Research Article

Nonlinear Vibration Characteristics of the Involute Spline Coupling in Aeroengine with the Parallel Misalignment

Xiangzhen Xue¹, ², Qixin Huo, ¹ and Jian Liu¹

¹Shaanxi University of Science & Technology, Xi’an 710021, China
²Mechanical Innovation and Tribology Group, School of Engineering, University of Birmingham, Edgbaston Birmingham B15 2TT, UK

Correspondence should be addressed to Xiangzhen Xue; a_zheny@163.com

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In order to control the vibration of the involute spline coupling in aeroengine well and reduce the fretting wear, a bending–torsion coupling nonlinear vibration model of the involute spline coupling with the misalignment was proposed, and a dynamic meshing stiffness function with multiteeth engagement was established. Then, the influence of different misalignment, wear, and rotation speeds with different misalignment on the nonlinear vibration characteristics of the involute aviation spline coupling was explored. The result shows that with an increase of the parallel misalignment, the system experienced the state of a single period, a quasiperiod, multiperiod, and chaos but finally only alternated between the quasiperiod and the chaos state. The uneven wear of each tooth of the spline displayed a significant influence on the vibration of the spline coupling, and the influence of the uniform wear was smaller under given conditions here. Furthermore, with an increase of the speed, the larger the misalignment was, the more times the system entered or left the chaos state were. The model proposed here is found to be closer to the actual working conditions, and the analysis results can provide more accurate external load conditions for the prediction of the fretting damage of the spline coupling in aeroengine.

1. Introduction

There should be no relative displacement between the involute spline couplings of aeroengine. As the involute spline couplings are not only subjected to a strong external excitation of cycle loads but also the internal excitation introduced by the time-varying meshing stiffness and the error produced in the process of manufacturing or assembly while taking off, cruising, and landing, they exhibit a heavy nonlinear vibration [1–4]. Especially, for the internal and external spline shafts, they are misaligned during installation or misaligned due to heating, loading, and foundation deformation, or there is a misalignment between the involute spline couplings before assembly. Generally, there are three kinds of misalignment in the involute spline coupling: parallel misalignment, angular misalignment, and combined misalignment. The parallel misalignment means two connected rotor axes are connected with some parallel offset in the radial direction; the angular misalignment means two connected rotor axes are connected with an angle; the combined misalignment means two connected rotor axis lines have both an angle and a parallel offset. All types of misalignment can cause the spline coupling bending, as well as introduce additional loads to the spline shaft. Then, the loads between the teeth is redistributed, which results in a strong bending–torsion coupling nonlinear vibration of the involute spline coupling in aeroengine [5, 6]. On the one hand, due to the bending torsional coupling nonlinear vibration, the smoothness of system transmission will get worse, so there will be much noise [7]; on the other hand, the fretting will be introduced between the two contact surfaces. Further, the fretting wear alters the backlash and the surface roughness of spline teeth, and then affects the vibration excitation of the system and intensifies its vibration characteristics of the system [8–10].
Therefore, there must be some relationship between the fretting wear and the vibration of the spline coupling [11, 12]. Finally, the repeated fretting movement leads to a serious failure of the fretting wear in the spline coupling of aeroengine [13–16]. At present, there are a large number of studies on fretting wear and the vibration characteristics of aviation involute splines, but both of them are two unrelated issues.

In terms of spline coupling vibration, there have been several studies focused on predicting the free and forced vibration characteristics of rotor-coupling systems [17–19]. These studies proposed the vibration model of the system using the finite-element method and analyzed the effect of the spline force on the stability of the rotor system. Theoretical and experimental analyses were conducted the effects of vibration parameters on the vibration characteristics and the stability of the system. However, these studies did not analyze the influence of the misalignment on the vibration characteristics of the system. Though there are numerous studies on the effect of the misalignment on the vibration characteristics of the system, the multiteeth meshing was not in these works [20, 21]. Consequently, the results are not accurate enough for the design of aeroengine involute spline couplings. Only Zhang and Cuffaro take into account the misalignment and investigated the vibration characteristics [22, 23]. Kahraman assumed both the shaft and the gear (or any other component splined to the shaft) deflect only at the spline teeth and investigated the vibration characteristics of involute spline couplings [24].

In terms of the fretting wear of the spline coupling, Leen et al. conducted a study on the characteristic of the friction contact of the helical spline coupling using the finite-element method considering the axial load and torque load and analyzed the influence of the tooth profile modification on the contact stress, sliding distance, and friction factor of the spline teeth [25]. Medina and Olver studied the elastic contact model of the spline coupling based on the boundary finite-element method and explored the influence of large-scale design parameters, torque, and eccentricity error on the contact pressure and the slip distance distribution of the spline coupling [26]. Mcoll proposed a finite-element method to calculate the fretting wear based on the modified Archard’s equation and measured the friction coefficient and the wear coefficient by using the friction and wear experiment [27]. In 2003, Ding et al. also proposed a numerical method based on the modified Archard’s equation to simulate the fretting wear of a pin disk under the partial sliding and complete sliding, which not only calculated the surface contact pressure and the sliding distance but also calculated the subsurface contact pressure [28]. According to these analyses, the above studies did not investigate the fretting wear on the vibration characteristics of bending–torsion under the internal and external excitation of the system, and Zhao and others only considered the relationship between the fretting wear and the vibration displacement [13, 29, 30].

Therefore, based on the above studies, a bending–torsion coupling nonlinear vibration model of the involute spline coupling with a misalignment was proposed; the working conditions of the parallel misalignment and the displacement on the meshing line in different quadrants of the involute spline coupling were analyzed, and the meshing stiffness of a single tooth under an ideal condition with the parallel misalignment was calculated. Then, the vibration characteristics of the involute spline coupling in the aeroengine system on different parallel misalignment, fretting wear, and speed rotation were studied. It is of great significance for controlling the vibration and reducing the fretting wear failure to study the bending–torsion coupling nonlinear vibration characteristics of the involute splines considering the parallel misalignment.

2. Vibration Model of the System considering the Misalignment

2.1. Establishment of the Vibration Model. The vibration model of the bending–torsion coupling of the involute spline coupling in the aeroengine system in the three-dimensional plane is shown in Figure 1(a). The four concentrated mass blocks marked as 1, 2, 3, and 4 in Figure 1(a) represent the prime mover, external spline, internal spline, and load, respectively. The prime mover with the torsional stiffness $k_{T1}$ and the torsional damping $c_{T1}$ is connected with the external spline using a massless elastic shaft; the internal spline with the torsional stiffness $k_{T2}$ and the torsional damping $c_{T2}$ is connected with the load using a massless elastic shaft; the internal and external splines are connected with the involute teeth, considering the meshing stiffness $k_m$, meshing damping $c_m$, and clearance $c_i$ between the working teeth profile and clearance $c_i'$ between the non-working teeth profile. It is further considered that there are supporting stiffness and supporting damping on the internal and external splines. $k_{p11}, k_{p12}, k_{p21}, k_{p22}$ represent the supporting stiffness of the external spline and the internal spline along the x and the y coordinate axes, respectively; $c_{p11}, c_{p12}, c_{p21}, c_{p22}$ represent the supporting damping of the external spline and the internal spline along the x and the y coordinate axes, respectively; the input torque and the load torque of the system are $T_3, T_4$, respectively.

The vibration model of the bending–torsion coupling of the involute spline coupling in the aeroengine system in the two-coordinate plane is shown in Figure 1(b). In Figure 1(b), the influence of the misalignment is taken into account.

2.2. Analysis of the Working Conditions of the Misalignment. The misalignment of the involute spline coupling in aeroengine includes parallel misalignment, angular misalignment, and combined misalignment, as shown in Figures 2–4. As the vibration of the involute spline coupling is only analyzed in the xoy plane in this work, the vibration of the z-axis freedom is not considered, and the parallel misalignment in the x and y directions is mainly analyzed. As shown in Figure 3, the black lines represent the external involute spline and its coordinate system, and the red lines represent the internal involute spline and its coordinate system. Here, it is assumed that the external spline is stationary under the parallel misalignment condition. Therefore, the misalignment is introduced from the movement of the internal spline. From
Figure 1: The vibration model of the bending–torsion coupling of the involute spline coupling: (a) in the three-dimensional plane; (b) in the xoy plane.

Figure 2: Diagram of the parallel misalignment.

Figure 3: Diagram of the angular misalignment.

Figure 4: Diagram of the combined misalignment.

Figure 3, it can be inferred that the parallel misalignment is formed by moving a distance \( l \) along a certain coordinate axis from the initial location. Therefore, in the xoy plane, the coordinate of the y and x direction misalignment of the original internal spline can be expressed as follows:

\[
j'_2 = y_2 + l_y, \quad (1)
\]

\[
x'_2 = x_2 + l_x. \quad (2)
\]

2.3. Establishment of Vibration Equations. According to Newton’s second law, the vibration equations of the spline coupling in aeroengine are shown in the following equation.

\[
\begin{align*}
J_M \ddot{\theta}_M + k_{T1} (\theta_M - \theta_1) + c_{T1} (\dot{\theta}_M - \dot{\theta}_1) &= T_d \\
m_1 \ddot{x}_1 + k_{p1} x_1 + c_{p1} \dot{x}_1 &= F_{mx} \\
m_1 \ddot{y}_1 + k_{p1} y_1 + c_{p1} \dot{y}_1 &= F_{my} - m_1 g \\
j_1 \ddot{\theta}_1 + k_{T1} (\theta_1 - \theta_M) + c_{T1} (\dot{\theta}_1 - \dot{\theta}_M) &= T_m \\
m_2 \dddot{x}_2' + k_{p2} x_2' + c_{p2} \dddot{x}_2' &= -F_{mx} \\
m_2 \dddot{y}_2' + k_{p2} y_2' + c_{p2} \dddot{y}_2' &= -F_{my} - m_2 g \\
j_2 \dddot{\theta}_2 + k_{T2} (\theta_2 - \theta_L) + c_{T2} (\dot{\theta}_2 - \dot{\theta}_L) &= -T_m \\
j_1 \dddot{\theta}_L + k_{T2} (\theta_L - \theta_2) + c_{T2} (\dot{\theta}_L - \dot{\theta}_2) &= -T_L,
\end{align*}
\]

where \( x_1, y_1, \) and \( \theta_1 \) are the vibration displacements along the x-axis, y-axis (m), and the torsional displacement around the axis of the external spline (rad), respectively; \( x_2, y_2, \) and \( \theta_2 \) are the vibration displacements along the x-axis, y-axis (m), and the torsional displacement around the axis of the internal spline, respectively (rad); \( \theta_M \) and \( \theta_1 \) are the torsional displacements of the prime mover and loads (rad), respectively; \( \theta_M \) and \( \theta_1 \) and \( \theta_L \) are all not the same due to the torsional deformation of the shaft and the spine teeth. Then, it is assumed that the supporting stiffness and the supporting damping of the external spline and the internal spline are the same in the x and y directions, respectively. It means
\[ k_{px1} = k_{py1} = k_{p1}, \]
\[ k_{px2} = k_{py2} = k_{p2}, \]
\[ c_{px1} = c_{py1} = c_{p1}, \]
\[ c_{px2} = c_{py2} = c_{p2}. \]

Then, the rigid displacement of Equation (5) requires to be eliminated. Thereafter, Equation (5) is expressed as follows:

\[
\begin{align*}
\ddot{x}_1 + \frac{k_{p1}}{m_1} x_1 + \frac{c_{p1}}{m_1} \dot{x}_1 &= \frac{F_{mx}}{m_1}, \\
\ddot{y}_1 + \frac{k_{p1}}{m_1} y_1 + \frac{c_{p1}}{m_1} \dot{y}_1 &= \frac{F_{my}}{m_1} - g, \\
\ddot{x}_2' + \frac{k_{p1}}{m_2} x_2' + \frac{c_{p1}}{m_2} \dot{x}_2' &= -\frac{F_{mx}}{m_2}, \\
\ddot{y}_2' + \frac{k_{p1}}{m_2} y_2' + \frac{c_{p1}}{m_2} \dot{y}_2' &= -\frac{F_{my}}{m_2} - g, \\
\Delta_1 + k_{T1} \left( \frac{1}{f_1 + \frac{1}{J_{M1}}} \right) \Delta_1 + c_{T1} \left( \frac{1}{f_1 + \frac{1}{J_{M1}}} \right) \Delta_1 &= -r_b T_d \frac{1}{J_{M1}} + r_b T_m \\
\Delta_2 + k_{T2} \Delta_3 - k_{T1} \frac{1}{f_2} \Delta_1 + c_{T2} \frac{1}{f_2} \Delta_3 - c_{T1} \frac{1}{f_1} \Delta_1 &= -r_b T_m \frac{1}{f_2} + r_b T_L \frac{1}{J_L}, \\
\Delta_3 + k_{T2} \left( \frac{1}{f_2 + \frac{1}{J_L}} \right) \Delta_3 + c_{T2} \left( \frac{1}{f_2 + \frac{1}{J_L}} \right) \Delta_3 &= -r_b T_m \frac{1}{f_2} - r_b T_L \frac{1}{J_L}, \\
\end{align*}
\]

In Equation (5), \( r_b \) is the radius of the base circle of internal and external splines (rad). The angular velocity \( \dot{\theta}_M \) of the prime mover is considered to be a constant value, and

\[
\begin{align*}
\Delta_1 &= r_b (\dot{\theta}_1 - \dot{\theta}_M), \\
\Delta_2 &= r_b (\dot{\theta}_2 - \dot{\theta}_1), \\
\Delta_3 &= r_b (\dot{\theta}_3 - \dot{\theta}_L). \\
\end{align*}
\]

Then, after introducing the dimensionless reference parameters \( \omega, l \), Equation (5) becomes dimensionless:

\[
\begin{align*}
\ddot{x}_1 + \frac{\overline{k}_{p1}}{\overline{m}_1} \overline{x}_1 + \frac{\overline{c}_{p1}}{\overline{m}_1} \overline{x}_1 &= \frac{\overline{F}_{mx}}{\overline{m}_1}, \\
\ddot{y}_1 + \frac{\overline{k}_{p1}}{\overline{m}_1} \overline{y}_1 + \frac{\overline{c}_{p1}}{\overline{m}_1} \overline{y}_1 &= \frac{\overline{F}_{my}}{\overline{m}_1} - \overline{g}, \\
\ddot{x}_2' + \frac{\overline{k}_{p1}}{\overline{m}_2} \overline{x}_2' + \frac{\overline{c}_{p1}}{\overline{m}_2} \overline{x}_2' &= -\frac{\overline{F}_{mx}}{\overline{m}_2}, \\
\ddot{y}_2' + \frac{\overline{k}_{p1}}{\overline{m}_2} \overline{y}_2' + \frac{\overline{c}_{p1}}{\overline{m}_2} \overline{y}_2' &= -\frac{\overline{F}_{my}}{\overline{m}_2} - \overline{g}, \\
\overline{\Delta}_1 + \overline{k}_{T1} \overline{\Delta}_1 + \overline{c}_{T1} \overline{\Delta}_1 &= -\overline{r}_b \overline{T}_d \frac{1}{\overline{J}_{M1}} + \overline{r}_b \overline{T}_m \\
\overline{\Delta}_2 + \overline{k}_{T2} \overline{\Delta}_3 - \overline{k}_{T1} \frac{1}{\overline{f}_2} \overline{\Delta}_1 + \overline{c}_{T2} \frac{1}{\overline{f}_2} \overline{\Delta}_3 - \overline{c}_{T1} \frac{1}{\overline{f}_1} \overline{\Delta}_1 &= -\overline{r}_b \overline{T}_m \frac{1}{\overline{f}_2} + \overline{r}_b \overline{T}_L \frac{1}{\overline{J}_L}, \\
\overline{\Delta}_3 + \overline{k}_{T2} \left( \frac{1}{\overline{f}_2 + \frac{1}{\overline{J}_L}} \right) \overline{\Delta}_3 + \overline{c}_{T2} \left( \frac{1}{\overline{f}_2 + \frac{1}{\overline{J}_L}} \right) \overline{\Delta}_3 &= -\overline{r}_b \overline{T}_m \frac{1}{\overline{f}_2} - \overline{r}_b \overline{T}_L \frac{1}{\overline{J}_L}. \\
\end{align*}
\]

In Equation (7),

\[
\begin{align*}
\overline{k}_{p1} &= \frac{k_{p1}}{m_1 \omega^2}, \overline{c}_{p1} = \frac{c_{p1}}{m_1 \omega}, \overline{k}_{p2} = \frac{k_{p2}}{m_2 \omega^2}, \overline{c}_{p2} = \frac{c_{p2}}{m_2 \omega}, \\
\overline{k}_{T1} &= \frac{k_{T1}}{f_1 \omega^2}, \overline{k}_{T2} = \frac{k_{T2}}{f_2 \omega^2}, \overline{k}_{T3} = \frac{k_{T3}}{f_2 \omega^2}, \overline{c}_{T1} = \frac{c_{T1}}{f_1 \omega}, \overline{c}_{T2} = \frac{c_{T2}}{f_2 \omega}, \overline{c}_{T3} = \frac{c_{T3}}{f_2 \omega}, \\
\overline{L} &= \frac{r_b}{f_1}, \\
\overline{x}_1 &= \frac{x_1}{\omega^2}, \overline{\dot{x}_1} = \frac{\dot{x}_1}{\omega}, \overline{x}_1 = \overline{x}_1 \frac{1}{T}; \overline{\dot{x}_2} = \frac{\dot{x}_2}{\omega^2}, \overline{\dot{x}_2} = \frac{\dot{x}_2}{\omega} \\
\overline{y}_1 &= \frac{y_1}{\omega^2}, \overline{\dot{y}_1} = \frac{\dot{y}_1}{\omega}, \overline{y}_1 = \overline{y}_1 \frac{1}{T}; \overline{\dot{y}_2} = \frac{\dot{y}_2}{\omega^2}, \overline{\dot{y}_2} = \frac{\dot{y}_2}{\omega} \\
\overline{\Delta}_1 &= \frac{\Delta_1}{\omega}, \overline{\dot{\Delta}_1} = \frac{\dot{\Delta}_1}{\omega}, \overline{\dot{\Delta}_1} = \overline{\Delta}_1 \frac{1}{T}, \\
\overline{F}_{mx} &= \frac{F_{mx}}{m_1 \omega^2}; \overline{F}_{my} = \frac{F_{my}}{m_1 \omega^2}; \overline{F}_{mx} = \frac{F_{mx}}{m_2 \omega^2}; \overline{F}_{my} = \frac{F_{my}}{m_2 \omega^2}; \\
\overline{T}_1 &= \frac{r_b T_m}{\overline{J}_1 \omega^2} - \frac{r_b T_d}{\overline{J}_{M1} \omega^2}; \overline{T}_2 = r_b T_m \left( \frac{1}{f_1 \omega^2} + \frac{1}{f_2 \omega^2} \right) + \overline{r}_b \overline{T}_m \frac{1}{\overline{J}_1 \omega^2} - \overline{r}_b \overline{T}_L \frac{1}{\overline{J}_1 \omega^2}. \\
\end{align*}
\]

3. Analysis of the Nonlinear Vibration Force

The component of the total meshing force along the coordinate axis can be obtained by a discrete summation of the forces along the meshing line of a single pair of teeth. The total meshing torque is the total meshing force multiplied by the radius of the base circle \( r_b \). The calculation is shown in the following equation:
\[
\begin{align*}
F_{mx} &= -\sum_{i=1}^{c} F_{mi} \sin \phi_i \\
F_{my} &= \sum_{i=1}^{c} F_{mi} \cos \phi_i \\
T_m &= r_b \sum_{i=1}^{c} F_{mi},
\end{align*}
\]

where \(F_{mi}\) is the meshing force of a single pair of teeth along the meshing line as shown in the following equation:

\[
F_{mi} = k_m g_i [\Delta n_i(t)] + c_m g_i [\Delta n_i(t)].
\]

Here, \(g_i [\Delta n_i(t)]\) is the meshing deformation function. It is assumed that the initial clearance on both sides of each tooth of internal and external splines is \(c_i\) and \(c'_i\), respectively (if the center deviation of the spline shaft is not considered, \(c_i = c'_i\)), and the meshing deformation function and its differential function (Equations (11) and (12)) are all piecewise functions.

When there is no fretting wear between the spline teeth, the meshing deformation function and its differential function are expressed as follows:

\[
g_i [\Delta n_i(t)] = \begin{cases} 
\Delta n_i(t) - c'_i, & \Delta n_i(t) > c'_i \\
0, & -c_i \leq \Delta n_i(t) \leq c'_i \\
\Delta n_i(t) + c_i, & \Delta n_i(t) < -c_i,
\end{cases}
\]

\[
\dot{g}_i [\Delta n_i(t)] = \begin{cases} 
\dot{\Delta n}_i(t), & \Delta n_i(t) > c'_i & \Delta n_i(t) < -c_i \\
0, & -c_i \leq \Delta n_i(t) \leq c'_i
\end{cases}
\]

(11)

However, the involute spline coupling of an aeroengine is accompanied by severe fretting wear during its operation. When the wear depth is considered, the original backlash of the spline coupling changes, that is, if the wear depth of each tooth is \(h_i\), \(i = 1, 2, 3 \cdots, z\), then the initial backlash of one side on-meshing of the aviation involute spline considering the fretting wear becomes \(c_i + h_i\). Hence, the expression of \(g_i [\Delta n_i(t)]\) becomes

\[
g_i [\Delta n_i(t)] = \begin{cases} 
\Delta n_i(t) - c'_i, & \Delta n_i(t) > c'_i \\
0, & -(c_i + h_i) \leq \Delta n_i(t) \leq c'_i \\
\Delta n_i(t) + (c_i + h_i), & \Delta n_i(t) < -(c_i + h_i),
\end{cases}
\]

\[
\dot{g}_i [\Delta n_i(t)] = \begin{cases} 
\dot{\Delta n}_i(t), & \Delta n_i(t) > c'_i & \Delta n_i(t) < -(c_i + h_i) \\
0, & -(c_i + h_i) \leq \Delta n_i(t) \leq c'_i.
\end{cases}
\]

(12)

(13)

If the misalignment of the spline shaft is not considered, \(c_i = c'_i\), according to the deformation formula of a single tooth (Equation (11)), the relative displacement on the meshing line \(\Delta n_i(t)\) is also required to obtain the vibration meshing force. Thus, the involute profile is simplified as trapezoid. It is observed that it is negative while moving inward the tooth along the meshing line on the working profile and it is positive while moving outward the tooth along the meshing line on the working profile. All the lengths of the segment involved below are positive as well. Then, \(\alpha_0\) is the pressure angle on the reference circle (rad); \(z\) is the total number of teeth; \(i\) is the tooth number (set the tooth on the \(x\)-axis at the initial time as the first pair of teeth, and the numbers of the rest of the teeth are 1, 2, and 3 in sequence); \(\omega\) is the angle velocity (rad/s); \(t\) is the time (s); \(\theta_0\) is the half angle of the tooth thickness of the reference circle (rad), \(\theta_0 = \pi/2z\); the angle \(\varphi_i\) between the working profile and the \(x\) direction is defined as follows:

\[
\varphi_i = \theta_i - \alpha_0.
\]

(14)

\(\theta_i\) is the angle of a tooth at a certain time which can be expressed as follows:

\[
\theta_i = 2\pi i/z + \omega_0 t + \theta_0 (\theta_0 = \pi/2z),
\]

(15)
as shown in Figure 5.

The formula of the displacement along the meshing line is expressed as follows:

\[
\Delta n_1 = -x_1 \sin \varphi_i + y_1 \cos \varphi_i.
\]

(16)

It can be concluded that the displacement formulas on the meshing line of the external and internal splines are the same for the \(\varphi_i\) of different quadrants and for different modes of movement; when considering the misalignment, they are expressed as follows:

\[
\Delta n_1 = -x_1 \sin \varphi_i + y_1 \cos \varphi_i,
\]

\[
\Delta n_2 = -x_2 \sin \varphi_i + y_2 \cos \varphi_i.
\]

(17)

Based on the above discussion, the formula for the relative displacement on meshing lines between the internal and external splines without considering the misalignment is expressed as follows:
\[
\Delta n_i(t) = \Delta n_2(t) - \Delta n_1(t) + r_b(\theta_2 - \theta_1) \\
= (x_1 - x'_{2}) \sin \varphi_i - (y_1 - y'_{2}) \cos \varphi_i + r_b(\theta_2 - \theta_1) \\
= (x_1 - x'_{2}) \sin \varphi_i - (y_1 - y'_{2}) \cos \varphi_i + \Delta_2.
\]

(18)

In this work, the influence of the nonlinear vibration characteristics under the misalignment of the involute spline coupling system is studied. Therefore, the relative displacement \( \Delta n_i(t) \) on the meshing line between the involute spline couplings is calculated considering the misalignment. Then, the corresponding equations are expressed by Equations (19) and (20), respectively, as follows:

\[
\Delta n_i(t) = (x_1 - x'_{2}) \sin \varphi_i - (y_1 - y'_{2}) \cos \varphi_i + \Delta_2,
\]

(19)

\[
\dot{\Delta n}_i(t) = \left(\dot{x}_1 - \dot{x'}_{2}\right) \sin \left(\frac{2\pi}{z}t + \omega_0 t + \theta_0 - \alpha_0\right) \\
+ \left(x_1 - x'_{2}\right) \omega_0 \cos \left(\frac{2\pi}{z} t + \omega_0 t + \theta_0 - \alpha_0\right) \\
- \left(y_1 - y'_{2}\right) \cos \left(\frac{2\pi}{z} t + \omega_0 t + \theta_0 - \alpha_0\right) \\
+ \left(y_1 - y'_{2}\right) \omega_0 \sin \left(\frac{2\pi}{z} t + \omega_0 t + \theta_0 - \alpha_0\right) + \Delta_2.
\]

(20)

4. Meshing Stiffness of the Spline Coupling

4.1. Meshing Stiffness of the Spline Coupling without Misalignment. It is well known that the loaded position is located at the pitch circle under ideal condition, as shown in Figure 6.

First, the length from the root of the tooth to the location of loading should be evenly divided into an identical microsegment, and then each parameter without considering the transverse vibration is calculated as follows:

The microsegment height is shown in the following:

\[
h'_{ik} = \frac{h_{ik}(h_{x2})}{n},
\]

(21)

where \( k = 1, 2 \); when it is the external spline, \( k = 1 \); when it is the internal spline, \( k = 2 \). The tooth root height \( h_{1i} \) of the external spline and the tooth top height \( h_{x2} \) of the internal spline are shown in the following:

\[
\begin{align*}
\hat{h}_{1i} &= 0.5(d_1 - d_{1\text{ min}}) \\
\hat{h}_{x2} &= 0.5(d_{2\text{ max}} - d_2),
\end{align*}
\]

(22)

where \( d_1 \) is the diameter of the reference circle of the external spline, \( d_{1\text{ min}} \) is the small diameter of the external spline, \( d_2 \) is the diameter of the reference circle of the internal spline, and \( d_{2\text{ max}} \) is the large diameter of the internal spline.

The thickness of the half tooth of the \( j \text{th} \) segment for the external and internal splines is expressed by Equation (23) as follows (for the external spline, \( Y_{j,1k} \) is the thickness of the upper half tooth of the \( j \text{th} \) segment; for the internal spline, \( Y_{j,1k} \) is the thickness of the lower half tooth of the \( j \text{th} \) segment):

\[
Y_{j,1k} = 0.5 \left( \frac{S}{r} \cdot \frac{R_{j,1k}}{r} \pm 2R_{j,1k} \left( \text{inv}(a_0) - \text{inv}(a_{j,1k}) \right) \right),
\]

(23)

where \( S \) is the tooth thickness of the reference circle, \( r \) is the radius of the reference circle, and \( \text{inv}(a_0) \) is the involute function of a pressure angle of the reference circle; for a pressure angle of 30°, the formula is shown as follows:

\[
\text{inv}(a_0) = \tan \left( \frac{\pi}{6} \right) - \frac{\pi}{6}.
\]

(24)

If \( \text{inv}(a_{j,1k}) \) is the involute function of the pressure angle corresponding to the upper surface of the \( j \text{th} \) segment, it can be expressed as follows:

\[
\text{inv}(a_{j,1k}) = \tan \left( a_{j,1k} \right) - a_{j,1k},
\]

\[
a_{j,1k} = \arccos \left( \frac{r_b}{R_{j,1k}} \right),
\]

(25)

where \( r_b \) is the radius of base circle and \( R_{j,1k} \) is the upper surface radius of the \( j \text{th} \) segment.
The average areas $A_{jk}$ of the upper and lower surfaces of the $j$th segment are shown by the following equation:

\[
A_{j1} = L(Y_{j,1} + Y_{j-1,1}) \\
A_{j2} = L(Y_{j,2} + Y_{j+1,2}),
\]

where $L$ is the tooth width.

For the external spline, the distance from the pitch circle to the upper surface of the $j$th segment can be expressed as follows:

\[
S_{j,1} = r - R_{j,1}.
\]

For the internal spline, the distance from the pitch circle to the lower surface of the $j$th segment is given by

\[
S_{j,2} = R_{j,2} - r.
\]

The moment of inertia of the $j$th segment $I_{jk}$ is given by

\[
I_{j,1} = \frac{1}{3} L (Y_{j,1}^3 + Y_{j-1,1}^3) \\
I_{j,2} = \frac{1}{3} L (Y_{j,2}^3 + Y_{j+1,2}^3).
\]

The elastic modulus is given by

\[
E_\mu = \frac{E}{1 - \mu^2}.
\]

For the involute spline meshing with side, only the tangential stiffness is considered. The deformation of the axial matrix and yield extrusion is ignored. Thus, after the bending and shear flexibility of the single tooth model of the internal and external splines being obtained, the total flexibility is obtained.
Figure 10: When the misalignment in the x direction is $2.73 \times 10^{-4}$ m: (a) relative displacement velocity and (b) the corresponding diagram of Poincare.

Figure 11: When the misalignment in the x direction is $2.83 \times 10^{-4}$ m: (a) relative displacement velocity and (b) the corresponding diagram of Poincare.

Figure 12: When the misalignment in the x direction is $3.05 \times 10^{-4}$ m: (a) relative displacement velocity and (b) the corresponding diagram of Poincare.
by summing them, and then its derivative is obtained. Finally, the stiffness of the single tooth of the internal and external splines is obtained.

The formula for calculating the bending flexibility is expressed by the following equation:

\[
\Delta_{b,k} = \frac{1}{E\mu} \left[ \frac{\cos^2 \varphi}{3} \sum_{j=1}^{n} h_k \left( h_k^2 + 3S_{j,k} h_k + 3S_{j,k}^2 \right) \right. \\
- Y_F \cos \varphi \sin \varphi \sum_{j=1}^{n} \frac{h_k}{I_{j,k}} (h_k + 2S_{j,k}) \\
+ Y_F^2 \sin^2 \varphi \sum_{j=1}^{n} \frac{h_k}{I_{j,k}} \left. \right]. 
\]

(31)

where \( \varphi \) is the contact angle of the meshing point, \( \varphi = \alpha_0 \), and \( Y_F \) is the thickness of the half tooth of the reference circle, \( Y_F = 0.25 \pi m \). The formula for calculating the shear flexibility is given by

\[
\Delta_{s,k} = \frac{1.2 \cos^2 \varphi}{G} \sum_{j} h_k A_{j,k}. 
\]

(32)

The total flexibility of a pair of teeth is given by

\[
D_T = D_{b,1} + D_{b,2} + D_{s,1} + D_{s,2}. 
\]

(33)

The total tangential stiffness of a pair of teeth is given by

\[
K_T = \frac{1}{D_T}. 
\]

(34)

4.2. Meshing Stiffness of the Spline Coupling with the Misalignment. However, the position of the meshing point (i.e., the position of the loading point) will be changed with the transverse vibration and the parallel misalignment, and thus, the meshing angle, the height of the microtooth, and the half-tooth thickness of the microtooth will also be changed, as well as the stiffness will inevitably change, as shown in Figure 7.

Therefore, the microsegment height changes into the following expression:

\[
h_k' = h_k + L_{ed}, 
\]

(35)

where \( L_{ed} \) is the increment of the root (or top) height of the external spline and the internal spline along the microsegment.
Figure 17: When the wear is $1.59 \times 10^{-5}$ m (the misalignment in the $x$ direction is $2.8 \times 10^{-4}$ m): (a) relative displacement velocity and (b) the corresponding diagram of Poincare.

Figure 18: When the wear is $3.975 \times 10^{-5}$ m (the misalignment in the $x$ direction is $2.8 \times 10^{-4}$ m): (a) relative displacement velocity and (b) the corresponding diagram of Poincare.

Figure 19: When the wear is $5.168 \times 10^{-5}$ m (the misalignment in the $x$ direction is $2.8 \times 10^{-4}$ m): (a) relative displacement velocity and (b) the corresponding diagram of Poincare.
direction after the change of the engagement point, which can be expressed as follows:

\[ L_{ed} = \Delta x \cos \theta_i + \Delta y \sin \theta_i. \]  

(36)

It is shown in Figure 8.

When considering the existence of the transverse vibration, for the external spline, the upper and lower surface radii of the \( j \)th segment can be expressed as follows:

\[ R'_{j,1} = R_{j,1} + L_{ed}, \]  

(37)

\[ R'_{j-1,1} = R_{j-1,1} + L_{ed}. \]  

(38)

For the internal spline, the upper and lower surface radii of the \( j \)th segment are given by

\[ R'_{j,2} = R_{j,2} + L_{ed}, \]  

(39)

\[ R'_{j+1,2} = R_{j+1,2} + L_{ed}. \]  

(40)

Accordingly, for external and internal splines, the distance from the pitch circle to the upper and lower surfaces of the \( j \)th segment is given by

\[ S'_{j,k} = r + L_{ed} - R'_{j,k}. \]  

(41)

By substituting Equations (35) and (37)–(42) into the corresponding calculation equations, respectively, the single tooth stiffness considering the transverse vibration can be obtained. Finally, the meshing force of the involute spline is obtained too.

5. Nonlinear Vibration Characteristics of the Involute Spline Coupling

In this work, the calculation process of the torsional stiffness \( k_T \) and the torsional damping and meshing damping of the spline shaft is adapted from a previous study [31]; and the equivalent diameter of internal and external splines is 30 mm for all; the equivalent length of external and internal splines is 81 and 82 mm, respectively; and the unilateral side clearance of splines is \( c_i = c_i' = 7.95 \times 10^{-5} \) m. The rest of the parameters are given in Table 1, the detailed analysis of the nonlinear vibration characteristics of the system under different working conditions is presented in the next sections.

5.1. The Influence of the Misalignment on the Nonlinear Vibration Characteristics of the System. In this work, the relative velocity of a single pair of teeth was analyzed, and it was assumed that there is no wear between the teeth.
rotational speed is 4000 r/min, and the parallel misalignment in the y direction is unchanged with the value of $1 \times 10^{-5}$ m, and then, the nonlinear vibration characteristics of the system are studied with the misalignment changing in the x direction. Figure 9 shows the bifurcation characteristics of the system as the misalignment changes. From the bifurcation diagram, it can be seen that when the misalignment in the x direction is between $2.6 \times 10^{-4}$ m and $2.82 \times 10^{-4}$ m, the system is in a single period, and Figure 10 shows the phase diagram and the Poincare section diagram with these misalignments of x. When the misalignment is $2.83 \times 10^{-4}$ m, the system is considered to be in a quasiperiodic state, as shown in Figure 11. After a short period of a single period state, when the misalignment is $3.05 \times 10^{-4}$ m, the system suddenly enters a multiperiod state, as shown in Figure 12, and there are some discrete points in the diagram of the Poincare section. When the misalignment increases to $3.14 \times 10^{-4}$ m, the system enters a short period of chaos, until the misalignment is $3.19 \times 10^{-4}$ m, and the system exits the chaos state and enters a single period. When the misalignment is $3.33 \times 10^{-4}$ m, the system returns back to the chaos state again and then returns to the single cycle motion, until the misalignment increases to $3.6 \times 10^{-4}$ m, and finally the system enters a long-term chaos, with the corresponding interval range of $3.6 \times 10^{-4}$ m to $4.18 \times 10^{-4}$ m. In other words, for the parameters used here, it is not found that the system exits from chaos. The motion state of the system is found to be very unstable in this region. For example, with some misalignment, the system returns to a single period or a quasiperiod state, but in the misalignment of $3.6 \times 10^{-4}$ m to $3.79 \times 10^{-4}$ m, the system alternates its state between the single period, multiperiod, and chaos. In the misalignment of $3.79 \times 10^{-4}$ m to $4.18 \times 10^{-4}$ m, the system changes its state between the quasiperiod state and the chaos state.

It can be concluded that with an increase in the parallel misalignment, the state of the system becomes more and more complex from the beginning of the single period and then changes its state between the single period, quasiperiod, multiperiod, and chaos state but finally changes its state between the quasiperiod and the chaos state. The number of times the system enters and exits the chaos increases obviously. Therefore, too much misalignment displays a significant impact on the motion state of the system, which makes the system very unstable and then causes damage.

5.2 The Effect of Wear on the Nonlinear Vibration Characteristics of the System with Different Misalignment

5.2.1 The Situation of the Uniform Wear. When the spline is worn, the backlash between the teeth changes, which affects the nonlinear vibration characteristics. Therefore, the nonlinear vibration characteristics with different $h_i$ are analyzed in this work. Here, the rotational speed is 4300 r/min, the misalignment in the y direction is $0.1 \times 10^{-4}$ m, the support damping in the x and y directions is 2 N s/m for all, and the torsional damping coefficient is 0.005.

At first, it was assumed that the wear amount of each tooth is equal. It means in Equation (12), $h_i = h_x = h_1 \cdots = h_n$, and then the influence of the uniform wear on the nonlinear vibration characteristics of the system was analyzed. As shown in Figure 13, when the misalignment in the x direction is $3.2 \times 10^{-4}$ m, the bifurcation diagram of the system is a straight line under this condition, which indicates that with an increase in the uniform wear of each tooth, the movement state of the system is a single cycle, and there are no obvious
changes. When the misalignment in the x direction increases to 3.4 × 10⁻⁴ m, the state of the system, as shown in Figure 14, shows no obvious changes as compared to Figure 13. However, as shown in Figure 15, when the misalignment in the x direction approaches some specific values (for example, the misalignment is 3.32 × 10⁻³ m), the system appears in chaos with an increase in the amount of wear. However, the increase in the wear amount of the teeth in the uniform wear shows no obvious effect on the system state with the working condition given above.

5.2.2. The Situation of the Uneven Wear. In fact, due to various factors such as the misalignment, mass eccentricity, and manufacturing error, the load on each spline tooth is not the same; in addition to the influence of the vibration, the wear of each spline tooth is not the same consequently. Therefore, it is necessary to analyze the nonlinear vibration characteristics of the involute spline with the nonuniform wear of each tooth. In order to simplify the analysis, it is assumed that the rest of the teeth were not worn; only the first pair of tooth was worn, which took 0.05 times of clearance with an increase in the bifurcation parameter; the wear of the first pair of tooth varied from 0 to 1.5 times of clearance, the rotational speed is 6000 r/min, the misalignment in the x direction is 0.1 × 10⁻⁴ m, the support damping in the x and y directions is 5N/s/m for all, and the torsional damping coefficient is 0.007. Then, when the misalignment in the x direction is 2.8 × 10⁻³ m, the bifurcation diagram of the system with an increase in the amount of wear is shown in Figure 16. It can be seen when the wear is in the range of 0–3.578 × 10⁻⁵ m (i.e., the wear is in the range of 0–0.45 times of the clearance), the system is in a single period state, as shown in Figure 17; when the wear approaches 3.975 × 10⁻⁵ m (i.e., the wear is 0.5 times of the clearance), the system appears in a multiple period bifurcation phenomenon, as shown in Figure 18. From the diagram of Poincare, it can be seen that there are three points on it, and the system is in a three-multiple period state; when the amount of wear is 4.77 × 10⁻⁵ m (0.6 times of the clearance), the system returns to the single period state; when the wear increases to 5.168 × 10⁻⁵ m (0.65 times of the side clearance), as shown in Figure 19, there are several discrete points on the Poincare section, and the system is in a multiperiod state; and when the wear continues to increase to 5.565 × 10⁻⁵ m (0.7 times the clearance) as shown in Figure 20, the system begins to enter chaos, and the corresponding Poincare section is scattered. Then, the chaos interval of the system lasts to 7.95 × 10⁻⁵ m (1 time of the clearance), which is mixed with a single period and a multiperiod state; when the wear amount increases to 8.348 × 10⁻⁵ m (1.05 times of the clearance), the system finally returns to the current periodic state and no longer changes.

Figure 21 shows the bifurcation diagram of the system with an increase in the amount of wear when the misalignment in the x direction increases to 3.434 × 10⁻⁴ m. From this figure, it can be seen that when the wear amount is 7.95 × 10⁻⁶ m (0.1 times of the clearance), the system appears in a two-period bifurcation; when the wear amount is 1.59 × 10⁻⁵ m (0.2 times of the clearance), the system is in a three-period state, as shown in Figure 22. Then, the bifurcation of the system increases gradually, and the state of the system changes from a four-period to a nine-period state. When the wear amount is 8.745 × 10⁻⁵ m (1.1 times of the...
clearance), the system enters chaos. Finally, after the bifurcation point \((9.142 \times 10^{-5})\) of the multiperiod state, the system is in the chaos state all the time.

When the misalignment in the \(x\) direction increases to \(3.7 \times 10^{-4}\) m, the bifurcation diagram of the system with an increase in the amount of wear is shown in Figure 23. As compared to Figures 16 and 21, it indicates that the system state of Figure 23 is obviously more complex. It can be seen that with an increase in the amount of wear, the system has experienced the state of the single period, quasi period,
chaos, multiperiod, quasiperiod, and multiperiod and chaos from the initial quasi-periodic state. Within a given range of bifurcation parameters (i.e., the amount of wear), the system finally stays in a chaotic state. It can be concluded that the system behavior is very complex under the condition of large misalignment; the complex movement form causes the fretting effect on the surface of teeth on spline to intensify, which significantly increases the fretting wear failure; and the increase in the amount of wear makes the system movement behavior more complex and thus forms a vicious cycle that is prone to safety accidents. And, when the wear is $4.77 \times 10^{-5}$ m as shown in Figure 24, the system is getting into quasiperiod.

5.3. The Effect of the Rotational Speed on the Nonlinear Vibration Characteristics of the System with Different Misalignment. When the misalignment in the y direction is $2.4 \times 10^{-4}$ m and in the x direction is $2.7 \times 10^{-4}$ m, Figure 25 shows the bifurcation situation of the relative speed of the 22nd teeth on the spline coupling with changes in the rotation speed. In Figure 25, the rotation speed increases from 4000 r/min to 8000 r/min with an increment of 50 r/min. Under this condition of the misalignment, the state of the system is dominated by a single period motion, but sometimes quasiperiod, double period, and chaos appear at some specific speeds. Figure 26 shows the diagram of the relative displacement velocity on the engagement line and the diagram of Poincare when the rotation speed of the system is 4300 r/min. It can be seen that, in this case, the diagram of the relative displacement velocity is a closed circle, and the corresponding diagram of Poincare is a single point. Obviously, it is a single period motion. With an increase in the rotation speed, the relative velocity on the engagement line shows an upward trend. When the rotation speed of the system is 4800 r/min, as shown in Figure 28, the system is in a quasiperiodic state. When the rotating speed continues to increase, the system returns to the single period state at first, and then the system enters into the chaotic state quickly when the rotating speed is $4800 \text{ r/min}$, as shown in Figure 28. When the rotating speed is $5600 \text{ r/min}$ as shown in Figure 29, the system enters into the quasiperiodic state again. Moreover, under other rotating speed conditions, the system is in a single period state.

Figure 30 shows the bifurcation situation of the relative speed of the 22nd teeth on the spline coupling with changes in the rotation speed when the misalignment in the y direction is $2.4 \times 10^{-4}$ m and in the x direction is $2.9 \times 10^{-4}$ m. From the figure, it can be seen that the system is mainly quasiperiodic (Figure 31 is one of them), and the relative vibration speed between spline pairs increases with an increase in the rotating speed. When the rotating speed is $4200 \text{ r/min}$, $4250 \text{ r/min}$, $4700 \text{ r/min}$, $4900 \text{ r/min}$, $5200 \text{ r/min}$, and $5650 \text{ r/min}$, the system is in a chaotic state for all, as shown in Figure 32 (due to the space limitation, only the diagram when the rotation speed of the system is $4200 \text{ r/min}$ is provided in this work).

When the misalignment in the y direction is $2.4 \times 10^{-4}$ m and in the x direction is $2.9 \times 10^{-4}$ m, Figure 33 shows the bifurcation situation of the relative speed of the 22nd teeth on the spline coupling with changes in the rotation speed. From the figure, it can be seen that the system is still dominated by a quasiperiodic state, but by comparing with
It can be seen that the number of times the system enters into and leaves from the chaotic state has increased obviously. At a specific speed, the system presents a single period state (such as at a speed of 5450 r/min). Therefore, from Figures 25, 30, and 33 it can be concluded that, as the speed continues to increase, the larger the misalignment is, and the more times the system enters or leaves the chaos state, which easily causes the system to be damaged due to large vibration displacement and speed after suddenly entering the chaos state in the acceleration and deceleration process.

### 6. Conclusion

In this work, a bending–torsion coupling nonlinear vibration model of the involute spline coupling with the misalignment was proposed, and a dynamic meshing stiffness function with multiteeth engagement was established. Then, the influence of different misalignment and different wear and rotation speeds with different misalignment on the nonlinear vibration characteristics of the involute aviation spline coupling was explored. From the above analysis, the following conclusions are drawn:

1. For the involute spline coupling without the fretting wear, with an increase in the misalignment, the system gradually becomes unstable, and the number of entering and leaving chaos increases, which easily damages the spline system. However, for the involute spline coupling system with the fretting wear, when
the fretting wear of each tooth is even, it shows a little effect on the behavior of the system

(2) And, when the fretting wear on each tooth increases evenly, the load on each tooth has not changed much, so the excitation generated by the system is not obvious. However, while only one tooth is worn, the increase in the fretting wear of this tooth induces a load difference between this tooth and the rest of the teeth, and then the spline coupling system exhibits larger internal excitation as well as produces complex vibration behavior. This is very detrimental to the stability of the system. Additional, the increase in the misalignment also complicates the vibration of the system under the same fretting wear condition

(3) Furthermore, the increase in the misalignment suddenly increases the number of the state changes as the rotating speed increases. As it is well known, for the mechanical system, the system inevitably experiences increase and decrease in the speed in the process of starting and stopping. When the amount of the misalignment is too large, the system frequently changes its state, so it is easy to induce the large vibration displacement and speed. It demonstrates serious security risks.

Abbreviations

- \( k_{T1} \): Torsional stiffness of external spline
- \( c_{T1} \): Torsional damping of external spline
- \( k_{T2} \): Torsional stiffness of internal spline
- \( c_{T2} \): Torsional damping of internal spline
- \( k_m \): Meshing stiffness
- \( c_{ni} \): Meshing damping
- \( c_j \): Working teeth profiles
- \( c_f \): Nonworking teeth profiles
- \( k_{p1} \): Supporting stiffness of external spline
- \( k_{p2} \): Supporting stiffness of internal spline
- \( c_{p1} \): Supporting damping of external splines
- \( c_{p2} \): Supporting damping of internal splines
- \( T_d \): Input torque
- \( T_L \): Loading torque
- \( x_1 \): Vibrational displacements along the x-axis of external spline
- \( y_1 \): Vibrational displacements along the y-axis of external spline
- \( \theta_1 \): Torsional displacement around the axis of the external spline
- \( x_2 \): Vibrational displacements along the x-axis of internal spline
- \( y_2 \): Vibrational displacements along the y-axis of internal spline.
- \( \theta_2 \): Torsional displacement around the axis of the internal spline.
- \( \theta_M \): Torsional displacements of the prime motor
- \( \theta_l \): Torsional displacements of the load
- \( r_b \): The radius of the base circle of internal and external splines (rad)
- \( \dot{\theta}_M \): Constant angular velocity for the prime mover
- \( u \): Nondimensionalizing parameters
- \( l \): Nondimensionalizing parameters
- \( F_{mx} \): Components of the dynamic force along the x
- \( F_{my} \): Components of the dynamic force along the y
- \( m_1 \): Mass of external spline
- \( m_2 \): Mass of internal spline
- \( I_M \): Moment of inertia of prime mover
- \( I_L \): Moment of inertia of load
- \( I_1 \): Moment of inertia of external spline
- \( I_2 \): Moment of inertia of internal spline
- \( g \): Gravitational acceleration
- \( \Delta \): Deflection
- \( L \): Tooth width
- \( T_d \): Input torque
- \( T_{\text{m}} \): Meshing torque
- \( T_L \): Load torque
- \( F_{n} \): Meshing force of a single pair of teeth along the meshing line
- \( g_2[\Delta n(t)] \): Meshing deformation function
- \( \Delta n_1(t) \): Equation of relative displacement
- \( h_j \): Wear depth of each tooth
- \( a_0 \): Pressure angle of the reference circle
- \( z \): Number of teeth
- \( i \): The tooth number
- \( \tau \): Time
- \( \theta_0 \): Half angle of the tooth thickness
- \( \theta_i \): Rotation angle of each tooth
- \( \phi \): Angle between the active tooth profile and the x-axis
- \( \omega_0 \): Angular velocity
- \( h_{j1} \): The tooth root height
- \( h_{j2} \): The tooth top height
- \( d_i \): The diameter of the reference circle of the external spline
- \( d_{1\text{min}} \): The small diameter of the external spline
- \( d_{2\text{max}} \): The large diameter of the internal spline,
- \( d_{2\text{max}} \): The large diameter of the internal spline.
- \( S \): The tooth thickness of the reference circle
- \( r \): The radius of the reference circle
- \( \text{inv}(a_0) \): The involute function of a pressure angle of the reference circle
- \( R_{j\text{k}} \): The upper surface radius of the jth segment
- \( a_{j\text{k}} \): Pressure angle between the upper surface and tooth profile of the jth segment
- \( A_{j\text{k}} \): The average areas
- \( q \): The contact angle of the meshing point
- \( h_{j} \): Tooth height of each segment
- \( Y_{j} \): The thickness of the half tooth of the reference circle
- \( l_{\text{sad}} \): The increment of the root (or top) height of the external spline
- \( I_{j\text{k}} \): Section moment of inertia of the jth segment
- \( E \): Elastic modulus
- \( E' \): Young’s modulus.
Data Availability
All data, models, and code generated or used during the study appear in the submitted article.

Conflicts of Interest
The authors declare that there are no conflicts of interest regarding the publication of this paper.

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