Lie and Noether point Symmetries for a Class of Nonautonomous Dynamical Systems

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Abstract
We prove two general theorems which determine the Lie and the Noether point symmetries for the equations of motion of a dynamical system which moves in a general Riemannian space under the action of a time dependent potential \( W(t, x) = \omega(t)V(x) \). We apply the theorems to the case of a time dependent central potential and the harmonic oscillator and determine all Lie and Noether point symmetries. Finally we prove that these theorems also apply to the case of a dynamical system with linear dumping and study two examples.

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1 Introduction
Newtonian dynamical systems share two fundamental characteristics: the equation of motion follows from Newton’s Second Law and its particular form depends upon the applied force or potential and the motion occurs in Euclidian three-dimensional space. This scenario goes over to Special Relativity (for pure forces) by the generalization of Newton’s Second Law to Minkowski space. In the case of General Relativity Newton’s Law is replaced by the Principle of Inertia according to which free fall takes place along geodesics. We note that in all these nonquantum theories we have two common elements: A second-order equation of motion and a (finite-dimensional) Riemannian manifold in which the motion is considered.

This remark takes us to the next step. The equation of motion as a second-order differential equation (ODE) admits certain Lie/Noether point symmetries which characterize this differential equation \([1, 2]\). On the other hand the geometry (metric) of the space is fixed (to a certain extent) by its collineations (Killing vectors (KV), Homothetic vectors (HV), Conformal Killing vectors (CKV), Affine collineations (AC) and Projective collineations (PC)) \([3]\). Hence what follows is to see if there exists a relation between the point symmetries of the equations of motion of a particular system and the collineations which define the underlying geometry.

This has been studied in the recent literature where one finds works which relate the collineations of the metric with the Lie and the Noether point symmetries of a general (usually conservative) dynamical system.

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We consider a nonautonomous dynamical system with time-dependent potential, while for some systems of partial differential equations in a Riemannian manifold the connection between Lie/Noether symmetries and collineations was the subject of study of [13, 14, 15, 16].

In the present paper we follow on the spirit of our previous works [17, 18, 19] and we extend our results to the case of a dynamical system which moves under the action of a certain type of time-dependent potential. We derive two theorems which allow the direct explicit calculation of the Lie and the Noether point symmetries of a dynamical system in terms of the collineations of the metric and the damping function. In Section 5 we consider and give two theorems which determine the Lie and the Noether point symmetries of a damped dynamical system in terms of the KVs and the HV of the metric and their degeneracies together with conditions / constraints which involve the potential function.

Comparing (1) and (2) we find that the results of the general case apply for \( P^i \neq 0 \) and the remaining \( P \)-terms zero. We infer that the Lie point symmetry conditions for a general \( P^i \) are the following:

\[
L_\eta P^i + \xi P^i,tt + 2\xi,tt + \eta^i,tt = 0 \tag{3}
\]

\[
(\xi,_{ik} \delta^i_j + 2\xi,_{ij} \delta^i_k) P^k + 2\eta^i,_{ijl} - \xi,tt \delta^i_j = 0 \tag{4}
\]

\[
L_\eta \Gamma^i_{(jk)} = 2\xi,tt(\delta^i_k) \tag{5}
\]

\[
\xi,_{(ijl)} \delta^i_k = 0, \tag{6}
\]

where \( X = \xi(t, x^i) \partial_t + \eta^i(t, x^j) \partial_{x^i} \) is the Lie point symmetry vector. For the specific case we are considering \( P(t, x^k)^i = \omega(t) V^i(x^k) \) the system of the Lie point symmetry conditions for the equation of motion (1) reduces to the following:

\[
\omega L_\eta V^i + \xi,tt V^i + 2\omega,tt V^i + \eta^i,tt = 0 \tag{7}
\]

\[
\omega (\xi,_{ik} \delta^i_j + 2\xi,_{ij} \delta^i_k) V^k + 2\eta^i,_{ijl} - \xi,tt \delta^i_j = 0 \tag{8}
\]

\[
L_\eta \Gamma^i_{(jk)} = 2\xi,tt(\delta^i_k) \tag{9}
\]

In Section 2 we prove Theorem 1 which gives the Lie point symmetries in terms of the special PCs of the metric and their degeneracies together with conditions / constraints which involve the potential function defining the specific dynamical system. In Section 3 we prove a second Theorem 2 which determines the Noether conditions in terms of the KVs and the HV of the metric and conditions which involve the potential function.

In Section 4 we show the equivalence of the linearly damped dynamical system with the time-dependent one and we consider and give two theorems which determine the Lie and the Noether point symmetries of a damped dynamical system in terms of the collineations of the metric and the damping function. In Section 5 we consider the application of the general theorems to various well-known and new cases. In Section 6 we discuss our conclusions.

2 Lie point symmetries for a class of nonautonomous systems

We consider a nonautonomous dynamical system with time-dependent potential, \( W(t, x) = \omega(t)V(x) \), the equations of motion of which are:

\[
\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k + \omega(t)V^i = 0, \quad \omega^i,tt \neq 0, V(x^i). \tag{1}
\]

In [17] the Lie point symmetry conditions for the system of ODEs:

\[
\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k + \sum_{m=0}^{n} P^i_{j_1...j_m} \dot{x}^{j_1} \ldots \dot{x}^{j_m} = 0, \tag{2}
\]

where \( \Gamma^i_{jk} \) are the connection coefficients in an affine space and \( P^i_{j_1...j_m}(t, x) \) are smooth polynomials completely symmetric in all lower indices have been derived.

Comparing (1) and (2) we find that the results of the general case apply for \( P^i \neq 0 \) and the remaining \( P \)-terms zero. We infer that the Lie point symmetry conditions for a general \( P^i \) are the following:
\[ \xi_{(i,j)} \delta^j_3 = 0. \quad (\text{10}) \]

We omit the solution of the determining equations. The steps that we follow are those described in [17] for the case of the autonomous system.

**Theorem 1** The Lie point symmetries of the time-dependent dynamical system:

\[ \ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k + \omega(t) V^i = 0, \quad \omega, t \neq 0, \quad (\text{11}) \]

are generated by the special PCs of the metric \( g_{ij} \) of the Riemannian space in which the motion takes place, as follows:

**Case I**

The Lie point symmetries generated by the Affine algebra \( Y^i \) of the metric. There are two cases to consider:

**Case I.1** The time function \( \omega(t) = \frac{1}{(d_1 t + d_2)^2} \). The Lie point symmetry for any potential \( V(x^i) \) is:

\[ X = (d_1 t + d_2) \partial_t \quad (\text{12}) \]

**Case I.2** The Lie point symmetry for \( \omega(t) \neq \frac{1}{(d_1 t + d_2)^2} \) is of the form:

\[ X = (d_1 t + d_2) \partial_t + a_1 Y^i, \quad (\text{13}) \]

where the constants \( a_1, d_1 \) are computed from the condition:

\[ (d_1 t + d_2) (\ln \omega)_t = d_1 - 2d_2 \]

and the potential satisfies the condition:

\[ a_1 LV^i + d_1 V^i = 0. \quad (\text{14}) \]

**Case II**

The Lie point symmetries are generated by the gradient KVs and/or gradient HV \( Y^i \) of the metric and \( Y^i \neq V^i \).

The Lie symmetry vector is:

\[ X = D(t) \partial_t + T(t) Y^i \partial_i, \quad (\text{15}) \]

where the functions \( D(t), T(t) \) are computed from the equations:

\[ D(\ln \omega)_t + 2D_t = d_0 T \quad (\text{16}) \]

\[ T_{tt} = m \omega T \quad (\text{17}) \]

and the potential \( V^i \) satisfies the condition:

\[ L_Y V^i + d_0 V^i + m Y^i = 0. \quad (\text{18}) \]

**Case III**

The Lie point symmetries are generated by the vector \( Y^i = kV^i \) which is a gradient KV or a gradient HV of the metric.

The Lie symmetry vector is:

\[ X = D(t) \partial_t + T(t) Y^i \partial_i \quad (\text{19}) \]

where the functions \( D(t), T(t) \) are computed from the equations:

\[ D(\ln \omega)_t + 2D_t + \frac{k}{\omega} T = 0 \quad (\text{20}) \]

\[ D_{tt} = 2\psi T_{tt} \quad (\text{21}) \]

and \( \psi = 0 \) for a KV and \( \psi = 1 \) for a HV.
Case IV
The Lie point symmetries are generated from the proper special projective vector $Y^i$. The Lie symmetry vector is:

$$X = (C(t)S_{ij} + D(t)) \partial_t + T(t)S_{ij} V^i \partial_i$$ (22)

where the functions $C(t), D(t), T(t)$ satisfy the following system of equations:

$$D (\ln \omega) t + \frac{2D_t}{T} = 0$$ (23)
$$D_{tt} = 0$$ (24)
$$\frac{C}{T} (\ln \omega) t + \frac{2C_t}{T} + \frac{\lambda}{\omega T} T_{tt} = \lambda_1$$ (25)
$$T_{tt} = a_7 \omega C$$ (26)
$$C_{tt} = a_0 T$$ (27)

and the potential $V^i$ satisfies the following conditions:

$$V^i$$ is a gradient HV

$$S_{ij}V^j - \lambda_1 S_j = 0$$ (28)
$$S_{ij} \delta^j_k + 2S_{ij} \delta^j_k V^k + a_7 \left(2V^i_j, ij - a_0 S_j \delta^i_k \right) = 0$$ (29)

and $S^i_j$ are the gradient KV's of the metric.

We continue our analysis with the Noether point symmetries.

3 Noether point symmetries for a class of nonautonomous systems

Consider a particle moving in a space with metric $g_{ij}$ under the influence of the potential $V(t, x^k)$. Then the Lagrangian describing the motion of the particle is:

$$L = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j - V(t, x^k).$$ (30)

A vector field in the space, $X = \xi (t, x^k) \partial_t + \eta^i (t, x^k) \partial_{x^i}$, is a Noether point symmetry of the Lagrangian if the following condition is satisfied:

$$X^{[1]}L + \frac{df}{dt},L = \frac{df}{dt},$$ (31)

where $X^{[1]} = \xi (t, x^k) \partial_t + \eta^i (t, x^k) \partial_{x^i} + \left(\frac{d\eta^i}{dt} - \dot{x}^i \frac{d\xi}{dt}\right) \partial_{x^i}$ is the first prolongation of $X$. We compute:

$$X^{[1]}L = \left(\xi \partial_t + \eta^i \partial_{x^i} + \left(\frac{d\eta^i}{dt} - \dot{x}^i \frac{d\xi}{dt}\right) \partial_{x^i} \right) \left(\frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j - V(t, x^k)\right)$$

$$= \frac{1}{2} \left(\eta^k g_{ij,k} \dot{x}^i \dot{x}^j + 2g_{ij} \dot{x}^i \dot{x}^j \frac{d\eta^j}{dt} + \frac{\partial \eta^j}{\partial x^k} g_{ij,k} \dot{x}^i \dot{x}^j + \frac{\partial \eta^j}{\partial x^k} g_{ij,k} \dot{x}^i \dot{x}^j \right).$$ (32)

The second term gives:

$$(\xi, t + \dot{x}^r \xi_r) \left(\frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j - V\right) = \frac{1}{2} \left(\xi, t g_{ij} \dot{x}^i \dot{x}^j - 2V \xi, t + g_{ij} \xi, \dot{x}^i \dot{x}^j \dot{x}^k - 2\dot{x}^k \xi, kV\right)$$ (33)

and similarly the third term:

$$\frac{df}{dt} = f_t + \dot{x}^k f_{,k}.$$ (34)
When we substitute into (31) and note that the resulting equation is an identity in \( \dot{x}^k \), we set the coefficient of each power of \( \dot{x}^k \) equal to zero and find the following set of equations:

\[
V_{,k}\eta^k + V_\xi + \xi V_t = -f_t
\]  
(35)

\[
\frac{\partial \eta^i}{\partial t} g_{ij} - \xi_j V = f_i
\]  
(36)

\[
L_\eta g_{ij} = 2 \left( \frac{1}{2} \frac{\partial \xi}{\partial t} \right) g_{ij}
\]  
(37)

\[
\frac{\partial \xi}{\partial x^k} = 0.
\]  
(38)

Equation (37) implies \( \xi = \xi(t) \) and the system of the equations reduces as follows:

\[
L_\eta g_{ij} = 2 \left( \frac{1}{2} \xi_t \right) g_{ij}
\]  
(39)

\[
V_{,k}\eta^k + V_\xi + \xi V_t = -f_t
\]  
(40)

\[
\eta_{i,t} = f_i.
\]  
(41)

The system of equations (39) - (41) for potential functions of the form:

\[
V(t, x^k) = \omega(t) V(x^k)
\]  
(42)

becomes:

\[
L_\eta g_{ij} = 2 \left( \frac{1}{2} \xi_t \right) g_{ij}
\]  
(43)

\[
V_{,k}\eta^k + \left( \xi_t + (\ln \omega)_{,t} \xi \right) V = -\frac{f_t}{\omega}
\]  
(44)

\[
\eta_{i,t} = f_i.
\]  
(45)

The solution of the system of equations (43)-(45) which leads to the following theorem.

**Theorem 2** For a dynamical system with Lagrangian:

\[
L = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j - \omega(t) V(x^k), \quad \omega, \xi \neq 0,
\]  
(46)

the Noether point symmetries are generated by the KVs and the HV of the metric as follows:

**Case I**

Noether point symmetries generated by the KVs and the HV \( Y_i \).

The Noether point symmetry vector and the Noether function are:

\[
X = (2a_1 \psi Y + d_1) \partial_t + a_1 Y^i \partial_i, \quad f = c_2 \int \omega dt,
\]  
(47)

where the constants are computed form the relation:

\[
(\ln \omega)_{,t} (2\psi Y + d_2) = d_1
\]  
(48)

and the potential function \( V(x^i) \) satisfies the condition:

\[
V_{,k}Y^k + d_2 V + c_2 = 0.
\]  
(49)

\( \psi \) is a constant which equals zero for a KV and nonzero constant for a HV.

**Case II**

Noether point symmetries produced by gradient KVs and/or the gradient HV \( Y^i \):
We have two cases:

**Case II.a** $V^i$ is not a gradient HV

The Noether point symmetry vector and the Noether function are:

\[ X = \xi(t)\partial_t + T(t)S_j\partial_j \ , \ f = T_{,t}S, \]  

where the index $J$ runs through all the gradient KVs. The functions $\xi(t), T(t)$ and the constants are computed from the equations:

\[ \xi_{,t} = 2\psi Y T \]  
\[ \frac{1}{T} \frac{T_{,tt}}{\omega} = m \]  
\[ \frac{1}{T} (\ln \omega)_{,t} \xi = d_1 \]  
\[ \frac{1}{T} \frac{K_{,t}}{\omega} = k \]  

and the potential function satisfies the condition:

\[ V_k Y^k + 2\psi Y V + d_1 V + mS + k = 0. \]  

**Case II.b** $V^i$ is a gradient KV

The Noether point symmetry vector and the Noether function are:

\[ X = \xi(t)\partial_t + T(t) V^i \partial_i \ , \ f = T_{,t} S + K(t), \]  

where the functions $\xi(t), T(t)$ are computed from the equations:

\[ \xi_{,t} = 2\psi Y T \]  
\[ \lambda \frac{1}{T} \frac{T_{,tt}}{\omega} + \frac{(\ln \omega)_{,t} \xi}{T} = d_2 \]  
\[ \frac{1}{T} \frac{K_{,t}}{\omega} = k \]  

and the potential function satisfies the condition:

\[ V_k Y^k + (2\psi Y + d_2) V + k = 0. \]  

4 The equivalence of a linearly damped dynamical system and a time-dependent dynamical system

Consider a particle moving in a Riemannian space with metric function $g_{ij}$ under the combined action of a linear damping force, $\phi(t) \dot{x}^i$, and a conservative force with potential function, $V(x^i)$. The equation of motion of the particle is:

\[ \frac{d^2 x^i}{dt^2} + \Gamma_{jk}^i \frac{dx^j}{dt} \frac{dx^k}{dt} + \phi(t) \frac{dx^i}{dt} + V^i = 0. \]  

If we consider the change of parameter $t \to s = S(t)$, the equation of motion is written as follows:

\[ \frac{d^2 x^i}{ds^2} + \Gamma_{jk}^i \frac{dx^j}{ds} \frac{dx^k}{ds} + \frac{1}{\left( \frac{dS(t)}{dt} \right)^2} \left( \frac{d^2 S(t)}{dt^2} \right) + \phi(t) \frac{dS(t)}{dt} + \frac{1}{\left( \frac{dS(t)}{dt} \right)^2} V^i = 0. \]  

We define the function $S(t)$ by the requirement:

\[ \left( \frac{d^2 S(t)}{dt^2} \right) + \phi(t) \frac{dS(t)}{dt} = 0 \]  

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and the equation of motion becomes:

\[ \frac{d^2 x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} + \frac{1}{(dS(t)/dt)^2} V^i = 0. \] (64)

The general solution of (63) is \( S(t) = \int e^{-\int \phi(t) dt} dt. \) Because the function \( S(t) \) is a point transformation, it must have an inverse so that \( t = S^{-1}(s) \), \( S^{-1}(s) = s \). Hence the equation of motion is written as follows:

\[ \frac{d^2 x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} + \left( \frac{dS^{-1}(s)}{ds} \right)^2 V^i = 0. \] (65)

We define the function

\[ \omega(s) = \left( \frac{dS^{-1}(s)}{ds} \right)^2 \] (66)

and the equation of motion takes the form:

\[ \frac{d^2 x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} + \omega(s) V^i = 0, \] (67)

which is equation (1) we studied in the previous sections. Therefore by applying the inverse transformation we are able to compute all Lie and Noether point symmetries of the linear damped time-dependent dynamical systems with equation of motion (61).

One application of the above general result is the time-dependent linearly damped harmonic oscillator in a space of constant curvature. The equation of motion of the time-dependent linear system is:

\[ \frac{d^2 x^i}{dt^2} + \omega(t) x^i = 0, \quad \omega(t) = \frac{\gamma^2}{t^2}. \] (68)

In this case we have from (66) the transformation:

\[ \left( \frac{dS^{-1}(t)}{dt} \right) = \frac{\gamma}{t} \to S^{-1}(t) = \gamma \ln t = s. \] (69)

The function \( S(t) \) is 1 : 1 and the inverse is:

\[ t = e^{\frac{s}{\gamma}}. \] (70)

Therefore:

\[ \frac{d}{dt} \left( \frac{dx^i}{ds} \right) \frac{ds}{dt} = \frac{d}{dt} \left( \frac{dx^i}{ds} \right) \left( \frac{\gamma}{t} \right) + \frac{dx^i}{ds} \left( \gamma \frac{d^2 (\ln t)}{dt^2} \right). \]

Hence equation (68) becomes:

\[ \frac{d^2 x^i}{ds^2} \gamma^2 - \frac{dx^i}{ds} \frac{\gamma}{t^2} x + \frac{\gamma^2}{t^2} x = 0, \]

that is:

\[ \frac{d^2 x^i}{ds^2} - \frac{1}{\gamma} \frac{dx^i}{ds} + x^i = 0 \] (71)

which is the damped time-dependent harmonic oscillator. We conclude that the time-dependent harmonic oscillator has the same Lie and Noether point symmetries with the linear damped time-dependent harmonic oscillator [24] [25].

5 Applications

In order to demonstrate our results in this section we determine the point symmetries of some well-known systems.
5.1 The Lie and the Noether point symmetries of a generalized time-dependent central potential

We determine the Lie and the Noether point symmetries of the time-dependent central potential \( V(t, r) = \omega(t) \frac{1}{n} r^n \) in Newtonian space. The equation of motion is:

\[
\ddot{x}^i + \omega(t) \frac{1}{n} r^n x^i = 0, \quad (n \neq 0, -2, 2).
\] (72)

The case \( n = 2 \) corresponds to the harmonic oscillator and the case \( n = -2 \) is a singular case. The Kepler potential follows for the value \( n = 1 \).

5.1.1 Lie point symmetries

We apply Theorem 1 in which only Case II is applied. Condition (16) is satisfied by the nongradient KVs \( X_{IJ} \) for \( d_1 = 0 \) and by the gradient HV \( H^i, d'_1 = (2 - n) \). Therefore we have the following Lie point symmetries:

\[
X = a_1 (d_2 t + d_3) \frac{\partial}{t} + (a_0 X_{IJ} + a_1 H^i) \frac{\partial}{i},
\]

where the constants \( d_2, d_3 \) must satisfy the condition:

\[
(d_2 t + d_3) (\ln \omega)_t = (2 - n) - 2d_2.
\]

This condition is satisfied only for \( \omega(t) = t^a \), where \( a \neq -2 \), that is, for \( (\ln \omega)_t = \frac{a}{t} \). In this case \( d_3 = 0 \) and \( d_2 = \frac{(2 - n)}{a + 2} \). We conclude that Case B possesses the Lie point symmetries:

\[
X = a_1 \left( 2 t + d_2 \right) \frac{\partial}{t} + (a_0 X_{IJ} + a_1 H^i) \frac{\partial}{i}
\]

only when \( \omega(t) = t^a \), where \( a \neq -2 \). It is a standard calculation to show that the finite point transformation generated by the Lie point symmetry vector corresponding to the parameter \( a_1 \) leaves invariant the quantity:

\[
\frac{r^{(2-n)}}{t^{(a+2)}} = \text{constant}. \quad (73)
\]

We conclude that for \( \omega(t) = t^a \) where \( a \neq -2 \) Kepler’s Third Law is generalized to the time-dependent central potential, \( V(t, r) = t^a r^{-2} \), \( a \neq -2, n \neq 0, -2, 2 \).

5.1.2 Noether point symmetries

For the Lagrangian function,

\[
L = \frac{1}{2} \delta_{ij} \dot{x}^i \dot{x}^j - \frac{1}{n} r^n,
\] (74)

we apply Theorem 2 and from Case I we have that condition (19) is met by the nongradient KVs \( X_{IJ} \) for \( d_1 = 0 \) and by the gradient HV \( H^i, d_1 = (n + 2) \). Therefore we have the Noether point symmetries:

\[
X = a_1 (2t + d_2) \frac{\partial}{t} + (a_0 X_{IJ} + a_1 H^i) \frac{\partial}{i}
\]

provided the constant \( d_2 \) satisfies the condition:

\[
(\ln \omega)_t (2t + d_2) = -(n + 2).
\]

This is possible only when \( \omega(t) = t^{-\frac{n+2}{2}} \), that is \( (\ln \omega)_t = -\frac{n+2}{2} \) and \( d_2 = 0 \). For this particular case we have the Noether point symmetries:

\[
X = a_1 2t \frac{\partial}{t} + (a_0 X_{IJ} + a_1 H^i) \frac{\partial}{i}.
\]

We note that the Noether point symmetries are Lie point symmetries when \( a = -\frac{n+2}{2} \) and \( n \neq 4 \).
For the special case, $\omega (t) = t^{-\frac{n+2}{2}}$, the Noether currents are:

$$ I_0 = X_{ij} \dot{x}_j $$

$$ I_1 = 2t \left( \frac{1}{2} \delta_{ij} \dot{x}^i \dot{x}^j + \frac{1}{n} r^n \right) - H^i \dot{x}_i. $$

The Noether point symmetry corresponding to the parameter $a_1$ gives the first integral (conserved quantity):

$$ \frac{r^2}{t} = \text{constant} \quad (75) $$

which follows form (73) if we set $a = -\frac{n+2}{2}$.

### 5.2 The Lie and the Noether point symmetries of the exceptional case $W(r) = \omega (t) r^{-4}$

This is the one special case we did not study in the last example. The equation of motion is:

$$ \ddot{x}^i + \omega (t) x^i r^{-4} = 0. \quad (76) $$

#### 5.2.1 Lie point symmetries

Following Theorem 1 and from Case II it follows that condition (16) is satisfied by the nongradient KVs $X_{IJ}$ for $d_1 = 0$ and the gradient HV $H^i$, $d'_1 = 4$. Therefore we have the Lie point symmetries:

$$ X = a_1 (d_2 t + d_3) \partial_t + (a_0 X_{IJ} + a_1 H^i) \partial_i $$

provided the constants $d_2, d_3$ satisfy the relation:

$$(d_2 t + d_3) (\ln \omega)_t = 4 - 2d_2. $$

This is the case only when $\omega (t) = t^a, a \neq -2$ and $d_3 = 0, d_2 = \frac{4}{a+2}$. In this case we have the Lie point symmetries:

$$ X = a_1 \left( \frac{4}{a+2} t \right) \partial_t + (a_0 X_{IJ} + a_1 H^i) \partial_i. \quad (77) $$

The Lie point symmetry corresponding to the parameter $a_1$ gives the conserved quantity (generalized Kepler Law):

$$ \frac{r^4}{t(a+2)} = \text{constant}. \quad (78) $$

In Case III we have the extra Lie point symmetry which is due to the gradient HV $H^i$ for the values $d_1 = 4, m = 0$. Then from the solution of the system of equations (19) - (21) we compute the functions $D (t), T (t)$ and find the Lie point symmetry (18). In the special case $\omega (t) = t^a, a \neq -2$ we compute:

$$ T (t) = c_1, \quad D (t) = \frac{4c_1}{a+2} t $$

which is rejected because in this case it is assumed that $T, t \neq 0$.

We note that the Lie point symmetries of the potential $V(r) = t^a r^{-4}, a \neq -2$ follow the same structure with the central potential $V(r) = t^a r^{-n}$.

#### 5.2.2 Noether point symmetries

Consider the Lagrangian

$$ L = \frac{1}{2} \delta_{ij} \dot{x}^i \dot{x}^j + \frac{1}{2} r^{-2}. \quad (79) $$
From Theorem 2 for Case I we have that condition (49) is satisfied by the nongradient KVs \(X_{IJ}\) for \(d_1 = 0\) and by the HV \(H^i\) for \(d'_1 = 0\). Therefore we have the Noether point symmetries:

\[ X = a_1 (2t + d_2) \partial_t + \left(a_0 X_{IJ} + a_1 H^i\right) \partial_i \]

provided the constant \(d_2\) satisfies the relation:

\[(\ln \omega)_t (2t + d_2) = 0\]

which is impossible. Therefore no Noether symmetry exists in this case.

For Case II we obtain a Noether point symmetry from the gradient HV \(H^i\) for \(d_1 = 0\), \(m = 0\). Then the system of equations (51) - (53) has the trivial solution unless \(\omega(t) = \text{constant}\). Therefore we do not have a Noether point symmetry in this case. We conclude that the Lagrangian (79) has the Noether symmetries:

\[ X = X_{IJ} \]

with corresponding Noether first integrals:

\[ I_0 = X_{IJ} \dot{x}_j. \]

Contrary to the other central potentials in this case we do not have a special \(\omega(t)\) for which a Noether point symmetry is admitted.

5.3 The time-dependent Kepler potential \(W(r) = \omega(t)r^{-2}\)

This is the second singular case to the general central potential. The Lie point symmetries of the Kepler potential in Euclidian space have been determined in [23]. The Kepler potential has also been considered in the case of spaces of constant curvature [26] for which the Lie and the Noether point symmetries have not been determined.

5.3.1 Lie point symmetries

The equation of motion is:

\[ \ddot{x}^i + x^i \frac{\omega(t)}{r^2} = 0 \]  

(80)

We apply theorem [1] and consider cases.

Case II

Condition (16) is satisfied by the nongradient KVs \(X_{IJ}\) for \(d_1 = 0\) and by the HV \(H^i\) for \(d'_1 = 3\). Hence we have the Lie point symmetries:

\[ X = a_1 (d_2 t + d_3) \partial_t + \left(a_0 X_{IJ} + a_1 H^i\right) \partial_i, \]

where the constants \(d_2, d_3\) must satisfy the condition:

\[(d_2 t + d_3) (\ln \omega)_t = 3 - 2d_2.\]

This is the case only if \(\omega(t) = t^a\) \((a \neq -2)\), \(d_3 = 0, d_2 = \frac{3}{a+2}\). Therefore we have the Lie point symmetry:

\[ X = a_1 \left(\frac{3}{a+2} t\right) \partial_t + \left(a_0 X_{IJ} + a_1 H^i\right) \partial_i \]

provided \(\omega(t) = t^a\) \((a \neq -2)\).

One shows easily that under the action of the point transformation generated by the parameter \(a_1\) the following quantity is conserved (generalized Kepler’s Third Law):

\[ \frac{r^3}{t(a+2)} = \text{constant}. \]  

(81)

Case III and Case IV do not give more Lie symmetries.
5.3.2 Noether point symmetries

The Lagrangian is:

\[ L = \frac{1}{2} \delta_{ij} \dot{x}^i \dot{x}^j + r^{-1}. \]  \hspace{1cm} \text{(82)}

We apply theorem 2 and consider cases.

**Case I**

Condition (49) is satisfied by the nongradient KVs \( X_{IJ} \) for \( d_1 = 0 \) and by the HV \( H^i \) for \( d_3 = q \). Hence we have the Noether point symmetries:

\[ X = a_1 (2t + d_2) \partial_t + (a_0 X_{IJ} + a_1 H^i) \partial_i \]

provided the parameter \( d_2 \) satisfies the condition:

\[ (\ln \omega)_t (2t + d_2) = -1. \]

This is the case only when \( \omega(t) = t^{-\frac{1}{2}} \), that is \( (\ln \omega)_t = -\frac{1}{2t} \) and \( d_2 = 0 \). The resulting Noether point symmetry is:

\[ X = a_1 2t \partial_t + (a_0 X_{IJ} + a_1 H^i) \partial_i \]

provided \( \omega(t) = t^{-\frac{1}{2}} \).

**Case II** does not give more Noether symmetry vectors.

The First Integrals corresponding to the Noether symmetries are:

\[ I_0 = X^i_{IJ} \dot{x}^j \]  \hspace{1cm} \text{(83)}

\[ I_1 = 2t \left( \frac{1}{2} \delta_{ij} \dot{x}^i \dot{x}^j - r^{-1} \right) - H^i \dot{x}_i. \]  \hspace{1cm} \text{(84)}

The Noether symmetry corresponding to the parameter \( a_1 \) gives the additional conserved quantity (Kepler’s Third Law):

\[ \frac{r^2}{t} = \text{constant} \]  \hspace{1cm} \text{(85)}

which is compatible with \( \text{(81)} \) when \( a = -\frac{1}{2} \).

It is important to note that, when \( \omega(t) = t^{\frac{1}{2}} \), Kepler’s potential gives the same number of Lie and Noether point symmetries as the time independent Kepler potential, \( \omega(t) = \text{constant} \). Furthermore the Noether point symmetries are the same as the Lie point symmetries for both cases.

5.4 The time dependent harmonic oscillator in Euclidean space

Lastly we consider the time-dependent harmonic oscillator in Euclidean space which has been extensively studied using the standard Lie approach \([21],[22]\). Here we use Theorem 1 and retrieve these results directly using the projective algebra of Euclidean space. The equation of motion is:

\[ \ddot{x}^i + \omega(t) x^i = 0, \quad \omega(t) \neq \frac{1}{4t^2}. \]  \hspace{1cm} \text{(86)}

The projective algebra of Euclidean space is:

\[ S^i_i : \text{gradient KV}, \quad X_{IJ} : \text{nongradient KV} \]

\[ H^i : \text{gradient HV}, \quad A^i_j : \text{Affine Collineation} \]

\[ P^i_j : \text{special PC}. \]
In Cartesian coordinates these vectors are:

\[ S_I = S^i_I \partial_i \Rightarrow S_I = \delta^i_I i = 1, 2, ..., n \]
\[ X_{IJ} = \delta^a_I \delta^b_J x_a \partial_b \]
\[ H = x^i \partial_i \Rightarrow H = \frac{1}{2} (x^i x_i) \]
\[ A_I = S_I S^i_I \partial_i \Rightarrow A_I = x_I \delta^i_I \]
\[ P_I = S_I H = x_I x^i \partial_i \Rightarrow P_I = x_I x^i. \]

Following Theorem 1 we consider cases.

**Case II**

Condition (16) is satisfied by \( X_{IJ} \) for \( d_1 = 0 \), by \( H_i \) for \( d_1' = 0 \) and by \( A^i \) for \( d_1'' = 0 \). From (17) follows \( D(t) = 0 \). Therefore (15) gives the following Lie point symmetry vectors:

\[ X = (l_1 H^i + l_2 X_{IJ} + l_3 A^i) \partial_i. \]  

**Case III**

**Case III.1**

We check that the gradient KVs \( S^i_J \) \( J = 1, 2, ..., n \) provides a symmetry vector for \( m = -1, d_0 = 0 \). We compute \( D(t) = 0 \) and that \( T(t) \) is found from the ODE:

\[ T_{tt} = -\omega(t)T. \]  

The resulting Lie point symmetry vectors are:

\[ X = T(t) S^i_J \partial_i. \]  

**Case III.2**

Because the potential of the harmonic oscillator is the gradient HV we have the Lie point Symmetry:

\[ X = D(t) \partial_i + T(t) V^i \partial_i, \]  

where the functions \( D(t), T(t) \) are the solution of the system of equations:

\[ D \left( \ln \omega \right)_t + 2D_t + \frac{1}{\omega} T_{tt} = 0 \]  
\[ 2T_t - D_{tt} = 0. \]

**Case IV.**

From equation (23) it follows that \( D(t) = 0 \), and \( \lambda = 1, \lambda_1 = 1, \lambda_2 = -1 \). In this case we also have \( a_0 = 1 \). Hence the system of equations (23)-(24) which yields the functions \( C(t), T(t) \) becomes:

\[ \frac{C}{T} \left( \ln \omega \right)_t + \frac{2C_t}{T} + \frac{1}{\omega(t)} \frac{T_{tt}}{T} = 1 \]  
\[ T_t = -\omega(t)C \]  
\[ C_t = T. \]

The resulting Lie point symmetry vectors are:

\[ X = C(t) S_J \partial_t + T(t) S_J V^i \partial_i. \]

Counting the symmetry vectors we see that they are \( N^2 - 1 \), \( N = n + 2 \) where \( n \) is the dimension of the space. Hence the admitted Lie algebra is the \( sl(n + 2, R) \), which means that the system is maximally symmetric and equivalent with the free particle [21, 22].
6 Conclusions

This work extends our previous analysis for the relation between symmetries of differential equations and
collineations of the underlying manifold. Specifically in [17] we studied the connection between projective
collineations and point symmetries of the equations of motion of a particle moving in a general affine space
under the action of an autonomous potential, we called that $(1 \times m)$ system. Similar analyses have been done
and for linear partial differential equation $(n \times 1)$ for which a connection with the conformal algebra of the space
of the independent variables was found [18]. Recently for a system of quasilinear partial differential equation,
$(n \times m)$ systems, and for $m > 1$, it was found that that the point symmetries are related with the conformal
algebra of the space of the independent variables and the affine collineations of the space which defines the evolution
of the dependent variables [19]. However, the system of $(1 \times m)$ nonautonomous equations could not
be covered from the previous works.

We have proved two general theorems which prove that the Lie and the Noether point symmetries of a
dynamical system moving in Riemannian space under the action of the time-dependent potential
$W(t, x) = \omega(t)V(x)$ are given in terms of the collineations of the space and a set of constraint conditions involving the
potential and the collineation vectors. We have applied the theorems to equations of motion in the Euclidian
space and derived the Lie and the Noether point symmetries for the time-dependent central motion and the
time-dependent harmonic oscillator. We have also proved that these theorems can be used to determine the
point symmetries of dynamical systems with linear dumping. It is interesting to note that this result may be
applied to any second-order equation where the connection coefficients is taken as a general connection $\Gamma^{j}_{ik}$.

That geometric approach is an alternate method for the study of the symmetries of differential equations. It
is useful because we can study the possible admitted Lie algebras by just using results from differential geometry
and derive with minimal calculation the collineations of the underlying manifold. However, except from the
technical part it provides a strong relation between geometry and dynamical systems.

In a forthcoming work we plan to investigate the geometric properties for the evolution of the dynamical
systems by using that geometric point of view.

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