Detecting the entanglement by the mean value of spin on a quantum computer

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(Dated: March 3, 2020)

We propose the protocol to determine the value of entanglement between qubit and the rest of the system prepared on a quantum computer. This protocol is tested on a 5-qubit superconducting quantum processor called ibmq-ourense. We determine the values of entanglement of the Schrödinger cat and the Werner states prepared on this device and compare them with the theoretical ones. Also the protocol to determine the entanglement of rank-2 mixed states is proposed. We apply this protocol for determination of entanglement of the mixed state which consists of two Bell states prepared on the ibmq-ourense quantum device.

INTRODUCTION

Entanglement is an inherent property of a quantum system\textsuperscript{[1, 2]}. It plays a crucial role in processes related to quantum information and quantum computation\textsuperscript{[3, 4]}. The presence of entanglement in a system allows to implement various quantum information schemes and devices that cannot be realized by classical systems. The application of this phenomenon to the implementation of various quantum algorithms began after paper\textsuperscript{[5]}, where Aspect et al. tested Bell’s inequality\textsuperscript{[6]} and experimentally solved the EPR paradox \textsuperscript{[1]}. The preparation of entangled state is an indispensable step in the realization of such tasks as quantum cryptography\textsuperscript{[7]}, super-dense coding\textsuperscript{[8]}, teleportation\textsuperscript{[9]}, etc.

Quantum entanglement is a key resource that is crucial in the efficient and fast modeling of many-body quantum systems on quantum computers\textsuperscript{[3, 10, 12]}. Due to this feature quantum computers are much more efficient than classical ones for studying different problems relating to the behavior of quantum systems in various fields, including condensed-matter physics, high-energy physics, atomic physics, and quantum chemistry.

In recent years, physical implementation of quantum computers on superconducting circuits have achieved significant progress. The IBM company has developed a cloud service called the IBM Q Experience\textsuperscript{[13]}. It has access to different quantum devices based on processors containing from 1 up to 20 superconducting qubits. Recently, it was shown that the 16-qubit\textsuperscript{[19]} and 20-qubit\textsuperscript{[15]} IBM Q quantum processors can be fully entangled. The authors made the quantum tomography on each pair of connected qubits prepared in the graph states, then calculated the negativity as a measure of entanglement between them. They obtain that the state is inseparable with respect to any fixed pair. Also, it was prepared and measured the entangled state on other systems with full qubit control, namely, 20-qubit ion trap system\textsuperscript{[12, 17]}, system of photons\textsuperscript{[18, 20]} and superconducting system\textsuperscript{[21, 22]}

In this paper we propose the protocol for measuring the entanglement of one qubit with other system on quantum computers. For this purpose in the case of pure state we use the definition of the geometric measure of entanglement by the mean value of spin obtained in paper\textsuperscript{[23]}. We test this protocol on the ibmq-ourense quantum device\textsuperscript{[13]}. So, the entanglement of the Schrödinger cat and Werner states are defined. Also we propose and test the protocol to determine the entanglement of rank-2 mixed quantum states.

PROTOCOL FOR MEASURING THE ENTANGLEMENT OF A PURE STATE BY THE MEAN VALUE OF SPIN

We propose the method which allows to measure the entanglement of the prepared state on quantum computer. For this purpose we use the definition of the geometric measure of entanglement by the mean value of spin obtained in paper\textsuperscript{[23]}. So, if we have the spin which can be entangled with the rest of the system in a pure state

\[
|\psi\rangle = a|0\rangle|\phi_1\rangle + b|1\rangle|\phi_2\rangle
\]  

then the entanglement between these subsystems can be defined by the mean value of this spin as follows

\[
E\left(|\psi\rangle\right) = \frac{1}{2} \left(1 - |\langle\psi|\sigma|\psi\rangle|\right),
\]

where \(a\) and \(b\) are some complex constants which satisfy the normalization condition \(|a|^2 + |b|^2 = 1\); \(|\phi_1\rangle\) and \(|\phi_2\rangle\) are state vectors which define the quantum system entangled with spin, and which satisfy the normalization conditions \(|\langle\phi_1|\phi_1\rangle = 1, \langle\phi_2|\phi_2\rangle = 1\); modulus of the mean value of a qubit is determined by the expression

\[
|\langle\psi|\sigma|\psi\rangle| = \sqrt{\langle\psi|\sigma|\psi\rangle^2},
\]

and operator \(\sigma\) is defined by the Pauli matrices as follows \(\sigma = i\sigma^x + j\sigma^y + k\sigma^z\). Note that in a general case functions \(|\phi_1\rangle\) and \(|\phi_2\rangle\) are not orthogonal \(|\langle\phi_1|\phi_2\rangle \neq 1\). Since any two-level quantum system is described by the Pauli matrices, expression\textsuperscript{[23]} can be applied to the determination of the value of entanglement of any quantum system that consists of a set of two-level systems. Thus, to determine the entanglement of one qubit with the rest of the system the mean value of
In a standard basis, we represent them as follows.

So, to measure the remaining mean values of the spin of measure of qubit on a standard basis in the form

\[ \langle \psi | \sigma^z | \psi \rangle = |\langle \psi | 0 \rangle|^2 - |\langle \psi | 1 \rangle|^2. \]  

(3)

So, to measure the remaining mean values of the spin in a standard basis, we represent them as follows

\[ \sigma^z = e^{-i \frac{\pi}{2} e^{i \frac{\pi}{4}}} |0\rangle \langle 0| e^{i \frac{\pi}{4}} - |1\rangle \langle 1|. \]  

In turn, this allows us to express the mean value of this operator in state (1) by the probabilities which define the result of measure of qubit on a standard basis in the form

\[ \langle \psi | \sigma^z | \psi \rangle = |\langle \psi | 0 \rangle|^2 - |\langle \psi | 1 \rangle|^2. \]  

(3)

Then, the mean value of the Pauli operator corresponding to this qubit in state (1) can be measured. It is convenient to use a standard basis \(|0\rangle, |1\rangle\) for measuring certain value of this operator. For instance, the tomography process of the IBM quantum computer is provided by the measurements of a state on a standard basis. For this purpose, we represent the mean values of the Pauli operators of a qubit by the values which can be measured using a standard basis.

This state can be prepared by applying to the initial state \(|\psi\rangle\) the matrix representation of the single-qubit gate \( U_{3}(\theta, \phi, \lambda) \) has the following form

\[ U_{3}(\theta, \phi, \lambda) = \begin{pmatrix} \cos \frac{\theta}{2} & -e^{i \lambda} \sin \frac{\theta}{2} \\ e^{i \phi} \sin \frac{\theta}{2} & e^{i (\lambda + \phi)} \sin \frac{\theta}{2} \end{pmatrix}, \]  

(7)

where \( \lambda \) is a real parameter which can take the values from the range \([0, 2\pi]\). This gate has the effect of rotating a qubit in the initial state \(|0\rangle\) to an arbitrary one-qubit state

\[ U_{3}(\theta, \phi, \lambda)|0\rangle = \cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} e^{i \phi} |1\rangle. \]  

(8)

The entanglement of any qubit with the remaining system in state (4) according to (2) is

\[ E(|\psi_{cat}\rangle) = \frac{1}{2} (1 - |\cos \theta|). \]  

(9)

In this section we use the protocol proposed above to measure the entanglement of states prepared on the ibmq-ourense quantum computer. This is one of the quantum devices to which the IBM company provides free access via its cloud service. This device consists of five superconducting qubits in the way shown in Fig. 2.

So, we prepare some pure states, namely, the Schrödinger cat states and the Werner-like states, and apply to them the protocol considered in the previous section.

**Schrödinger cat states**

In general, the Schrödinger cat state for \(n\) qubits reads

\[ |\psi_{cat}\rangle = \cos \frac{\theta}{2} |00\ldots 0\rangle + \sin \frac{\theta}{2} e^{i \phi} |11\ldots 1\rangle, \]  

(6)

where \( \theta \in [0, \pi] \) and \( \phi \in [0, 2\pi] \) are some real parameters. This state can be prepared by applying to the initial state \(|00\ldots 0\rangle\) the single-qubit gate \( U_{3}(\theta, \phi, \lambda) \) and a sequence of controlled-NOT operators as it is shown in Fig. 3. In the standard basis \(|0\rangle, |1\rangle\) the matrix representation of \( U_{3}(\theta, \phi, \lambda) \) gate has the following form

\[ U_{3}(\theta, \phi, \lambda) = \begin{pmatrix} \cos \frac{\theta}{2} & -e^{i \lambda} \sin \frac{\theta}{2} \\ e^{i \phi} \sin \frac{\theta}{2} & e^{i (\lambda + \phi)} \sin \frac{\theta}{2} \end{pmatrix}, \]  

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(8)

On the ibmq-ourense quantum device we measure the dependence of the geometry measure of entanglement on...
the parameter $\theta$ for two-qubit Schrödinger cat state with $\phi = 0$. The results are obtained for two qubits (Fig. 3). In Figs. 4a and 4b the circuits and the dependences of entanglement in the case of measuring of first $q[0]$ and second $q[1]$ qubits, respectively, are presented. These results are slightly different. So, the values of entanglement obtained measuring the $q[1]$ qubit are a little bit worse than obtained measuring the $q[0]$ qubit. This behavior may be caused by the fact that the $q[1]$ qubit is in contact with two other qubits, but the $q[0]$ qubit interacts with only one qubit (see Fig. 3). Thus, the state of $q[1]$ qubit is more destroyed by the environment than the $q[0]$ one.

Also we prepare and measure the entanglement of the 3- and 4-qubit Schrödinger cat states (Fig. 5). Despite the fact that in these cases we have more qubits in the system, the results obtained by the quantum computer are in good agreement with the theoretical ones. This is because that the gates that generate the Schrödinger cat state are basis operators of the ibmq-ourense quantum device. This fact allows to reduce the error of preparation of this state.

**Werner state**

By this point we study the entanglement of symmetric states in context that the measure of entanglement by the mean value of spin are the same with respect to any qubit from the system. Now let us study on the quantum computer the entanglement of the Werner-like state

$$|\psi_W\rangle = \sin \frac{\theta}{2} |001\rangle + \frac{1}{\sqrt{2}} \cos \frac{\theta}{2} (|010\rangle + |100\rangle), \quad (10)$$

which is not symmetrical. In order to prepare this state, one should additionally use the Hadamard gate ($H$), the $R_\pi$ gate which provides a single-qubit rotation around the $x$-axis by the angle $\pi$, the controlled-Hadamard gate ($cH$) which performs an $H$ on the target qubit whenever the control qubit is in state $|1\rangle$, and the Toffoli gate. The circuit of the preparation of state $|\psi_W\rangle$ on the $q[1]$, $q[3]$ and $q[4]$ qubits is presented in Fig. 6.

Due to asymmetric nature of state $|\psi_W\rangle$ the expression which defines the entanglement of the first or second qubit with other qubits is different from the expression which defines the entanglement of the third qubit with the first two. This expressions have the following form

$$E_{1,2} (|\psi_W\rangle) = \frac{1}{2} \left( 1 - \sin^2 \frac{\theta}{2} \right),$$

$$E_3 (|\psi_W\rangle) = \frac{1}{2} \left( 1 - |\cos \theta| \right). \quad (11)$$
The results of measuring the values of entanglement of each qubit with the two other by the ibmq-ourense quantum device are presented in Fig. 7. Due to the greater influence of the environment on the q[1] qubit then on the q[3] qubit and in turn on q[4] once the dependence in Fig. 7 is closer to theoretical than in Figs. 11 and 12 respectively. Also, the number of operators which should be applied to the system to prepare the Werner state reduces the accuracy of achievement of this state which in turn affects the value of entanglement.

Figure 7: The dependence of value of entanglement of Werner state \( \| \) on parameter \( \theta \) in the cases of measuring the mean values of q[1] (a), q[3] (b) and q[4] (c) spins.

MEASURING THE ENTANGLEMENT OF MIXED STATES PREPARED ON THE IBMQ-OURENSE QUANTUM COMPUTER

In paper [2] the geometry measure of entanglement of rank-2 mixed states by correlations between qubits was studied. The authors obtained the expression which allows to calculate the entanglement of any qubit with other in the mixed state defined by density matrix \( \rho = \sum_\alpha \omega_\alpha | \psi_\alpha \rangle \langle \psi_\alpha | \), where \( | \psi_\alpha \rangle \) are given on the subspace spanned by vectors \( | 0 \rangle = | 00 \ldots 0 \rangle, | 1 \rangle = | 11 \ldots 1 \rangle \), and \( \sum_\alpha \omega_\alpha = 1 \). In the case of an N-qubit state the value of entanglement of ith qubit with other is determined by the expression

\[
E(\rho) = \frac{1}{2} \left( 1 - \sqrt{1 - (\Sigma^x)^2 - (\Sigma^y)^2} \right). \tag{12}
\]

The operators \( \Sigma^x = \sigma_1^x \sigma_2^x \ldots \sigma_i^x \ldots \sigma_N^x, \Sigma^y = \sigma_1^y \sigma_2^y \ldots \sigma_i^y \ldots \sigma_N^y \), \( I_1 I_2 \ldots I_j \ldots I_N \) are the analogs of the Pauli operators acting on the subspace spanned by \( | 0 \rangle, | 1 \rangle \). Here \( I_j \) is unity single-qubit operator and \( \sigma_j^z \) is dependent on the qubit number \( j \). Note that in the similar way the analog of the Pauli operators for any two-dimensional Hilbert subspace can be introduced. Then the value of entanglement of a 2-rank mixed state is defined by expression [2] with the Pauli operators defined on this subspace.

\[
\langle \sigma_i^a \sigma_j^b \rangle = \sum_\alpha \omega_\alpha | \psi_\alpha \rangle \langle \sigma_i^a \sigma_j^b | \psi_\alpha \rangle,
\]

where \( a, b = x, y \). This expression contains the correlation functions constructed on pure states \( | \psi_\alpha \rangle \). These functions can be expressed by the probabilities measured on the experiment as follows

\[
\langle \psi_\alpha \rangle | \sigma_i^a \sigma_j^b | \psi_\alpha \rangle = | \langle \sigma_i^a \sigma_j^b \rangle |^2 - | \langle \sigma_i^a \rangle |^2 - | \langle \sigma_j^b \rangle |^2 + | \langle \sigma_i^a \sigma_j^b \rangle |^2 + | \langle \sigma_i^a \sigma_j^b \rangle |^2,
\]

\[
\langle \psi_\alpha \rangle | \sigma_i^a \sigma_j^b \sigma_k^c \rangle | \psi_\alpha \rangle = | \langle \sigma_i^a \sigma_j^b \sigma_k^c \rangle |^2 - | \langle \sigma_i^a \rangle |^2 - | \langle \sigma_j^b \rangle |^2 - | \langle \sigma_k^c \rangle |^2 + | \langle \sigma_i^a \sigma_j^b \sigma_k^c \rangle |^2 + | \langle \sigma_i^a \sigma_j^b \sigma_k^c \rangle |^2,
\]

where \( | \langle \sigma_i^a \rangle |^2 = e^{i2\pi} | \langle \sigma_i^a \sigma_j^b \rangle |^2 \). As an example, we consider the mixed state which consists of two Bell states \( | \Phi^+ \rangle = 1/\sqrt{2} (| 00 \rangle \pm | 11 \rangle) \) as follows

\[
\rho_{Bell} = \omega | \Phi^+ \rangle \langle \Phi^+ | + (1 - \omega) | \Phi^- \rangle \langle \Phi^- |\tag{15}
\]

where \( \omega \in [0, 1] \). The exact expression of geometric measure of entanglement of this system according to [12] is

\[
E(\rho_{Bell}) = \frac{1}{2} \left( 1 - 2\sqrt{\omega(1 - \omega)} \right). \tag{16}
\]

So, on the quantum device ibmq-ourense we separately prepare and measure the correlations of the \( | \Phi^+ \rangle \) (Fig. 8a) and \( | \Phi^- \rangle \) (Fig. 8b). Note that the number of measurements of pure states should be made according to their weights defined by \( \omega \). Namely, to find the entanglement of state [10] it should be measured the states \( | \Phi^+ \rangle \) and \( | \Phi^- \rangle \) as shown in Fig. 8a and Fig. 8b respectively. For instance, in the case of the maximally mixed state \( (\omega = 1/2) \) the numbers of measurements are the same for both states. The exact behavior [10] and behavior of entanglement obtained by quantum computer.

Figure 8: In Figs. (a) and (b) the circuits for the preparation and measurement of the correlations of \( | \Phi^+ \rangle \) (a) and \( | \Phi^- \rangle \) (b) Bell states are presented. Here operators \( R_\theta \) provide the rotations of the qubit state around the \( a = x, y \) axes by the angle \( \pi/2 \).
with different value of $\omega$ are presented in Fig. 9a. The deviation between theoretical $E$ and measured by quantum computer $E_i$ values of entanglement $\Delta = |E_i - E|$ for certain $\omega$ is depicted in Fig. 9a. As we can see, for more mixed states the entanglement is determined more precisely. It should be noted that the same trend is observed when determining the relative deviation $\delta = |E_i - E|/E$. In our opinion, this is because the maximally mixed state ($\omega = 1/2$) state [15] which has the form $1/2(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)$ is spanned by the basis vectors $|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle$. In turn, these vectors are the basis states on which the quantum computer makes the measurements. Therefore, the closer the state is to maximally mixed state, the more accurate it is measured.

![Figure 9](image)

**CONCLUSIONS**

We have proposed the protocol to determine the value of entanglement between qubit and the rest of the system prepared on a quantum computer in the pure quantum state. This protocol is based on the calculation of the mean value of spin. We have applied this protocol to states, namely, the Schrödinger cat and Werner-like states, prepared by the ibmq-ourense quantum computer. In the case of the Schrödinger cat states the results obtained on a quantum computer are in good agreement with the theoretical ones. The error increases a little bit with respect to the number of qubits in the system. This is because the gates that generate the Schrödinger cat state are basis operators of the ibmq-ourense quantum device. Another situation we have in the case of three-qubit Werner states, where its preparation requires additional operators. This leads to an accumulation of errors and worse agreement with theoretical predictions. It should be noted that the measurement of only one spin of the system allows to minimize the measurement error. This fact allows to determined the entanglement of pure state in good agreement with the theory. Also we have generalized the protocol to determine the value of entanglement of a rank-2 mixed states. In this case the correlations of all spins of the system should be measured. As an example, we have obtained on the ibmq-ourense quantum computer the value of entanglement of the mixed state which consists of two Bell states $|\Phi^\pm\rangle = 1/\sqrt{2}(|00\rangle \pm |11\rangle)$ (see Fig. 9a). In this case we also obtained good agreement with the theoretical prediction. For more mixed states we obtain the smaller deviation from the theory. So, the more mixed states are measured more accurately. This is because that maximally mixed state [15] is spanned by the basis vectors of quantum computer that reduces measurement error.

**ACKNOWLEDGEMENTS**

The authors thank Dr. Andrij Rovenchak for useful comments. This work was supported by Project FF-83F (No. 0119U002203) from the Ministry of Education and Science of Ukraine.