A minisuperspace model of compact phase space gravity

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The kinematical phase space of classical gravitational field is flat (affine) and unbounded. Because of this, field variables may tend to infinity leading to appearance of singularities, which plague Einstein’s theory of gravity. The purpose of this article is to study the idea of generalizing the theory of gravity by compactification of the phase space. We investigate the procedure of compactification of the phase space on a minisuperspace gravitational model with two dimensional phase space. In the affine limit, the model reduces to the flat de Sitter cosmology. The phase space is generalized to the spherical case, and the case of loop quantum cosmology is recovered in the cylindrical phase space limit. Analysis of the dynamics reveals that the compactness of the phase space leads to both UV and IR effects. In particular, the phase of re-collapse appears, preventing the universe from expanding to infinite volume. Furthermore, the quantum version of the model is investigated and the quantum constraint is solved. As an example, we analyze the case with the spin quantum number $s = 2$, for which we determine transition amplitude between initial and final state of the classical trajectory. The probability of the transition is peaked at $\Lambda = 0$.

I. INTRODUCTION

Compact phase spaces emerge in the semi-classical description of quantum system with finite dimensional Hilbert spaces. An important property of compactness is that values of phase space variables are bounded. As a consequence, physical quantities such as energy density may be constrained from above, resolving the problem of divergences appearing in case of affine phase spaces. Compactification of phase spaces may, therefore, serve as a way to impose the Principle of finiteness, introduced by Max Born and Leopold Infeld [1].

Following this reasoning, in Ref. [2] a research program of Nonlinear Field Space Theory (NFST) has been initiated with the goal of generalizing the known types of physical fields to the case of compact phase spaces. In the original article [2], the procedure of compactification has been investigated at the level of the Fourier space representation of a scalar field theory. For the standard scalar field, each Fourier mode is associated with a two dimensional $\mathbb{R}^2$ phase space, which in the NFST has been considered as a local approximation to the spherical phase space $S^2$. The procedure has been thereafter applied for the field defined in the position space in Ref. [3]. It has been shown that, thanks to the fact that the spherical phase space is a phase space of angular momentum (spin), a new possibility of relating spin systems with field theories emerges. In particular, it has been shown in Ref. [3] that small excitations of the continuous version of the Heisenberg model are described by non-relativistic scalar field theory, if the large spin limit ($S \to \infty$) is considered. Furthermore, the scalar field theory recovered satisfies the so-called Born reciprocity symmetry between generalized positions and conjugate momenta [4]. The next step was to show that the construction can be generalized to the case of relativistic Klein-Gordon scalar field theory [5]. It was demonstrated that such theory is recovered in the large spin limit of the XX Heisenberg model (XXZ Heisenberg model in the limit of the vanishing anisotropy parameter $\Delta \to 0$). Possible consequences of the compact phase space scalar field theory have been investigated in the cosmological context, by applying the compactness to the inflationary scalar field [6]. It has been shown that the compactness of the inflaton field may have implications on amplitudes of cosmological primordial inhomogeneities as well leads to the phase of cosmological re-collapse [6].

While the case of scalar field theory is a good testing ground for the procedure of compactification of the phase spaces, the ambition of the NFST is to ultimately apply the method to the gravitational interactions. The expectation is that compact phase space extension of general relativity (GR) may resolve the problem of singularities (simply by restricting field variables to take finite values) and pave a way to a finite Hilbert space quantum version of the theory of gravity. A possibility of such approach to quantum theory of gravity has already been discussed in the context of Loop Quantum Gravity (LQG) in Ref. [7]. In the current formulation, LQG is a theory with $su(2) \times SU(2)$ phase space per link of the spin network. Part of the phase space, associated with the $SU(2)$ holonomies is already compact. However, the remaining contribution is affine and is described by elements of the $su(2)$ algebra. The idea pushed forward in Ref. [7] was that generalization of the theory to the compact phase space $SU(2) \times SU(2)$ may resolve certain problems (e.g. IR divergences) present in the current formulation. Furthermore, non-trivial phase spaces are considered in the Relative locality approach to quantum gravity [8]. However, at the current stage of development, the phase space of particles rather than field (including gravitational field) are considered in this approach.

The purpose of this article is to make a step towards construction of the compact phase space version of GR by studying the procedure of compactification of the gravitational degrees of freedom in a minisuper-
space model. More specifically, our objective is to introduce phase space compactness to the flat Friedmann-Robertson-Walker (FRW) cosmological model with positive cosmological constant $\Lambda$. There is of course a freedom of choices of possible compact phase space extensions of initial affine phase space theory. In our studies, we explore the spherical $S^2$ phase space, which will allow us to build relation with the spin physics. However, in general, also other possibilities, such as toroidal phase space $S \times S$, can also be considered.

We study the classical and the quantum theory for which exact analytical solutions are found. We show that loop quantum cosmology is recovered in a certain limit. We demonstrate how transition amplitude can be explicitly computed in the quantum theory using the projector onto solutions of the quantum Hamiltonian constraint and coherent boundary states in the kinematical Hilbert space.

The organization of the article is as follows. In Sec. [II] we introduce the standard de Sitter model and notation used through this article is established. Then in Sec. [III] the spherical phase space is introduced and the affine large spin limit at the kinematical level is discussed. Based on this, in Sec. [IV] the standard FRW dynamics with positive cosmological constant in generalized to the case with compact phase space. The Hamiltonian constraint we obtain is expressed in terms of the spin variable, which parametrizes the phase space. In Sec. [V] we show that the theory reduces to loop quantum cosmology if the phase space is elongated in the direction of one of the canonical variables. This yields a cylindrical phase space. In Sec. [VI] we derive analytical solutions of the full model. The quantum analysis of the model is performed in Sec. [VII] where the quantum constraint is explicitly solved and exemplary transition amplitudes associated with end points of the classical trajectory are calculated. The results are summarized in Sec. [VIII].

II. DE SITTER MODEL

The phase space is a symplectic manifold equipped with closed 2-form $\omega$. In case of the FRW cosmology, $\omega$ can be written in the Darboux form:

$$\omega = dp \wedge dq,$$

where $q$ is the generalized coordinate and $p$ is the canonically conjugated momentum. $q$ is a volume element, related to the scale factor $a$ and fiducial volume $V_0$ as follows: $|q| = V_0 a^3$. Keeping in mind different possible triad orientations, we allow both positive and negative real values of $q$. As a consequence, the phase space for the system is $\mathbb{R}^2$.

In terms of the $q$ and $p$ variables, the Hamiltonian for the flat FRW cosmology with cosmological constant $\Lambda$ takes the form

$$H_{GR} = N q \left( -\frac{3}{4} \kappa p^2 + \frac{\Lambda}{\kappa} \right)$$

where $\kappa := 8\pi G = 8\pi l_P^2$ and $N$ is the lapse function. $l_P \approx 1.62 \cdot 10^{-35}$ is the Planck length.

By inverting the symplectic form (1) the Poisson bracket

$$\{f, g\} = (\omega^{-1})^{ij} \partial_i f \partial_j g = \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial q}.$$ (3)

can be introduced, where $f$ and $g$ are phase space functions. The Poisson bracket allows us to introduce the Hamilton equation $\dot{f} = \{f, H_{GR}\}$. In the cosmological context it is useful to introduce the Hubble factor $H$, which quantifies the rate of cosmological expansion:

$$H := \frac{1}{3} \frac{\dot{q}}{q}.$$ (4)

where the time derivative of $q$ is defined in terms of Hamilton equation $\dot{q} = \{q, H_{GR}\} = -\frac{3}{2} N \kappa q p$. Fixing, from now on, the gauge $N = 1$, we can express the Hubble factor as follows:

$$H = -\frac{1}{2} \kappa p.$$ (5)

Plugging this relation into the scalar constraint, we have

$$0 = \frac{\partial H_{GR}}{\partial N} = q \left( -\frac{3}{4} \kappa p^2 + \frac{\Lambda}{\kappa} \right),$$ (6)

and the Friedmann equation in the well known form

$$H^2 = \frac{\Lambda}{3}$$

is recovered, with exponential solutions $q(t) = C e^{\pm \sqrt{3\Lambda} t}$ for $N = 1$.

III. SPHERICAL PHASE SPACE

For the spherical phase space $S^2$ the natural candidate for the symplectic 2-form is the area form:

$$\omega = S \sin \theta \, d\phi \wedge d\theta,$$

where $\theta$ and $\phi$ are spherical angles and $S$ has been introduced for dimensional reasons. The volume (area) of the phase space is now finite and equal to

$$A = \int_\Omega \omega = 4\pi S.$$ (9)

The affine limit corresponds to $S \to \infty$. As showed in Ref. [3], it is convenient to perform a change of coordinates in the form:

$$\phi = \frac{p}{R_1} \in (-\pi, \pi],$$

$$\theta = \frac{\pi}{2} + \frac{q}{R_2} \in (0, \pi),$$ (11)
such that the 2-form \( \omega \) rewrites as
\[
\omega = \cos \left( \frac{q}{R_2} \right) dp \wedge dq, \tag{12}
\]
where we have set that \( R_1 R_2 = S \). This guarantees that in the large \( S \) limit \( (R_{1,2} \to \infty) \), the symplectic form \( \omega \) simplifies to the \( \mathbb{R}^2 \) case with symplectic form \( \omega_0 \). The form \( \omega \) differs from the one introduced in Ref. \( \omega_0 \) by the change of variables \((q \to p, p \to -q)\), which does not change physics but the convention used here will allow us to make direct connection with polymerization of momentum \( p \). Based on the symplectic form \( \omega \), the Poisson bracket becomes
\[
\{f, g\} = \frac{1}{\cos(q/R_2)} \left( \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial q} \right). \tag{13}
\]

The difference with the \( \mathbb{R}^2 \) case \( \omega_0 \) is the presence of the factor \( 1/\cos(q/R_2) \). Both the \( q \) and \( p \) variables are only locally well defined on the sphere, but it is justified to introduce globally defined functions to study the dynamics also away from the origin of the coordinate system \((q, p) = (0, 0)\). A choice that guarantees that the new variables are globally defined and the algebra of the variables takes a simple form is associated with the parametrization of the sphere in a Cartesian coordinate system. Namely, we introduce a vector \( S = (S_x, S_y, S_z) \), with components expressed in terms of the \( q \) and \( p \) variables as follows:
\[
S_x = S \cos \left( \frac{p}{R_1} \right) \cos \left( \frac{q}{R_2} \right), \tag{14}
\]
\[
S_y = S \sin \left( \frac{p}{R_1} \right) \cos \left( \frac{q}{R_2} \right), \tag{15}
\]
\[
S_z = -S \sin \left( \frac{q}{R_2} \right). \tag{16}
\]

Differentiating the components with respect to the \( q \) and \( p \) variables we get
\[
\frac{\partial S_i}{\partial q} = \begin{cases} \partial S_x/\partial q = -R_1 \cos(p/R_1) \sin(q/R_2) \\ \partial S_y/\partial q = -R_1 \sin(p/R_1) \sin(q/R_2) \\ \partial S_z/\partial q = -R_1 \cos(q/R_2) \end{cases}, \tag{17}
\]
and
\[
\frac{\partial S_i}{\partial p} = \begin{cases} \partial S_x/\partial p = -S_y/R_1 \\ \partial S_y/\partial p = S_x/R_1 \\ \partial S_z/\partial p = 0 \end{cases}, \tag{18}
\]
when applied to the Poisson bracket \( \{S_i, S_j\} = \varepsilon_{ijk} S_k \). We find that the \( S_i \) components satisfy the \( \mathfrak{so}(3) \) algebra
\[
\{S_i, S_j\} = \varepsilon_{ijk} S_k. \tag{19}
\]

The vector \( \vec{S} = (S_x, S_y, S_z) \) is therefore a vector of angular momentum (spin) with magnitude equal to \( S \). The affine limit \( R_{1,2} \to \infty \) is, therefore, a large spin limit.

### IV. COMPACT PHASE SPACE FRW MODEL

In the previous section we have shown that, at the kinematical level, compactification of the affine phase space \( \mathbb{R}^2 \) of the FRW model to the case of \( S^2 \) can be performed replacing the symplectic form \( \omega_0 \) with \( \omega \). The second step is to introduce the compactness at the level of the dynamics, by suitable modification of the minisuperspace Hamiltonian \( \omega_0 \). Since the original \( q \) and \( p \) phase space variables are not globally defined on the sphere, we have to replace them with the \( S_i \) variables introduced in the previous section. The consistency requirement is that in the large spin limit \( (R_{1,2} \to \infty) \) the classical FRW Hamiltonian \( \omega_0 \) would be recovered. We take here the additional requirement that in the \( R_2 \to \infty \) limit the case of the polymer quantization is recovered.

The simplest way to satisfy the above conditions is to perform the following replacements:

\[
p \to p_S := \frac{S_y}{R_2} = R_1 \sin \left( \frac{p}{R_1} \right) \cos \left( \frac{q}{R_2} \right), \tag{20}
\]
\[
q \to q_S := -\frac{S_z}{R_1} = R_2 \sin \left( \frac{q}{R_2} \right), \tag{21}
\]
where the new variables are defined such that \( \lim_{R_1,2 \to \infty} p_S = p \) and \( \lim_{R_1,2 \to \infty} q_S = q \). Applying the above replacements in Eq. \( \omega_0 \) a new Hamiltonian defined on spherical phase space can be introduced:
\[
H_S = N S_z \left[ 3 \frac{S_y^2}{R_1 R_2} - \frac{\Lambda}{4} \right], \tag{22}
\]
such that \( H_S \to H_{GR} \) in the \( R_{1,2} \to \infty \) limit.

The Hamiltonian \( \omega_0 \) and the Poisson bracket \( \omega_0 \) yield the equations of motion:
\[
\dot{S}_x = \frac{3 N \kappa}{4 R_1 R_2} S_y (2S_x^2 - S_y^2) + \frac{N \Lambda}{R_1 \kappa} S_y, \tag{23}
\]
\[
\dot{S}_y = \frac{3 N \kappa}{4 R_1 R_2} S_x S_y^2 - \frac{N \Lambda}{R_1 \kappa} S_x, \tag{24}
\]
\[
\dot{S}_z = -\frac{3 N \kappa}{2 R_1 R_2} S_x S_y S_z. \tag{25}
\]

With the use of the set of equations above, one can check that
\[
\frac{dS^2}{dt} = 2(S_x \dot{S}_x + S_y \dot{S}_y + S_z \dot{S}_z) = 0. \tag{26}
\]

Let us now derive the Friedmann equation. For this purpose, using the Poisson bracket \( \omega_0 \), we calculate
\[
\dot{q} = \frac{1}{\cos(q/R_2)} \frac{\partial H_S}{\partial p} = \frac{3 N \kappa}{2} \frac{S_x S_y S_z}{S^2 \cos(q/R_2)}, \tag{27}
\]
so that (fixing as above the gauge \( N = 1 \)) the Hubble factor \( \omega_0 \) takes the following form
\[
H = \frac{1}{2} \frac{\kappa}{S^2 q} S_x S_y S_z. \tag{28}
\]
or equivalently

$$H = -\frac{S\kappa}{2\varrho} \sin\left(\frac{p}{R_1}\right) \cos\left(\frac{p}{R_1}\right) \sin\left(\frac{q}{R_2}\right) \cos\left(\frac{q}{R_2}\right),$$

(29)

such that in the \(R_{1,2} \to \infty\) limit we recover \(H = -\frac{1}{2} \kappa p\), as expected (see Eq. 5).

The \(\frac{\partial H}{\partial N} = 0\) condition implies that the scalar constraint takes the form:

$$\Lambda = \frac{3}{4} \kappa^2 \frac{S_y^2}{R_1^2},$$

(30)

which reduces to

$$\sin^2\left(\frac{p}{R_1}\right) \cos^2\left(\frac{q}{R_2}\right) = \frac{4}{3} \Lambda \kappa^2 R_1^2.$$  

(31)

Therefore, the Friedmann equation takes the form:

$$H^2 = \frac{\Lambda}{3} \left(\sin\left(\frac{q}{R_2}\right) \frac{q}{R_2}\right)^2 \left[\cos^2\left(\frac{q}{R_2}\right) - \delta\right],$$

(32)

in the \(N = 1\) gauge. The real solutions to the equation can be obtained if the condition

$$\cos^2\left(\frac{q}{R_2}\right) \geq \delta$$

(33)

is satisfied, where for convenience we have defined

$$\delta := \frac{4}{3} \frac{\Lambda}{R_1^2 \kappa^2}.$$  

(34)

V. POLYMER LIMIT

While the spherical case imposes restrictions on the values of \(p\) and \(q\), we now study the limits where the constraints on the phase space variables are released. This is equivalent to elongating the spherical phase space into a cylindrical shaped phase space, by taking either the \(R_2 \to \infty\) or \(R_1 \to \infty\) limit (see Fig. 1). The cylindrical phase space obtained corresponds to the so-called polymerization [3], playing a crucial role in loop quantum cosmology (LQC) [10, 11]. A preliminary analysis of the relation between spherical phase space and the polymerization of momentum at the kinematical has been performed in Ref. [5]. Here, we consider the polymer limit in both canonical directions and explore consequences on dynamics.

A. Momentum polymerisation

We first consider the \(R_2 \to \infty\) limit, so that the symplectic form [12] reduces to the Darboux form [1] and the Poisson bracket [13] reduces to the affine case [3].

FIG. 1. Momentum and position polymerisation (cylindrical) limits of the spherical phase space.

Under this limit, the spin components reduce to:

$$S_x = R_1 R_2 \cos\left(\frac{p}{R_1}\right),$$  

(35)

$$S_y = R_1 R_2 \sin\left(\frac{p}{R_1}\right),$$  

(36)

$$S_z = -R_1 q,$$  

(37)

together with \(R_2 \to \infty\). Based on the \(so(3)\) algebra, this one reduces to:

$$\left\{\sin\left(\frac{p}{R_1}\right), \cos\left(\frac{p}{R_1}\right)\right\} = 0,$$  

(38)

$$\left\{q, R_1 \sin\left(\frac{p}{R_1}\right)\right\} = \cos\left(\frac{p}{R_1}\right),$$  

(39)

$$\left\{q, \cos\left(\frac{p}{R_1}\right)\right\} = -\frac{1}{R_1} \sin\left(\frac{p}{R_1}\right),$$  

(40)

which is the cylindrical algebra on the \(S^1 \times \mathbb{R}\) phase space [12]. If one takes the \(R_1 \to \infty\) limit the only nontrivial contribution to the algebra which remains is (39), which simplifies to \(\{q, p\} = 1\) as expected.

The Hamiltonian of the spherical phase space (22) reduces now to:

$$H_C = Nq \left[-\frac{3}{4} \kappa^2 R_1^2 \sin^2\left(\frac{p}{R_1}\right) + \frac{\Lambda}{\kappa}\right],$$  

(41)

where the \(C\) index denotes that we now deal with cylindrical phase space. We can simplify the Hamiltonian \(H_C\) to:

$$H_C = Nq \left[-\frac{3}{4} \sin^2\left(\frac{\lambda p}{\lambda}\right) + \frac{\Lambda}{\kappa}\right],$$  

(42)

where we have introduced the scale of polymerization \(\lambda := \frac{1}{R_1}\), so that in the limit \(R_1 \to \infty\) (\(\lambda \to 0\)) the FRW Hamiltonian \(H_{GR}\) is recovered.
The expression \([12]\) is equivalent to the Hamiltonian of LQC, where we have the following polymerization of the momentum:

\[
p \rightarrow p_\lambda := \frac{\sin(\lambda p)}{\lambda},
\]

(43)
such that \(\lim_{\lambda \to 0} p_\lambda = p\).

In the \(R_2 \to \infty\) limit, the Friedmann equation \([32]\) simplifies to:

\[
H^2 = \frac{\Lambda}{3} (1 - \delta).
\]

(44)

for \(N = 1\), and a real solution to the equation can be found only if \(\delta \in [0,1]\). Hence, the dynamics imposes bounds on the possible values of cosmological constant. This effect has already been noticed before in the polymerized cosmology \([13-15]\).

Using the expression for the energy density of the cosmological constant \(\rho_\Lambda := \frac{\Lambda}{\kappa}\), the Friedmann equation \([44]\) can be rewritten as

\[
H^2 = \frac{\kappa}{3} \rho_\Lambda \left(1 - \frac{\rho_\Lambda}{\rho_c}\right),
\]

(45)

where the \(\rho_c\) is the critical energy density considered in LQC:

\[
\rho_c := \frac{3}{4} \frac{\kappa}{\lambda^2}.
\]

(46)

The leading contribution in Eq. \([45]\) matches with the Friedmann equation for a flat phase space \([7]\), whereas the second contribution implies correction to the cosmological evolution due to the cylindrical phase space. Eq. \([45]\) has solution in the exponential form

\[
q(t) = C e^{\pm \sqrt{3\Lambda}\, \omega t},
\]

(47)

where we have introduced an effective cosmological constant \(\Lambda_{\text{eff}} := \Lambda (1 - \delta)\).

### B. Position polymerisation

We now proceed as above, but considering the \(R_1 \to \infty\) limit. The symplectic form associated to a spherical phase space is unchanged, and so is the corresponding Poisson bracket, while the spin components reduce to:

\[
S_x = S \cos \left(\frac{q}{R_2}\right),
\]

(48)

\[
S_y = p R_2 \cos \left(\frac{q}{R_2}\right),
\]

(49)

\[
S_z = -S \sin \left(\frac{q}{R_2}\right).
\]

(50)

The \(\mathfrak{so}(3)\) algebra reduces to

\[
\left\{\sin \left(\frac{q}{R_2}\right), \cos \left(\frac{q}{R_2}\right)\right\} = 0,
\]

(51)

\[
\left\{R_2 \sin \left(\frac{q}{R_2}\right), p\right\} = 1,
\]

(52)

\[
\left\{\cos \left(\frac{q}{R_2}\right), p\right\} = -\frac{1}{R_2} \tan \left(\frac{q}{R_2}\right),
\]

(53)

which is different from the cylindrical algebra because of the presence of the non-vanishing \(\cos(q/R_2)\) factor in the symplectic form \([12]\). However, if instead of the \(q\) variable one considers \(q_S\) given by Eq. \([21]\), the symplectic 2-form \([12]\) can be expressed in a Darboux form

\[
\omega = dp \wedge dq_S,
\]

(54)

and we obtain \(\{q_S, p\} = 1\), which agrees with \([52]\).

Rewriting the hamiltonian \(H_S\) \([22]\) under this limit gives

\[
H_C = N \left(R_2 \sin \left(\frac{q}{R_2}\right)\right) \left[-\frac{3}{4} S^2 \cos^2 \left(\frac{q}{R_2}\right) + \frac{\Lambda}{\kappa}\right].
\]

(55)

Regarding the Friedmann equation, we can put equation \([28]\) under the form:

\[
H^2 = \frac{\Lambda}{3} \left(\frac{\sin(q/R_2)}{q/R_2}\right)^2.
\]

(56)

Exact solution to this equation can be found again:

\[
q(t) = 2 R_2 \arctan \left(C e^{\pm \sqrt{3\Lambda} \, t}\right),
\]

(57)

where \(C\) is a constant of integration.

### VI. COSMOLOGICAL EVOLUTION

The aim of this section is to study the consequences of the modified Friedmann equation \([32]\) obtained in case of the spherical phase space. The equation can be rewritten in the form of the integral

\[
\int \frac{\cos x \, dx}{\sin(x) \sqrt{\cos^2(x) - \delta}} = \pm \sqrt{3\Lambda_{\text{eff}}} (t - t_0) + c,
\]

(58)

where \(x := q/R_2 \in [-\pi/2, \pi/2]\). The integral can be performed leading to

\[
\cos(2x) = 2 (1 - \delta) \tanh^2 \left(\pm \sqrt{3\Lambda_{\text{eff}}} (t - t_0)\right)
\]

\[
+ 2 \delta - 1,
\]

(59)

where \(t_0\) is a constant of integration which, without loss of generality, can be fixed as \(t_0 = 0\). The solution is then symmetric with respect to the mirror symmetry \(t \to -t\).
and by examining the limits of Eq. 59 we find that in the $t \to \pm \infty$ limits the value of $q$ tends to its minimal value

$$q_{\text{min}} := 0. \quad (60)$$

On the other hand, at $t = 0$ the value of $q$ reaches value $q_{\text{max}}$ (or symmetrically $-q_{\text{max}}$), where

$$q_{\text{max}} := \frac{R_2}{2} \arccos (2\delta - 1). \quad (61)$$

There are two symmetric branches of solution, related by the mirror symmetry $q \to -q$. The solution has a form of re-collapsing universe with asymptotical de Sitter solutions at $t \to \pm \infty$ with the effective cosmological constant $\Lambda_{\text{eff}}$ (given by Eq. 47). We present the solution in Fig. 2.

A qualitatively similar solution has been studied in the context of quantum reduced loop gravity [16].

In Fig. 3 we show phase trajectories corresponding to the dynamics. Because the kinematical phase space is two-dimensional, by imposing the constraint the physical subspace is just one dimensional subspace (curve). In this figure, the solid line represents the trajectory associated with the solution shown in Fig. 2 and positive values of $p$.

The other three symmetric solutions, obtained by applying the reflections $q \to -q$ and $p \to -p$, are shown as the dashed lines in Fig. 3 (for the same value of $\delta$). The black dots represent beginning and ends to the trajectories, corresponding to times $t \to \pm \infty$. They are located at the equator of the phase space, where $\theta = \pi/2$.

While the $q$ reaches its maximal value, the canonically conjugated variable tends to either $p = \frac{\pi}{2} R_1$ or (for symmetric solution) to $p = -\frac{\pi}{2} R_1$. The minimal positive value of $p$, associated with the ends of the trajectory is

$$p_{\text{min}} = R_1 \arcsin \left( \sqrt{\delta} \right). \quad (62)$$

and the maximal value is

$$p_{\text{max}} = \pi R_1 - p_{\text{min}}. \quad (63)$$

FIG. 2. Evolution of the canonical variable $q$.

FIG. 3. Phase trajectories for the system under consideration.

Based on Eq. 32 the leading $q$-dependent correction to the Friedmann equation is

$$H^2 = \frac{\Lambda_{\text{eff}}}{3} - \frac{\Lambda}{9} \left( 1 + 2\delta \right) \left( \frac{q}{R_2} \right)^2 + O(q^4). \quad (64)$$

Because $|q| := V_0 a^3$, the leading correction scales as $a^6$.

We can define an effective energy density:

$$\rho_{\text{eff}} := \rho_{\Lambda} + \rho_S = \frac{\Lambda_{\text{eff}}}{\kappa} - \frac{\Lambda}{3\kappa} (1 + 2\delta) \left( \frac{q}{R_2} \right)^2. \quad (65)$$

Using the continuity equation:

$$\frac{d}{dt} \rho_{\text{eff}} + 3H (\rho_{\text{eff}} + P_{\text{eff}}) = 0, \quad (66)$$

we find that the universe behaves effectively as filled by a fluid with effective pressure

$$P_{\text{eff}} := P_{\Lambda} + P_S. \quad (67)$$

The effective equation of state implied by the compact nature of the phase space is:

$$P_S = -3\rho_S, \quad (68)$$

where the equation of state for the effective cosmological constant remains in the classical form $P_{\Lambda} = -\rho_{\Lambda}$.

The barotropic index for the correction is $w_S = -3 < -1$, which can be interpreted as a special kind of the phantom matter [17].

A. Evolution of the vector $\mathbf{S}$

Let us now discuss evolution of the three spin components $S_i$. Because of the scalar constraint (30), which
can be written in the form
\[ \frac{\dot{S}_y}{S} = \pm \sqrt{\delta} = \text{const}, \]
the equations of motion \[ \text{(23-25)} \] (on the surface of the constraint) can be written as:
\[ \dot{S}_x = \frac{3}{2} \frac{N \kappa}{R_1 R_2^2} S_y S_x^2, \]
\[ \dot{S}_y = 0, \]
\[ \dot{S}_z = -\frac{3}{2} \frac{N \kappa}{R_1 R_2^2} S_x S_y S_z, \]
with \( S_y = \text{const} \). Because of the equation of the sphere \( S_x^2 + S_y^2 + S_z^2 = S^2 \), only one independent differential equations remains:
\[ \dot{S}_x = -\alpha S_x^2 + \alpha \left( S^2 - S_y^2 \right), \]
where \( \alpha := \frac{3}{2} \frac{N \kappa}{R_1 R_2^2} \). Analytical solution to Eq. \[ \text{(73)} \] can be found leading to:
\[ S_x(t) = \sqrt{S^2 - S_y^2} \tanh \left( \alpha \sqrt{S^2 - S_y^2} (t - t_0) \right), \]
\[ S_y(t) = \pm S \sqrt{\delta}, \]
\[ S_z(t) = \pm \frac{\sqrt{S^2 - S_y^2}}{\cosh \left( \alpha \sqrt{S^2 - S_y^2} (t - t_0) \right)}. \]

VII. QUANTUM THEORY

In the quantum theory the Hamiltonian \[ \text{(22)} \] is promoted to an operator \( \hat{H}_S = N \hat{C} \). The appropriately symmetrized\(^1\) and normalized constraint can be written as:
\[ \hat{C} := \frac{4 S^2}{3 \kappa R_1} \hat{C} = \frac{1}{3} \left( \hat{S}_x \hat{S}_y \hat{S}_y + \hat{S}_y \hat{S}_z \hat{S}_y + \hat{S}_y \hat{S}_y \hat{S}_z \right) - \delta \hat{S}_x^2 \hat{S}_z \approx 0. \]

It is important to stress that there are two ways of looking at the minisuperspace model. The first is that the model provides description of the universe at the largest possible (cosmological) scale, describing averaged degrees of freedom. However, there is also a second interpretation, which is especially interesting in the Belinsky-Khalatnikov-Lifschitz (BKL) \[ \text{[21, 22]} \] limit characterized by decoupling off the space points. In this case, evolution at each space point is described by a homogeneous (in general anisotropic) minisuperspace model. Furthermore, by introducing interactions between the minisuperspace models at points, one can try to study spatial properties of the field configuration \[ \text{[23]} \]. From this perspective, the small spin case can have relevance for the very early universe, before the semiclassical regime has been entered. Therefore, let us start our consideration with the simplest case of spin-1/2.

\(^1\) For simplicity, we do not decompose \( \hat{S}^4 \) and do not include factorial powers of the operators \( \hat{S}_i \) in the symmetrization.
by $S$ component remains constant during the evolution and is given as solutions in the three others quadrants of the sphere. The $\delta$ curve), left to right): $\delta = 0, 1$ (red thick curve), $\delta = 0.9$ (blue thick curve). There are symmetric solutions in the three others quadrants of the sphere. The $S_y$ component remains constant during the evolution and is given by $S_y = S \sqrt{\delta}$. The $t \to \pm \infty$ limits are located at the ends of the curves (on equator). The maximal value of the $S_z$ component corresponds to the maximal value of the $q$ variable that is reached on a trajectory.

A. Spin-1/2

In this case, the spin components are expressed as

$$S_i = \frac{\hbar}{2} \sigma_i$$  \hspace{1cm} (79)

where $\sigma_i$ are Pauli matrices and $i = x, y, z$. Therefore, $\hat{S}^2 = \frac{3\hbar^2}{4} \mathbb{1}$ and the spherical phase space has an area operator $A = 2\pi \sqrt{3} \hbar$. The scalar constraint (77) reduces to:

$$\dot{c} = \frac{1}{3} (1 - 9\delta) \hat{S}_z^2 = \beta \sigma_z$$  \hspace{1cm} (80)

defining $\beta = \frac{1}{3} (1 - 9\delta) \hbar^3$. We directly conclude that there are no physical states in this case: $\dim(H_{\text{phys}}) = 0$. This is because $\hat{C} |\Psi_{\text{phys}}\rangle \sim \sigma_z |\Psi_{\text{phys}}\rangle$, where $\sigma_z = \text{diag}(1, -1)$, does not have nontrivial solutions. In agreement with this, the projector operator $\hat{P}$ is null:

$$\hat{P} := \lim_{T \to \infty} \frac{\sin(\beta T)}{\beta T} \mathbb{1} = 0.$$  \hspace{1cm} (81)

B. General spin $s$

Let us now proceed to the general quantum number $s$. In order to find matrix elements of the $\hat{c}$ in the basis $|s, m\rangle$, where $m = -s, \ldots, s$, it is useful to introduce spin ladder operators:

$$\hat{S}_\pm = \hat{S}_x \pm i \hat{S}_y.$$  \hspace{1cm} (82)

The action of the relevant spin operators on the states $|s, m\rangle$ is given as follows:

$$\hat{S}_z |s, m\rangle = s(s + 1) \hbar |s, m\rangle,$$

$$\hat{S}_x |s, m\rangle = m \hbar |s, m\rangle,$$

$$\hat{S}_y |s, m\rangle = s(s + 1) - m(m + 1) \hbar |s, m, \pm 1\rangle.$$

Applying the above formulas to Eq. (77) we find the matrix elements:

$$\langle s, m' | \hat{c} | s, m \rangle = c_1^m \delta_{m', m+2} + c_2^m \delta_{m', m} + c_3^m \delta_{m', m-2}$$

where $\delta$ denotes the Kronecker delta, and the $c_i^m$ coefficients are defined as

$$\begin{cases}
    c_1^m = -\frac{\hbar^3}{4} (m + 1) \sqrt{(s(s + 1) - m(m + 1))(s(s + 1) - (m + 1)(m + 2))}
    \\
    c_2^m = \hbar^3 m \left(\frac{1}{2} (s(s + 1) - m^2) - \frac{1}{6} - \delta(s + 1)\right)
    \\
    c_3^m = -\frac{\hbar^3}{4} (m - 1) \sqrt{(s(s + 1) - m(m - 1))(s(s + 1) - (m - 1)(m - 2))}
\end{cases}$$  \hspace{1cm} (87)

where we notice the following properties:

$$\begin{cases}
    c_1^m = -c_3^m
    \\
    c_2^m = -c_2^m
    \\
    c_0^{m+2} = c_2^m
\end{cases}$$  \hspace{1cm} (88)

And $\dot{c}$ reduces then to an anti-persymmetric Hankel matrix with only three non-zero diagonals.

For the particular case $s = 1/2$, then $c_0^m$ and $c_2^m$ are not defined, and $\langle s, m' | \hat{c} | s, m \rangle = \frac{m}{12} (1 - 9\delta) \hbar^3 \delta_{m', m}$ which in agreement with Eq. (80).

An important property of the matrix

$$c_{m', m} := \langle s, m' | \hat{c} | s, m \rangle$$

is that its determinant is always equal to zero only for
bosonic representations (s ∈ \mathbb{N}_+). For the fermionic representations with s half integers (2s ∈ \mathbb{N}_+) the determinant is some function of δ and can be equal to zero only for some special values of δ. As a consequence, for arbitrary δ there are nontrivial vectors belonging to the kernel of the \(c_{m',m}\) matrix only in the bosonic case. Furthermore, the matrix is symmetric and therefore has real eigenvalues.

C. Solving the quantum constraint

The general solutions of the constraint equation

\[ \hat{c}|\Psi\rangle = 0, \quad (90) \]

can be expressed in terms of the basis states as follows:

\[ |\Psi\rangle = \sum_{m=-s}^{s} a_m |s,m\rangle, \quad (91) \]

together with the normalization condition \(\sum_{m=-s}^{s} |a_m|^2 = 1\). With the use of the matrix elements, we obtain the following recursive equation

\[ a_{m-2} c_{1}^{m-2} + a_{m-1} c_{1}^{m-1} + a_{m} c_{3}^{m+2} = 0, \quad (92) \]

for \(m = -s + 2, \ldots, 2s - 3, s - 2\), together with

\[ a_{s-2} c_{s}^{s-2} + a_{s-1} c_{s}^{s-1} = 0, \quad (93) \]
\[ a_{s-3} c_{3}^{s-3} + a_{s-1} c_{2}^{s-1} = 0, \quad (94) \]
\[ a_{s+3} c_{3}^{s+3} + a_{s+1} c_{2}^{s+1} = 0, \quad (95) \]
\[ a_{s} c_{3}^{s} + a_{s-2} c_{s}^{s} = 0. \quad (96) \]

Combining these conditions together and using the equation \(92\), we get the following conditions:

\[ a_{s} a_{s-2} = a_{s} a_{s+2}, \quad (97) \]
\[ a_{s} a_{s-3} = a_{s} a_{s+3}, \quad (98) \]
\[ a_{s+1} a_{s-3} = a_{s+1} a_{s+3}, \quad (99) \]
\[ a_{s+1} a_{s-5} = a_{s+1} a_{s+5}. \quad (100) \]

We can now prove by recurrence that, for any \(m \in [-s+2, s-2]\):

\[ a_{s} a_{m} = a_{s} a_{-m}, \quad (101) \]
\[ a_{s+1} a_{m} = a_{s+1} a_{-m}. \quad (102) \]

which allows us, using another recurrence, to show that

- for the bosonic case \(s \in \mathbb{N}_+\), \(\forall m \in [-s, s]\):
  \[ a_{m} = a_{-m} \quad (103) \]
- for the fermionic case \(2s \in \mathbb{N}_+\), \(\forall m \in [-s, s]\):
  \[ a_{m} = ± a_{-m} \quad (104) \]

Let \(\hat{c}\) be an arbitrary \(N \times N\) matrix with \(N = 2k + 1\) for the bosonic case or \(N = 2k\) for the fermionic one, \(k \in \mathbb{N}_+\). The matrix determinant can be represented as:

\[ \det \hat{c} = \begin{vmatrix} c_2^{-s} & c_3^{s+2} & 0 \\ 0 & c_2^{s+1} & 0 \\ c_1^{-s} & c_2^{s+2} & 0 \end{vmatrix} \quad (105) \]

By property of the \(N\) linear alternating form of the determinant, we can exchange columns by pairs and do the same with rows to rewrite:

\[ \det \hat{c} = (-1)^{2k} \begin{vmatrix} c_2^{-s} & 0 & c_1^{s-2} & 0 \\ 0 & c_2^{s+1} & 0 & \ddots \\ c_3^{-s} & 0 & c_2^{s-2} & 0 \\ 0 & \ddots & 0 & \ddots \end{vmatrix} \quad (106) \]

Using the properties of Eq. (88) we finally get:

\[ \det \hat{c} = (\det -|N|) \det \hat{c} = (\det -|N|) \det \hat{c}. \quad (107) \]

Therefore, in the bosonic case (odd \(N\)), the determinant is always 0, ensuring that at least one eigenvalue is equal 0.

From this we conclude that for bosons, in the eigenbasis of \(\hat{c}\) the projection operator can be written as:

\[ \text{diag}(\hat{P}) = (1, \ldots, 1, 0, \ldots, 0) \quad (108) \]

where we have the 1 components originating from each (degenerated) eigenvalue equal to 0 in \(\hat{c}\), whereas any eigenvalue \(\lambda_i \neq 0\) implies a component \(\lim_{T \to \infty} \sin(\lambda_i T) = 0\) in the diagonalized projection operator. Therefore, as long as the projection operator is non 0, there exists a nontrivial solution to equation \(\hat{c}|\Psi\rangle = 0\).

In the fermionic case, non-trivial solutions may exist only for certain values of the parameter \(\delta\). In particular, for \(s = 1/2\) there are no non-trivial solutions (except in the case \(\delta = \frac{1}{3}\) for which the constraint is identically equal zero), for \(s = 3/2\) non-trivial solution exists for \(\delta = \frac{12}{7}\), for \(s = 5/2\) there are two values for which the determinant of \(\hat{c}\) is vanishing: \(\delta = \frac{7}{9}\) and \(\delta = \frac{1}{105}\). Every further case has to be investigated individually.

D. Example: \(s = 2\)

As an example of the solution of the constraint let us examine the case of \(s = 2\). In this case the constraint \(\hat{c}\) takes the following matrix form:

\[ c = \hbar^3 \begin{pmatrix} \frac{5}{3} - 12\delta & 0 & -\sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & \frac{7}{3} - 6\delta & 0 & 0 & 0 \\ -\sqrt{\frac{3}{2}} & 0 & 0 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 6\delta - \frac{7}{3} & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}} & 0 & 12\delta - \frac{5}{3} \end{pmatrix} \quad (109) \]
For $\delta \neq \frac{7}{18}$, there is a single state which satisfies the constraint and is given by

$$|\Psi_{\text{phys}}\rangle = a_{-2}|2, -2\rangle + a_{0}|2, 0\rangle + a_{2}|2, 2\rangle,$$

where appropriately normalized coefficients are

$$a_{2} = a_{-2} = \frac{3}{2} \sqrt{\frac{3}{2}} \sqrt{324\delta^{2} - 90\delta + 13},$$

$$a_{0} = \frac{1}{2} \sqrt{324\delta^{2} - 90\delta + 13} \quad (112)$$

On the other hand, for $\delta = \frac{7}{18}$, the constraint $\hat{c}$ has three linearly independent solutions:

$$|\Psi_{\text{phys},1}\rangle = \frac{1}{2\sqrt{2}}|2, -2\rangle - \frac{\sqrt{3}}{2}|2, 0\rangle + \frac{1}{2\sqrt{2}}|2, 2\rangle,$$

$$|\Psi_{\text{phys},2}\rangle = |2, 1\rangle,$$

$$|\Psi_{\text{phys},3}\rangle = |2, -1\rangle.$$  

The matrix $[109]$ can be diagonalized so that $\mathbf{c} = \mathbf{U}^{-1} \Lambda \mathbf{U}$, where the diagonal matrix $\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$, where the eigenvalues:

$$\lambda_1 = 0,$$  

$$\lambda_2 = \hbar^{3} \frac{1}{3}(7 - 18\delta),$$  

$$\lambda_3 = \hbar^{3} \frac{1}{3}(18\delta - 7),$$  

$$\lambda_4 = -\hbar^{3} \frac{2}{3} \sqrt{324\delta^{2} - 90\delta + 13},$$  

$$\lambda_5 = \hbar^{3} \frac{2}{3} \sqrt{324\delta^{2} - 90\delta + 13}.$$  

With the use of this, the exponentiation of the constraint can be written as

$$e^{ir\mathbf{c}} = \mathbf{U}^{-1} e^{ir\Lambda} \mathbf{U} \quad (122)$$

As a consequence, the projection operator takes the form

$$\hat{P} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} d\tau e^{ir\mathbf{c}}$$

$$= \mathbf{U}^{-1} \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix} \mathbf{U} \quad (124)$$

in the $\delta \neq \frac{7}{18}$ case. The factor 1 in the middle matrix comes directly from the $\lambda_1 = 0$ eigenvalue, whereas any eigenvalue $\lambda_i \neq 0$ implies a component $\lim_{T \to \infty} \frac{\sin(\lambda_i T)}{\lambda_i T} = 0$ in the projection operator.

Using the expression of $\mathbf{U}$, the projection operator finally reads:

$$\hat{P} = \begin{pmatrix}
P_1 & 0 & P_2 & 0 & P_3 \\
0 & 0 & 0 & 0 & 0 \\
P_2 & 0 & P_3 & 0 & P_4 \\
0 & 0 & 0 & 0 & 0 \\
P_3 & 0 & P_4 & 0 & P_1
\end{pmatrix}$$  

(125)

in the $\delta \neq \frac{7}{18}$ case, where we defined:

$$P_1 := \frac{27}{8(324\delta^{2} - 90\delta + 13)} = a_{0}^{2},$$

$$P_2 := \frac{3}{4} \frac{\sqrt{3}}{2} \frac{5 - 36\delta}{\sqrt{324\delta^{2} - 90\delta + 13}} = a_{0}a_{2},$$

$$P_3 := \frac{(5 - 36\delta)^{2}}{4(324\delta^{2} - 90\delta + 13)} = a_{0}^{2}.$$  

The form of the projection operator we have obtained agrees with the expression given as a dyadic product of the physical states $|\Psi_{\text{phys}}\rangle$, i.e.

$$\hat{P} = |\Psi_{\text{phys}}\rangle \langle \Psi_{\text{phys}}|$$  

(129)

taking the physical state given by Eq. $[110]$. The property $\hat{P}^{2} = |\Psi_{\text{phys}}\rangle \langle \Psi_{\text{phys}}| |\Psi_{\text{phys}}\rangle \langle \Psi_{\text{phys}}| = |\Psi_{\text{phys}}\rangle \langle \Psi_{\text{phys}}| = \hat{P}$ is satisfied in a consequence of the proper normalization of the state $|\Psi_{\text{phys}}\rangle$.

If we consider the degenerated case with $\delta = \frac{7}{18}$, for which $\lambda_1 = \lambda_2 = \lambda_3 = 0$, the projection operator reads

$$\hat{P} = \mathbf{U}^{-1} \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix} \mathbf{U}.$$  

(130)

Let’s now take arbitrary $\delta$ and consider sequences from an initial state $|m\rangle := |s, m\rangle$ to a final state $|n\rangle$, together with the transition amplitudes $\langle n | \hat{P} | m \rangle$. From the expressions of $[125]$, we directly observe that states $|-1\rangle$ and $|1\rangle$ do not belong to the sequence if $\delta \neq 7/18$. For the basis states, the transition amplitudes $\langle n | \hat{P} | m \rangle$ are:

- $P_1$: each time we transit from $|m\rangle$ to $|n\rangle$ where $m, n = -2, 2$ (i.e. the state is possibly unchanged).
• $P_2$ each time we transit from $| - 2 \rangle$ to $| 0 \rangle$ and vice versa, or from $| 2 \rangle$ to $| 0 \rangle$ and vice versa.

• $P_3$ each time the state $| 0 \rangle$ is unchanged.

• 1 each time the states $| - 1 \rangle$ and $| 1 \rangle$ remain unchanged for the $\delta = 7/18$ case.

In Sec. [7] we have shown that the cosmological evolution from $t \to - \infty$ to $t \to + \infty$ is associated with the rotation of the spin vector $\vec{S}$ from the position

$$\vec{S}_{\text{in}} = S(-\sqrt{1 - \delta}, \sqrt{\delta}, 0)$$

(131)

to

$$\vec{S}_{\text{out}} = S(\sqrt{1 - \delta}, \sqrt{\delta}, 0).$$

(132)

See the bolded trajectories in Fig. [4]. There are also symmetric solutions in the three other quartets of the sphere, but let us now focus on this representative one. Namely, we want to find what is the quantum transition amplitude associated with this evolution as a function of the parameter $\delta$

$$W(\delta) := \langle \text{out} | \hat{P} | \text{in} \rangle,$$

(133)

for the $s = 2$ case, considered in this section. At the level of quantum mechanics, the boundary conditions [131] and [132] translate into the properties of the mean values of the components of the spin operator $\hat{S}$ in the initial and final states, i.e.

$$\langle \text{in} | \hat{S}_z | \text{in} \rangle = -S\sqrt{1 - \delta},$$

(134)

$$\langle \text{in} | \hat{S}_y | \text{in} \rangle = S\sqrt{\delta},$$

(135)

$$\langle \text{in} | \hat{S}_z | \text{in} \rangle = 0,$$

(136)

and

$$\langle \text{out} | \hat{S}_z | \text{out} \rangle = S\sqrt{1 - \delta},$$

(137)

$$\langle \text{out} | \hat{S}_y | \text{out} \rangle = S\sqrt{\delta},$$

(138)

$$\langle \text{out} | \hat{S}_z | \text{out} \rangle = 0,$$

(139)

where for the case considered, $S = 2\hbar$. Furthermore, it is assumed that dispersion relations are minimized, which is satisfied by $SU(2)$ coherent states. These can be obtained by rotations of the $|2, 2\rangle$ state into directions of the vectors [131] and [132]. An appropriate rotation operator takes the form

$$\hat{R}(\phi, \theta, 0) = e^{-i\hat{\phi}\hat{S}_z}e^{-i\hat{\theta}\hat{S}_y},$$

(140)

where $\phi$ and $\theta$ are the spherical angles introduced in Sec. [III]. What is to be performed is the rotation of the state $|2, 2\rangle$ (for which the spin vector is precessing around the $z$ axis) first by angle $\theta = \pi/2$ around $y$ axis (using the operator $e^{-i\hat{\theta}\hat{S}_y}$). This aligns the vector to the $x - y$ plane. Then we rotate the vector around the $z$ axis (using the operator $e^{-i\hat{\phi}\hat{S}_z}$) to the initial and final orientations of the spin vector, with $\phi_{\text{in}}$ and $\phi_{\text{out}}$. For both initial and final state, we have $\theta_{\text{in}} = \theta_{\text{out}} = \pi/2$ and

$$\phi_{\text{out}} = \arctan \left(\sqrt{\frac{\delta}{1 - \delta}}\right),$$

(141)

$$\phi_{\text{in}} = \pi - \phi_{\text{out}}.$$  

(142)

Using this, the initial and final states can be written as

$$|\text{in/out}\rangle = \hat{R}(\phi_{\text{in/out}}, \pi/2, 0) |2, 2\rangle$$

$$= \sum_{m=-2}^{2} b_m(\phi_{\text{in/out}}, \pi/2, 0) |2, m\rangle,$$

(143)

where the coefficients

$$b_m(\phi_{\text{in/out}}, \pi/2, 0) = \langle 2, m | \hat{R}(\phi_{\text{in/out}}, \pi/2, 0) |2, 2\rangle$$

$$= e^{-i m \phi_{\text{in/out}}} d_{m2}(\pi/2),$$

(144)

where $d_{m2}(\pi/2)$ are components of the Wigner $d$-matrices:

$$d_{22}(\pi/2) = \frac{1}{4},$$

(145)

$$d_{12}(\pi/2) = \frac{1}{2},$$

(146)

$$d_{02}(\pi/2) = \frac{\sqrt{3}}{8},$$

(147)

$$d_{-2}(\pi/2) = \frac{1}{2},$$

(148)

$$d_{-12}(\pi/2) = \frac{1}{4}.$$  

(149)

With the use of these results, we obtain the following initial and final states:

$$|\text{in}\rangle = \left( \frac{1}{4} - \frac{\delta}{2} - \frac{i}{2} \sqrt{\delta(1 - \delta)} \right) |2, -2\rangle$$

$$+ \left( -\frac{1}{2} \sqrt{1 - \delta} + \frac{i}{2} \sqrt{\delta} \right) |2, -1\rangle$$

$$+ \sqrt{\frac{3}{8}} |2, 0\rangle$$

(150)

$$+ \left( -\frac{1}{2} \sqrt{1 - \delta} - \frac{i}{2} \sqrt{\delta} \right) |2, 1\rangle$$

$$+ \left( \frac{1}{4} - \frac{\delta}{2} + \frac{i}{2} \sqrt{\delta(1 - \delta)} \right) |2, 2\rangle$$

and

$$|\text{out}\rangle = \left( \frac{1}{4} - \frac{\delta}{2} + \frac{i}{2} \sqrt{\delta(1 - \delta)} \right) |2, -2\rangle$$

$$+ \left( \frac{1}{2} \sqrt{1 - \delta} + \frac{i}{2} \sqrt{\delta} \right) |2, -1\rangle$$

$$+ \sqrt{\frac{3}{8}} |2, 0\rangle$$

(151)

$$+ \left( \frac{1}{2} \sqrt{1 - \delta} - \frac{i}{2} \sqrt{\delta} \right) |2, 1\rangle$$

$$+ \left( \frac{1}{4} - \frac{\delta}{2} - \frac{i}{2} \sqrt{\delta(1 - \delta)} \right) |2, 2\rangle$$
Then, it is straightforward to calculate the transition amplitude \( W(\delta) \) using the projection operator given by Eq. 125 for \( \delta \neq \frac{7}{18} \) and by Eq. 130 for \( \delta = \frac{7}{18} \). For \( \delta \neq \frac{7}{18} \), the final form of the transition amplitude is

\[
W(\delta) = \frac{3(4 - 21\delta)^2}{8(324\delta^2 - 90\delta + 13)},
\]

which is purely real function. In the special case \( \delta = 7/18 \) the transition amplitude is \( W(7/18) = \frac{337}{193} \approx 0.130 \). We plot modulus square of the amplitude in Fig. 5. The probability of transition is maximized for \( \Lambda = 0 \), where \( |W(0)|^2 = \left(\frac{6}{13}\right)^2 \approx 0.213 \). Then, the probability is decreasing to zero at \( \delta = \frac{4}{21} \) and is increasing to the value \( |W(1)|^2 = \left(\frac{367}{1770}\right)^2 \approx 0.193 \) at the maximal value of \( \delta \). It is interesting that for the values of the cosmological constant near to zero the probability of the transition between two coherent states takes the maximal value. It requires further analysis to be performed to check whether this feature is preserved in the semi-classical \( s \to \infty \) limit.

**VIII. SUMMARY**

The research program of Nonlinear Field Space Theory (NFST) aims at generalizing the known types of field theory to the case of compact phase space. This goal has already been achieved in the case of scalar field theory.

The aim of this program is the compactification of the phase space of gravity. Thanks to the compactness of the phase space, the field variables become constrained, which may eliminate singularities. The compactness of the phase space is a consequence of the finite dimensionality of the Hilbert space. Therefore, the compact phase space of gravitational field theory is expected to be associated with a quantum theory of gravity characterized by the finite number of basis states. In this article, we have examined the possibility of compactifying a minisuperspace gravitational model with a single degree of freedom - a scale factor.

We have focused our attention on the vacuum case with positive cosmological constant. The phase space has been generalized from the affine case \( R^2 \) to a spherical phase space \( S^2 \). This choice has been suggested by the fact that the spherical phase space is a phase space of angular momentum (spin). This enabled us to describe kinematics and dynamics of the model in terms of the angular momentum (spin) vector \( \vec{S} \), which components satisfy the \( so(3) \) \((su(2)) \) algebra. The affine case is recovered in the large spin limit \( R_{1,2} \to \infty \). At the quantum level, the large-spin limit \( s \to \infty \) is known to be associated with semi-classical limit, which is consistent with our discussion.

We have investigated semi-classical dynamics of the system and we have derived the modified Friedmann equation. The ambiguity associated with introduction of the compact phase space extension of the Hamiltonian has been reduced by requirement that the known case of loop quantum cosmology with cylindrical phase space is recovered in the limit \( R_{1,2} \to \infty \). In such case, the so-called polymerization of the momentum \( p \), with the polymerization scale \( \lambda = \frac{1}{R_{1,2}} \), is obtained.

Furthermore, equations of motion for the minisuperspace model with the spherical phase space can be solved analytically. Analysis of the solutions confirmed that both UV and IR results are expected due to compactness of the phase space. The UV effects are associated with the \( p \) variable (related to the Hubble factor) while the IR effects are associated with the \( q \) variable (related to the scale factor). In the considered model, manifestation of the IR effects is the maximal possible value of \( q \) associated with the phase of re-collapse of the universe. On the other hand, the UV effects lead to “renormalization” of the cosmological constant and existence of their maximal value \( \Lambda_+ \).

Thereafter, we have analyzed fully quantum version of the theory and promoted the constraint to be a quantum operator. It turned out that the constraint can be solved recursively for any value of the quantum number \( s \). However, the nontrivial solutions exist only in the bosonic case, while in the fermionic case the solutions may exist only for some special values of the cosmological constant \( \Lambda \). As an example we have analyzed the case with \( s = 2 \). We have found both physical states (one for \( \delta \neq 7/18 \) and three for \( \delta = 7/18 \) which satisfy the constraint as well we derived the form of the projector operator \( \hat{P} \). We used the projection operator to evaluate transition amplitude between two coherent states which are associated with the endpoints of the classical trajectory. We have found that the probability of transition is maximal for \( \Lambda = 0 \).

Some of the future steps in the direction of the research are:
• Analysis of the de Sitter model with positive curvature. This can resolve problem with the interpretation of the IR limit since, in this case, the spatial volume is finite and well defined. Furthermore, the model is expected to lead to oscillatory cosmological evolution.

• Analysis of the toroidal compact phase space $S \times S$. This case can be viewed as a symmetry reduced version of the $SU(2) \times SU(2)$ phase space, being a possible generalization of the loop quantum gravity phase space.

• Reconstruction of the physical Hamiltonian. It would be worth to reconstruct and study properties of the physical Hamiltonian associated with the constrained system under investigation. The resulting physical spin Hamiltonian can be used to implement (in laboratory) analog spin models corresponding to the gravitational dynamics under investigation. For this purpose, e.g. tunable magnetic metamaterials with controlled spins could be used. Such systems have been successfully implemented experimentally in 2D case [24].

• Taking into account matter. Our focus on this article was on the gravitational degrees of freedom and contribution from the cosmological constant has only been taken into account. This needs to be generalized by investigating contributions from different forms of matter. Especially interesting in the cosmological context is the case of a scalar field. Two effects have to be taken into account. The first is an appropriate modification of the scalar field Hamiltonian such that the compactness associated with the $q$ variable is introduced in a consistent way. Another issue is the compactness of the scalar field itself. This second issue has be investigated in Ref. [6]. However, consistent merging of the both types of compactness (gravitational and scalar field) is an open issue to be addressed in future studies.

• Semi-classical limit. In Sec. VII the quantum counterpart of the compact phase space minisuperspace model has been introduced and investigated. A physical quantum state belonging to the kernel by the constraint can be found. One can expect that from the state, the classical cosmological dynamics should be recovered in the (semi-classical) large spin limit, $s \to \infty$. In particular, transition amplitudes between two $SU(2)$ coherent states in the $s \to \infty$ limit have to be investigated. Furthermore, analysis of the transitions amplitudes should be extended beyond the case of boundaries of a classical trajectory, e.g. to the case of arbitrary two points on the phase space.

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