Semigroup wellposedness and asymptotic stability of a compressible Oseen–structure interaction via a pointwise resolvent criterion

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Abstract
In this study, we consider the Oseen structure of the linearization of a compressible fluid–structure interaction (FSI) system for which the interaction interface is under the effect of material derivative term. The flow linearization is taken with respect to an arbitrary, variable ambient vector field. This process produces extra “convective derivative” and “material derivative” terms, which render the coupled system highly nondissipative. We show first a new well-posedness result for the full incorporation of both Oseen terms, which provides a uniformly bounded semigroup via dissipativity and perturbation arguments. In addition, we analyze the long time dynamics in the sense of asymptotic (strong) stability in an invariant subspace (one-dimensional less) of the entire state space, where the continuous semigroup is uniformly bounded. For this, we appeal to the pointwise resolvent condition introduced in Chill and Tomilov [Stability of operator semigroups: ideas and results, perspectives in operator theory Banach center publications, 75 (2007), Institute of Mathematics Polish Academy of Sciences, Warszawa, 71–109], which avoids an immensely technical and challenging spectral analysis and provides a short and relatively easy-to-follow proof.

KEYWORDS
compressible flows, flow–structure interaction, material derivative, resolvent, stability, uniformly bounded semigroup

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1 | INTRODUCTION

The mathematical analysis of fluid–structure interaction (FSI) problems constitutes a broad area of research with applications in aeroelasticity, biomechanics, and fluid dynamics [3–14, 19–22, 29, 30, 38]. Such interactive dynamics between flow/fluid and a plate (or shell) are mathematically realized by coupled partial differential equation (PDE) systems with...
Compressible flow and elastic plate components. The analysis of these PDE systems is considered from many points of view [7, 8, 10, 21, 29, 30].

In this work, we consider a linearized flow–structure PDE model with respect to some reference state, which results in the appearance of an arbitrary spatial flow field. In contrast to the incompressible case, having a compressible flow component presents many difficulties due to the increase in the number of unknown variables. By the nature and physics of compressible flows, density can change by pressure forces and a new set of governing equations are necessarily derived along with the equations for the conservation of mass and momentum. These equations should be valid for the flows (compressible) whose range of Mach number is

\[
\text{Mach Number} = \frac{\text{velocity}}{\text{local speed of sound}} > 0.3.
\]

The cases \( M < 0.3 \) and \( 0.3 < M < 0.8 \) are subsonic/incompressible and subsonic/compressible regimes, respectively. Compressible flows can be either transonic (\( 0.8 < M < 1.2 \)) or supersonic (\( 1.2 < M < 3.0 \)). In supersonic flows, pressure effects are only transported downstream; the upstream flow is not affected by conditions downstream.

Our principle aim is to consider the long-time behavior of the corresponding coupled FSI system with a focus of (asymptotic) strong stability properties of the \( C_0 \)-semigroup generated by the solution. This asymptotic decay for solutions of the compressible flow–structure PDE model will be stated within the context of the associated semigroup formulation and “frequency domain” approach.

### 1.1 The FSI geometry

Let the flow domain \( \mathcal{O} \subset \mathbb{R}^3 \) with Lipschitz boundary \( \partial \mathcal{O} \). We assume that \( \partial \mathcal{O} = \overline{S} \cup \overline{\Omega} \), with \( S \cap \Omega = \emptyset \), and the (structure) domain \( \Omega \subset \mathbb{R}^3 \) is a flat portion of \( \partial \mathcal{O} \) with \( C^2 \)-boundary. In particular, \( \partial \mathcal{O} \) has the following specific configuration:

\[
\Omega \subset \{ x = (x_1, x_2, 0) \} \quad \text{and} \quad S \subset \{ x = (x_1, x_2, x_3) : x_3 \leq 0 \}.
\]

Additionally, the flow domain \( \mathcal{O} \) should be a curvilinear polygonal domain, which satisfies the following conditions:

1. Each corner of the boundary \( \partial \mathcal{O} \) —if any— is diffeomorphic to a convex cone.
2. Each point on an edge of the boundary \( \partial \mathcal{O} \) is diffeomorphic to a wedge with opening \( < \pi \).

We note that these additional conditions on the flow domain \( \mathcal{O} \) are necessary for the application of some elliptic regularity results for solutions of second-order boundary value problems on corner domains [24, 27]. We denote the unit outward normal vector to \( \partial \mathcal{O} \) by \( \mathbf{n}(x) \), where \( \mathbf{n}|_{\partial \Omega} = [0, 0, 1] \), and the unit outward normal vector to \( \partial \Omega \) by \( \mathbf{v}(x) \). Some examples of geometries can be seen in Figure 1.

### 1.2 Linearization and the PDE model

In what follows, we provide some information about the linearization process and the PDE description of the compressible flow–structure interaction system under consideration. First, we note since the linear flow problem here is already of
great technical complexity and mathematical challenge, we assume that the pressure is a linear function of the density; \( p(x, t) = C \rho(x, t) \), as is typically seen in the compressible flow literature, and it is chosen as a primary variable to solve. For further and detailed explanations of the physical background concerning the relationship between pressure and density, the reader is referred to \([10, 30]\).

The linearization takes place around an equilibrium point of the form \( \{ p_*, U, \varphi_* \} \) where the pressure and density components \( p_*, \varphi_* \) are assumed to be scalars (for simplicity, assume \( p_* = \varphi_* = 1 \)), and a generally nonzero, fixed, ambient vector field \( U : \mathcal{O} \rightarrow \mathbb{R}^3 \)

\[
U(x_1, x_2, x_3) = \left[ U_1(x_1, x_2, x_3), U_2(x_1, x_2, x_3), U_3(x_1, x_2, x_3) \right].
\]

At this point, we emphasize that flow linearization is taken with respect to some inhomogeneous compressible Navier–Stokes system; thus, \( U \) does not need to be divergence free, generally.

Now, with respect to the above linearization, the small perturbations give the following physical equations by generalizing the forcing functions:

\[
(\partial_t + U \cdot \nabla)p + \text{div}(u) + (\text{div} U)p = f(x) \quad \text{in} \quad \mathcal{O} \times \mathbb{R}_+,
\]

\[
(\partial_t + U \cdot \nabla)u - \nu \Delta u - (\nu + \lambda)\text{div} u + \nabla p + \nabla U \cdot u + (U \cdot \nabla U)p = F(x) \quad \text{in} \quad \mathcal{O} \times \mathbb{R}_+.
\]

(For further discussion, see also \([10, 21]\).) When we ignore the noncritical lower order term \((U \cdot \nabla U)p\) as well as the inhomogeneous terms in the above equations, this linearization gives rise to the following system of structure–Oseen interaction PDEs, in solution variables \( u(x_1, x_2, x_3, t) \) (flow velocity), \( p(x_1, x_2, x_3, t) \) (pressure), \( w_1(x_1, x_2, t) \) (elastic plate displacement), and \( w_2(x_1, x_2, t) \) (elastic plate velocity):

\[
\begin{aligned}
& p_t + U \cdot \nabla p + \text{div}(u) + \text{div}(U)p = 0 \quad \text{in} \quad \mathcal{O} \times (0, \infty) \\
& u_t + U \cdot \nabla u + u \cdot \nabla U - \text{div} \sigma(u) + \eta u + \nabla p = 0 \quad \text{in} \quad \mathcal{O} \times (0, \infty) \\
& (\sigma(u)n - pn) \cdot \tau = 0 \quad \text{on} \quad \partial \mathcal{O} \times (0, \infty) \\
& u \cdot n = 0 \quad \text{on} \quad S \times (0, \infty) \\
& u \cdot n = w_2 + U \cdot \nabla w_1 \quad \text{on} \quad \Omega \times (0, \infty)
\end{aligned}
\]

(1.2)

\[
\begin{aligned}
& w_{1t} - w_2 - U \cdot \nabla w_1 = 0 \quad \text{on} \quad \Omega \times (0, \infty) \\
& w_{2t} + \Delta^2 w_1 + \left[ 2\nu \partial_{x_3}^3(u_3) + \lambda \text{div}(u) - p \right]_{\Omega} = 0 \quad \text{on} \quad \Omega \times (0, \infty) \\
& w_1 = \frac{\partial w_1}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega \times (0, \infty)
\end{aligned}
\]

(1.3)

\[
[p(0), u(0), w_1(0), w_2(0)] = [\overline{p}, \overline{u}, \overline{w_1}, \overline{w_2}] \in H_0.
\]

(1.4)

Here, \( H_0 \) is given as follows:

\[
H_0 = \{ [p_0, u_0, w_1, w_2] \in H : \int_{\mathcal{O}} p_0 d\mathcal{O} + \int_{\Omega} w_1 d\Omega = 0 \},
\]

(1.5)

where

\[
H \equiv L^2(\mathcal{O}) \times L^2(\mathcal{O}) \times H^2_0(\Omega) \times L^2(\Omega)
\]

(1.6)

is the associated finite energy (Hilbert) space, topologized by the standard inner product:

\[
(y_1, y_2)_H = (p_1, p_2)_{L^2(\mathcal{O})} + (u_1, u_2)_{L^2(\mathcal{O})} + (\Delta w_1, \Delta w_2)_{L^2(\Omega)} + (v_1, v_2)_{L^2(\Omega)}
\]

(1.7)

for any \( y_i = (p_i, u_i, w_i, v_i) \in H, \ i = 1, 2. \)
Remark 1.1. We note that a change of variable \( w(t) = w_1(t), \) \( w = w_2 + U \cdot \nabla w_1 \) can be invoked to formulate the PDE model \((1.2)-(1.4)\) as follows:

\[
\begin{align*}
\frac{\partial p}{\partial t} + U \cdot \nabla p + \text{div} u + \text{div}(U)p &= 0 \quad \text{in} \quad \mathcal{O} \times (0, \infty) \\
\frac{\partial u}{\partial t} + U \cdot \nabla u - \text{div} \sigma(u) + \eta u + \nabla p &= 0 \quad \text{in} \quad \mathcal{O} \times (0, \infty) \\
\sigma(u)n - \frac{p}{\eta} n - \tau &= 0 \quad \text{on} \quad \partial \mathcal{O} \times (0, \infty) \\
u \cdot n &= 0 \quad \text{on} \quad S \times (0, \infty) \\
u \cdot n &= w_t \quad \text{on} \quad \Omega \times (0, \infty) \\
\end{align*}
\]

where \( w(0) = w_1(0) = w_1 \) and \( w_t(0) = w_2(0) + U \cdot \nabla w_1(0) = w_2 + U \cdot \nabla w_1 \).

At this point, we should emphasize that the change of variable defined above passes the nondissipativity effect of the boundary material derivative term \( U \cdot \nabla w_1 \) in \((1.2)\) to the inside of the plate equation with the term \( \frac{\partial \sigma}{\partial y} \) in \((1.3)\). This additional term in \((1.3)\) is indeed necessary and crucial to have well-posedness in an invariant subspace, which is suitable for bounded semigroup generation. Without this term, there is no stability analysis.

It was shown in [7] and [29] that \( H_0^\perp \) is the null space of the flow–structure semigroup generator closely associated with \((2)-(4)\). It will be shown below that solutions of \((2)-(4)\), with initial data drawn from \( H_0 \), decay asymptotically to the zero state. Also, the terms \( U \cdot \nabla u + u \cdot \nabla U \) constitute the so-called Oseen (linear) approximation of the Navier–Stokes equations [42].

The quantity \( \eta > 0 \) represents a drag force of the domain on the viscous flow. In addition, the quantity \( \tau \) in \((1.2)\) is in the space \( TH^{1/2}(\partial \mathcal{O}) \) of tangential vector fields of Sobolev index \( 1/2 \); that is,

\[
\tau \in TH^{1/2}(\partial \mathcal{O}) = \{v \in H^1(\partial \mathcal{O}) : v|_{\partial \mathcal{O}} \cdot n = 0 \text{ on } \partial \mathcal{O}\}. \tag{1.11}
\]

(See, e.g., p. 846 of [17].) In addition, we take ambient field \( U \in V_0 \cap W \) where

\[
V_0 = \{v \in H^1(\mathcal{O}) : v|_{\partial \mathcal{O}} \cdot n = 0 \text{ on } \partial \mathcal{O}\}, \tag{1.12}
\]

\[
W = \{v \in H^1(\mathcal{O}) : v \in L^\infty(\mathcal{O}), \text{ div}(v) \in L^\infty(\mathcal{O}), \text{ and } v|_{\Omega} \in C^2(\overline{\Omega})\}. \tag{1.13}
\]

(The vanishing of the boundary for ambient fields is a standard assumption in compressible flow literature; see [1, 23, 32, 41].) Moreover, the stress and strain tensors in the flow PDE component of \((1.2)-(1.4)\) are defined, respectively, as

\[
\sigma(\mu) = 2\nu \varepsilon(\mu) + \lambda I_3 \cdot \varepsilon(\mu) I_3; \quad \varepsilon(\mu) = \frac{1}{2} \left( \frac{\partial \mu_j}{\partial x_i} + \frac{\partial \mu_i}{\partial x_j} \right), \quad 1 \leq i, j \leq 3,
\]

where Lamé Coefficients \( \lambda \geq 0 \) and \( \nu > 0 \).

Here, we impose the so-called impermeability condition on \( \Omega \); namely, we assume that no fluid passes through the elastic portion of the boundary during deflection [15, 28]. Also, note that the FSI problem under consideration has a material derivative term on the deflected interaction surface, which computes the time rate of change of any quantity such as temperature or velocity (and hence also acceleration) for a portion of a material in motion. For further details and the physical explanation of the material derivative boundary conditions, see [10, 30].

**2 | PREVIOUS CONSIDERATIONS**

The long-time behavior, in particular the stability properties of incompressible/compressible fluid/flow–structure interaction (FSI) systems have been a popular topic treated by many authors at different levels [3–7, 9, 13, 19, 20, 22, 29]. The
PDE systems under consideration are generally quite complicated, due to their unbounded hyperbolic–parabolic coupling mechanisms, both being intrinsic to the underlying physics.

In contrast to the large body of literature on incompressible FSI [3–6, 9, 11–14, 19, 20, 22, 38], existing work on compressible flows that interact with elastic solids is relatively limited due to the inherent mathematical challenges presented by the extra density (pressure) variable. In analyzing these compressible FSI PDE models, there are the following challenges:

(i) Given that the flow-plate variables are coupled via boundary interfaces, the FSI geometry is inherently nonsmooth, [21, 23]. Although such geometries are physically relevant, these domains also give rise to regularity issues for solutions. (ii) The linearization of the compressible FSI system under consideration takes place around a rest state, which includes a generally nonzero (not constant) ambient flow field $U$, and the said pressure PDE will contain a “convective derivative” term $UVp$, which is strictly above the level of finite energy. (iii) The boundary conditions, which couple flow and structure contain nondissipative and unbounded terms, which complicate the PDE analysis.

A preliminary linearized compressible FSI model with $U \equiv 0$ (and so without material derivative) was derived by I. Chueshov [21]. In this work, the author showed the wellposedness and the existence of global attractors in the case that the structure equation has a von Karman nonlinearity. However, in this pioneering work, the author noted that his methods (Galerkin approach and Lyapunov functionals) to show the wellposedness and long-term behavior of the corresponding system would not accommodate the case of interest, namely, linearization about $U \neq 0$.

The main reason for the difficulty is the need to control the “convective derivative” term $UVp$, which requires a decomposition of the fluid and pressure solution pair $\{u, p\}$ with an eventual appeal to some elliptic regularity results on nonsmooth domains. Subsequently, the suggested model that was analyzed in [8] via a semigroup formulation and a wellposedness result with additional interior terms associated to the $U \neq 0$ under a pure velocity matching condition at the interface was obtained. Then in [10], the authors revisited the same problem after a careful derivation of fluid–structure interface conditions written in terms of “material derivative” $(\partial_t + U \cdot \nabla)w$. This material derivative computes the time rate of change of any quantity such as temperature or velocity (and hence also acceleration) for a portion of a material in motion. However, since the material derivative term $UVw$ is unbounded and nondissipative, it adds an additional challenge to the analysis. In [10], this lack of boundedness and dissipativity were ultimately overcome by retopologizing the finite energy space, and so the desired wellposedness result was obtained by an appropriate semigroup formulation. At this point, we should note that in both papers [8, 10], the obtained semigroup is unfortunately not uniformly bounded (in time), which prevents us to seek long-term behavior of the solution to these corresponding problems.

With a view of looking into stability properties of the flow–structure PDE models considered in [8, 10], it appeared natural to ask: Is it possible to obtain a semigroup wellposedness and subsequently a stability result, with the semigroup being bounded uniformly in time, at least in some (inherently invariant) subspace of the finite energy space? This was indeed a very important departure point in order to analyze the stability of these FSI problems since there was not any long-time behavior result in the literature for such classes of FSI systems. Motivated by this question, an initial result of uniform stability result for the solution to linearized compressible FSI model (without material derivative) in an appropriate subspace was shown in [7, 29] by using linear perturbation theory and also some novel multipliers.

### 3 | Novelty

As discussed in the previous section, having established the uniform decay result for the solution to the “material free” compressible FSI system in [7, 29], our next goal was to analyze the long-time behavior of the system, when under the effect of “material term” on the interaction interface. In order to embark on this stability work, we first needed to have a uniformly bounded (in time) semigroup, again in some (inherently invariant) subspace of the finite energy space. This result was obtained in [30]. Now, in this work, we provide a novel result on the asymptotic (strong) stability of compressible FSI model under said “material derivative” boundary conditions. This result ascertains that solution to the FSI model decays to zero state for all initial data taken from a subspace $H_0$, which is “almost” the entire phase space $H$. This is, to the knowledge of the author, the first such result obtained with “material derivative” BC.

By way of obtaining the aforesaid asymptotic (strong) stability result, we operate in the frequency domain, which requires us to deal with some static equations, analogous to (2)–(3), and the resolvent of the generator of the dynamical system. While we have already been aware of the need to control the “convective derivative” term $UVp$ from our previous works, in this paper, the primary challenge is to have tight control of the material derivative term $UVw$ in the coupling
condition at the interface since it destroys the dissipative nature of the dynamics. In this regard, the main challenges associated with the analysis and the novelties are as follows:

(i) **Inner product adjustment for dissipativity:** In order to establish a long-time behavior result for the given FSI system, we expect that the energy of this system is decreasing. However, the presence of the problematic convective and material derivative terms $\mathbf{U}\nabla p$ and $\mathbf{U}\nabla w$ adds a great challenge to the long-term analysis since they cannot be treated as a perturbation and moreover cause a lack of dissipativity property in the system. To deal with these issues and get the necessary dissipativity estimate, we change the inner-product structure of the state space and we retopologize the inherently invariant subspace with an inner product, which is equivalent to the standard inner product defined for the entire state space. In this construction, we make use of a multiplier, which exploits the characterization of this invariant subspace and the Dirichlet map that extends boundary data defined on the interaction interface to a harmonic function in the flow domain. See Theorem 5.1 and [30].

(ii) **Wellposedness of the full linearization with Oseen terms:** One of the main contributions of this work is to incorporate the previously discarded fluid terms appearing in the full Oseen linearization of the problem. While those terms seemed to be benign in the previous works [7, 8, 10], being of lower order with respect to the energy space, they gain a critical importance for the boundedness of the semigroup on the invariant subspace and long-time behavior analysis of this work. In addition, in this work, we consider the full linearization of the compressible Navier–Stokes equations, which produces the so-called Oseen system. Since one of the goals of this work is also to investigate the stability properties of the FSI system under consideration, a question of bounded semigroup wellposedness for the Oseen FSI system arises. By appealing to linear perturbation theory, we ultimately show that the new (full) system generates a strongly continuous semigroup, which is indeed uniformly bounded in time on the invariant subspace.

(iii) **Use of pointwise resolvent criterion for stability:** Since the domain of the associated flow–structure generator is not compactly embedded into finite energy space $\mathcal{H}$ (see (4.1)–(4.2) below), it would seem natural to appeal to the well-known strong stability result introduced by Arend–Batty [2]. However, this does not seem practicable for our model since the spectral analysis of the semigroup generator will require a great amount of technicalities. For example, to analyze the residual spectrum of the generator, we need to check that none of the points on the imaginary axis are the eigenvalues of the adjoint operator of this generator. Unfortunately, it is not going to be straightforward to get this result since the adjoint, itself, is given as the sum of three complicated matrix (see Lemma 6.2). Instead, we will use the very interesting pointwise resolvent criterion developed by Chill and Tomilov [18, 40] for the stability of bounded semigroups, which perfectly works for our model (see also [16]), which gives a preliminary version of the resolvent criterion for strong stability). Hence, we provide a very clean, short, and easy-to-follow proof, which does not touch the—very complicated—adjoint operator and the challenges it might cause.

4 | PRELIMINARIES

Throughout, for a given domain $D$, the norm of corresponding space $L^2(D)$ will be denoted as $\| \cdot \|_D$ (or simply $\| \cdot \|$ when the context is clear). Inner products in $L^2(\Omega)$ or $L^2(\partial \Omega)$ will be denoted by $(\cdot, \cdot)_{\Omega}$, whereas inner products $L^2(\partial \Omega)$ will be written as $(\cdot, \cdot)_{\partial \Omega}$. We will also denote pertinent duality pairings as $(\cdot, \cdot)_{X' \times X}$ for a given Hilbert space $X$. The space $H^s(D)$ will denote the Sobolev space of order $s$, defined on a domain $D$; $H^s_0(D)$ will denote the closure of $C_0^\infty(D)$ in the $H^s(D)$-norm $\| \cdot \|_{H^s(D)}$. We make use of the standard notation for the boundary trace of functions defined on $\Omega$, which are sufficiently smooth: that is, for a scalar function $\phi \in H^s(\Omega), \frac{1}{2} < s < \frac{3}{2}$, $\gamma(\phi) = \phi|_{\partial \Omega}$, which is a well-defined and surjective mapping on this range of $s$, owing to the Sobolev Trace Theorem on Lipschitz domains (see, e.g., [39], or Theorem 3.38 of [37]). Also, $C > 0$ will denote a generic constant.

In order to reduce the PDE system (1.2)–(1.4) to an ordinary differential equation (ODE) setting in the Hilbert space $H$, define the operators

$$
\mathcal{A} = \begin{bmatrix}
-\mathbf{U} \cdot \nabla (\cdot) & -\operatorname{div}(\cdot) & 0 & 0 \\
-\nabla (\cdot) & \operatorname{div} \sigma(\cdot) - \eta I - \mathbf{U} \cdot \nabla (\cdot) - \nabla \mathbf{U} \cdot (\cdot) & 0 & 0 \\
0 & 0 & 0 & I \\
\mathcal{L}|_{\Omega} & -[2\nu \delta^1_{\mathcal{X}}(\cdot) + \lambda \operatorname{div}(\cdot)]_{\Omega} & -\Delta^2 & 0 \\
\end{bmatrix}
$$

(4.1)
where \( D(A + B) \subseteq H \) is given by

\[
D(A + B) = \{(p_0, u_0, w_1, w_2) \in L^2(\Omega) \times H^1(\Omega) \times H^2_0(\Omega) \times L^2(\Omega) : \text{ properties (A.i)-(A.vi) hold} \}
\]

(A.i) \( \mathbf{U} \cdot \nabla p_0 \in L^2(\Omega) \).
(A.ii) \( \text{div } \sigma(u_0) - \nabla p_0 \in L^2(\Omega) \) (so, \( [\sigma(u_0)n - p_0 n]_{\partial \Omega} \in H^{-\frac{1}{2}}(\partial \Omega) \)).
(A.iii) \( -\Delta^2 w_1 - [2\nu \partial_{\gamma_3}(u_0)_3 + \lambda \text{div}(u_0)]_{\Omega} + p_0 |_{\Omega} \in L^2(\Omega) \) (by elliptic regularity theory \( w_1 \in H^3(\Omega) \)).
(A.iv) \( (\sigma(u_0)n - p_0 n) \perp TH^{1/2}(\partial \Omega) \). That is,

\[
\langle (\sigma(u_0)n - p_0 n, \tau) \rangle_{H^{-\frac{1}{2}}(\partial \Omega) \times H^{\frac{1}{2}}(\partial \Omega)} = 0 \text{ in } D'(\Omega) \text{ for every } \tau \in TH^{1/2}(\partial \Omega).
\]

(A.v) \( w_2 + \mathbf{U} \cdot \nabla w_1 \in H^2_0(\Omega) \) (and so \( w_2 \in H^1_0(\Omega) \)).
(A.vi) The flow velocity component \( u_0 = f_0 + \bar{f}_0 \), where \( f_0 \in V_0 \) and \( \bar{f}_0 \in H^1(\Omega) \) satisfies

\[
\bar{f}_0 = \begin{cases} 
0 & \text{on } S \\
(w_2 + \mathbf{U} \cdot \nabla w_1)n & \text{on } \Omega
\end{cases}
\]

(and so \( f_0 |_{\partial \Omega} \in TH^{1/2}(\partial \Omega) \)).

Then, the function \( \Phi(t) = [p, u, w_1, w_2] \in C([0,T]; H) \) that solves the problem (1.2)–(1.4) satisfies

\[
\frac{d}{dt} \Phi(t) = (A + B)\Phi(t);
\]

\[
\Phi(0) = \Phi_0.
\]

(4.3)
With the above notation, let us take \( \varphi = [p_0, u_0, w_1, w_2] \in H_0, \tilde{\varphi} = [\tilde{p}_0, \tilde{u}_0, \tilde{w}_1, \tilde{w}_2] \in H_0. \) Then, the new inner product is given as

\[
((\varphi, \tilde{\varphi}))_{H_0} = (p_0, \tilde{p}_0)_0 + (u_0 - \alpha D(g \cdot \nabla w_1)e_3 + \xi \nabla \psi(p_0, w_1), \tilde{u}_0 - \alpha D(g \cdot \nabla \tilde{w}_1)e_3 + \xi \nabla \psi(\tilde{p}_0, \tilde{w}_1))_0 \\
+ (\Delta w_1, \Delta \tilde{w}_1)_{\Omega} + (w_2 + h_\alpha \cdot \nabla w_1 + \xi w_1, \tilde{w}_2 + h_\alpha \cdot \nabla \tilde{w}_1 + \xi \tilde{w}_1)_{\Omega},
\]
(4.4)

and in turn the norm

\[
\|\varphi\|^2_{H_0} = ((\varphi, \varphi))_{H_0} = \|p_0\|^2_0 + \|u_0 - \alpha D(g \cdot \nabla w_1)e_3 + \xi \nabla \psi(p_0, w_1)\|^2_0 \\
+ \|\Delta w_1\|^2_{\Omega} + \|w_2 + h_\alpha \cdot \nabla w_1 + \xi w_1\|^2_{\Omega},
\]
(4.5)

for every \( \varphi = [p_0, u_0, w_1, w_2] \in H_0. \) Here,

(a) \( \xi = \left(\frac{1}{2} - C_2r_U\right) - \frac{\sqrt{\left(\frac{1}{2} - C_2r_U\right)^2 - 4C_2(C_1 + C_2r_U)r_U}}{2(C_1 + C_2r_U)}, \)
(b) \( r_U = \|U\|_s + \|U\|_2^s + \|U\|_3^s, \)
(4.6)

where \( C_1, C_2 > 0 \) are independent of measurement \( \|U\|_s \) which is given as

\[
\|U\|_s = \|U\|_{L^\infty(\Omega)} + \|\text{div}(U)\|_{L^\infty(\Omega)} + \|U\|_{C^2(\overline{\Omega})}. 
\]
(4.7)

Also,

(i) the function \( \psi = \psi(f, \chi) \in H^1(\Omega) \) is considered to solve the following Boundary Value Problem (BVP) for data \( f \in L^2(\Omega) \) and \( \chi \in L^2(\Omega) : \)

\[
\begin{cases}
-\Delta \psi = f & \text{in } \Omega \\
\frac{\partial \psi}{\partial n} = 0 & \text{on } S \\
\frac{\partial \psi}{\partial n} = \chi & \text{on } \Omega
\end{cases}
\]
(4.8)

with the compatibility condition

\[
\int_{\partial \Omega} f d\Omega + \int_{\Omega} \chi d\Omega = 0.
\]
(4.9)

We should note that by known elliptic regularity results for the Neumann problem on Lipschitz domains—see, for example, [31]—we have

\[
\|\psi(f, \chi)\|_{H^{3/2}(\partial \Omega)} \leq \left[\|f\|_\Omega + \|\chi\|_{2\partial \Omega}\right].
\]
(4.10)

(ii) The map \( D(\cdot) \) is the Dirichlet map that extends boundary data \( \varphi \) defined on \( \Omega \) to a harmonic function in \( \Omega \) satisfying:

\[
D\varphi = f \iff \begin{cases}
\Delta f = 0 & \text{in } \Omega \\
f|_{\partial \Omega} = \varphi|_{\text{ext}} & \text{on } \partial \Omega,
\end{cases}
\]
(4.11)

where

\[
\varphi|_{\text{ext}} = \begin{cases}
0 & \text{on } S \\
\phi & \text{on } \Omega.
\end{cases}
\]

Then, by, for example, [37, Theorem 3.3.8], and Lax–Milgram Theorem, we deduce that

\[
D \in L(H^{1/2+\varepsilon}_0(\Omega); H^1(\Omega)).
\]
(4.12)
(iii) The vector field \( h_\alpha(\cdot) \) is defined as \( h_\alpha(\cdot) = U|_\Omega - \alpha g \), where \( g(\cdot) \) is a \( C^2 \) extension of the normal vector \( \nu(x) \) (recall, with respect to \( \Omega \)) and we specify the parameter \( \alpha \) to be

\[
\alpha = 2\|U\|_*.
\]

where \( \|U\|_* \) is as defined in (4.7).

5 RESULT I: BOUNDED SEMIGROUP WELLPOSEDNESS IN \( H_0 \)

The bounded semigroup wellposedness is given as follows:

**Theorem 5.1.** With reference to problem (1.2)–(1.4), assume that \( U \in V_0 \cap W \) with

\[
\|U\|_* = \|U\|_{L^\infty(\Omega)} + \|\text{div}(U)\|_{L^\infty(\Omega)} + \|U|_\Omega\|_{C^2(\bar{\Omega})}
\]

sufficiently small. Then, we have the following:

(i) The operator \((A + B) : D(A + B) \cap H_0 \rightarrow H_0\) is maximal dissipative. In particular, it obeys the following inequality for all \( \phi = [p_0, u_0, w_1, w_2] \in D(A + B) \cap H_0 \):

\[
\text{Re}((A + B)\phi, \phi)_{H_0} \leq \left( -\frac{1}{4} + C_\delta r_U \right) \left[ (\sigma(u_0), \epsilon(u_0))_\Omega + \|u_0\|_{\partial\Omega}^2 \right] + \left( -\frac{1}{2} + \delta C^* \right) \xi \left[ \|p_0\|_{\partial\Omega}^2 + \|\Delta w_1\|_{\Omega}^2 \right],
\]

where \( 0 < \delta < \frac{1}{2C^*} \), and \( r_U \) is as given in (4.6).

(ii) Consequently, the operator \((A + B) : D(A + B) \cap H_0 \rightarrow H_0\) generates a strongly continuous semigroup \( \{e^{(A+B)t}\}_{t \geq 0} \) on \( H_0 \). Hence, for every initial data \([\overline{p}, \overline{u}, \overline{w}_1, \overline{w}_2] \in H_0 \), the solution \([p(t), u(t), w_1(t), w_2(t)]\) of problem (1.2)–(1.4) is given continuously by

\[
\begin{bmatrix}
  p(t) \\
  u(t) \\
  w_1(t) \\
  w_2(t)
\end{bmatrix} = e^{(A+B)t}
\begin{bmatrix}
  \overline{p} \\
  \overline{u} \\
  \overline{w}_1 \\
  \overline{w}_2
\end{bmatrix} \in C([0, T]; H_0).
\]

Moreover, this semigroup is uniformly bounded in time with respect to the standard \( H \)-inner product. With respect to the special norm in (4.5), the semigroup \( \{e^{(A+B)t}\}_{t \geq 0} \) is in fact a contraction.

**Proof.** Let

\[
A_0 \equiv A + L_U,
\]

where

\[
L_U =
\begin{bmatrix}
  0 & 0 & 0 & 0 \\
  0 & \nu \cdot (\cdot) & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix}
\]

with \( D(A_0 + B) = D(A + B) \). Therewith, it is established in [30] that \( A_0 + B \) is maximal dissipative in \( H_0 \), for all \( \|U\|_* \) sufficiently small. In particular, we have for \( \phi = [p_0, u_0, w_1, w_2] \in D(A_0 + B) \cap H_0 \):

\[
\text{Re}((A_0 + B)\phi, \phi)_{H_0} \leq -\frac{1}{4} (\sigma(u_0), \epsilon(u_0))_\Omega - \frac{\eta}{4} \|u_0\|_{\partial\Omega}^2 - \frac{\xi}{2} \|p_0\|_{\partial\Omega}^2 - \frac{\xi}{2} \|\Delta w_1\|_{\Omega}^2,
\]

(5.5)
where parameter $\xi$ is as given in (4.6). (For the sake of completion, the proof of this estimate is given in the Appendix.)

Given $\phi = [p_0, u_0, w_1, w_2] \in D(A + B) \cap H_0$, invoking (5.5) we have

\[
((A + B) \phi, \phi)_{H_0} = ((A + B) \phi, \phi)_{H_0} - ((L_U \phi, \phi))_{H_0} \\
\leq -\frac{1}{4}(\sigma(u_0), \varepsilon(u_0))_{\partial \Omega} - \frac{\eta}{4}\|u_0\|^2_{\Omega} - \frac{\xi}{2}\|p_0\|^2_{\partial \Omega} - \frac{\xi}{2}\|\Delta w_1\|^2_{\Omega} \\
- (\nabla U \cdot (u_0), u_0 - \alpha D(g \cdot \nabla w_1) e_3 + \xi \nabla \psi(p_0, w_1))_{\partial \Omega} \\
= -\frac{1}{4}(\sigma(u_0), \varepsilon(u_0))_{\partial \Omega} - \frac{\eta}{4}\|u_0\|^2_{\partial \Omega} - \frac{\xi}{2}\|p_0\|^2_{\partial \Omega} - \frac{\xi}{2}\|\Delta w_1\|^2_{\Omega} \\
- \int_{\partial \Omega} (u_0 \cdot n) U \cdot [u_0 - \alpha D(g \cdot \nabla w_1) e_3 + \xi \nabla \psi(p_0, w_1)] d\partial \Omega \\
+ \int_{\partial \Omega} \text{div}(u_0) U \cdot [u_0 - \alpha D(g \cdot \nabla w_1) e_3 + \xi \nabla \psi(p_0, w_1)] d\partial \Omega \\
+ \int_{\partial \Omega} U \cdot (u_0 \cdot \nabla [u_0 - \alpha D(g \cdot \nabla w_1) e_3 + \xi \nabla \psi(p_0, w_1)]) d\partial \Omega. 
\] (5.6)

We estimate the right-hand side (RHS) implicitly using the regularity results of [25–27], which are valid under the assumptions made on the geometry.

(I) The mapping in (4.11) satisfies

\[
\|D(g \cdot \nabla w_1)\|_{H^1(\partial \Omega)} \leq C\|w_1\|_{H^2(\Omega)}. 
\] (5.7)

Thus, we have

(I-a):

\[
\left| \alpha \int_{\partial \Omega} (u_0 \cdot n) U \cdot D(g \cdot \nabla w_1) e_3 d\partial \Omega \right| \leq \alpha \|U\|_{\partial \Omega} \|u_0\|_{H^1(\partial \Omega)} \|w_1\|_{H^2(\Omega)} \\
= \sqrt{\|U\|_{\partial \Omega}} \|u_0\|_{H^1(\partial \Omega)} \frac{\alpha \sqrt{\|U\|_{\partial \Omega}}}{\sqrt{\xi}} \|\Delta w_1\|_{\Omega} \\
\leq C_\delta \|U\|_{\partial \Omega} \|u_0\|_{H^1(\partial \Omega)}^2 + \frac{\delta \alpha^2 \|U\|_{\partial \Omega} \xi}{\xi} \|\Delta w_1\|_{\Omega}^2 \\
\leq C_\delta \|U\|_{\partial \Omega} \|u_0\|_{H^1(\partial \Omega)}^2 + \delta C \frac{\|U\|_{\partial \Omega}}{\xi} \|\Delta w_1\|_{\Omega}^2 \\
\leq C_\delta \|U\|_{\partial \Omega} \|u_0\|_{H^1(\partial \Omega)}^2 + \delta C \xi \|\Delta w_1\|_{\Omega}^2. \] (5.8)

At this point, we should emphasize that $\frac{\|U\|_{\partial \Omega}}{\xi}$ will be bounded if $\|U\|_{\partial \Omega}$ is small enough and so $C > 0$ is independent of $\|U\|_{\partial \Omega}$. Also, we implicitly used the Sobolev Trace Theorem for the first inequality.

(I-b): Again by (5.7), similarly

\[
\left| \alpha \int_{\partial \Omega} U \cdot (u_0 \cdot \nabla D(g \cdot \nabla w_1) e_3) d\partial \Omega \right| \leq C_\delta \|U\|_{\partial \Omega} \|u_0\|_{H^1(\partial \Omega)}^2 + \delta C \xi \|\Delta w_1\|_{\Omega}^2. \] (5.9)
(II) Using (4.10) and (5.7), we have as in (I-a),

\[
\left| \int_{\partial} \text{div}(u_0)U \cdot [u_0 - \alpha D(g \cdot \nabla w_1)e_3 + \xi \nabla \psi(p_0, w_1)]d\partial \right| \leq C_\delta \left( \|U\|_{L^2(\partial)}^2 + \|U\|_{L^2(\partial)} \|u_0\|_{H^1(\partial)}^2 \right) + \delta C_\xi \left( \|p_0\|_{\partial}^2 + \|\Delta w_1\|_{\Omega}^2 \right). \tag{5.10}
\]

(III) The mapping (4.8) satisfies

\[
|\psi(p_0, w_1)|_{H^2(\partial)} \leq \left[ \|p_0\|_{\partial} + \|w_1\|_{H^2(\partial)} \right]. \tag{5.11}
\]

Consequently, we have

\[
\left| \frac{\xi}{\delta} \int_{\partial} (u_0 \cdot n)U \cdot \nabla \psi(p_0, w_1)d\partial \right| + \left| \frac{\xi}{\delta} \int_{\partial} U \cdot (u_0 \cdot \nabla \psi(p_0, w_1))d\partial \right| \leq C_\delta \left( \|U\|_{L^2(\partial)} \|u_0\|_{H^1(\partial)}^2 \right) + \delta C_\xi \left( \|p_0\|_{\partial}^2 + \|\Delta w_1\|_{\Omega}^2 \right). \tag{5.12}
\]

Applying the estimates (5.8),(5.9),(5.10), and (5.12) to RHS of (5.6), we now have

\[
\text{Re}((\mathcal{A} + \mathcal{B})\varphi, \varphi)_{H_0} \leq -\frac{1}{4}(\sigma(u_0), \epsilon(u_0))_{\partial} - \frac{\eta}{4}\|u_0\|_{\partial}^2 - \frac{\xi}{2}\|p_0\|_{\partial}^2 - \frac{\xi}{2}\|\Delta w_1\|_{\Omega}^2 + C_\delta \|U\|_{L^2(\partial)}^2 \|u_0\|_{H^1(\partial)}^2 \|u_0\|_{H^1(\partial)}^2 \tag{5.13}
\]

which yields the estimate (5.1), upon a use of Korn’s inequality and this gives the dissipativity of \(\mathcal{A} + \mathcal{B}\), for \(U\) sufficiently small and \(0 < \delta < \frac{1}{2C^*}\). Moreover, it is shown in [30] that \(\mathcal{A}_0 + \mathcal{B}\) is maximal dissipative, then by a perturbation argument—see, for example, p. 211 of [33]—\(\mathcal{A} + \mathcal{B}\) is likewise maximal dissipative. This concludes the proof of Theorem 5.1.

\[\square\]

6 RESULT II: ASYMPTOTIC (STRONG) STABILITY

This section is devoted to address the issue of asymptotic behavior of the solution whose existence-uniqueness is guaranteed by Theorem 5.1. In this regard, we show that the system given in (1.2)–(1.4) is strongly stable in \(H_0\). That is, given any \([\bar{p}, \bar{u}, \bar{w}_1, \bar{w}_2] \in H_0\), the corresponding solution \([p(t), u(t), w_1(t), w_2(t)] \in C([0, T]; H_0)\) of (1.2)–(1.4) satisfies

\[
\lim_{t \to \infty} \|\|p(t), u(t), w_1(t), w_2(t)\|_{H_0} = 0.
\]

Our proof will be based on an ultimate appeal to the pointwise criterion [18, p. 26, Theorem 8.4 (i)]:

**Theorem 6.1.** Let \(A\) be the generator of a bounded \(C_0\)-semigroup on a Hilbert space \(X\). If there exists a dense set \(M \subseteq X\) such that

\[
\lim_{\alpha \to 0^+} \sqrt{\alpha R(\alpha + i\beta; A)x} = 0
\]

for every \(x \in M\) and every \(\beta \in \mathbb{R}\), then the semigroup is stable.
There are reasons why we use here the resolvent criterion in [18], instead of the more well-known approaches in [36] and [2]. For one, the domain of the flow–structure generator \( \mathcal{A} + B \) is not compactly embedded into \( H_0 \), and so the classical stabilizability approach in [36] is not possible. In addition, the spectrum criterion for stability in [2] is not applicable here since it would entail the elimination of all three parts of \( \sigma(A + B) \); particularly, the analysis of residual spectrum will be of great technical complexity and mathematical challenge since it requires one to prove that \( i\beta (\beta \neq 0) \) is not an eigenvalue of the adjoint of the generator \( \mathcal{A} + B \). In order to justify our concern related to a residual spectrum analysis and to see how technical calculations the adjoint of the generator will require, we provide here the adjoint operator \((\mathcal{A} + B)^*\) (for a detailed proof, the reader is referred to [30, Lemma 13]):

**Lemma 6.2.** With reference to problem (1.2)–(1.4), the adjoint operator

\[
(\mathcal{A} + B)^* : D((\mathcal{A} + B)^*) \cap H_0 \subset H_0 \rightarrow H_0
\]

of the semigroup generator \( \mathcal{A} + B \) is defined as

\[
(\mathcal{A} + B)^* = \mathcal{A}^* + B^*
\]

\[
= \begin{bmatrix}
U \cdot \nabla (\cdot) & \text{div}(\cdot) & 0 & 0 \\
0 & \text{div}(\cdot) & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-\text{div}(\mathcal{U})(\cdot) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
= L_1 + L_2 + B^*.
\]

(6.1)

Here, \( V^* \in \mathcal{L}(L^2(\Omega), [H^1(\Omega)]') \) is the adjoint of the gradient operator \( V \in \mathcal{L}(H^1(\Omega), L^2(\Omega)) \) and the domain of \( (\mathcal{A} + B)^* |_{H_0} \) is given as

\[
D((\mathcal{A} + B)^*) \cap H_0 = \{(p_0, u_0, w_1, w_2) \in L^2(\Omega) \times H^1(\Omega) \times H_0^2(\Omega) \times L^2(\Omega) : \text{properties (A*.i)–(A*.vii) hold},
\]

where

1. \( \text{(A*.i)} \) \( U \cdot \nabla p_0 \in L^2(\Omega) \).
2. \( \text{(A*.ii)} \) \( \text{div} \sigma(p_0) + \nabla p_0 \in L^2(\Omega) \) (so, \( \sigma(u_0) n + p_0 n |_{\partial \Omega} \in H^{-\frac{1}{2}}(\partial \Omega) \)).
3. \( \text{(A*.iii)} \) \( \Delta^2 \omega_1 - \left[2\nu \partial_{x_3}(u_0) + \lambda \text{div}(u_0)\right]_{\partial \Omega} - p_0 |_{\partial \Omega} \in L^2(\Omega) \).
4. \( \text{(A*.iv)} \) \( \sigma(u_0) n + p_0 n \perp TH^{1/2}(\partial \Omega) \). That is,

\[
\langle \sigma(u_0) n + p_0 n, \tau \rangle_{H^{-\frac{1}{2}}(\partial \Omega) \times H^{\frac{1}{2}}(\partial \Omega)} = 0 \text{ in } D'(\partial \Omega) \text{ for every } \tau \in TH^{1/2}(\partial \Omega).
\]
5. \((\mathbf{A}^* \cdot \mathbf{v})\) The flow velocity component \(u_0 = \mathbf{f}_0 + \tilde{\mathbf{f}}_0\), where \(\mathbf{f}_0 \in \mathbf{V}_0\) and \(\tilde{\mathbf{f}}_0 \in \mathbf{H}^1(\mathcal{O})\) satisfies

\[
\tilde{\mathbf{f}}_0 = \begin{cases} 
0 & \text{on } S \\
w_2 \mathbf{n} & \text{on } \Omega 
\end{cases}
\]

(and so \(\mathbf{f}_0 \mid_{\partial \mathcal{O}} \in \mathbf{TH}^{1/2}(\partial \mathcal{O})\).

6. \((\mathbf{A}^* \cdot \mathbf{v} \cdot \mathbf{i})\left[-w_2 + \mathbf{U} \cdot \nabla w_1 + \Delta \mathbf{A}^{-1} \mathbf{V}^* (\nabla \cdot (\mathbf{U} \cdot \nabla w_1))\right] \in \mathbf{H}^2_0(\Omega)\) (and so \(w_2 \in \mathbf{H}^1(\Omega)\)).

7. \((\mathbf{A}^* \cdot \mathbf{v}^2)\)

\[
\int_{\mathcal{O}} (\mathbf{U} \cdot \nabla p_0 + \nabla \cdot \mathbf{u}_0) \, d\mathcal{O} + \int_{\Omega} \mathbf{A}^{-1} \left\{ \left( \nabla \cdot (\mathbf{U}_1, \mathbf{U}_2) + \mathbf{U} \cdot \nabla \right) \left[ p_0 + 2\nu \partial x_3 (u_0)_3 + \lambda \nabla \cdot \mathbf{u}_0 \right] \right\} \, d\mathcal{O} \\
- \int_{\Omega} \mathbf{A}^{-1} \left\{ \left( \nabla \cdot (\mathbf{U}_1, \mathbf{U}_2) + \mathbf{U} \cdot \nabla \right) \Delta^2 \mathbf{w}_1 \right\} \, d\mathcal{O} + \int_{\Omega} \left( \mathbf{U} \cdot \nabla \mathbf{w}_1 + \Delta \mathbf{A}^{-1} \mathbf{V}^* (\nabla \cdot (\mathbf{U} \cdot \nabla \mathbf{w}_1)) \right) \, d\mathcal{O} = 0.
\]

Looking at the expression for \(\mathbf{A}^* + \mathbf{B}^*\) in (6.1), it seems that showing this adjoint operator has empty null space—necessary for ruling out residual spectrum of \(\mathbf{A} + \mathbf{B}\)—would be a difficult exercise. Now, our stability result is as follows:

**Theorem 6.3.** The bounded \(C_0\)-semigroup \(\{e^{(\mathbf{A}^* + \mathbf{B}^*) t}\}_{t \geq 0}\) given in Theorem 5.1 is strongly stable under the condition that

\[
\| \mathbf{U} \|_e = \| \mathbf{U} \|_{L^\infty(\mathcal{O})} + \| \nabla \cdot \mathbf{U} \|_{L^\infty(\mathcal{O})} + \| \mathbf{U} \|_{C^2(\Omega)}
\]

is small enough. That is, the solution \(\phi(t) = [p(t), u(t), \omega_1(t), \omega_2(t)]\) of the PDE system (1.2)–(1.4) tends asymptotically to the zero state for all initial data \(\phi_0 \in \mathcal{H}_0\).

**Proof.** The proof relies on the pointwise resolvent criterion given in Theorem 6.1, and hence to obtain the strong stability result it will be enough to show that the resolvent operator

\[
R(a + ib; [\mathbf{A} + \mathbf{B}]) = ((a + ib)I - [\mathbf{A} + \mathbf{B}])^{-1}
\]

obeys the limit estimate:

\[
\lim_{a \to 0^+} \| \sqrt{a} R(a + ib; [\mathbf{A} + \mathbf{B}]) \phi^* \|_{\mathcal{H}_0} = 0,
\]

where \(a > 0, b \in \mathbb{R}\) and given \(\phi^* \in \mathcal{H}_0\). Here, we invoke the special inner product \((\cdot, \cdot)\) (defined in (4.4)) to get the necessary estimates but since the norms \(|| \cdot ||_{\mathcal{H}_0} \) and \(|| \cdot ||_{\mathcal{H}_0} \) are equivalent, we obtain the strong stability with respect to the standard inner product as well. Let

\[
\phi = \begin{bmatrix} p_0 \\ u_0 \\ w_1 \\ w_2 \end{bmatrix} = R(a + ib; [\mathbf{A} + \mathbf{B}]) \phi^* \in D(\mathbf{A} + \mathbf{B}) \cap \mathcal{H}_0, \quad \phi^* = \begin{bmatrix} p_0^* \\ u_0^* \\ w_1^* \\ w_2^* \end{bmatrix} \in \mathcal{H}_0.
\]

Then, \(\phi\) solves the following static PDE system:

\[
\begin{align*}
(a + ib)p_0 + \mathbf{U} \cdot \nabla p_0 + \text{div} u_0 + \text{div}(\mathbf{U}) p_0 &= p_0^* \quad \text{in } \mathcal{O} \\
(a + ib)u_0 + \mathbf{U} \cdot \nabla u_0 + \nabla \cdot \mathbf{u}_0 - \text{div} \sigma(u_0) + \eta u_0 + \nabla p_0 &= u_0^* \quad \text{in } \mathcal{O} \\
\sigma(u_0) \mathbf{n} - p_0 \mathbf{n} \cdot \mathbf{t} &= 0 \quad \text{on } \partial \mathcal{O} \\
u_0 \cdot \mathbf{n} &= 0 \quad \text{on } S \\
u_0 \cdot \mathbf{n} &= w_2 + \mathbf{U} \cdot \nabla w_1 \quad \text{on } \Omega,
\end{align*}
\]

\[
\begin{align*}
(a + ib)w_1 - w_2 - \mathbf{U} \cdot \nabla w_1 &= w_1^* \quad \text{on } \Omega \\
(a + ib)w_1 + \Delta^2 w_1 + \left[ 2\nu \partial x_3 (u_0)_3 + \lambda \text{div}(u_0) - p_0 \right] &= w_2^* \quad \text{on } \Omega \\
w_1 &= \frac{\partial w_1}{\partial \nu} = 0 \quad \text{on } \partial \Omega.
\end{align*}
\]
We invoke the dissipativity estimate (5.1) in Theorem 5.1 (i):

$$(a + ib) \| \phi \|_{H_0}^2 - (\langle [A + B] \phi, \phi \rangle)_{H_0} = ((\phi^*, \phi))_{H_0}$$

or

$$a \| \phi \|_{H_0}^2 - \text{Re}(\langle [A + B] \phi, \phi \rangle)_{H_0} = \text{Re}((\phi^*, \phi))_{H_0},$$

which gives

$$\left( \frac{1}{4} - C r u \right) \left[ \sigma(u_0), c(u_0) \right] + \| u_0 \|_{\mathcal{O}}^2 \leq \text{Re}((\phi^*, \phi))_{H_0}. \tag{6.5}$$

In turn, from the boundary condition $w_2 = u_0 \cdot n - U \cdot \nabla w_1$ and (6.5), we get

$$\| w_2 \|_{\Omega} \leq C_{\xi, U} \sqrt{\text{Re}((\phi^*, \phi))_{H_0}}, \tag{6.6}$$

where we have also used implicitly the Sobolev Trace Theorem. Now, combining (6.5) and (6.6), and using the equivalence of norms $\| \cdot \|_{H_0}$ and $\| \cdot \|_{H}$ on $H_0$, we have then

$$\| \phi \|_{H_0} \leq C_{\xi, U} \sqrt{\text{Re}((\phi^*, \phi))_{H_0}}.$$

Scaling the above inequality by $\sqrt{a}$, followed by Young’s Inequality gives now

$$\sqrt{a} \| \phi \|_{H_0} \leq \sqrt{a} C_{\xi, U} \| \phi^* \|_{H_0} \| \phi \|_{H_0} \leq \frac{\sqrt{a}}{2} C_{\xi, U}^2 \| \phi^* \|_{H_0} + \frac{\sqrt{a}}{2} \| \phi \|_{H_0}.$$

From the last inequality, we have

$$\frac{\sqrt{a}}{2} \| \phi \|_{H_0} \leq \frac{\sqrt{a}}{2} C_{\xi, U}^2 \| \phi^* \|_{H_0},$$

and so

$$\lim_{a \to 0^+} \sqrt{a} \| \phi \|_{H_0} = 0.$$

This gives the desired limit estimate (6.2) and concludes the proof of Theorem 6.3.

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\section*{Conflict of Interest}

The authors declare no potential conflict of interests.

\section*{Endnote}

\footnote{The existence of an $H^1(\mathcal{O})$-function $\tilde{f}_0$ with such a boundary trace on Lipschitz domain $\mathcal{O}$ is assured; see, for example, Theorem 3.33 of [37].}

\section*{References}

[1] R. Aoyama and Y. Kagei, Spectral properties of the semigroup for the linearized compressible Navier-Stokes equation around a parallel in a cylindrical domain, Adv. Differ. Equ. 21 (2016), no. (3/4), 265–300.

[2] W. Arendt and C. J. K. Batty, Tauberian theorems and stability of one-parameter semigroups, Trans. Am. Math. Soc. 306 (1988), no. 2, 837–852.
[3] G. Avalos and T. Clark, A mixed variational formulation for the wellposedness and numerical approximation of a PDE model arising in a 3-D fluid-structure interaction, Evol. Equ. Control Theory. 3 (2014), no. 4, 557–578.

[4] G. Avalos and M. Dvorak, A new maximality argument for a coupled fluid-structure interaction, with implications for a divergence free finite element method, Appl. Math. 35 (2008), no. 3, 259–280.

[5] G. Avalos and F. Bucci, Exponential decay properties of a mathematical model for a certain flow-structure interaction, New Prospects in Direct, Inverse and Control Problems for Evolution Equations, Springer International Publishing, 2014, pp. 49–78.

[6] G. Avalos and F. Bucci, Rational rates of uniform decay for strong solutions to a flow-structure PDE system, J. Differ. Equ. 258 (2015), no. 12, 4398–4423.

[7] G. Avalos and P. G. Geredeli, Exponential stability of a nondissipative, compressible flow-structure PDE model, J. Evol. Equ. 20 (2020), 1–38.

[8] G. Avalos, P. G. Geredeli, and J. T. Webster, Semigroup well-posedness of a linearized, compressible flow with an elastic boundary, Discrete Contin. Dyn. Syst. Ser. B 23 (2018), no. 3, 1267–1295.

[9] G. Avalos, P. G. Geredeli, and B. Muha, Wellposedness, spectral analysis and asymptotic stability of a multilayered heat-wave-wave system, J. Differ. Equ. 269 (2020), 7129–7156.

[10] G. Avalos, P. G. Geredeli, and J. T. Webster, A linearized viscous, compressible flow-plate interaction with non-dissipative coupling, J. Math. Anal. Appl. 477 (2019), no. 1, 334–356.

[11] G. Avalos and R. Triggiani, The coupled PDE system arising in fluid-structure interaction, part I: explicit semigroup generator and its spectral properties, Contemp. Math. 440 (2007), 15–54.

[12] G. Avalos and R. Triggiani, Semigroup wellposedness in the energy space of a parabolic-hyperbolic coupled Stokes-Lamé PDE of fluid-structure interactions, Discrete Contin. Dyn. Syst. 2 (2009), no. 3, 417–447.

[13] G. Avalos, R. Triggiani and I. Lasiecka, Heat-wave interaction in 2 or 3 dimensions: optimal decay rates, J. Math. Anal. Appl. 437 (2016), no. 2, 782–815.

[14] L. Bociu, D. Toundykov, and J. P. Zolasio, Well-posedness analysis for a linearization of a fluid-elasticity interaction, SIAM J. Math. Anal. 47 (2015), no. 3, 1958–2000.

[15] V.V. Bolotin, Nonconservative problems of the theory of elastic stability, Macmillan, Pergamon Press, 1963.

[16] K. N. Boyadzhiev and N. Levan, Strong stability of Hilbert space contraction semigroups, Stud. Sci. Math. Hung. 30 (1995), 165–182.

[17] A. Buffa, M. Costabel, and D. Sheen, On traces for $H(curl, \Omega)$ in Lipschitz domains, J. Math. Anal. Appl. 276 (2002), no. 2, 845–867.

[18] R. Chill and Y. Tomilov, Stability of operator semigroups: ideas and results, Vol. 75, Perspectives in Operator Theory Banach Center PublicationsInstitute of Mathematics Polish Academy of Sciences, Warszawa, 2007, pp. 71–109.

[19] I. Chueshov, I. Lasiecka, and J. T. Webster, Flow-plate interactions: well-posedness and long-time behavior, Discrete Contin. Dyn. Syst. - S 7 (2014), no. 5, 925–965.

[20] I. Chueshov and I. Ryzhkova, Wellposedness and long time behavior for a class of fluid-plate interaction models, IFIP Conference on System Modeling and Optimization Springer, Berlin, Heidelberg, 2011, pp. 328–337.

[21] I. Chueshov, Dynamics of a nonlinear elastic plate interacting with a linearized compressible viscous flow, Nonlinear Anal. Theory Methods Appl. 95 (2014), 650–665.

[22] I. Chueshov, Interaction of an elastic plate with a linearized inviscid incompressible fluid, Commun. Pure Appl. Anal. 13 (2014), no. 5, 1459–1778.

[23] H. B. da Veiga, Stationary motions and incompressible limit for compressible viscous flows, Houst. J. Math. 13 (1987), no. 4, 527–544.

[24] M. Dauge, Stationary Stokes and Navier Stokes systems on two or three dimensional domains with corners, part I: linearized equations, Siam J. Math. Anal. 20 (1989), no. 1.

[25] M. Dauge, Elliptic boundary value problems on corner domains, Lecture Notes in Mathematics, Vol. 1341 Springer, New York, 1988.

[26] M. Dauge, Neumann and mixed problems on curvilinear polyhedral domains Integr. Equat. Oper. Th 15 (1992), 227–261.

[27] M. Dauge, Regularity and singularities in polyhedral domains. The case of Laplace and Maxwell equations. Slides d’un mini-cours de 3 heures, Karlsruhe 7 (2008) https://perso.univ-rennes1.fr/monique.dauge/publis/Talk-Karlsruhe08.html.

[28] E. Dowell, A modern course in aeroelasticity, Kluwer Academic Publishers, 2004.

[29] P. G. Geredeli, A time domain approach for the exponential stability of a nondissipative linearized compressible flow-structure PDE system, Math. Methods Appl. Sci. 44 (2021), no. 2, 1326–1342.

[30] P.G. Geredeli, Bounded Semigroup Wellposedness for a linearized compressible flow structure PDE interaction with material derivative, SIAM J. Math. Anal. 53 (2021), no. 2, 1711–1744.

[31] D.S. Jerison and C. E. Kenig, The Neumann problem on Lipschitz domains, Bull. New Ser. Am. Math Soc. 4 (1981), 203–207.

[32] Y. Kagei, Decay estimates on solutions of the linearized compressible Navier-Stokes equation around a parallel flow in a cylindrical domain, Kyushu J. Math. 69 (2015), 293–343.

[33] S. Kesavan, Topics in functional analysis and applications, Wiley Eastern Publisher, India, 1989.

[34] J. Lagnese, Boundary stabilization of thin plates, SIAM, Philadelphia, 1989.

[35] I. Lasiecka and R. Triggiani, Control theory for partial differential equations: volume 1, abstract parabolic systems: continuous and approximation theories, Cambridge University Press, Cambridge, 2000.

[36] N. Levan, The stabilizability problem: “A Hilbert Space Operator Decomposition Approach,” IEEE Trans. Circuits Syst. CAS-25 (1978), no. 9, 721–727.

[37] W. C. H. McLean, Strongly elliptic systems and boundary integral equations, Cambridge University Press, Cambridge, 2000.
APPENDIX A

In this paper, one of the key points used to analyze the stability properties of the PDE system (1.2)–(1.4) is the dissipativity estimate (5.1) in the inherently invariant subspace of the finite energy space. To show this, we appeal to the dissipativity result given in [30] for an operator, which is closely associated with our generator $A + B$. For the readers’ convenience, we also provide its proof.

**Lemma A.1.** Let $A_0$ be the operator defined in (5.3). Then, with reference to problem (1.2)–(1.4), the semigroup generator $(A_0 + B) : D(A_0 + B) \cap H_0 \subset H_0 \to H_0$ is dissipative with respect to inner product $(\cdot, \cdot)_{H_0}$ for $\|U\|_s = \|U\|_{L^\infty(\Omega)} + \|\text{div}(U)\|_{L^\infty(\Omega)} + \|U\|_{C^0(\overline{\Omega})}$ small enough. In particular, for $\varphi = [p_0, u_0, w_1, w_2] \in D(A_0 + B) \cap H_0$,

$$
\text{Re}((A_0 + B)\varphi, \varphi)_{H_0} \leq -\frac{(\sigma(u_0), \varepsilon(u_0))_\Omega}{4} - \frac{\eta\|u_0\|_{C^2(\Omega)}^2}{4} - \frac{\xi\|p_0\|_{C^0(\Omega)}^2}{2} - \frac{\xi\|\Delta w_1\|_{C^0(\Omega)}^2}{2},
$$

(A.1)

where $\xi$ is specified in (A.34).

**Proof.** Given $\varphi = [p_0, u_0, w_1, w_2] \in D(A_0 + B) \cap H_0$, we have

$$
((A_0 + B)\varphi, \varphi)_{H_0} = (-\nabla p_0 - \text{div}(u_0) - \text{div}(U)p_0, p_0)_\Omega \\
+(-\nabla p_0 + \alpha D(g \cdot \nabla w_1))e_3, u_0 - \alpha D(g \cdot \nabla w_1)e_3)_\Omega \\
+(-\nabla p_0 + \alpha D(u_0 - \nabla u_0, u_0 - \alpha D(g \cdot \nabla w_1))e_3)_\Omega \\
+(-\nabla p_0 + \alpha D(u_0 - \nabla u_0, \xi \psi(p_0, w_1))e_3)_\Omega \\
+(-\nabla p_0 + \alpha D(u_0 - \nabla u_0, \xi \psi(p_0, w_1))e_3)_\Omega \\
+\xi(\Delta \psi(\nabla p_0 - \nabla u_0 - \nabla(U)p_0, w_2 + \nabla w_1), w_0 - \alpha D(g \cdot \nabla w_1)e_3)_\Omega \\
+\xi(\Delta \psi(\nabla p_0 - \nabla u_0 - \nabla(U)p_0, w_2 + \nabla w_1), \psi(p_0, u_1))_\Omega \\
+\xi(\Delta \psi(\nabla p_0 - \nabla u_0 - \nabla(U)p_0, w_2 + \nabla w_1), \psi(p_0, u_1))_\Omega \\
+\Delta w_2, \Delta w_1)_{\Omega} + (\Delta(\nabla(U)w_1), \Delta w_1)_{\Omega} \\
+(p_0|_\Omega - [2\nu\partial_{x_3}(u_0)_3 + \lambda \text{div}(u_0)]|_{\Omega}, w_2 + h_x \cdot \nabla w_1 + \xi w_1)_{\Omega} \\
+(h_x \cdot U|_{\Omega}, w_2 + h_x \cdot \nabla w_1 + \xi w_1)_{\Omega} \\
-(\Delta^2 w_1, w_2 + h_x \cdot \nabla w_1 + \xi w_1)_{\Omega} \\
+\xi(\Delta w_2, w_2 + h_x \cdot \nabla w_1 + \xi w_1)_{\Omega} \\
+\xi(\Delta^2 w_1, w_2 + h_x \cdot \nabla w_1 + \xi w_1)_{\Omega}.
$$

After integration by parts, we then arrive at
\[ \begin{align*}
((A_0 + B \varphi, \varphi))_{H_0} &= -\langle \sigma(u_0), \xi \rangle_{\partial \Omega} - \eta \| u_0 \|_{H^1(\Omega)}^2 + \frac{1}{2} \int_{\partial \Omega} \text{div}(U) |u_0|^2 - |p_0|^2 \, d\partial \Omega \\
&+ 2i \text{Im}[(p_0, \text{div}(u_0))_{\partial \Omega} + (\Delta w_2, \Delta w_1)_{\Omega}] - i \text{Im}[(U \nabla p_0, p_0)_{\partial \Omega} + (U \nabla u_0, u_0)_{\partial \Omega}] + \sum_{j=1}^8 I_j, \quad (A.2)
\end{align*} \]

where above the \( I_j \) are given by:

\[ I_1 = (\nabla p_0 - \text{div} \sigma(u_0) + \eta u_0 + U \nabla u_0, \alpha D(g \cdot \nabla w_1) e_3 \rangle_{\partial \Omega} - \alpha(p_0 |_{\partial \Omega} - [2\nu \partial x_j(u_0)_3 + \lambda \text{div}(u_0)] |_{\partial \Omega}, g \cdot \nabla w_1)_{\Omega}, \quad (A.3) \]

\[ I_2 = (\nabla p_0 + \text{div} \sigma(u_0) - \eta u_0 - U \nabla u_0, \xi \nabla \psi(p_0, w_1))_{\partial \Omega} - \xi(\Delta^2 w_1, w_1)_{\Omega} + (p_0 |_{\partial \Omega} - [2\nu \partial x_j(u_0)_3 + \lambda \text{div}(u_0)] |_{\partial \Omega}, \xi w_1)_{\Omega}, \quad (A.4) \]

\[ I_3 = -\alpha(D(g \cdot \nabla(w_2 + U \nabla w_1)) e_3, u_0 - \alpha D(g \cdot \nabla w_1) e_3 + \xi \nabla \psi(p_0, w_1))_{\partial \Omega}, \quad (A.5) \]

\[ I_4 = \xi(\nabla \psi(-U \nabla p_0 - \text{div}(u_0)) - \text{div}(U)p_0, w_2 + U \nabla w_1), u_0 - \alpha D(g \cdot \nabla w_1) e_3)_{\partial \Omega}, \quad (A.6) \]

\[ I_5 = \xi^2(\nabla \psi(-U \nabla p_0 - \text{div}(u_0)) - \text{div}(U)p_0, w_2 + U \nabla w_1), \nabla \psi(p_0, w_1))_{\partial \Omega}, \quad (A.7) \]

\[ I_6 = (\Delta(U \nabla w_1), \Delta w_1)_{\Omega} - (\Delta^2 w_1, h_x \cdot \nabla w_1)_{\Omega}, \quad (A.8) \]

\[ I_7 = (h_x \cdot \nabla w_2 + U \nabla w_1, w_2)_{\Omega}, \quad (A.9) \]

\[ I_8 = (h_x \cdot \nabla w_2 + U \nabla w_1, h_x \cdot \nabla w_1 + \xi w_1)_{\Omega} + \xi(w_2 + U \nabla w_1, w_2 + h_x \cdot \nabla w_1 + \xi w_1)_{\Omega}, \quad (A.10) \]

where we also recall the definition \( h_x = U|_{\Omega} - \alpha g \). In the course of estimating the terms \( (A.3) - (A.10) \) above, we will invoke the polynomial

\[ r(a) = a + a^2 + a^3, \]

and for the simplicity, we set

\[ r_U = r(\| U \|_{L^2}). \]

We start with \( I_1 \); integrating by parts, we have

\[ I_1 = -\alpha(p_0, \text{div}(D(g \cdot \nabla w_1)e_3))_{\partial \Omega} + \alpha(\sigma(u_0), \xi (D(g \cdot \nabla w_1)e_3))_{\partial \Omega} + \alpha \eta(u_0, D(g \cdot \nabla w_1)e_3)_{\partial \Omega} + \alpha(U \nabla u_0, D(g \cdot \nabla w_1)e_3)_{\partial \Omega}. \quad (A.11) \]

Using the fact that Dirichlet map \( D \in L(H^{\frac{1}{2}}_{0,(\gamma)}(\Omega), H^1(\partial \Omega)) \), we have

\[ I_1 \leq r_U C \left\{ \| u_0 \|_{H^{\frac{1}{2}}(\Omega)}^2 + \| p_0 \|_{\partial \Omega}^2 + \| \Delta w_1 \|_{\Omega}^2 \right\}. \quad (A.12) \]
We continue with $I_2$; using the definition of the map $\psi(\cdot, \cdot)$ in (4.8) and integrating by parts, we get

$$
I_2 = -\xi \int_{\partial} |p_0|^2 d\partial - \xi (\sigma(u_0), \varepsilon(\nabla \psi(p_0, w_1)))_{\partial}
+ \xi (\sigma(u_0) n - p_0 n, (\nabla \psi(p_0, w_1), n)n)_{\partial} - \eta(u_0, \xi \nabla \psi(p_0, w_1))_{\partial} \\
\times (-U \nabla u_0, \xi \nabla \psi(p_0, w_1))_{\partial} - (\Delta^2 w_1, \xi w_1)_{\Omega}
+ (p_0 |_{\Omega} - \left[ 2\nu s_3(u_0) + \lambda \text{div}(u_0) \right]_{\Omega}, \xi w_1)_{\Omega},
$$

whence we obtain

$$
I_2 \leq -\xi \|p_0\|^2_{\partial} - \xi \|\Delta w_1\|^2_{\Omega} + \xi r_U C \left\{ \|u_0\|^2_{H^1(\Omega)} + \|p_0\|^2_{\partial} + \|\Delta w_1\|^2_{\Omega} \right\}
+ \xi C \left\{ \|u_0\|_{H^1(\partial)} \left[ \|p_0\|_{\partial} + \|\Delta w_1\|_{\Omega} \right] \right\}. \quad (A.13)
$$

For $I_3$ : recalling the boundary condition

$$(u_0)_3 |_{\partial} = w_2 + U \nabla w_1,$$

making use of Lemma 6.1 of [10] and considering the assumptions made on the geometry, we have

$$I_3 \leq \alpha C \|g \cdot \nabla (u_0)_3\|_{H^{-\frac{1}{2}}(\Omega)} \|u_0 - \alpha D(g \cdot \nabla w_1) e_3\|_{\partial}
\leq C \left\{ r_U \left\{ \|u_0\|^2_{H^1(\partial)} + \|\Delta w_1\|^2_{\Omega} \right\} + \xi \left\{ \|p_0\|^2_{\partial} + \|\Delta w_1\|^2_{\Omega} \right\} \right\}, \quad (A.14)
$$

where we have also implicitly used the Sobolev Embedding Theorem. To continue with $I_4$ :

$$
I_4 = \xi (\nabla \psi(-U \nabla p_0 - \text{div}(U)p_0, 0), u_0 - \alpha D(g \cdot \nabla w_1) e_3)_{\partial}
+ \xi (\nabla \psi(-\text{div}(u_0), u_0 \cdot n), u_0 - \alpha D(g \cdot \nabla w_1) e_3)_{\partial}
= I_4a + I_4b. \quad (A.15)
$$

Since $U \cdot n |_{\partial} = \mathbf{0}$, we have that $(U \nabla p_0 + \text{div}(U)p_0) \in [H^1(\partial)]'$ with

$$
\|U \nabla p_0 + \text{div}(U)p_0\|_{[H^1(\partial)]'} \leq C \|U\|_{\sigma} \|p_0\|_{\partial}. \quad (A.16)
$$

By Lax–Milgram Theorem, we then have

$$I_4a \leq C \xi \|\nabla \psi(-U \nabla p_0 - \text{div}(U)p_0, 0)\|_{\partial} \|u_0 - \alpha D(g \cdot \nabla w_1) e_3\|_{\partial}
\leq C \xi r_U \left\{ \|u_0\|^2_{H^1(\partial)} + \|p_0\|^2_{\partial} + \|\Delta w_1\|^2_{\Omega} \right\} \quad (A.17)
$$

and similarly

$$I_4b \leq C \xi r_U \left\{ \|u_0\|^2_{H^1(\partial)} + \|\Delta w_1\|^2_{\Omega} \right\}. \quad (A.18)
$$

Now, applying (A.17)–(A.18) to (A.15) gives

$$I_4 \leq C \xi r_U \left\{ \|u_0\|^2_{H^1(\partial)} + \|p_0\|^2_{\partial} + \|\Delta w_1\|^2_{\Omega} \right\}. \quad (A.19)
$$

Estimating $I_5$ : We proceed as before done for $I_4$ and invoke (A.16), Lax–Milgram Theorem and the estimate (4.10) to have

$$I_5 \leq C \xi^2 \left[ \|U\|_{\sigma} \left\{ \|p_0\|^2_{\partial} + \|\Delta w_1\|^2_{\Omega} \right\} + \|u_0\|_{H^1(\partial)}^2 \right]. \quad (A.20)$$
For $I_6$, in order to estimate the second term in (A.8), we follow the standard calculations used for the flux multipliers and the commutator symbol given by

$$[P, Q]f = P(Qf) - Q(Pf)$$

(A.21)

for the differential operators $P$ and $Q$. Hence,

$$-(\Delta^2 w_1, h_\alpha \cdot \nabla w_1)_\Omega = (\nabla \Delta w_1, \nabla (h_\alpha \cdot \nabla w_1))_\Omega$$

(A.22)

$$= -(\Delta w_1, \Delta(h_\alpha \cdot \nabla w_1))_\Omega + \int_{\partial \Omega} (h_\alpha \cdot \nu) |\Delta w_1|^2 d\partial \Omega,$$

(A.23)

where, in the first identity, we have directly invoked the clamped plate boundary conditions, and in the second, we have used the fact that $w_1 = \delta_\gamma w_1 = 0$ on $\partial \Omega$, which yields that

$$\frac{\partial}{\partial \nu}(h_\alpha \cdot \nabla w_1) = (h_\alpha \cdot \nu) \frac{\partial^2 w_1}{\partial \nu} = (h_\alpha \cdot \nu)(\Delta w_1|_{\partial \Omega}).$$

(See [34] or [35, p. 305]). Using the commutator bracket $[\cdot, \cdot]$, we can rewrite the latter relation as

$$-(\Delta^2 w_1, h_\alpha \cdot \nabla w_1)_\Omega = -(\Delta w_1, [\Delta, h_\alpha \cdot \nabla] w_1)_\Omega - (\Delta w_1, h_\alpha \cdot \nabla (\Delta w_1))_\Omega + \int_{\partial \Omega} (h_\alpha \cdot \nu) |\Delta w_1|^2 d\partial \Omega.$$

With Green’s relations once more:

$$-(\Delta^2 w_1, h_\alpha \cdot \nabla w_1)_\Omega = -(\Delta w_1, [\Delta, h_\alpha \cdot \nabla] w_1)_\Omega - \frac{1}{2} \int_{\partial \Omega} (h_\alpha \cdot \nu) |\Delta w_1|^2 d\partial \Omega$$

$$+ \frac{1}{2} \int_{\Omega} \text{div}(h_\alpha) |\Delta w_1|^2 d\Omega - i \text{Im}(\Delta w_1, h_\alpha \cdot \nabla (\Delta w_1))_\Omega$$

$$+ \int_{\partial \Omega} (h_\alpha \cdot \nu) |\Delta w_1|^2 d\partial \Omega.$$  

(A.24)

Thus,

$$-(\Delta^2 w_1, h_\alpha \cdot \nabla w_1)_\Omega = -(\Delta w_1, [\Delta, h_\alpha \cdot \nabla] w_1)_\Omega + \frac{1}{2} \int_{\partial \Omega} (h_\alpha \cdot \nu) |\Delta w_1|^2 d\partial \Omega$$

$$+ \frac{1}{2} \int_{\Omega} \text{div}(h_\alpha) |\Delta w_1|^2 d\Omega - i \text{Im}(\Delta w_1, h_\alpha \cdot \nabla (\Delta w_1)).$$  

(A.25)

Since $h_\alpha = U|_{\Omega} - \alpha g$, where $g$ is an extension of $\nu(x)$, we will have then

$$-\text{Re}(\Delta^2 w_1, h_\alpha \cdot \nabla w_1)_\Omega = \frac{1}{2} \int_{\partial \Omega} (U \cdot \nu - \alpha)|\Delta w_1|^2 d\partial \Omega + \frac{1}{2} \int_{\Omega} \text{div}(h_\alpha) |\Delta w_1|^2 d\Omega - \text{Re}(\Delta w_1, [\Delta, h_\alpha \cdot \nabla] w_1)_\Omega.$$  

(A.26)
Since we can explicitly compute the commutator
\[
[\Delta, h_\alpha \cdot \nabla] w_1 = (\Delta h_1)(\partial_{x_1} w_1) + 2(\partial_{x_1} h_1)(\partial_{x_1}^2 w_1) + (\Delta h_2)(\partial_{x_2} w_1) + 2\text{div}(h_\alpha)(\partial_{x_1} \partial_{x_2} w_1),
\]
and
\[
|||[\Delta, h_\alpha \cdot \nabla] w_1|||_{L^2(\Omega)} \leq r_U ||\Delta w_1||_{L^2(\Omega)}.
\] (A.27)
Combining (A.26)–(A.27), we eventually get
\[
-\text{Re}(\Delta^2 w_1, h_\alpha \cdot \nabla w_1) \leq \frac{1}{2} \int_{\Omega} [\nabla \cdot \nu - \alpha]|\Delta w_1|^2 d\Omega + C r_U ||\Delta w_1||_{L^2(\Omega)}^2.
\] (A.28)
Moreover, for the first term of (A.8), we have
\[
(\Delta (U\nabla w_1), \Delta w_1) = (\nabla \cdot U) \cdot \partial_{x_1} w_1 - \int_{\Omega} \text{div}(U)|\Delta w_1|^2 d\Omega
\]
where we also use the commutator expression in (A.21). This gives us
\[
\text{Re}(\Delta (U\nabla w_1), \Delta w_1) \leq \frac{1}{2} \int_{\Omega} (U \cdot \nabla) |\Delta w_1|^2 d\Omega + C r_U ||\Delta w_1||_{L^2(\Omega)}^2.
\] (A.29)
Now applying (A.28)–(A.29) to (A.8), we obtain
\[
\text{Re} I_6 \leq \frac{1}{2} \int_{\Omega} [U \cdot \nabla - \alpha]|\Delta w_1|^2 d\Omega + C r_U ||\Delta w_1||_{L^2(\Omega)}^2.
\] (F)
To estimate \( I_7 \) : since \( w_2 \in H^1_0(\Omega) \), we have
\[
\text{Re}(h_\alpha \cdot \nabla w_2, w_2) = -\frac{1}{2} \int_{\Omega} \text{div}(h_\alpha)|w_2|^2 d\Omega
\]
\[
= -\frac{1}{2} \int_{\Omega} \text{div}(h_\alpha)(u_0)_3 - U \nabla w_1 |^2 d\Omega
\]
after using the boundary condition in (A.9). Applying the last relation to RHS of (A.9) and recalling that \( h_\alpha = U|_{\Omega} - \alpha g \), we get
\[
\text{Re} I_7 = \text{Re}(h_\alpha \cdot \nabla w_2, w_2) + \text{Re}(h_\alpha \cdot \nabla (U\nabla w_1), (u_0)_3 - U \nabla w_1)
\]
\[
\leq C r_U \left\{ ||u_0||_{L^2(\Omega)}^2 + ||\Delta w_1||_{L^2(\Omega)}^2 \right\},
\] (A.30)
where we also implicitly use Sobolev Trace Theorem. Lastly, for the term \( I_8 \), we proceed in a manner similar to that adopted for \( I_7 \) and we have
\[
I_8 = (h_\alpha \cdot \nabla (u_0)_3, h_\alpha \cdot \nabla w_1 + \xi w_1) + \xi ((u_0)_3, (u_0)_3 - U \cdot \nabla w_1 + h_\alpha \cdot \nabla w_1 + \xi w_1)
\]
\[ \leq C [r_U + \xi^2] \left\{ \|u_0\|_{H^1(\Omega)}^2 + \|\Delta w_1\|_{\Omega}^2 \right\} \]
\[ + C \xi \left\{ \|u_0\|_{H^1(\Omega)}^2 + r_U \left\{ \|u_0\|_{H^1(\Omega)}^2 + \|\Delta w_1\|_{\Omega}^2 \right\} \right\}. \]  
(A.31)

Now, if we apply (A.12)–(A.31) to RHS of (A.2), we obtain
\[ \text{Re}((\mathcal{A}_0 + B[\varphi, \varphi]))_{H_0} \leq -\langle \sigma(u_0), \varepsilon(u_0) \rangle_{\partial \Omega} - \eta \|u_0\|_{\partial \Omega}^2 - \xi \|p_0\|_{\partial \Omega}^2 - \xi \|\Delta w_1\|_{\Omega}^2 \]
\[ + \int_{\partial \Omega} [\mathbf{U} \cdot \eta - \frac{\alpha}{2} \|\Delta w_1\|^2] d\partial \Omega \]
\[ + C [r_U + \xi r_U + \xi^2 + \xi^2 \eta] \left\{ \|u_0\|_{H^1(\Omega)}^2 \right\} \]
\[ + C [r_U + \xi r_U + \xi^2 + \xi^2 r_U] \left\{ \|p_0\|_{\partial \Omega}^2 + \|\Delta w_1\|_{\Omega}^2 \right\} \]
\[ + C \xi \|u_0\|_{H^1(\Omega)}^2 \left\{ \|p_0\|_{\partial \Omega} + \|\Delta w_1\|_{\Omega} \right\}. \]  
(A.32)

We recall now the value of \( \alpha = 2 \|\mathbf{U}\|_* \) to get
\[ \text{Re}((\mathcal{A}_0 + B[\varphi, \varphi])_{H^1_{\Omega}}) \leq -\langle \sigma(u_0), \varepsilon(u_0) \rangle_{\partial \Omega} - \eta \|u_0\|_{\partial \Omega}^2 - \xi \|p_0\|_{\partial \Omega}^2 - \xi \|\Delta w_1\|_{\Omega}^2 \]
\[ + \left\{ (C_1 + C_2 r_U) \xi^2 + C_2 r_U \xi + C_2 r_U \right\} \left\{ \|p_0\|_{\partial \Omega}^2 + \|\Delta w_1\|_{\Omega}^2 \right\} \]
\[ + \|p_0\|_{\partial \Omega}^2 + \|\Delta w_1\|_{\Omega}^2 \right\} \]
\[ + \left\{ (\sigma(u_0), \varepsilon(u_0))_{\partial \Omega} + \eta \|u_0\|_{\partial \Omega}^2 \right\} + C_2 [r_U + \xi r_U + \xi^2 + \xi] \|u_0\|_{H^1(\Omega)}^2, \]  
(A.33)

where the positive constants \( C_1, C_2, \) and \( C_3 \) are obtained with the application of Holder–Young and Korn’s inequalities and \( C_2 \) depends on the constant in Korn’s inequality. We now specify \( \xi \) to be a zero of the equation
\[ (C_1 + C_2 r_U) \xi^2 + (C_2 r_U - \frac{1}{2}) \xi + C_2 r_U = 0. \]

Namely,
\[ \xi = \frac{1}{2} - C_2 r_U \frac{\sqrt{(\frac{1}{2} - C_2 r_U)^2 - 4C_2(C_1 + C_2 r_U) r_U}}{2(C_1 + C_2 r_U)}, \]  
(A.34)

where the radicand is nonnegative for \( \|\mathbf{U}\|_* \) sufficiently small. Then, (A.33) becomes
\[ \text{Re}((\mathcal{A}_0 + B[\varphi, \varphi])_{H_0}) \leq -\frac{\langle \sigma(u_0), \varepsilon(u_0) \rangle_{\partial \Omega}}{4} - \frac{\eta}{4} \|u_0\|_{\partial \Omega}^2 - \frac{\xi}{2} \|p_0\|_{\partial \Omega}^2 - \frac{\xi}{2} \|\Delta w_1\|_{\Omega}^2 \]
\[ - \frac{\langle \sigma(u_0), \varepsilon(u_0) \rangle_{\partial \Omega}}{4} - \frac{\eta}{4} \|u_0\|_{\partial \Omega}^2 \]
\[ + C_K [r_U + \xi r_U + \xi^2 + \xi] \left\{ (\sigma(u_0), \varepsilon(u_0))_{\partial \Omega} + \eta \|u_0\|_{\partial \Omega}^2 \right\}. \]

With \( \xi \) as prescribed in (A.34), we now have the dissipativity estimate (A.1), for \( \|\mathbf{U}\|_* \) small enough. (Here, we also implicitly reuse Korn’s inequality and \( C_K \) is the constant there). This concludes the proof of Lemma A1. \( \square \)