Construction of continuous-state branching processes in varying environments

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Abstract: A continuous-state branching process in varying environments is constructed by the pathwise unique solution to a stochastic integral equation driven by time-space noises. The process arises naturally in the limit theorem of Galton–Watson processes in varying environments established by Bansaye and Simatos (2015). In terms of the stochastic equation we clarify the behavior of the continuous-state process at its bottlenecks, which are the times when it arrives at zero almost surely by negative jumps.

Key words and phrases: Branching process; continuous-state; varying environments; cumulant semigroup; stochastic integral equations; Gaussian white noise; Poisson random measure.

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1 Introduction

Continuous-state branching processes (CB-processes) are often used to model the stochastic evolution of large populations with small individuals. The branching property means intuitively that different individuals in the population propagate independently of each other. The study of such processes was initiated by Feller (1951), who noticed that a diffusion process may arise in a limit theorem of rescaled Galton–Watson branching processes (GW-processes). The basic structures of general CB-processes were discussed in Jiřina (1958). It was proved in Lamperti (1967a) that the class of CB-processes with homogeneous transition semigroups coincides with that of scaling limits of classical GW-processes; see also Aliev and Shchurenkov (1982) and Grimvall (1974). The connection between CB-processes and time changed Lévy processes was established by Lamperti (1967b). A general existence theorem for homogeneous CB-processes was proved in Silverstein (1968); see also Watanabe (1969) and Ryzhov and Skorokhod (1970). The

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approach of stochastic equations for CB-processes without or with immigration has been
developed by Bertoin and Le Gall (2006), Dawson and Li (2006, 2012), Fittipaldi and
Fontbona (2012), Fu and Li (2010), Li (2011, 2019+), Pardoux (2016) and many others.

There have also been some attempts at the understanding of inhomogeneous CB-
processes. Let $X = \{X(t) : t \in I\}$ be a Markov process with state space $[0, \infty]$ and
inhomogeneous transition semigroup $\{Q_{r,t} : t \geq r \in I\}$, where $I \subseteq \mathbb{R}$ is an interval.
We call $X$ a \textit{CB-process in varying environments} (CBVE-process) if there is a family of
continuous mappings $\{v_{r,t} : t \geq r \in I\}$ on $(0, \infty)$ so that
\begin{equation}
\int_{[0,\infty]} e^{-\lambda y} Q_{r,t}(x, dy) = e^{-x v_{r,t}(\lambda)}, \quad \lambda > 0, x \in [0, \infty]
\end{equation}
with $e^{-\lambda y} = 0$ for $y = \infty$ by convention. It is natural to expect that the processes defined by (1.1)
are scaling limits of \textit{GW-processes in varying environments} (GWVE-processes), where individuals in different generations may have different reproduction distributions.
The understanding of the CBVE-processes is important since they provide the bases of
further study of \textit{CB-processes in random environments} (CBRE-processes). The reader
may refer to Bansaye et al. (2013, 2019), Bansaye and Simatos (2015), He et al. (2018),
Helland (1981), Kurtz (1978), Li and Xu (2018), Palau et al. (2016), Palau and Pardo
(2017, 2018) and the references therein for some progresses in the study. In particular,
a scaling limit theorem for a sequence of GWVE-processes was proved by Bansaye and
Simatos (2015), who provided a general sufficient condition for the weak convergence of
the sequence and showed a CBVE-process indeed arises as the limit. Their condition
allows infinite variance of the reproduction distributions and extends considerably the
results in this line established before. But the general existence theorem for the CBVE-
process was not provided in Bansaye and Simatos (2015). In fact, with their approach
they need to avoid the \textit{bottlenecks}, which are the times when the process arrives at zero
a.s. by negative jumps. The determination of the behavior of the CBVE-process at the
bottlenecks was left open in Bansaye and Simatos (2015).

The purpose of this work is to give a construction of the CBVE-process under rea-
sonably general assumptions and clarify its behavior at the bottlenecks. Let $b_1$ and $c$
be càdlàg functions on $[0, \infty)$ satisfying $b_1(0) = c(0) = 0$ and having locally bounded
variations. Let $m$ be a $\sigma$-finite measure on $(0, \infty)^2$ satisfying
\begin{equation}
m_1(t) := \int_0^t \int_0^\infty (1 \wedge z^2)m(ds, dz) < \infty, \quad t \geq 0.
\end{equation}
Here and in the sequel, we understand, for $t \geq r \in \mathbb{R}$,
\begin{align*}
\int_r^t &= \int_{[r,t]} = -\int_t^r, \quad \int_r^\infty = \int_{(r,\infty)} = -\int_\infty^r.
\end{align*}
Let us consider the backward integral evolution equation:
\begin{equation}
v_{r,t}(\lambda) = \lambda - \int_r^t v_{s,t}(\lambda)b_1(ds) - \int_r^t v_{s,t}(\lambda)^2c(ds) - \int_r^t \int_0^\infty K_1(v_{s,t}(\lambda), z)m(ds, dz),
\end{equation}
where \( K_1(\lambda, z) = e^{-\lambda z} - 1 + \lambda z 1_{\{z \leq 1\}} \). This is an equivalent reformulation of the equation in Theorem 2.2 of Bansaye and Simatos (2015). We say the parameters \((b_1, c, m)\) are weakly admissible provided:

(1.A) \( t \mapsto c(t) \) is increasing and continuous;

(1.B) for every \( t > 0 \) we have
\[
\Delta b_1(t) + \int_0^1 zm(\{t\}, dz) \leq 1,
\]
where \( \Delta b_1(t) = b_1(t) - b_1(t-) \).

It is natural to introduce condition (1.4) to ensure that the solution of (1.3) stays positive (= nonnegative). In fact, from (1.3) we have, for any \( v \) and hence \( \int_0^1 zm(\{t\}, dz) \)

\[
\int_0^1 zm(\{t\}, dz) + m(\{t\} \times (\varepsilon, \infty)),
\]

and hence \( v_{r,t}(\lambda) < 0 \) for sufficiently large \( \lambda > 0 \) if (1.4) is not satisfied. Let \( J = \{ s > 0 : \Delta b_1(s) = 1 \} \) and \( K = \{ s \in J : m(\{s\} \times (0, \infty)) = 0 \} \). We say the parameters \((b_1, c, m)\) are admissible if they are weakly admissible and \( K \) is an empty set.

**Theorem 1.1** Let \((b_1, c, m)\) be admissible parameters. Then for \( t \geq 0 \) and \( \lambda > 0 \) there is a unique bounded and strictly positive solution \([0, t] \ni r \mapsto v_{r,t}(\lambda)\) to the integral evolution equation (1.3) and a transition semigroup \((Q_{r,t})_{t \geq r}\) on \([0, \infty)\] is defined by (1.1).

**Theorem 1.2** Let \((b_1, c, m)\) be admissible parameters. Then for any \( t \geq 0 \), \( r \mapsto v_{r,t}(0) := \lim_{\lambda \downarrow 0} v_{r,t}(\lambda)\) is the largest positive solution to (1.3) with \( \lambda = 0 \) and \( r \mapsto v_{r,t}(\infty) := \lim_{\lambda \uparrow \infty} v_{r,t}(\lambda)\) is the smallest positive solution to (1.3) with \( \lambda = \infty \).

Suppose that \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})\) is a filtered probability space satisfying the usual hypotheses. Let \( W(ds, du) \) be a time-space \((\mathcal{F}_t)\)-Gaussian white noise on \((0, \infty)^2\) with intensity \(2c(ds)du\). Let \( M(ds, dz, du) \) be a time-space \((\mathcal{F}_t)\)-Poisson random measure on \((0, \infty)^3\) with intensity \(m(ds, dz, du)\). Denote by \( \tilde{M}(ds, dz, du) \) the compensated measure of \( M(ds, dz, du) \). Given an \( \mathcal{F}_0\)-measurable random variable \( X(0) \geq 0 \), we consider the stochastic integral equation:
\[
X(t) = X(0) + \int_0^t \int_0^1 X(s-) W(ds, du) + \int_0^t \int_0^1 \int_0^1 X(s-) z \tilde{M}(ds, dz, du),
- \int_0^t X(s-) b_1(ds) + \int_0^t \int_0^\infty \int_0^1 z M(ds, dz, du).
\]

By saying the positive càdlàg process \( \{ X(t) : t \geq 0 \} \) in \([0, \infty)\) is a solution to (1.5) we mean the equation holds a.s. if \( t \) is replaced by \( t \wedge \tau_k \) for every \( t \geq 0 \), where \( \tau_k = \inf\{ t \geq 0 : X(t) \geq k \} \), and the states 0 and \( \infty \) are traps for \( \{ X(t) : t \geq 0 \} \).
Theorem 1.3 Let \((b_1, c, m)\) be admissible parameters. Then there is a pathwise unique solution \(\{X(t) : t \geq 0\}\) to \((1.5)\) and the solution is a CBVE-process with transition semigroup \((Q_{r,t})_{t \geq r}\) defined by \((1.1)\) and \((1.3)\).

The CBVE-process constructed by \((1.3)\) and \((1.5)\) is a generalization of the model studied in Jiřina (1958), where a smoothness was assumed for \((1.3)\). We shall first treat special forms of \((1.3)\) and \((1.5)\) by imposing an integrability condition stronger than \((1.2)\), which implies the CBVE-process has finite first moments. The existence of the cumulant semigroup is constructed by an iteration argument combined with an inhomogeneous nonlinear \(h\)-transformation. A suitably chosen transformation of this type changes the CBVE-process into a positive martingale and plays an important role in the establishment of the stochastic equation under the first moment assumption. The solutions to the general equations \((1.3)\) and \((1.5)\) are then obtained by increasing limits. The Poisson random measure in \((1.5)\) does not fit immediately into the framework of single valued general equations \((1.3)\) and \((1.5)\) are then obtained by increasing limits. The Poisson measure developed in standard references such as Ikeda and Watanabe (1989), Jacod and Shiryaev (2003) and Situ (2005). In fact, at a fixed discontinuity \(t > 0\) the jump size \(\Delta X(t)\) of the CBVE-process is identified by a composite Lévy–Itô representation as the position at time \(X(t–)\) of a spectrally positive Lévy process constructed from the random measure \(M(\{t\}, dz, du)\), which typically has infinitely many atoms. This is essentially different from its homogeneous version discussed in Bertoin and Le Gall (2006) and Dawson and Li (2006, 2012), where \(M(\{t\}, dz, du)\) has no more than one atom. The complexity of jumps of the solution makes the treatment of \((1.5)\) much more difficult than the homogeneous equations. The time-space noises in the stochastic equation yield natural interactions among the solutions started from different initial states, which are essential in the analysis of the model. By Theorem 1.2, the uniqueness of solutions to \((1.3)\) holds for \(\lambda \geq 0\) if and only if it holds for \(\lambda = 0\). This verifies an observation of Ryzhov and Skorokhod (1970, p.706) in our setting. The probabilistic meanings of the quantities \(v_{r,t}(0)\) and \(v_{r,t}(\infty)\) are given in \((2.11)\) and \((2.12)\), respectively.

Let \((b_1, c, m)\) be weakly admissible parameters. We call any moment \(s \in K\) a bottleneck following the terminology of Bansaye and Simatos (2015). Since \(b_1\) is a càdlàg function, we can rearrange \(K\) into an increasing (finite or infinite) sequence \(\{s_1, s_2, \cdots\}\). For \(t > 0\) let \(\varphi(t) = \max\{s \in K : s \leq t\}\) with \(\max\emptyset = 0\) by convention. By Theorem 1.1, for any \(\lambda > 0\) there is a unique bounded and strictly positive solution \(r \mapsto v_{r,t}(\lambda)\) to \((1.3)\) on the interval \([\varphi(t), t]\). By setting \(v_{r,t}(\lambda) = 0\) for \(0 \leq r < \varphi(t)\) we can extend \(r \mapsto v_{r,t}(\lambda)\) into a solution to \((1.3)\) on \([0, t]\). In this case, we may not be able to define the whole transition semigroup \(\{Q_{r,t} : t \geq r \in [0, \infty)\}\) simultaneously by \((1.1)\). However, for each \(i = 0, 1, 2, \cdots\) we can use \((1.1)\) and \((1.3)\) to define a transition semigroup \(\{Q_{r,t} : t \geq r \in [s_i, s_{i+1})\}\) on \([0, \infty]\), where we understand \(s_0 = 0\).

In terms of the stochastic equation, the behavior of the CBVE-process at the bottlenecks is clarified as follows. For weakly admissible parameters, we can use Theorem 1.3 to see there is still a pathwise unique solution \(\{X_0(t) : t \geq 0\}\) to \((1.5)\) and its restriction to the time interval \([0, s_1]\) is a CBVE-process with transition semigroup \(\{Q_{r,t} : t \geq r \in [0, s_1)\}\). Let \(\tau_{0,k} = \inf\{t \geq 0 : X_0(t) \geq k\}\) and let \(\tau_{0,\infty} = \lim_{k \to \infty} \tau_{0,k}\) be the explosion time of
\{X_0(t) : t \geq 0\}. Then we have \(X_0(s_1) = 0\) on the event \(\{s_1 < \tau_{0,\infty}\}\) and \(X_0(s_1) = \infty\) on the event \(\{\tau_{0,\infty} \leq s_1\}\). In fact, for any \(r_i \in [s_i, s_{i+1}), i = 1, 2, \ldots\), given the initial value \(X_i(r_i) \geq 0\), we can construct a process \(\{X_i(t) : t \geq r_i\}\) by the pathwise unique solution to a time-shift of (1.3). The restriction of the solution to \([r_i, s_{i+1})\) is a CBVE-process with transition semigroup \(\{Q_{r,t} : t \geq r \in [r_i, s_{i+1})\}\). The behavior of \(\{X_i(t) : t \in [r_i, s_{i+1})\}\) at \(s_{i+1} \in K\) is similar to that of \(\{X_0(t) : t \in [0, s_1)\}\) at \(s_1 \in K\).

The remaining part of the paper is organized as follows. In Section 2, some preliminary results are presented. In Section 3, we exploit the existence and uniqueness of solutions to some special cases of (1.3). The corresponding CBVE-process is constructed in Section 4 by solving a special form of (1.5). The general results for admissible parameters are proved in Section 5.

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## 2 Preliminaries

Given a càdlàg function \(\alpha\) on \([0, \infty)\) with locally bounded variations and \(\alpha(0) = 0\), we write \(\Delta \alpha(t) = \alpha(t) - \alpha(t-)\) for the size of its jump at \(t > 0\) and \(\|\alpha\|(t)\) for the total variation of \(\alpha\) on \([0, t]\). It is well-known the set \(J_\alpha := \{s > 0 : \Delta \alpha(s) \neq 0\}\) is at most countable. The function \(\alpha\) has the decomposition \(\alpha(t) = \alpha_c(t) + \alpha_d(t)\), where \(\alpha_d(t) := \sum_{0<s \leq t} \Delta \alpha(s)\) is the jump part and \(\alpha_c(t) := \alpha(t) - \alpha_d(t)\) is the continuous part.

**Proposition 2.1** Suppose that \(\alpha\) and \(G\) are càdlàg functions on \([0, \infty)\) with locally bounded variations such that \(\Delta \alpha(t) > -1\) for every \(t > 0\). Let \(\zeta\) be the càdlàg function on \([0, \infty)\) such that \(\zeta_c(t) = \alpha_c(t)\) and \(\Delta \zeta(t) = \log[1 + \Delta \alpha(t)]\) for every \(t > 0\). Then we have:

(i) (Forward equation) There is a unique locally bounded solution to:

\[
F(t) = G(t) + \int_0^t F(s-) \alpha(ds), \quad t \geq 0,
\]

which is given by

\[
F(t) = e^{\zeta(t)-\zeta(0)}G(0) + \int_0^t e^{\zeta(t)-\zeta(s)}G(ds).
\]

(ii) (Backward equation) For every \(t \geq 0\) there is a unique bounded solution to:

\[
H(r) = G(r) + \int_r^t H(s) \alpha(ds), \quad r \in [0, t],
\]
which is given by

\[ H(r) = e^{\zeta(t)-\zeta(r)}G(t) - \int_r^t e^{\zeta(s)-\zeta(r)}G(ds). \]  

(2.4)

\textbf{Proof.} The uniqueness of the solution to (2.1) or (2.3) follows by standard applications of Gronwall’s inequalities and is left to the reader. By (2.2) and integration by parts, we have

\[
F(t) = G(0) + e^{-\zeta(0)}G(0) \int_0^t e^{\zeta(s)}ds + \int_0^t G(ds) + \int_0^t e^{\zeta(s)} \int_s^t e^{-\zeta(v)}G(dv)
= e^{-\zeta(0)}G(0) \int_0^t e^{\zeta(s)}d\zeta_c(s) + e^{-\zeta(0)}G(0) \sum_{s \in [0,t]} (e^{\zeta(s)} - e^{\zeta(s-)} + G(t)
+ \int_0^t e^{\zeta(s-)}d\zeta_c(s) \int_0^s e^{-\zeta(v)}G(dv) + \sum_{s \in [0,t]} (e^{\zeta(s)} - e^{\zeta(s-)} \int_0^s e^{-\zeta(v)}G(dv)
= e^{-\zeta(0)}G(0) \int_0^t e^{\zeta(s-)}d\zeta_c(s) + e^{-\zeta(0)}G(0) \sum_{s \in [0,t]} e^{\zeta(s-)}(e^{\zeta(s)} - 1) + G(t)
+ \int_0^t e^{\zeta(s-)}d\zeta_c(s) \int_0^s e^{-\zeta(v)}G(dv) + \sum_{s \in [0,t]} e^{\zeta(s-)}(e^{\zeta(s)} - 1) \int_0^s e^{-\zeta(v)}G(dv)
= e^{-\zeta(0)}G(0) \int_0^t e^{\zeta(s-)}d\alpha(s) + G(t) + \int_0^t e^{\zeta(s-)}d\alpha(s) \int_0^s e^{-\zeta(v)}G(dv)
= G(t) + \int_0^t F(s-)d\alpha(s).
\]

Then \( t \mapsto F(t) \) is a solution to (2.1). Similarly, by (2.4) and integration by parts,

\[
H(t) = H(r) + e^{\zeta(t)}G(t) \int_r^t e^{-\zeta(s)} + \int_r^t G(ds) - \int_r^t e^{-\zeta(s)} \int_s^t e^{\zeta(v-)}G(dv)
= H(r) - e^{\zeta(t)}G(t) \int_r^t e^{-\zeta(s)}d\alpha(ds) + \int_r^t G(ds) + \int_r^t e^{-\zeta(s)}d\alpha(s) \int_s^t e^{\zeta(v-)}G(dv)
= H(r) - \int_r^t H(s)\alpha(ds) + G(t) - G(r).
\]

Then \( r \mapsto H(r) \) is a solution to (2.3) on \([0, t] \). \( \square \)

\textbf{Corollary 2.2} Let \( \alpha \) be a càdlàg function on \([0, \infty) \) with locally bounded variations such that \( \Delta \alpha(t) > -1 \) for \( t > 0 \). Then for \( \lambda \in \mathbb{R} \) we have:

(i) (Forward equation) There is a unique locally bounded solution to:

\[ \pi_t(\lambda) = \lambda + \int_0^t \pi_{s-}(\lambda)\alpha(ds), \quad t \geq 0, \]  

(2.5)
which is given by
\[ \pi_t(\lambda) = \lambda \prod_{s \in [0,t]} (1 + \Delta \alpha(s)) \exp\{\alpha_c(t) - \alpha_c(0)\}. \] \tag{2.6}

(ii) (Backward equation) For every \( t \geq 0 \) there is a unique bounded solution to:
\[ \pi_{r,t}(\lambda) = \lambda + \int_r^t \pi_{s,t}(\lambda) \alpha(ds), \quad r \in [0,t], \] \tag{2.7}
which is given by
\[ \pi_{r,t}(\lambda) = \lambda \prod_{s \in [r,t]} (1 + \Delta \alpha(s)) \exp\{\alpha_c(t) - \alpha_c(r)\}. \] \tag{2.8}

We next discuss briefly the structures of the transition semigroup \( \{Q_{r,t} : t \geq r \in I \} \) defined by (1.1). A family of mappings \( \{v_{r,t} : t \geq r \in I \} \) on \((0, \infty)\) is called a cumulant semigroup if the following conditions are satisfied:

(2.A) (Semigroup property) for \( \lambda > 0 \) and \( t \geq s \geq r \in I \),
\[ v_{r,t}(\lambda) = v_{r,s} \circ v_{s,t}(\lambda) = v_{r,s}(v_{s,t}(\lambda)); \] \tag{2.9}

(2.B) (Lévy–Khintchine representation) for \( \lambda > 0 \) and \( t \geq r \in I \),
\[ v_{r,t}(\lambda) = a_{r,t} + h_{r,t}\lambda + \int_0^\infty (1 - e^{-\lambda y})l_{r,t}(dy), \] \tag{2.10}
where \( a_{r,t} \geq 0, h_{r,t} \geq 0 \) and \((1 \wedge y)l_{r,t}(dy)\) is a finite measure on \((0, \infty)\).

Given a cumulant semigroup \( \{v_{r,t} : t \geq r \in I \} \), we can define the transition semigroup \( \{Q_{r,t} : t \geq r \in I \} \) on \([0, \infty)\) using (1.1). Clearly, the CBVE-process with this transition semigroup has both 0 and \( \infty \) as traps. By (1.1) we have
\[ Q_{r,t}(x, [0, \infty)) = e^{-xv_{r,t}(0)}, \quad x \in [0, \infty), \] \tag{2.11}
and
\[ Q_{r,t}(x, \{0\}) = e^{-xv_{r,t}(\infty)}, \quad x \in (0, \infty), \] \tag{2.12}
where \( v_{r,t}(0) := \lim_{\lambda \downarrow 0} v_{r,t}(\lambda) = a_{r,t} \in [0, \infty) \) and \( v_{r,t}(\infty) := \lim_{\lambda \uparrow \infty} v_{r,t}(\lambda) \in (0, \infty] \). We say a cumulant semigroup \( \{v_{r,t} : t \geq r \in I \} \) is conservative if \( v_{r,t}(0) = a_{r,t} = 0 \) for all \( t \geq r \in I \). In this case, we can restrict \( \{Q_{r,t} : t \geq r \in I \} \) to a conservative transition semigroup on \([0, \infty)\) and rewrite (1.1) into
\[ \int_{[0,\infty)} e^{-\lambda y} Q_{r,t}(x, dy) = e^{-xv_{r,t}(\lambda)}, \quad x \geq 0, \lambda \geq 0. \] \tag{2.13}
To conclude this section, we prove some useful upper and lower bounds for the solutions to the integral evolution equation (1.3). Let \((b_1, c, m)\) be admissible parameters. For \(\lambda > 0\) and \(t \geq r \geq 0\) let
\[
U_{r,t}(\lambda) = [\lambda + m((0, t] \times (1, \infty))] \exp\{\|b_t\|(t) - \|b_t\|(r)\}. \tag{2.14}
\]
By the admissibility of the parameters we have \(m(\{s\} \times (0, 1]) = 0\) and \(m(\{s\} \times (1, \infty)) > 0\) for \(s \in J\). For \(t \geq 0\) choose a sufficiently large constant \(\eta_t > 1\) so that \(m(\{s\} \times (1, \eta_t]) > 0\) when \(s \in (0, t] \cap J\). Let
\[
F_t(\lambda) = U_{0,t}(\lambda)^{-1}(1 - e^{-U_{0,t}(\lambda)}), \quad H_t(\lambda) = [\eta_t U_{0,t}(\lambda)]^{-1}(1 - e^{-\eta_t U_{0,t}(\lambda)}).
\]
Let \(r > 0\) and \(\lambda > 0\). Let \(\alpha(r) = \alpha(r, t, \lambda)\) be the càdlàg function on \([0, t]\) defined by
\[
\alpha(r) = \frac{-1}{2} U_{0,t}(\lambda) \int_0^r \int_0^{\varepsilon_t(\lambda)} z^2 m(ds, dz) + H_t(\lambda) \int_0^r \int_0^{\eta_t} z m(ds, dz) - b_1(r) - U_{0,t}(\lambda)c(r) - [1 - F_t(\lambda)] \int_0^r \int_{\varepsilon_t(\lambda)}^{1} z m(ds, dz), \tag{2.15}
\]
where \(\varepsilon_t(\lambda) = 1 \wedge [U_{0,t}(\lambda)^{-1} F_t(\lambda)]\). Let
\[
l_{r,t}(\lambda) = \lambda \prod_{s \in (r, t]} [1 + (0 \wedge \Delta \alpha(s))] \exp\{\|\alpha\|(r) - \|\alpha\|(t)\}. \tag{2.16}
\]

**Proposition 2.3** Suppose that \(r \mapsto v_{r,t}(\lambda)\) is a bounded positive solution to (1.3) with \(\lambda > 0\). Then we have
\[
l_{r,t}(\lambda) \leq v_{r,t}(\lambda) \leq U_{r,t}(\lambda), \quad r \in [0, t]. \tag{2.17}
\]

**Proof.** The upper bound in (2.17) follows by Gronwall’s inequality since (1.3) implies
\[
v_{r,t}(\lambda) \leq \lambda + \int_0^t \int_1^\infty m(ds, dz) + \int_r^t v_{s,t}(\lambda)\|b_t\|(ds).
\]
Let \(r \mapsto \pi_{r,t}(\lambda)\) be the solution to (2.7) with \(\alpha\) given by (2.15). Then we have
\[
v_{r,t}(\lambda) - \pi_{r,t}(\lambda) = G_{r,t}(\lambda) + \int_r^t [v_{s,t}(\lambda) - \pi_{s,t}(\lambda)]\alpha(ds), \tag{2.18}
\]
where
\[
G_{r,t}(\lambda) = \int_r^t v_{s,t}(\lambda)[U_{0,t}(\lambda) - v_{s,t}(\lambda)]c(ds) + \int_r^t \int_{\eta_t}^{\infty} (1 - e^{-v_{s,t}(\lambda)} z) m(ds, dz)
\]
\[
+ \int_r^t \int_{\varepsilon_t(\lambda)}^{1} \left[\frac{1}{2} U_{0,t}(\lambda)v_{s,t}(\lambda)z^2 - K(v_{s,t}(\lambda), z)\right] m(ds, dz).
\]
Given \( t \alpha \) bounded variations and satisfying \( \Delta \) measure on \((0, \infty)\), in this section, we take

\[
\Box (2.17).
\]

we see \( v \) defines a measure \( K(\lambda, z) = e^{-\lambda z} - 1 + \lambda z \). In view of \([1.4]\), for any \( s \in (0, t] \) we have

\[
\int_0^1 zm_d(\{s\}, dz) \leq 1 - \Delta b_1(s).
\]

It follows that

\[
\Delta \alpha(s) = -\frac{1}{2} U_{0,t}(\lambda) \int_0^{\varepsilon_t(\lambda)} z^2 m_d(\{s\}, dz) - [1 - F_t(\lambda)] \int_0^1 zm_d(\{s\}, dz)
\]

\[
- \Delta b_1(s) + H_t(\lambda) \int_1^{\eta_\lambda} zm(\{s\}, dz)
\]

\[
\geq -\frac{1}{2} U_{0,t}(\lambda) \varepsilon_t(\lambda) \int_0^1 zm_d(\{s\}, dz) - [1 - F_t(\lambda)] \int_0^1 zm_d(\{s\}, dz)
\]

\[
- \Delta b_1(s) + H_t(\lambda) \int_1^{\eta_\lambda} zm(\{s\}, dz)
\]

\[
\geq -\frac{1}{2} U_{0,t}(\lambda) \varepsilon_t(\lambda) \int_0^1 [1 - \Delta b_1(s)] - [1 - F_t(\lambda)] [1 - \Delta b_1(s)]
\]

\[
- \Delta b_1(s) + H_t(\lambda) \int_1^{\eta_\lambda} zm(\{s\}, dz).
\]

By the admissibility of the parameters we have \( 1 - \Delta b_1(s) > 0 \) when \( s \in (0, t) \setminus J \) and \( m(\{s\} \times (1, \eta_\lambda)) > 0 \) when \( s \in (0, t] \cap J \), so \( \Delta \alpha(s) > -1 \) for each \( s \in (0, t] \). Then Proposition \([2.4]\) applies to \([2.18]\). Since \( r \mapsto G_{r,t}(\lambda) \) is a decreasing function, from \([2.4]\) we see \( \nu_{r,t}(\lambda) - \pi_{r,t}(\lambda) \geq 0 \). By comparing \([2.8]\) and \([2.16]\) we have the lower bound in \([2.17]\).

\[\Box\]

## 3 Conservative cumulant semigroups

In this section, we take \( I = [0, \infty) \). Let \( \alpha \) be a càdlàg function on \([0, \infty)\) having locally bounded variations and satisfying \( \Delta \alpha(t) > -1 \) for \( t > 0 \). Let \( \mu(ds, dz) \) be a \( \sigma \)-finite measure on \((0, \infty)^2\) satisfying

\[
\int_0^t \int_0^\infty z \mu(ds, dz) < \infty, \quad t \geq 0.
\]

Given \( t \geq 0 \) and \( \lambda \geq 0 \), we first consider the backward integral evolution equation:

\[
u_{r,t}(\lambda) = \lambda + \int_r^t u_{s,t}(\lambda) \alpha(ds) + \int_r^t \int_0^\infty (1 - e^{-u_{s,t}(\lambda)z}) \mu(ds, dz), \quad r \in [0, t].\]

This is clearly spacial case of \([1.3]\).
Proposition 3.1 For \( t \geq 0 \) and \( \lambda \geq 0 \), there is a unique bounded positive solution \( r \mapsto u_{r,t}(\lambda) \) on \([0,t]\) to (3.2) and \((u_{r,t})_{t \geq r}\) is a conservative cumulant semigroup. Moreover, for \( \lambda \geq 0 \) we have
\[
u_{r,t}(\lambda) \leq \lambda e^{\|\rho\|(r,t)} \leq \lambda e^{\|\rho\|(t)},
\] (3.3)
where
\[
\rho(t) = \alpha(t) + \int_0^t \int_0^\infty z \mu(ds, dz).
\]

Proof. Step 1. Let \( r \mapsto u_{r,t}(\lambda) \) be a bounded positive solutions to (3.2). From the equation it is easy to see that
\[
u_{r,t}(\lambda) \leq \lambda + \int_r^t \nu_{s,t}(\lambda) \|\rho\|(ds).
\]
Then (3.3) follows by Gronwall’s inequality. Suppose that \( r \mapsto w_{r,t}(\lambda) \) is also a bounded positive solution to (3.2). Then we have
\[
|\nu_{r,t}(\lambda) - w_{r,t}(\lambda)| \leq \int_r^t |\nu_{s,t}(\lambda) - w_{s,t}(\lambda)| \|\rho\|(ds).
\]
By Gronwall’s inequality we see \( |\nu_{r,t}(\lambda) - w_{r,t}(\lambda)| = 0 \) for every \( r \in [0,t] \).

Step 2. Consider the case where \( \alpha \) vanishes. Let \( t \geq 0 \) and \( \lambda \geq 0 \) be fixed. For \( r \in [0,t] \) set \( v_{r,t}^{(0)}(\lambda) = 0 \) and define \( v_{r,t}^{(k)}(\lambda) \) inductively by
\[
v_{r,t}^{(k+1)}(\lambda) = \lambda + \int_r^t \int_0^\infty (1 - e^{-v_{s,t}^{(k)}(\lambda)z}) \mu(ds, dz).
\] (3.4)
By Proposition 4.2 of Silverstein (1968) one can use (3.4) to see inductively that each \( v_{r,t}^{(k)}(\lambda) \) has the Lévy–Khintchine representation (2.10). Moreover, we have \( v_{r,t}^{(k)}(0) = 0 \) and
\[
0 \leq v_{r,t}^{(k)}(\lambda) \leq v_{r,t}^{(k+1)}(\lambda) \leq \pi_{r,t}(\lambda),
\]
where \( r \mapsto \pi_{r,t}(\lambda) \) is the solution to (2.7) with
\[
\alpha(t) = \int_0^t \int_0^\infty z \mu(ds, dz).
\]
Then the limit \( v_{r,t}(\lambda) = \lim_{k \to \infty} v_{r,t}^{(k)}(\lambda) \) exists and the convergence is uniform in \((r, \lambda) \in [0,t] \times [0,B] \) for every \( t \geq 0 \) and \( B \geq 0 \). In fact, setting \( u_k(r, t, \lambda) = \sup_{r \leq s \leq t} |v_{s,t}^{(k)}(\lambda) - v_{s,t}^{(k-1)}(\lambda)| \), we have
\[
u_k(r, t, \lambda) \leq \int_r^t u_{k-1}(t_1, t, \lambda) \alpha(dt_1)
\]
and hence \( \sum_{k=1}^{\infty} u_k(t, \lambda) \leq Be^{\alpha(t)} < \infty \). By Corollary 1.33 in Li (2011) we infer that \( v_{r,t}(\lambda) \) has representation (2.10) with \( v_{r,t}(0) = 0 \). By (3.4) and monotone convergence we see \( r \mapsto v_{r,t}(\lambda) \) is a solution to (3.2) with \( \alpha \equiv 0 \). The semigroup property (2.9) follows from the uniqueness of the solution. Then \( (u_{r,t})_{t \geq r} \) is a conservative cumulant semigroup.

Step 3. Let \( \zeta \) be the càdlàg function on \([0, \infty)\) such that \( \zeta(t) = \alpha_c(t) \) and \( \Delta \zeta(t) = \log(1 + \Delta \alpha(t)) \) for every \( t \geq 0 \). By the second step, there is a unique bounded positive solution \( r \mapsto u_{r,t}(\lambda) \) on \([0, t]\) to

\[
\lambda = e^{-\zeta(t)}u_{r,t}(\lambda) + \int_{r}^{t} u_{s,t}(\lambda)e^{-\zeta(s)}ds - \int_{r}^{t} e^{-\zeta(s)}du_{s,t}(\lambda).
\]

Then \( r \mapsto v_{r,t}(\lambda) \) is a solution to (3.2) on \([0, t]\).

We next consider a more interesting spacial case of (1.3). Let \( (b_1, c, m) \) be admissible parameters given as in the introduction. Instead of (1.2), we here assume the stronger integrability condition:

\[
m(t) := \int_{0}^{t} \int_{0}^{\infty} (z \wedge z^2)m(ds, dz) < \infty, \quad t \geq 0.
\]

Then we can rewrite (1.3) equivalently into:

\[
v_{r,t}(\lambda) = \lambda - \int_{r}^{t} v_{s,t}(\lambda)b(ds) - \int_{r}^{t} v_{s,t}(\lambda)^2c(ds) - \int_{r}^{t} \int_{0}^{\infty} K(v_{s,t}(\lambda), z)m(ds, dz),
\]

where \( K(\lambda, z) = e^{-\lambda z} - 1 + \lambda z \) and

\[
b(t) = b_1(t) - \int_{0}^{t} \int_{1}^{\infty} zm(ds, dz).
\]
Let $B[0, \infty)^+$ be the set of locally bounded positive Borel functions on $[0, \infty)$ and $M[0, \infty)$ the set of Radon measures on $[0, \infty)$. By a branching mechanism with parameters $(b, c, m)$ we mean the functional on $B[0, \infty)^+ \times M[0, \infty)$ defined by

$$
\phi(f, B) = \int_B f(s)b(ds) + \int_B f(s)^2c(ds) + \int_B \int_0^\infty K(f(s), z)m(ds, dz),
$$

where $f \in B[0, \infty)^+$ and $B \in \mathcal{B}[0, \infty)$. Using this notation, we can rewrite (3.7) equivalently into

$$
v_{r,t}(\lambda) = \lambda - \phi(v, t(\lambda), (r, t)), \quad r \in [0, t]. \quad (3.9)
$$

For any integer $n \geq 1$ we define the branching mechanism $\phi_n$ on by

$$
\phi_n(f, B) = \int_B f(s)b(ds) + 2n^2\int_B (e^{-f(s)/n} - 1 + f(s)/n)c(ds)
\begin{align*}
&+ \int_B \int_0^1 (e^{-f(s)z} - 1 + f(s)z)(1 \wedge nz)m(ds, dz) \\
&+ \int_B \int_1^\infty (e^{-f(s)z} - 1 + f(s)z)m(ds, dz)
\end{align*}
= -\int_B f(s)\alpha_n(ds) - \int_B \int_0^\infty (1 - e^{-f(s)z})\mu_n(ds, dz), \quad (3.10)
$$

where

$$
\alpha_n(ds) = -b(ds) - 2nc(ds) - \int_0^1 z(1 \wedge nz)m(ds, dz)
\begin{align*}
&- \int_1^\infty zm(ds, dz) \\
&= -b_1(ds) - 2nc(ds) - \int_0^1 z(1 \wedge nz)m(ds, dz)
\end{align*}
$$

and

$$
\mu_n(ds, dz) = 2n^2c(ds)\delta_{1/n}(dz) + 1_{\{z \leq 1\}}(1 \wedge nz)m(ds, dz)
\begin{align*}
&+ 1_{\{z > 1\}}m(ds, dz).
\end{align*}
$$

Then $\Delta\alpha_n(t) > -1$ for every $t > 0$ since $(b_1, c, m)$ are admissible parameters.

**Lemma 3.2** The branching mechanisms $\phi$ and $\phi_n$ have the following properties:

(i) For $t \geq r \geq 0$ and $f \in B[0, \infty)^+$, we have $\phi(f, (r, t]) = \lim_{n \uparrow \infty} \phi_n(f, (r, t])$.

(ii) For $t \geq s \geq r \geq 0$ and $f \leq g \in B[0, \infty)^+$,

$$
\phi(f, (s, t]) - \phi_n(f, (s, t]) \leq \phi(g, (r, t]) - \phi_n(g, (r, t]).
$$
(iii) For \( t \geq s \geq r \geq 0 \) and \( f, g \in B[0, \infty)^+ \),
\[
|\phi(f, (s, t]) - \phi(g, (s, t])| \leq \left[ C_1(t) + 1 \right] \int_r^t |f(s) - g(s)| C_2(ds),
\]
where \( C_1(t) = \sup_{s \in [0, t]} |f(s) + g(s)| \) and
\[
C_2(ds) = \|b\|(ds) + c(ds) + \int_0^\infty (z \wedge z^2)m(ds, dz).
\] (3.11)

**Proof.** By (3.10) we obtain immediately (i) and (ii). For any \( t \geq s \geq r \geq 0 \) and \( f, g \in B[0, \infty)^+ \), we have
\[
|\phi(f, (s, t]) - \phi(g, (s, t])| \leq \int_r^t |f(u) - g(u)| \|b\|(du) + C_1(t) \int_r^t |f(u) - g(u)|c(du)
+ C_1(t) \int_r^t \int_r^1 |f(u) - g(u)| z^2 m(du, dz)
+ \int_r^t \int_1^\infty |f(u) - g(u)| z m(du, dz).
\]
Then (iii) follows. □

**Theorem 3.3** For every \( t \geq 0 \) and \( \lambda \geq 0 \) there is a unique bounded positive solution \( r \mapsto v_{r,t}(\lambda) \) to the integral evolution equation (3.7) or (3.9) and \((v_{r,t})_{t \geq r}\) is a conservative cumulant semigroup.

**Proof.** Let \( \phi_n \) be defined by (3.10). It is easy to see that \( \alpha_n \) and \( \mu_n \) satisfy the conditions of Proposition 3.1. In particular, for any \( s > 0 \) we have
\[
\Delta \alpha_n(s) \geq -\Delta b(s) - (1 - e^{-n}) \int_0^\infty z m(\{s\}, dz) > -1.
\]
Then a conservative cumulant semigroup \((v_{r,t}^{(n)})_{t \geq r}\) is defined by the evolution integral equation:
\[
v_{r,t}^{(n)}(\lambda) = \lambda - \phi_n(v_{s,t}^{(n)}(\lambda), (r, t]), \quad \lambda \geq 0, r \in [0, t].
\] (3.12)

By (3.3) we have \( v_{r,t}^{(n)}(\lambda) \leq Ae^{\|b\|(t)} \) for \( r \in [0, t] \) and \( \lambda \in [0, A] \). For \( n \geq k \geq 1 \) let
\[
D_{k,n}(r, t, \lambda) = \sup_{s \leq \lambda} \left| v_{s,t}^{(n)}(\lambda) - v_{s,t}^{(k)}(\lambda) \right|.
\]
By Lemma 3.2 we have
\[
D_{k,n}(r, t, \lambda) \leq 2|\phi(Ae^{\|b\|(t)}, (0, t]) - \phi_k(Ae^{\|b\|(t)}, (0, t])| + \left[ C_1(t) + 1 \right] \int_r^t D_{k,n}(s, t, \lambda) C_2(ds),
\]

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where $C_1(t) = 2Ae^{\|b\|(t)}$ and $C_2(ds)$ is given by (3.11). By Gronwall’s inequality,

$$D_{k,n}(r, t, \lambda) \leq 2|\phi(Ae^{\|b\|(t)}, (0, t)) − \phi_k(Ae^{\|b\|(t)}, (0, t))|e^{[C_1(t)+1]C_2(t)}.$$

By Lemma 3.2 it is easy to see the limit $v_{r,t}(\lambda) := \lim_{k \to \infty} v_{r,t}(k)$ exists and convergence is uniform in $(r, \lambda) \in [0, t] \times [0, A]$ for every $A \geq 0$. By Corollary 1.33 in Li (2011) we have the Lévy–Khintchine representation (2.10) with $\alpha(t) = -b(t)$.

**Proposition 3.4** Let $r \mapsto v_{r,t}(\lambda)$ be the unique bounded positive solution to (3.7) on $[0, t]$. Then we have $v_{r,t}(\lambda) \leq \pi_{r,t}(\lambda)$, where

$$r \mapsto \pi_{r,t}(\lambda) := \lambda \prod_{r<s\leq t} (1 − \Delta b(s)) \exp\{b_c(r) − b_c(t)\}$$

(3.13)

is the solution to (2.7) with $\alpha(t) = −b(t)$.

**Proof.** Fix $t \geq 0$ and $\lambda \geq 0$ and let $H(r) = \pi_{r,t}(\lambda) − v_{r,t}(\lambda)$. From (2.7) and (3.7) we see that $r \mapsto H(r)$ satisfies (2.3) with $\alpha(t) = −b(t)$ and

$$G(r) = \int_r^t v_{s,t}(\lambda)^2 c(ds) + \int_r^t \int_0^\infty K(v_{s,t}(\lambda), z) m(ds, dz).$$

By Proposition 2.1 we have the representation (2.4) for $H(r)$, which implies $H(r) \geq 0$ since $r \mapsto G(r)$ is a decreasing function on $[0, t]$. □

**Proposition 3.5** For $t \geq r \geq 0$ let $\pi_{r,t}(1)$ be defined by (3.13) with $\lambda = 1$. Then we have

$$h_{r,t} + \int_0^\infty yl_{r,t}(dy) = \pi_{r,t}(1)$$

(3.14)

and

$$\int_{[0,\infty)} yQ_{r,t}(x, dy) = x\pi_{r,t}(1), \quad x \geq 0.$$

(3.15)

**Proof.** By Proposition 3.4 we have $\lambda^{-1}v_{r,t}(\lambda) \leq \pi_{r,t}(1)$. Then we can differentiate both sides of (3.7) and use bounded convergence to see that $r \mapsto \frac{\partial}{\partial \lambda} v_{r,t}(0+)$ is a solution to (2.7) with $\alpha(t) = −b(t)$ and $\lambda = 1$. It follows that $\frac{\partial}{\partial \lambda} v_{r,t}(0+) \equiv \pi_{r,t}(1)$. By differentiating both sides of (2.10) we obtain (3.14). Similarly we get (3.15) from (2.13). □
The transformation of the cumulant semigroup used in the proof of Proposition 3.1 is an inhomogeneous nonlinear variation of the classical \( h \)-transformation and has been used in the study of CB-processes; see, e.g., Bansaye et al. (2013), He et al. (2018) and Li (2011, Section 6.1). A generalized form of the transformation is given below, which will be useful in the next section.

**Theorem 3.6** Let \((v_{r,t})_{r \geq t}\) be the conservative cumulant semigroup defined by (3.7) or (3.9). Let \( t \mapsto \zeta(t) \) be a locally bounded function on \([0, \infty)\). Then another conservative cumulant semigroup \((u_{r,t})_{r \geq t}\) is defined by:

\[
u_{r,t} (\lambda) = e^{c(r)} v_{r,t} (e^{-\zeta(t)} \lambda), \quad \lambda \geq 0. \tag{3.16}\]

Moreover, if \( \zeta \) is a càdlàg function on \([0, \infty)\) with locally bounded variations, then \([0, t] \ni r \mapsto \nu_{r,t}(\lambda)\) is the unique bounded positive solution to

\[
u_{r,t} (\lambda) = \lambda - \int_r^t v_{s,t}(\lambda) d\beta(s) - \int_r^t u_{s,t}(\lambda) e^{-\Delta \zeta(s)} b(ds) - \int_r^t u_{s,t}(\lambda)^2 e^{-\zeta(s)} c(ds) \nonumber \]

\[
- \int_r^t \int_0^\infty K(v_{s,t}(\lambda), z)e^{c(s-)} m(ds, e^{c(s)} dz), \tag{3.17}
\]

where

\[
\beta(t) = \zeta(t) + \sum_{s \in (0,t]} (1 - e^{-\Delta \zeta(s)}).
\]

**Proof.** The arguments are generalizations of those in the last step of the proof of Proposition 3.1. Clearly, the family \((u_{r,t})_{r \geq t}\) defined by (3.16) is a conservative cumulant semigroup. If \( \zeta \) is a càdlàg function with locally bounded variations, we can use integration by parts to get

\[
\lambda = \int_r^t v_{s,t}(e^{-\zeta(t)} \lambda) d\zeta(s) + \sum_{s \in (0,t]} v_{s,t}(e^{-\zeta(t)} \lambda) (e^{c(s)} - e^{c(s-)}).
\]

\[
u_{r,t} (\lambda) + \int_r^t v_{s,t}(e^{-\zeta(t)} \lambda) e^{c(s)} \zeta c(ds) + \sum_{s \in (0,t]} v_{s,t}(e^{-\zeta(t)} \lambda) (e^{c(s)} - e^{c(s-)})
\]

\[
+ \int_r^t e^{c(s-)} v_{s,t}(e^{-\zeta(t)} \lambda) b(ds) + \int_r^t e^{c(s-)} v_{s,t}(e^{-\zeta(t)} \lambda)^2 c(ds)
\]

\[
+ \int_r^t \int_0^\infty e^{c(s-)} K(v_{s,t}(e^{-\zeta(t)} \lambda), z)m(ds, dz)
\]

\[
u_{r,t} (\lambda) + \int_r^t v_{s,t}(e^{-\zeta(t)} \lambda) \zeta c(ds) + \int_r^t e^{c(s-)} v_{s,t}(e^{-\zeta(t)} \lambda) b(ds)
\]

\[
+ \int_r^t e^{c(s-)} v_{s,t}(e^{-\zeta(t)} \lambda)^2 c(ds) + \int_r^t \int_0^\infty e^{c(s-)} K(v_{s,t}(e^{-\zeta(t)} \lambda), z)m(ds, dz)
\]

\[
u_{r,t} (\lambda) + \int_r^t v_{s,t}(\lambda) \zeta c(ds) + \int_r^t e^{-\Delta \zeta(s)} v_{s,t}(\lambda) b(ds) + \int_r^t e^{-\zeta(s)-\Delta \zeta(s)} v_{s,t}(\lambda)^2 c(ds)
\]
where we have used the continuity of $s \mapsto c(s)$ for the last equality. Then $r \mapsto u_{r,t}(\lambda)$ solves \eqref{3.17}. The uniqueness of the solution holds by Theorem 3.3. \hfill \Box

\section{Stochastic equations for CBVE-processes}

Let $(b, c, m)$ be given as in the last section. Recall that $m$ also denotes the increasing function defined by \eqref{3.6}. Then $J_m := \{ s > 0 : \Delta m(s) > 0 \}$ is at most a countable set. Let $m_d(ds, dz) = 1_{J_m}(s)m(ds, dz)$ and $m_c(ds, dz) = m(ds, dz) - m_d(ds, dz)$. Suppose that $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is a filtered probability space satisfying the usual hypotheses. Let $W(ds, du)$ and $M(ds, dz, du)$ be $(\mathcal{F}_t)$-noises given as in the introduction. One can see that $M_c(ds, dz, du) := 1_{J_m}(s)M(ds, dz, du)$ and $M_d(ds, dz, du) := 1_{J_m}(s)M(ds, dz, du)$ are $(\mathcal{F}_t)$-Poisson random measures with intensities $m_c(ds, dz)du$ and $m_d(ds, dz)du$, respectively. Those random measures are independent of each other as they have disjoint supports. We can rewrite \eqref{1.5} equivalently into:

\begin{align*}
X(t) & = X(0) + \int_0^t \int_0^X(s-) W(ds, du) + \int_0^t \int_0^\infty \int_0^X(s-) z\tilde{M}_c(ds, dz, du) \\
& \quad - \int_0^t X(s-)b(ds) + \int_0^t \int_0^\infty \int_0^X(s-) z\tilde{M}_d(ds, dz, du), \tag{4.1}
\end{align*}

where $b$ is defined by \eqref{3.8}.

**Proposition 4.1** Let $\{X(t) : t \geq 0\}$ be a solution to \eqref{1.1} and let $\tau_k = \inf\{t \geq 0 : X(t) \geq k\}$ for $k \geq 1$. Then $\tau_k \to \infty$ almost surely as $k \to \infty$. Moreover, for $t \geq 0$ and $k \geq 1$ we have

\begin{align*}
k\mathbb{P}\{\tau_k \leq t\} & \leq \mathbb{P}[X(0)]e^{\|b\|(t)} \quad \text{and} \quad \mathbb{P}[X(t)] \leq \mathbb{P}[X(0)]e^{\|b\|(t)} \tag{4.2}
\end{align*}

We omit the proof of the above proposition, which is based on an application of Gronwall’s inequality. The comparison property of the solutions to \eqref{1.1} plays an important role in the analysis of the stochastic equation. In the proof, some special care has to be taken for the negative jumps brought about by the compensator of the Poisson random measure. For simplicity we only give a treatment of the property under a stronger integrability condition.

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Proposition 4.2 The pathwise uniqueness of solution holds for (4.1) under the additional integrability condition

\[
\int_0^t \int_0^\infty z^2 m(ds, dz) < \infty, \quad t \geq 0.
\] (4.3)

Moreover, under the above condition, if \( \{X_1(t) : t \geq 0\} \) and \( \{X_2(t) : t \geq 0\} \) are two solutions to (4.1) satisfying \( P\{X_1(0) \leq X_2(0)\} = 1 \), then we have \( P\{X_1(t) \leq X_2(t) \text{ for every } t \geq 0\} = 1 \).

Proof. It suffices to prove the second assertion. For each integer \( n \geq 0 \) define \( a_n = \exp\{-n(n+1)/2\} \). Then \( \int_0^{a_n-1} z^{-1} dz = n \) and \( a_n \to 0 \) decreasingly as \( n \to \infty \). Let \( x \mapsto g_n(x) \) be a positive continuous function supported by \( (a_n, a_n) \) so that \( \int_0^{a_n-1} g_n(x) dx = 1 \) and \( g_n(x) \leq 2(nx)^{-1} \) for every \( x > 0 \). For \( n \geq 0 \) and \( z \in \mathbb{R} \) let

\[
f_n(z) = \int_0^z dy \int_0^y g_n(x) dx.
\] (4.4)

From (4.1) we have

\[
X(t) = X(0) + \int_0^t \int_0^\infty W(ds, du) + \int_0^t \int_0^\infty \int_0^\infty zM_c(ds, dz, du)
- \int_0^t X(s-)1_{J_m}(s)b(ds) + \sum_{s \in [0,t]} 1_{J_m}(s)g_s(X(s-)),
\] (4.5)

where

\[
g_s(x) = \int_0^\infty \int_0^x zM_d(\{s\}, dz, du) - x \left[ \Delta b(s) + \int_0^\infty zm_d(\{s\}, dz) \right].
\]

It is easy to see that \( x \mapsto x + g_s(x) \) is an increasing function on \([0, \infty)\). Let \( l_s(x_1, x_2) = g_s(x_1) - g_s(x_2) \). Suppose that \( \{X_1(t) : t \geq 0\} \) and \( \{X_2(t) : t \geq 0\} \) are two solutions to (4.1) satisfying \( X_1(0) \leq X_2(0) \). Let \( Y(t) = X_1(t) - X_2(t) \) for \( t \geq 0 \). Then

\[
Y(t) = Y(0) + \int_0^t \int_{X_1(s-) \wedge X_2(s-) \leq \infty} (1_{\{Y(s-) > 0\}} - 1_{\{Y(s-) < 0\}}) W(ds, du)
+ \int_0^t \int_{X_1(s-) \wedge X_2(s-) \leq \infty} z(1_{\{Y(s-) > 0\}} - 1_{\{Y(s-) < 0\}}) M_c(ds, dz, du)
- \int_0^t Y(s-)1_{J_m}(s)b(ds) + \sum_{s \in [0,t]} l_s(X_1(s-), X_2(s-))1_{J_m}(s).
\]

By Itô’s formula,

\[
f_n(Y(t)) = \int_0^t Y(s-)f_n'(Y(s-))1_{\{Y(s-) > 0\}}c(ds) + \text{local mart}.
\]

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where $D_z f_n(x) = f_n(x + z) - f_n(x) - f_n(x)z$. Following the arguments in the proof of Theorem 8.2 in Li (2019+) one sees $|f_n''(Y(s-))| \leq 2n^{-1}|Y(s-)|^{-1}$ and $|D_z f_n(Y(s-))| \leq 2n^{-1}|Y(s-)|^{-1}z^2$. By Lemma 3.1 in Li and Pu (2012) we have, for $s \in J_b \setminus J_m$,

$$|D_{-Y(s-)}\Delta b(s) f_n(Y(s-))| \leq 2n^{-1}|Y(s-)|^{-1}|Y(s-)| \Delta b(s)|^2 \leq 2n^{-1}|Y(s-)||\Delta b(s)|^2$$

and, for $s \in J_m$,

$$|D_{l_s(x_1(s-), x_2(s-))} f_n(Y(s-))| \leq 2n^{-1}|Y(s-)|^{-1}\left( \int_0^\infty \int_{X_{1(s-)}}^\Delta z M_d(\{s\}, dz, du) - Y(s-)|\Delta b(s)| \right)^2$$

$$\leq 4n^{-1}\left[ |Y(s-)|^{-1}\left( \int_0^\infty \int_{X_{1(s-)}}^\Delta z M_d(\{s\}, dz, du) \right)^2 + |Y(s-)| |\Delta b(s)|^2 \right].$$

It follows that, for $s \in J_b \setminus J_m$,

$$P\left[ |D_{-Y(s-)}\Delta b(s) f_n(Y(s-))| 1_{\{Y(s-) > 0\}} \right] \leq 2n^{-1}P(Y(s-) \lor 0)|\Delta b(s)|^2$$

and, for $s \in J_m$,

$$P\left[ |D_{l_s(x_1(s-), x_2(s-))} f_n(Y(s-))| 1_{\{Y(s-) > 0\}} \right]$$

$$\leq 4n^{-1}\left[ \int_0^\infty z^2 m_d(\{s\}, dz) + P(Y(s-) \lor 0)|\Delta b(s)|^2 \right].$$

By taking the expectations in (4.6) we get

$$P[f_n(Y(t))] \leq \int_0^t P(Y(s-) \lor 0)||b||(ds) + 2n^{-1}c(t) + 2n^{-1} \int_0^t \int_0^\infty z^2 m_c(ds, dz)$$

$$+ 4n^{-1} \int_0^t \int_0^\infty z^2 m_d(ds, dz) + 4n^{-1} \sum_{s \in \{0, t\}} P(Y(s-) \lor 0)|\Delta b(s)|^2.$$
By Gronwall’s inequality one can see \( P(Y(t) \vee 0) = 0 \) for every \( t \geq 0 \). That proves the desired result. □

The following result gives a characterization of the conditional distribution of the jump of the CBVE-process at any moment \( t \in J_b \cup J_m \).

**Proposition 4.3** The CBVE-process with transition semigroup \((Q_{r,t})_{t \geq r}\) given by (2.13) and (3.7) has a càdlàg semimartingale realization \( \{(X(t), \mathcal{F}_t) : t \geq 0\} \) with the filtration satisfying the usual hypotheses. For such a realization and \( t \in J_b \cup J_m \) we have

\[
P(e^{-\lambda \Delta X(t) | \mathcal{F}_{t-}}) = e^{(\lambda - v_{t-,t}(\lambda))X(t-)}, \quad \lambda \geq 0,
\]

where \( \Delta X(t) = X(t) - X(t-) \) and

\[
\lambda - v_{t-,t}(\lambda) = \Delta b(t) \lambda + \int_0^\infty K(\lambda, z)m(\{t\}, dz).
\]

**Proof.** Let \( \{(X(t), \mathcal{G}_t) : t \geq 0\} \) be a realization of the CBVE-process defined on a complete probability space \((\Omega, \mathcal{F}, P)\). In view of (3.7), for any \( \lambda \geq 0 \) we have \( v_{r,t}(\lambda) \to \lambda \) as \( t \downarrow r \). Then (2.13) implies \( \lim_{t \downarrow r} Q_{r,t}(x, dy) = \delta_x(dy) \) by weak convergence and so \( \lim_{t \downarrow r} Q_{r,t}(x, \{y \geq 0 : |y - x| \geq \varepsilon\}) = 0 \) for every \( \varepsilon > 0 \). By dominated convergence,

\[
\lim_{t \downarrow r} P(|X(t) - X(r)| > \varepsilon) = \lim_{t \downarrow r} P[Q_{r,t}(x, \{y \geq 0 : |y - x| \geq \varepsilon\}) |_{x=X(r)}] = 0.
\]

Then \( \{X(t) : t \geq 0\} \) is stochastically right continuous. From (3.7) we see \( r \mapsto v_{r,t}(\lambda) \) is right-continuous on \([0, t]\), so \( \{e^{-X(r)v_{r,t}(\lambda)} : r \in [0, t]\} \) is stochastically right-continuous. Let \( \mathcal{G}_t \) be the augmentation of \( \mathcal{G}_t \) and let \( \mathcal{F}_t = \mathcal{G}_{t+} \) for \( t \geq 0 \). The Markov property implies

\[
P[e^{-\lambda X(t) | \mathcal{G}_r}] = e^{-X(r)v_{r,t}(\lambda)}, \quad t \geq r \geq 0.
\]

This means \( \{e^{-X(r)v_{r,t}(\lambda)} : r \in [0, t]\} \) is a positive bounded martingale, so it has a càdlàg \((\mathcal{F}_r)\)-martingale modification. By Proposition 2.3 we have \( v_{r,t}(\lambda) \geq l_0,\lambda) > 0 \) for \( \lambda > 0 \). Then \( \{X(r) : r \in [0, t]\} \) has a càdlàg modification. It follows that \( \{X(t) : t \geq 0\} \) has a càdlàg semimartingale modification; see, e.g., Dellacherie and Meyer (1982, pp.219-221).

Using such a modification we can replace \( \mathcal{G}_r \) by \( \mathcal{F}_r \) in (4.9). Then \( \{(X(t), \mathcal{F}_t) : t \geq 0\} \) is a càdlàg semimartingale realization of the CBVE-process with the filtration satisfying the usual hypotheses. By letting \( r \uparrow t \) in (3.7) and (4.9) we get

\[
P[e^{-\lambda X(t) | \mathcal{F}_{t-}}] = e^{-X(t-v_{t-,t}(\lambda))}, \quad \lambda \geq 0,
\]

where

\[
v_{t-,t}(\lambda) = (1 - \Delta b(t))\lambda - \int_0^\infty K(\lambda, z)m(\{t\}, dz).
\]
Then (4.7) follows. □

In view of (4.7) and (4.8), for any $t \in J_b \cup J_m$ it is natural to expect that the jump $\Delta X(t)$ of the CBVE-process should be given by the position at time $X(t-)$ of a spectrally positive Lévy process with Lévy measure $m\{\{t\}, dz\}$. It can be realized by an extension of the probability space. For this purpose we first establish a composite Lévy–Itô representation as follows.

**Proposition 4.4** Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with the sub-$\sigma$-algebra $\mathcal{G} \subset \mathcal{F}$. Suppose that $(\xi, Z)$ is a random vector taking values in $[0, \infty) \times \mathbb{R}$ such that $\xi$ is $\mathcal{G}$-measurable and, for every $\lambda \geq 0$,

$$P(e^{-\lambda Z}|\mathcal{G}) = \exp \left\{ \xi \left[ \beta \lambda + \int_0^\infty (e^{-\lambda z} - 1 + \lambda z) \gamma(dz) \right] \right\},$$

(4.10)

where $\beta \in \mathbb{R}$ and $\gamma(dz)$ is a $\sigma$-finite measure on $(0, \infty)$ satisfying

$$\int_0^\infty (z \land z^2) \gamma(dz) < \infty.$$

Then on an extension of the probability space there exists a Poisson random measure $N(dz, du)$ on $(0, \infty)^2$ with intensity $\gamma(dz)du$ such that $N$ is independent of $\mathcal{G}$ and a.s.

$$Z = -\beta \xi + \int_0^\infty \int_0^z z \tilde{G}(dz, du).$$

(4.11)

**Proof.** This proof also makes precise the statements of the proposition. Let $\rho(z, u) = (z \land z^2)(1 + u^2)^{-1}$ for $z > 0$ and $u > 0$. Let $M_\rho$ denote the space of all $\sigma$-finite Borel measures $\nu$ on $(0, \infty)^2$ so that

$$\int_0^\infty \int_0^\infty \rho(z, u) \nu(dz, du) < \infty.$$

We equip $M_\rho$ with the $\sigma$-algebra $M_\rho$ generated by the mappings $\nu \mapsto \nu((a, \infty) \times B)$ for all $a > 0$ and bounded $B \in \mathcal{B}(0, \infty)$. It is well-known that there is a spectrally positive Lévy process $\{Y_s : s \geq 0\}$ such that

$$E(e^{-\lambda Y_s}) = \exp \left\{ s \left[ \beta \lambda + \int_0^\infty (e^{-\lambda z} - 1 + \lambda z) \gamma(dz) \right] \right\}, \quad \lambda \geq 0.$$

By Lévy–Itô representation, there is a Poisson random measure $G = G(dz, du)$ on $(0, \infty)^2$ with intensity $\gamma(dz)du$ such that

$$Y_s = -\beta s + \int_0^s \int_0^z z \tilde{G}(dz, du), \quad s \geq 0.$$
Let \( P(s, dy, d\nu) \) be the joint distribution of the random vector \((Y_s, G)\) on \( \mathbb{R} \times M_\rho \). Let \( P_1(s, dy) \) and \( P_2(s, d\nu) \) denote the marginal distributions of \( Y_s \) and \( G \), respectively. Let \( \kappa_1(s, y, d\nu) \) be a regular conditional distribution of \( G \) given \( Y_s \). Then \( \kappa_1(s, y, d\nu) \) is a kernel from \([0, \infty) \times \mathbb{R}\) to \( M_\rho \) and \( P(s, dy, d\nu) = P_1(s, dy)\kappa_1(s, y, d\nu) \). Let \( \bar{\Omega} = \Omega \times M_\rho \) and \( \bar{\mathcal{F}} = \mathcal{F} \times M_\rho \). Let \( \bar{P} \) be the probability law on \((\bar{\Omega}, \bar{\mathcal{F}})\) defined by \( \bar{P}(d\bar{\omega}) = P(d\omega)\kappa_1(\xi(\omega), Z(\omega), d\mu), \) where \( \bar{\omega} = (\omega, \mu) \in \bar{\Omega} \). For any random variable \( X \) on \((\Omega, \mathcal{F}, P)\), write \( X(\bar{\omega}) = X(\omega) \) for \( \bar{\omega} = (\omega, \mu) \in \bar{\Omega} \), which extends \( X \) to a random variable on \((\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})\). It is easy to see that \( \bar{\mathcal{G}} := \mathcal{G} \times \{\emptyset, M_\rho\} \subset \bar{\mathcal{F}} \) and \( \xi \) is \( \bar{\mathcal{G}} \)-measurable as a random variable on \((\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})\). Let \( N(\bar{\omega}) = \mu \) for \( \bar{\omega} = (\omega, \mu) \in \bar{\Omega} \). By \((4.11)\) we have \( \bar{P}(Z \in dy|\bar{\mathcal{G}}) = P(Z \in dy|\xi) = P_1(\xi, dy). \) From the definition of \( \bar{P} \) it follows that

\[
\bar{P}(Z \in dy, N \in dv|\bar{\mathcal{G}}) = \bar{P}(Z \in dy, N \in dv|\xi) = P_1(\xi, dy)\kappa_1(\xi, y, dv) = P(\xi, dy, dv).
\]

Then \( N \) is a Poisson random measure on \([0, \infty)^2\) with intensity \( \gamma(dz)du \) and \((4.11)\) a.s. holds. Let \( F \) be a bounded \( \mathcal{G} \)-measurable random variable on \((\Omega, \mathcal{F}, P)\). For any positive Borel function \( f \) on \([0, \infty)^2\) bounded above by \( \rho \cdot \text{const.}, \) we have

\[
\bar{P}(F \exp \left\{ - \int_0^\infty \int_0^\infty f(z, u)N(dz, du) \right\}) \\
= \bar{P}(F \bar{P} \left( \exp \left\{ - \int_0^\infty \int_0^\infty f(z, u)N(dz, du) \right\} |\bar{\mathcal{G}} \right)) \\
= \bar{P}(F \int_{M_\rho} \exp \left\{ - \int_0^\infty \int_0^\infty f(z, u)\nu(dz, du) \right\} P_2(\xi, d\nu) \\
= \bar{P}(F) \exp \left\{ - \int_0^\infty \gamma(dz) \int_0^\infty (1 - e^{-f(z, u)}) du \right\} = \bar{P}(F) \bar{P} \left( \exp \left\{ - \int_0^\infty \int_0^\infty f(z, u)N(dz, du) \right\} \right).
\]

Then \( N \) is independent of \( \bar{\mathcal{G}} \) on \((\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})\).

**Theorem 4.5** There is a pathwise unique solution \( \{X(t) : t \geq 0\} \) to \((4.1)\) and the solution is a CBVE-process with transition semigroup \((Q_{r,t})_{t \geq r} \) defined by \((2.13)\) and \((3.7)\).

**Proof.** Step 1. Consider the case where \((4.3)\) holds and \( b(t) = 0 \) for every \( t \geq 0 \). Let \((v_{r,t})_{t \geq r} \) be the conservative cumulant semigroup defined by \((3.7)\) in this special case. Suppose that \( \{ (X(t), \mathcal{F}_t) : t \geq 0 \} \) is the realization of the corresponding CBVE-process provided by Proposition \((4.3)\). Then the process is actually a martingale by Proposition \((3.5)\).

Let \( N_0(ds, dz) \) be the optional random measure on \((0, \infty) \times \mathbb{R}\) by

\[
N_0(ds, dz) := \sum_{s > 0} 1_{\{\Delta X(s) \neq 0\}} \delta_{(s, \Delta X(s)}(ds, dz).
\]

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Let $\tilde{N}_0(ds, dz)$ denote the predictable compensator of $N_0(ds, dz)$ and let $\tilde{N}_0(ds, dz) = N_0(ds, dz) - N_0(ds, dz)$ be the compensated measure. We can write

$$X(t) = X(0) + M(t) + \int_0^t \int_{\mathbb{R}} z\tilde{N}_0(ds, dz),$$

(4.12)

where $\{M(t) : t \geq 0\}$ is a continuous local martingale. Let $\{C(t) : t \geq 0\}$ be its quadratic variation process. Let $f(x, \lambda) = e^{-x\lambda}$ for $x, \lambda \geq 0$. Then

$$f'_1(x, \lambda) = -\lambda f(x, \lambda), \quad f'_2(x, \lambda) = -xf(x, \lambda), \quad f''_1(x, \lambda) = \lambda^2 f(x, \lambda).$$

By Itô’s formula, for $t \geq 0$ and $\lambda \geq 0$,

$$e^{-X(r)\nu_{r,t}(\lambda)} = e^{-X(r)\nu_{r,t}(\lambda)} + \int_r^t f'_1(X(s-), v_{s,t}(\lambda)) dX(s) + \int_r^t f'_2(X(s-), v_{s,t}(\lambda)) dv_{s,t}(\lambda)$$

$$+ \frac{1}{2} \int_r^t f''_1(X(s-), v_{s,t}(\lambda)) dC(s)$$

$$+ \sum_{s \in [r,t] \cap J_m} [f(X(s), v_{s,t}(\lambda)) - f(X(s), v_{s-}(\lambda)) - f'_1(X(s), v_{s,t}(\lambda)) \Delta X(s) - f'_2(X(s), v_{s,t}(\lambda)) \Delta v_{s,t}(\lambda)]$$

$$+ \sum_{s \in [r,t] \cap J_m} [f(X(s), v_{s,t}(\lambda)) - f(X(s), v_{s-}(\lambda)) - f'_1(X(s), v_{s,t}(\lambda)) \Delta X(s)]$$

$$= e^{-X(r)\nu_{r,t}(\lambda)} - \int_r^t e^{-X(s-\nu_{s,t}(\lambda))} v_{s,t}(\lambda) dX(s)$$

$$- \int_r^t e^{-X(s-\nu_{s,t}(\lambda))} X(s-\nu_{s,t}(\lambda))^2 c(ds)$$

$$- \int_r^t 0 \int_0^{\infty} e^{-X(s-\nu_{s,t}(\lambda))} X(s-\nu_{s,t}(\lambda)) K(v_{s,t}(\lambda), z) m(ds, dz)$$

$$+ \frac{1}{2} \int_r^t e^{-X(s-\nu_{s,t}(\lambda))} v_{s,t}(\lambda)^2 C(ds) + \sum_{s \in [r,t] \cap J_m} [e^{-X(s-\nu_{s,t}(\lambda))} - e^{-X(s-\nu_{s-}(\lambda))}$$

$$+ e^{-X(s-\nu_{s,t}(\lambda))} v_{s,t}(\lambda) \Delta X(s) + e^{-X(s-\nu_{s,t}(\lambda))} X(s) \Delta v_{s,t}(\lambda)]$$

$$+ \sum_{s \in [r,t] \cap J_m} [e^{-X(s-\nu_{s,t}(\lambda))} - e^{-X(s-\nu_{s-}(\lambda)) + e^{-X(s-\nu_{s,t}(\lambda)) v_{s,t}(\lambda) \Delta X(s)}$$

$$= e^{-X(r)\nu_{r,t}(\lambda)} + \int_0^r e^{-X(s-\nu_{s,t}(\lambda))} v_{s,t}(\lambda) dX(s)$$

$$+ \int_0^r e^{-X(s-\nu_{s,t}(\lambda))} X(s-\nu_{s,t}(\lambda))^2 c(ds)$$

$$+ \int_0^r 0 \int_0^{\infty} e^{-X(s-\nu_{s,t}(\lambda))} X(s-\nu_{s,t}(\lambda)) K(v_{s,t}(\lambda), z) m(ds, dz) + Z(t)$$

$$- \frac{1}{2} \int_0^r e^{-X(s-\nu_{s,t}(\lambda))} v_{s,t}(\lambda)^2 C(ds) - \sum_{s \in [0, r] \cap J_m} [e^{-X(s-\nu_{s,t}(\lambda))} - e^{-X(s-\nu_{s-}(\lambda))}$$

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Taking the conditional expectation in (4.13) we obtain

\[ t \mathbb{P}\left[ dZ + 1 + C(s) \right] \]

where

\[ Z(t) = -\int_0^t e^{-X(s-t)v_{s-t}(\lambda)} v_{s-t}(\lambda) dX(s) - \int_0^t e^{-X(s-t)v_{s-t}(\lambda)} X(s-t)v_{s-t}(\lambda)^2 c(ds) \]

\[ - \int_0^t \int_0^\infty e^{-X(s-t)v_{s-t}(\lambda)} X(s-t)K(v_{s-t}(\lambda), z) m(ds, dz) \]

\[ + \frac{1}{2} \int_0^t e^{-X(s-t)v_{s-t}(\lambda)} v_{s-t}(\lambda)^2 C(ds) + \sum_{s \in (0, t] \cap J_m} \left[ e^{-X(s-v_{s-t}(\lambda)} - e^{-X(s-t)v_{s-t}(\lambda)} \right] \]

\[ + \left[ \Delta X(s)v_{s-t}(\lambda) + X(s-t)\Delta v_{s-t}(\lambda) \right] \]

\[ + \sum_{s \in (0, t] \cap J_m} e^{-X(s-t)v_{s-t}(\lambda)} \left[ e^{-\Delta X(s)v_{s-t}(\lambda)} - 1 + v_{s-t}(\lambda)\Delta X(s) \right] \]

Taking the conditional expectation in (4.13) we obtain

\[ \mathbb{P}[Z(t) \mid \mathcal{F}_r] = \int_0^r e^{-X(s-t)v_{s-t}(\lambda)} X(s-t)v_{s-t}(\lambda)^2 c(ds) + \text{mart.} \]

\[ + \int_0^r \int_0^\infty e^{-X(s-t)v_{s-t}(\lambda)} X(s-t)K(v_{s-t}(\lambda), z) m(ds, dz) \]

\[ - \frac{1}{2} \int_0^r e^{-X(s-t)v_{s-t}(\lambda)} v_{s-t}(\lambda)^2 C(ds) \]

\[ - \sum_{s \in (0, t] \cap J_m} e^{-X(s-t)v_{s-t}(\lambda)} \left[ \Delta X(s)v_{s-t}(\lambda) + X(s-t)\Delta v_{s-t}(\lambda) \right] \]

\[ - \sum_{s \in (0, t] \cap J_m} e^{-X(s-t)v_{s-t}(\lambda)} \left[ e^{-\Delta X(s)v_{s-t}(\lambda)} - 1 + v_{s-t}(\lambda)\Delta X(s) \right] \]

\[ = \int_0^r e^{-X(s-t)v_{s-t}(\lambda)} X(s-t)v_{s-t}(\lambda)^2 c(ds) + \text{mart.} \]

\[ + \int_0^r \int_0^\infty e^{-X(s-t)v_{s-t}(\lambda)} X(s-t)K(v_{s-t}(\lambda), z) m_c(ds, dz) \]

\[ - \frac{1}{2} \int_0^r e^{-X(s-t)v_{s-t}(\lambda)} v_{s-t}(\lambda)^2 C(ds) \]

\[ - \sum_{s \in (0, t] \cap J_m} e^{-X(s-t)v_{s-t}(\lambda)} \Delta X(s)v_{s-t}(\lambda) \]

\[ - \sum_{s \in (0, t] \cap J_m} e^{-X(s-t)v_{s-t}(\lambda)} \left[ e^{-\Delta X(s)v_{s-t}(\lambda)} - 1 + v_{s-t}(\lambda)\Delta X(s) \right] , \]

Then the uniqueness of canonical decompositions of martingales yields

\[ dC(s) = 2X(s-)c(ds), \]

\[ 1_{J_m}(s) \hat{N}_0(ds, dz) = X(s-)m_c(ds, dz). \]

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By El Karoui and Méliéard (1990, Theorem III.6), on an extension of the original probability space there exists a Gaussian white noise \( W(ds, du) \) on \((0, \infty)^2\) with intensity \(2c(ds)du\) such that

\[
M(t) = \int_0^t \int_0^{X(s-)} W(ds, du).
\]

By Kabanov et al. (1981, Theorem 1), on a further extension of the original probability space we can define a Poisson random measure \( M_c(ds, dz, du) \) with intensity \( m_c(ds, dz)du\) so that

\[
\int_0^t \int_0^{\infty} z1_{J_m}(s)\check{N}_0(ds, dz) = \int_0^t \int_0^{\infty} \int_0^{X(s-)} z\check{M}_c(ds, dz, du);
\]

see also El Karoui and Lepeltier (1977). By (4.12) we see the process a.s. makes a jump at time \( s \in J_m \) with the representation:

\[
\Delta X(s) = \int_\mathbb{R} z\check{N}_0(\{s\}, dz).
\]

From Proposition 4.3 it follows that

\[
\mathbb{P}(e^{-\lambda \Delta X(s)}|\mathcal{F}_{s-}) = e^{(\lambda - v_{s-,s}(\lambda))X(s-)}, \quad \lambda \geq 0,
\]

where

\[
\lambda - v_{s-,s}(\lambda) = \int_0^{\infty} (e^{-\lambda z} - 1 + \lambda z)m_d(\{s\}, dz).
\]

By Proposition 4.3 we can make another extension of the probability space and define a Poisson random measure \( N_s(dz, du) \) on \((0, \infty)^2\) with intensity \( m_d(s, dz)du\) such that \( N_s \) is independent of \( \mathcal{F}_{s-} \) and

\[
\Delta X(s) = \int_0^{\infty} \int_0^{X(s-)} z\check{N}_d(dz, du).
\]

Let \( M_d(ds, dz, du) \) be the random measure on \((0, \infty)^3\) defined by

\[
M_d((0, t] \times A) = \sum_{s \in (0, t] \cap J_m} N_s(A), \quad t \geq 0, \ A \in \mathcal{B}((0, \infty)^2).
\]

Then \( M_d(ds, dz, du) \) is a Poisson random measure with intensity \( m_d(ds, dz)du\). From (4.12) we see \( \{X(t) : t \geq 0\} \) is a solution to (4.1) with \( b = 0 \).

Step 2. Let us show that the noises \( W, M_c \) and \( M_d \) constructed in the first step are independent. Let \( g \) be a positive continuous function on \((0, \infty)\) and let \( f \) and \( h \) be positive continuous functions on \((0, \infty)^2\). We assume all those functions have compact supports. For \( t \geq 0 \) write

\[
F_t(g, f, h) = \int_0^t \int_0^{\infty} g(u)W(ds, du) + \int_0^t \int_0^{\infty} \int_0^{\infty} f(z, u)M_c(ds, dz, du)
\]

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By Itô’s formula, we have

\[
\begin{align*}
\exp(-F_t(g,f,h)) &= 1 + \int_0^t \exp(-F_s(g,f,h)) c(ds) \int_0^\infty g(u)^2 du + \text{mart.} \\
&+ \int_0^t \int_0^\infty \int_0^\infty \left[ \exp(-F_s(g,f,h)) - \exp(-F_s(g,f,h)) \right] M_c(ds,dz,du) \\
&+ \sum_{s \in (0,t] \cap J_m} \left[ \exp(-F_s(g,f,h)) - \exp(-F_s(g,f,h)) \right] \\
&= 1 + \int_0^t \exp(-F_s(g,f,h)) c(ds) \int_0^\infty g(u)^2 du + \text{mart.} \\
&+ \int_0^t \int_0^\infty \int_0^\infty \exp(-F_s(g,f,h)) (e^{-f(z,u)} - 1) M_c(ds,dz,du) \\
&+ \sum_{s \in (0,t] \cap J_m} \exp(-F_s(g,f,h)) (e^{-M_d(s,h)} - 1),
\end{align*}
\]

where

\[
M_d(s,h) = \int_0^\infty \int_0^\infty h(z,u) M_d(s\{s\},dz,du)
\]

and

\[
m_d(s,h) = \int_0^\infty m_d(s\{s\},dz) \int_0^\infty (1 - e^{-h(z,u)}) du.
\]

Then we can use Proposition 2.1 to obtain

\[
\mathbb{P}[\exp(-F_t(g,f,h))] = \exp \left\{ \int_0^t c(ds) \int_0^\infty g(u)^2 du - \sum_{s \in (0,t] \cap J_m} m_d(s,h) \right. \\
- \left. \int_0^t \int_0^\infty m_c(ds,dz) \int_0^\infty (1 - e^{-f(z,u)}) du \right\}.
\]

That gives the independence of $W$, $M_c$ and $M_d$.

**Step 3.** Consider the case where (4.3) holds. Let $(v_r,t)_{t \geq r}$ be the conservative cumulant semigroup defined by (3.7). Let $\zeta$ be the càdlàg function on $[0,\infty)$ such that $\zeta_c(t) = -b_c(t)$ and $\Delta \zeta(t) = \log[1 - \Delta b(t)]$ for every $t \geq 0$. By Theorem 3.6 we can define another
cumulant semigroup \((u_{r,t})_{t \geq r}\) by (3.16) and \(r \mapsto u_{r,t}(\lambda)\) is the unique bounded positive solution to

\[
u_{r,t}(\lambda) = \lambda - \int_r^t u_{s,t}(\lambda)^2 e^{-\zeta(s)c(ds)} - \int_r^t \int_0^\infty K(u_{s,t}(\lambda), z)e^{\zeta(s)m(ds, e^{\zeta(s)}dz)}.
\]

Let \(W(ds, du)\) and \(M(ds, dz, du)\) be given as in the introduction. One can see that \(W_0(ds, du) := e^{-\zeta(s)}W(ds, e^{\zeta(s)}du)\) is a Gaussian white noise on \((0, \infty)^2\) with intensity \(2e^{-\zeta(s)}c(ds)du\) and \(M_0(ds, dz, du) := M(ds, e^{\zeta(s)}dz, e^{\zeta(s)}du)\) is a Poisson random measure on \((0, \infty)^3\) with intensity \(e^{\zeta(s)}m(ds, e^{\zeta(s)}dz)du\). By Proposition 4.2 and the first step of the proof, we can construct a CBVE-process with cumulant semigroup \((u_{r,t})_{t \geq r}\) by the pathwise unique solution to

\[
Z(t) = X(0) + \int_0^t \int_0^{Z(s-)} W_0(ds, du) + \int_0^t \int_0^\infty \int_0^{Z(s-)} z\tilde{M}_0(ds, dz, du).
\]

It is easy to see that \(t \mapsto X(t) := e^{\zeta(t)}Z(t)\) is a CBVE-process with cumulant semigroup \((v_{r,t})_{t \geq r}\). By integration by parts we have

\[
X(t) = X(0) + \int_0^t Z(s- \text{de}^{\zeta(s)}) + \int_0^t e^{\zeta(s)}dZ(s)
\]

\[
= X(0) - \int_0^t e^{\zeta(s)}Z(s-)b(ds) + \int_0^t \int_0^{Z(s-)} e^{\zeta(s)}W_0(ds, du) + \int_0^t \int_0^\infty \int_0^{Z(s-)} e^{\zeta(s)}z\tilde{M}_0(ds, dz, du)
\]

\[
= X(0) - \int_0^t X(s-)b(ds) + \int_0^t \int_0^{X(s-)} e^{\zeta(s)}W_0(ds, e^{-\zeta(s)}du) + \int_0^t \int_0^\infty \int_0^{X(s-)} z\tilde{M}_0(ds, e^{-\zeta(s)}dz, e^{-\zeta(s)}du).
\]

Then \(\{X(t) : t \geq 0\}\) solves (4.11).

**Step 4.** Let us consider the general case. By Proposition 4.2 and the second step of the proof, for each \(k \geq 1\) we can construct a CBVE-process \(\{X_k(t) : t \geq 0\}\) by the pathwise unique solution to:

\[
X(t) = X(0) + \int_0^t \int_0^{X(s-)} W(ds, du) + \int_0^t \int_0^\infty \int_0^{X(s-)} (z \wedge k)\tilde{M}(ds, dz, du)
\]

\[
- \int_0^t X(s-)b(ds) - \int_0^t \int_k^\infty X(s-)(z-k)m(ds, dz).
\]

The cumulant semigroup \((v^{(k)}_{r,t})_{t \geq r}\) of \(\{X_k(t) : t \geq 0\}\) is defined by:

\[
v_{r,t}(\lambda) = \lambda - \int_r^t v_{s,t}(\lambda)^2 c(ds) - \int_r^t \int_0^\infty K(v_{s,t}(\lambda), z \wedge k)m(ds, dz)
\]

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We can rewrite (4.14) into the equivalent form:

\[
- \int_r^t v_{s,t}(\lambda)b(ds) - \int_r^t \int_k^\infty v_{s,t}(\lambda)(z-k)m(ds,dz).
\] (4.15)

This is obtained from (4.1) by modifying the magnitudes of the large jumps. Let \( \zeta = 0 \) and for \( i \geq 0 \) inductively define

\[
\zeta_{i+1,k} = \inf \left\{ t > 0 : \int_{\zeta_i,k}^{\zeta_{i+1,k}} \int_k^\infty \int_0^\infty M(ds,dz,du) \geq 1 \right\}.
\]

From (4.16) it is easy to see that \( X_{k+1}(t) = X_k(t) \) for \( 0 \leq t < \zeta_{1,k} \) and \( X_{k+1}(\zeta_{1,k}) \geq X_k(\zeta_{1,k}) \). By applying Proposition 4.2 successively at the stopping times \( \eta_{n,k} := \sum_{i=1}^n \zeta_{i,k} \), \( n \geq 1 \) we infer that \( X_{k+1}(t) \geq X_k(t) \) for every \( t \geq 0 \). For \( r, x \geq 0 \) let \( \{ X_k(r, x, t) : t \geq r \} \) be the pathwise unique solution to:

\[
X(t) = x + \int_r^t \int_0^\infty \int_0^\infty W(ds,du) + \int_r^t \int_0^\infty \int_0^\infty \int_0^\infty M(ds,dz,du)
\]

\[
- \int_r^t X(s-)b(ds) - \int_r^t \int_k^\infty X(s-)(z-k)m(ds,dz).
\]

By the preceding arguments we have \( X_{k+1}(r, x, t) \geq X_k(r, x, t) \), which implies

\[
v_{r,t}^{(k+1)}(\lambda) = -\log P \exp \{-\lambda X_{k+1}(r, 1, t)\}
\]

\[
\geq -\log P \exp \{-\lambda X_k(r, 1, t)\} = v_{r,t}^{(k)}(\lambda).
\]

From (4.15) we see that \( r \mapsto v_{r,t}(\lambda) := \lim_{t\to\infty} v_{r,t}^{(k)}(\lambda) \) is the unique bounded positive solution to (3.7). Observe that \( \zeta_{1,k} \geq \tau_{k/2} := \inf \{ t \geq 0 : X_k(t) \geq k/2 \} \). By Proposition 4.1 it is easy to show that \( \tau_{k/2} \to \infty \) as \( k \to \infty \). Then \( \{ X_k(t) : t \geq 0 \} \) converges increasingly as \( k \to \infty \) to a càdlàg process \( \{ X(t) : t \geq 0 \} \) which is a CBVE-process with cumulant semigroup \( \{ v_{r,t} \}_{t \geq r} \). From (4.14) we infer that \( \{ X(t) : t \geq 0 \} \) is a solution to (4.14). The pathwise uniqueness of the solution holds by Proposition 4.2.

5 Extensions to the general case

In this section, we extend the results established in the last two sections to the general equations (1.3) and (1.5). Suppose that \( (b_1, c, m) \) are admissible parameters defined as in the introduction.
Proposition 5.1 For any $\lambda > 0$ and $t \geq 0$ the uniqueness of bounded positive solutions holds for (1.3).

Proof. Suppose that both $r \mapsto v_{r,t}(\lambda)$ and $r \mapsto w_{r,t}(\lambda)$ are bounded positive solutions to (1.3). Then $v_{r,t}(\lambda) \wedge w_{r,t}(\lambda) \geq l_{0,t}(\lambda) > 0$ by Corollary 2.2 and Proposition 2.3. From (1.3) it follows that

$$|v_{r,t}(\lambda) - w_{r,t}(\lambda)| \leq \int_r^t |v_{s,t}(\lambda) - w_{s,t}(\lambda)||b_1||(ds) + \int_r^t |v_{s,t}(\lambda)^2 - w_{s,t}(\lambda)^2|c(ds)$$

$$+ \int_r^t \int_0^1 |K(v_{s,t}(\lambda), z) - K(w_{s,t}(\lambda), z)|m(ds, dz)$$

$$+ \int_r^t \int_1^{\infty} |e^{-v_{s,t}(\lambda)z} - e^{-w_{s,t}(\lambda)z}|m(ds, dz)$$

$$\leq \int_r^t |v_{s,t}(\lambda) - w_{s,t}(\lambda)||b_1||(ds) + 2U_{0,t}(\lambda) \int_r^t |v_{s,t}(\lambda) - w_{s,t}(\lambda)|c(ds)$$

$$+ 2U_{0,t}(\lambda) \int_r^t \int_0^1 |v_{s,t}(\lambda) - w_{s,t}(\lambda)|z^2 m(ds, dz)$$

$$+ \int_r^t \int_1^{\infty} |v_{s,t}(\lambda) - w_{s,t}(\lambda)|ze^{-l_{0,t}(\lambda)z}m(ds, dz).$$

By Gronwall’s inequality we have $|v_{r,t}(\lambda) - w_{r,t}(\lambda)| = 0$ for $r \in [0, t]$.

Proof of Theorems 1.1 and 1.3. By Theorem 3.3 for each $k \geq 1$ there is a cumulant semigroup $(v_{r,t}^{(k)})_{t \geq r}$ defined by the integral evolution equation:

$$v_{r,t}(\lambda) = \lambda - \int_r^t v_{s,t}(\lambda)^2c(ds) - \int_r^t \int_0^\infty K(v_{s,t}(\lambda), z \wedge k)m(ds, dz)$$

$$- \int_r^t v_{s,t}(\lambda)b_1(ds) + \int_r^t \int_0^1 v_{s,t}(\lambda)(z \wedge k)m(ds, dz). \quad (5.1)$$

By Theorem 4.5 we can construct a CBVE-process $\{X_k(t) : t \geq 0\}$ with cumulant semigroup $(v_{r,t}^{(k)})_{t \geq r}$ by the pathwise unique solution to:

$$X(t) = X(0) + \int_0^t \int_0^X(s-) W(ds, du) + \int_0^t \int_0^\infty \int_0^X(s-) (z \wedge k)\bar{M}(ds, dz, du)$$

$$- \int_0^t X(s-)b_1(ds) + \int_0^t \int_0^\infty X(s-)(z \wedge k)m(ds, dz).$$

The above stochastic equation is equivalent to

$$X(t) = X(0) + \int_0^t \int_0^X(s-) W(ds, du) + \int_0^t \int_0^1 \int_0^X(s-) z\bar{M}(ds, dz, du)$$

$$- \int_0^t X(s-)b_1(ds) + \int_0^t \int_0^\infty \int_0^X(s-) (z \wedge k)\bar{M}(ds, dz, du). \quad (5.2)$$
Let $\zeta_{1,k}$ and $\tau_{k/2}$ be defined as in the last step of the proof of Theorem 4.5. By the arguments in that proof we have $X_{k+1}(t) = X_k(t)$ for $0 \leq t < \zeta_{1,k}$ and both $\{X_k(t) : t \geq 0\}$ and $(v^{(k)}_{t})_{t \geq r}$ are increasing in $k \geq 1$. By Proposition 2.3 we have $b_{0,t}(\lambda) \leq v^{(k)}_{r,t}(\lambda) \leq U_{0,t}(\lambda)$. Then for $\lambda > 0$ the limit $v_{r,t}(\lambda) := \lim_{k \to \infty} v^{(k)}_{r,t}(\lambda)$ exists and strictly positive. By letting $k \to \infty$ in (5.4) we see $r \mapsto v_{r,t}(\lambda)$ is a solution to (1.3). The uniqueness of the solution is guaranteed by Proposition 5.1. Clearly, the family $(v_{r,t})_{t \geq r}$ is a cumulant semigroup. It is easy to see that $\lim_{k \to \infty} \zeta_{1,k} = \tau_{\infty} := \lim_{k \to \infty} \tau_{k/2}$. Let $\{X(t) : t \geq 0\}$ be the càdlàg process such that $X(t) = X_k(t)$ for $0 \leq t < \zeta_{1,k}$ and $X(t) = \infty$ for $t \geq \tau_{\infty}$. Then $\{X(t) : t \geq 0\}$ is a CBVE-process with cumulant semigroup $(v_{r,t})_{t \geq r}$. From (5.2) we see that $\{X(t) : t \geq 0\}$ is a solution to (1.5). The pathwise uniqueness for (1.5) follows from that for (5.2).

Proof of Theorem 1.2. It is easy to see that $r \mapsto v_{r,t}(0)$ is indeed a bounded positive solution to (1.3) with $\lambda = 0$. Suppose that $r \mapsto u_{r,t}(0)$ is another positive solution to (1.3) with $\lambda = 0$ and $u_{r,t}(0) > 0$ for some $r \in [0, t]$. Let $t_0 = \inf \{r \in [0, t] : u_{r,t}(0) = 0\}$. We clearly have $u_{r,t}(0) = 0$ for $r \in [t_0, t]$, and hence $u_{t_0,t}(0) = 0$ by (1.3). Then for any $\lambda > 0$ we can choose $r_0 \in [0, t_0]$ so that $u_{r,t}(0) \leq b_{0,t}(\lambda) \leq u_{r,t}(\lambda)$ when $r \in [r_0, t_0]$. The definition of $t_0$ yields the existence of some $t_1 \in [r_0, t_0]$ so that $0 < u_{t_1,t}(0) \leq u_{t_1,t}(\lambda)$. For $r \in [0, t_1]$ we see from (1.3) that

$$u_{r,t}(0) = u_{t_1,t}(0) - \int_{r}^{t_1} u_{s,t}(0) a_1(ds) - \int_{r}^{t_1} u_{s,t}(0)^2 c(ds) - \int_{r}^{t_1} \int_{0}^{\infty} K_1(u_{s,t}(0), z) m(ds, dz).$$

By the uniqueness of the solution we have $u_{r,t}(0) = v_{r,t}(u_{t_1,t}(0)) \leq v_{r,t}(u_{t_1,t}(\lambda)) = v_{r,t}(\lambda)$. Then $u_{r,t}(0) \leq v_{r,t}(\lambda)$ for every $r \in [0, t]$, implying $u_{r,t}(0) \leq \lim_{\lambda \downarrow 0} v_{r,t}(\lambda) = v_{r,t}(0)$ for every $r \in [0, t]$. Similarly, one can show $r \mapsto v_{r,t}(\infty)$ is the smallest positive solution to (1.3) with $\lambda = \infty$. 

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