ON REFLECTION ORDERS COMPATIBLE WITH A COXETER ELEMENT

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Abstract. In this article we give a simple, almost uniform proof that the lattice of noncrossing partitions associated with a well-generated complex reflection group is lexicographically shellable. So far a uniform proof is available only for Coxeter groups. In particular we show that, for any complex reflection group \( W \) and any element \( x \in W \), every \( x \)-compatible reflection order is a recursive atom order of the corresponding interval in absolute order. Since any Coxeter element \( \gamma \) in any well-generated complex reflection group admits a \( \gamma \)-compatible reflection order, the lexicographic shellability follows from a well-known result due to Björner and Wachs.

1. Introduction

In the past few years the field of Coxeter-Catalan combinatorics has become a very active and fruitful research area that influences many branches of mathematics, such as group theory, topology, free probability, representation theory of quivers, or the theory of cluster algebras. One of the most prominent objects relating all these branches is the lattice of noncrossing partitions associated with a well-generated complex reflection group, see \([4,7,9,10,16,21,27]\). The study of these objects was initiated by Kreweras’ investigation of noncrossing set partitions in \([22]\). He showed that these set partitions are enumerated by the classical Catalan numbers, and that they form a lattice when ordered by refinement. More- over, Kreweras computed the values of their Möbius function. It was also shown that these lattices have several other nice properties, see for instance \([31,32]\).

Biane observed in \([8]\) that the lattice of noncrossing set partitions can be seen as an interval in the absolute order on the symmetric group. This connection was then used by Brady and Watt as well as Bessis to define \( W \)-noncrossing partition lattices, denoted by \( \mathcal{NC}_W \), for any well-generated complex reflection group \( W \), see \([4,5,17]\). In the last years many explicit bijections between the \( W \)-noncrossing partitions and other Coxeter-Catalan objects have been found, provided that \( W \) is a Coxeter group. These Coxeter-Catalan objects include \( W \)-nonnesting partitions, \( W \)-clusters, sortable elements or facets of certain subword complexes, see \([1,21,26,27]\).

In this article we provide a new and simple proof of the lexicographic shellability of the lattice \( \mathcal{NC}_W \), where \( W \) is some well-generated complex reflection group.
The main motivation for the study of the lexicographic shellability of posets comes from its deep topological impact. The order complex of a lexicographically shellable poset is shellable and hence Cohen-Macaulay. Moreover, it is homotopy equivalent to a wedge of spheres, and we can compute the Betti numbers of this poset from the labeling. Another nice property of such posets is a certain “connectedness”. More precisely, whenever we have two maximal chains \( C \) and \( C' \) in a lexicographically shellable poset that differ in more than one element, then we can find a sequence of maximal chains starting with \( C \) and ending with \( C' \) such that two consecutive chains in this sequence differ by exactly one element. In the present situation of \( NC_W \), this connectedness implies the transitivity of the Hurwitz action on the set of reduced words of any Coxeter element. This connection will be made precise in an upcoming article. For other proofs of the transitivity of the Hurwitz action, see [5, Proposition 7.5] or [3] and the references given there.

The lexicographic shellability of \( NC_W \) was shown in type \( A \) by Björner and Edelman [11], in type \( B \) by Reiner [28], and uniformly for all Coxeter groups by Athanasiadis, Brady and Watt [2]. The latter paper also introduced the notion of a reflection order compatible with a Coxeter element, and showed that these orders have a close connection to the lexicographic shellability of \( NC_W \). For the remaining well-generated complex reflection groups, the lexicographic shellability of \( NC_W \) was shown case-by-case in [25]. Again the crucial tool was a certain total order on the reflections that satisfies a compatibility condition similar to that used in [2].

The results mentioned in the previous paragraph were obtained by using a special form of lexicographic shellability, called EL-shellability. In this article we consider a slightly different form of lexicographic shellability, called CL-shellability. This concept was introduced in [13,14]. It was shown in [14] that EL-shellability implies CL-shellability, and that all topological properties of EL-shellable posets hold as well for CL-shellable posets. However, it is not known whether CL-shellability is indeed a weaker concept than EL-shellability. The advantage of CL-shellability is that it admits a recursive formulation in terms of a certain total order on the atoms of the poset in consideration. Our main theorem states that every reflection order of \( W \) that is compatible with some group element \( x \in W \), see Definition 3.1 below, is indeed a recursive atom order of the absolute order interval between the identity and \( x \).

**Theorem 1.1.** Let \( W \) be a complex reflection group, let \( e \in W \) denote the identity and let \( x \in W \). Let \( T_x \) denote the set of reflections of \( W \) lying below \( x \) in absolute order. Every \( x \)-compatible reflection order of \( T_x \) is a recursive atom order of the interval \([e,x]\) in absolute order.

We will see later on that the proof of Theorem 1.1 requires only few simple properties of complex reflection groups, mainly that the set of reflections is closed under conjugation. In particular, we want to emphasize that the proof of Theorem 1.1 is uniform, and its core idea can be generalized to any group with a distinguished generating set that is closed under conjugation. This generalization will be addressed in a future paper.
It was shown in [2,25] that compatible reflection orders exist for any Coxeter element in any well-generated complex reflection group, see also Proposition 3.4 below. (Again this is known uniformly for Coxeter groups.) Thus the desired result on the lexicographic shellability of NCW is an immediate consequence of the main result in [14].

Theorem 1.2. The lattice NCW(γ) is CL-shellable for every well-generated complex reflection group W and every Coxeter element γ ∈ W.

In Section 2 we recall the necessary definitions. In particular we recall the definition of recursive atom orders and CL-shellability (Section 2.1), we briefly introduce complex reflection groups (Section 2.2), and we recall the definition of noncrossing partitions (Section 2.3). In Section 3, we prove Theorems 1.1 and 1.2. We conclude this article with a short example in Section 4.

2. Preliminaries

An extensive introduction to complex reflection groups is [24], and a recent exposition on the theory of Coxeter elements in well-generated complex reflection groups is [29]. The seminal work on CL-shellable posets and recursive atom orders is [14].

2.1. Recursive Atom Orders and CL-Shellability. Let $\mathcal{P} = (P, \leq)$ be a bounded poset, i.e. a poset with a least element $\hat{0}$ and a greatest element $\hat{1}$. A chain of $\mathcal{P}$ is a totally ordered subset $C \subseteq P$, i.e. we can uniquely write $C = \{p_1, p_2, \ldots, p_s\}$ with $p_1 < p_2 < \cdots < p_s$. A cover relation (or an edge) of $\mathcal{P}$ is a pair $(p, q)$ with $p, q \in P$ such that $p < q$ and there is no element $x \in P$ with $p < x < q$. In this case we usually write $p < q$. An element $p \in P$ with $\hat{0} \ll p$ is called an atom of $\mathcal{P}$. A maximal chain of $\mathcal{P}$ is a chain $C = \{p_1, \ldots, p_s\}$ such that $\hat{0} = p_1 < p_2 < \cdots < p_s = \hat{1}$, and we denote by $\mathcal{M}(\mathcal{P})$ the set of maximal chains of $\mathcal{P}$. The poset $\mathcal{P}$ is graded if all maximal chains have the same cardinality, and in this case the length of $\mathcal{P}$ is the cardinality of a maximal chain minus one.

Let $\mathcal{E}^*(\mathcal{P}) = \{(C; p, q) \ | \ C \in \mathcal{M}(\mathcal{P}), p, q \in C, p < q\}$. A chain-edge labeling of $\mathcal{P}$ is a map $\lambda : \mathcal{E}^*(\mathcal{P}) \to \mathbb{N}$ such that the following condition is satisfied: if two maximal chains coincide along their first $d$ edges, then their labels also coincide along these edges. A rooted interval of $\mathcal{P}$ is a pair $([p, q], R)$, where $[p, q] = \{x \in P \ | \ p \leq x \leq q\}$, and where $R$ is a maximal chain from $\hat{0}$ to $p$, the so-called root of $[p, q]$. A maximal chain $C = \{p_1, \ldots, p_s\}$ in a rooted interval $([p, q], R)$ is increasing (with respect to $\lambda$) if the tuple

$$
\lambda(R; C) = (\lambda(R \cup C; p_1, p_2), \lambda(R \cup C; p_2, p_3), \ldots, \lambda(R \cup C; p_{s-1}, p_s))
$$

is strictly increasing. Moreover, if $C'$ is another maximal chain in $([p, q], R)$, then $C$ precedes $C'$ (with respect to $\lambda$) if $\lambda(R; C)$ is lexicographically smaller than $\lambda(R; C')$.

Definition 2.1 ([14, Definition 2.2]). Let $\mathcal{P}$ be a graded poset. A chain-edge labeling is a CL-labeling of $\mathcal{P}$ if for every rooted interval $([p, q], R)$ of $\mathcal{P}$ there exists a unique increasing maximal chain $C$ in $([p, q], R)$, and this chain precedes every other maximal chain in this rooted interval.
Figure 1. A bounded poset that admits a CL-labeling.

Consequently we call a graded poset \textit{CL-shellable} if it admits a CL-labeling. Figure 1 shows a poset with a CL-labeling. Observe that there are two different labels assigned to the edge $c \preccurlyeq 1$ depending on the maximal chain it lies on. It was shown in [14, Proposition 2.3] that the order complex of a CL-shellable poset is shellable, and hence Cohen-Macaulay. In particular it is homotopy equivalent to a wedge of spheres. It was also shown in [14] that there is an equivalent, recursive formulation of CL-shellability. Given any total order $\prec$ of the atoms of a bounded graded poset $P$, define for each atom $a$ of $P$ the set

$$F_\prec(a) = \{ x \in P \mid a \prec x \text{ and there exists some } a' \prec a \text{ with } a' \prec x \}.$$  

\textbf{Definition 2.2 ([14, Definition 3.1])}. Let $P = (P, \leq)$ be a graded poset with greatest element $\hat{1}$, and let $a_1, a_2, \ldots, a_s$ denote its atoms. $P$ is said to admit a \textit{recursive atom order} if and only if either $P$ has length 1 or there exists a total order $a_1 \prec a_2 \prec \cdots \prec a_s$ that satisfies the following conditions.

\begin{itemize}
  \item[(R1)] For all $j \in \{1, 2, \ldots, s\}$, the interval $[a_j, \hat{1}]$ admits a recursive atom order $\sqsubset a_j$ such that there are no two atoms $x, x'$ of $[a_j, \hat{1}]$ with $x \sqsubset x'$ and $x \in F_\prec(a_j)$, but $x' \notin F_\prec(a_j)$.
  \item[(R2)] For all $i, j \in \{1, 2, \ldots, s\}$ and any $y \in P$ with $i < j$ and $a_i, a_j \leq y$, there exists some $k < j$ and some $z \leq y$ with $a_k, a_j \prec z$.
\end{itemize}

Any such order is called a \textit{recursive atom order}.

\textbf{Theorem 2.3 ([14, Theorem 3.2])}. A graded poset admits a recursive atom order if and only if it is CL-shellable.

2.2. \textbf{Complex Reflection Groups}. Let $V$ be an $n$-dimensional complex vector space and let $U(V)$ denote the group of unitary transformations on $V$. A \textit{complex reflection} is a unitary transformation on $V$ that has finite order and fixes a hyperplane pointwise. A \textit{complex reflection group} is a finite subgroup $W \leq U(V)$ generated by complex reflections. If $W$ can be realized as a group of transformations acting on a real vector space, then $W$ is a real reflection group. The first classification of real reflection groups was given by Coxeter in [20], and we thus refer to these groups as \textit{Coxeter groups} rather than real reflection groups. The group $W$ is \textit{irreducible} if it does not fix a proper subspace of $V$, and it is \textit{well-generated} if it can be generated by $n$ reflections. In particular, any Coxeter group is well-generated.
By definition, any \( w \in W \) can be written as a product of reflections. Let \( T \) denote the set of all reflections of \( W \). We define the absolute length on \( W \) by

\[
\ell_T : W \to \mathbb{N}, \quad w \mapsto \min \{ k \mid w = t_1 t_2 \cdots t_k \text{ where } t_j \in T \text{ for } 1 \leq j \leq k \}
\]

and we define the absolute order on \( W \) by

\[
u \preceq_T v \quad \text{if and only if} \quad \ell_T(v) = \ell_T(u) + \ell_T(u^{-1}v).
\]

If \( \ell_T(w) = k \), then a product \( w = t_1 t_2 \cdots t_k \), where \( t_j \in T \) for \( j \in \{1, 2, \ldots, k\} \), is a reduced \( T \)-decomposition of \( w \). Since \( T \) is closed under conjugation [24, Lemma 1.9], it follows easily that \( u \preceq_T v \) implies that any reduced \( T \)-decomposition of \( u \) is a prefix of some reduced \( T \)-decomposition of \( v \).

In [30] the irreducible complex reflection groups were completely classified. There exists one infinite family, indexed by three parameters \( d, e, n \) where \( n \) denotes the dimension of \( V \) and \( d \) and \( e \) are integers such that \( e \) divides \( d \), as well as 34 exceptional groups, denoted by \( G_4, G_5, \ldots, G_{37} \). This classification contains Coxeter’s classification of the real reflection groups from [20]. In particular we have the following isomorphisms, see [24, Example 2.11] or [19]:

- the group \( G(1,1,n) \) for \( n \geq 2 \) is isomorphic to the Coxeter group \( A_{n-1} \),
- the group \( G(2,1,n) \) for \( n \geq 2 \) is isomorphic to the Coxeter group \( B_n \),
- the group \( G(2,2,n) \) for \( n \geq 4 \) is isomorphic to the Coxeter group \( D_n \),
- the group \( G(d,d,2) \) for \( d \geq 3 \) is isomorphic to the Coxeter group \( I_2(d) \).

Moreover, the exceptional irreducible Coxeter groups can be found among the 34 exceptional complex reflection groups. The irreducible well-generated complex reflection groups are \( G(d,1,n) \) and \( G(d,d,n) \) for \( d, n \geq 2 \), as well as \( G_4, G_5, G_6, G_8, G_9, G_{10}, G_{14}, G_{16}, G_{17}, G_{18}, G_{20}, G_{21}, G_{23}, G_{24}, G_{25}, G_{26}, G_{27}, G_{28}, G_{29}, G_{30}, G_{32}, G_{33}, G_{34}, G_{35}, G_{36} \), and \( G_{37} \).

2.3. Coxeter Elements and Noncrossing Partitions. For every irreducible complex reflection group \( W \) there exists a set of basic invariants, called the degrees of \( W \). See for instance [29, Section 1] for a precise definition. If \( W \) is irreducible well-generated, then the largest degree is called the Coxeter number of \( W \), and it is usually denoted by \( h \). According to [33], an element \( w \in W \) is regular if it has an eigenvector \( v \) that does not lie in one of the reflection hyperplanes of \( W \). If \( \zeta \) is the eigenvalue of \( w \) with respect to \( v \), then we say that \( w \) is \( \zeta \)-regular. If \( \zeta \) is of order \( d \), then it follows from [33, Theorem 4.2] that the order of \( w \) is also \( d \), and we call \( d \) a regular number. If \( W \) is irreducible and well-generated, then it follows from [23, Theorem C] that the Coxeter number \( h \) is always a regular number.

**Definition 2.4 ([29, Definition 1.1]).** Let \( W \) be an irreducible well-generated complex reflection group. A Coxeter element is a regular element of \( W \) that has order \( h \).

In the remainder of this article, unless otherwise stated, let \( W \) denote an irreducible well-generated complex reflection group, let \( \varepsilon \) denote the identity of \( W \), and let \( \gamma \) denote a Coxeter element of \( W \). Let

\[
NC_W(\gamma) = \{ w \in W \mid \varepsilon \preceq_T w \preceq_T \gamma \}
\]

be the set of \( W \)-noncrossing partitions. The poset \( NC(W,\gamma) = (NC_W(\gamma), \preceq_T) \) is called the lattice of \( W \)-noncrossing partitions. The fact that this poset is indeed a
lattice was shown by a collaborative effort of several authors, see [4–6, 15, 17]. In [18] Brady and Watt gave a uniform proof of the lattice property of $\mathcal{NC}_W(\gamma)$ for the Coxeter groups. Moreover, $\mathcal{NC}_W(\gamma)$ is graded and self-dual. For more properties of the lattice $\mathcal{NC}_W(\gamma)$ we refer for instance to [29] and the references given there. The next result states that the structure of the lattice $\mathcal{NC}_W(\gamma)$ does not depend on the choice of Coxeter element $\gamma$.

**Proposition 2.5** ([29, Corollary 1.6]). Let $W$ be an irreducible well-generated complex reflection group and let $\gamma, \gamma' \in W$ be two Coxeter elements. The posets $\mathcal{NC}_W(\gamma)$ and $\mathcal{NC}_W(\gamma')$ are isomorphic.

**Remark 2.6.** In fact, Coxeter elements and noncrossing partition lattices are so far only defined for irreducible well-generated complex reflection groups. However, if $W$ is reducible, then by definition we can write $W \cong W_1 \times W_2 \times \cdots \times W_k$, where for each $i \in \{1,2,\ldots,k\}$ the factor $W_i$ is an irreducible well-generated complex reflection group. For $i \in \{1,2,\ldots,k\}$ let $\gamma_i$ denote a Coxeter element of $W_i$. The product $\gamma = \gamma_1 \gamma_2 \cdots \gamma_k$ can be taken as a substitute for a Coxeter element of $W$, and it follows then immediately that

$$\mathcal{NC}_W(\gamma) \cong \mathcal{NC}_{W_1}(\gamma_1) \times \mathcal{NC}_{W_2}(\gamma_2) \times \cdots \times \mathcal{NC}_{W_k}(\gamma_k),$$

where $\mathcal{NC}_W(\gamma)$ denotes the interval $[\varepsilon, \gamma]$ in $(W, \leq_T)$.

### 3. Proof of the Main Theorems

The definition of a reflection order is standard in the theory of Coxeter groups, see for instance [12, Section 5.2]. In [2, Definition 3.1] a more specialized notion was introduced, namely that of a reflection order compatible with a Coxeter element. However, this notion was defined using the root system of a Coxeter group. We generalize this concept by dropping the dependence on a root system, and hence allowing a generalization to all complex reflection groups.

**Definition 3.1.** Let $W$ be a complex reflection group, and let $x \in W$. Let $T_x$ denote the set of reflections of $W$ lying below $x$ in absolute order. A total order $\prec$ of $T_x$ is an $x$-compatible reflection order if for every $w \leq_T x$ with $\ell_T(w) = 2$ there exists a unique reduced $T$-decomposition $w = rt$ with $r < t$.

A reduced $T$-decomposition $w = rt$ is increasing if $r \prec t$. It is immediate that for any $w \leq_T x$ the restriction of an $x$-compatible reflection order of $T_x$ to $T_w$ is a $w$-compatible reflection order.

**Lemma 3.2.** Let $W$ be a complex reflection group, and let $x \in W$. Let $w \leq_T x$ with $\ell_T(w) = 2$, and let $\prec$ denote the restriction of an $x$-compatible reflection order of $T_x$ to $T_w$. If $w = rt$ is the unique increasing reduced $T$-decomposition of $w$, then $r$ is minimal and $t$ is maximal with respect to $\prec$.

**Proof.** Let $r_{\min}$ denote the minimal reflection below $w$ with respect to $\prec$. By definition there exists a reduced $T$-decomposition $w = r_{\min}t_1$ for some $t_1 \in T_w$. Since $r_{\min}$ is minimal it follows that $r_{\min} \prec t_1$ and hence $r = r_{\min}$. Now let $r_{\max}$ denote the maximal reflection below $w$ with respect to $\prec$. Again, by definition, there exists a reduced $T$-decomposition $w = r_{\max}t_2$ for some $t_2 \in T_w$. Since $T$ is closed under conjugation there exists another reduced $T$-decomposition $w =$
Lemma 3.3. Let $W$ be a complex reflection group, and let $u,v \in W$ with $u \leq_T v$. The map $f(x) = ux$ is a poset isomorphism from $[e, u^{-1}v]$ to $[u, v]$.

Proof. Let $x \in W$. By definition, we have

$$\ell_T(v) = \ell_T(u) + \ell_T(u^{-1}v) \leq \ell_T(u) + \ell_T(x^{-1}u^{-1}v) \geq \ell_T(ux) + \ell_T((ux)^{-1}v) \geq \ell_T(v).$$

Hence we have $x \leq_T u^{-1}v$ if and only if $ux \leq_T v$ as desired. \qed

Now we are ready to prove Theorem 1.1.

of Theorem 1.1. We proceed by induction on $\ell_T(x)$. If $\ell_T(x) \leq 2$, then the claim is trivially true. So let $\ell_T(x) > 2$, and suppose that the claim is true for all elements $w <_T x$. Let $<$ be a $x$-compatible reflection order of $T_x$, and label the elements of $T_x$ accordingly, i.e. $T_x = \{t_1, t_2, \ldots, t_N\}$ with $t_i < t_j$ if and only if $i < j$. In what follows “minimal” and “maximal” are always used with respect to $<$.

First we show that $<$ satisfies (R1). Fix $t_j \in T_x$. In view of Lemma 3.3 it follows that $[t_j, x] \cong [e, t_j^{-1}x]$, and we write $w = t_j^{-1}x$. By induction hypothesis we conclude that the isomorphism from Lemma 3.3 yields a recursive atomic order of $[t_j, x]$, which we denote by $\sqsubset$. Let $a_1, a_2, \ldots, a_s$ denote the atoms of $[t_j, x]$ indexed increasingly with respect to $\sqsubset$. Let $F(t_j)$ denote the set of atoms of $[t_j, x]$ that cover some $t_l \in T_x$ with $t_l < t_j$, as defined in (1). We need to show that the elements in $F(t_j)$ come first in $\sqsubset$, and we proceed by contradiction. Suppose that there are indices $k, l \in \{1, 2, \ldots, s\}$ such that $a_k \sqsubset a_l$ but $a_k \notin F(t_j)$ and $a_l \in F(t_j)$.

In particular, there exists some $t_i \in T_x$ with $t_i <_T a_l$ and $t_i < t_j$. Since $t_j <_T a_k$ we can write $a_k = t_j r$ for some $r \in T_x$, and since $a_k \notin F(t_j)$ we conclude $t_j < r$. Analogously, since $t_j <_T a_l$ we can write $a_l = t_j r'$ for some $r' \in T_x$. It follows from $t_i < r$ that $t_j$ is not the minimal reflection below $a_l$, and hence Lemma 3.2 implies $r' < t_j$. Now $a_k \sqsubset a_l$ implies together with Lemma 3.3 that $t_j^{-1}a_k < t_j^{-1}a_l$. This can be summarized as

$$t_j < r = (t_j^{-1}t_j)r = t_j^{-1}a_k < t_j^{-1}a_l = (t_j^{-1}t_j)r' = r' < t_j,$$

which is a contradiction. Hence (R1) is satisfied.

Now we show that $<$ satisfies (R2). Fix $t_i, t_j \in T_x$ with $t_i < t_j$, and pick some $w \leq_T x$ with $t_i, t_j \leq_T w$. If $w <_T x$, then the claim follows from the induction hypothesis. Thus let $w = x$, and assume the following.

(A) For every $r \in T_x$ with $r < a_j$ there does not exist an element $z \leq_T x$ with $r, a_j \leq_T z$.

Moreover choose $t_j$ maximal with respect to this property. Define

$$C = \{ t \in T_x \mid \text{there exists some } z \leq_T x \text{ with } t, t_j \leq_T z \}.$$

Since the interval $[e, x]$ is graded there exists some $z \leq_T x$ with $z = t_j r$ for some $r \in T_x$. Hence $C$ is not empty, and (A) implies $j < N$. Let $t$ be the minimal reflection in $C$. We have $t_j < t$ by (A). Since $t \in C$ we can find some $y \leq_T x$
with \( t, t_j \prec_T y \), and we can write \( y = tr \) for some \( r \in T_x \). (This \( r \) is not necessarily related to the \( r \) used above).

If \( t \prec r \), then this is the only increasing reduced \( T \)-decomposition of \( y \) since \( \prec \) is \( x \)-compatible. However, we can also write \( y = t_t' \) for some \( t' \prec_T y \), and the previous implies \( r' \prec t_j \), which contradicts (A). Thus \( r \prec t \), which implies with the minimality of \( t \) that \( r \notin \mathcal{C} \). The only reflection below \( y \) that is not in \( \mathcal{C} \) is \( t_j \), which implies \( r = t_j \). Hence we have \( y = tt_j \). Let \( \sqsubset \) denote the total order of the atoms of \( [t, x] \), which is induced by \( \prec \) under the isomorphism from Lemma 3.3. Further let \( y' \leq_T x \) satisfy \( t, t_j \leq_T y' \). (Such an element exists since \([e, x]\) is bounded.) We distinguish two cases.

(i) Assume \( \ell_T(y') < \ell_T(x) \). By induction hypothesis, we can find some reflection \( r \leq_T y' \) with \( r \prec t \) and some \( z \leq_T x \) with \( r, t \leq_T z \leq_T y' \). We write \( z = tr' \). Since \( t_j \prec t \), it follows that \( t \) is not minimal below \( z \). Thus Lemma 3.2 implies \( r' \prec t \). See Figure 2 for an illustration of the following three cases.

(a) Assume \( r' = t_j \). It follows that \( z = tt_j = y \). Thus we have \( t_i \leq_T y' \) and \( t_j \leq_T z \leq_T y' \). Since \( y' \prec x \), we can find some \( s \prec t_j \) and some \( q \leq_T x \) such that \( s, t_j \prec_T q' \) by induction hypothesis. This contradicts (A).

(b) Assume \( r' \neq t_j \). It follows that \( z \sqsubset y \). By induction hypothesis we can find some atom \( s' \in [t, x] \) with \( s' \sqsubset y \), and some \( q' \leq_T x \) with \( s', y \prec_T q' \). In particular, we can write \( s' = ts \) and \( q' = tz \). Now Lemma 3.3 implies \( s \prec t_j \) and \( s, t_j \prec_T x \). Hence \( s \in \mathcal{C} \), which contradicts (A).

(c) Assume \( t_j \prec r' \). By induction hypothesis and by (A) \( t_j \) is the minimal reflection below \( t^{-1}x \). Let \( s \) denote the second smallest reflection below \( t^{-1}x \). By induction hypothesis there exists an element \( q \leq_T t^{-1}x \) with \( t_j, s \prec_T q \). Thus \( s \in \mathcal{C} \) and hence \( t \leq s \). Since \( r' = t^{-1}z \leq_T t^{-1}x \) and \( r' \neq t_j \) we conclude \( s \leq r' \prec t \leq s \), which is a contradiction.

(ii) Assume \( \ell_T(y') = \ell_T(x) \). If for every \( r \prec t \) there does not exist some \( z \leq_T x \) with \( r, t \leq_T z \), then since \( t_j \prec t \), we obtain a contradiction to the maximality of \( t_j \). Hence we can find such \( r \) and \( z \). Once more we can write \( z = tr' \), and since \( t \) is not minimal below \( z \), Lemma 3.2 implies \( r' \prec t \). Now we obtain a contradiction analogously to case (i).

Hence in both cases we obtain a contradiction, which implies that (A) is false. We can thus find some \( r \in T_x \) with \( r \prec t_j \) and some element \( z \leq_T x \) with \( r, t_j \leq_T z \leq_T x \) as desired. Hence \( \prec \) satisfies (R2), and we are done. \( \square \)

The converse of Theorem 1.1 does not hold, i.e. there are recursive atom orders of \([e, x]\) that are not \( x \)-compatible. Consider for instance \( W = G(3,3,2) \), namely the dihedral group of order 6, or equivalently the symmetric group on \( \{1,2,3\} \). Let \( t_1, t_2 \) and \( t_3 \) denote its reflections. We can interpret these reflections as transpositions, say \( t_1 = (1 \ 2), t_2 = (2 \ 3) \) and \( t_3 = (1 \ 3) \). Consider the Coxeter element \( \gamma = t_1t_2 \), which admits three reduced \( T \)-decompositions, namely

\[
\gamma = t_1t_2 = t_2t_3 = t_3t_1.
\]

We notice that among the six total orders of \( T_\gamma = \{t_1, t_2, t_3\} \) exactly three are \( \gamma \)-compatible, namely

\[
t_1 \prec t_3 \prec t_2, \quad t_2 \prec t_1 \prec t_3, \quad t_3 \prec t_2 \prec t_1.
\]
while the other three total orders yield two increasing reduced $T$-decompositions of $\gamma$. However, since $\mathcal{N}C_{G(3,3,2)}(\gamma)$ has rank 2, every total order of $T_\gamma$ is a recursive atom order of $\mathcal{N}C^\alpha_{G(3,3,2)}(\gamma)$.

Note that Theorem 1.1 is true for any element in any well-generated complex reflection group. In fact, this statement can be generalized to any group with a distinguished conjugation-closed set of generators. This generalization will be addressed in a follow-up paper. In general, however, not every element of a well-generated complex reflection group admits a compatible reflection order.
Consider for instance the group $G(2, 1, 2)$, which is isomorphic to the hyperoctahedral group of rank 2. It can thus be realized as the group of permutations $\pi$ of $\{1, 2, -1, -2\}$ that satisfy $\pi(-i) = -\pi(i)$ for $i \in \{1, 2\}$. The reflections of this group are

$$t_1 = (1 \, 2)(-1 \, -2), \quad t_2 = (1 \, -2)(-1 \, 2), \quad t_3 = (1 \, -1), \quad t_4 = (2 \, -2),$$

and the remaining elements are

$$\varepsilon = (1), \quad w_1 = (1 \, -1)(2 \, -2), \quad w_2 = (1 \, -2 \, -1 \, 2), \quad w_3 = (1 \, 2 \, -1 \, -2).$$

We have $w_1^2 = w_2^4 = w_3^4 = \varepsilon$. According to \cite[Table 1]{2} the Coxeter number of $G(2, 1, 2)$ is 4. Hence $w_1$ cannot be a Coxeter element since its order is only 2. We have the following reduced decompositions of $w_1$:

$$w_1 = t_1 t_2 = t_2 t_1 = t_3 t_4 = t_4 t_3,$$

and we see that there exists no total order of $\{t_1, t_2, t_3, t_4\}$ that yields only one increasing reduced decomposition of $w_1$.

The next result states, however, that any Coxeter element in any well-generated complex reflection group admits a compatible reflection order.

**Proposition 3.4.** Let $W$ be a well-generated complex reflection group, and let $\gamma \in W$ be a Coxeter element. There exists a $\gamma$-compatible reflection order of $T_\gamma$.

**Proof.** Suppose first that $W$ is irreducible. Theorem 4.1 in \cite{2} states (uniformly) that for every Coxeter group there exists a Coxeter element $\gamma$ such that we can find a $\gamma$-compatible reflection order in the sense of \cite[Definition 3.1]{2}. One consequence of \cite[Definition 3.1]{2} is that for any length-2 element $w \leq_T \gamma$ there exists a unique increasing reduced $T$-decomposition $w = rt$, where $r$ and $t$ are the reflections corresponding to the simple roots of the respective induced rank-2 root system. Hence any $\gamma$-compatible reflection order in the sense of \cite[Definition 3.1]{2} is also $\gamma$-compatible in our sense. In view of Proposition 2.5 we conclude that the claim is true for all irreducible Coxeter groups.

It is well known that $NC_{G(d,1,n)} \cong NC_{B_n}$, see for instance \cite[p. 42]{6} or \cite[Proposition 4.1]{25}. Hence in view of the previous paragraph, we conclude that the claim is true for the groups $G(d, 1, n)$, where $d, n \geq 2$.

If $W = G(d, d, n)$ for $d, n \geq 3$, then \cite[Lemma 3.15]{25} implies the claim together with Proposition 2.5.

In the case where $W$ is an exceptional well-generated complex reflection group, the claim has been checked in \cite[Section 4]{25} by computer.

Now suppose that $W$ is reducible. This means that we can write $W \cong W_1 \times W_2 \times \cdots \times W_k$, where for each $i \in \{1, 2, \ldots, k\}$ the factor $W_i$ is an irreducible well-generated complex reflection group. In view of Remark 2.6, we can write

$$NC_W(\gamma) \cong NC_{W_1}(\gamma_1) \times NC_{W_2}(\gamma_2) \times \cdots \times NC_{W_k}(\gamma_k),$$

where $\gamma_i$ is a Coxeter element of $W_i$ for $i \in \{1, 2, \ldots, k\}$. In view of the first part of this proof we can find a $\gamma_i$-compatible reflection order for each $i \in \{1, 2, \ldots, k\}$. If we concatenate these orders, then we clearly obtain a $\gamma$-compatible reflection order for $W$. \qed
Proposition 3.4 implies that for every well-generated complex reflection group $W$ and every Coxeter element $\gamma \in W$ there exists a $\gamma$-compatible reflection order of $T_\gamma$. Theorem 1.1 implies that this order is a recursive atom order of $\mathcal{N}_W(\gamma)$, and Theorem 2.3 implies that $\mathcal{N}_W(\gamma)$ is CL-shellable. \qed

We wish to emphasize that the only obstruction for the uniformity of the proof of Theorem 1.2 is its dependence on Proposition 3.4. In other words: if we can find a uniform definition of a $\gamma$-compatible reflection order of $W$ for some “uniform” choice of $\gamma$, we would immediately obtain a uniform proof of Theorem 1.2.

4. An Example

We illustrate Theorem 1.1 on the complex reflection group $G(1,1,4)$, which is isomorphic to the symmetric group on $\{1,2,3,4\}$. The reflections in $G(1,1,4)$ are the transpositions $(i \ j)$ for $1 \leq i < j \leq 4$, and we choose the long cycle $\gamma = (1 \ 2 \ 3 \ 4)$ as a Coxeter element. The lattice $\mathcal{N}^*_G(1,1,4)(\gamma)$ is shown in Figure 3. We consider the lexicographic order on the transpositions, namely

$$(1 \ 2) \prec (1 \ 3) \prec (1 \ 4) \prec (2 \ 3) \prec (2 \ 4) \prec (3 \ 4).$$

This is indeed a $\gamma$-compatible reflection order, since we have the following reduced $T$-decompositions of the permutations of length 2:

$$(1 \ 2 \ 3) = (1 \ 2)(2 \ 3) = (2 \ 3)(1 \ 3) = (1 \ 3)(1 \ 2),$$

$$(2 \ 3 \ 4) = (2 \ 3)(3 \ 4) = (3 \ 4)(2 \ 4) = (2 \ 4)(2 \ 3),$$

$$(1 \ 3 \ 4) = (1 \ 3)(3 \ 4) = (3 \ 4)(1 \ 4) = (1 \ 4)(1 \ 3),$$

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$$(1 \ 2)(3 \ 4) = (1 \ 2)(3 \ 4) = (3 \ 4)(1 \ 2)$$

$$(1 \ 4)(2 \ 3) = (1 \ 4)(2 \ 3) = (2 \ 3)(1 \ 4),$$

and in each of these cases the first decomposition is the unique increasing one with respect to $\prec$.  

If we consider for instance \( t = (1\ 4) \), then the atoms in the interval \([t, \gamma]\) are 
\( a_1 = (1\ 2\ 4), a_2 = (1\ 3\ 4) \) and \( a_3 = (1\ 4)(2\ 3) \). The induced order under the isomorphism from Lemma 3.3 is 
\[
(1\ 2\ 4) \sqsubset (1\ 3\ 4) \sqsubset (1\ 4)(2\ 3).
\]
Let \( F_t(a_i) \) denote the set of reflections that are smaller than \( t \) (with respect to \( \prec \)) and that are covered by \( a_i \). We have 
\[
F_t(a_1) = \{(1\ 2)\}, \quad F_t(a_2) = \{(1\ 3)\}, \quad F_t(a_3) = \emptyset.
\]
Hence the atoms of \([t, \gamma]\) for which \( F_t \) is not empty come first in \( \sqsubset \). Since the interval \([t, \gamma]\) has rank 2 it follows that it trivially satisfies (R1). We can quickly check that the same is true for the other reflections.

There are only two reflections \( r \) and \( t \) that are not covered by a common element, namely \( r = (1\ 3) \) and \( t = (2\ 4) \). We have \( r \prec t \), and if we take \( t' = (2\ 3) \), then we have \( t' \prec t \) and \( t', t \vartriangleleft (2\ 3\ 4) \). Hence (R2) is satisfied.

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