Coupled Whispering Gallery Modes under Kerr-Nonlinearity

Jeong-Bo Shim,1 Peter Schlagheck,2 Martina Hentschel,3 and Jan Wiersig1

1 Institut für Theoretische Physik, Otto-von-Guericke-Universität Magdeburg, D-39016, Magdeburg, Germany
2 Département de Physique, Université de Liège, 4000, Liège, Belgium
3 Institut für Physik, Technische Universität Ilmenau, D-98693, Ilmenau, Germany

(Dated: March 12, 2014)

The effect of Kerr-nonlinearity on coupled whispering gallery modes is studied. As a model system, we choose an add-drop filter, which comprises a circular microcavity and two side-coupled waveguides. The coupling of counter propagating whispering gallery modes is achieved by evanescent field’s scattering on the waveguides, and this coupling leads to doublets of whispering gallery modes. By numerically computing the transmission of the add-drop filter, the formation of a doublet whispering gallery mode is verified. With increase of Kerr-nonlinearity in the cavity, the doublets of whispering gallery modes exhibit the bistable transition in the beginning. However, when the nonlinearity surpasses a certain level, an unexpectedly low transmission is generated between the two transmission peaks of the doublet, which we name ‘anomalous transmission’. In order to analyse the anomalous transmission, the system is modelled by coupled mode theory and analysed in terms of nonlinear dynamics. This analysis reveals that the stable- and the unstable branches of a WGM doublet are merged at the frequency where the anomalous transmission occurs, as the nonlinearity exceeds a certain limit. From this observation, we can find a clue, that this merger can locally break the stability and cause the observed anomalous transmission.

PACS numbers:

I. INTRODUCTION

A whispering gallery mode (WGM) in a microcavity is recognized for its easy experimental realization and high cavity quality factor (cavity $Q$-factor). Using such properties of WGMs, high intensity can be easily induced in a micrometer-scale optics, and the induced high intensity allows for nonlinear optical effects such as Kerr-nonlinear effect [1], which requires high intensity in conventional optics. The Kerr-nonlinear effect is a sort of $\chi^{(3)}$-nonlinear optical effect, in which the change of refractive index is proportional to the squared value of electric field.

A pronounced effect of Kerr-nonlinearity on an optical mode is the bistability of a resonance. If a resonant mode is excited in an optical resonator filled with Kerr-nonlinear material, every point on the resonance profile would see different refractive index, due to the enhanced intensity. This difference in refractive index results in the shift of the resonant frequency. If the shift surpass a certain threshold, the whole resonance profile appears to lean to one side, and a part of it would obtain bistability, i.e. two stable amplitudes correspond to a given driving frequency [1]. This process that a whole resonance profile obtains the bistability by increasing nonlinearity is called “bistable transition”.

A microcavity filled with Kerr-nonlinear material can of course exhibit the bistability and the bistable transition of a WGM. Up to now, the bistability of a WGM has been studied for various applications, such as optical switch or filter [2, 3]. In this work, however, we are going to focus on some other aspect of the Kerr-effect in a microcavity, namely the relationship between the dynamical tunneling and the cubic nonlinearity. As other solutions of Maxwell’s equations, a WGM has an intrinsic time-reversal symmetry, which manifests itself in two counter-propagating waves. Owing to the high cavity-$Q$ factor again, the degeneracy of a WGM can be lifted by a small perturbation, such as Rayleigh scattering [4, 5]. The doublet of a WGM can be interpreted as a mode coupled by tunneling between two separate dynamical states, i.e. clockwise and counterclockwise propagating waves. Since these two coupled waves occupy the same spatial domain, we can find an analogy with dynamical tunneling, in contrast to a conventional tunneling between spatially separate domains. Such a bistable transition of a WGM doublet is already experimentally observed [5].

For a conventional resonant tunneling, it is well known that a cubic nonlinearity such as Kerr-nonlinearity can suppress the tunneling rate, because the nonlinear effect can break the symmetry when the whole system is unevenly excited. However, some different effect of nonlinearity is expected for the situation of dynamical tunneling, because dynamical tunneling can occur between spatially overlapped domains with different dynamical status. Hence, even if one domain involved in a dynamical tunneling is selectively excited, the nonlinearity can spatially affect both of them. In the context of the contrast to conventional tunneling, the influence of cubic nonlinearity on dynamical tunneling is also worthwhile being investigated. To our best knowledge, there has been no study on this topic.

As the Kerr-nonlinearity provides a cubic nonlinear term of amplitude in an optical wave equation, it can be classified as a cubic nonlinearity. Such a cubic nonlinearity in a wave equation also appears when a many-body interaction is involved. For instance, a Bose-Einstein condensate with a high density can be described by
Schrödinger equation with a cubic nonlinearity induced by the mean-field approximation. Hence, the subject of this work can be generalized in other physical systems.

To investigate the coupled WGMs in a microcavity, we choose an ‘add-drop filter’\cite{6} as a model system. Then, we analyze the mode dynamics of the system under the influence of Kerr-nonlinearity, using numerical and theoretical models. For a better comparative study with BEC systems, we set a negative value for $\chi^{(3)}$ in this work, which induces defocusing Kerr-effect. The negative Kerr effect is experimentally feasible currently in optics\cite{8}.

This article is organised as follows: in the next section, the add-drop filter will be first introduced. Then, the numerically computed transmission and the feature of doublets in the numerical result will be presented. In Section III the effect of nonlinearity will be investigated. Concerning the emergence of an unexpected transmission peak, we will build up a theoretical model, using ‘Coupled-Mode Theory (CMT)’\cite{6}, and by analyzing the dynamics of this model, the physical background behind the observed phenomena will be tackled.

II. ADD-DROP FILTER AND CMT MODELING

The add-drop filter consists of a microcavity and two optical fibers side-coupled to the cavity through evanescent field. The two wave guide provides effectively four channels. In Fig. 1 the four channels are indexed by $\pm i$ ($i = 1, 2, 3, 4$), where the sign $\pm$ denotes in-coming (+) and out-going (−) waveguide modes of each channel.

In this add-drop filter, the shortest distance between the cavity boundary and the wave guide is given by $0.1 R$, where $R$ is the radius of the cavity, and the refractive index of the optical components is given by $n_0 = 2.0$. So as that a strong backscattering of evanescent field is induced and the degeneracy of a whispering gallery mode is thus more easily lifted, very narrow wave guides with the width of $0.04 R$ are put at the upper and the lower sides of the cavity.

In order to investigate the transmission of this system numerically, a continuous wave source with frequency $\Omega$ is set at channel 1, and the flux arriving in channel 2 is computed by means of ‘Finite Difference Time Domain’ method (FDTD)\cite{7}.

If a mode with an extremely high cavity-$Q$ factor is excited in the microcavity, the coupling via scattering can lift the intrinsic degeneracy of the doublet state and split the modes\cite{2,4}. By the FDTD simulation, such a coupling is observed. Figure 2 shows that a whispering gallery mode with the radial mode number $l = 1$ and the azimuthal mode number $m = 28$ is coupled to the wave guide. As the cavity mode has an extremely high $Q$ factor, the transmission curve exhibits two split downward peaks, and each peak corresponds to a different parity with respect to reflectional symmetric axis. Figures 2(b) and (c) show the two modes corresponding to the left and the right peaks in the transmission curve in Fig. 2(a).

![FIG. 1: (color online) Structure of the add-drop filter: a circular microcavity with radius $R$ is coupled to two wave guides. The coupling is through overlapped evanescent fields. The width of the waveguides is $0.04 R$ and the closest gap between the cavity and the waveguides is $0.1 R$.](image1)

![FIG. 2: (color online) (a) Transmission at channel 2. Due to the scattering on the waveguides and the high-$Q$ factor of the resonance, the transmission peak is split. Each peak holds different parity with respect to the symmetric axes of the whole system ((b) odd (c) even).](image2)
which show even and odd symmetry, respectively. The splitting of a whispering gallery mode is analyzed by using ‘Coupled Mode Theory’. For the sake of intuitive modeling, two counter-rotating modes with the same resonant frequency $\omega$ are first adopted as a basis.

$$\frac{da_c}{dt} = (i(\omega + g) - \gamma - \frac{1}{2}\Gamma)a_c + (ig - \frac{1}{2}\Gamma)a_{cc} + \eta_cs_{1+}, \tag{1}$$

$$\frac{da_{cc}}{dt} = (i(\omega + g) - \gamma - \frac{1}{2}\Gamma)a_{cc} + (ig - \frac{1}{2}\Gamma)a_c + \eta_cs_{1+}, \tag{2}$$

where the subscript $c$ and $cc$ stand for clockwise and counterclockwise modes, $g$ is the coupling parameter given by the scattering, and $\gamma$ and $\Gamma$ are the characteristic attenuation of the WGM and the additional attenuation induced by the scattering, respectively. Here, the characteristic attenuation refers to an intrinsic attenuation of an WGM, given by the mode configuration $^{10}$ $^{12}$. Although the source is put only in channel 1, it is necessary to assume the driving $\eta_c$, because the clockwise mode is driven through the scattering.

Since the modes associated with the two transmission peaks are two standing wave modes, the basis are transformed correspondingly to

$$\psi_+ = \psi_c + \psi_{cc} \quad \text{and} \quad \psi_- = \psi_c - \psi_{cc}. \tag{3}$$

For the circular microcavity, these modes are represented by the product of a Bessel function $J_m$ and a sin or cos function as

$$\psi_+ = J_m(nkR) \cos(m\phi)\hat{e}_z,$$

$$\psi_- = J_m(nkR) \sin(m\phi)\hat{e}_z. \tag{4}$$

Corresponding to the change of basis, the equations of motions in Eqs. (1) and (2) must be also rearranged in the form

$$\frac{da_+}{dt} = (i(\omega + 2g) - \gamma - \Gamma)a_+ + \eta_+s_{1+} \tag{5},$$

$$\frac{da_-}{dt} = (i\omega - \gamma)a_- + \eta_-s_{1+}, \tag{6}$$

where the driving strengths are given by

$$\eta_+ = \frac{\eta_c + \eta_{cc}}{2} \quad \text{and} \quad \eta_- = \frac{\eta_c - \eta_{cc}}{2i}. \tag{7}$$

Then, the transmission at the channel 2 can be formulated as

$$t = 1 - \frac{\eta_+^2 a_+ - \eta_-^2 a_-}{s_{1+}} \tag{8}$$

$$= 1 - \frac{|\eta_+|^2}{i(\omega + 2g - \Omega) - \gamma - \Gamma} - \frac{|\eta_-|^2}{i(\omega - \Omega) - \gamma}. \tag{9}$$

The parameters in Eq. (8) cannot be analytically evaluated due to the scattering. Thus, the values of the couplings and the attenuations are extracted by fitting the numerical result in Fig. 2 with Eq. (9).

$$|\eta_+|^2 = 5.043 \times 10^{-5},$$

$$|\eta_-|^2 = 1.159 \times 10^{-4},$$

$$\gamma = 1.118 \times 10^{-4},$$

$$\Gamma = 1.135 \times 10^{-4}. \tag{10}$$

In the process of fitting, the contribution of the neighboring modes to the transmission is carefully considered. In the spectrum of a circular microcavity, ($l, m$) = (20, 3)- and (24, 2)-modes ($l$: azimuthal mode number, $m$: radial mode number) are standing on the left-hand and the right-hand sides of the doublet, respectively. Their contributions to the transmission are extracted prior to the fitting.

### III. KERR NONLINEARITY IN NUMERICAL COMPUTATION AND CMT MODELING

In a Kerr-medium, electric field and electric displacement are related by the nonlinear equation

$$D = e_0(1 + \chi^{(3)})E + e_0\chi^{(3)}E^3. \tag{11}$$

By substituting a monochromatic electromagnetic wave $E = E_0 e^{i\omega t}$ into Eq. (11), the third-order nonlinear term in Eq. (11) leads to

$$(\text{Re}[E_0 e^{-i\omega t}])^3 = (\frac{1}{4})\text{Re}[E_0^3 e^{-3i\omega t}] + (\frac{3}{4})|E_0|^2\text{Re}[E_0 e^{-i\omega t}]. \tag{12}$$

By neglecting the third-harmonic term, the effective refractive index can be written in the following well-known form

$$n = n_0 + \frac{3\chi^{(3)}}{4e_0^2} |E|^2. \tag{13}$$

The effect of the Kerr-nonlinearity on the split whispering gallery modes is first numerically investigated. Figure 3 shows the evolution of the doublet with increase of the nonlinearity. As the nonlinearity increases, the resonance profiles in the transmission curves lean more toward the right-hand side. When the nonlinearity exceeds a certain value, it is noticed that an abrupt upward shift occurs in both of transmission peaks. This is an evidence of the bistable transition, because unstable mode dynamics is involved in the parts where the abrupt shifts occur.

The bistable transition progresses gradually and continuously with increase of nonlinearity, up to $\chi^{(3)} = 4.0 \times 10^{-7}$. However, from $\chi^{(3)} = 5.0 \times 10^{-7}$, an unexpected downward transmission peak emerges around the middle point between the two peaks of the doublet (denoted by red arrows in Fig. 2 (e) and (f)). A similar anomalous transmission is observed also in a focusing Kerr-nonlinear material.
FIG. 3: (color online) FDTD simulation of the transmission associated with the split whispering gallery modes. (a) $\chi^{(3)} = -1.0 \times 10^{-7}$ (b) $-2.0 \times 10^{-7}$ (c) $-3.0 \times 10^{-7}$ (d) $-4.0 \times 10^{-7}$ (e) $-5.0 \times 10^{-7}$ (f) $-6.0 \times 10^{-7}$. As $\chi^{(3)}$ surpasses $-4.0 \times 10^{-7}$, an anomalous peak emerges on the transmission curve (red arrows).

FIG. 4: (color online) Results of the CMT-analysis for the split whispering gallery modes under $\chi^{(3)}$ nonlinearity. With increase of nonlinearity, stationary states are identified and plotted. Stable and unstable branches of the profile are denoted by blue and red colors, respectively. (a) $\chi^{(3)} = -1.0 \times 10^{-7}$ (b) $-2.0 \times 10^{-7}$ (c) $-3.0 \times 10^{-7}$ (d) $-4.0 \times 10^{-7}$ (e) $-5.0 \times 10^{-7}$ (f) $-6.0 \times 10^{-7}$. As $\chi^{(3)}$ surpasses $-5.0 \times 10^{-7}$, a noticeable change at the point where the anomalous transmission starts occurring is that the unstable and the stable branch between two downward peaks get to meet and forms a kink (black arrows). The superimposed black dotted lines are transmission curves expected to be numerically observed in Fig. 3.
The numerically observed phenomena are analyzed by the CMT modeling. For this purpose, the Kerr effect is incorporated in the model by treating it as a perturbation. For other values necessary for the model, such as attenuations and resonant frequencies, the result of fitting in Eq. (10) are used. The electric field in the cavity has to be first represented with the basis functions, \( \psi_+ \) and \( \psi_- \), as follows:

\[
E = a_+(t)\psi_+(r) + ia_-(t)\psi_-(r). \tag{14}
\]

Then, the field intensity formed in the cavity is proportional to

\[
|E|^2 = (a_+^*\psi_+ - i a_-^*\psi_-) \cdot (a_+ \psi_+ + i a_- \psi_-). \tag{15}
\]

By applying the perturbation approach to Eq. (13), the equations of motions in Eqs. (5) and (6) are modified to

\[
\frac{da_+}{dt} = (i(\omega + 2g) - \gamma - \Gamma)a_+ + i \omega \kappa_+ a_+^2 + i \omega \kappa_-(2a_+|a_-|^2 - a_+^*a_-^2) + \eta_+ s_{1+}, \tag{16}
\]

\[
\frac{da_-}{dt} = (i(\omega - \gamma)a_- + i \omega \kappa_- |a_-|^2 a_- + i \omega \kappa_+(2a_-|a_+|^2 - a_-^*a_+^2) - i \eta_- s_{1+}, \tag{17}
\]

where \( \kappa_+, \kappa_- \) and \( \kappa_- \) are given by

\[
\kappa_{pq} = \frac{3\chi^{(3)}}{4\pi^2} \int |\psi_p|^2|\psi_q|^2dV, \quad p, q = + \text{ or } -. \tag{18}
\]

The result of the computation reveals the underlying profile of the full numerical solution. Figure 4 shows a bistable transition in the transmission profiles, following the increase of nonlinearity. As well known of the bistable transition, when the \( \chi^{(3)} \) nonlinearity exceeds a certain threshold, three stationary states correspond to one frequency in a bistable range. In this case, the two states with the highest and the lowest amplitudes have stable mode dynamics around themselves, whereas one in the middle has unstable dynamics.

By substituting Eqs. (20) and (21) into Eqs. (10) and (11), and separating the real and the imaginary parts, the equations of motion are reduced to

\[
i \Omega A_+ = (i(\omega + 2g) - \gamma - \Gamma)A_+ + i 3 \omega \kappa A_+^3 + i \omega \kappa(2A_+ A_-^2 - A_+ A_-^2 e^{-i(\phi_+ - \phi_-)}) + \eta_+ Ke^{-i\phi_+}, \tag{22}
\]

\[
i \Omega A_- = (i(\omega - \gamma)A_- + i 3 \omega \kappa A_-^3 + i \omega \kappa(2A_- A_+^2 - A_- A_+^2 e^{-i(\phi_- - \phi_+)}) - i \eta_- Ke^{-i\phi_-}. \tag{23}
\]

By numerically optimizing Eqs. (22) and (23) with changing nonlinearity \( \kappa \), stationary states are computed at a given driving frequency. To make the comparison to the full numerical solution easier, the obtained stationary states are converted to transmission amplitude by using Eq. (4). The result of the computation reveals the underlying profile of the full numerical solution. Figure 4 shows a bistable transition in the transmission profiles, following the increase of nonlinearity. As well known of the bistable transition, when the \( \chi^{(3)} \) nonlinearity exceeds a certain threshold, three stationary states correspond to one frequency in a bistable range. In this case, the two states with the highest and the lowest amplitudes have stable mode dynamics around themselves, whereas one in the middle has unstable dynamics.

In order to measure the stability of stationary states, another ansätze are introduced for \( a_+ \) and \( a_- \):

\[
a_+ = (\alpha_+ + i \beta_+) e^{i \Omega t} \quad \text{and} \quad a_- = (\alpha_- + i \beta_-) e^{i \Omega t}, \tag{24}
\]

where the real and the imaginary parts of complex amplitudes, \( \alpha \) and \( \beta \) are time-dependent unlike the ansätze in Eqs. (20) and (21).

By substituting these ansätze, the following four coupled equations are derived.
The stability of a coupled mode can be judged by the Jacobian matrix of the above four equations. If the eigenvalues of the Jacobian matrix are all negative, the associated stationary state has a stable mode dynamics around it. Otherwise, an unstable dynamics is involved [11]. The result of the stability test is encoded on the resonance profile with a color code. In Fig. 4, stable and unstable branches are denoted by blue and red colors, respectively.

Among the two stable branches in the bistable regime, the lower stable branch in the transmission curve has a relative smaller stability, thus the upper branch is more likely to be numerically observed. In Fig. 4, the curves connecting lower stable branches are superimposed in black dotted lines. The curves demonstrate a good agreement with numerical results in Fig. 3 except the anomalous transmission peaks.

However, at the position where the anomalous transmission occurs, an interesting change is noticed, namely the merger of stable and unstable branches (Fig. 4). As the nonlinearity passes between $\kappa = -4.0 \times 10^{-7}$, and $-5.0 \times 10^{-7}$, the stable and the unstable branches between the two peaks get overlapped and form a kink in between (denoted by black arrows in Fig. 4). By increasing nonlinearity further, the overlap gets broader. From the overlapped branches, we can expect that two different effects, which might disturb the transmission and lead to the anomalous transmission. First one is that the overlapped unstable branch can disturb the stability of transmission, and the second one is that the different parities of the overlapped branches can result in an interference.
The mode dynamics behind the phenomenon can be more clearly seen by plotting the amplitudes of the modes $A_+$ and $A_-$ separately. In Fig. 5, the same color code is used as in Fig. 4 and two mode pairs are distinguished by the tone of colors (bright for $(+)$ and dark for $(-)$). In the bistable transition, the two separate modes reveals an interesting property, that one mode shows a twist in its resonance profile, while the other mode leans to the right-hand side in the same spectral range. As the leaning resonance profile, the twisted parts also have two stable and one unstable branch. With increase of nonlinearity, the twisted part gets larger. In the end, two unstable branches in between two resonance peaks (denoted by thick lines in Fig. 5) meet and get connected, as $\chi^{(3)}$ surpasses $-4 \times 10^{-7}$. As the two unstable branches get connected, they are also crossed by two other stable branches. Therefore, the two different mode pairs gets to have the same amplitudes with different stability. The predicted curves in Fig. 4 (black dotted curves in Fig. 4) are also decomposed and superimposed in Fig. 5. As seen in Fig. 5, the feet of the curves meet the crossing points of stable and unstable branches, as $\chi^{(3)}$ surpasses $-4.0 \times 10^{-7}$. Thus, if the cavity is driven at the frequency where the crossing of branches occurs, the mode dynamics can diverge by the influence of the connected unstable branches. From this analysis, we can see the physics behind the numerically observed anomalous transmission in the Kerr-nonlinear microcavity.

**IV. CONCLUSION**

The effect of Kerr nonlinearity on coupled whispering gallery modes is studied numerically and theoretically. In this study, an optical add-drop filter which consists of a microcavity and two side-coupled wave guides. When a whispering gallery mode with extremely high $Q$ factor is induced in the microcavity, the intrinsic degeneracy of the whispering gallery mode can be lifted by coupling provided by scattering of evanescent field, and the corresponding transmission exhibits two split downward peaks. The evolution of these split peaks with increase of nonlinearity is first numerically investigated by using FDTD algorithm. In this computation, the transmission shows a typical bistable transition in the beginning. However, as the nonlinear parameter $\chi^{(3)}$ surpasses a certain limit, it shows a gradual appearance of an extraordinary low transmission. We call it an anomalous transmission, because it deviates from the normal bistable transition of a single resonance. The computation result of the bistable transition does not show a conspicuous change in tunneling time, i.e. the separation of two transmission peaks.

This anomalous transmission is theoretically addressed by using a CMT model. Using the perturbation approach, we incorporate the nonlinear optical effect in the CMT model, and obtain the dynamics of modes in microcavity. This theoretical approach displays a quantitative consistency with numerical observations, which is that a stable and an unstable branch get merged at the point where the anomalous transmission starts occurring. Therefore, a complex mode dynamics can arise at this point and lead to such an anomalous behavior.

Using the theoretical approach developed in this work, we anticipate that further issues in transport phenomena under cubic nonlinear effects can be tackled.

**Acknowledgments**

This work is financially supported by Deutsche Forschungsgemeinschaft (DFG) within the framework of Forschergruppe FOR760.

[1] Robert W. Boyd. *Nonlinear Optics*. Academic Press, 3rd edition, 2008.
[2] T. J. Kippenberg, S. M. Spillane, and K. J. Vahala. Modal coupling in traveling-wave resonators. *Optics Letters*, 27(19):1669, October 2002.
[3] Alejandro Rodriguez, Marin Soljacic, J. D. Joannopoulos, and Steven G. Johnson. $\chi^{(2)}$ and $\chi^{(3)}$ harmonic generation at a critical power in inhomogeneous doubly resonant cavities. *Optics Express*, 15(12):7303, June 2007.
[4] A. Mazzei, S. Götzinger, L. de S. Menezes, G. Zumofen, O. Benson, and V. Sandoghdar. Controlled Coupling of Counterpropagating Whispering-Gallery Modes by a Single Rayleigh Scatterer: A Classical Problem in a Quantum Optical Light. *Physical Review Letters*, 90(17):173603, October 2007.
[5] F Treussart, V. S Ilenko, J.-F Roch, J Hare, V Lefeuvre-Seguin, J-M Raimond, and S Haroche. Evidence for intrinsic Kerr bistability of high-Q microsphere resonators in superfluid helium. *Eur. Phys. J. D*, 1:235–238, 1998.
[6] C Manolatou, M J Khan, Shanhui Fan, Pierre R Villeneuve, H A Haus, Life Fellow, and J D Joannopoulos. Coupling of Modes Analysis of Resonant Channel Add Drop Filters. *IEEE Journal of Quantum Electronics*, 35(9):1322–1331, 1999.
[7] Ardavan F. Oskooi, David Roundy, Mihai Ibanescu, Peter Bermel, J.D. Joannopoulos, and Steven G. Johnson. Meep: A flexible free-software package for electromagnetic simulations by the FDTD method. *Computer Physics Communications*, 181(3):687–702, March 2010.
[8] Yan Li, Yuan Chen, Jin Yan, Yifan Liu, Jianpeng Cui, Qionghua Wang, and Shin-Tson Wu. Polymer-stabilized blue phase liquid crystal with a negative Kerr constant. *Optical Materials Express*, 2(8):1135, July 2012.
[9] Victor Grigoriev and Fabio Biancalana. Coupled-mode theory for on-channel nonlinear microcavities. *Journal of the Optical Society of America B*, 28(9):2165, August 2011.
[10] Jens Uwe Nöckel. *Resonances in Nonintegrable Open Sys-
tems. PhD thesis, 1997.

[11] Edward Ott. *Chaos in Dynamical Systems*. Cambridge University Press, London, 2nd edition, 2002.

[12] Jeong-bo Shim and Jan Wiersig. Semiclassical evaluation of frequency splittings in coupled optical microdisks. *Optics express*, 21(20):341–354, 2013.