CLASSIFICATION OF REAL TRIVECTORS IN DIMENSION NINE

MIKHAIL BOROVOI, WILLEM A. DE GRAAF, AND HONG VAN LÊ

Abstract. In this paper we classify real trivectors in dimension 9. The corresponding classification over the field \( \mathbb{C} \) of complex numbers was obtained by Vinberg and Elashvili in 1978. One of the main tools used for their classification was the construction of the representation of \( \text{SL}(9, \mathbb{C}) \) on the space of complex trivectors of \( \mathbb{C}^9 \) as a theta-representation corresponding to a \( \mathbb{Z}/3\mathbb{Z} \)-grading of the simple complex Lie algebra of type \( \mathfrak{E}_8 \). This divides the trivectors into three groups: nilpotent, semisimple, and mixed trivectors. Our classification follows the same pattern. We use Galois cohomology, first and second, to obtain the classification over \( \mathbb{R} \).

Contents

1. Introduction 2
1.1. Notation and conventions 4
2. The tables 5
2.1. The nilpotent orbits 5
2.2. The semisimple orbits 8
2.3. The mixed orbits 12
3. Real Galois cohomology 17
3.1. Group cohomology for a group of order 2 17
3.2. Real structures on complex algebraic groups and algebraic varieties 18
3.3. Using \( H^1 \) for finding real orbits in homogeneous spaces 19
3.4. Using \( H^2 \) for finding a real point in a complex orbit 20
4. A \( \mathbb{Z}_3 \)-grading of the simple complex Lie algebra of type \( \mathfrak{E}_8 \) 22
4.1. Constructing \( \mathfrak{E}_8 \) with a \( \mathbb{Z}_3 \)-grading 22
4.2. Nilpotent elements and homogeneous \( \mathfrak{sl}_2 \)-triples 24
4.3. Semisimple elements in \( \mathfrak{g}^\mathbb{C}_1 \) 25
5. Classification of the orbits of \( \text{SL}(9, \mathbb{R}) \) on \( \bigwedge^3 \mathbb{R}^9 \) 27
5.1. The nilpotent orbits 28
5.2. The semisimple orbits 29
5.3. The mixed orbits 33
References 37

Date: August 3, 2021.
2020 Mathematics Subject Classification. Primary: 15A21. Secondary: 11E72, 20G05, 20G20.
Key words and phrases. Trivector, graded Lie algebra, real Galois cohomology.

Borovoi was partially supported by the Israel Science Foundation (grant 870/16). De Graaf was partially supported by an Australian Research Council grant, identifier DP190100317. Research of Lê was supported by GAČR-project 18-01953J and RVO: 67985840.
1. Introduction

Let $V$ be an $n$-dimensional vector space over a field $k$. The group $\text{GL}(V)$ naturally acts on the spaces $\wedge^m V$. The elements of $\wedge^2 V$ are called bivectors. The elements of $\wedge^3 V$ are called trivectors. If the ground field $k$ is $\mathbb{R}$ or $\mathbb{C}$, then the orbits of $\text{GL}(V)$ on the space of bivectors can be listed for all $n$ (see Gurevich [19], §34). The situation for trivectors is much more complicated and a lot of effort has gone into finding orbit classifications for particular $n$. For $n$ up to 5 it is straightforward to obtain the orbits of $\text{GL}(V)$ on the space of trivectors (see [19], §35). The classification for higher $n$ and $k = \mathbb{C}$ started with the thesis of Reichel [35], who classified the orbits for $n = 6$ and with a few omissions also for $n = 7$. In 1931 Schouten [37] published a classification for $n = 7$. In 1935 Gurevich [18] obtained a classification for $n = 8$ (see also [19], §35). In these cases the number of orbits is finite. This ceases to be the case for $n \geq 9$. Vinberg and Elashvili [46] classified the orbits of trivectors for $n = 9$ under the group $\text{SL}(V)$. In this classification there are several parametrized families of orbits. The maximum number of parameters of such a family is 4.

More recently classifications have appeared for different fields. For $n = 6$, Revoy [36] gave a classification for arbitrary field $k$. For $n = 7$, Westwick [18] classified the trivectors for $k = \mathbb{R}$, Cohen and Helminck [10] treated the case of a perfect field $k$ of cohomological dimension $\leq 1$ (which includes finite fields), and Noui and Revoy [33] treated the cases of a perfect field of cohomological dimension $\leq 1$ and of a $p$-adic field. For $n = 8$, Djoković [16] treated the case $k = \mathbb{R}$, Noui [32] treated the case of an algebraically closed field $k$ of arbitrary characteristic, and Midoune and Noui [29] treated the case of a finite field. For $n = 9$, Hora and Pudlák [22] treated the case of the finite field $\mathbb{F}_2$ of two elements. In the present paper we give a classification of trivectors under the action of $\text{SL}(V)$ for $n = 9$ and $k = \mathbb{R}$.

For their classification, Vinberg and Elashvili used a particular construction of the action of $\text{SL}(V)$ on $\wedge^3 V$ $(\dim V = 9$, $k = \mathbb{C})$. They considered the complex Lie algebra $g^c_1$ of type $E_8$ and the corresponding adjoint group $G$ (here $G$ is equal to the automorphism group of $g^c_1$). This Lie algebra has a $\mathbb{Z}_3$-grading $g^c_1 = g^c_{-1} \oplus g^c_0 \oplus g^c_1$ such that $g^c_0 \cong \mathfrak{sl}(9, \mathbb{C})$. Let $G_0$ be the connected algebraic subgroup of $G$ with Lie algebra $g^c_0$. Since the Lie algebra $g^c_0$ preserves $g^c_1$ when acting on $g^c_1$, so does the group $G_0$. An isomorphism $\psi^c : \mathfrak{sl}(9, \mathbb{C}) \to g^c_0$ lifts to a surjective morphism of algebraic groups $\Psi^c : \text{SL}(9, \mathbb{C}) \to G_0$. This defines an action of $\text{SL}(9, \mathbb{C})$ on $g^c_1$, and it turns out that $g^c_1 \cong \wedge^3 C^9$ as $\text{SL}(9, \mathbb{C})$-modules. This construction pertains to Vinberg’s theory of $\theta$-groups ([43], [44], [45]). We use this theory to study the orbits of trivectors when $n = 9$. Among the technical tools that this makes available, we mention the following:

- The elements of $g^c_1$ have a Jordan decomposition, that is, every $x \in g^c_1$ can be uniquely written as $x = s + n$ where $s \in g^c_{-1}$ is semisimple (i.e., its $G_0$-orbit is closed), $n \in g^c_1$ is nilpotent (i.e., the closure of its orbit contains 0) and $[s, n] = 0$. This naturally splits the orbits into three types according to whether the elements of the orbit are nilpotent, semisimple, or mixed (that is, neither semisimple nor nilpotent). Thus the classification problem splits into three subproblems. We remark that an element $y \in g^c_1$ is semisimple (respectively nilpotent) if and only if the linear map $\text{ad} y : g^c_1 \to g^c_1$ is semisimple (respectively nilpotent).
- Any nilpotent $e \in g^c_1$ lies in a homogeneous $\mathfrak{sl}_2$-triple, meaning that there are $h \in g^c_0$, $f \in g^c_{-1}$ with $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. The $G_0$-orbits of $e$ and of the triple $(h, e, f)$ determine each other. Furthermore, $e$ lies in a carrier algebra and the theory of these algebras can be used to classify the nilpotent orbits (see Vinberg [45]).
- A Cartan subspace is a maximal subspace $\mathcal{C}$ of $g^c_1$ consisting of commuting semisimple elements. Any other Cartan subspace is conjugate to $\mathcal{C}$ under the action of $G_0$. Each
semisimple orbit has a point in \( C \). Furthermore, two elements of \( C \) are \( G_0 \)-conjugate if and only if they are conjugate under the finite group \( W := N_{G_0}(C)/Z_{G_0}(C) \) called the Weyl group of the graded Lie algebra \( g^C \), or the little Weyl group.

* Fix a semisimple element \( s \in g_1^+ \), let \( Z_s \) denote the stabilizer of \( s \) in \( SL(9, \mathbb{C}) \), and let \( Z_{g^c}(s) \) denote the centralizer of \( s \) in \( g^C \). Then the grading of \( g^C \) induces a grading of its Lie subalgebra \( Z_{g^c}(s) \), and the classification of the orbits of mixed elements \( s + e \), where \( e \) is nilpotent, reduces to the classification of the nilpotent \( Z_s \)-orbits in the graded Lie algebra \( Z_{g^c}(s) \).

Over the real numbers we use a similar construction of the action of \( SL(V) \) on \( \bigwedge^3 V \). There is a Chevalley basis of \( g^C \) such that the \( Z_3 \)-grading is defined over \( \mathbb{R} \) (that is, the spaces \( g_i^C \) for \( i = -1, 0, 1 \) all have bases whose elements are \( \mathbb{R} \)-linear combinations of the given Chevalley basis of \( g^C \)). For \( i = -1, 0, 1 \) we denote by \( g_i \) the real spans of these bases. Then \( g_0 \) is isomorphic to \( \mathfrak{so}(9, \mathbb{R}) \). We can define \( \psi^C \) in such a way that it restricts to an isomorphism \( \psi: \mathfrak{so}(9, \mathbb{R}) \rightarrow g_0 \). Then \( \Psi^C \) is defined over \( \mathbb{R} \) and restricts to a surjective homomorphism \( \Psi: SL(9, \mathbb{R}) \rightarrow G_0(\mathbb{R}) \). In this way \( g_1 \) becomes an \( SL(9, \mathbb{R}) \)-module isomorphic to \( \bigwedge^3 \mathbb{R}^9 \). In Section 4 we give details of this construction as well as a summary of a number of results that are useful for the classification of nilpotent and semisimple elements.

As in the earlier works on classification of trivectors [36] and [16], our main workhorse for classifying the \( SL(9, \mathbb{R}) \)-orbits on \( \bigwedge^3 \mathbb{R}^9 \) is Galois cohomology. Very briefly this amounts to the following. Let \( O \) be an \( SL(9, \mathbb{C}) \)-orbit in \( \bigwedge^3 \mathbb{C}^9 \). We are interested in determining the \( SL(9, \mathbb{R}) \)-orbits that are contained in \( O \cap \bigwedge^3 \mathbb{R}^9 \). It can happen that this intersection is empty (we say that \( O \) has no real points). In that case we discard \( O \). On the other hand, if we know a real point \( x \) of \( O \), then we consider the stabilizer \( Z_x \) of \( x \) in \( SL(9, \mathbb{C}) \). The \( SL(9, \mathbb{R}) \)-orbits contained in \( O \) are in bijection with the first Galois cohomology set \( H^1 \mathbb{Z}_x := H^1(\mathbb{R}, Z_x) \). Moreover, from an explicit cocycle representing a cohomology class in \( H^1 \mathbb{Z}_x \), we can effectively compute a representative of the corresponding real orbit. Section 3 has further details on Galois cohomology over \( \mathbb{R} \).

In Section 5 we describe the methods that we have developed to obtain the classification over \( \mathbb{R} \). For the nilpotent \( SL(9, \mathbb{R}) \)-orbits we calculate as described above. From the explicit representatives in [16], it follows immediately that every nilpotent \( SL(9, \mathbb{C}) \)-orbit in \( \bigwedge^3 \mathbb{C}^9 \) has a real point. However, instead of a real point \( e \) in the orbit, we work with a real homogeneous \( \mathfrak{sl}_2 \)-triple \( t = (h, e, f) \) containing it. Since nilpotent orbits correspond bijectively to orbits of homogeneous \( \mathfrak{sl}_2 \)-triples, it suffices to compute the stabilizer \( Z_t \) of the triple \( t \) and to determine the set \( H^1 \mathbb{Z}_t \).

In [16] an explicit Cartan subspace \( C \) of \( g_1^+ \) is given. In \( C \) seven canonical sets are defined with the property that every semisimple orbit has a point in one of the canonical sets. Moreover, two elements of the same canonical set have the same stabilizer in \( SL(9, \mathbb{C}) \). Let \( F \) be such a canonical set. Let \( N \) be its normalizer in \( SL(9, \mathbb{C}) \) (\( N \) is the set of \( g \) in \( SL(9, \mathbb{C}) \) with \( gp \in F \) for all \( p \) in \( F \)) and let \( Z \) be its centralizer in \( SL(9, \mathbb{C}) \) (\( Z \) is the set of all \( g \) in \( SL(9, \mathbb{C}) \) with \( gp = p \) for all \( p \) in \( F \)). Write \( A = N/Z \). In Section 5.2 we show how to divide the complex orbits of the elements of \( F \) having real points into several classes that are in bijection with the Galois cohomology set \( H^1 A \). For each class we can explicitly find a real representative of each orbit in this class. Let \( p \) be such a real representative, and let \( Z_p \) be its centralizer in \( SL(9, \mathbb{C}) \). Then the real orbits contained in the \( SL(9, \mathbb{C}) \)-orbit of \( p \) are in bijection with \( H^1 Z_p \).

Section 5.3 is devoted to the elements of mixed type. To classify the orbits consisting of mixed elements, we fix a real semisimple element \( p \) and consider the problem of listing the \( SL(9, \mathbb{R}) \)-orbits of mixed elements having a representative of the form \( p + e \) where \( e \) is nilpotent and \( [p, e] = 0 \). Let \( Z_p(\mathbb{R}) \) denote the stabilizer of \( p \) in \( SL(9, \mathbb{R}) \). Let \( a = Z_p(p) \) be
the centralizer of \( p \) in \( \mathfrak{g} \). Note that the grading of \( \mathfrak{g} \) induces a grading on \( \mathfrak{a} \). Then our problem amounts to classifying the nilpotent \( \mathbb{Z}p(\mathbb{R}) \)-orbits in \( \mathfrak{a}_1 \). In principle this can be done with Galois cohomology in the same way as for the nilpotent \( SL(9, \mathbb{R}) \)-orbits in \( \mathfrak{g}_1 \). In this case, however, there is one extra problem. If the reductive subalgebra \( \mathfrak{a} \) is not split over \( \mathbb{R} \), then it can happen that certain complex nilpotent orbits in \( \mathfrak{a}_C^1 \) (where \( \mathfrak{a}_C = \mathfrak{z}_{\mathfrak{g}}(\mathbb{C}) \)) do not have real points. In Section 5.3 we develop a method for checking whether a complex nilpotent orbit has real points and for finding one in the affirmative case. Among other things, this method uses calculations with second (abelian) Galois cohomology.

The main results of this paper are the tables in Section 2 containing representatives of the orbits of \( SL(9, \mathbb{R}) \) in \( \bigwedge^3 \mathbb{R}^9 \).

In order to obtain the results of this paper, essential use of the computer has been made. In [6, Section 5] we give details of the computational methods that we used. Here we just mention that we have used the computer algebra system GAP [17] and its package SLA [13]. For the computations involving Gröbner bases we have used the computer algebra system Singular [14].

In order to write a paper of reasonable length, we have omitted the details of our computations, focussing instead on the theoretical background and the methods that we have developed. The computations are described in detail in our paper [6]. That paper contains also much more material on Galois cohomology and graded Lie algebras.

Finally we would like to add a few more motivations for the problem of classification of trivectors of \( \mathbb{R}^8 \) and put our method in a broader perspective. Special geometries defined by differential forms are of central significance in Riemannian geometry and in physics; see [20], [21], [24]; see also [27] for a survey. One of important problems in geometry defined by differential forms is to understand the orbit space of the standard \( GL(n, \mathbb{R}) \)-action on the vector space \( \bigwedge^k(\mathbb{R}^n)^\ast \) of alternating \( k \)-forms on \( \mathbb{R}^n \), which is in a bijection with the orbit space of the standard \( GL(n, \mathbb{R}) \)-action on the space of \( k \)-vectors \( \bigwedge^k \mathbb{R}^n \). For general \( k \) and \( n \), this classification problem is intractable. In the case of trivectors of \( \mathbb{R}^8 \) and \( \mathbb{R}^9 \), this classification problem is tractable thanks to its formulation as an equivalent problem for graded semisimple Lie algebras, discovered first by Vinberg in his works [43], [44]. Semisimple graded Lie algebras and their adjoint orbits play an important role in geometry and supersymmetries; see [49], [50], [31], [25], [9], [42].

Acknowledgements. We are indebted to Alexander Elashvili for his discussions with us on his work with Vinberg [46] and related problems in graded semisimple Lie algebras over years; it was he who brought us together for working on this paper. We are grateful to Domenico Fiorenza for his helpful comments on an early version of this paper.

1.1. Notation and conventions. Here we briefly describe some notation and conventions that we use in the paper.

By \( \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C} \) we denote, respectively, the ring of integers and the fields of rational, real, and complex numbers. By \( i \in \mathbb{C} \) we denote an imaginary unit.

By \( \mu_n \) we denote the cyclic group consisting of the \( n \)-th roots of unity in \( \mathbb{C} \). Occasionally we denote by \( \mu_n \) the set of matrices consisting of scalar matrices with an \( n \)-th root of unity on the diagonal. From the context it will be clear what is meant.

If \( G \) is an algebraic group and \( A \) is a subset in the Lie algebra \( \mathfrak{g} \) of \( G \) then by \( \mathcal{N}_{\mathfrak{g}}(A) \), \( \mathcal{Z}_{\mathfrak{g}}(A) \) we denote the normalizer and centralizer in \( \mathfrak{g} \) of \( A \). If \( v \) is an element of a \( G \)-module, then \( \mathcal{Z}_{\mathfrak{g}}(v) \) denotes the stabilizer of \( v \) in \( G \). Furthermore, \( \mathfrak{z}_{\mathfrak{g}}(A) \) denotes the centralizer of \( A \) in \( \mathfrak{g} \).
We write $\Gamma$ for the group $\text{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \gamma\}$, where $\gamma$ is the complex conjugation. By a $\Gamma$-group we mean a group $A$ on which $\Gamma$ acts by automorphisms. Clearly, a $\Gamma$-group is a pair $(A, \sigma)$, where $A$ is a group and $\sigma: A \to A$ is an automorphism of $A$ such that $\sigma^2 = \text{id}_A$. If $a \in A$, we write $\gamma a$ for $\sigma(a)$.

By a linear algebraic group over $\mathbb{R}$ (for brevity $\mathbb{R}$-group) we mean a pair $M = (M, \sigma)$, where $M$ is a linear algebraic group over $\mathbb{C}$, and $\sigma$ is an anti-regular involution of $M$; see Subsection 3.2.1. Then $\Gamma$ naturally acts on $M$ (namely, the complex conjugation $\gamma$ acts as $\sigma$). In other words, $M$ naturally is a $\Gamma$-group.

In this paper the spaces $\wedge^3 \mathbb{C}^9$ and $\wedge^3 \mathbb{R}^9$ play a major role. We denote by $e_1, \ldots, e_9$ the standard basis of $\mathbb{C}^9$ (or of $\mathbb{R}^9$). Then by $e_{ijk}$ we denote the basis vector $e_i \wedge e_j \wedge e_k$ of $\wedge^3 \mathbb{C}^9$ (or of $\wedge^3 \mathbb{R}^9$).

2. The tables

In this section we list the representatives of the orbits of $\text{SL}(9, \mathbb{R})$ in $\wedge^3 \mathbb{R}^9$. Except for the semisimple orbits, we list the representatives in tables. We also give various centralizers of the complex orbit. Let $\mathfrak{sl}(9, \mathbb{C})$ be the real form of $\mathfrak{sl}(9, \mathbb{R})$. By $t$ we denote a 1-dimensional subalgebra of $\mathfrak{sl}(9, \mathbb{C})$ spanned by a semisimple matrix all of whose eigenvalues are real. By $u$ we denote a 1-dimensional subalgebra of $\mathfrak{sl}(9, \mathbb{R})$ spanned by a semisimple matrix all of whose eigenvalues are imaginary.

In the sequel we occasionally write about elements without specifying the set to which they belong and also about orbits without further specification. By an element we usually mean an element of $\wedge^3 \mathbb{C}^9$. By a real element we mean an element of $\wedge^3 \mathbb{R}^9$. By a complex orbit we mean an orbit of $\text{SL}(9, \mathbb{C})$ in $\wedge^3 \mathbb{C}^9$, and by a real orbit we mean an orbit of $\text{SL}(9, \mathbb{R})$ in $\wedge^3 \mathbb{R}^9$. We say that an orbit (real or complex) is nilpotent or semisimple or of mixed type, if its elements are of the given type.

2.1. The nilpotent orbits. Table 1 contains representatives of the nilpotent orbits of $\text{SL}(9, \mathbb{R})$ on $\wedge^3 \mathbb{R}^9$. In the first column we list the number of the complex orbit as contained in Table 6 of [46]. The second column has the characteristic of the complex orbit (see Remark 4.2.4). In the third column we give the dimension $d$ of the complex orbit. The fourth column lists the rank $d_e$ of a trivector $e$ in the complex orbit, that is, the minimal dimension of a subspace $U$ of $\mathbb{C}^9$ such that $e \in \wedge^3 U$. Let $t = (e, h, f)$ be a homogeneous $\mathfrak{sl}_2$-triple containing a representative $e$ of the complex orbit. Let $Z_{\text{SL}(9, \mathbb{C})}(t)$ denote the stabilizer in $\text{SL}(9, \mathbb{C})$ of $t$. The fifth and sixth columns have, respectively, the size of the component group $K$ of $Z_{\text{SL}(9, \mathbb{C})}(t)$ and a description of the structure of $K$. We omit the latter when the component group has order 1 or prime order. The seventh column has representatives of all $\text{SL}(9, \mathbb{R})$-orbits contained in the complex orbit. The last column has a description of the structure of the centralizer in $\mathfrak{sl}(9, \mathbb{R})$ of a real homogeneous $\mathfrak{sl}_2$-triple $t = (e, h, f)$ containing the real representative $e$ in the same line.

| No. | Char. | $d$ | $d_e$ | $|K|$ | $K$ | Representatives | $3(t)$ |
|-----|-------|-----|-------|-------|-----|----------------|---------|
| 1   | 6 6 6 6 6 6 6 12 | 80 | 9     | 3     | $e_{126} + e_{135} - e_{234} + e_{279} + e_{369} + e_{459} + e_{478} + e_{568}$ | 0       |
| 2   | 6 6 6 6 6 6 6 6 | 79 | 9     | 3     | $e_{126} + e_{145} - e_{234} + e_{279} + e_{369} - e_{378} + e_{478} + e_{568}$ | 0       |
| 3   | 6 6 6 6 6 6 6 6 | 78 | 9     | 3     | $e_{126} + e_{145} - e_{234} + e_{279} + e_{369} - e_{378} + e_{478} + e_{568}$ | 0       |
| 4   | 6 0 6 0 6 6 6 6 | 78 | 9     | 6     | $a_6$ | $e_{127} + e_{145} - e_{234} + e_{279} + e_{369} - e_{378} + e_{478} + e_{568}$ | 0       |
\[
\begin{array}{c|c|c|c}
\mu_6 & \mu_3 \times S_5 & \mu_3 \\
\end{array}
\]
| 37 | 1 1 4 1 0 1 0 4 | 69 | 1 | \(-e_{135}+e_{147}+e_{236}+e_{379}+e_{459}+e_{578}+e_{678}\) | t |
| 38 | 2 1 1 1 1 1 1 2 | 68 | 3 | \(e_{127}+e_{146}+e_{236}+e_{245}+e_{379}+e_{569}+e_{578}\) | t |
| 39 | 1 0 1 0 4 0 1 1 | 68 | 6 | \(\mu_6\) | t |
| 40 | 0 1 0 1 4 0 1 0 | 69 | 2 | \(e_{137}+e_{236}+e_{245}+e_{468}+e_{478}+e_{569}\) | t |
| 41 | 1 0 1 2 1 1 1 | 67 | 3 | \(e_{137}+e_{457}+e_{236}+e_{479}+e_{569}+e_{678}\) | t |
| 42 | 0 3 0 0 0 3 0 3 | 67 | 6 | \(e_{127}+e_{146}+e_{236}+e_{245}+e_{379}+e_{478}+e_{569}+e_{578}\) | t |
| 43 | 3 0 0 3 0 0 3 0 | 67 | 3 | \(e_{137}+e_{146}+e_{236}+e_{245}+e_{379}+e_{478}+e_{569}+e_{578}\) | t |
| 44 | 1 1 0 1 3 1 1 0 | 67 | 3 | \(e_{137}+e_{146}+e_{236}+e_{245}+e_{379}+e_{478}+e_{569}+e_{578}\) | t |
| 45 | 1 1 1 1 1 1 1 1 | 66 | 9 | \(e_{137}+e_{146}+e_{236}+e_{245}+e_{379}+e_{478}+e_{569}+e_{578}\) | t |
| 46 | 0 2 0 0 2 2 0 2 | 66 | 2 | \(e_{136}+e_{147}+e_{236}+e_{245}+e_{379}+e_{478}+e_{569}+e_{678}\) | t |
| 47 | 2 0 0 4 0 2 0 2 | 66 | 2 | \(e_{136}+e_{236}+e_{245}+e_{478}+e_{569}+e_{678}\) | t |
| 48 | 3 0 1 0 2 1 2 0 | 66 | 1 | \(e_{136}+e_{145}+e_{236}+e_{245}+e_{379}+e_{478}+e_{569}+e_{578}\) | t |
| 49 | 2 0 4 2 0 0 4 | 66 | 1 | \(e_{136}+e_{145}+e_{236}+e_{245}+e_{379}+e_{478}+e_{569}+e_{578}\) | t |
| 50 | 1 1 1 1 1 0 1 2 | 65 | 3 | \(e_{137}+e_{146}+e_{236}+e_{245}+e_{379}+e_{569}+e_{678}\) | t |
| 51 | 1 1 0 1 2 1 1 1 | 65 | 1 | \(e_{137}+e_{146}+e_{236}+e_{245}+e_{379}+e_{478}+e_{569}+e_{678}\) | t |
| 52 | 2 0 1 3 1 1 0 | 65 | 1 | \(e_{137}+e_{146}+e_{236}+e_{245}+e_{379}+e_{478}+e_{569}+e_{678}\) | t |
| 53 | 0 3 0 0 3 0 0 | 64 | 9 | \(e_{127}+e_{146}+e_{236}+e_{245}+e_{379}+e_{478}+e_{569}+e_{578}\) | t |
| 54 | 2 0 2 0 2 0 2 | 64 | 9 | \(e_{127}+e_{146}+e_{236}+e_{245}+e_{379}+e_{478}+e_{569}+e_{578}\) | t |
| 55 | 0 0 3 0 3 0 0 | 64 | 3 | \(e_{136}+e_{146}+e_{237}+e_{458}+e_{478}+e_{569}\) | t |
| 56 | 0 0 0 0 0 0 6 | 64 | 6 | \(\mu_6\) | t |
| 57 | 1 1 1 1 1 0 1 1 | 63 | 9 | \(e_{137}+e_{246}+e_{345}+e_{479}+e_{569}+e_{578}\) | t |
| 58 | 0 4 0 0 2 0 0 | 63 | 1 | \(e_{137}+e_{246}+e_{345}+e_{479}+e_{569}+e_{578}\) | t |
| 59 | 0 3 0 0 0 3 0 | 63 | 6 | \(\mu_6\) | t |
| 60 | 0 0 0 1 0 0 5 | 63 | 8 | \(e_{136}+e_{146}+e_{236}+e_{245}+e_{379}+e_{478}+e_{569}\) | t |
| 61 | 1 1 0 1 0 2 0 1 | 62 | 9 | \(e_{137}+e_{246}+e_{345}+e_{479}+e_{569}+e_{578}\) | t |
| 62 | 0 1 2 0 3 0 1 | 62 | 2 | \(e_{136}+e_{146}+e_{237}+e_{458}+e_{478}+e_{569}\) | t |
| 63 | 1 1 0 1 0 1 2 1 | 62 | 1 | \(e_{137}+e_{146}+e_{236}+e_{245}+e_{379}+e_{478}+e_{569}+e_{578}\) | t |
| 64 | 0 1 2 0 1 2 0 | 62 | 2 | \(e_{137}+e_{246}+e_{345}+e_{479}+e_{569}+e_{578}\) | t |
| 65 | 0 2 0 0 2 0 2 | 61 | 18 | \(\mu_4 \times S_3\) | t |
| 66 | 1 2 1 1 0 0 1 1 | 61 | 1 | \(e_{137}+e_{246}+e_{345}+e_{479}+e_{569}+e_{578}\) | t |
| 67 | 0 0 1 0 0 0 1 4 | 61 | 2 | \(e_{137}+e_{146}+e_{237}+e_{458}+e_{478}+e_{569}\) | t |
| 68 | 1 0 1 0 3 1 0 | 60 | 9 | \(e_{137}+e_{146}+e_{236}+e_{245}+e_{379}+e_{478}+e_{569}+e_{578}\) | t |
| 69 | 1 1 0 1 1 0 1 1 | 60 | 9 | \(e_{137}+e_{146}+e_{236}+e_{245}+e_{379}+e_{478}+e_{569}+e_{578}\) | t |
2.2. The semisimple orbits. A Cartan subspace of $\mathfrak{g}^\vee_\mathbb{F}$ is a maximal subspace of $\mathfrak{g}^\vee_\mathbb{F}$ consisting of commuting semisimple elements. In [46], it is shown that

\[
p_1 = e_{123} + e_{456} + e_{789}
\]
\[
p_2 = e_{147} + e_{258} + e_{369}
\]
\[
p_3 = e_{159} + e_{267} + e_{348}
\]
\[
p_4 = e_{168} + e_{249} + e_{357}
\]
span a Cartan subspace in \( g_1^c \). In this paper we denote it by \( C^c \). Each complex semisimple orbit has a point in this space. This can still be refined a bit: in [46] seven canonical sets \( F^c_i \), \( i = 1, \ldots, 7 \) in \( C^c \) are described with the property that each complex semisimple orbit has a point in precisely one of the \( F^c_i \). By \( F_i \) we denote the set of real elements of \( F^c_i \).

In this paper we divide the real semisimple orbits into two groups, the orbits of canonical and noncanonical semisimple elements. We say that a real semisimple element is canonical if it lies in one of the \( F_i \), or more generally, if it is SL(9, \( \mathbb{R} \))-conjugate to an element of one of the sets \( F_i \). The noncanonical real semisimple elements are those that are not SL(9, \( \mathbb{R} \))-conjugate to elements of some \( F_i \).

Similarly to the complex case, we define a Cartan subspace of \( g_1 \) to be a maximal subspace consisting of commuting semisimple elements [20 §2]. Clearly, any semisimple element of \( g_1 \) is contained in some Cartan subspace of \( g_1 \). By \( C \) we denote the space consisting of real linear combinations of \( p_1, \ldots, p_4 \), that is, the set of real points of \( C^c \). It is clear that this is a Cartan subspace of \( g_1 \). We show that all Cartan subspaces in \( g_1 \) are conjugate under SL(9, \( \mathbb{R} \)) (Theorem 5.2.6). It follows that every semisimple SL(9, \( \mathbb{R} \))-orbit has a point in \( C \).

Below we give the representatives of the orbits of canonical and noncanonical semisimple elements. The representatives of the noncanonical orbits do not lie in \( C \). However, as just observed, these representatives are SL(9, \( \mathbb{R} \))-conjugate to elements of \( C \). In each case we also give these elements of \( C \). Note that noncanonical semisimple elements are SL(9, \( \mathbb{R} \))-conjugate to elements of \( C \), but not to elements of some \( F_i \).

Except for \( F^c_7 \), the sets \( F^c_i \) are parametrized families of semisimple elements with at least one complex parameter. The real semisimple elements also occur in parametrized families. For each parametrized family we give a finite matrix group that determines when different elements of the same family are conjugate. More precisely, consider a family of semisimple elements \( p(\lambda_1, \ldots, \lambda_m) \), where \( \lambda_i \in \mathbb{R} \) must satisfy certain polynomial conditions. Write \( \lambda \) for the column vector consisting of the \( \lambda_i \). Let \( \mu \) be a second column vector with coordinates \( \mu_1, \ldots, \mu_m \). Then we describe a finite matrix group \( G \) with the property that \( p(\lambda_1, \ldots, \lambda_m) \) and \( p(\mu_1, \ldots, \mu_m) \) are SL(9, \( \mathbb{R} \))-conjugate if and only if there is an element \( g \in G \) with \( g \cdot \lambda = \mu \). We express this by saying that the conjugacy of the elements \( p(\lambda_1, \ldots, \lambda_m) \) is determined by \( G \).

**Canonical set \( F^c_1 \):** It consists of trivectors

\[
P^{1,1}_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} = \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \lambda_4 p_4
\]

where the \( \lambda_i \in \mathbb{C} \) are such that

\[
\begin{align*}
\lambda_1 \lambda_2 \lambda_3 \lambda_4 & \neq 0 \\
(\lambda_2^3 + \lambda_3^3 + \lambda_4^3)^3 - (3\lambda_2 \lambda_3 \lambda_4)^3 & \neq 0 \\
(\lambda_3^3 + \lambda_4^3 + \lambda_1^3)^3 + (3\lambda_1 \lambda_3 \lambda_4)^3 & \neq 0 \\
(\lambda_4^3 + \lambda_2^3 + \lambda_3^3)^3 - (3\lambda_1 \lambda_2 \lambda_4)^3 & \neq 0 \\
(\lambda_1^3 + \lambda_3^3 + \lambda_4^3)^3 + (3\lambda_1 \lambda_2 \lambda_3)^3 & \neq 0.
\end{align*}
\]

The real elements in \( F_1 \) consist of \( P^{1,1}_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} \) with \( \lambda_i \in \mathbb{R} \) satisfying the same polynomial conditions.

There are no noncanonical real semisimple elements that are SL(9, \( \mathbb{C} \))-conjugate to elements of \( F^c_1 \).
The conjugacy of the real elements $p_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}^{1\,1} \in F_1$ is determined by a group isomorphic to $GL(2, F_3)$ and generated by

$$
\begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}.
$$

Canonical set $F_2^c$: It consists of trivectors

$$p_{\lambda_1,\lambda_2,\lambda_3}^{2\,1} = \lambda_1 p_1 + \lambda_2 p_2 - \lambda_3 p_3$$

where the $\lambda_i \in \mathbb{C}$ are such that

$$\lambda_1 \lambda_2 \lambda_3 (\lambda_1^3 - \lambda_2^3)(\lambda_2^3 - \lambda_3^3)(\lambda_3^3 - \lambda_1^3)[(\lambda_1^3 + \lambda_2^3 + \lambda_3^3)^3 - (3\lambda_1 \lambda_2 \lambda_3)^3] \neq 0.$$

The real elements in $F_2$ consist of $p_{\lambda_1,\lambda_2,\lambda_3}^{1\,1}$ where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ satisfy the same polynomial conditions.

Here we have noncanonical real semisimple elements

$$p_{\lambda_1,\lambda_2,\lambda_3}^{2\,2} = \lambda_1((-\frac{1}{2} e_{126} - \frac{1}{3} e_{349} + 2 e_{358} - e_{457} + e_{789}) + \lambda_2(-2 e_{137} - \frac{1}{4} e_{249} - e_{258} - \frac{1}{2} e_{456} - \frac{1}{2} e_{689})
- \lambda_3(-e_{159} - e_{238} - \frac{1}{4} e_{247} - \frac{1}{4} e_{346} - e_{678}), \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}.$$

The trivector $p_{\lambda_1,\lambda_2,\lambda_3}^{2\,2}$ is $SL(9, \mathbb{C})$-conjugate to the purely imaginary trivector $p_{i\lambda_1,i\lambda_2,i\lambda_3}^{2\,1} \in F_2^c$. So here the $\lambda_i \in \mathbb{R}$ are required to satisfy

$$\lambda_1 \lambda_2 \lambda_3 (\lambda_1^3 - \lambda_2^3)(\lambda_2^3 - \lambda_3^3)(\lambda_3^3 - \lambda_1^3)[(\lambda_1^3 + \lambda_2^3 + \lambda_3^3)^3 + (3\lambda_1 \lambda_2 \lambda_3)^3] \neq 0.$$

The trivector $p_{\lambda_1,\lambda_2,\lambda_3}^{2\,2}$ is $SL(9, \mathbb{R})$-conjugate to

$$\frac{1}{3} \sqrt{3}((\lambda_2 - \lambda_3)p_1 + (\lambda_1 - \lambda_3)p_2 + (\lambda_1 + \lambda_2 + \lambda_3)p_3 + (\lambda_1 - \lambda_2)p_4)$$

which lies in $\mathfrak{c}$.

The conjugacy of the elements $p_{\lambda_1,\lambda_2,\lambda_3}^{2\,1} \in F_2$ is determined by a group isomorphic to the dihedral group of order 12 and generated by

$$
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & -1 \\
0 & -1 & 0
\end{pmatrix}.
$$

The conjugacy of the elements $p_{\lambda_1,\lambda_2,\lambda_3}^{2\,2}$ is determined by exactly the same group.

Canonical set $F_3^c$: It consists of trivectors

$$p_{\lambda_1,\lambda_2}^{3\,1} = \lambda_1 p_1 + \lambda_2 p_2$$

where the $\lambda_i \in \mathbb{C}$ are such that $\lambda_1 \lambda_2 (\lambda_1^6 - \lambda_2^6) \neq 0$. The real elements in $F_3$ consist of $p_{\lambda_1,\lambda_2}^{3\,1}$ where $\lambda_1, \lambda_2 \in \mathbb{R}$ satisfy the same polynomial conditions.

Here we have noncanonical real semisimple elements

$$p_{\lambda_1,\lambda_2}^{3\,2} = \lambda_1(-\frac{1}{2} e_{126} - \frac{1}{3} e_{349} + 2 e_{358} - e_{457} + e_{789}) + \lambda_2(-2 e_{137} - \frac{1}{4} e_{249} - e_{258} - \frac{1}{2} e_{456} - \frac{1}{2} e_{689})
- \lambda_3(-e_{159} - e_{238} - \frac{1}{4} e_{247} - \frac{1}{4} e_{346} - e_{678}), \lambda_1, \lambda_2 \in \mathbb{R}.$$

$$p_{\lambda_1,\lambda_2}^{3\,3} = \lambda_1(12 e_{129} - e_{138} + 2 e_{237} - \frac{1}{4} e_{456} - 2 e_{789}) + \lambda_2(-\frac{1}{4} e_{147} - e_{258} + 2 e_{369})
- \lambda_3(12 e_{147} - 2 e_{245} + e_{289} - e_{356} - \frac{1}{4} e_{378}) + \mu(-e_{124} - e_{136} - \frac{1}{4} e_{238} + e_{457} - 2 e_{569} + e_{789}),$$

where $\lambda_1, \lambda_2, \lambda, \mu \in \mathbb{R}$. These are $SL(9, \mathbb{C})$-conjugate, respectively, to the nonreal elements of $F_3^c$.

$$i\lambda_1 p_1 + i\lambda_2 p_2, \quad i\lambda_1 p_1 + \lambda_2 p_2, \quad (\lambda + i\mu)p_1 + (\lambda - i\mu)p_2.$$

So for $p_{\lambda_1,\lambda_2}^{3\,2}$ the $\lambda_i$ are required to satisfy $\lambda_1 \lambda_2 (\lambda_1^6 - \lambda_2^6) \neq 0$. For $p_{\lambda_1,\lambda_2}^{3\,3}$ the $\lambda_i$ are required to satisfy $\lambda_1 \lambda_2 (\lambda_1^6 + \lambda_2^6) \neq 0$. For $p_{\lambda_1,\mu}^{3\,4}$ the $\lambda, \mu$ are required to satisfy $\lambda \mu (3\lambda^4 - 10\lambda^2 \mu^2 + 3\mu^4) \neq 0$. 

10 MIKHAIL BOROVOI, WILLEM A. DE GRAAF, AND HÔNG VÂN LÊ
We have that $p_{\lambda_1, \lambda_2}^{3,2}, p_{\lambda_1, \lambda_2}^{3,3}, p_{\lambda_1, \lambda_2}^{3,4}$ are $\text{SL}(9, \mathbb{R})$-conjugate to respectively

\[
\frac{1}{3} \sqrt{3}(\lambda_2 p_1 + \lambda_1 p_2 + (\lambda_1 + \lambda_2)p_3 + (\lambda_1 - \lambda_2)p_4),
\]

\[-\lambda_2 p_1 + \frac{1}{3} \sqrt{3} \lambda_1 (p_2 + p_3 + p_4),
\]

\[(\lambda + \frac{1}{3} \sqrt{3} \mu)p_2 + (-\lambda + \frac{1}{3} \sqrt{3} \mu)p_3 - \frac{2}{3} \sqrt{3} \mu p_4,
\]

which lie in the Cartan subspace $\mathfrak{c}$.

The conjugacy of the elements $p_{\lambda_1, \lambda_2}^{3,1} \in F_3$ is determined by a group isomorphic to the dihedral group of order 8 generated by

\[
\begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\]

The conjugacy of the elements $p_{\lambda_1, \lambda_2}^{3,2}$ is determined by exactly the same group. The conjugacy of the elements $p_{\lambda_1, \lambda_2}^{3,3}$ is determined by a group of order 4 generated by

\[
\begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}.
\]

The conjugacy of the elements $p_{\lambda_1, \lambda_2}^{3,4}$ is determined by exactly the same group of order 4.

**Canonical set $F_4^4$:** It consists of trivectors

\[p_{\lambda_1, \lambda_2}^{4,1} = \lambda p_1 + \mu (p_3 - p_4)\]

where the $\lambda, \mu \in \mathbb{C}$ are such that $\lambda \mu (\lambda^3 - \mu^3)(\lambda^3 + 8 \mu^3) \neq 0$. The real elements in $F_4$ consist of $p_{\lambda, \mu}^{4,1}$ were $\lambda, \mu \in \mathbb{R}$ satisfy the same polynomial conditions. Here there are no noncanonical real semisimple elements.

The conjugacy of the elements $p_{\lambda_1, \lambda_2}^{4,1} \in F_4$ is determined by the group consisting of $\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$.

**Canonical set $F_5^5$:** It consists of trivectors

\[p_{\lambda_1, \lambda_2}^{5,1} = \lambda (p_3 - p_4)\]

where the $\lambda \in \mathbb{C}$ is nonzero. The real elements in $F_5$ consist of $p_{\lambda}^{5,1}$ with $\lambda \in \mathbb{R}$ nonzero.

Here we have noncanonical real semisimple elements

\[p_{\lambda_1, \lambda_2}^{5,2} = \lambda (e_{148} - e_{159} - e_{238} + \frac{1}{2} e_{239} - \frac{1}{2} e_{247} + e_{257} - \frac{1}{2} e_{346} - e_{356} - e_{678} - \frac{1}{2} e_{679})\]

where $\lambda \in \mathbb{R}$ is nonzero. We have that $p_{\lambda}^{5,2}$ is $\text{SL}(9, \mathbb{C})$-conjugate to the purely imaginary trivector $p_{\lambda}^{5,1} \in F_5$.

Furthermore, $p_{\lambda}^{5,2}$ is $\text{SL}(9, \mathbb{R})$-conjugate to

\[\frac{1}{3} \sqrt{3} \lambda (2p_2 - p_3 - p_4)\]

which lies in the Cartan subspace $\mathfrak{c}$.

The conjugacy of the elements $p_{\lambda}^{5,1} \in F_5$ is determined by the group consisting of $1, -1$. The conjugacy of the elements $p_{\lambda}^{5,2}$ is determined by the same group.

**Canonical set $F_6^6$:** It consists of trivectors

\[p_{\lambda_1, \lambda_2}^{6,1} = \lambda p_1\]

where the $\lambda \in \mathbb{C}$ is nonzero. The real elements in $F_6$ consist of $p_{\lambda}^{6,1}$ where $\lambda \in \mathbb{R}$ is nonzero.
Here we have noncanonical real semisimple elements

\[ p^{6,2}_\lambda = \lambda(-\frac{1}{2}e_{126} - \frac{1}{2}e_{349} + 2e_{358} - e_{457} + e_{789}) \]

where \( \lambda \in \mathbb{R} \) is nonzero. We have that \( p^{6,2}_\lambda \) is \( SL(9, \mathbb{C}) \)-conjugate to the purely imaginary trivector \( p^{6,1}_\lambda \in \mathcal{F}_6 \).

We have that \( p^{6,2}_\lambda \) is \( SL(9, \mathbb{R}) \)-conjugate to

\[ \frac{1}{3}\sqrt{3} \lambda (p_2 + p_3 + p_4) \]

which lies in the Cartan subspace \( \mathcal{C} \).

The conjugacy of the elements \( p^{6,1}_\lambda \) is determined by the group consisting of \( 1, -1 \). The conjugacy of the elements \( p^{6,2}_\lambda \) is determined by the same group.

**Canonical set** \( \mathcal{F}^C_7 \): this set consists just of 0.

### 2.3. The mixed orbits.

Here we give tables of representatives of the real orbits of mixed type. The representatives of those orbits have a semisimple part which is equal to one of the semisimple elements listed above. So the semisimple parts are of the form \( p^{2,1}_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} p^{3,2}_{\lambda_1, \lambda_2, \lambda_3} \cdots \).

For each such semisimple part we have a table listing the possible nilpotent parts. Note that the centralizer of \( p^{2,1}_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} \) in \( g^c \) is equal to the unique Cartan subalgebra of \( g^c \) that contains \( \mathcal{C} \), so there are no mixed elements with \( p^{2,1}_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} \) as semisimple part.

In the previous subsection we also computed elements of the Cartan subspace \( \mathcal{C} \) that are \( SL(9, \mathbb{R}) \)-conjugate to the elements \( p^{2,2}_{\lambda_1, \lambda_2, \lambda_3} p^{3,2}_{\lambda_1, \lambda_2} \cdots \). However we do not work with those, because the nilpotent parts tend to become rather bulky. For example, the mixed element \( p^{6,2}_\lambda - 2e_{137} \) is \( SL(9, \mathbb{R}) \)-conjugate to

\[ \frac{1}{3}\sqrt{3} \lambda (p_2 + p_3 + p_4) + \frac{1}{9}\sqrt{3}(e_{123} + e_{126} + e_{129} - e_{135} - e_{138} + e_{156} + e_{159} - e_{168} + e_{189} + e_{234} + e_{237} - e_{246} - e_{249} + e_{267} - e_{279} + e_{345} + e_{348} - e_{357} + e_{378} + e_{456} + e_{459} - e_{468} + e_{489} + e_{567} - e_{579} + e_{678} + e_{789}). \]

The caption of each table has the semisimple part \( p \) of the mixed elements whose nilpotent parts are listed in the table. All tables have three columns. The first column has the number of the complex nilpotent orbit, which corresponds to the numbering in the tables in [46]. The second column has the representatives of the real nilpotent orbits. In the third column we display the isomorphism type of the centralizer in \( \mathfrak{g}_0(p) \) of a homogeneous \( \mathfrak{sl}_2 \)-triple in \( \mathfrak{g}(p) \) containing the nilpotent element on the same line. Among the nilpotent elements we also include 0; in that case the centralizer is equal to \( \mathfrak{g}_0(p) \).

**Table 2:** Nilpotent parts of mixed elements with semisimple part \( p^{2,1}_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} \)

| No. | Reps. of nilpotent parts | \( \mathfrak{g}_0(p, h, e, f) \) |
|-----|--------------------------|-----------------|
| 1   | \( e_{168} + e_{249} \)   | 0               |
| 2   | \( e_{168} \)             | 1               |
| 3   | 0                         | 2t              |
Table 3: Nilpotent parts of mixed elements with semisimple part \( p_{\lambda_1, \lambda_2, \lambda_3}^{2,2} \)

| No. | Reps. of nilpotent parts | \( \mathfrak{z}_0(p, h, e, f) \) |
|-----|--------------------------|-------------------------------|
| 1   | \( e_{235} + e_{279} + \frac{1}{2} e_{369} - e_{567} \) | 0 |
| 2   | \(-e_{148}\) | u |
| 3   | 0 | t+u |

Table 4: Nilpotent parts of mixed elements with semisimple part \( p_{\lambda_1, \lambda_2}^{3,1} \)

| No. | Reps. of nilpotent parts | \( \mathfrak{z}_0(p, h, e, f) \) |
|-----|--------------------------|-------------------------------|
| 1   | \( e_{159} + e_{249} + e_{267} \) | 0 |
| 2   | \( e_{159} + e_{249} + e_{267} \) | t |
| 3   | \( e_{159} + e_{249} + e_{267} \) | t |
| 4   | \( e_{159} + e_{249} + e_{267} \) | 2t |
| 5   | \( e_{159} + e_{267} \) | 2t |
| 6   | \( e_{168} + e_{249} \) | 2t |
| 7   | \( e_{159} \) | 3t |
| 8   | \( e_{168} \) | 3t |
| 9   | 0 | 4t |

Table 5: Nilpotent parts of mixed elements with semisimple part \( p_{\lambda_1, \lambda_2}^{3,2} \)

| No. | Reps. of nilpotent parts | \( \mathfrak{z}_0(p, h, e, f) \) |
|-----|--------------------------|-------------------------------|
| 1   | \( e_{235} + e_{238} + \frac{1}{2} e_{234} + e_{279} + e_{368} + e_{467} \) | 0 |
| 2   | \( e_{159} + e_{235} + e_{234} + e_{279} + e_{368} + e_{467} \) | u |
|     | \(-e_{159} - e_{235} + e_{234} + e_{279} + e_{368} + e_{467} \) | u |
| 3   | \(-e_{148} + \frac{1}{2} e_{234} + e_{278} + e_{368} + e_{467} \) | u |
|     | \( e_{148} + e_{234} + e_{278} + e_{368} + e_{467} \) | u |
| 4   | \(-e_{148} + e_{159} \) | 2u |
|     | \(-e_{148} - e_{159} \) | 2u |
|     | \( e_{148} + e_{159} \) | 2u |
|     | \( e_{148} - e_{159} \) | 2u |
| 5   | \( e_{234} + e_{278} + e_{368} + e_{467} \) | t+u |
| 6   | \( e_{235} + e_{234} + e_{278} + e_{368} + e_{467} \) | t+u |
| 7   | \(-e_{159} \) | t+2u |
|     | \( e_{159} \) | t+2u |
| 8   | \(-e_{148} \) | t+2u |
|     | \( e_{148} \) | t+2u |
| 9   | 0 | 2t+2u |
Table 6: Nilpotent parts of mixed elements with semisimple part $p_{\lambda_1, \lambda_2}^{3,3}$

| No. | Reps. of nilpotent parts | $z_0(p, h, e, f)$ |
|-----|-------------------------|------------------|
| 1   | $e_{159} - 2e_{249} - 2e_{348} + 2e_{357}$ | 0 |
| 4   | $e_{159} + 2e_{357}$ | $t + u$ |
| 9   | 0                       | $2t + 2u$ |

Table 7: Nilpotent parts of mixed elements with semisimple part $p_{\lambda, \mu}^{3,4}$

| No. | Reps. of nilpotent parts | $z_0(p, h, e, f)$ |
|-----|-------------------------|------------------|
| 1   | $e_{123} - 2e_{179} - 2e_{259} + \frac{1}{2}e_{349} + e_{357}$ | 0 |
| 2   | $e_{123} - 2e_{179} - 2e_{259} + \frac{1}{2}e_{349} + e_{357}$ | 0 |
| 3   | $-2e_{267} - \frac{1}{2}e_{349} + \frac{1}{4}e_{349} - \frac{1}{4}e_{468}$ | 0 |
| 4   | $-2e_{349} + e_{468}$ | $t + u$ |
| 5   | $-2e_{349}$ | $t + u$ |
| 6   | $e_{123} - 2e_{179} - 2e_{259} + e_{357}$ | $2t$ |
| 7   | $e_{349}$ | $2t + u$ |
| 8   | $-e_{468}$ | $2t + u$ |
| 9   | 0                       | $3t + u$ |

Table 8: Nilpotent parts of mixed elements with semisimple part $p_{\lambda, \mu}^{4,1}$

| No. | Reps. of nilpotent parts | $z_0(p, h, e, f)$ |
|-----|-------------------------|------------------|
| 1   | $e_{149} + e_{167} + e_{258} + e_{347}$ | 0 |
| 2   | $e_{149} + e_{158} + e_{167} + e_{248} + e_{257} + e_{347}$ | 0 |
| 3   | $e_{147} + e_{258}$ | 0 |
| 4   | $e_{147} - e_{158} - e_{248} - e_{257}$ | 0 |
| 5   | $e_{148} + e_{157} + e_{247}$ | $t$ |
| 6   | 0                       | $sl(2, \mathbb{R})$ |

Table 9: Nilpotent parts of mixed elements with semisimple part $p_{\lambda}^{3,1}$

| No. | Reps. of nilpotent parts | $z_0(p, h, e, f)$ |
|-----|-------------------------|------------------|
| 1   | $e_{123} + e_{149} + e_{167} + e_{258} + e_{347} + e_{456}$ | 0 |
| 2   | $e_{123} + e_{149} + e_{158} + e_{167} + e_{248} + e_{257} + e_{347} + e_{456}$ | 0 |
| 3   | $e_{123} + e_{149} + e_{167} + e_{258} + e_{347}$ | $t$ |
| 4   | $e_{123} + e_{147} + e_{258} + e_{456}$ | 0 |
| 5   | $e_{123} + e_{148} - 2e_{157} - 2e_{247} + 2e_{258} + e_{456}$ | 0 |
| 6   | $e_{123} + e_{149} + e_{158} + e_{167} + e_{248} + e_{257} + e_{347}$ | $t$ |
| No. | Reps. of nilpotent parts                                                                 | $\mathfrak{sl}(p, h, e; f)$               |
|-----|------------------------------------------------------------------------------------------|------------------------------------------|
| 1   | $2e_{113} + e_{119} + e_{147} + 2e_{157} + e_{237} + e_{345} - e_{389} - e_{468} + \frac{1}{3}e_{469} + \frac{1}{2}e_{479} + 2e_{568} - e_{569} - 2e_{578}$ | 0                                        |
| 2   | $-\frac{1}{12}e_{112} + e_{135} - e_{178} + e_{248} + e_{249} + \frac{1}{2}e_{258} - \frac{1}{2}e_{259} + e_{345} + e_{389} - e_{390} + \frac{1}{2}e_{456} + \frac{1}{2}e_{479} - 2e_{578} + \frac{1}{2}e_{689}$ | 0                                        |
| 3   | $-\frac{1}{12}e_{112} + e_{134} - 2e_{135} + 2e_{178} - e_{179} - e_{248} - \frac{1}{2}e_{249} - 2e_{258} - e_{259} - 2e_{367}$ | u                                        |
|     | $\frac{1}{12}e_{126} + e_{134} + 2e_{135} + 2e_{178} - e_{179} - e_{248} - \frac{1}{2}e_{249} - 2e_{258} - e_{259} - 2e_{367}$ | u                                        |
| 4   | $\frac{1}{12}e_{126} - e_{134} - 2e_{135} + 2e_{178} - e_{179} - e_{248} - \frac{1}{2}e_{249} - 2e_{258} - e_{259} - 2e_{367}$ | u                                        |
| 5   | $\frac{1}{12}e_{126} - 2e_{134} + e_{135} + e_{147} + e_{148} - e_{149} + 2e_{157} - e_{158} + e_{159} + 2e_{237} + \frac{1}{3}e_{346} + \frac{1}{3}e_{356} + \frac{1}{3}e_{379} - e_{389} - \frac{1}{3}e_{679}$ | u                                        |
|     | $\frac{1}{12}e_{126} + 2e_{134} + e_{135} + e_{147} + e_{148} - e_{149} + 2e_{157} - e_{158} + e_{159} + 2e_{237} + \frac{1}{3}e_{346} + \frac{1}{3}e_{356} + \frac{1}{3}e_{379} - e_{389} - \frac{1}{3}e_{679}$ | u                                        |
| 6   | $2e_{118} + e_{119} + e_{147} + 2e_{157} - e_{389} + e_{468} + \frac{1}{3}e_{469} + 2e_{568} + e_{569} - 2e_{578}$ | t + u                                    |
| 7   | $e_{134} - 2e_{135} + 2e_{178} - e_{179} - e_{389} + 2e_{468} + \frac{1}{2}e_{479} - 2e_{568} + e_{569}$ | t                                        |
| 8   | $-\frac{1}{2}e_{126} + 2e_{149} - e_{258} + e_{346} + \frac{1}{2}e_{356} - e_{678} - \frac{1}{2}e_{679}$ | u                                        |
|     | $\frac{1}{2}e_{126} + e_{137} + e_{468} - e_{469} - 2e_{568} + e_{569}$ | u                                        |
|     | $\frac{1}{2}e_{126} + 2e_{149} - e_{258} + e_{346} + \frac{1}{2}e_{356} - e_{678} - \frac{1}{2}e_{679}$ | u                                        |
| 9   | $2e_{126} + e_{130} + e_{147} + e_{148} + e_{149} + 2e_{157} - e_{158} + 2e_{159} + 2e_{237} + \frac{1}{3}e_{346} + \frac{1}{3}e_{356} + \frac{1}{3}e_{379} - e_{389} - \frac{1}{3}e_{679}$ | t + u                                    |
|     | $-\frac{1}{2}e_{126} + 2e_{138} + e_{139} + e_{147} + 2e_{157} + 2e_{237}$ | t + u                                    |
| 10  | $\frac{1}{2}e_{126} + 2e_{138} + e_{139} + e_{147} + 2e_{157} + 2e_{237}$ | t + u                                    |
| 11  | $e_{248} + \frac{1}{2}e_{249} + 2e_{258} + e_{259} + \frac{1}{2}e_{346} - \frac{1}{4}e_{348} + \frac{1}{4}e_{349} + \frac{1}{2}e_{358} + \frac{1}{2}e_{369}$ | t + u                                    |
| 12  | $e_{345} + e_{359} + e_{369} + e_{379} + \frac{1}{3}e_{379} + 2e_{568} - e_{569} - 2e_{578}$ | t + u                                    |
| 13  | $e_{134} - e_{135} - e_{178} - e_{179} - e_{389} + 2e_{468} + \frac{1}{3}e_{469} + 2e_{568} + e_{569} - 2e_{578}$ | 2t + u                                  |
| 14  | $-\frac{1}{2}e_{126} - 2e_{137}$ | s$\mathfrak{sl}(2, R) + u$ |
| 15  | $-e_{468} + \frac{1}{3}e_{469} + 2e_{568} + e_{569}$ | s$\mathfrak{sl}(2, R) + t + u$ |
| 16  | $e_{345} + e_{359} + \frac{1}{2}e_{379} - 2e_{578}$ | s$\mathfrak{sl}(3, R)$ |
| 17  | $-e_{126}$ | s$\mathfrak{sl}(3, R) + u$ |
| 18  | 0 | s$\mathfrak{sl}(3, R) + t + u$ |

Table 10: Nilpotent parts of mixed elements with semisimple part $p^5_{\lambda}$.  

CLASSIFICATION OF REAL TRIVECTORS IN DIMENSION NINE 15
Table 11: Nilpotent parts of mixed elements with semisimple part $p^{6,1}_\lambda$

| No. | Reps. of nilpotent parts                                      | $\zeta(p, h, e, f)$ |
|-----|---------------------------------------------------------------|---------------------|
| 1   | $\epsilon_{159} + \epsilon_{168} + e_{249} + e_{258} + e_{267} + e_{347}$ | 0                   |
| 2   | $\epsilon_{159} + \epsilon_{168} + e_{249} + e_{257} + e_{258} + e_{347}$ | 0                   |
| 3   | $\epsilon_{149} + \epsilon_{158} + e_{167} + e_{248} + e_{259} + e_{347}$ | 0                   |
|     | $-\epsilon_{148} - e_{159} - e_{249} + e_{258} - e_{267} - e_{357}$ | 0                   |
| 4   | $\epsilon_{149} + \epsilon_{158} + e_{248} + e_{257} + e_{367}$ | t                   |
| 5   | $\epsilon_{149} + \epsilon_{167} + e_{168} + e_{257} + e_{348}$ | t                   |
| 6   | $\epsilon_{149} + \epsilon_{158} + e_{248} + e_{267} + e_{357}$ | t                   |
| 7   | $\epsilon_{149} + \epsilon_{158} + e_{167} + e_{248} + e_{357}$ | t                   |
| 8   | $\epsilon_{149} + \epsilon_{167} + e_{258} + e_{347}$ | $2t$                |
| 9   | $\epsilon_{147} + \epsilon_{158} + e_{258} + e_{269}$ | $2t$                |
|     | $2e_{147} - \epsilon_{159} + e_{168} - e_{249} - 2e_{257}$ | $t + u$             |
| 10  | $\epsilon_{149} + \epsilon_{158} + e_{167} + e_{248} + e_{257} + e_{347}$ | 0                   |
| 11  | $\epsilon_{149} + \epsilon_{167} + e_{248} + e_{357}$ | $2t$                |
| 12  | $\epsilon_{149} + \epsilon_{167} + e_{247} + e_{258}$ | $2t$                |
| 13  | $\epsilon_{149} + \epsilon_{158} + e_{167} + e_{248} + e_{257}$ | t                   |
| 14  | $\epsilon_{149} + \epsilon_{157} + e_{168} + e_{247} + e_{348}$ | $sl(2, R)$          |
| 15  | $\epsilon_{158} + e_{169} + e_{247}$ | $sl(2, R) + 2t$    |
| 16  | $\epsilon_{149} + \epsilon_{158} + e_{167} + e_{247}$ | $2t$                |
| 17  | $\epsilon_{148} + \epsilon_{157} + e_{249} + e_{267}$ | $sl(2, R) + t$     |
| 18  | $\epsilon_{147} + \epsilon_{158} + e_{248} + e_{259}$ | $sl(2, R) + t$     |
| 19  | $\epsilon_{149} + \epsilon_{157} + e_{248}$ | $3t$                |
| 20  | $\epsilon_{147} + e_{258}$ | $4t$                |
|     | $2e_{147} - 2e_{158} - 2e_{248} - 2e_{257}$ | $2t + u$            |
| 21  | $\epsilon_{148} + \epsilon_{157} + e_{247}$ | $3t$                |
| 22  | $\epsilon_{147} + \epsilon_{158} + e_{169}$ | $sl(2, R) + sl(3, R)$ |
| 23  | $\epsilon_{147} + e_{158}$ | $2sl(2, R) + 2t$   |
| 24  | $\epsilon_{147}$ | $3sl(2, R) + 2t$   |
| 25  | 0 | $3sl(3, R)$          |

Table 12: Nilpotent parts of mixed elements with semisimple part $p^{6,2}_\lambda$

| No. | Reps. of nilpotent parts                                      | $\zeta(p, h, e, f)$ |
|-----|---------------------------------------------------------------|---------------------|
| 1   | $\epsilon_{139} - \epsilon_{148} + 2\epsilon_{157} + e_{245} + e_{289} + 2e_{367}$ | 0                   |
| 2   | $-2\epsilon_{135} - \epsilon_{179} + \frac{1}{2}e_{234} + 2e_{278} + e_{368} + \frac{1}{2}e_{467} - e_{659}$ | 0                   |
| 3   | $-2\epsilon_{135} - \epsilon_{145} - \epsilon_{179} - \epsilon_{189} + \frac{1}{2}e_{234} + 2e_{278} + e_{368} + \frac{1}{2}e_{467}$ | 0                   |
|     | $\epsilon_{145} + \epsilon_{189} - e_{237} - e_{248} + e_{346} + 2e_{678}$ | 0                   |
| 4   | $\epsilon_{159} + 2e_{237} - e_{248} - \frac{1}{2}e_{249} - 2e_{258} - \frac{1}{2}e_{456} - \frac{1}{2}e_{689}$ | u                   |
|     | $-4\epsilon_{159} + \frac{1}{2}e_{237} - \frac{1}{2}e_{238} - \frac{1}{2}e_{299} + \frac{1}{2}e_{247} + 20e_{248} + 5e_{249} - \frac{1}{4}e_{257} + 20e_{258} + 4e_{259} - \frac{1}{4}e_{356} + 2e_{456} - \frac{1}{8}e_{679} + 2e_{689}$ | u                   |
| 5   | $\epsilon_{159} - \frac{1}{2}e_{239} - e_{248} - e_{257} + e_{365} + e_{679}$ | u                   |
|     | $-\epsilon_{159} - \frac{1}{2}e_{239} - e_{248} - e_{257} + e_{365} + e_{679}$ | u                   |
| 6   | $-\epsilon_{134} + 2e_{178} + 2e_{235} + e_{379} - e_{609}$ | t                   |
| 7   | $\epsilon_{139} - \epsilon_{148} + 2\epsilon_{157} + e_{238} + \frac{1}{2}e_{247} + \frac{1}{2}e_{346} + e_{678}$ | u                   |
|     | $-4\epsilon_{135} - \frac{1}{2}e_{148} - 2e_{179} - \frac{1}{2}e_{238} - \frac{1}{2}e_{247} + \frac{1}{2}e_{346} + \frac{1}{2}e_{678}$ | u                   |
| 8   | $-\epsilon_{145} + 2\epsilon_{189} + 2e_{237} + 4e_{688}$ | $t + u$             |
### 3. Real Galois Cohomology

#### 3.1. Group Cohomology for a Group of Order 2

In this section we give an alternative definition of $H^1(\Gamma, A)$ and $H^2(\Gamma, A)$ and establish their properties.

**Definition 3.1.1.** Let $A$ be a $\Gamma$-group, not necessarily abelian, where $\Gamma = \{1, \gamma\}$ is a group of order 2. We define

$$Z^1A = \{c \in A \mid c \cdot \gamma c = 1\}.$$  \tag{3.1.2}

We say that an element $c$ as in (3.1.2) is a 1-cocycle of $\Gamma$ in $A$. The group $A$ acts on $Z^1A$ on the right by

$$c * a = a^{-1} \cdot c \cdot \gamma a \quad \text{for } c \in Z^1A, \ a \in A.$$  

We say that the cocycles $c$ and $c * a$ are cohomologous or equivalent. We denote by $H^1(\Gamma, A)$, or for brevity $H^1A$, the set of equivalence classes, that is, the set of orbits of $A$ in $Z^1A$. If $c \in Z^1A$, we denote by $[c] \in H^1A$ its cohomology class.

In general $H^1A$ has no natural group structure, but it has a *neutral element* denoted by $[1]$, the class of the unit element $1 \in Z^1A \subseteq A$.

**Definition 3.1.3.** Let $A$ be an abelian $\Gamma$-group, where $\Gamma$ is a group of order 2. We define abelian groups

$$Z^1A = \{c \in A \mid c \cdot \gamma c = 1\}, \quad B^1A = \{a^{-1} \cdot \gamma a \mid a \in A\}, \quad H^1A = Z^1A/B^1A.$$
Definition 3.1.4. Let $A$ be an abelian $\Gamma$-group, where $\Gamma$ is a group of order 2. We define abelian groups

$$Z^2 A = A^\Gamma := \{c \in A \mid \gamma c = c\}, \quad B^2 A = \{a \cdot \gamma a \mid a \in A\}, \quad H^2 A = Z^2 A/B^2 A.$$ 

Remark 3.1.5. Definitions 3.1.3 and 3.1.4 (given for a group $\Gamma$ of order 2 only!) are equivalent to the standard ones. Namely, for $c \in Z^1 A$ we construct a function of one variable

$$f_c : \Gamma \to A, \quad f_c(1) = 1, \quad f_c(\gamma) = c,$$

which is a 1-cocycle in the sense of [39, Section I.5.1]. Similarly, for $c \in Z^2 A$ we construct a function of two variables

$$\phi_c : \Gamma \times \Gamma \to A, \quad \phi_c(1, 1) = \phi_c(\gamma, 1) = \phi_c(1, \gamma) = 0, \quad \phi_c(\gamma, \gamma) = c,$$

which is a 2-cocycle in the sense of Serre [39, Section I.2.2]. In this way we obtain canonical isomorphisms of pointed sets (for Definition 3.1.1), and of abelian groups (for Definitions 3.1.3 and 3.1.4) between the cohomology sets and groups defined above and the corresponding cohomology sets and groups defined in [39].

We shall need the following result of Borel and Serre:

Proposition 3.1.6 ([39, Section I.5.4, Corollary 1 of Proposition 36]). Let $B$ be a $\Gamma$-group, $A \subseteq B$ be a $\Gamma$-subgroup, and $Y = B/A$, which has a natural structure of a $\Gamma$-set. Then $B^\Gamma$ naturally acts on $Y^\Gamma$, and the connecting map $\delta$ in the cohomology exact sequence

$$1 \to A^\Gamma \to B^\Gamma \to Y^\Gamma \xrightarrow{\delta} H^1 A \to H^1 B$$

induces a canonical bijection between the set of orbits $Y^\Gamma/B^\Gamma$ and the kernel $\ker [H^1 A \to H^1 B]$. 

3.2. Real structures on complex algebraic groups and algebraic varieties.

3.2.1. Let $G$ be a real linear algebraic group. In the coordinate language, the reader may regard $G$ as a subgroup in the general linear group $GL_n(\mathbb{C})$ (for some integer $n$) defined by polynomial equations with real coefficients in the matrix entries; see Borel [2, Section 1.1]. More conceptually, the reader may assume that $G$ is an affine group scheme of finite type over $\mathbb{R}$; see Milne [30, Definition 1.1]. With any of these two equivalent definitions, $G$ defines a covariant functor

$$A \mapsto G(A)$$

from the category of commutative unital $\mathbb{R}$-algebras to the category of groups. Applying this functor to the $\mathbb{R}$-algebra $\mathbb{R}$, we obtain a real Lie group $G(\mathbb{R})$. Applying this functor to the $\mathbb{R}$-algebra $\mathbb{C}$ and to the morphism of $\mathbb{R}$-algebras

$$\gamma : \mathbb{C} \to \mathbb{C}, \quad z \mapsto \bar{z} \quad \text{for} \quad z \in \mathbb{C},$$

we obtain a complex Lie group $G(\mathbb{C})$ together with an anti-holomorphic involution $G(\mathbb{C}) \to G(\mathbb{C})$, which will be denoted by $\sigma_G$. The Galois group $\Gamma$ naturally acts on $G(\mathbb{C})$; namely, the complex conjugation $\gamma$ acts by $\sigma_G$. We have $G(\mathbb{R}) = G(\mathbb{C})^\Gamma$ (the subgroup of fixed points).

We shall consider the complex linear algebraic group $G_C := G \times_{\mathbb{R}} \mathbb{C}$ obtained from $G$ by extension of scalars from $\mathbb{R}$ to $\mathbb{C}$. In this article we shall denote $G_C$ by $G$, the same Latin letter, but non-boldface. By abuse of notation we shall identify $G$ with $G(\mathbb{C})$; in particular, we shall write $g \in G$ meaning that $g \in G(\mathbb{C})$.

Since $G$ is an affine group scheme over $\mathbb{C}$, we have the ring of regular function $\mathbb{C}[G] = \mathbb{R}[G] \otimes_{\mathbb{R}} \mathbb{C}$. Our anti-holomorphic involution $\sigma_G$ of $G(\mathbb{C})$ is anti-regular in the following
sense: when acting on the ring of holomorphic functions on $G$ by sending a holomorphic function $f$ to the holomorphic function

$$(\gamma f)(g) = \gamma(f(\gamma g)) \quad \text{for } g \in G,$$

it preserves the subring $\mathbb{C}[G]$ of regular functions. An anti-regular involution of $G$ is called also a real structure on $G$.

**Remark 3.2.2.** If $G$ is a reductive algebraic group over $\mathbb{C}$ (not necessarily connected), then any anti-holomorphic involution of $G$ is anti-regular. See Adams and Taibi [11, Lemma 3.1].

3.2.3. A morphism of real linear algebraic groups $G \to G'$ induces a morphism of pairs $(G, \sigma_G) \to (G', \sigma_{G'})$. In this way we obtain a functor

$$G \mapsto (G, \sigma_G)$$

from the category of real linear algebraic groups to the category of pairs $(G, \sigma)$, where $G$ is a complex linear algebraic group and $\sigma$ is a real structure on $G$. By Galois descent, see [38, Section V.4.20, Corollary 2 of Proposition 12] or [23, Theorem 2.2], or [17, Section 6.2, Example B], this functor is an equivalence of categories. In particular, any pair $(G, \sigma)$, where $G$ is a complex linear algebraic group and $\sigma$ is a real structure on $G$, is isomorphic to a pair coming from a real linear algebraic group $G$, and any morphism of pairs $(G, \sigma) \to (G', \sigma')$ comes from a unique morphism of the corresponding real algebraic groups.

3.2.4. Let $Y$ be a real quasi-projective variety. The reader may regard $Y$ as a locally closed subvariety in the complex projective space $\mathbb{P}^n_\mathbb{C}$ (for some integer $n$) defined using homogeneous polynomials with real coefficients. As above, by functoriality we obtain a complex analytic space $Y(\mathbb{C})$ together with a real structure (anti-regular involution)

$$\mu_Y : Y(\mathbb{C}) \to Y(\mathbb{C}).$$

We consider the base change $Y := Y \times_{\mathbb{R}} \mathbb{C}$, which by abuse of notation we shall identify with $Y(\mathbb{C})$. We obtain a functor

$$Y \mapsto (Y, \mu_Y)$$

from the category of real quasi-projective varieties to the category of pairs $(Y, \mu)$, where $Y$ is a complex quasi-projective variety and $\mu$ is a real structure on $Y$. By Galois descent this functor is an equivalence of categories. Here it is important that we consider quasi-projective varieties. Note that any homogeneous space of a linear algebraic group is quasi-projective; see, for instance, [3, Theorem 6.8].

3.2.5. From now on, when mentioning a real algebraic group $G$, we shall actually work with a pair $(G, \sigma)$, where $G$ is a complex algebraic group and $\sigma$ is a real structure on $G$. We shall write $G = (G, \sigma)$. We shall shorten “real linear algebraic group” to “$\mathbb{R}$-group”. Similarly, when mentioning a real algebraic variety $Y$, we shall actually work with a pair $(Y, \mu)$, where $Y$ is a complex algebraic variety and $\mu$ is a real structure on $Y$. We shall write $Y = (Y, \mu)$.

3.3. Using $H^1$ for finding real orbits in homogeneous spaces.

3.3.1. Let $G$ be a real algebraic group acting (over $\mathbb{R}$) on a real algebraic variety $Y$. We write $G = (G, \sigma)$, $Y = (Y, \mu)$. Here $\mu$ is an anti-regular involution of $Y$ given by $\mu(y) = \gamma y$. The assertion that $G$ acts on $Y$ over $\mathbb{R}$ means that $G$ acts on $Y$ by $(g, y) \mapsto g \cdot y$, and this action is $\Gamma$-equivariant: for $g \in G$, $y \in Y$ we have

$$\gamma(g \cdot y) = \gamma g \cdot \gamma y,$$

that is,

$$\mu(g \cdot y) = \sigma(g) \cdot \mu(y).$$

We assume that $G$ acts on $Y$ transitively, that is, $G$ acts on $Y$ transitively.
We have $G(\mathbb{R}) = G^\sigma$, $Y(\mathbb{R}) = Y^\mu$. The group $G(\mathbb{R})$ naturally acts on $Y(\mathbb{R})$, and this action might not be transitive. Assuming that $Y(\mathbb{R})$ is non-empty, we describe the set of orbits $Y(\mathbb{R})/G(\mathbb{R})$ in terms of Galois cohomology.

Let $e \in Y(\mathbb{R}) = Y^\mu$ be an R-point. Let $C = \text{Stab}_G(e)$. Since $\sigma(e) = e$, we have $\sigma(C) = C$. We consider the real algebraic subgroup

$$C = \text{Stab}_G(e) = (C, \sigma_C),$$

where $\sigma_C = \sigma|_C$, and the homogeneous space $G/C$, on which $\Gamma$ acts by $\gamma(gC) = \gamma g \cdot C$. We have a canonical bijection

$$G/C \sim Y, \quad gC \mapsto g \cdot e,$$

and an easy calculation shows that this bijection is $\Gamma$-equivariant. Taking into account Proposition 3.1.6, we obtain bijections

$$\ker [\pi^1 C \to \pi^1 G] \sim (G/C)^\Gamma / G(\mathbb{R}) \sim Y(\mathbb{R})/G(\mathbb{R}).$$

Construction 3.3.4. We describe explicitly the composite bijection (3.3.3). Let $c \in \mathbb{Z} C$ be such that $[c] \in \ker [\pi^1 C \to \pi^1 G]$. Then there exists $g \in G$ such that $c = g^{-1} \cdot \gamma g$, that is, $\sigma(g) = gc$. We set $e_c = g \cdot e \in Y$. We compute

$$\mu(e_c) = \mu(g \cdot e) = \sigma(g) \cdot \mu(e) = gc \cdot e = g \cdot (c \cdot e) = g \cdot e = e_c.$$

Thus $e_c \in Y(\mathbb{R})$. To $[c]$ we associate the $G(\mathbb{R})$-orbit $G(\mathbb{R}) \cdot e_c \subset Y(\mathbb{R})$.

Proposition 3.3.5. The correspondence $c \mapsto e_c$ of Construction 3.3.4 induces a bijection

$$\ker [\pi^1 C \to \pi^1 G] \sim Y(\mathbb{R})/G(\mathbb{R}).$$

Proof. The proposition follows from Proposition 3.1.6.

3.3.6. Let $G_0$ be a real algebraic group embedded into a larger real algebraic group $G$, and let $h \in g := \text{Lie } G$. Then $G_0$ acts on $g$ via the adjoint representation of $G$. Let $Y_h$ be the $G_0$-orbit of $h$, that is, the real algebraic variety defined by

$$Y_h = \{h' \in g^C \mid h' = \text{Ad}(g) \cdot h \text{ for some } g \in G_0\},$$

where $g^C = g \otimes_R C$. Let $Z_h$ denote the centralizer of $h$ in $G_0$. The group $G_0$ acts transitively (over $C$) on the left on $Y_h$ by

$$(g, h') \mapsto \text{Ad}(g) \cdot h', \quad g \in G, \ h' \in Y_h$$

with stabilizer $Z_h$ of $h$.

Corollary 3.3.7 (from Proposition 3.3.5). The set of the $G_0(\mathbb{R})$-conjugacy classes in $Y(\mathbb{R})$ is in a canonical bijection with

$$\ker [\pi^1 Z_h \to \pi^1 G_0].$$

Corollary 3.3.8. If $\pi^1 Z_h = 1$, then any element $h' \in g$ that is $G_0$-conjugate to $h$ over $C$, is $G_0$-conjugate to $h$ over $\mathbb{R}$.

3.4. Using $\pi^2$ for finding a real point in a complex orbit.

3.4.1. In order to use Construction 3.3.4 we need a real point $e \in Y(\mathbb{R})$. We describe a general method of finding a real point using second Galois cohomology, following an idea of Springer [40], Section 1.20; see also [4] Section 7.7 and [15] Section 2].

Let $G = (G, \sigma)$ and $Y = (Y, \mu)$ be as in Subsection 3.3.1. We choose a complex point $e \in Y$ and set $C = \text{Stab}_G(e) \subset G$. In this article we consider only the special case when the complex algebraic group $C$ is abelian; this suffices for our applications.
Consider $\mu(e) \in Y$. We have
\[ e = g \cdot \mu(e) \quad \text{for some } g \in G, \]
because $Y$ is homogeneous. Since $\mu^2 = \text{id}$, we have
\[ e = \mu(\mu(e)) = \mu(g^{-1} \cdot e) = \sigma(g^{-1}) \cdot \mu(e) = \sigma(g^{-1}) \cdot g^{-1} \cdot e = (g \cdot \sigma(g))^{-1} \cdot e. \]
Thus $g \cdot \sigma(g) \in C$. We set
\[ d = g \cdot \sigma(g) \in C. \]

We define an anti-regular involution $\nu$ of $C$. Let $c \in C$. We calculate:
\[ c \cdot e = e \quad \Rightarrow \quad \sigma(c) \cdot \mu(e) = \mu(e) \quad \Rightarrow \quad g \cdot \sigma(c) \cdot \mu(e) = g \cdot \mu(e) \quad \Rightarrow \quad g \sigma(c) g^{-1} \cdot g \cdot \mu(e) = g \cdot \mu(e) \quad \Rightarrow \quad g \sigma(c) g^{-1} \cdot e = e. \]
We see that $g \sigma(c) g^{-1} \in C$. We set
\begin{equation}
(3.4.2) \quad \nu(c) = g \sigma(c) g^{-1} \in C.
\end{equation}
We compute
\[ \nu^2(c) = \nu(\nu(c)) = g \cdot \sigma(\nu(c)) \cdot g^{-1} = g \cdot \sigma(g \sigma(c) g^{-1} \cdot g^{-1}) = (g \sigma(g)) \cdot c \cdot (g \sigma(g))^{-1} = c, \]
because $g \sigma(g) \in C$ and $C$ is abelian. Thus $\nu^2 = 1$, that is, $\nu$ is an involution, and it is clear from (3.4.2) that it is anti-regular. We have
\[ \nu(d) = g \sigma(d) g^{-1} = g \sigma(g) gg^{-1} = g \sigma(g) = d. \]
This $d \in C^\nu$. We have constructed an anti-regular involution
\[ \nu : C \to C, \quad c \mapsto g \sigma(c) g^{-1} \]
and an element
\[ d = g \sigma(g) \in C^\nu. \]
\subsection{3.4.3.} We consider the real algebraic group $\mathbf{C} = (C, \nu)$. Recall that
\[ H^2 \mathbf{C} = Z^2 \mathbf{C} / B^2 \mathbf{C}, \]
where
\[ Z^2 \mathbf{C} = C^\nu := \{ c \in C \mid \nu(c) = c \}, \quad B^2 \mathbf{C} = \{ c' \cdot \nu(c') \mid c' \in C \}. \]

We define the class of $Y$
\[ \text{Cl}(Y) = [d] \in H^2 \mathbf{C}. \]

\textbf{Proposition 3.4.4.} If $Y$ has a $\mu$-fixed point, then $\text{Cl}(Y) = 1$.

\textit{Proof.} Assume that $Y$ has a $\mu$-fixed point $y$, that is, $y = \mu(y)$. Write $y = u^{-1} \cdot e$ with $u \in G$. Then $\mu(u^{-1} \cdot e) = u^{-1} \cdot e$, whence
\[ u \cdot \sigma(u)^{-1} \cdot g^{-1} \cdot e = e. \]
Set $c = u \sigma(u)^{-1} g^{-1}$; then $c \cdot e = e$, whence $c \in C$. We calculate:
\[ c \cdot \nu(c) \cdot d = u \sigma(u)^{-1} g^{-1} \cdot g \cdot \sigma(u) u^{-1} \sigma(g)^{-1} \cdot g^{-1} \cdot g \sigma(g) = 1. \]
Thus $\text{Cl}(Y) = [d] = 1 \in H^2 \mathbf{C}$, as required. \hfill \square

\textbf{Proposition 3.4.5.} If $\text{Cl}(Y) = 1$ and $H^1 \mathbf{G} = 1$, then $Y$ has a $\mu$-fixed point.
Proof. Choose $e \in Y$ and choose $g \in G$ such that $e = g \cdot \mu(e)$. Write
\[ d = g \cdot \sigma(g) \in \mathbb{Z}^2 \mathbb{C} = C^v. \]

By assumption $[d] = 1$, that is,
\[ c \cdot \nu(c) \cdot d = 1 \quad \text{for some } c \in C. \]

Set $g' = cg$. We have
\[ 1 = c \cdot \nu(c) \cdot d = c \cdot g\sigma(c)g^{-1} \cdot g\sigma(g) = cg \cdot \sigma(cg) = g' \cdot \sigma(g'). \]

Thus $g'$ is a 1-cocycle, $g' \in Z^1 G$.

By assumption $H^1 G = 1$. It follows that there exists $u \in G$ such that
\[ u^{-1}g'\sigma(u) = 1. \]

Set $y = u^{-1} \cdot e \in Y$. Then
\[
\mu(y) = \sigma(u^{-1}) \cdot \mu(e) = \sigma(u)^{-1}g^{-1} \cdot e = \sigma(u)^{-1}g^{-1} \cdot c^{-1} \cdot e.
\]

\[ = \sigma(u)^{-1}(g')^{-1} \cdot e = \sigma(u)^{-1}(g')^{-1}u \cdot y = (u^{-1}g'\sigma(u))^{-1} \cdot y = y. \]

Thus $y$ is a $\mu$-fixed point in $Y$, as required. \(\square\)

Remark 3.4.6. Propositions 3.4.5 and 3.4.4 and their proofs give a method of finding a real (that is, $\mu$-fixed) point in $Y$ or proving that $Y$ has no real points, assuming that $H^1 G = 1$ and that $C$ is abelian. The general case (without these two assumptions) is treated in [3].

4. A $\mathbb{Z}_3$-grading of the simple complex Lie algebra of type $E_8$

4.1. Constructing $E_8$ with a $\mathbb{Z}_3$-grading. We write $Z_3 = \mathbb{Z}/3\mathbb{Z}$. We have
\[ Z_3 = \{ 0 = 0 + 3\mathbb{Z}, \bar{1} = 1 + 3\mathbb{Z}, \bar{-1} = -1 + 3\mathbb{Z} \}. \]

Generalizing and simplifying a construction of Vinberg and Elashvili [6], here we construct a $\mathbb{Z}_3$-grading of a split simple Lie algebra $g$ of type $E_8$ over a field $k$ of characteristic 0. This grading plays a pivotal role in our classification of trivectors of a real 9-dimensional space. We follow [26, Example 3.3.v].

Consider the free abelian group $\mathbb{Z}^9$ with the standard basis $\bar{e}_1, \ldots, \bar{e}_9$. Set
\[ Q = \mathbb{Z}^9/(\bar{e}_1 + \cdots + \bar{e}_9). \]

For $i = 1, \ldots, 9$, let $\varepsilon_i \in Q$ denote the image of $\bar{e}_i$ in $Q$. Then
\[ \varepsilon_1 + \cdots + \varepsilon_9 = 0. \]

We construct a subset $\Phi_\kappa \subset Q$ for each $\kappa \in Z_3$. We write $\Phi_0$ for $\Phi_0$ etc. Set
\[ \Phi_0 = \{ \varepsilon_i - \varepsilon_j \mid i \neq j \}, \]
\[ \Phi_1 = \{ \varepsilon_i + \varepsilon_j + \varepsilon_k \mid i, j, k \ \text{pairwise different} \}, \]
\[ \Phi_{-1} = -\Phi_1 = \{ -(\varepsilon_i + \varepsilon_j + \varepsilon_k) \mid i, j, k \ \text{pairwise different} \}. \]

We set
\[ \Phi = \Phi_0 \cup \Phi_1 \cup \Phi_{-1}, \]
\[ \alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \ldots, \alpha_7 = \varepsilon_7 - \varepsilon_8, \alpha_8 = \varepsilon_6 + \varepsilon_7 + \varepsilon_8, \]
\[ \Pi = \{ \alpha_1, \ldots, \alpha_7, \alpha_8 \} \subset \Phi, \]
\[ V = Q \otimes \mathbb{R}. \]

Then $\Phi$ is a root system of type $E_8$ in $V$, and $\Pi$ is a basis of $\Phi$; see Onishchik and Vinberg [34, Table 1]. Note that $\Phi = \Phi_0 \cup \Phi_1 \cup \Phi_{-1}$ is a $\mathbb{Z}_3$-grading of $\Phi$ in the following sense:
\[ (\Phi_i + \Phi_\kappa) \cap \Phi \subset \Phi_{i+\kappa} \quad \text{for } i, \kappa \in Z_3, \ i \neq \kappa. \]
We specify an isomorphism $g \sim \mathfrak{sl}_n(C \mathbb{Z})$. Consider the dual root system $\Phi^\vee = \{ \eta^\vee \in V^* \mid \eta \in \Phi \}$, where $\eta^\vee$ is the coroot corresponding to the root $\eta$. Set $Q^\vee = \langle \Phi^\vee \rangle \subset V^*$, the lattice generated by $\Phi^\vee$. Then $\{ \alpha^\vee \mid \alpha \in \Pi \}$ is a basis of $Q^\vee$.

Let $k$ be a field of characteristic 0. Consider the pair $(\mathfrak{g}, \mathfrak{h})$ defined by $(\Phi, \Pi)$; see Bourbaki [8, Section VIII.4.3, Theorem 1]. Here $(\mathfrak{g}, \mathfrak{h})$ is a split simple Lie algebra of type $E_8$ over $k$ (namely, $\mathfrak{g}$ is a simple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ is a split Cartan subalgebra). They come with a generating family

$$(x_{-\alpha}, h_\alpha, x_\alpha)_{\alpha \in \Pi}, \text{ where } h_\alpha \in \mathfrak{h} \subset \mathfrak{g}, \quad x_\alpha, x_{-\alpha} \in \mathfrak{g}.$$ These elements $x_{-\alpha}, h_\alpha, x_\alpha$ satisfy conditions (16 – 22) of Bourbaki [8, Section VIII.4.3], in particular,

$$[h_\alpha, h_\beta] = 0, \quad [h_\alpha, x_\beta] = n(\beta, \alpha) x_\beta, \quad [h_\alpha, x_{-\beta}] = -n(\beta, \alpha) x_{-\beta}, \quad [x_\alpha, x_{-\alpha}] = -h_\alpha$$

for $\alpha, \beta \in \Pi$, where $n(\beta, \alpha) = \langle \beta, \alpha^\vee \rangle$. The family $(h_\alpha)_{\alpha \in \Pi}$ is a basis of $\mathfrak{h}$. Let $\mathfrak{h}^*$ denote the dual vector $k$-space to $\mathfrak{h}$. The homomorphism $\eta \mapsto \eta h_1: Q \rightarrow \mathfrak{h}^*$ such that $\langle \eta h_1, h_\alpha \rangle = \langle \eta, \alpha^\vee \rangle$, takes $\Phi$ to the root system of $(\mathfrak{g}, \mathfrak{h})$. We have a root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\eta \in \Phi} \mathfrak{g}_\eta,$$

where in $\mathfrak{g}_\eta$ we write $\eta$ instead of $\eta h_1$, and where $g_\alpha = k x_\alpha$ for $\alpha \in \Pi \subset \Phi$.

Set

$$\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\eta \in \Phi_0} \mathfrak{g}_\eta, \quad \mathfrak{g}_1 = \bigoplus_{\eta \in \Phi_1} \mathfrak{g}_\eta, \quad \mathfrak{g}_{-1} = \bigoplus_{\eta \in \Phi_{-1}} \mathfrak{g}_\eta.$$

It follows from (4.1.1) that

$$[g_i, g_{\kappa}] \subset g_{i+\kappa} \quad \text{for } i, \kappa \in \mathbb{Z}_3.$$ In other words,

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_{-1}$$

is a $\mathbb{Z}_3$-grading of $\mathfrak{g}$. In particular, $\mathfrak{g}_0$ is a Lie subalgebra of $\mathfrak{g}$, and the subspaces $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are naturally $\mathfrak{g}_0$-modules.

Set

$$\alpha_{8i} = \varepsilon_8 - \varepsilon_9, \quad \alpha_i = \alpha_i \text{ for } i = 1, \ldots, 7, \quad \Pi_0 = \{ \alpha_{81}, \ldots, \alpha_{87}, \alpha_8 \}.$$ Clearly, $\Phi_0$ is a root system of type $A_8$, and $\Pi_0$ is a basis of $\Phi_0$. Therefore, $\mathfrak{g}_0 \simeq \mathfrak{s}(9, k)$.

For the simple root $\alpha_8^A$ of $(\mathfrak{g}_0, \mathfrak{h})$, we choose a nonzero element $x_{\alpha_8^A} \in \mathfrak{g}_{\alpha_8^A}$. We define $h_{8}^A \in \mathfrak{h}$ by the formula

$$[h_{8}^A, x_\beta] = n(\beta, \alpha_8^A) x_\beta \quad \text{for } \beta \in \Pi_0,$$

and we define $x_{-\alpha_8^A} \in g_{-\alpha_8^A}$ by the formula

$$[x_{\alpha_8^A}, x_{-\alpha_8^A}] = -h_{8}^A.$$ We set

$$h_i^A = h_{\alpha_i} \text{ for } i = 1, \ldots, 7.$$ We specify an isomorphism $\mathfrak{g}_0 \xrightarrow{\sim} \mathfrak{s}(9, k)$ by sending

$$h_i^A \mapsto E_{i,i} - E_{i+1,i+1}, \quad x_{\alpha_i^A} \mapsto E_{i,i+1}, \quad x_{-\alpha_i^A} \mapsto -E_{i+1,i} \text{ for } i = 1, \ldots, 8,$$

where $E_{i,j}$ is the matrix with the $(i,j)$-entry 1 and all other entries 0. This isomorphism defines a structure of $\mathfrak{s}(9, k)$-module on $\mathfrak{g}_1$. From the formula for $\Phi_1$ we see
that \( \mathfrak{g}_1 \) is an irreducible \( \mathfrak{sl}(9, \mathbb{k}) \)-module with highest weight \( \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \). Therefore, it is isomorphic to \( \bigwedge^3 \mathbb{k}^9 \). We specify an isomorphism

\begin{equation}
\mathfrak{g}_1 \xrightarrow{\sim} \bigwedge^3 \mathbb{k}^9
\end{equation}

by sending \( x_\alpha \) to \( \varepsilon_6 \wedge \varepsilon_7 \wedge \varepsilon_8 \) (recall that \( \alpha_8 = \varepsilon_6 + \varepsilon_7 + \varepsilon_8 \)).

Set \( G = \text{Aut}(\mathfrak{g}) \). It is a split simple algebraic \( \mathbb{k} \)-group of type \( E_8 \), simply connected and of adjoint type, with Lie algebra \( \mathfrak{g} \). The Lie algebra \( \mathfrak{sl}(9, \mathbb{k}) \) is the Lie algebra of the simply connected simple algebraic \( \mathbb{k} \)-group \( \text{SL}_{9, \mathbb{k}} \). Since \( \text{SL}_{9, \mathbb{k}} \) is connected and simply connected, the homomorphism of Lie algebras

\[ \psi: \mathfrak{sl}(9, \mathbb{k}) \xrightarrow{\sim} \mathfrak{g}_0 \xrightarrow{\sim} \mathfrak{g} \]

is the differential of a uniquely defined homomorphism of algebraic \( \mathbb{k} \)-groups

\begin{equation}
\Psi: \text{SL}_{9, \mathbb{k}} \to G.
\end{equation}

Let \( G_0 \subset G \) denote the image of this homomorphism, which is an algebraic \( \mathbb{k} \)-subgroup of \( G \) with Lie algebra \( \mathfrak{g}_0 \). From the formulas for \( \Phi_0 \), \( \Phi_1 \), \( \Phi_{-1} \), and \( \Phi \) we see that the kernel of the homomorphism (4.1.3) is

\[ \mu_3 = \{ \text{diag}(z, \ldots, z) \mid z \in \mathbb{k}, z^3 = 1 \}, \]

where \( \mathbb{k} \) denotes an algebraic closure of \( \mathbb{k} \), which is a central algebraic \( \mathbb{k} \)-subgroup of \( \text{SL}_{9, \mathbb{k}} \). Thus we may and shall identify the algebraic \( \mathbb{k} \)-groups \( G_0 \) and \( \text{SL}_{9, \mathbb{k}}/\mu_3 \).

Since the algebraic \( \mathbb{k} \)-subgroup \( G_0 \subset G = \text{Aut}(\mathfrak{g}) \) is connected, and its Lie algebra \( \mathfrak{g}_0 \) preserves \( \mathfrak{g}_1 \), we see that \( G_0 \) itself preserves \( \mathfrak{g}_1 \). Thus we can regard \( \mathfrak{g}_1 \) as a \( G_0 \)-module and as an \( \text{SL}_{9, \mathbb{k}} \)-module. Our isomorphism (4.1.2) \( \mathfrak{g}_1 \xrightarrow{\sim} \bigwedge^3 \mathbb{k}^9 \) is \( \mathfrak{g}_0 \)-equivariant, and hence \( G_0 \)-equivariant, because \( G_0 \) is connected. Thus the isomorphism (4.1.2) is \( \text{SL}_{9, \mathbb{k}} \)-equivariant.

The homomorphism (4.1.3) induces homomorphism on \( \mathbb{k} \)-points

\[ \Psi: \text{SL}(9, \mathbb{k}) \to G_0(\mathbb{k}). \]

If \( \mathbb{k} = \mathbb{C}, \mathbb{R} \), then this homomorphism is bijective (for \( \mathbb{k} = \mathbb{R} \) by [6, Corollary 3.3.14]).

An element \( x \in \mathfrak{g}_1 \) is called semisimple (respectively nilpotent) if the linear operator \( \text{ad} x: \mathfrak{g} \to \mathfrak{g} \) is semisimple (respectively nilpotent) in \( \mathfrak{g} \). By [26, §2] the homogeneous Jordan decomposition holds: any \( x \in \mathfrak{g}_1 \) has a unique decomposition \( x = x_s + x_n \) with \( x_s, x_n \in \mathfrak{g}_1 \), where \( x_s \) is semisimple, \( x_n \) is nilpotent, and \( [x_s, x_n] = 0 \).

It follows that the elements of \( \mathfrak{g}_1 \) are naturally divided into three classes consisting, respectively, of the nilpotent elements, the semisimple elements, and the elements that are neither semisimple nor nilpotent. We say that the elements of the third class are of mixed type. Thus each of the problems of classification of the \( \text{SL}(9, \mathbb{C}) \)-orbits in \( \mathfrak{g}_1^\mathbb{C} \) and the \( \text{SL}(9, \mathbb{R}) \)-orbits in \( \mathfrak{g}_1 = \mathfrak{g}_1^\mathbb{R} \) naturally splits into three subproblems.

4.2. Nilpotent elements and homogeneous \( \mathfrak{sl}_2 \)-triples. A triple \((h, e, f)\) of elements in \( \mathfrak{g} \) is called an \( \mathfrak{sl}_2 \)-triple if

\[ [e, f] = h, \ [h, e] = 2e, \ [h, f] = -2f. \]

We say that an \( \mathfrak{sl}_2 \)-triple \((h, e, f)\) is homogeneous if \( h \in \mathfrak{g}_0 \), \( e \in \mathfrak{g}_1 \), \( f \in \mathfrak{g}_{-1} \).

The element \( h \) is called the characteristic of the triple \((h, e, f)\).

**Proposition 4.2.1** (Jacobson-Morozov-Vinberg theorem for a \( \mathbb{Z}_m \)-graded semisimple Lie algebra). We write \( \mathfrak{g}_k^\mathbb{R} \) for \( \mathfrak{g} \). For \( \mathbb{k} = \mathbb{C}, \mathbb{R} \), let \( e \in \mathfrak{g}_k^\mathbb{R} \) be a nonzero nilpotent element.

(i) There is a semisimple element \( h \in \mathfrak{g}_0^\mathbb{R} \) and a nilpotent element \( f \in \mathfrak{g}_{-1}^\mathbb{R} \) such that

\[ [h, e] = 2e, \ [h, f] = -2f, \ [e, f] = h. \]
(ii) The element $h$ is defined uniquely up to conjugacy via an element in the centralizer $Z_{G_0(k)}(e)$ of $e$ in $G_0(k)$.

(iii) Given $e$ and $h$, the element $f$ is defined uniquely.

**Proof.** For $k = \mathbb{C}$ see Vinberg [45, Theorem 1]. For $k = \mathbb{R}$ see [26, Theorem 2.1]. See also [12, Lemma 8.3.5].

For $k = \mathbb{C}, \mathbb{R}$, let $\mathcal{T}^k$ be the set of homogeneous $\mathfrak{sl}_2$-triples in $g^k$. We also write $\mathcal{T}$ for $\mathcal{T}^k$.

**Corollary 4.2.2.** For $k = \mathbb{C}, \mathbb{R}$, let $(h, e, f)$ and $(h', e', f')$ be two homogeneous $\mathfrak{sl}_2$-triples in $g^k$. Then $e, e'$ are $\text{SL}(9, k)$-conjugate if and only if there exists $g \in \text{SL}(9, k)$ with $(g \cdot h, g \cdot e, g \cdot f) = (h', e', f')$.

**Proof.** Suppose that there is $g_1 \in \text{SL}(9, k)$ such that $g_1 \cdot e = e'$. Then the image of $g_1$ in $G_0(k)$ also maps $e$ to $e'$. From Proposition 4.2.1(ii) and (iii), it now follows that there exists $g_0 \in G_0(k)$ with $g_0 \cdot h = h', g_0 \cdot e = e', g_0 \cdot f = f'$. The homomorphism $\Psi: \text{SL}(k) \to G_0(k)$ is bijective (see above). So the preimage $g$ of $g_0$ in $\text{SL}(9, k)$ does the job.

We note the following important fact: each $\text{SL}(9, \mathbb{C})$-orbit in $\mathcal{T}^\mathbb{C}$ has a real representative. This follows immediately from the classification of these orbits in [46], where a real representative is given for each orbit. It can also be proved more conceptually and more generally, see [6, Proposition 4.3.22].

**Theorem 4.2.3.** Let $e \in \mathfrak{g}_1$ be a nilpotent element and $t = (h, e, f)$ be a homogeneous $\mathfrak{sl}_2$-triple containing $e$. Write

$$Z_{\text{SL}(9, \mathbb{C})}(t) = \{ g \in \text{SL}(9, \mathbb{C}) \mid g \cdot f = f, g \cdot h = h, g \cdot e = e \}.$$ 

The $\text{SL}(9, \mathbb{R})$-orbits in $(\text{SL}(9, \mathbb{C}) \cdot e) \cap \mathfrak{g}_1$ correspond bijectively to the elements of $H^1 Z_{\text{SL}(9, \mathbb{C})}(t)$.

**Proof.** By Corollary 4.2.2 the $\text{SL}(9, \mathbb{R})$-orbits in $(\text{SL}(9, \mathbb{C}) \cdot e) \cap \mathfrak{g}_1$ correspond bijectively to the $\text{SL}(9, \mathbb{R})$-orbits in $(\text{SL}(9, \mathbb{C}) \cdot t) \cap \mathcal{T}$. By Proposition 3.3.3 the latter correspond bijectively to $\ker \left[ H^1 Z_{\text{SL}(9, \mathbb{C})}(t) \to H^1 \text{SL}(9, \mathbb{C}) \right]$. Since $H^1 \text{SL}(9, \mathbb{C}) = 1$, the theorem follows.

**Remark 4.2.4.** Let $(h, e, f)$ be a homogeneous $\mathfrak{sl}_2$-triple in $g^\mathbb{C}$ and consider the element $h' = (\psi^\mathbb{C})^{-1}(h) \in \text{sl}(9, \mathbb{C})$. It is not difficult to see that $h'$ has rational eigenvalues. So $h'$ is $\text{SL}(9, \mathbb{C})$-conjugate to a unique real diagonal matrix $h''$ with weakly decreasing diagonal entries. Furthermore, the differences of two eigenvalues of $h'$ are integral. In the second column of Table 1 we write the *indices* of $h'$, i.e., the value $(\varepsilon_i - \varepsilon_{i+1})(h')$, where $\varepsilon_i - \varepsilon_{i+1}$ are the simple roots of $\mathfrak{sl}(9, \mathbb{C})$ with respect to the Cartan subalgebra consisting of the diagonal matrices.

### 4.3. Semisimple elements in $g^C$.

For $k = \mathbb{C}, \mathbb{R}$, a *Cartan subspace* in $g^k$ is, by definition, a maximal subspace in $g^k$ consisting of commuting semisimple elements (cf. [44] and [26, §2]). In [44] a specific Cartan subspace $\mathfrak{C}^C \subset g^C_1$ is given (it is described in Section 2.2 above).

Define

$$\mathcal{N}_{\text{SL}(9, \mathbb{C})}(\mathfrak{C}^C) = \{ g \in \text{SL}(9, \mathbb{C}) \mid g \cdot \mathfrak{C}^C = \mathfrak{C}^C \},$$

$$Z_{\text{SL}(9, \mathbb{C})}(\mathfrak{C}^C) = \{ g \in \text{SL}(9, \mathbb{C}) \mid g \cdot p = p \text{ for all } p \in \mathfrak{C}^C \}.$$ 

Then $W = \mathcal{N}_{\text{SL}(9, \mathbb{C})}(\mathfrak{C}^C)/Z_{\text{SL}(9, \mathbb{C})}(\mathfrak{C}^C)$ is called the Weyl group (or also little Weyl group) of the $\mathbb{Z}_2$-graded Lie algebra $g^C$.

In [44] the following is shown:
• Every semisimple SL(9, C)-orbit in \( g^c \) has a point in \( c^c \).
• Two elements of \( c^c \) are SL(9, C)-conjugate if and only if they are W-conjugate.

In \[46\] it is shown how to refine these statements. Seven canonical subsets \( F_1^c, \ldots, F_7^c \) of \( c^c \) along with finite groups \( \Gamma_1, \ldots, \Gamma_7 \) are determined such that

• Every semisimple SL(9, C)-orbit has a point in precisely one of the \( F_k^c \).
• Each group \( \Gamma_k \) acts on \( F_k^c \). Furthermore, two elements of \( F_k^c \) are SL(9, C)-conjugate if and only if they are \( \Gamma_k \)-conjugate.

Here we summarize some of the constructions of \[46\] that are used to establish this.

Let \( c^c \) be a Cartan subalgebra of \( g^c \) containing \( c^c \). It turns out that such a Cartan subalgebra \( c^c \) is unique and that \( c^c = (c^c \cap g_{-1}^c) \oplus c^c \). Let \( \Pi \) be the root system of \( g^c \) with respect to \( c^c \).

The \( \mathbb{Z}_3 \)-grading \( g^c = g_{-1}^c \oplus g_0^c \oplus g_1^c \) yields an automorphism \( \theta : g^c \to g^c \) defined by \( \theta(x) = \zeta^i x \) for \( x \in g_c^c \), where \( \zeta \in \mathbb{C} \) is a fixed primitive third root of unity. We see that \( c^c \) is \( \theta \)-stable and we consider the dual map on the dual space \( c^{c,*} \) to \( c^c \):

\[
\theta^* : c^{c,*} \to c^{c,*}, \quad (\theta^* \gamma)(c) = \gamma(\theta^{-1} c) \quad \text{for} \quad \gamma \in c^{c,*}, c \in c^c.
\]

We have \( \theta^* + \theta + 1 = 0 \), so that for \( \alpha \in \Pi \) the set \( \Pi(\alpha) := \{ \pm \alpha, \pm \theta \alpha, \pm \theta^2 \alpha \} \) lies in a 2-dimensional space. Furthermore, it is a root subsystem of type \( A_2 \). The sets \( \Pi(\alpha) \) form a partition of \( \Pi \), and hence there are 40 such sets. Note that for \( \alpha \in \Pi \subset c^{c,*} \) and \( c \in c^c \subset c^c \) we have \( \theta^{i*}(\alpha)(c) = \zeta^i \alpha(c) \). Hence the restrictions of the elements of \( \Pi(\alpha) \subset c^{c,*} \) to \( c^c \subset c^c \) are scalar multiples of the restriction of \( \alpha \).

By definition a complex reflection is a linear transformation \( r \) of a complex vector space \( V \) such that there is a basis of \( V \) with respect to which \( r \) has the matrix \( \text{diag}(\omega, 1, \ldots, 1) \), where \( \omega \) is a primitive \( m \)-th root of unity for some \( m \geq 2 \) (see \[28\] Definition 1.7, \[47\] Definition 3.6)). A complex reflection group is a subgroup of \( \text{GL}(V) \) generated by complex reflections. Vinberg has shown that the Weyl group \( W \) is a complex reflection group in \( \text{GL} (c^c) \) (\[44\] Theorem 8, \[47\] Theorem 3.69)). In \[46\] a complex reflection \( w_\alpha \) of order 3 is constructed corresponding to each \( \Pi(\alpha) \). So we have 40 of these complex reflections, and together they generate the Weyl group \( W \). (In fact, \( W \) is already generated by four of those complex reflections.) In \[46\] shown that there exist \( p_\alpha \in c^c \) such that

\[
(4.3.1) \quad w_\alpha(h) = h - \alpha(h)p_\alpha \quad \text{for} \quad h \in c^c.
\]

Now let \( p \in c^c \). The centralizer \( z(p) \) of \( p \) in \( g^c \) is given by

\[
(4.3.2) \quad z(p) = c^c \oplus \bigoplus_{\alpha \in \Pi, \alpha(p) = 0} g_\alpha^c,
\]

where \( g_\alpha^c \) denotes the root subspace in \( g^c \) corresponding to a root \( \alpha \).

Let \( W_p \) be the subgroup of \( W \) generated by the complex reflections \( w_\alpha \) where \( \alpha \) is such that \( \alpha(p) = 0 \), or equivalently \( w_\alpha(p) = p \). From \(4.3.1\) and \(4.3.2\) it follows that \( z(p) \) and \( W_p \) determine each other. Indeed, if we know the group \( W_p \), then we know the set of \( w_\alpha \) contained in it, and hence we know all \( \alpha \in \Pi \) such that \( \alpha(p) = 0 \); this in turn determines \( z(p) \). (Also note that if \( \alpha(p) = 0 \), then \( \beta(p) = 0 \) for all \( \beta \in \Pi(\alpha) \).) The argument for the converse is similar. By \[44\] Proposition 14 the stabilizer in \( W \) of an element \( p \in c^c \) is generated by complex reflections. Hence \( W_p \) coincides with the stabilizer of \( p \) in \( W \), that is

\[
W_p = \{ w \in W \mid w \cdot p = p \}.
\]
Define
\[ C^c_p = \{ h \in C^c \mid w \cdot h = h \text{ for all } w \in W_p \} \]
\[ C^c_p \cdot v = \{ q \in C^c_p \mid W_q = W_p \} \].

Then \( C^c_p \cdot v \) is a Zariski-open subset of \( C^c_p \) because it is defined by the inequalities \( w \cdot q \neq q \) for all \( w \in W \setminus W_p \). It is straightforward to check that for \( p, q \in C^c_p \) and \( v \in W \) we have
\[
(4.3.3) \quad C^c_q = v \cdot C^c_p \text{ if and only if } W_q = vW_p v^{-1}.
\]

This yields the following. Let \( R = \{ w_\alpha \in W \mid \alpha \in \Pi \} \). Then \( R \) is a single conjugacy class in \( W \); this is stated in [46, Section 3.3, page 84 of the English version; we have also checked it by computer. Note that for \( \beta \in \Pi(\alpha) \) we have \( w_\beta = w_\alpha \), so \( R \) consists of 40 elements. Let \( w \) be a complex reflection in \( W \), then it is known that either \( w \in R \) or \( w^{-1} \in R \); see [28, Table D.2]. A subgroup of \( W \) is said to be a reflection subgroup if it is generated by complex reflections. Since the set \( R \) along with the inverses of the elements of \( R \) exhaust all complex reflections in \( W \), it follows that the reflection subgroups of \( W \) are the subgroups generated by elements of \( R \). Because a conjugate of a complex reflection is a complex reflection we have that a conjugate of a reflection subgroup is a reflection subgroup as well. It can be shown that each reflection subgroup arises as \( W_p \) for an element \( p \in C^c_p \); see [46, Section 3.4]. So there are \( q_1, \ldots, q_m \in C^c_p \) such that the \( W_{q_i} \) are representatives of the conjugacy classes of reflection subgroups. Then each \( q \in C^c_p \) is \( W \)-conjugate to an element of precisely one of the \( C^c_{q_i} \). It turns out that \( m = 7 \), so there are seven “canonical sets” of semisimple elements \( F_k = C^c_{q_k} \). Each semisimple element of \( g_k^c \) is \( SL(9, C) \)-conjugate to an element of precisely one canonical set. In [46] explicit descriptions of the sets \( F_k \) are given; see also Section 2.2 above.

Now define
\[ N(W_p) = \{ v \in W \mid vW_p v^{-1} = W_p \} \].

By (4.3.3) it is clear that \( N(W_p) = \{ v \in W \mid v \cdot C^c_p = C^c_p \} \).

**Lemma 4.3.4.** Let \( p_1, p_2 \in C^c_p \cdot v \) and let \( w \in W \) be such that \( w \cdot p_1 = p_2 \). Then \( w \in N(W_p) \).

**Proof.** From \( wp_1 = p_2 \) it follows that \( W_{p_2} = wW_{p_1} w^{-1} \). Since \( p_i \in C^c_p \cdot v \), we have \( W_{p_2} = W_{p_1} = W_{p_2} \). Hence \( w \in N(W_p) \).

Define \( \Gamma_p = N(W_p)/W_p \). Then \( \Gamma_p \) acts naturally on \( C^c_p \). Let \( p_1, p_2 \in C^c_p \cdot v \). Then \( p_1, p_2 \) are \( SL(9, C) \)-conjugate if and only if they are \( W \)-conjugate, if and only if they are \( N(W_p) \)-conjugate (by Lemma 4.3.4), if and only if they are \( \Gamma_p \)-conjugate. For \( 1 \leq k \leq 7 \) set \( \Gamma_k = \Gamma_{q_k} \). In [46] in each of the seven cases, the group \( \Gamma_k \) is determined.

5. **Classification of the orbits of \( SL(9, R) \) on \( \wedge^3 R^9 \)**

In this section we describe the methods that we used to classify the orbits of \( SL(9, R) \) on \( \wedge^3 R^9 \). We use the setup of Section 4. Throughout we write \( \tilde{G}_0 = \tilde{G}_0(C) = SL(9, C) \) and \( G_0(R) = SL(9, R) \). For \( k = C, R \) we identify the vector spaces \( g_k^c \) and \( \wedge^3 k^9 \) on which the group \( G_0(K) \) acts. By **complex orbits** we mean the \( \tilde{G}_0(C) \)-orbits in \( g_k^c = \wedge^3 C^9 \), and by **real orbits** we mean the \( G_0(R) \)-orbits in \( g_k^R = \wedge^3 R^9 \). Then any real orbit is contained in a complex orbit, and any complex orbit contains finitely many real orbits. As seen in Section 4, the orbits are divided into three groups: nilpotent, semisimple and mixed. For each we have a subsection.
5.1. The nilpotent orbits. As seen in Section 4.2 the nilpotent orbits over $\mathbb{k}$ are in bijection with the orbits of homogeneous $\mathfrak{sl}_2$-triples over $\mathbb{k}$, for $\mathbb{k} = \mathbb{C}, \mathbb{R}$. The complex nilpotent orbits are listed in [46]. As noted in Section 4.2, any complex nilpotent orbit has a real representative $e$. By Proposition 4.2.1(i), there exists a real homogeneous $\mathfrak{sl}_2$-triple $t = (h, e, f) \in T^r$ containing $e$. By Theorem 4.2.3 the $\tilde{G}_0(\mathbb{R})$-orbits contained in $\tilde{G}_0(\mathbb{C}) \cdot e$ are in a canonical bijection with the elements of the Galois cohomology set $H^1 Z_{\tilde{G}_0}(t)$. So the classification of these orbits involves the following steps:

1. Find $t = (h, e, f) \in T^r$ containing $e$.
2. Determine $Z_{\tilde{G}_0}(t)$.
3. Determine $H^1 Z_{\tilde{G}_0}(t)$.
4. For each $[a] \in H^1 Z_{\tilde{G}_0}(t)$, where $a \in Z^1 Z_{\tilde{G}_0}(t)$, find an element $g \in \tilde{G}_0$ with $g^{-1}g = a$. Then $g \cdot e$ is a representative of the $\tilde{G}_0(\mathbb{R})$-orbit corresponding to $[a]$.

In Step 1 we first find an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ that contains $e$. This is done by solving linear equations, according to an algorithm that closely follows the proof of the existence of such an $\mathfrak{sl}_2$-triple; we refer to [12] Section 2.13 for the details. If the constructed $\mathfrak{sl}_2$-triple is not homogeneous, then we find a homogeneous $\mathfrak{sl}_2$-triple containing $e$ following the steps outlined in the proof of [12] Lemma 8.3.5.

In Step 2 we need to find a description of $Z_{\tilde{G}_0}(t)$ that is as detailed as possible in order to be able to execute Step 3. It is straightforward to obtain polynomials in 81 indeterminates whose zero locus is $Z_{\tilde{G}_0}(t)$. By themselves they do not give a useful description of the group. However, by computing a Gröbner basis (see [11]) of the ideal that these polynomials generate, we are often able to find a set of polynomials defining the same group and from which it is straightforward to read off the group structure. There are also quite a few cases for which it is computationally too hard to compute a Gröbner basis, or for which the Gröbner basis does not yield the desired information. For those cases we have developed ad hoc computational methods. We refer to [5] for details.

Step 3 is essentially carried out by hand, considering each case individually. The paper [6] contains complete descriptions of the centralizers $Z_{\tilde{G}_0}(t)$ as well as detailed computations of the sets $H^1 Z_{\tilde{G}_0}(t)$.

For Step 4 we use a computational method. The equation $g^{-1}g = a$ is the same as $\tilde{g} = ag$. The latter is equivalent to a set of linear equations over $\mathbb{R}$ for the coefficients of $g$. We solve these equations and in the solution space we look for an element lying in $\tilde{G}_0$.

Example 5.1.1. Consider the nilpotent orbit with representative $e = e_{136} + e_{147} - e_{245} + e_{379} + e_{569} + e_{678}$ (this is the orbit with number 47 in Table 1). We compute a homogeneous $\mathfrak{sl}_2$-triple $t = (h, e, f)$ containing $e$; let $Z_0 = Z_{\tilde{G}_0}(t)$ denote its stabilizer. Computer calculations show that the identity component $Z_0^0$ consists of

$$X(a, b) = \text{diag}(a^{-1}b^{-1}, a^{-2}, a, a^2b^2, b^{-2}, b, a^{-1}b^{-1}, a, b), \text{ for } a, b \in \mathbb{C}^\times.$$ 

The component group $C$ is of order 2 and generated by the image $c_1$ of

$$g_0 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$ 

We have

$$g_0^2 = X(1, -1), \quad g_0 X(a, b) g_0^{-1} = X(a, a^{-1}b^{-1}).$$
Set \( g_1 = g_0 \cdot X(-1,1); \) then \( g_1^2 = 1. \) In particular, \( g_1 \) is a cocycle with image \( c_1 \) in \( C. \) We have a short exact sequence

\[ 1 \to Z_0^c \to Z_0 \xrightarrow{j} C \to 1. \]

By [39, Section I.5.5, Proposition 38] this yields an exact sequence

\[ H^1 Z_0^c \to H^1 Z_0 \xrightarrow{j} H^1 C. \]

Since \( C \) is a group of order 2, we have \( H^1 C = \{ [1], [c_1] \}. \) So

\[ H^1 Z_0 = j_*^{-1}([1]) \cup j_*^{-1}([c_1]). \]

We have \( H^1 Z_0^c = 1, \) hence \( j_*^{-1}([1]) = \{ [1] \}. \) By twisting the above exact sequence by the cocycle \( g_1 \) it can be shown that \( j_*^{-1}([c_1]) = \{ [g_1] \}. \) We refer to [6, Section 3.1] for a description of this technique. Furthermore, [6, Section 6] has the details of the proof in this case. It follows that \( H^1 Z_0 = \{ [1], [g_1] \}. \)

By computer we compute a basis of the real vector space consisting of all \( 9 \times 9 \) complex matrices \( u \) with \( \pi = g_1 u. \) In this space we select 9 elements that have a matrix of maximal rank in their span, and that commute with the image of \( h \) in \( sl(9, C). \) By some random tries we find an element of determinant 1 in the space spanned by these 9 elements. It is

\[ u_0 = \begin{pmatrix}
  \begin{array}{cccccccc}
  i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & -2i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & -\frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & -\frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
  \end{array}\end{pmatrix}.\]

We have

\[ u_0 \cdot e = -e_{136} - e_{147} + e_{157} - e_{235} - e_{379} - e_{469} - e_{569} - e_{678}. \]

We conclude that the \( SL(9, C) \)-orbit of \( e \) contains two \( SL(9, R) \)-orbits with representatives \( e \) and \( u_0 \cdot e. \)

5.2. The semisimple orbits. In this section we consider the problem to classify the semisimple \( \tilde{G}_0(R) \)-orbits in \( g_1. \) We use the notation and results described in Section 4.3.

Let \( \mathcal{F}^c = \mathcal{F}^c_\mathcal{C} \) be one of the canonical subsets of \( \mathcal{C}^c. \) Then there is a trivector \( r \in \mathcal{C}^c \) with \( \mathcal{F} = \mathcal{C}^c_\mathcal{C}. \) Let \( p \in \mathcal{F}^c \) and let \( O = \tilde{G}_0(C) \cdot p \) denote the \( \tilde{G}_0(C) \)-orbit of \( p. \) We wish to know whether \( O \) contains a real point and whether \( O \cap \mathcal{F}^c \) contains a real point. If \( O \) has a real point, we wish to classify the real orbits in \( O. \)

For \( q \in \mathcal{C} \) define

\[ Z_{\tilde{G}_0}(q) = \{ g \in \tilde{G}_0(C) | g \cdot q = q \}. \]

Lemma 5.2.1. Let \( p, q \in \mathcal{F} \) then \( Z_{\tilde{G}_0}(q) = Z_{\tilde{G}_0}(p). \)

Proof. Since \( p, q \in \mathcal{F} \) we have \( W_p = W_q \) (notation as in Section 4.3). As seen in Section 4.3 the centralizer \( Z_{\mathcal{C}}(p) \) can be determined from \( W_p. \) Hence \( Z_{\mathcal{C}}(p) = Z_{\mathcal{C}}(q). \) We recall the following general fact: let \( H^c \) be a reductive algebraic group over \( C \) with Lie algebra \( \mathfrak{h}^c, \) and let \( s \in \mathfrak{h}^c \) be a semisimple element; then the algebraic subgroup \( \{ h \in H^c | \text{Ad}(h)(s) = s \} \subset H \) is connected; see Steinberg [11, Theorem 3.14]. It follows that \( p \) and \( q \) have the same stabilizer in \( G, \) and hence they have the same stabilizer in \( G_0. \) The inverse images of these stabilizers in \( \tilde{G}_0 \) are \( Z_{\tilde{G}_0}(q), Z_{\tilde{G}_0}(p), \) which are therefore equal as well. \( \square \)
Define

\[ Z_{G_0}(F^c) = \{ g \in \widetilde{G}_0 \mid g \cdot q = q \text{ for all } q \in F^c \}, \]

\[ N_{G_0}(F^c) = \{ g \in \widetilde{G}_0 \mid g \cdot q \in F^c \text{ for all } q \in F^c \}. \]

**Lemma 5.2.2.** Let \( p_1, p_2 \in F^c \) and let \( g \in \widetilde{G}_0 \) be such that \( g \cdot p_1 = p_2 \). Then \( g \in N_{G_0}(F^c) \).

*Proof.* Let \( q \in F \). Since \( p_1, p_2 \in \mathcal{E}^c \) are \( \widetilde{G}_0 \)-conjugate they are \( W \)-conjugate ([14], Theorem 2)). So by Lemma 4.3.1 there is \( w \in N(W_r) \) such that \( w \cdot p_1 = p_2 \), where \( r \in \mathcal{E}^c \) was introduced in the beginning of this subsection. Let \( \hat{w} \in N_{\widetilde{G}_0}(F^c) \) be a preimage of \( w \). Then \( g^{-1} \hat{w} \in Z_{\widetilde{G}_0}(p_1) \). Hence by Lemma 5.2.1 we see that \( g^{-1} \hat{w} \in Z_{\widetilde{G}_0}(q) \), from which it follows that \( g \cdot q = \hat{w} \cdot q \in F^c \). \( \square \)

From Section 4.3 we recall that \( \Gamma_r = N(W_r)/W_r \). We define a map \( \varphi : N_{\widetilde{G}_0}(F^c) \to \Gamma_r \). Let \( g \in N_{\widetilde{G}_0}(F^c) \). Then \( g \cdot r \in F^c \), hence there is \( w \in N(W_r) \) such that \( w \cdot r = g \cdot r \). We set \( \varphi(g) = wW_r \). Note that \( \varphi \) is well defined: if \( w' \in N(W_r) \) also satisfies \( w' \cdot r = g \cdot r \), then \( w^{-1}w' \in W_r \), so that \( w'W_r = wW_r \).

**Lemma 5.2.3.** The map \( \varphi : N_{\widetilde{G}_0}(F^c) \to \Gamma_r \) is a surjective group homomorphism with kernel \( Z_{\widetilde{G}_0}(F^c) \). Furthermore, for \( g \in N_{\widetilde{G}_0}(F^c) \) and \( q \in F^c \) we have \( g \cdot q = \varphi(g) \cdot q \).

*Proof.* First we claim the following: let \( g \in N_{\widetilde{G}_0}(F^c) \) and let \( w \in N(W_r) \) be such that \( w \cdot r = g \cdot r \); then \( g \cdot q = w \cdot q \) for all \( q \in F^c \). Indeed, let \( \hat{w} \in N_{\widetilde{G}_0}(F^c) \) be a preimage of \( w \). Then \( \hat{w}^{-1}g \in Z_{\widetilde{G}_0}(r) \). So by Lemma 5.2.1 it follows that \( \hat{w}^{-1}g \in Z_{\widetilde{G}_0}(q) \), or \( g \cdot q = \hat{w} \cdot q = w \cdot q \).

Our claim immediately implies the last statement of the lemma.

Now let \( g_1, g_2 \in N_{\widetilde{G}_0}(F^c) \) and let \( w_1, w_2 \in N(W_r) \) be such that \( w_i \cdot r = g_i \cdot r \), \( i = 1, 2 \). Then by our claim we see that \( g_1g_2 \cdot r = w_1w_2 \cdot r \) implying that \( \varphi \) is a group homomorphism.

Let \( \hat{w} \in N_{\widetilde{G}_0}(F^c) \) be a preimage of \( w \in N(W_r) \). Then \( \hat{w} \in N_{\widetilde{G}_0}(F^c) \) and \( \varphi(\hat{w}) = wW_r \), so that \( \varphi \) is surjective. If \( g \in \ker \varphi \), then \( g \cdot r = r \), and by our claim we see that \( g \in Z_{\widetilde{G}_0}(F^c) \). \( \square \)

It follows that \( \varphi \) induces an isomorphism, which we also denote by \( \varphi \), between \( A = N_{\widetilde{G}_0}(F^c)/Z_{\widetilde{G}_0}(F^c) \) and \( \Gamma_r \).

**Proposition 5.2.4.** As before, let \( \mathcal{O} = \widetilde{G}_0 \cdot p \). Write \( \mathcal{N} = N_{\widetilde{G}_0}(F^c) \), \( \mathcal{Z} = Z_{\widetilde{G}_0}(F^c) \).

(i) \( \mathcal{O} \) has an \( R \)-point if and only if \( \overline{p} = n^{-1} \cdot p \) for some \( n \in Z_{\mathcal{N}, A} \).

(ii) Assume that \( \mathcal{O} \) has an \( R \)-point and let \( n \) be as in (i). Write \( a = nZ \in Z_{\mathcal{N}, \mathcal{A}} \) and \( \xi = [a] \in H_{\mathcal{N}, \mathcal{A}} \). Then \( \xi \) only depends on \( \mathcal{O} \) and not on the choices of \( p \in \mathcal{O} \cap F^c \) and \( n \in Z_{\mathcal{N}, \mathcal{N}} \).

(iii) With the hypothesis and notation of (ii), the orbit \( \mathcal{O} \) has a real point in \( F^c \) if and only if \( \xi = 1 \).

*Proof.* (i) Assume that \( \mathcal{O} \) has an \( R \)-point \( p_R = g \cdot p \). Then \( \overline{g \cdot p} = g \cdot p \), whence

\[ \overline{p} = \overline{g^{-1}} \cdot g \cdot p = (g^{-1} \overline{g})^{-1} \cdot p. \]

Write

\[ (5.2.5) \quad n = g^{-1} \overline{g}. \]

Since \( p, \overline{p} \in F^c \), we see that \( n \in \mathcal{N} \) by Lemma 5.2.2. It follows from (5.2.5) that \( n \in Z_{\mathcal{N}, \mathcal{N}} \).

Conversely, assume that \( \overline{p} = n^{-1} \cdot p \) where \( n \in Z_{\mathcal{N}, \mathcal{N}} \). Since \( H_{\mathcal{N}, \mathcal{G}_0} = \{1\} \), there exists \( g \in \widetilde{G}_0 \) such that \( n = g^{-1} \overline{g} \). Set

\[ p_R = g \cdot p \in \mathcal{O}. \]
Then
\[ \overline{pr} = \overline{g} \cdot \overline{p} = gn \cdot n^{-1} \cdot p = g \cdot p = \overline{pr}. \]
Thus \( pr \) is an \( R \)-point of \( O \), which proves (i).

(ii) Assume that \( O \) has an \( R \)-point and let \( n \) be as in (i). We show that \( \xi \) depends only on \( O \). First suppose that \( \overline{p} = \hat{n}^{-1} \cdot p \) for some \( \hat{n} \in N \). Then \( \hat{n} n^{-1} \in Z_{\hat{G}_0}(p) \), which is equal to \( Z \) by Lemma 5.2.1. Hence \( \xi \) does not depend on the choice of \( n \).

We show that \( \xi \) does not depend on the choice of \( p \). Indeed, if \( p' \in O \cap \mathcal{F}^c \), then \( p' = a' \cdot p \) for some \( a' \in A \), and we have
\[ \overline{p'} = \overline{a'} \cdot \overline{p} = \overline{a'} \cdot a^{-1} \cdot p = \overline{a'} \cdot \overline{a^{-1}(a')^{-1} \cdot a'} \cdot p = \overline{(a'a^{-1})^{-1} \cdot p'}. \]
We obtain the 1-cocycle
\[ a'a^{-1} \sim a. \]
Thus \( \xi = [a] \) does not depend on the choice of \( p \).

(iii) Now assume that \( \xi = 1 \). We have
\[ \overline{p} = a^{-1} \cdot p \quad \text{and} \quad a = (a')^{-1} \overline{a} \quad \text{for some} \quad a' \in A. \]
Set \( p_R = a' \cdot p \in O \cap \mathcal{F}^c \). Then
\[ \overline{p_R} = \overline{a'} \cdot \overline{p} = \overline{a'} \cdot a^{-1} \cdot p = \overline{a'} \cdot \overline{a^{-1}} \cdot a' \cdot p = a' \cdot p = p_R. \]
Thus \( p_R \) is real.

Conversely, if \( O \cap \mathcal{F}^c \) contains a real point \( p_R \), then clearly \( \xi = 1 \), which proves (iii). \( \square \)

Assume that \( O \) contains a real point \( p_R = g \cdot p \). We wish to classify real orbits in \( O \). We write \( q \) for \( p_R \). Write \( C_q = Z_{\hat{G}_0}(q) \) and set
\[ C_q = (C_q, \sigma_q), \quad \text{where} \quad \sigma_q(c) = \overline{c}. \]
Similarly we write \( C_p = Z_{\hat{G}_0}(p) \). Since \( q = g \cdot p \), we have an isomorphism
\[ t_q : C_p \sim \rightarrow C_q, \quad c \mapsto gcg^{-1}. \]
We transfer the real structure \( \sigma_q \) on \( C_q \) to \( C_p \) using \( t_q \). We obtain a real structure \( \sigma_p \) on \( C_p \):
\[ \sigma_p : C_p \xrightarrow{t_q} C_q \xrightarrow{\sigma_q} C_q \xrightarrow{t_q^{-1}} C_p, \quad c \mapsto gcg^{-1} \mapsto \overline{gcg^{-1}} \mapsto g^{-1}gcg^{-1}c. \]
Let \( n = g^{-1}q (\text{see also the proof of Proposition} \ 5.2.4) \), then \( \sigma_p(c) = n \overline{c}n^{-1} \) for \( c \in C_p \). We obtain a real algebraic group
\[ C_p = (C_p, \sigma_p), \quad \text{where} \quad \sigma_p(c) = n \overline{c}n^{-1} \quad \text{for} \quad c \in C_p, \]
and an isomorphism
\[ t_q : C_p \sim \rightarrow C_q, \quad c \mapsto gcg^{-1} \]
inducing a bijection on cohomology
\[ H^1 C_p \sim \rightarrow H^1 C_q. \]

By Proposition 5.3.5, the real orbits in \( O \) are classified by \( H^1 C_q \), and hence by \( H^1 C_p \).

The map is as follows. To \( c \in Z^1 C_p \) we associate \( gcg^{-1} \in Z^1 C_q \). We find \( g_1 \in \hat{G}_0 \) such that \( g_1^{-1}q = gcg^{-1} \) and set \( r = g_1 \cdot q = g_1 g \cdot p \). To \([c]\) we associate the real orbit \( \overline{G}_0(\mathbb{R}) \cdot r \subseteq O \).

In order to check our calculations, we show that \( r = g_1 g \cdot q \) is real. We calculate:
\[ r = g_1 g \cdot q = g_1 g \cdot \overline{q} = g_1 g \cdot g \cdot p = g_1 g \cdot p = r, \]
because \( c \in C_p = Z_{\hat{G}_0}(p) \). Thus \( r \) is real.
This leads to the following procedure to list the real semisimple orbits having a representative that is \( \bar{G}_0 \)-conjugate to an element of \( F^c \). First we compute \( H^1A \), and then for every \([a] \in H^1A\) we do the following:

1. Find all \( p \in F^c \) such that \( \bar{p} = a^{-1} \cdot p \).
2. Lift \( a \) to a cocycle \( n \in Z_1N_{\bar{G}_0}(F^c) \) and compute an element \( g \in \bar{G}_0 \) with \( g^{-1} \bar{g} = n \).
3. Set \( C_p = Z_{\bar{G}_0}(p) \) and define \( \sigma_p: C_p \to C_p \) by \( \sigma_p(c) = ncn^{-1} \) and set \( C_p = (C_p, \sigma_p) \).
4. Compute \( H^1C_p \) and for each \([c] \in H^1C_p\) find an element \( g_1 \in \bar{G}_0 \) with \( g_1^{-1} \bar{g}_1 = gcg^{-1} \).

Then \( g_1g \cdot p \) is a representative of the semisimple \( \bar{G}_0(\mathbb{R}) \)-orbit corresponding to \([c]\).

On this procedure we remark the following. We have that \( \varphi: A \to \Gamma_r \) is a \( \Gamma \)-equivariant isomorphism. So it induces a bijection between \( H^1A \) and \( H^1\Gamma_r \). The latter can be computed by brute force because \( \Gamma_r \) is a known finite group. In the cases that are relevant to our classification, the lifting in Step 2 always turned out to be possible, but we cannot prove this a priori.

The resulting classification of the semisimple orbits is described in Section 2.2. For the details of the computations we refer to [6]. Using some of the results of our computations we can also prove the following result.

**Theorem 5.2.6.** All Cartan subspaces in \( g_1 \) are conjugate under \( SL(9, \mathbb{R}) \).

**Proof.** The short exact sequence

\[
1 \to Z_{\bar{G}_0}(\mathbb{C}) \to N_{\bar{G}_0}(\mathbb{C}) \to W \to 1,
\]

gives rise to a cohomology exact sequence

\[
H^1Z_{\bar{G}_0}(\mathbb{C}) \to H^1N_{\bar{G}_0}(\mathbb{C}) \to H^1W.
\]

By brute force (computer) computations it is easily established that \( H^1W = 1 \). We have explicitly determined \( Z_{\bar{G}_0}(\mathbb{C}) \), which is a group of order \( 3^5 \). By [6 Corollary 3.2.5] we have \( H^1A = 1 \) for any finite group \( A \) of order \( p^m \), where \( p \) is an odd prime. It follows that \( H^1Z_{\bar{G}_0}(\mathbb{C}) = 1 \) and hence \( H^1N_{\bar{G}_0}(\mathbb{C}) = 1 \). By [6 Theorem 4.4.9], the conjugacy classes of Cartan subspaces in \( g_1 \) are in bijection with \( \text{ker} [H^1N_{\bar{G}_0}(\mathbb{C}) \to H^1\bar{G}_0] \), and the theorem follows. \( \square \)

**Example 5.2.7.** Let \( F^c = F_3^c \) be the third canonical set consisting of

\[
p_{x_1,x_2}^1 = \lambda_1p_1 + \lambda_2p_2
\]

where \( p_1, p_2 \) are given in Section 2.2 and the \( \lambda_i \in \mathbb{C} \) are such that \( \lambda_1\lambda_2(\lambda_1^6 - \lambda_2^6) \neq 0 \). The group \( \Gamma_3 \) is of order 72 and generated by

\[
\left( \begin{array}{cc} 0 & -1 \\ -1 & 0 \end{array} \right), \quad \left( \begin{array}{cc} 1 & 0 \\ 0 & \zeta \end{array} \right),
\]

where \( \zeta \) is a primitive third root of unity. Here the elements of \( \Gamma_3 \) are given as linear transformations of the space spanned by \( p_1, p_2 \).

A small brute force computation shows that \( H^1\Gamma_3 = \{[1], [-1], [u_1], [u_2]\} \), where

\[
u_1 = \left( \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right), \quad \nu_2 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right).
\]

So the elements of \( F^c \) whose orbits have real points are divided into four classes consisting of \( q \in F^c \) with \( \bar{q} = q, \bar{q} = -q, \bar{q} = u_1^{-1} \cdot q, \) and \( \bar{q} = u_2^{-1} \cdot q \), respectively.

We consider the fourth class consisting of \( q = \lambda_1p_1 + \lambda_2p_2 \) with \( \lambda_1 = x + iy, \lambda_2 = x - iy, \) \( x, y \in \mathbb{R} \). The polynomial conditions on \( \lambda_1, \lambda_2 \) translate to \( xy(x^2 - 3y^2)(x^2 - \frac{1}{3}y^2) \neq 0 \). Let
\[ n_3 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \]

Then \( n_3 \) is a cocycle (in fact, \( n_3^2 = 1 \)) in \( \mathcal{N}_{\tilde{G}_0} (F^c) \) projecting to \( u_2 \). Furthermore, \( g_3^{-1}g_3 = n_3 \).

So a real representative of \( O_{x,y} \) is

\[ p_{x,y} = g_3 \cdot q = x(e_{147} - 2e_{169} - e_{245} + e_{289} - e_{356} - \frac{1}{2}e_{378}) + y(e_{124} + e_{136} + \frac{1}{2}e_{238} - e_{457} + 2e_{569} - e_{789}), \]

where \( x, y \in \mathbb{R} \) satisfy the polynomial condition written above.

In [6] the stabilizer \( C_q \) of \( q \) in \( SL(9, \mathbb{C}) \) is explicitly determined. We have \( C_q = T_4 \rtimes H \), where \( H \) is a group of order 9 and \( T_4 \) a 4-dimensional torus consisting of

\[ T_4(t_1, t_2, t_3, t_4) = \text{diag}(t_1, t_2, (t_1t_2)^{-1}, t_3, (t_3t_4)^{-1}, (t_1t_3)^{-1}, (t_2t_4)^{-1}, t_1t_2t_3t_4), \]

for \( t_1, t_2, t_3, t_4 \in \mathbb{C}^* \). For \( g \in C_q \) we set \( \sigma_g(q) = n_3 g n_3^{-1} \). Then

\[ \sigma_g(T_4(t_1, t_2, t_3, t_4)) = T_2(\tilde{t}_1, \tilde{t}_1^{-1}\tilde{t}_3^{-1}, \tilde{t}_1^{-1}\tilde{t}_2^{-1}, \tilde{t}_1\tilde{t}_2\tilde{t}_3\tilde{t}_4). \]

Using this formula it can be shown (see [6] for the details) that \( H^1(T_4, \sigma_q) = 1 \). Since the component group \( H \) is of order 9, this implies that \( H^1(C_q, \sigma_q) = 1 \) as well ([6] Proposition 3.3.16]). It follows that \( O_{x,y} \) contains one \( SL(9, \mathbb{R}) \)-orbit with representative \( p_{x,y} \).

5.3. The mixed orbits. For \( k = \mathbb{C}, \mathbb{R} \), let \( p + e \) in \( \mathfrak{g}_k^c \) be a mixed element, that is, \( p \) is semisimple, \( e \) is nilpotent, both are nonzero and \([p, e] = 0 \). After acting with \( \tilde{G}_0(\mathbb{R}) \), we may assume that \( p \) is a representative that we obtained when classifying the semisimple orbits. If we fix such a semisimple element \( p \), then the classification of the mixed elements with semisimple part equal to \( p \) amounts to the classification of the nilpotent elements in the intersection of the centralizer of \( p \) and \( \mathfrak{g}_k^c \), up to the action of the stabilizer of \( p \) in \( \tilde{G}_0(\mathbb{R}) \).

First we briefly comment on the classification of the elements of mixed type in \( \mathfrak{g}_k^c \). This classification was carried out in [46]. From Section 4.3 we recall that the Cartan subspace contains seven canonical sets \( \mathcal{F}^c_k \), \( 1 \leq k \leq 7 \) such that every semisimple element of \( \mathfrak{g}_k^c \) is \( \tilde{G}_0 \)-conjugate to an element in precisely one of the \( \mathcal{F}^c_k \). For the classification of the elements of mixed type, the canonical sets \( \mathcal{F}^c_1 \) and \( \mathcal{F}^c_2 \) are not relevant, because the elements of \( \mathcal{F}^c_1 \) do not centralize non-unit nilpotent elements and because \( \mathcal{F}^c_2 = \{0\} \).

Now fix a semisimple element \( p \in \mathcal{F}^c_k \) and write \( a = i_\mathfrak{sl}^c(p) \). Then \( a \) inherits the grading from \( \mathfrak{g}_k^c \). The nilpotent elements \( e \) such that \( p + e \) is of mixed type lie in \( a_1 \). Furthermore, \( p + e_1, p + e_2 \) are \( \tilde{G}_0 \)-conjugate if and only if \( e_1, e_2 \) are \( Z_{\tilde{G}_0}(p) \)-conjugate. The Lie algebra of \( Z_{\tilde{G}_0}(p) \) is \( a_0 \). Hence by using the method of [45] (see also [12], Chapter 8) we can determine the \( Z_{\tilde{G}_0}(p) \)-orbits of nilpotent elements in \( a_1 \). Finally we reduce the list by considering conjugacy under the elements of the component group of \( Z_{\tilde{G}_0}(p) \). The paper [46] contains lists of representatives of nilpotent parts of elements of mixed type, for \( p \in \mathcal{F}^c_k \), \( 2 \leq k \leq 6 \). Let \( \mathcal{T}^c_p \) be the set of homogeneous \( \mathfrak{sl}_2 \)-triples in \( a_1 \). Note that by Proposition 4.2.1 the nilpotent \( Z_{\tilde{G}_0}(p) \)-orbits in \( a_1 \) correspond bijectively to the \( Z_{\tilde{G}_0}(p) \)-orbits in \( \mathcal{T}^c_p \). Note also that \( Z_{\tilde{G}_0}(p) \) and \( i_\mathfrak{sl}^c(p) \) only depend on \( \mathcal{F}^c_k \), not on the particular element \( p \) (Section 4.3 and Lemma 5.2.1). Hence the orbits of the nilpotent parts also only depend on \( \mathcal{F}^c_k \).

Now we turn to the problem of classifying \( \tilde{G}_0(\mathbb{R}) \)-orbits of mixed type. As in Section 2.2 we denote the set of real points of \( \mathcal{F}^c_k \) by \( \mathcal{F}_k \). We divide the real semisimple elements into two groups: those that are \( \tilde{G}_0(\mathbb{R}) \)-conjugate to an element of some \( \mathcal{F}_k \) (these are called canonical
semisimple elements) and those that are not (which are called noncanonical semisimple elements).

Let \( p \) be a representative of a semisimple \( \tilde{G}_0(\mathbb{R}) \)-orbit and again let \( a = \mathfrak{g}^{\mathbb{C}}(p) \). Similarly to the complex case, we need to determine the \( Z_{\tilde{G}_0(\mathbb{R})}(p) \)-orbits of real nilpotent elements in \( a_1 \). By Proposition 4.2.4 these correspond bijectively to the \( Z_{\tilde{G}_0(\mathbb{R})}(p) \)-orbits in the set \( T_p \) of real homogeneous \( \mathfrak{sl}_2 \)-triples in \( a \). Let \( t = (h,e,f) \) be a real homogeneous \( \mathfrak{sl}_2 \)-triple in \( a \) containing \( e \). Let

\[
Z_0(p,t) = \{ g \in \tilde{G}_0(p) \mid g \cdot h = h, g \cdot e = e, g \cdot f = f \}.
\]

Then by Proposition 4.3.5 there is a bijection between the real \( Z_{\tilde{G}_0(\mathbb{R})}(p) \)-orbits contained in the complex \( Z_{\tilde{G}_0}(p) \)-orbit of \( t \) and \( \ker [H^1Z_0(p,t) \to H^1Z_{\tilde{G}_0}(p)] \). In [6] it has been established that \( H^1Z_{\tilde{G}_0}(p) = 1 \) in all cases. Hence the orbits we are interested in here correspond bijectively to \( H^1Z_0(p,t) \).

The procedure that we use to classify the mixed elements whose semisimple part is noncanonical is markedly more complex than the procedure for classifying those with canonical semisimple part. If the semisimple element \( p \) is canonical, then we consider a real nilpotent \( e \in a = \mathfrak{g}^{\mathbb{C}}(p) \) lying in the homogeneous real \( \mathfrak{sl}_2 \)-triple \( t = (h,e,f) \). In order to compute representatives of the real \( \tilde{G}_0(\mathbb{R}) \)-orbits contained in the \( \tilde{G}_0 \)-orbit of \( p + e \), we compute the centralizer \( Z_0(p,t) \) and its Galois cohomology \( H^1Z_0(p,t) \). The elements of the latter set correspond to the real orbits that we are looking for.

It is possible to take the same approach when \( p \) is not canonical. However, in that case the groups \( Z_0(p,t) \) tend to be difficult to describe (they can be nonsplit, for example) and therefore difficult to work with. Moreover, in these cases it can happen that the complex \( Z_{\tilde{G}_0}(p) \)-orbit of a nilpotent \( e \in a_1 \) has no real points.

In the next paragraphs we describe a different method for this case. The main idea is the following. We have that \( p \) is conjugate over \( C \) to a canonical semisimple element \( q \). So if \( t_1 \) is a homogeneous \( \mathfrak{sl}_2 \)-triple in \( \mathfrak{g}^{\mathbb{C}}(p) \) then \( t_1 \) is \( \tilde{G}_0 \)-conjugate to a homogeneous \( \mathfrak{sl}_2 \)-triple \( t \) in \( \mathfrak{g}^{\mathbb{C}}(q) \). Also the stabilizers \( Z_0(p,t_1) \) and \( Z_0(q,t) \) are conjugate. We define a conjugation on the latter so that these two groups are \( \Gamma \)-equivariantly isomorphic. By doing computations in \( \mathfrak{g}^{\mathbb{C}}(q) \), we decide whether the complex orbit \( Z_{\tilde{G}_0}(p) \cdot t_1 \) has a real point. If it does, we work with the preimage \( t' \) in \( \mathfrak{g}_1^{\mathbb{C}}(q) \) of such a real point and compute \( H^1Z_0(q,t') \) where we use the modified conjugation.

Let \( p \in \mathfrak{g}_1 \) be a real noncanonical semisimple element and of the form \( p = g \cdot q \), where \( g \in \tilde{G}_0 \) and \( q \in F_k^{\mathbb{C}} \) where \( 2 \leq k \leq 6 \). From our construction (Proposition 5.2.4) it follows that setting \( n = g^{-1}q \) entails \( n \in Z^1N_{\tilde{G}_0}(F_k^{\mathbb{C}}) \) and \( n^{-1} \cdot q = \overline{q} \). The cocycles \( n \) are explicitly given in [6]. In all cases it turns out that \( n \) is real, so that \( n^2 = 1 \). In the sequel these \( g \) and \( n \) are fixed.

Also define

\[
\varphi : \mathfrak{g}_1^{\mathbb{C}}(q) \cap \mathfrak{g}_1^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}(p) \cap \mathfrak{g}_1^{\mathbb{C}}, \quad x \mapsto g \cdot x.
\]

Because \( Z_{\tilde{G}_0}(p) = gZ_{\tilde{G}_0}(q)g^{-1} \), this is a bijection between the sets of nilpotent elements in the respective spaces mapping \( Z_{\tilde{G}_0}(q) \)-orbits to \( Z_{\tilde{G}_0}(p) \)-orbits. For \( x \in \mathfrak{g}_1^{\mathbb{C}}(q) \cap \mathfrak{g}_1^{\mathbb{C}} \), the point \( g \cdot x \) is real (that is, \( g \cdot x = g \cdot x \)) if and only if \( n \cdot x = x \). Since \( n \in N_{\tilde{G}_0}(F_k^{\mathbb{C}}) \), we have that \( q \) and \( n \cdot q \) both lie in \( F_k^{\mathbb{C}} \) and therefore have the same centralizer in \( g^{\mathbb{C}} \) (see Section 1.13); hence \( n \cdot \mathfrak{g}_1^{\mathbb{C}}(q) = \mathfrak{g}_1^{\mathbb{C}}(q) \). From \( n^{-1} \cdot q = \overline{q} \) it follows that \( \mathfrak{g}_1^{\mathbb{C}}(q) \) is stable under complex conjugation \( x \mapsto \overline{x} \). We set \( u = \mathfrak{g}_1^{\mathbb{C}}(q) \cap \mathfrak{g}_1^{\mathbb{C}} \) and define \( \mu : u \to u \) by \( \mu(x) = n \overline{x} \). So for \( x \in u \) we have that \( \varphi(x) \) is real if and only if \( \mu(x) = x \). Because \( n \) is a cocycle, we have \( \mu^2(x) = x \) for all \( x \in u \).
Now fix a nilpotent $e \in \mathfrak{sl}_2(q) \cap \mathfrak{g}_1^C = \mathfrak{u}$ lying in a homogenous $\mathfrak{sl}_2$-triple $t = (h, e, f)$. We let $Y = Z_{\tilde{G}_0}(q) \cdot e \subset \mathfrak{u}$ be its orbit. Then $\varphi(Y)$ is a $Z_{\tilde{G}_0}(p)$-orbit in $\mathfrak{z}_{\tilde{G}_0}(p) \cap \mathfrak{g}_1^C$ (and all nilpotent $Z_{\tilde{G}_0}(p)$-orbits in $\mathfrak{z}_{\tilde{G}_0}(p) \cap \mathfrak{g}_1^C$ are obtained in this way). We want to determine the real $Z_{\tilde{G}_0(R)}(p)$-orbits contained in $\varphi(Y)$.

Lemma 5.3.1. Let $y_0$ be any element of $Y$. We have $\mu(Y) = Y$ if and only if $\mu(y_0) \in Y$.

Proof. Only one direction needs a proof, so suppose that $n \cdot \overline{y_0} = g_1 \cdot y_0$ for some $g_1 \in Z_{\tilde{G}_0}(q)$.

Note that $\psi: Z_{\tilde{G}_0}(q) \to Z_{\tilde{G}_0}(p), \psi(h) = g h g^{-1}$, is an isomorphism. As $p$ is real, i.e., $\overline{p} = p$, we have that $Z_{\tilde{G}_0}(p)$ is closed under conjugation. So for $h \in Z_{\tilde{G}_0}(q)$ we see that $\psi^{-1}(\psi(h))$ lies in $Z_{\tilde{G}_0}(q)$. But the latter element is equal to $n\overline{h}n^{-1}$. We conclude that $Z_{\tilde{G}_0}(q)$ is closed under $h \mapsto n\overline{h}n^{-1}$.

Let $g_2 \in Z_{\tilde{G}_0}(q)$, then

$$
\mu(g_2 \cdot y_0) = n \cdot \overline{g_2 y_0} = n g_2^{-1} n \cdot y_0 = n \overline{g_2 n}^{-1} \cdot g_1 \cdot y_0 = g_3 \cdot y_0
$$

with $g_3 \in Z_{\tilde{G}_0}(q)$. We conclude that $\mu(Y) = Y$. \hfill $\square$

Now there are two possibilities. If $\mu(Y) \neq Y$, then $\varphi(Y)$ has no real points by the previous lemma. In this case there are no real mixed elements of the form $p + e_1$ conjugate to $q + e$. So we do not consider this $e$. Note that we can check whether $\varphi(Y) = Y$ using Lemma 5.3.1 along with computational methods for determining conjugacy of nilpotent elements that are outlined in [6].

On the other hand, if $\mu(Y) = Y$ then we consider the restriction of $\mu$ to $Y$. We set $Y = (Y, \mu)$. With the methods of Section 3.4, we establish whether $Y$ has a real point (that is, a point $y \in Y$ with $\mu(y) = y$) and, if so, we find one. If, on the other hand, $Y$ does not have a real point, then $\varphi(Y)$ has no real points either and also in this case we do not consider this $e$.

We briefly summarize the main steps of the method of Section 5.4 to find a real point in $\varphi(Y)$. We set $H = Z_{\tilde{G}_0}(q)$ and define the conjugation $\tau: H \to H, \tau(h) = n\overline{h}n^{-1}$. Set $H = (H, \tau)$ and we assume that $H^1H = 1$. Then we do the following

1. Compute $h_0 \in H$ such that $\mu(e) = h_0^{-1} e$.
2. Set $C = Z_{H}(e)$ and let $\nu: C \to C$ be the conjugation defined by $\nu(c) = h_0 \tau(c) h_0^{-1}$. Set $C = (C, \nu)$.
3. Set $d = h_0 \tau(h_0)$ and consider the class $[d] \in H^2 C$. If $[d] \neq 1$, then $Y$ has no real point and we stop.
4. Otherwise we can find an element $c \in C$ with $c \nu(c) d = 1$. Set $h_1 = c h_0$. (Then $\mu(e) = h_0^{-1} e$ and $h_1 \tau(h_1) = 1$.)
5. Because $H^1H = 1$, we can find $u \in H$ such that $u h_1 \tau(u)^{-1} = 1$. Then $y = u \cdot e$ is a real point of $Y$.

Now assume that $Y$ has a real point $e'$. Then we set $e_1 = g \cdot e'$ and find a real homogeneous $\mathfrak{sl}_2$-triple $t_1 = (h_1, e_1, f_1)$ in $\mathfrak{z}_{\tilde{G}_0}(p)$. Set $t' = (h', e', f')$ where $h' = g^{-1} \cdot h_1, f' = g^{-1} \cdot f_1$. Furthermore, we determine an element $g' \in Z_{\tilde{G}_0}(q)$ such that $g' \cdot t = t'$, then $gg' \cdot t = t_1$. We set $g_0 = gg'$ and $n_0 = g_0^{-1} \overline{g_0}$. Then $n_0 \in Z^* N_{\tilde{G}_0}(\mathfrak{f}_k^C)$. We compute $Z_q = Z_{\tilde{G}_0}(q, t)$ (the stabilizer of $t$ in $Z_{\tilde{G}_0}(q)$). We define $\sigma: Z_q \to Z_q$ by $\sigma(z) = n_0 \overline{n_0}^{-1}$ (in the same way as in the proof of Lemma 5.3.1 it is seen that $Z_q$ is closed under $\sigma$). We set $Z_q = (Z_q, \sigma)$. Also, we set $Z_q = Z_{\tilde{G}_0}(p, t_1)$ and $Z_0 = (Z_q, \overline{\sigma})$. Then $\psi: Z_q \to Z_q, \psi(z) = g_0 z g_0^{-1}$ is a $\Gamma$-equivariant isomorphism. Therefore the real orbits in $\varphi(Y)$ correspond bijectively to $H^1Z_q$. 
The most straightforward way to find representatives of the orbits is to use the group $Z^{-1}_{G_0}(q)$, the conjugation $\sigma$ and the elements of $H^1Z_q$. For a class $[c] \in H^1Z_q$ we find an element $a \in Z^{-1}_{G_0}(q)$ with $ac = \sigma(a) = n_0\tau n_0^{-1}$. Then $g_0a \cdot e$ is a nilpotent element in $3g_3^c(p) \cap g_1^c$ and $p + g_0a \cdot e$ is a representative of the orbit of mixed elements corresponding to the class $[c]$.

**Remark 5.3.2.** We have used an ad hoc method by which it is possible to show in many cases that $Y$ has a real point. The map $\mu: u \to u$ is an $R$-linear involution. Set $u_R = \{x \in u \mid \tau = x\}$. In all cases that we have considered we have $\tau = n$. Hence we have $\mu(u_R) = u_R$. We have $u_R = u_+ \oplus u_-$, where the latter are the eigenspaces of $\mu$ corresponding to the eigenvalues 1 and $-1$ respectively. We compute bases of these spaces and consider elements of three forms: $u \in u_+$, $iu$ for $u \in u_-$ and $u_1 + iv_2$ for $u_1 \in u_+$, $u_2 \in u_-$. All these elements are fixed by $\mu$. We list many of these elements that are nilpotent and check to which $Z^{-1}_{G_0}(q)$-orbit they belong. This way we have found real points in most cases.

**Example 5.3.3.** Here we consider the same situation as in Example 5.2.7. We wish to list the mixed orbits whose semisimple part is $p_{x,y}$. We have $p_{x,y} = g_0 \cdot q$ where $q \in F_3$. Furthermore $g_3^{-1}\tau_3 = n_3$. As above we set $u = 3g_3^c(q) \cap g_1^c$ and we define $\mu: u \to u$ by $\mu(x) = n_3\tau$.

We know the classification of the nilpotent $Z^{-1}_{G_0}(q)$-orbits in $u$. This classification is given in [40] Table 2 and also in Table [4] (this table contains the real classification, but here this happens to coincide with the complex classification). We consider the third nilpotent element $e = e_{159} + e_{168} + e_{267}$, which lies in a homogeneous $sl_2$-triple $t = (h, e, f)$ (which we do not explicitly describe). By the ad hoc method of the previous remark we found an element $e'$ in the $Z^{-1}_{G_0}(q)$-orbit of $e$ with $\mu(e') = e'$. It is $e' = -e_{159} + ie_{168} - e_{267}$.

Now we set $e_1 = g_3 \cdot e' = -2e_{267} - \frac{1}{2}e_{349} + \frac{1}{2}e_{468}$. We compute a homogeneous $sl_2$-triple $t_1 = (h_1, e_1, f_1)$ and set $h' = g_3^{-1}\cdot h_1$, $f' = g_3^{-1}\cdot f_1$ (here we do not give explicit expressions for these elements). We determine an element $g' \in Z^{-1}_{G_0}(q)$ such that $g' \cdot t = t'$ and set $g_0 = gg'$. Then $g_0 \cdot t = t_1$. Furthermore, we set $n_0 = g_0^{-1}\tau_0$, so that $n_0 \in Z^1N^-_{G_0}(F_3^c)$. We have

$$g_0 = \begin{pmatrix} 0 & 0 & i & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{i}{2} & 0 & 0 & 0 & 0 \\ -\frac{i}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \end{pmatrix}, \quad n_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \end{pmatrix}.$$

Let $Z_q = Z^{-1}_{G_0}(t, q)$. The identity component of this group is a 1-dimensional torus $T_1$ consisting of the elements

$$T_1(s) = \text{diag}(1, s, s^{-1}, s, s^{-1}, 1, s^{-1}, 1, s).$$

The component group is of order 9 (in [6] it is explicitly given). We define the conjugation $\sigma: Z_q \to Z_q$ by $\sigma(z) = n_0\tau n_0^{-1}$. A small calculation shows that $\sigma(T_1(s)) = T_1(\tau^{-1})$. This implies that $H^1(T_1, \sigma) = \{[1], [T_1(-1)]\}$ (see [6] Examples 3.1.7(3)). By [6] Proposition 3.3.16 it follows that $H^1(Z_q, \sigma) = \{[1], [T_1(-1)]\}$.

Next we set $c = T_1(-1)$ and find an element $a \in Z^{-1}_{G_0}(q)$ with $ac = n_0\tau n_0^{-1}$. It is

$$a = \text{diag}(-1, -i, -i, i, 1, -i, 1, i).$$

We have $g_0a \cdot e = 2e_{267} - \frac{1}{2}e_{349} - \frac{1}{2}e_{468}$. So we get two real mixed orbits with representatives $q_{x,y} + e_1$ and $q_{x,y} + g_0a \cdot e$. They are both $SL(9, C)$-conjugate to $p + e$. 

REFERENCES

[1] J. Adams and O. Taibî. Galois and Cartan cohomology of real groups. Duke Math. J., 167(6):1057–1097, 2018.
[2] A. Borel. Linear algebraic groups. In Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965), pages 3–19. Amer. Math. Soc., Providence, R.I., 1966.
[3] A. Borel. Linear algebraic groups, volume 126 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1991.
[4] M. Borovoi. Abelianization of the second nonabelian Galois cohomology. Duke Math. J., 72(1):217–239, 1993.
[5] M. Borovoi. Real points in a homogeneous space of a real algebraic group. [arXiv:2106.14871 [math.AG]]
[6] M. Borovoi, W. A. de Graaf, and H. V. Lé. Real graded Lie algebras, Galois cohomology and classification of trivectors in $\mathbb{R}^9$. [arXiv:2106.00246 [math.RT]].
[7] S. Bosch, W. Lütkebohmert, and M. Raynaud. Néron models, volume 21 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3). Springer-Verlag, Berlin, 1990.
[8] N. Bourbaki. Lie groups and Lie algebras. Chapters 7–9. Elements of Mathematics. Springer-Verlag, Berlin, 2005. Translated from the 1975 and 1982 French originals by Andrew Pressley.
[9] A. Čap and J. Slovák. Parabolic geometries. I, volume 154 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2009. Background and general theory.
[10] A.M. Cohen and A.G. Helminck. Trilinear alternating forms on a vector space of dimension 7. Comm. Algebra, 16(1):1–25, 1988.
[11] D.A. Cox, J. Little, and D. O’Shea. Ideals, varieties, and algorithms. Undergraduate Texts in Mathematics. Springer, Cham, fourth edition, 2015. An introduction to computational algebraic geometry and commutative algebra.
[12] W. A. de Graaf. Computation with linear algebraic groups. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 2017.
[13] W. A. de Graaf and T. GAP Team. SLA, computing with simple Lie algebras, Version 1.5.3. [https://gap-packages.github.io/jlal/], Nov 2019. Refereed GAP package.
[14] W. Decker, G.-M. Greuel, G. Pfister, and H. Schönenmann. SINGULAR 4-2-1 — A computer algebra system for polynomial computations. [http://www.singular.uni-kl.de] 2021.
[15] C. Demarche and G. Lucchini Arteche. Le principe de Hasse pour les espaces homogènes: reduction au cas des stabilisateurs finis. Compos. Math., 155(8):1568–1593, 2019.
[16] D.Ž. Djoković. Classification of trivectors of an eight-dimensional real vector space. Linear and Multilinear Algebra, 13(1):3–39, 1983.
[17] The GAP Group. GAP – Algorithms, and Programming, Version 4.11.1, 2021.
[18] G.B. Gurevich. Classification des trivecteurs ayant le rang huit. C. R. (Dokl.) Acad. Sci. URSS, n. Ser., 1935(2):353–356, 1935.
[19] G.B. Gurevich. Foundations of the theory of algebraic invariants. P. Noordhoff Ltd., Groningen, 1964. Translated by J. R. M. Radok and A. J. M. Spencer.
[20] R. Harvey and H.B. Lawson. Jr. Calibrated geometries. Acta Math., 148:47–157, 1982.
[21] N. Hitchin. The geometry of three-forms in six dimensions. J. Differential Geom., 55(3):547–576, 2000.
[22] J. Hora and P. Pudlák. Classification of 9-dimensional trilinear alternating forms over GF(2). J. Lie Theory, 21(2):285–305, 2011.
[23] H.V. Lé. Orbits in real $\mathbb{Z}_n$-graded semisimple Lie algebras. J. Lie Theory, 21(2):285–305, 2011.
[24] H.V. Lé and J. Vanžura. Classification of $k$-forms on $\mathbb{R}^n$ and the existence of associated geometry on manifolds. Chebyshevski Sb., 21(2):362–382, 2020.
[25] G.I. Lehrer and D.E. Taylor. Unitary reflection groups, volume 20 of Australian Mathematical Society Lecture Series. Cambridge University Press, Cambridge, 2009.
[26] N. Midoune and L. Noui. Trilinear alternating forms on a vector space of dimension 8 over a finite field. Linear Multilinear Algebra, 61(1):15–21, 2013.
[27] J.S. Milne. Algebraic groups, volume 170 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2017. The theory of group schemes of finite type over a field.
[28] W. Nahm. Supersymmetries and their representations. Nuclear Phys. B, 135(1):149–166, 1978.
[32] L. Noui. Transvecteur de rang 8 sur un corps algébriquement clos. C. R. Acad. Sci. Paris Sér. I Math., 324(6):611–614, 1997.
[33] L. Noui and P. Revoy. Formes multilineaires alternées. Ann. Math. Blaise Pascal, 1(2):43–69 (1995), 1994.
[34] A. L. Onishchik and E. B. Vinberg. Lie groups and algebraic groups. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1990. Translated from the Russian and with a preface by D. A. Leites.
[35] W. Reichel. Über trilineare alternierende Formen in sechs und sieben Veränderlichen und die durch sie definierten geometrischen Gebilde. Dissertation, Greifswald. 59 S., 1907.
[36] P. Revoy. Trivecteurs de rang 6. Bull. Soc. Math. France Mém., (59):141–155, 1979. Colloque sur les Formes Quadratiques, 2 (Montpellier, 1977).
[37] J. A. Schouten. Klassifizierung der alternierenden Größen dritten Grades in 7 Dimensionen. Rend. Circ. Mat. Palermo, 55:137–156, 1931.
[38] J.-P. Serre. Algebraic groups and class fields, volume 117 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1988. Translated from the French.
[39] J.-P. Serre. Galois cohomology. Springer Monographs in Mathematics. Springer-Verlag, Berlin, english edition, 2002. Translated from the French by Patrick Ion and revised by the author.
[40] T. A. Springer. Nonabelian $H^2$ in Galois cohomology. In Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965), pages 164–182. Amer. Math. Soc., Providence, R.I., 1966.
[41] R. Steinberg. Torsion in reductive groups. Advances in Math., 15:63–92, 1975.
[42] A. Swann. Hyper-Kähler and quaternionic Kähler geometry. Math. Ann., 289(3):421–450, 1991.
[43] É. B. Vinberg. Linear groups that are connected with periodic automorphisms of semisimple algebraic groups. Dokl. Akad. Nauk SSSR, 221(4):767–770, 1975. English translation: Soviet Math. Dokl. 16, no. 2, 406–409 (1975).
[44] É. B. Vinberg. The Weyl group of a graded Lie algebra. Izv. Akad. Nauk SSSR Ser. Mat., 40(3):488–526, 1976. English translation: Math. USSR-Izv. 10, 463–495 (1976).
[45] É. B. Vinberg. Classification of homogeneous nilpotent elements of a semisimple graded Lie algebra. Trudy Sem. Vektor. Tenzor. Anal., (19):155–177, 1979. English translation: Selecta Math. Sov. 6, 15–35 (1987).
[46] É. B. Vinberg and A. G. Elashvili. A classification of the three-vectors of nine-dimensional space. Trudy Sem. Vektor. Tenzor. Anal., 18:197–233, 1978. English translation: Selecta Math. Sov., 7, 63–98, (1988).
[47] N. R. Wallach. Geometric invariant theory. Universitext. Springer, Cham, 2017. Over the real and complex numbers.
[48] R. Westwick. Real trivectors of rank seven. Linear and Multilinear Algebra, 10(3):183–204, 1981.
[49] J. A. Wolf and A. Gray. Homogeneous spaces defined by Lie group automorphisms. I. J. Differential Geometry, 2:77–114, 1968.
[50] J. A. Wolf and A. Gray. Homogeneous spaces defined by Lie group automorphisms. II. J. Differential Geometry, 2:115–159, 1968.