Investigation of Financial Track Records by Using Some Novel Concepts of Complex q-Rung Orthopair Fuzzy Information

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ABSTRACT The involvment of complex numbers in the theory of fuzzy sets (FSs) opened the gates for many new ideas. In a complex fuzzy set (CFS), the level of membership attains values from the unit circle in a complex plane. Since the level of membership is a complex number, it is expressed in a form consisting of two parts called the amplitude term and the phase term. This complex structure allows modeling multivariable problems such as problems with periodicity and phase changes. This article studies the complex q-rung orthopair fuzzy sets (CqROFSs) and discovers the innovative concept of complex q-rung orthopair fuzzy relations (CqROFRs) which can deal with a wide range of information, including: fuzzy, complex fuzzy, complex intuitionistic, complex Pythagorean and q-rung orthopair fuzzy information. Moreover, the types of relations are defined with examples and interesting properties. Furthermore, this article also proposes a method based on CqROFRs for modeling the financial track records of business companies. In addition, the applications of the proposed concepts have been presented, which discuss the internal effects of different parameters and factors on the business that might help the sponsors to make the most out of their funds and investments. Another application deliberates the external impacts, i.e., influences of one business over other businesses and provides valuable information to stakeholders which will enable them to identify the key factors for making their business efficient. The results acquired by using the CqROFRs were excellent and more pleasing than other structures in the literature. This flexibility of the proposed framework and the verification of its advantages for solving the application problems is verified through a comprehensive comparative study.

INDEX TERMS Complex q-rung orthopair composite fuzzy relation, complex q-rung orthopair equivalence fuzzy relation, complex q-rung orthopair fuzzy relation, complex q-rung orthopair fuzzy set, financial track record.

I. INTRODUCTION

In the field of mathematics, one of the challenges is to cope with uncertain, imprecise, inaccurate, vague and unclear information. Several theories have been introduced to handle such information. Fuzzy set (FS) theory introduced by Zadeh [1] is one of the outstanding theories for modeling unclear information. FSs are widely used in different real-world scenarios where information is vague. Many researchers studied, applied, and expanded this theory. It helps in describing real-world situations in numerical form using the concept of membership. Membership is a function whose values range between 0 and 1 i.e., [0, 1]. FSs only talk about the level of membership which indicates the level of satisfaction, approval, support, or truth value of the object but does not talk about the level of non-membership. Henceforth, Atanassov [2] enhanced the concept of FSs and...
initiated the concept of an intuitionistic fuzzy set (IFS). The advantage of an IFS over an FS is that IFS discusses both the level of membership and the level of non-membership of an object. Each of the levels is independently defined and belongs to $[0, 1]$. However, there is a constraint that says the sum of level of membership and level of non-membership must be in $[0, 1]$. To overcome this constraint, a new concept of Pythagorean fuzzy set (PFS) was introduced by Yager [3]. Like IFS, the PFS also assigns an object a level of membership, as well as a level of non-membership, both of these values belong to $[0, 1]$. However, the codomain of PFS is slightly broader as in PFS the sum of the squares of both the levels must be contained within the fuzzy set, i.e., $[0, 1]$. Then again, PFS is not enough to handle many problems as sometimes there are cases where both the level of membership and level of non-membership are close to 1, and the conditions of PFS are not satisfied. For example, if the level of membership is $\frac{8}{9}$ and the level of non-membership is $\frac{4}{5}$ then according to PFS, the sum of squares of these values, i.e., $\left(\frac{8}{9}\right)^2 + \left(\frac{4}{5}\right)^2$ must not exceed 1. But, these values do not hold up with the constraints of a PFS. In order to overcome this obstruction, once again, Yager [4] came forward to offer the innovative notion of a q-rung orthopair fuzzy set (qROFS). Like IFS and PFS, the qROFS also discusses the level of membership and the level of non-membership of the objects, but the constraints are modified. In qROFS, the levels of membership and non-membership can be raised to very large powers so that the sum of the $n$th power of level of membership and the $n$th power of level of non-membership lies within the fuzzy set. Thus, it is the most efficient tool to cope the imprecise information. If the level of membership is symbolized by the letter $M$ and the level of non-membership by $N$, then Table I shows the comparison among the constraints on the sum of $M$ and $N$ in all the aforementioned structures. In addition, Figure 1 is also given for a visual demonstration of the ranges of IFSs, PFSs, and qROFSs for $n = 10$. Jan et al. [5] analyzed the communication networks, social networks and shortest path problems in interval-valued q-rung orthopair fuzzy information, Khan et al. [6] presented axiomatically supported divergence measures for qROFSs and Akram et al. [7] made multi-attribute decision making (MADM) with q-rung picture fuzzy information. Wang and Garg [8] proposed the algorithm for MADM with interactive Archimedean norm operations under Pythagorean fuzzy uncertainty.

| Constraint | Set |
|------------|-----|
| $M + N \in [0,1]$ | IFS |
| $M^2 + N^2 \in [0,1]$ | PFS |
| $M^n + N^n \in [0,1]$ for $n \in \mathbb{N}$ | qROFS |

FIGURE 1. Comparison among the ranges of IFS, PFS and qROFS for $n = 10$.

Ramot et al. [9] initiated a new concept by modifying the FSs. They brought in the complex numbers into the fuzzy set as the level of membership of an object and called it complex fuzzy set (CFS). Since the level of membership in CFS is a complex number so instead of belonging to $[0, 1]$, the level belongs to the unit circle in the complex plane. Thus, CFSs have greater codomain for the levels of membership than FS. Moreover, the simple FSs are unable to model varying information such as periodicity or phase change. Since, these phenomena are used in many fields of sciences, which makes the Ramot’s CFS theory the right tool to handle such problems. Actually, the level of membership in CFS is of the form $\frac{\alpha(k)}{\rho(k)2\pi n + 1}$, where $\alpha(k)$ is said to be the amplitude term and its values remain in $[0, 1]$ and $\rho(k)$ also belongs to $[0, 1]$ and is called the phase terms. The phase term refers to the phase change or periodicity. Thus, the CFSs are capable of handling problems with two dimensions. While the FSs, IFSs, PFSs and other related theories cannot model the multidimensional problems. Ramot et al. [10] described the usage of CFSs in future commission merchants. Moreover, Xueling et al. [11] gave the model for identifying the reference signal using CFSs. Complex fuzzy theory can also model the traffic congestions where real-valued set theories tend to fail. The drawback of CFSs is the lack of a level of non-membership. Therefore, Alkouri et al. [12] generalized CFSs and gave the idea of CIFS, which discusses the level of membership as well as the level of non-membership of an object. These levels of membership and non-membership are
again complex numbers. CIFS also restricts the sum of mods of these complex-valued levels of membership and non-membership between 0 and 1. They [12] also discussed some operations on CIFSs, like union, intersection and complement. Other scientists also added in the field of complex fuzzy theory. For the solution of problems with multiple periodic factors, Ma et al. [13] gave the method based on the theory of CFS. The operations of CFSs were discussed by Dick et al. [14]. Liu and Zeng [15] extended and enhanced the work of Dick et al., and Greenfield et al. [16] generalized the notion of CFSs and proposed the idea of complex interval-valued fuzzy sets (CIVFSs). Due to restrictions on CIFSs, the need for complex Pythagorean fuzzy sets (CPFSs) rose. So, Ullah et al. [17] fulfilled this need and introduced CPFSs by modifying the constraints in CIFSs. Like CIFS, the CPFS also consists of complex-valued mappings as levels of membership and non-membership that belong to a unit circle in a complex plane. Actually, the constraint is modified to increase the codomain of CPFSs as compared to the codomain of CIFSs. According to the new constraints, the sum of squares of mods of complex-valued levels of membership and complex-valued non-membership must be in [0, 1]. Ullah et al. [17] also defined some distance measures of CPFSs. Moreover, Akram et al. [18,19] discussed the decision-making and competition graphs under CPFSs. They also studied the complex Pythagorean Dombi fuzzy operators by aggregation operators [20]. Besides these concepts, they also worked on the prioritized weighted aggregation operators under CPFSs [21]. Garg et al. [22] devised the complex qROFS (CqROFS). They modified the constraints on CIFSs and CPFSs such that the sum of mods of the complex-valued level of membership and complex-valued level of non-membership when raised to the powers of \( n \in \mathbb{N} \), lies within [0, 1]. Table II shows the comparison among the constraints on the CIFSs, CPFSs, and CqROFs. M and N symbolize the complex-valued level of membership and the complex-valued level of non-membership.

| Table II | Comparison among the Constraints of CIFS, CPFS and CqROFS |
|----------|----------------------------------------------------------|
| Set      | Constraint                                               |
| CIFS     | \(|M| + |N| \in [0,1]\)                                    |
| CPFS     | \(|M|^2 + |N|^2 \in [0,1]\)                               |
| CqROFS   | \(|M|^n + |N|^n \in [0,1]\) for n \in \mathbb{N}\)       |

From Table II, it clearly observed that the codomain of CqROFS is much greater than the other two set structures.

In various areas of sciences, the notion of relations is considered an important concept. Hence, the relations are used in several sciences, including different fields of mathematics, engineering, social sciences, and many more. In mathematics, a relation defines the relationship between a pair of non-empty sets. There are many types of relations such as reflexive, irreflexive, symmetric, anti-symmetric, asymmetric, transitive, composite relation, order and equivalence relation. Relations are also defined in fuzzy set theory which is applied in the reasoning of vagueness, fuzzy clustering and control.

The crisp relation (CR) introduced by Klir [23] shows the existence or nonexistence of some relationship between some classical sets. But it does not define the level of the relationship, i.e., the strength or weakness of any relation. For that reason, the fuzzy relation (FR), which is a generalization of a CR, was introduced by Mendel [24]. Thus, FRs not only show the existence or nonexistence of a relationship between the sets but also tells the level of strength of a relationship. These relations are of great concern in a fuzzy logic system. Moreover, to determine both the weakness and strength levels of a relationship between sets, Burillo et al. [25] introduced intuitionistic fuzzy relation (IFR). In addition, Triapathi et al. [26] achieved the diagnostic conclusion about diabetes by using FRs. Majid [27] predicted the score of cricket sport using FRs.

Since CRs and FRs cannot handle two-dimensional problems, so, Ramot et al. [9] provided the idea of complex fuzzy relations (CFRs). These relations discuss the existence or nonexistence of the relationship along with the phase of the relationship between the parameters. Furthermore, Ramot et al. [10] applied the CFRs in Future Commission Merchant. Jan et al. [28] initiated the notion of complex intuitionistic fuzzy relation (CIFR) and applied their method to investigate the cyber-security and cyber-attacks in oil and gas industries. Jan et al. [29] also described the complex Pythagorean fuzzy relations (CPFRs). Singh et al. [30] discussed the lattices and granular decompositions of interval-valued complex fuzzy sets. Hu et al. [31] studied the distances of CFSs and the continuity of CF operations. Nasir et al. [32] introduced the interval-valued complex intuitionistic fuzzy relations with an application to study the loopholes in industrial control systems, Quek and Selvachandran [33] proposed the algebraic structures of CIFS with groups. Feng et al. [34] applied CFS to E-commerce. Garg et al. concluded information measures of CIFSs [35]. Rani et al. [36] defined the distance measures between the CIFSs and applied this concept in the decision-makings. Yaqoob et al. [37] applied CIFSs and graphs to cellular network providers. For more research on complex structures and their applications, consider [38-46].
However, this paper overcomes many obstructions regarding set theory and its branches. The novel concepts of CqROFSs and CqROFRs are devised. These notions discuss both the levels, i.e., the level of membership and the level of non-membership. Also, both the levels are independent complex-valued mappings. Ultimately, these concepts are capable of modeling phenomena involving multidimensional variables, whereas other available techniques and methods cannot handle such problems. Furthermore, the range of CqROFSs is much more extensive than CPFSs and CIFSs, as can be seen in Table II. This allows the decision-makers to make accurate and better decisions by assigning the values of levels of membership and non-membership without severe constraints. Moreover, this study explains different types of CqROFRs with examples such as CqRO-reflexive-FR, CqRO-irreflexive-FR, CqRO-symmetric-FR, CqRO-asymmetric-FR, CqRO-antisymmetric-FR, CqRO-transitive-FR, CqRO-equivalence-FR, CqRO-order-FR and CqRO-composite-FR. In addition, a couple of applications of CqRO-equivalence-FR and CqRO-composite-FR are proposed for inspecting the effects of certain financial parameters of a business company on the other financial parameters of the company. The CqROFRs are the generalization of FRs, CFRs, IFRs, CIFRs, PFRs and CPFRs. The proposed methods can cope with all sorts of fuzzy information whether it be the fuzzy, complex fuzzy, intuitionistic, complex intuitionistic or complex Pythagorean fuzzy information. The CqROFRs are capable of dealing with level of membership as well as the level of non-membership. Since the structure of proposed concepts involves complex numbers, they have the ability to model the multivariable problems; the problems involving time, phase changes, cycles or periodicity. Another advantage is that the values of levels of membership and non-membership can be assigned independently with extremely mild restrictions. Thus, allowing the decision-makers to make a better-quality decision. Also, an application is discussed that scrutinized several companies that had some financial relationships. The progressive and regressive effects of different parameters of one business company over the other business companies with respect to the time phase are also studied. These applications might help an investor to make the most beneficial decision while investing his assets. The other advantage of approaching the financial track record of a business using the proposed process will help the stakeholders to identify the most critical factors that are the reason behind some sort of loss, or they can be focused upon to earn more profits. Further, the indirect impacts of a business on the financial track records of other related businesses might be learnt from the applications. This lesson would assist the businessmen to take their business companies to the next level and stay ahead of the other business companies. As an overview of the proposed method, Figure 2 clarifies organization and process.

![Figure 2](image-url)

**FIGURE 2.** Overview of the process of the proposed method.

The proposed methods can be further worked out to obtain interesting and powerful structures. Since the structure of the proposed notions is so versatile and flexible, they can be used to solve many single and multidimensional problems with fuzziness. Some modifications in the structure will further improve their abilities, such as the inclusion of a level of neutrality. Also, keeping the least constraints on the assignment of values of levels might open up the gates for wide range of applications. The milder constraints will also facilitate the decision-makers.

The remaining paper is organized in a way that Section II reviews some fundamental concepts such as FSs, CFSs, IFSs, CIFSs, qROFSs with examples. Section III defines some novel concepts such as CqROFSs, CIFRs, CPFRs, CqROFRs and Cartesian products in CqROFSs with examples. Also, the types of CqROFRs are defined with suitable examples in Section III. Section IV produces some valuable results and discusses the properties of CqROFRs. Section V applies these novel concepts to study the financial structures of business companies. Section VI compares the proposed technique with preexisting techniques in the literature. Finally, Section VII concludes the paper.
II. PRELIMINARIES

This section enlightens some related prerequisites such as fuzzy set (FS), complex fuzzy set (CFS), intuitionistic fuzzy set (IFS), complex intuitionistic fuzzy set (CIFS), q-rung orthopair fuzzy set (qROFS), complex q-rung orthopair fuzzy set (CqROFS). Moreover, the Cartesian product and relations in the above sets are also described with some examples.

Definition 1. [1] In a universe \( N \), the collection \( K \) of the following form is a fuzzy set (FS),

\[
K = \{ k, m_k(k) : k \in N \}
\]

where \( m_k(k) \) is a fuzzy valued mapping, i.e., \( m_k : N \rightarrow [0,1] \) which symbolizes the level of membership of \( K \).

Definition 2. [9] In a universe \( N \), the collection \( K \) of the following form is a complex fuzzy set (CFS),

\[
K = \{ k, m_k(k) : k \in N \}
\]

where \( m_k(k) \) symbolizes the level of membership of \( K \) which is a complex-valued mapping defined as,

\[
m_k : N \rightarrow \{ z : \in C, |z| \leq 1 \}
\]

where \( C \) denotes the collection of complex numbers and implies,

\[
z(k) = a + b\sqrt{-1} \text{ or } z(k) = \frac{a(k)}{e^{\rho(k)2\pi\sqrt{-1}}} \text{ where } a(k), \rho(k) \in [0,1] .
\]

Thus, a CFS has the following form,

\[
K = \left\{ k, \frac{a(k)}{e^{\rho(k)2\pi\sqrt{-1}}} : k \in N \right\}
\]

Definition 3. [9] If \( K = \left\{ k, \frac{a(k)}{e^{\rho(k)2\pi\sqrt{-1}}} : k \in N \right\} \) and \( J = \left\{ j, \frac{a(j)}{e^{\rho(j)2\pi\sqrt{-1}}} : j \in N \right\} \) are any two CFSs in a universal \( N \). Then the Cartesian product between \( K \) and \( J \) is symbolized and defined as,

\[
K \times J = \left\{ (k,j), \left( \frac{a(k,j)}{e^{\rho(k,j)2\pi\sqrt{-1}}} : k \in K, j \in J \right) \right\}
\]

where \( a(k,j) = \min\{a(k),a(j)\} \) and \( \rho(k,j) = \min\{\rho(k),\rho(j)\} \)

Definition 4. [9] For any two CFSs in \( N \), \( K = \left\{ k, \frac{a(k)}{e^{\rho(k)2\pi\sqrt{-1}}} : k \in N \right\} \) and \( J = \left\{ j, \frac{a(j)}{e^{\rho(j)2\pi\sqrt{-1}}} : j \in N \right\} \), a complex fuzzy relation (CFR) \( R \) is a subset of their Cartesian product, i.e., \( R \subseteq K \times J \).

Example 1. Consider the following CFS \( K \) in universe \( N \),

\[
K = \left\{ \left( k, \frac{4}{5} e^{(\frac{1}{2})2\pi\sqrt{-1}} \right), \left( j, \frac{1}{3} e^{(\frac{1}{2})2\pi\sqrt{-1}} \right), \left( l, \frac{0}{0} e^{(\frac{1}{2})2\pi\sqrt{-1}} \right) \right\}
\]

The Cartesian product \( K \times K \) is,

\[
K \times K = \left\{ \left( (k,k), \frac{4}{5} e^{(\frac{1}{2})2\pi\sqrt{-1}} \right), \left( (k,j), \frac{1}{3} e^{(\frac{1}{2})2\pi\sqrt{-1}} \right), \left( (k,l), \frac{0}{0} e^{(\frac{1}{2})2\pi\sqrt{-1}} \right), \right. \left. \left( (j,j), \frac{1}{3} e^{(\frac{1}{2})2\pi\sqrt{-1}} \right), \left( (j,l), \frac{0}{0} e^{(\frac{1}{2})2\pi\sqrt{-1}} \right), \left( (l,l), \frac{0}{0} e^{(\frac{1}{2})2\pi\sqrt{-1}} \right) \right\}
\]

The CFR \( R \) is,

\[
R = \left\{ \left( (k,k), \frac{4}{5} e^{(\frac{1}{2})2\pi\sqrt{-1}} \right), \left( (k,j), \frac{1}{3} e^{(\frac{1}{2})2\pi\sqrt{-1}} \right), \left( (j,j), \frac{1}{3} e^{(\frac{1}{2})2\pi\sqrt{-1}} \right), \left( (j,l), \frac{0}{0} e^{(\frac{1}{2})2\pi\sqrt{-1}} \right), \left( (l,l), \frac{0}{0} e^{(\frac{1}{2})2\pi\sqrt{-1}} \right) \right\}
\]

Definition 5. [2] In a universe \( N \), the collection \( K \) of the following form is an intuitionistic fuzzy set (IFS),

\[
K = \{ k, m_k(k), n_k(k) : k \in N \}
\]

where \( m_k(k) \) and \( n_k(k) \) are fuzzy valued mappings provided that \( m_k(k) + n_k(k) \in [0,1] \). \( m_k(k) \) and \( n_k(k) \) symbolize the level of membership and the level of non-membership of \( K \), respectively.

Definition 6. [12] In a universe \( N \), the collection \( K \) of the following form is a complex intuitionistic fuzzy set (CIFS),

\[
K = \left\{ \left( k, m_k(k), n_k(k) : k \in N \right) \right\}
\]
where \( m_c(k) \) and \( n_c(k) \) symbolize the level of membership and the level of non-membership of \( K \), respectively. These mappings are complex-valued, defined as,

\[
m_c: N \rightarrow \{z_1: z_1 \in \mathbb{C}, |z_1| \leq 1\} \quad \text{and} \quad n_c: N \rightarrow \{z_2: z_2 \in \mathbb{C}, |z_2| \leq 1\}
\]

where \( \mathbb{C} \) denotes the collection of complex numbers and implies,

\[
z_1(k) = a_1 + b_1 \sqrt{-1} \quad \text{and} \quad z_2(k) = a_2 + b_2 \sqrt{-1}
\]

provided that \((|z_1(k)|^2 + |z_2(k)|^2) \in [0, 1]\) or \(z_1(k) = \alpha_m(k) e^{\rho_m(k)2\pi i} \) and \(z_2(k) = \alpha_n(k) e^{\rho_n(k)2\pi i} \)

where \( \alpha_m(k), \rho_m(k), \alpha_n(k), \rho_n(k) \in [0, 1] \), provided that \((\alpha_m(k))^2 + (\alpha_n(k))^2 \in [0, 1] \) and \((\rho_m(k))^2 + (\rho_n(k))^2 \in [0, 1] \).

Thus a CPFS has the following form,

\[
K = \left\{k, \frac{\alpha_m(k)}{e^{\rho_m(k)2\pi i}}, \frac{\alpha_n(k)}{e^{\rho_n(k)2\pi i}} : k \in N \right\}
\]

**Definition 7.** [3] In a universe \( N \), the collection \( K \) of the following form is a Pythagorean fuzzy set (PFS),

\[
K = \{k, m_c(k), n_c(k) : k \in N\}
\]

where \( m_c(k) \) and \( n_c(k) \) are fuzzy valued mappings provided that \((m_c(k))^2 + (n_c(k))^2 \in [0, 1]\). \( m_c(k) \) and \( n_c(k) \) symbolize the level of membership and the level of non-membership of \( K \), respectively.

**Definition 8.** [17] In a universe \( N \), the collection \( K \) of the following form is a complex Pythagorean fuzzy set (CPFS),

\[
K = \left\{(k, m_c(k), n_c(k)) : k \in N \right\}
\]

where \( m_c(k) \) and \( n_c(k) \) symbolize the level of membership and the level of non-membership of \( K \), respectively. These mappings are complex-valued, defined as,

\[
m_c: N \rightarrow \{z_1: z_1 \in \mathbb{C}, |z_1| \leq 1\} \quad \text{and} \quad n_c: N \rightarrow \{z_2: z_2 \in \mathbb{C}, |z_2| \leq 1\}
\]

where \( \mathbb{C} \) denotes the collection of complex numbers and implies,

\[
z_1(k) = a_1 + b_1 \sqrt{-1} \quad \text{and} \quad z_2(k) = a_2 + b_2 \sqrt{-1}
\]

provided that \((|z_1(k)|^2 + |z_2(k)|^2) \in [0, 1]\) or \(z_1(k) = \alpha_m(k) e^{\rho_m(k)2\pi i} \) and \(z_2(k) = \alpha_n(k) e^{\rho_n(k)2\pi i} \)

where \( \alpha_m(k), \rho_m(k), \alpha_n(k), \rho_n(k) \in [0, 1] \), provided that \((\alpha_m(k))^2 + (\alpha_n(k))^2 \in [0, 1] \) and \((\rho_m(k))^2 + (\rho_n(k))^2 \in [0, 1] \).

Thus a CqROFS has the following form,

\[
K = \left\{k, \frac{\alpha_m(k)}{e^{\rho_m(k)2\pi i}}, \frac{\alpha_n(k)}{e^{\rho_n(k)2\pi i}} : k \in N \right\}
\]

**Definition 9.** [4] In a universe \( N \), the collection \( K \) of the following form is a q-rung orthopair fuzzy set (qROFS),

\[
K = \{k, m_q(k), n_q(k) : k \in N\}
\]

where \( m_q(k) \) and \( n_q(k) \) are fuzzy valued mappings provided that \((m_q(k))^n + (n_q(k))^n \in [0, 1]\). \( m_q(k) \) and \( n_q(k) \) symbolize the level of membership and the level of non-membership of \( K \), respectively.

**Definition 10.** [22] In a universe \( N \), the collection \( K \) of the following form is a complex q-rung orthopair fuzzy set (CqROFS),

\[
K = \left\{(k, m_c(k), n_c(k)) : k \in N \right\}
\]

where \( m_c(k) \) and \( n_c(k) \) symbolize the level of membership and the level of non-membership of \( K \), respectively. These mappings are complex-valued, defined as,

\[
m_c: N \rightarrow \{z_1: z_1 \in \mathbb{C}, |z_1| \leq 1\} \quad \text{and} \quad n_c: N \rightarrow \{z_2: z_2 \in \mathbb{C}, |z_2| \leq 1\}
\]

where \( \mathbb{C} \) denotes the collection of complex numbers and implies,

\[
z_1(k) = a_1 + b_1 \sqrt{-1} \quad \text{and} \quad z_2(k) = a_2 + b_2 \sqrt{-1}
\]

provided that \((|z_1(k)|^2 + |z_2(k)|^2) \in [0, 1]\) or \(z_1(k) = \alpha_m(k) e^{\rho_m(k)2\pi i} \) and \(z_2(k) = \alpha_n(k) e^{\rho_n(k)2\pi i} \)

where \( \alpha_m(k), \rho_m(k), \alpha_n(k), \rho_n(k) \in [0, 1] \), provided that \((\alpha_m(k))^n + (\alpha_n(k))^n \in [0, 1] \) and \((\rho_m(k))^n + (\rho_n(k))^n \in [0, 1] \).

Thus a CqROFS has the following form,

\[
K = \left\{k, \frac{\alpha_m(k)}{e^{\rho_m(k)2\pi i}}, \frac{\alpha_n(k)}{e^{\rho_n(k)2\pi i}} : k \in N \right\}
III. COMPLEX Q-RUNG ORTHOPAIR FUZZY RELATIONS AND THEIR TYPES

This section presents the definitions of some novel concepts such as the Cartesian product of CIFSs, Cartesian product of CPFSs, Cartesian product of CqROFs, CIFR, CPFR, and CqROFR. Moreover, some examples are also given to clarify the concepts. This section also presents the types of relations in a CqROFR along with examples, such as the inverse of CqROFR, CqRO-reflexive-FR, CqRO-symmetric-FR, CqRO-asymmetric-FR, CqRO-transitive-FR, CqRO-equivalence-FR, CqRO-irreflective-FR, CqRO-antisymmetric-FR, CqRO-order-FR, CqRO-composite-FR and CqRO equivalence class.

Definition 11. If \( K = \left\{ k, \frac{a_m(k)}{e^{\rho_m(k)3\pi/4}}, \frac{a_n(k)}{e^{\rho_n(k)3\pi/4}} : k \in N \right\} \) and \( J = \left\{ j, \frac{a_m(j)}{e^{\rho_m(j)3\pi/4}}, \frac{a_n(j)}{e^{\rho_n(j)3\pi/4}} : j \in N \right\} \) are any two CqROFs in a universe \( N \). Then the Cartesian product of \( K \) and \( J \) is symbolized and defined as,

\[
K \times J = \left\{ (k, j), \frac{a_m(k, j)}{e^{\rho_m(k,j)3\pi/4}}, \frac{a_n(k, j)}{e^{\rho_n(k,j)3\pi/4}} : k \in K, j \in J \right\}
\]

where \( a_{m,n}(k,j) = \min\{a_{m,n}(k), a_{m,n}(j)\} \), \( \rho_{m,n}(k,j) = \min\{\rho_{m,n}(k), \rho_{m,n}(j)\} \), \( a_{m,n}(k,j) = \max\{a_{m,n}(k), a_{m,n}(j)\} \) and \( \rho_{m,n}(k,j) = \max\{\rho_{m,n}(k), \rho_{m,n}(j)\} \).

Definition 12. For any two CqROFs in \( N \), \( K = \left\{ k, \frac{a_m(k)}{e^{\rho_m(k)3\pi/4}}, \frac{a_n(k)}{e^{\rho_n(k)3\pi/4}} : k \in N \right\} \), and \( J = \left\{ j, \frac{a_m(j)}{e^{\rho_m(j)3\pi/4}}, \frac{a_n(j)}{e^{\rho_n(j)3\pi/4}} : j \in N \right\} \), a complex q-rung orthopair fuzzy relation (CqROFR) \( R \) is a subset of their Cartesian product, i.e., \( R \subseteq K \times J \).

Remark: As qROFS is a generalization of IFS and PFS. Similarly, the CqROFS is a generalization of CIFS and CPFS. Henceforth, for \( n = 1 \), the CqROFS becomes a CIFS, and for \( n = 2 \), the CqROFS becomes a CPFS. Therefore, the definition of CqROFRs also covers the definition of CIFRs and CPFRs for \( n = 1 \) and \( n = 2 \), respectively.

Example 2. Consider the following CIFS \( K \) in universe \( N \),

\[
K = \begin{cases}
\left( k, \frac{3}{10}, e^{\frac{3}{10}2\pi \sqrt{-1}}, \frac{3}{5} e^{\frac{1}{10}2\pi \sqrt{-1}} \right), \\
\left( j, \frac{7}{10}, e^{\frac{7}{10}2\pi \sqrt{-1}}, \frac{1}{10} e^{\frac{1}{10}2\pi \sqrt{-1}} \right)
\end{cases}
\]

The Cartesian product \( K \times K \) is,

\[
K \times K = \begin{cases}
\left( (k, j), \frac{3}{10}, e^{\frac{3}{10}2\pi \sqrt{-1}}, \frac{3}{5} e^{\frac{1}{10}2\pi \sqrt{-1}} \right), \\
\left( (k, j), \frac{3}{10}, e^{\frac{3}{10}2\pi \sqrt{-1}}, \frac{3}{5} e^{\frac{1}{10}2\pi \sqrt{-1}} \right), \\
\left( (j, k), \frac{3}{10}, e^{\frac{3}{10}2\pi \sqrt{-1}}, \frac{3}{5} e^{\frac{1}{10}2\pi \sqrt{-1}} \right), \\
\left( (j, k), \frac{7}{10}, e^{\frac{7}{10}2\pi \sqrt{-1}}, \frac{1}{10} e^{\frac{1}{10}2\pi \sqrt{-1}} \right)
\end{cases}
\]

The CIFR \( R \) is,

\[
R = \begin{cases}
\left( (k, j), \frac{3}{10}, e^{\frac{3}{10}2\pi \sqrt{-1}}, \frac{3}{5} e^{\frac{1}{10}2\pi \sqrt{-1}} \right), \\
\left( (k, j), \frac{3}{10}, e^{\frac{3}{10}2\pi \sqrt{-1}}, \frac{3}{5} e^{\frac{1}{10}2\pi \sqrt{-1}} \right), \\
\left( (j, k), \frac{7}{10}, e^{\frac{7}{10}2\pi \sqrt{-1}}, \frac{1}{10} e^{\frac{1}{10}2\pi \sqrt{-1}} \right)
\end{cases}
\]

Example 3. Consider the following CPFS \( K \) in universe \( N \),

\[
K = \begin{cases}
\left( k, \frac{3}{5}, e^{\frac{3}{5}2\pi \sqrt{-1}}, \frac{7}{10} e^{\frac{7}{10}2\pi \sqrt{-1}} \right), \\
\left( j, \frac{3}{5}, e^{\frac{3}{5}2\pi \sqrt{-1}}, \frac{4}{5} e^{\frac{4}{5}2\pi \sqrt{-1}} \right)
\end{cases}
\]

The Cartesian product \( K \times K \) is,
The CPFR $R$ is,

$$R = \left\{ \begin{array}{l}
(k, j), \frac{3}{10} e^{\frac{4}{5}2\pi\sqrt{-1}}, \frac{4}{5} e^{\frac{4}{5}2\pi\sqrt{-1}}, \\
(j, k), \frac{3}{10} e^{\frac{9}{10}2\pi\sqrt{-1}}, \frac{4}{5} e^{\frac{9}{10}2\pi\sqrt{-1}}, \\
(j, j), \frac{3}{10} e^{\frac{9}{10}2\pi\sqrt{-1}}, \frac{4}{5} e^{\frac{9}{10}2\pi\sqrt{-1}}, \end{array} \right\}$$

The inverse CPFR $R^{-1}$ in $K$ is defined as denoted as,

$$R^{-1} = \left\{ \begin{array}{l}
(k, j), \frac{\alpha_m(j, k)}{e^{\theta_m(j,k)2\pi\sqrt{-1}}}, \frac{\alpha_n(j, k)}{e^{\theta_n(j,k)2\pi\sqrt{-1}}}, (k, j) \in R \\
(j, k), \frac{\alpha_m(j, k)}{e^{\theta_m(k,j)2\pi\sqrt{-1}}}, \frac{\alpha_n(j, k)}{e^{\theta_n(k,j)2\pi\sqrt{-1}}}, (j, k) \in K \end{array} \right\}$$

Example 4. Consider the following CIFS $K$ for $n = 7$ in universe $N$,

$$K = \left\{ k, \frac{9}{10} e^{\frac{4}{5}2\pi\sqrt{-1}}, \frac{9}{10} e^{\frac{4}{5}2\pi\sqrt{-1}}, \\
(j), \frac{2}{5} e^{\frac{4}{5}2\pi\sqrt{-1}}, \frac{7}{10} e^{\frac{4}{5}2\pi\sqrt{-1}}, \right\}$$

The Cartesian product $K \times K$ is,

$$K \times K = \left\{ \begin{array}{l}
(k, k), \frac{9}{10} e^{\frac{4}{5}2\pi\sqrt{-1}}, \frac{9}{10} e^{\frac{4}{5}2\pi\sqrt{-1}}, \\
(k, j), \frac{2}{5} e^{\frac{4}{5}2\pi\sqrt{-1}}, \frac{9}{10} e^{\frac{4}{5}2\pi\sqrt{-1}}, \\
(j, k), \frac{2}{5} e^{\frac{4}{5}2\pi\sqrt{-1}}, \frac{9}{10} e^{\frac{4}{5}2\pi\sqrt{-1}}, \\
(j, j), \frac{2}{5} e^{\frac{4}{5}2\pi\sqrt{-1}}, \frac{7}{10} e^{\frac{4}{5}2\pi\sqrt{-1}}, \end{array} \right\}$$

The CqROFR $R$ is,

$$R = \left\{ \begin{array}{l}
(k, k), \frac{9}{10} e^{\frac{4}{5}2\pi\sqrt{-1}}, \frac{9}{10} e^{\frac{4}{5}2\pi\sqrt{-1}}, \\
(k, j), \frac{3}{10} e^{\frac{4}{5}2\pi\sqrt{-1}}, \frac{4}{5} e^{\frac{4}{5}2\pi\sqrt{-1}}, \\
(j, k), \frac{3}{10} e^{\frac{9}{10}2\pi\sqrt{-1}}, \frac{4}{5} e^{\frac{9}{10}2\pi\sqrt{-1}}, \\
(j, j), \frac{3}{10} e^{\frac{9}{10}2\pi\sqrt{-1}}, \frac{4}{5} e^{\frac{9}{10}2\pi\sqrt{-1}}, \end{array} \right\}$$

Definition 13. Consider a CqROFR $R$ on a CqROFS $K$,

$$R = \left\{ (k, j), \frac{\alpha_m(k, j)}{e^{\theta_m(k,j)2\pi\sqrt{-1}}}, \frac{\alpha_n(k, j)}{e^{\theta_n(k,j)2\pi\sqrt{-1}}}, (k, j) \in K \times K \right\}$$

The inverse CqROFR $R^{-1}$ in $K$ is defined as denoted as,

$$R^{-1} = \left\{ (k, j), \frac{\alpha_m(j, k)}{e^{\theta_m(j,k)2\pi\sqrt{-1}}}, \frac{\alpha_n(j, k)}{e^{\theta_n(j,k)2\pi\sqrt{-1}}}, (j, k) \in R \right\}$$

Definition 14. A CqROFR $R$ is called a complex q-rung orthopair reflexive fuzzy relation (CqRO-reflexive-FR) if

$$\forall (k, \frac{\alpha_m(k)}{e^{\theta_m(k)2\pi\sqrt{-1}}}, \frac{\alpha_n(k)}{e^{\theta_n(k)2\pi\sqrt{-1}}}) \in K \text{ implies the following,}$$

$$\left( (k, k), \frac{\alpha_m(k, k)}{e^{\theta_m(k,k)2\pi\sqrt{-1}}}, \frac{\alpha_n(k, k)}{e^{\theta_n(k,k)2\pi\sqrt{-1}}}, \right) \in R$$

Definition 15. A CqROFR $R$ is called a CqRO-symmetric-FR if

$$\left( (k, j), \frac{\alpha_m(j, k)}{e^{\theta_m(k,j)2\pi\sqrt{-1}}}, \frac{\alpha_n(j, k)}{e^{\theta_n(k,j)2\pi\sqrt{-1}}}, \right) \in R$$

then

$$\left( (j, k), \frac{\alpha_m(j, k)}{e^{\theta_m(j,k)2\pi\sqrt{-1}}}, \frac{\alpha_n(j, k)}{e^{\theta_n(j,k)2\pi\sqrt{-1}}}, \right) \in R.$$

Definition 16. A CqROFR $R$ is called a CqRO-antisymmetric-FR if

$$\left( (k, j), \frac{\alpha_m(j, k)}{e^{\theta_m(k,j)2\pi\sqrt{-1}}}, \frac{\alpha_n(j, k)}{e^{\theta_n(k,j)2\pi\sqrt{-1}}}, \right) \in R$$

and

$$\left( (j, k), \frac{\alpha_m(j, k)}{e^{\theta_m(j,k)2\pi\sqrt{-1}}}, \frac{\alpha_n(j, k)}{e^{\theta_n(j,k)2\pi\sqrt{-1}}}, \right) \in R.$$
Definition 17. A CqROFR $R$ is called a CqRO-transitive-FR if
\[
\begin{pmatrix}
k, j, & \alpha_n(k, j) - e^{\rho_n(k, j)(2\pi/\sqrt{5} - 1)} - e^{\rho_n(k, j)(2\pi/\sqrt{5} - 1)} \\
(j, l), & \alpha_n(j, l) - e^{\rho_n(j, l)(2\pi/\sqrt{5} - 1)} - e^{\rho_n(j, l)(2\pi/\sqrt{5} - 1)} \\
k, l, & \alpha_n(k, l) - e^{\rho_n(k, l)(2\pi/\sqrt{5} - 1)} - e^{\rho_n(k, l)(2\pi/\sqrt{5} - 1)}
\end{pmatrix} \in R
\]
and
\[
\begin{pmatrix}
k, k, & \alpha_n(k, k) - e^{\rho_n(k, k)(2\pi/\sqrt{5} - 1)} - e^{\rho_n(k, k)(2\pi/\sqrt{5} - 1)} \\
(j, j), & \alpha_n(j, j) - e^{\rho_n(j, j)(2\pi/\sqrt{5} - 1)} - e^{\rho_n(j, j)(2\pi/\sqrt{5} - 1)} \\
k, l, & \alpha_n(k, l) - e^{\rho_n(k, l)(2\pi/\sqrt{5} - 1)} - e^{\rho_n(k, l)(2\pi/\sqrt{5} - 1)}
\end{pmatrix} \in R.
\]

Definition 18. A CqROFR $R$ is called a CqRO-reflexive-FR if $\forall (k, \alpha_n(k) - e^{\rho_n(k)(2\pi/\sqrt{5} - 1)} - e^{\rho_n(k)(2\pi/\sqrt{5} - 1)}) \in R$ implies that
\[
\begin{pmatrix}
k, k, & \alpha_n(k, k) - e^{\rho_n(k, k)(2\pi/\sqrt{5} - 1)} - e^{\rho_n(k, k)(2\pi/\sqrt{5} - 1)} \\
(j, j), & \alpha_n(j, j) - e^{\rho_n(j, j)(2\pi/\sqrt{5} - 1)} - e^{\rho_n(j, j)(2\pi/\sqrt{5} - 1)} \\
k, l, & \alpha_n(k, l) - e^{\rho_n(k, l)(2\pi/\sqrt{5} - 1)} - e^{\rho_n(k, l)(2\pi/\sqrt{5} - 1)}
\end{pmatrix} \in R.
\]

Definition 19. A CqROFR $R$ is called a CqRO-asymmetric-FR if $\forall (k, \alpha_n(k) - e^{\rho_n(k)(2\pi/\sqrt{5} - 1)} - e^{\rho_n(k)(2\pi/\sqrt{5} - 1)}) \in R$ implies that
\[
\begin{pmatrix}
k, k, & \alpha_n(k, k) - e^{\rho_n(k, k)(2\pi/\sqrt{5} - 1)} - e^{\rho_n(k, k)(2\pi/\sqrt{5} - 1)} \\
(j, j), & \alpha_n(j, j) - e^{\rho_n(j, j)(2\pi/\sqrt{5} - 1)} - e^{\rho_n(j, j)(2\pi/\sqrt{5} - 1)} \\
k, l, & \alpha_n(k, l) - e^{\rho_n(k, l)(2\pi/\sqrt{5} - 1)} - e^{\rho_n(k, l)(2\pi/\sqrt{5} - 1)}
\end{pmatrix} \in R.
\]

Definition 20. A CqROFR $R$ is called a CqRO-equivalence-FR if $R$ is CqRO-reflexive-FR, CqRO-symmetric-FR and CqRO-transitive-FR.

Definition 21. A CqROFR $R$ is called a CqRO-order-FR if $R$ is CqRO-reflexive-FR, CqRO-antisymmetric-FR and CqRO-transitive-FR.

Definition 22. Consider a CqROFR $R$ on a CqROFS $K$. For $\alpha_n(k) - e^{\rho_n(k)(2\pi/\sqrt{5} - 1)} - e^{\rho_n(k)(2\pi/\sqrt{5} - 1)} \in K$, an equivalence class of $k$ modulo $R$ is denoted and defined as,
\[
R(k) = \{ j : (j, k) \in R \}
\]

Definition 23. If $R$ is a CqROFR on a CqROFS $K$. Then the CqRO-composite-FR $R \circ R$ is defined as,
\[
\begin{pmatrix}
k, j, & \alpha_n(k, j) - e^{\rho_n(k, j)(2\pi/\sqrt{5} - 1)} - e^{\rho_n(k, j)(2\pi/\sqrt{5} - 1)} \\
(j, l), & \alpha_n(j, l) - e^{\rho_n(j, l)(2\pi/\sqrt{5} - 1)} - e^{\rho_n(j, l)(2\pi/\sqrt{5} - 1)} \\
k, l, & \alpha_n(k, l) - e^{\rho_n(k, l)(2\pi/\sqrt{5} - 1)} - e^{\rho_n(k, l)(2\pi/\sqrt{5} - 1)}
\end{pmatrix} \in R \circ R \text{ for all } k, j, l \in N.
\]

Example 5. Consider the following CqROFS $K$ for $n = 5$, in universe $N$, 

\[
K = \begin{pmatrix}
7/10, & 1/2, & 3/5 \\
9/10, & 4/5, & 7/10 \\
2/5, & 3/5, & 2/5
\end{pmatrix}
\]

The Cartesian product $K \times K$ is,

\[
K \times K = \begin{pmatrix}
7/10, & 1/2, & 3/5 \\
9/10, & 4/5, & 7/10 \\
2/5, & 3/5, & 2/5
\end{pmatrix}
\]

1. The CqRO-reflexive-FR on a CqROFS $K$ is,
\[
R_1 = \begin{cases}
(k, k), & \left(\frac{7}{10} e^{\frac{1}{2} \pi \sqrt{-1}}, \frac{3}{5} e^{\frac{4}{5} \pi \sqrt{-1}}\right)
\end{cases}
\]

\[
R_2 = \begin{cases}
(k, j), & \left(\frac{7}{10} e^{\frac{1}{2} \pi \sqrt{-1}}, \frac{4}{5} e^{\frac{4}{5} \pi \sqrt{-1}}\right)
\end{cases}
\]

\[
R_3 = \begin{cases}
(j, j), & \left(\frac{9}{10} e^{\frac{1}{2} \pi \sqrt{-1}}, \frac{4}{5} e^{\frac{4}{5} \pi \sqrt{-1}}\right)
\end{cases}
\]

\[
R_4 = \begin{cases}
(k, l), & \left(\frac{7}{10} e^{\frac{1}{2} \pi \sqrt{-1}}, \frac{3}{5} e^{\frac{4}{5} \pi \sqrt{-1}}\right)
\end{cases}
\]

\[
R_5 = \begin{cases}
(j, l), & \left(\frac{7}{10} e^{\frac{1}{2} \pi \sqrt{-1}}, \frac{4}{5} e^{\frac{4}{5} \pi \sqrt{-1}}\right)
\end{cases}
\]

\[
R_6 = \begin{cases}
(j, l), & \left(\frac{2}{5} e^{\frac{1}{2} \pi \sqrt{-1}}, \frac{3}{5} e^{\frac{4}{5} \pi \sqrt{-1}}\right)
\end{cases}
\]

\[
R_7 = \begin{cases}
(j, l), & \left(\frac{2}{5} e^{\frac{1}{2} \pi \sqrt{-1}}, \frac{4}{5} e^{\frac{4}{5} \pi \sqrt{-1}}\right)
\end{cases}
\]
Consider the following CqRO equivalence classes:

\[
R_7 = \begin{cases} 
(k, k), & \frac{7}{10} \left( e^{\frac{1}{2} \pi \sqrt{-1}} \right), \frac{3}{5} \left( e^{\frac{4}{5} \pi \sqrt{-1}} \right), \\
(k, l), & \frac{2}{5} \left( e^{\frac{1}{3} \pi \sqrt{-1}} \right), \frac{3}{5} \left( e^{\frac{4}{5} \pi \sqrt{-1}} \right), \\
(j, j), & \frac{9}{10} \left( e^{\frac{1}{2} \pi \sqrt{-1}} \right), \frac{4}{5} \left( e^{\frac{4}{5} \pi \sqrt{-1}} \right), \\
(l, k), & \frac{2}{5} \left( e^{\frac{1}{3} \pi \sqrt{-1}} \right), \frac{3}{5} \left( e^{\frac{4}{5} \pi \sqrt{-1}} \right), \\
(l, l), & \frac{2}{5} \left( e^{\frac{1}{3} \pi \sqrt{-1}} \right), \frac{1}{2} \left( e^{\frac{4}{5} \pi \sqrt{-1}} \right).
\end{cases}
\]

8. The CqRO-order-FR on a CqROFS \( K \) is,

\[
R_8 = \begin{cases} 
(k, k), & \frac{7}{10} \left( e^{\frac{1}{2} \pi \sqrt{-1}} \right), \frac{3}{5} \left( e^{\frac{4}{5} \pi \sqrt{-1}} \right), \\
(k, j), & \frac{7}{10} \left( e^{\frac{1}{2} \pi \sqrt{-1}} \right), \frac{4}{5} \left( e^{\frac{4}{5} \pi \sqrt{-1}} \right), \\
(j, j), & \frac{9}{10} \left( e^{\frac{1}{2} \pi \sqrt{-1}} \right), \frac{4}{5} \left( e^{\frac{4}{5} \pi \sqrt{-1}} \right), \\
(l, l), & \frac{2}{5} \left( e^{\frac{1}{3} \pi \sqrt{-1}} \right), \frac{1}{2} \left( e^{\frac{4}{5} \pi \sqrt{-1}} \right).
\end{cases}
\]

Then,

i. The CqRO equivalence class of \( k \) modulo \( R \) is,

\[
R_{(k)} = \begin{cases} 
k, & \frac{7}{10} \left( e^{\frac{1}{2} \pi \sqrt{-1}} \right), \frac{3}{5} \left( e^{\frac{4}{5} \pi \sqrt{-1}} \right), \\
l, & \frac{2}{5} \left( e^{\frac{1}{3} \pi \sqrt{-1}} \right), \frac{1}{2} \left( e^{\frac{4}{5} \pi \sqrt{-1}} \right).
\end{cases}
\]

ii. The CqRO equivalence class of \( j \) modulo \( R \) is,

\[
R_{(j)} = \begin{cases} 
(j), & \frac{9}{10} \left( e^{\frac{1}{2} \pi \sqrt{-1}} \right), \frac{4}{5} \left( e^{\frac{4}{5} \pi \sqrt{-1}} \right),
\end{cases}
\]

iii. The CqRO equivalence class of \( l \) modulo \( R \) is,

\[
R_{(l)} = \begin{cases} 
k, & \frac{7}{10} \left( e^{\frac{1}{2} \pi \sqrt{-1}} \right), \frac{3}{5} \left( e^{\frac{4}{5} \pi \sqrt{-1}} \right), \\
l, & \frac{2}{5} \left( e^{\frac{1}{3} \pi \sqrt{-1}} \right), \frac{1}{2} \left( e^{\frac{4}{5} \pi \sqrt{-1}} \right).
\end{cases}
\]

IV. RESULTS ON COMPLEX Q-RUNG ORTHOPAIR FUZZY RELATIONS

In this section, some interesting properties and important results of the CqRO-inverse-FR, CqRO-symmetric-FR, CqRO-transitive-FR, CqRO-equivalence-FR, CqRO-order-FR and CqRO equivalence class are discussed.
Theorem 1. The intersection of two CqRO-symmetric-FRs $R_1$ and $R_2$ on a CqROFS $K$ is also a CqRO-symmetric-FR on $K$.

Proof. Let $R_1$ and $R_2$ be any two CqRO-symmetric-FR on a CqROFS $K$. Since $R_1$ and $R_2$ are CqROFSs, according to the definition, it follows that, $R_1 \subseteq K \times K$ and $R_2 \subseteq K \times K$ implies that $R_1 \cap R_2 \subseteq K \times K$. Thus, $R_1 \cap R_2$ is also a CqROFS on $K$.

Now to show that $R_1 \cap R_2$ is a CqRO-symmetric-FR on $K$, suppose that $\left( (k, j), \frac{a_{m}(j, j)}{e^{\rho_{m}(j, j)2\pi\sqrt{-1}}}, \frac{a_{n}(j, j)}{e^{\rho_{n}(j, j)2\pi\sqrt{-1}}} \right) \in R_1 \cap R_2$.

Then, $\left( (k, j), \frac{a_{m}(j, j)}{e^{\rho_{m}(j, j)2\pi\sqrt{-1}}}, \frac{a_{n}(j, j)}{e^{\rho_{n}(j, j)2\pi\sqrt{-1}}} \right) \in R_1$ and $\left( (k, j), \frac{a_{m}(j, j)}{e^{\rho_{m}(j, j)2\pi\sqrt{-1}}}, \frac{a_{n}(j, j)}{e^{\rho_{n}(j, j)2\pi\sqrt{-1}}} \right) \in R_2$.

But it is given that $R_1$ and $R_2$ are the two CqRO-symmetric-FRs on $K$.

So, $\left( (k, j), \frac{a_{m}(j, j)}{e^{\rho_{m}(j, j)2\pi\sqrt{-1}}}, \frac{a_{n}(j, j)}{e^{\rho_{n}(j, j)2\pi\sqrt{-1}}} \right) \in R_1$ and $\left( (k, j), \frac{a_{m}(j, j)}{e^{\rho_{m}(j, j)2\pi\sqrt{-1}}}, \frac{a_{n}(j, j)}{e^{\rho_{n}(j, j)2\pi\sqrt{-1}}} \right) \in R_2$.

which implies that,

$\left( (k, j), \frac{a_{m}(j, j)}{e^{\rho_{m}(j, j)2\pi\sqrt{-1}}}, \frac{a_{n}(j, j)}{e^{\rho_{n}(j, j)2\pi\sqrt{-1}}} \right) \in R_1 \cap R_2$.

Hence, $R_1 \cap R_2$ is a CqRO-symmetric-FR on $K$ if and only if $R_1$ and $R_2$ are CqRO-symmetric-FRs on $K$.

Theorem 2. A CqROFR $R$ is a CqRO-symmetric-FR on a CqROFS $K$ iff $R = R^{-1}$.

Proof. Let $R = R^{-1}$, then $\left( (k, j), \frac{a_{m}(j, j)}{e^{\rho_{m}(j, j)2\pi\sqrt{-1}}}, \frac{a_{n}(j, j)}{e^{\rho_{n}(j, j)2\pi\sqrt{-1}}} \right) \in R$.

$\Leftrightarrow \left( (j, k), \frac{a_{m}(j, j)}{e^{\rho_{m}(j, j)2\pi\sqrt{-1}}}, \frac{a_{n}(j, j)}{e^{\rho_{n}(j, j)2\pi\sqrt{-1}}} \right) \in R^{-1}$

$\Leftrightarrow \left( (j, k), \frac{a_{m}(j, j)}{e^{\rho_{m}(j, j)2\pi\sqrt{-1}}}, \frac{a_{n}(j, j)}{e^{\rho_{n}(j, j)2\pi\sqrt{-1}}} \right) \in R$

Hence, $R$ is a CqRO-symmetric-FR on a CqROFS $K$.

Conversely, assume that the CqROFR $R$ is a CqRO-symmetric-FR on a CqROFS $K$, then according to the definition it follows that, $\left( (k, j), \frac{a_{m}(j, j)}{e^{\rho_{m}(j, j)2\pi\sqrt{-1}}}, \frac{a_{n}(j, j)}{e^{\rho_{n}(j, j)2\pi\sqrt{-1}}} \right) \in R$.

$\Leftrightarrow \left( (j, k), \frac{a_{m}(j, j)}{e^{\rho_{m}(j, j)2\pi\sqrt{-1}}}, \frac{a_{n}(j, j)}{e^{\rho_{n}(j, j)2\pi\sqrt{-1}}} \right) \in R$.

Theorem 3. The CqROFR $R$ is a CqRO-transitive-FR on a CqROFS $K$, iff $R \circ R \subseteq R$.

Proof. Let the relation $R$ be a CqRO-transitive-FR on a CqROFS $K$.

Suppose that $\left( (k, j), \frac{a_{m}(j, j)}{e^{\rho_{m}(j, j)2\pi\sqrt{-1}}}, \frac{a_{n}(j, j)}{e^{\rho_{n}(j, j)2\pi\sqrt{-1}}} \right) \in R \circ R$.

Then by CqRO-transitivity of $R$

$\left( (k, j), \frac{a_{m}(j, j)}{e^{\rho_{m}(j, j)2\pi\sqrt{-1}}}, \frac{a_{n}(j, j)}{e^{\rho_{n}(j, j)2\pi\sqrt{-1}}} \right) \in R$ and $\left( (j, l), \frac{a_{m}(j, j)}{e^{\rho_{m}(j, j)2\pi\sqrt{-1}}}, \frac{a_{n}(j, j)}{e^{\rho_{n}(j, j)2\pi\sqrt{-1}}} \right) \in R$

$\Rightarrow \left( (k, l), \frac{a_{m}(k, l)}{e^{\rho_{m}(k, l)2\pi\sqrt{-1}}}, \frac{a_{n}(k, l)}{e^{\rho_{n}(k, l)2\pi\sqrt{-1}}} \right) \in R$.

$\Rightarrow R \circ R \subseteq R$.

Conversely, assume that $R \circ R \subseteq R$, then for $\left( (k, j), \frac{a_{m}(j, j)}{e^{\rho_{m}(j, j)2\pi\sqrt{-1}}}, \frac{a_{n}(j, j)}{e^{\rho_{n}(j, j)2\pi\sqrt{-1}}} \right) \in R$ and $\left( (j, l), \frac{a_{m}(j, j)}{e^{\rho_{m}(j, j)2\pi\sqrt{-1}}}, \frac{a_{n}(j, j)}{e^{\rho_{n}(j, j)2\pi\sqrt{-1}}} \right) \in R$

$\Rightarrow \left( (k, l), \frac{a_{m}(k, l)}{e^{\rho_{m}(k, l)2\pi\sqrt{-1}}}, \frac{a_{n}(k, l)}{e^{\rho_{n}(k, l)2\pi\sqrt{-1}}} \right) \in R \circ R \subseteq R$

$\Rightarrow \left( (k, l), \frac{a_{m}(k, l)}{e^{\rho_{m}(k, l)2\pi\sqrt{-1}}}, \frac{a_{n}(k, l)}{e^{\rho_{n}(k, l)2\pi\sqrt{-1}}} \right) \in R$.

Hence, $R$ is CqRO-transitive-FR on a CqROFS $K$.

Theorem 4. If $R$ is a CqRO-equivalence-FR on a CqROFS $K$, then $R \circ R = R$.

Proof. Let $\left( (k, j), \frac{a_{m}(j, j)}{e^{\rho_{m}(j, j)2\pi\sqrt{-1}}}, \frac{a_{n}(j, j)}{e^{\rho_{n}(j, j)2\pi\sqrt{-1}}} \right) \in R$. Then, by the CqRO-symmetry property of a CqRO-equivalence-FR it follows that, $\left( (j, k), \frac{a_{m}(j, k)}{e^{\rho_{m}(j, k)2\pi\sqrt{-1}}}, \frac{a_{n}(j, k)}{e^{\rho_{n}(j, k)2\pi\sqrt{-1}}} \right) \in R$.

Now by using CqRO-transitive property of a CqRO-equivalence-FR, $\left( (k, k), \frac{a_{m}(k, k)}{e^{\rho_{m}(k, k)2\pi\sqrt{-1}}}, \frac{a_{n}(k, k)}{e^{\rho_{n}(k, k)2\pi\sqrt{-1}}} \right) \in R$. 
The definition of CqRO-composite-FR implies that,
\[
(k, k), \frac{\alpha_m(k, k)}{e^{\rho_m(k,k)2\pi\nu^{-1}}} , \frac{\alpha_n(k, k)}{e^{\rho_n(k,k)2\pi\nu^{-1}}} \in R \circ R
\]
Thus, \( R \subseteq R \circ R \) \hspace{1cm} (1)

Conversely, assume that \( (k, j), \frac{\alpha_m(k, j)}{e^{\rho_m(k,j)2\pi\nu^{-1}}} , \frac{\alpha_n(k, j)}{e^{\rho_n(k,j)2\pi\nu^{-1}}} \in R \circ R \), then there exists and \( l \in N \) such that,
\[
(k, l), \frac{\alpha_m(k, l)}{e^{\rho_m(k,l)2\pi\nu^{-1}}} , \frac{\alpha_n(k, l)}{e^{\rho_n(k,l)2\pi\nu^{-1}}} \in R \quad \text{and} \quad (l, j), \frac{\alpha_m(l, j)}{e^{\rho_m(l,j)2\pi\nu^{-1}}} , \frac{\alpha_n(l, j)}{e^{\rho_n(l,j)2\pi\nu^{-1}}} \in R
\]
But since \( R \) is a CqRO-equivalence-FR on \( K \), so \( R \) is also a CqRO-transitive-FR. Thus,
\[
(k, j), \frac{\alpha_m(k, j)}{e^{\rho_m(k,j)2\pi\nu^{-1}}} , \frac{\alpha_n(k, j)}{e^{\rho_n(k,j)2\pi\nu^{-1}}} \in R
\]
\[\Rightarrow R \circ R \subseteq R \] \hspace{1cm} (2)

Hence by (1) and (2) \( R \circ R = R \).

**Theorem 5.** The inverse of a CqRO-order-FR \( R \) on a CqROFS \( K \) is also a CqRO-order-FR on \( K \).

**Proof.** To prove the given statement, it is sufficient to verify that a CqRO-order-FR \( R^{-1} \) fulfills the three conditions of a CqRO-order-FR.

i. As \( R \) is a CqRO-reflexive-FR. Therefore, for any \( k \in N \) it follows that,
\[
(k, k), \frac{\alpha_m(k, k)}{e^{\rho_m(k,k)2\pi\nu^{-1}}} , \frac{\alpha_n(k, k)}{e^{\rho_n(k,k)2\pi\nu^{-1}}} \in R
\]
\[\Rightarrow (k, k), \frac{\alpha_m(k, k)}{e^{\rho_m(k,k)2\pi\nu^{-1}}} , \frac{\alpha_n(k, k)}{e^{\rho_n(k,k)2\pi\nu^{-1}}} \in R^{-1}
\]
Hence, \( R^{-1} \) is a CqRO-reflexive-FR.

ii. Suppose \( (k, j), \frac{\alpha_m(k, j)}{e^{\rho_m(k,j)2\pi\nu^{-1}}} , \frac{\alpha_n(k, j)}{e^{\rho_n(k,j)2\pi\nu^{-1}}} \in R^{-1} \) and \( (j, k), \frac{\alpha_m(j, k)}{e^{\rho_m(j,k)2\pi\nu^{-1}}} , \frac{\alpha_n(j, k)}{e^{\rho_n(j,k)2\pi\nu^{-1}}} \in R^{-1} \)

Then \( (k, j), \frac{\alpha_m(k, j)}{e^{\rho_m(k,j)2\pi\nu^{-1}}} , \frac{\alpha_n(k, j)}{e^{\rho_n(k,j)2\pi\nu^{-1}}} \in R \) and \( (j, k), \frac{\alpha_m(j, k)}{e^{\rho_m(j,k)2\pi\nu^{-1}}} , \frac{\alpha_n(j, k)}{e^{\rho_n(j,k)2\pi\nu^{-1}}} \in R \)

But according to the definition of a CqRO-order-FR, \( R \) is also a CqRO-antisymmetric-FR and therefore,
\[
(k, j), \frac{\alpha_m(k, j)}{e^{\rho_m(k,j)2\pi\nu^{-1}}} , \frac{\alpha_n(k, j)}{e^{\rho_n(k,j)2\pi\nu^{-1}}} \in R \circ R
\]
\[\Rightarrow (k, j), \frac{\alpha_m(k, j)}{e^{\rho_m(k,j)2\pi\nu^{-1}}} , \frac{\alpha_n(k, j)}{e^{\rho_n(k,j)2\pi\nu^{-1}}} \in R^{-1}
\]

Hence, \( R^{-1} \) is also a CqRO-antisymmetric-FR.

From i, ii and iii, it is proved that \( R^{-1} \) is also a CqRO-order-FR.

**Theorem 6.** If \( R \) is a CqRO-equivalence-FR on a CqROFS \( K \), then \( (k, j), \frac{\alpha_m(k, j)}{e^{\rho_m(k,j)2\pi\nu^{-1}}} , \frac{\alpha_n(k, j)}{e^{\rho_n(k,j)2\pi\nu^{-1}}} \in R \) iff \( R(k) = R(j) \).

**Proof.** Suppose \( (k, j), \frac{\alpha_m(k, j)}{e^{\rho_m(k,j)2\pi\nu^{-1}}} , \frac{\alpha_n(k, j)}{e^{\rho_n(k,j)2\pi\nu^{-1}}} \in R \) and \( (l, m_K(l)), n_K(l) \in R(k) \)

Then \( (l, k), \frac{\alpha_m(l, k)}{e^{\rho_m(l,k)2\pi\nu^{-1}}} , \frac{\alpha_n(l, k)}{e^{\rho_n(l,k)2\pi\nu^{-1}}} \in R \) Now by the CqROF-transitive property of a CqRO-equivalence-FR R, it follows that,
\[
(l, j), \frac{\alpha_m(l, j)}{e^{\rho_m(l,j)2\pi\nu^{-1}}} , \frac{\alpha_n(l, j)}{e^{\rho_n(l,j)2\pi\nu^{-1}}} \in R
\]
\[\Rightarrow (l, m_K(l), n_K(l) \in R(j). \]

Thus, \( R(k) \subseteq R(j) \) \hspace{1cm} (3)
\[
\left( j, k \right), \frac{\alpha_n(j,k)}{e^{\rho_n(j,k)2\pi\sqrt{-1}}} \in \mathbb{R} \quad \text{and} \quad \frac{\alpha_n(j,k)}{e^{\rho_n(j,k)2\pi\sqrt{-1}}} \in \mathbb{R} \quad \text{also suppose that} \quad \left( l, m_{\mathcal{C}}(l), n_{\mathcal{C}}(l) \right) \in \mathbb{R}.
\]

Then, \[
\left( l, j \right), \frac{\alpha_n(l,j)}{e^{\rho_n(l,j)2\pi\sqrt{-1}}} \in \mathbb{R} \quad \text{and} \quad \frac{\alpha_n(l,j)}{e^{\rho_n(l,j)2\pi\sqrt{-1}}} \in \mathbb{R}.
\]

Now by CqROF-transitive property of a CqRO-equivalence-\( FR \), it follows that,

\[
\left( l, k \right), \frac{\alpha_n(l,k)}{e^{\rho_n(l,k)2\pi\sqrt{-1}}} \in \mathbb{R}
\]

\[
\Rightarrow \left( l, n_{\mathcal{C}}(l), n_{\mathcal{C}}(l) \right) \in R(k).
\]

Thus, \( R(i) \subseteq R(k) \) \( \text{(4)} \).

Hence, (3) and (4) imply that \( R(k) = R(j) \).

Conversely, assume that \( R(k) = R(j) \), \( \left( l, n_{\mathcal{C}}(l), n_{\mathcal{C}}(l) \right) \in R(k) \) and \( \left( l, m_{\mathcal{C}}(l), n_{\mathcal{C}}(l) \right) \in R(j) \).

\[
\Rightarrow \left( l, k \right), \frac{\alpha_n(l,k)}{e^{\rho_n(l,k)2\pi\sqrt{-1}}} \in \mathbb{R}
\]

\[
\Rightarrow \left( l, j \right), \frac{\alpha_n(l,j)}{e^{\rho_n(l,j)2\pi\sqrt{-1}}} \in \mathbb{R}
\]

Using the CqROF-symmetric property of a CqRO-equivalence-\( FR \), it follows that,

\[
\left( l, k \right), \frac{\alpha_n(l,k)}{e^{\rho_n(l,k)2\pi\sqrt{-1}}} \in \mathbb{R}
\]

\[
\Rightarrow \left( k, l \right), \frac{\alpha_n(k,l)}{e^{\rho_n(k,l)2\pi\sqrt{-1}}} \in \mathbb{R}
\]

Now, by CqROF-transitive property of a CqRO-equivalence-\( FR \), it follows that,

\[
\left( k, l \right), \frac{\alpha_n(k,l)}{e^{\rho_n(k,l)2\pi\sqrt{-1}}} \in \mathbb{R}
\]

\[
\Rightarrow \left( k, j \right), \frac{\alpha_n(k,j)}{e^{\rho_n(k,j)2\pi\sqrt{-1}}} \in \mathbb{R}
\]

which completes the proof.

V. APPLICATIONS

This section presents the use of CqROFR in modeling some practical problems. The method is applied to investigate the financial track records of business companies and the effects of one business on the other businesses are also determined. First of all, the procedure of the method is discussed. Then a couple of applications are given that use the proposed methods in two different situations.

A. PROCEDURE

\[
\left( j, k \right), \frac{\alpha_n(j,k)}{e^{\rho_n(j,k)2\pi\sqrt{-1}}} \in \mathbb{R} \quad \text{and} \quad \frac{\alpha_n(j,k)}{e^{\rho_n(j,k)2\pi\sqrt{-1}}} \in \mathbb{R} \quad \text{also suppose that} \quad \left( l, m_{\mathcal{C}}(l), n_{\mathcal{C}}(l) \right) \in \mathbb{R}.
\]

A financier is attracted by the profits. So, it is essential for him/her to spot successful business companies. The identification of successful businesses is a challenging task. It requires rich financial knowledge and market experience. Moreover, for the future prediction of a business company, the effects and impacts of different businesses and companies must be analyzed.

For the investigation of the financial track record of a business company, it is essential to define some parameters. In this application, the most critical parameters that define the financial structure and position of any business company, are considered. The parameters taken into account for the problem are listed and explained in Table III. The flowchart of the procedure is portrayed in Figure 3.

**FIGURE 3. Flowchart of the procedure.**

| S. No. | Parameter | Notation | Description |
|-------|-----------|----------|-------------|
| 1     | Earnings Per Share | \( EPS \) | Apparently, earnings and profits are the main concerns of a financier. So, the first parameter is \( EPS \) that describes the profit made per share. |
| 2     | Net Sales | \( NS \) | \( NS \) is directly proportional to the profit, i.e., more profit will be made if sales are high and lesser profit will be made if sales are low. So, \( EPS \) and \( NS \) are somehow related to each other. |
| 3     | Net Operating Cash Flow | \( NOCF \) | \( NOCF \) is the money required to runs the daily operations of a company. A better \( NOCF \) is the indication of a good company which can run its operations on its own. |
| 4     | Book Value Per Share | \( BVPS \) | Whenever a business company increases its capacity by investing its profit in the business, it grows and develops. Thus, \( BVPS \) tells about the expansion of a company. |
| 5     | Return On Invested Capital | \( ROIC \) | The business companies earn money and spend those earnings to run and maintain the business. The efficiency of a company means the capability of the company to generate more revenue than its investments and operational expenses. \( ROIC \) is the parameter to determine the efficiency of a company. |
| 6     | Debt to Net Profit Ratio | \( D = -NPR \) | The capability of a business to pay back its debts and loans within a reasonable time frame without severe |
troubles is known as debt to net profit ratio.

The next step after defining the parameters is the construction of CqROFS which is composed of these parameters. Each parameter is assigned the value for the level of membership and the level of non-membership. The third step is to find out the Cartesian product of the CqROFS. The level of membership and the level of non-membership are complex-valued, and thus, they consist of two parts, i.e., the real part called the amplitude term and the complex part called the phase term. The final step is to interpret the numerical data and infer some useful results. Each element of the CqROFR or Cartesian product of CqROFSs is an ordered pair of financial parameters. The relation tells the effects of the first parameter on the second parameter. The level of membership indicates the level of supportive, helpful, and positive impacts, while the level of non-membership indicates the level of unhelpful and negative impacts. As mentioned before that each of the membership and non-membership has two parts; the amplitude term refers to the level and grade of the effect, whereas the phase term refers to the time lag, i.e., the time period of that effect. The lower values (closer to 0) of amplitude terms mean little effects, and the higher values (closer to 1) mean strong effects.

**B. ANALYSIS OF FINANCIAL TRACK RECORD OF A BUSINESS COMPANY**

This application uses the CqROFR to learn the influence of a financial parameter on the other financial parameters of a business company. Assume that \( N \) is the collection of the above-mentioned parameters.

\[
N = \{ \text{EPS}, \text{NS}, \text{NOCF}, \text{BVPS}, \text{ROIC}, D - \text{NPR} \}
\]

Now, defining a CqROFS \( K \) based on \( N \) as,

\[
K = \left\{ \left( \text{EPS}, \frac{3}{5}, \frac{1}{2} \right), \left( \text{NS}, \frac{9}{10}, \frac{3}{10} \right), \left( \text{NOCF}, \frac{9}{10}, \frac{2}{5} \right), \left( \text{BVPS}, \frac{4}{5}, \frac{1}{2} \right), \left( \text{ROIC}, \frac{9}{10}, \frac{1}{2} \right), \left( D - \text{NPR}, \frac{3}{5}, \frac{1}{2} \right) \right\}
\]

The frameworks of CIFS and CPFS cannot deal with the information in the above set \( K \) because \( \frac{3}{5} + \frac{2}{5} \notin [0, 1] \) and \( \left( \frac{9}{10} \right)^2 + \left( \frac{1}{2} \right)^2 \notin [0, 1] \). But it is a CqROFS for \( n = 4 \), i.e., the highest valued pair: \( \left( \frac{22}{25} \right)^4 + \left( \frac{3}{4} \right)^4 \in [0, 1] \). So, the CqROFS has a wider range as compared to the CIFS and CPFS. So it is convenient to stick with the wide-ranging tool.

In the CqROFS \( K \), the amplitude terms of the level of membership and level of non-membership are \( \alpha_m(k) \) and \( \alpha_n(k) \), respectively. The phase terms of the level of membership and level of non-membership are \( \rho_m(k) \) and \( \rho_n(k) \), respectively.

Now, to learn the level of effectiveness of a parameter on the others, a CqROFR \( R \) on \( K \) is found, which is given in Table IV.

| TABLE IV | CARTESIAN PRODUCT ON CqROFS K |
|----------|-----------------------------|
| \( R = K \times K \) | \( \frac{\alpha_m(k, k)}{e^{\rho_m(k, k)2\pi\nu^{-1}}} \) | \( \frac{\alpha_n(k, k)}{e^{\rho_n(k, k)2\pi\nu^{-1}}} \) |
| \( (\text{EPS, EPS}) \) | \( e^{\left( \frac{3}{5} \right)2\pi\nu^{-1}} \) | \( e^{\left( \frac{1}{2} \right)2\pi\nu^{-1}} \) |
| \( (\text{EPS, NS}) \) | \( e^{\left( \frac{3}{5} \right)2\pi\nu^{-1}} \) | \( e^{\left( \frac{1}{2} \right)2\pi\nu^{-1}} \) |
| \( (\text{EPS, NOCF}) \) | \( e^{\left( \frac{3}{5} \right)2\pi\nu^{-1}} \) | \( e^{\left( \frac{1}{2} \right)2\pi\nu^{-1}} \) |
| (EPS, BVPS) | 3/5 | 1/2 |
|------------|-----|-----|
| (EPS, ROIC) | 3/5 | 1/2 |
| (EPS, D − NPR) | 3/5 | 1/2 |
| (NS, EPS) | 3/5 | 1/2 |
| (NS, NS) | 9/10 | 3/10 |
| (NS, NOCF) | 9/10 | 3/5 |
| (NS, BVPS) | 3/5 | 1/2 |
| (NS, ROIC) | 3/5 | 1/2 |
| (NS, D − NPR) | 3/5 | 1/2 |
| (NOCF, EPS) | 3/5 | 1/2 |
| (NOCF, NS) | 9/10 | 2/5 |
| (NOCF, NOCF) | 9/10 | 2/5 |
| (NOCF, BVPS) | 9/10 | 2/5 |
| (NOCF, ROIC) | 9/10 | 2/5 |
| (NOCF, D − NPR) | 9/10 | 2/5 |
| (BVPS, EPS) | 4/5 | 1/2 |
| (BVPS, NS) | 4/5 | 1/2 |
| (BVPS, NOCF) | 4/5 | 1/2 |
| (BVPS, BVPS) | 4/5 | 1/2 |
| (BVPS, ROIC) | 4/5 | 1/2 |
| (BVPS, D − NPR) | 3/5 | 1/2 |
| (ROIC, EPS) | 3/5 | 1/2 |

| (ROIC, NS) | 3/5 | 1/2 |
| (ROIC, NOCF) | 3/5 | 1/2 |
| (ROIC, BVPS) | 3/5 | 1/2 |
| (ROIC, ROIC) | 3/5 | 1/2 |
| (D − NPR, EPS) | 3/5 | 1/2 |
| (D − NPR, NS) | 3/5 | 1/2 |
| (D − NPR, NOCF) | 3/5 | 1/2 |
| (D − NPR, BVPS) | 3/5 | 1/2 |
| (D − NPR, ROIC) | 3/5 | 1/2 |
| (D − NPR, D − NPR) | 3/5 | 1/2 |

Obviously, the Cartesian product on a CqROFS $K$ is a CqRO-equivalence-FR $R$. This property of $R$ is beneficial whenever direct relationships and effects of one parameter on the other parameters are not known. So, in such cases, the CqROF-symmetry and CqROF-transitive properties of a CqRO-equivalence-FR are noteworthy.

The CqROFR $R$ in Table IV determines the effects of every parameter on the others. For instance, in

$$\left(\text{(NS, BVPS)}, \frac{4}{5} e^{\frac{1}{e^{\frac{1}{5}}} - 1}, \frac{1}{2} e^{\frac{1}{e^{\frac{1}{5}}} - 1}\right) \in R,$$

the level of membership is interpreted as; fraction $\frac{4}{5}$ represents the positive effects of net sales on the book value per share with respect to half a month. The level of non-membership is interpreted as; the fraction $\frac{1}{2}$ represents the negative effects of net sales on book value per share with respect to a month. Similarly, this practice can be carried out for all the other ordered pairs in the relation. Additionally, these parameters can be compared to get some useful conclusions, like which parameter affects the net sales the most and which parameters have the worst effects on the net sale.
This application shows the use of CqROFRs to inspect the direct and indirect impacts of any financial parameters of a business company on some other financial parameters of the company. In this fashion, a business can deal with the reasons behind its decline. Also, it can further progress by spotting the reasons that regulate some of the important business factors. The three properties of a CqRO-equivalence-FR can be used to learn the statuses of unidentified parameters.

C. INVESTIGATION OF FINANCIAL TRACK RECORDS OF MULTIPLE BUSINESS COMPANIES

Another important application of CqROFRs is the assessment of different businesses or companies in the race of success. In this application, three business companies are considered, and the effects of their financial track records on each other are discussed. It utilizes the CqRO-composite-FR to find out the strength of an unknown relation when the strengths of some other relations are known. The rules to interpret the numerical results are similar to the ones discussed in the precious application.

Assume that $K$, $J$ and $L$ are the CqROFSs. Each of these three sets represents three different business companies. All the sets are composed of similar parameters, but their values are different. These parameters are explained in Table III.

\[
K = \begin{pmatrix}
\left(\frac{3}{5}, \frac{1}{2} \right)_{e^{\left(\frac{1}{2}\right)2\pi\sqrt{-1}T}} \left(\frac{5}{4} \right)_{e^{\left(\frac{1}{4}\right)2\pi\sqrt{-1}T}} \\
\left(\frac{9}{10}, \frac{3}{10} \right)_{e^{\left(\frac{1}{10}\right)2\pi\sqrt{-1}T}} \left(\frac{3}{4} \right)_{e^{\left(\frac{3}{4}\right)2\pi\sqrt{-1}T}} \\
\left(\frac{9}{10}, \frac{2}{5} \right)_{e^{\left(\frac{1}{5}\right)2\pi\sqrt{-1}T}} \left(\frac{2}{4} \right)_{e^{\left(\frac{2}{4}\right)2\pi\sqrt{-1}T}} \\
\left(\frac{4}{5}, \frac{1}{2} \right)_{e^{\left(\frac{1}{2}\right)2\pi\sqrt{-1}T}} \left(\frac{5}{4} \right)_{e^{\left(\frac{1}{4}\right)2\pi\sqrt{-1}T}} \\
\left(\frac{9}{10}, \frac{1}{2} \right)_{e^{\left(\frac{1}{2}\right)2\pi\sqrt{-1}T}} \left(\frac{2}{4} \right)_{e^{\left(\frac{2}{4}\right)2\pi\sqrt{-1}T}} \\
\left(\frac{3}{10}, \frac{1}{2} \right)_{e^{\left(\frac{1}{2}\right)2\pi\sqrt{-1}T}} \left(\frac{5}{4} \right)_{e^{\left(\frac{1}{4}\right)2\pi\sqrt{-1}T}}
\end{pmatrix}
\]

\[
J = \begin{pmatrix}
\left(\frac{4}{5}, \frac{2}{5} \right)_{e^{\left(\frac{1}{2}\right)2\pi\sqrt{-1}T}} \left(\frac{5}{4} \right)_{e^{\left(\frac{1}{4}\right)2\pi\sqrt{-1}T}} \\
\left(\frac{7}{10}, \frac{1}{5} \right)_{e^{\left(\frac{1}{5}\right)2\pi\sqrt{-1}T}} \left(\frac{5}{4} \right)_{e^{\left(\frac{1}{4}\right)2\pi\sqrt{-1}T}} \\
\left(\frac{7}{10}, \frac{1}{5} \right)_{e^{\left(\frac{1}{5}\right)2\pi\sqrt{-1}T}} \left(\frac{5}{4} \right)_{e^{\left(\frac{1}{4}\right)2\pi\sqrt{-1}T}} \\
\left(\frac{3}{10}, \frac{1}{2} \right)_{e^{\left(\frac{1}{2}\right)2\pi\sqrt{-1}T}} \left(\frac{5}{4} \right)_{e^{\left(\frac{1}{4}\right)2\pi\sqrt{-1}T}} \\
\left(\frac{3}{10}, \frac{1}{2} \right)_{e^{\left(\frac{1}{2}\right)2\pi\sqrt{-1}T}} \left(\frac{5}{4} \right)_{e^{\left(\frac{1}{4}\right)2\pi\sqrt{-1}T}} \\
\left(\frac{3}{10}, \frac{1}{2} \right)_{e^{\left(\frac{1}{2}\right)2\pi\sqrt{-1}T}} \left(\frac{5}{4} \right)_{e^{\left(\frac{1}{4}\right)2\pi\sqrt{-1}T}}
\end{pmatrix}
\]

\[
L = \begin{pmatrix}
\left(\frac{9}{10}, \frac{1}{2} \right)_{e^{\left(\frac{1}{2}\right)2\pi\sqrt{-1}T}} \left(\frac{5}{4} \right)_{e^{\left(\frac{1}{4}\right)2\pi\sqrt{-1}T}} \\
\left(\frac{9}{10}, \frac{1}{2} \right)_{e^{\left(\frac{1}{2}\right)2\pi\sqrt{-1}T}} \left(\frac{5}{4} \right)_{e^{\left(\frac{1}{4}\right)2\pi\sqrt{-1}T}} \\
\left(\frac{9}{10}, \frac{1}{2} \right)_{e^{\left(\frac{1}{2}\right)2\pi\sqrt{-1}T}} \left(\frac{5}{4} \right)_{e^{\left(\frac{1}{4}\right)2\pi\sqrt{-1}T}} \\
\left(\frac{9}{10}, \frac{1}{2} \right)_{e^{\left(\frac{1}{2}\right)2\pi\sqrt{-1}T}} \left(\frac{5}{4} \right)_{e^{\left(\frac{1}{4}\right)2\pi\sqrt{-1}T}} \\
\left(\frac{9}{10}, \frac{1}{2} \right)_{e^{\left(\frac{1}{2}\right)2\pi\sqrt{-1}T}} \left(\frac{5}{4} \right)_{e^{\left(\frac{1}{4}\right)2\pi\sqrt{-1}T}} \\
\left(\frac{9}{10}, \frac{1}{2} \right)_{e^{\left(\frac{1}{2}\right)2\pi\sqrt{-1}T}} \left(\frac{5}{4} \right)_{e^{\left(\frac{1}{4}\right)2\pi\sqrt{-1}T}}
\end{pmatrix}
\]

The CqROFS $K$ is of order 4, whereas the CqROFSs $J$ and $L$ are of order 3. Now, finding the Cartesian products $K \times J$ and $J \times L$, which are represented in the form of tables, i.e., Tables V and VI, respectively.
Table V holds valuable information about the first two companies. It tells the impacts of decline or rise of one business company on the second company. For example, consider these two events from Table V:

\[
\left( NS_{K_{1}}, NS_{J_{1}} \right), \frac{7}{10} \frac{3}{10} \frac{1}{e^{(\frac{7}{2})e^{2\pi i}}}, \frac{1}{e^{(\frac{3}{2})e^{2\pi i}}},
\]

and

\[
\left( BVPS_{K_{1}}, EPS_{J_{1}} \right), \frac{4}{5} \frac{2}{5} \frac{1}{e^{(\frac{4}{2})e^{2\pi i}}}, \frac{1}{e^{(\frac{2}{2})e^{2\pi i}}}.\]

In the first ordered pair, the relationships between net sales of both the business companies are expressed. It states that the impacts of the nets sales of the first company on the net sales of the second
company are \( \frac{7}{10} e^{\left(\frac{1}{2}\right) 2\pi n^2/1}, \frac{3}{10} e^{\left(\frac{1}{2}\right) 2\pi n^2/1} \). The level of membership \( \frac{7}{10} e^{\left(\frac{1}{2}\right) 2\pi n^2/1} \) conveys that the grade of positive impacts of net sales of first company on the second is \( \frac{1}{7} \) with respect to \( \frac{1}{2} \) time unit (or half a month because \( \left(\frac{1}{2}\right) 2\pi n^2/1 = \frac{n}{2} \)). And the level of non-membership \( \frac{3}{10} e^{\left(\frac{1}{2}\right) 2\pi n^2/1} \) conveys that the grade of negative effects of the net sale of the first company on the second is \( \frac{1}{3} \) with respect to a month. In the same way, the second event \( \left( BVPS_K, EPS_J \right) \frac{4}{e^{\left(\frac{1}{2}\right) 2\pi n^2/1}}, \frac{1}{2} e^{\left(\frac{1}{2}\right) 2\pi n^2/1} \) expresses that the changes in the book value per share of the first company will improve the earning per share of the second company up to \( \frac{4}{5} \) grades with respect to a month and will negatively affect the earning per share of the second company up to \( \frac{1}{2} \) grades with respect to one and half month.

### TABLE VI

| Cartesian product between the CoRoFSS \( J \) and \( L \) | \( a_J(j,l) \) | \( a_J(j,l) \) |
|----------------------------------------------------------|----------------|----------------|
| \( (EPS_J, EPS_L) \) | \( 4/5 e^{\left(\frac{1}{2}\right) 2\pi n^2/1} \) | \( 1/2 e^{\left(\frac{1}{2}\right) 2\pi n^2/1} \) |
| \( (EPS_J, NS_L) \) | \( 3/5 e^{\left(\frac{1}{2}\right) 2\pi n^2/1} \) | \( 7/10 e^{\left(\frac{1}{2}\right) 2\pi n^2/1} \) |
| \( (EPS_J, NOCF_L) \) | \( 4/5 e^{\left(\frac{1}{2}\right) 2\pi n^2/1} \) | \( 3/5 e^{\left(\frac{1}{2}\right) 2\pi n^2/1} \) |
| \( (EPS_J, BVPS_L) \) | \( 3/5 e^{\left(\frac{1}{2}\right) 2\pi n^2/1} \) | \( 7/10 e^{\left(\frac{1}{2}\right) 2\pi n^2/1} \) |
| \( (EPS_J, ROIC_L) \) | \( 2/5 e^{\left(\frac{1}{2}\right) 2\pi n^2/1} \) | \( 7/10 e^{\left(\frac{1}{2}\right) 2\pi n^2/1} \) |
| \( (NS_J, EPS_L) \) | \( 10/7 e^{\left(\frac{1}{2}\right) 2\pi n^2/1} \) | \( 1/2 e^{\left(\frac{1}{2}\right) 2\pi n^2/1} \) |
| \( (NS_J, NS_L) \) | \( 5/7 e^{\left(\frac{1}{2}\right) 2\pi n^2/1} \) | \( 1/2 e^{\left(\frac{1}{2}\right) 2\pi n^2/1} \) |
| \( (NS_J, NOCF_L) \) | \( 3/5 e^{\left(\frac{1}{2}\right) 2\pi n^2/1} \) | \( 7/10 e^{\left(\frac{1}{2}\right) 2\pi n^2/1} \) |
| \( (NS_J, BVPS_L) \) | \( 3/5 e^{\left(\frac{1}{2}\right) 2\pi n^2/1} \) | \( 7/10 e^{\left(\frac{1}{2}\right) 2\pi n^2/1} \) |
| \( (NS_J, ROIC_L) \) | \( 1/2 e^{\left(\frac{1}{2}\right) 2\pi n^2/1} \) | \( 7/10 e^{\left(\frac{1}{2}\right) 2\pi n^2/1} \) |
Table VI has the same characteristics as Table V. The difference is that Table V describes the relationships between the first two companies, and Table VI describes the relationships between second and third companies.

Since Table V states the relationships between first and second companies, and Table VI gives the relationship between second and third companies, i.e., $K \times J$ relates first company to second company and $J \times L$ relates second company to the third company. So a CqRO-composite-FR proves to be useful in finding out the indirect relationships between first and third companies, i.e., $R = (K \times J) \circ (J \times L) = K \times L$. Table VII expresses the CqRO-composite-FR $R$.

### Table VII

| CqRO-composite-FR $R$ between the relations in Table V and VI | $\alpha_n(k,l)$ | $\frac{1}{e_n(k,l)}$ | $\alpha_n(k,l)$ | $\frac{1}{e_n(k,l)}$ |
|---------------------------------------------------------------|-----------------|-------------------|-----------------|-------------------|
| $(D-NPR_{K,BVPS_L})$                                          | $\frac{1}{2}$   | $\frac{7}{10}$   | $\frac{3}{5}$   | $\frac{7}{10}$   |
| $(D-NPR_{K,ROIC_L})$                                          | $\frac{1}{2}$   | $\frac{7}{10}$   | $\frac{3}{5}$   | $\frac{7}{10}$   |
| $(D-NPR_{K,D}$                                          | $\frac{2}{5}$   | $\frac{7}{10}$   | $\frac{7}{10}$  | $\frac{7}{10}$   |
| $-NPR_L$)                                            | $\frac{1}{2}$   | $\frac{7}{10}$   | $\frac{3}{5}$   | $\frac{7}{10}$   |

| $(NS_{K,D-NPR_L})$                                            | $\frac{2}{5}$   | $\frac{7}{10}$   | $\frac{7}{10}$  | $\frac{7}{10}$   |
| $(NOCF_{K,EPS_L})$                                            | $\frac{9}{10}$  | $\frac{1}{2}$    | $\frac{1}{2}$   | $\frac{7}{10}$   |
| $(NOCF_{K,NS_L})$                                             | $\frac{4}{5}$   | $\frac{7}{10}$   | $\frac{3}{5}$   | $\frac{7}{10}$   |
| $(NOCF_{K,NOCF_L})$                                           | $\frac{3}{5}$   | $\frac{7}{10}$   | $\frac{3}{5}$   | $\frac{7}{10}$   |
| $(NOCF_{K,BVPS_L})$                                           | $\frac{4}{5}$   | $\frac{7}{10}$   | $\frac{3}{5}$   | $\frac{7}{10}$   |
| $(NOCF_{K,ROIC_L})$                                           | $\frac{3}{5}$   | $\frac{7}{10}$   | $\frac{3}{5}$   | $\frac{7}{10}$   |
| $(NOCF_{K,D-NPR_L})$                                          | $\frac{2}{5}$   | $\frac{7}{10}$   | $\frac{7}{10}$  | $\frac{7}{10}$   |
| $(BVPS_{K,EPS_L})$                                            | $\frac{4}{5}$   | $\frac{7}{10}$   | $\frac{3}{5}$   | $\frac{7}{10}$   |
| $(BVPS_{K,NS_L})$                                             | $\frac{4}{5}$   | $\frac{7}{10}$   | $\frac{3}{5}$   | $\frac{7}{10}$   |
| $(BVPS_{K,NOCF_L})$                                           | $\frac{3}{5}$   | $\frac{7}{10}$   | $\frac{3}{5}$   | $\frac{7}{10}$   |
| $(BVPS_{K,BVPS_L})$                                           | $\frac{4}{5}$   | $\frac{7}{10}$   | $\frac{3}{5}$   | $\frac{7}{10}$   |
| $(BVPS_{K,ROIC_L})$                                           | $\frac{3}{5}$   | $\frac{7}{10}$   | $\frac{3}{5}$   | $\frac{7}{10}$   |
The CqRO-composite-FR \( R \) uses the data of relationships between the first two companies and the relationships between the last two companies to describe the impacts of the parameters of the first company on the parameters of the third company.

This application verified the significance of CqRO-composite-FR and CqRO-equivalence-FR. These two types of CqROFRs provide much valuable information, such as highlighting the key factors affecting the business companies. Ultimately, helping the growth of the companies. Moreover, in competitive circumstances or in the race of success, these types of CqROFRs are praiseworthy tools. In addition, these concepts have the ability to efficiently model multidimensional problems. They also spot the central features that lead to the success of businesses.

**VI. COMPARATIVE ANALYSIS**

The purpose of this section is to compare the proposed framework of CqROFRs with the preexisting ones, such as FRs, CFRs, IFRs, CIFRs, PFRs, CPFrs and qROFRs.

**A. COMPARISON WITH FR AND CFR**

FR and CFR are used to study relationships among the FSs and CFSs, respectively. The structures of FSs and CFSs are limited to the level of membership only. FRs deal with the real-valued levels of membership which limit them to only single-dimensional problems. CFRs can model multivariable problems, but they still lack the levels of non-membership. So these structures completely fail to solve the above problem.

**B. COMPARISON WITH IFR, PFR AND QROFR**

The notions of IFR, PFR, and qROFR are used to investigate the relations among the IFSs, PFSs, and qROFSs, respectively. All three of these frameworks argue about the level of membership as well as the level of non-membership. The qROFRs are the most powerful among these three frameworks. Since all of these frameworks deal with real-valued levels, they lag behind the proposed CqROFRs. The proposed methods can model multivariable problems, but IFRs, PFRs, and qROFRs fail to do so.

Since qROFRs are the strongest among these three concepts, we try to solve the problem in the above application by using the qROFRs to highlight the drawbacks of these structures.

Consider the following qROFS for \( n = 3 \)

\[
K = \begin{pmatrix}
\left(\text{EPS}, \frac{3}{5}, \frac{1}{2}\right), & \left(\text{NS}, \frac{9}{10}, \frac{3}{10}\right), & \left(\text{NOCF}, \frac{9}{10}, \frac{2}{5}\right), \\
\left(\text{BVPS}, \frac{4}{5}, \frac{1}{2}\right), & \left(\text{ROIC}, \frac{9}{10}, \frac{1}{2}\right), & \left(D - \text{NPR}, \frac{3}{10}, \frac{1}{2}\right)
\end{pmatrix}
\]

Now, by following the algorithm presented in Figure 3, we find out Cartesian product,

\[ K \times K = \]

\[
\begin{pmatrix}
\left(\text{EPS, EPS}, \frac{3}{5}, \frac{1}{2}\right), & \left(\text{EPS, NS}, \frac{3}{5}, \frac{1}{2}\right), \\
\left(\text{EPS, NOCF}, \frac{3}{5}, \frac{1}{2}\right), & \left(\text{EPS, BVPS}, \frac{3}{5}, \frac{1}{2}\right), \\
\left(\text{EPS, ROIC}, \frac{3}{5}, \frac{1}{2}\right), & \left(\text{EPS, D - NPR}, \frac{3}{5}, \frac{1}{2}\right), \\
\left(\text{NS, EPS}, \frac{3}{5}, \frac{1}{2}\right), & \left(\text{NS, NS}, \frac{9}{10}, \frac{3}{10}\right), \\
\left(\text{NS, NOCF}, \frac{9}{10}, \frac{2}{5}\right), & \left(\text{NS, BVPS}, \frac{3}{5}, \frac{1}{2}\right), \\
\left(\text{NS, ROIC}, \frac{9}{10}, \frac{2}{5}\right), & \left(\text{NS, D - NPR}, \frac{3}{10}, \frac{1}{2}\right), \\
\left(\text{NOCF, EPS}, \frac{3}{5}, \frac{1}{2}\right), & \left(\text{NOCF, NS}, \frac{9}{10}, \frac{2}{5}\right), \\
\left(\text{NOCF, NOCF}, \frac{3}{5}, \frac{1}{2}\right), & \left(\text{NOCF, BVPS}, \frac{4}{5}, \frac{1}{2}\right), \\
\left(\text{NOCF, ROIC}, \frac{9}{10}, \frac{2}{5}\right), & \left(\text{NOCF, D - NPR}, \frac{3}{10}, \frac{1}{2}\right)
\end{pmatrix}
\]

The above Cartesian product is a qROF-equivalence-FR. It is observed that a qROFR has given outputs, but they do not contain much-needed details, such as the time frames. So the IFR, PFR, and qROFR fail to achieve the required results.

**C. COMPARISON WITH CIFR AND CPFFR**

The CIFRs and CPFrs are used to find out the relations among the CIFs and CPFs, respectively. Both the structures involve the level of membership as well as the level of non-membership. Moreover, they also deal with complex-valued
levels. Therefore, they can successfully model many single and multivariable problems. In addition, they can produce similar kinds of results as the proposed CqROFRs. But they have certain limitations, as one cannot assign the values freely to the level of membership and the level of non-membership. These constraints limit them to a certain space. The achieved results may not be as practical and exact as required. The CIFRs and CPFRs are applied to the data given in the application. So that a direct comparison can be carried out among CIFR, CPFRs and the proposed CqROFRs. Tables VIII and IX contain the comparisons of CqROFSs with CIFSs and CPFSs, respectively.

### TABLE VIII
CIF tools applied to the CqROF information in application

| Parameter | \( \frac{\alpha_m(k,l)}{e^{\theta_0(k,l)2\pi/n}} \) | \( \frac{\alpha_q(k,l)}{e^{\theta_0(k,l)2\pi/n}} \) | \( \alpha_m + \alpha_q \) | Remarks |
|-----------|---------------------------------|---------------------------------|-----------------|--------|
| EPS       | \( 5 \)                          | \( 2 \)                          | 11              | 3      | ✓      |
| NS        | \( 9/10 \)                       | \( 9/10 \)                       | 5               | 4      | ✓      |
| NOCF      | \( 9/10 \)                       | \( 9/10 \)                       | 5               | 4      | ✓      |
| BVPS      | \( 4/5 \)                        | \( 4/5 \)                        | 10              | 2      | ✓      |
| ROIC      | \( 9/10 \)                       | \( 9/10 \)                       | 5               | 4      | ✓      |
| D – NPR   | \( 3/10 \)                       | \( 3/10 \)                       | 5               | 2      | ✓      |

In Table VIII, the intentions are to apply the CIFRs to analyze the application problem. For that reason, the same values of levels of membership and non-membership are taken for the comparison, that were used in the application. It is clearly seen that the structure of CIFR cannot handle that information. The sums of the values of levels of membership and non-membership do not follow the restrictions of the CIFR, and thus the choice of these values does not fall in the range of CIFSs.

### TABLE IX
CPF tools applied to the CqROF information in application

| Parameter | \( \frac{\alpha_m(k,l)}{e^{\theta_0(k,l)2\pi/n}} \) | \( \frac{\alpha_q(k,l)}{e^{\theta_0(k,l)2\pi/n}} \) | \( (\alpha_m)^2 + (\alpha_q)^2 \) | Remarks |
|-----------|---------------------------------|---------------------------------|----------------------------|--------|
| EPS       | \( e^{(1/2)2\pi/10} \)         | \( e^{(1/2)2\pi/10} \)         | 100, 5           | ✓      |
| NS        | \( e^{(1/2)2\pi/10} \)         | \( e^{(1/2)2\pi/10} \)         | 100, 5           | ✓      |
| NOCF      | \( e^{(1/2)2\pi/10} \)         | \( e^{(1/2)2\pi/10} \)         | 100, 8           | ✓      |
| BVPS      | \( e^{(1/2)2\pi/10} \)         | \( e^{(1/2)2\pi/10} \)         | 100, 8           | ✓      |
| ROIC      | \( e^{(1/2)2\pi/10} \)         | \( e^{(1/2)2\pi/10} \)         | 100, 1000        | ✓      |
| D – NPR   | \( e^{(1/2)2\pi/10} \)         | \( e^{(1/2)2\pi/10} \)         | 50              | 8      |

Similarly, the CPFRs are used for the analysis of the problem in the application section. Table IX demonstrates this comparison. Here too, similar values of levels of membership and non-membership are considered for the comparison, that were used previously. Like, CIFRs, the structure of CPFRs is also unable to hold that information. They too failed to contain the values of levels of membership and non-membership within their range, i.e., the sums of these values breaks the limits of the structure. The decision makers feel to assign certain values for better and accurate assessment. Therefore, in order to use CIFRs or CPFRs, the professionals would have to compromise on the accuracy levels of the results. To avoid inaccuracies, the proposed CqROFRs are the best choice among all the available tools.

![Comparison among the ranges of CIFS, CPFS and CqROFS.](image-url)
A graphical comparison among the ranges of CIFSs, CPFSs and CqROFSs is given by Figure 4. The inmost light green colored diamond represents the range of all complex numbers that can be assigned as the level of membership and non-membership. The middle rounded diamond (colored with slightly darker green) shows the range of all those complex numbers that can be assigned as the levels of membership and non-membership of an object in a CPFS. It also includes the numbers in the CIFS. Likewise, the outmost circle highlights the range of CqROFSs. Thus, it is observed that a CqROFS provides the most flexibility to decision makers. Ultimately, it implies the superiority of CqROFRs, because the other relation finding techniques are unable to process that wide-ranged information.

Table X contains the summary of this section. The term “multi-variation” refers to the ability to deal with multivariable problems, i.e., having complex-valued levels. “Independence” means having no constraints.

| Structure | Membership | Non-membership | Multi-variation | Independence |
|-----------|------------|----------------|----------------|--------------|
| FR        | ✓          | x              | x              | ✓            |
| CFR       | ✓          | x              | ✓              | ✓            |
| IFR       | ✓          | ✓              | x              | ✓            |
| CIFR      | ✓          | ✓              | ✓              | ✓            |
| PFR       | ✓          | ✓              | x              | ✓            |
| CPFR      | ✓          | ✓              | ✓              | ✓            |
| qROFR     | ✓          | ✓              | ✓              | ✓            |
| CqROFR    | ✓          | ✓              | ✓              | ✓            |

It is seen that each of the competing structures has one or more drawbacks. The only structure that tick all the compartments is CqROFR. All of the above assessments and comparisons verify the strength and omnipotence of proposed CqROFRs.

VII. CONCLUSION
This research discussed the concepts of complex q-rung orthopair fuzzy sets (CqROFSs). The theory of CqROFS is the most efficient theory to model vague information. It discusses both the levels of membership and non-membership. Their complex structure permits them to model multivariable problems such as phase-altering events, time involving and cyclic problems. Besides this, for the first time, this article defined the notion of complex q-rung orthopair fuzzy relations (CqROFRs) in the literature of fuzzy set theory. In addition, several types of CqROFRs are defined with appropriate examples, including CqRO-equivalence-FR, CqRO-order-FR, CqRO-composite-FR, the inverse of a CqROFR and many more. Some interesting properties of CqROFRs and their types are discussed. Further, a couple of applications of the proposed methods are presented that use the CqROFRs and their types to analyze the financial track records of business companies. One of the applications offered a modeling technique to investigate the valuable information about the financial structure of a business company. This technique allows the investors to identify the most critical factors that can perhaps earn them more profit. In contrast, the second application revealed the hidden and indirect financial impacts of one business company over some other company. The goal in the second application is to help the stakeholders lift their business and take their company to higher levels compared to the competing companies. Finally, a comparative study has been carried out among the proposed and existing methods. The CqROFR has been proved to be a powerful tool. Henceforth, the proposed modeling techniques can be used to model many situations that involve some sort of relationships. Moreover, these techniques can be further nourished to get more powerful techniques that might be used effectively in different fields of sciences such as engineering, economics, computer science, and technology, which deal with multivariable problems.

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