Torsional chiral magnetic effect due to skyrmion textures in a Weyl superfluid $^3$He-A

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We investigate chiral anomaly phenomena induced by skyrmion-like textures of the $\ell$-vector field in the A phase of the superfluid $^3$He, which is a prototype of Weyl superfluids. In particular, we focus on torsional chiral magnetic effect (TCME), which gives rise to mass current flow due to a torsional magnetic field arising from $\ell$-textures. We find that skyrmion $\ell$-textures induce chiral fermion states with spectral asymmetry, leading to the equilibrium currents of Weyl-Bogoliubov quasiparticles. It is also demonstrated that skyrmion $\ell$-textures possess spatially inhomogeneous structures of Weyl bands in the real coordinate space. Solving the Bogoliubov-de Gennes equation in the quantum regime, we clarify the contribution of the TCME to the mass current. From the viewpoint of the TCME, furthermore, we examine the mass current density generated by $\ell$ textures in $^3$He-A, which is involved with the so-called angular momentum paradox.

I. INTRODUCTION

Weyl semimetals have been attracting much attention because of the realization of chiral anomaly in condensed matter systems, which is experimentally detectable in various exotic transport phenomena such as the anomalous Hall effect, chiral magnetic effect, and negative magnetoresistance.\textsuperscript{1-14} Chiral anomaly is the violation of conservation law of axial currents in the case with both electric and magnetic fields which are not orthogonal to each other. Its origin is attributed to monopole charge carried by Weyl points in the momentum space, which generate the Berry curvature, and are sources and drains of momentum generation. Recent experimental studies revealed the realization of chiral anomaly in Weyl semimetal materials via the observation of negative magnetoresistance.\textsuperscript{15-17}

The notion of Weyl semimetals is naturally generalized to superconducting systems.\textsuperscript{18} In superconductors with broken time-reversal symmetry such as chiral pairing states and non-unitary odd-parity pairing states, nodal excitations from point-nodes of the superconducting gap behave as Weyl fermions accompanying the Berry curvature. There are several candidate systems of Weyl superconductors and superfluids such as the A-phase of the superfluid $^3$He,\textsuperscript{19-22} URu$_2$Si$_2$,\textsuperscript{23} the B-phase of UPt$_3$, UCoGe,\textsuperscript{24,25} and the B-phase of U$_{1-x}$Th$_x$Be$_2$.\textsuperscript{26-28} Since the Bogoliubov quasiparticles are the superposition of electrons and holes, the usual coupling with electromagnetic fields does not directly lead to chiral anomaly. However, it is still possible that in Weyl superconductors and Weyl superfluids, emergent electromagnetic fields arising from spatially inhomogeneous textures of the superconducting order parameter and its dynamics give rise to chiral anomaly phenomena. As a matter of fact, in 1997, more than ten years before the invention of the notion of Weyl semimetals, Bevan et al. observed momentum generation due to chiral anomaly in $^3$He-A with skyrmion textures of the $\ell$-vector field,\textsuperscript{29} which was motivated by pioneering theoretical works of Volovik and his collaborators.\textsuperscript{19,30-33} In the experiment,\textsuperscript{29} the chiral anomaly was detected via the measurement of an extra force on skyrmion-vortices.

In this paper, we consider another chiral anomaly effect which is referred to as the torsional chiral magnetic effect (TCME). The TCME was originally proposed for magnetic Weyl semimetals with lattice dislocations.\textsuperscript{34} Lattice dislocations give rise to torsion fields which cause emergent magnetic fields acting on Weyl fermions, and result in equilibrium currents flowing along the dislocation lines. The current induced by the torsion field is given by,

$$ J_{\text{TCME}} = \frac{\nu F A}{4 \pi} \sum_{a=x,y,z} T^a (p_{L a} - p_{R a}) , $$

(1)

where $\nu F$ is the Fermi velocity, $A$ is the momentum cut-off, $p_{L a}[\mu]$ ($\mu = x,y,z$) is the position of the Weyl point with left(right)-handed chirality in momentum space, and

$$ (T^a)^\mu = \frac{e \mu v_L}{2} T^a_{\nu \lambda} , $$

(2)

where $T^a_{\nu \lambda}$ is torsion which can be realized in condensed matter systems by topological defects such as lattice dislocation and a skyrmion texture of magnetic order. In the case of superconductors, torsional magnetic fields arise from vortex textures of the superconducting order parameter or lattice strain, and hence, the negative thermal magnetoresistivity, that is, the anomalous enhancement of longitudinal thermal conductivity along the vortex line.\textsuperscript{35}

Here we show that the TCME of Weyl-Bogoliubov quasiparticles is realized as the equilibrium currents induced by an order parameter texture in Weyl superfluids. We focus on the A-phase of $^3$He having a skyrmion-vortex as a promising platform for the TCME. The order parameter tensor, $A_{\mu \nu}$, that transforms as a vector with respect to index $\mu = x,y,z$ ($i = x,y,z$) under spin (orbital) rotations, is given by the complex form\textsuperscript{36,37}

$$ A_{\mu \nu} = \Delta_\lambda (T) d_\mu (\hat{\mathbf{r}} + i \hat{n})_\nu e^{\imath \phi} , $$

(3)

where $d$ and $(\hat{\mathbf{r}}, \hat{n})$ are unit vectors representing spin and orbital degrees of freedom in the superfluid vacuum, respectively. This is the Cooper-pair state with a definite orbital angular momentum represented by $\hat{l} \equiv \hat{\mathbf{r}} \times \hat{n}$. The $\ell$-vector also points to the nodal orientation at which Weyl-Bogoliubov quasiparticles reside. Owing to the spontaneously broken gauge-orbit symmetry, the rotation of the orbital part, $\hat{\mathbf{r}} + i \hat{n}$, about $\hat{l}$ is equivalent to the U(1) phase rotation $\phi$. This implies that the superfluid current can be generated by the texture of the triad $(\hat{\mathbf{r}}, \hat{n}, \hat{l})$ without U(1) phase singularities.
For rotating $^3$He-A, therefore, the $\ell$-vector field spontaneously forms a skyrmion-like texture as a ground state, which is known as the Anderson-Toulouse vortex and the Mermin-Ho vortex (see Fig. 1).\textsuperscript{22,41-43} The $\ell$-texture fields also appear in $^3$He confined in a narrow cylinder.\textsuperscript{44,45}

To clarify the TCME in $^3$He-A with skyrmion-vortices, we utilized two approaches: the semiclassical theory for Weyl-Bogoliubov quasiparticles and Bogoliubov-de Gennes (BdG) equation. The former clarifies that a "torsion" field emerges from a nontrivial $\ell$-texture as $T^3 = \text{curl}\hat{\ell}$ and gives rise to the TCME as in Eq. (1). The BdG equation provides a full quantum mechanical approach to the TCME and Weyl-Bogoliubov quasiparticles. Under skyrmion textures, the Bogoliubov quasiparticle spectrum possesses chiral fermion branches with spectral asymmetry, and the chiral fermions carry the macroscopic equilibrium current along the torsional magnetic field. Furthermore, we demonstrate that the skyrmion textures of $\ell$-vector give rise to spatially inhomogeneous structures of the Weyl fermion band in addition to the torsional magnetic field\textsuperscript{46}; the position of Weyl points in the momentum space exhibits inhomogeneous textures in the real space.

From the viewpoint of the TCME, we also revisit the superfluid current generated by $\ell$ textures. The expression of the current density in spatially inhomogeneous $^3$He-A possesses a longstanding issue involved with the angular momentum paradox. McClure and Takagi\textsuperscript{27} showed that the ground state with an axially symmetric $\ell$-textures possesses the total angular momentum $L_z = Na/2$, where $N$ is the total number of particles. The mass current density at zero temperatures was derived by Mermin and Muzikar\textsuperscript{48} as:

$$j^{\text{MM}} = \rho v_s \hat{\rho} + \frac{\hbar}{4M} \text{curl}\rho \hat{\ell} - \frac{\hbar}{2M} C_0 (\hat{\ell} \cdot \text{curl}\hat{\ell}),$$

(4)

where $\rho$ is the mass density of $^3$He atoms, $M$ is the mass, and $C_0 \approx \rho$ in the weak coupling approximation. The first term results from the superfluid velocity. The second term in Eq. (4) represents current due to a variation of orbital angular momentum of the Cooper pair, $\hbar\hat{\ell}$, which resembles the electric current induced by a variation of the magnetization density in materials. The third term is the anomalous term referred to as $j^{\text{an}}$. The derivation of Eq. (4) is based on the configuration-space form of the BCS variational ground state wavefunction.\textsuperscript{48} The similar approach was also used by Ishikawa, Miyake, and Usui but came to the different conclusion that $j^{\text{an}}$ is absent and $j^{\text{IMU}} = j^{\text{MM}} - j^{\text{an}}$.\textsuperscript{49} The anomalous term violates the McClure-Takagi relation when $\hat{\ell} \cdot \text{curl}\hat{\ell} \neq 0$. The discrepancy between $j^{\text{MM}}$ and $j^{\text{IMU}}$ is referred to as the McClure-Takagi paradox\textsuperscript{48,50,51}.

The physical origin of $j^{\text{an}}$ was addressed by Combescot and Dombre\textsuperscript{22,52,53} and Balatsky et al.\textsuperscript{19,32-33} where the latter unveiled that $j^{\text{an}}$ is attributed to the chiral anomaly of Weyl-Bogoliubov quasiparticles so as to violate the law of momentum conservation. In this paper, we show that in addition to $j^{\text{an}}$, the second term in Eq. (4) is related to the TCME, i.e., Weyl-Bogoliubov quasiparticle current generated by a torsion field. We further demonstrated that our numerical calculation for twisted $\ell$-textures coincides to $j^{\text{MM}}$ with $j^{\text{an}}$ in the weak coupling limit.

The organization of this paper is as follows. In Sec. II, we present semiclassical analysis for TCME in the case of Weyl superconductors/superfluids. In Sec. III, we describe the numerical method for solving the Bogoliubov-de-Gennes (BdG) equation for our purpose, and show the intrinsic features of Weyl-Bogoliubov quasiparticles in the presence of the $\ell$-texture, such as spectral asymmetry and spatially inhomogeneous structures of the Weyl fermion band. In Sec. IV, we discuss the mass current density induced by the skyrmion texture from the viewpoint of the TCME. The final section is devoted to conclusion and discussion.

II. TORSIONAL CHIRAL MAGNETIC EFFECT AND SEMICLASSICAL ANALYSIS

A. Semiclassical equation of motion for Weyl-Bogoliubov quasiparticles

We consider the Bogoliubov-de Gennes (BdG) Hamiltonian for the Bogoliubov quasiparticles in the A-phase of the superfluid $^3$He, i.e., a 3D chiral $p + ip$ superfluid, with spatially varying gap structures such as skyrmion-like $\ell$-vector textures of the Anderson-Toulouse vortex and the Mermin-Ho vortex. We apply the path integral formulation in a curved space with nonzero torsion which is induced by a vortex structure. In the Lagrangian in the Feynman kernel, the spatially varying gap function is expressed as $A_{\mu} p_{/p}$ with the tensorial field $A_{\mu}$ in Eq. (3). Here $m$ and $n$ are unit vectors for a local orthogonal frame which are perpendicular to the direction of the point nodes at $p = sp_0 \equiv sp_0\ell$, where $s \geq 1$ is the chirality of the Weyl points, and $\ell = m \times n$ is the $\ell$-vector. $\Delta$ is the superconducting gap, and $p_{/p}$ is the Fermi momentum. Then, the effective Lagrangian for Bogoliubov quasiparticles around $p = sp_0$ is given by:

$$\mathcal{L}_s = p \cdot \dot{r} - \mathcal{H}_s(p, \mathbf{r}),$$

(5)

with the Weyl-Bogoliubov Hamiltonian:

$$\mathcal{H}_s(p, \mathbf{r}) = se_{\alpha}^{\mu} v_{\beta}^{\nu} e_{\mu}^{\nu} (p_{/p} - sp_{/p}),$$

(6)
where $V^\mu = \text{diag}[\frac{\partial}{\partial p^x}, \frac{\partial}{\partial p^y}, v]$ with $v$ the Fermi velocity, $\tau^\mu$ is the Pauli matrix in the particle-hole space, and the vielbein $e^\mu_a$ is given by
\[
(e^\mu_1, e^\mu_2, e^\mu_3) = (m^\mu, n^\mu, \ell^\mu).
\]
(7)

We use greek letter indices $\mu = 1, 2, 3$ as space indices for the laboratory frame, and roman letters $a = 1, 2, 3$ as indices for a local orthogonal frame. The Weyl Hamiltonian must obey the particle-hole symmetry
\[
\mathcal{H}(p, r) = -\mathcal{H}(-p, r),
\]
(8)
where $\mathcal{H} = K\tau_1$ is the particle-hole conversion operator with $\tau_\mu$ the Pauli matrices in the particle-hole space. The particle-hole symmetry guarantees that the Weyl point appears as a pair of $p_0$ and $-p_0$ and the pairwise Weyl points have opposite chirality, $s = \pm 1$.

The Berry connection and the Berry curvature in the momentum space characterizing Weyl fermions appear when one projects the state into the one of the two energy band of $\mathcal{H}(p, r)$. This approach is justified for Weyl semimetals, when the Fermi level crosses only one band, and the other band is well separated from the Fermi level. However, in the case of Weyl superconductors and Weyl superfluids, the Fermi level crosses the Weyl point at which the lower band touches the upper band, and thus, the projection procedure is not justified. Nevertheless, we exploit this approach to see qualitatively how the Berry curvature plays the role in the response against torsion fields.

Following the method developed in ref. 54, we obtain the effective Lagrangian for the upper band:
\[
\mathcal{L}_s = p_\mu i^\mu + \mathcal{A}_s + \mathcal{A}_s p_\mu + \mathcal{A}_s i^\mu,
\]
(9)
where the Berry connections are $\mathcal{A}_s = i\langle u_+ | \partial_\mu | u_+ \rangle$ and $\mathcal{A}_s = i\langle u_+ | \partial_\mu | u_+ \rangle$ with $\mathcal{H}(u_+|u_+)\rangle = e_\mu |u_+\rangle$ and $e_\mu$ is the single-particle energy of Weyl-Bogoliubov quasiparticles with chirality $s$.

In some cases with a spatially varying structure of the gap function such as a vortex, and the Mermin-Ho texture or the Anderson-Toulouse texture of the $\ell$-vector, nonzero torsion appears. The torsion field is defined by
\[
T^{\ell a}_{\mu\nu} = \frac{\partial e_\ell^\mu}{\partial p^\mu} - \frac{\partial e_\ell^\nu}{\partial p^\nu},
\]
(10)
where $e_\ell^\mu$ is the inverse of $e^\mu_\ell$.

In the case with nonzero torsion, the Euler-Lagrangian equation for $r$ and $\dot{r}$ is modified as:55
\[
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}^\mu}\right) - \frac{\partial \mathcal{L}}{\partial r^\mu} = T^\ell_{\mu\lambda} \dot{r}^\lambda \frac{\partial \mathcal{L}}{\partial \dot{r}^\nu},
\]
(11)
with $T^\ell_{\mu\lambda} = e_\ell^\nu T^\nu_{\mu\lambda}$, while that for $p$ and $\dot{p}$ is not changed.

Then, we obtain the equation of motion for the Weyl-Bogoliubov quasiparticles:
\[
\frac{dp}{dt} = -\frac{\partial e_s}{\partial p} + \dot{\Omega}^+_{pr}\cdot \frac{dr}{dt} \times \Omega^+_{pp} + \Omega^+_{ps},
\]
(12)
\[
dpdt = -\frac{\partial e_s}{\partial p} + \dot{\Omega}^+_{pr}\cdot \frac{dr}{dt} \times \Omega^+_{pp} + \Omega^+_{ps} + \frac{dr}{dt} \times \Omega^+_{pp},
\]
\[
\frac{dp}{dt} + \frac{dr}{dt} \times \Omega^+_{pp} = \Omega^+_{ps} + \dot{\Omega}^+_{pr}\cdot \frac{dr}{dt} \times \Omega^+_{pp},
\]
where the Berry curvatures are,
\[
\Omega^+_{X_+} = i\langle \nabla_X u_+ | \nabla_X u_+ \rangle,
\]
(14)
\[
\Omega^+_{r_+} = i\langle \partial_\mu u_+ | \partial_\mu u_+ \rangle,
\]
(15)
\[
(\dot{\Omega}^+_{pr})_{\alpha\beta} = i\langle \partial_\mu u_+ | \partial_\nu u_+ \rangle - \langle \partial_\mu u_+ | \partial_\nu u_+ \rangle,
\]
(16)
\[
\dot{\Omega}^+_{pr} = - (\dot{\Omega}^+_{pr}),
\]
(17)
with $X = r, p$, and $(T^{\mu})^\nu = \frac{1}{2} \epsilon^{\nu\lambda\rho} T^\mu_{\lambda\rho}$. In Eqs. (12) and (13), all components of vectors are expressed in the laboratory frame. We can obtain similar equations also for the lower band. The Berry curvature is $\Omega^+_{XX} = -\Omega^+_{XX}$. It is noted that, as seen from (13), the torsion generates an effective magnetic field acting on quasiparticles. We denote it as
\[
\mathcal{B} = T^{\mu}(p_\mu + \Omega^+_{pr}),
\]
(18)
In the following, we consider static inhomogeneity of the order parameter, and neglect $\Omega^+_{pr}$ and $\Omega^+_{rr}$. Then, it follows from Eqs. (12) and (13),
\[
\frac{dp}{dt} = -\frac{\partial e_s}{\partial p} + \frac{dr}{dt} \times \mathcal{B} + \left(\frac{\partial e_s}{\partial p} \cdot \mathcal{B}\right) \Omega^+_{pp},
\]
(19)
\[
\frac{dp}{dt} = -\frac{\partial e_s}{\partial p} + \frac{dr}{dt} \times \mathcal{B} + \left(\frac{\partial e_s}{\partial p} \cdot \mathcal{B}\right) \Omega^+_{pp} + \dot{\Omega}^+_{pr}\cdot \frac{dp}{dt} + (\Omega^+_{pp} \cdot \mathcal{B}) \frac{dp}{dt},
\]
(20)
where $v_{ps} = \partial e_s/\partial p$, and $\mathcal{B} = \Omega^+_{rr} + \mathcal{B}$. In Eq. (19), the third term of the right-hand side represents the chiral magnetic effect, and the contribution proportional to $\mathcal{B}$ corresponds to the torsional chiral magnetic effect found in ref. 34. The third term of the right-hand side of Eq. (20) is associated with chiral anomaly; i.e., the momentum is generated or annihilated at the Weyl points when both an effective magnetic field and a bias potential are applied in the same direction. It has recently been found that from Eqs. (19) and (20) the torsional chiral magnetic effect arising from a U(1) phase vortex or lattice strain leads to the negative thermal magnetoresistivity, that is, the anomalous enhancement of longitudinal thermal conductivity along the vortex line.35

B. Torsional chiral magnetic effect due to $\ell$-textures

The current density induced by the torsional field is written as
\[
J^\mu_{\text{TCME}} = \sum_{s} \int \frac{d^3k}{(2\pi)^3} \left( v^+_{ks} \cdot \Omega^+_{ks} \right) f(\epsilon_s) \mathcal{B}^\mu,
\]
(21)
where \( f(\epsilon) = 1/(\epsilon^{\epsilon/T} + 1) \) is the Fermi distribution function. Equation (21) reproduces the current expression based on the linear response of the effective action for Weyl-Bogoliubov quasiparticles with respect to the torsional magnetic field.34 The effective Hamiltonian in Eq. (6) possesses anisotropic dispersion of Weyl-Bogoliubov quasiparticles and the Berry curvature is

\[
\Omega_{ks}^z = \pm s \left( \frac{1}{P^v} \right)^2 \frac{k - s k_0}{2|k - s k_0|^3},
\]

where we have introduced \( k_0 = \frac{v^g k_0}{v_F} \) and \( s = v_F/\Lambda \). We also introduce a momentum cutoff \( |\hat{k} - s\hat{k}_0| < b\Delta_s/v_F \equiv \Lambda \) for Weyl-Bogoliubov quasiparticles with the chirality \( s \), where \( b \sim 1 \). A nonzero torsional magnetic field leads to the equilibrium current

\[
j_{\text{TCME}} = \frac{v_F P F \Lambda}{2\pi^2} T^3,
\]

where we utilize the particle-hole symmetry in Eq. (8).

As seen from Eqs. (10) and (18), in the A-phase of \(^3\)He, a torsional magnetic field is generated by the rotation of the Weyl points, \( p_0 = P F \ell \), as

\[
T^3 = \text{curl} \hat{\ell}.
\]

The real space texture of the \( \ell \) vector field is thermodynamically stable in superfluid \(^3\)He-A under rotation. This is a consequence of the broken gauge-orbit symmetry in the superfluid vacuum; the U(1) phase rotation in Eq. (3) is equivalent to the rotation of the orbital part \( \hat{m} + i\hat{n} \) about \( \hat{\ell} \). Therefore, the supercurrent can be generated by a variation of \( \langle \hat{m}, \hat{n}, \hat{\ell} \rangle \) without a U(1) phase singularity. The \( \ell \)-texture spontaneously emerges in \(^3\)He-A under rotation, and continuous skyrmion-like textures provide an elementary building-block for a variety of coreless vortices with the spatially uniform superfluid density.38,39 Here we consider skyrmion-vortices with the radial and circular \( \hat{\ell} \) textures as in Fig. 1.

Let us now clarify the torsional field induced by skyrmion vortices. It is convenient to express the orbital part of the order parameter tensor in Eq. (3) in terms of Euler angles \( \alpha, \beta, \gamma \) as

\[
\hat{m}' = \cos \beta \cos \alpha \hat{x} + \cos \beta \sin \alpha \hat{y} - \sin \beta \hat{z},
\]

\[
\hat{n}' = -\sin \alpha \hat{x} + \cos \alpha \hat{y},
\]

and \( \hat{\ell} = \cos \alpha \sin \beta \hat{x} + \sin \alpha \sin \beta \hat{y} + \cos \beta \hat{z} \). The texture is assumed to be translationally invariant along the \( z \) axis and be axially symmetric. The axially symmetric skyrmion texture requires the Euler angles to obey \( \alpha \equiv \theta + \alpha_0(r) \), \( \gamma = -n\theta \) \((n \in \mathbb{Z})\), and \( \beta \equiv \beta(r) \), where \( r \) is the distance from the center of the vortex and \( \theta \) is the azimuthal angle. For the real-space \( \ell \)-vector field with axial symmetry, therefore, the parametrization reduces to

\[
\hat{\ell}(r) = \sin \beta(r) \cos \alpha_0(r) \hat{r} + \sin \beta(r) \sin \alpha_0(r) \hat{\theta} + \cos \beta(r) \hat{z}.
\]

The bending angle is a monotonic function on \( r \) which obeys \( \beta(r) = 0 \) at \( r = 0 \) and \( \beta(r) = \pi/2 \) at \( r = R \), where \( R \) determines the size of the skyrmion texture. In Fig. 1, we present the texture of \( \langle \hat{m}, \hat{n}, \hat{\ell} \rangle \) in the skyrmion-vortex with \( n = 1 \): (a) the radial skyrmion with \( \alpha = 0 \) and (b) the circular skyrmion with \( \alpha = \pi/2 \). From Eqs. (27) and (7), the nonzero torsion field in Eq. (10) can be generated by a skyrmion \( \ell \) texture of continuous vortices as

\[
T^3 = b' \sin \beta \hat{\theta}
\]

\[
+ \left[ \left( \beta' \cos \beta + \frac{1}{r} \sin \beta \right) \sin \alpha + \frac{\alpha'}{r} \sin \beta \cos \alpha \right] \hat{z},
\]

where \( \alpha' = \partial \alpha/\partial r \) and \( \beta' = \partial \beta/\partial r \). As shown in Fig. 1, the radial skyrmion texture for \( \alpha = 0 \) generates the toroidal torsional magnetic field in the \( xy \) plane, while the circular skyrmion for \( \alpha \neq 0 \) is accompanied by the nonzero torsion field along the \( z \) axis.

Using the particle density, \( \rho = p_F^3/3\pi^2 \), the TCME is recast into

\[
j_{\text{TCME}} = \left( b' \left( \frac{v_F}{P F} \right) \right) \frac{\hbar}{4 M} \rho \text{curl} \hat{\ell},
\]

where \( b' \sim 1 \) is the dimensionless quantity associated with the cutoff of the Weyl cone. The skyrmion texture with Eq. (27) therefore gives rise to the in-plane circulating current and the out-of-plane current, depending on \( \alpha \). We note that \( j_{\text{TCME}} \) in Eq. (29) corresponds to the Weyl-Bogoliubov quasiparticle contributions to the second term of the right-hand side of the equilibrium current density in Eq. (4). This is distinct from the chiral anomaly effect due to the emergent electromagnetic fields considered in previous studies.19,30 In Sec. IV, we will discuss the temperature dependence of the equilibrium current density based on the full quantum mechanical calculation.

### III. CHIRAL FERMIONS IN SKYRMION-VOXET

In this section, we show the emergence of chiral fermions in superfluid \(^3\)He-A with skyrmion-\( \ell \)-textures. Beyond the semiclassical effective theory, we here utilize the BdG equation which is the fully quantum-mechanical equation for Bogoliubov quasiparticles in superfluid \(^3\)He-A.

We start with the Hamiltonian for the equal spin pairing state, \( A_{\mu}(r) = \Delta(r) d^\dagger_\mu \)

\[
\mathcal{H} = \int dr \psi^\dagger(r) \epsilon(-i\nabla) \psi(r)
\]

\[
+ \frac{1}{2} \int dr \left[ \psi^\dagger(r) \left\{ \Delta(r), \frac{\Lambda}{P_F} \partial_r \right\} \psi(r) + \text{h.c.} \right].
\]

where \( \psi_\nu \) and \( \psi^\dagger_\nu \) denote the fermionic field operators. As \(^3\)He-A is the equal spin pairing state, we omit the spin degrees of freedom. The single-particle Hamiltonian density represents fermions with mass \( M, \epsilon(-i\nabla) = -\nabla^2/2M - \mu \). The repeated Roman and Greek indices imply the sum over the spin degrees of freedom and \( x, y, z \), respectively. Now, let us introduce the Bogoliubov transformation of the fermion operator \( \Psi \equiv [\psi, \psi^\dagger]^T \),

\[
\Psi(r) = \sum_{E > 0} \left[ \varphi_E(r) \eta_E + \varphi_E^\dagger(r) \eta_E^\dagger \right],
\]
where \( \eta_E \) and \( \eta_E^\dagger \) stand for Bogoliubov quasiparticle operators with the energy \( E \) that satisfy fermionic anticommutation relations. We notice that Eq. (31) obeys the particle-hole symmetry, \( \mathcal{C} \Psi = \Psi^\dagger \), with \( \mathcal{C} = e^{i\tau_1 K} \) where \( K \) is the complex conjugation operator. Substituting Eq. (31) into the Hamiltonian (30), one obtains the Bogoliubov-de Gennes (BdG) equations

\[
\mathcal{H}_{\text{BdG}}(r) \varphi_E(r) = E \varphi_E(r),
\]
with

\[
\mathcal{H}_{\text{BdG}}(r) = \left( \begin{array}{cc} \varepsilon(-i \nabla) & \frac{1}{2} \sum_j \left\{ \Delta_j, \frac{n}{p} \partial_j \right\} \\ \frac{1}{2} \sum_j \left\{ \Delta_j^\dagger, \frac{n}{p} \partial_j \right\} & -\varepsilon(-i \nabla) \end{array} \right).
\]

To make the Bogoliubov transformation canonical, the quasiparticle wavefunction must satisfy the orthonormal conditions \( \int dr \varphi_E^*(r) \varphi_{E'}(r) = \delta_{EE'} \).

### A. Symmetry of Weyl-Bogoliubov quasiparticles in skyrmion-vortices

To clarify the symmetry of Bogoliubov quasiparticles in the presence of a skyrmion \( \ell \)-texture, we start with the symmetry classification of axisymmetric vortices in superfluid \( ^3 \text{He} \),

\[
G_v = D_{\omega h} \times T \times T \times U(1)_\phi,
\]
where \( D_{\omega h} \) contains the group of rotations about the vortex line (\( \hat{z} \)), rotations about an axis perpendicular to \( \hat{z} \), and space inversion, and \( T \) represents the translational symmetry of the skyrmion-vortex along \( \hat{z} \). The time reversal symmetry, \( T \), transforms the order parameter tensor as \( A_{\mu i} \rightarrow A_{\mu i}^\dagger \).

The generator of the continuous rotation symmetry about \( \hat{z} \) is expressed as \( e^{i\theta \hat{\alpha}} \), where \( \hat{\alpha} \equiv \hat{L}_z - n \hat{l} \) is the combination of the orbital angular momentum operator \( \hat{L}_z \) and the \( U(1) \) phase rotation operator. The order parameter for axisymmetric vortices satisfies \( \Delta_{\alpha i}(r) = 0 \). For the skyrmion-vortex parameter given by

\[
\Delta(r) = \Delta_{\ell}(r) e^{i \theta} [\hat{r}^\dagger (r) + i \hat{r}(r)],
\]
with Eqs. (25) and (26). In this paper, we focus on the \( n = 1 \) case.

It is convenient to transform Eq. (35) into the eigenstates of the orbital angular momentum \( \ell = +1, 0, -1 \) as \( (\Delta_{+1}, \Delta_0, \Delta_{-1}) \). This transforms the off-diagonal component of Eq. (33) as \( \sum_j \{ \Delta_j, \frac{p}{n} \partial_j \} \rightarrow \sum_j \{ \Delta_j(r), \mathcal{H}_{\ell j}(r) \} \), where \( \mathcal{H}_{\ell j}(r) \) is the spherical harmonic function of degree \( \ell \) obtained by replacing \( \hat{p} \rightarrow -i \partial / \partial \phi \). The phase factor \( e^{i\theta} \), which appears in \( \mathcal{H}_{\ell j} \), is compensated by the winding of \( (\hat{r}^\dagger, \hat{r}) \) in \( \Delta_{\ell}(r) \) and the \( U(1) \) phase factor \( e^{i\theta} \) is factorized from the off-diagonal component in Eq. (33) at all. The quasiparticle wavefunctions are then factorized in terms of the azimuthal quantum number \( m \in \mathbb{Z} \) and axial quantum number \( k \) as

\[
\varphi_E(r) = e^{ikz} \left( \begin{array}{c} u_{nmk}(r) e^{im\theta} \\ v_{nmk}(r) e^{(m-1)\theta} \end{array} \right).
\]

Here we impose the periodic boundary condition along the axial direction, \( \varphi_E(x, y, z) = \varphi_E(x, y, z + \ell) \), which implies \( k = 2\pi n_z / \ell \) with \( n_z \in \mathbb{Z} \). By using the factorization in Eq. (36), Eq. (33) is reduced to the one-dimensional differential equation for \( [u_{nmk}(r), v_{nmk}(r)] \), where the set of the quantum numbers is given as \((n, m, k)\). The differential equation is solved by expanding the wavefunctions with the orthonormal basis. The set of the basis functions is constructed with the Bessel function, \( J_{\ell}(x) \). The Bessel function expansion reduces Eq. (33) to the \( 2N \times 2N \) eigenvalue equation,\(^{56-58}\) where \( N \) is the number of the orthonormal functions.

Apart from the continuous symmetry, there are discrete symmetries, which leave axisymmetric vortex order parameter invariant. First we note that the BdG Hamiltonian in Eq. (33) always satisfies the particle-hole symmetry

\[
\mathcal{C} \mathcal{H}_{\text{BdG}}(r) \mathcal{C}^{-1} = -\mathcal{H}_{\text{BdG}}(r).
\]

This results in the relation (8). The symmetry guarantees that the eigenstate of the BdG equation must appear as a pair of the positive and negative energy states. The positive energy state with \( E_n(m, k) \) and \( \varphi_{n, m, k}(r) \equiv [u_{nmk}(r), v_{nmk}(r)] \)\(^T\) has the particle-hole symmetric partner with \(-E_n(-m + 1, -k)\) and \( \mathcal{C} \varphi_{n, m, k}(r) \).

The other discrete symmetries relevant to the vortex classification are given by three operators, \( \{P_v, P_w, P_{uvw}\}\),\(^{22,38,59,60}\) \( P_v \) is the space inversion operator. The order-two antunitary operator, \( P_v \), is the combination of the time reversal symmetry and \( \pi \)-rotation about any axis perpendicular to the vortex line. The \( P_w \) symmetry is defined as \( P_w = P_v P_3 \), which is the combination of the time-reversal symmetry and mirror reflection symmetry in a plane that contains the vortex line. Axisymmetric continuous vortices with skyrmion-like \( \ell \)-texture are classified in terms of these discrete symmetries into three categories: \( v \)-, \( w \)-, and \( uvw \)-vortices.\(^{38}\) The order parameters of the \( v \)- and \( w \)-vortices are invariant under the \( P_v \) and \( P_w \) symmetry, respectively, while the \( uvw \)-vortex class spontaneously breaks all the discrete symmetries.

Let us define the operator \( M = i \sigma_z \) that denotes the mirror reflection in the \( xy \) plane. The operator flips the spin, momentum, and spatial coordinate as \( \sigma \rightarrow (-\sigma_x, \sigma_y, -\sigma_z) \), \( k \rightarrow (k_x, -k_y, k_z) \), and \( r \rightarrow (x, -y, z) \). The \( P_v \) operator is defined as the combination of \( M \) and the time-reversal operator \( \mathcal{F} = -i \sigma_z K \). For \( d \parallel \hat{z} \), the \( P_v \) operator flips the triad as \( \hat{n}(r, \theta, z) \rightarrow [\hat{m}_x(r, -\theta, z), -\hat{m}_y(r, -\theta, z), \hat{m}_z(r, -\theta, z)] \), \( \hat{n}(r, \theta, z) \rightarrow [-\hat{n}_x(r, -\theta, z), \hat{n}_y(r, -\theta, z), -\hat{n}_z(r, -\theta, z)] \), and \( \hat{E}(r, \theta, z) \rightarrow [\hat{E}_x(r, -\theta, z), -\hat{E}_y(r, -\theta, z), \hat{E}_z(r, -\theta, z)] \). Similarly,

---

**TABLE I. Classification of skyrmion-vortex textures in terms of the discrete symmetries.** The “radial”, “circular”, and “twist” textures correspond to \( \alpha = 0 \), \( \alpha = \pi / 2 \), and \( \alpha(r) \), respectively. \( \mathcal{B} \) is the torsional magnetic field due to the \( \ell \)-texture, and “ZES” denotes the distribution of the zero energy states.

| Class | \( \ell \)-texture | \( P_v \) | \( P_w \) | \( P_{uvw} \) | \( \mathcal{B} \parallel \hat{z} \) | ZES |
|-------|------------------|---------|---------|-----------|-----------------|-----|
| v     | radial           | \( - \) | \( \sqrt{\ } \) | \( \sqrt{\ } \) | flat             |     |
| w     | circular         | \( - \) | \( \sqrt{\ } \) | \( \sqrt{\ } \) | point            |     |
| uvw   | twist            | \( - \) | \( - \) | \( \sqrt{\ } \) | point            |     |
The $P_1$ symmetry relates the triad at $(r, \theta, z)$ to $(\hat{m}_x, -\hat{m}_y, \hat{m}_z)$, $(-\hat{m}_x, \hat{m}_y, -\hat{m}_z)$, and $(\hat{\ell}_x, -\hat{\ell}_y, -\hat{\ell}_z)$ at $(r, \pi - \theta, -z)$. The radial skyrmion-vortex-($v$-)vortex with $\alpha = 0$ in Fig. 1(a) spontaneously breaks the $P_1$ symmetry but maintains the $P_2$ symmetry. The vortex state with the circular skyrmion texture ($\alpha = \pi/2$) in Fig. 1(b) belongs to the $u$-vortex class with the $P_3$ symmetry. The $uvw$-vortex class can be realized by twisting the skyrmion $\ell$-texture so as to satisfy the conditions $\alpha(0) = \pi/2$ and $\alpha(R) = 0$. In Table I, we summarize the possible classification of skyrmion-vortices in terms of the discrete symmetries.

The $P_2$ symmetry imposes an important constraint on the energy spectrum of the Bogoliubov quasiparticles so as to prohibit the equilibrium current along the axial direction. Axisymmetric $v$-vortices must satisfy the relation

$$E_n(m, k) = E_n(m, -k) = -E_n(-m + 1, -k).$$

The first equality results from the $P_2$ symmetry, while the second equality reflects the particle hole symmetry. Hence, the Bogoliubov quasiparticle spectrum in the radial skyrmion-vortex with $\alpha = 0$ is an even function on $k$ and the current flow along $\hat{z}$ is prohibited. In contrast, as the $P_3$ symmetry does not impose any constraints on the eigenvalues, the circular and twisted skyrmion $\ell$-textures may generate the equilibrium current along the axial direction.

B. Chiral fermions and real space texture of Weyl points

Let us consider the superfluid $^3$He-A confined in a cylinder with radius $R$. The triad $(\hat{m}, \hat{n}, \hat{\ell})$ parameterized as Eqs. (25), (26), and (27) slowly varies from $\ell = \hat{z}$ at $r = 0$ to $\ell = \hat{r}$ at $r = R$. Hence, the bending angle is given by $\beta(r) = \frac{\pi r}{R}$. We here set the angle as $\alpha_0 = 0$ for the radial skyrmion ($v$-vortex) and $\alpha_0 = \frac{\pi}{2}(1 - r/R)$ for the twisted skyrmion ($uvw$-vortex). The size of the half-skyrmion is set to be larger than the superfluid coherence length $\xi_s R \gtrsim 10 \xi_s$ where $\xi_s = \sqrt{\frac{\hbar}{2m^*}}$.

In Fig. 2, we show the Bogoliubov quasiparticle spectra obtained by diagonalizing Eq. (33) with the radial skyrmion textures ($v$-vortex). The Bogoliubov spectrum is asymmetric with respect to the azimuthal quantum number $m$ and lowest branch ($n = 0$) crosses the zero energy. As mentioned in Sec. II, the Weyl-Bogoliubov quasiparticles around point nodes experience the torsional magnetic field $T^3 = \nabla \times \ell$. For the radial $\ell$ skyrmion with $\alpha_0 = 0$, the toroidal torsional magnetic field, $T^3 \propto \hat{\theta}$, leads to the emergence of the Landau levels linearly dispersing from $m = 0$. The lowest energy branch crossing the Fermi level in Fig. 2(a) is asymmetric with respect to $m$

$$E_0(m, k) = -v \left( m - \frac{1}{2} \right),$$

which is identified as the chiral fermion due to the emergent toroidal field. The group velocity, $-v < 0$, is an order of $\Delta_L/k_F$. Figure 2(b) shows the dispersion of the Bogoliubov spectrum with respect to the axial momentum $k = 0$, which satisfies the $P_2$ symmetry constraint in Eq. (38). The almost flat dispersion of the lowest eigenstates indicates that the chiral branch in Eq. (39) exists within $|k| < k_F$. Hence, the spectral asymmetry in the radial skyrmion-vortex leads to the equilibrium current along the azimuthal direction and the $P_2$ symmetry prohibits the flow along the axial direction.

To capture the spatial distribution of the chiral fermions, we show in Fig. 2(c) the $k_z$-resolved zero-energy local density of states, $N(k_z, r, E)$. \(61\)

$$N(k, r, E) = \sum_{E > 0} \left[ |u_{nmk}(r)|^2 \delta (E - E_n(m, k)) + |v_{nmk}(r)|^2 \delta (E + E_n(m, k)) \right],$$

where $\sum_{E > 0}$ stands for the sum over $(n, m)$ that satisfies $E_n(m, k) > 0$. The peak amplitude in the plane $(k, r)$ shifts from $r = R$ at $k = 0$ to $r = 0$ at $k = \pm k_F$. The spectral evolution reflects the spatial profiles of the radial skyrmion $\ell$-texture that smoothly tilts from the axial direction to the radial direction. Therefore, the asymmetric branch crossing $E = 0$ in Fig. 2 is attributed to the Weyl-Bogoliubov quasiparticles bound to the Weyl points and the $\ell$-vector texture leads to spatially inhomogeneous structures of Weyl bands in the real coordinate space.

Figure 3 shows the Bogoliubov quasiparticle spectra for the twisted skyrmion $\ell$-texture. It is seen that the chiral fermion branch with the spectral asymmetry exists in the $k$ direction as well as the azimuthal momentum $m$. In order to satisfy the
rigid wall boundary condition, \( \ell = \hat{r} \), at \( r = R \), we set the azimuthal angle as \( \alpha(r) = (1 - r/R) \pi/2 \). The resulting \( \ell \) texture generates the torsional magnetic field long the axial direction in addition to the azimuthal direction. This is categorized to the \( \nu \omega \)-vortex class which holds neither the \( P_3 \) symmetry nor \( P_3 \) symmetry. The symmetry relation in Eq. (38) can be violated and the spectral asymmetry along \( k \) is responsible for the equilibrium current along the axial direction.

IV. TORSIONAL CHIRAL MAGNETIC EFFECT AND MASS CURRENT IN SKYRMION-VORTEX

In the previous section, we have demonstrated that the chiral fermion branches emerge in the Bogoliubov quasiparticle spectrum under skyrmion-like \( \ell \)-textures. Using the full quantum mechanical BdG equation, in this section, we show that low-lying Weyl-Bogoliubov quasiparticles dominantly contribute to the current density in the weak coupling regime, while the contributions from continuum states become significant as the quantum regime is approached. Here we introduce the dimensionless parameter \( p_F \xi = 2E_F/\Delta_A \) so as to quantify the quantum corrections to the quasiclassical limit (\( p_F \xi \gg 1 \)).

We define the mass current density \( j(r) \) as the linear response of the thermodynamic potential with respect to an infinitesimal flow \( v \). \( j_\mu = (\delta \mathcal{H}/\delta \nu_\mu)_{\nu=0} \), where the Hamiltonian under a homogeneous velocity field is given by a Galilean transformation \( -i \nabla \rightarrow -i \nabla - Mv \). The current density is then given by \( j_\mu(r) = -i(\psi^\dagger(r)\partial_\mu \psi(r) - \psi(r) \partial_\mu \psi^\dagger(r)) \). In terms of the Bogoliubov quasiparticle wavefunctions \( \varphi_E = [u_E, v_E]^T \), this is rewritten to

\[
 j_\mu(r) = 2 \sum_{E > 0} \left[ \text{Im} \left\{ u_E(r) \partial_\mu u_E(\bar{r}) \right\} f(E) + \text{Im} \left\{ v_E(r) \partial_\mu v_E(\bar{r}) \right\} f(-E) \right],
\]

where the factor “2” arises from the spin degeneracy in the equal spin pairing state and \( f(E) = 1/\left(e^{E/T} + 1\right) \) is the Fermi distribution function at temperature \( T \). Owing to the particle-hole symmetry, the azimuthal current density in Eq. (41), or the angular momentum density \( \langle r \times j \rangle_z = rj_\theta(r) \), is recast into

\[
 rj_\theta(r) = -2 \sum_{E > 0} (m - 1)|v_E(r)|^2 = 2 \sum_{E < 0} m|u_E(r)|^2,
\]

at \( T = 0 \), where \( n(r) = 2(\psi^\dagger \psi) \) is the particle density and \( \sum_{E > 0} \) stands for the sum over \( (n, m, k) \) within \( E_n(m, k) > 0 \). As shown in Fig. 2(a), the chiral fermion states with \( m \geq 1 \) have negative energy and thus are occupied at \( T = 0 \). These chiral fermion states make a positive contribution to the mass current density along the azimuthal direction, \( j_\theta(r) \). In the radial skyrmion-vortex, therefore, they produce an azimuthal mass current in the same sense as the torsional field, \( T^3 = \text{curl} \hat{\ell} \).

In Fig. 4(a), we plot the current density, \( j_\theta(r) \), in the radial skyrmion-vortex (\( \nu \)-vortex) at \( T = 0 \) for \( p_F \xi = 20 \), 6, 7, and 4.0. For comparison, we plot the current density obtained from the gradient expansion, \( j^{\text{MM}} \), with \( C_0 = 0 \). As the radial skyrmion-vortex always satisfies \( \hat{\ell} \perp \text{curl} \hat{\ell} \), this configuration is free from the issue on the anomalous term \( j^{\text{MM}} \). It is seen from Fig. 4(a) that the current density obtained from the BdG equation is in good agreement with \( j^{\text{MM}} \) in the weak coupling regime \( p_F \xi = 20 \), while \( j_\theta(r) \) deviates from \( j^{\text{MM}} \) as \( p_F \xi \) decreases. To clarify the Weyl-Bogoliubov quasiparticle
contributions, we introduce the \( E \)-resolved current density as

\[
j_\mu(r, E) = 2 \sum_{E_i > 0} \left[ \text{Im} \left\{ u_{E_i}^\dagger(r) \partial_\mu u_{E_i}(r) \right\} \delta(E - E_i) + \text{Im} \left\{ v_{E_i}(r) \partial_\mu v_{E_i}^\dagger(r) \right\} \delta(E + E_i) \right],
\]

where \( E_i = E_\mu(m,k) \). The current density is obtained by integrating \( j(r, E) \) over \( E \) as

\[
j_\mu = \int dE j_\mu(r, E) f(E).
\]

For numerical calculations, the \( \delta \)-function in Eq. (43) is replaced to the Lorentzian function with the width \( 0.025 \Delta_A \). Figure 4(b) shows the \( E \)-resolved current density for the radial skyrmion-vortex at \( p_F \xi = 20 \). The mass current density may be decomposed into two contributions, \( \tilde{j} = j^{\text{Weyl}} + j^{\text{cont}} \). The contribution arising from Weyl-Bogoliubov quasiparticles, \( j^{\text{Weyl}} \), is defined as

\[
j^{\text{Weyl}}(r) \equiv \int_{-\Delta_A}^{\Delta_A} dE j(r, E) f(E),
\]

and \( j^{\text{cont}} = j - j^{\text{Weyl}} \) is the current carried by continuum states. It is seen from Fig. 4(b) that Weyl-Bogoliubov quasiparticle states, including the chiral branch, dominantly contribute to the mass current. However, the continuum states within \( |E| > \Delta_A \) make non-vanishing contributions to the mass current. They satisfy \( j^{\text{cont}}(r) \approx -j^{\text{Weyl}}/2 \) for \( p_F \xi \gg 1 \), and lead to the counter flow to the Weyl-Bogoliubov quasiparticle flow.

The backflow of the continuum states is also pointed out in the edge mass current with spatially polarized \( \hat{\ell} \)-vectors and the surface spin current in the B-phase of the superfluid \( ^3\text{He} \). The main contributions to the mass/spin currents originate in chiral/helicical fermion states that are the topologically protected Andreev bound states at the edge. As pointed out by Stone and Roy, however, the bound states are not the only contribution. Another contribution results from the continuum states affected by the formation of the Andreev bound states. The contribution cancels the bound states, and the edge current arising from the bound states alone differs from the actual edge current by a factor of 2. The \( E \)-resolved current density in Fig. 4(b) resembles to the behavior of the edge mass/spin current, where the continuum states are affected by the existence of the nontrivial \( \hat{\ell} \)-texture and weakens the flow arising from the chiral fermion states.

To capture the contribution of the Weyl-Bogoliubov quasiparticles more systematically, we calculate the total angular momentum per particle at \( T = 0 \). The total angular momentum is defined as

\[
L_z = \int (r \times j)_z d^3r,
\]

and the total particle number is given by \( N = \int n(r) d^3r \). The angular momentum arising from the Weyl-Bogoliubov quasiparticles within \( |E| < \Delta_A \) is given by

\[
L_z^{\text{Weyl}} = \int (r \times j^{\text{Weyl}})_z d^3r
\]

and the contribution of the continuum states is

\[
L_z^{\text{cont}} = L_z - L_z^{\text{Weyl}}.
\]

In Fig. 5, we plot the \( p_F \xi \)-dependence of the total angular momentum at \( T = 0 \).

\[
\begin{align*}
L_z &= \hbar N/2, \\
L_z^{\text{Weyl}} &= \hbar N + L_z^{\text{TCME}}, \\
L_z^{\text{cont}} &= \hbar N - L_z^{\text{Weyl}} - L_z^{\text{TCME}}.
\end{align*}
\]

In Fig. 5, however, the numerical calculation of the BdG equation in the quantum regime shows that \( L_z/N \) gradually increases from \( \hbar/2 \) as \( p_F \xi \) decreases. Although \( L_z/N \) almost stays constant for \( p_F \xi \geq 5 \), we find two characteristic behaviors in \( L_z^{\text{Weyl}} \) and \( L_z^{\text{cont}} \): (i) the Weyl-Bogoliubov quasiparticle contribution, \( L_z^{\text{Weyl}} \), exhibits the nonmonotonic behavior as a function of \( p_F \xi \) and has a maximum around \( p_F \xi = 10 \). (ii) The continuum contribution, \( L_z^{\text{cont}} \), changes its sign around \( p_F \xi = 3 \) and makes a dominant contribution to \( L_z \) in \( p_F \xi \lesssim 1 \). As for (i), the characteristic \( p_F \xi \)-dependence of \( L_z^{\text{Weyl}} \) enables one to discriminate the contribution of TCME from other low-lying quasiparticle contributions. For \( p_F \xi \gtrsim 10 \), the lower energy part of the angular momentum, \( L_z^{\text{Weyl}} \), is approximately decomposed into \( L_z^{\text{Weyl}} \sim \hbar N + L_z^{\text{TCME}} \), where

\[
L_z^{\text{TCME}} = \int \left( r \times j^{\text{TCME}} \right)_z d^3r \propto 1/(p_F \xi)
\]

is the angular momentum arising from the torsion-induced current \( j^{\text{TCME}} \) in Eq. (23). The TCME vanishes at the weak coupling limit \( p_F \xi \gg 1 \), while it makes a significant contribution to \( L_z^{\text{Weyl}} \) as \( p_F \xi \) decreases. The contribution of \( j^{\text{TCME}} \) is consistent with the increase behavior of \( L_z^{\text{Weyl}} \) with decreasing \( p_F \xi \) within \( p_F \xi \gtrsim 10 \). We note that the anomalous enhancement of \( L_z^{\text{Weyl}} \) is compensated by \( L_z^{\text{cont}} \) and the resultant \( L_z/N \) stays constant at \( L_z/N = \hbar/2 \) for \( p_F \xi \gtrsim 10 \).

As Eq. (23) is derived from the semiclassical equations of motion for Weyl-Bogoliubov quasiparticles, however, we must be careful about the applicability of Eq. (23). Equation (23) is applicable only to the large \( p_F \xi \) regime and fails down in the quantum regime \( p_F \xi \sim O(1) \). Figure 5 indeed shows...
is understandable with an extra contribution of the torsional chiral magnetic effect, $L_z^\text{TCME}$, as discussed in Fig. 5. Such extra contribution vanishes as the topological phase transition ($p_F^z = 0$) is approached and the mass current is dominated by the contribution arising from continuum states.

Lastly, we discuss the mass current induced by twisted-skyrmion textures in connection to the paradox of the anomalous current in Eq. (4). Using the particle-hole symmetry, one obtains the current along the axial direction, $j_z(r)$, from Eq. (41) as

$$j_z(r) = 2 \sum_{k<0} |u_{nmk}(r)|^2,$$

at $T = 0$. As shown in Fig. 3(b), the twisted skyrmion $\ell$-texture breaks the $P_z$ symmetry and induces the chiral fermion branches along axial momentum. As the branch has the negative group velocity with respect to $k$ and the $k > 0$ region is occupied at $T = 0$, the asymmetry branch makes a positive contribution to $j_z(r)$. In Fig. 7, we plot $j_\theta(r)$ and $j_z(r)$ in the twisted skyrmion-vortex $\ell$-texture. It is seen that in weak coupling regime, both the current profiles are in good agreement with $j_{\ell}^\text{IMU}$ including the anomalous term, rather than $j_{\ell}^\text{MM}$. For $p_F^z = 20$, we find that the total angular momentum is $L_z^\text{IMU}(R)/N = 0.43\hbar$, which is inconsistent to the McClure-Takagi prediction that $L_z^\text{IMU}(R)/N = \hbar/2$ is independent of the $\ell$-texture. The depletion of $L_z^\text{IMU}(R)$ is understandable with the existence of the extra contribution arising from the anomalous
term, which is \( j_{\ell}^a < 0 \) in the case of the twisted skyrmion \( \ell \)-texture. Hence, our results show that in contrast to the McClure-Takagi prediction, \( L_z(0)/N \) is not fixed to \( \hbar/2 \) but sensitive to the \( \ell \)-texture even in the weak coupling regime.

As shown in Fig. 7, the current density profiles are deviated from \( j_{\ell}^{\text{MM}} \) as \( p_{\ell} \xi \) decreases. To capture the change of the spatial profiles in \( j_{\ell} \) and \( j_{z} \), we plot in Figs. 7(c) and 7(d) the rescaled mass current densities, where \( n^\prime \) is a fitting parameter to rescale \( j_{\ell}^\prime(r) \) to \( j_{\ell}^{\text{MM}}(R) = j_{\ell}^{\text{MU}}(R) \). Figures 7(c) and 7(d) show that the rescaled profiles gradually shift from \( j_{\ell}^{\text{MM}} \) to \( j_{\ell}^{\text{MU}} \) as \( p_{\ell} \xi \) decreases, implying that \( j_{\ell} \) becomes negligible around \( p_{\ell} \xi \sim 0 \). As shown in Fig. 5, the contribution arising from Weyl-Bogoliubov quasiparticles vanishes at \( p_{\ell} \xi = 0 \). Hence, the shift of the current density profiles indicates that Weyl-Bogoliubov quasiparticles make significant contributions in the weak coupling regime. Our results shown above are consistent with the prediction by Balatsky et al.\(^{19,32,33} \) that the anomalous term originates in the chiral anomaly of Weyl-Bogoliubov quasiparticles induced by twisted skyrmion \( \ell \)-textures.

V. SUMMARY

We have investigated chiral anomaly phenomena induced by skyrmion-like \( \ell \)-textures in the superfluid \( ^3\text{He}-\text{A} \) which is a prototype of Weyl superfluids. Using the semiclassical theory, we find the torsional chiral magnetic effect that torsion fields induced by skyrmion-like \( \ell \)-textures act on Weyl-Bogoliubov quasiparticles as emergent magnetic fields, and result in equilibrium mass current flowing along the emergent magnetic fields. Using the full quantum mechanical BdG equation, we have shown that in skyrmion vortices a chiral fermion branch with spectral asymmetry appears in the low-lying quasiparticle spectrum. The chiral fermion states are responsible for the equilibrium mass flow. Our numerical results, however, show that the total mass current or the total angular momentum differs from that arising from the chiral fermions alone by a factor of \( 1/2 \). The discrepancy is compensated by the backflow arising from the continuum states, and for radial skyrmion vortices, our numerical results in the quasiclassical limit coincide with the prediction by McClure and Takagi.\(^ {47} \) \( L_z = \hbar N/2 \). Furthermore, it has been demonstrated that the angular momentum associated with the Weyl-Bogoliubov quasiparticles increase as the quantum regime is approached. This anomalous behavior is understandable with the extra contribution of the current due to the torsional chiral magnetic effect in Eq. (23). We have clarified the chiral anomaly aspect of the mass current density in Eq. (4); the curl\( \ell \) term in Eq. (4) is associated with the TCME of Weyl-Bogoliubov quasiparticles induced by a skyrmion-vortex.

The appearance of a chiral branch in \( ^3\text{He}-\text{A} \) was first pointed out by Combescot and Dombre,\(^ {52,53} \) who demonstrated that in the case of a twisted (non-skyrmionic) \( \ell \)-texture \( (\ell \parallel \text{curl}\ell) \) the BdG equation for low-lying quasiparticles reduces to the Dirac-type equation with a fictitious magnetic field generated by a variation of the \( \ell \)-field. Balatsky et al.\(^ {31} \) clarified that the chiral branch is topologically protected by the Atiyah-Singer index theorem and the chiral fermion carries uncompensated current at \( T = 0 \). Although our result for the mass current qualitatively agrees with that in Ref. 33, it differs from Ref. 33 because they consider only the chiral fermion contributions. As mentioned above, we find that in the quasiclassical limit, the continuum states bring about backflow to the quasiparticle flow, i.e., \( L_z^{\text{Weyl}} \approx -2L_z^{\text{cont}} \approx \hbar n \). As the quantum regime is approached, the mass current carried by the continuum states changes its sign and makes a dominant contribution. In the vicinity of the topological phase transition \( (p_{\ell} \xi = 0) \), indeed, the total angular momentum is governed by the continuum states and the contribution from chiral fermions becomes negligible. We have also shown that the contribution of \( j_{\ell} \) is crucial for the weak coupling regime, while it vanishes as \( p_{\ell} \xi \) decreases. Although \( L_z/N \) is composed of the composite contributions of Weyl quasiparticles and continuum states and it is difficult to extract the TCME contribution solely, our results may put a new aspect on the paradox of the mass current and the intrinsic angular momentum.

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1 H. Nielsen and M. Ninomiya, The Adler-Bell-Jackiw anomaly and Weyl fermions in a crystal, Phys. Lett. B 130, 389 (1983).
2 X. Wan, A. M. Turner, A. Vishwanath, and S. Y. Savrasov, Topological semimetal and Fermi-arc surface states in the electronic structure of pyrochlore iridates, Phys. Rev. B 83, 205101 (2011).
3 A. A. Burkov and L. Balents, Weyl Semimetal in a Topological Insulator Multilayer, Phys. Rev. Lett. 107, 127205 (2011).
4 G. B. Halász and L. Balents, Time-reversal invariant realization of the Weyl semimetal phase, Phys. Rev. B 85, 035103 (2012).
5 S. Murakami, Phase transition between the quantum spin Hall and insulator phases in 3D: emergence of a topological gapless phase, New Journal of Physics 9, 356 (2007).
6 A. A. Zyuzin and A. A. Burkov, Topological response in Weyl semimetals and the chiral anomaly, Phys. Rev. B 86, 115133 (2012).
7 P. Goswami and S. Tewari, Axionic field theory of (3 + 1)-dimensional Weyl semimetals, Phys. Rev. B 88, 245107 (2013).
8 D. T. Son and B. Z. Spivak, Chiral anomaly and classical negative magnetoresistance of Weyl metals, Phys. Rev. B 88, 104412 (2013).
9 C.-X. Liu, P. Ye, and X.-L. Qi, Chiral gauge field and axial anomaly in a Weyl semimetal, Phys. Rev. B 87, 235306 (2013).
10. M. M. Vazifeh and M. Franz, *Electromagnetic Response of Weyl Semimetals*, Phys. Rev. Lett. **111**, 027201 (2013).
11. P. Hosur and X.-L. Qi, *Recent developments in transport phenomena in Weyl semimetals*, C. R. Phys. **14**, 857 (2013), [arXiv:1309.4464].
12. O. Vafek and A. Vishwanath, *Dirac Fermions in Solids: From High-Tc Cuprates and Graphene to Topological Insulators and Weyl Semimetals*, Annu. Rev. Condens. Matter Phys. **5**, 83 (2014).
13. P. Goswami, J. H. Pixley, and S. Das Sarma, *Axial anomaly and longitudinal magnetoresistance of a generic three-dimensional metal*, Phys. Rev. B **92**, 075205 (2015).
14. N. Yamamoto, *Generalized Bloch theorem and chiral transport phenomena*, Phys. Rev. D **92**, 085011 (2015).
15. S. Jia, S.-Y. Xu, and M. Z. Hasan, *Weyl semimetals, Fermi arcs and chiral anomalies*, Nat. Mater. **15**, 1140 (2016).
16. X. Huang, L. Zhao, Y. Long, P. Wang, D. Chen, Z. Yang, H. Liang, M. Xue, H. Weng, Z. Fang, X. Dai, and G. Chen, *Observation of the Chiral-Anomaly-Induced Negative Magnetoresistance in 3D Weyl Semimetal TaAs*, Phys. Rev. X **5**, 031023 (2015).
17. C.-L. Zhang, S.-Y. Xu, I. Belopolski, Z. Yuan, Z. Lin, B. Tong, G. Bian, N. Alidoust, C.-C. Lee, S.-M. Huang, T.-R. Chang, G. Chang, C.-H. Hsu, H.-T. Jeng, M. Neupane, D. S. Sanchez, H. Zheng, J. Wang, H. Lin, C. Zhang, H.-Z. Lu, S.-Q. Shen, T. Neupert, M. Z. Hasan, and S. Jia, *Signatures of the Adler-Bell-Jackiw chiral anomaly in a Weyl fermion semimetal*, Nat. Commun. **7**, 10735 (2016).
18. T. Meng and L. Balents, *Weyl superconductors*, Phys. Rev. B **86**, 054504 (2012).
19. A. V. Balatsky, G. E. Volovik, and V. A. Konysev, *On the chiral anomaly in superfluid 3He-A*, Zh. Eksp. Teor. Fiz. **90**, 2038 (1986), [Sov. Phys. JETP **63**, 1194 (1986)].
20. G. E. Volovik, *The Universe in a Helium Droplet* (Clarendon, Oxford, 2003).
21. G. E. Volovik, *Topology of chiral superfluid: skyrmions, Weyl fermions and chiral anomaly*, JETP Lett. **103**, 140 (2016).
22. T. Mizushima, Y. Tsutsumi, T. Kawakami, M. Sato, M. Ichioda, and K. Machida, *Symmetry-Protected Topological Superfluids and Superconductors—From the Basics to the 3He-A*, J. Phys. Soc. Jpn. **85**, 022001 (2016).
23. P. Goswami and L. Balicas, *Topological properties of possible Weyl superconducting states of URu_{2}Si_{2}*, arXiv:1312.3632 (2013).
24. M. Sato and S. Fujimoto, *Majorana Fermions and Topology in Superconductors*, J. Phys. Soc. Jpn. **85**, 072001 (2016).
25. P. Goswami and A. H. Nevidomskyy, *Topological Weyl superconductor to diffusive thermal Hall metal crossover in the B phase of UPt_{3}*, Phys. Rev. B **92**, 214504 (2015).
26. Y. Shimizu, S. Kittaka, S. Nakamura, T. Sakakibara, D. Aoki, Y. Homma, A. Nakamura, and K. Machida, *Quasiparticle excitations and evidence for superconducting double transitions in monocrystalline U_{0.97}Th_{0.03}Be_{11}*, Phys. Rev. B **96**, 100505 (2017).
27. T. Mizushima and M. Nitta, *Topology and symmetry of surface Majorana arcs in cyclic superconductors*, Phys. Rev. B **97**, 024506 (2018).
28. K. Machida, *Spin Triplet Nematic Pairing Symmetry and Superconducting Double Transition in U_{1-x}Th_{x}Be_{13}*, J. Phys. Soc. Jpn. **87**, 033703 (2018).
29. T. D. C. Bevan, A. J. Manninen, J. B. Cook, J. R. Hook, H. E. Hall, T. Vachaspati, and G. E. Volovik, *Momentum creation by vortices in superfluid 3He as a model of primordial baryogenesis*, Nature **386**, 689 (1997).
30. G. E. Volovik and V. P. Mineev, *Orbital angular momentum and orbital dynamics: 3He-A and the Bose liquid*, Zh. Eksp. Teor. Fiz. **81**, 989 (1981), [Sov. Phys. JETP **56**, 579 (1982)].
31. G. E. Volovik, *Normal Fermi liquid in a superfluid 3He-A at T = 0 and the anomalous current*, Pi’sma Zh. Eksp. Teor. Fiz. **42**, 294 (1985), [JETP Lett. **42**, 363 (1985)].
32. G. E. Volovik, *Chiral anomaly and the law of conservation of momentum in 3He-A*, Pi’sma Zh. Eksp. Teor. Fiz. **43**, 428 (1986), [JETP Lett. **43**, 551 (1986)].
33. A. V. Balatsky and V. A. Konysev, *The anomalous superfluid current in 3He-A and the index theorem*, Zh. Eksp. Teor. Fiz. **92**, 841 (1987), [Sov. Phys. JETP **65**, 474 (1987)].
34. H. Sumiyoshi and S. Fujimoto, *Toroidal Chiral Magnetic Effect in a Weyl Semimetal with a Topological Defect*, Phys. Rev. Lett. **116**, 166601 (2016).
35. T. Kobayashi, T. Matsushita, T. Mizushima, A. Tsuruta, and S. Fujimoto, *Negative Thermal Magnetoresistivity as a Signature of Chiral Anomaly in Weyl Superconductors*, arXiv:1806.00993 (2018).
36. P. W. Anderson and P. Morel, *Generalized Bardeen-Cooper-Schrieffer States and the Proposed Low-Temperature Phase of Liquid He*³, Phys. Rev. **123**, 1911 (1961).
37. P. W. Anderson and W. F. Brinkman, *Anisotropic Superfluidity in 3He: A Possible Interpretation of its Stability as a Spin-Fluctuation Effect*, Phys. Rev. Lett. **30**, 1108 (1973).
38. M. M. Saloma and G. E. Volovik, *Quantized vortices in superfluid 3He*, Rev. Mod. Phys. **59**, 533 (1987).
39. O. V. Lounasmaa and E. Thuneberg, *Vortices in rotating superfluid 3He*, Proc. Natl. Acad. Sci. **96**, 7760 (1999).
40. V. Eltsov, M. Krusius, and G. Volovik, *Vortex formation and dynamics in superfluid 3He and analogies in quantum field theory*, in *Progress in Low Temperature Physics*, Vol. 15, edited by W. Halperin (Elsevier, 2005) pp. 1 – 137.
41. D. Vollhardt and P. Wölfle, *The Superfluid Phases of Helium 3* (Taylor and Francis, London, 1990).
42. N. D. Mermin and T.-L. Ho, *Circulation and Angular Momentum in the A Phase of Superfluid Helium-3*, Phys. Rev. Lett. **36**, 594 (1976).
43. P. W. Anderson and G. Toulouse, *Phase Slippage without Vortex Cores: Vortex Textures in Superfluid 3He*, Phys. Rev. Lett. **38**, 508 (1977).
44. J. J. Wiman and J. A. Sauls, *Superfluid phases of 3He in nanoscale channels*, Phys. Rev. B **92**, 144515 (2015).
45. J. J. Wiman and J. A. Sauls, *Spontaneous helical order of superfluid 3He confined in nano-scale channels*, arXiv:1802.08719 (2018).
46. R. Takashima and S. Fujimoto, *Supercurrent-induced skyrmion dynamics and tunable Weyl points in chiral magnet with superconductivity*, Phys. Rev. B **94**, 235117 (2016).
47. M. G. McClure and S. Takagi, *Angular Momentum of Anisotropic Superfluids*, Phys. Rev. Lett. **43**, 596 (1979).
48. N. D. Mermin and P. Muzikar, *Cooper pairs versus Bose condensed molecules: The ground-state current in superfluid 3He-A*, Phys. Rev. B **21**, 980 (1980).
49. M. Ishikawa, K. Miyake, and T. Usui, *Intrinsic Angular Momentum and Mass Current in Superfluid 3He-A*, Prog. Theor. Phys. **63**, 1083 (1980).
50. T. Kita, *The Angular Momentum Paradox of 3He-A*, J. Phys. Soc. Jpn. **65**, 664 (1996).
51. A. Tsuruta, S. Yukawa, and K. Miyake, *Intrinsic Angular Momentum and Intrinsic Magnetic Moment of Chiral Superconductor on Two-Dimensional Square Lattice*, J. Phys. Soc. Jpn. **84**, 094712 (2015).
52. R. Combescot and T. Dombre, *Superfluid current in 3He-A at T =
0. Phys. Rev. B 28, 5140 (1983).
53. R. Combescot and T. Dombre, Twisting in superfluid $^3$A and consequences for hydrodynamics at T=0, Phys. Rev. B 33, 79 (1986).
54. M. A. Stephanov and Y. Yin, Chiral Kinetic Theory, Phys. Rev. Lett. 109, 162001 (2012).
55. H. Kleinert, Classical and Fluctuating Paths in Spaces with Curvature and Torsion, arXiv:quant-ph/9606001 (1996).
56. T. Mizushima and K. Machida, Vortex structures and zero-energy states in the BCS-to-BEC evolution of p-wave resonant Fermi gases, Phys. Rev. A 81, 053605 (2010).
57. M. Matsumoto and R. Heeb, Vortex charging effect in a chiral $p_x \pm ip_y$-wave superconductor, Phys. Rev. B 65, 014504 (2001).
58. F. Gygi and M. Schlüter, Self-consistent electronic structure of a vortex line in a type-II superconductor, Phys. Rev. B 43, 7609 (1991).
59. M. M. Salomaa and G. E. Volovik, Symmetry and structure of quantized vortices in superfluid $^3$B, Phys. Rev. B 31, 203 (1985).
60. M. M. Salomaa and G. E. Volovik, Vortices with Ferromagnetic Superfluid Core in $^3$He-B, Phys. Rev. Lett. 51, 2040 (1983).
61. M. Ichioka, T. Mizushima, and K. Machida, Skyrmion lattice and intrinsic angular momentum effect in the A phase of superfluid $^3$He under rotation, Phys. Rev. B 82, 094516 (2010).
62. J. A. Sauls, Surface states, edge currents, and the angular momentum of chiral p-wave superfluids, Phys. Rev. B 84, 214509 (2011).
63. Y. Tsutsumi and K. Machida, Edge mass current and the role of Majorana fermions in A-phase superfluid $^3$He, Phys. Rev. B 85, 100506 (2012).
64. H. Wu and J. A. Sauls, Majorana excitations, spin and mass currents on the surface of topological superfluid $^3$He-B, Phys. Rev. B 88, 184506 (2013).
65. Y. Tsutsumi and K. Machida, Edge Current due to Majorana Fermions in Superfluid $^3$He A- and B-Phases, J. Phys. Soc. Jpn. 81, 074607 (2012).
66. T. Mizushima, Y. Tsutsumi, M. Sato, and K. Machida, Symmetry protected topological superfluid $^3$He-B, J. Phys.: Condens. Matter 27, 113203 (2015).
67. M. Stone and R. Roy, Edge modes, edge currents, and gauge invariance in $p_x+ip_y$ superfluids and superconductors, Phys. Rev. B 69, 184511 (2004).
68. K. Nagai, Y. Nagato, and S. Higashitani, Low Temperature Properties of the Mermin-Ho Texture of Superfluid $^3$He-A in a Cylinder, J. Phys.: Conf. Ser. 400, 012054 (2012).
69. K. Nagai, Quasi-classical Theory of the A-phase of Superfluid $^3$He-A in a Cylinder, J. Low Temp. Phys. 175, 44 (2014).
70. M. Cross, A generalized Ginzburg-Landau approach to the superfluidity of helium 3, J. Low Temp. Phys. 21, 525 (1975).