Postulates of quantum mechanics and phenomenology

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Abstract

We describe a system of axioms that, on one hand, is sufficient for constructing the standard mathematical formalism of quantum mechanics and, on the other hand, is necessary from the phenomenological standpoint. In the proposed scheme, the Hilbert space and linear operators are only secondary structures of the theory, while the primary structures are the elements of a noncommutative algebra (observables) and the functionals on this algebra, associated with the results of a single observation.

1 Introduction

Hilbert space and the linear operators in it are the basic concepts in modern quantum mechanics. Von Neumann [1] gave a transparent mathematical formulation of quantum mechanics based on these concepts. The Hilbert space formalism became the mathematical basis of the immense achievements in quantum mechanics. But such a seemingly perfect construction of the theory is not free from shortcomings, which was noted by [2]: ”These axioms are technically simple but intuitively, they are completely unclear and seem to be ad hoc”.

The same can be expressed in other words as follows. The axioms of the standard quantum mechanics are phenomenologically sufficient, i.e., we can describe practically all observable effects on the basis of these axioms. At the same time, their phenomenological necessity is not clear. Such a situation may lead to undesirable consequences. A theory constructed based on these axioms may prove to be ”overdetermined”. Numerous paradoxes such as the Einstein-Podolsky-Rosen paradox [3], the ”Schrödinger cat” [4, 5], and so on are indirect evidence supporting the existence of such a danger in the standard quantum mechanics. We mention at once that there are different opinions about the paradoxes. The most active adherents of the standard quantum mechanics deny the existence of any paradoxes.

In any case, it is extremely desirable that only statements that can be directly verified in an experiment are chosen as postulates. Otherwise, when remote consequences are verified, there always exists a danger that we have not verified all of them, and the unverified ones may contradict the experiment. Although such a system of axioms may be less effective technically, it will be more reliable. In this article, we try to give such a formulation of the main postulates of quantum mechanics. The proposed system of axioms is not the ideal one in this respect; we only want to minimize the technical part.
Another disadvantage of the approach based on using the Hilbert space is that the Hilbert space is foreign to classical physics. In classical physics, the phase space is used, which in turn is foreign to the standard quantum mechanics. A philosopher would say that classical and quantum physics use different paradigms. This term can be understood as some rules of the game that are given a priori. It would be extremely desirable to construct a scheme covering both classical and quantum mechanics. In this article, a corresponding attempt is made. This by no means implies that quantum mechanics is reduced to classical mechanics. At the same time, the construction is performed in the framework of the classical paradigm.

The classical paradigm is primarily the classical formal logic and the idea about the existence of a causal connection between both physical phenomena and logical statements. Secondarily, it is the assumption about the existence of physical realities that are the bearers of the causes of physical phenomena and the assumption that probabilistic statements satisfy the classical Kolmogorov probability theory.

It is commonly assumed that these postulates are incompatible with the mathematical scheme adopted in quantum mechanics. Here, we try to prove the contrary. This article is a further development of the ideas formulated in [6, 7]. The contents of this article correspond to paper [8].

2 Observables, states, and measurements

We begin by considering a classical physical system. For such a system, "observable" is a basic notion. This notion seems self-evident, requiring no exact definition. Heuristically, an observable is an attribute of the studied physical system that can be given a numerical value by means of a measurement. In what follows, we assume that the measurement is ideal, i.e., is performed with ideal accuracy.

The main property of observables is that they can be multiplied by real numbers, added, and multiplied by each other. In other words, they form a real algebra $A_{cl}$. To define a physical system, we must establish the relations between different observables, i.e., to fix the algebra $A_{cl}$. In the classical case, the algebra turns out to be commutative.

Fixing an observable $\hat{A}$ still tells nothing about the value $A$ that will be obtained as a result of a measurement in a concrete situation. Fixing the values of observables is realized by fixing the state of a physical system. In mathematical terms, this corresponds to fixing a functional $\tilde{\varphi}(\hat{A})$ on the algebra $A_{cl}$.

We know from experience that the sum and product of observables correspond to the sum and product of the measurement results:

$$\hat{A}_1 + \hat{A}_2 \rightarrow A_1 + A_2, \quad \hat{A}_1 \hat{A}_2 \rightarrow A_1 A_2.$$ 

In this connection, the following definition [9] is useful in what follows.

**Definition 1.** Let $B$ be a real commutative algebra and $\tilde{\varphi}$ be a linear functional on $B$. If $\tilde{\varphi}(\hat{B}_1 \hat{B}_2) = \tilde{\varphi}(\hat{B}_1) \tilde{\varphi}(\hat{B}_2)$ for all $\hat{B}_1 \in B$ and $\hat{B}_2 \in B$, then the functional $\tilde{\varphi}$ is called a real homomorphism on the algebra $B$.

Using this definition, we can say that a state is a real homomorphism on the algebra of observables. Physically, a state is some attribute of the physical system separated from the environment. A system can be either isolated or nonisolated. In the latter case, we assume
that the external influence is reduced to the action of external fields. We suppose that a state is determined by some local reality including the internal structure of the physical system as well as the structure of the external field in the localization domain of the system.

In the measurement process, the physical system is influenced by the measuring instrument. According to the character of this influence, measurements can be divided into two types: reproducible and nonreproducible. The characteristic feature of reproducible measurements is that a repeated measurement of an observable gives the same result in spite of the perturbation suffered by the system in each measurement. We assume that the system is isolated from external influences in the interval between measurements and that we are able to take a possible change of the observable in the process of free evolution into account. In what follows, we are mainly interested in reproducible measurements. We therefore include the reproducibility requirement, if the contrary is not explicitly mentioned, in the term "measurement".

The reproducibility problem is particularly interesting if the measurements of several observables for the same physical system are performed. For example, we suppose that we first measure an observable $\hat{A}$, then an observable $\hat{B}$, then again the observable $\hat{A}$, and finally the observable $\hat{B}$. If the results of the repeated measurements coincide with the results of the initial measurements for each observable, then we say that such measurements are compatible. If there exist devices for compatible measurements of $\hat{A}$ and $\hat{B}$, then these observables are said to be compatible or simultaneously measurable.

We know from experience that all observables are compatible for classical physical systems. In contrast, there are both compatible and incompatible observables in the quantum case. In the standard quantum mechanics, this fact is qualified as the "complementarity principle". We regard it simply as evidence of the fact that measuring two incompatible observables requires mutually exclusive instruments.

Although the observables also have algebraic properties in the quantum case, it is impossible to construct a closed algebra from them that would be real, commutative, and associative. On the other hand, it is extremely desirable from the technical standpoint to have a possibility to use the advanced mathematical apparatus of the theory of algebras. We therefore adopt the following compromise variant of the first postulate.

**Postulate 1.**

The observables form a set $\mathcal{A}_+$ of Hermitian elements of an involutive, associative, and (generally) noncommutative algebra $\mathcal{A}$ satisfying the conditions that for every element $\hat{R} \in \mathcal{A}$ there is a Hermitian element $\hat{A}$ ($\hat{A}^* = \hat{A}$) such that $\hat{R}^* \hat{R} = \hat{A}^2$ and that if $\hat{R}^* \hat{R} = 0$, then $\hat{R} = 0$.

We assume that the algebra has a unit element $\hat{1}$. The elements of the algebra $\mathcal{A}$ are called dynamical quantities.

Postulate 1 is not completely free from a technical component, but the latter is essentially smaller than in the corresponding postulate of the standard quantum mechanics. There, it is additionally postulated that dynamical quantities are linear operators in a Hilbert space. This fact can be hardly considered self-evident.

The following postulate is an immediate consequence of quantum measurements.
Postulate 2.
Compatible (simultaneously measurable) observables correspond to mutually commuting elements of the set $\mathfrak{A}_+$. 

In connection with this postulate, real commutative subalgebras of the algebra $\mathfrak{A}$ are especially important for us. Let $\mathfrak{Q}$ ($\mathfrak{Q} \equiv \{Q\} \subset \mathfrak{A}_+$) denote a maximal real commutative subalgebra of the algebra $\mathfrak{A}$. This is the subalgebra of compatible observables. If the algebra $\mathfrak{A}$ is commutative (an algebra of classical dynamical quantities), then such a subalgebra is unique. If the algebra $\mathfrak{A}$ is noncommutative (an algebra of quantum dynamical quantities), then there are many different subalgebras $\mathfrak{Q}_\xi$ ($\xi \in \Xi$). Moreover, the set $\Xi$ has the cardinality of the continuum. Indeed, even if the algebra $\mathfrak{A}$ has two noncommutative Hermitian generators $\hat{A}_1$ and $\hat{A}_2$, then every real algebra $\mathfrak{Q}_\alpha$ with the generator $\hat{B}(\alpha) = \hat{A}_1 \cos \alpha + \hat{A}_2 \sin \alpha$ is an algebra of type $\mathfrak{Q}$.

In both the classical and quantum cases, the value of a variable is revealed in an experiment. In other words, the result of a measurement is described by some functional whose domain is the set $\mathfrak{A}_+$. But the structure of this functional turns out to be much more complicated in the quantum case than in the classical case.

We can separate a subsystem described by the subalgebra $\mathfrak{Q}_\xi$ (the corresponding dynamical variables) from the quantum physical system described by the algebra $\mathfrak{A}$. Because the subalgebra $\mathfrak{Q}_\xi$ is commutative, the separated subsystem can be considered classical. Its state can be described by a functional $\varphi_\xi$ that is a real homomorphism on the algebra $\mathfrak{Q}_\xi$. Of course, this classical subsystem is not isolated from the rest of the quantum system, but isolatedness is not a necessary condition for separating a classical subsystem. A state of a classical system is determined by some local physical reality; it would be strange if the same were not true for the classical subsystem.

**Definition 2.** A multilayer functional $\varphi = \{\varphi_\xi\}$ is the totality of functionals $\varphi_\xi$, where $\xi$ ranges the set $\Xi$ and the domains of the functionals $\varphi_\xi$ are the subalgebras $\mathfrak{Q}_\xi$.

In quantum measurements in each individual experiment, we deal only with observables belonging to one of the subalgebras $\mathfrak{Q}_\xi$. The result of such a measurement is determined by the functional $\varphi_\xi$. Fixing a functional $\varphi$, we fix all such functionals. We can therefore call $\varphi$ a state of the quantum system. To avoid confusion with the term used in the standard quantum mechanics, we call $\varphi$ a elementary state. As a result, we formulate the following postulate.

**Postulate 3.**
The result of each individual measurement of observables of the physical system is determined by the elementary state of this system. The elementary state is described by a multilayer functional $\varphi = \{\varphi_\xi\}$, $\xi \in \Xi$, defined on $\mathfrak{A}_+$ whose restriction $\varphi_\xi(\hat{A})$ to each subalgebra $\mathfrak{Q}_\xi$ is a real homomorphism on the algebra $\mathfrak{Q}_\xi$.

We note that we make no additional assumptions about the properties of the functionals $\varphi_\xi$. In particular, we do not suppose that

$$\varphi_\xi(\hat{A}) = \varphi_\xi'(\hat{A})$$ (1)
for $\hat{A} \in \mathfrak{Q}_\xi \cap \mathfrak{Q}_{\xi'}$. Of course, equality (11) may be satisfied for some functionals $\varphi$. We say that a functional $\varphi$ is stable on an observable $\hat{A}$ if equality (11) holds for all allowable $\xi$ and $\xi'$.

Equality (11) seems self-evident. The possibility of its violation therefore requires a special comment. A measurement is the result of an interaction of two systems: the studied physical system (classical or quantum) and the classical measuring instrument. A priori, the result can depend on both the studied system and the instrument. The dependence on the instrument is parasitic. To exclude it, the instrument is calibrated. The calibration procedure is essentially as follows. A test physical system is taken and subjected to a reproducible measurement. Then a repeated measurement of this system is performed by the instrument to be calibrated. The results of both measurements should coincide (within the allowable error). Only an instrument that passes such a test many times deserves the name of a measuring instrument.

We can thus exclude the dependence of the result on the microstate of the instrument. But we suppose that according to peculiarities of the construction, spatial orientation, or some other macroscopic properties, every measuring instrument belongs to one of the types that can be labeled by the parameter $\xi$ ($\xi \in \Xi$). We assume that the following properties determine whether the instrument belongs to the type $\xi$. First, the instrument is designed for measuring an observable (observables) belonging to the subalgebra $\mathfrak{Q}_\xi$. Second, for the system in the elementary state $\varphi$, the measurement of an observable $\hat{A} \in \mathfrak{Q}_\xi$ gives the result $A_\xi = \varphi_\xi(\hat{A})$.

It turns out that the dependence on the parameter $\xi$ cannot be excluded. To prove this statement, it suffices to give at least one example of a scenario of the interaction of the instrument with the studied object in which the dependence on $\xi$ is not excluded by any calibration.

We construct such an example. Let an instrument perform a reproducible measurement and its influence on the elementary state of the studied object have the following properties: the functional $\varphi_\xi$ remains unchanged, the multilayer functional $\varphi$ becomes stable on the observable $\hat{A}$, and if $\hat{B} \in \mathfrak{Q}_\xi$ and $\varphi$ is stable on $\hat{B}$, then it remains stable in the future. In other respects, $\varphi$ changes uncontrollably. An elementary check shows that instruments with such properties are compatible. On the other hand, these properties are necessary for the compatibility of the instruments.

Let $\hat{A} \in \mathfrak{Q}_\xi \cap \mathfrak{Q}_{\xi'}$ ($\xi \neq \xi'$). We show that for the instruments of types $\xi$ and $\xi'$, the possibility of violation of equality (11) cannot be excluded by any calibration. To verify equality (11), we must perform two measurements for the same physical system: first by the instrument of type $\xi$ and then by the instrument of type $\xi'$, or vice versa. It is impossible to perform measurements by two different instruments absolutely simultaneously.

Let the instrument of type $\xi$ be used first. We then obtain the result $A_\xi = \varphi_\xi(\hat{A})$, and the functional $\varphi$ is transformed to the functional $\varphi'$. According to the scenario, this functional should have the property $\varphi'_\xi(\hat{A}) = \varphi'_\xi(\hat{A}) = \varphi_\xi(\hat{A})$. The second measurement (by the instrument of type $\xi'$) therefore again gives $A_\xi$.

If we first use the instrument of type $\xi'$ and act identically, then we obtain the answer $A_{\xi'}$. But it is impossible to find out if the values $A_\xi$ and $A_{\xi'}$ coincide because we obtain only one answer in every variant of consecutive measurements.

We emphasize that the classification of instruments according to the types $\xi$ is the classification according to the character of interaction between the instrument and the studied system. It is therefore determined not only by the properties of the instrument but also by the studied system (the algebra $\mathfrak{A}$). Moreover, the instrument can influence the values of
observables that are global for the whole system even if it interacts only with an isolated part of the system.

The dependence of the measurement result on the type of the instrument can be considered a realization and concretization of Bohr’s concepts about the dependence of the result on the general context of the experiment. At the same time, the proposed variant of dependence contradicts neither the causality principle nor the notion about the existence of local reality. But local reality is not a definite value of every observable for the considered physical system but the reaction of a measuring instrument of a definite type to the elementary state of the system. We can speak about a definite value of an observable only if the corresponding elementary state is stable for this observable. The commutative algebra $\mathfrak{A}$ has only one maximal real commutative subalgebra. Therefore, in the classical case, all measuring instruments have the same type, and all elementary states are stable for all observables.

In view of the above, we can newly interpret the result obtained by Kochen and Specker in [10], where a no-go theorem was proved. The essence of this theorem is that a particle of spin 1 has no internal characteristic (physical reality [3]) uniquely determining the values of the squares of the projections of the spin on three mutually orthogonal directions.

In the approach described in this article, the conditions of the Kochen-Specker theorem are not satisfied. Indeed, the observables $(\hat{S}_x^2, \hat{S}_y^2, \hat{S}_z^2)$ used in [10] are compatible. The observables $(\hat{S}_x^2, \hat{S}_{y'}^2, \hat{S}_z^2)$, where the directions $x, y'$, and $z'$ are mutually orthogonal but the directions $y$ and $z$ are not parallel to the directions $y'$ and $z'$, are also compatible. The observables $(\hat{S}_{y'}^2, \hat{S}_z^2)$ are not compatible with the observables $(\hat{S}_y^2, \hat{S}_z^2)$. The instruments compatible with the observables $(\hat{S}_x^2, \hat{S}_{y'}^2, \hat{S}_z^2)$ and $(\hat{S}_x^2, \hat{S}_y^2, \hat{S}_z^2)$ have different types. These instruments therefore need not necessarily give the same result for the measurement of the projection of the spin on the direction $x$ but this is implicitly assumed in the Kochen-Specker theorem.

It can be shown [9] that the functionals $\varphi_\xi(\hat{A})$ have the properties

\begin{align}
a) \quad & \varphi_\xi(0) = 0; \\
b) \quad & \varphi_\xi(\hat{I}) = 1; \\
c) \quad & \varphi_\xi(\hat{A}^2) \geq 0; \\
d) \quad & \text{if } \lambda = \varphi_\xi(\hat{A}), \text{ then } \lambda \in \sigma(\hat{A}; \Omega_\xi); \\
e) \quad & \text{if } \lambda \in \sigma(\hat{A}; \Omega_\xi), \text{ then } \lambda = \varphi_\xi(\hat{A}) \text{ for some } \varphi_\xi(\hat{A}).
\end{align}

Here, $\sigma(\hat{A}; \Omega_\xi)$ is the spectrum of the element $\hat{A}$ in the subalgebra $\Omega_\xi$. Because the subalgebra $\Omega_\xi$ is maximal, the spectrum in the algebra $\mathfrak{A}$ is the same (see, e.g., [9]). We recall that, by definition, $\lambda \in \sigma(\hat{A}; \mathfrak{A})$ if and only if the element $\hat{A} - \lambda \hat{I}$ has no inverse in the algebra $\mathfrak{A}$.

In the standard quantum mechanics, we must introduce a special postulate to take the properties of measurements described by relations (2) into account. Relations (2) have a large constructive potential. In particular, properties (2d) and (2e) allow constructing the total set of elementary states allowable for the considered physical system.

To construct a multilayer functional $\varphi$, it is clearly necessary and sufficient to construct all its components $\varphi_\xi$. Each functional $\varphi_\xi$ can be constructed as follows. In the subalgebra $\Omega_\xi$, we choose an arbitrary system $G(\Omega_\xi)$ of independent generators. We construct $\varphi_\xi$ as some map $G(\Omega_\xi)$ onto the set of the points of the spectra of corresponding elements.
rest of the elements of the subalgebra $Q_\xi$, the functional $\varphi_\xi$ is determined by its properties of linearity and multiplicativity.

The procedure for constructing the functionals $\varphi_\xi$ is certainly consistent if the functionals are constructed independently for different $\xi$. Of course, equality (1) may be violated in this case. But we can always construct a multilayer functional $\varphi$ that is stable on all observables belonging to one of the subalgebras $Q_\xi$. For this, it suffices to begin constructing the functional $\varphi$ from this very subalgebra using the procedure just described. On another subalgebra $Q_{\xi'}$, we construct the functional $\varphi_{\xi'}$ as follows. Let $Q_\xi \cap Q_{\xi'} = Q_{\xi\xi'}$ and $G(Q_{\xi\xi'})$ be independent generators of the subalgebra $Q_{\xi\xi'}$. Let $\bar{G}(Q_{\xi\xi'})$ be the complement of these generators up to the set of generators of the subalgebra $Q_{\xi'}$. If $\hat{A} \in Q_{\xi\xi'}$, then we set $\varphi_{\xi'}(\hat{A}) = \varphi_\xi(\hat{A})$. If $\hat{A} \in \bar{G}(Q_{\xi\xi'})$, then we construct $\varphi_{\xi'}(\hat{A})$ as a map $\hat{A}$ to one of the points of its spectrum. On the rest of the elements of the subalgebra $Q_{\xi'}$, the functional $\varphi_{\xi'}$ is constructed by linearity and multiplicativity.

In quantum measurement, the elementary state cannot be fixed unambiguously. Indeed, in one experiment, we can measure observables belonging to the same maximal commutative subalgebra $Q_\xi$ because instruments measuring incompatible observables are incompatible. As a result, we find only the values of the functional $\varphi_\xi$. The rest of the multilayer functional $\varphi$ remains undetermined. A repeated measurement using an instrument of another type will give new information but will uncontrollably perturb the elementary state that arose after the first measurement. The information obtained in the first measurement will therefore become useless.

In this connection, it is convenient to adopt the following definition.

**Definition 3.** Multilayer functionals $\varphi$ are said to be $\varphi_\xi$-equivalent if they have the same restriction $\varphi_\xi$ to the subalgebra $Q_\xi$.

In quantum measurement, we can thus find only the equivalence class to which the studied elementary state belongs. In a reproducible measurement of observables belonging to the subalgebra $Q_\xi$, we easily recognize the state preparation procedure of the standard quantum mechanics. In what follows, we call this state a quantum state and adopt the following definition.

**Definition 4.** A quantum state $\Psi_{\varphi_\xi}(\cdot)$ is an equivalence class $\varphi \{ \varphi \}$ of $\varphi_\xi$-equivalent elementary states that are stable for the subalgebra $Q_\xi$.

In fact, such a definition of quantum state is convenient only for systems without identical particles. The point is that the measuring instrument cannot distinguish which of the identical particles it has registered. It is therefore convenient to somewhat generalize the notion of the equivalence. We say that a functional $\varphi$ is weakly $\varphi_\xi$-equivalent to a functional $\varphi'$ if the restriction $\varphi_\xi$ of the functional $\varphi$ to the subalgebra $Q_\xi$ coincides with the restriction $\varphi'_{\xi'}$ of the functional $\varphi'$ to the subalgebra $Q_{\xi'}$. Here, we assume that the subalgebra $Q_{\xi'}$ is obtained from the subalgebra $Q_\xi$ by replacing the observables of one of the identical particles with the corresponding observables of another one.

For systems with identical particles, the equivalence should be replaced with the weak equivalence in the definition of the quantum state. In what follows, we assume that such a replacement is made if necessary.
3 Quantum ensemble and probability theory

The Kolmogorov probability theory \[11\] is nowadays the most developed mathematically. It is commonly assumed that a special quantum probability theory is required for quantum systems. In this article, we defend the opinion that the classical Kolmogorov probability theory is also quite sufficient for the quantum case if we take the peculiarity of quantum measurements into account \[7\].

The Kolmogorov probability theory (see, e.g., \[11, 12\]) is based on the notion of the so-called probability space \((\Omega, \mathcal{F}, P)\). The first component \(\Omega\) is the set (space) of elementary events. The physical meaning of elementary events is not explicitly specified, but it is assumed that they are mutually exclusive and that one and only one elementary event is realized in each trial. In our case, the role of an elementary event is played by a elementary state \(\varphi\). Along with elementary event, the notion of a “random event” or simply ”event” is introduced. Each event \(F\) is identified with some subset of the set \(\Omega\). The event \(F\) is assumed to be realized if one of the elementary events belonging to this set \((\varphi \in F)\) is realized. It is assumed that we can find out whether an event is realized or not in each trial. For elementary events, this requirement is not imposed. Sets of subsets of the set \(\Omega\) (including the set \(\Omega\) itself and the empty set \(\emptyset\)) are endowed with the structure of Boolean algebras. The algebraic operations are the intersection of subsets, the union of subsets, and the complement of a subset up to \(\Omega\). A Boolean algebra that is closed with respect to countable unions and intersections is called a \(\sigma\)-algebra.

The second component of a probability space is some \(\sigma\)-algebra \(F\). The set \(\Omega\), where a fixed \(\sigma\)-algebra \(F\) is chosen, is called a measurable space.

Finally, the third component of a probability space is a probability measure \(P\). This is a map from the algebra \(F\) to the set of real numbers satisfying the conditions (a) \(0 \leq P(F) \leq 1\) for all \(F \in \mathcal{F}\), \(P(\Omega) = 1\) and (b) \(P(\sum_j F_j) = \sum_j P(F_j)\) for every countable family of disjoint subsets \(F_j \in \mathcal{F}\). We note that the probability measure is defined only for the events belonging to the algebra \(F\). The probability is generally not defined for elementary events.

We now consider the application of the basic principles of probability theory to the problem of quantum measurements. The main purpose of a quantum experiment is to find the probability distributions for some observable quantities. Using a definite measuring instrument, we can obtain such a distribution for a set of compatible observables. From the probability theory standpoint, choosing a certain measuring instrument corresponds to fixing the \(\sigma\)-algebra \(F\).

We suppose that we conduct a typical quantum experiment. We have an ensemble of quantum systems in a definite quantum state. For example, we consider particles with the spin 1/2 and the projection of the spin on the x axis equal to 1/2. We suppose that we want to investigate the distribution of the projections of the spin on the directions having the angles \(\theta_1\) and \(\theta_2\) to the x axis. The corresponding observations are incompatible, and we cannot measure both observables in one experiment. We should therefore conduct two groups of experiments using different measuring instruments. In our concrete case, the magnets in the Stern–Gerlach instrument should have different spatial orientations.

These two groups of experiments can be described by the respective probability spaces \((\Omega, \mathcal{F}_1, P_1)\) and \((\Omega, \mathcal{F}_2, P_2)\). Although the space of elementary events \(\Omega\) is the same in both cases, the probability spaces are different. To endow these spaces with the measurability property, they are given different \(\sigma\)-algebras \(\mathcal{F}_1\) and \(\mathcal{F}_2\).

Formally and purely mathematically \[12\], we can construct a \(\sigma\)-algebra \(\mathcal{F}_{12}\) including
both the algebras \( F_1 \) and \( F_2 \). Such an algebra is called the algebra generated by \( F_1 \) and \( F_2 \). In addition to the subsets \( F^{(1)}_i \in F_1 \) and \( F^{(2)}_j \in F_2 \) of the set \( \Omega \), it also contains all intersections and unions of these subsets. But such a \( \sigma \)-algebra is unacceptable from the physical standpoint. Indeed, the event \( F_{ij} = F^{(1)}_i \cap F^{(2)}_j \) means that the values of two incompatible observables lie in a strictly fixed domain for one quantum object. For a quantum system, it is impossible in principle to conduct an experiment that could distinguish such an event. For such an event, the notion "probability" therefore does not exist at all, i.e., the subset \( F_{ij} \) does not correspond to any probability measure, and the \( \sigma \)-algebra \( F_{12} \) cannot be used to construct the probability space. Here, an important peculiar feature of the application of probability theory to quantum systems is revealed: not every mathematically possible \( \sigma \)-algebra is physically allowable.

An element of the measurable space \((\Omega, \mathcal{F})\) thus corresponds in the experiment to a pair consisting of a quantum object (for example, in a definite quantum state) and certain type of measuring instrument that allows fixing an event of a certain form. Each such instrument can separate events corresponding to some set of compatible observable quantities, i.e., belonging to the same subalgebra \( \Omega_\xi \). If we assume that each measuring instrument has some type \( \xi \), then the \( \sigma \)-algebra \( \mathcal{F} \) depends on the parameter \( \xi \): \( \mathcal{F} = \mathcal{F}_\xi \).

In view of the peculiarity of quantum experiments, we should take care in defining one of the basic notions of probability theory – the real random variable. A real random variable is usually defined as a map from the space \( \Omega \) of elementary events to the extended real axis \( \bar{R} = [-\infty, +\infty] \). But such a definition does not take peculiarities of quantum experiments, where the result may depend on the type of measuring instrument, into account. We therefore adopt the following definition.

**Definition 5.** A real random variable is a map from the measurable space \((\Omega, \mathcal{F}_\xi)\) of elementary events to the extended real axis.

For an observable \( \hat{A} \), this means that

\[
\varphi \xrightarrow{\hat{A}} A_\xi(\varphi) \equiv \varphi_\xi(\hat{A}) \in \bar{R}.
\]

We call a set of physical systems of the same type (described by one algebra \( \mathfrak{A} \)) that are in some quantum state a quantum ensemble. Experiment gives unambiguous evidence confirming that such an ensemble has probabilistic properties. We therefore adopt the following postulate.

**Postulate 4.** A quantum ensemble can be equipped with the structure of a probability space.

We consider an ensemble of physical systems in a quantum state \( \Psi_{\varphi\eta}(\cdot) \) (\( \eta \in \Xi \)). Correspondingly, we consider the equivalence class \( \{\varphi\}_{\varphi\eta} \) the space \( \Omega(\varphi_\eta) \) of elementary events \( \varphi \). We suppose that the value of an observable \( \hat{A} \in \Omega_\xi \) is measured in an experiment and an instrument of type \( \xi \) is used. Let \( (\Omega(\varphi_\eta), \mathcal{F}_\xi) \) denote the corresponding measurable space. Let \( P_\xi \) be a probability measure on this space, i.e., \( P_\xi(F) \) is the probability of the event \( F \in \mathcal{F}_\xi \).

We assume that an event \( \hat{A} \) is realized in the experiment if the registered value of the observable \( \hat{A} \) does not exceed \( \hat{A} \). Let \( P_\xi(\hat{A}) = P(\varphi : \varphi_\xi(\hat{A}) \leq \hat{A}) \) denote the probability of...
this event. If we know the probabilities \( P_\xi(F) \), then we can find the probability \( P_\xi(\hat{A}) \) using the corresponding summations and integrations; the distribution \( P_\xi(\hat{A}) \) is marginal for the probabilities \( P_\xi(F) \) (see, e.g., [13]). The observable \( \hat{A} \) may belong not only to the subalgebra \( \mathfrak{Q}_\xi \) but also to another maximal subalgebra \( \mathfrak{Q}_{\xi'} \). To find the probability of the event \( \tilde{A} \), we can therefore use an instrument of type \( \xi' \). In this case, we might obtain a different value \( P_{\xi'}(\tilde{A}) \) for the probability. But experience shows that probabilities are independent of the measuring instrument used. We must therefore adopt one more postulate.

**Postulate 5.**

Let an observable \( \hat{A} \in \mathfrak{Q}_\xi \cap \mathfrak{Q}_{\xi'} \). Then the probability of the event \( \tilde{A} \) for the system in the quantum state \( \Psi_{\varphi_\eta}(\cdot) \) is independent of the type of instrument used, \( P(\varphi : \varphi_\xi(\hat{A}) \leq \tilde{A}) = P(\varphi : \varphi_{\xi'}(\hat{A}) \leq \tilde{A}) \).

Therefore, although \( \varphi \) is a multilayer functional, we can use the notation \( P(\varphi : \varphi(\hat{A}) \leq \tilde{A}) \) for the probability of the event \( \tilde{A} \).

We suppose that we have an ensemble of quantum systems in a quantum state \( \Psi_{\varphi_\eta}(\cdot) \) and conduct a series of experiments with this ensemble in which an observable \( \hat{A} \) is measured. In every real series, we deal with a finite set of elementary states. In the ideal series, this set can be countable. Let \( \{\varphi\}^A_{\varphi_\eta} \) denote a random countable sample in the space \( \Omega(\eta) \) containing all elementary states belonging to the real series. By the law of large numbers (see, e.g., [12]), the probability measure \( P_{\hat{A}} \) in this sample is uniquely determined by the probabilities \( P(\varphi : \varphi(\hat{A}) \leq \tilde{A}) \). The probability measure \( P_{\hat{A}} \) determines the average of the observable \( \hat{A} \) in the sample \( \{\varphi\}^A_{\varphi_\eta} \):

\[
\langle \hat{A} \rangle = \int_{\{\varphi\}^A_{\varphi_\eta}} P_{\hat{A}}(d\varphi) \varphi(\hat{A}) \equiv \Psi_{\varphi_\eta}(\hat{A}).
\]

This average is independent of the concrete sample and is completely determined by the quantum state \( \Psi_{\varphi_\eta}(\cdot) \). Formula (3) defines a functional (quantum average) on the set \( \mathfrak{A}_+ \). This functional is also denoted by \( \Psi_{\varphi_\eta}(\cdot) \). The totality of quantum experiments indicate that we should adopt the following postulate.

**Postulate 6.**

The functional \( \Psi_{\varphi_\eta}(\cdot) \) is linear on the set \( \mathfrak{A}_+ \).

This means that

\[
\Psi_{\varphi_\eta}(\hat{A} + \hat{B}) = \Psi_{\varphi_\eta}(\hat{A}) + \Psi_{\varphi_\eta}(\hat{B}) \quad \text{also in the case } [\hat{A}, \hat{B}] \neq 0.
\]

Each element also in the case of the algebra \( \mathfrak{A} \) is uniquely represented in the form \( \hat{R} = \hat{A} + i\hat{B} \), where \( \hat{A}, \hat{B} \in \mathfrak{A}_+ \). The functional \( \Psi_{\varphi_\eta}(\cdot) \) can therefore be uniquely extended to a linear functional on the algebra \( \mathfrak{A} \): \( \Psi_{\varphi_\eta}(\hat{R}) = \Psi_{\varphi_\eta}(\hat{A}) + i\Psi_{\varphi_\eta}(\hat{B}) \).

We define the seminorm of the element \( \hat{R} \) by the equality

\[
||\hat{R}||^2 = \sup_{\varphi_\xi} \varphi_\xi(\hat{R}^* \hat{R}) = \rho(\hat{R}^* \hat{R}),
\]
where \( \rho(\hat{R}^*\hat{R}) \) is the spectral radius of the element \( \hat{R}^*\hat{R} \) in the algebra \( \mathfrak{A} \). Such a definition is allowable. First, \( \|\hat{R}\|^2 \geq 0 \) by property (2c). Further, the definition of the probability measure gives

\[
\Psi_{\varphi_\xi}(\hat{R}^*\hat{R}) = \int_{\{\varphi_\xi \in \mathfrak{A}^*\}} P_{\hat{R}^*\hat{R}}(d\varphi)[\hat{R}^*\hat{R}](\varphi) \leq \sup_{\varphi_\xi} \varphi_\xi(\hat{R}^*\hat{R}) = \rho(\hat{R}^*\hat{R}). \tag{5}
\]

Let \( \eta \in \Xi \) be such that \( \hat{R}^*\hat{R} \in \Omega_\eta \). Then we have \( \Psi_{\varphi_\eta}(\hat{R}^*\hat{R}) = \varphi_\eta(\hat{R}^*\hat{R}) \). For such \( \eta \), we obtain

\[
\sup_{\varphi_\eta} \Psi_{\varphi_\eta}(\hat{R}^*\hat{R}) = \sup_{\varphi_\eta} \varphi_\eta(\hat{R}^*\hat{R}) = \rho_\eta(\hat{R}^*\hat{R}), \tag{6}
\]

where \( \rho_\eta(\hat{R}^*\hat{R}) \) is the spectral radius in \( \Omega_\eta \). Because the subalgebra \( \Omega_\eta \) is maximal, we have \( \rho_\eta(\hat{R}^*\hat{R}) = \rho(\hat{R}^*\hat{R}) \). In view of equalities (4), (5), and (6), it hence follows that

\[
\|\hat{R}\|^2 = \sup_{\varphi_\xi} \varphi_\xi(\hat{R}^*\hat{R}) = \sup_{\varphi_\xi} \Psi_{\varphi_\xi}(\hat{R}^*\hat{R}). \tag{7}
\]

Because \( \Psi_{\varphi_\xi}(\cdot) \) is a linear positive functional, the Cauchy-Schwarz-Bunyakovskii inequality holds:

\[
|\Psi_{\varphi_\xi}(\hat{R}^*\hat{S})\Psi_{\varphi_\xi}(\hat{S}^*\hat{R})| \leq \Psi_{\varphi_\xi}(\hat{R}^*\hat{R})\Psi_{\varphi_\xi}(\hat{S}^*\hat{S}). \tag{8}
\]

It hence follows that the axioms of a seminorm of the element \( \hat{R} \) are satisfied for \( \|\hat{R}\| \) (see, e.g., [14]):

\[
\|\hat{R} + \hat{S}\| \leq \|\hat{R}\| + \|\hat{S}\|, \quad \|\lambda \hat{R}\| = |\lambda|\|\hat{R}\|, \quad \|\hat{R}^*\\| = \|\hat{R}\|.
\]

We now consider the set \( J \) of the elements \( \hat{R} \) of the algebra \( \mathfrak{A} \) such that \( \|\hat{R}\|^2 = 0 \). It follows from inequality [3] that \( J \) is a two-sided ideal of \( \mathfrak{A} \). We can therefore form the quotient algebra \( \mathfrak{A}' = \mathfrak{A}/J \). In the algebra \( \mathfrak{A}' \), the relation \( \|\hat{R}\|^2 = 0 \) implies that \( \hat{R} = 0 \). Equality [4] therefore defines not a seminorm but a norm in the algebra \( \mathfrak{A}' \). On the other hand, we can verify that the algebra \( \mathfrak{A}' \) contains the same physical information as \( \mathfrak{A} \). For this, we consider two observables \( \hat{A} \) and \( \hat{B} \) that either both belong or both do not belong to each subalgebra \( \Omega_\xi \). Let \( \hat{A} \) and \( \hat{B} \) satisfy the additional condition \( \|\hat{A} - \hat{B}\| = 0 \). Then it follows from equality [4] that

\[
\varphi_\xi(\hat{A}) = \varphi_\xi(\hat{B}) \tag{9}
\]

for all \( \Omega_\xi \) containing these observables. Equality [4] means that no experiment can distinguish between these observables and they can therefore be identified from the phenomenological standpoint. Passing from the algebra \( \mathfrak{A} \) to the algebra \( \mathfrak{A}' \) allows realizing this identification mathematically. To deal with an algebra of type \( \mathfrak{A}' \) from the beginning, it is necessary to adopt the following postulate.

**Postulate 7.**

If \( \sup_{\varphi_\xi} |\varphi_\xi(\hat{A} - \hat{B})| = 0 \), then \( \hat{A} = \hat{B} \).

Postulate 7 has a technical character. At the same time, it imposes no additional restrictions from the phenomenological standpoint but only simplifies the mathematical description of physical systems. In what follows, we assume that Postulate 7 is satisfied and that equality [7] therefore defines a norm of the element \( \hat{R} \).
The multiplicative properties of the functional $\varphi_\xi$ imply that $\varphi_\xi([\hat{R}^*\hat{R}]^2) = [\varphi_\xi(\hat{R}^*\hat{R})]^2$. This means that $\|\hat{R}^*\hat{R}\| = \|\hat{R}\|^2$. Therefore, if we complete the algebra $\mathfrak{A}$ with respect to the norm $\| \cdot \|$, then $\mathfrak{A}$ becomes a $C^*$-algebra [13]. The algebra of quantum dynamical quantities can thus be equipped with the structure of a $C^*$-algebra. In the standard algebraic approach to quantum theory, this statement is accepted as an initial axiom. Of course, this is very convenient mathematically. But the necessity of such an axiom from the phenomenological standpoint remains completely unclear.

4 $C^*$-algebra and Hilbert space

A remarkable property of $C^*$-algebras is that every $C^*$-algebra is isometrically isomorphic to a subalgebra of linear bounded operators in a suitable Hilbert space [14, 15]. This allows using the customary Hilbert space formalism below. The connection between $C^*$-algebras and the Hilbert space is realized by the so-called Gelfand-Naimark-Segal (GNS) construction (see, e.g., [14]). In brief, it is as follows.

Let $\mathfrak{A}$ be a $C^*$-algebra and $\Psi_0$ be a linear positive functional on this algebra. We assume that two elements $\hat{R}, \hat{R}' \in \mathfrak{A}$ are equivalent if the equality $\Psi_0(\hat{K}^*(\hat{R} - \hat{R}')) = 0$ holds for every $\hat{K} \in \mathfrak{A}$. We let $\Phi(\hat{R})$ denote the equivalence class of the element $\hat{R}$ and consider the set $\mathfrak{A}(\Psi_0)$ of all equivalence classes in $\mathfrak{A}$. We convert the set $\mathfrak{A}(\Psi_0)$ into a linear space by setting $a\Phi(\hat{R}) + b\Phi(\hat{S}) = \Phi(a\hat{R} + b\hat{S})$. We define a scalar product in $\mathfrak{A}(\Psi_0)$ by the formula

$$\left(\Phi(\hat{R}), \Phi(\hat{S})\right) = \Psi_0(\hat{R}^*\hat{S}).$$

(10)

This scalar product determines the norm $\|\Phi(\hat{R})\|^2 = \Psi_0(\hat{R}^*\hat{R})$ in $\mathfrak{A}(\Psi_0)$. The completion with respect to this norm converts $\mathfrak{A}(\Psi_0)$ into a Hilbert space. Each element $\hat{S}$ of the algebra $\mathfrak{A}$ is uniquely represented in this space by the linear operator $\Pi(\hat{S})$ acting by the rule

$$\Pi(\hat{S})\Phi(\hat{R}) = \Phi(\hat{S}\hat{R}).$$

(11)

We consider a subalgebra $\mathfrak{Q}_\xi \xi \in \Xi$. Without loss of generality, we can consider the elements of this subalgebra as mutually commuting linear self-adjoint operators in some Hilbert space. In this case, for each element $\hat{A} \in \mathfrak{Q}_\xi$, we have a spectral decomposition

$$\Psi(\hat{A}) = \int \Psi(\hat{\rho}(d\lambda)) A(\lambda),$$

(12)

where $\Psi$ is an arbitrary bounded positive linear functional and $\hat{\rho}(d\lambda)$ are the projectors of the orthogonal decomposition of unity. In what follows, we write the integrals of type (12) in the form

$$\hat{A} = \int \hat{\rho}(d\lambda) A(\lambda)$$

(13)

and assume that all integrals (and limits) on the algebra $\mathfrak{A}$ should be understood in the weak topology. Correspondingly, the projectors $\hat{\rho}(d\lambda)$ satisfy the relation

$$\hat{I} = \int \hat{\rho}(d\lambda).$$

(14)

All elements of the subalgebra $\mathfrak{Q}_\xi$ have a common decomposition of unity. Relations (12)-(14) are purely algebraic and are independent of the concrete realization of the elements of the algebra.
Let \( \hat{p}_r \) be a one-dimensional projector. In the operator representation, this is the projector on a one-dimensional Hilbert subspace. Such a projector has the properties
\[
\hat{p}_r^* = \hat{p}_r, \quad \hat{p}_r^2 = \hat{p}_r
\]
and cannot be represented in the form
\[
\hat{p}_r = \sum \alpha \hat{p}_\alpha, \quad \hat{p}_r \hat{p}_\alpha = \hat{p}_\alpha \hat{p}_r = \hat{p}_\alpha \neq \hat{p}_r.
\]
Properties (15) and (16) can be used as the definition of a one-dimensional projector as an element of the algebra.

We consider a one-dimensional projector \( \hat{p}_r \in \mathfrak{Q}_\xi \). Because (14) is an orthogonal decomposition of unity, we can write
\[
\hat{p}_r \hat{p}(d\lambda) = \hat{p}_r \delta(\lambda - \tau) d\lambda.
\]
Let \( \{ \varphi \}_{(\tau)} \) be a set of multilayer functionals such that \( \varphi_\xi(\hat{p}_r) = 1 \). For each observable \( \hat{A} \in \mathfrak{Q}_\xi \), the value \( \varphi_\xi(\hat{A}) \) is the same for all such functionals. Indeed, using (13) and (17), we obtain
\[
\varphi_\xi(\hat{A}) = \int A(\lambda) \left[ \varphi_\xi(\hat{p}_r) \delta(\lambda - \tau) d\lambda + \varphi_\xi(1 - \hat{p}_r) \hat{p}(d\lambda) \right] = A(\tau).
\]
The set \( \{ \varphi \}_{(\tau)} \) is therefore a class of \( \varphi_\xi \)-equivalent functionals. We separate its subset \( \{ \varphi \}_{(\xi \tau)} \) of multilayer functionals that are stable for all elements of the subalgebra \( \mathfrak{Q}_\xi \). This subset determines a quantum state that is denoted by \( \Psi_{\varphi_\xi}(\cdot) \) or, briefly, by \( \Psi_\tau(\cdot) \).

We now consider the GNS construction, where \( \Psi_{\varphi_\xi}(\cdot) \) plays the role of the functional generating the representation. Let \( \Phi_\tau(\hat{p}_r) \) and \( \Phi_\tau(\hat{I}) \) be the respective equivalence classes of the elements \( \hat{p}_r \) and \( \hat{I} \). We verify that \( \Phi_\tau(\hat{p}_r) = \Phi_\tau(\hat{I}) \). Indeed, by (8), we have
\[
|\Psi_\tau(\hat{R}^*(\hat{I} - \hat{p}_r))|^2 \leq \Psi_\tau(\hat{R}^* \hat{R}) \Psi_\tau(\hat{I} - \hat{p}_r),
\]
but \( \Psi_\tau(\hat{I} - \hat{p}_r) = 0 \) because \( \varphi_\xi(\hat{p}_r) = 1 \). By (10) and (11), we have
\[
\left( \Phi_\tau(\hat{I}), \Pi(\hat{S}) \Phi_\tau(\hat{I}) \right) = \Psi_\tau(\hat{S}).
\]
for every \( \hat{S} \in \mathfrak{A} \).

The quantum average over the quantum ensemble \( \{ \varphi \}_{\xi \tau} \) can therefore be represented in the form of the expectation of the linear operator \( \Pi(\hat{S}) \) in the state described by the vector \( \Phi_\tau(\hat{I}) = \Phi_\tau(\hat{p}_r) \) of the Hilbert space. In the proposed approach, this allows using the mathematical formalism of the standard quantum mechanics to the full extent for calculating quantum averages. But there is an essential difference here between the proposed approach and the standard quantum mechanics. In the latter, the relations of type (18) are postulated (the Born postulate). This postulate is sufficient for quantum mechanical calculations, but its necessity is unclear. In contrast, equality (18) is a consequence of phenomenologically necessary postulates in our case.

We now verify that the functional in Postulate 6 exists. We first show that the functional \( \Psi_{\varphi_\xi}(\cdot) \) must satisfy the relation
\[
\Psi_{\varphi_\xi}(\hat{S}) = \Psi_{\varphi_\xi}(\hat{p}_r \hat{S} \hat{p}_r)\]
for every $\hat{S} \in \mathfrak{A}$. Indeed, we have
\[
\Psi_\tau(\hat{p}_\tau\hat{S}\hat{p}_\tau) = \left(\Phi_\tau(\hat{I}), \Pi(\hat{p}_\tau)\Pi(\hat{S})\Pi(\hat{p}_\tau)\Phi_\tau(\hat{I})\right) = \left(\Phi_\tau(\hat{p}_\tau), \Pi(\hat{S})\Phi_\tau(\hat{p}_\tau)\right) = \Psi_\tau(\hat{S}).
\]

We now prove the following statement.

**Statement.** If $\hat{A} \in \mathfrak{A}_+$, then $\hat{A}_\tau \equiv \hat{p}_\tau\hat{A}\hat{p}_\tau$ has the form $\hat{A}_\tau = \hat{p}_\tau \Psi_\tau(\hat{A})$, where $\Psi_\tau(\hat{A})$ is a linear positive functional satisfying the normalization condition $\Psi_\tau(\hat{I}) = 1$.

**Proof.** We assume that the elements $\hat{A}_\tau$ and $\hat{p}_\tau$ are realized by linear self-adjoint operators. Because $[\hat{A}_\tau, \hat{p}_\tau] = 0$, the operators $\hat{A}_\tau$ and $\hat{p}_\tau$ have a common spectral decomposition of unity. Because the projector $\hat{p}_\tau$ is one-dimensional, the spectral decomposition of $\hat{A}_\tau$ should have the form $\hat{A}_\tau = \hat{p}_\tau \Psi_\tau(\hat{A}) + (\hat{I} - \hat{p}_\tau)\hat{A}_\tau$. In view of the relation $\hat{p}_\tau\hat{A}_\tau = \hat{A}_\tau$, it hence follows that
\[
\hat{A}_\tau = \hat{p}_\tau \Psi_\tau(\hat{A}).
\]
The equality
\[
\hat{p}_\tau \Psi_\tau(\hat{A} + \hat{B}) = \hat{p}_\tau(\hat{A} + \hat{B})\hat{p}_\tau = \hat{p}_\tau \Psi_\tau(\hat{A}) + \hat{p}_\tau \Psi_\tau(\hat{B})
\]
implies the linearity $\Psi_\tau(\hat{A} + \hat{B}) = \Psi_\tau(\hat{A}) + \Psi_\tau(\hat{B})$. By linearity, the functional $\Psi_\tau(\hat{A})$ can be extended to the algebra $\mathfrak{A}$: $\Psi_\tau(\hat{A} + i\hat{B}) = \Psi_\tau(\hat{A}) + i\Psi_\tau(\hat{B})$, where $\hat{A}, \hat{B} \in \mathfrak{A}_+$.

The positivity of the functional follows from the relation
\[
\Psi_\tau(\hat{S}^*\hat{S}) = \varphi_\xi(\hat{p}_\tau \Psi_\tau(\hat{S}^*\hat{S})) = \varphi_\xi(\hat{p}_\tau\hat{S}^*\hat{S}\hat{p}_\tau) \geq 0.
\]
Here, we use property (2c). The normalization condition holds by the equalities
\[
\Psi_\tau(\hat{I}) = \varphi_\xi(\hat{p}_\tau \Psi_\tau(\hat{I})) = \varphi_\xi(\hat{p}_\tau\hat{I}\hat{p}_\tau) = 1.
\]

The functional $\Psi_{\varphi_\xi}(\cdot)$ describing the quantum average therefore has the property required by Postulate 6. Moreover, it is positive and satisfies the normalization condition. These are exactly the conditions that should be satisfied for the functional describing a quantum state.

Because equality [20] is purely algebraic, it holds regardless of the concrete realization of $\hat{A}_\tau$ and $\hat{p}_\tau$, and the value of the functional depends only on two factors: on $\hat{p}_\tau$ (quantum state) and on $\hat{A}$ as an element of the algebra $\mathfrak{A}$ but not on some particular commutative subalgebra ($\hat{A}$ may belong to several such subalgebras). This means that the functional $\Psi_{\varphi_\xi}(\cdot)$ satisfies Postulate 5.

A functional satisfying relation [19], where the projector $\hat{p}_\tau$ is one-dimensional, corresponds to a pure quantum state. If the projector $\hat{p}_\tau$ were multidimensional, the state would be mixed.

## 5 Illustration

We now apply the general arguments to a concrete physical system – the one-dimensional harmonic oscillator. We are interested in the Green’s functions of this system. Of course, we can pass to the standard scheme using the Hilbert space via the GNS construction. But we can propose a more direct approach from the standpoint of the approach developed here.
We therefore assume that the harmonic oscillator is a physical system described by the algebra \( \mathfrak{A} \) of dynamical quantities with two noncommuting Hermitian generators \( \hat{Q} \) and \( \hat{P} \) satisfying the commutation relation

\[
[\hat{Q}, \hat{P}] = i.
\]

The time evolution in the algebra \( \mathfrak{A} \) is governed by the equation

\[
\frac{d\hat{A}(t)}{dt} = i[\hat{H}, \hat{A}(t)], \quad \hat{A}(0) = \hat{A},
\]

where the Hamiltonian has the form \( \hat{H} = \frac{1}{2}(\hat{P}^2 + \omega^2 \hat{Q}^2) \). The quantities \( \hat{Q}, \hat{P}, \hat{A}, \) and \( \hat{H} \) are considered elements of the abstract algebra \( \mathfrak{A} \). The elements \( \hat{Q}, \hat{P} \) and \( \hat{H} \) are unbounded and therefore do not belong to the \( C^* \)-algebra. But their spectral projectors are elements of the \( C^* \)-algebra, i.e., \( \hat{Q}, \hat{P} \) and \( \hat{H} \) are elements adjoined to the \( C^* \)-algebra. In this case, the algebra \( \mathfrak{A} \) can therefore be considered a \( C^* \)-algebra completed by adjoined elements.

It is convenient to pass from the Hermitian elements \( \hat{Q}, \hat{P} \) to the elements

\[
\hat{a}^- = \frac{1}{\sqrt{2\omega}}(\omega \hat{Q} + i\hat{P}), \quad \hat{a}^+ = \frac{1}{\sqrt{2\omega}}(\omega \hat{Q} - i\hat{P})
\]

with the commutation relation

\[
[\hat{a}^-, \hat{a}^+] = 1 \tag{21}
\]

and the simple time dependence

\[
\hat{a}^-(t) = \hat{a}^- \exp(-i\omega t), \quad \hat{a}^+(t) = \hat{a}^+ \exp(+i\omega t).
\]

We calculate the generating functional of the Green’s functions. In the standard quantum mechanics, the n-time Green’s function is defined by the formula

\[
G(t_1, \ldots t_n) = \langle 0 | T(\hat{Q}(t_1) \ldots \hat{Q}(t_n)) | 0 \rangle,
\]

where \( T \) is the operator of chronological ordering and \( | 0 \rangle \) is the quantum ground state.

By the statement proved in the end of the preceding section, the Green’s function in the proposed approach is defined by the formula

\[
\hat{p}_0 T(\hat{Q}(t_1) \ldots \hat{Q}(t_n))\hat{p}_0 = G(t_1, \ldots t_n)\hat{p}_0, \tag{22}
\]

where \( \hat{p}_0 \) is the spectral projector of \( \hat{H} \) corresponding to the minimal energy value. It is easy to see that \( \hat{p}_0 \) can be represented in the form

\[
\hat{p}_0 = \lim_{r \to \infty} \exp(-r\hat{a}^+\hat{a}^-). \tag{23}
\]

As mentioned above, the limit should be understood in the sense of the weak topology of the \( C^* \)-algebra.

We first prove the auxiliary statement:

\[
\hat{E} = \lim_{r_1, r_2 \to \infty} \exp(-r_1\hat{a}^+\hat{a}^-)(\hat{a}^+)^k(\hat{a}^-)^l \exp(-r_2\hat{a}^+\hat{a}^-) = 0 \quad k, l \geq 0, \quad k + l > 0. \tag{24}
\]

Let \( \Psi \) be a bounded positive linear functional. Then we have

\[
\Psi(\hat{E}) = \lim_{r_1, r_2 \to \infty} \exp(-r_1k - r_2l)\Psi((\hat{a}^+)^k \exp(-r_1\hat{a}^+\hat{a}^-) \exp(-r_2\hat{a}^+\hat{a}^-)(\hat{a}^-)^l).
\]
Here, we use the continuity of the functional $\Psi$ and commutation relation (21). Taking the inequality $\|\exp(-r\hat{a}^+\hat{a}^-)\| \leq 1$ into account, we further obtain

$$\left| \Psi(\hat{E}) \right| \leq \lim_{r_1, r_2 \to \infty} \exp(-r_1 k - r_2 l) |\Psi((\hat{a}^+)^k \exp(-2r_1 \hat{a}^+\hat{a}^-)(\hat{a}^-)^k)|^{1/2}$$

$$\times |\Psi((\hat{a}^+)^l \exp(-2r_2 \hat{a}^+\hat{a}^-)(\hat{a}^-)^l)|^{1/2}$$

$$\leq \lim_{r_1, r_2 \to \infty} \exp(-r_1 k - r_2 l) |\Psi((\hat{a}^+)^k(\hat{a}^-)^k)|^{1/2} |\Psi((\hat{a}^+)^l(\hat{a}^-)^l)|^{1/2} = 0.$$

This proves equality (24).

We now verify equality (23). In terms of the elements $\hat{a}^+$ and $\hat{a}^-$, the Hamiltonian $\hat{H}$ has the form $\hat{H} = \omega(\hat{a}^+\hat{a}^- + 1/2)$. By (21), we have

$$\lim_{r_1, r_2 \to \infty} \exp(-r_1 \hat{a}^+\hat{a}^-)\hat{H} \exp(-r_2 \hat{a}^+\hat{a}^-) = \frac{\omega}{2} \lim_{r_1, r_2 \to \infty} \exp(-(r_1 + r_2)\hat{a}^+\hat{a}^-).$$

This proves equality (23).

It follows from formula (22) that

$$G(t_1, \ldots t_n)\hat{p}_0 = \left( \frac{1}{i} \right)^n \frac{\delta^n}{\delta j(t_1) \ldots \delta j(t_n)} \hat{p}_0 T \exp \left( i \int_{-\infty}^{\infty} dt j(t)\hat{Q}(t) \right) \hat{p}_0 \bigg|_{j=0}. \quad (25)$$

By the Wick theorem (16), we have

$$T \exp \left( i \int_{-\infty}^{\infty} dt j(t)\hat{Q}(t) \right) =$$

$$= \exp \left( \frac{1}{2i} \int_{-\infty}^{\infty} dt_1 dt_2 \frac{\delta}{\delta \hat{Q}(t_1)} D^c(t_1 - t_2) \frac{\delta}{\delta \hat{Q}(t_2)} \right) : \exp \left( i \int_{-\infty}^{\infty} dt j(t)\hat{Q}(t) \right) : . \quad (26)$$

Here, $: :$ is the normal ordering operation and

$$D^c(t_1 - t_2) = \frac{1}{2\pi} \int dE \exp \left( -i(t_1 - t_2)E \right) \frac{1}{\omega^2 - E^2 - i0}.$$

Varying the right-hand side of (26) with respect to $\hat{Q}$ and taking (21) into account, we obtain

$$\hat{p}_0 T \exp \left( i \int_{-\infty}^{\infty} dt j(t)\hat{Q}(t) \right) \hat{p}_0 = \exp \left( -\frac{1}{2i} \int_{-\infty}^{\infty} dt_1 dt_2 j(t_1)D^c(t_1 - t_2)j(t_2) \right)$$

$$\times \hat{p}_0 : \exp \left( i \int_{-\infty}^{\infty} dt j(t)\hat{Q}(t) \right) : \hat{p}_0 = \hat{p}_0 \exp \left( -\frac{1}{2i} \int_{-\infty}^{\infty} dt_1 dt_2 j(t_1)D^c(t_1 - t_2)j(t_2) \right).$$

Comparison with formula (23) yields

$$G(t_1 \ldots t_n) = \left( \frac{1}{i} \right)^n \frac{\delta^n Z(j)}{\delta j(t_1) \ldots \delta j(t_n)} \bigg|_{j=0},$$

where

$$Z(j) = \exp \left( \frac{i}{2} \int_{-\infty}^{\infty} dt_1 dt_2 j(t_1)D^c(t_1 - t_2)j(t_2) \right)$$

is the generating functional.

It is well known that considering quantum field systems in the framework of perturbation theory can be reduced to considering the multidimensional harmonic oscillator. Therefore, the proposed method for calculating the generating functional of Greens functions is immediately generalizable to quantum field models.
6 Conclusion

We have attempted to formulate the postulates of quantum mechanics in a maximally weak form. Almost all postulates are formulated such that the nonfulfillment of each of them not only violates the mathematical structure of the theory but also leads to a direct contradiction of experiments. Perhaps only the first postulate, whose formulation contains unobservable quantities, does not completely correspond to this assertion.

Despite their weakness, the proposed postulates suffice for constructing a mathematical formalism including the formalism of the standard quantum mechanics. At the same time, the proposed approach clearly shows the applicability domain of this formalism. It can be used to describe the properties of quantum ensembles. In exactly this case, quantum state is an adequate notion. The formalism of the standard quantum mechanics is most closely connected with this notion.

In contrast, "elementary state" is an adequate notion in the case of a single phenomenon. This notion lies beyond the framework of the standard quantum mechanics. Its application to such objects as the "Schrödinger cat" or the Universe does not lead to any paradoxes. In this case, we have no need in such exotic constructions as the superposition of the live and dead cat or the multiworld construction of Everett [17]. It should be noted that the problem of quantum paradoxes is completely absent in the proposed approach.

An important feature of the proposed approach is that it is equally applicable to both quantum and classical systems although almost all attention was given to quantum systems in this article. Nowadays, no serious physicist tries to reduce quantum physics to the classical one. At the same time, the vast majority believe that the classical physics is only the limiting case of the quantum physics, i.e., the classical physics is reducible to the quantum one. A desire hence arises to quantize all fields encountered in nature. In the proposed approach, quantum and classical fields are considered on the same footing, just as Abelian and non-Abelian fields in the theory of gauge fields. This opens new possibilities for constructing models in which both classical and quantum fields are present.

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