SOLUTIONS TO DEGENERATE COMPLEX HESSIAN EQUATIONS

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Abstract. Let \((X, \omega)\) be an \(n\)-dimensional compact Kähler manifold. We study degenerate complex Hessian equations of the form \((\omega + dd^c \varphi)^m \wedge \omega^{n-m} = F(x, \varphi)\omega^n\). Under some natural conditions on \(F\), this equation has a unique continuous solution. When \((X, \omega)\) is rational homogeneous we further show that the solution is H"older continuous.

1. Introduction

Let \((X, \omega)\) be a compact Kähler manifold of complex dimension \(n\). Fix an integer \(m\) between 1 and \(n\), and let \(d, d^c\) denote the usual real differential operators \(d := \partial + \bar{\partial}\), \(d^c := \sqrt{-1} \pi (\bar{\partial} - \partial)\) so that \(dd^c = \frac{i}{\pi} \partial \bar{\partial}\). We are studying degenerate complex Hessian equations of the form

\[(1.1) \quad (\omega + dd^c \varphi)^m \wedge \omega^{n-m} = F(x, \varphi)\omega^n,\]

where the density \(F : X \times \mathbb{R} \to \mathbb{R}^+\) satisfies some natural integrability conditions (see Theorem A below).

The case \(m = 1\) corresponds to the Laplace equation and the case \(m = n\) corresponds to degenerate complex Monge-Ampère equations which have been studied intensively in recent years (see [Bl03, Bl05, Bl12, BGZ08, BK07, EGZ09, GKZ08, GZ05, GZ07, Kol98, Kol02, Kol03, Kol05]). So, equation \((1.1)\) is a generalization of both Laplace and Monge-Ampère equations.

The non degenerate complex Hessian equation on compact Kähler manifold, where \(F(x, \varphi) = f(x)\), with \(0 < f \in C^\infty(X)\), has been studied recently in [H09, HMW10, Jb10, DK12]. In [H09] and [Jb10], the authors independently solved this equation with a strong additional hypothesis, assuming \((X, \omega)\) has non negative holomorphic bisectional curvature. Later on, in [HMW10] an a priori \(C^2\) estimate was obtained without curvature assumption. Recently, using this estimate and a blowing up analysis suggested in [HMW10], Dinew and Kolodziej solved the equation in full generality.

Following Blocki [Bl05] we develop a potential theory for the complex Hessian equation on compact Kähler manifold. We define the class of \((\omega, m)\)-subharmonic functions which is a generalization of the class of \(\omega\)-plurisubharmonic functions when \(m = n\). The definition of the complex Hessian operator on bounded \((\omega, m)\)-subharmonic functions is delicate due to difficulties in regularization process.

To go around this difficulty, we introduce a capacity and use it to define the concept of quasi-uniform convergence. This allows us to define a suitable class of bounded and quasi-continuous \((\omega, m)\)-subharmonic functions on which the complex Hessian operator is well defined and continuous under quasi-uniform convergence.
We show that this definition coincides with the definition in the spirit of Bedford and Taylor method for the complex Monge-Ampère operator. A comparison principle and convergence results for this operator are also established.

With these potential tools in hand, we then consider the degenerate complex Hessian equation (1.1). The first main result of this paper is the following:

**Theorem A.** Let \((X,\omega)\) be a \(n\)-dimensional compact Kähler manifold. Fix \(1 \leq m \leq n\). Let \(F: X \times \mathbb{R} \to [0, +\infty)\) be a function satisfying the following conditions:

- (F1) for all \(x \in X\), \(t \mapsto F(x, t)\) is non-decreasing and continuous,
- (F2) for any fixed \(t \in \mathbb{R}\), there exists \(p > n/m\) such that the function \(x \mapsto F(x, t)\) belongs to \(L^p(X)\),
- (F3) there exists \(t_0 \in \mathbb{R}\) such that \(\int_X F(., t_0) \omega^n = \int_X \omega^n\).

Then there exists a function \(\varphi \in P^m(X,\omega) \cap C^0(X)\), unique up to an additive constant, such that

\[(\omega + dd^c \varphi)^m \wedge \omega^{n-m} = F(x, \varphi) \omega^n.\]

Moreover if \(\forall x \in X\), \(t \mapsto F(x, t)\) is increasing, then the solution is unique.

Note that the condition (F3) is automatically satisfied if \(F(\cdot, -\infty) = 0\) and \(F(\cdot, +\infty) = +\infty\). An important particular case is the exponential function \(F(x, t) = f(x)e^t\).

A particular case of this result has been obtained in [DK11]. The key point in their proof is a domination between volume and capacity. Our main result is proved using this technique and the recent result in the smooth case [DK12].

When \((X,\omega)\) is rational homogeneous with \(\omega\) being invariant under the Lie group action, we can easily regularize \((\omega, m)\)-subharmonic. Adapting the techniques in [EGZ09] we obtain Hölder continuity of the solution:

**Theorem B.** Under the same assumption as in Theorem A, assume further that \((X,\omega)\) is rational homogeneous and \(\omega\) is invariant under the Lie group action. Then the unique solution is Hölder continuous with exponent \(0 < \gamma < \frac{2(mp-n)}{mnp+2mp-2n}\).

When \(m = n\) we get the same exponent \(\gamma\) as in [EGZ09].

**Acknowledgement.** The paper is part of my Ph.D Thesis. I would like to express my deep gratitude to my advisor, Professor Ahmed Zeriahi, for sacrificing his very valuable time for me. I wish to express my sincere gratitude to Professor Vincent Guedj for his very useful suggestions and discussions to improve the paper. I also wish to say a special word of thanks to Professor Sébastien Boucksom for his kind invitation to IMJ and useful discussions. This paper owes much to their help and constant encouragement.

2. Preliminaries

In this section we introduce the notion of \((\omega, m)\)-subharmonic functions following Blocki’s ideas [Bl05] (see also [DK11]). Using classical techniques for plurisubharmonic functions we obtain similar results.
2.1. **Elementary symmetric functions.** First, we recall some basic properties of elementary symmetric functions (see [Bl05], [CW01], [Ga59]). We use the notations in [Bl05]. Let $S_k$, $k = 1, \ldots, n$ be the $k$-elementary symmetric function, that is, for $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$,

$$S_k(\lambda) = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_k}.$$ 

Let $\Gamma_k$ denote the closure of the connected component of $\{S_k(\lambda) > 0\}$ containing $(1, \ldots, 1)$. It is easy to show that

$$\Gamma_k = \{ \lambda \in \mathbb{R}^n \mid S_j(\lambda) \geq 0, \forall 1 \leq j \leq k \}.$$

We have an obvious inclusion $\Gamma_n \subset \ldots \subset \Gamma_1$. By Gårding [Ga59] the set $\Gamma_k$ is a convex cone in $\mathbb{R}^n$ and $S_k^{1/k}$ is concave on $\Gamma_k$.

Let $H$ denote the vector space (over $\mathbb{R}$) of complex hermitian $n \times n$ matrices. For $A \in H$ we set $\tilde{S}_k(A) = S_k(\lambda(A))$, where $\lambda(A) \in \mathbb{R}^n$ are the $n$ eigenvalues of $A$. The function $\tilde{S}_k$ can also be defined as the sum of all principal minors of order $k$,

$$\tilde{S}_k(A) = \sum_{|I|=k} A_{II}.$$ 

From the latter we see that $\tilde{S}_k$ is a homogeneous polynomial of order $k$ on $H$ which is hyperbolic with respect to the identity matrix $I$ (that is for every $A \in \tilde{S}$ the equation $\tilde{S}_k(A + tI) = 0$ has $n$ real roots; see [Ga59]). As in [Ga59] (see also [Bl05]), the cone

$$\tilde{\Gamma}_k := \{ A \in H \mid \tilde{S}_k(A + tI) \geq 0, \forall t \geq 0 \} = \{ A \in H \mid \lambda(A) \in \Gamma_k \}$$

is convex and the function $\tilde{S}_k^{1/k}$ is concave on $\tilde{\Gamma}_k$.

2.2. **$\omega$-subharmonic functions.** In this section, we consider $\Omega \subset X$ an open subset contained in a local chart.

**Definition 2.1.** A function $u \in L^1(\Omega)$ is called weakly $\omega$-subharmonic if

$$dd^c u \wedge \omega^{n-1} \geq 0,$$

in the weak sense of currents.

Thanks to Littman [Lit63] we have the following approximation properties.

**Proposition 2.2.** Let $u$ be a weakly $\omega$-subharmonic function in $\Omega$. Then there exists a one parameter family of functions $u_h$ with the following properties: for every compact subset $\Omega' \subset \Omega$

a) $u_h$ is smooth in $\Omega'$ for $h$ sufficiently large,

b) $dd^c u_h \wedge \omega^{n-1} \geq 0$ in $\Omega'$,

c) $u_h$ is non increasing with increasing $h$, and $\lim_{h \to \infty} u_h(x) = u(x)$ almost every where in $\Omega'$,
d) $u_h$ is given explicitly as

$$u_h(y) = \int_{\Omega} K_h(x,y)u(x)dx,$$

where $K_h$ is a smooth non negative function and

$$\int_{\Omega} K_h(x,y)dy \to 1,$$

uniformly in $x \in \Omega'$.

**Definition 2.3.** A function $u$ is called $\omega$-subharmonic if it is weakly $\omega$-subharmonic and for every $\Omega' \Subset \Omega$, $\lim_{h \to \infty} u_h(x) = u(x), \forall x \in \Omega'$, where $u_h$ is constructed as in Proposition 2.2.

**Remark 2.4.** Any continuous weakly $\omega$-subharmonic function is $\omega$-subharmonic.

If $(u_j)$ is a sequence of continuous $\omega$-subharmonic functions decreasing to $u$ and if $u \neq -\infty$ then $u$ is $\omega$-subharmonic.

Let $(u_j)$ be a sequence of $\omega$-subharmonic functions and $(u_j)$ is uniformly bounded from above. Then $u := (\limsup_j u_j)^*$ is $\omega$-subharmonic. Where for a function $v$, $v^*$ denotes the upper semicontinuous regularization of $v$.

The following Hartogs lemma can be proved in the same way as in the case of subharmonic functions.

**Lemma 2.5.** Let $u_t(x), t > 0$ be a family of non positive $\omega$-subharmonic functions in $\Omega$ and $u_t$ is uniformly bounded in $L^1_{\text{loc}}(\Omega)$. Suppose that for compact subset $K$ in $\Omega$ there exists a constant $C$ such that $v(x) = [\limsup_t u_t(x)]^* \leq C$ on $K$. Then for every $\epsilon > 0$, there exists $T_\epsilon$ such that $u_t(x) \leq C + \epsilon$ for $t \geq T_\epsilon$ and $x \in K$.

2.3. $(\omega, m)$-subharmonic functions. We associate real (1,1)-forms $\alpha$ in $\mathbb{C}^n$ with hermitian matrices $[a_{j,k}]$ by

$$\alpha = \frac{i}{\pi} \sum_{j,k} a_{j,k} dz_j \wedge d\bar{z}_k.$$

Then the canonical Kähler form $\beta$ is associated with the identity matrix $I$. It is easy to see that

$$\binom{n}{k} \alpha^k \wedge \beta^{n-k} = \tilde{S}_k(A) \beta^n.$$

**Definition 2.6.** Let $\alpha$ be a real $(1, 1)$-form on $X$. We say that $\alpha$ is $(\omega, m)$-positive at a given point $P \in X$ if at this point we have

$$\alpha^k \wedge \omega^{n-k} \geq 0, \quad \forall k = 1, \ldots, m.$$

We say that $\alpha$ is $(\omega, m)$-positive if it is $(\omega, m)$-positive at any point of $X$.

**Remark 2.7.** Locally at $P \in X$ with local coordinates $z_1, \ldots, z_n$, we have

$$\alpha = \frac{i}{\pi} \sum_{j,k} \alpha_{j,k} dz_j \wedge d\bar{z}_k,$$

and

$$\omega = \frac{i}{\pi} \sum_{j,k} g_{j,k} dz_j \wedge d\bar{z}_k.$$
Then $\alpha$ is $(\omega, m)$-positive at $P$ if and only if the eigenvalues $\lambda(g^{-1}\alpha) = (\lambda_1, \ldots, \lambda_n)$ of the matrix $\alpha_{j,k}(P)$ with respect to the matrix $g_{j,k}(P)$ is in $\Gamma_m$. These eigenvalues are independent of any choice of local coordinates.

We can show easily the following result:

**Proposition 2.8.** Let $\alpha \in A^{1,1}(X)$ be a real $(1,1)$-form on $X$. Then $\alpha$ is $(\omega, m)$-positive if and only if
\[
\alpha \wedge \beta_1 \wedge \ldots \wedge \beta_{m-1} \wedge \omega^{n-m} \geq 0,
\]
for all $(\omega, m)$-positive forms $\beta_1, \ldots, \beta_{m-1}$.

**Definition 2.9.** A current $T$ of bidegree $(p, p)$ is said to be $(\omega, m)$-positive if
\[
\alpha_1 \wedge \ldots \wedge \alpha_{n-p} \wedge T \geq 0,
\]
for all smooth $(\omega, m)$-positive $(1,1)$-forms $\alpha_i$.

Following Blocki \cite{Bl05} we can define $(\omega, m)$-subharmonicity for non-smooth functions.

**Definition 2.10.** A function $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ is called $(\omega, m)$-subharmonic if the following conditions hold

(i) In any local chart $\Omega$, given $\rho$ the local potential of $\omega$ and set $u := \rho + \varphi$, then $u$ is $\omega$-subharmonic,

(ii) for every smooth $(\omega, m)$-positive forms $\beta_1, \ldots, \beta_{m-1}$ we have, in the weak sense of distributions,
\[
(\omega + dd^c \varphi) \wedge \beta_1 \wedge \ldots \wedge \beta_{m-1} \wedge \omega^{n-m} \geq 0.
\]

Let $SH_m(X, \omega)$ be the set of all $(\omega, m)$-subharmonic functions on $X$. Observe that, by definition, any $\varphi \in SH_m(X, \omega)$ is upper semicontinuous.

The following properties of $(\omega, m)$-subharmonic functions are easy to show.

**Proposition 2.11.** (i) A $C^2$ function $\varphi$ is $(\omega, m)$-subharmonic if and only if the form $(\omega + dd^c \varphi)$ is $(\omega, m)$-positive or equivalently
\[
(\omega + dd^c \varphi) \wedge (\omega + dd^c u_1) \wedge \ldots \wedge (\omega + dd^c u_{m-1}) \wedge \omega^{n-m} \geq 0,
\]
for all $C^2$ $(\omega, m)$-subharmonic functions $u_1, \ldots, u_{m-1}$.

(ii) If $\varphi, \psi \in SH_m(X, \omega)$ then $\max(\varphi, \psi) \in SH_m(X, \omega)$.

(iii) If $\varphi, \psi \in SH_m(X, \omega)$ and $\lambda \in [0, 1]$ then $\lambda \varphi + (1 - \lambda) \psi \in SH_m(X, \omega)$.

Thanks to Hartogs Lemma\cite{GZ01} the following proposition can be proved in the same way as in the case of $\omega$-plurisubharmonic function (see \cite{GZ03}).

**Proposition 2.12.** Let $(\varphi_j)$ be a sequence of functions in $SH_m(X, \omega)$.

(i) If $(\varphi_j)$ is uniformly bounded from above on $X$, then either $\varphi_j$ converges uniformly to $-\infty$ or the sequence $(\varphi_j)$ is relatively compact in $L^1(X)$.

(ii) If $\varphi_j \to \varphi$ in $L^1(X)$ then
\[
\sup_X \varphi = \lim_j \sup_X \varphi_j.
\]

The compactness result can be deduced easily from Proposition 2.12.
Lemma 2.13. There exists a constant $C_0 > 0$ such that for all $\varphi \in SH_m(X, \omega)$ satisfying $\sup_X \varphi = 0$ we have
\[ \int_X \varphi \omega^n \geq -C_0. \]

It then follows that
\[ C := \{ \varphi \in SH_m(X, \omega) \mid \sup_X \varphi \leq 0; \int_X \varphi \omega^n \geq -C_0 \} \]
is a convex compact subset of $L^1(X)$.

2.4. Non degenerate complex Hessian equations. We summarize here some recent results on the non degenerate complex Hessian equation on compact Kähler manifolds,
\[
(\omega + dd^c \varphi)^m \wedge \omega^{n-m} = f \omega^n,
\]
where $0 < f$ is smooth such that
\[
\int_X f \omega^n = \int_X \omega^n.
\]

The following existence result was solved by Dinew and Kolodziej:

Theorem 2.14. [DK12] If $(X, \omega)$ is a compact Kähler manifold and $0 < f \in C^\infty(X)$ satisfies (2.2) then equation (2.1) has a unique (up to an additive constant) smooth solution.

This result was known to hold when $(X, \omega)$ has non negative holomorphic bisectional curvature [H09, Jb10].

The complex Hessian equation in domains of $\mathbb{C}^n$, i.e. equations of the form
\[
(dd^c u)^m \wedge \beta^{n-m} = f \beta^n,
\]
where $\beta$ is the canonical Kähler form in $\mathbb{C}^n$, was considered by Li [Li04] and Blocki [Bl05]. Existence and uniqueness of smooth solution to the Dirichlet problem in smoothly bounded domains with $(m-1)$-pseudocovex boundary was obtained in [Li04]. In [Bl05], a potential theory for $m$-subharmonic functions was developed and the corresponding degenerate Dirichlet problem was solved. Recently, Sadullaev and Abdullaev studied capacities and polar sets for $m$-subharmonic functions [SA12]. Note that the corresponding problem when $\beta$ is not the euclidean Kähler form is fully open.

It is important to mention that the study of real Hessian equations is a classical subject which has been developed previously in many papers, for example [CNS85, CW01, ITW04, Kr95, La02, Tr95, TW99, Ur01, W09].

3. Complex Hessian operators.

One of the key points in pluripotential theory is the smooth approximation which holds for quasi plurisubharmonic functions [BK07, De82]. Such a result for $(\omega, m)$-subharmonic functions seems to be very difficult. To overcome this difficulty we work in an (a priori) restrictive class which is defined by means of uniform convergence with respect to capacity.
3.1. Capacity.

**Definition 3.1.** Let \( E \in X \) be a Borel subset. We define the inner \((\omega, m)\)-capacity of \( E \) by

\[
\cap_{\omega,m}(E) := \sup \left\{ \int_E \omega^m \wedge \omega^{n-m} \mid \varphi \in SH_m(X,\omega) \cap C^2(X), 0 \leq \varphi \leq 1 \right\}.
\]

The outer \((\omega, m)\)-capacity of \( E \) is defined to be

\[
Cap_{\omega,m}(E) := \inf \left\{ \cap_{\omega,m}(U) \mid E \subset U, \text{ } U \text{ is an open subset of } X \right\}.
\]

It follows directly from the definition that \(Cap_{\omega,m}\) is monotone and \(\sigma\)-sub-additive.

Observe that if \( \varphi \in SH_m(X,\omega) \cap C^2(X) \), \( 0 \leq \varphi \leq M \) then, for any Borel subset \( E \subset X \),

\[
(3.1) \quad \int_E \omega^m \wedge \omega^{n-m} \leq M^m \cap_{\omega,m}(E).
\]

**Definition 3.2.** A sequence \((\varphi_j)\) converges in \(\cap_{\omega,m}\) to \(\varphi\) if for any \( \delta > 0 \) we have

\[
\lim_{j \to \infty} \cap_{\omega,m}(|\varphi_j - \varphi| > \delta) = 0.
\]

**Definition 3.3.** A sequence of functions \((\varphi_j)\) converges quasi-uniformly to \(\varphi\) on \(X\) (w.r.t \(Cap_{\omega,m}\)) if for every \( \epsilon > 0 \) there exists an open subset \( U \subset X \) such that \(Cap_{\omega,m}(U) \leq \epsilon\) and \(\varphi_j\) converges uniformly to \(\varphi\) in \(X \setminus U\).

This convergence is almost equivalent to the convergence in capacity as the following result shows.

**Proposition 3.4.** (i) If \(\varphi_j\) converges quasi-uniformly to \(\varphi\), then for each \( \delta > 0 \),

\[
\lim_{j \to \infty} Cap_{\omega,m}(|\varphi_j - \varphi| > \delta) = 0.
\]

(ii) Conversely, assume that \((\varphi_j)\) is a sequence of functions and \(\varphi\) is a function such that, for every \( \delta > 0 \),

\[
\lim_{j \to \infty} Cap_{\omega,m}(|\varphi_j - \varphi| > \delta) = 0.
\]

Then there exists a subsequence \((\varphi_{j_k})\) converging quasi-uniformly to \(\varphi\).

**Proof.** The first part is obvious, so we only prove the second part. We can find a subsequence (and for convenience we still denote it by \((\varphi_j)\)) such that

\[
Cap_{\omega,m}(|\varphi_j - \varphi| > 1/j) \leq 2^{-j}, \forall j.
\]

For each \( j \), let \( U_j \) be an open subset of \( X \) such that \( (|\varphi_j - \varphi| > 1/j) \subset U_j \) and \( Cap_{\omega,m}(U_j) \leq 2^{-j+1} \). Then for each \( \epsilon > 0 \), we can find \( k \in \mathbb{N} \) such that \( \cup_{j \geq k} U_j \) is the open subset of \( Cap_{\omega,m} \) less than \( \epsilon \) and \(\varphi_j\) converges uniformly to \(\varphi\) on its complement. \(\Box\)

**Definition 3.5.** We denote \(\mathcal{P}_m(X,\omega)\) the set of all functions \(\varphi \in SH_m(X,\omega)\) such that there exists a sequence of \(C^2\), \((\omega, m)\)-subharmonic functions \((\varphi_j)\) converging quasi-uniformly to \(\varphi\) on \(X\). Equivalently, we can replace quasi-uniform convergence by convergence in Capacity thanks to Proposition 3.4.
Proposition 3.6. (i) Any \( \varphi \in \mathcal{P}_m(X, \omega) \) is quasi continuous, that means, for any \( \epsilon > 0 \) there exists an open subset \( U \subset X \) of \( \text{Cap}_{\omega,m} \) less than \( \epsilon \) such that \( \varphi \) is continuous on \( X \setminus U \).
(ii) If \( \varphi_j \downarrow \varphi \) in \( \mathcal{P}_m(X, \omega) \) then \( (\varphi_j) \) converges quasi-uniformly to \( \varphi \).

Proof. The first statement follows directly from the definition. From (i), for each \( \epsilon > 0 \), there exists an open subset \( U \) of \( \text{cap}_{\omega,m} \) less than \( \epsilon \) such that \( \varphi_j, \varphi \) are continuous on \( X \setminus U \) which is compact. By Dini’s Theorem, \( \varphi_j \) converges uniformly to \( \varphi \) on \( X \setminus U \). \( \square \)

We have obvious inclusions
\[
\text{SH}_m(X, \omega) \cap C^2(X) \subset \mathcal{P}_m(X, \omega) \subset \text{SH}_m(X, \omega),
\]
and
\[
\text{PSH}(X, \omega) \subset \mathcal{P}_m(X, \omega).
\]

Remark 3.7. Quasi-uniform convergence implies convergence point wise outside a subset of \( \text{Cap}_{\omega,m} \) zero. Moreover, if \( \varphi_j \) is uniformly bounded and converges quasi-uniformly to \( \varphi \), then we have convergence in \( L^p \) for every \( p > 0 \). Indeed, for any \( \epsilon > 0 \) and an open subset \( U \) as in definition 3.3, we have
\[
\int_X |\varphi_j - \varphi|^p \omega^n \leq \int_{X \setminus U} |\varphi_j - \varphi|^p \omega^n + \int_U |\varphi_j - \varphi|^p \omega^n
\]
\[
\leq \int_{X \setminus U} |\varphi_j - \varphi|^p \omega^n + \sup_{X,j} |\varphi_j - \varphi|^p \cap \text{cap}_{\omega,m}(U)
\]
\[\leq \int_{X \setminus U} |\varphi_j - \varphi|^p \omega^n + C \epsilon.
\]
Taking the limsup over \( j \) and then letting \( \epsilon \to 0 \) we obtain
\[
\limsup_j \|\varphi_j - \varphi\|_p = 0.
\]

Lemma 3.8. If \( \varphi, \psi \) belong to the class \( \mathcal{P}_m(X, \omega) \) then so does \( \max(\varphi, \psi) \).

Proof. Let \( (\varphi_j), (\psi_j) \) be uniformly bounded sequences of functions in \( \text{SH}_m(X, \omega) \cap C^2(X) \) converging quasi-uniformly to \( \varphi, \psi \) respectively. Set
\[
u := \max(\varphi, \psi); \quad u_j := \max(\varphi_j, \psi_j); \quad v_j := \frac{1}{j} \log(e^{\varphi_j} + e^{\psi_j}).
\]
For each \( \epsilon > 0 \) there exists an open subset \( U \) of \( \text{cap}_{\omega,m} \) less than \( \epsilon \) and \( \varphi_j, \psi_j \) converges uniformly on \( X \setminus U \) to \( \varphi, \psi \) respectively. Since \( u_j \leq v_j \leq \log(2)/j + u_j \) and \( u_j \) converges uniformly to \( u \) on \( X \setminus U \) we deduce that \( v_j \) converges uniformly to \( u \) on \( X \setminus U \). \( \square \)

3.2. Hessian measure. In this section we define complex Hessian measure for functions in \( \text{SH}_m(X, \omega) \) which can be approximated in \( \text{Cap}_{\omega,m} \) by \( C^2 \)-functions in \( \text{SH}_m(X, \omega) \). In particular, for functions in \( \mathcal{P}_m(X, \omega) \cap L^\infty(X) \) this notion of Hessian measure can be defined by Bedford-Taylor’s method.

Theorem 3.9. Let \( \varphi \in \text{SH}_m(X, \omega) \) such that there exists a uniformly bounded sequence \( (\varphi_j) \) of \( C^2(\omega, m) \)-subharmonic functions converging in \( \text{cap}_{\omega,m} \) to \( \varphi \). Then the sequence of measures
\[
H_m(\varphi_j) := (\omega + dd^c \varphi_j)^m \wedge \omega^{n-m}
\]
converges (weakly in the sense of measures) to a positive Radon measure $\mu$. Moreover, the measure $\mu$ does not depend on the choice of the approximating sequence $(\varphi_j)$. We define the Hessian measure of $\varphi$ to be $H_m(\varphi) := \mu$.

**Proof.** Since all the measures $H_m(\varphi_j)$ have uniformly bounded mass (which is $\int_X \omega^n$), they stay in a weakly compact subset. It suffices to show that all accumulation points of this sequence are just the same. To do this it is enough to show that for every test function $\chi \in C^\infty(X)$,

$$\lim_{j,k \to \infty} \int_X \chi[H_m(\varphi_j) - H_m(\varphi_k)] = 0.$$ 

By integration by part formula we have

$$\int_X \chi[H_m(\varphi_j) - H_m(\varphi_k)] = \int_X \chi dd^c(\varphi_j - \varphi_k) \land T$$

where

$$T = \sum_{l=0}^{m-1} (\omega + dd^c \varphi_j)^l \land (\omega + dd^c \varphi_k)^{m-1-l} \land \omega^{n-m}.$$ 

Fix $\epsilon > 0$, and set $U = U(j, k, \epsilon) = \{ |\varphi_j - \varphi_k| \geq \epsilon \}$. By $C$ we will denote a constant that does not depend on $j, k, \epsilon$. Then by (3.2) and (3.1) there exists $C > 0$ such that

$$\left| \int_X \chi[H_m(\varphi_j) - H_m(\varphi_k)] \right| \leq \int_U |\varphi_j - \varphi_k| C_\omega \land T + \int_{X \setminus U} |\varphi_j - \varphi_k| C_\omega \land T$$

$$\leq C \text{cap}_{\omega,m}(U) + C\epsilon \sup_{X \setminus U} |\varphi_j - \varphi_k| \int_{X \setminus U} \omega \land T.$$ 

Now, it follows that

$$\lim_{j,k \to \infty} \sup \left| \int_X \chi[H_m(\varphi_j) - H_m(\varphi_k)] \right| \leq C\epsilon.$$ 

The result follows by letting $\epsilon \downarrow 0$. For the independence in the choice of the sequence it is enough to repeat the above arguments. $\square$

**Lemma 3.10.** Let $U \subset X$ be an open subset and $\varphi$ be a bounded function in $P_m(X, \omega)$. Then

$$\int_U H_m(\varphi) \leq 2(\sup_X |\varphi| + 1) \text{cap}_{\omega,m}(U).$$ 

**Proof.** Let $\varphi_j$ be a sequence of $C^2$ functions in $SH_m(X, \omega)$ converging quasi uniformly to $\varphi$. We can assume that

$$-\sup_X |\varphi| - 1 \leq \varphi_j \leq \sup_X |\varphi| + 1, \forall j.$$ 

Then

$$\int_U H_m(\varphi) \leq \liminf_{j \to +\infty} \int_U H_m(\varphi_j) \leq 2(\sup_X |\varphi| + 1) \text{cap}_{\omega,m}(U).$$ 

$\square$
For functions in $\mathcal{P}_m(X, \omega) \cap L^\infty(X)$ we can also define the Hessian measure in a weak sense following Bedford-Taylor method.

**Lemma 3.11.** Let $\varphi_1, \varphi_2 \in \mathcal{P}_m(X, \omega) \cap L^\infty(X)$. Then the current $\omega_{\varphi_1} \wedge \omega_{\varphi_2} \wedge \omega^{n-m}$ is well defined in the weak sense (Bedford-Taylor), symmetric and $\omega, m$-positive. Then we can define inductively the Hessian measure of $\varphi \in \mathcal{P}_m(X, \omega) \cap L^\infty(X)$,

$$H_m(\varphi) := (\omega + dd^c \varphi)^m \wedge \omega^{n-m}.$$  

Moreover, this definition coincides with the one in Theorem 3.9.

**Proof.** It follows from definition of $(\omega, m)$-subharmonic functions that $T_1 = (\omega + dd^c \varphi_1) \wedge \omega^{n-m}$ is a $(\omega, m)$-positive current. If $\varphi_2 \in \mathcal{P}_m(X, \omega) \cap L^\infty(X)$ then $dd^c \varphi_2 \wedge T_1$ is the current defined by

$$dd^c \varphi_2 \wedge T_1 = dd^c (\varphi_2 T_1).$$

We denote by $T_2 = \omega_{\varphi_1} \wedge \omega_{\varphi_2} \wedge \omega^{n-m}$. Since $\varphi_1, \varphi_2$ are in $\mathcal{P}_m(X, \omega) \cap L^\infty(X)$, there exist uniformly bounded sequences $(\varphi'_1), (\varphi'_2)$ in $\mathcal{P}_m(X, \omega) \cap C^2(X)$ converging quasi-uniformly to $\varphi_1, \varphi_2$ respectively. The sequence of currents $T'_2 = \omega_{\varphi'_1} \wedge \omega_{\varphi'_2} \wedge \omega^{n-m}$ converges to $T_2$ and hence $T_2$ is $(\omega, m)$-positive and

$$\omega_{\varphi_1} \wedge \omega_{\varphi_2} \wedge \omega^{n-m} = \omega_{\varphi_2} \wedge \omega_{\varphi_1} \wedge \omega^{n-m}.$$  

To prove that $T'_2$ converges to $T_2$, let us choose some test form $\chi$ and prove the following convergence

$$\lim_{j \to \infty} \int_X \chi \wedge dd^c (\varphi'_2 - \varphi_2) \wedge T_1 = 0.$$  

We have

$$\left| \int_X \chi \wedge dd^c (\varphi'_2 - \varphi_2) \wedge T_1 \right| = \left| \int_X (\varphi'_2 - \varphi_2)(dd^c \chi \wedge T_1) \right| \leq C \int_X |\varphi'_2 - \varphi_2| \omega_{\varphi_1} \wedge \omega^{n-1},$$  

where the constant $C$ depends only on $\chi, \omega$. Now (3.3) follows from the last inequality in view of

$$\int_U \omega_{\varphi_1} \wedge \omega^{n-1} \leq C \text{cap}_{\omega, m}(U),$$  

for every open subset $U \subset X$.  

We can prove inductively that the current

$$T_k = \omega_{\varphi_1} \wedge ... \wedge \omega_{\varphi_k} \wedge \omega^{n-m}$$

is well-defined, symmetric, $(\omega, m)$-positive, for each $k \leq m$ and $\varphi_i \in \mathcal{P}_m(X, \omega) \cap L^\infty(X)$. The Hessian measure of $\varphi \in \mathcal{P}_m(X, \omega) \cap L^\infty(X)$ is defined in this way

$$H_m(\varphi) = \omega_{\varphi} \wedge ... \wedge \omega_{\varphi} \wedge \omega^{n-m}.$$  

Now, given $\varphi \in \mathcal{P}_m(X, \omega) \cap L^\infty(X)$, it is easy to see that the Hessian measure of $\varphi$ defined by the above construction coincides with the Hessian measure $H_m(\varphi)$ defined in Theorem 3.9.
3.3. Some Convergence results. In this section we state some convergence results and the comparison principle for functions in $\mathcal{P}_m(X, \omega) \cap L^\infty(X)$. The proofs are nearly the same as for the Monge-Ampère operator and hence are omitted.

**Proposition 3.12.** Let $(\varphi_j^1), ..., (\varphi_j^n)$ be uniformly bounded sequence of functions in $\mathcal{P}_m(X, \omega) \cap L^\infty(X)$ converging quasi-uniformly to $\varphi^1, ..., \varphi^n$ respectively. Assume that $(f_j)$ is a uniformly bounded sequence of quasi continuous functions converging quasi uniformly to $\varphi$. Then we have the weak convergence of measures

$$f_j \omega_{\varphi_j^1} \wedge ... \wedge \omega_{\varphi_j^n} \wedge \omega^{n-m} \to f \omega_{\varphi^1} \wedge ... \wedge \omega_{\varphi^n} \wedge \omega^{n-m}.$$  

**Proof.** Thanks to Lemma 3.10 we can follow the lines in [Kol05]. □

The integration by parts formula is valid for $C^2$ functions (by Stokes). By Proposition 3.12 we see that it is also valid for functions in $\mathcal{P}_m(X, \omega) \cap L^\infty(X)$.

**Theorem 3.13** (Integration by parts). Let $\varphi, \psi \in \mathcal{P}_m(X, \omega) \cap L^\infty(X)$ and $T$ be a current of the form

$$T = \omega_{\varphi_1} \wedge ... \wedge \omega_{\varphi_{m-1}} \wedge \omega^{n-m},$$  

with $\varphi_i \in \mathcal{P}_m(X, \omega) \cap L^\infty(X)$. Then

$$\int_X \varphi d\psi \wedge T = \int_X \psi d\varphi \wedge T.$$  

The maximum principle for functions in $\mathcal{P}_m(X, \omega)$ can be proved by the same way as in the classical case.

**Theorem 3.14** (Maximum principle). If $\varphi, \psi$ be two functions in $\mathcal{P}_m(X, \omega) \cap L^\infty(X)$ then

$$\mathbb{I}_{\{\varphi > \psi\}} H_m(\max(\varphi, \psi)) = \mathbb{I}_{\{\varphi > \psi\}} H_m(\varphi).$$

From Theorem 3.14 we easily get

**Corollary 3.15** (Comparison principle). If $\varphi, \psi \in \mathcal{P}_m(X, \omega) \cap L^\infty(X)$ then

$$\int_{\{\varphi > \psi\}} H_m(\varphi) \leq \int_{\{\varphi > \psi\}} H_m(\psi).$$

**Lemma 3.16.** Let $\varphi, \psi$ be two non positive functions in $\mathcal{P}_m(X, \omega) \cap L^\infty(X)$. If $s > 0$ and $0 < t < 1$ then we have

$$(3.4) \quad t^m \text{Cap}_{\omega, m}(\varphi - \psi < -t - s) \leq (1 + M)^m \int_{\{\varphi - \psi < -s\}} H_m(\varphi),$$

where $M = \|\psi\|_{L^\infty(X)}$.

**Proof.** We can assume that $\psi$ is continuous on $X$. For the general case we can approximate $\psi$ quasi-uniformly by sequence of $C^2$ functions in $\text{SH}_m(X, \omega)$. In [3.3] we can replace Cap_{\omega, m} by cap_{\omega, m} since they coincide on open sets. Now, it suffices to repeat the arguments in [ECZ09]. □

**Proposition 3.17** (Chern-Levine-Nirenberg inequality). Let $T$ be any current of the form $T = \omega_{u_1} \wedge ... \wedge \omega_{u_{m-1}} \wedge \omega^{n-m}$ with $u_1, ..., u_{m-1} \in \mathcal{P}_m(X, \omega) \cap L^\infty(X)$, and $\varphi, \psi$ be two functions in $\mathcal{P}_m(X, \omega) \cap L^\infty(X)$. Then

$$(3.5) \quad \int_X |\psi| \omega_{\varphi} \wedge T \leq \int_X |\psi| T \wedge \omega + \left(2 \sup_X |\psi| + \inf_X \varphi - \inf_X \varphi\right) \int_X \omega^n.$$
Proof. The proof is nearly the same as in [GZ05] and is omitted. □

Applying (3.5) for $T_i = \omega_i \wedge \omega^{n-m+i}$ for $i = m-1, \ldots, 0$ we obtain

**Corollary 3.18.** Let $\varphi, \psi$ be two functions in $\mathcal{P}_m(X, \omega)$ such that $0 \leq \varphi \leq 1$. Then
\[
\int_X |\psi| H_m(\varphi) \leq \int_X |\psi| \omega^n + m \left( 2 \sup_X |\psi| + 1 \right) \int_X \omega^n.
\]

Applying Corollary 3.18 we obtain:

**Corollary 3.19.** There exists a constant $C > 0$ such that for all $\psi \in \mathcal{P}_m(X, \omega)$ satisfying $\sup_X \psi = -1$ and for every $t > 0$ we have
\[
\text{Cap}_{\omega,m}(\psi < -t) \leq C/t.
\]

We end this section by showing that the class $\mathcal{P}_m(X, \omega)$ is stable under decreasing sequences.

**Proposition 3.20.** Let $(\varphi_j)$ be a decreasing sequence of functions in $\mathcal{P}_m(X, \omega)$ converging to $\varphi \neq -\infty$. Then $\varphi_j$ converges to $\varphi$ quasi-uniformly. In particular, $\varphi \in \mathcal{P}_m(X, \omega)$.

**Proof.** It is easy to see that $\varphi$ is $(\omega, m)$-subharmonic. It suffices to show that there exists a subsequence of $(\varphi_j)$ converging quasi-uniformly to $\varphi$.

In view of Corollary 3.19 we can assume that $\varphi$ is bounded.

Fix $k \in \mathbb{N}$. For each $j > k \in \mathbb{N}$, by applying Lemma 3.16 with $\varphi = \varphi_j, \psi = \varphi_k, s = t$ we obtain
\[
(3.6) \quad t^m \text{Cap}_{\omega,m}(\varphi_j - \varphi_k < -2t) \leq (1 + M)^m \int_{(\varphi_j - \varphi_k < -t)} H_m(\varphi_j) \leq (1 + M)^m \int_X (\varphi_k - \varphi_j) H_m(\varphi_j).
\]

After extracting a subsequence if necessary we can assume that $H_m(\varphi_j) \rightharpoonup \mu$ in the weak sense of measures. We apply Lemma 3.21 below to get
\[
\lim_{j \to +\infty} \int_X \varphi_j H_m(\varphi_j) = \int_X \varphi d\mu.
\]

From the quasi-continuity of the functions $\varphi_j, j \in \mathbb{N}$ and the $\sigma$-subadditivity of Cap$_{\omega,m}$ we deduce that for each fixed $\epsilon > 0$ there exists an open subset $U$ such that Cap$_{\omega,m}(U) < \epsilon$ and there exists a subsequence $(\tilde{\varphi}_j)$ of continuous functions on $X$ such that for any $j$, $\varphi_j = \tilde{\varphi}_j$ on $X \setminus U$.

From basic properties of Cap$_{\omega,m}$ we have
\[
(3.7) \quad t^m \text{Cap}_{\omega,m}(\varphi_j - \tilde{\varphi}_k < -2t) \leq t^m \text{Cap}_{\omega,m}(\varphi_j - \varphi_k < -2t) + t^m \epsilon \\
\leq \frac{(1 + M)^m}{t} \int_X (\varphi_k - \varphi_j) H_m(\varphi_j) + t^m \epsilon.
\]

Recall that cap$_{\omega,m}$ is continuous under increasing sequence. Note also that Cap$_{\omega,m}$ and cap$_{\omega,m}$ coincide on open sets. By taking the limit when $j \to +\infty$ in (3.7), we obtain
\[
t^m \text{Cap}_{\omega,m}(\varphi - \tilde{\varphi}_k < -2t) \leq \frac{(1 + M)^m}{t} \int_X (\varphi_k - \varphi) d\mu + t^m \epsilon.
\]
It follows that

\[ t^m \text{Cap}_{\omega,m}(\varphi - \varphi_k < -2t) \leq \frac{(1 + M)^m}{t} \int_X (\varphi_k - \varphi) d\mu + 2t^m \epsilon, \]

and hence,

\[ \lim_{k \to +\infty} \text{Cap}_{\omega,m}(\varphi - \varphi_k < -2t) = 0. \]

Now, by Proposition 3.4 there exists a subsequence of \((\varphi_j)\) converging quasi-uniformly to \(\varphi\). To complete the proof it remains to prove the following lemma. □

**Lemma 3.21.** Assume that \((\varphi_j)\) is a sequence in \(P_m(X,\omega)\) decreasing to \(\varphi \in L^\infty(X)\). If \(H_m(\varphi_j)\) converges weakly to \(\mu\) in the sense of measures then

\[ \lim_{j \to +\infty} \int_X \varphi_j H_m(\varphi_j) = \int_X \varphi d\mu. \]

**Proof.** We prove this lemma by induction. It obviously holds when \(m = 1\). Remark also that

\[ \lim \sup_{j \to +\infty} \int_X \varphi_j H_m(\varphi_j) \leq \int_X \varphi d\mu. \]

Thus, it suffices to prove that

\[ \lim \inf_{j \to +\infty} \int_X \varphi_j H_m(\varphi_j) \geq \int_X \varphi d\mu. \]

Fix \(k \in \mathbb{N}\). For each \(j > k\), By integration by parts we get

\[ \int_X \varphi_k [H_m(\varphi_k) - H_m(\varphi_j)] = \int_X \varphi_k dd^c(\varphi_k - \varphi_j) \wedge T \wedge \omega^{n-m} \]

\[ = \int_X (\varphi_k - \varphi_j) dd^c \varphi_k \wedge T \wedge \omega^{n-m} \]

\[ \geq - \int_X (\varphi_k - \varphi_j) T \wedge \omega^{n-m+1}, \]

where

\[ T = \sum_{p=0}^{m-1} (\omega + dd^c \varphi_k)^p \wedge (\omega + dd^c \varphi_j)^{m-1-p}. \]

By setting \(\psi_j = \frac{\varphi_k + \varphi_j}{2}\), we get

\[ T \wedge \omega^{n-m+1} \leq 2^{m-1} (\omega + dd^c \psi_j)^{m-1} \wedge \omega^{n-m+1}. \]

As a consequence, (3.8) yields

\[ \int_X \varphi_k [H_m(\varphi_k) - H_m(\varphi_j)] \geq -2^{m-1} \int_X (\varphi_k - \psi_j) H_{m-1}(\psi_j) \]

After extracting a subsequence if necessary, we can assume that \(H_{m-1}(\psi_j) \to \nu\) in the weak sense of measures. By letting \(j \to +\infty\) in (3.9), the induction hypothesis gives us

\[ \int_X \varphi_k [H_m(\varphi_k) - \mu] \geq -2^{m-1} \int_X (\varphi_k - \psi) d\nu. \]

We then infer that

\[ \lim \inf_{j \to +\infty} \int_X \varphi_j H_m(\varphi_j) \geq \int_X \varphi d\mu, \]

and the result follows. □
4. Stability results

In this section we use the volume-capacity estimate in [DK11] and mimic the arguments in [EGZ09] to prove stability results for the complex Hessian equation.

Using Blocki’s technique [Bl03] we obtain the following stability results.

**Theorem 4.1.** Let \( \varphi, \psi \in SH_m(X, \omega) \cap C^2(X, \omega) \), \( r \geq 2 \), and set \( \rho = \varphi - \psi \). Then

\[
\int_X |\rho|^{r-2} \rho \wedge d^\omega \rho \wedge \omega^{n-1} \leq C \left( \int_X |\rho|^{r-2} \rho(H_m(\psi) - H_m(\varphi)) \right)^{2^{1-m}},
\]

where \( C \) is a positive constant depending only on \( n, m, r \), and upper bounds of \( \|\varphi\|_{L^\infty(X)}, \|\psi\|_{L^\infty(X)} \), and \( \int_X \omega^n \).

From Theorem 4.1 and Corollary 3.2 we thus get

**Corollary 4.2.** Let \( \varphi, \psi \in P_m(X, \omega) \cap L^\infty(X) \), and set \( \rho = \varphi - \psi \). Then

\[
\int_X dp \wedge d^\omega \rho \wedge \omega^{n-1} \leq C \left( \int_X \rho(H_m(\psi) - H_m(\varphi)) \right)^{2^{1-m}},
\]

where \( C \) is a positive constant depending only on \( n, m, r \), and upper bounds of \( \|\varphi\|_{L^\infty(X)}, \|\psi\|_{L^\infty(X)} \), and \( \int_X \omega^n \).

Corollary 4.2 is useful to prove uniqueness results as we will see in the proof of Theorem A.

**Definition 4.3.** Let \( \alpha > 0, A > 0 \). A Borel measure \( \mu \) on \( X \) satisfies condition \( Q_m(\alpha, A, \omega) \) if for all Borel subsets \( K \) of \( X \),

\[
\mu(K) \leq ACap_{\omega,m}(K)^{1+\alpha}.
\]

**Proposition 4.4.** Let \( \mu \) be a Borel measure satisfying condition \( Q_m(\alpha, A, \omega) \). Suppose that \( \varphi \in P_m(X, \omega) \) solves \( H_m(\varphi) = \mu \), and \( \sup_X \varphi = -1 \). Then there exists a constant \( C = C(\alpha, A, \omega, n, m) \) such that

\[
\sup_X |\varphi| \leq C.
\]

**Sketch of proof.** Set

\[
f(s) := [\text{Cap}_{\omega,m}(\varphi < -s)]^{1/m}.
\]

Observe that \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) is right continuous, decreasing with \( \lim_{s \to +\infty} f = 0 \). Since \( \mu \) satisfies condition \( Q_m(\alpha, A, \omega) \), it follows from Lemma 3.16 applied to the function \( \psi \equiv 0 \) that \( f \) satisfies the condition in Lemma 2.4 in [EGZ09]. Moreover it follows from Corollary 3.2 that

\[
f(s) \leq C_1 s^{-1/m},
\]

for some constant \( C_1 \) which only depends on \( \omega \). Thus, by following the lines in [EGZ09], page 615, we have the desired uniform estimate.

**Theorem 4.5.** Suppose that \( \varphi, \psi \in P_m(X, \omega) \cap L^\infty(X) \) satisfy

\[
\sup_X \varphi = \sup_X \psi = -1.
\]

Assume that \( H_m(\varphi), H_m(\psi) \) satisfy condition \( Q_m(\alpha, A, \omega) \) for some \( \alpha, A > 0 \). Then there exists \( C = C(\alpha, A, \omega, \|\varphi\|_{L^\infty(X)}, \|\psi\|_{L^\infty(X)}) > 0 \) such that, for any \( \epsilon > 0 \),

\[
\sup_X (\psi - \varphi) \leq \epsilon + C[\text{Cap}_{\omega,m}(\varphi - \psi < -\epsilon)]^{\alpha/m}.
\]
Proof. The same as in [EGZ09], Proposition 2.6. □

The following Proposition is due to Kolodziej and Dinew [DK11 Proposition 2.1]. We include here a slightly different proof.

**Proposition 4.6.** [DK11] Let $1 < p < \frac{n}{n-m}$. There exists a constant $C = C(p, \omega)$ such that for every Borel subset $K$ of $X$, we have

$$V(K) \leq C \text{Cap}_{\omega,m}(K)^p,$$

where $V(K) := \int_K \omega^n$. Proof. Fix an open subset $U$ such that $K \subset U$. Solve the complex Monge-Ampère equation to find $u \geq 0$ such that $\omega^n_u = f \omega^n$ on $X$, with $f = V(U)^{-1} \chi_U$. From [BGZ08], Corollary 3.2, the solution $u$ is continuous and moreover, for each $r > 1$,

$$\sup_X u \leq C \|f\|_r^{1/n},$$

where the constant $C = C(r, \omega)$ does not depend on $K$. The inequality between mixed complex Monge-Ampère measures [Di09] tells us that

$$\omega_u^m \wedge \omega^{n-m} \geq f^{m/n} \omega^n.$$

Thus since $u \in \mathcal{P}_m(X, \omega) \cap L^\infty(X)$, we obtain

$$\text{Cap}_{\omega,m}(U) \geq (\sup_X u)^{-m} \int_U H_m(u) \geq (\sup_X u)^{-m} \int_U f^{m/n} \omega^n \geq C^{-m} V(U)^{1 - \frac{m}{n}}.$$

Thus, for every $r > 1$, there exists a constant $C$ not depending on $K$ such that $V(K) \leq C \text{Cap}_{\omega,m}(K)^{\frac{m}{n-r}}$. The proof is complete. □

As a consequence of Proposition 4.6 we have some examples of measures satisfying condition $Q_m(\alpha, A, \omega)$.

**Lemma 4.7.** Assume $\mu = f \omega^n$ is a Borel measure with $0 \leq f \in L^p(X)$ for some $p > n/m$. Then for any $0 < \alpha < \frac{mp-n}{n-mp}$ there exists $A_\alpha > 0$ such that $\mu$ satisfies $Q_m(\alpha, A, \omega)$.

The following stability theorem was established in [EGZ09] for the Monge-Ampère equation.

**Theorem 4.8.** Assume $H_m(\varphi) = f \omega^n$, $H_m(\psi) = g \omega^n$, where $\varphi, \psi \in \mathcal{P}_m(X, \omega) \cap C^0(X)$ and $f, g \in L^p(X)$ with $p > n/m$. Fix $r > 0$. Then if $\gamma$ small enough such that $\frac{\gamma m a}{r - \gamma(r + mq)} < \frac{mp-n}{n-mp}$, we have

$$\|\varphi - \psi\|_{L^\infty(X)} \leq C \|\varphi - \psi\|_{L^r(X)},$$

where $q = \frac{p}{p-1}$ denotes the conjugate exponent of $p$, and the constant $C$ depends only on $n, m, p, r$ and upper bounds of $\|f\|_p, \|g\|_p$.

Proof. Fix $\epsilon > 0$, and $\alpha > 0$ to be chosen later. It follows from Theorem 4.5 and Proposition 4.4.4 that

$$\|\varphi - \psi\|_{L^\infty(X)} \leq \epsilon + C_1 \text{Cap}_{\omega,m}(\|\varphi - \psi\| > \epsilon)^{\alpha/m}.$$

Applying Lemma 4.10 we see that

$$\text{Cap}_{\omega,m}(\|\varphi - \psi\| > \epsilon) \leq \frac{C_2}{\epsilon^{m+r/q}} \int_X |\varphi - \psi|^{r/q}(f + g) \omega^n.$$
It follows thus from Hölder’s inequality that
\[ \text{Cap}_{\omega, m}(|\varphi - \psi| > \epsilon) \leq \frac{C_3 \| f + g \|_p}{\epsilon^{m+r/q}} \| \varphi - \psi \|_{L^{r/q}}. \]
Choose \( \epsilon := \| \varphi - \psi \|_{L^2}^2 \). Then
\[ \text{Cap}_{\omega, m}(|\varphi - \psi| > \epsilon) \leq C_4 \|[\varphi - \psi]_{L^r}^{r/q - \gamma(m+r/q)}. \]
We infer that
\[ \| \varphi - \psi \|_{L^\infty(X)} \leq \| \varphi - \psi \|_{L^r(X)}^r + C_5 \| \varphi - \psi \|_{L^r(X)}^{\gamma'}, \]
where \( \gamma' = \frac{\alpha}{m}[r/q - \gamma(m+r/q)] \). We finally choose \( \alpha \) so that \( \gamma = \gamma' \): this yields the desired estimate. \( \square \)

5. Proof of the main results

5.1. Proof of Theorem A. We first prove the uniqueness. Suppose that \( \varphi \) and \( \psi \) are two continuous solutions of \((1.1)\). Set \( \rho := \varphi - \psi \). It follows from Corollary \[1.2\] that
\[ \int_X d\rho \wedge d^c \rho \wedge \omega^{n-1} \leq C \left( \int_X \rho(H_m(\psi) - H_m(\varphi)) \right)^{2^{1-m}}, \]
where \( C \) is a positive constant. Since \( F \) is non decreasing in the second variable, it follows from Stokes formula that
\[ 0 \leq \int_X \rho(H_m(\psi) - H_m(\varphi)) = \int_X (\varphi - \psi)(F(\cdot, \psi) - F(\cdot, \varphi))\omega^n \leq 0. \]
Thus,
\[ \int_X d\rho \wedge d^c \rho \wedge \omega^{n-1} = 0, \]
which implies that \( \rho \) is constant. If moreover \( t \mapsto F(x, t) \) is increasing for every \( x \in X \), it is easy to see that \( \rho = 0 \).

To prove the existence, we consider three cases.

Case 1: \( F \) does not depend on the second variable, \( F(x, t) = f(x), \forall x, t. \)
Take a sequence of smooth strictly positive functions \( (f_j) \) converging to \( f \) in \( L^p(X) \). We can assume that \( \int_X f_j \omega^n = \int_X \omega^n \), for every \( j \). We use the existence result in Theorem \[2.14\] to produce a sequence of smooth solutions \( (\varphi_j) \) normalized by \( \text{sup}_X \varphi_j = 0, \forall j \). By passing to a subsequence we can assume that \( (\varphi_j) \) converges in \( L^1(X) \). Since \( \| f_j \|_p \) is uniformly bounded, by Lemma \[4.7\] we can find \( \alpha, A \) which do not depend on \( j \) such that all the measures \( f_j \omega^n \) satisfy condition \( Q_m(\alpha, A, \omega) \). By Proposition \[4.4\] the sequence \( (\varphi_j) \) is uniformly bounded. Now it follows from Theorem \[1.8\] that \( \varphi_j \) converges uniformly to a continuous function \( \varphi \in \mathcal{P}_m(X, \omega) \) which solves equation \( H_m(\varphi) = f \omega^n \).

In the next two cases we will use the Schauder fixed point Theorem.

Case 2: There exists \( t_1 \in \mathbb{R} \) such that \( \int_X F(x, t_1)\omega^n > \int_X F(x, t_0)\omega^n \).
We set
\[ \mathcal{C} := \{ \varphi \in SH_m(X, \omega) \mid \int_X \varphi \omega^n \geq -C_0; \text{sup}_X \varphi \leq 0 \}, \]
where \( C_0 \) is the constant introduced in Lemma \[2.14\]. It follows that \( \mathcal{C} \) is a compact convex subset of \( L^1(X) \).
Take $\psi \in \mathcal{C}$, we use the result in case 1 to find $\varphi \in \mathcal{P}_m(X, \omega) \cap C^0(X)$ such that $\sup_X \varphi = 0$ and

$$H_m(\varphi) = F(., \psi + c_\psi)\omega^n,$$

where $c_\psi \geq t_0$ is a constant such that

$$\int_X F(., \psi + c_\psi)\omega^n = \int_X \omega^n.$$  (5.1)

This can be done because $F$ satisfies conditions (F2) and (F3). Indeed, by Fatou’s Lemma we have

$$\liminf_{t \to +\infty} \int_X F(., \psi + t)\omega^n \geq \int_X F(., \psi_1)\omega^n > \int_X \omega^n.$$

Moreover $\int_X F(., \psi + t_0)\omega^n \leq \int_X F(., \psi_0) = \int_X \omega^n$. Thus by continuity of $t \mapsto \int_X F(., \psi + t)\omega^n$ we can find $c_\psi$ satisfying (5.1). Observe that $\varphi$ is well-defined and does not depend on $c_\psi$. Indeed, assume that $c_1, c_2$ are two constants such that

$$\int_X F(., \psi + c_1)\omega^n = \int_X F(., \psi + c_2)\omega^n = \int_X \omega^n,$$

and $\varphi_1, \varphi_2$ are two continuous functions in $\mathcal{P}_m(X, \omega)$ such that

$$H_m(\varphi_1) = F(., \psi + c_1), \quad H_m(\varphi_2) = F(., \psi + c_2).$$

Since $t \mapsto F(x, t)$ is non decreasing for every $x \in X$, we have $F(., \psi + c_1) = F(., \psi + c_2)$ almost everywhere on $X$. Thus by the uniqueness result above, $\varphi_1 = \varphi_2 + c$ for some constant $c$ which must be 0 by the normalization. Then we define the map $\Phi : \mathcal{C} \to \mathcal{C}$, $\psi \mapsto \varphi$.

Now we prove that $\Phi$ is continuous on $\mathcal{C}$. Suppose that $(\psi_j)$ is a sequence in $\mathcal{C}$ converging to $\psi \in \mathcal{C}$ in $L^1(X)$ and let $\varphi_j = \Phi(\psi_j)$. We set $c_j := c_\psi$, and prove that $(c_j)$ is uniformly bounded. Suppose in the contrary that $c_j \uparrow +\infty$. By subtracting a subsequence if necessary we can assume that $\psi_j \to \psi$ almost everywhere in $X$. Then by Fatou’s lemma we have

$$\int_X \omega^n = \lim_{j \to +\infty} \int_X F(., \psi_j + c_j)\omega^n \geq \int_X F(., \psi_1)\omega^n,$$

which is impossible. Therefore the sequence $(c_j)$ is bounded. This implies that the sequence $(F(., \psi_j + c_j))_j$ is bounded in $L^p(X)$, for some $p > n/m$ which does not depend on $j$. To prove the continuity of $\Phi$ it suffices to show that any cluster point of $(\varphi_j)$ satisfies $\Phi(\psi) = \varphi$. Suppose that $\psi_j \to \varphi$ in $L^1(X)$. It follows from Theorem 1.31 that the sequence $(\varphi_j)$ is Cauchy in $C^0(X)$. Thus $(\varphi_j)$ converges to $\varphi$ in $C^0(X)$ and $\varphi \in \mathcal{P}_m(X, \omega) \cap C^0(X)$. By subtracting a subsequence if necessary we can assume that $\psi_j \to \psi$ almost everywhere on $X$ and $c_j \to c$. Since $t \mapsto F(x, t)$ is continuous we see that $F(., \psi_j + c_j) \to F(., \psi + c)$ almost everywhere. Thus $H_m(\varphi) = F(., \psi + c)$ which means $\Phi(\psi) = \varphi$ and hence $\Phi$ is continuous on $\mathcal{C}$.

By the Schauder fixed point Theorem, it follows that $\Phi$ has a fixed point in $\mathcal{C}$, say $\varphi$. By definition of $\Phi$, the function $\varphi$ must be in the class $\mathcal{P}_m(X, \omega) \cap C^0(X)$ and we have

$$H_m(\varphi) = F(., \psi + c_\psi)\omega^n.$$  

The function $\varphi + c_\varphi$ is the required solution.
Case 3: \( \int_X F(.,t)\omega^n = \int_X F(.,t_0)\omega^n, \forall t \geq t_0 \). In this case we have \( F(x,t) = F(x,t_0) \) for all \( t \geq t_0 \) and for almost \( x \in X \). Thus, for every \( t \geq t_0 \),

\[
\|F(.,t_0)\|_{L^p(X)} = \|F(.,t)\|_{L^p(X)}.
\]

From Proposition 4.4 we can find a positive constant \( C_1 \) such that for any \( \varphi \in \mathcal{P}_m(X,\omega) \cap \mathcal{C}^0(X) \) satisfying \( \sup_X \varphi = 0 \) and

\[
H_m(\varphi) = f \omega^n,
\]

with \( \|f\|_p \leq \|F(.,t_0)\|_p \) then

\[
\varphi \geq -C_1.
\]

We set

\[
C' := \{ \varphi \in SH_m(X,\omega) \mid -C_1 \leq \varphi \leq 0 \}.
\]

Then \( C' \) is a compact convex subset of \( L^1(X) \).

Take \( \psi \in C' \), we use the result in case 1 to find \( \varphi \in \mathcal{P}_m(X,\omega) \cap \mathcal{C}^0(X) \) such that \( \sup_X \varphi = 0 \) and

\[
H_m(\varphi) = F(.,\psi + c_\psi)\omega^n,
\]

where \( t_0 \leq c_\psi \leq t_0 + C_1 \) is a constant such that

\[
\int_X F(.,\psi + c_\psi)\omega^n = \int_X \omega^n.
\]

This can be done because \( F \) satisfies the condition (F2) and (F3). Indeed,

\[
\int_X F(.,\psi + t_0)\omega^n \leq \int_X \omega^n \leq \int_X F(.,\psi + t_0 + C_1)\omega^n.
\]

Thus by continuity we can find \( c_\psi \) as above. As in case 2, \( \varphi \) is well-defined and does not depend on the choice of \( c_\psi \). By the choice of \( C_1 \), we see that \( \varphi \in C' \). So, we can define a map \( \Phi : C' \rightarrow C' \) by setting \( \Phi(\psi) = \varphi \).

Now we prove that \( \Phi \) is continuous on \( C' \). Suppose that \( (\psi_j) \) is a sequence in \( C' \) converging to \( \psi \in C' \) in \( L^1(X) \) and let \( \varphi_j = \Phi(\psi_j) \). We set \( c_j := c_{\psi_j} \). For each \( j \in \mathbb{N} \),

\[
\int_X [F(.,\psi_j + c_j)]p\omega^n \leq \int_X [F(.,c_j)]p\omega^n = \int_X [F(.,t_0)]p\omega^n.
\]

Therefore, the sequence \( (F(.,\psi_j + c_j))_j \) is bounded in \( L^p(X) \).

As in case 2, we can assume that \( \varphi_j \rightarrow \varphi \) in \( L^1(X) \). It follows from Theorem 4.8 that the sequence \( (\varphi_j) \) is Cauchy in \( \mathcal{C}^0(X) \). Thus \( \varphi_j \) converges to \( \varphi \) in \( \mathcal{C}^0(X) \) and \( \varphi \in \mathcal{P}_m(X,\omega) \cap \mathcal{C}^0(X) \). By subtracting a subsequence if necessary we can assume that \( \psi_j \rightarrow \psi \) in \( L^1(X) \) and \( c_j \rightarrow c \). Then \( H_m(\varphi) = F(.,\psi + c) \) and \( \Phi(\psi) = \varphi \) which implies that \( \Psi \) is continuous on \( C' \).

By the Schauder fixed point Theorem, it follows that \( \Phi \) has a fixed point in \( C' \), say \( \varphi \). By definition of \( \Phi \), the function \( \varphi \) must be in the class \( \mathcal{P}_m(X,\omega) \cap \mathcal{C}^0(X) \) and we have

\[
H_m(\varphi) = F(.,\varphi + c_{\varphi})\omega^n.
\]

The function \( \varphi + c_{\varphi} \) is the required solution.
5.2. **Proof of Theorem B.** In this section we consider a special class of compact Kähler manifolds. We assume that \((X, \omega)\) is a rational homogeneous manifold. That means \(X = G/H\), where \(G\) is a complex semi-simple algebraic group and \(H\) is a parabolic subgroup. Let \(K\) be a maximal compact subgroup of \(G\). Then \(K\) acts transitively on \(X\). We assume moreover that \(\omega\) is fixed by action of \(K\). In this case we can regularize singular \((\omega, m)\)-subharmonic functions by using the group action which preserves the metric.

Let \(\varphi\) be a continuous \((\omega, m)\)-subharmonic function on \(X\). We consider the following regularizing sequence

\[ \varphi_\epsilon(x) := \int_K \varphi(g^{-1}x) \chi_\epsilon(g) dg, \]

where \(dg\) is the Haar measure on \(K\) and \(\chi_\epsilon\) are cut-off functions whose supports decreases to \(\{e\}\) (the identity of \(K\)), and \(\int_K \chi_\epsilon(g) dg = 1, \forall \epsilon > 0\).

It follows from \([G99], [Hu94]\) that \(\varphi_\epsilon\) is smooth for every \(\epsilon > 0\).

**Theorem 5.1.** Let \(\varphi\) be a continuous \((\omega, m)\)-subharmonic function on \(X\). Then for each \(\epsilon > 0\), \(\varphi_\epsilon\) is smooth \((\omega, m)\)-subharmonic and

\[ \lim_{\epsilon \to 0} \varphi_\epsilon = \varphi \]

uniformly on \(X\).

**Proof.** The uniform convergence always holds for continuous functions. Let us show the second assertion. Let \(\alpha_1, ..., \alpha_{m-1}\) be \((\omega, m)\)-positive closed \((1,1)\)-forms on \(X\), and denote (for short) \(\alpha = \alpha_1 \wedge ... \alpha_{m-1}\). Let \(\mathcal{L}_g\) denote the left action of \(g \in K\), i.e.

\[ \mathcal{L}_g(x) = g.x, \quad x \in X. \]

Then \(\mathcal{L}_g^*\alpha_j\) is also \((\omega, m)\)-positive for every \(j\), since \(\mathcal{L}_g^*\omega = \omega\), and

\[ \mathcal{L}_g^*\alpha_j \wedge \omega^{n-k} = \alpha_k \wedge \omega^{n-k}. \]

Fix a positive test function \(\psi\). We have

\[ \int_X \psi(\omega + dd^c \varphi) \wedge \alpha \wedge \omega^{n-m} = \int_X \psi \alpha \wedge \omega^{n-m+1} + \int_X \varphi \psi \wedge \alpha \wedge \omega^{n-m} \]

\[ = \int_X \psi \alpha \wedge \omega^{n-m+1} + \int_X \left( \int_K \mathcal{L}_g^* \varphi \chi_\epsilon(g) dg \right) \psi \wedge \alpha \wedge \omega^{n-m} \]

\[ = \int_K \left( \int_X \psi(\omega + dd^c \varphi) \wedge \alpha \wedge \omega^{n-m} \right) \chi_\epsilon(g) dg \]

\[ = \int_K \left( \int_X \psi(\omega + \mathcal{L}_g^* dd^c \varphi) \wedge \alpha \wedge \omega^{n-m} \right) \chi_\epsilon(g) dg \]

\[ = \int_K \left( \int_X \psi \mathcal{L}_g^* \left[ (\omega + dd^c \varphi) \wedge \mathcal{L}_g^* \alpha \wedge \omega^{n-m} \right] \right) \chi_\epsilon(g) dg \geq 0. \]

\(\Box\)

**Remark 5.2.** Thanks to Theorem 5.1 every continuous \((\omega, m)\)-subharmonic function belongs to \(\mathcal{P}_m(X, \omega)\).
Proof of Theorem B. Let $\varphi$ be the unique continuous solution to (1.1). For $h \in K$, let $\varphi_h(x) := \varphi(h,x)$, $x \in X$. If $u$ is smooth then

$$\|u_h - u\|_{L^2}^2 \leq C \text{dist}^2(h,e) \int_X (-u) dd^c u \wedge \omega^{n-1},$$

where $C$ is some universal constant. Then, it follows from the approximation theorem (Theorem 5.1) that

$$\|\varphi_h - \varphi\|_{L^2(X)} \leq C \text{dist}(h,e).$$

For fixed $h \in K$, observe that $\varphi_h$ is $(\omega, m)$-subharmonic and satisfies

$$H_m(\varphi_h) = F(h,x,\varphi(h,x)) \omega^n.$$

Thus, by applying Theorem 4.8 with $r = 2$ we obtain

$$\|\varphi_h - \varphi\|_{L^\infty(X)} \leq C', \|\varphi_h - \varphi\|_{L^2(X)}^\gamma,$$

where $0 < \gamma < \frac{2(mp-n)}{mnp + 2mp - 2n}$ is a given constant and $C' > 0$ is another constant which does not depend on $h$. We thus get

$$\|\varphi_h - \varphi\|_{L^\infty(X)} \leq C C' \text{dist}(h,e) \gamma, \forall h \in K.$$

This yields the $\gamma$-Hölder continuity of $\varphi$ (see [EGZ09]).

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