ON THE DEPENDENCE ON $p$ OF THE VARIATIONAL EIGENVALUES OF THE $p$-LAPLACE OPERATOR

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Abstract. We study the behavior of the variational eigenvalues of the $p$-Laplace operator, with homogeneous Dirichlet boundary condition, when $p$ is varying. After introducing an auxiliary problem, we characterize the continuity answering, in particular, a question raised in [18].

1. Introduction

Let $\Omega$ be a connected and bounded open subset of $\mathbb{R}^N$ and let $1 < p < \infty$. The study of the nonlinear eigenvalue problem

$$\begin{cases}
-\Delta_p u = \lambda |u|^{p-2}u & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}$$

has been the object of several papers, starting from [17], where it has been proved that the first eigenvalue is simple and is the unique eigenvalue which admits a positive eigenfunction. Alternative proofs and more general equations have been the object of the subsequent papers [3, 4, 14, 15, 20, 21], while the existence of a diverging sequence of eigenvalues has been proved under quite general assumptions in [21, 24].

If we denote by $\lambda_p^{(1)}$ the first eigenvalue of (1.1) and by $u_p$ the associated positive eigenfunction such that

$$\int_{\Omega} u_p^p \, dx = 1,$$

a challenging question concerns the behavior of $\lambda_p^{(1)}$ and $u_p$ with respect to $p$. As shown in [18], about the dependence from the right one has in full generality

$$\lim_{s \to p^+} \lambda_s^{(1)} = \lambda_p^{(1)},$$

$$\lim_{s \to p^+} \int_{\Omega} |\nabla u_s - \nabla u_p|^p \, dx = 0,$$

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while the “corresponding” assertions from the left
\[
\lim_{s \to p^-} \lambda^{(1)}_s = \lambda^{(1)}_p ,
\]
\[
\lim_{s \to p^-} \int_\Omega |\nabla u_s - \nabla u_p|^s \, dx = 0
\]
are true under some further assumption about \( \partial \Omega \). A counterexample in the same paper [18] shows that otherwise in general they are false.

A related question concerns the equivalence, without any assumption on \( \partial \Omega \), between the two assertions. In [18] it is proved that, if
\[
\lim_{s \to p^-} \int_\Omega |\nabla u_s - \nabla u_p|^s \, dx = 0,
\]
then
\[
\lim_{s \to p^-} \lambda^{(1)}_s = \lambda^{(1)}_p ,
\]
while the converse is proposed as an open problem. Subsequent papers have considered more general situations (see [10, 11]), but the previous question seems to be still unsolved (see also [19]).

The main purpose of this paper is to introduce an auxiliary problem which allows to describe the behaviour of \( \lambda^{(1)}_s \) and \( u_s \) as \( s \to p^- \) (see the next Theorem 3.2). Then in Theorem 4.1 we provide several equivalent characterizations of the fact that
\[
\lim_{s \to p^-} \lambda^{(1)}_s = \lambda^{(1)}_p .
\]
In particular, in Corollary 4.4 we give a positive answer to the mentioned open problem.

We also consider the dependence on \( s \) of the full sequence \( (\lambda^{(m)}_s) \) of the variational eigenvalues, defined according to some topological index \( i \). In particular, in Corollary 6.2 we prove that
\[
\lim_{s \to p^-} \lambda^{(m)}_s = \lambda^{(m)}_p \quad \forall m \geq 1
\]
if and only if
\[
\lim_{s \to p^-} \lambda^{(1)}_s = \lambda^{(1)}_p .
\]
The convergence of \( \lambda^{(m)}_s \) has been already studied in [6], under the \( \Gamma \)-convergence of the associated functionals. More specifically, in [22] it has been proved the continuity of \( \lambda^{(m)}_s \) with respect to \( s \), provided that \( \partial \Omega \) is smooth enough.

2. The First Eigenvalue with Respect to a Larger Space

Throughout the paper, \( \Omega \) will denote a bounded and open subset of \( \mathbb{R}^N \). No assumption will be imposed \( a \) priori about the regularity of \( \partial \Omega \). We will also denote by \( \mathcal{L}^N \) the Lebesgue measure in \( \mathbb{R}^N \).

If \( u \in W^{1,p}(\Omega) \), the condition “\( u = 0 \) on \( \partial \Omega \)” is usually expressed by saying that \( u \in W^{1,p}_0(\Omega) \). If \( \partial \Omega \) is smooth enough, this is perfectly reasonable; if not, other (nonequivalent) formulations can be proposed. In the line of the approach of [18], if \( 1 < p < \infty \) we set
\[
W^{1,p}_0(\Omega) = W^{1,p}(\Omega) \cap \left( \bigcap_{1 < s < p} W^{1,s}_0(\Omega) \right) = \bigcap_{1 < s < p} \left( W^{1,p}(\Omega) \cap W^{1,s}_0(\Omega) \right) .
\]
Proposition 2.1. The following facts hold:
(a) $W^{1,p}_0(\Omega)$ is a closed vector subspace of $W^{1,p}(\Omega)$ satisfying
$$W^{1,p}_0(\Omega) \subseteq W^{1,p}_0(\Omega);$$
(b) for every $u \in W^{1,p}_0(\Omega)$, the function
$$\hat{u} = \begin{cases} u & \text{on } \Omega; \\ 0 & \text{on } \mathbb{R}^N \setminus \Omega \end{cases}$$
belongs to $W^{1,p}(\mathbb{R}^N)$; in particular,
$$\left( \int_\Omega |\nabla u|^p \, dx \right)^{1/p}$$
is a norm on $W^{1,p}_0(\Omega)$ equivalent to the one induced by $W^{1,p}(\Omega)$;
(c) if $p < N$, we have $W^{1,p}_0(\Omega) \subseteq L^{p'}(\Omega)$ and
$$\inf \left\{ \frac{\int_\Omega |\nabla u|^p \, dx}{(\int_\Omega |u|^{p^{*}} \, dx)^{p/p^{*}}} : u \in W^{1,p}_0(\Omega) \setminus \{0\} \right\}$$
$$= \inf \left\{ \frac{\int_\Omega |\nabla u|^p \, dx}{(\int_\Omega |u|^{p^{*}} \, dx)^{p/p^{*}}} : u \in W^{1,p}_0(\Omega) \setminus \{0\} \right\};$$
(d) if $p > N$, we have $W^{1,p}_0(\Omega) = W^{1,p}(\Omega)$;
(e) if $\Omega$ has the segment property, we have $W^{1,p}_0(\Omega) = W^{1,p}(\Omega)$ for any $p$.

Proof. Since $W^{1,p}(\Omega) \cap W^{1,s}(\Omega)$ is a closed vector subspace of $W^{1,p}(\Omega)$ containing $W^{1,p}_0(\Omega)$, assertion (a) follows.

If $u \in W^{1,p}_0(\Omega)$, the function
$$\hat{u} = \begin{cases} u & \text{on } \Omega; \\ 0 & \text{on } \mathbb{R}^N \setminus \Omega \end{cases}$$
begins to $W^{1,s}(\mathbb{R}^N)$ for any $s < p$ and
$$- \int_{\mathbb{R}^N} \hat{u} D_j v \, dx = \int_\Omega D_j u v \, dx \quad \forall v \in C^1_c(\mathbb{R}^N).$$

It follows $\hat{u} \in W^{1,p}(\mathbb{R}^N)$, whence assertion (b).

If $p < N$, let $U$ be a bounded open subset of $\mathbb{R}^N$ with $\overline{\Omega} \subseteq U$ and let $u \in W^{1,p}_0(\Omega)$. Then $\hat{u} \in W^{1,p}_0(U) \subseteq L^{p'}(U)$ (see e.g. [5, Lemma 9.5]) and
$$\frac{\int_U |\nabla \hat{u}|^p \, dx}{(\int_U |\hat{u}|^{p^{*}} \, dx)^{p/p^{*}}} = \frac{\int_\Omega |\nabla u|^p \, dx}{(\int_\Omega |u|^{p^{*}} \, dx)^{p/p^{*}}}.$$

Assertion (c) follows from the fact that
$$\inf \left\{ \frac{\int_\Omega |\nabla v|^p \, dx}{(\int_\Omega |v|^{p^{*}} \, dx)^{p/p^{*}}} : v \in W^{1,p}_0(\Omega) \setminus \{0\} \right\}$$
is independent of $\Omega$ (see e.g. [25]).
If \( p > N \) and \( u \in W^{1,p}_0(\Omega) \), we have \( \hat{u} \in C(\overline{\Omega}) \cap W^{1,p}(\Omega) \) with \( \hat{u} = 0 \) on \( \partial \Omega \). It follows that \( \hat{u} \in W^{1,p}_0(\Omega) \) (see e.g. [5, Theorem 9.17], where the proof of \((i) \Rightarrow (ii)\) does not use the regularity of \( \partial \Omega \)), whence assertion \((d)\).

Assertion \((e)\) is taken from [11, Theorem 2.1]. □

Let us point out that the counterexample in [18] shows that \( W^{1,p}_0(\Omega) \) can be strictly larger than \( W^{1,p}_0(\Omega) \) in the case \( 1 < p \leq N \) (see also the next Remark 3.3).

Now we set

\[
\lambda_p^{(1)} = \inf \left\{ \frac{\int_{\Omega} |\nabla u|^p \, dx}{\int_{\Omega} |u|^p \, dx} : u \in W^{1,p}_0(\Omega) \setminus \{0\} \right\},
\]

\[
\lambda_p^{(1)} = \inf \left\{ \frac{\int_{\Omega} |\nabla u|^p \, dx}{\int_{\Omega} |u|^p \, dx} : u \in W^{1,p}_0(\Omega) \setminus \{0\} \right\}.
\]

It is easily seen that also the infimum defining \( \lambda_p^{(1)} \) is achieved and we clearly have \( 0 < \lambda_p^{(1)} \leq \lambda_p^{(1)} \).

More precisely, there exists \( v \in W^{1,p}_0(\Omega) \) such that

\[
v \geq 0 \quad \text{a.e. in } \Omega, \quad \int_{\Omega} v^p \, dx = 1, \quad \int_{\Omega} |\nabla v|^p \, dx = \lambda_p^{(1)}.
\]

According to [17, 18], if \( \Omega \) is connected we also denote by \( u_p \) the positive eigenfunction in \( W^{1,p}_0(\Omega) \) associated with \( \lambda_p^{(1)} \) such that

\[
\int_{\Omega} u_p^p \, dx = 1.
\]

In the next result we will see that something similar can be done with respect to the space \( W^{1,p}_0(\Omega) \).

**Theorem 2.2.** If \( \Omega \) is connected, the following facts hold:

(a) there exists one and only one \( u_p \in W^{1,p}_0(\Omega) \) such that

\[
u_p \geq 0 \quad \text{a.e. in } \Omega, \quad \int_{\Omega} u_p^p \, dx = 1, \quad \int_{\Omega} |\nabla u_p|^p \, dx = \lambda_p^{(1)};
\]

moreover, we have \( u_p \in L^\infty(\Omega) \cap C^1(\Omega) \) and \( u_p > 0 \) in \( \Omega \);

(b) the set of \( u \)'s in \( W^{1,p}_0(\Omega) \) such that

\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx = \lambda_p^{(1)} \int_{\Omega} |u|^{p-2}uv \, dx \quad \forall v \in W^{1,p}_0(\Omega)
\]

is a vector subspace of \( W^{1,p}_0(\Omega) \) of dimension 1;

(c) if \( \lambda \in \mathbb{R} \) and \( u \in W^{1,p}_0(\Omega) \) satisfy

\[
\begin{cases}
u \geq 0 \quad \text{a.e. in } \Omega, \\
\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx = \lambda \int_{\Omega} u^{p-1}v \, dx \quad \forall v \in W^{1,p}_0(\Omega),
\end{cases}
\]

then \( \lambda = \lambda_p^{(1)} \) and \( u = t u_p \) for some \( t > 0 \).

Since the proof follows the same lines of [17], we postpone it to the Appendix.
3. Behavior from the left of the first eigenvalue

Proposition 3.1. If $1 < s < p < \infty$, it holds

$$s \left( \Delta_{s}^{(1)} \right)^{1/s} \leq \lambda_{s}^{(1)} \leq p \left( \Delta_{p}^{(1)} \right)^{1/p}.$$  

Proof. Let $s < t < p$. For every $u \in W^{1,p}_{0}(\Omega) \cap L^{\infty}(\Omega)$, we have $|u|^{p/t} \in W^{1,t}_{0}(\Omega)$, hence

$$\lambda_{t}^{(1)} \int_{\Omega} |u|^{p} \, dx \leq \int_{\Omega} |\nabla |u|^{p/t}|^{t} \, dx = \left( \frac{p}{t} \right)^{t} \int_{\Omega} |u|^{p-t} |\nabla u|^{t} \, dx \leq \left( \frac{p}{t} \right)^{t} \left( \int_{\Omega} |u|^{p} \, dx \right)^{\frac{p-t}{p}} \left( \int_{\Omega} |\nabla u|^{p} \, dx \right)^{\frac{t}{p}}.$$  

It follows

$$\lambda_{t}^{(1)} \left( \int_{\Omega} |\nabla u|^{p} \, dx \right)^{\frac{1}{p}} \leq \left( \frac{p}{t} \right)^{t} \left( \int_{\Omega} |u|^{p} \, dx \right)^{\frac{1}{p}} \quad \forall u \in W^{1,p}_{0}(\Omega) \cap L^{\infty}(\Omega).$$  

For every $u \in W^{1,p}_{0}(\Omega)$, the function

$$v_{k} = \max\{\min\{u, -k\}, k\}$$

belongs to $W^{1,p}_{0}(\Omega) \cap L^{\infty}(\Omega)$ and is convergent to $u$ in $W^{1,p}(\Omega)$. It follows

$$\lambda_{t}^{(1)} \left( \int_{\Omega} |u|^{p} \, dx \right)^{\frac{1}{p}} \leq \left( \frac{p}{t} \right)^{t} \left( \int_{\Omega} |\nabla u|^{p} \, dx \right)^{\frac{1}{p}} \quad \forall u \in W^{1,p}_{0}(\Omega),$$

whence

$$\lambda_{t}^{(1)} \leq \left( \frac{p}{t} \right)^{t} \left( \lambda_{p}^{(1)} \right)^{\frac{1}{p}}.$$  

On the other hand, in [18, Remark p. 204] it is proved that

$$s \left( \lambda_{s}^{(1)} \right)^{1/s} \leq t \left( \lambda_{t}^{(1)} \right)^{1/t}$$

and the argument does not require $\Omega$ to be connected. We conclude that

$$s \left( \lambda_{s}^{(1)} \right)^{1/s} \leq t \left( \lambda_{t}^{(1)} \right)^{1/t} \leq p \left( \lambda_{p}^{(1)} \right)^{1/p}.$$  

For the sake of completeness we have also recalled the other (easy) inequalities. □

Now we can prove the main result of this section.

Theorem 3.2. If $1 < p < \infty$, it holds

$$\lim_{s \to p^{-}} \lambda_{s}^{(1)} = \lim_{s \to p^{-}} \lambda_{s}^{(1)} = \lambda_{p}^{(1)}.$$  

If $\Omega$ is connected, we also have

$$\lim_{s \to p^{-}} \int_{\Omega} |\nabla u_{s} - \nabla u_{s}^{*}|^{s} \, dx = \lim_{s \to p^{-}} \int_{\Omega} |\nabla u_{s} - \nabla u_{s}^{*}|^{s} \, dx = 0.$$  

Proof. By Proposition 3.1 it is clear that

$$\lim_{s \to p^{-}} \Delta_{s}^{(1)} \leq \lim_{s \to p^{-}} \lambda_{s}^{(1)} \leq \Delta_{p}^{(1)}.$$
Let \((p_k)\) be a sequence strictly increasing to \(p\), let \(v_k \in W^{1,(p_k)-}_0(\Omega)\) with
\[
v_k \geq 0 \quad \text{a.e. in } \Omega, \quad \int_\Omega v_k^{p_k} \, dx = 1, \quad \int_\Omega |\nabla v_k|^{p_k} \, dx = \lambda^{(1)}(p_k)
\]
and let \(1 < t < p\). In particular, it holds
\[
\lim_{k \to \infty} \int_\Omega |\nabla v_k|^{p_k} \, dx < +\infty.
\]
Up to a subsequence, we have \(p_k > t\) and \((v_k)\) is weakly convergent to some \(u\) in \(W^{1,t}_0(\Omega)\). Since the sequence \((v_k)\) is eventually bounded in \(W^{1,s}_0(\Omega)\) for any \(s < p\), it follows
\[
u \in \bigcap_{1 < s < p} W^{1,s}_0(\Omega).
\]
Moreover, it holds
\[
u \geq 0 \quad \text{a.e. in } \Omega, \quad \int_\Omega u^{p} \, dx = 1
\]
and, for every \(s < p\),
\[
\int_\Omega |\nabla u|^s \, dx \leq \liminf_{k \to \infty} \int_\Omega |\nabla v_k|^s \, dx \leq \liminf_{k \to \infty} \left[ \mathcal{L}^N(\Omega)^{1 - \frac{s}{p_k}} \left( \int_\Omega |\nabla v_k|^{p_k} \, dx \right)^{\frac{s}{p_k}} \right]
\]
\[
= \lim_{k \to \infty} \left[ \mathcal{L}^N(\Omega)^{1 - \frac{s}{p_k}} \left( \lambda^{(1)}(p_k)^{\frac{1}{p_k}} \right)^{\frac{s}{p_k}} \right] = \mathcal{L}^N(\Omega)^{1 - \frac{s}{p}} \left( \lim_{k \to \infty} \lambda^{(1)}(p_k) \right)^{\frac{s}{p}}.
\]
By the arbitrariness of \(s\), we infer that \(u \in W^{1,p}(\Omega)\), hence \(u \in W^{1,p-}_0(\Omega)\), with
\[
\lambda^{(1)} \leq \int_\Omega |\nabla u|^p \, dx \leq \lim_{k \to \infty} \lambda^{(1)}(p_k).
\]
It follows
\[
\lim_{s \to p-} \lambda^{(1)} = \lim_{s \to p-} \lambda^{(1)} = \lambda^{(1)} = \int_\Omega |\nabla u|^p \, dx.
\]
Now assume that \(\Omega\) is connected. From (a) of Theorem 2.2, we infer that \(v_k = u_{p_k}\), \(u = u_p\) and
\[
\lim_{s \to p-} u_s = u_p \quad \text{weakly in } W^{1,t}_0(\Omega) \quad \text{for any } t < p.
\]
In particular, it holds
\[
\lim_{s \to p-} \int_\Omega \left| \frac{u_s + u_p}{2} \right|^s \, dx = 1,
\]
whence
\[
\liminf_{s \to p-} \int_\Omega \left| \frac{\nabla u_s + \nabla u_p}{2} \right|^s \, dx \geq \lambda_p.
\]
If \(2 < s < p\), Clarkson’s inequality yields
\[
\int_\Omega \left| \frac{\nabla u_s + \nabla u_p}{2} \right|^s \, dx + \int_\Omega \left| \frac{\nabla u_s - \nabla u_p}{2} \right|^s \, dx \leq \frac{1}{2} \int_\Omega |\nabla u_s|^s \, dx + \frac{1}{2} \int_\Omega |\nabla u_p|^s \, dx,
\]
whence
\[
\lim_{s \to p-} \int_\Omega |\nabla u_s - \nabla u_p|^s \, dx = 0 \quad \text{if } p > 2.
\]
When $1 < s < p \leq 2$, Clarkson’s inequality becomes
\[
\left( \int_{\Omega} \left| \frac{\nabla u_s + \nabla u_p}{2} \right|^s \, dx \right)^{\frac{1}{s-1}} + \left( \int_{\Omega} \left| \frac{\nabla u_s - \nabla u_p}{2} \right|^s \, dx \right)^{\frac{1}{s-1}} \leq \left( \frac{1}{2} \int_{\Omega} |\nabla u_s|^s \, dx + \frac{1}{2} \int_{\Omega} |\nabla u_p|^s \, dx \right)^{\frac{1}{s-1}},
\]
but the argument is the same.

Finally, in a similar way one can prove that
\[
\lim_{s \to p^-} \int_{\Omega} |\nabla u_s - \nabla u_p|^s \, dx = 0.
\]

\[\square\]

Remark 3.3. Since the paper [18] contains a counterexample with
\[
\lim_{s \to p^-} \lambda_s^{(1)} < \lambda_p^{(1)},
\]
in that case we have $\lambda_2^{(1)} < \lambda_p^{(1)}$, hence $W^{1,p}_{-0}(\Omega) \neq W^{1,p}_{0}(\Omega)$.

Now we aim also to describe the behavior as $s \to p^-$ in the terms of the variational convergence of [1, 7].

Definition 3.4. Let $X$ be a metrizable topological space, $f : X \to [-\infty, +\infty]$ a function and let $(f_h)$ be a sequence of functions from $X$ to $[-\infty, +\infty]$. According to [7, Proposition 8.1], we say that $(f_h)$ is $\Gamma$-convergent to $f$ and we write
\[
\Gamma - \lim_{h \to \infty} f_h = f,
\]
if the following facts hold:
(a) for every $u \in X$ and every sequence $(u_h)$ converging to $u$ in $X$ it holds
\[
\liminf_{h \to \infty} f_h(u_h) \geq f(u);
\]
(b) for every $u \in X$ there exists a sequence $(u_h)$ converging to $u$ in $X$ such that
\[
\lim_{h \to \infty} f_h(u_h) = f(u).
\]

If $1 < p < \infty$, we define two functionals $\mathcal{E}_p, \mathcal{E}_{-p} : L^1_{\text{loc}}(\Omega) \to [0, +\infty]$ as
\[
\mathcal{E}_p(u) = \begin{cases} 
\left( \int_{\Omega} |\nabla u|^p \, dx \right)^{1/p} & \text{if } u \in W^{1,p}_0(\Omega), \\
+\infty & \text{otherwise},
\end{cases}
\]
\[
\mathcal{E}_{-p}(u) = \begin{cases} 
\left( \int_{\Omega} |\nabla u|^p \, dx \right)^{1/p} & \text{if } u \in W^{1,p}_{-0}(\Omega), \\
+\infty & \text{otherwise}.
\end{cases}
\]

Theorem 3.5. For every sequence $(p_h)$ strictly increasing to $p$, with $1 < p < \infty$, it holds
\[
\Gamma - \lim_{h \to \infty} \mathcal{E}_{p_h} = \Gamma - \lim_{h \to \infty} \mathcal{E}_{-p_h} = \mathcal{E}_p.
\]
Proof. Define, whenever $1 < s < \infty$, $f_s, f_s^\prime: L_{\text{loc}}^1(\Omega) \to [0, +\infty]$ as
\[
f_s(u) = \mathcal{L}^N(\Omega)^{-1/s} \mathcal{E}_s(u), \quad f_s^\prime(u) = \mathcal{L}^N(\Omega)^{-1/s} \mathcal{E}_s(u).
\]
Then $f_s, f_s^\prime$ are lower semicontinuous and the sequences $(f_{p_h})$, $(f_{p_h}^\prime)$ are both increasing and pointwise convergent to $f_p^\prime$. From [7, Proposition 5.4] we infer that
\[
\Gamma - \lim_{h \to \infty} f_{p_h} = \Gamma - \lim_{h \to \infty} f_{p_h}^\prime = f_p^\prime
\]
and the assertion easily follows. □

4. SOME CHARACTERIZATIONS

Without imposing any assumption on $\partial \Omega$, we aim to characterize the fact that
\[
\lim_{s \to p^-} \lambda_s^{(1)} = \lambda_p^{(1)}.
\]

Theorem 4.1. If $1 < p < \infty$ and $\Omega$ is connected, the following facts are equivalent:

(a) $\lim_{s \to p^-} \lambda_s^{(1)} = \lambda_p^{(1)}$;
(b) for every sequence $(p_h)$ strictly increasing to $p$, it holds
\[
\Gamma - \lim_{h \to \infty} \mathcal{E}_{p_h} = \mathcal{E}_p;
\]
(c) $W_0^{1,p^-}(\Omega) = W_0^{1,p}(\Omega)$;
(d) $\lambda_p^{(1)} = \lambda_p^{(1)}$;
(e) $\underline{u}_p = u_p$;
(f) $\underline{w}_p \in W_0^{1,p}(\Omega)$;
(g) the solution $u$ of
\[
\begin{cases}
  u \in W_0^{1,p^-}(\Omega), \\
  \int \Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx = \int \Omega v \, dx \quad \forall v \in W_0^{1,p^-}(\Omega),
\end{cases}
\]
given by the next Theorem 7.1, belongs to $W_0^{1,p}(\Omega)$.

Proof. By Theorem 3.2 it is clear that (a) $\iff$ (d), while we have (b) $\iff$ (c) by Theorem 3.5. Now we consider the assertions from (e) to (g).

It is clear that (c) $\implies$ (d). If $\lambda_p^{(1)} = \lambda_p^{(1)}$, we have that $u_p \in W_0^{1,p}(\Omega) \subseteq W_0^{1,p^-}(\Omega)$ satisfies
\[
u_p \geq 0 \text{ a.e. in } \Omega, \quad \int \Omega u_p \, dx = 1, \quad \int \Omega |\nabla u_p|^p \, dx = \lambda_p^{(1)}.
\]
From (a) of Theorem 2.2 we infer that $u_p = \underline{w}_p$. Therefore (d) $\implies$ (e).

Of course, (e) $\implies$ (f). If $\underline{w}_p \in W_0^{1,p}(\Omega)$, let
\[
z_k = \min \{ \lambda_p^{(1)} (k\underline{w}_p)^{p-1}, 1 \}
\]
and let $w_k \in W_0^{1,p^-}(\Omega)$ be the solution of
\[
\int \Omega |\nabla w_k|^{p-2} \nabla w_k \cdot \nabla v \, dx = \int \Omega z_k v \, dx \quad \forall v \in W_0^{1,p^-}(\Omega)
\]
according to Theorem 7.1. Since $0 \leq z_k \leq \lambda_p^{(1)}(ku_p)^{p-1}$ a.e. in $\Omega$, we have $0 \leq w_k \leq ku_p$ a.e. in $\Omega$. From $w_k \in W^{1,p}(\Omega)$ and $ku_p \in W^{1,p}_0(\Omega)$, we infer that $w_k \in W^{1,p}_0(\Omega)$.

Since $(z_k)$ is convergent to 1 in $L^p(\Omega)$, we also have

$$\lim_{k \to \infty} \int_{\Omega} |\nabla w_k - \nabla u|^p \, dx = 0,$$

whence $u \in W^{1,p}_0(\Omega)$. Therefore $(f) \Rightarrow (g)$.

Finally, assume that $(g)$ holds and let $u$ be as in assertion $(g)$. If $z \in L^\infty(\Omega)$ and $w \in W^{1,p}_0(\Omega)$ is the solution of

$$\int_{\Omega} |\nabla w|^{p-2} \nabla w \cdot \nabla v \, dx = \int_{\Omega} z v \, dx \quad \forall v \in W^{1,p}_0(\Omega),$$

we have $-M^{p-1} \leq z \leq M^{p-1}$ for some $M > 0$, whence $-Mu \leq w \leq Mu$ a.e. in $\Omega$. It follows $w \in W^{1,p}_0(\Omega)$.

Now let $w \in W^{1,p}_0(\Omega)$. Let $z \in L^p(\Omega)$ and $Z \in L^p(\Omega; \mathbb{R}^N)$ be such that

$$\int_{\Omega} |\nabla w|^{p-2} \nabla w \cdot \nabla v \, dx = \int_{\Omega} (zv + Z \cdot \nabla v) \, dx \quad \forall v \in W^{1,p}(\Omega).$$

Then let $(z_k)$ and $(Z_k)$ be two sequences in $C^\infty_c$ converging to $z$ and $Z$, respectively, in $L^p$. Since $(z_k - \div Z_k) \in L^\infty(\Omega)$, there exists $w_k \in W^{1,p}_0(\Omega)$ such that

$$\int_{\Omega} |\nabla w_k|^{p-2} \nabla w_k \cdot \nabla v \, dx = \int_{\Omega} (z_k - \div Z_k)v \, dx \quad \forall v \in W^{1,p}_0(\Omega).$$

Since

$$\int_{\Omega} |\nabla w|^{p-2} \nabla w \cdot \nabla v \, dx = \int_{\Omega} (zv + Z \cdot \nabla v) \, dx \quad \forall v \in W^{1,p}_0(\Omega),$$

$$\int_{\Omega} |\nabla w_k|^{p-2} \nabla w_k \cdot \nabla v \, dx = \int_{\Omega} (z_kv + Z_k \cdot \nabla v) \, dx \quad \forall v \in W^{1,p}_0(\Omega),$$

it follows

$$\lim_{k \to \infty} \int_{\Omega} |\nabla w_k - \nabla w|^p \, dx = 0,$$

whence $w \in W^{1,p}_0(\Omega)$. Therefore $(g) \Rightarrow (c)$. \hfill \Box

**Remark 4.2.** If $\Omega$ is not assumed to be connected, it holds

$$\begin{align*}
(b) & \iff (c) \iff (g) \\
(a) & \iff (d)
\end{align*}$$

In fact the same proof shows that

$$\begin{align*}
(b) & \iff (c) \iff (g) \\
(a) & \iff (d)
\end{align*}$$

and it is obvious that $(c) \Rightarrow (g)$.

On the other hand, let $U$ be a bounded open set as in Remark 3.3, with $W^{1,p}_0(U) \neq W_0^{1,p}(U)$, and let $\Omega = U \cup B$, where $B$ is an open ball with $\overline{U \cap B} = \emptyset$. Then $W^{1,p}_0(\Omega) \neq W^{1,p}_0(\Omega)$, so that $(b)$, $(c)$ and $(g)$ are false. However, if the ball $B$ is large enough, the first eigenvalue associated with $\Omega$ coincides with that
associated with $B$, which has the segment property, so that assertions (a) and (d) are true.

**Remark 4.3.** Let us stress, in Theorem 4.1, the assertion $(a) \Rightarrow (b)$. When $\Omega$ is connected, the convergence of the first eigenvalue implies the $\Gamma$-convergence of the full functional. This fact will be on the basis of the next Corollary 6.2.

**Corollary 4.4.** If $\Omega$ is connected and

$$
\lim_{s \to p^-} \chi^{(1)}_s = \lambda^{(1)}_p,
$$

then it holds

$$
\lim_{s \to p^-} \int_{\Omega} |\nabla u_s - \nabla u_p|^s \, dx = 0.
$$

**Proof.** From Theorem 4.1 we infer that $u_p = u_p$. By Theorem 3.2 we conclude that

$$
\lim_{s \to p^-} \int_{\Omega} |\nabla u_s - \nabla u_p|^s \, dx = \lim_{s \to p^-} \int_{\Omega} |\nabla u_s - \nabla u_p|^s \, dx = 0.
$$

□

**Remark 4.5.** The converse of the previous Corollary was known since a long time (see [18, Theorem 3.11]), while Corollary 4.4 was proposed as an open problem. Corollary 4.4 also answers a question raised in [18] concerning the formulation of Lemma 3.12 in that paper.

5. Behavior from the right of the first eigenvalue

The next results are essentially contained in [18, 8]. We mention them for the sake of completeness.

**Theorem 5.1.** If $1 < p < \infty$, it holds

$$
\lim_{s \to p^+} \lambda^{(1)}_s = \lim_{s \to p^+} \lambda^{(1)}_s = \lambda^{(1)}_p.
$$

If $\Omega$ is connected, we also have

$$
\lim_{s \to p^+} \int_{\Omega} |\nabla u_s - \nabla u_p|^p \, dx = \lim_{s \to p^+} \int_{\Omega} |\nabla u_s - \nabla u_p|^p \, dx = 0.
$$

**Proof.** The assertions concerning $\lambda^{(1)}_s$ and $u_s$ are proved in [18, Theorems 3.5 and 3.6], but the same arguments apply also to $\lambda^{(1)}_s$ and $u_s$. □

**Theorem 5.2.** For every sequence $(p_h)$ strictly decreasing to $p$, with $1 < p < \infty$, it holds

$$
\Gamma - \lim_{h \to \infty} \mathcal{E}_{p_h} = \Gamma - \lim_{h \to \infty} \mathcal{E}_{p_h} = \mathcal{E}_p.
$$

**Proof.** The assertion concerning $\mathcal{E}_{p_h}$ is proved in [8, Theorem 5.3] when $1 < p < N$, but the same argument applies to the other cases. □
6. Behavior of Higher Eigenvalues

Let \( i \) be an index with the following properties:

(i) \( i(K) \) is an integer greater or equal than 1 and is defined whenever \( K \) is a nonempty, compact and symmetric subset of a topological vector space such that \( 0 \not\in K \);

(ii) if \( X \) is a topological vector space and \( K \subseteq X \setminus \{0\} \) is compact, symmetric and nonempty, then there exists an open subset \( U \) of \( X \setminus \{0\} \) such that \( K \subseteq U \) and

\[
\hat{i}(K) \leq i(K) \quad \text{for any compact, symmetric and nonempty } \hat{K} \subseteq U;
\]

(iii) if \( X, Y \) are two topological vector spaces, \( K \subseteq X \setminus \{0\} \) is compact, symmetric and nonempty and \( \pi : K \to Y \setminus \{0\} \) is continuous and odd, we have

\[
i(\pi(K)) \geq i(K);
\]

(iv) if \( X \) is a normed space with \( 1 \leq \dim X < \infty \), we have

\[
i(\{ u \in X : ||u|| = 1 \}) = \dim X.
\]

Well known examples are the Krasnosel’skiǐ genus (see e.g. [16, 23, 28]) and the \( \mathbb{Z}_2 \)-cohomological index (see [12, 13]). More general examples are contained in [2].

If \( 1 < p < \infty \), we consider

\[
M = \left\{ u \in W^{1,p}_0(\Omega) : \int_\Omega |u|^p \, dx = 1 \right\},
\]

\[
\overline{M} = \left\{ u \in W^{1,p-}_0(\Omega) : \int_\Omega |u|^p \, dx = 1 \right\},
\]

endowed with the \( W^{1,p}(\Omega) \)-topology, and we define for every \( m \geq 1 \) the variational eigenvalues of the \( p \)-Laplace operator as

\[
\lambda_p^{(m)} = \inf \left\{ \max_{u \in K} \int_\Omega |\nabla u|^p \, dx : K \text{ is a nonempty, compact and symmetric subset of } M \text{ with } i(K) \geq m \right\},
\]

\[
\overline{\lambda}_p^{(m)} = \inf \left\{ \max_{u \in K} \int_\Omega |\nabla u|^p \, dx : K \text{ is a nonempty, compact and symmetric subset of } \overline{M} \text{ with } i(K) \geq m \right\}.
\]

It is easily seen that the new definitions of \( \lambda_p^{(1)} \) and \( \overline{\lambda}_p^{(1)} \) are consistent with the previous ones and we clearly have

\[
\lambda_p^{(m)} \leq \lambda_p^{(m+1)},
\]

\[
\overline{\lambda}_p^{(m)} \leq \overline{\lambda}_p^{(m+1)},
\]

\[
\lambda_p^{(m)} \leq \lambda_p^{(m)}.
\]
Theorem 6.1. If $1 < p < \infty$, for every $m \geq 1$ we have

$$\lim_{s \to p^-} \lambda_s^{(m)} = \lim_{s \to p^-} \lambda^{(m)} = \lambda^{(m)},$$

$$\lim_{s \to p^+} \lambda_s^{(m)} = \lim_{s \to p^+} \lambda^{(m)} = \lambda^{(m)}.$$

Proof. Taking into account Theorems 3.5 and 5.2, the assertions follow from the results of [6, 8]. Let us give some detail following the approach of [8].

If we define $g_p : L^1_{loc}(\Omega) \to \mathbb{R}$ as

$$g_p(u) = \begin{cases} 
\left( \int_{\Omega} |u|^p \, dx \right)^{1/p} & \text{if } u \in L^p(\Omega), \\
0 & \text{otherwise},
\end{cases} \quad (6.1)$$

it is easily seen that $g_p$ is $L^1_{loc}(\Omega)$-continuous on

$$\{ u \in L^1_{loc}(\Omega) : \mathcal{E}_p(u) \leq b \}$$

for any $b \in \mathbb{R}$.

If we consider

$$\widehat{M} = \{ u \in L^1_{loc}(\Omega) : g_p(u) = 1 \}$$

dowered with the $L^1_{loc}(\Omega)$-topology, by [8, Corollary 3.3] we have

$$\left( \lambda_p^{(m)} \right)^{1/p} = \inf \left\{ \sup_{u \in K} \mathcal{E}_p(u) : K \text{ is a nonempty, compact and symmetric subset of } \widehat{M} \text{ with } i(K) \geq m \right\},$$

$$\left( \lambda^{(m)} \right)^{1/p} = \inf \left\{ \sup_{u \in K} \mathcal{E}_p(u) : K \text{ is a nonempty, compact and symmetric subset of } \widehat{M} \text{ with } i(K) \geq m \right\},$$

(see also [8, Theorem 5.2]). Then the assertions follow from Theorems 3.5, 5.2 and [8, Corollary 4.4] (see also [8, Theorem 6.4]). \hfill \Box

Corollary 6.2. Let $1 < p < \infty$ and assume that $\Omega$ is connected. Then we have

$$\lim_{s \to p} \lambda^{(m)} = \lambda_p^{(m)} \quad \forall m \geq 1$$

if and only if

$$\lim_{s \to p^-} \lambda_s^{(1)} = \lambda_p^{(1)}.$$ 

Proof. If

$$\lim_{s \to p^-} \lambda_s^{(1)} = \lambda_p^{(1)},$$

from Theorem 4.1 we infer that $W^{1,p}_0(\Omega) = W^{1,p}_0(\Omega)$, whence $\lambda_p^{(m)} = \lambda_p^{(m)}$ for any $m \geq 1$. Then the assertion follows from Theorem 6.1.

The converse is obvious. \hfill \Box
7. Appendix

In this appendix we see that several well-known properties of $W^{1,p}_0(\Omega)$ are still valid for $W^{1,p}_0(\Omega)$.

**Theorem 7.1.** If $1 < p < \infty$, the following facts hold:

(a) For every $z \in L^p(\Omega)$ and $Z \in L^p(\Omega; \mathbb{R}^N)$, there exists one and only one $w \in W^{1,p}_0(\Omega) \subseteq W^{1,p}(\Omega)$ such that

\[
\int_{\Omega} |\nabla w|^{p-2} \nabla w \cdot \nabla v \, dx = \int_{\Omega} (zv + Z \cdot \nabla v) \, dx \quad \forall v \in W^{1,p}_0(\Omega)
\]

and the map

\[
L^p(\Omega) \times L^p(\Omega; \mathbb{R}^N) \rightarrow W^{1,p}(\Omega)
\]

\[
(z, Z) \mapsto w
\]

is continuous;

(b) If $z_1, z_2 \in L^p(\Omega)$ with $z_1 \leq z_2$ a.e. in $\Omega$ and $w_1, w_2 \in W^{1,p}_0(\Omega)$ are the solutions of

\[
\int_{\Omega} |\nabla w_k|^{p-2} \nabla w_k \cdot \nabla v \, dx = \int_{\Omega} z_k v \, dx \quad \forall v \in W^{1,p}_0(\Omega),
\]

then it holds $w_1 \leq w_2$ a.e. in $\Omega$.

**Proof.** Assertion (a) easily follows from Proposition 2.1. Since $(w_1 - w_2)^+ \in W^{1,p}_0(\Omega)$, we have

\[
\int_{\Omega} |\nabla w_1|^{p-2} \nabla w_1 \cdot \nabla (w_1 - w_2)^+ \, dx = \int_{\Omega} z_1(w_1 - w_2)^+ \, dx,
\]

\[
\int_{\Omega} |\nabla w_2|^{p-2} \nabla w_2 \cdot \nabla (w_1 - w_2)^+ \, dx = \int_{\Omega} z_2(w_1 - w_2)^+ \, dx,
\]

hence

\[
0 \leq \int_{\{w_1 > w_2\}} \left( |\nabla w_1|^{p-2} \nabla w_1 - |\nabla w_2|^{p-2} \nabla w_2 \right) \cdot (\nabla w_1 - \nabla w_2) \, dx
\]

\[
= \int_{\Omega} (z_1 - z_2)(w_1 - w_2)^+ \, dx \leq 0.
\]

It follows $w_1 \leq w_2$ a.e. in $\Omega$. \hfill \Box

**Lemma 7.2.** If $\lambda \in \mathbb{R}$ and $u \in W^{1,p}_0(\Omega) \setminus \{0\}$ satisfy

\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx = \lambda \int_{\Omega} |u|^{p-2} u v \, dx \quad \forall v \in W^{1,p}_0(\Omega),
\]

then $u \in L^\infty(\Omega) \cap C^1(\Omega)$.

Moreover, if $\Omega$ is connected and $u \geq 0$ a.e. in $\Omega$, it holds $u > 0$ in $\Omega$.

**Proof.** If $p > N$ we have $W^{1,p}_0(\Omega) = W^{1,p}_0(\Omega)$ and the assertion is proved in [17]. Therefore assume that $1 < p \leq N$. If we set

\[
R_k(t) = \begin{cases} t + k & \text{if } t < -k, \\ 0 & \text{if } -k \leq t \leq k, \\ t - k & \text{if } t > k, \end{cases}
\]
we have $R_k(u) \in W^{1,p-}_0(\Omega)$, hence
\[
\int_\Omega |\nabla R_k(u)|^p \, dx = \lambda \int_\Omega |u|^{p-1}|R_k(u)| \, dx.
\]

Let $1 < s < p$ with $s^* \geq p$. If we set
\[
A_k = \{ x \in \Omega : |u(x)| > k \} = \{ x \in \Omega : R_k(u(x)) \neq 0 \},
\]
it follows
\[
\int_{A_k} (|u| - k)^p \, dx \leq \mathcal{L}^N(A_k)^{1-\frac{s}{p}} \left( \int_\Omega |R_k(u)|^{s^*} \, dx \right)^\frac{p}{s^*}
\]
\[
\leq c(N,s)^p \mathcal{L}^N(A_k)^{1-\frac{s}{p}} \left( \int_\Omega |\nabla R_k(u)|^s \, dx \right)^\frac{p}{s^*}
\]
\[
\leq c(N,s)^p \mathcal{L}^N(A_k)^{1-\frac{s}{p}} \int_\Omega |\nabla R_k(u)|^p \, dx
\]
\[
= c(N,s)^p \mathcal{L}^N(A_k)^{1-\frac{s}{p}} \lambda \int_{A_k} |u|^{p-1}(|u| - k) \, dx.
\]

Then the same argument of [18, Lemma 4.1] shows that $u \in L^\infty(\Omega)$. By the results of [9], [26], we infer that $u \in C^1(\Omega)$.

If $\Omega$ is connected and $u \geq 0$ a.e. in $\Omega$, by [27, Theorem 5] we conclude that $u > 0$ in $\Omega$. \qed

**Proof of Theorem 2.2.**

If $u \in W^{1,p-}_0(\Omega) \setminus \{0\}$ satisfies
\[
\int_\Omega |\nabla u|^p \, dx = \lambda^{(1)}_p \int_\Omega |u|^p \, dx,
\]
we claim that $u \in L^\infty(\Omega) \cap C^1(\Omega)$ and either $u > 0$ in $\Omega$ or $u < 0$ in $\Omega$.

Actually, by the minimality of $\lambda^{(1)}_p$ it follows that
\[
\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx = \lambda^{(1)}_p \int_\Omega |u|^{p-2}uv \, dx \quad \forall v \in W^{1,p-}_0(\Omega)
\]
and from Lemma 7.2 we infer that $u \in L^\infty(\Omega) \cap C^1(\Omega)$. Moreover, also $w = |u|$ has the same properties and, by Lemma 7.2, satisfies $w > 0$ in $\Omega$. Since $\Omega$ is connected, we have either $u = w$ or $u = -w$ and the claim is proved.

In particular, there exists $w \in W^{1,p-}_0(\Omega) \cap L^\infty(\Omega) \cap C^1(\Omega)$ such that $w > 0$ in $\Omega$ and
\[
\int_\Omega |\nabla w|^{p-2}\nabla w \cdot \nabla v \, dx = \lambda^{(1)}_p \int_\Omega |w|^{p-2}wv \, dx \quad \forall v \in W^{1,p-}_0(\Omega).
\]

Now let $\lambda \in \mathbb{R}$ and $u \in W^{1,p-}_0(\Omega) \setminus \{0\}$ satisfy
\[
\begin{cases}
  u \geq 0 \text{ a.e. in } \Omega,
  \\
  \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx = \lambda \int_\Omega |u|^{p-2}v \, dx \quad \forall v \in W^{1,p-}_0(\Omega).
\end{cases}
\]

Again, from Lemma 7.2 we infer that $u \in L^\infty(\Omega) \cap C^1(\Omega)$ with $u > 0$ in $\Omega$, so that $\tilde{u} = \log u$ and $\tilde{w} = \log w$ also belong to $C^1(\Omega)$. If we set $u_k = u + (1/k)$,
\[ w_k = w + (1/k), \; \tilde{u}_k = \log u_k \quad \text{and} \quad \tilde{w}_k = \log w_k, \] we have
\[ \frac{1}{u_k} (u_k^p - w_k^p), \quad \frac{1}{w_k} (w_k^p - u_k^p) \in W_0^{1,p} (\Omega). \]

The first function can be used as a test in the equation of \( u \) and the second one in that of \( w \). As in [17, Lemma 3.1], it follows
\[
\int_{\Omega} \left( \lambda \frac{w^{p-1}}{u^{p-1}} - \lambda_p^{(1)} \frac{w^{p-1}}{w^{p-1}} \right) (u_k^p - w_k^p) \, dx
= \int_{\Omega} u_k^p \left( |\nabla \tilde{u}_k|^p - |\nabla \tilde{w}_k|^p - p |\nabla \tilde{w}_k|^{p-2} \nabla \tilde{w}_k \cdot (\nabla \tilde{u}_k - \nabla \tilde{w}_k) \right) \, dx
+ \int_{\Omega} w_k^p \left( |\nabla \tilde{w}_k|^p - |\nabla \tilde{u}_k|^p - p |\nabla \tilde{u}_k|^{p-2} \nabla \tilde{u}_k \cdot (\nabla \tilde{w}_k - \nabla \tilde{u}_k) \right) \, dx \geq 0.
\]
Passing to limit as \( k \to \infty \) and applying Lebesgue’s theorem and Fatou’s lemma, we infer that
\[
(\lambda - \lambda_p^{(1)}) \int_{\Omega} (u^p - w^p) \, dx
\geq \int_{\Omega} u^p \left( |\nabla \tilde{u}|^p - |\nabla \tilde{w}|^p - p |\nabla \tilde{w}|^{p-2} \nabla \tilde{w} \cdot (\nabla \tilde{u} - \nabla \tilde{w}) \right) \, dx
+ \int_{\Omega} w^p \left( |\nabla \tilde{w}|^p - |\nabla \tilde{u}|^p - p |\nabla \tilde{u}|^{p-2} \nabla \tilde{u} \cdot (\nabla \tilde{w} - \nabla \tilde{u}) \right) \, dx \geq 0.
\]
Since \( u \) can be replaced by \( tu \) for any \( t > 0 \), it follows \( \lambda = \lambda_p^{(1)} \). Then the strict convexity of \( \{ \xi \mapsto |\xi|^p \} \) implies that \( \nabla (\tilde{u} - \tilde{w}) = 0 \) in \( \Omega \). Since \( \Omega \) is connected, we infer that \( \tilde{u} = \tilde{w} + c \), hence \( u = e^c w \).

On the other hand, if \( u \in W_0^{1,p} (\Omega) \setminus \{0\} \) satisfies
\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx = \lambda_p^{(1)} \int_{\Omega} |u|^{p-2} uv \, dx \quad \forall v \in W_0^{1,p} (\Omega),
\]
it follows
\[
\int_{\Omega} |\nabla u|^p \, dx = \lambda_p^{(1)} \int_{\Omega} |u|^p \, dx,
\]
hence \( u \in C^1(\Omega) \) with either \( u > 0 \) in \( \Omega \) or \( u < 0 \) in \( \Omega \). We infer that \( u = tw \) for some \( t \neq 0 \). \( \square \)

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