IDENTICAL PARTICLES IN QUANTUM MECHANICS: OPERATIONAL AND
TOPOLOGICAL CONSIDERATIONS

by

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This dissertation reports our investigation into the existence of anyons, which interpolate between bosons and fermions, in light of the Symmetrization Postulate, which states that only the two extremes exist. The Symmetrization Postulate can be understood as asserting that there are only two consistent ways of combining the behavior of distinguishable particles to obtain the behavior of identical ones. We showed that anyonic behavior then arises because of the way in which the probability amplitudes of distinguishable particles in two dimensions are affected by the topology of the space. These can then be combined in one of the ways arising from the Symmetrization Postulate, to form identical anyons. Therefore, anyons do not invalidate the Symmetrization Postulate, but are entirely compatible with it. In order to show that anyons can arise without particle identity, we investigated how distinguishable particles can gain particular properties from the topology of their configuration space. We managed to do so, and in the process, discovered a novel approach for quantizing multiply-connected spaces, in a way that is more suited to anyons than the standard approach.
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In Chapter 1 we provide an introduction to the conceptual and topological background needed for placing our main results in context. First we present the more traditional approach to identical particles, where quantization precedes symmetrization. The restriction to bosons and fermions was eventually recognized to require a Symmetrization Postulate. Section 2.5, in particular, refers to an operational proof of this postulate published in Goyal [1]. Preliminary results focusing on the non-interacting, two-particle case were presented as “On the Origin of the Quantum Rules for Identical Particles” at QTRF-6‡ and MaxEnt 2012§, and published in the proceedings of the latter as Neori and Goyal [2]. As explained in Ref. [1], this result concerns the combination of distinguishable-particle transition amplitudes into those of identical particles, which is important for the application to anyons.

The reduced configuration space approach to identical particles attempts to derive the Symmetrization Postulate by considering particles to be identical before quantization, and then using topological tools to restrict their behavior, rather than identity being imposed as a symmetry after quantization [3–5]. This approach leads to additional possibilities, called anyons, in two dimensions. Using the operational understanding of the Symmetrization Postulate, we show that anyonic behavior is solely due to topology, rather than particle identicality. An overview of these results was presented as “Anyons in the Operational Formalism” at MaxEnt 2014¶, published in its proceedings as Neori and Goyal [6]. The derivation of quantization in multiply-connected spaces using the fundamental groupoid, taken up in Chapter 3, was submitted for publication as Neori and Goyal [7]. This is followed by an explicit formation of distinguishable anyons, and their combination into identical anyons, in Chapter 4. Chapter 5 closes the novel contributions of this dissertation with a discussion of the historical and philosophical implications of the work. The latter two chapters comprise the contents of Neori and Goyal [8], which is in draft form.

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Elementary particles are usually assumed to belong to one of two classes: *bosons*, which are limited to symmetric multiple-particle wave-functions; and *fermions*, which are limited to anti-symmetric multiple-particle wave-functions.

Initial works in the new quantum mechanics did not venture beyond fermions and bosons [9, 10]. However, it was soon noted that there is more to be said about the subject. Wigner [11, 12] was the first to explore other possibilities. He treated particle exchanges as generators of the symmetric group $S_N$ of all permutations over $N$ particles, and used group representation theory to perform a more general analysis. He found that, indeed, for two particles only bosons and fermions are possible, but even three particles [11] make the situation more complicated: while bosons and fermions emerge as one-dimensional representations of $S_N$, there are two more two-dimensional representations, which might correspond to a different type of particle. Pauli [13] provided a similar development, and considered the fact that these other options do not come about to be a mystery yet to be resolved. Dirac [14] similarly used group representation theory to treat identical particles, but never actually developed even proto-paraparticles. While he acknowledged that other possibilities are mathematically possible, he noted that only bosons and fermions were so far needed to explain natural phenomena.

For a few decades, only a limited number of *ad hoc* attempts to push beyond these two possibilities were published. Gentile [15] posited identical particle types allowing for an arbitrary number $d$ in each single-particle state. However, this formulation depends on the choice of basis, so it was not pursued further. Green [16] made a similar extension in quantum field theory, but this, again, was somewhat *ad hoc* and did not immediately lead to further developments.

The issue was only seriously and rigorously taken up in the 1960s. At first, several authors attempted to prove that there are only two types of identical particles, using the basic rules of quantum mechanics combined with the assumption that the particles are indistinguishable [17–20]. However, Messiah and Greenberg [21] responded by showing that this was not a direct consequence of particle indistinguishability. Instead, limiting the situation to bosons and fermions requires an additional assumption, which they termed the *Symmetrization Postulate*. In the context of non-relativistic quantum mechanics, they proposed to call particles which, as anticipated by Refs. [11–13], live in multi-dimensional representations of the symmetric group, *paraparticles*. They then showed how difficult it would be to test for this kind of behavior with most common experiments, while suggesting new ones which would nevertheless allow deviations to be detected. The impetus for this investigation was the introduction by Greenberg [22] of the idea that quarks are paraparticles: specifically, that they live a two-dimensional irreducible representation of $S_3$, which helps explain how they are able to share a ground state in nucleons, even though they otherwise share properties with fermions. Greenberg [23] later argued that this was an important step on the way to the ultimate resolution of this conundrum, namely, the introduction of color as an additional internal degree of freedom for quarks [24]. This direction continued to be explored and extended into a field-theoretical generalization called *quons* [25].

In addition to these extensions, attempts to prove the Symmetrization Postulate, or at least to break it down to less blatant subsidiary assumptions, continued. One notable example is Tikochinsky [26]. This work proved the Symmetrization Postulate for non-interacting particles by working from the Feynman rules for transition amplitudes [27]. We were able to improve upon the rigor
of that result by starting from the operational approach to reconstructing the Feynman rules de-
veloped in Goyal et al. [28]. We published preliminary results, for two non-interacting particles,
in Neori and Goyal [2]. A generalization to $N$ interacting particles can be found in Goyal [1].

An important alternative attempt at proving the Symmetrization Postulate, which takes an
entirely different tack, creates a challenge for the operational approach. A common feature of the
non-relativistic approaches described so far is that they start from particles which are labeled, and
whose exchange is then treated as a physical process, akin to rotation or translation. Particle iden-
tity is then imposed, just like spherical symmetry or space homogeneity. Dirac [14] is particularly
explicit about this point. However, as was most clearly expressed by Mirman [29], if particles are
identical, then there is no physical meaning to their labels, nor to the exchange thereof. Conse-
quently, there is no reason to assume the existence of unitary symmetry operators corresponding
to particle label permutations. One way to resolve this is by treating exchanges as the end results
of paths undertaken by particles. That is, the particles start out in a certain state, move through
their configuration space, and then end up in an end-state which differs from their initial state only
by a permutation of the particles. Particle identity can then be expressed by taking this classical
configuration space and identifying points which only differ by such a permutation. The resulting,
reduced configuration space, can now be quantized. The hope is that then the Symmetrization
Postulate can be proven as resulting from quantization over this space. This reduced configuration
space approach was popularized by Leinaas and Myrheim [5], although it was preceded by Souriau
[3] and Laidlaw and DeWitt-Morette [4].

In the reduced configuration space, exchange paths correspond to certain types of paths starting
and ending at the same point, that is, to loops. The degree of freedom which allows for bosons and
fermions now comes from the fact that there are several types of paths, called homotopy classes.
Each class consists of all paths which can be continuously deformed to one another without changing
the end points. If we restrict ourselves to loops, paths starting and ending at the same point, then
we have a group structure called the fundamental group, which is an algebraic topological property
of the space. The representations of this group are what corresponds to statistics in the label-based
formalism. Unlike there, the possible statistics depend on the dimension of the space in which the
particles dwell.

All three of the original works [3–5] came to the same conclusions about identical particles in
three dimensions. As long as particle incidence points are removed, the fundamental group is simply
the symmetric group, meaning that there are two one-dimensional representations, corresponding to
bosons and fermions. However, Ref. [5] found that in two dimensions or less, there is a whole family
of particle types interpolating between bosons and fermions. Wilczek [30] coined the term anyons to
describe this new range of possibilities in two dimensions. When anyons are exchanged, their wave-
function accrues a certain complex amplitude, which depends on the particular loop of exchange,
and is of the form $e^{i\kappa}$, where $e^{i\phi}$ is the generator of the statistics of the anyons in question, in
analogy to the $\pm 1$ accrued by exchanged bosons and fermions, respectively.\footnote{Girardeau [31] did anticipate the importance of topology as early as 1965, but only treated connectivity issues. He focused on the contrast between one- and three-dimensional spaces, skipping over two.}

On the surface, this could be considered as simply a failure to prove the Symmetrization Post-
tulate. However, anyonic behavior is not just theoretical: the Fractional Quantum Hall Effect,
which occurs in thin layers in condensed matter [32–36], results in excitations which have anyonic
statistics [37, 38]. Therefore, any attempt to deal with identical particles must provide for this
kind of behavior in two dimensions. This, however, does not mean that anyonic behavior necessar-
ily requires identical particles. As was noted by Goldin et al. [39], even distinguishable particles
may have topological exchange amplitudes, as it is the multiple-connectedness of the configuration
space that creates them, and that happens in two dimensions as soon as the coincidence points are removed. It is very common to generate the topological amplitudes through the use of non-local interaction \cite{40–42}.

Nevertheless, the references to \cite{5} have imbued anyons with identicality, and connected them to the Symmetrization Postulate. An overwhelming portion of the literature about anyons, such as Refs. \cite{30, 43–46}, takes reducing the configuration space as their starting point (Ref. \cite{30} resorts to it in implying complications in generalization for more than two particles, even though the author does not need this for two particles).

Now that we have provided a general motivational overview, we will dive into individual issues that build up the understanding necessary for appreciating the novel results reported herein (Chapter 2). We will explore the strangeness of quantum mechanics (Section 2.1), and the reconstruction efforts it motivated (Section 2.2). We will then study the relation between the Schrödinger and Feynman formalisms (Section 2.3), followed by the operational reconstruction which uses the latter (Section 2.4), and its application to the issue of identical particles (Section 2.5).

Following that we will discuss the reduced configuration space approach to identical particles (Section 2.6), which lead to anyons, followed by an explanation of the most striking evidence for their existence, the Fractional Quantum Hall Effect (Section 2.7). Moving through the Aharonov-Bohm Effect (Section 2.8), we will explain homotopy, an important topological tool, and will specifically discuss a relatively unusual tool, the fundamental groupoid (Section 2.9).

This background chapter concluded, there will come our novel results. First, we explore the possibilities of quantization in multiply-connected spaces through the representations of the fundamental groupoid (Chapter 3). This can be used to choose specific topological amplitudes for paths, not just for loops as with the fundamental group. We use this to provide consistent amplitudes for distinguishable particle exchanges, which, in the special case of the punctured plane, gives us distinguishable anyons. These can be combined using the operational results mentioned in Section 2.5 to create identical anyons (Chapter 4). This cleanly breaks up the behavior of identical anyons into the topological level, which is what grants them the ability to interpolate between bosons and fermions, and the operational level, which gives rise to the bosons and fermions due to identicality.

Finally, we will explore the persistence of the view that the reduced configuration space is vital for anyons and their exploration, providing some philosophical background having to do with the idea of identity (Chapter 5). We will find that the attempt to avoid the use of labels as unmeasurable in themselves simply leads to other kinds of unmeasurable superstructure, whether it is in covering spaces or fiber bundles. This approach offers no escape from auxiliary degrees of freedom.
2.1 The Strangeness of Quantum Mechanics

Quantum mechanics has been a counter-intuitive theory since its inception. The link between its abstract mathematical formalism and experienced reality is through measurements, for which it only provides a statistical distribution [47], yet it does not include an underlying deterministic theory. It satisfies complementarity [48], so that attempts to ascribe qualities to systems outside of measurements lead to paradoxes, like that of Einstein et al. [49]. Identical quantum-mechanical particles behave strangely, either satisfying the Pauli Exclusion Principle [50] or Bose-Einstein statistics [51, 52]. Nevertheless, it has been extremely successful at explaining phenomena which were not amenable to classical physical models, and was able to generate many successful predictions, which then lead to additional phenomena worth investigating. As a result, most of the intellectual energy dedicated to quantum mechanics has been focused upon solving problems with it, rather than better understanding its foundations.

For those who did wish for quantum mechanics to be more intuitive, there was a sense that in the background to quantum mechanics there might lie a theory that is classical, albeit statistical and very complicated. From the Bohm-de Broglie pilot wave theory [53, 54] to Nelson’s stochastic mechanics [55], it seemed like there must be a way to dig in and find an underlying deterministic theory. However, results such as the theorems of Bell [56] and Kochen and Specker [57] showed that, even if such a theory were to be found, it would be highly non-local, as well as context-dependent. Therefore, it would be nothing like classical models which we previously found conducive to our understanding.

2.2 Reconstructing Quantum Mechanics

These issues have lead researchers to investigate other routes of rebuilding or reconstructing quantum mechanics, in ways which bring into focus the physical intuitions required to understand it. Hardy’s 5 axioms [58] are an attempt to use continuous unitary transformations as the highlight of quantum behavior, while working within the realm of states and operators. Caticha’s Entropic Dynamics [59] tries to go as far as possible with ideas stemming from information processing, before adding physical assumptions, resulting in the wave-function formalism through a Hamilton-Jacobi formulation.

Our work follows upon the operational approach introduced in Goyal et al. [28] to reconstruct specifically the Feynman rules for quantum mechanics [27]. Before we delve into it, let us provide a brief overview of how the Feynman rules relate to the more familiar Dirac-Schrödinger formulation.

2.3 Schrödinger vs. Feynman Rules in Quantum Mechanics

The usual entity of importance in quantum mechanics is the wave-function, or, more generally, the wave-vector, which is a description of the physical state of a system. We start with the state in
which a system is prepared. In Dirac notation, we have:

$$|\psi_0\rangle ; \quad (2.1)$$

the wave-function for a particle is written as $\psi_0(x) = \langle x | \psi_0 \rangle$. The prepared system then evolves with time, with an optional interaction with an outside field, which is expressed by transforming this wave-vector or wave-function through the application of a unitary transformation generated by a Hamiltonian $H$, which, for simplicity, we take to be time-independent:

$$|\psi(t)\rangle = e^{-iHt/\hbar} |\psi_0\rangle ; \quad (2.2)$$

the current wave-function is then $\psi(x,t) = \langle x | \psi(t) \rangle$. Finally, the system is measured, using a complete orthonormal set of wave-functions $|e\rangle$ representing the possible classical outcomes. The probability for each outcome $e$ is then the absolute value squared of the component of the resulting wave-function on the member of the unitary basis reflecting that outcome (Born’s Rule):

$$\text{Pr}(e) = |\langle e | \psi(t) \rangle|^2 . \quad (2.3)$$

The Feynman approach focuses on the relation between preparation and measurement. We can rewrite the inner product between the final wave-function and a measurement state as a matrix element of a propagator indexed by the initial state and the final state:

$$A(e) = \langle e | \psi(t) \rangle = \langle e | e^{-iHt/\hbar} |\psi_0\rangle \quad (2.4)$$

so that:

$$\text{Pr}(e) = |A(e)|^2 . \quad (2.3')$$

Full calculation comes through the Feynman path-integral approach, in which one would write:

$$\psi(x,t) = \langle x | \psi(t) \rangle = \langle x | e^{-iHt/\hbar} |\psi_0\rangle = \int dx' \langle x | e^{-iHt/\hbar} |x'\rangle \langle x' | \psi_0 \rangle$$

$$= \int dx' K(x,t ; x') \psi_0(x'), \quad (2.5)$$

and the propagator $K(x,t ; x')$ becomes the interface with physical theory. Feynman’s rules then relate to how these propagators act when additional measurements are interposed between the preparation and the final measurement.

Acting through transition amplitudes or propagators makes many quantum mechanical phenomena easier to express. For example, if we have a transition amplitude $A_U$ for a particle passing from a point $x$ on the left of a barrier to a point $y$ to the right of it when only one hole is open, and another transition amplitude $A_D$ for passing from one to the other when only another hole is open, then the amplitude for this transitions when both holes are open is $A_U + A_D$. This is generally called the sum rule for transition amplitudes. The fact that it is the amplitudes rather than the probabilities which add is what causes an interference pattern to appear on a screen to the right. On the other hand, if a measurement were to decide whether a particle was going through one hole or the other, then the probabilities would add, as would be expected for classical particles.

The other Feynman rules are the product rule, which states that the amplitudes of two processes done in succession multiply (which can easily be seen from the propagator expression, since successive developments in time simply multiply by $e^{-iHt_n/\hbar}$); reciprocity, which states that the amplitude of a set of measurements or a path taken in reverse order (within a short enough timeframe, so that $e^{-iHt/\hbar} \rightarrow I$) is the complex conjugate; and, finally, Born’s rule for calculating the probabilities from the amplitudes, which states that the transition probability is given by the modulus-squared of the corresponding amplitude.
2.4 The Operational Approach to Quantum Mechanics

The aim of the operational approach in Ref. [28] is to derive the Feynman rules for these transition amplitudes from more fundamental principles. Focusing on amplitudes rather than states or measurement operators has the advantage of inter-relating physically measurable results, namely measurement outcomes. The idea is that sufficiently many physical assumptions may result in some constraints on how these amplitudes can be combined, and then we will end up with Feynman’s rules, from which the rest of Quantum Mechanics can be derived.

This seems to have first been attempted by Tikochinsky [60, 61], and independently by Caticha [62]. The idea was simple: start from sequences of measurement outcomes as your basic objects, and assume that their probability is derived from complex transition amplitudes. Then use the ways in which these sequences can be combined to find symmetries in the functions relating the respective amplitudes, which lead to functional equations. Solutions to these equations lead to the sum, product, and complex conjugate rules. Three main distinctions between Refs. [60, 61] and Ref. [62] were that the latter, unlike the former, had a more careful definition of “and” and “or”, did not assume determinism, and did not use reciprocity, which corresponds to negation from Cox [63], and which they would rather avoid. The third distinction is acknowledged in the retrospective Tikochinsky and Gull [64].

Nevertheless, both of these derivations assume complex numbers, and moreover, that the functions in their functional equations are analytic, so that it makes sense to differentiate them by their complex arguments. In both cases, analyticity does a lot of the heavy lifting for them, and there is a risk that the importance of other physical or logical assumptions is neglected as a result. Indeed, Ref. [60] itself notes that, once analyticity has been used, one of the algebraic symmetries is automatically satisfied. Analyticity also does most of the work in Tikochinsky [26].

Goyal et al. [28] is notable for starting with weaker hypotheses: instead of assuming analyticity, or even complex numbers, they posit that two real numbers are fundamentally needed to calculate the probability of a process, an assumption which they argue is a formalization of complementarity [48]. That, combined with several other assumptions that are physically or logically motivated, leads to the Feynman rules, without assuming analyticity.

2.5 Behavior of Identical Particles in Quantum Mechanics

Once the basics of quantum mechanics have been derived by a certain framework, the best way to bolster its credibility is to apply it to additional problems in quantum mechanics, and to see whether it can shed any light on them. Indeed, Tikochinsky [26] did so for the issue of identical particles in quantum mechanics.

Current elementary particles fall into two types: bosons and fermions. Bosons can be pushed into the same state without limit. Photons are bosons, which is why electromagnetic waves are so easily treated classically: the overabundance of photons in the same state masks the quantum nature of light. In wave-function terms, this property is expressed in their overall wave-function being symmetric; for example:

$$\Psi(x_2, x_1, x_3) = +\Psi(x_1, x_2, x_3). \quad (2.6)$$

Fermions, meanwhile, refuse to be in the same state. Electrons are fermions, which is why atoms are built up by progressively filling up shells of electrons. This property is expressed by their wave-function being anti-symmetric:

$$\Psi(x_2, x_1, x_3) = -\Psi(x_1, x_2, x_3). \quad (2.7)$$
The Symmetrization Postulate, as noted in Chapter 1, states that these are the only types possible. Ref. [26] proved that this is the case for non-interacting particles, starting with the Feynman rules Tikochinsky previously derived in Refs. [60, 61], and using the same framework. Here, too, analyticity played a key role, as did several implicit assumptions.

The research reported in Neori and Goyal [2] and Goyal [1] was able to overcome the dependence on analyticity. Additionally, one of the implicit assumptions in Ref. [26], namely, that the amplitude for an identical particle transition had a functional dependence on the single-particle amplitudes, was made explicit in the special case of two, non-interacting particles in Ref. [2]. In Ref. [1], this was generalized to a form which allows the incorporation of interactions: the amplitude of the identical-particle transition was now assumed to have a functional dependence on the amplitudes for the corresponding transitions of the distinguishable particle system. This leads to a different view of what the Symmetrization Postulate means: instead of limiting to two types of particle behaviors, it limits to two types of combinations of distinguishable-particle behaviors.

2.6 The Reduced Configuration Space

Now comes a significant jump in subject and tone from the preceding, which focused on the standard, textbook approach to dealing with identical particles in non-relativistic quantum mechanics: one starts by assuming that one knows how to quantize a system of distinguishable particles; one then takes the elements of this quantum theory, whether these were wave-functions or transition amplitudes, and imposes particle identity as a form of additional symmetry, which leads to constraints as to the types of statistics possible. This is the most common way of addressing identical particles.

However, as we noted in Chapter 1, three foundational papers, Souriau [3], Laidlaw and DeWitt-Morette [4] and Leinaas and Myrheim [5], took a different approach. They assumed that the classical particles were themselves identical, and then derived the resulting properties of identical quantum particles from the nuances of the quantization of an already fully symmetrized system. Ref. [5] argued most strongly for this point, responding to the questioning of the very idea of particle labels raised in Mirman [29], and citing the Gibbs paradox [65] as additional evidence. These papers initiated the the reduced configuration space approach. This was presaged by a discussion at the 5th Solvay Conference in 1927, where Einstein and Lorentz noted a problem with using the full configuration space for \( N \) particles, and Pauli responded with a solution involving field theory (see Bacciagaluppi and Valentini [66, pp. 181–183, 442–443]). Nevertheless, our discussion will continue to restrict itself to the non-relativistic regime, with a constant number of particles.

Since the developments in this approach rely heavily on the properties of the space of the classical particles, they had to make that an explicit parameter of their analysis, choosing: \( \mathbb{R}^n \), for \( n \geq 1 \). All three papers reproduce the symmetrization postulate for identical particles in \( \mathbb{R}^3 \), considered the most physical single-particle configuration space. In fact, the Symmetrization Postulate is recovered for any \( n \geq 3 \). However, the situation is different in two and one dimensions. This is merely referred to in passing in Refs. [3, 4], but is developed at length in Ref. [5]. Their treatment shows that in those cases, there is a continuous interpolation between fermions and bosons. The two-dimensional case in particular leads to particles dubbed anyons. They capture the properties of excitations in two-dimensional quantum systems, and their physical manifestation is found in the Fractional Quantum Hall Effect [32–36].

As already presaged in Schulman [67], where it was noted that the fundamental group for a twice-punctured plane is non-abelian, further theoretical development of this approach included the incorporation of parastatistics, in this case particles existing in multiple-dimensional representations.
of the braid group. Another direction was the extension from \( \mathbb{R}^n \) to circles, tori and spheres. These, and the use of more advanced mathematical tools, are summarized in Imbo et al. [68].

In the next section, we will examine the clearest expression of anyons in Nature, the fractional quantum Hall effect for low-temperature two-dimensional systems.

### 2.7 The Fractional Quantum Hall Effect and Anyons

The Fractional Quantum Hall Effect furnishes the most reliable evidence for the existence of anyons in Nature. We will build up to them, starting from the Classical Hall Effect, for which the only form of quantization needed is the fact that charges are discrete, passing through the Integer Quantum Hall Effect, in which electrons are quantum-mechanical, but non-interacting, and finally to the Fractional Quantum Hall Effect, in which electron-electron interaction is fundamental, and over which anyonic excitations can form.

The Hall effect [69] is the existence of a transverse voltage drop over an approximately flat conductor in the direction perpendicular to a magnetic field passing through it. After charges realign so that the system is in a steady state, charges are expected to move along the wire, so that the forces in the transverse direction, being the Lorentz force from the magnetic field in one direction, and the electrostatic force from the inhomogeneous charge distribution in the other, will cancel out. This results in a Hall voltage:

\[
V_{\text{Hall}} = \frac{IB}{qnh},
\]

where \( I \) is the current, \( B \) is the magnetic field component perpendicular to the plane of the wire, \( q \) is the unit charge of the current carriers, \( n \) is the density of the carriers in the material, and \( h \) is the wire’s height. (See Figure 2.1)

![Figure 2.1: Negative charge carriers passing horizontally through a conductor subject to a magnetic field pointing towards the page create a vertical Hall voltage.](image)

This can be rewritten as the Hall coefficient:

\[
R_{\text{Hall}} \triangleq \frac{V_{\text{Hall}}h}{IB} = \frac{1}{nq},
\]

so that macroscopic measurements in the middle expression is a measure of an important property of the material on the right: namely, the charge density of the carriers, and very importantly, their sign.

In usual conductors, such as copper, the sign is negative, as clearly current is carried by electrons, so \( q = -e \). One indication that the situation in semiconductors is peculiar, and requires the use of the quantum theory of solids, is the existence of positive Hall coefficients, meaning that they act as
if they have positively-charged current carriers, even though the positive ions do not move around. The solution is that holes, or absences of electrons in almost-filled energy bands, act like positive carriers.

Nevertheless, this is insufficient for understanding the Quantum Hall Effects. As their name would suggest, they depend on the explicit quantum behavior of electrons in solids under a magnetic field. Let us start with the Integer Quantum Hall Effect.

In the classical case, we expect the Hall conductance $\sigma_{Hall}$ to depend linearly on the reciprocal of the external magnetic field $1/B$:

$$\sigma_{Hall} = \frac{I}{V_{Hall}} = \frac{qnh}{B}; \quad (2.10)$$

however, at low temperatures, and under sufficiently strong magnetic fields, the situation becomes quite different. Let us rewrite the expression in terms of the London unit of flux ($\Phi_q \triangleq 2\pi hc/q$):

$$\sigma_{Hall} = \frac{qnh}{B} = \frac{q(N/V)h}{B} = \frac{qN}{BA} = \frac{qN}{k\frac{2\pi hc}{q}} = \frac{N}{k} \cdot \frac{q^2}{2\pi hc}, \quad (2.11)$$

where $A$ is the area of the conductor in the direction perpendicular to the magnetic field.

The ratio $\nu \triangleq N/k$, loosely the number of electrons per flux quantum, is called the filling factor. When this is an integer, it means that a Landau level is full. Now, if the flat wire were completely ordered and pure, then we would not expect there to be any difference between this situation and a classical one, as is argued in Laughlin [36], since we should be able to substitute any current with a relativistic boost of a resting wire, so quantum phenomena would not enter into it. However, if there are imperfections in the sample, these cause there to be localized states, which create gaps between the Landau levels. This leads to plateaus, as there are regions in which an increase in the magnetic field does not lead to additional empty states in the next Landau level. These plateaus are the manifestation of the integer quantum Hall effect measured by von Klitzing et al. [70]. Notice that the discussion so far has treated electrons as non-interacting fermions (hence the importance of filled or unfilled Landau levels). Under these circumstances, the integer effect is the most we can comprehend. In order to explain the fractional effect, we will need to incorporate collective behavior.

Following the exposition in Stormer [35], for a non-integer $\nu = N/k$, we can think of the system in the two-dimensional conductor as consisting of $N$ electrons per $k$ vortices generated by flux quanta; furthermore, the vortices represent a many-particle configuration in which there is a gap in the electron cloud. If we take Coulomb interactions into account, it is energetically favorable for electrons to coincide with these vortices, and the more vortices surrounding each electron, the farther away are the rest of the electrons, so the lower the energy. The lowest-energy configuration will have the vortices spread evenly between the electrons. For example, for a filling factor of $\nu = \frac{1}{3}$, the case measured in Tsui et al. [32] and theoretically explained by Laughlin [33, 71], the favored state has three vortices surrounding each electron. The resulting electrons with fluxes are called composite particles.

The flux quanta generating these vortices have important consequences for statistics. Each flux quantum changes the sign of an exchange. So an electron (a fermion) with an odd number of flux quanta becomes a boson, while if it has an even number of flux quanta, it remains a fermion. In the $\nu = \frac{1}{3}$ case, the number of flux quanta is odd, so we get composite bosons. That means that they can condensate, and we get a very stable multi-particle ground state. Like in the integer case, there is a plateau, hence the fractional quantum Hall effect. Note that for $\nu = \frac{1}{2}$, each electron would be coupled with two flux quanta at the ground state; this would make the composite particle
a fermion, and these fill up the Landau levels until the Fermi energy. The situation is similar to
that at zero field, so the system behaves as then: without a plateau.

So far we have only discussed composite particles, which are fermions with flux quanta transform-
ing them either into bosons or back into fermions. The objects which have fractional statistics
are quasiparticles which are excitations of the condensate. We again discuss the $\nu = \frac{1}{3}$ case.
Suppose we change the magnetic field a little. Then we may increase it, adding a flux quantum,
equivalent to a deficit of one-third of an electron, or a quasiparticle of charge $+\frac{1}{3}e$. Alternately,
if the magnetic field is slightly less than necessary for this filling factor, then a few electrons will
be missing their third flux, so there will be surpluses of thirds of electrons, or quasiparticles of
charge $-\frac{1}{3}$. As shown in Halperin [72, 73] and Arovas et al. [34], these particles have an anyonic
phase of $e^{i\pi/3}$ (and generally, for an excitation over filling factor $\nu$, the anyonic phase is $e^{i\pi\nu}$). It is
also there that the connection with Ref. [30] and, in the latter, with Ref. [5], is made.

Since the fractional quasiparticles are excitations of collective states of a material, they cannot
simply be extracted, interfered with one another, and then detected in convenient bubble chambers
in order to establish their statistics. Instead, interference must happen within the same two-
dimensional material, at low temperature, and the magnetic field needed to establish the condensate
over which they arise. It took more than two decades, but this was eventually achieved by two
groups: Camino et al. [37] and Kim et al. [38]. The former provides particular credence to promoting
distinguishable anyons, as the evidence is provided through the interference of one $\nu = \frac{1}{3}$ particle
around a tube full of $\nu = \frac{2}{5}$ particles. This is seen as evidence of anyon “statistics” even though these
particles are distinguishable. This is another example of the fact that, at least in the experimental
literature, “statistics” is used interchangeably with a topological interaction between the particles,
rather than some property hinging on particle identity, or the reduction of the configuration space.
We will pursue anyonic behavior for distinguishable particles through strictly topological tools
in Section 4.1.

But is this evidence of anyons, or just of excitations which act anyonically? After all, these
are not fundamental particles, but quasiparticles over a complicated ground-state. Their existence
depends upon interactions between more fundamental particles, namely electrons and ions in a solid,
as well as an external magnetic field. However, even in Quantum Field Theory, which is the theory
where many of what we call fundamental particles are expressed, there is an underlying assumption
that it is an effective theory of excitations over something more fundamental. That is the basis
for renormalization: the theory does not express behavior all the way to arbitrarily high energies
or arbitrarily short distances, so it is valid to simply stop integration at a certain maximal energy,
in order to avoid the ultra-violate divergences. In that sense, the fractional statistics excitations
over the Fractional Quantum Hall Effect ground state are explained by an anyonic effective theory,
and are measurable, so they are evidence of such particles existing, and therefore any foundational
approach to quantum mechanics would have to be able to take that into account.

In order to facilitate our ability to do so, we will introduce a simpler example of the importance
of topology in quantum mechanics, the Aharonov-Bohm effect, followed by an introduction to
relevant mathematical tools necessary for the rest of this Part, namely path homotopy and the
fundamental groupoid.

### 2.8 The Aharonov-Bohm Effect

We now step back from the 1980s to the late 1950s. One of the first hints that global aspects of
the configuration space create curious situations in quantum mechanics came from the Aharonov-
Bohm effect [74]. Unlike their lesser-known predecessor Ehrenberg and Siday [75], covering similar
ground, Ref. [74] cited topological concerns explicitly. The issue in question was a peculiar property of electromagnetism in the quantum formalism. Recall that in classical electromagnetism, it is the fields $E$ and $H$ which are fundamental and appear in the equations of motion:

$$m\ddot{\mathbf{r}} = e\mathbf{E} + \frac{e}{c}(\dot{\mathbf{r}} \times \mathbf{B}),$$

(2.12)

while the potentials $\phi$ and $A$, related to the former through:

$$\mathbf{E} = -\nabla\phi \text{ and } \mathbf{A} = \nabla \times \mathbf{A},$$

(2.13)

are relegated to formal convenience when expressing such systems in the Lagrange or Hamilton formalisms. In quantum mechanics, however, it is in fact the Hamiltonian which appears in the main dynamical equation, whether that of Schrödinger or of Heisenberg, and similarly it is the Lagrangian which appears in the Feynman path integral approach. Both of these feature the potentials directly.

Now, quantum mechanics is gauge invariant; in the Schrödinger formalism, this is expressed by saying that if $\Psi$ is a solution of:

$$i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{r}) = \left[\frac{1}{2m} \left(-i\hbar \nabla - \frac{q}{c} \mathbf{A}(\mathbf{x},t)\right)^2 + q\phi(\mathbf{x},t)\right] \Psi(\mathbf{r}),$$

(2.14)

then $\Psi' = e^{i\frac{q}{c}\Lambda} \Psi$ is a solution of:

$$i\hbar \frac{\partial}{\partial t} \Psi'(\mathbf{r}) = \left[\frac{1}{2m} \left(-i\hbar \nabla - \frac{q}{c} \mathbf{A}'(\mathbf{x},t)\right)^2 + q\phi'(\mathbf{x},t)\right] \Psi'(\mathbf{r}),$$

(2.14')

with:

$$\mathbf{A}' = \mathbf{A} + \nabla \Lambda \text{ and } \phi' = \phi - \frac{1}{c} \frac{\partial}{\partial t} \Lambda,$$

(2.15)

for arbitrary $\Lambda(\mathbf{x},t)$. Since the only effect on the wave-function is multiplying by a phase, the potentials themselves are not significant directly; however, in some cases one is forced to have a non-zero potential in a region where no field operates; for example, around an isolated solenoid containing a magnetic field. Does this field then have a physical effect, even though the electron is excluded from the region in which the field itself acts?

The surprising answer is that it does, in fact, have an effect: the Aharonov-Bohm effect. The interference pattern created by an electron scattering around an isolated solenoid depends in a measurable way on the flux through this same solenoid. Let us show how this comes about.

We start with the Schrödinger formalism. Let us use a setup as in Figure 2.2. We have an electron emitter $E$ on the left, and an array $D$ of electron detectors on the right. In between, we have a screen $S$, presumed to extend arbitrarily high and low, with two slits, $A$ and $B$, between which is embedded a solenoid which is isolated from the electrons, and through which passes a magnetic flux, controlled by adjusting the current running through it. Finally, we single out some point $x_i$ on the left side of the screen, and a representative point $x_f$ at one of the detectors.

The wave-function value for a point $x_f$ on the screen for the packet created when $B$ is closed and there is no flux is $\Psi^{(0)}_A(x_f)$, while when $A$ is closed we have $\Psi^{(0)}_B(x_f)$. Due to the superposition principle, when both slits are open, the total wave-function value will be:

$$\Psi^{(0)}(x_f) = \Psi^{(0)}_A(x_f) + \Psi^{(0)}_B(x_f).$$

(2.16)

Now we perform the same experiment with a current running through the solenoid, creating a total flux $\Phi_{\text{exc}}$, while there is still no field outside of the excluded area. With $B$ blocked, there are
Figure 2.2: $E$ is an electron emitter, $D$ is an array of electron detectors on the right, $S$ is a screen with two slits, $A$ and $B$, and a solenoid which is isolated from the electrons, and through which there passes a magnetic flux $\Phi$ controlled via the current. $x_i$ is an arbitrary point near the emitter, and $x_f$ is the location of a detector.

no longer any holes in the region allowed for the electrons; furthermore, there is no magnetic field in this region, so that the vector potential satisfies:

$$\nabla \times \mathbf{A} = 0;$$  \hspace{1cm} (2.17)

this means that we can apply Stokes’ theorem to show that the line integral of the vector potential between two points only depends on the end-points, or, equivalently, that the vector potential is a conservative field, and is generated by a scalar function:

$$\mathbf{A} = \nabla \Lambda_A,$$  \hspace{1cm} (2.18)

where we choose:

$$\Lambda_A(x) \triangleq \int_{x_i}^{x_f} \mathbf{A}_{exc}(x') \cdot dx'.$$  \hspace{1cm} (2.19)

We can then use a gauge transformation to express the effect on the wave-function at the detector through multiplication by an appropriate phase depending on this generating function:

$$\Psi_A^{(\Phi_{exc})}(x_f) = e^{i\frac{\Phi}{\hbar c} \Lambda_A(x_f)} \Psi_A^{(0)}(x_f);$$  \hspace{1cm} (2.20)

similarly:

$$\Psi_B^{(\Phi_{exc})}(x_f) = e^{i\frac{\Phi}{\hbar c} \Lambda_B(x_f)} \Psi_B^{(0)}(x_f),$$  \hspace{1cm} (2.21)

where:

$$\Lambda_B(x) \triangleq \int_{x_i}^{x_f} \mathbf{A}_{exc}(x') \cdot dx'.$$  \hspace{1cm} (2.22)

Using simple superposition, we get for the total wave-function:

$$\Psi_T^{(\Phi_{exc})}(x_f) = e^{i\frac{\Phi}{\hbar c} \Lambda_A(x_f)} \Psi_A^{(0)}(x_f) + e^{i\frac{\Phi}{\hbar c} \Lambda_B(x_f)} \Psi_B^{(0)}(x_f) =$$

$$= e^{i\frac{\Phi}{\hbar c} \Lambda_A(x_f)} \left[ \Psi_A^{(0)}(x_f) + e^{i\frac{\Phi}{\hbar c} \Lambda_B(x_f) - \Lambda_A(x_f)} \Psi_B^{(0)}(x_f) \right].$$  \hspace{1cm} (2.23)

Now (see Figure 2.3):

$$\Lambda_B(x_f) - \Lambda_A(x_f) = \int_{x_i = x_f} \mathbf{A}_{exc}(x) \cdot dx = \iint_{S} \nabla \times \mathbf{A}_{exc}(x) \cdot da = \iint_{S} \mathbf{B} \cdot da = \Phi_{exc},$$  \hspace{1cm} (2.24)
so:

\[ \Psi_T^{\Phi_{\text{exc}}}(x_f) = e^{\frac{i\pi}{\hbar} \Lambda_A(x_i)} \left[ \Psi_A^{(0)}(x_f) + e^{2\pi i (\Phi_{\text{exc}}/\Phi_q)} \Psi_B^{(0)}(x_f) \right] ; \]  

(2.25)

we see that the magnetic flux in the solenoid, which the electron does not encounter directly, nevertheless can be probed by the electron through the interference pattern on the detector array. This effect has the period \( \Phi_0 \equiv \frac{2\pi \hbar c}{q} \), called London’s unit of flux, which we mentioned in Section 2.7.

Figure 2.3: Subtracting \( \Lambda_A \) from \( \Lambda_B \) results in a closed line integral around the flux, which leads to a surface integral including the flux via Stokes’ theorem.

Let us now show how the path integral formalism, which we will be using extensively later, is applied to this case. Recall that there, the focus is on calculating propagators:

\[ \Psi_f(x_f, t_f) = \int K(x_f, t_f; x_i, t_i) \Psi_f(x_i, t_i) d^3x, \]  

(2.26)

that is, \( K \) is the transition amplitude between one point to the other. This is calculated through the path integral:

\[ K(x_f, t_f; x_i, t_i) = \int \exp \left( \frac{iS[x(t)]}{\hbar} \right) \mathcal{D}\{x(t)\}, \]  

(2.27)

where \( \mathcal{D}\{\cdot\} \) is used instead of \( d\cdot \) in the integral in order to emphasize the sum over paths. The action functional \( S \) is defined as:

\[ S[x(t)] = \int_{t_i}^{t_f} L(x(t), \dot{x}(t), t) dt. \]  

(2.28)

For a charged particle in a general electromagnetic field affecting it directly, denoted by the scalar potential \( \phi_0 \) and the vector potential \( A_0 \), along with an additional time-constant magnetic field excluded from the space of the particle, denoted by \( A_{\text{exc}} \):

\[ L(x, \dot{x}, t) = \frac{m}{2} \dot{x}^2 - q\phi_0(x, t) + \frac{q}{c} [A_0(x, t) + A_{\text{exc}}(x)] \cdot \dot{x}, \]  

(2.29)

Since the vector potential for the excluded flux is time-independent, if we integrate the Lagrangian into an action, we get an additive term depending only on the path in space, rather than
on the path in space-time:

\[ S[x(t)] = \int_{t_i}^{t_f} \left( \frac{m}{2} \dot{x}^2 - q\phi_0(x, t) + \frac{q}{c} A_0(x, t) \cdot \dot{x} + \frac{q}{c} A_{exc}(x) \cdot \dot{x} \right) dt = \]

\[ = \int_{t_i}^{t_f} \left( \frac{m}{2} \dot{x}^2 - q\phi_0(x, t) + \frac{q}{c} A_0(x, t) \cdot \dot{x} \right) dt + \frac{q}{c} \int_{x_i}^{x_f} A_{exc}(x) \cdot dx. \] (2.30)

If we now define the first term, which only depends on the particle’s behavior and the fields acting upon it, as \( S_0[x(t)] \), and see the integral in the second term as a generalization of the \( \Lambda \) functions from earlier into a functional \( \Lambda[x(t)] \), and insert these into the exponent in Eq. (2.27), we get the following suspiciously familiar result:

\[ K(x_f, t_f; x_i, t_i) = \hat{\exp} \left( i \hbar S_0[x(t)] \right) \cdot \exp \left( i q \hbar c \Lambda[x(t)] \right) D\{x(t)\}; \] (2.31)

Now, in Figure 2.4, we return to the setup above, this time dispensing with the screen and slits.

Figure 2.4: Electrons emanate from the emitter \( E \) to the detectors \( D \), while being excluded from a solenoid containing a flux \( \Phi \). Two distinct families of paths are illustrated: the line integral for those above is \( \Lambda_A \), for those below \( \Lambda_B \).

Suppose we take a path \( x_A(t) \) which passes above the solenoid, without winding around it, like one of those included in \( \Lambda_A \) in Figure 2.4, and compare it to another path, \( x_A'(t) \), which can be smoothly deformed into it, while still having the same end-points. Then if we look at the surface \( S_A \) between them, it does not contain any net flux, meaning that, using a process similar to Eq. (2.24):

\[ \Lambda[x_A'(t)] - \Lambda[x_A(t)] = \oint_{\partial S_A} A_{exc}(x) \cdot dx = \int_{S_A} B \cdot da = \Phi(S_A) = 0. \] (2.32)

Therefore, we can ascribe a constant value, which we will call \( \Lambda_A \), to all paths which can be smoothly deformed into one that goes from above. Similarly, we can do the same for all paths that pass below without winding, overall, that is, which can be smoothly deformed into one of those marked by \( \Lambda_B \) in Figure 2.4, again, while retaining the end-points \( x_i \) and \( x_f \).

However, if we compare a path from below to a path from above, we will get, as in Eq. (2.24), that:

\[ \Lambda[x_B(t)] - \Lambda[x_A(t)] = \Phi_{exc}. \] (2.33)
In fact, there is an infinity of classes of paths. Each class is defined by the winding number it has in relation to the paths with $\Lambda_A$, or number of times that it is going around the flux in the center, equivalent to the times it counts the excluded flux in its net flux. If we define $K^A_0(x_f, t_f; x_i, t_i)$, $K^B_0(x_f, t_f; x_i, t_i)$, etc., to be the path integral:

$$\int \exp \left( \frac{i}{\hbar} S_0[x(t)] \right) \mathcal{D}\{x(t)\}$$

(2.34)

limited to paths of the class $A$, $B$, etc., respectively, then we can break up Eq. (2.31) into a sum by these classes:

$$K(x_f, t_f; x_i, t_i) = e^{i\alpha A} K^A_0(x_f, t_f; x_i, t_i) + e^{i\alpha B} K^B_0(x_f, t_f; x_i, t_i) + \cdots.$$  

(2.35)

If we remove the explicit dependence on start and end points for brevity, we get a result similar to that in Eq. (2.25):

$$K = e^{i\alpha A} \left[ K^A_0 + e^{i\alpha (\Lambda_B-\Lambda_A)} K^B_0 \right] + \cdots = e^{i\alpha A} \left[ K^A_0 + e^{2\pi i (\Phi_{\text{exc}}/\Phi^0_{\text{exc}})} K^B_0 \right] + \cdots.$$  

(2.36)

the magnitude of the probability amplitude therefore oscillates with period $\Phi^0_{\text{exc}}$, as we saw from the Schrödinger picture.

An advantage of the path integral approach is that it is easier to get to the crux of the matter. For let us return to the reason that the vector potential $A$ cannot be identically zero outside the solenoid. The field is zero, and $B = \nabla \times A$, so a null vector potential would be consistent with this requirement. Furthermore, Stokes’ theorem, relating the line integral of a vector field around a region with the surface integral of the curl of this field through that region would suggest that a zero potential will be valid wherever this theorem would apply, as the line integral should be zero anyway. However, when we take a path around the solenoid, regardless of whether the electron passes through it, the electromagnetic field does exist within it, so that a use of Stokes’ theorem results in the line integral equaling the total flux through the solenoid, times the number of turns of the path. Furthermore, this total flux is a parameter that affects the interference pattern of an electron scattered through this system. In effect, the electron is probing a portion of the physical electromagnetic field with which it does not interact directly, simply because this field goes through a “hole” in this region, so that without it the region is no longer simply-connected, a term we shall explain in the next section.

Now, in itself this is merely another piece of evidence for the non-locality of quantum mechanics. Moreover, the closed line integral itself, depending as it does on field values, is already a gauge-invariant quantity in the classical theory as well, so that is not an issue. However, it does raise the question of what happens when the configuration space of a system is really multiply-connected. What if we find a space in which there are real, intrinsic holes, where absolutely nothing, neither fields nor matter, exists? Then there seems to be an ambiguity in the definition of the problem. If we take the physical fields and extend them to the hole, then depending on what flux this creates in the now “filled-out” region, we could have different constraints on the behavior outside. Initial conditions and fields in the configuration space are no longer enough information; we must add a new piece of information, in this case, the line integral of the vector potential around each “hole”. But is that always the case? What if the space cannot be described as a plane with holes in it? For example, the topological group of rotations in three dimensions, SO(3), is doubly-connected; that is, if you take a path once around an axis, Stokes’ theorem does not apply, but compounding this path with itself leads to one in which this can again be done. What is needed is a language which generalizes the notion of “holes”, and then an approach to quantum mechanics which will allow us to quantize systems with such generalized spaces. That language is homotopy theory, and these spaces are multiply-connected configuration spaces.
2.9 Homotopy and the Fundamental Groupoid

In the previous section, there was a property of paths in the configuration space which made it possible to conclude that the line integral of the potential around the them was zero. It seems to have had to do with the fact that the surface that they captured between them was completely within the space, and we knew we could apply Stokes’ theorem to show that the properties of the curl inside that surface are what matters. Another way of putting it is that these paths could be gradually deformed one into the other, while retaining the same endpoints. This is not the case for the paths on opposite sides of the solenoid. This is a special case of a more general connection between paths which is called homotopy (see Refs. [76, 77]). Here we will give a short review of the subject, while focusing on an uncommonly used algebraic structure, the fundamental groupoid, to be defined below (see Ref. [78]).

We start with a space $X$ which has some topology which we will keep implicit. Let us single out two points in that space, $a$ and $b$, which may be identical. A path from $a$ to $b$ is a smooth function from the unit interval to the space:

$$q : [0, 1] \rightarrow X$$

such that $q(0) = a$ and $q(1) = b$. We then write:

$$s(q) \triangleq q(0), \quad t(q) \triangleq q(1),$$

the former being the path’s source, from which it is coming, the latter its target, to which it is going. We say that two paths, $q$ and $q'$, with the same source and target are homotopic to each other if they can be smoothly deformed one into the other, while retaining the same end-points. Mathematically speaking, this means that there is a smooth function:

$$F : [0, 1] \times [0, 1] \rightarrow X$$

such that $F(t, 0) = a$, $F(t, 1) = b$, $F(0, t) = q(t)$, and $F(1, t) = q'(t)$. This is an equivalence relation, and we can talk about the homotopy classes of paths between $a$ and $b$; that of the path $q$ is $[q]$, so this property can be written as:

$$[q] = [q']$$

that is, the homotopy classes of $q$ and $q'$ are equal (see Figure 2.5).

When the two paths are not homotopic, we instead write:

$$[q] \neq [q'],$$

that is, the homotopy classes of the two paths are different (see Figure 2.6).
Figure 2.6: Non-homotopic paths from a to b: \([q] \neq [q']\).

For a homotopy class \([q]\), we extend the definitions for \(s\) and \(t\) in Eq. (2.38) to:

\[
s([q]) \triangleq s(q), \quad t([q]) \triangleq t(q);
\]

(2.42)

these are well-defined because paths in the same homotopy class share their source and target.

For convenience, if \(x\) is any point in \(X\), then \([x]\) is the homotopy class of the constant path at \(x\).

Given two paths, \(q\) and \(p\), such that \(t(q) = s(p)\), their concatenation \(qp\) is defined as:

\[
(qp)(t) = \begin{cases} 
q(2t), & 0 \leq t \leq \frac{1}{2} \\
p(2t - 1), & \frac{1}{2} < t \leq 1
\end{cases}
\]

(2.43)

with \([qp] = [q][p]\); this operation can be shown to be associative (see Figure 2.7). The homotopy

classes of a space \(X\) with concatenation form an algebraic structure, called its fundamental groupoid, and written as \(\Pi(X, X)\). Let us go on a short detour to explain this structure, as it is not commonly known.

The shortest definition of a groupoid is that it is a category where every morphism is invertible. In our case, the objects of the category would be the points of the space \(X\), and the morphisms between two points are the homotopy classes of paths between them.

In more group-theoretic terms, a groupoid is a collection \(\{\alpha\}\) on which a partial operation is defined, usually designated by a product, as are a source function \(s(\bullet)\) and a target function \(t(\bullet)\) projecting to another space \(X\), such that:

1. \(\alpha \beta\) is defined if and only if \(s(\beta) = t(\alpha)\).

2. The operation is associative: whenever \((\alpha \beta) \gamma\) is defined, then so is \(\alpha(\beta \gamma)\), and they are equal.

3. For each \(\alpha\) there is a reverse \(\alpha^{-1}\), such that \(\alpha \alpha^{-1} = \I_{s(\alpha)}\) and \(\alpha^{-1} \alpha = \I_{t(\alpha)}\), where \(\I_{x}\) is an element called the identity at \(x\), so that \(s(\I_{x}) = t(\I_{x}) = x\), and \(\I_{t(\beta)} \beta = \beta \I_{s(\beta)} = \beta\).

Normally we would call this the inverse, but inversion is also a symmetry of some very important spaces, one example of which is in Section 3.2. The term reverse has the advantage of being suitable for a discussion of paths.
For a more thorough discussion, see Higgins [78]. A proof of the necessity of $s$ and $t$ is in the Appendix.

We have already shown that the operation is well-defined and associative. To complete the list of requirements for a groupoid, we note that aside from concatenation, we also have reversal: for each path $q$ we may define a reverse, $q^{-1}$, as follows:

$$(q^{-1})(t) = q(1 - t),$$

and the two satisfy:

$$[qq^{-1}] = [s(q)]; \quad [q^{-1}q] = [t(q)],$$

so we may define $[q]^{-1} \triangleq [q^{-1}]$, in the sense that $[x]$ acts as $I_x$ (see Figure 2.8).

Let us now choose a point $x_0$ in $X$. The fundamental group based at that point is:

$$\Pi(X, x_0) = \{ [q] | s(q) = t(q) = x_0 \};$$

that is, the classes of all the paths which start and finish at that point. They will also be called loops (see Figure 2.9). Generally, for a subset $A \subseteq X$, $\Pi(X, A)$ is the subgroupoid of all $[q]$ where $s([q]), t([q]) \in A$. Another special case used in Ref. [4] is $\Pi(X, a, b)$, the collection of homotopy classes $[q]$ with $s([q]) = a$ and $t([q]) = b$.

Fundamental groups at different points are not entirely independent, as long as there is a path connecting them. This is always the case when the space is path-connected, which is a postulate common to Ref. [4] and to their successors. Indeed, let $x_1$ and $x_2$ be arbitrary points in $X$. Let $l$ be a loop based at $x_1$, that is, $s(l) = t(l) = x_1$, and let $q$ be a path such that $s(q) = x_1$ and $t(q) = x_2$. Then $q^{-1}lq$ is a loop based at $x_2$, and $[q^{-1}l][q]$ is an element of the fundamental group there, so $[q]$ induces a homomorphism between the fundamental groups, since if $l$ and $k$ are loops based at $x_1$, then:

$$[q^{-1}lkq] = [q^{-1}l][k][q] = [q^{-1}]l[q][q^{-1}][k][q] = [q^{-1}lq][q^{-1}kq];$$

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similarly, the reverse homotopy class $[q]^{-1}$ induces a homomorphism in the opposite direction, and:

$$([q]^{-1})^{-1}([q^{-1}]l[q])[q]^{-1} = [qq^{-1}lqq^{-1}] = [l];$$  (2.48)

therefore, the homotopy class $[q]$ induces an isomorphism between the two fundamental groups. For any two homotopy classes, $[q]$ and $[p]$, such that $s([q]) = s([p]) = x_1$ and $t([q]) = t([p]) = x_2$, the respective isomorphisms induced by them are related by:

$$[p^{-1}lp] = [p^{-1}qq^{-1}lqq^{-1}p] = [p^{-1}q][q^{-1}lq][p^{-1}q]^{-1},$$  (2.49)

which is an inner isomorphism, that is, of the form $\alpha \to \beta\alpha\beta^{-1}$, since $[p^{-1}q] \in \Pi(X, x_2)$. We will see in Chapter 3 that this leads to a simplification for abelian groups and for one-dimensional representations.

Finally, we can explain what we mean by simply-connected, multiply-connected, etc. Always speaking of connected, path-connected spaces, a simply-connected space is one whose fundamental group is trivial, that is, every closed loop in it can be contracted to a point, while a multiply-connected space is one whose fundamental group is non-trivial. So the two-dimensional plane without “holes” is simply-connected. A space with a hole, such as the space to which electrons are restricted in the Aharonov-Bohm setup, is multiply-connected. In fact, it is infinitely-connected, in the sense used in e.g. Ref. [79]: its fundamental group is isomorphic to an infinite group, in this case the infinite cyclic group $\mathbb{Z}$ of integers.

Note that the Feynman path integral approach is very well-suited for exploring these topological properties of spaces, as paths, rather than points, are its building blocks.
3.1 Motivation and Relation to Homotopy

Before we present the result of this chapter, let us provide a review of some work with which it contrasts, and from which it extracts some vital tools.

The path integral approach to multiply-connected spaces was originally introduced in Schulman [80], and further disseminated through Schulman [79]. Those works developed a justification for discrete spin under the path integral formalism. The configuration space which is quantized for that purpose, that of SO(3) taken as a topological group, is not simply-connected, and that is where the discussion of path integrals in multiply-connected spaces becomes necessary. The standard definition of the path-integral becomes ambiguous under these circumstances: in a singly-connected space, all paths between two given points are in the same homotopy class, so they are smoothly deformable into each other, and thus their contributions also go smoothly into each other. Therefore, aside from an overall phase ambiguity, the sum over paths making up the transition amplitude is defined as:

$$K(b,t_b;a,t_a) = \hat{D}\{x(t)\} \exp \left( \frac{iS\{x(t)\}}{\hbar} \right), \quad (3.1)$$

where $x(t)$ runs over all smooth paths between $a$ and $b$.

If there is more than one homotopy class, then the situation is different: paths in one class can no longer be smoothly deformed into those in another, so that neither do the phases, and we in fact have to split the integral into a sum over integrals of these components:

$$K(b,t_b;a,t_a) = \sum_{[q] \in \Pi(X,a,b)} \tilde{\chi}([q]) \tilde{K}^{[q]}(b,t_b;a,t_a), \quad (3.2)$$

with the partial amplitudes being defined as the path integral limited to the homotopy classes:

$$\tilde{K}^{[q]}(b,t_b;a,t_a) \triangleq \int \mathcal{D}\{x(t)\} : [x(t)] = [q] \exp \left( \frac{iS\{x(t)\}}{\hbar} \right), \quad (3.3)$$

and the weights $\tilde{\chi}([q])$ yet to be determined. Note that, unlike in Section 2.9, the parameterizations of the paths matter, which we indicate by writing them as $x(t)$. The homotopy class of the path $[x(t)]$ is then that of the path reparamaterized to the unit interval:

$$q(t) = x((t_b - t_a)t + t_a); \quad (3.4)$$

we will use this notation in Section 3.2, as well.

In our adapted notation, Eq. (3.2) is the result in Ref. [79] from which Refs. [4, 81] start their analysis. However, they do not parameterize the partial amplitudes with homotopy classes directly, but instead through members of the fundamental group at some arbitrary point $x_0$; hence the use of the tilde. Now, we have seen in Section 2.9 that the fundamental groups at different points for a path-connected space are related; this paper goes a step further, capturing all homotopy classes, that is, members of the fundamental groupoid, by just the fundamental group at a single point, $x_0$. This is done using a mesh, a choice of path from $x_0$ to each point $a$:

$$\mathcal{C} : X \to \{q | q : [0,1] \to X\} \quad (3.5)$$
\[ C(a)(0) = x, C(a)(1) = a. \]

This can be done because \( X \) is path-connected. Once a mesh has been chosen, a natural one-to-one correspondence between elements of the fundamental group and those of the fundamental groupoid, given the end-points, is given by:

\[
\begin{align*}
    f_{ab} : \Pi(X, x) &\rightarrow \Pi(X, a, b) \\
    f_{ab}(\alpha) &= [C^{-1}(a)]\alpha[C(b)],
\end{align*}
\]

where \( C^{-1}(a) \) is the reverse of the path \( C(a) \) (See Figure 3.1).

![Figure 3.1: Loops become paths through a mesh: \([q_i] = f_{ab}([\ell_i]) = [C^{-1}(a)][\ell_i][C(b)].\) ](image)

This works for any \( a \) and \( b \), allowing a consistent definition of the partial amplitudes, as well as their coefficients in the expression for the total amplitude, indexed by the fundamental group:

\[
\begin{align*}
    K^\alpha(b, t_b ; a, t_a) &\triangleq \tilde{K}f_{ab}(\alpha)(b, t_b ; a, t_a) \\
    \chi(\alpha) &\triangleq \tilde{\chi}(f_{ab}(\alpha))
\end{align*}
\]

providing them with an amended version of Eq. (3.2):

\[
K(b, t_b ; a, t_a) = \sum_{\alpha \in \Pi(X, x_0)} \chi(\alpha)K^\alpha(b, t_b ; a, t_a),
\]

as their new starting point. For them the partial amplitudes \( K^\alpha \) are black boxes: they do not use the formulas Eq. (3.1) or Eq. (3.3), so a significant portion of the paper is dedicated to investigating their properties as a stepping-stone to the more relevant results constraining \( \chi(\alpha) \). We will see in Section 3.2 that a direct investigation of the integrand of Eq. (3.1) will save us much of this trouble.

Returning to their method, we note that \( f \) respects the groupoid operation:

\[
f_{ab}(\alpha)f_{bc}(\beta) = f_{ac}(\alpha\beta); \tag{3.11}
\]

we find it helpful to see it as a parametrized local inverse to a homomorphism from the fundamental groupoid to the fundamental group at \( x_0 \):

\[
\begin{align*}
    g : \Pi(X, X) &\rightarrow \Pi(X, x_0) \\
    g([q]) &\triangleq [C(x([q]))][q][C^{-1}(t([q]))],
\end{align*}
\]

as their new starting point. For them the partial amplitudes \( K^\alpha \) are black boxes: they do not use the formulas Eq. (3.1) or Eq. (3.3), so a significant portion of the paper is dedicated to investigating their properties as a stepping-stone to the more relevant results constraining \( \chi(\alpha) \). We will see in Section 3.2 that a direct investigation of the integrand of Eq. (3.1) will save us much of this trouble.

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we find it helpful to see it as a parametrized local inverse to a homomorphism from the fundamental groupoid to the fundamental group at \( x_0 \):

\[
\begin{align*}
    g : \Pi(X, X) &\rightarrow \Pi(X, x_0) \\
    g([q]) &\triangleq [C(x([q]))][q][C^{-1}(t([q]))],
\end{align*}
\]
so \( g([q][q']) = g([q])g([q']) \) and:

\[
g(f_{ab}(\alpha)) = \alpha ; \quad f_{s([q])t([q])}(g([q])) = [q]. \tag{3.14}
\]

The main result of the work in Refs. [4, 81] is a proof that the weights \( \chi(\alpha) \) are a fundamental group representation. They then show that the choice of mesh does not change the physical predictions, but rather only the transition amplitudes by an overall phase, so that it is the choice of group representation which embodies the additional, physically meaningful topological degree of freedom.

However, this result has a limitation, which we have had to overcome, in order to apply these methods to the problem of distinguishable anyons. These anyons would be required to have a consistent topological phase for every simple, counter-clockwise exchange. Since such an exchange would not be a closed loop, methods which only provide topological phases representing members of the fundamental group will not be suitable.

To see this explicitly, let \( D(\bullet) \) be the fundamental group representation, and let there be a mesh \( C \) and base-point \( x_0 \) chosen as well, so that \( g \) is well defined. Then:

\[
\tilde{\chi}([q]) = D(g([q])), \tag{3.15}
\]

that is, a fundamental groupoid representation has implicitly been chosen as well.

We will now show that this choice does not generally allow the incorporation of symmetries of the space. We will do so for the important special case of the punctured plane, \( X = \mathbb{R}^2 \setminus \{0\} \), an example we will return to in Section 3.2, and will use in the next chapter. Let us define an inversion, “minus” operation, with \(-\langle a, b \rangle \equiv (-a, -b)\), where \(\langle a, b \rangle\) are Cartesian coordinates. We break a circular loop counter-clockwise around the origin into two pieces, \([q]\) and \([-q]\), which are related by a simple inversion symmetry (not to be confused with the reversed path, \(q^{-1}\), defined in Section 2.9):

\[
(-q)(t) = -(q(t)), \tag{3.16}
\]

as seen in Figure 3.2.

![Figure 3.2: A counter-clockwise circle is broken up into a half-circle, q, and its inverse, −q.](image)

We then get:

\[
\tilde{\chi}([q(-q)]) = \tilde{\chi}([q])\tilde{\chi}([-q]). \tag{3.17}
\]
Now, \([q(-q)]\) generates the fundamental group at \(x_0\). We can then treat it as isomorphic to \(\mathbb{Z}\) with addition, \([q(-q)]\) then corresponding to 1. Let us write: \(\tilde{\chi}([q(-q)]) = z\). This is the generator of the image of \(\mathbb{Z}\) under \(\chi(\bullet)\). In order for this representation to respect inversion symmetry, we require:

\[
\tilde{\chi}([-q]) = \tilde{\chi}([q]) = w,
\]

so:

\[
z = w^2. \tag{3.19}
\]

However, since \(z\) generates the group, we must have \(w = z^k\), giving

\[
z = z^{2k}, \tag{3.20}
\]

or

\[
1 = z^{2k-1}. \tag{3.21}
\]

Therefore, this is only possible for special choices of \(z\), namely odd roots of unity. If we would instead like to incorporate symmetries of the space for \(\text{any}\) value of \(z\), we need to explore more degrees of freedom for the groupoid representation, other than re-ordering the group representations through a different choice of mesh.

The next section (Section 3.2) will feature one of our novel results. We will start out by proving that the phases are a groupoid representation directly. We will then explore what the additional degrees of freedom are, beyond the choice of a fundamental group at a point. While they create physically indistinguishable solutions, they do allow the incorporation of more of the symmetries of the space at hand. The main simplifying tool in our treatment is that it starts from paths and their phases, rather than from partial propagators. It is bottom-up rather than top-down, and does not treat propagators as black boxes.

### 3.2 A Direct Path to the Fundamental Groupoid

The aim of this section is to improve upon the work in Refs. [4, 81], reviewed in the previous section, in a way which uses the fundamental groupoid, bearing in mind its most general representations. It will be seen that the result in Refs. [4, 81] is a special case of ours: we add a generalized gauge freedom. We will also provide an example of a groupoid representation which incorporates the rotational symmetry of the punctured plane, \(\mathbb{R}^2 \setminus \{0\}\).

We start by exploring the origins of the topological degrees of freedom. Instead of starting from the top, that is, from the propagator and the quantum formalism, as do Schulman [80] and its successors, we begin from the bottom, with Hamilton’s Principle in classical physics: the least action principle, or, more accurately, the extremal action principle.

Given an initial time \(t_1\) and point \(x_1\), and a final time \(t_2\) and point \(x_2\), we wish to find the path \(x_c(t)\) which is extremal (that is, a local minimum, maximum, possibly saddle) for the action functional:

\[
S\{x(t)\} = \int_{t_1}^{t_2} \mathcal{L}(x(t), \dot{x}(t), t) dt.
\]

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We use \{•\} to denote a functional, rather than the usual \([•]\), since we have already reserved the latter for homotopy classes. More formally:

\[
\delta S\{x_c(t)\} = 0, \tag{3.23}
\]

with the variation keeping \(t_1, t_2, x_1, x_2\) fixed. Therefore, the same path would be extremal, even if a value depending only on these constants was added:

\[
\delta S\{x(t)\} = \delta (S\{x(t)\} + \varsigma(x_2, t_2 ; x_1, t_1)). \tag{3.24}
\]

We can go even further: extrema are only modified if the action functional is changed in such a way that its value for two paths which can be smoothly deformed into each other is altered; therefore, we can have this parameter also depend on the homotopy class of \(x(t)\), and write

\[
\delta S\{x(t)\} = \delta (S\{x(t)\} + \varsigma_x(t)(x_2, t_2 ; x_1, t_1)), \tag{3.25}
\]

and this is independent of the specific Lagrangian, so long as it has the same singularity behavior (if applicable). In fact, we can write

\[
\varsigma_x(t)(x_2, t_2 ; x_1, t_1) = \varsigma_x(t)(t_2 ; t_1), \tag{3.26}
\]

as the end-points are implicit through \(s([x(t)]) = x_1\) and \(t([x(t)]) = x_2\), where the source and target functions \(s\) and \(t\) are as defined in Section 2.9.

Finally, let us address time. Earlier literature widely agrees that the topological parameter should be time-independent, with [41, 43, 45, 82–90] all making similar assumptions. However, returning to the punctured plane, the topological degree of freedom is equivalent to a flux through a solenoid at the origin. That can be slowly changed, leading from one representation of the fundamental group at a point to another. Gaveau et al. [91] go further to provide a solution for general changes of flux as a function of time. They find that the only strictly topological parameter is the enclosed flux at an arbitrary point in time. Since this result depends explicitly on the particular space being quantized, we expect a general solution to be more involved, and leave it to future work. We therefore choose to focus on investigating a parameter which only depends on the homotopy class of the path:

\[
\varsigma = \varsigma([x(t)]); \tag{3.27}
\]

Now we may calculate the appropriate Feynman amplitude for the path:

\[
\exp \left( \frac{i}{\hbar} (S\{x(t)\} + \varsigma([x(t)])) \right) = \exp \left( \frac{i}{\hbar} S\{x(t)\} \right) \cdot \chi([x]), \tag{3.28}
\]

where, to remain consistent with the terms in Refs. [4, 81], \(\chi([x(t)])\) is called the weight corresponding to the homotopy class \([x(t)]\), although we remove the “scare tilde”. In fact, since this total amplitude must obey the Feynman product rule, and since we already know that the first factor does, as it is simply that corresponding to the original action, then these weights also obey the Feynman rules.

We are no longer interested in the time-dependence of the paths, so we can revert to the lowercase \([q]\) notation introduced in Section 2.9. Combining concatenation, which gives \([qp] = [q][p]\), with the product rule of the weights gives us:

\[
\chi([q][p]) = \chi([qp]) = \chi([q])\chi([p]); \tag{3.29}
\]
meaning that $\chi(\bullet)$ is a representation of the fundamental groupoid. Working directly with the groupoid rather than the fundamental group allows us to more generally discuss the situation for arbitrary paths, instead of being forced to work strictly through loops, as in Refs. [4, 8]. As a subgroupoid of the fundamental groupoid, we have seen that fundamental groups in two points $x_1$ and $x_2$ are isomorphic, with isomorphisms induced by paths, and the relation between different isomorphisms being

$$[p^{-1}lp] = [p^{-1}q[q^{-1}l][q^{-1}]p^{-1}]^{-1};$$

(2.49)

this is an inner isomorphism, as we stated in Section 2.9. If the group is abelian, or if we limit ourselves to a one-dimensional representation $\chi$, all inner isomorphisms collapse into the identity. In the former case:

$$[p^{-1}q][q^{-1}l][p^{-1}q]^{-1} = [p^{-1}q][p^{-1}q]^{-1}[q^{-1}l]q = [q^{-1}l]q

(3.30)

while in the latter case:

$$\chi([p^{-1}q][q^{-1}l][p^{-1}q]^{-1}) = \chi([p^{-1}q])\chi([q^{-1}l])\chi([p^{-1}q]^{-1})
= \chi([p^{-1}q]^{-1})\chi([p^{-1}q])\chi([q^{-1}l]) = \chi([q^{-1}l]),$$

(3.31)

so all isomorphisms between $x_1$ and $x_2$ induced by paths are identical. Moreover:

$$\chi([q^{-1}l]q) = \chi([q]^{-1}[l][q]) = \chi([q]^{-1})\chi([l])\chi([l])
= \chi([q][q]^{-1}[l]) = \chi([q][q]^{-1})
= \chi([x_1][l]) = \chi([l]).$$

(3.32)

This means that once a fundamental group representation is chosen at one point, it has been set for all other points. However, this is not sufficient for describing the groupoid representation, since we have only set the representations for loops, while we have to assign values to $\chi([q])$ when $s([q]) \neq t([q])$. As we noted in Eq. (3.15), the framework in Ref. [4] provides one way of assigning these weights. We intend to generalize that framework, using the fact that we already have the groupoid representation structure, in order to catalog all possible representations, allowing us to choose one that implements the symmetry of the space. To that end, we must introduce the notion of compatibility of representations. By saying that two representations $\chi$ and $\chi'$ of the fundamental groupoid are compatible we mean that, for any given $a$ and $b$, there is phase $e^{i\delta(a,b)}$ such that:

$$\chi'([q]) = e^{i\delta(a,b)}\chi([q]),$$

(3.33)

for all $[q]$ such that $s([q]) = a$ and $t([q]) = b$; that is, $e^{i\delta(a,b)}$ only depends on the end-points.

When two representations are compatible, each one induces a propagator which results in the same physical predictions as the other, as an overall phase is removed by taking the absolute value:

$$\left|\sum_{[q]} \chi'([q])\mathcal{K}([q])\right| = \left|\sum_{[q]} e^{i\delta}\chi([q])\mathcal{K}([q])\right| = \left|\sum_{[q]} e^{i\delta}\chi([q])\mathcal{K}([q])\right| = \left|\sum_{[q]} \chi([q])\mathcal{K}([q])\right|$$

(3.34)

Let us now explore exactly what kind of leeway we have with a fundamental groupoid representation once we have chosen a fundamental group representation, $D(\bullet)$. As we saw earlier, only paths between distinct points $a$ and $b$ can have any ambiguity to them. Let us choose a homotopy class $[p_0]$ from $a$ to $b$. Then any path class $[q]$ between these two points can be related to
a loop around \( \mathbf{a} \) through \([q] \mapsto [qp^{-1}]\). The weight of this object is set by the fundamental group representation:

\[
D([qp^{-1}]) = \chi([qp^{-1}]) = \chi([q])\chi([p^{-1}]),
\]  

(3.35)

or:

\[
\chi([q]) = D([qp^{-1}])\chi([p]);
\]  

(3.36)

note that this is self-consistent, as \( D([pq^{-1}]) = D(I_a) = 1. \)

Naïvely it would seem that, for each pair of points \( \mathbf{a} \) and \( \mathbf{b} \), we are free to choose an arbitrary class \([p_{a \rightarrow b}]\), followed by a value \( \chi([p_{a \rightarrow b}]) \).

This is not the case, however. First, this would lead to over-counting: as we have already chosen a fundamental group representation, a choice of \( \chi([p_{a \rightarrow b}]) \) is equivalent, for any other class \([p'_{a \rightarrow b}]\) between these two points, to a choice of \( \chi([p'_{a \rightarrow b}]) \):

\[
\chi([p'_{a \rightarrow b}]) = D([p'_{a \rightarrow b}]^{-1}[p_{a \rightarrow b}])\chi([p_{a \rightarrow b}]).
\]  

(3.37)

Furthermore, the groupoid structure imposes severe limitations on our freedom. If we made the following choices:

\[
[p_{a \rightarrow b}], \chi([p_{a \rightarrow b}]); \ [p_{b \rightarrow c}], \chi([p_{b \rightarrow c}]); \ [p_{a \rightarrow c}], \chi([p_{a \rightarrow c}]);
\]  

(3.38)

then they would be subject to the constraint:

\[
\chi([p_{a \rightarrow b}])\chi([p_{b \rightarrow c}]) = \chi([p_{a \rightarrow b}][p_{b \rightarrow c}]) = \chi([p_{a \rightarrow c}])D([p_{a \rightarrow c}]^{-1}[p_{a \rightarrow b}][p_{b \rightarrow c}]),
\]  

(3.39)

so even with proper counting, only two of the three can be freely chosen.

In short, we have a big set of degrees of freedom, but they are subject to over-counting and interdependencies. Fortunately, Ref. [4] furnishes us with a way of organizing them: by using the base-point and mesh formalism we reviewed in Section 3.1, but in a more generalized fashion.

If we single out any one point \( \mathbf{x}_0 \), and create a mesh from it to any other point, \( C(x) \), then we can write, for any homotopy class \([q]\) between \( \mathbf{a} \) to \( \mathbf{b} \):

\[
\chi([q]) = \chi([C(a)]^{-1}[C(a)][q][C^{-1}(b)][C(b)])
\]  

\[
= \chi([C(a)]^{-1})\chi([C(a)qC^{-1}(b)])\chi([C(b)])
\]  

\[
= \chi([C(a)]^{-1})D([C(a)qC^{-1}(b)])\chi([C(b)]);
\]  

(3.40)

we retrieve Eq. (3.15) when we choose \( \chi([C(x)]) \equiv 1 \). Generally, if we pick a different base-point, \( \mathbf{x'}_0 \), and any mesh \( C'(x) \), we have:

\[
\chi([C'(x)]) = \chi([C'(x'_0)]^{-1})D([C'(x'_0)C'(x)C^{-1}(x)])\chi([C(x)]),
\]  

(3.41)

so once this has been chosen for one base-point and mesh, we have set it up for all base-points and meshes. A special case of this is the one discussed in Ref. [4], a change of mesh with the same base-point. Using our tools, we would also have to change the phases of the mesh so that \( \chi([C(x)]) \equiv 1 \), which leads to a physically equivalent situation.

Conversely, if we choose a mesh and phases for each path in the mesh (as well as the reciprocal for the reversed path: \( \chi([C(x)]^{-1}) = \chi^{-1}(\chi([C(x)])) \)), and a representation of the fundamental group at \( \mathbf{x}_0 \), Eq. (3.40) defines a groupoid representation; let \( q \) and \( p \) be such that \( s(q) = \mathbf{a}, s(p) = t(q) = \mathbf{b}, t(p) = \mathbf{c} \); then:

\[
\chi([q])\chi([p])
\]
Note that the coefficients in larger parentheses are independent of \( r \). Therefore, there is a phase \( \psi \) that incorporates the symmetry of a space, by returning to the case of the punctured plane, \( \mathbb{R}^2 \) with holes. We will make a choice which will allow for a single Aharonov-Bohm fluxes for the special case of spaces which can be represented as multiply-connected spaces. Had this been a simply-connected space, this would be the same as saying we had some space-dependent phase ambiguity (as then all paths between two given points would be of the same homotopy class). More generally, there are additional degrees of freedom having to do with representations of the fundamental group at any point, which correspond to Aharonov-Bohm fluxes for the special case of spaces which can be represented as \( \mathbb{R}^2 \) with holes.

Now, we have seen that every choice of phases for the mesh provides us with a valid groupoid representation. We have also seen that once these phases are set for one choice of base-point and mesh, the choice of phases for any other choice is also set. We therefore proceed to prove that, given the fundamental group representation, a base-point, and a mesh, the choice of phases for the mesh has no physical effect. Let us have two sets of mesh-weights, \( \chi(\bullet) \) and \( \chi'(\bullet) \), and calculate:

\[
\chi([q]) = \chi([C(a)]^{-1}) D([C(a) q C^{-1}(b)]) \chi([C(b)]) \\
= (\chi([C(a)]^{-1}) \chi([C(b)]) ) D([C(a) q C^{-1}(b)])
\]

and:

\[
\chi'([q]) = \chi'([C'(a)]^{-1}) D([C'(a) q C'^{-1}(b)]) \chi'([C'(b)]) \\
= (\chi'([C'(a)]^{-1}) \chi'([C'(b)]) ) D([C'(a) q C'^{-1}(b)])
\]

However:

\[
D([C'(a) q C'^{-1}(b)]) = D([C'(a) C'^{-1}(a) C(a) q C^{-1}(b) C(b) C'^{-1}(b)]) \\
= D([C'(a) C'^{-1}(a) C(a) q C^{-1}(b) C(b) C'^{-1}(b)]) \\
= D([C'(a) C'^{-1}(a)]) D([C(a) q C^{-1}(b)]) D([C(b) C'^{-1}(b)]) \\
= (D([C'(a) C'^{-1}(a)]) D([C(b) C'^{-1}(b)])) D([C(a) q C^{-1}(b)]).
\]

Note that the coefficients in larger parentheses are independent of \([q]\), so long as \( s([q]) \) and \( t([q]) \) remain unchanged. Therefore, there is a phase \( e^{i\delta} \), independent of \([q]\), such that:

\[
\chi'([q]) = e^{i\delta} \chi([q]),
\]

meaning that, unless we alter the fundamental group representation, an arbitrary change in the weights for any mesh starting from any base-point will create physically indistinguishable groupoid representations. In particular, the choice of \( \chi([q]) \equiv 1 \) in Ref. [4] is, in fact, a legitimate special choice of gauge.

Finally, let us show that we can use this additional degree of freedom in order to directly incorporate the symmetry of a space, by returning to the case of the punctured plane, \( \mathbb{R}^2 \setminus \{0\} \) from the end of the last section. We will make a choice which will allow for a single phase, \( e^{i\phi} \), where \( \phi = \phi/2 \) and \( D(1) = e^{i\phi} \), for any half-circle counter-clockwise path around the origin.
We will (ab)use complex polar coordinates to describe points in $\mathbb{R}^2 \setminus \{0\}$:

$$re^{i\theta} \triangleq (r \cos \theta, r \sin \theta),$$

where $0 \leq \theta < 2\pi$.

Let $\mathbf{x}_0 = (1, 0) = 1e^{i0}$, and the mesh be as follows (see Figure 3.3):

$$C(re^{i\theta})(t) = (re^{i\theta})^t = r^te^{it\theta}$$

with the weights:

$$\chi(C(re^{i\theta})) = e^{i\phi\theta/2\pi}. \quad (3.49)$$

Figure 3.3: Mesh for fundamental groupoid representation with rotational symmetry.

We will now show that, under this scheme, the amplitude of every counter-clockwise half-circle is equal to $e^{i\phi/2}$. Let $re^{i\omega}$ be the starting point, and let $q_\omega$ be the counter-clockwise half-circle between it and $(-re^{i\omega})$. Then:

$$\chi(q_\omega) = \chi([C(re^{i\omega})^{-1}]D([C(re^{i\omega})q_\omega C^{-1}(-re^{i\omega})])\chi([C(-re^{i\omega})]). \quad (3.50)$$

Now, there are two possibilities: either $0 \leq \omega < \pi$, in which case $0 \leq \omega + \pi < 2\pi$, so we can write:

$$-re^{i\omega} = re^{i(\omega+\pi)}, \quad (3.51)$$

and:

$$\chi(q_\omega) = e^{-i\phi\omega/2\pi} D([C(re^{i\omega})q_\omega C^{-1}(-re^{i(\omega+\pi)})])e^{i\phi(\omega+\pi)/2\pi}$$

$$= e^{i\phi\omega/2\pi} D([C(re^{i\omega})q_\omega C^{-1}(re^{i(\omega+\pi)})])$$

$$= e^{i\phi/2} D([C(re^{i\omega})q_\omega C^{-1}(re^{(\omega+\pi)})]); \quad (3.52)$$

the path $C(re^{i\omega})q_\omega C^{-1}(re^{i(\omega+\pi)})$ is a path which starts at $(1, 0)$, goes to $(r, 0)$, circles to $re^{i\omega}$, then retraces this path back to $(1, 0)$ (see Figure 3.4); therefore, it can be contracted to a point, and:

$$\chi(q_\omega) = e^{i\phi/2} \cdot 1 = e^{i\phi/2}. \quad (3.53)$$
The second possibility is that \( \pi \leq \omega < 2\pi \), so \( 0 \leq \omega - \pi < \pi \), meaning we can write:

\[
-re^{i\omega} = re^{i(\omega - \pi)},
\]

so:

\[
\chi(q_\omega) = e^{-i\phi/2\pi} D([C(re^{i\omega})q_\omega C^{-1}(re^{i(\omega - \pi)})])e^{i\phi(\omega - \pi)/2\pi} \\
= e^{-i\phi/2\pi} D([C(re^{i\omega})q_\omega C^{-1}(re^{i(\omega - \pi)})]) \\
= e^{-i\phi/2} D([C(re^{i\omega})q_\omega C^{-1}(re^{i(\omega - \pi)})]);
\]

(3.55)

the path \( C(re^{i\omega})q_\omega C^{-1}(re^{i(\omega - \pi)}) \) is a path which starts at \((1, 0)\), goes to \((r, 0)\), circles clockwise to \( re^{i\omega} \), which causes it to cross the positive \( x \)-axis, and so it is only the segment beyond which it retraces to \((1, 0)\) (see Figure 3.5); therefore, it has winding number 1, and:

\[
\chi(q_\omega) = e^{-i\phi/2} e^{i\phi} = e^{i\phi/2}.
\]

(3.56)

Figure 3.4: Mesh takes a non-crossing half-circle into a contractible loop.

Therefore, in both cases, which cover all counter-clockwise half-circles, the accrued phase is \( e^{i\phi/2} \), as expected. Furthermore, any fundamental groupoid representation which agrees with this one on the fundamental group is compatible with it. Finally, it embodies rotational invariance around the origin, so it is a natural choice. In the next chapter, we will use this representation to create distinguishable anyons.

### 3.3 Conclusion

In this chapter, we have found a direct derivation of the result in Ref. [79] for the Feynman path integral in multiply-connected spaces, and generalized upon the treatment in Refs. [4, 81] by presenting a novel way of generating the topological degrees of freedom as representations of the fundamental groupoid. This provided a well-organized way of assigning topological phases to paths rather than loops, which allowed us to directly incorporate the symmetry of a space into the representation.
Figure 3.5: Mesh takes a crossing half-circle into a loop with winding number 1.

In the next chapter, we will use this result to prove the strictly topological origin of anyons. The example of the punctured plane, running throughout this chapter, and culminating in our ability to provide a consistent topological phase for the exchange of distinguishable particles, will prove essential.
4.1 Distinguishable Anyons

As we have mentioned in Section 2.7, the experimental evidence for anyons in Camino et al. [37] involves interference between different types of particles, while that in Kim et al. [38] involves ones of the same type. Therefore, the aim of this chapter is to establish anyonic “statistics” between two distinguishable particles, and then to apply the results of Goyal [1] to arrive at identical anyons. This is in contrast with approaches described in Section 2.6, where anyons were required to be identical. Since the exchange of distinguishable particles is not a closed loop, representations of the fundamental group, from which Ref. [5] get their topological phases, are insufficient. We will therefore make use of the result of Chapter 3, where we showed that a topologically non-trivial space has an additional degree of freedom, which, for the case in question, corresponds to a choice of anyonic exchange amplitude.

The focus on two particles is in the interest of concreteness and simplicity of exposition. Since the operational results in Ref. [1] apply to an arbitrary number of particles, and since the topological situation for many particles involves the “colored braids” (see comment in Goldin and Sharp [92]), which are generated by two-particle exchanges, we do not expect the generalization to $N > 2$ particles to introduce any significant complications.

The main assumption we must make is that the particles do not coincide. We will take from Ref. [5] the transition from the two-particle configuration space to the center of mass and relative spaces, thus helping us focus on the latter, as the former is topologically trivial.

Let us start from the original space:

$$\mathbb{R}^2 \times \mathbb{R}^2 \setminus \{(r, r) \mid r \in \mathbb{R}^2\}. \quad (4.1)$$

If the original variables are of the form $(r_a, r_b)$, then we may define the center of mass and relative coordinates, respectively, as (see Figure 4.1):

$$R_{CoM} \triangleq \frac{r_b + r_a}{2} \quad (4.2)$$

and:

$$r_{rel} \triangleq r_b - r_a. \quad (4.3)$$

In the variables $(R_{CoM}, r_{rel})$, the total space is now:

$$\mathbb{R}^2 \times (\mathbb{R}^2 \setminus \{0\}). \quad (4.4)$$

The center of mass space is thus simply-connected, so all features of topological interest will appear in the relative space, the punctured plane $\mathbb{R}^2 \setminus \{0\}$. We shall subsequently focus on physical systems where we can perform an appropriate separation of variables, for example, those where all interactions are between the two particles.
At the end of Section 3.2, we characterized the topological degree of freedom of this very space through representations of the fundamental groupoid of the space. Therefore, for any path $x(t)$, the amplitude associated with it is:

$$\mathcal{A}\{x(t)\} = \chi([x(t)]) \exp\left(\frac{i}{\hbar} S\{x(t)\}\right),$$

(4.5)

where $[x(t)]$ is the homotopy class of the path, and $\chi([x(t)])$ is a representation of the fundamental groupoid. In this case, the representation of the fundamental group at an arbitrary point is parameterized by the amplitude corresponding to a single rotation around the origin, $e^{i\phi}$, and we can choose a groupoid representation such that the amplitude corresponding to any half-circular, counter-clockwise path $q_{ex}$ is:

$$\chi([q_{ex}]) = e^{i\phi},$$

(4.6)

where $\varphi = \phi/2$ (see Figure 4.2). For our two-particle system, this corresponds to a simple, counter-clockwise exchange.
If we choose to perform an exchange in the opposite direction, then the path is $q_{\text{ex}}^{-1}$, so:

$$\chi([q_{\text{ex}}^{-1}]) = (\chi([q_{\text{ex}}]))^{-1} = e^{-i\phi} \quad (4.7)$$

Generally, if we have a path which half-circles the origin $k$ times, $q_{\Xi}$, we note that even if the exchange path does not pass through either the initial or final point other than during the start or the end, it is homotopic to a path which does, so that the product rule can be used:

$$[q_{\Xi}] = [q_{\text{ex}}]^k, \quad (4.8)$$

and we end up with:

$$\chi([q_{\Xi}]) = \chi([q_{\text{ex}}]^k) = \chi([q_{\Xi}])^k = e^{ik\phi}. \quad (4.9)$$

Now, we have referred to this as an “exchange”, but at this point we only have a representation for a homotopy class. An actual exchange would be a path $x(t)$ such that $[x(t)] = [q_{\Xi}]$, and the exchange amplitude would be:

$$A_D^{\Xi} = \exp\left(\frac{i}{\hbar}S\{x(t)\}\right) \chi([x(t)]). \quad (4.10)$$

For simplicity let us treat the free-particle case:

$$S\{x(t)\} = \int_{t_1}^{t_2} \frac{\mu}{2} (\dot{x}(t))^2 dt, \quad (4.11)$$

meaning that if we take the quasi-static limit, $S \to 0$, and we get:

$$A_D^{\Xi} \to e^{ik\phi}. \quad (4.12)$$

So anyonic behavior is entirely topological, having nothing to do with identical particles. In the rest of the chapter we will combine distinguishable anyons into identical anyons.

### 4.2 Applying the Operational Results to Anyons

We now turn to show that identical particles can behave as anyons, in a way that is compatible with the operational results in Ref. [1], which provide only two forms of combination of distinguishable-particle amplitudes into identical ones, and therefore appear to only admit bosons and fermions.

Let us recall the situation for two identical particles. Suppose that we had a way of finding the transition amplitudes for a system of two distinguishable particles, which are individually dynamically the same as one of the identical particles. For a single, given transition of the identical particles, we would have two transitions of the distinguishable system that would be indistinguishable if we were to ignore the particle labels: an original transition $\Pi$, whose amplitude is $\alpha_{\Pi}$, and a permuted transition $X$, for the situation where the two final particle labels are permuted with each other, whose amplitude is $\alpha_X$ (see Figure 4.3). Then there are two possible expressions for the total amplitude for two subsequent measurements of a system of two identical particles:

$$A_{\text{total}} = H(\alpha_{\Pi}, \alpha_X) = \alpha_{\Pi} \pm \alpha_X, \quad (4.13)$$

where $+$ and $-$ correspond to operational bosons and fermions, respectively. Note that this expression does not, at first, seem to allow for anyons. However, as we saw in the previous section, anyonic
behavior arises from topological considerations for distinguishable particles. So both \( \alpha_{II} \) and \( \alpha_X \) separately incorporate this degree of freedom. Here we will show how these ideas are combined to form a pair of identical anyons.

Let us write more explicit expressions for these amplitudes. We recall that we are discussing two particles in two dimensions, which cannot coincide. Now let us see how identity translates into the relative space. Given a pair of different locations in \( \mathbb{R}^2 \):

\[
\mathbf{r}_a \neq \mathbf{r}_b; \quad (4.14)
\]

if \((\mathbf{r}_a, \mathbf{r}_b)\) is, as in Section 4.1, the outcome in which the first particle is in \( \mathbf{r}_a \) and the second is in \( \mathbf{r}_b \), then:

\[
(\mathbf{r}_a \leftrightarrow \mathbf{r}_b) \quad (4.15)
\]

is an outcome in which one particle is found in each of these locations, without being able to distinguish the two. By the definition of the center of mass and relative coordinates in Eq. (4.2) and Eq. (4.3), exchanging the particles:

\[
(\mathbf{r}_a, \mathbf{r}_b) \mapsto (\mathbf{r}_b, \mathbf{r}_a) \quad (4.16)
\]

translates into a transformation reversing \( \mathbf{r}_{rel} \), without changing \( \mathbf{R}_{CoM} \):

\[
(\mathbf{R}_{CoM}, \mathbf{r}_{rel}) \mapsto (\mathbf{R}_{CoM}, -\mathbf{r}_{rel}); \quad (4.17)
\]

see Figure 4.4. Therefore, when we focus on the relative space, distinguishable particle states may be designated by a single relative coordinate, \((\mathbf{r}_{rel})\), while the symmetrized space may be written as \((\mathbf{r}_{rel} \leftrightarrow -\mathbf{r}_{rel})\).

To simplify the notation, we can suppress the \( rel \) in the subscript, and simply use a subscript number for time-ordering purposes. Then a measurement outcome for the distinguishable-particle system is written as \((\mathbf{r}_n)\), while the corresponding outcome for the identified system is \((\mathbf{r}_n \leftrightarrow -\mathbf{r}_n)\); however, the latter contains an ambiguity, as both \((\mathbf{r}_n \leftrightarrow -\mathbf{r}_n)\) and \((\mathbf{r}_n \leftrightarrow \mathbf{r}_n)\) refer to the same outcome of the identical particle system. As we can see from Eq. (4.13), in the case of fermions, this would lead to a multiply-defined value. We therefore must choose a subspace of the punctured plane of the “Real” \((+\mathbf{r})\) values, so that they, along with the “Inverted” values \((-\mathbf{r})\), cover the whole space. We choose: \( \{y > 0\} \cup \{y = 0, x > 0\} \) (see Figure 4.5). In this we are borrowing from the reduced configuration space approach in Ref. [5]; note, however, that we are dealing with particle identity after quantization, as we assume that we have amplitudes to work with from the appropriate distinguishable-particle system.
If we then write, for two consecutive measurements indexed by 1 and 2:

\[ \alpha_{\Pi} = \mathcal{A}( (r_1)(r_2) ) \]  
\[ \alpha_{X} = \mathcal{A}( (r_1)(-r_2) ) \]
for the original transition, and:

\[ \mathcal{A}_{\text{total}} = \mathcal{A}( (r_1 \leftrightarrow -r_1)(r_2 \leftrightarrow -r_2) ) ; \]
for the identical particle transition, we can then express Eq. (4.13) as:

\[ \mathcal{A}( (r_1 \leftrightarrow -r_1)(r_2 \leftrightarrow -r_2) ) = H( \mathcal{A}( (r_1)(r_2) ), \mathcal{A}( (r_1)(-r_2) ) ) \]
\[ = \mathcal{A}( (r_1)(r_2) ) \pm \mathcal{A}( (r_1)(-r_2) ) . \]  

4.3 Exchanging Identical Anyons

4.3.1 Circular Exchanges

We turn to exchanging particles. In the \( \Delta t \to 0 \) limit, the only contribution to the amplitude comes from around the classical path [93]. So let us have the particles take very short steps, in a half-arc, around their center of mass (see Figure 4.6):

\[ r_1 = (\rho \cos \theta, \rho \sin \theta) \mapsto (\rho \cos(\theta + \Delta \theta), \rho \sin(\theta + \Delta \theta)) = r_1 + \rho \Delta \theta (\sin \theta, -\cos \theta) \]
\[ = r_1 + \rho \Delta \theta \hat{\theta} ; \]  
we are dealing then with the \( \Delta t \to 0 \) limit of two distinguishable-particle amplitudes; direct:

\[ \mathcal{A}_d(\theta + \Delta \theta, t + \Delta t ; \theta, t) \triangleq \mathcal{A} \left( (r_1)(r_1 + \rho \Delta \theta \hat{\theta}) \right) , \]
over a classical line-segment we will call \( r_d \), and opposite:

\[ \mathcal{A}_o(\theta + \Delta \theta, t + \Delta t ; \theta, t) \triangleq \mathcal{A} \left( (r_1)(-r_1 - \rho \Delta \theta \hat{\theta}) \right) . \]
Figure 4.5: Complete symmetrized space is \( \{ y > 0 \} \cup \{ y = 0, x > 0 \} \). Unless an arbitrary choice is made, there is an ambiguity regarding which transition is original and which is permuted. This is resolved by deciding that the transition between \( r \)'s in “Real” space is original, while that from “Real” to “Inverted” space is permuted.

Figure 4.6: The direct and opposite transitions for a small angular change. Note that \( r_o \) does not pass through the hole at the origin.

Classically, outside the origin, the two particles are free, so in both cases we are looking at particles moving uniformly, that is, in straight lines at constant speeds. In the case of the opposite transition, the straight line is within the relevant space because of the slight displacement from a diameter of the circle. For the direct transition, the velocity is:

\[
v_d = \frac{r_1 + \rho \Delta \theta \hat{\theta} - r_1}{\Delta t} = \frac{\rho \Delta \theta \hat{\theta}}{\Delta t},
\]

(4.25)
while the velocity for the opposite transition is very different:

\[
v_o = \frac{-r_1 - \rho \Delta \dot{\theta} - r_1}{\Delta t} = \frac{-2r_1}{\Delta t} - \rho \Delta \dot{\theta}; \quad (4.26)
\]

if we then take \( \Delta t, \Delta \theta \to 0 \) with \( \Delta \theta/\Delta t = \dot{\theta} \), a constant, then:

\[
v_d = \rho \dot{\theta} \dot{\theta}, \quad (4.27)
\]

while:

\[
v_o = -\frac{2r_1}{\Delta t}. \quad (4.28)
\]

We will find the misbehavior of the opposite transition’s velocity very useful in the following. We now calculate the action functional for both cases (in this case \( m_1 = m_2 = m \)), using the standard free-particle Lagrangian:

\[
S_d(\theta + \Delta \theta, t + \Delta t; \theta, t) = \int_t^{t+\Delta t} \frac{m}{2} v_d^2 \, dt = \int_t^{t+\Delta t} \frac{m}{2} \rho^2 \dot{\theta}^2 = \frac{m}{2} \rho^2 \dot{\theta}^2 \Delta t, \quad (4.29)
\]

which is well-behaved, while:

\[
S_o(\theta + \Delta \theta, t + \Delta t; \theta, t) = \int_t^{t+\Delta t} \frac{2m \rho^2}{\Delta t^2} \Delta t' = \frac{2m \rho^2}{\Delta t}, \quad (4.30)
\]

is not. When passing to the amplitudes, recall that the distinguishable system incorporates the fundamental groupoid representation \( \chi(\bullet) \) defined in Section 4.1. Additionally, while the action functionals are time-independent, we will want to distinguish between the respective line segments, so we will index them:

\[
A_{dn} = \chi([r_{dn}]) \exp \left[ \frac{im \rho^2 \dot{\theta}^2}{2\hbar} \Delta t \right] \quad (4.31)
\]

and

\[
A_{on} = \chi([r_{on}]) \exp \left[ \frac{2m \rho^2}{i\hbar \Delta t} \right]. \quad (4.32)
\]

Now, let us concatenate \( N \) such transitions, with \( N \Delta t = T, \quad N \Delta \theta = \pi \), so that \( \dot{\theta} = \pi/T \), and we create an overall exchange amplitude:

\[
A_{\Xi}^d = (A_{d1} \pm A_{o1})(A_{d2} \pm A_{o2}) \cdots (A_{d[N-1]} \pm A_{o[N-1]}) (A_{oN} \pm A_{dN}); \quad (4.33)
\]

the change in order in the last transition is not accidental: since it crosses out of the domain of definition of \( r \leftrightarrow -r \), the order of terms is then, and only then, reversed. This will always happen an odd number of times in an overall exchange, regardless of where the line of demarcation is chosen (see Figure 4.7).
Figure 4.7: Accumulation of the direct amplitudes, with illustration of the opposites. Note that in the last transition, the direct transition, while retaining the same physical amplitude, crosses the boundary between “real” and “inverted” space, while the opposite transition stays within “real” space, so that the former is “permuted” and the latter is “original”. Therefore, the total exchange amplitude acquires a possible sign, depending on whether the particles are originally bosons (+) or fermions (−).

Now, as seen in Eq. (4.32), $A_{on}$ depends on $\Delta t$ as $\exp(\tilde{C}i\Delta t^{-1})/\hbar$, or equivalently on $N$ as $\exp(CiN/\hbar)$, with $C > 0$. If we follow the procedure in Feynman [27] and replace $\hbar \rightarrow (1 - i\delta)\hbar$, with the limit $\delta \rightarrow 0^+$ to be taken after our calculations, then $1/\hbar \rightarrow (1 + i\delta)\hbar$, and:

$$\exp(CiN/\hbar) \rightarrow e^{CiN(1+i\delta)/\hbar} = e^{CiN/\hbar} \cdot e^{-CN\delta/\hbar}.$$  

(4.34)

Therefore, in each factor of $A_1^n$, $A_{on}$ will become exponentially smaller than $A_{dn}$ as $N \rightarrow \infty$, so that we can focus on the latter at the expense of the former. Another way of understanding this mathematical trick is by noting that in the $A_{on}$, the phase varies wildly as $N \rightarrow \infty$, so that it will tend to cancel out, while the $A_{dn}$ remain manageable, and lead to a finite product among themselves.

Therefore, we are left with the amplitude of a circular path $r_\Xi$ in the positive direction:

$$A_\Xi = \pm \prod_{n=1}^{N} A_{dn} = \pm \prod_{n=1}^{N} \chi([r_{dn}]) \exp \left[ i \frac{m\rho^2 \dot{\theta}^2}{2\hbar} \Delta t \right]$$

$$= \pm \chi([r_{d1} \cdots r_{dN}]) \exp \left[ i \frac{m\rho^2 \dot{\theta}^2}{2\hbar} N \Delta t \right] = \pm \chi([r_\Xi]) e^{\frac{i m\rho^2 \dot{\theta}^2}{2\hbar}}$$

(4.35)

factoring nicely to an operational factor $\pm$, a topological factor $\chi([r_\Xi])$, and a dynamical factor $e^{\frac{i m\rho^2 \dot{\theta}^2}{2\hbar}}$. We can finally take the quasi-static limit, $T \rightarrow \infty$, removing the dynamical factor to find that:

$$A_\Xi = \pm \chi([r_\Xi]) = \pm e^{i\phi},$$  

(4.36)

that is, an exchange in the positive direction multiplies by the operational coefficient, as well as an Aharonov-Bohm phase. Since $-1 = e^{i\pi}$, we can incorporate the operational degree of freedom into a new phase, with angle $\varphi$, and we have anyons:

$$A_\Xi = e^{i\varphi}.$$  

(4.37)
This was presented as a single exchange in the positive direction. The treatment in the negative direction only requires us to choose \( N \Delta \theta = -\pi \), and to now have the segments and the exchange itself be in the opposite direction. Then the exchange amplitude would be:

\[
A^1_{\Xi} = (A_{o1} \pm A_{d1})(A_{o2} \pm A_{d2}) \cdots (A_{d[N-1]} \pm A_{o[N-1]})(A_{dN} \pm A_{oN}),
\]

(4.38)

where the change in order is now in the first transition, which is the one crossing the line of demarcation, despite being shorter locally (see Figure 4.8).

![Figure 4.8: Situation is as for Figure 4.7, but in the reverse direction. Here it is the first transition in which “original” and “permuted” must be reversed.](image)

The total amplitude would then be:

\[
A^1_{\Xi \text{rev}} = \pm \chi([r_{\Xi \text{rev}}]) e^{i\frac{m^2 \rho^2}{\hbar^2}} = \pm e^{-i\phi} e^{i\frac{m^2 \rho^2}{\hbar^2}},
\]

(4.39)

or in the quasi-static limit (and noting that \( e^{i\pi} = e^{-i\pi} \)):

\[
A^1_{\Xi \text{rev}} = e^{-i\phi}.
\]

(4.40)

Generally, if one wishes to perform a \( k \)-fold exchange, then if \( k \geq 0 \), one gets:

\[
(A^1_{\Xi})^k = e^{ik\phi},
\]

(4.41)

while if \( k < 0 \), one would have:

\[
\left(A^1_{\Xi \text{rev}}\right)^{|k|} = e^{-i|k|\phi} = e^{ik\phi}.
\]

(4.42)

So, for the general \( k \)-fold circular exchange:

\[
A^1_{\Xi} = e^{ik\phi},
\]

(4.43)

as in Eq. (4.12).
4.3.2 More General Paths

Now, let us generalize further. Suppose the exchange were performed along a more general path, \( r(t) \), such that:

\[
 r(0) = -r(T) = x_0 \neq 0. \tag{4.44}
\]

We will still impose a few regularity conditions on this path: it is smooth, with finite velocity, it stays at a minimal distance \( \rho_{\text{min}} > 0 \) from the excluded origin, and for simplicity, the angular velocity is never zero. We choose some \( \Delta t = T/N \), and look at short steps in time:

\[
 r_n \rightarrow r_{n+1} = r_n + \Delta \rho_n \hat{\rho}_n + \rho_n \Delta \theta_n \hat{\theta}_n = r_n + \left( \dot{\rho}_n \hat{\rho}_n + \rho_n \dot{\theta}_n \hat{\theta}_n \right) \Delta t, \tag{4.45}
\]

where:

\[
 \{ \text{variable} \}_n \triangleq \{ \text{variable} \}(n \Delta t). \tag{4.46}
\]

At the \( \Delta t \to 0 \) limit, the two relevant paths — which do not cross the excluded origin, as we have specified that \( \dot{\theta} \neq 0 \) — have the following direct straight-line velocity:

\[
 v_{dn} = \dot{\rho}_n \hat{\rho}_n + \rho_n \dot{\theta}_n \hat{\theta}_n, \tag{4.47}
\]

with the opposite velocity being:

\[
 v_{on} = -2 \frac{r_n}{\Delta t} - v_{d}; \tag{4.48}
\]

as in Section 4.3.1, the \( \Delta t^{-1} \) term dominates, and we may write:

\[
 v_{on} = -2 \frac{r_n}{\Delta t}. \tag{4.49}
\]

The action functionals for the two cases are:

\[
 S_d(\rho_{n+1}, \theta_{n+1}, t_{n+1}; \rho_n, \theta_n, t_n) = \int_{t_n}^{t_{n+\Delta t}} \frac{m}{2} \nu_{dn}^2 \text{d}t' = \int_{t_n}^{t_{n+\Delta t}} \frac{m}{2} \left[ \dot{\rho}_n^2 + \rho_n^2 \dot{\theta}_n^2 \right] \text{d}t' = \frac{m}{2} r_n^2 \Delta t, \tag{4.50}
\]

which is well-behaved, while:

\[
 S_o(\rho_{n+1}, \theta_{n+1}, t_{n+1}; \rho_n, \theta_n, t_n) = \int_{t_n}^{t_{n+\Delta t}} \frac{2m \rho_n^2}{\Delta t^2} \text{d}t' = \frac{2m \rho_n^2}{\Delta t} \geq \frac{2m \rho_{\text{min}}^2}{\Delta t} \tag{4.51}
\]

is not. we can then write the amplitudes more succinctly as follows:

\[
 A_{dn} = \chi([r_{dn}]) \exp \left[ \frac{i m}{2\hbar} r_n^2 \Delta t \right], \tag{4.52}
\]

and

\[
 A_{on} = \chi([r_{on}]) \exp \left[ \frac{4m \rho_n^2}{\hbar \Delta t} \right]. \tag{4.53}
\]
where $\chi(\bullet)$ is defined as in Section 4.3.1.

The next step is more subtle than for a simple, circular path. We have a product of mostly:

$$\mathcal{A}_{dn} \pm \mathcal{A}_{on}, \quad (4.54)$$

where this is replaced by $\mathcal{A}_{on} \pm \mathcal{A}_{dn}$ whenever we have a path segment crossing the boundary of the region of definition of the symmetrized space. If the exchange involves $k$ half-rotations of the particles around each other, then there will be $k$ such terms. Since, as in Section 4.3.1, a $\mathcal{A}_{on}$ factor suppresses any term which contains even one instance thereof, the total amplitude of exchange is:

$$\mathcal{A}_{\Xi}^k = (\pm 1)^k \prod_n \mathcal{A}_{dn} = (\pm 1)^k \left\{ \prod_n \chi([r_{dn}]) \right\} \exp \left[ \sum_n i \frac{m \dot{r}_n^2}{2\hbar} \Delta t \right]$$

$$= (\pm 1)^k \chi([r]) e^{S_{\text{free}}[r(t)]}; \quad (4.55)$$

now, the $k$ half-rotations have to be in the same direction, as, again, we assume $\dot{\theta} \neq 0$, and the sign of $k$ can be taken to signify the direction of rotation, so that the total topological phase is:

$$\chi([r]) = e^{ik\phi}. \quad (4.56)$$

Finally, we can take the quasi-static limit, in which the free particle’s action functional is zero, and we define $\varphi = \phi$ or $\varphi = \phi + \pi$, depending on whether we started from a fermionic or bosonic total amplitude, so we find:

$$\mathcal{A}_{\Xi}^k = e^{ik\varphi}, \quad (4.57)$$

which is again the same expression in Eq. (4.12), for distinguishable particles.

### 4.4 One Dimension Without Incidence Point

Before moving on to the discussion, we will note that the appropriate “anyonic” behavior is also found in one dimension, without the need for symmetrizing the space prior to quantization. For two particles in one dimension, if the incidence point is taken out, the particles cannot change places. Therefore, $\alpha_X = 0$, and the total amplitude is:

$$\mathcal{A} = \alpha_{II}; \quad (4.58)$$

that is, the situation is the same as that for distinguishable particles without incidence. Nevertheless, whether or not the particles are identical, the removal of the incidence point creates a one-dimensional family of self-adjoint extensions for this problem, allowing for an interpolation between fermion- and boson-like behavior on each side, as described in Ref. [5], as well as more rigorously, and with an incorporation of Feynman path integrals, in Farhi and Gutmann [94]. The importance of non-connectivity was originally noted by Girardeau [31].

### 4.5 Conclusion

In this chapter we were able to use the groupoid-based approach to quantization of multiply-connected spaces developed in Chapter 3 to express anyonic behavior in two distinguishable particles in the path integral formalism. We then showed how to use the identical particle amplitude combination results of Ref. [1] to turn distinguishable anyons into identical anyons. This should put to rest the idea that anyonic behavior is intrinsically tied to particle identicality; instead, it is a result of the topology of particles in two-dimensional space which cannot coincide.
CHAPTER 5

HISTORICAL AND PHILOSOPHICAL DISCUSSION OF PARTICLE IDENTITY

Now that we have shown that anyonic behavior is a topological phenomenon rather than having to do with identical particles, let us look back at the historical development of this issue, compare it to our own results in Chapter 4, and see why anyons and identicality are so often intertwined in the literature.

All theoretical discussions and most experimental papers which use anyons see the origins of their topic in Leinaas and Myrheim [5]. In that paper, the original, classical configuration space of a set of identical particles was symmetrized: coordinates connected by a permutation were simply identified, and the resulting, non-Euclidean space, was quantized. They noted that this quantization would be difficult if the incidence points were retained, as the symmetrized space would then not be a manifold, so they took them out. They then found that in two dimensions, the path of exchange between particles, not just the fact that they are exchanged, matters. Not every two paths can be continuously deformed into each other, so that the space has become multiply-connected. This makes it the first paper to make use of the fact that the configuration space for two non-intersecting particles in two dimensions is multiply-connected, and the result was anyonic behavior.

The reduced configuration space project itself came about as a response to several issues raised with the way identical particles were tackled at the time: criticism in Mirman [29], which noted that particle labels are physically meaningless, and therefore should be abandoned in a more rigorous theory; the existence of the Gibbs paradox [65], showing that it is necessary to reduce the size of the configuration space of a system of indistinguishable particles by the number of permutations even in classical systems; and, most importantly, the seeming arbitrariness of the Symmetrization Postulate (see Section 2.5). Ref. [5] decided to resolve all of these issues in one fell swoop, by imposing the identity of particles on a classical system, and then proceeding to quantize it with the tools dedicated to geometrically or topologically non-trivial spaces, with the hope that something like the symmetrization postulate would come up naturally. The failure of this program in two dimensions led to the discovery of anyons.

However, as we have seen in Section 4.1, anyons can be dealt with without referring to the reduced configuration space project, or indeed, to identical particles at all, and this has already been recognized in the past. Ref. [5] was published in 1977; as early as 1985, we already find Goldin et al. [39] pointing out that unusual phases on exchange are possible even for distinguishable particles in two dimensions, since it is the connectivity of the space, rather than the identity of the particles, that is relevant. Zee [42], in notes from a conference in 1994, acknowledges that the “statistics” involved here can be generated from a phase interaction between the particles, rather than by requiring them to be a type of identical particles.

Despite these early admissions of the separability of anyons from particle identity, the vast majority of papers and reviews which were aware of either Ref. [5] or Wilczek [30] (which cited the former while popularizing the term “anyon”), introduced the issue in the context of the reduced configuration space, even though in many cases they proceeded to use a particle-flux composite or Chern-Simons potential when performing calculations [40, 46, 95, 96].

Ref. [30] itself represents a clear hybrid phenomenon. On the one hand, the two-anyon case is treated with an Aharonov-Bohm potential, without using a reduced configuration space. On the other hand, the generalization to \( N > 2 \) particles is said to be difficult because the reduced configuration space would then become more complicated. More recently, Nayak et al. [97], while
starting out with reduced configuration space ideas, acknowledge that there could be many different “types” of anyons, with arbitrary phases accrued by twisting them around each other, implicitly admitting non-identical anyons. They, with Zee [42], accepted that what is presented as “statistics” of anyons could be instead understood as a special kind of interaction.

Another curious phenomenon is the fact that none of the papers or books cited above seem to seriously address the geometrical (rather than topological) consequences of reducing the configuration space in two dimensions. As noted in Ref. [5], the reduced space is equivalent to that of a cone, which has non-zero curvature at the origin. Had this been important, we would expect to find many more citations and discussions of works dealing with this kind of system.

So the conceptual question is why, despite the fact that the reduced configuration space approach is ill-suited to applications, and would lead to fundamental additional problems if taken seriously, is it nevertheless invoked without causing any practical problems?

The solution involves a tool essential for the conventional quantization of multiply-connected spaces: the use of covering spaces (see Figure 5.1; for a more extensive discussion, see Singer and Thorpe [77]). These are spaces that are themselves simply-connected, contain several copies of the underlying multiply-connected space, properly patched together, and which project in a canonical manner down to the space in question. A wave-function or propagator is calculated for the covering space, and then further restricted so that the existence of a space larger than the physical space cannot be detected. While one may cite Laidlaw and DeWitt-Morette [4] as bucking this trend, the more elaborate development of this work in Laidlaw [81] concedes that the covering space approach is necessary in order to make any meaningful calculations, and from Dowker [82] on, this is the method normally used.

Figure 5.1: $X^*$ is the covering space of $X$. $x_i^*$ all project to the same $x$.

The relevance of this issue can be found in French and Krause [105], through an analysis of an argument between Grünbaum [106] and Stein [107, 108]. The salient point is that the passage to

---

1 This is usually known as the “cosmic string”; see Refs. [98–102] for in-depth discussion.

2 The other two resolutions in the literature are the use of multivalued functions and of fiber bundles. Multivalued functions themselves are best accommodated through the universal covering space [103], or its complex function analogue, the Riemann surface [104], while fiber bundles are a different form of covering space, and in any event, are applied to the non-symmetrized, electromagnetic problem anyway [41].
a covering space is not innocent. We see this most clearly when we look at Ref. [5]'s treatment of the three-dimensional quantum case. After going to the trouble of removing the origin and identifying exchanged point, the symmetrization postulate in three dimensions is retrieved by using the universal covering space, which, since the configuration space is now doubly-connected, is a doubling-up of the reduced space: in other words, the original space, unnecessary particle labels and all. In two dimensions the situation is more difficult, but there is nothing preventing a practitioner from “unreducing” the configuration space to that of distinguishable particles before applying the tools needed to deal with an Aharonov-Bohm effect. We ourselves approached this topologically in Section 4.1. Previous treatments sometimes involved a universal covering space, but this special case is more easily handled through a composite flux or or Chern-Simons field, instead [40, 41]. Most importantly for our argument, these tools do not concern themselves with whether the particles are intrinsically identical. Identity can be imposed in parallel or in concert with these other tools.

We can also see evidence for the work that the covering space implicitly performs by asking why issues of curvature hardly arise. Indeed, while the symmetrized space of two particles in two dimensions is, in fact, a cone, and thus has a non-trivial curvature at the removed origin, the particular form of the cone belongs to a family of special cases. In these special cases, the reduced range of the azimuthal angle is a whole fraction of $2\pi$ [98, 101]. These special cases are equivalent to solving the problem in a flat space, and then imposing an n-fold symmetry condition on the probability density: that is, requiring that $|\Psi(\rho, \theta + 2\pi/n)| = |\Psi(\rho, \theta)|$. For the relative space of two particles in two dimensions, this is seen to be equivalent to the classical way of deriving the symmetrization postulate.

The basic idea of approaching identical particles from Refs. [1, 2] completely sidesteps any criticism of the legitimacy of using particle labels, and does not result in a curved configuration space. Instead, it assumes that a certain problem has been solved for distinguishable particles, that is, with particle labels, and sees what consistent ways there are to combine these solutions into a solution to the identified system. So it is compatible both with the idea that particle labels are necessary, but then suppressed, and with the idea that there is a useful covering space using particle labels which can be harnessed for the purpose of also dealing with identical particles.

To conclude, we challenged the fundamental impetus for the reduced configuration space approach, namely that quantization after symmetrizing the space is more natural because it removes the need for using unphysical labels, by showing that successful quantization in the reduced space requires the addition of redundant structure, so that one unmeasurable family of parameters is exchanged with another. Reflecting on the results in Chapter 4, we found that this is also not necessary in order to provide the additional richness that is anyonic behavior.
We began our research asking whether the Symmetrization Postulate was valid. The results in Neori and Goyal [2], and their generalization in Goyal [1], only leave room for bosons and fermions.

On its surface, this seemed to run counter to the existence of anyons, which interpolate between bosons and fermions. Since there are theoretical models as well as physical evidence standing behind anyons, we were not satisfied to simply rule them out. Instead, we investigated the issue of identical particles in two dimensions. We found that the apparent paradox stems from the aforementioned assumption that anyons must be identical, when this is demonstrably not the case. We discovered a novel approach for quantization in general multiply-connected spaces using the fundamental groupoid, which is based upon the concatenation of classes of paths, rather than loops. This allowed us to provide consistent topological amplitudes for exchanges of distinguishable particles, which lead us to distinguishable anyons. Using the operational results from Ref. [1], we showed that the resulting probability amplitudes can then be combined to form identical anyons.

Looking to future work, there are at least two important issues that have not yet been resolved, and await further developments.

The first is the question of paraparticles. While the operational formalism, with the assumptions we added to incorporate identical particles into it, rules them out, it would be interesting to see what would be needed to change in order to allow for them. One possibility is to weaken atomic measurements so we would allow additional parameters, which cannot be directly measured. That may result in vectors of numbers needed to express a measurement, rather than a single parameter, leading to the possibility of unitary operators instead of numerical amplitudes. Similar work will be needed to incorporate measurable internal degrees of freedom such as spin, so it may well be a good exercise even if it does not result in paraparticles. Additionally, generalizing the groupoid results to multiple-dimensional representations may have applications to non-abelian anyons, of the form used in topological quantum computing [97].

Another issue is the removal of incidence points. This is a recurring problem in the literature. The operational results are general, meaning that they should apply even to particles with small, discrete configuration spaces. In those situations, particle incidence become a significant part of the space, and a serious challenge for bosons, which should be able to occupy them in an unlimited manner. We hope future work will resolve these issues.

Finally, a potentially minor point regarding our anyon proofs: strictly speaking, we only discussed systems with two anyons, and have not generalized to any number of particles. However, since the operational results do generalize, and since the braid group, which forms the foundation for anyonic behavior, is generated by individual exchanges, we do not expect there to be serious challenges when more than two particles are involved.
APPENDIX: GROUPOID SOURCE AND TARGET PROOF

Let us show that any groupoid will have \( s \) and \( t \) functions as described in Chapter 3.

Minimally, a groupoid is a set with a sometimes-defined operation, satisfying:

(I) If \((ab)c\) is defined then, in particular, \(ab\) is defined; furthermore, \((bc)\) is defined, so that \(bc\) is also defined. Finally, \((ab)c = a(bc)\).

(II) For each \(a\) there is always a \(a^{-1}\) such that \(aa^{-1}\) and \(a^{-1}a\) are defined, and:

\[
\begin{align*}
(i) \quad & (a^{-1}a)b = b \text{ when } ab \text{ is defined} \\
(ii) \quad & (aa^{-1})b = b \text{ when } a^{-1}b \text{ is defined} \\
(iii) \quad & b(aa^{-1}) = b \text{ when } ba \text{ is defined} \\
(iv) \quad & b(a^{-1}a) = b \text{ when } ba^{-1} \text{ is defined}
\end{align*}
\]

If we replace (II) by a weaker condition:

(II') For each \(a\) there are \(a^{-1}_L\) and \(a^{-1}_R\) such that \(a^{-1}_La\) and \(aa^{-1}_R\) are defined, and:

\[
\begin{align*}
(i') \quad & (a^{-1}_La)b = b \text{ when } ab \text{ is defined} \\
(ii') \quad & b(aa^{-1}_R) = b \text{ when } ba \text{ is defined}
\end{align*}
\]

we can retrieve it as follows. We note that:

\[
a^{-1}_L = a^{-1}_L(aa^{-1}_R) = (a^{-1}_La)a^{-1}_R = a^{-1}_R, \tag{A.1}
\]

so \(a^{-1}_L = a^{-1}_R\) and we can write both as \(a^{-1}\), giving us (i) and (iii). Now, there also exists a \((a^{-1})^{-1}\), and we find that:

\[
a = a(a^{-1}(a^{-1})^{-1}) = (aa^{-1})(a^{-1})^{-1} = (a^{-1})^{-1}, \tag{A.2}
\]

and we get (ii) and (iv).

Now, let us define:

\[
\begin{align*}
t(a) &= \{b : ab \text{ is defined}\} \\
s(b) &= \{c : c^{-1}b \text{ is defined}\} = t(b^{-1}). \tag{A.4}
\end{align*}
\]

We can prove that \(t(a) = s(b)\) iff \(ab\) is defined. Note that \(a^{-1} = t(a)\), since \(aa^{-1}\) is defined, and \(b \in s(b)\), since \(b^{-1}b\) is defined. Then:

\[
\Rightarrow: \text{ If } t(a) = s(b), \text{ then } b \in t(a), \text{ so } ab \text{ is defined.}
\]

\[
\Leftarrow: \text{ If } ab \text{ is defined, then:}
\]

\[
\subseteq: \text{ If } c \in t(a) \text{ then } ac \text{ is defined, so } a(cc^{-1}) = a \text{ is defined, and:}
\]

\[
ab = (a(cc^{-1}))b = a((cc^{-1})b) = a(c(c^{-1}b)), \tag{A.5}
\]

meaning \(c^{-1}b\) is defined, so \(c \in s(b)\).
Therefore, \( t(a) = s(b) \).

\( \supseteq \): If \( c \in s(b) \) then \( c^{-1}b \) is defined, so \( (cc^{-1})b = b \) is defined, and:

\[
ab = a((cc^{-1})b) = (a(cc^{-1}))b = ((ac)c^{-1})b,
\]

meaning \( ac \) is defined, so \( c \in t(a) \).

And we are done. \( \blacksquare \)
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