Estimation of coefficient of rolling friction by the evolvent pendulum method

S Alaci¹, F C Ciornei¹, A Ciogole¹ and M C Ciornei²

¹“Stefan cel Mare” University of Suceava, Mechanics and Technologies Department, Str. Universitatii no.13, 720229, Suceava, Romania
²“Carol Davila” University of Medicine and Pharmacy, Department 2 Physiology I, Bd. ErolilorSanitari no.8, Bucuresti, Romania

E-mail: alaci@fim.usv.ro

Abstract. The paper presents a method for finding the coefficient of rolling friction using an evolvent pendulum. The pendulum consists in a fixed cylindrical body and a mobile body presenting a plane surface in contact with a cylindrical surface. The mobile body is placed over the fixed one in an equilibrium state; after applying a small impulse, the mobile body oscillates. The motion of the body is video recorded and afterwards the movie is analyzed by frames and the decrease with time of angular amplitude of the pendulum is found. The equation of motion is established for oscillations of the mobile body. The equation of motion, differential nonlinear, is integrated by Runge–Kutta method. Imposing the same damping both to model’s solution and to theoretical model, the value of coefficient of rolling friction is obtained. The last part of the paper presents results for actual pairs of materials. The main advantage of the method is the fact that the dimensions of contact regions are small, of order a few millimeters, and thus is substantially reduced the possibility of variation of mechanical characteristic for the two surfaces.

1. Introduction

Three types of friction may occur between two bodies: sliding, spinning and rolling, the last happening only in higher pairs when the contact between the boundary surfaces of the contacting bodies is a concentrated (Hertzian) contact. In order to characterize the rolling friction, finding the rolling friction torque is required, [1–4]. For the general case this is an extremely difficult task due to complex motion from higher pair, on one side, and to the surfaces shape of contacting bodies, on the other side. To overcome these difficulties, bodies with simple geometries are involved, between which plane-parallel motion exists. The most usual pair of surfaces is the cylinder and plane pair [5]. When between the two bodies the plane-parallel motion is present, the motion can be studied in a plane normal to the axis of rotation, as the relative motion between a circle and a straight line. The higher pairs are irreversible. Thus, if the straight line is maintained immobile and the circle is mobile, the points from the circle describe a family of cycloids, and the pendulum is a cycloid pendulum [6]. The method is applicable for studying materials with significant gradients of elastic characteristics, when regions implied in tests require small dimensions. A further example refers to biomechanics, for finding the tribological characteristics of biological tissues with complex geometry, like teeth or bones [7–10].
2. Pendulum dynamic equation

The contact between a circle and a straight line is considered. Maintaining fixed the circle, a basis circle \( C_b \), and obliging the straight line \( \Delta \) to roll without sliding over the fixed circle, Figure 1, the points from the straight line will describe a family of evolvents (common—the point on the circle, extended—the point outside the circle and shortened –the points inside the circle [11]). Based on Figure 1, the equation of motion of the pendulum is deduced under the hypothesis of pure rolling and proportionality between normal force \( N \) and the moment of rolling friction \( M_r \).

![Figure 1. Principle of evolvent pendulum](image1)

![Figure 2. Common, extended and shortened involute generated using the same basic circle](image2)

It is considered that the contact between the straight line and the circle is initially in \( B_0 \) point. At a certain time, after the straight line rolled in trigonometric direction, the contact point is in \( B \), whose position is précised by the polar angle \( \varphi \). A ground frame of reference is chosen with the origin in the centre of the base circle and a moving coordinate system, attached to the straight line, with the origin in the initial contact point, the current position of the moving system is given by the polar radius \( \overline{OO'} \) and polar angle \( \varphi \). The pure rolling condition between the straight line \( \Delta \) and the base circle \( C_b \) is obtained by perceiving that the length of the arch \( B_0B \) should be equal to the length of segment, \( BO' \) mathematically expressed as:

\[
r_b \varphi = r_b \tan(\varphi - \theta)
\]

Additionally:

\[
\rho = r_b / \cos(\varphi - \theta)
\]

Considering that the pendulum is symmetric, the centre of mass \( G \) will be positioned on the axis of symmetry and having in the attached reference frame the coordinates \((0, y'_G)\). The vector relation:

\[
r_G = \overline{OO'} + y'_G \mathbf{j}
\]

is projected on the axis of stationary coordinate system. The coordinates of the centre of mass of the pendulum are:
\[
\begin{align*}
x_G &= -(y'_G + r_b \sin \varphi + r_b \varphi \cos \varphi) \\
y_G &= -(y'_G + r_b \cos \varphi + r_b \varphi \sin \varphi)
\end{align*}
\] (4)

The equations (4) describe the common, \( y'_G = 0 \), extended \( y'_G > 0 \) and shortened \( y'_G < 0 \) involute, as in Figure 2.

To figure out the equations of motion, the mass of the pendulum is symbolized by \( M \) and the moment of inertia with respect to a centric axis normal to the plane of motion is denoted by \( J_z \). The only external force action upon the pendulum is its own weight. In the contact point there are acting:

- The normal reaction \( N \), on the direction of radius of contact point;
- The tangential reaction \( T \), tangent in the contact point;
- The rolling friction torque, \( M_r \), normal to the plane of motion and with the sense opposite to angular velocity \( \omega = \dot{\varphi} \) of the pendulum;

The equations of motion of the pendulum are represented by the theorem of motion of the centre of mass:

\[Ma_G = G + N + T\] (5)

and the moment of momentum theorem written with respect to the centre of mass:

\[J_z \ddot{\omega} = M_r \dot{k} + r_{BG} \times (N + T)\] (6)

The system of scalar equations generated by equations (5) and (6) is:

\[
\begin{align*}
M \dddot{x}_G &= -N \sin \varphi + T \cos \varphi \\
M \dddot{y}_G &= N \cos \varphi + T \sin \varphi - Mg \\
J_z \dddot{\varphi} &= -N \dot{r}_b \dot{\varphi} + Ty'_G - M_r
\end{align*}
\] (7)

Under the hypothesis of pure rolling, the position of centre of mass of the pendulum is fully determined by the \( \varphi \) angle, equation (4), the unknowns of the system being the magnitude of normal reaction \( N \), the magnitude of tangential reaction \( T \), the magnitude of moment of friction \( M_r \) and the angle \( \varphi \). An extra equation is required in order to make a compatible system (3). It is assumed that the friction torque is proportional to the normal force \( N \) by means of a coefficient of proportionality \( s_r \) (rolling friction coefficient):

\[M_r = -s_r N \text{sgn} (\dot{\varphi})\] (8)

The differential equation of motion is obtained from equations (4), (7) and (8):

\[
\dddot{\varphi} = -M \frac{\left(1 - \frac{y'_G}{r_b}\right) \frac{s_r \text{sgn} \dot{\varphi} + \varphi}{r_b} \dot{r}_b^2 - Mg \frac{s_r \text{sgn} \dot{\varphi} + \varphi}{r_b} \cos \varphi - \frac{y'_G}{r_b} \sin \varphi}{J_z + \frac{\dot{r}_b^2 + \frac{s_r \text{sgn} \dot{\varphi} + \varphi}{r_b} \dot{r}_b^2}{M r_b^2}} - Mg \frac{s_r \text{sgn} \dot{\varphi} + \varphi}{r_b} \cos \varphi - \frac{y'_G}{r_b} \sin \varphi}\] (9)

With the known law of motion \( \varphi = \varphi(t) \), the other two unknowns are found using the relations:

\[
\begin{align*}
N &= M \dddot{y}_G \cos \varphi - M \dddot{x}_G \sin \varphi + Mg \cos \varphi \\
T &= M \dddot{y}_G \sin \varphi + M \dddot{x}_G \cos \varphi + Mg \sin \varphi
\end{align*}
\] (10)

The relations (10) are necessary to validate the condition:
$-\mu N < T < \mu N \quad (11)$

condition necessary for pure rolling existence.

3. **Experimental device: description and function**

The experimental set-up consists of a base 1 that allows for positioning the cylindrical body 2 which materialises the base circle. On top of it, a rod 3 is placed. This rod that materializes the mobile straight line has square cross-section. In order to be able to modify the position of the centre of mass $y_G'$ of the pendulum, two threaded shafts 4 are attached at the ends of the rod, with identical bodies supported. To mention the position of the rod is sufficient to specify the angle that the rod makes with horizontal line. To avoid the occurrence of supplementary forces in the system, a small mirror 5 was fixed on top of the rod (Figure 4.a), inclined at an angle $\gamma = 45^\circ$ with respect to the rod, the theoretical point $M$ where the mirror is fixed being situated at a distance $h = 30\text{mm}$ from the initial contact point, Figure 4b.

![Figure 3. Experimental set-up](image1)

![Figure 4. Details of experimental set-up](image2)

A laser beam, produced by a vertical laser level generator 6 is reflected by the mirror and the image is formed in the point $E$ on a screen 7 (Figure 4.c) placed at distance from the vertical in the initial contact point. The coordinates of mounting point $M$ in the stationary frame are found using relations (4), where $y_G'$ must be replaced by $h$. It is shown that the ordinate of image point from the screen is:
The pendulum was modelled in a specialised software, Figure 5, in order to find the parameters $J_z$ and $y'_G$. When the pendulum oscillates, the laser fascicle reflected by the mirror will produce a mobile spot on the screen, whose motion is video captured, Figure 6.

$y_E(\varphi) = \frac{x_M(\varphi)}{\tan(\varphi + \gamma)} + \frac{d}{\tan[2(\varphi + \gamma)]}$ (12)

4. Experimental results. Discussions
In order to find the rolling friction coefficient between two materials, two bodies are necessary: a cylinder made from the first material and a small prism made from the second material. The prism is interposed between the cylinder and the plane inferior surface of the pendulum rod.

The pendulum is set into motion and let to oscillate free. Using a video camera, the motion of the reflected laser beam on the screen is filmed. The movie is afterwards analyzed frame by frame and the instants of maximum amplitude are identified. Using the relation (12), the angular amplitudes of the pendulum are found. Thus it is obtained the experimental amplitude decrease of involute pendulum.

The differential equation of the pendulum is integrated, applying the Runge-Kutta methodology [12] with the initial conditions for the equation: $\varphi = \varphi_0$, $\varphi = 0$ (where $\varphi_0$ is one of the experimental amplitudes). The obtained theoretical and experimental results are plotted, as in Figure 8.

By suitable choice of coefficient $s_r$ in the theoretical model, the experimental data and theoretical ones will correspond. In figure 9 there are represented the experimental results and the theoretical interpolation curve. The interpolation was made by variation of the value of coefficient of rolling friction. For the obtained value of coefficient of rolling friction $s_r = 3.5 \mu m$ (steel-steel pair), applying...
equation (10), the forces $N$ and $T$ were found and it was validated the pure rolling condition (11). The time variation of the ratio of reaction forces $T/N$ is represented in Figure 10 and it is noticed that this ratio is much smaller than the coefficient of sliding friction (for steel-steel, about 0.4 for dry sliding [13], [14]).

![Figure 8](image1.png) **Figure 8.** Unfitted experimental results and a solution of the proposed model

![Figure 9](image2.png) **Figure 9.** Fitted experimental data and theoretical model for steel–steel pair

![Figure 10](image3.png) **Figure 10.** Pure rolling condition test for a steel–steel pair

For the pair rubber plate-steel cylinder, the results are analyzed similarly, as presented in Figures 11 and 12, with the mention that the value found for the coefficient of rolling friction is $s_r = 42 \, \mu m$.

![Figure 11](image4.png) **Figure 11.** Fitted experimental data and theoretical model for steel–rubber pair

![Figure 12](image5.png) **Figure 12.** Pure rolling condition test for a steel–rubber pair
5. Conclusions
The paper presents a method for finding the coefficient of rolling friction using the principle of involute pendulum. The pendulum consists of a prismatic rod in contact with an immobile cylinder.

The oscillation non-linear differential equation of the pendulum is deduced for pure rolling condition and then integrated using a numerical procedure. For a chosen pair of materials to be tested, one has to be an immobile base cylinder and the other one, a plate attached to the pendulum rod.

The motion of the pendulum was analyzed via a non-contact method using a laser beam reflected by the rod and the equation of motion of the reflected spot was deduced. The experimental amplitudes of the rod, decreasing in time, were compared to the results generated by the theoretical model. The coincidence between experimental data and theoretical amplitudes was obtained by optimum choice of coefficient of rolling friction. After establishing the coefficient of rolling friction, the pure rolling condition was verified and the validation of the method is therefore ensured.

Two experimental cases are presented, for steel-steel and rubber-steel pair of materials. The values of the coefficient of rolling friction obtained are in good agreement to the ones from technical literature.

The main advantage of the method is the fact that the regions implied in tests have small dimensions and so it can be used for analyzing materials with significant gradients of elastic characteristics. Another advantage refers to the studies from biomechamcs: for finding the tribological characteristics of biological tissues with complex geometry and/or reduced dimensions, these will play the part of the cylinder and the moving part (pendulum) will have a simpler geometry.

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