Dispersive response of atoms trapped near the surface of an optical nanofiber with applications to quantum nondemolition measurement and spin squeezing

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We study the strong coupling between photons and atoms that can be achieved in an optical nanofiber geometry when the interaction is dispersive. While the Purcell enhancement factor for spontaneous emission into the guided mode does not reach the strong-coupling regime for individual atoms, one can obtain high cooperativity for ensembles of a few thousand atoms due to the tight confinement of the guided modes and constructive interference over the entire chain of trapped atoms. We calculate the dyadic Green’s function, which determines the scattering of light by atoms in the presence of the fiber, and thus the phase shift and polarization rotation induced on the guided light by the trapped atoms. The Green’s function is related to a full Heisenberg-Langevin treatment of the dispersive response of the quantized field to tensor polarizable atoms. We apply our formalism to quantum nondemolition (QND) measurement of the atoms via polarimetry. We study shot-noise-limited detection of atom number for atoms in a completely mixed spin state and the squeezing of projection noise for atoms in clock states. Compared with squeezing of atomic ensembles in free space, we capitalize on unique features that arise in the nanofiber geometry including anisotropy of both the intensity and polarization of the guided modes. We use a first principles stochastic master equation to model the squeezing as function of time in the presence of decoherence due to optical pumping. We find a peak metrological squeezing of ~5 dB is achievable with current technology for ~2500 atoms trapped 180 nm from the surface of a nanofiber with radius a = 225 nm.

I. INTRODUCTION

Strong coupling between atoms and photons is at the heart of many quantum information processing protocols including efficient generation of remote entanglement [1, 2], quantum data storage and retrieval [3], and QND measurements [4]. From a general perspective, strong coupling arises when atoms radiate predominantly into the electromagnetic field mode that defines the quantum atom-light interface. For an individual atom, the strong coupling regime is attained via the Purcell effect, whereby the boundary conditions of nearby dielectrics and/or conductors enhance radiation into a desired mode relative to all other modes. This can be achieved with Fabry-Perot cavities (cavity QED) [5] and/or via nanophotonic structures engineered such that the radiation is predominantly into a specified mode [6–8]. The Purcell enhancement factors for emission into a guided or cavity mode scale respectively as $\Gamma_{\text{vac}}/\Gamma_{\text{vac}} \sim \sigma_0/A$ and $\Gamma_{\text{vac}}/\Gamma_{\text{vac}} \sim Q\lambda^3/V \sim F\sigma_0/A$. Here $\Gamma_{\text{vac}}$ is the free space spontaneous emission rate, $\sigma_0 \propto \lambda^2$ is the resonant absorption cross section, $Q$, $V$, and $F$, are the cavity quality factor, volume, and finesse respectively, and $A$ is the effective area of the cavity or guided mode that couples to the atom. The strongest coupling occurs on resonance, and thus much effort has been devoted to developing the largest possible $\Gamma_{\text{vac}}$ and $\Gamma_{\text{vac}}$ through ultrahigh-$Q$, small-volume resonators [5, 9, 10] and through nanophotonic plasmonic [11, 12], metamaterial [13], and dielectric [8, 14] waveguides.

In free space, where there is no Purcell enhancement, strong coupling can be achieved via the cooperativity of atomic ensembles. This is most naturally implemented in a dispersive regime, off resonance, where light elastically scattered from the ensemble constructively interferes to match the mode of an exciting paraxial probe [15]. The cooperativity per atom in a typical paraxial beam is small, $\Gamma_{\text{1D}}/\Gamma_{\text{vac}} \sim \sigma_0/A \sim 10^{-6}$. The total cooperativity, however, can be significant for sufficiently large ensembles, e.g. $N_A \sim 10^7$ atoms. The key parameter that characterizes cooperativity is the total resonant optical density of the ensemble, $OD = N_A(\sigma_0/A)$. Such strong cooperativity in free space has been employed in a variety of applications including quantum memory for storage of photonic states [16] and the generation of squeezed states of the collective spin of the ensemble via quantum nondemolition (QND) measurement [17–20].

A particular system that combines the elements above consists of cold atoms trapped in the evanescent field of the guided mode of a tapered optical nanofiber with a subwavelength diameter [21–25] (see Fig. 1). The typical resonant OD per atom, or OD$/N_A$, in the fiber ($\sigma_0/A \sim 10^{-2}$) is boosted by orders of magnitude over free space for paraxial beams. However, one cannot reach the strong coupling regime where $\Gamma_{\text{1D}}$ is on the order of $\Gamma_{\text{vac}}$ as is possible in engineered nanophotonic waveguides, such as those arising in photonic crystals [8], where atoms can be trapped at positions of peak intensity of the field. One can, however, achieve strong cooperativity in the dispersive regime with a moderately sized ensemble. When compared to free space, all light scattered into the guided mode is automatically mode matched, and thus, given the relatively large ratio $\sigma_0/A$, one can achieve high OD with only a few thousand atoms (see Fig. 1). Such strong cooperativity opens the door to new regimes to create non-Gaussian quantum states of the ensemble [26].
and potentially to implement nonlinear optics at the level of a few photons [27–29].

One-dimensional optical lattices in nanofibers based on multiple co- and counter-propagating trapping beams have been loaded with up to several thousand alkali atoms [21, 22]. This has proved a fruitful platform for quantum information processing. The anisotropic nature of the strong atom-light coupling has been exploited for control of internal atomic states [30], enhanced coupling into a preferred propagation direction [31, 32], and optical switching [29]. Off resonance, dispersive coupling has allowed for non-destructive atom counting [33, 34] and storage of fiber-guided light [35, 36]. Recent demonstrations of photonic crystal cavities fabricated on the nanofiber [37–39] promise further enhanced atom-light coupling.

In this paper we study the quantum atom-light interface in the dispersive regime for an optical nanofiber geometry. We focus here on the coupling between the atomic spin and light polarization induced by the elastic scattering of photons by tensor-polarizable cesium atoms trapped near the surface of the nanofiber. This provides an entangling interaction that can be employed to generate spin squeezing via QND measurement. Our analysis unifies a variety of different approaches found in the literature, including direct calculation of the dyadic Green’s function for photon scattering [6, 11, 40–45] and the input-output formalism studied for one-dimensional field theories based on Heisenberg-Langevin equations [46–51].

The remainder of this article is organized as follows. In Sec. II we solve for the mode decomposition of the dyadic Green’s function which determines the electric field scattered by a point dipole near the surface of the nanofiber. This allows us to calculate the phase shift and polarization transformation for fiber-guided photons induced by tensor-polarizable atoms in the dispersive regime. We connect this with a fully quantum mechanical treatment based on a Heisenberg-Langevin picture in Sec. III. The formalism we develop is used in Sec. IV to study QND measurement of atoms based on polarization spectroscopy. We consider shot-noise-limited atom detection as well as measurement-backaction-induced squeezing of spin projection noise. We study squeezing of the collective pseudospin associated with ensembles of atoms in the atomic clock state and calculate its dynamics based on a first principles stochastic master equation that includes both the effects of QND measurement as well as decoherence due to optical pumping. We conclude with a summary and outlook for future research in Sec. V.

II. DYADIC GREEN’S FUNCTION AND INPUT-OUTPUT FIELD RESPONSE

Given a point particle with tensor polarizability $\tilde{\alpha}$ at position $\mathbf{r}'$ near the surface of a nanofiber, the field at frequency $\omega_0$ is given by the solution to the wave equation,

$$[-\nabla \times \nabla \times + \alpha^2(r)k_0^2] \mathbf{E}(\mathbf{r}) = -4\pi \kappa_0^2 \delta^{(3)}(\mathbf{r} - \mathbf{r}') \mathbf{\tilde{\alpha}} \cdot \mathbf{E}(\mathbf{r}),$$

where $k_0 = \omega_0/c$ and $n(\mathbf{r})$ is the spatially varying index of refraction that describes the fiber; Gaussian-cgs units are used throughout. For an asymptotic input field $\mathbf{E}_{in}(\mathbf{r})$, the scattering solution to Eq. (1) is given by the Lippmann-Schwinger equation [44],

$$\mathbf{E}_{out}(\mathbf{r}) = \mathbf{E}_{in}(\mathbf{r}) + \tilde{\mathbf{G}}^{(+)}(\mathbf{r}, \mathbf{r}'; \omega_0) \cdot \mathbf{\tilde{\alpha}} \cdot \mathbf{E}_{out}(\mathbf{r}') \quad (2a)$$

$$\approx \mathbf{E}_{in}(\mathbf{r}) + \tilde{\mathbf{G}}^{(+)}(\mathbf{r}, \mathbf{r}'; \omega_0) \cdot \mathbf{\tilde{\alpha}} \cdot \mathbf{E}_{in}(\mathbf{r}'), \quad (2b)$$

where in Eq. (2b) we have made the first Born approximation valid for weak scattering. The fundamental object that fully characterizes the scattered radiation as well as the energy level shift and modified decay rate of a scatterer near the dielectric is the dyadic Green’s function, $\tilde{\mathbf{G}}(\mathbf{r}, \mathbf{r}'; \omega_0)$. This determines the scattered field from a point dipole at $\mathbf{r}'$, $\mathbf{E}_{scat}(\mathbf{r}) = \tilde{\mathbf{G}}^{(+)}(\mathbf{r}, \mathbf{r}'; \omega_0) \cdot \mathbf{d}$, and satisfies the equation of motion,

$$[-\nabla \times \nabla \times + \alpha^2(r)k_0^2] \tilde{\mathbf{G}}(\mathbf{r}, \mathbf{r}'; \omega_0) = -4\pi \kappa_0^2 \delta^{(3)}(\mathbf{r} - \mathbf{r}') \mathbf{\tilde{1}},$$

where $\mathbf{\tilde{1}}$ is the unit tensor.

The solution for the Green’s function $\tilde{\mathbf{G}}(\mathbf{r}, \mathbf{r}'; \omega_0)$, following from Maxwell’s equations, has been studied previously [40, 42, 44]. As we are interested here in the forward-scattered components that lead to phase shifts and polarization transformations, we directly calculate $\tilde{\mathbf{G}}(\mathbf{r}, \mathbf{r}'; \omega_0)$ through a decomposition into normal modes. A complete set of eigenmodes in the presence of lossless, spatially inhomogeneous dielectric are defined according to the procedure of Glauber and Lewenstein [52]. We seek...
the eigenmodes \( f_\eta(r) \), indexed by \( \eta \), that satisfy the homoge-
neous wave equation in the absence of sources, i.e., Eq. (1) for \( \mathbf{\hat{k}} = 0 \) with the eigen-wavenumber \( k_0 \rightarrow k_\eta \).

To do so, one defines functions \( g_\eta(r) \equiv n(r)f_\eta(r) \) that form a complete basis, as they are eigenfunctions of the Hermitian operator, \( \mathcal{H}(k_0) = -\frac{1}{n(r)}\nabla \cdot \nabla \times \nabla \cdot \frac{k_0^2}{n(r)} \), according to \( \mathcal{H}(k_0)g_\eta(r) = \lambda_\eta g_\eta(r) \). The eigenvalue, \( \lambda_\eta = (\omega_\eta^2 - \omega_0^2)/c^2 \), determines the wavenumber for a given mode at frequency \( \omega_\eta \). We are interested specifically in the general transverse functions satisfying \( \nabla \cdot [n(r)g_\eta(r)] = 0 \) with eigenvalues \( \lambda_\eta \neq 0 \). These fall into two categories, guided (\( \eta = \mu \)) and unguided (\( \eta = \nu \)) modes, which together form a complete, orthonormal set for transverse vector functions,

\[
\int d^3 r g_\eta^*(r) \cdot g_\eta'(r) = \int d^3 r n^2(r)f_\eta^*(r)f_\eta'(r) = \delta_{\eta\eta'}, \quad (4)
\]

\[
\sum_\eta g_\eta(r)g_\eta^*(r') + \sum_\nu g_\nu(r)g_\nu^*(r') = \delta^{(T)}(r-r')I, \quad (5)
\]

where \( \delta^{(T)}(r-r') \) is the delta function for generalized transverse vector fields [53]. It follows that the general transverse dyadic Green’s function can be decomposed in terms of the eigenfunctions [40, 42]

\[
\tilde{G}^{(T)}(r, r'; \omega_0) = -4\pi \sum_\eta \frac{\omega_\eta^2 f_\eta^*(r)f_\eta'(r')}{\omega_0^2 - \omega_\eta^2}, \quad (6)
\]

where the eigenvalues appear as \( \omega_\eta^2 = c^2k_\eta^2 \). The sum includes both guided and unguided contributions. We focus here on the guided-mode contribution to the Green’s function.

We treat an optical nanofiber of radius \( a \) with step-index profile,

\[
n(r_\perp) = \begin{cases} n_1 & r \leq a \\ n_2 & r > a \end{cases}, \quad (7)
\]

for a silica core (\( n_1 = 1.4469 \)) [54] and infinite vacuum cladding (\( n_2 = 1 \)). For a cylindrically symmetric dielectric the guided modes are \( f_\mu(r) = u_\mu(r_\perp)e^{i\beta_r} / \sqrt{2\pi} \), with indices \( \mu = \{ j, \beta, p \} \) for the \( j^{th} \) guided mode with propagation constant \( \beta \) at frequency \( \omega_\mu = \omega(\beta) \) and polarization \( p \). The transverse mode functions are normalized according to \( \int d^2 r_\perp n^2(r_\perp)u_{\mu p}^*(r_\perp) \cdot u_{\nu q}^*(r_\perp)\big|_{\beta = \beta'} = \delta_{\mu\nu}\delta_{p,q} \) and have units \( 1/\sqrt{\lambda} \). Two convenient guided-mode bases are the quasilinear and quasicircular polarization modes, described in Appendix A [54].

We consider nanofibers that support the lowest \( HE_{11} \) guided modes at the relevant frequency \( \omega_0 \) [50], and thus we drop the mode index \( j \). In this case there are four guided modes: two polarizations \( p \), each with propagation constant \( \beta(\omega_0) = \pm \beta_0 \) corresponding to forward and backward propagation. The guided-mode contribution to the dyadic Green’s function is then

\[
\tilde{G}_{\eta}(r, r'; \omega_0) = \int d^3 r \sum_p \frac{-2\omega_0^2}{\omega_0^2 - \omega_0^2(\beta)} u_{\eta p}(r_\perp)L_{\eta p}(r'_\perp)e^{i\beta_0(z'-z)}, \quad (8)
\]

where \( \omega(\beta) \) is the frequency of the guided \( HE_{11} \) for a given \( \beta \).

For \( z > z' \) (\( z < z' \)), the contribution of the guided modes to the retarded (causal) Green’s function is found by the usual displacement of the pole on the positive (negative) \( \beta \)-axis into the upper (lower) half of the complex plane. The result for \( z \neq z' \) is [6]

\[
\tilde{G}^{(+)}_{\eta}(r, r'; \omega_0) = 2\pi \sum_{b, p} \text{Res}_{\beta=\beta_0} \left[ \frac{-2\omega_0^2}{\omega_0^2 - \omega_0^2(\beta)} \right] u_{b\eta p}(r_\perp)L_{b\eta p}(r'_\perp)e^{i\beta_0(z'-z)} \]

\[
-2\pi i \frac{\omega_0}{v_g} \sum_{b, p} u_{b\eta}(r_\perp)L_{b\eta p}(r'_\perp)e^{i\beta_0(z'-z)} \Theta(z-z')b, \quad (9)
\]

where \( b = \pm \) indicates the propagation direction, \( v_g = |d\omega/d\beta| = \beta_0 \) is the group velocity at \( \omega_0 \), and \( \Theta(b - z') \) is a Heaviside function enforcing causality for the forward- and backward-scattered fields. In the second line, we have suppressed the label \( \beta_0 \) as it is implicit in the definition of the guided modes at frequency \( \omega_0 \).

Radiative properties of a scatterer (the decay rate and energy level shift) are determined by evaluation of the dyadic Green’s function at the source point \( r = r' \) [45]. However, for \( z = z' \) we cannot close the contour. Instead, we expand the resonant denominator in Eq. (8) with the poles moved to yield the retarded (causal) response,

\[
\frac{1}{(\omega_0 + i\epsilon)^2 - \omega_0^2(\beta)} = \frac{1}{2\omega(\beta)} \left[ \frac{1}{\omega_0 + i\epsilon - \omega(\beta)} + \frac{1}{\omega_0 - i\epsilon + \omega(\beta)} \right],
\]

and employ the usual distribution identities [42],

\[
\lim_{\epsilon \rightarrow 0^+} \frac{1}{\omega_0 + i\epsilon + \omega(\beta)} = \mathcal{P}\left[ \frac{1}{\omega_0 + i\epsilon + \omega(\beta)} \right] + \pi i \delta(\omega_0 + \omega(\beta)). \quad (10)
\]

Only the positive-frequency component contributes to the \( \delta \)-function, and it follows that the imaginary part of the Green’s function at \( r = r' \) that determines the resonant Purcell enhancement of spontaneous emission into the guided modes is [41, 45, 57]

\[
\text{Im}[\tilde{G}_{\eta}^{(+)}(r', r; \omega_0 = \omega_{eg})] = \frac{\pi \omega_0}{v_g} \sum_{b, p} u_{b\eta}(r'_\perp)L_{b\eta p}(r'_\perp), \quad (11)
\]

where \( \omega_{eg} \) is resonance frequency of the atomic scatterer.

The energy level shift of the scatterer due to its proximity to the dielectric is found from the real part of the Green’s function at \( r = r' \). To find the total modified spontaneous emission rate and energy level shift one must include the unguided radiation modes [49] or employ other representations of the Green’s function [43].

Equation (9) is the central result from which we can calculate the dispersive response. Consider a forward-propagating input field in the guided modes with frequency \( \omega_0 \), positive-frequency amplitude \( F_0^{(+)} \), and arbitrary polarization, \( E_{\text{in}}^{(+)}(r) = F_0^{(+)} u_{\text{in}}(r_\perp)e^{i\beta_0 z} \) dispersively coupled to an atom at position \( r' \). The effective
mode area at the atom’s position is determined from the total cycle-averaged power transported along the nanofiber, \( P_{in,z} = (\nu_g/2\pi) \int d^2 r n^2(r_\perp)|\mathbf{E}_{in}(r_\perp)|^2 \), and the intensity at the atom, \( I_{in}(r') = (e/2\pi)|\mathbf{E}_{in}(r')|^2 \), via the relation \( 1/F_{in}(r') = \frac{1}{n_g|u_{in}(r_\perp)|^2} \), \( n_g \equiv c/v_g \) is the group index of refraction.

Substitution of the guided-mode Green’s function, Eq. (9), into the Lippman-Schwinger equation, Eq. (2b), yields the transmitted (forward-scattered) and reflected (backward-scattered) output fields, \( \mathbf{E}_\text{out}(r) = \mathcal{F}_0[|\mathbf{u}_{+,out}(r_\perp)e^{i\beta_0 z} + \mathbf{u}_{-,out}(r_\perp)e^{-i\beta_0 z}|] \),

\[
\begin{align*}
\mathbf{u}_{+,out}(r_\perp) &= \sum_{p,p'} c_{pp'} \mathbf{u}_{+,p'}(r_\perp), \\
\mathbf{u}_{-,out}(r_\perp) &= \sum_{p,p'} c_{pp'} \mathbf{u}_{-,p'}(r_\perp),
\end{align*}
\]

where we have decomposed the input into the polarization eigenmodes, \( \mathbf{u}_a(r_\perp) = \sum_p e_p \mathbf{u}_{+,p}(r_\perp) \). For \( z > z' \), the transmission and reflection matrices are

\[
\begin{align*}
t_{pp'} &= 2\pi i k_0 n_g u^*_{+,p'}(r'_\perp) \cdot \hat{\alpha} \cdot u_{+,p}(r'_\perp), \\
r_{pp'} &= 2\pi i k_0 n_g u^*_{-,p'}(r'_\perp) \cdot \hat{\alpha} \cdot u_{+,p}(r'_\perp) e^{2i\beta_0 z'},
\end{align*}
\]

We focus here on the transmitted fields whose interference for input field for \( z > z' \) results in a phase shift and a polarization transformation. For weak scattering the diagonal terms, \( t_{pp} \approx \sqrt{1 - R_p e^{i\delta \phi_p}} \), determine the phase shift and attenuation induced on each polarization mode,

\[
\delta \phi_p = \frac{2\pi i k_0}{A_{in}} \text{Re}(\alpha_{pp}), \quad R_p = \frac{4\pi i k_0}{A_{in}} \text{Im}(\alpha_{pp}).
\]

Here, the \( \{p, p'\} \)-element of the tensor polarizability is given by, \( \alpha_{pp'} \equiv \mathbf{e}_{p'} \cdot \hat{\alpha} \cdot \mathbf{e}_p \), with unit vectors for each of the forward-propagating modes, \( \mathbf{e}_p \equiv \mathbf{u}_{+,p}(r'_\perp)/|\mathbf{u}_{+,p}(r'_\perp)| \).

The phase shift per atom, Eq. (15a), is modified over free space in two ways, both of which are captured by the effective mode area \( A_{in} \). First, although material dispersion in an optical fiber is negligible over the distances we consider, additional waveguide dispersion can lead to a significant reduction in the group velocity [8, 14]. Such “slow light” enhances the atom-photon coupling strength. In the nanofiber geometry this effect is moderate – we calculated the group index to be \( n_g \approx 1.40 \). Second and more importantly, the tight spatial confinement as measured by OD/\( N_A \) significantly increases the coupling strength over free space for every atom along the nanofiber, which yields strong cooperativity. In contrast, in free space diffraction restricts the collective phase shift for an ensemble of atoms [15, 59].

For a Gaussian beam with beam waist \( w_0 \), the total phase shift induced by a collection of polarizable atoms will be \( \delta_0 = N_{eff} 2\pi k_0 \text{Re}(\alpha)/A \), where \( A = \pi w_0^2/2 \) is the beam area at the focus and \( N_{eff} \) is the effective number of atoms that radiate into this mode. One can couple strongly to few atoms at the center by tightly focusing the beam or couple weakly to many atoms by choosing a larger focal volume, but hence, smaller cooperativity per atom.

The off-diagonal terms in the transmission matrix, Eq. (14a), describe the polarization transformation. For example, if we take the polarization of the modes to be the quasilinear, \( p = \{H, V\} \) as defined in Eq. (A4), then \( t_{HV} \equiv \chi_{Far} \) is the rotation angle of the Stokes vector on the Poincaré sphere corresponding to the Faraday effect [60, 61]. The phase difference in that basis, \( \delta \phi_H - \delta \phi_V \), corresponds to birefringence induced on the guided mode and \( t_{HV} \) to Faraday rotation. Analyzed in the quasircular polarization modes \( (p = \pm) \), given in Eq. (A3), the differential phase \( \delta \phi_\pm - \delta \phi_- \) corresponds to Faraday rotation and \( t_{\pm} \) to birefringence. We make use of such polarization transformations as a means to nondestructively measure the atoms and generate collective spin squeezing.

### III. HEISENBERG-LANGEVIN-PICTURE

#### SOLUTION AND ATOMIC RESPONSE

The Lippmann-Schwinger solution, Eq. (2b), determines the input-output relation for linear atomic response given by the polarization tensor \( \hat{\alpha} \). In this section we connect this with the fully quantum mechanical description of dispersive atomic response and input-output relations for the quantized guided modes. Following Ref. [49], we use a Heisenberg-Langevin approach for one-dimensional systems.

The positive frequency component of the quantized electric field operator decomposes into guided and radiation (unguided) modes, \( \hat{\mathbf{E}}^{(+)} = \hat{\mathbf{E}}^{(+)}_g + \hat{\mathbf{E}}^{(+)}_r \), where

\[
\begin{align*}
\hat{\mathbf{E}}^{(+)}_g(r) &= \sum_{b,p} \int_{0}^{\infty} d\omega \sqrt{\frac{\hbar \omega}{v_g}} \hat{a}_{b,p}(\omega) \mathbf{u}_\mu(r_\perp)e^{i\beta(\omega)z}, \\
\hat{\mathbf{E}}^{(+)}_r(r) &= \sum_{m,p} \int_{-\infty}^{\infty} d\omega d\beta \sqrt{\hbar \omega} \hat{a}_{m,p}(\omega, \beta) \mathbf{u}_\mu(r_\perp)e^{i\beta(\omega)z}.
\end{align*}
\]

The HE\( \mu \) guided modes are specified by \( \mu = (\omega, b, p) \), where \( \omega \) is the mode frequency, \( p \) is the polarization, and the propagation direction \( b = \pm \) corresponds to wavenumber \( b/\beta(\omega) \). The radiation modes are specified by \( \nu = (\omega, \beta, m, p) \), where \( m \) is the azimuthal (angular momentum) quantum number, \( p \) labels the two orthogonal polarizations, and longitudinal propagation constant \( \beta \) can vary continuously from \( -kn_2 \) to \( kn_2 \), with \( k = \omega/c \) [42, 49]. The creation/annihilation operators satisfy the usual continuous-mode commutation relation.
where the coupling constants for guided/radiation modes are

\[ \delta_{ij} = \langle i\rangle^{(e)} \hat{d}_{eg}^{(e)} \langle r' \rangle^{(e)} \hat{E}^{(e)}(r') \].

Integrating the field equations,

\[ \hat{a}_\mu(t) = \hat{a}_\mu(t_0) e^{-i\omega(t-t_0)} \]

\[ + i \sum_{e,g} g_{\mu,e,g}^* \int_{t_0}^{t} dt' e^{-i\omega(t-t')} \hat{\sigma}_{ge}(t'), \]

\[ \hat{a}_\nu(t) = \hat{a}_\nu(t_0) e^{-i\omega(t-t_0)} \]

\[ + i \sum_{e,g} g_{\nu,e,g}^* \int_{t_0}^{t} dt' e^{-i\omega(t-t')} \hat{\sigma}_{ge}(t'), \]

substituting into Eq. (23c), and making the usual Markov approximation [49] gives an expression for the ground-excited coherences. This yields

\[ \frac{d\hat{\sigma}_{ge}}{dt} = -i\omega_{eg} \hat{\sigma}_{ge} - \sum_{e'} \frac{\Gamma_{ee'}}{2} \hat{\sigma}_{ge} + \sum_{e'} \sum_{g'} \left( \hat{\sigma}_{ee'} \hat{\sigma}_{gg'} - \delta_{g,e} \hat{\sigma}_{ee'} \right) \]

\[ \int_0^{\infty} \left( \sum_{b,p} g_{\mu,e,g}^* \hat{a}_{\mu}(t_0) + \sum_{m,p} d_{b,p} \hat{g}_{\nu,e,g} \hat{a}_{\nu}(t_0) \right) e^{-i\omega(t-t_0)}, \]

where the decay rates of excited-populations and coherences are given by

\[ \Gamma_{ee'} = 2\pi \sum_{\mu,g} g_{\mu,e,g}^* g_{\nu,e',g} |\omega=\omega_{eg}| \]

\[ + 2\pi \sum_{m,p} \int_{-kn_2}^{kn_2} d_{b,p} \hat{g}_{\nu,e,g} \hat{g}_{\nu,e',g} |\omega=\omega_{eg}|, \]

and the small energy shift is absorbed into the transition frequency \( \omega_{eg} = (E_e - E_g) / \hbar \). Equation (25) captures the modification of the spontaneous emission rate due to the nanofiber. The first sum describes decay into the guided modes and the second into the unguided radiation modes [43, 49, 62–64]. The decay rate of a given excited state into all guided modes is given by

\[ \Gamma_{ee'} = 2\pi \sum_{b,p,g} |g_{\mu,e,g}|^2 |\omega=\omega_{eg}| \sum_{e'} \sum_{g'} |\langle e|d|g\rangle u_{b,p}(r')|^2. \]

This is in agreement with the expected expression from the guided-mode contribution to the dyadic Green’s function in Eq. (11),

\[ \Gamma_{ee'} = \frac{2}{\hbar} \sum_{b,p,g} |\langle e|d|g\rangle \cdot \text{Im} \left[ G_{gg'}^{(e)}(r', r'; \omega_{eg}) \right] \cdot \langle e|d|g\rangle, \]

which is enhanced over the free-space rate by the Purcell factor.

Here we are interested in linear response for excitation far from resonance. We follow Ref. [65] and consider an atom sufficiently far from the fiber surface such that the modification of the spontaneous emission rate is small. In this case the decay rate is approximated as \( \Gamma_{ee'} \approx \delta_{e,e'} \Gamma_e \), where \( \Gamma_e \) is the total decay rate from excited state \( |e\rangle \), given by the diagonal elements of Eq. (25). In steady...
state, the dipole operator in the linear regime (\(\hat{\sigma}_{ee'} \to 0\)) is approximately

\[
\hat{\sigma}_{ge} \approx -\sum_{g'} \hat{\sigma}_{gg'} \int_0^\infty d\omega \sum_{b, p} \frac{g_{b, p, g'} g_{\mu, e, g'}}{\omega - \omega_{eg} + i\Gamma_e/2} \hat{a}_\mu(t_0)
+ \sum_{m, p} \int_{-k_2}^{k_2} d\delta \frac{g_{\mu, e, g'}}{\omega - \omega_{eg} + i\Gamma_e/2} \hat{a}_\mu(t_0) e^{-i\omega(t-t_0)}.
\]

(28)

By substituting this into Eq. (24a) and defining asymptotic modes, \(\hat{\sigma}^{in}_e(\omega) = \lim_{t_0 \to -\infty} \hat{a}(\omega)e^{i\omega t_0}\), \(\hat{\sigma}^{out}_e(\omega) = \lim_{t_0 \to +\infty} \hat{a}(\omega)e^{i\omega t_0}\) [51], we obtain the input-output relationship for the guided modes,

\[
\hat{\sigma}^{out}_\mu(\omega) = \hat{\sigma}^{in}_\mu(\omega) - 2\pi i \sum_{b', p, e, g, g'} g_{b', p, e, g} g_{\mu, e, g'} \hat{a}^{in}_\mu(\omega) - 2\pi i \sum_{m, p, e, g, g'} g_{m, p, e, g} g_{\mu, e, g'} \hat{a}^{in}_\mu(\omega).
\]

(29)

This input-output relation contains the phase shift on forward scattered modes as well as attenuation due to elastic scattering into all other modes. For a probe with frequency \(\omega_0\), Eq. (29) agrees with the expected form given by the Lippmann-Schwinger equation in the first Born approximation [66],

\[
\hat{E}^{(+)}_{out, g}(r, \omega_0) = \hat{E}^{(+)}_{in, g}(r, \omega_0) + G^{(+)}_g(r, r', \omega_0) \cdot \hat{\alpha} \cdot \hat{E}^{(+)}_{in, g}(r', \omega_0) + \hat{E}^{(+)}_{in, g}(r', \omega_0) \hat{\alpha} \cdot \hat{E}^{(+)}_{in, g}(r', \omega_0),
\]

(30)

by noting that for the guided-mode dyadic Green’s function given in Eq. (9),

\[
\int d^2r \hat{u}^*_\mu(r_\perp) \cdot \hat{\alpha} \cdot \hat{u}_\mu(r_\perp) = \frac{2\pi \omega_0}{v_g} \hat{u}^*_{\mu, p}(r_\perp) \cdot \hat{\alpha} \cdot \hat{u}_{\mu, p}(r_\perp).
\]

(31)

Here, the atomic polarizability operator [61, 67, 68], is given by

\[
\hat{\alpha} = -\frac{1}{\hbar} \sum_{\mu, e, g} |g\rangle \langle e| \langle \hat{d}| \langle g'| \Delta_{eg} + i\Gamma_e/2 \langle g'|,
\]

(32)

and \(\Delta_{eg} = \omega_0 - \omega_{eg}\) is the laser detuning from the atomic transition. For an atom in ground state \(|g\rangle\) and polarization \(p\), the phase shift can be expressed as [65]

\[
\delta \phi_{\mu, g} = 2\pi \frac{\omega_0}{v_g} \hat{u}^*_{\mu, p}(r_\perp) \cdot \text{Re}[\langle g| \hat{\alpha} \hat{d}| e \rangle] \cdot \hat{u}_{\mu, p}(r_\perp)
= -\frac{\omega_0}{v_g} \sum_e 2\pi \left| \langle e| \hat{d}| g \rangle \cdot \hat{u}^*_{\mu, p}(r_\perp) \right|^2 \frac{1}{\hbar \Delta_{eg}}.
\]

(33)

We employ this dispersive response for QND measurement of atoms, as we describe in the next section.

### IV. QND MEASUREMENT OF ATOMS

The dispersive interface between the atoms and nanofiber guided photons provides the entangling mechanism necessary to perform a QND measurement on the atoms. We restrict here to the quasilinear modes, \(p = \{H, V\}\), of a single HE\(_{11}\) guided mode at frequency \(\omega_0\), whose form is given explicitly in Eq. (A4). In typical experimental configurations, two one-dimensional arrays of atoms are trapped on either side of the nanofiber, see Fig. 2. We define coordinate axes \((x, y, z)\) with \(z\) oriented along the fiber axis for forward propagation, and the two chains of atoms lie in the \(x-z\) plane at azimuthal angles \(\phi = \{0, \pi\}\). In the evanescent region, the \(H\)-mode is purely \(e_z\)-polarized at \(\phi = \pm \pi/2\) and the \(V\)-mode is purely \(e_y\)-polarized at \(\phi = \{0, \pi\}\). At other azimuthal angles the electric field is generally rotating along an ellipse in the \(x-z\) plane. The atoms at \(\phi' = 0\) experience \(H\) and \(V\) fields,

\[
\hat{u}_{\mu, H}(r_\perp, \phi = 0) = \sqrt{2} \left[ e_x u_x(r_\perp) + ie_y u_y(r_\perp) \right],
\]

(34a)

\[
\hat{u}_{\mu, V}(r_\perp, \phi = 0) = \sqrt{2} \left[ e_y u_y(r_\perp) \right],
\]

(34b)

where the real-valued functions \(u_{\mu}(r_\perp)\), given in Eq. (A5), depend only on the radial coordinate. On the opposite side of the fiber at \(\phi' = \pi\), atoms experience the same transverse electric field, but the \(z\)-component changes sign. This broken symmetry has been used to selectively address and separately control the two atomic arrays [30, 32, 36].

We consider quasi-monochromatic fields at carrier frequency \(\omega_0\) that are sufficiently narrowband, \(\Delta \omega \ll \omega_0\).
For each guided mode we define input propagating, continuous-mode field operators in the interaction picture [46, 47, 50],
\[
\hat{a}_{b,p}(z,t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty d\omega \; \hat{a}_{b,p}(\omega) e^{i\hbar k_z z - (\omega - \omega_0) t},
\]
(35)
that satisfy the free-field commutation relations,
\[
[\hat{a}_{b,p}(z,t), \hat{a}_{b',p'}(z',t')] = \delta_{bb'}\delta_{pp'}\delta(t-t'-(z-z'))/v_g.
\]
(36)
In terms of these propagating modes the quantized electric field operator, Eq. (16a), becomes
\[
\hat{E}^{(+)}(r_\perp,z;\phi;\pm) = \sum_{b,p} \sqrt{2\pi\hbar \omega_0/v_g} \mathbf{u}_{b,p}(r_\perp,\phi) \hat{a}_{b,p}(\pm,\pm) e^{i\hbar k_z z},
\]
(37)
Considering here only the forward-propagating guided modes (\(b = +\)), we drop the \(b\) index. The propagating electric field, Eq. (37), interacts with the trapped atoms via the dispersive light-shift Hamiltonian [61, 68, 69],
\[
\hat{H}_{LS} = -\sum_{n=1}^{N_A} \sum_{\phi = 0,\pi} \hat{\mathbf{\hat{A}}}^{(n)}(r_\perp^\prime;\phi) \cdot \hat{\mathbf{\hat{A}}}^{(+)}(r_\perp;\phi),
\]
(38)
where \(\hat{\mathbf{\hat{A}}}^{(n)}\) is the atomic tensor polarizability operator, given in Eq. (32), for the \(n\)th atom trapped near the nanofiber surface at position \(r_\perp^\prime\). We ignore here any effects of atomic motion and treat the atoms as localized at fixed positions in space.

The Lippmann-Schwinger scattering equation, Eq. (30), follows in the time domain as the evolution of coarse-grained input-output modes [46, 51, 65]. Since multiple scattering is negligible and the propagation time across the ensemble is small compared to the atomic dynamics, we drop the position label \(z\) and index the propagating fields by time alone; \(\hat{a}_{b,p}(z,t) \rightarrow \hat{a}_{b,p}(t)\) as is standard in input-output theory [46, 70]. It follows that the effects of retardation can be ignored, in which case each term in the sum over atoms contributes equally for all atoms (for details see [50, 69]). The forward-propagating output fields are then given by the Fourier transform of Eq. (29), yielding [50]
\[
\hat{a}^{\text{out}}_p(t) = \hat{a}^{\text{in}}_p(t) + \frac{i}{2\pi \hbar \omega_0} \sum_{p'} \left[ N_0 \mathbf{u}^*_p(r_\perp^\prime,0) \cdot \hat{\mathbf{\hat{A}}} \cdot \mathbf{u}_{p'}(r_\perp^\prime,0) + N_\pi \mathbf{u}^*_p(r_\perp^\prime,\pi) \cdot \hat{\mathbf{\hat{A}}} \cdot \mathbf{u}_{p'}(r_\perp^\prime,\pi) \right] \hat{a}^{\text{in}}_p(t),
\]
(39)
where \(\{N_0, N_\pi\}\) are the total number of atoms trapped at \(\phi' = \{0, \pi\}\). The quantum effects from the first term give rise to shot noise in the transmitted field at the detector. The second term represents scattering into the guided modes, as described by the dyadic Green’s function, Eq. (27).

We introduce the vector Stokes operators that describe the polarization of the propagating fields in the quasilinear \(HV\)-basis,
\[
\begin{align*}
\hat{S}_1(t) &= \frac{1}{2} \left[ \hat{a}^\dagger_H(t) \hat{a}_V(t) - \hat{a}^\dagger_V(t) \hat{a}_H(t) \right], \\
\hat{S}_2(t) &= \frac{1}{2} \left[ \hat{a}^\dagger_H(t) \hat{a}_V(t) + \hat{a}^\dagger_V(t) \hat{a}_H(t) \right], \\
\hat{S}_0(t) &= \frac{i}{2} \left[ \hat{a}^\dagger_H(t) \hat{a}_V(t) - \hat{a}^\dagger_V(t) \hat{a}_H(t) \right],
\end{align*}
\]
(40a-40c)
that satisfy equal-\(z\) commutation relations following from Eq. (36),
\[
\left[ \hat{S}_i(t), \hat{S}_j(t') \right] = i\epsilon_{ijk}(t-t')\hat{S}_k(t).
\]
(41)
These, along with the total photon flux operator,
\[
\hat{S}_0(t) = \frac{1}{2} \left[ \hat{a}^\dagger_H(t) \hat{a}_H(t) + \hat{a}^\dagger_V(t) \hat{a}_V(t) \right],
\]
(42)
are used to reexpress the Hamiltonian, Eq. (38), in the \(HV\)-basis,
\[
\hat{H}_{LS} = -2\pi \hbar g_n \sum_{\phi = 0,\pi} N_{\phi'} \left\{ \frac{1}{2} \left[ \hat{K}_H(\phi') + \hat{K}_V(\phi') \right] \hat{S}_0(t) + \left[ \hat{K}_H(\phi') - \hat{K}_V(\phi') \right] \hat{S}_2(t) + i \left[ \hat{K}_H(\phi') + \hat{K}_V(\phi') \right] \hat{S}_1(t) \right\},
\]
(43)
The atomic couplings to the \{\(H, V\)\} modes,
\[
\hat{K}_{pp'}(\phi') \equiv |\mathbf{u}^*_p(r_\perp^\prime,\phi)| |\mathbf{u}_{p'}(r_\perp^\prime,\phi')| \hat{a}_{pp'}(\phi'),
\]
(44)
are determined by components of the quantum mechanical tensor operator weighted by the transverse mode functions at the atomic position, \(\hat{a}_{pp'}(\phi') = \mathbf{e}_{p'}(\phi') \cdot \hat{\mathbf{\hat{A}}} \cdot \mathbf{e}_p(\phi')\), whose classical analog appeared in Eq. (15a).

We explore a QND measurement of \(^{133}\text{Cs}\) atoms in the electronic ground state, \(6S_{1/2}\), via polarization spectroscopy based on the collective atom-light coupling described by the dispersive light-shift Hamiltonian in Eq. (43). Polarization transformations occur due to the tensor nature of the atomic response,
\[
\hat{\mathbf{\hat{A}}} = \sum_{f,f'} \alpha_0(\Delta f'f) \sum_{i,j} \hat{\mathbf{\hat{A}}}(f,f'),
\]
(45)
where the operator \(\hat{\mathbf{\hat{A}}}(f,f') = \sum_{i,j} \hat{\mathbf{\hat{A}}}_{ij}(f,f') \mathbf{e}_i \otimes \mathbf{e}_j\) decomposes into irreducible components within each ground hyperfine multiplet \(f\) for light detuned near excited multiplet \(f'\),
\[
\hat{\mathbf{\hat{A}}}_{ij}(f,f') = C_{ij}^{(0)} \delta_{i,j} + iC_{ij}^{(1)} \mathbf{e}_g \hat{f}_k + C_{ij}^{(2)} \left[ \frac{1}{2}(\hat{f}_k \hat{f}_k + 2 \hat{f}_k) - \frac{i}{2} \hat{\mathbf{\hat{f}}} \delta_{i,j} \right],
\]
(46)
Here, \(\alpha_0(\Delta f') = -\frac{\sigma_0}{\Delta \omega_{f'f}} \Delta \omega_{f'f}/\pi\) is the characteristic dynamic polarizability where \(\sigma_0 = 3\lambda\omega_c/2\pi\) is the resonant scattering cross section, \(\hat{\mathbf{\hat{f}}}\) is the atomic spin operator.
in hyperfine multiplet \( f \), and \( C^{(K)}_{ij} \) are coefficients for irreducible rank-\( K \) components defined in [61].

In addition to the atomic tensor response, the nanofiber geometry gives rise to unique features of polarisation spectroscopy not present in free space. The spatial anisotropy of the intensity for the quasi-linearly polarized guided modes leads to unequal scattering of the \( H \) and \( V \) modes, producing intrinsic birefringence even for a purely scalar atomic polarizability. In particular, atoms trapped on the quasi-\( H \) axis leads to a phase delay of this mode relative to the fast quasi-\( V \) axis. This birefringence was exploited by Dawkins et al. [33] as a mechanism for implementing a dispersive QND measurement of the number of atoms trapped around the nanofiber, as we treat in the next section.

A. Dispersive atom number measurement

The anisotropy of the guided modes provides a mechanism for counting the number of atoms trapped around the nanofiber based on polarization spectroscopy. We consider \( N_A \) atoms, each in a completely mixed hyperfine spin state. In this case the atomic polarizability tensor in Eq. (46) reduces to \( \langle A_{ij}(f,f') \rangle = C^{(0)}_{ijf} \delta_{ij} \), and the collective interaction is determined entirely by the scalar (rank-0) terms. With the atoms trapped along the quasi-\( H \) axis, while \( \langle \hat{K}_{HH} \rangle \neq \langle \hat{K}_{VV} \rangle \), the off-diagonal elements in Eq. (43) do not contribute to the Birefringent interaction we are interested in and actually vanish \( \langle \hat{K}_{HV} \rangle = \langle \hat{K}_{VH} \rangle = 0 \) when \( x-, y- \) or \( z- \) axis is chosen as the quantization axis which includes the one close to the optimal choice of quantization axis for spin squeezing we will discuss in the next section. Atoms on either side of the nanofiber experience the same scalar light shift yielding from Eq. (43) the Hamiltonian for QND measurement of atom number,

\[
\hat{H}_N = -2\pi \hbar k_0 n_g \sum_{\phi'} N_{\phi'} \{ \langle \hat{K}_{HH}(\phi') \rangle - \langle \hat{K}_{VV}(\phi') \rangle \} \hat{S}_1(t) = \hbar \chi_N N_A \hat{S}_1(t). \tag{47}
\]

This birefringent interaction induces a rotation of the Stokes vector around the \( S_1 \)-axis on the Poincaré sphere through an angle,

\[
\chi_N = \frac{\sigma_0}{A_N} \sum_{j,j'} C_{jj'}^{(0)} \frac{\Gamma}{2\Delta_{jj'}}, \tag{48}
\]

characterized by an effective area, \( A_N^2 = (n_{j'}/2)(|\mathbf{u}_H|_i^2 - |\mathbf{u}_V|_i^2)^2 \).

Dawkins et al. [33] used this interaction to make a dispersive measurement of \( N_A \) via birefringence polarimetry in the usual way: launching linearly polarized light at 45° to the quasi-\( H \) axis, \( \mathbf{u}_{in} = (\mathbf{u}_H + \mathbf{u}_V)/\sqrt{2} \), and measuring the differential power between the guided right- and left-circularly polarized photons. Thus, the integrated measurement is described by the operator \( \hat{M} = \int_0^T dt' \hat{S}_3^{out}(t') \). The shot-noise variance of the polarimeter, \( \Delta \hat{M}^2 |_{SN} = \chi_N^2 N_L T \) for integration time \( T \), determines the fundamental resolution of the polarimeter. The smallest detectable atom number using this dispersive measurement is thus, \( \delta N_A \sim (\chi_N^2 N_L T)^{-1/2} \) [71]. In an ideal setting, \( \delta N_A \) can always be reduced by increasing the integration time, but in practice this time is limited by atom loss. As a coarse approximation we take this time to be \( T = \gamma_s^{-1} \), where \( \gamma_s \) is the photon scattering rate in free space, and assume perfect quantum efficiency of the detectors. For detuning \( \Delta \) large compared to the excited hyperfine splitting on the D1- or D2-line \( (j' = 1/2 \text{ or } 3/2) \), the unit-oscillator scattering rate is \( \gamma_s = \frac{\sigma_0}{\hbar \alpha} \left( \frac{\Gamma}{\Delta} \right)^2 N_L \), with effective area determined by the probe at the atomic position, Eq. (12). In this limit, the rotation angle \( \chi_N = C_{jj'}^{(0)} (\sigma_0/A_N)(\Gamma/2\Delta) \), Eq. (48), yields a shot noise-limited atom number resolution,

\[
\delta N_A \sim \frac{1}{C_{jj'}^{(0)}} \sqrt{\frac{A_N^2}{\alpha_0}} \tag{49}
\]

where \( C_{jj'}^{(0)} = \sum_{f,f'} C_{jj'}^{(0)} \) are the far-detuned, rank-0 coefficients on a \( j \rightarrow j' \) transition [61]. Using the parameters reported by Dawkins et al. [33], we find the shot-noise limited minimum detectable atom-number \( \delta N_A \sim 10 \) for atoms trapped at 1.8a ~ 2.0a from the fiber axis with a D2-line probe light.

In practice, loss and decoherence limit the atom-number resolution [33, 72]. The experimental implementation reported by Dawkins et al. [33] implies a resolution of a few tens of atoms for 200 ~ 1000 trapped atoms. A similar experiment based on a two-color QND measurement in a nanofiber geometry was recently carried out by Béguin et al. [34] to squeeze the uncertainty in the number of trapped atoms. They achieved an atom number uncertainty of \( \delta N_A = 8 \) for \( N_A \sim 2500 \) atoms, well below standard quantum limit, \( \delta N_A = \sqrt{N_A} \).

B. Collective spin squeezing via QND measurement

The same birefringent interaction, Eq. (43), can be utilized in a QND measurement to squeeze the projection noise of the collective atomic spin. We consider squeezing of the uncertainty associated with the “clock states” of cesium, \( \{|\uparrow\rangle = |6S_{1/2}, f = 4, m_f = 0 \rangle \) and \( |\downarrow\rangle = |6S_{1/2}, f = 3, m_f = 0 \rangle \), which define a pseudospin within each atom and associated Pauli operators \( \{\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3\} \). The quantum uncertainty in the collective pseudospin,

\[
\hat{J}_3 = \frac{1}{2} \sum_{n=1}^{N_A} \hat{\sigma}_3^{(n)}, \tag{50}
\]
fundamentally limits the precision of atomic clocks [73]. For atoms prepared in a spin coherent state (SCS) the projection noise, $\Delta J^2_{||}\text{SCS}=N_A/4$, sets the standard quantum limit for spin measurements. A spin squeezed state (SSS) exhibits reduced fluctuations, $\Delta J^2_{||}\text{SSS}<N_A/4$, due to negative pairwise correlations between the atoms [74]. Spin squeezing is typically quantified with the metrological squeezing parameter defined by Wineland et al. [73],

$$\xi^2 \equiv N_A \frac{\Delta J^2_{||}}{(J||)^2}, \quad (51)$$

where $\langle J|| \rangle$ is the mean collective spin along the direction of spin polarization.

The clock states are defined to have zero projection of angular momentum with respect to a bias magnetic field that defines a quantization axis, $e_z$. Within the clock-state subspace the rank-1 vector light shift in the dispersive Hamiltonian, Eq. (43), vanishes since $\langle \hat{f}|e_k|\uparrow \rangle = \langle \hat{f}|e_k|\downarrow \rangle = 0$ for any direction of the spin, $k$, and any quantization axis, $e_z$. Furthermore, as shown below, atoms on either side of the nanofiber experience the same birefringent coupling. The resulting Hamiltonian, restricted to the clock subspace, couples the guided field of the nanofiber to the $J_3$-component of the collective pseudospin. The interaction has contributions from both the scalar and tensor light shifts,

$$\hat{H}_{J_3} = \hbar \left\{ \left[ (\chi_{H,\uparrow} + \chi_{V,\uparrow}) - (\chi_{H,\downarrow} + \chi_{V,\downarrow}) \right] \hat{J}_3 \hat{S}_0(t) \right\} + \left\{ \left[ (\chi_{H,\uparrow} - \chi_{V,\uparrow}) - (\chi_{H,\downarrow} - \chi_{V,\downarrow}) \right] \hat{J}_3 \hat{S}_1(t) \right\}, \quad (52)$$

where the coupling strength between an atom in the clock subspace and a photon with polarization $p = \{H,V\}$ is

$$\chi_{p,f} \equiv -2\pi k_0 n_g |\mathbf{u}_p(\mathbf{r})|^2 (f,0)|\vec{\alpha}_{pp}(f,0)|, \quad (53)$$

and $f = \{4,3\}$ labels $\{\uparrow, \downarrow\}$. The diagonal terms in the polarizability tensor are the same for atoms at positions above and below the nanofiber, and thus all atoms contribute equally. In addition, a constant birefringence proportional to $J_0 \hat{S}_1$ is neglected here as it can be canceled with a compensating waveplate. Finally, the first term in Eq. (52) does not affect polarization spectroscopy, but will act to rotate the pseudo-spin around the $J_3$-axis of the generalized Bloch sphere proportional to classical intensity fluctuations. While this does not affect the squeezing of projection noise in $J_3$, it affects the metrologically relevant squeezing by adding uncertainty to the direction of the mean spin. By choosing a “magic frequency” at which the light shifts on the two clock states are equal,

$$\chi_{H,\uparrow} + \chi_{V,\uparrow} = \chi_{H,\downarrow} + \chi_{V,\downarrow}, \quad (54)$$

this term can be canceled [75], where we have ignored the imaginary part of the coupling strengths in the dispersive regime. Using the D1-line of $^{133}$Cs atoms as the probe light, there are two magic-frequency solutions, $\tilde{\omega}_3$ and $\tilde{\omega}_4$, shown in Fig. 3(a).

Because the guided probe light at the position of the atom will generally be elliptical, the light-shift interaction coherently couples different magnetic sublevels in a given manifold $f$, and thus does not conserve $J_3$. For example, the ellipticity of the probe light leads to a fictitious magnetic field proportional to $i \mathbf{E}_{in}^{(\pm)}(r') \times \mathbf{E}_{in}^{(\mp)}(r')$ that causes a precession of the spin within hyperfine manifold $f$. This can be mitigated by a sufficiently strong bias magnetic field compared to the fictitious field [76].

The remaining QND interaction Hamiltonian is

$$\hat{H}_{J_3} = \hbar \chi_{J_3} \hat{J}_3 \hat{S}_1(t), \quad (55)$$

where the rotation angle on the Poincaré sphere at the magic wavelength is,

$$\chi_{J_3} = (\chi_{H,\uparrow} - \chi_{V,\uparrow}) - (\chi_{H,\downarrow} - \chi_{V,\downarrow}) = 2(\chi_{H,\uparrow} - \chi_{H,\downarrow}). \quad (56)$$

In the standard way, squeezing the uncertainty in $\hat{J}_3$ by QND measurement can be generated by preparing the atoms in a SCS along $\hat{J}_1$, passing a probe prepared along $\hat{S}_2$ with photon flux $N_L$, and continuously monitoring the
atomic polarizability is azimuthal symmetry of clock state around the ef of the atom, set by the magnetic field. Because of the two fundamental sources. The anisotropy of the polarizability tensor is diagonal in that basis. Noting Eq. (46). Let reducible tensor decomposition of the atomic polarizability, onpendence on the two clock states of the atom. This spin-dependent coupling will depend on the choice of quantization axis that defines the clock state with projection \( m_f = 0 \).

We combine these two effects and obtain a compact expression for the coupling strength \( \chi_{js} \) using the irreducible tensor decomposition of the atomic polarizability, Eq. (46). Let \( \{ e_x, e_y, e_z \} \) be a space-fixed Cartesian coordinate system, where \( e_z \) defines the quantization axis of the atom, set by the magnetic field. Because of the azimuthal symmetry of clock state around the \( e_z \)-axis, the polarizability tensor is diagonal in that basis. Noting that \( \langle f, 0 \mid f, f \rangle = 0 \) and \( \langle f, 0 \mid f, f \rangle = \langle f, 0 \rangle \langle f, 0 \rangle = \langle f, 0 \rangle \langle f, 0 \rangle = \langle f, 0 \rangle \langle f, 0 \rangle = f(f + 1)/2 \), it follows that the expectation value of the irreducible rank-2 component of the atomic polarizability is

\[
\langle f, 0 \rangle \langle 0 | C_{ff'} f(f + 1)/2 = \delta(f, f') \langle f | e \rangle (3 e_z \otimes e_z). \tag{59}
\]

The combined scalar and tensor light shifts yield a coupling strength, Eq. (53),

\[
\chi_{p,f} = n_g \sigma_0 \left( a_f |u_p(r'_\perp)|^2 - b_f |e_z \cdot u_p(r'_\perp)|^2 \right), \tag{60}
\]

with coefficients that depend on detunings and atomic structure,

\[
a_f = \sum_{f'} \left( C_{ff'}^{(0)} + \frac{f(f + 1)}{6} C_{ff'}^{(2)} \right) \frac{\Gamma}{4 \Delta_{ff'}}, \tag{61}
\]

\[
b_f = \frac{f(f + 1)}{2} \sum_{f'} C_{ff'}^{(2)} \frac{\Gamma}{4 \Delta_{ff'}}. \tag{62}
\]

At the magic wavelength set by Eq. (54),

\[
\frac{a_4 - a_3}{b_4 - b_3} = \frac{|e_z \cdot u_p(r'_\perp)|^2 + |e_z \cdot u_H(r'_\perp)|^2}{|u_H(r'_\perp)|^2 + |u_V(r'_\perp)|^2}, \tag{63}
\]

which depends on the choice of quantization axis.

We write the effective rotation angle in the Hamiltonian, Eq. (55), as

\[
\chi_{js} = \frac{\sigma_0}{A_{js}} \frac{\Gamma}{2 \Delta_{js}}, \tag{64}
\]

with an “effective detuning” set by the magic-wavelength condition,

\[
\Delta_{js}^{-1} = \frac{4}{\Gamma} (b_4 - b_3) = \sum_{f'} \left( \frac{C_{ff'}^{(2)}}{4 \Delta_{ff'}} - \frac{C_{3f'}^{(2)}}{\Delta_{3f'}} \right), \tag{65}
\]

and an effective area given by

\[
A_{js}^{-1} = n_g \frac{|e_z \cdot u_p(r'_\perp)|^2 |u_H(r'_\perp)|^2 - |e_z \cdot u_H(r'_\perp)|^2 |u_V(r'_\perp)|^2}{|u_H(r'_\perp)|^2 + |u_V(r'_\perp)|^2}. \tag{66}
\]

We see here the explicit dependence of the coupling strength on both the anisotropy of the modes and on the tensor atomic polarizability. The dependence of the magic detunings on the direction of quantization axis is shown in Fig. 3(b) for atoms trapped at a typical distance of \( r'_\perp = 1.8a \) on the x-axis. In typical operating regimes, the magic frequencies are hundreds of MHz from resonance with either excited state, placing the interaction in the off-resonant, dispersive regime. Using these magic detunings, in Fig. 3(c) we show the variation in \( \chi_{js} \) as a function of \( \varphi \). This suggests that, based solely on the strength of the coherent interaction, the x-axis is the optimal quantization axis. As we will see in the next section, the optimal quantization axis is significantly modified when decoherence due to optimal pumping is included.

C. Decoherence due to optical pumping

The treatment above considers an idealized QND interaction. The coupling of the atoms to the probe, however, will always lead to scattering of photons into modes other than the forward-scattered guided mode. This is accompanied by optical pumping that destroys the entanglement associated with spin squeezing. In addition it reduces the metrologically useful signal. The maximum achievable metrologically relevant squeezing is determined by the balance of this decoherence with the QND measurement.
We model this using a first-principles stochastic master equation description (SME) \[15, 77,\]
\[d\hat{\rho} = s\sqrt{\frac{\kappa}{4}} \mathcal{H}[\hat{\rho}]dW + \frac{\kappa}{4} \mathcal{L}[\hat{\rho}]dt + \sum_n D_n[\hat{\rho}]dt, \tag{68}\]
where \(s = \text{sign}(\chi_{\alpha})\) and \(\hat{\rho}\) is the collective atomic state. The measurement strength \(\kappa = |\chi_{\alpha}|^2 N_L\) determines the rate of the spin squeezing in the absence of decoherence. The first two terms describe the QND measurement, where \(dW\) is a stochastic Weiner increment satisfying \(dW^2 = dt\). The conditional dynamics that result from the measurement are generated by the superoperator
\[\mathcal{H}[\hat{\rho}] = \hat{J}_3 \hat{\rho} + \hat{\rho} \hat{J}_3 - 2\langle \hat{J}_3 \rangle \hat{\rho}, \tag{69}\]
and the collective Lindblad map is
\[\mathcal{L}[\hat{\rho}] = -\frac{1}{2}(\hat{\rho} \hat{J}_3^2 + \hat{J}_3^2 \hat{\rho}) + \hat{J}_3 \hat{\rho} \hat{J}_3. \tag{70}\]
The final term in Eq. (68) describes the effect of optical pumping acting locally on each atom along the nanofiber.

The optical pumping map is governed by a standard master equation \[61\]. Restricting to the two-dimensional subspace associated with the clock states, the action on the \(n^{th}\) atom is
\[D_n[\hat{\rho}] = \sum_{f=3,4} \left\{ \gamma_f \frac{1}{2} \left[ \hat{\rho} \langle f, 0 | \langle f, 0 \rangle \rangle^{(n)} + | f, 0 \rangle \langle f, 0 | \hat{\rho} \langle f, 0 | \langle f, 0 \rangle \rangle^{(n)} \right] \right. \]
\[+ \sum_{f=3,4} \gamma_f \langle f, 0 | \langle f, 0 | \hat{\rho} \langle f, 0 | \langle f, 0 \rangle \rangle^{(n)} \}, \tag{71}\]
Here, \(\gamma_f\) is the total rate of photon scattering by atoms in state \(| f, 0 \rangle\) and \(\gamma_{f \rightarrow \bar{f}}\) is the rate of optical pumping between the clock states, \(| f, 0 \rangle \rightarrow | \bar{f}, 0 \rangle\) (see Appendix B). Expressed in terms of Pauli operators on the clock-state pseudospin, the map acts as
\[D_n[\hat{\rho}] = -\left[ \frac{2(\gamma_{\uparrow\downarrow} + \gamma_{\downarrow\uparrow} - \gamma_{\uparrow\uparrow} - \gamma_{\downarrow\downarrow})}{4} \right] \hat{\rho} \]
\[+ \gamma_{\uparrow\downarrow} - \gamma_{\downarrow\uparrow} + \gamma_{\uparrow\uparrow} - \gamma_{\downarrow\downarrow} \left[ \frac{1}{4} \hat{\sigma}_3^{(n)} \hat{\rho} \hat{\sigma}_3^{(n)} + \hat{\sigma}_3^{(n)} \hat{\rho} \hat{\sigma}_3^{(n)} \right] \]
\[+ \gamma_{\uparrow\downarrow} \hat{\sigma}_3^{(n)} \hat{\rho} \hat{\sigma}_3^{(n)} + \gamma_{\downarrow\uparrow} \hat{\sigma}_3^{(n)} \hat{\rho} \hat{\sigma}_3^{(n)} \]. \tag{72}\]

There are three important features of this map that are not typical in a QND measurement of ideal spin-\(\frac{1}{2}\) particles. First, the map is not trace preserving because atoms can be pumped out of the clock states. Second, unequal rates of optical pumping for \(| \uparrow \rangle\) and \(| \downarrow \rangle\) polarize the mean \(\langle \hat{J}_3 \rangle\) towards a value different from that found in the QND measurement. Third, owing to the large ground hyperfine splitting, photons arising from optical pumping of \(f \rightarrow \bar{f} = 3\) and \(f \rightarrow \bar{f} = 4\) are distinguishable, thus these processes destroy coherences between \(| \uparrow \rangle\) and \(| \downarrow \rangle\).

We calculate the squeezing parameter as a function of time based on the evolution of atomic correlation functions, where operators evolve according to the adjoint form of the SME in Eq. (68). The collective atomic variables obey the following stochastic equations of motion (see Appendix C),
\[dN_C = -\gamma_{00} N_C dt + 2\gamma_{03} \langle \hat{J}_3 \rangle dt \tag{73a}\]
\[d\langle \hat{J}_1 \rangle = -\gamma_{11} \langle \hat{J}_1 \rangle dt \tag{73b}\]
\[d\langle \hat{J}_3 \rangle = s\sqrt{\kappa \Delta J_3^2} dW - \gamma_{33} \langle \hat{J}_3 \rangle dt + \frac{1}{2} \gamma_{30} N_C dt \tag{73c}\]
\[d\Delta J_3^2 = -\kappa (\Delta J_3^2)^2 dt - 2\gamma_{33} \Delta J_3^2 dt \]
\[+ \frac{1}{4} \left( 2 \gamma_{33} - \gamma_{00} \right) N_C dt + \frac{1}{2} \left( \gamma_{03} - 2 \gamma_{30} \right) \langle \hat{J}_3 \rangle dt, \tag{73d}\]
where the decay and feeding rates are given in Eq. (C4). The total number of atoms in the clock-state subspace is given by \(N_C\), which primarily decays at rate \(\gamma_{00}\). The final term in Eq. (73d), proportional to \(\langle \hat{J}_3 \rangle\), is typically negligible since in most applications \(\langle \hat{J}_3 \rangle \ll N_C\). We retain this small correction since unbalanced optical pumping acts to polarize the atoms and alters the rate of atom loss.

To find the peak squeezing in the presence of optical pumping, we numerically integrate Eqs. (73a–73d) and then use Eq. (51) to calculate the metrological squeezing parameter, \(\xi^2\), as a function of time. We choose here the magic frequency close to the \(f = 4\leftrightarrow f' = 4\) transition, \(\omega_m\), which is furthest from resonance with both excited hyperfine transitions. Typical time evolution is shown in Fig. 4 for 2500 atoms trapped a distance \(r'_{\perp} = 1.8a\) from the center of the nanofiber, where time is scaled to the characteristic scattering rate,
\[\gamma_s = \frac{\sigma_0}{A_{in}} \frac{\Gamma^2}{4\Delta^2 J_n} N_L. \tag{74}\]
We study the dynamics for two choices of quantization axis: (i) along the \(x\)-axis and (ii) along the numerically determined optimal axis. Figure 4(a) shows the time evolution of the squeezing parameter. We achieve a maximum squeezing of 4.7 dB when the clock states are chosen along the optimal axis: \(\varphi_{opt} \approx 86^\circ\) in Eq. (67).

The peak squeezing is ultimately limited by the combined effects of optical pumping on both \(\langle \hat{J}_1 \rangle\) and \(\Delta J_3^2\). Here, as in a free-space model \[15\], the primary factor that limits metrological squeezing is the decay of the collective mean spin \(\langle \hat{J}_1 \rangle\). A scattered photon eliminates the initial coherence between \(| \uparrow \rangle\) and \(| \downarrow \rangle\) within a single atom, thus depolarizing \(\langle \hat{J}_1 \rangle\). Atoms optically pumped to magnetic sublevels outside of the clock subspace decay \(N_C\), further reducing \(\langle \hat{J}_1 \rangle\). These effects are captured by the depolarization rate \(\gamma_{11}\) in the equation for \(\langle \hat{J}_1 \rangle\), Eq. (73b), whose solution is plotted in Fig. 4(b).

We can gain deeper understanding in the microscopic effects of optical pumping on spin squeezing by looking at the evolution of the one and two-body correlation functions. In terms of its constituent pseudospins, the col-
The single-body variance takes the form
\[ \Delta J_3^2 = N_A(\Delta J_3^{(1)})^2 + N_A(N_A - 1)(\Delta J_3^{(1)}\Delta J_3^{(2)}) \] (75)
for permutationally symmetric states considered here, where (1) and (2) label any two atoms in the ensemble. Loss of atoms affects the first (single-body) variance term, which scales as \(N_A\). The two-body correlations which contribute as \(N_A^2\) to the collective fluctuations,
\[ \langle \Delta J_3^{(1)}\Delta J_3^{(2)} \rangle = \frac{1}{4}\langle [\sigma_3^{(1)} \otimes \sigma_3^{(2)}] - [\sigma_3^{(1)}]^2 \rangle, \] (76)
have a much larger influence on the total variance. Spin-spin correlations at the heart of spin squeezing, \(\langle [\sigma_3^{(1)} \otimes \sigma_3^{(2)}] \rangle\), rapidly generated by the measurement backaction decohere by optical pumping according to Eqs. (C2–C3),
\[ \frac{d}{dt} (\sigma_3^{(1)} \otimes \sigma_3^{(2)})_{\text{op}} = -2\gamma_3([\sigma_3^{(1)} \otimes \sigma_3^{(2)}] + [\sigma_3^{(1)}]^2), \] (77)
where \(\hat{1}_C^{(n)} \equiv (|\uparrow\rangle + |\downarrow\rangle)(|\downarrow\rangle)^n\) is the single-body projector onto the clock states. In addition, atoms that return to the clock subspace after scattering a photon inject additional noise into \(\Delta J_3^2\). All of these effects are included in the equation for \(\Delta J_3^2\), Eq. (73d), whose overall and decomposed dynamical evolutions are shown in Fig. 4(c).

With our model, we explore optimal conditions for generating spin squeezing. The choice of quantization axis \(e_z\) that defines clock states affects both the measurement strength and the relative rates of optical pumping. We plot the peak squeezing as a function of the direction of \(e_z\) in the \(x-y\) plane in Fig. 5(b). We gain insight into the tradeoffs between QND entangling interaction and decoherence by independent inspection of the measurement strength and optical pumping rates. First, the rate of squeezing is determined by the effective optical density per atom on resonance,
\[ \text{OD}/N_A \equiv \kappa/\gamma_s = \frac{\sigma_0 A_{in}}{A_{J_3}^2}, \] (78)
which peaks when \(e_z\) is along the \(y\)-axis, as seen in Fig. 5(c). Choosing \(e_z\) along \(y\), the OD/\(N_A\) is about 50% larger than along \(x\)-axis. The various forms of decoherence similarly vary with quantization axis, as seen in Fig. 5(d), where we plot the dominant rate of atom loss,
\( \gamma_{01} \), and the depolarization rate of the mean pseudospin \( \langle J_1 \rangle \), \( \gamma_{11} \). Because the magic frequency \( \tilde{\omega}_m \) is nearly equidistant from \( f' = 3 \) and \( f' = 4 \) when the quantization axis is near the \( y \)-axis (see Fig. 3(b)) this choice of quantization axis provides more protection from decoherence. While the decoherence rates in Fig. 5(d) are largest near the \( y \)-axis, the increase in \( \kappa \) more than compensates to provide optimal peak squeezing.

Finally, we explore the optimal conditions as a function of the trapping geometry. The dispersive entangling interaction is based on the collective atomic coupling to the evanescent guided-mode fields, which decay exponentially away from the nanofiber surface, as seen in OD/\( N_A \) plotted in Fig. 6(a). From Eq. (73d), the optimal choice of quantization axis depends not only on distance from the fiber but also weakly on the atom number, Fig. 6(b) because of the competition between squeezing and decoherence. At the optimal quantization axis, the strong dependence of peak achievable squeezing on distance from the fiber is as seen in Fig. 6(c) along with the expected increase as more atoms contribute to the atom-light interface.

Several effects limit the reliability of the simulations for atoms trapped very near the fiber surface as \( r'_{\perp} \rightarrow a \).

First, strong van der Waals interactions modify the light shifts and magic frequencies [22, 78]. Second, the optical pumping model used here breaks down when the local density of states is significantly modified by the presence of the dielectric nanofiber [49, 79]. At distances \( r'_{\perp} > 1.5a \) the atoms’ local environment is roughly that of unmodified vacuum [49] and a free-space optical pumping model in Eq. (72) suffices.

V. SUMMARY AND OUTLOOK

We studied the strong cooperativity in the atom-light interface that can be achieved based on atoms trapped in the evanescent field surrounding an optical nanofiber, and interacting with a guided mode in the dispersive regime. The key parameter that determines the coupling is the resonant optical density per atom. Due to the tight confinement of the guided mode over the entire chain of atoms this parameter is OD/\( N_A \) \( \sim 10^{-2} \) for typical geometries used in current experiments, which approaches that achieved for atomic ensembles trapped inside optical cavities of moderate finesse [72, 80]. In contrast, the atom-light coupling for atoms in free space is typically orders of magnitude smaller, OD/\( N_A \) \( \sim 10^{-6} \). Under ideal conditions the atom-light interaction is entirely symmetric along the nanofiber, providing a platform for long-range correlations independent of distance between the atoms. As the light is entirely guided, fiber- or waveguide-coupled atomic ensembles can be networked together or coupled to other physical systems in a hybrid platform [81–84] for truly long-range entanglement generation and distribution.

We calculated the dispersive response based on a modal decomposition of the dyadic Green’s function, which provides a general method to calculate the induced phase shifts and polarization rotations of the guided modes. With this we studied the QND measurement of atoms via polarization spectroscopy. In particular, we studied squeezing of the collective pseudospin associated the atomic clock states of cesium. The atoms induce a birefringent index of refraction on the light, conditional on the spin state, which provides a mechanism for measuring the atomic spin projection and thus squeezing its uncertainty. Based on our formalism we calculated the nanofiber-enhanced measurement strength that determines the rate of squeezing.

The peak squeezing one can generate depends on a detailed balance between the reduction of spin projection noise based on QND measurement and the damage done to the spin ensemble due to optical pumping. Both measurement and optical pumping arise from the same physical mechanism – scattering of photons by atoms. The former corresponds to cooperative forward scattering into the guided mode whereas the latter corresponds to local scattering into all other modes, primarily the unguided “radiation” modes. The cooperativity, specified by the effective OD/\( N_A \), determines the ratio of these two effects and thus the ultimate power of the quantum atom-light interface.

We studied QND measurement using a first-principles stochastic master equation model, which allowed us to track the atomic correlation functions that define the metrologically relevant squeezing parameter. These include the atomic projection noise uncertainty as well as the length of the collective spin vector that defines the metrological signal. We find that decoherence acts primarily to depolarize the mean pseudospin and optically pump atoms out of the clock subspace, which we treat as loss. In addition, optical pumping decoheres the spin correlations at the heart of spin squeezing, but at a reduced rate compared with the effect on the mean pseudospin. The combined effect of QND measurement and decoherence yields a peak squeezing approaching 5 dB with \( \sim 2500 \) atoms. Larger enhancements in atom-light coupling and QND squeezing are possible with modest increases in the number of trapped atoms and/or for atoms trapped closer to the nanofiber surface.

Whereas we have assumed here that atoms can be prepared in a desired clock state defined by a particular quantization axis, in practice such preparation will require optical pumping that may be challenging for atoms near the surface of the nanofiber. In addition, though we have treated the atoms as localized at well defined points, in practice the atoms’ thermal motion can reduce the strong coupling described here. Our formalism provides a starting point for developing models necessary to study the dynamics of optical pumping, including the possibility of cooling atoms to the vibrational ground state, where thermal motion is negligible.

Finally, though we have treated here the case of strong coupling due solely to tight confinement of the guided
mode for atoms near the surface of the nanofiber, we can achieve even greater enhancement by combining this effect with longitudinal confinement provided by fiber-based optical cavities [37, 38, 85–87]. The coupling can be further improved under EIT conditions that substantially slow the group velocity [35, 36, 88, 89]. In addition, quantum control of the internal hyperfine state [90] can greatly enhance the entangling power of the atom-light interface [91, 92]. For large enough coupling, QND measurement should allow production of highly entangled spin states beyond the Gaussian regime [70, 93].

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FIG. 6. Parameters that define squeezing as a function of trapping distance \( r'_\perp \) and initial atom number \( N_a \). (a) OD/\( N_a \). Inset is the corresponding magic detuning in units of MHz (see text). (b) Optimal quantization axis orientation angle \( \theta_z \) for different atom numbers. The line with 500 atoms terminates when the squeezing effect is too weak to be observed \((r'_\perp > 2.3a)\). (c) Peak metrological squeezing at the optimal \( \theta_z \) for different atom numbers.

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Appendix A: Guided-mode functions for the optical nanofiber

In this Appendix we provide, for reference, the fundamental HE_{11} solutions to the homogeneous wave equation, Eq. (1) with $\mathbf{\Omega} = 0$, for a cylindrical nanofiber of radius $a$ and index of refraction given by Eq. (7). At a given frequency, $\omega_0 = ck_0$, the magnitudes of the longitudinal and transverse wave vectors for a guided mode are related by $n^2 k_0^2 = \beta_0^2 + k_\perp^2$. The positive propagation constant, $\beta_0 \equiv \beta(\omega_0)$, is determined from the eigenvalue equation that results from enforcing physical boundary conditions at the fiber surface [56],

$$\frac{J_0(\beta_0 \rho)}{\rho J_1(\beta_0 \rho)} = -\frac{n_1^2 + n_2^2}{2n_1^2 - q a K_1(q a)} + \frac{1}{h^2 a^2}$$

$$- \left[ \frac{n_1^2 - n_2^2}{2n_1^2 - q a K_1(q a)} \right] \frac{\beta_0}{n_1^2 k_0^2} + \frac{1}{h^2 a^2} \right]^{1/2}. \quad \text{(A1)}$$

Inside the nanofiber the transverse wavevector is real, $k_\perp = q$, where $q = \sqrt{\beta_0^2 - n_2^2 k_0^2}$, and outside the nanofiber it is purely imaginary, $k_\perp = i h$, where $h = \sqrt{n_1^2 k_0^2 - \beta_0^2}$. The vector eigenfunctions are expressed as $f_\perp(r) = (2\pi)^{-1/2} u_{b,p}(r_\perp) e^{ibz}$, where the modes are indexed by frequency $\omega_0$, propagation direction $b = \pm$, and polarization $p$.

A relatively simple form for the guided-mode functions can be expressed in a cylindrical basis ($r_\perp, \phi, z$) with longitudinal unit vector $e_z$, oriented along the fiber axis. The transverse unit vectors are related to their fixed Cartesian counterparts via the relations

$$e_\perp = e_z \cos \phi + e_y \sin \phi, \quad (A2a)$$
$$e_\phi = -e_z \sin \phi + e_y \cos \phi. \quad (A2b)$$

The transverse profile for the quasicircular guided modes, $p = \pm$, is

$$u_{b,\pm}(r_\perp) = [e_z u_{b,\pm}(r_\perp) \mp ie_\phi u_{b,\pm}(r_\perp) + ib e_z u_z(r_\perp)] e^{\pm i\phi}, \quad (A3)$$

and for the quasilinear guided modes, $p = \{H, V\}$, is

$$u_{b,H}(r_\perp) = \sqrt{2} [e_z u_{b,H}(r_\perp) \cos \phi - e_\phi u_{b,H}(r_\perp) \sin \phi + ib e_z u_z(r_\perp) \cos \phi], \quad (A4a)$$
$$u_{b,V}(r_\perp) = \sqrt{2} [e_z u_{b,V}(r_\perp) \sin \phi + e_\phi u_{b,V}(r_\perp) \cos \phi + ib e_z u_z(r_\perp) \sin \phi]. \quad (A4b)$$

The modes are expressed in terms of real-valued functions that depend only on the radial coordinate $r_\perp$, $u_{b,r}(r_\perp) = u_0[(1-s)K_0(q r_\perp) + (1+s)K_2(q r_\perp)]$, $u_{b,q}(r_\perp) = u_0[(1-s)K_0(q r_\perp) - (1+s)K_2(q r_\perp)]$, $u_{b,z}(r_\perp) = 2q K_1(q a) / \beta_0 J_1(h a)$, $J_1(h a)$, (A5c)

where $u_0$ is set by the normalization condition, $\int d^2 r_\perp n(r_\perp)|u_0(r_\perp)|^2 = 1$, $J_n$ and $K_n$ are the $n^{th}$ Bessel functions of the first and second kind, $f'(x)$ indicates a derivative with respect to the argument $x$, and

$$s = \frac{1/(q ^2 a^2)^2 + 1/(h ^2 a^2)^2}{J_1(h a)/J_1(h a) + K_1(q a)/qa K_1(q a)}. \quad \text{(A6)}$$

Of particular interest is the $z$-component, Eq. (A5c), which can become appreciable. Note that the phase convention in Eqs. (A3-A5) has been chosen to emphasize properties of the quasilinear modes and differs from that of Le Kien et al. – for instance in Ref. [65]. Further details about the guided-mode fields inside the nanofiber ($r_\perp \leq a$), the radiation (unguided) modes, and the quantized form of both can be found in Refs. [42, 49, 54, 78, 95].

Appendix B: Photon scattering and optical pumping rates

In this Appendix we give the explicit expressions for the photon scattering rates used in Sec. IV following the formalism given in [61]. The total rate of photon scattering by an atom in the clock state $|f, 0\rangle$ is

$$\gamma_f = -\frac{2}{h} \text{Im}[\langle f, 0|\hat{h}_{\text{eff}}|f, 0\rangle], \quad (B1)$$
where the effective non-Hermitian light-shift Hamiltonian for one atom is
\[ \hat{h}_{\text{eff}} = -\hat{E}_{\text{in}}^{-}(r'; t) \cdot \hat{\alpha} \cdot \hat{E}_{\text{in}}^{+}(r'; t) \] (B2)
as follows from Eq. (38), where $\alpha_0(\Delta_{ff'}) = -\frac{\sigma_0}{\pi \varepsilon_0} \frac{\Gamma}{\sqrt{\Gamma^2 + \eta^2}}$ is the complex polarizability and the irreducible tensor operator $\hat{\alpha}(f, f')$ is given in Eq. (46).

The rate of optical pumping between clock states $|f, 0\rangle \rightarrow |f, 0\rangle$ is
\[ \gamma_{f\rightarrow f} = \sum_q |\langle f, 0| \hat{W}^f_q |f, 0\rangle|^2, \] (B3)
where $\hat{W}^f_q = \sum_{f'} \frac{1}{\sqrt{2}} \sum_q (e^*_{\text{in}} \hat{D}_{ff'})(e_{\text{in}} \hat{D}^+_{f'f})$ are the Lindblad jump operators for optical pumping between ground levels $f \rightarrow f'$ [61]. Each jump operator $\hat{W}^f_q$ is associated with absorption of the probe photon polarized along $\mathbf{e}_{\text{in}}$ followed by spontaneous emission of a photon with polarization $\mathbf{e}_q$, where $q = \{0, \pm 1\}$ labels spherical basis elements for $\pi$ and $\sigma$ transitions.

To find the dependence on the input field intensity, we define a characteristic photon scattering rate, $\gamma_s = \frac{\gamma \Omega}{4\Delta_{ff'}} = \frac{\gamma_0}{\alpha_0} \frac{1}{\Lambda_{\text{in}}} N_L$, with Rabi frequency $\Omega = 2|j|d|j'| \mathcal{E}^{\text{in}}_{\text{in}} / \hbar$, reduced optical dipole matrix element $|j|d|j'|$, and field amplitude $\mathcal{E}^{\text{in}}_{\text{in}} = |\mathbf{E}^{\text{in}}_{\text{in}}(r')|$. Eqs. (B1) and (B3) yield,
\[ \gamma_\tau = \sum_{f'} \sum_{f''} \left| \sigma(\Delta_{ff'}) \mathbf{u}_{\text{in}}^*(r') \cdot \langle f, 0| \hat{\alpha}(f, f') |f, 0\rangle \cdot \mathbf{u}_{\text{in}}(r') \right|^2 \] (B4a)
\[ \gamma_{\pi} \approx \sum_{f'} \sum_{f''} \sum_q \left| \sigma(\Delta_{ff'}) \mathbf{u}_{\text{in}}^*(r') \cdot \mathbf{e}^q \langle f, 0| \hat{\alpha}(f, f') |f, 0\rangle \cdot \mathbf{u}_{\text{in}}(r') \right|^2 \] (B4b)
\[ \gamma_{\sigma} \approx \sum_{f'} \sum_{f''} \sum_q \left| \sigma(\Delta_{ff'}) \mathbf{u}_{\text{in}}^*(r') \cdot \mathbf{e}^q \langle f, 0| \hat{\alpha}(f, f') |f, 0\rangle \cdot \mathbf{u}_{\text{in}}(r') \right|^2 \] (B5)
where $\sigma(\Delta_{ff'}) = \sigma_0 \Gamma^2 / 4 \Delta_{ff'}^2$ is the scattering cross section at the probe detuning, $C_{f'f}^{0:1} = \langle f' | f; 0 \rangle$ are the Clebsch-Gordan coefficients, and
\[ \left| \sigma(\Delta_{ff'}) \right|^2 = (2j' + 1)(2j + 2) \left( \begin{array}{c} j' \ 7/2 \\ j \ 1 \end{array} \right)^2 \] (B6)
are the relative oscillator strengths determined by the relevant Wigner 6-J symbol.

**Appendix C: Derivation of the equations of motion for the moments**

In this Appendix we derive the equations of motion for the correlation functions that define the metrologically relevant squeezing parameter, $\zeta^2 = N_A \Delta J_1^2 / \langle J_1 \rangle^2$.

We seek the time evolution of the one and two-body correlation functions:
\[ \langle \hat{N}_C \rangle = \sum_n \langle \hat{z}^{(n)}_C \rangle \] (C1a)
\[ \langle \hat{J}_3 \rangle = \frac{1}{2} \sum_n \langle \hat{z}^{(n)}_3 \rangle \] (C1b)
\[ \langle \hat{J}_9 \rangle = \frac{1}{2} \sum_n \langle \hat{z}^{(n)}_3 \rangle \] (C1c)
\[ \langle \hat{J}_4 \rangle = \frac{\langle \hat{N}_C \rangle}{4} + \frac{1}{4} \sum_{m \neq n} \langle \hat{z}^{(m)}_3 \rangle \hat{z}^{(n)}_3 \] (C1d)
where $\hat{1}_C \equiv |\uparrow\rangle \langle \uparrow| + |\downarrow\rangle \langle \downarrow|$ is the single-atom projector onto the clock states. To include optical pumping, we apply the following equations of motion. For a collective, single-body operator, $X = \sum_n \hat{z}^{(n)}$, the evolution due to optical pumping is $d(\hat{X})_{\text{op}} = \sum_n \text{Tr}[D_n [\hat{\sigma}] \hat{X}] dt = \sum_n \langle D_n [\hat{z}^{(n)}] \rangle dt$, where the map, which acts locally on atoms along the nanofiber, is given in Eq. (72). Two-body microscopic operators decay by optical pumping according to [15]
\[ \frac{d}{dt} \langle \hat{D}_m [\hat{z}^{(n)}] \hat{g}^{(m)} \rangle_{\text{op}} = \langle \hat{D}_m [\hat{z}^{(n)}] \hat{g}^{(m)} \rangle + \langle \hat{z}^{(n)} \hat{D}_m \hat{g}^{(m)} \rangle, \] (C2)
where the superscripts refer to the $m^{th}$ and $n^{th}$ atoms.

Applying the adjoint map to the single-atom operators yields
\[ \hat{D}_m^\dagger \hat{1}_C = -\gamma_{00} \hat{1}_C + \gamma_{03} \hat{z}_3 \] (C3a)
\[ \hat{D}_m^\dagger \hat{z}_3 = -\gamma_{33} \hat{z}_3 + \gamma_{30} \hat{1}_C \] (C3b)
\[ \hat{D}_m^\dagger \hat{z}_1 = -\gamma_{11} \hat{z}_1, \] (C3c)
with rates
\[ \gamma_{00} = \frac{\gamma_\tau + \gamma_\pi - \gamma_{\sigma} + \gamma_{\tau} + \gamma_{\pi} - \gamma_{\sigma}}{2} \] (C4a)
\[ \gamma_{03} = \frac{-\gamma_\tau + \gamma_\pi + \gamma_{\sigma} + \gamma_{\tau} + \gamma_{\pi} + \gamma_{\sigma}}{2} \] (C4b)
\[ \gamma_{33} = \frac{\gamma_\tau + \gamma_\pi + \gamma_{\sigma} + \gamma_{\tau} + \gamma_{\pi} + \gamma_{\sigma}}{2} \] (C4c)
\[ \gamma_{30} = \frac{-\gamma_\tau + \gamma_\pi + \gamma_{\sigma} + \gamma_{\tau} + \gamma_{\pi} + \gamma_{\sigma}}{2} \] (C4d)
\[ \gamma_{11} = \frac{\gamma_\tau + \gamma_\pi}{2}. \] (C4e)

Given Eqs. (C2, C3), the equations for the two-body spin correlations, Eq. (77), follow. Similarly, one can derive equations of motion for the remaining two-body microscopic operator correlations $\langle \hat{1}_C^{(m)} \hat{1}_C^{(n)} \rangle$ and $\langle \hat{1}_C^{(m)} \hat{z}_3^{(n)} \hat{z}_3^{(m)} \rangle$ when $m \neq n$ and from these, the macroscopic operator expectation values $\langle \hat{J}_4 \rangle$, $\langle \hat{N}_C \rangle$, and $\langle \hat{N}_C \hat{J}_3 \rangle$. As we have examined numerically, on the time scale of the QND measurement, the correlation between atom number in the clock state subspace and the pseudospin moment is weak, and one can thus treat the
atom number operator in the clock state subspace as a $c$-number. We therefore set $\langle \hat{N}_C \hat{J}_3 \rangle - \langle \hat{N}_C \rangle \langle \hat{J}_3 \rangle = 0$ and $\langle \hat{N}_C^2 \rangle - \langle \hat{N}_C \rangle^2 = 0$ and define $N_C \equiv \langle \hat{N}_C \rangle$.

The equations of motion for the moments of $\hat{J}_3$ are now found from the SME, Eq. (68),

$$d\langle \hat{J}_3 \rangle = s\sqrt{\kappa} \Delta J_3^2 dW - \gamma_{33} \langle \hat{J}_3 \rangle dt + \frac{1}{2} \gamma_{30} N_C dt, \quad (C5a)$$

$$d\langle \hat{J}_3^2 \rangle = 2s\sqrt{\kappa} \langle \hat{J}_3 \rangle \Delta J_3^2 dW - 2\gamma_{33} \langle \hat{J}_3^2 \rangle dt + \frac{1}{4} (2\gamma_{33} - \gamma_{00}) N_C dt + \gamma_{30} \langle \hat{J}_3 \rangle N_C dt + \frac{1}{2} (\gamma_{03} - 2\gamma_{30}) \langle \hat{J}_3 \rangle dt. \quad (C5b)$$

The stochastic term in $d\langle \hat{J}_3^2 \rangle$ was simplified by assuming Gaussian statistics [77], $\langle \hat{J}_3^2 \rangle = 3\langle \hat{J}_3 \rangle^2$. Finally, the Itô calculus governing the stochastically evolving moments requires that differentials be taken to second order, and the evolution of the variance is given by $d\Delta J_3^2 = d\langle \hat{J}_3^2 \rangle - 2\langle \hat{J}_3 \rangle d\langle \hat{J}_3 \rangle - (d\langle \hat{J}_3 \rangle)^2$. The equation of motion for the conditional variance, Eq. (73d), then follows from Eqs. (C5).