ON \( W_2 \)-LIFTING OF FROBENIUS OF ALGEBRAIC SURFACES

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Abstract. We completely decide which minimal algebraic surfaces in positive characteristics allow a lifting of their Frobenius over the truncated Witt rings of length 2.

1. Introduction

To have Frobenius morphism makes schemes in characteristic \( p \) distinct from those in characteristic 0. For a scheme in characteristic \( p \) of dimension \( r \), the Frobenius morphism defined on it is a finite morphism of degree \( p^r \). However, if the base field is of characteristic 0, Beauville proved in [2] that a smooth hypersurface in \( \mathbb{P}^m \) of degree \( d \) has no endomorphism of degree larger than 1 provided \( m, d \geq 3 \).

One can show easily Beauville’s result combined with Zariski’s main theorem implies the following corollary. Suppose we are given a smooth hypersurface in mixed characteristic satisfying the numerical conditions above, then the Frobenius morphism on the special fiber cannot be lifted to a finite endomorphism of the whole hypersurface. In spite of this implication, one can ask further whether the Frobenius morphism can be lifted to the truncated Witt rings. To be more precise, let \( X \) be a projective smooth hypersurface over an algebraically closed field \( k \) with positive characteristic and \( n \geq 2 \), is there a flat lifting \( X_n \) of \( X \) over \( W_n(k) \) and a morphism \( F_{X_n} : X_n \to X_n \) such that the restriction of \( F_{X_n} \) to \( X \) coincides with \( F_X \)?

It is a general belief that such liftings rarely exist, however it is difficult to verify such judgement for a given variety even when \( n = 2 \). If \( X \) a projective smooth curve, it is easy to prove the liftability of \( F_X \) over \( W_2(k) \) implies the genus of \( X \) cannot be \( \geq 2 \). A similar discussion given in [10] shows that there exist no liftings of Frobenius to mixed characteristic for a variety with Kodaira dimension \( \geq 1 \). For varieties with negative Kodaira dimension, the only known proved cases seem to be certain classes of flag varieties [7] based on Bott non-vanishing theorems for these varieties. It is noticeable that the result in [7] strengthens earlier results in [21], where such flag varieties over \( p \)-adic numbers are proved to have no endomorphisms lifting the Frobenius. On the positive side, the only known varieties with liftable Frobenius seem to be ordinary varieties with trivial tangent bundles [19] and toric varieties [7].

It is the aim of this paper to investigate this problem for minimal algebraic surfaces. We obtain a complete answer to the above question for \( n = 2 \) in this situation.

Theorem 1. Let \( X \) be a smooth projective minimal algebraic surface over a field \( k \) of characteristic \( p \), then the Frobenius of \( X \) is liftable over \( W_2(k) \) iff \( X \) belongs to one of the following classes.

(1) \( \kappa(X) = 0 \)
   (a) Ordinary abelian surfaces,
   (b) Ordinary hyperelliptic surfaces of type a), b), c), or d) if \( p \neq 2, 3 \) and type a) if \( p = 2 or 3 \) such that \( \omega_{X/k}^{(p-1)} \cong \mathcal{O}_X \).

(2) \( \kappa(X) = -1 \)
   (a) \( \mathbb{P}^2 \), \( \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)), n \geq 0, n \neq 1 \).
   (b) Ruled surfaces over an ordinary elliptic curve \( C \).

The proof of nonexistence of liftings of Frobenius for those minimal surfaces not appearing in the above list is done by contradiction, which is drawn out from a morphism induced by a lifted Frobenius, see Proposition 2.8. The positive part of this theorem is obtained either by proving the vanishing of the obstruction space (1)(b), or constructing the liftings directly (2)(b).

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In the preliminary section we give some definitions and prove some lemma which are necessary for later development. The contents of the remaining section gives the proof our main result. The division of this section is partially based on the classification of minimal algebraic surfaces in characteristic $p$, see for instance [1, Appendix A].

Conventions and Notations.

$k$: an algebraically closed field of characteristic $p > 0$.
$W(k)$: the Witt ring with coefficients in $k$.
$W_n(k)$: the truncated Witt ring of length $n$ with coefficients in $k$.
$\kappa(X)$: the Kodaira dimension of $X$.
$h^i(F) = \dim H^i(X, F)$.

2. Preliminaries

The main objective of this section is to fix terminologies and prove several results, among which Proposition 2.8 will be used later as an obstruction for the existence of liftings of Frobenius. Along the way, we will also review some facts on deformation theory.

Definition 2.1. Let $X$ be a smooth variety defined over $k$, a flat lifting of $X$ to $W_n(k)$ (resp. $W(k)$) is a scheme $\tilde{X}$ which is flat over $W_n(k)$ (resp. $W(k)$) such that $\tilde{X} \times_{W_n(k)} k \cong X$ (resp. $\tilde{X} \times_{W(k)} k \cong X$). $X$ is said to have a $W_n$-lifting (resp. $W(k)$-lifting or a lifting to characteristic 0) of its Frobenius, if there exists a flat lifting $\tilde{X}$ of $X$ to $W_n(k)$ (resp. $W(k)$) and a morphism $F_\tilde{X} : \tilde{X} \to \tilde{X}$ such that the following diagram is commutative

$$
\begin{array}{ccc}
X & \xrightarrow{i} & \tilde{X} \\
F_\tilde{X} \downarrow & & \downarrow F_{\tilde{X}} \\
X & \xrightarrow{i} & \tilde{X}
\end{array}
$$

where $i : X \to \tilde{X}$ is the closed immersion defined by the ideal $(p)$.

As the above definition manifests, to study liftability of Frobenius, the deformation theory of schemes and morphisms over local artin rings is an indispensable tool. Related results can be found in [13, Theorem 5.9] and we reproduce it here for convenience.

Proposition 2.2. Let $X$ be a $S$-scheme and $j : X_0 \to X$ be a closed immersion defined by an ideal $J$ such that $J^2 = 0$. Let $Y$ be a smooth $S$-scheme and $g : X_0 \to Y$ be an $S$-morphism. There is an obstruction $o(g, j) \in H^1(X_0, J \otimes_{\mathcal{O}_{X_0}} g^* T_{Y/S})$ whose vanishing is necessary and sufficient for the existence of an $S$-morphism $h : X \to Y$ extending $g$, i.e. such that $hj = g$. When $o(g, j) = 0$, the set of extensions $h$ of $g$ is an affine space under $H^0(X_0, J \otimes_{\mathcal{O}_{X_0}} g^* T_{Y/S})$.

Let $S_0 \to S$ be a closed immersion defined by an ideal $I$ of square 0. Let $X_0$ be an $S_0$-scheme, then there is an obstruction $o(X_0, i) \in H^2(X_0, f_0^* I \otimes T_{X_0/S_0})$ (where $f_0 : X_0 \to S_0$ is the structure morphism) whose vanishing is the sufficient and necessary condition for the existence of a deformation of $X_0$ over $S$. When $o(X_0, i) = 0$, the set of isomorphic classes of such deformations is an affine space under $H^1(X_0, f_0^* I \otimes T_{X_0/S_0})$.

Corollary 2.3. Let $X, Y$ be smooth varieties over $k$ such that $Y$ is an étale cover of $X$. If $F_X$ is liftable to $W_2(k)$, then $F_Y$ is liftable to $W_2(k)$.

Proof. Let $X$ be a lifting of $X$ to $W_2(k)$ such that $F_X$ is liftable to $X$. By the second part of the above proposition, there exists a lifting $Y \to X$ of $Y$ as an $X$-scheme. In particular, $Y$ is liftable to $W_2(k)$. To find a lifting of $F_Y$ to $Y$, it suffices to find a lifting of the relative Frobenius $F_{Y/X} : Y \to Y \times_X X$ to a morphism $Y \to Y \times_X X$. By the first part of the above proposition, the obstruction space for such lifting is 0.

The following useful lemma is self-evident and we omit its proof.
Lemma 1. Let $M$ be a flat $\mathbb{Z}/p^2\mathbb{Z}$-module, then the ‘multiplication by $p$’ map on $M$ defines an isomorphism $M/pM \xrightarrow{p} pM$.

The following corollary can be deduced from proposition 2.2 directly and we omit its proof.

Corollary 2.4. Let $A$ be commutative $k$-algebra, $A$ be a flat lifting of $A$ over $W_2(k)$. Then for any two liftings $F_A$, $F_A'$ of $F_A$, we have $F_A'(a) = F_A(a) + p\eta(a)$, where $\eta: A \rightarrow A$ is a unique function satisfying

\begin{equation}
\begin{aligned}
(1) \quad \eta(a_1 + a_2) &= \eta(a_1) + \eta(a_2) \\
(2) \quad \eta(a_1a_2) &= a_1^2\eta(a_2) + a_2^2\eta(a_1)
\end{aligned}
\end{equation}

The following result will be useful later.

Corollary 2.5. Let $A$ be a commutative $k$-algebra, and $X = \mathbb{P}^1_A$. Then to give a lifting of $F_X$ over $W_2(k)$ is equivalent to give a lifting of $F_A$ over $W_2(k)$ together with a polynomial with coefficients in $A$ of degree $\leq 2p$.

Proof. By the second part of Proposition 2.2, a flat lifting of $X$ over $W_2(k)$ is unique up to isomorphism. In particular, let $A$ be a flat lifting of $A$ over $W_2(k)$, it suffices to prove the corollary for all lifting of $F_X$ to $X = \mathbb{P}^1_A$. For any lifting of $F_X$, there is an induced endomorphism of $\Gamma(X, \mathcal{O}_X) \cong A$ hence a lifting of $F_A$. On the other hand, let $U = \text{Spec } A[x], V = \text{Spec } A[y]$ be two open subschemes of $X$ glued by the relation $xy = 1$. Then a lifting $F_X$ induces a lifting on $U$, on which the image of $x$ under $F_X$ can be written as $x^p + pf$ with $f \in A[x]$. In order that this morphism can be extended to $V$, one checks easily deg $f \leq 2p$. 

In [19] Appendix Proposition 1], the authors give another version of obstruction for liftings of Frobenius. Now we recall their results.

Proposition 2.6. Let $X$ be a smooth variety, $(X_n, F_{X_n})/W_{n+1}(k)$ a lifting of $X$ to $W_{n+1}(k)$ together with a lifting of Frobenius. Then the obstruction to the existence of a pair $(X_{n+1}, F_{X_{n+1}})$ over $W_{n+2}(k)$ consisting of a lifting $X_{n+1}$ of $X_n$ and a lifting of Frobenius $F_{X_{n+1}}$ such that $F_{X_{n+1}}|_{X_n} \cong F_{X_n}$ is given by a class in $H^1(X, T_X \otimes B^1_X)$, where $B^1_X := F_X^*\mathcal{O}_X/\mathcal{O}_X$. The various liftings form a principal homogeneous space under $H^0(X, T_X \otimes B^1_X)$.

Note that the obstruction space above takes into consideration of pairs $(X_{n+1}, F_{X_{n+1}})$ up to isomorphism, which is different from Proposition 2.2.

The liftability of Frobenius is a fairly strong property for an algebraic variety, one can show [13] Proposition 8.6] a proper smooth variety with liftable Frobenius is ordinary in the following sense.

Definition 2.7. Let $X$ be a proper, smooth variety over $k$, $X$ is said to be ordinary if $H^j(X, B^i_X) = 0$ for all $(i, j)$, where $B^i_X = \text{d}\Omega^i_{X/k}$ is the $i$-th coboundary of the de Rham complex $\Omega^i_{X/k}$.

Next we prove the main result of this section, which serves as an obstruction in disproving the existence of $W_2$-lifting of Frobenius for most surfaces.

Let $X$ be a smooth variety over $k$, $X$ be a flat lifting of $X$ over $W_2(k)$ and $F_X$ be a lifting of the Frobenius of $X$ to $X$. Then we will have an induced morphism

$$dF_X : F_X^*\Omega^1_{X/W_2(k)} \to \Omega^1_{X/W_2(k)}.$$ 

Since the reduction of the above morphism modulo $p$ is 0 and the sheaf $\Omega^1_{X/W_2(k)}$ is $W_2(k)$-flat, $dF_X$ factors through $p\Omega^1_{X/W_2(k)}$. By lemma 1 we have an isomorphism of $\mathcal{O}_X$-modules $p\Omega^1_{X/W_2(k)} \cong \Omega^1_{X/k}$. Thus $dF_X$ induces an $\mathcal{O}_X$-linear morphism

$$\varphi_{F_X} : F_X^*\Omega^1_{X/k} \to \Omega^1_{X/k}.$$ 

Proposition 2.8. Let $X$ be a smooth variety over $k$ admitting a $W_2$-lifting of its Frobenius, then the morphism $\varphi_{F_X}$ is generically bijective.
Proof. It is easy to see the claim in the proposition can be reduced to the affine case and it suffices to prove the determinant of $\varphi_{F_X}$ is nonzero. To be more precise, let $X = \text{Spec } A$, $X \to \mathbb{A}^n$ be an étale cover and $\{d t_i\}_{1 \leq i \leq n}$ be a basis of $\Omega^1_{X/k}$ with $t_i \in A$. Then to give a flat lifting of $X$ over $W_2(k)$ is equivalent to give a flat $W_2(k)$-algebra $A$ such that $A/pA \cong A$. Now we choose a set of liftings $t_i \in A$ of $t_i$, $1 \leq i \leq n$, then for any lifting $F_X$ of $F_X$ to $X = \text{Spec } A$ we can find $f_i \in A$ such that

$$F_X(t_i) = t_i^p + pf_i.$$  

Then the morphism $\varphi_{F_X}$ with respect to the basis $\{d t_1, \ldots, d t_n\}$ is given by the matrix

$$(2.2) \text{Diag}(t_1^{p-1}, \ldots, t_n^{p-1}) + \left(\frac{\partial f_i}{\partial t_j}\right),$$

where $f_i$ is the reduction of $f_i$ modulo $p$. In order to prove the determinant of the above matrix is nonzero, we need th following lemma.

Lemma 2. Let $A$ be a regular local ring over $k$ and $\hat{A}$ be the completion of $A$ with respect to its maximal ideal. Then for any derivation $D \in \text{Der}_k(A, A)$, we can find $\hat{D} \in \text{Der}_k(\hat{A}, \hat{A})$ such that $\rho(D(a)) = \hat{D}(\rho(a))$, where $a \in A$ and $\rho : A \to \hat{A}$ is the natural inclusion.

Proof. Let $d : A \to \Omega^1_{A/k}$ be the Kähler differential, by [16] Definition 11.4, 12.2, Proposition 12.4 there exists a universally finite $\rho$-extension of $d$. Moreover, the universally finite module of differentials is isomorphic to $\Omega^1_{A/k} \otimes_A \hat{A}$ and the differential $\hat{d} : \hat{A} \to \Omega^1_{A/k} \otimes_A \hat{A}$ is nothing but taking differentials term-by-term then summing up the results. To be more explicit, we have the following commutative diagram

$$\begin{array}{ccc}
A & \xrightarrow{\rho} & \hat{A} \\
\downarrow{d} & & \downarrow{d} \\
\Omega_{A/k} & \xrightarrow{d} & \Omega_{A/k} \otimes_A \hat{A}
\end{array}$$

By the universal property for the pair $(\Omega_{A/k} \otimes_A \hat{A}, \hat{d})$ [16] Definition 11.4 (b), one can see easily the derivation $\hat{D} = D \otimes \text{id}_\hat{A}$ satisfies the requirement in the lemma. 

Therefore, if we denote by $\hat{f}_i$ the image of $f_i$ in $\hat{A}$ and $\hat{\partial}_j \in \text{Der}_k(\hat{A}, \hat{A})$ be derivation associated to $\frac{\partial}{\partial t_j}$ as in lemma[2] then the image of the determinant of the matrix $(2.2)$ in $\hat{A}$ under the inclusion $\rho$ is nothing but

$$(2.3) \det \left(\text{Diag}(t_1^{p-1}, \ldots, t_n^{p-1}) + (\hat{\partial}_j(\hat{f}_i))\right).$$

To prove the proposition, it suffices to show the coefficient of the monomial $t_1^{p-1} \cdots t_n^{p-1}$ in the formal power series $(2.3)$ is 1. Note that the determinant $(2.3)$ is equal to

$$\sum_{1 \leq N \leq n} \sum_{i_1 < \cdots < i_N} t_1^{p-1} \cdots t_n^{p-1} \det M_{i_1, \ldots, i_N},$$

where $M_{i_1, \ldots, i_N}$ is the minor of $(\hat{\partial}_j(\hat{f}_i))$ obtained by deleting the $i_1$-th, \ldots, $i_N$-th rows and columns. Therefore, it suffices to prove for any $m$ power series $\hat{f}_i \in k[[t_1, \ldots, t_n]]$, $1 \leq i \leq m$, $1 \leq m \leq n$, the coefficient of $t_1^{p-1} \cdots t_m^{p-1}$ in $\det(\hat{\partial}_j(\hat{f}_i))$ is 0, where $1 \leq j \leq m$.

Since $\hat{f}_i \in k[[t_1, \ldots, t_n]]$, it can be written as (might not be uniquely)

$$(2.4) \sum_{0 \leq s_1, \ldots, s_n \leq p-1} a_{i_1, \ldots, s_n} t_1^{s_1} \cdots t_n^{s_n} + \sum_{1 \leq i \leq n} t_i^{p} g_i, \quad a_{i_1, \ldots, s_n} \in k, \quad g_i \in k[[t_1, \ldots, t_n]].$$

Note that the terms $\hat{\partial}_j(t_i^{p} g_i)$, $1 \leq i, j \leq m$, $1 \leq s \leq n$ have no contribution in computing the coefficient of $t_1^{p-1} \cdots t_m^{p-1}$ in $\det(\hat{\partial}_j(\hat{f}_i))$, we may just assume $g_i = 0$, for all $1 \leq i \leq m, 1 \leq s \leq n$ in proving the claim above.

By the multilinearity of determinant, we are reduced to prove the following claim.
Let \( f_i = t_1^{k_{i1}} \cdots t_{n}^{k_{in}}, 1 \leq i \leq n \) such that \( 0 \leq k_{i1}, \ldots, k_{in} \leq p - 1 \), then the coefficient of \( t_1^{p-1} \cdots t_{n}^{p-1} \) in the determinant of \( \left( \frac{\partial f_i}{\partial t_j} \right) \) is 0.

By assumption, one checks easily

\[
\det \left( \frac{\partial f_j}{\partial t_i} \right) = \det \left( k_{ij} \right) n \prod_{j=1}^{n} t_j^{s_j}, \quad s_j = -1 + \sum_{i=1}^{n} k_{ij}.
\]

Let \( s_j = p - 1 \) for \( 1 \leq j \leq n \), then \( \sum_{i=1}^{n} k_{ij} = p \) consequently \( \det(k_{ij}) = 0 \). Thus the claim above hence the proposition is proved.

\[\square\]

### 3. Proof of the Theorem

#### Surfaces with \( \kappa \geq 1 \).

**Proposition 3.1.** Let \( X \) be a proper smooth variety over \( k \) such that \( \kappa(X) \geq 1 \), then \( F_X \) cannot be lifted to \( W_2(k) \).

**Proof.** Otherwise, by Proposition 2.8 we will have a generically bijective morphism \( \varphi_{F_X} : F_X^* \Omega^1_{X/k} \to \Omega^1_{X/k} \).

By taking determinant, we obtain an injective morphism \( \wedge \varphi_{F_X} : \omega_{X/k}^{(p)} \to \omega_{X/k} \). Iterating this morphism with its pullbacks via Frobenius, we get an injective morphism from \( \omega_{X/k}^{(p^n)} \to \omega_{X/k} \) hence a nonzero global section of \( \omega_{X/k}^{(1-p^n)} \) for all \( n \geq 1 \). This is impossible when \( \kappa(X) \geq 1 \), since a sufficiently high positive power of the canonical divisor is linearly equivalent to an effective divisor.

\[\square\]

**Corollary 3.3.** If \( X \) is a K3 surface, an Enriques surface or a quasi-hyperelliptic surface, then the Frobenius morphism \( F_X \) cannot be lifted to \( W_2(k) \).

#### Surfaces with \( \kappa = 0 \).

**K3 surface, Enriques surfaces and Quasi-hyperelliptic Surfaces.**

**Proposition 3.2.** If \( X \) has a torsion canonical line bundle and admits a lifting of \( F_X \) over \( W_2(k) \), then \( \varphi_{F_X} \) is an isomorphism. Moreover,

1. \( \Omega^1_{X/k} \) is étale trivializable,
2. \( X \) contains no rational curves.

**Proof.** Suppose we are given a lifting \( F_X \) of \( F_X \), then by Proposition 2.8 there is an induced generically bijective morphism \( \varphi_{F_X} : F_X^* \Omega^1_{X/k} \to \Omega^1_{X/k} \).

By taking determinant, we have an injective morphism \( \wedge \varphi_{F_X} : \omega_{X/k}^{(p)} \to \omega_{X/k} \). Iterating this morphism with its pullbacks via Frobenius, we get an injective morphism from \( \omega_{X/k}^{(p^n)} \to \omega_{X/k} \) hence a nonzero global section of \( \omega_{X/k}^{(1-p^n)} \) for all \( n \geq 1 \). This is impossible when \( \kappa(X) \geq 1 \), since a sufficiently high positive power of the canonical divisor is linearly equivalent to an effective divisor.

\[\square\]

**Corollary 3.3.** If \( X \) is a K3 surface, an Enriques surface or a quasi-hyperelliptic surface, then the Frobenius morphism \( F_X \) cannot be lifted to \( W_2(k) \).
Proof. Note that the canonical line bundles of all these surfaces are torsion, it suffices prove they do not satisfy one of the properties in Proposition [3, 2]. If $X$ is a K3 surface, it doesn’t satisfy (1) since it is simply connected and has nontrivial tangent bundle. By a recent result [3, Proposition 17], a K3 surface doesn’t satisfy property (2) either.

If $X$ is an Enriques surface in characteristic $p \neq 2$ or a singular Enriques surface in characteristic 2 [9, Theorem 2.7], then it has an étale cover by a K3 surface. If $X$ is a classical or supersingular Enriques surface in characteristic 2, then it is unirational by [3, Theorem 2]. Therefore, all Enriques surfaces do not satisfy property (2).

For all quasi-hyperelliptic surfaces, there exist a fibration by cuspidal rational curves [6, Proposition 5] hence they do not satisfy property (2).

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Hyperelliptic Surfaces.

A complete classification of hyperelliptic surfaces is given in the list [6, page 37], in which each surface is isomorphic to a quotient of the product of two elliptic curves by a finite subgroup scheme. Moreover, the canonical line bundle of a hyperelliptic surface is torsion of order equals to 1, 2, 3, 4 or 6 [6, page 37].

By Corollary 2.3, if $X$ admits a lifting of its Frobenius to $W_2(k)$, then the aforementioned elliptic curves are both ordinary since their Frobenius can be lifted to $W_2(k)$. Therefore, if $X$ is of type b) c) or d) in characteristic 2 or 3 then $F_X$ cannot be lifted, since one of the two elliptic curves is supersingular.

Lemma 3. Let $f : Y \to X$ be a Galois cover belonging to one of the following types

1. the Galois group is $\mathbb{Z}/2\mathbb{Z}$ and $p \neq 2, 2 | p - 1$,
2. the Galois group is $\mathbb{Z}/3\mathbb{Z}$ and $p \neq 3, 3 | p - 1$,
3. the Galois group is $\mathbb{Z}/4\mathbb{Z}$ and $p \neq 2, 4 | p - 1$.

Suppose $h^i(B_X^1) = 0$ for all $i$, then $h^i(B_X^1) = 0$ for all $i$.

Proof. It is easy to see we only need to treat the case when the cover is nontrivial. Suppose we are in case (1), then by [5, proposition 0.1.6, 0.1.8], we always have $f_*\mathcal{O}_Y \cong \mathcal{O}_X \oplus \mathcal{L}$, where $\mathcal{L}$ is a torsion line bundle on $X$ of order 2. Therefore,

$$ f_*F_Y^*\mathcal{O}_Y \cong F_X^*f_*\mathcal{O}_Y \cong F_X^*\mathcal{O}_X \oplus F_X^*\mathcal{L}. $$

We claim the images of sub line bundles $\mathcal{O}_X$ and $\mathcal{L}$ of $f_*\mathcal{O}_Y$ under the inclusion $f_*\mathcal{O}_Y \to f_*F_Y^*\mathcal{O}_Y$ fall in $F_X^*\mathcal{O}_X$ and $F_X^*\mathcal{L}$ respectively. Indeed, since $\mathcal{L}$ is nontrivial, we have $H^0(F_X^*\mathcal{L}) = 0$ hence there exists no nonzero morphism from $\mathcal{O}_X$ to $F_X^*\mathcal{L}$. On the other hand, let $\mathcal{L} \to F_X^*\mathcal{O}_X$ be a nonzero morphism, then by adjointness, we are given a nonzero morphism $F_X^*\mathcal{L} \to \mathcal{O}_X$. As $\mathcal{L}$ is odd, $F_X^*\mathcal{L} \cong \mathcal{L}$, thus we are given a nonzero morphism $\mathcal{L} \to \mathcal{O}_X$, again impossible. Therefore, the quotient $B_X^1 \cong F_X^*\mathcal{O}_X / \mathcal{O}_X$ is a direct summand of $f_*B_Y^1 = f_*F_Y^*\mathcal{O}_Y / f_*\mathcal{O}_Y$ hence the lemma is proved in this case.

The remaining cases can be proved similarly by using [20, Proposition 7.1] and [11, 7.2 Case 1]. It suffices to observe $f_*\mathcal{O}_Y$ decomposes into direct sum of pairwise non-isomorphic torsion line bundles with order a factor of 3 and 4 respectively.

\[ \]

Proposition 3.4. Let $X$ be a hyperelliptic surface of type b), c) or d) in characteristic $p \neq 2, 3$, then the Frobenius $F_X$ can be lifted to $W_2(k)$ if and only if

1. the associated elliptic curves $E_0, E_1$ are ordinary,
2. $\omega_{X/k}^{\otimes(p-1)} \cong \mathcal{O}_X$.

Proof. The necessity of condition (1) follows from Proposition 2.3 and necessity of condition (2) is contained in the proof of Proposition 3.2. Next we prove the sufficiency. By Proposition 2.6 it suffices to show the obstruction space $H^1(T_X \otimes B_X^1)$ is 0. Since the tangent bundle $T_X$ splits as $T_X \cong \mathcal{O}_X \oplus \omega_{X/k}^{-1}$ [17, page 495], thus we are reduced prove $h^1(B_X^1) = h^1(\omega_{X/k}^{-1} \otimes B_X^1) = 0$.

For any hyperelliptic surface of type b), c), or d), one can find an abelian surface or a hyperelliptic surface $Y$ and a cyclic Galois cover $Y \to X$ of order 2, 3, or 4, see [6, page 37]. Then by the orders of $\omega_{X/k}$ listed on [6, page 37] and condition (2) of this proposition, we have the divisibility conditions required in lemma 3. Therefore, the vanishing of $H^1(B_X^1)$ follows from lemma 3.
On the other hand, if $X$ is of type b), c) or d), by [17] Theorem 4.9] we have $h^1(\omega_{X/k}^{-1}) = h^2(\omega_{X/k}^{-1}) = 0$. By projection formula and condition (2) we obtain $h^1(F_X, \mathcal{O}_X \otimes \omega_{X/k}^{-1}) = h^1(F_X, \mathcal{O}_X \otimes \omega_{X/k}^{-1}) = h^1(\omega_{X/k}^{-1}) = 0$. Therefore, by studying the long exact sequence derived from

$$0 \to \omega_{X/k}^{-1} \to F_X, \mathcal{O}_X \otimes \omega_{X/k}^{-1} \to B_X^1 \otimes \omega_{X/k}^{-1} \to 0,$$

one can see easily $h^1(\omega_{X/k}^{-1} \otimes B_X^1) = 0$ hence the proposition is proved.

\[\square\]

**Proposition 3.5.** Let $X$ be a hyperelliptic surface of type a) in characteristic $p \neq 2$ such that the associated elliptic curves $E_0, E_1$ are ordinary, then the Frobenius $F_X$ can be lifted to $W_2(k)$.

**Proof.** By Proposition 2.6 it suffices to prove $H^1(T_X \otimes B_X) = 0$. In this case we still have $T_X \cong \mathcal{O}_X \oplus \omega_{X/k}$ and $\omega_{X/k}$ is of order 2 [18] Theorem 4.9]. Thus we are reduced to prove $h^1(B_X^1) = h^1(B_X^1 \otimes \omega_{X/k}) = 0$. By lemma 3 one can see easily $h^1(B_X^1) = 0$ in both cases a1) and a2).

Now we prove $h^1(B_X^1 \otimes \omega_{X/k}) = 0$. Since $\omega_{X/k}$ is of order 2 and $p$ is odd, we have $F_X, \mathcal{O}_X \otimes \omega_{X/k} \cong F_X, \mathcal{O}_X \otimes \omega_{X/k} \cong F_X, \omega_{X/k}$. Therefore, $B_X^1 \otimes \omega_{X/k} \cong F_X, \omega_{X/k}$. Let $Y = E_0 \times E_1$ and $f : Y \to X$ be the étale cover described on [6] page 37], if we can prove $F_X, \omega_{X/k} \otimes \omega_{X/k}$ is isomorphic to a direct summand of $f_*B_Y^1$ then we are done.

If $X$ is of type a1), then by [8] Proposition 0.1.3], we have

$$\omega_{\bar{Y}/k} \cong f^*(\omega_{X/k} \otimes L^{-1})$$

where $L$ is the line bundle such that $f_*\mathcal{O}_Y \cong \mathcal{O}_X \otimes L$. On the other hand since $f$ is étale, $\omega_{\bar{Y}/k} \cong f^*\omega_{X/k}$ hence $f^*L \cong f^*\omega_{X/k}$ is trivial. By [15] Theorem 2.1] we have $L$ is isomorphic either to $\mathcal{O}_X$ or $\omega_{X/k}$. Note the cover $f$ is nontrivial, hence $L$ is also nontrivial so $\omega_{X/k} \otimes \omega_{X/k} \cong L$. By the proof of lemma 3] we can see $F_X, L \otimes L$ is a direct summand of $f_*B_Y^1$, hence so is true for $F_X, \omega_{X/k} \otimes \omega_{X/k}$.

If $X$ is of type a2), let $Z = Y/Z/2Z$ and $g : Y \to Z$, $h : Z \to X$ be the intermediate étale covers such that $f = hg$. Moreover, let $g_*\mathcal{O}_Y \cong \mathcal{O}_Z \otimes L_Z$ and $h_*\mathcal{O}_Z \cong \mathcal{O}_X \otimes L_X$. Then by the proof for type a1) case, we have $L_Z$ satisfies $L_Z \cong h_*\omega_{X/k} \cong h^*\omega_{X/k}$. Therefore, we have the following decomposition

$$f_*\mathcal{O}_Y \cong \mathcal{O}_X \otimes L_X \otimes \omega_{X/k} \otimes \omega_{X/k} \otimes L_X$$

Moreover, by formula [32] $\omega_{X/k} \otimes \omega_{X/k} \cong L_X$ since $\omega_{X/k}$ is nontrivial. Therefore, the four line bundles on the right side are pairwise non-isomorphic. One the other hand, as these line bundles are of order 1 or 2, one can prove similarly as in lemma 3] that $F_X, \omega_{X/k} \otimes \omega_{X/k}$ is a direct summand of $F_X, f_*\mathcal{O}_Y / f_*\mathcal{O}_Y \cong f_*B_Y^1$.

\[\square\]

**Proposition 3.6.** Let $X$ be a hyperelliptic surface of type a) in characteristic 2, then $F_X$ is liftable to $W_2(k)$ if and only if $E_0, E_1$ are ordinary.

**Proof.** If $X$ is of type a1) and satisfies the remaining conditions in the proposition, then by [19] Theorem 2, Lemma 1.1] $X$ is ordinary. In particular $H^1(B_X^1) = 0$. Note that in this case the tangent bundle of $X$ is trivial, hence by Proposition 2.6] the Frobenius $F_X$ can be lifted to $W_2(k)$.

Now assume $X$ is of type a3), and the associated elliptic curves $E_0, E_1$ are ordinary. Let $Y = E_0 \times E_1/(Z/2Z)$, then $Y$ is a $\mu_2$-torsor over $X$ in the flat topology. By [3] Proposition 0.18], we have $f_*\mathcal{O}_Y \cong \mathcal{O}_X \otimes L$, where $L$ is nontrivial torsion line bundle of order 2. As in the proof of lemma 3] $f_*\mathcal{O}_Y \otimes \mathcal{O}_Y$ factors as $F_X, \mathcal{O}_X \otimes F_X, L$. One can prove easily the images of the sub line bundles $\mathcal{O}_X$ and $L$ under the inclusion $f_*\mathcal{O}_Y \hookrightarrow f_*\mathcal{O}_Y \otimes \mathcal{O}_Y$ now both fall in $F_X, \mathcal{O}_X$. Then we have

$$f_*B_Y^1 \cong f_*F_Y, \mathcal{O}_Y / f_*\mathcal{O}_Y \cong F_X, \mathcal{O}_X / \mathcal{O}_X \otimes L \bigcup F_X, L.$$
The necessity of the ordinary of $E_0, E_1$ for case a1) follows from Corollary 2.3. For case a3) let $Z = E_0 \times_{C} \mu_{2,k}$, then we have an étale cover $Z \to X$. Again by Corollary 2.3 $Z$ has also a liftable Frobenius. Thus $Z$ hence $E_0, E_1$ are ordinary.

To make things more clear, we list the results above ($\checkmark$ for existence of lifting of Frobenius under suitable condition and $\times$ for nonexistence) in the following table.

| Case | char $\neq 2,3$ | char $= 3$ | char $= 2$ |
|------|-----------------|-------------|-------------|
| a    | $\checkmark$    | $\checkmark$| $\checkmark$|
| b    | $\checkmark$    | $\times$   | $\times$   |
| c    | $\checkmark$    | $\times$   | $\times$   |
| d    | $\times$        | $\times$   | $\times$   |

Table 1. Hyperelliptic Surfaces

Ruled Surfaces.

The following proposition is the first step in proving the ruled surface part of Theorem 1. We will prove it by using deformation theory directly.

**Proposition 3.7.** Let $X$ be a ruled surface over a curve $C$, if $F_X$ is liftable to $W_2(k)$, so is $F_C$. In particular, $C$ must be an ordinary elliptic curve or the projective line.

Before proving this proposition, we first recall some facts on ruled surfaces. Let $X$ be a ruled surface over a smooth projective curve $C$, then by [12, V, Proposition 2.2] $X$ is isomorphic to the projective bundle $\mathbb{P}(E)$, where $E$ is a rank 2 vector bundle over $C$. Since any rank 2 vector bundle on a curve is an extension by line bundles [12, V, Corollary 2.7] and $\mathbb{P}(E) \cong \mathbb{P}(E \otimes N)$ for any line bundle $N$, we can assume $E$ fits into the following exact sequence

$$0 \to \mathcal{O}_C \to E \to L \to 0$$

on $C$, where $L$ is a line bundle on $C$. Let $U, V$ be two open subsets of $C$ over which the vector bundle $E$ is trivialized by

$$\alpha_U : E|_U \cong \mathcal{O}_U e \oplus \mathcal{O}_U f_U \quad \text{and} \quad \alpha_V : E|_V \cong \mathcal{O}_V e \oplus \mathcal{O}_V f_V,$$

where $f_U \in E_U, f_V \in E_V$ and $e$ is the global section of $E$ defined by the exact sequence (3.3). Then we have the following induced isomorphisms over $U$ and $V$ respectively

$$\beta_U : X_U := \pi^{-1}(U) \cong \text{Proj} \ \mathcal{O}_U [e, f_U], \quad \beta_V : X_V := \pi^{-1}(V) \cong \text{Proj} \ \mathcal{O}_V [e, f_V].$$

Now let $t = \frac{f_U}{f_V}, \ s = \frac{f_V}{f_U}, \ x = \frac{f_U}{e}$ and $y = \frac{f_V}{e}$, then it is easy to see $X$ is covered by four open affine subschemes isomorphic to $\text{Spec} \ \mathcal{O}_U [t], \ \text{Spec} \ \mathcal{O}_V [s], \ \text{Spec} \ \mathcal{O}_U [x]$ and $\text{Spec} \ \mathcal{O}_V [y]$ respectively. Suppose the transition matrix of $E$ from the base \{ $f_U, e$ \} to \{ $f_V, e$ \} over $U \cap V$ is given by \( \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \), where $a, b \in \mathcal{O}_{U \cap V}$ and $a$ is invertible. Then the gluing rule between the four open subschemes is given by $tx = sy = 1$ and $x = ay + b$.

**Proof of Proposition 3.7.** Given a lifting $X$ of $X$ to $W_2(k)$, we claim $X$ induces a lifting $C$ of $C$ to $W_2(k)$. This is equivalent to say any lifting of $X$ as a $k$-scheme is a lifting of $X$ as a $C$-scheme. By Proposition 2.2 there is a one-to-one correspondence between the above two spaces of liftings and the vector spaces $H^1(X, T_{X/C})$ and $H^1(X, T_{X/k})$ respectively. On the other hand, there is a one-to-one correspondence between the space of liftings of $C$ to $W_2(k)$ and $H^1(C, T_{C/k})$. Therefore, it’s enough to show $h^1(T_{X/C}) + h^1(T_{C/k}) \geq h^1(T_{X/k})$. This inequality follows easily from the long exact sequence derived from

$$0 \to T_{X/C} \to T_X \to \nu^* T_C \to 0.$$

Let $Y_U$ (resp. $Y_V$) be the open subscheme of $X_U$ (resp. $X_V$) isomorphic to $\text{Spec} \ k[x]$ (resp. $\text{Spec} \ k[y]$). Let $Y_U$ (resp. $Y_V$) be the open subschemes of $X$ with the underlying open subset $i(Y_U)$ (resp. $i(Y_V)$), where $i$
is the homeomorphism $X \to X$. Let $C$ be the lifting of $C$ induced by $X$, then the gluing rule between the two open subschemes $Y_U, Y_V$ can be written as

$$x \mapsto ay + b + py^2f,$$

where $a, b \in \mathcal{O}_C$ are liftings of $a, b$ and $f \in \mathcal{O}_{U \cup V}[y]$.

Suppose we are given a lifting $F_X$ of $F_X$, then $F_X$ induces a lifting of the Frobenius of any open subscheme of it. In particular, we are given a lifting $F_{Y_U}$ of $F_{Y_U}$ (resp. $F_{Y_V}$ of $F_{Y_V}$). Now given $\lambda \in \mathcal{O}_U$, then the image of $\lambda$ under $F_{Y_U}$ can be written uniquely as $\sum_i F_i(\lambda)x^i, F_i(\lambda) \in \mathcal{O}_U$. Moreover, for $i \geq 1$, we have $pF_i(\lambda) = 0$. One can see easily the function $F_0 : \mathcal{O}_U \to \mathcal{O}_U$ is a lifting of $F_0$. Similarly, let $\mu \in \mathcal{O}_V$, if we write the image of $\mu$ under $F_X$ as $\sum_i G_i(\mu)y^i$ with $G_i(\mu) \in \mathcal{O}_V$, then $G_0$ defines a lifting of $F_V$.

Now given a section $\lambda \in \mathcal{O}_{U \cup V}$, by (3.5) we have

$$\sum_i F_i(\lambda)(ay + b + py^2f)^i = \sum_i G_i(\lambda)y^i$$

By comparing the terms with degree 0 on the two sides we get

$$\sum_i F_i(\lambda)b^i = \sum_i G_i(\lambda).$$

As we note above, $pF_i(\lambda) = 0$ so $p \sum_{i \geq 1} F_i(\lambda)b^i = 0$ hence the left side of the above equality can be written as $F_0(\lambda) + p\eta(\lambda)$, where the function $\eta : \mathcal{O}_U \to \mathcal{O}_U$ is uniquely defined by $p\eta(\lambda) = \sum_{i \geq 1} F_i(\lambda)b^i$. Moreover, as $F_0$ and $G_0$ are liftings of $F_{U \cup V}$, therefore the restriction of $\eta$ to $\mathcal{O}_{U \cup V}$ satisfies the condition listed in Corollary 2.4. As the base scheme $U$ is integral $\eta$ as a function from $\mathcal{O}_U$ to itself also satisfies these conditions. By applying Corollary 2.4 again, $F_0(\lambda) + p\eta(\lambda)$ is also a lifting of $F_U$ hence we are given a lifting of $F_C$ and the proposition is proved.

If the base curve is $\mathbb{P}^1$, then $X$ is a toric surface, hence endowed with a lifting of Frobenius to $W_2(k)$. Next we consider the case when $C$ is an ordinary elliptic curve.

Given a lifting $C$ of $C$ to $W_2(k)$ and a lifting $E$ of $E$ over $C$, then it is easy to see $X := \mathbb{P}(E)$ is a lifting of $X$. Next we will prove the Frobenius of $X$ can be lifted to $X$.

**Proposition 3.8.** Let $C$ be an ordinary elliptic curve over $k$, $E$ be a vector bundle of rank 2 over $C$, and $X = \mathbb{P}(E)$ be the associated ruled surface, then $F_X$ can be lifted to $W_2(k)$

**Proof.** Following the notations used earlier in this subsection, let $U, V$ be open subschemes of $X$ with underlying open subset $U$ and $V$, $e, f_U, f_V$ be sections of $E$ lifting $e, f_U$ and $f_V$ respectively. Furthermore, let $t = \frac{f_U}{f_V}, s = \frac{f_U}{f_V}, x = \frac{t + y}{e}$ and $y = \frac{t - y}{e}$, then $X$ is covered by the open subschemes $\text{Spec } \mathcal{O}_U[t], \text{Spec } \mathcal{O}_U[x], \text{Spec } \mathcal{O}_V[s]$ and $\text{Spec } \mathcal{O}_V[y]$. Moreover, the gluing rules between these open subschemes are given by $tx = 1, sy = 1$ and $x = ay + b$.

We fix a lifting $F_C : C \to C$, and define a lifting of $F_X$ over $\text{Spec } \mathcal{O}_U[x]$ by sending $x$ to $x^p$. We take $f$ to be 0, then by the relation $x = ay + b$, the image of $y$ under the lifted Frobenius is given by

$$F_X(y) = F_X \left( \frac{x - b}{a} \right) = \frac{(ay + b)^p - F_C(b)}{F_C(a)}.$$

The right side of the above equality can be written as $y^p + ph, h \in \mathcal{O}_{U \cup V}[y]$ and it is easy to see $\deg h \leq p$. By Corollary 2.4 this lifting of Frobenius over $\text{Spec } \mathcal{O}_U[x]$ (resp. $\text{Spec } \mathcal{O}_V[y]$) can be extended to $\text{Spec } \mathcal{O}_U[t]$ (resp. $\text{Spec } \mathcal{O}_V[s]$). It is easy to see these liftings glue well hence the proposition is proved. \qed

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