \) and \( \Phi(x) \). We omit the detail of calculation and show the result:

**Theorem 3.** \( \tilde{\Psi}(x) \) satisfies the q-difference equation
\[
(1 - Q_1 Q_2 q^{-x_0}) (1 - Q_1 Q_2 q^{-x_0}) \tilde{\Psi}(q^{-1}x) 
+ Q_1 (1 + Q_2 Q_1) q^{-1/2} (1 - Q_1 Q_2 q^{-x_0}) \tilde{\Psi}(q^{-1}x) + Q_1^2 Q_2 Q_3 q^{-1} x^2 \tilde{\Psi}(q^{-1}x)
= (1 - Q_1 Q_2 q^{-x_0}) (1 - Q_1 Q_2 q^{-x_0}) \tilde{\Psi}(x) 
+ (1 + Q_1 Q_2) q^{-1/2} (1 - Q_1 Q_2 q^{-x_0}) \tilde{\Psi}(x) + Q_1 Q_3 q^{-1} x^2 \tilde{\Psi}(x). \tag{5.16}
\]
Remark 4. Both sides of (5.13) and (5.16) can be rewritten as

\[
(1 - Q_1 Q_2 q^{-x_0} - Q_1 q^{1/2}) (1 - Q_1 Q_2 q^{-q^{x_0}} - Q_1 Q_2 q^{1/2} x) \Psi(q x)
\]

and

\[
(1 - Q_1 Q_2 q^{x_0} + Q_1 q^{-1/2} x) (1 - Q_1 Q_2 q^{q^{x_0}} + Q_1 Q_2 q^{-1/2} x) \tilde{\Psi}(q^{-1} x)
\]

This expression corresponds to writing q-difference equations for \( \Phi(x) \) and \( \tilde{\Phi}(x) \) as

\[
(1 - Q_1 q^{1/2} x)(1 - Q_1 Q_2 q^{1/2} x) \Phi(q x) = (1 - q^{1/2} x)(1 - Q_1 Q_2 q^{1/2} x) \Phi(x)
\]

and

\[
(1 + Q_1 q^{-1/2} x)(1 + Q_1 Q_2 q^{-1/2} x) \tilde{\Phi}(q^{-1} x) = (1 + q^{-1/2} x)(1 + Q_1 Q_2 q^{-1/2} x) \tilde{\Phi}(x).
\]

These q-difference equations are transformed to the foregoing ones for \( \Psi(x) \) and \( \tilde{\Psi}(x) \) by the transformation (5.6) and (5.15) of the coefficients.

5.2. Structure of q-difference operators

Let us rewrite (5.13) as

\[
H(x, q^{x_0}) \Psi(x) = 0
\]

and examine the structure of the q-difference operator \( H \). This operator reads

\[
H(x, q^{x_0}) = (1 - Q_1 Q_2 q^{-2} x_0)(1 - Q_1 Q_2 q^{-q x_0})
- (1 + Q_1 Q_2 q^{1/2} x)(1 - Q_1 Q_2 q^{-q x_0}) + Q_1 Q_2 q x^2
- (1 - Q_1 Q_2 q^{-2} x_0)(1 - Q_1 Q_2 q^{-q x_0}) q x^2
+ Q_1 (1 + Q_2 Q_3) q^{1/2} x (1 - Q_1 Q_2 q^{-q x_0}) q x^2 - Q_1^2 Q_2 Q_3 q x^2 q x_0.
\]

Remarkably, \( H(x, q^{x_0}) \) can be factorized as

\[
H(x, q^{x_0}) = (1 - Q_1 Q_2 q^{-2} x_0) K(x, q^{x_0}),
\]

where

\[
K(x, q^{x_0}) = (1 - Q_1 Q_2 q^{-1} x_0)(1 - q^{x_0}) - (1 + Q_1 Q_3) q^{1/2} x
+ Q_1 (1 + Q_2 Q_3) q^{1/2} x q x_0 + Q_1 Q_3 q x^2.
\]

This is also the case for the q-difference equation (5.16) for \( \tilde{\Psi}(x) \). The q-difference operator \( \tilde{H}(x, q^{x_0}) \) in the expression

\[
\tilde{H}(x, q^{x_0}) \tilde{\Psi}(x) = 0
\]
This operator can be factorized as
\[ \tilde{H}(x, q^{x_0}) = \left( 1 - Q_1 Q_2 q^2 q^{-x_0} \right) \tilde{K}(x, q^{x_0}), \quad (5.27) \]
where
\[ \tilde{K}(x, q^{x_0}) = \left( 1 - Q_1 Q_2 q^{-x_0} \right) \left( 1 - q^{-x_0} \right) + (1 + Q_1 Q_2) q^{1/2} x \]
- \[ Q_1 \left( 1 + Q_2 Q_3 \right) q^{1/2} x q^{-x_0} + Q_1 Q_3 q x^2. \quad (5.28) \]

Let us note here that the action of \( 1 - Q_1 Q_2 q^{-2} q^{x_0} \) and \( 1 - Q_1 Q_2 q^2 q^{-x_0} \) on the space of power series of \( x \) is invertible as far as \( Q_1 \) and \( Q_2 \) take generic values, i.e., apart from the exceptional cases where \( Q_1 Q_2 = q^n, n \in \mathbb{Z} \). Therefore these factors can be removed from the \( q \)-difference equations (5.21) and (5.25). Actually, this genericity is implicitly assumed in the transformations (5.6) and (5.15) of these generating functions. Thus we find the following refinement of theorems 2 and 3.

**Theorem 4.** For generic values of \( Q_1 \) and \( Q_2 \), the \( q \)-difference equations (5.13) and (5.16) can be reduced to
\[ K(x, q^{x_0}) \Psi(x) = 0, \quad \tilde{K}(x, q^{x_0}) \tilde{\Psi}(x) = 0. \quad (5.29) \]

This result fits well into the perspectives of mirror geometry of topological string theory on non-compact toric Calabi–Yau threefolds \([6, 7]\). \( \Psi(x) \) and \( \tilde{\Psi}(x) \) may be thought of as wave functions of a probe D-brane. In this interpretation, a \( q \)-difference equation satisfied by these functions defines a quantum mirror curve. The \( q \)-difference equation (5.29) indeed have such a characteristic. In the classical limit as \( q \to 1 \), the non-commutative polynomials \( K(x, q^{x_0}) \) and \( \tilde{K}(x, q^{-x_0}) \) turn into the ordinary polynomials
\[ K_{cl}(x, y) = \left( 1 - Q_1 Q_2 y \right) (1 - y) - (1 + Q_1 Q_3) x \]
+ \[ Q_1 \left( 1 + Q_2 Q_3 \right) x y + Q_1 Q_3 x^2 \quad (5.30) \]
in \((x, y)\) and
\[ \tilde{K}_{cl}(x, y) = \left( 1 - Q_1 Q_2 y^{-1} \right) (1 - y^{-1}) + (1 + Q_1 Q_3) \]
- \[ Q_1 \left( 1 + Q_2 Q_3 \right) x y^{-1} + Q_1 Q_3 x^2 \quad (5.31) \]
in \((x, y^{-1})\). As expected from the perspectives of mirror geometry, the Newton polygons of these polynomials have the same shape as the toric diagram in figure 1.
Let us examine the flop transition from figure 1 to 6. After this move, the previous setup for defining the amplitude $Z_{\alpha_1 \alpha_2 \alpha_3}$ turns into the setup shown in figure 7. Note that the Kähler parameters after the flop transition are denoted by $P_1, P_2, P_3$; they are expected to be related to the Kähler parameters $Q_1, Q_2, Q_3$ before the transition by birational transformations.

Our method for calculating $Z_{\alpha_1 \alpha_2 \alpha_3}$ can be extended to the amplitude $\hat{Z}_{\alpha_1 \alpha_2 \alpha_3}$ of figure 7 as follows.

The sum over $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{P}$ can be decomposed to a partial sum $\hat{Z}_{\beta_1 \beta_2}$ with respect to $\alpha_1, \alpha_2$ at the first stage and a sum with respect to $\alpha_3$ at the next stage as

$$\hat{Z}_{\beta_1 \beta_2} = \sum_{\alpha_3 \in \mathcal{P}} \hat{Z}_{\beta_1 \beta_2 | \alpha_3} ( -P_3)^{|\alpha_3|} q^{-\kappa(\alpha_3)/2} C_{\alpha_3 \beta_1 \beta_2}.$$

(6.1)

The extra factor $(-1)^{|\alpha_3|} q^{-\kappa(\alpha_3)/2}$ is inserted by the gluing rule. The framing number (2.3) along the internal line carrying $\alpha_3$ is equal to 1.

The partial sum $\hat{Z}_{\beta_1 \beta_2 | \alpha_3}$ is an open string amplitude of the double-$\mathbb{P}^1$ diagram shown in figure 8. Since this is an on-strip diagram, the amplitude can be calculated explicitly as

Figure 6. Web and toric diagrams after flop.

Figure 7. Setup for open string amplitude $\hat{Z}_{\beta_1 \beta_2}$.
This amplitude is related to its counterpart $Z_{\beta_{1}\beta_{2}\alpha_{3}}$ by the same flop operation as the move from figure 1 to 6. One see form (2.6) and (6.2) that $\hat{Z}_{\beta_{1}\beta_{2}\alpha_{3}}$ is almost identical to $Z_{\beta_{1}\beta_{2}\alpha_{3}}$ if the Kähler parameters are related as

$$P_{1}P_{2} = Q_{2}, \quad P_{2} = Q_{1}Q_{2}. \quad (6.3)$$

The only discrepancy lies in the infinite products $\prod_{i,j=1}^{\infty} (1 - Q_{i} q^{-i})$ in (2.6) and $\prod_{i,j=1}^{\infty} (1 - P_{i} q^{-i})$ in (6.2).

Substituting (6.2) and (2.5) in (6.1), we obtain the following expression of $\hat{Z}_{\beta_{1}\beta_{2}}^{\text{ctv}}$:

$$\hat{Z}_{\beta_{1}\beta_{2}}^{\text{ctv}} = s_{\beta_{1}}(q^{-})s_{\beta_{2}}(q^{-}) \prod_{i,j=1}^{\infty} \left( 1 - P_{2}q^{-\beta_{1} - \beta_{2} + i + j - 1} \right)^{-1}$$

$$\times \prod_{i,j=1}^{\infty} \left( 1 - P_{1}q^{-\alpha_{k} - \beta_{2} + i + j - 1} \right) \prod_{i,j=1}^{\infty} \left( 1 - P_{1}P_{2}q^{-\alpha_{k} + i + j - 1} \right). \quad (6.4)$$

Note that we have used the identity (4.4) as well to rewrite the first part of the summand as

$$s_{\beta_{1}}(q^{-})s_{\beta_{2}}(q^{-})q^{-\beta_{1}\beta_{2}/2} = s_{\beta_{1}}(q^{-})^{2}. \quad$$

Thus, in contrast with (2.7), the sum in this case resembles the partition function of the ordinary melting crystal model [9, 10] for which the main part of the Boltzmann weight is $s_{\alpha_{k}}(q^{-})^{2}$ rather than $s_{\alpha_{k}}(q^{-})s_{\alpha_{k}}(q^{-})$.

The sum in (6.4) can be calculated in more or less the same way as the case of (2.7). Let us show the final result only.
Theorem 5. The open string amplitude $Z_{\alpha_1,\alpha_2}$ can be expressed as

$$Z_{\alpha_1,\alpha_2} = q^{\alpha(\beta_1)/2 + \alpha(\beta_2)/2} \prod_{i,j=1}^{\infty} \left( 1 - P_2 q^{-\beta_2 + \beta_1 + i+j-1} \right)^{-1}$$

$$\times \left\{ \beta_1 \right\} \Gamma^\prime_+ \left( q^{-\rho} \right) \Gamma_+ \left( q^{-\rho} \right) \left( -P_1 \right)^{L_0} \Gamma_+ \left( q^{-\rho} \right) \Gamma_+ \left( q^{-\rho} \right) P_3^{L_0}$$

$$\times \Gamma_+ \left( q^{-\rho} \right) \Gamma_+ \left( q^{-\rho} \right) \left( -P_1 P_2 \right)^{L_0} \Gamma_+ \left( q^{-\rho} \right) \Gamma_+ \left( q^{-\rho} \right) \left\{ \beta_2 \right\}.$$

(6.5)

The main part $\left\{ \beta_1 \right\} \cdot \left\{ \beta_2 \right\}$ of this expression is essentially the open string amplitude of the web diagram shown in figure 9. This web diagram can be derived from the web diagram of figure 5 by the same flop operation as the move from figure 1 to 6.

To see how this part is related to the main part of (4.10), let us use the commutation relations (5.10) to exchange the order of the first four vertex operators therein as

$$\left\{ \beta_1 \right\} \Gamma^\prime_+ \left( q^{-\rho} \right) \Gamma_+ \left( q^{-\rho} \right) \left( -P_1 \right)^{L_0} \Gamma_+ \left( q^{-\rho} \right) \Gamma_+ \left( q^{-\rho} \right) P_3^{L_0}$$

$$= \left\{ \beta_1 \right\} \left( -P_1 \right)^{L_0} \left( -P_1^{-1} q^{-j} \right)^{L_0} \Gamma^\prime_+ \left( q^{-\rho} \right) \Gamma_+ \left( q^{-\rho} \right) P_3^{L_0}$$

$$= \left( -P_1 \right)^{L_0} \prod_{i,j=1}^{\infty} \left( 1 - P_2 q^{i+j-1} \right) \left( 1 - P_1^{-1} q^{i+j-1} \right)^{-1}$$

$$\times \left\{ \beta_1 \right\} \Gamma^\prime_+ \left( q^{-\rho} \right) \Gamma_+ \left( q^{-\rho} \right) \left( -P_1^{-1} q^{-j} \right)^{L_0} \left( -P_1 q^{-\rho} \right) P_3^{L_0}$$

$$= \left( -P_1 \right)^{L_0} \prod_{i,j=1}^{\infty} \left( 1 - P_2 q^{i+j-1} \right) \left( 1 - P_1^{-1} q^{i+j-1} \right)^{-1}$$

$$\times \left\{ \beta_1 \right\} \Gamma^\prime_+ \left( q^{-\rho} \right) \Gamma_+ \left( q^{-\rho} \right) \left( -P_1^{-1} q^{-j} \right)^{L_0} \Gamma^\prime_+ \left( q^{-\rho} \right) \Gamma_+ \left( q^{-\rho} \right) \left( -P_1 P_3 \right)^{L_0}.$$

This shows that if the two sets of Kähler parameters are matched as

$$P_1^{-1} = Q_1, \quad P_1 P_3 = Q_3, \quad P_1 P_2 = Q_2,$$

(6.6)

$Z_{\alpha_1,\alpha_2}$ and $Z_{\alpha_1,\alpha_2}$ are related as

$$Z_{\alpha_1,\alpha_2} = q^{\alpha(\beta_1)/2} \left( -P_1 \right)^{\beta_1} \prod_{i,j=1}^{\infty} \left( 1 - P_2 q^{i+j-1} \right) \left( 1 - P_1^{-1} q^{i+j-1} \right)^{-1} \cdot Z_{\alpha_1,\alpha_2}^{\text{CTV}}.$$

(6.7)

Note that (6.6) is consistent with (6.3). These matching rules of parameters agree with the known result for the partition functions [2, 5, 31].

Remark 5. It is instructive to examine a different cut-and-glue procedure in this case. Let us try to cut the middle internal line (to which $Q_1$ and $\alpha_1$ are assigned) of the web diagram (see figure 7). The cutting procedure yields two subdiagrams of the on-strip type. They are glued
together with the edge weight \((-P)\)\(^{\alpha_1}\). Note that the framing number in this case is equal to 0. Thus the total amplitude can be expressed as

\[
\hat{Z}^{\text{ctv}}_{\beta_i,\beta_2} = \sum_{\alpha_1 \in \mathcal{P}} \hat{Z}'_{\alpha_1} \left( -P \right)^{\alpha_1} \hat{Z}''_{\alpha_1|\beta_i,\beta_2}, \tag{6.8}
\]

where \(\hat{Z}'_{\alpha_1}\) and \(\hat{Z}_{\alpha_1|\beta_i,\beta_2}\) are contributions of the two on-strip subdiagrams, i.e.

\[
\hat{Z}'_{\alpha_1} = \langle 0 | \Gamma'_{\alpha_1} (q^{-\rho}) \Gamma'_{\alpha_1} (q^{-\rho}) P_3^{L_0} \Gamma'_- (q^{-\rho}) \Gamma'_+ (q^{-\rho}) | \alpha_1 \rangle
\]

and

\[
\hat{Z}_{\alpha_1|\beta_i,\beta_2} = q^{-\kappa (\alpha_1)^2} s_{\beta_i} (q^{-\rho}) s_{\beta_2} (q^{-\rho}) \times \langle \alpha_1 | \Gamma'_- (q^{-\beta_i - \rho}) \Gamma'^+ (q^{-\beta_2 - \rho}) P_2^{L_0} \Gamma'_- (q^{-\beta_i - \rho}) \Gamma'^+ (q^{-\beta_2 - \rho}) | 0 \rangle.
\]

Plugging these expressions into (6.8) leads to yet another fermionic expression of \(\hat{Z}^{\text{ctv}}_{\beta_i,\beta_2}\):

\[
\hat{Z}^{\text{ctv}}_{\beta_i,\beta_2} = s_{\beta_i} (q^{-\rho}) s_{\beta_2} (q^{-\rho}) \times \langle 0 | \Gamma'_- (q^{-\rho}) \Gamma'^+ (q^{-\rho}) P_3^{L_0} \Gamma'^+ (q^{-\rho}) \Gamma'_- (q^{-\rho}) q^{-K/2} (q^{-\rho}) q^{-K/2} (q^{-\rho}) | 0 \rangle
\]

\[
\times \Gamma'_- (q^{-\beta_i - \rho}) \Gamma'^+ (q^{-\beta_2 - \rho}) P_2^{L_0} \Gamma'_- (q^{-\beta_i - \rho}) \Gamma'^+ (q^{-\beta_2 - \rho}) | 0 \rangle. \tag{6.9}
\]

This expression looks very similar to an on-strip amplitude. The operator product in this expression, however, contains the operator \(q^{-K/2}\) that does not appear in on-strip amplitudes. Because of this operator, one cannot calculate this expression directly. In contrast, if one applies the cut-and-glue procedure to an on-strip amplitude, operators of the form \(q^{\pm K/2}\) do not appear or cancel out in the outcome of calculation\(^9\). This cancellation mechanism is a consequence of the \textit{linear} shape of the on-strip diagram. In this respect, the web diagram of figure 6 is a \textit{chain} of on-strip diagrams, and its web diagram is \textit{bent} to ninety degrees in the middle. It is this bend that generates the operator \(q^{-K/2}\). Actually, the present case is special in the sense that this difficulty can be circumvented by the foregoing different cut-and-glue description\(^10\). In a general case, such an escape route is not prepared.

7. Conclusion

Let us summarize what we have done in this paper.

\textit{Calculation of open string amplitudes} We reformulated the open string amplitude \(Z^{\text{ctv}}_{\beta_i,\beta_2}\) of figure 2 in the partially summed form (2.4), and derived the reduced expression (2.7). The main part of (2.7) turns out to be similar to the partition function of the modified melting crystal model. Firstly, the main part \(s_{\beta_i} (q^{-\rho}) s_{\beta_2} (q^{-\rho})\) of the summand is exactly the same. Secondly, the other part can be described by matrix elements of the \textit{diagonal} operators \(V_0^{(\pm)}\) in the quantum torus algebra. This is also a characteristic of the external potentials in the melting crystal models. We could thereby apply the method for the melting crystal models to derive the fermionic expression (4.1) of \(Z^{\text{ctv}}_{\beta_i,\beta_2}\). This expression was further converted to the final expression (4.10) of \(Z^{\text{ctv}}_{\beta_i,\beta_2}\), which is a product of a simple prefactor and the open string amplitude \(Y_{\beta_i,\beta_2}\) of a new on-strip diagram.

\(^9\) This is a key to prove the fermionic formula (A.1) of on-strip amplitudes by induction.

\(^10\) One can also convert (6.9) to a more tractable form with the aid of techniques used in sections 3 and 4. This eventually leads to the same result as presented therein.
Derivation of q-difference equations

We derived q-difference equations for the generating functions $\Psi(x), \tilde{\Psi}(x)$ of the normalized amplitudes $Z_{y_0, y_1}^{\text{civ}}/Z_{y_0, y_1}^{\text{civ}}$ specialized to $\beta_1 = (1^k), k = 0, 1, 2, \ldots$, and $\beta_2 = \emptyset$. The derivation makes full use of the factorized form of (4.10). Namely, we first derived the q-difference equations (5.8) and (5.14) for the generating functions $\Phi(x), \tilde{\Phi}(x)$ obtained from $Y_{y_0, y_1}/Y_{y_0, y_1}$. These equations are transformed to the q-difference equations (5.13) and (5.16) for $\Psi(x), \tilde{\Psi}(x)$. This is the place where the prefactor of $Y_{y_0, y_1}$ in (4.10) plays a role. We examined the structure of these q-difference equations and found that they can be reduced to the simpler equation (5.29). It is these reduced equations that should be interpreted as the defining equation of a quantum mirror curve.

Flop transition

We considered the flop transition from figure 1 to 6. The open string amplitude $Z_{y_0, y_1}^{\text{civ}}$ after the transition can be calculated in much the same way as in the case of $Z_{y_0, y_1}^{\text{civ}}$. We confirmed that $Z_{y_0, y_1}^{\text{civ}}$ can be matched to the amplitude $Z_{y_0, y_1}^{\text{civ}}$ by the birational transformations (6.6) of the Kähler parameters.

On the other hand, we have been unable to derive q-difference equations in other configurations of partitions on the external lines of the web diagram (except for those that can be derived from the setup of section 2 by symmetries or specializations of the amplitude). A major obstacle is the emergence of $q^{\pm 1/2}$ that do not cancel out in a fermionic expression of the amplitude as opposed to the case of (4.9). Because of this obstacle\(^{11}\), the fermionic expression in such a case cannot be converted to a form from which a q-difference equation can be read out.

We have encountered the same difficulty in an attempt to extend our results to more general tree-like web diagrams studied by Karp et al [4]. Our attempt has been unsuccessful not only for open string amplitudes, but also for the closed string partition function. We believe that this difficulty is of technical nature and can be overcome by a new computational idea.

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\(^{11}\) The situation presented in remark 5 is similar, but the difficulty in that case can be circumvented.
Appendix A. Amplitudes of on-strip geometry

The toric diagram of on-strip geometry is a triangulation of the strip of height 1 to triangles of area \(1/2\) (see figure 10). The associated web diagram is a connected acyclic graph. If the toric graph comprises \(N\) triangles, the web diagram has \(N\) vertices, \(N - 1\) internal lines and \(2N\) external lines. The \(N\) external lines other than the leftmost and rightmost ones are vertical. For brevity, the external lines are also referred to as ‘legs’.

We assign the Kähler parameters \(Q_1, \ldots, Q_{N-1}\) to the internal lines, the partitions \(\beta_1, \ldots, \beta_N\) to the vertical external lines, and the partitions \(\alpha_0, \alpha_N\) to the leftmost and rightmost external lines. Let \(Z_{\alpha_0,\alpha_N}^{\beta_1,\ldots,\beta_N}\) denote the open string amplitude in this setup. This amplitude is defined as a sum of the product of vertex and edge weights with respect to the partitions \(\beta_1, \ldots, \beta_N\) on the internal lines. In the case of \(\alpha_0 = \alpha_N = \emptyset\), Iqbal and Kashani-Poor [2] calculated this sum in a closed form by skillful use of the Cauchy identities for skew Schur functions. Their result can be reformulated, without restriction to \(\alpha_0 = \alpha_N = \emptyset\), in the language of fermions [32–34].

Following the notations of Nagao [33] and Sulkowski [34], let us define the sign (or type) \(\sigma_n = \pm 1\) of the \(n\)th vertex as:

(i) \(\sigma_n = +1\) if the vertical leg points up,

(ii) \(\sigma_n = -1\) if the vertical leg points down.

For example, in the case of the web diagram of figure 10,

\[
\sigma_1 = -1, \quad \sigma_2 = +1, \quad \sigma_3 = +1, \quad \sigma_4 = -1, \quad \sigma_5 = -1.
\]

These data are used to show the types of vertex operators as

\[
\Gamma_\sigma^\tau(\alpha) = \begin{cases} \Gamma_\sigma, & \text{if } \sigma = +1, \\ \Gamma_\tau, & \text{if } \sigma = -1. \end{cases}
\]

Let us further introduce the auxiliary notations

\[
\beta^{(n)} = \begin{cases} \beta_n, & \text{if } \sigma_n = +1, \\ \beta_n, & \text{if } \sigma_n = -1, \end{cases}
\]

With these notations, the fermionic expression of \(Z_{\beta_1,\ldots,\beta_N}^{\alpha_0,\alpha_N}\) read

\[
Z_{\beta_1,\ldots,\beta_N}^{\alpha_0,\alpha_N} = q^{(1-\sigma_1)c(\alpha_0)4q(1+\sigma_2)c(\alpha_2)4s_{\beta_1}(q^{-\rho})s_{\beta_2}(q^{-\rho})} \times \left(\Gamma_0^\sigma(q^{-\rho})\Gamma_\tau^\tau(q^{-\rho})\right)^{\sigma_2},
\]

\[
\times \Gamma_{N-1}^\sigma(q^{-\rho})\Gamma_0^\tau(q^{-\rho})\left(\sigma_N-1\sigma_0\right)^{\sigma_0},
\]

\[
\times \left(\Gamma_0(q^{-\rho})\Gamma_\tau(q^{-\rho})\right)^{\alpha_0} \Gamma_\sigma(q^{-\rho})^{\alpha_N} \Gamma_\sigma(q^{-\rho})^{\beta_1}, \ldots, \Gamma_\sigma(q^{-\rho})^{\beta_N} \left(\sigma_0\sigma_N-1\right). \tag{A.1}
\]

In the case where \(N = 1\), this formula reduces to the fermionic expression (4.6) of the vertex weight itself. Starting from (4.6), one can prove this formula by induction. If \(\alpha_0 = \alpha_N = \emptyset\), one can use the commutation relations (5.10) to move \(\Gamma^-\)’s to the left and \(\Gamma^+\)’s to the right until they hit \(0\) and \(0\) and disappear. This yields the explicit formula

\[
Z_{\beta_1,\ldots,\beta_N}^{\alpha_0,\alpha_N} = s_{\beta_1}(q^{-\rho})s_{\beta_2}(q^{-\rho}) \times \prod_{1 \leq m < n \leq N} \prod_{i,j=1}^{\infty} \left(1 - Q_{mn}q^{-\rho_{i+1} - \rho_{i} - \rho_{i,j} - \rho_{i,j+1}}\right)^{\sigma_n} \tag{A.2}
\]

of Iqbal and Kashani-Poor [2].
Appendix B. Direct proof of two-leg cyclic symmetry

As another application of the techniques used in section 3, we present a direct proof of the identities (4.7) and (4.8) that amounts to the cyclic symmetry of two-leg vertices. Actually, these two identities are equivalent, and can be reduced to the following one:

\[ s_{\lambda}(q^{-\rho})s_{\mu}(q^{-\lambda}) = \langle \mu | q^{-K/2} \Gamma_+^{-}(q^{-\rho})\Gamma_+^{-}(q^{-\mu})q^{-K/2} | \lambda \rangle. \]  

(B.1)

It is this identity that we prove here. Note that this identity implies the non-trivial relation

\[ s_{\lambda}(q^{-\rho})s_{\mu}(q^{-\lambda}) = s_{\mu}(q^{-\rho})s_{\lambda}(q^{-\mu}). \]  

(B.2)

from which the equivalence of (4.7) and (4.8) follows.

We prove (B.1) by generating functions. Namely, we construct generating functions of both sides by the Schur functions \( s_{\lambda}(x) \), \( x = (x_1, x_2, \ldots) \), and confirm that these generating functions are identical.

It is easy to calculate the generating function of the left side of (B.1). By the Cauchy identity

\[ \sum_{\mu \in \mathcal{P}} s_{\mu}(x) s_{\mu}(y) = \prod_{i,j=1}^{\infty} \left( 1 - x_i y_j \right)^{-1}, \quad y = (y_1, y_2, \ldots), \]  

(B.3)

of the Schur functions [28], the generating function of the left side of (B.1) can be expressed as

\[ \sum_{\mu \in \mathcal{P}} s_{\mu}(x)s_{\lambda}(q^{-\rho})s_{\mu}(q^{-\lambda}) = s_{\lambda}(q^{-\rho}) \prod_{i,j=1}^{\infty} \left( 1 - x_i q^{-\lambda + j - 1/2} \right)^{-1}. \]  

(B.4)

On the other hand, constructing the generating function of the left side of (B.1) amounts to inserting \( \Gamma_+^{-}(x) \) to the right of \( \langle 0 | \) as

\[
\langle 0 | \sum_{\mu \in \mathcal{P}} s_{\mu}(x) \langle \mu | q^{-K/2} \Gamma_+^{-}(q^{-\rho})\Gamma_+^{-}(q^{-\mu})q^{-K/2} | \lambda \rangle = \langle 0 | \Gamma_+^{-}(q^{-\lambda})\Gamma_+^{-}(q^{-\rho})q^{-K/2} | \lambda \rangle

\]  

The subsequent calculation is very similar to section 3. One can use (3.6) and (3.8) to rewrite the last quantity as

\[
\langle 0 | \exp \left( \sum_{i,k=1}^{\infty} \frac{x_i^k}{k} \right) q^{-K/2} \Gamma_+^{-}(q^{-\rho})\Gamma_+^{-}(q^{-\mu})q^{-K/2} | \lambda \rangle

\]  

Note that the order of \( \exp(\cdots) \) and \( q^{K/2} \) has been exchanged because \( V_0^{(-k)} \) commutes with \( q^{K/2} \). By (3.1), the action of \( \exp(\cdots) \) on |\( \lambda \rangle \) can be expressed as
Thus the generating function of the right side of (B.1) turns out to take such a form as

\[
\sum_{\mu \in \mathcal{P}} s_\mu(x) \langle \mu | q^{-K/2} \Gamma_{-}^\nu(q^{-\rho}) \Gamma_{+}^\nu(q^{-\rho}) q^{-K/2} | \lambda \rangle = \langle 0 | \Gamma_{-}^\nu(q^{-\rho}) q^{-K/2} | \lambda \rangle \prod_{i,j=1}^{\infty} \left( 1 - x_i q^{-\lambda_j+1/2} \right)^{-1} = q^{-k(\lambda_1^2)} s_{\lambda}(q^{-\rho}) \prod_{i,j=1}^{\infty} \left( 1 - x_i q^{-\lambda_j+1/2} \right)^{-1}.
\]

By (4.4), this coincides with (B.4).

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