Capital Regulation under Price Impacts and Dynamic Financial Contagion

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Abstract

We construct a continuous time model for price-mediated contagion precipitated by a common exogenous stress to the trading book of all firms in the financial system. In this setting, firms are constrained so as to satisfy a risk-weight based capital ratio requirement. We use this model to find analytical bounds on the risk-weights for an asset as a function of the market liquidity. Under these appropriate risk-weights, we find existence and uniqueness for the joint system of firm behavior and the asset price. We further consider an analytical bound on the firm liquidations, which allows us to construct exact formulas for stress testing the financial system with deterministic or random stresses. Numerical case studies are provided to demonstrate various implications of this model and analytical bounds.

Key words: Financial contagion; fire sales; risk-weighted assets; stress testing

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1 Introduction

Financial contagion occurs when the negative actions of one bank or firm causes the distress of a separate bank or firm. Such events are of critical importance due to their relation to systemic risk. In this work we consider price-mediated contagion that occurs through impacts to mark-to-market wealth as firms hold overlapping portfolios. Price-mediated contagion can occur due to the price impacts of liquidations in a crisis and can be exacerbated by pro-cyclical regulations. Importantly, this kind of contagion can be self-reinforcing, causing extreme events and ultimately a systemic crisis as witnessed in, e.g., the 2007-2009 financial crisis.

Systemic risk and financial contagion has been studied in a network of interbank payments by [15]. We refer to [25, 24] for a review of this payment network model and extensions thereof to include, e.g., bankruptcy costs. The focus of this paper is on price-mediated contagion and fire sales. This single contagion channel causes impacts globally to all other firms due to mark-to-market accounting. As prices drop due to the liquidations of one bank, the value of the assets of all other banks are also impacted. The model from [15] has been extended to consider fire sales and price-mediated contagion in a static, one-period, system by works such as [11, 23, 19, 11, 10, 25, 2, 16, 18, 17, 5]. Price-mediated contagion and fire sales have been studied in other works without the inclusion of interbank payment networks. This has been undertaken in a static setting by [21, 12, 7, 6], in a discrete time setting in [8, 9], and in continuous time by [13, 14].

In this work we will be extending the model of [7, 6] to incorporate true time dynamics. Those works present a static price-mediated contagion due to deleveraging and the need to satisfy a capital ratio requirement. In particular, we will focus on the case in which firms liquidate assets during a crisis due to risk-weighted capital requirement constraints. These capital requirements will be described by the ratio of equity over risk-weighted assets. We focus on those works as they include methodology for calibrating the model to public data, but also include equilibrium liquidations and prices that in reality occur over time.
Herein we will consider a continuous time model for these equilibrium liquidations and price movements. We will demonstrate that such a model has useful mathematical properties, notably uniqueness of the clearing prices in time. This is in contrast to the static models of, e.g., [11, 18, 6] in which fire sales due to capital ratios can result in multiple equilibria. Further, by incorporating time dynamics, we are able to consider the first-mover advantage in which the first firm to engage in the fire sale will receive a higher price than later firms. This is not accounted for in any of the static models discussed previously.

Briefly, the risk-weighted capital ratio that we consider in this work is featured in, e.g., the Basel Accords and is defined by a firm’s capital divided by its risk-weighted assets. For more details, we refer to [7, 6]. Officially, in Basel III, the total capital is defined as the sum of Tier 1 and Tier 2 capital. In this work, we do not consider a distinction between different types of capital. The risk-weighted assets are defined as being a weighted sum of the mark-to-market assets. Conceptually, the riskier an asset the greater its risk-weight. The risk-weights of credit portfolios are given by, e.g., the Basel Accords or national laws, and often determined by internal models of each institution [4]. Basel regulations state that the risk-based capital ratio must never be below 8%. When a firm is constrained by this ratio, the firm will typically need to liquidate assets in order to reduce liabilities as issuing equity in such a scenario is often untenable or excessively costly [22, 20]. However, these liquidations can and will cause price impacts on the risk-weighted assets. This causes feedback effects which causes that same firm to liquidate further assets as well as negatively effects the risk-weighted capital ratio of all other firms.

As stated, the static model was studied in, e.g., [11, 7, 6, 18]. In those works, uniqueness of the prices and liquidations cannot be guaranteed for most financial systems. Further, if the price impact is too large it is found that banks can no longer satisfy their capital ratio requirement even after selling all assets. In contrast, we will demonstrate that, in continuous time, a firm will never need to sell all assets, though may asymptote its asset holdings to 0. In fact, we will relate the price impacts of the illiquid asset to appropriate risk-weights. We
will also demonstrate that if the risk-weight were set too low in relation to price impacts, the firm will be forced to purchase assets to drive up the price rather than liquidate.

The primary goal of this paper is to model the behavior of banks so that they satisfy this capital ratio requirement continuously in time under price impacts. The use of a dynamic model is important as the Basel regulations enforce the risk-weighted capital ratio to exceed the threshold at all times. In particular, we will consider the situation in which multiple banks may be at the regulatory threshold and behaving in the required manner so as to consider the implications of financial contagion to systemic risk. In utilizing the proposed model, we will consider appropriate choices for the risk-weight as a function of market liquidity. Additionally, in proposing the continuous time model for bank behavior, we find an analytical bound to the firm behavior. This is particularly of value as it allows us to consider a distribution of outcomes for the health of the system directly under randomized stress tests.

This paper is organized as follows. Section 2 presents the risk-weighted capital ratio and its relation to both leverage requirements and capital requirements based on positive homogeneous risk measures. Section 3 proposes the differential model for the actions of a single bank system. This is extended in Section 4 to provide existence and uniqueness results in a\( n \) bank financial system. As this model has no closed-form solution in general, we propose an analytical approximation that bounds the system response for stress testing purposes in Section 5. These analytical results allow for a bound on, e.g., the probability that the terminal asset price is above some threshold in a probabilistic setting. Numerical case studies are provided in Section 6 to demonstrate simple insights from this model and provide numerical accuracy of the analytical bounds from Section 5. The proofs of all theoretical results are presented in the appendix.
2 The Risk-Weighted Capital Ratio

In this section we provide mathematical descriptions of the risk-weighted capital ratio (henceforth often referred to merely as the capital ratio) and provide two remarks to relate other notions of regulatory requirements to the risk-weighted capital ratio we consider throughout this work. We do this to demonstrate that the results are generalizable to other settings as well. We first consider the leverage ratio. Second we relate capital requirements based on positive homogeneous risk measures, e.g., value-at-risk and expected shortfall, to the capital ratio.

2.1 Risk-Weighted Capital Ratios

Consider a firm with stylized trading book depicted in Figure 1. That is, at time 0, the firm has assets split between liquid investments (e.g., cash or otherwise zero risk-weighted assets) denoted by $x \geq 0$ and illiquid investments (e.g., tradable credit positions) denoted by $s \geq 0$. For simplicity and without loss of generality, we will assume that the initial price of all assets is 1, thus the mark-to-market assets for the bank at time 0 is equal to $x + s$. The firm has liabilities in the total amount of $\bar{p} \geq 0$. For simplicity in this work, we will assume...
that all liabilities are long term and not held by any other firms in this system. The capital of the firm, at time 0, is thus provided by \( x + s - \bar{p} \). For further simplicity, throughout this work we will assume a single, representative, illiquid asset only. This is along the lines of the undertaking in, e.g., [11, 2, 6].

Capital ratios are used for regulatory purposes to bound the risk of financial institutions. We will assume that this illiquid asset has risk-weight \( \alpha > 0 \) and price process \( q : [0, T] \to \mathbb{R}^+ \) (with \( q(0) = 1 \)). The bank may liquidate assets over time. We will assume that, at time \( t \), they liquidate the illiquid asset at a rate of \( \gamma(t) \). The total amount of cash gained from liquidations up to time \( t \) is provided by \( \Psi(t) = \int_0^t \gamma(u)q(u)du \) and the total number of units liquidated up to time \( t \) is provided by \( \Gamma(t) = \int_0^t \gamma(u)du \). Thus, as depicted in Figure 1, at time \( t \), the liquid assets for the firm are provided by \( x + \Psi(t) \) and the illiquid assets by \( (s - \Gamma(t))q(t) \). Throughout this work we will assume that prices drop over time and as a function of the liquidations, so the total assets and therefore also capital will drop over time as shown by the crossed out portions of the trading book in Figure 1. More discussion on the price changes will be provided in Section 3 below.

The capital ratio for a firm at time \( t \) is given by total capital divided by the risk-weighted assets. Mathematically, this is formulated as

\[
\theta(t) = \frac{(x + \Psi(t) + [s - \Gamma(t)]q(t) - \bar{p})^+}{\alpha [s - \Gamma(t)] q(t)}. \tag{1}
\]

The capital ratio requirement specifies that all institutions must satisfy the condition that \( \theta(t) \geq \theta_{\text{min}} \) for all times \( t \) for some minimal threshold \( \theta_{\text{min}} > 0 \).

We will assume throughout that \( \alpha, \theta_{\text{min}} > 0 \) with \( \alpha \theta_{\text{min}} < 1 \). Additionally, we will assume that the firm satisfies the capital ratio at the initial time 0, i.e., \( \theta(0) \geq \theta_{\text{min}} \).

**Remark 2.1.** If \( \alpha \theta_{\text{min}} \geq 1 \), the assumption that the capital ratio at time 0 is above the regulatory threshold guarantees that the risk-weighted capital ratio is nonincreasing in the price of the asset. To see this we note that, at time \( t = 0 \) (i.e., before any intervention from...
the bank):  

\[ \frac{(x + sq(0) - \bar{p})^+}{sq(0)} \geq \alpha \theta_{\min} \geq 1 \Rightarrow \frac{x - \bar{p}}{sq(0)} \geq 0 \Leftrightarrow x \geq \bar{p}. \]

From this we can immediately conclude that the capital ratio is nonincreasing in the price, which is contrary to the understanding of how a regulatory threshold usually works. In particular, for the considerations of this paper, this monotonicity implies that, as the price drops (without the intervention of the firm), the bank will always satisfy the capital regulation, and thus no rebalancing of assets will ever need to occur.

### 2.2 Leverage Requirements

We now wish to consider how leverage requirements relate with the capital ratio requirements discussed above. As with the capital ratio, we will assume that in a crisis situation issuing new equity will be prohibitively expensive or not feasible. As such, the only method for reducing leverage would be to use liquid (cash) assets to pay down liabilities. Using the same notation and trading book notions from above, the leverage of a firm at time \( t \) is given by the total assets divided by equity. As we assume the only way to delever is to pay down liabilities with liquid assets, we will consider the case in which \( x^* \in [0, x] \) and \( \Psi^*(t) \in [0, \Psi(t)] \) of the initial liquid endowment and cash obtained from liquidations is used to reduce liabilities. The leverage ratio at time \( t \) is thus, mathematically, given by

\[
\lambda(t) = \frac{[x - x^*] + [\Psi(t) - \Psi^*(t)] + [s - \Gamma(t)]q(t)}{[x - x^*] + [\Psi(t) - \Psi^*(t)] + [s - \Gamma(t)]q(t) - [\bar{p} - x^* - \Psi^*(t)]}.
\]

Importantly, as we impose no reserve requirements, there is no penalty for increasing \( x^* \) and \( \Psi^*(t) \). Further, the leverage for a firm is only improved as \( x^* \) and \( \Psi^*(t) \) are increased. Thus we find that the leverage ratio \( \lambda(t) \) can be considered explicitly with \( x^* = x \) and \( \Psi^*(t) = \Psi(t) \). With this modification, the leverage ratio and capital ratio are related through

\[
\lambda(t) = \frac{1}{\alpha \theta(t)}. \quad \text{That is, a leverage requirement of } \lambda(t) \leq \lambda_{\max} \text{ is equivalent to } \theta(t) \geq \frac{1}{\alpha \lambda_{\max}}. \quad \text{In particular, these two requirements are equivalent so long as } \lambda_{\max} = \frac{1}{\alpha \theta_{\min}}.
\]
2.3 Positive Homogeneous Risk Measures

Consider capital regulations provided by a positive homogeneous risk measure \( \rho \) (e.g., value-at-risk or any coherent risk measure such as expected shortfall). Risk measures, as defined in [3], map random future wealth to capital requirements and satisfy three main properties:

1. **Normalization**: \( \rho(0) = 0 \);

2. **Monotonicity**: \( \rho(X) \leq \rho(Y) \) if \( X \geq Y \) almost surely; and

3. **Cash Invariance**: \( \rho(X + m) = \rho(X) + m \) if \( m \in \mathbb{R} \).

Here we will also assume positive homogeneity, i.e. \( \rho(\eta X) = \eta \rho(X) \) for any \( \eta > 0 \).

As described previously and shown in Figure 1, the mark-to-market wealth of a firm at time \( t \) is given by \( x + \Psi(t) + [s - \Gamma(t)]q(t) - \bar{p} \). For the purposes of risk management, we wish to consider “random” stresses to the asset price \( q \), i.e. let \( Q(t) = \beta q(t) \) be a stress scenario set for a future date that evolves from the time \( t \) price \( q(t) \) via a multiplier \( \beta \in L^2 \) such that \( \beta \in [0, 1] \) almost surely. This assumption of a constant (random) multiplier for any time \( t \) follows from the practice of considering the same historical stress scenario for the evaluation of risk. The capital requirements at time \( t \) would thus be given by the constraint

\[
\rho(x + \Psi(t) + [s - \Gamma(t)]Q - \bar{p}) := [s - \Gamma(t)]q(t)\rho(\beta) - [x + \Psi(t) - \bar{p}] \leq 0.
\]

In this construction, the risk measure applied to \( \beta \) (by the definition of a risk measure) will satisfy the constraint \( \rho(\beta) \in [-1, 0] \). In fact, if \( \rho(\beta) = 0 \) then the firm must hold cash at least equal to liabilities and therefore will not have capital requirements affected by fire sales or price movements. In contrast, if \( \rho(\beta) = -1 \) then this can be considered a system without risk, i.e., the illiquid asset has the same risk profile as the cash asset. Thus we will only consider the case when \( \rho(\beta) \in (-1, 0) \). By rearranging terms, it can be shown that satisfying the risk measure based capital requirements is equivalent to satisfying the risk-weighted capital ratio so long as the relationship \( \alpha \theta_{\min} = 1 + \rho(\beta) \) is satisfied.
3 Capital Ratio Requirements for a Single Bank System

In this section we consider a single firm attempting to satisfy its risk-weighted capital ratio when subject to price impacts. We will consider this in continuous time and determine conditions that provide unique liquidations for the bank to satisfy the capital requirement. In particular, we determine a condition relating the risk-weight and the price impacts.

Consider a single bank with 1 (liquid) cash asset and 1 illiquid asset. As the crisis model we wish to consider is generically on a short time horizon, we will consider all price impacts to be permanent for the duration of the considered time $[0, T] \subseteq \mathbb{R}_+$. Further, we will assume the price of the illiquid asset is subject to market impacts given by a nonincreasing inverse demand function $F : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^{++}$ such that $F(0, 0) = 1$. That is, $F(t, \Gamma)$ is a function of time and units sold; the inclusion of time allows for exogenous shocks, e.g., $F(t, \Gamma) = \exp(-at\mathbb{1}_{\{t<T\}} - aT\mathbb{1}_{\{t \geq T\}})f_T(\Gamma)$ for some inverse demand function $f_T : \mathbb{R} \to \mathbb{R}^{++}$. For mathematical simplicity we will restrict ourselves to the situation in which we can decouple the exogenous effects from time and the endogenous effects from firm behavior.

**Assumption 3.1.** Throughout this work we assume that $F(t, \Gamma) = f_t(t)f_T(\Gamma)$ for continuously differentiable and nonincreasing function $f_t : \mathbb{R}_+ \to (0, 1]$ and twice continuously differentiable and nonincreasing function $f_T : \mathbb{R} \to \mathbb{R}^{++}$ where $f_t(0) = f_T(0) = 1$.

**Remark 3.2.** Throughout this paper we assume Assumption 3.1 i.e., the clearing prices follow the path $f_t(t)f_T(\Gamma(t))$. However, realistically the price effects from time occur due to asset liquidations outside of the firm of interest, i.e., $F(t, \Gamma) = f_T(\eta(t) + \Gamma)$ for some (nondecreasing) exogenous liquidation function $\eta$. Letting the full inverse demand function be defined by the exponential inverse demand function with strictly positive price impact, i.e. $f_T(\Gamma) := \exp(-b\Gamma)$ with $b > 0$, then $F(t, \Gamma) = \exp(-b\eta(t))\exp(-b\Gamma)$. In particular, we can provide the one-to-one correspondence: $f_t(t) = \exp(-b\eta(t))$ and $\eta(t) = -\frac{1}{b}\log(f_t(t))$. If the inverse demand function $f_T$ were chosen otherwise, the exogenous liquidations would...
need to be defined as a function of bank liquidations $\Gamma$ as well.

Recall the setting described in Section 2.1. That is, consider the firm with initial trading book made up of liquid assets of $x \geq 0$, liabilities of $\bar{p} \geq 0$, and illiquid holdings of $s \geq 0$ at time 0 with a no-short selling constraint. The capital ratio is given by (1). As mentioned previously, we will assume that $\theta(0) \geq \theta_{\text{min}} > 0$ so that the firm satisfies the capital ratio requirement at time 0. The change in $\theta$ over time is thus given by:

$$\dot{\theta}(t) = \dot{q}(t)[s - \Gamma(t)][\bar{p} - x - \Psi(t)] + \dot{\Gamma}(t)q(t)([s - \Gamma(t)]q(t) - [\bar{p} - x - \Psi(t)])\frac{\alpha([s - \Gamma(t)]q(t))^2}{\alpha([s - \Gamma(t)]q(t))^2} \mathbb{1}_{\{\theta(t) \geq 0\}} \tag{2}$$

where

$$\dot{q}(t) = f'_t(t)f_t(\Gamma(t)) + \dot{\Gamma}(t)f_t(t)f'_{\Gamma}(\Gamma(t)), \tag{3}$$

$$\dot{\Psi}(t) = \dot{\Gamma}(t)q(t). \tag{4}$$

As a simplifying assumption, no liquidations will occur except if $\theta(t) \leq \theta_{\text{min}}$. Therefore the first time that the firm takes actions is at time $\tau$ such that $f_{\tau}(\tau) = \bar{q} := \frac{\bar{p} - x}{(1 - \alpha\theta_{\text{min}})^2}$. If $\inf_{t \in [0, \tau]} f_t(t) > \bar{q}$ then no fire sale will occurs. Once the firm starts acting, we assume that it does so only to the extent that it remains at the capital ratio requirement. Assuming it is possible (as proven later in this section) that a firm is capable of remaining at the regulatory requirement for all times through liquidations alone, i.e. $\theta(t) \geq \theta_{\text{min}}$ for all times $t$, we can drop the indicator function on the firm’s capital being positive in $\dot{\theta}(t)$ as it is always satisfied for $\theta(t) \geq \theta_{\text{min}}$. Thus by solving for $\dot{\theta}(t) = 0$ (with the indicator function in (2) set equal to 1), we can conclude that:

$$\dot{\Gamma}(t) = -\frac{\dot{q}(t)[s - \Gamma(t)][\bar{p} - x - \Psi(t)]}{q(t)([s - \Gamma(t)]q(t) - [\bar{p} - x - \Psi(t)])} \mathbb{1}_{\{\theta(t) \leq \theta_{\text{min}}\}}. \tag{5}$$
For notational simplicity, we will construct the mapping:

\[
Z(t, \Gamma(t), q(t), \Psi(t)) = \frac{[s - \Gamma(t)][\bar{p} - x - \Psi(t)]}{q(t)([s - \Gamma(t)]q(t) - [\bar{p} - x - \Psi(t)])} \mathbb{1}_{\{\theta(t) \leq \theta_{\text{min}}\}}.
\]

In fact, by the monotonicity of the inverse demand function, we further have that a firm will remain at the \(\theta(t) = \theta_{\text{min}}\) boundary for any time \(t \geq \tau = \inf\{t \in [0, T] \mid \theta(t) \leq \theta_{\text{min}}\}\) provided it does not run out of illiquid assets to sell. Therefore, by solving for the price as a function of liquidations for the equation \(\theta(t) = \theta_{\text{min}}\), we find that

\[
q(t) = \frac{\bar{p} - x - \Psi(t)}{(1 - \alpha \theta_{\text{min}})(s - \Gamma(t))} \in \mathbb{R}^{++}
\]

for \(t \geq \tau\). This provides the price directly as a function of the bank’s trading book.

With the representation of the price \(q(t)\) from (6), we find that

\[
[s - \Gamma(t)]q(t) - [\bar{p} - x - \Psi(t)] = \frac{\alpha \theta_{\text{min}}[\bar{p} - x - \Psi(t)]}{1 - \alpha \theta_{\text{min}}}
\]

for any time \(t \geq \tau\) (equivalently if \(\theta(t) \leq \theta_{\text{min}}\)). Therefore we can rewrite \(Z\) to only depend on time and the liquidations via

\[
Z(t, \Gamma) = \frac{(1 - \alpha \theta_{\text{min}})[s - \Gamma]}{\alpha \theta_{\text{min}} f_t(t) f_{\Gamma}(\Gamma)} \mathbb{1}_{\{t \geq \tau\}}.
\]

In fact, we can decouple \(\dot{\Gamma}(t)\) from \(\dot{q}(t)\) and thus consider \(q(t) = f_t(t)f_{\Gamma}(\Gamma(t))\) directly and \(\dot{\Gamma}(t)\) to solve the differential equation:

\[
\dot{\Gamma}(t) = -\frac{Z(t, \Gamma(t))f_t'(t)f_{\Gamma}(\Gamma(t))}{1 + Z(t, \Gamma(t)) f_t'(t)f_{\Gamma}'(\Gamma(t))}
\]

(7)

**Remark 3.3.** Of particular interest is that \(\dot{\Gamma}(t) \geq 0\) in general. By Assumption 3.1 we have that \(f_t'(t), f_{\Gamma}'(\Gamma) \leq 0\) for all times \(t\) and liquidations \(\Gamma\). As previously discussed for any time \(t \geq \tau\) we can conclude that \((s - \Gamma(t))q(t) = \frac{\bar{p} - x - \Psi(t)}{1 - \alpha \theta_{\text{min}}} \geq \bar{p} - x - \Psi(t) > 0\) which can be
used to show that \( Z(t, \Gamma(t)) \geq 0 \). Therefore \( \dot{\Gamma}(t) = -\frac{Z(t, \Gamma(t))f_t(t)f_t(\Gamma(t))}{1+Z(t, \Gamma(t))f_t(t)f_t(\Gamma(t))} \geq 0 \) if and only if \( f_t(t)f_t'(\Gamma(t)) \geq -\frac{1}{Z(t, \Gamma(t))} \), otherwise \( \dot{\Gamma}(t) < 0 \) and the bank will purchase assets at the given price \( q(t) \). As both financial theory and practice indicate such purchasing does not occur in times of a crisis, we utilize the following results in order to calibrate the risk-weights of our model so as to appropriately consider fire sales.

Formally, as above, let \( \tau := \inf \{ t \mid \theta(t) \leq \theta_{\min} \} = \inf \{ t \mid f_t(t) \leq \bar{q} \} \) be the first time the firm hits the regulatory boundary. Define the mapping \( \Lambda(t) := 1 + Z(t, \Gamma(t))f_t(t)f_t'(\Gamma(t)) \), which will be utilized throughout much of this work.

**Lemma 3.4.** Let the inverse demand function \( f_\Gamma \) be such that \( (s - \Gamma)f_\Gamma'(\Gamma)/f_\Gamma(\Gamma) \leq 0 \) is nondecreasing for all \( \Gamma \in [0, s) \). If \( \alpha \in (-\frac{sf_\Gamma'(0)}{(1-sf_\Gamma'(0))\theta_{\min}}, \frac{1}{\theta_{\min}}) \) then any solution \( \Gamma : [\tau, T] \rightarrow \mathbb{R} \) of (7) is such that \( \Gamma(t) \in [0, s) \) and \( \dot{\Gamma}(t) \geq 0 \) for all times \( t \).

**Remark 3.5.** In the prior theorem we require a monotonicity condition on \( (s - \Gamma)f_\Gamma'(\Gamma)/f_\Gamma(\Gamma) \). This term is the “equivalent” marginal change in units held to the price change when \( \Gamma \) units are liquidated (with the next marginal unit is liquidated externally). That is, the firm’s wealth drops by the same amount under the marginal change in price as if the firm held \( \lvert \frac{(s - \Gamma)f_\Gamma'(\Gamma)}{f_\Gamma(\Gamma)} \rvert \) fewer illiquid assets in their trading book. In this sense, this term provides the number of units needed to be sold at the current price in order to counteract the price movement. Therefore the assumed monotonicity property implies that the firm need not increase the speed it is selling the illiquid assets solely to counteract its own market impacts.

**Theorem 3.6.** Consider the setting of Lemma 3.4 with \( \alpha \in (-\frac{sf_\Gamma'(0)}{(1-sf_\Gamma'(0))\theta_{\min}}, \frac{1}{\theta_{\min}}) \). There exists a unique solution \( (\Gamma, q, \Psi) : [0, T] \rightarrow [0, s) \times \mathbb{R}_{++} \times [0, \bar{p} - x) \) to the differential system (7), (3), and (4) (and thus for \( \theta \) as well for (2)).

**Remark 3.7.** Noting that \( \frac{sf_\Gamma'(0)}{1-sf_\Gamma'(0)} \in [0, 1) \) because \( f_\Gamma'(0) \leq 0 \) by Assumption 3.1 we are now able to determine the appropriate risk-weight from Lemma 3.4, i.e., \( \alpha \in (-\frac{sf_\Gamma'(0)}{(1-sf_\Gamma'(0))\theta_{\min}}, \frac{1}{\theta_{\min}}) \).

If the risk-weight were set too low, i.e. \( \alpha \in [0, \frac{sf_\Gamma'(0)}{(1-sf_\Gamma'(0))\theta_{\min}}) \), then the bank would instead purchase assets to remain at the regulatory threshold rather than liquidating as is expected.
and observed in practice. The existence and uniqueness results follow for $\alpha < -\frac{s f'_1(0)}{(1-s f'_1(0))\theta_{\text{min}}}$ as well, though we will only focus on the risk-weights that match with reality. In fact, this lower threshold on the risk-weight $\alpha$ can be viewed as a function to map the illiquidity of the asset (measured by $f'_1(0)$) to an acceptable risk-weight, rather than choosing based on heuristics.

We will conclude this section by considering three example inverse demand functions $f_1$. Specifically we will consider the setting without market impacts, linear price impacts, and exponential price impacts.

**Example 3.8.** Consider the special case that there are no effects on the price from the firm’s actions, i.e., $F(t, \Gamma) = f_1(t)$ or $f_1(\Gamma) = 1$. Therefore we can see that any $\alpha \in (0, \frac{1}{\theta_{\text{min}}})$ is an acceptable risk-weight. That is, the firm will only take part in a fire sale and the proposed model is relevant for any choice of risk-weight.

**Example 3.9.** Consider the case in which the firm’s actions impact the price linearly, i.e., $F(t, \Gamma) = f_1(t)(1 - b\Gamma)$ for $b \in [0, \frac{1}{s})$. The condition on the inverse demand function for Lemma 3.4 is satisfied for any choice $b \in [0, \frac{1}{s})$. Further, the risk-weight condition, $\alpha > -\frac{s f'_1(0)}{(1-s f'_1(0))\theta_{\text{min}}}$, is satisfied if and only if $\alpha > \frac{sb}{(1+sb)\theta_{\text{min}}}$. In particular, if $\alpha \geq \frac{1}{2\theta_{\text{min}}}$ then the fire sale situation is always actualized without dependence on the price impact parameter $b$.

**Example 3.10.** Consider the case in which the firm’s actions impact the price exponentially, i.e., $F(t, \Gamma) = f_1(t) \exp(-b\Gamma)$ for $b \geq 0$. The condition on the inverse demand function for Lemma 3.4 is satisfied for any choice $b \geq 0$. Further, the risk-weight condition, $\alpha > -\frac{s f'_1(0)}{(1-s f'_1(0))\theta_{\text{min}}}$, is satisfied if and only if $\alpha > \frac{sb}{(1+sb)\theta_{\text{min}}}$.

## 4 Capital Ratio Requirements in an $n$ Bank System

Consider the same setting as in Section 3 but with $n \geq 1$ banks. Throughout this section we will let firm $i$ have initial trading book defined by $x_i$ units of liquid asset, $s_i$ units of illiquid
asset, and \( \bar{p}_i \) in obligations. Further, we will consider the (pre-fire sale) market cap for the illiquid asset to be given by \( M \geq \sum_{i=1}^n s_i \). The inverse demand function will still be assumed to follow Assumption [3.1].

Immediately, as before in the \( n = 1 \) bank setting of Section 3, we can conclude that the change in the capital ratio \( \theta_i \) over time is given by:

\[
\dot{\theta}_i(t) = \frac{\dot{q}(t)[s_i - \Gamma_i(t)][\bar{p}_i - x_i - \Psi_i(t)] + \dot{\Gamma}_i(t)q(t)([s_i - \Gamma_i(t)]q(t) - [\bar{p}_i - x_i - \Psi_i(t)])}{\alpha([s_i - \Gamma_i(t)]q(t))^2}
\]  (8)

where

\[
\dot{q}(t) = f'_t(t)f_T\left(\sum_{i=1}^n \Gamma_i(t)\right) + \left[\sum_{i=1}^n \dot{\Gamma}_i(t)\right]f_t(t)f_T^2\left(\sum_{i=1}^n \Gamma_i(t)\right),
\]  (9)

\[
\dot{\Psi}_i(t) = \dot{\Gamma}_i(t)q(t).
\]  (10)

With the assumption that \( \theta_i(0) \geq \theta_{\min} \), we know that firm \( i \) will not take any actions unless \( \theta_i(t) \leq \theta_{\min} \). As in the 1 bank case, this first occurs at \( \bar{q}_i = \frac{\bar{p}_i - x_i}{(1 - \alpha \theta_{\min})s_i} \). If \( \inf_{t \in [0,T]} f_t(t) > \max_i \bar{q}_i \) then no fire sale occurs. When a firm does need to take action, we will make the assumption that it is only enough so that the firm remains at the capital ratio requirement. Thus by solving for \( \dot{\theta}_i(t) = 0 \) when \( \theta_i(t) \leq \theta_{\min} \), we can conclude that, as in the \( n = 1 \) bank setting:

\[
\dot{\Gamma}_i(t) = -\frac{\dot{q}(t)[s_i - \Gamma_i(t)][\bar{p}_i - x_i - \Psi_i(t)]}{q(t)([s_i - \Gamma_i(t)]q(t) - [\bar{p}_i - x_i - \Psi_i(t)])} \mathbb{1}_{\{\theta_i(t) \leq \theta_{\min}\}}
\]

As in the prior section (after consideration of how the prices must evolve so that the firms remain at the required capital ratio), let us consider the mapping

\[
Z_i(t, \Gamma) = \frac{(1 - \alpha \theta_{\min})[s_i - \Gamma_i(t)]}{\alpha \theta_{\min} f_t(t)f_T(\sum_{j=1}^n \Gamma_j)} \mathbb{1}_{\{\theta_i(t) \leq \theta_{\min}\}}.
\]
With this mapping, we can consider the joint differential equation of $\Gamma$ and $q$:

$$
\dot{\Gamma}(t) = - \left( I + (Z(t, \Gamma(t)) \cdots Z(t, \Gamma(t))) f(t) f_\Gamma'(\sum_{j=1}^n \Gamma_j(t)) \right)^{-1} 
\times \left( Z(t, \Gamma(t)) f_t'(t) f_\Gamma'(\sum_{j=1}^n \Gamma_j(t)) \right)
$$

(11)

$$
\dot{q}(t) = f_t'(t) f_\Gamma'(\Gamma(t)) \frac{f_t'(t) f_\Gamma'(\Gamma(t))}{1 + [\sum_{i=1}^n Z_i(t, \Gamma(t))] f_t'(t) f_\Gamma'(\sum_{i=1}^n \Gamma_i(t))}
$$

(12)

Let $\tau_0 = 0$, $\tau_{k+1} := \inf\{t \in [\tau_k, T] \mid \exists \tilde{\gamma}: \theta_i(t) \leq \theta_{\min}, \theta_i(\tau_k) > \theta_{\min}\}$, and $\tau_{n+1} = T$. For the remainder, we will order the banks so that $\bar{q}_i \geq \bar{q}_{i+1}$ for every $i$. Due to the monotonicity property this implies that bank $k$ hits the regulatory threshold only after the first $k - 1$ banks.

**Lemma 4.1.** Assume the ordering of banks by decreasing $\bar{q}$. Let the inverse demand function $f_\Gamma$ be such that $(M - \Gamma)f_\Gamma'(\Gamma)/f_\Gamma(\Gamma) \leq 0$ is nondecreasing and $f_\Gamma(\Gamma)f''_\Gamma(\Gamma) \leq f'_\Gamma(\Gamma)^2$ for any $\Gamma \in [0, M)$. If $\alpha \in \left(-\frac{Mf_\Gamma'(0)}{(1-Mf_\Gamma'(0))\theta_{\min}}, \frac{1}{\theta_{\min}}\right)$ then any solution $\Gamma : [0, T] \rightarrow \mathbb{R}$ of (11) is such that $\Gamma(t) \in [0, s)$, $\dot{\Gamma}(t) \in \mathbb{R}_+^n$, and $\dot{q}(t) \geq 0$ for all times $t$.

Using this result on monotonicity of the processes, we are able to determine a result on the existence and uniqueness of the system under financial contagion.

**Corollary 4.2.** Consider the setting of Lemma 4.1 with $\alpha \in \left(-\frac{Mf_\Gamma'(0)}{(1-Mf_\Gamma'(0))\theta_{\min}}, \frac{1}{\theta_{\min}}\right)$. There exists a unique solution $(\Gamma, q, \Psi) : [0, T] \rightarrow [0, s) \times \mathbb{R}_+ \times [0, \tilde{p} - x)$ to the differential system (11), (12), and (10) (and thus for $\theta$ as well for (8)).

**Remark 4.3.** As in the single bank $n = 1$ setting, we can consider a situation in which the risk-weight was set too low. Under such parameters eventually one bank may begin purchasing assets rather than liquidating in order to satisfy the capital requirements. Existence of a solution would still exist in this setting for the $n$ bank case, but uniqueness will no longer hold.
5 Analytical Stress Test Bounds

As described in the proofs of Lemmas 3.4 and 4.1, we are able to determine upper bounds for the number of assets being sold for each firm in the system. In the following results we will refine these estimates and use this to determine simple analytical worst-case results for the health of the financial system. As such, given the initial trading book for each firm, a heuristic for the health of the system can be determined with ease. Mathematically this is provided by Lemma 5.1. Following this result, we will present a quick example to demonstrate the value of these bounds to consider a stochastic stress test. Throughout, we will be recalling that firm \( i \) hits the regulatory threshold \( \theta_{\text{min}} \) when \( q(t) = \bar{q}_i := \bar{p}_i - x_i (1 - \alpha \theta_{\text{min}}) s_i \) and that firms are ordered so that \( \bar{q}_i \) is a nonincreasing sequence.

Lemma 5.1. Consider the setting of Corollary 4.2 with \( n \geq 1 \) banks. Define approximate hitting times \( \tilde{\tau}_k \) and bounds on the firm behavior \( t \mapsto \tilde{\Gamma}_k(t) \) for \( k = 1, \ldots, n \):

\[
\tilde{\Gamma}_i^k(t) = \begin{cases} 
1_{\{t < \tilde{\tau}_k\}} \tilde{\Gamma}_i^{k-1}(t) + 1_{\{t \geq \tilde{\tau}_k\}} \left[ s_i - (s_i - \tilde{\Gamma}_i^{k-1}(\tilde{\tau}_k)) \left( \frac{f_i(t)}{f_i(\tilde{\tau}_k)} \right)^{1-\alpha \theta_{\text{min}}} \right] & \text{if } i \leq k \\
0 & \text{else}
\end{cases}
\]

\[
\tilde{\tau}_k = \inf \left\{ t \in [\tilde{\tau}_{k-1}, T] \mid f(t) f_T \left( \sum_{i=1}^{k-1} \tilde{\Gamma}_i^{k-1}(t) \right) \leq \bar{q}_k \right\}
\]

\[
\tilde{\Lambda}_k = 1 + \frac{1 - \alpha \theta_{\text{min}}}{\alpha \theta_{\text{min}}} \sum_{j=1}^{k} (s_j - \tilde{\Gamma}_j^{k-1}(\tilde{\tau}_k)) \left( \frac{f_j'(t)}{f_j'(\tilde{\tau}_k)} \right) \cdot \left( \frac{\sum_{j=1}^{k-1} \tilde{\Gamma}_j^{k-1}(\tilde{\tau}_k)}{\sum_{j=1}^{k-1} \tilde{\Gamma}_j^{k-1}(\tilde{\tau}_k)} \right)
\]

where \( \tilde{\tau}_0 = 0, \tilde{\tau}_{n+1} = T, \) and \( \tilde{\Gamma}_i^0(t) \equiv 0 \). Then \( \Gamma_i(t) \leq \tilde{\Gamma}_i^n(t) \) for all times \( t \in [0, T] \) and all firms \( i = 1, \ldots, n \).

With this general analytical construction, we now wish to turn our attention to a specific choice of inverse demand function to provide some additional results. In particular, as noted in Remark 3.2, we will choose the exponential inverse demand function considered in Example 3.10 to deduce exact analytical formulations. For the remainder of this section we
will make use of the Lambert W function $W : [-\exp(-1), \infty) \to [-1, \infty]$, i.e. the inverse mapping of $x \mapsto x \exp(x)$.

**Corollary 5.2.** Consider the setting of Lemma 5.1. Further, consider an exponential inverse demand function $f_\Gamma(\Gamma) := \exp(-b\Gamma)$ as in Example 3.10 for $b \geq 0$. The analytical stress test bounds can be explicitly provided for any $i = 1, \ldots, n$:

$$
\tilde{\Gamma}_i^n(t) = s_i \left(1 - \prod_{j=i}^{n} \left(\frac{f_t(t \wedge \tilde{\tau}_j+1)}{f_t(t \wedge \tilde{\tau}_j)}\right)^{\frac{1-\alpha \theta_{\min}}{\alpha \theta_{\min}} \tilde{\Lambda}}\right)
$$

$$
\tilde{\tau}_i = \begin{cases} 
    f^{-1}_t(\tilde{q}_1) & \text{if } i = 1 \\
    f^{-1}_t\left(\frac{\Lambda_{i-1} W\left(\frac{\alpha \theta_{\min}}{1-\alpha \theta_{\min}} \log(\tilde{q}_i) + b \sum_{j=1}^{i-1} s_j \right)}{m_{i-1}}\right) & \text{if } i \in \{2, \ldots, n\}
\end{cases}
$$

$$
\tilde{\Lambda}_i = 1 - b \frac{1 - \alpha \theta_{\min}}{\alpha \theta_{\min}} \left[\sum_{j=1}^{i} s_j \prod_{k=j}^{i-1} \left(\frac{f_t(\tilde{\tau}_k+1)}{f_t(\tilde{\tau}_k)}\right)^{\frac{1-\alpha \theta_{\min}}{\alpha \theta_{\min}} \tilde{\Lambda}}\right]
$$

$$
m_i = \frac{1 - \tilde{\Lambda}_i}{f_t(\tilde{\tau}_i)^{\frac{1-\alpha \theta_{\min}}{\alpha \theta_{\min}} \tilde{\Lambda}_i}}
$$

where $\wedge$ denotes the minimum operator.

**Remark 5.3.** The expanded form $\tilde{\Gamma}_i^n$ provided in Corollary 5.2 holds for any inverse demand function $f_\Gamma$ and is not dependent on the choice of the exponential form. However, the forms of $\tilde{\tau}_i$ and $\tilde{\Lambda}_i$ are specific to the exponential inverse demand function considered in Corollary 5.2.

Further, this analytical stress test bound has significant value in considering probability distributions. All results in this paper, up until now, would require Monte Carlo simulations in order to approximate the distribution of the health of the financial system if there is uncertainty in the parameters. However, with this analytical bound, we are able to determine analytical worst-case distributions that would be almost surely worse than the actualized results due to the results of Lemma 5.1. Thus if the system is deemed healthy enough under this analytical results, it would be pass the stress test under the true dynamics as well.
Corollary 5.4. Consider the setting of Corollary 5.2 with exponential price response in time \( f_t(t) := \exp(-at\mathbb{1}_{t<T} - aT\mathbb{1}_{t\geq T}) \). Consider a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and let the parameter \(a\) be random with known distribution. Fix a time \( t \in [0, T] \), the distribution of the price \( q(t) \) at time \( t \) is bounded by:

\[
\mathbb{P}(q(t) \geq q^*) \geq \mathbb{P}
\left(\frac{1}{t} \Phi^{-1}_k \left( \log(q^*) + b \sum_{i=1}^{k} s_i \right) \right)
\]

\[
\Phi^{-1}(x) = \frac{\alpha \theta_{\min} \Lambda_k}{1 - \alpha \theta_{\min}} W \left( \frac{m_k}{\Lambda_k} \exp \left( \frac{1 - \alpha \theta_{\min}}{\alpha \theta_{\min} \Lambda_k} x \right) \right) - x.
\]

where \( q^* \in [\bar{q}_{k+1}, \bar{q}_k) \) for some \( k = 0, 1, \ldots, n \) (where \( \Lambda_0 = 1, m_0 = 0, \bar{q}_0 = 1, \) and \( \bar{q}_{n+1} = 0 \) for \( k = 1, \ldots, n \)).

Remark 5.5. We can generalize the bound for any random price response in time \( f_t \) from Corollary 5.4 by considering

\[
\mathbb{P}(q(t) \geq q^*) \geq \mathbb{P}(f_t(t) \geq \left[ \frac{\Lambda_k W \left( \frac{m_k}{\Lambda_k} \exp \left( \frac{1 - \alpha \theta_{\min}}{\alpha \theta_{\min} \Lambda_k} \left[ \log \left( \frac{q^*}{b} \sum_{j=1}^{k} s_j \right) \right) \right) \right]}{m_k} \right]^{\alpha \theta_{\min} \Lambda_k})
\]

where \( q^* \in [\bar{q}_{k+1}, \bar{q}_k) \) for some \( k = 1, \ldots, n \). The case with \( k = 0 \) is trivially given by \( \mathbb{P}(f_t(t) \geq q^*) \).

This result allows us to consider the case for jointly random price response \( f_t \) and price impact parameter \( b \in [0, \frac{\alpha \theta_{\min}}{1 - \alpha \theta_{\min} M}) \) with marginal density \( g_b \):

\[
\mathbb{P}(q(t) \geq q^*) \geq \int_0^{\frac{\alpha \theta_{\min}}{1 - \alpha \theta_{\min} M}} \mathbb{P}(f_t(t) \geq \left[ \frac{\Lambda_k W \left( \frac{m_k}{\Lambda_k} \exp \left( \frac{1 - \alpha \theta_{\min}}{\alpha \theta_{\min} \Lambda_k} \left[ \log \left( \frac{q^*}{b} \sum_{j=1}^{k} s_j \right) \right) \right) \right]}{m_k} \right]^{\alpha \theta_{\min} \Lambda_k} b db
\]

\[
\Xi(b) = \mathbb{P}(f_t(t) \geq \left[ \frac{\Lambda_k W \left( \frac{m_k}{\Lambda_k} \exp \left( \frac{1 - \alpha \theta_{\min}}{\alpha \theta_{\min} \Lambda_k} \left[ \log \left( \frac{q^*}{b} \sum_{j=1}^{k} s_j \right) \right) \right) \right]}{m_k} \right]^{\alpha \theta_{\min} \Lambda_k} b)
\]

where \( q^* \in [\bar{q}_{k+1}, \bar{q}_k) \) for some \( k = 1, \ldots, n \). The upper bound on the price impact parameter \( b \) is so as to guarantee the selected risk-weight satisfies the sufficient conditions considered.
within this work.

6 Case Studies

In this section we will consider four numerical case studies to consider implications of the proposed model. First, we will consider a 20 bank system and determine the effects of the market impacts on the health of the financial system. Second, we will consider a system with random parameters to study a probabilistic stress test. Third, we will determine some simple implications for viewing the system as a single institution compared with a system of heterogeneous institutions. Finally, we will consider the effects of changing the regulatory capital ratio threshold. In these numerical examples we will consider both the numerical solutions to the differential system introduced in Sections 3 and 4 and the stress test bounds considered in Section 5. This allows us to consider the accuracy of the analytical stress test bounds.

Example 6.1. Consider a financial system with \( n = 20 \) banks and a crisis that lasts until the terminal time \( T = 1 \). Assume that each bank has liabilities \( \bar{p}_i = 1 \) and liquid assets \( x_i = \frac{2(i-1)}{475} \) for \( i = 1, \ldots, 20 \). Additionally, each bank is given \( s_i = 2 \) units of the illiquid asset; accordingly we set the market capitalization \( M = \sum_{i=1}^{20} s_i = 40 \). We will consider the regulatory environment with threshold \( \theta_{\text{min}} = 0.10 \) and risk-weight \( \alpha = \frac{1}{2\theta_{\text{min}}} = 5 \). Finally, we will take the inverse demand function to have an exponential form, i.e., \( F(t, \Gamma) = \exp(-at1_{t<1} - a1_{t\geq1} - b\Gamma) \) with \( a = -\log(0.95) \approx 0.0513 \) and varied market impact parameter \( b \in [0, \frac{1}{M}] \) which satisfies the conditions of Corollary 4.2. In this example we will demonstrate the nonlinear response that market impacts \( b \) introduce to the health of the firms and clearing prices.

First we wish to consider the impact over time that the market impacts can cause. To do so we compare the asset prices without market impacts \( (b = 0) \) to those with high market impacts \( (b \approx \frac{1}{M}) \). As depicted in Figure 2a we see that the prices with and without price
impacts are comparable for (approximately) $t \in [0, 0.29]$. After that time the two systems diverge, drastically so after $t \approx 0.80$. At that point 18 of the 20 firms (90%) have hit the regulatory threshold and the feedback effects of their actions are quite evident. We wish to note the distinction between this steep drop in the prices to the subtle price drop for $t \in [0, 0.29]$ when only the first 3 banks have hit their regulatory threshold. The times at which the firms hit the regulatory threshold at different liquidity situations (i.e. no, medium, and high market impacts) are summarized in Table 1.

With the notion of how high market impacts effect the prices over time, and how the feedback effects can cause virtual jumps in the price, we now wish to consider these effects in more detail by studying only the final state of the system. In Figure 2b we see that, as more banks hit the threshold capital ratio, the range of price impact thresholds that match
that state shrink. That is, the system becomes more sensitive to the price impact parameter as more banks are at the regulatory threshold. This is due to the same feedback effects seen in the high price impact scenario of Figure 2a. Further, we see that until about 90% of the banks (18 out of the 20 firms) hit the regulatory threshold (at about $b \lesssim 0.7/M$), the terminal price is principally affected by the price change in time ($f_t(1) = 0.95$). At market impacts above this level ($b \gtrsim 0.7/M$) the feedback effects of firm liquidations on each other causes the terminal price to drop drastically. Thus, providing only a small amount of liquidity to the market can have outsized effects on the health of the system by decreasing the price impacts, though this type of response to a financial crisis would have quickly decreasing marginal returns as evidenced by Figure 2b.

Finally, we wish to consider the analytical stress test bounds. We see the response of the stress test bounds in the high market impact scenario ($b \approx \frac{1}{M}$) in Figure 2a. This is not depicted in the setting without market impacts as there is no distinction between the exact price process and the bounded price process in this case. In the high market impact scenario, we see that the exact price process and the stress test bounds provide virtually indistinguishable results for the $t \in [0, 0.84]$. After that time the stress test bound results in a significantly larger shock than the real solution. As seen in Table 1, the times that firms hit the regulatory threshold are robust between the exact numerical solution and the analytical approximations in all market impact environments (no market impacts, medium impacts, and high impacts). Finally, we see that the terminal health of the system is replicated with extreme accuracy so long as the price impacts are $b \in [0, \frac{0.9}{M}]$.

**Example 6.2.** Consider the setting of Example 6.1 with exponential inverse demand function $F(t, \Gamma) = \exp(-at1_{t<1} - a1_{t\geq1} - b\Gamma)$ for $a \sim \text{Exp}(\mu)$, $\mu = \frac{\log(20)}{\log(20) - \log(19)} \approx 58.4$, and $b = \frac{0.9}{M}$. The choice of the exponential distribution for $a$ with parameter $\mu$ is so that $\mathbb{P}(F(1, 0) \leq 0.95) = 0.05$. We wish to compare the true distribution for $q(1)$ to the analytical stress test bound given in Corollary 5.4. In comparison to the analytical cumulative distribution function given in Corollary 5.4, the true distribution was found numerically through
(a) A comparison of the asset price over time in a low \( b = 0 \) and high \( b = \frac{1}{M+10^{-8}} \) market impact environment.

(b) The final price \( q(1) \) and percentage of firms that have hit their regulatory threshold as a function of the price impact parameter \( b \).

Figure 2: Example 6.1 The effects of price impacts on market response in a 20 bank system under the exact differential equation and the analytical stress test bounds.

repeated computation on a (log scaled) regular interval. Figure 3 displays the cumulative distribution functions \( \mathbb{P}(q(t) \leq q^*) \) without market impacts (black dashed line), with market impacts (black solid line), and the analytical stress test bound (blue solid line). Notably, the analytical bound, as seen in Figure 3a, is a very accurate estimate of the true distribution while the market without price impacts distinctly underestimates large price drops. This is more pronounced in Figure 3b which is the same figure but focused on the region for \( q^* \in [0.8, 0.9] \). Here we can see that the true distribution is bounded by the analytical stress test distribution, but gives a distribution significantly above the market without price impacts. In particular, without market impacts the probability \( \mathbb{P}(f_t(1) \leq 0.9) \approx 0.002 \) whereas \( \mathbb{P}(q(1) \leq 0.9) \approx 0.055 \). On the other hand, without market impacts the probability \( \mathbb{P}(f_t(1) \leq 0.8) \approx 0 \) whereas \( \mathbb{P}(q(1) \leq 0.8) \approx 0.014 \) and \( \mathbb{P}(f_t(1) f_r (\sum_{i=1}^{n} \tilde{\Gamma}_i^r(1)) \leq 0.8) \approx 0.02 \). Thus the analytical stress test is a bound for the true distribution, but an accurate one (as seen in Figure 3a).

While considering the probabilistic setting, we can also consider and plot the response to varying the stress scenario given by \( a \). This is depicted in Figure 4 by plotting the terminal price \( q(1) \) as a function of the price without market impacts \( f_t(1) \). The setting without
(a) The distribution of terminal prices $q(1)$ with and without price impacts. (b) Zoomed in view of the distribution of terminal prices $q(1)$.

Figure 3: Example 6.2. True and analytical stress test distributions for the terminal price $q(1)$ under a randomly stressed financial system of 20 banks.

market impacts is the diagonal line by definition. Market impacts cause feedback effects that drive the price below $f_t(1)$. All settings coincide for low stress scenarios ($f_t(1) \gtrsim 0.98$) as few banks are driven to the regulatory threshold. Further, the analytical stress test bound is demonstrably worse than the numerical terminal value for most stresses; however, these occur at typically unrealistic stresses.

Example 6.3. Consider the setting of Example 6.1. In order to study the effects of system heterogeneity, we will compare the 20 bank heterogeneous system to an aggregated system with only a single institution. As the aggregated system, this firm has liquid assets $x = \sum_{i=1}^{20} x_i$, illiquid assets $s = \sum_{i=1}^{20} s_i$, and liabilities $\bar{p} = \sum_{i=1}^{20} \bar{p}_i$. Note that, this single aggregated system will result in the same system response as a 20 bank setting with all banks constructed from identical balance sheets (each exactly $1/20$th of the size of this aggregated system). Figure 5 displays the impact that changing the system heterogeneity has on the market response. In particular, due to the cross-subsidies, the aggregated system does not hit the regulatory threshold until $t = 0.796$. However, as shown in Figure 5a, under high price impacts ($b = \frac{0.98}{M}$) due to the effect that the entire system must liquidate, the price drops faster once the threshold has been hit than in the heterogeneous system of 20
Figure 4: Example 6.2. The impacts of the stress scenario on market response in a 20 bank system under the exact differential equation and the analytical stress test bounds.

banks. In fact, the terminal price is lower in the single bank aggregated system than in the original 20 bank heterogeneous system. Figure 5A shows that when price impacts are low, the aggregated system outperforms the heterogeneous system, but the opposite is true for high price impacts. This leads us to conclude that mergers and bail-ins to rescue troubled banks may only be a wise policy response to rescue the system if the market is sufficiently liquid.

Example 6.4. Consider a single bank system \( n = 1 \) with crisis that lasts until terminal time \( T = 1 \). For simplicity, assume that this bank holds no liquid assets, i.e. \( x = 0 \). Further, we will directly consider the setting of a leverage constrained firm (i.e., the setting of Section 2.2) with varying maximal leverage \( \lambda_{\max} > 1 \). As we change this leverage requirement, we will assume that the initial trading book for the firm is such that they begin (at time 0) exactly at the leverage constraint and have a single unit of capital, i.e. \( s = \lambda_{\max} \) and \( \bar{p} = \lambda_{\max} - 1 \). For comparison we will fix the inverse demand
(a) A comparison of the asset price over time in the heterogeneous system of 20 banks and in the aggregated 1 bank system in a high price impact ($b = \frac{0.98}{M}$) environment.

(b) The difference in the terminal price in the heterogeneous system of 20 banks ($q^{20}(1)$) and in the aggregated 1 bank system ($q^1(1)$) as a function of the price impact parameter $b$.

Figure 5: Example 6.3. The impacts of system heterogeneity and price impacts on market response in a financial system.

function to have an exponential form, i.e., $F(t, \Gamma) = \exp(-atI_{t<1} - aI_{t\geq1} - b\Gamma)$ with $a = -\log(0.95) \approx 0.0513$ and $b = -\frac{\log(0.9)}{1-\frac{1}{\log(0.9)}} \approx 0.0100$ which satisfies the conditions of Theorem 3.6 for $\lambda_{\text{max}} \in (1, 1-\frac{1}{\log(0.9)}) \approx (1, 10.50)$. In this example we will demonstrate the nonlinear response that higher leverage has on the firm behavior and health.

In Figure 6a, we clearly see that if the leverage requirement is nearly $\lambda_{\text{max}} \approx 1$ then, even though the firm has a trading book that is leverage constrained, very few asset liquidations are necessary and the final portfolio is nearly nearly identical to the original portfolio. However, as the leverage requirement is relaxed the firm must liquidate a larger percentage of their (larger number of) assets, up to nearly 70% of all assets. In fact, once the leverage requirement exceeds 7.15 the firm has a decreasing number of terminal assets as the leverage requirement increases; this is despite the firm having a greater number of initial assets. Thus the combination of increasing percentage of assets liquidated and increasing number of initial assets as the leverage requirement $\lambda_{\text{max}}$ increases, the terminal prices decrease as the leverage requirement increases (as depicted in Figure 6b). Finally, we notice that the analytical stress test bounds are accurate for $\lambda_{\text{max}} \lesssim 5.5$. However, for leverage requirements
above that threshold the analytical stress test bounds stop performing well, though clearly are a worst-case bound for the health of the financial system.

(a) The asset holdings and percentage of assets the firm has remaining in its trading book at time 1 as a function of the leverage constraint $\lambda_{\text{max}}$.

(b) The terminal price time 1 of the illiquid asset as a function of the leverage constraint $\lambda_{\text{max}}$.

Figure 6: Example 6.4: The impacts of the leverage requirement on asset holdings and prices under the exact differential equation and the analytical stress test bounds.

7 Conclusion

In this paper we considered a dynamic model of price-mediated contagion that extends the work of [11, 18, 7, 6]. The focus of this work was on capital ratio requirements and risk-weighted assets. In analyzing this model, we determine bounds for appropriate risk-weights for an asset that is dependent on the liquidity of the asset itself, as modeled through the price impacts of liquidating the asset. Under the appropriate risk-weights, we find existence and uniqueness for the firm behavior and system health. However, though the output of the model can be computed with standard methods, an analytical solution cannot be found; an analytical bound on the health of the system in a stressed scenario was provided. This analytical stress test bound can be used to analyze random stresses and find the probability for the system health.
A Proofs from Section 3

A.1 Proof of Lemma 3.4

Proof. We will demonstrate that if a solution exists then it must satisfy the monotonicity property. To do so, first, we note that the condition on the risk-weight $\alpha$ is equivalent to $\alpha \theta_{\min} < 1$ and $\Lambda(\tau) > 0$. Therefore we find that $\dot{\Gamma}(\tau) > 0$. Now we wish to show that $\dot{\Lambda}(t)$ has the same sign as $\dot{\Gamma}(t)$, i.e., $\dot{\Gamma}(t)\dot{\Lambda}(t) \geq 0$.

$$
\Lambda(t) = 1 + Z(t, \Gamma(t)) f_t(t) f'_\Gamma(\Gamma(t)) \\
= 1 + \frac{1 - \alpha \theta_{\min}}{\alpha \theta_{\min}} \frac{s - \Gamma(t)}{q(t)} f_t(t) f'_\Gamma(\Gamma(t)) \\
= 1 + \frac{1 - \alpha \theta_{\min}}{f_\Gamma(\Gamma(t))} s - \Gamma(t) f'_\Gamma(\Gamma(t)) \\
\dot{\Lambda}(t) = \frac{1 - \alpha \theta_{\min}}{\alpha \theta_{\min}} \frac{d}{dt} \left[ \frac{(s - \Gamma)}{f_\Gamma(\Gamma)} \right]_{\Gamma=\Gamma(t)} \\
= (1 - \alpha \theta_{\min}) \dot{\Gamma}(t) \frac{d}{d\Gamma} \left[ \frac{(s - \Gamma)}{f_\Gamma(\Gamma)} \right]_{\Gamma=\Gamma(t)} \\
$$

Therefore, by $\alpha \theta_{\min} \in (0, 1)$, we find that $\dot{\Gamma}(t)\dot{\Lambda}(t) \geq 0$ if and only if

$$
\frac{d}{d\Gamma} \left[ \frac{(s - \Gamma)}{f_\Gamma(\Gamma)} \right]_{\Gamma=\Gamma(t)} \geq 0
$$

which is true by assumption. We will now use an induction argument to prove $\Lambda(t) > 0$ for all times $t \in [\tau, T]$:

- At time $\tau$ we have (by assumption) that $\Lambda(\tau) > 0$.

- For any time $t \in [\tau, T)$ such that $\Lambda(t) > 0$ then it must be that $\Lambda(u) > 0$ for every $u \in [t, t + \epsilon]$ for some $\epsilon > 0$ by continuity of $\Gamma$ and therefore of $\Lambda$ (and that $\Lambda$ is strictly above 0).

- For any time $t \in (\tau, T]$ such that $\Lambda(u) > 0$ for every $u \in [\tau, t)$ then $\dot{\Gamma}(u) > 0$ and, as a consequence, $\dot{\Lambda}(u) \geq 0$ for every $u \in [\tau, t)$. This implies $\Lambda(t) > 0$ as well.
Therefore, if $\Lambda(\tau) > 0$ it must hold that $\Lambda(t) \geq \Lambda(\tau)$ for all times $t \geq \tau$ (which implies $\dot{\Gamma}(t) \geq 0$ for all times $t \geq \tau$).

Finally, we will demonstrate that, if a solution $\Gamma : [0, T] \to \mathbb{R}$ exists, then $\Gamma(t) < s$ for all times $t \in [0, T]$. By definition $\Gamma(t) = 0$ for all times $t \in [0, \tau]$, so we begin with $\Gamma(\tau) = 0$. Take $T^* = \inf\{t \in [\tau, T] \mid \Gamma(t) \geq s\}$ and assume this infimum is taken over a nonempty set. On $u \in [\tau, T^*)$ we have that:

$$
\dot{\Gamma}(u) = -\frac{Z(u, \Gamma(u)) f_t'(u) f_t(\Gamma(u))}{\Lambda(u)} \leq -\frac{(1 - \alpha \theta_{\min}) \inf_{t \in [\tau, T^*]} f_t'(t)}{\alpha \theta_{\min} f_t(T^*) \Lambda(\tau)} (s - \Gamma(u))
$$

and $\inf_{t \in [\tau, T^*]} f_t'(t)$ is attained as we are infimizing a continuous function over a compact space. This differential equation coupled with the initial condition $\Gamma(\tau) = 0$ implies

$$
\Gamma(u) \leq s \left[ 1 - \exp \left( \frac{(1 - \alpha \theta_{\min}) \inf_{t \in [\tau, T^*]} f_t'(t)}{\alpha \theta_{\min} f_t(T^*) \Lambda(\tau)} (u - \tau) \right) \right] < s
$$

for any time $u \in [\tau, T^*)$. In particular, by continuity, this implies that $\Gamma(T^*) < s$. 

\[\square\]

A.2 Proof of Theorem 3.6

Proof. We will use Lemma 3.4 to prove the existence and uniqueness of a solution $(\Gamma, q, \Psi)$. First, for all times $t \in [0, \tau]$ there exists a unique solution given by $\Gamma(t) = 0$, $q(t) = f_t(t)$, and $\Psi(t) = 0$.

Now consider the initial value problem with initial condition at $t = \tau$. We will consider the differential equation for $\Gamma$ given in (7). As this equation is no longer dependent on either $q$ or $\Psi$ we can consider the existence and uniqueness of the liquidations $\Gamma$ separately. Indeed, from $\Gamma$, we can define $q(t) = f_t(t) f_t(\Gamma(t))$ for all times $t$, thus the existence and uniqueness of $\Gamma$ provides the same results for $q$. The results for $\Psi$ follow from the same logic as $\Gamma$ and thus will be omitted herein. In our consideration of (7) we will consider a modification of the function $\Lambda(t)$ to be given by $\Lambda(\Gamma)$ so that its dependence on the liquidations is made
explicit.

\[
\dot{\Gamma}(t) = -\frac{(1 - \alpha \theta_{\min})(s - \Gamma(t))}{\alpha \theta_{\min} f_{\Gamma}(t) \Lambda(\Gamma(t))} f'_{\Gamma}(t) =: g(t, \Gamma(t))
\]

\[
\Lambda(\Gamma) = 1 + \frac{(1 - \alpha \theta_{\min})(s - \Gamma)}{\alpha \theta_{\min} f_{\Gamma}(\Gamma)} =: g(t, \Gamma(t)) \bar{\Lambda}(\Gamma) = 1 + \frac{(1 - \alpha \theta_{\min})(s - \Gamma)}{\alpha \theta_{\min} f_{\Gamma}(\Gamma)}
\]

Now we wish to consider the initial value problem for \( \Gamma \) with dynamics given by \( g \) and initial value \( \Gamma(\tau) = 0 \). Before continuing we wish to note that the function \( \bar{\Lambda} \) is constant in time, i.e., only depends on the total number of units liquidated \( \Gamma \) and not on the time.

Define the domain

\[
U = \left\{ \Gamma \in [0, s) \mid \bar{\Lambda}(\Gamma) > \frac{1}{2} \Lambda(\tau) = \frac{1}{2} \left[ 1 + \frac{(1 - \alpha \theta_{\min})s}{\alpha \theta_{\min} f'_{\Gamma}(0)} \right] \right\}.
\]

We wish to note from the above that by \( \Lambda \) nondecreasing in time, any solution must lie in \( U \), i.e., it must satisfy \( \Gamma(t) \in U \) for all times \( t \in [\tau, T] \). From the definition of \( U \) as well as the property that \( \bar{\Lambda} \) is constant in time, we can conclude

\[
\alpha \theta_{\min} f_{\Gamma}(t) \bar{\Lambda}(\Gamma) > \frac{1}{2} \alpha \theta_{\min} f_{\Gamma}(t) \Lambda(\tau) > 0
\]

for any \( \Gamma \in U \) and any time \( t \in [\tau, T] \), and thus the denominator in \( g \) is always strictly greater than 0. From this we can conclude that \( g \) and \( \frac{\partial}{\partial \Gamma} g \) are continuous mappings over \([\tau, T] \times U \) and thus \( \Gamma \in U \mapsto g(t, \Gamma) \) is locally Lipschitz for any time \( t \in [\tau, T] \). This implies there exists some \( \delta > 0 \) such that \( \Gamma : [\tau, \tau + \delta] \to U \) is the unique solution satisfying \( \dot{\Gamma}(t) = g(t, \Gamma(t)) \) for all times \( t \in [\tau, \tau + \delta] \). From a sequential application of this approach (i.e., consider now an initial value problem starting at time \( \tau + \delta \)) we can either conclude that there exists a unique solution over the entire time range \( \Gamma : [\tau, T] \to U \) or there exists some maximal domain \( [\tau, T^*) \subsetneq [\tau, T] \) over which we can conclude the existence and uniqueness. We will finish by focusing on this second case to prove a contradiction to this maximal domain being strictly contained in the full time domain. To do this we will first show that \( g \) is bounded.
on \([\tau, T] \times U\). By definition, we have that \(g(t, \Gamma) \geq 0\) for any \(t \in [\tau, T]\) and \(\Gamma \in U\). In fact, we find that

\[
0 \leq g(t, \Gamma) \leq -\frac{2(1 - \alpha \theta_{\min}) s \inf_{u \in [\tau, T]} f'_t(u)}{\alpha \theta_{\min} f_t(T) \Lambda(\tau)}
\]

where \(\inf_{u \in [\tau, T]} f'_t(u)\) is attained as this is optimizing a continuous function over a compact space. With the boundedness of \(g\) we find that the limit \(\Gamma(T^*) := \lim_{t \to T^*} \Gamma(t)\) exists. Furthermore, \(\Lambda(\Gamma(T^*)) \geq \Lambda(\tau) > \frac{1}{2} \Lambda(\tau)\) and \(\Gamma(T^*) < s\) (by the result of Lemma 3.4). Thus we can continue our solution to \(\Gamma : [\tau, T^*] \to U\) and as such we found a contradiction to \([\tau, T^*]\) being the maximal domain.

\(\Box\)

### B Proofs from Section 4

#### B.1 Proof of Lemma 4.1

*Proof.* We will consider this argument by induction. In the \(n\) bank case, define

\[
\Lambda(t, \Gamma) := 1 + \sum_{j=1}^{n} Z_j(t, \Gamma) f_t(t) f'_t(\sum_{j=1}^{n} \Gamma_j) \\
= \det \left[ I + (Z(t, \Gamma) \cdots Z(t, \Gamma)) f_t(t) f'_t(\sum_{j=1}^{n} \Gamma_j) \right].
\]

By the ordering of banks and the assumption that no firm will modify its portfolio until it hits the regulatory threshold we know that \(\Gamma_i(t) = 0\) if \(t < \tau_i\). We will consider this proof by induction. Note first that \(\tau_0 = 0\) by construction. By \(\Gamma_i(t) = 0\) if \(t < \tau_i\), the results are trivial for \(t \in [0, \tau_1]\). Now take \(k \in \{1, 2, \ldots, n\}\). For any time \(t \in [\tau_k, \tau_{k+1})\) (or \(t \in [\tau_k, T]\) if \(\tau_{k+1} \geq T\))

\[
\Lambda(t, \Gamma(t)) = 1 + \sum_{j=1}^{n} Z_j(t, \Gamma(t)) f_t(t) f'_t(\sum_{j=1}^{n} \Gamma_j(t)) \\
= 1 + \frac{1 - \alpha \theta_{\min}}{\alpha \theta_{\min}} \left[ \sum_{j=1}^{k} \frac{s_j - \Gamma_j(t)}{q(t)} \right] f_t(t) f'_t(\sum_{j=1}^{k} \Gamma_j(t))
\]

30
Using the same logic as in Lemma 3.4, we recover that \( \dot{\Lambda}(t, \Gamma(t)) \geq 0 \) if and only if

\[
\sum_{i=1}^{k} \dot{\Gamma}_i(t) \geq 0
\]

so long as

\[
\dot{\Gamma}(t) = 1 + \frac{1 - \alpha}{\alpha \theta_{\min}} \sum_{j=1}^{k} \left( s_j - \Gamma_j(t) \right) f'_\Gamma(\sum_{j=1}^{k} \Gamma_j(t))
\]

\[
\dot{\Lambda}(t, \Gamma(t)) = 1 - \frac{1 - \alpha}{\alpha \theta_{\min}} \sum_{i=1}^{k} \dot{\Gamma}_i(t) \frac{\left[ \sum_{j=1}^{k} (s_j - \Gamma_j(t)) \right] f'_\Gamma(\sum_{j=1}^{k} \Gamma_j(t))}{f_\Gamma(\sum_{j=1}^{k} \Gamma_j(t))}
\]

To prove this sufficient condition, consider the assumptions on the inverse demand function

\( f_\Gamma \) and assume \( \Gamma = \sum_{i=1}^{k} \Gamma_i(t) \in [0, \sum_{i=1}^{k} s_i] \subseteq [0, M) \):

\[
\frac{d}{d\Gamma} \left[ \frac{\left( \sum_{j=1}^{k} s_j \right) - \Gamma}{f_\Gamma(\Gamma)} \right]_{\Gamma = \sum_{i=1}^{k} \Gamma_i(t)} \geq 0.
\]

Further, by construction, if \( \Lambda(\tau_k, \Gamma(\tau_k)) > 0 \) then \( q(\tau_k) \leq 0 \) and \( \dot{\Gamma}(\tau_k) \) exists. We now want to demonstrate that \( \Lambda(\tau_k, \Gamma(\tau_k)) > 0 \). By construction this is true if and only if
\[ \alpha > -\left(\sum_{i=1}^{k} \left[s_i - \Gamma_i(\tau_k) + s_k \right] f_i'(\sum_{j=1}^{n} \Gamma_j(\tau_k)) / f_i'\left(\sum_{j=1}^{n} \Gamma_j(\tau_k)\right) \right) \theta_{\text{min}}. \]

With \( \Gamma_k(\tau_k) = 0 \) by definition and by the assumption on the inverse demand function

\[
0 \geq \frac{\left(\sum_{i=1}^{k} s_i - \sum_{i=1}^{k} \Gamma_i(\tau_k)\right) f_i'(\sum_{i=1}^{k} \Gamma_i(\tau_k))}{f_i'(\sum_{i=1}^{k} \Gamma_i(\tau_k))} \geq \frac{\left(\sum_{i=1}^{k} s_i\right) f_i'(0)}{f_i'(0)} \geq M f_i'(0) = M f_i'(0).
\]

Therefore, if \( \alpha > -\frac{M f_i'(0)}{(1-M f_i'(0)\theta_{\text{min}})\theta_{\text{min}}} \) then \( \Lambda(\tau_k, \Gamma(\tau_k)) > 0 \).

Next we wish to show that \( \dot{\Gamma}(t) \in \mathbb{R}_+^n \) for all times \( t \). As originally constructed we have that for any \( i \leq k \):

\[
\dot{\Gamma}_i(t) = \frac{\dot{q}(t) \left[ s_i - \Gamma_i(t) \right] \left[ \bar{p}_i - x_i - \Psi_i(t) \right]}{q(t) \left[ (s_i - \Gamma_i(t)) q(t) - (\bar{p}_i - x_i - \Psi(t)) \right]} \mathbb{1}_{\{\theta_i(t) \leq \theta_{\text{min}}\}}.
\]

If a bank is brought above the regulatory threshold they will not perform any transactions, i.e., \( \dot{\Gamma}_i(t) = 0 \), but this can only occur if \( \dot{q}(t) > 0 \). Otherwise \( (1 - \alpha \theta_{\text{min}})(s_i - \Gamma_i(t)) q(t) = \bar{p}_i - x_i - \Psi(t) \) as the firm will need to remain at the regulatory threshold. As such we can simplify \( \dot{\Gamma}_i(t) \) as

\[
\dot{\Gamma}_i(t) = -\frac{\dot{q}(t)(1 - \alpha \theta_{\text{min}})(s_i - \Gamma_i(t))}{\alpha \theta_{\text{min}} q(t)} \mathbb{1}_{\{\theta_i(t) \leq \theta_{\text{min}}\}}.
\]

This allows us to conclude that \( \dot{\Gamma}_i(t) \) has the opposite sign of \( \dot{q}(t) \), i.e., \( \dot{\Gamma}(t) \in \mathbb{R}_+^n \).

Finally, we will now demonstrate that \( \Gamma(t) \in [0, s) \) for all times \( t \in [\tau_k, \tau_{k+1}] \) (or \( t \in [\tau_k, T] \) if \( \tau_{k+1} \geq T \)) by induction for any \( k \in \{0, 1, \ldots, n\} \). As noted above, we find that \( \dot{q}(t) \leq 0 \) for all times \( t \in [\tau_k, \tau_{k+1}] \). By assumption \( \Gamma(t) \in [0, s) \) for all times \( t \in [0, \tau_k] \), so we begin with \( \Gamma(\tau_k) \in [0, s) \). Take \( T^* = \inf\{t \in [\tau_k, \tau_{k+1}] \mid \exists i: \Gamma_i(t) \geq s_i\} \) and assume this infimum is taken over a nonempty set. On \( u \in [\tau_k, T^*) \) we have that:

\[
\dot{q}(u) = \frac{f_i'(u) f_i(\sum_{j=1}^{n} \Gamma_j(u))}{\Lambda(u)} \geq \inf_{t \in [\tau_k, T^*]} f_i'(t) f_i(\sum_{j=1}^{n} \Gamma_j(u)) / \Lambda(\tau_k, \Gamma(\tau_k)) \geq -\frac{(1 - \alpha \theta_{\text{min}}) \dot{q}(u) \left[ s_i - \Gamma_i(u) \right]}{\alpha \theta_{\text{min}} f_i'(u) f_i(\sum_{j=1}^{n} \Gamma_j(u))} \leq -\frac{(1 - \alpha \theta_{\text{min}}) \inf_{t \in [\tau_k, T^*]} f_i'(t)}{\alpha \theta_{\text{min}} f_i(T^*) \Lambda(\tau_k, \Gamma(\tau_k))} \left( s_i - \Gamma_i(u) \right).
\]
coupled with the initial condition $\Gamma_i(\tau_k) \in [0, s_i)$ implies

$$\Gamma_i(u) \leq s_i - (s_i - \Gamma_i(\tau_k)) \exp \left( \frac{(1 - \alpha \theta_{\min})}{\alpha \theta_{\min} f_i(T^*) \Lambda(\tau_k, \Gamma(\tau_k))} (u - \tau_k) \right) < s_i$$

for any time $u \in [\tau_k, T^*)$. As in the proof of Lemma 3.4 we note that $\inf_{t \in [\tau, T^*]} f_i'(t)$ is attained as we are infimizing a continuous function over a compact space. Thus, by continuity, this implies that $\Gamma_i(T^*) < s_i$ for all banks $i$. □

B.2 Proof of Corollary 4.2

Proof. We will use the Lemma 4.1 to prove the existence and uniqueness of a solution. First, for all times $t \in [0, \tau_1]$ there exists a unique solution given by $\Gamma(t) = 0$, $q(t) = f_t(t)$, and $\Psi(t) = 0$. As in the Proof of Theorem 3.6, we will consider the differential equation for $\Gamma$ given by (11). We note that, though we considered the joint differential equation for $\Gamma$ and $q$ previously, (11) only depends on $q$ through the collection of indicator functions on $\theta_i(t) \leq \theta_{\min}$; for the purposes of this proof we will replace the $i$th condition with $f_t(t) f_s \Gamma(\sum_{j=1}^n \Gamma_j(t)) \leq \bar{q}_i$. From the solution $\Gamma$ we can immediately define $q(t) = f_t(t) f_s \Gamma(\sum_{i=1}^n \Gamma_i(t))$ for all times $t$, thus the existence and uniqueness of $\Gamma$ provides the same results for $q$. The results for $\Psi$ follow from the same logic as $\Gamma$ and thus will be omitted herein. We will consider an inductive argument to prove the existence and uniqueness. Assume that we have the existence and uniqueness of the solution $\Gamma(t)$ up to time $\tau_k$ for some $k \in \{1, 2, \ldots, n\}$, then we wish to show we can continue this solution until $\tau_{k+1} \in [\tau_k, T]$.

By Lemma 4.1 $\dot{\Gamma}_i(\tau_k) \geq 0$ for all banks $i$. Define the process $\Gamma^*(t) = \sum_{i=1}^n \Gamma_i(t) = \sum_{i=1}^k \Gamma_i(t)$ with initial condition $\Gamma^*(\tau_k) = \sum_{i=1}^{k-1} \Gamma_i(\tau_k)$. Then following the initial formulation for $\dot{\Gamma}_i(t)$ we find

$$\dot{\Gamma}^*(t) = -\frac{Z_k(t, \Gamma^*(t)) f_t'(t) f_t(\Gamma^*(t))}{\Gamma^*(t)}$$

$$Z_k(t, \Gamma^*) = \frac{(1 - \alpha \theta_{\min})}{\alpha \theta_{\min} f_t(t) f_t(\Gamma^*)} \mathbb{1}_{\{t \geq \tau_k\}}.$$

33
We note that this follows the differential equation of the 1 bank setting (with possibly non-zero initial value). Therefore we can conclude that $\Gamma^*(t)$ exists and is unique for $t \in [\tau_k, \tau_{k+1}]$ (where $\tau_{k+1}$ is a stopping time determined solely by $\Gamma^*$) via an application of Theorem 3.6. Utilizing this unique process $\Gamma^*$ we find that for any bank $i = 1, ..., k$:

$$\dot{\Gamma}_i(t) = g_i(t, \Gamma) = (1 - \alpha \theta_{\min}) \left[ f'_t(t) f_t(\Gamma^*(t)) + \dot{\Gamma}_1(t) f'_t(\Gamma^*(t)) \right] [s_i - \Gamma_i(t)].$$

As $\Gamma^*(t)$ and $\dot{\Gamma}^*(t)$ are bounded in finite time we are able to deduce that $g_i$ is uniformly Lipschitz in $\Gamma$ and thus the existence and uniqueness of $\Gamma_i$ is guaranteed on the domain $[\tau_k, \tau_{k+1}]$. \hfill $\Box$

C Proofs from Section 5

C.1 Proof of Lemma 5.1

Proof. We will prove this inductively. Recall the definition of $\Lambda$ from the proof of Lemma 4.1, i.e. $\Lambda(t, \Gamma) = 1 + \sum_{j=1}^{n} Z_j(t, \Gamma) f_i(t) f_t(\sum_{j=1}^{n} \Gamma_j)$.

1. First, by definition it is clear that $\tilde{\tau}_1 = \tau_1$ and $\tilde{\Gamma}(t) = \Gamma(t) = 0$ for all times $t \in [0, \tilde{\tau}_1]$. Thus $\tilde{\Lambda}_1 = \Lambda(\tau_1, 0)$ as well. By the proof of Lemma 3.4 we know

$$\dot{\tilde{\Gamma}}_1(t) \leq \frac{(1 - \alpha \theta_{\min}) f'_1(t)}{\alpha \theta_{\min} f_t(\Gamma^*(t))} [s_1 - \Gamma_1(t)].$$

for $t \in [\tau_1, \tau_2]$. As expressed in the proof of Lemma 3.4 we can conclude $\tilde{\Gamma}^k_1(t) \geq \Gamma_1(t)$ for all times $t \in [\tilde{\tau}_1, \tau_2]$ and for any iteration $k = 1, ..., n$ by construction as $\tilde{\Gamma}_1^k$ is the maximal solution to this differential inequality.

2. Fix $k \in \{1, 2, ..., n-1\}$ and assume $\tilde{\Gamma}^k_1(t) \geq \Gamma_i(t)$ for all times $t \in [0, \tau_{k+1}]$ and any firm $i = 1, ..., k$. This implies, for any $k \geq k$, $\tilde{\Gamma}^k_1(t) \geq \Gamma_i(t)$ for all times $t \in [0, \tau_{k+1}]$ as well. Assume $\tau_{k+1} < T$ or else the proof is complete. By monotonicity of the inverse demand
function, \( \tilde{\tau}_{k+1} \leq \tau_{k+1} \) with \( \tilde{\Gamma}_i^{k}(\tilde{\tau}_{k+1}) \geq \Gamma_i(\tilde{\tau}_{k+1}) \) for any \( i = 1, \ldots, k \). In particular, this implies \( \Lambda_{k+1} \geq \Lambda(\tilde{\tau}_{k+1}, \Gamma(\tilde{\tau}_{k+1})) \). By the proof of Lemma 4.1 we can show

\[
\dot{\Gamma}_i(t) \leq -\frac{(1 - \alpha \theta_{\min}) f'_i(t)}{\alpha \theta_{\min} f_i(t) \Lambda_{k+1}} (s_i - \Gamma_i(t))
\]

for \( t \in [\tilde{\tau}_{k+1}, \tau_{k+2}) \) and firm \( i = 1, \ldots, k + 1 \). We note that this is a stricter bound than that given in Lemma 4.1, but exists using the same logic. Solving for the maximal solution to this differential inequality provides the solution \( \tilde{\Gamma}_{k+1} \) which must satisfy \( \tilde{\Gamma}_i^{k+1}(t) \geq \Gamma_i(t) \) for all times \( t \in [0, \tau_{k+2}, T] \).

\[\square\]

C.2 Proofs of Corollary [5.2]

**Proof.** First, we will demonstrate that \( \tilde{\Gamma}_n^i(t) \) has the expanded form given above.

\[
\tilde{\Gamma}_i^n(t) = \mathbb{1}_{(t < \tilde{\tau}_n)} \tilde{\Gamma}_i^{n-1}(t) + \mathbb{1}_{(t \geq \tilde{\tau}_n)} \left[ s_i \left( 1 - \frac{f_i(t)}{f_i(\tilde{\tau}_n)} \right)^{1 - \alpha \theta_{\min} / \alpha \theta_{\min} \Lambda_n} + \tilde{\Gamma}_i^{n-1}(\tilde{\tau}_n) \left( \frac{f_i(t)}{f_i(\tilde{\tau}_n)} \right)^{1 - \alpha \theta_{\min} / \alpha \theta_{\min} \Lambda_n} \right]
\]

\[
= \mathbb{1}_{(t < \tilde{\tau}_n)} \tilde{\Gamma}_i^{n-2}(t) + \mathbb{1}_{(t \geq \tilde{\tau}_n)} \left[ s_i \left( 1 - \frac{f_i(t)}{f_i(\tilde{\tau}_n)} \right)^{1 - \alpha \theta_{\min} / \alpha \theta_{\min} \Lambda_n} + \tilde{\Gamma}_i^{n-2}(\tilde{\tau}_n) \left( \frac{f_i(t)}{f_i(\tilde{\tau}_n)} \right)^{1 - \alpha \theta_{\min} / \alpha \theta_{\min} \Lambda_n} \right]
\]

\[
= \mathbb{1}_{(t < \tilde{\tau}_n)} \tilde{\Gamma}_i^{n-2}(t) + \mathbb{1}_{(t \geq \tilde{\tau}_n)} \left[ s_i \left( 1 - \frac{f_i(t)}{f_i(\tilde{\tau}_n)} \right)^{1 - \alpha \theta_{\min} / \alpha \theta_{\min} \Lambda_n} + \tilde{\Gamma}_i^{n-2}(\tilde{\tau}_n) \left( \frac{f_i(t)}{f_i(\tilde{\tau}_n)} \right)^{1 - \alpha \theta_{\min} / \alpha \theta_{\min} \Lambda_n} \right]
\]

\[
= \mathbb{1}_{(t < \tilde{\tau}_n)} \tilde{\Gamma}_i^{n-2}(t) + \mathbb{1}_{(t \geq \tilde{\tau}_n)} \left[ s_i \sum_{j=n-1}^{n-1} \left( 1 - \frac{f_i(t \wedge \tilde{\tau}_{j+1})}{f_i(t \wedge \tilde{\tau}_j)} \right)^{1 - \alpha \theta_{\min} / \alpha \theta_{\min} \Lambda_j} \prod_{k=j+1}^{n} \left( \frac{f_i(t \wedge \tilde{\tau}_{k+1})}{f_i(t \wedge \tilde{\tau}_k)} \right)^{1 - \alpha \theta_{\min} / \alpha \theta_{\min} \Lambda_k} \right]
\]

\[
= \ldots
\]
\[ 1_{(t<\bar{\tau}_i)} \tilde{\Gamma}_i^{i-1}(t) + 1_{(t\geq\bar{\tau}_i)} s_i \sum_{j=i}^{n} \left( 1 - \left( \frac{f_i(t \wedge \bar{\tau}_{j+1})}{f_i(t \wedge \bar{\tau}_j)} \right)^{1-\alpha \theta_{\min}} \right) \prod_{k=j+1}^{n} \left( \frac{f_i(t \wedge \bar{\tau}_{k+1})}{f_i(t \wedge \bar{\tau}_k)} \right)^{1-\alpha \theta_{\min}} \] 

\[ = s_i \sum_{j=i}^{n} \left( 1 - \left( \frac{f_i(t \wedge \bar{\tau}_{j+1})}{f_i(t \wedge \bar{\tau}_j)} \right)^{1-\alpha \theta_{\min}} \right) \prod_{k=j+1}^{n} \left( \frac{f_i(t \wedge \bar{\tau}_{k+1})}{f_i(t \wedge \bar{\tau}_k)} \right)^{1-\alpha \theta_{\min}} \] 

\[ = s_i \left( 1 - \prod_{j=i}^{n} \left( \frac{f_i(t \wedge \bar{\tau}_{j+1})}{f_i(t \wedge \bar{\tau}_j)} \right)^{1-\alpha \theta_{\min}} \right). \]

The penultimate line uses the fact that \( \tilde{\Gamma}_i^{i-1}(t) = 0 \) for all times \( t \) by construction and \( f_i(t \wedge \bar{\tau}_{j+1})/f_i(t \wedge \bar{\tau}_j) = 1 \) for every \( j \geq i \) if \( t < \bar{\tau}_i \).

Now, let us consider the form of \( \tilde{\Lambda}_i \). Here we will take advantage of the exponential form for the inverse demand function \( f_i \).

\[ \tilde{\Lambda}_i = 1 + \frac{1-\alpha \theta_{\min}}{\alpha \theta_{\min}} \left[ \sum_{j=1}^{i} (s_j - \tilde{\Gamma}_j^{i-1}(\bar{\tau}_i)) \right] \frac{f_i \left( \sum_{j=1}^{i-1} \tilde{\Gamma}_j^{i-1}(\bar{\tau}_i) \right)}{f_i \left( \sum_{j=1}^{i-1} \tilde{\Gamma}_j^{i-1}(\bar{\tau}_i) \right)} \]

\[ = 1 - b \frac{1-\alpha \theta_{\min}}{\alpha \theta_{\min}} \left[ \sum_{j=1}^{i} (s_j - \tilde{\Gamma}_j^{i-1}(\bar{\tau}_i)) \right] \]

\[ = 1 - b \frac{1-\alpha \theta_{\min}}{\alpha \theta_{\min}} \left[ \sum_{j=1}^{i} \left( f_i(\bar{\tau}_i \wedge \bar{\tau}_{j+1})/f_i(\bar{\tau}_i \wedge \bar{\tau}_j) \right)^{1-\alpha \theta_{\min}} \prod_{k=j+1}^{n} \left( f_i(\bar{\tau}_i \wedge \bar{\tau}_{k+1})/f_i(\bar{\tau}_i \wedge \bar{\tau}_k) \right)^{1-\alpha \theta_{\min}} \right] \]

Finally, let us consider the time at which the analytical worst-case pricing process hits \( \bar{q}_i := \frac{\bar{\theta}_i - \bar{\tau}_i}{(1-\alpha \theta_{\min}) s_i} \), i.e. the time when firm \( i \) reaches the regulatory threshold \( \theta_{\min} \) provided all firms follow the worst-case path. As no firms act before \( \bar{\tau}_1 = \tau_1 \), this can easily be computed as \( \bar{\tau}_1 = f_{\tau}^{-1}(\bar{q}_1) \). Consider now \( i = 2, \ldots, n \), recall that \( \bar{q}_1 \geq \bar{q}_2 \geq \ldots \geq \bar{q}_n \), and assume
\( t \geq \bar{r}_{i-1}: \)

\[
\tilde{q}_i = f_t(t) f_T \left( \sum_{j=1}^{i-1} \tilde{\Gamma}_j(t) \right)
\]

\[
\Leftrightarrow \tilde{q}_i = f_t(t) f_T \left( \sum_{j=1}^{i-1} s_j - \sum_{j=1}^{i-1} s_j \prod_{k=j}^{i-1} \left( \frac{f_t(t \land \tilde{r}_{k+1})}{f_t(\tilde{r}_k)} \right) \right) \left( 1 + \frac{1 - \alpha \theta_{\min}}{1 - \alpha \theta_{\min} \Lambda_{i-1}} \right)
\]

\[
\Leftrightarrow \log(\tilde{q}_i) - \sum_{j=1}^{i-1} s_j = \log(f_t(t)) + b \sum_{j=1}^{i-1} s_j \prod_{k=j}^{i-1} \left( \frac{f_t(t \land \tilde{r}_{k+1})}{f_t(\tilde{r}_k)} \right) \left( 1 + \frac{1 - \alpha \theta_{\min}}{1 - \alpha \theta_{\min} \Lambda_{i-1}} \right)
\]

\[
\Leftrightarrow \log(\tilde{q}_i) - \sum_{j=1}^{i-1} s_j = \log(f_t(t)) + \left( \frac{bf_t(t)}{f_t(\tilde{r}_{i-1})} \right) \left[ \sum_{j=1}^{i-1} s_j \prod_{k=j}^{i-1} \left( \frac{f_t(\tilde{r}_{k+1})}{f_t(\tilde{r}_k)} \right) \right] \left( 1 + \frac{1 - \alpha \theta_{\min}}{1 - \alpha \theta_{\min} \Lambda_{i-1}} \right)
\]

\[
\Leftrightarrow \log(\tilde{q}_i) - \sum_{j=1}^{i-1} s_j = \log(f_t(t)) + \left( \frac{\alpha \theta_{\min}}{1 - \alpha \theta_{\min}} \right) m_{i-1} f_t(t) \left( 1 + \frac{1 - \alpha \theta_{\min} \Lambda_{i-1}}{1 - \alpha \theta_{\min}} \right)
\]

\[
f_t(t) = \frac{\Lambda_{i-1} W \left( m_{i-1} \exp \left( \frac{1 - \alpha \theta_{\min}}{\alpha \theta_{\min} \Lambda_{i-1}} \left[ \log(\tilde{q}_i) + b \sum_{j=1}^{i-1} s_j \right] \right) \right)}{m_{i-1} \left( 1 - \alpha \theta_{\min} \right)}
\]

\[\square\]

### C.3 Proof of Corollary 5.4

**Proof.** First, before we prove the bound provided in Corollary 5.4 we need to demonstrate that \( \tilde{A}_k \) and \( m_k \) do not depend on the parameter \( a \) of the inverse demand function \( f_t \), i.e. they are constants in this problem. We will do this by induction jointly on \( \tilde{A}_k \), \( m_k \), and \( f(\tilde{r}_k) \) for \( k = 1, \ldots, n \) (trivially this is the case for the assumed values \( \tilde{A}_0 = 1, m_0 = 0 \), and \( f(\tilde{r}_0) = 1 \)).

1. Fix \( k = 1 \), then \( \tilde{A}_1 = 1 - b \frac{1 - \alpha \theta_{\min}}{\alpha \theta_{\min} \Lambda} s_1 \), \( f(\tilde{r}_1) = \tilde{q}_1 \), and \( m_1 = \frac{1 - \tilde{A}_1}{f_t(\tilde{r}_1)} \frac{1 - \alpha \theta_{\min}}{\alpha \theta_{\min} \Lambda} \). Since \( \tilde{A}_1 \) and \( f(\tilde{r}_1) \) do not depend on the parameter \( a \) then neither does \( m_1 \).

2. Fix \( k \in \{2, \ldots, n\} \) and assume \( (\tilde{A}_i, m_i, f(\tilde{r}_i)) \) \( i=1 \) do not depend on the parameter \( a \). By Corollary 5.2 \( f(\tilde{r}_k) \) only depends on \( \tilde{A}_{k-1} \) and \( m_{k-1} \), and thus does not depend on the parameter \( a \). Additionally, \( \tilde{A}_k \) only depends on \( (f(\tilde{r}_i)) \) \( i=1 \), which from the prior
statement we know does not depend on \(a\). Finally, \(m_k\) only depends on \((\tilde{A}_i, f_t(\tilde{\tau}_i))_{i=1}^k\), thus it does not depend on \(a\) either.

We will prove the bound on the probability by induction:

1. Let \(q^* \in [\bar{q}_1, 1]\) (i.e. \(k = 0\)). For such an event to occur, no firms will have hit the regulatory threshold and thus it must be the case that \(q(t) = f_t(t)\). Therefore,

\[
P(q(t) \geq q^*) = P(f_t(t) \geq q^*) = P\left(a \leq -\frac{1}{t} \log(q^*)\right) = P\left(a \leq \frac{1}{t} \Phi^{-1}_0(\log(q^*))\right).
\]

2. Assume the provided bound is true for any \(q^* \in [\bar{q}_k, 1]\). Now let \(q^* \in [\bar{q}_k+1, \bar{q}_k]\).

\[
P(q(t) \geq q^*) = P(q(t) \geq \bar{q}_k) + P(q(t) \in [q^*, \bar{q}_k])
\]

\[
\geq P\left(a \leq \Phi^{-1}_{k-1}(\log(q^*) + b \sum_{i=1}^{k-1} s_i)\right) + P\left(f_t(t) f_t\left(\sum_{i=1}^k \tilde{\Gamma}^n_i(t)\right) \in [q^*, \bar{q}_k]\right)
\]

\[
= P\left(a \leq \Phi^{-1}_{k-1}(\log(q^*) + b \sum_{i=1}^{k-1} s_i)\right)
\]

\[
+ P\left(-at - b \sum_{i=1}^k \tilde{\Gamma}^n_i(t) \in [\log(q^*), \log(\bar{q}_k)]\right)
\]

\[
= P\left(a \leq \Phi^{-1}_{k-1}(\log(q^*) + b \sum_{i=1}^{k-1} s_i)\right)
\]

\[
+ P\left(-at + \left(\frac{\alpha \theta_{\min}}{1 - \alpha \theta_{\min}}\right) m_k f_t(t) \left(1 - \frac{1}{\theta_{\min} \Lambda_k}\right) \exp\left(-\frac{1}{\theta_{\min} \Lambda_k}\right) \in [\log(q^*) + b \sum_{i=1}^k s_i, \log(\bar{q}_k) + b \sum_{i=1}^k s_i]\right)
\]

\[
= P\left(a \leq \Phi^{-1}_{k-1}(\log(q^*) + b \sum_{i=1}^{k-1} s_i)\right)
\]

\[
+ P\left(-at + \left(\frac{\alpha \theta_{\min}}{1 - \alpha \theta_{\min}}\right) m_k \exp\left(-\frac{1}{\theta_{\min} \Lambda_k}\right) \in [\log(q^*) + b \sum_{i=1}^k s_i, \log(\bar{q}_k) + b \sum_{i=1}^k s_i]\right)
\]

\[
= P\left(a \leq \Phi^{-1}_{k-1}(\log(q^*) + b \sum_{i=1}^{k-1} s_i)\right)
\]

38
\[ + \mathbb{P} \left( a \in \Phi_k^{-1} \left( \log(\bar{q}_k) + b \sum_{i=1}^k s_i \right), \Phi_k^{-1} \left( \log(q^*) + b \sum_{i=1}^k s_i \right) \right) \]

\[ = \mathbb{P} \left( a \leq \Phi_k^{-1}(\log(q^*) + b \sum_{i=1}^k s_i) \right). \]

The last line follows from \( \Phi_{k-1}^{-1}(\log(\bar{q}_k) + b \sum_{i=1}^{k-1} s_i) = \Phi_k^{-1}(\log(\bar{q}_k) + \sum_{i=1}^k s_i) \) which is proven by:

\[
\Phi_{k-1}^{-1} \left( \log(\bar{q}_k) + b \sum_{i=1}^{k-1} s_i \right) = \left( \frac{\alpha \theta_{\text{min}}}{1 - \alpha \theta_{\text{min}}} \right) m_{k-1} f_t(\tilde{\tau}_k) \frac{1-\alpha \theta_{\text{min}}}{\alpha \theta_{\text{min}}} \lambda_k - \left[ \log(\bar{q}_k) + b \sum_{i=1}^{k-1} s_i \right] \\
\]

\[
= \left( \frac{\alpha \theta_{\text{min}}}{1 - \alpha \theta_{\text{min}}} \right) m_k f_t(\tilde{\tau}_k) \frac{1-\alpha \theta_{\text{min}}}{\alpha \theta_{\text{min}}} \lambda_k - \left[ \log(\bar{q}_k) + b \sum_{i=1}^k s_i \right] \\
\]

\[
= \left( \frac{\alpha \theta_{\text{min}} \lambda_k}{1 - \alpha \theta_{\text{min}}} \right) W \left( \frac{m_k}{\lambda_k} \right) f_t(\tilde{\tau}_k) \frac{1-\alpha \theta_{\text{min}}}{\alpha \theta_{\text{min}}} \lambda_k \exp \left[ \frac{1 - \Lambda_k}{\Lambda_k} \right] - \left[ \log(\bar{q}_k) + b \sum_{i=1}^k s_i \right] \\
\]

\[
= \Phi_k^{-1} \left( \log(\bar{q}_k) + b \sum_{i=1}^k s_i \right). \]

\[ \square \]

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