Complexity analysis of dynamical cylinder in massive Brans–Dicke gravity

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Abstract In this paper, a complexity factor is devised for a non-static cylindrical system in the framework of massive Brans–Dicke theory. The definition of complexity is developed by taking into account the essential physical characteristics (such as anisotropy and inhomogeneity) of the system. In order to determine the complexity factor of the self-gravitating object, we acquire structure scalars from the orthogonal splitting of the Riemann tensor. Moreover, we discuss two patterns of evolution and choose the homologous mode as the simplest pattern under the influence of massive scalar field. We derive solutions in the absence as well as the presence of heat dissipation for a specific form of the scalar field. The factors that induce complexity in an initially complexity-free system are also examined. It is concluded that the massive scalar field as well as heat dissipation contribute to the complexity of the celestial system. Thus, a dynamical cylinder is more complex as compared to its static counterpart.

1 Introduction

Self-gravitating cosmic structures provide significant information regarding the origin and evolution of the universe. For this purpose, astrophysicists have performed detailed surveys of these large-scale structures (Large Synoptic Survey Telescope, Sloan Digital Sky Survey, Two-degree Field Galaxy Redshift Survey) to investigate the mechanism of the cosmos. However, the physical features of their intricate interior regions depend on different factors such as mass and matter composition. A slight fluctuation in any of the matter variables may lead to a fundamental change in the behavior of celestial systems. Therefore, it is necessary to define a complexity factor that effectively represents the complicated nature and physics of astrophysical components. The formulation of this factor is based on the inter-relationship of state parameters (mass, pressure, density, etc.) which helps in determining the extent to which internal or external disturbances influence the characteristics of the cosmic system. Such a factor also provides a comparison of complexities in different astrophysical objects through a stability criterion.

The concept of complexity has been explored in different avenues, but researchers have failed to agree upon a standard definition [1–4]. The notion of complexity first arose when...
structures of two physical models (ideal gas and perfect crystal) were compared. The molecules of ideal gas frequently change their positions, while the perfect crystal has a symmetric arrangement in which atoms occupy fixed places. Thus, maximum information is required to specify a probable state of ideal gas, whereas a perfect crystal is completely specified by the least amount of information. Despite the differences in the atomic arrangements, the two models are allotted zero complexity. The previous definitions accommodated the concepts of information regarding atomic arrangement and symmetries as well as quantification of geometrical attributes. Lopez-Ruiz et al. [5–8] also examined the complexity of self-gravitating structures by examining their disequilibrium. They detected the differences between probable states of the matter distribution and the equiprobable configuration of the system. Researchers used energy density instead of probability distribution in the definition to examine the complexity of compact dense objects like white dwarfs and neutron stars [9–12].

The definition proposed by Lopez-Ruiz et al. is considered incomplete as it only encompasses the density of the matter source and the contribution of significant state variables is neglected. The astrophysical study of celestial systems has revealed that the constituent particles are compactly arranged within the internal configurations of dense stellar objects. Consequently, the radial motion of particles is limited which generates anisotropy in pressure [13]. Other factors of anisotropic interiors include pion condensation [14], phase transition [15] and superfluid [16]. Thus, anisotropy is an essential feature of compact distributions. Herrera [17] incorporated anisotropy as well as energy density to determine the complexity of a static sphere in the context of general relativity (GR). He put forward the new definition on the assumption that an isotropic and homogeneous distribution has zero complexity. According to this approach, an effective measure of complexity was devised via splitting the Riemann tensor in terms of structure scalars.

Herrera et al. [18] computed the structure scalars and formulated the complexity factor for a homologously evolving sphere. This notion of complexity was also extended to axially symmetric cosmic objects, and three complexity factors were evaluated through the orthogonal splitting of the Riemann tensor [19]. In the same work, a possible relation between complexity and symmetry of the spacetime was explored. Herrera and his collaborators [20] also established a hierarchy from the complex fluids to the simpler Minkowski spacetime. Sharif and Butt [21] adopted Herrera’s approach to examine the impact of charge on the complexity of a static sphere and deduced that the electromagnetic field enhances the complexity of a system. Sharif and Tariq [22] investigated the complex structure of a charged spherical system evolving in a homologous pattern. Recently, the conditions which led to a complexity-free system following a quasi-homologous mode of evolution were also determined [23].

The solution of vacuum cylindrical spacetime by Levi-Civita motivated researchers to explore different astrophysical phenomena with cylindrical geometry. The study of strong gravitational waves emitted at the end of gravitational collapse of a stellar object also attracted astrophysicists to examine salient features and behavior of cylindrical cosmic bodies [24]. Herrera et al. [25] inspected the regularity of static cylinders and deduced that a spacetime matched to the Levi-Civita exterior does not admit conformally flat solutions. The phenomenon of gravitational collapse was discussed for a non-static cylinder through junction conditions, and it was found that radial pressure vanishes at the boundary of the object [26]. Sharif and Abbas [27] considered a charged radiating cylinder and computed the gravitational mass of the anisotropic setup. Herrera et al. [28] specified the fundamental properties and structure of an evolving self-gravitating cylinder through structure scalars. Recently, complexity factors corresponding to uncharged [29] as well as charged [30] cylindrical systems were also developed.
The modified theories of relativity, obtained by modifying the Einstein–Hilbert action, have opened the pathway to new avenues of cosmological and astronomical phenomena. Moreover, these modifications are aimed at providing solutions to two cosmic problems: cosmic coincidence and fine-tuning. Scalar-tensor theories include a wide range of modified theories obtained via modification in the geometric structure of GR. Brans–Dicke (BD) theory, the prototype of this family of modifications, is based on Machian principle and Dirac hypothesis [31]. In BD gravity, a dynamical scalar field \( \psi(t) = \frac{1}{\Phi(t)} \) acts as a mediator of gravity, while a coupling parameter \( \omega_{BD} \) gauges the influence of the massless scalar field on matter distribution. Larger values of scalar field correspond to smaller values of coupling parameter.

The rapid expansion of the universe in the inflationary era is effectively described by a large scalar field. Thus, small values of coupling parameter efficiently explain the inflation of the cosmos [32], while the weak-field tests are consistent with BD theory for larger values of \( \omega_{BD} \) [33]. This discrepancy is resolved through a self-interacting potential \( (V(\Phi)) \) which establishes a standard domain for the parameter through a restriction on the mass of the scalar field (\( \Phi \)). Brans–Dicke theory incorporating a potential function is termed as massive BD (MBD) gravity. Sharif and Manzoor analyzed the effect of the massive scalar field on spherical [34,35] as well as cylindrical [36,37] configurations by formulating structure scalars in the background of MBD theory. Recently, Herrera’s definition was adopted to check the variation in the complexity of different configurations under the influence of massive scalar field [38–40]. Researchers have utilized Herrera’s definition in other modified theories to construct complexity factors for different geometries as well [41–47].

This paper focuses on devising a complexity factor for a non-static radiating cylindrical self-gravitating structure in the context of MBD theory. The paper is arranged in the following format. In the next section, the MBD field equations and relations between different physical aspects of the dissipative cylinder are established. We split the Riemann tensor to derive structure scalars in Sect. 3. The pattern of evolution is discussed in Sect. 4. In Sect. 5, solutions for non-dissipative and dissipative scenarios are obtained by evaluating kinematical as well as dynamical quantities. We inspect the stability of the zero complexity condition in Sect. 6. Section 7 provides a summary of the main results.

2 Massive Brans–Dicke theory and matter variables

The MBD field equations (in relativistic units) are obtained by varying the action

\[
S = \int \sqrt{-g} \left( R \Phi - \frac{\omega_{BD}}{\Phi} \nabla^{\gamma} \nabla_{\gamma} \Phi - V(\Phi) + L_m \right) \, d^4x,
\]

with respect to the metric tensor as

\[
G_{\lambda\mu} = T^{(\text{eff})}_{\lambda\mu} = \frac{1}{\Phi} \left( T^{(m)}_{\lambda\mu} + T^{\Phi}_{\lambda\mu} \right),
\]

Here, the matter Lagrangian and Ricci scalar are represented by \( L_m \) and \( R \), respectively, while \( g = |g_{\lambda\mu}| \). Moreover, the energy-momentum tensor \( T^{(m)}_{\lambda\mu} \) specifies the matter source, whereas \( T^{\Phi}_{\lambda\mu} \) incorporates the effects of the massive scalar field as

\[
T^{\Phi}_{\lambda\mu} = \Phi_{,\lambda;\mu} - g_{\lambda\mu} \Box \Phi + \frac{\omega_{BD}}{\Phi} \left( \Phi_{,\lambda} \Phi_{,\mu} - \frac{g_{\lambda\mu} \Phi_{,\alpha} \Phi_{,\alpha}}{2} \right) - \frac{V(\Phi) g_{\lambda\mu}}{2}.
\]
where $\square \Phi = \Phi_{,i;^i}$. The equation of motion for the scalar field is derived via action (1) as

$$
\square \Phi = \frac{T^{(m)}}{3 + 2\omega_{BD}} = \frac{1}{3 + 2\omega_{BD}} \left( \Phi \frac{dV(\Phi)}{d\Phi} - 2V(\Phi) \right),
$$

(4)

where $T^{(m)} = g_{\lambda\mu} T^{(m)\lambda\mu}$. We consider a cylindrical cosmic object bounded by a hypersurface $\Sigma$ and defined by the following line element

$$
ds^2 = -X^2(t, r)dt^2 + Y^2(t, r)dr^2 + Z^2(t, r)(d\theta^2 + dz^2).
$$

(5)

We assume that the cylinder is filled with anisotropic fluid dissipating in the form of heat flux $(q)$. The radial $(p_r)$/transverse $(p_\perp)$ pressures and energy density $(\rho)$ of the matter distribution are determined by the following energy-momentum tensor

$$
T^{(m)}_{\lambda\mu} = (\rho + p_\perp)u_\lambda u_\mu + p_\perp g_{\lambda\mu} + (p_r - p_\perp)\delta_\lambda^s \delta_\mu^t + q_\lambda u_\mu + u_\lambda q_\mu,
$$

where the heat flux $(q_\lambda = (0, qY, 0, 0))$, radial 4-vector $(s_\lambda = (0, Y, 0, 0))$ and 4-velocity $(u_\lambda = (-X, 0, 0, 0))$ satisfy the following relations

$$
s^i u_\lambda = 0, \quad s^i s_\lambda = 1, \quad u^i u_\lambda = -1, \quad u^i q_\lambda = 0.
$$

The energy-momentum tensor can be rewritten in a simplified form in terms of the quantities $\Pi_{\lambda\mu} = \Pi(s_\lambda s_\mu - h_{\lambda\mu}/3)$, $P = \frac{1}{3}(p_r + 2p_\perp)$, $\Pi = p_r - p_\perp$, and $h_{\lambda\mu} = g_{\lambda\mu} + u_\lambda u_\mu$ as

$$
T^{(m)}_{\lambda\mu} = \rho u_\lambda u_\mu + Ph_{\lambda\mu} + \Pi \lambda_{\mu} + q(s_\lambda u_\mu + u_\lambda s_\mu).
$$

(6)

The field equations are obtained through Eqs.(2)–(6) as

$$
\frac{1}{\Phi} (X^2 \rho - T_{00}^{(\Phi)}) = \frac{\dot{Z}}{Z} \left( \frac{2Y}{Y} + \frac{\dot{Z}}{Z} \right) - \frac{X^2}{Y^2} \left( \frac{Z^2}{Z^2 - 2Y'Z' + 2Z''} \right),
$$

(7)

$$
\frac{1}{\Phi} (-qXY + T_{01}^{(\Phi)}) = \frac{2X'Z}{XZ} + \frac{2\dot{Y}Z^2}{YZ} - \frac{2\dot{Z}'}{Z},
$$

(8)

$$
\frac{1}{\Phi}(Y^2 p_r + T_{11}^{(\Phi)}) = -\frac{Y^2}{X^2} \left( \frac{2\dot{Z}}{Z} - \frac{\dot{Z} \left( \frac{2X}{X} - \frac{\dot{Z}}{Z} \right)}{Z} \right) + \frac{Z'}{Z} \left( \frac{2X'}{X} + \frac{Z'}{Z} \right),
$$

(9)

$$
\frac{1}{\Phi}(Z^2 p_r + T_{22}^{(\Phi)}) = -\frac{Z^2}{X^2} \left( -\frac{\dot{X}}{X} \left( \frac{2\dot{Y}}{Y} + \frac{\dot{Z}}{Z} \right) \right) + \frac{\dot{Y}Z}{Y} + \frac{\dot{Z}}{Z} + \frac{\dot{X}}{X} \left( \frac{X'}{X} - \frac{Z'}{Z} \right)
$$

$$
+ \frac{Z^2}{Y^2} \left( \frac{X' \left( \frac{X'}{X} - \frac{Y'}{Y} \right)}{Y} - \frac{X'Y'}{XY} + \frac{X''}{X} + \frac{Z''}{Z} \right),
$$

(10)

where

$$
T_{00}^{\Phi} = -\Phi \left( \frac{\dot{Y}}{Y} + \frac{2\dot{Z}}{Z} \right) + \frac{X^2\Phi'}{Y^2} - \frac{2XZ''}{Z^2} - \frac{\omega_{BD}}{2} \left( \frac{X^2\Phi'^2}{Y^2} + \Phi^2 \right)
$$

$$
+ \frac{X^2\Phi''}{Y^2} + \frac{X^2}{2} V(\Phi),
$$

$$
T_{01}^{\Phi} = -\frac{X'\Phi}{X} - \frac{\dot{Y} \Phi'}{Y} + \frac{\omega_{BD}}{2} \Phi \Phi' + \Phi',
$$

\[\square \Phi = \Phi_{,i;^i}\]


\[ T_{11}^\Phi = - \Phi' \left( \frac{X'}{X} + 2 \frac{Z'}{Z} \right) - \frac{Y^2 \Phi \left( \frac{\dot{X}}{X} - \frac{2 \dot{Z}}{Z} \right)}{X^2} + \frac{\omega_{BD} \left( \frac{Y^2 \Phi^2}{X^2} + \Phi'^2 \right)}{2 \Phi} + \frac{Y^2 \Phi}{X^2} - \frac{Y^2}{2} V(\Phi), \]

\[ T_{22}^\Phi = - \frac{Z^2 \Phi' \left( \frac{X}{X} - \frac{Y'}{Y} + \frac{Z'}{Z} \right)}{Y^2} - \frac{Z^2 \Phi \left( \frac{\dot{X}}{X} - \frac{\dot{Y}}{Y} - \frac{\dot{Z}}{Z} \right)}{X^2} - \frac{\omega_{BD} Z^2 \left( \frac{\Phi'^2}{Y^2} - \frac{\Phi^3}{X^2} \right)}{2 \Phi} + \frac{Z^2 \Phi''}{X^2} - \frac{Z^2 \Phi'}{Y^2} - \frac{Z^2}{2} V(\Phi). \]

Here, prime and dot represent differentiation with respect to \( r \) and \( t \), respectively. The conservation of dissipative fluid is expressed in the form of following equations

\[ \dot{j}_0^{(\text{eff})} + (T_0^{(\text{eff})} - T_1^{(\text{eff})}) \frac{\dot{Y}}{Y} + 2(T_0^{(\text{eff})} - T_2^{(\text{eff})}) \frac{\dot{Z}}{Z} + (T_0^{(\text{eff})} + T_1^{(\text{eff})}) \right) = 0, \]

\[ \dot{j}_0^{(\text{eff})} + (T_1^{(\text{eff})}) + T_0^{(\text{eff})} \left( \frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + 2 \frac{\dot{Z}}{Z} \right) - (T_0^{(\text{eff})} - T_1^{(\text{eff})}) \frac{X'}{X} + 2(T_1^{(\text{eff})} - T_2^{(\text{eff})}) \frac{Z'}{Z} = 0. \]

Furthermore, the wave equation corresponding to the cylindrical setup is given as

\[ \Box \Phi = \frac{\Phi' \left( \frac{X'}{X} - \frac{Y'}{Y} + \frac{2 \dot{Z}}{Z} \right)}{Y^2} - \frac{\Phi' \left( \frac{\dot{X'}}{X} + \frac{\dot{Y'}}{Y} + \frac{2 \dot{Z}}{Z} \right)}{X^2} - \frac{\Phi''}{X^2} = \frac{1}{3 + 2 \omega_{BD}} \left[ - \rho + 3 P + \left( \Phi \frac{dV(\Phi)}{d \Phi} - 2 V(\Phi) \right) \right]. \]

In case of non-rotating fluid, the three kinematical variables required to describe the motion of celestial system are 4-acceleration \( (\mathbf{a}) \), expansion scalar \( (\Theta) \) and shear tensor \( (\sigma_{\lambda\mu}) \) which are, respectively, defined as

\[ a_\lambda = u_{\lambda\mu} u^\mu, \quad \Theta = u^\lambda, \quad \sigma_{\lambda\mu} = u_{\lambda;\mu} + a_{(\lambda} u_{\mu)} - \frac{1}{3} \Theta h_{\lambda\mu}. \]

The above quantities are evaluated corresponding to the cylindrical setup as

\[ a_1 = \frac{X'}{X}, \quad a_2^2 = a_\lambda a^\lambda = \left( \frac{X'}{XY} \right)^2, \]

\[ \Theta = \frac{1}{X} \left( \frac{\dot{Y}}{Y} + 2 \frac{\dot{Z}}{Z} \right), \]

\[ \sigma_{11} = \frac{2}{3} Y^2 \sigma, \quad \sigma_{22} = - \frac{1}{3} Z^2 \sigma, \]

with \( a_\lambda = a_{\lambda\mu} \) and \( \sigma = \sqrt{\frac{3}{5}} \sigma_{\lambda\mu} \). Thorne [48,49] proposed the C-energy formula to calculate the mass of a cylinder as

\[ m(t, r) = l \dot{E} = \frac{l}{8} \left( 1 - \frac{1}{l^2} \nabla_\lambda \hat{r} \nabla^\lambda \hat{r} \right), \]
where

\[ \hat{r} = \mathcal{N}, \quad l^2 = \chi(2)u \chi^2, \quad \mathcal{N}^2 = \chi(1)u \chi^2. \]

Here, \( \chi_1 = \frac{\partial}{\partial \theta}, \chi_2 = \frac{\partial}{\partial z} \) and \( \hat{E} \) is the gravitational energy per unit specific length \((l)\). The \( C \)-energy corresponding to Eq. (5) is computed as

\[ m(t, r) = \frac{Z}{2} \left( \frac{1}{4} + \frac{\dot{Z}^2}{X^2} - \frac{Z'^2}{Y^2} \right). \]

The variation in energy of the anisotropic celestial object with respect to time is evaluated via the proper time derivative \((D_T = \frac{1}{X} \frac{\partial}{\partial t})\) as

\[ D_T m = -\frac{Z^2}{2} \left( T_1^{(\text{eff})} + \frac{1}{4Z^2} \right) U - \frac{T_0^{(\text{eff})}}{2Y} Z^2 E. \]

where \( E \equiv \frac{Z'}{T} \). Moreover, the velocity of the collapsing cylinder \((U = D_T Z < 0)\) is related to \( C \)-energy as

\[ E = \left( \frac{1}{4} + U^2 - \frac{2m}{Z} \right)^{\frac{1}{2}}. \]

It is noted that the term \( \left( T_1^{(\text{eff})} + \frac{1}{4Z^2} \right) U \) in Eq.(18) contributes to the energy of the system if \( \left( T_1^{(\text{eff})} + \frac{1}{4Z^2} \right) > 0 \). In order to inspect the behavior of \( C \)-energy within the adjacent walls of the cylinder, we compute its proper radial derivative \((D_R = \frac{1}{Z} \frac{\partial}{\partial r})\) as

\[ D_R m = \frac{Z^2}{2} \left( T_0^{(\text{eff})} + \frac{T_1^{(\text{eff})}}{Y} \frac{U}{E} \right) - \frac{1}{4Z^2}, \]

which implies that if \( \frac{T_0^{(\text{eff})}}{Y} > 0 \) the energy of the system decreases due to the second term within the brackets as \( U < 0 \). Integrating of Eq.(20) provides

\[ \frac{3m}{Z^3} = -\frac{1}{2} \frac{T_0^{(\text{eff})}}{2} + \frac{1}{2Z^3} \int_0^r Z' Z^3 \left( D_R T_0^{(\text{eff})} - \frac{3T_0^{(\text{eff})}}{XY} \frac{U}{E} \right) dr + \frac{3}{4Z^2}. \]

The tidal forces experienced by a celestial system due to nearby gravitational field play a vital role in determining its significant physical features. In order to incorporate the effects of tidal forces in our work, we evaluate the Weyl tensor defined as

\[ C_{\alpha \beta \lambda \sigma} = R_{\alpha \beta \lambda \sigma} - \frac{R_{\alpha \beta}}{2} g_{\sigma \lambda} + \frac{R_{\alpha \beta}}{2} \delta_{\lambda \sigma} - \frac{R_{\alpha \sigma}}{2} \delta_{\beta \lambda} + \frac{R_{\lambda \sigma}}{2} g_{\alpha \beta} + \frac{1}{6} \delta_{\lambda \beta} g_{\alpha \sigma} + g_{\alpha \beta} \delta_{\lambda \sigma}, \]

where \( R_{\alpha \beta \lambda \sigma} \) and \( R_{\alpha \beta} \) represent the Riemann and Ricci tensors, respectively. The trace-free magnetic \((H_{\lambda \mu})\) and electric \((\xi_{\lambda \mu})\) parts of the Weyl tensor are obtained through \( u^\nu \) as

\[ H_{\lambda \mu} = \frac{1}{2} \eta_{\lambda \nu \epsilon \beta} C_{\mu \nu} \epsilon^\mu u^\nu u^\gamma, \]

\[ \xi_{\lambda \mu} = C_{\lambda \beta \mu \sigma} u^\beta u^\sigma. \]
In case of cylindrical symmetry, the magnetic and electric parts are non-vanishing in general. However, the magnetic part corresponding to the considered setup vanishes, whereas the electric part turns out to be

$$\xi_{\lambda\mu} = C_{\lambda\gamma\mu\delta} u^\gamma u^\delta = \varepsilon \left( g_{\lambda\mu} + \frac{h_{\lambda\mu}}{3} \right),$$  \(\text{(25)}\)

where

$$\varepsilon = \left( \frac{Z'}{Z} - \frac{X'}{X} \right) \left( \frac{Z'}{Z} + \frac{X'}{X} - \frac{Z''}{Z} \right) + \frac{Z'}{Z} - \frac{\ddot{Z}}{Z} - \left( \frac{\ddot{X}}{X} + \frac{Z'}{Z} \right) \left( \frac{\ddot{X}}{X} - \frac{\ddot{Z}}{Z} \right).$$  \(\text{(26)}\)

Furthermore, the impact of the massive scalar field on pressure, Weyl tensor and energy density is demonstrated in the following relation

$$\frac{\partial}{\partial t} \left[ \varepsilon - \frac{1}{2} \left( - T_0^{0(\text{eff})} - T_1^{1(\text{eff})} + T_2^{2(\text{eff})} \right) \right] = \frac{3\dot{Z}}{Z} \left[ \frac{1}{2} \left( - T_0^{0(\text{eff})} + T_2^{2(\text{eff})} \right) - \varepsilon \right] - \frac{3Z'}{2Z} T_0^{1(\text{eff})}. $$  \(\text{(27)}\)

### 3 Structure scalars

In this section, we formulate a complexity factor for the dynamical object in terms of structure scalars which are determined via the orthogonal splitting of Riemann tensor. The procedure of splitting Riemann tensor was first applied by Herrera [51] by expressing the Riemann tensor in terms of trace and trace-free parts as follows

$$\mathcal{R}_{\beta\gamma} = C_{\beta\gamma} + 2T_{[\beta}^{(\text{eff})}[\alpha \delta]_{\gamma]} + T^{(\text{eff})} \left( \frac{1}{3} \delta_{[\rho}^{\alpha} \delta_{\gamma]}^{\delta} - \delta_{[\beta}^{\alpha} \delta_{\gamma]}^{\delta} \right).$$  \(\text{(28)}\)

Employing the expressions

$$\mathcal{R}_{(I)\beta\gamma} = \frac{2}{\Phi} \left[ \rho u^{[\alpha} u_{[\beta}^{\delta]} - PH_{[\beta}^{\alpha} \delta_{\gamma]} + (\rho - 3P) \left( \frac{1}{3} \delta_{[\rho}^{\alpha} \delta_{\gamma]}^{\delta} - \delta_{[\beta}^{\alpha} \delta_{\gamma]}^{\delta} \right) \right],$$

$$\mathcal{R}_{(II)\beta\gamma} = \frac{2}{\Phi} \left[ \Pi_{[\beta}^{\alpha} \delta_{\gamma]} + u_{[\alpha}^{\delta} \delta_{[\beta}^{\gamma]} \right],$$

$$\mathcal{R}_{(III)\beta\gamma} = 4u_{[\alpha}^{[\beta} \delta_{\gamma]} - \varepsilon_{\alpha}^{\beta} \varepsilon_{\beta\gamma}^{\delta} \xi_{\gamma}^{\delta},$$

$$\mathcal{R}_{(IV)\beta\gamma} = \frac{2}{\Phi} \left[ \Phi_{[\alpha}^{[\beta} \delta_{\gamma]} + \frac{\omega BD}{\Phi} \Phi_{[\alpha}^{[\beta} \delta_{\gamma]} - \left( \Box \Phi + \frac{\omega BD}{2\Phi} \Phi_{,\lambda}^{,\lambda} + \frac{V(\Phi)}{2} \right) \right] \times \delta_{[\beta}^{\alpha} \delta_{\gamma]}^{\delta},$$

$$\mathcal{R}_{(V)\beta\gamma} = \frac{1}{\Phi} \left[ \left( - \frac{\omega BD}{\Phi} \Phi_{,\lambda}^{,\lambda} - 2V(\Phi) - 3\Box \Phi \right) \left( \frac{1}{3} \delta_{[\beta}^{\alpha} \delta_{\gamma]}^{\delta} - \delta_{[\beta}^{\alpha} \delta_{\gamma]}^{\delta} \right) \right],$$

the Riemann tensor is decomposed as

$$\mathcal{R}_{\beta\gamma} = \mathcal{R}_{(I)\beta\gamma} + \mathcal{R}_{(II)\beta\gamma} + \mathcal{R}_{(III)\beta\gamma} + \mathcal{R}_{(IV)\beta\gamma} + \mathcal{R}_{(V)\beta\gamma}. $$  \(\text{(29)}\)

As per Herrera’s technique, we introduce the following tensors

$$Y_{\lambda\mu} = \mathcal{R}_{\lambda\delta\mu\gamma} u^\gamma u^\delta,$$
\[ X_{\lambda\mu} = R_{\lambda\delta\mu\gamma}^* u^\delta u^\gamma = \frac{1}{2} \eta_{\lambda\delta}^* \eta_{\alpha\mu}^* u^\delta u^\gamma, \]

which are computed for the non-static cylinder as

\[ X_{\lambda\mu} = \frac{1}{\Phi} \left( \frac{\rho h_{\lambda\mu}}{3} + \frac{\Pi_{\lambda\mu}}{2} \right) - \xi_{\lambda\mu} + \frac{1}{2\Phi} \left( \Phi_{\lambda,\delta} h_{\delta\mu} + \frac{\omega_{BD}}{2\Phi} \Phi_{\lambda,\delta} \Phi_{\delta\mu}^* \right) + \frac{h_{\lambda\mu}}{4\Phi}(\Box\Phi + 7V(\Phi)), \]

(30)

\[ Y_{\lambda\mu} = \frac{1}{\Phi} \left( \frac{(\rho + 3P) h_{\lambda\mu}}{6} + \frac{\Pi_{\lambda\mu}}{2} \right) + \xi_{\lambda\mu} + \frac{1}{2\Phi} \left( -\Phi_{\lambda,\delta} - \Phi_{\lambda,\lambda} u_{\mu} u^\delta \right. \]

\[ -\Phi_{\delta,\mu} u_{\lambda} u^\delta + \Phi_{\gamma,\delta} u_{\lambda} u^\delta u^\gamma g_{\mu\lambda} + \frac{\omega_{BD}}{2\Phi^2} \left( -\Phi_{\lambda,\mu} - \Phi_{\lambda,\lambda} u_{\mu} u^\delta \right. \]

\[ \left. \left. -\Phi_{\delta,\mu} u_{\lambda} u^\delta - \Phi_{\gamma,\delta} u_{\lambda} u^\delta u^\gamma g_{\mu\lambda} + \frac{h_{\lambda\mu}}{6\Phi} \left( \frac{\omega_{BD}}{\Phi} \Phi_{\delta,\delta} - V(\Phi) \right) \right) \right). \]

(31)

Here, \( R_{\lambda\mu\delta\gamma}^* = \frac{1}{2} \eta_{\alpha\epsilon\delta\gamma} R_{\alpha\epsilon\mu\lambda}^* \) and \( ^* R_{\lambda\mu\delta\gamma} = \frac{1}{2} \eta_{\lambda\mu\alpha\epsilon} R_{\alpha\epsilon\delta\gamma}^* \) are the right and left duals, respectively.

The required scalar quantities are obtained by splitting the tensors \( X_{\lambda\mu} \) and \( Y_{\lambda\mu} \) in the following form

\[ X_{\lambda\mu} = X_T h_{\lambda\mu} + X_{<\lambda\mu>}, \]

\[ Y_{\lambda\mu} = Y_T h_{\lambda\mu} + Y_{<\lambda\mu>}, \]

where

\[ X_T = X_T^\lambda, \quad X_{<\lambda\mu>} = h_{\lambda}^\alpha h_{\mu}^\beta \left( X_{\alpha\beta} - \frac{X_T^\lambda}{3} h_{\alpha\beta} \right), \]

\[ Y_T = Y_T^\lambda, \quad Y_{<\lambda\mu>} = h_{\lambda}^\alpha h_{\mu}^\beta \left( Y_{\alpha\beta} - \frac{Y_T^\lambda}{3} h_{\alpha\beta} \right). \]

The structure scalars incorporating the essential characteristics of the anisotropic fluid turn out to be

\[ X_T = X_T^{(m)} + X_T^\Phi = \frac{1}{\Phi} (\rho + \frac{5}{2}\Box\Phi + \Phi_{,\lambda\alpha} u^\alpha u^\lambda + \frac{\omega_{BD}}{2\Phi} (\Phi_{,\lambda} \Phi_{\lambda}^*) \]

\[ + \Phi_{,\lambda} \Phi_{,\alpha} u^\alpha u^\lambda + \frac{21}{2} V(\Phi)), \]

(32)

\[ X_{TF} = X_{TF}^{(m)} + X_{TF}^\Phi = -\frac{1}{\Phi} \left( \frac{\Pi}{2} + \varepsilon\Phi \right) + \frac{1}{2\Phi} \left( \Box\Phi + \Phi_{,\alpha\lambda} u^\alpha u^\lambda \right. \]

\[ \left. \left. + \frac{\omega_{BD}}{2\Phi} (\Phi_{,\alpha} \Phi_{,\alpha} + \Phi_{,\lambda} \Phi_{,\alpha} u^\alpha u^\lambda) \right) \right), \]

(33)

\[ Y_T = Y_T^{(m)} + Y_T^\Phi = \frac{1}{\Phi} (\rho + 3P - 2\Pi) - \frac{1}{2\Phi} \left( \Box\Phi + \Phi_{,\alpha\gamma} u^\gamma u^\alpha \right. \]

\[ \left. + \frac{\omega_{BD}}{\Phi} (\Phi_{,\gamma} \Phi_{,\alpha} u^\gamma u^\alpha) + V(\Phi)), \]

(34)

\[ Y_{TF} = Y_{TF}^{(m)} + Y_{TF}^\Phi = \frac{1}{\Phi} \left( \varepsilon\Phi - \frac{\Pi}{2} \right) - \frac{1}{2\Phi} \left( \Box\Phi + \frac{\omega_{BD}}{\Phi} (\Phi_{,\alpha} \Phi_{,\alpha}) \right) \]

The required scalar quantities are obtained by splitting the tensors \( X_{\lambda\mu} \) and \( Y_{\lambda\mu} \) in the following form

\[ X_{\lambda\mu} = X_T h_{\lambda\mu} + X_{<\lambda\mu>}, \]

\[ Y_{\lambda\mu} = Y_T h_{\lambda\mu} + Y_{<\lambda\mu>}, \]

where

\[ X_T = X_T^\lambda, \quad X_{<\lambda\mu>} = h_{\lambda}^\alpha h_{\mu}^\beta \left( X_{\alpha\beta} - \frac{X_T^\lambda}{3} h_{\alpha\beta} \right), \]

\[ Y_T = Y_T^\lambda, \quad Y_{<\lambda\mu>} = h_{\lambda}^\alpha h_{\mu}^\beta \left( Y_{\alpha\beta} - \frac{Y_T^\lambda}{3} h_{\alpha\beta} \right). \]
\[+ \Phi_{\gamma \beta} u^\gamma u^\beta + \Phi_{\lambda \gamma} u^\gamma u^\lambda\).  

The principal stresses and total energy density of the dynamical system are controlled by the scalars \(Y_T\) and \(X_T\), respectively. Moreover, the remaining scalars govern the local anisotropy of the fluid in the presence of the massive scalar field. Furthermore, \(Y_{TF}\) determines the evolution of the cylinder filled with anisotropic and inhomogeneous fluid as

\[
Y_{TF} = T_2^{2(\text{eff})} - T_1^{1(\text{eff})} - \frac{1}{2Z^3} \int_0^r Z' Z^3 \left(D_R T_0^{\Phi(\text{eff})} + \frac{3U}{\Phi E Z} \left(q - T_{01}^{\Phi XY}\right)\right) dr + \frac{1}{2\Phi} \left[3\Phi' \frac{Z'}{X^2 Z} - 2\Phi'' - \frac{3\Phi' Z'}{Y^2 Z} + \frac{\omega_{BD}}{\Phi} \Phi^2\right].
\]

(36)

4 Complexity and evolution of the system

According to the definition put forward by Herrera [18], the complexity of a configuration depends on its different physical characteristics. The description of a less complex structure (such as dust or vacuum) requires fewer matter variables as compared to a more complex fluid (such as perfect fluid distribution). Thus, in order to determine the complexity of anisotropic, inhomogeneous and dissipative self-gravitating structures we require a scalar function incorporating these essential features. Equation (36) shows that the structure scalar \(Y_{TF}\) includes the effects of complexity inducing features as well as the massive scalar field. Moreover, \(Y_{TF}\) has been treated as a suitable candidate of complexity factor in the case of a static regime [40]. Thus, \(Y_{TF}\) can adequately represent the complexity of the dynamical cylinder. Furthermore, the pattern of evolution is an important aspect of non-static celestial systems. In this section, we discuss two possible evolution patterns of the anisotropic fluid.

4.1 The homologous evolution

If the density of the collapsing configuration is the same throughout, i.e., matter falls into the core at the same rate in the internal region then the celestial body evolves homologously. However, if the velocity with which the matter collapses is not proportional to the radial distance, then the setup follows a non-homologous pattern. Employing Eqs. (8) and (19), the heat flux is expressed as

\[
\frac{1}{2E} \Phi \left(q - \frac{T_{01}^{\Phi}}{XY}\right) = \frac{1}{3} D_R (\Theta - \sigma) - \frac{\sigma}{Z},
\]

(37)

which leads to

\[
D_R \left(\frac{U}{Z}\right) = \frac{1}{2E} \Phi \left(q - \frac{T_{01}^{\Phi}}{XY}\right) + \frac{\sigma}{Z}.
\]

(38)

Consequently, the velocity of the collapsing cylinder is obtained as

\[
U = Z \int_0^r Z' \left[\frac{1}{2E} \Phi \left(q - \frac{T_{01}^{\Phi}}{XY}\right) + \frac{\sigma}{Z}\right] dr + f(t) Z,
\]

(39)
where \( f(t) = \frac{U}{Z^2} \) is an integration function. The condition of homologous evolution \((U \sim Z [52–54])\) is obtained if the integral in the above equation vanishes. Thus, the condition

\[
\frac{1}{2E \Phi} \left( q - \frac{\tau_{01}}{XY} \right) + \frac{\sigma}{Z} = 0, 
\]

must hold for homologous evolution. This condition suggests that for two shells of fluids, the ratio of the aerial radii is constant. We proceed by considering \( Z(t, r) \) as a separable function of \( t \) and \( r \).

### 4.2 The homogeneous expansion

Another simplest pattern of evolution is the homogeneous expansion which corresponds to \( \Theta' = 0 \). In other words, if the rate at which the self-gravitating system evolves is independent of the radial co-ordinate, then the system collapses or expands homogeneously. Applying the homogeneous condition to Eq.(37) yields

\[
\frac{1}{2E \Phi} \left( q - \frac{\tau_{01}}{XY} \right) = -\frac{1}{3} D_R(\sigma) - \frac{\sigma}{Z}.
\]

If the fluid is homologous as well, then the homogeneous condition reduces to \( D_R \sigma = 0 \) or \( \sigma = 0 \) (because of the regularity conditions at \( r = 0 \)). Thus, Eq.(37) can be written as

\[
q = \frac{\tau_{01}}{XY}.
\]

It must be noted that under the influence of the scalar field, homogeneous expansion does not lead to a dissipation-free matter source. However, in GR, a matter distribution with \( \sigma = \Theta' = 0 \) is non-dissipative as well as homologous.

### 5 Kinematical and dynamical variables

In this section, we choose the simplest pattern of evolution based on the analysis of some important kinematical quantities. If a fluid follows the homologous pattern of evolution, then Eq.(37) provides

\[
(\Theta - \sigma)' = \left( \frac{3 \dot{Z}}{XZ} \right)' = 0.
\]

Imposing the condition \( Z(t, r) = Z_1(t)Z_2(r) \) implies that \( X' = 0 \). Consequently, the fluid is geodesic \((a = 0)\). Without loss of generality, we consider \( X = 1 \). Conversely, consider the geodesic condition

\[
\Theta - \sigma = \frac{3 \dot{Z}}{Z}.
\]

Successively differentiating the above equation with respect to \( r \) and assuming that \((\Theta - \sigma)'\) equals its Taylor series close to the center leads to a homologous fluid [18]. Thus, the cylindrical object evolves homologously if and only if the fluid is geodesic. For this reason, we consider the homologous pattern as the simplest pattern of evolution. If the self-gravitating...
structure is non-dissipative, then the shear tensor under the effect of the massive scalar field takes the form
\[ \sigma = \frac{Z T_0^\Phi}{2Z'} \].

If the celestial system adopts the mode of homogeneous expansion, then the non-dissipative scenario produces \( T_0^\Phi = 0 \). Further, the shear scalar is computed as
\[ \sigma = \frac{1}{2Z^3} \int_0^r \frac{Z^3}{X} T_0^\Phi dr + \frac{h(t)}{Z^3} = \frac{h(t)}{Z^3}, \]
where \( h(t) \) is an arbitrary integration function. Since \( Z \to 0 \) as \( r \to 0 \), therefore, \( h(t) \) must approach to zero. It is deduced that homogeneous expansion implies homologous evolution when \( q = 0 \) (since \( \sigma = 0 \Rightarrow U \sim Z \)). On the other hand, if the non-dissipative fluid evolves homologously, then \( \sigma = \frac{Z T_0^\Phi}{2Z'} \) which implies
\[ \Theta' = \left( \frac{Z T_0^\Phi}{2Z'} \right)'. \]
Thus, homologous evolution does not imply homogeneous expansion. The C-energy of a homologously evolving cylinder is related to the rate of collapse as
\[ D_T U = -\frac{m}{Z^2} - \frac{Z}{2} T_1^{(eff)} + \frac{1}{8Z}, \tag{42} \]
which leads to
\[ \frac{3D_T U}{Z} = \frac{1}{2} (T_0^{0(\text{eff})} - T_1^{(\text{eff})} + 2T_2^{(e\text{ff})}) + Y_{TF} - \frac{1}{2\Phi} \left[ 3\Phi \frac{\dot{Z}}{Z} - 2\Phi' \frac{\ddot{Y}}{Y^2} - 3\Phi' \frac{Z'}{Z} \right. \]
\[ + \left. \frac{\omega_{BD}}{\Phi} \frac{\Phi'^2}{Y^2} \right]. \tag{43} \]
Employing the field equations and the definition of \( U \) in the above relation, the complexity factor is expressed as
\[ Y_{TF} = \frac{\dot{Z}}{Z} - \frac{\ddot{Y}}{Y} + \frac{1}{2\Phi} \left[ 3\Phi \frac{\dot{Z}}{Z} - 2\Phi' \frac{\ddot{Y}}{Y^2} - 3\Phi' \frac{Z'}{Z} + \frac{\omega_{BD}}{\Phi} \frac{\Phi'^2}{Y^2} \right]. \]
In the next sections, we utilize the constraints corresponding to vanishing complexity and homologous fluid to obtain solutions for \( q = 0 \) and \( q \neq 0 \) by assuming \( X = 1 \). Since the number of unknowns exceeds the number of equations, we consider the massive scalar field as
\[ \Phi(t, r) = \Phi(t) = \Phi_0 t^m, \tag{44} \]
where \( \Phi_0 \) is the present day value of the scalar field and \( m \) is a constant. Moreover, for this choice of the scalar field \( T_0^\Phi = 0 \). Thus, the homogeneous and homologous evolution conditions are same for \( q = 0 \) which implies a unique mode of evolution.

5.1 Case 1: non-dissipative fluid

In the non-dissipative scenario, the homologous condition becomes
\[ Y(t, r) = Z(t, r) h_1(r), \tag{45} \]
where $h_1(r)$ is an arbitrary function of integration. The wave equation and condition of vanishing complexity corresponding to Eqs.(44) and (45) generate the following relations

$$
\Phi_{0t}^{m-1} \frac{1}{2(\omega_{BD} + 3)h_1(r)} \frac{1}{Z^2} \left( -4tZ(h_1(r)^3(Z'((m-2)m\omega_{BD} + t^2V'(\Phi)) - 2t(m\omega_{BD} + 1)Z' + tZ) - 2t^2h_1(r)Z'(h_1(r)^2Z^2 - 1) \right) = 0,
$$

$$
V(\Phi) = \frac{\Phi_{0t}^{m-2}}{h_1(r)^3Z^2} \left( 2t^2h_1(r)^3Z'Z'' + 4t^2h_1(r)^3Z'\dot{Z} + m^2\omega_{BD}h_1(r)^3ZZ' + 2m^2h_1(r)^3ZZ' - 2mh_1(r)^3ZZ' - mth_1(r)^3Z'\dot{Z} + 2t^2h_1'(r) \right).
$$

A suitable choice of $h_1(r)$ determines the complete solution.

5.2 Case 2: dissipative fluid

In the non-dissipative case, the homologous, wave equation and complexity-free condition, respectively, read

$$
Y = h_2(r) \exp \left( \int_1^t \frac{\Phi_{0t}^m Z'Z'' - Z'Z}{(\Phi_{0t}^m - 1) Z} dt \right),
$$

$$
\frac{\Phi_{0t}^{m-1}}{2(\omega_{BD} + 3)YZ} \left( -4t^2Y'ZZ' - 2tY^2Z(2tY\dot{Z} + Z(m\omega_{BD} + 1)Y + tY) \right) + Y^3 \left( Z^2((m-2)m\omega_{BD} + t^2V'(\Phi)) - 2t(\dot{Z})^2 - 4tZ(m\omega_{BD} + 1)\dot{Z} + t\ddot{Z}) \right) + 2t^2Y \left( Z^2 + 2ZZ'' \right) = 0,
$$

$$
V(\Phi) = \frac{\Phi_{0t}^{m-2}}{Y^3Z} \left( 2t^2Y'\dot{Z} + 2t^2Y'Z'' - 2t^2YZ'' + Y^3(t(4t\dot{Z} - m\dot{Z}) + m(m\omega_{BD} + 2) - 2Z) \right),
$$

where $h_2(r)$ is an integration function whose appropriate form completely specifies the above system of equations for the chosen scalar field.

6 Stability of $Y_{TF} = 0$ condition

It is possible that a system that initiates with complexity-free interior develops complex nature at a later time, i.e., the condition of vanishing complexity may be disturbed during the evolution of the system. In this section, we investigate if the vanishing complexity condition can sustain throughout the homologous evolution of matter configuration corresponding to $\Phi(t, r) = \Phi(t) = \Phi_{0t}^m$. Equations (11) and (27) determine the evolution of the complexity as

$$
\dot{Y}_{TF} + \frac{\Pi}{\Phi} + 3\frac{\dot{Z}}{Z}Y_{TF} + (\rho + P_r)\frac{\sigma}{2\Phi} + \frac{1}{2Y\Phi} \left( \frac{q'}{Z} - \frac{q'Z'}{Z} \right) + \frac{2\Pi\dot{Z}}{Z\Phi} + S_1 = 0, \quad (46)
$$

where the term $S_1$ contains the effects of scalar field and is given as

$$
S_1 = \frac{(T_1^{\phi} - T_2^{\phi})}{2\Phi} - \frac{(T_0^{\phi})'}{\Phi} - \frac{(T_1^{\phi})'}{2\Phi} \left( \frac{Y'}{Y} - \frac{Z'}{Z} \right) - \frac{(T_0^{\phi} - T_1^{\phi})\dot{Y}}{2Y\Phi}
$$
\[- \frac{5(T_0^0 - T_2^2)\dot{Z}}{2Z\Phi} - \dot{Y}_F - Y_{TF}. \]

We first consider \( q = 0 \) with \( \Pi = \sigma = Y_{TF} = 0 \) at \( t = 0 \). Equation (46) and its derivative with respect to \( t \) are, respectively, expressed as

\[ S_1 = - (\dot{Y}_{TF} + \dot{\Pi}), \tag{47} \]
\[ \ddot{Y}_{TF} + \frac{\dot{\Pi}}{\Phi} - \frac{\dot{\Phi}}{\Phi^2} = 3S_1 - \dot{S}_1 + \frac{\dot{\Pi}\dot{Z}}{Z\Phi}, \tag{48} \]

which leads to the following forms of first and second \( t \)-derivatives of Eq.(36)

\[ S_1 + 3\left( \frac{\dot{\Phi}}{\Phi} \right) \frac{\dot{Z}}{Z} = \frac{\partial}{\partial t} \left( \int_0^r Z^3(T_0^{0\text{eff}})' dr \right), \]
\[ 3S_1 - \dot{S}_1 + \frac{\dot{\Pi}\dot{Z}}{Z\Phi} - 3\left( \frac{\dot{\Phi}}{\Phi} \right) \frac{\dot{Z}}{Z} = \frac{\partial^2}{\partial t^2} \left( \int_0^r -Z^3(T_0^{0\text{eff}})' dr \right). \]

The higher order derivatives can be obtained by proceeding in the same manner. It is observed that the state variables (anisotropic pressure and inhomogeneous energy density) along with the scalar field induce complexity in the system. Thus, the stability of \( Y_{TF} = 0 \) condition depends on these factors. For \( q \neq 0 \), heat dissipation also affects the condition of zero complexity.

7 Summary

Self-gravitating systems are complicated and intriguing cosmic objects. Researchers examine the origin and evolution of these systems to gain insight into the structure of the universe. In this paper, we have formulated a complexity factor to investigate the relations between different state parameters of a dynamical cylinder in the context of MBD gravity. The complexity of the dissipative setup has been determined through structure scalars obtained via the orthogonal splitting of the Riemann tensor. The dynamics of the cylindrical regime have been incorporated in the definition of complexity by considering the pattern of evolution. We have examined two evolution modes namely, homologous and homogeneous. Solutions corresponding to \( q = 0 \) and \( q \neq 0 \) have been developed by applying the homologous and zero complexity conditions. Finally, the criteria under which the self-gravitating cylinder departs from the initial state of zero complexity have also been discussed.

The splitting of the Riemann tensor has yielded structure scalars that govern the mechanism and inhomogeneous structure of the anisotropic system. It has been noted that heat dissipation, anisotropy and inhomogeneity are the main factors contributing to the complexity of the non-static model in GR. However, the presence of massive scalar field and potential function in structure scalars indicates that complexity of the MBD model depends on the scalar field as well. Thus, the cylindrical system in GR is less complicated as compared to its MBD analog. We have chosen the structure scalar \( Y_{TF} \) to represent the complexity of the system based on the following reasons.

- It incorporates the effects of anisotropic pressure, heat dissipation and inhomogeneous energy density of the configuration.
- The complexity of the static cylinder has been adequately measured through \( Y_{TF} [40] \). Thus, choosing \( Y_{TF} \) for the dynamical system ensures that the current definition can be restored in the static regime.
Furthermore, the homologous pattern of evolution fulfills the condition of geodesic fluid and conversely the geodesic fluid evolves homologously. Therefore, we have chosen the homologous pattern as the least complex mode of evolution. It is noted that the scalar field influences the evolution of the system. Consequently, the condition $q = 0$ does not lead to a shear-free self-gravitating model. The implementation of vanishing complexity and homologous pattern for $\Phi(t, r) = \Phi(t) = \Phi_0^{\text{GR}}$ leads to open systems corresponding to non-dissipative as well as dissipative scenarios. Suitable choices of integration functions specify the system completely. Moreover, a perturbation in the scalar field or state determinants (heat flux, pressure, density) may disturb the system from its state of zero complexity. It is noteworthy that if $\Phi = \text{constant}$ and $\omega_{\text{BD}} \to \infty$, then all the results derived in this paper reduce to their GR counterparts.

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