AN ANALYST’S TRAVELLING SALESMAN THEOREM FOR GENERAL SETS IN $\mathbb{R}^n$

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ABSTRACT. In his 1990 paper, Jones proved the following: given $E \subseteq \mathbb{R}^2$, there exists a curve $\Gamma$ such that $E \subseteq \Gamma$ and

$$\mathcal{H}^1(\Gamma) \sim \text{diam } E + \sum_{Q} \beta_E(3Q)^2 \ell(Q).$$

Here, $\beta_E(Q)$ measures how far $E$ deviates from a straight line inside $Q$. This was extended by Okikiolu to subsets of $\mathbb{R}^n$ and by Schul to subsets of a Hilbert space.

In 2018, Azzam and Schul introduced a variant of the Jones $\beta$-number. With this, they, and separately Villa, proved similar results for lower regular subsets of $\mathbb{R}^n$. In particular, Villa proved that, given $E \subseteq \mathbb{R}^n$ which is lower content regular, there exists a ‘nice’ $d$-dimensional surface $F$ such that $E \subseteq F$ and

$$\mathcal{H}^d(F) \sim \text{diam } (E)^d + \sum_{Q} \beta_E(3Q)^2 \ell(Q)^d.$$ 

In this context, a set $F$ is ‘nice’ if it satisfies a certain topological non-degeneracy condition, first introduced in a 2004 paper of David.

In this paper we drop the lower regularity condition and prove an analogous result for general $d$-dimensional subsets of $\mathbb{R}^n$. To do this, we introduce a new $d$-dimensional variant of the Jones $\beta$-number that is defined for any set in $\mathbb{R}^n$.

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1. Introduction

1.1. Background. The 1-dimensional Analyst’s Travelling Salesman Theorem was first proven by Peter Jones [Jon90] for subsets of $\mathbb{C}$, with the motivation of studying the boundedness of a certain class of singular integral operators. Roughly speaking, he proved that if $E$ is flat enough at most scales and locations, then there is a curve $\Gamma$ of finite length (with quantitative control on its length) such that $E \subseteq \Gamma$. Conversely, he proved that if $E$ is a curve of finite length, then there is quantitative control over how often $E$ can be non-flat. The Jones $\beta$-number is the quantity he introduced to measure flatness.

Define for $E, B \subseteq \mathbb{R}^n$, 

$$\beta_{E, \infty}^d(B) = \frac{1}{r_B} \inf_{L} \sup \{ \text{dist}(y, L) : y \in E \cap B \}$$

where $L$ ranges over $d$-planes in $\mathbb{R}^n$. Thus $\beta_{E, \infty}^d(B)r_B$ is the width of the smallest tube containing $E \cap B$.

Jones’ result was then extended by Okikiolu [Oki92] to subsets of $\mathbb{R}^n$ and further by Schul [Sch07b] to subsets of Hilbert spaces. We state the version for Euclidean spaces below.

**Theorem 1.1** (Jones: $\mathbb{R}^2$ [Jon90]; Okikiolu: $\mathbb{R}^n$ [Oki92]; Schul [Sch07b]). Let $n \geq 2$. There is a $C = C(n)$ such that the following holds. Let $E \subseteq \mathbb{R}^n$. Then there is a connected set $\Gamma \supseteq E$ such that

$$\mathcal{H}^1(\Gamma) \lesssim_n \text{diam } E + \sum_{Q \in \Delta} \beta_{E, \infty}^1(3Q)^2 \ell(Q),$$

where $\Delta$ denotes the collection of all dyadic cubes in $\mathbb{R}^n$. Conversely, if $\Gamma$ is connected and $\mathcal{H}^1(\Gamma) < \infty$, then

$$\text{diam } \Gamma + \sum_{Q \in \Delta} \beta_{\Gamma, \infty}^1(3Q)^2 \ell(Q) \lesssim_n \mathcal{H}^1(\Gamma).$$

As a corollary of Theorem 1.1, if $E \subseteq \mathbb{R}^n$ and the right hand side of (1.1) is finite, then there exists a curve $\Gamma \supseteq E$ such that

$$\mathcal{H}^d(\Gamma) \sim \text{diam } E + \sum_{Q \in \Delta} \beta_{E, \infty}^d(3Q)^2 \ell(Q).$$

Pajot [Paj96] proved an analogous result to the first half of Theorem 1.1 for 2-dimensional sets in $\mathbb{R}^n$. In this, he gave a sufficient condition in terms of the Jones $\beta$-number for when a set $E \subseteq \mathbb{R}^n$ can be contained in a surface $f(\mathbb{R}^2)$ for some smooth $f : \mathbb{R}^2 \to \mathbb{R}^n$. It is natural to ask whether a $d$-dimensional analogue of the above Jones’ theorem is true. That is:
(1) Given a set \( E \), can we find a ‘nice’ set \( F \) containing \( E \) such that
\[
\mathcal{H}^d(F) \lesssim \text{diam}(E)^d + \sum_{Q \in \Delta} \beta_{E,\infty}^d(3Q)^2 \ell(Q)^d?
\]
(2) Given that \( E \) is a ‘nice’ set, can we say that
\[
\text{diam}(E)^d + \sum_{Q \in \Delta} \beta_{E,\infty}^d(3Q)^2 \ell(Q)^d \lesssim \mathcal{H}^d(E)?
\]

The most natural candidate for a ‘nice’ set is a Lipschitz graph. However, in his PhD thesis, Fang \([\text{Fan90}]\) constructed a 3-dimensional Lipschitz graph whose \( \beta_{E,\infty}^3 \) sum was infinite. Thus, with the \( \beta \)-numbers as defined by Jones, a \( d \)-dimensional analogue of the second half of Theorem 1.1 was proven to be false. This issue was resolved by David and Semmes \([\text{DS91}]\) who introduced a new \( \beta \)-number and proved a Travelling Salesman Theorem for Ahlfors \( d \)-regular sets in \( \mathbb{R}^n \). A set \( E \subseteq \mathbb{R}^n \) is said to be \textit{Ahlfors \( d \)-regular} if there is \( A > 0 \) such that
\[
r^d/A \leq \mathcal{H}^d(E \cap B(x, r)) \leq Ar^d
\]
for all \( x \in E, \ r \in (0, \text{diam}E) \).

They defined their \( \beta \)-number as follows. For \( E \subseteq \mathbb{R}^n \) and \( B \) a ball, set
\[
\hat{\beta}_{E}^{d,p}(B) = \inf_L \left( \frac{1}{r_B^d} \int_B \left( \frac{\text{dist}(y,L)}{r_B} \right)^p d\mathcal{H}^d|_{E}(y) \right)^{\frac{1}{p}},
\]
where \( L \) ranges over all \( d \)-planes in \( \mathbb{R}^n \). Thus, \( \hat{\beta}_{E}^{d,p}(B) \) measures the \( L^p \)-average deviation of \( E \) from a plane. Let
\[
p(d) := \begin{cases} 
2d/(d-2), & d < 2 \\
\infty, & d \leq 2. 
\end{cases}
\]

Their result goes as follows:

\textbf{Theorem 1.2} (David, Semmes \([\text{DS91}]\)). \textit{Let \( E \subseteq \mathbb{R}^n \) be Ahlfors \( d \)-regular. The following are equivalent:}

1. The set \( E \) has big pieces of Lipschitz images, meaning, there are constant \( L, c > 0 \) such that for all \( x \in E \) and \( r \in (0, \text{diam}E) \), there is an \( L \)-Lipschitz map \( f : \mathbb{R}^d \to \mathbb{R}^n \) satisfying \( \mathcal{H}^d(f(\mathbb{R}^d) \cap B(x, r)) \geq cr^d \).
2. For \( 1 \leq p < p(d) \), \( \beta_{E}^{d,p}(x, r)^2 \frac{dr}{r} \) is a Carleson measure on \( E \times (0, \infty) \).

Recall that \( \sigma \) is a Carleson measure on \( E \cap (0, \infty) \) if \( \sigma(B(x,r) \times (0,r)) \lesssim r^d \).

\textbf{Remark 1.3.} An Ahlfors \( d \)-regular sets satisfying condition (1) of the above theorem is said to be \textit{uniformly rectifiable} (UR). This is one of many characterizations of UR sets.

More recently Azzam and Schul \([\text{AS18}]\) proved a Traveling Salesman Theorem for \( d \)-dimensional sets in \( \mathbb{R}^n \), under the weakened assumption of \( (c, d) \)-lower content regularity. A set \( E \subseteq \mathbb{R}^n \) is said to be \( (c, d) \)-\textit{lower content regular} in a ball \( B \) if for all \( x \in E \cap B \) and \( r \in (0, r_B) \),
\[
\mathcal{H}_{\infty}^d(E \cap B(x, r)) \geq cr^d.
\]

Here, \( \mathcal{H}_{\infty} \) denotes that Hausdorff content. See \([\text{Mat99}]\) for more details. Notice, under this relaxed condition, the measure \( \mathcal{H}_{\infty}^d|_E \) may not be locally finite. As a result, Azzam and Schul were required to introduce a new \( \beta \)-number. The \( \beta \)-number
they defined is analogous to that of David and Semmes but they instead ‘integrate’ with respect to Hausdorff content. For $E \subseteq \mathbb{R}^n$ and a ball $B$, they defined

$$\hat{\beta}^{d,p}_E(B) = \inf_L \left( \frac{1}{r_B^d} \int_0^1 \mathcal{H}_\infty^d(\{x \in E \cap B : \text{dist}(x, L) > tr_B\}) t^{p-1} \, dt \right)^{\frac{1}{p}},$$

where $L$ ranges over all $d$-planes in $\mathbb{R}^n$.

To state their results, we need some additional notation. For closed sets $E, F \subseteq \mathbb{R}^n$ and a set $B$, we define

$$d_B(E, F) = 2 \text{diam}(B) \max\left\{ \sup_{y \in E \cap B} \text{dist}(y, F), \sup_{y \in F \cap B} \text{dist}(y, E) \right\}.$$ 

In the case where $B = B(x, r)$, we may write $d_{x,r}(E, F)$ to denote the above quantity.

For $C_0 > 0$ and $\varepsilon > 0$, let

$$\text{BWGL}(C_0, \varepsilon) = \{Q \in \mathcal{D} : d_{C_0 B_Q}(E, P) \geq \varepsilon \text{ for all } d\text{-planes } P\}.$$ 

**Remark 1.4.** Above, BWGL stands for bi-lateral weak geometric lemma. David and Semmes [DS93] gave another characterization of UR sets in terms of BWGL. They showed that an Ahlfors $d$-regular set is UR if and only if for every $C_0 \geq 1$, there exists $\varepsilon > 0$ such that BWGL($C_0, \varepsilon$) satisfies a Carleson condition with constant depending on $\varepsilon$.

We now state the result from [AS18]. In fact, we state the reformulation presented in [AV19].

**Theorem 1.5** (Azzam, Schul [AS18]). Let $1 \leq d < n$ and $E \subseteq \mathbb{R}^n$ be a closed set. Suppose that $E$ is $(c, d)$-lower content regular and let $\mathcal{D}$ denote the Christ-David cubes for $E$. Let $C_0 > 1$. Then there is $\varepsilon > 0$ small enough so that the following holds. Let $1 \leq p < p(d)$. For $R \in \mathcal{D}$, let

$$\text{BWGL}(R) = \text{BWGL}(R, \varepsilon, C_0) = \sum_{Q \in \text{BWGL}(\varepsilon, C_0) : Q \subseteq R} \ell(Q)^d$$

and

$$\hat{\beta}_{E,A,p}(R) := \ell(R)^d + \sum_{Q \subseteq R} \hat{\beta}^{d,p}_E(AB_Q)^2 \ell(Q)^d.$$ 

Then, for $R \in \mathcal{D}$,

$$(1.2) \quad \mathcal{H}^d(R) + \text{BWGL}(R, \varepsilon, C_0) \sim_{A,n,c,p,C_0,\varepsilon} \hat{\beta}_{E,A,p}(R).$$

We should mention the work of Edelen, Naber and Valtorta [ENV16], who describe how well the size of a Radon measure $\mu$ can be bounded from above by the corresponding $\hat{\beta}^{d,p}_\mu$-number (these are defined analogously to $\hat{\beta}^{d,p}_E$ with the integral taking over $\mu$ instead of $\mathcal{H}^d$). We state a corollary of their results for Hausdorff measure and compare this to Theorem 1.5.

**Theorem 1.6.** Let $E \subseteq \mathbb{R}^n$. Set $\mu = \mathcal{H}^d|_E$ and assume

$$\int_0^2 \beta_{\mu,2}^2(x, r) \frac{dr}{r} \leq M \quad \text{for } \mu\text{-a.e } x \in \mathbb{B}.$$
Then $E$ is rectifiable and for every $x \in \mathcal{B}$ and $0 < r \leq 1$, we have
\[ \mathcal{H}^d(E \cap B(x, r)) \lesssim_n (1 + M)r^d. \]

As mentioned, this is just one corollary of much more general theorem for general Radon measures (see [ENV16, Theorem 1.3]). Both Theorem 1.5 and Theorem 1.6 do not require $E$ to be Ahlfors $d$-regular. Instead, Theorem 1.5 requires that $E$ must be lower content regular, whereas Theorem 1.6 requires the existence of a locally finite measure. Furthermore, Theorem 1.5 also provides lower bounds for Hausdorff measure.

Azzam and Villa further generalise Theorem 1.5 in [AV19]. Here, they introduce the notion of a quantitative property which is a way of splitting the surface cubes of a set $E$ into “good” and “bad” parts. They prove estimates of the form of Theorem 1.5, where BWGL($R$) is instead replaced with other quantitative properties which ‘guarantee uniform rectifiability’. Here, a quantitative property is said to guarantee uniform rectifiability if whenever the bad set of cubes is small (quantified by a Carleson packing condition) then $E$ is uniform rectifiability. The BWGL condition is an example of a quantitative property which guarantees uniform rectifiability. We direct the reader to [AV19] for a more precise description and more example of these quantitative properties. We state one of their results for the bilateral approximation uniformly by planes (BAUP) condition, which we explain below (this will be used later to prove the second main result of the paper).

**Theorem 1.7** (Azzam, Villa [AV19]). Let $E \subseteq \mathbb{R}^n$ be a $(c, d)$-lower content regular set with Christ-David cubes $\mathcal{D}$. Let
\[ \text{BAUP}(C_0, \varepsilon) = \{Q \in \mathcal{D} : d_{C_0 B_Q}(E, U) \geq \varepsilon, U \text{ is a union of } d\text{-plane}\}. \]

For $R \in \mathcal{D}$, define
\[ \text{BAUP}(R, C_0, \varepsilon) = \sum_{Q \subseteq R} \ell(Q)^d. \]

and
\[ \tilde{\beta}_E(R) = \ell(R)^d + \sum_{Q \subseteq R} \tilde{\beta}_E^2(3B_Q)^2 \ell(Q)^d. \]

Then, for all $R \in \mathcal{D}$, $C_0 > 1$, and $\varepsilon > 0$ small enough depending on $C_0$ and $c$,
\[ \mathcal{H}^d(R) + \text{BAUP}(R, C_0, \varepsilon) \sim \tilde{\beta}_E(R). \]

Notice, Theorem 1.5 is more concerned with establishing quantitative estimates of the form seen in Theorem 1.1, rather than studying what types of surfaces could mimic the role of finite length curves, as in the 1-dimensional case. Very recently, Villa [Vil19] proved a Travelling Salesman Theorem for lower content regular sets more closely resembling that of Jones’ original theorem. The ‘nice’ sets he used were a certain class of topological non-degenerate surfaces, first introduced by David [Dav04].

**Definition 1.8.** Let $0 < \alpha_0 < 1$. Consider a one parameter family of Lipschitz maps $\{\varphi_t\}_{0 \leq t \leq 1}$, defined on $\mathbb{R}^n$. We say $\{\varphi_t\}_{0 \leq t \leq 1}$ is an allowed Lipschitz deformation with parameters $\alpha_0$, or an $\alpha_0$-ALD, if it satisfies the following condition:

1. $\varphi_t \subseteq \mathcal{B}(x, r)$ for each $t \in [0, 1]$;
2. for each $y \in \mathbb{R}^n$, $t \mapsto \varphi_t(y)$ is a continuous function on $[0, 1]$;
(3) $\varphi_0(y) = y$ and $\varphi_t(y) = y$ for $t \in [0, 1]$ whenever $y \in \mathbb{R}^n \setminus B(x, r)$;
(4) $\text{dist}(\varphi_t(y), E) \leq \alpha_0 r$ for $t \in [0, 1]$ and $y \in E \cap B(x, r)$, where $0 < \alpha_0 < 1$.

**Definition 1.9.** Fix parameters $r_0$, $\alpha_0$, $\delta_0$ and $\eta_0$. We say $E \subseteq \mathbb{R}^n$ is a *topologically stable $d$-surface* with parameters $r_0$, $\alpha_0$, $\delta_0$ and $\eta_0$, if for all $\alpha_0$-ALD $\{\varphi\}$, and for all $x_0 \in E$ and $0 < r < r_0$, we have

$$\mathcal{H}^d(B(x, (1 - \eta_0)r) \cap \varphi_1(E)) \geq \delta_0 r^d.$$

Amongst other things, Villa proved the following.

**Theorem 1.10.** Let $E \subseteq \mathbb{R}^n$ be a $(c, d)$-lower content regular set and let $Q_0 \in \mathcal{D}$. Given two parameters $0 < \varepsilon, \kappa < 1$, there exists a set $\Sigma = \Sigma(\varepsilon, \kappa, Q_0)$ such that

1. $Q_0 \subseteq \Sigma$,
2. $\Sigma$ is topologically stable $d$-surface with parameters $r_0 = \text{diam}(Q_0)/2$, $0 < \eta_0 > 1/100$, and $\alpha_0$ and $\delta_0$ sufficiently small with respect to $\varepsilon$ and $\kappa$.
3. We have the estimate

$$\mathcal{H}^d(\Sigma) \sim c_0 n, d, \varepsilon \text{diam}(Q_0)^d + \sum_{Q \in \mathcal{D}} \beta_{d,p}^E(C_0 B_Q)^2 \ell(Q)^d$$

Before stating our main results, we mention that Travelling Salesman type problems have been considered in a variety of other setting outside of $\mathbb{R}^n$. For such results in the Heisenberg group see [FFP+07], [LS16b], [LS16a] and for general metric spaces see [Hahl05], [Hahl08], [Sch07a], [DS19].

1.2. **Main Results.** We prove a $d$-dimensional analogue of Theorem 1.1 for general sets in $\mathbb{R}^n$. In particular, we do not assume $E$ to be Ahlfors regular or lower content regular and we do not assume the existence of a locally finite measure on $E$ (as was the case in Theorem 1.2, Theorem 1.5 and Theorem 1.6, respectively). We should emphasize that while Theorem 1.5 and Theorem 1.6 concentrate on proving bounds for measures, our result differs in the sense that we construct a nice surface which contains our set, and the measure of this surface is controlled by our $\beta$-numbers.

Observe that if $E$ does not satisfy any lower regularity condition, it may be that $\mathcal{H}_Q^d(E) = 0$. Thus, $\hat{\beta}_E^{d,p}$ may trivially return a zero value even if there is some inherent non-flatness, for example if $E$ is a dense collection of point in some purely rectifiable set. We shall introduce a new $\beta$-numbers, $\beta_{d,p}^E$, to deal with this. We first define a variant of the Hausdorff content, where we ‘force’ sets to have some lower regularity with respect to this content, and define $\beta_{d,p}^E$ (analogously to Azzam and Schul) by integrating with respect to this new content.

**Definition 1.11.** Let $E \subseteq \mathbb{R}^n$, $B$ a ball and $0 < c_1 \leq c_2 < \infty$ be constants to be fixed later. We say a collection of balls $\mathcal{B}$ which covers $E \cap B$ is *good* if $r_B' \leq 100 r_B$ for every $B' \in \mathcal{B}$ and for all $x \in E \cap B$ and $0 < r < r_B$, we have

$$\sum_{B' \in \mathcal{B}} r_B'^d \geq c_1 r^d$$

and

$$\sum_{B' \in \mathcal{B}} r_B'^d \leq c_2 r^d.$$


Then, for $A \subseteq E \cap B$, define
\[
\mathcal{H}_{B,\infty}^{d,E}(A) = \inf \left\{ \sum_{B' \in \mathcal{B}} r_{B'}^d : \mathcal{B} \text{ is good for } E \cap B \right\}
\]

**Definition 1.12.** Let $1 \leq p < \infty$, $E \subseteq \mathbb{R}^n$, $B$ a ball centered on $E$ and $L$ a $d$-plane. Define
\[
\beta_{E}^{d,p}(B, L)^p = \frac{1}{r_B^d} \int_0^1 \left( \frac{\text{dist}(x, L)}{r_B} \right)^p d_{\mathcal{H}_{B,\infty}^{d,E}}
\]
\[
= \frac{1}{r_B^d} \int_0^1 \mathcal{H}_{B,\infty}^{d,E} \big( \{ x \in E \cap B : \text{dist}(x, L) > tr_B \} \big) t^{p-1} dt,
\]
and
\[
\beta_{E}^{d,p}(B) = \inf \{ \beta_{E}^{d,p}(B, L) : L \text{ is a } d\text{-plane} \}.
\]

In Section 2 we study the above definitions. We are more explicit about the constant $c_1, c_2$ appearing in Definition 1.11 and we shall prove some basic properties of $\mathcal{H}_{B,\infty}^{d,E}$ and $\beta_{E}^{d,p}$.

Our main results read as follows:

**Theorem 1.13.** Let $1 \leq d < n$, $C_0 > 1$ and $1 \leq p < p(d)$. There exists a constant $c_1 > 0$ such that the following holds. Suppose $E \subseteq F \subseteq \mathbb{R}^n$, where $F$ is $(c, d)$-lower content regular for some $c \geq c_1$. Let $\mathcal{D}^E$ and $\mathcal{D}^F$ be the Christ-David cubes for $E$ and $F$ respectively. Let $Q_0^E \in \mathcal{D}^E$ and let $Q_0^F$ be the cube in $\mathcal{D}^F$ with the same center and side length as $Q_0^E$. Then
\[
\text{diam}(Q_0^E)^d + \sum_{Q \in \mathcal{D}^E} \beta_{E}^{d,p}(C_0Q)^2 \ell(Q)^d
\]
\[
\lesssim_{C_0, c, n, p} \text{diam}(Q_0^F)^d + \sum_{Q \in \mathcal{D}^F} \beta_{F}^{d,p}(C_0Q)^2 \ell(Q)^d.
\]

**Theorem 1.14.** Let $1 \leq d < n$, $C_0 > 1$ and $1 \leq p < p(d)$. Let $E \subseteq \mathbb{R}^n$, $\mathcal{D}^E$ denote the Christ-David cubes for $E$ and let $Q_0^E \in \mathcal{D}^E$ be such that diam$(Q_0^E) \geq \lambda (Q_0^E)$ for some $0 < \lambda \leq 1$. Then there exists a $(c_1, d)$-lower content regular set $F$ (with $c_1$ as in the previous theorem) such that the following holds. Let $\mathcal{D}^F$ denote the Christ-David cubes for $F$ and let $Q_0^F$ denote the cube in $\mathcal{D}^F$ with the same center and side length as $Q_0^E$. Then
\[
\text{diam}(Q_0^F)^d + \sum_{Q \in \mathcal{D}^F} \beta_{F}^{d,p}(C_0Q)^2 \ell(Q)^d
\]
\[
\lesssim_{C_0, c, n, p, \lambda} \text{diam}(Q_0^F)^d + \sum_{Q \in \mathcal{D}^E} \beta_{E}^{d,p}(C_0Q)^2 \ell(Q)^d.
\]

**Remark 1.15.** The condition that diam$(Q_0^E) \geq \lambda (Q_0)$ is not too stringent. It simply ensures that the top cube is suitably sized for the set we are considering.
As an immediate corollary of Theorem 1.13 and Theorem 1.14, along with Theorem 1.10, we obtain a Travelling Salesman Theorem for general sets in \( \mathbb{R}^n \) resembling that of Jones original theorem.

**Corollary 1.16.** Let \( 1 \leq d \leq n \), \( 1 \leq p \leq p(d) \), and \( C_0 > 1 \). Suppose \( E \subseteq \mathbb{R}^n \) and \( Q_0 \in \mathcal{Q} \). Then there exists \( r_0, \alpha_0, \delta_0 \) and \( \eta_0 \) and a topologically stable \( d \)-surface \( \Sigma \), with parameters \( r_0, \alpha_0, \delta_0 \) and \( \eta_0 \) such that \( E \subseteq \Sigma \) and

\[
\mathcal{H}^d(\Sigma) \sim \text{diam}(Q_0)^d + \sum_{Q \in \mathcal{Q}, Q \subseteq Q_0} \beta^d_P(C_0 B_Q)^2 \ell(Q)^d.
\]

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2. **Preliminaries**

2.1. **Notation.** If there exists \( C > 0 \) such that \( a \leq Cb \), then we shall write \( a \lesssim b \). If the constant \( C \) depends on a parameter \( t \), we shall write \( a \lesssim_t b \). We shall write \( a \sim b \) if \( a \lesssim b \) and \( b \lesssim a \), similarly we define \( a \sim_i b \).

For \( A, B \subseteq \mathbb{R}^n \), let

\[
\text{dist}(A, B) = \inf\{|x - y| : x \in A, y \in B\}
\]

and

\[
\text{diam}(A) = \sup\{|x - y| : x, y \in A\}.
\]

We recall the \( d \)-dimensional Hausdorff measure and content. For \( A \subseteq \mathbb{R}^n \), \( d \geq 0 \) and \( 0 < \delta \leq \infty \) define

\[
(2.1) \quad \mathcal{H}^d_\delta(A) = \inf \left\{ \sum_i \text{diam}(A_i)^d : A \subseteq \bigcup_i A_i \text{ and } \text{diam } A_i \leq \delta \right\}.
\]

The \( d \)-dimensional Hausdorff content of \( A \) is defined to be \( \mathcal{H}^d_\infty(A) \) and the \( d \)-dimensional Hausdorff measure of \( A \) is defined to be

\[
\mathcal{H}^d(A) = \lim_{\delta \to 0} \mathcal{H}^d_\delta(A).
\]

2.2. **Chirst-David cubes.** For a set \( E \subseteq \mathbb{R}^n \) we shall need a version of “dyadic cubes”. These were first introduced by David [Dav88] and generalised in [Chr90] and [HM12].

**Lemma 2.1.** Let \( X \) be a doubling metric space and \( X_k \) be a sequence of maximal \( \rho^k \)-separated nets, where \( \rho = 1/1000 \) and let \( c_0 = 1/500 \). Then, for each \( n \in \mathbb{Z} \), there is a collection \( \mathcal{Q}_k \) of cubes such that the following hold.

1. For each \( k \in \mathbb{Z} \), \( X = \bigcup_{Q \in \mathcal{Q}_k} Q \).
2. If \( Q_1, Q_2 \in \mathcal{Q} \) and \( Q_1 \cap Q_2 \neq \emptyset \), then \( Q_1 \subseteq Q_2 \) or \( Q_2 \subseteq Q_1 \).
3. For \( Q \in \mathcal{Q} \), let \( k(Q) \) be the unique integer so that \( Q \in \mathcal{Q}_k \) and set \( \ell(Q) = 5\rho^k \). Then there is \( x_Q \in X_k \) such that

\[
B(x_Q, c_0 \ell(Q)) \subseteq Q \subseteq B(x_Q, \ell(Q)).
\]

Given a collection of cubes \( \mathcal{Q} \) and \( Q \in \mathcal{Q} \), define

\[
\mathcal{Q}(Q) = \{ R \in \mathcal{Q} : R \subseteq Q \}.
\]
Let $\text{Child}_k(Q)$ denote the $k^{th}$ generational descendants of $Q$ (where we often write $\text{Child}(Q)$ to mean $\text{Child}_1(Q)$) and $Q^{(k)}$ denote the $k^{th}$ generational ancestor. We shall denote the descendants up to the $k^{th}$ by $\text{Des}_k(Q)$, that is,

\begin{equation}
\text{Des}_k(Q) = \bigcup_{i=0}^{k} \text{Child}_i(Q).
\end{equation}

Finally define a distance function, $d_\mathcal{C}$, to a collection of cubes $\mathcal{C} \subseteq \mathcal{D}$ by setting

\[d_\mathcal{C}(x) = \inf \{ \ell(R) + \text{dist}(x, R) : R \in \mathcal{C} \},\]

and for $Q \in \mathcal{D}$, set

\[d_\mathcal{C}(Q) = \inf \{d_\mathcal{C}(x) : x \in Q\}.
\]

The following lemma is standard and can be found in, for example, [AS18].

**Lemma 2.2.** Let $\mathcal{C} \subseteq \mathcal{D}$ and $Q, Q' \in \mathcal{D}$. Then

\begin{equation}
d_\mathcal{C}(Q) \leq 2\ell(Q) + \text{dist}(Q, Q') + 2\ell(Q') + d_\mathcal{C}(Q').
\end{equation}

### 2.3. Theorem of David and Toro

The surface $F$ from Theorem 1.14 will be a union of surfaces constructed using the following Reifenberg parametrization theorem of David and Toro [DT12].

**Theorem 2.3** ([DT12, Sections 1 - 9]). Let $P_0$ a plane. Let $k \in \mathbb{N}$ and set $r_k = 10^{-k}$. Let $\{x_{j,k}\}_{j \in J_k}$ be an $r_k$-separated net. To each $x_{j,k}$, associate a ball $B_{j,k} = B(x_{j,k}, r_k)$ and a plane $P_{j,k}$ containing $x_{j,k}$. Assume

\[\{x_{j,0}\}_{j \in J_0} \subseteq P_0,
\]

and

\[x_{i,k} \in V_k^n,
\]

where $V_k^n := \bigcup_{j \in J_k} \lambda B_{j,k}$. Define

\[\varepsilon_k(x) = \sup \{d_{x_{j,k}10^{-r_j}(P_{j,k}, P_0)} : j \in J_k, |l - k| \leq 2, i \in J_k, x \in 100B_{j,k} \cap 100B_0\}.
\]

Then, there is $\varepsilon_0 > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$ and $\varepsilon_k(x_{j,k}) < \varepsilon$, for all $k \geq 0$ and $j \in J_k$,

then there is a bijection $f : \mathbb{R}^n \to \mathbb{R}^n$ such that:

1. We have

\[E_\infty := \bigcap_{K=1}^{\infty} \bigcup_{k=K}^{\infty} \{x_{j,k}\}_{j \in J_k} \subseteq \Sigma := f(\mathbb{R}^n).
\]

2. $f(x) = x$ when $\text{dist}(x, P_0) > 2$.

3. For $x, y \in \mathbb{R}^n$,

\[\frac{1}{4} |x - y|^{1+\tau} \leq |f(x) - f(y)| \leq 10|x - y|^{1-\tau}.
\]

4. $|f(x) - x| \leq \varepsilon$ for $x \in \mathbb{R}^n$.

5. For $x \in P_0$, $f(x) = \lim_k \sigma_k \circ \cdots \circ \sigma_0$, where

\[\sigma_k(y) = \psi_k(y) + \sum_{j \in J_k} \theta_j,k(y)[\pi_{j,k}(y) - y].
\]

Here, $\{x_{j,k}\}_{j \in L_k}$ is a maximal $\frac{7}{10}$-separated set in $\mathbb{R}^n \setminus V_k^n$,

\[B_{j,k} = B(x_{j,k}, r_k/10) \quad \text{for} \ j \in L_k.
\]
Remark 2.4. We conjecture the main results hold for subsets of an infinite dimensional Hilbert space. With this in mind, we have tried to make as much of our work here dimension free. We shall indicate the places where the estimates depend on the ambient dimension. As such, many of the volume arguments rely on the following result. Put simply, it states that a collection of disjoint balls lying close enough to a $d$-dimensional plane will satisfy a $d$-dimensional packing condition.
Lemma 2.5 ([ENV18, Lemma 3.1]). Let $V$ be an affine $d$-dimensional plane in a Banach space $X$, and $\{B(x_i, r_i)\}_{i \in I}$ be a family of pairwise disjoint balls with $r_i \leq R$, $B(x_i, r_i) \subseteq B(x, R)$, for some $x \in \mathbb{R}^n$ and $\operatorname{dist}(x_i, V) < r_i / 2$. Then, there is a constant $\kappa = \kappa(d)$ such that

\[
\sum_{i \in I} r_i^d \leq \kappa R^d. \tag{2.9}
\]

Lemma 2.6. Let $E \subseteq \mathbb{R}^n$, $Q \in \mathcal{Q}$ and $M \geq 1$. If $0 < \varepsilon \leq \frac{c_0 \rho}{2M}$ and

\[
\beta_{E, \infty}(MB_Q) \leq \varepsilon,
\]

then $Q$ has at most $K = K(M, d)$ children, i.e. independent of $n$.

Proof. The balls $\{c_0 B_R\}_{R \in \text{Child}(Q)}$ are pairwise disjoint (recall $c_0$ from Lemma 2.1), contained in $MB_Q$, and have radius less than or equal to $r_{MB_Q}$. Since

\[
\beta_{E, \infty}(MB_Q) \leq \varepsilon,
\]

there exists a $d$-plane $P_Q$ such that

\[
\operatorname{dist}(y, P_Q) \leq \varepsilon M \ell(Q)
\]

for all $y \in MB_Q$. In particular, for any $R \in \text{Child}(Q)$, we have

\[
\operatorname{dist}(x_R, P_Q) \leq \varepsilon M \ell(Q) \leq r_{c_0 B_R} / 2.
\]

By Lemma 2.5,

\[
\#\{R : R \in \text{Child}(Q)\} c_0 \rho^d \ell(Q)^d = \sum_{R \in \text{Child}(Q)} (c_0 \ell(R))^d \leq \kappa(M \ell(Q))^d,
\]

from which the lemma follows by dividing through by $c_0 \rho^d \ell(Q)^d$. \hfill \square

As a simple corollary of the above lemma, we also get a bound on the number of descendants up to a specified generation. The constant here ends up also depending on the generation.

Lemma 2.7. Let $E \subseteq \mathbb{R}^n$, $Q \in \mathcal{Q}$, $M \geq 1$ and $k \geq 0$. If $0 < \varepsilon < \frac{c_0 \rho}{2M}$ and

\[
\beta_{E, \infty}(MB_Q) \leq \varepsilon
\]

for all $R \in \text{Des}_k(Q)$, then

\[
\sum_{R \in \text{Des}_k(Q)} \ell(R)^d \lesssim_{d, M, k} \ell(Q)^d. \tag{2.10}
\]

2.4. Hausdorff-type content. In this section we study the Hausdorff content $\mathcal{H}_{B, \infty}^{d, E}$ that we defined in the introduction. For the convenience of the reader, we state the definition again.

Remark 2.8. Let us be explicit about the constant $c_1$ appearing in Theorem 1.13 and Theorem 1.14. We fix now

\[
c_1 := \frac{\omega_d}{2} \min\{4^{-d-1}, \rho / 6\}
\]

where $\rho$ is the constant appearing in Theorem 2.1 and $\omega_d$ is the volume of the unit ball in $\mathbb{R}^d$. We fix another constant

\[
c_2 := 18^d \kappa,
\]
with \( \kappa \) as in Lemma 2.5. We comment on this choice of constants in Remark 2.22. Basically, we have chosen \( c_1 \) sufficiently small and \( c_2 \) sufficiently large.

**Definition 2.9.** Let \( E \subseteq \mathbb{R}^n \), \( B \) a ball. We say a collection of balls \( \mathcal{B} \) which covers \( E \cap B \) is **good** if \( r_{B'} \leq 100r_B \) for every \( B' \in \mathcal{B} \) and for all \( x \in E \cap B \) and \( 0 < r < r_B \), we have

\[
\sum_{B' \in \mathcal{B}} r^d_{B'} \geq c_1 r^d \\
B' \cap B(x,r) \cap E \neq \emptyset
\]

and

\[
\sum_{B' \in \mathcal{B}} r^d_{B'} \leq c_2 r^d.
\]

Then, for \( A \subseteq E \cap B \), define

\[
\mathcal{H}^{d,E}_{B,\infty}(A) = \inf \left\{ \sum_{B' \in \mathcal{B}} r^d_{B'} : \mathcal{B} \text{ is good for } E \cap B \right\}
\]

See Figure 1 for an example of a good cover.

**Remark 2.10.** For the usual Hausdorff content, \( \mathcal{H}^d_{\infty} \), all coverings of a set are permissible (see (2.1)). In defining our new content, we restrict the permissible coverings to ensure all sets will have a lower regularity property with respect to this content (this is the role of (2.11)). In addition, we require an upper regularity condition, (2.12). This is to ensure any cover we choose is sensible. In particular, it stops us constructing lower regular covers by just repeatedly adding the same ball over and over again.

**Remark 2.11.** If \( E \subseteq \mathbb{R}^n \) and \( B \) is a ball, then \( \mathcal{B} = \{ B \} \) is a good cover for \( E \cap B \). In particular, every set has a good cover.

**Remark 2.12.** The constant 100 is not important, it can be chosen to be any large number.

**Remark 2.13.** It is easy to see that for any \( A \subseteq E \cap B \), we have \( \mathcal{H}^d_{\infty}(A) \leq \mathcal{H}^{d,E}_{B,\infty}(A) \).

We shall now prove some basic properties of \( \mathcal{H}^{d,E}_{B,\infty} \). Before doing so we need some preliminary lemmas. The first is [Mat99, Lemma 2.5] and the second is modification of [Mat99, Lemma 2.6], whose proof is essentially the same.

**Lemma 2.14.** Suppose \( a, b \in \mathbb{R}^2 \), \( 0 < |a| \leq |a - b| \) and \( 0 < |b| \leq |a - b| \). Then the angle between the vectors \( a \) and \( b \) is at least \( 60^\circ \), that is,

\[
|a|/|a - b|/|b| \geq 1.
\]

**Lemma 2.15.** There is \( N(n) \in \mathbb{N} \) with the following property. Let \( B \) be a ball and suppose there are \( k \) disjoint balls \( B_1, \ldots, B_k \) such that \( r_{B_i} \geq r_B \) and \( B \cap B_i \neq \emptyset \) for all \( i = 1, \ldots, k \). Then \( k \leq N(n) \).
Figure 1. Example of a good cover for $E \cap B$. Notice we need to cover the star segment with a large ball to ensure the cover is lower regular. We could equally have added many smaller balls as long as the upper regularity condition is not violated.

Proof. We may assume $B$ is centered at the origin. If one of the $B_i$ is centered at the origin then $k = 1$ so assume this is not the case. Let $B_i = B(x_i, r_i)$. Since $B_i \cap B_j \neq \emptyset$, we have

$$|x_i - x_j| > r_i + r_j > r_i + r_B,$$

and so

$$0 < |x_i| \leq r_B + r_i \leq |x_i - x_j| \quad \text{for } i \neq j.$$

Applying Lemma 2.14 with $a = x_i$ and $b = x_j$ for $i \neq j$ in the two dimensional plane containing $0, x_i, x_j$, we obtain

$$|x_i/|x_i| - x_j/|x_j|| \geq 1 \quad \text{for } i \neq j.$$

Since the unit sphere $S^{n-1}$ is compact there are at most $N(n)$ such points. \qed

Lemma 2.16. Let $E \subseteq \mathbb{R}^n$, and $B$ be a ball. Then,

1. $H_{B, \infty}^d(E \cap B(x, r)) \geq c_1 r^d$ for all $x \in E \cap B$, $0 < r \leq r_B$.
2. If $A_1 \subseteq A_2 \subseteq E \cap B$, then $H_{B, \infty}^d(A_1) \leq H_{B, \infty}^d(A_2)$.
3. If $B' \subseteq B$ and $A \subseteq E \cap B'$, then $H_{B', \infty}^d(A) \leq H_{B, \infty}^d(A)$.
4. Suppose $E \cap B = E_1 \cup E_2$. Then $H_{B, \infty}^d(E \cap B) \leq H_{B, \infty}^d(E_1) + H_{B, \infty}^d(E_2)$.
Proof. Property (1) is an immediate consequence of Definition 2.9 since any good cover \( \mathcal{B} \) of \( E \cap B \) satisfies (2.11). Property (2) is also clear from Definition 2.9. If \( B' \subseteq B \) then any good cover for \( B \) is also a good cover for \( B' \), and (3) follows.

To prove (4), let \( \varepsilon > 0 \) and suppose \( \mathcal{B}_i, \ i = 1, 2 \) are good covers of \( E \cap B \) such that

\[
(2.13) \quad \sum_{B' \in \mathcal{B}} r_{B'}^d \leq \mathcal{H}^d_{E, \infty}(E_i) + \varepsilon/2.
\]

Let \( \mathcal{B}'_1 \) be the collection of balls \( B' \in \mathcal{B}_1 \) such that \( B' \cap E_1 = \emptyset \). We partition \( \mathcal{B}'_1 \) into two further collections. Define

\[
\mathcal{B}'_{1,1} = \{ B' \in \mathcal{B}_1 : \text{there is } B'' \in \mathcal{B}_2 \text{ with } B' \cap B'' \cap E \neq \emptyset \text{ and } r_{B''} \geq r_{B'} \}
\]

and

\[
\mathcal{B}'_{1,2} = \{ B' \in \mathcal{B}_1 : r_{B''} < r_{B'} \text{ for all } B'' \in \mathcal{B}_2 \text{ such that } B' \cap B'' \cap E \neq \emptyset \}.
\]

For \( B' \in \mathcal{B}'_{1,1} \) let \( \tilde{B} \) be the ball in \( \mathcal{B}_2 \) such that \( \tilde{B} \cap B' \cap E \neq \emptyset \) and \( r_{\tilde{B}} \geq r_{B'} \). Since \( B' \cap E_1 = \emptyset \) and \( E \cap B = E_1 \cup E_2 \), it must be that \( B' \cap E \cap B \subseteq E_2 \), hence \( \tilde{B} \cap E_2 \neq \emptyset \). If there is more than one such ball we can choose \( \tilde{B} \) arbitrarily. Then

\[
(2.12) \quad \sum_{B' \in \mathcal{B}'_{1,1}} r_{B'}^d = \sum_{B'' \in \mathcal{B}_2} \sum_{B' \in \mathcal{B}'_{1,1}} \sum_{B' \cap E_2 \neq \emptyset} r_{B'}^d \leq \sum_{B'' \in \mathcal{B}_2} \sum_{B' \cap E_2 \neq \emptyset} r_{B'}^d = c_2 \sum_{B'' \in \mathcal{B}_2} r_{B''}^d.
\]

We turn our attention to \( \mathcal{B}'_{1,2} \). Let \( B_1 \) be the largest ball in \( \mathcal{B}'_{1,2} \). Then, given \( B_1, \ldots, B_k \), define \( B_{k+1} \) to be the largest ball \( B' \in \mathcal{B}'_{1,2} \) such that

\[
E \cap B \cap B' \cap \bigcup_{i=1}^{k} B_i = \emptyset.
\]

Let \( \{B_i\}_{i=1}^{\infty} \) be the resulting disjoint collection of balls. Any ball \( B' \in \mathcal{B}'_{1,2} \) such that \( E \cap B \cap B' \neq \emptyset \) is contained in 201B (recall from Definition 2.9 that each ball \( B' \) has radius at most 100r_B). So by compactness, for any \( R > 0 \) there at most a finite number of disjoint balls \( B' \in \mathcal{B}'_{1,2} \) such that \( r_{B'} \geq R \) and \( E \cap B \cap B' \neq \emptyset \). Hence, for any \( B' \in \mathcal{B}'_{1,2} \) there exists \( B_i \) such that \( B' \cap B_i \cap E \neq \emptyset \). Moreover, for any \( B_i \) such that \( B' \cap B_i \cap E \neq \emptyset \), we have \( r_{B'} \leq r_{B_i} \). Thus,

\[
(2.14) \quad \#\{B_i : B' \cap B_i \cap E \neq \emptyset\} \lesssim_{c_1} 1.
\]
As before, since \( B_1 \cap E_1 = \emptyset \), if \( B' \in \mathcal{B}_2 \) satisfies \( B' \cap B_i \cap E \cap B \neq \emptyset \) for some \( i \), then \( B' \cap E_2 \neq \emptyset \). Hence

\[
\sum_{B' \in \mathcal{B}_{1,2}} r_{B'}^d \leq c_2 \sum_{i=1}^{\infty} r_{B_i}^d \overset{(2.11)}{\leq} \frac{c_2}{c_1} \sum_{B' \in \mathcal{B}_2} \sum_{B' \cap B_i \cap E \cap B \neq \emptyset} r_{B'}^d \leq \frac{c_2}{c_1} \sum_{B' \in \mathcal{B}_2} \sum_{B' \cap E_2 \neq \emptyset} r_{B'}^d \overset{(2.14)}{\leq} \frac{c_2}{c_1} \sum_{B' \in \mathcal{B}_2} \sum_{B' \cap E_2 \neq \emptyset} r_{B'}^d.
\]

We concluding by noting that, since \( \mathcal{B}_1 \) is a good cover, we have

\[
\mathcal{H}_{B,\infty}^{d,E}(E \cap B) \leq \sum_{B' \in \mathcal{B}_2} r_{B'}^d = \sum_{B' \in \mathcal{B}_2} r_{B'}^d + \sum_{B' \in \mathcal{B}_1} r_{B'}^d \leq \sum_{B' \in \mathcal{B}_1} r_{B'}^d + \sum_{B' \in \mathcal{B}_2} r_{B'}^d \leq \mathcal{H}_{B,\infty}^{d,E}(E_1) + \mathcal{H}_{B,\infty}^{d,E}(E_2) + \varepsilon.
\]

Since \( \varepsilon \) was arbitrary, (4) follows. \( \square \)

For a function \( f : \mathbb{R}^n \to [0, \infty) \), define integration with respect to \( \mathcal{H}_{B,\infty}^{d,E} \) via the Choquet integral:

\[
\int f \, d\mathcal{H}_{B,\infty}^{d,E} := \int_0^\infty \mathcal{H}_{B,\infty}^{d,E}(\{x \in E \cap B : f(x) > t\}) \, dt.
\]

We state some basic properties of the above Choquet integral, see [Wan11] for more details. Note, in [Wan11] there are additional upper and lower continuity assumption, but these are not required for the following lemma.

**Lemma 2.17.** Let \( f, g : \mathbb{R}^n \to [0, \infty) \) such that \( f \leq g \) and \( \alpha > 0 \). Then

1. \( \int f \, d\mathcal{H}_{B,\infty}^{d,E} \leq \int g \, d\mathcal{H}_{B,\infty}^{d,E} \);
2. \( \int (f + \alpha) \, d\mathcal{H}_{B,\infty}^{d,E} = \int f \, d\mathcal{H}_{B,\infty}^{d,E} + \alpha \mathcal{H}_{B,\infty}^{d,E}(E \cap B) \);
3. \( \int \alpha f \, d\mathcal{H}_{B,\infty}^{d,E} = \alpha \int f \, d\mathcal{H}_{B,\infty}^{d,E} \).

The Choquet integral also satisfies a Jensen-type inequality. The proof is based on the proof of the usual Jensen’s inequality for general measures, see [Rud06].

**Lemma 2.18.** Suppose \( E, B \subseteq \mathbb{R}^n \), \( \phi : \mathbb{R} \to \mathbb{R} \) is convex and \( f : \mathbb{R}^n \to \mathbb{R} \) is bounded. Then,

\[
\phi \left( \frac{1}{\mathcal{H}_{B,\infty}^{d,E}(E \cap B)} \int f \, d\mathcal{H}_{B,\infty}^{d,E} \right) \leq \frac{1}{\mathcal{H}_{B,\infty}^{d,E}(E \cap B)} \int \phi \circ f \, d\mathcal{H}_{B,\infty}^{d,E}. \tag{2.15}
\]

**Proof.** Since \( f \) is bounded and \( \mathcal{H}_{B,\infty}^{d,E}(E \cap B) \geq r_{B'}^d \), we can set

\[
t = \frac{1}{\mathcal{H}_{B,\infty}^{d,E}(E \cap B)} \int f \, d\mathcal{H}_{B,\infty}^{d,E} < \infty.
\]
Since $\phi$ is convex, if $-\infty < s < t < u < \infty$, then
\begin{align}
\frac{\phi(t) - \phi(s)}{t - s} \leq \frac{\phi(u) - \phi(t)}{u - t}.
\end{align}
(2.16)

Let $\gamma$ be the supremum of the left hand side of (2.16) taken over all $s \in (-\infty, t)$. It is clear then that
\begin{align}
\phi(t) \leq \phi(s) + \gamma \cdot (t - s)
\end{align}
for all $s \in \mathbb{R}$. By rearranging the above inequality we have that for any $x \in \mathbb{R}^n$,
\begin{align}
\gamma f(x) + \phi(t) \leq \phi(f(x)) + \gamma t.
\end{align}

Integrating both with respect to $x$, and using Lemma 2.17, we have
\begin{align}
\gamma \int f \, d\mathcal{H}^{d,E}_{B,\infty} + \mathcal{H}^{d,E}_{B,\infty}(E \cap B) \phi(t) \leq \int \phi \circ f \, d\mathcal{H}^{d,E}_{B,\infty} + \gamma \mathcal{H}^{d,E}_{B,\infty}(E \cap B)t,
\end{align}
thus,
\begin{align}
\mathcal{H}^{d,E}_{B,\infty}(E \cap B) \phi(t) \leq \int \phi \circ f \, d\mathcal{H}^{d,E}_{B,\infty} + \gamma \left( \mathcal{H}^{d,E}_{B,\infty}(E \cap B)t - \int f \, d\mathcal{H}^{d,E}_{B,\infty} \right).
\end{align}

By definition of $t$, the second term on the right hand side of the above inequality is zero, from which the lemma follows.

Integration with respect to $\mathcal{H}^{d}_{\infty}$ can be defined similarly and satisfies identical properties. In the special case of integration with respect to $\mathcal{H}^{d}_{\infty}$ we also have the following:

**Lemma 2.19 ([AS18, Lemma 2.1]).** Let $0 < p < \infty$. Let $f_i$ be a countable collection of Borel functions in $\mathbb{R}^n$. If the sets $\text{supp} f_i = \{ f_i > 0 \}$ have bounded overlap, meaning there exists a $C < \infty$ such that
\begin{align}
\sum_1 \text{supp} f_i \leq C,
\end{align}
then
\begin{align}
\left( \int \left( \sum f_i \right)^p \, d\mathcal{H}^{d}_{\infty} \right) \leq C^p \sum \int f_i^p \, d\mathcal{H}^{d}_{\infty}.
\end{align}

2.5. **Preliminaries with $\beta$-numbers.** In this section we prove some basic properties of $\beta_{E}^{d,p}$. Again, we restate the definition for the readers convenience.

**Definition 2.20.** Let $1 \leq p < \infty$, $E \subseteq \mathbb{R}^n$, $B$ a ball centered on $E$ and $L$ a $d$-plane. Define
\begin{align}
\beta_{E}^{d,p}(B, L)^p = \frac{1}{r_B^p} \int \left( \frac{\text{dist}(x, L)}{r_B} \right)^p \, d\mathcal{H}^{d,E}_{B,\infty} = \frac{1}{r_B^p} \int_{0}^{r_B} \mathcal{H}^{d,E}_{B,\infty}(\{ x \in E \cap B : \text{dist}(x, L) > tr_B \}) t^{p-1} \, dt,
\end{align}
and
\begin{align}
\beta_{E}^{d,p}(B) = \inf \{ \beta_{E}^{d,p}(B, L) : L \text{ is a } d\text{-plane} \}.
\end{align}

**Remark 2.21.** It is easy to show that $\mathcal{H}^{d}_{\infty} \leq \mathcal{H}^{d,E}_{B,\infty}$ which implies $\beta_{E}^{d,p} \leq \beta_{E}^{d,p}$. If $c \geq c_1$ and $E$ is a $(c, d)$-lower regular set, we get the reverse inequality up to a constant, see Corollary 2.26.
Remark 2.22. It is possible to define a $\beta$-number $\beta_{E}^{d,p,c}$ where the additional parameter $c$ replaces the fixed lower regularity constant $c_2$ from (2.11). We could prove a version of Theorem 1.14 for $\beta_{d,p,c}$, that is, if $E \subseteq F$ and the sum of these $\beta_{d,p,c}$ coefficients for $E$ is finite, then we can find a $(c_1, d)$-lower content regular set $F$ such that $E \subseteq F$ (notice that the regularity constant for $F$ is independent of $c$). Thus to prove Corollary 1.16 we could just as well have used $\beta_{d,p,c}$ instead of $\beta_{d,p}$ for any $c \leq c_1$. We have fixed our lower regularity parameter for clarity.

Lemma 2.23. Let $1 \leq p < \infty$, $E \subseteq \mathbb{R}^n$ and $B$ a ball centred on $E$. Then

\begin{equation}
\beta_{E}^{d,1}(B) \lesssim \beta_{E}^{d,p}(B)
\end{equation}

Proof. This is a direct consequence of Lemma 2.18 and Definition 2.20. The inequality obviously holds for $p = 1$, so assume $p > 1$. Then the map $x \mapsto x^p$ is convex. Let $P$ be the $d$-plane such that $\beta_{E}^{d,p}(B) = \beta_{E}^{d,p}(B, P)$. Since $\text{dist}(x, P) \leq 2r_B$ for all $x \in E \cap B$ and $\mathcal{H}_{d,E}(E \cap B) \sim r_B^d$, we have, by Lemma 2.18,

\begin{align*}
\beta_{E}^{d,1}(B) &\leq \frac{1}{r_B^d} \int \left( \frac{\text{dist}(x, P)}{r_B} \right)^d \mathcal{H}_{d,E}(x, P) \lesssim \left( \frac{1}{r_B^d} \int \left( \frac{\text{dist}(x, P)}{r_B} \right)^p \mathcal{H}_{d,E}(x, P) \right)^{\frac{1}{p}} \\
&= \beta_{E}^{d,p}(B).
\end{align*}

Lemma 2.24. Let $1 \leq p < \infty$ and $E \subseteq \mathbb{R}^n$. Then for all balls $B' \subseteq B$ centered on $E$,

$\beta_{E}^{d,p}(B') \leq \left( \frac{r_B}{r_{B'}} \right)^{1+p} \beta_{E}^{d,p}(B)$.

Proof. Let $P$ be the $d$-plane such that $\beta_{E}^{d,p}(B) = \beta_{E}^{d,p}(B, P)$. By Lemma 2.16 (2),(3) and a change of variables, we have

\begin{align*}
\beta_{E}^{d,p}(B')^p &\leq \frac{1}{r_{B'}^d} \int_0^1 \mathcal{H}_{d,E}^{d,1}(\{x \in E \cap B' : \text{dist}(x, P) > tr_B \}) t^{p-1} dt \\
&\leq \frac{r_B^d}{r_{B'}^d} \frac{1}{r_{B'}^d} \int_0^1 \mathcal{H}_{d,E}^{d,1}(\{x \in E \cap B : \text{dist}(x, P) > tr_B \}) t^{p-1} dt \\
&\leq \left( \frac{r_B}{r_{B'}} \right)^{d+p} \frac{1}{r_{B'}^d} \int_0^1 \mathcal{H}_{d,E}^{d,1}(\{x \in E \cap B : \text{dist}(x, P) > tr_B \}) t^{p-1} dt \\
&= \beta_{E}^{d,p}(B')^p.
\end{align*}
Figure 2. \( \beta^{d,p} \) is not monotone.

Figure 2). Since \( F \) is lower 1-regular and we have comparability for \( \beta^{d,p} \) and \( \beta^{d,p} \) on lower regular sets (see Corollary 2.26), it is easy to show

\[
\beta_{F}^{1,1}(B, L) \lesssim \beta_{F}^{1,1}(2B, L) \sim \epsilon^2.
\]

On the other hand, by Lemma 2.16 (1), we must have

\[
\beta_{E}^{1,1}(B, L) = \int_0^1 \mathcal{H}^{1,1}_{B, \infty}(\{x \in E \cap B : \text{dist}(x, L) \geq t\}) \, dt
\]

\[
= \int_0^{\epsilon} \mathcal{H}^{1,1}_{B, \infty}(E \cap B) \, dt \quad (1)
\]

which for \( \epsilon \) small enough implies

\[
\beta_{F}^{1,1}(B, L) \leq \beta_{E}^{1,1}(B, L).
\]

We do however have the following result at least in the case where the larger set is lower regular. Roughly speaking it states that if \( E \subseteq F \) and \( F \) is lower regular, then we can control the \( \beta \)-number of \( E \) by the \( \beta \)-number of \( F \) with some error term dependent on the average distance of \( F \) from \( E \).

Lemma 2.25. Let \( F \subseteq \mathbb{R}^n \) be \((c, d)\)-lower content regular for some \( c \geq c_1 \) and let \( E \subseteq F \). Let \( B \) be a ball and \( L \) a \( d \)-dimensional plane. Then

\[
\beta_{E}^{d,p}(B, L) \lesssim_{c,d,p} \beta_{F}^{d,p}(2B, L) + \left( \frac{1}{r_B^d} \int_{F \cap 2B} \left( \frac{\text{dist}(x, E)}{r_B} \right)^p \, d\mathcal{H}_\infty^d(x) \right)^{\frac{1}{p}}.
\]
Proof. We actually prove a stronger version of the above statement, with \( B^d \) replaced by \( \beta^d \). Note, by re-scaling and translating, we may assume \( B = \mathbb{B} \). For \( t > 0 \) let

\[
E_t = \{ x \in E \cap \mathbb{B} : \text{dist}(x, L) > t \}.
\]

To prove the lemma, it suffices to show

\[
(2.18) \quad \mathcal{H}^d_{\infty}(E_t) \lesssim \mathcal{H}^d_{\infty}(\{ x \in F \cap 2\mathbb{B} : \text{dist}(x, L) > t/150 \}) + \mathcal{H}^d_{\infty}(\{ x \in F \cap 2\mathbb{B} : \text{dist}(x, E) > t/150 \})
\]

since if the above were true,

\[
\beta^d_E(\mathbb{B}, L)^p = \int_0^1 \mathcal{H}^d_{\infty}(E_t)^{t/p-1} dt 
\]

\[
\lesssim \int_0^1 \mathcal{H}^d_{\infty}(\{ x \in F \cap 2\mathbb{B} : \text{dist}(x, L) > t/150 \})^{t/p-1} dt 
+ \int_0^1 \mathcal{H}^d_{\infty}(\{ x \in F \cap 2\mathbb{B} : \text{dist}(x, L) > t/150 \})^{t/p-1} dt
\]

\[
\lesssim \int_0^1 \mathcal{H}^d_{\infty}(\{ x \in F \cap 2\mathbb{B} : \text{dist}(x, L) > 2t \})^{t/p-1} dt 
+ \int_0^1 \mathcal{H}^d_{\infty}(\{ x \in F \cap 2\mathbb{B} : \text{dist}(x, L) > 2t \})^{t/p-1} dt
\]

\[
\sim \beta^d_E(2\mathbb{B}, L)^p + \int_{F \cap 2\mathbb{B}} \text{dist}(x, E)^p d\mathcal{H}_d(x).
\]

For the rest of the proof we focus on (2.18). Fix \( t > 0 \). We must first construct a suitable good cover for \( E \cap \mathbb{B} \). For \( x \in F \cap \mathbb{B} \), let

\[
(2.19) \quad \delta(x) = \max \{ \text{dist}(x, L), \text{dist}(x, E) \} + t/120
\]

and let \( X = \{ x_i \}_{i \in I} \) be a maximal net in \( F \cap \mathbb{B} \) such that

\[
(2.20) \quad |x_i - x_j| \geq 4 \max \{ \delta(x_i), \delta(x_j) \}
\]

for all \( i \neq j \). For each \( i \in I \) let \( B'_i = B(x_i, 4\delta(x_i)) \) and \( \mathscr{B}' = \{ B'_i \}_{i \in I} \). By (2.19), \( \delta(x) > 0 \) for all \( x \in F \cap \mathbb{B} \), so the balls \( B'_i \) are non-degenerate. Furthermore, define

\[
\mathscr{B} = \{ 3B'_i \}_{i \in I} = \{ B_i \}_{i \in I}.
\]

Claim: \( \mathscr{B} \) is a good cover for \( E \cap \mathbb{B} \).

Since \( E \subseteq F \), it follows that \( \mathscr{B} \) covers \( E \) by maximality. We are left to verify (2.11) and (2.12). Let \( x \in E \cap \mathbb{B} \) and \( 0 < r < 1 \). We look first at (2.11).

Assume \( B'_i \cap B(x, r/3) \cap F \neq \emptyset \) for some \( i \in I \). If \( 4\delta(x_i) \geq r/3 \) then \( B(x, r/3) \subseteq 3B'_i \), hence \( 3B'_i \cap B(x, r/3) \cap E \neq \emptyset \). If \( 4\delta(x_i) < r/3 \), then \( B'_i \subseteq B(x, r) \) and since \( \text{dist}(x_i, E) \leq \delta(x_i) \) there exists some \( y \in E \cap B(x, r) \cap B'_i \), in particular, \( B'_i \cap B(x, r) \cap E \neq \emptyset \). In either case, we conclude \( E \cap B(x, r) \cap 3B'_i \neq \emptyset \). Then, since \( F \) is \( (c, d) \)-lower content regular in \( \mathbb{B} \), we have

\[
\sum_{B \in \mathscr{B}} \left( \frac{r_B}{3} \right)^d \sum_{B' \in \mathscr{B}'} \frac{r_B^d}{3} \geq c \left( \frac{r}{3} \right)^d,
\]

hence \( \mathscr{B} \) satisfies the lower bound (2.11).
Now for the upper bound (2.12). If \( B \in \mathcal{B} \) satisfies \( B \cap B(x, r) \cap E \neq \emptyset \) and \( r_B \leq r \), then \( B \subseteq B(x, 3r) \). Furthermore, since the balls \( \{ \frac{1}{3} B \}_{B \in \mathcal{B}} \) are disjoint and satisfy \( \text{dist}(x, B) \leq \frac{r}{2} \), we have by Lemma 2.5

\[
\sum_{B \in \mathcal{B}} r_B^d \leq 6^d \sum_{B \in \mathcal{B}} r_B^d \leq 18^d \kappa r^d \tag{2.9}
\]

Since \( c \geq c_1 \) and, recalling that \( c_2 = 18^d \kappa \), it follows that \( \mathcal{B} \) satisfies (2.12), thus, \( \mathcal{B} \) is a good cover for \( E \cap \mathcal{B} \) which proves the claim.

We partition the balls in \( \mathcal{B} \) as follows, let

\[
\mathcal{B}_E = \{ B_i \in \mathcal{B} : r_i = 12 \text{dist}(x_i, E) + t/10 \},
\]

\[
\mathcal{B}_L = \{ B_i \in \mathcal{B} : r_i = 12 \text{dist}(x_i, L) + t/10 \}.
\]

If \( \text{dist}(x_i, E) = \text{dist}(x_i, L) \) then we put \( B_i \) in \( \mathcal{B}_E \) or \( \mathcal{B}_L \) arbitrarily. Then, since \( \mathcal{B} \) is good for \( E \cap \mathcal{B} \), we have

\[
\mathcal{H}^d_{E, \infty}(E_i) \leq \sum_{B \in \mathcal{B}} r_B^d = \sum_{B \in \mathcal{B}_E} r_B^d + \sum_{B \in \mathcal{B}_L} r_B^d. \tag{2.21}
\]

If we can show that

\[
\sum_{B \in \mathcal{B}_E \cap E_i \neq \emptyset} r_B^d \leq \mathcal{H}^d_{\infty}\left( \{ x \in F \cap 2 \mathcal{B} : \text{dist}(x, E) > t/150 \} \right) \tag{2.22}
\]

and

\[
\sum_{B \in \mathcal{B}_L \cap E_i \neq \emptyset} r_B^d \leq \mathcal{H}^d_{\infty}\left( \{ x \in F \cap 2 \mathcal{B} : \text{dist}(x, L) > t/150 \} \right), \tag{2.23}
\]

then (2.18) follows from the above two inequalities and (2.21). We first prove (2.22). The proof of (2.23) is similar and we shall comment on the necessary changes after we are done with (2.22).

Let

\[
A := \{ x \in F \cap 2 \mathcal{B} : \text{dist}(x, E) > t/150 \}
\]

and let \( \mathcal{B}_A \) be a cover of \( A \) such that each ball \( B \in \mathcal{B}_A \) is centered on \( A \), has \( r_B \leq r_B \geq r_B \) and

\[
\mathcal{H}^d_{\infty}(A) \sim \sum_{B \in \mathcal{B}_A} r_B^d. \tag{2.24}
\]

Let \( B_i \in \mathcal{B}_E \) satisfy \( E_i \cap B_i \neq \emptyset \) and \( y_i \in E_i \cap B_i \). Recall that since \( B_i \in \mathcal{B}_E \) we have \( \text{dist}(x_i, L) \leq \text{dist}(x_i, E) \) and \( B_i = B(x_i, 12 \text{dist}(x_i, E) + t/10) \). It follows that

\[
t < \text{dist}(y_i, L) \leq |y_i - x_i| + \text{dist}(x_i, E) \\
\leq 12 \text{dist}(x_i, E) + t/10 + \text{dist}(x_i, E) \\
\leq 13 \text{dist}(x_i, E) + t/2.
\]

Rearranging, we find that

\[
\text{dist}(x_i, E) > t/26. \tag{2.25}
\]
This implies that
\[(2.26) \quad F \cap \frac{1}{24} B_i \subseteq A,\]
since for any \( y \in F \cap \frac{1}{24} B_i \) we have
\[
\text{dist}(y, E) \geq \text{dist}(x_i, E) - \frac{1}{24} r_{B_i} = \frac{1}{2} \text{dist}(x_i, E) - \frac{1}{240} t \geq \frac{t}{150}.
\]
By (2.26), since \( \mathcal{B}_A \) covers \( A \), there is \( B \in \mathcal{B}_A \) such that \( \frac{1}{24} B_i \cap B \neq \emptyset \). We partition \( \mathcal{B}_E \) further by setting
\[
\mathcal{C}_1 = \{ B_i : \text{there exists } B \in \mathcal{B}_A \text{ such that } \frac{1}{24} B_i \cap B \neq \emptyset \text{ and } r_B \geq r_{B_i}/24 \},
\]
\[
\mathcal{C}_2 = \mathcal{B}_E \setminus \mathcal{C}_1.
\]
See Figure 3. We first control the sum over balls in \( \mathcal{C}_1 \). Assume \( B \in \mathcal{B}_A \) is such that there exists some \( B_i \in \mathcal{C}_1 \) such that \( \frac{1}{24} B_i \cap B \neq \emptyset \) and \( r_B \geq r_{B_i}/24 \). For \( y \in B_i \), we have
\[
|y - x_B| \leq |y - x_i| + |x_i - x_B| \leq r_{B_i} + r_B + r_{B_i}/24 \leq 26 r_B,
\]
from which we conclude $B_i \subseteq 26B$. So, if
\[
C^B_i = \{ B_i \in C_1 : \frac{1}{24} B_i \cap B \neq \emptyset \text{ and } r_B \geq r_{B_i}/24 \},
\]
the balls $\{ \frac{1}{6} B_i \} \subset C^B_i$ are pairwise disjoint, contained in $26B$ and satisfy
\[
\text{dist}(x_i, L) \leq r_{B_i}/2
\]
(recall that $B_i = B(x_i, 12\delta(x_i))$). By Lemma 2.5 and because $\mathcal{B}$ is a good cover, this gives
\[
\sum_{B_i \in C_i} r_{B_i}^d \leq \sum_{B_i \in C_i} r_{B_i}^d \overset{(2.29)}{\lesssim} r_B^d.
\]
Thus,
\[
(2.27) \quad \sum_{B_i \in C_i} r_{B_i}^d \leq \sum_{B \in \mathcal{B}_A} \sum_{B_i \in C_i} r_{B_i}^d \overset{(2.24)}{\lesssim} \mathcal{H}_\infty^d(A).
\]
Now, the sum over balls in $C_2$. If $B_i \in C_2$ and $B \in \mathcal{B}_A$ is such that $\frac{1}{24} B_i \cap B \neq \emptyset$, then $r_B < r_{B_i}/24$. Furthermore $B \cap \frac{1}{24} B_j = \emptyset$ for all $B_j \in C_2$, $j \neq i$, since otherwise
\[
|x_i - x_j| \leq |x_i - x_B| + |x_B - x_j| \leq r_{B_i}/24 + 2r_B + r_{B_i}/24
\]
\[
\leq \left( \frac{1}{24} + \frac{1}{12} + \frac{1}{24} \right) \max\{ r_{B_i}, r_B \} \leq \frac{1}{6} \max\{ r_{B_i}, r_B \}
\]
\[
\leq 4 \max\{ \delta(x_i), \delta(x_j) \},
\]
contradicting (2.20). Thus,
\[
(2.28) \quad \# \{ B_i \in C_2 : B \cap \frac{1}{24} B_i \neq \emptyset \} \leq 1.
\]
Since $\mathcal{B}_A$ forms a cover for $A$ and $F \cap \frac{1}{24} B_i \subseteq A$ by (2.26), it follows that $\{ B \in \mathcal{B}_A : B \cap \frac{1}{24} B_i \neq \emptyset \}$ forms a cover for $F \cap \frac{1}{24} B_i$. Using then that $F$ is $(c, d)$-lower content regular, we have
\[
(2.29) \quad \sum_{B_i \in C_2} r_{B_i}^d \lesssim \mathcal{H}_\infty^d(F \cap \frac{1}{24} B_i) \leq \sum_{B_i \in C_2} \sum_{B \in \mathcal{B}_A} r_{B}^d
\]
\[= \sum_{B \in \mathcal{B}_A} \sum_{B_i \in C_2} \sum_{B \cap \frac{1}{24} B_i \neq \emptyset} r_{B}^d \overset{(2.28)}{\leq} \sum_{B \in \mathcal{B}_A} r_{B}^d \lesssim \mathcal{H}_\infty^d(A).
\]
Combining (2.27) and (2.29) completes the proof of (2.22). The proof of (2.23) follows exactly the same reasoning: For each $B_i \in \mathcal{B}_L$ we have
\[
\text{dist}(x_i, L) > t/26,
\]
the proof of which is the same as (2.25). Analogously to (2.26), this implies
\[
F \cap \frac{1}{24} B_i \subseteq A',
\]
where
\[
A' = \{ x \in F \cap 2\mathcal{B} : \text{dist}(x, L) > t/150 \}.
\]
The rest of the proof is identical. This completes the proof of (2.18) which in turn completes the proof of the lemma.
As a corollary of the above proof of Lemma 2.25 and Remark 2.21, we have the following:

**Corollary 2.26.** Suppose \( c \geq c_1 \) and \( E \subseteq \mathbb{R}^n \) is \((c,d)\)-lower content regular. For any ball \( B \) and \( d \)-plane \( L \), we have

\[
\bar{\beta}^{d,p}_E(B, L) \leq \beta^{d,p}_E(B, L) \lesssim \beta^{d,p}_E(2B, L).
\]

**Lemma 2.27.** Assume \( E \subseteq \mathbb{R}^n \) and there is \( B \) centered on \( E \) so that for all \( B' \subseteq B \) centered on \( E \) we have \( \mathcal{H}^d_\infty(E \cap B') \geq c r_{B'}^d \). Then

\[
\beta^{d,\infty}_{E,\infty}\left(\frac{1}{2}B\right) \lesssim \beta^{d,1}_{E,1}(B)^{\frac{1}{d+1}}. \tag{2.30}
\]

**Proof.** Azzam and Schul prove the same inequality for \( \bar{\beta}^{d,p}_E \) (see [AS18, Lemma 2.12]). Then, by Corollary 2.26,

\[
\beta^{d,\infty}_{E,\infty}\left(\frac{1}{2}B\right) \lesssim \beta^{d,1}_{E,1}(B)^{\frac{1}{d+1}} \lesssim \beta^{d,1}_{E,1}(B)^{\frac{1}{d+1}}.
\]

\[\square\]

**Remark 2.28.** By (2.30) and Lemma 2.6, if \( \varepsilon > 0 \) is small enough (depending on \( M \)) and \( \beta^{d,p}_E(MB_Q) \leq \varepsilon \) for some \( Q \in \mathcal{D} \), then \( Q \) has at most \( K \) children where \( K \) depends only on \( M \) and \( d \) and not the ambient dimension \( n \).

The following is analogous to Lemma 2.21 in [AS18]. It says the \( \beta \)-number of a lower regular set can be controlled by the \( \beta \)-number of a nearby set, with an error depending on the average distance between the two. The proof is very similar to the proof of Lemma 2.25.

**Lemma 2.29.** Let \( 1 \leq p < \infty \). Suppose \( E,F \subseteq \mathbb{R}^n \), \( B^1 \) is a ball centered on \( E \) and \( B^2 \) is a ball of same radius but centered on \( F \) such that \( B^1 \subseteq 2B^2 \). Suppose for all balls \( B \subseteq 2B^1 \) centered on \( E \) we have \( \mathcal{H}^d_\infty(B \cap E) \geq c r_B^d \) for some \( c > 0 \). Then

\[
(2.31) \quad \bar{\beta}^{d,p}_E(B^1, P) \lesssim_{c,p,d} \beta^{d,p}_F(2B^2, P)
\]

\[+ \left( \frac{1}{r_{B^1}^d} \int_{E \cap 2B^1} \left( \frac{\operatorname{dist}(y,F)}{r_{B^1}} \right)^p d\mathcal{H}^d_\infty(y) \right)^{\frac{1}{p}}. \]

**Proof.** By scaling, we can assume that \( B^1 = \mathbb{B} \). For \( t > 0 \), set

\[ E_t = \{ x \in E \cap \mathbb{B} : \operatorname{dist}(x,L) > t \}. \]

To prove (2.31), it suffices to show

\[
(2.32) \quad \mathcal{H}^d_\infty(E_t) \lesssim \mathcal{H}^{d,F}_{28,\infty}(\{ x \in F \cap 2\mathbb{B} : \operatorname{dist}(x,L) > tr_B/2 \})
\]

\[+ \mathcal{H}^{d}_{\infty}(\{ x \in E \cap 2\mathbb{B} : \operatorname{dist}(x,F) > tr_B/32 \}), \]
which gives
\begin{align*}
\beta_{E}^{d,p}(B, L)^p &= \int_0^1 \mathcal{H}^d_{\infty} (E_t) t^{p-1} dt \\
&\lesssim \int_0^1 \mathcal{H}^{d,F}_{2B, \infty}(\{x \in F \cap 2B : \text{dist}(x, L) > t/2\}) t^{p-1} dt \\
&\quad + \int_0^1 \mathcal{H}^d_{\infty}(\{x \in E \cap 2B : \text{dist}(x, F) > t/32\}) t^{p-1} dt \\
\lesssim &\int_0^1 \mathcal{H}^{d,F}_{2B, \infty}(\{x \in F \cap 2B : \text{dist}(x, L) > 2t\}) t^{p-1} dt \\
&\quad + \int_0^1 \mathcal{H}^d_{\infty}(\{x \in E \cap 2B : \text{dist}(x, F) > 2t\}) t^{p-1} dt \\
\sim &\beta_{E}^{d,p}(2B, L)^p + \int_{E \cap 2B} \text{dist}(x, F)^p \, d\mathcal{H}^d_{\infty}(x).
\end{align*}

So, let us prove (2.32). We first need to construct a suitable cover for $E_t$. For $x \in E_t$, let
\[ \delta(x) = \max\{\text{dist}(x, L), 16\text{dist}(x, F)\} \]
and set $X_t$ to be a maximally separated net in $E_t$ such that, for $x, y \in X_t$, we have
\[ |x - y| \geq 4 \max\{\delta(x), \delta(y)\}. \]

Enumerate $X_t = \{x_i\}_{i \in I}$. For $x_i \in X_t$ denote $B_i = B(x_i, \delta(x_i))$. Notice, these balls are non-degenerate since $x_i \in X_t \subset E_t$. By maximality we know $\{4B_i\}$ covers $E_t$ so,
\[ \mathcal{H}^d_{\infty}(E_t) \lesssim \sum_{i \in I} (r_{4B_i})^d. \]

We partition $I = I_1 \cup I_2$, where
\[ I_1 = \{i \in I : \delta(x_i) = \text{dist}(x_i, L)\} \]
and
\[ I_2 = \{i \in I : \delta(x_i) = 16\text{dist}(x_i, F)\}. \]

If $\text{dist}(x_i, L) = 16\text{dist}(x_i, F)$, we put $i$ in $I_1$ or $I_2$ arbitrarily. By (2.34), it follows that
\[ \mathcal{H}^d_{\infty}(E_t) \lesssim \sum_{i \in I_1} (r_{4B_i})^d + \sum_{i \in I_2} (r_{4B_i})^d. \]

We will show that
\[ \sum_{i \in I_1} (r_{4B_i})^d \lesssim \mathcal{H}^{d,E}_{2B, \infty}(\{x \in F \cap 2B : \text{dist}(x, L) > t/2\}) \]
and
\[ \sum_{i \in I_2} (r_{4B_i})^d \lesssim \mathcal{H}^d_{\infty}(\{x \in E \cap 2B : \text{dist}(x, F) > t/32\}) \]
from which (2.32) follows. The rest of the proof is dedicated to proving (2.35) and (2.36). Let us begin with (2.35). Let
\[ F_t = \{x \in F \cap 2B : \text{dist}(x, L) > t/2\} \]
and $\mathcal{B}$ be a good cover for $F \cap 2\mathcal{B}$ such that
\begin{equation}
\mathcal{H}^{d,F}_{2\mathcal{B},\infty}(F_t) \sim \sum_{\mathcal{B} \in \mathcal{B}, B \cap F_t \neq \emptyset} r^d_B.
\end{equation}

If $i \in I_1$ then
\begin{equation}
F \cap \frac{1}{2}B_i \neq \emptyset \quad \text{and} \quad F \cap \frac{1}{2}B_i \subseteq F_t
\end{equation}
since dist$(x_i, L) > t$ (by virtue of that fact that $x_i \in E_t$) and
\[
\frac{1}{2}B_i = B(x_i, \text{dist}(x_i, L)/2) \supseteq B(x_i, 8\text{dist}(x_i, F)).
\]

Since $\mathcal{B}$ forms a cover for $F$ there exists at least one $B \in \mathcal{B}$ such that $B \cap \frac{1}{2}B_i \neq \emptyset$. We further partition $I_1$. Let
\[
I_{1,1} = \{i \in I_1 : \text{there exists } B \in \mathcal{B} \text{ such that } \frac{1}{2}B_i \cap B \neq \emptyset \text{ and } r_B \geq r_{B_i}\},
\]
\[
I_{1,2} = I_1 \setminus I_{1,1}.
\]

We first control the sum over $I_{1,1}$. If $B \in \mathcal{B}$ is such that there is $B_i$ satisfying $\frac{1}{2}B_i \cap B \neq \emptyset$ and $r_B \geq r_{B_i}$ (which by definition implies $i \in I_{1,1}$), then $2B_i \subseteq 4B$. By (2.33) we know the $\{2B_i\}$ are disjoint and satisfy dist$(x_i, L) \leq r_{2B_i}/2$. By Lemma 2.5, we have
\[
\sum_{i \in I_{1,1}, \frac{1}{2}B_i \cap B \neq \emptyset} \sum_{r_{B_i} \leq r_B} r^d_{4B_i} \lesssim \sum_{i \in I_{1,1}, \frac{1}{2}B_i \cap B \neq \emptyset} \sum_{r_{B_i} \leq r_B} r^d_{2B_i} \lesssim \sum_{r_{B_i} \leq r_B} r^d_B.
\]

Thus,
\begin{equation}
\sum_{i \in I_{1,1}} r^d_{4B_i} \lesssim \sum_{B \in \mathcal{B}, B \cap F_t \neq \emptyset} \sum_{i \in I_{1,1}, \frac{1}{2}B_i \cap B \neq \emptyset} \sum_{r_{B_i} \leq r_B} r^d_{4B_i} \lesssim \sum_{B \in \mathcal{B}, B \cap F_t \neq \emptyset} r^d_B.
\end{equation}

We now turn our attention to $I_{1,2}$. For $i \in I_{1,2}$, let $x'_i$ be the point in $F$ closest to $x_i$ and set $B'_i = B(x'_i, \text{dist}(x_i, L)/4)$. Note that $B'_i \subseteq \frac{1}{2}B_i$, since for $y \in B'_i$ we have
\[
|y - x_i| \leq \frac{1}{4} \text{dist}(x_i, L) + |x - x_i| \leq \left(\frac{1}{4} + \frac{1}{16}\right) \text{dist}(x_i, L) \leq \frac{1}{2} \text{dist}(x_i, L).
\]

Since $F \cap \frac{1}{2}B_i \subseteq F_t$ by (2.38), and $\mathcal{B}$ forms a cover for $F_t$, the balls
\[
\{B \in \mathcal{B} : B \cap F \cap \frac{1}{2}B_i \neq \emptyset\}
\]
form a cover for $F \cap \frac{1}{2}B_i$. Furthermore, if $B \cap \frac{1}{2}B_i \neq \emptyset$ then $B \cap \frac{1}{2}B_j = \emptyset$ for all $i \neq j$, that is
\begin{equation}
\#\{i \in I_{1,2} : B_i \cap B \neq \emptyset\} \leq 1,
\end{equation}

since otherwise $|x_i - x_j| < 4 \max\{\delta(x_i), \delta(x_j)\}$, contradicting (2.33). By Lemma 2.16 (1), we know
\[
\mathcal{H}^{d,F}_{2\mathcal{B},\infty}(F \cap B'_i) \gtrsim r^d_{B'_i} \gtrsim r^d_{B_i},
\]
and since $\mathcal{B}$ is a good cover for $F \cap 2B$, we have

\begin{equation}
\sum_{i \in I_{1,2}} (r_{4B_i})^d \lesssim \sum_{i \in I_{1,2}} \mathcal{H}^{d,F}_{2B_i \cap \infty}(F \cap B_i') \leq \sum_{i \in I_{1,2}} \mathcal{H}^{d,F}_{2B_i \cap \infty}(F \cap {B_i}/2) \nonumber
\end{equation}

\begin{equation}
\leq \sum_{i \in I_{1,2}} \sum_{B \in \mathcal{B}} r_B^d = \sum_{i \in I_{1,2}} \sum_{B \in \mathcal{B}} r_B^d \quad (2.40) \nonumber
\end{equation}

Combining (2.39) and (2.41), we conclude

\begin{equation}
\sum_{i \in I_{1}} r_{4B_i}^d \lesssim \sum_{B \in \mathcal{B}} r_B^d \quad (2.37) \nonumber
\end{equation}

which is (2.35).

We turn our attention to proving (2.36), the proof of which follows much the same as that for (2.35). Let

\begin{equation}
E_i' = \{ x \in E \cap 2B : \text{dist}(x, F) > t/32 \} \nonumber
\end{equation}

and $\mathcal{B}'$ be a collection of balls covering $E_i'$ such that each $B \in \mathcal{B}'$ is centered on $E_i'$, has $r_B \leq r_B$ and

\begin{equation}
\mathcal{H}^d(F') \sim \sum_{B \in \mathcal{B}'} r_B^d. \tag{2.42} \nonumber
\end{equation}

As before, we partition $I_2$. If $i \in I_2$, since $x_i \in E_i$, we have $\text{dist}(x_i, L) > t$ and

\begin{equation}
\text{dist}(x_i, F) \geq \text{dist}(x_i, L)/16 \geq t/16, \nonumber
\end{equation}

hence

\begin{equation}
E \cap \frac{1}{32} B_i = E \cap B(x_i, \text{dist}(x_i, F)/2) \subseteq E_i'. \tag{2.43} \nonumber
\end{equation}

Thus for each $B_i$, since $\mathcal{B}'$ forms a cover for $E_i'$, there exists $B \in \mathcal{B}'$ such that $B \cap \frac{1}{32} B_i \neq \emptyset$. We partition $I_2$ by letting

\begin{align}
I_{2,1} &= \{ i \in I_2 : \text{there exists } B \in \mathcal{B}' \text{ such that } \frac{1}{32} B_i \cap B \neq \emptyset \text{ and } r_B \geq r_B \}, \\
I_{2,2} &= I_2 \setminus I_{2,1}. \nonumber
\end{align}

If $B \in \mathcal{B}'$ and $B \cap \frac{1}{32} B_i \neq \emptyset$ with $r_B \geq r_B$, then $2B_i \subseteq 4B$. Furthermore, by (2.33), we know the $\{2B_i\}$ are disjoint and satisfy $\text{dist}(x_i, L) < \text{dist}(x_i, F)/16 = r_{2B_i}/2$, so by Lemma 2.5, we have

\begin{equation}
\sum_{i \in I_{2,1}} r_{4B_i}^d \lesssim \sum_{i \in I_{2,1}} r_{2B_i}^d \quad (2.9) \nonumber
\end{equation}

Thus,

\begin{equation}
\sum_{i \in I_{2,1}} r_{4B_i}^d \lesssim \sum_{i \in I_{2,1}} r_{2B_i}^d \lesssim r_B. \nonumber
\end{equation}

Thus,

\begin{equation}
\sum_{i \in I_{2,1}} r_{4B_i}^d \lesssim \sum_{i \in I_{2,1}} r_{2B_i}^d \lesssim r_B. \tag{2.44} \nonumber
\end{equation}
We now deal with $I_{2,2}$. Since by (2.43), $E \cap \frac{1}{32} B_i \subseteq E'$, the balls
\[
\{ B \in \mathcal{B}' : B \cap \frac{1}{32} B_i \cap E \neq \emptyset \}
\]
form a cover for $E \cap \frac{1}{32} B_i$. As before, if $B \cap \frac{1}{32} B_i \neq \emptyset$ then $B \cap \frac{1}{32} B_j = \emptyset$ for all $i \neq j$ by (2.33). By lower regularity of $E$, we know $\mathcal{H}_E^d(E \cap \frac{1}{32} B_i) \gtrsim r^{d}_{B_i}$, from which we conclude
\[
\sum_{i \in I_{2,2}} r^{d}_{B_i} \lesssim \sum_{i \in I_{2,2}} \mathcal{H}_E^d(E \cap \frac{1}{32} B_i) \leq \sum_{i \in I_{2,2}} \sum_{B \in \mathcal{B}'} r^{d}_{B_i} \cap \frac{1}{32} B_i \cap E \neq \emptyset \]
\[
= \sum_{B \in \mathcal{B}'} \sum_{i \in I_{2,2} \cap \frac{1}{32} B_i \cap E \neq \emptyset} r^{d}_{B_i} \lesssim \sum_{B \in \mathcal{B}'} \sum_{i \in I_{2,2}} r^{d}_{B_i}.
\]
(2.45)

The proof of (2.36) (and hence the proof of the lemma) is completed since
\[
\sum_{i \in I_{2}} r^{d}_{B_i} = \sum_{i \in I_{2,1}} r^{d}_{B_{i,1}} + \sum_{i \in I_{2,2}} r^{d}_{B_{i,2}} \lesssim \sum_{B \in \mathcal{B}'} \sum_{i \in I_{2}} r^{d}_{B_i} \lesssim \mathcal{H}_E^d(E').
\]

In Section 3 and Section 4, we want to apply the construction of David and Toro (Theorem 2.3). We wish to do this by controlling the angles between pairs of planes, by their corresponding $\beta$-numbers. The following series of lemmas, culminating in Lemma 2.34, will allow us to do so. We first introduce some more notation.

For two planes $P, P'$ containing the origin, we define
\[
\angle(P, P') = d_{B(0,1)}(P, P').
\]
If $P, P'$ are general affine planes with $x \in P$ and $y \in P'$, we define
\[
\angle(P, P') = \angle(P - x, P' - y).
\]
For planes $P_1, P_2$ and $P_3$, it is not difficult to show that
\[
\angle(P_1, P_3) \leq \angle(P_1, P_2) + \angle(P_2, P_3).
\]

**Lemma 2.30** ([AT15, Lemma 6.4]). Suppose $P_1$ and $P_2$ are $d$-planes in $\mathbb{R}^n$ and $X = \{x_0, \ldots, x_d\}$ are points so that

1. $\eta \in (0, 1)$, where $\eta = \eta(X) = \min\{\text{dist}(x_i, \text{span}(X \setminus \{x_i\})) / \text{diam}(X)\}$,
2. $\text{dist}(x_i, P_j) < \varepsilon \text{diam}(X)$ for $i = 0, \ldots, d$ and $j = 1, 2$, where $\varepsilon < \eta d^{-1}/2$.

Then
\[
\text{dist}(y, P_1) \leq \varepsilon \left(\frac{2d}{\eta} \text{dist}(y, X) + \text{diam}(X)\right) \text{ for all } y \in P_2.
\]

In order to control angles between $d$-planes, we need to know that $E$ is sufficiently spread out in at least $d$ directions. This is quantified below.

**Definition 2.31.** Let $0 < \alpha < 1$. We say a ball $B$ has $(d + 1, \alpha)$-separated points if there exist points $X = \{x_0, \ldots, x_d\}$ in $E \cap B$ such that, for each $i = 1, \ldots, d$, we have
\[
\text{dist}(x_{i+1}, \text{span}\{x_0, \ldots, x_d\}) \geq \alpha r_B.
\]
Lemma 2.32. Suppose $E \subseteq \mathbb{R}^n$ and there is $B'$ and $B$ both centered on $E$ with $B' \subseteq B$. Suppose further that there exists $0 < \alpha < 1$ such that $B'$ has $(d + 1, \alpha)$-separated points. Let $P$ and $P'$ be two $d$-planes. Then

$$d_B(P, P') \lesssim \frac{1}{\alpha^{d+2}} \left[ \left( \frac{r_B}{r_{B'}} \right)^{d+1} \beta^d_E(2B, P) + \beta^d_E(2B', P') \right].$$

Proof. Since $B'$ has $(d + 1, \alpha)$-separated points, we can find $X = \{x_0, \ldots, x_d\}$ satisfying (2.46). This implies that $\alpha < \eta(X) \leq 1$. Let $B_i = B(x_i, \alpha r_{B'}/4)$ and for $t > 0$ let

$$E_{t,i} = \{x \in E \cap B_i : \text{dist}(x, P) > tr_{B'}, \text{ or dist}(x, P') > tr_{B'}\}.$$ 

Let $T > 0$ and suppose $E_{t,i} = E \cap B_i$ for all $t \leq T$. We shall bound $T$. By Lemma 2.16 (1),

$$\mathcal{H}_{B_i}^d(E \cap B_i) \geq c_1^d t_B = \frac{c_1 t_B}{4^d} r_{B'}. $$

Using this, along with Lemma 2.16 (4), we get

$$T \leq \mathcal{H}_{B_i}^d(E \cap B_i)^{-1} \int_0^T \mathcal{H}_{B_i}^d(E_{t,i}) dt \lesssim \frac{1}{\alpha r_{B'}} \int_0^T \mathcal{H}_{B_i}^d(E_{t,i}) dt 
\lesssim \left( \frac{1}{\alpha r_{B'}} \int_0^T \mathcal{H}_{2B_i}^d(E_{t,i}) dt \right) \lesssim \left( \frac{1}{\alpha r_{B'}} \int_0^T \mathcal{H}_{2B_i}^d(E_{t,i}) dt \right) \lesssim \frac{1}{\alpha r_{B'}} \left[ \left( \frac{r_B}{r_{B'}} \right)^{d+1} \beta^d_E(2B, P) + \beta^d_E(2B', P') \right].$$

Note, we define $\lambda$ like this for convenience in the forthcoming estimates. Thus, there is a constant $C$ such that $T \leq C \lambda \alpha^2$. This implies for each $i = 0, 1, \ldots, d$, there exists some $y_i \in (E \cap B_i) \setminus E_{2\lambda^2 \alpha^2, i}$. Let $Y = \{y_0, \ldots, y_d\}$. Since $|x_i - x_j| \geq \alpha r_{B'}$ for all $i \neq j$, and $y_i \in B_i$, it follows that

$$\text{diam}(Y) \geq \alpha r_{B'}/2.$$ 

Thus,

$$\text{dist}(y_i, P_j) \leq 2C \lambda \alpha^2 \alpha r_{B'} = \frac{2C \alpha^2 \alpha r_{B'}}{\text{diam}(Y)} \text{ diam}(Y) \leq 4C \alpha \text{ diam}(Y).$$

Because $d_B(P, P') \leq 1$, if $\lambda \geq \frac{1}{16C \alpha^2}$ then the lemma follows. Assume instead that $\lambda < \frac{1}{16C \alpha^2}$. By (2.47) we can show that

$$\alpha / 2 \leq \eta(Y) \leq 1,$$ 

which gives

$$4C \lambda \alpha \leq \frac{\alpha \alpha^2}{4} \leq \eta(Y) \alpha / 2.$$
so, taking $\varepsilon = 4C\lambda \alpha$ in Lemma 2.30, we get
\[
d_{B'}(P, P') \leq \varepsilon \left( \frac{2d}{\eta(Y)} + 1 \right) \leq 4C\lambda \alpha \left( \frac{4d}{\alpha} + 1 \right) \leq 20Cd\lambda,
\]
which proves the lemma.

\[\square\]

**Remark 2.33.** The following lemma is essentially Lemma 2.18 from [AS18] and the proof is the same. The main difference is that since $E$ is not necessarily lower regular, we need to assume that $E$ has $(d+1, \alpha)$-separated points in each cube. The final constant then also ends up depending on $\alpha$.

**Lemma 2.34.** Let $\mathcal{M} > 1$, $\alpha > 0$ and $E$ a Borel set. Let $\mathcal{D}$ be the cubes for $E$ from Lemma 2.1 and $Q_0 \in \mathcal{D}$. Let $P_Q$ satisfy $\beta_{E}^{d,1}(MB_Q) = \beta_{E}^{d,1}(MB_Q, P_Q)$. Let $Q, R \in \mathcal{D}$, $Q, R \subseteq Q_0$ and suppose for all cubes $T \subseteq Q_0$ such that $T$ contains either $Q$ or $R$ that $\beta_{E}^{d,1}(MB_T) < \varepsilon$ and $T$ has $(\alpha, d+1)$-separated points. Then for $\Lambda > 0$, if $\text{dist} (Q, R) \leq \Lambda \max \{ \ell(Q), \ell(R) \} \leq \Lambda^2 \min \{ \ell(Q), \ell(R) \}$, then
\[
\angle(P_Q, P_R) \lesssim_{M, \Lambda} \frac{\varepsilon}{\alpha^{d+2}}.
\]

3. **Proof of Theorem 1.13**

Let $X^E_k \subseteq X^F_k$ be sequences of maximally $\rho^k$-separated nets in $E$ and $F$ respectively. Let $\mathcal{D}^E$ and $\mathcal{D}^F$ be the cubes from Theorem 2.1 with respect to $X^E_k$ and $X^F_k$. Let $Q^E_0 \in \mathcal{D}^E$ and let $Q^F_0 \in \mathcal{D}^F$ be the cube with the same center and side length as $Q^E_0$. To simplify notation we will write $\mathcal{D} = \mathcal{D}^F$ and $Q_0 = Q^F_0$. We first reduce to proof of (1.4) to the proof of (3.1) below.

**Lemma 3.1.** If
\[
\sum_{Q \subseteq Q_0} \beta_{E}^{d, p}(C_0B_Q)^2 \ell(Q)^d \lesssim \mathcal{H}^d(Q_0) + \sum_{Q \subseteq Q_0} \beta_{E}^{d, 1}(AB_Q)^2 \ell(Q)^d
\]
for some $A \geq C_0$, then (1.4) holds.

**Proof.** Assume (3.1) holds. First, it is easy to see that
\[
diam (Q^E_0)^d + \sum_{Q \subseteq Q^E_0} \beta_{E}^{d, p}(C_0B_Q)^2 \ell(Q)^d \lesssim diam (Q_0)^d + \sum_{Q \subseteq Q_0} \beta_{E}^{d, p}(C_0B_Q)^2 \ell(Q)^d.
\]
By (3.1), Lemma 2.23 and Theorem 1.5 (using the fact that $F$ is lower content regular), we have
\[
\sum_{Q \subseteq Q_0} \beta_{E}^{d, p}(C_0B_Q)^2 \ell(Q)^d \lesssim \mathcal{H}^d(Q_0) + \sum_{Q \subseteq Q_0} \beta_{E}^{d, 1}(AB_Q)^2 \ell(Q)^d
\]
\[
\lesssim \mathcal{H}^d(Q_0) + \sum_{Q \subseteq Q_0} \beta_{F}^{d, p}(AB_Q)^2 \ell(Q)^d
\]
\[
\lesssim \text{diam}(Q_0)^d + \sum_{Q \subseteq Q_0} \beta_{F}^{d, p}(AB_Q)^2 \ell(Q)^d.
\]
Now, let \( K = K(C_0, A) \) be the smallest integer such that \( 1 + A\rho^K \leq C_0 \). Let \( Q \in \mathcal{D}_k \) for some \( k \geq K \) and \( y \in AB_Q \). Then
\[
|y - x_{Q(K)}| \leq A\ell(Q) + \ell(Q(K)) = (A\rho^K + 1)\ell(Q(K)) \leq C_0\ell(Q(K)).
\]
Hence \( AB_Q \subseteq C_0B_Q(K) \). Each cube \( Q \in \mathcal{D} \) has at most \( C = C(n) \) children (notice the number of descendants is dependent on the the ambient dimension since we do not necessarily know \( \beta_{d,p}^{*,1} \) is small for an arbitrary cube in \( \mathcal{D} \)). It follows that each \( Q \) has at most \( KC \) descendants up to the \( K^{th} \) generation. In particular, this is also true for \( Q_0 \). By this and Lemma 2.24, we have
\[
\sum_{Q \in \mathcal{D}, Q \subseteq Q_0} \tilde{\beta}_{d,p}^d(AB_Q)^2\ell(Q)^d \lesssim \sum_{k=0}^{K-1} \sum_{Q \in \mathcal{D}_k, Q \subseteq Q_0} \tilde{\beta}_{d,p}^d(AB_Q)^2\ell(Q)^d
\]
\[
+ \sum_{k=K}^{\infty} \sum_{Q \in \mathcal{D}_k, Q \subseteq Q_0} \tilde{\beta}_{d,p}^d(AB_Q)^2\ell(Q)^d
\]
\[
\lesssim \text{diam}(Q_0)^d + \sum_{Q \in \mathcal{D}} \tilde{\beta}_{d,p}^d(C_0B_Q)^2\ell(Q)^d.
\]
Combing each of the above sets of inequalities gives (1.5).

The rest of this section is devoted to proving (3.1) holds for \( A \geq C_0 \).

**Definition 3.2.** A collection of cubes \( S \subseteq \mathcal{D} \) is called a stopping time region if the following hold.

1. There is a cube \( Q(S) \in S \) such that \( Q(S) \) contains all cubes in \( S \).
2. If \( Q \in S \) and \( Q \subseteq R \subseteq Q(S) \), then \( R \in S \).
3. If \( Q \in S \), then all siblings of \( Q \) are also in \( S \).

We let:
- \( Q(S) \) denote the maximal cube in \( S \).
- \( \min(S) \) denote the cubes in \( S \) which have a child not contained in \( S \).
- \( S(Q) \) denote the unique stopping time regions \( S \) such that \( Q \in S \).

We split \( \mathcal{D}(Q_0) \) into a collection of stopping time regions \( \mathcal{S} \) where in each stopping time region, \( F \) is well-approximated by \( E \) and there is good control on a certain Jones type function. Observe that if \( Q \in \mathcal{D} \) and \( C_0B_Q \cap E = \emptyset \) then \( \beta_{d,p}^d(C_0B_Q) = 0 \), and so we will not restart our stopping times on these cubes.

Let \( M > 1 \) be a large constant (to be fixed later) and \( \varepsilon > 0 \) be small (also fixed later). For each \( Q \in \mathcal{D}(Q_0) \) such that \( E \cap C_0B_Q \neq \emptyset \), we define a stopping time region \( S_Q \) as follows. Begin by adding \( Q \) to \( S_Q \) and inductively, on scales, add cubes \( R \) to \( S_Q \) if each of the following holds,

1. \( R^{(1)} \in S_Q \).
2. for every sibling \( R' \) of \( R \), if \( x \in R' \) then \( \text{dist}(x, E) \leq \varepsilon \ell(R') \).
3. every sibling \( R' \) of \( R \) satisfies
\[
\sum_{R' \subseteq T \subseteq Q} \tilde{\beta}_{d,p}^d(MB_T)^2 < \varepsilon^2.
\]
Remark 3.3. If $\beta_F^{d,1}(MB_Q) \geq \varepsilon$ or if there exists $x \in Q$ such that $\text{dist}(x, E) > \varepsilon \ell(Q)$ then $S_Q = \{Q\}$.

Remark 3.4. Firstly, $\varepsilon$ will be chosen sufficiently small so that each cube $R$ contained in some stopping time region has at most $K$ children, where $K$ depends only on $\varepsilon$ and $d$. See Remark 2.28 for why this is possible.

We partition $\{Q \in \mathcal{D}(Q) : E \cap C_0B_Q \neq \emptyset\}$ as follows. First, add $S_{Q_0}$ to $\mathcal{S}$. Then, if $S$ has been added to $\mathcal{S}$ and if $Q \in \text{Child}(R)$ for some $R \in \min(S)$ such that $E \cap C_0B_Q \neq \emptyset$, also add $S_Q$ to $\mathcal{S}$. Let $\mathcal{S}$ be the collection of stopping time regions obtained by repeating this process indefinitely. Note that

$$\sum_{Q \in \mathcal{D}(Q_0)} \beta_E^{d,p}(C_0B_Q)^2 \ell(Q)^d = \sum_{S \in \mathcal{S}} \sum_{Q \in S} \beta_E^{d,p}(C_0B_Q)^2 \ell(Q)^d.$$ 

For each $S \in \mathcal{S}$ which is not a singleton (i.e., $S \neq \{Q\}$) we plan to find a bi-Lipschitz surface which well approximates $F$ inside $S$. With some additional constraints, the surfaces produced by Theorem 2.3 will be bi-Lipschitz.

Theorem 3.5 ([DT12, Theorem 2.5]). With the same notation and assumptions as Theorem 2.3, assume additionally that there exists $K < \infty$ such that

$$\sum_{k \geq 0} \varepsilon_k'(f_k(z))^2 \leq K \text{ for } z \in \Sigma_0$$

with

$$\varepsilon_k'(x) = \sup \{d_{x_i,10^s r_j(P_{j,k}, P_{i,\ell})} : j \in J_k, \ |l-k| \leq 2, \ i \in J_k, x \in 10B_{j,k} \cap 10B_{\ell}\}.$$ 

Then $f = \lim f_N = \lim_{N} \sigma_0 \circ \cdots \circ \sigma_N : \Sigma_0 \to \Sigma$ is $C(K)$-bi-Lipschitz.

Lemma 3.6. There exists $\varepsilon > 0$ small enough so that for each $S \in \mathcal{S}$, which is not a singleton, there is a surface $\Sigma_S$ such that

$$\text{dist}(y, \Sigma_S) \lesssim \varepsilon \frac{d}{\ell} \ell(R)$$

for each $y \in F \cap \frac{M}{2}B_R$ where $R \in S$. Also, for each ball $B$, centered on $\Sigma_S$ and contained in $MB_Q(S)$, we have

$$\frac{\omega}{2} r_B^d \leq \mathcal{H}^d(\Sigma_S \cap B) \lesssim r_B^d.$$ 

Proof. For $k \geq 0$ let $s(k)$ be such that $5\rho_s(k) \leq r_k \leq 5\rho_s(k)^{-1}$. For each $Q \in \mathcal{S}$, let $L_Q$ be the $d$-plane through $x_Q$ such that

$$\beta_F^{d,1}(MB_Q, L_Q) \leq 2\beta_F^{d,1}(MB_Q).$$

For each $k$, let $\mathcal{C}'_k$ be a maximal $r_k$-separated net for

$$\mathcal{C}'_k = \{x_Q : Q \in \mathcal{D}(s(k)) \cap S\}.$$ 

By Lemma 2.34, $\mathcal{C}'_k$, with planes $\{L_Q\}_{Q \in \mathcal{C}'_k}$ satisfy the assumptions of Theorem 2.3 with

$$\varepsilon_k(x) \lesssim \beta_F^{d,1}(MB_Q) < \varepsilon$$

for $x \in Q \in \mathcal{C}'_k$. Let $\Sigma_S'$ be the resulting surface from Theorem 2.3. By (2.8), we can choose $\varepsilon$ small enough so that for all balls $B$ centered on $\Sigma_S'$ we have

$$\mathcal{H}^d(\Sigma_S' \cap B) \geq \frac{\omega}{2} r_B^d.$$
We verify that the additional assumption of Theorem 3.5 is satisfied, thus $\Sigma_S$ is in fact a bi-Lipschitz surface. Let $x = f(z) \in \Sigma'_S$ and set $x_k = f_k(z)$. By the triangle inequality, we have

$$|x - x_k| \leq \sum_{j \geq k} |x_{j+1} - x_j| \leq \sum_{j \geq k} |\sigma_j(x_j) - x_j| \tag{2.6}$$

$$\lesssim \sum_{j \geq k} \varepsilon_j r_j \lesssim \sum_{j \geq k} \varepsilon r_j \lesssim \varepsilon r_k. \tag{3.4}$$

Thus, for $\varepsilon$ small enough, $x_k \in B(x, 2r_k)$. Let $Q \in \mathcal{D}_{\lambda(k)} \cap S$ such that $x \in Q$. Suppose $\ell \in \{k, k-1\}$ and $x_{Q'} \in \mathcal{C}'_S$ is such that $x_k \in B(x_{Q'}, r_\ell)$ (the existence of $x_{Q'}$ is guaranteed by maximality). It follows that

$$|x_{Q'} - x_Q| \leq |x_{Q'} - x_k| + |x_k - x| + |x - x_Q| \leq r_\ell + 2r_k + r_k \leq 13r_k$$

For $M$ large enough, this gives

$$100B(x_{Q'}, r_\ell) \subseteq B(x_Q, 13r_k + 100r_\ell) \subseteq B(x_Q, 1100r_k) \subseteq B(x_Q, M\ell(Q)).$$

Then by Lemma 2.32, $\varepsilon'_k(x_k) \lesssim \beta^{-1}_F(MB_Q)$. Since, by our stopping time condition, we have control of the sum of $\beta^{-1}_F(MB_Q)$, we have verified the additional assumption. We define

$$\Sigma_S = \Sigma'_S \cap MB_Q(S).$$

Thus, for all balls $B$ centered on $\Sigma_S$ such that $B \subseteq MB_Q(S)$, left-most inequality in (3.3) follows by (3.5) and the right-most inequality follows since $\Sigma_S$ is the bi-Lipschitz image of $\mathbb{R}^d$.

We check (3.2). Let $R \in \mathcal{D}_{\lambda(k)} \cap S$ and $y \in \frac{M}{4}B_R$. Let $R' \in \mathcal{D}_{\lambda(k)} \cap S_Q$ be a cube such that $x_{R'} \in \mathcal{C}'_S$ and $|x_R - x_{R'}| \leq r_k$. By the triangle inequality

$$|y - x_{R'}| \leq |y - x_R| + |x_R - x_{R'}| \leq \frac{M}{4} \ell(R) + r_k \leq \left(\frac{M}{4} + \rho^{-1}\right) \ell(R).$$

Choosing $M \geq 4\rho^{-1}$ gives $\frac{M}{4}B_R \subseteq \frac{M}{4}B_{R'}$. Since $\beta^{-1}_F(MB_{R'}, L_{R'}) < 2\varepsilon$, by (2.30), $\beta^{-1}_E,\infty(MB_{R'}, L_{R'}) \lesssim \varepsilon \frac{1}{\ell(R)}$, hence

$$\text{dist}(y, L_{R'}) \lesssim \varepsilon \frac{1}{\ell(R)} r_k \lesssim \varepsilon \frac{1}{\ell(R)} \ell(R).$$

By (2.5), there exists $z \in \Sigma_S$ such that $|\pi_{L_{R'}}(y) - z| \lesssim \varepsilon r_k$. Furthermore, by (2.7), $\text{dist}(z, \Sigma_S) \lesssim \varepsilon r_k$. Combing the previous estimates we see that (3.2) holds. \hfill \Box

**Lemma 3.7.** For $M \geq 2C_0$,

$$\sum_{s \in \mathcal{S}} \sum_{Q \in S} \beta^{-p}_E(C_0B_Q) \ell(Q)^d \lesssim \sum_{s \in \mathcal{S}} \sum_{Q \in S} \beta^{-p}_F(MB_Q)^2 \ell(Q)^d + \sum_{s \in \mathcal{S}} \ell(Q(S))^d.$$

To prove Lemma 3.7 we will need apply the smoothing procedure of David and Semmes (see for example [DS91, Chapter 8]). Let us introduce these smoothed cubes and prove a general fact about them. Let

$$\tau = \frac{1}{\pi^{(1+\rho)}}$$

and, for each $S \in \mathcal{S}$, let $\text{Stop}(S)$ be the collection of maximal cubes in $\mathcal{D}$ such that $\ell(Q) < \tau d_S(Q)$, that is,

$$\text{Stop}(S) = \{Q \in \mathcal{D} : Q \text{ is maximal so that } \ell(Q) < \tau d_S(Q)\}.$$
Lemma 3.8. Let $S \in \mathcal{S}$ and suppose $R \in \text{Stop}(S)$ is such that $R \cap 2C_0B_Q(S) \neq \emptyset$. Then there exists $Q = Q(R) \in S$ such that

\begin{equation}
\tau \text{ dist}(R, Q) \leq \frac{4}{\rho} \ell(R)
\end{equation}

and

\begin{equation}
\frac{\tau \rho}{4} \ell(Q) \leq \ell(R) \leq 3C_0 \tau \ell(Q).
\end{equation}

Proof. Let $Q' \in S$ be a cube such that

\begin{equation}
2d_S(R) \geq \ell(Q') + \text{dist}(Q', R).
\end{equation}

By maximality,

\begin{equation}
\frac{1}{\rho} \ell(R) = \ell(R^{(1)}) > \tau d_S(R^{(1)}) \geq \tau \left( d_S(R) - 2\ell(R) - 2\ell(R^{(1)}) \right)
= \tau d_S(R) - 2(1 + \rho^{-1})\tau \ell(R).
\end{equation}

By our choice of $\tau$ (see (3.6)), this gives $\tau d_S(R) \leq \frac{2}{\rho} \ell(R)$. Then,

\begin{equation}
\tau \left( \ell(Q') + \text{dist}(Q', R) \right) \geq 2\tau d_S(R) \leq \frac{4}{\rho} \ell(R).
\end{equation}

From here, we see that (3.7) and the left hand inequality in (3.8) are true. If $\ell(R) \leq 3C_0 \tau \ell(Q')$, we can set $Q = Q'$ and the lemma follows. Otherwise, since $R \cap 2C_0Q(S) \neq \emptyset$, we have

\begin{equation}
\ell(R) < \tau(\ell(Q(S)) + \text{dist}(R, Q(S))) \leq 3C_0 \tau \ell(Q(S)).
\end{equation}

We can then choose $Q$ to be the smallest ancestor of $Q'$ contained in $Q(S)$ such that (3.8) holds. The existence of such a $Q$ is guaranteed by the above inequality. □

Proof of Lemma 3.7. First, by Lemma 2.25, we have

\begin{equation}
\sum_{S \in \mathcal{S}} \sum_{Q \in S} \beta^d_{\ell_e}(C_0B_Q)^2 \ell(Q)^d \lesssim \sum_{S \in \mathcal{S}} \sum_{Q \in S} \beta^d_{\ell_e}(2C_0B_Q)^2 \ell(Q)^d
\end{equation}

\begin{equation}
+ \sum_{S \in \mathcal{S}} \sum_{Q \in S} \left( \frac{1}{\ell(Q)^d} \int_{F \cap 2C_0B_Q} \left( \frac{\text{dist}(x, E)}{\ell(Q)} \right)^p \, d\mathcal{H}_\infty^d(x) \right)^{\frac{2}{p}} \ell(Q)^d.
\end{equation}

Since $M \geq 2C_0$, we have the desired bound on the first term. To deal with the second term, let

\begin{equation}
I^p_S := \sum_{Q \in S} \left( \frac{1}{\ell(Q)^d} \int_{F \cap 2C_0B_Q} \left( \frac{\text{dist}(x, E)}{\ell(Q)} \right)^p \, d\mathcal{H}_\infty^d(x) \right)^{\frac{2}{p}} \ell(Q)^d.
\end{equation}

If we can show that

\begin{equation}
I^p_S \lesssim \ell(Q(S))^d
\end{equation}

for each $S \in \mathcal{S}$ then the lemma follows. The rest of the proof is devoted to proving (3.10). Note, we may assume that $p \geq 2$, since for $p < 2$, we have $I^p_S \lesssim I^2_S$ by Jensen’s inequality (Lemma 2.18).

Let $S \in \mathcal{S}$. In the case that $S = \{Q\}$ is a singleton, since $C_0B_Q \cap E \neq \emptyset$, we have $\text{dist}(x, E) \lesssim \ell(Q)$ for each $x \in F \cap 2C_0B_Q$. Then, $I^p_S$ reduces to $\ell(Q)$ and (3.10) follows.
Assume then that $S$ is not a singleton. First, by Lemma 2.19, we can write

$$I^p_S \lesssim \sum_{Q \in S} \left( \frac{1}{\ell(Q)^d} \sum_{R \in \text{Stop}(S)} \sum_{R \cap 2C_0B_Q \neq \emptyset} \int_R \left( \frac{\text{dist}(x, E)}{\ell(Q)} \right)^p d\mathcal{H}^d_\infty(x) \right)^{\frac{2}{p}} \ell(Q)^d.$$  

Let $R \in \text{Stop}(S)$ be such that $R \cap 2C_0B_Q \neq \emptyset$ for some $Q \in S$. By Lemma 3.8, we can find a cube $Q' \in S$ such that

$$(3.11) \quad \text{dist}(Q', R) \lesssim \ell(R) \quad \text{and} \quad \ell(R) \sim \ell(Q').$$

By our stopping time condition (2), since $Q' \in S$, we have

$$(3.12) \quad \text{dist}(y', E) \leq \varepsilon \ell(Q')$$

for all $y' \in Q'$. Let $y'^Q \in Q'$ and $y^R \in R$ be points such that

$$(3.13) \quad \text{dist}(R, Q') = |y'^Q - y^R|.$$  

Then, for any $y \in R$, we have

$$\text{dist}(y, E) \leq |y - y'^Q| + \text{dist}(y'^Q, E) \leq |y - y'^Q| + |y'^Q - y^R| + \varepsilon \ell(Q')$$

$$(3.14) \quad \lesssim \ell(R) + \text{dist}(R, Q') + \varepsilon \ell(Q') \lesssim \ell(R).$$

Using this, along with the that fact that $\frac{2}{p} \leq 1$, we get

$$I^p_S \lesssim \sum_{Q \in S} \left( \sum_{R \in \text{Stop}(S)} \frac{\ell(R)^{d+p}}{\ell(Q)^{d+p}} \right)^{\frac{2}{p}} \ell(Q)^d \lesssim \sum_{Q \in S} \sum_{R \in \text{Stop}(S)} \frac{\ell(R)^{d+p}}{\ell(Q)^{d+p}} \ell(Q)^d \left( \frac{2}{p} - 1 \right)^{\frac{2}{p}}.$$  

**Claim:** For each $k \in \mathbb{N}$, we have

$$(3.15) \quad \# \{Q \in \mathcal{D}_k \cap S : R \cap 2C_0B_Q \neq \emptyset\} \lesssim 1.$$  

**Proof of Claim.** Let $Q'$ be the cube from Lemma 3.8 for $R$. By (3.11), we can choose $M$ large enough so that $R \subseteq \frac{M}{4}B_Q$ (taking $M \geq 100C_0/\rho$ is sufficient). In particular, $x_R \in \frac{M}{4}B_Q$. Then, by Lemma 3.6,

$$(3.16) \quad \text{dist}(x_R, \Sigma_S) \lesssim \varepsilon \frac{\ell(Q')}{\ell(Q)} \lesssim \frac{\varepsilon}{\tau} \ell(R).$$

Recall the definition of $\tau$ from (3.6). For any $Q \in \mathcal{D}_k \cap S$ such that $R \cap 2C_0B_Q \neq \emptyset$, we have

$$(3.17) \quad \ell(R) < \tau d_S(R) \leq \tau (\ell(Q) + \text{dist}(Q, R)) \lesssim \tau \ell(Q).$$

Let $x_R'$ be the point in $\Sigma_S$ closest to $x_R$. By (3.15) and (3.16) there exists $A > 0$ so that if $Q \in \mathcal{D}_k \cap S$ is such that $R \cap 2C_0B_Q \neq \emptyset$ then

$$Q \subseteq B(x_R', A\ell(Q)) := B.$$  

Since $\Sigma_S$ is $(C\varepsilon, d)$-Reifenberg flat we can find a plane $P$ through $x_R'$ such that

$$(3.18) \quad d_B(P, \Sigma_S) \lesssim \varepsilon.$$
Let \( x'_Q \) be the point in \( \Sigma_S \) which is closest to \( x_Q \), then

\[
\text{dist}(x_Q, P) \leq |x_Q - x'_Q| + \text{dist}(x'_Q, P) \overset{(3.2)}{\lesssim} (\varepsilon \frac{1}{r^H} + \varepsilon)\ell(Q).
\]

So, for \( \varepsilon \) small enough, we have \( \text{dist}(x_Q, P) \leq c_0\ell(Q)/2 \). Then (3.14) follows from Lemma 2.5.

Returning to \( I^p_S \), since by assumption, \( p < 2d/(d - 2) \), or equivalently, \( d(2/p - 1) + 2 > 0 \), we can swap the order of integration, apply (3.14), and sum over a geometric series and obtain

\[
I^p_S \lesssim \sum_{R \in \text{Stop}(S)} \sum_{Q \in S} \frac{\ell(R)^{d^2+2}}{\ell(Q)^{d(\frac{2}{p}-1)+2}} \lesssim \sum_{R \in \text{Stop}(S)} \ell(R)^d.
\]

By (3.15), for \( \varepsilon \ll \tau^{d+1} \), each \( c_0 B_R \) carves out a large proportion of \( \Sigma_S \) i.e. \( \mathcal{H}^d(c_0 B_R \cap \Sigma_S) \gtrsim \ell(R)^d \). Since the \( c_0 B_R \) are disjoint and \( \Sigma_S \) is bi-Lipschitz, we have

\[
I^p_S \lesssim \sum_{R \in \text{Stop}} \mathcal{H}^d(c_0 B_R \cap \Sigma_S) \leq \mathcal{H}^d(B_{Q(S)} \cap \Sigma_S) \lesssim \ell(Q(S))^d,
\]

completing the proof of (3.10) and hence the proof of the lemma.

**Lemma 3.9.**

\[
\sum_{S \in \mathcal{F}} \ell(Q(S))^d \lesssim \mathcal{H}^d(Q_0) + \sum_{Q \in \mathcal{Q}(Q_0)} \tilde{\beta}^{d,1}_F (MB_Q)^2 \ell(Q)^d.
\]

Lemma 3.9 along with Lemma 3.7 finishes the proof of (3.1) (and hence finishes the proof of Theorem 1.13). We prove Lemma 3.9 via the following two lemma. Let \( \min F \) be the collection of minimal cubes from \( \mathcal{F} \), i.e.

\[
\min F = \bigcup_{S \in \mathcal{F}} \{Q \in \min(S)\}.
\]

**Lemma 3.10.**

\[
\sum_{Q \in \min \mathcal{F}} \ell(Q)^d \lesssim \mathcal{H}^d(Q_0) + \sum_{Q \in \mathcal{Q}(Q_0)} \tilde{\beta}^{d,1}_F (MB_Q)^2 \ell(Q)^d.
\]

**Proof of Lemma 3.10.** We split \( \min \mathcal{F} \) into two sub families, \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) where \( \mathcal{F}_1 \) is the collection of cubes \( R \) in \( \min \mathcal{F} \) such that \( R \) has a child \( R' \) with

\[
\sum_{Q \in S(R)} \tilde{\beta}^{d,1}_F (MB_Q)^2 \geq \varepsilon^2
\]

and \( \mathcal{F}_2 = \min \mathcal{F} \setminus \mathcal{F}_1 \) (recall the definition of \( S(R) \) from Definition 3.2). We deal with \( \mathcal{F}_1 \) first. Let \( R \in \mathcal{F}_1 \) and let \( R' \) be its child satisfying (3.18). Note that if \( \tilde{\beta}^{d,1}_F (MB_{R'}) \leq \varepsilon^2/2 \), then

\[
\sum_{Q \in S(R)} \tilde{\beta}^{d,1}_F (MB_Q)^2 \geq \varepsilon^2/2.
\]
If, instead, \( \beta_{p,F}^{d,1}(MB_R) > \varepsilon^2/2 \), we have
\[
\sum_{Q \in \mathcal{S}(R) \cap R \geq R} \beta_{p,F}^{d,1}(MB_Q)^2 \geq \beta_{p,F}^{d,1}(MB_R)^2 \geq \beta_{p,F}^{d,1}(MB_R')^2 \geq \varepsilon^2/2.
\]
In either case it follows that
\[
\sum_{Q \in \mathcal{S}(R) \cap R \geq R} \beta_{p,F}^{d,1}(MB_Q)^2 \geq \varepsilon^2.
\]
Thus,
\[
\sum_{R \in \mathcal{F}_1} \ell(R)^d \leq \frac{1}{\varepsilon^2} \sum_{R \in \mathcal{F}_1} \ell(R)^d \sum_{Q \in \mathcal{S}(R) \cap R \geq R} \beta_{p,F}^{d,1}(MB_Q)^2 \leq \sum_{S \in \mathcal{F}} \sum_{Q \in S} \beta_{p,F}^{d,1}(MB_Q)^2 \sum_{R \in \mathcal{S}(Q) \cap R \leq Q} \ell(R)^d.
\]
(3.19)
In the case that \( S \) is not a singleton, by Lemma 3.6, \( \text{dist}(x_R, \Sigma_{S(R)}) \leq \varepsilon \ell(R) \). Thus, for \( \varepsilon > 0 \), small enough \( c_0B_R \) carves out a large proportion of \( \Sigma_S \), hence \( \mathcal{H}^d(c_0B_R \cap \Sigma_{S(R)}) \approx \ell(R)^d \). The balls \( c_0B_R \) are disjoint and contained in \( B_Q \), so
\[
(3.20) \sum_{R \in \mathcal{S}(Q) \cap R \leq Q} \ell(R)^d \leq \sum_{R \in \mathcal{S}(Q) \cap R \leq Q} \mathcal{H}^d(c_0B_R \cap \Sigma_{S(R)}) \leq \mathcal{H}^d(B_Q \cap \Sigma_{S(R)}) \leq \ell(Q)^d.
\]
If \( S \) is a singleton, \( \Sigma_S \) was not defined but we have \( S = \{Q\} = \{R\} \), so the above estimate holds trivially. In either case, combining (3.19) and (3.20) gives
\[
(3.21) \sum_{R \in \mathcal{F}_1} \ell(R)^d \leq \sum_{S \in \mathcal{F}} \sum_{Q \in S} \beta_{p,F}^{d,1}(MB_Q)^2 \ell(Q)^d.
\]
Let us consider \( \mathcal{F}_2 \). For \( R \in \mathcal{F}_2 \), we know there exists a child \( R' \) of \( R \) and a point \( x' \in R' \) such that \( \text{dist}(x', E) \geq \varepsilon \ell(R') \). We let \( C_R \) be the maximal cube containing \( x' \) such that \( \ell(C_R) \leq \frac{\varepsilon^2}{C_0} \ell(R) \). This implies that
\[
(3.22) \frac{\varepsilon^2}{C_0} \ell(R) \leq \ell(C_R) \leq \frac{\varepsilon \rho}{C_0} \ell(R).
\]
In this way, we have \( \text{dist}(y, E) > 0 \) for all \( y \in 2C_0B_{C_R} \) since
\[
\varepsilon \ell(R) \leq \text{dist}(x', E) \leq |x' - y| + \text{dist}(y, E) \leq 2C_0 \ell(C_R) + \text{dist}(y, E) \\
\leq 2\varepsilon \ell(R) + \text{dist}(y, E).
\]
In particular,
\[
(3.23) 2C_0B_{C_R} \cap E = \emptyset.
\]
Now, let \( x \in F, N \in \mathbb{N} \) and suppose \( \{R_i\}_{i=1}^N \) is a finite collection of distinct cubes in \( \mathcal{F}_2 \) such that
\[
x \in \bigcap_{i=1}^N C_{R_i}.
\]
We assume without loss of generality that \( \ell(R_i) > \ell(R_j) \) for all \( i < j \). It follows that
\[
\ell(R_j) \geq \frac{\varepsilon^2}{C_0} \ell(R_1)
\]
for all \( j = 2, \ldots, N \). Otherwise, by (3.22) we have \( R_N \subseteq C_{R_1} \), which by (3.23) gives that \( 2C_0B_{R_N} \cap E = \emptyset \). Thus, we arrive at a contradiction since \( E \cap 2C_0B_Q \neq \emptyset \) for any \( Q \in \mathcal{F}_2 \) by our stopping time condition (2). From (3.24), it follows that
\[
N \lesssim \varepsilon, C_0 1.
\]
(3.25)

Now, since \( F \) is lower content regular, we get
\[
\sum_{R \in \mathcal{F}_2} \ell(Q)^d \overset{(3.22)}{\lesssim} \sum_{R \in \mathcal{F}_2} \mathcal{H}^d(C_R) \overset{(3.25)}{\lesssim} \mathcal{H}^d(Q_0). \tag{3.26}
\]
Combining (3.21) and (3.26) completes the proof of the lemma. \( \square \)

4. Proof of Theorem 1.14

4.1. Constructing \( F \) and its properties. Let \( X_k^E \) be a sequence of maximal \( \rho^k \)-separated nets in \( E \) and let \( \mathcal{D}^E \) be the cubes from Theorem 2.1 with respect to these maximal nets. By scaling and translation, we can assume there is \( Q_0 \in \mathcal{D}_0^E \) which contains 0. Let \( M \geq 1 \) and \( \varepsilon, \alpha > 0 \).

**Remark 4.1.** The constants \( M \) and \( \varepsilon \) may be different from the previous section. We shall choose \( M \) sufficiently large, \( \varepsilon \) and \( \alpha \) shall be chosen sufficiently small.

We wish to construct \( F \) using the Reifenberg parametrisation theorem of David and Toro. Since \( E \) is not lower regular, we need the condition that we have \( (d+1, \alpha) \)-separated points to apply the construction. This will be added to our stopping time conditions.

For \( Q \in \mathcal{D}^E(Q_0) \), we define a stopping time region \( S_Q \) as follows. First, add \( Q \) to \( S_Q \). Then, inductively on scales, we add a cube \( R \) to \( S_Q \) if \( R^{(1)} \in S_Q \) and each sibling \( R' \) of \( R \) satisfies:

1. \[ \sum_{Q \supseteq T \supseteq R'} \beta^{d+1}_E(MB_Q)^2 \leq \varepsilon^2. \]
2. \( MB_{R'} \) has \( (d + 1, \alpha) \) separated points (in the sense of Definition 2.31).

We partition \( \mathcal{D}^E(Q_0) \) into a collection of stopping time regions \( \mathcal{S} \): Begin by adding \( S_{Q_0} \) to \( \mathcal{S} \). Then, if \( S \) has been added to \( \mathcal{S} \), add \( S_Q \) to \( \mathcal{S} \) if \( Q \in \text{Child}(R) \) for some \( R \in \min(S) \). We continue in this way to generate \( \mathcal{S} \). Let
\[
\min \mathcal{S} = \bigcup_{S \in \mathcal{S}} \{Q \in \min(S)\}
\]
denote the collection of all minimal cubes in \( \mathcal{S} \).

**Remark 4.2.** The following is essentially Lemma 3.6. The proof follows the same if we take \( \varepsilon \leq \alpha^{2d+4} \).
Lemma 4.3. There exists \( \varepsilon > 0 \) small enough so that for each \( S \in \mathcal{S} \), which is not a singleton, there is a surface \( \Sigma_S \) which satisfies the following. Firstly, if \( \Sigma_S \) denotes the bi-Lipschitz surface from Theorem 3.5, then

\[
\Sigma_S = \Sigma'_S \cap MB_{Q(S)}.
\]

Second, for each \( R \in S \) and \( y \in F \cap \frac{M}{4} B_R \),

\[
\text{dist}(y, \Sigma_S) \lesssim \varepsilon \pi^{\frac{d}{2}} (R)
\]

Finally, for each ball \( B \) centered on \( \Sigma_S \) and contained in \( MB_{Q(S)} \),

\[
\frac{\omega_d}{2} r_B^d \leq \mathcal{H}^d(\Sigma_S \cap B) \lesssim r_B^d.
\]

Remark 4.4. If \( S = \{Q\} \) is a singleton, we define \( \Sigma_S = P_Q \cap MB_Q \), where \( P_Q \) is some \( d \)-plane through \( x_Q \) such that \( \beta_{E}^{d,p}(MB_Q, P_Q) \leq 2 \beta_{E}^{d,p}(MB_Q) \).

Let \( \mathcal{B}_\alpha \subseteq \mathcal{P}^E(Q_0) \) be the set of cubes \( Q \) which have a sibling \( Q' \) which fails the stopping time condition (2) i.e. \( MB_{Q'} \) does not have \( (d+1, \alpha) \)-separated points. Consider those points in \( Q_0 \) for which we stopped a finite number of times or we never stopped. That is, we consider

\[
G := \left\{ x \in Q_0 : \sum_{Q \in \mathcal{B}_\alpha} \beta_{E}^{d,p}(MB_Q)^2 \chi_Q(x) < \infty \text{ and } \sum_{Q} \chi_Q(x) < \infty \right\}.
\]

Observe that \( G \subseteq \bigcup_{S \in \mathcal{S}} \Sigma_S \). We define \( E' = Q_0 \setminus G \) and set

\[
F := E' \cup \bigcup_{S \in \mathcal{S}} \Sigma_S.
\]

Lemma 4.5. \( F \) is \((c_1, d)\)-lower content regular.

Proof. Fix \( x \in F \) and \( r > 0 \). Assume first of all that \( x \in \Sigma_S \) for some \( S \in \mathcal{S} \). Let \( Q = Q(S) \). For any \( k \geq 0 \), recalling that \( Q^{(k)} \) is the \( k^{\text{th}} \) generational ancestor of \( Q \), we have

\[
|x - x_{Q^{(k)}}| \leq |x - x_Q| + |x_Q - x_{Q^{(k)}}| \leq M\ell(Q) + \ell(Q^{(k)}) \leq 2 M\ell(Q^{(k)})
\]

Assume first that \( r \geq 3 M\ell(Q) \) and let \( k = k(r) \geq 0 \) be the integer such that

\[
3 M\ell(Q^{(k)}) \leq r \leq 3 M\ell(Q^{(k+1)}).
\]

Then,

\[
MB_{Q^{(k)}} \subseteq B(x, 2 M\ell(Q^{(k)}) + M\ell(Q^{(k)})) \subseteq B(x, r).
\]

The lower regularity follow since

\[
\mathcal{H}^d(\mathcal{F} \cap B(x, r)) \geq \mathcal{H}^d(\Sigma_{S(Q^{(k)})}) \geq \frac{\omega_d}{2} (M\ell(Q^{(k)}))^d \geq \frac{\omega_d \rho}{6} r^d \geq c_1(2r)^d.
\]

Assume now that \( r < 3 M\ell(Q(S)) \). If \( B(x, r) \subsetneq MB_{Q(S)} \) we can trivially apply the lower regularity estimates for \( \Sigma_S \), so it suffices to consider the case when \( B(x, r) \supsetneq MB_{Q(S)} \). We split this into two further sub-cases: either \( B(x, r) \cap 10 B_{Q(S)} = \emptyset \) or \( B(x, r) \cap 10 B_{Q(S)} \neq \emptyset \).

In the first sub-case, since by (2.4), we have \( \Sigma_S \setminus 10 B_{Q(S)} = P_{Q(S)} \setminus 10 B_{Q(S)} \), the portion of \( \Sigma_S \) contained in \( B(x, r) \) is just a \( d \)-plane through \( x \). Since \( x \) is contained
in $MB_{Q(S)}$, we can find $y \in B(x, r) \cap \Sigma_S$ such that $B(y, r/4) \subseteq MB_{Q(S)} \cap B(x, r)$ and apply the lower regularity estimates for $\Sigma_S$ inside $B(y, r/4)$ to obtain
\[
\mathcal{H}_\infty^d(F \cap B(x, r)) \geq \mathcal{H}_\infty^d(\Sigma_S \cap B(y, r/4)) \geq \frac{\omega_d}{4^{d+1}} r^d \geq c_1(2r)^d.
\]
See for example the left image in Figure 4.

In the second sub-case, since $B(x, r) \not\subseteq MB_{Q(S)}$ but $B(x, r) \cap 10B_{Q(S)} \neq \emptyset$, it must be that $r$ is comparable with $\ell(Q)$. For $M$ sufficiently large ($M \geq 100$ is sufficient), we can certainly find $y \in B(x, r) \cap \Sigma_S$ such that $B(y, r/10) \subseteq MB_{Q(S)} \cap B(x, r)$ and apply the lower regularity estimates for $\Sigma_S$ inside $B(y, r/10)$ to get
\[
\mathcal{H}_\infty^d(F \cap B(x, r)) \geq \mathcal{H}_\infty^d(\Sigma_S \cap B(y, r/10)) \geq \frac{\omega_d}{10^{d+1}} r^d \geq c_1(2r)^d.
\]
See for example the right image in Figure 4.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{Sub-cases 1 and 2}
\end{figure}

Suppose now that $x \in E'$. Since $E'$ is the collection of points where we stopped an infinite number of times, we may find a sequence of stopping time regions $S_i$ such that $x \in Q(S_i)$ and $\ell(Q(S_i)) \downarrow 0$. We denote by $S$ the stopping time region for which $x \in S$ and $\ell(Q(S)) \leq r/10$. Then, by (4.2),
\[
dist(x, \Sigma_S) \lesssim \varepsilon \frac{1}{\ell(Q(S))} \leq \varepsilon \frac{1}{\ell(Q(S)) \frac{r}{10}}.
\]
Let $x'$ be the point in $\Sigma_S$ closest to $x$. For $\varepsilon$ small enough, $B(x', r/2) \subseteq B(x, r)$, which gives
\[
\mathcal{H}^d(F \cap B(x, r)) \geq \mathcal{H}^d(F \cap B(x', r/2)) \geq c_1 r^d,
\]
where the final inequality follows by the first case we considered. \hfill \square

To finish the proof of Theorem 1.14, it remains to show (1.5). If the right hand side of (1.5) is infinite then this proves Theorem 1.14. Hence from now on, we shall assume the right hand side of (1.5) is finite. We have the following.

**Lemma 4.6.** Assume that
\[
\sum_{Q \in \mathcal{D}(Q_0)} \beta_{E_i}^1(MB_Q)^2 \ell(Q)^d < \infty.
\]
Let \( \mathcal{H}^d(E') = 0 \), from which it follows that \( E \) is rectifiable and

\[
\mathcal{H}^d(F) \lesssim \ell(Q_0)^d + \sum_{Q \in \mathcal{D}(Q_0)} \beta_{E}^{d,1}(MB_{Q})^2\ell(Q)^d.
\]

Before proving Lemma 4.6, we shall need two preliminary results.

**Lemma 4.7.** Let \( S \in \mathcal{I} \) and \( Q \in S \). For \( \varepsilon > 0 \) small enough, we have

\[
\sum_{R \in \min S \atop R \subseteq Q} \ell(R)^d \lesssim \ell(Q)^d.
\]

**Proof.** Let \( R \in \min S \) satisfy \( R \subseteq Q \). By (4.2), the ball \( c_0B_R \) carves out a large proportion of \( \Sigma_S \), in particular, \( \mathcal{H}^d(\Sigma_S \cap c_0B_R) \gtrsim \ell(R)^d \). Furthermore, the balls \( \{c_0B_R\} \) are disjoint and contained in \( B_Q \). Using this, we have

\[
\sum_{R \in \min S \atop R \subseteq Q} \ell(R)^d \lesssim \sum_{R \in \min S \atop R \subseteq Q} \mathcal{H}^d(\Sigma_S \cap c_0B_R) \leq \mathcal{H}^d(\Sigma_S \cap B_Q) \lesssim \ell(Q)^d.
\]

\[\square\]

**Lemma 4.8.**

\[
\sum_{R \in \min \mathcal{I}} \ell(R)^d \lesssim \ell(Q_0)^d + \sum_{S \in \mathcal{I}} \sum_{Q \in S} \beta_{E}^{d,1}(MB_{Q})^2\ell(Q)^d.
\]

**Proof.** We split \( \min \mathcal{I} \) into two sub families, Type1 and TypeII, where Type1 is the collection of cubes \( R \in \min \mathcal{I} \) such that each child \( R' \) of \( R \) has \( (d+1, \alpha) \)-separated points, but there is a child \( R'' \) such that

\[
\sum_{Q \in S(R) \atop Q \supseteq R''} \beta_{E}^{d,1}(MB_{Q})^2 \geq \varepsilon^2
\]

and TypeII = \( \min \mathcal{I} \setminus \text{Type1} \). Controlling cubes in Type1 is done in a similar way to how we controlled the cubes in \( \mathcal{F}_1 \) in the proof of Lemma 3.10, the only difference being that we construct the surfaces \( \Sigma_S \) by Lemma 4.3 instead of Lemma 3.6. This is because \( E \) is not necessary lower regular but we know each cube in a stopping time region (which is not a singleton) has \( (d+1, \alpha) \)-separated points. This gives

\[
\sum_{Q \in \text{Type1}} \ell(Q)^d \lesssim \sum_{Q \in \mathcal{D}(Q_0)} \beta_{E}^{d,1}(MB_{Q})^2\ell(Q)^d.
\]

We now consider cubes in TypeII. We will in fact show that

\[
\sum_{Q \in \text{TypeII}} \ell(Q)^d \lesssim \ell(Q_0)^d + \sum_{Q \in \text{TypeI}} \ell(Q)^d
\]

from which the lemma follows immediately. We state and prove three preliminary claims before we proceed with the proof of (4.7).
Claim 1: For all $\delta > 0$ there exists $\alpha > 0$ such that if $Q \in \text{Type}_{II}$ and
\[
k^* = \left\lfloor \frac{\log(2\alpha M \rho/c_0)}{\log \rho} \right\rfloor,\]
then
\[
\sum_{R \in \text{Child}_{k^*}(Q)} \ell(R)^d \leq \delta \ell(Q)^d.
\]
(4.8)

Let $\delta > 0$ and assume that $Q \in \text{Type}_{II}$. By definition, there exists $Q' \in \text{Child}(Q)$ such that $\beta_{E,\infty}^{d-1}(MB_{Q'}) \leq \alpha$. If $L$ is the $(d - 1)$-plane such that $\beta_{E,\infty}^{d-1}(MB_{Q'}, L)$,

then each $R \subseteq Q$ satisfies
\[
dist(x_R, L) \leq \alpha M \ell(Q') = \alpha M \rho \ell(Q).
\]

By our choice of $k^*$, if $R \subseteq Q$ is such that $\ell(R) = \rho^{k^*} \ell(Q)$ then $dist(x_R, L) \leq \frac{\alpha}{2} \ell(R)$. Thus, by Lemma 2.5,
\[
\sum_{R \in \text{Child}_{k^*}(Q)} \ell(R)^{d-1} \leq C \ell(Q)^{d-1}.
\]

Multiplying both sides by $\ell(R)$, we obtain
\[
\sum_{R \in \text{Child}_{k^*}(Q)} \ell(R)^d \leq C \ell(Q)^{d-1} \ell(R) \leq C \rho^{k^*} \ell(Q)^d.
\]

By taking $\alpha > 0$ small enough, we can ensure that $C \rho^{k^*} < \delta$. This proves the claim.

Let $Q \in \text{Type}_{II}$. Define $\text{Type}_1(Q)$ to be the maximal collection of cubes in $\text{Type}_1$ contained in $Q$. Let $\text{Tree}(Q)$ be the collection of cubes $R \in \mathcal{D}^E$ such that $R \subseteq Q$ and $R$ is not properly contained in any cube from $\text{Type}_1(Q)$ and
\[
\text{Type}_{II}(Q) = \text{Tree}(Q) \cap \text{Type}_{II}.
\]

We also define sequences of subsets of $\text{Tree}(Q)$, $\mathcal{T}_k$ and $\mathcal{M}_k$ as follows: Let
\[
\mathcal{T}_0 = \{Q\} \quad \text{and} \quad \mathcal{M}_0 = \min S_Q.
\]

Then, supposing that $\mathcal{T}_k$ and $\mathcal{M}_k$ have been defined for some integer $k \geq 0$, we let
\[
\mathcal{T}_{k+1} = \{T \in \text{Tree}(Q) : T \in \text{Child}_{k^*}(R) \text{ for some } R \in \mathcal{M}_k\}
\]
and
\[
\mathcal{M}_{k+1} = \{T \in \text{Type}_{II}(Q) : T \text{ is max so that } T \subseteq R \text{ for some } R \in \mathcal{T}_{k+1}\}
\]

Recalling the definition of $\text{Des}_{k^*}(R)$ from (2.2), we have the following:

Claim 2:
\[
\text{Type}_{II}(Q) \subseteq \{Q\} \cup \bigcup_{k=0}^{\infty} \bigcup_{R \in \mathcal{M}_k} \text{Des}_{k^*-1}(R).
\]
(4.9)
Let $R \in \text{Type}_{II}(Q)$. The claim is clearly true for $R = Q$ so let us assume $R \neq Q$. Let $k_R$ be the largest integer $k \geq 0$ such that there is a cube $T \in \mathcal{M}_k$ with $R \subseteq T$. The existence of such a $k$ is guaranteed since there exists $T \in \mathcal{M}_0$ such that $R \subseteq T$ and each cube $T \in \mathcal{M}_k$ satisfies $\ell(T) \leq \rho^{kk^*} \ell(Q)$. Let $T_R \in \mathcal{M}_{k_R}$ be such that $R \subseteq T_R$. Assume $R \in \text{Child}_k(T_R)$ for some $k \geq k^*$. If this is the case then we can find a cube $R' \in \text{Child}_{k^*}(T_R)$ such that $R \subseteq R'$, recall by definition that $R' \in \mathcal{T}_{k_R+1}$.

By maximality this implies that there exists some $R'' \in \mathcal{M}_{k_R+1}$ such that $R \subseteq R''$, which contradicts the definition of $k_R$. It follows that $R \in \text{Des}_{k^*-1}(T_R)$ which completes the proof of the claim.

**Claim 3:** There exists $\alpha > 0$ small enough so that for each $k \geq 0$,

$$
(4.10) \quad \sum_{R \in \mathcal{M}_k} \ell(R)^d \lesssim \left(\frac{1}{2}\right)^k \ell(Q)^d.
$$

The results holds for $k = 0$ by Lemma 4.7. Assume $k \geq 1$ and assume the result holds for $k - 1 \geq 0$. We will show that it holds for $k$. Let $R \in \mathcal{M}_k$ and $S(R)$ be the stopping time region $S \in \mathcal{S}$ such that $R \in S$. By maximality, then there exists $T \in \mathcal{T}_k \cap S(R)$ such that $R \subseteq T$. By Lemma 4.7, Claim 1 and the definitions of $\mathcal{T}_k$ and $\mathcal{M}_k$, we have

$$
\sum_{R \in \mathcal{M}_k} \ell(R)^d \leq \sum_{T \in \mathcal{T}_k} \sum_{R \in \mathcal{M}_k} \ell(T)^d \lesssim C \sum_{T \in \mathcal{T}_k} \ell(T)^d \leq C \delta \sum_{T \in \mathcal{M}_{k-1}} \ell(T)^d \lesssim C \delta \left(\frac{1}{2}\right)^{k-1} \ell(Q)^d.
$$

Choosing $\delta$ (and hence $\alpha$) small enough so that $C \delta < 1/2$ we get the required result.

We return to the proof of (4.7). Our first goal is to find a bound on the sum over cubes in Type$_{II}(Q)$. Suppose $R, R' \in \text{Tree}(Q)$ are such that $R' \subset R$. Any cube $T \in \mathcal{T}$ such that $R' \subset T \subset R$ is either the child of a Type$_{II}$ cube or is contained in some stopping that is not a singleton. In either case

$$
\beta_{E,\infty}^{d-1}(MB_T) \leq \alpha \quad \text{or} \quad \beta_{E}^{d,1}(MB_T) \leq \varepsilon.
$$

So, by Lemma 2.7 (for $\alpha$ and $\varepsilon$ small enough),

$$
\sum_{T \in \text{Des}_{k^*-1}(R)} \ell(T)^d \lesssim \alpha \ell(R)^d.
$$

Combining all the above, we get

$$
\sum_{R \in \text{Type}_{II}(Q)} \ell(R)^d \lesssim \ell(Q)^d + \sum_{k=0}^{\infty} \sum_{R \in \mathcal{M}_k} \sum_{T \in \text{Des}_{k^*-1}(R)} \ell(T)^d \lesssim \ell(Q)^d + \sum_{k=0}^{\infty} \sum_{R \in \mathcal{M}_k} \ell(R)^d \lesssim \ell(Q)^d + \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \ell(Q)^d \lesssim \ell(Q)^d.
$$
Now, for each \(Q \in \text{Type}_{II}\), let \(R(Q)\) be the smallest cube in Type\(_1\) which contains \(Q\). By construction, if \(T \in \text{Tree}(Q)\) for some \(Q \in \text{Type}_{II}\) then \(R(T) = R(Q)\). Given this, if \(R \in \text{Type}_1\),

\[
\sum_{Q \in \text{Type}_{II}} \ell(Q)^d = \sum_{Q \in \text{Child}(R) \cap \text{Type}_{II}} \sum_{T \in \text{Type}_{II}(Q)} \ell(T)^d \\
\lesssim \sum_{Q \in \text{Child}(R) \cap \text{Type}_{II}} \ell(Q)^d \lesssim \ell(R)^d.
\]

It may be that there is a cube \(Q\) in Type\(_1\) which is not contained in any cube from Type\(_1\). If this is the case, then \(Q_0 \in \text{Type}_{II}\) and \(Q \in \text{Type}_{II}(Q_0)\). In any case, we get

\[
\sum_{Q \in \text{Type}_{II}} \ell(Q)^d \leq \sum_{Q \in \text{Type}_{II}(Q_0)} \ell(Q)^d + \sum_{R \in \text{Type}_1} \sum_{Q \in \text{Type}_{II}} \ell(Q)^d \\
\lesssim \ell(Q_0)^d + \sum_{Q \in \text{Type}_1} \ell(Q)^d.
\]

\(\square\)

**Proof of Lemma 4.6.** Let \(x \in E'\). By definition, for each \(r > 0\) there exists \(Q \in \min \mathcal{J}\) such that \(\ell(Q) < r\) and \(x \in Q\). By induction, we can construct a sequence of distinct covers \(\mathcal{G}_k \subseteq \min \mathcal{J}\) of \(E'\) such that \(\ell(Q) < \frac{1}{k}\) for each \(Q \in \mathcal{G}_k\). With the finite assumption (4.4), it must be that

(4.11) \[\lim_{k \to \infty} \sum_{Q \in \mathcal{G}_k} \ell(Q)^d = 0.\]

Assume towards a contradiction that there exists \(\delta > 0\) and \(K \in \mathbb{N}\) such that \(\sum_{Q \in \mathcal{G}_k} \ell(Q)^d > \delta\) for all \(k \geq K\). By this and Lemma 4.8 it follows that

\[
\ell(Q_0)^d + \sum_{S \in \mathcal{J}} \sum_{Q \in S} \beta_E^{d,1} (MB_Q)^2 \ell(Q)^d \geq \sum_{Q \in \min \mathcal{J}} \ell(Q)^d \geq \sum_{k=1}^{\infty} \sum_{Q \in \mathcal{G}_k} \ell(Q)^d = \infty
\]

which contradicts (4.4) and proves (4.11). The fact that \(\mathcal{H}^d(E') = 0\) follows from (4.11) since

\[
\mathcal{H}^d(E') = \lim_{k \to \infty} \mathcal{H}^d_{\mathcal{J}}(E') \leq \lim_{k \to \infty} \sum_{Q \in \mathcal{G}_k} \ell(Q)^d = 0.
\]

Furthermore

\[
\mathcal{H}^d(F) \leq \mathcal{H}^d \left( \bigcup_{S \in \mathcal{J}} \Sigma_S \right) \leq \sum_{S \in \mathcal{J}} \ell(Q(S))^d \\
\lesssim \ell(Q_0)^d + \sum_{Q \in \min \mathcal{J}} \sum_{R \in \text{Child}(Q)} \ell(R)^d \\
\lesssim \ell(Q_0)^d + \sum_{Q \in \min \mathcal{J}} \ell(Q)^d \\
\lesssim \ell(Q_0)^d + \sum_{Q \in \mathcal{J}} \beta_E^{d,1} (MB_Q)^2 \ell(Q)^d,
\]

(4.12)
which proves (4.5). The inequality from the second the third lines follows because any $Q \in \min \mathcal{F}$ has at most $K = K(M, d)$ children by Lemma 2.6.

4.2. Proof of (1.5). Recall the definition of $Q_0^F$ from the statement of Theorem 1.14. Just like at the beginning of the proof of Theorem 1.13, we can reduce proving (1.5) to proving

\[
\text{(4.13)} 
\begin{align*}
\diam(Q_0^F)^d + \sum_{Q \in \mathcal{D}(Q_0)} \beta^d_F(C_0B_Q)^2 \ell(Q)^d
\lesssim \diam(Q_0)^d + \sum_{Q \in \mathcal{D}(Q_0)} \beta_1^d(AB_Q)^2 \ell(Q)^d,
\end{align*}
\]

that is, we can replace the constant $C_0$ by some larger constant $A$ and set $p = 1$ on the right-hand side. The rest of this section is devoted to proving (4.13). In the statement of Theorem 1.14 we assume $\diam(Q_0) \geq \lambda \ell(Q_0)$. We have the following bound on the first term:

\[
\diam(Q_0^F) \leq \ell(Q_0^F) = \ell(Q_0) \lesssim \lambda \diam(Q_0).
\]

So, in order to prove (4.13), it suffices to bound the second term. Let $\{S_i\}$ be an enumeration of the stopping time regions in $\mathcal{F}$ which are not singletons. First, we observe that

\[
\sum_{Q \in \mathcal{D}(Q_0)} \beta^d_F(C_0B_Q)^2 \ell(Q)^d = \sum_i \sum_{Q \in S_i} \beta^d_F(C_0B_Q)^2 \ell(Q)^d + \sum_{S \in \mathcal{F}} \beta^d_F(C_0B_Q)^2 \ell(Q)^d.
\]

If $S \in \mathcal{F}$ is such that $S = \{Q\}$ then $Q \in \min \mathcal{F}$. Then, since $\beta^d_F(\cdot) \lesssim 1$, the second term on the right hand side of the above equation is at most some constant multiple of

\[
\sum_{Q \in \min \mathcal{F}} \ell(Q)^d,
\]

which we bound by Lemma 4.8. Thus, to prove (4.13) it suffices to bound the first term. Using the $\beta$-error estimate (Lemma 2.29), we obtain

\[
\sum_i \sum_{Q \in S_i} \beta^d_F(C_0B_Q)^2 \ell(Q)^d \overset{(2.31)}{\lesssim} \sum_i \sum_{Q \in S_i} \beta^d_F(2C_0B_Q)^2 \ell(Q)^d
\]

\[
+ \sum_i \sum_{Q \in S_i} \left( \frac{1}{\ell(Q)^d} \int_{F \cap 2C_0B_Q} \left( \frac{\dist(x, E)}{\ell(Q)^d} \right)^p d\mathcal{H}_\infty(x) \right)^{\frac{1}{p}} \ell(Q)^d.
\]

We have a trivial bound for the first term so we now focus on bounding the second term, the proof of which is very similar to the proof of Lemma 3.7. First, let

\[
\mathcal{D}' = \bigcup_i S_i.
\]
We split $\mathcal{D}'$ into two families. Let $\delta > 0$ be small and define

$$\mathcal{G} = \{ Q \in \mathcal{D}' : \text{dist}(x, E) \leq \delta \ell(Q) \text{ for all } x \in F \cap \frac{M}{2} B_Q \},$$
$$\mathcal{B} = \mathcal{D}' \setminus \mathcal{G}.$$

To each cubes $Q \in \mathcal{B}$ we shall assign a patch $C_Q$ of $F$. By definition, if $Q \in \mathcal{B}$ then there exists a point $y_Q \in F \cap MB_Q$ such that $\text{dist}(y_Q, E) > \delta \ell(Q)$. We define $C_Q = B(y_Q, \delta \ell(Q)/2)$.

We claim that the balls $\{C_Q\}_{Q \in \mathcal{B}}$ have bounded overlap in $F$. This is the content of the following lemma.

**Lemma 4.9.** The collection of balls $\{C_Q\}_{Q \in \mathcal{B}}$ have bounded overlap in $F$.

**Proof.** Let $x \in F$. We will show

$$\mathcal{B}_x = \# \{ Q \in \mathcal{B} : x \in C_Q \} \lesssim 1. \quad (4.14)$$

First, note that

$$C_Q \subseteq \frac{M}{2} B_Q$$

for all $Q \in \mathcal{B}$. Let $Q, Q' \in \mathcal{B}$. If $\ell(Q') \leq \frac{\delta}{2M} \ell(Q)$ and $y \in C_Q$ we have

$$|y - x_Q| \geq |y_Q - x_Q| - |y - y_Q| \geq \delta \ell(Q) - \frac{\delta \ell(Q)}{2} \geq \frac{\ell(Q)}{2},$$

in particular, $C_Q \cap \frac{M}{2} B_{Q'} = \emptyset$. Since $C_Q \subseteq \frac{M}{2} B_{Q'}$, we must have that $C_Q \cap C_{Q'} = \emptyset$. Reversing the role of $Q$ and $Q'$ above, we conclude that if $C_Q \cap C_{Q'} = \emptyset$ then $\ell(Q) \sim_{M, \delta} \ell(Q')$. In particular, (4.14) follows if we can show that for each $k$,

$$\{ Q \in \mathcal{B} \cap S_k : x \in C_Q \} \lesssim 1, \quad (4.15)$$

with constant independent of $k$. Fix $k$ and let $Q, Q' \in \mathcal{B} \cap S_k$ such that $C_Q \cap C_{Q'} \neq \emptyset$. Then $\frac{M}{2} B_Q \cap \frac{M}{2} B_{Q'} \neq \emptyset$ which implies $|x_Q - x_{Q'}| \leq M \ell(Q)$. Since, $\beta_{E}^{d, p}(MB_Q) \leq \varepsilon$, we have $\text{dist}(x_{Q'}, P_Q) \lesssim \varepsilon^{\frac{1}{d+1}} \ell(Q) = \varepsilon^{\frac{1}{d+1}} \ell(Q')$. For $\varepsilon$ small enough, (4.15) follows from Lemma 2.5. \hfill $\square$

**Lemma 4.10.** We have

$$\sum \sum_{Q \in \mathcal{S}_i} \left( \frac{1}{\ell(Q)^d} \int_{F \cap 2C_0 B_Q} \left( \frac{\text{dist}(x, E)}{\ell(Q)} \right)^p \text{d}\mathcal{H}_{\infty}^d (x) \right)^{\frac{2}{p}} \ell(Q)^d \lesssim \ell(Q_0)^d + \sum_{Q \in \mathcal{D} \cap \mathcal{S}_i} \beta_{E}^{d, 1}(MB_Q)^2 \ell(Q)^d. \quad (4.14)$$

**Proof.** By Jensen’s inequality (Lemma 2.18), we may assume $p \geq 2$. Let $\mathcal{D}', \mathcal{G}$ and $\mathcal{B}$ be as above Lemma 4.9. First, since $F$ is lower regular and the $C_Q$ have bounded overlap, we get

$$\sum_{Q \in \mathcal{B}} \left( \frac{1}{\ell(Q)^d} \int_{F \cap 2C_0 B_Q} \left( \frac{\text{dist}(x, E)}{\ell(Q)} \right)^p \text{d}\mathcal{H}_{\infty}^d (x) \right)^{\frac{2}{p}} \ell(Q)^d \leq \sum_{Q \in \mathcal{B}} \ell(Q)^d \lesssim \sum_{Q \in \mathcal{B}} \mathcal{H}^d(C_Q) \lesssim \mathcal{H}^d(F) \quad (4.14)$$

$$\lesssim \ell(Q_0)^d + \sum_{Q \in \mathcal{D} \cap \mathcal{S}_i} \beta_{E}^{d, 1}(MB_Q)^2 \ell(Q)^d. \quad (4.5)$$
Consider a single $S = S_i$. Let $S_{\mathcal{G}}$ denote $S \cap \mathcal{G}$. Let $\mathscr{D}^F$ denote the Christ-David cubes for $F$ and let $S_{\mathcal{G}}^F$ be the smoothed out cubes in $F$ with respect to $S_{\mathcal{G}}$. Define $\tau = 1/2(1 + \rho)$ (as in (3.6)) then let

$$S_{\mathcal{G}}^F = \{Q \in \mathscr{D}^F : Q \text{ is maximal with } \ell(Q) < \tau d_{S_{\mathcal{G}}}(Q)\}.$$  

Let $R \in S_{\mathcal{G}}^F$.

**Claim 1.** For each $y \in R$, $\text{dist}(y, E) \lesssim \ell(R)$.

By a direct analogue of Lemma 3.8 (whose proof is exactly the same), there exists $Q \in S_{\mathcal{G}}$ such that $\tau \text{dist}(Q, R) \lesssim \ell(R)$ and $\tau \ell(Q) \sim \ell(R)$. Let $y^Q$ be the point in $Q$ closest to $y$. Then,

$$\text{dist}(y, E) \leq \text{dist}(y, y^Q) + \text{dist}(y^Q, E) \lesssim \frac{\ell(R)}{\tau} + \delta \ell(Q) \lesssim \ell(R).$$

**Claim 2.** We have $\text{dist}(x_R, \Sigma_S) \lesssim \frac{\epsilon}{\tau} \ell(R)$.

Let $Q \in S_{\mathcal{G}}$ be as in the proof of Claim 1. We can chose $M$ large enough (depending on $\tau$) so that $R \subseteq M^4 B_Q$. Then, by Lemma 4.3 and the fact that $\ell(R) \sim \tau \ell(Q)$,

$$(4.16) \quad \text{dist}(x_R, \Sigma_S) \lesssim \frac{\epsilon}{\tau} \ell(Q) \lesssim \frac{\epsilon}{\tau} \ell(R).$$

**Claim 3.** For each $k \in \mathbb{N}$,

$$\#\{Q \in \mathscr{D}_k \cap S_{\mathcal{G}} : R \cap 2C_0 B_Q \neq \emptyset\} \lesssim 1.$$  

If $Q \in \mathscr{D}_k \cap S_{\mathcal{G}}$ is such that $R \cap 2C_0 B_Q \neq \emptyset$, then

$$\ell(R) \leq \tau(\ell(Q) + \text{dist}(R, Q)) \lesssim \ell(Q).$$

Let $x'_R$ be the point in $\Sigma_S$ closest to $x_R$. By the above and (4.16) there exists a constant $A > 0$ so that if $Q \in \mathscr{D}_k \cap S$ is such that $R \cap 2C_0 B_Q \neq \emptyset$ then

$$Q \subseteq B(x'_R, A\ell(Q)) := B.$$  

Since $\text{dist}(x_Q, \Sigma_S) \lesssim \epsilon \frac{1}{\tau} \ell(Q)$, the balls $c_0 B_Q$ carve out a large proportion of $\Sigma_S$. Then, by (4.3),

$$\sum_{Q \in S_{\mathcal{G}} : R \cap 2C_0 B_Q \neq \emptyset} \ell(Q)^d \lesssim \sum_{Q \in S_{\mathcal{G}} \cap c_0 B_Q : R \cap 2C_0 B_Q \neq \emptyset} \mathcal{H}^d(\Sigma_S \cap c_0 B_Q) \leq \mathcal{H}^d(\Sigma_S \cap B) \lesssim \ell(Q)^d,$$

which proves the claim.
Since we have assumed $p > 2$, we apply Jensen’s inequality (Lemma 2.18) and Claim 1 to get

\[
I := \sum_{Q \in S_S} \left( \frac{1}{\ell(Q)^d} \int_{F \cap 2C_0B_Q} \left( \frac{\text{dist}(x, E)}{\ell(Q)} \right)^p dH^d_\infty(x) \right)^\frac{2}{p} \ell(Q)^d
\]

\[
\leq \sum_{Q \in S_S} \left( \sum_{R \in S_F \cap 2C_0B_Q \neq \emptyset} \frac{\ell(R)^{d+p}}{\ell(Q)^{d+p}} \right)^\frac{2}{p} \ell(Q)^d \approx \sum_{Q \in S_S} \sum_{R \in S_F \cap 2C_0B_Q \neq \emptyset} \frac{\ell(R)^{\frac{d}{p}+2}}{\ell(Q)^{(\frac{d}{p}-1)+2}} \ell(Q)^d.
\]

By Claim 3, we swap the order of integration and sum over a geometric series to get,

\[
I \lesssim \sum_{R \in S_F \cap 2C_0B_Q \neq \emptyset} \sum_{Q \in S_S \cap 2C_0B_Q \neq \emptyset} \frac{\ell(R)^{\frac{d}{p}+2}}{\ell(Q)^{(\frac{d}{p}-1)+2}} \lesssim \sum_{R \in S_F \cap 2C_0B_Q \neq \emptyset} \ell(R)^d.
\]

By (4.16), for $\varepsilon$ small enough, the ball $c_0B_R$ carves out a large proportion of $\Sigma_S$ for each $R \in S_F$, i.e. $H^d(c_0B_R \cap \Sigma_S) \gtrsim \ell(R)^d$. By (4.3), using the fact the $c_0B_R$ are disjoint, we have

\[
I \lesssim \sum_{R \in S_F \cap 2C_0B_Q \neq \emptyset} H^d(c_0B_R \cap \Sigma_S) \leq H^d(B_{Q(S)} \cap \Sigma_S) \lesssim \ell(Q(S))^d.
\]

Hence,

\[
\sum_{Q \in S_S} \left( \frac{1}{\ell(Q)^d} \int_{F \cap 2C_0B_Q} \left( \frac{\text{dist}(x, E)}{\ell(Q)} \right)^p dH^d_\infty(x) \right)^\frac{2}{p} \ell(Q)^d \lesssim \sum_{S \in \mathcal{F}} \ell(Q(S))^d \overset{(4.12)}{\lesssim} \ell(Q_0)^d + \sum_{Q \in \mathcal{F}^E} \beta_{E,1}(MBQ)^2 \ell(Q)^d.
\]

We have proved so far that

\[
\text{diam}(Q_0^F)^d + \sum_{Q \in \mathcal{F}^E(Q_0)} \beta_{E,F,q}(C_0B_Q)^2 \ell(Q)^d \lesssim \text{diam}(Q_0)^d + \sum_{Q \in \mathcal{F}^E(Q_0)} \beta_{E,1}(MBQ)^2 \ell(Q)^d.
\]

In order to finish the proof of Theorem 1.14, we wish to prove the same inequality but with the sum on the left hand side of (4.17) taken over all cubes in $\mathcal{F}^E(Q_0^F)$. We do this by partitioning the cubes in $F$ into those which lie close to $E$ and those which do not. We control the sum over $F$-cubes which lie close to $E$ by the corresponding sum over $E$-cubes and shall control the sum over the $F$-cubes far away from $E$ by Theorem 1.7.
We define a Whitney decomposition of $F \setminus E$. For $k \geq 0$, we let
\[
\mathcal{B}_k^E = \bigcup_{Q \in \mathcal{D}_k^F} MB_Q.
\]

**Lemma 4.11.** Define
\[
\text{Top}_F = \{Q \in \mathcal{D}_1^F(Q_0^F) : Q \text{ is max such that } (C_0 + M)\ell(Q) < \text{dist}(x_Q, E)\}.
\]

Let $k \in \mathbb{N}$ and $Q \in \text{Top}_F \cap \mathcal{D}_k^F$. Then
\[
\begin{align*}
C_0B_Q \cap \mathcal{B}_l^F &= \emptyset \quad \text{for all } l \geq k \\
\text{and} & \\
C_0B_Q \subseteq 3\mathcal{B}_l^F \quad \text{for all } 0 \leq l \leq k - 1.
\end{align*}
\]

**Proof.** Let $Q \in \text{Top}_F \cap \mathcal{D}_k^F$. By definition, (4.18) is immediate so let us prove (4.19). By maximality, we have
\[
\text{dist}(x_Q, E) \leq |x_Q - x_{Q(1)}| + \text{dist}(x_{Q(1)}, E) \leq (1 + C_0 + M)\rho^{-1}\ell(Q).
\]

Let $z_Q$ be the point in $E$ closest to $x_Q$ and let $Q'$ be the cube in $\mathcal{B}_{k-1}^E$ such that $z_Q \in Q'$. Then, for $y \in C_0B_Q$, we have
\[
|y - x_Q| \leq |y - x_Q| + |x_Q - z_Q| + |z_Q - x_{Q'}| 
\leq \ell(Q) + (1 + C_0 + M)\rho^{-1}\ell(Q) + \ell(Q') 
\leq 3M\rho^{-1}\ell(Q) = 3M\ell(Q')
\]

This implies $y \in 3MB_{Q'}$. Since $y$ is arbitrary point in $C_0B_Q$, we have
\[
C_0B_Q \subseteq 3MB_{Q'} \subseteq 3\mathcal{B}_{k-1}^F.
\]

Clearly, this also implies that $C_0B_Q \subseteq 3\mathcal{B}_l^F$ for all $0 \leq l \leq k - 1$. \qed

**Lemma 4.12.** The collection of balls $\{B_Q\}_{Q \in \text{Top}_F}$ have bounded overlap with constant dependent on $n$.

**Proof.** Let $x \in \mathbb{R}^n$ and let
\[
\mathcal{D}_x = \{Q \in \text{Top}_F : x \in B_Q\}.
\]

We first show that each cube in $\mathcal{D}_x$ has comparable size. Let $Q, Q' \in \mathcal{D}_x$ and assume without loss of generality that $\ell(Q) \leq \ell(Q')$. Since $Q' \in \text{Top}_F$ and $B_Q \cap B_{Q'} \neq \emptyset$ we have
\[
\ell(Q') \leq (C_0 + M)^{-1}\text{dist}(x_{Q'}, E) \leq (C_0 + M)^{-1}(|x_Q - x_{Q'}| + \text{dist}(x_Q, E)) 
\leq (C_0 + M)^{-1}(2\ell(Q') + \text{dist}(x_Q, E)).
\]

Taking $M$ large enough so that $2(C_0 + M)^{-1} \leq \frac{1}{2}$, we can rearrange the above equation to give
\[
\ell(Q') \leq \frac{2M}{C_0 + M} \text{dist}(x_Q, E) \overset{4.20}{\lesssim} \ell(Q)
\]

which proves that
\[
\ell(Q) \sim \ell(Q').
\]
By a standard volume argument, for each $k \in \mathbb{N}$ we have
$$\#(Q_x \cap D_k) \lesssim n,$$
which when combined with (4.21) finishes the proof of the lemma. \qed

**Lemma 4.13.**
\[
\sum_{Q \in \text{Top}_F} \sum_{R \subseteq Q} \beta^{d,p}_F (C_0 B_R)^2 \ell(R)^d \lesssim \ell(Q_0^E)^d + \sum_{Q \in \mathcal{E}} \beta^{d,p}_E (MB_Q)^2 \ell(Q)^d.
\]

**Proof.** Let $Q \in \text{Top}_F \cap \mathcal{D}_k$ and let $\mathcal{S}_Q \subseteq \mathcal{D}$ be the collection of stopping time regions such that $C_0 B_Q \cap \Sigma_{Q(S)} \neq \emptyset$. Since $C_0 B_Q \cap \mathcal{B} = \emptyset$ for all $l \geq k$, by (4.18), it must be that $Q(S) \in \mathcal{D}^E$ for some $0 \leq l \leq k - 1$. By (4.19) it follows that
\[
C_0 B_Q \subseteq 3MB_Q(S)
\]
for each $S \in \mathcal{S}_Q$. We wish to use that fact that in each of these balls, $F$ is well approximated by a union of planes. At the minute this is not quite the case. It could be that $C_0 B_Q$ intersects $MB_Q(S)$ at the boundary for some $S \in \mathcal{S}_Q$ (see for example Figure 5). As such we must extend each of the surfaces $\Sigma_S$. The resulting union of these extended surfaces still has comparable measure to $F$.

**Figure 5.** $F$ is not necessarily well approximated by a union of planes.

Recall from Lemma 4.3 that $\Sigma_S'$ is the unbounded bi-Lipschitz surface from Theorem 3.5. For each $S \in \mathcal{S}$, let $\tilde{\Sigma}_S$ be the surface obtained by restricting $\Sigma_S'$ to $6MB_Q(S)$, i.e.
\[
\tilde{\Sigma}_S = \Sigma_S' \cap 6MB_Q(S).
\]

Compare this to how we define $\Sigma_S$ in (4.1). This ensures that each $\tilde{\Sigma}_S$ is $(C\varepsilon,d)$-Reifenberg flat in $3MB_Q(S)$. Clearly we have
\[
F \cap C_0 B_Q \subseteq E' \cup \left( \bigcup_{S \in \mathcal{S}_Q} \tilde{\Sigma}_S \right) \cap C_0 B_Q.
\]
Define

\[ F_Q = E' \cup \left( \bigcup_{S \in \mathcal{J}_Q} \tilde{\Sigma}_S \right) \cup \left( \bigcup_{S \in \mathcal{J} \setminus \mathcal{J}_Q} \Sigma_S \right). \]

We can show that \( F_Q \) is lower regular in exactly the same way as we did for \( F \) (we shall omit the details). Let \( \mathcal{Q}^F \) denote the cubes for \( F_Q \) from Theorem 2.1. In this way, for each \( R \in \mathcal{Q}^F \) there exists a corresponding cube \( \tilde{R} \in \mathcal{Q}^F \) such that \( x_R = x_{\tilde{R}} \) and \( \ell(R) = \ell(\tilde{R}) \). It is clear then, that

\[ C_0 B_R = C_0 B_{\tilde{R}}. \]

Now, let \( R \subseteq Q \) and let \( \tilde{R} \in \mathcal{Q}^F \) be the cube described above. If \( S \in \mathcal{J}_Q \) and \( \tilde{\Sigma}_S \cap C_0 B_{\tilde{R}} \neq \emptyset \), let \( x_S \in \tilde{\Sigma}_S \cap C_0 B_{\tilde{R}} \) be a point of intersection. Since

\[ C_0 B_{\tilde{R}} \subseteq B(x_S, 3C_0 \ell(\tilde{R})) \subseteq 3MB_Q(S) \]

and \( \tilde{\Sigma}_S \) is \((C\varepsilon, d)\)-Reifenberg flat in \( 3MB_Q(S) \), we can find a plane \( L_S \) through \( x_S \) such that

\[ d_{C_0 B_{\tilde{R}}} (\tilde{\Sigma}_S, L_S) \leq 3C\varepsilon. \]

Let

\[ U_{\tilde{R}} := \bigcup_{S \in \mathcal{J}_Q \atop C_0 B_{\tilde{R}} \cap \tilde{\Sigma}_S \neq \emptyset} L_S. \]

Since by construction we have

\[ F_Q \cap C_0 B_{\tilde{R}} = \bigcup_{S \in \mathcal{J}_Q \atop C_0 B_{\tilde{R}} \cap \tilde{\Sigma}_S \neq \emptyset} \tilde{\Sigma}_S, \]

it follows that

\[ d_{C_0 B_{\tilde{R}}} (F_Q, U_{\tilde{R}}) \leq 3C\varepsilon, \]

i.e. \( \tilde{R} \notin BAUP(C_0, 3C\varepsilon) \). See the below Figure 6. Since \( Q \in \text{Top}_F \) and \( R \subseteq Q \)

\[ \text{Figure 6. An illustration of the above argument for } R = Q. \]

Compare the extended surface shown above to the original surface shown in Figure 5.

were arbitrary, we have \( BAUP(\tilde{Q}, C_0, 3C\varepsilon) = 0 \) for all \( Q \in \text{Top}_F \). Using that fact
that $F \subseteq F_Q$, the correspondence between cubes in $\mathcal{G}^F$ and $\mathcal{G}^{F_Q}$ and Theorem 1.7, we get

$$\sum_{R \subseteq Q} \tilde{\beta}^{d,p}_F(C_0B_R)^2\ell(R)^d \leq \sum_{R \subseteq Q} \beta^{d,p}_F(C_0B_R)^2\ell(R)^d \overset{(1.3)}{\lesssim} \mathcal{H}^d(\tilde{Q}).$$

Since by Lemma 4.12 the collection of balls $\{B_Q\}_{Q \in \mathcal{T}_{op}^F}$ have bounded overlap, the same is true for the cubes $\{\tilde{Q}\} Q \in \mathcal{T}_{op}^F$. If we define

$$\tilde{F} = E' \cup \bigcup_{S \in \mathcal{S}} \tilde{S},$$

then $\tilde{Q} \subseteq \tilde{F}$ for all $Q \in \mathcal{T}_{op}^F$, which gives

$$\sum_{Q \in \mathcal{T}_{op}^F} \sum_{R \subseteq Q} \tilde{\beta}^{d,p}_F(C_0B_Q)^2\ell(Q)^d \lesssim \sum_{Q \in \mathcal{T}_{op}^F} \mathcal{H}^d(\tilde{Q}) \lesssim \mathcal{H}^d(\tilde{F})$$

$$\lesssim \sum_{S \in \mathcal{S}} \ell(Q(S))^d$$

$$\lesssim \ell(Q_0)^d + \sum_{Q \in \mathcal{G}^E(Q_0)} \tilde{\beta}^{d,1}_E(MB_Q)^2\ell(Q)^d,$$

where the last inequality follows from (4.12).

\begin{lemma}
Let $\mathcal{U}_{op}$ be the collection of cubes which are not properly contained in any cube from $\mathcal{T}_{op}^F$. Then

$$\sum_{Q \in \mathcal{U}_{op}} \tilde{\beta}^{d,p}_F(C_0B_Q)^2\ell(Q)^d \lesssim \ell(Q_0)^d + \sum_{Q \in \mathcal{G}^E(Q_0)} \beta^{d,1}_E(MB_Q)^2\ell(Q)^d.$$

\end{lemma}

\begin{proof}
Let $Q \in \mathcal{U}_{op} \cap \mathcal{G}^E$. By construction, we have $\text{dist}(x_Q, E) < (C_0 + M)\ell(Q)$ so there exists a cube $Q' \in \mathcal{G}^E$ such that $C_0B_Q \cap MB_{Q'} \neq \emptyset$. In particular $C_0B_Q \subseteq 2MB_{Q'}$, which by Lemma 2.24 implies

$$\beta^{d,p}_F(C_0B_Q) \lesssim \beta^{d,p}_F(MB_{Q'}).$$

For $Q \in \mathcal{G}^E$, a standard volume argument gives

$$\{R \in \mathcal{U}_{op} : R' = Q\} \lesssim 1.$$

Then,

$$\sum_{Q \in \mathcal{U}_{op}} \tilde{\beta}^{d,p}_F(C_0B_Q)^2\ell(Q)^d \lesssim \sum_{Q \in \mathcal{G}^E(Q_0)} \sum_{R \subseteq \mathcal{U}_{op}} \tilde{\beta}^{d,p}_F(C_0B_R)^2\ell(R)^d$$

$$\lesssim \sum_{Q \in \mathcal{G}^E(Q_0)} \tilde{\beta}^{d,p}_F(2MB_Q)^2\ell(Q)^d$$

$$\lesssim \ell(Q_0)^d + \sum_{Q \in \mathcal{G}^E(Q_0)} \tilde{\beta}^{d,p}_F(C_0B_Q)^2\ell(Q)^d$$

$$\lesssim \ell(Q_0)^d + \sum_{Q \in \mathcal{G}^E(Q_0)} \beta^{d,1}_E(MB_Q)^2\ell(Q)^d.$$

\end{proof}
The third inequality follows for the following reason: For $Q$ small enough (depending on $C_0$ and $M$) we can find a larger cube $Q'$ such that $MB_Q \subseteq C_0 B_{Q'}$. We use this along with the fact that any cube has a bounded number of descendants up to the $K^{th}$ generation, say, with constant dependent on $n$ and $K$. The sum of the larger cubes is absorbed into the first term since again we can control the number of these cubes. This is what we did in the proof of Lemma 3.1, see there for more details.

The proof of Theorem 1.14 is finished by noting that

$$\sum_{Q \in \mathcal{P}} \hat{\beta}^{d,p}_F(C_0 B_Q)^2 \ell(Q)^d = \sum_{Q \in \mathcal{P}} \hat{\beta}^{d,p}_F(C_0 B_Q)^2 \ell(Q)^d + \sum_{Q \in \mathcal{P}} \sum_{R \subseteq Q} \hat{\beta}^{d,p}_F(C_0 B_R)^2 \ell(R)^d$$

and applying Lemma 4.13 and Lemma 4.2.

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