On Baxter’s $Q$ operator of the higher spin XXZ chain at the Razumov-Stroganov point

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Based on the conjecture for the exact eigenvalue of the transfer matrix of the higher half-integer spin XXZ chain at the Razumov-Stroganov point, we evaluate the corresponding Baxter’s $Q$ operator in closed form by solving the $TQ$ equation. The combination of the $Q$ operators on the “right side” and the “wrong side” is shown to produce the hierarchy of functional relations.

1 Introduction

Among the various methods to analyze one-dimensional quantum integrable models, the Bethe ansatz is one of the traditional and most powerful methods. There are also many variants of the Bethe ansatz itself today. The coordinate Bethe ansatz was invented by Bethe himself to diagonalize the Hamiltonian of the Heisenberg XXX chain, leading to the derivation of the Bethe ansatz equation [1]. Later on, alternative techniques to obtain the Bethe ansatz equation developed, such as the algebraic Bethe ansatz (quantum inverse scattering method) [2, 3] and the analytic Bethe ansatz [4, 5]. The spirit of these new types of the Bethe ansatz is to diagonalize the transfer matrix associated with the model, instead of dealing the Hamiltonian directly.

One version of the Bethe ansatz invented by Baxter now called the $Q$ operator method [6] is a way to diagonalize the transfer matrix by making gauge transformation with the help of the $Q$ operator. The so-called $TQ$ equation is essentially the Bethe ansatz equation. By setting the spectral parameter to the zeros of the $Q$ operator turns the $TQ$ equation to the Bethe ansatz equation, which implies the zeros of the $Q$ operator are the Bethe roots, i.e., the $Q$ operator contains the information of the eigenstates of the model. Among the works on the $TQ$ equation and the $Q$ operator [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27], there has been a large advance on the representation theoretical construction for the last twenty years. One of its motivations was to apply this method originally developed for the integrable spin chains to conformal field theory. It also gained interest recently from the particle physics to check the validity of the AdS/CFT correspondence, the conjectural duality between a string theory with gravity on the anti-de Sitter space and a gauge theory on its boundary, from the gauge theory side.

Progresses in the $TQ$ equation and the $Q$ operator on the XXZ chain have also been made. One of them is the speciality at some point of the anisotropy parameter. Taking the anisotropy parameter to the special point now called the Razumov-Stroganov point, the transfer matrix eigenvalue corresponding to the ground state was conjectured to be expressed in an particularly simple form using the exact form of the transfer matrix [28], the $Q$ operator was expressed in an explicit form [29, 30, 31]. Furthermore, various exact quantities on the groundstate components and correlation functions of the XXZ chain were conjectured to be related with those of the alternating sign matrix [32, 33, 34]. The symmetries of the partition function of the six vertex model was also clarified [35, 36].

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and there is an extensive study of the related loop models by making using of the \(q\)-KZ equations \([37, 38, 39, 40, 41]\) for example.

In this paper, we generalize some of the analysis previously done for the spin-1/2 XXZ chain at the Razumov-Stroganov point to higher spins. Contrary to the spin-1/2 case, not is much known for the higher spins at the Razumov-Stroganov point. Several works of them are: the exact eigenvalue of the transfer matrix of the higher spin XXZ chain \([12]\), the sum rule of the fused loop model \([43]\), some groundstate components for the spin-1 XXZ chain \([44]\) and the Macdonald polynomial description of the partition function of the fused vertex model \([45]\). Starting from the conjectured eigenvalue of the transfer matrix \([42]\), we investigate the properties of the \(TQ\) operator in closed form by use of the interpolation formula, generalizing the case for the spin-1/2 XXZ chain with \(M\) quantum space is the anisotropy parameter associated with the XXZ chain \((\eta)\). Furthermore, the combination of the two \(TQ\) operator in closed form by use of the interpolation formula, generalizing the case for the spin-1/2 XXZ chain \([40]\). Furthermore, the combination of the two \(TQ\) operators into the form of discrete Wronskian is shown to produce a hierarchy of functional relations \([46, 47, 48, 49]\).

This paper is organized as follows. In the next section, we review the Baxter’s \(TQ\) equation of the integrable higher spin-\(s\) XXZ chain with \(M\) sites under the periodic boundary condition

\[
T(u)Q(u) = \sh^M(u + s\eta)Q(u - \eta) + \sh^M(u - s\eta)Q(u + \eta),
\]

where \(T(u)\) is the eigenvalue of the transfer matrix \(\hat{T}(u)\) whose auxiliary space has spin-1/2 and the quantum space is the \(M\)-fold tensor product of spin-\(s\) spaces. \(u\) is the spectral parameter and \(\eta\) is the anisotropy parameter associated with the XXZ chain \((\eta = 0\) corresponds to the isotropic XXX point). \(Q(u)\) is the eigenvalue of the \(Q\) operator \(Q(u)\) \([5]\) which was introduced to diagonalize the transfer matrix. \(Q(u) = \prod_{j=1}^{p} \sh(u - u_j)\) encodes the information of the eigenstate of the transfer matrix \(\hat{T}(u)\). Indeed, setting the spectral parameter \(u\) to \(u = u_j\), the \(TQ\) equation \((1)\) reduces to the Bethe ansatz equation of the integrable higher spin XXZ chain

\[
\left(\frac{\sh(u_j + s\eta)}{\sh(u_j - s\eta)}\right)^M = \prod_{j < k} \sh(u_j - u_k + \eta) \prod_{k < j} \sh(u_j - u_k - \eta),
\]

which implies that the parameters \(\{u_j\}\) correspond to the Bethe roots. Namely, the \(TQ\) equation implies the Bethe ansatz equation.

For the half-integer spin \(s = L/2\) \((L = 1, 3, 5, \cdots)\) XXZ chain with odd number of total sites \(M = 2N + 1\) \((N = 1, 2, 3, \cdots)\), the transfer matrix was found to have simple eigenvalues at the so-called Razumov-Stroganov point \(\eta = -i(L + 1)/2(L + 2)\). In the sector of \(p\) Bethe roots where \(NL \leq p \leq (N + 1)L\) (in the sector of total spin \(S^z = ML/2 - p\)), the transfer matrix was conjectured to have an exact eigenvalue in the following form \([12]\) (obtained by setting \(N_k = 0, M = 0, \ell = 1\) in

\[
\]
Ref. 42)

\[ T(u) = 2\text{ch} \left( \frac{(L + 1)(ML - 2p)\pi i}{2(L + 2)} \right) \text{sh}^M(u), \]  

or simply \( T(u) = 2\text{ch}(\eta S^z)\text{sh}^M(u) \). We examine the eigenvalue of the \( Q \) operator corresponding to the eigenvalue \( \text{e}^{\pi i/(L + 2)} \) of the transfer matrix in two ways. First, we make an analysis by expanding the \( Q \) operator in terms of the symmetric polynomials of the Bethe roots, regarding them as unknown parameters to be solved. For simplicity, we consider the sector \( p = N + M(L - 1)/2 \) \((S^z = 1/2)\). The other sectors \( NL \leq p \leq (N + 1)L \) can be examined in the same way.

We introduce \( z = \exp(2u), z_j = \exp(2u_j), q = \exp(\pi i/(L + 2)) \) and redefine the eigenvalue of the \( Q \) operator as

\[ Q(z) = \prod_{j=1}^{p} (z - z_j), \]  

for convenience. The \( TQ \) equation \( \text{(1)} \) can be rewritten as

\[ -2\text{ch} \left( \frac{(L + 1)\pi i}{2(L + 2)} \right) (z - 1)^{2N+1} \prod_{j=1}^{p} (z - z_j) \]

\[ + q^{-(L+1)/2}(z - q^2)^{2N+1} \prod_{j=1}^{p} (z - q^2 z_j) + q^{(L+1)/2}(z - q^{-2})^{2N+1} \prod_{j=1}^{p} (z - q^{-2} z_j) = 0. \]  

Making the expansion of the product of polynomials in terms of the symmetric polynomials

\[ \prod_{j=1}^{K} (z - z_j) = \sum_{j=1}^{K} (-1)^j z^j e_j, \]  

\[ e_j = \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq K} z_{i_1}z_{i_2}\cdots z_{i_j} \ (j = 1, 2, \cdots, K), \ e_0 = 1, \]  

\[ (z - 1)^K = \sum_{j=0}^{K} (-1)^j z^j \binom{K}{j}, \]

and inserting into \( \text{(3)} \), we have

\[ 2N+1 \sum_{k=0}^{p} (-1)^j z^{2N+1} (2N+1)_{j} \binom{2N+1}{k} \left\{ -\text{ch} \left( \frac{(L + 1)\pi i}{2(L + 2)} \right) + \text{ch} \left( \frac{2\pi i}{L + 2} \left( j + k - \frac{L + 1}{4} \right) \right) \right\} = 0. \]  

Changing the summation from \( j \) and \( k \) to \( j \) and \( \ell = j + k \), one gets

\[ 2N+1 \sum_{\ell=0}^{2N+1} \left\{ -\text{ch} \left( \frac{(L + 1)\pi i}{2(L + 2)} \right) + \text{ch} \left( \frac{2\pi i}{L + 2} \left( \ell - \frac{L + 1}{4} \right) \right) \right\} \sum_{j=\max(0,\ell-2N-1)}^{\min(p,\ell)} (2N+1)_{\ell-j} e_j = 0. \]  

Since this equation must hold for arbitrary \( z \), each coefficient of \( z^{2N+1} \) has to be zero, leading to

\[ \left\{ -\text{ch} \left( \frac{(L + 1)\pi i}{2(L + 2)} \right) + \text{ch} \left( \frac{2\pi i}{L + 2} \left( \ell - \frac{L + 1}{4} \right) \right) \right\} \sum_{j=\max(0,\ell-2N-1)}^{\min(p,\ell)} (2N+1)_{\ell-j} e_j = 0. \]
To obtain the $Q$ operator means to evaluate the symmetric polynomials of the Bethe roots, and the problem reduces to solving the linear equations. Note we have $p$ parameters $e_j$ ($j = 1, 2, \cdots, p$) to be solved. Taking into account the factor

$$-\text{ch} \left( \frac{(L + 1)\pi i}{2(L + 2)} \right) + \text{ch} \left( \frac{2\pi i}{L + 2} \left( \frac{\ell}{4} - \frac{L + 1}{4} \right) \right), \quad (12)$$

in front of the equation (11), we find the equations (11) with $\ell = (L + 2)k, (L + 2)k + (L + 1)/2$ ($k = 0, 1, \cdots, N$) are automatically satisfied. Moreover, this factor depends only on $\ell$, and we find the problem of computing the $Q$ operator eigenvalue is to solve the set of $p$ linear equations

$$\sum_{j=\min(0,\ell-2N-1)}^{\min(p,\ell)} (2N + 1) (\ell - j) e_j = 0, \quad (13)$$

for $\ell = 0, 1, \cdots, N(L + 2) + (L + 1)/2$ ($\ell \neq (L + 2)k, (L + 2)k + (L + 1)/2$ ($k = 0, 1, \cdots, N$)). The number of linear equations is exactly $p$, the same with that of the parameters to be solved. Although the problem of linear dependence remains, many examples convince us that the equations are linearly independent. For example, for $L = 3$ and $N = 3$, the set of linear equations are

$$7 + e_1 = 0,$$
$$35 + 21e_1 + 7e_2 + e_3 = 0,$$
$$35 + 35e_1 + 21e_2 + 7e_3 + e_4 = 0,$$
$$7 + 21e_1 + 35e_2 + 35e_3 + 21e_4 + 7e_5 + e_6 = 0,$$
$$e_1 + 7e_2 + 21e_3 + 35e_4 + 35e_5 + 21e_6 + 7e_7 + e_8 = 0,$$
$$e_2 + 7e_3 + 21e_4 + 35e_5 + 35e_6 + 21e_7 + 7e_8 + e_9 = 0,$$
$$e_4 + 7e_5 + 21e_6 + 35e_7 + 35e_8 + 21e_9 + 7e_{10} = 0,$$
$$e_6 + 7e_7 + 21e_8 + 35e_9 + 35e_{10} = 0,$$
$$e_7 + 7e_8 + 21e_9 + 35e_{10} = 0,$$
$$e_9 + 7e_{10} = 0.$$

Solving these equations, we easily have

$$Q(z) = \prod_{j=1}^{10} (z - z_j) = \sum_{j=0}^{10} (-1)^j z^{10-j} e_j$$

$$= z^{10} + 7z^9 + \frac{609}{26} z^8 + \frac{1351}{26} z^7 + \frac{1064}{13} z^6 + \frac{1229}{13} z^5$$
$$+ \frac{1064}{13} z^4 + \frac{1351}{26} z^3 + \frac{609}{26} z^2 + 7z + 1. \quad (14)$$

We can see that the coefficients of the $Q$ operator are all positive. Another interesting point we observe is that all the coefficients of the linear equations have the form $\binom{2N+1}{j} (j = 0, 1, \cdots, 2N + 1)$, which is independent of the spin-$L/2$ but depends on the total number of sites $M = 2N + 1$. The approach made in this section can solve any $Q$ operator in principle since the problem reduces to solving the linear equations whose number is the same as that of the parameters to be solved. This approach might also be useful for making a guess on the eigenvalue of the transfer matrix. In general, the number of linear equations is larger than that of the parameters to be solved. However, by taking appropriate eigenvalue which can produce factors such as (12), the number of linear equations reduces to that of the parameters, which can be solved in a unique way. So making a guess on the transfer matrix eigenvalue can be reduced to that on the set of factors like (12) which give the correct number of zeros. In the next section, we evaluate the $Q$ operator in a different way.
3 Spin-3/2

We can evaluate the $Q$ operator in closed form by using the interpolation formula which has been done for the spin-1/2 XXZ chain \cite{29}. We first consider the spin-3/2 ($S = 3/2$) case. Since this is the simplest nontrivial and illustrative example to treat the $Q$ operators of the higher half-integer spin XXZ chain. The other cases can be treated in the same way. First, let us set $f(u) = \text{sh}(2N+1)(u) \prod_{j=1}^{3N+1} \text{sh}(u - u_j)$. The $TQ$ equation can be written as

$$-2 \text{ch} \left( \frac{2 \pi i}{5} \right) f(u) + f \left( u + \frac{4}{5} \pi i \right) + f \left( u + \frac{6}{5} \pi i \right) = 0. \quad (15)$$

$f(u)$ is a trigonometric polynomial of degree $5N + 2$ and satisfies $f(u + \pi i) = (-1)^N f(u)$. We make an assumption on $Q(u) = \prod_{j=1}^{3N+1} \text{sh}(u - u_j)$ that it is an even function of $u$ and therefore $f(u)$ is an odd function. In terms of the symmetric polynomials $e_k$ of $z_j$, this is equivalent to the assumption $e_{p-j} = (-1)^p e_j$. The fact that this assumption holds is supported by solving a set of linear equations \cite{13} in the last section. Let us make some comment on this assumption. We can show $e_j = e_j |_{z_k \rightarrow z_k^{-1}}$ which follows from the $z \leftrightarrow z^{-1}$, $z_j \leftrightarrow z_j^{-1}$ invariance of the $TQ$ equation. Namely, rewriting (5) as

$$-2 \text{ch} \left( \frac{(L+1)\pi i}{2(L+2)} \right) (z^{-1} - 1)^{2N+1} \prod_{j=1}^{p} (z^{-1} - z_j^{-1})$$

$$+ q^{-(L+1)/2} (z^{-1} - q^{2})^{2N+1} \prod_{j=1}^{p} (z^{-1} - q^{2} z_j^{-1}) + q^{(L+1)/2} (z^{-1} - q^{-2})^{2N+1} \prod_{j=1}^{p} (z^{-1} - q^{-2} z_j^{-1}) = 0. \quad (16)$$

we find the symmetric polynomials $e_j |_{z_k \rightarrow z_k^{-1}}$ satisfy exactly the same set of linear equations as $e_j$ by performing the same analysis in the last section. Using this property, we find the problem of showing the assumption $e_{p-j} = (-1)^p e_j$ reduces to showing just only one of them $e_p = (-1)^p$ since $e_p^{-1} e_j = e_{p-j} |_{z_k \rightarrow z_k^{-1}} = e_{p-j}$.

From the above properties, we find $f(u)$ has the following form

$$f(u) = \sum_{j=-5N/2N_{\text{even}}}^{1} a_j \text{sh}(5N + 2j)u. \quad (17)$$

Inserting this expression into (15), one gets

$$2 \sum_{j=-5N/2N_{\text{even}}}^{1} \left\{ - \text{ch} \left( \frac{2 \pi i}{5} \right) + \text{ch} \left( \frac{2 \pi j i}{5} \right) \right\} a_j \text{sh}(5N + 2j)u = 0. \quad (18)$$

We easily find

$$- \text{ch} \left( \frac{2 \pi i}{5} \right) + \text{ch} \left( \frac{2 \pi j i}{5} \right) \neq 0, \quad j = 0, 2, 3 \ (\text{mod } 5), \quad (19)$$

which implies $a_j = 0$ for $j = 0, 2, 3 \ (\text{mod } 5)$, and the number of coefficients to be determined reduces. The function $f(u)$ can be expressed as

$$f(u) = \sum_{k=0}^{N/2} \alpha_k \text{sh}(5N + 2 - 10k)u + \sum_{k=0}^{N/2-1} \beta_k \text{sh}(5N - 2 - 10k)u, \quad (20)$$
for $N$ even and
\[ f(u) = \sum_{k=0}^{(N-1)/2} \alpha_k \text{sh}(5N + 2 - 10k)u + \sum_{k=0}^{(N-1)/2} \beta_k \text{sh}(5N - 2 - 10k)u, \tag{21} \]

for $N$ odd. From now on, we only consider the case $N$ even. The case $N$ odd can be treated in the same way.

The function $f(u)$ is proportional to $\text{sh}^M u$ and therefore
\[ \left( \frac{\partial^\mu}{\partial u^\mu} f(u) \right) \bigg|_{u=0} = 0, \quad \mu = 0, 1, \cdots, 2N, \tag{22} \]

follows. Writing down this condition explicitly, we have
\[ \frac{N}{2} \sum_{k=0}^{N/2-1} \alpha_k (5N + 2 - 10k)^{2\mu+1} + \frac{N}{2} \sum_{k=0}^{N/2-1} \beta_k (5N - 2 - 10k)^{2\mu+1} = 0, \quad \mu = 0, 1, \cdots, N - 1. \tag{23} \]

This condition [23] is equivalent to the one that the parameters $\alpha_k$ ($k = 0, 1, \cdots, N/2$), $\beta_k$ ($k = 0, 1, \cdots, N/2 - 1$) must satisfy the relation
\[ \frac{N}{2} \sum_{k=0}^{N/2-1} \alpha_k (5N + 2 - 10k)W((5N + 2 - 10k)^2) + \frac{N}{2} \sum_{k=0}^{N/2-1} \beta_k (5N - 2 - 10k)W((5N - 2 - 10k)^2) = 0, \tag{24} \]

for any polynomial $W(x)$ of degree equal to or less than $N - 1$.

We now consider the more general problem [29]: given a set of different complex numbers $\{x_1, x_2, \ldots, x_K\}$, is there any set of complex numbers $\{\gamma_1, \gamma_2, \ldots, \gamma_K\}$ ($\gamma_i \neq 0$ for some $i$) satisfying the relation
\[ \sum_{k=1}^{K} \gamma_k W(x_k) = 0, \tag{25} \]

for any polynomial $W(x)$ of degree equal to or less than $K - 2$? The answer [29] is that up to an overall constant $C$, there is a unique set of complex numbers $\gamma_k$ ($k = 1, 2, \cdots, K$) which can be expressed as
\[ \gamma_k = \frac{C}{\prod_{j=1, j \neq k}^{K} (x_k - x_j)}. \tag{26} \]

This formula comes from the equality
\[ \sum_{k=1}^{K} \frac{x_k^\ell}{\prod_{j=1, j \neq k}^{K} (x_k - x_j)} = 0, \quad 0 \leq \ell \leq K - 2, \tag{27} \]

which can be shown as follows [50, 51]. Consider a polynomial $V(x) = \prod_{j=1}^{K} (x - x_j)$ and the following integral
\[ \frac{1}{2\pi i} \oint \frac{z^{\ell+1}dz}{V(z)(z-x)}, \quad 0 \leq \ell \leq K - 2, \tag{28} \]

where the contour surrounds $x$ and all the zeros of $V(x)$. Taking the residues, one gets
\[ \frac{1}{2\pi i} \oint \frac{z^{\ell+1}dz}{V(z)(z-x)} = x^{\ell+1} V(x) + \sum_{k=1}^{K} \frac{x_k^{\ell+1}}{V'(x_k)(x_k - x)}. \tag{29} \]
On the other hand, regarding the contour as a circle at infinity, the integral becomes zero for $0 \leq \ell \leq K - 2$, and we have

$$\frac{x^\ell + 1}{V(x)} + \sum_{k=1}^K \frac{\beta_k^\ell + 1}{V(x_k)(x_k - x)} = 0. \quad (30)$$

Taking $x = 0$, one has $\mathbf{27}$.

Applying the above fact to the special case $\mathbf{24}$ which we consider, one gets

$$\alpha_0(5N + 2 - 10k) = \frac{C}{\prod_{j \neq k}^{N/2} \{(5N + 2 - 10k)^2 - (5N + 2 - 10j)^2\} \prod_{j=0}^{N/2 - 1} \{(5N + 2 - 10k)^2 - (5N + 2 - 10j)^2\}}.$$

$$\beta_k(5N - 2 - 10k) = \frac{C}{\prod_{j=0}^{N/2} \{(5N - 2 - 10k)^2 - (5N + 2 - 10j)^2\} \prod_{j \neq k}^{N/2 - 1} \{(5N - 2 - 10k)^2 - (5N + 2 - 10j)^2\}}, \quad (31)$$

with some factor $C$, which can be determined by the coefficient of $\exp(5u)$ of $\mathbf{f(u)}$

$$\alpha_0 = \prod_{j=1}^{3N+1} e^{-w_j} \sum_{k=0}^{2N+1} \left( \prod_{j=1}^{3N+1} e^{w_j} \right) \alpha_k \left( \prod_{j=1}^{3N+1} e^{w_j} \right) \alpha_k = \frac{(-1)^k}{2^{5N+1}} \binom{N}{k} \frac{2 + 5j}{2 - 5k + 5j}, \quad (33)$$

and the function $\mathbf{f(u)}$ has the following form

$$\left( \prod_{j=1}^{3N+1} e^{w_j} \right) f(u) = \sum_{k=0}^{N/2} \left( \prod_{j=1}^{3N+1} e^{w_j} \right) \alpha_k \left( \prod_{j=1}^{3N+1} e^{w_j} \right) \alpha_k = \sum_{k=0}^{N/2 - 1} \left( \prod_{j=1}^{3N+1} e^{w_j} \right) \alpha_k \left( \prod_{j=1}^{3N+1} e^{w_j} \right) \alpha_k.$$

The expression $\mathbf{35}$ leads to the following expression for the $\mathbf{Q}$ operator $\mathbf{34}$ for $N$ even

$$Q(z) = (z - 1)^{-(2N+1)} \left\{ \sum_{k=0}^{N/2} \left( \prod_{j=0}^{N} \frac{2 + 5j}{2 - 5k + 5j} \right) (z^{5N+2-5k} - z^{5k}) \right. \left. + \sum_{k=0}^{N/2 - 1} \left( \prod_{j=0}^{N} \frac{2 + 5j}{2 - 5k + 5j} \right) (z^{5N-5k} - z^{5k+2}) \right\}. \quad (36)$$

The $\mathbf{Q}$ operator for $N$ odd is

$$Q(z) = (z - 1)^{-(2N+1)} \left\{ \sum_{k=0}^{(N-1)/2} \left( \prod_{j=0}^{N} \frac{2 + 5j}{2 - 5k + 5j} \right) (z^{5N+2-5k} - z^{5k}) \right. \left. + \sum_{k=0}^{(N-1)/2} \left( \prod_{j=0}^{N} \frac{2 + 5j}{2 - 5k + 5j} \right) (z^{5N-5k} - z^{5k+2}) \right\}. \quad (37)$$
One can read off the $Q$ operator at special points of $z$ from this expression. For example, $Q(0) = 1$ for both $N$ even and odd, and

$$Q(-1) = \begin{cases} 
4^{-N} \sum_{k=0}^{(N-1)/2} \binom{N}{k} \left( \prod_{j=0}^{N} \frac{2+5j}{2-5k+5j} + \prod_{j=0}^{N} \frac{-2+5j}{2-5k+5j} \right), & N \text{ even}, \\
0, & N \text{ odd}. 
\end{cases} \tag{38}
$$

### 4 Spin-$L/2$

The steps to calculate the $Q$ operator in the last section can be applied to the general case of the higher half-integer spin XXZ chain as well. Recall that we are considering the half-integer spin-$s = L/2$ ($L = 1, 3, 5, \cdots$) XXZ chain with odd sites $M = 2N + 1$ ($N = 1, 2, 3, \cdots$) at the Razumov-Stroganov point $\eta = -i(L + 1)\pi/(L + 2)$ on the sector of $p$ Bethe roots where $NL \leq p \leq (N + 1)L$.

Repeating the same steps as the last section, we find

$$Q(z) = (z - 1)^{-(2N + 1)} \left\{ \sum_{k=0}^{N/2} (-1)^k \binom{N}{k} \prod_{j=0}^{N} \frac{p + 1 - LN + (L + 2)j}{p + 1 - LN + (L + 2)(j - k)} \right( z^{p + 1 + 2N - (L + 2)k} - z^{(L + 2)k} \right)$$

$$+ \sum_{k=0}^{N/2 - 1} (-1)^k \binom{N}{k} \prod_{j=0}^{N} \frac{p + 1 - LN + (L + 2)j}{LN - p - 1 + (L + 2)(j - k)} \right( z^{(L + 2)(N - k)} - z^{p + 1 - LN + (L + 2)k} \right), \tag{39}$$

for $N$ even and

$$Q(z) = (z - 1)^{-(2N + 1)} \left\{ \sum_{k=0}^{(N-1)/2} (-1)^k \binom{N}{k} \prod_{j=0}^{N} \frac{p + 1 - LN + (L + 2)j}{p + 1 - LN + (L + 2)(j - k)} \right( z^{p + 1 + 2N - (L + 2)k} - z^{(L + 2)k} \right)$$

$$+ \sum_{k=0}^{(N-1)/2} (-1)^k \binom{N}{k} \prod_{j=0}^{N} \frac{p + 1 - LN + (L + 2)j}{LN - p - 1 + (L + 2)(j - k)} \right( z^{(L + 2)(N - k)} - z^{p + 1 - LN + (L + 2)k} \right), \tag{40}$$

for $N$ odd. From these expressions, we can see $Q(0) = 1$ for both $N$ even and odd. One can also find that the $Q(z)$ has a zero at $z = -1$ for $L$ and $p$ odd.

### 5 Functional relations

One can derive the hierarchy of functional relations by combining the $Q$ operator on the “right side” and its corresponding “wrong side” as the spin-1/2 XXZ chain and the conformal field theory [S][10][13] by using Plücker relations. The “right side” (“wrong side”) means the sector with positive (negative) total spin. From now on, we denote the $Q$ operator on the “right side” by $Q(z)$, and the $Q$ operator on the corresponding “wrong sector” by $P(z)$. The two sectors have the same exact transfer matrix eigenvalue. We rewrite the two $TQ$ equations in the following form

$$2\text{ch} \left( \frac{(L + 1)(ML - 2p)\pi i}{2(L + 2)} \right) (z - 1)^{2N + 1} Q(z) = q^{(L + 1)(ML - 2p)/2}(zq^2 - 1)^{2N + 1}Q(zq^{-2})$$

$$+ q^{-(L + 1)(ML - 2p)/2}(zq^{-2} - 1)^{2N + 1}Q(zq^2), \tag{41}$$

$$2\text{ch} \left( \frac{(L + 1)(ML - 2p)\pi i}{2(L + 2)} \right) (z - 1)^{2N + 1} P(z) = q^{-(L + 1)(ML - 2p)/2}(zq^{-2} - 1)^{2N + 1}P(zq^{-2})$$

$$+ q^{(L + 1)(ML - 2p)/2}(zq^2 - 1)^{2N + 1}P(zq^2). \tag{42}$$
Eliminating the transfer matrix eigenvalue $2\text{ch} \frac{(L+1)(ML-2p)\pi i}{2L}$ in (11) by combining the two equations (11) and (12) leads to

\[
(q^{-2} - 1)^{2N+1} \left\{ q^{(L+1)(ML-2p)/2} Q(q^{-2}) P(z) - q^{-(L+1)(ML-2p)/2} P(z) Q(q^{-2}) \right\} = (q^{-2} - 1)^{2N+1} \left\{ q^{(L+1)(ML-2p)/2} P(z) Q(z) - q^{-(L+1)(ML-2p)/2} Q(z) P(z) \right\},
\]

therefore we can define a polynomial $\Psi_1(z)$ of degree $(2N + 1)(L - 1)$ satisfying

\[
q^{(L+1)(ML-2p)/2} P(z) Q(q^{-2}) P(q^{-1}) = (q^{-2} - 1)^{2N+1} \Psi_1(z),
\]

\[
q^{(L+1)(ML-2p)/2} Q(z) P(z) Q(q^{-2}) P(q^{-1}) = (q^{-2} - 1)^{2N+1} \Psi_1(z).
\]

Shifting $z \to q^{-1}$ in (11) and $z \to q$ in (15) leads to

\[
q^{(L+1)(ML-2p)/2} P(q) Q(q^{-1}) = (q^{-2} - 1)^{2N+1} \Psi_1(z^{-1})
\]

\[
= (q^{-3} - 1)^{2N+1} \Psi_1(q).
\]

The equality (46) leads to the existence of the polynomial $\Psi_2(z)$ of degree $(2N + 1)(L - 2)$ satisfying

\[
\Psi_1(z^{-1}) = (z^{-3} - 1)^{2N+1} \Psi_2(z),
\]

\[
\Psi_1(z) = (z^{-2} - 1)^{2N+1} \Psi_2(z).
\]

Inserting these expressions into (46), we get

\[
q^{(L+1)(ML-2p)/2} P(q) Q(q^{-1}) = (q^{-2} - 1)^{2N+1} (z^{-3} - 1)^{2N+1} \Psi_2(z)
\]

\[
= (q^{-3} - 1)^{2N+1} (z^{-5} - 1)^{2N+1} \Psi_2(q^{-2})
\]

\[
= (q^{-3} - 1)^{2N+1} (z^{-5} - 1)^{2N+1} \Psi_2(q^{-2}).
\]

For the case of spin-3/2 ($L = 3$), (48) leads to the determination of $\Psi_2(z)$ as $\Psi_2(z) = C(z^{-5} - 1)^{2N+1}$ which finally results in the following equality

\[
q^{2(ML-2p)} P(z) Q(z^{-1}) - q^{-2(ML-2p)} P(z) Q(z^{-1}) = C(z^{-3} - 1)^{2N+1} (z^{-5} - 1)^{2N+1} (z^{-7} - 1)^{2N+1}.
\]

For generic $L$, one can repeat the same argument to find

\[
q^{(L+1)(ML-2p)/2} P(q) Q(q^{-1}) - q^{-(L+1)(ML-2p)/2} Q(z) P(z^{-1}) = C \prod_{j=1}^{L} (z^{-j+1} - 1)^{2N+1}.
\]

Evaluating the both hand sides of this equality at $z = 0$ leads to the determination of the constant factor $C$ as

\[
C = 2\text{sh} \left( \frac{(L+1)(2p - ML)\pi i}{2(L+2)} \right).
\]

Thus we have

\[
q^{(L+1)(ML-2p)} P(z) Q(z^{-1}) - q^{-(L+1)(ML-2p)/2} Q(z) P(z^{-1}) = 2\text{sh} \left( \frac{(L+1)(2p - ML)\pi i}{2(L+2)} \right) \prod_{j=1}^{L} (z^{-j+1} - 1)^{2N+1}.
\]
which is the first fundamental relation between the $Q$ operator and the $P$ operator.

We next proceed to find the second fundamental relation which, together with the first one, leads to the construction of the hierarchy of the functional relations. Comparing (46) and (52), we have

$$
\Psi_1(z) = 2\mathrm{sh} \left( \frac{(L + 1)(2p - ML)\pi i}{2(L + 2)} \right) \prod_{j=2}^{L} (zq_{j}^{2} - 1)^{2N+1}.
$$

(Multiplying the both hand sides of the second $TQ$ equation (52) by $\Psi_1(z)$ and using the relations (43) and (45), we find the following relation

$$
2\mathrm{ch} \left( \frac{(L + 1)(2p - ML)\pi i}{2(L + 2)} \right) (z - 1)^{2N+1} \Psi_1(z) = q^{(L+1)(ML-2p)}P(zq^{2})Q(zq^{-2}) - q^{-(L+1)(ML-2p)}Q(zq^{2})P(zq^{-2}).
$$

(54)

Inserting (53), one gets the second fundamental functional relation

$$
q^{(L+1)(ML-2p)}P(zq^{2})Q(zq^{-2}) - q^{-(L+1)(ML-2p)}Q(zq^{2})P(zq^{-2}) = 2\mathrm{sh} \left( \frac{(L + 1)(2p - ML)\pi i}{2(L + 2)} \right) (z - 1)^{2N+1} \prod_{j=2}^{L} (zq_{j}^{2} - 1)^{2N+1}.
$$

(55)

The left hand sides of the functional relations (52) and (55) can be regarded as the discrete analogue to $\tilde{Q}(z) = z^{(L+1)(p/2-(ML-1)/4)}Q(z)$ and $\tilde{P}(z) = z^{(L+1)(ML+1)/4-p/2}P(z)$ to rewrite the relations (52) and (55) in a more symmetric form

$$
\tilde{P}(zq)\tilde{Q}(zq^{-1}) - \tilde{Q}(zq)\tilde{P}(zq^{-1}) = 2\mathrm{sh} \left( \frac{(L + 1)(2p - ML)\pi i}{2(L + 2)} \right) z^{(L+1)/2} \prod_{j=1}^{L} (zq_{j}^{2} - 1)^{1} \prod_{j=1}^{L} (zq_{j}^{2} - 1)^{2N+1},
$$

(56)

$$
\tilde{P}(zq^{2})\tilde{Q}(zq^{-2}) - \tilde{Q}(zq^{2})\tilde{P}(zq^{-2}) = 2\mathrm{sh} \left( \frac{(L + 1)(2p - ML)\pi i}{L + 2} \right) z^{(L+1)/2}(z - 1)^{2N+1} \prod_{j=2}^{L} (zq_{j}^{2} - 1)^{2N+1}.
$$

(57)

We now define the following family of functions

$$
t_{s}(z) = \tilde{P}(zq^{2s+1})\tilde{Q}(zq^{-(2s+1)}) - \tilde{P}(zq^{-(2s+1)})\tilde{Q}(zq^{2s+1}),
$$

(58)

($s = 0, \pm 1/2, \pm 1, \cdots$) which is a family of quantum Wronskian. From the definition and (56), (57), one finds the function $t_{s}(z)$ satisfies

$$
t_{-s-1}(z) = -t_{s}(z), \quad t_{-1/2}(z) = 0,
$$

(59)

$$
t_{0}(z) = 2\mathrm{sh} \left( \frac{(L + 1)(2p - ML)\pi i}{2(L + 2)} \right) z^{(L+1)/2} \prod_{j=1}^{L} (zq_{j}^{2} - 1)^{2N+1},
$$

(60)

$$
t_{1/2}(z) = 2\mathrm{sh} \left( \frac{(L + 1)(2p - ML)\pi i}{L + 2} \right) z^{(L+1)/2}(z - 1)^{2N+1} \prod_{j=2}^{L} (zq_{j}^{2} - 1)^{2N+1}.
$$

(61)

We define the function $\Delta(a, b)$ as

$$
\Delta(a, b) = \tilde{P}(a)\tilde{Q}(b) - \tilde{Q}(a)\tilde{P}(b),
$$

(62)

which satisfies the Plücker relation

$$
\Delta(a, b)\Delta(c, d) - \Delta(a, c)\Delta(b, d) + \Delta(a, d)\Delta(b, c) = 0.
$$

(63)
Noting the function $t_s(z)$ can be expressed as $t_s(z) = \Delta(zq^{2s+1}, zq^{-(2s+1)})$, one can see that setting $a = z$, $b = zq^{-2(2s+1)}$, $c = zq^{-2(2s_2+1)}$, $d = zq^{-(2s_3+1)}$ in the relation (63) leads to the hierarchy of functional relations

\[
t_{s_1}(zq^{-(2s_1+1)})t_{s_3-s_2-1/2}(zq^{-(2s_2+s_3+1)}) - t_{s_2}(zq^{-(2s_1+1)})t_{s_3-s_1-1/2}(zq^{-2(s_1+s_3+1)}) \\
+ t_{s_3}(zq^{-(2s_3+1)})t_{s_2-s_1-1/2}(zq^{-2(s_1+s_2+1)}) = 0.
\]

In particular, we have the following fundamental fusion relations

\[
2\operatorname{ch}\left(\frac{(L+1)(2p-ML)\pi i}{2(L+2)}\right)(z-1)^{2N+1}t_s(zq^{-(2s+1)}) \\
= q^{(L+1)/2} \prod_{j=1}^{L}(zq^{-2} - 1)^{2N+1}t_{s-1/2}(zq^{-2(s+1)}) + q^{-(L+1)/2} \prod_{j=1}^{L}(zq^{-2} - 1)^{2N+1}t_{s+1/2}(zq^{-2s}),
\]

from $s_1 = s$, $s_2 = -1$, $s_3 = 0$.

### 6 Summary and Discussion

In this paper, we investigated the $TQ$ equation and the $Q$ operator of higher half-integer integer spin XXZ chain at the Razumov-Stroganov point. First, by making expansion of the $Q$ operator in terms of the symmetric polynomials of the Bethe roots and regarding them as unknown parameters, we showed that solving the $TQ$ equations reduces to solving a simple set of linear equations. This approach might be useful for making a guess on the Razumov-Stroganov point of other models, boundary conditions and so on. Next, rewriting the conditions that the $Q$ operator must satisfy to a form such that the interpolation formula can be applied, the $Q$ operator is evaluated in closed form. By combining the $Q$ operators on the “right side” and the “wrong side”, we showed they produce the hierarchy of functional relations.

The analysis made in this paper is for the models with the periodic boundary condition. It should be straightforward to extend the analysis to open boundary conditions. It is interesting to lift the analysis to the elliptic case. For the spin-1/2 case, the groundstate transfer matrix eigenvalue of the XYZ chain is found to be simply expressed as a theta function [28], which reduces to a trigonometric function at the trigonometric limit. This fact and the one for the higher half-integer XXZ chain [42] would be reasons to believe that the exact transfer matrix eigenvalue can be lifted to the elliptic higher half-integer XYZ spin. Making a guess on the transfer matrix eigenvalue and investigating the $TQ$ equation and the $Q$ operators is an interesting problem to study.

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### A Relation between the $Q$ operators

In this appendix, we show the general construction of the relation between the $Q$ operator $Q(z)$ on the “right side” and its corresponding $Q$ operator $P(z)$ on the “wrong side”. First, we divide the both
hand sides of the first fundamental functional relation \[ \text{by } Q(z)Q(z^{-1}) \text{ as} \]

\[
q^{(L+1)(ML-2p)/2} \frac{P(zq)}{Q(zq)} - q^{-(L+1)(ML-2p)/2} \frac{P(zq^{-1})}{Q(zq^{-1})} = 2\text{sh} \left( \frac{(L+1)(2p-ML)\pi}{2(L+2)} \right) \prod_{j=1}^{L} (zq^{2j+1} - 1)^{2N+1}.
\]

(A1)

By partial fraction decomposition, the right hand side of (A1) can be expressed as

\[
2\text{sh} \left( \frac{(L+1)(2p-ML)\pi}{2(L+2)} \right) \prod_{j=1}^{L} (zq^{2j+1} - 1)^{2N+1}
\]

\[
\frac{Q(zq)Q(z^{-1})}{Q(zq^2)Q(z^{-2})} = q^{(L+1)(ML-2p)/2} \frac{P(zq^2)}{Q(zq^2)} - q^{-(L+1)(ML-2p)/2} \frac{P(zq^{-2})}{Q(zq^{-2})}
\]

\[
= q^{(L+1)(ML-2p)/2} R(z) + q^{-(L+1)(ML-2p)/2} R(z^{-1}) + q^{(L+1)(ML-2p)/2} A(z) - q^{L+1)(ML-2p)/2} B(z) Q(z)
\]

(A3)

If the last term of the right hand side of (A3) is not zero, the zeros of \(Q(z)\) would be poles of this equality, which does not exist in the left hand side, leading to contradiction. Thus the polynomials \(A(z)\) and \(B(z)\) are related as

\[
A(z) = q^{(L+1)(ML-2p)/2} C(z),
\]

\[
B(z) = q^{-(L+1)(ML-2p)/2} C(z),
\]

(A4)

with some polynomial \(C(z)\) essentially the same with \(A(z)\) and \(B(z)\). We also assume that the polynomial \(R(z)\) can be decomposed as

\[
R(z) = q^{(L+1)(ML-2p)/2} F(z) - q^{-(L+1)(ML-2p)/2} F(z^{-1}),
\]

(A5)

with some polynomial \(F(z)\) whose degree is the same with \(R(z)\). The combination of (A1) and (A2) now becomes

\[
q^{(L+1)(ML-2p)/2} \frac{P(zq)}{Q(zq)} - q^{-(L+1)(ML-2p)/2} \frac{P(zq^{-1})}{Q(zq^{-1})}
\]

\[
= q^{(L+1)(ML-2p)/2} \left( F(z) + \frac{C(z)}{Q(z)} \right) - q^{-(L+1)(ML-2p)/2} \left( F(z^{-1}) + \frac{C(z^{-1})}{Q(z^{-1})} \right),
\]

(A6)

from which one finds the \(Q\) operator on the “wrong side” \(P(z)\) should be expressed by the \(Q\) operator on the “right side” \(Q(z)\) as

\[
P(z) = F(z)Q(z) + C(z)
\]

\[
= F(z)Q(z) + q^{-(L+1)(ML-2p)/2} A(z)
\]

\[
= F(z)Q(z) + q^{(L+1)(ML-2p)/2} B(z),
\]

(A7)

where \(F(z)\) and \(A(z)\) are obtained by the partial fraction decomposition (A2) of the \(Q\) operator \(Q(z)\).
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