Dual Prices for Frank–Wolfe Algorithms
—a note—

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Abstract
In this note we observe that for constrained convex minimization problems \( \min_{x \in \mathcal{P}} f(x) \) over a polytope \( \mathcal{P} \), dual prices for the linear program \( \min_{z \in \mathcal{P}} \nabla f(z) \) obtained from linearization at approximately optimal solutions \( x \) have a similar interpretation of rate of change in optimal value as for linear programming, providing a convex form of sensitivity analysis. This is of particular interest for Frank–Wolfe algorithms (also called conditional gradients), forming an important class of first-order methods, where a basic building block is linear minimization of gradients of \( f \) over \( \mathcal{P} \), which in most implementations already compute the dual prices as a by-product.

1 Introduction
We consider the constrained convex minimization problem

\[
\min_{x \in \mathcal{P}} f(x),
\]

where \( f \) is a smooth convex function and \( \mathcal{P} \) is a compact convex feasible region. Our primary interest is first-order algorithms where access to \( f \) is provided by computing gradients \( \nabla f(x) \) and function values \( f(x) \) at any feasible point \( x \). An important class of first-order methods is formed by conditional gradient algorithms (also known as Frank–Wolfe algorithms), which access the feasible region \( \mathcal{P} \) solely through a linear minimization oracle, i.e., presented with a linear objective \( c \) the oracle returns \( \arg\min_{x \in \mathcal{P}} c \cdot x \). This class of algorithms has several advantages two of which are (1) Projection-freeness: no projection to the domain \( \mathcal{P} \) is needed, and (2) Sparsity: iterates are represented as convex combination of a small number of vertices, usually at most one vertex per iteration. As an example, the simplest algorithm, namely, the (vanilla) Frank–Wolfe Algorithm (Frank and Wolfe, 1956; Levitin and Polyak, 1966) is recalled in Algorithm 1, which however will not be used in the rest of the paper.

Many implementations of a linear minimization oracle already compute dual prices for an optimal solution, to verify optimality. Therefore it is of interest to make use of this extra information. The role of conditional gradient algorithms in this paper is only as a practical example: making available dual prices at no extra cost in most implementations. We will provide an interpretation for dual prices similar to sensitivity analysis for linear optimization: dual prices at an optimal solution \( x \) are the rate of change in optimal value under small changes to the right-hand side \( b \) of constraints over a domain \( \{z : A z \leq b\} \) defined by linear inequalities. We shall see this

\begin{algorithm}
1 \( x_0 \in \mathcal{P} \) arbitrary;
2 for \( t = 0 \) to \( \ldots \) do
3 \hspace{1em} \( v_t \leftarrow \arg\min_{z \in \mathcal{P}} \nabla f(x_t) \cdot z \)
4 \hspace{1em} \( \gamma_t \leftarrow \arg\min_{0 \leq \gamma \leq 1} f(x_t + \gamma (v_t - x_t)) \)
5 \hspace{1em} \( x_{t+1} \leftarrow x_t + \gamma_t (v_t - x_t) \)
6 end
\end{algorithm}
interpretation holds even for approximately optimal solutions \( x \) while retaining the additive error of the accuracy of \( x \). Thus dual prices can then be used as customary, e.g., in sensitivity analysis, to compute risk-free state probabilities, (economic) shadow prices in e.g., energy systems, etc.

\subsection{1.1 Preliminaries}

We briefly recall basic definitions. Recall that a differentiable function \( f : P \rightarrow \mathbb{R} \) is convex if for all \( x, y \in P \)

\[ f(y) - f(x) \geq \nabla f(x)(y - x), \tag{1.1} \]

We denote by \( \nabla f(x) \) the gradient of \( f \) as a row vector, while \( P \) is considered to be contained in a vector space of column vectors. This will be convenient for dual prices, but we note that the formalism is inherently free of coordinate choices in the space of \( P \) (treating \( \nabla f(x) \) as a linear function).

Further the function \( f \) is \textit{L-smooth} if for all \( x, y \in P \)

\[ f(y) - f(x) \leq \nabla f(x)(y - x) + \frac{L}{2} \| y - x \|^2. \tag{1.2} \]

Here \( \| \cdot \| \) is any norm on the vector space containing \( P \) and the value of \( L \) depends on the norm \( \| \cdot \| \).

Conditional gradient algorithms often use the \textit{Frank–Wolfe gap} defined as:

\[ \max_{z \in P} \nabla f(x)(x - z) = \nabla f(x)(x - v), \tag{1.3} \]

with \( v \) a point of \( P \) minimizing \( \min_{z \in P} \nabla f(x)z \). In the context of Frank–Wolfe algorithms, these minimizers are called \textit{Frank–Wolfe vertices} at \( x \) (even though not all are vertices, but only vertex minimizers are used in practice). They are usually used to define the next iterate as in the vanilla variant in Algorithm 1. Recall that by convexity

\[ 0 \leq f(x) - f(x^*) \leq \nabla f(x)(x - x^*) \leq \nabla f(x)(x - v). \tag{1.4} \]

Here and below \( x^* \) is an optimal solution to \( \min_{x \in P} f(x) \). Thus the Frank–Wolfe gap is an upper bound to the primal gap \( f(x) - f(x^*) \) and it is 0 at optimal solutions to \( \min_{z \in P} f(z) \). As such the Frank–Wolfe gap is useful as a proxy for the primal gap, while it also provides a lower bound to the optimal value:

\[ f(x) - \nabla f(x)(x - v) \leq f(x) - \nabla f(x)(x - x^*) \leq f(x^*). \]

\section{2 Dual Prices in convex minimization}

We recall dual prices for linear optimization applied to our context here. Let \( P = \{ z : Az \leq b \} \) with \( A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^n \) be a polytope and let \( f \) be a convex function and let \( x, v \in P \) be arbitrary. Recall that \textit{strong duality} states that \( v \in P \) is a minimizer for the linear program \( \min_{z : Az \leq b} \nabla f(x)z \), i.e., \( v = \arg \min_{z : Az \leq b} \nabla f(x)z \) if and only if there is a nonnegative combination of constraints certifying optimality, i.e., a vector \( 0 \leq \lambda \in \mathbb{R}^m \) whose entries are multipliers called \textit{dual prices} satisfying

\[ \nabla f(x) = -\lambda A, \quad \lambda \geq 0, \tag{2.1a} \]

\[ \nabla f(x)v = \min_{z : Az \leq b} \nabla f(x)z = -\lambda b. \tag{2.1b} \]

The second equality can be replaced with \textit{complementary slackness}, namely, that \( v \) satisfies with equality the constraints of \( Az \leq b \) whose multiplier in \( \lambda \) is positive (i.e., \( a_i v = b_i \) for \( \lambda_i > 0 \), where \( a_i \) is row \( i \) of \( A \), and \( \lambda_i \) and \( b_i \) are entry \( i \) of \( \lambda \) and \( b \) respectively), or short \( \lambda(b - Av) = 0 \). A \textit{primal-dual pair} is a pair \((v, \lambda)\) satisfying the strong duality conditions stated in Equation system (2.1). Obviously \( v \) is a Frank–Wolfe vertex \( v \) at \( \nabla f(x) \) and the \textit{Frank–Wolfe gap} at \( x \) equals the \textit{complementarity gap}, i.e.,

\[ \nabla f(x)(x - v) = -\lambda A(x - v) = \lambda(b - Ax). \tag{2.2} \]

Common implementations of a linear optimization oracle naturally compute dual prices for a Frank–Wolfe vertex \( v \), which is indeed a vertex of \( P \). For example, the widely used simplex algorithm internally operates
with data providing a candidate \((v, \lambda)\) for a primal-dual pair, where \(v\) is a vertex of \(P\) but \(\lambda\) may violate the nonnegativity condition, which is then incrementally improved to a primal-dual pair.

Recall that the celebrated Slater’s condition of optimality (a special case of the Karush–Kuhn–Tucker condition for convex functions) is the strong duality form of the optimality condition for \(\min_{z : A z \leq b} f(z)\): a point \(x\) is an optimal solution to \(\min_{z : A z \leq b} f(z)\) if and only if \(x\) is an optimal solution to \(\min_{z : A z \leq b} \nabla f(x) z\), i.e., \((x, \lambda)\) is a primal-dual pair for the linear program \(\min_{z : A z \leq b} \nabla f(x) z\) for some \(\lambda\); equivalently, there are dual prices \(\lambda\) for \(x\) under the linear objective \(\nabla f(x)\).

In practice implementations, e.g., of Frank–Wolfe algorithms, this means that dual prices \(\lambda\) for the optimal solution \(x^*\) can be obtained as dual prices for the Frank–Wolfe vertex \(v\) associated with \(\nabla f(x^*)\).

### 2.1 Dual Prices for approximately optimal solutions: Sensitivity in \(b\)

In practice we rarely have exact optimal solutions to convex minimization problems (even within the limit of numerical accuracy), and we are usually satisfied with a good approximate solution with, e.g., an additive error in function value of at most \(\epsilon\). For Frank–Wolfe algorithms the usual stopping criterion is an upper bound on the Frank–Wolfe gap (sometimes also called dual gap) \(\max_{z : A z \leq b} \nabla f(x)(x - z) \leq \epsilon\) as due to the linear minimizations the Frank–Wolfe gap is essentially computed anyway. As such we will now consider the case of approximately optimal solutions. To this end let \(v = \arg\min_{z : A z \leq b} \nabla f(x) z\) be the Frank–Wolfe vertex at \(x\) and let \(0 \leq \lambda \in \mathbb{R}^m\) be associated dual prices as in Equation system (2.1) above.

A common interpretation of \(\lambda\) in the context of linear programs is the rate of change in optimal value as a function of change to the constant term \(b\), i.e., \(\min_{z : A z \leq b'} \nabla f(x) z = \lambda b'\) for \(b'\) close to \(b\); if \(\lambda\) is not a unique dual solution it needs to be chosen depending on \(b'\). Morally, the optimal value changes by \(\lambda(b - b')\), while the dual solution \(\lambda\) does not change.

The next observation carries this sensitivity analysis over to smooth convex functions and approximately optimal solutions: in this case we will incur additional error terms due to (1) non-linearity of the objective function, and (2) approximate optimality (with no error for optimal solutions). There are many common assumptions on the objective convex function \(f\) to bound its non-linearity. For the sake of exposition, we assume the most common one, namely, smoothness, which is only needed for the last inequality in Equations (2.4) and (2.5). For other assumptions the error term \(L\|v' - v\|^2/2\) should be replaced accordingly.

**Observation 2.1.** Let \(f\) be an \(L\)-smooth convex function over a convex domain containing the polytopes \(P = \{z : Az \leq b\}\) and \(P' = \{z : Az \leq b'\}\). Let \(x \in P\) and \(v = \arg\min_{z \in P} \nabla f(x) z\). Similarly, let \(v' = \arg\min_{z \in P'} \nabla f(x) z\). Assume that \(x' := x - v + v' \in P'\). Then

\[
\begin{align*}
  f(x) - \nabla f(x)(x - v) &\leq \min_{z \in P} f(z) \\
  f(x) - \nabla f(x)(x - v) + \nabla f(x)(v' - v) &\leq \min_{z \in P'} f(z) \\
  f(x) - \nabla f(x)(x - v) + \nabla f(x)(v' - v) + L/2 \|v' - v\|^2
\end{align*}
\]

(2.4)

When \(\lambda\) is a common dual solution for both \(v\) in \(P\) and \(v'\) in \(P'\)

\[
\begin{align*}
  f(x) - \nabla f(x)(x - v) + \lambda(b - b') &\leq \min_{z \in P'} f(z) \\
  f(x) - \nabla f(x)(x - v) + \lambda(b - b') &\leq \min_{z \in P'} f(z) + \lambda(b - b') + L/2 \|v' - v\|^2.
\end{align*}
\]

(2.5)

To justify the assumption \(x' \in P'\), we note that it holds when \(b'\) is sufficiently close to \(b\) and \(x\) is sufficiently close to the optimal solution \(x^*\) (with \(v\) and \(v'\) appropriately chosen depending on \(b'\)). Intuitively, a neighborhood of \(v'\) in \(P'\) is just a translation of a neighborhood of \(v\) in \(P\) and \(x\) is well inside the neighborhood to be preserved by translation. Let us split the defining linear inequalities \(Az \leq b\) for \(P\) into two: let \(A_1 z \leq b_1\) be the subsystem which \(x\) satisfies with equality (describing the boundary of the neighborhood at \(v\)), and \(A_2 z \leq b_2\) be the subsystem with inequalities which \(x\) satisfies with strict inequality (describing far away parts of \(P\)), i.e., \(A_1 x = b_1\) and \(A_2 x < b_2\). We claim that when \(x\) is close enough to \(x^*\) then \(A_2 v = b_2\) (regardless of the choice of \(v\)). In geometrical terms the claim means that \(v\) is contained in the minimal face containing \(x\). To verify it, let \(F\) be the face of \(P\) containing \(x^*\) in its relative interior (allowing \(F = P\)), which is a minimal solution to \(\min_{z \in P} \nabla f(x) z\) by optimality. When \(x\) is close to \(x^*\) then (1) all minimal solutions to \(\min_{z \in P} \nabla f(x) z\) lie in \(F\), too, and (2) every hyperface of \(P\) containing \(x\) also contains \(x^*\) and hence \(F\) is contained in the minimal face containing \(x\).
As for linear programs, if $b'$ is sufficiently close to $b$ then for some choice of optimal solutions $v$ and $v'$ (recall they need not be unique) they have a common dual solution $\lambda$ and $v'$ is sufficiently close to $v$.

After these preliminaries, we verify $x' \in P'$. First we deal with the inequalities $x$ satisfy with equality for $P$, which turns out to be the easy case: $A_c x' = A_c x - A_c v + A_c v' \leq b_\geq - b_\geq + b_\geq' = b_\geq'$. For the other inequalities note that $b_\geq' - A_c x' = (b_\geq - A_c x) + (b_\geq' - b_\geq) - A_c (v' - v)$. As $b_\geq - A_c x$ is strictly positive, with $b'$ close enough to $b$ (and hence $v'$ close enough to $v$), the other terms on the right-hand side are small enough for the right-hand side remaining positive, i.e., $A_c x' < b_\geq'$.

Proof. The bounds for the polytope $P$ in (2.3) are well-known and presented for comparison only; they easily follow from convexity and minimality of $v$.

Equation (2.4) provides the same bounds for the polytope $P'$, even though the points at which we compute the lower and the upper bound might differ. The left-hand side of the first inequality is $f(x) - \nabla f(x)(x - v')$ (written in a form to ease comparison with Equation (2.3)). While $x$ might not be contained in $P'$, it does not affect the validity of the inequality. The second inequality explicitly uses the assumption $x' \in P'$, and the last inequality is just the smoothness inequality for $f$, using $x' - x = v' - v$.

Finally, observe that $\lambda(b - b') = \nabla f(x) (v' - v)$ by the definition of dual prices, leading to Equation (2.5).

References

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E. S. Levitin and B. T. Polyak. Constrained minimization methods. USSR Computational Mathematics and Mathematical Physics, 6(5):1–50, 1966.