Existence, Regularity and Compactness Properties in the $\alpha$-Norm for Some Partial Functional Integrodifferential Equations with Finite Delay

Boubacar Diao · Khalil Ezzinbi · Mamadou Sy

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Abstract In this work, we study in the $\alpha$-norm, the existence, the continuity dependence, regularity and compactness of solutions for some partial functional integro-differential equations by using the operator resolvent theory. We suppose that the linear part has a resolvent operator in the sense of Grimmer and Pritchard (J Diff Equ 50:234–259, 1983). The nonlinear part is assumed to be continuous with respect to a fractional power of the linear part in the second variable. An application is provided to illustrate our results.

Keywords Integrodifferential · Mild solution · Resolvent operator · Fractional power operator

Introduction

The main purpose of this work is to study the existence, regularity and compactness properties of a class for partial functional integrodifferential equations of retarded type with deviating arguments in terms involving spatial partial derivatives in the form

\begin{align}
\frac{d u(t)}{d t} &= -A u(t) + \int_{0}^{t} B(t - s) u(s) d s + F(t, u_{t}) \quad \text{for} \quad t \geq 0,
\quad u_{0} = \varphi \in C_{\alpha} = C([-r, 0], D(A^{\alpha})),
\end{align}

(1.1)
where \(-A\) is the infinitesimal generator of an analytic semigroup \((T(t))_{t \geq 0}\) on a Banach space \(X\). \(B(t)\) is a closed linear operator with domain \(D(B(t)) \supset D(A)\) time-independent. For \(0 < \alpha < 1\), \(A^\alpha\) is the fractional power of \(A\) which will be precised in the sequel. The domain \(D(A^\alpha)\), endowed with the norm \(\|x\|_\alpha = \|A^\alpha x\|\) is a Banach space. \(C_\alpha\) is the Banach space \(C([-r, 0], D(A^\alpha))\) of continuous functions from \([-r, 0]\) to \(D(A^\alpha)\) endowed with the following norm

\[
\|\phi\|_\alpha = \sup_{-r \leq \theta \leq 0} \|\phi(\theta)\|_\alpha \quad \text{for} \quad \phi \in C_\alpha.
\]

\(F : \mathbb{R}_+ \times C_\alpha \to X\) is a continuous function and as usual, the history function \(u_t \in C_\alpha\) is defined by

\[
u_t(\theta) = u(t + \theta) \quad \text{for} \quad \theta \in [-r, 0].
\]

As a model for this class one may take the following Lotka–Volterra equation

\[
\begin{aligned}
\frac{\partial u(t, x)}{\partial t} &= \frac{\partial^2 u(t, x)}{\partial x^2} + \int_0^t h(t - s) \frac{\partial^2 u(s, x)}{\partial x^2} ds + \int^0_{-r} g(t, \frac{\partial}{\partial x} u(t + \theta, x)) d\theta \\
&\quad \text{for} \quad t \geq 0 \quad \text{and} \quad x \in [0, \pi], \\
u(t, 0) &= \nu(t, \pi) = 0 \quad \text{for} \quad t \geq 0, \\
u(\theta, x) &= u_0(\theta, x) \quad \text{for} \quad \theta \in [-r, 0] \quad \text{and} \quad x \in [0, \pi],
\end{aligned}
\]

\[(1.2)\]

where

\[
u_0 : [-r, 0] \times [0, \pi] \to \mathbb{R}, \quad g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}
\]

and \(h : \mathbb{R}_+ \to \mathbb{R}\) are appropriate functions.

The theory of partial functional integrodifferential equations and its applications are an active area of research, see for instance [1, 10, 17, 20, 21] and the references therein. In the particular case where \(\alpha = 0\), that is we have the following integrodifferential Eq.

\[
\begin{aligned}
\frac{du(t)}{dt} &= Au(t) + \int_0^t B(t - s)u(s)ds + F(t, u_t) \quad \text{for} \quad t \geq 0, \\
u_0 &= \varphi \in C([-r, 0]; X),
\end{aligned}
\]

\[(1.3)\]

many results are obtained in the literature under various hypotheses concerning \(A, B\) and \(F\). In [5], Ezzinbi et al. investigated the existence and regularity of solutions of Eq. (1.3). The authors obtained also the uniqueness, the representation of solutions via a variation of constant formula and other properties of the resolvent operator were studied. In [4], Ezzinbi et al. studied a local existence and regularity of Eq. (1.3). To achieve their goal, the authors used the variation of constant formula, the theory of resolvent operator and the principle contraction method. Ezzinbi et al. in [6] studied the local existence and global continuation for Eq. (1.3). Recall that the resolvent operator plays an important role in solving Eq. (1.3) in the weak and strict sense, it replaces the role of the \(c_0\)-semigroup theory. For more details in this topic, here are the papers of Chen and Grimmer [1], Hannsgen [11], Smart [18], Miller [12, 13], and Miller and Wheeler [14, 15]. In the case where the non-linear part involves spatial derivative, the above obtained results become invalid. To overcome this difficulty, we shall restrict our problem in a Banach space \(Y_\alpha \subset X\), to obtain our main results for Eq. (1.1).

Considering the case where \(B = 0\), Travis et al. in [19] obtained results on the existence, stability, regularity and compactness of Eq. (1.1). To achieve their goal, the authors assumed that \(-A\) is the infinitesimal generator of a compact analytic semigroup and \(F\) is only continuous with respect to a fractional power of \(A\) in the second variable. The present paper is motivated by the paper of Travis et al. in [19].
The paper is organized as follows. In “Fractional Power of Closed Operators and Resolvent Operator for Integrodifferential Equations” section, we recall some fundamental properties of the resolvent operator and fractional powers of closed operators. The global existence, uniqueness and continuous dependence with respect to the initials data are studied in “Global Existence, Uniqueness and Continuous Dependence with Respect to the Initials Data” section. In “Local Existence, Blowing Up Phenomena and the Compactness of the Flow” section, we study the local existence and bowing up phenomena. In “Regularity of the Mild Solutions” section we prove under some conditions, the regularity of the mild solutions. And finally we illustrate our main results in “Application” section by examining an example.

Fractional Power of Closed Operators and Resolvent Operator for Integrodifferential Equations

In this section, we first introduce some notations and definitions which we will use in the whole work.

We shall write $Y$ for $D(A)$ endowed with the graph norm $\|x\|_Y = \|x\| + \|Ax\|$, $Y_\alpha$ for $D(A^\alpha)$ and $L(Y_\alpha, X)$ will denote the space of bounded linear operators from $Y_\alpha$ to $X$ and for $Y_0 = X$, we write $L(X)$ with norm $\|\cdot\|_{L(X)}$. We also frequently use the Laplace transform of $f$ which is denoted by $f^\star$. If we assume that $-A$ generates an analytic semigroup and, without loss of generality that $0 \in \rho(A)$, then one can define the fractional power $A^\alpha$ for $0 < \alpha < 1$, as a closed linear operator on its domain $Y_\alpha$ with its inverse $A^{-\alpha}$ given by

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} T(t) \, dt,$$

where $\Gamma$ is the Gamma function

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} \, dt.$$

We have the following known results.

**Theorem 2.1** [16] The following properties are true.

(i) $Y_\alpha = D(A^\alpha)$ is a Banach space with the norm $|x|_\alpha = \|A^\alpha x\|$ for $x \in Y_\alpha$.

(ii) $A^\alpha$ is a closed linear operator with domain $Y_\alpha = \text{Im}(A^{-\alpha})$ and $A^\alpha = (A^{-\alpha})^{-1}$.

(iii) $A^{-\alpha}$ is a bounded linear operator in $X$.

(iv) If $0 < \alpha \leq \beta$ then $D(A^\beta) \hookrightarrow D(A^\alpha)$. Moreover the injection is compact if $T(t)$ is compact for $t > 0$.

Now, we collect the definition and basic results about the theory of resolvent operator for integrodifferential equations in Banach spaces.

**Definition 2.2** [7] A family of bounded linear operators $(R(t))_{t \geq 0}$ in $X$ is called resolvent operator for the following equation

$$\begin{cases}
\frac{d}{dt} u(t) = -Au(t) + \int_0^t B(t-s)u(s) \, ds \quad \text{for} \quad t \geq 0, \\
u(0) = u_0 \in X,
\end{cases}$$

(2.1)

if

(a) $R(0) = I$ and $\|R(t)\| \leq M_1 \exp(\sigma t)$ for some $M_1 \geq 1$ and $\sigma \in \mathbb{R}$,

(b) For all $x \in X$, $t \mapsto R(t)x$ is continuous for $t \geq 0$,
(c) \( R(t) \in \mathcal{L}(\mathbb{Y}) \) for \( t \geq 0 \). For \( x \in \mathbb{Y} \), \( R(.)x \in C^1(\mathbb{R}_+, \mathbb{X}) \cap C(\mathbb{R}_+, \mathbb{Y}) \) and for \( t \geq 0 \) we have
\[
R'(t)x = -AR(t)x + \int_0^t B(t-s)R(s)xds \\
= -R(t)Ax + \int_0^t R(t-s)B(s)xds.
\]

It what follows, we assume the hypothesis taken from [8] which implies the existence of an analytic resolvent operator \( (R(t))_{t \geq 0} \).

(V1) \(-A \) generates an analytic semi-group on \( \mathbb{X} \). \( (B(t))_{t \geq 0} \) is a closed operator on \( \mathbb{X} \) with domain at least \( D(A) \) a.e \( t \geq 0 \) with \( B(t)x \) strongly measurable for each \( x \in \mathbb{Y} \) and \( \|B(t)x\| \leq b(t)\|x\|_{\mathbb{Y}} \), for \( b \in L^1_{loc}(0, \infty) \) with \( b^*(\lambda) \) absolutely convergent for \( \Re \lambda > 0 \).

(V2) \( \rho(\lambda) = (\lambda I + A - B^*(\lambda))^{-1} \) exists as a bounded operator on \( \mathbb{X} \) which is analytic for \( \lambda \in \Lambda = \{ \lambda \in \mathbb{C} : |\arg \lambda| < \pi/2 + \delta \} \), where \( 0 < \delta < \pi/2 \). In \( \Lambda \) if \( |\lambda| \geq \epsilon > 0 \) there exists \( M = M(\epsilon) > 0 \) so that \( \|\rho(\lambda)\| \leq M/|\lambda| \).

(V3) \( A\rho(\lambda) \in \mathcal{L}(\mathbb{X}) \) for \( \lambda \in \Lambda \) and is analytic from \( \Lambda \) to \( \mathcal{L}(\mathbb{X}) \). \( B^*(\lambda) \rho(\lambda) \in \mathcal{L}(\mathbb{Y}, \mathbb{X}) \) for \( \lambda \in \Lambda \). Given \( \epsilon > 0 \), there exists a positive constant \( M = M(\epsilon) \) so that \( \|A\rho(\lambda)x\| + \|B^*(\lambda)\rho(\lambda)x\| \leq (M/|\lambda|)\|x\|_{\mathbb{Y}} \) for \( x \in \mathbb{Y} \) and \( \lambda \in \Lambda \) with \( \lambda \geq \epsilon \) and \( \|B^*(\lambda)\| \to 0 \) as \( |\lambda| \to \infty \) in \( \Lambda \). In addition, \( \|A\rho(\lambda)x\| \leq M|\lambda|^n\|x\| \) for some \( n > 0 \), \( \lambda \in \Lambda \) with \( \lambda \geq \epsilon \). Further, there exists \( D \subset D(A^2) \) which is dense in \( \mathbb{Y} \) such that \( A(D) \) and \( B^*(\lambda)(D) \) are contained in \( \mathbb{Y} \) and \( \|B^*(\lambda)x\|_{\mathbb{Y}} \) is bounded for each \( x \in D \) and \( \lambda \in \Lambda \) with \( |\lambda| \geq \epsilon \).

**Theorem 2.3** [8] Assume that conditions (V1)–(V3) are satisfied. Then there exists an analytic resolvent operator \( (R(t))_{t \geq 0} \) for the linear equation (2.1). Moreover, there exist positive constants \( N, N_\alpha \) such that \( \|R(t)\| \leq N \) and \( \|A^\alpha R(t)\| \leq N_\alpha /t^{\alpha} \) for \( t > 0 \) and \( 0 \leq \alpha < 1 \).

We assume the following hypothesis.

(H0) The semigroup \( (T(t))_{t \geq 0} \) is compact for \( t > 0 \).

**Theorem 2.4** [3] Assume that the hypotheses (VI)–(V3) and (H0) hold. Then the corresponding resolvent operator \( (R(t))_{t \geq 0} \) of Eq. (2.1) is also compact for \( t > 0 \).

We give in next, the definition of the so-called strict and mild solutions. Consider the following nonhomogeneous equation
\[
\begin{aligned}
\frac{du(t)}{dt} &= -Au(t) + \int_0^t B(t-s)u(s)ds + f(t) \quad \text{for } t \in [0, b], \\
u(0) &= u_0 \in \mathbb{X}.
\end{aligned}
\]

**Definition 2.5** [7] A continuous function \( u : [0, b] \to \mathbb{X} \) is called a strict solution of the Eq. (2.3) if
(i) \( t \to u(t) \) is continuously differentiable on \([0, b]\),
(ii) \( u(t) \in \mathbb{Y} \) for \( t \in [0, b] \),
(iii) \( u \) satisfies Eq. (2.3) on \([0, b]\).

**Theorem 2.6** [7] If \( u \) is a strict solution of Eq. (2.3), then \( u \) is given by the following formula
\[
u(t) = R(t)u_0 + \int_0^t R(t-s)f(s)ds \quad \text{for } t \in [0, b].
\]
Remark 2.7 If \( u \) satisfies formula (2.4), \( u \) is not generally a strict solution of Eq. (2.3).

Definition 2.8 A continuous function \( u : [0, b] \to \mathbb{Y} \) is called a mild solution of Eq. (2.3) if \( u \) satisfies Eq. (2.4).

Global Existence, Uniqueness and Continuous Dependence with Respect to the Initials Data

This section is asserted to the results of global existence and continuous dependence with respect to the initials data. We give the definitions of the so-called mild and strict solutions of Eq. (1.1).

Definition 3.1 A function \( u : [0, b] \to \mathbb{Y}_\alpha \) is called a strict solution of Eq. (1.1), if

(i) \( t \to u(t) \) is continuously differentable on \([0, b]\),
(ii) \( u(t) \in \mathbb{Y} \) for \( t \in [0, b] \),
(iii) \( u \) satisfies Eq. (1.1) on \([0, b]\).

Definition 3.2 A continuous function \( u : [0, b] \to \mathbb{Y}_\alpha \) is called a mild solution of Eq. (1.1) if

\[
\begin{aligned}
\begin{cases}
  u(t) = R(t)\varphi(0) + \int_0^t R(t-s)F(s, u_s)ds & \text{for } t \in [0, b], \\
u_0 = \varphi \in \mathbb{C}_\alpha.
\end{cases}
\end{aligned}
\]

(H1) There exists a constant \( L_F > 0 \) such that

\[
\|F(t, \varphi_1) - F(t, \varphi_2)\| \leq L_F \|\varphi_1 - \varphi_2\|_\alpha \quad \text{for } t \geq 0 \text{ and } \varphi_1, \varphi_2 \in \mathbb{C}_\alpha.
\]

Theorem 3.3 Assume that (VI)–(V3) and (H1) hold. Then for \( \varphi \in \mathbb{C}_\alpha, \) Eq. (1.1) has an unique mild solution which is defined for all \( t \geq 0 \).

Proof Let \( a > 0 \). For \( \varphi \in \mathbb{C}_\alpha, \) we define the set \( \land \) by

\[
\land = \{ y \in C([0, a]; \mathbb{Y}_\alpha) : y(0) = \varphi(0) \}.
\]

The set \( \land \) is a closed subset of \( C([0, a]; \mathbb{Y}_\alpha) \) where \( C([0, a]; \mathbb{Y}_\alpha) \) is the space of continuous functions from \([0, a]\) to \( \mathbb{Y}_\alpha \) equipped with the uniform norm topology

\[
\|y\|_\alpha = \sup_{0 \leq t \leq a} \|y(t)\|_\alpha \quad \text{for } y \in C([0, a]; \mathbb{Y}_\alpha).
\]

For \( y \in \land \), we introduce the extension \( \bar{y} \) of \( y \) on \([-r, a] \) defined by \( \bar{y}(t) = y(t) \) for \( t \in [0, a] \) and \( \bar{y}(t) = \varphi(t) \) for \( t \in [-r, 0] \). We consider the operator \( \Gamma \) defined on \( \land \) by

\[
\Gamma y(t) = R(t)\varphi(0) + \int_0^t R(t-s)F(s, \bar{y}_s)ds \quad \text{for } t \in [0, a].
\]

We claim that \( \Gamma(\land) \subseteq \land \). In fact for \( y \in \land \) we have \( (\Gamma y)(0) = \varphi(0) \) and by continuity of \( F \) and \( R(t)x \) for \( x \in \mathbb{X} \), we deduce that \( \Gamma y \in \land \). In order to obtain our result, we apply the strict contraction principle. In fact, let \( u, v \in \land \) and \( t \in [0, a] \). Then

\[
(\Gamma(u) - \Gamma(v))(t) = \int_0^t R(t-s)(F(s, \bar{u}_s) - F(s, \bar{v}_s))ds.
\]
Using the $\alpha$-norm, we have
\[
\left\| A^\alpha (\Gamma(u) - \Gamma(v))(t) \right\| \leq L_F N_\alpha \int_0^t \frac{1}{(t-s)^\alpha} \| \bar{u}_s - \bar{v}_s \|_\alpha \ ds
\]
\[
\leq L_F N_\alpha \int_0^t \frac{1}{(t-s)^\alpha} \sup_{0 \leq \tau \leq a} \| u(\tau) - v(\tau) \|_\alpha \ ds
\]
\[
\leq \left( L_F N_\alpha \int_0^a \frac{ds}{s^\alpha} \right) \| u - v \|_\alpha.
\]
Now we choose $\alpha$ such that
\[
L_F N_\alpha \int_0^a \frac{ds}{s^\alpha} < 1.
\]
Then $\Gamma$ is a strict contraction on $\&$ and it has an unique fixed point $y$ which is the unique mild solution of Eq. (1.1) on $[0, a]$. To extend the solution of Eq. (1.1) in $[a, 2a]$, we show that the following equation has an unique mild solution.
\[
\begin{cases}
\frac{d}{dt} z(t) = -A z(t) + \int_a^t B(t-s)z(s) \ ds + F(t, z_t) & \text{for } t \in [a, 2a], \\
z_a = y_a \in C \left( [r, a]; \mathbb{R}_+ \right).
\end{cases}
\tag{3.2}
\]
Notice that the solution of Eq. (3.2) is given by
\[
z(t) = R(t-a)z(a) + \int_a^t R(t-s)F(s, z_s) \ ds & \text{for } t \in [a, 2a].
\]
Let $\tilde{z}$ be the function defined by $\tilde{z}(t) = z(t)$ for $t \in [a, 2a]$ and $\tilde{z}(t) = y(t)$ for $t \in [-r, a]$. Consider now again the set $\&$ defined by
\[
\& = \{ z \in C([a, 2a]; \mathbb{R}_+) : z(a) = y(a) \},
\]
provided with the induced topological norm. We define the operator $\Gamma_a$ on $\&$ by
\[
(\Gamma_a z)(t) = R(t-a)z(a) + \int_a^t R(t-s)F(s, z_s) \ ds & \text{for } t \in [a, 2a].
\]
We have $(\Gamma_a z)(a) = y(a)$ and $\Gamma_a z$ is continuous. Then it follows that $\Gamma_a \& \subset \&$. Moreover, for $u, v \in \&$, one has
\[
\left\| A^\alpha (\Gamma_a(u) - \Gamma_a(v))(t) \right\| \leq L_F N_\alpha \int_a^t \frac{1}{(t-s)^\alpha} \| \bar{u}_s - \bar{v}_s \|_\alpha \ ds.
\]
Since $\bar{u} = \bar{v} = \varphi$ in $[-r, 0]$, we deduce that
\[
\left\| A^\alpha (\Gamma_a(u) - \Gamma_a(v)) \right\| \leq \left( L_F N_\alpha \int_0^a \frac{ds}{s^\alpha} \right) \| u - v \|_\alpha.
\]
Then we deduce that $\Gamma_a$ has an unique fixed point in $\&$ which extends the solution $y$ in $[a, 2a]$. Proceeding inductively, $y$ is uniquely and continuously extended to $[na, (n+1)a]$ for all $n \geq 1$ and this ends the proof. \qed

Now we show the continuous dependence of the mild solutions with respect to the initial data.
Theorem 3.4 Assume that (VI)–(V3) and (HI) hold. Then the mild solution \( u(., \varphi) \) of equation (1.1) defines a continuous Lipschitz operator \( U(t) \), \( t \geq 0 \) in \( C_\alpha \) by \( U(t)\varphi = u_t(., \varphi) \). That is \( U(t)\varphi \) is continuous from \( [0; \infty) \) to \( C_\alpha \) for each fixed \( \varphi \in C_\alpha \). Moreover there exist a real number \( \delta \) and a scalar function \( P \) such that for \( t \geq 0 \) and \( \varphi_1, \varphi_2 \in C_\alpha \) we have

\[
\| U(t)\varphi_1 - U(t)\varphi_2 \| \leq P(\delta) e^{\delta t} \| \varphi_1 - \varphi_2 \|_\alpha. \tag{3.3}
\]

Proof We use the Gamma formula

\[
\Gamma(1 - \alpha)k^{\alpha-1} = \int_0^\infty e^{-ks}s^{\alpha-1} ds,
\]

where \( k > 0 \) (see [9], p. 265). The continuity is obvious on what the map \( t \to u_t(., \varphi) \) is continuous. Now, let \( \varphi_1, \varphi_2 \in C_\alpha \). If we pose \( w(t) = u(t, \varphi_1) - u(t, \varphi_2) \), then we have

\[
\| w(t) \|_\alpha \leq M_1 e^{\alpha t} \| \varphi_1 - \varphi_2 \|_\alpha + L_F N_\alpha \int_0^t \frac{e^{\sigma(t-s)}}{(t-s)^\alpha} \| w_s \|_\alpha ds. \tag{3.4}
\]

Let \( \delta \) a real number be such that

\[
\sigma - \delta < 0 \quad \text{and} \quad N_\alpha \max\{e^{-\delta r}, 1\} L_F \Gamma(1 - \alpha)(\delta - \sigma)^{\alpha-1} < 1.
\]

We define the function \( P \) by

\[
P(\delta) = M_1 M_3 M_4 (1 - N_\alpha M_4 L_F \Gamma(1 - \alpha)(\delta - \sigma)^{\alpha-1})^{-1},
\]

where

\[
M_3 = \max\{e^{\delta r}, 1\}, \quad M_4 = \max\{e^{-\delta r}, 1\}.
\]

Fix \( \bar{t} > 0 \) and let \( E = \sup_{0 \leq s \leq \bar{t}} e^{-\delta s} \| w_s \| \). If \( 0 \leq \tau \leq \bar{t} \), then from (3.4), we have

\[
e^{-\delta \tau} \| w(\tau) \|_\alpha \leq M_1 e^{(\sigma-\delta)\tau} \| \varphi_1 - \varphi_2 \|_\alpha + L_F N_\alpha \int_0^\tau \frac{e^{\sigma(t-s)}}{(t-s)^\alpha} e^{-\delta s} \| w_s \|_\alpha ds
\]

\[
\leq M_1 \| \varphi_1 - \varphi_2 \|_\alpha + L_F N_\alpha E \Gamma(1 - \alpha)(\delta - \sigma)^{\alpha-1}. \tag{3.5}
\]

If \( -r \leq \tau \leq 0 \) we have

\[
e^{-\delta \tau} \| w(\tau) \|_\alpha \leq \| \varphi_1 - \varphi_2 \|_\alpha M_3. \tag{3.6}
\]

Therefore, (3.5) and (3.6) imply that

\[
\sup_{-r \leq \tau \leq \bar{t}} e^{-\delta \tau} \| w(\tau) \|_\alpha \leq M_1 M_3 \| \varphi_1 - \varphi_2 \|_\alpha + L_F N_\alpha E \Gamma(1 - \alpha)(\delta - \sigma)^{\alpha-1}. \tag{3.7}
\]

For \( 0 \leq t \leq \bar{t} \) we have

\[
e^{-\delta t} \| w_t \|_\alpha = \sup_{-r \leq \theta \leq 0} e^{\delta \theta} e^{-\delta(t+\theta)} \| w(t + \theta) \|_\alpha
\]

\[
\leq M_4 \sup_{-r \leq \theta \leq 0} e^{-\delta(t+\theta)} \| w(t + \theta) \|_\alpha
\]

\[
\leq M_4 \sup_{-r \leq \tau \leq \bar{t}} e^{-\delta \tau} \| w(\tau) \|_\alpha. \tag{3.8}
\]

Then from (3.7) and (3.8) we deduce that for \( 0 \leq t \leq \bar{t} \)

\[
e^{-\delta t} \| w_t \|_\alpha \leq M_1 M_3 M_4 \| \varphi_1 - \varphi_2 \|_\alpha + L_F N_\alpha M_4 E \Gamma(1 - \alpha)(\delta - \sigma)^{\alpha-1},
\]
which implies that
\[ E \leq M_1M_3M_4\|\varphi_1 - \varphi_2\|_a + L_F N_\alpha M_4 E \Gamma(1 - \alpha)(\delta - \sigma)^{a - 1}. \]

Then the result follows. \qed

**Local Existence, Blowing up Phenomena and the Compactness of the Flow**

In this section, we establish the local existence and blowing up phenomena under assumption that \( F \) is continuous. We also study the compactness of the flow. To achieve our goal, we need the following Lemma used in [3] in the case of the usual norm on \( \mathbb{X} \) (\( \alpha = 0 \)). It will be seen in the case of \( \alpha \)-norm. We take the following assumption.

\((H2)\) \( B(t) \in \mathcal{L}(\mathbb{X}_\beta, \mathbb{X}) \) for some \( 0 < \beta < 1 \), a.e \( t \geq 0 \) and \( \|B(t)x\| \leq b(t)\|x\|_\beta \) for \( x \in \mathbb{X}_\beta \), with \( b \in L^q_{loc}(0, \infty) \) where \( q > 1/(1 - \beta) \).

**Theorem 4.1** Assume that \( (V1)-(V3) \) and \( (H2) \) holds. Then for any \( a > 0 \), there exists a positive constant \( M = M(a) \) such that for \( x \in \mathbb{X} \) we have
\[ \left\| A^a \left( R(t+h)x - R(h)R(t)x \right) \right\| \leq M \int_0^h \frac{ds}{s^a}\|x\| \quad \text{for} \quad 0 \leq h < t \leq a. \]

**Proof** Let \( a > 0 \) and \( x \in \mathbb{X} \). Then
\[
\frac{d}{dt} R(t+h)x = -AR(t+h)x + \int_t^{t+h} B(t+s)R(s)x \, ds \\
= -AR(t+h)x + \int_t^t B(t-s)R(s+h)x \, ds \\
+ \int_t^{t+h} B(t+s)R(s)x \, ds.
\]

We deduce that \( R(t+h)x \) satisfies the equation of the form
\[
\frac{d}{dt} y(t) = -Ay(t) + \int_0^t B(t-s)y(s) \, ds + f(t).
\]

Then by Theorem 2.4, it follows that
\[
R(t+h)x = R(t)R(h)x + \int_0^t R(t-s) \int_s^{s+h} B(s+h-u)R(u)x \, du \, ds \\
= R(h)R(t)x + \int_0^h R(h-s) \int_0^t B(u)R(s+t-u)x \, du \, ds,
\]

which yields that
\[
R(t+h)x - R(h)R(t)x = \int_0^h R(h-s) \int_0^t B(u)R(s+t-u)x \, du \, ds.
\]
Taking the $\alpha$-norm, we obtain that
\[
\left\| A^\alpha \left( R(t+h)x - R(h)R(t)x \right) \right\| \leq N_\alpha \int_0^h \frac{1}{(h-s)^\alpha} \int_0^t B(u) R(s + t - u)x \, du \, ds \\
\leq N_\alpha \int_0^h \frac{1}{(h-s)^\alpha} \int_0^t b(u) \| A^\beta R(s - u)x \| \, du \, ds \\
\leq N_\alpha N_\beta \int_0^h \frac{ds}{(h-s)^\alpha} \int_0^t b(u) \|x\| \, du.
\]

Let $p$ such that $1/q + 1/p = 1$, so $p < 1/\beta$. Then it follows that
\[
\left\| A^\alpha \left( R(t+h)x - R(h)R(t)x \right) \right\| \leq N_\alpha N_\beta \|b\|_{L^q(0_a)} \|u^{\beta}\|_{L^p(0_a)} \int_0^h \frac{ds}{s^\alpha} \|x\|.
\]

And the proof is complete. \hfill \Box

The local existence result is given by the following Theorem.

**Theorem 4.2** Suppose that (V1)–(V3), (H0) and (H2) hold. Moreover, assume that $F$ defined from $J \times \Omega$ into $\mathbb{X}$ is continuous where $J \times \Omega$ is an open set in $\mathbb{R}_+ \times C_\alpha$. Then for each $\varphi \in \Omega$, Eq. (1.1) has at least one mild solution which is defined on some interval $[0, b]$.

**Proof** Let $\varphi \in \Omega$. For any real $\zeta \in J$ and $p > 0$, we define the following sets
\[I_\zeta = \{ t : 0 \leq t \leq \zeta \} \text{ and } H_p = \{ \phi \in C_\alpha : \|\phi\|_\alpha \leq p \}.
\]

For $\phi \in H_p$, we choose $\zeta$ and $p$ such that $(t, \phi + \varphi) \in I_\zeta \times H_p$ and $H_p \subseteq \Omega$. By continuity of $F$, there exists $N_1 \geq 0$ such that $\|F(t, \phi + \varphi)\| \leq N_1$ for $(t, \phi)$ in $I_\zeta \times H_p$. We consider $\bar{\varphi} \in C([-r, \zeta]; \mathbb{Y}_\alpha)$ be the function defined by $\bar{\varphi}(t) = R(t)\varphi(0)$ for $t \in I_\zeta$ and $\bar{\varphi}_0 = \varphi$. Suppose that $\bar{p} < p$ and choose $0 < b < \zeta$ such that
\[N_\alpha N_1 \int_0^b \frac{ds}{s^\alpha} < \bar{p} \text{ and } \|\bar{\varphi}_t - \varphi\|_\alpha \leq p - \bar{p} \text{ for } t \in I_b.
\]

Let $K_0 = \{ \eta \in C([-r, b]; \mathbb{Y}_\alpha) : \eta_0 = 0 \text{ and } \|\eta_t\|_\alpha \leq \bar{p} \text{ for } 0 \leq t \leq b \}$. Then we have $\|F(t, \bar{\varphi}_t + \eta)\| \leq N_1$ for $0 \leq t \leq b$ and $\eta \in K_0$, since $\|\eta_t + \bar{\varphi}_t - \varphi\|_\alpha \leq p$. Consider the mapping $S$ from $K_0$ to $C([-r, b]; \mathbb{Y}_\alpha)$ defined by $(S\eta)(0) = 0$
\[S\eta(t) = \int_0^t R(t - s) F(s, \eta_s + \bar{\varphi}_s) \, ds \text{ for } 0 \leq t \leq b. \quad (4.1)
\]

Notice that finding a fixed point of $S$ in $K_0$ is equivalent to find a mild solution of Eq. (1.1) in $K_0$. Furthermore, $S$ is a mapping from $K_0$ to $K_0$, since if $\eta \in K_0$ we have $(S\eta)_0 = 0$ and
\[
\| (S\eta)(t) \|_\alpha \leq \int_0^t \left\| A^\alpha R(t - s) F(s, \eta_s + \bar{\varphi}_s) \right\| \, ds.
\]

Then
\[
\| (S\eta)(t) \|_\alpha \leq N_\alpha N_1 \int_0^t \frac{ds}{(t - s)^\alpha} \leq N_\alpha N_1 \int_0^b \frac{ds}{s^\alpha} < \bar{p}.
\]

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which implies that \( S(K_0) \subset K_0 \). We claim that \( \{(S\eta)(t) : \eta \in K_0\} \) is compact in \( \mathcal{Y}_\alpha \) for fixed \( t \in [-r, b] \). In fact, let \( \beta \) be such that \( 0 < \alpha < \beta < 1 \). The above estimate show that \( \{A^{\beta}(S\eta)(t) : \eta \in K_0\} \) is bounded in \( \mathcal{Y} \). Since \( A^{\alpha-\beta} \) is compact operator, we infer that \( \{A^{\alpha-\beta}A^{\beta}(S\eta)(t) : \eta \in K_0\} \) is compact in \( \mathcal{Y} \), hence \( \{(S\eta)(t) : \eta \in K_0\} \) is compact in \( \mathcal{Y}_\alpha \). Next, we show that \( \{(S\eta)(t) : \eta \in K_0\} \) is equicontinuous. The equicontinuity of \( \{(S\eta)(t) : \eta \in K_0\} \) at \( t = 0 \) follows from the above estimation of \( (S\eta)(t) \). Now let \( 0 < t_0 < t < b \) with \( t_0 \) be fixed. Then we have

\[
\left\| A^\alpha \left( (S\eta)(t) - (S\eta)(t_0) \right) \right\| \leq \int_0^{t_0} \left\| A^\alpha (R(t-s) - R(t-t_0)R(t_0-s)) F(s, \eta_s + \bar{\varphi}_s) \right\| ds
\]

\[
+ \left\| A^\alpha (R(t-t_0) - I) \int_0^{t_0} R(t_0-s) F(s, \eta_s + \bar{\varphi}_s)ds \right\|
\]

\[
+ \int_0^t \left\| A^\alpha R(t-s) F(s, \eta_s + \bar{\varphi}_s) \right\| ds.
\]

(4.2)

Using Theorem 4.1, it follow that

\[
\left\| A^\alpha \left( (S\eta)(t) - (S\eta)(t_0) \right) \right\| \leq t_0N_1M \int_0^{t-t_0} \frac{ds}{s^\alpha} + \left\| (R(t-t_0) - I) A^\alpha \int_0^{t_0} R(t_0-s) F(s, \eta_s + \bar{\varphi}_s)ds \right\|
\]

\[
+ N_\alpha N_1 \int_0^{t-t_0} \frac{1}{s^\alpha}ds.
\]

As the set \( \{(S\eta)(t_0) : \eta \in K_0\} \) is compact in \( \mathcal{Y}_\alpha \), we have that

\[
\lim_{t \to t_0} \left\| (S\eta)(t) - (S\eta)(t_0) \right\|_\alpha = 0 \quad \text{uniformly in} \quad \eta \in K_0.
\]

We obtain the same results by taking \( t_0 \) be fixed with \( 0 < t < t_0 \leq b \). Then we claim that

\[
\lim_{t \to t_0} \left\| (S\eta)(t) - (S\eta)(t_0) \right\|_\alpha = 0
\]

uniformly in \( \eta \in K_0 \) which means that \( \{(S\eta)(t) : \eta \in K_0\} \) is equicontinuous. Then by Ascoli–Arzela Theorem, \( \{S\eta : \eta \in K_0\} \) is relatively compact in \( K_0 \). Finally, we prove that \( S \) is continuous. Since \( F \) is continuous, given \( \varepsilon > 0 \), there exists \( \delta > 0 \), such that

\[
\sup_{0 \leq s \leq b} \left\| \eta(s) - \hat{\eta}(s) \right\|_\alpha < \delta \quad \text{implies that} \quad \left\| F(s, \eta_s + \bar{\varphi}_s) - F(s, \hat{\eta}(s) + \bar{\varphi}_s) \right\| < \varepsilon.
\]

Then for \( 0 \leq t \leq b \), we have

\[
\left\| (S\eta)(t) - (S\hat{\eta})(t) \right\|_\alpha \leq N_\alpha \int_0^t \frac{1}{(t-s)^\alpha} \left\| F(s, \eta_s + \bar{\varphi}_s) - F(s, \hat{\eta}(s) + \bar{\varphi}_s) \right\| ds
\]

\[
\leq N_\alpha \varepsilon \int_0^t \frac{ds}{s^\alpha}.
\]

This yields the continuity of \( S \) and using Schauder’s Fixed Point Theorem we deduce that \( S \) has a fixed point. Then the proof of the Theorem is complete.

The following result gives the blowing up phenomena of the mild solution in finite times.
Theorem 4.3 Assume that (V1)–(V3), (H0) and (H2) hold and $F$ is a continuous and bounded mapping. Then for each $\varphi \in C$, Eq. (1.1) has a mild solution $u(., \varphi)$ on a maximal interval of existence $[-r, b_\varphi]$. Moreover if $b_\varphi < \infty$ then $\lim_{t \to b_\varphi^-} u(t, \varphi) \|_a = +\infty$.

Proof Let $u(., \varphi)$ be the mild solution of Eq. (1.1) defined on $[0, b]$. Similar arguments used in the local existence result can be used for the existence of $b_1 > b$ and a function $u(., u_b)$ defined from $[b, b_1]$ to $\mathbb{R}^n$ satisfying

$$u(t, u_b(., \varphi)) = R(t)u(b, \varphi) + \int_b^t R(t - s) F(s, u_s) ds \text{ for } t \in [b, b_1].$$

By a similar proceeding, we show that the mild solution $u(., \varphi)$ can be extended to a maximal interval of existence $[-r, b_\varphi]$. Assume that $b_\varphi < +\infty$ and $\lim_{t \to b_\varphi^-} u(t, \varphi) \|_a < +\infty$. There exists $N_2 > 0$ such that $\|F(s, u_s)\| \leq N_2$, for $s \in [0, b_\varphi)$. We claim that $u(., \varphi)$ is uniformly continuous. In fact, let $0 < h \leq t \leq t + h < b_\varphi$. Then

$$u(t + h) - u(t) = (R(t + h) - R(t))\varphi(0) + \int_0^t (R(t + h - s) - R(t - s)) F(s, u_s) ds$$

$$+ \int_t^{t+h} R(t + h - s) F(s, u_s) ds.$$

By continuity of $A^\alpha R(t)$, we claim that $A^\alpha (R(t + h) - R(t))\varphi(0)$ is uniformly continuous on each compact set. Moreover, Theorem 4.1 implies that $A^\alpha (R(t + h - s) - R(t - s)) F(s, u_s) \to 0$ uniformly in $t$ when $h \to 0$. In fact we have

$$\int_0^t \left\| (R(t + h - s) - R(t - s)) F(s, u_s) \right\|_a ds$$

$$\leq \int_0^t \left\| (R(t + h - s) - R(h)R(t - s)) F(s, u_s) \right\|_a ds$$

$$+ \left\| (R(h) - I) A^\alpha \int_0^t R(t - s) F(s, u_s) ds \right\|.$$

Then using Theorem 4.1, we obtain that

$$\int_0^t \left\| (R(t + h - s) - R(t - s)) F(s, u_s) \right\|_a ds$$

$$\leq b_\varphi N_2 M \int_0^h ds \int_0^s \left\| (R(h) - I) A^\alpha \int_0^t R(t - s) F(s, u_s) ds \right\|.$$

We claim that the set

$$\left\{ A^\alpha \int_0^t R(t - s) F(s, u_s) ds : t \in [0, b_\varphi) \right\}$$

is relatively compact. In fact, let $(t_n)_{n \geq 0}$ be a sequence of $[0, b_\varphi)$. Then there exist a subsequence $(t_{n_k})_k$ and a real number $t_0$ such that $t_{n_k} \to t_0$. Using Dominate Convergence Theorem we deduce that

$$\int_0^{t_{n_k}} A^\alpha R(t_{n_k} - s) F(s, u_s) ds \to \int_0^{t_0} A^\alpha R(t_0 - s) F(s, u_s) ds.$$
This implies that
\[
\left\{ A^\alpha \int_0^t R(t-s) F(s, u_s) ds : t \in [0, b_\varphi) \right\}
\]
is relatively compact. Now using Banach Steinhauss’s theorem we deduce that
\[
(R(h) - I) A^\alpha \int_0^t R(t-s) F(s, u_s) ds \to 0
\]
uniformly when \( h \to 0 \) with respect to \( t \in [0, b_\varphi) \). Moreover we have
\[
\left\| \int_t^{t+h} R(t + h - s) F(s, u_s) ds \right\|_{\alpha} \leq N_2 N_\alpha \int_0^h \frac{ds}{s^\alpha}.
\]
Consequently \( \|u(t + h) - u(t)\|_{\alpha} \to 0 \) as \( h \to 0 \) uniformly in \( t \in [0, b_\varphi) \). If \( h \leq 0 \), that is for \( t \leq t_0 \), we have
\[
u(t) - u(t_0) = (R(t) - R(t_0)) \varphi(0) - \int_0^t (R(t_0) - R(t_0 - t) R(t - s)) F(s, u_s) ds
\]
\[- (R(t_0 - t) - I) \int_0^t R(t - s) F(s, u_s) ds - \int_0^{t_0} R(t_0 - s) F(s, u_s) ds,
\]
one can show a similar results by using the same reasoning. This implies that \( u(., \varphi) \) is uniformly continuous. Therefore \( \lim_{t \to b_\varphi} u(t, \varphi) \) exists in \( \mathbb{V}_\alpha \). And consequently, \( u(., \varphi) \) can be extended to \( b_\varphi \), which contradicts the maximality of \( [0, b_\varphi) \).

Next result gives the global existence of the mild solutions under weak conditions of \( F \). To achieve our goal, we introduce a following necessary result which is a consequence of Lemma 7.1.1 given in [2](p. 197, Exo 4).  

**Lemma 4.4** [2] Let \( \alpha, a, b \geq 0, \beta < 1 \) and \( 0 < d < \infty \). Also assume that \( v \) is nonnegative and locally integrable on \( [0, d) \) with
\[
v(t) \leq \frac{a}{t^\alpha} + b \int_0^t \frac{v(s)}{(t-s)^\beta} ds \quad \text{for} \quad t \in (0, d).
\]
Then there exists constant \( M_2 = M_2(\alpha, b, \alpha, \beta, d) < \infty \) such that \( v(t) \leq M_2/t^\alpha \) on \( (0, d) \).

**Theorem 4.5** Assume that (V1)–(V3), (H0) and (H2) hold and \( F \) is a completely continuous function on \( \mathbb{R}^+ \times C_\alpha \). Moreover suppose that there exist continuous non-negative functions \( f_1 \) and \( f_2 \) such that \( \|F(t, \varphi)\| \leq f_1(t) \|\varphi\|_\alpha + f_2(t) \) for \( \varphi \in C_\alpha \) and \( t \geq 0 \). Then Eq. (1.1) has a mild solution which is defined for \( t \geq 0 \).

**Proof** Let \( u(., \varphi) \) be the mild solution on the maximal intervall of existence \( [0, b_\varphi) \). Assume that \( b_\varphi < +\infty \). By Theorem 4.3 we have \( \lim_{t \to b_\varphi} \|u(t, \varphi)\|_{\alpha} = +\infty \). Recall that the solution of equation (1.1) is given by \( u_0 = \varphi \) and
\[
u(t, \varphi) = R(t) \varphi(0) + \int_0^t R(t-s) F(s, u_s(., \varphi)) ds \quad \text{for} \quad t \in [0, b_\varphi).
\]
Then taking the \( \alpha \)-norm, we obtain
\[
\|u(t, \varphi)\|_{\alpha} \leq \|R(t)\| \|\varphi(0)\|_{\alpha} + k_2 N_\alpha \int_0^{b_\varphi} \frac{ds}{s^\alpha} ds + k_1 N_\alpha \int_0^t \frac{1}{(t-s)^\alpha} \|u_s(., \varphi)\|_{\alpha} ds.
\]
where \( k_1 = \max_{0 \leq t \leq b_\varphi} |f_1(t)| \) and \( k_2 = \max_{0 \leq t \leq b_\varphi} |f_2(t)| \). Then we deduce that
\[
\|u(t, \varphi)\|_\alpha \leq N \|\varphi(0)\|_\alpha + k_1 N_\alpha \int_0^t \frac{1}{(t-s)^\alpha} \sup_{-r \leq \tau \leq s} \|u(\tau, \varphi)\|_\alpha \, ds + k_2 N_\alpha \int_0^{b_\varphi} \frac{ds}{s^{\alpha}}.
\]

Now we claim that the function
\[
t \mapsto \int_0^t \frac{1}{(t-s)^\alpha} \sup_{-r \leq \tau \leq s} \|u(\tau, \varphi)\|_\alpha \, ds,
\]
is non-decreasing. In fact, let \( 0 \leq t_1 \leq t_2 \). Then
\[
\int_0^{t_1} \frac{1}{(t_1-s)^\alpha} \sup_{-r \leq \tau \leq s} \|u(\tau, \varphi)\|_\alpha \, ds = \int_0^{t_1} \frac{1}{s^\alpha} \sup_{-r \leq \tau \leq t_1-s} \|u(\tau, \varphi)\|_\alpha \, ds
\leq \int_0^{t_2} \frac{1}{s^\alpha} \sup_{-r \leq \tau \leq t_2-s} \|u(\tau, \varphi)\|_\alpha \, ds
= \int_0^{t_2} \frac{1}{(t_2-s)^\alpha} \sup_{-r \leq \tau \leq s} \|u(\tau, \varphi)\|_\alpha \, ds,
\]
which yields the result. Then it follow from (4.3) that
\[
\sup_{-r \leq s \leq t} \|u(s, \varphi)\|_\alpha \leq N \|\varphi(0)\|_\alpha + k_2 N_\alpha \int_0^{b_\varphi} \frac{ds}{s^{\alpha}} + k_1 N_\alpha \int_0^t \frac{1}{(t-s)^\alpha} \sup_{-r \leq \tau \leq s} \|u(\tau, \varphi)\|_\alpha \, ds.
\]

Then using Lemma 4.4, we deduce that \( u(., \varphi) \) is bounded in \( [0, b_\varphi] \). Consequently \( \lim_{t \to b_\varphi} \|u(t, \varphi)\|_\alpha < \infty \), which contradicts our hypothesis. Then the mild solution is global.

We focus now to the compactness of the flow defined by the mild solutions.

**Theorem 4.6** Assume that (V1)–(V3) and (H0)–(H2) hold. Then the flow \( U(t) \) defined from \( C_\alpha \) to \( C_\alpha \) by \( U(t) \varphi = u_t(., \varphi) \) is compact for \( t > r \), where \( u_t(., \varphi) \) denotes the mild solution starting from \( \varphi \).

**Proof** We use Ascoli–Arzela’s theorem. Let \( E = \{ \varphi_\gamma : \gamma \in \Gamma \} \) be a bounded subset of \( C_\alpha \) and let \( t > r \) be fixed, but arbitrary. We will prove that \( \overline{U(t)E} \) is compact. It follows from (H1) and inequality (3.3) that there exists \( N_5 \) such that
\[
\|F(t, u_t(\varphi_\gamma))\| \leq N_2 \|u_t(\varphi_\gamma))\| + \|F(t, 0)\| = N_5 \quad \text{for } \gamma \in \Gamma.
\]
For each \( \gamma \in \Gamma \), we define \( f_\gamma \in C_\alpha \) by \( f_\gamma = u_t(., \varphi_\gamma) \). We show now that for fixed \( \theta \in [-r, 0] \), the set \( \{ f_\gamma(\theta) : \gamma \in \Gamma \} \) is precompact in \( \mathbb{V}_\alpha \). For any \( \gamma \in \Gamma \), we have
\[
f_\gamma(\theta) = R(t + \theta) \varphi_\gamma(0) + \int_0^{t+\theta} R(t + \theta - s) F(s, u_s(., \varphi_\gamma)) \, ds.
\]
As \( R(t) \) is compact for \( t > 0 \), we need only to prove that the set
\[
\left\{ \int_0^{t+\theta} R(t + \theta - s) F(s, u_s(., \varphi_\gamma)) \, ds : \gamma \in \Gamma \right\}
\]
Thus, if $X$ is a bounded subset of $\mathbb{X}$ with $123$ compact. Also we have $260$

\[
\mu \left( \left\{ R(\varepsilon) \int_0^{t+\theta-s} R(t+\theta-s) F(s, u_s(., \varphi_\gamma)) \, ds : \gamma \in \Gamma \right\} \right) = 0,
\]

where $\mu$ is the measure of non-compactness. Moreover, using Theorem 4.1 we have

\[
\begin{align*}
&\left\| A^\alpha \left( \int_0^{t+\theta-s} R(t+\theta-s) - R(\varepsilon) R(t+\theta-s) F(s, u_s(., \varphi_\gamma)) \, ds \right) \right\| \\
&\leq \int_0^{t+\theta-s} \left\| (R(t+\theta-s) - R(\varepsilon) R(t+\theta-s)) F(s, u_s(., \varphi_\gamma)) \right\|_\alpha \, ds \\
&\leq N_5 M \int_0^E ds s^\alpha \to 0 \text{ as } \varepsilon \to 0.
\end{align*}
\]

We deduce that

\[
\mu \left( \left\{ \int_0^{t+\theta-s} R(t+\theta-s) F(s, u_s(., \varphi_\gamma)) \, ds : \gamma \in \Gamma \right\} \right) = 0.
\]

On the other hand, for $0 < \alpha \leq \beta < 1$ we have

\[
\begin{align*}
&\left\| A^\beta \left( \int_{t+\theta-s}^{t+\theta} R(t+\theta-s) F(s, u_s(., \varphi_\gamma)) \, ds \right) \right\| \\
&\leq \int_{t+\theta-s}^{t+\theta} \left\| R(t+\theta-s) F(s, u_s(., \varphi_\gamma)) \right\|_\beta \, ds \\
&\leq N_\beta N_5 \int_{t+\theta-s}^{t+\theta} \frac{ds}{(t+\theta-s)^\beta} \\
&= N_\beta N_5 \int_0^E \frac{ds}{s^\beta} \to 0 \text{ as } \varepsilon \to 0.
\end{align*}
\]

Thus

\[
\left\{ A^\beta \left( \int_{t+\theta-s}^{t+\theta} R(t+\theta-s) F(s, u_s(., \varphi_\gamma)) \, ds : \gamma \in \Gamma \right) \right\}
\]

is a bounded subset of $\mathbb{X}$. The precompactness in $\mathbb{Y}_\alpha$ now follows from the compactness of $A^{-\beta} : \mathbb{Y} \to \mathbb{Y}_\alpha$. Then the set $\{(U(t)E)(\theta) : -r \leq \theta \leq 0\}$ is precompact in $\mathbb{Y}_\alpha$. We prove that the family $\{f_\gamma : \gamma \in \Gamma\}$ is equicontinuous. Let $\gamma$ in $\Gamma$, $0 < \varepsilon < t - r$, $-r \leq \hat{\theta} \leq \theta \leq 0$ with $\hat{\theta}$ fixed and $h = \theta - \hat{\theta}$. Then

\[
\begin{align*}
&\left\| A^\alpha (f_\gamma(h + \hat{\theta}) - f_\gamma(\hat{\theta})) \right\| \\
&\leq \left\| R(t + \hat{\theta} + h) - R(t + \hat{\theta}) \varphi_\gamma(0) \right\|_\alpha \\
&\quad + \int_0^{t+\hat{\theta}} \left\| A^\alpha \left( R(t + \hat{\theta} + h - s) - R(h) R(t + \hat{\theta} - s) \right) F(s, u_s(., \varphi_\gamma)) \right\| \, ds \\
&\quad + \left\| \left( R(h) - I \right) A^\alpha \left( \int_0^{t+\hat{\theta}} R(t + \hat{\theta} - s) F(s, u_s(., \varphi_\gamma)) \, ds \right) \right\| \\
&\quad + \int_{t+\hat{\theta}}^{t+\hat{\theta}+h} \left\| A^\alpha R(t + \hat{\theta} + h - s) F(s, u_s(., \varphi_\gamma)) \right\| \, ds.
\end{align*}
\]
Then it follows that
\[
\| A^\alpha (f_{\gamma} (h + \hat{\theta}) - f_{\gamma} (\hat{\theta})) \|
\leq \| (R(t + \hat{\theta} + h) - R(t + \hat{\theta})) A^\alpha \varphi_{\gamma} (0) \| + MN_5(t + \hat{\theta}) \int_0^h \frac{ds}{s^\alpha} \\
+ \| (R(h) - I) A^\alpha \int_0^{t+\theta} R(t + \hat{\theta} - s) F(s, u_s(., \varphi_{\gamma})) ds \|
\]
\[
+ N_5 N_\alpha \int_0^h \frac{ds}{s^\alpha}.
\]

Using the compactness of the set
\[
\{ A^\alpha \int_0^{t+\theta} R(t + \theta - s) F(s, u_s(., \varphi_{\gamma})) ds : \gamma \in \Gamma \}
\]
and the continuity of \( t \to R(t)x \) for \( x \in \mathbb{X} \) the right side of the above inequality can be made sufficiently small for \( h > 0 \) small enough. Then we conclude that \( \{ f_{\gamma} : \gamma \in \Gamma \} \) is equicontinuous. Consequently, by Ascoli–Arzela’s Theorem we conclude that the set \( \{ U(t) \varphi : \varphi \in E \} \) is compact, which means that the operator \( U(t) \) is compact for \( t > r \).

\[ \blacksquare \]

**Regularity of the Mild Solutions**

This section is devoted to the regularity of the mild solution. We define the set \( C^1_\alpha \) by \( C^1_\alpha = C^1([-r, 0]; Y_\alpha) \) as the set of continuously differentiable functions from \([-r, 0]\) into \( Y_\alpha \). We assume the following hypothesis.

\( \text{(H3)} \quad F \) is continuously differentiable and the partial derivatives \( D_r F \) and \( D_\varphi F \) are locally Lipschitz in the classical sense with respect to the second argument.

**Theorem 5.1** Assume that \( (V1)-(V3), (H1) \) and \( (H3) \) hold. Let \( \varphi \) in \( C^1_\alpha \) be such that \( \varphi(0) \in \mathbb{X} \) and \( \dot{\varphi}(0) = -A \varphi(0) + F(0, \varphi) \). Then the corresponding mild solution \( u \) becomes a strict solution of Eq. (1.1).

**Proof** Let \( a > 0 \). Take \( \varphi \in C^1_\alpha \) be such that \( \varphi(0) \in \mathbb{X} \) and \( \dot{\varphi}(0) = -A \varphi(0) + F(0, \varphi) \) and let \( u \) be the mild solution of Eq. (1.1) which is defined on \([0, +\infty[\). Consider the following equation

\[
\begin{cases}
\dot{v}(t) = R(t) \dot{\varphi}(0) + \int_0^t R(t - s) \left[ D_r F(s, u_s) + D_\varphi F(s, u_s) v_s \right] ds \\
\quad + \int_0^t R(t - s) B(s) \varphi(0) ds & \text{for } t \geq 0, \\
v_0 = \dot{\varphi}.
\end{cases}
\]  
(5.1)

Using the strict contraction principle, we can show that there exists an unique continuous function \( v \) solution in \([0, a]\) of Eq. (5.1). We introduce the function \( w \) defined by

\[
w(t) = \begin{cases}
\varphi(0) + \int_0^t v(s) ds & \text{if } t \geq 0, \\
\varphi(t) & \text{if } -r \leq t \leq 0.
\end{cases}
\]
Then it follows
\[ w_t = \varphi + \int_0^t v_s ds \quad \text{for} \ t \in [0, a]. \]

Consequently, the map \( t \mapsto w_t \) and \( t \mapsto \int_0^t R(t - s)F(s, w_s)ds \) are continuously differentiable and the following formula holds
\[
\frac{d}{dt} \int_0^t R(t - s)F(s, w_s)ds = R(t)F(0, w_0) + \int_0^t R(t - s)[D_t F(s, w_s) + D_\varphi F(s, w_s)v_s]ds
\]
\[ = R(t)F(0, \varphi) + \int_0^t R(t - s)[D_t F(s, w_s) + D_\varphi F(s, w_s)v_s]ds. \]

This implies that
\[
\int_0^t R(s)F(0, \varphi)ds = \int_0^t R(t - s)F(s, w_s)ds - \int_0^t \int_0^s R(s - \tau)[D_t F(\tau, w_\tau) + D_\varphi F(\tau, w_\tau)v_\tau]d\tau ds.
\]

On the other hand, from equality (2.2), we have
\[
- \int_0^t R(s)A\varphi(0)ds = R(t)\varphi(0) - \varphi(0) - \int_0^t \int_0^s R(s - \tau)B(\tau)\varphi(0)d\tau ds.
\]

We rewrite \( w \) as following
\[
w(t) = \varphi(0) - \int_0^t R(s)A\varphi(0)ds + \int_0^t R(s)F(0, \varphi)ds
\]
\[ + \int_0^t \int_0^s R(s - \tau)[D_t F(\tau, u_\tau) + D_\varphi F(\tau, u_\tau)v_\tau]d\tau ds
\]
\[ + \int_0^t \int_0^s R(s - \tau)B(\tau)\varphi(0)d\tau ds.
\]

Then it follows that
\[
w(t) = R(t)\varphi(0) + \int_0^t R(t - s)F(s, w_s)ds
\]
\[ + \int_0^t \int_0^s R(s - \tau)[(D_t F(\tau, u_\tau) - D_\varphi F(\tau, w_\tau))]d\tau ds
\]
\[ + \int_0^t \int_0^s (D_\varphi F(\tau, u_\tau)v_\tau - D_\varphi F(\tau, w_\tau)v_\tau)d\tau ds.
\]

We deduce, for \( t \in [0, a] \), that
\[
\|u(t) - w(t)\|_\alpha \leq \int_0^t \|A^\alpha R(t - s)(F(s, u_s) - F(s, w_s))\| ds
\]
\[ + \int_0^t \int_0^s \|A^\alpha R(s - \tau)(D_t F(\tau, u_\tau) - D_\varphi F(\tau, w_\tau))\| d\tau ds
\]
\[ + \int_0^t \int_0^s \|A^\alpha R(s - \tau)(D_\varphi F(\tau, u_\tau) - D_\varphi F(\tau, w_\tau))v_\tau\| d\tau ds. \quad (5.2)
\]
The set $H = \{u_s, w_s : s \in [0, a]\}$ is compact in $C_\alpha$. Since the partial derivatives of $F$ are locally Lipschitz with respect to the second argument, it is well-known that they are globally Lipschitz on $H$. Then we deduce that

$$
\|u(t) - w(t)\|_\alpha \leq N_\alpha h(a) \int_0^t \frac{1}{(t-s)^\alpha} \|u_s - w_s\|_\alpha \, ds
\leq N_\alpha h(a) \int_0^t \frac{1}{(t-s)^\alpha} \sup_{0 \leq \tau \leq a} \|u(\tau) - w(\tau)\|_\alpha \, ds,
$$

where $h(a) = L_F N_\alpha + a N_\alpha \text{Lip}(D_t F) + a N_\alpha \text{Lip}(D_\varphi F)$, which implies that

$$
\|u - w\|_\alpha \leq (N_\alpha h(a)) \int_0^a \frac{ds}{s^\alpha} \|u - w\|_\alpha.
$$

If we choose $a$ such that

$$
N_\alpha h(a) \int_0^a \frac{ds}{s^\alpha} < 1,
$$

then $u = w$ in $[0, a]$. Now we will prove that $u = w$ in $[0, +\infty)$. Assume that there exists $t_0 > 0$ such that $u(t_0) \neq w(t_0)$. Let $t_1 = \inf\{t > 0 : \|u(t) - w(t)\| > 0\}$. By continuity, one has $u(t) = w(t)$ for $t \leq t_1$ and there exists $\varepsilon > 0$ such that $\|u(t) - w(t)\| > 0$ for $t \in (t_1, t_1 + \varepsilon)$. Then it follows that, for $t \in (t_1, t_1 + \varepsilon)$

$$
\|u(t) - w(t)\|_\alpha \leq N_\alpha h(\varepsilon) \int_0^\varepsilon \frac{ds}{s^\alpha} \sup_{\varepsilon \leq \tau \leq t_1 + \varepsilon} \|u(\tau) - w(\tau)\|_\alpha.
$$

Now choosing $\varepsilon$ such that

$$
N_\alpha h(\varepsilon) \int_0^\varepsilon \frac{ds}{s^\alpha} < 1,
$$

then $u = w$ in $[t_1, t_1 + \varepsilon]$ which gives a contradiction. Consequently, $u(t) = w(t)$ for $t \geq 0$.

We conclude that $t \to u_t$, from $[0, +\infty)$ to $Y$, and $t \to F(t, u_t)$ from $[0, +\infty) \times C_\alpha$ to $X$ are continuously differentiables. Thus, we claim that $u$ is a strict solution of Eq. (1.1) on $[0, +\infty)$.

**Application**

For illustration, we propose to study the model (1.2) given in the Introduction. We recall that this is defined by

$$
\begin{aligned}
\frac{\partial}{\partial t} w(t, x) &= \frac{\partial^2}{\partial x^2} w(t, x) + \int_0^t h(t-s) \frac{\partial^2}{\partial x^2} w(s, x) ds \\
&\quad + \int_{-r}^0 g \left( t, \frac{\partial}{\partial x} w(t+\theta, x) \right) d\theta \quad \text{for } t \geq 0 \quad \text{and } x \in [0, \pi], \\
w(t, 0) &= w(t, \pi) = 0 \quad \text{for } t \geq 0, \\
w(\theta, x) &= w_0(\theta, x) \quad \text{for } \theta \in [-r, 0] \quad \text{and } x \in [0, \pi],
\end{aligned}
$$

(6.1)

where $w_0 : [-r, 0] \times [0, \pi] \to \mathbb{R}$, $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R}_+ \to \mathbb{R}_+$ are appropriates functions. To study this equation, we choose $X = L^2([0, \pi])$, with its usual norm $\| \cdot \|$. We define the operator $A : Y = D(A) \subset X \to X$ by

$$
A w = -w'' \quad \text{with domain } D(A) = H^2(0, \pi) \cap H^1_0(0, \pi),
$$

where $H^k(0, \pi) = \{ f \in L^2(0, \pi) : (1 + \partial^2) \cdots (1 + \partial^2)^{k-1} f \in L^2(0, \pi) \}$.
and \( B(t)x = h(t)Ax \in \mathbb{X} \), for \( t \geq 0, x \in \mathbb{Y} \). For \( \alpha = 1/2 \), we define \( \mathbb{Y}_{1/2} = (D(A^{1/2}), | \cdot |_{1/2}) \) where \( |x|_{1/2} = \| A^{1/2}x \| \) for each \( x \in \mathbb{Y}_{1/2} \). We define \( \mathcal{C}_{1/2} = C([-r, 0], \mathbb{Y}_{1/2}) \) equipped with norm \( | \cdot |_{\infty} \) and the functions, \( u \) and \( \varphi \) and \( F \) by \( u(t) = w(t, x), \varphi(\theta)(x) = w_0(\theta, x) \) for a.e \( x \in [0, \pi] \) and \( \theta \in [-r, 0], t \geq 0 \) and finally

\[
F(t, \varphi)(x) = \int_{-r}^{0} g(t, \frac{\partial}{\partial x} \varphi(\theta)(x))
\]

for a.e \( x \in [0, \pi] \) and \( \varphi \in \mathcal{C}_{1/2} \).

Then the Eq. (6.1) takes the abstract form

\[
\begin{aligned}
\frac{du(t)}{dt} &= -Au(t) + \int_{0}^{t} B(t-s)u(s)ds + F(t, u_t) \quad \text{for } t \geq 0, \\
u_0 &= \varphi \in \mathcal{C}_{1/2} = C([-r, 0], D(A^{1/2})).
\end{aligned}
\]

The operator \(-A\) is closed operator and generates an analytic compact semigroup \((T(t))_{t \geq 0}\) on \( \mathbb{X} \). Thus, there exists \( \delta \) in \((0, \pi/2)\) and \( M \geq 0 \) such that \( \Lambda = \{ \lambda \in \mathbb{C} : |\arg \lambda| < \frac{\pi}{2} + \delta \} \cup \{0\} \) is contained in \( \rho(-A) \), the resolvent set of \(-A\) and \( \|R(\lambda, -A)\| \leq M/|\lambda| \) for \( \lambda \in \Lambda \).

The operator \( B(t) \) is closed and for \( x \in \mathbb{Y} \), \( \| B(t)x \| \leq h(t) \| x \|_{\mathbb{Y}} \). The operator \( A \) has a discrete spectrum, the eigenvalues are \( n^2 \) and the corresponding normalized eigenvectors are \( e_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx), n = 1, 2, \ldots \). Moreover the following formula hold.

(i) \( Au = \sum_{n=1}^{\infty} n^2 \langle u, e_n \rangle e_n \quad u \in D(A), \)

(ii) \( A^{-1/2}u = \sum_{n=1}^{\infty} \frac{1}{n} \langle u, e_n \rangle e_n \quad \text{for } u \in \mathbb{X}, \)

(iii) \( A^{1/2}u = \sum_{n=1}^{\infty} n \langle u, e_n \rangle e_n \quad \text{for } u \in D(A^{1/2}) = \{ u \in \mathbb{X} : \sum_{n=1}^{\infty} \frac{1}{n^2} \langle u, e_n \rangle e_n \in \mathbb{X} \}. \)

One also have the following result.

**Lemma 6.1** [19] Let \( \varphi \in \mathbb{Y}_{1/2} \). Then \( \varphi \) is absolutely continuous, \( \varphi' \in \mathbb{X} \) and

\[ \| \varphi' \| = \| A^{1/2} \varphi \|. \]

We assume the following assumptions.

(H4) The scalar function \( h(.) \in L^1(0, \infty) \) and satisfies \( g_1(\lambda) = 1 + h^*(\lambda) \neq 0 \) and \( \lambda g_1^{-1}(\lambda) \in \Lambda \) for \( \lambda \in \Lambda \). Further, \( h^*(\lambda) \to 0 \) as \( |\lambda| \to \infty \), for \( \lambda \in \Lambda \) and \( (h^*(\lambda))^{-1} = o(|\lambda|^n) \).

(H5) The function \( g : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R} \) is a continuous and Lipschitz with respect to the second variable.

By assumption (H4), the operator \( \rho(\lambda) = (\lambda I + g_1(\lambda)A)^{-1} = g_1^{-1}(\lambda)(\lambda g_1^{-1}(\lambda)I + A)^{-1} \) exists as a bounded operator on \( \mathbb{X} \), is analytic in \( \Lambda \), and satisfies \( \| \rho(\lambda) \| < M/|\lambda| \). Other hand, for \( x \in \mathbb{X} \) we have

\[
A \rho(\lambda)x = A(\lambda I + g_1(\lambda)A)^{-1} x = (A + \lambda g_1^{-1}(\lambda)I - \lambda g_1^{-1}(\lambda)I) (\lambda I + g_1(\lambda)A)^{-1} x
= g_1^{-1}(\lambda) I - \lambda g_1^{-1}(\lambda)I (\lambda g_1^{-1}(\lambda)I + A)^{-1} x.
\]
Since \( \lambda g^{-1}_1(\lambda)(\lambda g^{-1}_1(\lambda)I + A)^{-1} \) is bounded because \( g^{-1}_1(\lambda) \in \Lambda \), then \( \|A\rho(\lambda)x\| \) has the growth properties of \( g^{-1}_1(\lambda) \) which tends to 1 if \( |\lambda| \) goes to infinity. Then we deduce that \( A\rho(\lambda) \in \mathcal{L}(\mathbb{Y}) \). Moreover, it is analytic from \( \Lambda \) to \( \mathcal{L}(\mathbb{Y}) \). Now, for \( x \in \mathbb{Y} \), one has

\[
A\rho(\lambda)x = g^{-1}_1(\lambda)(\lambda g^{-1}_1(\lambda)I + A)^{-1}Ax \quad \text{and} \quad B^*(\lambda)\rho(\lambda)x = h^*(\lambda)\rho(\lambda)Ax.
\]

Then it follows that

\[
\|A\rho(\lambda)x\| \leq M/|\lambda|\|x\|_{\mathbb{Y}} \quad \text{and} \quad \|B^*(\lambda)\rho(\lambda)\| \leq M/|\lambda|\|x\|_{\mathbb{Y}}.
\]

We deduce that \( A\rho(\lambda) \in \mathcal{L}(\mathbb{Y}, \mathbb{X}) \), \( B^*(\lambda) = h^*(\lambda)A \in \mathcal{L}(\mathbb{Y}, \mathbb{X}) \) and \( B^*(\lambda)\rho(\lambda) \in \mathcal{L}(\mathbb{Y}, \mathbb{X}) \). Considering \( D = C^\infty_0([0, \pi]) \), we see that the condition (V1)–(V3) and (H0) are verified. Hence the homogeneous linear equation of equation (6.1) has an analytic compact resolvent operator \( (R(t))_{t \geq 0} \). The function \( F \) is continuous in the first variable from the fact that \( g \) is continuous in the first variable. Moreover from Lemma 6.1 and the continuity of \( g \), we deduce that \( F \) is continuous with respect to the second argument. This yields the continuity of \( F \) in \( \mathbb{R}_+ \times C^{1/2}_1 \). In addition, by assumption (H5) we deduce that

\[
\|F(t, \varphi_1) - F(t, \varphi_2)\| \leq rL_f\|\varphi_1 - \varphi_2\|_{C^{1/2}_1}.
\]

Then \( F \) is a continuous globally Lipschitz function with respect to the second argument. We obtain the following important result.

**Proposition 6.2** Suppose that the assumptions (H4)–(H5) hold. Then the Eq. (6.2) has a mild solution which is defined for \( t \geq 0 \).

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