Objections to the Unified Approach to the Computation of Classical Confidence Limits

Günter Zech*
Universität Siegen, D-57068 Siegen, Germany

March 31, 2022

Abstract

Conventional classical confidence intervals in specific cases are unphysical. A solution to this problem has recently been published by Feldman and Cousins[1]. We show that there are cases where the new approach is not applicable and that it does not remove the basic deficiencies of classical confidence limits.

1 Introduction

Feldman and Cousins propose a new approach to the computation of classical confidence bounds which avoids the occurrence of unphysical confidence regions, one of the most problematic features of the conventional classical confidence limits. In addition it unifies the two procedures “computation of confidence intervals” and “computation of confidence limits”. The unified treatment represents a considerable improvement compared to the conventional classical method and has already been adopted by several experiments and is recommended by the Particle Data Group [2]. However, it has serious deficiencies.

*E-mail: zech@physik.uni-siegen.de
2 Basic idea of the unified approach

We consider the example of section B of Ref. [1]. For a Gaussian resolution function \( P(x; \mu) \) we define for each mean \( \mu \) an interval \( x_1 < x < x_2 \) with the property

\[
\int_{x_1}^{x_2} P(x; \mu) \, dx = \alpha
\]

where \( \alpha \) is the confidence level. For a measurement \( \hat{x} \) all values \( \mu \) with the property \( x_1(\mu) < \hat{x} < x_2(\mu) \) form the confidence interval. The intervals have the property that the true values are covered in the fraction \( \alpha \) of a large number of experiments. The freedom in the choice of the interval inherent in the relation (1) is used to avoid unphysical limits. (Usually the limits \( x_1, x_2 \) are fixed by choosing central intervals.) In case that only one limit can be placed inside the allowed parameter space, upper (or lower) limits are computed. The data and the selected level \( \alpha \) unambiguously fix the bounds and whether bounds or limits are given. The probability bounds are defined by an ordering scheme based on the likelihood ratio. In the case of discrete parameters an analogous procedure is applied with some additional plausible conventions. The complete recipe is too complicated to be discussed in a few words. The reader has to consult the original publication.

3 Objections to the unified approach

The new approach has very attractive properties, however, there are also severe limitations most of which are intrinsic in the philosophy of classical statistics.

3.1 Inversion of significance

In some cases less significant data can provide more stringent limits than more informative data.

As an example we present in the following table the 90% confidence upper limits for a Poisson distributed signal from data with no event found \( (n = 0) \) for different background expectations of mean \( b \).

The experimental information on the signal \( s \) is the same in all four cases independent of the background expectation since no background is present.
For the case $n = 0, b = 3$ the unified approach avoids the unphysical negative limit of the conventional classical method but finds a limit which is more significant than that of an experiment with no background expected and twice the flux.

If in the $n = 0, b = 3$ experiment by an improved analysis the background expectation is reduced, the limit becomes worse.

The reason for this unsatisfactory behavior is related to the violation of the likelihood principle\(^1\) by the classical methods. All four cases presented in the table have common likelihood functions $L \sim e^{-s}$ of the unknown signal up to an irrelevant multiplicative constant depending on $b$.

### 3.2 Difficulties with two-sided bounds

Let us assume a measurement $\hat{x} = 0$ of a parameter $x$ with a physical bound $-1 < x < 1$ and a Gaussian resolution of $\sigma = 1.1$. (This could be for example a track measurement by a combination of a proportional wire chamber and a position detector with Gaussian resolution.) The unified approach fails to give 68.3% confidence bounds or limits.

### 3.3 Difficulties with certain probability distributions

The prescription for the definition of the probability intervals may lead to disconnected interval pieces. A simple example for such a distribution is the superposition of a narrow and a wide Gaussian

$$P(x; \mu) = \frac{1}{\sqrt{2\pi}} \left\{ 0.9 \exp \left( -(x-\mu)^2 / 2 \right) + \exp \left( -(x-\mu)^2 / 0.02 \right) \right\}$$

\(^1\)A detailed discussion of the likelihood principle and references can be found in \[3\] and \[4\].
with the additional requirement of positive parameter values $\mu$. It will produce quite odd confidence intervals.

Another simple example is the linear distribution

$$P(x; \theta) = \frac{1}{2}(1 + \theta x)$$

where the parameter $\theta$ and the variate $x$ are bound by $|\theta| \leq 1$ and $|x| \leq 1$. (The variable $x$ could be the cosine of a polar angle.) Values of $\theta$ outside its allowed range produce negative probabilities. Thus the likelihood ratio which is used as an ordering scheme for the choice of the probability interval is undefined for $|\theta| > 1$. Remark that also the conventional classical confidence scheme fails in this case.

Similarly all digital measurements like track measurements with proportional wire chambers or TDC time registration cannot be treated. Since the probability distributions are delta-functions the bounds are undefined.

### 3.4 Restriction due to unification

Let us assume that in a search for a Susy particle a positive result is found which however is compatible with background within two standard deviations. Certainly one would prefer to publish an upper limit to a measurement contrary to the prescription of the unified method.

### 3.5 Difficulty to use the error bounds

Errors associated to a measurement usually are used to combine results from different experiments or to compute other parameters depending on them. There is no prescription how this can be done in the unified approach. Averaging of data will certainly be difficult due to the bias introduced by asymmetric probability contours used to avoid unphysical bounds. Feldman and Cousins propose to use the conventional classical limits for averaging. Thus two sets of errors have to be documented.

### 3.6 Restriction to continuous variables

It is not possible to associate a classical confidence to discrete hypothesis.
3.7 Subjectivity

The nice property of a well defined coverage depends on pre-experimental analysis criteria: The choice of the selection criteria and of the confidence level as well as the decision to publish have to be done independently of the result of the experiment. This requirement is rather naive.

4 Conclusions

There are additional difficulties to those discussed above: The elimination of nuisance parameters and the treatment of upper Poisson limits with uncertainty in the background predictions pose problems. These may be tractable but certainly introduce further complications. The computation of the limits will be very computer time consuming in most cases. The essential objections, however, are those mentioned in sections 3.1, 3.3 and 3.5. It is absolutely intolerable that significant limits can be obtained with poor data and it is also essential to have useful error intervals. Feldman and Cousins are aware of the difficulties related to the inversion of significance and to biased errors and propose to publish additional information. This certainly is a sensible advice but does not justify classical limits. Most of the deficiencies of the conventional classical method remain unresolved in the unified approach.

The experimental information relative to a parameter can be documented by its likelihood function. The log-likelihood functions of different experiments can easily be combined without introducing biases simply by adding them. In most cases the likelihood function can be parametrized in a sensible way, as is common practice, by the parameters which maximize the likelihood and the values at $1/\sqrt{e}$ of the maximum. The latter define an error interval. In the case of Poisson limits the Bayesian limits with constant prior (see Table 1) provide a useful parametrization which avoids the difficulties of section 3.1. These pragmatic procedures, however, do not allow to associate a certain coverage to the intervals or limits. Coverage is the magic objective of classical confidence bounds. It is an attractive property from a purely esthetic point of view but it is not obvious how to make use of this concept.

References
[1] G. J. Feldman, R. D. Cousins, Phys. Rev. D 57 (1998) 1873.

[2] C. Caso et al., Europ. Phys. J. C 3 (1998) 1.

[3] J. O. Berger and R. P. Wolpert, The likelihood principle, Lecture notes of Institute of Mathematical Statistics, Hayward, Ca, edited by S. S. Gupta (1984)

[4] D. Basu, Statistical Information and Likelihood, Lecture notes in Statistics, Springer (1988)