POLES OF NON-ARCHIMEDEAN ZETA FUNCTIONS FOR
NON-DEGENERATE RATIONAL FUNCTIONS

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Abstract. In this article, we study local zeta functions over non-Archimedean
locals fields of arbitrary characteristic attached to rational functions and charac-
ters $\chi$ of the units of the ring of integers $\mathcal{O}_K$ with conductor 1. When the
rational function is non-degenerate with respect to its Newton polyhedron,
we give an explicit formula for the local zeta function and a list of the pos-
sible poles in terms of the normal vectors of the supporting hyperplanes of
the Newton polyhedron attached to the rational function and their expected
multiplicities. Furthermore, we obtain the conditions under which the local
zeta function attached to the trivial character has at least one real pole by
describing the largest negative real pole and the smallest positive one.

1. Introduction

Let $K$ be a non-trivially valued non-Archimedean local field of arbitrary charac-
teristic. By a well-known classification theorem, a non-Archimedean local field is
a finite extension of $\mathbb{Q}_p$, the field of $p$-adic numbers, or the field of formal Laurent
series $\mathbb{F}_q((T))$ over a finite field $\mathbb{F}_q$. For further details the reader may consult [2,
Chapter 1]. We denote by $\mathcal{O}_K$ the ring of integers of $K$ and let $\mathbf{F}_q$ be the residue
field of $K$, the finite field with $q = p^m$ elements, where $p$ is a prime number.
For $z \in K \setminus \{0\}$, let $\nu(z) \in \mathbb{Z} \cup \{+\infty\}$ denote the valuation
of $z$, let $|z|_K = q^{-\nu(z)}$ denote the normalized absolute value (or norm), and let $ac(z) = z\pi^{-\nu(z)}$ denote
the angular component, where $\pi$ is a fixed uniformizing parameter of $K$. We extend
the norm $|\cdot|_K$ to $K^n$ by taking $||(x_1, \ldots, x_n)||_K := \max \{|x_1|_K, \ldots, |x_n|_K\}$. Then
$(K^n, ||\cdot||_K)$ is a complete metric space and the metric topology is equal to the
product topology.

Let $\mathcal{O}_K^\times$ be the multiplicative group of $\mathcal{O}_K$. A character of $\mathcal{O}_K^\times$ is a continuous
homomorphism $\chi : \mathcal{O}_K^\times \to S^1$ where $S^1$ is the circle in $\mathbb{C}$ considered as a multiplica-
tive group. We will call the characters of $\mathcal{O}_K^\times$ the multiplicative characters of $K$,
because given any character $\chi$ of $\mathcal{O}_K^\times$, the mapping $x \to \chi(ac(x))$ gives rise to a
character of the multiplicative group $K^\times$ of $K$. We formally put $\chi(0) = 0$. We will
denote by $\chi_{\text{triv}}$ the trivial character of $\mathcal{O}_K^\times$. Since $\mathcal{O}_K^\times$ is a totally disconnected,
compact, abelian group, there exists $e \in \mathbb{N}$ such that $|\chi|_{1+\tau K^\times} = 1$. The smallest
positive integer satisfying this condition $e = e(\chi)$ is called the conductor of the
character $\chi$. Notice that $e(\chi_{\text{triv}}) = 1$ and $e(\chi) = e(\chi^{-1})$.

Let $f, g \in \mathcal{O}_K[x_1, \ldots, x_n] \setminus \pi\mathcal{O}_K[x_1, \ldots, x_n]$ be two non-constant co-prime poly-
nomials, $n \geq 2$. The twisted local zeta function associated to a rational function

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\[ f/g \text{ and a character } \chi \text{ is defined as} \]
\[
Z(s, \chi, \frac{f}{g}) = \int_{\mathcal{O}_K^* \setminus D_K} \chi \left( ac \left( \frac{f(x)}{g(x)} \right) \right) \left| \frac{f(x)}{g(x)} \right|^s |dx|; \\
\]
where \( D_K = \{ x \in \mathcal{O}_K : f(x) = 0 \} \cup \{ x \in \mathcal{O}_K : g(x) = 0 \} \) and \( |dx|_K \) denotes the Haar measure on \((\mathbb{K}^n, +)\), normalized such that the measure of \( \mathcal{O}_K^* \) is equal to one.

If \( \chi = \chi_{\text{triv}} \) we use the notation \( Z(s, \frac{f}{g}) \) instead of \( Z(s, \chi_{\text{triv}}, \frac{f}{g}) \).

These local zeta functions were established in [14] by W. Veys and W. A. Zúñiga-Galindo. They extended Igusa’s theory [10], to the case of meromorphic functions \( f/g \), with coefficients in a local field of characteristic zero. In this generalization new geometric phenomena appear, for example, in the classical local zeta functions all the possible poles only have negative real parts, but in this case also positive real parts appear.

In the general theory of local zeta functions, two basic problems are to find an explicit formula for the local zeta functions and to determine the poles of their meromorphic continuation because the most of the candidate poles are not actually poles. In the case of \( K \) having characteristic zero, the strategy used to calculate the candidate poles of local zeta functions for rational functions is to use resolution of singularities. In [14], it was determined that the real parts of the poles are contained in the set of ratios \( \left\{ -\frac{v_i}{N_i} \right\} \cup \{-1, 1\} \) where \( \{(N_i, v_i)\} \) are the numerical data of an embedded resolution of singularities of the divisor \( f^{-1}(0) \cup g^{-1}(0) \), however in the general case of positive characteristic, this method cannot be applied. Thus, in this article we use a technique based on Newton polyhedra to give a geometric description of the candidate poles and an explicit formula for \( Z(s, \chi, \frac{f}{g}) \) when \( K \) has arbitrary characteristic and \( f/g \) is non-degenerate with respect to certain Newton polyhedron. Furthermore, we study the conditions under which some of the candidate poles are actually poles.

The local zeta functions attached to a class of non-degenerate rational functions (Laurent polynomials) were first studied in [5] and [6]. The notion of non-degeneracy used here allows us to study the twisted local zeta functions attached to much larger class of rational functions, see [12, Definition 1]. In [12], the author and W. A. Ziga Galindo studied the most simple case of local zeta functions attached to non-degenerate rational functions, thus this work generalizes those results. Moreover, these results extend the corresponding results of K. Hoornaert [11], to the case of rational functions.

This article is organized as follows. In Section 2 we review some basic aspects about polyhedral subdivisions and Newton polyhedra, we also recall the notion of non-degeneracy for polynomial mappings established in [12]. In Section 3, we study the meromorphic continuation for twisted multivariate local zeta functions attached to non-degenerate mappings. These local zeta functions were introduced by L. Loeser in [7], he showed, in the case of local fields of zero characteristic, that they admit meromorphic continuations as rational functions by using resolution of singularities. In this paper, we use a geometric method to compute explicitly the meromorphic continuations of multivariate local zeta functions over non-archimedean fields of arbitrary characteristic, see Theorem 3.7. In Section 4 we show that the local zeta functions attached to non-degenerate rational functions
have meromorphic continuations as rational functions, see Theorem \[4.1\] In Section 5 we give a geometric description of the all possible poles of these twisted local zeta functions and their expected order. We will notice that the real parts of the poles can be positive or negative rational numbers. Furthermore, we give the upper and lower bounds for the possible negative and positive real parts of the poles in terms of \((t_0, \ldots, t_0)\), the intersection point of the diagonal with the boundary of the Newton polyhedron of \(f/g\). In the classical case, when \(\chi = \chi_{\text{triv.}}\), it is well known that the largest real negative pole different from \(-1\) is \(-1/t_0\) if \(t_0 > 1\), in this case this remains true if \((t_0, \ldots, t_0)\) is the intersection point with the boundary of the Newton polyhedron of \(f\), see Theorem \[5.8\] Similarly, we prove that the smallest real positive pole different from \(1\) is \(1/t_0\) if \(t_0 > 1\) and \((t_0, \ldots, t_0)\) is the intersection point of the diagonal with the boundary of the Newton polyhedron of \(g\). In Section 6 we show that, under certain conditions, these local zeta functions attached to the trivial character have a real pole, by describing the largest and the smallest negative and positive real poles, respectively.

1.1. Notation. Along this article, vectors will be written in boldface, so for instance we will write \(\mathbf{b} := (b_1, \ldots, b_l)\) where \(l\) is a positive integer. For polynomials we will use \(x = (x_1, \ldots, x_n)\), thus \(h(x) = h(x_1, \ldots, x_n)\). For each \(n\)-tuple of natural numbers \(k = (k_1, \ldots, k_n) \in \mathbb{N}^n\), we will denote by \(\sigma(k)\) the sum of all its components i.e. \(\sigma(k) = k_1 + k_2 + \ldots + k_n\). Furthermore, \(|dx|_K\) denotes the Haar measure on \((K^n, +)\), normalized so that the measure of \(O^*_K\) is equal to one. In dimension one we will use the notation \(|dx|_K\).

Denote by \(\overline{x}\) the image of an element of \(O^*_K\) under the canonical homomorphism \(O^*_K \to (\pi O_K)^n \cong \mathbb{F}^n_q\), we call \(\overline{x}\) the reduction modulo \(\pi\) of \(x\). Given \(h(x) \in O_K[x_1, \ldots, x_n]\), we denote by \(\overline{h}(x)\) the polynomial obtained by reducing modulo \(\pi\) the coefficients of \(h(x)\). Furthermore if \(f = (f_1, \ldots, f_r)\) is a polynomial mapping with \(f_i \in O_K[x_1, \ldots, x_n]\) for all \(i\), then \(\overline{f} := (\overline{f}_1, \ldots, \overline{f}_r)\) denotes the polynomial mapping obtained by reducing modulo \(\pi\) all the components of \(f\).

2. Polyhedral Subdivisions of \(\mathbb{R}^n_+\) and Non-degeneracy Conditions

In this section we review, without proofs, some well-known results about Newton polyhedra and non-degeneracy conditions that we will use along the article. Our presentation follows closely \[15\], \[13\].

2.1. Newton polyhedra. We set \(\mathbb{R}^+_n := \{x \in \mathbb{R}^n; x \geq 0\}\). Let \(G\) be a non-empty subset of \(\mathbb{N}^n\). The Newton polyhedron \(\Gamma = \Gamma(G)\) associated to \(G\) is the convex hull in \(\mathbb{R}^n_+\) of the set \(\bigcup_{m \in G} \{m + \mathbb{R}^n_+\}\). For instance classically one associates a Newton polyhedron \(\Gamma(h)\) (at the origin) to a polynomial function \(h(x) = \sum_{m \in \mathbb{N}^n} c_m x^m\) \((x = (x_1, \ldots, x_n), h(0) = 0\), where \(G^\text{supp}(h) := \{m \in \mathbb{N}^n; c_m \neq 0\}\). Further we will associate more generally a Newton polyhedron to a polynomial mapping.

We fix a Newton polyhedron \(\Gamma\) as above. We first collect some notions and results about Newton polyhedra that will be used in the next sections. Let \(\langle \cdot , \cdot \rangle\) denote the usual inner product of \(\mathbb{R}^n\), and identify the dual space of \(\mathbb{R}^n\) with \(\mathbb{R}^n\) itself by means of it.

Let \(H\) be the hyperplane \(H = \{x \in \mathbb{R}^n; \langle x, b \rangle = c\}\), \(H\) determines two closed half-spaces

\[H^+ = \{x \in \mathbb{R}^n; \langle x, b \rangle \geq c\} \quad \text{and} \quad H^- = \{x \in \mathbb{R}^n; \langle x, b \rangle \leq c\}.
\]
We say that \( H \) is a \textit{supporting hyperplane} of \( \Gamma(h) \) if \( \Gamma(h) \cap H \neq \emptyset \) and \( \Gamma(h) \) is contained in one of the two closed half-spaces determined by \( H \). By a \textit{proper face} \( \tau \) of \( \Gamma(h) \), we mean a non-empty convex set \( \tau \) obtained by intersecting \( \Gamma(h) \) with one of its supporting hyperplanes. By the \textit{faces} of \( \Gamma(h) \) we will mean the proper faces of \( \Gamma(h) \) and the whole the polyhedron \( \Gamma(h) \). By \textit{dimension} of a face \( \tau \) of \( \Gamma(h) \) we mean the dimension of the affine hull of \( \tau \), and its \textit{codimension} is \( \text{cod} \tau = n - \dim(\tau) \), where \( \dim(\tau) \) denotes the dimension of \( \tau \). A face of codimension one is called a \textit{facet}.

For \( \mathbf{k} \in \mathbb{R}_+^n \), we define
\[
d(\mathbf{k}, \Gamma) = \min_{x \in \Gamma} \langle \mathbf{k}, x \rangle,\]
and the \textit{first meet locus} \( F(\mathbf{k}, \Gamma) \) of \( \mathbf{k} \) as
\[
F(\mathbf{k}, \Gamma) := \{ x \in \Gamma; \langle \mathbf{k}, x \rangle = d(\mathbf{k}, \Gamma) \}.\]
The first meet locus is a face of \( \Gamma \). Moreover, if \( \mathbf{k} \neq 0 \), \( F(\mathbf{k}, \Gamma) \) is a proper face of \( \Gamma \).

If \( \Gamma = \Gamma(h) \), we define the \textit{face function} \( h_k(x) \) of \( h(x) \) with respect to \( k \) as
\[
h_k(x) = h_{F(\mathbf{k}, \Gamma)}(x) = \sum_{m \in F(\mathbf{k}, \Gamma)} c_m x^m.\]

In the case of functions having subindices, say \( h_i(x) \), we will use the notation \( h_{i,k}(x) \) for the face function of \( h_i(x) \) with respect to \( k \).

2.2. Polyhedral Subdivisions Subordinate to a Polyhedron. We define an equivalence relation in \( \mathbb{R}_+^n \) by taking \( \mathbf{k} \sim \mathbf{k}' \Leftrightarrow F(\mathbf{k}, \Gamma) = F(\mathbf{k}', \Gamma) \). The equivalence classes of \( \sim \) are sets of the form
\[
\Delta_\tau = \{ \mathbf{k} \in \mathbb{R}_+^n; F(\mathbf{k}, \Gamma) = \tau \},
\]
where \( \tau \) is a face of \( \Gamma \).

We recall that the \textit{cone strictly spanned} by the vectors \( \mathbf{w}_1, \ldots, \mathbf{w}_r \in \mathbb{R}_+^n \setminus \{0\} \) is the set \( \Delta = \{ \lambda_1 \mathbf{w}_1 + \cdots + \lambda_r \mathbf{w}_r; \lambda_i \in \mathbb{R}_+, \lambda_i > 0 \} \). If \( \mathbf{w}_1, \ldots, \mathbf{w}_r \) are linearly independent over \( \mathbb{R} \), \( \Delta \) is called a \textit{simplicial cone}. If \( \mathbf{w}_1, \ldots, \mathbf{w}_r \in \mathbb{Z}_n \), we say \( \Delta \) is a \textit{rational cone}. If \( \{ \mathbf{w}_1, \ldots, \mathbf{w}_r \} \) is a subset of a basis of the \( \mathbb{Z} \)-module \( \mathbb{Z}^n \), we call \( \Delta \) a \textit{simple cone}.

A precise description of the geometry of the equivalence classes modulo \( \sim \) is as follows. Each \textit{facet} \( \gamma \) of \( \Gamma \) has a unique vector \( \mathbf{w}(\gamma) = (w_{\gamma,1}, \ldots, w_{\gamma,n}) \in \mathbb{N}^n \setminus \{0\} \), whose nonzero coordinates are relatively prime, which is perpendicular to \( \gamma \). We denote by \( D(\Gamma) \) the set of such vectors. The equivalence classes are rational cones of the form
\[
\Delta_\tau = \{ \sum_{i=1}^r \lambda_i \mathbf{w}(\gamma_i); \lambda_i \in \mathbb{R}_+, \lambda_i > 0 \},
\]
where \( \tau \) runs through the set of faces of \( \Gamma \), and \( \gamma_i, i = 1, \ldots, r \) are the facets containing \( \tau \). We note that \( \Delta_\tau = \{0\} \) if and only if \( \tau = \Gamma \). The family \( \{ \Delta_\tau \}_{\tau} \), with \( \tau \) running over the proper faces of \( \Gamma \), is a partition of \( \mathbb{R}_+^n \setminus \{0\} \); we call this partition a \textit{polyhedral subdivision of} \( \mathbb{R}_+^n \) \textit{subordinate} to \( \Gamma \). We call \( \{ \Delta_\tau \}_{\tau} \), the family formed by the topological closures of the \( \Delta_\tau \), a \textit{fan subordinate} to \( \Gamma \).

Each cone \( \Delta_\tau \) can be partitioned into a finite number of simplicial cones \( \Delta_{\tau,i} \). In addition, the subdivision can be chosen such that each \( \Delta_{\tau,i} \) is spanned by part
of $\mathcal{D}(\Gamma)$. Thus from the above considerations we have the following partition of $\mathbb{R}_+^n \setminus \{0\}$:

$$\mathbb{R}_+^n \setminus \{0\} = \bigcup_{\tau} \left( \bigcup_{i=1}^{l_\tau} \Delta_{\tau,i} \right),$$

where $\tau$ runs over the proper faces of $\Gamma$, and each $\Delta_{\tau,i}$ is a simplicial cone contained in $\Delta_\tau$. We will say that $\{\Delta_{\tau,i}\}$ is a simplicial polyhedral subdivision of $\mathbb{R}_+^n$ subordinate to $\Gamma$, and that $\{\Sigma_{\tau,i}\}$ is a simplicial fan subordinate to $\Gamma$.

By adding new rays, each simplicial cone can be partitioned further into a finite number of simple cones. In this way we obtain a simple polyhedral subdivision of $\mathbb{R}_+^n$ subordinate to $\Gamma$, and a simple fan subordinate to $\Gamma$ (or a complete regular fan) (see e.g. [8]).

2.3. The Newton polyhedron associated to a polynomial mapping. Let $h = (h_1, \ldots, h_r)$, $h(0) = 0$, be a non-constant polynomial mapping. In this article we associate to $h$ a Newton polyhedron $\Gamma(h) := \Gamma(\prod_{i=1}^r h_i(x))$. From a geometrical point of view, $\Gamma(h)$ is the Minkowski sum of the $\Gamma(h_i)$, for $i = 1, \ldots, r$ (see e.g. [13], [4], [15]). By using the results previously presented, we can associate to $\Gamma(h)$ a simplicial polyhedral subdivision $\mathcal{F}(h)$ of $\mathbb{R}_+^n$ subordinate to $\Gamma(h)$.

Remark 2.1. A basic fact about the Minkowski sum operation is the additivity of the faces. From this fact follows:

1. $F(k, \Gamma(h)) = \sum_{j=1}^r F(k, \Gamma(h_j))$, for $k \in \mathbb{R}_+^n$;
2. $d(k, \Gamma(h)) = \sum_{j=1}^r d(k, \Gamma(h_j))$, for $k \in \mathbb{R}_+^n$;
3. let $\tau$ be a proper face of $\Gamma(h)$, and let $\tau_j$ be proper face of $\Gamma(h_j)$, for $i = 1, \ldots, r$.

If $\tau = \sum_{j=1}^r \tau_j$, then $\Delta_{\tau} \subseteq \bigcup_{j=1}^r \Delta_{\tau_j}$, for $i = 1, \ldots, r$.

Remark 2.2. Note that the equivalence relation,

$$k \sim k' \iff F(k, \Gamma(h)) = F(k', \Gamma(h)),$$

used in the construction of a polyhedral subdivision of $\mathbb{R}_+^n$ subordinate to $\Gamma(h)$ can be equivalently defined in the following form:

$$k \sim k' \iff F(k, \Gamma(h_j)) = F(k', \Gamma(h_j)),$$

for each $j = 1, \ldots, r$.

This last definition is used in Oka’s book [13].

The following result is a generalization of the one given in [9] Lemma 2.6] to the case of polynomial mappings. The proof follows from the definition of the Newton polyhedron associated to a polynomial mapping $h = (h_1, \ldots, h_r)$ as the Newton polyhedron of the polynomial $\prod_{i=1}^r h_i$.

Lemma 2.3. Let $h$ be a polynomial mapping as above. Let $\tau$ be a proper face of $\Gamma(h)$ and $\gamma_1, \ldots, \gamma_c$ be all the facets of $\Gamma(h)$ such that $\tau \subseteq \gamma_i$. Let $w_1, \ldots, w_c$ be the unique primitive vectors in $\mathbb{N} \setminus \{0\}$ which are perpendicular to respectively $\gamma_1, \ldots, \gamma_c$. Then the cone $\Delta_{\tau}$ associated to $\tau$ is the following convex cone

$$\Delta_{\tau} = \{ \lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_c w_c : \lambda_i \in \mathbb{R}, \lambda_i > 0 \}$$

and its dimension is equal to $n - \dim(\tau)$.

Let $h = (h_1, \ldots, h_c)$ be a polynomial mapping. If $h_i = \sum_{m} c_{m,i} x^m$ and $k \in \mathbb{R}_+^n$, we set

$$h_{i,k}(x) = \sum_{m \in F(k, \Gamma(h_i))} c_{m,i} x^m$$
2.4. Non-degeneracy Conditions. For $K = \mathbb{Q}_p$, Hoornaert in [11] Theorem 3.4 gave an explicit formula for $Z(s, \chi, h)$, in the case $r = 1$ with $h$ a non-degenerate polynomial with respect to its Newton polyhedron $\Gamma(h)$. This explicit formula can be generalized to the case $r \geq 1$ by using the condition of non-degeneracy for polynomial mappings introduced here.

**Definition 2.4.** Let $h = (h_1, \ldots, h_r)$, $h(0) = 0$, be a polynomial mapping with $r \leq n$ and let $\Gamma(h)$ be the Newton polyhedron of $h$ at the origin. The mapping $h$ is called non-degenerate over $\mathbb{F}_q$ with respect to $\Gamma(h)$, if for every vector $k \in \mathbb{R}_+^n$ and for any non-empty subset $I \subseteq \{1, \ldots, r\}$, it verifies that

$$\text{rank}_{k_i} \left[ \frac{\partial h_i}{\partial x_j}(\zeta) \right]_{i,j \in \{1, \ldots, n\}} = \text{Card}(I)$$

for any

$$\zeta \in \{ \zeta \in (\mathbb{F}_q^\times)^n ; h_i(\zeta) = 0 \Leftrightarrow i \in I \}.$$

The above non-degeneracy definition was introduced in [12].

Let $\Delta$ be a rational simplicial cone spanned by $S = \{s_1, \ldots, s_r\} \subseteq \mathbb{R}_+^n$. We define the barycenter of $\Delta$ as $b(\Delta) = \sum_{i=1}^r w_i$. Set $b(\{0\}) := 0$.

**Remark 2.5.** (i) Let $F(h)$ be a simplicial conical subdivision of $\mathbb{R}_+^n$ subordinate to $\Gamma(h)$. Then, it is sufficient to verify the condition given in Definition 2.4 for $k = b(\Delta)$ with $\Delta \in F(h) \cup \{0\}$.

(ii) Notice that our notion of non-degeneracy agrees, in the case $K = \mathbb{Q}_p$, $r = 1$, with the corresponding notion in [12] and with the notion given in [3, Definition 4.2] when $r = 2$.

3. Meromorphic continuation of multivariate local zeta functions

In this section we will give a formula for the multivariate local zeta functions $Z(s, \chi, h)$ (see Theorem 3.7) that holds if the polynomial mapping $h$ is non-degenerate over $\mathbb{F}_q$ with respect to all faces of its Newton polyhedron. In [12], we obtain the result for $\chi = \chi_{\text{triv}}$, thus in this work we extend the results to the case of characters $\chi$ with conductor $e(\chi) = 1$.

3.1. Multivariate Igusa’s local zeta functions. Let $n \geq 2$, $r \leq n$, and

$$h = (h_1, \ldots, h_r) : K^n \rightarrow K^r$$

be a polynomial mapping such that each $h_i(x) \in \mathcal{O}_K[x_1, \ldots, x_n] \cap \mathcal{O}_K[x_1, \ldots, x_n]$.

The twisted multivariate Igusa local zeta function associated to $s = (s_1, \ldots, s_r) \in \mathbb{C}^r$, $\chi = (\chi_1, \ldots, \chi_r)$, where $\chi_i$ a multiplicative character of $\mathcal{O}_K^\times$, and $h$ is defined as

$$Z(s, \chi, h) = \int_{\mathcal{O}_K^n \setminus D_K} \prod_{i=1}^r \chi_i(a(c(h_i(x)))) |h_i(x)|_{K}^{|s_i|} \, dx |K$$

where $D_K := \bigcup_{i \in \{1, \ldots, r\}} \{x \in \mathcal{O}_K^n ; h_i(x) = 0\}$. Notice that $Z(s, \chi, h)$ converges for $\text{Re}(s_i) > 0$ for all $i = 1, \ldots, r$. We write $Z(s, \chi_{\text{triv}}, h)$ when $\chi = \chi_{\text{triv}} := (\chi_{\text{triv}}, \ldots, \chi_{\text{triv}})$. By $\chi \neq \chi_{\text{triv}}$ we mean that there exists at least one index $i$ such that $\chi_i \neq \chi_{\text{triv}}$. 

3.2. Some π-adic integrals. Given \( s = (s_1, \ldots, s_l) \in \mathbb{C}^l \) with \( \text{Re}(s_i) > 0 \), \( i = 1, \ldots, l \) and \( \chi = (\chi_1, \ldots, \chi_l) \) as above, we set

\[
I(s) = I(s, \chi) := \int_{\mathcal{O}_K' \setminus \{0\}} \prod_{i=1}^l \chi_i(ac(x_i))|x_i|_K^n |dx|_K.
\]

Proposition 3.1. With the preceding notation,

\[
I(s, \chi) = \begin{cases} 
0 & \text{if } \chi \neq \chi_{\text{triv}} \\
\frac{L((q^{-s_i})_{i=1}^{l}x, \chi)}{(1-q^{-s_i})} \prod_{i=1}^l (1-q^{-s_i}) & \text{if } \chi = \chi_{\text{triv}} 
\end{cases}
\]

where \( L(q^{-s_i}, \chi) \) is a polynomial in \( q^{-s_i}, \ i \in \{1, \ldots, l\} \) with complex coefficients.

Proof. If \( \chi \neq \chi_{\text{triv}} \), there exists at least one index \( i \) such that \( \chi_i \neq \chi_{\text{triv}} \). Without loss of generality we may assume that \( i = 1 \). Thus there exists an element \( u \in \mathcal{O}_K' \) such that \( \chi_1(u) \neq 1 \). By making the following change of variables

\[
x_1 \mapsto ux_i',
\]

we obtain

\[
\int_{\mathcal{O}_K' \setminus \{0\}} \prod_{i=1}^l \chi_i(ac(x_i))|x_i|_K^n |dx|_K = \chi_1(u) \int_{\mathcal{O}_K' \setminus \{0\}} \prod_{i=1}^l \chi_i(ac(x_i'))|x_i'|_K^n |dx'|_K.
\]

Since \( (1 - \chi_1(u))I(s, \chi) = 0 \), it follows that \( I(s, \chi) = 0 \).

Now, if \( \chi = \chi_{\text{triv}} \), we set \( I(s, \chi_{\text{triv}}) = I(s) \). For a nonempty subset \( J \subseteq \{1, \ldots, l\} \), we set

\[
W_J = \{ x \in \mathcal{O}_K : \nu(x_i) = 0 \leftrightarrow i \in J \}.
\]

Thus, \( \mathcal{O}_K' \) admits the partition

\[
\mathcal{O}_K' = (\pi \mathcal{O}_K') \bigcup_{J \subset \{1, \ldots, l\}} \bigcup_{J \neq \emptyset} W_J.
\]

Then

\[
I(s) = \int_{(\pi \mathcal{O}_K') \setminus \{0\}} \prod_{i=1}^l |x_i|_K^n |dx|_K + \sum_{J \subset \{1, \ldots, l\}} I(s, J)
\]

where

\[
I(s, J) := \int_{W_J} \prod_{i=1}^l |x_i|_K^n |x|_K.
\]

Notice that a direct calculation shows that

\[
\int_{(\pi \mathcal{O}_K') \setminus \{0\}} \prod_{i=1}^l |x_i|_K^n |dx|_K = q^{-\sum_{i=1}^l s_i} I(s)
\]
On the other hand, by the definition of $W_J$,

$$I(s, J) = \left( \prod_{i \in J} \int_{\mathcal{O}_K^\times} |x_i|^n_K |dx_i|_K \right) \left( \prod_{i \in \mathcal{J}(\pi \mathcal{O}_K)^n(0)} |x_i|^n_K |dx_i|_K \right)$$

$$= \frac{(1 - q^{-1})q^{-(l - |J|)} - \sum_{i \notin J} s_i}{\prod_{i \notin J} (1 - q^{-1 - s_i})}$$

Let $f = (f_1(x), f_2(x), \ldots, f_l(x)) : K^n \to K^l$, $\mathbf{x} = (x_1, \ldots, x_n)$ be a polynomial mapping with $f_i(x) \in \mathcal{O}_K[\mathbf{x}] \setminus \pi \mathcal{O}[\mathbf{x}]$, $i = 1, \ldots, n$, and $l \leq n$. Let $\mathbf{a} \in \mathcal{O}_K^n$, the Jacobian matrix of $f$ at $\mathbf{a}$ is $J(f, \mathbf{a}) = \left[ \frac{\partial f_i}{\partial x_j} (\mathbf{a}) \right]_{1 \leq i \leq l}^{1 \leq j \leq n}$. In a similar way we define the Jacobian matrix of $\overline{f}$ at $\overline{\mathbf{a}}$.

**Lemma 3.2.** With the above notation, let $\chi \neq \chi_{triv}$. Suppose $\mathbf{a} \in \mathcal{O}_K^n$ such that $\overline{f}(\mathbf{a}) = 0$ and such that the Jacobian matrix $J(\overline{f}, \overline{\mathbf{a}})$ has rank $l$. Then

$$J(s, \chi, f) := \int_{\mathcal{O}_K^n} \prod_{i=1}^{l} \chi_i(ac(f_i(\mathbf{x}))) |f_i(\mathbf{x})|_K^n |dx|_K = 0$$

**Proof.** We consider the following change of variables

$$\phi : \mathcal{O}_K^n \to \mathcal{O}_K^n \quad (x_1, \ldots, x_n) \mapsto (y_1, \ldots, y_n)$$

where

$$y_i = \phi_i(x) := \begin{cases} \frac{f_i(a + \pi x) - f_i(\mathbf{a})}{\pi} & \text{if } i = 1, \ldots, l \\ x_i & \text{if } l + 1 \leq i \leq n. \end{cases}$$

By using that rank of $Jac(\overline{f}, \overline{\mathbf{a}})$ is $l$ we get that $det \left[ \frac{\partial \phi_i}{\partial x_j}(0) \right]_{1 \leq i \leq n} \neq 0 \mod \pi$, which implies that $y = \phi(x)$ gives a measure-preserving map from $\mathcal{O}_K^n$ to itself (see e.g. [10] Lemma 7.4.3)), therefore $J(s, \chi, f)$ equals

$$= q^{-n} \int_{\mathcal{O}_K^n \setminus \bigcup_{i=1}^{l} \{ x \in \mathcal{O}_K^n : f_i(x + a) = 0 \}} \prod_{i=1}^{l} \chi_i(ac(f_i(a + \pi x))) |f_i((a + \pi x))|_K^n |dx|_K$$

$$= q^{-n} \int_{\mathcal{O}_K^n \setminus \{ y \in \mathcal{O}_K^n : y + f(a) = 0 \}} \prod_{i=1}^{l} \chi_i(ac(y_i + f_i(a))) |\pi y_i + f_i(a)|_K^n |dy_i|_K$$

$$= q^{-n} \sum_{i=1}^{l} s_i \int_{\mathcal{O}_K^n(0)} \prod_{i=1}^{l} \chi_i(ac(y_i)) |y_i|_K^n |dy_i|_K.$$
Lemma 3.3. Let $h = (h_1, h_2, \ldots, h_r)$ be a polynomial mapping with $h_i(x) \in \mathcal{O}_K[x] \setminus \mathcal{O}_K$, $h_i(x) \neq 0$, $i = 1, \ldots, n$, and $r \leq n$, and let $\chi = (\chi_1, \ldots, \chi_r)$ with $\chi_i \neq \chi_{\text{triv}}$ and $e(\chi_i) = 1$ for $i = 1, \ldots, r$. Suppose that for any non-empty subset $I \subseteq \{1, \ldots, r\}$ the Jacobian matrix $J(h_I, \pi)$ has rank $\text{Card}(I)$ for any $z \in \{z \in (F_q^n) : h_i(z) = 0 \iff i \in I\}$.

Then

$$ L(s, \chi, h) := \int_{(O_K^n \setminus D_K)} \prod_{i=1}^r \chi_i(ac(h_i(x))) |h_i(x)|_K^n |dx|_K $$

$$ = \sum_{\pi \in (F_q^n) \setminus \pi | \pi \neq 0} \prod_{i=1}^r \chi_i(h_i(a)) $$

Proof. In order to prove the proposition, we set

$$ V := \left\{ z \in (O_K^n) : \pi \in \pi^{-1}(0) \right\}, $$

and for every non-empty subset $I \subseteq \{1, \ldots, r\}$ we set

$$ V_I := \left\{ z \in (O_K^n) : \nu(h_i(z)) = 0 \iff i \in I \right\}. $$

Denote by $\overline{V}$, $\overline{V}_I$ the image of $V$, $V_I$ under the canonical homomorphism $O_K^n \rightarrow F_q^n$.

With this notation, $(O_K^n)^n$ can be partitioned as

$$ (O_K^n)^n = V \bigcup_{I \subseteq \{1, \ldots, r\}} \bigcup_{I \neq \emptyset} V_I, $$

and thus

$$ \int_{(O_K^n \setminus D_K)} \bullet |dx|_K = \int_{V \setminus D_K} \bullet |dx|_K + \sum_{I \subseteq \{1, \ldots, r\}} \int_{V_I \setminus D_K} \bullet |dx|_K $$

We notice that

$$ \int_{V \setminus D_K} \prod_{i=1}^r \chi_i(ac(h_i(x))) |h_i(x)|_K^n |dx|_K $$

$$ = \sum_{\pi \in V} \prod_{i=1}^r \chi_i(ac(h_i(x))) |h_i(x)|_K^n |dx|_K $$

$$ = 0, $$

it follows by the condition in the statement of Lemma 3.3 that the Jacobian matrix $J(h, \pi)$ has rank $r$ for any $\pi \in \overline{V}$ and Lemma 3.2. On the other hand, for any
non-empty $I \subseteq \{1, \ldots, r\}$

\begin{equation}
\int_{V_i \setminus D_K} \prod_{i=1}^{r} \chi_i(ac(h_i(x)))|h_i(x)|_{K}^{n_i}|dx|_{K}
\end{equation}

\begin{align*}
= & \sum_{\pi \in \mathcal{V}_i} \int_{\mathcal{O}_K \setminus \bigcup_{c=1}^{r} \{ x \in K^n : h_i(x) \neq c \}} \prod_{i=1}^{r} \chi_i(ac(h_i(x)))|h_i((x + \pi x)|_{K}^{n_i}|dx|_{K}
\end{align*}

\begin{equation}
= \sum_{\pi \in \mathcal{V}_i} q^{-n} \int_{\mathcal{O}_K \setminus \bigcup_{c=1}^{r} \{ x \in K^n : h_i(x) = c \} } \prod_{i \in I} \chi_i(ac(h_i(x)))
\end{equation}

\begin{align*}
& \times \prod_{i \notin I} \chi_i(ac(h_i(x)))|h_i((x + \pi x)|_{K}^{n_i}|dx|_{K}.
\end{align*}

\textbf{Claim 3.1.} If $h_i(a) \in \mathcal{O}_K^\times$ and $e(\chi_i) = 1$ then

\begin{equation}
\chi_i(ac(h_i(a + \pi x))) = \chi_i(h_i(a))
\end{equation}

In effect, by applying the Taylor formula

\begin{equation}
\begin{aligned}
& h_i(a + \pi x) = h_i(a) + \pi \sum_j \frac{\partial h_i}{\partial x_j}(a)x_j + \pi^2 (\text{degree } \geq 2)

& = h_i(a) \left(1 + \pi(h_i(a))^{-1} \left(\sum_j \frac{\partial h_i}{\partial x_j}(a)x_j + \pi^2 (\text{degree } \geq 2)\right)\right).
\end{aligned}
\end{equation}

The announced formula follows by the condition $e(\chi_i) = 1$ and equation 3.10.

Thus,

\begin{equation}
\int_{V_i \setminus D_K} \prod_{i=1}^{r} \chi_i(ac(h_i(x)))|h_i(x)|_{K}^{n_i}|dx|_{K} = \sum_{\pi \in \mathcal{V}_i} q^{-n} \prod_{i \in I} \chi_i(h_i(a))L(\chi, a, I)
\end{equation}

where

\begin{equation}
L(\chi, a, I) := \int_{\mathcal{O}_K \setminus \bigcup_{i \in I} \{ x \in K^n : h_i(x + \pi x) = 0 \} } \prod_{i \notin I} \chi_i(ac(h_i(a + \pi x)))|h_i((a + \pi x)|_{K}^{n_i}|dx|_{K}.
\end{equation}

The definition of $V_i$ implies that for any $\pi \in \mathcal{V}_i$, $\bar{h}_i(\pi) = 0$ if and only if $i \in I^c$. Without loss of generality we may assume that $I^c = \{1, \ldots, l\}$. Notice that $l < r$ and $I^c \neq \emptyset$ because $I \neq \emptyset$ and $I \subseteq \{1, \ldots, r\}$, respectively. Thus the condition of the rank that the Jacobian matrix $J(\bar{h}_I, \overline{z})$ has rank $\text{Card}(I^c)$ for any

\begin{equation}
\overline{z} \in \{ \overline{z} \in \mathbb{R}_q^n : \bar{h}_i(\overline{z}) = 0 \iff i \in I^c \}
\end{equation}
gives a measure-preserving map from $O_K^n$ to itself given by $y = \phi(x)$ with
\[
y_i = \phi_i(x) := \begin{cases} \frac{h_i(ax) - h_i(x)}{\pi} & \text{if } 1 \leq i \leq l \\ x_i & \text{if } l + 1 \leq i \leq n.
\end{cases}
\]
In a similar way as in the proof of Lemma 3.2 and the condition $\chi_i \neq \chi_{\text{triv}}$ for any $i$, and Proposition 3.1,

\[
L(\chi, \mathbf{a}, I) = q^{-n - \sum_{i=1}^l s_i} \int_{O_K^n \setminus \{0\}} \prod_{i=1}^l \chi_i(ac(y_i))|y_i|_K^n dy_i_K = 0.
\]

From equations 3.7, 3.11, and 3.12,

\[
\int_{v_1 \setminus D_K} \prod_{i=1}^r \chi_i(ac(h_i(x)))|h_i(x)|_K^n dx_K = 0.
\]

Finally, if $I = \{1, \ldots, r\}$, then $h_i(a) \in O_K^n$ for all $i$, hence by the hypothesis that $e(\chi_i) = 1$ for all $i$ and Claim 3.1,

\[
\sum_{\mathbf{a} \in V_1} q^{-n} \int_{O_K^n \setminus \bigcup_{i=1}^r \{x \in O_K^n; h_i(ax+\mathbf{a})=0\}} \prod_{i=1}^r \chi_i(ac(h_i(a)))|dx_K| = 0.
\]

The result follows from 3.6, 3.13, and 3.14. \qed

**Remark 3.4.** (i) In integral $L(s, \chi, \mathbf{h})$ we can replace $\mathbf{h}$ by $\mathbf{h} + \pi \mathbf{g}$, where $\mathbf{g}$ is a polynomial mapping over $O_K$, and the formulas given in Lemma 3.3 remain valid.

(ii) A character satisfying $e(\chi_i) = 1$ is just a multiplicative character of $F_q^n$. Thus expressions of type $\sum_{\mathbf{a} \in V_1} \prod_{i=1}^r \chi_i(ac(h_i(a)))$ are ‘types’ of exponential sums over $F_q$ along $F_q$-rational points of an algebraic set.

Let $\Delta \in F(h) \cup \{0\}$ and $I \subseteq \{1, \ldots, r\}$, with the convention that $h_{i,0} := h_i$, we put

\[
N_{\Delta,I} := \text{Card } \{ \mathbf{x} \in (F_q^n); \, \, \, \widetilde{h}_{i,b(\Delta)}(\mathbf{x}) = 0 \, \Leftrightarrow \, i \in I \}.
\]

In particular,

\[
N_{\Delta,\emptyset} = \text{Card } \{ \mathbf{x} \in (F_q^n); \, \, \, \widetilde{h}_{i,b(\Delta)}(\mathbf{x}) \neq 0, \, i = 1, \ldots, r \}.
\]

**Remark 3.5.** If $\mathbf{h} = (h_1, \ldots, h_r)$ is a non-degenerated polynomial mapping over $F_q$ with respect to $\Gamma(h)$, then Lemma 3.3 is true for $h_{b(\Delta)} = (h_{1,b(\Delta)}, \ldots, h_{r,b(\Delta)})$.

**Remark 3.6.** In order to give an explicit formula for twisted local zeta functions attached to rational functions when the conductor of the character is one, in the following theorem, we only consider $r$-tuple of characters $\chi = (\chi_1, \ldots, \chi_r)$ where $\chi = \chi_{\text{triv}}$ or $\chi_i \neq \chi_{\text{triv}}$ and $e(\chi_i) = 1$ for all $i$. 
Theorem 3.7. Assume that $h = (h_1, \ldots, h_r)$ is non-degenerated polynomial mapping over $\mathbb{F}_q$ with respect to $\Gamma(h)$, with $r \leq n$. Let $\chi = (\chi_1, \ldots, \chi_r)$ as in Remark 3.2. Fix a simplicial polyhedral subdivision $F(h)$ subordinate to $\Gamma(h)$. Then $Z(s, \chi, h)$ has a meromorphic continuation to $\mathbb{C}^r$ as a rational function in the variables $q^{-s_i}$, $i = 1, \ldots, r$. In addition, the following explicit formula holds:

$$Z(s, \chi, h) = \sum_{\Delta \in F(h) \cup \{0\}} L_\Delta(s, \chi, h) S_\Delta,$$

where

$$S_\Delta = \sum_{k \in \mathbb{N}^{n \cap \Delta}} p^{-\sigma(k) - \sum_{i=1}^r d(k, \Gamma(h_i)) s_i},$$

and

$$L_\Delta(s, \chi, h) = \begin{cases} q^{-n} \sum_{\Delta \subseteq \{1, \ldots, r\}} N_{\Delta, I} \prod_{i \in I} \left( \frac{q-1-s_i}{1-q^{-s_i}} \right), & \text{if } \chi = \chi_{\text{triv}}, \\ \frac{\pi \in (\mathbb{F}_q^*)^n}{\prod_{i=1}^r \chi_i(h_i, b(\Delta))(a)}, & \text{if } \chi_i \neq \chi_{\text{triv}},\\ \tilde{h}_{i, b(\Delta)}(\pi) \neq 0, & \text{if } \chi_i = 1, \quad i = 1, \ldots, r, \end{cases}$$

(with the convention that if $I = \emptyset$, then $\prod_{i \in \emptyset} \bullet = 1$) and

$$S_\emptyset = \begin{cases} 1, & \text{if } S_\emptyset = 1. \end{cases}$$

Let $\Delta$ be the cone strictly positively generated by linearly independent vectors $w_1, \ldots, w_l \in \mathbb{N}^n \setminus \{0\}$, then

$$S_\Delta = \sum_{k \in \mathbb{N}^{n \cap \Delta}} p^{-\sigma(k) - \sum_{i=1}^r d(k, \Gamma(h_i)) s_i},$$

where $t$ runs through the elements of the set

$$\mathbb{Z}^n \cap \left\{ \sum_{i=1}^l \lambda_i w_i; \ 0 < \lambda_i \leq 1 \text{ for } i = 1, \ldots, l \right\}.\tag{3.15}$$

Proof. For the case $\chi_{\text{triv}}$ we refer to [12] Theorem 1. Furthermore, the proof for the other case is similar to the trivial case $\chi = \chi_{\text{triv}}$ by using Lemma 3.4 instead of Lemma 2 established in [12]. Notice that in Lemma 3.3 the condition $e(\chi_i) = 1$, for all $i$, implies that the integral in the proof of Theorem 1 in [12]

$$\int_{(\mathcal{O}_K^n)^n} \prod_{i=1}^r \chi_i(ac(h_i, b(\Delta)) \langle u \rangle + \pi h_{i, k}(u)) |h_i, b(\Delta)(u) + \pi h_{i, k}(u)|_K d\langle u \rangle_K$$

is independent of $h_{i, k}$. \hfill $\square$

4. An explicit formula for local zeta functions associated to rational functions

From now on, we fix two co-prime polynomials $f(x), g(x) \in \mathcal{O}_K[x] \setminus \pi \mathcal{O}_K[x]$ with $n \geq 2$, and set $D_K := \{x \in K^n; f(x) = g(x) = 0\}$, and

$$\frac{f}{g} : K^n \setminus D_K \to K.$$

Furthermore, we define the Newton polyhedron $\Gamma \left( \frac{f}{g} \right)$ of $\frac{f}{g}$ to be $\Gamma(fg)$, and assume that the mapping $\left( \frac{f}{g} \right) : K^n \to K^2$ is non-degenerate over $\mathbb{F}_q$ with respect to $\Gamma \left( \frac{f}{g} \right)$.
Theorem 4.1. The function attached to the polynomial mapping \( \left( \frac{f}{g} \right) \) is non-degenerate over \( F_q \) with respect to \( \Gamma \left( \frac{f}{g} \right) \). We fix a simplicial polyhedral subdivision \( \mathcal{F} \left( \frac{f}{g} \right) \) of \( \mathbb{R}_+^n \) subordinate to \( \Gamma \left( \frac{f}{g} \right) \). For \( \Delta \in \mathcal{F} \left( \frac{f}{g} \right) \cup \{0\} \), we put

\[
N_{\Delta,f}(f) := \text{Card} \left\{ \overline{\alpha} \in (F_q^n)_{\mathbb{T}} ; \overline{\mathcal{T}}_{\Delta}(\overline{\alpha}) = 0 \right\},
\]

\[
N_{\Delta,g}(g) = \text{Card} \left\{ \overline{\alpha} \in (F_q^n)_{\mathbb{T}} ; \overline{\mathcal{T}}_{\Delta}(\overline{\alpha}) \neq 0 \right\},
\]

\[
N_{\Delta,f,g}(f,g) = \text{Card} \left\{ \overline{\alpha} \in (F_q^n)_{\mathbb{T}} ; \overline{\mathcal{T}}_{\Delta}(\overline{\alpha}) = 0 \right\},
\]

with the convention that \( f_{b(0)} = f \) and \( g_{b(0)} = g \). We also define \( D \left( \frac{f}{g} \right) = D(f,g) \), which is the set of primitive vectors in \( \mathbb{N}^n \setminus \{0\} \) perpendicular to the facets of \( \Gamma \left( \frac{f}{g} \right) \). We set

\[
T_+ := \left\{ w \in D \left( \frac{f}{g} \right) ; d(w,\Gamma(g)) - d(w,\Gamma(f)) > 0 \right\},
\]

\[
T_- := \left\{ w \in D \left( \frac{f}{g} \right) ; d(w,\Gamma(g)) - d(w,\Gamma(f)) < 0 \right\},
\]

\[
\alpha := \alpha \left( \frac{f}{g} \right) = \begin{cases} \min_{w \in T_+} \left\{ \frac{s(w)}{d(w,\Gamma(g)) - d(w,\Gamma(f))} \right\} & \text{if } T_+ \neq \emptyset, \\ +\infty & \text{if } T_+ = \emptyset, \end{cases}
\]

\[
\beta := \beta \left( \frac{f}{g} \right) = \begin{cases} \max_{w \in T_-} \left\{ \frac{s(w)}{d(w,\Gamma(g)) - d(w,\Gamma(f))} \right\} & \text{if } T_- \neq \emptyset, \\ -\infty & \text{if } T_- = \emptyset, \end{cases}
\]

and

\[
\bar{\alpha} := \bar{\alpha} \left( \frac{f}{g} \right) = \min \{1, \alpha\}, \quad \bar{\beta} := \bar{\beta} \left( \frac{f}{g} \right) = \max \{-1, \beta\}.
\]

Notice that \( \alpha > 0 \) and \( \beta < 0 \).

We define the local zeta function attached to \( \left( \frac{f}{g}, \chi \right) \), where \( \chi \) is a multiplicative character of \( \mathcal{O}_K^* \), as

\[
Z \left( s, \chi, \frac{f}{g} \right) = Z(s, -s, \chi, \chi^{-1}, f, g), \quad s \in \mathbb{C},
\]

where \( Z(s_1, s_2, \chi_1, \chi_2; f, g) \) denotes the meromorphic continuation of the local zeta function attached to the polynomial mapping \( (f,g) \), see Theorem 3.7.

**Theorem 4.1.** Assume that \( \frac{f}{g} \) is non-degenerate over \( F_q \) with respect to \( \Gamma \left( \frac{f}{g} \right) \), with \( n \geq 2 \) as before. Let \( \chi \) be a character with conductor \( e(\chi) = 1 \). We fix a simplicial polyhedral subdivision \( \mathcal{F} \left( \frac{f}{g} \right) \) of \( \mathbb{R}_+^n \) subordinate to \( \Gamma \left( \frac{f}{g} \right) \). Then the following assertions hold:

(i) \( Z\left(s, \chi, \frac{f}{g}\right) \) has a meromorphic continuation to the whole complex plane as a rational function of \( q^{-s} \) and the following explicit formula holds:

\[
Z\left(s, \chi, \frac{f}{g}\right) = \sum_{\Delta \in \mathcal{F} \left( \frac{f}{g} \right) \cup \{0\}} L_\Delta \left( s, \chi, \frac{f}{g} \right) S_\Delta(s),
\]
where

\[ L_\Delta \left( s, \chi, \frac{f}{g} \right) = q^{-n} \left[ (q - 1)^n - N_\Delta(f) \frac{1 - q^{-s}}{1 - q^{-1-s}} - N_\Delta(g) \frac{1 - q^s}{1 - q^{-1+s}} \right] \]

if \( \chi = \chi_{\text{triv}} \) or

\[ L_\Delta \left( s, \chi, \frac{f}{g} \right) = q^{-n} \sum_{\mathbf{a} \in (\mathbb{F}_q^n)^n \atop \mathbf{f}(\mathbf{a}) \neq 0 \atop \mathbf{g}(\mathbf{a}) \neq 0} \chi \left( \frac{f_b(\Delta)(\mathbf{a})}{g_b(\Delta)(\mathbf{a})} \right), \]

if \( \chi \neq \chi_{\text{triv}} \), \( S_0(s) = 1 \), and if \( \Delta \in \mathcal{F}(\frac{f}{g}) \) is a cone strictly positively generated by linearly independent vectors \( \mathbf{w}_1, \ldots, \mathbf{w}_l \in \mathcal{D}(\frac{f}{g}) \), then

\[ S_\Delta(s) = \frac{\sum \rho_\sigma(t) - (d(t, \Gamma(f)) - d(t, \Gamma(g)))s}{\prod_{i=1}^l (1 - q^{-\sigma(w_i)} - (d(w_i, \Gamma(f)) - d(w_i, \Gamma(g)))s)}, \]

where \( t \) runs through the elements of the set

\[ \mathbb{Z}^n \cap \left\{ \sum_{i=1}^l \lambda_i \mathbf{w}_i; \ 0 < \lambda_i \leq 1 \text{ for } i = 1, \ldots, l \right\}. \]

(ii) \( Z \left( s, \chi, \frac{f}{g} \right) \) is a holomorphic function on \( \beta < \Re(s) < \alpha \), and on this band it verifies that

\[ Z \left( s, \chi, \frac{f}{g} \right) = \int_{\mathcal{O}_K \setminus \mathcal{D}_K} \chi \left( \frac{f(x)}{g(x)} \right) \left| \frac{f(x)}{g(x)} \right|^s |dx|; \]

\[ \left( f(x) \right)^s = \int_{\mathcal{O}_K \setminus \mathcal{D}_K} \chi \left( \frac{f(x)}{g(x)} \right) \left| \frac{f(x)}{g(x)} \right|^s |dx|. \]

Proof. (i) For the case \( \chi = \chi_{\text{triv}} \) we refer to [12, Theorem 2]. The another case follows from Theorem 3.7 as follows: we take \( r = 2, \chi_1 = \chi, \chi_2 = \chi^{-1}, h_1 = f_b(\Delta) \) and \( h_2 = g_b(\Delta) \) for \( \Delta \in \mathcal{F}(\frac{f}{g}) \cup \{0\} \), with the convention that if \( b(\Delta) = b(0) = 0 \), then \( h_1 = f \) and \( h_2 = g \).

(ii) From the explicit formula given in (i), follows that the rational function \( Z(s, \chi, \frac{f}{g}) = Z(s, -s, \chi, -\chi^{-1}, f, g) \) is holomorphic on the band \( \beta < \Re(s) < \alpha \), and then \( Z(s, \chi, \frac{f}{g}) \) is given by integral (4.1) because \( Z(s_1, s_2, \chi_1, \chi_2, f, g) \) agrees with an integral on its domain of holomorphy.

5. The Candidate Poles of \( Z \left( s, \chi, \frac{f}{g} \right) \) and the Pole Determined by the Diagonal

In this section we use all the notation introduced in Section 4. We fix two co-prime polynomials \( f(x), g(x) \in \mathcal{O}_K[x_1, \ldots, x_n] \setminus \pi \mathcal{O}_K[x_1, \ldots, x_n] \) with \( n \geq 2 \) and \( f(0) = g(0) = 0 \), and also we fix a simplicial polyhedral subdivision \( \mathcal{F} \left( \frac{f}{g} \right) \) of \( \mathbb{R}_+^n \) subordinate to \( \Gamma \left( \frac{f}{g} \right) \).
Proposition 5.1. Suppose that \( f, g \) satisfy the conditions of Theorem 4.1. Let \( \gamma_1, \ldots, \gamma_l \) be all the facets of \( \Gamma \left( \frac{f}{g} \right) \) and let \( w_1, \ldots, w_l \in \mathbb{N} \setminus \{0\} \) be the unique primitive vectors that are perpendicular to \( \gamma_1, \ldots, \gamma_l \) respectively. Then

(i) If \( \chi = \chi_{\text{triv}} \) and \( s \) is a pole of the meromorphic continuation of \( Z(s, \chi_{\text{triv}}, \frac{f}{g}) \), then

\[
s = 1 + \frac{2\pi \sqrt{-1} k}{\ln q} \quad \text{with } k \in \mathbb{Z} \quad \text{or}
\]

\[
s = -1 + \frac{2\pi \sqrt{-1} k}{\ln q} \quad \text{with } k \in \mathbb{Z} \quad \text{or}
\]

\[
s = \frac{\sigma(w_i)}{d(w_i, \Gamma(g)) - d(w_i, \Gamma(f))} + \frac{2\pi \sqrt{-1} k}{\ln q}
\]

with \( k \in \mathbb{Z}, i \in \{1, 2, \ldots, l\} \) and \( d(w_i, \Gamma(g)) - d(w_i, \Gamma(f)) \neq 0 \).

(ii) If \( \chi \neq \chi_{\text{triv}} \) and \( s \) is a pole of the meromorphic continuation of \( Z(s, \chi, \frac{f}{g}) \), then

\[
s = \frac{\sigma(w_i)}{d(w_i, \Gamma(g)) - d(w_i, \Gamma(f))} + \frac{2\pi \sqrt{-1} k}{\ln q}
\]

with \( k \in \mathbb{Z}, i \in \{1, 2, \ldots, l\} \) and \( d(w_i, \Gamma(g)) - d(w_i, \Gamma(f)) \neq 0 \).

Proof. This result is a direct consequence of the explicit formula in Theorem 4.1 and Lemma 2.3. \(\square\)

We will call to each \( s \) describe above a **candidate pole** of \( Z(s, \chi, \frac{f}{g}) \).

5.1. The expected order of a candidate pole. Let \( f, g \), and \( \chi \) as in Theorem 4.1

For any \( k \in \mathbb{Q} \setminus \{0\} \) we put

\[
\mathcal{P}(k) := \left\{ \begin{array}{ll}
\{ w \in T_-; \frac{\sigma(w)}{d(w, \Gamma(g)) - d(w, \Gamma(f))} = k \} & \text{if } k < 0, \\
\{ w \in T_+; \frac{\sigma(w)}{d(w, \Gamma(g)) - d(w, \Gamma(f))} = k \} & \text{if } k > 0,
\end{array} \right.
\]

and for \( m \in \mathbb{N} \) with \( 1 \leq m \leq n \),

\[
\mathcal{M}_m(k) := \left\{ \Delta \in \mathcal{F} \left( \frac{f}{g} \right); \Delta \text{ has exactly } m \text{ generators belonging to } \mathcal{P}(k) \right\},
\]

\[
\rho(k) := \max \{ m; \mathcal{M}_m(k) \neq \emptyset \}.
\]

If \( s \) is a candidate pole of \( Z(s, \chi, \frac{f}{g}) \), we set \( \mathcal{P}(s) := \mathcal{P}(\text{Re}(s)), \mathcal{M}_m(s) := \mathcal{M}_m(\text{Re}(s)) \) and \( \rho(s) := \rho(\text{Re}(s)). \)

Definition 5.2. The expected order of a candidate pole \( s \) is \( \rho(s) \) if \( \text{Re}(s) \neq -1, 1 \) and \( \chi = \chi_{\text{triv}} \) or \( \chi \neq \chi_{\text{triv}} \); otherwise if \( \text{Re}(s) = -1 \) or \( 1 \) and \( \chi = \chi_{\text{triv}} \), it will be \( 1, \rho(s) \) or \( \rho(s) + 1 \).

This definition follows from the explicit form of \( Z(s, \chi, \frac{f}{g}) \) given in Theorem 4.1

Notice that the actual order of a pole is less than or equal than the expected order.

We recall that in the case \( T_- \neq \emptyset \),

\[
\beta := \max_{w \in T_-} \left\{ \frac{\sigma(w)}{d(w, \Gamma(g)) - d(w, \Gamma(f))} \right\}
\]
is the largest possible negative real part of the poles of $Z(s, χ, \frac{f}{g})$ coming from $S_\Delta$ and we set $ρ := ρ(β)$. Similarly, in the case $T_+ \neq \emptyset$,

$$\alpha := \min_{w \in T_+} \left\{ \frac{σ(w)}{d(w, Γ(g)) − d(w, Γ(f))} \right\}$$

is the smallest possible positive real part of the poles of $Z(s, χ, \frac{f}{g})$ coming from $S_\Delta$ and we set $κ := ρ(α)$.

5.2. The pole determined by the diagonal $D = \{(t, \ldots , t) : t \in \mathbb{R}\}$. It is well known that in the case of local zeta functions attached to one polynomial, the ‘intersection point’ of the diagonal $D = \{(t, \ldots , t) : t \in \mathbb{R}\}$ with the boundary of the Newton polyhedron is the largest real candidate pole different from $−1$, see e.g. [11], [1]. But in this case of local zeta functions for rational functions, due to the forms of the poles, this occur only for special cases, see Theorems 5.3 and 5.4.

**Definition 5.3.** Let $f$, $g$ be non-zero polynomials as above. Let $(t_0, \ldots , t_0)$ be the unique intersection point of the diagonal $D = \{(t, \ldots , t) : t \in \mathbb{R}\}$ with the boundary of the Newton polyhedron $Γ(\frac{f}{g})$, $τ_0$ be the smallest face of $Γ(\frac{f}{g})$ such that $(t_0, \ldots , t_0) \in τ_0$. Let $γ_1, \ldots , γ_r$ be all the facets such that $τ_0 \subseteq γ_i$ for all $i$, and $w_1, \ldots , w_r \in \mathbb{N}\{0\}$ be the unique primitive vectors that are perpendicular to $γ_1, \ldots , γ_r$, respectively. We denote by $D(t_0)$ the set of all of these primitive vectors and we set $D_−(t_0) := D(t_0) \cap T_−$ and $D_+(t_0) := D(t_0) \cap T_+$.

The following Proposition is similar to Lemma 5.3 in [9].

**Proposition 5.4.** For any $a \in \mathbb{R}^n_+$ we have $σ(a) − (d(a, Γ(g)) + d(a, Γ(f)))(1/t_0) ≥ 0$ with equality if and only if $τ_0 \subseteq F\left(\frac{a}{a, Γ(\frac{f}{g})}\right)$.

**Proof.** Because of $(t_0, \ldots , t_0) \in Γ((f, g))$ and by definition of $d(a, Γ((f, g)))$, it follows that $(t_0, \ldots , t_0) \cdot a ≥ d(a, Γ((f, g)))$. Hence, $σ(a) − d(a, Γ((f, g)))(1/t_0) ≥ 0$. The first result follows from the fact $d(a, Γ((f, g))) = d(a, Γ(g)) + d(a, Γ(f))$, see Remark 2.1. Furthermore, $σ(a) − d(a, Γ((f, g)))(1/t_0) = 0$ if and only if $(t_0, \ldots , t_0) \in F(a, Γ((f, g)))$ if and only if $τ_0 \subseteq F\left(\frac{a}{a, Γ((f, g))}\right)$, because $τ_0$ is the smallest face that contains $(t_0, \ldots , t_0)$. □

**Corollary 5.5.** Let $w ∈ D(t_0)$. Then, $d(w, Γ(g)) = 0$ if and only if $(t_0, \ldots , t_0) \in F(w, Γ(f))$. Similarly, $d(w, Γ(f)) = 0$ if and only if $(t_0, \ldots , t_0) \in F(w, Γ(g))$.

**Proof.** By using Proposition 5.4 and the definition of $F(w, Γ)$, the result follows from the following equivalent propositions, $d(w, Γ(g)) = 0$ if and only if $σ(w) − d(w, Γ(f))(1/t_0) = 0$ if and only if $(t_0, \ldots , t_0) \in F(w, Γ(f))$. A similar argument shows $d(w, Γ(f)) = 0$ if and only if $(t_0, \ldots , t_0) \in F(w, Γ(g))$. □

The following propositions stablsh that all the possible negative (resp., positive) real parts of the poles coming from $S_\Delta$ have $−1/t_0$ as an upper bound (resp., $1/t_0$ as a lower bound). Furthermore, the result stablsh in Proposition 5.6 is similar to the one given in the case of local zeta functions attached to one polynomial, see [11] Proposition 4.6.

**Proposition 5.6.** Let $f$, $g$ be non-zero polynomials and the character $χ$ satisfy the conditions of Theorem 4.1. Then for every pole $s$ of $Z(s, χ, \frac{f}{g})$ with $Re(s) < 0$ one has $Re(s) ≤ −1/t_0$ if $χ ≠ χ_{triv}$; otherwise or $Re(s) = −1 or Re(s) ≤ −1/t_0$. 
boundary of the Newton polyhedron $\Gamma(f)$. Assume that $(t, g) \leq 0$.
Moreover if $\chi = 1$ or $\chi = \chi_{\text{triv}}$, then $-1/t_0$ is a pole of $Z(s, \chi, g)$, otherwise the expected order is $\rho$. The second part is exactly the result established in [12, Theorem 3] by using that $\beta = -1/t_0$.

Theorem 5.9. Let $f, g$ be non-zero polynomials and the character $\chi$ satisfy the conditions of Theorem 4.1. Assume that $(t_0, \ldots, t_0) \in F(w, \Gamma(f))$ for some $w \in \mathcal{D}(t_0)$, it means that $(t_0, \ldots, t_0)$ is the intersection point of the diagonal with the boundary of the Newton polyhedron $\Gamma(f)$. Then $\alpha = 1/t_0$ of expected order $\kappa$ if $t_0 \neq 1$ and $\chi = \chi_{\text{triv}}$ or $\chi \neq \chi_{\text{triv}}$; otherwise $\kappa + 1$ if $t_0 = 1$ and $\chi = \chi_{\text{triv}}$. Moreover if $\chi = \chi_{\text{triv}}$ and $t_0 > 1$, then $1/t_0$ is a pole of $Z(s, \chi, g)$ of order $\kappa$.

Proof. A similar argument given in Theorem 5.8 shows the first part of Theorem 5.9. The second part follows from [12, Theorem 4], because $\alpha = 1/t_0$, and $t_0 > 1$ implies $1/t_0 < 1$.

Example 5.10. Let $f(x, y) = x^2 - y$, $g(x, y) = x^2 y$ polynomials in $\mathcal{O}_K[x, y]$. Then a simplicial polyhedral subdivision subordinate to $\Gamma(g)$ is given by
Furthermore, \( Z \) is non-degenerate over \( F_q \) with respect to \( \Gamma \left( \frac{f}{g} \right) \). Notice that the intersection point of the diagonal with \( \Gamma \left( \frac{f}{g} \right) \) is the point \( (1, 0) \). Thus, \( t_0 = 2 \) and \( D(t_0) = \{(1,0),(1,2)\} \). Furthermore, the hypothesis of Theorem 5.9 holds: \( d((1,0), \Gamma(f)) = 0 \), the intersection point \((2, 2) \in F((1,0), \Gamma(g)), \) and \( t_0 > 1 \). Hence, Theorem 5.9 implies that \( 1/2 \) is a pole of \( Z \left( s, \frac{f}{g} \right) \) of order 1 and it is the smallest positive real part of the poles of the local zeta function. With the notation of Theorem 4.1.

| Cone             | \( L_\Delta \)                  | \( S_\Delta \)                  |
|------------------|---------------------------------|---------------------------------|
| \( \emptyset \)  | \( q^2((q - 1)^2 - (q - 1) \frac{1 - q^{-1}}{1 - q^{-2}}) \) | \( \frac{q^{1+3}}{1 - q^{1+2}} \) |
| \( \Delta_1 \)   | \( q^2(q - 1)^2 \)              | \( \frac{q^{-1+3}}{1 - q^{-1+2}} \) |
| \( \Delta_2 \)   | \( q^2(q - 1)^2 \)              | \( \frac{q^{-1+3}}{1 - q^{-1+2}} \) |
| \( \Delta_3 \)   | \( q^2((q - 1)^2 - (q - 1) \frac{1 - q^{-1}}{1 - q^{-2}}) \) | \( \frac{q^{-1+3}}{1 - q^{-1+2}} \) |
| \( \Delta_4 \)   | \( q^2(q - 1)^2 \)              | \( \frac{q^{-1+3}}{1 - q^{-1+2}} \) |
| \( \Delta_5 \)   | \( q^2(q - 1)^2 \)              | \( \frac{q^{-1+3}}{1 - q^{-1+2}} \) |

**Table 2.** \( f(x, y) = x^2 - y \), \( g(x, y) = x^2 y \)

the local zeta function for the rational function \( f/g \) is

\[
Z \left( s, \frac{f}{g} \right) = \frac{(q-1)^2 L(q^{-s})}{(1 - q^{s-1})(1 - q^{-1-s})(1 - q^{2s-1})(1 - q^{2s-3})},
\]

where

\[
L(q^{-s}) = q - q^{-1} - 2 - q^{2s-4} + q^{s-3} - q^{s-2} + q^{2s-2} + q^{3s-3} + 2q^{2s-1} - q^{3s-2} - q^{3s-1} + q^{-s-1}.
\]

Furthermore, \( Z(s, \frac{f}{g}) \) has poles with real parts belonging to \( \{-1, 1/2, 1, 3/2\} \).

6. **The largest and smallest real poles of \( Z \left( s, \frac{f}{g} \right) \) (negative and positive, respectively)**

In this section we determine the conditions under which \( Z \left( s, \frac{f}{g} \right) \) has a real pole if \( f, g \) satisfy the conditions of Theorem 4.1. Furthermore, we obtain information about the largest negative real pole and the smallest positive real pole and they
orders, see Theorems 6.1 and 6.2. Also, we notice that if the conditions given in Theorems 5.8 or 5.9 are satisfied then \(Z(s, \frac{f}{g})\) has always a real pole determined by the intersection point of the diagonal with the boundary of the Newton polyhedron \(\Gamma\left(\frac{f}{g}\right): -1/t_0\) or \(1/t_0\), respectively.

6.1. The largest negative real pole of \(Z(s, \frac{f}{g})\). We remark that if the conditions of Theorem 5.8 are satisfied then \(\beta = -\frac{1}{t_0}\), and the results given in Theorem 6.1 are similar to the ones given in the classical case of local zeta functions attached to one polynomial, see 11 Theorem 4.10).

**Theorem 6.1.** (1) Assume that \(T_- \neq \emptyset\). Then the following hold:

(1.a) Suppose that \(\beta > -1\). Then \(\beta\) is the largest real negative pole of \(Z\left(s, \frac{f}{g}\right)\) with order \(\rho\).

(1.b) Suppose that \(\beta < -1\). Then if there exists a cone \(\Delta \in \mathcal{F}\left(\frac{f}{g}\right)\) such that \(N_{\Delta,\{f\}} \neq 0\) or \(N_{\Delta,\{f,g\}} \neq 0\) then \(-1\) is the largest real negative pole of \(Z\left(s, \frac{f}{g}\right)\) and its order will be 1. Otherwise, \(\beta\) will be the largest real negative pole of \(Z\left(s, \frac{f}{g}\right)\) and its order will be \(\rho\).

(1.c) if \(\beta = -1\), then \(\beta\) will be the largest real negative pole of \(Z\left(s, \frac{f}{g}\right)\). If there is a cone \(\Delta \in \mathcal{M}_\rho(\beta)\) such that \(N_{\Delta,\{f\}} \neq 0\) or \(N_{\Delta,\{f,g\}} \neq 0\) then its order will be \(\rho + 1\). Otherwise, its order will be \(\rho\).

(2) If \(T_- = \emptyset\) and \(N_{\Delta,\{f\}} \neq \emptyset\) or \(N_{\Delta,\{f,g\}} \neq \emptyset\), then \(-1\) is the unique real negative pole of \(Z\left(s, \frac{f}{g}\right)\) and its order is 1.

**Proof.** First we proof Part (1). Case (1.a) is given in 12 Theorem 3]. For case (1.b) we recall that

\[
Z\left(s, \frac{f}{g}\right) = \sum_{\Delta \in \mathcal{F}\left(\frac{f}{g}\right) \cup \{0\}} L_\Delta \left(s, \frac{f}{g}\right) S_\Delta(s),
\]

\(\Delta \in \mathcal{F}\left(\frac{f}{g}\right) \cup \{0\}\),

\[
L_\Delta \left(s, \frac{f}{g}\right) = q^{-n} \left[ (q - 1)^n - N_{\Delta,\{f\}} \frac{1 - q^{-s}}{1 - q^{-1-s}} - N_{\Delta,\{g\}} \frac{1 - q^s}{1 - q^{-1+s}} \right.
-
\left. N_{\Delta,\{f,g\}} \frac{(1 - q^{-s})(1 - q^s)}{q(1 - q^{-1-s})(1 - q^{-1+s})} \right]
\]

where \(S_0(s) = 1\), and if \(\Delta \in \mathcal{F}\left(\frac{f}{g}\right)\) is a cone strictly positively generated by linearly independent vectors \(w_1, \ldots, w_l \in D\left(\frac{f}{g}\right)\), then

\[
S_\Delta(s) = \frac{\sum_{i=1}^l q^{-\sigma(t) - (d(t,\Gamma(f)) - d(t,\Gamma(g)))s}}{\prod_{i=1}^l (1 - q^{-\sigma(w_l) - (d(w_l,\Gamma(f)) - d(w_l,\Gamma(g)))s})}.
\]

In order to proof the first part of case (1.b) we assume that there exist a cone \(\Delta_0 \in \mathcal{F}\left(\frac{f}{g}\right)\) such that \(N_{\Delta_0,\{f\}} \neq 0\) or \(N_{\Delta_0,\{f,g\}} \neq 0\).
To prove that \(-1\) is a pole of \(Z\left(s, \frac{L}{g}\right)\) of order 1, it is sufficient to show that

\[
\text{Res} (\Delta, -1) := \lim_{s \to -1} (1 - q^{-1-s})L_\Delta \left(s, \frac{f}{g}\right) S_\Delta(s) \geq 0
\]

for every cone \(\Delta \in \mathcal{F}(\frac{L}{g})\) and \(\text{Res} (\Delta_0, -1) > 0\).

Notice that

(6.1) \[ \lim_{s \to -1} S_\Delta(s) > 0 \]

for all cones \(\Delta \in \mathcal{F}(\frac{L}{g}) \cup \{0\} \). Inequality (6.1) follows from

\[
\lim_{s \to -1} \sum_t q^{-\sigma(t)-(d(t, \Gamma(f))-d(t, \Gamma(g)))s} > 0,
\]

and

\[1 - q^{-\sigma(w_i)-(d(w_i, \Gamma(f))-d(w_i, \Gamma(g)))(-1)} > 0\]

because \(\beta < -1\) implies \(-\sigma(w_i) - (d(w_i, \Gamma(f)) - d(w_i, \Gamma(g)))(-1) < 0\) for any \(w_i \in T_+ \cup T_-\). From these observations, we have

\[
\lim_{s \to -1} \frac{\sum_t q^{-\sigma(t)-(d(t, \Gamma(f))-d(t, \Gamma(g)))s}}{\prod_{i=1}^t (1 - q^{-\sigma(w_i)-(d(w_i, \Gamma(f))-d(w_i, \Gamma(g)))s})} > 0.
\]

Now, we prove that

(6.2) \[ \lim_{s \to -1} (1 - q^{-1-s})L_\Delta \left(s, \frac{f}{g}\right) \geq 0. \]

By definition of \(L_\Delta \left(s, \frac{f}{g}\right)\), for each cone such that \(N_{\Delta,\{f\}} = 0 = N_{\Delta,\{f,g\}}\), we have

\[
\lim_{s \to -1} (1 - q^{-1-s})L_\Delta \left(s, \frac{f}{g}\right) = 0
\]

On the other hand, we assume that \(N_{\Delta,\{f\}} \neq 0\) and \(N_{\Delta,\{f,g\}} \neq 0\). The other cases when one of these is zero are treated in a similar way. Hence

\[
\lim_{s \to -1} (1 - q^{-1-s})L_\Delta \left(s, \frac{f}{g}\right) > q^n \left((q-1)^n + N_{\Delta,\{f\}}(q-1)ight)
\]

\[
- N_{\Delta,\{g\}} + N_{\Delta,\{f,g\}}(1-q^{-1}) > 0.
\]

In particular, \(\text{Res} (\Delta_0, -1) > 0\). This shows that \(-1\) is a pole of order 1.

Now, the second part of case (1.b) and \(\beta < 0\) implies that

\[
L_\Delta \left(\beta, \frac{f}{g}\right) = q^{-n} \left((q-1)^n - N_{\Delta,\{g\}} \frac{1-q^{\beta}}{1-q^{-1+\beta}}\right) > q^{-n} \left((q-1)^n - N_{\Delta,\{g\}}\right) > 0.
\]

Thus, it is sufficient to prove that \(\lim_{s \to \beta}(1 - q^{s-\beta})^\rho S_\Delta(s) \geq 0\) for every cone \(\Delta \in \mathcal{F}(\frac{L}{g})\) and that there exists a cone \(\Delta_0 \in \mathcal{M}_\rho(\beta)\) such that \(\text{Res} (\Delta_0, \beta) > 0\).

We show that for at least one cone \(\Delta_0\) in \(\mathcal{M}_\rho(\beta)\), \(\text{Res} (\Delta_0, \beta) > 0\), because for any cone \(\Delta \notin \mathcal{M}_\rho(\beta)\), \(\text{Res} (\Delta, \beta) = 0\). This last assertion can be verified by using the argument that we give for the cones in \(\mathcal{M}_\rho(\beta)\). We first note that there exists at least one cone \(\Delta_0\) in \(\mathcal{M}_\rho(\beta)\). Let \(w_1, \ldots, w_{\rho}, w_{\rho+1}, \ldots, w_l\) its generators with \(w_i \in \mathcal{P}(\beta) \iff 1 \leq i \leq \rho\).
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We notice that
\[
\lim_{s \to \beta} \sum_{t} q^{-\sigma(t) - (d(t, \Gamma(f)) - d(t, \Gamma(g)))s} > 0.
\]

Hence in order to show that \( \text{Res} (\Delta_0, \beta) > 0 \), it is sufficient to show that
\[
\lim_{s \to \beta} \frac{(1 - q^{s-\beta})^\rho}{\prod_{i=1}^{\rho} (1 - q^{-\sigma(w_i) - (d(w_i, \Gamma(f)) - d(w_i, \Gamma(g)))s})} > 0.
\]

Now, notice there are positive integer constants \( c_i \) such that
\[
\prod_{i=1}^{\rho} (1 - q^{-\sigma(w_i) - (d(w_i, \Gamma(f)) - d(w_i, \Gamma(g)))s}) = \prod_{i=1}^{\rho} (1 - q^{(s-\beta)c_i})
\]
\[
= (1 - q^{s-\beta})^\rho \prod_{i=1}^{\rho} \prod_{\varsigma \neq 1} \left( 1 - \varsigma q^{s-\beta} \right).
\]

In addition, for \( i = \rho + 1, \ldots, l \),
\[
1 - q^{-\sigma(w_i) - (d(w_i, \Gamma(f)) - d(w_i, \Gamma(g)))\beta} > 0
\]
because \(-\sigma(w_i) - (d(w_i, \Gamma(f)) - d(w_i, \Gamma(g)))\beta < 0\) for any \( w_i \in T_+ \cup T_- \) with \( i = \rho + 1, \ldots, l \). From these observations, we have
\[
\lim_{s \to \beta} \frac{(1 - q^{s-\beta})^\rho}{\prod_{i=1}^{\rho} (1 - q^{-\sigma(w_i) - (d(w_i, \Gamma(f)) - d(w_i, \Gamma(g)))s})} = \lim_{s \to \beta} \frac{(1 - q^{s-\beta})^\rho}{\prod_{i=1}^{\rho} (1 - q^{\varsigma q^{s-\beta}})} \times \lim_{s \to \beta} \frac{1}{\prod_{i=\rho+1}^{l} (1 - q^{-\sigma(w_i) - (d(w_i, \Gamma(f)) - d(w_i, \Gamma(g)))s})} > 0.
\]

The case (1.c) and the proof of part (2) are similar to the case (1.b) and the first case of (1.b), respectively. \( \square \)

6.2. The smallest positive real pole of \( Z \left( s, \frac{f}{g} \right) \). The following theorem is the ‘posivite’ counterpart of the classical theory of local zeta functions attached to one polynomial.

**Theorem 6.2.** (1’) Assume that \( T_+ \neq \emptyset \). Then the following hold:

(1.a’) Suppose that \( \alpha < 1 \). Then \( \alpha \) is the smallest real positive pole of \( Z \left( s, \frac{f}{g} \right) \) with order \( \kappa \).

(1.b’) Suppose that \( \alpha > 1 \). Then if there exists a cone \( \Delta \in F \left( \frac{f}{g} \right) \) such that \( N_{\Delta, \{g\}} \neq 0 \) or \( N_{\Delta, \{f, g\}} \neq 0 \) then \( 1 \) is the smallest real positive pole of \( Z \left( s, \frac{f}{g} \right) \) of order 1. Otherwise, \( \alpha \) will be the smallest real positive pole of \( Z \left( s, \frac{f}{g} \right) \) and its order will be \( \kappa \).

(1.c’) If \( \alpha = 1 \), then \( \alpha \) will be the smallest real positive pole of \( Z \left( s, \frac{f}{g} \right) \). If there is a cone \( \Delta \in F \left( \frac{f}{g} \right) \) such that \( N_{\Delta, \{g\}} \neq 0 \) or \( N_{\Delta, \{f, g\}} \neq 0 \) then its order will be \( \kappa + 1 \). Otherwise, its order will be \( \kappa \).
(2') If \( T_+ = \emptyset \) and \( N_{\Delta, \{g\}} \neq \emptyset \) or \( N_{\Delta, \{f, g\}} \neq \emptyset \), then 1 is the unique real positive pole of \( Z \left( s, \frac{1}{f} \right) \) and its order is 1.

Proof. The case (1.a') is well known, see [12, Theorem 4]. The other cases are similar to cases (1.b), (1.c) and (2) in Theorem 6.1. □

Remark 6.3. It follows by definition of \( d(k, \Gamma) \), see Section 2, that if \( \Gamma(g) \subset \Gamma(f) \), then \( T_- = \emptyset \), then -1 could be the only negative pole of \( Z \left( s, \frac{1}{g} \right) \), see (2) in Theorem 6.1 and its smallest positive real pole is given by Theorem 6.2.

Analogously, if \( \Gamma(f) \subset \Gamma(g) \), then \( T_+ = \emptyset \), then 1 could be the only positive pole of \( Z \left( s, \frac{1}{g} \right) \), see (2') in Theorem 6.1 and its largest negative real pole is given by Theorem 6.2. In addition if \( \Gamma(f) = \Gamma(g) \), then \( T_+ = T_- = \emptyset \). Thus the only real poles of \( Z \left( s, \frac{1}{g} \right) \) could be the trivial ones -1 or 1.

Example 6.4. In Example 5.10, \( \Gamma(g) \subset \Gamma(f) \), hence by the last remark, \( T_- = \emptyset \). Furthermore, \( N_{\Delta, \{f\}} \neq \emptyset \), then by Theorem 6.1, \(-1\) is the only negative real pole of \( Z(s, \frac{1}{g}) \) and its order is 1. Also, by Theorem 6.2, the smallest real pole of \( Z(s, \frac{1}{g}) \) is \( \alpha = 1/2 \) with order 1.

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