Adaptive Convex Combination of APA and ZA-APA algorithms for Sparse System Identification

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Abstract

In general, one often encounters the systems that have sparse impulse response, with time varying system sparsity. Conventional adaptive filters which perform well for identification of non-sparse systems fail to exploit the system sparsity for improving the performance as the sparsity level increases. This paper presents a new approach that uses an adaptive convex combination of Affine Projection Algorithm (APA) and Zero-attracting Affine Projection Algorithm (ZA-APA) algorithms for identifying the sparse systems, which adapts dynamically to the sparsity of the system. Thus works well in both sparse and non-sparse environments and also the usage of affine projection makes it robust against colored input. It is shown that, for non-sparse systems, the proposed combination always converges to the APA algorithm, while for semi-sparse systems, it converges to a solution that produces lesser steady state EMSE than produced by either of the component filters. For highly sparse systems, depending on the value of the proportionality constant ($\rho$) in ZA-APA algorithm, the proposed combined filter may either converge to the ZA-APA based filter or produce a solution similar to the semi-sparse case i.e., outperforms both the constituent filters.

Index terms-Sparse Systems, $l_1$ Norm, Compressive Sensing, Excess Mean Square Error.

I. INTRODUCTION

Usually, many real-life systems exhibit sparse representation i.e., their system impulse response is characterized by small number of non zero taps in the presence of large number of inactive taps. Sparse systems are encountered in many important practical applications such as network and acoustic echo cancelers [1]-[2], HDTV channels [3], wireless multipath channels [4], underwater acoustic communications [5]. The conventional system identification algorithms such as LMS and NLMS are sparsity agnostic i.e., they are unaware of underlying sparsity of the system impulse response. Recent studies have shown that the a priori knowledge about the system sparsity, if utilized properly by the identification algorithm, can result in substantial improvement in its estimation performance. This resulted in a flurry of research activities in the last decade or so towards developing sparsity aware adaptive filter algorithms, notable amongst them being the Proportionate Normalized LMS (PNLMS) algorithm [6] and its variants [7]-[8]. These current regressor based algorithms exhibit slower convergence rate for the correlated input. As a solution, these proportionate-type concepts were extended to data reuse case to provide the uniform convergence rate for both white and correlated input [9].
On the other hand, drawing from the ideas of compressed Sensing (CS) \cite{10,12}, several new sparse adaptive filters have been proposed in the last few years, notably, the zero attracting LMS (ZALMS) \cite{13} derived by incorporating the $l_1$ norm penalty into the LMS cost function which was later extended to ZA-APA \cite{14}. The simplicity of using $l_1$ norm penalty makes their implementation extremely simple. These algorithms perform well for the systems that are highly sparse, but struggle as the system sparsity decreases. That means these algorithms cannot perform well as the system sparsity varies widely over time.

In \cite{15}, an improved PNLMS (IPNLMS) algorithm, which is a controlled mixture of the PNLMS and the NLMS is proposed to handle the variable system sparsity. In \cite{16}, an adaptive convex combination of two IPNLMS filters is proposed that can adapt to situations where the system sparsity is time varying and unknown. However, it provides the steady state MSD which is same as that of conventional sparse agnostic filters. Reweighted ZALMS (RZALMS) \cite{13} addresses this issue by selecting the shrinkage parameter $\rho$, in a tricky fashion at the cost of increased complexity. As a solution to this problem, in \cite{17}, a convex combination of LMS and ZALMS algorithm has been proposed that switches between ZALMS and LMS adaptive filters depending on the sparsity level, thus enjoying the robustness against time-varying system sparsity with reduced complexity compared to the RZALMS. However, the aforementioned mechanisms that support time varying sparsity could not perform well in colored (correlated) input signal condition.

In this paper, we propose an alternative method that enjoys the robustness against time-varying system sparseness as well as coloredness of (correlated) input signal by using an adaptive convex combination of the Affine Projection Algorithm (APA) and Zero Attracting Affine Projection Algorithm (ZA-APA). The proposed algorithm uses the general framework of convex combination of adaptive filters and requires less complexity. The performance of the proposed scheme is first evaluated analytically by carrying out the convergence analysis. This requires evaluation of the steady state cross correlation between the output \textit{a priori} errors generated by the APA and the ZA-APA based filters and then relating it to the respective steady state EMSE of the two filters. In our analysis, we have carried out this exercise for three different kinds of systems, namely, non-sparse, semi-sparse and sparse. The analysis shows that the proposed combined filter always converges to the APA based filter for a non-sparse system, while for semi-sparse systems, it converges to a solution that can outperform both the constituents. For sparse systems, the proposed scheme usually converges to the ZA-APA based filter. However, by adjusting the regularized parameter $\rho$, a solution like the semi sparse case can be achieved i.e., outperforming both the filters. Finally, the robustness of the proposed methods against variable sparsity and coloredness is verified by the detailed simulation studies.

II. Problem Formulation, Proposed Algorithm and Performance Analysis

We consider the problem of identifying an unknown system (supposed to be sparse), modeled by the $L$ tap coefficient vector $w_{opt}$ which takes a signal $u(n)$ with variance $\sigma_u^2$ as the input and produces the observable output $d(n) = u^T(n)w_{opt} + \epsilon(n)$, where $u(n) = [u(n), u(n-1), ..., u(n-M+1)]^T$ is the input data vector at time $n$, and $\epsilon(n)$ is the observation noise with zero mean and variance $\xi_0$. In order to identify the system, the approach mentioned in \cite{18} is followed to deploy an adaptive convex combination of two adaptive filters as shown in the Fig. 1, where $t^{th}$ filter uses the APA/ZA-APA algorithm to adapt
a filter coefficient vector \( w_l(n) \) as follows,

\[
w_l(n + 1) = w_l(n) + \mu U(n) \left( e_{M} + U^T(n) U(n) \right)^{-1} e_l(n) - \rho_l \text{sgn}(w_l(n))
\]  

(1)

where \( \mu \) is the step size (common for both the filters), controller constant \( \rho_l = 0 \), when \( l = 1 \) and \( \rho_l = \rho \), for \( l = 2 \), also \( e_l(n) = d(n) - y_l(n) = [e_1(n), e_1(n-1), ..., e_1(n-M+1)]^T \) is the respective filter output error vectors with \( y_l(n) = U^T(n) w_l(n) \) denoting the respective filter output vectors, and \( U(n) = [u(n), u(n-1), ..., u(n-M+1)]^T \) is the input data matrix. The convex combination generates a combined output vector \( y(n) = \lambda(n) y_l(n) + [1 - \lambda(n)] y_2(n) \). The variable \( \lambda(n) \in [0, 1] \), is a mixing parameter, which is to be adapted by the following gradient descent method to minimize the quadratic error function of the overall filter, namely \( e^2(n) \), where \( e(n) = d(n) - y(n) \). However, such adaptation does not guarantee that \( \lambda(n) \) will lie between 0 and 1. Therefore, instead of \( \lambda(n) \), an equivalent variable \( a(n) \) is updated which expresses \( \lambda(n) \) as a sigmoidal function, i.e., \( \lambda(n) = \frac{1}{1 + \exp(-a(n))} \). The update equation of \( a(n) \) is given by [18],

\[
a(n + 1) = a(n) - \frac{\mu}{2} \frac{\partial e^2(n)}{\partial a(n)} = a(n) + \mu a(n) e(n) [y_1(n) - y_2(n)] \lambda(n) [1 - \lambda(n)]
\]  

(2)

In practice, \( \lambda(n) \approx 1 \) for \( a(n) \gg 0 \) and conversely, \( \lambda(n) \approx 0 \) for \( a(n) \ll 0 \). Therefore, instead of updating the \( a(n) \) up to \( \pm \infty \), it is sufficient to restrict it to a range \([-a^+, +a^+] \) (\( a^+ \) : a large finite number) which limits the permissible range of \( \lambda(n) \) to \([1 - \lambda^+, \lambda^+]\), where \( \lambda^+ = \frac{1}{1 + \exp(-a^+)} \).

![Diagram](image)

Figure 1. The proposed adaptive convex combination of two adaptive filters (Filter 1 : ZA-APA and Filter 2 : APA)

A. Important Assumptions and Definitions

In [19], [20] convergence analysis of APA was presented using an equivalent form of APA known as Normalized LMS with Orthogonal Correction Factors (NLMS-OCF), along with the assumptions proposed by Slock in [21] for modeling the system input. The analysis in [19] provides the basis for the work presented in this paper. Some introduction about the NLMS-OCF algorithm is presented below. For the given input signal \( u(n) \), observation noise \( e(n) \) and the reference signal \( d(n) \), the NLMS-OCF algorithm uses orthogonal correction factors based on the past \( M - 1 \) input vectors (each of length \( L \)), to update the
weights \( w(n) \), in every iteration. The weight update equation is

\[
    \mathbf{w}(n + 1) = \mathbf{w}(n) + \mu_0 \mathbf{u}(n) + \mu_1 \mathbf{u}^1(n) + \ldots + \mu_{M-1} \mathbf{u}^{M-1}(n)
\]

(3)

where the notation \( \mathbf{u}^k(n) \) refers to the component of vector \( \mathbf{u}(n-k) \) that is orthogonal to the vectors: \( \mathbf{u}(n), \mathbf{u}(n-1), \ldots, \mathbf{u}(n-k+1) \), where \( k = 0, 1, \ldots, M - 1 \) is the previous input signal vector index.

And also \( \mu_k \) for \( k = 1, 2, \ldots, M - 1 \) is chosen as,

\[
    \mu_k = \begin{cases} 
        \frac{\mu u(n)}{\mathbf{u}^T(n) \mathbf{u}(n)}, & \text{for } k = 0, \text{ if } \|\mathbf{u}(n)\| \neq 0 \\
        \frac{\mu u^k(n)}{(\mathbf{u}^k(n))^T \mathbf{u}^k(n)}, & \text{for } k = 1, 2, \ldots, M - 1, \text{ if } \|\mathbf{u}^k(n)\| \neq 0
    \end{cases}
\]

(4)

where

\[
    \begin{align*}
        e(n) &= d(n) - \mathbf{w}^T(n)\mathbf{u}(n) \\
        e^k(n) &= d(n-k) - (\mathbf{w}^k(n))^T\mathbf{u}(n-k) & \text{for } k = 1, 2, \ldots, M - 1 \\
        \mathbf{w}^k(n) &= \mathbf{w}(n) + \mu_0 \mathbf{u}(n) + \mu_1 \mathbf{u}^1(n) + \ldots + \mu_{k-1} \mathbf{u}^{k-1}(n)
    \end{align*}
\]

(5)

The algorithm can be seen as the process of computing weight updates, using NLMS, based on the current input data vector, \( \mathbf{u}(n) \), as well as the orthogonal components from each of the previous \( M - 1 \) input data vectors.

In addition to NLMS-OCF, as described in [19] the performance analysis is done based on the following assumptions on the signal and the underlying system

A1) The input signal vectors \( \mathbf{u}(n) \) are assumed to be zero mean with covariance matrix

\[
    \mathbf{R} = E[\mathbf{u}(n)\mathbf{u}^T(n)] = \mathbf{V} \mathbf{A} \mathbf{V}^T
\]

(6)

where \( \mathbf{A} = diag(\lambda_1, \lambda_2, \ldots, \lambda_L) \), and \( \mathbf{V} = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_L) \). Here, \( \lambda_1, \lambda_2, \ldots, \lambda_L \) are the eigenvalues of input covariance matrix \( \mathbf{R} \) and \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_L \) are the corresponding orthonormal eigenvectors (\( \mathbf{V}^T \mathbf{V} = \mathbf{I} \)) i.e., \( \mathbf{V} \) is a unitary matrix.

A2) Observation noise \( e(n) \) (i.e., zero mean white noise, with variance \( \xi^0 \)) is independent of \( \mathbf{u}(n) \) and, the initial conditions \( w_1(0) \) and \( a(0) \) are also independent of \( \mathbf{u}(n) \), \( d(n) \).

A3) The random signal vector \( \mathbf{u}(n) \) is the product of three i. i. d. random variables, that is

\[
    \mathbf{u}(n) = s(n) r(n) \mathbf{v}(n)
\]

(7a)

where

\[
    \begin{align*}
        P\{s(n) = \pm 1\} &= \frac{1}{2} \\
        r(n) &\sim \|\mathbf{u}(n)\| \\
        p\{\mathbf{v}(n) = \mathbf{v}_i\} &= p_i = \frac{\lambda_i}{\text{tr}(\mathbf{R})}, \quad i = 1, 2, \ldots, L
    \end{align*}
\]

(7b)

here \( r(n) \sim \|\mathbf{u}(n)\| \) i.e., \( r(n) \) has the same distribution as the norm of the true input signal vectors.
Assumption A3 was first introduced by Slock in [19], which leads to a simple distribution for the vectors $u(n)$ while following the actual first and second order statistics of the input signal in A1 consistently. Assumption A3 was used in [19, 20] as well as here, to simplify the convergence analysis.

As mentioned in [19], using Assumption A3, the weight update in (3)-(5) can be simplified, since the computation of orthogonal components $u^k(n)$ becomes unnecessary i.e., each $u(n)$ is already chosen from the orthogonal set. Hence, NLMS-OCF update equation can be rewritten as follows

$$\mathbf{w}(n+1) = \mathbf{w}(n) + \mu_0 \mathbf{u}(n) + \mu_1 \mathbf{u}(n-1) + \ldots + \mu_{M-1} \mathbf{u}(n-M+1)$$

$$e(n) = d(n) - \mathbf{w}^T(n)\mathbf{u}(n), \quad \text{and}$$

$$e^k(n) = d(n-k) - \mathbf{w}^T(n)\mathbf{u}(n-k) \quad \text{for } k = 1, 2, \ldots, M-1$$

where

$$\mu_k = \begin{cases} \frac{\mu e(n)}{u^T(n)u(n)}, & \text{for } k = 0, \text{ if } ||u(n)|| \neq 0 \\ \frac{\mu e^k(n)}{u^T(n-k)u(n-k)}, & \text{for } k = 1, 2, \ldots, M-1, \text{ if } ||u^k(n)|| \neq 0 \end{cases}$$

Therefore, using the above NLMS-OCF approximation, weight update equations of Filter1 and Filter2 are as follows

APA weight update Equation:

$$\mathbf{w}_1(n+1) = \mathbf{w}_1(n) + \mu_0 \mathbf{u}(n) + \mu_1 \mathbf{u}(n-1) + \ldots + \mu_{M-1} \mathbf{u}(n-M+1)$$

and ZA-APA weight update equation:

$$\mathbf{w}_2(n+1) = \mathbf{w}_2(n) + \mu_0 \mathbf{u}(n) + \mu_1 \mathbf{u}(n-1) + \ldots + \mu_{M-1} \mathbf{u}(n-M+1) - \rho sgn(\mathbf{w}_2(n))$$

where $\mu_k$ can be calculated using (9).

Next, as in [18], certain definitions that are useful in the analysis are presented below. For $l = 1, 2$, we thus define,

a) Weight Error Vectors: $\tilde{\mathbf{w}}_l(n) = \mathbf{w}_{opt} - \mathbf{w}_l(n)$;

b) Equivalent Weight Vector for the combined filter: $\mathbf{w}_c(n) = \lambda(n)\mathbf{w}_1(n) + [1 - \lambda(n)]\mathbf{w}_2(n)$;

c) Equivalent Weight Error Vector for the combined filter: $\tilde{\mathbf{w}}_c(n) = \mathbf{w}_{opt} - \mathbf{w}_c(n) = \lambda(n)\tilde{\mathbf{w}}_1(n) + [1 - \lambda(n)]\tilde{\mathbf{w}}_2(n)$;

d) A Priori Errors: $e_{a,l}(n) = \mathbf{u}^T(n)\tilde{\mathbf{w}}_l(n)$ and $e_a(n) = \mathbf{u}^T(n)\tilde{\mathbf{w}}_c(n)$. Clearly, $e_a(n) = \lambda(n)e_{a,1}(n) + [1 - \lambda(n)]e_{a,2}(n)$ and $e(n) = e_a(n) + \varepsilon(n)$;

e) Excess Mean Square Error (EMSE): $J_{ex,l}(n) = E[e_{a,l}^2(n)]$, for $l = 1, 2$ and $J_{ex}(n) = E[e_a^2(n)]$;

f) Cross EMSE: $J_{ex,12}(n) = E[e_{a,1}(n)e_{a,2}(n)] \leq \sqrt{J_{ex,1}(n)}\sqrt{J_{ex,2}(n)}$ [from Cauchy-Schwartz inequality]. This means $J_{ex,12}(n)$ cannot be greater than both $J_{ex,1}(n)$ and $J_{ex,2}(n)$ simultaneously.

From (2), one can write,

$$E[a(n+1)] = E[a(n)] + \mu_a E[e(n) (y_1(n) - y_2(n))\lambda(n)(1 - \lambda(n))]$$

(12)
As shown in [18], the convergence of $E[a(n)]$ in [12] depends on the steady state values of the individual filter EMSE and the cross EMSE, namely, $J_{ex,i}(\infty) = \lim_{n \to \infty} J_{ex,i}(n)$ and $J_{ex,12}(\infty) = \lim_{n \to \infty} J_{ex,12}(n)$ respectively. In practice, both $J_{ex,i}(n)$ and $J_{ex,12}(n)$, however, take only finite number of steps to reach their steady state values as both the APA and the ZA-APA algorithms converge in finite number of iterations. Substituting $e(n) = e_a(n) + \varepsilon(n)$ in [12], where $e_a(n)$ is defined above, noting that $y_1(n) - y_2(n) = e_{a,2}(n) - e_{a,1}(n)$ and also that $E[\varepsilon(n)] = 0$, and assuming like [18] that in steady state, $\lambda(n)$ is independent of the a priori errors $e_{a,l}(n)$, it is easy to verify that for large $n$ (theoretically for $n \to \infty$),

$$E[a(n + 1)] = E[a(n)] + \mu_a E[\lambda(n) | 1 - \lambda(n)|^2] \Delta J_1 - \mu_a E[\lambda^2(n) | 1 - \lambda(n)|] \Delta J_2$$

(13)

where $\Delta J_1 = J_{ex,1}(\infty) - J_{ex,12}(\infty)$ and $\Delta J_2 = J_{ex,2}(\infty) - J_{ex,12}(\infty)$. By assuming $\Delta J_1$ and $\Delta J_2$ are constant (i.e., APA and ZA-APA algorithms converge) Eq. (13), can be used to yield the dynamics of the evolution of $E[a(n)]$. To analyze the convergence of $E[a(n)]$, we need to evaluate $\Delta J_1$ and $\Delta J_2$ of the proposed combination.

III. PERFORMANCE ANALYSIS OF THE COMBINATION

In this section, we examine the convergence behavior of the proposed convex combination for various levels of sparsity i.e., how the filter coefficients vector $w_e(n)$ of the proposed combination adapts dynamically to an optimum value as dictated by the sparsity of the system. For this, first we evaluate $J_{ex,2}(\infty)$ and $J_{ex,12}(\infty)$.

A. Mean Convergence Analysis of the ZA-APA Algorithm

From Eq.(11), the recursion for the weight error vector of the ZA-APA algorithm can be written as follows:

$$\tilde{w}_2(n + 1) = [I_L - \mu \sum_{j \in J_n} u(n - j) u_T(n - j)] \tilde{w}_2(n) - \mu \sum_{j \in J_n} \frac{u(n - j) \varepsilon(n - j)}{u_T(n - j) u(n - j)} + \rho sgn(\tilde{w}_2(n))$$

(14)

where $J_n \subseteq \{0,1,\ldots,M-1\}$ is the set of $M$ or fewer indices $j$ for which the input regressor vectors $u(n - j)$ are orthogonal to each other. The orthogonalization process determines the indices forming the set $J_n$. The equation (14) forms the basis for the performance analysis of the ZA-APA algorithm. Using the statistical independence between $w_2(n)$ and $u(n)$ (i.e., “independence assumption”), and recalling that $\varepsilon(n)$ is zero-mean i. i. d random variable which is independent of $u(n)$ and thus of $\tilde{w}_2(n)$, one can write

$$E[\tilde{w}_2(n + 1)] = [I_L - \mu E\left[ \sum_{j \in J_n} \frac{u(n - j) u_T(n - j)}{u_T(n - j) u(n - j)} ] \right] E[\tilde{w}_2(n)] + \rho E[sgn(\tilde{w}_2(n))]$$

(15)

Using A3), we can write the outer to inner product as,

$$\frac{u(n - j) u_T(n - j)}{u_T(n - j) u(n - j)} = \frac{s(n - j) \varepsilon(n - j) v_T(n - j) \varepsilon(n - j) s(n - j) r(n - j)}{s^2(n - j) v^2(n - j) ||v(n - j)||^2}$$

$$= \frac{v(n - j) v_T(n - j)}{r(n - j)}$$

(16)
where \( v(n-j) \in \{ v_1, v_2, ..., v_L \} \). Note that the above result is independent of the norm of \( u(n-j) \).

Using the result presented in (16), the Eq. (15) can be rewritten as

\[
E[\tilde{w}_2(n+1)] = \left[ I_L - \mu E \left( \sum_{k \in K_n} v_k v_k^T \right) \right] E[\tilde{w}_2(n)] + \rho E \left[ sgn(w_2(n)) \right]
\]  

where

\[
K_n = \{ k : \exists j \in J_n \ni \frac{u(n-j)u^T(n-j)}{u^T(n-j)u(n-j)} = v_k v_k^T \}
\]  

By defining the vector \( \alpha(n) \) as the representation of \( E[\tilde{w}_2(n)] \) in terms of the orthonormal vectors \( \{ v_1, v_2, ..., v_L \} \). That is \( \alpha(n) \equiv V^T E[\tilde{w}_2(n)] \) and \( \alpha_{i}(n) = v_i^T E[\tilde{w}_2(n)] \).

Using this notation, pre-multiplying (17) by \( v_i^T \) results in

\[
v_i^T E[\tilde{w}_2(n+1)] = \left[ I_L - \mu E \left( v_i^T \sum_{k \in K_n} v_k v_k^T \right) \right] E[\tilde{w}_2(n)] + \rho v_i^T E \left[ sgn(w_2(n)) \right]
\]  

From the orthonormality of \( v_k \)'s we have

\[
v_i^T \sum_{k \in K_n} v_k v_k^T = \begin{cases} v_i^T, & \text{if } i \in K_n \\ 0, & \text{if } i \notin K_n \end{cases}
\]  

Using the above result and substituting \( P(i \in K_n) = \beta_i \), Eq.(19) becomes

\[
\alpha_{i}(n+1) = [1 - \mu \beta_i] \alpha_{i}(n) + \rho b_i(n)
\]  

where \( b_i(n) = v_i^T E \left[ sgn(w_2(n)) \right] \) and the term \( Pr(i \in K_n) = \beta_i \) in (21) is the probability of drawing an eigen vector \( v_i \) from the eigen vectors set \( \{ v_1, v_2, ..., v_L \} \) at most in \( M \) trials. If we assume \( p_i \) is the probability of drawing an eigen vector \( v_i \), then \( Pr(i \in K_n) = \beta_i = 1 - (1 - p_i)^M \).

In steady state i.e., as \( n \to \infty \), from (21), it is possible to write

\[
\lim_{n \to \infty} \alpha_{i}(n) = \frac{\rho}{\mu \beta_i} \lim_{n \to \infty} b_i(n)
\]  

To evaluate \( \lim_{n \to \infty} b_i(n) \), we classify the unknown parameters (i.e., filter taps) into two disjoint subsets same as in [13], denoting, \( NZ \) and \( Z \) for active and inactive filter taps respectively. If the \( i^{th} \) tap of the optimum filter is active, then \( i \in NZ \), else \( i \in Z \). For sufficiently small controller constant \( \rho \), for every \( i \in NZ \), we have \( |\tilde{w}_{2,i}(\infty)| \ll |w_{opt,i}| \). On the other hand for every \( i \in Z \), we have \( |\tilde{w}_{2,i}(\infty)| \gg |w_{opt,i}| \). Since \( w_{opt,i} = 0 \) for every \( i \in Z \), so for zero mean Gaussian input, in steady state we can assume \( w_{2,i}(\infty) \) to be Gaussian with zero mean.
Therefore, from the aforementioned remarks and for the sufficiently small controller constant \( \rho \), we can approximate,

\[
\lim_{n \to \infty} sgn(w_{2,i}(n)) = sgn(w_{\text{opt},i}) \quad \text{for } i \in NZ
\]

\[
\lim_{n \to \infty} E[sgn(w_{2,i}(n))] = 0 \quad \text{for } i \in Z
\]

(23)

These approximations are helpful to evaluate the expectations of non linear function \( \lim_{n \to \infty} b_i(n) \), involved in (22) for both white and correlated inputs. To derive the closed form expressions for the steady state mean of the weight deviation (\( \alpha_{2,i}(\infty) \)) of active and inactive coefficients, at this stage, we assume that the ZA-APA algorithm is operating on white input. This assumption simplifies the approach and the eigen vectors set \( \{v_1, v_2, ..., v_L\} \) becomes the trivial basis \( \{e_1, e_2, ..., e_L\} \). Therefore, in steady state for the ZA-APA algorithm, we have,

\[
\lim_{n \to \infty} \alpha_{2,i}(n) = \lim_{n \to \infty} E[\tilde{w}_{2,i}(n)] = \begin{cases} \\
\mu \cdot sgn(w_{\text{opt},i}) & \text{for } i \in NZ \\
0 & \text{for } i \in Z 
\end{cases}
\]

(24)

B. Excess Mean Square Error Analysis of the ZA-APA Algorithm

The EMSE of ZA-APA algorithm can be written in the following form,

\[
J_{ex,2}(n) = E[e_{2,2}^2(n)] = E[\tilde{w}_2^T(n) R \tilde{w}_2(n)] = Tr(R K_2(n)) = Tr(\Lambda V^T K_2(n) V)
\]

(25)

where \( E[\tilde{w}_2(n)\tilde{w}_2^T(n)] = K_2(n) \).

Let us define the diagonal elements of the transformed covariance matrix \( V^T K_2(n) V \) as \( \tilde{\lambda}_{2,i}(n) \) for \( i = 1, 2, ..., L \). That is

\[
[V^T K_2(n) V]_{ii} = \nu_i^2 K_2(n) v_i = \tilde{\lambda}_{2,i}(n)
\]

(26)

with the above notation EMSE of the ZA-APA algorithm can be written as follows:

\[
J_{ex,2}(n) = \sum_{i=1}^{L} \lambda_i \tilde{\lambda}_{2,i}(n)
\]

(27)

Using the statistical independence between \( w_1(n) \) and \( u(n) \) (i.e., “independence assumption”), and recalling that \( \varepsilon(n) \) is of zero-mean and also independent of \( u(n) \) and thus of \( \tilde{w}_1(n) \), one can write the recursion for the mean square deviation of ZA-APA algorithm as follows:

\[
K_2(n+1) = K_2(n) - \mu \sum_{j \in J_n} \frac{u(n-j)u^T(n-j)}{u^T(n-j)u(n-j)} K_2(n) - \mu \sum_{m \in J_n} u(n-m)u^T(n-m) K_2(n) \\
+ \mu^2 \sum_{j \in J_n} \frac{u(n-j)u^T(n-j)}{u^T(n-j)u(n-j)} K_2(n) \left( \sum_{m \in J_n} \frac{u(n-m)u^T(n-m)}{u^T(n-m)u(n-m)} \right) \\
+ \mu^2 \sum_{j \in J_n} \frac{u(n-j)\varepsilon(n-j)}{u^T(n-j)u(n-j)} \left( \sum_{m \in J_n} \frac{u(n-m)\varepsilon(n-m)}{u^T(n-m)u(n-m)} \right)^T
\]
\[ + \rho E \left[ I_L - \mu \left( \sum_{j \in J_n} \frac{u(n-j)u^T(n-j)}{u^T(n-j)u(n-j)} \right) \right] E \left[ w_2(n) \text{sgn}(w_2^T(n)) \right] \]

\[ + \rho E \left[ \text{sgn}(w_2(n)) w_2^T(n) \right] E \left[ I_L - \mu \left( \sum_{m \in J_n} \frac{u(n-m)u^T(n-m)}{u^T(n-m)u(n-m)} \right) \right] \]

\[ + \rho^2 E \left[ \text{sgn}(w_2(n)) \text{sgn}(w_2^T(n)) \right] \]

Using (18) to replace the outer-to-inner product ratios with \( v_k v_k^T \), one can have

\[ K_2(n+1) = K_2(n) - \mu E \left[ \sum_{k \in K_n} v_k v_k^T \right] K_2(n) - \mu K_2(n) E \left[ \sum_{m \in K_n} v_m v_m^T \right] \]

\[ + \mu^2 E \left[ \left( \sum_{k \in K_n} v_k v_k^T \right) K_2(n) \left( \sum_{m \in K_n} v_m v_m^T \right) \right] + \mu^2 \xi^0 E \left[ \frac{1}{v^2} \right] E \left[ \sum_{k \in K_n} v_k v_k^T \right] \]

\[ + \rho E \left[ I_L - \mu \sum_{k \in K_n} v_k v_k^T \right] E \left[ w_2(n) \text{sgn}(w_2^T(n)) \right] \]

\[ + \rho E \left[ \text{sgn}(w_2(n)) w_2^T(n) \right] E \left[ I_L - \mu \sum_{m \in K_n} v_m v_m^T \right] \]

\[ + \rho^2 E \left[ \text{sgn}(w_2(n)) \text{sgn}(w_2^T(n)) \right] \]

With the notation presented in (26), pre and post multiplication of (29) by \( v_i^T \) and \( v_i \) respectively, results in

\[ \tilde{\lambda}_{2,i}(n+1) = \tilde{\lambda}_{2,i}(n) - \mu E \left[ v_i^T \left( \sum_{k \in K_n} v_k v_k^T \right) \right] K_2(n) v_i - \mu v_i^T K_2(n) E \left[ \left( \sum_{m \in K_n} v_m v_m^T \right) v_i \right] \]

\[ + \mu^2 E \left[ v_i^T \left( \sum_{k \in K_n} v_k v_k^T \right) K_2(n) \left( \sum_{m \in K_n} v_m v_m^T \right) v_i \right] + \mu^2 \xi^0 E \left[ \frac{1}{v^2} \right] E \left[ v_i^T \left( \sum_{k \in K_n} v_k v_k^T \right) v_i \right] \]

\[ + \rho E \left[ v_i^T \left( I_L - \mu \sum_{k \in K_n} v_k v_k^T \right) \right] E \left[ (w_2(n) \text{sgn}(w_2^T(n)) v_i \right] \]

\[ + \rho E \left[ v_i^T \left( \text{sgn}(w_2(n)) w_2^T(n) \right) \right] E \left[ \left( I_L - \mu \sum_{m \in K_n} v_m v_m^T \right) v_i \right] \]

\[ + \rho^2 E \left[ v_i^T \left( \text{sgn}(w_2(n)) \text{sgn}(w_2^T(n)) \right) v_i \right] \]

From the orthonormality of \( v_k \)'s, using the result presented in (20) and substituting \( P(i \in K_n) = \beta_i \), Eq. (30) becomes

\[ \tilde{\lambda}_{2,i}(n+1) = \left[ 1 - \mu (2 - \mu) \beta_i \beta_i \right] \tilde{\lambda}_{2,i}(n) + \mu^2 \xi^0 E \left[ \frac{1}{v^2} \right] \beta_i \]

\[ + \rho (1 - \mu \beta_i) \left[ v_i^T \Phi(n) v_i \right] + \rho (1 - \mu \beta_i) \left[ v_i^T \Phi^T(n) v_i \right] + \rho^2 \left[ v_i^T \Psi(n) v_i \right] \]

\[ \Delta_{2,i}(n) \]

where \( \Phi(n) = E \left[ \tilde{w}_2(n) \text{sgn}(w_2^T(n)) \right] \) and \( \Psi = E \left[ \text{sgn}(w_2(n)) \text{sgn}(w_2^T(n)) \right] \)
In steady state i.e., as \( n \to \infty \), from (31), it is possible to write

\[
\tilde{\lambda}_{2,i}(\infty) = \frac{\mu}{2 - \mu} \xi^0 E\left[\frac{1}{r^2}\right]
+ \frac{\rho(1-\mu\beta_i)}{\mu(2-\mu)\beta_i} \left[ v_i^H \Phi(\infty) v_i \right] + \frac{\rho^2(2-\mu\beta_i)}{\mu^2 \beta_i^2 [2-\mu]} \left[ v_i^T \Psi(\infty) v_i \right]
\]

(32)

where \( \Phi(\infty) = \lim_{n \to \infty} E[\tilde{w}_{2,i}(n) \text{sgn}(w_2^n(\infty))] \) and \( \Psi(\infty) = \lim_{n \to \infty} E[\text{sgn}(w_2(n)) \text{sgn}(w_2^n(\infty))] \).

To evaluate \( \Phi(\infty) \) and \( \Psi(\infty) \), as mentioned in the convergence in mean section the unknown parameters (i.e., filter taps) are classified into two disjoint subsets denoting, \( N, Z \) for active and inactive filter taps respectively. Under small misadjustment condition (i.e., the steady state standard deviation of \( w_{2,i} \) (\( \sigma_{w_{2,i}} \)) is very small), also following the lines of (32), we can approximate \( \lim_{n \to \infty} E[w_{2,i}(n)\text{sgn}(w_{2,j}(n))] = \lim_{n \to \infty} E[w_{2,i}(n)E[\text{sgn}(w_{2,j}(n))] \) and \( \lim_{n \to \infty} E[\text{sgn}(w_{2,i}(n)) \text{sgn}(w_{2,j}(n))] = \lim_{n \to \infty} E[\text{sgn}(w_{2,i}(n))]E[\text{sgn}(w_{2,j}(n))] \) for all \( i \neq j \). These approximations are helpful to evaluate the expectations of non linear functions involved in (32) for both white and correlated input. To derive the closed form expressions for the steady state mean square deviation \( \tilde{\lambda}_{2,i}(\infty) \) of active and inactive coefficients, we assume that the ZA-APA algorithm is operating on white input. This assumption simplifies the approach and the eigen vectors set \( \{v_1,v_2,...,v_L\} \) becomes the trivial basis \( \{e_1,e_2,...,e_L\} \). Therefore, in steady state for the ZA-APA algorithm, we have,

\[
\Phi_{i,i}(\infty) = \begin{cases} 
E[\tilde{w}_{2,i}(\infty) \text{sgn}(w_{opt,i})] & \text{for } i \in N, Z \\
E[\tilde{w}_{2,i}(\infty) \text{sgn}(w_{2,i}(\infty))] & \text{for } i \in Z 
\end{cases}
\]

(33)

and

\[
\Psi_{i,i}(\infty) = \begin{cases} 
E[\text{sgn}(w_{opt,i})^2] = 1 & \text{for } i \in N, Z \\
E[\text{sgn}(w_{2,i}(\infty))^2] = 1 & \text{for } i \in Z 
\end{cases}
\]

(34)

Using these results for every \( i \in N, Z \) we can write,

\[
\tilde{\lambda}_{2,i}(\infty) = \frac{\mu}{2 - \mu} \xi^0 E\left[\frac{1}{r^2}\right]
+ \frac{\rho(1-\mu\beta_i)}{\mu(2-\mu)\beta_i} \left[ v_i^H \Phi(\infty) v_i \right] + \frac{\rho^2(2-\mu\beta_i)}{\mu^2 \beta_i^2 [2-\mu]} \left[ v_i^T \Psi(\infty) v_i \right]
\]

(35)

On the other hand for \( i \in Z \), \( \tilde{w}_{2,i}(\infty) = w_{opt,i} - w_{2,i}(\infty) = -w_{2,i}(\infty) \). Assuming \( w_{1,i}(\infty) \) and \( w_{2,i}(\infty) \) to be jointly Gaussian (having mean zero in the steady state, as \( w_{opt,i}(\infty) = 0 \)), using Price’s theorem we can write,

\[
E[\text{sgn}(w_{2,i}(\infty)) w_{2,i}(\infty)] = -E[\text{sgn}(w_{2,i}(\infty)) w_{2,i}(\infty)] = -gE[w_{2,i}(\infty) w_{2,i}(\infty)] = -gE[\tilde{w}_{2,i}(\infty) \tilde{w}_{2,i}(\infty)], \text{ where } g = \sqrt{\frac{2}{\pi \sigma^2_{w_{2,i}}}} = \sqrt{\frac{2}{\pi \tilde{\lambda}_{2,i}(\infty)}}.
\]

With these results for \( i \in Z \), the equation (32) can be written as,

\[
\tilde{\lambda}_{2,i}(\infty) = \frac{\mu^2 \beta_i \xi^0 E[1/r]}{\mu(2-\mu)\beta_i + 2\rho(1-\mu\beta_i) \sqrt{\frac{1}{\pi \tilde{\lambda}_{2,i}(\infty)}}}
+ \frac{\rho^2(2-\mu\beta_i)}{\mu^2 \beta_i^2 [2-\mu]} \left[ v_i^T \Psi(\infty) v_i \right]
\]

(36)
the above equation is a quadratic of the form,

$$a \tilde{\lambda}_{1,i}(\infty) + b \sqrt{\tilde{\lambda}_{1,i}(\infty)} + c = 0$$  \hspace{1cm} (37a)$$

where the coefficients are,

$$a = 1$$

$$b = \sqrt{\frac{8}{\pi} \frac{\rho(1 - \mu \beta_i)}{\mu(2 - \mu) \beta_i}}$$

$$c = - \left( \frac{\mu}{2 - \mu} \xi^0 E[\frac{1}{r^2}] + \frac{\rho^2}{\mu(2 - \mu) \beta_i} \right)$$

Then, for $i \in Z$, we have (since $\sigma_{w_{2,i}}$ is positive, note that only real, positive root has to be considered.)

$$\sqrt{\tilde{\lambda}_{2,i}(\infty)} = \frac{1}{\mu(2 - \mu) \beta_i} \left[ - \frac{2}{\sqrt{\pi}} \rho(1 - \mu \beta_i) + \sqrt{\frac{2}{\pi} \rho^2(1 - \mu \beta_i)^2 + \left( \frac{\mu^2 \beta_i \xi^0 E[\frac{1}{r^2}] + \rho^2}{\mu(2 - \mu) \beta_i} \right) \mu(2 - \mu) \beta_i} \right]$$

$$= \tilde{\lambda}_{1,i}(\infty) - \sqrt{\frac{8}{\pi} \frac{\rho(1 - \mu \beta_i)}{\mu(2 - \mu) \beta_i} \tilde{\lambda}_{1,i}(\infty)} \hspace{1cm} \rho^2 \ll \rho$$

Squaring both the sides of the above equation, the steady state mean square deviation of single inactive tap can be written as,

$$\tilde{\lambda}_{2,i}(\infty) = \tilde{\lambda}_{1,i}(\infty) + \left[ - \frac{\rho^2}{\mu(2 - \mu) \beta_i} \sqrt{\frac{2}{\pi} \rho^2(1 - \mu \beta_i)^2 + \left( \frac{\mu^2 \beta_i \xi^0 E[\frac{1}{r^2}] + \rho^2}{\mu(2 - \mu) \beta_i} \right) \mu(2 - \mu) \beta_i} \right]$$

$$= \tilde{\lambda}_{1,i}(\infty) - \sqrt{\frac{8}{\pi} \frac{\rho(1 - \mu \beta_i)}{\mu(2 - \mu) \beta_i} \tilde{\lambda}_{1,i}(\infty)} \hspace{1cm} \rho^2 \ll \rho$$

From the above equation, it can be observed that for having $\tilde{\lambda}_{2,i}(\infty) \leq \tilde{\lambda}_{1,i}(\infty)$, the controller constant $\rho$ has to be chosen very carefully. By using (35) and (39) EMSE of ZA-APA can be calculated.

1) $\rho$ Range: For a system with given length $L$ and projection order $M$, $J_{ex,2}(\infty) \leq J_{ex,1}(\infty)$, if and only if

$$\rho^2 \leq \frac{8}{\pi} \frac{(L - K)^2 (1 - \mu \beta_i)^2 \mu^2 \beta_i^2 \xi^0 E[\frac{1}{r^2}]}{C}$$

where

$$C = \left( K(L - K)(2 - \mu \beta_i) \left( \frac{2}{\mu \beta_i} (1 - \mu \beta_i)^2 + 2 \mu(2 - \mu) \beta_i \right) + \left( K^2(2 - \mu \beta_i)^2 (2 - \mu) \right) - \left( (L - k)^2(2 - \mu) \mu^2 \beta_i^2 \right) \right)$$

This bound shows the dependence of regularization parameter $\rho$ value on the projection order $M$.

C. Cross Excess Mean Square Error Analysis of ZA-APA and APA Algorithms

The cross EMSE of the proposed combination can be written in the following form,

$$J_{ex,12}(n) = E[\bar{w}_{2}^T(n) R \bar{w}_1(n)] = Tr(R K_{12}(n)) = Tr(A V^T K_{12}(n) V)$$

$$= \mathbf{K}_{12}(n).$$

where $E[\bar{w}_1(n)\bar{w}_{2}^T(n)] = \mathbf{K}_{12}(n)$. 

By defining the diagonal elements of the transformed covariance matrix $V^T K_{12}(n) V$ as $\tilde{\lambda}_{12,i}(n)$ for $i = 1, 2, ..., L$. That is

$$[V^T K_{12}(n) V]_{ii} = v_i^T K_{12}(n) v_i = \tilde{\lambda}_{12,i}(n)$$  \hspace{1cm} (42)$$

With the above notation cross EMSE of the combination can be written as follows:

$$J_{ex,12}(n) = \sum_{i=1}^{L} \lambda_i \tilde{\lambda}_{12,i}(n)$$  \hspace{1cm} (43)$$

Post multiplying $\tilde{w}_1(n+1)$ by $\tilde{w}_2^T(n+1)$, taking expectation, and using assumptions A2 and A3 we get

$$K_{12}(n+1) = K_{12}(n) - \mu E \left[ \sum_{k \in K_n} v_k v_k^T \right] K_{12}(n) - \mu K_{12}(n) E \left[ \sum_{m \in K_n} v_m v_m^T \right]$$

$$+ \mu^2 E \left[ \sum_{k \in K_n} v_k v_k^T \right] K_{12}(n) \left( \sum_{m \in K_n} v_m v_m^T \right) v_i$$

$$+ \mu^2 E \left[ \sum_{k \in K_n} v_k v_k^T \right] K_{12}(n) \left( \sum_{m \in K_n} v_m v_m^T \right) v_i$$

$$+ \rho E \left[ I_L - \mu \sum_{k \in K_n} v_k v_k^T \right] E \left[ \tilde{w}_1(n) \text{sgn}(\tilde{w}_2^T(n)) \right] v_i$$  \hspace{1cm} (44)$$

With the above notation, the pre and post multiplication of (44) by $v_i^T$ and $v_i$ respectively, results in

$$\tilde{\lambda}_{12,i}(n+1) = \tilde{\lambda}_{12,i}(n) - \mu E \left[ v_i^T \left( \sum_{k \in K_n} v_k v_k^T \right) \right] K_{12}(n) v_i - \mu v_i^T K_{12}(n) E \left[ \sum_{k \in K_n} v_k v_k^T \right] v_i$$

$$+ \mu^2 E \left[ v_i^T \left( \sum_{k \in K_n} v_k v_k^T \right) \right] K_{12}(n) \left( \sum_{m \in K_n} v_m v_m^T \right) v_i$$

$$+ \mu^2 E \left[ v_i^T \left( \sum_{k \in K_n} v_k v_k^T \right) \right] K_{12}(n) \left( \sum_{m \in K_n} v_m v_m^T \right) v_i$$

$$+ \rho E \left[ v_i^T \left( I_L - \mu \sum_{k \in K_n} v_k v_k^T \right) \right] E \left[ \tilde{w}_1(n) \text{sgn}(\tilde{w}_2^T(n)) \right] v_i$$  \hspace{1cm} (45)$$

From the orthonormality of $v_k$’s and substituting $Pr(i \in K_n) = \beta_i$, Eq. (45) can be written as,

$$\tilde{\lambda}_{12,i}(n+1) = \underbrace{\tilde{\lambda}_{12,i}(n)[1 - \mu(2 - \mu)\beta_i]}_{\tilde{\lambda}_{12,i}(n+1)} + \mu^2 \xi^0 E \left[ \frac{1}{r^2} \right] \beta_i + \rho \underbrace{[1 - \mu \beta_i] E \left[ v_i^T \left( \tilde{w}_1(n) \text{sgn}(\tilde{w}_2^T(n)) \right) \right] v_i}_{\Delta_{12,i}(n)}$$  \hspace{1cm} (46)$$

In steady state i.e., as $n \to \infty$, the term $\tilde{\lambda}_{12,i}(n)$ becomes time invariant. Therefore, we have

$$\tilde{\lambda}_{12,i}(\infty) = \frac{\mu}{2 - \mu} \xi^{0} E \left[ \frac{1}{r^2} \right] + \frac{\rho (1 - \mu \beta_i)}{\mu(2 - \mu)\beta_i} E \left[ v_i^T \left( \tilde{w}_1(\infty) \text{sgn}(\tilde{w}_2^T(\infty)) \right) \right] v_i$$  \hspace{1cm} (47)$$

Using the same approximations used in EMSE analysis of ZA-APA, we can solve the term $E \left[ v_i^T \left( \tilde{w}_1(\infty) \text{sgn}(\tilde{w}_2^T(\infty)) \right) \right] v_i$ in the above equation for white and correlated input. However, to derive the closed form expressions for both active and inactive coefficients, we are assuming that the algorithms are operating on white input.

Using the fact that $\lim_{n \to \infty} E \left[ \tilde{w}_{1,i}(n) \right] = 0$, for every $i \in NZ$ one can write,

$$\Delta_{12,i}(\infty) = \frac{\rho (1 - \mu \beta_i)}{\mu(2 - \mu)\beta_i} E \left[ \tilde{w}_{1,i}(\infty) \text{sgn}(\tilde{w}_{2,i}(\infty)) \right]$$

$$= \frac{\rho (1 - \mu \beta_i)}{\mu(2 - \mu)\beta_i} E \left[ \tilde{w}_{1,i}(\infty) \text{sgn}(\tilde{w}_{opt,i}(\infty)) \right] = 0$$  \hspace{1cm} (48)$$
and using Price’s theorem for every \( i \in Z \) we can write,

\[
\Delta_{12,i}(\infty) = \frac{\rho (1 - \mu \beta_i)}{\mu (2 - \mu) \beta_i} E \left[ \tilde{w}_{1,i}(\infty) \text{sgn}(\tilde{w}_{2,i}(\infty)) \right]
\]

\[
= -\frac{\rho (1 - \mu \beta_i)}{\mu (2 - \mu) \beta_i} \sqrt{\frac{2}{\pi \sigma_{w_{2,i}}^2}} \tilde{\lambda}_{12,i}(\infty)
\]

\[
= -\frac{\rho (1 - \mu \beta_i)}{\mu (2 - \mu) \beta_i} \sqrt{\frac{2}{\pi}} \frac{\tilde{\lambda}_{12,i}(\infty)}{\sqrt{\lambda_{2,i}(\infty)}}
\]

From (47), for every \( i \in NZ \) the mean square deviation can be written as,

\[
\tilde{\lambda}_{12,i}(\infty) = \frac{\mu}{2 - \mu} \xi^0 E \left[ \frac{1}{\tau^2} \right]
\]

and for \( i \in Z \),

\[
[\mu (2 - \mu) \beta_i] \tilde{\lambda}_{12,i}(\infty) = \mu^2 \xi^0 \beta_i E \left[ \frac{1}{\tau^2} \right] - \rho (1 - \mu \beta_i) \sqrt{\frac{2}{\pi}} \frac{\tilde{\lambda}_{12,i}(\infty)}{\sqrt{\lambda_{2,i}(\infty)}}
\]

By rearranging the terms in the above equation, we get

\[
\tilde{\lambda}_{12,i}(\infty) = \frac{\mu^2 \xi^0 \beta_i E \left[ \frac{1}{\tau^2} \right]}{\left( [\mu (2 - \mu) \beta_i] + \rho (1 - \mu \beta_i) \sqrt{\frac{2}{\pi}} \right)}
\]

From (38) and (50)

\[
\tilde{\lambda}_{12,i}(\infty) = \frac{\mu^2 \xi^0 \beta_i E \left[ \frac{1}{\tau^2} \right] \sqrt{\lambda_{2,i}(\infty)}}{\left( [\mu (2 - \mu) \beta_i \sqrt{\lambda_{2,i}(\infty)}] + \rho (1 - \mu \beta_i) \sqrt{\frac{2}{\pi}} \right)}
\]

\[
= \tilde{\lambda}_{1,i}(\infty) \left[ \frac{\sqrt{\frac{2}{\pi}} \rho (1 - \mu \beta_i)}{\sqrt{\frac{2}{\pi} \rho^2 (1 - \mu \beta_i)^2 + \left( \mu^2 \beta_i \xi^0 E \left[ \frac{1}{\tau^2} \right] + \rho^2 \right) \mu (2 - \mu) \beta_i}} \right]
\]

\[
= \tilde{\lambda}_{1,i}(\infty) \left[ \frac{\sqrt{\frac{2}{\pi}} \rho (1 - \mu \beta_i)}{\sqrt{\frac{2}{\pi} \rho^2 (1 - \mu \beta_i)^2 + \left( \mu^2 \beta_i \xi^0 E \left[ \frac{1}{\tau^2} \right] + \rho^2 \right) \mu (2 - \mu) \beta_i}} \right]
\]

\[
\approx \tilde{\lambda}_{1,i}(\infty) \left[ \frac{\sqrt{\frac{2}{\pi}} \rho (1 - \mu \beta_i)}{\sqrt{\frac{2}{\pi} \rho^2 (1 - \mu \beta_i)^2 + \left( \mu^2 \beta_i \xi^0 E \left[ \frac{1}{\tau^2} \right] + \rho^2 \right) \mu (2 - \mu) \beta_i}} \right]
\]

By using (50) and (53), cross EMSE of the proposed combination can be calculated.
IV. CONVERGENCE ANALYSIS OF $a(n)$ FOR VARIOUS CASES OF SPARSITY LEVEL OF THE SYSTEM

1) Non-Sparse Systems: For non-sparse systems, the set $Z$ (set of zero taps) contains very small number of coefficients, Therefore from (35) and (50) we have $\tilde{\lambda}_{1,i}(\infty) \approx \tilde{\lambda}_{12,i}(\infty)$ and $\tilde{\lambda}_{1,i}(\infty) < \tilde{\lambda}_{2,i}(\infty)$ or equivalently, $J_{ex,1}(\infty) \approx J_{ex,12}(\infty)$ and $J_{ex,1}(\infty) < J_{ex,2}(\infty)$. These imply $\Delta J_1 \approx 0$ and $\Delta J_2 = J_{ex,2}(\infty) - J_{ex,12}(\infty) = J_{ex,2}(\infty) - J_{ex,1}(\infty) > 0$. Therefore, Eq. (13) leads to

$$E[a(n+1)] = E[a(n)] + \mu_a E [\lambda(n)(1 - \lambda(n))^2] \Delta J_2 \quad (54)$$

Note that $\forall \lambda(n) \in [0, 1]$, the function $f(\lambda(n)) = \lambda^2(n)(1 - \lambda(n)) \geq 0$, with a maxima at $\lambda(n) = \frac{2}{3}$ and with $f(\lambda(n)) = 0$ at $\lambda(n) = 0, 1$. Assume that at the $n$-th index, $-a^+ < E[a(n)] < a^+$, meaning that $a(n)$ has not converged to $a^+$ or $-a^+$(in all trials), but is taking values from $[-a^+, a^+]$, or equivalently, $\lambda(n)$ has not converged to $\lambda^+$ or $1 - \lambda^+$ but assuming values from $[(1 - \lambda^+), \lambda^+]$. For $1 - \lambda^+ \leq \lambda(n) \leq \lambda^+$, $f(\lambda(n)) \geq f(1 - \lambda^+)$ = $C$. Substituting in (54), $E[a(n+1)] = E[a(n)] + \mu_a C \Delta J_2$. Since $\Delta J_2 > 0$, the above implies $\lim_{n \to \infty} E[a(n)] = a^+$ and thus $\lim_{n \to \infty} a(n) = a^+$ almost surely. The results in $\lim_{n \to \infty} \lambda(n) = \lambda^+$ (almost surely) $\approx 1$, and therefore the proposed combination switches to APA algorithm (Filter 1) which performs better than ZA-APA algorithm (Filter 2) (i.e., $J_{ex,1}(\infty) < J_{ex,2}(\infty)$).

2) Semi-Sparse Systems: In semi-sparse systems the zero taps set $Z$ contains significant amount of elements when compared to non-sparse system. For the given controller constant $\rho$ and $0 < \mu < 2$, the term $\left[ \frac{\sqrt{2} \rho}{\mu^3 \rho^2 (2-\rho) 0.5 \sqrt{\frac{3}{\mu}}} \right]$ in the equation (53) becomes positive. Therefore, for every $i \in Z$, we have $\tilde{\lambda}_{12,i}(\infty) \leq \tilde{\lambda}_{1,i}(\infty)$ i.e., $J_{ex,1}(\infty) - J_{ex,12}(\infty) > 0$. And also, due to the presence of large number of non-zero coefficients still APA performs better than ZA-APA algorithm i.e., $J_{ex,2}(\infty) > J_{ex,1}(\infty)$. So, we have $J_{ex,2}(\infty) > J_{ex,1}(\infty) > J_{ex,12}(\infty)$ and thus both $\Delta J_1 > 0$ and $\Delta J_2 > 0$. This is analogous to the case (3), section III of [18]. Under this, a stationary point is obtained by setting the update term in (13) to zero as $n \to \infty$, leading to $E[\lambda(\infty)(1 - \lambda(\infty))^2] \Delta J_2 = E[\lambda^2(\infty)(1 - \lambda(\infty))] \Delta J_1$. Assuming a negligibly small variance for $\lambda(\infty)$, i.e., assuming $E[\lambda^2(\infty)] \to 0$ which implies that $\lambda(\infty) \to$ a constant (almost surely) as $n \to \infty$ one can then obtain from the above

$$(1 - E[\lambda(\infty)]) \Delta J_2 = E[\lambda(\infty)] \Delta J_1$$

or equivalently,

$$E[\lambda(\infty)] = \left[ \frac{\Delta J_2}{\Delta J_1 + \Delta J_2} \right]^{\lambda^+} \quad (55)$$

It follows that

$$\lambda^+ \geq \tilde{\lambda}(\infty) > 0.5; \quad \text{if} \quad J_{ex,1}(\infty) < J_{ex,2}(\infty) \quad (56)$$

$$0.5 \geq \tilde{\lambda}(\infty) > 1 - \lambda^+; \quad \text{if} \quad J_{ex,1}(\infty) > J_{ex,2}(\infty)$$

As proved in [18], this case is not sub-optimal. Rather it leads to

$$J_{ex}(\infty) \leq \min[J_{ex,1}(\infty), J_{ex,2}(\infty)], \quad (57)$$

which means in this case, the proposed convex combination works even better than each of its component filters.
3) Sparse Systems: For sparse systems, Z (set of in active taps) contains the majority of the coefficients and NZ (set of active taps) contains very negligible number of elements. From (39), we have $\tilde{\lambda}_{2,i}(\infty) < \tilde{\lambda}_{1,i}(\infty)$, or equivalently, $J_{ex,2}(\infty) < J_{ex,1}(\infty)$, i.e., the ZA-APA algorithm in this case outperforms the APA algorithm. Depending on the value of the controller constant $\rho$, $J_{ex,2}(\infty) < J_{ex,12}(\infty)$ and $J_{ex,2}(\infty) > J_{ex,12}(\infty)$ are possible. From (39) and (53), we have

$$J_{ex,2}(\infty) \leq J_{ex,12}(\infty) \quad \text{for} \quad 0 < \rho^2 \leq \frac{\tilde{\mu}^2(1 - \mu\beta_i)^2\lambda\xi_0 E[|h|]}{\mu(2 - \mu)\beta_i + \tilde{\mu}(1 - \mu\beta_i)^2}$$

(58)

Case I is the consequence of the above situation as we discuss in the following paragraph.

Case I: $J_{ex,2}(\infty) < J_{ex,12}(\infty) < J_{ex,1}(\infty)$, that imply $\triangle J_2 \leq 0$ and $\triangle J_1 > 0$

This is analogous to the case (1), section III of [18]. Equation (13) in this case leads to

$$E[a(n+1)] = E[a(n)] + \mu_q E[g(\lambda(n))]\triangle J_2 - \mu_q E[f(\lambda(n))]\triangle J_1,$$

(59)

where $g(\lambda(n)) = \lambda(n)[1 - \lambda(n)]^2 = f(1 - \lambda(n))$. Like $f(\lambda(n))$, $g(\lambda(n)) \geq g(\lambda^+) = f(1 - \lambda^+) = C, \forall \lambda(n) \in [1 - \lambda^+, \lambda^+]$. From the arguments used for the non-sparse case above, we have, $E[a(n+1)] \leq E[a(n)] - \mu_q C(\triangle J_2 - \triangle J_1)$, which means

$$\lim_{n \to \infty} E[a(n)] = -a^+ \quad \text{and thus} \quad \lim_{n \to \infty} a(n) = -a^+ \quad (\text{almost surely}), \quad \text{or equivalently} \quad \lim_{n \to \infty} E[\lambda(n)] = 1 - \lambda^+ \quad (\text{almost surely}) \approx 0.$$

The combination filter in this case will be converged to the ZA-APA based filter which is better of the two filters for sparse system.

For higher values of $\rho$, the cross EMSE eventually becomes relatively less than the EMSE of ZA-APA which gives case II.

Case II: $J_{ex,12}(\infty) < J_{ex,2}(\infty) < J_{ex,2}(\infty)$, meaning $\triangle J_1 > 0$

This is again analogous to the case (3), section III of [18]. Using the arguments used for the semi-sparse case above, (55) and thus (57) will be satisfied in this case, meaning the proposed combined filter will perform better than both filter1 and filter2.

V. A NEW APPROACH TO INCREASE THE CONVERGENCE RATE OF THE COMBINATION IN SPARSE SYSTEM CASE

In sparse system case, ZA-APA provides better steady-state EMSE performance, however, cannot improve the convergence rate. The proportionate-type concepts can be incorporated into the ZA-APA algorithm to attain both improved convergence rate and steady-state EMSE performance simultaneously. Therefore, we are using Zero Attracting Proportionate Affine Projection Algorithm (ZA-PAPA) as filter2 in the proposed combination. The weight update equation of the ZA-PAPA algorithm is as follows,

$$w_{2}(n+1) = w_{2}(n) + \mu \ G(n) \ U(n) \left[ \varepsilon I_{M} + U^T(n)G(n)U(n) \right]^{-1} e_{2}(n) - \rho \ sgn(w_{2}(n))$$

(60)

where $G(n)$ is a diagonal gain matrix that distributes the adaptation energy unevenly over the filter taps by modifying the step size of each tap.

The gain matrix $G(n)$ is evaluated as,

$$G(n) = diag(g_0(n), g_1(n), ... g_{L-1}(n))$$

(61)
where,
\[ g_l(n) = \frac{\gamma_l(n)}{\frac{1}{L}\sum_{l=0}^{L-1} \gamma_l(n)}, \quad 0 \leq l \leq (L - 1) \]  
(62)

with,
\[ \gamma_l(n) = \max[\rho \gamma_{\min}(n), F[|w_l(n)|]] \]
(63)

\[ \gamma_{\min}(n) = \max[\delta, F[|w_0(n)|], ..., F[|w_{L-1}(n)|]] \]

where \( \rho \) is a very small positive constant which together with \( \gamma_{\min}(n) \), ensures that \( \gamma_l(n) \) and thus \( g_l(n) \) do not turn out to be zero for the inactive taps and thus the corresponding updation does not stall. The parameter \( \delta \) is again a small positive constant employed to avoid stalling of the weight updation at the start of the iterations when the tap weights are initialized to zero.

A full understanding of the APA and ZA-PAPA combination requires a steady state EMSE analysis for \( a(n) \) or \( \lambda(n) \), which is beyond the scope of this correspondence. Detailed simulations of the convex combination of ZA-PAPA and APA are presented hereunder.

VI. SIMULATION STUDIES AND DISCUSSION

Here the analytical results presented in section IV are compared with the simulations for system identification example. The proposed algorithm has been simulated for identifying the system \((w_{\text{opt}})\) of length \( L = 256 \) taps. Initially the system is taken to be non-sparse with all coefficients (randomly chosen) having significant magnitude. After 6000 time samples, the system is changed to a semi-sparse system with 80 active taps and the remaining coefficients magnitude equal to zero. Finally, after 12000 time samples the system is changed to a highly-sparse system having 16 active taps with the remaining coefficients being inactive.

Simulations were performed using zero mean, Gaussian white noise and first-order auto-regressive (AR(1)) process having a pole at 0.8 with unit variance \((\sigma_w^2 = 1)\). The observation noise \( \epsilon(n) \) was taken to be zero-mean Gaussian white noise with variance \( \sigma_\epsilon^2 = 0.001 \). Projection order \((M)\) was taken to be 8 for both the APA and the ZA-APA algorithms and the initial taps were chosen to be zero. The controller constant \( \rho \) was taken to be \( 8 \times 10^{-6} \) and \( 3 \times 10^{-5} \) for white and colored input cases respectively. The simulation results shown in Fig. 2 and Fig. 3 are obtained by plotting the EMSE against the iteration index \( n \), by averaging over 1000 experiments.

Several interesting observations follow from Fig. 2 and Fig. 3. The plots confirm our conjuncture that for non-sparse system \( J_{ex,1}(\infty) \approx J_{ex,12}(\infty) \) while for all other cases (i.e., sparse and semi-sparse systems), \( J_{ex,1}(\infty) \geq J_{ex,12}(\infty) \). Secondly, for the chosen value of \( \rho \), it is seen that \( J_{ex,2}(\infty) < J_{ex,12}(\infty) \) when the system is highly-sparse. As per our analysis above, the proposed combination in this case should converge to the ZA-APA based filter, meaning we should have \( J_{ex}(\infty) \approx J_{ex,2}(\infty) \). Similarly, for the non-sparse system, it is seen from Fig. 2 and Fig. 3 that \( J_{ex}(\infty) \approx J_{ex,1}(\infty) \), meaning the convex combination in this case favors filter1, which confirms our arguments above. Lastly, for the semi-sparse system, the plots confirm that \( J_{ex,2}(\infty) > J_{ex,1}(\infty) > J_{ex,12}(\infty) \). As per the discussions of the previous section, the proposed combination in this case is
likely to produce a solution that performs better than both filter1 and filter2.

Later, simulations were performed for the proposed convex combination of APA and ZA-PAPA. The learning curves of the proposed combination for white and colored input signal are presented in Fig. 4 and Fig. 5 respectively. From these figures, it can be observed that usage of proportionate concepts in ZA-APA provides high convergence rate in sparse system case.

VII. CONCLUSIONS

An algorithm that is robust against the correlated input is presented for identifying sparse systems with the degree of sparsity varying with time and context. The proposed method uses an adaptive convex combination of the sparsity aware ZA-APA algorithm and the standard APA algorithm. The algorithm adapts dynamically to the level of sparseness and exhibits robustness against the correlated input.
Figure 4. Learning curves of ZA-PAPA and APA combination for White input with $M = 8$

Figure 5. Learning curves of ZA-PAPA and APA combination for AR input with $M = 8$

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