VUST’S THEOREM AND HIGHER LEVEL SCHUR-WEYL DUALITY FOR TYPES B, C AND D

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Abstract. Let $G$ be a complex linear algebraic group, $\mathfrak{g} = \text{Lie}(G)$ its Lie algebra and $e \in \mathfrak{g}$ a nilpotent element. Vust’s theorem says that in case of $G = \text{GL}(V)$, the algebra $\text{End}_{G_e}(V^{\otimes d})$, where $G_e \subset G$ is the stabilizer of $e$ under the adjoint action, is generated by the image of the natural action of $d$-th symmetric group $\mathfrak{S}_d$ and the linear maps $\{1^{\otimes(i-1)} \otimes e \otimes 1^{\otimes(d-i)}| i = 1, \ldots, d\}$. In this paper, we generalize this theorem to $G = \text{O}(V)$ and $\text{SP}(V)$ for nilpotent element $e$ with $G \cdot e$ being normal. As an application, we study the higher Schur-Weyl duality in the sense of [BK2] for types $B, C$ and $D$, which establishes a relationship between $W$-algebras and degenerate affine braid algebras.

1. Introduction

The classical Schur-Weyl duality, named after two pioneers of representation theory, shows a double centralizer property between the general linear group $\text{GL}(V)$ and the symmetric group $\mathfrak{S}_d$. Precisely, the $d$-fold tensor space $V^{\otimes d}$ admits a $(\mathbb{C}\text{GL}(V), \mathbb{C}\mathfrak{S}_d)$-bimodule structure, where $\mathfrak{S}_d$ acts by permuting the tensor positions and $\text{GL}(V)$ acts naturally in each tensor position. If we name the representations as follows

$$\mathbb{C}\text{GL}(V) \xrightarrow{\varphi} V^{\otimes d} \xleftarrow{\sigma} \mathbb{C}\mathfrak{S}_d$$

then

$$\text{End}_{\text{GL}(V)}(V^{\otimes d}) = \sigma(\mathbb{C}\mathfrak{S}_d);$$

$$\varphi(\mathbb{C}\text{GL}(V)) = \text{End}_{\mathfrak{S}_d}(V^{\otimes d}).$$

Differentiating the action of $\text{GL}(V)$, we obtain an action (denoted by $\phi$) of its Lie algebra $\mathfrak{gl}(V)$ on $V^{\otimes d}$. The following is an alternative statement of Schur-Weyl duality:

$$\text{End}_{\mathfrak{gl}(V)}(V^{\otimes d}) = \sigma(\mathbb{C}\mathfrak{S}_d);$$

$$\phi(\mathbb{U}(\mathfrak{gl}(V))) = \text{End}_{\mathfrak{S}_d}(V^{\otimes d}).$$

Nowadays there are varieties of generalizations for this duality. Its quantum analogue was studied by Jimbo [Ji] where symmetric groups and universal enveloping algebras are replaced by Iwahori-Hecke algebras and quantum groups, respectively. The super version was achieved by Sergeev [S], who established a double centralized property between the Lie superalgebra $\mathfrak{gl}_{m|n}$ and $\mathfrak{S}_d$.

For other classical algebraic groups $G = \text{O}(V)$ or $\text{SP}(V)$, Brauer [Br] introduced a series of algebras (now named Brauer algebras) and showed that $G$ and Brauer algebras form an analogue of Schur-Weyl duality for types $B, C$ and $D$. 
Moreover, Vust considered another interesting generalization of Schur-Weyl duality. Let $G = \text{GL}(V)$, $\mathfrak{g} = \mathfrak{gl}(V)$ its Lie algebra and $e \in \mathfrak{g}$ a nilpotent element. Denote the centralizer of $e$ in $G$ by

$$G_e := \{ g \in G | g^{-1}eg = e \}.$$

For any $1 \leq i \leq d$, write

$$e^{(i)} := 1 \otimes (i-1) \otimes e \otimes 1 \otimes (d-i) \in \text{End}(V \otimes d). \quad (1.1)$$

Denote by $\mathfrak{S}_d[e]$, the subalgebra of $\text{End}(V \otimes d)$ generated by $\sigma(\mathfrak{S}_d) \cup \{ e^{(i)} | 1 \leq i \leq d \}$. Vust’s Theorem (c.f. [KP1]) says that

$$\text{End}_{G_e}(V \otimes d) = \mathfrak{S}_d[e]. \quad (1.2)$$

Its arbitrary characteristic version was proved by Donkin in [D].

Based on Vust’s Theorem, Brundan and Kleshchev [BK2] established a duality between $\mathfrak{g}_e$ and $\mathfrak{S}_d[e]$. Then they developed its filtered deformation, which is called higher level Schur-Weyl duality. This duality shows a double centralizer property between the $W$-algebras of type $A$ and the cyclotomic Hecke algebras.

In this paper, we will investigate the Vust’s theorem for types $B$, $C$ and $D$, and then study the higher level Schur-Weyl duality for these types. The main results of this present paper are Theorem 2.10 and 3.9. Throughout this paper, the base field is the complex number field $\mathbb{C}$ (any algebraically closed field of characteristic zero is fine, too).

We would like to point out here that there is also another kind of Schur-Weyl duality different from Brauer’s setting. Note that the symmetric group $\mathfrak{S}_d$ is the Weyl group of type $A$. It is natural to consider the duality when $\mathfrak{S}_d$ is replaced by Weyl groups of other types. We refer to Green’s work [Gre] about this issue. Furthermore, its quantum analogue, developed by Bao and Wang [BW], can be used to give a new approach to Kazhdan-Lusztig Theory. Chen, Guay and Ma’s work [CGM] about the duality between Yangians and affine Hecke algebras is also with this taste. We will study the higher level Schur-Weyl duality for this different setting in a subsequent paper, which may provide a relationship between $W$-algebras and Yangians for type $B/C$.

The paper is organized as follows. Section 2 is devoted to generalizing Vust’s Theorem. In Section 3 we study the higher level Schur-Weyl duality for types $B$, $C$ and $D$.

2. Vust’s Theorem for Types $B$, $C$ and $D$

This section is mainly devoted obtaining Vust’s Theorem for types $B$, $C$ and $D$ (i.e. Theorem 2.10).
2.1. Trace function. Let $G = O(V)$ or $SP(V)$, and $\langle , \rangle$ be the defining quadratic form on $V$ for $G$. For each $X \in \text{End}(V)$, denote by $X^i \in \text{End}(V)$ the unique element satisfying $\langle Xv, u \rangle = \langle v, X^iu \rangle$ for any $u, v \in V$. In particular, $(X^i)^i = X$. Furthermore,

$$X \in \mathfrak{g} = \text{Lie}(G) \iff X^i = -X. \quad (2.1)$$

There is a bijection $\theta : V^{\otimes 2} \to \text{End}(V)$ determined by

$$\theta(u \otimes w)(v) := \langle w, v \rangle u, \quad (\forall u, w, v \in V). \quad (2.2)$$

It is clear that

$$\text{Trace}(\theta(u \otimes w)) = \langle w, u \rangle$$

and hence

$$\text{Trace}(\theta(Xu \otimes w)) = -\text{Trace}(\theta(u \otimes X^iw)). \quad (2.3)$$

Lemma 2.1. (1). If $X = \theta(u \otimes w) \in \text{End}(V)$, then $X^i = \theta(w \otimes u)$.

(2). Let $X_i = \theta(u_i \otimes w_i)$ where $u_i, w_i \in V$ for $i = 1, 2, \ldots, k$. Then

$$X_1X_2 \cdots X_k = \langle w_1, u_2 \rangle \langle w_2, u_3 \rangle \cdots \langle w_{k-1}, u_k \rangle \theta(u_1 \otimes w_k),$$

and hence

$$\text{Trace}(X_1X_2 \cdots X_k) = \langle w_1, u_2 \rangle \langle w_2, u_3 \rangle \cdots \langle w_{k-1}, u_k \rangle \theta(u_1 \otimes w_k).$$

Proof. The first statement follows from the following computation:

$$\langle \theta(u \otimes w)(v_1), v_2 \rangle = \langle w, v_1 \rangle \langle u, v_2 \rangle = \langle v_1, \theta(w, u) v_2 \rangle, \quad \forall v_1, v_2 \in V.$$

For the second statement, we can show that for any $v \in V$,

$$X_1X_2 \cdots X_k(v) = \langle w_k, v \rangle X_1X_2 \cdots X_{k-1}(u_k)$$

$$= \langle w_k, v \rangle \langle w_{k-1}, u_k \rangle X_1X_2 \cdots X_{k-2}(u_{k-1})$$

$$= \cdots$$

$$= \langle w_k, v \rangle \langle w_{k-1}, u_k \rangle \cdots \langle w_1, u_2 \rangle (u_1)$$

$$= \langle w_1, u_2 \rangle \langle w_2, u_3 \rangle \cdots \langle w_{k-1}, u_k \rangle \theta(u_1 \otimes w_k)(v).$$

2.2. $G$-invariant ring. Let $\mathbb{C}[\text{End}(V)^{\oplus d}]$ be the polynomial function ring of $\text{End}(V)^{\oplus d}$. The conjugation action of $G$ on $\text{End}(V)$ induces an action of $G$ on $\mathbb{C}[\text{End}(V)^{\oplus d}]$.

Write

$$\mathbb{C}[\text{End}(V)^{\oplus d}]^G := \left\{ f \in \mathbb{C}[\text{End}(V)^{\oplus d}] \mid f(X_1, X_2, \ldots, X_d) = f(g^{-1}X_1g, g^{-1}X_2g, \ldots, g^{-1}X_dg), \quad X_1, X_2, \ldots, X_d \in \text{End}(V) \right\}$$

to be the invariant ring for the action of $G$ on $\mathbb{C}[\text{End}(V)^{\oplus d}]$.

Theorem 2.2 (c.f. Theorem 7.1 in [P]). For $G = O(V)$ or $SP(V)$, the invariant ring $\mathbb{C}[\text{End}(V)^{\oplus d}]^G$ is generated by functions $f$ in form of

$$f(X_1, X_2, \ldots, X_d) = \text{Trace}(U_{i_1} \cdots U_{i_k}),$$

where $U_j = X_j$ or $X_j^*$, $k \in \mathbb{N}$ and $1 \leq i_1, \ldots, i_k \leq d$. 

□
2.3. **Action of Brauer algebra on** $V^\otimes d$. The original definition of Brauer algebras involves $d$-diagrams with $2d$ vertices and $d$ edges. Since it would occupy too much space but will never be used in this paper, we refer to [B] (also c.f. [Gro]) for this definition. Instead, we describe the image of Brauer algebra in $\text{End}(V^\otimes d)$ in the following.

Take a basis $\{v_p \mid 1 \leq p \leq n\}$ of $V$, and let $\{v^p \mid 1 \leq p \leq n\}$ be the dual basis (i.e. $\langle v_p, v^q \rangle = \delta_{ij}$). Define $\gamma_{ij} \in \text{End}(V^\otimes d)(i \neq j)$ by

$$
\gamma_{ij}(u) = \langle u_i, u_j \rangle \sum_{p=1}^{n} u_1 \otimes \cdots \otimes v_p \otimes \cdots \otimes v^p \otimes \cdots \otimes u_d
$$

for any $u = u_1 \otimes \cdots \otimes u_d \in V^\otimes d$. It is known that $\gamma_{ij}$ is independent on the choice of $\{v_p \mid 1 \leq p \leq n\}$.

Let $B_d$ be the subalgebra of $\text{End}(V^\otimes d)$ generated by $\{\gamma_{ij} \mid 1 \leq i \neq j \leq n\}$ and $\sigma(\mathfrak{g}_d)$. It is known (c.f. Proposition 10.1.3 in [GW]) that $B_d$ is the image of Brauer algebra in $\text{End}(V^\otimes d)$.

**2.4. Some technical lemmas.** For any $l = (l_1, \ldots, l_d) \in \mathbb{Z}_{\geq 0}^d$ and $X \in \mathfrak{g}$, set

$$
X(l) := X^{l_1} \otimes \cdots \otimes X^{l_d} \in \text{End}(V^\otimes d).
$$

**Lemma 2.3.** Take $Y = \theta(u_1 \otimes w_1) \otimes \cdots \otimes \theta(u_d \otimes w_d) \in \text{End}(V^\otimes d)$ where $u_i, w_i \in V, (i = 1, 2, 3, \ldots, d)$. For any $b \in B_d, l = (l_1, \ldots, l_d) \in \mathbb{Z}_{\geq 0}^d$ and $X \in \mathfrak{g}$, we have

$$
\text{Trace}(X(l) \circ b \circ Y) = (-1)^{\sum_{i=1}^{d} l_i} \text{Trace}(b \circ Y')
$$

where

$$
Y' = \theta(u_1 \otimes X^{l_1}w_1) \otimes \cdots \otimes \theta(u_d \otimes X^{l_d}w_d).
$$

**Proof.** For any $s \in \sigma(\mathfrak{g}_d) \subset B_d$ and $v_1, \ldots, v_d \in V$,

$$
s \circ Y(v_1 \otimes \cdots \otimes v_d) = s(\langle w_1, v_1 \rangle u_1 \otimes \cdots \otimes \langle w_d, v_d \rangle u_d)
$$

$$
= \langle w_1, v_1 \rangle u_{s(1)} \otimes \cdots \otimes \langle w_d, v_d \rangle u_{s(d)}
$$

$$
= (\theta(u_{s(1)} \otimes w_1) \otimes \cdots \otimes \theta(u_{s(d)} \otimes w_d))(v_1 \otimes \cdots \otimes v_d).
$$

That is,

$$
s \circ Y = \theta(u_{s(1)} \otimes w_1) \otimes \cdots \otimes \theta(u_{s(d)} \otimes w_d).
$$

Similarly, for $\gamma_{ij} \in B_d$ we have

$$
\gamma_{ij} \circ Y = \langle u_i, u_j \rangle \sum_{p=1}^{n} \theta(u_1 \otimes w_1) \otimes \cdots \otimes \theta(u_p \otimes w_i) \otimes \cdots \otimes \theta(u_p \otimes w_j) \otimes \cdots \otimes \theta(u_d \otimes w_d).
$$

Hence we can assume that for any $b \in B_d$,

$$
b \circ Y = \sum \theta(\square_1 \otimes w_1) \otimes \cdots \otimes \theta(\square_d \otimes w_d).
$$
Therefore by (2.1) and (2.3), we have
\[
\text{Trace}(X(l) \circ b \circ Y) = \text{Trace}(\sum \theta((X^l \cdot \square_1) \otimes w_1) \otimes \cdots \otimes \theta((X^{l_d} \cdot \square_d) \otimes w_d))
\]
\[
= (-1)^{\sum_{i=1}^l} \text{Trace}(\sum \theta(\square_1 \otimes (X^l \cdot w_1)) \otimes \cdots \otimes \theta(\square_d \otimes (X^{l_d} \cdot w_d)))
\]
\[
= (-1)^{\sum_{i=1}^l} \text{Trace}(b \circ Y).
\]
\[\square\]

Lemma 2.4. For any \(F \in [\text{End}(V^\otimes d)^*]^G\), there exists a \(b_F \in B_d\) such that
\[
F(X_1 \otimes X_2 \otimes \cdots \otimes X_d) = \text{Trace}(b_F \circ X_1 \otimes X_2 \otimes \cdots \otimes X_d).
\] (2.4)

Proof. Define a linear map \(J : B_d \to \text{End}(V^\otimes d)^*\) by
\[
J(b)(X_1 \otimes X_2 \cdots \otimes X_d) = \text{Trace}(b \circ X_1 \otimes X_2 \cdots \otimes X_d).
\]
For any \(g \in G\), we check that
\[
(g \cdot J(b))(X_1 \otimes X_2 \cdots \otimes X_d) = J(g^{-1} \cdot (X_1 \otimes X_2 \cdots \otimes X_d))
\]
\[
= \text{Trace}(g \circ X_1 \otimes X_2 \cdots \otimes X_d \circ g^{-1})
\]
\[
= \text{Trace}(g \circ b \circ X_1 \otimes X_2 \cdots \otimes X_d \circ g^{-1})
\]
\[
= \text{Trace}(b \circ X_1 \otimes X_2 \cdots \otimes X_d)
\]
\[
= J(b)(X_1 \otimes X_2 \cdots \otimes X_d),
\]
where the third equality holds because the actions of \(G\) and \(B_d\) on \(V^\otimes d\) commute with each other. Therefore \(J(B_d) \in [\text{End}(V^\otimes d)^*]^G\).

Non-degeneracy of \(\text{Trace}(\cdot \circ \cdot)\) on \(\text{End}(V)^\otimes d\) implies that \(J\) is injective. So
\[
\dim B_d = \dim \text{End}_G(V^\otimes d) = \dim[\text{End}(V^\otimes d)^*]^G
\]
implies that
\[
J(B_d) = [\text{End}(V^\otimes d)^*]^G.
\]
The lemma then follows by taking \(b_F = J^{-1}(F)\). \[\square\]

Any \(F \in [\text{End}(V^\otimes d)^*]^G\) can be viewed as a function \(\hat{F} \in \mathbb{C}[\text{End}(V)^\otimes d]\) by
\[
\hat{F}(X_1, X_2, \ldots, X_d) := F(X_1 \otimes X_2 \cdots \otimes X_d).
\]
Thanks to Theorem 2.2 and the fact that \(F\) is linear in variables \(X_1, \ldots, X_d\), we know that \(\hat{F}\) should be a sum of functions in terms of
\[
\text{Trace}(U_{j_1} \cdots U_{j_s}) \text{Trace}(U_{j_{s+1}} \cdots U_{j_k}) \cdots \text{Trace}(U_{j_{t+1}} \cdots U_{j_d}),
\]
where \(U_{j_i} = X_{j_i}\) or \(X_{j_i}^t\), and \((j_1, \ldots, j_s, j_{s+1}, \ldots, j_k, \ldots, j_{t+1}, \ldots, j_d)\) is an arrangement of \(\{1, 2, \ldots, d\}\).

Lemma 2.5. Assume
\[
F(X_1 \otimes \cdots \otimes X_d) = \text{Trace}(U_{j_1} \cdots U_{j_s}) \cdots \text{Trace}(U_{j_{t+1}} \cdots U_{j_d}) \in [\text{End}(V^\otimes d)^*]^G
\]
where \(U_{j_i} = X_{j_i}\) or \(X_{j_i}^t\), and \((j_1, \ldots, j_s, \ldots, j_{t+1}, \ldots, j_d)\) is an arrangement of \(\{1, 2, \ldots, d\}\). Let \(l_1 = (l_1^{(1)}, \ldots, l_d^{(1)})\), \(l_2 = (l_1^{(2)}, \ldots, l_d^{(2)})\) \(\in \mathbb{Z}_{\geq 0}^d\) such that \(l_i^{(1)} =
Lemma 2.7. If $\psi$ is said to be normal, then $\omega \in \mathfrak{g}$ satisfies that $G \cdot \omega \subseteq \mathfrak{a}$ is normal; and

2. $\dim(G \cdot e \setminus G \cdot e) \leq \dim(G \cdot e) - 2$.
then for any \( m \in \mathbb{M}^G \), there exists a \( G \)-equivariant morphism \( \Psi : \mathbb{A} \rightarrow \mathbb{M} \) such that \( \Psi(e) = m \).

Specify \( G = O(V) \) or \( SP(V), \mathbb{A} = \text{Lie}(G) = \mathfrak{g} \subset \text{End}(V) \) and \( \mathbb{M} = \text{End}(V^\otimes_d) \).

**Remark 2.8.** For any nilpotent element \( e \in \mathfrak{g} \), the second condition in Lemma 2.7 always holds (c.f. Lemma 8.4 [Ja]).

2.6. \( G \)-equivariant morphisms \( \text{Mor}_G(\mathfrak{g}, \text{End}(V^\otimes_d)) \). Let \( R := \mathbb{C}[\mathfrak{g}]^G \) and denote by \( \text{Mor}_G(\mathfrak{g}, \text{End}(V^\otimes_d)) \) the set of all \( G \)-equivariant morphism (of varieties) from \( \mathfrak{g} \) to \( \text{End}(V^\otimes_d) \). There is an \( R \)-module structure on \( \text{Mor}_G(\mathfrak{g}, \text{End}(V^\otimes_d)) \) given by

\[(r \circ f)(X) = r(X)f(X), \quad (\forall r \in R, f \in \text{Mor}_G(\mathfrak{g}, \text{End}(V^\otimes_d)), X \in \mathfrak{g}).\]

Let \( S \subset \text{Mor}_G(\mathfrak{g}, \text{End}(V^\otimes_d)) \) be the subset consisting of those \( \Psi \in \text{Mor}_G(\mathfrak{g}, \text{End}(V^\otimes_d)) \) such that

\[\Psi(X) = (X^{l_1} \otimes \cdots \otimes X^{l_d}) \circ b \circ (X^{l_1} \otimes \cdots \otimes X^{l_d}), \quad (\forall X \in \mathfrak{g})\]

for some \( b \in B_d \) and \( l_1, \ldots, l_d, l_1, \ldots, l_d \in \mathbb{Z}_{\geq 0} \).

**Proposition 2.9.** As an \( R \)-module, \( \text{Mor}_G(\mathfrak{g}, \text{End}(V^\otimes_d)) \) is generated by \( S \).

**Proof.** Set \( N = \mathfrak{g} \oplus \text{End}(V)^{\oplus d} \). The embedding \( R \hookrightarrow \mathbb{C}[N]^G \) induces an \( R \)-module structure on \( \mathbb{C}[N]^G \). Consider the \( R \)-module homomorphism

\[J : \text{Mor}_G(\mathfrak{g}, \text{End}(V)^{\oplus d}) \rightarrow \mathbb{C}[N]^G, \quad \Psi \mapsto J(\Psi)\]

defined by

\[J(\Psi)(X, X_1, X_2, \ldots, X_d) = \text{Trace}(\Psi(X) \circ (X_1 \otimes X_2 \otimes \cdots \otimes X_d))\]

for any \( X \in \mathfrak{g} \) and \( X_1, X_2, \ldots, X_d \in \text{End}(V) \). Observe that \( J(\Psi) \) is linear in variables \( X_1, X_2, \ldots, X_d \).

Non-degeneracy of \( \text{Trace}(\text{End}(V^\otimes_d) \circ \text{End}(V^\otimes_d)) \) implies that \( J \) is injective. Therefore we only need to prove \( RJ(S) = J(\text{Mor}_G(\mathfrak{g}, \text{End}(V^\otimes_d))) \).

**Claim:** \( J(\Psi)(X, X_1, X_2, \ldots, X_d) \) is in form of

\[\sum r(X)\text{Trace}(X^{l_1}U_{j_1}X^{l_2}U_{j_2}\cdots X^{l_k}U_{j_k})\cdots \text{Trace}(X^{l_{t+1}}U_{j_{t+1}}X^{l_{t+2}}U_{j_{t+2}}\cdots X^{l_k}U_{j_k})\]

where \( l_i \in \mathbb{Z}_{\geq 0}, U_i = X_i \) or \( X_i^t \), \( (j_1, \ldots, j_k, \ldots, j_{t+1}, \ldots, j_d) \) is an arrangement of \( \{1, 2, \ldots, d\} \) and \( r \in R \).

**Proof of the claim:**

Choose a \( G \)-equivariant extension \( \Psi' : \text{End}(V) \rightarrow M \) of \( \Psi \) by Lemma 2.6. Then \( J(\Psi') \) can be viewed as a \( G \)-invariant function on \( \text{End}(V)^{\oplus(d+1)} \). Thus by (2.1) and Theorem 2.2 we can see that \( J(\Psi')(X, X_1, \ldots, X_d) \) is in form of

\[\sum r(X)\text{Trace}(X^{l_1}U_{j_1}X^{l_2}U_{j_2}\cdots X^{l_k}U_{j_k})\cdots \text{Trace}(X^{l_{t+1}}U_{j_{t+1}}X^{l_{t+2}}U_{j_{t+2}}\cdots X^{l_k}U_{j_k})\]

with \( l_i \in \mathbb{Z}_{\geq 0}, U_i = X_i \) or \( X_i^t, r \in R \) and \( j_i \in \{1, 2, \ldots, d\} \) for \( i = 1, 2, \ldots, k \). Notice that \( J(\Psi')(X, X_1, \ldots, X_d) \) is linear in variables \( X_1, \ldots, X_d \). So we have \( k = d \) and \( j_{i_1} \neq j_{i_2} \) if \( i_1 \neq i_2 \). We complete the proof of the claim.
Thanks to the claim, we only need to show that
\[ \text{Trace}(X^l_1 U_{j_1} \cdots X^l_s U_{j_s}) \cdots \text{Trace}(X^{l+1}_t U_{j_{t+1}} \cdots X^{l+2}_t U_{j_{t+2}} \cdots X^{l_d}_d U_{j_d}) \in J(S), \]
which is obvious by Lemma 2.5. \hfill \square

2.7. Vust’s Theorem for \(O(V)\) and \(SP(V)\). Let \(e \in \mathfrak{g}\) be a nilpotent element and recall the notation \(e^{(i)} \in \text{End}(V^{\otimes d})\) in (1.1). Denote by \(B_d[e]\) the subalgebra of \(\text{End}(V^{\otimes d})\) generated by \(B_d \cup \{e^{(i)}| 1 \leq i \leq d\}\). The following is a generalization of Vust’s Theorem (1.2) for the cases other than type \(A\).

**Theorem 2.10.** Let \(G = O(V)\) or \(SP(V)\). If a nilpotent element \(e \in \mathfrak{g} = \text{Lie}(G)\) satisfies that the nilpotent orbit closure \(G \cdot e\) is normal, then
\[ \text{End}_{G_e}(V^{\otimes d}) = B_d[e]. \] (2.5)

**Proof.** For any \(m \in \text{End}(V^{\otimes d})^{G_e}\), by Lemma 2.7 we have a \(G\)-equivariant morphism \(\Psi : \mathfrak{g} \to (\text{End}(V^{\otimes d}))^{G_e}\) such that \(\Phi(e) = m\). So
\[ (\text{End}(V^{\otimes d}))^{G_e} = \{\Psi(e) | \Psi \in \text{Mor}_G(\mathfrak{g}, \text{End}(V^{\otimes d}))\}. \]

Notice that for any \(\Psi \in S\),
\[ \Psi(e) = (e^{l_1} \otimes \cdots \otimes e^{l_d}) \circ b \circ (e^{l_1} \otimes \cdots \otimes e^{l_d}) \in B_d[e]. \]
Hence Proposition 2.9 implies that
\[ (\text{End}(V^{\otimes d}))^{G_e} = \{\Psi(e) | \Psi \in \text{Mor}_G(\mathfrak{g}, \text{End}(V^{\otimes d}))\} \subset B_d[e]. \]

On the other hand, it can be checked directly that
\[ B_d[e] \subset (\text{End}(V^{\otimes d}))^{G_e}. \]

So we finally obtain that
\[ B_d[e] = (\text{End}(V^{\otimes d}))^{G_e} = \text{End}_{G_e}(V^{\otimes d}). \] \hfill \square

**Remark 2.11.** A criteria on the normality of \(G \cdot e\) for any nilpotent element \(e \in \mathfrak{g}\) can be found in [KP2].

2.8. Description of \(G_e\). It can be found in Section 3 of [Ja] that
\[ G_e = C_e \rtimes R_e \]
where \(C_e\) is the reductive part and \(R_e\) is the unipotent radical. Moreover, \(R_e\) is connected (c.f. Proposition 3.12 in [Ja]). Suppose that \(e \in \mathfrak{g}\) corresponds to a partition \([1^{r_1} 2^{r_2} \cdots]\) of \(\dim(V)\) (by Jordan blocks), then we have an isomorphism of algebraic groups (c.f. §3.8 in [Ja])
\[ \rho_{O(V)} : \prod_{s \geq 1; s \text{ odd}} O_{r_s} \times \prod_{s \geq 1; s \text{ even}} SP_{r_s} \to C_e, \text{ if } G = O(V) \]
while
\[ \rho_{SP(V)} : \prod_{s \geq 1; s \text{ even}} O_{r_s} \times \prod_{s \geq 1; s \text{ odd}} SP_{r_s} \to C_e, \text{ if } G = SP(V). \]
We only describe the isomorphism $\rho_{O(V)}$. Choose $v_1, v_2, \ldots, v_r \in V$ such that $e^{d_i}v_i = 0$ and $\{e^j \cdot v_i \mid 0 \leq j \leq d_i - 1, 1 \leq i \leq r\}$ forms a basis of $V$. Here each number $d_i$ corresponds to the order of a Jordan block. Set

$$W_s = \sum_{i; d_i = s} C v_i.$$ 

The orthogonal group $O_s$ is defined on $W_s$ by a non-degenerate symmetric bilinear form. For any $g \in O_s$, its image under $\rho_{O(V)}$ is given by

$$\rho_{O(V)}(g)(e^j \cdot v_i) = \begin{cases} e^j \cdot gv_i, & \text{if } d_i = s; \\ e^j \cdot v_i, & \text{otherwise.} \end{cases}$$

Therefore as an $O_s$-module,

$$V \simeq W_s^{\oplus s} \oplus W'_s$$

where $W_s$ is the standard $O_s$-module and $O_s$ acts on $W'_s$ trivially.

Furthermore, the above construction shows that $W_{s_1} \subset W_{s_2}$ for any $s_1 \neq s_2$.

2.9. Vust’s Theorem for $\mathfrak{so}(V)$ and $\mathfrak{sp}(V)$. The following lemma comparing $[V^{\otimes k}]^{SO(V)}$ and $[V^{\otimes k}]^{O(V)}$ will be used in the proof of Theorem 2.13.

Lemma 2.12. (1). If $\dim(V)$ is odd, then we have $[V^{\otimes k}]^{SO(V)} = [V^{\otimes k}]^{O(V)}$ for all $k \in \mathbb{N}$.

(2). If $\dim(V)$ is even, then we have $[V^{\otimes k}]^{SO(V)} = [V^{\otimes k}]^{O(V)}$ for all $k < \dim(V)$.

Proof. Statement (1) follows from the fact $O(V) = SO(V) \cup (-1)SO(V)$.

Suppose $\dim(V) = 2r$ for some $r \in \mathbb{N}$. If $k$ is odd, since $-\text{id}_V \in SO(V)$ we have $[V^{\otimes k}]^{SO(V)} = [V^{\otimes k}]^{O(V)} = 0$. If $k$ is even, we identify $V^{\otimes k}$ with $\text{End}(V^{\otimes k/2})$ similar to (2.2). Then Theorem 1.4 (2) in [Gro] implies that $[\text{End}(V^{\otimes k/2})]^{SO(V)} = [\text{End}(V^{\otimes k/2})]^{O(V)}$. Thus we have proved statement (2). \qed

Now we can obtain the Lie algebra version of Vust’s Theorem for cases other than type $A$.

Theorem 2.13. Let $G = O(V)$ or $SP(V)$, and $e \in g = \text{Lie}(G)$ be a nilpotent element with partition $[1^{r_1}2^{r_2}\cdots]$ of $\dim(V)$ by Jordan blocks. Assume $e$ satisfies that

(1) the nilpotent orbit closure $\overline{G \cdot e}$ is a normal variety;

(2) if $G = O(V)$, either $r_s = odd$ or $r_s > 2d$ for all odd $s$; if $G = SP(V)$, either $r_s = odd$ or $r_s > 2d$ for all even $s$.

Then we have

$$\text{End}_{U(g_e)}(V^{\otimes d}) = B_d[e].$$

Proof. Here we will only prove the theorem for $G = O(V)$ since a similar argument works for $G = SP(V)$. Denote by $G_e^c$ the connected component of $G_e$ containing $\text{id}_V$. By the relation between representation of connected algebraic group and its Lie algebra, we need to show $\text{End}_{G_e^c}(V^{\otimes d}) = B_d[e]$. 

Set

\[ O_e := O_{r_1} \times O_{r_3} \times O_{r_5} \times \cdots, \]
\[ SO_e := SO_{r_1} \times SO_{r_3} \times SO_{r_5} \times \cdots, \]
\[ SP_e := SO_{r_2} \times SO_{r_4} \times SO_{r_6} \times \cdots. \]

Thus

\[ G_e = O_e \rtimes (SP_e \rtimes R_e) \quad \text{and} \quad G^e = SO_e \rtimes (SP_e \rtimes R_e). \]

We claim that

\[ \text{End}_{O_{rs}}(V \otimes d) = \text{End}_{SO_{rs}}(V \otimes d) \quad \text{for all even } s. \]

Indeed we have

\[ \text{End}_{O_{rs}}(V \otimes d) = \left[ \text{End}(V \otimes d) \right]^{O_{rs}} \]
\[ \simeq [V^{\otimes 2d}]^{O_{rs}} \quad \text{(by bijection } \theta^{\otimes d} : V^{\otimes 2d} \to \text{End}(V^{\otimes d}) \text{ similar to (2.2)}) \]
\[ = \bigoplus_{k=0}^{2d} \left( W^{\otimes k}_{s} \otimes W^{\otimes (2d-k)}_{s} \right)^{O_{rs}} \]
\[ = \bigoplus_{k=0}^{2d} \left( W^{\otimes k}_{s} \otimes W^{\otimes (2d-k)}_{s} \right)^{O_{rs}} \oplus c_k \]
\[ = \bigoplus_{k=0}^{2d} \left( W^{\otimes k}_{s} \otimes W^{\otimes (2d-k)}_{s} \right)^{O_{rs}} \oplus c_k \]

where \( c_k = s^k(2^d). \) By the same procedure we have

\[ \text{End}_{SO_{rs}}(V \otimes d) \simeq \bigoplus_{k=0}^{2d} \left( W^{\otimes k}_{s} \otimes W^{\otimes (2d-k)}_{s} \right)^{SO_{rs}} \oplus c_k. \]

Therefore, the claim follows from Lemma 2.12.

Using the above claim repeatedly, we get that

\[ \text{End}_{G_e}(V^{\otimes d}) = \text{End}_{SO_e}(V^{\otimes d}), \]

and hence

\[ \text{End}_{G_e \times (SP_e \rtimes R_e)}(V^{\otimes d}) = \text{End}_{SO_e \times (SP_e \rtimes R_e)}(V^{\otimes d}). \]

Thus we obtain

\[ \text{End}_{G_e}(V^{\otimes d}) = \text{End}_{G_e}(V^{\otimes d}) = B_d[e]. \quad \Box \]

2.10. Double centralizer property. Denote by \( \phi \) the action of \( U(\mathfrak{g}) \) on \( V^{\otimes d}. \) Though we do not give a double centralizer property for \( U(\mathfrak{g}_e) \) and \( B_d[e], \) instead we have the following proposition.

**Proposition 2.14.** Let \( \mathfrak{g} = \mathfrak{sp}(V) \) or \( \mathfrak{so}(V) \) be a simple Lie algebra of type B or C. If the nilpotent element \( e \in \mathfrak{g} \) satisfies the assumption in Theorem 2.13. Then the following double centralizer property holds:

\[ \text{End}_{\phi(U(\mathfrak{gl}(V),e)) \cap \phi(U(\mathfrak{g}))}(V^{\otimes d}) = B_d[e], \quad (2.6) \]
\[ \phi(U(\mathfrak{gl}(V),e)) \cap \phi(U(\mathfrak{g})) = \text{End}_{B_d[e]}(V^{\otimes d}). \quad (2.7) \]
Proof. Firstly, it is clear that actions of $B_d[e]$ and $\phi(U(\mathfrak{gl}(V_e))) \cap \phi(U(\mathfrak{g}))$ commute with each other. Thus Equation (2.6) follows from Theorem 2.13 and the fact that $\phi(U(\mathfrak{gl}(V_e))) \cap \phi(U(\mathfrak{g})) \supseteq \phi(U(\mathfrak{g}_e))$.

The following duality can be found in Theorem 2.4 in [BK2]:

$$\text{End}_{B_d[e]}(V \otimes d^e) = S_d[e];$$
$$\phi(U(\mathfrak{gl}(V_e))) = \text{End}_{S_d[e]}(V \otimes d).$$

Note that $B_d[e] \supseteq S_d[e]$ and $B_d[e] \supseteq B_d$. Thus

$$\phi(U(\mathfrak{gl}(V_e))) \cap \phi(U(\mathfrak{g})) \subset \text{End}_{B_d[e]}(V \otimes d) \subset \text{End}_{S_d[e]}(V \otimes d) = \phi(U(\mathfrak{g}))$$
and

$$\phi(U(\mathfrak{gl}(V_e))) \cap \phi(U(\mathfrak{g})) \subset \text{End}_{B_d[e]}(V \otimes d) \subset \text{End}_{B_d}(V \otimes d) = \phi(U(\mathfrak{g})).$$

Therefore there comes Equation (2.7). □

Remark 2.15. It is natural to ask whether

$$\phi_d(U(\mathfrak{gl}_e)) \cap \phi_d(U(\mathfrak{g})) = \phi_d(U(\mathfrak{g}_e)).$$

Though we can not answer this question in general, a direct calculation shows that the above equality holds when $d = 2$ and $\text{rank}(\mathfrak{g}) \leq 3$.

3. Centralizer of $W$-algebra action on $V \otimes d$

In this section, take $\mathfrak{g} = \mathfrak{so}_{2r}, \mathfrak{so}_{2r+1}$ or $\mathfrak{sp}_{2r}$. For convenience, entries of matrices in $\mathfrak{g}$ are indexed by $I \times I$ where

$$I = \begin{cases} \{-r, \ldots, -1, 0, 1, \ldots, r\} & \text{if } \mathfrak{g} = \mathfrak{so}_{2r+1}; \\ \{-r, \ldots, -1, 1, \ldots, r\} & \text{if } \mathfrak{g} = \mathfrak{so}_{2r} \text{ or } \mathfrak{sp}_{2r}. \end{cases}$$

3.1. Gradings. Assume that $\Gamma : \mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ is a $\mathbb{Z}$-grading of $\mathfrak{g}$. We say $\Gamma$ is good for nilpotent $e \in \mathfrak{g}$ if it satisfies that

1. $e \in \mathfrak{g}_2$;
2. $\text{ad}_e : \mathfrak{g}_j \to \mathfrak{g}_{j+2}$ is injective for $j \leq -1$; and
3. $\text{ad}_e : \mathfrak{g}_j \to \mathfrak{g}_{j+2}$ is surjective for $j \geq -1$.

We call $\Gamma$ is even if $\mathfrak{g}_j = 0$ for all odd $j$.

Refer to the literature [BK] for classification of nilpotent elements which admit even good gradings for classical Lie algebras. In this section we always assume that $e$ admits an even good grading. Moreover, an even good grading $\Gamma$ induces a grading for $U(\mathfrak{g})$, which is called a loop grading.

For any $\mathbb{Z}$-grading $\Gamma$, there exists a semisimple element $h_\Gamma \in \mathfrak{g}$ such that $\Gamma$ coincides with the eigenspace decomposition of $\text{ad}_{h_\Gamma}$ (c.f. [W]), i.e.

$$\mathfrak{g}_j = \{x \in \mathfrak{g} | [h_\Gamma, x] = jx\}.$$

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $h_\Gamma$. 




Convention: Without loss of generality, we assume that $h_\Gamma$ is diagonal (by a conjugate transformation if necessary), and hence we take $\mathfrak{h}$ to be the standard Cartan subalgebra consisting of all diagonal matrices in $\mathfrak{g}$.

Write $F_{i,j} := E_{i,j} - \theta_{i,j} E_{-j,-i}$ with $i, j \in I$ and

\[
\theta_{i,j} = \begin{cases} 
1, & \text{if } \mathfrak{g} = \mathfrak{so}_{2r+1} \text{ or } \mathfrak{so}_{2r}; \\
\text{sgn}(i)\text{sgn}(j), & \text{if } \mathfrak{g} = \mathfrak{sp}_{2r}.
\end{cases}
\]

The following set

\[
\mathbb{B} = \begin{cases} 
\{ F_{i,i} \mid 0 < i \leq r \} \cup \{ F_{\pm i,\pm j} \mid 0 < i < j \leq r \} \cup \{ F_{0,\pm i} \mid 0 < i \leq r \}, & \text{if } \mathfrak{g} = \mathfrak{so}_{2r+1}; \\
\{ F_{\pm i,\pm j} \mid 0 < i < j \leq r \} \cup \{ F_{i,i}, F_{-i,i}, F_{i,-i} \mid 0 < i \leq r \}, & \text{if } \mathfrak{g} = \mathfrak{sp}_{2r}; \\
\{ F_{\pm i,\pm j} \mid 0 < i < j \leq r \} \cup \{ F_{i,i} \mid 0 < i \leq r \}, & \text{if } \mathfrak{g} = \mathfrak{so}_{2r},
\end{cases}
\]

forms a basis of $\mathfrak{g}$. The subset $\{ F_{i,i} = E_{i,i} - E_{-i,-i} \mid 0 < i \leq r \} \subset \mathbb{B}$ forms a basis of $\mathfrak{h}$.

Define a map

\[
col : I \rightarrow \mathbb{Z}, \ i \mapsto \col(i) \quad \text{such that} \quad h_\Gamma \cdot v_i = \col(i)v_i.
\]

Equip $V$ a $\mathbb{Z}$-grading by $\gr(v_i) := \col(i)$. It is easy to check that $V$ is a graded $\mathfrak{g}$-module under this grading.

The set

\[
\{ v_{i_1} \otimes \cdots \otimes v_{i_d} \mid (i_1, \ldots, i_d) \in I^d \}
\]

forms a homogeneous basis of graded $\mathfrak{g}$-module $V^\otimes d$ with

\[
\gr(v_{i_1} \otimes \cdots \otimes v_{i_d}) = \sum_{k=1}^d \col(i_k).
\] (3.1)

Set $\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{g}(i)$ and $\mathfrak{m} = \bigoplus_{i < 0} \mathfrak{g}(i)$. The map col satisfies the following proposition.

**Proposition 3.1.**

1. $\col(i) + \col(-i) = 0, \ (\forall 1 \leq i \leq r);$
2. $F_{i,j} \in \mathfrak{p} \iff \col(j) \leq \col(i), \ (\forall 1 \leq i, j \leq r);$
3. $F_{i,j} \in \mathfrak{m} \iff \col(j) > \col(i), \ (\forall 1 \leq i, j \leq r).$

**Proof.** Assume $h_\Gamma = \sum_{1 \leq i \leq r} a_i (E_{i,i} - E_{-i,-i})$. It is clear that

\[
\col(i) = a_i, \quad \col(-i) = -a_i \quad (\forall 1 \leq i \leq r)
\]

and

\[
\gr(F_{i,j}) = \col(i) - \col(j).
\]

So the proposition follows. \qed
3.2. **W-algebra** $W_\chi$. There are several equivalent definitions for W-algebras. Here we adapt the following definition for those nilpotent element $e \in \mathfrak{g}$ admitting an even good grading.

Let $\chi \in \mathfrak{g}^*$ be the linear function on $\mathfrak{g}$ uniquely determined by

$$\chi(g) = \text{Trace}(\text{ad}_e \circ \text{ad}_X), \quad (\forall X \in \mathfrak{g}).$$

Let $I_\chi$ be the left ideal of $U(\mathfrak{m})$ generated by $a - \chi(a)$ for all $a \in \mathfrak{m}$.

The **$W$-algebra associated to** $e$ is defined as $W_\chi := \{ y \in U(\mathfrak{p}) | [a, y] \in I_\chi, \forall a \in \mathfrak{m} \}$.

The $W$-algebra $W_\chi$ is a filtration subalgebra of graded algebra $U(\mathfrak{g})$ (with loop grading). By restriction, $V^\otimes d$ has a $W_\chi$-module structure which is compatible with the above filtration of $W_\chi$.

It is clear by the definition of good grading that $\mathfrak{g}_e \subset \mathfrak{p}$. So there is an embedding $U(\mathfrak{g}_e) \hookrightarrow U(\mathfrak{p})$.

**Theorem 3.2** (c.f. Theorem 3.8 in [BGK]). The embedding $U(\mathfrak{g}_e) \hookrightarrow U(\mathfrak{p})$ induces a graded algebra isomorphism

$$U(\mathfrak{g}_e) \cong \text{gr}(W_\chi).$$

3.3. **Tensor identities.** All statements in this subsection can be found in [BK1, BK3]. Though Brundan and Kleshchev dealt with case of type $A$ only, their proofs are still valid for types $B, C$ and $D$ when $e$ admits an even good grading.

Set the quotient space $Q_\chi := U(\mathfrak{g})/I_\chi$. Denote by $1_\chi$ the coset of $1 \in U(\mathfrak{g})$ in $Q_\chi$. The vector space $Q_\chi$ possesses a $(U(\mathfrak{g}), W_\chi)$-bimodule structure, where the left action of $U(\mathfrak{g})$ is given by

$$u \circ u'1_\chi = (uu')1_\chi \quad (\forall u, u' \in U(\mathfrak{g}))$$

while the right action of $W_\chi$ is given by

$$(u'1_\chi)w = (u'w)1_\chi \quad (\forall w \in W_\chi, u' \in U(\mathfrak{g})).$$

We also have an isomorphism $W_\chi \rightarrow \text{End}_{U(\mathfrak{g})}(Q_\chi)$. It has been known (c.f. [BK3]) that $Q_\chi$ is a free $W_\chi$-module and there exist $a_1, \ldots, a_h \in \mathfrak{p}$ such that $\{a_1^{i_1} \cdots a_h^{i_h} 1_\chi | i_1, \ldots, i_h \geq 0\}$ forms a basis of $Q_\chi$ as a free $W_\chi$-module.

Denote by $\mathcal{C}(\chi)$ the category consisting of all $\mathfrak{g}$-modules on which $a - \chi(a)$ acts locally nilpotently for all $a \in \mathfrak{m}$. Skrybian’s equivalence theorem says that the functor

$$Q_\chi \otimes_{W_\chi}? : W_\chi\text{-mod} \rightarrow \mathcal{C}(\chi), \quad M \mapsto Q_\chi \otimes_{W_\chi} M$$

is an equivalence of categories.

Given $M \in \mathcal{C}(\chi)$, the subspace

$$\text{Wh}(M) := \{ v \in M | xv = \chi(x)v, \forall x \in \mathfrak{m} \}$$
has a natural $\mathcal{W}_\chi$-module structure. Thus we have a functor
\[
\text{Wh} : \mathcal{C}(\chi) \to \mathcal{W}_\chi\text{-mod},
\]
\[M \mapsto \text{Wh}(M),\]
which is the inverse of $Q_\chi \otimes \mathcal{W}_\chi$.\]

Let $W$ be an arbitrary finite dimensional $\mathfrak{g}$-module. Suppose that $W$ has a basis $\{w_1, \ldots, w_r\}$. Define a functor
\[
? \otimes W : \mathcal{W}_\chi\text{-mod} \to \mathcal{W}_\chi\text{-mod},
\]
\[M \mapsto M \otimes W := \text{Wh}((Q_\chi \otimes \mathcal{W}_\chi M) \otimes W).\]

Define $c_{i,j} \in U(\mathfrak{g})^*$ via the equation
\[
uw_j = \sum_{i=1}^r c_{i,j}(u)w_i \quad \text{for any } u \in U(\mathfrak{g}).
\]

Take a projection $p : Q_\chi \twoheadrightarrow \mathcal{W}_\chi$ with $p(1_\chi) = 1$. Define a linear map of vector space by
\[
\chi_{M,W} : M \otimes W \to M \otimes W, \quad (u_1 \chi \otimes m) \otimes w \mapsto p(u_1 \chi) m \otimes w.
\]

**Theorem 3.3** (c.f. Theorem 8.1 in [BK1]). For any left $\mathcal{W}_\chi$-module $M$ and finite dimensional $\mathfrak{g}$-module $W$, the linear map $\chi_{M,W}$ is an isomorphism of vector space and
\[
\chi_{M,W}^{-1}(m \otimes w_j) = \sum_{i=1}^r (x_{i,j} \cdot 1_\chi \otimes m) \otimes w_i,
\]
where $(x_{i,j})_{1 \leq i,j \leq r}$ is a matrix with entries in $U(p)$ determined uniquely by the properties

1. $p(x_{i,j} 1_\chi) = \delta_{i,j}$; and
2. $[a, x_{i,j}] + \sum_{s=1}^r c_{i,s}(a) x_{s,j} \in U(\mathfrak{g}) I_\chi$ for any $a \in \mathfrak{m}$.

Any $U(p)$-module $M$ can be viewed as a $\mathcal{W}_\chi$ module by restriction. For any $\mathfrak{g}$-module $W$, define a linear map
\[
\mu_{M,W} : M \otimes W \to M \otimes W, \quad (u_1 \chi \otimes m) \otimes w \mapsto um \otimes w.
\]

for all $u \in p, m \in M$ and $w \in W$.

**Corollary 3.4** (c.f. Corollary 8.2 in [BK1]). For any $U(p)$-module $M$ and finite dimensional $\mathfrak{g}$-module $W$, $\mu_{M,W}$ is an isomorphism of $\mathcal{W}_\chi$-modules and
\[
\mu_{M,W}^{-1}(m \otimes v_k) = \sum_{i,j=1}^r (x_{i,j} \cdot 1_\chi \otimes y_{j,k} m) \otimes v_i
\]
where $(x_{i,j})_{1 \leq i,j \leq r}$ is the matrix defined in Theorem 3.3 and $(y_{i,j})_{1 \leq i,j \leq r}$ is its inverse matrix.

**Theorem 3.5** (c.f. Lemma 3.2 in [BK3]). Let $M = \mathbb{C}1_M$ be a one dimensional $U(p)$-module. There exist $x_{i,j} \in U(p)$ (1 ≤ i, j ≤ r) such that

1. $[a, x_{i,j}] + \sum_{s=1}^r c_{i,s}(a) x_{s,j} \in U(\mathfrak{g}) I_\chi$ for any $a \in \mathfrak{m}$;
For any such choice of $x_{i,j}$, we have

$$\mu_{M,V}(1_M \otimes v) = \sum_{i=1}^{r} x_{i,j} 1_x \otimes 1_M \otimes v_i.$$ 

Proof. Denote by $c$ the linear function on $p$ determined by

$$a \cdot 1_M = c(a) 1_M \quad (\forall a \in p).$$

Specify the projection $p$ in Theorem 3.3 by $p(a_1^{i_1} \cdots a_h^{i_h}) = c(a_1^{i_1}) \cdots c(a_h^{i_h})$. Then the statement follows from Theorem 3.3 and Corollary 3.4. □

3.4. Degenerate affine braid algebra. For any $g \in B$, denote by $g^* \in g$ its dual with respect to the Killing form. Let $\kappa = \sum_{g \in B} gg^* \in U(g)$ be the Casimir element.

Definition 3.6. Degenerate affine braid algebra $B_d$ is defined by generators $\tilde{s}_1, \ldots, \tilde{s}_{d-1}, \tilde{r}_0, \ldots, \tilde{r}_d$ and $\tilde{r}_{i,j}$ ($0 \leq i \neq j \leq d$) with some relations (refer to Theorem 1.1 in [DRV] since it occupies too much space and will not be used in this paper).

Let $V$ be the natural $g$-module with a standard basis $\{v_i | i \in I\}$, and $M$ be any $g$-module. There is an action $\tilde{\Phi} : B_d \to \text{End}(M \otimes V^{\otimes d})$ as follows.

$$\begin{cases}
\tilde{\Phi}(\tilde{s}_i) = 1_{\otimes i} \otimes P \otimes 1_{\otimes (d-1-i)}, (i = 1, \ldots, d);
\tilde{\Phi}(\tilde{r}_i) = 1_{\otimes i} \otimes \kappa \otimes 1_{\otimes (d-i)}, (i = 0, \ldots, d);
\tilde{\Phi}(\tilde{r}_{i,j}) = \sum_{g \in B} 1_{\otimes i} \otimes g \otimes 1_{\otimes (j-i-1)} \otimes g^* \otimes 1_{\otimes (d-j)}, (0 \leq i < j \leq d),
\end{cases}$$

where $P$ is the linear operator such that $P(u \otimes v) = v \otimes u$. This action of $B_d$ on $M \otimes V^{\otimes d}$ commutes with the action of $U(g)$ (c.f. Theorem 1.2 [DRV]).

3.5. Action of $B_d$ on $V^{\otimes d}$. Let $C_e$ be the trivial $U(p)$-module, which can be viewed as a $W_x$-module by restriction. Hence $Q_x \otimes W_x C_e$ is a $g$-module due to Skrybien’s equivalence theorem. Then there is a $B_d$ action on $(Q_x \otimes W_x C_e) \otimes V^{\otimes d}$ via $\tilde{\Phi}$. The subspace $\text{Wh}((Q_x \otimes W_x C_e) \otimes V^{\otimes d})$ is invariant under $\tilde{\Phi}(B_d)$ since the action of $a - x(a)$ $(\forall a \in m)$ commutes with $\tilde{\Phi}(B_d)$. Thus we have an action of $B_d$ on $\text{Wh}((Q_x \otimes W_x C_e) \otimes V^{\otimes d}) = C_e \otimes V^{\otimes d}$, which commutes with the action of $W_x$.

Thanks to the following isomorphisms of $W_x$-modules

$$V^{\otimes d} \simeq C_e \otimes V^{\otimes d} \simeq C_e \otimes V^{\otimes d}$$

we obtain a $B_d$ action (denoted by $\tilde{\Phi}$) which commutes with the action of $W_x$.

The following lemma can be obtained by a straightforward calculation.

Lemma 3.7. We have

$$\tilde{\Phi}(\tilde{s}_i) = 1_{\otimes (i-1)} \otimes P \otimes 1_{\otimes (d-1-i)}, \quad (1 \leq i \leq d)$$

and

$$\Phi(\tilde{r}_{i,j}) = \sum_{g \in B} 1_{\otimes (i-1)} \otimes g \otimes 1_{\otimes (j-i-1)} \otimes g^* \otimes 1_{\otimes (d-j)} = -\gamma_{i,j} + s_{i,j}, \quad (0 < i < j \leq d),$$
where \( s_{i,j} \) is the endomorphism of \( V^{\otimes d} \) permuting the \( i \)-th and \( j \)-th tensor positions.

Write \( v_i := v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_d} \) for any \( i = (i_1, i_2, \ldots, i_d) \in I^d \).

**Lemma 3.8.** For any \( 1 \leq k \leq d \) and \( i \in I^d \), we have
\[
\Phi(\tilde{\gamma}_{0,k}) \cdot v_i = e^{(k)} \cdot v_i + \text{lower terms associated to the grading } (3.1).
\]

**Proof.** Recall \( e^{(k)} \) in (1.1). The notation \( F_{q,p}^{(k)} \) used in this proof is defined similarly. Write \( \mu := \mu_{C_{e,V^{\otimes d}}} \) for short. We have
\[
\Phi(\tilde{\gamma}_{0,k}) \cdot v_i = \mu(\Phi(\tilde{\gamma}_{0,k}) \cdot \sum_{j \in I^d} (x_{j,i} 1 \chi_1 \otimes v_j) = \sum_{F_{p,q} \in \mathcal{B}, j \in I^d} \mu((F_{p,q} x_{j,i} 1 \chi_1 \otimes 1) \otimes (F_{p,q}^*)^{(k)} v_j)
\]
where \( x_{j,i} \) (\( \forall i, j \in I^d \)) are determined by theorem 3.5. The first equality comes from Theorem 3.5 (3). The second one follows from the action of \( \tilde{\gamma}_{0,k} \) constructed in Equation (3.2).

If \( \col(q) \leq \col(p) \), then by Proposition 3.1 (2) we have \( F_{p,q} \in p \). By Theorem 3.5 (2) we have
\[
\mu((F_{p,q} x_{j,i} 1 \chi_1 \otimes 1) \otimes (F_{p,q}^*)^{(k)} v_j) = F_{p,q} x_{j,i} \cdot 1 \otimes 1 \otimes (F_{p,q}^*)^{(k)} v_j = 0.
\]

If \( \col(q) > \col(p) \) then by Proposition 3.1 (2), we have \( F_{p,q} \in m \). Thus Theorem 3.5 (1) implies that
\[
\mu((F_{p,q} x_{j,i} 1 \chi_1 \otimes 1) \otimes (F_{p,q}^*)^{(k)} v_j) = \mu((x_{j,i} F_{p,q} 1 \chi_1 \otimes 1 \otimes (F_{p,q}^*)^{(k)} v_j + \sum_{s \in I^d} c_{j,s}(F_{p,q}) x_{s,i} 1 \chi_1 \otimes 1 \otimes (F_{p,q}^*)^{(k)} v_j).
\]
Since \( F_{p,q} 1 \chi_1 = \chi(F_{p,q}) \), we have
\[
\mu((x_{j,i} F_{p,q} 1 \chi_1 \otimes 1) \otimes (F_{p,q}^*)^{(k)} v_j) = \begin{cases} 0, & \text{if } j \neq i; \\ \chi(F_{p,q})(F_{p,q}^*)^{(k)} v_j, & \text{if } j = i \end{cases} \tag{3.3}
\]
and
\[
\mu(c_{j,s}(F_{p,q}) x_{s,i} 1 \chi_1 \otimes 1 \otimes (F_{p,q}^*)^{(k)} v_j) = \begin{cases} 0, & \text{if } s \neq i \text{ or } c_{j,s}(F_{p,q}) = 0, \\ c_{j,i}(F_{p,q})(F_{p,q}^*)^{(k)} v_j, & \text{otherwise}. \end{cases} \tag{3.4}
\]
A direct calculation shows that
\[
F_{p,q}^* = F_{q,p} \text{ if } p = q; \quad F_{p,q}^* = \frac{1}{2} F_{q,p} \text{ if } p = -q.
\]
Finally, we obtain the term \( e^{(k)} \cdot v_i \) by summing up Equation (3.3) over all \( F_{p,q} \in \mathcal{B} \) with \( \col(p) > \col(q) \) and \( j \in I^d \), while the lower terms come from summing up Equation (3.4) over all \( F_{p,q} \in \mathcal{B} \) with \( \col(p) > \col(q) \) and \( j \in I^d \).

3.6. **Higher level Schur-Weyl duality.** Following is a half of the higher level Schur-Weyl duality for types \( B, C \) and \( D \).

**Theorem 3.9.** Let \( G = O(V) \) or \( SP(V) \), and \( e \) be a nilpotent element in \( g = \mathfrak{g} = \text{Lie}(G) \) with partition \([1^{r_1} 2^{r_2} \cdots]\) of \( \dim(V) \) by Jordan blocks. Assume \( e \) satisfies that

1. the nilpotent orbit closure \( \overline{G \cdot e} \) is a normal variety;
if $G = O(V)$, either $r_s$ is odd or $r_s > 2d$ for all for odd $s$; if $G = SP(V)$, either $r_s$ is odd or $r_s > 2d$ for all for even $s$.

(3) $e$ admits an even good grading $\Gamma : g = \bigoplus_{i \in \mathbb{Z}} g(i)$.

Then

$$\operatorname{End}_{\mathcal{W}_\chi}(V^\otimes d) = \Phi(B_d).$$

(3.5) Proof. Notice that the action of $\mathcal{W}_\chi$ on $V^\otimes d$ is compatible with the filtration of $\mathcal{W}_\chi$. Hence we have an action of $\operatorname{gr}(\mathcal{W}_\chi)$ on $V^\otimes d$. The canonical isomorphism $\operatorname{gr}(\mathcal{W}_\chi) \cong U(g_e)$ given in Theorem 3.2 implies that the above action of $\operatorname{gr}(\mathcal{W}_\chi)$ coincides with the action of $U(g_e)$ on $V^\otimes d$.

As a subalgebra of the graded algebra $\operatorname{End}(V^\otimes d)$, $\Phi(B_d)$ admits a natural filtrated algebra structure. And hence there is a natural embedding $\operatorname{gr}(\Phi(B_d)) \hookrightarrow \operatorname{End}(V^\otimes d)$. Without confusion, we also denote the image of this embedding by the same notation $\operatorname{gr}(\Phi(B_d))$. Since $\Phi(B_d) \subseteq \operatorname{End}_{\mathcal{W}_\chi}(V^\otimes d)$, we can calculate that

$$\operatorname{gr}(\Phi(B_d)) \subseteq \operatorname{End}_{\operatorname{gr}(\mathcal{W}_\chi)}(V^\otimes d) = \operatorname{End}_{U(g_e)}(V^\otimes d) = B_d[e].$$

On the other hand, Lemmas 3.7 and 3.8 show that $B_d[e] \subseteq \operatorname{gr}(\Phi(B_d))$.

So $\operatorname{gr}(\Phi(B_d)) = \operatorname{End}_{\operatorname{gr}(\mathcal{W}_\chi)}(V^\otimes d) \supseteq \operatorname{gr}(\operatorname{End}_{\mathcal{W}_\chi}(V^\otimes d))$, which together with the fact $\operatorname{End}_{\mathcal{W}_\chi}(V^\otimes d) \supseteq \Phi(B_d)$ implies $\operatorname{End}_{\mathcal{W}_\chi}(V^\otimes d) = \Phi(B_d)$. \hfill $\square$

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