Contractibility of fixed point sets of auter space

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Abstract We show that for every finite subgroup $G$ of $\text{Aut}(F_n)$, the fixed point subcomplex $X_n^G$ is contractible, where $F_n$ is the free group on $n$ letters and $X_n$ is the spine of “auter space” constructed by Hatcher and Vogtmann in [6]. In more categorical language, $X_n = E\text{Aut}(F_n)$. This is useful because it allows one to compute (see, for example, [7, 8]) the cohomology of normalizers or centralizers of finite subgroups of $\text{Aut}(F_n)$ based on their actions on fixed point subcomplexes. The techniques used to prove it are largely those of Krstic and Vogtmann in [10], who in turn used techniques similar to Culler and Vogtmann in [4]

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1 Introduction

Let $F_n$ denote the free group on $n$ letters and let $\text{Aut}(F_n)$ and $\text{Out}(F_n)$ denote the automorphism group and outer automorphism group, respectively, of $F_n$. In [4] Culler and Vogtmann defined a space on which $\text{Out}(F_n)$ acts nicely called “outer space”. By studying the action of $\text{Out}(F_n)$ on this space, various people have been able to calculate the cohomology of $\text{Out}(F_n)$ in specific cases. More recently, Hatcher in [5] and Hatcher and Vogtmann in [6] have defined a space on which $\text{Aut}(F_n)$ acts nicely called “auter space” and have used this to calculate the cohomology of $\text{Aut}(F_n)$ in specific cases.

We review some basic properties and definitions of auter space. Most of these can be found in [4], [6], [13], or [14]. Let $(R_n, v_0)$ be the $n$-leafed rose, a wedge of $n$ circles. We say a pointed graph $(G, x_0)$ is admissible if it has no free edges, all vertices except the basepoint have valence at least three, and there is a basepoint-preserving continuous map $\phi: R_n \to G$ which induces an isomorphism on $\pi_1$. The triple $(\phi, G, x_0)$ is called a marked graph. Two marked graphs $(\phi_1, G_1, x_1)$ for $i = 0, 1$ are equivalent if there is a homeomorphism $\alpha: (G_0, x_0) \to (G_1, x_1)$ such that $(\alpha \circ \phi_0)_# = (\phi_1)_# : \pi_1(R_n, v_0) \to \pi_1(G_1, x_1)$. Define a partial order on the set of all equivalence classes of marked graphs by setting $(\phi_0, G_0, x_0) \leq (\phi_1, G_1, x_1)$ if $G_1$ contains a forest (a disjoint union of trees in $G_1$ which contains all of the vertices of $G_1$) such that collapsing each tree in the forest to a point yields $G_0$, where the collapse is compatible with the maps $\phi_0$ and $\phi_1$. 

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From [5] and [6] we have that $\text{Aut}(F_n)$ acts with finite stabilizers on a contractible space $X_n$. The space $X_n$ is the geometric realization of the poset of marked graphs that we defined above. Let $Q_n$ be the quotient of $X_n$ by $\text{Aut}(F_n)$. Note that the CW-complex $Q_n$ is not necessarily a simplicial complex. Since $\text{Aut}(F_n)$ has a torsion free subgroup of finite index [5] and it acts on the contractible, finite dimensional space $X_n$ with finite stabilizers and finite quotient, $\text{Aut}(F_n)$ has finite vcd. From [16] (cf. [3]), any finite subgroup $G$ of $\text{Aut}(F_n)$ fixes a point of $X_n$. Our goal is to show 

**Theorem 1.1** Auter space is an $E\text{Aut}(F_n)$-space. That is, for any finite subgroup $G$ of $\text{Aut}(F_n)$, the fixed point subcomplex $X_n^G$ is contractible.

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# 2 Norms and Absolute Values

We strongly recommend that the reader study [10] by Krstic and Vogtmann, where they prove the analog of Theorem 1.1 for $\text{Out}(F_n)$ and outer space. This paper is essentially a modification of their results on fixed point spaces of outer space to fixed point spaces of auter space, and we will often omit details which are similar to work already done in [10]. White [15] also proved the result for fixed point subcomplexes of outer space, but we do not know to what extent his work can be applied to auter space.

In particular, Krstic and Vogtmann define a complex $L^G$ of “essential marked $G$-graphs” that the fixed point set $X^G_n$ in outer space deformation retracts to. Then they order the reduced marked $G$-graphs in $L^G$ using a norm $\| \cdot \|_{\text{out}}$. Using this norm to determine which reduced marked $G$-graphs should be considered next, Krstic and Vogtmann performed a transfinite induction argument to show that $L^G$ is contractible, by building $L^G$ up as the union of stars of reduced marked $G$-graphs.

We will follow a similar approach, and define norms $\| \cdot \|_{\text{aut}}$ and $\| \cdot \|_{\text{tot}} = \| \cdot \|_{\text{out}} \times \| \cdot \|_{\text{aut}}$ to order the reduced marked essential $G$-graphs in auter space. For technical reasons, $\| \cdot \|_{\text{tot}}$ will be the appropriate norm to use when performing the transfinite induction argument to show the contractibility of the corresponding $L^G$ in auter space.

The norm $\| \cdot \|_{\text{out}}$ was defined by Krstic and Vogtmann as follows. Order the set $W$ of conjugacy classes of elements of $F_n$ as $W = \{w_1, w_2, \ldots \}$. Totally order $Z^W$ by the lexicographic order. Let $\sigma = [s, \Gamma]$ be a marked graph and define $\| \sigma \|_{\text{out}} \in Z^W$ by letting $\| \sigma \|_{\text{out}}(s)$ be the sum over all $x \in G$ of the lengths in $\Gamma$ of the reduced loops (given by the marking $s$) corresponding to $xw$. Equivalently,
they define an absolute value $|·|_{out} \in \mathbb{Z}^W$ on the edges of $\Gamma$ and set

$$||\sigma||_{out} = \frac{1}{2} \sum_{e \in E(\Gamma)} |e|_{out}. $$

The $i$th coordinate of $|e|_{out}$ is simply the sum for all $x \in G$ of the contributions of $e$ or $\bar{e}$ to the loop $xw_i$ in $\Gamma$. In other words, it is the sum over all $x \in G$ of the number of times $e$ or $\bar{e}$ appears in the cyclically reduced edge path representing $xw_i$. For $A, B \subseteq E(\Gamma)$ define $(A.B)_{out} \in \mathbb{Z}^W$ to be the function whose $i$th coordinate is the sum over all $x \in G$ of the number of times $ab$ or $\bar{b}a$ appears in the reduced loop in $\Gamma$ corresponding to $xw_i$. Finally, for $C \subseteq E(\Gamma)$, define $|C|_{out}$ inductively by the formula

$$|A \square B|_{out} = |A|_{out} + |B|_{out} - 2(A.B)_{out}$$

for disjoint subsets $A$ and $B$ of $E(\Gamma)$. Note that with the above definition, $|A|_{out} = (A.E(\Gamma) - A)_{out} = |E(\Gamma) - A|_{out}$.

The corresponding quantities for $\text{Aut}(F_n)$ are defined in much the same way, the basic difference being that we think of the lengths of reduced paths rather than reduced loops. Order $F_n$ as $F_n = \{\alpha_1, \alpha_2, \ldots\}$, and give $\mathbb{Z}^{F_n}$ the lexicographic order. For a finite subgroup $G$ of $\text{Aut}(F_n)$, consider a pointed marked $G$-graph $\sigma = [s, \Gamma]$. Define the norm $||\sigma||_{out} \in \mathbb{Z}^{F_n}$ to be $|G| \cdot L$, where $L : F_n \to \mathbb{Z}$ is the Lyndon length function of the marked graph. In other words, the pointed marked graph $\sigma$ corresponds to an action of $F_n$ on a rooted $\mathbb{Z}$-tree $T$. Define

$$L(\alpha_i) = \{\text{the distance } \alpha_i \text{ moves the root of } T\}.$$ 

Equivalently, the $i$th coordinate of $||\sigma||_{out}$ is the sum over all $x \in G$ of the lengths in $\Gamma$ of the reduced (but not cyclically reduced) paths corresponding to $x\alpha_i \in \pi_1(\Gamma, \ast)$. As before in the case of $\text{Out}(F_n)$, we can define an absolute value $|·|_{out} \in \mathbb{Z}^{F_n}$ on the edges of $\Gamma$ and set

$$||\sigma||_{aut} = \frac{1}{2} \sum_{e \in E(\Gamma)} |e|_{aut} $$

The $i$th coordinate of $|e|_{aut}$ is simply the sum of for all $x \in G$ of the contributions of $e$ or $\bar{e}$ to the reduced (but not cyclically reduced) path $x\alpha_i \in \pi_1(\Gamma, \ast)$. Hence it is the sum over all $x \in G$ of the number of times $e$ or $\bar{e}$ appears in the reduced edge path representing $x\alpha_i$. For $A, B \subseteq E(\Gamma)$ define $(A.B)_{aut} \in \mathbb{Z}^W$ to be the function whose $i$th coordinate is the sum over all $x \in G$ of the number of times $ab$ or $\bar{b}a$ appears in the reduced path in $\Gamma$ corresponding to $x\alpha_i$. Finally, for $C \subseteq E(\Gamma)$, define $|C|_{aut}$ inductively by the formula

$$|A \square B|_{aut} = |A|_{aut} + |B|_{aut} - 2(A.B)_{aut}$$

for disjoint subsets $A$ and $B$ of $E(\Gamma)$. In contrast to the case with $\text{Out}(F_n)$ the formula $|A|_{aut} = (A.E(\Gamma) - A)_{aut}$ certainly does not hold any longer.

Our final norm $||·||_{tot}$ is just the product of the previous two. That is, let $\sigma = [s, \Gamma]$ be a pointed marked $G$-graph for a finite subset $G$ of $\text{Aut}(F_n)$ and...
totally order $\mathbb{Z}^W \times \mathbb{Z}^F_n$ by the lexicographic order. Define $\|\sigma\|_{\text{tot}} \in \mathbb{Z}^W \times \mathbb{Z}^F_n$ as $\|\sigma\|_{\text{tot}} = \|\sigma\|_{\text{out}} \times \|\sigma\|_{\text{out}}$, where to calculate $\|\sigma\|_{\text{out}}$ we just forget that $\Gamma$ has a basepoint. The functions $\|\cdot\|_{\text{out}}$, $(A, B)_{\text{tot}}$, and $\|\cdot\|_{\text{tot}}$ are defined similarly.

For a vertex $v$, let $E_v$ be the set of oriented edges ending at $v$. We call certain subsets $\alpha \subseteq E_v$ ideal edges and think of them as corresponding to new edges created when we blow up the original graph at the vertex $v$ by pulling away the edges in $\alpha$. Formally, the notion of ideal edges is defined as in [10], with the exception that if the ideal edge $\alpha \subseteq E_*\sigma$ then condition (i) of their definition should be changed to:

1. $\text{card}(\alpha) \geq 2$ and $\text{card}(E_* - \alpha) \geq 1$.

That is, ideal edges at the basepoint can contain all except one of the edges of $E_*$. The definition of blowing up an ideal edge is taken exactly as defined in [10]. Hence if we are blowing up an ideal edge $\alpha \subseteq E_*$ then we are pulling the edges of $\alpha$ away from the basepoint along a new edge $e(\alpha)$ we just constructed. If $\text{card}(E_* - \alpha) = 1$, this will result in a graph whose basepoint has valence 2.

Let $\alpha$ be an ideal edge of $\sigma = [s, \Gamma]$ and $\sigma^{G\alpha} = [s^{G\alpha}, \Gamma^{G\alpha}]$ be the result of blowing up the ideal edge $\alpha$. Then it is easy to show that $\text{card}(\alpha)$ in $\Gamma$ is equal to $|e(\alpha)|_{\text{out}}$ in $\Gamma^{G\alpha}$ (which was the whole point of defining $|\cdot|_{\text{out}}$ on subsets of edges.) Hence $|\alpha|_{\text{tot}} = (\bar{\alpha})_{\text{tot}}$ also, as Krstic and Vogtmann show the corresponding formula for $|\cdot|_{\text{out}}$. From this, the analogs of Proposition 6.4 about Whitehead moves in [10] are true for the norms $|\cdot|_{\text{out}}$ and $|\cdot|_{\text{tot}}$. That is, for an ideal edge $\alpha$ define $D(\alpha)$ by

$$D(\alpha) = \{ a \in \alpha : \text{stab}(a) = \text{stab}(\alpha) \text{ and } \bar{a} \notin \bigcup G\alpha \}.$$ 

Then the Whitehead move $(G\alpha, G\alpha)$ is the result of first blowing up $\alpha$ in $\Gamma$ to get $\sigma^{G\alpha}$ and then collapsing $G\alpha$ in $\Gamma^{G\alpha}$ to get $\sigma'$. Proposition 6.4 of [10] states that

$$|\sigma'|_{\text{out}} = |\sigma|_{\text{out}} + [G : \text{stab}(\alpha)](\alpha|_{\text{out}} - |\alpha|_{\text{out}}).$$

As mentioned before, this remains true if out-norms and absolute values are replaced by aut- or tot-norms and absolute values.

The value $[G : \text{stab}(\alpha)](\alpha|_{\text{out}} - |\alpha|_{\text{out}})$ is called the out-reductivity of $(\alpha, a)$ and is denoted $\text{red}_{\text{out}}(\alpha, a)$. Similar notions of aut-reductivity and tot-reductivity are defined as well. A Whitehead move reduces the norm if the corresponding reductivity is greater than zero, in which case the Whitehead move is called reductive. The $x$-reductivity of an ideal edge $\alpha$ is the maximum over all elements $a \in D(\alpha)$ of $\text{red}_{x}(\alpha, a)$, where $x$ is out, aut, or tot. It thus makes sense to talk of an ideal edge $\alpha$ as being out-reductive, etc. The norm $|\cdot|_{\text{tot}}$ will be useful to us because:

**Proposition 2.1** Let $\alpha \subseteq E_\sigma$ be a tot-reductive ideal edge of a reduced marked $G$-graph $\rho$. Suppose $\alpha$ is invertible (that is, $E_\sigma - \alpha \nsucccurlyeq G\alpha$ and $E_\sigma - \alpha$ is an ideal edge.) Then $\alpha^{-1} = E_\sigma - \alpha$ is tot-reductive.
Proof Assume \( v = * \), else the proof is trivial. Say \( (\alpha, a) \) is the reductive ideal edge. Since \( \text{stab}(\ast) = G \) and \( \alpha \) is invertible, the analog of Lemma 5.1 of [10] gives us that \( \text{stab}(\alpha) = \text{stab}(a) = G \). Say \( \rho = [s, \Gamma] \). As before, let \( \rho^{G\alpha} = [s^{G\alpha}, \Gamma^{G\alpha}] \) be the result of blowing up the ideal edge \((\alpha, a)\). Then let \( \rho' = [s', \Gamma'] \) be the result of collapsing \( a \) in \( \Gamma^{G\alpha} \). We know that \( \|\rho'\|_{\text{tot}} < \|\rho\|_{\text{tot}} \) as \((\alpha, a)\) is tot-reductive.

Assuming the claim below, it will be easy to complete the proof as follows: Let \( \rho'' \) be the result of doing the Whitehead move \((\alpha^{-1}, a^{-1})\) to \( \rho \). Because \( \text{red}_{\text{out}}(\alpha, a) = \text{red}_{\text{out}}(\alpha^{-1}, a^{-1}) \) (see the comments in [10] following the proof of §6.4), it follows that \( \|\rho''\|_{\text{out}} = \|\rho'\|_{\text{out}} \). So \( \|\rho''\|_{\text{out}} < \|\rho\|_{\text{out}} \) and hence \( \|\rho''\|_{\text{tot}} < \|\rho\|_{\text{tot}} \). Thus \( \alpha^{-1} \) is reductive.

\[ \square \]

Claim 2.2 \( \|\rho'\|_{\text{out}} \neq \|\rho\|_{\text{out}} \).

Proof Since \( \text{stab}(a) = G \) and \( \rho = [s, \Gamma] \) is reduced, the edge \( a \) must both begin and end at \(*\). Enumerate the edges of \( \alpha - \{a\} \) and \( \alpha^{-1} - \{a^{-1}\} \) as \( b_0, \ldots, b_r \) and \( c_0, \ldots, c_s \), respectively, where \( r, s \geq 0 \). We have three cases, which are not disjoint but are exhaustive.

1. Some \( b_i \) is a loop at \( \ast \) and \( b_i^{-1} \not\in \alpha \). Let \( w_k \in W \) be an element that maps to the loop \( b_i \). Then \( |a|_{\text{out}} k = 0 \) and \( |\alpha|_{\text{out}} k \geq 1 \) since the loop \( b_i \) is sent to \( b_i e(\alpha) \).

2. Some \( b_i \) starts at another vertex \( v \neq \ast \). Since \( G \) acts nontrivially on \( b_i \) and because \( b_i \) must be elliptic (as it is clearly not bent hyperbolic), there must be another \( b_j \neq b_i \) also going from \( \ast \) to \( v \). (For the definitions of elliptic and bent hyperbolic see §4A in the paper by Krstic and Vogtmann.) There are two subcases:

   - There is an edge \( c_i \) in \( \alpha^{-1} - \{a^{-1}\} \) that begins and ends at \( \ast \). We can assume \( c_i^{-1} \in \alpha^{-1} - \{a^{-1}\} \) also, else we are in case 1. Choose a \( w_k \in W \) that maps to the loop \( b_i b_j^{-1} c_i \). Now \( |a|_{\text{out}} k = 0 \) and \( |\alpha|_{\text{out}} k \geq 1 \) since \( w_k \) is sent to \( e(\alpha)^{-1} b_i b_j^{-1} e(\alpha) c_i \).

   - There is an edge \( c_i \) in \( \alpha^{-1} - \{a^{-1}\} \) that begins at \( v_2 \neq \ast \) and ends at \( \ast \). Because \( G \) acts nontrivially on \( c_i \) and \( c_i \) is elliptic, there is another edge \( c_m \neq c_i \) also going from \( v_2 \) to \( \ast \). Choose a \( w_k \in W \) that maps to the loop \( b_i b_j^{-1} c_i c_m^{-1} \). Then \( |a|_{\text{out}} k = 0 \) but \( |\alpha|_{\text{out}} k \geq 1 \) as \( b_i b_j^{-1} c_i c_m^{-1} \) is sent to \( e(\alpha)^{-1} b_i b_j^{-1} e(\alpha) c_i c_m^{-1} \).

3. Some \( b_i \) is a loop at \( \ast \) and \( b_i^{-1} \not\in \alpha \) also. As in case 2, above, there are two subcases:

   - Same as in case 2, above. Choose a \( w_k \in W \) that maps to \( b_i c_i \). Then \( |a|_{\text{out}} k = 0 \) but \( |\alpha|_{\text{out}} k \geq 1 \) as \( b_i c_i \) is sent to \( e(\alpha)^{-1} b_i e(\alpha) c_i \).

   - Same as in case 2, above. Choose a \( w_k \in W \) that maps to \( b_i c_i c_m^{-1} \). Hence \( |a|_{\text{out}} k = 0 \) and \( |\alpha|_{\text{out}} k \geq 1 \) because \( b_i c_i c_m^{-1} \) is mapped to \( e(\alpha)^{-1} b_i e(\alpha) c_i c_m^{-1} \).
In each case we have \( |a|_{\text{out}} \neq |\alpha|_{\text{out}} \); therefore, \( \text{red}_{\text{out}}(\alpha, a) \neq 0 \) and \( \|\rho\|_{\text{out}} = \|\rho\|_{\text{out}} \).

Because of Proposition 2.1, tot-reductivity will be the most useful of the three types of reductivity (out, aut, and tot) for us. From now on when we say that \( \rho \) is reductive, this is just shorthand for saying \( \rho \) is tot-reductive.

**Proposition 6.1** states that

\[
|A \bigsqcup B|_{\text{out}} = |A|_{\text{out}} + |B|_{\text{out}} - 2(A \cdot B)_{\text{out}}
\]

for disjoint subsets \( A \) and \( B \) of \( E(\Gamma) \). This also holds for aut-norms because it is our definition of the absolute values \( | \cdot |_{\text{aut}} \) for sets of edges and can be inductively shown to be well-defined. It is important that this property holds for aut-norms because it is used by many of the later propositions in Krstic and Vogtmann (e.g., Proposition 6.2 of [10] which will correspond to our Proposition 2.4.)

Proposition 6.2 of [10] states that:

**Proposition 2.3** (Krstic-Vogtmann) Let \( K \) be a subgroup of \( G \), let \( A \) be a \( K \)-invariant subset of \( E(\Gamma) \), and let \( e \) be an edge of \( \Gamma \) with \( \text{stab}(e) \) contained in \( K \). Then

\[
((Ke) \cdot A)_{\text{aut}} = [K : \text{stab}(e)](e \cdot A)_{\text{aut}}.
\]

We now show Proposition 6.2 of [10] also holds for the aut-norm, which will be useful in some combinatorial lemmas later in this section. Once we show that the analog of Proposition 2.3 is true for the aut-norm, it will be true for both the out- and aut-norms on a component-by-component basis. In other words, the equality stated in the proposition is true for each component of \( \mathbb{Z}^W \) or \( \mathbb{Z}^F_n \) and does not use the total (lexicographic) order on those sets. Hence it is automatically true for the tot-norm, as the tot-norm is just the product of the out-norm and the aut-norm. We will be able to use the same approach (that of just showing something to be true for the aut-norm) in some lemmas later on in this section.

**Proposition 2.4** Let \( K \) be a subgroup of \( G \), \( A \) be a \( K \)-invariant subset of \( E(\Gamma) \), and \( e \) be an edge of \( \Gamma \) with \( \text{stab}(e) \) contained in \( K \). Then

\[
((Ke) \cdot A)_{\text{aut}} = [K : \text{stab}(e)](e \cdot A)_{\text{aut}}.
\]

**Proof** To simplify the notation in the proof below, we write (just for this proof)

\( \| \cdot \| \) for \( \| \cdot \|_{\text{aut}} \), \( | \cdot | \) for \( | \cdot |_{\text{aut}} \), reductive for aut-reductive, etc.

Examine \( ((Ke) \cdot A)_{\text{i}} \). It is the number of times one of the strings \((ke)a^{-1}\) or \(a(ke)^{-1}\) appears in one of the \( x\alpha_i\), for all \( k \in K, a \in A, \) and \( x \in G \).

Now \( \text{stab}(e) \subseteq K \) and we can write

\[
K = \text{stab}(e) \bigsqcup k_1 \text{stab}(e) \bigsqcup \ldots \bigsqcup k_{[K:\text{stab}(e)]} \text{stab}(e)
\]

using coset representatives \( k_i \). Further note that the number of times one of the strings \( ea^{-1} \) or \( ae^{-1} \) appears in one of the strings \( x\alpha_i \) for \( a \in A, x \in G \) is
exactly the same as the number of times one of the strings $k_i e a^{-1}$ or $a(k_i e)^{-1}$ appears in one of the $x \alpha_i$ for $a \in A$, $x \in G$. This is because each $k_i$ is in $G$ and $A$ is $K$-invariant so if $e a^{-1}$ is in $x \alpha_i$ then $(k_i e)(k_i a)^{-1}$ is in $(k_i x)\alpha_i$. So $((Ke)A)_i = [K : stab(e)](e.A)_i$.

**Proposition 2.5** The set of pointed marked $G$-graphs is well-ordered by the tot-norm.

**Proof** Let $\mathcal{A}$ be a nonempty collection of pointed marked $G$-graphs. We must find a least element of $\mathcal{A}$. Let $[\mathcal{A}]$ be the set of equivalence classes of marked $G$-graphs in $\mathcal{A}$ obtained by forgetting the basepoint $*$. From Proposition 6.3 of [10] the out-norm well orders marked $G$-graphs, and $[\mathcal{A}]$ has a least element $U \subseteq \mathcal{A}$.

Say $\sigma = [s, \Gamma]$ is the marked $G$-graph representing this $U$. The marked graph $\sigma$ corresponds to an action of $F_n$ on the tree $\hat{\Gamma} = \Lambda$. From [4] $\sigma$ corresponds to a free, minimal (there are no invariant proper subtrees), and not abelian (an action is abelian if every element of the commutator $[F_n, F_n]$ has length 0) action without inversions on the tree $\hat{\Gamma} = \Lambda$.

The action has an associated nonabelian (see Alperin and Bass in [1]) length function $l$ on $F_n$. By Theorem 7.4 of [1], there exist hyperbolic elements $\alpha_n, \alpha_m \in F_n$, $n < m$, such that the characteristic subtrees $A_{\alpha_n}$ and $A_{\alpha_m}$ are linear and disjoint.

Recall that we wish to find the least element of $U$ in the tot-norm. Following the proof of Proposition 6.3 in [10], we set $U_0 = U$ and define $U_i$ inductively for $i \geq 1$. Let $\gamma_i = \min\{\|\delta\|_{aut} : \delta \in U_{i-1}\}$. Next define $U_i$ to be the subset of $U_{i-1}$ consisting of $\delta$ with $\|\delta\|_{aut} = \gamma_i$. To finish our proof, it suffices to show that $U_m$ has only finitely many elements.

Each element of $U$ corresponds to an action of $F_n$ on a pointed tree. In each case, if we forget the basepoint then the tree is homeomorphic to $\Lambda$. The map from $U$ to Lyndon length functions on $F_n$, given by seeing how far the basepoint is moved under the corresponding action, is injective (see [6], [1].) Note that in each case, the action of $F_n$ on the underlying non-pointed tree $\Lambda$ is the same. We are only varying where we place the basepoint on $\Lambda$ and seeing how far elements of $F_n$ move this basepoint.

The elements of $U_1$ are those where the basepoint is located closest to the linear subtree $A_{\alpha_n} \subset \Lambda$, and $U_1$ could be infinite. Let $B$ be the bridge joining $A_{\alpha_n}$ and $A_{\alpha_m}$. To show that $U_m$ is finite, it suffices to show that there are only finitely many points at fixed distances $d_1$ and $d_2$ from $A_{\alpha_n}$ and $A_{\alpha_m}$, respectively. If $d(x, A_{\alpha_n}) = d_1$ and $d(x, A_{\alpha_m}) = d_2$, then choose paths $p_1$ and $p_2$ of lengths $d_1$ and $d_2$ from $x$ to $q_1 \in A_{\alpha_n}$ and $q_2 \in A_{\alpha_m}$, respectively. The union of these two paths $p_1$ and $p_2$ contains the bridge $B$. Consequently, $d(x, B) \leq d_1 + d_2$. Since the tree is locally finite and $B$ is finite, $x$ is one of a finite number of vertices.

A few definitions are in order at this point. Basically, we are trying to find the appropriate parallels of definitions in [10]. Fix a reduced marked $G$-graph
\[ \rho = [s, \Gamma]. \] Let \((\mu, m)\) be a maximally reductive ideal pair of \(\rho\). That is, \(\mu\) is the maximally reductive ideal edge in \(\rho\) and \(m \in D(\mu)\) is an edge in \(\mu\) which allows the Whitehead move \((\mu, m)\) to realize this maximum.

Let \(\alpha \subset E_v\) and \(\beta \subset E_v\) be ideal edges of \(\rho\). Then the ideal edge orbits \(G\alpha\) and \(G\beta\) are compatible if one of the following holds:
1. \(G\alpha \subseteq G\beta\).
2. \(G\beta \subseteq G\alpha\).
3. \(G\alpha \cap G\beta = \emptyset\) and \(\alpha \neq \beta^{-1}\).
4. \(G\alpha \cap G\beta = \emptyset\) and \(u = v = \ast\).

The ideal edge orbits \(G\alpha\) and \(G\beta\) are pre-compatible if one of the following holds:
1. They are compatible.
2. \(\alpha\) is invertible and \(\alpha^{-1} \subseteq \beta\).
3. \(\beta\) is invertible and \(\beta^{-1} \subseteq \alpha\).

Note that 2. and 3. above would be equivalent if we did not need to consider ideal edges of the form \(\gamma = E_v - \{c^{-1}\}\) which have \(\text{stab}(\gamma) = G\) but are not invertible.

An oriented ideal forest is a collection of pairwise compatible ideal edge orbits. These can be blown up to obtain marked graphs in the star in \(L_G\) of \(\rho\). The correspondence is not unique, however, as two different oriented ideal forests can be blown up to yield the same marked graph. This problem is solved by defining ideal forests. There is a poset isomorphism between the poset of ideal forests and the star of \(\rho\) in \(L_G\).

An ideal forest is a collection \(\Phi = \Phi_1 \coprod \Phi_2\) where \(\Phi_1\) are the edges at \(\ast\) and \(\Phi_2\) are the edges not at \(\ast\), such that
1. The elements of \(\Phi_2\) are pairwise pre-compatible and \(\Phi_2\) contains the inverse of each of its invertible edge orbits; and
2. The elements of \(\Phi_1\) are pairwise compatible.

With respect to a particular reduced marked \(G\)-graph \(\rho\) and maximally reductive ideal edge \((\mu, m)\), the following definitions will be used frequently in the next section (which contains the core proof of the contractibility of \(L_G\)).

- \(\mathcal{R} = \{\text{reductive ideal edges}\}\).
- If \(\mathcal{C}\) is a set of ideal edges, then let \(\mathcal{C}^\pm\) denote the set obtained by adjoining to \(\mathcal{C}\) the inverses of its invertible elements that are not at the basepoint.
- Let \(S(\mathcal{C})\) be the subcomplex of the star \(st(\rho)\) spanned by ideal forests of \(\rho\), all of whose edges are in \(\mathcal{C}\). Note: The empty forest should not be taken to be in \(S(\mathcal{C})\).
- \(\mathcal{C}_0 = \{\alpha \in \mathcal{R} : \alpha\text{ is compatible with }\mu\}\).
- \(\mathcal{C}_0' = \mathcal{C}_0 \cup \{\alpha \in \mathcal{R} : \text{ if } \alpha \subset E_v \text{ then } \text{stab}(\alpha) = \text{stab}(v)\}\) (cf. Lemma 5.1 of \([10]\)).
- \(\mathcal{C}_1 = \mathcal{C}_0' \cup \{\alpha \in \mathcal{R} : m \in G\alpha \text{ and } N(G\alpha, G\mu) = 1\}.\)
The definition of the crossing number \( N(G\alpha, G\beta) \) comes from §7 of [10] where it and other combinatorial notions are defined. For the reader’s convenience, we briefly state their definitions again here. Say \( \alpha \) and \( \beta \) are two ideal edges at some vertex \( v \), with stabilizers \( P \) and \( Q \), respectively, of indices \( p \) and \( q \) in \( G \). Choose double coset representatives \( x_1, \ldots, x_k \) of \( P \backslash G / Q \). The intersection \( \delta = \alpha \cap G\beta \) breaks up as a disjoint union

\[ \delta = \gamma_1 \sqcup \cdots \sqcup \gamma_k \]

with each \( \gamma_i = \alpha \cap P x_i \beta \). The \( \gamma_i \) are called the intersection components of \( \alpha \) with \( \beta \) and the number \( N(G\alpha, G\beta) \) of nonempty intersection components is called the crossing number. If \( N(G\alpha, G\beta) = 1 \) then \( G\alpha \) and \( G\beta \) are said to cross simply.

The following two lemmas are stated for the out-norm by Krstic and Vogtmann. We will show them for the aut-norm. The proofs will be routine, although they are not the same as the proofs given in [10]. This is because their proofs use the fact that \( |A|_{\text{out}} = (A.E(\Gamma) - A)_{\text{out}} \), which is no longer true with the new norms. As with Proposition 2.4, the lemmas are true for both the out- and aut-norms on a component-by-component basis. That is, the inequalities stated in the lemmas are true for each component of \( Z_i \) or \( Z_i \) and do not use the total (lexicographic) order on those sets. Hence it suffices to show them for the aut-norm, as the tot-norm is the product of the out-norm and the aut-norm.

**Lemma 2.6** Suppose \( G\alpha \) and \( G\beta \) cross simply, with \( P \leq Q \), then

\[ p|\alpha \cap \beta|_{\text{aut}} + q|\beta \cup Q\alpha|_{\text{aut}} \leq p|\alpha|_{\text{aut}} + q|\beta|_{\text{aut}}. \]

**Proof** To simplify the notation in the proof below, we write (just for this proof) \( \| \cdot \| \) for \( \| \cdot \|_{\text{aut}} \), \( | \cdot | \) for \( | \cdot |_{\text{aut}} \), reductive for aut-reductive, etc.

Let \( [Q: P] = n \). Then \( p = nq \). Dividing by \( q \), we see that we want to show that

\[ n|\alpha \cap \beta| + |\beta \cup Q\alpha| \leq n|\alpha| + |\beta|. \]

Let \( q_1, \ldots, q_n \) be a set of coset representatives for \( P \) in \( Q \). Let \( \delta = \alpha \cap \beta \), \( A = \alpha - \delta \), and \( B = \beta - Q\delta \). Since

\[ n|\delta| + |B \bigcup Q\alpha| = n|\delta| + |B| + |Q\delta| + |QA| - 2Q\delta.QA - 2B.Q\alpha, \]

and

\[ n|\delta| \bigcup A| + |B \bigcup Q\delta| = n|\delta| + n|A| - 2n\delta.A + |B| + |Q\delta| - 2B.Q\delta, \]

we have reduced the problem to showing that

\[ |QA| - 2Q\delta.QA - 2B.Q\alpha \leq n|A| - 2n\delta.A - 2B.Q\delta. \]

Note that \( -2B.Q\alpha \leq -2B.Q\delta \) as \( Q\delta \subseteq Q\alpha \). Also note that by decomposing \( QA \) into a disjoint union of \( q_i A \)'s, we have \( Q\delta.QA \geq n\delta.A \). Similarly, we could use induction to show that \( |QA| \leq n|A| \). 

\[ \square \]
Lemma 2.7 Suppose $G\alpha$ and $G\beta$ cross (i.e., $N(G\alpha, G\beta) \neq 0$). Just as $\delta$ breaks up into intersection components of $\alpha$ with $\beta$, let $\delta' = \beta \cap G\alpha$ give the analogous disjoint components

$$\delta' = \gamma_1' \cdots \gamma_k'$$

with $\gamma_i' = \beta \cap Q\alpha_i^{-1} \alpha$. Then for all $i$,

$$p|\alpha - \gamma_i'|_{\text{out}} + q|\beta - \gamma_i'|_{\text{out}} \leq p|\alpha|_{\text{aut}} + q|\beta|_{\text{aut}}.$$

Proof To simplify the notation in the proof below, we write (just for this proof) $\| \cdot \|$ for $\| \cdot \|_{\text{aut}}$, $| \cdot |$ for $| \cdot |_{\text{aut}}$, reductive for aut-reductive, etc. Let $A = \alpha - \gamma_i$ and $B = \beta - \gamma_i'$. We must show that

$$p|\gamma_i'| + q|\gamma_i''| \geq 2|\gamma_i| + 2|\gamma_i'|B.$$

Note that $G\gamma_i = G\gamma_i'$. Choose coset representatives $y_1, \ldots, y_p$ for $P$ in $G$ and $z_1, \ldots, z_q$ for $Q$ in $G$. Then

$$p|\gamma_i| + q|\gamma_i'| = \sum_{n=1}^p |\gamma_i| + \sum_{m=1}^q |\gamma_i'| \geq 2|G\gamma_i| = 2|G\gamma_i'|.$$

and

$$2|G\gamma_i|A + 2|\gamma_i'|B \leq 2|G\gamma_i|(E(\Gamma) - G\gamma_i).$$

So to prove the lemma it suffices to show

$$|G\gamma_i| \geq G\gamma_i.(E(\Gamma) - G\gamma_i),$$

which follows from induction on $|G\gamma_i|$.

Next we review the Pushing and Shrinking Lemmas of Krstic and Vogtmann hold in the context of aut-norms and absolute values. Unlike the proofs of the previous two lemma, the proofs for the next two follow exactly the same lines as the original proofs by Krstic and Vogtmann for out-norms and absolute values. The only way that the new proofs differ from the old ones is that the new cardinality conditions for ideal edges $\alpha_0 \subseteq E_v$ should be verified, namely that:

- If $v = *$ then $\text{card}(\alpha_0) \geq 2$ and $\text{card}(E_v - \alpha_0) \geq 1$.
- If $v \neq *$ then $\text{card}(\alpha_0) \geq 2$ and $\text{card}(E_v - \alpha_0) \geq 2$.

As before, it is easily seen from the proofs of the lemmas that since they hold for both the out- and aut-norms and absolute values, they also hold for the tot-norms and absolute values.

Lemma 2.8 (Pushing Lemma) Let $(\mu, m)$ be a maximally aut-reductive ideal edge of a reduced pointed marked $G$-graph with $m \in D(\mu)$. Let $(\alpha, a)$ be an aut-reductive ideal edge containing $m$ which simply crosses $\mu$, and set $P = \text{stab}(\alpha)$. Then either both $\mu - \alpha$ and $\alpha - \mu$ are aut-reductive or both $\alpha \cup P \mu$ and $\alpha \cap \mu$ are aut-reductive.

Proof To simplify the notation in the proof below, we write (just for this proof) $\| \cdot \|$ for $\| \cdot \|_{\text{aut}}$, $| \cdot |$ for $| \cdot |_{\text{aut}}$, reductive for aut-reductive, etc.

Note that since $m \in \alpha$, $\text{stab}(\mu) \subseteq P$. As in [10], there are four cases depending upon where $a^{-1}$ and $m^{-1}$ are located. Since this follows the proof by Krstic and Vogtmann so closely, the only real detail will be put into the first case.
Case 1. $a^{-1} \notin G_{\mu}$. From Lemma 2.6,

$$[G : \text{stab}(\mu)]|\alpha \cap \mu| + [G : \text{stab}(\alpha)]|\alpha \cup P_{\mu}| \leq [G : \text{stab}(\mu)]|\mu| + [G : \text{stab}(\alpha)]|\alpha|.$$ 

Consequently,

$$[G : \text{stab}(\mu)](|m| - |\alpha \cap \mu|) + [G : \text{stab}(\alpha)](|\alpha| - |\alpha \cup P_{\mu}|)$$

is greater than or equal to

$$[G : \text{stab}(\mu)](|m| - |\mu|) + [G : \text{stab}(\alpha)](|\alpha| - |\alpha|).$$

In other words,

$$\text{red}(\alpha \cap \mu, m) + \text{red}(\alpha \cup P_{\mu}) \geq \text{red}(\mu, m) + \text{red}(\alpha, \mu).$$

Since $(\mu, m)$ is maximally reductive and $(\alpha, \mu)$ is reductive, both of $(\alpha \cup P_{\mu}, a)$ and $(\alpha \cap \mu, m)$ are reductive. As mentioned above in the discussion preceding this lemma, we must verify the cardinality conditions on these two prospective ideal edges.

First we deal with $(\alpha \cup P_{\mu}, a)$. The edge $a$ is either bent hyperbolic or elliptic (see Corollary 4.5 of [10].) Assume it is bent hyperbolic. Then as in [10] we can choose $x \in G$ such that $xa^{-1} \in E_{v} - (G_{\alpha} \cup G_{\mu})$. If $v \neq *$ and $xa^{-1}$ is the only edge in $E_{v} - (\alpha \cup P_{\mu})$ then

$$|\alpha \cup P_{\mu}| = |xa^{-1}| = |a^{-1}| = |a|,$$

where the first equality holds because $v \neq *$, the second is by the $G$-invariance of $|\cdot|$, and the third follows from our definition of $|\cdot|$ for edges.

In more detail, the first equality $|\alpha \cup P_{\mu}| = |xa^{-1}|$ holds since

$$E_{v} = (\alpha \cup P_{\mu}) \bigsqcup \{xa^{-1}\}$$

and $v \neq *$. For a particular coordinate $i$, both $|\alpha \cup P_{\mu}|$ and $|xa^{-1}|$ are measuring the number of times one of the paths $y_{\alpha_{i}}$ enters $v$ via $\alpha \cup P_{\mu}$ and leaves via the reverse of $xa^{-1}$ (i.e., $xa$) or enters $v$ via $xa^{-1}$ and leaves it via the reverse of something in $\alpha \cup P_{\mu}$. There would be problems if $v = *$ since the above paths could then enter $v$ and not have to leave it again.

But $|\alpha \cup P_{\mu}| = |a|$ contradicts the fact that $(\alpha \cup P_{\mu}, a)$ is reductive because

$$[G : \text{stab}(\alpha)](|\alpha| - |\alpha \cup P_{\mu}|) > 0.$$ 

So if $v \neq *$ then $xa^{-1}$ is not the only edge in $E_{v} - (\alpha \cup P_{\mu})$.

For the next possibility, that $a$ is elliptic, the proof by Krstic and Vogtmann can be used verbatim.

Second we deal with $(\alpha \cap \mu, m)$. The set $\alpha \cap \mu$ must contain more than two edges because it is reductive:

$$[G : \text{stab}(\mu)](|m| - |\alpha \cap \mu|) > 0.$$ 

The condition on the cardinality of $E_{v} - (\alpha \cap \mu)$ is easily satisfied because $\alpha$ is an ideal edge and so satisfies the corresponding condition with $E_{v} - \alpha$. 

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Case 2. $a^{-1} \in G\mu$ and $m^{-1} \in G\alpha$. Krstic and Vogtmann show that both $(\alpha - \mu, ym^{-1})$ and $(\mu - \alpha, xa^{-1})$ are reductive.

Case 3. $a^{-1} \in G\mu$, $m^{-1} \notin G\alpha$, and $a \in \mu$. Both $(\alpha \cup \mu, m)$ and $(\alpha \cap \mu, a)$ are reductive.

Case 4. $a^{-1} \in G\mu$, $m^{-1} \notin G\alpha$, and $a \notin \mu$. Both $(\alpha \cup \mu, m)$ and $(\alpha \cap \mu, m)$ are reductive.

Lemma 2.9 \textbf{(Shrinking Lemma)} Let $(\mu, m)$ be a maximally aut-reductive ideal edge of a reduced pointed marked $G$-graph with $m \in D(\mu)$. Let $\alpha$ be an ideal edge with $N(G\alpha, G\mu) \neq 0$. Let $\gamma_{i_1}, \ldots, \gamma_{i_k}$ be the intersection components of $\alpha$ with $\mu$ which contain no translate of $m$ and let $\beta = \alpha - \bigcup \gamma_{i_j}$. Then $\beta$ or one of the sets $\gamma_{i_j}$ is an aut-reductive ideal edge.

\textbf{Proof} See verbatim the proof by Krstic and Vogtmann. If $\alpha_0$ is one of the above sets, we know it is aut-reductive, and we want to show it is an ideal edge, then the cardinality checks are easy. The set $\alpha_0$ contains more than one edge because it is aut-reductive. Moreover, the cardinality checks on $E_v - \alpha_0$ follow from similar ones on $E_v - \alpha$, because $\alpha_0 \subset \alpha$ for each possibility of $\alpha_0$.

The following proposition will also be useful in the next section.

Proposition 2.10 Let $(\mu, m)$ be a maximally aut-reductive ideal edge of a reduced pointed marked $G$-graph with $m \in D(\mu)$. There is at most one reductive ideal edge $(\gamma, c)$ at $*$ with $\text{stab}(\gamma) = G$ but where $\gamma$ is not invertible. The Whitehead move $(\gamma, c)$ is just conjugation by $c$, and $||\gamma||_{out} = 0$. If $\gamma$ is not compatible with $\mu$, then $c = m^{-1}$ and $\mu$ is invertible.

\textbf{Proof} Since $\text{stab}(\gamma) = G$ and $\gamma$ is not invertible, $E_* - \gamma = \{c^{-1}\}$ must contain just one element. The Whitehead move $(\gamma, c)$ consists of first blowing up $\gamma$ and then collapsing $c$, as illustrated in Figure 1. The Whitehead move has no effect on the out-norm. The effect on the aut-norm can be calculated as follows. Recall that $F_n = \{\alpha_1, \alpha_2, \ldots\}$. The Whitehead move $(\gamma, c)$ conjugates each $\alpha_i$ by $c$; i.e., each $\alpha_i \mapsto c^{-1}\alpha_i c$ (or $e(c)^{-1}\alpha_i e(c)$ more accurately, but in the final graph we can just relabel $e(c)$ as $c$.)

$$
\begin{array}{c}
\text{c} \\
\text{blow up } \gamma \\
\end{array} \quad \Rightarrow \quad 
\begin{array}{c}
\text{e(c)} \\
\text{collapse c} \\
\end{array}
$$

Figure 1: The Whitehead move $(\gamma, c)$.

Since $\gamma$ is reductive, there exists an $n$ such that (i) each of $\alpha_1, \ldots, \alpha_{n-1}$ either begins with $c$ or ends with $c^{-1}$; and (ii) $\alpha_n$ begins with $c$ and ends in $c^{-1}$.
Thus for any other $\gamma' = E_* - \{d^{-1}\}, d \neq c, \gamma'$ will increase the length of one of the $\alpha_1, \ldots, \alpha_n$ and it will not be reductive. So $(\gamma, c)$ is the only reductive edge at $*$ with $\text{stab}(\gamma) = G$ but where $\gamma$ is not invertible.

If we further suppose that $\gamma$ is not compatible with $\mu$, then $c^{-1} \in \mu$ else $\mu \subseteq \gamma$ and they are compatible. As $\text{stab}(c^{-1}) = G, \text{stab}(\mu) = G$. Since $\mu$ is reductive and not equal to $\gamma$, $\mu$ is invertible. By way of contradiction, assume $m \in \gamma$. Then $m \neq c$ else $\gamma$ and $\mu$ are compatible. We apply the Pushing Lemma to $\gamma$ and $\mu$. Case 2 is the relevant case and so $(\mu - \gamma, a^{-1})$ is reductive. This contradicts the fact that $\mu - \gamma = \{a^{-1}\}$ has just one edge in it. So $m \notin \gamma$ and hence $c = m^{-1}$.

3 The contractibility lemmas

Proof of Theorem 1.1: The space $X^G$ deformation retracts to $L_G$. Following the proof of Theorem 8.1 by Krstic and Vogtmann in [10], we show that the complex $L_G$ is contractible by setting

$$L_{<\rho} = \bigcup_{\|\rho'\|_{\text{tot}} < \|\rho\|_{\text{tot}}} \text{st}(\rho')$$

and letting

$$S_{\rho} = \text{st}(\rho) \cap L_{<\rho}.$$  

As in [10], we show that $S_{\rho}$ is contractible when it is non-empty, so that a transfinite induction argument then yields that for all $\rho$, all of the components of $L_{<\rho}$ are contractible. Krstic’s work in [9] shows that any two reduced graphs in $L_G$ can be connected by Whitehead moves, so that $L_G$ is connected. Thus $L_G$ is contractible if we can perform the above transfinite induction.

As in [10], the first step is to deformation retract $S_{<\rho}$ to $S(\mathcal{R})$ by the Poset Lemma (stated in [10], deriving from Quillen in [12]). We can do this for the case of $\text{Aut}(F_n)$ rather than $\text{Out}(F_n)$ without any significant modifications of the arguments in the previous case. This is because the Factorization Lemma and Proposition 6.5 of [10] let us identify $S_{<\rho}$ with the poset of ideal forests which contain a reductive ideal edge (where we must, of course, use the newly modified definition of an ideal forest.) (The Factorization Lemma gives a certain isomorphism between forest that does not preserve basepoints, but the fact that basepoints are not preserved is not relevant to Proposition 6.5.)

After contracting $S_{<\rho}$ to $S(\mathcal{R})$, Krstic and Vogtmann then use a series of lemmas to deformation retract from $S(\mathcal{R})$ to $S(C_1)$, from there to $S(C_0)$, and finally to a point. We more or less follow this, except there is an additional intermediate step where we deformation retract from $S(C_1)$ to $S(C_0)$ and from there to $S(C_0)$.

The rest of this section will be devoted to proving the aforementioned series of lemmas which show that $S(\mathcal{R})$ deformation retracts to a point.

We assume that the maximally reductive ideal edge $(\mu, m)$ is at the basepoint in all that follows, else the arguments of Krstic and Vogtmann directly give the
contractibility of $S(R)$. Moreover, if $(\mu, m) = (\gamma, c)$ where $\gamma = E_\ast - \{c^{-1}\}$, then $\mu$ is not out-reductive at all. Since $\mu$ is maximally reductive, Proposition 2.10 implies that $\mu$ is the only reductive ideal edge. So in this case $R = C_0 = \{\mu\}$ and $S(R)$ is contractible. Assume $\mu \neq \gamma$ from now on.

Note the slight difference in our definition of $C_1$ from that of [10], where here it is phrased to include $\alpha \subset E_v$ which have $\text{stab}(\alpha) = \text{stab}(v)$, rather than just invertible $\alpha$. In other words, from Proposition 2.10, there is at most one reductive ideal edge $(\gamma, c)$ at the basepoint which has $\text{stab}(\gamma) = G$ and yet is not invertible. This $\gamma$ would be in both $C_1$ and $C_0$.

The next lemma (unlike the ones which follow it) is essentially the corresponding lemma in [10] with minimal modifications. We repeat their arguments here for the sake of convenience.

**Lemma 3.1** The complex $S(R)$ deformation retracts onto $S(C_1)$.

**Proof** Let $C = C^\pm$ be a subset of $R$ which contains $C_1$. We show that $S(C)$ deformation retracts to $S(C_1)$ by induction on the cardinality of $C - C_1$.

Choose $\alpha \in C - C_1$ which satisfies both of:
1. The cardinality $|\alpha \cap G\mu|$ is minimal (recall that $\mu$ is the maximally reductive ideal edge.)
2. The ideal edge $\alpha$ is minimal with respect to property 1.

Using the Shrinking Lemma 7.4 of [10] with $\alpha$ and $\mu$, we obtain a reductive ideal edge $\alpha_0 \subset \alpha$ which is compatible with $\mu$. Let $\gamma_i$ be the intersection components of $\alpha$ with $\mu$ and index them so that $m \in \gamma_0$. Now from the Shrinking Lemma, we can choose $\alpha_0$ so that it is either one of the intersection components $\gamma_i$ of $\alpha$ with $\mu$ which contain no translate of $m$, or it is $\alpha - \cup \gamma_i$. Because $\alpha \in C - C_1$, $\text{stab}(\alpha) \neq G$ and $\alpha$ is neither invertible nor equal to $\gamma = E_\ast - \{c^{-1}\}$.

**Claim 3.2** For every $\beta \in C$, if $G\beta$ is compatible with $G\alpha$, then $G\beta$ is compatible with $G\alpha_0$.

**Proof** The three cases are
1. $G\alpha \subseteq G\beta$. In this case, $G\alpha_0 \subseteq G\beta$ as $G\alpha_0 \subseteq G\alpha$.
2. $G\alpha \cap G\beta = \varnothing$. It follows that $G\alpha_0 \cap G\beta = \varnothing$ since $G\alpha_0 \subseteq G\alpha$.
3. $G\beta \subseteq G\alpha$. Without loss of generality $\beta \subseteq \alpha$. If $\beta \notin C_1$, then the minimality conditions on $\alpha$ imply that $\beta = \alpha$, in which case $\beta$ is clearly compatible with $\alpha_0$. So assume $\beta \in C_1$. As $G\beta \subseteq G\alpha$, $\text{stab}(\beta) \neq G$ since $\text{stab}(\alpha) \neq G$. So either $\beta \in C_0$ or $m \in G\beta$ and $N(G\beta, G\mu) = 1$. If $\beta \in C_0$ then either $G\beta \subseteq G\mu$ (in which case $\beta$ is in some $\gamma_i$ and thus compatible with $\alpha_0$), $G\mu \subseteq G\beta$ (which can not happen as then $\alpha$ would be in $C_0$), or $G\beta \cap G\mu = \varnothing$ (in which case $\beta \subseteq \alpha - \cup \gamma_i$ and thus compatible with $\alpha_0$.) Finally, if $m \in G\beta$ and $N(G\beta, G\mu) = 1$ then $\beta \cap G\mu$ is not in any of the $\gamma_i$’s as those are the intersection components of $\alpha$ with $\mu$ that do not contain a translate of $m$. In fact, $\beta \cap G\mu$ is in $\gamma_0$ and $\beta \subseteq \alpha - \cup \gamma_i$. Thus $\beta$ is compatible with every choice of $\alpha_0$.\[14\]
Define a poset map \( f : S(C) \rightarrow S(C) \) by sending an ideal forest \( \Phi \) to \( \Phi \cup \{G\alpha\} \) if \( \Phi \) contains \( \alpha \) and to itself otherwise. By the Poset Lemma, the image of \( f \) is a deformation retract of \( S(C) \), because \( \Phi \subseteq f(\Phi) \) for all \( \Phi \). Define another poset map \( g : f(S(C)) \rightarrow f(S(C)) \) by sending an ideal forest \( \Psi \) to \( \Psi - \{G\alpha\} \) if \( \Psi \) contains \( \alpha \) and to itself otherwise. By the Poset Lemma, the image of \( g \) is a deformation retract of \( f(S(C)) \), as \( g(\Psi) \subseteq \Psi \) for all \( \Psi \). Hence \( S(C) \) deformation retracts to \( S(C - \{G\alpha\}) \), completing the induction step.

**Lemma 3.3** The complex \( S(C_1) \) deformation retracts onto \( S(C_0') \).

**Proof** Let \( C \) be a subset of \( C_1 \) which contains \( C_0' \). We show that \( S(C) \) deformation retracts to \( S(C_0') \) by induction on the cardinality of \( C - C_0' \).

Choose \( \alpha \in C - C_0' \) such that \( m \in \alpha \) and

1. The cardinality \( |\alpha \cap G\mu| \) is minimal.
2. The ideal edge \( \alpha \) is minimal with respect to property 1.

We apply the Pushing Lemma 7.3 of [10] to get a reductive edge \( \alpha_0 \) with \( \alpha_0 = \alpha \cap \mu \) or \( \alpha_0 = \alpha - \mu \). Note that \( \alpha_0 \in C_0 \).

**Claim 3.4** For every \( \beta \in C \), if \( G\beta \) is compatible with \( G\alpha \), then \( G\beta \) is compatible with \( G\alpha \).

**Proof** The three cases are

1. \( G\alpha \subseteq G\beta \). In this case, \( G\alpha \) and \( G\beta \) as \( G\alpha \subseteq G\beta \).
2. \( G\alpha \cap G\beta = \emptyset \). It follows that \( G\alpha \cap G\beta = \emptyset \) since \( G\alpha \subseteq G\alpha \).
3. \( G\beta \subseteq G\alpha \). Without loss of generality \( \beta \subseteq \alpha \). If \( \beta \not\subseteq C_0' \), then the minimality conditions on \( \alpha \) imply that \( \beta = \alpha \), in which case \( \beta \) is clearly compatible with \( \alpha_0 \). So assume \( \beta \in C_0' \). Since \( \alpha \in C - C_0' \), \( stab(\alpha) \neq G \). As \( G\beta \subseteq G\alpha \), \( stab(\beta) \neq G \) also, which means that \( \beta \in C_0 \). The three ways in which \( \beta \) could be compatible with \( \mu \) are:
   - \( G\beta \cap G\mu = \emptyset \). Then \( G\beta \) is disjoint from \( G(\alpha \cap \mu) \) and contained in \( G(\alpha - \mu) \).
   - \( G\beta \subseteq G\mu \). Then \( G\beta \subseteq G(\alpha \cap \mu) \) and \( G\beta \) is disjoint from \( G(\alpha - \mu) \).
   - \( G\mu \subseteq G\beta \). Then \( G\mu \subseteq G\alpha \) and so \( G\mu \) and \( G\alpha \) are compatible, a contradiction.

Define a poset map \( f : S(C) \rightarrow S(C) \) by sending an ideal forest \( \Phi \) to \( \Phi \cup \{G\alpha\} \) if \( \Phi \) contains \( \alpha \) and to itself otherwise. By the Poset Lemma, the image of \( f \) is a deformation retract of \( S(C) \), because \( \Phi \subseteq f(\Phi) \) for all \( \Phi \). Define another poset map \( g : f(S(C)) \rightarrow f(S(C)) \) by sending an ideal forest \( \Psi \) to \( \Psi - \{G\alpha\} \) if \( \Psi \) contains \( \alpha \) and to itself otherwise. By the Poset Lemma, the image of \( g \) is a deformation retract of \( f(S(C)) \), as \( g(\Psi) \subseteq \Psi \) for all \( \Psi \). Hence \( S(C) \) deformation retracts to \( S(C - \{G\alpha\}) \), completing the induction step.
Now we are left with the task of showing that $S(C_0')$ deformation retracts to $S(C_0)$ and from there to a point. The methods used will be analogous to those in Lemmas 3.1 and 3.3, and we will omit unnecessary detail from the remaining proofs. From Proposition 2.10, we see that this can be handled in three separate cases:

1. The ideal edge $\mu$ is invertible and the reductive ideal edge $\gamma = E_0 - \{c^{-1}\}$ is not compatible with $\mu$. In this case, the proposition gives us that $c = m^{-1}$.
2. The ideal edge $\mu$ is invertible and the reductive ideal edge $\gamma = E_0 - \{c^{-1}\}$ is compatible with $\mu$.
3. The ideal edge $\mu$ is not invertible.

**Lemma 3.5** Suppose $\mu$ is invertible and $\gamma = E_0 - \{m\}$ is reductive. Then $S(C_0')$ is contractible.

**Proof** We first contract $S(C_0')$ to $S(C_0 \cup \{\gamma\})$. Let $C$ be a subset of $C_0'$ which contains $C_0 \cup \{\gamma\}$. Also assume that if $\alpha \in C$ is not pre-compatible with $\mu$, then $\alpha^{-1} \in C$ also. We will use induction on $|C - (C_0 \cup \{\gamma\})|$ to show that $S(C)$ deformation retracts to $S(C_0 \cup \{\gamma\})$.

Choose $\alpha \in C - (C_0 \cup \{\gamma\})$ such that $m \in \alpha$ and

1. The cardinality $|\alpha \cap \mu|$ is maximal.
2. The edge $\alpha$ is maximal with respect to property 1.

There are two main cases, and two subcases in the second case.

**Case 1.** $\alpha^{-1}$ is compatible with $\mu$.

Then $\mu \not\subseteq \alpha^{-1}$ because $m \in \mu$ and $m \in \alpha$. Also, $\mu \cap \alpha^{-1} \neq \emptyset$ else $\mu \subseteq \alpha$ and $\alpha$ is compatible with $\mu$. So $\alpha^{-1} \subseteq \mu$. Let $\alpha_0 = \alpha^{-1}$ and note that $\alpha_0 \in C$.

For every $\beta \in C$, if $G\beta$ is compatible with $\alpha$, then $G\beta$ is compatible with $\alpha_0$. Hence we can replace occurrences of $G\alpha$ in ideal forests with $G\alpha_0$, and retract $S(C)$ to $S(C - \{\alpha\})$.

**Case 2.** $\alpha^{-1}$ is not compatible with $\mu$.

Since $\alpha$ and $\mu$ cross simply (this is automatic because $\alpha$ is invertible), the Pushing Lemma applies. Thus one of the sets $\alpha_0 = \mu - \alpha$ or $\alpha_0 = \alpha \cup \mu$ is a reductive ideal edge. As $\mu - \alpha \subseteq \mu$ and $\mu \subseteq \alpha \cup \mu$, $\alpha_0 \in C_0$ in either case.

**Subcase 1.** $\alpha_0 = \mu - \alpha$. For every $\beta \in C$, if $G\beta$ is pre-compatible with $\alpha$, then $G\beta$ is compatible with $\alpha_0$. Now replace occurrences of $\alpha$ or $\alpha^{-1}$ with $G\alpha_0$ to retract $S(C)$ to $S(C - \{\alpha, \alpha^{-1}\})$.

**Subcase 2.** $\alpha_0 = \alpha \cup \mu$. Since $m \in \alpha_0$, $\alpha_0 \neq \gamma$. Accordingly, $\alpha_0$ is invertible and both $\alpha_0$ and $\alpha_0^{-1}$ are in $C_0$.

For every $\beta \in C$, if $G\beta$ is compatible with $\alpha^{-1}$ then $G\beta$ is compatible with $\alpha_0^{-1}$. Substitute $\alpha_0^{-1}$ for $\alpha^{-1}$ to retract $S(C)$ to $S(C - \{\alpha^{-1}\})$.

For every $\beta \in C - \{\alpha^{-1}\}$, if $G\beta$ is compatible with $\alpha_0$ then $G\beta$ is compatible with $\alpha_0$. Now substitute $\alpha_0$ for $\alpha$ to retract $S(C - \{\alpha^{-1}\})$ to $S(C - \{\alpha, \alpha^{-1}\})$. 

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This concludes our argument that \( S(C_0') \) contracts to \( S(C_0 \cup \{ \gamma \}) \). To eliminate \( \gamma \), note that \( \gamma \) is compatible with \( \mu^{-1} \in C_0 \) and verify that for every \( \beta \in C_0 \), if \( G\beta \) is compatible with \( \gamma \) then \( G\beta \) is compatible with \( \mu^{-1} \). Now replace \( \gamma \) with \( \mu^{-1} \) to deformation retract \( S(C_0 \cup \{ \gamma \}) \) to \( S(C) \).

The final step of contracting \( S(C_0) \) to a point is done by adding \( \mu \) to all ideal forest and then removing everything else.

**Lemma 3.6** Suppose \( \mu \) is invertible and the reductive \( \gamma = E_* - \{ c^{-1} \} \) is compatible with \( \mu \). Then \( S(C_0) \) is contractible.

**Proof** The proof of the more complicated case in Lemma 3.5 carries over to this one, with the exception that the penultimate step of deformation retracting from \( S(C_0 \cup \{ \gamma \}) \) to \( S(C) \) is unnecessary, because \( \gamma \) is already compatible with \( \mu \). In addition, various other minor changes need to be made because \( \gamma \) is now compatible with \( \mu \).

**Lemma 3.7** Suppose \( \mu \) is not invertible. Then \( S(C_0') \) is contractible.

**Proof** As before, let \( \gamma = E_* - \{ c^{-1} \} \) be the reductive edge that Proposition 2.10 gives us (if it exists). We know that \( \gamma \) is compatible with \( \mu \) because \( \text{stab}(\mu) \neq G = \text{stab}(c^{-1}) \).

We first contract \( S(C_0') \) to \( S(C_0) \). Let \( C \) be a subset of \( C_0' \) which contains \( C_0 \). Also assume that if \( \alpha \in C \) and \( \alpha \) is invertible, then \( \alpha^{-1} \in C \) also. We will use induction on \( |C - C_0| \) to show that \( S(C) \) deformation retracts to \( S(C_0) \).

Choose \( \alpha \in C - (C_0) \) such that \( m \in \alpha \) and

1. The cardinality \( |\alpha \cap G\mu| \) is maximal.

2. The edge \( \alpha \) is maximal with respect to property 1.

Since \( \alpha \) and \( \mu \) cross simply (this is automatic because \( \alpha \) is invertible), the Pushing Lemma applies. Say \( \alpha = (\alpha, a) \). Neither \( a \) nor \( a^{-1} \) is in \( \mu \) since \( \text{stab}(a) = G \) and \( \mu \) is not invertible (and not equal to \( \gamma \).) So case 1 of the Pushing Lemma shows that \( \alpha_0 = \alpha \cup G\mu \) is a reductive ideal edge. As \( \mu \subseteq \alpha \cup G\mu \), \( \alpha_0 \in C_0 \). The ideal edge \( \alpha_0 \) is not equal to \( \gamma \) because it is out-reductive by the proof of the Pushing Lemma (as both \( \alpha \) and \( \mu \) are out-reductive.) So \( \alpha_0 \) is invertible and \( \alpha_0^{-1} \in C_0 \) also.

As in Subcase 2, Case 2 of Lemma 3.5 above, first retract to \( S(C - \{ \alpha^{-1} \}) \), then to \( S(C - \{ \alpha, \alpha^{-1} \}) \), and finally to \( S(C_0) \).

The sequence of lemmas above concludes our proof of Theorem 1.1.

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