RADIATION FROM PERFECT MIRRORS FOLLOWING PRESCRIBED RELATIVISTIC TRAJECTORIES (*)

A. Calogeracos
Division of Theoretical Mechanics
Hellenic Air Force Academy TG1010
Dhekelia Air Force Base
Dhekelia, Greece

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Abstract

The question is examined of a mirror which starts from rest and either (i) accelerates for some time and eventually reverts to motion at constant velocity, (ii) continues accelerating forever. A sharp distinction is made between cases (i) and (ii) concerning the spectrum of the emitted radiation, and the qualitative difference between the two cases is pointed out. The Bogolubov coefficients are calculated for a trajectory of type (i). A type (ii) trajectory is entirely unphysical as far as any realistic mirror is concerned, however it is of interest in that it has been used as a simple analog of black hole collapse. The spectrum emitted for the type (ii) trajectory \( z = - \ln (\cosh t) \) is examined and it is shown that it is indeed that of a black body. Inconsistencies in previous derivations of the above result are pointed out.

I. INTRODUCTION

The subject of mirror induced radiation has received great attention in the last ten years. Nonrelativistic calculations have been performed for imperfect mirrors (dielectric or dispersive) using the Hamiltonian formalism. In this talk, rather than report on recent progress along these lines, I shall go about twenty five years back and examine the old problem of an one-sided perfect mirror following a prescribed relativistic trajectory. The papers by Fulling and Davies [1], [2] are early important contributions on the subject, and a good review is provided by Birrell and Davies [3]. I consider a mirror that starts from rest and accelerate along the trajectory \( z = - \ln (\cosh \kappa t) \) (I take \( \kappa = 1 \) thus fixing the energy scale). I make the distinction between a type (i) (asymptotically inertial) trajectory where the mirror accelerates till it reaches a space-time point \( P \) and then reverts to motion at uniform velocity \( B_P \) and a type (ii) trajectory where the mirror accelerates ad infinitum. The results presented here are contained in the two preprints [4], [5].

Concerning the type (i) case an exact calculation of the Bogolubov \( \beta (\omega, \omega') \) amplitude in terms of hypergeometric functions is presented, the analytic properties of the \( \alpha (\omega, \omega') \) and \( \beta (\omega, \omega') \) amplitudes are discussed and the constraints imposed by unitarity are stressed. There are several advantages in considering asymptotically inertial trajectories: (a) Acceleration continuing for an infinite time implies mathematical singularities and also entails physical pathologies associated, for example, with the infinite energy that has to be imparted to the mirror. (b) The mirror’s rest frame eventually (after acceleration stops) becomes an inertial frame and the standard description in terms of in and out states is possible. One may choose either the lab frame or the mirror’s rest frame to describe the photons produced. (c) One avoids statements about photons produced while the mirror is accelerated; rather one makes unambiguous statements pertaining to times \( t = \pm \infty \) when acceleration vanishes. We show in general that for an asymptotically inertial trajectory with the velocity being everywhere continuous the amplitude squared \( |\beta (\omega, \omega')|^2 \) goes as \( (\omega')^{-5} \) (or faster) for large \( \omega' \). This is in contrast to a type (ii) trajectory where \( |\beta (\omega, \omega')|^2 \) goes as \( 1/\omega' \) for large \( \omega' \). This radical difference is due to a subtle cancellation in the type (i) case between two contributions, one arising from the initial and the other from the final asymptotic part of the trajectory.

The type (ii) trajectory mentioned previously has been considered extensively in the past, starting with the classic papers by Fulling and Davies op. cit.. The problem is of interest because of its connection to radiation emitted from a collapsing black hole (Hawking [6], DeWitt [7]) and to the attendant thermal spectrum. In the present context one can counter that such a trajectory is unrealistic on the grounds of (b) above; then one may decide to stick to type (i) trajectories. In the second part of this talk I ignore such questions, take the premises of the early papers on the subject for granted, and concentrate on the calculation of the Bogolubov amplitude \( \beta (\omega, \omega') \) (and thus of \( \alpha (\omega, \omega') \) as well) and of the spectrum

\[
N(\omega) = \int_0^{\infty} d\omega' |\beta (\omega, \omega')|^2 e^{-\sigma (\omega+\omega')} \tag{1}
\]
(a is a convergence factor). We shall show that contrary to folklore a correct derivation of the black body spectrum via the calculation of the Bogolubov amplitudes requires consideration of the whole trajectory and not just of its asymptotic part.

II. THE BOGOLUBOV AMPLITUDES FOR A TYPE (I) TRAJECTORY

A. Calculation of $\beta(\omega, \omega')$

Let us introduce coordinates

$$u = t - z, v = t + z$$

(2)

The point $P$ at which the mirror reverts to uniform motion has $v$ coordinate given by $v = r$. The velocity at $P$ is given by $B_P = 1 - e^r$ (see the Appendix of [4] for details). The trajectory equation $u = f(v)$ is defined in a piecewise manner via

$$u = v, v < 0$$

(3)

corresponding to a mirror at rest for $t < 0$,

$$u = f_{acc}(v) \equiv -\ln \left(2 - e^v\right), 0 < v < r$$

(4)

corresponding to the accelerating part of the trajectory, and

$$u = f_0(v) \equiv C + \frac{1 - B_P}{1 + B_P} v$$

(5)

describing uniform motion after point $P$. The constant $C$ is fixed by the requirement that the trajectory be continuous at $P$

$$C + \frac{1 - B_P}{1 + B_P} r = -\ln \left(2 - e^r\right)$$

(6)

Let us take everything to exist to the right of the mirror. There are two sets of modes

$$\varphi_\omega(u, v) = \frac{i}{2\sqrt{\pi}\omega} \left(\exp(-i\omega v) - \exp(-i\omega p(u))\right)$$

(7)

and

$$\bar{\varphi}_\omega(u, v) = \frac{i}{2\sqrt{\pi}\omega} \left(\exp(-i\omega f(v)) - \exp(-i\omega u)\right)$$

(8)

which satisfy the free wave equation and the condition that the field should vanish on the mirror. The modes $\varphi_\omega(u, v)$ of (7) describe waves incident from the right as it is clear from the sign of the exponential in the first term; the second term represents the reflected part which has a rather complicated behaviour depending on the motion of the mirror. These modes constitute the $in$ space and should obviously be unoccupied before acceleration starts,$$

a_i |0in\rangle = 0

Similarly the modes $\bar{\varphi}_\omega(u, v)$ describe waves travelling to the right (emitted by the mirror) as can be seen from the exponential of the second term. Correspondingly the first term is complicated. These modes define the $out$ space and

$$\bar{a}_i |0out\rangle = 0$$

The state $|0out\rangle$ corresponds to the state where nothing is produced by the mirror. The two representations are connected by the Bogolubov transformation

$$\bar{a}_i = \sum_j \left(\alpha_{ji} a_j + \beta_{ji}^* a_j^\dagger\right)$$

(9)
The fact that the in and out vacua are not identical lies at the origin of particle production. In our notation the matrix $\beta_{ji}$ is given by the overlap (see Birrell and Davies op. cit. equations (2.9), (3.36))

$$\beta(\omega, \omega') = -i \int_0^\infty dz \varphi_\omega(z, 0) \frac{\partial}{\partial t} \varphi_\omega(z, 0) + i \int_0^\infty dz \left( \frac{\partial}{\partial t} \varphi_\omega(z, 0) \right) \varphi_\omega(z, 0) \tag{10}$$

The integration in (10) can be over any spacelike hypersurface and the choice $t = 0$ is convenient. After we substitute expressions (7), (8) in (10) we get the $\beta(\omega, \omega')$ amplitude in the form

$$\beta(\omega, \omega') = \frac{1}{4\pi \sqrt{\omega\omega'}} \int_0^\infty dz \left\{ e^{i\omega' z} - e^{-i\omega' z} \right\} \left\{ \omega e^{-i\omega f} - f e^{-i\omega f} \right\} +$$

$$+ \frac{(\omega - \omega')}{4\pi \sqrt{\omega\omega'}} \int_0^\infty dz \left\{ e^{i\omega' z} - e^{-i\omega' z} \right\} e^{iz} \tag{11}$$

Notice that the second integral is $f$-independent (i.e. trajectory independent) and that its origin is purely kinematic. Some of the integrals involved are conveniently expressed in terms of the $\zeta$ function and its complex conjugate $\zeta^*$ defined in Heitler [8], pages 66-71:

$$\zeta(x) \equiv -i \int_0^\infty e^{i\kappa x} d\kappa = P\frac{1}{x} - i\pi \delta(x) \tag{12}$$

Amplitude (11) is split to three contributions

$$\beta(\omega, \omega') \equiv \beta_I(\omega, \omega') + \beta_{II}(\omega, \omega') + \beta_{III}(\omega, \omega') \tag{13}$$

originating as follows. The $\beta_{III}(\omega, \omega')$ term stands for the second ($f$-independent) integral in (11), is always there for a mirror starting from rest (or initially moving at uniform velocity), and does not depend on the subsequent form of the trajectory. The $\beta_{II}(\omega, \omega')$ contribution results from the accelerating part of the trajectory and corresponds to the $0 < z < r$ integration range in (11). Its evaluation is mathematically somewhat involved. The $\beta_{II}(\omega, \omega')$ amplitude results from the $z > r$ part of the integration range in (11), is readily evaluated, and is specific to a mirror that reverts to the state of uniform motion (it does not arise in the case of a type (ii) trajectory). Thus

$$\beta_{II}(\omega, \omega') =$$

$$= \frac{1}{4\pi i \sqrt{\omega\omega'}} \left( \frac{1}{\omega - 1 + B_p} - \omega' \right) \exp \left[ i \left( \frac{1}{1 + B_p} + \omega' \right) r \right] \zeta \left( \omega \frac{1}{1 + B_p} + \omega' \right) e^{-i\omega C}$$

$$- \frac{1}{4\pi i \sqrt{\omega\omega'}} \exp \left[ i \left( -\omega' + \frac{1}{1 + B_p} \right) r \right] e^{-i\omega C} \tag{15}$$

$$\beta_{III}(\omega, \omega') = \frac{1}{4\pi i \sqrt{\omega\omega'}} - \frac{1}{4\pi i \sqrt{\omega\omega'}} (\omega - \omega') \zeta(\omega + \omega') \tag{16}$$

Notice that the arguments of the $\zeta$ functions in (15) and (16) never vanish. Hence as far as the evaluation of $\beta(\omega, \omega')$ is concerned we observe that it is only the first term in (12) that is operative, and that $\beta(\omega, \omega')$ does not have any $\delta$-type singularities. Giving the result in the form (13), (16) is useful because we can obtain the other Bogolubov amplitude $\alpha(\omega, \omega')$ via the substitution $\omega \rightarrow -\omega$ in accordance with (22) below. The $\alpha(\omega, \omega')$ is of course expected to have $\delta$-type singularities, and in fact reduces to just $\delta(\omega - \omega')$ in the trivial case when the in and out modes coincide. Comparing with Davies and Fulling [2] note that the $\beta_{III}$ term is unaccountably missing from the latter reference. It will be shown in the following section that this term is in fact crucial in containing the thermal spectrum for the type (ii) trajectory.

The amplitude $\beta_I$ may be expressed in terms of hypergeometrics (for details on the calculation see [4]).
\[ \beta_f(\omega, \omega') = \frac{1}{2 \pi \sqrt{\omega \omega'}} \sin (\omega' r) \left( 2 - e^r \right)^{i\omega} - \]

\[ - \frac{1}{2 \pi i \sqrt{\omega \omega'}} \left( F \left( -i\omega, -i\omega'; -i\omega' + 1; \frac{1}{2} \right) + \right. \]

\[ + F \left( -i\omega, -i\omega'; -i\omega' + 1; \frac{e^r}{2} \right) e^{-i\omega' r} \}

It is interesting to observe that the value \( r = \ln 2 \) corresponding to the asymptote of the type (ii) trajectory (see (4)) falls exactly on the radius of convergence of the series as one can see from the argument of the second hypergeometric appearing above.

**B. On the Bogolubov coefficients \( \alpha(\omega, \omega'), \beta(\omega, \omega') \)**

The Bogolubov \( \alpha \) coefficients appearing in (9) are given by

\[ \alpha(\omega, \omega') = i \int_0^\infty dz \varphi_{\omega'}(z, 0) \frac{\partial}{\partial t} \bar{\varphi}_\omega(z, 0) - i \int_0^\infty dz \left( \frac{\partial}{\partial t} \varphi_{\omega'}(z, 0) \right) \bar{\varphi}_\omega(z, 0) \] (18)

Recall also the unitarity condition ((3.39) of Birrell and Davies op. cit.)

\[ \int_0^\infty d\tilde{\omega} \left( \alpha(\tilde{\omega}, \omega_1) \alpha^*(\tilde{\omega}, \omega_2) - \beta(\tilde{\omega}, \omega_1) \beta^*(\tilde{\omega}, \omega_2) \right) = \delta(\omega_1 - \omega_2) \] (19)

and its partner

\[ \int_0^\infty d\tilde{\omega} \left( \alpha(\omega_1, \tilde{\omega}) \alpha^*(\omega_2, \tilde{\omega}) - \beta(\omega_1, \tilde{\omega}) \beta^*(\omega_2, \tilde{\omega}) \right) = \delta(\omega_1 - \omega_2) \] (20)

Relations (19), (20) above are direct consequences of the fact that the sets \( \varphi_\omega \) and \( \varphi_{\omega'} \) respectively are orthonormal and complete. They also guarantee that the operators \( a_j, a_j^\dagger \) and \( \bar{a}_i, \bar{a}_i^\dagger \) obey the standard equal time commutation relations that creation and annihilation operators do.

We find it convenient to isolate the square roots in (11) and introduce quantities \( A(\omega, \omega'), B(\omega, \omega') \) that are analytic functions of the frequencies (without the branch cuts attendant to square roots) via

\[ \alpha(\omega, \omega') = A(\omega, \omega') \frac{\omega}{\sqrt{\omega \omega'}} \]

\[ \beta(\omega, \omega') = B(\omega, \omega') \frac{\omega}{\sqrt{\omega \omega'}} \] (21)

The quantity \( B(\omega, \omega') \) is read off (11) (and \( A(\omega, \omega') \) from the corresponding expression for \( \alpha(\omega, \omega') \)). From the expressions for the Bogolubov coefficients one can immediately deduce that

\[ B^*(\omega, \omega') = A(-\omega, \omega'), A^*(\omega, \omega') = B(-\omega, \omega') \] (22)

Observe that identities (19), (20) are a result of the completeness of the basis \textit{out} and \textit{in} wavefunctions respectively.

**III. A TRAJECTORY ACCELERATING AD INFINITUM AND THE BLACK BODY SPECTRUM**

It was mentioned in the Introduction that one major mathematical difference between type (i) and type (ii) trajectories lies in the behaviour of the \( \beta(\omega, \omega') \) amplitude in the \( \omega' \to \infty \) limit. To see this let us consider the integral

\[ I = \int_0^\infty dz e^{i\omega' z - \alpha z} e^{-i\omega f(z)} \] (23)

(a being a convergence factor), which is one of the integrals appearing in the amplitude (11) (the other integrals are handled in a similar way). To examine the asymptotic behaviour of (23) I split the integral \( \int_0^\infty \) to \( \int_0^r + \int_r^\infty \) and apply
simple integration by parts twice to each one of them (see Bender and Orszag [9], p. 278). The endpoint contributions from infinity vanish due to the convergence factor. The endpoint contributions at \( z = r \) cancel out in pairs: \( f_0(r) \) cancels with \( f_{acc}(r) \) (both accompanied by a factor \( 1/\omega' \)) and \( f'_0(r) \) cancels with \( f'_{acc}(r) \) (both accompanied by a factor \( (\omega')^{-2} \)). These cancellations are hardly fortuitous. The first cancellation reflects the fact that the trajectory itself \( u = f(v) \) is continuous, and the second reflects the continuity of the velocity at \( P \). Similarly endpoint contributions at \( z = 0 \) cancel with corresponding terms originating from a large \( \omega' \) expansion of the second (trajectory independent) term in (11) on the same continuity grounds as above. Thus integral \( I \) goes as \( (\omega')^{-3} \) and observing the prefactors in (11) we deduce that its contribution to \( \beta(\omega, \omega') \) goes as \( (\omega')^{-5/2} \). The same behaviour is obtained for the other terms making up \( \beta(\omega, \omega') \).

In the case of a type (ii) trajectory the amplitude is again given by (11) provided we replace the upper limit of the first integration by \( \ln 2 \). It consists of the \( \beta_I \) term given by (14) (again with \( \omega \) replaced by \( \ln 2 \)) and of the \( \beta_{II} \) term given by (16) (the \( \beta_{II} \) term relates to the motion of the mirror after point \( P \) and so it simply does not appear in a type (ii) trajectory). The cancellation mechanism mentioned above does not exist now, and integration by parts is of no help since the resulting integral diverges. I thus adopt a different approach and transform the \( \beta_I \) term to

\[
\beta_I(\omega, \omega') = -\frac{2i(\omega-\omega')}{2\pi} \sqrt{\frac{\omega'}{\omega}} \int_0^{\ln 2} \frac{d\rho e^{i\omega'\rho}}{\rho} (1 - e^{-\rho})^i\omega.
\]  

To obtain the asymptotic behaviour of the above integral I employ the standard technique of deforming the integration path to a contour in the complex plane (see [9] chapter 6; see also Morse and Feshbach [10], p. 610 where a parallel to the real axis vanishes exponentially in the limit \( iT + \ln 2 \), and then down again parallel to the imaginary axis from \( iT + \ln 2 \) to \( \ln 2 \). The contribution of the segment parallel to the real axis vanishes exponentially in the limit \( T \to \infty \). One is thus led to two integrals of the form

\[
\int_0^\infty ds e^{-\omega's} \times (...) 
\]

Hence the \( \omega' \to \infty \) limit is equivalent to the \( s \to 0 \) limit. Eventually in the large \( \omega' \) limit

\[
\beta_I(\omega, \omega') \simeq -\frac{i2i(\omega-\omega')}{2\pi \sqrt{\omega\omega'}} (\omega')^{-i\omega} e^{-\pi\omega/2} \Gamma(1 + i\omega) + \frac{i}{2\pi \sqrt{\omega\omega'}}
\]  

The second term in (25) exactly cancels with the asymptotic form of \( \beta_{III} \) given by (16), the end result being

\[
\beta(\omega, \omega') \simeq -\frac{i2i(\omega-\omega')}{2\pi \sqrt{\omega\omega'}} (\omega')^{-i\omega} e^{-\pi\omega/2} \Gamma(1 + i\omega)
\]  

By taking the modulus of (26) and squaring we get the black body result

\[
|\beta(\omega, \omega')|^2 \simeq \frac{1}{2\pi\omega'} e^{2\pi\omega} - 1
\]

**IV. CONCLUSIONS**

I will try to summarize the main conclusions that can be drawn from the above treatment, and also correct some misconceptions in the literature.

(a) It was shown that in the case of a type (ii) trajectory the amplitude square \( |\beta(\omega, \omega')|^2 \) behaves as \( 1/\omega' \) (hence the corresponding spectrum integral diverges logarithmically and a cut-off is required). This fact has been known for some time, and is usually stated in the form that "large frequencies dominate". However this statement is also unfortunately coupled to the incorrect one that the important contributions to the black body spectrum arise from the asymptotic part of the trajectory (i.e. the part lying close to the asymptote \( v = \ln 2 \)). It is shown in the Appendix of [9] that if one expands in powers of distance from the asymptote, then the contributions from the various terms are all of the same order of magnitude and thus the expansion cannot be truncated. Statements to the effect that short distances dominate have been made by Hawking [9] in the context of radiation from black holes; their applicability in the present different problem of an accelerating mirror should be doubted. In the present case the misleading assertion
that short distances dominate may be in accordance with one’s classical instincts, in the sense that roughly speaking the small amount of time that the mirror spends near the origin ought to have an insignificant effect compared to the infinitely long time spent near the asymptote. It does however go counter to quantum mechanical orthodoxy, according to which one cannot make statements as to where and when photons have been produced. It is certainly true that were it not for the singularity on the v asymptote the thermal spectrum would not arise, but this does not provide any guidance as to the origin of the dominant contribution to the amplitude. This remark can be illustrated by the crucial role of the $\beta_{\text{III}}$ term (missing in [2]) which appears as if it originates at $t = 0$.

One further danger in throwing away contributions has to do with possible violations of unitarity. As already pointed out after (16), the $\zeta$ functions featuring in $\beta_{\text{II}}, \beta_{\text{III}}$ lead to the delta function $\delta(\omega - \omega')$ when one calculates the $\alpha(\omega, \omega')$ amplitudes. These delta functions are essential in ensuring the validity of the unitarity relations (19), (20).

(b) Fulling and Davies produce perfectly good arguments in favour of the black body spectrum for the type (ii) trajectory based on the calculation of matrix elements of local field quantities. In this talk I have adopted a rather different approach (also tackled in [1], [2]) based on the calculation of the Bogolubov coefficients. These quantities are by definition time-independent, and in this context the question as to where and when the photons are produced simply does not arise (this remark goes in parallel to (a) above). Similarly attempts to distinguish between “transient” and “steady state” radiation at the level of the $\alpha$ and $\beta$ amplitudes are bound to fail; the emphasis in the literature on the importance of the asymptotic part of the trajectory has unfortunately led to the use of such terminology (see e.g. [7]).

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