A new definition for parameterizing perturbed Kepler dynamics

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Abstract. Event rate estimations show that parabolic and hyperbolic encounters of compact objects will become significant sources for advanced ground-based gravitational wave detectors. To analytically describe the motion of the objects and the emitted gravitational waveforms a suitable parameterization is needed to be introduced. In the following paper such a parameterization is defined which is valid for both bound and open orbits. First post-Newtonian and spin-orbit corrections are described in detail.

1. Introduction
Event rate estimations (O'Leary at al., 2008) results that parabolic and hyperbolic encounters of compact objects provide significant sources for advanced ground-based gravitational wave detectors (GWs) for ground-based interferometric detectors LIGO, VIRGO, GEO, TAMA and advLIGO. A source like these provides characteristic burst-like wave signal, and can be described analytically. Furthermore, due to radiational reaction open orbits may become closed. In this case the first quasi-parabolic burst-like signal further bursts periodically, until it reaches the validity region of the usual elliptic description of the binary. The characteristic behavior and the possibility to describe them analytically greatly increases the detectability of these systems.

When investigating different corrections to the evolution, similarly to eccentric binaries these sources are described by the post-Newtonian (PN) approximation. In the case of emitted gravitational waves this method is combined with the multipolar expansion of the post-Minkowskian approximation of the radiation field (Blanchet, 2006). Although the method is developed for elliptic and circular orbits, it can be applied for open orbits, too.

Former works (Turner, 1977) use only the Keplerian description of open orbits, and neglecting relativistic corrections and interactions. To give the appropriate description of these systems the next step is to investigate the first relativistic correction to the dynamics, and the effects of the spin-orbit interaction. The general description is well known for these corrections, but a detailed investigation is still under construction.

Since the perturbed Kepler motion cannot be investigated fully analytically, an appropriate parameterization of the orbit is needed to be introduced. This question is highly detailed in many works for closed orbits (Keresztes et al., 2005) and (Damour and Deruelle, 1985). Furthermore, there are some works introducing parameterizations for 1PN corrections in the case of open orbits, too. An other important line of the investigations is the description of the
energy and angular momentum losses for eccentric and hyperbolic orbits (Peters and Mathews, 1963), (Peters, 1964) and (Hansen, 1972).

When investigating spin interactions there arise a freedom in defining the spin vector, or, equivalently the center of mass of the rotating body. This gauge freedom is treated with gauge fixing properties called spin supplementary conditions (SSC) (Pirani, 1956), (Dixon, 1964), (Newton and Wigner, 1949) and (Corinaldesi and Papapetrou, 1951). These SSC conditions are physically equivalent, but the explicit evaluations highly depend on the choice. In the following the SSC choice in (Gergely et al., 1998) is used.

In the following paper a generally applicable parameterization is defined in a way independent on the type of the orbit. Most true anomalous parameterizations are defined by the orbital angle. This is not straightforwardly applicable for spin-orbit coupling, where the orbital plane precess. The following definition does not depend on the exact notion of the "orbital angle". Besides the general method given for linear corrections first post-Newtonian and spin-orbit terms are detailed. The general description of the motion, and the radial and angular equations of motion can be found in (Majá and Vasút, 2008).

Throughout the paper the $G = c = 1$ units are used, and $\delta$ denotes the different perturbative corrections to the expressions in a general way.

2. General definition for the parameterization of the orbit

The main goal of introducing different parameterizations is to solve the radial equation of the perturbed Keplerian motion

$$r^2 = \frac{2E}{\mu} + \frac{2m}{r} - \frac{L^2}{\mu^2 r^2} + \delta \left( \dot{r}^2 \right).$$

The total energy $E$ and the magnitude $L$ of the orbital angular momentum are constants of motion which follows from a Lagrangian description, and $m = m_1 + m_2$ and $\mu = m_1 m_2 / m$ denotes the total and relative mass of the binary. The turning points, defined by $\dot{r}^2 = 0$, are solved as

$$r_{\text{max}}^\text{min} = \frac{m \mu \pm A_0}{-2E} + \delta r_{\text{min}}^\text{max},$$

where $A_0$ denotes the magnitude of the Laplace-Runge-Lenz vector ($A_0^2 = m_2^2 \mu^2 + 2E L^2 / \mu$). To introduce the radial parameterization first we investigate the following integral:

$$\tilde{\Upsilon}(r') = \Upsilon(r)|_{r_{\text{min}}}^{r'} = \int_{r_{\text{min}}}^{r'} \frac{\tilde{\Upsilon} dr}{\dot{r}},$$

where the equation for the polar angle $\Upsilon$ can be evaluated following the description of the angular evolution in (Majá and Vasút, 2008).

It is straightforward to evaluate $\tilde{\Upsilon}(r_{\text{min}}) = 0$. In the case of unbounded orbits $r_{\text{min}}$ is the only positive root of the equation, and $r_{\text{max}} = \infty$. For elliptic orbits $r_{\text{min}}$ and $r_{\text{max}}$ are the turning points of the radial motion. With the use of these facts we have

$$\tilde{\Upsilon}(r_{\text{max}}) = \chi_{\text{max}} + \delta \tilde{\Upsilon},$$

where in the case of elliptic and parabolic orbits $\chi_{\text{max}} = \pi$, and for hyperbolic orbits $\chi_{\text{max}} = \arccos (-m \mu / A_0)$. Furthermore, $\chi_{\text{max}}$ is the maximum value of $\chi$ for open orbits in the Newtonian case.

With the use of the above quantity the new definition of the parameterization can be given as

$$\frac{dr}{d(\cos \chi)} = -\gamma r^2, \quad \chi_{\text{min}}^{\text{max}} = \tilde{\Upsilon}(r_{\text{max}}^{\text{min}}) / K,$$
where $K$ is one plus the periastron precession for the orbit.

The general solution of the radial motion for all types of orbits is
\[
    r(\chi) = \frac{L^2}{\mu (\mu m + A_0 \cos \chi)} + \delta r,
\]
and the relation between the generalized true anomaly parameter and the coordinate time is given by
\[
    \frac{dt}{d\chi} = \frac{1}{\dot{r}} \frac{dr}{d\chi} = \frac{\mu r^2}{L} + \delta \left( \frac{dt}{d\chi} \right).
\]

It can be shown that this parameterization is equivalent to the results of Damour and Deruelle for 1PN corrections, and to the generalized true anomaly parameterization in (Keresztes et al., 2005).

The type of orbital motion, namely elliptic, parabolic and hyperbolic, is determined by the value of the total energy $E$. Although the formal expressions of $r(\chi)$ and $dt/d\chi$ are identical for all cases, the numerical integration of $\chi(t)$ and the determination of $r(t)$ lead to different results, which show the different behavior of the three types of orbit.

### 3. First post-Newtonian and Spin-Orbit corrections

The 1PN correction for the radial equation of motion Eq.(1) is
\[
    \delta \left( \dot{r}^2 \right)_{PN} = 3(3\eta - 1) \frac{E^2}{\mu r^2} + 2(7\eta - 6) \frac{Em}{\mu r} - 2(3\eta - 1) \frac{EL^2}{\mu r^2} + (5\eta - 10) \frac{m^2}{r^2} - (3\eta - 8) \frac{mL^2}{\mu r^3},
\]
where $\eta = \mu/m$, and the spin-orbit correction is
\[
    \delta \left( \dot{r}^2 \right)_{SO} = \frac{2EL\sigma}{m^2 \mu^2 r^2} - \frac{2}{\mu^3} (2LS + L\sigma),
\]
where $S = S_1 + S_2$ and $\sigma = S_1 m_2/m_1 + S_2 m_1/m_2$. Henceforth the corrections for the $\delta r_{\text{max}}^{PN}$ turning points are
\[
    \delta r_{\text{max}}^{PN} = (\eta - 7) \frac{m}{4} \pm (\eta + 9) \frac{m^2 \mu}{8A_0} + (3\eta - 1) \frac{A_0}{8\mu},
\]
\[
    \delta r_{\text{max}}^{SO} = \frac{2LS + L\sigma}{L^2 A_0} (A_0 + m\mu) \pm \frac{EL\sigma}{A_0 m\mu}.
\]

The most important step when determining the perturbative terms of the parameterization is to evaluate the periastron precession corrections at 1PN order and for the spin-orbit interaction. The result is
\[
    K = \left( 1 + \frac{3m^2 \mu^2}{L^2} - \frac{m\mu^3 (3L\sigma + 4LS)}{L^4} \right).
\]

With the use of the above definition the corrections to $r(\chi)$ are evaluated as
\[
    \delta r_{PN} = - \frac{2(6 - \eta) m^4 \mu^6 + 2(10 - 3\eta) m^2 \mu^3 + (1 - 3\eta) E^2 L^4}{2A_0 m^3 (\mu m + A_0 \cos \chi)^2} \cos \chi
\]
\[
    = - \frac{2(2 - \eta) m E L^2 + (6 - \eta) m^3 \mu^3}{\mu (\mu m + A_0 \cos \chi)^2},
\]
\[
    \delta r_{SO} = \frac{2(2LS + L\sigma) [A_0 (m^2 \mu^3 + E L^2) + m\mu (2m^2 \mu^3 + 3E L^2) \cos \chi]}{A_0 L^2 (\mu m + A_0 \cos \chi)^2}
\]
\[
    - \frac{2E L\sigma [A_0 m \mu^2 + (m^2 \mu^3 + E L^2) \cos \chi]}{A_0 m \mu^2 (\mu m + A_0 \cos \chi)^2}. 
\]
Again, the above expressions are valid for all the three types of orbits. To complete the evaluations the 1 PN order and spin-orbit terms in the Eq.(5) differential equations become

\[
\left(\frac{dt}{d\chi}\right)_{PN} = -\frac{\mu r^2}{2L^3} \left[ (\eta - 13)m^2 \mu^2 + (3\eta - 1)A_0^2 + (3\eta - 8)m\mu A_0 \cos \chi \right],
\]

\[
\left(\frac{dt}{d\chi}\right)_{SO} = -\frac{\mu r^2}{mL^5} \left[ (2LS + L\sigma)m\mu^2(3m\mu + A_0 \cos \chi) - EL^2L\sigma \right].
\]

4. Detectable gravitational waveform

As it has been detailed the above definition gives rise to expressions formally independent of the value of the energy, or, equivalently of the type of the orbit. Since all the methods used to evaluate the detectable waveform are independent of it too, the \( h(\chi) \) expressions are formally the same for elliptic, parabolic and hyperbolic cases.

When generating realistic gravitational waveform templates (Kidder, 1995) and (Will and Wiseman, 1996) the numerical integration of the inverse of Eq.(5) is needed. During this procedure there arise three completely different results for elliptic, parabolic and hyperbolic orbits. These differences leads to highly different gravitational wave signals, too. In the case of elliptic orbits the expected waveform signals are periodic (or, including radiational reaction, quasi-periodic), in the case of hyperbolic orbits the signal is burst-like, and in the case of parabolic orbits the burst-like signal evolves into a quasi-periodic waveform if the radiational reaction is included in the description.

The above definition gives rise to the opportunity to determine the higher order corrections to the parameterization of the orbit in a way independent of the type of the perturbed Keplerian orbit. Henceforth, the corrections of further effects (namely, quadruple-monopole, magnetic dipole-dipole interactions, etc.) can be easily evaluated. This method provides the possibility to give the most realistic theoretical waveform signals for the gravitational wave detector template banks.

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