The Logarithmic Sobolev Inequality in Infinite dimensions for Unbounded Spin Systems on the Lattice with non Quadratic Interactions.

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Abstract

We are interested in the Logarithmic Sobolev Inequality for the infinite volume Gibbs measure with no quadratic interactions. We consider unbounded spin systems on the one dimensional Lattice with interactions that go beyond the usual strict convexity and without uniform bound on the second derivative. We assume that the one dimensional single-site measure with boundaries satisfies the Log-Sobolev inequality uniformly on the boundary conditions and we determine conditions under which the Log-Sobolev Inequality can be extended to the infinite volume Gibbs measure.

Keywords: Logarithmic Sobolev inequality, Gibbs measure, Infinite dimensions, Spin systems.

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1 Introduction

We are interested in the q Logarithmic Sobolev Inequality (LSq) for measures related to systems of unbounded spins on the one dimensional Lattice with nearest neighbour interactions that are not strictly convex. Suppose that the Log-Sobolev Inequality is true for the single site measure with a constant uniformly bound on the boundary conditions. The aim of this paper is to present a criterion under which the inequality can be extended to the infinite volume Gibbs measure. More specifically, we extend the already know results for interactions V that satisfy \( \|\nabla_i \nabla_j V(x_i, x_j)\|_\infty < \infty \) to the more general case of interactions with \( \|\nabla_i \nabla_j V(x_i, x_j)\|_\infty = \infty \).

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Regarding the Log-Sobolev Inequality for the local specification \( \{E^\Lambda,\omega\}_{\Lambda \subset \subset \mathbb{Z}^d, \omega \in \Omega} \) on a d-dimensional Lattice, criterions and examples of measures \( E^\Lambda,\omega \) that satisfy the Log-Sobolev inequality with a constant uniformly on the set \( \Lambda \) and the boundary conditions \( \omega - \) are investigated in [Z2], [B-E], [B-L], [Y] and [B-H].

For \( \|\nabla_i \nabla_j V(x_i, x_j)\|_\infty < \infty \) the Log-Sobolev is proved when the phase \( \phi \) is strictly convex and convex at infinity. Furthermore, in [G-R] the Spectral Gap Inequality is proved to be true for phases beyond the convexity at infinity, while in [M-M] and [B-J-S] the Decay of Correlation is studied.

For the measure \( E^{\{i\}, \omega} \) on the real line, necessary and sufficient conditions are presented in [B-G], [B-Z] and [R-Z], so that the Log-Sobolev Inequality is satisfied uniformly on the boundary conditions \( \omega \).

The problem of the Log-Sobolev inequality for the Infinite dimensional Gibbs measure on the Lattice is examined in [G-Z], [Z1] and [Z2]. The first two study the LS for measures on a d-dimensional Lattice for bounded spin systems, while the third one looks at continuous spins systems on the one dimensional Lattice.

In [M] and [O-R], criterions are presented in order to pass from the Log-Sobolev Inequality for the single-site measure \( E^{\{i\}, \omega} \) to the LS\(_2\) for the Gibbs measure \( \nu_N \) on a finite N-dimensional product space. Furthermore, using these criterions one can conclude the Log-Sobolev Inequality for the family \( \{\nu_N, N \in \mathbb{N}\} \) with a constant uniformly on \( N \). Concerning the same problem for the LS\(_q\) (\( q \in (1, 2) \)) inequality in the case of Heisenberg groups with quadratic interactions in [I-P] a similar criterion is presented for the Gibbs measure based on the methods developed in [Z1] and [Z2].

All the pre mentioned developments refer to measures with interactions \( V \) that satisfy \( \|\nabla_i \nabla_j V(x_i, x_j)\|_\infty < \infty \). The question that arises is whether similar assertions can be verified for the infinite dimensional Gibbs measure in the case where \( \|\nabla_i \nabla_j V(x_i, x_j)\|_\infty = \infty \) and in this paper we present a strategy to solve this problem.

Consider the one dimensional measure

\[
E^{\{i\}, \omega}(dx_i) = \frac{e^{-\phi(x_i) - \sum_{j \sim i} J_{ij}V(x_i, \omega_j)}dX_i}{Z^{\{i\}, \omega}} \quad \text{with} \quad \|\partial_x \partial_y V(x, y)\|_\infty = \infty
\]

Assume that \( E^{\{i\}, \omega} \) satisfies the (LS) inequality with a constant uniformly on \( \omega \). Our aim is to set conditions, so that the infinite volume Gibbs measure \( \nu \) for the local specification \( \{E^\Lambda,\omega\}_{\Lambda \subset \subset \mathbb{Z}^d, \omega \in \Omega} \) satisfies the LS inequality. We will focus on measures on the one dimensional Lattice, but our result can also be easily extended on trees.

Our general setting is as follows:

The Lattice. When we refer to the Lattice we mean the 1-dimensional Lattice \( \mathbb{Z} \).
The Configuration space. We consider continuous unbounded random variables in $\mathbb{R}$, representing spins. Our configuration space is $\Omega = \mathbb{R}^\mathbb{Z}$. For any $\omega \in \Omega$ and $\Lambda \subset \mathbb{Z}$ we denote

$$\omega = (\omega_i)_{i \in \mathbb{Z}}, \omega_\Lambda = (\omega_i)_{i \in \Lambda}, \omega_{\Lambda^c} = (\omega_i)_{i \in \Lambda^c} \quad \text{and} \quad \omega = \omega_\Lambda \circ \omega_{\Lambda^c}$$

where $\omega_i \in \mathbb{R}$. When $\Lambda = \{i\}$ we will write $\omega_i = \omega_{\{i\}}$. Furthermore, we will write $i \sim j$ when the nodes $i$ and $j$ are nearest neighbours, that means, they are connected with a vertex, while we will denote the set of the neighbours of $k$ as $\{\sim k\} = \{r : r \sim k\}$.

The functions of the configuration. We consider integrable functions $f$ that depend on a finite set of variables $\{x_i\}, i \in \Sigma_f$ for a finite subset $\Sigma_f \subset \subset \mathbb{Z}$. The symbol $\subset \subset$ is used to denote a finite subset.

The Measure on $\mathbb{Z}$. For any subset $\Lambda \subset \subset \mathbb{Z}$ we define the probability measure

$$E^{\Lambda, \omega}(dx_\Lambda) = \frac{e^{-H^{\Lambda, \omega}} dx_\Lambda}{Z^{\Lambda, \omega}}$$

where

- $x_\Lambda = (x_i)_{i \in \Lambda}$ and $dx_\Lambda = \prod_{i \in \Lambda} dx_i$
- $Z^{\Lambda, \omega} = \int e^{-H^{\Lambda, \omega}} dx_\Lambda$
- $H^{\Lambda, \omega} = \sum_{i \in \Lambda} \phi(x_i) + \sum_{i \in \Lambda, j \sim i} J_{ij} V(x_i, z_j)$

and

- $z_j = x_\Lambda \circ \omega_{\Lambda^c} = \begin{cases} x_j, & i \in \Lambda \\ \omega_j, & i \notin \Lambda \end{cases}$

We call $\phi$ the phase and $V$ the potential of the interaction. For convenience we will frequently omit the boundary symbol from the measure and will write $E^\Lambda \equiv E^{\Lambda, \omega}$.

The Infinite Volume Gibbs Measure. The Gibbs measure $\nu$ for the local specification $\{E^{\Lambda, \omega}\}_{\Lambda \subset \mathbb{Z}, \omega \in \Omega}$ is defined as the probability measure which solves the Dobrushin-Lanford-Ruelle (DLR) equation

$$\nu E^{\Lambda, \ast} = \nu$$

for finite sets $\Lambda \subset \mathbb{Z}$ (see [P]). For conditions on the existence and uniqueness of the Gibbs measure see e.g. [B-HK] and [D]. In this paper we consider local specfications for which the Gibbs measure exists and it is unique. It should be noted that $\{E^{\Lambda, \omega}\}_{\Lambda \subset \subset \mathbb{Z}, \omega \in \Omega}$ always satisfies the DLR equation, in the sense that

$$E^{\Lambda, \omega} E^{M, \ast} = E^{\Lambda, \omega}$$

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for every $M \subset \Lambda$. [P].

The gradient $\nabla$ for continuous spins systems. For any subset $\Lambda \subset \mathbb{Z}$ we define the gradient

$$|\nabla f|^q = \sum_{i \in \Lambda} |\nabla_i f|^q,$$

where $\nabla_i = \frac{\partial}{\partial x_i}$

When $\Lambda = \mathbb{Z}$ we will simply write $\nabla = \nabla_{\mathbb{Z}}$. We denote

$$E^{\Lambda, \omega} f = \int f dE^{\Lambda, \omega}(x_{\Lambda})$$

We can define the following inequalities

The $q$ Log-Sobolev Inequality ($LS_q$). We say that the measure $E^{\Lambda, \omega}$ satisfies the $q$ Log-Sobolev Inequality for $q \in (1, 2]$, if there exists a constant $C_{LS}$ such that for any function $f$, the following holds

$$E^{\Lambda, \omega} |f|^q \log \frac{|f|^q}{E^{\Lambda, \omega} |f|^q} \leq C_{LS} E^{\Lambda, \omega} |\nabla f|^q$$

with a constant $C_{LS} \in (0, \infty)$ uniformly on the set $\Lambda$ and the boundary conditions $\omega$.

The $q$ Spectral Gap Inequality. We say that the measure $E^{\Lambda, \omega}$ satisfies the $q$ Spectral Gap Inequality for $q \in (1, 2]$, if there exists a constant $C_{SG}$ such that for any function $f$, the following holds

$$E^{\Lambda, \omega} |f|^q - E^{\Lambda, \omega} f|^q \leq C_{SG} E^{\Lambda, \omega} |\nabla f|^q$$

with a constant $C_{SG} \in (0, \infty)$ uniformly on the set $\Lambda$ and the boundary conditions $\omega$.

Remark 1.1. We will frequently use the following two well known properties about the Log-Sobolev and the Spectral Gap Inequality. If the probability measure $\mu$ satisfies the Log-Sobolev Inequality with constant $c$ then it also satisfies the Spectral Gap Inequality with a constant $\hat{c} = \frac{4c}{\log 2}$. More detailed, in the case where $q = 2$ the optimal constant is less or equal to $\frac{c}{2} < \hat{c}$, while in the case $1 < q < 2$ it is less or equal to $\frac{4c}{\log 2}$. The constant $\hat{c}$ does not depend on the value of the parameter $q \in (1, 2]$.

Furthermore, if for a family $I$ of sets $\Lambda_i \subset \mathbb{Z}$, $\text{dist}(\Lambda_i, \Lambda_j) > 1$, $i \neq j$ the measures $E^{\Lambda_i, \omega}$, $i \in I$ satisfy the Log-Sobolev Inequality with constants $c_i$, $i \in I$, then the probability measure $E^{(\bigcup_{i \in I} \Lambda_i), \omega}$ also satisfies the (LS) Inequality with constant $c = \max_{i \in I} c_i$. The last result is also true for the Spectral Gap Inequality. The proofs of these two properties can be found in [G] and [G-Z] for $q = 2$ and in [B-Z] for $1 < q < 2$. 

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2 The Main Result

We want to extend the Log-Sobolev Inequality from the single-site measure $\mathbb{E}^{i,\omega}$ to the Gibbs measure for the local specification $\{\mathbb{E}^{\Lambda,\omega}\}_{\Lambda \subset \mathbb{Z}, \omega \in \Omega}$ on the entire one dimensional Lattice.

**Hypothesis** We consider four main hypothesis:

(H0): The one dimensional measures $\mathbb{E}^{i,\omega}$ satisfies the Log-Sobolev-q Inequality with a constant $c$ uniformly with respect to the boundary conditions $\omega$.

(H1): The restriction $\nu^{\Lambda(k)}$ of the Gibbs measure $\nu$ to the $\sigma-$algebra $\Sigma^{\Lambda(k)}$,

$$\Lambda(k) = \{k - 2, k - 1, k, k + 1, k + 2\}$$

satisfies the Log-Sobolev-q Inequality with a constant $C \in (0, \infty)$.

(H2): For some $\epsilon > 0$ and $K > 0$

$$\nu^{\Lambda(i)}e^{2\epsilon^2 + 2\epsilon}V(x_r, x_s) \leq e^K \quad \text{and} \quad \nu^{\Lambda(i)}e^{2\epsilon^2 + 2\epsilon}\|\nabla_r V(x_r, x_s)\|^q \leq e^K$$

for $r, s \in \{i - 2, i - 1, i, i + 1, i + 2\}$

(H3): The coefficients $J_{i,j}$ are such that $|J_{i,j}| \in \{0, J\}$ for some $J < 1$ sufficiently small.

**Remark 2.1.** From Hypothesis (H2) and Jensen’s inequality it follows that

$$\nu e^{\epsilon(|F(r)| + E^{S(r)} \omega_r |F(r)|)^q} \leq e^K,$$

where the functions $F(r)$ are defined by

$$F(r) = \begin{cases} \nabla_r V(x_{i-1}, x_i) + \nabla_r V(x_{i+1}, x_i) & \text{for } r = i - 1, i, i + 1 \\ \nabla_r V(x_s, x_r)\mathbb{I}_{s = r, s \in \{i-3, i+3\}} & \text{for } r = i - 2, i + 2 \end{cases}$$

and the sets $S(r)$ by

$$S(r) = \begin{cases} \{\sim i\} & \text{for } r = i - 1, i, i + 1 \\ \{i + 3, i + 4, \ldots\} & \text{for } r = i + 2 \text{ and } s = i + 3 \\ \{\ldots, i - 4, i - 3\} & \text{for } r = i - 2 \text{ and } s = i - 3 \end{cases}$$

These bounds will be frequently used throughout the paper.
Remark 2.2. Throughout this paper we will consider differentiable functions that satisfy
\[ \nu |f|^q < \infty \text{ and } \nu |\nabla f|^q < \infty \]

The main theorem follows.

Theorem 2.3. If hypothesis (H0)-(H3) are satisfied, then the infinite dimensional Gibbs measure \( \nu \) for the local specification \( \{E^{\Lambda, \omega}\}_{\Lambda \subset Z, \omega \in \Omega} \) satisfies the \( q \) Log-Sobolev inequality
\[ \nu |f|^q \log \left( \frac{|f|^q}{\nu f} \right) \leq C \nu |\nabla f|^q \]
for some positive constant \( C \).

Proof. For the proof of the theorem it is sufficient to consider \( f \geq 0 \). This is an assumption that we will make through all the proofs presented in this paper.

We want to extend the Log-Sobolev Inequality from the single-site measure \( E^{\{i\}, \omega} \) to the Gibbs measure for the local specification \( \{E^{\Lambda, \omega}\}_{\Lambda \subset Z, \omega \in \Omega} \) on the entire one dimensional lattice. To do so, we will follow the iterative method developed by Zegarlinski in [Z1] and [Z2]. Define the following sets
\[ \Gamma_0 = \text{even integers}, \quad \Gamma_1 = \mathbb{Z} \setminus \Gamma_0 \]

One can notice that \( \{\text{dist}(i, j) > 1, \ \forall i, j \in \Gamma_k, k = 0, 1\} \), \( \Gamma_0 \cap \Gamma_1 = \emptyset \) and \( \mathbb{Z} = \Gamma_0 \cup \Gamma_1 \). For convenience we will write \( \mathcal{P} = E^{\Gamma_1, \Gamma_0} \)

In order to prove the Log-Sobolev Inequality for the measure \( \nu \), we will express the entropy with respect to the measure \( \nu \) as the sum of the entropies of the measures \( E^{\Gamma_0} \) and \( E^{\Gamma_1} \) which are easier to handle. We can write
\[
\nu(f^q \log \frac{f^q}{\nu f}) = \nu E^{\Gamma_0}(f^q \log \frac{f^q}{E^{\Gamma_0} f^q}) + \nu E^{\Gamma_1}(E^{\Gamma_0} f^q \log E^{\Gamma_1} f^q) + \nu(E^{\Gamma_1} f^q \log E^{\Gamma_1} f^q) - \nu(f^q \log \nu f^q) \tag{2.1}
\]

According to hypothesis (H0), the Log-Sobolev Inequality is satisfied for the single-state measures \( E^{\{i\}} \) and the sets \( \Gamma_0 \) and \( \Gamma_1 \) are unions of one dimensional sets of distance greater than the length of the interaction one. Thus, as we mentioned in Remark 1.1 in the introduction, the (LS) holds for the product measures \( E^{\Gamma_0} \) and \( E^{\Gamma_1} \) with the same constant \( c \). If we use the LS for \( E^{\Gamma_i}, i = 0, 1 \) we get
\[
(2.1) \leq c \nu(E^{\Gamma_0} |\nabla f|^q) + c \nu E^{\Gamma_1} \left| \nabla f \right|^q + \nu(E^{\Gamma_1} f^q \log E^{\Gamma_1} f^q) - \nu(f^q \log f) \tag{2.2}
\]
For the third term of (2.2) we can write

\[
\nu(P f^q \log f^q) = \nu\mathbb{E}_{\Gamma_0} \left( \mathcal{P} f^q \log \frac{\mathcal{P} f^q}{\mathbb{E}_{\Gamma_0} f^q} \right) + \nu\mathbb{E}_{\Gamma_1} \left( \mathbb{E}_{\Gamma_0} \mathcal{P} f^q \log \frac{\mathcal{E}_{\Gamma_0} \mathcal{P} f^q}{\mathbb{E}_{\Gamma_1} \mathcal{E}_{\Gamma_0} \mathcal{P} f^q} \right)
\]

\[+ \nu(\mathbb{E}_{\Gamma_1} \mathcal{E}_{\Gamma_0} \mathcal{P} f^q \log \mathbb{E}_{\Gamma_1} \mathcal{E}_{\Gamma_0} \mathcal{P} f^q)\]

If we use again the Log-Sobolev Inequality for the measures \(\mathbb{E}_{\Gamma_i}, i = 0, 1\) we get

\[\nu(P f^q \log P f^q) \leq c \nu \left| \nabla_{\Gamma_0} (P f^q) \right|^q + c \nu \left| \nabla_{\Gamma_1} \left( \mathbb{E}_{\Gamma_0} P f^q \right) \right|^q + \nu(P^2 f^q \log P^2 f^q)\]  (2.3)

If we work similarly for the last term \(\nu(P^2 f^q \log P^2 f^q)\) of (2.3) and inductively for any term \(\nu(P^k f^q \log P^k f^q)\), then after \(n\) steps (2.2) and (2.3) will give

\[
\nu(f^n f^q \log f^n f^q) \leq \nu(P^n f^q \log P^n f^q) - \nu(f^q \log f^q) + c \nu \left| \nabla_{\Gamma_0} f^n \right|^q
\]

\[+ c \sum_{k=0}^{n-1} \nu \left| \nabla_{\Gamma_0} (P^k f^q) \right|^q + c \sum_{k=0}^{n-1} \nu \left| \nabla_{\Gamma_1} \left( \mathbb{E}_{\Gamma_0} P^k f^q \right) \right|^q\]  (2.4)

In order to calculate the fourth and fifth term on the right-hand side of (2.4) we will use the following proposition

**Proposition 2.4.** Suppose that hypothesis (H0)-(H3) are satisfied. Then the following bound holds

\[\nu \left| \nabla_{\Gamma_i} \left( \mathbb{E}_{\Gamma_0} |f|^q \right) \right|^{\frac{1}{q}} \leq C_1 \nu \left| \nabla_{\Gamma_i} f \right|^q + C_2 \nu \left| \nabla_{\Gamma_i} f \right|^q\]  (2.5)

for \(\{i, j\} = \{0, 1\}\) and constants \(C_1 \in (0, \infty)\) and \(0 < C_2 < 1\).

The proof of Proposition 2.4 will be the subject of Section 4. If we apply inductively relationship (2.5) \(k\) times to the fourth and the fifth term of (2.4) we obtain

\[\nu \left| \nabla_{\Gamma_0} (P^k f^q) \right|^q \leq C_2^{k-1} C_1 \nu \left| \nabla_{\Gamma_0} f \right|^q + C_2^k \nu \left| \nabla_{\Gamma_0} f \right|^q\]  (2.6)

and

\[\nu \left| \nabla_{\Gamma_1} \left( \mathbb{E}_{\Gamma_0} P^k f^q \right) \right|^q \leq C_2^{k+1} C_1 \nu \left| \nabla_{\Gamma_1} f \right|^q + C_2^{k+1} \nu \left| \nabla_{\Gamma_0} f \right|^q\]  (2.7)

If we plug (2.6) and (2.7) in (2.4) we get

\[\nu(f^n f^q \log f^n f^q) \leq \nu(P^n f^q \log P^n f^q) - \nu(f^q \log f^q)\]

\[+ c \sum_{k=0}^{n-1} C_2^{k-1} C_1 \nu \left| \nabla_{\Gamma_1} f \right|^q + c \sum_{k=0}^{n-1} C_2^k \nu \left| \nabla_{\Gamma_0} f \right|^q\]

\[+ c \sum_{k=0}^{n-1} C_2^k C_1 \nu \left| \nabla_{\Gamma_1} f \right|^q + c \sum_{k=0}^{n-1} C_2^{k+1} \nu \left| \nabla_{\Gamma_0} f \right|^q\]  (2.8)
If we take the limit of $n$ to infinity in (2.8), the first two terms on the right hand side cancel with each other, as explained on the proposition below.

**Proposition 2.5.** Under hypothesis (H0)-(H3), $\mathcal{P}^n f$ converges $\nu$-almost everywhere to $\nu f$.

The proof of this proposition will be presented in Section 3. So, taking the limit of $n$ to infinity in (2.8) leads to

$$\nu(|f|^q \log \frac{|f|^q}{\nu|f|^q}) \leq cA \left( \sum_{k=0}^{n-1} C_2^{-2k} \right) \nu |\nabla \Gamma_1 f|^q + cA \nu |\nabla \Gamma_0 f|^q$$

where $A = \lim_{n \to \infty} \sum_{k=0}^{n-1} C_2^{-2k} < \infty$ for $C_2 < 1$, and the theorem follows for a constant $C = \max\{cA \left( \sum_{k=0}^{n-1} C_2^{-2k} \right), cA\}$.

### 3 Proof of Proposition 2.5

Before proving Proposition 2.5, we will present three useful lemmata. These lemmata will also be used in the next section 4 where Proposition 2.4 is proved.

In the case of quadratic interactions $V(x, y) = (x - y)^2$ one can calculate

$$\mathbb{E}_i,\omega \left( f^2 (\nabla_j V(x_i - x_j) - \mathbb{E}_i,\omega \nabla_j V(x_i - x_j))^2 \right)$$

(see [B-H] and [H]) with the use of the Deuschel-Stroock relative entropy inequality (see [D-S]) and the Herbst argument (see [L] and [H]). Herbst’s argument states that if a probability measure $\mu$ satisfies the LS2 inequality and a function $F$ is Lipschitz continuous with $\|F\|_{Lips} \leq 1$ and such that $\mu(F) = 0$, then for some small $\epsilon$ we have

$$\mu e^{\epsilon F^2} < \infty$$

For $\mu = \mathbb{E}_i,\omega$ and $F = \frac{\nabla_j V(x_i - x_j) - \mathbb{E}_i,\omega \nabla_j V(x_i - x_j)}{2}$ we then obtain

$$\mathbb{E}_i,\omega e^{\epsilon \mathbb{E}_i,\omega (\nabla_j V(x_i - x_j) - \mathbb{E}_i,\omega \nabla_j V(x_i - x_j))^2} < \infty$$

uniformly on the boundary conditions $\omega$, because of hypothesis (H0). In the more general case however of non quadratic interactions that we examine in this work, the Herbst argument cannot be applied. In this and next sections we show how one can bound exponential quantities like the last one with the use of the projection of the infinite dimensional Gibbs measure and hypothesis (H1) and (H2).

For every probability measure $\mu$, we define the correlation function

$$\mu(f; g) \equiv \mu(fg) - \mu(f)\mu(g)$$
If for the set $M(k) = \mathbb{Z} \setminus \Lambda(k)$ and $h_k := f - \mathbb{E}^{(\sim k)} f$ we define

$$Q(u, k) \equiv \nu_{\Lambda(u)} \left| \nabla_{\Lambda(u)} (\mathbb{E}^{M(u)} | h_k |^q) \right|^q$$

then the following lemma presents an estimate for the correlation function, in terms of $Q(k, k)$.

**Lemma 3.1.** For any functions $u$ localised in $\Lambda(k)$ for which $\nu_{\Lambda(k)} e^{2\epsilon |u|^q} < \infty$ the following inequalities are satisfied

(a) under hypothesis (H1)

$$\nu |\mathbb{E}^{k-1}\mathbb{E}^{k+1}(f; u)|^q \leq \frac{C}{\epsilon} Q(k, k) + \frac{1}{\epsilon} \left( \log \nu_{\Lambda(k)} e^{\epsilon |u - \mathbb{E}^{k-1}\mathbb{E}^{k+1} u|^q} \right) \nu |f - \mathbb{E}^{k-1}\mathbb{E}^{k+1} f|^q$$

(b) under hypothesis (H0) and (H1)

$$\nu |\mathbb{E}^{k-1}\mathbb{E}^{k+1}(f; u)|^q \leq \frac{C}{\epsilon} Q(k, k) + \frac{c}{\epsilon} \left( \log \nu_{\Lambda(k)} e^{\epsilon |u - \mathbb{E}^{k-1}\mathbb{E}^{k+1} u|^q} \right) \sum_{i=k-1,k+1} \nu |\nabla_i f|^q$$

where $c = \frac{4\epsilon}{\log 2}$.

**Proof.** From the definition of the correlation function we can write

$$\nu |\mathbb{E}^{k-1}\mathbb{E}^{k+1}(f; u)|^q = \nu \left| \mathbb{E}^{k-1}\mathbb{E}^{k+1}((f - \mathbb{E}^{k-1}\mathbb{E}^{k+1} f)(u - \mathbb{E}^{k-1}\mathbb{E}^{k+1} u)) \right|^q$$

$$\leq \nu \mathbb{E}^{k-1}\mathbb{E}^{k+1} \left( |f - \mathbb{E}^{k-1}\mathbb{E}^{k+1} f|^q |u - \mathbb{E}^{k-1}\mathbb{E}^{k+1} u|^q \right)$$

$$= \nu \left( |f - \mathbb{E}^{k-1}\mathbb{E}^{k+1} f|^q |u - \mathbb{E}^{k-1}\mathbb{E}^{k+1} u|^q \right)$$

where above we first used the Jensen’s Inequality and then the fact that the Gibbs measure $\nu$ satisfies the DLR equation. Because the function $u$ is localised in $\Lambda(k)$ and the measure $\mathbb{E}^{(k-1,k+1),\omega}$ $= \mathbb{E}^{k-1}\mathbb{E}^{k+1}$ has boundary in $\{k-2, k, k+2\} \subset \Lambda(k)$, we have that $u - \mathbb{E}^{k-1}\mathbb{E}^{k+1} u$ is also localised in $\Lambda(k)$ and so for $M(k)$ being the complementary of $\Lambda(k)$ we can write

$$\nu(|f - \mathbb{E}^{k-1}\mathbb{E}^{k+1} f|^q |u - \mathbb{E}^{k-1}\mathbb{E}^{k+1} u|^q) = \nu_{\Lambda(k)} \left( (\mathbb{E}^{M(k)} |f - \mathbb{E}^{k-1}\mathbb{E}^{k+1} f|^q) |u - \mathbb{E}^{k-1}\mathbb{E}^{k+1} u|^q \right)$$

(3.2)

On the right hand side of (3.2) we can use the following entropic inequality (see [D-S])

$$\forall t > 0, \mu(u y) \leq \frac{1}{t} \log (\mu(e^{t u})) + \frac{1}{t} \mu(u \log y)$$

(3.3)
for any probability measure \( \mu \) and \( y \geq 0, \mu y = 1 \). Then from (3.1) and (3.2) we will obtain
\[
\nu \left| E_{-1}^{k+1}(f; u) \right|^q \leq \frac{1}{\epsilon} \nu_{\lambda(k)} \nu_{\lambda(k)} \frac{E_{-1}^{M}(k) \left| f - E_{-1}^{k+1} f \right|^q}{\nu_{\lambda(k)} \nu_{\lambda(k)} \left| f - E_{-1}^{k+1} f \right|^q} + \frac{1}{\epsilon} \left( \log \nu_{\lambda(k)} e^{\mu E_{-1}^{u \cdot E_{-1}^{k+1} u}} \right) \nu_{\lambda(k)} \nu_{\lambda(k)} \left| f - E_{-1}^{k+1} f \right|^q
\]

The first term on the right hand side of (3.4) can be bounded from hypothesis (H1) by the Log-Sobolev inequality for \( \nu \)
\[
\nu_{\lambda(k)} \nu_{\lambda(k)} \left| f - E_{-1}^{k+1} f \right|^q \leq C \nu_{\lambda(k)} \left| \nabla_{\lambda(k)} \left( E_{-1}^{M}(k) \left| f - E_{-1}^{k+1} f \right|^q \right) \right|^q = C Q(k, k)
\]

Using (3.4) and (3.5) we get
\[
\nu \left| E_{-1}^{k+1}(f; u) \right|^q \leq \frac{C}{\epsilon} Q(k, k) + \frac{1}{\epsilon} \left( \log \nu e^{\mu E_{-1}^{u \cdot E_{-1}^{k+1} u}} \right) \nu \left| f - E_{-1}^{k+1} f \right|^q
\]

which proves (a). If we assume hypothesis (H0), then we can bound the second term on the right hand side of (3.6) from the \( SG_q \) for the measures \( E_{-1}^{k-1}, E_{-1}^{k+1} \) from hypothesis (H0) and the product property for the \( SG_q \) (Remark 1.1), to obtain
\[
\nu \left| f - E_{-1}^{k-1} E_{-1}^{k+1} f \right|^q = \nu E_{-1}^{k-1} E_{-1}^{k+1} \left| f - E_{-1}^{k+1} f \right|^q \leq \hat{c} \sum_{i=k-1,k+1} \nu \left| \nabla_i f \right|^q
\]

where \( \hat{c} = \frac{k}{\log 2} \). Using (3.6) and (3.7) we finally get (b)
\[
\nu \left| E_{-1}^{k-1} E_{-1}^{k+1}(f; u) \right|^q \leq \frac{C}{\epsilon} Q(k, k) + \frac{\hat{c}}{\epsilon} \left( \log \nu e^{\mu E_{-1}^{u \cdot E_{-1}^{k+1} u}} \right) \sum_{i=k-1,k+1} \nu \left| \nabla_i f \right|^q
\]

The following lemma gives an explicit bound for the quantity \( Q(k, k) \).

**Lemma 3.2.** Suppose that hypothesis (H0)-(H3) are satisfied. Then
\[
Q(k, k) \leq D \sum_{r=k-2}^{k+2} \nu \left| \nabla_r f \right|^q + D \sum_{n=0}^{\infty} J^{(n+1)(q-1)} \sum_{r=0}^{3} \left( \nu \left| \nabla_{k+3+4n+r} f \right|^q + \nu \left| \nabla_{k-3-4n-r} f \right|^q \right)
\]

for some positive constant \( D \).
The proof of this lemma will be the subject of Section 5.

**Lemma 3.3.** Suppose that hypothesis (H0)-(H3) are satisfied. Then for \(\{i, j\} = \{0, 1\}\)
\[
\nu |\nabla_{\Gamma_{i}}(E^{f_{0}})|^{q} \leq D_{1}\nu |\nabla_{\Gamma_{i}}f|^{q} + D_{2}\nu |\nabla_{\Gamma_{i}}f|^{q}
\]
holds for constants \(D_{1} \in (0, \infty)\) and \(0 < D_{2} < 1\).

**Proof.** Assume \(i = 1, j = 0\). We have
\[
\nu |\nabla_{\Gamma_{i}}(E^{f_{0}})|^{q} = \sum_{i \in \Gamma_{1}} \nu |\nabla_{i}(E^{f_{0}})|^{q} \leq \sum_{i \in \Gamma_{1}} \nu |\nabla_{i}(E^{i-1}E^{i+1}f)|^{q} \tag{3.8}
\]

If we denote \(\rho_{i} = \frac{e^{-H(x_{i-1})}e^{-H(x_{i+1})}}{e^{-H(x_{i-1})}dx_{i}f_{e^{-H(x_{i+1})}dx_{i}}}\) the density of the measure \(E^{i-1}E^{i+1}\) we can then write
\[
\nu |\nabla_{i}(E^{i-1}E^{i+1}f)|^{q} = \nu \left| \nabla_{i} \left( \int \int \rho_{i}dx_{i-1}dx_{i+1} \right) \right|^{q} \leq 2^{q-1} \nu \left| \int \int (\nabla_{i}f)\rho_{i}dx_{i-1}dx_{i+1} \right|^{q} + 2^{q-1} \nu \left| \int \int f(\nabla_{i}\rho_{i})dx_{i-1}dx_{i+1} \right|^{q} \leq \tag{3.9}
\]
\[
c_{1}\nu \left| E^{i-1}E^{i+1}(\nabla_{i}f) \right|^{q} + c_{1}J^{q}\nu \left| E^{i-1}E^{i+1}(f; \nabla_{i}V(x_{i-1}, x_{i}) + \nabla_{i}V(x_{i+1}, x_{i})) \right|^{q} \tag{3.10}
\]

where in (3.10) we used hypothesis (H3) to bound the coefficients \(J_{i,j}\) and we have denoted \(c_{1} = 2^{4i}\). If we apply the Hölder Inequality to the first term of (3.10) and Lemma 3.1 (b) to the second term, we obtain
\[
\nu |\nabla_{i}(E^{i-1}E^{i+1}f)|^{q} \leq c_{1}\nu |\nabla_{i}f|^{q} + \frac{J^{q}c_{1}C}{\epsilon}Q(i, i) + \frac{J^{q}c^{c_{1}}K}{\epsilon} \sum_{k=i-1,i+1} \nu |\nabla_{k}f|^{q} \tag{3.11}
\]

where the constant \(K\) as in hypothesis (H2). From (3.8) and (3.11) we have
\[
\nu |\nabla_{\Gamma_{i}}(E^{f_{0}})|^{q} \leq c_{1}\nu |\nabla_{\Gamma_{i}}f|^{q} + \frac{J^{q}c_{1}C}{\epsilon} \sum_{i \in \Gamma_{1}} Q(i, i) + \frac{J^{q}c^{c_{1}}K}{\epsilon} \sum_{i \in \Gamma_{1}} \sum_{k=i-1,i+1} \nu |\nabla_{k}f|^{q} \tag{3.12}
\]

If we use Lemma 3.2 to replace \(Q(k, k)\) in the above expression we get
\[
\nu |\nabla_{\Gamma_{i}}(E^{f_{0}})|^{q} \leq c_{1}\nu |\nabla_{\Gamma_{i}}f|^{q} + \frac{J^{q}c^{c_{1}}K}{\epsilon} \nu |\nabla_{\Gamma_{0}}f|^{q} + \frac{J^{q}c_{1}DC}{\epsilon} \sum_{i \in \Gamma_{1}} \sum_{r=i-2}^{i+2} \nu |\nabla_{r}f|^{q} + \frac{J^{q}c_{1}DC}{\epsilon} \sum_{i \in \Gamma_{1}} \sum_{n=0}^{\infty} J^{(n+1)(q-1)} \left( \sum_{r=0}^{3} (\nu |\nabla_{i+3+4n+r}f|^{q} + \nu |\nabla_{i-3-4n-r}f|^{q}) \right)
\]
for constant $D > 0$ as in Lemma 3.2. For coefficients $J_{i,j}$ sufficiently small such that $J < 1$ in (H3) we finally obtain

$$
\nu \left| \nabla_{\Gamma_1} (\mathbb{E}^{\Gamma_0} f) \right|^q \leq J q \left( \frac{2c_1 K}{\epsilon} + 2c_1 CD \epsilon + 2D \frac{c_1 C}{\epsilon} \frac{J^{(q-1)}}{1 - J^{(q-1)}} \right) \nu \left| \nabla_{\Gamma_0} f \right|^q
$$

$$
+ \left( c_1 q + \frac{J q c_1 C}{\epsilon} 3D + D \frac{2J q c_1 C}{\epsilon} \frac{J^{(q-1)}}{1 - J^{(q-1)}} \right) \nu \left| \nabla_{\Gamma_1} f \right|^q
$$

and the lemma follows for $J$ sufficiently small such that

$$
D_2 = J q \left( \frac{c_1 K}{\epsilon} 2 + 2c_1 CD \epsilon + D \frac{c_1 C}{\epsilon} \frac{J^{(q-1)}}{1 - J^{(q-1)}} \right) < 1
$$

Now we can prove Proposition 2.5.

**Proof of Proposition 2.5.** Following [G-Z] we will show that in $L^1(\nu)$ we have

$$
\lim_{n \to \infty} \mathcal{P}_n f = \nu.
$$

For $i \neq j$ we have that

$$
\nu \left| \mathbb{E}^{\Gamma_j} f - \mathbb{E}^{\Gamma_i} \mathbb{E}^{\Gamma_j} f \right|^q = \nu \mathbb{E}^{\Gamma_i} \left| \mathbb{E}^{\Gamma_j} f - \mathbb{E}^{\Gamma_i} \mathbb{E}^{\Gamma_j} f \right|^q
$$

$$
\leq c \nu \left| \nabla_{\Gamma_i} (\mathbb{E}^{\Gamma_j} f) \right|^q
$$

(3.12)

The last inequality due to the fact that both the measures $\mathbb{E}^{\Gamma_0}$ and $\mathbb{E}^{\Gamma_1}$ satisfy the Log-Sobolev Inequality and the Spectral Gap inequality with constants independently of the boundary conditions. If we use Lemma 3.3 we get

$$
\nu \left| \mathbb{E}^{\Gamma_j} f - \mathbb{E}^{\Gamma_i} \mathbb{E}^{\Gamma_j} f \right|^q \leq c D_1 \nu \left| \nabla_{\Gamma_j} f \right|^q + c D_2 \nu \left| \nabla_{\Gamma_i} f \right|^q
$$

From the last inequality we obtain that for any $n \in \mathbb{N}$,

$$
\nu \left| \mathcal{P}_n f - \mathbb{E}^{\Gamma_0} \mathcal{P}_n f \right|^q \leq c D_1 \nu \left| \nabla_{\Gamma_0} (\mathbb{E}^{\Gamma_0} \mathcal{P}_n f) \right|^q + c D_2 \nu \left| \nabla_{\Gamma_1} (\mathbb{E}^{\Gamma_0} \mathcal{P}_n f) \right|^q
$$

$$
= c D_2 \nu \left| \nabla_{\Gamma_1} (\mathbb{E}^{\Gamma_0} \mathcal{P}_n f) \right|^q
$$

If we use Lemma 3.3 to bound the last expression we have the following

$$
\nu \left| \mathcal{P}_n f - \mathbb{E}^{\Gamma_0} \mathcal{P}_n f \right|^q \leq c D_2 \left( D_1 \nu \left| \nabla_{\Gamma_1} f \right|^q + D_2 \nu \left| \nabla_{\Gamma_0} f \right|^q \right)
$$

(3.13)

Similarly we obtain

$$
\nu \left| \mathbb{E}^{\Gamma_0} \mathcal{P}_n f - \mathcal{P}_n f \right|^q \leq c D_2 \left( D_1 \nu \left| \nabla_{\Gamma_1} f \right|^q + D_2 \nu \left| \nabla_{\Gamma_0} f \right|^q \right)
$$

(3.14)
Consider the sequence \( \{Q^n\}_{n \in \mathbb{N}} \) defined as
\[
Q^n f = \begin{cases} 
\mathcal{P}_f^n f & \text{if } n \text{ even} \\
\mathbb{E}^{\Gamma_0} \mathcal{P}_{\frac{n-1}{2}} f & \text{if } n \text{ odd}
\end{cases}
\]
for every \( n \in \mathbb{N} \). Hence, if we define the sets
\[
A_n = \{|Q^n f - Q^{n+1} f| \geq \left(\frac{1}{2}\right)^n\}
\]
we obtain
\[
\nu(A_n) = \nu \left( \{|Q^n f - Q^{n+1} f| \geq \left(\frac{1}{2}\right)^n\} \right) \leq 2^m \nu |Q^n f - Q^{n+1} f|^q
\]
by Chebyshev inequality. If we use (3.13) and (3.14) to bound the last we have
\[
\nu(A_n) \leq (2^q D_2^{\frac{1}{2}})^n \hat{c} (D_1 \nu |\nabla \Gamma_1 f|^q + D_2 \nu |\nabla \Gamma_0 f|^q)
\]
We can choose \( J \) sufficiently small such that \( 2^q D_2^{\frac{1}{2}} < \frac{1}{2} \) in which case we get that
\[
\sum_{n=0}^{\infty} \nu(A_n) \leq \left( \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \right) \hat{c} (D_1 \nu |\nabla \Gamma_1 f|^q + D_2 \nu |\nabla \Gamma_0 f|^q) < \infty
\]
From the Borel-Cantelli lemma, only finite number of the sets \( A_n \) can occur, which implies that the sequence
\[
\{Q^n f\}_{n \in \mathbb{N}}
\]
is a Cauchy sequence and that it converges \( \nu \)-almost surely. Say
\[
Q^n f \rightarrow \theta(f) \quad \nu \text{ -- a.e.}
\]
We will first show that \( \theta(f) \) is a constant, i.e. it does not depend on variables on \( \Gamma_0 \) or \( \Gamma_1 \). To show that, first notice that \( Q^n f \) is a function on \( \Gamma_1 \) and \( \Gamma_0 \) when \( n \) is odd and even respectively, which implies that the limits
\[
\theta_o(f) = \lim_{n \text{ odd}, n \rightarrow \infty} Q^n f \quad \text{and} \quad \theta_e(f) = \lim_{n \text{ even}, n \rightarrow \infty} Q^n f
\]
do not depend on variables on \( \Gamma_0 \) and \( \Gamma_1 \) respectively. Since both the subsequences \( \{Q^n f\}_{n \text{ even}} \) and \( \{Q^n f\}_{n \text{ odd}} \) converge to \( \theta(f) \) \( \nu \)-a.e. we have that
\[
\theta_o(f) = \theta(f) = \theta_e(f)
\]
which implies that \( \theta(f) \) is a constant. From that we obtain that
\[
\nu(\theta(f)) = \theta(f)
\]
(3.15)
Since the sequence \( \{Q_n f\}_{n \in \mathbb{N}} \) converges \( \nu \)-almost, the same holds for the sequence \( \{Q_n f - \nu Q^n f\}_{n \in \mathbb{N}} \). We have
\[
\lim_{n \to \infty} (Q^n f - \nu Q^n f) = \theta(f) - \nu(\theta(f)) = \theta(f) - \theta(f) = 0
\]
where above we used (3.15). On the other side, we also have
\[
\lim_{n \to \infty} (Q^n f - \nu Q^n f) = \lim_{n \to \infty} (Q^n f - \nu f) = \theta(f) - \nu(f)
\]
From (3.15) and (3.16) we get that
\[
\theta(f) = \nu(f)
\]
We finally get
\[
\lim_{n \to \infty} P_n f = \lim_{n \text{ even}, n \to \infty} Q_n f = \nu f, \ \nu \text{ a.e.}
\]

4 Proof of Proposition [2.4]

Before we prove Proposition [2.4] we present some useful lemmata. First we define
\[
W_k = \nabla_k V(x_k, x_{k-1}) + \nabla_k V(x_k, x_{k+1}) \quad \text{and} \quad U_k = |W_k|^q + \mathbb{E}^{\sim k} |W_k|^q
\]
where \( \{\sim k\} \equiv \{j : j \sim k\} = \{k-1, k+1\} \).

Lemma 4.1. The following inequality holds
\[
\mathbb{E}^{\sim k}(f^q; W_k) \leq c_0 \left( \mathbb{E}^{\sim k} |f|^q \right)^{\frac{1}{q}} \left( \mathbb{E}^{\sim k} (|f - \mathbb{E}^{\sim k} f|^q U_k) \right)^{\frac{1}{q}}
\]
for some constant \( c_0 \) uniformly on the boundary conditions and \( \frac{1}{q} + \frac{1}{p} = 1 \).

Proof. We can write
\[
\mathbb{E}^{\sim k}(f^q; W_k) = \frac{1}{2} \mathbb{E}^{\sim k} \otimes \tilde{\mathbb{E}}^{\sim k} \left( (f^q - \tilde{f}^q)(W_k - \tilde{W}_k) \right)
\]
where \( \tilde{\mathbb{E}}^{\sim k} \) is an isomorphic copy of \( \mathbb{E}^{\sim k} \). If we define the function \( F \) to be \( F(s) = sf + (1-s)\tilde{f} \) then
\[
[1.2] = \frac{1}{2} \mathbb{E}^{\sim k} \otimes \tilde{\mathbb{E}}^{\sim k} \left( \left( \int_0^1 ds \frac{d}{ds} F(s)^q \right) (W_k - \tilde{W}_k) \right) = \frac{1}{2} \mathbb{E}^{\sim k} \otimes \tilde{\mathbb{E}}^{\sim k} \left( \left( \int_0^1 ds q F(s)^{q-1} \frac{d}{ds} F(s) \right) (W_k - \tilde{W}_k) \right) = \frac{1}{2} \mathbb{E}^{\sim k} \otimes \tilde{\mathbb{E}}^{\sim k} \left( q \int_0^1 ds F(s)^{q-1} (f - \tilde{f}) \right) (W_k - \tilde{W}_k)
\]
If we use the Holder inequality for the conjugate numbers $p$ and $q$, then the last quantity can be bounded by
\[
\frac{q}{2} \left\{ E^{(\sim k)} \otimes \tilde{E}^{(\sim k)} \left( \int_0^1 ds F(s)^{q-1} \right)^p \right\}^{\frac{1}{p}} \times \left\{ E^{(\sim k)} \otimes \tilde{E}^{(\sim k)} \left| (f - \tilde{f})(W_k - \tilde{W}_k) \right|^q \right\}^{\frac{1}{q}} \quad (4.3)
\]
For the first term in the above product, by Jensen’s Inequality and $\frac{1}{q} + \frac{1}{p} = 1$, we obtain
\[
\left\{ E^{(\sim k)} \otimes \tilde{E}^{(\sim k)} \left( \int_0^1 ds F(s)^{q-1} \right)^p \right\}^{\frac{1}{p}} \leq \left\{ E^{(\sim k)} \otimes \tilde{E}^{(\sim k)} \int_0^1 ds F(s)^q \right\}^{\frac{1}{p}} = \left( \int_0^1 ds E^{(\sim k)} \otimes \tilde{E}^{(\sim k)} F(s)^q \right)^{\frac{1}{p}} \leq \left( 2^q \int_0^1 ds E^{(\sim k)} \otimes \tilde{E}^{(\sim k)} (sf^q + (1 - s)\tilde{f}^q) \right)^{\frac{1}{p}} = 2^q \left( E^{(\sim k)} f^q \right)^{\frac{1}{p}} \quad (4.4)
\]
If we plug (4.4) into (4.3) we finally get
\[
E^{(\sim k)}(f^q; W_k) \leq \frac{2^q q}{2} \left( E^{(\sim k)} f^q \right)^{\frac{1}{p}} \left\{ E^{(\sim k)} \otimes \tilde{E}^{(\sim k)} \left( \left| f - \tilde{f} \right| |W_k - \tilde{W}_k| \right)^q \right\}^{\frac{1}{q}} \leq 2^q 2^q q \left( E^{(\sim k)} f^q \right)^{\frac{1}{p}} \left\{ E^{(\sim k)} \left( \left| f - E^{(\sim k)} f \right| |W_k|^q + E^{(\sim k)} |W_k|^q \right) \right\}^{\frac{1}{q}}
\]
The lemma follows for constant $c_0 = 2^q 2^q q$. \hfill \square

Define now the quantity
\[
A(k) = \nu \left( E^{(\sim k)} \right)^\frac{q}{p} \left( E^{(\sim k)} |f|^q \right)^{-\frac{q}{p}} \left| E^{(\sim k)} (|f|^q; W_k) \right|^q
\]
The next lemma presents an estimate of $A(k)$ involving $Q(k, k)$.

**Lemma 4.2.** Suppose that that hypothesis (H0)-(H2) are satisfied. Then
\[
A(k) \leq \frac{c_0 C}{\epsilon} Q(k, k) + \frac{c_0 \tilde{C}}{\epsilon} \sum_{i=k-1, k+1} \nu |\nabla_i f|^q
\]
where the constants $\epsilon$ and $K$ are as in hypothesis (H2).

**Proof.** We can initially bound $A(k)$ with the use of Lemma 4.1
\[
A(k) = \nu \left( E^{(\sim k)} f^q \right)^{-\frac{q}{p}} \left| E^{(\sim k)} (f^q; W_k) \right|^q \leq c_0 \nu E^{k-1} E^{k+1} \left( \left| f - E^{k-1} E^{k+1} f^q \right| U_k \right) = c_0 \nu A(k) \left( \left| E^{M(k)} f - E^{(\sim k)} f^q \right| U_k \right)
\]
(4.5)
because $U_k$ is localized in $\Lambda(k)$. If we use the entropy inequality (5.3) and hypothesis (H1) for $\nu_{\Lambda(k)}$ as well as (H2), as we did in Lemma 3.1, then for $K$ as in (H2), we can bound (4.5) by

$$
(4.5) \leq \frac{c_0^2 C}{\nu} Q(k, k) + \frac{c_0^2 K}{\nu} \mathbb{E}(|\nabla_i f| + f f^q) \leq \frac{c_0^2 C}{\nu} Q(k, k) + \frac{c_0^2 \hat{c} K}{\nu} \sum_{i=k-1,k+1} \nu |\nabla_i f|^q
$$

where above we used that $\mathbb{E}(|\nabla_i f| + f f^q) \leq \nu |\nabla_i f|^q$ as well as (4.5).

Lemma 4.3. The following inequality holds

$$
\nu |\nabla_i (\mathbb{E}^{i-1} \mathbb{E}^{i+1} |f|^q)^{\frac{1}{q}}| \leq c_1 \nu |\nabla_i f|^q + \frac{f^q c_1}{q^q} A(i)
$$

Proof. We have

$$
\nu |\nabla_i (\mathbb{E}^{i-1} \mathbb{E}^{i+1} f^q)^{\frac{1}{q}}| = \nu \left| \frac{1}{q} (\mathbb{E}^{i-1} \mathbb{E}^{i+1} f^q)^{\frac{1}{q} - 1} \nabla_i (\mathbb{E}^{i-1} \mathbb{E}^{i+1} f^q) \right|^q = \frac{1}{q^q} \nu (\mathbb{E}^{i-1} \mathbb{E}^{i+1} f^q)^{-\frac{q}{q}} |\nabla_i (\mathbb{E}^{i-1} \mathbb{E}^{i+1} f^q)|^q \quad (4.6)
$$

But from relationship (3.3) of Lemma 3.3 for $\rho_i$ being the density of $\mathbb{E}^{i-s}$ we have

$$
|\nabla_i (\mathbb{E}^{i-1} \mathbb{E}^{i+1} f^q)|^q = |\nabla_i (\int \int \rho_i f^q dx_{i-1} dx_{i+1})|^q \leq 2^{2q-2} \int \int \nabla_i (f^q) \rho_i dx_{i-1} dx_{i+1} |^q + 2^{2q-2} \int \int f^q |\nabla_i \rho_i| dx_{i-1} dx_{i+1} |^q \quad (4.7)
$$

For the second term in (4.7) we have

$$
\int \int f^q |\nabla_i \rho_i| dx_{i-1} dx_{i+1} \leq J^q \mathbb{E}^{i-1} \mathbb{E}^{i+1} (f^q, \nabla_i V(x_{i-1}, x_i) + \nabla_i V(x_{i+1}, x_i)) |^q
$$

While for the first term of (4.7) the following bound holds

$$
\int \int \nabla_i (f^q) \rho_i dx_{i-1} dx_{i+1} |^q = q^q \mathbb{E}^{i-1} \mathbb{E}^{i+1} (f^{q-1} (\nabla_i f)) |^q \leq q^q \mathbb{E}^{i-1} \mathbb{E}^{i+1} (f^{q-1}) |\nabla_i f|^q \leq q^q \mathbb{E}^{i-1} \mathbb{E}^{i+1} f^q |\nabla_i f|^q \quad (4.9)
$$
where above we used the Hölder inequality and that $p$ is the conjugate of $q$. If we plug (4.8) and (4.9) in (4.7) we get

$$
|\nabla_i (E^{i-1}E^{i+1}f^q)|^q \leq 2^{2q-2}q^q (E^{i-1}E^{i+1}f^q)^\frac{q}{p} (E^{i-1}E^{i+1} |\nabla_if|^q)
+ 2^{2q-2}q^q |E^{i-1}E^{i+1}(f^q, \nabla_i V(x_i-1, x_i) + \nabla_i V(x_i+1, x_i))|^q
$$

From the last relationship and (4.6) the lemma follows. □

Now we can prove Proposition 2.4.

**Proof of Proposition 2.4.** We have

$$
\nu \left| \nabla_\Gamma_1 (E^{\Gamma_0} f^q)^\frac{1}{q} \right|^q = \sum_{i \in \Gamma_1} \nu \left| \nabla_i (E^{\Gamma_0} f^q)^\frac{1}{q} \right|^q \leq \sum_{i \in \Gamma_1} \nu \left| \nabla_i (E^{\Gamma_0} f^q)^\frac{1}{q} \right|^q
\leq \sum_{i \in \Gamma_1} c_1 \nu |\nabla_i f|^q + \frac{c_0^2 c_1}{q^q} \sum_{i \in \Gamma_1} A(i)
$$

where the last inequality is due to Lemma 4.3. If we use Lemma 4.2 to bound $A(i)$ we get

$$
\nu \left| \nabla_\Gamma_1 (E^{\Gamma_0} f^q)^\frac{1}{q} \right|^q \leq \sum_{i \in \Gamma_1} c_1 \nu |\nabla_i f|^q + \frac{c_0^2 c_1}{q^q} \sum_{i \in \Gamma_1} \sum_{r=i-1}^{i+1} \sum_{r=i-1}^{i+1} \nu |\nabla_r f|^q
+ \frac{J^q c_1}{q} \sum_{i \in \Gamma_1} Q(i, i)
$$

Furthermore, if we use Lemma 3.2 to bound $Q(i, i)$ we obtain

$$
\nu \left| \nabla_\Gamma_1 (E^{\Gamma_0} f^q)^\frac{1}{q} \right|^q \leq \sum_{i \in \Gamma_1} c_1 \nu |\nabla_i f|^q + \frac{c_0^2 c_1}{q^q} \sum_{i \in \Gamma_1} \sum_{r=i-1}^{i+1} \sum_{r=i-1}^{i+1} \nu |\nabla_r f|^q
+ \frac{J^q c_1}{q} \sum_{i \in \Gamma_1} \sum_{k=0}^{k+2} \nu |\nabla_k f|^q
+ \frac{J^q c_1}{q} \sum_{i \in \Gamma_1} \sum_{n=0}^{\infty} J^{(n+1)(q-1)} \sum_{r=0}^{3} \nu |\nabla_{i+3+4n+r} f|^q
+ \frac{J^q c_1}{q} \sum_{i \in \Gamma_1} \sum_{n=0}^{\infty} J^{(n+1)(q-1)} \sum_{r=0}^{3} \nu |\nabla_{i-3-4n-r} f|^q
\tag{4.10}
$$

If we set $R = c_1 + \frac{c_0^2 c_1}{q^q} (\frac{c_0^2 c_1}{q^q} + \frac{c_0^2 c_1}{q})$ and we choose $J < 1$, relationship (4.10) gives

$$
\nu \left| \nabla_\Gamma_1 (E^{\Gamma_0} f^q)^\frac{1}{q} \right|^q \leq (R + J^q 4 + \frac{R^8 J^q}{1 - J^q}) \nu |\nabla_\Gamma_1 f|^q + R^4 (4 + \frac{8}{1 - J^q}) \nu |\nabla_\Gamma_0 f|^q
$$
For $J$ sufficiently small (H3) such that $RJ^q(4 + \frac{8}{1-J^{q-1}}) < 1$ the lemma follows for constants $C_1 = R + RJ^q4 + \frac{R^qJ^q}{1-J^{q-1}}$ and $C_2 = RJ^q(4 + \frac{8}{1-J^{q-1}}) < 1$. \hfill \Box

5 Proof of Lemma 3.2

This section is dedicated in the proof of Lemma 3.2 under the assumptions (H0)-(H3). We begin by showing the weaker result of Lemma 5.1 under the weaker assumptions (H1)-(H3).

Lemma 5.1. Suppose that hypothesis (H1)-(H3) are satisfied. Then $Q(k, k) \leq J^qS\nu |f - \mathbb{E}^{k-1}\mathbb{E}^{k+1}f|^q + S \sum_{r=0}^{k+2} \nu |\nabla_r f|^q$ $+ S \sum_{n=0}^{\infty} J^{(n+1)(q-1)} \sum_{r=0}^{3} (\nu |\nabla_{k+3+4n+r} f|^q + \nu |\nabla_{k-3-4n-r} f|^q)$ for some positive constant $S$.

Lemma 3.2 follows for some constant $D > 0$ directly from the last lemma and the Spectral Gap inequality implied from (H0). The remaining of this section is dedicated to the proof of Lemma 5.1. At first we prove some lemmata. To start, for any $k \in \mathbb{Z}$, we define the sets $M_s(k)$ for $s = k - 3, k + 3$ as

$$M_s(k) = \begin{cases} \{j \in \mathbb{Z} : j \geq k + 3\} = \{k + 3, k + 4, ...\} & \text{if } s = k + 3 \\ \{j \in \mathbb{Z} : j \leq k - 3\} = \{..., k - 4, k - 3\} & \text{if } s = k - 3 \end{cases} \quad (5.1)$$

Remark 5.2. Since $\Lambda(k) = \{k - 2, k - 1, k, k + 1, k + 2\}$ and $M(k) = \mathbb{Z} \setminus \Lambda(k)$, with the use of the definition (5.1) we can write $M(k) = \{j \in \mathbb{Z} : j \leq k - 3\} \cup \{j \in \mathbb{Z} : j \geq k + 3\} = M_{k-3}(k) \cup M_{k+3}(k)$

Since the sets $M_{k-3}(k)$ and $M_{k+3}(k)$ are disjoint we obtain that $\mathbb{E}^{M(k)}$ is a product measure, and for every function $f$ we can write

$$\mathbb{E}^{M(k)} f = \mathbb{E}^{M_{k-3}(k)} \otimes \mathbb{E}^{M_{k+3}(k)} f \quad (5.2)$$

Accordingly, for functions, say $f_{k-3}$ and $f_{k+3}$, that depend on variables $x_i$ with $i \notin M_{k+3}(k)$ and $i \notin M_{k-3}(k)$ respectively, we obtain

$$\mathbb{E}^{M(k)} f_{k-3} = \mathbb{E}^{M_{k-3}(k)} f_{k-3}$$

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and
\[ \mathbb{E}^{M(k)} f_{k+3} = \mathbb{E}^{M_{k+3}(k)} f_{k+3} \]

For instance, for \( r = k - 2, k + 2 \) and \( s \in \{k - 3, k + 3\} : s \sim r \), that is for the couples \((r, s) = (k - 2, k - 3)\) and \((r, s) = (k + 2, k + 3)\), we have
\[ \mathbb{E}^{M(k)} \nabla_r V(x_s, x_r) = \mathbb{E}^{M_s(k)} \nabla_r V(x_s, x_r) \quad (5.3) \]

**Remark 5.3.** Consider couples \((r, s)\) that take the values \((k - 2, k - 3)\) and \((k + 2, k + 3)\). We then have that \( \nabla_r V(x_s, x_r) \) is localised in \( \Lambda(k - 4) \) when \((r, s) = (k - 2, k - 3)\) and in \( \Lambda(k + 4) \) when \((r, s) = (k + 2, k + 3)\). Furthermore, from Remark 5.2, for \((r, s) = (k - 2, k - 3)\) we get that
\[ \mathbb{E}^{M_{k-3}(k)} \nabla_{k-2} V(x_{k-3}, x_{k-2}) = \mathbb{E}^{(k-4,k-3)} \nabla_{k-2} V(x_{k-3}, x_{k-2}) \]
is localised in \( \Lambda(k - 4) \), while for \((r, s) = (k + 2, k + 3)\) we get that
\[ \mathbb{E}^{M_{k+3}(k)} \nabla_{k+2} V(x_{k+3}, x_{k+2}) = \mathbb{E}^{(k+3,k+4,...)} \nabla_{k+2} V(x_{k+3}, x_{k+2}) \]
is localised in \( \Lambda(k + 4) \). So, if we set
\[ Y_s(x_s, x_r) = [\nabla_r V(x_s, x_r) - \mathbb{E}^{M_s(k)} \nabla_r V(x_s, x_r)] \]
we then have that \( Y_{k+3}(x_{k+2}, x_{k+3}) \) and \( Y_{k-3}(x_{k-2}, x_{k-3}) \) are localised in \( \Lambda(k + 4) \) and \( \Lambda(k - 4) \) respectively. Thus, we have
\[ \nu(f^qY_{k+3}^q(x_{k+3}, x_{k+2})) = \nu_{\Lambda(k+4)} \left( (\mathbb{E}^{M(k+4)} f^q)Y_{k+3}^q(x_{k+3}, x_{k+2}) \right) \]
\[ \nu(f^qY_{k-3}^q(x_{k-3}, x_{k-2})) = \nu_{\Lambda(k-4)} \left( (\mathbb{E}^{M(k-4)} f^q)Y_{k-3}^q(x_{k-3}, x_{k-2}) \right) \]
If we combine the last two together we can write
\[ \nu(f^qY_s^q(x_s, x_r)) = \nu_{\Lambda(t)} \left( (\mathbb{E}^{M(t)} f^q)Y_s^q(x_s, x_r)\mathcal{I}_{t \in \{k-4,k+4\}\cap M_s(k)} \right) \]
for \((r, s) \in \{(k + 2, k + 3), (k - 2, k - 3)\}\).

**Lemma 5.4.** Suppose conditions \((H1)\) and \((H2)\) are satisfied. Then for \( r = k - 2, k + 2 \) and \( s \in \{k - 3, k + 3\} : s \sim r \) the following inequality is true
\[ \nu_{\Lambda(k)} \left( (\mathbb{E}^{M(k)} |f|^q)^{-\frac{q}{p}} \right) \mathbb{E}^{M(k)} |\nabla_r V(x_s, x_r)|^q \leq \frac{C}{\epsilon} \nu_{\Lambda(t)} \left[ \nabla_{\Lambda(t)} \left( (\mathbb{E}^{M(t)} |f|^q)^{\frac{1}{2}} \right) \mathcal{I}_{t \in \{k-4,k+4\}\cap M_s(k)} + \frac{K}{\epsilon} |f|^q \right] \]
where \( \mathcal{I}_A \) denotes the characteristic function of a set \( A \) and the set \( M_s(k) \) as in (5.7).
Proof. For any two functions \(f\) and \(g\) the covariance with respect to a measure \(\mu\) can be computed as follows:

\[
\mu(f; g) = \mu((f - \mu f)(g - \mu g)) = \mu(f(g - \mu g)) - \mu(\mu f(g - \mu g)) = \mu(f(g - \mu g)) - (\mu f)\mu(g - \mu g) = \mu(f(g - \mu g))
\]

Using this expression we can write

\[
\mathbb{E}^M(f^q; \nabla_r V(x_s, x_r)) = \mathbb{E}^M(f^q(\nabla_r V(x_s, x_r) - \mathbb{E}^M(\nabla_r V(x_s, x_r))))
\]  

(5.4)

If we use (5.3) from Remark 5.2 (5.4) becomes

\[
\mathbb{E}^M(f^q; \nabla_r V(x_s, x_r)) = \mathbb{E}^M(f^q(\nabla_r V(x_s, x_r) - \mathbb{E}^M(\nabla_r V(x_s, x_r))))
\]

(5.5)

If we set

\[
Y_s(x_s, x_r) = |\nabla_r V(x_s, x_r) - \mathbb{E}^M(\nabla_r V(x_s, x_r))|
\]

then for (5.5) we can write

\[
\left| \mathbb{E}^M(f^q; \nabla_r V(x_s, x_r)) \right| \leq \mathbb{E}^M(f^{q-1}Y_s(x_s, x_r)) \\
\leq \left(\mathbb{E}^M(f^{(q-1)p}) \right)^{\frac{1}{p}} \left(\mathbb{E}^M(f^q Y_s^q(x_s, x_r)) \right)^{\frac{1}{q}} \\
= \left(\mathbb{E}^M(f^q) \right)^{\frac{1}{p}} \left(\mathbb{E}^M(f^q Y_s^q(x_s, x_r)) \right)^{\frac{1}{q}}
\]

(5.6)

where above we used the Hölder inequality and that \(\frac{1}{p} + \frac{1}{q} = 1\). So, for \(s = k+3, k-3\) from relationship (5.3) we obtain

\[
\nu_{\lambda(k)} \left(\mathbb{E}^M(f^q) \right)^{\frac{1}{p}} \left| \mathbb{E}^M(f^q; \nabla_r V(x_s, x_r)) \right| \leq \nu_{\lambda(k)} \mathbb{E}^M(f^q Y_s^q(x_s, x_r)) \\
= \nu(f^q Y_s^q(x_s, x_r))
\]

If we combine the last inequality together with Remark 5.3 we finally obtain

\[
\nu_{\lambda(k)} \left(\mathbb{E}^M(f^q) \right)^{\frac{1}{p}} \left| \mathbb{E}^M(f^q; \nabla_r V(x_s, x_r)) \right| \leq \nu_{\lambda(t)} \left(\mathbb{E}^M(f^q) \right)^{\frac{1}{p}} \left(\mathbb{E}^M(Y_s^q(x_s, x_r)I_{t\in\{k-4,k+4\}\cap M_s}) \right)
\]

(5.7)

If in (5.7) we use the Entropy Inequality and the \(LS_q\) for \(\nu_{\lambda(s)}\) from hypothesis (H1) and (H2), we get

\[
\nu_{\lambda(k)} \left(\mathbb{E}^M(f^q) \right)^{\frac{1}{p}} \left(\mathbb{E}^M(f^q; \nabla_r V(x_s, x_r)) \right)^{\frac{1}{q}} \leq \frac{C}{\epsilon} \nu_{\lambda(t)} \left(\mathbb{E}^M(f^q) \right)^{\frac{1}{p}} \left(\mathbb{E}^M(Y_s^q(x_s, x_r)I_{t\in\{k-4,k+4\}\cap M_s}) \right) + \frac{K}{\epsilon} \nu f^q
\]

and \(s \in \{k-3, k+3\} : s \sim r\) and \(K\) and \(\epsilon\) as in hypothesis (H2).

\[\Box\]
Lemma 5.5. Suppose $P$ and $G$ are positive functions with domain on $\mathbb{N}$ such that for constants $J,K' > 0$

$$P(4) \leq G(4) + J^q K' P(8) \quad (5.8)$$

and for $n = 4k$ for $k \in \mathbb{N} \cap [2, \infty)$

$$P(n) \leq G(n) + J^q K' P(n - 4) + J^q K' P(n + 4) \quad (5.9)$$

Then for $J$ sufficiently small such that

$$J \leq 1 \quad \text{and} \quad JK' + J^q K' J^{q-1} \leq 1 \quad (5.10)$$

the following inequality holds

$$P(4n) \leq \frac{1}{1 - J^q K' J^{q-1}} \sum_{m=0}^{n-2} J^{mq-m} G(4n - 4m) + J^{(n-1)q-(n-1)} G(4)$$

$$+ J^q P(4n + 4) \quad (5.11)$$

for any $n \in \mathbb{N}, n \geq 2$.

Proof. In order to show (5.11) we will work inductively.

Step 1: The base case of the induction ($n=2$).

We prove (5.11) for $n = 2$. For $k = 8$ in (5.9) we have

$$P(8) \leq G(8) + J^q K' P(12) + J^q K' P(4)$$

If we bound $P(4)$ in the above inequality by (5.8) we obtain

$$P(8) \leq G(8) + J^q K' P(12) + J^q K' G(4) + (J^q K')^2 P(8) \Rightarrow$$

$$P(8) \leq \frac{1}{1 - (J^q K')^2} G(8) + \frac{J^q K'}{1 - (J^q K')^2} G(4) + \frac{J^q K'}{1 - (J^q K')^2} P(12) \quad (5.12)$$

For $J$ satisfying properties (5.10), we have $JK' + J^q K' J^{q-1} \leq 1$ and $JK' < 1$ which implies

$$JK' + (J^q K')^2 \leq 1 \Rightarrow \frac{J^q K'}{1 - (J^q K')^2} \leq J^{q-1} \quad (5.13)$$

From (5.12) and (5.13) we have

$$P(8) \leq \frac{1}{1 - (J^q K')^2} G(8) + J^{q-1} G(4) + J^{q-1} P(12)$$

$$\leq \frac{1}{1 - J^q K' J^{q-1}} G(8) + J^{q-1} G(4) + J^{q-1} P(12)$$

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because of (5.10). This proves (5.11) for \( n = 2 \).

Step 2: The induction step. Suppose the inequality (5.11) is true for \( n = k \). Then we will show it is also true for \( n = k + 1 \).

If we use (5.9) for \( n = 4k + 4 \) we have

\[
P(4k + 4) \leq G(4k + 4) + J^q K' P(4k + 8)
\]

If we use (5.11) for \( n = k \) to bound \( P(4k) \) in (5.14) we get

\[
P(4k + 4) \leq G(4k + 4) + \sum_{m=0}^{k-2} J^{mq-m} G(4k - 4m) + J^q K' J^{(k-1)q - (k-1)} G(4) + J^q K' J^{q-1} P(4k + 4) + J^q K' P(4k + 8)
\]

This implies

\[
P(4k + 4) \leq \frac{1}{1 - J^q K' J^{q-1}} G(4k + 4) + \frac{J^q K'}{1 - J^q K' J^{q-1}} \sum_{m=0}^{k-2} J^{mq-m} G(4k - 4m) + J^q K' J^{(k-1)q - (k-1)} G(4) + \frac{J^q K'}{1 - J^q K' J^{q-1}} P(4k + 8)
\]

If we use condition (5.10) for \( J \), (5.15) becomes

\[
P(4k + 4) \leq \frac{1}{1 - J^q K' J^{q-1}} \sum_{m=0}^{k-1} J^{mq-m} G(4k + 4 - 4m) + J^q K' J^{q-1} P(4k + 8)
\]

which proves (5.11) for \( n = k + 1 \). This finishes the proof of (5.11).

\[\square\]

**Lemma 5.6.** Suppose \( P \) and \( G \) are positive functions with domain on \( \mathbb{N} \) such that for constants \( J, K' > 0 \) one has

\[
\sup_{n \in \mathbb{N}} P(n) < \infty
\]

as well as (5.8) and (5.9) for \( n = 4k \) for \( k \in \mathbb{N} \cap [2, \infty) \). Then for \( J \) sufficiently small such that (5.10) is true, the following inequality holds

\[
P(4) \leq \frac{1}{1 - J^{2q-2}} \sum_{n=0}^{+\infty} J^{mq-n} G(4n + 4)
\]
Proof.
We can use relationship (5.11) from Lemma 5.5 to prove the lemma. We first replace the bound of $P(8)$ from (5.11) in (5.8), to obtain

$$P(4) \leq G(4) + J^q K' \frac{1}{1 - J^q K' J^q - 1} G(8) + J^q K' J^q - 1 P(12)$$

$$\leq (1 + J^q K' J^q - 1) G(4) + J^{2q - 2} G(8) + J K' J^{2q - 2} P(12)$$

where at the last inequality we used (5.10). If we now bound in the above expression $P(12)$ from (5.11), then $P(16)$ from (5.11) and so on, we will finally obtain

$$P(4) \leq (1 + J^q K' \sum_{n=0}^{+\infty} J^{2n+1} q - (2n+1)) G(4)$$

$$+ \frac{J^q K'}{1 - J^q K' J^q - 1} \sum_{n=1}^{+\infty} J^{(n-1)q - (n-1)} \left( \sum_{s=0}^{+\infty} J^{2q - 2s} G(4n + 4) \right)$$

$$= (1 + J^{2q - 1} K' \sum_{n=0}^{+\infty} J^{2n+1} q - (2n+1)) G(4) + \frac{J^q K'}{1 - J^q K' J^q - 1} \frac{1}{1 - J^{2q - 2}} \sum_{n=1}^{+\infty} J^{(n-1)q - (n-1)} G(4n + 4)$$

where above we used that $J < 1$, as well as that

$$\lim_{n \to \infty} J^{nq - n} P(8 + 4n) = 0$$

since (5.16) is true. Furthermore, if we use again (5.10) we then get

$$P(4) \leq \frac{1}{1 - J^{2q - 2}} \sum_{n=0}^{+\infty} J^{nq - n} G(4n + 4)$$

\[ \square \]

The next lemma presents a bound for

$$Q(u, k) = \nu_{\Lambda(u)} \left| \nabla_{\Lambda(u)} \left( E^{M(u)} | h_k |^q \right) \right|^q$$

in terms of $Q(t, k) I_{\text{dist}(u, t) = 4}$.

**Lemma 5.7.** Under hypothesis (H1) and (H2) the following bound for $Q(u, k)$ holds

$$Q(u, k) \leq \nu_{\Lambda(u)} | \nabla_u h_k |^q + \sum_{r=u-1, u+1} \nu_{\Lambda(u)} | \nabla_r h_k |^q + \frac{J^q c_1 2cK}{\epsilon} \nu | h_k |^q$$

$$+ c_1 \sum_{r=u-2, u+2} \nu_{\Lambda(u)} | \nabla_r h_k |^q + \frac{J^q c_1 C}{\epsilon} \sum_{\text{dist}(u, t) = 4} Q(t, k)$$

where $h_k = f - E^{\sim k} f$. 

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Proof. We have
\[
Q(u, k) = \nu_{\Lambda(u)} \left| \nabla_{\Lambda(u)} \left( \mathbb{E}^M(u) | h_k |^q \right)^{\frac{1}{q}} \right| q = \nu_{\Lambda(u)} \left| \nabla_u \left( \mathbb{E}^M(u) | h_k |^q \right)^{\frac{1}{q}} \right| q 
\]
\[
+ \sum_{r=u-1, u+1} \nu_{\Lambda(u)} \left| \nabla_r \left( \mathbb{E}^M(u) | h_k |^q \right)^{\frac{1}{q}} \right| q + \sum_{r=u-2, u+2} \nu_{\Lambda(u)} \left| \nabla_r \left( \mathbb{E}^M(u) | h_k |^q \right)^{\frac{1}{q}} \right| q
\]
(5.17)

For \( r = u - 1, u, u + 1 \)
\[
\nu_{\Lambda(u)} \left| \nabla_r \left( \mathbb{E}^M(u) | h_k |^q \right)^{\frac{1}{q}} \right| q \leq \nu_{\Lambda(u)} \left| \nabla_r h_k \right|^q
\]
(5.18)

For \( r = u - 2, u + 2 \)
\[
\nu_{\Lambda(u)} \left| \nabla_r \left( \mathbb{E}^M(u) | h_k |^q \right)^{\frac{1}{q}} \right| q \leq c_1 \nu_{\Lambda(u)} \left| \nabla_r h_k \right|^q + \frac{J q c_1}{q^q} \nu_{\Lambda(u)} \left( \mathbb{E}^M(u) | h_k |^q \right)^{-\frac{2}{q}} \left( \mathbb{E}^M(u) \left( | h_k |^q ; \nabla_r V(x_r, x_s) \right) q \right) I_{s \in \{u-3, u+3\} : s \sim r}
\]
(5.19)

For \( s \in \{u - 3, u + 3\} : s \sim r \), if we use Lemma 5.4 we obtain
\[
\nu_{\Lambda(u)} \left( \mathbb{E}^M(u) | h_k |^q \right)^{-\frac{2}{q}} \left( \mathbb{E}^M(u) \left( | h_k |^q ; \nabla_r V(x_r, x_s) \right) q \right) I_{s \in \{u-3, u+3\} : s \sim r} \leq \frac{C}{\epsilon} \nu_{\Lambda(t)} \left| \nabla_{\Lambda(t)} \left( \mathbb{E}^M(t) | h_k |^q \right)^{\frac{1}{q}} \right| q I_{s(t) = (u+1, u+4) \cup (u-1, u-4) : s \sim r} + \frac{K}{\epsilon} \nu | h_k |^q
\]
(5.20)

From (5.19) and (5.20) we get
\[
\nu_{\Lambda(u)} \left| \nabla_r \left( \mathbb{E}^M(u) | h_k |^q \right)^{\frac{1}{q}} \right| q \leq c_1 \nu_{\Lambda(u)} \left| \nabla_r h_k \right|^q + \frac{J q c_1}{\epsilon} Q(t, k) I_{s(t) = (u+1, u+4) \cup (u-1, u-4) : s \sim r} I_{s \in \{u-1, u+1\} : s \sim r} + \frac{J q c_1 K}{\epsilon} \nu | h_k |^q I_{s \in \{u-1, u+1\} : s \sim r}
\]
(5.21)

To summarise, if we plug (5.18) and (5.21) in (5.17) we finally obtain
\[
Q(u, k) \leq \frac{J q c_1 C}{\epsilon} \sum_{\text{dist}(u, t) = 4} Q(t, k) + \frac{J q c_1 2 K}{\epsilon} \nu \left| f - \mathbb{E}^{\langle k \rangle} f \right|^q + \nu | \nabla_u h_k |^q \\
+ \sum_{r=u-1, u+1} \nu_{\Lambda(u)} | \nabla_r h_k |^q + c_1 \sum_{r=u-2, u+2} \nu_{\Lambda(u)} | \nabla_r h_k |^q
\]
\[
\square
\]

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Lemma 5.8. Suppose conditions (H1) is satisfied. Then for $r \in \Lambda(k)$, the following statements are true

(a) When $r = \{k - 2, k, k + 2\}$

$$\nu |\nabla_r h_k|^q \leq c_1 \nu |\nabla_r f|^q + \frac{J^q C c_1}{\epsilon} Q(k, k) + \frac{J^q c_1 K}{\epsilon} \nu |f - E^{(\sim k)} f|^q$$

(b) When $r \in \{k - 1, k, k + 1\}$

$$\nu |\nabla_r h_k|^q = \nu |\nabla_r f|^q$$

where $h_k = f - E^{(\sim k)} f$.

Proof. We will show (a). For general $r \in \Lambda(k) \setminus \{k - 1, k + 1\}$ we have

$$\nu |\nabla_r h_k|^q \leq 2^{q-1} \nu |\nabla_r f|^q + 2^{q-1} \nu |\nabla_r E^{(\sim k)} f|^q$$

(5.22)

We will now compute $\nu |\nabla_r E^{(\sim k)} f|^q$ for the separate cases of $r \in \{k - 2, k + 2\}$ and $r = k$.

Consider $r = \{k - 2, k + 2\}$. In this case

$$\nu |\nabla_r E^{(\sim k)} f|^q \leq 2^{q-1} \nu |\nabla_r f|^q$$

$$+ \frac{J^q 2^{q-1} \nu}{\epsilon} \left| E^{(\sim k)} (f; \nabla_r V(x_s, x_r)) \right|^q I_{s \in \{k-1, k+1\}; s \sim r}$$

(5.23)

If we use Lemma 3.1 (a) to bound the second term on the right hand side of (5.23) we obtain

$$\nu |\nabla_r E^{(\sim k)} f|^q \leq 2^{q-1} \nu |\nabla_r f|^q + \frac{J^q 2^{q-1} C}{\epsilon} Q(k, k)$$

$$+ \frac{J^q 2^{q-1} K}{\epsilon} \nu |f - E^{(\sim k)} f|^q$$

(5.24)

Combining (5.22) and (5.24) together we derive

$$\nu |\nabla_r h_k|^q \leq c_1 \nu |\nabla_r f|^q + \frac{J^q C c_1}{\epsilon} Q(k, k) + \frac{J^q c_1 K}{\epsilon} \nu |f - E^{(\sim k)} f|^q$$

for $K$ as in (H2).

Consider $r = k$. In this case

$$\nu |\nabla_k E^{(\sim k)} f|^q \leq 2^{q-1} \nu |\nabla_k f|^q + \frac{J^q 2^{q-1} \nu}{\epsilon} \left( E^{(\sim k)} (f; W_k) \right)^q$$

$$\leq 2^{q-1} \nu |\nabla_k f|^q + \frac{J^q C 2^{q-1}}{\epsilon} Q(k, k) + \frac{J^q 2^{q-1} K}{\epsilon} \nu |f - E^{(\sim k)} f|^q$$

(5.25)
where in the last inequality the Lemma 3.1 (a) was used for $K$ as in (H2).

From (5.22) and (5.25)

\[ \nu |\nabla_k h_k|^q \leq c_1 \nu |\nabla_k f|^q + \frac{J^q C c_1}{\epsilon} Q(k,k) + \frac{J^q c_1 K}{\epsilon} |f - \mathbb{E}^{(\sim k)} f|^q \]

We can now prove Lemma 5.1.

**Proof of Lemma 5.1.** If we combine the bound for $Q(k,k)$ from Lemma 5.7, together with the bounds for $\nu |\nabla_r h_k|^q$, $r = k - 2, k - 1, k, k + 1, k + 2$ from Lemma 5.8 we obtain

\[
Q(k,k) \leq \sum_{r=k-1,k+1} \nu |\nabla_r f|^q + c_1 \nu |\nabla_k f|^q + \frac{J^q C c_1}{\epsilon} Q(k,k) + \frac{J^q c_1 K}{\epsilon} |f - \mathbb{E}^{(\sim k)} f|^q \\
+ c_1 \sum_{r=k-2,k+2} \left( c_1 \nu |\nabla_r f|^q + \frac{J^q C c_1}{\epsilon} Q(k,k) + \frac{J^q c_1 K}{\epsilon} |f - \mathbb{E}^{(\sim k)} f|^q \right) \\
+ \frac{J^q c_1 C}{\epsilon} \sum_{\text{dist}(k,t)=4} Q(t,k) + \frac{J^q c_1 K}{\epsilon} |f - \mathbb{E}^{(\sim k)} f|^q \\
= \sum_{r=k-1,k+1} \nu |\nabla_r f|^q + c_1 \nu |\nabla_k f|^q + \frac{J^q (c_1^3 + c_1^2)}{\epsilon} |f - \mathbb{E}^{(\sim k)} f|^q \\
+ c_1^2 \sum_{r=k-2,k+2} \nu |\nabla_r f|^q + \frac{J^q 2C (c_1^2 + c_1^3)}{\epsilon} Q(k,k) + \frac{J^q c_1 C}{\epsilon} \sum_{\text{dist}(k,t)=4} Q(t,k) \\
\tag{5.26}
\]

In order to bound $\sum_{\text{dist}(k,t)=4} Q(t,k)$ in the above quantity the lemma bellow will be used.

**Lemma 5.9.** Under conditions (H1)-(H3) the following inequality

\[
\sum_{t:\text{dist}(t,k)=4} Q(t,k) \leq J^q T Q(k,k) + J^q T \nu |f - \mathbb{E}^{(\sim k)} f|^q + T \sum_{r=k-2,k+2} \nu |\nabla_r f|^q \\
+ T \sum_{n=0}^{\infty} J^{q(n-1)} \sum_{r=0}^{3} \left( \nu |\nabla_{k+3+4n+r} f|^q + \nu |\nabla_{k-3-4n-r} f|^q \right)
\]

is satisfied for some positive constant $T$ independent of $k$. 

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The proof of Lemma 5.9 will be presented later in the section. If we use the bound of Lemma 5.9 in (5.26), we obtain

\[
Q(k, k) \leq J^q \left( \frac{T J^q c_1 C}{\epsilon} + \frac{c_1 3 c K}{\epsilon} + \frac{c_1^2 2 c K}{\epsilon} \right) \nu \left| f - \mathbb{E}^{(\sim k)} f \right|^q \\
+ J^q \left( \frac{2 C c_1}{\epsilon} + \frac{C c_1}{\epsilon} \right) J^q(k, k) \\
+ \sum_{r=k-1, k+1} \nu \left| \nabla_r f \right|^q + c_1 \nu \left| \nabla_k f \right|^q + \left( \frac{J^q c_1 C}{\epsilon} T + c_1^2 \right) \sum_{r=k-2, k+2} \nu \left| \nabla_r f \right|^q \\
+ \frac{J^q c_1 C}{\epsilon} T \sum_{n=0}^{\infty} J^n(q-1) \sum_{r=0}^{3} \left( \nu \left| \nabla_{k+3+4n+r} f \right|^q + \nu \left| \nabla_{k-3-4n-r} f \right|^q \right) \tag{5.27}
\]

If we choose \( J \) sufficiently small such that

\[
1 - J^q \left( \frac{2 C c_1}{\epsilon} + \frac{J^q c_1 C T}{\epsilon} + \frac{C c_1}{\epsilon} \right) > \frac{1}{2}
\]

then from (5.27) we have

\[
Q(k, k) \leq 2 J^q \left( \frac{T J^q c_1 C}{\epsilon} + \frac{c_1 3 c K}{\epsilon} + \frac{c_1^2 2 c K}{\epsilon} \right) \nu \left| f - \mathbb{E}^{(\sim k)} f \right|^q + 2 c_1 \nu \left| \nabla_k f \right|^q \\
+ 2 \sum_{r=k-1, k+1} \nu \left| \nabla_r f \right|^q + 2 \left( \frac{J^q c_1 C}{\epsilon} T + c_1^2 \right) \sum_{r=k-2, k+2} \nu \left| \nabla_r f \right|^q \\
+ \frac{2 J^q c_1 C}{\epsilon} T \sum_{n=0}^{\infty} J^n(q-1) \sum_{r=0}^{3} \left( \nu \left| \nabla_{k+3+4n+r} f \right|^q + \nu \left| \nabla_{k-3-4n-r} f \right|^q \right)
\]

and the lemma follows for an appropriate positive constant \( D \).

It remains to show Lemma 5.9. For this we will need the following lemmata.

**Lemma 5.10.** Under conditions (H1)-(H3) the following two bounds for \( Q(u, k) \) hold.

(a) For \( u \) such that \( \text{dist}(u, k) \geq 8 \)

\[
Q(u, k) \leq c_1 \nu_{\Lambda(u)} \left| \nabla_u f \right|^q + c_1 \sum_{r=u-1, u+1} \nu_{\Lambda(u)} \left| \nabla_r f \right|^q + c_1^2 \sum_{r=u-2, u+2} \nu_{\Lambda(u)} \left| \nabla_r f \right|^q \\
+ \frac{J^q c_1 C}{\epsilon} \sum_{\text{dist}(u, t) = 4} Q(t, k) + \frac{J^q c_1 2 K}{\epsilon} \nu \left| f - \mathbb{E}^{(\sim k)} f \right|^q
\]
(b) For $u$ such that $\text{dist}(u, k) = 4$

\[
Q(u, k) \leq c_1 \nu |\nabla u f|^q + c_1^2 \sum_{r = u-2, u+2} \nu |\nabla f|^q + J^q \left( \frac{c_1 2K}{\epsilon} + \frac{c_1^2 K}{\epsilon} \right) \nu |f - E^{\{\sim k\}} f|^q 
+ c_1 \sum_{r = u-1, u+1} \nu |\nabla f|^q + \frac{J^q c_1^2}{\epsilon} Q(k, k) + \frac{J^q c_1 C}{\epsilon} \sum_{\text{dist}(u, t) = 4, t \neq k} Q(t, k)
\]

Proof. The lemma follows from the bound of $Q(u, k)$ in Lemma 5.7. In the case where $\text{dist}(u, k) \geq 8$, for $r = u - 2, u - 1, u, u + 1, u + 2$ we have that

\[
\nu |\nabla r h_k|^q \leq 2^{q-1} 2^q \nu |\nabla r f|^q \tag{5.28}
\]

Substituting (5.28) in the expression from Lemma 5.7 we immediately obtain (a).

Consider the case where $\text{dist}(u, k) = 4$. Then for $r = u - 1, u, u + 1$

\[
\nu |\nabla r h_k|^q \leq 2^{q-1} 2^q \nu |\nabla r f|^q \tag{5.29}
\]

While for $r = \{u - 2, u + 2\}$ we can bound $\nu |\nabla r h_k|^q$ from Lemma 5.8 (a). If we plug the bounds from (5.29) and Lemma 5.8 (a) into the expression from Lemma 5.7, we obtain

\[
Q(u, k) \leq J^q \left( \frac{c_1 2K}{\epsilon} + \frac{c_1^2 K}{\epsilon} \right) \nu |f - E^{\{\sim k\}} f|^q + \frac{J^q c_1^2}{\epsilon} Q(k, k) + c_1 \nu |\nabla u f|^q 
+ c_1 \sum_{r = u-1, u+1} \nu |\nabla f|^q + \frac{J^q c_1 C}{\epsilon} \sum_{\text{dist}(u, t) = 4} Q(t, k)
\]

Before proving Lemma 5.9, we will also need to show that for any $k \in \mathbb{N}$

\[
\sup_{n \in \mathbb{N}} \sum_{\text{dist}(u, k) = n} Q(u, k) < C_f < \infty
\]

for $C_f$ a constant which depends on the function $f$ but not on $n, u$ and $k$. To show this we first need the following lemma.

**Lemma 5.11.** For any $r, k \in \mathbb{Z}$ we have

\[
\nu |\nabla r h_k|^q \leq \tilde{C}_f < \infty
\]

where $\tilde{C}_f$ depends on the function $f$ but not on $r$ and $k$. 

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Proof. For general \( r \in \{ k - 2, k, k + 2 \} \)
\[
\nu |\nabla r h_k|^q \leq 2^{q-1} \nu |\nabla r f|^q + 2^{q-1} \nu |\nabla r \mathbb{E}^{(\sim k)} f|^q
\]  
(5.30)
since \( h_k = f - \mathbb{E}^{(\sim k)} f \). For the second term on the right hand side of (5.30) we have
\[
\nu |\nabla r \mathbb{E}^{(\sim k)} f|^q \leq 2^{q-1} \nu |\nabla r f|^q + J^q 2^{q-1} \nu |\mathbb{E}^{(\sim k)}(f; Z_k)|^q
\]  
(5.31)
where
\[
Z_k = \nabla_{k-2} V(x_{k-2}, x_{k-1}) I_{r=k-1} + \nabla_{k+2} V(x_{k+2}, x_{k+1}) I_{r=k+1} + W_k I_{r=k}
\]
where \( W_k \) as in (4.1). We will now compute the last term on the right hand side of (5.31)
\[
\nu |\mathbb{E}^{(\sim k)}(f; Z_k)|^q \leq \nu |\mathbb{E}^{(\sim k)}(f - \mathbb{E}^{(\sim k)} f)(Z_k - \mathbb{E}^{(\sim k)} Z_k)|^q
\]
\[
\leq \nu f^q |Z_k - \mathbb{E}^{(\sim k)} Z_k|^q
\]
If we use the entropic inequality (3.3) we obtain
\[
\nu |\mathbb{E}^{(\sim k)}(f; Z_k)|^q \leq \frac{1}{\epsilon} \nu f^q \log \frac{f^q}{\nu f^q} + \frac{1}{\epsilon} \nu f^q \log \nu e^{\epsilon |Z_k - \mathbb{E}^{(\sim k)} Z_k|^q}
\]
\[
\leq \frac{1}{\epsilon} \nu f^q \log \frac{f^q}{\nu f^q} + \frac{K}{\epsilon} \nu f^q
\]  
(5.32)
where \( K \) as in (H2). If we combine (5.30), (5.31) and (5.32) we get that for \( r \in \{ k - 2, k, k + 2 \} \)
\[
\nu |\nabla r h_k|^q \leq 2^{q} \nu |\nabla r f|^q + \frac{J^q 2^q - 2}{\epsilon} \nu f^q \log \frac{f^q}{\nu f^q} + \frac{J^q 2^q - 2}{\epsilon} \nu f^q
\]  
(5.33)
For \( r \notin \{ k - 2, k, k + 2 \} \) we have
\[
\nu |\nabla r h_k|^q \leq 2^q \nu f^q
\]  
(5.34)
From (5.33) and (5.34) the lemma follows since functions \( f \) are as in Remark 2.2. \( \Box \)

Lemma 5.12. If (H2) is satisfied, then for any \( k \in \mathbb{N} \)
\[
\sup_{n \in \mathbb{N}} \sum_{\text{dist}(u, k) = n} Q(u, k) < C_f < \infty
\]
where \( C_f \) is a constant which depends on the function \( f \) but not on \( u \) and \( k \).
Proof. Since we work on the one dimensional lattice, it is sufficient to show that

\[ \sup_{n \in \mathbb{N}} Q(u, k) < C_f' < \infty \]

for \( C_f' \) depends only on the functions \( f \). To compute \( Q(u, k) \) we can use \((5.17)\) and \((5.18)\) to obtain

\[ Q(u, k) \leq \sum_{r = u-1, u+1} \nu |\nabla_r h_k|^q + \sum_{r = u-2, u+2} \nu_{\Lambda(u)} |\nabla_r (E^{M(u)}|h_k|^q)^{\frac{1}{q}}|^q \]  

(5.35)

Furthermore, from \((5.19)\) for \( r = u - 2, u + 2 \) we have

\[ \nu_{\Lambda(u)} |\nabla_r (E^{M(u)}|h_k|^q)^{\frac{1}{q}}|^q \leq c_1 \nu |\nabla_r h_k|^q + \frac{J^q c_1}{q^q} I_0 \]  

(5.36)

where

\[ I_0 := \nu_{\Lambda(u)} \left( E^{M(u)}|h_k|^q \right)^{-\frac{q}{p}} \left( E^{M(u)}|h_k|^q; \nabla_r V(x_r, x_s) \right)^q I_{s \in \{u-3, u+3\}; s \sim r} \]

In order to bound the second term on the right hand side of \((5.36)\) we compute

\[ E^{M(u)}|h_k|^q; \nabla_r V(x_r, x_s) = E^{M(u)}|h_k|^{p(q-1)+1} (V(x_r, x_s) - E^{M(u)} \nabla_r V(x_r, x_s)) \]

\[ \leq \left( E^{M(u)}|h_k|^{p(q-1)+1} \right)^{\frac{q}{p}} \left( E^{M(u)}|\nabla_r V(x_r, x_s) - E^{M(u)} \nabla_r V(x_r, x_s)|^q \right)^{\frac{1}{q}} \]

From the last bound, since \( p \) and \( q \) are conjugate, we get

\[ I_0 \leq \nu_{\Lambda(u)} E^{M(u)} \left( |h_k|^q; \nabla_r V(x_r, x_s) - E^{M(u)} \nabla_r V(x_r, x_s)|^q \right) I_{s \in \{u-3, u+3\}; s \sim r} \]

\[ = \nu(\nu h_k^q N_r \leq 2^{q-1} \nu(f^q N_r) + 2^{q-1} \nu((E^{(\sim k)} f^q) N_r) \]

where above we denoted \( N_r = \left| \nabla_r V(x_r, x_s) - E^{M(u)} \nabla_r V(x_r, x_s) \right|^2 I_{s \in \{u-3, u+3\}; s \sim r} \).

If we again use the entropic inequality \((5.3)\) we obtain

\[ I_0 \leq \frac{2^{q-1} \nu f^q \log f^q}{\nu f^q} + \frac{2^{q-1} \nu f^q \log \nu \epsilon N_r + 2^{q-1} \nu \epsilon E^{(\sim k)} f^q \log \frac{E^{(\sim k)} f^q}{\nu E^{(\sim k)} f^q}}{\nu \epsilon E^{(\sim k)} f^q} \]

\[ + \frac{2^{q-1} \nu f^q \log \nu \epsilon N_r}{\nu f^q} \]

\[ \leq \frac{2^{q-1} \nu f^q \log f^q}{\nu f^q} + \frac{2^{q-1} \nu f^q \log \nu \epsilon E^{(\sim k)} f^q \log \frac{E^{(\sim k)} f^q}{\nu E^{(\sim k)} f^q}}{\nu \epsilon E^{(\sim k)} f^q} \]

(5.37)

where \( K \) as in (H2). For the last term on the right hand side of \((5.37)\) we can write

\[ \nu \epsilon E^{(\sim k)} f^q \log \frac{E^{(\sim k)} f^q}{\nu \epsilon E^{(\sim k)} f^q} = \nu f^q \log \frac{E^{(\sim k)} f^q}{\nu f^q} \leq \nu f^q \log f^q \]

(5.38)
Combining together (5.37) and (5.38) we obtain

\[
I_0 \leq 2 \frac{q}{\epsilon} \nu f^q \log \frac{f}{\nu f^q} + 2 \frac{q}{\epsilon} K \nu f^q \tag{5.39}
\]

From (5.36), and (5.39) we then get that for \( r = u - 2, u + 2 \)

\[
\nu \Lambda(u) \left| \nabla_r (E^{M(u)} \lambda h_k) \right|^q \leq c_1 \nu \left| \nabla_r f \right|^q + \frac{J^q c_1 2K}{\epsilon} \nu f^q \log \frac{f}{\nu f^q} + \frac{J^q c_1 2K}{\epsilon} \nu f^q \tag{5.40}
\]

If we combine (5.35) and (5.40) together with Lemma 5.11 we conclude that for any function \( f \) there is a bound of \( \nu \Lambda(u) \left| \nabla_r (E^{M(u)} \lambda h_k) \right|^q \) uniformly with respect to the set \( M(u) \) depending only on \( \nu f^q, \max_{i \in \mathbb{Z}} \nu \left| \nabla_i f \right|^q \) and \( \nu f^q \log \frac{f}{\nu f^q} \).

We can now prove Lemma 5.9.

**Proof of Lemma 5.9.** For every \( u \) s.t. \( \text{dist}(u, k) \geq 8 \) define

\[
G(u, k) := c_1 \nu \Lambda(u) \left| \nabla_u f \right|^q + c_1 \sum_{r = u - 1, u + 1} \nu \left| \nabla_r f \right|^q + c_2 \sum_{r = u - 2, u + 2} \nu \Lambda(u) \left| \nabla_r f \right|^q + \frac{J^q c_1 2K}{\epsilon} \nu f^q \left| f - E^{(\sim k)} f \right|^q
\]

and for every \( u \) s.t. \( \text{dist}(u, k) = 4 \) define

\[
G(u, k) := c_1 \nu \left| \nabla_u f \right|^q + c_1 \sum_{r = u - 1, u + 1} \nu \left| \nabla_r f \right|^q + \frac{J^q c_1 2K}{\epsilon} Q(k, k) + c_1 \sum_{i = u - 2, u + 2} \nu \left| \nabla_i f \right|^q + J^q \left( \frac{c_1 2K}{\epsilon} + \frac{c_2 K}{\epsilon} \right) \nu f^q \left| f - E^{(\sim k)} f \right|^q
\]

If we set \( K' = \frac{c_2 C}{\epsilon} \), then from Lemma 5.10 (a) and (b) respectively we can write

\[
Q(u, k) \leq G(u, k) + J^q K' \sum_{\text{dist}(u, t) = 4} Q(t, k), \quad \text{for } \text{dist}(u, k) \geq 8 \tag{5.41}
\]

and

\[
Q(u, k) \leq G(u, k) + J^q K' Q(t, k) I_{\text{dist}(t, u) = 4, t \neq k}, \quad \text{for } \text{dist}(u, k) = 4 \tag{5.42}
\]
From equation (5.41) we obtain
\[ \sum_{\text{dist}(u,k)=n} Q(u,k) \leq \sum_{\text{dist}(u,k)=n} G(u,k) + J^q K' \sum_{\text{dist}(t,k)=n} \sum_{\text{dist}(t,u)=4} Q(t,k) \]
or equivalently
\[ \sum_{\text{dist}(u,k)=n} Q(u,k) \leq \sum_{\text{dist}(u,k)=n} G(u,k) + J^q K' \sum_{\text{dist}(t,k)=n+4} Q(t,k) + J^q K' \sum_{\text{dist}(t,k)=n-4} Q(t,k) \]
which implies
\[ \tilde{Q}(n) \leq \tilde{G}(n) + J^q K' \tilde{Q}(n - 4) + J^q K' \tilde{Q}(n + 4) \quad (5.43) \]
where we denote
\[ \tilde{Q}(n) = \sum_{\text{dist}(u,k)=n} Q(u,k) \quad \text{and} \quad \tilde{G}(n) = \sum_{\text{dist}(u,k)=n} G(u,k) \]

While from equation (5.42), we have
\[ \sum_{\text{dist}(u,k)=4} Q(u,k) \leq \sum_{\text{dist}(u,k)=4} G(u,k) + J^q K' \sum_{\text{dist}(t,k)=4} Q(t,k) \sum_{\text{dist}(t,u)=4,t \neq k} \]
This implies
\[ \sum_{\text{dist}(u,k)=4} Q(u,k) \leq \sum_{\text{dist}(u,k)=4} G(u,k) + J^q K' \sum_{\text{dist}(t,k)=8} Q(t,k) \]
which is equivalent to
\[ \tilde{Q}(4) \leq \tilde{G}(4) + J^q K' \tilde{Q}(8) \quad (5.44) \]
Choose \( J \) in (H3) sufficiently small such that hypothesis (5.10) of Lemma 5.6 is satisfied. Then, since relationships (5.43), (5.44) and Lemma 5.12 are true, the conditions of Lemma 5.6 are satisfied for \( P = \tilde{Q} \) and \( G = \tilde{G} \) and so we obtain
\[ \tilde{Q}(4) \leq \hat{J} \sum_{n=0}^{+\infty} J^{nq-n} \tilde{G}(4n + 4) \]
where \( \hat{J} = \frac{1}{1-J^{2q-2}} \). This is equivalent to
\[ \sum_{t: \text{dist}(t,k)=4} Q(t,k) \leq \hat{J} \sum_{\text{dist}(u,k)=4} G(u,k) + \hat{J} \sum_{n=1}^{+\infty} J^{nq-n} \sum_{\text{dist}(u,k)=4n+4} G(u,k) \quad (5.45) \]
Substituting $G(u, k)$ leads to

$$
\sum_{t: \text{dist}(t, k) = 4} Q(t, k) \leq \frac{J q C_1^2}{\epsilon} \sum_{\text{dist}(u, k) = 4} Q(k, k)
+ \hat{J} c_1 \sum_{n=0}^{+\infty} J_{q}^{-n} \sum_{\text{dist}(u, k) = 4n+4} \nu_{\Lambda(u)} |\nabla f|^q
+ \hat{J} c_1 \sum_{n=0}^{+\infty} J_{q}^{-n} \sum_{\text{dist}(u, k) = 4n+4} \sum_{r=u-1, u+1} \nu_{\Lambda(u)} |\nabla r f|^q
+ \hat{J} c_1^2 \sum_{n=0}^{+\infty} J_{q}^{-n} \sum_{\text{dist}(u, k) = 4n+4} \sum_{r=u-2, u+2} \nu_{\Lambda(u)} |\nabla r f|^q
+ J q \frac{c_1 K}{\epsilon} (c_1 + 2) \sum_{n=0}^{+\infty} J_{q}^{-n} \sum_{\text{dist}(u, k) = 4n+4} \nu |f - \mathbb{E}^{\sim k} f|^q
$$

(5.46)

But for $J$ in (H3) we have $J^{q-1} < 1$ which implies $\hat{J} = \sum_{n=0}^{+\infty} J_{q}^{-n} < \infty$. (5.46)

then implies

$$
\sum_{t: \text{dist}(t, k) = 4} Q(t, k) \leq \frac{J q C_1^2}{\epsilon} 2 \hat{J} Q(k, k)
+ \hat{J} c_1 \hat{J} \sum_{\text{dist}(u, k) = 4n+4} \sum_{r=u-1, u+1} \nu |\nabla r f|^q
+ \hat{J} c_1^2 \hat{J} \sum_{\text{dist}(u, k) = 4n+4} \sum_{r=u-2, u+2} \nu |\nabla r f|^q
+ J q \frac{c_1 K}{\epsilon} (c_1 + 2) \frac{2}{1 - J^{q-1}} \nu |f - \mathbb{E}^{\sim k} f|^q
$$

and the lemma follows for appropriate constant $T \geq 0$. \hfill \square

### 6 Conclusion

In the present work, we have determined conditions for the infinite volume Gibbs measure to satisfy the Log-Sobolev Inequality. As explained in the introduction, the criterion presented in Theorem 2.3 can in particular be applied in the case of local specifications $\{E^{\Lambda, \omega}\}_{\Lambda \subset \subset Z, \omega \in \Omega}$ with no quadratic interactions for which

$$
\|\nabla_i \nabla_j V(x_i, x_j)\|_\infty = \infty
$$

Thus, we have shown that our results can go beyond the usual uniform boundness of the second derivative of the interactions considered in [Z1], [Z2], [M] and [O-R].
Concerning the additional conditions (H1) and (H2) placed here to handle the exotic interactions, they refer to finite dimensional measures with no boundary conditions which are easier to handle than the \( \{E^{\Lambda,\omega}\}_{\Lambda \subset \mathbb{Z}, \omega \in \Omega} \) measures or the infinite dimensional Gibbs measure \( \nu \).

In fact, the following results concerning the conditions can be proven. This is a work in progress that will consist the material of a forthcoming paper.

**Proposition 6.1.** The hypothesis (H0), (H3) and (H2) imply hypothesis (H1).

Consequently, the main result of Theorem 2.3 is then reduced to the following

**Theorem 6.2.** If hypothesis (H0), (H3) and (H2) are satisfied, then the infinite dimensional Gibbs measure \( \nu \) for the local specification \( \{E^{\Lambda,\omega}\}_{\Lambda \subset \mathbb{Z}, \omega \in \Omega} \) satisfies the \( q \) Log-Sobolev inequality

\[
\nu |f|^q \log \frac{|f|^q}{\nu |f|^q} \leq \mathcal{C} \nu |\nabla f|^q
\]

for some positive constant \( \mathcal{C} \) independent of \( f \).

Concerning examples of measures that satisfy the above conditions, one can consider measures with phase \( \phi(x) = |x|^t \) with \( t \geq \frac{q}{q-1} \) and interaction \( V(x,y) = |x-y|^r \), with \( \max\{r, (r-1)q\} < t \). The main idea of the proof of the Proposition 6.1 follows in main lines the method followed in the current paper. Although some of the details are more involved because of the lack of hypothesis (H1), the fact that in Proposition 6.1 the Gibbs measure is localised and thus the approximation procedure starts from a finite set compensates for the loss of the \( \text{LS}_q \) for \( \nu^{\Lambda(i)} \).

In this paper we have been concerned with the \( q \) Logarithmic Sobolev inequality for measures on the 1 dimensional Lattice \( \mathbb{Z} \). It is interesting to try to extend the current result to a higher dimensional lattice on \( \mathbb{Z}^d, d \geq 2 \), although this does not appear to be immediate. In a different direction, we can consider the following class of modified Logarithmic Sobolev inequalities presented in [G-G-M]:

\[
\nu |f|^2 \log \frac{|f|^2}{\nu |f|^2} \leq \mathcal{C} \int H_{a,c} \left( \frac{\nabla f}{f} \right) f^2 d\nu
\]

(6.1)

for some positive constant \( \mathcal{C} \), where

\[
H_{a,c}(x) = \begin{cases} 
\frac{a^2}{2} & \text{if } |x| \leq a \\
\frac{a^2}{\beta} \frac{|x|^{\beta}}{\beta} + \frac{\alpha^2 - \beta}{2x^{2\beta}} & \text{if } |x| \geq a \text{ and } c \neq 1 \\
\infty & \text{if } |x| \geq a \text{ and } c = 1 
\end{cases}
\]

for \( c \in [1,2], a > 0 \) and \( \beta \) satisfying \( \frac{1}{c} + \frac{1}{\beta} = 1 \) (\( \beta \geq 2 \)). This new class of inequalities is an interpolation between Log-Sobolev (LS2) and Spectral Gap inequalities.
(SG2), which retains the basic properties of the Log-Sobolev inequalities mentioned in Remark [1.1]. Some preliminary results suggest that on $\mathbb{Z}^d$, $d \geq 2$, the infinite dimensional Gibbs measure satisfies a $[G-G-M]$ type inequality with $\beta = 2q$, under hypothesis (H0) for LSq ($1 < q < 2$) and some hypothesis stronger than (H2). This is work in early stages, but hopefully a modified LS inequality comparable to the $[G-G-M]$ inequalities can be obtained in the case of the higher dimensional lattice.

In addition, it is interesting to investigate whether the result presented in this paper can be extended to the family of weaker inequalities presented in [G-G-M], assuming (H0) and (H1) for the $[G-G-M]$ inequality instead of the LSq. However, this does not seem to be immediate especially in showing the sweeping out relationships and so more work needs to be done towards this direction.

Furthermore, concerning the hypothesis on the single-site measure, the main hypothesis (H0) for $\mathbb{E}^{i\omega}$ can be reduced to the same assumption for the boundary free single-site measure, that is

\[(H0'):\text{ The single-site measure } \frac{e^{-\phi(x)dx}}{\int e^{-\phi(x)dx}} \text{ satisfies the LSq Inequality.}\]

Measures as in $(H0')$ do not involve boundary conditions and for this reason it is easier to show that they satisfy the Log-Sobolev inequality. For instance, when in $\mathbb{R}$ one can think of phases that are convex and increase sufficiently fast, like $\phi(x) = |x|^p$ for $p > 2$ (see [B-Z]). In the case of the Heisenberg group $\mathbb{H}$ one can consider $\phi(x) = \beta d(x)^p$ with $p$ conjugate of $q$ (see [H-Z]).

However, that does not mean that condition $(H0')$ is in general weaker than condition (H0) as there are examples of single-site boundary free measures $\frac{e^{-\phi(x)dx}}{\int e^{-\phi(x)dx}}$ that do not satisfy the LSq inequality, which when perturbed with interactions, give new measures $\mathbb{E}^{i\omega}$ that satisfy the Log-Sobolev-q inequality uniformly on the boundary conditions, that is condition (H0) is satisfied. In addition, in the case of hypothesis $(H0')$, it seems that the analogues of Proposition 6.1 and Theorem 6.2 will be more difficult to be shown.

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