UNIVERSALLY IRREDUCIBLE SUBVARIETIES
OF SIEGEL MODULI SPACES

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Abstract. A subvariety of a quasi-projective complex variety $X$ is called "universally irreducible" if its preimage inside the universal cover of $X$ is irreducible. In this paper we investigate sufficient conditions for universal irreducibility. We consider in detail complete intersection subvarieties of small codimension inside Siegel moduli spaces of any finite level. Moreover we show that, for $g \geq 3$, every Siegel modular form is the product of finitely many irreducible analytic functions on the Siegel upper half-space $\mathbb{H}_g$. We also discuss the special case of singular theta series of weight $\frac{1}{2}$ and of Schottky forms.

1. Introduction

1.1. Motivation. The moduli space $A_g$ of complex principally polarized Abelian varieties of dimension $g$ can be obtained as the quotient of the Siegel upper half-space

$$\mathbb{H}_g := \{ \tau \in \text{Mat}_{g \times g}(\mathbb{C}) \text{ symmetric, with } \Im(\tau) > 0 \}$$

by $\Gamma_g := \text{Sp}_{2g}(\mathbb{Z})$ that acts with finite stabilizers. Hence, in the orbifold sense, the projection $\mathbb{H}_g \to A_g$ can be seen as a universal covering space and every finite-index subgroup $\Gamma$ of $\Gamma_g$ determines a finite étale cover $A_g(\Gamma) := \mathbb{H}_g/\Gamma$ of $A_g$. Since holomorphic line bundles on $\mathbb{H}_g$ are trivializable, sections of holomorphic line bundles (in the orbifold sense) on $A_g(\Gamma)$ lift to Siegel modular forms with respect to $\Gamma$, namely holomorphic functions $F : \mathbb{H}_g \to \mathbb{C}$ that enjoy a suitable equivariance property with respect to the action of $\Gamma$ on $\mathbb{H}_g$ (see Section 5.1).

The zero locus of a modular form with respect to $\Gamma$ determines a $\Gamma$-invariant divisor in $\mathbb{H}_g$, which is the pull-back of an effective divisor on $A_g(\Gamma)$. Vice versa, for $g \geq 3$, every effective divisor in any $A_g(\Gamma)$ pulls back to $\mathbb{H}_g$ to the zero locus of a modular form.

A basic example of modular forms are the theta constants with semi-integral characteristic $[\frac{\varepsilon}{2}]$ defined as

$$\theta[\frac{\varepsilon}{2}] (\tau) := \sum_{n \in \mathbb{Z}^g} \exp{\pi i \left[ (n + \varepsilon/2)^t \tau (n + \varepsilon/2) + (n + \varepsilon/2)^t \delta \right]}.$$

where $\varepsilon, \delta \in (\mathbb{Z}/2)^g$. Several years ago E. Freitag and the second named author discussed at length about the problem of factorizing such modular forms.
In [F91, Theorem 4.6], Freitag proved that these modular forms are absolutely irreducible, i.e. they are nonzero and their divisors are irreducible in \( \mathcal{A}_g(\Gamma) \) with respect to arbitrary small finite-index subgroups \( \Gamma \) of \( \Gamma_g \). We call universally irreducible a non-zero modular form that has a stronger property, namely of being irreducible as an analytic function on \( \mathbb{H}_g \). Such irreducibility properties have a geometric counterpart.

**Definition 1.1** (Absolute and universal irreducible subvarieties). A subvariety \( Z \) of \( \mathcal{A}_g(\Gamma) \) is *absolutely irreducible* if the preimage of \( Z \) in every finite étale cover (in the orbifold sense) of \( \mathcal{A}_g(\Gamma) \) is irreducible, and it is *universally irreducible* if the preimage of \( Z \) in \( \mathbb{H}_g \) is irreducible.

In this paper we investigate sufficient conditions for universal irreducibility of subvarieties of \( \mathcal{A}_g(\Gamma) \).

1.2. **Setting and conventions.** For every \( n \geq 2 \) we let

\[ \Gamma_g(n) := \{ \sigma \in \Gamma_g \mid \sigma \equiv I_{2g} \pmod{n} \} \]

denote the \( n \)-th principal congruence subgroup of \( \Gamma_g \) and we briefly denote \( \mathcal{A}_g(\Gamma_g(n)) \) by \( \mathcal{A}_g(n) \).

Now fix \( n \geq 3 \). The moduli space \( \mathcal{A}_g(n) \) is a smooth quasi-projective variety and \( \mathbb{H}_g \to \mathcal{A}_g(n) \) is its universal cover in the standard sense; moreover, the same holds for every finite-index subgroup of \( \Gamma_g(n) \).

Let now \( \Gamma \) be any finite-index subgroup of \( \Gamma_g \). Then \( \mathcal{A}_g(\Gamma_g(n) \cap \Gamma) \) is a smooth quasi-projective variety. Thus \( \mathcal{A}_g(\Gamma) \) is the quotient of \( \mathcal{A}_g(\Gamma_g(n) \cap \Gamma) \) by the finite group \( \Gamma/(\Gamma_g(n) \cap \Gamma) \). Hence \( \mathcal{A}_g(\Gamma) \) always has the structure of smooth complex-analytic orbifold and of smooth Deligne-Mumford stack.

In order to have a more uniform treatment for all finite-index subgroups of \( \Gamma_g \), we will always take the orbifold point of view. This means that the words smooth, singular, étale cover, fundamental group must be understood in the orbifold sense.

1.3. **Main results.** Let \( \Gamma \) be any finite-index subgroup of \( \Gamma_g \). Our first main result is the following.

**Theorem A.** Let \( g \geq 3 \) and let \( D \subset \mathcal{A}_g(\Gamma) \) be an effective divisor. Then the preimage \( \tilde{D} \) in \( \mathbb{H}_g \) of \( D \) is connected. Moreover the following hold.

(i) If \( D \) is locally irreducible, then \( D \) is universally irreducible.

(ii) If \( D \) is absolutely irreducible, then \( D \) is universally irreducible.

The connectedness claim (proven in Theorem 5.2) relies on a generalization (Theorem 3.1) of the homotopical Lefschetz hyperplane section theorem for the fundamental group of smooth quasi-projective varieties that have a projective model with small boundary. We mention that, for \( g = 2 \), connectedness of \( \tilde{D} \) holds for divisors \( D \) whose closures intersect the boundary of the Satake compactification of \( \mathcal{A}_g(\Gamma) \) in a finite set of points, e.g. divisors defined by Eisenstein series.
Assertion (i) (proven in Theorem 5.2 too) is a direct consequence of the connectedness of $\tilde{D}$. Here we recall that normality implies local irreducibility (see Lemma 2.4(iii)) and so claim (i) also applies to normal divisors. Finally, assertion (ii) is a special case of the following.

**Theorem B.** Let $g \geq 3$ and let $Z \subset \mathcal{A}_g(\Gamma)$ be an absolutely irreducible complete intersection subvariety of codimension at most $g - 2$. Then $Z$ is universally irreducible.

Actually, the above Theorem B is a consequence of the following (see Theorem 6.5 in the body of the paper).

**Theorem B’.** Let $g \geq 3$ and let $Z \subset \mathcal{A}_g(\Gamma)$ be any complete intersection subvariety of codimension at most $g - 2$. Then its preimage $\tilde{Z}$ in $\mathbb{H}_g$ has finitely many irreducible components.

As noted by Freitag [F91, Theorem 4.7], for $g \geq 3$ all modular forms can be factorized into absolutely irreducible ones (Lemma 5.4). Such result heavily relies on the fact that $\text{Pic}(\mathbb{H}_g/\Gamma) \otimes \mathbb{Q} = \mathbb{Q} \cdot \lambda$ and the divisibility of the integral class $\lambda$ in $\text{Pic}(\mathbb{H}_g/\Gamma)_{\text{tors}}$ is uniformly bounded from above for every finite-index subgroup $\Gamma$ of $\Gamma_g$ (see Section 5.2).

Combining Freitag’s observation with Theorem A, we immediately have the following.

**Corollary C.** For $g \geq 3$, every modular form is a finite product of universally irreducible ones.

Even with Theorem B in our hands, it is not always easy to verify whether a subvariety $Z$ of $\mathcal{A}_g(\Gamma)$ is absolutely irreducible. Thus, in our last main result, we provide a criterion for connectedness and irreducibility of the preimage in $\mathbb{H}_g$ of $Z$, which sometimes turns useful.

In order to state it, we recall that the Jacobian locus (resp. the hyperelliptic locus) in $\mathbb{H}_g$ is the locally closed locus of period matrices corresponding to Jacobians of smooth curves (resp. of smooth hyperelliptic curves) of genus $g$. The Jacobian and the hyperelliptic loci in $\mathcal{A}_g(\Gamma)$ are the image of the Jacobian and hyperelliptic loci of $\mathbb{H}_g$ via the projection $\mathbb{H}_g \to \mathcal{A}_g$.

**Proposition D.** Let $Z \subset \mathcal{A}_g(\Gamma)$ be an irreducible subvariety and let $\tilde{Z}$ be its preimage in $\mathbb{H}_g$.

(i) If a Zariski-open subset $Y$ of the Jacobian locus is contained in $Z$, then $\tilde{Z}$ is connected, with finitely many irreducible components. Moreover, if $Y$ is not entirely contained inside the singular locus of $Z$, then $\tilde{Z}$ is irreducible.

(ii) Assume $\Gamma \subseteq \Gamma_g(2)$. If a Zariski-open subset $Y$ of the hyperelliptic locus is contained in $Z$, then $\tilde{Z}$ is connected, with finitely many irreducible components. Moreover, if $Y$ is not entirely contained inside the singular locus of $Z$, then $\tilde{Z}$ is irreducible.
The above result (Proposition 5.11 in the body of the article) is essentially a consequence of the topological considerations recalled in Section 2 together with the surjectivity of the symplectic representations of the mapping class group (Fact 1.1) and of the hyperelliptic mapping class group at level 2 (Proposition 5.8).

As applications of the above technique, we show that even theta-nulls are universally irreducible for \( g \geq 3 \) (Corollary 5.6), that each component of the moduli space of intermediate Jacobians of cubic threefolds in \( \mathcal{A}_5(2) \) is universally irreducible (Corollary 7.2), and that the Schottky form is universally irreducible for \( g \geq 4 \) (Corollary 7.4). Finally, we also describe how different the situation with even theta-nulls is in genus two (Proposition 7.1).

1.4. **Structure of the paper.** Besides the present introduction, the paper has seven more sections and one appendix.

In Section 2 we collect some standard facts about the topology of complex-analytic spaces (Sections 2.1-2.2) and we prove some criteria for connectedness and irreducibility of liftings (Sections 2.3-2.4).

In Section 3 we prove the version of the Lefschetz hyperplane theorem mentioned in Section 1.3.

In Section 4 we distill some topological properties needed in a more general setting to have the existence of a factorization into absolutely irreducible divisors and to rephrase Theorem B and Theorem B’.

In Section 5 we recall some topological properties of \( \mathcal{A}_g \) or of its finite covers \( \mathcal{A}_g(\Gamma) \). Using the tools developed in Section 2, we prove the connectedness claim and part (i) of Theorem A and Proposition D. Moreover, we discuss universal irreducibility of subvarieties of \( \mathcal{A}_g(\Gamma) \) that contain the Jacobian or the hyperelliptic locus. In particular we analyze the case of even theta constants.

In Section 6 we prove Theorem 6.5 (which is a simultaneous formulation of Theorem B and Theorem B’), using the results contained in Section 2 and Section 8, and in Appendix I. In particular, we review the construction of a rational translate of a subvariety of \( \mathcal{A}_g(\Gamma) \).

In Section 7 we discuss three examples: the zero locus of even theta constants in genus 2, the locus of intermediate Jacobians of cubic threefolds (in genus 5) and the Schottky form.

In Section 8 we prove an arithmeticity criterion for subgroups of \( \Gamma_g \) needed in the proof of Theorem 6.5.

In Appendix I we prove some easy facts about the symplectic monodromy at infinity of subvarieties of the moduli space of curves.

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2. Covering spaces, connectedness and irreducibility

In the following section we collect sufficient conditions that ensure that, via a covering map of a smooth variety $X$, the preimage of an irreducible subvariety $Z \subset X$ is connected or irreducible.

For connectedness we have to estimate the fundamental group of the subvariety $Z$ and, in particular, the image of $\pi_1(Z) \to \pi_1(X)$. For irreducibility we have to deal with the fundamental group of the smooth locus of $Z$, which can be more subtle if $Z$ is not normal and in particular is not locally irreducible.

We introduce a simple technique that allows us to gain some control on such fundamental groups: it consists in finding an irreducible, locally irreducible subvariety $Y \subset Z$ for which we have better understanding of the image of $\pi_1(Y) \to \pi_1(X)$.

We state the results we are interested in for complex-analytic spaces. We then explain the needed modifications in the case of orbispaces.

2.1. Complex varieties and links. Here we collect some classical and basic facts about the topology of complex analytic spaces. Unless differently specified, we work with the classical topology.

Since part of the results we will recall involve the topology of neighbourhoods of (possibly singular) subvarieties of (possibly singular) analytic spaces, we consider stratifications whose locally closed strata are locally as simple as possible.

**Definition 2.1** (Locally trivial analytic subspaces). Let $X$ be a reduced analytic space. An analytic subspace $Y \subseteq X$ is **locally trivial** if there exists a neighbourhood $U_{X/Y}$ of $Y$ and a projection $U_{X/Y} \to Y$ such that

- the fiber $(U_{X/Y})_y$ over $y \in Y$ is the cone over the link $(L_{Y/X})_y := \partial U_y$ of $Y$ inside $X$ at the point $y$
- $U_{X/Y} \to Y$ and $L_{X/Y} := \bigcup_{y \in Y} (L_{X/Y})_y \to Y$ are topologically locally trivial fiber bundles.

Let $X$ be a reduced analytic space, and let $\{X_i\}_{i \in I}$ be a stratification of $X$, namely $I$ is a partially ordered set, $\bigcup_i X_i = X$ and $\overline{X}_i = \bigcup_{j \leq i} X_j$, where each $\overline{X}_i$ is an analytic subspace of $X$ and $X_i$ is Zariski open inside $\overline{X}_i$.

**Definition 2.2** (Good stratifications). A stratification $\{X_i\}_{i \in I}$ of $X$ is good if

- each locally closed stratum $X_i$ is connected and non-singular
• $X$ admits a locally finite triangulation such that every open simplex is contained in a unique $X_i$.
• every $X_i$ is locally trivial inside $X$.

In the setting of the above Definition 2.2, we call $U_i := U_{X_i/X}$ the tubular neighbourhood of $X_i$ and $L_i := L_{X_i/X}$ the link of $X_i$ inside $X$, and we denote by $\hat{U}_i$ the complement of $X_i$ inside $U_i$, which is a locally trivial $(0,1]$-bundle over $L_i$. Note that the tubular neighbourhood and the link of $X_i$ inside $X$ admit the natural stratifications $U_{X_i/X} = \bigsqcup_{j \geq i} (U_{X_i/X} \cap X_j)$ and $L_{X_i/X} = \bigsqcup_{j \geq i} (L_{X_i/X} \cap X_j)$.

It is well-known that complex-analytic spaces admit a Whitney stratification and that Whitney stratifications are good in the above sense. Also the above-mentioned stratifications of links and of tubular neighbourhoods of locally closed strata of a Whitney stratification are themselves Whitney stratifications. We refer to [GM88, Part I, section 1.2] for the definition of Whitney stratifications and for a list of its main properties, and to [GM88, Part I, section 1.4] for further references on the above-mentioned result.

Another consequence of triangulability of complex-analytic spaces is that they are locally contractible, and so for such spaces connectedness is equivalent to path-connectedness.

It can be shown that, given a collection $\{Y_\alpha\}$ of reduced analytic subspaces of $X$, every good stratification of $X$ can be refined to a good stratification which is compatible with $\{Y_\alpha\}$ in the following sense, cf. [C89].

**Definition 2.3 (Compatibility).** Let $\{Y_\alpha\}$ be a collection of reduced analytic subspaces of $X$. A stratification of $X$ is compatible with $\{Y_\alpha\}$ if every finite intersection $Y_{\alpha_1} \cap \cdots \cap Y_{\alpha_k}$ is a union of strata.

2.2. **Irreducibility and fundamental group.** Here we collect some remarks about fundamental groups and local irreducibility of analytic varieties. We underline that “locally irreducible” is to be understood with respect to the classical topology, and that such condition is equivalent to being “unibranch” (i.e. locally irreducible with respect to the étale topology).

The results collected in the following lemma are classical.

**Lemma 2.4 (On irreducibility and local irreducibility).** Let $X$ be a reduced analytic space.

(i) $X$ is irreducible if and only if its smooth locus $X_{\text{sm}}$ is connected.
(ii) If $X$ is locally irreducible and connected, then it is irreducible.
(iii) If $X$ is normal, then it is locally irreducible.
(iv) If $X$ is locally irreducible along an irreducible locally trivial subspace $W \subset X$, then the bundles $L_{W/X} \rightarrow W$ and $\hat{U}_{W/X}$ have connected fibers.
(v) If $X$ is normal and irreducible (resp. locally irreducible and connected), then so is its universal cover.
Consider now the class in $\pi_1 \times$ that joins two points a path entirely contained inside $L_d$. (iv) Since $X$ is locally irreducible along $W$, the subset $U_{W/X}$ is irreducible and so is $\hat{U}_{W/X}$. By (i) the smooth locus of $\hat{U}_{W/X}$ is dense and connected. Since $\hat{U}_{W/X} \to W$ is a locally trivial bundle, the smooth locus of every fiber is dense and connected, and so every fiber is connected. The same conclusion holds for $L_{W/X}$, since $\hat{U}_{W/X}$ is homeomorphic to an $\mathbb{R}$-bundle over $L_{W/X}$.

(v) By hypothesis $X$ is connected and so its universal cover $\tilde{X}$ is connected. Normality and local irreducibility are local properties, so they are inherited by $\tilde{X}$. The conclusion follows by (ii) and (iii). □

We believe that the following should be well-known. Since we have been unable to find a proper reference, we include a proof for completeness.

**Lemma 2.5** (Fundamental groups of locally irreducible varieties). Let $X$ be a connected, locally irreducible analytic variety of positive dimension.

(i) If $W \subseteq X$ is a closed analytic subspace, then $\tilde{X} = X \setminus W$ is connected and the inclusion $\tilde{X} \rightarrow X$ induces a surjection $\pi_1(\tilde{X}) \twoheadrightarrow \pi_1(X)$.

(ii) $X_{sm}$ is connected and $\pi_1(X_{sm}) \rightarrow \pi_1(X)$ is surjective.

**Proof.** Note first that $X$ is irreducible by Lemma 2.4(ii).

(i) Since $X$ is irreducible, $\tilde{X}$ is irreducible too and so it is connected. Consider a good stratification of $X$ such that $W$ is a union of strata (see Section 2.1) and let $W_d$ be the union of strata inside $W$ of dimension at most $d$. Since $W = W_N$ for $N$ large and $W_{-1} = \emptyset$, it is enough to show that $\pi_1(X \setminus W_d) \rightarrow \pi_1(X \setminus W_{d-1})$ is surjective for all $d \geq 0$. Fix $d \geq 0$ and let $X' = X \setminus W_{d-1}$ and $W' = W_d \setminus W_{d-1}$, and let $\iota : X' \setminus W' \hookrightarrow X'$ be the inclusion. We need to show that $\iota_* : \pi_1(X' \setminus W') \rightarrow \pi_1(X')$ (the basepoint being picked anywhere in $X' \setminus W'$). This is an immediate consequence of the fact that the fibers of $L_{W'/X'} \rightarrow W'$ are connected. For sake of completeness, here we give a thorough proof.

Note that $W'$ is a closed subset of $X'$ and a disjoint union of locally trivial smooth subvarieties of dimension $d$. We recall that $U_{W'/X'} \rightarrow W'$ and $\hat{U}_{W'/X'} \rightarrow W'$ are locally trivial fibrations, though the topology of the fiber might depend on the connected component of $W'$. Moreover, by Lemma 2.4(iv), the fibers of $\hat{U}_{W'/X'} \rightarrow W'$ are connected. Hence, a path in $U_{W'/X'}$ that joins two points $x_0, x_1$ of $L_{W'/X'}$ is homotopic (inside $U_{W'/X'}$) to a path entirely contained inside $L_{W'/X'}$ that goes from $x_0$ to $x_1$.

Consider now the class in $\pi_1(X')$ of a loop $\alpha$: we want to show that $[\alpha]$ is in the image of $\iota_*$. By the above observation, any portion of $\alpha$ that is contained inside $U_{W'/X'}$ can be replaced by a path with the same endpoints and homotopic to it, with support entirely contained in $L_{W'/X'}$. As a result,
we produce another loop $\alpha'$ in $X'$ homotopic to $\alpha$, whose support avoids $W'$, and so $[\alpha] = \iota_*[\alpha']$.

(ii) follows from (i) by taking $W = X_{\text{sing}}$. \hfill \Box

In the following lemma we estimate the fundamental group of the smooth locus of a variety $Z$ in terms of the fundamental group of a subvariety $Y \subset Z$, thus employing the first technique mentioned at the beginning of the section.

**Lemma 2.6** (Estimating the fundamental group of $Z_{\text{sm}}$). Suppose that $Z$ is an irreducible analytic variety and $Y \subset Z$ an irreducible, locally irreducible subvariety with $\dim(Y) < \dim(Z)$. Let $H$ be the image of $\pi_1(Y) \to \pi_1(Z)$ and $K$ be the image of $\pi_1(Z_{\text{sm}}) \to \pi_1(Z)$. Then $H \cap K$ has finite index in $H$.

Note that, if $Z$ is not locally irreducible, the image of $\pi_1(Z_{\text{sm}})$ need not be of finite index inside $\pi_1(Z)$.

**Proof of Lemma 2.6.** Note preliminarily that $Z_{\text{sm}}$ is smooth and connected by Lemma 2.4(i), and so $Z_{\text{sm}} \setminus \overline{Y}$ is connected by Lemma 2.5(i). A Whitney stratification of $Y$ inside $Z$ has one Zariski-open stratum $\overline{Y}$. Such $\overline{Y}$ is then smooth and locally trivial inside $Z$, and so it is connected by Lemma 2.4(i).

Observe that $\pi_1(\overline{Y})$ maps surjectively onto $\pi_1(Y)$ by Lemma 2.5(i).

Let $k$ be the number of branches of $Z$ at each point of $\overline{Y}$, that is the number of connected components of $\overline{U} = \overline{U}_{\overline{Y}/Z}$. Pick a connected component $\overline{U}'$ of $\overline{U}$. The fiber $F'$ of $\overline{U}' \to \overline{Y}$ has $c \leq k$ connected components and the exact sequence $\pi_1(\overline{U}') \to \pi_1(\overline{Y}) \to \pi_0(F') \to \{\ast\}$ shows that the image of $\pi_1(\overline{U}') \to \pi_1(\overline{Y})$ has index $c$.

Let $\overline{U}'_{\text{sm}} := \overline{U}' \cap Z_{\text{sm}}$. The fibers of $\overline{U}'_{\text{sm}} \to \overline{Y}$ are dense in those of $\overline{U}' \to \overline{Y}$ and so they have $c$ connected components too. As a consequence, the images of $\pi_1(\overline{U}'_{\text{sm}}) \to \pi_1(\overline{Y})$ and of $\pi_1(\overline{U}') \to \pi_1(\overline{Y})$ coincide.

The conclusion follows from the commutativity of the diagram

$$
\begin{array}{ccc}
\pi_1(\overline{U}'_{\text{sm}}) & \longrightarrow & \pi_1(Z_{\text{sm}}) \\
\pi_1(\overline{Y}) & \longrightarrow & \pi_1(\overline{Y}) \\
\pi_1(Z_{\text{sm}}) & \longrightarrow & \pi_1(Z)
\end{array}
$$

since the image of $\pi_1(\overline{U}'_{\text{sm}}) \to \pi_1(Z)$ is contained inside $H \cap K$ and has index at most $c$ in $H$.

\hfill \Box

Here is a very useful consequence of the above lemma.

**Corollary 2.7** (Image of fundamental groups of smooth loci). Let $Y$ and $Z$ be irreducible algebraic varieties and let $f : Y \to Z$ be morphism which is generically finite onto its image. Let $H$ be the image of $\pi_1(Y_{\text{sm}}) \to \pi_1(Z)$ and $K$ the image of $\pi_1(Z_{\text{sm}}) \to \pi_1(Z)$. Then

(i) $K \cap H$ is a finite-index subgroup of $H$. 

(ii) if $Y$ is unirational, then $K \cap H$ is a torsion-free subgroup of $H$.
(ii) If \( \dim(Y) = \dim(Z) \), then \( H \) is a finite-index subgroup of \( K \).

Proof. Let \( d \) be the cardinality of the general fiber of \( f : Y \to f(Y) \) and note that \( f(Y) \) is irreducible. Let \( W \subset f(Y) \) be the union of \( f(Y)_{\text{sing}} \), of \( f(Y_{\text{sing}}) \) and of the locus of \( z \in f(Y) \) such that \( f^{-1}(z) \) does not consist of \( d \) distinct points. Up to taking a larger \( W \), we can assume that \( f(Y) \setminus W \) is locally trivial.

Since \( Y, Z, f(Y) \) are irreducible, their smooth loci are connected, and so the smooth \( \bar{Y} = Y \setminus f^{-1}(W) \) and \( f(\bar{Y}) = f(Y) \setminus W \) are connected. Moreover, the restriction \( \bar{f} : \bar{Y} \to f(\bar{Y}) \) of \( f \) is a topological cover of degree \( d \).

Observe that the diagram

\[
\begin{array}{ccc}
\pi_1(Y) & \xrightarrow{i^*} & \pi_1(Y_{\text{sm}}) \\
\downarrow{j_*} & & \downarrow{\pi_1(f(Y'))} \\
\pi_1(f(Y')) & \xrightarrow{\pi_1(Y)} & \pi_1(Z)
\end{array}
\]

is commutative and let \( \bar{H} \) be the image of \( \pi_1(f(\bar{Y})) \to \pi_1(Z) \). Since \( H \) coincides with the image of \( \pi_1(\bar{Y}) \to \pi_1(Z) \), the subgroup \( H \) is contained inside \( \bar{H} \) with \( [\bar{H} : H] \leq d \).

(i) Suppose first that \( \dim(Y) < \dim(Z) \) and let \( k \) be the number of connected components of the fibers of \( U_{f(\bar{Y})/Z} \to f(\bar{Y}) \). By Lemma 2.6 applied to \( f(\bar{Y}) \subset Z \), the subgroup \( K \cap \bar{H} \) has finite index in \( \bar{H} \); in particular, the proof of Lemma 2.6 shows that \( [\bar{H} : K \cap \bar{H}] \leq k \). It follows that \( [H : K \cap H] \leq [H : K] \cdot [\bar{H} : K \cap \bar{H}] \leq dk \).

(ii) Suppose now that \( \dim(Y) = \dim(Z) \). Since \( Z \) is irreducible, \( f(Y) \) is a Zariski-dense open subset of \( Z \) and so \( \bar{Y} := f^{-1}(Z_{\text{sm}}) \) is a Zariski-open subset of \( Y \). Thus \( \pi_1(\bar{Y}) \to \pi_1(Z) \) factors through the surjective map \( \pi_1(f(\bar{Y})) \to \pi_1(Z_{\text{sm}}) \). It follows that \( \bar{H} = K \) and so \( H \subset K \). Hence, \( [K : H] = [\bar{H} : H] \leq d \).

\[\square\]

2.3. Liftings and connectedness. Let \( X, Z \) be connected and locally arc-connected topological spaces with universal covers \( p : \bar{X} \to X \) and \( \bar{Z} \to Z \). Given a map \( f : Z \to X \), consider the following diagram

where the rectangle is Cartesian and \( \bar{Z}_i \) is a connected component of \( Z \times_X \bar{X} \). We denote by \( f_* : \pi_1(Z) \to \pi_1(X) \) the homomorphism induced by \( f \).
Lemma 2.8 (Liftings). Let \( f : Z \to X \) be a map of connected topological spaces.

(i) The group \( \pi_1(X) \) acts transitively on the set of liftings \( \hat{f} : \hat{Z} \to \hat{X} \).

(ii) The \( \pi_1(X) \)-set of connected components of the fiber product \( Z \times_X \hat{X} \) is isomorphic to \( \pi_1(X)/f_*\pi_1(Z) \). For each component \( \hat{Z}_i \) of \( Z \times_X \hat{X} \), its fundamental group satisfies \( \pi_1(\hat{Z}_i) = \ker(f_*) \) and there exists a lift \( \hat{f}_i \) as in (i) that factors through the cover \( \hat{Z} \to \hat{Z}_i \).

(iii) \( Z \times_X \hat{X} \) is connected if and only if \( f_* : \pi_1(Z) \to \pi_1(X) \) is surjective.

The above statement follows from standard arguments in the theory of covering spaces. We include a proof for convenience.

Proof of Lemma 2.8. Fix a point \( z \in Z \) and choose \( \hat{z} \in \hat{Z} \) a lift of \( z \). Let also \( x = f(z) \in X \) and choose \( \hat{x} \in \hat{X} \) a lift of \( x \).

(i) A lifting \( \hat{f} \) is uniquely determined by the choice of \( \hat{f}(\hat{z}) \in \pi^{-1}(x) \). The conclusion follows since the set \( \pi^{-1}(x) \) is acted on simply transitive by \( \pi_1(X, x) \).

(ii) The group \( \pi_1(X, x) \) acts by simply and transitively permuting the elements in \( \pi^{-1}(x) \), and so \( \pi_1(X, x) \cdot \hat{x} = \pi^{-1}(x) \). It can be easily seen that the surjective map \( \pi_1(X, x) \cdot (z, \hat{x}) = \pi^{-1}(z) \to \pi_0(Z \times_X \hat{X}) \) is a map of \( \pi_1(X, x) \)-sets. Moreover, two elements \( (z, \hat{x}), (z, \hat{x}') \) in \( \pi^{-1}(z) \) belong to the same connected component of \( Z \times_X \hat{X} \) if and only if there exists a path \( \alpha \in \pi_1(Z, z) \) such that \( f \circ \alpha \) lifts to a path in \( \hat{X} \) that joins \( \hat{x} \) and \( \hat{x}' \), namely \( f_*(\alpha) \cdot \hat{x} = \hat{x}' \). Hence, \( \pi_0(Z \times_X \hat{X}) \) can be identified to \( \pi_1(X, x)/f_*\pi_1(Z, z) \).

By the universal property, any lifting \( \hat{f} \) factors through \( Z \times_X \hat{X} \) and covers a connected component of \( Z \times_X \hat{X} \). If the component \( \hat{Z}_i \) contains the point \( \hat{z}_i := (z, \hat{x}) \), then the lift that satisfies \( \hat{f}(\hat{z}_i) = \hat{x} \) induces a cover \( \hat{Z} \to \hat{Z}_i \).

Pick a connected component \( \hat{Z}_i \) of \( Z \times_X \hat{X} \). It covers \( Z \), and so \( \pi_1(\hat{Z}_i, \hat{z}_i) \hookrightarrow \pi_1(Z, z) \) is injective. Moreover, the composition \( \pi_1(\hat{Z}_i, \hat{z}_i) \to \pi_1(Z, z) \to \pi_1(X, x) \) vanishes, as it factors through \( \pi_1(\hat{X}, \hat{x}) = \{e\} \), and so \( \pi_1(\hat{Z}_i, \hat{z}_i) \) injectively maps into \( \ker(f_*) \). It also maps surjectively, since all elements of \( \ker(f_*) \) lift to closed loops in \( \hat{Z}_i \).

(iii) follows from (ii). \( \square \)

An argument analogous to Lemma 2.8(ii) also shows the following.

Lemma 2.9 (Liftings via finite covers). Let \( p' : X' \to X \) be a covering space and \( f : Z \to X \) a map of connected topological spaces.

(i) If \( p' \) is a finite cover, then the number of connected components of \( Z \times_X X' \) is \( |\pi_1(X) : p'_*\pi_1(X')|/|f_*\pi_1(Z) : p'_*\pi_1(X') \cap f_*\pi_1(Z)| \).

(ii) \( Z \times_X X' \) is connected if and only if the induced map \( \pi_1(Z) \to \pi_1(X)/p'_*\pi_1(X') \) is surjective.

Out of the above lemmas, we can already draw a first consequence, which is an example of the strategy outlined at the beginning of the section.
Corollary 2.10 (Connected lifting of analytic subspaces). Let $X' \to X$ be a cover of connected complex-analytic spaces, $Z \subset X$ a connected analytic subspace and $Z'$ its preimage inside $X'$. Suppose that there exists an analytic subspace $Y \subseteq Z$ such that $\pi_1(Y) \to \pi_1(X)$. Then $Z'$ is connected.

Proof. In view of Lemma 2.8, it is enough to show that $\pi_1(Z) \to \pi_1(X)$ is surjective. This is immediate because of the factorization $\pi_1(Y) \to \pi_1(Z) \to \pi_1(X)$. □

2.4. Liftings and irreducibility. Let $X, Z$ be as in Section 2.3 and assume that both $X$ and $Z$ are connected analytic spaces. We begin by stating our fundamental irreducibility criterion, which relies on Lemma 2.8 and Lemma 2.9.

Corollary 2.11 (Irreducibility criterion of $Z \times_X \tilde{X}$). Let $f : Z \to X$ be a map of complex-analytic varieties, with $Z$ irreducible.

(i) The analytic space $Z \times_X \tilde{X}$ is irreducible if and only if $f_* : \pi_1(Z_{\text{sm}}) \to \pi_1(X)$ is surjective.

(ii) If $p' : X' \to X$ is a covering space, then the irreducible components of $Z \times_X X'$ correspond bijectively to the cokernel of the map $\pi_1(Z_{\text{sm}}) \to \pi_1(X)/p'_*\pi_1(X')$ induced by $f_*$. 

Proof. (i) It is enough to observe that $Z_{\text{sm}} \times_X \tilde{X}$ is smooth and Zariski-dense in $Z \times_X \tilde{X}$ and that $Z_{\text{sm}} \times_X \tilde{X}$ is irreducible if and only if it is connected. The conclusion follows from Lemma 2.8(iii).

(ii) is analogous to (i), by applying Lemma 2.9 instead of Lemma 2.8(iii). □

The following proposition is an incarnation of the strategy mentioned at the beginning of the section, namely to prove the irreducibility of $Z \times_X \tilde{X}$ by using certain subvarieties of $Z$ with large fundamental group.

Proposition 2.12 (Irreducibility of $Z \times_X \tilde{X}$ via subvarieties of $Z$). Let $Z$ be an irreducible complex-analytic variety, and let $f : Z \to X$ be a morphism.

(i) Suppose that $Z$ is locally irreducible and $f_* : \pi_1(Z) \to \pi_1(X)$ is surjective. Then $Z \times_X \tilde{X}$ is irreducible.

Let now $Y \subset Z$ be an irreducible and locally irreducible subvariety.

(ii) If $\pi_1(Y) \to \pi_1(X)$ is surjective, then $Z \times_X \tilde{X}$ has finitely many irreducible components.

(iii) If the image of $\pi_1(Y) \to \pi_1(X)$ has finite index, and $Z \times_X X'$ is irreducible for all finite étale covers $p' : X' \to X$, then $Z \times_X \tilde{X}$ is irreducible.

Proof. By Lemma 2.4(i), the smooth locus $Z_{\text{sm}}$ is connected.

(i) By Lemma 2.5(ii), the map $\pi_1(Z_{\text{sm}}) \to \pi_1(Z)$ is surjective. Thus the composition $\pi_1(Z_{\text{sm}}) \to \pi_1(Z) \to \pi_1(X)$ is surjective too. We conclude by Corollary 2.11(i).
(ii) Since \( \pi_1(Y) \to \pi_1(X) \) is surjective, so is \( \pi_1(Z) \to \pi_1(X) \). Let \( k \) be the number of branches of \( Z \) at the general point of \( Y \). By Lemma 2.6, the image of \( \pi_1(Z_{\text{sm}}) \to \pi_1(Z) \) has index at most \( k \), and so the same holds for the image of \( \pi_1(Z_{\text{sm}}) \to \pi_1(X) \). The conclusion then follows from Corollary 2.11(ii) applied to the universal cover \( \tilde{X} \to X \).

(iii) Since \( Z \) has finitely many branches at the general point of \( Y \), the image of \( \pi_1(Z_{\text{sm}}) \to \pi_1(X) \) has finite index by Lemma 2.6: let \( d \) be such index. Let \( p': X' \to X \) be the étale cover of degree \( d \) such that \( p'_*\pi_1(X') = f_*\pi_1(Z_{\text{sm}}) \).

By Corollary 2.11(ii) the fiber product \( Z \times_X X' \) has \( d \) irreducible components, and our hypothesis implies that \( d = 1 \). It follows that \( \pi_1(Z_{\text{sm}}) \to \pi_1(X) \) is surjective and we conclude by Corollary 2.11(i). □

The following special case is a direct consequence of Proposition 2.12(ii).

**Corollary 2.13** (A criterion of irreducibility of \( Z \times_X \tilde{X} \)). Let \( Z \) be an irreducible complex-analytic variety and \( f : Z \to X \) be a morphism. Suppose that there exists an irreducible and locally irreducible subvariety \( Y \subset Z \) such that \( \pi_1(Y) \to \pi_1(X) \). Then \( Z \times_X \tilde{X} \) has finitely many irreducible components. Moreover, if \( Y \) intersects the smooth locus of \( Z \), then \( Z \times_X \tilde{X} \) is irreducible.

2.5. **The case of orbispaces.** Most of the above results hold in the category of complex-analytic orbispaces, namely objects locally modelled on \( [T/G] \), where \( T \) is a complex-analytic space and \( G \) is a finite group that acts on \( T \) via biholomorphisms. In this case, we must use open orbifold charts instead of open subsets, and we must modify the definition of neighborhoods accordingly. Moreover, the word smooth, singular, unramified cover, fiber bundle, universal cover and fundamental group must all be understood in the orbifold sense. Note in particular that an orbispace is smooth where it is locally modelled as \( [T/G] \) with \( T \) smooth. An analogous interpretation must be reserved to the words normal, irreducible, locally irreducible.

With the above caveat, the results in Sections 2.1-2.2-2.3-2.4 still hold in the setting of complex-analytic orbispaces.

3. **A hyperplane section theorem**

In the following section we present a generalization (Theorem 3.1) of the homotopical Lefschetz hyperplane section theorem (LHT) for \( \pi_0 \) and \( \pi_1 \) to smooth quasi-projective varieties that admit a projective model with small boundary. Such result then extended to complete intersections (Corollary 3.2) and to complete intersections inside orbifolds that are global quotients (Corollary 3.4).

We will often mention the following properties of a variety \( \overline{X} \) and of its algebraic loci \( \partial X \) and \( \overline{D} \).
(I) \( \overline{X} \) is a connected projective variety of dimension \( N \) and \( \partial X \) be a closed subscheme of dimension at most \( N - 1 \) such that \( X = \overline{X} \setminus \partial X \) is smooth and connected.

(II) \( \overline{D} \subset \overline{X} \) is the support of an effective, ample Cartier divisor.

(III) \( \text{codim}(\partial \overline{D}/\overline{D}) \geq h \), where \( \partial D = \overline{D} \cap \partial X \).

We remark that (III) is certainly implied by

(III') \( \text{codim}(\partial X/X) \geq h + 1 \).

The version of LHT we wish to prove is the following. In the typical application that we have in mind we will pick as \( X \) the moduli space \( A_g(n) \) for some level \( n \geq 3 \), as \( X \) its Satake compactification and as \( D \) the zero locus of a modular form.

**Theorem 3.1** (Improved LHT for smooth quasi-projective varieties). Let \( \overline{X} \) be a variety and \( \partial X, \overline{D} \) be subschemes of \( \overline{X} \) such that properties (I)-(II)-(III) above hold with \( h = 2 \) or \( h = 3 \). Then \( D = \overline{D} \setminus \partial D \) is connected and the natural map \( \pi_1(D) \to \pi_1(X) \) is an isomorphism if \( h = 3 \) (resp. is surjective, if \( h = 2 \)).

The key ingredient of the proof is the Lefschetz hyperplane type theorem (LHT) that appears at the very beginning of Part II, Section 5.1 of [GM88]. In particular, we will use the two versions of such theorem (see the “furthermore” below the statement of (LHT)) for general hyperplane (LHT-gen) and for compact hyperplane section (LHT-cpt). A similar idea is already in [AB96, Theorem 8.7].

**Proof of Theorem 3.1.** Consider then general very ample divisors \( \overline{D}_1, \ldots, \overline{D}_{N-h} \) in \( \overline{X} \) such that

(a) \( D_i := \overline{D}_i \setminus \partial D_i \) is smooth, where \( \partial D_i = \overline{D}_i \cap \partial X \)

(b) the intersection \( E = D_1 \cap \ldots \cap D_{N-h} \) is transverse, and so \( E \) is a smooth variety of dimension \( c \)

(c) \( S = E \cap D = E \cap \overline{D} \) is compact, of dimension \( h - 1 \)

(d) the singular locus of \( S \) is contained in the singular locus of \( D \).

Note that, in order to establish the existence of \( S \), we use property (I) in (a)-(b)-(d), property (III) in (c).

Suppose first that \( h = 3 \). Properties (II) and (b-c) together with (LHT-cpt) imply that \( S \) is a compact connected surface and \( \pi_1(S) \to \pi_1(E) \) is an isomorphism.

On the other hand, (b-d) and (LHT-gen) imply that \( E \) is connected and

\[ \pi_1(E) \to \pi_1(D_1 \cap \ldots \cap D_{N-h-1}) \to \cdots \to \pi_1(D_1 \cap D_2) \to \pi_1(D_1) \to \pi_1(X) \]

are isomorphisms. Similarly, \( S \) intersects every connected component of \( D \) and

\[ \pi_1(S) \to \pi_1(D \cap D_1 \cap \ldots \cap D_{N-h-1}) \to \cdots \to \pi_1(D \cap D_1) \to \pi_1(D) \]
are isomorphisms. It follows that $D$ is connected and $\pi_1(S) \to \pi_1(D) \to \pi_1(X)$ are isomorphisms.

If $h = 2$, then we can argue analogously as above and consider general very ample divisors $D_1, \ldots, D_{N-2}$ such that $E = D_1 \cap \ldots \cap D_{N-2}$ is a smooth surface and $C = D \cap E$ is a compact curve. A similar repeated application of (LHT-gen) and then of (LHT-cpt) gives

$$
\begin{array}{ccc}
\pi_1(C) & \longrightarrow & \pi_1(D) \\
\downarrow & & \downarrow \\
\pi_1(E) & \cong & \pi_1(X),
\end{array}
$$

from which we conclude that $\pi_1(D) \to \pi_1(X)$ is surjective. □

An analogous statement holds for complete intersections $Z$, by replacing properties (III)-(III$_h$) by

(III$^c$) $Z \subset X$ is the support of a complete intersection of Cartier divisors, whose classes in $X$ are all proportional to the same ample class in $H^2(X; \mathbb{Q})$,

(III$^c_h$) $\text{codim}(\partial Z/Z) \geq h$, where $\partial Z = Z \cap \partial X$.

More precisely, we have the following.

**Corollary 3.2** (Improved LHT for complete intersections). Suppose that $X$, $\partial X$ and $Z$ satisfy (I)-(III$^c$)-(III$^c_h$) above for $h = 2$ or $h = 3$. Then $Z = Z \setminus \partial Z$ is connected and $\pi_1(Z) \to \pi_1(X)$ is an isomorphism if $h = 3$ (resp. is surjective, if $h = 2$).

Proof. By (III$^c$) it is possible to embed $X$ inside some projective space $\mathbb{P}$ in such a way that $Z = X \cap H$ for some linear subspace $H \subset \mathbb{P}$ of codimension $\text{codim}(H/\mathbb{P}) = \text{codim}(Z/X)$. We then proceed analogously to the proof of Theorem 3.1, replacing the role of $\overline{D}$ there by $\overline{Z}$. □

**Remark 3.3.** Assuming that $X$, $\partial X$ and $Z$ satisfy (I)-(III$^c$)-(III$^c_h$) above, it is immediate that for higher values of $h$ we have similar results about the higher homotopy groups. In particular we obtain that $\pi_i(Z) \to \pi_i(X)$ is an isomorphism if $i \leq h - 2$, and $\pi_{h-1}(Z) \to \pi_{h-1}(X)$ is surjective.

**The case of orbispaces.** A version of LHT as in Section 3 can be also phrased for orbifolds that are global quotients. Let $\overline{X}$ be an analytic space and let $\partial X$ and $\overline{Z}$ be subspaces of $\overline{X}$. Suppose moreover that a finite group $G$ acts on $\overline{X}$, preserving $\partial X$ and $\overline{Z}$, and so preserving $Z = \overline{Z} \setminus \partial X$.

**Corollary 3.4** (Improved LHT for c.i. inside global quotients). Suppose that $\overline{X}$, $\partial X$ and $\overline{Z}$ satisfy properties (I)-(III$^c$)-(III$^c_h$) in Section 3 with $h = 2$ or $h = 3$. Then

(i) the locus $[Z/G]$ inside the orbifold $[X/G]$ is connected and the homomorphism $\pi_1([Z/G]) \to \pi_1([X/G])$ of orbifold fundamental groups is an isomorphism if $h = 3$ (resp. is surjective, if $h = 2$);
(ii) the preimage of $[Z/G]$ via an unramified cover of $[X/G]$ is connected.

Proof. Both claims follow from Corollary 3.2. □

4. Absolutely irreducible divisors

Recall that in Section 1.1 we called a modular form “absolutely irreducible” if it cannot be written as a product of two nonconstant modular forms. Such definition motivates the following generalizations.

Definition 4.1 (Absolutely irreducible and universally irreducible subvarieties). A subvariety $Z$ of $X$ is absolutely irreducible if its preimage through every finite étale cover $X' \to X$ is irreducible. Such $Z$ is universally irreducible if its preimage through the universal cover $\tilde{X} \to X$ is irreducible.

Obviously, a universally irreducible subvariety is absolutely irreducible. Below we exhibit a sufficient condition for an absolutely irreducible subvariety to be universally irreducible.

The key observation is the following.

Proposition 4.2 (When $\tilde{Z}$ has finitely many irreducible components). If the preimage $\tilde{Z}$ of $Z \subset X$ inside $\tilde{X}$ has finitely many irreducible components, then there exists a finite étale cover $X' \to X$ such that the preimage $Z'$ of $Z$ inside $X'$ has finitely many universally irreducible components.

Proof. Decompose then $\tilde{Z}$ into the union

$$\tilde{Z} = \bigcup_{i \in I} \tilde{Z}_i$$

of its irreducible components $\tilde{Z}_i$.

The group $\Lambda := \pi_1(X)$ acts on $\tilde{Z}$ via biholomorphisms, and permutes its irreducible components. Hence we get a homomorphism

$$\rho : \Lambda \longrightarrow \mathfrak{S}(I),$$

where $\mathfrak{S}(I)$ is the symmetric group on $I$. Consider the normal subgroup $\Lambda' = \ker(\rho)$ of $\Lambda$. Since $I$ is finite, $\Lambda'$ has finite index in $\Lambda$ and so $X' := \tilde{X}/\Lambda'$ is a finite étale cover of $X$. Let $Z' := \tilde{Z}/\Lambda'$ the subvariety of $X'$ obtained as inverse image of $Z$ through the cover $X' \to X$, and let $Z'_i := \tilde{Z}_i/\Lambda'$. Clearly, $Z'_i$ is universally irreducible and so $Z' = \bigcup_{i \in I} Z'_i$ is the decomposition of $Z'$ into its finitely many universally irreducible components. □

Corollary 4.3 (Promoting absolutely irreducible to universally irreducible). Let $X$ be a connected variety, with universal cover $\tilde{X}$, and let $Z \subset X$ be an absolutely irreducible subvariety whose inverse image $\tilde{Z} \subset \tilde{X}$ has a finite number of irreducible components. Then $Z$ is universally irreducible.

Proof. By Proposition 4.2 there exists a finite étale cover $X' \to X$ such that the preimage $Z'$ of $Z$ inside $X'$ is the union of finitely many universally irreducible subvarieties. Since $Z$ is absolutely irreducible, such $Z'$ is itself irreducible. It follows that $\tilde{Z}$ is irreducible. □
**Remark 4.4.** We observe that in this case we do not require any hypothesis on \( \pi_1(Z) \) and we do not assume that \( Z \) is locally irreducible.

Here is an immediate consequence of the above observations.

**Corollary 4.5 (Absolutely irreducible plus finite index implies universally irreducible).** Let \( Z \) be a subvariety of \( X \) such that the image of \( \pi_1(Z_{\text{sm}}) \rightarrow \pi_1(X) \) has finite index. Then the preimage \( \tilde{Z} \) of \( Z \) inside \( \tilde{X} \) has finitely many irreducible components. In particular, if \( Z \) is an absolutely irreducible subvariety of \( X \), then \( Z \) is universally irreducible.

**Proof.** The preimage \( \tilde{Z} \) of \( Z \) inside \( \tilde{X} \) has finitely many irreducible components by Lemma 2.8(ii). The conclusion follows from Proposition 4.2 and Corollary 4.3. \( \square \)

Now we focus on the case of divisors. In order to ensure that, up to a finite étale cover, a divisor splits into the union of absolutely irreducible divisors, we will consider projective varieties \( X \) with a subscheme \( \partial X \) that satisfy the following property:

\[
\begin{align*}
\text{(IV)} & \quad \begin{cases} 
\text{the codimension of } \partial X \text{ inside } X \text{ is at least two (III'$_1$);} \\
\text{Pic}(X)/\text{tors} = \mathbb{Z} \cdot \alpha \text{ with } \alpha \text{ ample;} \\
\text{the Picard number of } X' \text{ is 1 and the divisibility of } \alpha \text{ in } \text{Pic}(X')/\text{tors} \text{ is uniformly bounded from above, for every finite étale cover } X' \rightarrow X,
\end{cases}
\end{align*}
\]

where \( X = \overline{X} \setminus \partial X \). The following is essentially due to Freitag (see [F91, Theorem 4.7] for the case of a finite étale cover \( X \rightarrow \mathbb{A}_d \)).

**Proposition 4.6 (Existence of absolutely irreducible divisors).** Let \( \overline{X} \) be a projective variety with a subscheme \( \partial X \) that satisfy properties (I) and (IV). Then, for every effective Cartier divisor \( D \subset X \), there exists a finite étale cover \( X' \rightarrow X \) such that the preimage \( D' \subset X' \) of \( D \subset X \) is the union of finitely many absolutely irreducible divisors.

**Proof.** By property (IV) there an integer \( d > 0 \) such that \( [D] = d \cdot \alpha \). Let now \( X' \rightarrow X \) be any finite étale cover and let \( \overline{X}' \) be the normalization of \( \overline{X} \) in the function field of \( X' \). Since \( \overline{X}' \) is projective and \( \partial X' \) has codimension at least two in \( \overline{X}' \), property (IV) ensures that \( \text{Pic}(\overline{X}')/\text{tors} \subseteq \mathbb{Z} \cdot (\alpha/d_0) \) for some integer \( d_0 \geq 1 \). Thus, each effective Cartier divisor in \( \overline{X}' \) must be a positive multiple of \( \alpha/d_0 \). In particular, the divisor \( D' \) obtained by pulling back \( D \) via \( X' \rightarrow X \) has class \( d \cdot \alpha \), and so it can have at most \( d \cdot d_0 \) irreducible components. It is then enough to pick \( X' \) to be a finite étale cover of \( X \) on which the number of irreducible components of \( D' \) is maximal. \( \square \)

**The case of orbispaces.** The definition of absolutely irreducible and universally irreducible divisors can be verbatim understood in the orbispace setting. Thus, if \( (\overline{X}, \partial X) \) satisfy (I) and (IV), then the conclusion of Proposition 4.6 holds for every effective Cartier divisor in \( \overline{X}/G \). The assertion is
an immediate consequence of Proposition 4.6 and Corollary 4.3. As further examples, Proposition 4.2, Corollary 4.3 and Corollary 4.5 verbatim hold in the setting of orbispaces.

5. Subvarieties of \( A_g(\Gamma) \)

5.1. Siegel upper half-space, congruence subgroups and multipliers. We denote by \( \mathbb{H}_g = \{ \tau \in \text{Mat}_{g \times g}(\mathbb{C}) \mid \tau = \tau^t, \text{Im} \tau > 0 \} \) the Siegel upper half-space of symmetric matrices with positive definitive imaginary part. It is a homogeneous space for the action of \( \text{Sp}_{2g}(\mathbb{R}) \), where an element

\[
\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_{2g}(\mathbb{R})
\]

acts via

\[
\gamma \cdot \tau := (A\tau + B)(C\tau + D)^{-1}.
\]

Such action can be lifted to \( \tilde{\mathcal{L}} := \mathbb{H}_g \times \mathbb{C} \) as

\[
\gamma \cdot (\tau, u) := \left((A\tau + B)(C\tau + D)^{-1}, \det(C\tau + D) \cdot u\right),
\]

so that the projection \( \tilde{\mathcal{L}} \to \mathbb{H}_g \) onto the second factor descends to a holomorphic line bundle on \( \mathcal{L} \to A_g(\Gamma) \) for every subgroup \( \Gamma \) of \( \Gamma_g \).

**Notation.** Given a subvariety \( \mathcal{Y} \subset A_g \), we will write \( \tilde{\mathcal{Y}} \) to denote its preimage inside \( \mathbb{H}_g \), \( \mathcal{Y}(\Gamma) \) to denote its preimage in \( A_g(\Gamma) \) and briefly \( \mathcal{Y}(n) \) for its preimage in \( A_g(n) \).

We assume \( g \geq 2 \) and \( k \) half-integral. Consider now a subgroup \( \Gamma \) of finite index of \( \Gamma_g \).

**Definition 5.1.** A **multiplier of weight** \( k \) is a map \( \chi : \Gamma \to \mathbb{C}^* \) such that

\[
G(\gamma, \tau) := \chi(\gamma) \det(C\tau + D)^k
\]

satisfies the cocycle condition

\[
G(\gamma' \gamma, \tau) = G(\gamma', \gamma \cdot \tau) G(\gamma, \tau)
\]

for all \( \tau \in \mathbb{H}_g \) and all \( \gamma', \gamma \in \Gamma \). A function \( F : \mathbb{H}_g \to \mathbb{C} \) is called a **modular form of weight** \( k \) and **multiplier** \( \chi \) with respect to \( \Gamma \) if

\[
F(\gamma \circ \tau) = \chi(\gamma) \det(C\tau + D)^k F(\tau), \quad \forall \gamma \in \Gamma, \forall \tau \in \mathbb{H}_g.
\]

Note that, for every \( \gamma \) the function \( \tau \mapsto \det(C\tau + D) \) has a holomorphic square root, which we denote by \( \sqrt{\det(C\tau + D)} \). When \( k \) is half-integral, the expression \( \det(C\tau + D)^k \) must be interpreted as \( \sqrt{\det(C\tau + D)^{2k}} \). When \( k \) is integral, no square root is needed and the multiplier can be chosen to be just a character. We refer to [F91, Section I.3] for more details on modular forms and multipliers.

Denote by \( A(\Gamma) \) the \( \mathbb{C} \)-algebra of all modular forms (possibly with multiplier) with respect to \( \Gamma \), which is finitely generated (see [F83, Theorem 6.11]).
5.2. Some topological properties of $A_g$. We recall that $\Gamma_g$ is finitely generated (see [M65, Hilfssatz 2.1]) and so $\Gamma$ is. This in particular implies that the cohomology of $A_g(\Gamma)$ is finitely generated.

The Satake compactification $\overline{A}_g(\Gamma) := \text{Proj}(\mathcal{A}(\Gamma))$ of the coarse space of $A_g(\Gamma)$ is a normal projective variety (which may be singular for $g \geq 2$) and so the coarse space of $A_g(\Gamma)$ can be regarded as a normal quasi-projective variety. Moreover its boundary $\partial A_g(\Gamma) := \overline{A}_g(\Gamma) \setminus A_g(\Gamma)$ has pure codimension $g$. The boundary of $A_g$ has a natural stratification, whose locally closed strata can be identified to $A_k$ for $k = 0, \ldots, g - 1$ (see [FC91, Chapter V.2], for instance).

Now, assume $g \geq 2$ so that $\Gamma$ contains a principal congruence subgroup $\Gamma_g(n)$ by [BMS67], and so $A_g$ is covered by some $A_g(n)$. Let us fix such an $n \geq 3$.

We recall the following:

(a) $A_g(n)$ is smooth and connected (since $\mathcal{H}_g$ is smooth and connected and $\Gamma_g(n)$ acts freely).

(b) Since $\overline{A}_g(n)$ is normal and its boundary has codimension at least two, the (algebraic or analytic) Picard groups of $\overline{A}_g(n)$ and $A_g(n)$ coincide.

(c) The algebraic and analytic Picard groups of $\overline{A}_g(n)$ coincide (use [S56]), and the same holds for $A_g(n)$.

(d) $H^1(A_g(n); \mathbb{Q}) = 0$ (see [K67]) and so $H_1(A_g(n); \mathbb{Z})$ is a finite group; $H^2(A_g(n); \mathbb{Q})$ has dimension 1 for $g \geq 3$ (see [B74] and [B80]) and it is generated by the ample class $\lambda = c_1(L(n))$.

(e) For $g \geq 3$ the natural map $\text{Pic}(A_g(n)) \to H^2(A_g(n); \mathbb{Z})$ is an isomorphism (as in [F77]).

(f) Each effective divisor in $A_g(n)$ is the zero set of a modular form of half-integral weight $a/2$, possibly with a multiplier (cf. [F77], [D78] and [F91]). Its class in $H^2(A_g(n); \mathbb{Z})/\text{tors}$ is $a\lambda/2$.

(f') $H^2(A_g(n); \mathbb{Z})/\text{tors}$ is generated by $\lambda$ or by $\lambda/2$.

(g) Every effective divisor in $\overline{A}_g(n)$ is the closure of a divisor in $A_g(n)$ and it is ample. Moreover its boundary has codimension $g$ or $g - 1$ inside it.

Note that (f') is a consequence of (f) and that (g) depends on the fact that $\partial A_g(n)$ has codimension at least two.

Together with the above discussion, (a) and (f) imply that $(A_g(n), \partial A_g(n))$ satisfied property (I) introduced in Section 3 and property (IV) introduced in Section 4 for all $g \geq 3$ and $n \geq 3$.

Suppose now that $\Gamma$ is a subgroup of $\Gamma_g$ of finite index. Then all properties (b-g) still hold for $A_g(\Gamma)$. Moreover, if $\Gamma$ is contained in a $\Gamma_g(n)$ for some $n \geq 3$, then property (a) holds too.

5.3. Irreducible modular forms. We begin with a consequence of the Lefschetz hyperplane theorem as stated in Theorem 3.1.
Theorem 5.2 (Connectedness of zero loci of modular forms). Let \( g \geq 3 \) and let \( \mathcal{A}_g(\Gamma) \) for some finite-index subgroup \( \Gamma \) of \( \Gamma_g \). Then every divisor \( D \) in \( \mathcal{A}_g(\Gamma) \) pulls back to a connected divisor \( \tilde{D} \) in \( \mathbb{H}_g \). Moreover, if \( D \) is locally irreducible, then \( \tilde{D} \) is irreducible.

For \( g = 2 \) the same conclusions hold if \( \partial D \) consists of finitely many points.

Proof. Up to replacing \( \Gamma \) by \( \Gamma_g(\lfloor n \rfloor) \subset \Gamma \) for a suitable \( n \geq 3 \), and \( D \) by its pull-back to \( \mathcal{A}_g(\Gamma) \), we can assume that \( \mathcal{A}_g(\Gamma) \) is a smooth variety and we let \( \mathcal{A}_g(\Gamma) \) be its Satake compactification with \( \partial \mathcal{A}_g(\Gamma) = \mathcal{A}_g(\Gamma) \setminus \mathcal{A}_g(\Gamma) \). By what we have recalled in Section 5.2, \( D \) is an ample Cartier divisor in \( \mathcal{A}_g(\Gamma) \) and so, for \( g \geq 3 \), the triple \( (\mathcal{A}_g(\Gamma), \partial \mathcal{A}_g(\Gamma), D) \) satisfy properties (I)-(II)-(III\(_2^\prime\)) introduced in Section 3. By Theorem 3.1 it follows that \( \pi_1(D) \to \Gamma \) is surjective, and so \( \tilde{D} \) is connected by Lemma 2.8(iii). The second claim is a consequence of Proposition 2.12(i).

For \( g = 2 \) it is enough to note that the required condition ensures that \( D \) satisfies (III\(_2^2\)), so that Theorem 3.1 still applies.

Note that, for \( g \geq 4 \), the same argument as above shows that \( \pi_1(D) \cong \Gamma \) and so its lift \( \tilde{D} \) is simply connected.

Example 5.3 (Eisenstein series). For every \( g \) we write \( \Gamma_{g,0} \subset \Gamma \) for the subgroup defined by \( C = 0 \). For every integer \( k > \frac{g+1}{2} \), the Eisenstein series

\[
E_{2k}(\tau) := \sum_{\gamma \in \Gamma_{g}/\Gamma_{g,0}} \det(C\tau + D)^{-2k}, \quad \text{where} \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]

is a modular form of weight \( 2k \) and determines a divisor \( \overline{D} \) in \( \mathcal{A}_g \) that does not contain \( \partial \mathcal{A}_g \). It follows that \( \tilde{D} \subset \mathbb{H}_g \) is connected for \( g \geq 2 \).

In some cases the generic element \( \overline{E} \) in the linear system of a divisor \( \overline{D} \subset \mathcal{A}_g(\Gamma) \) has smooth internal part \( E = \overline{E} \cap \mathcal{A}_g(\Gamma) \): for example, this occurs by Bertini if \( \overline{D} \) is very ample away from the boundary. One can thus apply Theorem 5.2 to conclude that the preimage of \( E \) in \( \mathbb{H}_g \) is irreducible.

Since the Satake compactification of \( \mathcal{A}_g(\Gamma) \) is \( \text{Proj}(A(\Gamma)) \), the linear system of modular forms with respect to \( \Gamma \) of sufficiently divisible weight is certainly very ample on \( \mathcal{A}_g(\Gamma) \). On the other hand, given an arbitrary half-integral weight and a system of multipliers, there exists a cofinal subset \( \{ \Gamma_{g,n,2n} \}_{n} \) of finite-index subgroups of \( \Gamma_g \) such that the linear system of modular forms with the given weight and multipliers is very ample on every \( \mathcal{A}_g(\Gamma_{g,n,2n}) \) for \( n \geq 3 \) (see [M07, Theorem 10.14] and [SM96]). As consequence, we have that for high enough level (in a cofinal set), modular forms of weight \( 1/2 \) give an embedding of the modular variety.

Our aim now is to investigate the irreducibility of the preimage of any divisor of \( \mathcal{A}_g(\Gamma) \). In order to do that, we need to recall the following result from [F91, Theorem 4.7], which can be also obtained as a consequence of Proposition 4.6.
Lemma 5.4 (Factorization into absolutely irreducible forms). For \( g \geq 3 \) the following hold.

(i) Each non-vanishing modular form of weight \( 1/2 \) is absolutely irreducible.

(ii) Let \( F \) be a non-vanishing modular form with respect to a finite-index subgroup \( \Gamma \) of \( \Gamma_g \). There exists a factorization of \( F \) as

\[
F = f_1^{r_1} \cdots f_u^{r_u},
\]

where \( r_1, \ldots, r_u \) are positive integers and \( f_1, \ldots, f_u \) are distinct, absolutely irreducible, modular forms. Moreover, such factorization is unique up to reordering the \( f_i \)'s.

Note that Lemma 5.4 does not hold for \( g = 2 \): an example is discussed in Section 7.1.

As a preparation for a deeper investigation regarding liftings of subvarieties of \( \mathcal{A}_g(\Gamma) \) to \( \mathbb{H}_g \), we will first discuss the case of theta constants.

5.4. Theta functions. The (first-order) theta function \( \vartheta : \mathbb{H}_g \times \mathbb{C}^g \rightarrow \mathbb{C} \) is defined as

\[
\vartheta(\tau, z) := \sum_{n \in \mathbb{Z}^g} \exp \pi i (n^t \tau n + 2n^t z)
\]

and it is even in \( z \).

For \( \varepsilon, \delta \in (\mathbb{Z}/2)^g \) we define the (first order) theta function \( \vartheta[\underline{\varepsilon} \underline{\delta}] : \mathbb{H}_g \times \mathbb{C}^g \rightarrow \mathbb{C} \) with characteristic \( [\underline{\varepsilon} \underline{\delta}] \) as

\[
\vartheta[\underline{\varepsilon} \underline{\delta}](\tau, z) := \sum_{n \in \mathbb{Z}^g} \exp \pi i \left[ (n + \frac{\varepsilon}{2})^t \tau (n + \frac{\varepsilon}{2}) + 2 \left( n + \frac{\varepsilon}{2} \right)^t \left( z + \frac{\delta}{2} \right) \right]
\]

and note that \( \vartheta[\underline{0} \underline{0}] = \vartheta \).

The characteristic \( [\underline{\varepsilon} \underline{\delta}] \in (\mathbb{Z}/2)^{2g} \) is called even or odd depending on whether the scalar product \( \varepsilon \cdot \delta \) is zero or one as an element of \( \mathbb{Z}/2 \). It turns out that there are \( 2^{g-1}(2^g + 1) \) even characteristics and \( 2^{g-1}(2^g - 1) \) odd ones. As a function of \( z \), the theta function \( \vartheta[\underline{\varepsilon} \underline{\delta}](\tau, z) \) is even (resp. odd) if its characteristic is even (resp. odd).

The theta constant \( \theta[\underline{\varepsilon} \underline{0}] : \mathbb{H}_g \rightarrow \mathbb{C} \), already defined in Section 1.1, is the value of theta function \( \vartheta[\underline{\varepsilon} \underline{\delta}] \) at \( z = 0 \). Note that theta constant \( \theta[\underline{\delta}] \) vanish identically for \( [\underline{\delta}] \) odd.

Similarly to \( \vartheta = \vartheta[\underline{0} \underline{0}] \), the other even theta constants \( \vartheta[\underline{\varepsilon} \underline{0}] \) are examples of modular forms of weight \( 1/2 \) for the subgroup \( \Gamma_g(2) \) and their zero locus in \( \mathcal{A}_g(2) \) is denoted by \( \theta_{\text{null}}[\underline{\delta}] \).

The group \( \Gamma_g \) acts on characteristics, considered as elements of \( (\mathbb{Z}/2)^{2g} \), via an affine action of its quotient \( \text{Sp}_{2g}(\mathbb{Z}/2) = \Gamma_g/\Gamma_g(2) \). In particular, this is to say that \( \Gamma_g(2) \) is precisely equal to the subgroup of \( \Gamma_g \) that fixes each characteristic. Moreover, \( \Gamma_g/\Gamma_g(2) \) acts transitively on the subset of even characteristics. We consider the union \( \theta_{\text{null}}(2) \) of all \( \theta_{\text{null}}[\underline{\delta}] \subset \mathcal{A}_g(2) \), as \( [\underline{\delta}] \) ranges among the \( 2^{g-1}(2^g + 1) \) even characteristics. It is the pull-back of a
divisor $\theta_{null} \subset \mathcal{A}_g$. Thus $\theta_{null}$ is the image of a single $\theta_{null} [\tilde{\delta}]$ via the cover $\mathcal{A}_g(2) \to \mathcal{A}_g$ for any even $[\tilde{\delta}]$.

Geometrically, $\theta_{null}$ is the locus of ppav whose theta divisor has a singularity at an even two-torsion point. Thus we have

**Proposition 5.5** (Irreducible components of $\theta_{null}(2)$). Let $g \geq 3$ and let $[\tilde{\delta}] \in (\mathbb{Z}/2)^g$ be an even characteristic. Then

(i) the divisor $\theta_{null} [\tilde{\delta}]$ is irreducible, and so are its preimages in $\mathcal{A}_g(2n)$ for every $n > 1$;

(ii) the divisor $\theta_{null}(2n)$ is connected, not locally irreducible and consists of $2^{g-1}(2^g + 1)$ irreducible components;

(iii) the divisor $\theta_{null} \subset \mathcal{A}_g$ is irreducible but not locally irreducible.

**Proof.** (i) The divisor $\theta [\tilde{\delta}]$ is the zero locus inside $\mathcal{A}_g(2)$ of a modular form $\theta [\tilde{\delta}] (\tau)$ of minimal weight $\frac{1}{2}$, which then is absolutely irreducible by Lemma 5.4(i).

(ii) Since $g \geq 3$, every pair of irreducible components of $\theta_{null}(2)$ intersect. In fact, the closure of two distinct $\theta_{null} [\tilde{\delta}]$ and $\theta_{null} [\tilde{\delta}']$ inside the Satake compactification $\overline{\mathcal{A}_g}(2)$ must intersect each other, because theta-nulls are ample. Since $\partial \mathcal{A}_g(2)$ inside $\overline{\mathcal{A}_g}(2)$ has codimension $g \geq 3$, it follows that $\theta_{null} [\tilde{\delta}] \cap \theta_{null} [\tilde{\delta}']$ is not contained inside $\partial \mathcal{A}_g(2)$, and so they must intersect inside $\mathcal{A}_g(2)$. As a consequence, $\theta_{null}(2)$ is connected and not locally irreducible, and $\theta_{null}(2n)$ is so too for all $n$. Moreover, being each $\theta_{null} [\tilde{\delta}]$ absolutely irreducible by (i), every irreducible component of $\theta_{null}(2n)$ is a connected etale cover (in the orbifold sense) of some $\theta_{null} [\tilde{\delta}]$. Since there are exactly $2^{g-1}(2^g + 1)$ even characteristics, $\theta_{null}(2n)$ has $2^{g-1}(2^g + 1)$ irreducible components.

(iii) Since $\theta_{null}$ is the image of any even $\theta_{null} [\tilde{\delta}]$, it follows from (i) that $\theta_{null}$ is irreducible. Moreover, $\theta_{null}(2) \to \theta_{null}$ is an etale cover (in the orbifold sense), and so $\theta_{null}$ cannot be locally irreducible by (ii).

We then have the following

**Corollary 5.6** ($\theta [\tilde{\delta}]$ is universally irreducible for $g \geq 3$). For $g \geq 3$ and for any even characteristics $[\tilde{\delta}]$, the divisor $\theta_{null} [\tilde{\delta}] \subset \mathbb{H}_g$ is normal and irreducible.

**Proof.** We have seen that, since $\vartheta [\tilde{\delta}]$ is modular form of weight $1/2$, it is absolutely irreducible. Moreover, $\theta_{null} [\tilde{\delta}]$ is a divisor inside $\mathcal{A}_g(2)$ and regular in codimension 1 and so it is normal (see [CvG08]). Since normality is a local property, $\theta_{null} [\tilde{\delta}]$ is normal too. Hence $\theta_{null} [\tilde{\delta}]$ is irreducible by Theorem 5.2.

5.5. **Lifting subvarieties that contain $J_g$ or $H_g$**. We use $J_g \subset \mathcal{A}_g$ to denote the locus of Jacobians of smooth genus $g$ curves, and denote by $H_g \subset J_g$ the locus of hyperelliptic Jacobians. We recall that $J_g$ is the image
of the moduli space of curves $\mathcal{M}_g$ via the Torelli morphism $t_g: \mathcal{M}_g \to \mathcal{A}_g$ and that $\mathcal{H}_g$ is the image of the hyperelliptic locus $\mathcal{H}\mathcal{M}_g \subset \mathcal{M}_g$ via $t_g$. Moreover, as orbifolds $\mathcal{J}_g \setminus \mathcal{H}_g$ and $\mathcal{H}_g$ are smooth and $\mathcal{J}_g$ has unbranched singular locus $\mathcal{H}_g$ (for example, $\mathcal{J}_g(n) \setminus \mathcal{H}_g(n)$ and $\mathcal{H}_g(n)$ are smooth varieties for all $n \geq 3$).

We recall the following fact.

**Proposition 5.7** (Irreducible components of $\mathcal{H}_g(2)$ [T91]). For $g \geq 3$, the hyperelliptic locus $\mathcal{H}_g(2)$ has

$$2^{g^2} \prod_{k=1}^g (2^{2k} - 1)/(2g + 2)! = |\text{Sp}_{2g}(\mathbb{Z}/2)|/|\mathcal{S}_{2g+2}|$$

irreducible components. Moreover, the preimage in $\mathcal{A}_g(2n)$ of an irreducible component of $\mathcal{H}_g(2)$ is irreducible.

Since $\mathcal{H}_g$ is irreducible, the group $\Gamma_g$ transitively acts on the set of irreducible components of $\mathcal{H}_g(2)$. Hence the above Proposition 5.7 is a consequence of the following, cf. [M84, Lemma 8.12], or [AC79, Theorem 1].

**Proposition 5.8** (Fundamental group of the hyperelliptic locus). If $\iota: \mathcal{H}_g \to \mathcal{A}_g$ is the inclusion of the locus of hyperelliptic Jacobians with $g \geq 2$, then the image of $\pi_1(\iota): \pi_1(\mathcal{H}_g) \to \pi_1(\mathcal{A}_g) = \Gamma_g$ fits into the following exact sequence

$$1 \to \Gamma_g(2) \to \text{Im}(\pi_1(\iota)) \to \mathcal{S}_{2g+2} \to 1.$$

In particular, any irreducible component $\mathcal{H}$ of $\mathcal{H}_g(2n)$ satisfies

$$\pi_1(\mathcal{H}) \to \Gamma_g(2n).$$

In fact, using Proposition 5.8 and Lemma 2.8(iii) we can draw the following conclusion.

**Corollary 5.9** (Irreducible components of $\tilde{\mathcal{H}}_g$). The preimage in $\mathbb{H}_g$ of a connected component of $\mathcal{H}_g(2)$ is smooth and connected.

The preimage of $\mathcal{H}_g$ in $\mathbb{H}_g$ consists of $2^{g^2} \prod_{k=1}^g (2^{2k} - 1)/(2g + 2)!$ smooth irreducible components.

Similarly, for the Jacobian locus we have:

**Lemma 5.10** (Irreducibility of the Jacobian locus $\tilde{\mathcal{J}}_g$). The Jacobian loci $\mathcal{J}_g(\Gamma)$ are irreducible for all subgroups $\Gamma$ of $\Gamma_g$.

**Proof.** It is well-known that $\mathcal{M}_g$ is a connected orbifold. Since the homomorphism $\pi_1(\mathcal{M}_g) \to \pi_1(\mathcal{A}_g)$ of orbifold fundamental groups induced by $t_g$ is surjective (see Fact I.1), the irreducibility of the Jacobian locus $\tilde{\mathcal{J}}_g$ in $\mathbb{H}_g$ follows from Lemma 2.8(iii). We conclude, since $\mathcal{J}_g(\Gamma) = \tilde{\mathcal{J}}_g/\Gamma$. □

Here we specialize some of the results obtained in Section 2 to subvarieties of $\mathcal{A}_g(\Gamma)$ that contain the Jacobian or the hyperelliptic locus.
Proposition 5.11 (Subvarieties containing hyperelliptic or Jacobian locus). Let $Z \subset \mathcal{A}_g(\Gamma)$ be an irreducible subvariety.

(i) If a Zariski-open subset of $\mathcal{J}(\Gamma)$ is contained in $Z$, then $\tilde{Z} \subset \mathbb{H}_g$ is connected, with finitely many irreducible components. Moreover, if such Zariski-open subset of $\mathcal{J}(\Gamma)$ is not entirely contained inside the singular locus of $Z$, then $\tilde{Z}$ is irreducible.

(ii) Assume that $\mathcal{A}_g(\Gamma)$ dominates $\mathcal{A}_g(2)$. If a Zariski-open subset of $\mathcal{H}_g(\Gamma)$ is contained in $Z$, then $\tilde{Z} \subset \mathbb{H}_g$ is connected, with finitely many irreducible components. Moreover, if such Zariski-open subset of $\mathcal{H}_g(\Gamma)$ is not entirely contained inside the singular locus of $Z$, then $\tilde{Z}$ is irreducible.

Proof. (i) The inclusion $\mathcal{J}_g(\Gamma) \hookrightarrow \mathcal{A}_g(\Gamma)$ induces a surjection at the level of orbifold fundamental groups by Corollary I.2. The conclusion follows from Corollary 2.10 for the connectedness of $\tilde{Z}$ and Corollary 2.13 for the irreducibility of $\tilde{Z}$.

(ii) Since $\Gamma$ is contained inside $\Gamma_g(2)$, the inclusion $H \hookrightarrow \mathcal{A}_g(\Gamma)$ of a connected component of $\mathcal{H}_g(\Gamma)$ induces a surjection at the level of orbifold fundamental groups by Proposition 5.8. Again, we conclude by Corollary Corollary 2.10 and Corollary 2.13. \qed

As a first example, one can apply the above Proposition 5.11(ii) again to the case of $Z = \theta_{\text{null}} \left[ \frac{s}{\delta} \right]$ inside $\mathcal{A}_g(2)$ to deduce the irreducibility of $\theta_{\text{null}} \left[ \frac{s}{\delta} \right]$. In fact in Theorem 9.1 at page 137 of [M84], Mumford gives a characterization of a component $\tilde{H}$ of $\mathcal{H}_g(2)$ in terms of vanishing of theta constants. Moreover, in [SM03] it is proved that these equations cut $\tilde{H}$ smoothly, and so $\tilde{H}$ is contained in the smooth locus of $\theta_{\text{null}} \left[ \frac{s}{\delta} \right]$.

As a second example, we mention the irreducible component of $\mathcal{N}_k$ that contains the Jacobian locus. Assume $g \geq 4$. It was proven by Andreotti-Mayer [AM67] that the theta divisor of a Jacobian of dimension $g$ has singular locus of dimension at least $g - 4$. If $\mathcal{N}_k \subset \mathcal{A}_g$ is the locus of Abelian varieties whose theta divisor has singular locus of dimension at least $k$, then for all $k \leq g - 4$ there exists an irreducible component $\mathcal{N}'_k$ of $\mathcal{N}_k$ that contains the Jacobian locus. Then $\mathcal{N}'_k$ is connected with finitely many irreducible components by Proposition 5.11(i).

6. Universally irreducible subvarieties of $\mathcal{A}_g(\Gamma)$

6.1. Rational translates. We recall that the Siegel upper half-space $\mathbb{H}_g$ is transitively acted on by $\text{Sp}_{2g}(\mathbb{R})$. Given a finite-index subgroup $\Gamma$ of $\Gamma_g$,
the action by a rational element \( M \in \text{Sp}_{2g}(\mathbb{Q}) \) on \( \mathbb{H}_g \) induces a diagram

\[
\begin{array}{ccc}
A_g(\Gamma^M) & \xrightarrow{p} & A_g(\Gamma) \\
\searrow & & \nearrow \\
& q_M \sim & \\
& & A_g(\Gamma_M)
\end{array}
\]

for suitable finite-index subgroups \( \Gamma^M \) and \( \Gamma_M \) that only depend on \( \Gamma \) and \( M \). The multiplication by \( M \), that sends a subset of \( \mathbb{H}_g \) to its \( M \)-translate in \( \mathbb{H}_g \), descends to \( q_M \circ p^{-1} \), which sends a subset of \( A_g(\Gamma) \) to a subset of \( A_g(\Gamma_M) \). In this section we are describing the properties of such construction.

Let \( M \in \text{Sp}_{2g}(\mathbb{Q}) \). We denote by \( c_M \) the conjugation by \( M \) defined as \( c_M(\gamma) := M \gamma M^{-1} \). Given a finite-index subgroup \( \Gamma \) of \( \Gamma_g \), the conjugate \( c_M(\Gamma) \) is a subgroup of \( \text{Sp}_{2g}(\mathbb{Q}) \). We define \( \Gamma_M := \Gamma_g \cap c_M(\Gamma) \) and \( \Gamma^M := c_{M^{-1}}(\Gamma_M) \), which are subgroups of \( \Gamma_g \).

**Example 6.1.** If \( M = \begin{pmatrix} d & I_g \\ 0 & d^{-1} \cdot I_g \end{pmatrix} \) with \( d \geq 2 \) integer and \( \Gamma = \Gamma_g \), then

\[
\Gamma_M = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \mid C \equiv 0 \pmod{d^2} \right\}.
\]

**Lemma 6.2** (\( \Gamma_M \) and \( \Gamma^M \) have finite index in \( \Gamma_g \)). Let \( d \geq 1 \) be an integer and \( M \in \text{Sp}_{2g}(\mathbb{Q}) \) such that \( dM \) has integral entries.

(i) If \( \Gamma \supseteq \Gamma_g(\ell) \), then \( \Gamma_M \supseteq \Gamma_g(\ell d^2) \).

(ii) \( \Gamma_M \) has finite index in \( \Gamma_g \).

(iii) \( \Gamma^M \) is a finite-index subgroup of \( \Gamma \).

**Proof.** Note first that both \( d \cdot M \) and \( d^{-1} \cdot M \) have integral entries. This is immediate, since \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) is symplectic and so \( M^{-1} = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix} \).

(i) Let now \( \gamma \) be any element of \( \Gamma_g(\ell d^2) \). We want to show that \( \gamma \in \Gamma_M \). Since \( \Gamma_g(\ell d^2) \subseteq \Gamma_g(\ell) \subseteq \Gamma \), it is enough to show that \( \gamma \) belongs to \( c_M(\Gamma_g(\ell)) \subseteq c_M(\Gamma) \).

Since \( \gamma \in \Gamma_g(\ell d^2) \), we can write \( \gamma = I + \ell d^2 N \) for some integral \( N \). Then

\[
M^{-1} \gamma M = M^{-1}(I + nd^2 N)M = I + \ell(dM^{-1})N(dM).
\]

Hence \( M^{-1} \gamma M \) belongs to \( \Gamma_g(\ell) \) and so \( \gamma \in M \Gamma_g(\ell) M^{-1} \), as desired.

(ii) Since \( \Gamma \) has finite-index in \( \Gamma_g \), it contains a principal congruence subgroup \( \Gamma(\ell) \) for some \( \ell \geq 1 \). Hence, \( \Gamma_M \) contains \( \Gamma(\ell d^2) \) and so it has finite index in \( \Gamma_g \).

(iii) follows from (ii), since \( \Gamma^M = c_{M^{-1}}(\Gamma_M) = \Gamma \cap c_{M^{-1}}(\Gamma_g) \).

\[\square\]
By Lemma 6.2, we have the following diagram of finite-index subgroups of $\Gamma_g$

$$
\Gamma^M \xymatrix{ \sim \ar[dr]_{c_M} & \\
\Gamma & \Gamma_M \ar[lu]_{\sim} }
$$

The isomorphism of $\mathbb{H}_g$ given by the multiplication by $M$ descends to an isomorphism $q_M$ as in the following diagram

$$
\mathbb{A}_g(\Gamma^M) \xymatrix{ \sim \ar[dr]_{q_M} \ar[dl]_p & \\
\mathbb{H}_g & \mathbb{A}_g(\Gamma_M) \ar[lu]_{\sim} }
$$

**Definition 6.3** (Rational translate). Given $Z \subset \mathbb{A}_g(\Gamma)$ its $M$-translate inside $\mathbb{A}_g(\Gamma_M)$ is $Z_M := q_M(p^{-1}(Z))$.

Clearly, the $M$-translate of $Z$ can be pulled back to any level that dominates $\mathbb{A}_g(\Gamma_M)$.

The following statement gives a way to produce more absolutely or universally irreducible loci, starting from known ones.

**Lemma 6.4** (Absolute/universal irreducibility is translation invariant). Assume $g \geq 2$. Let $Z \subset \mathbb{A}_g(\Gamma)$ and $M \in \text{Sp}_{2g}(\mathbb{Q})$, and let $Z_M$ be the $M$-translate of $Z$.

(i) $Z$ is absolutely irreducible $\iff$ $Z_M$ is absolutely irreducible.

(ii) $Z$ is universally irreducible $\iff$ $Z_M$ is universally irreducible.

**Proof.** Let $p$ and $q_M$ be as above. Both claims (i-ii) are straightforward, since $(\mathbb{A}_g(\Gamma_M), Z_M)$ is isomorphic to $(\mathbb{A}_g(\Gamma^M), p^{-1}(Z))$ via $q_M$ and $p$ is a finite étale cover. $\square$

As an example of application of Lemma 6.4, for $g \geq 3$ all rational translates of $\mathcal{J}_g(n)$ and of $\theta_{\text{null}}[\frac{2}{3}]$ (for $n$ even) are universally irreducible by Corollary 5.6 and by Lemma 5.10.

**6.2. From absolutely irreducible to universally irreducible.** Let $\Gamma$ be a finite-index subgroup of $\Gamma_g$. The goal of this section is to prove the following.

**Theorem 6.5** (Absolutely irreducible of small codimension are universally irreducible). Let $g \geq 3$ and let $Z \subset \mathbb{A}_g(\Gamma)$ be a complete intersection subvariety of codimension at most $g - 2$. Then its preimage $\tilde{Z}$ in $\mathbb{H}_g$ has finitely many irreducible components. In particular, if $Z$ is absolutely irreducible, then it is universally irreducible.

The proof of Theorem 6.5 relies on the following main ingredients:
absolute and universal irreducibility are invariant under translation by elements of the rational symplectic group (Lemma 6.4)

• each complete intersection subvariety $Z \subset \mathcal{A}_g(\Gamma)$ of codimension at most $g - 2$ has a rational translate $Z_M$ that meets the Jacobian locus in a non-hyperelliptic Jacobian (Lemma 6.6)

• the image of $\pi_1((Z_M)_{\text{sm}}) \to \Gamma_g$ contains a pair of transvections along two linearly independent vectors of $\mathbb{Z}^{2g}$, and so does the image of $\pi_1(Z) \to \Gamma_g$ (Corollary 6.7)

• Zariski-dense subgroups of $\Gamma_g$ that contain commuting transvections associated to a pair of linearly independent vectors of $\mathbb{Z}^{2g}$ have finite index (Proposition 8.2).

We begin by analyzing how subvarieties of small codimension in $\mathcal{A}_g$ meet the boundary of the Jacobian locus: this is due to technical reasons, as it is easier to control the monodromy at infinity of a subvariety of the moduli space of curves $\mathcal{M}_g$ (as shown in Appendix I) rather than of a subvariety of $\mathcal{A}_g$.

Lemma 6.6. Let $g \geq 3$ and $Z \subset \mathcal{A}_g(\Gamma)$ be a complete intersection subvariety of codimension $k \leq g - 2$ and denote by $\pi' : \mathcal{A}_g(\Gamma_M) \to \mathcal{A}_g$ the natural projection. Then there exists a rational translate $Z_M \subset \mathcal{A}_g(\Gamma_M)$ of $Z$ such that

(i) the image $\pi'(Z_M) \subset \mathcal{A}_g$ meets the non-hyperelliptic Jacobian locus $\mathcal{J}_g \setminus \mathcal{H}_g$;

(ii) if $A_{g-h}$ is the largest boundary stratum of $\mathcal{A}_g$ that meets $\overline{\pi'(Z_M)} \cap \mathcal{J}_g$, then $\overline{\pi'(Z_M)} \cap \mathcal{J}_g \cap A_{g-h}$ is non-compact.

We denote by $\delta^\text{irr}_{\text{h}}$ the locally-closed locus inside the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$ that parametrizes curves whose normalization has genus $g - h$ (the genus of a smooth disconnected curve being the sum of the genera of its connected components).

Proof of Lemma 6.6. (i) Let $\tilde{Z}$ be the preimage of $Z$ in $\mathbb{H}_g$. Since $\text{Sp}_{2g}(\mathbb{R})$ transitively acts on $\mathcal{H}_g$, there exists $M \in \text{Sp}_{2g}(\mathbb{R})$ such that $M \cdot \tilde{Z}$ does not contain any component of $\tilde{\mathcal{H}}_g$. By density we can assume $M$ to be rational. Since $Z$ is a complete intersection, so is $Z_M$ and their rational classes are positive multiples of $\lambda^k$. Since the pull-back $t^*_g(\lambda^k)$ to $\mathcal{M}_g$ is not rationally trivial (see [F99]), we have that $\pi'(Z_M) \cap \mathcal{J}_g$ is not empty and its irreducible components have codimension at most $k$ inside $\mathcal{J}_g$. Moreover $\pi'(Z_M) \cap \mathcal{J}_g$ does not entirely contain $\mathcal{H}_g$. Since every component of $\pi'(Z_M) \cap \mathcal{J}_g$ has dimension at least $2g - 1 = \text{dim}(\mathcal{H}_g)$, it follows that $\pi'(Z_M)$ must contain the Jacobian of a smooth non-hyperelliptic curve.

(ii) Let $Y$ be a component of $t^{-1}_g(\pi'(Z_M)) \subset \mathcal{M}_g$, which thus has codimension at most $g - 2$ in $\mathcal{M}_g$. Let $\mathcal{M}_g^{\text{pt}} = \delta^\text{irr}_{\text{h}}$ be the locus inside $\overline{\mathcal{M}}_g$ consisting of curves of compact type, namely without non-disconnecting nodes, and recall that holomorphic subvarieties of $\mathcal{M}_g^{\text{pt}}$ of codimension less than $g$ are
non-compact [FP05]. Thus $\overline{\mathcal{Y}}$ must meet the locus $\overline{\delta_1^{\text{irr}}}$ of curves with at least one non-disconnecting node and $\overline{\mathcal{Y}} \cap \overline{\delta_1^{\text{irr}}}$ has codimension at most $g - 2$ inside $\overline{\delta_1^{\text{irr}}}$. Up to choosing a different component of $i_g^{-1}(p'(Z_{\text{irr}}))$ as $\mathcal{Y}$, we can assume that $t_g(\overline{\mathcal{Y}})$ meets $A_{g-h}$ and so the smallest number of non-disconnecting nodes in a curve parametrized by $\overline{\mathcal{Y}}$ is exactly $h \in [1, g - 1]$. Since $\overline{\delta_1^{\text{irr}}}$ has codimension $h - 1$ in $\overline{\delta_1^{\text{irr}}}$, the intersection $\overline{\mathcal{Y}} \cap \overline{\delta_1^{\text{irr}}}$ has codimension at most $(g - 2) - (h - 1) = g - 1 - h$ inside $\overline{\delta_1^{\text{irr}}}$. Consider now the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{M}_{g-h,2h}^{\text{cpt}} & \xrightarrow{\nu} & \delta_1^{\text{irr}} \\
\downarrow f & & \downarrow t_g \\
\mathcal{M}_{g-h}^{\text{cpt}} & \xrightarrow{t_{g-h}} & A_{g-h}
\end{array}
\]

in which $f$ sends the $2h$-marked curve $(C, x_1, \ldots, x_{2h})$ to $C$ and $\nu$ sends $(C, x_1, \ldots, x_{2h})$ to the curve obtained from $C$ by gluing each couple $(x_1, x_2), (x_3, x_4), \ldots, (x_{2h-1}, x_{2h})$ to a node. Since $\nu$ is finite surjective and $f$ is a surjective fibration with $2h$-dimensional fiber, $f(\nu^{-1}(\overline{\mathcal{Y}}))$ is non-empty of codimension at most $g - h - 1$ inside $\mathcal{M}_{g-h}^{\text{cpt}}$, and so is non-compact. Hence $t_g(\overline{\mathcal{Y}} \cap \overline{\delta_1^{\text{irr}}})$ is non-compact inside $A_{g-h}$. The conclusion follows, since $t_g(\overline{\mathcal{Y}} \cap \overline{\delta_1^{\text{irr}}})$ is closed and it is contained inside $\overline{p'(Z_{\text{irr}})} \cap \overline{\mathcal{Y}} \cap A_{g-h}$. \hfill \Box

The above lemma and the work done in Appendix I yield the following.

**Corollary 6.7.** Let $g \geq 3$ and $Z \subset A_g(\Gamma)$ be an absolutely irreducible complete intersection subvariety of codimension at most $g - 2$. Then the image of $\pi_1(Z_{\text{sm}}) \to \Gamma_g$ contains two commuting transvections associated to a pair of independent vectors.

**Proof.** Let $Z_M \subset A_g(\Gamma_M)$ be a rational translate of $Z$ as in Lemma 6.6. By Lemma 6.4(i) such $Z_M$ is absolutely irreducible. By Corollary 1.8 applied to the subvariety $Z_M$ of $A_g(\Gamma_M)$, the image of $\pi_1((Z_{\text{sm}})_M) \to \Gamma_g$ contains two commuting transvections $T_u, T_v$ associated to linearly independent vectors $u, v$. The isomorphism $Z_M \cong Z_M$ constructed in Section 6.1 shows that the image of $\pi_1(Z_{\text{sm}}^M) \to \Gamma_g$ is obtained from the image of $\pi_1((Z_{\text{sm}})_M) \to \Gamma_g$ by applying $c_{M^{-1}}$. As a consequence, the image of $\pi_1(Z_{\text{sm}}^M) \to \Gamma_g$ contains the commuting transvections $c_{M^{-1}}(T_u) = T_{M^{-1}u}$ and $c_{M^{-1}}(T_v) = T_{M^{-1}v}$, associated to the linearly independent vectors $M^{-1}u$ and $M^{-1}v$. The conclusion follows, since $Z_M$ covers $Z$ and so the image of $\pi_1(Z_{\text{sm}}) \to \Gamma_g$ contains the image of $\pi_1(Z_{\text{sm}}^M) \to \Gamma_g$. \hfill \Box

Using the previous steps and the arithmeticity criterion of Section 8, we can now prove our main result.
Proof of Theorem 6.5. By Corollary 6.7 the image of $\pi_1(Z_{\text{sm}}) \to \Gamma_g$ contains two linearly independent, commuting transvections. Since $Z$ is absolutely irreducible, the image of $\pi_1(Z_{\text{sm}}) \to \Gamma_g$ is Zariski-dense in $\Gamma_g$ by Lemma 8.6. As a consequence, the image of $\pi_1(Z_{\text{sm}}) \to \Gamma_g$ has finite-index in $\Gamma_g$ by Proposition 8.2. The conclusion follows from Corollary 4.5. □

The same argument also shows the following.

Proposition 6.8. Assume $g = 2$. Let $D \subset A_2(\Gamma)$ be an absolutely irreducible, ample divisor, whose image $D$ in $\overline{A}_2$ satisfies $\overline{D} \supset \partial A_2$. Then $D$ is universally irreducible.

Note that $\theta_{\text{null}}[\tilde{\delta}] \subset A_2(2)$ is not absolutely irreducible, as discussed in Section 7.1.

We conclude the present section with a few questions.

1) Do there indeed exist absolutely irreducible divisors in $A_2(\Gamma)$ with ample closure (and that contain an irreducible component of $\partial A_2(\Gamma)$)?

2) In Proposition 6.8, is the hypothesis $\overline{D} \supset \partial A_2$ necessary?

3) In Theorem 6.5, is the hypothesis $\text{codim}_{A_2(\Gamma)}(Z) \leq g - 2$ necessary?

4) Under what conditions on $X$, absolutely irreducible subvarieties of $X$ are always universally irreducible?

An interesting case in the above question (4) is $X = M_g$ with $g \geq 2$, whose universal cover (in the orbifold sense) is the Teichmüller space, which is isomorphic to a contractible bounded domain in $\mathbb{C}^{3g-3}$.

7. Examples

In this last section we treat three special examples.

7.1. Theta-nulls in genus two. Compared to what happens for genus at least 3, the situation for $\theta_{\text{null}}[\tilde{\delta}]$ in genus 2 is completely different.

Proposition 7.1 (Theta-nulls are not absolutely irreducible for $g = 2$).

The divisor $\theta_{\text{null}}[\tilde{\delta}]$ in $A_2(2)$ is not absolutely irreducible, and $\theta_{\text{null}}[\tilde{\delta}]$ has infinitely many smooth connected components, each isomorphic to $\mathbb{H}_1 \times \mathbb{H}_1$.

We remark that the hypotheses of Theorem 3.1 are not satisfied. Indeed, the divisor $\theta_{\text{null}} \subset A_2$ is irreducible and it coincides with the locus $A_2^{\text{dec}}$ of decomposable Abelian varieties. Hence, its closure contains $\partial A_2$, and so $\partial \theta_{\text{null}}$ has codimension 1 inside $\overline{\theta_{\text{null}}}$. As a consequence, the same happens for each of the 10 smooth irreducible components $\theta_{\text{null}}[\tilde{\delta}]$ of $\theta_{\text{null}}(2)$ inside $A_2(2)$.

Proof of Proposition 7.1. Note that $\pi_1(\theta_{\text{null}})$ can be identified to the subgroup $(\Gamma_1 \times \Gamma_1) \ltimes G_2$ of $\text{Sp}_4(\mathbb{Z})$, namely

$$\pi_1(\theta_{\text{null}}) \cong \left\{ \left( \begin{array}{cc} A & 0 \\ 0 & D \end{array} \right), \left( \begin{array}{cc} A \cdot \sigma & 0 \\ 0 & D \cdot \sigma \end{array} \right) \mid A, D \in \text{SL}_2(\mathbb{Z}) \right\}$$
where $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \text{GL}_2(\mathbb{Z})$. In view of Lemma 2.9(i), the number of connected components of the preimage of $\theta_{\text{null}}[\delta]$ inside $A_g(2n)$ is $[\text{Sp}_4(\mathbb{Z}/2n) : \text{SL}_2(\mathbb{Z}/2n) \times \text{SL}_2(\mathbb{Z}/2n)]$ for $n > 1$, which increases with $n$. Hence $\theta[\delta]$ is not absolutely irreducible.

The last claim is easy, since the reducible locus in $\mathbb{H}_2$ is obtained acting by $\text{Sp}_4(\mathbb{Z})$ on the locus of diagonal matrices, which can be naturally identified to $H_1 \times H_1$. □

Proposition 7.1 implies that, for $g = 2$, a factorization of $\theta[\delta]$ into absolutely irreducible forms does not exist (note that Lemma 5.4 does not apply).

### 7.2. Intermediate Jacobians of cubic threefolds

We recall that for an odd characteristic $[\delta]$ we can define $\text{grad}_{\text{null}}[\delta] \subset A_g(2)$ as the locus of all ppav’s at which the gradient $d_z \theta[\delta](\tau, 0)$ vanishes. Such locus has expected codimension $g$ in $A_g(2)$.

For $g \geq 5$, it is known that each $\text{grad}_{\text{null}}[\delta]$ has an irreducible component, which we denote by $C[\delta]$, that contains some irreducible component of $H_g(2)$; however, such $\text{grad}_{\text{null}}[\delta]$ has other irreducible components besides $C[\delta]$ (for example, inside the decomposable locus $A_1(2) \times A_{g-1}(2)$ we will find some components of type $A_1(2) \times \theta_{\text{null}}[\delta']$).

In genus $g = 5$ we have that, at level 1, this component is the closure of the moduli space $C$ of the intermediate Jacobians of cubic threefolds. In [ACT11, Section 4] it is proven that $H_5$ in $C$ is smooth (in an orbifold sense) and the period map

$$\pi : C \longrightarrow A_5$$

extends to a regular map at the generic point of $H_5$ in $C$. Hence we have

**Corollary 7.2 (Irreducibility of $\tilde{C}[\delta]$).** For $g = 5$ the locus $\tilde{C}[\delta]$ is irreducible in $\mathbb{H}_5$. For $g \geq 6$ the locus $C[\delta]$ lifts to the union $\tilde{C}[\delta]$ of finitely many irreducible divisors in $\mathbb{H}_g$.

**Proof.** The proof is an immediate consequence of the above discussion. Moreover, Julia Bernatska kindly communicated us that the above-mentioned component of $H_5(2)$ is contained inside the smooth locus of $C[\delta]$, as a consequence of [B19, Remark 9]. □

### 7.3. The Schottky form

In this section we are going to exhibit an absolutely irreducible form of weight greater than $1/2$. This will be the so-called Schottky form.

Let $S$ be an integral, positive-definite quadratic form on $\mathbb{Z}^m$, and consider the theta series

$$\theta_S(\tau) = \sum_{G \in \text{Mat}_{m,g}(\mathbb{Z})} \exp(\pi i \cdot \text{tr}(G^t S G \tau)).$$

Such $\theta_S$ is a modular form of weight $m/2$ with respect to a subgroup of finite index $\Gamma \subset \Gamma_g$. If $S$ is even and unimodular, then $\Gamma = \Gamma_g$. 
In the following lemma we wish to highlight an interesting property of modular forms that are linear combination of theta series.

**Lemma 7.3** (Absolute irreducibility of theta series propagates). Let \( S_1, \ldots, S_k \) be integral, positive-definite quadratic forms on \( \mathbb{Z}^m \) and let \( a_1, \ldots, a_k \in \mathbb{C} \), and consider for all \( g \geq 3 \) the following modular form

\[
h_g := \sum_{i=1}^{k} a_i \theta_{S_i}.
\]

If \( h_{g_0} \) is absolutely irreducible, then \( h_g \) is universally irreducible for all \( g \geq g_0 \).

**Proof.** We will prove that \( h_g \) is an absolutely irreducible form for all \( g \geq g_0 \). By Theorem 6.5 it will follow that \( h_g \) is universally irreducible for all \( g \geq g_0 \).

In order to show that \( h_g \) is an absolutely irreducible form for all \( g \geq g_0 \), we proceed by induction.

The initial case \( g = g_0 \) is given by hypothesis. In order to deal with the inductive step, we assume \( g > g_0 \) and we recall that the Siegel operator

\[
\Phi(h)(\tau') := \lim_{t \to +\infty} h \left( \begin{array}{cc} \tau' & * \\ * & it \end{array} \right)
\]

sends a modular form \( h \) of genus \( g \) at level \( n \) to a modular form \( \Phi(h) \) of genus \( g-1 \) at level \( n \), in such a way that \( \Phi(h_a h_b) = \Phi(h_a) \Phi(h_b) \).

By contradiction, suppose that the form \( h_g \) breaks as a product \( h_g = h_{g,a} h_{g,b} \) of modular forms \( h_{g,a} \) and \( h_{g,b} \) with respect to a finite-index subgroup \( \Gamma \) of \( \Gamma_g \).

Since \( \Gamma \) contains the full congruence subgroup \( \Gamma_g(n) \) for some \( n \), we can assume without loss of generality that \( \Gamma = \Gamma_g(n) \). Since \( h_g \) is a linear combination of theta series we have

\[
\Phi(h_g) = h_{g-1}.
\]

This implies that \( h_{g-1} = \Phi(h_{g,a}) \Phi(h_{g,b}) \) is a nontrivial factorization of modular forms with respect to \( \Gamma_{g-1}(n) \), and so \( h_{g-1} \) is not irreducible at level \( n \). Such contradiction proves that \( h_g \) is an absolutely irreducible form. \( \square \)

We now consider a special case of the above construction. We recall that, for \( m = 16 \), there are exactly two classes of integral, positive-definite quadratic forms on \( \mathbb{Z}^{16} \): those associated to the lattices \( E_8 \oplus E_8 \) and \( D_{16}^+ \). For any \( g \) we consider the form

\[
f_g = \theta_{E_8 \oplus E_8} - \theta_{D_{16}^+}.
\]

It is a well-known fact that \( f_g \) vanishes identically for \( g = 1, 2, 3 \). It is Schottky’s form when \( g = 4 \), cf. [GSM11] or [CSB14]. For \( g \geq 4 \) we denote by \( F_g \) the divisor defined by \( f_g \). We know that \( F_4 \) in \( \mathcal{A}_4 \) is universally irreducible by Lemma 5.10, since it coincides with the closure of the Jacobian locus \( \mathcal{J}_4 \).
The same Lemma 5.10 cannot be applied in higher genus: though $F_g$ contains the hyperelliptic locus $H_g$ (see [P96]), for $g \geq 5$ it does not contain the Jacobian locus (see [GSM11] or [CSB14]). Nevertheless, an immediate application of Lemma 7.3 gives

Corollary 7.4 (The Schottky form is universally irreducible). The divisor $F_g$ of $A_g$ is universally irreducible for all $g \geq 4$.

8. A criterion for arithmeticity in $\text{Sp}_{2g}(\mathbb{Z})$

We recall that the standard symplectic form $\omega$ on $\mathbb{Q}^{2g}$, defined by $\omega(e_i, e_j) = -\omega(e_j, e_i) := \delta_{j,g+i}$ for all $1 \leq i \leq j \leq 2g$, is nondegenerate and integral on $\mathbb{Z}^{2g}$.

If $W \subseteq \mathbb{Q}^{2g}$ is a vector subspace, we denote by $\text{Sp}(W, \omega)$ the group of linear automorphisms of $W$ that preserve the restriction of $\omega$ to $W$. We remark that $\text{Sp}(\mathbb{Z}^{2g}, \omega) = \text{Sp}_{2g}(\mathbb{Z})$ and that $\text{Sp}(\mathbb{Q}^{2g}, \omega) = \text{Sp}_{2g}(\mathbb{Q})$.

Definition 8.1 (Transvections in $(\mathbb{Z}^{2g}, \omega)$). The transvection associated to $w \in \mathbb{Z}^{2g}$ is the integral symplectic transformation $T_w : \mathbb{Z}^{2g} \rightarrow \mathbb{Z}^{2g}$ defined as $T_w(v) := v + \omega(v, w)w$. Two transvections $T_u, T_v$ are linearly independent if $u, v$ are linearly independent as vectors of $\mathbb{Q}^{2g}$.

Observe that $(T_w - I)(\mathbb{Z}^{2g}) = \mathbb{Z} \cdot aw$ for some integer $a \neq 0$.

The aim of this short section is to prove the following.

Proposition 8.2 (A criterion of arithmeticity). Let $G$ be a Zariski-dense subgroup of $\text{Sp}_{2g}(\mathbb{Z})$ that contains two linearly independent, commuting transvections $T_u, T_v$. Then $G$ has finite index in $\text{Sp}_{2g}(\mathbb{Z})$.

The proof of the above proposition will make essential use of

Theorem 8.3 (Singh-Venkataramana [SV14, Theorem 1.2]). Suppose that $G \subseteq \text{Sp}_{2g}(\mathbb{Z})$ is a subgroup that satisfies the following properties:

(i) $G$ is Zariski dense inside $\text{Sp}_{2g}(\mathbb{Z})$;

(ii) there exist three vectors $w_1, w_2, w_3 \in \mathbb{Z}^{2g}$ such that
- $W = \sum_{i=1}^{3} \mathbb{Q} \cdot w_i$ of $\mathbb{Q}^{2g}$ is 3-dimensional;
- $\Omega(w_i, w_j) \neq 0$ for some $i, j$;
- the group $G$ contains the transvections $T_{w_1}, T_{w_2}, T_{w_3}$;
- the subgroup of $\text{Sp}(W, \omega)$ generated by the restrictions $T'_{w_1}, T'_{w_2}, T'_{w_3}$ to $W$ of the transvections $T_{w_1}, T_{w_2}, T_{w_3}$ contains a non-trivial element of the unipotent radical of $\text{Sp}(W, \omega)$.

Then the group $G$ has finite index in $\text{Sp}_{2g}(\mathbb{Z})$.

Remark 8.4. The above result holds if $\omega$ is replaced by any non-degenerate symplectic form on $\mathbb{Q}^{2g}$, which is integral on $\mathbb{Z}^{2g}$.

We wish to apply Theorem 8.3 in the situation described in the following lemma.
Lemma 8.5 (Finding elements in the unipotent radical). Let \( w_1, w_2, w_3 \in \mathbb{Z}^{2g} \) be linearly independent and such that \( \omega(w_1, w_2), \omega(w_1, w_3) \neq 0 \) and \( \omega(w_2, w_3) = 0 \). Then the group generated by the restrictions of \( T_{w_1}, T_{w_2}, T_{w_3} \) to \( W = \text{Span}(w_1, w_2, w_3) \) contains a nontrivial element of the unipotent radical of \( \text{Sp}(W, \omega) \).

Proof. Note that it is enough to prove the statement with \( (w_1, w_2, w_3) \) replaced by nonzero multiples \( (k_1 w_1, k_2 w_2, k_3 w_3) \).

Let \( e \in \mathbb{Z}^n \) be a \( \mathbb{Q} \)-generator of \( \text{Rad}(W, \omega) = \mathbb{Q} \cdot e \), and write \( w_3 = a_1 w_1 + a_2 w_2 + b e \), with \( a_1, a_2, b \in \mathbb{Q} \) and \( b \neq 0 \). Up to replacing \( w_3 \) with a multiple, we can assume that \( a_1, a_2, b \in \mathbb{Z} \).

Let \( B = (w_1, w_2, e) \) be a \( \mathbb{Q} \)-basis of \( W \). Then elements of the unipotent radical of \( \text{Sp}(W, \omega) \) are represented by the matrices

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
s & t & 1
\end{pmatrix}, \quad \text{with } t, s \in \mathbb{Q}
\]

with respect to the basis \( (w_1, w_2, e) \) of \( W \). Call now \( \omega_{ij} = \omega(w_i, w_j) \) and note that \( \omega_{13} = a_2 \omega_{12} \neq 0 \) and \( \omega_{23} = a_1 \omega_{21} = 0 = a_1 \). Then the restrictions \( T_{w_1} \) of \( T_{w_i} \) to \( W \) can be written as

\[
T_{w_1}' = \begin{pmatrix}
1 & \omega_{21} & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad T_{w_2}' = \begin{pmatrix}
1 & 0 & 0 \\
\omega_{12} & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

\[
T_{w_3}' = \begin{pmatrix}
1 & 0 & 0 \\
a_2^2 \omega_{12} & 1 & 0 \\
ba_2 \omega_{12} & 0 & 1
\end{pmatrix}
\]

with respect to the basis \( B \). It follows that

\[
(T_{w_2}')^{-a_2^2} T_{w_3}' = \begin{pmatrix}
1 & 0 & 0 \\
-a_2^2 \omega_{12} & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
a_2^2 \omega_{12} & 1 & 0 \\
ba_2 \omega_{12} & 0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Since \( ba_2 \neq 0 \), the element \( (T_{w_2}')^{-a_2^2} T_{w_3}' \neq I \) is in the unipotent radical of \( \text{Sp}(W, \omega) \). \( \square \)

As a consequence, we easily obtain our wished criterion.

Proof of Proposition 8.2. The subset of \( \text{Sp}_{2g}(\mathbb{C}) \) consisting of elements \( \tau \in \text{Sp}_{2g}(\mathbb{C}) \) such that \( \omega(\tau(u), u) \omega(\tau(u), v) = 0 \) is a proper algebraic subvariety of \( \text{Sp}_{2g}(\mathbb{C}) \). Since \( G \) is Zariski-dense, there exists \( \gamma \in G \) such that \( \omega(\gamma(u), u) \neq 0 \) and \( \omega(\gamma(u), v) \neq 0 \).

Now, \( u, v \) are linearly independent, and \( T_u, T_v \) commute, which implies that \( \omega(u, v) = 0 \). Hence, the vectors \( w_1 = u, w_2 = \gamma(u), w_3 = v \) are linearly independent and \( \omega(w_1, w_2), \omega(w_2, w_3) \neq 0 \) whereas \( \omega(w_1, w_3) = 0 \). Moreover, \( G \) contains \( T_{w_1}, T_{w_2} = \gamma T_{w_1} \gamma^{-1}, T_{w_3} \). Lemma 8.5 then implies that the hypotheses of Theorem 8.3 are satisfied, and so \( G \) has finite index in \( \text{Sp}_{2g}(\mathbb{Z}) \). \( \square \)
In order to verify the hypothesis of Zariski-density in Proposition 8.2 when dealing with absolutely irreducible subvarieties in a finite étale cover of \( A_g \), we will make use of the following observation.

**Lemma 8.6.** Let \( \Gamma \) be a finite-index subgroup of \( \Gamma \) and let \( Z \subset A_g(\Gamma) \) be an absolutely irreducible subvariety. Then the image of \( \pi_1(Z_{\text{sm}}) \rightarrow \Gamma \) is Zariski dense.

**Proof.** Absolute irreducibility of \( Z \) implies that the image of \( \pi_1(Z_{\text{sm}}) \rightarrow \Gamma \) surjects onto every finite quotient of \( \Gamma \). Since \( \Gamma \) is residually finite, \( G \) is Zariski-dense in \( \Gamma \) by [MS81, Proposition 3], and so is Zariski-dense in \( \Gamma \). \( \square \)

**Appendix I. Subvarieties of \( \mathcal{M}_g \), Dehn twists and transvections**

We recall that the mapping class group \( \text{MCG}_g \) can be identified to the orbifold fundamental group of \( \mathcal{M}_g \). Moreover the homomorphism induced by the Torelli morphism \( t_g : \mathcal{M}_g \rightarrow \mathcal{A}_g \) at the level of fundamental groups \( \pi_1(t_g) : \pi_1(\mathcal{M}_g) \rightarrow \pi_1(\mathcal{A}_g) \) identifies to the standard symplectic representation \( \rho : \text{MCG}_g \rightarrow \Gamma_g = \text{Sp}_{2g}(\mathbb{Z}) \). We recall the following classical result (see, for instance, [FM12, Theorem 6.4]).

**Fact I.1.** \( \rho \) is surjective.

The following is an immediate consequence.

**Corollary I.2.** For every subgroup \( \Gamma \) of \( \Gamma_g \), the natural homomorphism \( \pi_1(J_g(\Gamma)) \rightarrow \Gamma \) is surjective, where \( J_g(\Gamma) \) is the Jacobian locus in \( \mathcal{A}_g(\Gamma) \).

We also recall that the Torelli morphism \( t_g \) can be extended to a map \( \overline{t}_g : \overline{\mathcal{M}}_g \rightarrow \overline{\mathcal{A}}_g \) from the Deligne-Mumford compactification \( \overline{\mathcal{M}}_g \) to the Satake compactification \( \overline{\mathcal{A}}_g \).

In order to state Proposition I.3, we denote by \( \delta^\text{irr}_h \) the locally-closed locus inside \( \overline{\mathcal{M}}_g \) that parametrizes curves whose normalization has genus \( g - h \) (the genus of a disconnected curve being the sum of the genera of its connected components): in other words, points in \( \delta^\text{irr}_h \) are isomorphism classes of stable curves with exactly \( h \) non-disconnecting nodes (and possibly other disconnecting nodes). We observe that \( \delta^\text{irr}_h \) is a subvariety of codimension \( h \).

In this appendix we want to show the following result (recall Definition 8.1).

**Proposition I.3** (Finding a pair of commuting transvections I). Let \( Y \) be an irreducible subvariety in \( \mathcal{M}_g \). Suppose that

(a) \( Y \) intersects the locally-closed boundary stratum \( \delta^\text{irr}_h \) for some \( h > 0 \);
(b) the intersection \( Y \cap \delta^\text{irr}_h \) is non-compact.

Then the image of \( \pi_1(Y_{\text{sm}}) \rightarrow \Gamma_g \) contains two linearly independent, commuting transvections.

In order to prove Proposition I.3, we first recall that \( \overline{\mathcal{M}}_g \) supports a universal family of stable curves with total space \( \overline{\mathcal{C}}_g \). Moreover, a path between \( b_0 \in \)
\( M_g \) and \( b \in \partial M_g \) induces a map \( C_{b_0} \to C_b \) that shrinks disjoint loops of \( C_{b_0} \) to nodes of \( C_b \); call them shrinking loops.

**Notation I.4.** If \( \gamma \subset C_{b_0} \) is a non-trivial simple loop, then we denote by \( \text{Tw}_\gamma \in \text{MCG}(C_{b_0}) \) the right Dehn twist along \( \gamma \) (see [FM12, Chapter 3]). If \( \gamma_1, \ldots, \gamma_k \) are non-trivial disjoint simple loops on \( C_{b_0} \) and \( a_1, \ldots, a_k > 0 \) are integers, then \( \text{Tw}_{\gamma_1}^{a_1} \cdots \text{Tw}_{\gamma_k}^{a_k} \) is the element of \( \text{MCG}(C_{b_0}) \) obtained by performing \( a_i \) Dehn twists along \( \gamma_i \) for \( i = 1, \ldots, k \).

The following statement is rather standard.

**Lemma I.5** (Boundary strata and Dehn twists). Let \( \bar{Y} \) be an irreducible (not necessarily closed) analytic subvariety of \( \overline{M}_g \), call \( Y := \bar{Y} \cap M_g \) and let \( b \in \partial Y \). Suppose \( b \) belongs to a locally-closed boundary stratum \( \delta \) of \( \overline{M}_g \). Fix \( b_0 \in Y_{\text{sm}} \) and a path in \( Y_{\text{sm}} \cup \{b\} \) from \( b_0 \) to \( b \) and let \( \gamma_1, \ldots, \gamma_k \subset C_{b_0} \) be the induced shrinking loops. Then there exists \( a_1, \ldots, a_k \geq 1 \) such that the image of \( \pi_1(Y_{\text{sm}}, b_0) \to \text{MCG}(C_{b_0}) \) contains \( \text{Tw}_{\gamma_1}^{a_1} \cdots \text{Tw}_{\gamma_k}^{a_k} \). Hence, the image of \( \pi_1(Y_{\text{sm}}, b_0) \to \text{Sp}(H_1(C_{b_0})) \) contains the transvection associated to the vector \([a_1 \gamma_1 + \cdots + a_k \gamma_k]\).

**Proof.** Pick a holomorphic map \( f : \Delta \to \bar{Y} \) such that \( f(0) = b \) and \( f(\Delta^*) \) is contained in \( Y_{\text{sm}} \). Since \( Y_{\text{sm}} \) is connected, we can choose \( b_0 = f(\tilde{b}_0) \) for some \( 0 \neq \tilde{b}_0 \in \Delta \). Every branch of \( \partial M_g \) that contains \( \delta \) corresponds to a loop \( \gamma_i \) on \( C_{b_0} \). If \( f \) has multiplicity \( a_i \geq 1 \) along the \( i \)-th branch at \( \tilde{b}_0 \), the image of \( \pi_1(\Delta^*, \tilde{b}_0) \to \text{MCG}(C_{b_0}) \) is generated by \( \prod_{i=1}^k \text{Tw}_{\gamma_i}^{a_i} \), and the conclusion follows. \( \Box \)

The claimed result is an easy consequence of the above lemma.

**Proof of Proposition I.3.** Let \( b_2 \) be a point in the boundary of \( \bar{Y} \cap \delta_{\text{H}} \). Pick a contractible neighbourhood \( U \) of \( b_2 \) inside \( \overline{M}_g \) such that \( U \cap \delta_{\text{H}} \) and \( U \cap \bar{Y} \) are contractible too. Let \( b_0 \in U \cap Y_{\text{sm}} \) and \( b_1 \in U \cap \delta_{\text{H}} \). Connect \( b_0 \) to \( b_1 \) and to \( b_2 \) through paths contained in \( Y_{\text{sm}} \) (except at \( b_1, b_2 \)). Call \( \beta_1, \ldots, \beta_k \) the disjoint loops in \( C_{b_0} \) that are shrunk by the map \( C_{b_0} \to C_{b_1} \). Up to isotopy, we can assume that the map \( C_{b_0} \to C_{b_2} \) shrinks \( \beta_1, \ldots, \beta_k \) and the other loops \( \beta_{k+1}, \ldots, \beta_m \) to nodes. By Lemma I.5 the image of \( \pi_1(Y_{\text{sm}}) \to \Gamma_g \) contains the commuting transvections \( T_v, T_w \), where \( v = a_1 \beta_1 + \cdots + a_k \beta_k \) and \( w = c_1 \beta_1 + \cdots + c_m \beta_m \) with \( a_i, c_i \geq 1 \). Since \( \delta \geq 1 \), at least one loop \( \beta_1, \ldots, \beta_k \) is non-disconnecting and so the vector \( v \) is non-zero. Since \( b_2 \in \delta_{\text{H}+1} \), there exists a loop \( \beta_i \) with \( k + 1 \leq i \leq m \) such that \( C_{b_0} \setminus (\beta_1 \cup \cdots \cup \beta_k) \) and \( C_{b_0} \setminus (\beta_1 \cup \cdots \cup \beta_k \cup \beta_i) \) have the same number of connected components. It follows that \( w \) is not a multiple of \( v \), and so \( v, w \) are linearly independent. \( \Box \)

We now want to use Proposition I.3 to obtain a similar statement but for subvarieties of \( A_0(\Gamma) \) instead of \( M_g \), where \( \Gamma \) is any finite-index subgroup of \( \Gamma_g \).

We begin with an elementary lemma.
Lemma I.6. Let $\iota : Z \to A_g$ the inclusion of a subvariety and $N, M$ be subgroups of $\pi_1(Z)$ such that $[N : M \cap N] < +\infty$. Denote by $\iota_* : \pi_1(Z) \to \pi_1(A_g) = \Gamma_g$ the homomorphism induced by $\iota$. If $\iota_*(N)$ contains a pair of linearly independent, commuting transvections, then $\iota_*(M)$ does.

Proof. Let $\alpha, \beta \in N$ such that $T_v = \iota_*(\alpha), T_w = \iota_*(\beta) \in \iota_*(N)$ are two linearly independent, commuting transvections. There exists an integer $m \geq 1$ such that $\alpha^m, \beta^m \in M \cap N$. It follows that $T_{mv} = \iota_*(\alpha^m), T_{mw} = \iota_*(\beta^m)$ is the wished pair of linearly independent, commuting transvections in $\iota_*(M)$. □

In the below lemma we prove the wished statement for subvarieties of $A_g$.

Lemma I.7 (Finding a pair of commuting transvections II). Let $Z \subset A_g$ be an irreducible subvariety that meets the Jacobian locus, and suppose that $\overline{Z} \cap \mathcal{J}_g$ meets the locally-closed boundary stratum $A_{g-h}$ in a non-compact subset. Then the image of $\pi_1(Z_{\text{sm}}) \to \Gamma_g$ contains two linearly independent, commuting transvections.

Proof. Since $\overline{Z} \cap \mathcal{J}_g$ inside $A_g$ meets the boundary stratum $A_{g-h}$ in a non-compact subset, the closure of $t^{-1}(Z)$ inside $\mathcal{M}_g$ meets the locally-closed boundary stratum $\delta^i_{\text{fr}}$ in a non-compact subset. Thus, an irreducible component $Y$ of $t^{-1}(Z)$ satisfies hypotheses (a) and (b) in Proposition I.3, and so the image of $\pi_1(Y_{\text{sm}}) \to \Gamma_g$ contains two linearly independent, commuting transvections.

Call $H$ the image of $\pi_1(Y_{\text{sm}}) \to \pi_1(Z)$ and $K$ the image of $\pi_1(Z_{\text{sm}}) \to \pi_1(Z)$. Since $Y \to Z$ is a finite map and $Y, Z$ are irreducible, Corollary 2.7(i) implies that $[H : K \cap H] < +\infty$. By Lemma I.6 applied to $N = H$ and $M = K$, the image of $\pi_1(Z_{\text{sm}}) \to \Gamma_g$ contains two linearly independent, commuting transvections. □

As a consequence, we obtain our criterion for subvarieties of $A_g(\Gamma)$.

Corollary I.8 (Finding a pair of commuting transvections III). Let $\Gamma$ be a finite-index subgroup of $\Gamma_g$, let $p : A_g(\Gamma) \to A_g$ be the natural projection and let $Z \subset A_g(\Gamma)$ be an irreducible subvariety. Suppose that the image $p(Z)$ meets the Jacobian locus inside $A_g$, and that $\overline{p(Z)} \cap \mathcal{J}_g$ intersects the boundary stratum $A_{g-h}$ in a non-compact subset. Then the image of $\pi_1(p(Z)_{\text{sm}}) \to \Gamma_g$ contains two linearly independent, commuting transvections.

Proof. Observe that $p(Z)$ is irreducible, and let $\iota : p(Z) \to A_g$ be the inclusion and $\iota_* : \pi_1(p(Z)) \to \Gamma_g$ the induced homomorphism. Moreover, denote by $H$ the image of $\pi_1(Z_{\text{sm}}) \to \pi_1(p(Z))$ and by $K$ the image of $\pi_1(p(Z)_{\text{sm}}) \to \pi_1(p(Z))$.

By Lemma I.7, the subgroup $\iota_*(K)$ contains two linearly independent, commuting transvections. Since $Z \to p(Z)$ has finite fibers, $H$ is a finite-index subgroup of $K$ by Corollary 2.7(ii) in the case $Y = Z$. Hence, $\iota_*(H)$ is a
finite-index subgroup of $\iota_*(K)$, and so it contains two linearly independent, commuting transvections by Lemma I.6 applied to $N = K$ and $M = H$. □

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