Selected elevation in quantum tunneling

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The tunneling through an opaque barrier with a strong oscillating component is investigated. It is shown, that in the strong perturbations regime (in contrast to the weak one), higher perturbations rate does not necessarily improve the activation. In fact, in this regime two rival factors play a role, and as a consequence, this tunneling system behaves like a sensitive frequency-shifter device: for most incident particles’ energies activation occurs and the particles are energetically elevated, while for specific energies activation is depressed and the transmission is very low. This effect is unique to the strong perturbation regime, and it is totally absent in the weak perturbation case. Moreover, it cannot be deduced even in the adiabatic regime. It is conjectured that this mechanism can be used as a frequency-dependent transistor, in which the device’s transmission is governed by the external field frequency.

When a quantum particle propagates through an opaque barrier in the classically forbidden domain, it tunnels. This conduct is manifested by the exponentially small transmission probability. Nevertheless, when some part of the barrier weakly oscillates the particle will be activated to higher energies, and the transmission will increase substantially. However, when the temporal change is not merely a perturbation but rather a strong variation the dynamics are too complicated to be described by such a simple rule of thumb, and most of the interesting phenomena belong to this category.

While it is well known that even a very weak external oscillating field may considerably increase the tunneling current, changes which are comparable to the initial system parameters can cause elevator resonant activation and activation assisted tunneling, charge quantization pumping, as well as wave function collapse. That is, large changes reveal a wealth of physical phenomena.

When the tunneling particle energy (Ω) is close to the potential barrier height (V), the dynamics become more complicated since the perturbation amplitude (∆V) can exceed V − Ω, i.e., the perturbation “strength” may be larger than the effective tunneling barrier. This case is extremely sensitive, since the dynamics are governed by two rival factors. On the one hand, the oscillation’s amplitude is so large, that for a finite segment of the oscillation period the alternating potential blocks the particles’ transmission. Hence, following this reasoning, the tunneling rate should decrease. On the other hand, energy quanta generated by the oscillating barrier can be absorbed by the tunneling particle to assist it in the due course of tunneling.

In this work, we investigate the interplay between these two rival factors, and show that the competition between them is responsible for the high sensitivity to the system’s parameters. For example, it so happens that the incident particles are not always activated to higher energy: when their initial energy is equal to one of a series of specific energies the activation is frustrated, elevation to higher energies does not occur and the transmission is decreased accordingly.

The problem of tunneling in the presence of an external time-dependent potential has been extensively studied. However, in this paper the strong perturbation regime is addressed and no approximate assumptions are made in the solution analysis. We give this case both an exact numerical solution, and an analytical solution, using an approximation, to show that the general conclusions can be deduced even in the slowly varying regime. However, we also show that the oscillations rate cannot be arbitrarily small, and in fact it should be larger than the spectral bandwidth of the resonances, otherwise (and, in particular, the adiabatic regime) the effect vanishes.

The apparent underlying physics can be described as follows: Due to the oscillations, the alternating barrier functions as an impenetrable barrier (or at least a very opaque one) only for a fraction of the oscillating period and therefore, a time-window (denoted τW) is formed in which the incident particle can propagate. Consequently, in order to prevent a destructive pattern, the incoming particles’ frequency Ω (in units where ħ = 1) should be equal to an integer of both the oscillations’ frequency ω and 2π/τW (i.e., Ω = 2πm1/τW and Ω = m2ω for integer m1 and m2). Therefore, a selection rule for activation should be expected. However, this selection rule is rather complicated since τW is a function of the incident particles’ frequency (Ω). It turns out that the energies, for which activation occurs, have a nontrivial dependence on the integer m1 (roughly like m1^{2/3}).

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A few words should be added here about the plausibility of the effect’s practical implementation. The industry has a special interest in resonant tunneling devices (RTD), which allow miniaturizing electronic circuits and improving their performance. Ordinary RTD are highly costly. The presented tunneling device allows for the fabrication of a low-cost device where any resonant refinement can be done by variations in the external field frequency, and no special geometry or manufacturing restrictions are needed.

In this paper we discuss the tunneling dynamics of a beam of particles which are activated by strong harmonic beam of particles which are activated by strong harmonic

FIG. 1. An illustration of the system: The incident particles’ energy \( \Omega \) (and the oscillating frequency \( \omega \)) determines whether the incident particles will be elevated to higher energy states (dark circles - most cases) or will remain in their initial state (open circles - specific cases).

In terms of the Schrödinger equation, the dynamics can be expressed by

\[
-\psi'' - \beta \delta(x) \cos(\omega t + \eta) \psi + V(x) \psi = i\delta \psi
\]

where we adopt the units \( 2m = 1, \hbar = 1 \), and the notations \( \psi'' = \partial^2 \psi / \partial x^2 \), \( \psi = \partial \psi / \partial t \), and \( V(x) \) is the potential barrier which vanishes quickly for \( |x| \to \infty \).

The time-dependent solution can be written as a discrete Fourier transform

\[
\psi(x < 0) = \varphi^+_{\Omega} e^{-i\Omega t} + \sum_n r_n \varphi^-_{\Omega+n\omega} e^{-i(\Omega+n\omega)t}
\]

\[
\psi(x > 0) = \sum_n l_n \varphi^+_{\Omega+n\omega} e^{-i(\Omega+n\omega)t}
\]

where \( \varphi^\pm_{\Omega} \) are the solutions of the stationary state Schrödinger equation, which does not include the oscillating term, i.e.

\[
-\varphi''_{\Omega} + [V(x) - \omega] \varphi_{\Omega} = 0
\]

The homogeneous solutions \( \varphi_{\Omega} \) describe waves that propagate from \( -\infty \) to \( +\infty \), while \( \varphi_{\Omega} \) describes the waves that are incoming from \( +\infty \) and outgoing to \( -\infty \). Thus, \( \varphi^+_{\Omega} \rightarrow \tau_{\alpha} e^{i\sqrt{\omega^2 - \Omega^2}x-i\omega t} \) (for \( x \to \infty \)) while \( \varphi^-_{\Omega} \rightarrow \tau_{\alpha} e^{-i\sqrt{\omega^2 - \Omega^2}x+i\omega t} \) (for \( x \to -\infty \)), and \( |\tau_{\alpha}|^2 \) is the probability of penetrating the barrier.

By taking care of the matching conditions of the solutions in eqs. 3 and 5 at \( x = 0 \), we easily obtain

\[
2s_n \alpha_n + \beta(s_n-1 + s_{n+1}) = 2\chi \varphi_{\Omega}^d \delta_{n0}.
\]

When using the following notations

\[
\chi_n \equiv \varphi^+_{\Omega} - \varphi^-_{\Omega}
\]

and

\[
s_n \equiv e^{i\eta} \varphi^+_{\Omega} l_n,
\]

where \( \varphi_{\Omega}^\pm = \varphi_{\Omega+n\omega}^\mp (x = 0) \), \( \varphi^\pm_{\Omega} \equiv \partial \varphi_{\Omega+n\omega}^\pm / \partial x |_{x=0} \), \( \delta_{n0} \) is the Kronecker delta and \( l_n \equiv l_{\Omega+n\omega} \). This difference equation can readily be solved. Notice that

\[
\chi_n = 1/g_n(0)
\]

where \( g_n(x) \) is the Green function of the equation

\[
-\psi'' + V(x) \psi = (\Omega + n\omega) \psi
\]

Thus, in the case of a perfectly symmetric rectangular barrier, \( \chi_n \) comes directly from \( 9 \)

\[
g_n(0) = -\coth[\rho_n L + i \arctan(k_n / \rho_n)] / (2\rho_n)
\]

where \( k_n \equiv \sqrt{\Omega + n\omega} \), \( \rho_n \equiv \sqrt{V - k_n^2} \), \( 2L \) is the barrier length and finally \( V \) is its potential height.

In Fig.2 we present the exact numerical solution of eq.8 in the case of a rectangular barrier (in the figure the absolute value of \( s_n \) is presented), which is the spectral solution of eq.8 at \( x = 0 \). This figures illustrates the solution’s sensitivity to the incoming particles’ energy: a 2% change in \( \Omega \) causes a severe reduction in activation.

To formulate an analytical expression for this solution, we take advantage of the fact that the perturbations are strong, i.e., we can assume that \( \beta^2 \gg V - \Omega \gg \omega \). Moreover, although in the numerical analysis we used the exact form of the Green function (eq.8), in the case of an opaque barrier, the approximation \( g_n(0) \approx - (2\rho_n)^{-1} \) may be used with great accuracy.

In this strong perturbation amplitude and low-frequency regime, the difference equation (eq.8) can be transformed to a differential equation. By using the definitions
and \( c(n) \equiv 1 + \chi_n / \beta \), one easily obtains the differential equation

\[
\frac{d^2}{dn^2} G(n) + c(n) G(n) = \delta(n)
\]  

(11)

where \( \delta(n) \) is the Dirac delta function.

Hence, one can regard \( G(n) \) as having a Green function properties (including the singularity at \( n = 0 \)).

\[
G(n) = \frac{\beta s_n}{\chi_0}
\]  

(10)

A plot of the transmission coefficient \(|s_n|\) (defined in Eq. [1]) as a function of the transmitted particles’ frequency (the activated energy) \( \Omega + n\omega \). The solid line represents the case \( \Omega/V = 0.625 \) and the dotted one represents the case \( \Omega/V = 0.6125 \). The other parameters are \( \omega/V = 0.0075 \), \( \sqrt{\gamma L} = 10.75 \) and \( \beta L = 9.35 \).

When the Green function \( G(n) \) is known the transmitted solution of Eq. [2] follows directly from Eqs. [3] and [11].

\[
\psi(x \geq 0) = \frac{\chi_0 \varphi_0^+}{\beta} e^{-i\eta n} \int dnnG(n) \frac{\varphi_\Omega+n\omega(x)}{\varphi_\Omega+n\omega(0)} e^{-i(n\omega + \eta)}
\]  

(12)

In particular, the scattered wave function at \( x = 0 \) is proportional to the Fourier transform of the Green function.

Since \( V - \Omega \gg \omega \), one can approximate

\[
c \simeq 1 - 2\rho/\beta + \omega n/ (\beta \rho),
\]  

(13)

where \( \rho \equiv \sqrt{V - \Omega} \).

Then, we can define for convenience the variable

\[
\xi \equiv \left( n + \frac{\beta - 2\rho}{\omega} \right) \left( \frac{\omega}{2\beta \rho} \right)^{1/3}
\]  

(14)

and the Green function is then

\[
G(\xi) = -i\pi \left( \frac{\beta \omega}{2\rho} \right)^{1/3} \times \left\{ \begin{array}{ll}
\text{Ai}(-\xi) [\text{Ai}(-\xi_0) + i\text{Bi}(-\xi_0)] & \text{for } \xi < \xi_0 \\
\text{Ai}(-\xi_0) [\text{Ai}(-\xi) + i\text{Bi}(-\xi)] & \text{for } \xi > \xi_0
\end{array} \right.
\]

(15)

where \( \xi_0 \equiv (\xi(n=0) = (1 - 2\rho/\beta) \left( \beta \rho / \omega \sqrt{2} \right)^{2/3} \) and \( \text{Ai} \) and \( \text{Bi} \) are the Airy functions of the first and second kind, respectively.

For frequencies which are lower than \( \Omega \) (negative \( n \)'s) the amplitude oscillates like a simple Airy function (see Fig. 2 for a typical plot of \(|s_n|\), which is related to the Green function by Eq. [10]).

\[
|G(\xi)|^2 \approx \pi \left( 2\beta \rho / \omega \right)^{2/3} \xi_0^{-1/2} \text{Ai}^2(-\xi) \quad \text{for } n < 0
\]

(16)

since \( |\text{Ai}(-\xi_0) + i\text{Bi}(-\xi_0)|^2 \approx \pi^{-1} \xi_0^{-1/2} \). That explains the insensitivity of the amplitude of \( G \) (for a specific \( n < 0 \) or \( \xi < \xi_0 \)) to small variations in the incoming energy \( \Omega \) (see, for example, Fig. 2). However, for a specific incoming energy \( \Omega \), the amplitude of \( G \) oscillates with respect to the transmitted energies (i.e., \( \Omega + n\omega \)). In this regime (i.e., \( \xi > \xi_0 \)),

\[
|G(\xi)|^2 \approx \pi \left( 2\beta \rho / \omega \right)^{2/3} \xi^{-1/2} \text{Ai}^2(-\xi_0) \quad \text{for } n > 0
\]

(17)

which means that for an incident particles’ energy \( \Omega \), the amplitude of the Green function has a very mild dependence on the transmitted particles’ energies (i.e., on \( n \)), while it is strongly dependent on the incident particles’ energies \( \Omega \). This can explain the result presented in Fig. 2, where a two percent change in the incoming particles’ energy was enough to frustrate the activation to higher energies.

It is clear from Eq. [17] and from the periodicity of the Airy function, that the probability of an incident particle being activated to higher energy is very sensitive to its initial energy. This sensitivity is manifested in the following calculation of the mean activation energy:

\[
\Omega_{\text{act}} = \Omega + \omega \left| \frac{\chi_0 \varphi_0^+}{\beta} \int dnnG(n) \frac{\varphi_\Omega+n\omega(x > L)}{\varphi_\Omega+n\omega(0)} \right|^2
\]

(18)

A plot of the mean activation energy (\( \Omega_{\text{act}} \)) as a function of the incident one (\( \Omega \)) is shown in Fig. 3 (an exact numeric solution).

The opacity of the barrier is the cause for the sharp changes in \( \Omega_{\text{act}} \). Since the tunneling coefficient \( \varphi_\Omega+n\omega(x > L)/\varphi_\Omega+n\omega(0) \approx \exp(-\sqrt{V - \Omega - n\omega L}) \) is
exponentially small for non-activated particles, there is a great advantage for a particle to be activated to higher energies \( \Omega + n \omega \simeq V \). However, since \( G(n) \) is an oscillating function of the incoming energy \( \Omega \), there are some energies \( \Omega_m \) for which \( G((V - \Omega_m)/\omega) \) vanishes. In these cases, according to eq. 17, not only is the transition to energy \( V \) forbidden, but the transition to all the other higher energies (which correspond to \( n > 0 \)) is, as well.

Therefore, the particles must tunnel out with energy which is very close to their initial one (i.e., \( \Omega_{act} = \Omega \)). That explains the source of these oscillations and the sharp transitions between the activated energies \( \simeq V \) and the non-activated ones \( \simeq \Omega \). The transitions occur when the Airy function of eq. 17 vanishes or, more accurately, when \( \text{Ai}^2(-\xi_0) \simeq \exp((-\sqrt{V - \Omega_m}L)). \) Taking the low-frequency limit (\( \omega \to 0 \)), the Airy function can be expressed by simple trigonometric functions to obtain the transition criterion \( \cos^2 \left[ \frac{\pi}{4} \left( 1 - \frac{2\rho_m}{\beta} \right) \right. \frac{3\beta}{2} \frac{\rho_m}{\omega} - \frac{\pi}{4} \right] \simeq \exp(-\rho_m L) \), where \( \rho_m \equiv \sqrt{V - \Omega_m} \). Therefore, the incoming energies for which \( \Omega_{act} \simeq \Omega \), and thus no activation occurs, are approximately

\[
\Omega_m \simeq V - \beta^2/4 + \frac{1}{2} \left( \frac{3}{2} \beta \omega \left( m + \frac{3}{4} \right) \pi \right)^{2/3} \tag{19}
\]

and the spectral width of these regions is exponentially small \( \Gamma \simeq \exp(-\beta L/2) \). In Fig.3 the non-activated energies \( \Omega_m \), for which \( \Omega_{act} \simeq \Omega \), are the minima of the plot while the maxima correspond to full activation (i.e., \( \Omega_{act} \simeq V \)).

To summarize, tunneling transmission through an opaque barrier with an oscillating section was investigated. It was shown that in the strong perturbation (and nonadiabatic) regime a new selection-rule appears. Not all the incident particles are activated, as could be anticipated in a tunneling process. Although in most cases the incident particles are elevated to higher energy states, for some incident particles’ energy activation is depressed and the particles remain approximately in their initial states. The spectral width of these energy domains is exponentially small.

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**FIG. 3.** Two characteristic plots (for arbitrary parameters) of the mean activation energy (\( \Omega_{act} \)) as a function of the incident particles’ energy \( \Omega \) (see Eq. 17). The two curves represents the same system except for \( \omega \), which is five times smaller in the dotted curve.

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