SHORT INTERVAL RESULTS FOR CERTAIN PRIME-INDEPENDENT
MULTIPlicative FUNCTIONS

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Abstract. Using recent results from the theory of integer points close to smooth curves, we
give an asymptotic formula for the distribution of values of a class of integer-valued prime-
independent multiplicative functions.

1. Introduction and result

A prime-independent multiplicative function is a multiplicative arithmetic function \( f \) satisfying
\( f(1) = 1 \) and such that there exists a map \( g : \mathbb{Z}_{\geq 0} \to \mathbb{R} \) such that \( g(0) = 1 \) and, for any prime
powers \( p^\alpha \)
\[
f (p^\alpha) = g(\alpha).
\]
In this article, we only consider integer-valued prime-independent multiplicative functions \( f \)
verifying \( f(p) = 1 \) for any prime \( p \). This is equivalent to the fact that \( g(1) = 1 \) and we also
assume that there exists \( r \in \mathbb{Z}_{\geq 2} \) such that
\[
g(1) = \cdots = g(r - 1) = 1 \quad \text{and} \quad \alpha \geq r \Rightarrow g(\alpha) > 1.
\]
One of the long-standing problems in number theory concerning these prime-independent mul-
tiplicative functions is the study of the distribution of their values. To this end, we fix \( k \in \mathbb{Z}_{\geq 1} \)
and set
\[
S_{f,k}(x) := \sum_{n \leq x \atop f(n) = k} 1
\]
and define the local density of \( f \) to be the real number
\[
d_{f,k} := \lim_{x \to \infty} \frac{S_{f,k}(x)}{x}
\]
whenever the limit exists.

The arithmetic function \( n \mapsto a(n) \), counting the number of finite, non-isomorphic abelian
groups of order \( n \), is one of the well-known examples of prime-independent multiplicative func-
tions, since \( a(p^\alpha) = P(\alpha) \), where \( P \) is the unrestricted partition function. The existence of the
local density \( d_{a,k} \) was first established in [7] and later Ivić [4] showed that
\[
S_{a,k}(x) = d_{a,k}x + O\left(x^{1/2}\log x\right).
\]
Further authors improved on this estimate, such as [6], [3] and [9] in which the best error term
to date was established. The general case was introduced by Ivić in [6] and improved in [9] for
a certain class of arithmetic functions.

The next step was the study of the distribution of values of \( f \) in short intervals. By 'short
intervals' we mean the study of sums of the shape
\[
S_{f,k}(x + y) - S_{f,k}(x) = \sum_{x < n \leq x + y \atop f(n) = k} 1
\]
where \( y = o(x) \) as \( x \to \infty \). In the case of \( f = a \), Ivić [5] first showed that
\[
\sum_{x<n \leq x+y \atop a(n)=k} 1 = d_{a,k}y + o(y)
\]
holds for \( y \geq x^{581/1744} \log x \). This value was successfully improved by many authors. For instance, by connecting the problem to the error term in certain divisor problems, Krätzel [2] showed that
\[
\sum_{x<n \leq x+y \atop a(n)=k} 1 = d_{a,k}y + o(y) + O \left( x^{369/1667+\varepsilon} \right).
\]

On the other hand, using results on gaps between squarefree numbers, Li [8] proved that the asymptotic formula
\[
\sum_{x<n \leq x+y \atop a(n)=k} 1 = d_{a,k}y + o(y)
\]
holds for \( y \geq x^{1/5+\varepsilon} \) uniformly for \( k \in \mathbb{Z}_{\geq 1} \). In the general case, Zhai [11, Theorem 2.5] showed that
\[
\sum_{x<n \leq x+y \atop f(n)=k} 1 = d_{f,k}y + o(y)
\]
holds for \( y \geq x^{31/113+\varepsilon} \), where \( r \) is given in (1). The purpose of this work is to establish an effective version of Zhai’s result by giving a fully effective error term. More precisely, we will show the following estimate.

**Theorem 1.** Let \( k \in \mathbb{Z}_{\geq 1} \) fixed and \( f \) be an integer-valued prime-independent multiplicative function such that \( f(p) = 1 \) for any prime \( p \) and let \( r \in \mathbb{Z}_{\geq 2} \) as in (1). Let \( x^{31/113+\varepsilon} \leq y \leq 4^{-2r^2}x \) be real numbers. Then
\[
\sum_{x<n \leq x+y \atop f(n)=k} 1 = d_{f,k}y + O_{r,\varepsilon} \left\{ \left( x^{r-1}y^{r+1} \right)^{1/2}x^\varepsilon + yx^{-\left( \frac{1}{r} - 1 \right) - \varepsilon} + y^{1-\frac{2(r-1)}{r(r+1)}-\varepsilon} \right\}.
\]

2. **Notation and preparation for the proof**

In what follows, \( k \in \mathbb{Z}_{\geq 1} \) is fixed and \( f \) is an integer-valued prime-independent multiplicative function satisfying the hypothesis of Theorem 1 with \( r \in \mathbb{Z}_{\geq 2} \) given in (1).

For any arithmetic function \( F \), \( L(s, F) \) is its formal Dirichlet series and \( F^{-1} \) is the Dirichlet convolution inverse of \( F \).

Let \( s_r \) be the characteristic function of the set of \( r \)-full numbers, \( \mu_r \) be that of the set of \( r \)-free numbers, so that \( \mu_r^{-1} \) is the multiplicative function such that \( \mu_r^{-1}(1) = 1 \) and given on prime powers \( p^\alpha \) by
\[
\mu_r^{-1}(p^\alpha) = \begin{cases} 1, & \text{if } r \mid \alpha; \\ -1, & \text{if } r \mid \alpha - 1; \\ 0, & \text{otherwise.} \end{cases}
\]

Finally, put
\[
1_{f,k}(n) = \begin{cases} 1, & \text{if } f(n) = k; \\ 0, & \text{otherwise.} \end{cases}
\]

Note that \( f(n) = 1 \) whenever \( n \) is \( r \)-free so that the Dirichlet series of \( 1_{f,k} \) may be formally written as
\[
L(s, 1_{f,k}) = \frac{\zeta(s)}{\zeta(rs)} H_{f,k,r}(s) := \frac{\zeta(s)}{\zeta(rs)} \sum_{n=1}^\infty \frac{h_{f,k,r}(n)}{n^s}
\]
and where the multiplicative function \( h_{f,k,r} \) is supported on \( r \)-full numbers. Indeed

\[
h_{f,k,r}(n) = \sum_{d|n \atop f(n/d)=k} \mu_r^{-1}(d)
\]

which implies that, for any prime powers \( p^\alpha \) with \( 1 \leq \alpha < r \)

\[
h_{f,k,r}(p^\alpha) = \sum_{j=0}^{\lfloor \alpha/r \rfloor} 1_{f,k}(p^{\alpha-j}) - \sum_{j=0}^{\lfloor (\alpha-1)/r \rfloor} 1_{f,k}(p^{\alpha-rj-1})
\]

\[
= 1_{f,k}(p^\alpha) - 1_{f,k}(p^{\alpha-1}) = 0
\]

since \( g(\alpha) = g(\alpha - 1) = 1 \). This in turn implies that the Dirichlet series \( H_{f,k,r} \) is absolutely convergent in the half-plane \( \sigma > \frac{1}{r} \) and also that

\[
|h_{f,k,r}(n)| \leq s_r(n)\tau(n)
\]

for any \( k, n \in \mathbb{Z}_{\geq 1} \) and \( r \in \mathbb{Z}_{\geq 2} \). The following bound will then be useful.

**Lemma 2.** Let \( r \in \mathbb{Z}_{\geq 2} \). Then

\[
\sum_{n \leq x} s_r(n)\tau(n) \ll x^{1/r}(\log x)^r.
\]

**Proof.** Every \( r \)-full integer \( n \) may be uniquely written as \( n = a_r a_r^1 \cdots a_r^2 a_r-1 \) with \( a_2 \cdots a_r \) squarefree and \( (a_i, a_j) = 1 \) for \( 2 \leq i < j \leq r \). Since the divisor function \( \tau \) is sub-multiplicative, we infer that the sum of the lemma does not exceed

\[
\ll \sum_{a_r \leq x^{1/r}} \tau(a_r^2 r-1) \sum_{a_{r-1} \leq \left(\frac{x}{a_r^{2 r-1}}\right)^{1/r}} \tau(a_r^2 r-2) \cdots \sum_{a_1 \leq \left(\frac{x}{a_2^{2 r-1} a_{r-1}^{r-1}}\right)^{1/r}} \tau(a_1^r).
\]

Now the well-known bound

\[
\sum_{a \leq z} \tau(a^r) \ll z (\log z)^r
\]

applied to the last inner sum, allows us to complete the proof. \( \square \)

The next result is an immediate consequence of Lemma 2.

**Lemma 3.** Let \( f \) be as in Theorem 1, \( r \) given in (1) and \( k \in \mathbb{Z}_{\geq 1} \) fixed.
1. Let \( \kappa \in \mathbb{R}_{\geq 0} \). Then

\[
\sum_{n \leq x} \frac{|h_{f,k,r}(n)|}{n^\kappa} \ll \begin{cases} x^{-\kappa+1/r}(\log x)^r, & \text{if } 0 \leq \kappa < \frac{1}{r}; \\ (\log x)^{r+1}, & \text{if } \kappa = \frac{1}{r}; \\ 1, & \text{if } \kappa > \frac{1}{r}. \end{cases}
\]

2. We also have

\[
\sum_{n > x} \frac{|h_{f,k,r}(n)|}{n} \ll x^{-1+1/r}(\log x)^r.
\]

**Proof.** Follows from Lemma 2, the inequality (3) and partial summation. \( \square \)
3. $r$-FREE NUMBERS IN SHORT INTERVALS

The following lemma plays a crucial part in Theorem 1. For a proof, see Lemma 3.2 and Corollary 5.1.

**Lemma 4.** Let $r \in \mathbb{Z}_{\geq 2}$. For any $X \in \mathbb{R}_{\geq 1}$ and $0 < Y < X$, define

$$R_r(X,Y) := X^{1/r+1} + YX^{-\frac{1}{e^{(r-1)(2r-1)}}} + Y^{1-\frac{2(r-1)}{r(3r-1)}}. \quad (4)$$

1. For any $X \in \mathbb{R}_{\geq 1}$, $0 < Y < X$ and any $\varepsilon > 0$

$$\sum_{2Y < n \leq 2X} s_r(n) \left( \left\lfloor \frac{X+Y}{n} \right\rfloor - \left\lfloor \frac{X}{n} \right\rfloor \right) \ll_{r,\varepsilon} R_r(X,Y)X^\varepsilon.$$

2. For any $X \in \mathbb{R}_{\geq 1}$, $4r \leq Y < X$ and any $\varepsilon > 0$

$$\sum_{X < n \leq X+Y} \mu_r(n) = \frac{Y}{\zeta(r)} + O_{r,\varepsilon}(R_r(X,Y)X^\varepsilon).$$

4. **Proof of Theorem 1**

From (2), we get

$$\sum_{x<n \leq x+y \atop f(n)=k} 1 = \sum_{d \leq x+y} h_{f,k,r}(d) \sum_{\frac{y}{d} < \ell \leq \frac{x+y}{d}} \mu_r(\ell)
= \left( \sum_{d \leq y(y/x)^{1/(2r)}} + \sum_{y(y/x)^{1/(2r)} < d \leq 2y} + \sum_{2y < d \leq x+y} \right) h_{f,k,r}(d) \sum_{\frac{y}{d} < \ell \leq \frac{x+y}{d}} \mu_r(\ell)
= S_1 + S_2 + S_3.$$

For $S_1$, which will provide the main term, we use the second estimate of Lemma 4 giving

$$S_1 = \sum_{d \leq y(y/x)^{1/(2r)}} h_{f,k,r}(d) \left\{ \frac{y}{d\zeta(r)} + O \left( R_r \left( \frac{x}{d}, \frac{y}{d} \right) x^\varepsilon \right) \right\}
= \frac{y}{\zeta(r)} \sum_{d=1}^{\infty} h_{f,k,r}(d) \frac{y}{d} + O \left( y \sum_{d>y(y/x)^{1/(2r)}} \frac{|h_{f,k,r}(d)|}{d} \right)
+ O \left( x^\varepsilon \sum_{d \leq y(y/x)^{1/(2r)}} |h_{f,k,r}(d)|R_r \left( \frac{x}{d}, \frac{y}{d} \right) \right)
= \frac{y}{\zeta(r)} H_{f,k,r}(1) + O \left( (x^{r-1}y^{r+1}) \frac{1}{d^{2r}} (\log x)^r \right)
+ O \left( x^\varepsilon \sum_{d \leq y(y/x)^{1/(2r)}} |h_{f,k,r}(d)|R_r \left( \frac{x}{d}, \frac{y}{d} \right) \right)$$
where we used Lemma 2 and where the error term $R_r$ is defined in (4). Using Lemma 3 again

$$
\sum_{d \leq y(x/y)^{1/(2r)}} |h_{f,k,r}(d)| R_r \left( \frac{x}{y} \frac{y}{d} \right) \ll \frac{x}{2^{r+1}} \sum_{d \leq y(x/y)^{1/(2r)}} \frac{|h_{f,k,r}(d)|}{d^{2r+1}} + y x^{-\frac{1}{6(4r-1)(2r-1)}} \sum_{d \leq y(x/y)^{1/(2r)}} \frac{|h_{f,k,r}(d)|}{d^{8r-36r+1/(2r-1)}} + y^{1-\frac{2(r-1)}{7(3r-1)}} \sum_{d \leq y(x/y)^{1/(2r)}} \frac{|h_{f,k,r}(d)|}{d^{1-\frac{2(r-1)}{7(3r-1)}}}.
$$

Hence

$$
S_1 = \frac{y}{\zeta(r)} H_{f,k,r}(1) + O \left\{ \frac{x^\epsilon}{y(x/y)^{1/(2r)}} \left(x^{r-1}y^{r+1}\right)^{\frac{1}{2r^2}} + y x^{-\frac{1}{6(4r-1)(2r-1)}} + y^{1-\frac{2(r-1)}{7(3r-1)}} \right\}.
$$

For $S_2$, we use the second point of Lemma 3 so that

$$
|S_2| \ll y \sum_{d > y(x/y)^{1/(2r)}} \frac{|h_{f,k,r}(d)|}{d} \ll \left(x^{r-1}y^{r+1}\right)^{\frac{1}{2r^2}} \left(\log x\right)^r.
$$

Now

$$
S_3 = \sum_{2y < d < x+y} h_{f,k,r}(d) \left( \left\lfloor \frac{x+y}{d} \right\rfloor - \left\lfloor \frac{x}{d} \right\rfloor \right)
$$

and using (3) and the first estimate of Lemma 4 we obtain

$$
|S_3| \ll x^\epsilon \sum_{2y < d < 2x} s_r(d) \tau(d) \left( \left\lfloor \frac{x+y}{d} \right\rfloor - \left\lfloor \frac{x}{d} \right\rfloor \right)
$$

$$
\ll x^\epsilon \sum_{2y < d < 2x} s_r(d) \left( \left\lfloor \frac{x+y}{d} \right\rfloor - \left\lfloor \frac{x}{d} \right\rfloor \right)
$$

$$
\ll x^{2\epsilon} \left(x^{\frac{1}{2r+1}} + y x^{-\frac{1}{6(4r-1)(2r-1)}} + y^{1-\frac{2(r-1)}{7(3r-1)}}\right).
$$

Collecting (5), (6) and (7) and noticing that

$$
\left(x^{r-1}y^{r+1}\right)^{\frac{1}{2r^2}} \geq x^{\frac{1}{2r+1}}
$$

whenever $y \geq x^{\frac{1}{2r+1}}$, we get

$$
\sum_{x < n \leq x+y \atop f(n)=k} 1 = \frac{y}{\zeta(r)} H_{f,k,r}(1) + O_{\epsilon,r} \left\{ x^\epsilon \left(x^{r-1}y^{r+1}\right)^{\frac{1}{2r^2}} + y x^{-\frac{1}{6(4r-1)(2r-1)}} + y^{1-\frac{2(r-1)}{7(3r-1)}} \right\}
$$

if $x^{\frac{1}{2r+1}} \leq y \leq 4^{-2r^2} x$. In order to prove the existence of the local density, we generalize [6] Theorem 1. Every positive integer $n$ may be uniquely written as $n = ab$, with $(a,b) = 1$, a $r$-free and $b$ $r$-full. Since $f$ is multiplicative, $f(n) = f(a)f(b) = f(b)$ and hence

$$
\sum_{n \leq x \atop f(n)=k} 1 = \sum_{b \leq x \atop f(b)=k} s_r(b) \sum_{a \leq x/b \atop (a,b)=1} \mu_r(a)
$$

$$
= \sum_{b \leq x \atop f(b)=k} s_r(b) \left\{ \frac{x}{\zeta(r)\Psi_r(b)} + O \left( \left( \frac{x}{b} \right)^{1/r} 2\omega(b) \right) \right\}
$$
where

$$\Psi_r(b) := \prod_{p \mid b} \left( 1 + \frac{1}{p} + \cdots + \frac{1}{p^{r-1}} \right).$$

Using the bound

$$\sum_{b \leq x} \frac{s_r(b) 2^n(b)}{b^{1/r}} \ll x^\varepsilon \sum_{b \leq x} \frac{s_r(b)}{b^{1/r}} \ll x^\varepsilon$$

we get

$$\sum_{n \leq x} 1 = \frac{x}{\zeta(r)} \sum_{b \leq x} \frac{s_r(b)}{\Psi_r(b)} + O_{r,\varepsilon} \left( x^{1/r+\varepsilon} \right).$$

Notice that

$$\sum_{b \leq x} \frac{b s_r(b)}{\Psi_r(b)} \ll \sum_{b \leq x} s_r(b) \ll x^{1/r}$$

so that the Dirichlet series of the multiplicative function $b \mapsto \frac{b s_r(b)\mathbf{1}_{f,k}(b)}{\Psi_r(b)}$ is absolutely convergent in the half-plane $\sigma > \frac{1}{r}$. Hence the series

$$\sum_{b \geq 1} \frac{s_r(b)}{\Psi_r(b)}$$

converges absolutely, which implies that the limit of

$$\frac{1}{x} \sum_{n \leq x} 1$$

exists as $x \to \infty$ and is equal to

$$d_{f,k} = \frac{1}{\zeta(r)} \sum_{b \leq x} \frac{s_r(b)}{\Psi_r(b)} = \text{Res}_{s=1} (L(s, \mathbf{1}_{f,k})) = \frac{H_{f,k,r}(1)}{\zeta(r)}$$

achieving the proof of Theorem 1. □

5. Applications

5.1. Abelian groups. As stated in Section 1, the most famous example of prime-independent multiplicative function $f$ satisfying $f(p) = 1$ is the arithmetic function $a$ counting the number of finite, non-isomorphic abelian groups of a given order. We have $a(p^\alpha) = P(\alpha)$ where $P$ is the unrestricted partition function and, from the generating function of $P$,

$$L(s, a) = \prod_{j=1}^\infty \zeta(js) \quad (\sigma > 1).$$

Hence Theorem 1 may be applied with $r = 2$ giving the following result.

Corollary 5. Let $k \in \mathbb{Z}_{>1}$ and $x^{1/8+\varepsilon} \leq y \leq 2^{-16}x$ be real numbers. Then

$$\sum_{x < n \leq x+y \atop a(n) = k} 1 = d_{a,k}y + O_{\varepsilon} \left( x^{1/8+\varepsilon}y^{3/8} + yx^{-1/42+\varepsilon} + y^{4/5}x^{4/5} \right).$$
5.2. **Plane partitions.** Let $P_2(n)$ be the number of plane partitions of $n$ (see [10] for instance) whose generating function is given by
\[
\sum_{n=0}^{\infty} P_2(n)x^n = \prod_{j=1}^{\infty} \left(1 - x^j\right)^{-j} \quad (|x| < 1).
\]

Let $f$ be the multiplicative function such that $f(1) = 1$ and $f(p^\alpha) = P_2(\alpha)$. We deduce from the generating function above that
\[
(f(p^\alpha))_{\alpha \in \mathbb{Z}_{\geq 0}} = (1, 1, 3, 6, 13, 24, 48, 86, 160, 282, 500, 859, 1479, \ldots)
\]
and also
\[
L(s, f) = \prod_{j=1}^{\infty} \zeta(js)^j \quad (\sigma > 1).
\]

Theorem [1] may be applied with $r = 2$ again.

**Corollary 6.** Let $k \in \mathbb{Z}_{\geq 1}$, the function $f$ defined as above and $x^{\frac{1}{4}+\varepsilon} \leq y \leq 2^{-16} x$ be real numbers. Then
\[
\sum_{x<n<x+y \atop f(n)=k} 1 = d_{f,k} y + O_x \left\{ x^{1/8+\varepsilon} y^{3/8} + y x^{-1/42+\varepsilon} + y^{4/5} x^{\varepsilon} \right\}.
\]

5.3. **Semisimple rings.** Another example, closely related to the function $\alpha$, is the multiplicative function $S$ counting the number of finite, non-isomorphic semisimple rings with a given number of elements. For any prime-powers $p^\alpha$, $S(p^\alpha) = P^*(\alpha)$ where $P^*$ is the number of partitions of $\alpha$ into parts which are square. Since the generating function of $P^*$ is
\[
\sum_{n=0}^{\infty} P^*(n)x^n = \prod_{q=1}^{\infty} \prod_{m=1}^{\infty} \left(1 - x^{qm^2}\right)^{-1} \quad (|x| < 1)
\]
we infer that
\[
(S(p^\alpha))_{\alpha \in \mathbb{Z}_{\geq 0}} = (1, 1, 2, 3, 6, 8, 13, 18, 29, 40, 58, 79, 115, 154, 213, \ldots)
\]
and
\[
L(s, S) = \prod_{q=1}^{\infty} \prod_{m=1}^{\infty} \zeta(qm^2 s) \quad (\sigma > 1).
\]

**Corollary 7.** Let $k \in \mathbb{Z}_{\geq 1}$ and $x^{\frac{1}{4}+\varepsilon} \leq y \leq 2^{-16} x$ be real numbers. Then
\[
\sum_{x<n<x+y \atop S(n)=k} 1 = d_{S,k} y + O_x \left\{ x^{1/8+\varepsilon} y^{3/8} + y x^{-1/42+\varepsilon} + y^{4/5} x^{\varepsilon} \right\}.
\]

5.4. **Exponential divisors.** A positive integer $d = p_1^{a_1} \cdots p_s^{a_s}$ is said to be an exponential divisor of a positive integer $n = p_1^{b_1} \cdots p_s^{b_s}$ if and only if, for all $i \in \{1, \ldots, s\}$, $a_i \mid \alpha_i$. It is customary to denote by $\tau(e)(n)$ the number of exponential divisors of $n$. The function $\tau(e)$ is multiplicative and satisfies $\tau(e)(p^\alpha) = \tau(\alpha)$. The same is true for the unitary exponential divisor function $\tau(e)^*$ for which $\tau(e)^*(p^\alpha) = 2^{\omega(\alpha)}$.

**Corollary 8.** Let $k \in \mathbb{Z}_{\geq 1}$ and $x^{\frac{1}{4}+\varepsilon} \leq y \leq 2^{-16} x$ be real numbers. If $f = \tau(e)$ or $f = \tau(e)^*$
\[
\sum_{x<n<x+y \atop f(n)=k} 1 = d_{f,k} y + O_x \left\{ x^{1/8+\varepsilon} y^{3/8} + y x^{-1/42+\varepsilon} + y^{4/5} x^{\varepsilon} \right\}.
\]
5.5. **The \( r \)-th power divisor function.** Let \( r \in \mathbb{Z}_{\geq 2} \) fixed and define the divisor function \( \tau^{(r)} \) by \( \tau^{(r)}(1) = 1 \) and, for any \( n \in \mathbb{Z}_{\geq 2} \)

\[
\tau^{(r)}(n) = \sum_{d|r} 1.
\]

Then \( \tau^{(r)} \) is multiplicative and

\[
\tau^{(r)}(p^\alpha) = 1 + \left\lfloor \frac{\alpha}{r} \right\rfloor \quad \text{and} \quad L(s, \tau^{(r)}) = \zeta(s)\zeta(rs).
\]

**Corollary 9.** Let \( k \in \mathbb{Z}_{\geq 1} \) and \( r \in \mathbb{Z}_{\geq 2} \) fixed, and let \( x^{\frac{1}{2}+\varepsilon} \leq y \leq 4^{-2r^2}x \) be real numbers. Then

\[
\sum_{x<n<x+y \atop \tau^{(r)}(n)=k} 1 = d_{\tau^{(r)}},_k y + O_{r,\varepsilon} \left\{ \left( x^{r-1}y^{r+1} \right)^\frac{1}{2r^2} x^\varepsilon + yx^{-\frac{1}{r(r-1)(2r-1)}+\varepsilon} + y^2 \frac{2(r-1)}{r(r-1)} x^\varepsilon \right\}.
\]

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