Constrained portfolio-consumption strategies with uncertain parameters and borrowing costs

Zhou Yang† Gechun Liang‡ Chao Zhou§

First version: September 2017; This version: November 9, 2017

Abstract

This paper studies the properties of the optimal portfolio-consumption strategies in a finite horizon robust utility maximization framework with different borrowing and lending rates. In particular, we allow for constraints on both investment and consumption strategies, and model uncertainty on both drift and volatility. With the help of the explicit solution, we quantify the impacts of uncertain market parameters, portfolio-consumption constraints and borrowing costs on the optimal strategies and their time monotone properties.

Keywords: Robust utility maximization, explicit solution, portfolio-consumption constraints, different borrowing and lending rates, model uncertainty.

AMS subject classifications (2000): 35R60, 47J20, 93E20

1 Introduction

One of the fundamental problems in mathematical finance is the construction of investment and consumption strategies \((\pi, c)\) that maximize the expected utility of a risk-averse investor:

\[
\max_{(\pi, c)} E \left[ \int_0^T U^c(s) ds + U(X^\pi,c; \mu, \sigma)_T \right],
\]

where \(U^c(\cdot)\) and \(U(\cdot)\) are the utilities of intertemporal consumption \(c\) and terminal wealth \(X^\pi,c; \mu, \sigma\), respectively. The market is described by a set of parameters \((\mu, \sigma)\)–the drift and volatility of the risky assets, and the investor’s utilities are often assumed to admit some homothetic properties (for example, the power and exponential types). Due to the market incompleteness arising from the randomness of the market parameters and the portfolio constraints, the resulting optimal portfolio

\[\tag{1}\]

\*Zhou Yang’s work is supported by NNSF of China (Grant No. 11371155, 11771158). Chao Zhou’s work is supported by Singapore MOE (Ministry of Education’s) AcRF grant R-146-000-219-112.

†School of Mathematical Sciences, South China Normal University, Guangzhou 510631, China; email: yangzhou@scnu.edu.cn

‡Department of Statistics, University of Warwick, Coventry, CV4 7AL, U.K.; email: g.liang@warwick.ac.uk

§National University of Singapore, Singapore; email: matzc@nus.edu.sg
is described as the sum a myopic strategy of Merton’s type and a hedging strategy. The latter is used to partially hedge the market risk stemming from the market incompleteness. Both the hedging strategy and the optimal consumption can be described via the solution of a backward stochastic differential equation (see [7] and [13]). However, the solution is in general not explicit, and consequently, there is limited information about the properties of the optimal strategies.

The purpose of this article is to study the properties of the optimal investment and consumption strategies when the investor optimally allocates her wealth among a risky stock, a riskless asset and her consumption. Our model takes consideration of several features including model uncertainty, constraints on both investment and consumption strategies, and borrowing costs. We derive an explicit solution to both the optimal portfolio-consumption strategy and the worst-case market parameters. The explicit forms further allow us to study the impacts of uncertain market parameters, portfolio-consumption constraints and different borrowing and lending rates on the optimal strategies and their time monotone properties.

In the vast majority of the literature, it is often assumed that the investor has a perfect knowledge of the market parameters, and is able to select her portfolio-consumption strategies without any constraints. However, constraints such as prohibition of short selling assets and the subsistence consumption are ubiquitous in reality. On the other hand, the paradigm of expected utility clearly has some deficiencies: it is not satisfactory in dealing with model uncertainty as predicted by the famous Ellsberg paradox. For the above reasons, it is desirable to take constraints on the portfolio-consumption strategies and uncertainty about the market parameters into account when studying the optimal strategies. We argue that the portfolio-consumption strategies must stay in an interval, and there are lots of probability models to describe the market, but none of them are really precise enough. This leads us to consider the so called robust utility maximization for which the investor worries about the worst-case scenario, and as opposed to (1), we solve the following maxmin problem

$$\max_{(\pi,c)\in B} \min_{(\mu,\sigma)\in A} E \left[ \int_0^T U(c_s)ds + U(X_T^{\pi,c,\mu,\sigma}) \right],$$

for an investor with power type utilities on both intertemporal consumption and terminal wealth. See (4) and (5) for further details.

As a first contribution, we obtain the optimal portfolio-consumption strategy and the worst-case parameters both in closed forms (see Theorems 4.1 and 4.2). Closed-form solutions seldom exist except for the standard Merton’s model with constant market parameters without portfolio-consumption constraints. We show that even with random market parameters and portfolio-consumption constraints, the closed form solutions will still be available in a robust utility framework. It is due to the fact that when the investor worries about the worst-case scenario, she will select either the upper or lower estimation of the market parameters, both of which are often assumed to be some constants. Eventually, this leads the investor to implement myopic strategies of Merton’s type to optimize her portfolios as in a complete market (see Theorem 3.2). Thus, there is no need for her to enforce the hedging strategy as opposed to the incomplete market situation. A similar phenomenon also occurs in [21], where the authors considered a market driven by Lévy processes with uncertain parameters but without consumption and borrowing costs.
The second contribution of our paper is a systematic study of the consumption plans in various situations. We argue that the consumption should stay above a minimum level for subsistence purpose, and be dominated by a reasonable upper bound for the sake of future consumption and investment. We show that the investor’s optimal consumption will degenerate to a deterministic process when she worries about the worst-case market scenario (see Theorem 3.2). By virtue of the closed form solution, we are able to obtain the time monotone properties of the optimal consumption plan (see Proposition 4.3), and quantify the impacts of different parameters (e.g. borrowing rate, uncertain market parameters and portfolio-consumption constraints) on the optimal consumption plan (see Proposition 5.1).

One of the striking results is that the optimal consumption is not necessarily increasing or decreasing when the investor lifts her upper bound for consumption. This is because the investor needs to balance her current consumption and future consumption and investment when she optimizes her consumption plans. Increasing the upper bound of consumption means the investor would consume in a larger constraint set in the future, and increase the weight of her future utility, thus the investor might decrease her current consumption level. On the other hand, lifting the upper bound for consumption also means a larger constraint set from which the investor makes her current consumption decisions, and in turn her current consumption level might increase. This two contradicting factors will offset their impacts by each other, and result in a non-monotone relationship of optimal consumption with respect to the upper bound of consumption plans.

Our third contribution is a classification of the optimal portfolio strategy in terms of borrowing and lending rates as well as the uncertain market parameters. We show that when the investor is optimistic about the market, her worst estimation of the stock’s return is still better than the borrowing rate, so she will implement a borrow-to-buy strategy to borrow as much as possible to approach the optimal strategy without constraint. When her worst estimation of the stock’s return is between the borrowing and lending rates, neither borrowing nor lending are attractive, and the investor will simply put all her money in the stock, i.e. performing a full-position strategy. When the lending rate is between the best and worst estimations of the stock’s return, the investor will simply put all her money in the bank account, i.e. performing a no-trading strategy. When the investor is pessimistic about the market, her best estimation of the stock’s return is still lower than the lending rate, so she will implement a shortsale strategy to short sell the stock as much as possible. See Theorem 4.1 for further details.

Turning to the literature, optimal portfolio-consumption problems in continuous time were first studied by Merton in 1970s (see [20] for a summary). In a sequence of papers [14], [15] and [17], the authors developed and generalized Merton’s model. In particular, [17] is one of the first arguing that the consumption must always be above a certain subsistence level, and sometimes neither borrowing nor shortsale are allowed for trading stocks, so they imposed constraints on both consumption and investment. Following this work, the optimal consumption with constraints was further studied in [6], [24], and more recently in [16], [28] in a complete market setting with constant market parameters. On the other hand, [8], [27], [29] and [30] among others studied constrained investment problems for models of varying generality.

Equal borrowing and lending rates is often assumed in the literature, and as a consequence, the
wealth equation is always linear. However, it is argued in [1] that such an assumption stands in contrast with reality. Subsequently, [10] introduced the borrowing cost for the utility maximization problem, and more recently in [3], the authors took borrowing costs into account in an optimal credit investment problem.

The early development of model uncertainty went back to [25] where the authors considered a worst-case risk management problem. Much of robust utility maximization in mathematical finance started with [4], [12] and [23], which mainly dealt with drift uncertainty. Volatility uncertainty is a much harder problem, and has been treated via various mathematical tools. To name a few, duality method was used in [9] where the uncertainty is specified by a family of semimartingales laws. G-expectation was employed in [11] in a stochastic volatility model to treat uncertain correlations. In contrast, [19] studied the robust utility maximization problem under volatility uncertainty via second-order backward stochastic differential equations, and [20] considered uncertain drift and volatility using mixed strategies and derived an explicit solution in a non-traded asset setting. More recently, the results have been further generalized in [21] to include drift, volatility and jump uncertainty, which are parameterized by a set of Lévy triples. However, consumption is not considered in the above works. Two exceptions are [18] and more recently [2], where the authors worked in a similar framework to our model, but portfolio-consumption constraints are not treated in those papers.

In summary, it seems the existing literature mainly focuses on the investment-consumption models with only parts of the above features: either with portfolio constraints and market uncertainty or with consumption constraints and borrowing costs. Although many elegant mathematical results are achieved in these papers, explicit solutions and the properties of the optimal strategy rarely exist except for some special cases. In particular, consumption constraints make it difficult to obtain an explicit solution, and almost all of the explicit solutions with consumption constraints are in the framework of infinite horizon.

In contrast, our paper systematically studies constrained portfolio-consumption strategies under model uncertainty and borrowing costs in a finite horizon. We provide the explicit solution and properties of the optimal strategies. Although only one risky asset is considered in this paper, our method can be applied to study the multiple risky assets setting as in [21], and similar results will still hold. However, to simplify our presentation and quantify the impacts of uncertainty parameters, constraints and borrowing cost on the optimal strategies, we focus on the setting with only one risky asset in this paper.

The paper is organized as follows. Section 2 presents a robust utility maximization model subject to borrowing costs and portfolio-consumption constraints. Section 3 solves the associated maxmin problem via a martingale argument, and characterizes the optimal portfolio-consumption strategy and the worst-case market parameters via the solution of a nonlinear ordinary differential equation (ODE). Section 4 further obtains its closed form solution in different cases. Section 5 studies the impacts of the various model parameters on the optimal strategies and the worst-case parameters. The proof of the explicit solution is given in the Appendix.
2 The utility maximization model

2.1 Uncertain parameters and borrowing costs

Let $W$ be a standard one dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathcal{F} := \{\mathcal{F}_t\}_{t \geq 0}$ be the augmented filtration generated by $W$. The market consists of a risky asset $S$ and a riskless bank account $B$. The risky asset’s price follows

$$dS_s = \mu_s S_s ds + \sigma_s S_s dW_s,$$

(2)

where $\mu$ and $\sigma$ represent the drift and volatility of the risky asset, respectively.

Consider a small investor in this market. She trades both risky asset and riskless bank account, yet she has limited information about the risky asset’s parameters ($\mu$, $\sigma$). The uncertainty about drift and volatility of the risky asset is parameterized by a nonempty set with the form

$$\mathcal{B} = \{(\mu_s, \sigma_s)_{s \geq 0} : (\mu, \sigma) \text{ are } \mathbb{F}\text{-progressively measurable}, \ (\mu_s, \sigma_s) \in [\underline{\mu}, \overline{\mu}] \times [\underline{\sigma}, \overline{\sigma}], \ \mathbb{P} \otimes ds\text{-a.e.}\},$$

where $\underline{\mu}, \overline{\mu}, \underline{\sigma}, \overline{\sigma}$ are constants satisfying $-\infty < \underline{\mu} \leq \overline{\mu} < +\infty$, $0 < \underline{\sigma} \leq \overline{\sigma} < +\infty$.

The width of the set $[\underline{\mu}, \overline{\mu}] \times [\underline{\sigma}, \overline{\sigma}]$ indicates the amount of uncertainty. The larger the width, the larger becomes the set of alternative models. The investor will then become more uncertain about the model parameters. Hence, $[\underline{\mu}, \overline{\mu}] \times [\underline{\sigma}, \overline{\sigma}]$ may be interpreted as the information set of the investor.

In terms of the bank account $B$, the standard assumption of equal borrowing and lending rates is in contrast with empirical evidence (see [1]). In reality, there always exists a spread between borrowing and lending rates. Let $R$ and $r$ be the constant borrowing and lending rates, respectively. When $B$ is positive, the investor lends with rate $r$. When $B$ is negative, the investor borrows with rate $R$. It is nature to assume that $R \geq r$. Consequently, the bank account $B$ follows

$$dB_s = \left(rB^+_s - RB^-_s\right) ds,$$

(3)

where $x^+ = \max\{0, x\}$, $x^- = \max\{0, -x\}$. Note that $rB^+_s - RB^-_s = rB_s - (R - r)B^-_s$, and therefore the spread $(R - r)$ represents the borrowing cost of the investor. The larger the spread, the more borrowing cost the investor has to bear. In the next section, we shall see the introduction of borrowing cost leads to a nonlinear wealth equation, which is concave in the portfolio strategies.

2.2 Portfolio and consumption constraints

Let $T > 0$ represent the trading horizon, and suppose that the investor has an initial wealth $x > 0$. Let $\pi$ be the proportion of her wealth invested in the risky asset, $c$ be her consumption rate proportional to her wealth, and $X^{x, \pi, c, \mu, \sigma}$ be the wealth process with initial value $x$, portfolio-consumption strategy $(\pi, c)$ and parameters $(\mu, \sigma)$. Using (2) and (3), it follows from the self-financing condition that

$$X^{x, \pi, c, \mu, \sigma}_s = x + \int_0^s \left[\mu_u \pi_u + r(1 - \pi_u) - (R - r)(1 - \pi_u) - c_u\right] X^{x, \pi, c, \mu, \sigma}_u du$$

$$+ \int_0^s \sigma_u \pi_u X^{x, \pi, c, \mu, \sigma}_u dW_u, \quad s \in [0, T].$$
Note that with the borrowing cost, the wealth equation is no longer linear but concave in the portfolio strategies $\pi$ in the case of $R > r$.

The investor will select her portfolio-consumption strategy from the following admissible set with constraints on both portfolio and consumption:

$$A = \{ (\pi_s, c_s)_{s \geq 0} : (\pi, c) \text{ are } \mathbb{F}\text{-progressively measurable, } (\pi_s, c_s) \in [\underline{\pi}, \bar{\pi}] \times [\underline{c}, \bar{c}], \mathbb{P} \otimes ds \text{-a.e.}, \int_0^T (|\pi_s|^2 + c_s) ds < +\infty, \text{ and } X^{x \pi, c, \mu, \sigma} \text{ satisfies the Class (D) condition}\},$$

where $\underline{\pi}, \bar{\pi}, \underline{c}, \bar{c}$ are constants satisfying $-\infty \leq \underline{\pi} \leq 0, 1 \leq \bar{\pi} \leq +\infty, 0 \leq \underline{c} \leq \bar{c} \leq +\infty$.

The integrability condition on $(\pi, c)$ is to guarantee that the wealth process is well defined, while the Class (D) condition imposed on the wealth process $X^{x \pi, c, \mu, \sigma}$ depends on the utility maximization problem that we want to solve, and will be specified in (7) in the next section.

The portfolio constraint interval $[\underline{\pi}, \bar{\pi}]$ includes several interesting examples. For instance, $(\pi - 1)$ represents the maximum proportion of wealth that the investor is allowed to borrow to invest in the risky asset, and $(-\underline{\pi})$ represents the largest shortsale position that the investor is allowed to take. In particular, if $\underline{\pi} = 0$ and $\bar{\pi} = 1$, it means prohibition of borrowing to invest and shortsale; if $\underline{\pi} = -\infty$ and $\bar{\pi} = +\infty$, it means no portfolio constraints.

On the other hand, the consumption constraint $[\underline{c}, \bar{c}]$ means that the investor should keep a minimal consumption level $\underline{c}$ for subsistence purpose, and at the same time, her consumption is also controlled by an upper bound $\bar{c}$ for the sake of future consumption and investment.

### 2.3 The robust utility maximization problem

The investor has a power utility of both intertemporal consumption and terminal wealth. Given a portfolio-consumption strategy $(\pi, c) \in A$, her expected utility is defined as

$$J(x; \pi, c, \mu, \sigma) := \mathbb{E}\left[ \int_0^T \lambda e^{-\rho s} U^c(c_s X_s^{x \pi, c, \mu, \sigma}) ds + e^{-\rho T} U(X_T^{x \pi, c, \mu, \sigma}) \right],$$

(4)

where $\lambda \geq 0$ represents the weight of the intertemporal consumption relative to the final bequest at maturity $T$, $\rho \geq 0$ represents the discount factor, and $U^c(x) = U(x) = \frac{1}{p} x^p$, with $p \in (0, 1)$ as the investor’s risk-aversion parameter\(^1\). Note that if $\lambda = 0$, the expected utility in (4) will reduce to the standard expected utility of the terminal wealth, which is equivalent to the case $\lambda > 0$ and $\bar{c} = $\underline{c}$. Therefore, we only concentrate on the case of $\lambda > 0$ in the rest of the paper.

Since the investor is uncertain about the model parameters $(\mu, \sigma)$, she will seek for an optimal portfolio-consumption strategy that is least affected by model uncertainty. In anticipation of the worst-case scenario, she solves the following maxmin problem: Find $(\pi^*, c^*) \in A$ and $(\mu^*, \sigma^*) \in B$ such that

$$J(x) := \sup_{(\pi, c) \in A} \inf_{(\mu, \sigma) \in B} J(x; \pi, c, \mu, \sigma) = J(x; \pi^*, c^*, \mu^*, \sigma^*),$$

(5)

where $J(\cdot)$ is the value function of the maxmin problem (5), i.e. the maximum worst-case expected utility.

\(^1\)When $p < 0$, we will have a similar result.
To robustify the portfolio-consumption strategy, the inner part of the above optimization problem is played by a so called mother nature who acts maliciously to minimize the expected utility by choosing the worst-case scenario, whereas the investor aims to select the best strategy that is least affected by the mother nature’s choice. For this reason, the maxmin problem (5) is also dubbed as the robust utility maximization problem in the literature (see [21] for example).

To solve the value function of the robust utility maximization problem (5) and its corresponding worst-case parameters and optimal portfolio-consumption strategy, we look for a saddle point strategy \( \{(\pi^*, c^*), (\mu^*, \sigma^*)\} \) of the expected utility \( J(x; \pi, c, \mu, \sigma) \) such that

\[
J(x; \pi, c, \mu^*, \sigma^*) \leq J(x; \pi^*, c^*, \mu^*, \sigma^*) \leq J(x; \pi^*, c^*, \mu, \sigma^*)
\]

for any admissible \((\pi, c) \in A\) and \((\mu, \sigma) \in B\). Then it follows that

\[
\sup_{(\pi, c) \in A} \inf_{(\mu, \sigma) \in B} J(x; \pi, c, \mu, \sigma) = \inf_{(\mu, \sigma) \in B} \sup_{(\pi, c) \in A} J(x; \pi, c, \mu, \sigma),
\]

and consequently, \( J(x) = J(x; \pi^*, c^*, \mu^*, \sigma^*) \) is the value function of the maxmin problem (5), with \((\mu^*, \sigma^*)\) and \((\pi^*, c^*)\) as the worst-case parameters and the optimal portfolio-consumption strategy, respectively.

To close this section, we further specify the Class (D) condition in the admissible set \( A \) associated with the maxmin problem (5):

Class (D) condition = \[ \left\{ E \left[ \int_0^\tau e^{\rho s} (X_s^{x, \pi, c, \mu, \sigma})^p ds \right] < +\infty, \ (X_\tau^{x, \pi, c, \mu, \sigma})^p \text{ is uniformly integrable} \right\}, \]

where \( \tau \in [0, T] \) is any \( F \)-stopping time. The Class (D) condition is required so as to include unbounded portfolio and consumption strategies, and it also appears in [7], where the authors solve a similar portfolio-consumption problem, but without model uncertainty, borrowing costs and consumption constraint.

3 A nonlinear ODE characterization of the value function

In this section, we apply a martingale argument, firstly introduced in [7] and [13], to construct a saddle point strategy \( \{(\mu^*, \sigma^*), (\pi^*, c^*)\} \) for the expected utility \( J(x; \pi, c, \mu, \sigma) \). This will in turn solve the original maxmin problem (5).

To this end, we aim to construct an \( F \)-adapted process \( J_{x; \pi, c, \mu, \sigma}^t, t \in [0, T] \), satisfying the following three conditions: For any \((\pi, c) \in A\) and \((\mu, \sigma) \in B\),

(C1) at the maturity \( T \),

\[
J_{x; \pi, c, \mu, \sigma}^T = \int_0^T e^{-\rho s} \frac{1}{p} \left( X_s^{x, \pi, c, \mu, \sigma} \right)^p ds + e^{-\rho T} \frac{1}{p} \left( X_T^{x, \pi, c, \mu, \sigma} \right)^p;
\]

(C2) at the initial time 0, \( J_{x; \pi, c, \mu, \sigma}^0 = J_0 \), which is a constant and is independent of \((\pi, c)\) and \((\mu, \sigma)\);

(C3) there exist \((\pi^*, c^*) \in A\) and \((\mu^*, \sigma^*) \in B\) such that the process \( J_{x; \pi^*, c^*, \mu^*, \sigma^*}^t \) is a martingale, \( J_{x; \pi^*, c^*, \mu^*, \sigma^*}^t \) is a supermartingale, and \( J_{x; \pi^*, c^*, \mu^*, \sigma^*}^t \) is a submartingale.
Following the above conditions (C1-C3), we then have

\[
\mathcal{J}(x; \pi, c, \mu^*, \sigma^*) = E[J_T^{x, \pi, c, \mu^*, \sigma^*}] \leq J_0^{x, \pi, c, \mu^*, \sigma^*} = J_0^x;
\]
\[
\mathcal{J}(x; \pi^*, c^*, \mu^*, \sigma^*) = E[J_T^{x, \pi^*, c^*, \mu^*, \sigma^*}] = J_0^{x, \pi^*, c^*, \mu^*, \sigma^*} = J_0^x;
\]
\[
\mathcal{J}(x; \pi^*, c^*, \mu, \sigma) = E[J_T^{x, \pi^*, c^*, \mu, \sigma}] \geq J_0^{x, \pi^*, c^*, \mu, \sigma} = J_0^x.
\]

Thus, the inequalities in \((6)\) hold, i.e., \(\{(\pi^*, c^*), (\mu^*, \sigma^*)\}\) is a saddle point strategy of the expected utility \(\mathcal{J}(x; \pi, c, \mu, \sigma)\), and the value function of the maximin problem \((5)\) is given by \(J(x) = J_0^x\).

Next, we construct the process \(\mathcal{J}^{x, \pi, c, \mu, \sigma}\). We start with the following lemma, which reduces the original maximin problem \((1)\), as an infinite dimensional optimization problem, to a finite dimensional one. To facilitate our discussions below, we introduce two functions \(f_1(\cdot, \cdot)\) and \(f_2(\cdot, \cdot, \cdot)\), which characterize the optimal portfolio-consumption and the worst-case parameters locally,

\[
f_1(x_q, x_c) := \frac{\lambda}{\rho} e^{-x_q} x_c^\rho - x_c;
\]
\[
f_2(x_\pi; x_\mu, x_\sigma) := \frac{p - 1}{2} x_\pi^2 + (x_\mu x_\pi + r(1 - x_\pi) - (R - r)(1 - x_\pi)^-)
\]

for \(x_q \in \mathbb{R}\) and \((x_c, x_\pi, x_\mu, x_\sigma) \in [\underline{c}, \overline{c}] \times [\underline{\pi}, \overline{\pi}] \times [\underline{\mu}, \overline{\mu}] \times [\underline{\sigma}, \overline{\sigma}]\).

**Lemma 3.1.** (i) There exists a continuous function \(x^*_c(\cdot)\) such that, for any \(x_q \in \mathbb{R}\), \(x^*_c(x_q)\) achieves the maximum value of the function \(f_1(x_q, \cdot)\):

\[
f_1(x_q, x^*_c(x_q)) = \max_{x_c \in [\underline{c}, \overline{c}]} f_1(x_q, x_c).
\]  

(ii) There exist constants \(x^*_\pi, x^*_\mu\) and \(x^*_\sigma\) such that \(\{(x^*_\pi, x^*_\mu, x^*_\sigma)\}\) is a saddle point of the function \(f_2(\cdot, \cdot, \cdot)\), i.e.,

\[
f_2(x_\pi, x^*_\mu, x^*_\sigma) \leq f_2(x^*_\pi, x_\mu, x_\sigma) \leq f_2(x_\pi, x_\mu, x_\sigma)
\]

for any \((x_\pi, x_\mu, x_\sigma) \in [\underline{c}, \overline{c}] \times [\underline{\pi}, \overline{\pi}] \times [\underline{\mu}, \overline{\mu}]\).

**Proof.** (i) For fixed \(x_q \in \mathbb{R}\), the function \(f_1(x_q, \cdot)\) is strictly concave in the interval \([\underline{c}, \overline{c}]\), which implies the existence of a unique maximizer \(x^*_c(x_q)\). It can be computed explicitly as

\[
x^*_c(x_q) = \mathbb{1}\{x_\pi(x_q) \leq \overline{\pi}\} + \hat{x}_c(x_q) \mathbb{1}\{\underline{\pi} < x_\pi(x_q) < \overline{\pi}\} + \mathbb{1}\{x_\pi(x_q) \geq \overline{\pi}\}, \quad \hat{x}_c(x_q) := \lambda^{\frac{1}{\rho - 1}} \exp\left(\frac{x_q}{p - 1}\right),
\]

where \(\mathbb{1}_A\) is the indicator function of \(A\). Moreover, the maximum value takes the form of

\[
f_1(x_q, x^*_c(x_q)) = \begin{cases} 
\frac{\lambda}{p} e^{-x_q} - \overline{c}, & \text{if } x_q < (p - 1) \ln \overline{\pi} + \ln \lambda; \\
\frac{(1 - p) \lambda^\rho \gamma}{p} e^{x_q/(p - 1)}, & \text{if } (p - 1) \ln \overline{\pi} + \ln \lambda \leq x_q \leq (p - 1) \ln \underline{\pi} + \ln \lambda; \\
\frac{\lambda}{p} e^{x_q} - \underline{c}, & \text{if } x_q > (p - 1) \ln \underline{\pi} + \ln \lambda.
\end{cases}
\]

(ii) The proof is given in Appendix A, where we will compute the explicit form of the saddle point \(\{(x^*_\pi, x^*_\mu, x^*_\sigma)\}\).
We are now ready to state our first main result, which is about a nonlinear ODE characterization of the value function $J(\cdot)$.

**Theorem 3.2.** Let $q(\cdot)$ be the unique solution to the following nonlinear ODE

$$q(t) = \int_t^T \left( pf_1(q(s), c^*(s)) + pf_2(x^{*}_\pi; x^{*}_\mu, x^{*}_\sigma) - \rho \right) \, ds, \quad \forall \, t \in [0, T],$$

(12)

where

$$c^*(t) := x^{*}_c(q(t)) = \xi_1(\zeta(t) \leq \xi) + \xi(t)1_{\xi < \zeta(t) < \theta} + \xi_1(\zeta(t) \geq \theta), \quad \xi(t) := \lambda \frac{1}{p - 1} \exp \left( \frac{q(t)}{p - 1} \right),$$

(13)

and $x^{*}_c(x_q), (x^{*}_\pi, x^{*}_\mu, x^{*}_\sigma)$ are given in Lemma 3.1.

Then the process

$$J^{x, c, \mu, \sigma}_t := \int_t^T \lambda e^{-\rho s} \frac{1}{p} \left( X^{x, c, \mu, \sigma}_s \right)^p ds + e^{-\rho t} \frac{1}{p} \left( X^{x, c, \mu, \sigma}_t \right)^p e^{q(t)},$$

(14)

together with $(\pi^*_t, c^*_t) = (x^{*}_\pi, c^*(t))$ and $(\mu^*_t, \sigma^*_t) = (x^{*}_\mu, x^{*}_\sigma)$, $t \in [0, T]$, satisfy the conditions (C1-C3). In particular, the value function of the maximin problem (5) is given as

$$J(x) = J^*_0 = \frac{1}{p} x^p e^{q(0)}.$$

Note that the exponential of the ODE’s solution $e^{q(t)}$ represents the investor’s extra utilities obtained by optimizing over all admissible portfolio-consumption strategies (least affected by model uncertainty) in the remaining horizon $[t, T]$, and in the literature, $e^{q(t)}$ is dubbed as a (deterministic) opportunity process (see [22]).

Moreover, ODE (12) and the definition of $f_1(\cdot, \cdot)$ imply

$$-\frac{(e^{q(t)})'}{e^{q(t)}} = -q'(t) = pf_1(q(s), c^*(s)) + pf_2(x^{*}_\pi; x^{*}_\mu, x^{*}_\sigma) - \rho,
$$

with

$$f_1(q(s), c^*(s)) = \lambda \frac{1}{p} e^{-q(s)} (c^*(s))^p - c^*(s).$$

Hence, we can further interpret ODE (12) as a description of the relative changing rate of the opportunity process $e^{q(t)}$, which consists of three factors: (i) the consumption contributing factor $pf_1(q(\cdot), c^*(\cdot))$, representing the change of the opportunity process due to the consumption optimization, and including two parts: current contribution $\lambda e^{-q(s)} (c^*(s))^p / p$ and future contribution $-c^*(s)$; (ii) the future investment contributing factor $pf_2(x^{*}_\pi; x^{*}_\mu, x^{*}_\sigma)$, representing the change of the opportunity process due to the portfolio optimization in the remaining horizon; and (iii) the discount rate $\rho$. Increasing the consumption and future investment contributing factors or decreasing the discount rate will lead to a larger opportunity process.

The current consumption contributing factor is the only one affecting the instantaneous utility, which is also reflected in the expression of the expected utility $J$. The future consumption contributing factor and the future investment contributing factor determine the future consumption and terminal utility through the channel of the future wealth. The player achieves the maximum utility through balancing the risky asset and riskless asset via the investment strategy, while balancing the current utility and future utility via the consumption strategy. Moreover, the definition
of \( f_1(q(\cdot), c^*(\cdot)) \) implies that \( \lambda e^{-q(s)} \) is the weight of the current consumption utility relative to the future utility, which is consistent with our intuition that increasing opportunity process will lead to a larger weight of the future utility, and decrease the current consumption.

**Proof.** First, with \( f_1(x, q_0(x)) \) given in (11), ODE (12) satisfies the monotonicity condition in [5], and admits a unique solution \( q(\cdot) \). In Appendix B, we will further compute its explicit solution in different cases, which will in turn provides explicit solution for the optimal consumption \( c^* \) in section 4.

Next, \( J^{t;\pi,\mu,\sigma} \) in (14) obviously satisfies the conditions (C1) and (C2), so it suffices to verify the martingale property (C3).

To this end, for any \( (\pi, c) \in A \) and \( (\mu, \sigma) \in B \), an application of Itô’s formula implies

\[
d(X_s^{t;\pi,\mu,\sigma}) = (X_s^{t;\pi,\mu,\sigma})^p \left[ (pf_2(\pi_s; \mu_s, \sigma_s) - pc_s)ds + \sigma_s \pi_s dW_s \right],
\]

and in turn,

\[
J^{t;\pi,\mu,\sigma} = J^{0;\pi,\mu,\sigma} + \int_0^t e^{q(s) - \rho s} (X_s^{0;\pi,\mu,\sigma})^p \left[ f_1(q(s), c_s) + f_2(\pi_s; \mu_s, \sigma_s) + \frac{1}{p}(q'(s) - \rho) \right] ds
\]

\[+ \int_0^t e^{q(s) - \rho s} (X_s^{0;\pi,\mu,\sigma})^\pi \sigma_s dW_s.
\]

By our construction, it is clear that \( c^*(\cdot) \) is a continuous and deterministic function, and \( \pi^*, \mu^* \) and \( \sigma^* \) are all constants, it follows that the stochastic exponential \( E(\int_0^t \rho \sigma_s^* \pi^*_s dW_s) \) is a uniformly integrable martingale, and moreover,

\[
\left( X_t^{t;\pi^*,\mu^*,\sigma^*} \right)^p = x^p E_t \left( \int_0^t \rho \sigma_s^* \pi^*_s dW_s \right) \exp \left( \int_0^t (pf_2(\pi_s^*; \mu_s^*, \sigma_s^*) - pc_s^*) ds \right)
\]

for \( t \in [0, T] \). Thus \( X^{t;\pi^*,\mu^*,\sigma^*} \) is in Class (D). On the other hand, there exists a constant \( C > 0 \) such that

\[
E \left[ \int_0^T (c_s^*)^p (X_s^{t;\pi^*,\mu^*,\sigma^*})^p ds \right] \leq CE \left[ \int_0^T E_s(\int_0^t \rho \sigma_s^* \pi_s^* dW_s) ds \right] = CT.
\]

Thus, \( (\pi^*, c^*) \in A \) and \( (\mu^*, \sigma^*) \in B \). In turn, \( J^{t;\pi^*,\mu^*,\sigma^*} \) is in Class (D). Together with ODE (12) for \( q(\cdot) \), we deduce

\[
E \left[ J^{t;\pi^*,\mu^*,\sigma^*} | F_t \right] = J^{t;\pi^*,\mu^*,\sigma^*}.
\]

for any \( 0 \leq t \leq s \leq T \).

With \( (\mu^*, \sigma^*) \) as in Lemma [4.1] the maximum condition (8) together with the first inequality in the saddle point condition (9) imply

\[
f_1(q(s), c_s) + f_2(\pi_s; \mu_s^*, \sigma_s^*) + \frac{1}{p}(q'(s) - \rho)
\]

\[
\leq f_1(q(s), c_s^*) + f_2(\pi_s^*; \mu_s^*, \sigma_s^*) + \frac{1}{p}(q'(s) - \rho) = 0, \forall (\pi, c) \in A.
\]

Thus \( J^{t;\pi,\mu,\sigma} \) is a positive local supermartingale, and therefore a supermartingale by Fatou’s lemma.
Finally, with \((\pi^*, c^*)\) as in Lemma 3.1, the second inequality in the saddle point condition (9) implies

\[
\begin{align*}
& f_1(q(s), c^*) + f_2(\pi^*_s; \mu_s, \sigma_s) + \frac{1}{p}(q'(s) - \rho) \\
& \geq f_1(q(s), c^*) + f_2(\pi^*_s; \mu_s, \sigma_s) + \frac{1}{p}(q'(s) - \rho) = 0, \forall (\mu, \sigma) \in \mathcal{B},
\end{align*}
\]

so \(J^{x, \pi^*, c^*, \mu, \sigma}\) is a local submartingale. Thus, there exist an increasing sequence of \(\mathbb{F}\)-stopping times \(\tau_n \uparrow T\) such that for any \(0 \leq t \leq s \leq T\),

\[
E \left[ J^{x, \pi^*, c^*, \mu, \sigma}_{s \wedge \tau_n} \mid \mathcal{F}_t \right] \geq E \left[ J^{x, \pi^*, c^*, \mu, \sigma}_{s \wedge \tau_n} \mid \mathcal{F}_t \right],
\]

i.e.

\[
E \left[ J^{x, \pi^*, c^*, \mu, \sigma, 1_A}_{s \wedge \tau_n} \right] \geq E \left[ J^{x, \pi^*, c^*, \mu, \sigma, 1_A}_{s \wedge \tau_n} \right]
\]

for any \(A \in \mathcal{F}_t\). By the Class (D) condition on \(J^{x, \pi^*, c^*, \mu, \sigma}\), we may let \(\tau_n \uparrow T\) in (15), which then implies that \(E[J^{x, \pi^*, c^*, \mu, \sigma, 1_A}] \geq E[J^{x, \pi^*, c^*, \mu, \sigma, 1_A}]\), i.e. \(J^{x, \pi^*, c^*, \mu, \sigma}\) is a submartingale. \(\square\)

## 4 Explicit solution of the optimal strategies

### 4.1 The worst-case parameters and the optimal portfolio

In this section, we further compute the saddle point \(\{(x^*_1, (x^*_\mu, x^*_\sigma)\}\) of the function \(f_2(\cdot, \cdot, \cdot, \cdot, \cdot)\) explicitly. By Theorem 3.2, the saddle point then provides an explicit solution for the worst-case parameters and the optimal portfolio of the maxmin problem (5) by letting \((\mu^*_s, \sigma^*_s) = (x^*_\mu, x^*_\sigma)\) and \(\pi^*_s = x^*_s\) for \(s \in [0, T]\).

**Theorem 4.1.** The worst-case parameters \((\mu^*, \sigma^*)\) and the optimal portfolio \(\pi^*\) are given as follows:

(i) the worst-case drift and volatility are

\[
(\mu^*_s, \sigma^*_s) = (\mu^1_{\mu>r} + [\mu, \sigma]1_{\mu \leq r \leq \sigma} + [\sigma, \infty]1_{\mu < r}),
\]

for \(s \in [0, T]\), where \([\mu, \sigma]\) means \(\mu^*_s\) may take any value in that interval;

(ii) the optimal portfolio is a constant process, which is summarized in Table 1, with \(\beta_1, \beta_2\) and \(\beta_3\) given as

\[
\beta^1 := \frac{\mu - \bar{R}}{(1 - p)\sigma^2}, \quad \beta^2 := \frac{\mu - r}{(1 - p)\sigma^2}, \quad \beta^3 := \frac{\sigma - r}{(1 - p)\sigma^2}.
\]

| \(\beta^1 \geq 1\) | \(\beta^1 \leq 1 \leq \beta^2\) | \(0 \leq \beta^2 \leq 1\) | \(\beta^2 \leq 0 \leq \beta^3\) | \(\beta^3 \leq 0\) |
|-------------------|-------------------|-------------------|-------------------|-------------------|
| \(\pi^*_s\) | \(\min\{\beta^1, \pi\}\) | \(\beta^2\) | \(0\) | \(\max\{\beta^3, \pi\}\) |

*Proof.* See Appendix A. \(\square\)
We note that the worst-case volatility $\sigma^*$ attains its upper bound $\overline{\sigma}$. This is due to the fact that the value function is monotone in volatility $\sigma$ in the one dimensional setting. A larger $\sigma$ means the investor faces more market risks, and therefore, she will have a smaller value function.

On the other hand, the worst-case drift is a bang-bang type. By the assertion (ii) about the optimal portfolio strategies, we know that $\underline{\mu} > r$ implies $\pi^* > 0$, i.e. the investor holds a long position of the stock. The worst drift is therefore its lower bound. Likewise, $\overline{\mu} < r$ implies $\pi^* < 0$, and therefore, the worst drift takes its upper bound. If $\underline{\mu} \leq r \leq \overline{\mu}$, then $\pi^* \equiv 0$, so the estimation of the drift is irrelevant in this situation.

From Table 1, we categorize five different optimal portfolio strategies $\pi^*$ according to various scenarios.

(i) Borrow-to-buy strategy. When $\beta^1 \geq 1$, the investor will borrow $(\min\{\beta^1, \overline{\pi}\} - 1)$ units of her wealth with borrowing rate $R$ to invest in the stock, and the optimal portfolio is $\min\{\beta^1, \overline{\pi}\}$. The reason is that in this situation, $\underline{\mu} \geq R + (1 - p)\overline{\sigma}^2$, i.e. the stock’s return even with the worst estimation of the drift is still higher than the borrowing cost. Hence, the stock’s high risk premium attracts the investor to borrow to invest as much as possible to approach the optimal strategy without constraint.

(ii) Full-position strategy. When $\beta^1 \leq 1 \leq \beta^2$, the investor will simply invest all her wealth in the stock with no additional borrowing or lending. In this case, since $\underline{\mu} \leq R + (1 - p)\overline{\sigma}^2$, there exists a possibility that the stock’s return may not be good enough to compensate for the borrowing cost. As a result, the investor would prefer not to borrow. On the other hand, since $\underline{\mu} \geq r + (1 - p)\overline{\sigma}^2$, the stock’s return even in the worst scenario is still better than the return from the bank account, and accordingly, the investor would put all her money in the stock rather than in the bank account.

(iii) Lend-and-buy strategy. When $0 \leq \beta^2 \leq 1$, the investor will invest $\beta^2$ proportion of her wealth in the stock, and the remaining proportion $(1 - \beta^2)$ in the bank account to earn the interest rate $r$. This is similar to the standard Merton’s strategy with Sharpe ratio $(\underline{\mu} - r)/\overline{\sigma}$.

(iv) No-trading strategy. When $\beta^2 \leq 0 \leq \beta^3$, the investor will put all her money in the bank account. In this case, $\underline{\mu} \leq r \leq \overline{\pi}$, so there is a risk that the return from buying the stock is not as good as holding the bank account, and the investor would prefer not to invest in the stock. On the other hand, the best estimation of the drift $\overline{\pi}$ is still better than the interest rate $r$, so implementing a shortsale strategy may incur a potential loss for the investor. This refrains her from short selling the stock.

(v) Shortsate strategy. When $\beta^3 \leq 0$, the investor will short sell her stock position as much as possible, which is $\max\{\beta^3, \overline{\mu}\}$ units of her wealth in this situation. Consequently, she keeps $(1 - \max\{\beta^3, \overline{\mu}\})$ units of her wealth in the bank account in order to earn the interest rate $r$.

We can further illustrate the above five optimal portfolio strategies via the following figure, where the horizontal axes represent the values of $\beta^1$, $\beta^2$ and $\beta^3$ from the top to the bottom, and the vertical axis represents the optimal portfolio.
4.2 The optimal consumption

In this section, we compute the explicit solution to ODE (12), which in turn allows us to construct the optimal consumption of the maxmin problem (5) (c.f. (13)). Note that if $\pi = \widetilde{c}$, the optimal consumption strategy $c^*_t = \pi = \omega$, so we concentrate on the case $\pi > \widetilde{c}$ in the rest of this section.

Theorem 4.2. Let $T > 0$ be a large enough number. The optimal consumption $c^*_t = c^*(t)$, $t \in [0, T]$, is a deterministic process defined in (13), which is summarized in Table 2.

Table 2: the optimal consumption in the case of $\pi > \widetilde{c}$

| $\pi - \rho K$ | $(-\infty, (1 - p)c)$ | $(1 - p)c$ | $(1 - p)c (1 - p\pi)$ | $(1 - p\pi)$ | $(1 - p\pi, +\infty)$ |
|----------------|-----------------------|------------|------------------------|-------------|------------------------|
| $\lambda \frac{1}{(1 - p)} < \pi$ | $\omega_{123}^3 + \epsilon(t)^{123} + \nu_{12}^1 + \tau_{12}^1$ | $\epsilon(t)^{12} + \nu_{12}^1 + \tau_{12}^1$ | $\pi$ | $\pi$ | $\pi$ |
| $\lambda \frac{1}{(1 - p)} = \pi$ | $\omega_{123}^3 + \epsilon(t)^{123} + \nu_{12}^1 + \tau_{12}^1$ | $\epsilon(t)^{12} + \nu_{12}^1 + \tau_{12}^1$ | $\pi$ | $\pi$ | $\pi$ |
| $\lambda \frac{1}{(1 - p)} < \pi$ | $\omega_{123}^3 + \epsilon(t)^{123} + \nu_{12}^1 + \tau_{12}^1$ | $\epsilon(t)^{12} + \nu_{12}^1 + \tau_{12}^1$ | $\pi$ | $\pi$ | $\pi$ |
| $\lambda \frac{1}{(1 - p)} \leq \pi$ | $\omega_{123}^3 + \epsilon(t)^{123} + \nu_{12}^1 + \tau_{12}^1$ | $\epsilon(t)^{12} + \nu_{12}^1 + \tau_{12}^1$ | $\pi$ | $\pi$ | $\pi$ |

The constant $K$ in the above table corresponds to the future investment contributing factor in (12).

In the case of $\widetilde{c} = 0$, the results are similar to those in Table 2 except that $\omega_{123}^3 + \epsilon(t)^{123} + \nu_{12}^1 + \tau_{12}^1$ and $\omega_{123}^3 + \epsilon(t)^{123}$ are replaced by $\tilde{\epsilon}(t)\nu_{12}^1 + \tau_{12}^1$ and $\tilde{\epsilon}(t)$, respectively. Note that when $\widetilde{c} = 0$, since $\lambda > 0$, the last two rows about the optimal consumption are then irrelevant.
and has the explicit form

\[
K := f_2(x^*_a, x^*_u, x^*_v) = \begin{cases} 
R + \frac{(\mu - R)^2}{2(1-p)\sigma^2}, & \beta^1 \geq \pi; \\
R + \frac{(\mu - R)^2}{2(1-p)\sigma^2}, & 1 \leq \beta^1 \leq \pi; \\
\mu - \frac{1-\rho}{\sigma^2}, & \beta^1 \leq 1 \leq \beta^2; \\
r + \frac{(\mu - r)^2}{2(1-p)\sigma^2}, & 0 \leq \beta^2 \leq 1; \\
r, & \beta^2 \leq 0 \leq \beta^3; \\
r + \frac{\pi(\pi - r)}{2(1-p)\sigma^2}, & \pi \leq \beta^3 \leq 0; \\
r + \frac{\pi(\pi - r)}{2(1-p)\sigma^2} - \frac{1-\rho}{\sigma^2} \pi^2, & \beta^3 \leq \pi.
\end{cases}
\]

and the indicator function \(I^a_i\) represents the time period \([T_0, T]\) with \(T_0 = 0\) and \(T_4 = T\), where the explicit forms of different time periods are given in Appendix B.

**Proof.** See Appendix B. \(\square\)

Table 2 lists all the possible consumption patterns under different parameter conditions. For example, \(\mathcal{L}_0^{12} + \hat{c}(t)I^{23}_1 + \tau I^{13}_0\) in the first row and the first column (left-top corner) is the optimal consumption when the market parameters satisfy \(\xi < \bar{\xi} < \lambda^{1/(1-p)}\) and \(\rho - pK \in (-\infty, (1-p)\xi)\). More specifically, in the time interval \([0, T_{12}]\), the investor will consume at the minimum rate \(\xi\). Then the investor will consume at the optimal rate \(\hat{c}(t) = \lambda^{1/(1-p)} \exp(q(t)/(p - 1))\) in the time interval \([T_{12}, T_{12}]\), since in this case \(\xi < \hat{c}(t) < \bar{\xi}\). Finally, in the remaining time interval \([T_{12}, T]\), the investor will consume at the maximum rate \(\bar{\xi}\).

In contrast, in the right-bottom corner, we obtain a reversed consumption pattern when \(\lambda^{1/(1-p)} < \xi < \bar{\xi}\) and \(\rho - pK \in ((1-p)\bar{\xi}, +\infty)\). That is, the consumption will be decreasing from the maximum rate \(\bar{\xi}\) in \([0, T_{321}]\), to \(\hat{c}(t)\) in \([T_{321}, T_{322}\), and finally to the minimal rate \(\xi\) in \([T_{322}, T]\).

In the following, we give some intuitive explanations of different consumption patterns. From Lemma 8.11 and Theorem 8.2, we know that the optimal consumption \(c^*_t = c^*(t)\) achieves the maximum of the concave function \(f_1(q(t), \cdot)\) in the interval \([\xi, \bar{\xi}]\). Moreover, note that \(\hat{c}(t) = \lambda^{1/(1-p)} \exp(q(t)/(p - 1))\) is as in (13) is the maximum point of \(f_1(q(t), \cdot)\) on \(\mathbb{R}_+\). Hence, \(c^*_t = \hat{c}(t)\) if \(\xi < \hat{c}(t) < \bar{\xi}\). Otherwise, \(c^*_t\) will be either \(\xi\) or \(\bar{\xi}\).

From the proof of Proposition 1.3 below, we know \(q(t)\) is monotone in time \(t\), so is \(\hat{c}(t)\). As a result, whether \(\hat{c}(t)\) stays in \([\xi, \bar{\xi}]\) or not only depends on its values at the two end points \(\hat{c}(T)\) and \(\hat{c}(0)\), and their relationship with \(\xi\) and \(\bar{\xi}\).

In fact, it follows from \(q(T) = 0\) that \(\hat{c}(T) = 1/\lambda^{1-p}\). By the continuity of \(\hat{c}(t)\), when \(t\) approaches maturity \(T\), \(c^*(t)\) will reach its upper bound \(\bar{\xi}\) if \(\xi < \bar{\xi} < \lambda^{1/(1-p)}\); \(c^*(t)\) will be precisely \(\hat{c}(t)\) if \(\xi < \lambda^{1/(1-p)} < \bar{\xi}\); \(c^*(t)\) will reach its lower bound \(\xi\) if \(\lambda^{1/(1-p)} < \xi < \bar{\xi}\). The above three situations thus determine the classification of the rows in Table 2.

On the other hand, we have the following asymptotic results for \(\lim_{T \to +\infty} \hat{c}(0)\) in Table 3 (see also Appendix B, in particular 8.11-8.14). By the continuity of \(\hat{c}(t)\), when \(T\) is large enough and \(t\) is near initial time 0, \(c^*(t) = \xi\) if \(\rho - pK \in (-\infty, (1-p)\xi)\); \(c^*(t) = \hat{c}\) if \(\rho - pK \in ((1-p)\xi, (1-p)\bar{\xi})\);
and \( c^*_t = \overline{\gamma} \) if \( \rho - pK \in ((1-p)\overline{\gamma}, +\infty) \). Consequently, the above three situations divide the columns in Table 2.

Table 3: the limit of \( \hat{c}(0) \) when \( T \to +\infty \)

| \( \rho - pK \)       | \(( -\infty, (1-p)\overline{\gamma} )\) | \(( (1-p)\overline{\gamma}, (1-p)\overline{\tau} )\) | \(( (1-p)\overline{\tau}, (1-p)\overline{\tau}, +\infty)\) |
|----------------------|-----------------------------------------|------------------------------------------|-----------------------------------------------|
| \( \lim_{T\to+\infty} \hat{c}(0) \) | \( < \overline{\gamma} \)             | \( = \overline{\gamma} \)               | \( \in (\overline{\gamma}, \overline{\tau}) \) | \( = \overline{\tau} \)                   | \( > \overline{\tau} \)                   |

Next, we further show that the optimal consumption admits some time monotone properties. As opposed to the unconstrained consumption case, the consumption constraints may force the optimal consumption to be either nonincreasing or nondecreasing no matter the value of \((\rho - pK)\).

**Proposition 4.3.** The optimal consumption \( c^*_t, t \in [0,T], \) has the following monotone properties in time \( t \), as specified in Table 4. The symbols \( \nearrow, \searrow \) and \( \perp \) represent nonincreasing, nonincreasing and independent of time \( t \), respectively.

Table 4: the optimal consumption in time

| \( \lambda^{1/(1-p)} - \overline{\gamma} \leq \gamma \leq \lambda^{1/(1-p)} \) | \( \nearrow \) |
| \( \lambda^{1/(1-p)} - \overline{\gamma} \leq \gamma \leq \lambda^{1/(1-p)} \) | \( \searrow \) |
| \( \lambda^{1/(1-p)} - \overline{\gamma} \leq \gamma \leq \lambda^{1/(1-p)} \) | \( \perp \) |

**Proof.** It follows from the expressions of \( c^*(t) \) and \( \hat{c}(t) \) in (13) that if \( q(t) \) is nonincreasing, then \( c^*(t) \) is nondecreasing; if \( q(t) \) is nondecreasing, then \( c^*(t) \) is nonincreasing. On the other hand, the expression (11) and ODE (12) lead to

\[
q''(t) = -p\bar{f}_1(q(t), c^*(t))q'(t),
\]

where

\[
\bar{f}_1(q(t), c^*(t)) = \begin{cases} 
-\frac{p}{\lambda^{1/(1-p)}}e^{-q(t)} < 0, & \text{if } q(t) < (p-1)\ln\overline{\gamma} + \ln \lambda; \\
-\frac{\lambda^{1/(1-p)}}{p}(q(t)/(p-1)) < 0, & \text{if } (p-1)\ln\overline{\gamma} + \ln \lambda \leq q(t) \leq (p-1)\ln\overline{\gamma} + \ln \lambda; \\
-\frac{p}{\lambda^{1/(1-p)}}e^{-q(t)} < 0, & \text{if } q(t) > (p-1)\ln\overline{\gamma} + \ln \lambda.
\end{cases}
\]

We claim that the sign of \( q'(t) \) does not change for \( t \in [0,T] \). Otherwise, suppose there exist \( 0 \leq t_1 < t_2 \leq T \) such that \( q'(t_1) > 0 \) and \( q'(t_2) < 0 \). By the continuity of \( q'(t) \), there exists \( t \in (t_1, t_2) \) such that \( q'(t) = 0 \). Now let \( t_3 := \inf \{ t > t_1 : q'(t) = 0 \} \). It follows that \( t_3 \in (t_1,t_2) \), \( q'(t_3) = 0 \), and \( q'(t) > 0 \) for \( t \in [t_1, t_3] \). By the Mean Value Theorem, there exits \( t_4 \in (t_1, t_3) \) such that \( q''(t_4) = \frac{q(t_2) - q(t_1)}{t_2 - t_1} < 0 \). However, \( q''(t) > 0 \) for \( t \in [t_1, t_3] \) according to (17). This is a contradiction.

We have shown that \( q(t) \) is either nonincreasing or nondecreasing for \( t \in [0,T] \). Thus, it suffices to consider the sign of \( q'(T) \).

Let us first consider the case \( \overline{\gamma} < \gamma < \lambda^{1/(1-p)} \). For this case, we have \( (p-1)\ln\overline{\gamma} + \ln \lambda > 0 = q(T) \), and therefore, (11) implies that ODE (12) at \( t = T \) reduces to

\[
q'(T) = -((\lambda^{1/(1-p)} - p\overline{\gamma}) - pK + \rho,
\]
where the constant $K$ is given in (16). However, Theorem 4.2 implies that $c^*(t) \equiv \tau$ if $\rho - pK \geq (1-p)\tau$ in this case, so we only need to consider the situation $\rho - pK < (1-p)\tau$ for the monotone property of $c^*(t)$. Together with $\tau < \lambda^{1/(1-p)}$, we further obtain that

$$q'(T) < -(\tau^{1-p}c^\tau - p\tau) + (1-p)\tau = 0.$$  

In turn, $q'(t) \leq 0$ for $t \in [0, T]$, which implies that $c^*(t)$ is nondecreasing for $t \in [0, T]$.

The other two cases $\zeta \leq \lambda^{1/(1-p)} \leq \tau$ and $\lambda^{1/(1-p)} < \zeta < \tau$ can be treated in a similar way, so their proofs are omitted. 

5 The impacts of model uncertainty, portfolio-consumption constraints and borrowing costs

In this section, we investigate the impacts of model uncertainty, portfolio-consumption constraints and borrowing costs on the worst-case parameters ($\mu^*, \sigma^*$) and the optimal portfolio-consumption strategy ($\pi^*, c^*$).

**Proposition 5.1.** The worst-case parameters and the optimal portfolio-consumption strategy admit the following monotone properties in terms of the borrowing rate $R$, the constraint set $[\bar{\pi}, \bar{\pi}] \times [\bar{\pi}, \bar{\pi}]$, and the uncertain parameter set $[\bar{\mu}, \bar{\pi}] \times [\bar{\sigma}, \bar{\sigma}]$, as specified in Table 5. The symbols $\downarrow$, $\uparrow$, $\perp$ and NM represent nonincreasing, nondecreasing, independent and non-monotone of the corresponding variable. For example, the bottom row and the first column (left-bottom corner) means $c^*_s$ is nondecreasing in the borrowing rate $R$.

Table 5: the comparative statistics

|       | $R$ | $\bar{\pi}$ | $\bar{\pi}$ | $\zeta$ | $\tau$ | $\bar{\mu}$ | $\bar{\pi}$ | $\bar{\sigma}$ | $\tau$ |
|-------|-----|-------------|-------------|---------|--------|-------------|-------------|-------------|--------|
| $\mu^*$ | $\perp$ | $\perp$ | $\perp$ | $\downarrow$ | $\uparrow$ | $\perp$ | $\perp$ | $\downarrow$ | $\uparrow$ |
| $\sigma^*$ | $\perp$ | $\perp$ | $\perp$ | $\downarrow$ | $\uparrow$ | $\perp$ | $\perp$ | $\downarrow$ | $\uparrow$ |
| $\pi^*$ | $\downarrow$ | $\uparrow$ | $\downarrow$ | $\perp$ | $\downarrow$ | $\uparrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ |
| $c^*_s$ | $\uparrow$ | $\downarrow$ | $\uparrow$ | NM | $\downarrow$ | $\uparrow$ | $\downarrow$ | $\uparrow$ | $\downarrow$ |

Before proceeding to the proof, we provide some intuitive explanations for the above results. The impacts of different parameters on the worst-case parameters ($\mu^*, \sigma^*$) and the optimal portfolio $\pi^*$ are obvious from the results in Theorem 4.1. So we only discuss about their impacts on the optimal consumption $c^*_s$.

By the expression (13), the parameters $(R, \bar{\pi}, \bar{\pi}, \bar{\mu}, \bar{\pi}, \bar{\sigma}, \tau)$ will effect the optimal consumption through the channel of the opportunity process $e^{\varphi(s)}$, which is the investor’s extra utilities obtained by optimizing over all the admissible portfolio-consumption strategies (least affected by model uncertainty) in the remaining horizon $[s, T]$. A closer look at the ODE (12) for $q(s)$ tells us that those parameters will only enter into the future investment contributing factor $f_2(x^*_s; x^*_\mu, x^*_\sigma)$ in (12). Increasing the borrowing cost $R$ will make the future investment contributing factor $f_2(x^*_s; x^*_\mu, x^*_\sigma)$ smaller, so the opportunity process will also become smaller, i.e. the investor will
obtain less utilities in the remaining horizon. In turn, her current optimal consumption will go up. Similarly, enlarging the uncertainty parameters interval \([\underline{\mu}, \overline{\mu}] \times [\underline{\sigma}, \overline{\sigma}]\) or shrinking the portfolio constraint interval \([\underline{\pi}, \overline{\pi}]\) will also make the future investment contributing factor \(f_2(x_\pi^*; x_\mu^*, x_\sigma^*)\) smaller, and therefore, the current optimal consumption will arise.

The more surprising result is probably the impact of the consumption constraint interval \([\underline{\pi}, \overline{\pi}]\) on the optimal consumption \(c_\pi^*\). Note that the constraint interval will only effect the consumption contributing factor \(f_1(q(s), c^*(s))\) as indicated in \([12]\). In turn, the investor will obtain less utilities in the remaining horizon \([s, T]\). This might suggest the current optimal consumption would increase. However, it is not always the case, as there is less chance for the unconstrained optimal consumption \(\hat{c}(s)\) to stay in the shrinking interval \([\underline{\pi}, \overline{\pi}]\). If \(\hat{c}(s)\) reaches the lower bound \(\underline{\pi}\), then the optimal consumption will further arise as \(\underline{\pi}\) increases. On the other hand, if \(\hat{c}(s)\) reaches the upper bound \(\overline{\pi}\), then the optimal consumption will go down for \(\overline{\pi}\) becomes smaller, thus offsets the previous increasing impact on the optimal consumption when \(\overline{\pi}\) is decreasing. This means the optimal consumption is non-monotone in its upper bound \(\overline{\pi}\).

**Proof.** (i) The monotone property of \(\mu_\pi^*\). According to Theorem 4.1, the worst-case drift can be rewritten as

\[
\mu_\pi^* = \mu 1_{\{R = r\}} + \overline{R} 1_{\{\overline{R} < r\}} = \mu 1_{\{\mu > r\}} + \overline{\mu} 1_{\{\mu < r\}}
\]

for \(s \in [0, T]\). The first line implies that \(\mu_\pi^*\) is nondecreasing in \(\mu\), and the second line implies it is also nondecreasing in \(\overline{\pi}\), and is irrelevant to the other parameters \((R, \underline{\pi}, \overline{\pi}, \underline{\sigma}, \overline{\sigma}, \overline{\pi})\).

(ii) The monotone property of \(\sigma_\pi^*\). The conclusion simply follows from the expression of the worst-case volatility \(\sigma_\pi^* = \overline{\pi}\) for \(s \in [0, T]\), in Theorem 4.1.

(iii) The monotone property of \(\pi_\mu^*\). First, the expressions of \(\beta_1, \beta_2, \beta_3\) in Theorem 4.1 imply that they are all nondecreasing in \(\mu, \overline{\pi}\), nonincreasing in \(R, \overline{\pi}\) and independent of \(\underline{\sigma}, \overline{\sigma}, \overline{\pi}\), so is the optimal portfolio \(\pi_\mu^*\), as \(\pi_\mu^*\) is nondecreasing with respect to \(\beta_1, \beta_2, \beta_3\) (c.f. Figure 4.1).

From Table 1 in Theorem 4.1, we further obtain

\[
\pi_\mu^* = \min\{\beta_1, \overline{\pi}\} 1_{\{\beta_1 \geq 1\}} + C_1 1_{\{\beta_1 < 1\}} = \max\{\beta_1, \overline{\pi}\} 1_{\{\beta_1 \leq 0\}} + C_2 1_{\{\beta_1 > 0\}}
\]

for some constants \(C_1\) independent of \(\overline{\pi}\), and \(C_2\) independent of \(\overline{\pi}\). Hence, \(\pi_\mu^*\) is nondecreasing in both \(\overline{\pi}\) and \(\overline{\pi}\).

(iv) The monotone property of \(c_\mu^*\). We first study the impacts of different parameters on the solution \(q(t)\) of ODE \([12]\). Note that \((R, \underline{\pi}, \overline{\pi}, \mu, \overline{\mu}, \underline{\sigma}, \overline{\sigma})\) will effect \(q(t)\) only through \(f_2(x_\pi^*; x_\mu^*, x_\sigma^*) = K\), where \(K\) is given in \([16]\).

It is obvious from the expression \([10]\) that \(K\) is nonincreasing in \(R\). Moreover, since \(K\) is the maximum value of \(f_2(x_\pi; x_\mu, x_\sigma)\) over \(x_\pi \in [\underline{\pi}, \overline{\pi}]\), \(K\) is nonincreasing in \(\overline{\pi}\) and nondecreasing in \(\overline{\pi}\). On the other hand, \(K\) is also the minimum value of \(f_2(x_\pi^*; x_\mu, x_\sigma)\) over \((x_\mu, x_\sigma) \in [\underline{\mu}, \overline{\mu}] \times [\underline{\sigma}, \overline{\sigma}]\).
Therefore, $K$ is nondecreasing in $\mu, \sigma$ and nonincreasing in $\pi, \sigma$. However, the expression of $K$ further implies that $K$ is independent of $\sigma$.

It then follows from the comparison theorem for ODE (12) that its solution $q(s)$ is nonincreasing in $R, \pi, \sigma$, nondecreasing in $\pi$ and $\mu$, and independent of $\sigma$. The conclusion about the optimal consumption $c_s^*$ then follows from Theorem 5.2 together with the expression (13).

In terms of the impacts of $\mu$ and $\sigma$, since $f_1(q(s), c^*(s))$ is the maximum value of $f_1(q(s), x_c)$ over $x_c \in [\underline{c}, \overline{c}]$, it is nonincreasing in $\mu$ and nondecreasing in $\sigma$. Following the comparison theorem for ODE (12) and the expression (13) once again, we conclude $c(s)$ is nondecreasing in $\mu$ and nonincreasing in $\sigma$.

In turn, the expression of $c^*(s)$ in (13) implies that the optimal consumption $c_s^*$ is also nondecreasing in $\mu$, but neither increasing nor decreasing in $\sigma$, for the second and last terms in $c^*(s)$ offset the effects of each other. Indeed, we show the non-monotonicity in the case of $0 \leq \underline{c} \leq \sigma_2 < \sigma_1 < \lambda^{1/(1-p)}$ and $\rho - pK \in ((1-p)\underline{c}, (1-p)\sigma_2)$. According to Theorem 4.2 both $c_1^*(t)$ and $c_2^*(t)$ take the form of $\hat{c}(t)(l_1^2+\pi I_{t_2}^2)$. When $t$ is close to $T$, then $c_1^*(t) = \sigma_1 > \sigma_2 = c_2^*(t)$.

On the other hand, when $T$ is large enough and $t$ is close to zero, we have

$$c_1^*(t) = \hat{c}_1(t) = \exp \left\{ \frac{q_1(t)}{p-1} \right\} < \exp \left\{ \frac{q_2(t)}{p-1} \right\} = \hat{c}_2(t) = c_2^*(t),$$

where the strict inequality can be derived from the comparison theorem for ODE.

### A Appendix: Proof of Theorem 4.1

**Proof of Theorem 4.1** According to Theorem 3.2 if $\{x_{\pi}^*, (x_{\mu}^*, x_{\sigma}^*)\}$ is a saddle point of the function $f_2(\cdot, \cdot, \cdot)$, then $x_{\pi}^*$ is the optimal portfolio, and $(x_{\mu}^*, x_{\sigma}^*)$ are the worst-case parameters. Thus, it is sufficient to show that $\{\pi^*; (\mu^*, \sigma^*)\}$ given in Theorem 4.1 is indeed a saddle point of the function $f_2(\cdot, \cdot, \cdot)$, which is also the conclusion (ii) in Lemma 3.3.

First, for fixed $x_{\pi} \in [\pi, \overline{\pi}]$, it is obvious to check that with

$$f_2(x_{\pi}; x_{\mu}, x_{\sigma}) = \frac{p-1}{2} x_{\pi}^2 + (x_{\mu} x_{\pi} + r(1-x_{\pi}) - (R-r)(1-x_{\pi}^{-})),
$$

we have

$$\min_{(x_{\mu}, x_{\pi}) \in [\underline{\mu}, \overline{\mu}] \times [\underline{\pi}, \overline{\pi}]} f_2(x_{\pi}; x_{\mu}, x_{\sigma}) = \begin{cases} f_2(x_{\pi}; \underline{\mu}, \overline{\sigma}), & \text{if } x_{\pi} > 0; \\ f_2(x_{\pi}; \overline{\mu}, \overline{\pi}), & \text{if } x_{\pi} = 0; \\ f_2(x_{\pi}; \overline{\mu}, \overline{\sigma}), & \text{if } x_{\pi} < 0, \end{cases} \tag{18}$$

where $[\underline{\mu}, \overline{\mu}]$ means that $x_{\mu}^*$ may take any value in that interval.

The above minimum function can be further written in a compact form by defining

$$f_3(x_{\pi}) := f_2(x_{\pi}; x_{\mu}^*, x_{\sigma}^*)$$

$$= \frac{p-1}{2} \overline{\sigma}^2 x_{\pi}^2 + (\underline{\mu}I_{\{x_{\pi} > 0\}} + \overline{\pi}I_{\{x_{\pi} < 0\}} - rI_{\{x_{\pi} < 1\}} - R I_{\{x_{\pi} \geq 1\}}) x_{\pi}$$

$$+ \left( rI_{\{x_{\pi} < 1\}} + R I_{\{x_{\pi} \geq 1\}} \right).$$

18
In the following, we study the maximum value of $f_3(x_\pi)$ in three different cases $x_\pi \geq 1$, $0 \leq x_\pi \leq 1$ and $x_\pi \leq 0$, then together with the constraint $\pi \leq x_\pi \leq \pi$, we will obtain the maximizer $x_\pi^*$ and the associated maximum value $f_3(x_\pi^*)$.

**Case (1) $x_\pi \geq 1$.**

$$f_3(x_\pi) = \frac{p-1}{2} \frac{1}{\sigma^2} \left[ x_\pi + \frac{\mu - R}{(p-1)\sigma^2} \right]^2 + R - \frac{(\mu - R)^2}{2(p-1)\sigma^2}.$$

If $\beta_1 = (\mu - R)/(1-p)\sigma^2 \geq \pi$, then

$$\max_{1 \leq x_\pi \leq \pi} f_3(x_\pi) = f_3(\pi) \geq f_3(1).$$

If $1 < \beta_1 < \pi$, then

$$\max_{1 \leq x_\pi \leq \pi} f_3(x_\pi) = f_3(\beta_1) > f_3(1).$$

If $\beta_1 \leq 1$, then

$$\max_{1 \leq x_\pi \leq \pi} f_3(x_\pi) = f_3(1).$$

**Case (2) $0 \leq x_\pi \leq 1$.**

$$f_3(x_\pi) = \frac{p-1}{2} \frac{1}{\sigma^2} \left[ x_\pi + \frac{\mu - r}{(p-1)\sigma^2} \right]^2 + r - \frac{(\mu - r)^2}{2(p-1)\sigma^2}.$$

If $\beta_2 = (\mu - r)/(1-p)\sigma^2 \geq 1$, then

$$\max_{0 \leq x_\pi \leq 1} f_3(x_\pi) = f_3(1) > f_3(0).$$

If $0 < \beta_2 < 1$, then

$$\max_{0 \leq x_\pi \leq 1} f_3(x_\pi) = f_3(\beta_2) > \{ f_3(1), f_3(0) \}.$$

If $\beta_2 \leq 0$, then

$$\max_{0 \leq x_\pi \leq 1} f_3(x_\pi) = f_3(0) > f_3(1).$$

**Case (3) $x_\pi \leq 0$.**

$$f_3(x_\pi) = \frac{p-1}{2} \frac{1}{\sigma^2} \left[ x_\pi + \frac{\overline{\mu} - r}{(p-1)\sigma^2} \right]^2 + r - \frac{(\overline{\mu} - r)^2}{2(p-1)\sigma^2}.$$

If $\beta_3 = (\overline{\mu} - r)/(1-p)\sigma^2 \geq 0$, then

$$\max_{\pi \leq x_\pi \leq 0} f_3(x_\pi) = f_3(0).$$

If $\pi < \beta_3 < 0$, then

$$\max_{\pi \leq x_\pi \leq 0} f_3(x_\pi) = f_3(\beta_3) > f_3(0).$$

If $\beta_3 \leq \pi$, then

$$\max_{\pi \leq x_\pi \leq 0} f_3(x_\pi) = f_3(\pi) \geq f_3(0).$$
Comparing the maximum values in the above three cases, and noting that the fact \( \beta_1 \leq \beta_2 \leq \beta_3 \), we see that 
\[
\max_{x^* \in [\mu, \bar{\mu}]} f_3(x^*_\mu) = f_3(x^*_\lambda) = K, \text{ where } K \text{ is defined in } \text{(10)} \]
and the optimal \( x^*_\mu \) is defined in Table 1. Thus, we have proved
\[
f_2(x^*_\mu; x^*_\mu, x^*_\sigma) \geq f_2(x^*_\mu; x^*_\mu, x^*_\sigma), \quad \forall x^*_\mu \in [\mu, \bar{\mu}].
\]

On the other hand, with \( x^*_\mu \) as in Table 1, it follows from \text{(13)} that
\[
f_2(x^*_\mu; x^*_\mu, x^*_\sigma) \leq f_2(x^*_\mu; x^*_\mu, x^*_\sigma), \quad \forall (x^*_\mu, x^*_\sigma) \in [\mu, \bar{\mu}] \times [\sigma, \bar{\sigma}].
\]
Hence, \( \{x^*_\mu; (x^*_\mu, x^*_\sigma)\} \) is a saddle point of the function \( f_2(\cdot; \cdot, \cdot) \).

\[\Box\]

B Appendix: Proof of Theorem 4.2

First, we give the explicit solution to ODE \text{(12)} in Table 6 when \( \xi > 0 \).

| Table 6: The explicit solution to ODE \text{(12)} in the case of \( \xi > 0 \) |
| --- |
| \( (\rho - p)K \) | \( (-\infty, 1 - p\xi) \) | \( (1 - p\xi, (1 - p)\Xi) \) | \( ((1 - p)\Xi, (1 - p)\bar{\Xi}) \) | \( ((1 - p)\bar{\Xi}, \infty) \) |
| \( \xi \leq \bar{\Xi} < \lambda^{1/(1 - p)} \) | \( q_{123}(t) \) | \( q_{122}(t) \) | \( q_{121}(t) \) | \( q_{11}(t) \) |
| \( \xi < \bar{\Xi} = \lambda^{1/(1 - p)} \) | \( q_{23}(t) \) | \( q_{22}(t) \) | \( q_{21}(t) \) | \( q_{11}(t) \) |
| \( \xi < \lambda^{1/(1 - p)} < \bar{\Xi} \) | \( q_{23}(t) \) | \( q_{22}(t) \) | \( q_{21}(t) \) | \( q_{21}(t) \) |
| \( \lambda^{1/(1 - p)} \leq \xi < \bar{\Xi} \) | \( q_{3}(t) \) | \( q_{3}(t) \) | \( q_{3}(t) \) | \( q_{21}(t) \) |
| \( \lambda^{1/(1 - p)} < \xi < \bar{\Xi} \) | \( q_{3}(t) \) | \( q_{3}(t) \) | \( q_{3}(t) \) | \( q_{321}(t) \) |

The solutions \( q_{123}, q_{122}, q_{121}, q_{23}, q_{22}, q_{21}, q_{3}, q_{32}, q_{321} \) have the explicit forms

\[
q_{123}(t) = q^{1}(t; 1/\lambda, T_{12}, T)I_{[T_{12}, T]} + q^{2}(t; \Xi^{p-1}, T_{123}, T_{12})I_{[T_{123}, T_{12}]} + q^{3}(t; \Xi^{p-1}, 0, T_{123})I_{[0, T_{123}]);
\]

\[
q_{122}(t) = q^{1}(t; 1/\lambda, T_{12}, T)I_{[T_{12}, T]} + q^{2}(t; \Xi^{p-1}, 0, T_{12})I_{[0, T_{12}]};
\]

\[
q_{121}(t) = q^{1}(t; 1/\lambda, 0, T); \quad q_{22}(t) = q^{2}(t; 1/\lambda, 0, T); \quad q_{3}(t) = q^{3}(t; 1/\lambda, 0, T); \]

\[
q_{23}(t) = q^{2}(t; 1/\lambda, T_{23}, T)I_{[T_{23}, T]} + q^{3}(t; \Xi^{p-1}, 0, T_{23})I_{[0, T_{23}]};
\]

\[
q_{22}(t) = q^{2}(t; 1/\lambda, 0, 0, T_{23})I_{[0, T_{23}]};
\]

\[
q_{32}(t) = q^{3}(t; 1/\lambda, T_{32}, T)I_{[T_{32}, T]} + q^{2}(t; \Xi^{p-1}, 0, T_{32})I_{[0, T_{32}]};
\]

\[
q_{321}(t) = q^{3}(t; 1/\lambda, 0, 0, T_{32})I_{[0, T_{32}]};
\]

where \( I_{[\lambda, \lambda]} \) is an indicator function of the set \([\lambda, \lambda] \), and the functions \( q^{1}(t; A, [\lambda, \lambda]), q^{2}(t; A, [\lambda, \lambda]), q^{3}(t; A, [\lambda, \lambda]) \),

\[\text{In the case of } \xi = 0 \text{, the results are similar to those in Table 10 except that } q_{123} \text{ and } q_{23} \text{ are replaced by } q_{12} \text{ and } q_{22}, \text{ respectively. Note that in this case, the forth and fifth rows in Table 10 and } q_{123}, q_{23}, q_{32}, q_{321}, q^{3}, T_{123}, T_{23}, T_{32}, T_{321} \text{ are irrelevant.} \]
\( q^3(t; A, T, T) \) in the interval \([T, T]\) are given as

\[
q^1(t; A, T, T) = \ln \lambda + \left\{ \ln \left[ \left( A - \frac{1}{\rho + pK} \right) e^{(\rho + pK)(t-T)} + \frac{1}{\rho + pK} \right], \quad \rho - pK \neq -\rho \epsilon; \right. \\
\left. \ln \left[ A + \rho \epsilon (T - t) \right], \quad \rho - pK = -\rho \epsilon; \right. \tag{19}
\]

\[
q^2(t; A, T, T) = \ln \lambda + \left\{ (1-p) \ln \left[ \left( A^{1/(1-p)} - \frac{1-p}{\rho + pK} \right) e^{\frac{1-p}{\rho + pK}(t-T)} + \frac{1-p}{\rho + pK} \right], \quad \rho - pK \neq 0; \right. \\
\left. (1-p) \ln \left[ A^{1/(1-p)} + T - t \right], \quad \rho - pK = 0; \right. \tag{20}
\]

\[
q^3(t; A, T, T) = \ln \lambda + \left\{ \ln \left[ \left( A - \frac{1}{\rho + pK} \right) e^{(\rho + pK)(t-T)} + \frac{1}{\rho + pK} \right], \quad \rho - pK \neq -\rho \epsilon; \right. \\
\left. \ln \left[ A + \rho \epsilon (T - t) \right], \quad \rho - pK = -\rho \epsilon. \right. \tag{21}
\]

and \( T_{12}, T_{13}, T_{23}, T_{21}, T_{32}, T_{31} \) are given as

\[
T_{12} = \left\{ T + \frac{1}{\rho + pK} \left[ \ln \left( e^{p-1} - \frac{1}{\rho + pK} \right) - \ln \left( \frac{1}{\rho + pK} \right) \right], \quad \rho - pK \neq -\rho \epsilon; \right. \\
\left. T - \frac{1}{\epsilon} + 1/\left(\lambda \rho \epsilon\right), \quad \rho - pK = -\rho \epsilon; \right. \tag{22}
\]

\[
T_{13} = \left\{ T_{12} + \frac{1-p}{\rho + pK} \left[ \ln \left( \frac{1}{\rho + pK} \right) - \ln \left( \frac{1}{\rho + pK} \right) \right], \quad \rho - pK \neq 0; \right. \\
\left. T_{12} + 1/\epsilon - 1/\lambda, \quad \rho - pK = 0; \right. \tag{23}
\]

\[
T_{23} = \left\{ T + \frac{1-p}{\rho + pK} \left[ \ln \left( \frac{1}{\rho + pK} \right) - \ln \left( \frac{1}{\rho + pK} \right) \right], \quad \rho - pK \neq 0; \right. \\
\left. T + \frac{1}{\rho + pK} - 1/\lambda, \quad \rho - pK = 0; \right. \tag{24}
\]

\[
T_{21} = T + \frac{1-p}{\rho + pK} \left[ \ln \left( \frac{1}{\rho + pK} \right) - \ln \left( \frac{1}{\rho + pK} \right) \right]; \tag{25}
\]

\[
T_{32} = T + \frac{1}{\rho + pK} \left[ \ln \left( \frac{1}{\rho + pK} \right) - \ln \left( \frac{1}{\rho + pK} \right) \right]; \tag{26}
\]

\[
T_{31} = T_{32} + \frac{1-p}{\rho + pK} \left[ \ln \left( \frac{1}{\rho + pK} \right) - \ln \left( \frac{1}{\rho + pK} \right) \right]. \tag{27}
\]

It is routine to check that for any \( A > 0 \) and \( 0 \leq T \leq T \), \( q^1(t; A, T, T), q^2(t; A, T, T), q^3(t; A, T, T) \) solve the following ODEs, respectively,

\[
q(t) = \ln \lambda + \ln A + \int_t^T \left[ - \rho + \lambda \rho \epsilon e^{-\rho(s)} - pK + pK \right] ds, \quad \forall t \in [T, T]; \tag{28}
\]

\[
q(t) = \ln \lambda + \ln A + \int_t^T \left[ - \rho + (1-p) \lambda \frac{1}{\rho + pK} \exp \left( \frac{q(s)}{p-1} \right) + pK \right] ds, \quad \forall t \in [T, T]; \tag{29}
\]

\[
q(t) = \ln \lambda + \ln A + \int_t^T \left[ - \rho + \lambda e^{p} e^{-\rho(s)} - pK + pK \right] ds, \quad \forall t \in [T, T]. \tag{30}
\]

When \( \lambda > 0 \), \( q^1(0; A, 0, T), q^2(0; A, 0, T) \) and \( q^3(0; A, 0, T) \) have the following asymptotic prop-
erties,
\[
\lim_{T \to \infty} q^1(0; A, 0, T) = \left\{ \begin{array}{ll}
\ln \lambda + \ln \left( \frac{\lambda}{\rho + pK - p\bar{c}} \right), & \rho - pK > -p\bar{c}; \\
+\infty, & \rho - pK \leq -p\bar{c};
\end{array} \right.
\]
\[
\lim_{T \to \infty} q^1(0; A, 0, T) \leq (p - 1) \ln \sigma + \ln \lambda \iff \rho - pK \geq (1 - p)\bar{c};
\]
\[
\lim_{T \to \infty} q^2(0; A, 0, T) = \left\{ \begin{array}{ll}
\ln \lambda + (1 - p) \ln \left( \frac{1 - p}{\rho - pK} \right), & \rho - pK > 0; \\
+\infty, & \rho - pK \leq 0;
\end{array} \right.
\]
\[
\lim_{T \to \infty} q^2(0; A, 0, T) \geq (p - 1) \ln \sigma + \ln \lambda \iff \rho - pK \leq (1 - p)\bar{c};
\]
\[
\lim_{T \to \infty} q^3(0; A, 0, T) = \left\{ \begin{array}{ll}
\ln \lambda + \ln \left( \frac{\lambda}{\rho + p\bar{c} - pK} \right), & \rho - pK > -p\bar{c}; \\
+\infty, & \rho - pK \leq -p\bar{c};
\end{array} \right.
\]
\[
\lim_{T \to \infty} q^3(0; A, 0, T) \geq (p - 1) \ln \bar{c} + \ln \lambda \iff \rho - pK \leq (1 - p)\bar{c}.
\]

**Proof of Theorem 4.2**

*Case (1) 0 ≤ \( \xi \) < \( \xi \) ≤ \( \lambda \)^{(1 - p)}.*

In this case, \( (p - 1) \ln \sigma + \ln \lambda > 0 \). Since \( q(T) = 0 \), then when \( t \) is close to \( T \), \( q(t) < (p - 1) \ln \sigma + \ln \lambda \) and \( c^*(t) = \sigma \). Thus \( q(t) \) satisfies ODE (29) with \( A = 1/\lambda \) and \( \bar{T} = T \) in the interval \([T, T]\), until \( T = 0 \) or \( q(T) = (p - 1) \ln \sigma + \ln \lambda \).

**1.1** If \( \rho - pK < (1 - p)\bar{c} \neq 0 \), solving ODE (29), we obtain \( q(t) = q^1(t; 1/\lambda, T, T) \) taking the form of (19) in the interval \([T, T]\). According to (31), \( q^1(0; 1/\lambda, 0, T) > (p - 1) \ln \sigma + \ln \lambda \) provided \( T \) is large enough. Thus, there exists a positive constant \( T_{12} \) such that \( q^1(T_{12}; 1/\lambda, T_{12}, T) = (p - 1) \ln \sigma + \ln \lambda \), and \( T_{12} \) is given in (22). Hence, we derive \( q(t) = q^1(t; 1/\lambda, T_{12}, T) \), and \( c^*(t) = \sigma \) in the interval \([T_{12}, T]\).

Since
\[
q^1(T_{12}) = \rho - pf_1(q(T_{12}), c^*(T_{12})) - pf_2(x^*_\mu, x^*_\sigma) = \rho - (1 - p)\bar{c} - pK < 0,
\]
then when \( t < T_{12} \) and \( t \) is close to \( T_{12} \), \( (p - 1) \ln \sigma + \ln \lambda < q(t) < (p - 1) \ln \bar{c} + \ln \lambda \), and \( c^*(t) = \sigma \) taking the form of (13). Thus \( q(t) \) satisfies ODE (29) with \( A = \bar{\sigma}^{(-1)} \) and \( \bar{T} = T_{12} \) in the interval \([T, T_{12}]\), until \( T = 0 \) or \( q(T) = (p - 1) \ln \bar{c} + \ln \lambda \) (where we have used the fact that the sign of \( q'(t) \) does not change (c.f. Proposition 4.3), and \( q(t) > (p - 1) \ln \sigma + \ln \lambda \) for any \( t < T_{12} \).

Solving ODE (29), we obtain \( q(t) = q^2(t; \bar{\sigma}^{(-1)}, T_{12}, T) \) taking the form of (20) in the interval \([T, T_{12}]\). According to (32), \( q^2(0; \bar{\sigma}^{(-1)}, 0, T_{12}) > (p - 1) \ln \bar{c} + \ln \lambda \) provided \( T \) is large enough. Thus, there exists a positive constant \( T_{123} \) such that \( q^2(T_{123}; \bar{\sigma}^{(-1)}, T_{123}, T_{12}) = (p - 1) \ln \bar{c} + \ln \lambda \), and \( T_{123} \) is given in (23). Hence, we derive \( q(t) = q^2(t; \bar{\sigma}^{(-1)}, T_{123}, T_{12}) \), and \( c^*(t) = \sigma \) in the interval \([T_{123}, T_{12}]\).
Recalling the fact that the sign of $q'(t)$ does not change (c.f. Proposition 4.3), we deduce that $q(t) \geq (p - 1) \ln \xi + \ln \lambda$, $c^*(t) = \xi$, and $q(t)$ satisfies ODE (30) with $A = \xi^{p - 1}$ and $\mathcal{T} = T_{123}$ in the interval $[0, T_{123}]$. Solving ODE (30), we have $q(t) = q^2(t; \xi^{p - 1}, 0, T_{123})$ as in (21).

(1.2) If $p - pK < (1 - p)\xi = 0$ or $(1 - p)\xi \leq p - pK < (1 - p)p$, repeating the same argument as in Case (1.1), we have $q(t) = q^1(t; 1/\lambda, T_{123}, T)$ as in (19), and $c^* = \pi$ in the interval $[T_{123}, T]$, and $q(t) = q^2(t; \pi^{p - 1}, \mathcal{T}, T_{123})$ as in (20) until $\mathcal{T} = 0$ or $q(\mathcal{T}) = (p - 1) \ln \xi + \ln \lambda$.

In the case of $p - pK \leq (1 - p)\xi = 0$, $(p - 1) \ln \xi + \ln \lambda = +\infty > q^2(t; \pi^{p - 1}, 0, T_{123})$, and $\mathcal{T} = 0$. In the other case, since $p - pK > 0$ and $p - pK - (1 - p)p < 0$, then (20) implies that $q^2(t; \pi^{p - 1}, 0, T_{123}) < \ln \lambda + (1 - p) \ln \frac{1 - p}{\rho - pK} \leq \ln \lambda + (1 - p) \ln \frac{1}{\xi}$.

Thus, we deduce that $\mathcal{T} = 0$. Therefore, $q(t) = q^2(t; \pi^{p - 1}, 0, T_{123})$ and $c^*(t) = \tilde{c}(t)$ in the interval $[0, T_{123}]$.

(1.3) If $(1 - p)p \leq p - pK < \lambda^{p - p} - p\pi$, solving ODE (28), we have $q(t) = q^1(t; 1/\lambda, \mathcal{T}, T)$ as in (19) until $\mathcal{T} = 0$ or $q(\mathcal{T}) = (p - 1) \ln \pi + \ln \lambda$. Since $q^1(t; 1/\lambda, 0, T) < \ln \lambda + \ln \frac{\pi^{p - 1}}{\rho + p\pi - pK} \leq \ln \lambda + \ln \pi^{p - 1} = (p - 1) \ln \pi + \ln \lambda$, $\forall t \in [0, T]$.

Then $q(t) = q^2(t; 1/\lambda, 0, T) < \ln \lambda + \ln \frac{\pi^{p - 1}}{\rho + p\pi - pK}$ and $c^*(t) = \pi$ in the interval $[0, T]$.

(1.4) If $p - pK \geq \lambda^{p - p} - p\pi$, solving ODE (28), we derive that $q(t) = q^1(t; 1/\lambda, \mathcal{T}, T)$ until $\mathcal{T} = 0$ or $q(\mathcal{T}) = (p - 1) \ln \pi + \ln \lambda$.

Since $p + p\pi - pK \geq 0$ and $p + p\pi - pK - \lambda^{p - p} \geq 0$, then $q^1(t; 1/\lambda, 0, T)$ is nondecreasing with respect to $t$, thus for $t \in [0, T]$, $q^1(t; 1/\lambda, 0, T) \leq q^1(T; 1/\lambda, 0, T) = (p - 1) \ln \pi + \ln \lambda$. Hence, $q(t) = q^1(t; 1/\lambda, 0, T)$ and $c^*(t) = \pi$ in the interval $[0, T]$.

Case (2) $0 \leq \xi < \pi = \lambda^{1/(1-p)}$. In this case, note that $(p - 1) \ln \pi + \ln \lambda = 0$.

(2.1) If $p - pK < (1 - p)\xi \neq 0$, since $q'(T - 0) = p - pf_1(q(T), c^*(T)) - pf_2(x_\pi^*, x_\pi^*, x_\sigma^*) = p - (1 - p)\pi - pK \leq p - pK - (1 - p)\xi < 0$, then when $t$ is close to $T$, $q(t) \geq (p - 1) \ln \pi + \ln \lambda$, $q(t) < (p - 1) \ln \xi + \ln \lambda$, and $c^*(t) = \tilde{c}(t)$.

Thus $q(t)$ satisfies ODE (24) with $A = 1/\lambda$ and $\mathcal{T} = T$ in the interval $[\mathcal{T}, T]$, until $\mathcal{T} = 0$ or $q(\mathcal{T}) = (p - 1) \ln \pi + \ln \lambda$.

Solving ODE (24), we obtain $q(t) = q^2(t; 1/\lambda, \mathcal{T}, T)$ as in (20) in the interval $[\mathcal{T}, T]$. According to (33), $q^2(0; 1/\lambda, 0, T) > (p - 1) \ln \xi + \ln \lambda$ provided that $T$ is large enough. Thus, there exists a positive constant $T_{23}$ such that $q^2(T_{23}; 1/\lambda, T_{23}, T) = (p - 1) \ln \xi + \ln \lambda$, and $T_{23}$ is given in (24).

Hence, we derive that $q(t) = q^2(t; 1/\lambda, T_{23}, T)$, and $c^*(t) = \tilde{c}(t)$ in the interval $[T_{23}, T]$.

Recalling the fact that the sign of $q'(t)$ does not change, we deduce that in the interval $[0, T_{23}]$, $q(t) \geq q(T_{23}) = (p - 1) \ln \xi + \ln \lambda$, $c^*(t) = \xi$, and $q(t)$ satisfies ODE (30) with $A = \xi^{p - 1}$ and $\mathcal{T} = T_{23}$.

Solving ODE (30), we have $q(t) = q^1(t; \xi^{p - 1}, 0, T_{23})$ as in (21) in the interval $[0, T_{23}]$.

(2.2) If $p - pK < (1 - p)\xi = 0$ or $(1 - p)\xi \leq p - pK < (1 - p)\pi$, since $q'(T - 0) = p - pf_1(q(T), c^*(T)) - pf_2(x_\pi^*, x_\pi^*, x_\sigma^*) = p - (1 - p)\pi - pK < 0$
still holds, then repeating the similar argument as in Case (2.1), we deduce that \( c^*(t) = \tilde{c}(t) \) and \( q(t) = q^2(t; 1/\lambda, T, T) \) in the interval \([T, T]\), until \( T \to 0 \) or \( q(T) = (p - 1)\ln \xi + \ln \lambda \).

In the case of \( \rho - pK \leq (1 - p)\xi = 0 \), \((p - 1)\ln \xi + \ln \lambda = +\infty > q^2(t; 1/\lambda, 0, T)\), and \( T \to 0 \). In the other case, since \( \rho - pK > 0 \) and \((p - pK) - (1 - p)\lambda^{1/(1-p)} = (p - pK) - (1 - p)\overline{c} < 0\), then for any \( t \in [0, T] \), we still have

\[
q^2(t; 1/\lambda, 0, T) < \ln \lambda + (1 - p)\ln \frac{1-p}{\rho - pK} \leq \ln \lambda + (1 - p)\ln \frac{1}{\xi} = (p - 1)\ln \xi + \ln \lambda.
\]

Therefore, \( q(t) = q^2(t; 1/\lambda, 0, T) \) and \( c^*(t) = \tilde{c}(t) \) in the interval \([0, T]\).

(2.3) If \( \rho - pK \geq (1 - p)\overline{c} \). We first discuss the case when \( \rho - pK > (1 - p)\overline{c} \). Combining the following calculation

\[
q'(T - 0) = \rho - pf_1(q(T), c^*(T)) - pf_2(x^*_\mu; x^*_\nu, x^*_\sigma) = \rho - (1 - p)\overline{c} - pK > 0,
\]

and the fact that the sign of \( q'(t) \) does not change, we drive that \( q(t) < q(T) = 0 = (p - 1)\ln \overline{c} + \ln \lambda, c^*(t) = \overline{c}, \) and \( q(t) \) satisfies ODE (28) with \( A = 1/\lambda \) and \( T = T \) in the interval \([0, T]\). Solving ODE (28), we have \( q(t) = q^1(t; 1/\lambda, 0, T) \) and \( c^*(t) = \overline{c} \) in the interval \([0, T]\).

On the other hand, if \( \rho - pK = (1 - p)\overline{c} \), then \( \rho - pK = \lambda \overline{c} - pK \), and for \( t \in [0, T] \), we have \( q(t) = 0 \), thus still have \( q(t) = q^1(t; 1/\lambda, 0, T) \).

Case (3) \( 0 \leq \xi < \lambda^{1/(1-p)} < \overline{c} \).

In this case, note that \((p - 1)\ln \overline{c} + \ln \lambda < 0 < (p - 1)\ln \xi + \ln \lambda \). Since \( q(T) = 0 \), then when \( t \) is close to \( T \), \((p - 1)\ln \overline{c} + \ln \lambda < q(t) < (p - 1)\ln \xi + \ln \lambda, c^*(t) = \tilde{c}(t) \) and \( q(t) \) satisfies ODE (29) with \( A = 1/\lambda \) and \( T = T \) in the interval \([T, T]\), until \( T \to 0 \) or \( q(T) = (p - 1)\ln \overline{c} + \ln \lambda \) or \( q(T) = (p - 1)\ln \xi + \ln \lambda \).

(3.1) If \( \rho - pK < (1 - p)\xi \neq 0 \), solving ODE (29), we have \( q(t) = q^2(t; 1/\lambda, T, T) \) and \( c^*(t) = \tilde{c}(t) \) in the interval \([T, T]\).

Since

\[
q'(T - 0) = \rho - pf_1(q(T), c^*(T)) - pf_2(x^*_\mu; x^*_\nu, x^*_\sigma) = \rho - (1 - p)\lambda^{1/(1-p)} - pK < \rho - (1 - p)\xi - pK < 0,
\]

then we deduce \( q(t) \) is nonincreasing with respect to \( t \) from the fact that the sign of \( q'(t) \) does not change. Hence, we have \( q(t) > (p - 1)\ln \xi + \ln \lambda \) for any \( t \in [0, T] \). Moreover, (33) implies that \( q^2(0; 1/\lambda, 0, T) > (p - 1)\ln \xi + \ln \lambda \) provided that \( T \) is large enough. Thus, there exists a positive constant \( T_{23} \) such that \( q^2(T_{23}; 1/\lambda, T_{23}, T) = (p - 1)\ln \xi + \ln \lambda \), and \( T_{23} \) is given in (23). Hence, we derive that \( q(t) = q^2(t; 1/\lambda, T_{23}, T) \), and \( c^*(t) = \tilde{c}(t) \) in the interval \([T_{23}, T]\).

Since

\[
q'(T_{23}) = \rho - pf_1(q(T_{23}), c^*(T_{23})) - pf_2(x^*_\mu; x^*_\nu, x^*_\sigma) = \rho - (1 - p)\xi - pK < 0,
\]

then for any \( t \in [0, T_{23}] \), we have \( q(t) > (p - 1)\ln \xi + \ln \lambda \), and \( q(t) \) satisfies ODE (30) with \( A = \xi^{p-1} \) and \( T = T_{23} \) in the interval \([0, T_{23}] \). Solving ODE (30), we obtain \( q(t) = q^3(t; \xi^{p-1}, 0, T_{23}) \) and \( c^*(t) = \xi \) in the interval \([0, T_{23}] \).
(3.2) If \( \rho - pK < (1 - p)\xi = 0 \) or \( (1 - p)\xi \leq \rho - pK < (1 - p)\lambda^{1/(1-p)} \), repeating the similar argument as in case (3.1), we deduce that \( q(t) = q^2(t; 1/\lambda, T, T) \), \( c^*(t) = \hat{c}(t) \) in the interval \([T, T]\), until \( T = 0 \) or \( q(T) = (p-1)\ln\xi + \ln\lambda \).

For the case of \( \rho - pK \leq (1 - p)\xi = 0 \), \( (p-1)\ln\xi + \ln\lambda = +\infty > q^2(t; 1/\lambda, 0, T) \), and \( T = 0 \). For the other case, since \( \rho - pK > 0 \) and \( \rho - pK - (1 - p)\lambda^{1/(1-p)} < 0 \), then (35) still holds. Therefore, \( q(t) = q^2(t; 1/\lambda, 0, T) \) and \( c^*(t) = \hat{c}(t) \) in the interval \([0, T]\).

(3.3) If \( (1 - p)\lambda^{1-p} \leq \rho - pK \leq (1 - p)\tau \), solving ODE (29), we have \( q(t) = q^2(t; 1/\lambda, T, T) \) and \( c^*(t) = \hat{c}(t) \) in the interval \([T, T]\). Since \( \rho - pK > 0 \) and \( \rho - pK - (1 - p)\lambda^{1-p} \geq 0 \), then \( q^2(t; 1/\lambda, 0, T) \) is nondecreasing and for \( t \in [0, T] \), we have

\[
(p-1)\ln\xi + \ln\lambda > q^2(T; 1/\lambda, 0, T) \geq q^2(t; 1/\lambda, 0, T) \geq (p-1)\ln\left(\frac{1-p}{\rho-pK}\right) + \ln\lambda \]

\[
\geq (p-1)\ln\frac{1}{\tau} + \ln\lambda = (p-1)\ln\tau + \ln\lambda.
\]

Therefore, \( q(t) = q^2(t; 1/\lambda, 0, T) \) and \( c^*(t) = \hat{c}(t) \) in the interval \([0, T]\).

(3.4) If \( \rho - pK > (1 - p)\tau \), solving ODE (29), we have \( q(t) = q^2(t; 1/\lambda, T, T) \) and \( c^*(t) = \hat{c}(t) \) in the interval \([T, T]\).

Since

\[
q'(T-0) = \rho - p f_1(q(T), c^*(T)) - p f_2(x^*_\mu; x^*_\sigma, x^*_\rho) = \rho - (1-p)\lambda^{1/(1-p)} - pK > \rho - (1-p)\tau - pK > 0,
\]

then we deduce \( q(t) \) is nondecreasing with respect to \( t \) from the fact that the sign of \( q'(t) \) does not change. Hence, we have \( q(t) < (p-1)\ln\xi + \ln\lambda \) for any \( t \in [0, T] \). Moreover, (32) implies that \( q^2(0; 1/\lambda, 0, T) < (p-1)\ln\tau + \ln\lambda \) provided that \( T \) is large enough. Thus, there exists a positive constant \( T_{21} \) such that \( q^2(T_{21}; 1/\lambda, T_{21}, T) = (p-1)\ln\tau + \ln\lambda \), and \( T_{21} \) is given in (26). Hence, we derive that \( q(t) = q^2(t; 1/\lambda, T_{21}, T) \), and \( c^*(t) = \hat{c}(t) \) in the interval \([T_{21}, T]\).

Since

\[
q'(T_{21}) = \rho - p f_1(q(T_{21}), c^*(T_{21})) - p f_2(x^*_\mu; x^*_\rho, x^*_\sigma) = \rho - (1-p)\tau - pK > 0,
\]

then for any \( t \in [0, T_{12}] \), we have \( q(t) < (p-1)\ln\tau + \ln\lambda \), and \( q(t) \) satisfies ODE (25) with \( A = \tau^{-1} \) and \( T = T_{12} \) in the interval \([0, T_{12}] \). Solving ODE (28), we obtain \( q(t) = q^1(t; \tau^{-1}, 0, T_{12}) \) and \( c^*(t) = \tau \) in the interval \([0, T_{12}] \).

Case (4) \( \lambda^{1/(1-p)} = \zeta < \tau \). In this case, note that \( (p-1)\ln\tau + \ln\lambda < 0 = (p-1)\ln\zeta + \ln\lambda \).

(4.1) If \( \rho - pK \leq (1 - p)\zeta \) we first consider the case where \( \rho - pK < (1 - p)\zeta \). Combining the following calculation

\[
q'(T-0) = \rho - p f_1(q(T), c^*(T)) - p f_2(x^*_\mu; x^*_\rho, x^*_\sigma) = \rho - (1-p)\zeta - pK < 0,
\]

and the fact that the sign of \( q'(t) \) does not change, we deduce that \( q(t) > 0 = (p-1)\ln\zeta + \ln\lambda, c^*(t) = \zeta \) and \( q(t) \) satisfies ODE (35) with \( A = 1/\lambda \) and \( T = T \) in the interval \([0, T] \). Solving ODE (30), we have \( q(t) = q^3(t; 1/\lambda, 0, T) \) and \( c^*(t) = \zeta \) in the interval \([0, T] \).

When \( \rho - pK = (1 - p)\zeta \), it is easy to see that for \( t \in [0, T] \), we have \( q(t) = 0 \), and we still have \( q(t) \) equal to \( q^3(t; 1, 0, T) \) and \( c^*(t) = \zeta \) in the interval \([0, T] \).
(4.2) If \((1 - p)\xi < \rho - pK \leq (1 - p)\overline{\sigma}\), since \(q(T) = 0 = (p - 1) \ln \xi + \ln \lambda\), and
\[
q'(T - 0) = \rho - pf_1(q(T), c^*(T)) - pf_2(x^*_\mu, x^*_\sigma) = \rho - (1 - p)\xi - pK > 0,
\]
then when \(t\) is close to \(T\), we have \((p - 1) \ln \xi + \ln \lambda < q(t) < (p - 1) \ln \xi + \ln \lambda\). Thus, \(q(t)\) satisfies ODE (29) with \(A = 1/\lambda\) and \(\overline{T} = T\) in the interval \([\xi, T]\), until \(\xi = 0\) or \(q(\xi) = (p - 1) \ln \xi + \ln \lambda\) or \(q(T) = (p - 1) \ln \xi + \ln \lambda\). Recalling the fact that the sign of \(q'(t)\) does not change, we deduce that \(q(t)\) is nondecreasing with respect to \(t\). Thus, it is impossible that \(q(T) = (p - 1) \ln \xi + \ln \lambda\) for some \(T \in [0, T]\).

Solving ODE (29), we have \(q(t) = q^2(t; 1/\lambda, \xi, T)\) and \(c^*(t) = \hat{c}(t)\) in the interval \([\xi, T]\). Since \((\rho - pK) - (1 - p)\lambda^{1/(1 - p)} = (\rho - pK) - (1 - p)\xi > 0\), then \(q^2(t; 1/\lambda, 0, T)\) is increasing with respect to \(t\), thus for \(t \in [0, T]\), we have
\[
(p - 1) \ln \xi + \ln \lambda = 0 = q^2(T; 1/\lambda, 0, T) > q^2(t; 1/\lambda, 0, T) > (p - 1) \ln \frac{1 - \rho}{\rho - pK} + \ln \lambda
\]
\[
\geq (p - 1) \ln \frac{1}{\xi} + \ln \lambda = (p - 1) \ln \xi + \ln \lambda.
\]
Therefore, \(q(t) = q^2(t; 1/\lambda, 0, T)\) and \(c^*(t) = \hat{c}(t)\) in the interval \([0, T]\).

(4.3) If \(\rho - pK > (1 - p)\xi\), repeating the similar argument as in case (4.2), we deduce that \(q(t) = q^2(t; 1/\lambda, \xi, T)\) and \(c^*(t) = \hat{c}(t)\) in the interval \([\xi, T]\), until \(\xi = 0\) or \(q(\xi) = (p - 1) \ln \xi + \ln \lambda\), and \(T_21\) is given in (26). Hence, we derive that \(q(t) = q^2(t; 1/\lambda, T_21, T)\), and \(c^*(t) = \hat{c}(t)\) in the interval \([T_21, T]\).

Combining
\[
q'(T_21) = \rho - pf_1(q(T_21), c^*(T_21)) - pf_2(x^*_\mu, x^*_\sigma) = \rho - pK - (1 - p)\xi > 0,
\]
and the fact that the sign of \(q'(t)\) does not change, we deduce that in the interval \([0, T_21]\), \(q(t) < (p - 1) \ln \xi + \ln \lambda\), \(c^*(t) = \xi\), and \(q(t)\) satisfies ODE (28) with \(A = \xi^{\mu - 1}\) and \(\overline{T} = T_21\). Solving ODE (28), we obtain \(q(t) = q^1(t; \xi^{\mu - 1}, 0, T_21)\) in the interval \([0, T_21]\).

Case (5) \(\lambda^{1/(1 - p)} < \xi < \overline{\sigma}\).

Since \(q(T) = 0 > (p - 1) \ln \xi + \ln \lambda\), then \(q(t)\) satisfies ODE (30) with \(A = 1/\lambda\) and \(\overline{T} = T\) in the interval \([\xi, T]\), until \(\xi = 0\) or \(q(\xi) = (p - 1) \ln \xi + \ln \lambda\). Solving ODE (30), we obtain \(q(t) = q^2(t; 1/\lambda, \overline{T}_21, T)\) and \(c^*(t) = \xi\) in the interval \([\xi, T]\).

(5.1) If \(\rho - pK < \lambda^{\mu - p} - p\xi\), then \(\rho + p\xi - pK - \lambda^{\mu - p} \leq 0\), and \(q^3(t; 1/\lambda, 0, T)\) is nonincreasing with respect to \(t\), thus for \(t \in [0, T]\), we have \(q^3(t; 1/\lambda, 0, T) \geq q^3(T; 1/\lambda, 0, T) = 0 > (p - 1) \ln \xi + \ln \lambda\).

Therefore, \(q(t) = q^3(t; 1/\lambda, 0, T)\) and \(c^*(t) = \xi\) in the interval \([0, T]\).

(5.2) If \(\lambda^{\mu - p} - p\xi - \rho - pK \leq (1 - p)\xi\), then \(\rho + p\xi - pK - \lambda^{\mu - p} > 0\), and
\[
q^3(t; 1/\lambda, 0, T) \geq \ln \lambda + \ln \frac{\xi^{\mu - p}}{\rho + p\xi - pK} \geq \ln \xi^{\mu - p} + \ln \lambda = (p - 1) \ln \xi + \ln \lambda, \quad \forall t \in [0, T].
\]

Therefore we still have \(q(t) = q^3(t; 1/\lambda, 0, T)\) and \(c^*(t) = \xi\) in the interval \([0, T]\).
(5.3) If $(1-p)_\xi < \rho - pK \leq (1-p)\overline{\xi}$, then (32) implies that $q^3(0; 1/\lambda, 0, T) < (p-1)\ln \xi + \ln \lambda$ provided that $T$ is large enough. Thus, there exists a positive constant $T_{32}$ such that $q^3(T_{32}; 1/\lambda, T_{32}, T) = (p-1)\ln \xi + \ln \lambda$, and $T_{32}$ is given in (26). Hence, we derive that $q(t) = q^3(t; 1/\lambda, T_{32}, T)$, and $c^*(t) = \xi$ in the interval $[T_{32}, T]$.

Since

$$q'(T_{32}) = \rho - pf_1(q(T_{32}), c^*(T_{32})) - pf_2(x^*_n; x^*_\mu, x^*_\sigma) = \rho - (1-p)\xi - pK > 0,$$

then when $t < T_{32}$ and $t$ is close to $T_{32}$, we have $q(t) < (p-1)\ln \xi + \ln \lambda$ and $q(t) > (p-1)\ln \overline{\tau} + \ln \lambda$, and $q(t)$ is nondecreasing with respect to $t$, and $q(t)$ satisfies ODE (29) with $A = \xi^{p-1}$ and $T = T_{32}$ in the interval $[\underline{T}, T_{32}]$, until $\underline{T} = 0$ or $q(T) = (p-1)\ln \overline{\tau} + \ln \lambda$. Solving ODE (29), we obtain $q(t) = q^2(t; \xi^{p-1}, T_{32})$ and $c^*(t) = \overline{\tau}(t)$ in the interval $[\underline{T}, T_{32}]$.

Since $\rho - pK - (1-p)\lambda^{1/(1-p)} > \rho - pK - (1-p)\xi > 0$, then in this case $q^2(t; \xi^{p-1}, 0, T_{32})$ is increasing with respect to $t$, thus for $t \in [0, T_{32})$, we have

$$(p-1)\ln \xi + \ln \lambda = q^2(T_{32}; \xi^{p-1}, 0, T_{32}) > q^2(t; \xi^{p-1}, 0, T_{32}) > \ln \lambda + (1-p)\ln \frac{1-p}{\rho - pK} \geq (p-1)\ln \frac{1-p}{\rho - pK} + \ln \lambda.$$

Therefore, $q(t) = q^2(t; \xi^{p-1}, 0, T_{32})$ and $c^*(t) = \overline{\tau}(t)$ in the interval $[0, T_{32}]$.

(5.4) If $\rho - pK > (1-p)\overline{\xi}$, repeating the similar argument as in case (5.3), we deduce that $q(t) = q^3(T_{32}; 1/\lambda, T_{32}, T)$, and $c^*(t) = \xi$ in the interval $[T_{32}, T]$, and $q(t) = q^2(t; \xi^{p-1}, T_{32})$ and $c^*(t) = \overline{\tau}(t)$ in the interval $[\underline{T}, T_{32}]$, until $\underline{T} = 0$ or $q(T) = (p-1)\ln \overline{\tau} + \ln \lambda$.

According to (32), $q^2(0; \xi^{p-1}, 0, T_{32}) < (p-1)\ln \overline{\tau} + \ln \lambda$ provided that $T$ is large enough. Thus, there exists a positive constant $T_{321}$ such that $q^2(T_{321}; \xi^{p-1}, T_{321}, T_{32}) = (p-1)\ln \overline{\tau} + \ln \lambda$, and $T_{321}$ is given in (27). Hence, we derive that $q(t) = q^2(t; \xi^{p-1}, T_{321}, T_{32})$, and $c^*(t) = \overline{\tau}(t)$ in the interval $[T_{321}, T_{32}]$.

Combining

$$q'(T_{321}) = \rho - pf_1(q(T_{321}), c^*(T_{321})) - pf_2(x^*_n; x^*_\mu, x^*_\sigma) = \rho - (1-p)\overline{\tau} - pK > 0,$$

and the fact that the sign of $q'(t)$ does not change sign, we deduce that $q(t)$ is nondecreasing with respect to $t$, and $q(t) < (p-1)\ln \overline{\tau} + \ln \lambda$ for any $t \in [0, T_{321})$. Thus, $c^*(t) = \overline{\tau}$ and $q(t)$ satisfies ODE (28) in the interval $t \in [0, T_{321})$. Solving ODE (28), we obtain $q(t) = q^1(t; \xi^{p-1}, 0, T_{321})$ in the interval $t \in [0, T_{321}]$.

\[\square\]

References

[1] BERGMAN, Y. Z. (1995): Option Pricing with Differential Interest Rates, The Review of Financial Studies 8(2), 475-500.

[2] BIAGINI, S. AND M. PINAR (2017): The Robust Merton Problem of an Ambiguity Averse Investor, Mathematics and Financial Economics 11(1), 1-24.
[3] Bo, L. and A. Capponi (2016): Optimal Credit Investment with Borrowing Costs, *Mathematics of Operations Research* 42(2), 546-575.

[4] Bordigoni, G., A. Matoussi, and M. Schweizer (2007): A Stochastic Control Approach to a Robust Utility Maximization Problem, *Stochastic Analysis and Applications* 2, 125-151.

[5] Briand, P., B. Delyon, Y. Hu, E. Pardoux and L. Stoica (2003): $L^p$ Solutions of Backward Stochastic Differential Equations, *Stochastic Processes and Their Applications* 108(1), 109-129.

[6] Cadenillas A. and S. Sethi (1997): The Consumption-Investment Problem with Subsistence Consumption, Bankruptcy, and Random Market Coefficients, *Journal of Optimization Theory and Applications* 93, 243-272.

[7] Cheridito, P. and Y. Hu (2011): Optimal Consumption and Investment in Incomplete Markets with General Constraints, *Stoch. Dyn.* 11, 283-299.

[8] Cvitanić, J. and I. Karatzas (1992): Convex Duality in Constrained Portfolio Optimization, *Ann. Appl. Probab.* 2(4), 767-818.

[9] Denis, L. and M. Kervarec (2013): Optimal Investment under Model Uncertainty in Nondominated Models, *SIAM J. Control Optim.* 51(3), 1803-1822.

[10] Fleming, W. H. and T. Zariphopoulou (1991): An Optimal Investment/Consumption Model with Borrowing, *Mathematics of Operations Research* 16(4), 802-822.

[11] Fouque, J. P., C. S. Pun, and H. Y. Wong (2016): Portfolio Optimization with Ambiguous Correlation and Stochastic Volatilities, *SIAM J. Control Optim.* 54(5), 2309-2338.

[12] Hernández-Hernández, D. and A. Schied (2006): Robust Utility Maximization in a Stochastic Factor Model, *Statistics and Decisions* 24, 109-125.

[13] Hu, Y., P. Imkeller, and M. Muller (2005): Utility Maximization in Incomplete Markets, *Ann. Appl. Probab.* 15(3), 1691-1712.

[14] Karatzas, I., J. Lehoczky, S. Sethi, and S. Shreve (1986): Explicit Solution of a General Consumption/Investment Problem, *Mathematics of Operations Research* 11(2), 261-294.

[15] Karatzas, I., J. Lehoczky, and S. Shreve (1987): Optimal Portfolio and Consumption Decisions for a Small Investor on a Finite Horizon, *SIAM J. Control Optim.* 25(6), 1557-1586.

[16] Jian, X., F. Yi, and J. Zhang (2017): Investment and Consumption Problem in Finite Time with Consumption Constraint, *ESAIM: Control, Optimization and Calculus of Variations* 23(4), 1601-1615.

[17] Lehoczky, J., S. Sethi, and S. Shreve (1983): Optimal Consumption and Investment Policies Allowing Consumption Constraints and Bankruptcy, *Mathematics of Operations Research* 8(4), 613-636.
[18] Lin, Q. and R. Riedel (2014): Optimal Consumption and Portfolio Choice with Ambiguity, Preprint [arXiv:1401.1639v1].

[19] Matoussi, A., D. Possamai, and C. Zhou (2015): Robust Utility Maximization in Non-Dominated Models with 2BSDEs: The Uncertain Volatility Model, Math. Finance 25(2), 258-287.

[20] Merton, R. C. (1992): Continuous-Time Finance, Oxford: Wiley-Blackwell.

[21] Neufeld, A. and M. Nutz (2016): Robust Utility Maximization with Lévy Processes, Math. Finance, to appear.

[22] Nutz, M. (2010): The Opportunity Process for Optimal Consumption and Investment with Power Utility, Mathematics and Financial Economics 3, 139-159.

[23] Schied, A. (2008): Robust Optimal Control for a Consumption-Investment Problem, Math. Methods Oper. Res. 67, 1-20.

[24] Sethi S., M. Taksar, and E. Presman (1992): Explicit Solution of a General Consumption/Portfolio Problem with Subsistence Consumption and Bankruptcy, Journal of Economic Dynamics and Control 16, 747-768.

[25] Talay, D., AND Z. Zheng (2002): Worst Case Model Risk Management, Finance Stoch. 6(4), 517-537.

[26] Tevzadze, R., T. Toronjadze, and T. Uzunashvili (2013): Robust Utility Maximization for a Diffusion Market Model with Misspecified Coefficients, Finance Stoch. 17(3), 535-563.

[27] Vila, J. L. and T. Zariphopoulou (1997): Optimal Consumption and Portfolio Choice with Borrowing Constraints, Journal of Economic Theory 77(2), 402-431.

[28] Xu, Z. and F. Yi (2016): An Optimal Consumption-Investment Model with Constraint on Consumption, Mathematical Control and Related Fields 6(3), 517-534.

[29] Yan, H., G. Liang, AND Z. Yang (2015): Indifference Pricing and Hedging in a Multiple-Priors Model with Trading Constraints, Science China Mathematics 58(4), 689-714.

[30] Zariphopoulou, T. (1994): Consumption-Investment Models with Constraints, SIAM J. Control Optim. 32(1), 59-85.