Observations on BPS observables in 6d

Nadav Drukker* and Maxime Trépanier†

Department of Mathematics, King’s College London,
London, WC2R 2LS, United Kingdom

Abstract

We study possible geometries and R-symmetry breaking patterns that lead to globally BPS surface operators in the six dimensional $\mathcal{N} = (2,0)$ theory. We find four main classes of solutions in different subspaces of $\mathbb{R}^6$ and a multitude of subclasses and specific examples. We prove that these constructions lead to supersymmetry preserving observables and count the number of preserved supercharges. We discuss the underlying geometry, calculate their anomalies and present analogous structures for the holographic dual M2-branes in $AdS_7 \times S^4$. We also comment on the dimensional reduction of these observables to line and surface operators in 4d and 5d theories. This rich spectrum of operators are proposed as the simplest and most natural observables of this mysterious theory.

*nadav.drukker@gmail.com
†maxime.trepanier@kcl.ac.uk
1 Introduction

In the quest to ascend the precipice of the 6d $\mathcal{N} = (2, 0)$ theory, we are in great need of edges or cracks to hold on to. Surface operators are ideal candidates, being the intrinsic, low dimensional, non-local observables of these theories [1–3]. Already, recent work combining the powerful techniques of the superconformal index [4, 5], holographic entanglement entropy [6–11] and defect CFT techniques [12–14] lead to the exact calculation of the anomaly coefficients (see Section 4), thus giving the leading property of surface operators: the logarithmic divergence in their expectation values, which is an analog of the anomalous dimensions of local operators [15–19].

A distinguished class of surface operators are BPS observables, i.e. those that preserve some of the supercharges of the theory. For these operators, supersymmetry affords many simplifications and, from the experience with BPS Wilson loops in lower dimensional theories [20–28], we expect BPS surface operators to be amenable to a variety of nonperturbative techniques, providing an ideal playground to learn more about the mysterious 6d $\mathcal{N} = (2, 0)$ theories.

The obvious examples of BPS surface operators are the plane [29,30] and the sphere [31], which preserve half of the supercharges and are analysed in detail in [6–11,13,32–35]. There are no explicit expressions for the surface operators beyond the abelian theory and the holographic theory at large $N$, but they should be labeled by a representation of the ADE algebra of the theory. As far as the global symmetries, the 1/2 BPS operator is defined by a choice of plane in 6d, which breaks the 6d conformal symmetry $\mathfrak{so}(2, 6)$ to $\mathfrak{so}(2, 2) \times \mathfrak{so}(4)$, and a breaking of the $\mathfrak{so}(5)$ R-symmetry to $\mathfrak{so}(4)$, which can be realised by choosing a point on $S^4$. Accounting for supercharges, the full preserved symmetry is an $\mathfrak{osp}(4^*|2)^2$ subalgebra of the $\mathfrak{osp}(8^*|4)$ symmetry algebra of the vacuum.

This analysis can be generalised to surface operators with an arbitrary geometry $\Sigma \to \mathbb{R}^6$ and a local breaking of R-symmetry $n : \Sigma \to S^4$ [19]. Pointwise these operators preserve
half of the supersymmetries, but typically the supersymmetries are not compatible over the surface, thus can be called “locally BPS”.

In this paper we present several large families of truly BPS surface operators, requiring the supercharges to be compatible over the full surface. As a simple illustration, consider two planes extended respectively along \((x^1, x^2)\) and \((x^1, x^3)\). For the first we take \(n^I = \delta^{I1}\), for the second, \(n^I = \delta^{I2}\). The two operators so defined share half of their supercharges, and as the planes intersect along a line, we can take half of each and form a corner (which is then 1/4 BPS). The operators presented below are massive generalisations of this example.

A word on notations: we use \(x^\mu\) with \(\mu = 1, \ldots, 6\) as coordinates of 6d Euclidean space with metric \(\delta_{\mu\nu}\). Spacetime spinor indices are \(\alpha\) and \(\bar{\alpha}\) respectively for chiral and anti-chiral spinors, and for the fundamental of \(\mathfrak{sp}(4)\) we use the indices \(A\), all take four values. For the R-symmetry vectors (fundamental of \(\mathfrak{so}(5)\)) we employ \(I, J = 1, \ldots, 5\). On the surface \(\Sigma\) we use coordinates \(\sigma^a = (u, v)\) with \(a = 1, 2\) and the metric induced from the embedding \(h_{ab} = \partial_a x^\mu \partial_b x^\mu\) with its inverse \(h^{ab}\). We also use \(\epsilon^{ab}\) to denote the Levi-Civita tensor density (including a factor \(\det(h_{ab})^{-1/2}\)).

Our construction relies on spinor geometry. To formulate the problem, recall that the supercharges of the \(\mathcal{N} = (2, 0)\) theory are parametrised by two 16-component constant spinors \(\varepsilon_0\) and \(\bar{\varepsilon}_1\). They are respectively chiral and anti-chiral in 6d and transform also under \(\mathfrak{sp}(4)\). Since we work in Euclidean signature our spinors do not satisfy a reality condition. \(\varepsilon_0\) parametrises the 16 super-Poincaré symmetries, \(Q\), and \(\varepsilon_1\) the 16 superconformal ones, \(S\). Together \(\varepsilon_0 + \bar{\varepsilon}_1(x \cdot \bar{\gamma})\) are the conformal Killing spinors in flat space.

We use \(\gamma^\mu\) and \(\bar{\gamma}^\mu\) for the spatial gamma matrices in the chiral basis and \(\rho^I\) for the \(\mathfrak{so}(5)\) gamma matrices. They satisfy

\[
\gamma_\mu \gamma_\nu + \bar{\gamma}_\nu \gamma_\mu = 2 \delta_{\mu\nu}, \quad \gamma_\mu \bar{\gamma}_\nu + \gamma_\nu \bar{\gamma}_\mu = 2 \delta_{\mu\nu}, \quad \{\rho_I, \rho_J\} = 2 \delta_{IJ},
\]

and \(\gamma_\mu\) and \(\rho_I\) commute. Explicitly the indices are \((\gamma_\mu)^{\alpha}_{\dot{\alpha}}, (\bar{\gamma}_\mu)^{\dot{\alpha}}_{\alpha}, (\rho_I)^{A}_{\alpha}, (\bar{\rho}_I)^{\dot{A}}_{\dot{\alpha}}, (\varepsilon_0)^{\alpha A}\) and \((\bar{\varepsilon}_1)^{\dot{\alpha} \dot{A}}\), but we suppress them as there is no ambiguity. We also use the standard shorthand notation \(\gamma_{\mu\nu} \equiv (\gamma_\mu \gamma_\nu + \bar{\gamma}_\nu \gamma_\mu)/2\) for antisymmetrised products.

With these notations in hand we can be more precise about the conditions required to preserve supersymmetries. At any point along the surface, the supercharges preserved by the tangent plane to \(\Sigma\) and the choice of \(n\) in \(S^4\) are those satisfying the projector equation

\[
[\varepsilon_0 + \bar{\varepsilon}_1(x^\mu \bar{\gamma}_\mu)][\frac{i}{2} \epsilon^{ab} \partial_a x^\mu \partial_b x^\nu \gamma_{\mu\nu} + n^I \rho_I] = 0.
\]

This is a set of coupled equations depending on the embedding of the surface \(x^\mu(u, v)\) and the scalar coupling along the surface \(n^I(u, v)\). If \(n\) is a unit vector, then this matrix is half-rank, so pointwise half the components of \(\varepsilon_0, \bar{\varepsilon}_1\) satisfy it, which is the “locally BPS” notion already mentioned above.

To find global solutions to these equations with the same \(\varepsilon_0, \bar{\varepsilon}_1\) along the surface, we do not attempt a full classification but present four main classes of solutions. These four classes have \(\Sigma\) in different subspaces of \(\mathbb{R}^6\) and/or satisfying different constraints and with appropriate choices of \(n^I\). Each of the four classes has many examples with extra properties. To avoid clutter, we give special names to only two of those subclasses, and all the other special examples are explained in the appropriate sections. The main examples are:

**Type-\(\mathbb{R}\):** Such surfaces are the product of a curve \(x^I(u) \subset \mathbb{R}^5\) \((I = 1, \ldots, 5)\) and the \(x^6 = v\) direction. To guarantee that they are BPS we should choose the scalar coupling to be tangent
to the curve

\[ n^I(u, v) = \frac{\partial_u x^I}{|\partial_u x|}. \quad (1.3) \]

The notation \( x^I \) instead of \( x^\mu \) is common in our examples, as in topological twisting, and is a crucial ingredient to get BPS observables. Examples of surfaces in this class were previously studied in [36].

This choice of identification allows for an immediate generalisation of multiplying the right hand side by a fixed \( SO(5) \) matrix (or indeed replace \( x^6 \) by any other line), so in total \( S^5 \times SO(5) \) of examples. Such modifications are also possible to all the classes below, but we do not worry about counting these generalisations and fix convenient representatives.

**Type-C:** Viewing \( \mathbb{R}^6 = \mathbb{C}^3 \) (with any standard complex structure), we can take the surface to be any holomorphic curve with a fixed scalar direction, say \( n^1 \).

**Type-H:** Restricting the surfaces to \( \mathbb{R}^4 \), we can take an arbitrary oriented surface and get a BPS observable, as long as

\[ n^I = \frac{1}{2} \eta_{\mu\nu} \partial_a x^\mu \partial_b x^\nu \varepsilon^{ab}, \quad (1.4) \]

where \( \eta \) is the 't Hooft chiral symbol (see (2.3)), and \( I = 1, 2, 3 \), so \( n^I \) are in \( S^2 \). This has a nice interpretation in terms of the Gauss map, which we explain in Section 2. Particular special subclasses are surfaces with fixed \( n^I = (0, 0, 1) \) which are holomorphic with respect to an appropriate choice of complex structure, so they overlap with **Type-C** (except for the exchange of \( n^1 \to n^3 \)). Another special subclass are Lagrangian submanifolds, for which \( n^3 = 0 \). We denote those as **Type-L**. Lastly we should mention that for conical surfaces, the conformal transformation to surfaces in \( \mathbb{R} \times S^5 \) have been described and studied in great detail in [37].

**Type-S:** Any surfaces within an \( S^3 \subset \mathbb{R}^6 \) is BPS provided we choose

\[ n^I = \frac{1}{2} \varepsilon^{IJKL} \varepsilon_{ab} \partial_a x^J \partial_b x^K x^L. \quad (1.5) \]

\( x^L \) appears explicitly and this expression assumes a sphere centred around the origin. We also assume a unit sphere, our results can be trivially extended to a sphere of arbitrary radius by dimensional analysis. These surfaces are different from the restriction of the **Type-H** ansatz to \( S^3 \subset \mathbb{R}^4 \), but they do overlap at the vicinity of the north pole, where \( x^\mu \sim \delta^\mu_4 \) and using the 3-index antisymmetric tensor, both (1.4) and (1.5) reduce at leading order in \( |x^\mu - \delta^\mu_4| \) to

\[ n^I = \frac{1}{2} \varepsilon^{IJKL} \partial_a x^J \partial_b x^K \varepsilon^{ab}. \quad (1.6) \]

We will refer to this subclass as **Type-N**.

The paper is organised by themes. In Section 2 we present the geometry underlying these constructions. The proof of supersymmetry for these classes of examples is contained in Section 3. We discuss the Weyl anomaly in Section 4, the reduction to line and surface operators in lower dimensional theories in Section 5, and the holographic duals in Section 6. The properties pertaining to each of the constructions above are scattered between these sections. To aid the reader we provided extensive cross referencing. Also, we summarize the main classification in Table 1 and some of the most important examples in Table 2, with links to the relevant pages where they are discussed.
Table 1: Main classes and subclasses of BPS surface operators. For each geometry we list the minimal number of preserved supercharges, which type it belongs to, and list the equation with its anomaly and the associated 3-form defined on a subspace of \( \text{AdS}_7 \times S^4 \) that is compatible with supersymmetry.

Table 2: Examples of BPS operators with enhanced supersymmetry.

2 Geometry

In all our examples, we construct BPS observables by appropriately choosing the surface and a vector \( n/I \) over it. The details of these choices rely on varied notions of geometry of surfaces, which we review in turn for each type.

**Type-\( \mathbb{R} \):** These surfaces are defined in terms of a curve in \( \mathbb{R}^5 \) and \( n/I (1.3) \) is its (normalised) tangent vector. Clearly for a general enough curve we can get any closed path \( n/I(u) \) on \( S^4 \). If the curve is restricted to an \( \mathbb{R}^d \) subspace, then \( n/I \) are constrained to an \( S^{d-1} \). As we show in the next section, such loops preserve \( \text{max}(1, 2^{4-d}) \) of the Poincaré supercharges \( Q \).
Type-N: To ease into the other examples, let us start with this subclass of surfaces in $\mathbb{R}^3$ where $n^I$ is the (unit) normal vector with a chosen orientation (1.6). This map from a surface to $S^2$ is known as the Gauss map and its degree is half of the Euler characteristic of the surface, which is important for the calculation of the anomaly below.

Type-H: The Gauss map has an interesting generalisation to surfaces in $\mathbb{R}^4$ [39], which we can use to pick our vector $n$. At any point on the surface the tangent space is a plane in $\mathbb{R}^4$. The space of these planes is the Grassmanian $G_2(\mathbb{R}^4)$, which is a homogenous space defined as the quotient of the orthogonal group $SO(n)$

$$G_k(\mathbb{R}^d) = \frac{SO(d)}{SO(k) \times SO(d-k)}.$$

(2.1)

For planes in 4d, the image of the Gauss map is thus a point on $G_2(\mathbb{R}^4) = S^2 \times S^2$.

The ansatz (1.4) restricts $n^I$ to $S^2$, which is just one of the two spheres of the Gauss map. To understand it, recall that 2-forms in 4d can be decomposed into self-dual and anti-self-dual 2-forms. $\eta^\mu_\nu$ projects to the self-dual part, which are in the $(3, 1)$ representation of $su(2)_L \times su(2)_R \simeq so(4)$. Another way to see it is that as a hyperkähler manifold, $\mathbb{R}^4$ has $S^2$ worth of Kähler forms $\omega^I$ (compatible with the orientation $\varepsilon_{1234} = 1$) and any surface is locally holomorphic with respect to one of them. We can write a basis for them in terms of the 't Hooft symbols

$$\omega^I = \frac{1}{2} \eta^I_\mu dx^\mu \wedge dx^\nu.$$

(2.2)

Explicitly we use

$$\omega^3 = dx^1 \wedge dx^2 + dx^3 \wedge dx^4,$$

$$\omega^2 = dx^3 \wedge dx^1 + dx^2 \wedge dx^4,$$

$$\omega^1 = dx^1 \wedge dx^4 + dx^2 \wedge dx^3.$$

(2.3)

If the map degenerates to a point $n^3 = 1$, the self-dual projection of the tangent space is a linear combination of the $(x^1, x^2)$ and $(x^3, x^4)$ planes, and in fact is holomorphic with respect the complex structure associated to $\omega^3$, hence of Type-C. The converse to this is a restriction to $n^3 = 0$ everywhere, which means that the pullback to $\Sigma$ of $\omega_3$ vanishes. Surfaces satisfying this condition are Lagrangian submanifold, as in Type-L, and the half of the Gauss map represented by $n^I$ reduces to an equator of $S^2$.

Type-S: These are also surfaces in $\mathbb{R}^4$, but now restricted to $S^3$ with $n^I$ as in (1.5). One way to understand this construction is that locally we have a surface in $\mathbb{R}^3$ with a version of the Type-N ansatz, so a local Gauss map, where the target $S^2$ is the one perpendicular to the vector $x$.

Note that restricting $\Sigma$ to any great $S^2 \subset S^3$ makes $n^I$ a constant, so these are 1/2 BPS spheres, conformal to the plane. A more general subclass are surfaces foliated by arcs between the north and south pole, which describe a geometry reminiscent of a banana. In this case $n^I$ belongs to an $S^2$, which is the sphere we get for the north pole Type-N surfaces. These “banana” surfaces can be obtained by a conformal transformation (stereographic projection) of the cones in $\mathbb{R}^3$ and in fact the value of $n^I$ exactly matches those of cones of Type-N also away from the north pole. These surfaces also have enhanced supersymmetry.

One more subclass of Type-S are non-maximal or “latitude” spheres. They resemble in some ways the latitude Wilson loops of [40,26] and their $n^I$ image is also a latitude.
sphere there. One can think of those as interpolating between the 1/2 BPS sphere and the Type-N sphere. Similar interpolations are also possible between Type-L and Type-C within Type-H, which are surfaces with fixed $n^3$, so the Gauss map is into a latitude on $S^2$.

**Type-C**: In this case we choose a complex structure on $\mathbb{R}^6$ and take any holomorphic curve. Regardless of how complicated the surface is, a constant $n^I = \delta^{I1}$ yields a BPS surface. For a generic surface, as we show in the next section, it preserves two $Q$’s. If it is located in a $\mathbb{C}^2$ subspace (so overlapping with Type-H) it preserves four, and the plane $\mathbb{C}$ preserves eight.

### 3 Supersymmetry

The approach for proving that our examples preserve some supersymmetries, or in fact for constructing the examples relies on a generalisation of the idea already presented in the introduction. For each class of examples we can reduce the projector equation (1.2) over the whole surface to a set of independent constraints on $\varepsilon_0$ and $\bar{\varepsilon}_1$ associated with each independent tangent plane of the surface (and its vector $n$).

The number of supercharges preserved by a given surface operator depends on the number of constraints needed to satisfy (1.2). Since each independent constraint is half-rank (this can be seen pointwise from the projector equation) and $\varepsilon$ has 32 components, in the most general case we could impose five independent conditions leaving a single Killing spinor. Each Killing spinor $\varepsilon$ satisfying the constraints parametrises a supersymmetry transformation given by

$$\varepsilon_0 Q + \bar{\varepsilon}_1 S. \quad (3.1)$$

The analysis of supersymmetry captures many geometrical properties of the surfaces and the associated map $n^I$ presented in Section 2.

**Type-R**: This case is particularly simple and closely resembles the beautiful construction of [41]. Choosing matching bases for the 2 unit $\mathfrak{so}(5)$ vectors $\partial_u x^I$ and $n^I$ and identifying them via (1.3), the projector (1.2) factorises as

$$[\varepsilon_0 + \bar{\varepsilon}_1 (x^\rho \bar{\gamma}_\rho)] (i \gamma_{I6} + \rho_I) n^I(u). \quad (3.2)$$

For a generic curve $x^I(u)$, all special supersymmetries are broken ($\bar{\varepsilon}_1 = 0$) and we focus only on $\varepsilon_0$. At any point, the constraint imposed on the Killing spinor is a linear combination of the five projectors acting on $\varepsilon_0$

$$i \gamma_{I6} + \rho_I, \quad I = 1, \cdots, 5. \quad (3.3)$$

To count the number of independent constraint we multiply them from the right by $\frac{1}{2} \gamma_{56}$ and define

$$a_I^\dagger = \frac{1}{2} (\gamma_{I5} + i \gamma_{56} \rho_I), \quad a_I = \frac{1}{2} (-\gamma_{I5} + i \gamma_{56} \rho_I), \quad I = 1, \cdots, 4. \quad (3.4)$$

These operators satisfy an oscillator algebra, which shows they are independent

$$\{a_I, a_J^\dagger\} = \delta_{IJ}, \quad \{a_I, a_J\} = \{a_I^\dagger, a_J^\dagger\} = 0. \quad (3.5)$$
Note that since our gamma matrices are chiral, they satisfy $\gamma_1\bar{\gamma}_2\gamma_3\bar{\gamma}_4\gamma_5\gamma_6 = i$, or equivalently $\gamma_1\bar{\gamma}_2\gamma_3\bar{\gamma}_4 = -i\gamma_5\gamma_6$. Likewise, $\rho_1\rho_2\rho_3\rho_4 = \rho_5$, so the spinor annihilated by all four conditions $a_I^I$ also satisfies $\varepsilon_0(i\gamma_{15} + \rho_5) = 0$.

Hence if the curve is a straight line, $n$ is a constant vector and the projector equation imposes a single condition. This is the plane, which preserves eight $Q$ supercharges (and exceptionally also eight $S$). For each extra dimension that the curve visits, the number of supercharges reduces by a half, except for the last dimension, that does not involve a new projector. Generically there are no solutions with nonzero $\varepsilon_1$, so no preserved special supersymmetries $S$.

**Type-C:** To make our discussion concrete it is convenient to choose an explicit complex structure for $\mathbb{R}^6$, represented here via the Kähler form

$$\frac{1}{2} \omega_{\mu\nu} dx^\mu dx^\nu = dx^1 \wedge dx^2 + dx^3 \wedge dx^4 + dx^5 \wedge dx^6.$$  \hfill (3.6)

Holomorphic curves with respect to this complex structure are those satisfying the Cauchy-Riemann equations

$$\partial_a x^\mu = \omega_{\mu\nu} \varepsilon_{ab} h^{bc} \partial_c x^\nu.$$  \hfill (3.7)

The natural constraints we impose on $\varepsilon_0$ are then

$$\varepsilon_0(i\gamma_\mu + \omega_\mu^\nu \gamma_\nu \rho_1) = 0.$$  \hfill (3.8)

Of those six equations only the ones with $\mu = 1, 3, 5$ are independent, as for example

$$\varepsilon_0(i\gamma_2 + \omega_2^\nu \gamma_\nu \rho_1) = \varepsilon_0(i\gamma_2 - \gamma_1 \rho_1) = \varepsilon_0(\gamma_2 \rho_1 + i\gamma_1) i\rho_1 = \varepsilon_0(i\gamma_1 + \omega_1^\nu \gamma_\nu \rho_1)i\rho_1 = 0.$$  \hfill (3.9)

Now for a generic holomorphic curve, we use the above conditions on $\varepsilon_0$ and then the Cauchy-Riemann equation to write

$$\varepsilon_0 i\gamma_\mu \nu \partial_a x^\mu \partial_a x^\nu \varepsilon_{ab} = -\varepsilon_0 \omega_\mu^\sigma \gamma_\sigma \bar{\gamma}_\nu \rho_1 \partial_a x^\mu \partial_b x^\nu \varepsilon_{ab} = -\varepsilon_0 \gamma_\sigma \bar{\gamma}_\nu \rho_1 \partial_a x^\sigma \partial_b x^\nu h^{ab}. $$ \hfill (3.10)

The symmetry of the term on the right under $\sigma \leftrightarrow \nu$ reduces the right hand side to $-\varepsilon_0 \rho_1$ (where we assume unit normalised tangent vectors), so the projector equation (1.2) is satisfied.

Again, each of the projectors (3.8) reduces the rank of $\varepsilon_0$ by half, giving eight, four and two $Q$ supercharges for the plane, holomorphic curves in $\mathbb{R}^4$ and $\mathbb{R}^6$ respectively.

**Type-R:** Here we have arbitrary surfaces in $\mathbb{R}^4$. Choosing $n$ according to (1.4), for each tangent plane we can assign a linear combination of the projectors (see (2.3))

$$\frac{1}{2}(i\gamma_{12}\rho_3 + 1), \quad \frac{1}{2}(i\gamma_{31}\rho_2 + 1), \quad \frac{1}{2}(i\gamma_{23}\rho_1 + 1),$$

$$\frac{1}{2}(i\gamma_{34}\rho_3 + 1), \quad \frac{1}{2}(i\gamma_{24}\rho_2 + 1), \quad \frac{1}{2}(i\gamma_{14}\rho_1 + 1).$$ \hfill (3.11)

In contrast to the Type-C oscillators above, all these commute rather than anticommute.

Not all the projectors are independent. For a generic surface in $\mathbb{R}^4$, we need to satisfy all of them. Taking any projector of the first line and its complement on the second line imposes $\varepsilon_0 = -\varepsilon_0\gamma_{1234}$, which is a restriction to the supersymmetries antichiral with respect to 4d chirality. On these supersymmetries, all the projectors of the second line and equal to those.
of the first, so we are left with 3 independent conditions on the antichiral spinors. Hence generic surfaces preserve a single $Q$.

For a Lagrangian submanifold of Type-L one component of $n$ is zero (e.g. $n^3 = 0$). In that case we only need to take 2 columns of projectors out of (3.11), and this imposes again 2 conditions on antichiral spinors, so leaving two $Q$.

Conversely if $n^3 = 1$ is constant (and all other components vanish), then the projector equation imposes only the first column of (3.11), and by the discussion above it preserves four $Q$. These are holomorphic curves, as in Type-C (except for the choice of $n^3$ instead of $n^1$).

The Type-N surfaces have the projectors on the first line of (3.11), which are all independent, so a generic surface of this type preserves two $Q$ supercharges. Note that the projectors on the second line are the same as (3.3) of Type-R if we replace $x^6 \leftrightarrow x^4$ in that construction, also preserving two $Q$s and manifesting another example of intersections between our examples.

**Conical surfaces:** The discussion so far focused only on the Poincaré supercharges $Q$, but if the surfaces are conical with the tip at the origin, they also preserve conformal supercharges $S$ (for other locations of the tip, they would preserve other combinations of $Q$ and $S$). We can express the conical surfaces by $x^\mu = u \xi^\mu(v)$ where $\xi$ is a curve on an appropriate sphere.

The simplest example is the plane, where $\xi$ is a circle centred around the origin. In addition to the eight $Q$’s, it is annihilated by eight $S$’s. The only other examples of Type-R are when the curve $x^\mu(u)$ is comprised of two rays meeting at any angle, which is a slight generalisation of the example mentioned in the introduction. The surface is then two half-planes glued along a line, so can be called crease. The base of the cone $\xi^\mu$ is comprised of two half-circles on $S^2$, so two longitude lines.

To see the extended supersymmetry, specializing to a conical surface over $\xi^\mu$, (1.2) becomes ($\dot{\xi} = d\xi/dv$)

$$[\varepsilon_0 + \bar{\varepsilon}_1(u \xi \cdot \bar{\gamma})] [i \varepsilon^{ab} \gamma_{\mu\nu} \xi^\mu \dot{\xi}^\nu + n_I \rho_I] = 0.$$  \hspace{1cm} (3.12)

Of course $\xi^a \bar{\gamma}_\rho \gamma_{\mu\nu} \xi^\mu \dot{\xi}^\nu = \xi^\nu \bar{\gamma}_\nu$, but more importantly, we can also permute the first gamma matrix through, such that the equation becomes

$$\varepsilon_0 \left[i \varepsilon^{ab} \gamma_{\mu\nu} \xi^\mu \dot{\xi}^\nu + n_I \rho_I \right] + \bar{\varepsilon}_1 \left[-i \varepsilon^{ab} \bar{\gamma}_{\mu\nu} \xi^\mu \dot{\xi}^\nu + n_I \rho_I \right] (u \xi \cdot \bar{\gamma}) = 0.$$  \hspace{1cm} (3.13)

So the equations for $\varepsilon_1$ involve the complementary projectors to $\varepsilon_0$, and certainly the dimension of the space of solutions for the two are the same.

**Tori:** Within the Type-L surfaces in $\mathbb{R}^4$, the simplest representatives are tori which are the product of circles in the $(x^1, x^2)$ and $(x^3, x^4)$ planes. In addition to the two $Q$ supercharges that any of the Lagrangian surfaces preserve, these also preserve a pair of $S$ generators. The derivation is similar to (3.13) and uses that for these surfaces $x^2 = 1$.

**Type-S:** To understand this class, recall that it includes 1/2 BPS spheres, which are not annihilated by any one $Q$ or $S$, but by 16 linear combinations of the form $Q + S$. We can get this by a stereographic map of the plane, and likewise we can understand the surfaces foliated by arcs between the north and south pole by their stereographic projection to cones of Type-N, which preserve two $Q$ and two $S$. So the analogous on $S^3$ preserve four independent
combinations $Q+S$. As a special case, the conformal map of the crease give surfaces comprised of two hemispheres of different great $S^2$ and preserves eight combinations $Q+S$ (to visualise this, think of two longitudinal lines, or hemicircles, meeting at the north and south pole, and replace them with hemispheres).

The most general surface on $S^3$ preserves two supercharges. To see this, write the projector equation (1.2) as

$$\frac{1}{2} \varepsilon^{ab} \partial_a x^\mu \partial_b x^\nu [\varepsilon_0 + \tilde{\varepsilon}_1 (x \cdot \tilde{\gamma})] [i \gamma_{\mu\nu} + \varepsilon_{\mu
u\sigma 1} x^\sigma \rho_1] = 0.$$  

We want this equation to hold for all orientations, meaning all $\mu, \nu$, so expanding and reorganising (assuming orthogonality) we find

$$[\varepsilon_0 i \gamma_{\mu\nu} + \tilde{\varepsilon}_1 \tilde{\gamma}^\sigma \varepsilon_{\mu\nu\sigma 1} \rho_1] + [\varepsilon_0 \varepsilon_{\mu\nu\sigma 1} \rho_1 + i \tilde{\varepsilon}_1 \tilde{\gamma}_{\mu\nu\sigma}] x^\sigma = 0.$$  

Since $\varepsilon_0, \tilde{\varepsilon}_1$ are constant the two terms in brackets need to vanish separately, and in fact imposing one implies the other.

To count the number of independent constraints, note that referring to the ansatz (1.5) every tangent plane on $S^3$ is associated to a vector $n$ on $S^3$. For the latter there is obviously four possible independent vectors, so for a generic surface we impose four independent conditions on $\varepsilon_0, \tilde{\varepsilon}_1$. This leaves 2 preserved supercharges.

## 4 Anomaly

Recall that surface operators have Weyl anomalies whose form is constrained by the Wess-Zumino consistency conditions to be a sum of conformal invariants [15–17, 19]. They can be expressed as the integral of the density

$$A_{\Sigma} = \frac{1}{4\pi} \left[ a_1 R^\Sigma + a_2 (H^2 + 4 \text{ tr } P) + b \text{ tr } W + c (\partial n)^2 \right].$$  

The first invariant is $R^\Sigma$, which is the Ricci scalar of the induced metric $h_{ab}$. The second piece includes $H^\mu$, the mean curvature vector and $\text{tr } P = h^{ab} P_{ab}$, the trace of the pullback of the Schouten tensor. $W$ is the pullback of the Weyl tensor and $(\partial n)^2 = h^{ab} \partial_a n^I \partial_b n^I$ is the norm of the variation of $n$ over the surface.

We work on flat euclidean space so $P = W = 0$. It is useful in the following to have an explicit expression for the other invariants, and they can be written nicely in terms of the second fundamental form $\Pi$ as

$$\Pi^\mu_{ab} = \partial_a \partial_b x^\nu (\delta^\mu_\nu - h^{cd} \partial_c x_\nu \partial_d x^\mu), \quad H^\mu = h^{ab} \Pi^\mu_{ab}, \quad R^\Sigma = \Pi^\mu_{ab} \Pi^a_{cd} \varepsilon^{bc}.$$  

The term $(\delta^\mu_\nu - h^{cd} \partial_c x_\nu \partial_d x^\mu)$ is a projector to the components normal to the surface.

Another simplification occurs in our case. While the Wess-Zumino consistency condition allow the four independent coefficients $a_1, a_2, b$ and $c$ above, supersymmetry imposes $a_2 = -c$ and $b = 0$ for locally BPS surface operators [42, 13]. The anomaly density then reduces to

$$A_{\Sigma} = \frac{1}{4\pi} \left[ a_1 R^\Sigma + c (\partial n^2 - H^2) \right].$$  

The constants $a_1$ and $c$ are observables that depend on the theory (the choice of ADE algebra underlying the $\mathcal{N} = (2, 0)$ theory), and the representation of the surface operator (at
least for the $A_N$ theories they are classified by representations of that algebra. According to [8–11, 5, 14] they are given by

$$a_1 = \frac{1}{2} (\Lambda, \Lambda), \quad c = C_2(\Lambda) - a_1,$$

(4.4)

where $\Lambda$ is the highest weight defining the representation and $C_2(\Lambda)$ is the quadratic Casimir of the representation. For the fundamental representation of $\mathfrak{su}(N)$ this is

$$a_1^{(N)} = \frac{1}{2} - \frac{1}{2N}, \quad c^{(N)} = N - \frac{1}{2} - \frac{1}{2N},$$

(4.5)

but we leave them as $a_1$ and $c$ in the following.

The above expressions give a precise expression for the anomaly of all the surfaces studied in this paper, requiring just the evaluation of the integrals of the expressions in (4.3). But as in all our examples the vector $n$ depends in varying ways on $x^\mu$, we can evaluate $(\partial n)^2$ relying on the geometry as studied in Section 2 and simplify the expression for the anomaly even further.

**Type-R:** For BPS surfaces satisfying (1.3), the anomaly vanishes since $(\partial n)^2 = H^2$ and the Ricci scalar is zero. In the abelian theory one can make a stronger statement, and using the field theory propagators given in [43–45, 19] one can check that the expectation value also vanishes identically. It is tempting to conjecture that this is true in general.

**Type-C:** In this case $n^I$ is a constant, so $(\partial n)^2$ vanishes. $H^2$ also vanishes, which can be seen by a direct computation, or by noting that the Kähler form $\omega$ (3.6) act as a calibration for these surfaces, so that any holomorphic curve is a minimal surface. The only contribution to the anomaly then comes from the Ricci scalar, so it is proportional to the genus of the curve

$$\int_\Sigma A^C_\Sigma \text{vol}_\Sigma = a_1 \chi(\Sigma).$$

(4.6)

Note that for non-compact surfaces the genus is defined by this integral, and may not match that expected for the analogous curve in $\mathbb{CP}^3$.

**Type-H:** Here the anomaly is more interesting. Using (1.4) and the definition of the geometric invariants (4.2), we can write

$$(\partial n)^2 - H^2 = \eta^L_{\mu\nu} \eta_{\rho\sigma} (\Pi_{ab}^\mu \partial_a x^\nu)(\Pi_{cd}^\nu \partial_f x^\sigma) \left( \varepsilon^{be} \varepsilon^{df} h_{ac} - \frac{1}{2} h^{ab} h^{cd} h^{ef} \right).$$

(4.7)

The ’t Hooft symbol (2.3) satisfy

$$\eta^L_{\mu\nu} \eta^L_{\rho\sigma} = \delta_{\mu\rho} \delta_{\nu\sigma} - \delta_{\mu\sigma} \delta_{\nu\rho} + \varepsilon_{\mu\rho\sigma}.$$  

(4.8)

If we focus on the $\delta$ contractions, only the first is nonzero. We identify $\delta_{\nu\sigma} \partial_a x^\nu \partial_f x^\sigma = h_{ef}$, and this reduces the last term to $-\varepsilon^{ac} \varepsilon^{bd}$, so we immediately recognize $-\mathcal{R}_\Sigma$ from (4.2).

The contraction involving $\varepsilon_{\mu\rho\sigma}$ gives us the Hodge dual of the 2-form $(\Pi_{cd}^\nu \partial_f x^\sigma) dx^\rho dx^\sigma$. This interchanges the tangent and normal space, and one can verify that it calculates the curvature of the normal bundle, which has a coordinate expression

$$\mathcal{R}_\perp = -\frac{1}{2} \varepsilon_{\mu\rho\sigma} (\Pi_{ab}^\mu \Pi_{cd}^\nu h^{bc} \varepsilon^{ad}) \varepsilon^{ef} \partial_a x^\rho \partial_f x^\sigma.$$  

(4.9)
Its integral, the Euler class of the normal bundle, is also equal in 4d to the self-intersection number (denoted $[\Sigma] \cdot [\Sigma]$), see e.g. \[46\].

The term proportional to $c$ in the anomaly density is then

$$(\partial n)^2 - H^2 = -\mathcal{R}^\Sigma + \mathcal{R}^\Sigma_\perp.$$  \hspace{1cm} (4.10)

Including the $a_1$ anomaly and integrating over the surface, we obtain a topological quantity

$$\int_\Sigma \mathcal{A}^\Sigma_\Sigma \text{vol}_\Sigma = (a_1 - c)\chi(\Sigma) + c[\Sigma] \cdot [\Sigma].$$  \hspace{1cm} (4.11)

This has a natural interpretation in terms of the Gauss map. Recall the image of the Gauss map is two $S^2$'s and $n^I$ is in one of these spheres. The topological invariant calculated by integrating $-\mathcal{R}^\Sigma + \mathcal{R}^\Sigma_\perp$ is (-2 times) the degree of this map $\Sigma \to \mathbb{S}^2$.

It is interesting to compare this result with our discussion for holomorphic surfaces in $\mathbb{C}^2$. As argued around (4.6), for holomorphic surfaces $(\partial n)^2 - H^2 = 0$. Indeed in this case the Gauss map gives a constant $n$ and certainly the degree of the map vanishes, and indeed the self-intersection is equal to the euler characteristic. The same is true also for Lagrangian submanifolds, as the image is on the equator of $S^2$, so again the degree vanishes.

**Type-S**: Here we simply expand $(\partial n)^2$ using (1.5) to obtain

$$(\partial n)^2 = \partial_\mu \partial_\nu x^\mu \left( \delta^{\mu \nu} - h^{\mu \nu} \partial_\alpha x^\alpha \partial_\beta x^\beta \right) \partial_\alpha \partial_\beta x^\nu h^{ae} h^{bf} - 2.$$  \hspace{1cm} (4.12)

The factor 2 seems surprising at first, as it leads to a term in the anomaly proportional to the area of the surface $\Sigma$. This in fact is not in contradiction to conformal invariance, as this construction has a dimensionful parameter, the radius of the sphere (which we set to one). As for the first term, we can identify the second fundamental form from (4.2) to rewrite this as

$$(\partial n)^2 - H^2 = -\mathcal{R}^\Sigma - 2.$$  \hspace{1cm} (4.13)

The anomaly is then

$$\int_\Sigma \mathcal{A}^\Sigma_\Sigma \text{vol}_\Sigma = (a_1 - c)\chi(\Sigma) - c\frac{\text{vol}(\Sigma)}{2\pi}.$$  \hspace{1cm} (4.14)

For infinitesimal surfaces on $S^2$ we should recover the anomaly of Type-N. In that limit the area of the surface vanishes giving (4.11), since in 3d the self-intersection number (4.9) is always zero.

### 4.1 Conical singularities

Surfaces with conical singularities have two sources of anomalies. First there are the usual anomalies discussed above but whose density may have a singular contribution at the apex of the cone. Second, the tips are distinguished points, so we may find new anomalies when integrating finite quantities over a scale-invariant cone. In fact, for non-BPS surfaces these two sources of divergences can conflate to log$^2\epsilon$ divergences [47–50, 19]. For BPS surfaces they split into two independent contributions. We discuss them in turn.

We start with a cone over a curve $\mathcal{K} \subset S^2$ (parametrised by $\xi^a$) of Type-N, which is a subclass of Type-$\mathcal{H}$. The self-intersection number (4.9) vanishes in three dimensions, so the anomaly (4.11) is determined solely by the Euler characteristic. The Ricci scalar vanishes
along the cone, but diverges at the tip. It can be regularised by smoothing the tip at distances $u \leq \epsilon$ to form the regularised surface $S_K$. Using the Gauss-Bonnet theorem

$$\frac{1}{4\pi} \int_{S_K} R^\Sigma \, \text{vol}_\Sigma = \chi(S_K) - \frac{1}{2\pi} \int_K \kappa_g \, \text{vol}_K.$$  \hfill (4.15)

Note that we defined here the anomaly for the non-compact cone without a boundary term at infinity, which would be on the left hand side, so instead it is on the right. If we choose the disc topology for $S_K$, the first term gives 1.

To evaluate the boundary contribution we can take our parametrisation $\xi^\mu$ to be unit speed, so that the geodesic curvature $\kappa_g$ is constant and equal to $1/\epsilon$ when evaluated on the sphere of radius $\epsilon$. The line integral then calculates the arc length of the curve $K$, which is $\epsilon L_K$ ($L_K$ being the arc length on a unit sphere).

Therefore the anomaly is

$$\int_\Sigma A_\Sigma \, \text{vol}_\Sigma = \left(1 - \frac{L_K}{2\pi}\right) (a_1 - c).$$  \hfill (4.16)

The story for general cones of Type-$\mathbb{H}$ is more interesting, since the curve $K$ can have the topology of a nontrivial knot on $S^3$. In the case of the unknot, we could choose the disc topology for our regularisation, but more generally this is a Seifert surface (see e.g. [51]). Given a knot there are many different possible such surfaces of different topology. This is easy to see already in the case of the disk to which we could add any number of handles, which would modify the anomaly above to

$$\left(1 - 2n - \frac{L_K}{2\pi}\right) (a_1 - c), \quad n \in \mathbb{N}. \hfill (4.17)$$

For a general knot there would be Seifert surfaces of lowest genus, whose Euler characteristic would replace the 1 in this equation, and other regularisations would still allow for the integer $n$. A slight generalisation to this result is to allow for multiple curves $K_i$ defining a link. There is again a Seifert surface of lowest genus whose boundary is the link, and in addition one needs to replace $L_K$ by $\sum_i L_{K_i}$.

The discussion thus far focused on regular surface anomalies. They are localized at the apex of the cone and can be calculated by choosing an appropriate regularisation of the conical singularity. The second type of divergences arising from cones come from terms that would be finite upon regularisation. Consider a cone with no anomaly density (apart from the apex) and restrict to an open subset $u \in (u_{\text{min}}, u_{\text{max}})$. While the contribution to the expectation value is finite for any such open subset, it would naturally scale like $\log(u_{\text{min}}/u_{\text{max}})$ and, after taking the limit $u_{\text{min}} \to \epsilon$, contribute a new term to the anomaly.

An example of such surfaces are the cones of Type-$\mathbb{H}$. They are related to BPS Wilson loops in 5d, see the discussion in Section 5.1 below.

We can consider also surfaces that are not globally conical, but have conical singularities. Examples are the surfaces foliated by hemicircles of Type-$S$, conformal to Type-$N$ cones. Those have the topology of the sphere, so we expect that the anomaly does not require special regularization.

### 4.2 Anomaless surfaces

Surfaces that are not anomalous can have finite expectation values, which one may hope to be able to calculate exactly, as is done for BPS line operators in 3d and 4d (as well as other observables).
In the examples above we found that those of Type-$\mathbb{R}$ are indeed anomalous, though we also expect their expectation value to vanish identically. This is certainly true in the free abelian theory, where using the propagators of $[19]$ and the ansatz (1.3) one can check that the propagator for the 2-form $B_{\mu\nu}$ exactly cancels the scalar $\Phi'$ propagator.

Both the anomaly of Type-$\mathbb{C}$ (4.6) and Type-$\mathbb{H}$ (4.11) are purely topological, so any surface with vanishing $\chi(\Sigma)$ and self-intersection number is anomalous. This includes all (topological) tori of Type-$\mathbb{H}$, with examples in Type-$\mathbb{L}$ and Type-$\mathbb{N}$.

The formula for the anomaly of Type-$\mathbb{H}$ cones, (4.17) vanishes only for $n = 0$ and $L_K = 2\pi$. This means it is a cone over the unknot with zero deficit angle and includes the crease of arbitrary opening angle (and in particular the plane). These surfaces cannot have finite expectation values, as they are non-compact, Note that by conformally transforming to a compact manifold like the sphere (or the crease generalisation of it discussed above equation (3.14)) changes the topology, so those examples are in fact anomalous.

Finally, we can look for cases where the surfaces are anomalous only for particular values of $a_1$ and $c$ (4.4). For example, if we restrict to surfaces in the fundamental representation, $c^{(N)} = (2N + 1)a_1^{(N)}$ (4.5). Now looking at the Type-$\mathbb{S}$ anomaly (4.14), this vanishes for

$$\frac{\text{vol}(\Sigma)}{4\pi} = -\chi(\Sigma)\frac{N}{2N + 1}.$$  \hspace{1cm} (4.18)

For example for $\chi = -2$, or genus 2 surfaces, the resulting area would be close to that of a maximal $S^2$. For higher genus, the surfaces would have to be even larger.

For Type-$\mathbb{H}$ surfaces the analogous equation (4.11) vanished for

$$[\Sigma] \cdot [\Sigma] = \chi(\Sigma)\frac{2N}{2N + 1}.$$  \hspace{1cm} (4.19)

The simplest possible solution would require Euler characteristic $\chi = -4N - 2$ and self-intersection number $[\Sigma] \cdot [\Sigma] = -4N$. We do not know whether there are solutions to these conditions. One can of course also go to other representations using the expressions in (4.4).

The expressions in (4.4) do not extend to the abelian theory, where $a_1 = c = 1/2$. In this case any surface of Type-$\mathbb{H}$ with vanishing self-intersection number is anomalous. In the large $N$ limit $a_1 \ll c$, so the classical supergravity calculation $[52,19]$ is only sensitive to the $c$ anomaly. In this limit all Type-$\mathbb{H}$ surfaces of equal Euler character and self-intersection number as well as all Type-$\mathbb{C}$ surfaces are effectively anomalous.

5 Dimensional reduction

The inspiration for this project comes from the rich history of BPS Wilson loops in $\mathcal{N} = 4$ SYM in 4d $[20–22,41,26]$. It is therefore natural to examine whether any of the families of surfaces uncovered here arise as the dimensional uplift of BPS loops in 4d or 5d. Likewise we would like to address the relation to BPS surfaces in lower dimensions.

5.1 Wilson and ’t Hooft loops

The classes of BPS loops in $\mathcal{N} = 4$ SYM have been completely classified $[53]$ and it is easy to check that they are indeed related to Type-$\mathbb{R}$ and conical Type-$\mathbb{H}$ surfaces, as follows.
To get line operators in lower dimensions we need to wrap the surface operator on a compact direction, so they need to be homogeneous in one direction. The simplest example is Type-$\mathbb{R}$ surfaces when the theory is compactified along the $x^6$ direction. This gives line operators in 5d and upon further reduction, lines in 4d. As the 4d gauge coupling is related to the modulus of the compactification torus, depending on which cycle of the torus is $x^6$, we end up with either Wilson, ’t Hooft or dyonic operator. The Wilson loops are clearly of the class identified by Zarembo in [41], the the ’t Hooft loops are their S-duals [54]. Note that Type-$\mathbb{H}$ surfaces extended along say $x^4$ are identical to Type-$\mathbb{R}$ ones in $\mathbb{R}^3$ with $x^6$ replaced by $x^4$.

Part of the reason we expect Type-$\mathbb{R}$ surfaces to have vanishing expectation value in 6d is that this is true for the loops in 4d [55–57].

The other example arises from conical surfaces extended along the radial direction. We can act with a conformal transformation that maps $\mathbb{R}^6 \to \mathbb{R} \times S^5$ by taking the log of the radial coordinate. Conical surfaces are mapped to surfaces extending along $\mathbb{R}$ and wrapping a curve in $S^5$. Compatifying, we end up with 5d Yang-Mills on $S^5$, and the surface operators are mapped to Wilson loops in that theory.

Recall that conical surfaces may have two types of divergences. Anomalies arising from singular curvature at the apex are now lost, as we changed the topology of the surface to a cylinder. The second type of divergences are easier to understand in this cylinder picture, as a finite expectation value for the 5d Wilson loops uplifts to a finite result for the compactified surfaces. The expression is necessarily extensive in the compactification radius and in the uncompactified limit, which is conformal to $\mathbb{R}^6$ studied in this paper, diverges. This is the source of the second type of cone anomalies alluded to in Section 4.1.

Wilson loops in 5d following the 4d ansatz in [26] and their uplift to surface operators in 6d were studied in [37]. That construction is also based on the chiral $\eta$ symbol and indeed maps to our Type-$\mathbb{H}$ cones. Using localization, one is able to evaluate the Wilson loops explicitly and the answer is proportional to that of Wilson loops in 3d topological Chern-Simons theory [58]. The proportionality constant can indeed be identified with the compactification radius.

Carrying this over to our setup, we find that for conical Type-$\mathbb{H}$ surfaces, in addition to the usual anomaly, which as explained in Section 4.1 is related to the arc-length of the base of the cone and the genus of an appropriate Seifert surface, there is another logarithmic divergence proportional to the topological Chern-Simons Wilson loop. Note that this localised divergence does not follow the structure of (4.1) and in particular the dependence on $N$ and the representation is not captured by (4.4).

The conical surfaces of Type-$\mathbb{H}$ realise the loops of [37] in 5d, but not the original ones in 4d [26]. The reason is that the preserved supersymmetries mix Poincaré and conformal supercharges (similar to Type-$\mathbb{S}$ surfaces, see (3.14)). As conformal invariance in 4d and 6d are not directly related, the uplift to surfaces on $S^3 \times \mathbb{R} \subset \mathbb{R}^6$ is not BPS but $S^3 \times S^1 \subset \mathbb{R}^4 \times T^2$ is (or as mentioned above, on $S^3 \times \mathbb{R} \subset S^5 \times \mathbb{R}$).

Compactifying the theory on other Riemann surfaces leads to class-$\mathcal{S}$ theories and BPS line operators in those theories [59] originate from BPS surfaces in 6d. The supercharges surviving the required topological twist are those satisfying (5.1). They are compatible with a surface operator along an appropriate cycle on the Riemann surface [59], an arbitrary curve in $\mathbb{R}^2 \subset \mathbb{R}^4$ and the adaptation or (1.3) to couple to $n^4$ and $n^5$. This give the class of mutually-BPS line operators in $\mathcal{N} = 2$ theories [60], which follow the same ansatz as [41], or our Type-$\mathbb{R}$, but restricted to a curve in $\mathbb{R}^2$.  

14
5.2 Surfaces in 4d

The other way to dimensionally reduce a surface is if it is at a fixed point along a compact direction. In that case we would get a surface operator in the lower dimensional theory. This can give surface operators in any dimension, but let us focus on 4d, either $\mathcal{N} = 4$ theory or theories of class-S. Theories with $\mathcal{N} = 1$ supersymmetry in 4d also have BPS surface operators, but we will not touch upon those. As the construction of surfaces of Type-H and Type-S is inherently four dimensional, we expect them to be realised in 4d as well.

A first comment is that when restricting to $\mathbb{R}^4$ both Type-C and Type-R become subclasses of Type-H, and Type-S is essentially the same as Type-N. This was discussed already below equations (2.3) and (3.11) respectively. So we need only discuss Type-H (and its subclasses: Type-C, Type-R, Type-L and Type-N) and separately Type-S.

A second comment is that within the AGT correspondence [61] it is very natural to distinguish the origin of a surface operator in 4d as arising from a codimension-4 operator in 6d, or from a a codimension-2 defect wrapping the Riemann surface. While our discussion is explicitly relevant only for the former case, it is only based on supersymmetry analysis, so is very likely to carry over, as to any other operator involving the coupling of an appropriate 2d superconformal field theory to the 4d theory.

Let us separate the discussion to theories of class-S and $\mathcal{N} = 4$ SYM.

$\mathcal{N} = 2$ theories:

There is an immense literature on surface operators in 4d class-S theories focusing mostly on operators breaking $\mathfrak{su}(2,2|2) \to \mathfrak{su}(2|1)^2$ (see, e.g. [62–65]). This is indeed the symmetry preserved by the 1/2 BPS surface operator of the 6d theory appropriately compactified to $\mathcal{N} = 2$ in 4d. In 2d language, these are $\mathcal{N} = (2,\bar{2})$ defects.

To make the relation between 6d and 4d explicit, it involves replacing the $x^5, x^6$ directions with a compact Riemann surface. In order to preserve supersymmetry away from flat space, one performs a topological twist which preserves the supercharges satisfying

$$\varepsilon (\gamma_{56} + \rho_{45}) = 0. \quad (5.1)$$

The resulting supersymmetries are the $Q$ and $\bar{Q}$ of the 4d theory. It is a simple check that (5.1) is automatically satisfied if we impose all the projectors (3.11).

Viewing things from the 4d point of view, the equations of BPS surface operators are very similar to (1.2), with only three $\rho$’s and the four dimensional Killing spinors. As mentioned above, all but Type-S are subclasses of Type-H, so we analyse the latter first in turn.

**Type-H**: For the most general surface of this type $n^I \in S^2$, and we need to solve all the equations of (3.11). The projectors in the first column give the 4d chirality projector $\gamma_{1234}$, eliminating the $\bar{Q}$ supercharges. This is not surprising as the Type-H ansatz (1.4) is chiral. In 4d the three projectors on the first line of (3.11) are not independent, as the gamma matrices of $\mathfrak{su}(2)_R$ symmetry satisfy $\rho_1 \rho_2 \rho_3 = -i$. We find that the generic surface of this type preserves a single $Q$.

**Type-R**: Here we have surfaces extended along $x^4$ and an arbitrary surface in $\mathbb{R}^3$ and generically $n^I \in S^2$. As already mentioned, these surfaces are now a subclass of Type-H and one should impose the projector equations on the second line of (3.11). Given that
\[ \varepsilon_0 \] is comprised of one 4-component chiral and one 4-component antichiral spinors, the two independent equations in the second line reduce to one \( Q \) and one \( \bar{Q} \).

**Type-N:** This case is equivalent to **Type-\( \mathbb{R} \)** in four dimensions, except that we need to impose the conditions on the first line of (3.11).

**Type-L:** In this case we need to impose the equations associated to two of the columns in (3.11). As mentioned above, these automatically imply that the third column is also satisfied, so there is no supersymmetry enhancement for Lagrangian surfaces.

**Type-C:** For holomorphic surfaces which are not a plane, we need to impose both equations in one of the columns in (3.11), so we find that the surfaces preserve a pair of \( Q \) supercharges. Indeed holomorphic surface operators in \( \mathcal{N} = 4 \) SYM were constructed in [66] and the analysis here suggests that this should extend to any class-\( \mathcal{S} \) theory.

**Type-S:** In this case the ansatz (1.5) requires \( n^I \in S^3 \), so cannot be fully realised for \( \mathcal{N} = 2 \) theories. Recall though that surfaces of **Type-N** are conformal to “banana” surfaces on \( S^3 \) and for those \( n^I = 0 \). We can implement this conformal transformation for \( \mathcal{N} = 2 \) theories in 4d as well, so they too preserve a pair of supercharges: one \( Q + S \) and one \( \bar{Q} + \bar{S} \).

\( \mathcal{N} = 4 \) SYM:

The dimensional reduction on a torus to \( \mathcal{N} = 4 \) theories does not break any supersymmetries, thus guaranteeing that our analysis carries over. The most studied surface operators are those of Gukov and Witten [67], and though they arise from codimension-2 defects in six dimensions, the 1/2 BPS plane preserves the same superalgebra as arising from 6d surface operators, \( \text{su}(2|2)^2 \), or in two dimension language \( \mathcal{N} = (4,4) \).

The projector equations in 4d take a slightly different form from (1.2), as is clear, since the R-symmetry breaking is \( \text{so}(6) \rightarrow \text{so}(4) \), which requires two \( \rho \) matrices. Also, the structure of the Killing spinors is slightly different. Yet, the dimensional reduction guarantees that solutions to all our classes exist. The number of preserved supercharges is also slightly different from 6d, and is double the number listed for the appropriate subclasses of **Type-\( \mathbb{H} \)** for \( \mathcal{N} = 2 \) theories above.

**Type-S:** In this case the we can realise the full ansatz (1.5) and the supersymmetry analysis in Section 3 is indeed consistent in 4d. The most general surface preserves two supercharges.

We should mention that this supersymmetry analysis does not provide a construction of these operators, but suggests which symmetries to try to realise. It would be interesting to extend the constructions of [67,68] as defect operators and those of [62,63] as coupled 2d-4d systems to arbitrary SUSY preserving geometries, generalising the **Type-\( \mathbb{H} \)** case of [66].

### 6 Holography

The large \( N \) limit of the \( \mathcal{N} = (2,0) \) theory has a holographic representation in terms of M-theory on \( AdS_7 \times S^4 \) [69]. Therein, the surface operators are described by M2-branes ending
along the surface at the boundary of space [30]. Globally BPS surface operators should be described by M2-brane embeddings which realise the same supersymmetries.

Here we take the first few steps to realise the different classes of BPS surfaces. The simplest examples of M2-branes embeddings are those whose boundary is a plane or a sphere, they were presented respectively in [30] and [31].

We take again inspiration from the holographic realisation of the large \( \mathcal{N} = 4 \) SYM. For each of the construction of [41, 26], the BPS conditions on the field theory could be extended to an almost complex structure on a subspace of \( AdS_5 \times S^5 \). In both cases the dual classical string surfaces are pseudo-holomorphic with respect to the almost complex structure. Furthermore, the action of the string can be evaluated by the pullback of an appropriate form in the bulk, which manifests as a (generalised) calibration. In the following we present some analogous structures for the four main classes of BPS surface operators. We refer to the original papers [57, 26], for more details on the string constructions.

Our starting point is the metric on \( AdS_7 \times S^4 \),

\[
ds^2 = \frac{y}{L} \, dx^\mu dx^\mu + \left( \frac{L}{y} \right)^2 dy^I dy^I, \tag{6.1}
\]

where \( y \equiv |y^I| \). To get the regular Poincaré patch metric redefine \( y = 4L^3/z^2 \) and parametrise \( y^I/y \) as angular coordinates on \( S^4 \). Note that here \( L \) is the radius of \( S^4 \) and the curvature radius of \( AdS_7 \) is \( 2L \). The background also has a 4-form field strength proportional to the volume form on \( S^4 \).

The Killing spinors in this coordinate system can be constructed from the same pair of constant spinors \( \varepsilon_0 \) and \( \bar{\varepsilon}_1 \) of Section 3. Using the 11-dimensional curved space \( \Gamma \)-matrices satisfying

\[
\{ \Gamma_\mu, \Gamma_\nu \} = \frac{2y}{L} \delta_{\mu\nu}, \quad \{ \Gamma_\mu, \Gamma_I \} = 0, \quad \{ \Gamma_I, \Gamma_J \} = 2 \left( \frac{L}{y} \right)^2 \delta_{IJ}, \tag{6.2}
\]

they are given by [70]

\[
\varepsilon = \left( \frac{y}{L} \right)^{1/4} \varepsilon_0 + \left( \frac{L}{y} \right)^{1/4} \bar{\varepsilon}_1 \left( x^\mu \Gamma_\mu - 2y^I \Gamma_I \right). \tag{6.3}
\]

In the limit of \( y \to \infty \), the last term drops out and we recover the conformal Killing spinors of flat space appearing in (1.2), \( (y/L)^{1/4}(\varepsilon_0 + \bar{\varepsilon}_1 x^\mu \gamma_\mu) \). This enables a very direct map between the BPS conditions that we found for the different types of surfaces in Section 3 to the bulk.

In the following we use the notation \( X^M \) for coordinates of \( AdS_7 \times S^4 \) and index \( m = 1, 2, 3 \) for the M2-brane. \( G_{MN} \) is the metric (6.1) and \( g_{mn} \) is the induced metric on the M2-brane.

The analog of the projector equation in supergravity is a consequence of the \( \kappa \)-invariance of the M2-brane action [71]. It reads

\[
- \frac{i}{6} \varepsilon \Gamma_{MNP} \partial_m X^M \partial_n X^N \partial_p X^P \varepsilon^{mnp} = \varepsilon, \tag{6.4}
\]

where \( \varepsilon^{mnp} \) is the Levi-Civita tensor density and includes \( 1/\sqrt{g} \). On the boundary this equation reduces to (1.2), so the supercharges preserved by the M2-brane are constructed from the same \( \varepsilon_0 \) and \( \bar{\varepsilon}_1 \) as in field theory.
Given a preserved supercharge represented by $\varepsilon$, we can construct a 3-form \[ \phi = -i \frac{\varepsilon \Gamma_{MNP} \varepsilon^{\dagger}}{\varepsilon \varepsilon^{\dagger}} dX^M \wedge dX^N \wedge dX^P. \] (6.5)

By construction, the pullback of this 3-form to an M2-brane satisfying (6.4) is its volume form. $\phi$ then serves a role analogous to the almost complex structure in string theory.

If $\phi$ is closed $d\phi = 0$, then the form is a calibration [72,57,73]; its integral is the same on all 3-surfaces of the same homology class and is equal to the volume of the minimal surface, so the classical M2-brane action.

If $\phi$ is exact then the action comes only from the boundary at large $y$ and is simply proportional to $y \text{vol}(\Sigma)$. This divergent term is removed by the Legendre transform of [74,9,19] or equivalently by renormalisation [75], and the expectation value of the surface operator vanishes.

In each of our main examples we construct the appropriate $\phi$ and comment on its properties. It is particularly useful when one can write an equation of the form

$$\partial_{m} X^{M} = \frac{1}{2} g_{m1} \varepsilon^{\mu \nu \rho} G^{M \nu \rho} \phi_{\lambda \mu \lambda} \partial_{\lambda} X^{N} \partial_{\rho} X^{P}.$$ (6.6)

This replaces the equations of motion with a set of first order nonlinear equations and is the generalisation of the pseudo-holomorphicity condition in string theory. The exact form of this equation varies between the examples presented below.

**Type-R:** Using $\varepsilon$ with $\varepsilon_0$ annihilated by all the $a_I$ oscillators in (3.5), equation (6.5) gives the 3-form

$$\phi^R = -d x^6 \wedge \omega^R, \quad \omega^R = \sum_{I=1}^{5} (d x^I \wedge dy^I).$$ (6.7)

$\phi^R = -d(x^6 \omega^R)$ is clearly exact, so the expectation value of the Type-R surface operators vanishes at large $N$.

By translation symmetry, the 3-surface should extend in the $x^6$ direction and identifying $\sigma^3 = x^6$, we expect the remaining coordinates to satisfy the pseudo-holomorphicity condition

$$\partial_{m} X^{M} = g_{m1} \varepsilon^{3 \mu \nu \rho} G^{M \nu \rho} \omega^R_{\lambda \mu \lambda} \partial_{\lambda} X^{N}.$$ (6.8)

**Type-C:** In this case there are two preserved supercharges, so (6.5) gives two choices of 3-form. The components shared by both are

$$\varphi^C = (dx^1 \wedge dx^2 + dx^3 \wedge dx^4 + dx^5 \wedge dx^6) \wedge dy^1.$$ (6.9)

This is obvious uplift of the the Kähler form $\omega^C$ of $\mathbb{R}^6$ (3.6) to $AdS_7$: $\varphi^C = \omega^C \wedge dy^1$. It is natural to augment it to

$$\phi^C = (dx^{12} + dx^{34} + dx^{56}) \wedge dy^1 + \left(\frac{y}{L}\right)^{3/2} (dx^{136} + dx^{145} + dx^{235} - dx^{246}),$$ (6.10)

which has the form of a $G_2$-structure (the shorthand notation is $dx^{\mu \nu \cdots} = dx^\mu \wedge dx^\nu \wedge \ldots$).

In this case the first order equation (6.6) is indeed satisfied. This is a consequence of properties of $G_2$-structures as described below, around (6.14), but this equation is also...
satisfied with the restricted version of $\varphi^C$ with just the three components (6.9). That 3-form is closed, which means that minimal volumes are parallel to it. Choosing a gauge where $X^7 = y^1 = \sigma^3$, this is the requirement that $\partial_3 x^\mu = 0$, so the pullback of $\varphi^C$ is

$$\varphi^C_{\mu\nu\tau} \partial_\mu x^\tau \partial_\nu x^\nu \partial_\mu x^\sigma \wedge \partial_\nu x^\sigma \wedge \partial_\sigma.$$  (6.11)

This is the pullback of the Kähler form $\omega^C$ to the surface at fixed $y^1$ times $d\sigma^3$. We can immediately integrate over $\sigma^3$ and, using (3.7), simply find a linear divergence times the area element of the 2-surface $\Sigma$. The expectation value of the surface operator therefore vanishes, which is also clear, since this version of the 3-form is exact. This is consistent with (4.6), as this calculation captures only $O(N)$ terms in the large $N$ limit [52,19] and the anomaly coefficient $a_1^{(N)} \rightarrow 1/2$ (4.5).

**Type-$\mathbb{H}$**: The natural 3-form extending (3.11) to $AdS_5 \times S^2$ is

$$\phi^H = \eta I_{\mu\nu} dx^\mu \wedge dx^\nu \wedge dy^I - \frac{L^3}{y^3} dy^{123}$$  (6.12)

$$= (dx^{12} + dx^{34}) \wedge dy^3 + (dx^{31} + dx^{24}) \wedge dy^2 + (dx^{23} + dx^{14}) \wedge dy^1 - \frac{L^3}{y^3} dy^{123}.$$  (6.14)

This is indeed the result of (6.5) with the supercharge preserving all **Type-$\mathbb{H}$** surfaces (3.11). This form is closed, so defines a calibration, and as in the **Type-$\mathbb{C}$** above (6.10), this is a $G_2$-structure [76–79] (see also [80] for a gentle introduction).

Let us prove this implies that the first order equation (6.6) is satisfied. Following [57], consider the quantity

$$a^M_m = \partial_m X^M - \frac{1}{2} g_{ml} \varepsilon^{lnp} G^{MN} \phi_{LNP} \partial_n X^N \partial_p X^P.$$  (6.13)

Squaring this we get

$$g^{mn} a^M_m a^n_M G_{MN} = g^{mn} \partial_m X^X \partial_n X^N G_{MN} - \varepsilon^{mpn} \partial_m X^X \partial_n X^N \partial_p X^P \phi_{MPN}$$

$$+ \frac{1}{2} g^{pq} g^{mp} \partial_m X^X \partial_n X^N \partial_p X^P \partial_q X^Q G^{RS} \phi_{RMP} \phi_{SNQ}.$$  (6.14)

The first term gives the trace of the induced metric and the second using (6.5), the contraction of two Levi-Civita tensors. Their sum is $-3$. The terms of the second line can be evaluated using properties of the 3-form $\phi$. In particular $G_2$-structures satisfy

$$G^{RS} \phi_{RMP} \phi_{SNQ} = G_{MN} G_{PQ} - G_{MQ} G_{PN} - (\ast \phi)_{MPNQ}.$$  (6.15)

The asymmetric term does not contribute and the contractions with the first two reduce everything to traces of the induced metric and the second line evaluates to 3, showing that $a^M_m = 0$, thus proving (6.6).

Surfaces analogous to **cones** of **Type-$\mathbb{H}$** were studied in [37] (see the discussion in Section 5.1), including their holographic realisation. There it was shown that the M2-branes are calibrated with respect to a 3-form, which should be the same as (6.12) in a different coordinate system. A reduction of (6.6) to conical surfaces (which looks similar to (6.8)) was also derived there.
**Type-S:** These surfaces are restricted to $S^3 \subset \mathbb{R}^4$ and $n^I \in S^3$, so the M2-brane duals are located within an $AdS_4 \times S^3 \subset AdS_7 \times S^4$. Using the metric (6.1), we take the eight coordinates $x^I$ and $y^I$ with $I = 1, 2, 3, 4$, set $x^5 = x^6 = y^5 = 0$ and impose the further constraint

$$|x|^2 + \frac{4L^3}{y} = 1.$$  

(6.16)

Using the two supercharges that preserve the **Type-S** surfaces (3.14) gives according to (6.5) two 3-forms. The terms appearing in both are

$$\phi^S = \frac{1}{6} \varepsilon_{IJKL} \left[ dx^I dx^J \left( 3x^K dy^L - 2y^K dx^L \right) - \left( \frac{L}{y} \right)^3 dy^I dy^J \left( x^K dy^L - 6y^K dx^L \right) \right].$$  

(6.17)

To simplify this, we define the complex vielbeins $\zeta^I$

$$\zeta^I = \sqrt{\frac{y}{L}} dx^I + i \frac{L}{y} dy^I, \quad \bar{\zeta}^I = \sqrt{\frac{y}{L}} dx^I - i \frac{L}{y} dy^I.$$  

(6.18)

They are null with respect to the metric (6.1), and $\frac{i}{2} \sum_i \zeta^I \wedge \bar{\zeta}^I$ defines an almost complex structure on $AdS_5 \times S^3$. Taking $\Xi$ to be the unit vector normal to (6.16), its contraction with $\zeta^I$ gives

$$\Xi \cdot \zeta^I = \sqrt{\frac{L}{y}} \left( x^J \partial_{x^J} - 2y^J \partial_{y^J} \right) \cdot \zeta^I = x^I - 2i \left( \frac{L}{y} \right)^{3/2} y^I,$$  

(6.19)

and the complex conjugate for $\Xi \cdot \bar{\zeta}^I$.

Now taking the (anti)holomorphic 4-forms on $AdS_5 \times S^3$

$$\Omega = \zeta^1 \wedge \zeta^2 \wedge \zeta^3 \wedge \zeta^4, \quad \bar{\Omega} = \bar{\zeta}^1 \wedge \bar{\zeta}^2 \wedge \bar{\zeta}^3 \wedge \bar{\zeta}^4,$$  

(6.20)

$\phi^S$ is the projection of their difference onto the hypersurface (6.16)

$$\phi^S = \frac{1}{2\pi} \Xi \cdot \left( \Omega - \bar{\Omega} \right) = \frac{i}{12} \varepsilon_{IJKL} \left( \zeta^I \wedge \zeta^J \wedge \zeta^K \left( \Xi \cdot \zeta^L \right) - \bar{\zeta}^I \wedge \bar{\zeta}^J \wedge \bar{\zeta}^K \left( \Xi \cdot \bar{\zeta}^L \right) \right).$$  

(6.21)

This representation makes it manifest that though we write $\phi$ in terms of eight coordinates, it is contained within (the cotangent to) the $AdS_4 \times S^3$ subspace defined by (6.16). Perhaps it is consistent with some notion of generalised calibrations as in [81, 82, 26].

**Minimal surfaces:** The discussion above allowed us to reduce the BPS equations to first order equations for **Type-R**, **Type-C** and **Type-H**. The structure of 3-form for **Type-S** in (6.17) is more complicated, and it is not clear what should be the form of (6.6) in this case.

The equations for **Type-R** and **Type-C** can be written as pseudo-holomorphicity equations (6.8), (6.11) and in the latter case the solutions are simply $\Sigma \times \mathbb{R}_+ $ where $\Sigma$ is the original surface and $y^1 \in \mathbb{R}_+$ for **Type-C** the equation can also be written in a form identical to (6.6) and this is also the form of the equation for **Type-H** surfaces.

We leave it to future work to find explicit solutions to those equations.
7 Discussion

This paper is meant to reveal some cracks in the steep granite face of the $\mathcal{N} = (2, 0)$ theory. They are unlikely to offer an easy path to the summit, but the anchors we placed should provide a good starting route for further exploration. At the very least, they promise to be a fun playground for those interested in this mysterious theory.

In the absence of a fully satisfying Lagrangian (or other traditional) formulation, we relied here on symmetry, geometry and algebra, the knowledge that these theories have planar surface operator observables and the large symmetry that they preserve. It is a very natural (and mild) assumption that such operators can have arbitrary geometries and an associated vector field $n$ representing the local R-symmetry breaking. Indeed this assumption can easily be checked and realised in both the abelian theory and the holographic realisation.

This philosophy leads to trying to find geometries and vector fields for which the projector equation (1.2) has global solutions. To our surprise we found a very intricate structure of solutions to this set of equations which we split into several main classes, subclasses and examples preserving different numbers (and types) of supersymmetries, see Tables 1 and 2.

The most obvious question is whether the examples we found are exhaustive. In one sense the answer is clearly no: acting with a global symmetry gives more examples and we chose to focus on single representatives of those. While changing $x^6$ to $x^5$ in Type-$R$ or the three components of $n^I$ that are turned on by the Type-$\mathbb{H}$ ansatz are rather trivial, acting with conformal transformations are less so. For example, they map $\mathbb{R}^4$ of Type-$\mathbb{H}$ to $S^4$ and for surfaces that extend to infinity, they change the topology, so can affect the anomaly, or in the absence of an anomaly the finite expectation value. A simple illustration of this is the relation between the non-anomalous plane and its anomalous counterpart, the sphere. Likewise for Type-$C$ surfaces, their compact versions could also be interesting.

Another generalisation is to allow $n$ to be complex, while keeping $(n)^2 = 1$ (which is required in order to satisfy (1.2)). In this case the analytic continuation to Lorentzian signature would lead to non-unitary operators, but the analog for those in the case of Wilson loops have been instrumental for localisation [22], the ladder limit [83] and the related fishnet theories [84–86].

Complexifying $n$ is somewhat akin to smearing chiral operators over the surface, but in fact one can consider combinations of surface operators and local chiral operators. Those can either be operators constrained to the surface such that they do not exist in its absence, or be away from it. The former case is studied extensively for maximally symmetric setups as part of the defect CFT programme [87–91, 42, 13]. This could be extended to general surfaces and indeed a natural approach to try to prove that Type-$R$ surfaces have trivial expectation value is to view all surfaces of this type as deformations of one another by the insertion of appropriate combinations of geometric and R-symmetry-changing local operators. In the latter case the question is which combination of surfaces and local operators are mutually BPS along the lines of [92–97].

Our approach here was to try auspicious ansätze and were fortunate to uncover some examples, but it should be possible to formalise the problem and treat it systematically. This could lead to a classification of BPS surface operators, as was done for Wilson loops in 4d [53].

Another generalisation would be to study surfaces when the theory itself is on curved space. We touch on that when relating to surface operators in class-$\mathcal{S}$ theories in Section 5.2.
One can likewise reduce the theory to 3d \([98]\) and 2d \([99,100]\), and one can look for other geometries it can be placed on and find compatible surface operators.

Lower dimensional line operator versions of \textbf{Type-R} and conical \textbf{Type-H} Wilson loops have been previously described in \([41,54,26,37]\) as were 4d surface operators similar to \textbf{Type-C} in \([66]\). It would be interesting to realise the surface operators of \textbf{Type-H} and \textbf{Type-S} there as well.

In four dimensional field theories entangling surfaces are also two dimensional and have been studied extensively in recent years \([18,48,49,10]\). They share many properties with surface operators, in particular the anomaly structure. Even more relevant for our discussion here is the supersymmetric Rényi entropy \([101,102]\), and it would be interesting to see whether more results on entropy can be gleaned from entangling surfaces inspired by our different classes.

In the cases where we can define the \((2,0)\) theory, namely the abelian theory and the large \(N\) holographic realisation, we can proceed to try to evaluate expectation values of surface operators explicitly. The anomaly has already been reproduced in both cases in \([52,43,45,19]\). To go beyond that, one can try to calculate finite expectation values for anomalyless surfaces in the abelian theory employing the free-field propagators \([19]\). In the holographic setup it would be interesting to utilise the first order equations in Section 6 to find explicit examples, at least for highly symmetric surfaces. In addition to providing finite expectation values for anomalyless surfaces, it would allow to calculate the correlation function with local operators as in \([31]\).

One may hope to be able to realise the BPS surface operators within some old and new approaches to the \(\mathcal{N} = (2,0)\) theory \([103–108]\). Many of those formulations address the presence of planar surfaces and it would be interesting to look for the richer spectrum of BPS observables found here.

Even limited to the abelian theory, we can use it to get clues to the nonabelian theory.

Following \([109]\) we could look for an effective theory calculating the BPS surface operators. An example is the proposal of \([37]\) that the (compactified) cones of \textbf{Type-H} are calculated by a 3d Chern-Simons theory. Attempting this for \textbf{Type-S}, we can write down an effective propagator which includes the contribution from the 2-form \(B\) and the scalars \(\Phi^I\) that couples to \(n^I\) in (1.5) \([19]\) to get an effective 2-form propagator

\[
\langle B^{\mu\nu}(x)B_{\rho\sigma}(y) \rangle = \frac{1}{4\pi^2 |x - y|^2} \left[ \frac{1}{2} \kappa^{\rho\sigma}_{\mu\nu} - \frac{4 (x - y)_\mu (x - y)_\nu (x - y)^\rho \delta^\sigma_\rho}{|x - y|^2} \right].
\]

This behaves somewhat like the propagator of a 2-form in 4d, and can also be dualised to a compact scalar. Integrating this over the surface easily reproduces the \textbf{Type-S} anomaly in Section 4 including the area term \((4.14)\). Finding an appropriate nonabelian generalisation could help in guessing an expression for the surface operators a generic \((2,0)\) theory.

Finally, the \(\mathcal{N} = (2,0)\) theory also contains co-dimension 2 defects, i.e. observables defined over 4-manifolds \([110–114]\) (see also \([115]\)). It would be interesting to study their supersymmetric embeddings in 6d, the corresponding anomalies and the compatibility with the surface operators presented here.

\textbf{Acknowledgments}

It is a pleasure to thanks Dima Panov and Simon Salamon for explaining to us a lot of the geometry underlying these constructions and to Malte Probst for continued discussion,
related collaborations, and just for being an awesome guy.

ND’s research is supported by the Science Technology & Facilities council under the grants ST/T000759/1 and ST/P000258/1. MT acknowledges the support of the Natural Sciences and Engineering Research Council of Canada (NSERC). Cette recherche a été financée par le Conseil de recherches en sciences naturelles et en génie du Canada (CRSNG).

References

[1] E. Witten, “Some comments on string dynamics,” in Future perspectives in string theory. Proceedings, Conference, Strings’95, Los Angeles, USA, March 13-18, 1995, pp. 501–523. 1995. hep-th/9507121.
[2] A. Strominger, “Open p-branes,” Phys. Lett. B383 (1996) 44–47, hep-th/9512059.
[3] O. J. Ganor, “Six-dimensional tensionless strings in the large N limit,” Nucl. Phys. B489 (1997) 95–121, hep-th/9605201.
[4] M. Bullimore and H.-C. Kim, “The superconformal index of the (2, 0) Theory with defects,” JHEP 05 (2015) 048, arXiv:1412.3872.
[5] A. Chalabi, A. O’Bannon, B. Robinson, and J. Sisti, “Central charges of 2d superconformal defects,” JHEP 05 (2020) 095, arXiv:2003.02857.
[6] E. D’Hoker, J. Estes, M. Gutperle, and D. Krym, “Exact half-BPS flux solutions in M-theory. I: Local solutions,” JHEP 08 (2008) 028, arXiv:0806.0605.
[7] E. D’Hoker, J. Estes, M. Gutperle, and D. Krym, “Exact half-BPS flux solutions in M-theory II: Global solutions asymptotic to AdS7 × S4,” JHEP 12 (2008) 044, arXiv:0810.4647.
[8] S. A. Gentle, M. Gutperle, and C. Marasinou, “Entanglement entropy of Wilson surfaces from bubbling geometries in M-theory,” JHEP 08 (2015) 019, arXiv:1506.00052.
[9] R. Rodgers, “Holographic entanglement entropy from probe M-theory branes,” JHEP 03 (2019) 092, arXiv:1811.12375.
[10] K. Jensen, A. O’Bannon, B. Robinson, and R. Rodgers, “From the Weyl anomaly to entropy of two-dimensional boundaries and defects,” Phys. Rev. Lett. 122 no. 24, (2019) 241602, arXiv:1812.08745.
[11] J. Estes, D. Krym, A. O’Bannon, B. Robinson, and R. Rodgers, “Wilson surface central charge from holographic entanglement entropy,” JHEP 05 (2019) 032, arXiv:1812.00923.
[12] N. Drukker, S. Giombi, A. A. Tseytlin, and X. Zhou, “Defect CFT in the 6d (2, 0) theory from M2 brane dynamics in AdS7 × S4,” JHEP 07 (2020) 101, arXiv:2004.04562.
[13] N. Drukker, M. Probst, and M. Trépanier, “Defect CFT techniques in the 6d $\mathcal{N} = (2, 0)$ theory,” arXiv:2009.10732.
[14] Y. Wang, “Surface Defect, Anomalies and b-Extremization,” arXiv:2012.06574.
[15] S. Deser and A. Schwimmer, “Geometric classification of conformal anomalies in arbitrary dimensions,” Phys. Lett. B309 (1993) 279–284, hep-th/9302047.
[16] N. Boulanger, “General solutions of the Wess-Zumino consistency condition for the Weyl anomalies,” JHEP 07 (2007) 069, arXiv:0704.2472.
[17] A. Schwimmer and S. Theisen, “Entanglement entropy, trace anomalies and holography,” *Nucl. Phys. B801* (2008) 1–24, arXiv:0802.1017.

[18] S. N. Solodukhin, “Entanglement entropy, conformal invariance and extrinsic geometry,” *Phys. Lett. B665* (2008) 305–309, arXiv:0802.3117.

[19] N. Drukker, M. Probst, and M. Trépanier, “Surface operators in the 6d $\mathcal{N} = (2,0)$ theory,” *J. Phys. A* 53 no. 36, (2020) 365401, arXiv:2003.12372.

[20] J. K. Erickson, G. W. Semenoff, and K. Zarembo, “Wilson loops in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory,” *Nucl. Phys. B582* (2000) 155–175, hep-th/0003055.

[21] N. Drukker and D. J. Gross, “An exact prediction of $\mathcal{N} = 4$ SUSY theory for string theory,” *J. Math. Phys. 42* (2001) 2896–2914, hep-th/0010274.

[22] V. Pestun, “Localization of gauge theory on a four-sphere and supersymmetric Wilson loops,” *Commun. Math. Phys. 313* (2012) 71–129, arXiv:0712.2824.

[23] A. Kapustin, B. Willett, and I. Yaakov, “Exact results for Wilson loops in superconformal Chern-Simons theories with matter,” *JHEP 03* (2010) 089, arXiv:0909.4559.

[24] N. Drukker and D. Trancanelli, “A supermatrix model for $\mathcal{N} = 6$ super Chern-Simons-matter theory,” *JHEP 02* (2010) 058, arXiv:0912.3006.

[25] M. Mariño and P. Putrov, “Exact results in ABJM theory from topological strings,” *JHEP 06* (2010) 011, arXiv:0912.3074.

[26] N. Drukker, S. Giombi, R. Ricci, and D. Trancanelli, “Supersymmetric Wilson loops on $S^3$,” *JHEP 05* (2008) 017, arXiv:0711.3226.

[27] D. Correa, J. Henn, J. Maldacena, and A. Sever, “An exact formula for the radiation of a moving quark in $\mathcal{N} = 4$ super Yang Mills,” *JHEP 06* (2012) 48, arXiv:1202.4455.

[28] B. Fiol, B. Garolera, and A. Lewkowycz, “Exact results for static and radiative fields of a quark in $\mathcal{N} = 4$ super Yang-Mills,” *JHEP 05* (2012) 093, arXiv:1202.5292.

[29] P. S. Howe, N. D. Lambert, and P. C. West, “The selfdual string soliton,” *Nucl. Phys. B515* (1998) 203–216, hep-th/9709014.

[30] J. M. Maldacena, “Wilson loops in large $N$ field theories,” *Phys. Rev. Lett. 80* (1998) 4859–4862, hep-th/9803002.

[31] D. E. Berenstein, R. Corrado, W. Fischler, and J. M. Maldacena, “The operator product expansion for Wilson loops and surfaces in the large $N$ limit,” *Phys. Rev. D59* (1999) 105023, hep-th/9809188.

[32] B. Chen, W. He, J.-B. Wu, and L. Zhang, “M5-branes and Wilson surfaces,” *JHEP 08* (2007) 067, arXiv:0707.3978.

[33] B. Chen and J.-B. Wu, “Wilson-Polyakov surfaces and M-theory branes,” *JHEP 05* (2008) 46, arXiv:0802.2173.

[34] H. Mori and S. Yamaguchi, “M5-branes and Wilson surfaces in $AdS_7/CFT_6$ correspondence,” *Phys. Rev. D90* no. 2, (2014) 26005, arXiv:1404.0930.

[35] C. A. Cremonini, P. A. Grassi, and S. Penati, “Surface operators in superspace,” *JHEP 11* (2020) 050, arXiv:2006.08633.

[36] K.-M. Lee and H.-U. Yee, “BPS String Webs in the 6-dim (2,0) Theories,” *JHEP 03* (2007) 057, hep-th/0606150.
[37] M. Mezei, S. S. Pufu, and Y. Wang, “Chern-Simons theory from M5-branes and calibrated M2-branes,” *JHEP* **08** (2019) 165, arXiv:1812.07572.

[38] N. B. Agmon and Y. Wang, “Classifying superconformal defects in diverse dimensions part I: superconformal lines,” arXiv:2009.06650.

[39] D. A. Hoffman and R. Osserman, “The Gauss map of surfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$,” *Proceedings of the London Mathematical Society* **3** no. 1, (1985) 27–56.

[40] N. Drukker, “1/4 BPS circular loops, unstable world-sheet instantons and the matrix model,” *JHEP* **09** (2006) 004, hep-th/0605151.

[41] K. Zarembo, “Supersymmetric Wilson loops,” *Nucl. Phys.* **B643** (2002) 157–171, hep-th/0205160.

[42] L. Bianchi and M. Lemos, “Superconformal surfaces in four dimensions,” *JHEP* **06** (2020) 056, arXiv:1911.05082.

[43] M. Henningson and K. Skenderis, “Weyl anomaly for Wilson surfaces,” *JHEP* **06** (1999) 12, hep-th/9905163.

[44] A. Gustavsson, “On the Weyl anomaly of Wilson surfaces,” *JHEP* **12** (2003) 59, hep-th/0310037.

[45] A. Gustavsson, “Conformal anomaly of Wilson surface observables: A field theoretical computation,” *JHEP* **07** (2004) 74, hep-th/0404150.

[46] R. Bott and L. W. Tu, *Differential forms in algebraic topology*, vol. 82. Springer Science & Business Media, 2013.

[47] I. R. Klebanov, T. Nishioka, S. S. Pufu, and B. R. Safdi, “On shape dependence and RG flow of entanglement entropy,” *JHEP* **07** (2012) 001, arXiv:1204.4160.

[48] R. C. Myers and A. Singh, “Entanglement entropy for singular surfaces,” *JHEP* **09** (2012) 13, arXiv:1206.5225.

[49] L. Bianchi, M. Meineri, R. C. Myers, and M. Smolkin, “Rényi entropy and conformal defects,” *JHEP* **07** (2016) 76, arXiv:1511.06713.

[50] H. Dorn, “On the logarithmic divergent part of entanglement entropy, smooth versus singular regions,” *Phys. Lett.* **B763** (2016) 134–138, arXiv:1608.04900.

[51] C. C. Adams, *The knot book*. American Mathematical Soc., 1994.

[52] C. R. Graham and E. Witten, “Conformal anomaly of submanifold observables in AdS/CFT correspondence,” *Nucl. Phys.* **B546** (1999) 52–64, hep-th/9901021.

[53] A. Dymarsky and V. Pestun, “Supersymmetric Wilson loops in $\mathcal{N} = 4$ SYM and pure spinors,” *JHEP* **04** (2010) 115, arXiv:0911.1841.

[54] A. Kapustin and E. Witten, “Electric-Magnetic Duality And The Geometric Langlands Program,” *Commun. Num. Theor. Phys.* **1** (2007) 1–236, hep-th/0604151.

[55] Z. Guralnik and B. Kulik, “Properties of chiral Wilson loops,” *JHEP* **01** (2004) 065, hep-th/0309118.

[56] Z. Guralnik, S. Kovacs, and B. Kulik, “Less is more: Non-renormalization theorems from lower dimensional superspace,” *Int. J. Mod. Phys. A* **20** (2005) 4546–4553, hep-th/0409091.

[57] A. Dymarsky, S. S. Gubser, Z. Guralnik, and J. M. Maldacena, “Calibrated surfaces and supersymmetric Wilson loops,” *JHEP* **09** (2006) 057, hep-th/0604058.

[58] E. Witten, “Quantum field theory and the Jones polynomial,” *Commun. Math. Phys.* **121** (1989) 351–399.
N. Drukker, D. R. Morrison, and T. Okuda, “Loop operators and S-duality from curves on Riemann surfaces,” JHEP 09 (2009) 031, arXiv:0907.2593.

A. Kapustin, “Holomorphic reduction of $\mathcal{N} = 2$ gauge theories, Wilson–t Hooft operators, and S-duality,” hep-th/0612119.

L. F. Alday, D. Gaiotto, and Y. Tachikawa, “Liouville correlation functions from four-dimensional gauge theories,” Lett. Math. Phys. 91 (2010) 167–197, arXiv:0906.3219.

D. Gaiotto, “Surface Operators in $\mathcal{N} = 2$ 4d Gauge Theories,” JHEP 11 (2012) 090, arXiv:0911.1316.

J. Gomis and B. Le Floch, “M2-brane surface operators and gauge theory dualities in Toda,” JHEP 04 (2016) 183, arXiv:1407.1852.

S. Gukov, “Surface operators,” in New dualities of supersymmetric gauge theories, J. Teschner, ed., pp. 223–259. 2016. arXiv:1412.7127.

B. Le Floch, “A slow review of the AGT correspondence,” arXiv:2006.14025.

E. Koh and S. Yamaguchi, “Holography of BPS surface operators,” JHEP 02 (2009) 012, arXiv:0812.1420.

S. Gukov and E. Witten, “Gauge theory, ramification, and the geometric Langlands program,” hep-th/0612073.

S. Gukov and E. Witten, “Rigid surface operators,” Adv. Theor. Math. Phys. 14 no. 1, (2010) 87–178, arXiv:0804.1561.

J. M. Maldacena, “The Large $N$ limit of superconformal field theories and supergravity,” Int. J. Theor. Phys. 38 (1999) 1113–1133, hep-th/9711200.

P. Claus and R. Kallosh, “Superisometries of the $\text{AdS} \times S^7$ superspace,” JHEP 03 (1999) 014, hep-th/9812087.

E. Bergshoeff, E. Sezgin, and P. K. Townsend, “Supermembranes and eleven-dimensional supergravity,” Phys. Lett. B189 (1987) 75–78.

R. Harvey and H. B. Lawson, “Calibrated geometries,” Acta Mathematica 148 no. 1, (1982) 47–157.

D. D. Joyce, Riemannian holonomy groups and calibrated geometry, vol. 12. Oxford University Press, 2007.

N. Drukker, D. J. Gross, and H. Ooguri, “Wilson loops and minimal surfaces,” Phys. Rev. D60 (1999) 125006, hep-th/9904191.

M. Bianchi, D. Z. Freedman, and K. Skenderis, “Holographic renormalization,” Nucl. Phys. B 631 (2002) 159–194, arXiv:hep-th/0112119.

E. Bonan, “Sur des variétés riemanniennes à groupe d’holonomie $G_2$ ou $\text{Spin}(7),” Comptes rendus hebdomadaires des séances de l’académie des sciences série A 262 no. 2, (1966) 127.

R. L. Bryant and S. M. Salamon, “On the construction of some complete metrics with exceptional holonomy,” Duke Math. J. 58 no. 3, (06, 1989) 829–850.

N. J. Hitchin, “The geometry of three-forms in six dimensions,” J. Diff. Geom. 55 no. 3, (2000) 547–576, arXiv:math/0010054.

D. D. Joyce, Compact manifolds with special holonomy. Oxford University Press on Demand, 2000.

S. Karigiannis, “Introduction to $G_2$ geometry,” Fields Institute Communications (2020) 3–50.
[81] J. Gutowski and G. Papadopoulos, “AdS calibrations,” Phys. Lett. B 462 (1999) 81–88, hep-th/9902034.
[82] J. Gutowski, G. Papadopoulos, and P. Townsend, “Supersymmetry and generalized calibrations,” Phys. Rev. D 60 (1999) 106006, hep-th/9905156.
[83] D. Correa, J. Henn, J. Maldacena, and A. Sever, “The cusp anomalous dimension at three loops and beyond,” JHEP 05 (2012) 098, arXiv:1203.1019.
[84] A. Zamolodchikov, “Fishnet’ diagrams as a completely integrable system,” Phys. Lett. B 97 (1980) 63–66.
[85] O. Mamroud and G. Torrents, “RG stability of integrable fishnet models,” JHEP 06 (2017) 012, arXiv:1703.04152.
[86] N. Gromov, V. Kazakov, G. Korchemsky, S. Negro, and G. Sizov, “Integrability of conformal fishnet theory,” JHEP 01 (2018) 095, arXiv:1706.04167.
[87] J. L. Cardy and D. C. Lewellen, “Bulk and boundary operators in conformal field theory,” Phys. Lett. B 259 (1991) 274–278.
[88] D. McAvity and H. Osborn, “Conformal field theories near a boundary in general dimensions,” Nucl. Phys. B 455 (1995) 522–576, cond-mat/9505127.
[89] P. Liendo, L. Rastelli, and B. C. van Rees, “The bootstrap program for boundary CFTd,” JHEP 07 (2013) 113, arXiv:1210.4258.
[90] D. Gaiotto, D. Mazac, and M. F. Paulos, “Bootstrapping the 3d Ising twist defect,” JHEP 03 (2014) 100, arXiv:1310.5078.
[91] M. Billò, V. Gonçalves, E. Lauria, and M. Meineri, “Defects in conformal field theory,” JHEP 04 (2016) 091, arXiv:1601.02883.
[92] G. W. Semenoff and K. Zarembo, “More exact predictions of SUSYM for string theory,” Nucl. Phys. B 616 (2001) 34–46, hep-th/0106015.
[93] N. Drukker and J. Plefka, “Superprotected n-point correlation functions of local operators in $\mathcal{N} = 4$ super Yang-Mills,” JHEP 04 (2009) 052, arXiv:0901.3653.
[94] S. Giombi and V. Pestun, “Correlators of Wilson loops and local operators from multi-matrix models and strings in AdS,” JHEP 01 (2013) 101, arXiv:1207.7083.
[95] C. Beem, M. Lemos, P. Liendo, W. Peelaers, L. Rastelli, and B. C. van Rees, “Infinite chiral symmetry in four dimensions,” Commun. Math. Phys. 336 no. 3, (2015) 1359–1433, arXiv:1312.5344.
[96] C. Beem, L. Rastelli, and B. C. van Rees, “W symmetry in six dimensions,” JHEP 05 (2015) 017, arXiv:1404.1079.
[97] S. Giombi and S. Komatsu, “Exact correlators on the Wilson loop in $\mathcal{N} = 4$ SYM: Localization, defect CFT, and integrability,” JHEP 05 (2018) 109, arXiv:1802.05201. [Erratum: JHEP 11, 123 (2018)].
[98] T. Dimofte, D. Gaiotto, and S. Gukov, “Gauge theories labelled by three-manifolds,” Commun. Math. Phys. 325 (2014) 367–419, arXiv:1108.4389.
[99] F. Benini and N. Bobev, “Two-dimensional SCFTs from wrapped branes and c-extremization,” JHEP 06 (2013) 005, arXiv:1302.4451.
[100] A. Gadde, S. Gukov, and P. Putrov, “Fivebranes and 4-manifolds,” Prog. Math. 319 (2016) 155–245, arXiv:1306.4320.
[101] T. Nishioka and I. Yaakov, “Supersymmetric Rényi entropy,” JHEP 10 (2013) 155, arXiv:1306.2958.
[102] T. Nishioka and I. Yaakov, “Supersymmetric Rényi entropy and defect operators,” JHEP 11 (2017) 071, arXiv:1612.02894.

[103] O. Aharony, M. Berkooz, and N. Seiberg, “Light cone description of (2,0) superconformal theories in six-dimensions,” Adv. Theor. Math. Phys. 2 (1998) 119–153, hep-th/9712117.

[104] N. Arkani-Hamed, A. G. Cohen, D. B. Kaplan, A. Karch, and L. Motl, “Deconstructing (2,0) and little string theories,” JHEP 01 (2003) 083, hep-th/0110146.

[105] C.-S. Chu and P. Vanichchapongjaroen, “Non-abelian self-dual string and M2-M5 branes intersection in supergravity,” JHEP 06 (2013) 028, arXiv:1304.4322.

[106] N. Lambert and T. Orchard, “Null reductions of the M5-brane,” JHEP 12 (2020) 037, arXiv:2005.14331.

[107] N. Lambert, A. Lipstein, R. Mouland, and P. Richmond, “Five-dimensional non-Lorentzian conformal field theories and their relation to six-dimensions,” arXiv:2012.00626.

[108] L. F. Alday, S. M. Chester, and H. Raj, “6d (2,0) and M-theory at 1-loop,” arXiv:2005.07175.

[109] N. Drukker, S. Giombi, R. Ricci, and D. Trancanelli, “Wilson loops: From four-dimensional SYM to two-dimensional YM,” Phys. Rev. D 77 (2008) 047901, arXiv:0707.2699.

[110] E. Witten, “Solutions of four-dimensional field theories via M theory,” Nucl. Phys. B 500 (1997) 3–42, hep-th/9703166.

[111] P. S. Howe, N. Lambert, and P. C. West, “The three-brane soliton of the M-fivebrane,” Phys. Lett. B 419 (1998) 79–83, hep-th/9710033.

[112] E. Bergshoeff, J. Gomis, and P. K. Townsend, “M-brane intersections from world volume superalgebras,” Phys. Lett. B 421 (1998) 109–118, hep-th/9711043.

[113] D. Gaiotto, “$\mathcal{N} = 2$ dualities,” JHEP 08 (2012) 034, arXiv:0904.2715.

[114] D. Gaiotto and J. Maldacena, “The gravity duals of $\mathcal{N} = 2$ superconformal field theories,” JHEP 10 (2012) 189, arXiv:0904.4466.

[115] Y. Zhou, “Supersymmetric Rényi entropy and Weyl anomalies in six-dimensional (2,0) theories,” JHEP 06 (2016) 064, arXiv:1512.03008.