On the high spin expansion in the $sl(2) \mathcal{N} = 4$ SYM theory

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Abstract

We study the high spin expansion of the anomalous dimension for long operators belonging to the $sl(2)$ sector of $\mathcal{N} = 4$ SYM. Keeping the ratio $j$ between the twist and the logarithm of the spin fixed, the anomalous dimensions expand as $\gamma = f(g, j) \ln s + f^{(0)}(g, j) + O(1/\ln s)$. This particular double scaling limit is efficiently described, up to the desired accuracy $O(\ln s^0)$, in terms of linear integral equations. By using them, we are able to evaluate both at weak and strong coupling the sub-leading scaling function $f^{(0)}(g, j)$ as series in $j$, up to the order $j^5$. Thanks to these results, the possible extension of the liaison with the $O(6)$ non-linear sigma model may be tackled on a solid ground.
1 Introduction

The set of composite operators

\[ \text{Tr}(D^s Z^L) + \ldots, \] (1.1)

where \( D \) is a covariant derivative acting in all possible ways on the complex scalar field \( Z \), constitutes the so-called \( sl(2) \) sector of \( \mathcal{N} = 4 \) SYM theory. The integer numbers \( s \) and \( L \) are called spin and twist, respectively. In the framework of the AdS-CFT correspondence [1] this set of operators received particular attention, also because of its connection [2, 3] to twist operators in QCD. In the high spin limit anomalous dimensions \( \gamma(g, L, s) \) of (1.1) show a logarithmic divergence,

\[ \gamma(g, L, s) = \ln s \ f(g) + f_c(g, L) + o(s^0). \] (1.2)

The function of the coupling \( f(g) \), which equals twice the cusp anomalous dimension of light-like Wilson loops, is known also as universal scaling function, where the term 'universal' takes into account its lack of dependance on the (fixed) twist \( L \). This property, in particular, must imply that \( f(g) \) is free from wrapping effects, which makes the asymptotic Bethe Ansatz equations [4, 5, 6, 7] predictive for its exact determination. On the other hand, the function \( f_c(g, L) \) is not 'universal', but, since it is related through functional relations [8, 9] to \( f(g) \), it contains a universal part. Finally, one has to mention that both \( f \) and \( f_c \) are related [10] to form factors in scattering amplitudes. The function \( f(g) \) can be found from the solution of a linear integral equation [11, 7] derived from asymptotic Bethe Ansatz. This equation was solved both at the weak [11, 7] and, more importantly, at the strong coupling [12, 13, 14] limit. Moreover, the function \( f_c(g, L) \) was studied in [15] (weak coupling), [9] (strong coupling from string theory) and in the contemporaneous paper [16, 17] (strong coupling, using linear integral equations derived from gauge theory Bethe Ansatz).

The structure (1.2) of the high spin anomalous dimension is preserved if one introduces the following scaling limit,

\[ s \to +\infty, \quad L \to +\infty, \quad j = L - 2 \ln s \ \text{fixed}, \] (1.3)

proposed\(^1\) in [18] for the strong coupling (string theory) case and then in [14] for the weak coupling \( \mathcal{N} = 4 \) SYM theory. Similar limits were also studied in one loop \( \mathcal{N} = 4 \) SYM [6, 20, 21, 22] and string [19, 23, 24] cases, when \( j = L \ln(s/L) \) is fixed - and in the strong coupling \( \mathcal{N} = 4 \) SYM [6, 20, 21, 22] and string [19, 23, 24] cases, when \( j = L \ln(s/L) \) is fixed.

As we will show, in the case of limit (1.3), relation (1.2) is replaced by

\[ \gamma(g, j, s) = f(g, j) \ln s + f^{(0)}(g, j) + \sum_{k=1}^{\infty} f^{(k)}(g, j)(\ln s)^{-k} + O((\ln s)^{-\infty}), \] (1.4)

\(^1\)To be precise the limit considered in [18, 15] is

\[ s \to +\infty, \quad L \to +\infty, \quad j = L \ln s \ \text{fixed}. \]

Referring to expansion (1.4), the different limit (1.3) does not affect \( f(g, j) \), but gives easier forms for \( f^{(0)}(g, j) \).
where the notation $O((\ln s)^{-\infty})$ stands for terms going to zero faster than any powers $(\ln s)^{-k}$, $k \in \mathbb{N}$. Importantly, we may conjecture that wrapping effects can affect none of the functions appearing in (1.4), which thus are still completely determined by the asymptotic Bethe Ansatz equations or (non-)linear integral equation [17]. One reason for that is the matching comparison with the string expansion of [9].

At small $j$ the structure of the function $f(g,j)$, called often generalised scaling function,

$$f(g,j) = \sum_{n=0}^{\infty} f_n(g)j^n,$$

was investigated at weak coupling [15] and at strong coupling [25 26 27 28]. The generalised scaling function $f(g,j)$ for $j \ll g$ was also shown [26] to coincide with the energy density [29] of the nonlinear $O(6)$ sigma model embedded into $AdS_5 \times S^5$, thus confirming the related previous proposal [18] by Alday and Maldacena. Results for $j \gg g$ are also available [30].

The aim of this paper is to investigate the structure of $f^{(0)}(g,j)$ at very small $j$, both at the weak and the strong coupling limit. We expect $f^{(0)}(g,j)$ to be an analytical function of $j$, so that the expansion

$$f^{(0)}(g,j) = \sum_{n=0}^{\infty} f_{n}^{(0)}(g)j^n$$

will capture all its properties. Even if we report explicit results on functions $f_n^{(0)}(g)$ for $n \leq 5$, we will show that in general the various $f_n^{(0)}(g)$ can be obtained using a linear integral equation - giving the density of Bethe roots and of the so-called holes - which is equivalent to the asymptotic Bethe Ansatz equations in the high spin limit if one neglects terms of order $(\ln s)^{-1}$. This equation follows from previous results, mainly contained in [31 32], and generalises the one we used previously [25 27 28] to determine the components $f_n(g)$ of the scaling function $f(g,j)$ and which is equivalent to the 'FRS' equation, proposed in [15].

The plan of this paper is as follows.

In Section 2 we report on the one loop results. We show that the density of Bethe roots and holes can be found by solving a linear integral equation, which is exact if one neglects terms of order $O(1/\ln s)$ and we give the expressions for $f_n^{(0)}(g=0)$, when $n = 1, \ldots, 5$.

In Section 3 we write the linear integral equation satisfied by the density of roots and holes at arbitrary values of the coupling constant. Using such density, we perform weak coupling computations and give $f_n^{(0)}(g)$ up to three loops and up to $n = 4$.

In Section 4 we re-write the linear integral equation for the density as a set of linear systems: from the solution to the $n$-th system one can get the $n$-th component in the expansion (1.6). The solution to the $n$-th system depends - very similarly to the systems describing the components $f_n(g)$ of the generalised scaling function - on the previous ones. This reformulation of the problem simplifies the analysis of the strong coupling limit, which we do up to $n = 5$. 
2 One loop results

We begin our study by considering the $sl(2)$ sector of $\mathcal{N} = 4$ SYM theory at one loop. In order to distinguish them from the exact (i.e. all loops) correspondents, all the one loop quantities (such as the density $\sigma$ and the roots-holes separator $c$) will be denoted by an index 0. As stated in the Introduction, we are interested in the limit

\[
s \to +\infty, \quad L \to +\infty, \quad j = \frac{L - 2}{\ln s} \text{ fixed.}
\]

In this limit, the density of Bethe roots and holes at one loop in perturbation theory, $\sigma_0(u)$, satisfies the linear equation

\[
\sigma_0(u) = L \left[ \psi \left( \frac{1}{2} - iu \right) + \psi \left( \frac{1}{2} + iu \right) \right] - \psi \left( 1 - iu - i \frac{s}{\sqrt{2}} \right) - \psi \left( 1 - iu + i \frac{s}{\sqrt{2}} \right) - \psi \left( 1 + iu - i \frac{s}{\sqrt{2}} \right) - \psi \left( 1 + iu + i \frac{s}{\sqrt{2}} \right) - 2 \ln 2 + \\
\int_{-c_0}^{c_0} \frac{dv}{2\pi} \left[ \psi(1 - iu + iv) + \psi(1 + iu + iv) \right] \sigma_0(v) + O \left( \frac{1}{\ln s} \right),
\]

which, as we wrote, is exact if we neglect terms of order $O(1/\ln s)$. Further, we have to impose the condition

\[
\int_{-c_0}^{c_0} du \sigma_0(u) = -2\pi j \ln s + O \left( \frac{1}{\ln s} \right),
\]

which comes from the fact that the parameter $c_0$ defines the interval $[-c_0, c_0]$ in which the $L - 2$ internal holes concentrate.

It is convenient to use the Fourier transform, $\hat{\sigma}_0(k)$, which satisfies the linear equation

\[
\hat{\sigma}_0(k) = -2\pi \frac{L}{\pi} \left( 1 - e^{-|k|/2} \right) + e^{-|k|/2} \left( 1 - \cos \frac{k s}{\sqrt{2}} \right) - 4\pi \delta(k) \ln 2 - \\
e^{-\frac{|k|}{2}} \sinh \frac{|k|}{2} \int_{-\infty}^{+\infty} \frac{dh}{2\pi} \hat{\sigma}_0(h) \left[ \frac{\sin(k - h)c_0}{k - h} - \frac{\sin hc_0}{h} \right] + O \left( \frac{1}{\ln s} \right),
\]

and the condition

\[
2 \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \hat{\sigma}_0(k) \frac{\sin kc_0}{k} = -2\pi j \ln s + O \left( \frac{1}{\ln s} \right).
\]

We can now give justifications of these formulæ. We start from (3.52) of [32], where the $L - 2$ in the last term of the first line is replaced by a sum on the internal holes $\sum_{h=1}^{L-2} e^{ikuh}$. This equation, for $\frac{d}{du} F_0(u) = \sigma_0(u) + O(1/s)$, is exact if we neglect terms of order $O(1/s)$. Now, in order to express the sum on internal holes in terms of the density $\sigma_0(u)$, we use results from Appendix A. Referring to formula (A.3), we realize that, if one neglects terms of order $O(1/\ln s)$, such sum is given by the first term in the right hand side of (A.3), which is linear in the density $\sigma_0(u) = \frac{d}{du} Z_0(u)$. After
doing this, we get expression (2.3) for the equation satisfied by the density of Bethe roots and holes in the Fourier space \(^2\). In a completely analogous way, when we express the counting of internal holes in terms of \(\sigma_0(u)\), we get (2.2) (2.4).

As a further verification of the correctness of our starting equations, in the first part of Appendix B we prove that equation (2.1) is compatible with the corresponding equation - valid for very small \(u\) - that can be deduced from the results of [19].

Having established on quite firm ground our starting point, we pass to analyze in detail the behaviour of the various quantities in the limit (1.3). Consistency considerations coming from the analysis of (2.3, 2.4) imply that in such limit the parameter \(c_0\), depending, via (2.3, 2.4), on \(\ln s\) and \(j\), expands, when \(j \ll 1\), as

\[
c_0 = \sum_{n=1}^{\infty} c_0^{(0,n)} j^n + \frac{1}{(\ln s)} \sum_{n=1}^{\infty} c_0^{(1,n)} j^n + O\left(\frac{1}{(\ln s)^2}\right). \tag{2.5}
\]

Expanding also the condition (2.4) we have

\[
2\sigma_0(0)c_0 + \frac{1}{3}\sigma_0''(0)c_0^3 + \ldots = -2\pi j \ln s, \tag{2.6}
\]

where we use the notation \(\sigma_0(0)\), \(\sigma_0''(0)\) to indicate the values at \(u = 0\) of the function \(\sigma_0\) and its second derivative, respectively. For such quantities expansions similar to (2.5) hold, e.g.

\[
\sigma_0(0) = \ln s\left[\sum_{n=0}^{\infty} \sigma_0^{(-1,n)}(0)j^n\right] + \left[\sum_{n=0}^{\infty} \sigma_0^{(0,n)}(0)j^n\right] + O\left(\frac{1}{\ln s}\right). \tag{2.7}
\]

Explicitly we have

\[
\sigma_0(0) = [-4 - 4j \ln 2 + O(j^3)] \ln s - [8 \ln 2 + 4\gamma_E + O(j^3)] + O\left(\frac{1}{\ln s}\right) \tag{2.8}
\]

and also that

\[
\sigma_0''(0) = [56\zeta(3) + O(j)] + O\left(\frac{1}{\ln s}\right). \tag{2.9}
\]

Now, inserting the expansion (2.5) in the condition (2.6) and using also (2.8) we can start finding, by equating equal powers in \(j\) and \(\ln s\), the various coefficients \(c_0^{(0,n)}\), \(c_0^{(1,n)}\).

We get, without much ado

\[
c^{(0,1)} = \frac{\pi}{4}, \quad c^{(0,2)} = -\frac{\pi}{4} \ln 2, \quad c^{(0,3)} = \frac{\pi}{4} (\ln 2)^2; \quad c^{(1,1)} = -\frac{\pi}{4} (2\ln 2 + \gamma_E) \tag{2.10}
\]

\[
c^{(1,2)} = \frac{\pi}{2} \ln 2 (2\ln 2 + \gamma_E), \quad c^{(1,3)} = -\frac{3}{4} \pi (\ln 2)^2 (2\ln 2 + \gamma_E) + \frac{7}{192} \pi^3 \zeta(3), \tag{2.11}
\]

in such a way that one can write

\[
c_0 = \left[\frac{\pi}{4} j - \frac{\pi}{4} \ln 2 j^2 + \frac{\pi}{4} (\ln 2)^2 j^3 + O(j^4)\right] + \left[-\frac{\pi}{4} (2\ln 2 + \gamma_E) j + \frac{\pi}{2} \ln 2 (2\ln 2 + \gamma_E) j^2 - \left(\frac{3}{4} \pi (\ln 2)^2 (2\ln 2 + \gamma_E) - \frac{7}{192} \pi^3 \zeta(3)\right) j^3 + O(j^4)\right] \frac{1}{\ln s} + O\left(\frac{1}{(\ln s)^2}\right). \tag{2.12}
\]

\(^2\)Actually, in order to get equation (2.3), we have to use once identity (2.4).
All this information can be used to compute the anomalous dimension at one loop $\gamma_{g^2}$ up to the desired order in $j$. From the formulæ - exact if we neglect terms $O\left(\frac{1}{\ln s}\right)$ -

\[
\gamma_{g^2} = g^2 E_0, \quad E_0 = -\int_{-\infty}^{\infty} \frac{du}{2\pi} e(u)\sigma_0(u) + \int_{-\infty}^{\infty} \frac{du}{2\pi} e(u)\sigma_0(u) = \int_{-\infty}^{\infty} \frac{dk}{4\pi^2} \hat{e}(k)\hat{\sigma}_0(k) + \int_{-\infty}^{\infty} \frac{dk}{4\pi^2} \hat{e}(k) \int_{-\infty}^{\infty} \frac{dh}{2\pi} \sigma_0(h) \left[ 2\frac{\sin(k-h)c_0}{k-h} - 2\frac{\sin hc_0}{h} \right] - \ln sj\epsilon(0),
\]

where $E_0$ denotes the energy of the spin $-1/2$ Heisenberg chain and the function

\[
e(u) = \frac{1}{u^2 + \frac{1}{4}} \Rightarrow \hat{e}(k) = 2\pi e^{-\frac{|k|}{2}}, \quad (2.14)
\]

we get, expanding in powers of $j$:

\[
\gamma_{g^2} = g^2 \ln s \left[ 4 - 4\ln 2 - j^3 - \frac{7\zeta(3)\pi^2}{24} j^3 - \frac{7\zeta(3)\pi^2 \ln 2}{12} j^4 + \left( \frac{7\pi^2(\ln 2)^2\zeta(3)}{8} - \frac{31\pi^4\zeta(5)}{640} \right) j^5 + O(j^6) \right] + \frac{g^2}{12} \left[ 4\gamma_E - \frac{7\zeta(3)\pi^2}{12} (2\ln 2 + \gamma_E) j^3 + \frac{7\zeta(3)\pi^2 \ln 2}{4} (2\ln 2 + \gamma_E) j^4 - \frac{7\pi^2(\ln 2)^2\zeta(3)}{2} - \frac{31\pi^4\zeta(5)}{160} \right] (2\ln 2 + \gamma_E) j^5 + \frac{49\pi^4\zeta(3)^2}{960} j^5 + O(j^6) + O\left(\frac{1}{\ln s}\right). \quad (2.15)
\]

As we show in Appendix B, one can also get such an expression improving the results of \cite{19} in order to include also contributions of order $(\ln s)^0 j^k$ (as far as we know, authors of \cite{19} are interested in order $(\ln s)^0 j^k$ contributions and develop their calculations accordingly). This reinforce our belief in the goodness of (2.1) as description of the leading and subleading high spin behaviour in one loop $sl(2)$ sector of $\mathcal{N} = 4$ SYM.

## 3 All loops equations

Let us now pass to study the anomalous dimension as a function of the coupling constant $g$. As usual, we split the density of roots and holes, $\sigma(u)$, as $\sigma(u) = \sigma_0(u) + \sigma_H(u)$, where $\sigma_0(u)$ is the one loop contribution and $\sigma_H(u)$ is the higher than one loop contribution to the actual density. It is convenient to introduce the quantity $S(k)$, related to the Fourier transforms of $\sigma_0(u)$ and $\sigma_H(u)$ as

\[
S(k) = \frac{2\sinh \frac{|k|}{2}}{2\pi |k|} \left[ \hat{\sigma}_H(k) - e^{-\frac{|k|}{2}} \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \hat{\sigma}_0(p) \frac{\sin(k-p)c_0}{k-p} + e^{\frac{|k|}{2}} \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \hat{\sigma}_0(p) \frac{\sin(k-p)c}{k-p} \right],
\]

where $c$ indicates the separator between the holes - concentrated in the interval $[-c, c]$ - and the Bethe roots.
The function \( S(k) \) satisfies the linear equation

\[
S(k) = \frac{L}{|k|} [1 - J_0(\sqrt{2}gk)] + \frac{1}{\pi|k|} \int_{-\infty}^{\infty} \frac{dh}{|h|} \left[ \sum_{r=1}^{\infty} r(-1)^{r+1} J_r(\sqrt{2}gk) J_r(\sqrt{2}gh) \frac{1 - \text{sgn}(kh)}{2} e^{-\frac{|h|}{2}} \right] + \\
\text{sgn}(h) \sum_{r=2}^\infty \sum_{\nu=0}^\infty c_{r,r+1+2\nu}(-1)^{r+\nu} e^{-\frac{|h|}{2}} \left( J_{r-1}(\sqrt{2}gk) J_{r+2\nu}(\sqrt{2}gh) - J_{r-1}(\sqrt{2}gh) J_{r+2\nu}(\sqrt{2}gk) \right) \\
- \frac{e^{\frac{|h|}{2}}}{\sinh \frac{|h|}{2}} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \hat{\sigma}(p) \left[ \frac{\sin(h-e+c h)}{h} - \frac{\sin pc}{p} \right] \right) + O \left( \frac{1}{\ln s} \right). \tag{3.2}
\]

And this relation has to be solved together with the condition

\[
2 \int_{-\infty}^{\infty} \frac{dk}{2\pi} \hat{\sigma}(k) \frac{\sin kc}{k} = -2\pi j \ln s + O \left( \frac{1}{\ln s} \right). \tag{3.3}
\]

To justify equation (3.2) we start from (4.11) of [32], where in the last term of the first line \( L - 2 \) is replaced by a sum on the internal holes \( \sum_{h=1}^{L-2} e^{iku_h} \) and the 0 in the arguments by \( u_h \). Moreover, in the integral in the second line we can replace the extremes \( \pm b_0 \) with \( \pm \infty \) and \( \frac{d}{dv} F_0(v) \) with \( \sigma_0(v) \) (2.1). The equation obtained in such a way for \( \frac{d}{dv} F^H(v) = \sigma_H(v) + O(1/s) \) is exact when neglecting terms of order \( O(1/s) \). Now, we can use results in Appendix A [3], which are valid for a general counting function \( Z(u) \) such that \( Z(c) = -\pi(L - 2) + O(1/\ln s) \), which is exactly condition (3.3). Using (A.3), we evaluate the sum over internal holes keeping only the first (linear) term in the right hand side of this formula, since we want to neglect \( O(1/\ln s) \) contributions. After passing to Fourier transforms and defining (5.1), we get eventually equation (3.2).

Finally, we notice that the relation of the function \( S(k) \) with the anomalous dimension \( \gamma(g) \) is a generalisation of an identity found in [33]:

\[
\gamma(g) = 2S(0). \tag{3.4}
\]

The proof of such identity follows by the simple comparison between (3.2) and the expression for the anomalous dimension

\[
\gamma(g) = -\int_{-\infty}^{\infty} \frac{du}{2\pi} q_2(u) \sigma(u) + \int_{-c}^{c} \frac{du}{2\pi} q_2(u) \sigma(u) , \quad q_2(u) = \frac{i}{x^+(u)} - \frac{i}{x^-(u)},
\]

\[
x^\pm(u) = \frac{u \pm 1/2}{2} \left[ 1 + \sqrt{1 - \frac{2g^2}{(u \pm 1/2)^2}} \right]. \tag{3.5}
\]

A little care is needed in order to interpret (3.4): the term proportional to \( g^{2n} \), which depends on the density at \( n + 1 \) loops, gives the \( n \)-th loop contribution to the anomalous dimension.

In the next subsection, we will compute at two and three loops the anomalous dimension in the limit (1.3). New results will be the subleading, \( O(\ln s^0) \) contribution, which we will evaluate as power series in \( j \), up to \( j^4 \).

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3We remember that \( \sigma(u) = \frac{d}{du} Z(u) \), where \( Z(u) \) is the counting function.
3.1 Two loops anomalous dimension

Let us expand the linear equation for $S(k)$ at order $g^4$, i.e., at three loops approximation for the density. Before doing this, it is convenient to write the expression for $S(k)$ at the order $g^2$. From the general expression we get

$$S_{g^2}(k) = \frac{g^2}{2} L|k| + \frac{1}{\pi|k|} \int_{-\infty}^{\infty} dh \frac{\sqrt{2gh} \sqrt{2gk} 1 - \text{sgn}(kh)}{|h|} \frac{e^{-|h|}}{2} \left\{ -4\pi \ln 2 \delta(h) - \frac{\pi |h|}{\sinh |h|} S_{g^2}(h) - \frac{e^{\frac{|h|}{2}}}{\sinh \frac{|h|}{2}} \int_{-\infty}^{\infty} dp \frac{\sigma(p)}{2\pi} \left[ \frac{\sin(h-p)c_0}{h-p} - \frac{\sin pc_0}{p} \right] \right\}.$$  \hspace{1cm} (3.6)

Comparing (3.4) with (3.6) we get

$$S_{g^2}(k) = \frac{g^2}{2} L|k| + \frac{1}{\pi} \gamma_{g^2}. \hspace{1cm} (3.7)$$

Let us pass now to the three loops (order $g^4$) density. In this case the relevant equation reads

$$S_{g^4}(k) = -\frac{g^4}{16} L|k|^3 + \frac{1}{\pi|k|} \int_{-\infty}^{\infty} dh \frac{\sqrt{2gh} \sqrt{2gk} 1 - \text{sgn}(kh)}{|h|^2} \frac{e^{-|h|}}{2} \left\{ \frac{\pi |h|}{\sinh |h|} S_{g^2}(h) - \frac{e^{\frac{|h|}{2}}}{\sinh \frac{|h|}{2}} \int_{-\infty}^{\infty} dp \frac{\sigma(p)}{2\pi} \left[ \frac{\sin(h-p)c_0}{h-p} - \frac{\sin pc_0}{p} \right] \right\} + \frac{1}{\pi|k|} \int_{-\infty}^{\infty} dh \frac{1}{|h|} \left[ -\frac{1}{8} g^4 k^2 h^2 - \frac{\sqrt{2gh}^3 \sqrt{2gk}}{16} - \frac{\sqrt{2gh}^3 \sqrt{2gk}}{16} 1 - \text{sgn}(kh) e^{-\frac{|h|}{2}} \left\{ -4\pi \ln 2 \delta(h) - \frac{\pi j \ln s}{\sinh |h|} \frac{1 - e^{-\frac{|h|}{2}} \cos \frac{\pi s}{\sqrt{2}}}{\sinh |h|} \right\} \frac{\int_{-\infty}^{\infty} dp \frac{\sigma(p)}{2\pi} \left[ \frac{\sin(h-p)c_0}{h-p} - \frac{\sin pc_0}{p} \right]}{2\pi} \right\}.$$  \hspace{1cm} (3.8)

It follows that the two loops contribution to the anomalous dimension, $\gamma_{g^4}$, is given by

$$\gamma_{g^4} = -\frac{g^2}{2\pi} \int_{-\infty}^{\infty} dh e^{-\frac{|h|}{2}} \left\{ \frac{\pi |h|}{\sinh |h|} S_{g^2}(h) - \frac{e^{\frac{|h|}{2}}}{\sinh \frac{|h|}{2}} \int_{-\infty}^{\infty} dp \frac{\sigma(p)}{2\pi} \left[ \frac{\sin(h-p)c_0}{h-p} - \frac{\sin pc_0}{p} \right] \right\} + \frac{g^4}{8\pi} \int_{-\infty}^{\infty} dh h^2 e^{-\frac{|h|}{2}} \left\{ -4\pi \ln 2 \delta(h) - \frac{\pi j \ln s}{\sinh |h|} \frac{1 - e^{-\frac{|h|}{2}} \cos \frac{\pi s}{\sqrt{2}}}{\sinh |h|} \right\} + \frac{g^4}{8\pi} \int_{-\infty}^{\infty} dh e^{-\frac{|h|}{2}} \left\{ -4\pi \ln 2 \delta(h) - \frac{\pi j \ln s}{\sinh |h|} \frac{1 - e^{-\frac{|h|}{2}} \cos \frac{\pi s}{\sqrt{2}}}{\sinh |h|} \right\}.$$  \hspace{1cm} (3.9)

It is not difficult to compute, neglecting orders $1/s$, the contributions to $\gamma_{g^4}$ coming from the first term in the first line, the terms in the second line and the first term in the third line. For these contribution we get

$$\gamma_{g^4,1} = -2g^4 \zeta(3)(j \ln s + 2) - \frac{g^2}{6} \pi^2 \gamma_{g^2} + 6g^4 \zeta(3) j \ln s - 2g^4 \zeta(3) = -6g^4 \zeta(3) + 4g^4 \zeta(3) j \ln s - \frac{g^2}{6} \pi^2 \gamma_{g^2}. \hspace{1cm} (3.10)$$
The computation of the contribution, which we indicate with \( \gamma_{g^4,2} \), coming from the last term in (3.9) relies on the one loop results. If one wants to restrict to terms containing powers of \( j \) not higher than 4, the relevant integral to compute is simple:

\[
\gamma_{g^4,2} = \frac{g^4}{24\pi} c_\sigma^3\sigma_0(0) \int_0^{+\infty} dh \frac{h^4}{\sinh \frac{h}{2}} = \frac{g^4}{\pi} 62\zeta(5)c_\sigma^3\sigma_0(0) .
\] (3.11)

\( c_\sigma^3\sigma_0(0) \) can be computed up to the desired order:

\[
c_\sigma^3\sigma_0(0) = \left[ -\frac{\pi^3}{16} j^3 + \frac{\pi^3}{8} j^4 \ln 2 \right] \ln s +
\]

\[
\quad + \left[ \frac{\pi^3}{8} (2 \ln 2 + \gamma_E) j^3 - 3\frac{\pi^3}{8} j^4 \ln 2(2 \ln 2 + \gamma_E) \right] + O \left( \frac{1}{\ln s} \right) .
\] (3.12)

From the one loop results we have, up to the desired order

\[
c_\sigma^3\sigma_0(0) = \left[ -\frac{2\pi^3}{16} j^3 + \frac{\pi^3}{8} j^4 \ln 2 \right] \ln s +
\]

\[
\quad + \left[ \frac{\pi^3}{8} (2 \ln 2 + \gamma_E) j^3 - 3\frac{\pi^3}{8} j^4 \ln 2(2 \ln 2 + \gamma_E) \right] + O \left( \frac{1}{\ln s} \right) .
\] (3.13)

We are now left with the calculation of the contribution - \( \gamma_{g^4,3} \) - to the anomalous dimension coming from the last term in the first line of (3.9). Since we restrict to terms containing powers of \( j \) not higher that 4, at desired order such contribution equals

\[
\gamma_{g^4,3} = -\frac{g^2}{6\pi} [c^3\sigma(0)]_{g^2} \int_0^{+\infty} dh \frac{h^2}{\sinh \frac{h}{2}} = -g^2 \frac{14}{3\pi} \zeta(3) [c^3\sigma(0)]_{g^2} .
\] (3.14)

We have now to compute \( \sigma(0) \) and \( c \) up to the order \( g^2 \). For what concerns \( \sigma(0) \) we have

\[
\sigma(0) = [-4 - 4j \ln 2] \ln s - [8 \ln 2 + 4 \gamma_E] + [\sigma_H(0)]_{g^2} + O \left( \frac{1}{\ln s} \right) ,
\] (3.16)

where, from (3.11) and (3.7), one has

\[
[\sigma_H(0)]_{g^2} = \int_{-\infty}^{\infty} dh \frac{|h|}{2\sinh \frac{|h|}{2}} [S(h)]_{g^2} + O(j^3) = 14(j \ln s + 2)g^2\zeta(3) + \frac{\pi^2}{2} g^2 + O \left( \frac{1}{\ln s} \right) .
\] (3.17)

On the other hand, the condition (3.3) for \( c \) at the desired order reads as

\[
2\sigma(0)c = -2\pi j \ln s .
\] (3.18)

Using such relation, we can get \( c \) at the order \( g^2 \) and up to the order \( j^2 \):

\[
[c]_{g^2} = \left[ \frac{\pi}{16 \ln s} j(1 - 2 \ln 2j) + \frac{\pi}{16(\ln s)^2} (\gamma_E + 2 \ln 2)(-2j + 6 \ln 2j^2) \right] .
\]

\[
\quad \cdot \left[ 28g^2\zeta(3) + \frac{\pi^2}{2} g^2 + 14jg^2\zeta(3) \ln s \right] + O \left( \frac{1}{\ln s} \right) .
\] (3.19)
Coming back to the energy, we have
\[ \gamma_{g^4,3} = g^2 \left( \frac{14}{3\pi} \zeta(3) j \ln s[^2]g^2 + O(j^5) \right), \]

which, after using the one loop expression for \( c_0 \) up to the order \( j^2 \),
\[ c_0 = \left[ \frac{\pi}{4} j - \frac{\pi}{4} \ln 2 j^2 + O(j^3) \right] + \left[ -\frac{\pi}{4} (2 \ln 2 + \gamma_E) j + \frac{\pi}{2} \ln 2 (2 \ln 2 + \gamma_E) j^2 + O(j^3) \right] \frac{1}{\ln s} + O \left( \frac{1}{(\ln s)^2} \right). \]

gives
\[ \gamma_{g^4,3} = \left[ \frac{7}{48} \zeta(3) g^2 \pi^2 j^3 \left( 1 - 3j \ln 2 \right) + \frac{2 \ln 2 + \gamma_E}{\ln s} (-3 + 12 \ln 2 j) \right] \cdot \left[ 28 g^2 \zeta(3) + \frac{\pi^2}{2} \gamma_g^2 + 14 j g^2 \zeta(3) \ln s \right] + O \left( \frac{1}{\ln s} \right). \]

Summing together (3.10, 3.14, 3.22) we get the energy at the order \( g^4 \)
\[ \gamma_{g^4} = -6g^4 \zeta(3) + 4g^4 \zeta(3) j \ln s - \frac{g^2}{6} \pi^2 \gamma_g^2 + \]
\[ + \frac{g^4}{6} \zeta(3)(5) \left[ -\frac{\pi^3}{16} j^3 + \frac{\pi^3}{8} j^4 \ln 2 \right] \ln s + \]
\[ + \frac{g^4}{6} \zeta(3)(5)(2 \ln 2 + \gamma_E) \left[ \frac{\pi^3}{8} j^3 - 3 \frac{\pi^3}{8} j^4 \ln 2 \right] + \]
\[ + \frac{7}{48} \zeta(3) g^2 \pi^2 j^3 \left( 1 - 3j \ln 2 \right) + \frac{2 \ln 2 + \gamma_E}{\ln s} (-3 + 12 j \ln 2) \cdot \left[ 28 g^2 \zeta(3) + \frac{\pi^2}{2} \gamma_g^2 + 14 j g^2 \zeta(3) \ln s \right] + O \left( \frac{1}{\ln s} \right). \]

Alternatively, we have
\[ \gamma_{g^4} = \frac{\pi^2}{6} g^2 \gamma_g^2 + g^4 \ln s \left[ 4 \zeta(3) j - \frac{31}{8} \pi^2 \zeta(5) j^3 + \frac{7}{24} \pi^4 \zeta(3) j^3 + \right] \]
\[ + \frac{31}{4} \pi^2 \ln 2 \zeta(5) j^4 + \frac{49}{24} \pi^2 \zeta(3)^2 j^4 - \frac{7}{6} \pi^4 \zeta(3) \ln 2 j^4 + O(j^5) \right] + \]
\[ + g^4 \left[ -6 \zeta(3) + \frac{31}{4} \pi^2 \zeta(5)(2 \ln 2 + \gamma_E) j^3 + \frac{49}{12} \pi^2 \zeta(3)^2 j^4 - \frac{7}{12} \pi^4 \zeta(3)(3 \ln 2 + \gamma_E) j^3 + \right] \]
\[ + \frac{7}{4} \pi^4 \zeta(3) (3 \ln 2 + 2 \gamma_E) j^4 - \frac{93}{4} \pi^2 \zeta(5) \ln 2 (2 \ln 2 + \gamma_E) j^4 - \]
\[ - \frac{49}{8} \zeta(3)^2 \pi^2 (4 \ln 2 + \gamma_E) j^4 + O(j^5) \right] + O \left( \frac{1}{\ln s} \right). \]

### 3.2 Three loops anomalous dimension

For the three loops anomalous dimension, the calculation follows the same route as in the two loops case. We simply report the final result. The three loops anomalous dimension \( \gamma_{g^6} \) expands
\[
\gamma_{g^6} = \ln s \sum_{n=0}^{\infty} f_{n,g^6} j^n + \sum_{n=0}^{\infty} f_{n,g^6}^{(0)} j^n + O(1/\ln s) \quad (3.25)
\]

where the coefficients \( f_{n,g^6} \) were already computed [15] (at least, explicitly up to \( n = 4 \)) and the new coefficients \( f_{n,g^6}^{(0)} \), still up to \( n = 4 \), read as

\[
\begin{align*}
  f_{0,g^6}^{(0)} &= 20\zeta(5) + \frac{2}{3}\pi^2\zeta(3) + \frac{11}{45}\pi^4\gamma_E \\
  f_{1,g^6}^{(0)} &= f_{2,g^6}^{(0)} = 0 \\
  f_{3,g^6}^{(0)} &= -\frac{331}{1080}\pi^6\zeta(3)(2\ln 2 + \gamma_E) + \frac{1}{144}\pi^6\zeta(3)(42\ln 2 + 49\gamma_E) + \frac{31}{3}\pi^4\zeta(5)(2\ln 2 + \gamma_E) - \\
  &\quad - \frac{31}{8}\pi^4\gamma_E\zeta(5) + \frac{385}{72}\pi^4\zeta(3)^2 - \frac{635}{8}\pi^2\zeta(7)(2\ln 2 + \gamma_E) - \frac{651}{8}\pi^2\zeta(3)\zeta(5) \\
  f_{4,g^6}^{(0)} &= \frac{1905}{8}\pi^2\ln 2(2\ln 2 + \gamma_E)\zeta(7) + \frac{1953}{16}\pi^2(2\ln 2 + \gamma_E)\zeta(3)\zeta(5) + \frac{1953}{8}\pi^2\ln 2\zeta(3)\zeta(5) - \\
  &\quad - \frac{341}{8}\pi^4\ln 2(2\ln 2 + \gamma_E)\zeta(5) - \frac{93}{4}\pi^4(\ln 2)^2\zeta(5) - \frac{385}{48}\pi^4(2\ln 2 + \gamma_E)\zeta(3)^2 - \frac{413}{12}\pi^4\ln 2\zeta(3)^2 + \\
  &\quad + \frac{343}{8}\pi^2\zeta(3)^3 + \frac{767}{720}\pi^6\ln 2(2\ln 2 + \gamma_E)\zeta(3) + \frac{35}{12}\pi^6(\ln 2)^2\zeta(3). \quad (3.28)
\end{align*}
\]

4 Systematics of the subleading term

We now want to put the linear integral equation (3.2) in a form which is more suitable for analysis of both the weak and the strong coupling limit. In brief, instead of working with one linear integral equation (3.2), we will be concerned with (two) linear infinite (i.e. containing infinite equations) systems. As far as the dependence on \( \ln s \) is concerned, equation (3.2) and its solution split in a part proportional to \( \ln s \) and a part proportional to \( (\ln s)^0 \).

We concentrate on the latter, which we call \( S^{(0)}(k) \), since the former has been extensively studied [15 28].

We now restrict to the domain \( k \geq 0 \) and expand \( S^{(0)}(k) \) in series of Bessel functions

\[
S^{(0)}(k) = \sum_{r=1}^{\infty} S_r^{(0)}(g) \frac{J_r(\sqrt{2gk})}{k}. \quad (4.1)
\]

As a consequence of (3.2), the coefficients \( S_r^{(0)}(g) \) satisfy the following system of equations,

\[
\begin{align*}
  S_{2p-1}^{(0)}(g) &= 2\sqrt{2g}\gamma_E\delta_{p,1} + 4(2p-1) \int_0^{\infty} \frac{dh}{h} \frac{J_{2p-1}(\sqrt{2gh})}{e^h - 1} + A_{2p-1}^{(0)}(g) - 2(2p-1) \sum_{m=1}^{\infty} Z_{2p-1,m}(g) S_{m}^{(0)}(g) \\
  S_{2p}^{(0)}(g) &= 4 + 8p \int_0^{\infty} \frac{dh}{h} \frac{J_{2p}(\sqrt{2gh})}{e^h - 1} + A_{2p}^{(0)}(g) + 4p \sum_{m=1}^{\infty} Z_{2p,2m-1}(g) S_{2m-1}^{(0)}(g) - 4p \sum_{m=1}^{\infty} Z_{2p,2m}(g) S_{2m}^{(0)}(g), \quad (4.2)
\end{align*}
\]
where, as usual, we introduced the notation
\[
Z_{n,m}(g) = \int_0^\infty \frac{dh}{h} \frac{J_n(\sqrt{2gh})J_m(\sqrt{2gh})}{e^h - 1}.
\] (4.3)
and where we introduced the function $\tilde{J}_{2p-1}(x)$ which coincides with the Bessel function $J_{2p-1}(x)$ for $p \geq 2$ and with $J_1(x) - \frac{x}{2}$ when $p = 1$. In (4.2) the terms $A_r^{(0)}(g)$ is the term proportional to $(\ln s)^0$ of the quantity:
\[
A_r(g) = r \int_0^{+\infty} \frac{dh}{2\pi h} \frac{J_r(\sqrt{2gh})}{\sinh \frac{h}{2}} \int_{-\infty}^{+\infty} \frac{dp}{2\pi} 2 \left[ \frac{\sin(h-p)c}{h-p} - \frac{\sin pc}{p} \right] \hat{\sigma}(p).
\] (4.4)
Eventually, anomalous dimension at order $(\ln s)^0$, $\gamma^{(0)}$, is extracted from $S_1^{(0)}(g)$ by means of the formula
\[
\gamma^{(0)} = \sqrt{2g}S_1^{(0)}(g).
\] (4.5)
To be complete, we write down also the equations satisfied by the part of $S(k)$, which is linear in $\ln s$. Let us call such part $S^{(-1)}(k)$. Expanding in series of Bessel functions
\[
S^{(-1)}(k) = \sum_{p=1}^{\infty} S_p^{(-1)}(g) \frac{J_p(\sqrt{2gk})}{k},
\] (4.6)
we find the following system of equations for the coefficients
\[
S_{2p-1}^{(-1)}(g) = 2\sqrt{2g}\delta_{p,1} - 2(2p - 1)J_2^{(0)}(\sqrt{2g}) \int_0^\infty \frac{dh}{h} \frac{J_{2p-1}(\sqrt{2gh})}{e^h + 1} + A_{2p-1}^{(-1)}(g) - 2(2p - 1) \sum_{m=1}^{\infty} Z_{2p-1,m}(g)S^{(-1)}_m(g)
\]
\[
S_{2p}^{(-1)}(g) = 2J_p(\sqrt{2g}) \int_0^{+\infty} \frac{dh}{h} \frac{J_p(\sqrt{2gh})}{e^h + 1} + A_{2p}^{(-1)}(g) + 4p \sum_{m=1}^{\infty} Z_{2p,2m-1}(g)S^{(-1)}_{2m-1}(g) - 4p \sum_{m=1}^{\infty} Z_{2p,2m}(g)S^{(-1)}_{2m}(g),
\] (4.7)
where $A_{r}^{(-1)}(g)$ is the term proportional to $\ln s$ of the quantity (4.4).

This system has been used in the series of papers \[25, 27, 28\] in order to compute the (strong coupling limit of) the generalised scaling function $f(g,j)$. In the next subsections we will adapt the steps of \[25, 27, 28\] to system (4.2) in order to study the subleading function $f^{(0)}(g,j)$ as series in $j$, up to the order $j^5$.

### 4.1 Slicing in powers of $j$

Since we are in the limit (1.3), the function $S^{(0)}(k)$ admits an expansion in powers of $j$,
\[
S^{(0)}(k) = \sum_{n=0}^{\infty} S^{(0,n)}(k)j^n.
\] (4.8)
Consequently, this way of expanding extends to the coefficients \( S_r^{(0)}(g) \) (4.1) as well:

\[
S_r^{(0)}(g) = \sum_{n=0}^{\infty} S_r^{(0,n)}(g) j^n. \quad (4.9)
\]

In expansion (4.9) the coefficients \( S_r^{(0,0)}(g) \) - which give the part of the function \( S^{(0)}(k) \) independent of \( j \) - satisfy the system of equations

\[
S_{2p-1}^{(0,0)}(g) = 2\sqrt{2}g\gamma E\delta_{p,1} + 4(2p-1) \int_0^\infty \frac{dh}{h} \frac{J_{2p-1}(\sqrt{2}gh)}{e^h - 1} - 2(2p-1) \sum_{m=1}^{\infty} Z_{2p-1,m}(g) S_m^{(0,0)}(g) \quad (4.10)
\]

\[
S_{2p}^{(0,0)}(g) = 4 + 8p \int_0^\infty \frac{dh}{h} \frac{J_{2p}(\sqrt{2}gh)}{e^h - 1} + 4p \sum_{m=1}^{\infty} Z_{2p,2m-1}(g) S_{2m-1}^{(0,0)}(g) - 4p \sum_{m=1}^{\infty} Z_{2p,2m}(g) S_{2m}^{(0,0)}(g).
\]

This system of equations is studied in the contemporaneous paper [17] - the coefficients \( S_r^{(0,0)}(g) \) being there denoted as \( S_r^{\text{extra}}(g) \). Therefore, we pass to study the function \( S_r^{(0,n)}(g) \), when \( n \geq 1 \). For simplicity's sake we limit to the cases \( n = 3, 4, 5 \) (for \( n = 1, 2, S_r^{(0,n)}(g) = 0 \)). The relevant forcing terms are obtained after developing (4.1) for small \( c \) up to \( c^5 \):

\[
A_r^{(0,n)}(g) = r \int_0^{+\infty} \frac{dh}{2\pi h} \frac{J_r(\sqrt{2}gh)}{\sinh \frac{h}{2}} \left[ -\frac{1}{3} h^2 c^3 \sigma(0) + \frac{1}{60} h^4 c^5 \sigma(0) - \frac{1}{10} h^2 c^2 \sigma(2) \right] \bigg|_{(\ln s)^n, j^n}, \quad 3 \leq n \leq 5. \quad (4.11)
\]

Now, from the expansions in powers of \( \ln s \):

\[
\sigma(0) = \ln s \sigma^{-1}(0) + \sigma^{(0)}(0) + O \left( \frac{1}{\ln s} \right), \quad c = c^{(0)} + (\ln s)^{-1} c^{(1)} + O \left( \frac{1}{(\ln s)^2} \right), \quad (4.12)
\]

we get (3 \( \leq n \leq 5 \)):

\[
A_r^{(0,n)}(g) = \sum_{n=1}^{\infty} \frac{c^{(k,n)} j^n}{n} = \sum_{n=0}^{\infty} \sigma^{(k-1,n)}(0) j^n, \quad k = 0, 1, \quad (4.11)
\]

we can now specialise (4.13) to the cases \( n = 3 \),

\[
A_r^{(0,3)}(g) = r \int_0^{+\infty} \frac{dh}{2\pi h} \frac{J_r(\sqrt{2}gh)}{\sinh \frac{h}{2}} \left[ -\frac{1}{3} h^2 \sigma^{(0,0)}(0) + 3c^{(0,2)}(0) \sigma^{(-1)}(0) \right], \quad (4.15)
\]
\( n = 4, \)

\[
A_r^{(0,4)}(g) = r \int_{0}^{+\infty} \frac{dh}{2\pi h} J_r(\sqrt{2gh}) \left[ -\frac{1}{3} h^2 \left( 3c^{(0,1)} c^{(0,2)} \sigma^{(0,0)}(0) + 3c^{(0,1)} c^{(1,1)} \sigma^{(-1,1)}(0) + 3c^{(1,2)} \sigma^{(-1,0)}(0) + 6c^{(0,1)} c^{(1,2)} \sigma^{(-1,0)}(0) \right) \right]
\]

\[
+ 3c^{(0,1)} c^{(1,2)} \sigma^{(-1,0)}(0) + 6c^{(0,1)} c^{(1,2)} \sigma^{(-1,0)}(0) + 6c^{(0,1)} c^{(1,3)} \sigma^{(-1,0)}(0) + 3c^{(0,2)} c^{(1,2)} \sigma^{(-1,0)}(0) + 6c^{(0,1)} c^{(1,2)} \sigma^{(-1,0)}(0) \right) \right] + \frac{1}{10} h^2 \left( c^{(1,1)} \sigma^{(0,1)}(0) + 5c^{(0,1)} c^{(1,1)} \sigma^{(-1,0)}(0) \right) - \frac{1}{10} h^2 \left( c^{(1,1)} \sigma^{(0,1)}(0) + 5c^{(0,1)} c^{(1,1)} \sigma^{(-1,0)}(0) \right) \right],
\]

(4.16)

and \( n = 5: \)

\[
A_r^{(0,5)}(g) = r \int_{0}^{+\infty} \frac{dh}{2\pi h} J_r(\sqrt{2gh}) \left[ -\frac{1}{3} h^2 \left( 3c^{(0,1)} c^{(0,2)} \sigma^{(0,0)}(0) + 3c^{(0,1)} c^{(1,1)} \sigma^{(0,0)}(0) + 3c^{(1,2)} \sigma^{(-1,0)}(0) + 6c^{(0,1)} c^{(1,2)} \sigma^{(-1,0)}(0) + 6c^{(0,1)} c^{(1,3)} \sigma^{(-1,0)}(0) + 3c^{(0,2)} c^{(1,2)} \sigma^{(-1,0)}(0) + 6c^{(0,1)} c^{(1,2)} \sigma^{(-1,0)}(0) \right) \right] + \frac{1}{10} h^2 \left( c^{(1,1)} \sigma^{(0,1)}(0) + 5c^{(0,1)} c^{(1,1)} \sigma^{(-1,0)}(0) \right) - \frac{1}{10} h^2 \left( c^{(1,1)} \sigma^{(0,1)}(0) + 5c^{(0,1)} c^{(1,1)} \sigma^{(-1,0)}(0) \right) \right].
\]

(4.17)

On the other hand, the \( c \) can be expressed in terms of the \( \sigma(0) \) by means of the condition \((3.3)\).

At the relevant order in \( c \) such condition reads

\[
2\sigma(0)c + \frac{1}{3} \sigma_2(0)c^3 = -2\pi j \ln s.
\]

(4.18)

The order \( \ln s^0 \) of such equation gives

\[
2\sigma^{(-1)}(0)c^{(1)} + 2\sigma^{(0)}(0)c^{(0)} + \sigma^{(-1)}(0)c^{(0)} c^{(1)} + \frac{1}{3} \sigma_2^{(0)}(0)c^{(0)}c^{(0)} = 0
\]

(4.19)

Specialising such equation at order \( j \), we get the condition \( c^{(1,1)} = -\frac{\sigma^{(0,0)}(0)}{\sigma^{(-1,0)}(0)} c^{(0,1)} = \frac{\sigma^{(0,0)}(0)}{\sigma^{(-1,0)}(0)} c^{(0,1)} \).

(4.21)

At order \( j^2 \) we have

\[
c^{(1,2)} = -\frac{\sigma^{(-1,1)}(0)}{\sigma^{(-1,0)}(0)} c^{(1,1)} - \frac{\sigma^{(0,0)}(0)}{\sigma^{(-1,0)}(0)} c^{(0,2)} = -2\pi \frac{\sigma^{(0,0)}(0)}{\sigma^{(-1,0)}(0)} \sigma^{(-1,1)}(0) = 2\pi \frac{\sigma^{(0,0)}(0)}{\sigma^{(-1,0)}(0)} \sigma^{(-1,1)}(0)
\]

(4.22)

Going at order \( j^3 \) we get

\[
c^{(1,3)} = -\frac{\sigma^{(-1,1)}(0)}{\sigma^{(-1,0)}(0)} c^{(1,2)} - \frac{\sigma^{(0,0)}(0)}{\sigma^{(-1,0)}(0)} c^{(0,3)} - \frac{1}{2} \frac{\sigma^{(-1,0)}(0)}{\sigma^{(-1,0)}(0)} c^{(0,1)} c^{(1,1)} - \frac{1}{6} \frac{\sigma^{(0,0)}(0)}{\sigma^{(-1,0)}(0)} c^{(1,1)} c^{(1,1)} = 3\pi \frac{\sigma^{(0,0)}(0)}{\sigma^{(-1,0)}(0)} \sigma^{(-1,1)}(0)^2 + \frac{2}{3} \pi^3 \frac{\sigma^{(0,0)}(0)}{\sigma^{(-1,0)}(0)} \sigma^{(-1,1)}(0)^3
\]

(4.23)

\[\text{We remember the formulae (relations (5.18-5.20) of [28]):}\]

\[
c^{(0,1)} = -\frac{\pi}{\sigma^{(-1,0)}(0)}, \quad c^{(0,2)} = \frac{\pi}{\sigma^{(-1,0)}(0)}, \quad c^{(0,3)} = \frac{\pi}{6} \frac{\sigma^{(-1,0)}(0)}{\sigma^{(-1,0)}(0)} - \frac{1}{3} \frac{\sigma^{(-1,0)}(0)}{\sigma^{(-1,0)}(0)}
\]

(4.20)
In such a way, we obtain

\[ A_r^{(0,3)}(g) = r \int_{0}^{+\infty} \frac{dh}{2\pi h} J_r(\sqrt{2gh}) \left(-\frac{2}{3} \pi^3 h^2 \right) \frac{\sigma^{(0,0)}(0)}{[\sigma^{(-1,0)}(0)]^3}. \]  

(4.24)

and, also,

\[ A_r^{(0,4)}(g) = r \int_{0}^{+\infty} \frac{dh}{2\pi h} J_r(\sqrt{2gh}) \frac{2\pi^3 h^2 \sigma^{(0,0)}(0) \sigma^{(-1,1)}(0) [\sigma^{(-1,0)}(0)]^4}{[\sigma^{(-1,0)}(0)]^4}. \]  

(4.25)

For what concerns the term proportional to \( j^5 \), we get, after some calculation:

\[ A_r^{(0,5)}(g) = r \int_{0}^{+\infty} \frac{dh}{2\pi h} J_r(\sqrt{2gh}) \left[ -\frac{1}{3} h^2 \left(\frac{\pi^5}{5} \frac{\sigma^{(0,0)}_2(0)}{[\sigma^{(-1,0)}(0)]^3} \right) + 12\pi^3 \sigma^{(0,0)}(0)[\sigma^{(-1,1)}(0)]^2 \right) - \pi^5 \sigma^{(0,0)}(0) \sigma^{(-1,0)}(0) \right] + \frac{1}{15} \pi^5 \sigma^{(0,0)}(0) \frac{\sigma^{(-1,0)}(0)}{[\sigma^{(-1,0)}(0)]^5} \]  

(4.26)

Now, in analogy to what done in [28], we introduce the 'reduced' coefficients \( \tilde{S}_r^{(k)}(g) \) which satisfy the system (4.23) of [28], i.e. the 'usual' system with the BES kernel and with forcing terms

\[ I_r^{(k)}(g) = r \int_{0}^{+\infty} \frac{dh}{2\pi h} h^{2k-1} J_r(\sqrt{2gh}) \frac{\sigma^{(0,0)}(0)}{\sinh h}. \]  

(4.27)

Using notations of [28], we can write the various \( f_n^{(0)} \), \( n = 3, 4, 5 \), in terms of \( \tilde{S}_1^{(k)}(g) \) and of the density and its derivatives in zero

\[ f_3^{(0)}(g) = -\frac{2}{3} \pi^3 \frac{\sigma^{(0,0)}(0)}{[\sigma^{(-1,0)}(0)]^3} \sqrt{2g} \tilde{S}_1^{(1)}(g), \]  

(4.28)

\[ f_4^{(0)}(g) = 2\pi^3 \frac{\sigma^{(0,0)}(0) \sigma^{(-1,1)}(0)}{[\sigma^{(-1,0)}(0)]^4} \sqrt{2g} \tilde{S}_1^{(1)}(g), \]  

(4.29)

\[ f_5^{(0)}(g) = \left[ -\frac{1}{3} \left(\frac{\pi^5}{5} \frac{\sigma^{(0,0)}_2(0)}{[\sigma^{(-1,0)}(0)]^5} \right) + 12\pi^3 \frac{\sigma^{(0,0)}(0)[\sigma^{(-1,1)}(0)]^2}{[\sigma^{(-1,0)}(0)]^5} - \pi^5 \frac{\sigma^{(0,0)}(0) \sigma^{(-1,0)}(0)}{[\sigma^{(-1,0)}(0)]^6} \right] \sqrt{2g} \tilde{S}_1^{(1)}(g) + \frac{\pi^5}{15} \frac{\sigma^{(0,0)}(0)}{[\sigma^{(-1,0)}(0)]^5} \sqrt{2g} \tilde{S}_1^{(2)}(g). \]  

(4.30)

For comparison, corresponding formul\( e \) for the coefficients \( f_n(g) \) of the function \( f(g, j) \) are, (5.30-32) of [28],

\[ f_3(g) = \frac{\pi^3}{3} \frac{1}{[\sigma^{(-1,0)}(0)]^2} \sqrt{2g} \tilde{S}_1^{(1)}(g), \]  

(4.31)

\[ f_4(g) = -\frac{2}{3} \pi^3 \frac{\sigma^{(-1,1)}(0)}{[\sigma^{(-1,0)}(0)]^2} \sqrt{2g} \tilde{S}_1^{(1)}(g), \]  

(4.32)

\[ f_5(g) = \left[ \pi^3 \frac{\sigma^{(-1,0)}(0)}{[\sigma^{(-1,0)}(0)]^4} - \frac{\pi^5}{15} \frac{\sigma^{(-1,0)}(0)}{[\sigma^{(-1,0)}(0)]^5} \right] \sqrt{2g} \tilde{S}_1^{(1)}(g) - \frac{\pi^5}{60} \frac{1}{[\sigma^{(-1,0)}(0)]^4} \sqrt{2g} \tilde{S}_1^{(2)}(g). \]  

(4.33)

Expressions (4.28) 4.29 4.30 interpolate from weak to strong coupling and for the moment lack of an explicit form as functions of \( g \). Their weak coupling expansion in powers of \( g^2 \) were given, up to \( g^6 \), in Section 3. In the next subsection, we will study and explicitly find their strong coupling limit.
4.2 Strong coupling

It is interesting to consider the strong coupling limit of the equations obtained in the last subsection. For that purpose, we need to know the values of the density and its derivatives in zero and the quantities $\tilde{S}^{(k)}_1(g)$ as well. Results concerning $\sigma^{(-1,n)}$, i.e. the part of the density proportional to $\ln s$, and $\tilde{S}^{(k)}_1(g)$ are reported in [28]. For what concerns the contribution to the anomalous dimension proportional to $\ln s$, i.e. $f(g, j)$, we remember [18, 26] that in the limit (1.3) and when $j \ll g$ it coincides with the energy density of the $O(6)$ sigma model. The nonperturbative (infrared) regime of the sigma model, $j \ll m(g)$, where

$$m(g) = k g^{\frac{1}{4}} e^{-\frac{\pi m(g)}{\sqrt{2}}} + \ldots$$

(4.34)

where the dots stand for subleading corrections, makes contact with the double limit $j \ll 1$, $g \to \infty$, considered in this subsection (and also in [28]). In such regime the scale of the mass is given by $m(g)$ and expansion of $f(g, j)$ for small $j$ and large $g$ was successfully checked [27, 28] against analogous nonperturbative expansions [29] in the sigma model.

It remains an open question if, as well as $f(g, j)$, also $f^{(0)}(g, j)$ shows connections with quantities of the $O(6)$ sigma model. This is a further motivation which leads to study the strong coupling limit of $f^{(0)}(g, j)$, which however is an important problem in itself, since its results can be checked against string theory data. We concentrate on $f^{(0)}_n(g)$, for $n = 3, 4, 5$: as follows from (4.28, 4.29, 4.30), the calculation of their strong coupling limit can be finalised if we know the large $g$ behaviour of the density and its second derivative in zero, $\sigma^{(0,0)}(0)$ and $\sigma^{(0,0)}(0)$. From results of [17], we know that at large $g$ and at the leading order

$$S^{(0,0)}_p(g) = -\ln g S^{(-1,0)}_p(g) + \ldots$$

(4.35)

where $S^{(0,0)}_p(g)$ is a solution of (4.10) and $S^{(-1,0)}_p(g)$ of (4.7) with $j = 0$ (i.e. the BES system). From this relation we can deduce that, at the leading order, the density in zero reads as

$$\sigma^{(0,0)}_H(0) = -\ln g \sigma^{(-1,0)}_H(0) + \ldots = -\ln g [4 - \pi m(g)] + \ldots \Rightarrow \sigma^{(0,0)}(0) = -4 \ln g + \ldots$$

(4.36)

Analogously, for what concerns the second derivative, we get

$$\sigma^{(0,0)}_2(0) = 56 \zeta(3) + \ldots$$

(4.37)

since the higher loops contributions, $\frac{\pi^3}{4} \ln g m(g)$, are exponentially depressed.

Using the results contained in [28], we get the strong coupling relations (which are valid only
at the leading order),
\[
\begin{align*}
    f_3^{(0)}(g) &= -\frac{8 \ln g}{\pi m(g)} f_3(g) + \ldots = -\frac{\pi \ln g}{3m(g)^2} + \ldots \\
    f_4^{(0)}(g) &= -\frac{12 \ln g}{\pi m(g)} f_4(g) + \ldots = \frac{\ln g}{m(g)^3} \left( \ln 2 + \frac{\pi}{2} \right) + \ldots \\
    f_5^{(0)}(g) &= \left[ -\frac{16 \ln g}{\pi^2 m(g)^5} \left( \ln 2 + \frac{\pi}{2} \right)^2 + \frac{\pi^2 \ln g}{3m(g)^5} \right] \sqrt{2} g S_1^{(1)}(g) + \\
    &+ \frac{16 \ln g}{60m(g)^5} \sqrt{2} g S_1^{(2)}(g) + \ldots = -\frac{16 \ln g}{\pi m(g)} f_5(g) + \frac{\pi^2 \ln g}{15m(g)^5} \sqrt{2} g S_1^{(1)}(g) + \ldots = \\
    &-\frac{2 \ln g}{\pi m(g)^4} \left( \ln 2 + \frac{\pi}{2} \right)^2 + \frac{\pi^3 \ln g}{30m(g)^4} + \ldots 
\end{align*}
\]

We see that there is a simple proportionality between \( f_n^{(0)}(g) \) and \((n-1)f_n(g)\) for \(n = 3, 4\), which, however, is lost when \(n = 5\). This could suggest a relation between \(f^{(0)}(g, j)\) and \(f(g, j)\), but at the moment our data are not conclusive about its form.

5 Conclusions

A systematic procedure for the computation of the subleading correction \(f^{(0)}(g, j)\) in the high spin limit to the anomalous dimensions in the \(sl(2)\) sector of \(\mathcal{N} = 4\) SYM is developed. The method is analogous to the one used in \[28\] to study the leading term \(f(g, j)\) and is based on a linear integral equation describing the behaviour of the density and the observables at high spin. We first found one loop (subleading) contributions to the anomalous dimension (Section 2), finding agreement with results obtained by another method, which is an improvement of the techniques of \[19\] (Appendix B). Then, we performed weak coupling expansions (two and three loops, Section 3). Finally, we wrote the linear integral equation for the density as linear infinite systems (Section 4). This was particularly convenient, for it simplifies the study of the strong coupling limit of \(f^{(0)}(g, j)\). In this respect, we explicitly found (Subsection 4.2) the large \(g\) limit of \(f_n^{(0)}(g)\), for \(n = 3, 4, 5\), comparing their expression with \(f_n(g)\). Although the string action may reduce to the \(O(6)\) non-linear sigma model, the question of its appearance in the sub-leading scaling function still stays as an open question.

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A Estimate of the holes contribution

We now give an estimate of the sum over the internal holes \( u_h \) of a generic function \( O(u_h) \). Results we present here are valid both at one and all loops cases. We start from the exact hole expression [33], depending on the so-called counting function \( Z(u) \)

\[
\sum_{h=1}^{L-2} O(u_h) = - \int_{c}^{c} \frac{d u}{2 \pi} O(u) Z'(u) + \text{Im} \int_{-c}^{c} \frac{d v}{\pi} O(v - i \epsilon) \frac{d}{d v} \ln [1 + \delta e^{i Z(v - i \epsilon)}] + \text{Im} \int_{c}^{c} \frac{d v}{\pi} O(v + i \epsilon) \frac{d}{d v} \ln [1 + \delta e^{i Z(v + i \epsilon)}].
\] (A.1)

The right hand side of (A.1) does not depend on \( \epsilon \) as far as no poles of the integrands lie in the region \( \text{Im} v < \epsilon, |\text{Re} v| < c \). The constant \( \delta \) is equal to \( \pm 1 \), depending on the parity of \( L \). Keeping \( \epsilon \) small, but finite, we suppose that

\[
|e^{i Z(z)}| \ll 1,
\] (A.2)

when \( z \) belongs to the integration contour of (A.1). In the large \( s \) limit this is justified, since \( Z'(v) \) is proportional to \( -\ln s \). Using such approximation, we can replace all the \( \ln [1 + e^{i Z(z)}] \) with \( e^{i Z(z)} \).

Then, since the expression we get is still independent of \( \epsilon \), we find convenient to evaluate it when \( \epsilon = 0 \). We obtain

\[
\sum_{h=1}^{L-2} O(u_h) = - \int_{c}^{c} \frac{d u}{2 \pi} O(u) Z'(u) + \delta \text{Im} \int_{-c}^{c} \frac{d v}{\pi} O(v) \frac{d}{d v} e^{i Z(v)}. \] (A.3)

The nonlinear term in this expression

\[
NL = \delta \text{Im} \int_{-c}^{c} \frac{d v}{\pi} O(v) \frac{d}{d v} e^{i Z(v)},
\] (A.4)

can be estimated after the change of variable \( x = Z(v) \) (we remind that \( Z'(v) < 0 \)) and after using the formula

\[
\int d x f(x) e^{a x} = \frac{e^{a x}}{a} \sum_{k=0}^{\infty} (-1)^k \frac{f^{(k)}(x)}{a^k},
\] (A.5)

where \( f^{(k)}(x) \) denotes the \( k \)-th derivative of \( f(x) \). We get

\[
NL = \delta \text{Im} \left[ \frac{e^{i x}}{\pi} \sum_{k=0}^{\infty} \frac{1}{i^k} \frac{\partial}{\partial x^k} O(Z^{-1}(x)) \right] Z(c)_{Z(-c)}. \] (A.6)

The first terms of such series are

\[
NL = \delta \frac{2}{\pi} O(c) \sin Z(c) + \delta \frac{2}{\pi} \frac{O'(c)}{Z'(c)} \cos Z(c) + \ldots
\] (A.7)

---

5 The counting function is connected to \( \sigma(u) \) by the relation \( Z'(u) = \sigma(u) \).
6 We can put directly \( \epsilon = 0 \) since, after the approximation (A.2), the integrand is regular on the real axis.
where the dots represent terms containing higher powers of \( Z'(c) \) in the denominator. Now, if \( c \) is chosen such that \( Z(c) = -\pi(L - 2) + O(1/\ln s) \) - which is exactly condition (2.4) for the one loop and (3.3) for the all loops case, respectively - the first term in (A.7) is \( O(1/\ln s) \). The second term is \( O(1/\ln s) \) as well, since \( Z'(c) \) - for generic \( g \) - is proportional to \( \ln s \). After carefully evaluating all the terms in the series, one can show by similar reasonings that the \( k \)-th term is \( O(1/\ln s^k) \): this ensures the convergence of the series in (A.6), which gives the nonlinear contribution to (A.3).

Evidently, such nonlinear contribution is \( O(1/\ln s) \).

B Contact with [19]

We now show that our one loop results of Section 1 can be obtained starting from the equations of [19]. In this paper, authors are interested to the contribution to the energy proportional to \( \ln s \) (i.e. the part \( \ln sf(j) \), where \( f(j) \) is the one loop contribution to the cusp anomalous dimension). However, their equations can be used to compute also the subleading part, proportional to \( \ln s^0 \), both at the level of equation for the density and at the level of explicit computations of energy eigenvalues.

Let us start from the equations describing the distribution of roots and holes. In paper [19], authors start from the equation

\[
2\delta_n \ln s - \frac{iL}{2} \ln \frac{\Gamma\left(\frac{1}{2} - i\delta_n\right)}{\Gamma\left(\frac{1}{2} + i\delta_n\right)} - \frac{i}{2} \sum_{j=2}^{L-1} \ln \frac{\Gamma(1 + i\delta_n - i\delta_j)}{\Gamma(1 - i\delta_n + i\delta_j)} = \frac{\pi}{2} k_n, \tag{B.1}
\]

which describe the distribution of internal holes, denoted as \( \delta_n, 2 \leq n \leq L - 1 \). This equation suggests the definition of the counting function

\[
Z_0(u) = -4 \ln s u + iL \ln \frac{\Gamma\left(\frac{1}{2} - iu\right)}{\Gamma\left(\frac{1}{2} + iu\right)} + i \sum_{j=2}^{L-1} \ln \frac{\Gamma(1 + iu - i\delta_j)}{\Gamma(1 - iu + i\delta_j)}. \tag{B.2}
\]

Indeed we have that \( Z_0(\delta_n) = \pi(2n - L - 1), \quad e^{iZ_0(\delta_n)} = (-1)^{L+1}. \)

In terms of the derivative \( \sigma_0(u) = \frac{d}{du} Z_0(u) \), we have the equation

\[
\sigma_0(u) = -4 \ln s + L \left[ \psi\left(\frac{1}{2} - iu\right) + \psi\left(\frac{1}{2} + iu\right) \right] - \frac{1}{2\pi} \int_{-c_0}^{c_0} dv \sigma_0(v)\left[ \psi(1 + iu - iv) + \psi(1 - iu - iv) \right]. \tag{B.3}
\]

which after expressing the sum on the holes in terms of the density of holes \( -\frac{1}{2\pi} \sigma_0(u) \) turns out into

\[
\sigma_0(u) = -4 \ln s + L \left[ \psi\left(\frac{1}{2} - iu\right) + \psi\left(\frac{1}{2} + iu\right) \right] + \int_{-c_0}^{c_0} \frac{dv}{2\pi} \sigma_0(v)\left[ \psi(1 + iu - iv) + \psi(1 - iu - iv) \right]. \tag{B.4}
\]

Such equation describes the distribution of holes and, consequently, it is valid only for \( -c_0 \leq u \leq c_0 \). And, indeed, one can verify that our starting equation (2.1) reduces exactly to (B.4) when \( -c_0 \leq u \leq c_0 \).
After having shown that our equation for the density of roots and holes agrees with the corresponding relation that can be deduced from results of [19], we pass to compare results for the eigenvalues of the energy: we verify that expression (2.15) can be obtained starting from formulæ of [19].

Referring still to equation (B.1), describing the distribution of internal holes, we develop such equation up to the order $\delta_n^3$. We obtain the relation

$$2\delta_n \ln s - L\psi \left( \frac{1}{2} \right) \delta_n + \frac{L}{6} \psi^{(2)} \left( \frac{1}{2} \right) \delta_n^3 + (L-2)\psi(1)\delta_n - \frac{L-2}{6} \psi^{(2)}(1)\delta_n^3 = \frac{2}{L} \psi^{(2)}(1) \delta_n \sum_{j=2}^{L-1} \delta_j^2 + O(\delta_n^5) = \frac{\pi}{2} k_n,$$

which we rewrite as

$$\delta_n = \frac{\pi k_n}{2 \ln s - L\psi \left( \frac{1}{2} \right) + (L-2)\psi(1) \left[ 1 - \frac{L}{6} \psi^{(2)} \left( \frac{1}{2} \right) \delta_n^2 - \frac{L-2}{6} \psi^{(2)}(1) \delta_n^2 - \frac{1}{2} \psi^{(2)}(1) \sum_{j=2}^{L-1} \delta_j^2 + O(\delta_n^4) \right] + O \left( \frac{\delta_n^4}{(\ln s)^2} \right)}.$$  

Therefore, we can write that

$$\delta_n = \frac{\pi k_n}{2 \ln s - L\psi \left( \frac{1}{2} \right) + (L-2)\psi(1) \left[ 1 - \frac{L}{6} \psi^{(2)} \left( \frac{1}{2} \right) \delta_n^2 - \frac{L-2}{6} \psi^{(2)}(1) \delta_n^2 - \frac{1}{2} \psi^{(2)}(1) \sum_{j=2}^{L-1} \delta_j^2 + O(\delta_n^4) \right] + O \left( \frac{\delta_n^4}{(\ln s)^2} \right)}.$$  

\[ \sum_{n=2}^{L-1} \delta_n^2 \]  

$$= \frac{\pi^2}{4} \left[ 2 \ln s - L\psi \left( \frac{1}{2} \right) + (L-2)\psi(1) \right]^2.$$  

\[ \sum_{n=2}^{L-1} \delta_n^4 \]  

$$= \frac{\pi^4}{16} \left[ 2 \ln s - L\psi \left( \frac{1}{2} \right) + (L-2)\psi(1) \right] \sum_{n=2}^{L-1} \delta_n^4.$$  

Now, for the ground state $k_n = L + 1 - 2n$. Since we want to compare this formula with (2.15), we have to consider only terms in the energy which go as $\ln s j^k$ and $j^k$, with $1 \leq k \leq 5$. In such an approximation the relevant sums appearing in (B.9) read as

\[ \sum_{n=2}^{L-1} \delta_n^2 \]  

$$= \frac{\pi^2}{4} \left[ 2 \ln s - L\psi \left( \frac{1}{2} \right) + (L-2)\psi(1) \right]^2.$$  

\[ \sum_{n=2}^{L-1} \delta_n^4 \]  

$$= \frac{\pi^4}{16} \left[ 2 \ln s - L\psi \left( \frac{1}{2} \right) + (L-2)\psi(1) \right] \sum_{n=2}^{L-1} \delta_n^4.$$  

\[ \sum_{n=2}^{L-1} \delta_n^6 \]  

$$= \frac{\pi^6}{128} \left[ 2 \ln s - L\psi \left( \frac{1}{2} \right) + (L-2)\psi(1) \right] \sum_{n=2}^{L-1} \delta_n^6.$$
Now, using the sums
\[ \sum_{n=2}^{L-1} k_n^2 = \frac{(L-2)^3}{3} + O(L-2) , \quad \sum_{n=2}^{L-1} k_n^4 = \frac{(L-2)^5}{5} + O(L-2^3) , \] (B.12)
and also the fact that
\[ 2 \ln s - L \psi \left( \frac{1}{2} \right) + (L - 2) \psi(1) = 2 \ln s \left[ 1 + \frac{2 \ln 2 + \gamma_E}{\ln s} + \ln 2 \right] , \] (B.13)
one gets, at the order of interest,
\[ \sum_{n=2}^{L-1} \delta_2^2 = \frac{\pi^2 (L-2)^3}{48 (\ln s)^2 \left( 1 + \frac{2 \ln 2 + \gamma_E}{\ln s} \right)^2} \left[ 1 - \frac{2 \ln 2}{\left( 1 + \frac{2 \ln 2 + \gamma_E}{\ln s} \right)^2} j + \frac{3 (\ln 2)^2}{\left( 1 + \frac{2 \ln 2 + \gamma_E}{\ln s} \right)^2} j^2 + \ldots \right] - \frac{\pi^4}{3840} \frac{\psi^{(2)} \left( \frac{1}{2} \right) (L - 2)^5}{(\ln s)^5} \] (B.14)
\[ \sum_{n=2}^{L-1} \delta_4^4 = \frac{\pi^4}{1280} \frac{\ln s}{\left( 1 + \frac{2 \ln 2 + \gamma_E}{\ln s} \right)^4} j^5 , \] (B.15)
where, in order to keep formulæ compact, we did not develop in inverse powers of \( \ln s \) the brackets in the denominators. With the help of the explicit values
\[ \psi^{(2)} \left( \frac{1}{2} \right) = -14 \zeta(3) , \quad \psi^{(4)} \left( \frac{1}{2} \right) = -744 \zeta(5) , \] (B.16)
we eventually plug all the results in (B.9), getting the expression
\[ E_0 = 4 \ln s - 4 \ln s \ln 2 j + 4 \gamma_E + \ln s \left[ \frac{7 \zeta(3) \pi^2}{24} \frac{j^3}{\left( 1 + \frac{2 \ln 2 + \gamma_E}{\ln s} \right)^2} - \ln s \frac{7 \zeta(3) \pi^2 \ln 2}{12} \frac{\ln z}{\left( 1 + \frac{2 \ln 2 + \gamma_E}{\ln s} \right)^3} + \ln s \left[ \frac{7 \pi^2 (\ln 2)^2 \zeta(3)}{8} - \frac{31 \pi^4 \zeta(5)}{640} \right] \right] \frac{j^5}{\left( 1 + \frac{2 \ln 2 + \gamma_E}{\ln s} \right)^4} + \frac{49 \pi^4 \zeta^2(3)}{960} j^5 + \ldots . \] (B.17)
Finally, developing the brackets in the denominators in their \( \ln s \) and \( (\ln s)^0 \) contribution, we get formula (2.15).

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