On a conformally invariant integral equation involving Poisson kernel

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Abstract

We study a prescribing functions problem of a conformally invariant integral equation involving Poisson kernel on the unit ball. This integral equation is not the dual of any standard type of PDE. As in Nirenberg problem, there exists a Kazdan-Warner type obstruction to existence of solutions. We prove existence in the antipodal symmetry functions class.

1 Introduction

Poisson integral and Riesz potential are basic objects in the singular integral theory; see Stein [17]. Riesz potential is the dual of (fractional) Poisson equations. In [14], Jin-Li-Xiong developed a blow up analysis procedure for critical nonlinear integral equations involving Riesz kernel and established a unified approach to the Nirenberg problem and its generalizations. The method is flexible; see Li-Xiong [16] for its application to compactness of fourth order constant $Q$-curvature metrics. In this paper, we extend some analysis further to a natural critical nonlinear integral equations involving Poisson kernel.

Let $B_1$ be the unit ball in $\mathbb{R}^n$, $n \geq 2$. For each $v \in L^p(\partial B_1)$, $p \geq 1$, the Poisson integral of $v$ is defined by

$$P_v(\xi) = \int_{\partial B_1} P(\eta, \xi)v(\eta) \, ds_\eta \quad \text{for } \xi \in B_1,$$

where $P(\eta, \xi) = \frac{1-|\xi|^2}{n\omega_n |\xi-\eta|^n}$ is the Poisson kernel and $\omega_n$ is the volume of the unit ball. Then $P_v$ is a harmonic function in $B_1$. If $n = 2$, a classical inequality of Carleman [2] asserts that

$$\int_{B_1} e^{2P_v} \, d\xi \leq \frac{1}{4\pi} \left( \int_{\partial B_1} e^v \, ds \right)^2$$

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and the equality holds if and only if \( v = c \) or \( v = -2 \ln |\xi - \xi_0| + c \) for some constant \( c \) and \( \xi_0 \in \mathbb{R}^2 \setminus \bar{B}_1 \). If \( n \geq 3 \), Hang-Wang-Yan [11] proved that
\[
\|Pv\|_{L^{2n}(B_1)} \leq S(n) \|v\|_{L^{\frac{2(n-1)}{n-2}}(\partial B_1)},
\]
where \( S(n) = n^{-\frac{n-2}{2(n-1)}} \omega_n^{-\frac{n-2}{2(n-1)}} \) and the equality holds if and only if \( v = 1 \) up to a conformal transform on the unit sphere \( \partial B_1 \). In [12], they studied (2) on Riemannian manifolds. See also the recent paper Dou-Guo-Zhu [7] and references therein for other related results. Motivated by the Nirenberg problem, starting from this paper we study positive solutions of the Euler-Larange equation of the functional
\[
I[v] = \int_{B_1} |Pv|^{\frac{2n}{n-2}} \, d\xi - \int_{\partial B_1} K|v|^{\frac{2(n-1)}{n-2}} \, ds,
\]
where \( v \in L^{\frac{2(n-1)}{n-2}}(\partial B_1) \) is not zero and \( K > 0 \) is a given continuous function. Namely,
\[
K(\eta)v(\eta)^{\frac{n}{n-2}} = \int_{B_1} P(\eta, \xi)Pv(\xi)^{\frac{n+2}{n-2}} \, d\xi, \quad v > 0 \text{ on } \partial B_1 .
\]
This equation is critical, conformally invariant and not always solvable. Indeed, a Kazdan-Warner type necessary condition was derived in [12]: For any conformal Killing vector field \( X \) on \( \partial B_1 \), endowed with the induced metric from \( \mathbb{R}^n \),
\[
\int_{\partial B_1} (\nabla_X K)v^{\frac{2(n-1)}{n-2}} \, ds = 0
\]
holds for any solution \( v \) of (3). For example, if \( K = \xi_n + 2 \), there is no solution of (3).

**Theorem 1.1.** Let \( n \geq 3 \) and \( K \in C^1(\partial B_1) \) be a positive function satisfying \( K(\xi) = K(-\xi) \). For every \( q > n - 1 \), there exists a constant \( \delta > 0 \), depending only on \( n \) and \( q \), such that if for a minimal point \( \xi_1 \) of \( K \) there holds \( K(\xi) - K(\xi_1) \leq \delta|\xi - \xi_1|^q \) for all \( \xi \in \partial B_1 \), then equation (3) has at least one positive solution.

The analogue of Theorem 1.1 for Nirenberg problem was established by Escobar-Schoen [8]. See Jin-Li-Xiong [14] and references therein for generalized Nirenberg problems. We prove Theorem 1.1 via subcritical approximation approach, which contains two steps. The first shows that if the supremum of \( I[\cdot] \) is greater than some threshold, then maximizers exist. Here we use a blow up analysis argument for integral equations, which was introduced by Jin-Li-Xiong [14]. Our current equation has a stronger nonlocal feature. New ingredients, such as boundary Harnack inequality, are incorporated in the proofs.

The second step verifies the strict inequality. Again due to the strong nonlocality, we introduce a trial function by gluing two bubbles along the equator of the sphere. This is
different from the Nirenberg problem case; see [13, 14] and references therein. These two bubbles do not affect each other in the boundary \( L^{\frac{2(n-1)}{n-2}} \) norm, but they do in the interior thanks to the harmonic extension. In particular, in the interior our trial function will be of the “a bubble plus a positive harmonic function of a linear growth” structure locally. Such structure was used by the author [19] to study boundary isolated singularity in a different context; see the proof of Proposition 4.2 in that paper. See also Jin-Xiong [15]. In [18], Sun-Xiong proved “bubbles plus polynomials” type classification theorems of higher order boundary conformally invariant problems.

In the future work, we will study existence and compactness of solutions beyond the antipodal symmetry functions class. We will also study the exponential nonlinearity problem of dimension two as the classical work Chang-Yang [3, 4] did.

At the end of this section, we note that the Poisson kernel on the upper half space (see section 2) coincides with the heat kernel of \( \partial_t + (-\Delta)^{1/2} \), see Blumenthal-Getoor [1]. Hence, our problem can also be interpreted through the 1/2 heat kernel. Within this in mind, one may draw an analogy to the studies of maximizers for the Strichartz inequality and Stein-Tomas inequality; see Foschi [9], Christ-Shao [5, 6], Frank-Lieb-Sabin [10] and references therein.

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2 A blow up analysis procedure

We denote \( x = (x', x_n) \), \( y = (y', y_n) \) as points in \( \mathbb{R}^n \), \( B_R(x) \) as the open ball of \( \mathbb{R}^n \) centered as \( x \) with radius \( R \), and \( B'_R(x') \) as the open ball in \( \mathbb{R}^{n-1} \) centered as \( x' \) with radius \( R \).

Let \( F : \mathbb{R}^n_+ \to B_1 \) be the Mobius transformation given by

\[
F(x) = \frac{2(x + e_n)}{|x + e_n|^2} - e_n,
\]

where \( e_n = (0, \ldots, 0, 1) \). For \( x = (x', 0) \in \partial\mathbb{R}^n_+ \), we see that

\[
F(x) = \left( \frac{2x'}{|x'|^2 + 1}, \frac{1 - |x'|^2}{|x'|^2 + 1} \right) \in \partial B_1
\]

is the inverse of the stereographic projection. For \( v \in L^{\frac{2(n-1)}{n-2}}(\partial B_1) \), let

\[
u(x') = \left( \frac{\sqrt{2}}{|x + e_n|} \right)^{n-2} v(F(x)) \quad \text{for } x = (x', 0).
\]

(5)

For saving notations, we still use \( P(\cdot, \cdot) \) to denote the Poisson kernel on the upper half space and

\[
P u(x) = \int_{\mathbb{R}^{n-1}} P(y', x) u(y') \, dy' = \frac{2}{n\omega_n} \int_{\mathbb{R}^{n-1}} \frac{x_n}{(|x' - y'|^2 + x_n^2)^{\frac{3}{2}}} u(y') \, dy'.
\]
It is easy to check that
\[ P u(x) = \left( \frac{\sqrt{2}}{|x + e_n|} \right)^{n-2} (P v)(F(x)). \] (6)

We will use the fact
\[ |\nabla^k x P(y', x)| = |\nabla^k y P(y', x)| \leq C(k)x_n(|x' - y|^2 + x_n^2)^{-\frac{n+k}{2}} \] (7)
for \( x' \neq y', k = 1, \ldots, \) to obtain regularity.

The main result of this section is the blowing up a bubble result as follows.

**Theorem 2.1.** Let \( \frac{n}{n-2} \leq p_i < \frac{n+2}{n-2} \) be a sequence of numbers with \( \lim_{i \to \infty} p_i = \frac{n}{n-2} \), and \( K_i \in C^1(B_1') \) be a sequence of positive functions satisfying
\[ K_i \geq \frac{1}{c_0}, \quad \|K_i\|_{C^1(B_1')} \leq c_0 \]
for some constant \( c_0 \geq 1 \) independent of \( i \). Suppose that \( u_i \in C^0(\mathbb{R}^{n-1}) \) is a sequence of nonnegative solutions of
\[ K_i(x')u_i(x')^{p_i} = \int_{\mathbb{R}^{n-1}_+} P(x', y)P u_i(y)^{\frac{n+2}{n-2}} \, dy \quad \text{for } x' \in B_1' \] (8)
and \( u_i(0) \to \infty \) as \( i \to \infty \). Suppose that \( R_i u_i(0)^{p_i-\frac{n+2}{n-2}} \to 0 \) for some \( R_i \to \infty \) and
\[ u_i(x') \leq bu_i(0) \quad \text{for } |x'| < R_i u_i(0)^{p_i-\frac{n+2}{n-2}}, \]
where \( b > 0 \) is independent of \( i \). Then, after passing to a subsequence, we have
\[ \phi_i(x') := \frac{1}{u_i(0)} u_i(0)^{p_i-\frac{n+2}{n-2}} x' \to \phi(x') \quad \text{in } C^{1/2}_{\text{loc}}(\mathbb{R}^{n-1}), \] (9)
where \( \phi > 0 \) satisfies
\[ K \phi(x')^\frac{n}{n-2} = \int_{\mathbb{R}^n_+} P(x', y)P \phi(y)^{\frac{n+2}{n-2}} \, dy \quad \text{for } x' \in \mathbb{R}^{n-1} \] (10)
and \( K = \lim_{i \to \infty} K_i(0) \) along the subsequence.

Solutions of (10) in \( L^{\frac{2(n-1)}{n-2}}_{\text{loc}}(\mathbb{R}^{n-1}) \) were classified in [11], which are \( (1 + |x|^2)^{-\frac{n-2}{2}} \) upon multiplying, translating and scaling.

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Proof. Note that
\[ H_i(x') \phi_i(x') p_i = \int_{\mathbb{R}^n} P(x', y) \mathcal{P} \phi_i(y) \frac{n+2}{n-2} dy \quad \text{for } |x'| < R_i, \quad (11) \]
where \( H_i(x') = K_i(u_i(0)p_i - n^{n+2}x') \). By the assumption, we have
\[ 0 \leq \phi_i(x') \leq b \quad \text{for } |x'| < R_i. \quad (12) \]

**Step 1.** Estimates of \( \phi_i \) and convergence.
For any fixed \( 0 < R < R_i/2 \), define
\[ \Phi_i' = \mathcal{P} (\chi_{B_R} \phi_i) \quad \text{and} \quad \Phi_i'' = \mathcal{P} (1 - \chi_{B_R} \phi_i), \]
where \( \chi_\Omega \) is the characterization function of the set \( \Omega \). Then \( \mathcal{P} \phi_i = \Phi_i' + \Phi_i'' \). Since the Poisson kernel is nonnegative, by (12) we have
\[ 0 \leq \Phi_i'(y) \leq b. \quad (13) \]

Since \( K_i \leq c_0 \) and (12), by (11) we have for any \( |x'| < R_i \),
\[ c_0 b p_i \geq \int_{B_{R/2}(x', e_n)} P(x', y) \mathcal{P} \phi_i(y) \frac{n+2}{n-2} dy \]
\[ \geq \frac{1}{C} \int_{B_{1/2}(x', e_n)} \mathcal{P} \phi_i(y) \frac{n+2}{n-2} dy \geq \frac{1}{C} \mathcal{P} \phi_i(\bar{y}) \frac{n+2}{n-2} \]
for some \( \bar{y} \in B_{1/2}(x', e_n) \), where we used the mean value theorem in the last inequality and \( C > 0 \) depends only on \( n \). It follows that
\[ \Phi_i''(\bar{y}) \leq \mathcal{P} \phi_i(\bar{y}) \leq C b^{\frac{p_i(n-2)}{n+2}}. \quad (14) \]

By the definition of \( \Phi_i''(\bar{y}) \), we immediately see the boundary Harnack inequality
\[ \frac{\Phi_i''(y)}{y_n} \leq C \frac{\Phi_i''(\bar{y})}{\bar{y}_n} \quad \text{for } y \in B'_1(x') \times (0, 2), \ |x'| < R - 1, \quad (15) \]
where \( C > 0 \) depends only on \( n \). Combining (13), (14) and (15) together, we have
\[ \mathcal{P} \phi_i(y) \leq C \quad \text{for every } y \in B'_{R-1} \times (0, 1]. \]

Using the above estimate, by direct computations we have
\[ \| \int_{B'_{R-1} \times (0, 1]} P(\cdot, y) \mathcal{P} \phi_i(y) \frac{n+2}{n-2} dy \|_{C^\alpha(B'_{R-2})} \]
\[ \leq C \int_{B'_{R-1} \times (0, 1]} (|x' - y'|^2 + y_n^2) \frac{n-1+\alpha}{2} dy \leq C(n, b, \alpha, R) \]

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for any \( \alpha \in (0, 1) \). On the other hand, for \( |x'| < R - 2 \), by (7) we have

\[
|\nabla_x'(\int_{\mathbb{R}^n_+ \setminus B_{R-1}^* \times [0,1]} P(x',y)\mathcal{P}(y)^{\frac{n+2}{n-2}}dy)| \leq C \int_{\mathbb{R}^n_+ \setminus B_{R-1}^* \times [0,1]} P(x',y)\mathcal{P}(y)^{\frac{n+2}{n-2}}dy 
\leq C(H_i(x')\mathcal{P}(x'))^{p_i} \leq Cb^{p_i}.
\]

where \( C > 0 \) depends only on \( n \) and \( c_0 \). Combining the above two estimates and using (11) we conclude that with \( \alpha = 3/4 \)

\[
\|\phi_i^{p_i}\|_{C^{3/4}(B_{R-1}')} \leq C(n,b,c_0,R).
\] (16)

Since \( \phi_i(0)^{p_i} = 1 \), by (16) one can find \( \delta > 0 \), depending only on \( n, b \) and \( c_0 \), such that \( \phi_i(x')^{p_i} \geq 1/2 \) for all \( |x'| < \delta \). Hence,

\[
\mathcal{P}(\phi_i(y) \geq y_n c(n) \int_{B^*_B} \frac{1}{(|x' - y'|^2 + y_n^2)^{n/2}}^{2-1/p_i} dx' \geq \frac{1}{C(n,b,c_0) (1 + |y|)^n}.
\]

for some \( C(n,b) > 0 \). Inserting the above estimate into (11), we see that for any \( 0 < |x'| < R - 1 \)

\[
\phi_i(x')^{p_i} \geq \frac{1}{C(n,b,c_0,R)} > 0.
\]

It follows from (16) that

\[
\|\phi_i\|_{C^{3/4}(B_{R-2}')} \leq C(n,b,c_0,R).
\] (17)

Therefore, (9) follows.

**Step 2.** \( \mathcal{P}(\phi_i) \) and the equation of \( \phi_i \) convergence.

The difficulty arises because there is no information about the behavior of \( \phi_i \) in the complement of \( B_{R_i}^* \). Here we adapt some idea from [14] by using monotonicity. For any \( 0 < R < R_i/2 \), we write equation (11) as

\[
H_i(x')\mathcal{P}(x')^{p_i} = \int_{B_{R_i}^*} P(y,x')\mathcal{P}(y)^{\frac{n+2}{n-2}}dy + h_i(R,x'),
\] (18)

where

\[
h_i(R,x') = \int_{\mathbb{R}^n_+ \setminus B_{R_i}^*} P(y,x')\mathcal{P}(y)^{\frac{n+2}{n-2}}dy.
\]

By (7), for any \( |x'| < R - 1 \) we have \( |\nabla h_i(R,x')| \leq C h_i(R,x') \leq C b^{p_i} \) for some \( C > 0 \) depending only on \( n \) and \( R \). Therefore, subject to subsequence \( h_i(R,x') \to h(R,x') \) for some nonnegative function \( h \in C^1(B_{R-1}^*) \).

Similar as in step 1, we split \( \mathcal{P}(\phi_i) \) as two parts \( \Phi_i' \) and \( \Phi_i'' \) with \( R \) replaced by \( R + 10 \). By (17) with \( R - 2 \) replaced by \( R + 8 \) and elementary estimates of Poisson integral, we
have $\|\Phi_i\|_{C^{3/4}(B_R^+)} \leq C(n, b, R)$. While $\|\Phi''_i\|_{C^{3/4}(B_R^+)} \leq C(n, b, R)$ follows from (14), (15) and interior estimates for harmonic functions. Therefore, subject to a subsequence,

$$\mathcal{P}\phi_i \rightarrow \tilde{\Phi} \quad \text{in} \quad C^{1/2}_{\text{loc}}(\mathbb{R}^n_+)$$

for some $\tilde{\Phi} \geq 0$ satisfies

$$-\Delta \tilde{\Phi} = 0 \quad \text{in} \quad \mathbb{R}^n_+ \quad \text{and} \quad \tilde{\Phi} = \phi \quad \text{on} \quad \partial\mathbb{R}^n_+.$$

Since $0 \leq \phi \leq b$, $\mathcal{P}\phi$ is bounded in $\mathbb{R}^n_+$. Hence, $\tilde{\Phi} - \mathcal{P}\phi$ is harmonic function bounded from below in $\mathbb{R}^n_+$ and satisfies the homogenous Dirichlet boundary condition. It follows from the Liouville theorem on the half space, see, e.g., Sun-Xiong [18], that

$$\tilde{\Phi}(x) = \mathcal{P}\phi(x) + ax_n \quad \text{for some constant} \quad a \geq 0. \quad (19)$$

Sending $i \rightarrow \infty$ in (18), we have

$$K\phi(x') \frac{n}{n-2} = \int_{B_R^+} P(y, x') \tilde{\Phi}(y) \frac{n+2}{n-2} \, dy + h(R, x') \quad (20)$$

If $a > 0$ in (19), sending $R \rightarrow \infty$ we see that

$$K\phi(0) \frac{n}{n-2} \geq \int_{B_R^+} P(y, 0) \tilde{\Phi}(y) \frac{n+2}{n-2} \, dy \rightarrow \infty.$$

This is impossible. Hence, $a = 0$ and $\tilde{\Phi} = \mathcal{P}\phi$. By (20), $h(R, x')$ is decreasing with respect to $R$. Note that for $R > |x'|$,

$$\frac{R^n}{(R + |x|)^n} h_i(R, 0) \leq h_i(R, x')$$

$$= \int_{\mathbb{R}^n_+ \setminus B_R^+} \frac{|y|^n}{(|y' - x'|^2 + y_n^2)^{n/2}} \frac{y_n}{|y|^n} \mathcal{P}\phi_i(y) \frac{n+2}{n-2} \, dy$$

$$\leq \frac{R^n}{(R - |x|)^n} h_i(R, 0).$$

It follows that

$$\lim_{R \rightarrow \infty} h(R, x') = \lim_{R \rightarrow \infty} h(R, 0) =: c_1 \geq 0.$$

Sending $R$ to $\infty$ in (20), by Lebesgue’s monotone convergence theorem we have

$$K\phi(x') \frac{n}{n-2} = \int_{\mathbb{R}^n_+} P(y, x') \mathcal{P}\phi(y) \frac{n+2}{n-2} \, dy + c_1.$$

If $c_1 > 0$, then $\phi \geq \frac{c_0}{c_1} > 0$ and thus $\mathcal{P}\phi \geq \frac{c_0}{c_1}$. This is impossible, otherwise the right hand side integration is infinity. Hence $c_1 = 0$.

Therefore, we complete the proof. \qed
3 A variational problem

Let $K \in C^1(\partial B_1)$ be a positive function satisfying $K(\xi) = K(-\xi)$, and $L^p_{as}(\partial B_1) \subset L^p(\partial B_1)$, $p \geq 1$, be the set of antipodally symmetric functions. For $p \geq \frac{n}{n-2}$, define

$$
\lambda_{as,p}(K) = \sup \left\{ \int_{B_1} |\mathcal{P}v|^{\frac{2n}{n-2}} \, d\xi : v \in L^p_{as}(\partial B_1) \text{ with } \int_{\partial B_1} K|v|^{p+1} \, ds = 1 \right\}.
$$

Denote $\lambda_{as,\frac{n}{n-2}} = \lambda_{as}$ for brevity.

**Proposition 3.1.** If

$$
\lambda_{as}(K) > \frac{S(n)^{\frac{2n}{n-2}}}{(\min_{\partial B_1} K)^{\frac{n}{n-2}2^{1/(n-1)}}};
$$

then $\lambda_{as}(K)$ is achieved.

**Proof.** We claim that $\liminf_{p \searrow \frac{n}{n-2}} \lambda_{as,p}(K) \geq \lambda_{as}(K)$.

Indeed, for any $\varepsilon > 0$, by the definition of $\lambda_{as}(K)$ one can find a function $v \in L^\infty_{as}(\partial B_1)$ such that

$$
\int_{B_1} |\mathcal{P}v|^{\frac{2n}{n-2}} \, d\xi > \lambda_{as}(K) - \varepsilon \quad \text{and} \quad \int_{\partial B_1} K|v|^{\frac{2(n-1)}{n-2}} \, ds = 1.
$$

Let $V_p := \int_{\partial B_1} K|v|^{p+1} \, ds$. Since $\lim_{p \searrow \frac{n}{n-2}} V_p = \int_{\partial B_1} K|v|^{\frac{2(n-1)}{n-2}} \, ds = 1$, we have, for $p$ close to $\frac{n}{n-2}$,

$$
\lambda_{as,p}(K) \geq \int_{B_1} |\mathcal{P}(v^{\frac{p}{p+1}})|^{\frac{2n}{n-2}} \, d\xi \geq \lambda_{as}(K) - 2\varepsilon.
$$

By the arbitrary choice of $\varepsilon$, the claim follows.

By the above claim, one can seek $p_i \searrow \frac{n}{n-2}$ as $i \to \infty$ such that $\lambda_{as,p_i}(K) \to \lambda \geq \lambda_{as}(K)$. Since $K \in C^1(\partial B_1)$ and $K$ is positive, it follows from the compact embedding result Corollary 2.2 of [11] that for $p_i > \frac{n}{n-2}$, $\lambda_{as,p_i}(K)$ is achieved, say, by $v_i$. Since $|\mathcal{P}v_i| \leq \mathcal{P}|v_i|$, we may assume $v_i$ is nonnegative. Noticing that $\|v_i\|_{L^{p_i+1}(\partial B_1)} \leq 1/\min_{\partial B_1} K$, by (2) we have $\|\mathcal{P}v_i\|_{L^{\frac{2n}{n-2}}(B_1)} \leq C$ for some $C$ independent of $i$. It is easy to see that $v_i$ satisfies the Euler-Lagrange equation

$$
\lambda_{as,p_i}(K)K(\xi)v_i(\xi)^{p_i} = \int_{B_1} \mathcal{P}(\xi, \eta)|\mathcal{P}v_i(\eta)|^{\frac{n+2}{n-2}} \, d\eta \quad \forall \xi \in \partial B_1.
$$

Hence, subject to a subsequence,

$$
v_i \rightharpoonup v \quad \text{weakly in } L^{\frac{2(n-1)}{n-2}}(\partial B_1)
$$

$$
\mathcal{P}v_i \rightharpoonup V \quad \text{weakly in } L^{\frac{2n}{n-2}}(B_1)
$$
for some nonnegative function \( v \in L^{\frac{2(n-1)}{n-2}}(\partial B_1) \) and \( V \in L^{\frac{2n}{n-2}}(B_1) \). By the compact embedding again, \( V = \mathcal{P} v \). Hence, \( v \) satisfies
\[
\lambda K(\xi)v(\xi)^{\frac{n}{n-2}} = \int_{B_1} P(\xi, \eta)Pv(\eta)^{\frac{n+2}{n-2}} \, d\eta. \tag{23}
\]

It follows that either \( v \equiv 0 \) or \( v > 0 \). If the later happens, then \( \lambda = \lambda_{as}(K) \) and we are done. Suppose now \( v \equiv 0 \).

By Proposition 5.2 of [11] and equation (22), \( v_i \in C(\partial B_1) \). By standard arguments (see the proof of Theorem 2.1), we have \( v_i \in C^\alpha(\partial B_1) \) for any \( 0 < \alpha < 1 \). Since \( v = 0 \), we must have \( v_i(\xi_i) = \max_{\partial B_1} v_i \to \infty \) as \( i \to \infty \). We may assume \( \xi_i \to \xi \) because \( \partial B_1 \) is compact. By stereographic projection with \( \xi_i \) as the south pole, equation (22) is transformed into
\[
\lambda_{as,p_i}(K_i)K_i(x')u_i(x')^{p_i} = \int_{\mathbb{R}^n_+} P(x', y)\mathcal{P}u_i(y)^{\frac{n+2}{n-2}} \, dy \quad \forall x' \in \mathbb{R}^{n-1}, \tag{24}
\]
where \( K_i(x') = K(F(x'))(\frac{2}{|x'|^2+1})^{(n-2)p_i-n)/2 \) and \( u_i(x') = (\frac{2}{|x'|^2+1})^{\frac{2}{n-2}}v_i(F(x')) \). Hence \( u_i(0) = \max_{\mathbb{R}^{n-1}} u_i \to \infty \) as \( i \to \infty \). By Theorem 2.1, we have, subject to a subsequence,
\[
\phi_i = \frac{1}{u_i(0)}u_i(u_i(0)^{p_i-\frac{n+2}{n-2}}x') \to \phi(x') \quad \text{in } C^{1/2}_{\text{loc}}(\mathbb{R}^{n-1})
\]
for some \( \phi \geq 0 \) satisfying
\[
\lambda K(\bar{\xi})\phi(x')^{\frac{n}{n-2}} = \int_{\mathbb{R}^n_+} P(x', y)\mathcal{P}\phi(y)^{\frac{n+2}{n-2}} \, dy. \tag{25}
\]
By [11], \( \phi \) is classified. Since \( v_i \) is nonnegative and antipodally symmetric, for any small \( \delta > 0 \) we have
\[
1 = \int_{\partial B_1} K v_i^{p_i+1} \, ds \geq 2 \int_{F(B'_1)} K v_i^{p_i+1} \, ds = 2 \int_{B'_1} K_i u_i^{p_i+1} \, dx' = 2 \int_{B'} K_i(u_i(0)^{p_i-\frac{n+2}{n-2}}y')\phi_i(y')^{p_i+1} \, dy' \\
\geq 2 \int_{B'_R} K_i(u_i(0)^{p_i-\frac{n+2}{n-2}}y')\phi_i(y')^{p_i+1} \, dy' \to 2K(\bar{\xi}) \int_{B'_R} \phi(y')^{\frac{2(n-1)}{n-2}} \, dy' \quad \text{as } i \to \infty \text{ for any fixed } R > 0.
\]
It follows that
\[
1 \geq 2K(\bar{\xi}) \int_{\mathbb{R}^{n-1}} \phi(y')^{\frac{2(n-1)}{n-2}} \, dy'. \tag{26}
\]
Hence, it follows from (2), (25) and (26) that

\[ S(n)^{\frac{2n}{n-2}} \geq \frac{\int_{\mathbb{R}^n_+} |P\phi|^\frac{2n}{n-2}}{(\int_{\mathbb{R}^{n-1}} |\phi|^\frac{2(n-1)}{n-2})^{\frac{n}{n-1}}} = \lambda K(\bar{\xi}) \left( \int_{\mathbb{R}^{n-1}} |\phi|^\frac{2(n-1)}{n-2} \right)^{-\frac{1}{n-1}} \geq \lambda K(\bar{\xi})^{\frac{n}{n-2}} 2^{\frac{1}{n-1}}. \]

This yields

\[ \lambda \leq \frac{S(n)^{\frac{2n}{n-2}}}{(\min K)^{\frac{n}{n-1}}}, \]

which contradicts the assumption (21). We complete the proof.

\[ \square \]

**Proposition 3.2.** Let \( n \geq 3 \) and \( K \in C^1(\partial B_1) \) be a positive function satisfying \( K(\xi) = K(-\bar{\xi}) \). For every \( q > n - 1 \), there exists a constant \( \delta > 0 \), depending only on \( n \) and \( q \), such that if for a minimal point \( \xi_1 \) of \( K \) there holds \( K(\xi) - K(\xi_1) \leq \delta |\xi - \xi_1|^q \) for all \( \xi \in \partial B_1 \), then (21) is valid.

**Proof.** Let \( \xi_1 \in \partial B_1 \) be a minimum point of \( K \) and \( \xi_2 = -\xi_1 \). Without loss of generality, we may assume \( \xi_1 \) is the south pole. For \( \beta > 1 \) and \( i = 1, 2 \), let

\[ v_{i,\beta}(\xi) = \begin{cases} \left( \frac{\sqrt{\beta^2-1}}{\beta - \cos r_i} \right)^{\frac{n-2}{2}} & \text{if } r_i \leq \frac{\pi}{2} \\ 0 & \text{if } r_i > \frac{\pi}{2} \end{cases} \]  

and

\[ v_\beta = v_{1,\beta} + v_{2,\beta}, \]

where \( r_i = d(\xi, \xi_i) \) is the geodesic distance between \( \xi \) and \( \xi_i \) on the sphere. Let

\[ u_{i,\lambda}(y') = \left( \frac{2}{1 + |y'|^2} \right)^{\frac{n-2}{2}} v_{i,\beta}(F(y')) \]

where \( F(y') \) is the inverse of stereographic projection and \( \lambda = \sqrt{\frac{\beta-1}{\beta+1}} \), and \( u_\lambda = u_{1,\lambda} + u_{2,\lambda} \). By direct computations, we have

\[ u_{1,\lambda}(y') = 2^{\frac{n-2}{2}} \left( \frac{\lambda}{\lambda^2 + |y'|^2} \right)^{\frac{n-2}{2}} \chi_{\{|y'| \leq 1\}} =: w_{1,\lambda}(y) \chi_{\{|y'| \leq 1\}} \]

\[ u_{2,\lambda}(y') = 2^{\frac{n-2}{2}} \left( \frac{\lambda}{1 + \lambda^2 |y'|^2} \right)^{\frac{n-2}{2}} \chi_{\{|y'| \geq 1\}}. \]
Hence,

\[ u_\lambda = w_{1,\lambda} + w_{2,\lambda}, \]

where

\[ w_{2,\lambda}(y') = 2^{\frac{n-2}{2}} \left( \left( \frac{\lambda}{1 + \lambda^2 |y'|^2} \right)^{\frac{n-2}{2}} - \left( \frac{\lambda}{\lambda^2 + |y'|^2} \right)^{\frac{n-2}{2}} \right) \chi(|y'| \geq 1). \]

Since \( 1 + \lambda^2 |y'|^2 - (\lambda^2 + |y'|^2) = (|y|^2 - 1)(\lambda^2 - 1) < 0 \) if \( |y| \geq 1 \) and \( \lambda < 1 \), we have \( w_{2,\lambda}(y') \geq 0 \). Let \( U_\lambda = P_{u_\lambda} = W_{1,\lambda} + W_{2,\lambda} \). We have

\[ W_{1,\lambda}(y) = 2^{\frac{n-2}{2}} \left( \frac{\lambda}{(y_n + \lambda)^2 + |y'|^2} \right)^{\frac{n-2}{2}}. \]

For \( |y| \leq \frac{1}{2} \), we have

\[
W_{2,\lambda}(y) = \frac{1}{n \omega_n} \int_{\mathbb{R}^{n-1} \setminus B_1} \frac{y_n}{(|x' - y'|^2 + y_n^2)^{\frac{n}{2}}} w_{2,\lambda}(x') \, dx' \\
\geq \frac{1}{C} y_n \int_{\mathbb{R}^{n-1} \setminus B_1} |x'|^{-n} w_{2,\lambda}(x') \, dx' \\
\geq \frac{1}{C} \lambda^{\frac{n-2}{2}} y_n
\]

for some \( C > 0 \) independent of \( \lambda \).

By the conformal invariance, antipodal symmetry and the fact

\[ \mathcal{P}_v(\beta)(\xi) \leq C \lambda^{-\frac{n-2}{2}} \text{dist}(\xi, \{\xi_1, \xi_2\})^{2-n}, \]

we have

\[
\int_{B_1} |\mathcal{P}_v|^{\frac{2n}{n-2}} \, d\xi = \int_{\mathbb{R}^n_+} U_{\lambda}^{\frac{2n}{n-2}} \, dy \\
= 2 \int_{B_{1/2}^n} U_{\lambda}^{\frac{2n}{n-2}} \, dy + O(\lambda^n) \\
= 2 \int_{B_{1/2}^n} W_{1,\lambda}^{\frac{2n}{n-2}} + \frac{2n}{n-2} W_{1,\lambda}^{\frac{n+2}{n-2}} W_{2,\lambda} \, dy + O(\lambda^n) \\
\geq 2(\omega_n + \frac{2^{\frac{n+4}{2}} n}{n-2} \lambda^{n-1} \int_{B_{1/2}^n \setminus \mathbb{R}^n_+} (\frac{1}{(z_n + 1)^2 + |z'|^2})^{\frac{n+2}{2}} z_n \, dz + O(\lambda^n)) \\
= 2(\omega_n + \frac{2^{\frac{n+4}{2}} n}{n-2} \lambda^{n-1} \int_{\mathbb{R}^n_+} (\frac{1}{(z_n + 1)^2 + |z'|^2})^{\frac{n+2}{2}} z_n \, dz + O(\lambda^n)) \\
=: 2(\omega_n + A \lambda^{n-1} + O(\lambda^n)) \quad (28)
\]
with $A > 0$. On the other hand, let $q > n - 1$ and suppose $K(\xi) - K(\xi_1) \leq \delta|\xi - \xi_1|^q$, where $\delta > 0$ is to be fixed. It follows that

$$
\int_{\partial B_1} K v_{1,\beta}^{2(n-1) \over n-2} = 2 \int_{\partial B_1 \cap \{x_n < 0\}} K v_{1,\beta}^{2(n-1) \over n-2} = 2(K(\xi_1) \int_{\partial B_1 \cap \{x_n < 0\}} v_{1,\beta}^{2(n-1) \over n-2} + \delta \int_{\partial B_1 \cap \{x_n < 0\}} |\xi - \xi_1|^q v_{1,\beta}^{2(n-1) \over n-2}) \leq 2K(\xi_1)(n\omega_n + \delta C(n, q) \lambda^{n-1}).
$$

Setting $\frac{n}{n-1} \delta C(n, q) < A$, for small $\lambda$ we have

$$
\int_{B_1} |P v_\beta|^{2n \over n-2} d\xi \geq {2\omega_n \over (\int_{\partial B_1} K v_{1,\beta}^{2(n-1) \over n-2})^{n \over n-1}} = {S(n)^{2n \over n-2} \over (\min_{\partial B_1} K)^{n \over n-1} 2^{1/(n-1)}}.
$$

Therefore, we complete the proof.

\[\square\]

**Proof of Theorem 1.1.** It follows immediately from Proposition 3.1 and Proposition 3.2.

\[\square\]

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