The classification of $n$-dimensional anticommutative algebras with $(n – 3)$-dimensional annihilator

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ABSTRACT
We give the classification of all $n$-dimensional anticommutative complex algebras with $(n – 3)$-dimensional annihilator. Namely, we describe all central extensions of all 3-dimensional anticommutative complex algebras.

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1. Introduction

Central extensions play an important role in quantum mechanics: one of the earlier encounters is by means of Wigner’s theorem which states that a symmetry of a quantum mechanical system determines an (anti-)unitary transformation of a Hilbert space.

Another area of physics where one encounters central extensions is the quantum theory of conserved currents of a Lagrangian. These currents span an algebra which is closely related to so-called affine Kac–Moody algebras, which are the universal central extension of loop algebras.

Central extensions are needed in physics, because the symmetry group of a quantized system usually is a central extension of the classical symmetry group, and in the same way the corresponding symmetry Lie algebra of the quantum system is, in general, a central extension of the classical symmetry algebra. Kac–Moody algebras have been conjectured to be symmetry groups of a unified superstring theory. The centrally extended Lie algebras play a dominant role in quantum field theory, particularly in conformal field theory, string theory and in M-theory.

In the theory of Lie groups, Lie algebras and their representations, a Lie algebra extension is an enlargement of a given Lie algebra $g$ by another Lie algebra $h$. Extensions arise in several ways. There is a trivial extension obtained by taking a direct sum of two Lie algebras. Other types are split extension and central extension. Extensions may arise naturally, for instance, when forming a Lie algebra from projective group representations. A central extension and an extension by a derivation of a polynomial loop algebra over finite-dimensional simple Lie algebra give a Lie algebra which is isomorphic with a non-twisted affine Kac–Moody algebra [1, Chapter 19]. Using the centrally extended loop algebra one may construct a current algebra in two spacetime dimensions. The Virasoro algebra is the universal central extension of the Witt algebra, the Heisenberg algebra is the central extension of a commutative Lie algebra [1, Chapter 18].
The algebraic study of central extensions of Lie and non-Lie algebras has a very big story (for more information, see [2–7]). So, Skjelbred and Sund used central extensions of Lie algebras for a classification of nilpotent Lie algebras [6]. Note that, the method of central extensions was used for the classification of nilpotent low dimensional algebras in Lie algebras [8], Jordan algebras [9], Malcev algebras [10] and evolution algebras [11]. Also, using the method described by Skjelbred and Sund were classified all $n$-dimensional Malcev (non-Lie) algebras with $(n-4)$-dimensional annihilator [4], all $n$-dimensional Jordan algebras with $(n-3)$-dimensional annihilator [3] and all $n$-dimensional algebras with $(n-2)$-dimensional annihilator [12].

The first attempt of the classification of 3-dimensional anticommutative algebras was given in [13]. After that, some more simple description of 3-dimensional anticommutative algebras was given in [14]. In the present paper, we use the classification given in [14] and describe all non-isomorphic $n$-dimensional anticommutative algebras with $(n-3)$-dimensional annihilator.

The main aim of the present paper is to prove the following result:

**Theorem 1** (Main Theorem). Let $(\mathfrak{A}, [, ,])$ be an $n$-dimensional anticommutative complex algebra with $(n-3)$-dimensional annihilator.

If $n = 3$ then $\mathfrak{A} \cong \mathfrak{A}_{3,i}$ for some $i = 1, \ldots, 6$, where $\mathfrak{A}_{3,1} = \mathfrak{g}_2, \mathfrak{A}_{3,2} = \mathfrak{g}_3^{x \neq 0}, \mathfrak{A}_{3,3} = \mathfrak{g}_4, \mathfrak{A}_{3,4} = \mathfrak{A}_2^3, \mathfrak{A}_{3,5} = \mathfrak{A}_2$ and $\mathfrak{A}_{3,6} = \mathfrak{A}_3$ (see Appendix Table 1).

If $n = 4$ then $\mathfrak{A} \cong \mathfrak{A}_{4,i}$ for some $i = 1, \ldots, 12$, where $\mathfrak{A}_{4,i}$ is one of the following non-isomorphic algebras:

\[
\mathfrak{A}_{4,1} = \mathfrak{A}_{3,1} \oplus \mathfrak{g}_4, \quad i = 1, \ldots, 6
\]

\[
\begin{align*}
\mathfrak{A}_{4,7} &= (\mathfrak{g}_1)_{4,7} : [e_1, e_2] = e_4, [e_1, e_3] = 0, [e_2, e_3] = e_1, e_2 = e_4 \\
\mathfrak{A}_{4,8} &= (\mathfrak{g}_2)_{4,8} : [e_1, e_2] = e_4, [e_1, e_3] = e_1, [e_2, e_3] = e_2 \\
\mathfrak{A}_{4,9}(x \in \mathbb{C}^* \cup \{0, 1\}) &= (\mathfrak{g}_3)_{4,9} : [e_1, e_2] = e_4, [e_1, e_3] = e_1 + e_2, [e_2, e_3] = x e_2 \\
\mathfrak{A}_{4,10} &= (\mathfrak{g}_3)_{4,10} : [e_1, e_2] = 0, [e_1, e_3] = e_1 + e_2, [e_2, e_3] = e_4 \\
\mathfrak{A}_{4,11} &= (\mathfrak{A}_4^3)_{4,11} : [e_1, e_2] = e_3, [e_1, e_3] = e_1 + e_3, [e_2, e_3] = e_4 \\
\mathfrak{A}_{4,12} &= (\mathfrak{A}_4)_4,12 : [e_1, e_2] = e_1, [e_1, e_3] = e_4, [e_2, e_3] = e_2
\end{align*}
\]

If $n = 5$ then $\mathfrak{A} \cong \mathfrak{A}_{5,i}$ for some $i = 1, \ldots, 15$, where $\mathfrak{A}_{5,i}$ is one of the following non-isomorphic algebras:

\[
\begin{align*}
\mathfrak{A}_{5,13} &= (\mathfrak{g}_1)_{5,13} : [e_1, e_2] = 0, [e_1, e_3] = e_4, [e_2, e_3] = e_5 \\
\mathfrak{A}_{5,14} &= (\mathfrak{g}_1)_{5,14} : [e_1, e_2] = e_4, [e_1, e_3] = e_5, [e_2, e_3] = e_1 \\
\mathfrak{A}_{5,15} &= (\mathfrak{g}_3^3)_{5,15} : [e_1, e_2] = e_4, [e_1, e_3] = e_1 + e_2, [e_2, e_3] = e_5
\end{align*}
\]

If $n = 6$ then $\mathfrak{A} \cong \mathfrak{A}_{6,i}$ for some $i = 1, \ldots, 16$, where $\mathfrak{A}_{6,i}$ is one of the following non-isomorphic algebras:

\[
\begin{align*}
\mathfrak{A}_{6,1} &= \mathfrak{A}_{5,1} \oplus \mathfrak{g}_6, \quad i = 1, \ldots, 15 \\
\mathfrak{A}_{6,16} &= (\mathfrak{g}_1)_{6,16} : [e_1, e_2] = e_4, [e_1, e_3] = e_5, [e_2, e_3] = e_6
\end{align*}
\]

if $n \geq 7$, then $\mathfrak{A} \cong \mathfrak{A}_{n,i}$ for some $i = 1, \ldots, 16$ where $\mathfrak{A}_{n,i} = \mathfrak{A}_{6,1} \oplus \mathfrak{g}_7 \oplus \cdots \oplus \mathfrak{g}_n$.

The paper is organized as follows. In Section 1 we review our method for classifying, up to isomorphisms, all $n$-dimensional anticommutative algebras with $(n-3)$-dimensional annihilator over any field $\mathbf{k}$ of characteristic not 2. This method is the analog of Skjelbred-Sund method for classifying nilpotent Lie algebras, (see [6]), and was introduced by Hegazi, Abdelwabad and the first author for classifying a certain class of Malcev algebras in [4]. The isomorphism problem will be solved by using cohomological methods. In Section 2 it is presented the classification of 3-dimensional anticommutative complex algebras given in [14] that will be used in the development of the next (main) section. Finally, in Section 3 we prove our above mentioned classification theorem.
2. A review of the method

Let \((A, [-,-])\) be an anticommutative algebra over an arbitrary base field \(k\) of characteristic not 2 and \(V\) a vector space over the same base field \(k\). Then the \(k\)-linear space \(Z^2(A, V)\) is defined as the set of all skew-symmetric bilinear maps \(\theta : A \times A \to V\). Its elements will be called cocycles. For a linear map \(f\) from \(A\) to \(V\), if we write \(\delta f : A \times A \to V\) by \(\delta f(x, y) = f([x, y])\), then \(\delta f \in Z^2(A, V)\). We define \(B^2(A, V) = \{\theta \in \text{Hom}(A, V)\}\). One can easily check that \(B^2(A, V)\) is a linear subspace of \(Z^2(A, V)\) which elements are called coboundaries. We define the second cohomology space \(H^2(A, V)\) as the quotient space \(Z^2(A, V)/B^2(A, V)\).

Let \(\text{Aut}(A)\) be the automorphism group of the anticommutative algebra \(A\) and let \(\phi \in \text{Aut}(A)\). For \(\theta \in Z^2(A, V)\) define \(\phi \theta(x, y) = \theta(\phi(x), \phi(y))\). Then \(\phi \theta \in Z^2(A, V)\). So, \(\text{Aut}(A)\) acts on \(Z^2(A, V)\). It is easy to verify that \(B^2(A, V)\) is invariant under the action of \(\text{Aut}(A)\) and so we have that \(\text{Aut}(A)\) acts on \(H^2(A, V)\).

Let \(A\) be an anticommutative algebra of dimension \(m < n\) over an arbitrary base field \(k\) of characteristic not 2, and \(V\) be an \(k\)-vector space of dimension \(n - m\). For any \(\theta \in Z^2(A, V)\) define on the linear space \(A_\theta := A \oplus V\) the bilinear product \(\ldots, \ldots\) by \([x + x', y + y']_{A_\theta} = [x, y] + \theta(x, y)\) for all \(x, y \in A, x', y' \in V\). The algebra \(A_\theta\) is an anticommutative algebra which is called an \((n - m)\)-dimensional annihilator extension of \(A\) by \(V\). Indeed, we have, in a straightforward way, that \(A_\theta\) is an anticommutative algebra if and only if \(\theta \in Z^2(A, V)\).

We also call to the set \(\text{rad}(\theta) = \{x \in A : \theta(x, A) = 0\}\) the radical of \(\theta\).

We recall that the annihilator of an anticommutative algebra \(A\) is defined as the ideal \(\text{Ann}(A) = \{x \in A : [x, A] = 0\}\) and observe that \(\text{Ann}(A_\theta) = (\text{rad}(\theta) \cap \text{Ann}(A)) \oplus V\).

We have the next key result:

**Lemma 2.** Let \(A\) be an \(n\)-dimensional anticommutative algebra such that \(\dim(\text{Ann}(A)) = m \neq 0\). Then there exists, up to isomorphism, a unique \((n - m)\)-dimensional anticommutative algebra \(A'\) and a bilinear map \(\theta \in Z^2(A, V)\) with \(\text{Ann}(A) \cap \text{rad}(\theta) = 0\), where \(V\) is a vector space of dimension \(m\), such that \(A \cong A'_\theta\) and \(A/\text{Ann}(A) \cong A'\).

**Proof.** Let \(A'\) be a linear complement of \(\text{Ann}(A)\) in \(A\). Define a linear map \(P : A \to A'\) by \(P(x + v) = x\) for \(x \in A'\) and \(v \in \text{Ann}(A)\) and define a multiplication on \(A'\) by \([x, y]_{A'} = P([x, y])\) for \(x, y \in A'\). For \(x, y \in A\) then

\[
P([x, y]) = P([x - P(x) + P(x), y - P(y) + P(y)]) = P([P(x), P(y)]) = [P(x), P(y)]_{A'}
\]

Since \(P\) is a homomorphism then \(P(A) = A'\) is an anticommutative algebra and \(A/\text{Ann}(A) \cong A'\), which give us the uniqueness. Now, define the map \(\theta : A' \times A' \to \text{Ann}(A)\) by \(\theta(x, y) = [x, y] - [x, y]_{A'}\). Thus, \(A'_\theta\) is \(A\) and therefore \(\theta \in Z^2(A, V)\) and \(\text{Ann}(A) \cap \text{rad}(\theta) = 0\).

However, in order to solve the isomorphism problem we need to study the action of \(\text{Aut}(A)\) on \(H^2(A, k)\). To do that, let us fix \(e_1, \ldots, e_s\) a basis of \(V\), and \(\theta \in Z^2(A, V)\). Then \(\theta\) can be uniquely written as \(\theta(x, y) = \sum_{i=1}^s \theta_i(x, y)e_i\), where \(\theta_i \in Z^2(A, k)\). Moreover, \(\text{rad}(\theta) = \text{rad}(\theta_1) \cap \text{rad}(\theta_2) \cdots \cap \text{rad}(\theta_s)\). Further, \(\theta \in B^2(A, V)\) if and only if all \(\theta_i \in B^2(A, k)\).

Given an anticommutative algebra \(A\), if \(A = I \oplus kx\) is a direct sum of two ideals, then \(kx\) is called an annihilator component of \(A\). It is not difficult to prove, (see [4, Lemma 13]), that given an anticommutative algebra \(A_\theta\), if we write as above \(\theta(x, y) = \sum_{i=1}^s \theta_i(x, y)e_i\) for \(\theta_i \in Z^2(A, V)\), and we have \(\text{rad}(\theta) \cap \text{Ann}(A) = 0\), then \(A_\theta\) has an annihilator component if and only if \([\theta_1], [\theta_2], \ldots, [\theta_s]\) are linearly dependent in \(H^2(A, k)\).

Let \(V\) be a finite-dimensional vector space over \(k\). The Grassmannian \(G_s(V)\) is the set of all \(k\)-dimensional linear subspaces of \(V\). Let \(G_s(H^2(A, k))\) be the Grassmannian of subspaces of dimension \(s\) in \(H^2(A, k)\). There is a natural action of \(\text{Aut}(A)\) on \(G_s(H^2(A, k))\). Let \(\phi \in \text{Aut}(A)\). For \(W = ([\theta_1], [\theta_2], \ldots, [\theta_s]) \in G_s(H^2(A, k))\) define \(\phi W = ([\phi \theta_1], [\phi \theta_2], \ldots, [\phi \theta_s])\). Then \(\phi W \in G_s(H^2(A, k))\). We denote the orbit of \(W \in G_s(H^2(A, k))\) under the action of \(\text{Aut}(A)\) by
Orb($W$). Since given
\[ W_1 = \langle [\theta_1], [\theta_2], \ldots, [\theta_s] \rangle, \quad W_2 = \langle [\vartheta_1], [\vartheta_2], \ldots, [\vartheta_s] \rangle \in G_s(H^2(A, k)) \]
we easily have that in case $W_1 = W_2$, then $\cap_{i=1}^s rad(\theta_i) \cap Ann(A) = \cap_{i=1}^s rad(\vartheta_i) \cap Ann(A)$, we can introduce the set
\[ T_s(A) = \left\{ W = \langle [\theta_1], [\theta_2], \ldots, [\theta_s] \rangle \in G_s(H^2(A, k)) : \bigcap_{i=1}^s rad(\theta_i) \cap Ann(A) = 0 \right\}, \]
which is stable under the action of $Aut(A)$.

Now, let $\mathbb{V}$ be an $s$-dimensional linear space and let us denote by $E(A, \mathbb{V})$ the set of all anticommutative algebras without annihilator components which are $s$-dimensional annihilator extensions of $A$ by $\mathbb{V}$ and have $s$-dimensional annihilator. We can write
\[ E(A, \mathbb{V}) = \left\{ A_\theta : \theta(x, y) = \sum_{i=1}^s \theta_i(x, y)e_i \text{ and } \langle [\theta_1], [\theta_2], \ldots, [\theta_s] \rangle \in T_s(A) \right\}. \]

Also we have the next result, which can be proved as [4, Lemma 17].

**Lemma 3.** Let $A_\theta, A_{\vartheta} \in E(A, \mathbb{V})$. Suppose that $\theta(x, y) = \sum_{i=1}^s \theta_i(x, y)e_i$ and $\vartheta(x, y) = \sum_{i=1}^s \vartheta_i(x, y)e_i$. Then the anticommutative algebras $A_\theta$ and $A_{\vartheta}$ are isomorphic if and only if $\text{Orb}((\{\theta_1\}, \{\theta_2\}, \ldots, \{\theta_s\})) = \text{Orb}((\{\vartheta_1\}, \{\vartheta_2\}, \ldots, \{\vartheta_s\}))$.

From here, there exists a one-to-one correspondence between the set of $Aut(A)$-orbits on $T_s(A)$ and the set of isomorphism classes of $E(A, \mathbb{V})$. Consequently we have a procedure that allows us, given the anticommutative algebras $A'$ of dimension $n - s$, to construct all of the anticommutative algebras $A$ of dimension $n$ with no annihilator components and with $s$-dimensional annihilator. This procedure would be:

**2.1. Procedure**

1. For a given anticommutative algebra $A'$ of dimension $n-s$, determine $H^2(A', k), Ann(A')$ and $Aut(A')$.
2. Determine the set of $Aut(A')$-orbits on $T_s(A')$.
3. For each orbit, construct the anticommutative algebra corresponding to a representative of it.

Finally, let us introduce some notation. Let $A$ be an anticommutative algebra with a basis $e_1, e_2, \ldots, e_n$. Then by $\Delta_{ij}$ we will denote the skew-symmetric bilinear form $\Delta_{ij} : A \times A \rightarrow k$ with $\Delta_{ij}(e_i, e_j) = -\Delta_{ij}(e_j, e_i) = 1$ and $\Delta_{ij}(e_i, e_m) = 0$ if $i, j \neq l, m$. Then the set $\{\Delta_{ij} : 1 \leq i < j \leq n\}$ is a basis for the linear space of skew-symmetric bilinear forms on $A$. Then every $\theta \in Z^2(A, k)$ can be uniquely written as $\theta = \sum_{1 \leq i < j \leq n} c_{ij}\Delta_{ij}$, where $c_{ij} \in k$.

We can apply this method to classify the $n$-dimensional anticommutative complex algebras with $(n-3)$-dimensional annihilator, because of the following result:

**3. The classification of 3-dimensional anticommutative complex algebras**

To give the classification of 3-dimensional anticommutative complex algebras we have to introduce some notation [14]. Let us consider the action of the cyclic group $\mathbb{Z}_2$ on $\mathbb{C}^* \setminus \{0\}$ defined by the equality $^{\pi}\!\!x = x^{-1}$ for $x \in \mathbb{C}^* \setminus \{0\}$. Let us fix some set of representatives of orbits under this action and denote it by $\mathbb{C}^*_{\geq 1}$. That is,
\[ \mathbb{C}^*_{\geq 1} = \{ x \in \mathbb{C}^* | |x| > 1 \} \cup \{ x \in \mathbb{C}^* | |x| = 1, 0 \arg(x) \leq \pi \}. \]
In the Appendix Table 1, we summarize the classification of 3-dimensional anticommutative complex algebras given by [14]. The products of basic elements whose values are zero or can be recovered from the anticommutativity were omitted.

4. Proof of the main result

Taking into account Section 1, we have that any $n$-dimensional anticommutative complex algebra with $(n - 3)$-dimensional annihilator $(A, [\cdot, \cdot])$ satisfies that $A/Ann(A)$ is isomorphic to one of the algebras in Appendix Table 1. First, observe that in case $n = 3$ then $Ann(A) = 0$ and so $A \in \{g_2, g_3, g_4, A_1^2, A_2, A_3\}$.

Hence, let us consider the cases $n \geq 4$. We will begin by studying those without annihilator components. From here, we are going to consider eight cases by distinguish to which algebra of Appendix Table 1 is $A/Ann(A)$ isomorphic.

4.1. Algebra $\mathfrak{R}$

It is easy to see that

$$Basis(H^2(\mathfrak{R}, \mathbb{C})) = \{[\theta_3], [\theta_2], [\theta_1]\}.$$ 

Now, since $Ann(\mathfrak{R}) = \langle e_1, e_2, e_3 \rangle$ and $rad(\theta) = e_i$ then we have:

- $T_3(\mathfrak{R}) = \{(\theta), 0 \in H^2(\mathfrak{R}, \mathbb{C}) : rad(\theta) = 0 \} = \emptyset$.
- $T_4(\mathfrak{R}) = Grass(H^2(\mathfrak{R}, \mathbb{C}))$ for $s = 2, 3$.

The action of an automorphism $(a_{ij}) \in Aut(\mathfrak{R}) = GL(3, \mathbb{C})$ on a subspace $\langle \gamma_1, ..., \gamma_s \rangle \in T_4(\mathfrak{R})$ is $\langle (a_{ij})^s \gamma_1(a_{ij}), ..., (a_{ij})^s \gamma_s(a_{ij}) \rangle$. Therefore, the action on $\langle a[\theta_3] + b[\theta_2] + c[\theta_1] \rangle$ is the following:

$$\langle (a_{11}(aa_{22} + a_{32}b) + a_{31}(-a_{12}b - a_{22}c) + a_{21}(-aa_{12} + a_{32}c))[\theta_3] + (a_{11}(aa_{23} + a_{33}b) + a_{31}(-a_{13}b - a_{23}c) + a_{21}(-aa_{13} + a_{33}c))[\theta_2] + (a_{12}(aa_{23} + a_{33}b) + a_{32}(-a_{13}b - a_{23}c) + a_{22}(-aa_{13} + a_{33}c))[\theta_1] \rangle.$$ 

It can be proved that for $s = 2, 3$ there is only one orbit. We choose good representatives and we obtain the following annihilator extensions:

- $\mathfrak{R}_{5,13} \colon [e_1, e_2] = 0 \quad [e_1, e_3] = e_4 \quad [e_2, e_3] = e_5$
- $\mathfrak{R}_{6,16} \colon [e_1, e_2] = e_4 \quad [e_1, e_3] = e_5 \quad [e_2, e_3] = e_6$

4.2. Algebra $\mathfrak{g}_1$

It is easy to see that

$$\delta e_i^* = \theta_1.$$ 

Moreover:

$$Basis(H^2(\mathfrak{g}_1, \mathbb{C})) = \{[\theta_3], [\theta_2]\}.$$ 

Since $Ann(\mathfrak{g}_1) = \langle e_1 \rangle$ and $e_1 \notin rad(\theta)$ for $[\theta] \in H^2(\mathfrak{g}_1, \mathbb{C})$ and $[\theta] \neq 0$ then $T_3(\mathfrak{g}_1) = Grass(H^2(\mathfrak{g}_1, \mathbb{C}))$.

Also, $Aut(\mathfrak{g}_1)$ sends $e_1$, $e_2$ and $e_3$ to $\delta e_1, a_{12}e_1 + a_{22}e_2 + a_{32}e_3$ and $a_{13}e_1 + a_{23}e_2 + a_{33}e_3$ respectively, where $\delta = a_{22}a_{33} - a_{23}a_{32} \neq 0$. Additionally, the action of $Aut(\mathfrak{g}_1)$ on a subspace $\langle a[\theta_3] + b[\theta_2] \rangle \in T_1(\mathfrak{g}_1)$ is the following:
\[\langle (\delta aa_{22} + \delta ba_{33})[\theta_3] + (\delta aa_{23} + \delta ba_{33})[\theta_2] \rangle.\]

Thus, for \(b = 0\) we have that the orbit of \(\langle a[\theta_3] \rangle \in T_1(g_1)\) is

\[\{ \langle \delta aa_{22}[\theta_3] + \delta aa_{23}[\theta_2] \rangle : a_{22}, a_{23} \in \mathbb{C}, \delta \neq 0 \} = \{ \langle a_{22}[\theta_3] + a_{23}[\theta_2] \rangle : a_{22}, a_{23} \in \mathbb{C} \}.\]

Note that for another subspace such that \(b \neq 0\), denoting \(\lambda = \frac{b}{a}\), we can choose \(a_{22} = \lambda\) and \(a_{23} = 1\) in the previous orbit. Therefore there is just one orbit and we can choose a good representative: \(\langle [\theta_3] \rangle\).

Therefore, the anticommutative annihilator extensions are the following:

\[
\begin{align*}
(g_1)_{4,7} & : [e_1, e_2] = e_4 \quad [e_1, e_3] = 0 \quad [e_2, e_3] = e_1 \\
(g_1)_{5,14} & : [e_1, e_2] = e_4 \quad [e_1, e_3] = e_5 \quad [e_2, e_3] = e_1
\end{align*}
\]

### 4.3. Algebra \(g_2\)

It is easy to see that

\[\delta e_1^* = \theta_2, \delta e_2^* = \theta_1.\]

Moreover:

\[\text{Basis}(H^2(g_2, \mathbb{C})) = \{[\theta_3]\}.\]

Since \(Ann(g_2) = 0\) then \(T_1(g_2) = \text{Grass}_1(H^2(g_2, \mathbb{C})) = H^2(g_2, \mathbb{C})\). Thus, there is only one orbit and therefore:

\[
\begin{align*}
(g_2)_{4,8} & : [e_1, e_2] = e_4 \quad [e_1, e_3] = e_1 \quad [e_2, e_3] = e_2
\end{align*}
\]

### 4.4. Algebra \(g_3\)

It is easy to see that

\[\delta e_1^* = \theta_2, \delta e_2^* = \theta_2 + x\theta_1.\]

If \(x \neq 0\) then

\[\text{Basis}(H^2(g_3, \mathbb{C})) = \{[\theta_3]\}.\]

Since \(Ann(g_3) = 0\) for \(x \neq 0\) then \(T_1(g_3) = \text{Grass}_1(H^2(g_3, \mathbb{C})) = H^2(g_3, \mathbb{C})\). Thus, there is only one orbit for this case.

If \(x = 0\) then

\[\text{Basis}(H^2(g_3, \mathbb{C})) = \{[\theta_3], [\theta_1]\}.\]

and \(Ann(g_3) = \langle e_2 \rangle\) but since \(e_2 \notin \text{rad}(\theta)\) for \([\theta] \in H^2(g_3, \mathbb{C})\) and \([\theta] \neq 0\) then \(T_1(g_3^0) = \text{Grass}_1(H^2(g_3^0, \mathbb{C}))\).

Additionally, \(Aut(g_3^0)\) sends \(e_1, e_2\) and \(e_3\) to \((a_{21} + a_{22})e_1 + a_{21}e_2, a_{22}e_2\) and \(a_{13}e_1 + a_{23}e_2 + e_3\) respectively, where \((a_{21} + a_{22})a_{22} \neq 0\).

Moreover, the action of \(Aut(g_3^0)\) on a subspace \(\langle a[\theta_3] + b[\theta_1] \rangle \in T_1(g_3^0)\) is the following:

\[\langle (aa_{22}(a_{21} + a_{22}))[\theta_3] + (a_{22}(b - aa_{13}))[\theta_1] \rangle.\]

For \(a = 0\), it shows that the orbit of \(\langle [\theta_1] \rangle\) is trivial.
If $a \neq 0$, denoting $\lambda = b/a$, the orbit of the subspace $\langle [\theta_3] + \lambda [\theta_1] \rangle$ is:
\[
\{ (a_{22}(a_{21} + a_{22})[\theta_3] + (a_{22}(\lambda - a_{13}))[\theta_1]) : a_{22}(a_{21} + a_{22}) \neq 0 \}.
\]
Choosing $a_{22} = 1/(a_{21} + a_{22})$ and $a_{13} = \lambda$, we obtain a good representative: $\langle [\theta_3] \rangle$.
Thus, we can conclude that the annihilator extensions of $g^*_3$ are the following:
\[
\begin{align*}
(g^*_3)^{x \neq 0} & : [e_1, e_2] = e_4 & [e_1, e_3] = e_1 + e_2 & [e_2, e_3] = xe_2 \\
(g^*_3)^{4,9} & : [e_1, e_2] = e_4 & [e_1, e_3] = e_1 + e_2 & [e_2, e_3] = 0 \\
(g^*_3)^{4,10} & : [e_1, e_2] = 0 & [e_1, e_3] = e_1 + e_2 & [e_2, e_3] = e_4 \\
(g^*_3)^{5,15} & : [e_1, e_2] = e_4 & [e_1, e_3] = e_1 + e_2 & [e_2, e_3] = e_5
\end{align*}
\]

### 4.5 Algebra $g_4$

Note that $\dim(Z^2(g_4, \mathbb{C})) = \dim(B^2(g_4, \mathbb{C})) = 3$. Therefore, the only annihilator extensions are constructed by adding annihilator components.

### 4.6 Algebra $A^*_1$

Note that $\dim(Z^2(A^*_1, \mathbb{C})) = \dim(B^2(A^*_1, \mathbb{C})) = 3$ for $\alpha \neq 0$. Therefore, there are no annihilator extensions without annihilator component for this case.

Now, for $\alpha = 0$ we have:
\[
\delta e^*_1 = \theta_2, \delta e^*_2 = \theta_3 + \theta_2.
\]

Moreover:
\[
\text{Basis}(H^2(A^*_1, \mathbb{C})) = \{ [\theta_1] \}.
\]

Since $\text{Ann}(A^*_1) = 0$ then $T_i(A^*_1) = \text{Grass}_i(H^2(A^*_1, \mathbb{C})) = H^2(A^*_1, \mathbb{C})$. Thus, we conclude that there is one annihilator extension for $\alpha = 0$ which is:
\[
(A^*_1)^{4,11} : [e_1, e_2] = e_3 & [e_1, e_3] = e_1 + e_3 & [e_2, e_3] = e_4
\]

### 4.7 Algebra $A_2$

It is easy to see that
\[
\delta e^*_1 = \theta_3, \delta e^*_2 = \theta_4.
\]

Moreover:
\[
\text{Basis}(H^2(A_2, \mathbb{C})) = \{ [\theta_2] \}.
\]

Thus, there is only one annihilator extension:
\[
(A_2)^{4,12} : [e_1, e_2] = e_4 & [e_1, e_3] = e_4 & [e_2, e_3] = e_2
\]

### 4.8 Algebra $A_3$

Note that $\dim(Z^2(A_3, \mathbb{C})) = \dim(B^2(A_3, \mathbb{C})) = 3$. Therefore, there are no annihilator extensions without annihilator component.
Finally, suppose $\mathbf{A}$ has an annihilator component. We begin by observing that in this case $\mathbf{A} = \mathbf{A}' \oplus \mathbf{C} e_n$ with $\mathbf{A}'$ an $(n-1)$-dimensional anticommutative algebra with an $((n-1)-3)$-dimensional annihilator. From here, if $n = 4$, then $\mathbf{A} = \mathbf{A}' \oplus \mathbf{C} e_4$ being $\mathbf{A}'$ a 3-dimensional anticommutative algebra with zero annihilator, which gives rise to algebras $\mathbf{A}_{4,i}, i = 1, ..., 6$ of the theorem. The cases $n > 4$ can be studied in a similar way to get the algebras $\mathbf{A}_{5,i}, \mathbf{A}_{6,j}$ and $\mathbf{A}_{n,k}, i = 1, ..., 12, j = 1, ..., 15, n \geq 7, k = 1, ..., 16$ of the theorem.

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**Appendix Table 1.**

| $\mathbf{A}$ | $e_1 e_2 = e_1$ | $e_1 e_3 = e_1$ | $e_2 e_3 = e_1$ |
|--------------|-----------------|-----------------|-----------------|
| $\mathbf{g}_1$ | $e_1 e_3 = e_1$ | $e_2 e_3 = e_2$ | $e_2 e_3 = e_2$ |
| $\mathbf{g}_2$ | $e_1 e_3 = e_1 + e_2$ | $e_2 e_3 = e_2$ | $e_2 e_3 = e_2$ |
| $\mathbf{g}_3, x \in \mathbb{C}_{>1} \cup \{0,1\}$ | $e_1 e_3 = -e_2$ | $e_2 e_3 = e_1$ | $e_2 e_3 = e_1$ |
| $\mathbf{A}_{4,i}, x \in \mathbb{C}_{>1} \cup \{0,1\}$ | $e_1 e_3 = e_1 + e_3$ | $e_2 e_3 = e_2$ | $e_2 e_3 = e_2$ |
| $\mathbf{A}_{5,i}$ | $e_1 e_2 = e_1$ | $e_1 e_3 = e_1$ | $e_2 e_3 = e_2$ |
| $\mathbf{A}_{6,j}$ | $e_1 e_2 = e_1$ | $e_1 e_3 = e_1$ | $e_2 e_3 = e_2$ |

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