RIGID HYPERHOLOMORPHIC SHEAVES REMAIN RIGID ALONG TWISTOR DEFORMATIONS OF THE UNDERLYING HYPERKÄHLER MANIFOLD

E. MARKMAN, S. MEHROTRA, AND M. VERBITSKY

Abstract. Let $S$ be a $K3$ surface and $M$ a smooth and projective $2n$-dimensional moduli space of stable coherent sheaves on $S$. Over $M \times M$ there exists a rank $2n-2$ reflexive hyperholomorphic sheaf $E_M$, whose fiber over a non-diagonal point $(F_1, F_2)$ is $\text{Ext}^1_S(F_1, F_2)$. The sheaf $E_M$ can be deformed along some twistor path to a sheaf $E_X$ over the cartesian square $X \times X$ of every Kähler manifold $X$ deformation equivalent to $M$. We prove that $E_X$ is infinitesimally rigid, and the isomorphism class of the Azumaya algebra $\text{End}(E_X)$ is independent of the twistor path chosen. This verifies conjectures in [MM1, MM2] and renders the results of these two papers unconditional.

Contents

1. Introduction 2
   1.1. Twistor families and hyperholomorphic sheaves 2
   1.2. The modular hyperholomorphic sheaf 3
   1.3. The characteristic class $\bar{c}_1(E)$ of the modular sheaf 5
   1.4. Outline of the proof of the main result 6
2. Twistor paths 6
   2.1. Spaces of twistor paths 6
   2.2. Twistor paths with fixed end points 7
      2.2.1. Fiber dimension estimates 7
      2.2.2. The well behaved open subset $\tilde{T}w^k_{\Lambda}$ 11
      2.2.3. Smooth connected spaces of twistor paths with fixed end points 16
   2.3. A universal twistor family 17
      2.3.1. Moduli of marked irreducible holomorphic symplectic manifolds with a Kähler-Einstein structure 19
   2.4. The universal twistor path and its universal family 20
   2.5. An equivalence relation for twistor paths 22
3. Hyperholomorphic sheaves 23
4. Rigid hyperholomorphic sheaves 27
   4.1. Rigidity is an open and closed condition in families of stable sheaves over a fixed variety 28
   4.2. The rigidity locus in the moduli space of marked pairs 30
   4.3. Monodromy invariance of the rigidity locus 32
5. Monodromy equivariance of the modular hyperholomorphic sheaf 33
   5.1. The polarized surface monodromy group of a moduli space of sheaves 34
   5.2. Invariance of $U_{\mathcal{E}}$ under the surface monodromy group of Douady spaces 34
   5.3. Stability preserving Fourier-Mukai functors 35

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1. INTRODUCTION

1.1. Twistor families and hyperholomorphic sheaves. An irreducible holomorphic symplectic manifold is a simply connected compact Kähler manifold $X$, such that $H^0(X,\Omega^2_X)$ is spanned by an everywhere non-degenerate holomorphic two form. The second cohomology $H^2(X,\mathbb{Z})$ of such a manifold is endowed with an integral non-degenerate symmetric bilinear pairing of signature $(3, b_2(X) - 3)$, known as the Beauville-Bogomolov-Fujiki paired [B1]. Fix a lattice $\Lambda$ isometric to the second integral cohomology of an irreducible holomorphic symplectic manifolds $X$. A marking of $X$ is an isometry $\eta : H^2(X,\mathbb{Z}) \to \Lambda$. Two marked pairs $(X_1, \eta_1)$ are said to be isomorphic, if there exists an isomorphism $g : X_1 \to X_2$, such that $\eta_2 = \eta_1 \circ g^*$. The moduli space $\mathfrak{M}_\Lambda$ of isomorphism classes of marked irreducible holomorphic symplectic manifolds is a non-Hausdorff complex manifold of dimension $\text{rank}(\Lambda) - 2$ [Hn].

Given a Kähler class $\omega$ on an irreducible holomorphic symplectic manifold $X$, denote by $\pi : \mathcal{X} \to \mathbb{P}_\omega^1$ the associated twistor family. A choice of a marking $\eta$ for $X$ determines one for each fiber of $\pi$, since the projective line $\mathbb{P}_\omega^1$ is simply connected. The associated classifying morphism $\mathbb{P}_\omega^1 \to \mathfrak{M}_\Lambda$ is an embedding [Hn 1.17]. We refer to the $d$-th fiber product $\Pi : \mathcal{X}_\pi^d \to \mathbb{P}_\omega^1$ of $\mathcal{X}$ over $\mathbb{P}_\omega^1$ as the diagonal twistor family of $X^d$ associated to $\omega$. Denote by $\tilde{\omega}$ the Kähler class $\sum_{i=1}^d \pi_i^*\omega$ over $X^d$, where $\pi_i : X^d \to X$ is the projection onto the $i$-th factor.

Definition 1.1. A reflexive sheaf of Azumaya $\mathcal{O}_X$-algebras of rank $r$ over a complex manifold $X$ is a sheaf $A$ of coherent $\mathcal{O}_X$-modules, with a global section $1_A$ and an associative multiplication $A \otimes_{\mathcal{O}_X} A \to A$ with identity $1_A$, admitting an open covering $\{U_\alpha\}$ of $X$ and an isomorphism of unital associative algebras of the restriction of $A$ to each $U_\alpha$ with $\mathcal{E}nd(E_\alpha)$, for some reflexive sheaf $E_\alpha$ of rank $r$ on $U_\alpha$.

We will use the term Azumaya algebra as an abbreviation for the term a reflexive sheaf of Azumaya $\mathcal{O}_X$-algebras.

Definition 1.2. Let $X$ be a $d$-dimensional compact Kähler manifold and $\omega$ a Kähler class on $X$. The $\omega$-degree of a coherent sheaf $\mathcal{E}$ on $X$ is $\text{deg}_\omega(\mathcal{E}) := \int_X \omega^{d-\ell} c_1(\mathcal{E})$. A twisted coherent torsion free sheaf $\mathcal{E}$ is $\omega$-slope-stable, if for every subsheaf $F$, satisfying $0 < \text{rank}(F) < \text{rank}(\mathcal{E})$, the inequality $\text{deg}_\omega(\mathcal{E}om(F), E)) < 0$ holds. $\mathcal{E}$ is $\omega$-semistable, if the latter holds with $<$ replaced by $\leq$. A reflexive torsion free sheaf $\mathcal{E}$ is $\omega$-slope-polystable, if it is $\omega$-slope-semistable and it decomposes as a direct sum of $\omega$-slope-stable sheaves.

The following is a slight generalization of [VI Theorem 3.19] of Verbitsky for twisted reflexive sheaves.

Theorem 1.3. [Ma3 Cor. 6.11] Let $E$ be a $\tilde{\omega}$-slope-stable reflexive possibly twisted sheaf on $X^d$. Assume that the parallel transport of $c_2(\mathcal{E}nd(E))$ remains of Hodge type along the local system $R^{11}\Pi_t\mathbb{Z}$ over $P^1_{\omega_t}$. Then there exists a sheaf $\mathcal{E}$ over $\mathcal{X}_\pi^d$, which restricts to $X^d$ as $E$, and whose restriction $E_t$ to the fiber of $\Pi$ over $t \in P^1_{\omega_t}$ is $\tilde{\omega}_t$-slope-stable, for all $t \in P^1_{\omega_t}$, where $\omega_t$ is the canonical Kähler class on the fiber $X_t$ of $\pi$ over $t$. Furthermore, the reflexive sheaf of Azumaya algebras $\mathcal{E}nd(\mathcal{E})$ depends canonically on $E$.

Definition 1.4. A reflexive sheaf $E$ satisfying the hypothesis of Theorem 1.3 is said to be $\omega$-hyperholomorphic.
The following important fact is known, unfortunately, only in the locally free case.

**Theorem 1.5.** [V3, Cor. 8.1] Let $E$ be a locally free $\omega$-hyperholomorphic possibly twisted sheaf over $X^d$. Then the dimension of $H^i(X^d, E\otimes (E_t))$ is independent of the point $t \in \mathbb{P}^1_\omega$, for all $i \geq 0$.

A coherent sheaf $E$ is said to be *infinitesimally rigid* if $\text{Ext}^1(E, E) = 0$. If $E$ is a locally free $\omega$-hyperholomorphic and infinitesimally rigid, then the sheaves $E_t$ are infinitesimally rigid, for all $t \in \mathbb{P}^1_\omega$, by Theorem 1.5. The goal of this paper is to establish the analogous fact for a certain class of reflexive non-locally free hyperholomorphic sheaves (Theorem 1.14 below).

**Definition 1.6.**

1. A *twistor path* from $(X_1, \eta_1)$ to $(X_2, \eta_2)$ consists of the following data.
   - A sequence $(Y_i, \eta_i), 1 \leq i \leq n$, of marked pairs in $M_A$, with $(Y_1, \eta_1) = (X_1, \eta_1)$ and $(Y_n, \eta_n) = (X_2, \eta_2)$.
   - A Kähler class $\omega_i$ over $Y_i$, $1 \leq i \leq n - 1$, each up to a positive scalar multiple.
   - This data is assumed to satisfy the condition that the twistor line $\mathbb{P}^1_{\omega_i}$ through $(Y_i, \eta_i)$, associated to the Kähler class $\omega_i$, passes through $(Y_{i+1}, \eta_{i+1})$, for $1 \leq i \leq n - 1$.

2. The twistor path is said to be *generic*, if Pic$(Y_i)$ is trivial, or cyclic generated by a class of non-negative self-intersection with respect to the Beauville-Bogomolov-Fujiki pairing, for $2 \leq i \leq n - 1$.

**Definition 1.7.** Let $\gamma := \{(Y_i, \eta_i)\}_{i=1}^n, \{(\omega_i)\}_{i=1}^{n-1}$ be a twistor path from $(X_1, \eta_1)$ to $(X_2, \eta_2)$. A reflexive sheaf $E$ over $X^d$ is said to be $\gamma$-hyperholomorphic, if $c_2(E\otimes (E_t))$ remain of Hodge type along $\gamma$, $E$ is $\tilde{\omega}_1$-slope-stable and the family $\mathcal{E}_i$, constructed recursively via Theorem 1.3 over the first $i$ twistor lines in $\gamma$, restricts to $Y_{i+1}$ as a $\tilde{\omega}_{i+1}$-slope-stable sheaf, for $1 \leq i \leq n - 1$.

Note that if $E$ is $\gamma$-hyperholomorphic, then it extends to a sheaf $\mathcal{E}$ over the twistor family over $\gamma$. Denote by

$$E_{\gamma}$$

the restriction of $\mathcal{E}$ to $X^d_\gamma$. The isomorphism class of the reflexive sheaf of Azumaya algebras $\mathcal{E}\otimes (E_{\gamma})$ depends canonically on $E$ and $\gamma$.

1.2. **The modular hyperholomorphic sheaf.** Let $S$ be a $K3$ surface. The *Mukai lattice* $\bar{H}(S, \mathbb{Z})$ of $S$ is its total integral cohomology ring $H^*(S, \mathbb{Z})$ endowed with the following symmetric Mukai pairing. Given a class $v := (r, c, s) \in \bar{H}^*(S, \mathbb{Z})$, with $r \in H^0(S)$, $c \in H^2(S)$, and $s \in H^4(S)$, set $(v, v) := (c, c) - 2rs$, where $(c, c)$ is the self-intersection pairing of $H^2(S, \mathbb{Z})$ and we identify $H^2(S, \mathbb{Z})$, $i = 0, 4$, with $\mathbb{Z}$ sending the unit and orientation classes to $1$. The *Mukai vector* of a coherent sheaf $F$ on $S$ is the class $c(F)\sqrt{\text{td}_S}$ in $\bar{H}(S, \mathbb{Z})$ [Mu1].

Let $v \in \bar{H}(S, \mathbb{Z})$ be a primitive Mukai vector of self-intersection $(v, v) \geq 2$ with $c_1(v)$ of Hodge type $(1, 1)$ and of non-negative rank. If the rank of $v$ is zero we assume further that the class $c_1(v)$ is effective. If $v$ is the Mukai vector $(1, 0, 1 - n)$ of the ideal sheaf of a length $n$ subscheme, we let $M$ be the Douady space $S^{[n]}$ of such sheaves. An irreducible holomorphic symplectic manifold is said to be of $K3^{[n]}$-type if it is deformation equivalent to such a Douady space. If $v \neq (1, 0, 1 - n)$, assume that $S$ is projective. Associated to $v$ is a locally finite collection of hyperplanes in the ample cone of a projective $S$, and a class not lying on any wall is called $v$-*generic* [HI1]. Let $H$ be a $v$-generic polarization. Then the moduli space $M := M_H(v)$ of $H$-stable sheaves with Mukai vector $v$ is a projective irreducible holomorphic symplectic manifold of $K3^{[n]}$-type, where $n = [(v, v) + 2]/2$, by a theorem due to Mukai, Huybrechts, O’Grady, and Yoshioka. It can be found in its final form in Yo.
Let $\mathcal{V}$ be a possibly twisted universal sheaf over $S \times M$. There exists a twisted locally free sheaf $W$ over $M$, such that the sheaf $\mathcal{Z} := \mathcal{V} \otimes \pi_M^*W$ is untwisted [Mu1, Appendix]. The Mukai homomorphism

\begin{equation}
\label{eq:12}
m_v : v^\perp \to H^2(M, \mathbb{Z})
\end{equation}

is given by $m_v(x) := \frac{1}{\text{rank}(W)} c_1(\pi_{M,*}(\pi_M^*[x^\vee/\text{td}_S] \cup ch(\mathcal{Z})))$, where $v^\perp$ is the sublattice of the Mukai lattice orthogonal to $v$ and $x^\vee$ is obtained from $x$ by changing the sign of the summand in $H^2(S, \mathbb{Z})$. The homomorphism $m_v$ is independent of the choice of $W$. It follows from the proof of the above mentioned result of O’Grady and Yoshioka that $m_v$ is an integral isometry, where $H^2(M, \mathbb{Z})$ is endowed with the Beauville-Bogomolov-Fujiki pairing.

**Definition 1.8.** The monodromy group $\text{Mon}(M)$ of a compact Kähler manifold $X$ is the subgroup of the automorphism group of the cohomology ring $H^*(X, \mathbb{Z})$ generated by monodromy operators $g$ of families $\mathcal{X} \to B$ (which may depend on $g$) of compact Kähler manifolds deforming $X$. Let $\text{Mon}^2(M)$ be the image of $\text{Mon}(M)$ in the automorphism group of $H^2(M, \mathbb{Z})$.

Let $\pi_{ij}$ be the projection from $M \times S \times M$ onto the product of the $i$-th and $j$-th factors. Let

\begin{equation}
\label{eq:13}
E := \text{Ext}^1_{\pi_{13}}(\pi_{12}^*\mathcal{V}, \pi_{23}^*\mathcal{V})
\end{equation}

be the relative extension sheaf over $M \times M$. Then $E$ is a reflexive sheaf of rank $2n - 2$, which is locally free away from the diagonal, by [Ma3, Prop. 4.1]. The class $c_2(\text{End}(E))$ is invariant under the diagonal action of $\text{Mon}(M)$, by [Ma3, Prop. 3.4]. The sheaf $E$ is $\bar{\omega}$-slope-stable with respect to every Kähler class of $M$, by [Ma5, Theorem 1.2]. Furthermore, $E$ is $\gamma$-hyperholomorphic with respect to every twistor path $\gamma$ staring at $(M, \eta)$, for any marking $\eta$, by [Ma5, Theorem 1.4]. Following is an abbreviated statement of the main result of this paper.

**Theorem 1.9.** Over the cartesian square $X \times X$ of a manifold $X$ of $K^3$-[n]-type, $n \geq 2$, there exists a canonical unordered pair $\{A_1, A_2\}$ of Azumaya algebras, with $A_2$ isomorphic to $A_1^\dagger$, and each $A_i$ is obtained from $\text{End}(E)$, for the sheaf $E$ in (1.3), via a deformation along some twistor path. Furthermore, if the rank of $\text{Pic}(X)$ is less than 21, then $A_i$ is infinitesimally rigid, for $i = 1, 2$.

Let $\theta_v' \in H^2(M, \mathbb{Z})/(2n - 2)H^2(M, \mathbb{Z})$ be the coset

\begin{equation}
\label{eq:14}
\theta_v' = m_v\left(\left\{w \in v^\perp : w - v \in (2n - 2)\bar{H}(S, \mathbb{Z})\right\}\right).
\end{equation}

The pair $\{\theta_v', -\theta_v'\}$ is $\text{Mon}^2(M)$-invariant, by [Ma3, Lemma 7.2], and $\text{Mon}^2(M)$ acts transitively on this set, by [Ma1, Theorem 1.6].

**Definition 1.10.** Let $M_{H_i}(v_i)$, $i = 1, 2$, be moduli spaces as above with $(v_i, v_i) = 2n - 2$, $n \geq 2$. Two markings $\eta_i : H^2(M_{H_i}(v_i), \mathbb{Z}) \to \Lambda$, $i = 1, 2$, are said to be compatible, if $(M_{H_i}(v_i), \eta_1)$ and $(M_{H_i}(v_2), \eta_2)$ belong to the same connected component of $\mathcal{M}_\Lambda$ and the isometry $\eta_2^{-1}\eta_1 : H^2(M_{H_i}(v_1), \mathbb{Z}) \to H^2(M_{H_i}(v_2), \mathbb{Z})$ takes $\theta_{v_1}'$ to $\theta_{v_2}'$.

Note that compatibility is an equivalence relation. Marked pairs $(M_H(v), \eta)$ in a fixed connected component $\mathcal{M}^\dagger_\Lambda$ of moduli space form two compatibility classes, if $n > 2$, and one compatibility class, if $n = 2$, by the monodromy invariance of the pair $\{\theta_v', -\theta_v'\}$ and the transitivity of the monodromy action on this set.

\footnote{The assumptions of [Ma5, Theorem 1.4] are preserved under twistor deformations of $E$, by [MM1, Prop. 3.2] and [MM1, Lemma 7.6]. The proofs of [MM1, Prop. 3.2] and [MM1, Lemma 7.6] are unconditional.}
Let $E$ be the sheaf given in (1.3). Choose a marking $\eta_0 : H^2(M, \mathbb{Z}) \to \Lambda$ and let $\mathfrak{M}_\Lambda^0$ be the component of the moduli space of marked pairs containing $(M, \eta_0)$. The Picard rank of $X$ is said to be maximal if $\text{rank}(\text{Pic}(X)) = \dim H^{1,1}(X)$. Let $O(\Lambda)$ be the isometry group of the lattice $\Lambda$ and $O^+(\Lambda)$ its index 2 subgroup, which is the kernel of the norm character $[C, \text{Mad Sec. } 4]$. Denote by $\text{Mon}(\Lambda)$ the subgroup of $O^+(\Lambda)$ acting on the discriminant group $\Lambda^*/\Lambda$ via multiplication by plus or minus one. The marking $\eta$, of every point $(X, \eta) \in \mathfrak{M}_\Lambda^0$, conjugates $\text{Mon}^2(X)$ to $\text{Mon}(\Lambda)$, by $[\text{Ma2}]$ Theorem 1.2 and Lemma 4.2. Let $\text{Mon}(\Lambda)_{\text{cov}}$ be the subgroup of $\text{Mon}(\Lambda)$ acting trivially on the discriminant group $\Lambda^*/\Lambda$. Following is the detailed statement of Theorem 1.11.

**Theorem 1.11.** Let $\gamma$ be a twistor path from $(M, \eta_0)$ to a point $(X, \eta)$ of $\mathfrak{M}_\Lambda^0$.

1. The sheaf $E_\gamma$ on $X \times X$ is infinitesimally rigid, if the Picard rank of $X$ is not maximal.
2. The Azumaya algebra $\text{End}(E_\gamma)$ depends only on the endpoint $(X, \eta)$ of $\gamma$ and is independent of the path $\gamma$, regardless of the Picard rank of $X$.
3. Let $\phi \in \text{Mon}(\Lambda)$, and let $\gamma'$ be a twistor path from $(M, \eta_0)$ to the translate $(X, \phi \circ \eta)$ of the end point $(X, \eta)$ of $\gamma$. The Azumaya algebras $\text{End}(E_\gamma)$ and $\text{End}(E_{\gamma'})$ are isomorphic, if $\phi$ belongs to $\text{Mon}(\Lambda)_{\text{cov}}$ and $\text{End}(E_{\gamma'})$ is isomorphic to $\text{End}(E_{\gamma})$ otherwise.
4. Let $\tilde{M} := M_\Lambda(\bar{v})$ be another smooth and projective such 2n-dimensional moduli space of stable sheaves on some polarized K3 surface $(\tilde{S}, \tilde{H})$, let $\tilde{E}$ be the corresponding sheaf over $\tilde{M} \times \tilde{M}$ given in (1.3), let $\tilde{\eta}_0$ be a marking for $\tilde{M}$, and let $\tilde{\gamma}$ be a twistor path from $(\tilde{M}, \tilde{\eta}_0)$ to the end point $(X, \eta)$ of $\gamma$. The Azumaya algebras $\text{End}(E_{\gamma})$ and $\text{End}(\tilde{E}_{\tilde{\gamma}})$ are isomorphic, if the markings of $(M, \eta_0)$ and $(\tilde{M}, \tilde{\eta}_0)$ are compatible and $\text{End}(E_{\gamma})$ is isomorphic to $\text{End}(E_{\gamma})$ otherwise.

The Theorem is proved in Section 5.4. Fix the compatibility class of $(M, \eta_0)$. We denote by $E_{(X, \eta)}$ the equivalence class of the twisted sheaf $E_\gamma$ under isomorphisms and tensor product by line bundles, as it is determined by the endpoint $(X, \eta)$ of $\gamma$ in view of the above Theorem. We will refer to the sheaf $E$ given in (1.3) as the modular sheaf and the sheaf $E_{(X, \eta)}$ of Theorem 1.11 as the deformed modular sheaf. Similarly, $\text{End}(E)$ will be called the modular Azumaya algebra and $\text{End}(E_{(X, \eta)})$ the deformed modular Azumaya algebra.

The conclusion of part (1) of Theorem 1.11 is established away from a closed and countable subset of $\mathfrak{M}_\Lambda^0$, since the set of isomorphism classes of marked pairs of maximal Picard rank $21$ is countable. We expect the conclusion to hold even when the Picard rank is maximal. Infinitesimal rigidity of $E$ was known when $v = (1, 0, 1 - n)$, so that $M$ is the Douady space, by $[\text{MM2}]$ Lemma 5.2. Infinitesimal rigidity of $E_{\gamma}$ was conjectured in $[\text{MM2}]$ Conj. 1.6 and $[\text{MM}]$ Conj. 1.7. The main results of these two papers, $[\text{MM}]$ Theorem 1.11 for $X$ of non-maximal Picard rank, and $[\text{MM2}]$ Theorem 1.8, thus hold unconditionally, by Theorem 1.11.

### 1.3. The characteristic class $c_1(E)$ of the modular sheaf.

We relate next the compatibility relation for markings (Definition 1.10) to a characteristic class $c_1(E)$ in $H^2(M \times M, \mu_{2n-2})$ of the modular sheaf. A holomorphic $\mathbb{P}^{r-1}$ bundle $\mathbb{P}$ over a complex manifold $X$ determines a class $[\mathbb{P}]$ in the first cohomology of the sheaf of holomorphic maps to $\text{PGL}(r)$. The connecting homomorphism associated to the short exact sequence $0 \rightarrow \mu_r \rightarrow \text{SL}(r) \rightarrow \text{PGL}(r) \rightarrow 0$ maps $[\mathbb{P}]$ to a class $\Theta(\mathbb{P})$ in $H^2(X, \mu_r)$. Now $H^2(M \times M, \mu_{2n-2})$ is isomorphic to $H^2(M \times M, \mu_{2n-2})$, since the diagonal $\Delta$ has complex codimension $2n \geq 4$. Hence, the projectivization of $E$ over $[M \times M] \setminus \Delta$ determines a class $c_1(E)$ in $H^2(M \times M, \mu_{2n-2})$. We recall next its computation. Set $\theta_v := \exp \left( -\frac{2n-2}{4} \theta_v' \right) \in H^2(M, \mu_{2n-2})$, where $\theta_v'$ is the coset
given in (1.4). Then
\[
(1.5) \quad \bar{c}_1(E) = \pi_1^*\theta_v^{-1}\pi_2^*\theta_{v'},
\]
by [Ma3, Lemma 7.3]. Consequently, the pair \(\{c_1(E), c_1(E)^{-1}\}\) is invariant under the diagonal action of \(\text{Mon}^2(M)\), and the latter acts transitively on this set, since the analogous result holds for \(\{\theta_v', -\theta_v'\}\), as mentioned above.

The markings \(\eta_{v_1}\) and \(\eta_{v_2}\) in Definition 1.10 are compatible, if and only if the cartesian square \(\eta_{v_2}^{-1} \circ \eta_{v_1}\) maps the characteristic class \(\bar{c}_1(E_{v_1})\) of the modular sheaf over \(M_{H_1}(v_1) \times M_{H_1}(v_1)\) to \(\bar{c}_1(E_{v_2})\), by Equation (1.5).

1.4. Outline of the proof of the main result. Let \((M, \eta_0)\) be the marked moduli space of Theorem 1.11 and \(E\) the modular sheaf given in (1.3). In Section 2 we construct a smooth and connected differentiable manifold \(Tw_k^\Lambda\) of twistor paths in \(\mathfrak{M}_\Lambda^0\) consisting of \(k - 1\) twistor lines, \(k \geq 10\), and a surjective map \(f_k : Tw_k^\Lambda \to \mathfrak{M}_\Lambda^0 \times \mathfrak{M}_\Lambda^0\) with smooth connected fibers, sending a twistor path to its start and end points. We then construct the universal twistor family over the universal twistor path over \(Tw_k^\Lambda\). Every twistor path in \(\mathfrak{M}_\Lambda^0\) is equivalent (Definition 2.21) to a path \(\gamma\) in \(Tw_k^\Lambda\), for all \(k\) sufficiently large. \(E_\gamma\) is isomorphic to \(E_{\gamma'}\), if \(\gamma\) and \(\gamma'\) are equivalent in that sense.

In Section 3 we construct the universal hyperholomorphic deformation of the modular sheaf \(E\) over the universal twistor path. We use the notions of differentiable families of holomorphic manifolds and bundles due to Kodaira and Spencer. Some of the basic tools of algebraic geometry, such as the Semi-Continuity Theorem and local triviality of rigid objects, hold in this setting [KS1].

Let \(\Gamma^{(X, \eta)}_{(M, \eta_0)}\) be the fiber of \(f_k\) over \(((M, \eta_0), (X, \eta))\). The first crucial observation is that the locus in \(\Gamma^{(X, \eta)}_{(M, \eta_0)}\), consisting of twistor paths \(\gamma\) such that \(E_\gamma\) is infinitesimally rigid, is both open (by the Semi-Continuity Theorem) and closed (by stability of \(E_\gamma\)), see Proposition 4.3. This locus is either empty or it consists of the entire fiber, by the connectedness of the latter.

Let \(U\) be the locus in \(\mathfrak{M}_\Lambda^0\) consisting of marked pairs \((X, \eta)\), such that \(E_\gamma\) is infinitesimally rigid for all twistor paths \(\gamma\) from \((M, \eta_0)\) to \((X, \eta)\). The second crucial observation is that \(U\) is an open subset, which is invariant under the monodromy group of the triple \((M, \eta_0, E)\) (Corollary 4.4).

In Section 5 we prove that the monodromy group of the triple \((M, \eta_0, E)\) is a subgroup of index 2 in the monodromy group of the pair \((M, \eta_0)\). \(U\) is non-empty, as it contains all marked Hilbert schemes of length \(n\) subschemes of a K3 surface. Thus, the monodromy invariance property of \(U\) implies that it contains every marked pair in \(\mathfrak{M}_\Lambda^0\), of non-maximal Picard rank, by a density theorem of Verbitsky [V3, Theorem 4.11] and [V6, Theorem 2.5].

Sections 2, 3, and 4 are written for general irreducible holomorphic symplectic manifolds. Theorem 4.16 is a version of Theorem 1.11 for a general irreducible holomorphic symplectic manifold. Section 5 specializes to manifolds of \(K3[\ell]\)-type and the deformed modular sheaf. We expect a similar result to hold for the modular sheaf over the cartesian square of generalized Kummer manifolds.

2. Twistor paths

Let \(\Lambda\) be a lattice isometric to the Beauville-Bogomolov-Fujiki lattice of some irreducible holomorphic symplectic manifold. Assume that the rank of \(\Lambda\) is greater than or equal to 7. Let \(\Omega_\Lambda := \{\ell \in \mathbb{P}(\Lambda_\mathbb{C}) : (\ell, \ell) = 0, (\ell, \bar{\ell}) > 0\}\) be the period domain. We consider in section 2.1 the space \(Tw_k^\Lambda\) of twistor paths in \(\Omega_\Lambda\) consisting of \(k - 1\) twistor lines. In Section 2.2 we identify
a smooth and dense open subset $T\omega^k_\Lambda$ of $T\omega^k_\Lambda$, such that the map $f_k : T\omega^k_\Lambda \to \Omega_\Lambda \times \Omega_\Lambda$, sending a twistor path to its initial point and end point, is submersive with smooth connected fibers (Proposition 2.6). In Section 2.4 we prove the analogous statement for an open subset $T\omega^k_\Lambda$ of the space of twistor paths in $\mathfrak{M}_\Lambda^0$ and the analogous map $f_k : T\omega^k_\Lambda \to \mathfrak{M}_\Lambda^0 \times \mathfrak{M}_\Lambda^0$ (Proposition 2.18). The relationship between these two statements involves the moduli space of marked irreducible holomorphic symplectic manifolds endowed with a Kähler-Einstein metric, which is described in Section 2.3 using the Global Torelli Theorem and recent results about the Kähler cone of such manifolds. In Section 2.5 we introduce an equivalence relation for twistor paths, which is a weak analogue of the homotopy relation for ordinary paths. Every twistor path is equivalent to a twistor path in $\tilde{T}\omega^k_\Lambda$, for all $k$ sufficiently large.

Given a family $\pi : \mathcal{F} \to \Sigma$ of irreducible holomorphic symplectic manifolds over a smooth connected analytic manifold $\Sigma$ with a marking $\eta : R^2 \pi_* \mathbb{Z} \to \Lambda$, we get a classifying morphism $\kappa : \Sigma \to \mathfrak{M}_\Lambda^0$. Given a marked pair $(Y, \psi)$, let $\Gamma_{\Sigma}^{(Y, \psi)}$ be the fiber product

$$
\begin{align*}
\Gamma_{\Sigma}^{(Y, \psi)} & \longrightarrow \tilde{T}\omega^k_\Lambda \\
\Sigma & \longrightarrow \mathfrak{M}_\Lambda^0 \\
& \cong \mathfrak{M}_\Lambda^0 \times \{(Y, \psi)\} \\
& \subset \mathfrak{M}_\Lambda^0 \times \mathfrak{M}_\Lambda^0.
\end{align*}
$$

We get a smooth differentiable fibration $\Gamma_{\Sigma}^{(Y, \psi)} \to \Sigma$ with connected fibers, whose fiber over $\sigma \in \Sigma$ consists of twistor paths in $\mathfrak{M}_\Lambda^0$ from $(X_\sigma, \eta_\sigma)$ to $(Y, \psi)$.

2.1. Spaces of twistor paths. Set $r := \text{rank}(\Lambda)$. The component $\mathfrak{M}_\Lambda^0$ determines an orientation of the positive cone of $\Lambda := \Lambda \otimes_\mathbb{Z} \mathbb{R}$, hence for any positive definite 3-dimensional subspace of $\Lambda$ [M4 Sec. 4]. The period domain $\Omega_\Lambda$ is naturally identified with the Grassmannian $Gr_{+++}(\Lambda)$ of oriented positive definite two dimensional subspaces of $\Lambda$. Given a point in $\Omega_\Lambda$, corresponding to an isotropic line spanned by a class $\sigma \in \Lambda_C$, we get the positive definite oriented subspace spanned by the real and imaginary parts of $\sigma$. Conversely, given a point $V$ of $Gr_{+++}(\Lambda)$, corresponding to a two dimensional positive definite subspace $\overline{V}$ and a choice of an orientation for $\overline{V}$, we get two isotropic lines in the complexification of $\overline{V}_C$, endowing $\overline{V}$ with two different orientations, and so the orientation singles out one of the isotropic lines. We will routinely identify $\Omega_\Lambda$ and $Gr_{+++}(\Lambda)$.

Let $Gr_{+++}(\Lambda)$ be the Grassmannian of positive three dimensional subspaces. A twistor path in $Gr_{+++}(\Lambda)$ from $V_i$ to $V_k$ consists of the data $\{(V_1, \ldots, V_k); (W_1, \ldots, W_{k-1})\}$, $V_j \in Gr_{+++}(\Lambda)$, $W_i \in Gr_{+++}(\Lambda)$, such that $W_j$ contains both $V_j$ and $V_{j+1}$, $1 \leq j \leq k - 1$. The twistor path is generic, if $\overline{V}_i \cap \Lambda$ is trivial, or cyclic generated by a class of non-negative self-intersection, for $2 \leq i \leq k - 1$.

Denote by

$$
\mathcal{I} \subset Gr_{+++}(\Lambda) \times Gr_{+++}(\Lambda)
$$

the incidence variety of pairs $(V, W)$, such that $\overline{V}$ is contained in $W$. We get the natural projections

$$
Gr_{+++}(\Lambda) \xrightarrow{p} \mathcal{I} \xrightarrow{q} Gr_{+++}(\Lambda).
$$

The fiber of $q$ over $W \in Gr_{+++}(\Lambda)$ is the complex plane conic $\mathbb{P}(W_C) \cap \Omega_\Lambda$. The fiber of $p$ over $V \in Gr_{+++}(\Lambda)$ is the open subset $Gr_{+++}(\Lambda/\overline{V})$ of positive lines in $\mathbb{P}(\Lambda/\overline{V}) \cong \mathbb{RP}^{r-3}$, where $r$ is the rank of $\Lambda$ and $\Lambda/\overline{V}$ is endowed with a pairing via the isomorphism $\overline{V}^\perp \cong \Lambda/\overline{V}$. We may and will view the fibers of $p$ as subsets of $Gr_{+++}(\Lambda)$. We have $\dim_{\mathbb{R}}(\mathcal{I}) = 3r - 7$. 

Let $\mathcal{H}$ be the tautological rank 2 subbundle of the trivial rank $r$ vector bundle over $Gr_{++}(\mathcal{V})$ with fiber $\Lambda$. Denote by $\mathcal{H}^{1,1}$ the orthogonal complement of $\mathcal{H}$. The identification $\Omega_{\Lambda} \cong Gr_{++}(\mathcal{V})$ pulls back $\mathcal{H}^{1,1}$ to the Hodge bundle of type $(1,1)$ over the period domain $\Omega_{\Lambda}$. Let $\mathcal{E}^{+} \subset \mathcal{H}^{1,1}$ be the positive cone. The fiber of $\mathcal{E}^{+}$ over $V$ consists of vectors $\omega \in V^{\perp}$ satisfying $(\omega, \omega) > 0$ and such that the orientation of the positive definite three dimensional subspace spanned by $V$ and $\omega$ and determined by $\mathcal{M}_{\Lambda}$ is equal to the orientation determined by a basis $\{v_1, v_2, \omega\}$, for a basis $\{v_1, v_2\}$ of $V$ compatible with the orientation of $V$. The projectivization $\mathbb{P}\mathcal{E}^{+}$ is a bundle of hyperbolic spaces over $Gr_{++}(\mathcal{V})$, which is naturally isomorphic to $\mathcal{I}$ 

$$\mathcal{I} \cong \mathbb{P}\mathcal{E}^{+}.$$ 

The isomorphism sends a pair $(V, W) \in \mathcal{I}$ to the point in the fiber of $\mathbb{P}\mathcal{E}^{+}$ over $V$ corresponding to the ray in $V^{\perp} \cap W$ compatible with the orientation of $W$ determined by $\mathcal{M}_{\Lambda}$.

A twistor path in $Gr_{++}(\mathcal{V})$ from $V_1$ to $V_k$, consisting of $(k - 1) \geq 1$ twistor lines, is a point in the fiber product of $2k - 2$ copies of $\mathcal{I}$ (alternating over $Gr_{++}(\mathcal{V})$ and over $\Omega_{\Lambda}$) 

$$Tw_{\Lambda}^k := \mathcal{I} \times_{Gr_{++}(\mathcal{V})} \mathcal{I} \times_{\mathcal{I}} \mathcal{I} \times_{Gr_{++}(\mathcal{V})} \mathcal{I} \cdots \mathcal{I} \times_{Gr_{++}(\mathcal{V})} \mathcal{I}.$$ 

**Lemma 2.1.** $Tw_{\Lambda}^k$ is a simply connected real analytic manifold of dimension $(k+1)(r-1)-2$. The map $Tw_{\Lambda}^k \to \Omega_{\Lambda}$, sending a twistor path to its starting point $V_1$, is surjective and submersive with simply connected fibers of dimension $(k + 1)(r - 1) - 2$.

**Proof.** Set $Tw_{\Lambda}^1 := Gr_{++}(\mathcal{V})$. Let $p_k : Tw_{\Lambda}^k \to Tw_{\Lambda}^{k-1}$, $k \geq 2$, be the map given by 

$$\{(V_1, \ldots, V_k); (W_1, \ldots, W_{k-1})\} \mapsto \{(V_1, \ldots, V_{k-1}); (W_1, \ldots, W_{k-2})\}.$$ 

The fiber of $p_k$ over $\{(V_1, \ldots, V_{k-1}); (W_1, \ldots, W_{k-2})\}$ is the conic bundle 

$$q^{-1}(p^{-1}(V_{k-1})) = q^{-1}(Gr_{+}(\mathcal{V}/V_{k-1})) \to Gr_{+}(\mathcal{V}/V_{k-1})$$

over the hyperbolic space $Gr_{+}(\mathcal{V}/V_{k-1}) \subset Gr_{++}(\mathcal{V})$. The dimension of $Tw_{\Lambda}^k$ is thus given by 

$$\dim_{\mathbb{R}}(Tw_{\Lambda}^k) = \dim_{\mathbb{R}}(\Omega_{\Lambda}) + (k - 1)[(r - 3) + 2] = 2(r - 2) + (k - 1)(r - 1) = (k + 1)(r - 1) - 2.$$ 

It follows, by induction on $k$, that $Tw_{\Lambda}^k$ is a simply connected manifold, being a fibration with simply connected fibers over a simply connected base. The proof of the statement for the map $Tw_{\Lambda}^k \to \Omega_{\Lambda}$ is similar. 

The tangent space to $Tw_{\Lambda}^k$ at $\{(V_1, \ldots, V_k); (W_1, \ldots, W_{k-1})\}$ is the kernel of the surjective homomorphism 

$$\bigoplus_{i=1}^k \text{Hom}(V_i, \mathcal{V}/V_i) \oplus \bigoplus_{j=1}^{k-1} \text{Hom}(W_j, \mathcal{V}/W_j) \to \bigoplus_{i=1}^{k-1} \text{Hom}(V_i, \mathcal{V}/W_i) \oplus \bigoplus_{j=2}^k \text{Hom}(V_j, \mathcal{V}/W_{j-1}^-)$$ 

given by $((a_i)_{i=1}^k; (b_j)_{j=1}^{k-1}) \mapsto ((\bar{a}_i - \bar{b}_i)_{i=1}^k; (\bar{a}_j - \bar{b}_{j-1})_{j=2}^k)$, where 

$$(-) : \text{Hom}(V_i, \mathcal{V}/V_i) \to \text{Hom}(V_i, \mathcal{V}/W_i),$$

$$(-) : \text{Hom}(V_i, \mathcal{V}/V_i) \to \text{Hom}(V_i, \mathcal{V}/W_{i-1}),$$

$$(-) : \text{Hom}(W_i, \mathcal{V}/W_i) \to \text{Hom}(V_i, \mathcal{V}/W_i),$$

$$(-) : \text{Hom}(W_i, \mathcal{V}/W_i) \to \text{Hom}(V_i, \mathcal{V}/W_i),$$

are the natural homomorphisms.
2.2. Twistor paths with fixed end points. Let $f_k : Tw_k^\Lambda \to Gr_{++}(\Lambda)^2$ be given by

$\{(V_1, \ldots, V_k); (W_1, \ldots, W_{k-1})\} \mapsto (V_1, V_k)$.

We describe in this section an open dense subset $\bar{Tw}_k^\Lambda$ of $Tw_k^\Lambda$, such that the restriction of $f_k$ to $\bar{Tw}_k^\Lambda$ is submersive with smooth connected fibers, for $k \geq 8$ (see Proposition 2.6 below).

The following example shows that $\bar{Tw}_k^\Lambda$ must be a proper subset of $Tw_k^\Lambda$.

Example 2.2. Let $t \in Tw_k^\Lambda$ be a twistor path such that all the $V_i$'s are equal to the same oriented two dimensional subspace $V$ and all the $W_j$'s are equal to the same positive definite three dimensional subspace $W$. Then the tangent space of $Tw_k^\Lambda$ at $t$ consists of elements $((a_i)_{i=1}^k; (b_j)_{j=1}^{k-1})$, such that $\bar{a}_1 = \bar{b}_1 = \bar{a}_2 = \cdots = \bar{b}_{k-1} = \bar{a}_k$. In particular, $\bar{a}_1 = \bar{a}_k$ and the differential of $f_k$ at $t$ has rank $2r - 2$ and is not surjective. Consequently, the fiber of $f_k$ is singular at $t$ or of dimension larger than that of the generic fiber. Such $t$ is contained in the $(k-1)(r-3)$-dimensional subset of the fiber $f_k^{-1}(V, V)$ consisting of all twistor paths with all $V_i$ equal $V$, while for $k \geq 3$ the generic fiber of $f_k$ has dimension $(k-3)(r-1) + 2$. Hence, such $t$ belongs to a fiber of dimension larger than that of the generic fiber if $k < r - 1$.

2.2.1. Fiber dimension estimates.

Lemma 2.3. The restriction of $f_k$ to the open subset of $Tw_k^\Lambda$, where $\nabla_1 \cap \nabla_k = 0$, is submersive.

Proof. Fix a twistor path $\{(V_1, \ldots, V_k); (W_1, \ldots, W_{k-1})\}$ such that $\nabla_1 \cap \nabla_k = 0$. Let $a_1$ be an element of Hom$(\nabla_1, \Lambda/\nabla_1)$ and $a_k$ of Hom$(\nabla_k, \Lambda/\nabla_k)$ so that $(a_1, a_k)$ is a tangent vector to $\Omega_\Lambda \times \Omega_\Lambda$ at $(V_1, V_k)$. Let $\hat{a}_i$ be an element of Hom$(\nabla_i, \Lambda)$ mapping to $a_i$ via the natural homomorphism, $i = 1, k$. The vanishing $\nabla_1 \cap \nabla_k = (0)$ enables us to choose a homomorphism $a : \Lambda \to \Lambda$ restricting to the subspace $\nabla_i$ as $\hat{a}_i$, $i = 1, k$. Given a subspace $Z$ of $\Lambda$ we have the natural homomorphism

$$\text{Hom}(\Lambda, \Lambda) \to \text{Hom}(Z, \Lambda/Z)$$

obtained by composition with the inclusion $Z \to \Lambda$ and projection $\Lambda \to \Lambda/Z$. We recover $a_i$ as the image of $a$ by choosing $Z$ to be $\nabla_i$, for $i = 1, k$. Define $a_i$ that way for $1 \leq i \leq k$. Define $b_j \in \text{Hom}(W_j, \Lambda/W_j)$ as the image of $a$ by choosing $Z = W_j$, $1 \leq j \leq k - 1$. Then $((a_i)_{i=1}^k; (b_j)_{j=1}^{k-1})$ is a tangent vector to $Tw_k^\Lambda$ which maps to $(a_1, a_k)$ via the differential of $f_k$. We conclude that $f_k$ is submersive at the given twistor path. \qed

Set

$$d_k := (k-3)(r-1) + 2,$$

$$\alpha_k := \max\{d_k, (k-2)(r-2)\},$$

$$\beta_k := \max\{d_k, (k-1)(r-3), (k-2)(r-2) + 1\}.$$

Lemma 2.4. Assume that $k \geq 3$. The dimension of $f_k^{-1}(V_1, V_k)$ is $d_k$, if $\nabla_1 \cap \nabla_k = 0$, and

$$\dim(f_k^{-1}(V_1, V_k)) \leq \begin{cases} \alpha_k & \text{if } \dim(\nabla_1 \cap \nabla_k) = 1, \\ \beta_k & \text{if } \dim(\nabla_1 \cap \nabla_k) = 2. \end{cases}$$

In particular, the fibers of $f_k$ all have the same dimension for $k \geq r + 1$.

Proof. If $\nabla_1 \cap \nabla_k = 0$, then $f_k$ is submersive along the fiber $f_k^{-1}(V_1, V_k)$, by Lemma 2.3, and so the dimension of the fiber is $\dim(Tw_k^\Lambda) - 2\dim(\Omega_\Lambda) = d_k$. If $k \geq r + 1$, then the right hand side of inequality (2.4) is $d_k$, and the equidimensionality of $f_k$ follows from the Semicontinuity Theorem, since $Tw_k^\Lambda$ is an open analytic subset of a real projective algebraic variety and $f_k$
is the restriction of a projective morphism. The proof of inequality (2.4) is by induction on $k$. Set $\epsilon := \dim(\nabla_1 \cap \nabla_k)$. We prove that for $\epsilon = 1, 2$, the dimension of the fiber $f_k^{-1}(V_1, V_k)$ is bounded by the function $\delta(k, \epsilon)$ defined recursively by $\delta(3, 1) = r - 2$, $\delta(3, 2) = 2r - 6$, and for $k \geq 3$

$$
\begin{align*}
\delta(k + 1, 1) &= \max\{d_{k+1}, \delta(k, 2), (r - 2) + \delta(k, 1)\}, \\
\delta(k + 1, 2) &= \max\{(r - 1) + \delta(k, 1), (r - 3) + \delta(k, 2)\}.
\end{align*}
$$

The proof will show also that the function $\delta(k, \epsilon)$ is bounded from above by the right hand side of inequality (2.4). As we need to prove simultaneously both facts we will not use the notation $\delta(k, \epsilon)$, but it explains the right hand side of inequality (2.4).

The case $k = 3$: If $\nabla_1 = \nabla_3 = V$, then $W_1$ and $W_2$ contain $V$. If $W_1 \neq W_2$, then $\nabla_2 = W_1 \cap W_2$, so that this subset of the fiber has dimension $2 \dim(\mathbb{P}(\Lambda/V)) = 2r - 6$. The subset of the fiber where $W_1 = W_2$ has dimension $\dim(\mathbb{P}(\Lambda/V)) + 2 = r - 1 < 2r - 6$. Hence, the dimension of the fiber is $2r - 6$. Assume next that $\ell := \nabla_1 \cap \nabla_3$ is one dimensional. Case 3.a: The subset of the fiber, where $\ell$ is not contained in $\nabla_2$, consists of twistor lines with $W_1 = \ell + \nabla_2 = W_2$. Now $\nabla_1 + \nabla_3$ is contained in $W_1 + W_2 = \ell + \nabla_2$. Hence, $\nabla_1 + \nabla_3 = \ell + \nabla_2$ and $\nabla_2$ is contained in $\nabla_1 + \nabla_3$. The dimension of this subset is 2. Case 3.b: If $\ell$ is contained in $\nabla_2$, $\nabla_2 \neq \nabla_1$, and $\nabla_2 \neq \nabla_3$, then $W_1 = \nabla_1 + \nabla_2$ and $W_2 = \nabla_2 + \nabla_3$. The corresponding subset of the fiber has dimension $\dim(\mathbb{P}(\Lambda/\ell)) = r - 2$. Case 3.c: If $\nabla_2 = \nabla_1$, then $W_2 = \nabla_2 + \nabla_3$. The corresponding subset of the fiber has dimension $\dim(\mathbb{P}(\Lambda/V_1)) = r - 3$. The case where $\nabla_2 = \nabla_3$ is analogous. We conclude that when $\dim(\nabla_1 \cap \nabla_3) = 1$ the fiber $f_k^{-1}(V_1, V_3)$ has dimension $r - 2$. Hence, inequality (2.4) holds for $k = 3$.

Assume that inequality (2.4) holds for $k$ and $\dim(\nabla_1 \cap \nabla_{k+1}) > 0$. We establish separately the two cases of inequality (2.4) for $k + 1$, according to the dimension of $\nabla_1 \cap \nabla_{k+1}$ being 1 or 2.

Assume that $\dim(\nabla_1 \cap \nabla_{k+1}) = 1$. The fiber $f_k^{-1}(V_1, V_{k+1})$ is the union of three subsets $\Sigma_d$, determined by the dimension $d$ of $\nabla_2 \cap \nabla_{k+1}$. If $d \neq 1$, then $\nabla_1 \neq \nabla_2$, and $W_1 = \nabla_1 + \nabla_2$. The dimension of $\Sigma_0$ is $(r - 1) + d_k = k_{k+1}$. The dimension of $\Sigma_2$ is $\leq \beta_k$, by the induction hypothesis, and $\delta_k \leq \alpha_k$. Hence, $\dim(\Sigma_2) \leq \alpha_{k+1}$. We claim that $\Sigma_1$ is the union of two set, $\Sigma'_1$, where $\nabla_1 \cap \nabla_2 = \nabla_1 \cap \nabla_{k+1}$, and $\Sigma'_2$, where $\nabla_2$ is contained in $\nabla_1 + \nabla_{k+1}$. It suffices to prove the implication

$$
\dim(\nabla_1 \cap \nabla_{k+1}) = 1 \text{ and } \dim(\nabla_2 \cap \nabla_{k+1}) = 1 \text{ and } \nabla_2 \not\subseteq \nabla_1 + \nabla_{k+1} \Rightarrow \nabla_1 \cap \nabla_{k+1} = \nabla_1 \cap \nabla_{k+1}.
$$

The assumption $\nabla_2 \not\subseteq \nabla_1 + \nabla_{k+1}$ implies that $\dim(\nabla_1 \cap \nabla_2) = 1$ and $\nabla_1 \cap \nabla_2 = \nabla_2 \cap \nabla_{k+1}$, which implies the conclusion of the displayed implication and verifies the claim. The dimension of $\Sigma'_1$ is $\leq (r - 2) + \alpha_k$, by the induction hypothesis, and the latter is $\leq \alpha_{k+1}$. The dimension of $\Sigma'_2$ is smaller than that of $\Sigma'_1$. Hence, $\dim(\Sigma_1) \leq \alpha_{k+1}$.

Assume that $\nabla_1 = \nabla_{k+1} = \nabla$. The fiber $f_k^{-1}(V_1, V_{k+1})$ is the union of two subsets $S_d$, determined by the dimension $d$ of $\nabla_2 \cap \nabla$. Note that inequality (2.4) is equivalent to the following inequality:

\[
\dim(f_k^{-1}(V_1, V_k)) \leq \begin{cases}
  d_k & \text{if } \dim(\nabla_1 \cap \nabla_k) = 1 \text{ and } k \geq r - 1, \\
  (k - 2)(r - 2) & \text{if } \dim(\nabla_1 \cap \nabla_k) = 1 \text{ and } k \leq r - 2, \\
  d_k & \text{if } \dim(\nabla_1 \cap \nabla_k) = 2 \text{ and } k \geq r, \\
  (k - 2)(r - 2) + 1 & \text{if } \dim(\nabla_1 \cap \nabla_k) = 2 \text{ and } r - 2 \leq k \leq r - 1, \\
  (k - 1)(r - 3) & \text{if } \dim(\nabla_1 \cap \nabla_k) = 2 \text{ and } k \leq r - 3.
\end{cases}
\]
If $d = 1$, then $V \neq V_2$, and $W_1 = V + V_2$. The dimension of $S_1$ satisfies

$$\dim(S_1) \leq (r - 1) + \alpha_k = \begin{cases} d_{k+1} & \text{if } k \geq r \\ (k-1)(r-2) + 2 & \text{if } k \leq r - 1, \end{cases}$$

by the induction hypothesis, and the right hand side is $\leq \beta_{k+1}$, by definition of the latter. The desired upper bound $\dim(S_1) \leq \beta_{k+1}$ follows. The dimension of $S_2$ satisfies

$$\dim(S_2) \leq (r - 3) + \beta_k$$

by the induction hypothesis. The latter is $\leq \beta_{k+1}$. The upper bound $\dim(S_2) \leq \beta_{k+1}$ follows.

Let $\pi_k : Tw^k_\Lambda \rightarrow \Omega_\Lambda$ send a twistor path to its endpoint $V_k$. Fix $V$ and $V_k$ in $\Omega_\Lambda$. Let $\Sigma_V$ be the open subset of the fiber $\pi^{-1}_k(V_k)$ consisting of twistor paths with starting point $V_1$ satisfying $V_1 \cap V = (0)$. Denote by $\Sigma_V^c$ the complement of $\Sigma_V$ in the fiber.

**Lemma 2.5.** If $k \geq 4$, then the codimension of $\Sigma_V^c$ in the fiber $\pi^{-1}_k(V_k)$ is greater than or equal to 2. Consequently, the set $\Sigma_V$ is connected.

**Proof.** The set $\Sigma_V^c$ is the union of the sets $S_{i,j,\ell}$ consisting of twistor paths satisfying

$$\dim(\overline{V} \cap \overline{V}_1) = i, \dim(\overline{V} \cap \overline{V}_k) = j, \dim(\overline{V}_1 \cap \overline{V}_k) = \ell,$$

$1 \leq i \leq 2, 0 \leq j, \ell \leq 2$. Note that if one of the indices is 2, then the other two are equal. Hence, we need to estimate the codimensions of $S_{1,0,0}, S_{1,0,1}, S_{1,1,0}, S_{1,1,1}, S_{2,2,2}, S_{2,0,0}, S_{1,2,1},$ and $S_{1,1,2}$. The dimension of the fiber $\pi^{-1}_k(V_k)$ is $(k - 1)(r - 1)$, by Lemma 2.1. If $\overline{V}_1 = \overline{V}$ ($i = 2$), or $\overline{V}_1 = \overline{V}_k$ ($\ell = 2$), then the dimension $s_{i,j,\ell}$ of $S_{i,j,\ell}$ equals that of $f^{-1}_k(V_1, V_k)$ and the statement easily follows from Lemma 2.4.

Let us estimate the dimension of $S_{1,2,1}$, where $\overline{V} = \overline{V}_k$ and $\dim(\overline{V}_1 \cap \overline{V}_k) = 1$. The space $\overline{V}_1$ varies in an $r - 1$ dimensional family, so that

$$s_{1,2,1} \leq r - 1 + \begin{cases} (k-3)(r-1) + 2 & \text{if } k \geq r - 1 \\ (k-2)(r-2) & \text{if } k \leq r - 2. \end{cases}$$

by Lemma 2.4. We conclude the inequality

$$\text{codim}(S_{1,2,1}) = (k - 1)(r - 1) - s_{1,2,1} \geq \begin{cases} r - 3, & \text{if } k \geq r - 1 \\ k - 2, & \text{if } k \leq r - 2. \end{cases}$$

The statement follows. The inequalities $s_{1,0,1} \leq s_{1,2,1}$ and $s_{1,1,1} \leq s_{1,2,1}$ hold, since the subspace $\overline{V}_1$ is restricted by two conditions when $\overline{V} \neq \overline{V}_k$ and only by one condition when $\overline{V} = \overline{V}_k$. Finally, for $j = 0, 1$, we have $s_{1,j,0} \leq r - 1 + d_k$ and its codimension is $\geq r - 3$.

**2.2.2. The well behaved open subset $\hat{T}w^k_\Lambda$.** Let $g_k : Tw^k_\Lambda \rightarrow Gr_{+++}(\Lambda)^{k - 1}$ be the natural map. Assume that $k \geq 5$. Let $U_i^k, 1 \leq i \leq k - 4$, be the open subset of $Gr_{+++}(\Lambda)^{k - 1}$, where $W_i \cap W_{i+3} = (0)$. The complement $D_k^i$ of $g^{-1}_k(U_i^k)$ has codimension 1 in $Tw^k_\Lambda$. Set $U^k := \bigcup_{i=1}^{k-4} U_i^k$ and

$$\hat{T}w^k_\Lambda := g^{-1}_k(U^k) = Tw^k_\Lambda \setminus \left( \bigcap_{i=1}^{k-4} D_i^k \right).$$

Denote the restriction of $f_k$ to $\hat{T}w^k_\Lambda$ by $\hat{f}_k : \hat{T}w^k_\Lambda \rightarrow Gr_{+++}(\Lambda)^2$. Following is the main result of Subsection 2.2.2.

**Proposition 2.6.** If $k \geq 6$, then $Tw^k_\Lambda$ is connected and the map $\hat{f}_k$ is submersive. If $k \geq 7$, then $\hat{f}_k$ is surjective. If $k \geq 8$, then every fiber of $\hat{f}_k$ is smooth and connected.
We will need the following preliminary lemmas.

**Lemma 2.7.** Let $V$ be a vector space with a non-degenerate bilinear pairing of signature $(1, \rho - 1)$, $\rho \geq 3$. Then the space $Z_d(V)$ of oriented negative definite $d$-dimensional subspaces of $V$ is connected for $d \leq \rho - 2$.

*Proof.* Take $V = \mathbb{R}^\rho$ with the quadratic form $x_0^2 - \sum_{i=1}^{\rho-1} x_i^2$. The proof is by induction on $d$. $Z_1(V)$ is the real affine variety cut out by the equation $x_0^2 - \sum_{i=1}^{\rho-1} x_i^2 = -1$, which is a bundle over $\mathbb{R}^1$ (the $x_0$-line) with $(\rho - 2)$-dimensional spheres as fibers. Let $F_d$ be the tautological rank $d$ subbundle over $Z_d$ and $F_d^\perp$ the orthogonal complement subbundle. Denote by $\pi_d : \mathcal{Z}(F_d^\perp) \to Z_d$ the bundle whose fiber over a negative definite oriented $d$-dimensional subspace $N$ is $Z_1(N^\perp)$. Assume that $d < \rho - 2$ and that $Z_d$ is connected. Then the fibers of $\pi_d$ are connected and hence so is $\mathcal{Z}(F_d^\perp)$. Now $\mathcal{Z}(F_d^\perp)$ maps naturally onto $Z_{d+1}$. Consequently, $Z_{d+1}$ is connected as well. \hfill $\Box$

Let $A \subset Gr^{++}(\Lambda)^2$ be the subset consisting of pairs $(V_1, V_2)$ such that $\mathbf{v}_1 \cap \mathbf{v}_2 = 0$ and the signature of $\mathbf{v}_1 + \mathbf{v}_2$ is $(3, 1)$. Note that the complement of $A$ contains the non-empty open subset of $Gr^{++}(\Lambda)^2$ of pairs $(V_1, V_2)$ such that $\mathbf{v}_1 + \mathbf{v}_2$ is four dimensional of signature $(2, 2)$.

**Lemma 2.8.** $A$ is an open and connected subset of $Gr^{++}(\Lambda)^2$. The open subset $\tilde{A} := f_3^{-1}(A)$ of $Tw^3_\Lambda$ has two connected components. Given $(V_1, V_3) \in A$, the intersection of the fiber $f_3^{-1}(V_1, V_2)$ with each connected component of $\tilde{A}$ is non-empty and connected.

*Proof.* Let $t := \{(V_1, V_2, V_3); (W_1, W_2)\}$ be a point of $Tw^3_\Lambda$ such that $\mathbf{v}_1 \cap \mathbf{v}_3 = 0$. The dimension of $W_1 + W_2$ is at most 4 and $\mathbf{v}_1 + \mathbf{v}_3$ is contained in $W_1 + W_2$, so both spaces are four dimensional and equal. $W_1^\perp$ is negative definite and it intersects $\mathbf{v}_1 + \mathbf{v}_3$ in a one dimensional subspace. We conclude that $(V_1, V_3)$ belongs to $A$. The subset $\tilde{A}$ is open, as it is equal to the subset of $Tw^3_\Lambda$ consisting of twistor paths where $\mathbf{v}_1 \cap \mathbf{v}_3 = 0$. The map $f_3$ restricts to $\tilde{A}$ as a submersive map into $Gr^{++}(\Lambda)^2$, by Lemma 2.3. Hence, its image $A$ is an open subset.

We prove next that $A$ is connected. The projection from $A$ to the first factor $Gr^{++}(\Lambda)$ is surjective. It suffices to prove that its fibers are connected. Fix $V_1 \in Gr^{++}(\Lambda)$ and denote by $A_{V_1}$ the fiber over $V_1$. Let $\mathbf{v}_{V_1}$ be the quotient of the fiber by the involution reversing the orientation of the second subspace. The set of four dimensional subspaces of $\Lambda$ of signature $(3, 1)$ containing $\mathbf{v}_1$ is isomorphic to the set $Gr^{+-}(\mathbf{v}_1^\perp)$ of two dimensional subspaces of $\mathbf{v}_1^\perp$ of signature $(1, 1)$. Denote by $Gr^{+-}(\mathbf{v}_1^\perp)$ its double cover corresponding to oriented two-dimensional subspaces. We get the cartesian diagram

$$
\begin{array}{ccc}
A_{V_1} & \longrightarrow & Gr^{+-}(\mathbf{v}_1^\perp) \\
\downarrow & & \downarrow \\
\overline{A}_{V_1} & \longrightarrow & \overline{Gr}^{+-}(\mathbf{v}_1^\perp).
\end{array}
$$

The space $Gr^{+-}(\mathbf{v}_1^\perp)$ is connected, by Lemma 2.7. It remains to prove that the fibers of the top horizontal map are connected. The fibers of the two horizontal maps are isomorphic, so we may prove connectedness of the fibers of the bottom horizontal map. It suffices to prove that given a four dimensional subspace $Z$ of signature $(3, 1)$ containing $\mathbf{v}_1$, the set $\overline{A}_Z$ of oriented positive definite two-dimensional subspaces $\mathbf{v}_3$ of $Z$ such that $\mathbf{v}_1 \cap \mathbf{v}_3 = 0$ is connected.
The orthogonal projection of the subspace $V_3$ to $V_1^\perp \cap Z$ is injective, hence an isomorphism. Composing its inverse with the orthogonal projection from $V_3$ to $V_1$ we get the linear homomorphism $\phi : V_1^\perp \cap Z \to V_1$, whose graph is $V_3$. Choose an orthogonal basis $\{e_1, e_2, e_3, e_4\}$ of $Z$, such that $\{e_1, e_2\}$ is an orthonormal basis of $V_1$, $(e_3, e_3) = 1$, and $(e_4, e_4) = -1$. Write $\phi(e_3) = a e_1 + c e_2$ and $\phi(e_4) = b e_1 + d e_2$. Then $\{e_3 + a e_1 + c e_2, e_4 + b e_1 + d e_2\}$ is a basis for $V_3$ with Gram matrix $G := \begin{pmatrix} a^2 + c^2 + 1 & ab + cd \\ ab + cd & b^2 + d^2 - 1 \end{pmatrix}$. The inequality $b^2 + d^2 > 1$ is a necessary condition for $V_3$ to be positive definite. Set $\phi_t(e_3) = t(a e_1 + c e_2)$ and $\phi_t(e_4) = b e_1 + d e_2$.

The determinant of the Gram matrix $G_t$ of the graph of $\phi_t$ with respect to the analogous basis satisfies

$$\det(G_t) = -(1 - t^2)(b^2 + d^2 - 1) + t^2 \det(G).$$

If $V_3$ is positive definite, then $\det(G) > 0$ and so is $\det(G_t)$, for $0 \leq t \leq 1$. We conclude that $\mathbb{A}_Z$ admits a deformation retract to the set of graphs of $\phi$, such that $a = c = 0$ and $b^2 + d^2 > 1$, which is a connected set. Hence $\mathbb{A}_Z$ is connected. Consequently, so is $A$.

The fiber of $f_3$ over $(V_1, V_3) \in A$ consists of oriented subspaces $V_2 \in Gr_++(\mathcal{A})$, such that $V_1 \cap V_2$ and $V_2 \cap V_3$ are both one-dimensional. The three dimensional subspaces $W_1$ of points in such fibers are determined by the $V_i$’s as follows: $W_1$ is spanned by $V_1$ and $V_2$ and $W_2$ is spanned by $V_2$ and $V_3$. Let $\Upsilon$ be the open subset of $\mathbb{R}P(V_1) \times \mathbb{R}P(V_3)$ consisting of pairs of lines $(\ell_1, \ell_3)$, such that

1. $W_1 := V_1 + \ell_3$ is positive definite.
2. $W_2 := V_3 + \ell_1$ is positive definite.

The fiber $f_3^{-1}(V_1, V_3)$ is a double cover of $\Upsilon$ corresponding to the two choices of orientation of $V_2 := V_1 + \ell_3$. For later use we note that we have a canonical isomorphism

$$\wedge^3 W_1 \otimes \wedge^3 W_2 \cong \wedge^2 V_1 \otimes \wedge^2 V_2 \otimes \wedge^2 V_3. \tag{2.5}$$

We claim that Condition (1) in the definition of $\Upsilon$ corresponds to a non-empty open connected proper subset of $\mathbb{R}P(V_3)$ (an “interval”) and Condition (2) corresponds to such a subset of $\mathbb{R}P(V_1)$, so that $\Upsilon$ is non-empty and simply connected. It suffices to verify the statement for Condition (1). Choose an orthonormal basis $\{e_1, e_2\}$ of $V_1$ and an orthogonal basis $\{e_3, e_4\}$ of $V_1^\perp \cap (V_1 + V_3)$ satisfying $(e_3, e_3) = 1$ and $(e_4, e_4) = -1$. We can choose a basis $\{b_1, b_2\}$ of $V_3$ of the form

$$b_1 = c_1 e_1 + c_2 e_2 + e_3$$
$$b_2 = d_1 e_1 + d_2 e_2 + e_4,$$

since $V_1 \cap V_3 = (0)$. Let $\ell_3 \subset V_3$ be spanned by $t_1 b_1 + t_2 b_2$. Then Condition (1) is equivalent to the inequality $t_1^2 > t_2^2$ verifying the assertion.

Let $S_1$ be the unit circle in $V_i$. Then $S_1 \times S_3$ is a $\mathbb{Z}/2 \times \mathbb{Z}/2$ covering of $\mathbb{R}P(V_1) \times \mathbb{R}P(V_3)$. A point $(u_1, u_3)$ in $S_1 \times S_3$ determines an orientation of span$\{u_1, u_2\}$. The fiber of $f_3$ is embedded in the quotient of $S_1 \times S_3$ by the orientation preserving involution $(u_1, u_3) \mapsto (-u_1, -u_3)$. The orientation and the metric of $V_i$ determine a complex structure on $V_i$, for $i = 1, 3$, so an action by the standard torus $U(1) \times U(1)$ on $S_1 \times S_3$. The above involution corresponds to a translation by the point of order two $(e^{\pi i}, e^{\pi i}) \in U(1) \times U(1)$. The fiber of $f_3$ over $(V_1, V_3)$ is thus the disjoint union of two open subsets $\Upsilon_1$ and $\Upsilon_2$ of a real two-dimensional torus. These two sheets are naturally labeled. The orientation of $\Upsilon$ determines an orientation of $W_1$ and $W_2$. Hence, both lines in Equation (2.5) are oriented. One of the sheets consists of twistor paths where the isomorphism (2.5) is orientation preserving and the other where is is
orientation reversing. We conclude that the subset $\tilde{A} := f_3^{-1}(A)$ is disconnected, consisting of two connected components $\tilde{A}_x$, where the isomorphism (2.5) is orientation preserving, and $\tilde{A}_\bar{x}$ where it is not. The two components are interchanged by the involution reversing the orientation of $V_2$.

Lemma 2.9. Let $W$ be a $d$-dimensional subspace of $\Lambda$, where $d \leq r - 4$, and $V$ a positive definite 2-dimensional subspace of $\Lambda$ such that $V \cap W = 0$. Then the subset of $Gr_{+++}(\Lambda)$, consisting of $W'$ containing $V$ such that $W \cap W' = 0$, is non-empty and connected.

Note: The above Lemma will be applied below with $d = 2$ and $d = 3$, which is the reason we assumed $r \geq 7$.

Proof. The set of $W' \in Gr_{+++}(\Lambda)$ containing $V$ is isomorphic to the hyperbolic space associated to $V^\perp$ and is hence connected. If $W'$ contains $V$ and $W' \cap W \neq 0$, then $W' \cap W$ is one dimensional and $W' = V + (W' \cap W)$. Hence, the set of $W' \in Gr_{+++}(\Lambda)$ containing $V$ and intersecting $W$ non-trivially is isomorphic to an open subset of $\mathbb{RP}(W)$, which is $(d - 1)$-dimensional, and its complement in the $(r - 3)$-dimensional hyperbolic space associated to $V^\perp$ is connected.

Lemma 2.10. Let $V_1, V_2$ be elements of $Gr_{+++}(\Lambda)$, such that $\overline{V_1} \cap \overline{V_2} = 0$. The subset of $Gr_{+++}(\Lambda)^2$, consisting of pairs $(W_1, W_2)$, such that $W_1$ contains $\overline{V_i}$, $i = 1, 2$, and $W_1 \cap W_2 = 0$, is non-empty and connected.

Proof. The subset of $Gr_{+++}(\Lambda)$ consisting of $W_1$ containing $\overline{V_1}$, such that $W_1 \cap \overline{V_2} = 0$, is non-empty and connected, by Lemma 2.9. Fix such a $W_1$. It suffices to show that the subset of $Gr_{+++}(\Lambda)$ consisting of $W_2$ containing $\overline{V_2}$, such that $W_1 \cap W_2 = 0$, is connected. This is the case by Lemma 2.9 again.

Lemma 2.11. Let $W_1$ and $W_2$ be two 3-dimensional positive definite subspaces of $\Lambda$ such that $W_1 \cap W_2 = 0$. Denote by $\tilde{\mathbb{RP}}(W_1)$ the universal cover of $\mathbb{RP}(W_1)$ parametrizing oriented two dimensional subspaces of $W_i$. Then the open subset $\Xi$ of $\tilde{\mathbb{RP}}(W_1) \times \tilde{\mathbb{RP}}(W_2)$, consisting of pairs $(V_1, V_2)$ with $\overline{V_1} + \overline{V_2}$ of signature $(3, 1)$, is non-empty and connected.

Proof. Let $p_i : \Xi \to \tilde{\mathbb{RP}}(W_i)$ be the natural projection. We prove that $p_i$ is surjective with connected fibers. It suffices to prove it for $i = 1$. Fix $V_1 \in \tilde{\mathbb{RP}}(W_1)$. Denote by $W_2$ the image of $W_2$ via the orthogonal projection into $\overline{V_1}^\perp$. Then $\overline{V_1} + W_2 = \overline{V_1} + \tilde{W}_2$. Note that $V_1 + W_2$ has signature $(3, 2)$. Hence, $\tilde{W}_2$ has signature $(1, 2)$. The fiber of $p_1^{-1}(V_1)$ is isomorphic to the open subset of $\tilde{\mathbb{RP}}(\tilde{W}_2)^*$ consisting of oriented 2-dimensional subspaces of $\tilde{W}_2$ of signature $(1, 1)$. Equivalently, $p_1^{-1}(V_1)$ is isomorphic to open subset of $\tilde{\mathbb{RP}}(\tilde{W}_2)$ consisting of oriented 1-dimensional negative definite subspaces of $\tilde{W}_2$. The latter is connected, by Lemma 2.7. Hence, $p_1$ is a surjective fibration with connected fibers and $\Xi$ is connected.

Denote by $\tau_i$, $1 \leq i \leq k$, the involution of $Tw_5^\Lambda$ taking a twistor path $\{(V_1, \ldots, V_k); (W_1, \ldots, W_{k-1})\}$ to the one obtained from it by reversing the orientation of $V_i$.

Lemma 2.12. The map $\tilde{f}_5$ is submersive. Its image consists of all pairs $(V_1, V_5)$ in $\Omega_\Lambda^2$, such that $\overline{V_1} \cap \overline{V_5} = 0$. Given a twistor path $\{V_1, \ldots, V_5; W_4, \ldots, W_k\}$ in $\tilde{Tw}_5^\Lambda$, there is a natural isomorphism

$$\Lambda^2 \overline{V_2} \otimes \Lambda^2 \overline{V_3} \otimes \Lambda^2 \overline{V_4} \cong \Lambda^3 W_2 \otimes \Lambda^3 W_3.$$
to the orientations of both lines above. The other component $\tilde{T}w^{5}_{A^g}$ consists of twistor paths where the isomorphism is orientation reversing. The two connected components are interchanged by $\tau_{i}$, if $2 \leq i \leq 4$, and each component is invariant with respect to $\tau_{1}$ and $\tau_{2}$. Each non-empty fiber of $\tilde{f}_{5}$ intersects each connected component of $\tilde{T}w^{5}_{A}$ along a connected set.

**Proof.** Let $\{(V_{1},\ldots,V_{5});(W_{1},\ldots,W_{4})\}$ be a point of $\tilde{T}w^{5}_{A}$. Then

$$\dim(V_{3}\cap W_{4}) \geq \dim(V_{3}\cap V_{4}) \geq 1, \quad \text{and}$$

$$\dim(V_{3}\cap W_{1}) \geq \dim(V_{3}\cap V_{2}) \geq 1.$$ 

Now $W_{1}\cap W_{4} = (0)$. Hence, $\ell_{4} := V_{3}\cap W_{4}$ and $\ell_{1} := V_{3}\cap W_{1}$ are both one dimensional, $V_{3} = \ell_{1} + \ell_{4}$, $W_{2} = V_{2} + \ell_{4}$, and $W_{3} = \ell_{1} + \ell_{4}$.

The map $\tilde{f}_{5}$ is submersive, by Lemma 2.3 since $\tilde{T}w^{5}_{A}$ is contained in the open subset of $Tw^{5}_{A}$, where $V_{1}\cap V_{5} = 0$.

We prove that the image of $\tilde{f}_{5}$ contains every pair $(V_{1},V_{5})$, such that $V_{1}\cap V_{5} = 0$. There exist three dimensional positive definite subspaces $W_{1}$ containing $V_{1}$ and $W_{4}$ containing $V_{5}$, such that $W_{1}\cap W_{4} = 0$, by Lemma 2.9. Now $W_{4}^{\perp}$ is negative definite. Hence $W_{4}\cap W_{4}^{\perp} = 0$, the orthogonal projection from $W_{4}$ to $W_{1}$ is an isomorphism, and composing its inverse with the orthogonal projection to $W_{1}^{\perp}$ yields an injective homomorphism $\phi : W_{1} \rightarrow W_{1}^{\perp}$, such that $W_{4}$ is its graph. Let $\{e_{1},e_{2},e_{3}\}$ be an orthonormal basis of $W_{1}$. Set

$$V_{2} := \text{span}\{e_{1},e_{3}\},$$

$$V_{3} := \text{span}\{e_{2},e_{2} + \phi(e_{2})\},$$

$$V_{4} := \text{span}\{e_{1} + \phi(e_{1}),e_{2} + \phi(e_{2})\},$$

$$W_{2} := \text{span}\{e_{1},e_{2} + \phi(e_{2}),e_{3}\},$$

$$W_{3} := \text{span}\{e_{1} + \phi(e_{1}),e_{2} + \phi(e_{2}),e_{3}\}.$$ 

$V_{3}$ is positive definite, since the basis given is orthogonal with elements of positive self intersection. $V_{3}$ is contained in $W_{1}$ and $V_{4}$ in $W_{4}$. The element $e_{2} + \phi(e_{2})$ has positive self intersection, it is orthogonal to $V_{3}$, and $W_{2} = V_{2} + \mathbb{R}(e_{2} + \phi(e_{2}))$. Hence, $W_{2}$ is positive definite. The element $e_{3}$ has positive self intersection, it is orthogonal to $V_{4}$, and $W_{3} = V_{3} + \mathbb{R}e_{3}$. Hence, $W_{3}$ is positive definite. Any choice of orientations for $V_{i}$, $i = 2,3,4$, yields a twistor path in $\tilde{T}w^{5}_{A}$, which is mapped to $(V_{1},V_{5})$ via $\tilde{f}_{5}$.

We prove next that $\tilde{T}w^{5}_{A}$ has precisely two connected components, which are interchanged by $\tau_{i}$, $2 \leq i \leq 4$, and are invariant under $\tau_{1}$ and $\tau_{2}$. Let

$$\mu : \tilde{T}w^{5}_{A} \rightarrow Tw^{3}_{A}\,$$

be the map given by $\{(V_{1},V_{2},V_{3},V_{4},V_{5});(W_{1},W_{2},W_{3},W_{4})\} \mapsto \{(V_{2},V_{3},V_{4});(W_{2},W_{3})\}$. We claim that the image of $\mu$ is precisely the subset $\tilde{A}$ of Lemma 2.8 and the fibers of $\mu$ are connected. The image is clearly contained in $\tilde{A}$. Given a point $t := \{(V_{2},V_{3},V_{4});(W_{2},W_{3})\}$ of $\tilde{A}$, the set $WW$ of pairs $(W_{1},W_{4})$, such that $W_{1}$ contains $V_{2}$ and $W_{4}$ contains $V_{4}$ and $W_{1}\cap W_{4} = 0$ is non-empty and connected, by Lemma 2.10. Over $WW$ we have the two pullbacks $\mathcal{W}_{1}$ and $\mathcal{W}_{4}$ of the tautological rank 3 real vector subbundle over $Gr_{+++}(A)$. Let $Gr_{+++}(\mathcal{W}_{i})$ be the bundle of oriented two dimensional subspaces in the fibers of $\mathcal{W}_{i}$, $i = 1,4$. Then $Gr_{+++}(\mathcal{W}_{i})$ is a bundle of two dimensional spheres over $WW$. The fiber of $\mu$ over $t$ is the fiber product $Gr_{+++}(\mathcal{W}_{1}) \times_{WW} Gr_{+++}(\mathcal{W}_{4})$, which is non-empty and connected. Lemma 2.8 implies that $\tilde{T}w^{5}_{A}$ has precisely two connected components, which are interchanged by $\tau_{i}$, $2 \leq i \leq 4$, and are invariant under $\tau_{1}$ and $\tau_{2}$.
It remains to prove that each non-empty fiber of $f_5$ has two connected components. Let $U \subset Gr_{+++}(\Lambda)^2$ be the subset of pairs $(W_1, W_2)$, such that $W_1 \cap W_2 = 0$. Let $\mathfrak{U} \subset X$ be the subset of pairs $(V_2, V_4; (W_1, W_4))$, such that $W_1 \cap W_4 = 0$, $V_2$ is contained in $W_1$, $V_4$ is contained in $W_4$, and $V_2 + V_4$ has signature $(3, 1)$. Denote by $a : \mathfrak{U} \to A$ and $u : \mathfrak{U} \to U$ the natural projections. Let $\mathcal{I}_1$ and $\mathcal{I}_4$ be the two pullbacks to $U$ of the rank 3 tautological subbundle over $Gr_{+++}(\Lambda)$. Let $Gr_{+++}(\Lambda)$ be the bundle of oriented two-dimensional subspaces in the fibers of $\mathcal{I}_i$, $i = 1, 4$. Then the connected component $\tilde{T}w_{A,=}^5$ is isomorphic to the fiber product

$$\tilde{T}w_{A,=}^5 \cong [Gr_{+++}(\mathcal{I}_1) \times_U Gr_{+++}(\mathcal{I}_4)] \times_U A \times A,$$

$$(V_1, \ldots, V_5); (W_1, \ldots, W_4) \mapsto [(V_1, V_3); (W_1, W_4)], [(V_2, V_4); (W_1, W_4)], [(V_2, V_3, V_4); (W_2, W_3)].$$

Let $Gr_{+++}^0(\Lambda)^2$ be the open subset of $Gr_{+++}(\Lambda)^2$ consisting of pairs $(V_1, V_5)$, such that $V_1 \cap V_5 = 0$. The restriction of $f_5$ to $\tilde{T}w_{A,=}^5$ factors as the composition of the natural maps

$$b : [Gr_{+++}(\mathcal{I}_1) \times_U Gr_{+++}(\mathcal{I}_4)] \to Gr_{+++}^0(\Lambda)^2,$$

$$c : \tilde{T}w_{A,=}^5 \to [Gr_{+++}(\mathcal{I}_1) \times_U Gr_{+++}(\mathcal{I}_4)].$$

The maps $a : \mathfrak{U} \to A$, $u : \mathfrak{U} \to U$, $A_\varnothing \to A$, and $[Gr_{+++}(\mathcal{I}_1) \times_U Gr_{+++}(\mathcal{I}_4)] \to U$ are all surjective with connected fibers. This is clear for the latter, for a use Lemma 2.10 for $u$ use Lemma 2.11 and for $A_\varnothing \to A$ use Lemma 2.3. Given two surjective continuous maps $X \to Y$ and $Z \to Y$ with connected fibers over a connected topological space $Y$, the fiber product $X \times_Y Z$ is connected and maps onto $X$ with connected fibers. Hence, $c$ is surjective with connected fibers. The map $b$ is surjective with connected fibers, by Lemma 2.10. Hence, so is $c \circ b$, which is the restriction of $f_5$ to $\tilde{T}w_{A,=}^5$. Similarly, the restriction of $f_5$ to $\tilde{T}w_{A,\neq}^5$ has connected fibers.

2.2.3. Smooth connected spaces of twistor paths with fixed end points.

Proof of Proposition 2.6. $\tilde{T}w_{A}^k$ is connected, since $T_{A}^k$ is a smooth connected manifold and the former is the complement of a subset of real codimension $\geq 2$, when $k \geq 6$.

The subset $g_k^{-1}(U_k^i)$ of $\tilde{T}w_{A}^k$ is the fiber product

$$T_{A}^i \times_{\Omega_A} T_{A}^5 \times_{\Omega_A} T_{A}^{k-i-3},$$

with respect to the following maps:

1. $\pi_i : T_{A}^i \to \Omega_A$ mapping $\{(V_1, \ldots, V_i), (W_1, \ldots, W_{i-1})\}$ to $V_i$,
2. the left map $\tilde{T}w_{A,=}^5 \to \Omega_A$ mapping $\{(V_1, \ldots, V_{i+4}), (W_i, \ldots, W_{i+3})\}$ to $V_i$,
3. the right map $\tilde{T}w_{A,=}^5 \to \Omega_A$ mapping $\{(V_i, \ldots, V_{i+3}), (W_i, \ldots, W_{i+3})\}$ to $V_{i+4}$, and
4. the map $\pi_{i+4} : T_{A}^{k-i-3} \to \Omega_A$ mapping $\{(V_{i+4}, \ldots, V_k), (W_{i+4}, \ldots, W_{k-1})\}$ to $V_{i+4}$.

Equivalently, $g_k^{-1}(U_k^i)$ is the fiber product over the cartesian square $\Omega_A^2$ of $\tilde{T}w_{A,=}^5$ and the cartesian product $[T_{A}^i \times T_{A}^{k-i-3}]$ with respect to the maps $f_5 : \tilde{T}w_{A,=}^5 \to \Omega_A^2$ and $(\pi_i, \pi_{i+4}) : [T_{A}^i \times T_{A}^{k-i-3}] \to \Omega_A^2$. The map $(\pi_i, \pi_{i+4})$ is surjective and submersive, by Lemma 2.1. The map $f_5$ is submersive, by Lemma 2.12. Hence, the projection

$$\psi : g_k^{-1}(U_k^i) \to [T_{A}^i \times T_{A}^{k-i-3}]$$

is submersive. The map $(\pi_1, \pi_k) : [T_{A}^i \times T_{A}^{k-i-3}] \to \Omega_A^2$, sending the pair of twistor paths to the pair $(V_1, V_k)$ consisting of the starting point $V_1$ of the first and the ending point $V_k$ of the
second, is submersive by Lemma 2.11. Hence, the map $f_k$ restricts to $g_k^{-1}(U_i^k)$, $1 \leq i \leq k - 4$, as a submersive map. It follows that the map $\tilde{f}_k$ is submersive and so all its fibers are smooth.

We prove next the surjectivity of $\tilde{f}_k$, for $k \geq 7$. It suffices to prove it for $k = 7$, since the concatenation of a path in $\tilde{T}w_7^5$ from $V_1$ to $V_7$ with any path in $T_{\Lambda}^{k-7+1}$ from $V_7$ to itself results in a path in $\tilde{T}w_7^5$. Given $(V_1, V_7)$ in $\Omega_\Lambda^2$ choose $V_2$ and $V_6$ in $\Omega_\Lambda$, such that $\nabla_2 \cap \nabla_6 = 0$, $W_1 := \nabla_1 + \nabla_2$ is a positive definite three dimensional subspace, and so is $W_6 := \nabla_6 + \nabla_7$. Then $(V_2, V_6)$ belongs to the image of $\tilde{f}_5$, by Lemma 2.11. If $\{(V_2, \ldots, V_6); (W_2, \ldots, W_6)\}$ belongs to $\tilde{T}w_7^5$, then $\{(V_1, V_2, \ldots, V_6, V_7); (W_1, W_2, \ldots, W_5, W_6)\}$ belongs to $\tilde{T}w_7^5$. Surjectivity of $\tilde{f}_7$ follows.

It remains to prove that all fibers of $\tilde{f}_k$ are connected, for $k \geq 8$. Recall that $T_{\Lambda}^{1} = \Omega_\Lambda$. Fix a pair $(V_1, V_k)$ in $\Omega_\Lambda^2$. Consider the above diagram with $i = 1$. Fibers of $(\pi_1, \pi_k)$ are isomorphic to fibers of $\pi_k : T_{\Lambda}^{k-4} \rightarrow \Omega_\Lambda$ and are thus connected, by Lemma 2.11. Consider the open subset $\Sigma_{V_1}$ of the fiber of $\pi_k : T_{\Lambda}^{k-4} \rightarrow \Omega_\Lambda$ over $V_k$, where $\nabla_1 \cap \nabla_5 = 0$. Similarly, we have the open subset $\Sigma$ of $T_{\Lambda}^{1} \times T_{\Lambda}^{k-4}$ consisting of pairs $(V_1, t)$, where $t$ is a twistor path from $V_5 \in \Omega_\Lambda$ to $V_k$, where $\nabla_1 \cap \nabla_5 = 0$. Fibers of $\psi$ are non-empty precisely over points of $\Sigma$, by the description of the image of $\tilde{f}_5$ in Lemma 2.12. $\Sigma_{V_1}$ forms a connected dense open subset of the fiber of $\pi_k$, by Lemma 2.5. Similarly, $\Sigma$ forms a dense open subset of $T_{\Lambda}^{1} \times T_{\Lambda}^{k-4}$. Hence, the fiber of $(\pi_1, \pi_k)$ intersects the image of $\psi$ in a connected set $\{V_1\} \times \Sigma_{V_1}$. Fibers of $\psi$ over points $(V_1, t) \in \Sigma$ are isomorphic to fibers of $\tilde{f}_5$, which have two connected components, by Lemma 2.12. Hence, the intersection of the fiber $f_k^{-1}(V_1, V_k)$ with $g_k^{-1}(U_i^k)$ has two connected components, as it maps to the connected set $\{V_1\} \times \Sigma_{V_1}$ with fibers each of which has two (labeled) connected components. The two connected components are interchanged under $\tau_2$.

Next consider the above diagram with $i = k - 4$. This case is analogous to the case $i = 1$, by reversing the twistor paths. Arguing as above, we get that the intersection of the fiber $f_k^{-1}(V_1, V_k)$ with $g_k^{-1}(U_i^k)$ has two connected components, each of which is invariant under $\tau_2$.

2.3. A universal twistor family. We have the period map

$$P : \mathfrak{M}_\Lambda^{0} \rightarrow \Omega_\Lambda,$$

given by $P(X, \eta) := \eta(H^{2,0}(X))$. The period map is surjective and it is injective over the locus $\Omega_\Lambda^{gen}$ of periods of manifolds $X$, such that Pic$(X)$ is trivial, or cyclic generated by a class of non-negative BBF-degree, by the Global Torelli Theorem [V5]. The Kähler cone of such an $X$ is equal to its positive cone in $H^{1,1}(X, \mathbb{R})$. 

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**Remark 2.13.** Let $X$ be an irreducible holomorphic symplectic manifold and $\omega$ a Kähler class on $X$. Set

$$V(X) := [H^{2,0}(X) \oplus H^{0,2}(X)] \cap H^2(X, \mathbb{R}).$$

Let $W \subset H^2(X, \mathbb{R})$ be the positive definite three dimensional subspace spanned by $V(X)$ and $\omega$. The base $P^1_\omega$ of the twistor family $\mathcal{Z} \to \mathbb{P}_\omega^1$ associated to $\omega$ is the conic in $P(W_C)$ of isotropic lines in $W_C$ [H13] Sec. 1.13 and 1.17]. The real subspace, of the direct sum of an isotropic line $t \in P^1_\omega$, and its complex conjugate, corresponds to a positive definite two dimensional subspace of $W$, which coincides with the subspace $V(\mathcal{Z}_t)$, associated to the fiber $\mathcal{Z}_t$ of $\mathcal{Z}$ over $t$, under the natural identification of $H^2(X, \mathbb{R})$ and $H^2(\mathcal{Z}_t, \mathbb{R})$ via the Gauss-Manin connection over the simply connected base $P^1_\omega$. We get the open ray $\rho_t$ in the line $V(\mathcal{Z}_t)^\perp \cap W$, consisting of classes $\omega_t$, such that a basis $\{v_1, v_2\}$ of $V(\mathcal{Z}_t)$, compatible with its orientation, extends to a basis $\{v_1, v_2, \omega_1\}$ compatible with the orientation of $W$. The fact we would like to recall is that the ray $\rho_t$ consists of Kähler classes of $\mathcal{Z}_t$ [H13].

**Lemma 2.14.** Given a marked pair $(X, \eta)$ in $\mathcal{M}_\Lambda^0$, and a generic twistor path\footnote{\cite[Sec. 1.13 and 1.17]{H13}.} $\{(V_1, \ldots, V_k); (W_1, \ldots, W_{k-1})\}$ satisfying $V_1 = P(X, \eta)$, such that $\eta^{-1}(V_1^\perp \cap W_1)$ is spanned by a Kähler class, there exists a unique lift of the path to a generic twistor path in $\mathcal{M}_\Lambda^0$ starting at $(X, \eta)$.

**Proof.** The statement follows from the surjectivity of the period map and its injectivity over the locus of periods $V \in Gr_{++}(\Lambda)$, such that $V^\perp \cap \Lambda$ is trivial, or cyclic generated by a class of non-negative self intersection, and the definitions of a generic twistor path in $Gr_{++}(\Lambda)$ and in $\mathcal{M}_\Lambda^0$. \hfill $\square$

We have seen in Equation (2.3) that the incidence variety $\mathcal{I}$ is the projectivization $\mathbb{RP}(\mathcal{C}^+)$ of the positive cone $\mathcal{C}^+$ in the Hodge bundle $\mathcal{H}^{1,1}$ over $Gr_{++}(\Lambda)$ and $p : \mathcal{I} \to Gr_{++}(\Lambda)$ is the bundle map. Let $\mathcal{K}$ be the universal Kähler cone in the Hodge bundle $P^*\mathcal{H}^{1,1}$ over $\mathcal{M}_\Lambda^0$. The fiber of the natural projection $\mathcal{K} \to \mathcal{M}_\Lambda^0$ over a marked pair $(X, \eta)$ is the image via $\eta$ of the Kähler cone of $X$. Recall that $\mathcal{K}$ is an open subset of $P^*\mathcal{H}^{1,1}$, by [KS2, Theorem 15]. The natural map from $P^*\mathcal{H}^{1,1}$ to $\mathcal{H}^{1,1}$ is a local homeomorphism, by Local Torelli. We get the local homeomorphism $\mathbb{RP}(P^*\mathcal{C}^+) \to \mathbb{RP}(\mathcal{C}^+) = \mathcal{I}$. Denote by $\hat{\mathcal{I}}$ the image in $\mathcal{I}$ of the projectivization of $\mathcal{K}$. The image $\hat{\mathcal{I}}$ is an open subset of $\mathcal{I}$, being the image of an open set via a local homeomorphism. Verbitsky’s Global Torelli Theorem implies that the map

$$\hat{P} : \mathbb{RP}(\mathcal{K}) \to \hat{\mathcal{I}}$$

is an isomorphism [Ma4, Theorem 5.16]. The complement of $\hat{\mathcal{I}}$ is the union of a countable collection of closed real analytic codimension 3 subsets in $\mathcal{I}$, consisting of projectivized walls of Kähler type chambers. More precisely, there is a subset $\Sigma \subset \Lambda$, consisting of a finite set of $\text{Mon}^2(\Lambda)$-orbits of classes $\lambda \in \Lambda$ with $(\lambda, \lambda) < 0$, such that the complement $\mathcal{I} \setminus \hat{\mathcal{I}}$ is the union

$$\bigcup_{\lambda \in \Sigma} \{(\ell, \alpha) : \ell \in \lambda^\perp \cap \Omega_\Lambda \text{ and } \alpha \in \lambda^\perp \cap \mathbb{RP}(\mathcal{C}^+(\ell))\},$$

where $\mathcal{C}^+(\ell)$ is the positive cone in the subspace of $\Lambda$ orthogonal to $\ell$ [AV, BHT, Mo]. Classes $\lambda$ in $\Sigma$ are called monodromy birationally minimal (MBM) classes in [AV]. The subset $\hat{\mathcal{I}}$ is connected, as its fiber over a generic point of $Gr_{++}(\Lambda)$ is the whole hyperbolic space (projectivization of the whole positive cone).
Let $\kappa : \mathbb{RP} \mathcal{K} \rightarrow \mathcal{M}_0^0$ be the natural projection. Over $\mathcal{M}_0^0$ we have a universal family, by [Ma0]. The pullback of the universal family via the map

\[ \tilde{\kappa} := \kappa \circ \bar{P}^{-1} : \check{\mathcal{I}} \rightarrow \mathcal{M}_0^0 \]

yields the family

\[ \pi : \mathcal{X} \rightarrow \check{\mathcal{I}}. \]

The map $\pi$ is clearly real analytic and in particular differentiable. The above is thus an example of a differentiable family of compact complex manifolds in the following sense. Denote by $GL(n, \mathbb{C}; m, \mathbb{R})$ the group of matrices of the form

\[
\begin{pmatrix}
A & B \\
0 & C
\end{pmatrix},
\]

where $A \in GL(n, \mathbb{C}), C \in GL(m, \mathbb{R}),$ and where $B$ is an arbitrary $n \times m$ matrix with complex entries. $GL(n, \mathbb{C}; m, \mathbb{R})$ is viewed as a subgroup of the group $GL(2n, m, \mathbb{R})$, of block upper triangular matrices, via the embedding

\[
\begin{pmatrix}
A & B \\
0 & C
\end{pmatrix} \mapsto \begin{pmatrix}
\text{Re}(A) & -\text{Im}(A) & \text{Re}(B) \\
\text{Im}(A) & \text{Re}(A) & \text{Im}(B) \\
0 & 0 & C
\end{pmatrix}.
\]

**Definition 2.15.** [KS1 Definition 1.1’ page 337]. A differentiable family of complex manifolds is a differentiable fiber bundle $\pi : \mathcal{X} \rightarrow \Sigma$ over a connected manifold $\Sigma$ with fiber a connected differentiable manifold $X$, $\text{dim}_\mathbb{R}(X) = 2n$, together with a differentiable reduction of the structure group $GL(2n, m, \mathbb{R})$ of the relative tangent bundle $T_\pi$ to $GL(n, \mathbb{C}; m, \mathbb{R})$, which imparts to each fiber a complex analytic structure. Differentiable above means of class $C^\infty$. There is a natural notion of an isomorphism of such families [KS1 Definition 1.2].

**Remark 2.16.** All differentiable families $\pi : \mathcal{X} \rightarrow \Sigma$ in this paper will be obtained from a holomorphic family $\check{\pi} : \check{\mathcal{X}} \rightarrow \check{\Sigma}$ as the pullback via a differentiable map $\kappa : \Sigma \rightarrow \check{\Sigma}$. Let $\beta : \check{\mathcal{Y}} \rightarrow \check{\mathcal{X}} \times_\pi \check{\mathcal{X}}$ be the blow-up centered along the relative diagonal. Pulling back $\beta$ via $\kappa$ we get the differentiable family $\mathcal{Y} \rightarrow \Sigma$ and the map $\beta : \mathcal{Y} \rightarrow \mathcal{X} \times_\pi \mathcal{X}$. We will refer to the latter as the blow-up centered along the relative diagonal of $\mathcal{X} \times_\pi \mathcal{X}$.

2.3.1. Moduli of marked irreducible holomorphic symplectic manifolds with a Kähler-Einstein structure. We have a natural section $\mathbb{RP} \mathcal{K} \rightarrow \mathcal{X}$, sending a ray in the Kähler cone to its unique Kähler class of self-intersection 1 with respect to the Beauville-Bogomolov-Fujiki pairing. Hence, the universal family (2.10) is endowed with a tautological Kähler class. For each Kähler class there exists a unique Kähler form representing it, such that the corresponding metric is Ricci flat, by Yau’s Theorem proving Calabi’s Conjecture [Ya]. Consequently, the relative tangent bundle of the universal family $\pi : \mathcal{X} \rightarrow \check{\mathcal{I}}$, given in (2.10), is endowed with a $C^\infty$ hermetian metric, which restricts to each fiber as the Ricci flat Kähler metric whose Kähler form represents the tautological Kähler class. See [Ko Theorem 10] for the construction in the case of the universal family of $K3$ surfaces. The general case follows via the same argument, using the isomorphism (2.8).
2.4. The universal twistor path and its universal family.

Given a point \( t := \{(V_1, \ldots, V_k); (W_1, \ldots, W_{k-1})\} \) of \( Tw^k_\Lambda \), denote by \( \rho_{i,i}(t) \) the open ray in the line \( \nabla_i^+ \cap W_i \) compatible with the orientations of \( V_i \) and \( W_i \), for \( 1 \leq i \leq k-1 \). Define \( \rho_{i,i-1}(t) \) similarly in terms of \( V_i \) and \( W_i-1 \), for \( 2 \leq i \leq k \). Let

\[
(2.11) \quad \tilde{Tw}^k_\Lambda \subset Tw^k_\Lambda
\]

be the open subset of points \( t = \{(V_1, \ldots, V_k), (W_1, \ldots, W_{k-1})\} \) satisfying the following three conditions:

1. The pairs \( (V_i, W_i) \) and \( (V_i, W_{i-1}) \) all belong to \( \tilde{J} \). In other words, each of the rays \( \rho_{i,i}(t) \) and \( \rho_{i,i-1}(t) \) is not orthogonal to any MBM class of Hodge type \( (1,1) \).
2. The rays \( \rho_{i,i-1}(t) \) and \( \rho_{i,i}(t) \) belong to the same Kähler type chamber in \( \nabla_i^+ \), for \( 2 \leq i \leq k-1 \). In other words, \( \tilde{\kappa}(V_i, W_i) = \tilde{\kappa}(V_i, W_{i-1}) \), where \( \tilde{\kappa} : \tilde{J} \to \mathfrak{M}^0_\Lambda \) is given in Equation \( (2.9) \).
3. \( W_i \cap W_{i+3} = 0 \), for some \( i \) in the range \( 2 \leq i \leq k-5 \).

Note that \( \tilde{Tw}^k_\Lambda \) is contained in \( \tilde{Tw}^k_\Lambda \). Condition \((1)\) excludes from \( \tilde{Tw}^k_\Lambda \) a countable union of closed codimension three subsets. Conditions \((1)\) and \((2)\) exclude from \( \tilde{Tw}^k_\Lambda \) a countable union of closed subsets of real codimension two, as a period which is not orthogonal to any MBM class has a unique Kähler type chamber. Condition \((3)\) excludes from \( \tilde{Tw}^k_\Lambda \) the closed subset where \( W_i \cap W_{i+3} \neq 0 \), for all \( i \) in the range \( 2 \leq i \leq k-5 \), which has codimension \( > 1 \) whenever \( k > 7 \).

**Remark 2.17.** Twistor paths in \( \Omega_\Lambda \), which satisfy Conditions \((1)\) and \((2)\) above, are in natural bijection with twistor paths in \( \mathfrak{M}^0_\Lambda \).

Let \( \tilde{\pi}_i : \tilde{Tw}^k_\Lambda \to \tilde{J} \) be the map sending a twistor path \( t := \{(V_1, \ldots, V_k); (W_1, \ldots, W_{k-1})\} \) to \( (V_1, W_1) \). Let \( \tilde{\pi}_k : \tilde{Tw}^k_\Lambda \to \tilde{J} \) be the map sending \( t \) to \( (V_k, W_{k-1}) \). Set

\[
(2.12) \quad \tilde{\kappa}_i := \tilde{\kappa} \circ \tilde{\pi}_i : \tilde{Tw}^k_\Lambda \to \mathfrak{M}^0_\Lambda,
\]

\( i = 1, k \). The restriction \( \tilde{Tw}^k_\Lambda \to Gr_{++}(\Lambda)^2 \) of \( \tilde{f}_k \), \( k \geq 7 \), admits a continuous lift

\[
(2.13) \quad \tilde{f}_k := (\tilde{\kappa}_1, \tilde{\kappa}_k) : \tilde{Tw}^k_\Lambda \to \mathfrak{M}^0_\Lambda \times \mathfrak{M}^0_\Lambda.
\]

Points in the fiber \( \tilde{f}_k^{-1}((X_1, \eta_1), (X_k, \eta_k)) \) represent twistor paths in \( \mathfrak{M}^0_\Lambda \) from \( (X_1, \eta_1) \) to \( (X_k, \eta_k) \).

**Proposition 2.18.** The map \( \tilde{f}_k \), \( k \geq 10 \), is surjective and submersive with smooth connected fibers of dimension \( (k-3)(r-1) + 2 \).

**Proof.** Given two points \( (X_1, \eta_1), (X_k, \eta_k) \) in \( \mathfrak{M}^0_\Lambda \) we can choose generic twistor lines through each, a point \( (X_2, \eta_2) \) on the first and \( (X_{k-1}, \eta_{k-1}) \) on the second, such that Pic\((X_i)\) is trivial, for \( i = 2, k-1 \). Any generic twistor path in \( \tilde{Tw}^k_\Lambda \) from \( P(X_2, \eta_2) \) to \( P(X_{k-1}, \eta_{k-1}) \) lifts to a unique twistor path from \( (X_2, \eta_2) \) to \( (X_{k-1}, \eta_{k-1}) \), by Lemma \((2.14)\) Surjectivity of \( \tilde{f}_k \), for \( k \geq 9 \), thus follows from that of \( \tilde{f}_{k-2} \) proven in Proposition \((2.6)\).

The map \( \tilde{f}_k \) is submersive and its fibers are of dimension \( (k-3)(r-1) + 2 \), since the same holds for \( \tilde{f}_{k-2} \). It remains to prove that the fibers are connected.

The projectivization \( \mathbb{P}(\eta_i(J_\Lambda)) \) of the image via \( \eta_i \) of the Kähler cone of \( X_i \) embeds in the fiber of \( p : J \to \Omega_\Lambda \) over \( P(X_i, \eta_i) \). Denote by \( J(X_i, \eta_i) \) the subset \( q^{-1}(q(\mathbb{P}(\eta_i(J_\Lambda)))) \) of \( J \). The subset \( J(X_i, \eta_i) \) is the union of twistor lines determined by Kähler classes of \( X_i \). A Kähler class is not orthogonal to any MBM class of Hodge type \( (1,1) \) [AV, Theorem 1.19], and so the
locus in the twistor line determined by it, consisting of complex structures which admit an MBM class of Hodge type \((1, 1)\), is zero dimensional. Hence, fibers of \(q : \mathcal{I} \to Gr_{+++}(\Lambda)\) over \(q(\mathbb{RP}(\eta_i(\mathcal{M}_{X_i}))\)) are connected. Consequently, \(\mathcal{I}(X_i, \eta_i)\) is a smooth and connected manifold.

We have the natural embedding

\[
(2.14) \quad \tilde{f}_k^{-1}((X_1, \eta_1), (X_2, \eta_2)) \to \left[\mathcal{I}(X_1, \eta_1) \times \mathcal{I}(X_2, \eta_2)\right] \times_{\Omega_\Lambda} Tw^k_{\Lambda}
\]
given by

\[
\{(V_1, \ldots, V_k); (W_1, \ldots, W_{k-1})\} \mapsto [(W_1, V_2), (W_{k-1}, V_k), \{(V_2, \ldots, V_{k-1}); (W_2, \ldots, W_{k-2})\}].
\]

The target space is smooth and connected, since \(\mathcal{I}(X_1, \eta_1) \times \mathcal{I}(X_2, \eta_2)\) is and the map \(\tilde{f}_k^{-1} : Tw^k_{\Lambda} \to \Omega_\Lambda^2\) is surjective and submersive with connected fibers, by Proposition 2.6. The complement of the image of the embedding \((2.14)\) has codimension \(\geq 2\). Hence, the fiber \(\tilde{f}_k^{-1}((X_1, \eta_1), (X_2, \eta_2))\) is connected.

Let \(\mathcal{W}\) be the tautological rank 3 real vector bundle over \(Gr_{+++}(\Lambda)\). Note that \(\mathcal{I}\) is naturally isomorphic to a conic subbundle of \(\mathcal{P}(\mathcal{W} \otimes \mathbb{C})\). Let \(q_i : \mathcal{W}_{\Lambda}^k \to Gr_{+++}(\Lambda)\) be the map sending a point \(\{(V_1, \ldots, V_k); (W_1, \ldots, W_{k-1})\}\) in \(\mathcal{W}_{\Lambda}^k\) to \(W_1\). Denote by \(\mathcal{W}\) the pullback of \(\mathcal{W} \otimes \mathbb{C}\) to \(\mathcal{W}_{\Lambda}^k\) via \(q_i\), \(1 \leq i \leq k - 1\). We get \(k - 1\) conic bundles \(\mathcal{I}_i\) over \(\mathcal{W}_{\Lambda}^k\). The fiber of \(\mathcal{I}_i\) over \(\{(V_1, \ldots, V_k); (W_1, \ldots, W_{k-1})\}\) is \(\Omega_\Lambda \cap \mathcal{P}[W_i \otimes \mathbb{C}]\). \(\mathcal{I}_i\) is the fiber product over \(Gr_{+++}(\Lambda)\) of \(\mathcal{I}\) and \(\mathcal{W}_{\Lambda}^k\) via \(q : \mathcal{I} \to Gr_{+++}(\Lambda)\) and \(q_i\). \(\mathcal{I}_i\) comes with a natural section \(s_i\), whose value at the above point corresponds to the first two dimensional oriented subspace \(V_1\). Similarly, \(\mathcal{I}_{i+1}\) comes with a natural section \(s_{i+1}\) associated to \(V_i\). For \(1 < i < k - 1\), \(\mathcal{I}_i\) comes with two sections, \(s_i'\) associated to \(V_i\) and \(s_{i+1}'\) associated to \(V_{i+1}\). We get a universal twistor path

\[
\mathcal{I} \subset Tw^k_{\Lambda} \times \Omega_\Lambda
\]
over \(Tw^k_{\Lambda}\) by gluing \(\mathcal{I}_i\) and \(\mathcal{I}_{i+1}\) along the two sections \(s_i'\) and \(s_{i+1}'\) corresponding to \(V_{i+1}\).

Let

\[
(2.15) \quad s_i : Tw^k_{\Lambda} \to \mathcal{I}
\]
be the natural section associated to \(s_{i-1}'\) and \(s_i', 2 \leq i \leq k\). Set \(s_1 := s_1'\). Let \(\text{per} : \mathcal{I} \to \Omega_\Lambda\) be the projection to the second factor. The value of \(\text{per} \circ s_i\) at the above point corresponds to the point \(V_i\) of \(Gr_{+++}(\Lambda)\). Let \(\mathcal{I} \subset Tw^k_{\Lambda} \times \Omega_\Lambda\) (resp. \(\mathcal{I}_i\)) be the restriction of \(\mathcal{I}\) (resp. \(\mathcal{I}_i\)) to \(Tw^k_{\Lambda}\).

**Lemma 2.19.** The restrictions of \(\text{per}\) to \(\mathcal{I}_i\) and \(\mathcal{I}\) admit natural lifts to continuous maps

\[
\text{Per}_i : \mathcal{I}_i \to \mathbb{RP},
\]

\[
\text{Per} : \mathcal{I} \to \mathbb{M}_0^0.
\]

**Proof.** Let \(\text{per}_i : \mathcal{I}_i \to \Omega_\Lambda\) be the restriction of \(\text{per}\). Let \(\tau_i : \mathcal{I}_i \to Tw^k_{\Lambda}\) be the natural projection. Let \(q_i : Tw^k_{\Lambda} \to Gr_{+++}(\Lambda)\) be the map sending a point \(\{(V_1, \ldots, V_k); (W_1, \ldots, W_{k-1})\}\) to \(W_i\), \(1 \leq i \leq k - 1\). The map \((\text{per}_i, q_i \circ \tau_i) : \mathcal{I}_i \to \mathcal{I} \subset \Omega_\Lambda \times Gr_{+++}(\Lambda)\) sends \(\mathcal{I}_i\) into \(\mathcal{I}\), by Remark 2.13. Set \(\tilde{\text{Per}}_i := \tilde{P}^{-1} \circ (\text{per}_i, q_i \circ \tau_i) : \mathcal{I}_i \to \mathbb{RP}\). The composition

\[
\text{Per}_i := \kappa \circ \tilde{\text{Per}}_i : \mathcal{I}_i \to \mathbb{M}_0^0
\]
is thus a well defined continuous map.

Each point of the smooth locus \(\mathcal{I} \setminus \bigcup_{i=2}^{k-1} s_i(Tw^k_{\Lambda})\) of \(\mathcal{I}\) belongs to the image of a unique universal twistor line \(\mathcal{I}_i\). The maps \(\text{Per}_{i-1}\) and \(\text{Per}_i\) agree along \(s_i(Tw^k_{\Lambda})\), by the condition
on \( \rho_{i,i-1}(t) \) and \( \rho_{i,i}(t) \) in the definition of \( Tw^k_\Lambda \). Hence, \( \{Per_i\}_{i=1}^{k-1} \) glue to a continuous map \( Per \).

There exists a universal family \( \overline{\mathcal{X}} \to \mathcal{M}^0_\Lambda \) over \( \mathcal{M}^0_\Lambda \), by [Ma6]. Pulling back the universal family via the map \( Per \) we obtain the universal twistor deformation

\[
\Pi : \mathcal{X} \to \hat{\mathcal{F}}.
\]

Let \( \overline{\mathcal{Y}} \to \overline{\mathcal{X}} \times_{\mathcal{M}^0_\Lambda} \overline{\mathcal{X}} \) be the blow-up of the image of the diagonal embedding of \( \overline{\mathcal{X}} \) in its fiber square. Pulling back \( \overline{\mathcal{Y}} \to \mathcal{M}^0_\Lambda \) via the map \( Per \) we obtain the commutative diagram

\[
\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{\beta} & \mathcal{X} \times_{\Pi} \mathcal{X} \\
\downarrow{\hat{\pi}} & & \downarrow{\hat{\tau}} \\
\overline{\mathcal{Y}} & \to & \overline{\mathcal{X}}
\end{array}
\]

A mild caveat: \( \hat{\mathcal{F}} \) is not a manifold, but is instead the gluing of \( k - 1 \) analytic manifolds. So Definition [2.15] does not apply to \( \Pi : \mathcal{X} \to \hat{\mathcal{F}} \) and \( \hat{\Pi} : \mathcal{Y} \to \hat{\mathcal{F}} \). We will use the following generalization of Definition [2.15]. Let \( \mathcal{G} \) be a connected oriented graph with vertices \( \{v\}_{v \in I} \), edges \( \{e\}_{e \in J} \), head function \( h : J \to I \), and tail function \( t : J \to I \).

**Definition 2.20.** A differentiable \( \mathcal{G} \)-family of complex manifolds consists of the following data.

1. An assignment of a differentiable family of complex manifolds \( \pi_v : \mathcal{X}_v \to \Sigma_v \), for each vertex \( v \in I \).
2. An assignment of connected submanifolds \( M'_e \subseteq \Sigma_{h(e)} \) and \( M''_e \subseteq \Sigma_{t(e)} \), and an isomorphism \( \varphi_e \) from the restriction of \( \pi_{h(e)} : \mathcal{X}_{h(e)} \to \Sigma_{h(e)} \) to \( M'_e \) onto the restriction of \( \pi_{t(e)} : \mathcal{X}_{t(e)} \to \Sigma_{t(e)} \) to \( M''_e \), for each edge \( e \in J \).

A differentiable \( \mathcal{G} \)-family of marked irreducible holomorphic symplectic manifolds consists, in addition, of trivializations \( \eta_e : \mathcal{R}^2\pi_{e,*} \mathcal{Z} \to \mathcal{\Lambda} \), whose restrictions to \( M'_e \) and \( M''_e \) are compatible with the isomorphisms \( \varphi_e \), for all edges \( e \).

Let \( \mathcal{G} \) be the graph with vertices \( I = \{v : 1 \leq v \leq k\} \) and edges \( J = \{e : 1 \leq e \leq k - 1\} \), with \( h : J \to I \) given by \( h(e) = e \), and \( t : J \to I \) given by \( t(e) = e + 1 \).

\[
1 \to 2 \to \cdots \to (k - 1) \to k.
\]

Then \( \Pi : \mathcal{X} \to \hat{\mathcal{F}} \) is a differentiable \( \mathcal{G} \)-family of complex manifolds via the assignment of the differentiable family \( \text{Per}_v(\mathcal{X}) \to \hat{\mathcal{F}}_v \), for \( v \in I \), \( M'_e := \mathcal{R}^1(Tw^k_\Lambda) \), \( M''_e := \mathcal{R}^1(Tw^k_\Lambda) \), and the gluing \( \varphi_e \), \( e \in J \), of the restrictions of \( \text{Per}_v(\mathcal{X}) \) to \( s'_e(\mathcal{F}^k) \) with the restriction of \( \text{Per}_{e+1}(\mathcal{X}) \) to \( s''_e(\mathcal{F}^k) \), arising from the equality \( \text{Per}_e \circ s'_e = \text{Per}_{e+1} \circ s''_e \). Similarly, \( \hat{\Pi} : \mathcal{Y} \to \hat{\mathcal{F}} \) is a differentiable \( \mathcal{G} \)-family of complex manifolds in an analogous way.

### 2.5. An equivalence relation for twistor paths

Let \( \mathcal{\gamma} = \{(V_1, \ldots, V_k), (W_1, \ldots, W_{k-1})\} \) be a twistor path in \( \mathcal{M}^0_\Lambda \) (Remark [2.17]). Set \( (X_i, \eta_i) := \tilde{\kappa}(V_i, W_i), 1 \leq i \leq k - 1 \), where \( \tilde{\kappa} \) is given in Equation (2.9). Set \( (X_k, \eta_k) := \tilde{\kappa}(V_k, W_{k-1}) \). If \( (X_{i+1}, \eta_{i+1}) = (X_i, \eta_i) \), for some \( 1 \leq i \leq k - 1 \), then omitting the \( i \)-th twistor line (i.e., omitting \( V_{i+1} \) and \( W_i \)) we obtain a twistor path \( \mathcal{\gamma}' \) in \( \mathcal{M}^0_\Lambda \). Conversely, given a twistor path \( \mathcal{\gamma} \) as above and a positive definite three space \( W \) containing \( \nabla_i \), such that \( (V_i, W) \) belongs to \( \hat{\mathcal{F}} \) given in (2.8) and corresponds to the same Kähler type chamber as \( (V_i, W_i) \), so that \( \tilde{\kappa}(V_i, W_i) = \tilde{\kappa}(V_i, W) \), we can repeat \( V_i \).
as the $i+1$ oriented positive plane and insert $W$ as the new $i$-th positive three space obtaining a twistor path

$$\{(V_1, \ldots, V_i, V_i, V_{i+1}, \ldots, V_k), (W_1, \ldots, W_{i-1}, W, W_i, \ldots, W_{k-1})\}$$

in $\mathfrak{M}_A^0$. Similarly, if $W_{j-1} = W_j$, for some $2 \leq j \leq k - 1$, then omitting $V_j$ and $W_j$, we obtain a twistor path $\gamma''$. Conversely, given an oriented plane $V$, such that $(V, W_{j-1})$ and $(V, W_j)$ both belong to $\mathcal{J}$ and correspond to the same Kähler type chamber, we can insert $V$ as the new $j$-th oriented plane and repeat $W_{j-1}$ as the new $j$-th positive three space obtaining a twistor path

$$\{(V_1, \ldots, V_{j-1}, V, V_j, \ldots, V_k), (W_1, \ldots, W_{j-1}, W_j, W_{j-1}, W_{j+1}, \ldots, W_{k-1})\}$$

in $\mathfrak{M}_A^0$.

**Definition 2.21.** We say that two twistor paths $\gamma_1$ and $\gamma_2$ in $\mathfrak{M}_A^0$ are equivalent, and write $\gamma_1 \sim \gamma_2$, if $\gamma_2$ can be obtained from $\gamma_1$ by a finite sequence of the above two types of omission or two types of insertion operations.

Given a twistor path $\gamma$ from $(X, \eta)$ to $(X', \eta')$ denote by $\gamma^{-1}$ the twistor path from $(X', \eta')$ to $(X, \eta)$ reversing the order of $V_i$’s and $W_j$’s. Given a twistor path $\tilde{\gamma}$ from $(X', \eta')$ to $(X'', \eta'')$ denote by $\tilde{\gamma}^{-1}$ the concatenated path from $(X, \eta)$ to $(X'', \eta'')$. Then $\gamma^{-1} \gamma$ is equivalent to the constant path from $(X, \eta)$ to itself. If $\gamma$ belongs to $Tw^m_A$ and $\tilde{\gamma}$ belongs to $Tw^k_A$, given in (2.11), then $\tilde{\gamma}^{-1} \gamma$ is a twistor path in $Tw^{m+2k-2}_A$ equivalent to $\gamma$. Note also that every twistor path in $Tw_A^k$ is equivalent to a twistor path in $Tw^{k+1}_A$, for example by inserting $W_k := W_{k-1}$ and $V_{k+1} := V_k$ at the end. We conclude the following.

**Lemma 2.22.** Every twistor path is equivalent to a twistor path in the space $Tw^k_A$, for all $k \geq k_0$ for some $k_0$.

## 3. Hyperholomorphic Sheaves

In this section holomorphic families of hyperholomorphic sheaves are extended to differentiable families of hyperholomorphic sheaves over the relative universal twistor family. Let $X_0$ be an irreducible holomorphic symplectic manifold and let $E$ be a twisted reflexive sheaf over $X_0 \times X_0$, which satisfies the following assumption.

**Assumption 3.1.**

1. $E$ is $\gamma$-hyperholomorphic for every twistor path $\gamma$ starting at $(X_0, \eta_0)$, for every marking $\eta_0$ (Definition 1.7).
2. Given a twistor path $\gamma$ from $X_0$ to $X$, denote by $\beta : Y \to X \times X$ the blow-up of the diagonal in $X \times X$ and let $\tilde{E}_\gamma$ be the quotient of $\beta^* E_\gamma$ by its torsion subsheaf. The sheaf $\tilde{E}_\gamma$ is locally free and $R^i \beta_* \tilde{E}_\gamma = 0$, for $i > 0$ and for every twistor path $\gamma$ from $X_0$ to $X$.

The sheaf $E$ given in Equation (1.3) is an example of such a sheaf, by [Ma5, Theorem 1.4]. Part (2) of the above assumption implies that the twisted sheaf $\mathcal{E}$ over the fiber-square of the twistor family associated to the path $\gamma$ is flat over the twistor path, as it is the direct image of a locally free sheaf over the blow-up of the relative diagonal, and the higher direct images vanish. Denote by

$$B_{\gamma}$$

the projectivization of the pullback of $E_\gamma$ to $Y$ modulo its torsion subsheaf. Clearly, $B_{\gamma_1}$ is isomorphic to $B_{\gamma_2}$, whenever $\gamma_1$ and $\gamma_2$ are equivalent.
Remark 3.2. Note that if $E$ satisfies Assumption 3.1 so does $E \otimes L$, for every line bundle $L$ over $X_0 \times X_0$. Let $U := X_0 \times X_0 \setminus \Delta$ be the complement of the diagonal and $i : U \to X_0 \times X_0$ the inclusion. Then $\iota_\ast i^\ast E \cong E$, so that the isomorphism class of the reflexive sheaf $E$ is determined by its locally free restriction $i^\ast E$ to $U$. The equivalence class of $E$, up to isomorphisms and tensorization by line bundles, is determined by the projectivization $\mathbb{P}(i^\ast E)$ of its restriction to $U$. Hence, Assumption 3.1 may be formulated in terms of $\mathbb{P}(i^\ast E)$. It would be convenient to reformulate the assumption in terms of a projective bundle over a compact manifold as follows.

Let $\beta_0 : Y_0 \to X_0 \times X_0$ be the blow-up centered along the diagonal and denote by $D_0 \subset Y_0$ the exceptional divisor. Let $B$ be a projective bundle over $Y_0$, such that $\tilde{c}_1(B) = \beta_0^\ast \theta$, for some class $\theta \in H^2(X_0 \times X_0, \mu_r)$ (see Section 1.3). Choose a locally free $\beta_0^\ast \theta$-twisted sheaf $\tilde{E}$ over $Y_0$, such that $B \cong \mathbb{P}(\tilde{E})$. The restriction of $\tilde{E}$ to $Y_0 \setminus D_0$ extends uniquely to a $\theta$-twisted reflexive sheaf $E$ over $X_0 \times X_0$, by the Main Theorem of [Sim]. We say that $B$ satisfies Assumption 3.1 if $E$ does and $B$ is isomorphic to the projectivization of the quotient of $\beta_0^\ast E$ by its torsion subsheaf.

Let $\pi : \mathcal{X} \to \Sigma$ be a differentiable family of complex analytic manifolds and let $G$ be a complex Lie group. The notion of a differentiable family $\mathcal{B} \to \mathcal{X} \stackrel{\eta}{\to} \Sigma$ of holomorphic fiber bundles with structure group $G$ was defined in [KST] Definition 1.8). The definition includes, in particular, the special cases of families of vector bundles and projective bundles relevant for us. Let $\mathcal{G}$ be a connected oriented graph with vertices $\{v\}_{v \in I}$, edges $\{e\}_{e \in J}$, and head and tail functions $h, t : J \to I$.

Definition 3.3. A differentiable $\mathcal{G}$-family of holomorphic fiber bundles with structure group $G$ consists of the data of a differentiable $\mathcal{G}$-family of complex manifolds as in Definition 2.20 together with differentiable families $\mathcal{B}_v \to \mathcal{X}_v \to \Sigma_v$, $v \in I$, of holomorphic fiber bundles with structure group $G$ and a lifting $\tilde{\varphi}_e$, $e \in J$, of the gluing isomorphisms in Definition 2.20 to isomorphisms of the restrictions of $\mathcal{B}_{h(e)}$ and $\mathcal{B}_{t(e)}$ to the subfamilies over $M'_e \subset \Sigma_{h(e)}$ and $M''_e \subset \Sigma_{t(e)}$.

Following is a relative version of Theorem 1.3. Let $\Sigma$ be a differentiable manifold and let $\psi : \Sigma \to \mathcal{J}$ be a differentiable map. Set $\tilde{\psi} := \tilde{\kappa} \circ \psi : \Sigma \to \mathfrak{M}_A^0$. Pulling back the universal family over $\mathfrak{M}_A^1$ and the relative metric over $\mathcal{J}$ (Section 2.3.1) we get the differentiable family $\pi : \tilde{\mathcal{X}} \to \Sigma$ of marked irreducible holomorphic symplectic manifolds admitting a marking $\eta : R^2 \pi_* \mathcal{Z} \to \Lambda$ and a $C^\infty$ hermitian metric $g$ on the relative tangent bundle of $\pi$, which restricts to a Kähler metric $g_\sigma$ on each fiber $X_\sigma$ of $\pi$ over $\sigma \in \Sigma$. Let $\mathbb{P}_g^1 \to \Sigma$ be the relative twistor line associated to the metric $g$ and let $\Pi : \tilde{\mathcal{X}} \to \mathbb{P}_g^1$ be the relative twistor family. We get the following diagram.

```
\begin{verbatim}
\xymatrix{
\tilde{\mathcal{X}} \ar[rr]^-{\Pi} \ar[d]_{\mathbb{P}_g^1} & & \mathcal{X} \ar[d]_{\mathbb{P}_g^1} \\
\tilde{\mathcal{X}} \ar[rr]^-{\tilde{\pi}_1} & & \mathcal{X} \ar[d]_{\mathbb{P}_g^1} \\
\tilde{\mathcal{X}} \ar[rrr]^-{\tilde{\pi}_2} & & & \mathcal{X} \ar[d]_{\mathbb{P}_g^1} \\
\tilde{\mathcal{X}} \ar[rrr]^-{\tilde{\pi}_3} & & & \mathcal{X} \ar[d]_{\mathbb{P}_g^1} \\
\mathcal{X} \ar[rr]^-{\psi} & & \mathcal{J} \\
\mathfrak{M}_A^0 \ar[uuuu]_{\tilde{\kappa}} & & }\end{verbatim}
```
where the top two squares and the bottom left parallelogram are cartesian, by definition. Let \( \mathcal{Y} \to \mathcal{X} \times_{\pi} \mathcal{X} \) be the blow-up centered along the relative diagonal \( \Delta \) (Remark 2.10). Let \( \mathcal{B} \to \mathcal{Y} \to \Sigma \) be a differentiable family of projective bundles satisfying Assumption 3.1 in the sense of Remark 3.2. Denote by \( B_\sigma \) the restriction of \( \mathcal{B} \) to the fiber \( Y_\sigma \) of \( \mathcal{Y} \) over \( \sigma \in \Sigma \). Let \( \tilde{\beta} : \mathcal{T} \to \mathcal{X} \times_{\Pi} \mathcal{X} \) be the blow-up of the relative diagonal, and let \( \tilde{\Pi} : \mathcal{T} \to \mathbb{P}^1_g \) be the natural map. Let \( s : \Sigma \to \mathbb{P}^1_g \) be the natural section. Composing the isomorphism \( \mathcal{Y} \cong s^* \mathcal{T} \) with the natural embedding \( s^* \mathcal{T} \subset \mathcal{T} \) we get the embedding \( \tilde{s} : \mathcal{Y} \to \mathcal{T} \).

**Proposition 3.4.** There exists a differentiable family \( \mathcal{B} \to \mathcal{T} \to \mathbb{P}^1_g \) of holomorphic projective bundles satisfying the following properties.

1. \( \tilde{s}^* \mathcal{B} \cong \mathcal{B} \).
2. The restriction of \( \mathcal{B} \) to the fiber of \( \mathcal{T} \) over \( \sigma \in \Sigma \) is the twistor deformation of the hyperholomorphic bundle \( B_\sigma \) along the twistor line associated to the Kähler class of \( g_\sigma \).

**Proof.** We provide only a sketch of the proof. Denote by \( \mathcal{A} \) the differentiable family of holomorphic Azumaya algebras over \( \mathcal{Y} \) associated to \( \mathcal{B} \) and by \( A_\sigma \) its restriction to \( Y_\sigma \). It suffices to construct the corresponding differentiable family \( \mathcal{A} \) of holomorphic Azumaya algebras over \( \mathcal{Y} \). We may regard \( X_\sigma \times X_\sigma \setminus \Delta_\sigma \) as an open subset of \( Y_\sigma \). Let \( A_\sigma \) be the reflexive Azumaya algebra over \( X_\sigma \times X_\sigma \) extending the restriction of \( A_\sigma \) to \( X_\sigma \times X_\sigma \setminus \Delta_\sigma \) via the Main Theorem of [Sin], as in Remark 3.2. Denote by \( \mathbb{P}^1_{g_\sigma} \) the fiber of \( \mathbb{P}^1_g \) over \( \sigma \in \Sigma \). Let us first recall the construction of the Azumaya algebra \( A_\sigma \) over the twistor line \( \mathbb{P}^1_{g_\sigma} \) associated to the hyperholomorphic Azumaya algebra \( A_\sigma \) over \( X_\sigma \times X_\sigma \) (corresponding to the projective bundle \( B_\sigma \) over \( Y_\sigma \)). Denote by \( \omega_\sigma \) the Kähler form on \( X_\sigma \) associated to the metric \( g_\sigma \) and let \( \tilde{\omega}_\sigma \) be the corresponding Kähler form on \( X_\sigma \times X_\sigma \). The \( \tilde{\omega}_\sigma \)-poly-stability of \( A_\sigma \) implies the existence of an admissible \( \tilde{\omega}_\sigma \)-Einstein-Hermitian metric \( h_\sigma \) on the restriction of \( A_\sigma \) to the complement \( X_\sigma \times X_\sigma \setminus \Delta_\sigma \) of the diagonal, unique up to a scalar factor on each stable summand [BS] Theorems 3 and 4. The \( h_\sigma \)-metric connection \( \nabla_\sigma \) is, by definition of \( h_\sigma \), the unique admissible \( \tilde{\omega}_\sigma \)-Einstein-Hermitian connection on \( A_\sigma \) away from \( \Delta_\sigma \), see [VI] Rem. 3.20, where the Einstein-Hermitian property is referred to as Yang-Mills [VI] Def. 3.6. The complex structure of \( A_\sigma \) is the \((0,1)\)-summand of the connection \( \nabla_\sigma \) with respect to the direct sum decomposition of the sheaf of \( C^\infty \) one-forms on \( X_\sigma \times X_\sigma \) into \((1,0)\) and \((0,1)\)-forms determined by the complex structure of \( X_\sigma \times X_\sigma \). Varying the complex structure of \( X_\sigma \) in the twistor family over \( \mathbb{P}^1_{g_\sigma} \), keeping \( \nabla_\sigma \) constant, varies the \((0,1)\)-summand \( \nabla_\sigma t \), \( t \in \mathbb{P}^1_{g_\sigma} \), of the connection \( \nabla_\sigma \). Verbitsky proved that \( \nabla_\sigma t \) is an integrable complex structure associated to a reflexive sheaf \( \mathcal{A}_{\sigma,t} \) and that these fit in a holomorphic sheaf \( \mathcal{A}_\sigma \) over the fiber square of the twistor family [VI] Theorem 3.19].

The metric connection \( \nabla_\sigma \) depends differentiably on \( \sigma \), as so does the metric \( h_\sigma \). In other words, a \( C^\infty \) metric \( h \) can be constructed on \( \mathcal{A} \) over \( \mathcal{X} \times \mathcal{X} \setminus \Delta \), which restricts to an admissible \( \omega_\sigma \)-Einstein-Hermitian metric \( h_\sigma \) over \( \sigma \in \Sigma \), as done in the proof of Theorem 4 in [ST]. Now the differentiable nature of the family \( \Pi : \mathcal{X} \to \mathbb{P}^1_g \) means that the decomposition of the relative complexified tangent bundle of \( \mathcal{X} \times_{\Pi} \mathcal{X} \to \mathbb{P}^1_g \) into its \((1,0)\) and \((0,1)\) summands varies differentiably. Combined with the differentiable dependence of \( \nabla_\sigma \) on the parameter \( \sigma \), we get the differentiable family \( \mathcal{A} \) of holomorphic vector bundles over \( \mathcal{X} \times_{\Pi} \mathcal{X} \setminus \Delta \).

Assumption 3.1 implies that it corresponds to a differentiable family of Azumaya algebras \( \mathcal{A} \), or equivalently of projective bundles \( \mathcal{B} \), over \( \mathcal{Y} \) (see Remark 3.2). □
Choose a marking $\eta_0$ for $X_0$, such that $(X_0, \eta_0)$ belongs to $\mathfrak{M}_\Lambda^0$. Fix an integer $k \geq 10$ and a pair $(X, \eta)$ in $\mathfrak{M}_\Lambda^0$. The fiber

$$\Gamma := \Gamma^{(X, \eta)} := f_k^{-1}((X_0, \eta_0), (X, \eta))$$

of the map $\hat{f}_k$ given in Equation (2.13) is a smooth and connected real analytic manifold, by Proposition 2.18. Given a twistor path $\gamma \in \Gamma$ and a sheaf $E$ over $X_0 \times X_0$ satisfying Assumption 3.1 we get a reflexive sheaf $E_\gamma$ on $X \times X$. Let $\mathcal{T}_\Gamma$ be the restriction of the universal twistor path $\mathcal{F}$ to $\Gamma \subset Tw^k_\Lambda$. Let

$$\Pi_\Gamma : \mathcal{T}_\Gamma \to \mathcal{F}_\Gamma$$

be the restriction of the universal twistor family, given in (2.16), to $\mathcal{F}_\Gamma$. We get the diagram

![Diagram](image)

by restriction of Diagram (2.17) to $\mathcal{F}_\Gamma$.

Let $Y$ be the blow-up of the diagonal in $X \times X$. Let $s_i : Tw^k_\Lambda \to \mathcal{F}$ be the restriction of the section in Equation (2.15), $1 \leq i \leq k$. We get the following commutative diagram.

![Diagram](image)

Let $\beta_0 : Y_0 \to X_0 \times X_0$ be the blow-up of the diagonal, let $E$ be a reflexive sheaf over $X_0 \times X_0$ satisfying Assumption 3.1, and denote by $B$ the projectivization of the quotient of $\beta_0^*E$ by its torsion subsheaf. Let $\mathcal{B}$ be the graph (2.18).

**Lemma 3.5.** There exists a differentiable $\mathcal{B}$-family $\mathcal{B} \to \mathcal{T}_\Gamma \to \mathcal{T}_\Gamma$ of holomorphic projective bundles with the following properties. The pull back of $\mathcal{B}$ via the section $s_1$ is the constant family over $\Gamma \times Y_0$ with the projective bundle $B$, and its pullback via the section $s_k$ is a family

$$\mathcal{B}_\Gamma \to \Gamma \times Y \to \Gamma$$

over $\Gamma \times Y$, which restricts over $\gamma \in \Gamma$ to the projective bundle $B_\gamma$.

**Proof.** The family is constructed vertex by vertex via a recursive application of Proposition 3.4. In the $i$-th iterate the map $\psi : \Sigma \to \mathcal{F}$ of Proposition 3.4 is the composition

$$\Gamma \xrightarrow{s_i} Tw^k_\Lambda \xrightarrow{\tilde{\Per}_i} \mathcal{F} \xrightarrow{\tilde{\Per}} \mathcal{F},$$

where $s_i$ is the restriction of the section in Equation (2.15), $\tilde{\Per}_i$ is the map in Lemma 2.19, and $\tilde{\Per}$ is the isomorphism in Equation (2.8). The twistor line $\mathbb{P}^1_g \to \Sigma$ of Proposition 3.4 is the restriction of $\mathcal{T}_i \to Tw^k_\Lambda$ to $\Gamma$. \qed
The manifold $\Gamma := \Gamma^{(X,\eta)}_{(X_0,\eta_0)}$, given in Equation (3.1), parametrizes twistor paths consisting of $k - 1$ twistor lines with fixed initial point $(X_0,\eta_0)$ and endpoint $(X,\eta)$. Its construction could be generalized as follows. Assume given a smooth complex manifold $\Sigma$, a complex analytic family $\zeta : \mathcal{Z} \to \Sigma$ of irreducible holomorphic symplectic manifolds, and an isometry $\eta$ of $H^2(\zeta^*\mathcal{Z})$ with the trivial local system with fiber the lattice $\Lambda$. Assume, further, that we are given a holomorphic family $E_{\mathcal{Z}} \to \mathcal{Z} \times \Sigma \mathcal{Z}$, flat over $\Sigma$, of twisted reflexive sheaves satisfying Assumption 3.1. Given a point $\sigma \in \Sigma$, denote by $(Z_\sigma,\eta_\sigma)$ the corresponding marked pair and let $E_\sigma$ be the restriction of $E_{\mathcal{Z}}$ to $Z_\sigma \times Z_\sigma$. The assumption requires that each $E_\sigma$, $\sigma \in \Sigma$, is $\gamma$-hyperholomorphic with respect to every twistor path $\gamma$ starting at $(Z_\sigma,\eta_\sigma)$. Denote by $\mathcal{Y}$ the blow-up of the relative diagonal of $\mathcal{Z} \times \Sigma \mathcal{Z}$. Let $\mathcal{B}$ be the family of projective bundles over $\mathcal{Y}$ corresponding to $E_{\mathcal{Z}}$. Denote by $B_\sigma$ the restriction of $\mathcal{B}$ to the fiber over $\sigma \in \Sigma$. We can then let the initial point vary in $\Sigma$ and let the initial projective bundle be $B_\sigma$ and replace the family $\Gamma^{(X,\eta)}_{(X_0,\eta_0)}$ by a relative version $\Gamma_{\Sigma} \to \Sigma$ whose fiber $\Gamma_\sigma$, $\sigma \in \Sigma$, is described by the manifold $\Gamma_{(Z_\sigma,\eta_\sigma)}$ of twistor paths in $\mathcal{T}_s^{(X,\eta)}$ with initial point $(Z_\sigma,\eta_\sigma)$. Explicitly, if we let $\kappa_{\Sigma} : \Sigma \to \mathcal{M}_{\Lambda}^0$ be the classifying morphism, then $\Gamma_{\Sigma}$ is the submanifold of $\Sigma \times \mathcal{T}_s^{(X,\eta)}$, which is the inverse image of the diagonal via $\kappa_{\Sigma} \times \tilde{\kappa}_1 : \Sigma \times \mathcal{T}_s^{(X,\eta)} \to \mathcal{M}_{\Lambda}^0 \times \mathcal{M}_{\Lambda}^0$, where $\tilde{\kappa}_1$ is given in Equation (2.12). We get the relative version of diagram (3.2) over $\Sigma$. Let $\mathcal{F}_{\Sigma}$ be the fiber product of $\Gamma_{\Sigma}$ and $\mathcal{F}$ over $\mathcal{T}_s^{(X,\eta)}$. The relative version of Lemma 3.5 is the following, and is again an immediate consequence of Proposition 3.4.

**Lemma 3.6.** There exists a differentiable $\mathcal{Y}$-family $\mathcal{B} \to \mathcal{Y}_{\Sigma} \to \mathcal{F}_{\Sigma}$ of holomorphic projective bundles with the following properties. The pull back of $\mathcal{B}$ via the section

$$(1_\Sigma \times s_1)|_{\Gamma_{\Sigma}} : \Gamma_{\Sigma} \to \mathcal{F}_{\Sigma},$$

is the pullback to $\Gamma_{\Sigma} \times \Sigma \mathcal{Y}$ of the original family $\mathcal{B}$ of holomorphic projective bundles with the following properties. The pull back of $\mathcal{B}$ via the section $(1_\Sigma \times s_\kappa)|_{\Gamma_{\Sigma}}$ restricts over $(\sigma,\gamma) \in \Gamma_{\Sigma}$ to the projective bundle $(B_\sigma)_{\gamma}$.

Let $\Gamma^{(X,\eta)}_{\Sigma} \subset \Gamma_{\Sigma}$ be the family over $\Sigma$ of twistor paths ending at $(X,\eta)$, given in Diagram (2.1). Note that the pullback of $\mathcal{B}$ via the section $(1_\Sigma \times s_\kappa)|_{\Gamma_{\Sigma}}$ restricts to $\Gamma^{(X,\eta)}_{\Sigma}$ as a family

$$(3.4) \quad \mathcal{B}_{\Gamma^{(X,\eta)}_{\Sigma}} \to \Gamma^{(X,\eta)}_{\Sigma} \times \mathcal{Y} \to \Gamma^{(X,\eta)}_{\Sigma}$$

over $\Gamma^{(X,\eta)}_{\Sigma} \times \mathcal{Y}$.

4. **Rigid hyperholomorphic sheaves**

Let $\mathcal{B} \to \mathcal{Y} \to \Sigma$ be a differentiable family of holomorphic projective bundles satisfying Assumption 3.1 in the sense of Remark 3.2. In Section 4.1 we prove that the locus of $\sigma \in \Sigma$, where the restriction $B_\sigma$ of $\mathcal{B}$ to $\mathcal{Y} \times \{\sigma\}$ is infinitesimally rigid, is both open and closed (Corollary 4.4). Let $\mathcal{B} \to \mathcal{Y} \to \Sigma$ be a differentiable family of holomorphic projective bundles satisfying Assumption 3.1 where $\mathcal{Y}$ is the blow-up of the relative diagonal of a marked differentiable family $\mathcal{X} \to \Sigma$. In Section 4.2 we associate to the family $\mathcal{B}$ an open subset $U_{\mathcal{B}}$ of the moduli space $\mathcal{M}_{\Lambda}^0$ of marked irreducible holomorphic symplectic manifolds consisting of pairs $(X,\eta)$, such that $(B_\sigma)|_{\mathcal{U}_{\mathcal{B}}}$ is infinitesimally rigid for every twistor path $\gamma$ from $(X_\sigma,\eta_\sigma)$ to $(X,\eta)$, for every $\sigma \in \Sigma$ (Corollary 4.7). In Section 4.3 we define the monodromy group of the family $\mathcal{B}$ and prove that the rigidity locus $U_{\mathcal{B}}$ is monodromy invariant (Corollary 4.14).
4.1. Rigidity is an open and closed condition in families of stable sheaves over a fixed variety. Let \( Y \) be a compact Kähler manifold, \( \Sigma \) a connected differentiable manifold, \( \sigma_0 \in \Sigma \) a point, and \( \Sigma_0 \subset \Sigma \) a contractible open subset containing \( \sigma_0 \). Let \( \mathcal{B} \to \Sigma \times Y \to \Sigma \) be a differentiable family of holomorphic \( \mathbb{P}^{r-1} \)-bundles. Denote by \( \mathcal{B}_\Sigma \) its restriction to \( \Sigma_0 \). The product \( \mathcal{C} := B^*_\sigma \times_Y \mathcal{B}_\Sigma \) is a differentiable family of holomorphic \( \mathbb{P}^{r-1} \times \mathbb{P}^{r-1} \)-bundles over \( \Sigma_0 \times Y \). Set \( \mathcal{C}_\sigma := B^*_\sigma \times_Y \mathcal{B}_\sigma, \sigma \in \Sigma_0 \). Let \( p : \mathcal{C} \to \Sigma_0 \times Y \) and \( p_\sigma : \mathcal{C}_\sigma \to Y \) be the natural projections.

**Lemma 4.1.** There exists a differentiable family \( \mathcal{L} \to \mathcal{C} \to \Sigma_0 \) of holomorphic line bundles with the following properties. Let \( \mathcal{L}_\sigma \) be the restriction of \( \mathcal{L} \) to \( \mathcal{C}_\sigma, \sigma \in \Sigma_0 \).

1. \( \mathcal{L} \) restricts to the \( \mathbb{P}^{r-1} \times \mathbb{P}^{r-1} \) fiber over each point \((\sigma, y) \in \Sigma_0 \times Y\) as the line bundles \( \mathcal{O}_{\mathbb{P}^{r-1} \times \mathbb{P}^{r-1}}(1, 1) \).
2. The equality \( c_1(p_\sigma, (\mathcal{L}_\sigma)) = 0 \) holds, for all \( \sigma \in \Sigma_0 \).

**Proof.**

Let \( \mathcal{O}_\mathcal{E} \) be the sheaf of germs of differentiable complex valued functions, which restrict to holomorphic functions on fibers of \( \mathcal{C} \to \Sigma_0 \). Denote by \( \mathcal{O}^*_\mathcal{E} \) the sheaf of invertible such germs of functions \(^\boxleftcite{KS1} \text{Sec. I.1}\). We have the standard short exact exponential sequence

\[
0 \to \mathcal{Z} \to \mathcal{O}_\mathcal{E} \to \mathcal{O}^*_\mathcal{E} \to 0
\]

and its long exact cohomology sequence

\[
\cdots H^1(\mathcal{C}, \mathcal{O}_\mathcal{E}) \to H^1(\mathcal{C}, \mathcal{O}^*_\mathcal{E}) \xrightarrow{c_1} H^2(\mathcal{C}, \mathcal{Z}) \to H^2(\mathcal{C}, \mathcal{O}_\mathcal{E}) \to \cdots
\]

The group \( H^1(\mathcal{C}, \mathcal{O}^*_\mathcal{E}) \) parametrizes equivalence classes of differentiable families of holomorphic line bundles over \( \mathcal{C} \to \Sigma_0 \), by \(^\boxleftcite{KS1} \text{Prop. 1.1}\).

Let \( \omega_{p_\sigma} \) be the relative canonical line bundle over \( \mathcal{C}_\sigma \) and let \( \omega_p \to \mathcal{C} \to \Sigma_0 \) be the corresponding differentiable family of holomorphic line bundles over \( \mathcal{C} \to \Sigma_0 \). The dual line bundle \( \omega_{p_\sigma}^* \) is isomorphic to \( \mathcal{O}_{\mathcal{C}_\sigma}(rD) \), where \( D \subset B^*_\sigma \times_Y \mathcal{B}_\sigma \) is the incidence divisor. In particular, the class \( c_1(\omega_{p_\sigma}^*) \) is equal to \( r\lambda_0 \), where \( \lambda_0 \in H^2(\mathcal{C}_\sigma, \mathcal{Z}) \) is the cohomology class of \( D \). The restriction homomorphism \( H^2(\mathcal{C}, \mathcal{Z}) \to H^2(\mathcal{C}_\sigma, \mathcal{Z}) \) is an isomorphism, for all \( \sigma \in \Sigma_0 \), since \( \mathcal{C} \to \Sigma_0 \) is a differentiable fibration over a contractible base. Consequently, \( c_1(\omega_p) = r\lambda \), for the class \( \lambda \in H^2(\mathcal{C}, \mathcal{Z}) \) restricting to \( \lambda_0 \). The class \( \lambda \) maps to zero in \( H^2(\mathcal{C}_\sigma, \mathcal{O}^*_\mathcal{E}) \), for all \( \sigma \in \Sigma_0 \), so does \( c_1(\omega_p) \). Hence, the image of \( \lambda \) in \( H^2(\mathcal{C}, \mathcal{O}^*_\mathcal{E}) \) vanishes, by \(^\boxleftcite{KS1} \text{Theorem 2.2(ii)}\). It follows that \( \lambda \) is the image of some class \( \lambda \) in \( H^1(\mathcal{C}, \mathcal{O}^*_\mathcal{E}) \), by the exactness of the long exact sequence above. The existence of a differentiable family \( \mathcal{L} \) of holomorphic line bundles over \( \mathcal{C} \to \Sigma_0 \) with \( c_1(\mathcal{L}) = \lambda \) follows, by \(^\boxleftcite{KS1} \text{Prop. 1.1}\).

2. The vector bundle \( p_{\sigma_0, *}(\mathcal{L}_{\sigma_0}) \) is isomorphic to the Azumaya algebra of \( B_{\sigma_0} \) and hence its first Chern class vanishes. Let \( f : \mathcal{C} \to \Sigma_0 \) be the natural map. The Chern classes \( c_1(p_{\sigma_0}, (\mathcal{L}_{\sigma_0})), \sigma \in \Sigma_0 \), define a continuous section of the local system \( R^2f_*\mathcal{Z} \) over \( \Sigma_0 \), which vanishes at \( \sigma_0 \) and hence vanishes globally.

**Definition 4.2.** A projective \( \mathbb{P}^r \)-bundle \( B \) is said to be infinitesimally rigid, if \( H^1(ad(P(B))) = 0 \), where \( P(B) \) is the principal \( \text{PGL}(r + 1, \mathbb{C}) \)-bundle associated to \( B \) and \( ad(P(B)) \) is the adjoint Lie algebra bundle.

**Proposition 4.3.** Assume that the differentiable family \( \mathcal{B} \to Y \times \Sigma \to \Sigma \) above has the following additional properties:

1. \( Y \) is the blow-up of a simply connected compact Kähler manifold \( Z \) along a smooth subvariety.
2. Each projective bundle \( B_\sigma \) is the projectivization of the pullback to \( Y \) of a twisted reflexive sheaf \( E_\sigma \) modulo its torsion subsheaf.
(3) There exists a Kähler class $\omega$ on $Z$, such that $E_\sigma$ is $\omega$-slope-stable, for every $\sigma \in \Sigma$. Then either $B_\sigma$ is infinitesimally rigid, for every $\sigma \in \Sigma$, or $B_\sigma$ is not infinitesimally rigid for any $\sigma \in \Sigma$. In the former case the set $\{B_\sigma, \sigma \in \Sigma\}$ consists of a single isomorphism class.

Proof. Assume that there exists a point $\sigma_0$ in $\Sigma$, such that $B_{\sigma_0}$ is infinitesimally rigid. Let $U \subset \Sigma$ be the subset
\[ \{\sigma \in \Sigma : B_\sigma \cong B_{\sigma_0}\}. \]

Denote its complement by $U^c$. The subset $U$ is open, by [KS1] Theorem 7.4. Let $\sigma_1$ be a point of $U^c$. Choose an open contractible subset $\Sigma_0 \subset \Sigma$, such that $\Sigma_0$ contains $\{\sigma_0, \sigma_1\}$. This is possible, since $\Sigma$ is a connected manifold. Let $\mathcal{L} \to \mathcal{E} \to \Sigma_0$ be a differentiable family of holomorphic line bundles with the properties of Lemma [4]. The characteristic classes $\theta_\sigma \in H^1(Y, \mu_r)$ of $B_\sigma$, $\sigma \in \Sigma$, define a continuous section of the trivial local system with fiber $H^1(Y, \mu_r)$ over $\Sigma$. Hence, $\tilde{\theta}_{\sigma_1} = \tilde{\theta}_{\sigma_2}$. Choose a cocycle $\theta$ representing this characteristic class and let $\tilde{B}_{\sigma_1}, i = 0, 1$, be $\theta$-twisted locally free sheaves over $Y$, such that $\mathbb{P}(\tilde{B}_{\sigma_1}) = B_{\sigma_1}$. Then the vector bundle $p_{\sigma_1, *}((\mathcal{L}_1))$ is isomorphic to the tensor product of $\mathcal{H}om(\tilde{B}_{\sigma_0}, \tilde{B}_{\sigma_1})$ by some line bundle over $Y$. The vector bundle $p_{\sigma_1, *}((\mathcal{L}_0))$ is isomorphic to $\mathcal{H}om(\tilde{B}_{\sigma_0}, \tilde{B}_{\sigma_0})$, since $c_1 : \text{Pic}(Y) \to H^2(Y, \mathbb{Z})$ is injective and $c_1(p_{\sigma_1, *}((\mathcal{L}_0))) = 0$, for all $\sigma \in \Sigma_0$, by the property of $\mathcal{L}$ mentioned in Lemma [4]. The lift $\tilde{B}_{\sigma_1}$ is determined by the lift $\tilde{B}_{\sigma_0}$ and the condition that $c_1\left(\mathcal{H}om(\tilde{B}_{\sigma_0}, \tilde{B}_{\sigma_1})\right) = 0$. Furthermore, with that lift $p_{\sigma_1, *}((\mathcal{L}_0))$ is isomorphic to $\mathcal{H}om(\tilde{B}_{\sigma_0}, \tilde{B}_{\sigma_1})$. If $B_{\sigma_0}$ is isomorphic to $B_{\sigma_1}$ then $p_{\sigma_1, *}((\mathcal{L}_1))$ is isomorphic to the Azumaya algebra of $B_{\sigma_0}$, since the first Chern classes of both vanish. Hence, dim $H^0(Y, p_{\sigma_1, *}((\mathcal{L}_1))) > 0$, whenever $B_{\sigma_0}$ is isomorphic to $B_{\sigma_1}$.

If $B_{\sigma_0}$ is not isomorphic to $B_{\sigma_1}$, then $H^0(Y, p_{\sigma_1, *}((\mathcal{L}_1)))$ vanishes. This is seen as follows. Let $\beta : Y \to Z$ be the blow-up morphism. By assumption, $B_{\tau}$ is the projectivization of the locally free part of the pullback $\beta^*E_\sigma$ of a twisted sheaf $E_\sigma$ over $Z$. We can choose $E_{\sigma_i}$ so that $\tilde{B}_i$ is the tensor product of the locally free part of $\beta^*E_{\sigma_i}$ with $\mathcal{O}_Y(j_iD)$, for some integer $j_i$, $i = 0, 1$. Similarly, each of the characteristic classes $\tilde{\theta}_\sigma$ is a pullback of the characteristic class over $Z$ of the reflexive twisted sheaf $E_{\sigma_i}$. We can thus choose the cocycle $\theta$ to be the pullback of a cocycle over $Z$. The first Chern class of $\mathcal{H}om(\tilde{B}_{\sigma_0}, \tilde{B}_{\sigma_1})$ vanishes, by the isomorphism of the latter with $p_{\sigma_1, *}((\mathcal{L}_1))$. The vanishing of $c_1\left(\mathcal{H}om(\tilde{B}_{\sigma_0}, \tilde{B}_{\sigma_1})\right)$ implies that $j_0 = j_1 = j$, for some integer $j$. Then $\beta_*(\tilde{B}_{\sigma_1}(-jD)) \cong E_{\sigma_i}, i = 0, 1$.

The functor $\beta_*$ induces a natural injective homomorphism
\[ \text{Hom}(\tilde{B}_{\sigma_0}, \tilde{B}_{\sigma_1}) \cong \text{Hom}(\tilde{B}_{\sigma_0}(-jD), \tilde{B}_{\sigma_1}(-jD)) \xrightarrow{\beta_*} \text{Hom}(E_{\sigma_0}, E_{\sigma_1}). \]

The sheaves $E_{\sigma_i}, i = 0, 1$, are $\omega$-slope-stable. The first Chern class of the sheaf $\mathcal{H}om(E_{\sigma_0}, E_{\sigma_1})$ is equal to the image via the Gysin map $\beta_* : H^2(Y, \mathbb{Z}) \to H^2(Z, \mathbb{Z})$ of that of $\mathcal{H}om(\tilde{B}_{\sigma_0}, \tilde{B}_{\sigma_1})$. We have seen that the first Chern class of $\mathcal{H}om(\tilde{B}_{\sigma_0}, \tilde{B}_{\sigma_1})$ vanishes. Hence, $c_1(\mathcal{H}om(E_{\sigma_0}, E_{\sigma_1})) = 0$. Consequently, Hom$(E_{\sigma_0}, E_{\sigma_1})$ does not vanish, if and only if $E_{\sigma_0}$ is isomorphic to $E_{\sigma_1}$. It follows that Hom$(\tilde{B}_{\sigma_0}, \tilde{B}_{\sigma_1})$ does not vanish, if and only if $B_{\sigma_0}$ is isomorphic to $B_{\sigma_1}$. The isomorphism Hom$(\tilde{B}_{\sigma_0}, \tilde{B}_{\sigma_1}) \cong H^0(Y, p_{\sigma_1, *}((\mathcal{L}_1)))$ implies that the space $H^0(Y, p_{\sigma_1, *}((\mathcal{L}_1)))$ does not vanish, if and only if $B_{\sigma_0}$ is isomorphic to $B_{\sigma_1}$.

We have shown that the intersection $U^c \cap \Sigma_0$ is precisely the subset of $\Sigma_0$ consisting of points $\sigma$ such that $H^0(Y, p_{\sigma_1, *}((\mathcal{L}_1)))$ vanishes. The latter is an open subset of $\Sigma_0$, by the Upper-Semi-Continuity Theorem [KS1] Theorem 2.1. We conclude that $U^c$ is an open subset of $\Sigma$. The space $\Sigma$ is a connected manifold. Hence, $U^c$ must be empty. \qed
Corollary 4.4. Assume given a differentiable $\mathcal{G}$-family of $\mathbb{P}^{r-1}$-bundles $\mathcal{B}_v \to Y \times \Sigma_v \to \Sigma_v$, $v \in I$, as in Definition 3.3 with constant fiber $Y$, each satisfying the hypotheses of Proposition 4.3. Assume further that $B_{\sigma_0}$ is infinitesimally rigid, for some $\sigma_0 \in \Sigma_{v_0}$, for some vertex $v_0$. Then the restrictions of $\mathcal{B}_v$ to $Y \times \{\sigma\}$, for all vertices $v$ and all $\sigma \in \Sigma_v$, are isomorphic to a single projective bundle over $Y$.

Proof. The statement follows from Proposition 4.3 by induction on the distance from a vertex $v$ to $v_0$ in the graph, since the graph $\mathcal{G}$ is assumed connected and the base manifolds $\Sigma_v$, $v \in I$, are assumed connected, by Definitions 2.15 and 2.20.

4.2. The rigidity locus in the moduli space of marked pairs. Assume given a differentiable family $\mathcal{B}$ as in Lemma 3.5. We get the differentiable family $\mathcal{B}_\Gamma \to \Gamma \times Y \to \Gamma$ in Equation 3.3. The assumptions of Proposition 4.3 then follow from Assumption 3.1. We can apply Proposition 4.3 with the Kähler class $\tilde{\omega}$ on $Z := X \times X$, for any Kähler class $\omega$ on $X$, as the sheaves $E_\gamma$ in Assumption 3.1 are required to be $\tilde{\omega}$-slope-stable, for every Kähler class $\omega$ and every $\gamma \in \Gamma$.

Corollary 4.5. Let $B$ be an infinitesimally rigid bundle over the blow-up $Y_0$ of the diagonal in $X_0 \times X_0$ associated to a sheaf $E$ over $X_0 \times X_0$ satisfying Assumption 3.1.

1. The bundle $B_\gamma$ is isomorphic to $B$, for every twistor path $\gamma$ from $(X_0, \eta_0)$ to itself.

2. The isomorphism class of the bundle $B_\gamma$ depends only on the endpoint of $\gamma$, and is independent of $\gamma$.

Proof. (1) The statement depends only on the equivalence class of $\gamma$, and so we may choose $k$ sufficiently large, such that $T w_\lambda^k$ contains both a path $\gamma_0$ equivalent to the constant twistor path from $(X_0, \eta_0)$ to itself, as well as the path $\gamma$, by Lemma 2.22 possibly after replacing $\gamma$ by an equivalent twistor path. Then $B_{\gamma_0} = B$. Hence, $B_{\gamma}$ is isomorphic to $B$, for all $\gamma \in \Gamma$, by Proposition 4.3 applied with the differentiable family $\mathcal{B}_\Gamma \to \Gamma \times Y_0 \to \Gamma$ given in (3.3).

(2) Let $\gamma_1$ and $\gamma_2$ be two twistor paths from $(X_0, \eta_0)$ to the same point $(X, \eta)$. Then $B_{\gamma_2} \cong B_{\gamma_1 \gamma_2} \cong (B_{\gamma_1}^{-1} \gamma_2^{-1}) \gamma_1 \cong B_{\gamma_1}$, where the last isomorphism follows from Part (1).

Let $\zeta_v : \mathcal{Z}_v \to \Sigma_v$, $v \in I$, be a differentiable $\mathcal{G}$-family of marked irreducible holomorphic symplectic manifolds. It extends to a differentiable $\mathcal{G}$-family $\mathcal{B}_v \to \mathcal{Y}_{\Sigma_v} \to \mathcal{F}_{\Sigma_v}$, $v \in I$, of $\mathbb{P}^{r-1}$-bundles as in the relative set-up of Lemma 3.6.

Lemma 4.6. Assume that there exists a point $(\sigma, \gamma)$ of $\Gamma_{\Sigma_0}$, for some vertex $v_0$, such that $(B_\sigma)_{\gamma}$ is infinitesimally rigid. Then $(B_{\sigma'})_{\gamma'}$ is isomorphic to $(B_\sigma)_{\gamma}$, for every $(\sigma', \gamma') \in \Gamma_{\Sigma_v}$, such that $\gamma'$ has the same endpoint as $\gamma$, for all vertices $v$ of $\mathcal{G}$. In particular, $B_{\sigma'}$ is isomorphic to $(B_\sigma)_{\gamma''}$, for some twistor path $\gamma''$.

Proof. Let $(X_k, \eta_k)$ be the endpoint of $\gamma$. The statement follows immediately from Corollary 4.4 applied to the restriction of each family $\mathcal{B}_v$ to the submanifold $\Gamma_{\Sigma_v}^{(X_k, \eta_k)}$ given in Equation (3.4). Taking $\gamma'' := \gamma'^{-1} \gamma$ we get the isomorphism $B_{\sigma'} \cong (B_\sigma)_{\gamma''}$.

We continue to consider a differentiable $\mathcal{G}$-family $\mathcal{B}_v \to \mathcal{Y}_{\Sigma_v} \to \mathcal{F}_{\Sigma_v}$, $v \in I$, each as in the relative set-up of Lemma 3.6. We denote the data of the $\mathcal{G}$-family by $\mathcal{B}$. Let $U_{\mathcal{B}}$ be the subset of $\mathcal{M}_{\mathcal{B}}^0$ consisting of all marked pairs $(X, \eta)$, such that there exists some point $\sigma$ of $\Sigma_v$, for some vertex $v$, and there exists a twistor path $\gamma$ in $\mathcal{M}_{\mathcal{B}}^0$ from $(X_\sigma, \eta_\sigma)$ to $(X, \eta)$, such that $(B_\sigma)_{\gamma}$ is infinitesimally rigid.
Corollary 4.7. \( U_\mathcal{G} \) is an open subset of \( \mathcal{M}_0^0 \). For every point \( (X, \eta) \in U_\mathcal{G} \), for every point \( \sigma \in \Sigma_v \), for every vertex \( v \) of \( \mathcal{G} \), and for every twistor path \( \gamma \) from \( (X_\sigma, \eta_\sigma) \) to \( (X, \eta) \), the bundle \( (B_\sigma)_\gamma \) is infinitesimally rigid.

Proof. The statement is vacuous if \( U_\mathcal{G} \) is empty. Assume otherwise. So \( (B_\sigma)_\gamma \) is infinitesimally rigid, for some point \( \sigma_0 \) of \( \Sigma_v \), for some vertex \( v_0 \), and for some twistor path \( \gamma_0 \) in \( \mathcal{M}_0^0 \) starting at \( (X_{\sigma_0}, \eta_{\sigma_0}) \). We may assume that the twistor path \( \gamma_0 \) belongs to \( Tw^k_{\Lambda} \) for some \( k \geq 10 \), possibly after replacing it by an equivalent twistor path, by Lemma 2.22. The pair \( (\sigma_0, \gamma_0) \) is then a point of \( \Gamma_{\Sigma_{v_0}} \). Denote by \( \kappa_{\Sigma_v,k} : \Gamma_{\Sigma_v} \to \mathcal{M}_0^0 \) the composition

\[ \Gamma_{\Sigma_v} \subset \Sigma_v \times Tw^k_{\Lambda} \to Tw^k_{\Lambda} \xrightarrow{\tilde{k}_k} \mathcal{M}_0^0, \]

where \( \tilde{k}_k \) is the map given in Equation (2.12).

Let \( U_v \subset \Gamma_{\Sigma_v} \) be the subset consisting of pairs \( (\sigma, \gamma) \in \Gamma_{\Sigma_v} \), such that \( (B_\sigma)_\gamma \) is infinitesimally rigid. The subset \( U_v \) is open in \( \Gamma_{\Sigma_v} \), by the Semi-Continuity Theorem [KS1 Theorem 2.1], and non-empty containing all twistor paths with the same endpoint as \( \gamma_0 \), by Lemma 4.6. The map \( \kappa_{\Sigma_v,k} \) is submersive and surjective, since its restriction to every fiber of \( \Gamma_{\Sigma_v} \to \Sigma_v \) is, by Proposition 2.18. Hence, \( \kappa_{\Sigma_v,k}(U_v) \) are open subsets of \( \mathcal{M}_0^0 \), for all \( v \in I \), and all are equal to \( U_{\mathcal{G}} \), by Lemma 4.6.

Assume given a point \( (X, \eta) \) in \( U_\mathcal{G} \) and a twistor path \( \gamma \) from \( (X_\sigma, \eta_\sigma) \) to \( (X, \eta) \). Choose a twistor path \( \gamma_1 \) in \( Tw^k_{\Lambda} \) from \( (X_\sigma, \eta_\sigma) \) to \( (X, \eta) \). Then \( (\sigma, \gamma_1) \) belongs to \( \Gamma_{\Sigma_v} \), and so the bundle \( (B_\sigma)_{\gamma_1} \) is infinitesimally rigid, by Lemma 4.6. There exists an integer \( k' \geq k \) and twistor paths \( \tilde{\gamma}_1 \) and \( \tilde{\gamma} \) in \( Tw^k_{\Lambda} \), such that \( \gamma \sim \tilde{\gamma} \) and \( \gamma_1 \sim \tilde{\gamma}_1 \). Then \( (B_\sigma)_{\tilde{\gamma}_1} \) is infinitesimally rigid, being isomorphic to \( (B_\sigma)_{\gamma_1} \). Hence, \( (B_\sigma)_{\tilde{\gamma}} \) is infinitesimally rigid, by Lemma 4.6. Consequently, \( (B_\sigma)_{\tilde{\gamma}} \), which is isomorphic to \( (B_\sigma)_{\gamma_1} \), is infinitesimally rigid.

Given a point \( (X, \eta) \) of the set \( U_\mathcal{G} \) of Corollary 4.7, a point \( \sigma \in \Sigma_v \), for some vertex \( v \), and a twistor path \( \gamma \) from \( (X_\sigma, \eta_\sigma) \) to \( (X, \eta) \), the isomorphism class of the bundle \( (B_\sigma)_\gamma \) over the blow-up \( Y \) of the diagonal in \( X \times X \) is independent of the choice of \( \sigma \) and \( \gamma \), by Lemma 4.6.

We denote this isomorphism class by

\[ B_{(X, \eta)}. \]

The following is a useful special case of Corollary 4.7. Let \( (X_0, \eta_0) \) be a marked pair, \( Y_0 \) the blow-up of the diagonal in \( X_0 \times X_0 \), and \( B_0 \) the projective bundle over \( Y_0 \) associated to a reflexive sheaf \( E_0 \) over \( X_0 \times X_0 \) satisfying Assumption 3.1. Let \( \mathcal{M}_A^0 \) be the component containing \( (X_0, \eta_0) \) and let \( U_{B_0} \) be the subset of \( \mathcal{M}_0^0 \) consisting of pairs \( (X, \eta) \), such that \( (B_0)_\gamma \) is infinitesimally rigid for every twistor path \( \gamma \) from \( (X_0, \eta_0) \) to \( (X, \eta) \).

Lemma 4.8. (1) \( U_{B_0} \) is an open subset of \( \mathcal{M}_0^0 \).

(2) For every \( (X, \eta) \in U_{B_0} \) and for every twistor path \( \gamma \) from \( (X_0, \eta_0) \) to \( (X, \eta) \) the bundle \( (B_0)_\gamma \) is infinitesimally rigid and its isomorphism class is independent of the choice of \( \gamma \).

(3) For every marked pair \( (X, \eta) \) in \( \mathcal{M}_0^0 \) and for every twistor path \( \gamma \) from \( (X_0, \eta_0) \) to \( (X, \eta) \) the equality \( U_{B_0} = U_{(B_0)_\gamma} \) holds.

Proof. Parts (1) and (2) form a special case of Corollary 4.7 where the graph \( \mathcal{G} \) is trivial, consisting of a single vertex \( v \) and no edges, and \( \Sigma_v = \{(X_0, \eta_0)\} \). Part (3) follows from the obvious equality \( ((B_0)_\gamma)_{\tilde{\gamma}} = (B_0)_{\tilde{\gamma}_\gamma} \).
Corollary 4.9. If the projective bundle $B_{\sigma_1}$ is infinitesimally rigid, then $B_{\sigma_2}$ is isomorphic to $\tilde{g}^*B_{\sigma_1}$.

Proof. There exists a twistor path $\gamma$ from $(X_{\sigma_1}, \eta_{\sigma_1})$ to $(X_{\sigma_2}, \eta_{\sigma_2})$, such that $(B_{\sigma_1})_\gamma \cong B_{\sigma_2}$, by Lemma 4.6. Now, $(X_{\sigma_1}, \eta_{\sigma_1}) \cong (X_{\sigma_2}, \eta_{\sigma_2})$, and so $(B_{\sigma_1})_\gamma \cong B_{\sigma_1}$, by Corollary 4.5 when they are considered as bundles on $Y_{\sigma_1}$, i.e., when the left hand side of the latter isomorphism is pulled back to $Y_{\sigma_1}$ via $\tilde{g}^{-1}$. □

Remark 4.10. Corollary 4.9 is false if we weaken the assumption that the marked pairs are isomorphic and assume only that $X_{\sigma_1}$ is isomorphic to $X_{\sigma_2}$, as demonstrated by Part 11 of Theorem 4.11.

4.3. Monodromy invariance of the rigidity locus. We continue to consider a differentiable $G$-family $\pi_v : \mathcal{X}_v \to \Sigma_v$ of $\Lambda$-marked irreducible holomorphic symplectic manifolds, for some graph $G$, which extends to a differentiable $G$-family $\mathcal{B}_v \to \mathcal{B}_{\Sigma_v} \to \mathcal{B}_{\Sigma_v}$, $v \in I$, each as in the relative set-up of Lemma 3.6. Let $\text{Mon}^2(X)$ be the image of the monodromy group $\text{Mon}(X)$ (Definition 1.8) in the isometry group of $H^2(X, \mathbb{Z})$. Set $\text{Mon}^\Lambda(\mathcal{M}_0) := \eta \circ \text{Mon}^2(X) \circ \eta^{-1}$, for some marked pair $(X, \eta) \in \mathcal{M}_0^\Lambda$. Then $\text{Mon}(\mathcal{M}_0) = \text{Mon}^\Lambda(\mathcal{M}_0)^\Lambda$. In particular, $\text{Mon}(\mathcal{M}_0)$ is independent of the choice of the point $(X, \eta)$.

Lemma 4.11. Let $\phi$ be an element of $\text{Mon}(\mathcal{M}_0^\Lambda)$. Assume that the following two conditions hold.

1. There exist vertices $v_i$, points $\sigma_i \in \Sigma_{v_i}$, $i = 1, 2$, and an isomorphism of the marked pairs $(X_{\sigma_1}, \phi \circ \eta_{\sigma_1})$ and $(X_{\sigma_2}, \eta_{\sigma_2})$, i.e., an isomorphism $g : X_{\sigma_1} \to X_{\sigma_2}$ satisfying $\phi \circ \eta_{\sigma_1} \circ g^* = \eta_{\sigma_2}$.

2. $B_{\sigma_1}$ is isomorphic to $\tilde{g}^*B_{\sigma_2}$, where $\tilde{g} : Y_{\sigma_1} \to Y_{\sigma_2}$ is the isomorphism induced by $g$.

Then $\phi(U_{G_\mathcal{B}}) = U_{G_\mathcal{B}}$, where $U_{G_\mathcal{B}} \subset \mathcal{M}_0^\Lambda$ is the open subset in Corollary 4.7. Furthermore, the bundles $B_{(X, \eta)}$ and $B_{(X, \phi \circ \eta)}$ are isomorphic, for every point $(X, \eta)$ of $U_{G_\mathcal{B}}$, where we used the notation 4.7.

Proof. Let $(X, \eta)$ be a point of $U_{G_\mathcal{B}}$. The isomorphism $g$ induces an isomorphism of the isomorphisms of the space $\Gamma(X_{\sigma_1}, \eta_{\sigma_1})$ to $\Gamma(X_{\sigma_2}, \eta_{\sigma_2})$, which lifts to an isomorphism of the space $\Gamma(X_{\sigma_1}, \eta_{\sigma_1})$. Composing with the automorphism $\phi$ of $\mathcal{M}_0^\Lambda$ we get the isomorphism, denoted by $g$ as well, from $\Gamma(X_{\sigma_1}, \eta_{\sigma_1})$ to $\Gamma(X_{\sigma_2}, \eta_{\sigma_2})$. Choose a path $\gamma \in \Gamma(X_{\sigma_1}, \eta_{\sigma_1})$. Then the isomorphism $B_{\sigma_1} \cong \tilde{g}^*B_{\sigma_2}$ extends to an isomorphism $B_{\sigma_1} \gamma \cong \text{Mon}^2(X, \mathbb{Z})$. The projective bundle $B_{\sigma_1}$ is infinitesimally rigid, by the definition of $U_{G_\mathcal{B}}$ and Lemma 4.6. Hence, so is $(B_{\sigma_2})_\gamma$ and $(X, \phi \circ \eta)$ belongs to $U_{G_\mathcal{B}}$. Furthermore, the isomorphism $B_{\sigma_1} \gamma \cong (B_{\sigma_2})_\gamma$ translates to the desired isomorphism $B_{(X, \eta)} \cong B_{(X, \phi \circ \eta)}$ via notation 4.7.

Remark 4.12. Assumption 2 of the above Lemma does not follow from assumption 1, as we saw in cautionary Remark 4.10. Assumption 1 stipulates that $\eta_{\sigma_1}^{-1} \circ \phi \circ \eta_{\sigma_1}$ is a monodromy operator. This is seen as follows. Consider the family $\pi_\mathcal{B} : \mathcal{B}_{\mathcal{B}} \to \Sigma_\mathcal{B}$ obtained from the $G$-family via the gluings associated to edges. As the family $\mathcal{B}_{\mathcal{B}} \to \Sigma_\mathcal{B}$ is marked, for each vertex, and the gluings are compatible with the markings, by Definition 2.20, then the local system $R^2\pi_{\mathcal{B}}_* \mathcal{Z}$ is trivial and the composition $\eta_{\sigma_2}^{-1} \eta_{\sigma_1}$ is the parallel transport operator for
any path $\gamma$ from $\sigma_1$ to $\sigma_2$. Further gluing the points $\sigma_1$ and $\sigma_2$ and the fibers $X_{\sigma_1}$ and $X_{\sigma_2}$ via the isomorphism $g$, the path $\gamma$ becomes a loop and its monodromy operator is $\eta_{\sigma_1}^{-1}\phi^{-1}\eta_{\sigma_1}$ (substitute $g^*\eta_{\sigma_1}^{-1}\phi^{-1}$ for $\eta_{\sigma_2}^{-1}$ in the parallel transport operator $\eta_{\sigma_1}^{-1}\eta_{\sigma_1}$ and drop $g_*$). Assumptions (1) and (2) may be regarded as stipulating that $\phi$ is a monodromy operator of the pair $(X_{\sigma_1}, B_{\sigma_1})$.

**Definition 4.13.** Let $\text{Mon}(\mathcal{B})$ be the subgroup of $\text{Mon}(\mathcal{M}^0_\Lambda)$ generated by elements $\phi$ satisfying assumptions (1) and (2) of Lemma 4.11.

**Corollary 4.14.** $U_{\mathcal{B}}$ is $\text{Mon}(\mathcal{B})$-invariant and $B_{(X,\eta)} = B_{(X,\phi\eta)}$, for every $(X,\eta)$ in $U_{\mathcal{B}}$ and every $\phi$ in $\text{Mon}(\mathcal{B})$.

**Example 4.15.** Let $\xi: \mathcal{X} \to \Sigma$ be a family of irreducible holomorphic symplectic manifolds, $\mathcal{E}_{\mathcal{X}} \to \mathcal{X} \times_{\Sigma} \mathcal{X}$ a family of reflexive sheaves satisfying Assumption 3.11 $\mathcal{Y} \to \mathcal{X} \times_{\Sigma} \mathcal{X}$ the blow-up of the relative diagonal, and $\mathcal{B} \to \mathcal{Y}$ the $\mathbb{P}^2$-bundle associated to $\mathcal{E}_{\mathcal{X}}$. Choose a point $\sigma_0 \in \Sigma$ and a marking $\eta_{\sigma_0}$ of $X_{\sigma_0}$ and let $\mathcal{M}_\Lambda^0$ be the component of the moduli space of marked pairs containing $(X_{\sigma_0}, \eta_{\sigma_0})$. We get the composite homomorphism

$$\pi_1(\Sigma,\sigma_0) \to \text{Mon}^2(\mathcal{X}) \to \text{Mon}(\mathcal{M}_\Lambda^0),$$

where the latter is conjugation by $\eta_{\sigma_0}$. Let $\tilde{\Sigma} \to \Sigma$ be the universal cover and $\tilde{\xi}: \tilde{\mathcal{X}} \to \tilde{\Sigma}$ the pulled back family. Choose a point $\tilde{\sigma}_0 \in \tilde{\Sigma}$ over $\sigma_0$ and let $\eta$ be the trivialization of the local system $\tilde{R}^2(\xi)_* \mathcal{Z}$ determined by the marking $\eta_{\tilde{\sigma}_0} := \eta_{\sigma_0}$. The trivialization endows each fiber of $\tilde{\xi}$ with a marking. Set $\mathcal{Y} := \mathcal{Y} \times_{\Sigma} \tilde{\Sigma}$ and let $\tilde{\mathcal{B}} \to \mathcal{Y}$ be the pulled back family of projective bundles. Then $\mathcal{B}$ is $\text{Gal}(\tilde{\Sigma}/\Sigma)$-equivariant, $\text{Gal}(\tilde{\Sigma}/\Sigma)$ is isomorphic to $\pi_1(\Sigma,\sigma_0)$, and so the image of $\pi_1(\Sigma,\sigma_0)$ in $\text{Mon}(\mathcal{M}_\Lambda^0)$ is contained in $\text{Mon}(\mathcal{B})$. Hence, $U_{\mathcal{B}}$ is $\pi_1(\Sigma,\sigma_0)$-invariant, by Corollary 4.11.

Given an irreducible holomorphic symplectic manifold $X$, let $r(X)$ be the rank of the lattice $[H^{2,0}(X) + H^{0,2}(X)] \cap H^2(\mathbb{Z})$. Note that $0 \leq r(X) \leq 2$, and $r(X) = 2$ if and only if the Picard rank of $X$ is maximal.

**Theorem 4.16.** Assume that $\text{Mon}(\mathcal{B})$ is a finite index subgroup of $\text{Mon}(\mathcal{M}_\Lambda^0)$ and $U_{\mathcal{B}}$ is non-empty. Then $U_{\mathcal{B}}$ contains every marked pair $(X,\eta)$ with $r(X) = 0$. If, furthermore, $U_{\mathcal{B}}$ contains some marked pair $(X,\eta)$ with $r(X) = 1$, then $U_{\mathcal{B}}$ contains every marked pair with non-maximal Picard rank.

**Proof.** Any non-empty open subset of $\mathcal{M}_\Lambda^0$, which is invariant under some finite index subgroup $G$ of $\text{Mon}^2(\mathcal{M}_\Lambda^0)$, necessarily contains all marked pairs $(X,\eta)$ in $\mathcal{M}_\Lambda^0$ with $r(X) = 0$, since the $G$-orbit of $(X,\eta)$ is dense in $\mathcal{M}_\Lambda^0$ by a result of Verbitsky [V4] Theorem 4.11 (which applies to $\mathcal{M}_\Lambda^0$ using the isomorphism of [V5] Cor. 4.31] between each component of Teichmüller space and the associated component of the moduli space of marked pairs). If, in addition, the open subset contains some marked pair $(X',\eta')$ with $r(X') = 1$, then it necessarily contains all marked pairs $(X,\eta)$ in $\mathcal{M}_\Lambda^0$ with $r(X) = 1$ as well, since the $G$-orbit of such a marked pair is dense in the locus of marked pairs with $r(X) > 0$, by [V6] Theorem 2.5].

5. **Monodromy equivariance of the modular hyperholomorphic sheaf**

We prove Theorem 4.11 in this section.
5.1. The polarized surface monodromy group of a moduli space of sheaves. Let $S_0$ be a $K3$ surface with a cyclic Picard group and $v = (r, kh_0, s) \in \tilde{H}(S, \mathbb{Z})$ a primitive Mukai vector, where $h_0$ is the ample generator of $H^{1,1}(S_0, \mathbb{Z})$ and $k$ is a non-zero integer. Assume that $r > 0$ or $k > 0$ and $(v, v) = 2n - 2$, where $n \geq 2$. Then $M_H(v)$ is a smooth projective manifold of $K3^{[n]}$-type. The second cohomology $H^2(S_0, \mathbb{Z})$ is a direct summand in $\tilde{H}(S, \mathbb{Z})$.

The sublattice $h_0^1$ of $H^2(S_0, \mathbb{Z})$ orthogonal to $h_0$ is contained in the sublattice $v^-$ of $\tilde{H}(S, \mathbb{Z})$ orthogonal to $v$. Let $\text{Mon}^2(S_0)_{h_0}$ be the subgroup of the monodromy group of $S_0$ stabilizing $h_0$. We regard $\text{Mon}^2(S_0)$ also as a subgroup of the isometry group of the Mukai lattice acting via the identity on $H^i(S_0, \mathbb{Z})$, $i = 0, 4$. Then $\text{Mon}^2(S_0)_{h_0}$ leaves the Mukai vector $v$ invariant and embeds in the isometry group of $v^-$. The latter is naturally isometric to $H^2(M_H(v), \mathbb{Z})$ via Mukai’s Hodge isometry $m_v$, given in $[1, 2]$. Denote by $\text{Mon}^2(S_0)_{h_0}^\vee$ the image of $\text{Mon}^2(S_0)_{h_0}$ in the isometry group of $H^2(M_H(v), \mathbb{Z})$ via conjugation by $m_v$.

**Proposition 5.1.** There exists a smooth quasi-projective curve $C$ and a family $p : \mathcal{S} \to C$ of $K3$ surfaces admitting a section $h$ of the local system $\mathbb{R}^2p_*\mathbb{Z}$ and a smooth proper morphism $\pi : \mathcal{M} \to C$ with the following properties. The class $h_t$ is of type $(1, 1)$ and ample, for all $t \in C$. The fiber $\mathcal{M}_t$ is a smooth and projective moduli space of $H^1_t$-stable sheaves on $S_t$ with Mukai vector $v_t := (r, kh_t, s)$, for some $v_t$-generic polarization $H^1_t$. $S_0$ is isomorphic to the fiber of $p$ over some point $t_0 \in C$, and the class $h_{t_0}$ corresponds to $h_0$ via this isomorphism. The fiber $\mathcal{M}_{t_0}$ of $\pi$ is thus isomorphic to $M_H(v)$. The image of $\pi_1(C, t_0)$ in $\text{Mon}^2(M_H(v))$ is equal to $\text{Mon}^2(S_0)_{h_0}^\vee$.

**Proof.** The statement is proven in the last paragraph of the proof of Theorem 6.1 in [Ma1]. $C$ is a curve in the moduli space of polarized $K3$ surfaces of degree $2n - 2$, whose fundamental group surjects onto that of the moduli space, and we use the fact that the construction of moduli spaces of sheaves works in families, as well as a result of Yoshioka [Yo, Prop. 5.1] which enables one to choose the $v_t$-generic polarizations $H^1_t$ in a non-continuous fashion.

Over $\mathcal{S} \times C$, $\mathcal{M}$ we have a relative twisted universal sheaf $\mathcal{U}$, yielding over $\mathcal{M} \times_C \mathcal{M}$ a flat family $\delta_C$ of reflexive modular sheaves $[1, 3]$. Let $\mathcal{Y} \to \mathcal{M} \times_C \mathcal{M}$ be the blow-up of the relative diagonal and $\mathcal{B} \to \mathcal{Y}$ the corresponding family of $\mathbb{P}^{2n-3}$-bundles. Next apply the construction of Example $[4, 15]$. Let $\tilde{C} \to C$ be the universal cover and $\mathcal{B} \to \mathcal{Y} := \mathcal{Y} \times_C \tilde{C} \to \tilde{C}$ the pulled back family. Choosing a marking $\eta_{t_0}$ for $\mathcal{M}_{t_0} \cong M_H(v)$ we get the subset $U_{\mathcal{B}}$ of the component $\mathcal{Y}^0$ containing $(M_H(v), \eta_{t_0})$.

**Proposition 5.2.** The subset $U_{\mathcal{B}}$ of $\mathcal{Y}^0$ is $\eta_{t_0} \left[ \text{Mon}^2(S_0)_{h_0}^\vee \right] \eta_{t_0}^{-1}$-invariant.

**Proof.** The construction is a special case of the one carried out in Example $[4, 15]$ and thus follows from Corollary $[1, 14]$. \hfill $\Box$

5.2. Invariance of $U_{\mathcal{B}}$ under the surface monodromy group of Douady spaces. Let $\Lambda_{K3}$ be the $K3$ lattice and let $\Sigma$ be a component $\mathcal{Y}^0$ of the moduli space of marked $K3$ surfaces. The Beauville-Bogomolov lattice of a manifold of $K3^{[n]}$-type is the orthogonal direct sum $\Lambda_{K3} \oplus \mathbb{Z}\delta$, where $(\delta, \delta) = 2 - 2n$, $n \geq 2$. Let $f : \mathcal{S} \to \Sigma$ be the universal $K3$ surface and let $\mathcal{F} := \mathcal{S}^{[n]} \to \Sigma$ be the relative Hilbert scheme of length $n$ subschemes of fibers of $f$. Denote by $\mathcal{U}$ the ideal sheaf of the universal subscheme in $\mathcal{S} \times \Sigma \mathcal{S}^{[n]}$, and let $\mathcal{E}_{\mathcal{F}}$ be the relative extension sheaf

$$\mathcal{E}_{\mathcal{F}} := \mathcal{E}_{\mathcal{F}^1_{\pi_{13}}} (\pi_{12}^* \mathcal{U}, \pi_{23}^* \mathcal{U}),$$
where \( \pi_{ij} \) is the projection from \( S[n] \times S \times S[n] \) onto the fiber product of the \( i \)-th and \( j \)-th factors. The restriction of the sheaf \( \mathcal{E}_\mathcal{F} \) to the fiber \( S_\sigma[n] \times S_\sigma[n] \) of \( S[n] \times S[n] \) over \( \sigma \in \Sigma \) is an example of the modular sheaf \([1.3]\) and so satisfies Assumption \([3.1]\) by \([Ma5, \text{Theorem } 1.4]\). The marking \( \eta_\sigma : H^2(S_\sigma, \mathbb{Z}) \rightarrow \Lambda_{K3} \), of each \( K3 \) surface, extends canonically to a marking \( \eta_\sigma : H^2(S_\sigma[n], \mathbb{Z}) \rightarrow \Delta \) of \( S_\sigma[n] \), by sending half the class of the divisor of non-reduced subschemes to \( \delta \) \([32]\). The monodromy group \( \text{Mon}(\mathfrak{M}_{\Lambda}^{0}) \) acts on \( \Sigma \) and the action lifts to an action on the universal \( K3 \) surface \( S \), since any automorphism of a \( K3 \) surface, which acts as the identity on its second cohomology, is the identity. Hence, the universal Hilbert scheme \( S[n] \) is \( \text{Mon}(\mathfrak{M}_{\Lambda}^{0}) \)-equivariant as well. It follows that the universal ideal sheaf \( \mathcal{U} \) is \( \text{Mon}(\mathfrak{M}_{\Lambda}^{0}) \)-equivariant with respect to the diagonal action on \( S \times S[n] \). Hence, the universal modular sheaf \( \mathcal{E}_\mathcal{F} \) is \( \text{Mon}(\mathfrak{M}_{\Lambda}^{0}) \)-equivariant.

Extending each element of \( \text{Mon}(\mathfrak{M}_{\Lambda}^{0}) \) to an isometry of \( \Lambda \), by acting as the identity on \( \delta \), we get an embedding \( \nu : \text{Mon}(\mathfrak{M}_{\Lambda}^{0}) \rightarrow \text{Mon}(\mathfrak{M}_{\Lambda}^{0}) \) in the monodromy group \( \text{Mon}(\mathfrak{M}_{\Lambda}^{0}) \) of the corresponding component \( \mathfrak{M}_{\Lambda}^{0} \) of marked manifolds of \( K3[n] \)-type.

Let \( \mathcal{U} \rightarrow S[n] \times S[n] \) be the blow-up of the relative diagonal. The torsion free quotient of the pullback of \( \mathcal{E}_\mathcal{F} \) to \( \mathcal{U} \) is locally free, by \([Ma3, \text{Prop. } 4.1]\). Denote its projectivization by \( \mathcal{B} \rightarrow \mathcal{U} \). The \( \text{Mon}(\mathfrak{M}_{\Lambda}^{0}) \)-equivariance of \( \mathcal{B} \) yields the inclusion \( \nu \left[ \text{Mon}(\mathfrak{M}_{\Lambda}^{0}) \right] \subset \text{Mon}(\mathcal{B}) \), by Definition \([4.13]\) Corollary \([4.13]\) thus yields the following statement.

**Proposition 5.3.** The open subset \( \mathcal{U}_\mathcal{B} \) of \( \mathfrak{M}_{\Lambda}^{0} \) is \( \nu \left[ \text{Mon}(\mathfrak{M}_{\Lambda}^{0}) \right] \)-invariant.

**Remark 5.4.** Let \( \nu = (1, 0, 1 - n) \) be the Mukai vector of an ideal sheaf of a length \( n \) subscheme of a \( K3 \) surface. The markings \( \eta_\sigma \) and \( \eta_\sigma \) conjugate the homomorphism \( \nu \) to an embedding \( \nu : \text{Mon}(\mathfrak{M}_{\Lambda}^{0}) \rightarrow \text{Mon}(\mathfrak{M}_{\Lambda}^{0}) \). The latter is the composition of the embedding of \( \text{Mon}(\mathfrak{M}_{\Lambda}^{0}) \) is the isometry group of \( \nu \) (extending the action on \( H^2(S_\sigma, \mathbb{Z}) \) to an action on the Mukai lattice via the trivial action on \( H^2(S_\sigma, \mathbb{Z}) \), \( i = 0, 4 \)), followed by conjugation via Mukai’s Hodge isometry \( m_\nu \), given in \([1.2]\). We denote the image of \( \nu \) by \( \text{Mon}(\mathfrak{M}_{\Lambda}^{0}) \) in analogy to the notation used in Proposition \([5.1]\).

### 5.3. Stability preserving Fourier-Mukai functors

Let \( S_i \) be a \( K3 \) surface, \( v_i \) a primitive Mukai vector in \( \overline{H}(S_i, \mathbb{Z}) \), and \( H_i \) a \( v_i \)-generic polarization, \( i = 1, 2 \). Assume that \( M_i := M_{H_i}(v_i) \) is non-empty, \( i = 1, 2 \). Let \( \Phi : D^b(S_1) \rightarrow D^b(S_2) \) be an equivalence of the bounded derived categories of coherent sheaves. Assume that the object \( \Phi(F) \) is represented by an \( H_2 \) stable sheaf of Mukai verctor \( v_2 \), for every \( H_i \) stable sheaf with Mukai verctor \( v_1 \). Then \( \Phi \) induces an isomorphism \( \phi : M_1 \rightarrow M_2 \), by \([Ma2, \text{Theorem } 1.6]\) (see also \([Ma1, \text{Lemma } 5.6]\)).

Let \( \pi_i \) be the projection from \( S_1 \times S_2 \rightarrow S_i \), \( i = 1, 2 \). There exists an object \( \mathcal{P} \) over \( S_1 \times S_2 \), known as a Fourier-Mukai kernel, such that \( \Phi \) is the integral transform \( R\pi_{2,*}(\mathcal{P} \otimes L\pi_{1}^*) \), where the tensor product is taken in the derived category \([\mathcal{O}]\). Let \( \pi_{ij} \) be the projection from \( S_1 \times M_1 \times S_2 \times M_2 \) onto the product of the \( i \)-th and \( j \)-th factors. Let \( \Gamma_\phi \subset M_1 \times M_2 \) be the graph of \( \phi \). The integral transform with respect to the object \( \pi_{13}(\mathcal{P}) \otimes \pi_{24}(\mathcal{P}) \) is an equivalence \( \tilde{\Phi} : D^b(S_1 \times M_1, \pi_{23}^*\alpha) \rightarrow D^b(S_2 \times M_2, \pi_{24}^*\phi^*\alpha) \), which takes a universal sheaf \( \mathcal{U}_{v_1} \) over \( S_1 \times M_1 \), twisted by a Brauer class \( \alpha \) of \( M_1 \), to an object represented by a universal sheaf \( \mathcal{U}_{v_2} \) over \( S_2 \times M_2 \), twisted by the Brauer class \( \phi_* (\alpha) \) on \( M_2 \), again by \([Ma2, \text{Theorem } 1.6]\). Consequently, the pullback via \( \phi \times \phi : M_1 \times M_1 \rightarrow M_2 \times M_2 \) of the modular sheaf \( E_{v_2} \) over \( M_2 \times M_2 \), given in \([1.3]\), is the modular sheaf \( E_{v_1} \) over \( M_1 \times M_1 \).

Let \( Y_i \) be the blow-up of the diagonal in \( M_i \times M_i \). The torsion free quotient of the pullback of \( E_{v_i} \) to \( Y_i \) is locally free, by \([Ma3, \text{Prop. } 4.1]\), and we denote its projectivization by \( B_{v_i} \).
We will refer to $B_{\nu_i}$ as the modular projective bundle. Choose a marking $\eta_1$ for $M_1$ and set $\eta_2 := \eta_1 \circ \phi^*$. The two markings are then compatible (Definition 1.10), by the computation of the characteristic classes of the modular sheaves (1.5). Denote by $\mathfrak{M}_1^0$ the connected component containing the marked pairs $(M_i, \eta_i)$, $i = 1, 2$. Let $U_{B_{\nu_i}}$ be the open subset of $\mathfrak{M}_1^0$ associated to the triple $(M_i, \eta_i, B_{\nu_i})$, $i = 1, 2$, in Lemma 4.8. Then $U_{B_{\nu_1}} = U_{B_{\nu_2}}$, since $\phi$ lifts to an isomorphism of the two triples. Propositions 5.2 and 5.3 yield the following conclusion.

**Corollary 5.5.** $U_{B_{\nu_1}}$ is invariant under the subgroup $G$ of Mon($\mathfrak{M}_1^0$) generated by the two subgroups $\eta_i \left[ \text{Mon}(S_1)^{m_{\nu_i}} \right] \eta_i^{-1}$, $i = 1, 2$. Furthermore, $B_{(X,\eta)} \cong B_{(X,\phi^*\eta)}$, for every $(X,\eta) \in U_{B_{\nu_1}}$ and every $\phi \in G$.

If, furthermore, $B_{\nu_1}$ is infinitesimally rigid, then $U_{B_{\nu_1}}$ contains $(M_1, \eta_1)$ and so is non-empty.

### 5.4. Proof of Theorem 1.11

**Step 1:** Let $(S,\bar{\eta})$ be a marked $K3$ surface and $(S^{[n]},\eta_1)$ its Hilbert scheme with the extended marking as in Section 5.2. Let $\mathfrak{M}_1^0$ be the connected component containing $(S^{[n]},\eta_1)$. Denote by $E_1$ the modular sheaf over $S^{[n]} \times S^{[n]}$. Let $Y_1$ be the blow-up of $S^{[n]} \times S^{[n]}$ along the diagonal and let $B_1$ be the modular projective bundle over $Y_1$. $E_1$ is infinitesimally rigid, by [MM2, Lemma 5.2]. The projective bundle $B_1$ is infinitesimally rigid, by [MM1, Lemma 4.3]. Hence, $U_{B_1}$ contains $(S^{[n]},\eta_1)$. Let $\text{Mon}^2(S^{[n]},\eta_1)$ be the subgroup of $\text{Mon}^2(S^{[n]})$ acting trivially on $H^2(S^{[n]},\mathbb{Z})/H^2(S^{[n]},\mathbb{Z})$. Then $\text{Mon}^2(S^{[n]},\eta_1)$ is an index 2 subgroup of $\text{Mon}^2(S^{[n]})$, if $n > 2$, and the whole of $\text{Mon}^2(S^{[n]})$ if $n = 2$, by [Ma2, Theorem 1.2 and Lemma 4.2]. Let $\text{Mon}(\mathfrak{M}_1^0)$ be the corresponding subgroup of $\text{Mon}(\mathfrak{M}_1^0)$. We first show the the subset $U_{B_1}$ of $\mathfrak{M}_1^0$ is $\text{Mon}(\mathfrak{M}_1^0)$ invariant.

Set $v_1 := (1,0,1-n)$. We already know that $U_{B_1}$ is $\eta_1 \left[ \text{Mon}^2(S)^{m_{\nu_1}} \right] \eta_1^{-1}$, invariant, by Proposition 5.3 using the notation of Remark 5.3. We may assume that $S$ admits an elliptic fibration with a section and that the rank of Pic$(S)$ is 2. Denote by $f$ the class in $H^2(S,\mathbb{Z})$ of an elliptic fiber, and let $\sigma$ be the class of the section. Let $v_2$ be the Mukai vector $(0,\sigma+nf,1)$. The line bundle $H$ with $c_1(H) = \sigma + kf$ is $v_2$-generic, for $k$ sufficiently large, and there exists over $S \times S$ an object $\mathscr{P}$, inducing an auto-equivalence $\Phi$ of $D_b(S)$ sending the ideal sheaf of a length $n$ subscheme (with Mukai vector $v_1$) to an $H$-stable sheaf over $S$ with Mukai vector $v_2$, by [Yo, Theorem 3.15]. We get the isomorphism $\phi : S^{[n]} \rightarrow M_H(v_2)$. The two subgroups $\text{Mon}^2(S)^{m_{\nu_1}}$ and $\phi^* \left[ \text{Mon}^2(S)^{m_{\nu_2}} \right] \phi_*$ generate the subgroup $\text{Mon}^2(S^{[n]},\eta_1)$ of $\text{Mon}^2(S^{[n]})$, by [Ma1, Prop. 7.1 and Prop. 8.6], as explained in [Ma1, Sec. 1.3.1, sub-step 1.2]. We conclude that $U_{B_1}$ is $\text{Mon}(\mathfrak{M}_1^0)$ invariant, and $B_{(X,\eta)} \cong B_{(X,\phi^*\eta)}$, for all $(X,\eta) \in U_{B_1}$ and $\phi \in \text{Mon}(\mathfrak{M}_1^0)\text{cov}$, by Corollary 5.5.

**Step 2:** Proof of Part 1 of Theorem 1.11 in the case where the sheaf $E$ in Equation (1.13) is $E_1$ and $M = S^{[n]}$. $U_{B_1}$ contains marked Hilbert schemes $(S^{[n]},\eta)$ with all possible values of $r(S^{[n]})$ (note that $r(S^{[n]}) = r(S)$). Hence, $U_{B_1}$ contains every marked pair $(X,\eta)$ in $\mathfrak{M}_1^0$, such that the Picard rank of $X$ is not maximal, by Theorem 1.10.

Let $\gamma$ be a twistor path from $(S^{[n]},\eta_1)$ to $(X,\eta) \in U_{B_1}$, let $(E_1)_\gamma$ be the sheaf over $X \times X$ obtained from the modular sheaf $E_1$ via $\gamma$, and let $(B_1)_\gamma$ be the associated projective bundle over the blow-up $Y$ of the diagonal in $X \times X$. Then $(B_1)_\gamma$ is infinitesimally rigid, by Lemma 4.8. Infinitesimal rigidity of $(B_1)_\gamma$ was shown to imply that of $(E_1)_\gamma$ in [MM1] as follows. Let $\mathcal{A}$ be the Azumaya algebra over $Y$ associated to $(B_1)_\gamma$. We have the left exact sequence

$$0 \to H^1(X \times X, \mathcal{A}) \to H^1(Y, \mathcal{A}) \to H^0(X \times X, R^1\mathcal{A}).$$

The isomorphism

$$\beta_\mathcal{A} \cong \text{End}(E_1)_\gamma$$

(5.1)
was established in [MM1] Step 3 of the proof of Prop. 3.2]. Hence, rigidity of \((B_1)_{\gamma}\), which is equivalent to the vanishing of \(H^1(Y, \mathcal{A}')\), implies the vanishing of \(H^1(X \times X, \mathcal{E}' \mathcal{N}d((E_1)_{\gamma}))\). Next, we have the left exact
\[ 0 \to H^1(X \times X, \mathcal{E}' \mathcal{N}d((E_1)_{\gamma})) \to \text{Ext}^1((E_1)_{\gamma}, (E_1)_{\gamma}) \to H^0(X \times X, \mathcal{E}' \text{xt}^1((E_1)_{\gamma}, (E_1)_{\gamma})). \]
The right space \(H^0(X \times X, \mathcal{E}' \text{xt}^1((E_1)_{\gamma}, (E_1)_{\gamma}))\) vanishes, by [MM1] Prop. 3.5], and the left space vanishes, as noted above. Hence, \(\text{Ext}^1((E_1)_{\gamma}, (E_1)_{\gamma})\) vanishes as well and \((E_1)_{\gamma}\) is infinitesimally rigid.

Step 3: Proof of Part 2 of Theorem 1.11 in the case where the sheaf \(E\) in Equation (1.3) is \(E_1\) and \(M = S[n]\). Assume first that the Picard rank of \(X\) is not maximal. The isomorphism class of the projective bundle \((B_1)_{\gamma}\) is independent of the choice of the path \(\gamma\) from \((S[n], \eta_1)\) to \((X, \eta)\), by its rigidity and Lemma 4.6. The isomorphism \(5.1\) implies the independence of \((E_1)_{\gamma}\) of the choice of \(\gamma\). Assume next that the Picard rank of \(X\) is maximal. Let \((X', \eta')\) be a marked pair with \(X'\) of non-maximal Picard rank. Let \(\gamma_i, i = 1, 2\), be two twistor paths from \((S[n], \eta_1)\) to \((X, \eta)\) and \(\gamma'\) a twistor path from \((X, \eta)\) to \((X', \eta')\). The isomorphism class of \((E_1)_{\gamma_1}\) and \((E_1)_{\gamma_2}\) are equal, since the Picard rank of \(X'\) is not maximal, and consequently so are \(((E_1)_{\gamma_1})_{\gamma_i}, i = 1, 2\), in which turn are isomorphic to \((E_1)_{\gamma_i}\), \(i = 1, 2\).

Step 4: Proof of Part 3 of Theorem 1.11 in the case where \((M, \eta_0) = (S[n], \eta_1)\) and \(\phi\) belongs to \(\text{Mon}(\Lambda)_{\text{cov}}\). The isomorphism \(B_{(X, \eta)} \cong B_{(X, \phi \eta_0)}\) was established in Step 1, for \((X, \eta) \in U_{B_1}\) and \(\phi \in \text{Mon}^2(\mathfrak{M}_\Lambda)_{\text{cov}}\). The isomorphism \(\mathcal{E}' \mathcal{N}d(E_{(X, \eta)}) \cong \mathcal{E}' \mathcal{N}d(E_{(X, \phi \eta_0)})\) follows for such marked pairs and \(\phi\) via the isomorphism \(5.1\). The equality \(\text{Mon}^2(\mathfrak{M}_\Lambda)_{\text{cov}} = \text{Mon}(\Lambda)_{\text{cov}}\) is established in [Ma2] Theorem 1.2 and Lemma 4.2].

Step 5: Proof of Part 3 of Theorem 1.11 in the case where the markings of \((\tilde{M}, \tilde{\eta_0})\) and \((M, \eta_0)\) are compatible. It suffices to prove the statement for \((M, \eta_0) = (S[n], \eta_1)\) and \((\tilde{M}, \tilde{\eta_0})\) with a compatible marking. It suffices to prove it for one marking \(\tilde{\eta_0}\) in the compatibility class, by Step 4. Denote by \(B_{\tilde{E}}\) the modular projective bundle over the blow-up of the diagonal in \(\tilde{M} \times M\). It suffices to prove that there exists a twistor path \(\gamma\) from \((S[n], \eta_1)\) to \((\tilde{M}, \tilde{\eta_0})\), such that \((B_1)_{\gamma}\) is isomorphic to \(B_{\tilde{E}}\), as the sets \(U_{B_1}\) and \(U_{B_{\tilde{E}}}\) would then be equal, by Lemma 1.8 (3). The latter statement would follow, if there exists a differentiable \(\mathcal{G}\)-family \(\mathcal{B}\), as in the set-up of Lemma 3.6, vertices \(v_i\) and points \(\sigma_i \in \Xi_{v_i}, i = 1, 2\), such that \((X_{\sigma_i}, \eta_1, B_{\sigma_i}) \cong (S[n], \eta_1, B_1)\), and \((X_{\sigma_2}, \eta_0, B_{\sigma_2}) \cong (\tilde{M}, \tilde{\eta_0}, B_{\sigma_2})\) by Lemma 4.6.

There exists a finite sequence of algebraic curves \(C_i, 1 \leq i \leq N, \) a family of K3 surfaces \(p_i : S_i \to C_i\), sections \(h_i\) of \(R^2p_i* \mathcal{L}\), and smooth proper morphisms \(\pi_i : \mathcal{M}_i \to C_i\), satisfying the properties of Proposition 5.1 points \(\sigma_i\) in \(C_i\), and stability preserving Fourier-Mukai functors
\[ \Phi_i : D^b(S_{\sigma_i}) \to D^b(S_{\sigma_{i+1}}) \]
inducing isomorphisms \(\phi_i : \mathcal{M}_{\sigma_i} \to \mathcal{M}_{\sigma_{i+1}}\), such that \(\phi_i \times \phi_i\) pulls back the modular sheaf \(E_{\sigma_{i+1}}\) over \(\mathcal{M}_{\sigma_{i+1}} \times \mathcal{M}_{\sigma_{i+1}}\) to the modular sheaf \(E_{\sigma_{i}}\) over \(\mathcal{M}_{\sigma_{i}} \times \mathcal{M}_{\sigma_{i}}\), as in Section 5.3 and such that \(\mathcal{M}_{\sigma_{i}} = S[n]\) and \(\mathcal{M}_{\sigma_{N}} = \tilde{M}\), by the work of Yoshioka [Yo] (see also [Ma1 Sec. 1.3.1]). We get the graph \(\mathcal{G}\) with vertices \(\{1, 2, \ldots, N\}\) and edges \(\{e_i\}_{i=1}^{N-1}\) induced by \(\phi_i\), and the family \(\mathcal{B}_i \to \mathcal{B}_i \to C_i\) of modular projective bundles over the blow-up \(\mathcal{B}_i\) of the relative diagonal in \(\mathcal{M}_i \times C_i\). Let \(\tilde{C}_i\) be the universal cover, \(\tilde{\sigma}_i \in \tilde{C}_i\) a point over \(\sigma_i, 1 \leq i \leq N\), and form the pulled back \(\mathcal{G}\)-family \(\mathcal{B}_{\tilde{C}} \to \mathcal{B}_{\tilde{C}} \to \tilde{C}_i\) gluing again via \(\phi_i\) the fibers of \(\mathcal{B}_i\) over \(\tilde{\sigma}_i\) and of \(\mathcal{M}_{\sigma_{i+1}}\) over \(\tilde{\sigma}_{i+1}\), \(1 \leq i \leq N - 1\). The marking \(\eta_1\) of \(\tilde{M}_{\tilde{\sigma}_1} := S[n]\) determines a marking \(\tilde{\eta}_0\) of \(\tilde{M}_{\tilde{\sigma}_1}\), since the curves \(\tilde{C}_i\) are simply connected and so is the reducible curve \(\tilde{C}\) obtained from their union by gluing \(\tilde{\sigma}_i\) to \(\tilde{\sigma}_{i+1}\). We claim that the marked pairs \((S[n], \eta_1)\) and \((\tilde{M}, \tilde{\eta}_0)\) are compatible. Indeed, \(\tilde{\eta}_0^{-1}\eta_1 : H^2(S[n], \mathcal{L}) \to H^2(\tilde{M}, \mathcal{L})\) is a parallel transport operator, by
construction, and so it maps the value of any flat section of a local system at \( \tilde{c}_1 \) to its value at \( \tilde{\sigma}_N \). The characteristic classes \( \tilde{c}_1(\mathcal{E}_\sigma), \sigma \in \tilde{C} \), of the relative (twisted) modular sheaf \( \mathcal{E}_\sigma \), given in \([L.5]\), form such a flat section. We conclude that the marked pairs \( (S^{[n]}, \eta_1) \) and \( (\tilde{M}, \tilde{\eta}_0) \) are compatible, by the characterization in Section \([L.3]\) of the compatibility relation in terms of these characteristic classes.

Step 6: Proof of Parts [3] and [4] of Theorem \([L.11]\). It remains to provide an example where \( \delta nd(E_\gamma) \) is isomorphic to \( \delta nd(E_\tilde{\gamma}) \), for \( \gamma \) a twistor path from \( (M, \eta_0) \) to \( (X, \eta) \) and \( \tilde{\gamma} \) a twistor path from \( (\tilde{M}, \tilde{\eta}_0) \) to \( (X, \eta) \). The two marked moduli spaces \( (M, \eta_0) \) and \( (\tilde{M}, \tilde{\eta}_0) \) would then necessarily have incompatible markings. Theorem 7.9 in \([Ma1]\) provides an example of two smooth and projective 2n-dimensional moduli spaces \( M \) and \( \tilde{M} \) of stable sheaves on a K3 surface and an isomorphism \( f : M \to \tilde{M} \), such that the pull back \( (f \times f)^*(\tilde{E}) \) of the modular sheaf over \( M \times \tilde{M} \) is isomorphic to the dual \( E^* \) of the modular sheaf over \( M \times M \), for every integer \( n \geq 2 \). Choose a marking \( \eta_0 \) for \( M \), set \( \tilde{\eta}_0 := \eta_0 \circ f^* \), so that \( (\tilde{M}, \tilde{\eta}_0) \cong (M, \eta_0) \). The desired example is provided, by choosing \( \gamma \) and \( \tilde{\gamma} \) to be the trivial paths.

Step 7: The general form of Parts [1] and [2] now follows from Part [3] of the Theorem. This completes the proof of Theorem \([L.11]\).

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**Department of Mathematics and Statistics, University of Massachusetts, Amherst, MA 01003**  
E-mail address: markman@math.umass.edu

**Facultad de Matemáticas, PUC Chile, 4860 Vicuña Mackenna, Santiago, Chile**  
E-mail address: smehrotra@mat.uc.cl

**Laboratory of Algebraic Geometry, National Research University, HSE, Department of Mathematics, 7 Vavilova Street, Moscow, Russia**  
E-mail address: verbit@mccme.ru