We obtain the asymptotic orders of entropy numbers of Sobolev classes on the unit sphere with Dunkl weight which associates with the finite reflection group. Moreover, the asymptotic order of entropy numbers of weighted Sobolev classes on the unit ball and on the standard simplex are discussed.

1. Introduction

The purpose of this paper is to study the entropy numbers of weighted Sobolev space on the sphere. Let $S^{d-1} := \{x : \|x\| = 1\}$ denote the unit sphere in $\mathbb{R}^d$ endowed with the rotation invariant measure $d\sigma$ normalized by $\int_{S^{d-1}} d\sigma(x) = 1$, where $\|\cdot\|$ denotes the usual Euclidean norm.

Given a nonzero vector $u$, the reflection $\sigma_u$ with respect to the hyperplane perpendicular to $u$ is defined by $x\sigma_u := x - \frac{2\langle x, u \rangle}{\langle u, u \rangle} u, x \in \mathbb{R}^d$, where $\langle \cdot, \cdot \rangle$ denotes the usual Euclidean inner product. A root system $\mathcal{R}$ is a finite subset of nonzero vectors in $\mathbb{R}^d$ such that $u, v \in \mathcal{R}$ implies $u\sigma_v \in \mathcal{R}$. For a fixed $u_0 \in \mathbb{R}^d$ such that $\langle u, u_0 \rangle \neq 0$ for all $u \in \mathcal{R}$, the set of positive roots $\mathcal{R}_+$ with respect to $u_0$ is defined by $\mathcal{R}_+ = \{u \in \mathcal{R} : \langle u, u_0 \rangle > 0\}$ and $\mathcal{R} = \mathcal{R}_+ \cup (-\mathcal{R}_+)$. A finite reflection group $G$ with root system $\mathcal{R}$ is a subgroup of orthogonal group $O(d)$ generated by $\{\sigma_u : u \in \mathcal{R}\}$.

Now we define a real function on $\mathcal{R}_+$, which is called multiplicity function, written as $\kappa_v : v \mapsto \kappa_v$ of $\mathcal{R}_+ \mapsto \mathbb{R}$ satisfying the property that $\kappa_u = \kappa_v$ whenever $\sigma_u$ is conjugate to $\sigma_v$ in $G$, that is, there is a $w$ in the reflection group $G$ generated by $\mathcal{R}_+$, such that $\sigma_u w = \sigma_v$. From the definition of multiplicity function, we can see that $\kappa_v$ is a $G$-invariant function.
In the following, we will consider the Dunkl weight function
\[ h_\kappa(x) = \prod_{v \in \mathcal{R}_+} |\langle x, v \rangle|^{\kappa_v}, \quad x \in \mathbb{R}^d, \quad \kappa_v \geq 0, \]
the function \(h_\kappa\) is a homogeneous function of degree \(\gamma_\kappa = \sum_{v \in \mathcal{R}_+} \kappa_v\) and invariant under the group \(G\). For simplicity of notation, we denote by
\[ h_\kappa(x) := \# \mathcal{R}_+ \prod_{i=1}^d |\langle x, v_j \rangle|^{\kappa_j} := m \prod_{i=1}^d |\langle x, v_j \rangle|^{\kappa_j}, \]
where \(\# \mathcal{E}\) is the number of the elements in \(\mathcal{E}\). The simplest example of \(h_\kappa\) is
\[ h_\kappa(x) = d \prod_{i=1}^d |x_i|^{\kappa_i} \quad \kappa_i \geq 0, \]
corresponding to the group \(G = \mathbb{Z}_d^2\).

Let \(1 \leq p \leq \infty\), we denote by \(L^p(h_\kappa^2)\) the usual weighted Lebesgue space on \(S^{d-1}\) with the finite norm
\[ \|f\|_{p, \kappa} := \left( \frac{1}{a_d^p} \int_{S^{d-1}} |f(x)|^p h_\kappa^2(x) d\sigma(x) \right)^{1/p} < \infty, \]
where \(a_d^p = \int_{S^{d-1}} h_\kappa^2(x) d\sigma(x)\) is the normalization constant. For \(p = \infty\), we assume that \(L_\infty\) is replaced by \(C(S^{d-1})\), the space of continuous function on \(S^{d-1}\) with the usual norm \(\| \cdot \|_\infty\). It is worthwhile to point out that the Dunkl weight plays an important role in the theory of multivariate orthogonal polynomials and will be of great use for the proofs of our results.

Let \(K\) be a compact subset of a Banach space \(X\). The \(n\)th entropy number \(e_n(K, X)\) is defined as the infimum of all positive \(\varepsilon\) such that there exist \(x_1, \ldots, x_{2^n}\) in \(X\) satisfying \(K \subset \bigcup_{k=1}^{2^n} (x_k + \varepsilon B_X)\), where \(B_X\) is the unit ball of \(X\), that is,
\[ e_n(K, X) = \inf\{\varepsilon > 0 : K \subset \bigcup_{k=1}^{2^n} (x_k + \varepsilon B_X), \ x_1, \ldots, x_{2^n} \in X\}. \]
Let \(T \in L(X, Y)\) be a bounded linear operator between the Banach spaces \(X\) and \(Y\). The entropy number \(e_n(T)\) is defined as
\[ e_n(T) := e_n(T : X \to Y) = e_n(T(B_X), Y), \]
Entropy number plays important roles in many related fields including the function space theory\([10]\), \(m\)-term approximation\([12, 25]\), and so on. In recent years, this classical approximation characterization draws more application in information-based complexity and tractability problem\([11, 16, 20]\), signal and image processing\([3, 7]\), learning theory\([2, 14]\). In this paper, we shall determine asymptotic orders of entropy numbers of Sobolev classes \(BW^r_p(h_\kappa^2)\) in \(L_q(h_\kappa^2)\) for all \(1 \leq p, q \leq \infty\) (see the definitions of \(W^r_p(h_\kappa^2)\) in Section 2). In the unweighted case, the exact orders of the entropy numbers of Sobolev classed \(BW^r_p\) on the sphere in \(L_q\) were obtained by Kushpel and Tozoni\([17]\) for \(1 < p, q < \infty\) and H. Wang, K. Wang and J.
Wang ([26]) for the remaining case (when \( p \) and/or \( q \) is equal to 1 or \( \infty \)). In the
special case \( G = Z_d^2 \), the Kolmogorov, linear, and Gelfand widths of the weighted
Sobolev classes on the sphere in weighted \( L_q \) space were obtained by Huang and
Wang ([13]). Our main result can be formulated as follows:

**Theorem 1.1.** Let \( r > (d - 1)(\frac{1}{p} - \frac{1}{q})_+ (2\gamma_\kappa + 1), \ 1 \leq p, q \leq \infty \), then we have

\[
e_n(BW^r_p(h^2_\kappa), \ L_q(h^2_\kappa)) \asymp n^{-\frac{\kappa}{d+r}},
\]

where \( (a)_+ := \max\{a, 0\} \), \( A(n) \asymp B(n) \) means that \( A(n) \ll B(n) \) and \( A(n) \gg B(n) \) means that there exists a positive constant \( c \) independent of \( n \) such that \( A(n) \leq cB(n) \).

Similarly, we can also consider entropy numbers of weighted Sobolev space
\( W^r_p(\omega^B_{\kappa,\mu}) \) (see the definitions of \( W^r_p(\omega^B_{\kappa,\mu}) \) in Section 5) on the unit ball \( \mathbb{B}^d = \{ x \in \mathbb{R}^d : \| x \| \leq 1 \} \), in which the weight function takes the form

\[
\omega^B_{\kappa,\mu}(x) = h^2_\kappa(x)(1 - \| x \|^2)^{\mu-1/2}, \ x \in \mathbb{B}^d,
\]

where \( h_\kappa \) is a reflection invariant weight function on \( \mathbb{R}^d \) and \( \mu > 0 \). The case \( h_\kappa = 1 \) corresponds to the classical weight function \( \omega^B_{\kappa,\mu}(x) = (1 - \| x \|^2)^{\mu-1/2} \). We have the following result:

**Theorem 1.2.** Let \( r > d(\frac{1}{p} - \frac{1}{q})_+ (2\gamma_\kappa + 1), \ 1 \leq p, q \leq \infty \), then we have

\[
e_n(BW^r_p(\omega^B_{\kappa,\mu}), \ L_q(\omega^B_{\kappa,\mu})) \asymp n^{-\frac{\mu}{d+r}}.
\]

Moreover, there is also a close relation between the unit ball and the simplex
\( \mathbb{T}^d = \{ x \in \mathbb{R}^d : x_1 \geq 0, \ldots, x_d \geq 0, 1 - | x | \geq 0 \} \), where \( | x | = x_1 + \cdots + x_d \), which allows us to further extend the results to the weighted functions of Sobolev space
\( W^r_p(\omega^T_{\kappa,\mu}) \) on \( \mathbb{T}^d \), in which the weight functions take the form

\[
\omega^T_{\kappa,\mu}(x) = h^2_\kappa(\sqrt{x_1}, \ldots, \sqrt{x_d})(1 - | x |)^{\mu-1/2}/\sqrt{x_1 \cdots x_d},
\]

where \( \mu \geq 1/2 \) and \( h_\kappa \) is a reflection invariant weight function defined on \( \mathbb{R}^d \) and \( h_\kappa \) is even in each of its variables. The case \( h_\kappa(x) = \prod_{i=1}^d | x_i |^{2\gamma_i} \), gives the classical
weight function on the simplex.

**Theorem 1.3.** Let \( r > d(\frac{1}{p} - \frac{1}{q})_+ (2\gamma_\kappa + 1), \ 1 \leq p, q \leq \infty \), then we have

\[
e_n(BW^r_p(\omega^T_{\kappa,\mu}), \ L_q(\omega^T_{\kappa,\mu})) \asymp n^{-\frac{\mu}{d+r}}.
\]

This paper is organized as follows. Section 2 is devoted to giving the preliminary
knowledge about \( h \)-harmonic analysis and weighted polynomial inequalities on the
sphere. In Section 3, we obtain the lemmas about discretization of the problem
of estimates of entropy numbers. Finally, we prove Theorems 1.1 in Section 4. In
Section 5, we deduce the result for entropy numbers on the unit ball from those on
the sphere. Finally, in Section 6, we deduce the result for entropy numbers on the
simplex from those on the ball.
2. *h*-HARMONIC ANALYSIS AND WEIGHTED POLYNOMIAL INEQUALITIES ON THE SPHERE

In the following, we consider the weighted $L_p$ best approximation with respect to the measure $h^2 \sigma$ on $S^{d-1}$, the theory of *h*-harmonic is necessary. As an extension of spherical harmonics, the usual Laplace operator is replaced by a sum of square Dunkl operators.

The Dunkl operators are defined by

$$D_i f(x) = \partial_i f(x) + \sum_{v \in R_+} \kappa_v \frac{f(x) - f(x \sigma_v)}{\langle x, v \rangle} \langle v, e_i \rangle, i = 1, \ldots, d,$$

where $e_1 = (1, 0, \ldots, 0), \ldots, e_d = (0, \ldots, 0, 1)$. From the definition of $D_i$ we can see that they are the first order differential-difference operators, for more properties of $D_i$ refer to [3][9].

The analogue of the Laplace operator, which is called *h*-Laplacian, is defined by

$$\Delta_h = D_1^2 + \cdots D_d^2.$$ 

We denote by $\Pi_n^d$ the space of polynomials of degree at most $n$ on $S^{d-1}$, i.e. the polynomials of degree at most $n$ restricted on $S^{d-1}$, $P_n^d$ the subspace of homogeneous polynomials of degree $n$ in $d$ variables. An *h*-harmonic polynomial $Y$ of degree $n$ is a homogeneous polynomial $Y \in P_n^d$ such that $\Delta_h Y = 0$.

A spherical *h*-harmonic $Y_n$ of degree $n$ is a homogeneous polynomial of degree $n$ restricted on $S^{d-1}$ and $\Delta_h Y_n = 0$. We denote by $\mathcal{H}_n^d(h^2)$ the space of all spherical *h*-harmonics of degree $n$ on $S^{d-1}$. Furthermore, spherical *h*-harmonic polynomials of different degree are orthogonal with respect to the inner product

$$\langle f, g \rangle_\kappa := \frac{1}{\omega_d} \int_{S^{d-1}} f(x)g(x)h^2(x)d\sigma(x),$$

that is, for $n \neq m$, $(Y_n, Y_m)_\kappa = 0$. $Y_n \in \mathcal{H}_n^d(h^2)$, $Y_m \in \mathcal{H}_m^d(h^2)$. We can follow from the standard Hilbert space theory that

$$L_2(h_n^2) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n^d(h_n^2), \quad \Pi_n^d = \bigoplus_{k=0}^{n} \mathcal{H}_k^d(h_n^2).$$

It is well known that $\dim \Pi_n^d \asymp n^{d-1}$, $\dim \mathcal{H}_n^d \asymp n^{d-2}$.

In terms of the polar coordinates $y = ry'$, $r = \|y\|$, the *h*-Laplacian operator takes the form [20]

$$\Delta_h = \frac{\partial^2}{\partial y'^2} + \frac{2\lambda_n + 1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{h,0},$$

where $\lambda_n := \frac{d-2}{2} + \gamma_n$, $\Delta_{h,0}$ is the Laplace-Beltrami operator on the sphere.

It is analogous with the usual harmonics that the spherical *h*-harmonics are eigenfunctions of Laplace-Beltrami operator $\Delta_{h,0}$ on the sphere, that is

$$\Delta_{h,0}Y_n(x) = -(n(n + 2\lambda_n))Y_n(x), \quad x \in S^{d-1}, \quad Y_n \in \mathcal{H}_n^d(h_n^2),$$
Denote by $\text{proj}_n^\kappa : L_2(h_k^2) \to \mathcal{H}_n^d(h_k^2)$ the orthogonal projection from $L_2(h_k^2)$ onto $\mathcal{H}_n^d(h_k^2)$, which can be expressed as

$$\text{proj}_n^\kappa(f)(x) = \frac{1}{\omega^*_d} \int_{S^{d-1}} f(y)P_n(h_k^2; x, y)h_k^2(y) d\sigma(y),$$

where $P_n(h_k^2; x, y)$ is the reproducing kernel of $\mathcal{H}_n^d(h_k^2)$. Moreover, we get that for $f \in L_2(h_k^2)$, $f(x) = \sum_{n=0}^{\infty} \text{proj}_n^\kappa f(x)$ in $L_2(h_k^2)$ norm.

We note that the kernel $P_n(h_k^2; x, y)$ has a compact formula in terms of the intertwining operator which acts between ordinary harmonics and $h$-harmonics and encodes essentially information on the action of reflection group. The intertwining operator $V_\kappa$ is a linear operator on the space of algebraic polynomials on $\mathbb{R}^d$ which satisfies

$$\mathcal{D}_iV_\kappa = V_\kappa \partial_i, \ 1 \leq i \leq d, \ V_\kappa 1 = 1, \ V_\kappa \mathcal{P}_n \subset \mathcal{P}_n, \ n \in \mathbb{N}_0.$$

One important property of the intertwining operator is that it is positive [23], that is, $V_\kappa p \geq 0$ if $p \geq 0$.

For the general reflection group $G$, the explicit formula of $P_n(h_k^2; x, y)$ is given by [28],

$$P_n(h_k^2; x, y) = \frac{n + \lambda}{\lambda} V_\kappa |C_\lambda^\kappa((\cdot, y))|(x).$$

In the case $G = \mathbb{Z}^d$, the kernel $P_n(h_k^2; x, y)$ has an explicit formula (see [8, 27, 28])

$$P_n(h_k^2; x, y) = \frac{n + \beta}{\beta} c_\kappa \int_{[-1, 1]^d} C_n^\beta(x_1 y_1 t_1 + \cdots + x_d y_d t_d) \prod_{i=1}^d (1 + t_i)(1 - t_i^{2\kappa})^{\kappa_i} - 1 dt,$$

where $\beta_\kappa = \frac{d-2}{2} + |\kappa|$, $|\kappa| = \sum_{i=1}^d \kappa_i$, $c_\kappa = c_{\kappa_1} \cdots c_{\kappa_d}$, $c_\lambda = \frac{\Gamma(\lambda + 1/2)}{\sqrt{\pi} \Gamma(\lambda)}$ and $C_\lambda$ denotes the Gegenbauer polynomial of degree $n$.

Given $r > 0$, we define the fractional order of Laplace-Beltrami operator $(-\Delta_{h,0})^{\frac{r}{2}}$ on $\mathbb{S}^{d-1}$ in a distribution sense by

$$(-\Delta_{h,0})^{\frac{r}{2}} f = \sum_{n=0}^{\infty} (n(n + 2\lambda_n))^{r/2} \text{proj}_n^\kappa(f).$$

where $f$ is a distribution on $\mathbb{S}^{d-1}$. We call $(-\Delta_{h,0})^{\frac{r}{2}} f$ the $r$-th order of distribution $f$.

For $f \in L_1(h_k^2)$, $r \in \mathbb{R}$, the Fourier series of $(-\Delta_{h,0})^{\frac{r}{2}} f$ can be written as

$$(-\Delta_{h,0})^{\frac{r}{2}} f = \sum_{n=1}^{\infty} (n(n + 2\lambda_n))^{r/2} \text{proj}_n^\kappa(f).$$

Let $r > 0$, $1 \leq p \leq \infty$, the Sobolev space $W^r_p(h_k^2, \mathbb{S}^{d-1})$ is defined by

$$W^r_p(h_k^2) := W^r_p(h_k^2, \mathbb{S}^{d-1}) := \left\{ f \in L_p(h_k^2) : \|f\|_{W^r_p(h_k^2)} < \infty, \right\}$$

where $g \in L_p(h_k^2)$, such that $g = (-\Delta_{h,0})^{\frac{r}{2}} f$.
where \( \| f \|_{W^r_p(Z)} := \| f \|_{p,\kappa} + \| \Delta h_0 f \|_{p,\kappa} < \infty \). While the Sobolev class \( BW^r_p(h^2_\alpha) \) is defined to be the unit ball of \( W^r_p(h^2_\alpha) \).

Let \( 1 \leq p \leq \infty, \ n \in \mathbb{Z}^+ \), the best approximation of \( f \in L_p(h^2_\alpha) \) is defined by

\[
E_n(f)_{p,\kappa} := \inf \{ \| f - P \|_{p,\kappa} : P \in \Pi^d_n \}.
\]

It is known that for \( f \in W^r_p(h^2_\alpha), \ 1 \leq p \leq \infty, \)

\[
(2.1) \quad E_n(f)_{p,\kappa} \leq n^{-r} \| (-\Delta h_0) f \|_{p,\kappa}.
\]

Let \( \eta \) be a \( C^\infty \) function on \( [0, \infty) \) satisfying \( \eta(t) = 1 \) for \( 0 \leq t \leq 1 \) and \( \eta(t) = 0 \) if \( t \geq 2 \). Define a sequence of operator \( \eta_n \) for \( n \in \mathbb{N} \) by

\[
\eta_n f(x) = \sum_{k=0}^{\infty} \eta_n \langle \frac{k}{n} \rangle \text{proj}_n f(x).
\]

The operator \( \eta_n \) shares the following properties:

1. \( \eta_n f \in \Pi^d_{n-1} \), and \( \eta_n p = p \) for \( p \in \Pi^d_0 \);
2. \( \| \eta_n f \|_{p,\kappa} \leq \| f \|_{p,\kappa}, \ n \in \mathbb{N} \);
3. \( \| f - \eta_n f \|_{p,\kappa} \leq E_n(f)_{p,\kappa}, \ n \in \mathbb{N} \).

Now define

\[
L^r_{n,\kappa}(x, y) = \sum_{k=0}^{\infty} \eta_n \langle \frac{k}{n} \rangle \text{proj}_n h^2_\kappa(x, y).
\]

From the property (1) of \( \eta_n \) we can see that for \( f \in \Pi^d_0 \)

\[
(2.3) \quad f(x) = \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} f(y) L^r_{n,\kappa}(x, y) h^2_\kappa(y) d\sigma(y).
\]

For \( f \in L_p(h^2_\alpha) \), we define

\[
A_0(f) = \eta_1(f), \quad A_s(f) = \eta_s(f) - \eta_{s-1}(f), \quad s \geq 1.
\]

Then \( A_s(f) \in \Pi^d_{s+1} \) and \( f = \sum_{s=0}^{\infty} A_s(f) \) in \( L_p(h^2_\alpha) \) norm. Furthermore, it follows from (2.1) that

\[
(2.4) \quad \| A_s(f) \|_{p,\kappa} \leq 2^{-sr} \| (-\Delta h_0) f \|_{p,\kappa}.
\]

Denote by \( d(x, y) = \arccos\langle x, y \rangle \) the geodesic distance between two points \( x \) and \( y \) on \( \mathbb{S}^{d-1} \), \( c(x, r) \) the spherical cap centered at \( x \in \mathbb{S}^{d-1} \) with radius \( r > 0 \), i.e.,

\[
c(x, r) = \{ y \in \mathbb{S}^{d-1} : d(x, y) \leq r \}.
\]

Given \( \varepsilon > 0 \), a subset \( \Lambda \subset \mathbb{S}^{d-1} \) is called \( \varepsilon \)-separable if

\[
\min_{x \neq x' \in \Lambda} d(x, x') \geq \varepsilon,
\]

furthermore, a maximal \( \varepsilon \)-separable set \( \Lambda \) is an \( \varepsilon \)-separable set satisfying

\[
\max_{x \in \mathbb{S}^{d-1}} \min_{u \in \Lambda} d(x, u) < \varepsilon.
\]
A weight function \( w \) on \( S^{d-1} \) is called a doubling weight if there exists a constant \( L > 0 \) such that for any \( x \in S^{d-1} \) and \( r > 0 \)
\[
(2.5) \quad \int_{c(x,2r)} w(x) d\sigma(x) \leq L \int_{c(x,r)} w(x) d\sigma(x),
\]
the least constant \( L \) for which (2.5) holds is called the doubling constant of \( w \) and is denoted by \( L_w \).

We write for a doubling weight \( w \) and measurable subset \( E \) of \( S^{d-1} \),
\[
w(E) = \int_E w(x) d\sigma(x).
\]

For a spherical cap \( B = c(x,r) \), integrating (2.5) shows that \( w(2^m B) \leq L_w^m w(B) = 2^m \log_2 L_w w(B) \). We will use the symbol \( s_w \) to denote a number in \([0, \log_2 L_w]\) such that
\[
\sup_{B \subseteq S^{d-1}} \left\{ \frac{w(2^m B)}{w(B)} \right\} \leq C_{L_w} 2^{m s_w}, \quad m = 1, 2, \ldots,
\]
where \( C_{L_w} \) is a constant depending only on \( L_w \), and the supremum is taken over all spherical caps \( B \subseteq S^{d-1} \).

It is also known that for \( x, y \in S^{d-1} \) and \( n = 0, 1, \ldots \),
\[
(2.6) \quad w(c(x, \frac{1}{n})) \leq C_{L_w} (1 + nd(x,y))^{s_w} w(c(y, \frac{1}{n})).
\]

By the definition of doubling weight, it is easily seen that the weight function \( w(x) = h^2_\frac{1}{n}(x) \) satisfies the doubling condition, and for \( x \in S^{d-1} \) and \( n \in \mathbb{N} \) (see [3])
\[
(2.7) \quad w(c(x, \frac{1}{n})) \approx n^{-(d-1)} \prod_{v \in \mathbb{R}_+} (|\langle x, v \rangle| + 1/n)^{2 \kappa_n} = n^{-(d-1)} \prod_{j=1}^m (|\langle x, v_j \rangle| + 1/n)^{2 \kappa_j}.
\]

For more properties of the doubling weight and weighted polynomial inequalities see [3] [18] [19].

The proof of our main results is based on the following positive cubature formulas and Marcinkiewicz-Zygmund inequalities (see [1] [4] [21] [22]).

**Theorem A.** Let \( w \) be a doubling weight on \( S^{d-1} \), there exists a positive constant \( \varepsilon \) depending only on \( d \) and \( s_w \), such that for any \( \delta \in (0, \varepsilon) \), and any maximal \( \frac{\delta}{n} \) separable subset \( \Lambda \subseteq S^{d-1} \), there exists a sequence of positive numbers \( \lambda_\xi \approx w(c(\xi, \frac{\delta}{n})) \), \( \xi \in \Lambda \), for which the following
\[
\int_{S^{d-1}} f(y) w(y) d\sigma(y) = \sum_{\xi \in \Lambda} \lambda_\xi f(\xi)
\]
holds for \( f \in \Pi_n^d \). Moreover, if the above equality is exact for \( f \in \Pi_{3n}^d \), then for \( 1 \leq p \leq \infty \), and \( f \in \Pi_n^d \),
\[
\|f\|_{p,\kappa} \propto \left\{ \sum_{\xi \in \Lambda} \lambda_\xi |f(\xi)|^p \right\}^{\frac{1}{p}} \quad \text{if } 1 \leq p < \infty,
\]
\[
\max_{\xi \in \Lambda} |f(\xi)| \quad \text{if } p = \infty,
\]
where the constants of equivalence depend only on $d$ and $s_w$.

3. The main lemmas and proofs

Let $\kappa = (\kappa_1, \cdots, \kappa_m) \in \mathbb{R}^m_+$, $v = (v_1, \cdots, v_m)$ with $v_j \in \mathbb{S}^{d-1}$, $1 \leq j \leq m$. The weight function $w(x) = \prod_{j=1}^{m} |\langle x, v_j \rangle|^{\kappa_j}$ satisfies the doubling condition on $\mathbb{S}^{d-1}$. Furthermore, we conclude from Theorem A that the corresponding results hold for $w(x)$. For simplicity of notation, we use the same signs as stated in Theorem A.

We have the following important lemma:

**Lemma 3.1.** Let $\kappa = (\kappa_1, \cdots, \kappa_m) \in \mathbb{R}^m_+$, $v = (v_1, \cdots, v_m)$ with $v_j \in \mathbb{S}^{d-1}$, $1 \leq j \leq m$. Then for the weight function $w(x) = \prod_{j=1}^{m} |\langle x, v_j \rangle|^{\kappa_j}$, there exists a constant $\beta \in (0, \frac{1}{n})$, such that

$$\sum_{\xi \in \Lambda} \lambda_{\xi}^{-\beta} \ll n^{(d-1)(1+\beta)}.$$ 

**Remark 3.2.** When the group $G = \mathbb{Z}^d_2$, Huang and Wang ([13]) used elementary polar coordinate method and complicated computation to give the above estimate. However, their method cannot adapt for the general finite reflection group. Instead, we use the properties of doubling weight and generalized Hölder inequality to get it.

In order to prove Lemma 3.1, we need the following two lemmas.

**Lemma 3.3.** ([5, Lemma 5.4.4.] Suppose that $\alpha$ is a fixed nonnegative number, $n$ is a positive integer and $f$ is a nonnegative function on $\mathbb{S}^{d-1}$ satisfying

$$f(x) \leq C_1 (1 + nd(x, y))^\alpha f(y) \text{ for all } x, y \in \mathbb{S}^{d-1}.$$ 

Then for any $0 < p < \infty$, there exists a nonnegative spherical polynomial $g \in \Pi_n^d$ such that

$$C^{-1} f(x) \leq g(x) \leq C f(x) \text{ for any } x \in \mathbb{S}^{d-1},$$

where $C > 0$ depends only on $d, C_1, p$ and $\alpha$.

For $n = 1, 2, \cdots$, it is often convenient to work with an approximation $w_n$ of weight function $w$, which is defined by

$$(3.1) \quad w_n(x) = n^{d-1} \int_{c(x, \frac{1}{n})} w(y) d\sigma(y).$$

**Lemma 3.4.** ([4, Corollary 3.4.] For $f \in \Pi_n^d$ and $0 < p < \infty$,

$$C^{-1} \|f\|_{p, w_n} \leq \|f\|_{p, w} \leq C \|f\|_{p, w_n},$$

where $C > 0$ depends only on $d, L$ and $p$ when $p$ is small.

Now we are in the position to the proof of Lemma 3.1:
Proof. It is easy to check that each $w_n(x)$ as defined in (3.1) is again a doubling weight. Hence, by (2.7), for any $x, y \in S^{d-1}$ and $n = 1, 2, \ldots$,

$$w_n(x) \ll (1 + nd(x, y))^s w_n(y).$$

We conclude from (3.2) and Lemma 3.3 that for some $\beta > 0$, whose range will be decided later, there exists a nonnegative spherical polynomial $g \in \Pi_d^n$ such that $g(x) \asymp (w_n(x))^{-(\beta+1)}$.

By the equivalent form above and Lemma 3.4 we can easily get that

$$\|g\|_{1,w} \asymp \|g\|_{1,w_n} \asymp \|w_n^{-\beta}\|_1$$

Thus, for some $\beta > 0$, we have

$$
\sum_{\xi \in \Lambda} \lambda_\xi^{-\beta} \asymp \sum_{\xi \in \Lambda} (w(c(\xi, \frac{1}{n}))^{-(\beta+1)} = \sum_{\xi \in \Lambda} n^{(d-1)(1+\beta)} (w_n(\xi))^{-(\beta+1)} \int_{c(\xi, 1/n)} w(x) d\sigma(x) \\
\asymp n^{(d-1)(1+\beta)} \sum_{\xi \in \Lambda} \int_{c(\xi, 1/n)} (w_n(x))^{-(\beta+1)} w(x) d\sigma(x) \\
\asymp n^{(d-1)(1+\beta)} \int_{S^{d-1}} (w_n(x))^{-(\beta+1)} w(x) d\sigma(x) \\
\asymp n^{(d-1)(1+\beta)} \int_{S^{d-1}} (w_n(x))^{-\beta} d\sigma(x),
$$

where in the two equalities, we used the definition of $w_n(x)$ in (3.1), in the second inequality, we used the fact that $w_n(x) \asymp w_n(\xi)$ for $x \in c(\xi, 1/n)$, in the last second inequality, we used the property of $\delta_n$-maximal separable set $\Lambda$, and in the last inequality, we used (3.3).

It remains to show that the integration in (3.4) is controlled by some constant independent of $n$ and $d$.

We note from (2.7) and (3.1) that for each $n \in \mathbb{N}$,

$$(w_n(x))^{-(\beta+1)} \asymp \prod_{i=1}^m (|\langle x, v_i \rangle| + n^{-1})^{-\kappa_i \beta},$$

Denote by

$$\widetilde{w_i}(x) = (|\langle x, v_i \rangle| + n^{-1})^{-\kappa_i \beta}, \quad i = 1, \ldots, m,$$

and $r_i = \sum_{j \neq i} \kappa_j / \kappa_i$, then $\sum_{i=1}^m \frac{1}{r_i} = 1$. 


We continue our proof, the generalized Hölder inequality shows that
\[
\int_{S^{d-1}} (\varpi_n(x))^{-\beta} \, d\sigma(x) \ll \int_{S^{d-1}} \prod_{i=1}^{m} (|\langle x, v_i \rangle| + n^{-1})^{-\kappa_i \beta} \, d\sigma(x) \\
= \int_{S^{d-1}} \tilde{\varpi}_1(x), \ldots, \tilde{\varpi}_m(x) \, d\sigma(x) \\
\leq \|\tilde{\varpi}_1\|_{r_1} \cdots \|\tilde{\varpi}_m\|_{r_m}.
\]
(3.5)

Notice that
\[
\|\tilde{\varpi}_i\|_{r_i} = \left( \int_{S^{d-1}} \left(|\langle x, v_i \rangle| + n^{-1}\right)^{-\kappa_i \beta} \, d\sigma(x) \right)^{1/r_i}
\]
(3.6)

By (3.6) and the rotation invariance of Lebesgue measure \(d\sigma(x)\),
\[
\|\tilde{\varpi}_1\|_{r_1} = \|\tilde{\varpi}_2\|_{r_2} = \cdots = \|\tilde{\varpi}_m\|_{r_m}
\]
(3.7)

It follows from (3.5), (3.6), and (3.7) that
\[
\int_{S^{d-1}} (\varpi_n(x))^{-\beta} \, d\sigma(x) \ll \|\tilde{\varpi}_1\|_{r_1} \cdots \|\tilde{\varpi}_m\|_{r_m} \\
\leq \left( \int_{S^{d-1}} \left(|\langle x, v_i \rangle| + n^{-1}\right)^{-|\kappa_i| \beta} \, d\sigma(x) \right)^{1/r_1+\cdots+1/r_m} \\
= \int_{S^{d-1}} \left(|\langle x, v_i \rangle| + n^{-1}\right)^{-|\kappa_i| \beta} \, d\sigma(x) = \omega_{d-1} \int_{-1}^{1} \frac{(1-t)^{\frac{d+1}{2}}}{(1+n^{-1})^{|\kappa_i| \beta}} \, dt \\
= 2\omega_{d-1} \int_{0}^{1} \frac{(1-t)^{\frac{d+1}{2}}}{(1+n^{-1})^{|\kappa_i| \beta}} \, dt \ll \sum_{l=1}^{n} \int_{l-1}^{l} \frac{dt}{(l+n^{-1})^{|\kappa_i| \beta}} \\
\leq \sum_{l=1}^{n} \frac{|l|^{-|\kappa_i| \beta}}{n^{|\kappa_i| \beta}} = n^{|\kappa_i| \beta-1} \sum_{l=1}^{n} l^{-|\kappa_i| \beta} \ll 1,
\]
(3.8)

where the last equality in (3.8) holds if \(\beta \in (0, \frac{1}{|\kappa_i|})\). The proof of Lemma 3.1 is completed. \(\square\)

It is well known that \(\# \Lambda \asymp n^{d-1}\). Now for each integer \(s\) and \(w(x) = h^2_\kappa(x)\), given \(\{w_{s,k} : k \in \Lambda_s^d\}\) of distinct points \(w_{s,k} \in S^{d-1}\) satisfying
\[
\min_{k \neq k' \in \Lambda_s^d} d(w_{s,k}, w_{s,k'}) \geq \frac{\delta}{2s+4} \quad \text{and} \quad \max_{x \in S^{d-1}} \min_{k \in \Lambda_s^d} d(x, w_{s,k}) < \frac{\delta}{2s+4},
\]

By Theorem A, there exists a sequence of numbers
\[
\lambda_{s,k} \asymp \int_{\text{int}(w_{s,k}, \frac{1}{2s+4})} h^2_\kappa(x) \, d\sigma(x), \quad k \in \Lambda_s^d,
\]
(3.9)
such that for any \(f \in \Pi_{2s+4}^d\),
\[
\frac{1}{\omega_{d}} \int_{S^{d-1}} f(y)h^2_\kappa(y) \, d\sigma(y) = \sum_{k \in \Lambda_s^d} \lambda_{s,k} f(w_{s,k}),
\]
(3.10)
and

\[
\|f\|_{p,\kappa} = \begin{cases} 
\left( \sum_{k \in \Lambda_d^f} \lambda_{s,k} |f(\omega_{s,k})|^p \right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty, \\
\max_{k \in \Lambda_d^f} |f(\omega_{s,k})|, & \text{if } p = \infty.
\end{cases}
\]

For \( w = (w_1, \ldots, w_m) \in \mathbb{R}^m \), we define as usual

\[
\|x\|_{\ell_p^m,w} = \left( \sum_{i=1}^m |x_i|^p w_i \right)^{1/p}
\]

for \( 1 \leq p < \infty \) and \( \|x\|_{\ell_\infty^m} = \max_{1 \leq i \leq m} |x_i| \) for \( p = \infty \).

We denote by \( \ell_p^m \) the set of vectors \( x \in \mathbb{R}^m \) equipped with the norm \( \| \cdot \|_{\ell_p^m} \), and \( B_{\ell_p^m} \) the unit ball of \( \ell_p^m \). In the case \( w = (1, \ldots, 1) \), it returns to the standard instance and we write \( \ell_p^m \), \( \| \cdot \|_{\ell_p^m} \), \( B_{\ell_p^m} \) instead of \( \ell_p^m \), \( \| \cdot \|_{\ell_p^m} \), \( B_{\ell_p^m} \).

**Lemma 3.5.** Let \( r > 0 \), \( 1 \leq p, q \leq \infty \). Then for \( 1 \leq p \leq q \leq \infty \), there exists a constant \( \beta \in (0, \frac{1}{2r}) \), such that

\[
e_n(BW_p^r(h_2^q, L_q^q(h_2^q))) \leq \sum_{s=0}^{\infty} 2^{-s} (r - (\frac{1}{2} - \frac{1}{4}) (d-1)) \sum_{k=0}^{s+1} \left( \frac{\# \Lambda_2^d}{2(k-1)(d-1)} \right)^{\frac{1}{2}(\frac{1}{2} - \frac{1}{4})} e_n(B_{\ell_p}^{m_s, k}, \ell_q^{m_s, k}),
\]

where \( \sum_{s=0}^{\infty} \sum_{k=0}^{s+1} n_{s,k} \leq n \), \( m_{s,1} = 2 \), \( m_{s,k} \asymp 2(k-1)(d-1) \), \( 2 \leq k \leq s \), and \( m_{s,s+1} = \# \Lambda_2^d - 2s(d-1) \).

The following lemma plays a key role in the proof of Lemma 3.5.

**Lemma 3.6.** Let \( w = (w_1, \ldots, w_m) \in \mathbb{R}^m \) satisfying \( w_i > 0 \), \( 1 \leq i \leq m \) and let

\[
\sum_{j=1}^m w_j^{-\gamma} \leq m, \quad \text{for some } \gamma > 0.
\]

Then for \( 1 \leq p \leq q \leq \infty \), there exists \( j_0 \in \mathbb{N} \) such that \( 2^{j_0} \leq m < 2^{j_0+1} \) and

\[
e_n(B_{\ell_p}^{m_{j_0}, w}, \ell_q^{m_{j_0}, w}) \leq \sum_{k=1}^{j_0} \left( \frac{m}{2^{k-1}} \right)^{\frac{1}{2}(\frac{1}{2} - \frac{1}{4})} e_n(B_{\ell_p}^{m_k}, \ell_q^{m_k}),
\]

where \( m_1 = 2 \), \( m_k = 2^{k-1} \), \( 2 \leq k \leq j_0 - 1 \), \( m_{j_0} = m - 2^{j_0-1}, \) and \( \sum_{k=1}^{j_0} n_k \leq n \).

**Proof.** Without loss of generality we assume that

\[
w_1 \leq w_2 \leq \cdots \leq w_m.
\]

First we claim that

\[
e_n(B_{\ell_p}^{m_{j_0}, w}, \ell_q^{m_{j_0}, w}) = e_n(B_{\ell_p}^{m_{j_0}}, \ell_q^{m_{j_0}}),
\]

where \( v = (w_1^{1-\frac{p}{q}}, \ldots, w_m^{1-\frac{p}{q}}) \).

In fact, for any \( x = (x_1, \ldots, x_m) \in B_{\ell_p}^{m_{j_0}, w} \), the mapping \( U \) defined by

\[
Ux = (x_1 w_1^{\frac{p}{q}}, \ldots, x_1 w_1^{\frac{p}{q}})
\]

satisfies that
yields an isometry of \( B_{ℓ_{p,w}}^m \) onto \( B_{ℓ_{p,v}}^m \) and \( U^{-1} \) yields an isometry of \( ℓ^m \) onto \( ℓ_{q,w}^m \).

As a consequence of the definition of the entropy numbers and the properties of \( U \) and \( U^{-1} \) we obtain (3.15).

Now for arbitrary \( x \in ℜ^m \), we set
\[
S_k x = (x_1, \ldots, x_k, 0, \ldots, 0), \quad k = 1, \ldots, m
\]
and
\[
δ_1 x = S_2 x, \quad δ_j x = S_2 j x - S_2 j - 1 x, \quad 2 \leq j \leq j_0 - 1, \quad \text{and} \quad δ_{j_0} x = x - S_2 j_0 - 1 x.
\]
Then \( x = \sum_{j=1}^{m} δ_j x \). It follows from (3.13) and (3.14) that for \( 1 \leq j \leq m \)
\[
j w_j^{-γ} \leq \sum_{i=1}^{j} w_i^{-γ} \leq \sum_{i=1}^{m} w_i^{-γ} \leq m,
\]
which implies
\[
(3.16) \quad w_j^{-1} \leq \left( \frac{m}{j} \right)^{\frac{1}{p}}, \quad 1 \leq j \leq m.
\]

Define the set
\[
A_k := \left\{ x \in ℜ^{mk} : \left( \sum_{i=1}^{mk} |x_i|^p v_{i+2k-1} \right)^{\frac{1}{p}} \leq 1 \right\}, 1 \leq k \leq j_0.
\]
We deduce from (3.16) that for any \( x \in A_k \)
\[
\sum_{i=1}^{mk} |x_i|^p = \sum_{i=1}^{mk} |x_i|^p v_{i+2k-1} w_{i+2k-1}^{-1} \leq \sum_{i=1}^{mk} x_i^p v_{i+2k-1} \left( \frac{m}{2k-1} \right)^{\frac{1}{p}} w_{i+2k-1}^{-1} \leq \left( \frac{m}{2k-1} \right)^{\frac{1}{p}} \left( 1 - \frac{1}{p} \right).
\]
By the above inequality we have
\[
(3.17) \quad A_k \subset \left( \frac{m}{2k-2} \right)^{\frac{1}{p}} \left( 1 - \frac{1}{p} \right) B_{ℓ_{p,k}}^m.
\]
Combine with (3.15), (3.17) and the definition and the properties of the entropy numbers we can get the conclusion of the lemma. \( \square \)

**Remark 3.7.** If we choose suitable \( n_k \), we can prove that for \( 0 < p < q < \infty \),
\[
e_n(B_{ℓ_{p,w}}^m, ℓ_{q,w}^m) \leq \left( \frac{m}{n} \right)^{\frac{1}{p} \left( 1 - \frac{1}{q} \right)}, \quad \text{if} \quad 1 \leq n \leq \log 2m,
\]
\[
e_n(B_{ℓ_{p,w}}^m, ℓ_{q,w}^m) \leq \left( \frac{m}{n} \right)^{\frac{1}{p} \left( 1 - \frac{1}{q} \right)} n^{-\left(1/p-1/q\right)}, \quad \text{if} \quad \log 2m \leq n \leq 2m,
\]
and
\[
e_n(B_{ℓ_{p,w}}^m, ℓ_{q,w}^m) \leq \left( \frac{m}{n} \right)^{\frac{1}{p} \left( 1 - \frac{1}{q} \right)} 2^{-\frac{m}{2m}} m^{-\left(1/p-1/q\right)}, \quad \text{if} \quad 2m \leq n.
\]
Remark 3.8. Let \( \omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n \) satisfying \( \omega_i > 0, 1 \leq i \leq n \) and let
\[
\sum_{j=1}^{n} \omega_j^{-\beta} \leq n, \text{ for some } \beta > 0.
\]

Then for \( 1 \leq m \leq n \) and \( 1 \leq p \leq q \leq \infty \),
\[
S_m(B_{\ell^p_{m,w}, \ell^q_{q,w}}) \leq 2\left(\frac{n}{m}\right)^{\frac{1}{p}} S_m(B_{\ell^p_p, \ell^q_q}),
\]
where \( S_m \) denotes one of the Kolmogorov m-width \( d_m \) or the linear m-width \( \delta_m \) (see [4]) or the Gelfand width \( d_m \) (see [13]). We remark that the similar result cannot hold for the entropy numbers. This is due to the fact that \( S_m(B_{\ell^p_{m,w}, \ell^q_{q,w}}, w) = 0 \) if \( m \geq n \), while \( m(B_{\ell^p_{p,w}, \ell^q_{q,w}}, w) > 0 \) for all \( n, m \in \mathbb{N} \).

Now we are ready to prove **Lemma 3.5**.

**Proof of Lemma 3.5:**

Denote by \( \text{id} : X \mapsto Y \) the identity operator from \( X \) to \( Y \), where \( X \) and \( Y \) are normed linear spaces. Then
\[
e_n(BW_p^r(h_\kappa^2), L_q(h_\kappa^2)) = e_n(\text{id} : W_p^r(h_\kappa^2) \mapsto L_q(h_\kappa^2)).
\]

Since for \( f \in L_q(h_\kappa^2), f = \sum_{s=0}^{\infty} A_s f \) in \( L_q(h_\kappa^2) \) norm, which implies that \( \text{id} = \sum_{s=0}^{\infty} A_s \).

It follows immediately that
\[
e_n(\text{id} : W_p^r(h_\kappa^2) \mapsto L_q(h_\kappa^2)) \leq \sum_{s=0}^{\infty} e_n(A_s : W_p^r(h_\kappa^2) \mapsto L_q(h_\kappa^2)),
\]
where \( \sum_{s=0}^{\infty} n_s \leq n \).

It is clearly from (2.3) that for \( f \in W_p^r(h_\kappa^2), 1 \leq p \leq \infty \) and \( s \geq 0 \),
\[
\|A_s(f)\|_{p, \infty} \ll 2^{-s\rho}\|f\|_{W_p^r(h_\kappa^2)}.
\]

Hence,
\[
e_n(A_s : W_p^r(h_\kappa^2) \mapsto L_q(h_\kappa^2)) = e_n(A_s(BW_p^r(h_\kappa^2)), L_q(h_\kappa^2)) \ll 2^{-s\rho}e_n(BL_p(h_\kappa^2) \cap \Pi_{2s+1}^d, L_q(h_\kappa^2))
\]

In order to prove (3.12), we proceed to show that
\[
e_n(BL_p(h_\kappa^2) \cap \Pi_{2s+1}^d, L_q(h_\kappa^2)) \ll 2^{s(d-1)(\frac{1}{p} + \frac{1}{q})} \sum_{k=1}^{\Pi_{2s+1}^d} \left(\frac{\#A_{d,k}^s}{2^{(k-1)(d-1)}}\right)^{\frac{1}{p}} e_{n,k}(B_{\ell^p_{p,w}, \ell^q_{q,w}}) \cap L_q(h_\kappa^2)
\]

Now we define the operators \( U_s : BL_p(h_\kappa^2) \cap \Pi_{2s+1}^d \mapsto \ell_{p,w}^{\#A_{d,k}^s} \) and \( V_s : \ell_{q,w}^{\#A_{d,k}^s} \mapsto L_q(h_\kappa^2) \) by
\[
U_s(f) = (f(\omega_{s,1}), \ldots, f(\omega_{s,\#A_{d,k}^s})), \quad f \in BL_p(h_\kappa^2) \cap \Pi_{2s+1}^d, \quad w = (\lambda_{s,k})_{k \in \mathbb{N}^d}
\]
and
\[ V_s(a)(x) = \sum_{k=1}^{\#\Lambda_s^d} a_{s,k} \lambda_{s,k} \mathcal{L}^{(s+1)}_\wedge(x,\omega_{s,k}), \quad x \in \mathbb{S}^{d-1}, \quad a = (a_{s,k}) \in \ell^\#\Lambda_s^d, \]

where \( \mathcal{L}^{(s+1)}_\wedge \) is defined as in (2.2), \#\Lambda_s^d, \lambda_{s,k}, \omega_{s,k} \) as in (3.11) and (3.14).

We conclude from (3.11) that for \( f \in BL_p(h^2_{\kappa}) \cap \Pi_{2^{s+1}}^d \),
\[ \|U_s(f)\|_{\ell^\#\Lambda_s^d} \asymp \|f\|_{p,\kappa}. \]  
(3.19)

Furthermore, we claim that for \( 1 \leq p \leq \infty \),
\[ \|V_s(a)\|_{q,\kappa} \ll \|a\|_{\ell^\#\Lambda_s^d}. \]  
(3.20)

As a matter of fact, for \( q = 1 \), (3.20) follows directly from Proposition 2.2 in [31].

For \( q = \infty \), by (3.11) with \( p = 1 \) and \( f(y) = L^*_\kappa(x,y) \), we have
\[ \|V_s(a)\|_{\infty} \leq \|a\|_{\ell^\#\Lambda_s^d} \sum_{k=1}^{\#\Lambda_s^d} \lambda_{s,k} \|L^*_\kappa(x,\omega_{s,k})\| \asymp \|a\|_{\ell^\#\Lambda_s^d} \|L^*_\kappa(x,\cdot)\|_{1,\kappa} \ll \|a\|_{\ell^\#\Lambda_s^d}. \]

For \( 1 < q < \infty \), (3.20) follows from the Riesz-Thorin theorem.

On account of (2.23) and (3.10), we check at once that for \( f \in L_p(h^2_{\kappa}) \cap \Pi_{2^{s+1}}^d \),
\[ f(x) = \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} f(y) L^*_\wedge \kappa(x,y) h^2_{\kappa}(y) d\sigma(y) = \sum_{k=1}^{\#\Lambda_s^d} \lambda_{s,k} f(\omega_{s,k}) L^*_\wedge \kappa(x,\omega_{s,k}) = V_s U_s(f)(x). \]

This implies that the operator \( id : L_p(h^2_{\kappa}) \cap \Pi_{2^{s+1}}^d \rightarrow L_q(h^2_{\kappa}) \) can be factored as follows:
\[ id : L_p(h^2_{\kappa}) \cap \Pi_{2^{s+1}}^d \xrightarrow{U_s} \ell^\#\Lambda_s^d \xrightarrow{id} \ell^\#\Lambda_s^d \xrightarrow{V_s} L_q(h^2_{\kappa}). \]

It then follows by (3.19), (3.20) and the properties of entropy numbers that
\[ e_{n_s}(id : L_p(h^2_{\kappa}) \cap \Pi_{2^{s+1}}^d \rightarrow L_q(h^2_{\kappa})) \leq \|V_s\| \leq |\Pi_{2^{s+1}}^d| \ll e_{n_s}(id : \ell^\#\Lambda_s^d \rightarrow \ell^\#\Lambda_s^d) \|U_s\| \ll e_{n_s}(id : \ell^\#\Lambda_s^d \rightarrow \ell^\#\Lambda_s^d) \|V_s\|. \]  
(3.21)

At last, it follows from (3.19) and Lemma 3.1 that for a constant \( \beta \in (0, \frac{1}{2^{s+1}}) \),
\[ \sum_{k \in \Lambda_s^d} (\#\Lambda_s^d \lambda_{s,k})^{-\beta} \ll \#\Lambda_s^d. \]

Hence by the inequality above we use Lemma 3.6 to the vector \( \hat{w} = (c\#\Lambda_s^d \lambda_{s,k})_{k \in \Lambda_s^d} \), we get that
\[ e_{n_s}(B\ell^\#\Lambda_s^d, \ell^\#\Lambda_s^d) \asymp (\#\Lambda_s^d)^{\frac{1}{2} - \frac{1}{q}} e_{n_s}(B\ell^\#\Lambda_s^d, \ell^\#\Lambda_s^d) \ll 2^{s(d-1)(\frac{1}{t} - \frac{1}{q})} \sum_{k=0}^{\#\Lambda_s^d} (\#\Lambda_s^d)^{-\frac{1}{2} - \frac{1}{q}} e_{n_s}(B\ell^\#\Lambda_s^d, \ell^\#\Lambda_s^d). \]  
(3.22)

which combines with (3.21), gives (3.18) and this finish the proof of Lemma 3.5. □
**Lemma 3.9.** Let \( r > 0, \ 1 \leq p, q \leq \infty \). Then there exists a positive integer \( N \) such that \( N \asymp n, \ N \geq 2n \), and

\[
e_n(BW_p^r(h_{n,2}^2), L_q(h_{n,2}^2)) \gg n^{-\frac{4}{mp}+\frac{1}{p}+\frac{1}{q}}e_n(B]\ell_p^N, \ \ell_q^N).
\]

**Proof.** For convenience, we denote again by \( h_{n}(x) = \prod_{v \in R_{+}}|\langle x, v \rangle|^{\kappa_v} = \prod_{j=1}^{m} |\langle x, v_j \rangle|^{\kappa_j} \).

To prove the lower estimate, let

\[
E_j = \{ x \in S^{d-1} : \frac{\pi}{2} - d(x, v_j) | \leq 2\varepsilon_{d,m}, \ 1 \leq j \leq m, \ \varepsilon_{d,m} \}
\]

and

\[
\tilde{E}_j = \{ x \in S^{d-1} : d(x, v_j) \leq \varepsilon_{d,m}, \ 1 \leq j \leq m, \ \varepsilon_{d,m} \}
\]

where \( \varepsilon_{d,m} \) is a sufficiently small positive constant depending only on \( d \) and \( m \).

Let \( |E| \) denote the Lebesgue measure of a set \( E \subset S^{d-1} \). A straightforward calculation shows that

\[
|\bigcup_{j=1}^{m} E_j| \leq \sum_{j=1}^{m} |E_j| \leq c_d m \varepsilon_{d,m} \leq \frac{1}{2}|S^{d-1}|
\]

provided that \( \varepsilon_{d,m} \) is small enough. If \( x \in S^{d-1}\backslash(\bigcup_{j=1}^{m} E_j) \), then

\[
|\langle x, v_j \rangle| \geq \sin(2\varepsilon_{d,m}, \ j = 1, \ldots, m.
\]

Hence

\[
h_{n}^2(x) = \prod_{j=1}^{m} |\langle x, v_j \rangle|^{2\kappa_j} \gg (\sin(2\varepsilon_{d,m}))^{2|\kappa|} \gg 1, \ x \in S^{d-1}\backslash(\bigcup_{j=1}^{m} E_j).
\]

We assume that \( l \in \mathbb{N} \) is sufficiently large and \( c_1 l^{d-1} \leq n \leq c_2 l^{d-1} \) with \( c_1, c_2 > 0 \) being independent of \( n \) and \( l \). We let \( \{ x_j \}_{j=1}^{N} \subset S^{d-1}\backslash(\bigcup_{j=1}^{m} E_j) \) such that \( N \asymp l^{d-1} \),

\[
c(x_i, \frac{1}{l}) \cap c(x_i, \frac{1}{l}) = \emptyset, \text{ if } i \neq j,
\]

and \( c(x_i, \frac{1}{l}) \subset S^{d-1}\backslash(\bigcup_{j=1}^{m} \tilde{E}_j) \). Obviously, such points \( x_j \) exist. We may take \( c_2 \) sufficiently small so that \( N \geq 2n \).

Let \( \varphi \) be a nonnegative \( C^\infty \)-function on \( \mathbb{R} \) supported in \( [0,1] \) and be equal to 1 on \( [0,\frac{1}{2}] \). We define

\[
\varphi_i(x) = \varphi(ld(x, x_i)), \ i = 1, \ldots, N
\]

and set

\[
A_N := \left\{ f_a(x) = \sum_{i=1}^{N} a_i \varphi_i(x) : a = (a_1, \ldots, a_N) \in \mathbb{R}^N \right\}.
\]

It is clearly that

\[
\text{supp } \varphi_i \subset c(x_i, \frac{1}{l}) \subset S^{d-1}\backslash(\bigcup_{j=1}^{m} \tilde{E}_j)
\]

and

\[
\| \varphi_i \|_{p,\kappa} := \left( \int_{c(x_i, \frac{1}{l})} |\varphi(ld(x, x_i))|^p d\sigma(x) \right)^{1/p} \asymp l^{-\frac{d-1}{p}}.
\]
and

\[ \text{supp } \varphi_i \bigcap \text{supp } \varphi_j = \emptyset, \quad (i \neq j). \]

Hence, for \( f_a \in A_N, \ a = (a_1, \ldots, a_N) \in \mathbb{R}^N, \)

\[ (3.23) \quad \| f_a \|_{p, \kappa} \asymp \left( l^{-(d-1)} \sum_{i=1}^{N} |a_i|^p \right)^{1/p} = l^{-(d-1)} \| a \|_{\ell^p}. \]

Next, we note that if \( f \in C_c^\infty(\mathbb{R}^d), \) then by the definition of \( D_k \) we have

\[ \text{supp } D_k f \subset \bigcup_{\rho \in G} \rho(\text{supp } f), \quad k = 1, \ldots, d. \]

where \( G \) is the finite reflection group generated by \( \mathcal{R}_+ \), \( \rho(E) = \{ \rho x : x \in E \} \). This, combining with the fact that the set \( \bigcup_{\rho \in G} \rho(E) \) is invariant under the action of the group \( G \), means that for \( v \in \mathbb{N}, \)

\[ \text{supp } (\Delta_h)^v f \subset \bigcup_{\rho \in G} \rho(\text{supp } f). \]

By the definition of \( h \)-Laplace-Beltrami operator, for \( \xi \in S^{d-1} \), we have

\[ \Delta_{h,0} \varphi_i(\xi) = \Delta_h \left( \varphi_i(m d(\frac{x}{\|x\|}, x_i)) \right) \mid_{x = \xi}, \]

which implies that

\[ (3.24) \quad \text{supp } \Delta_{h,0} \varphi_i \subset \bigcup_{\rho \in G} \rho(c(x_i, \frac{1}{l})). \]

Furthermore,

\[ \text{supp } (\Delta_{h,0})^v \varphi_i \subset \bigcup_{\rho \in G} \rho(c(x_i, \frac{1}{l})), \quad v = 1, 2, \ldots. \]

Finally, we can verify that

\[ \| (\Delta_h)^v \varphi_i \|_\infty \leq l^{2v}, \quad 1 \leq i \leq N, \quad v = 1, 2, \ldots, \]

which implies that

\[ (3.25) \quad \| (\Delta_{h,0})^v \varphi_i \|_\infty \ll l^{2v}, \quad v = 1, 2, \ldots. \]

By (3.24) and (3.25), we have

\[ (3.26) \quad \| (\Delta_{h,0})^v \varphi_i \|_{p, \kappa} \ll l^{2v-(d-1)/p}, \quad v = 1, 2, \ldots. \]

For \( f_a \in A_N, \ a = (a_1, \ldots, a_N) \), Then

\[ (\Delta_{h,0})^v f_a(x) = \sum_{i=1}^{N} a_i (\Delta_{h,0})^v \varphi_i(x). \]

Since \( \text{supp } (\Delta_{h,0})^v \varphi_i \subset \bigcup_{\rho \in G} \rho(c(x_i, \frac{1}{l})), \quad v \in \mathbb{N}, \) we get that for any \( x \in S^{d-1}, \)

\[ \# \{ i : a_i (\Delta_{h,0})^v \varphi_i(x) \neq 0, \quad 1 \leq i \leq N \} \leq \sum_{i=1}^{N} \chi \bigcup_{\rho \in G} \rho(c(x_i, \frac{1}{l}))(x), \quad v \in \mathbb{N}, \]
Proofs of Theorems 1.1

for $0 < p < q < p$ and in case $0 < q < p \leq \infty$, it holds

\[
\sum_{i=1}^{N} \chi \bigcup_{\rho \in G} \rho(c(x_i, \frac{1}{p})) (x) \leq \sum_{i=1}^{N} \sum_{\rho \in G} \chi_{\rho(c(x_i, \frac{1}{p}))} (x)
\]

\[
= \sum_{\rho \in G} \sum_{i=1}^{N} \chi_{\rho(c(x_i, \frac{1}{p}))} (x) \leq \sum_{\rho \in G} 1 = \#G,
\]

where in the last inequality we used the pairwise disjoint property of $\{\rho(c(x_i, \frac{1}{p}))\}_{i=1}^{N}$ for any $\rho \in G$. We deduce from (3.26) and (3.27) that for $1 \leq p \leq \infty$,

\[
\|(-\Delta_{h,0})^{v} f_a\|_{p,\kappa} \leq \left( \int_{\mathbb{R}^{d-1}} \left| \sum_{i=1}^{N} a_i (-\Delta_{h,0})^{v} \varphi_i (x) \right|^p \, d\sigma (x) \right)^{1/p} \leq \left( \#G \right)^{1-1/p} \left( \int_{\mathbb{R}^{d-1}} \left| \sum_{i=1}^{N} a_i (-\Delta_{h,0})^{v} \varphi_i (x) \right|^p \, d\sigma (x) \right)^{1/p} \leq \|f\|_{\ell^p_N}, \quad v \in \mathbb{N}.
\]

It then follows by Kolmogorov type inequality (see [6, Theorem 8.1]), (3.28) and (3.29) that for $v > r$, $v \in \mathbb{N}$,

\[
\|(-\Delta_{h,0})^{r/2} f_a\|_{p,\kappa} \leq \|(-\Delta_{h,0})^{v} f_a\|_{p,\kappa} \| f_a \|_{p,\kappa} \leq \|f\|_{\ell^p_N} \|f\|_{\ell^r_N} \leq \|f\|_{\ell^p_N}.
\]

Recall that $n \asymp p^{d-1}$, hence

\[
c n^{-\frac{d-1}{r}} (BL_p (h^2) \cap A_N) \subset BW_p (h^2) \cap A_N,
\]

where $c$ is a positive constant independent of $n$. We have

\[
e_n \left( BW_p (h^2), L_q (h^2) \right) \geq e_n \left( BW_p (h^2) \cap A_N, L_q (h^2) \right) \geq n^{-\frac{d-1}{r}} e_n \left( BW_p (h^2) \cap A_N, L_q (h^2) \cap A_N \right) \geq n^{-\frac{d-1}{r}} \cdot \frac{1}{2} e_n (B_l^N, \ell^N_q).
\]

The proof of Lemma 3.9 is finished. \hfill \Box

4. PROOF OF THEOREMS 1.1

Proofs of Theorems 1.1

First we consider the lower estimates. For all $k, m \in \mathbb{N}$, we have (see [10, 24, 15]): for $0 < p \leq q \leq \infty$

\[
e_k (B^m_p, \ell^m_q) \asymp \begin{cases} 1, & 1 \leq k < \log 2m, \cr \left( \frac{\log(1+p)}{k} \right)^{1/p-1/q}, & \log 2m \leq k \leq 2m, \cr 2^{-\frac{d-1}{r}} m^{1/q-1/p}, & 2m \leq k,
\end{cases}
\]

and in case $0 < q < p \leq \infty$, it holds

\[
e_k (B^m_p, \ell^m_q) \asymp 2^{-k/(2m)} m^{1/q-1/p}.
\]
This implies that if \( m \asymp k \), then for all \( 0 < p, q \leq \infty \),

\[
e_k(B_{p,q}^m, e_q^m) \asymp k^{1/q-1/p}.
\]

By Lemma 3.9 and (4.3), we obtain the lower estimates for \( e_n(B_{p,q}^r(h_k^2), L_q(h_k^2)) \).

The only point remaining concerns the upper estimates. We only need to consider the case \( 1 \leq p \leq q \leq \infty \), since for \( 1 \leq q < p \leq \infty \), we have the relation \( B_{p,q}^r(h_k^2) \subseteq B_{q,q}^r(h_k^2) \), which implies that \( e_n(B_{p,q}^r(h_k^2), L_q(h_k^2)) \leq e_n(B_{q,q}^r(h_k^2), L_q(h_k^2)) \).

Under the condition stated above, it follows from the proof in Lemma 3.5 that

\[
e_n(B_{p,q}^r(h_k^2), L_q(h_k^2)) \leq \sum_{s=0}^{\infty} e_n(A_s(B_{p,q}^r(h_k^2)), L_q(h_k^2)).
\]

For the convenience of estimation, we may take sufficiently small positive number \( \rho \) and define

\[
n_s := \begin{cases} \lfloor \rho \cdot 2^{1-(1-\rho)(d-1)(J-s)} \rfloor & \text{if } 0 \leq s \leq J, \\ \lfloor \rho \cdot 2^{1+(1-\rho)(d-1)(J-s)} \rfloor & \text{if } s > J,
\end{cases}
\]

here \( \lfloor x \rfloor \) is the largest integer less than the real number \( x \). Since

\[
\sum_{s=0}^{\infty} n_s \ll \sum_{0 \leq s \leq J} \lfloor \rho \cdot 2^{1-(1-\rho)(d-1)(J-s)} \rfloor + \sum_{s > J} \lfloor \rho \cdot 2^{1+(1-\rho)(d-1)(J-s)} \rfloor \ll 2^{J(d-1)},
\]

we can choose an integer \( J \) such that \( \sum_{s=0}^{\infty} n_s \leq n \) and \( 2^{J(d-1)} \asymp n \).

By Lemma 3.5, for arbitrary \( \varepsilon > 0 \), take \( \beta = \frac{1}{2(\gamma_k + \varepsilon)} \), we have

\[
e_n(B_{p,q}^r(h_k^2), L_q(h_k^2)) \ll \sum_{s=0}^{\infty} 2^{-s} (r - \left\lfloor \frac{1}{p} - \frac{1}{q} \right\rfloor (d-1)) \sum_{k=0}^{s} \left\lfloor \frac{\lfloor \rho \cdot 2^{1-(1-\rho)(d-1)(J-s)} \rfloor}{k^{d-1}} \right\rfloor \sum_{s,k}^n e_n(A_s(B_{p,q}^r, e_q^m), B_{p,q}^m, e_q^m, k)
\]

\[
\ll \sum_{s=0}^{\infty} 2^{-s} (r - \left\lfloor \frac{1}{p} - \frac{1}{q} \right\rfloor (d-1)) \sum_{k=0}^{s} 2^{(s-k)(d-1)} \left\lfloor \frac{1}{p} - \frac{1}{q} \right\rfloor \sum_{s,k}^n e_n(A_s(B_{p,q}^r, e_q^m), B_{p,q}^m, e_q^m, k)
\]

\[
= \sum_{0 \leq s < J} I_1 + \sum_{J \leq s \leq \frac{1}{p} \cdot J} I_2 + \sum_{s > \frac{1}{p} \cdot J} I_3.
\]

For \( 0 \leq s < J \), \( n_s = \left\lfloor \rho \cdot 2^{1-(1-\rho)(d-1)(J-s)} \right\rfloor \), consequently we define

\[
n_{s,k} = \left\lfloor 2^{(1-\rho)(d-1)(J-k)} m_{s,k} \right\rfloor,
\]

where \( m_{s,k} \) is the the same as in Lemma 3.5. A short computation shows that

\[
\sum_{k=1}^{s} n_{s,k} \leq 2^{(1-\rho)(J-s)} \left\lfloor \frac{\rho}{\rho} \right\rfloor \leq n_s \text{ and } n_{s,k} \geq 2m_{s,k}.
\]
Hence, by the third case in (4.1)

\[
I_1 \ll \sum_{s=0}^{J} 2^{-s} \sum_{k=0}^{s} 2^{(s-k)(d-1)(\frac{1}{p} - \frac{1}{q})} \cdot 2^{-k(d-1)(\frac{1}{p} - \frac{1}{q})} 2^{-2^{(1-\rho)(d-1)(J-k)}} 2^{-k(d-1)(\frac{1}{p} - \frac{1}{q})}
\]

which completes estimate of the first part.

Now we are ready to estimate \( I_2 \). For \( J < s \leq \frac{1+p}{p} J \), \( n_s = \lfloor \Lambda_x^d \cdot 2^{(1+\rho)(d-1)(J-s)} \rfloor \).

There is no loss of generality in assuming \( n_s \geq 2^{J_1(d-1)} \), and

\[
n_{s,k} := \begin{cases} \\
2^{(1-\rho)(d-1)(J_1-k)} m_{s,k} & \text{if } 0 \leq k \leq J_1, \\
2^{(1+\rho)(d-1)(J_1-k)} m_{s,k} & \text{if } k > J_1.
\end{cases}
\]

It also shows that \( \sum_{k=0}^{s} n_{s,k} \leq \Lambda_x^d \cdot 2^{(1+\rho)(d-1)(J-s)} \leq n_s \).

For convenience, denote by \( J_0 := \frac{1+p}{p} J \), we have

\[
I_2 \ll \sum_{s=0}^{J_0} 2^{-s} \sum_{k=0}^{s} 2^{(s-k)(d-1)(\frac{1}{p} - \frac{1}{q})} \cdot 2^{-k(d-1)(\frac{1}{p} - \frac{1}{q})} 2^{-\beta s(k-\frac{1}{p})} 2^{-2^{(1-\rho)(d-1)(J-k)}} 2^{-k(d-1)(\frac{1}{p} - \frac{1}{q})}
\]

In order to get the estimate of \( I_2 \), we need to compute \( P_1 \) and \( P_2 \). From the third case in (4.1), it follows

\[
P_1 \ll \sum_{k \leq J_1} 2^{-k(d-1)(\frac{1}{p} - \frac{1}{q})} \cdot 2^{-\beta s(k-\frac{1}{p})} 2^{-2^{(1-\rho)(d-1)(J_1-k)}} 2^{-k(d-1)(\frac{1}{p} - \frac{1}{q})}
\]

\[
P_2 \ll \sum_{k \leq J_1} 2^{-k(d-1)(\frac{1}{p} - \frac{1}{q})} \cdot 2^{-2^{(1-\rho)(d-1)(J_1-k)}} 2^{-k(d-1)(\frac{1}{p} - \frac{1}{q})}
\]

\[
\ll 2^{-J_1(d-1)(\frac{1}{p} - \frac{1}{q})(\frac{J}{J+1})}.
\]
The term $P_2$ need to be handled in a more complicated way, it deduces that

\[
P_2 = \sum_{J_1 < k \leq s} 2^{-k(d-1)(\frac{1}{p} - \frac{1}{q})} e_{n_s,k}(B_{\ell_p^m}, \ell_q^m) \]
\[
\quad = \sum_{J_1 < k \leq \frac{1+p}{p} J_1} + \sum_{\frac{1+p}{p} J_1 < k \leq s} =: Q_1 + Q_2. \tag{4.10}
\]

For the case $J_1 < k \leq \frac{1+p}{p} J_1$, recall that $n_{s,k} = \lceil 2^{(1+p)(d-1)(J_1-k)} \rceil$. It follows from the second case of (4.1) that

\[
Q_1 = \sum_{J_1 < k \leq \frac{1+p}{p} J_1} 2^{-k(d-1)(\frac{1}{p} - \frac{1}{q})} e_{n_s,k}(B_{\ell_p^m}, \ell_q^m) \]
\[
\quad \ll \sum_{J_1 < k \leq \frac{1+p}{p} J_1} 2^{-k(d-1)(\frac{1}{p} - \frac{1}{q})} \left\{ (1+p)(d-1)(k-J_1) \right\}^{\frac{1}{d} - \frac{1}{q}} 2^{(1+p)(d-1)(\frac{1}{p} - \frac{1}{q})(k-J_1)} \]
\[
\quad \ll 2^{-J_1(d-1)(\frac{1}{p} - \frac{1}{q})(\frac{1}{d} + 1)}. \tag{4.11}
\]

For the other case $\frac{1+p}{p} J_1 < k \leq s$, it follows from the first case of (4.1) that

\[
Q_2 = \sum_{\frac{1+p}{p} J_1 < k \leq s} 2^{-k(d-1)(\frac{1}{p} - \frac{1}{q})} e_{n_s,k}(B_{\ell_p^m}, \ell_q^m) \]
\[
\quad \ll 2^{-\frac{1+p}{p} J_1(d-1)(\frac{1}{p} - \frac{1}{q})(\frac{1}{d} + 1)} \ll 2^{-J_1(d-1)(\frac{1}{p} - \frac{1}{q})(\frac{1}{d} + 1)}. \tag{4.12}
\]

Combining with (4.10) - (4.12) we can get that

\[
P_2 \ll 2^{-J_1(d-1)(\frac{1}{p} - \frac{1}{q})(\frac{1}{d} + 1)}. \tag{4.13}
\]

By (4.7), and the estimates of (4.8), (4.13) and the definition of $J_1$, we finally get that

\[
I_2 \ll \sum_{s=j}^J 2^{-s} \left( r - (d-1)(\frac{1}{p} - \frac{1}{q})(\frac{1}{d} + 1) \right) 2^{-J_1(d-1)(\frac{1}{p} - \frac{1}{q})(\frac{1}{d} + 1)} \]
\[
\quad \ll \sum_{s \geq J} 2^{-sr} 2^{-(J-s)(1+p)(d-1)(\frac{1}{p} - \frac{1}{q})(\frac{1}{d} + 1)} \]
\[
\quad \ll 2^{-Jr} \ll n^{-\frac{\beta}{1+p}}, \tag{4.14}
\]

where the last second inequality holds if

\[
r > (1+p)(d-1)(\frac{1}{p} - \frac{1}{q})(\frac{1}{\beta} + 1) \]

for arbitrary $\varepsilon > 0$ and small $\rho$. 


We are left with the task of estimating $I_3$, it follows from the property of entropy numbers and the definition of $J_0$ that

\[ I_3 \ll \sum_{s \geq J_0} 2^{-s (r - (\frac{1}{p} - \frac{1}{q}) (d - 1))} \sum_{k=0}^{s} 2^{s-k} (d-1)(\frac{1}{p} - \frac{1}{q}) \frac{k}{2} \]

\[ \ll \sum_{s \geq J_0} 2^{-s (d-1)(\frac{1}{p} - \frac{1}{q}) (\frac{1}{q} + 1)} \]

\[ \ll 2^{-J_0 (r - d + 1)(\frac{1}{q} + 1)} \ll 2^{-J r} \ll n^{-\frac{1}{p - 1}}, \]

(4.16)

where in the last second inequality, we once again used (4.15).

By the inequalities (4.5)-(4.6) and (4.14), (4.16), we can deduce the upper estimate, which completes the proof.

□

5. Entropy numbers on the unit ball

Now we consider the analogous result of entropy numbers of weight ed Sobolev spaces on the unit ball $B^d$.

As introduced in the first section, take the weight function of the form

\[ \omega^{B}_{\kappa, \mu}(x) = h^{2}_{\kappa}(x)(1 - \|x\|^2)^{\mu - 1/2}, \quad x \in \mathbb{B}^d, \]

where $\mu > 0$, $h_{\kappa}$ is a reflection invariant weight function on $\mathbb{R}^d$.

Denote by $L^p(\omega^{B}_{\kappa, \mu})$, $1 \leq p < \infty$ the space of measurable functions defined on $B^d$ with the finite norm

\[ \|f\|_{p, \omega^{B}_{\kappa, \mu}} := \left( \frac{1}{b^{*}_{d}} \int_{B^d} |f(x)|^p \omega^{B}_{\kappa, \mu} d\sigma(x) \right)^{1/p} < \infty, \]

where $b^{*}_{d} = \int_{B^d} \omega^{B}_{\kappa, \mu} dx$ is the normalization constant. For $p = \infty$, we assume that $L^\infty$ is replaced by $C(\mathbb{B}^d)$, the space of continuous function on $\mathbb{B}^d$ with the usual norm $\| \cdot \|_\infty$.

Let $\mathcal{V}^{d}_{n}(\omega^{B}_{\kappa, \mu})$ denote the space of orthogonal polynomials of degree $n$ with respect to $\omega^{B}_{\kappa, \mu}$ on $\mathbb{B}^d$. There is a close relation between $h$-harmonics on the sphere $S^d$ and orthogonal polynomials on the unit ball $B^d$ (see [31] and reference in there). Denote by

\[ h_{\kappa, \mu}(x_1, \cdots, x_{d+1}) = h_{\kappa}(x_1, \cdots, x_d)|x_{d+1}|^\mu, \quad (x_1, \cdots, x_{d+1}) \in \mathbb{R}^{d+1}. \]

We can construct an one-to-one correspondence between the $h$-harmonics space $\mathcal{H}^{d}_{n}(h_{\kappa, \mu})$ with respect to the function $h_{\kappa, \mu}$ and orthogonal polynomial space $\mathcal{V}^{d}_{n}(\omega^{B}_{\kappa, \mu})$. That is, for an even $Y_n \in \mathcal{H}^{d}_{n}(h_{\kappa, \mu})$ satisfying $Y_n(x, x_{d+1}) = Y_n(x, -x_{d+1})$, we can write

\[ Y_n(y) = r^n P_{n}(x), \quad y = r(x, x_{d+1}) \in \mathbb{R}^{d+1}, \quad r = \|y\|, \quad (x, x_{d+1}) \in S^d \]

in polar coordinates. Then $P_{n}$ is in $\mathcal{V}^{d}_{n}(\omega^{B}_{\kappa, \mu})$. 

Moreover, by the elementary integral relation
\[
\int_{S^d} f(y) d\sigma(y) = \int_{B_d} \left\{ f(x, \sqrt{1-\|x\|^2}) + f(x, -\sqrt{1-\|x\|^2}) \right\} dx
\]
it follows that for \( Y_n \in \mathcal{H}_n^d(h_{\kappa,u}) \) and associated \( P_n \in \mathcal{V}_n^d(\omega_{\kappa,u}) \)
\[
\int_{S^d} Y_n(y) h_{\kappa,u}(y) d\sigma(y) = \int_{B_d} \left\{ P_n(x, \sqrt{1-\|x\|^2}) + P_n(x, -\sqrt{1-\|x\|^2}) \right\} \omega_{\kappa,u}(x) dx.
\]

Let \( \Delta_h^{\kappa,\mu} \) denote the \( h \)-Laplacian with respect to \( h_{\kappa,u} \) and \( \Delta_{h,0}^{\kappa,\mu} \) the corresponding spherical \( h \)-Laplacian. For \( y = r(x, x_{d+1}), (x, x_{d+1}) \in \mathbb{S}^d \), the spherical \( h \)-Laplacian can be written as [30]
\[
\Delta_{h,0}^{\kappa,\mu} = \Delta_h - (x, \nabla)^2 - 2\lambda (x, \nabla), \quad \lambda = \sum_{v \in \mathbb{R}^+} \kappa_v + \mu + \frac{d-1}{2},
\]
where \( \Delta_h \) is the \( h \)-Laplacian associated with \( h_{\kappa} \) on \( \mathbb{R}^d \). Define
\[
D_{h,\kappa}^{\kappa,\mu} := \Delta_h - (x, \nabla)^2 - 2\lambda (x, \nabla).
\]
It follows that the elements of orthogonal polynomial subspace \( \mathcal{V}_n^d(\omega_{\kappa,u}) \) are just the eigenfunctions of \( D_{h,\kappa}^{\kappa,\mu} \), that is
\[
D_{h,\kappa}^{\kappa,\mu}(P) = -n(n+2\lambda)P, \quad P \in \mathcal{V}_n^d(\omega_{\kappa,u}).
\]
For \( f \in L_2(\omega^B_{\kappa,u}) \), the orthogonal expansion can be represented as
\[
L_2(\omega^B_{\kappa,u}) = \sum_{k=0}^{\infty} \bigoplus \mathcal{V}_n^d(\omega^B_{\kappa,u}), \quad f = \sum_{k=0}^{\infty} \text{proj}^{\kappa,\mu}_k f,
\]
where \( \text{proj}^{\kappa,\mu}_k : L_2(\omega^B_{\kappa,u}) \to \mathcal{V}_n^d(\omega^B_{\kappa,u}) \) is the projection operator. The fractional power of \( D_{h,\kappa}^{\kappa,\mu} \) is defined by
\[
(D_{h,\kappa}^{\kappa,\mu})^{r/2}(f) \sim (n(n+2\lambda))^{r/2} P\text{proj}^{\kappa,\mu}_r(f), \quad f \in L_p(\omega^B_{\kappa,u}).
\]
Let \( r > 0, 1 \leq p \leq \infty \), the Sobolev space \( W_p^r(\omega^B_{\kappa,u}, \mathbb{S}^d) \) is defined by
\[
W_p^r(\omega^B_{\kappa,u}) := W_p^r(\omega^B_{\kappa,u}, \mathbb{S}^d) := \left\{ f \in L_p(\omega^B_{\kappa,u}) : \| f \|_{W_p^r(\omega^B_{\kappa,u})} < \infty, \quad \exists \ g \in L_p(\omega^B_{\kappa,u}), \text{ such that } g = (-D_{h,\kappa}^{\kappa,\mu})^{r/2} f \right\},
\]
where \( \| f \|_{W_p^r(\omega^B_{\kappa,u})} := \| f \|_{p,\omega^B_{\kappa,u}} + \| (-D_{h,\kappa}^{\kappa,\mu})^{r/2} f \|_{p,\omega^B_{\kappa,u}} \). The Sobolev class \( BW_p^r(\omega^B_{\kappa,u}) \) is defined to be the unit ball of \( W_p^r(\omega^B_{\kappa,u}) \).

For \( f \in W_p^r(\omega^B_{\kappa,u}, \mathbb{S}^d) \), define \( T(f)(y) = f(x), \quad y = (x, y_{d+1}) \in \mathbb{S}^d \), it follows from [31] Proposition 4.1 that
\[
\|(−D_{h,\kappa}^{\kappa,\mu})^{r/2} f\|_{p,\omega^B_{\kappa,u}} = \|(−\Delta_{h,0}^{\kappa,\mu})^{r/2} T(f)\|_{p,\kappa,u}.
\]

Under nearly the same argument, we can get exact order of the entropy numbers \( e_n(BW_p^r(\omega^B_{\kappa,u}), L_q(\omega^B_{\kappa,u})) \) of Sobolev classes \( BW_p^r(\omega^B_{\kappa,u}) \) in \( L_q(\omega^B_{\kappa,u}) \) as stated in Theorem 1.2.
6. Entropy numbers on the simplex

Now we consider the problems of entropy numbers on the simplex $T^d$ with respect to the weight

$$\omega^T_{\kappa,\mu}(x) = h^2_{\kappa}(\sqrt{x_1}, \ldots, \sqrt{x_d})(1 - |x|)^{\mu-1/2}/\sqrt{x_1 \cdots x_d},$$

where $\mu \geq 1/2$ and $h_\kappa$ is a reflection invariant weight function defined on $\mathbb{R}^d$ and $h_\kappa$ is even in each of its variables. The last requirement essentially limits the weight functions to the case of group $\mathbb{Z}_2^d$, for which

$$\omega^T_{\kappa}(x) = x_1^{\kappa_1-1/2} \cdots x_d^{\kappa_d-1/2}(1 - |x|)^{\kappa_{d+1}-1/2}$$

which is the classical weight function on $T^d$.

For the case on the simplex $T^d$ with respect to the weight $\omega^T_{\kappa,\mu}(x)$, the related results of entropy numbers can be deduced from the related results on the unit ball $B^d$ with respect to the weight $\omega^B_{\kappa,\mu}(x)$.

The background on orthogonal expansion and approximation on $T^d$ is similar to the case of the unit ball $B^d$. The definitions of various notions, such as $\| \cdot \|_{\omega^T_{\kappa,\mu}}$, $V_n(\omega^T_{\kappa,\mu})$, are exactly the same as in the previous section with $T^d$ in place of $B^d$. There is a close relation between orthogonal polynomials on $B^d$ and those on $T^d$. Let $P_{2n}$ be an element of $V_n(\omega^B_{\kappa,\mu})$ and assume that $P_{2n}$ is even in each of its variables. Then we can write $P_{2n}$ as $P_{2n}(x) = R_n(x_1^2, \ldots, x_d^2)$. It turns out that $R_n$ is an element of $V_n(\omega^T_{\kappa,\mu})$ and the relation is a one-to-one correspondence. In particular, applying $D^B_{\kappa,\mu}$ on $P_{2n}$ leads to a second-order differential-difference operator acting on $R_n$. Denote this operator by $D^T_{\kappa,\mu}$. Then

$$D^T_{\kappa,\mu}(P) = -n(n + \lambda)P, \quad P \in V_n(\omega^T_{\kappa,\mu}), \quad \lambda = \gamma_\kappa + \frac{d - 1}{2}.$$

Denote by $L_p(\omega^T_{\kappa,\mu})$, $1 \leq p < \infty$ the space of measurable functions defined on $T^d$ with the finite norm

$$\| f \|_{p,\omega^T_{\kappa,\mu}} := \left( \frac{1}{c_d^2} \int_{T^d} |f(x)|^p \omega^B_{\kappa,\mu} \, dx \right)^{1/p} < \infty,$$

where $c_d^2 = \int_{T^d} \omega^B_{\kappa,\mu} \, dx$ is the normalization constant. For $p = \infty$, we assume that $L_{\infty}$ is replaced by $C(T^d)$, the space of continuous function on $T^d$ with the usual norm $\| \cdot \|_{\infty}$.

Let $r > 0$, $1 \leq p \leq \infty$, under the standard argument, we can introduce the Sobolev spaces $W^r_p(\omega^T_{\kappa,\mu}, T^d)$ with the norm

$$\| f \|_{W^r_p(\omega^T_{\kappa,\mu})} := \| f \|_{p,\omega^T_{\kappa,\mu}} + \| (D^T_{\kappa,\mu})^r f \|_{p,\omega^T_{\kappa,\mu}}.$$

The Sobolev class $B^r_p(\omega^T_{\kappa,\mu})$ is defined to be the unit ball of $W^r_p(\omega^T_{\kappa,\mu})$.

The related results for the entropy numbers of weighted Sobolev classes $B^r_p(\omega^T_{\kappa,\mu})$ in $L_q(\omega^T_{\kappa,\mu})$ are obtained due to the corresponding results on the unit ball $B^d$. 
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