On Plane Cremona Transformations of Fixed Degree

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Received: 21 December 2012 / Published online: 2 October 2013
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Abstract We study the quasi-projective variety Bir$_d$ of plane Cremona transformations defined by three polynomials of fixed degree $d$ and its subvariety Bir$^\circ_d$ where the three polynomials have no common factor. We compute their dimension and the decomposition in irreducible components. We prove that Bir$_d$ is connected for each $d$ and Bir$^\circ_d$ is connected when $d < 7$.

Keywords Plane Cremona transformations · Homaloidal nets · De Jonquières transformations

Mathematics Subject Classification 14E07

1 Introduction

The birational geometry of algebraic varieties is governed by the group of birational self-maps. In general, it is very difficult to determine this group for an arbitrary variety, and as a matter of fact only few examples are completely understood. The special case of the projective plane has attracted lots of attention since the 19-th century. The pioneering work of Cremona and then the classical geometers of the Italian and German school were able to give partial descriptions of it, but it was only after Noether and Castelnuovo that generators of the group were described. The
Noether–Castelnuovo theorem (see [1], for instance) states that the group of birational self-maps of $\mathbb{P}^2$, usually called the \textit{plane Cremona group} and denoted by $\text{Cr}(2)$, is generated by linear automorphisms of $\mathbb{P}^2$ and a single birational non-biregular map, the so-called \textit{elementary quadratic} transformation $\sigma : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ defined by $\sigma ([x : y : z]) = [yz : xz : xy]$.

Even if the generators of $\text{Cr}(2)$ have been known for a century now, many other properties of this group are still mysterious. Only after decades, see [25], a complete set of relations has been described, and more recently the non-simplicity of $\text{Cr}(2)$ has been shown [9], and a good understanding of its finite subgroups has been achieved; see [16] and [5]. This brief and fairly incomplete list is only meant to stress the difficulties and the large unknown parts in the study of $\text{Cr}(2)$; for a more complete picture the interested reader should refer to [15] and [14]. Amid all its subgroups the one associated with polynomial automorphisms of the plane, $\text{Aut}(\mathbb{C}^2)$, attracted even more attention than $\text{Cr}(2)$ itself [3]. The generators of $\text{Aut}(\mathbb{C}^2)$ have been known since 1942 [28], and later on [31] it has been proved that $\text{Aut}(\mathbb{C}^2)$ is the amalgamated product of two of its subgroups, more precisely of the affine and elementary ones. Nevertheless, this group is not less mysterious and challenging than the entire Cremona group. Jung’s description yields a natural decomposition

\[ \text{Aut}(\mathbb{C}^2) = A \cup \text{G}[2] \cup \text{G}[3] \cup \text{G}[2, 2] \cup \text{G}[4] \cup \text{G}[5] \cdots \]

into sets of polynomial automorphisms of multidegree $(d_1, \ldots, d_m)$ and the affine subgroup $A$. In [21], Friedland and Milnor proved that $\text{G}[d_1, \ldots, d_m]$ is a smooth analytic manifold of dimension $(d_1 + d_2 + \cdots + d_m + 6)$. Later on, Furter [22] computed the number of irreducible components of polynomial automorphisms of $\mathbb{C}^2$ with fixed degree less than or equal to 9 and proved that the variety of polynomial automorphisms of the plane with degree bounded by the positive integer $n$ is reducible when $n \geq 4$. Edo and Furter [17] studied some degenerations of the multidegrees; for example, they were able to show that $\text{G}[3] \cap \overline{\text{G}[2, 2]} \neq \emptyset$ using the lower semi-continuity of the length of a plane polynomial automorphism as a word of the amalgamated product [23]. Contrary to what happens in the Cremona group $\text{Cr}(2)$, see [6], the group of polynomial automorphisms $G$ of the plane can be endowed with the structure of an infinite-dimensional algebraic group. Denoted by $G_d$ the set of polynomial automorphisms of fixed degree $d$, Furter proved in [24] that $G_d$ is a smooth and locally closed subset of $G$.

Inspired by these works on $\text{Aut}(\mathbb{C}^2)$ and by [10], we approach the study of $\text{Cr}(2)$ by forgetting its group structure. Our aim is to study an explicit set of functions generating a birational map. In this way the same birational map is associated with different triples of homogeneous polynomials.

Let $f_1, f_2, f_3$ be homogeneous polynomials of degree $d$. Whenever $f_1, f_2, f_3$ are not all zero, let us denote by $[f_1 : f_2 : f_3]$ the equivalence class of $(f_1, f_2, f_3)$ with respect to the relation $(f_1, f_2, f_3) \sim (\lambda f_1, \lambda f_2, \lambda f_3)$, for $\lambda \in \mathbb{C}^*$. Consider $[f_1 : f_2 : f_3]$ as an element of $\mathbb{P}^{3N-1}=3(4^2)=1-1=3d(d+3)/2+2$, where the homogeneous coordinates in $\mathbb{P}^{3N-1}$ are the coefficients of the $f_i$’s, up to multiplication by the same nonzero scalar. Setting

\[ \gamma : \mathbb{P}^2 \rightarrow \mathbb{P}^2, \quad \gamma ([x : y : z]) = [f_1(x, y, z) : f_2(x, y, z) : f_3(x, y, z)], \quad (1) \]
let us define
\[ \text{Bir}_d = \{ [f_1 : f_2 : f_3] \mid \gamma \text{ is birational} \} \subset \mathbb{P}^{3N-1}. \]

The natural subset to consider in Bir$_d$ is the one corresponding to triplets without common factors. The latter called Bir$_d^0$, which parameterizes birational maps of degree $d$, will be the main actor of this paper; see Sect. 2 for all the relevant definitions.

Among other things, in [10] the authors describe Bir$_d^0$ and Bir$_d$ for $d \leq 3$; see Sect. 2 for some of their results. Their description is essentially based on a set-theoretic analysis of the plane curves contracted by a Cremona transformation of degree $\leq 3$.

In this paper, we describe Cremona transformations $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ in the usual classical way, i.e., we consider the so-called homaloidal net of plane curves in the source plane corresponding to the net of lines in the target plane. Roughly speaking, this means studying the span \( \langle f_1, f_2, f_3 \rangle \subset \mathbb{C}^N = (d+2) \) in the above setting.

By looking at base points of homaloidal nets, we prove the following; cf. Sect. 3.

**Theorem 1** For each $d \geq 2$, Bir$_d^0$ is a quasi-projective variety of dimension $4d + 6$ in $\mathbb{P}^{3d(d+3)/2+2}$ and, if $d \geq 4$, it is reducible.

Each irreducible component of Bir$_d^0$ is rational.

Each element of the maximal dimension component of Bir$_d^0$ is a de Jonquières transformation, i.e., it is defined by a homaloidal net of plane curves of degree $d$ with a base point of multiplicity $d - 1$ and $2d - 2$ simple base points.

If $d \geq 4$, there are further irreducible components of Bir$_d^0$ having dimension at most $2d + 12$. Indeed, each irreducible component of Bir$_d^0$ is determined by a set of $(d - 1)$-tuples of non-negative integers $(v_1, v_2, \ldots, v_{d-2}, v_{d-1})$ such that $\sum_{i=1}^{d-1} i v_i = 3d - 3$ and $\sum_{i=1}^{d-1} i v_i^2 = d^2 - 1$. Each one of these components of Bir$_d^0$ has dimension $8 + 2 \sum_{i=1}^{d-1} v_i$ and its general element is a Cremona transformation given by a homaloidal net of plane curves of degree $d$ with $v_i$ base points of multiplicity $i$, in general position.

Quite surprisingly, in this setup, we may say that the general Cremona transformation in Bir$_d^0$ is de Jonquières. For small values of the degree $d$ something more can be said. Our approach yields an explicit description of Bir$_d^0$ and allows us to prove its connectedness, for $d \leq 6$.

**Theorem 2** For each $d \leq 6$, Bir$_d^0$ is connected.

Even if, from our point of view, Bir$_d^0$ is the right variety to consider, we complete the study extending the results to the whole Bir$_d$.

**Theorem 3** For each $d \geq 2$, Bir$_d$ is a connected quasi-projective variety of dimension
\[
\max \left\{ 4d + 6, \frac{d(d + 1)}{2} + 7 \right\} = \begin{cases} 
4d + 6 & \text{if } d \leq 6, \\
\frac{(d + 1)d}{2} + 7 & \text{if } d \geq 7
\end{cases}
\]
in $\mathbb{P}^{3d(d+3)/2+2}$. If $d \geq 3$, Bir$_d$ is reducible.
Each irreducible component of Bir$_d$ is rational.

If $d \geq 7$, each element of the maximal dimension component of Bir$_d$ is $[hl_1 : hl_2 : hl_3]$, where $\deg(h) = d - 1$ and $\deg(l_i) = 1$, $i = 1, 2, 3$.

Also in this case our approach allows a more precise representation of Bir$_d$ and an explicit description of it for low $d$; see Sect. 3. This also corrects an imprecision in [10] about Bir$_3$. In principle, one could describe the irreducible components of Bir$_d^\circ$ and Bir$_d$, for a fixed degree $d$. As a matter of fact, the combinatorial and computations become wild when the degree increases. For this reason we think that it would be interesting to find closed formulas for the numbers $N^\circ(d)$ and $N(d)$ of irreducible components of Bir$_d^\circ$ and Bir$_d$ and to study the connectedness of Bir$_d^\circ$ for each $d$.

This study of Bir$_d^\circ$, as pointed out by Cerveau and Déserti in [10], also allows one to build a bridge between the classical algebraic geometry of $\text{Cr}(2)$ and foliations on $\mathbb{P}^k$, $k \geq 2$. The reconstruction of a foliation from its singular set [7, 8], the study of irreducible components of foliations on $\mathbb{P}^k$, $k \geq 3$ [12, 13], and the description of the orbits by the action of $\text{PGL}(3, \mathbb{C})$ on the foliations of fixed degree 2 on $\mathbb{P}^2$ [11] have been extensively studied and suggest the possibility of using our methods also in this context. We will not dwell on this here. However, we investigate the dynamical behavior of plane Cremona transformations in a work in progress; see [4].

When this work was finished we were informed by J. Blanc of D. Nguyen Dat’s PhD thesis, [30]. Some of the results, and the techniques, in this paper overlap the content of his work. In particular, he was able to compute the dimension of Bir$_d^\circ$. Our approach allows one to prove his Conjecture 3 about the irreducible components of Bir$_d$ and corrects the wrong statement on the non-connectedness of $\mathcal{H}_d$ and Bir$_d^\circ$, in [30, Théorème 16, Sect. 6.2], for $d \geq 4$.

2 Notation, Definitions, and Known Results

We work over the complex field. Let $\mathbb{C}[x, y, z]_d$ be the set of homogeneous polynomials of degree $d$ in the variables $x, y, z$ with coefficients in $\mathbb{C}$, including the null polynomial. In particular, $\mathbb{C}[x, y, z]_d \cong \mathbb{C}^N$ as $\mathbb{C}$-vector spaces, where

$$N = \binom{d+2}{2} = \frac{d(d+3)}{2} + 1$$

is the number of coefficients of homogeneous polynomials of degree $d$ in 3 unknowns.

Whenever $f_1, f_2, f_3 \in \mathbb{C}[x, y, z]_d$ are not all zero, let us denote by $[f_1 : f_2 : f_3]$ the equivalence class of the triplets $(f_1, f_2, f_3)$ with respect to the relation $(f_1, f_2, f_3) \sim (\lambda f_1, \lambda f_2, \lambda f_3)$, for $\lambda \in \mathbb{C}^\times$. Consider $[f_1 : f_2 : f_3]$ as an element of $\mathbb{P}^{3N-1=3d(d+3)/2+2}$, where the homogeneous coordinates are all coefficients of the three polynomials $f_1, f_2, f_3$, up to multiplication by the same nonzero scalar for all of them. Setting

$$\gamma : \mathbb{P}^2 \longrightarrow \mathbb{P}^2, \quad \gamma([x : y : z]) = [f_1(x, y, z) : f_2(x, y, z) : f_3(x, y, z)],$$
let us define, according to [10],

\[ \text{Bir}_d = \{(f_1 : f_2 : f_3) | \gamma \text{ is birational} \} \subset \mathbb{P}^{3N-1}. \]

In [10], the integer \( d \) is called the \textit{degree} of the map \( \gamma \). However, in algebraic geometry, if \( f_1, f_2, f_3 \) have a common factor, i.e., their greatest common divisor \( \gcd(f_1, f_2, f_3) = h \) is a polynomial of positive degree \( d' \), then the degree of the map \( \gamma \) is usually considered as \( d - d' \). If \( \gcd(f_1, f_2, f_3) = 1 \), the authors of [10] say that \( \gamma \) has pure degree \( d \) and they define

\[ \text{Bir}_d^0 = \{(f_1 : f_2 : f_3) \in \text{Bir}_d | \gamma \text{ has pure degree } d \} \subset \text{Bir}_d \subset \mathbb{P}^{3N-1}. \]

If \( [f_1 : f_2 : f_3] \in \text{Bir}_d^0 \), let us identify it with the birational map \( \gamma \).

One may check that the subset \( \{(f_1 : f_2 : f_3) | \deg(\gcd(f_1, f_2, f_3)) > 0 \} \) of \( \mathbb{P}^{3N-1} \) is Zariski closed, e.g., by using resultants.

\textbf{Remark 4} When \( d = 1 \), write the \( 3 \times 3 \) nonzero matrix \( (a_{ij}) \), whose rows are the coefficients of \( f_i, i = 1, 2, 3 \), thus \( \text{Bir}_1^0 = \text{Bir}_1 = \text{PGL}(3) \) is the Zariski open subset of \( \mathbb{P}^8 \) where the determinant \( \det(a_{ij}) \) is nonzero and its Zariski closure is \( \text{Bir}_1 = \mathbb{P}^8 \).

Recall that \( \mathbb{P}^8 \setminus \text{Bir}_1 = Z(\det(a_{ij})) \) is a cubic, irreducible hypersurface, which is singular along the locus where \( \text{rank}(a_{ij}) = 1 \).

In [10], Cerveau and Déserti are interested in \( \text{Bir}_d \) also for applications to the study of \textit{foliations}. Among other things, they prove the following.

\textbf{Theorem 5} (Cerveau–Déserti) \textit{If} \( d = 2 \), \textit{the subset} \( \text{Bir}_2^0 \) [\( \text{Bir}_2 \), resp.] \textit{in} \( \mathbb{P}^{17} \) \textit{is a smooth} [singular, resp.], \textit{unirational, irreducible, quasi-projective variety of dimension} 14. Their Zariski closures in \( \mathbb{P}^{17} \) \textit{coincide.}

\textit{If} \( d = 3 \), \textit{the subset} \( \text{Bir}_3^0 \subset \mathbb{P}^{29} \) \textit{is an irreducible, rationally connected, quasi-projective variety of dimension} 18. \textit{Furthermore, the subset} \( \text{Bir}_3 \subset \mathbb{P}^{29} \) \textit{is singular and contains at least two irreducible components.}

Actually, in [10], the authors state that \( \text{Bir}_3 \) has exactly two irreducible components. We will see later, in Example 44, that \( \text{Bir}_3 \) has exactly three irreducible components.

Now we recall the classical notions which will be used later.

\textbf{Definition 6} Let \( \gamma = [f_1 : f_2 : f_3] \in \text{Bir}_d^0, d \geq 2 \). Setting \( W \) to be the linear span \( \langle f_1, f_2, f_3 \rangle \) of the polynomials \( f_1, f_2, f_3 \) in \( \mathbb{C}^N \), the plane \( \mathbb{P}(W) \subset \mathbb{P}^{N-1} \) is called the \textit{homaloidal net} associated with \( \gamma \), and we denote it by \( L_\gamma \).

The general element of \( L_\gamma \) defines an irreducible rational plane curve of degree \( d \) passing through some fixed points \( p_1, \ldots, p_r \) in \( \mathbb{P}^2 \), called \textit{set-theoretic base points} of \( L_\gamma \), with certain multiplicities. Let \( \phi : S \to \mathbb{P}^2 \) be the blowing up at \( p_1, \ldots, p_r \). The strict transform of the general element of \( L_\gamma \) may have further base points on the exceptional curves in \( S \), classically called \textit{infinitely near} base points; cf. Sect. 2 in [1]. Equivalently, \( \phi \circ \gamma \) may not be a morphism yet. Eventually, there is a birational
morphism \( \phi': S' \to \mathbb{P}^2 \) such that the strict transform of the net \( \mathcal{L}_\gamma \) is a complete 2-dimensional linear system on \( S' \), or equivalently \( \phi' \circ \gamma \) is a morphism.

Set \( v_i \) to be the number of base points of \( \mathcal{L}_\gamma \) with multiplicity \( i \), \( i = 1, \ldots, d-1 \), including infinitely near base points. The multi-index of the homaloidal net \( \mathcal{L}_\gamma \) is

\[
v_I = (v_1, v_2, \ldots, v_{d-2}, v_{d-1}).
\]  

(2)

It is an easy consequence of the definition of homaloidal net (cf., e.g., Sect. 2.5 in [1]) that

\[
\sum_{i=1}^{d-1} i^2 v_i = d^2 - 1, \quad \sum_{i=1}^{d-1} i v_i = 3(d-1),
\]

(3)

and \( v_i = 0, i \geq d \). Formulas (3) are usually called Noether’s equations.

We say that the multi-index \( v_I \) is the \( H \)-type of the homaloidal net \( \mathcal{L}_\gamma \) and of the Cremona transformation \( \gamma \).

One usually says that \( \gamma \) is symmetric if its \( H \)-type has a unique nonzero \( v_i \). By (3), \( \gamma \) is symmetric if and only if either \( d = 2 \) and \( v_I = (3) \), or \( d = 5 \) and \( v_I = (0, 6, 0, 0) \), or \( d = 8 \) and \( v_I = (0, 0, 7, 0, \ldots) \), or \( d = 17 \) and the unique nonzero term in \( v_I \) is \( v_6 = 8 \) (cf., e.g., Lemma 2.5.5 in [1]).

Finally, let us define the index \( i(\gamma) \) of \( \gamma \) as follows. We set \( i(\gamma) = 0 \) if and only if there is no infinitely near base point of \( \mathcal{L}_\gamma \), i.e., all base points of \( \mathcal{L}_\gamma \) belong to \( \mathbb{P}^2 \). If \( \mathcal{L}_\gamma \) has infinitely near base points, consider birational morphisms \( \psi: S \to \mathbb{P}^2 \) such that the strict transform \( \mathcal{L}' \) of \( \mathcal{L}_\gamma \) in \( S \) is a net with all base points belonging to \( S \), i.e., \( \mathcal{L}' \) has no infinitely near base point. Recalling that \( \psi \) is the composition of finitely many blowing ups, each one at the maximal ideal of a single point, define \( i(\gamma) \) as the minimum number of such blowing ups among all birational morphisms \( \psi \) with the above property.

Remark 7 If \( d \geq 3 \), the irreducibility of the general member of \( \mathcal{L}_\gamma \) implies that \( \sum_{i>d/2} v_i \leq 1 \), i.e., \( \mathcal{L}_\gamma \) has at most one base point of multiplicity \( > d/2 \).

Remark 8 Usually, the \( H \)-type of a linear system \( \mathcal{L} \) is encoded by listing the multiplicities \( m_i \) of the base points \( p_i \) of \( \mathcal{L} \), which are commonly written in nonincreasing order, say \( m_1 \geq m_2 \geq m_3 \geq \cdots \), instead of a multi-index like \( v_I \). Given a multi-index \( v_I = (v_1, \ldots, v_{d-1}) \) of a homaloidal net \( \mathcal{L}_\gamma \), the multiplicities of the base points of \( \mathcal{L}_\gamma \), in nonincreasing order, can be computed as follows:

\[
m_i = \max \left\{ j : \sum_{k \geq j} v_k > i - 1 \right\}, \quad \text{for each } i \geq 1.
\]

(4)

Equivalently, set \( m_1 = \max \{ j : v_j \neq 0 \} \) and then

\[
m_1 = m_2 = \cdots = m_{R_1} > m_{R_1+1} = \max \{ j < m_1 : v_j \neq 0 \}, \quad R_1 = v_{m_1},
\]

\[
m_{R_1+1} = \cdots = m_{R_1+R_2} > m_{R_1+R_2+1} = \max \{ j < m_{R_1+1} : v_j \neq 0 \}, \quad R_2 = v_{m_{R_1+1}},
\]
and so on, until \( m_R = \min \{ j : v_j \neq 0 \} \) with \( R = \sum_{i=1}^{d-1} v_i = R_1 + R_2 + \cdots + R_s \), \( s = | \{ j : v_j \neq 0 \} | \).

**Definition 9** Let \( v_I = (v_1, \ldots, v_{d-1}) \) be a multi-index with non-negative entries. We say that the positive numbers \( m_1, \ldots, m_r \), with \( r = \sum_{i=1}^{d-1} v_i \), computed in Remark 8 are the **multiplicities** associated with \( v_I \).

**Definition 10** One says that \( \gamma \in \text{Bir}_d^o \) is a **quadratic transformation**. By (3), its H-type is \( v_I = (3) \), namely, \( \gamma \) is defined by a homaloidal net of generically irreducible conics passing through three simple base points \( p_1, p_2, p_3 \). We say that \( \gamma \) is a quadratic transformation **centered at** \( p_1, p_2, p_3 \).

**Definition 11** The map \( \gamma \in \text{Bir}_d^o \) is called a **de Jonquières transformation** if there exists a base point of \( L_\gamma \) with multiplicity \( d - 1 \). If \( d \geq 3 \), irreducibility and (3) force \( v_{d-1} = 1, v_i = 0, 2 \leq i \leq d - 2 \), and \( v_1 = 2d - 2 \), hence the H-type of a de Jonquières transformation of degree \( d \geq 3 \) is \( (2d - 2, 0, \ldots, 0, 1) \).

A little work on (3) proves the following (cf., e.g., Proposition 2.6.4 in [1]).

**Theorem 12** (Noether’s Inequality) Let \( \gamma \in \text{Bir}_d^o, d \geq 2 \), and let \( m_1 \geq m_2 \geq m_3 \) be the maximal multiplicities of the base points of the homaloidal net \( L_\gamma \). Then

\[
m_1 + m_2 + m_3 \geq d + 1.
\]

Furthermore, equality holds if and only if \( \gamma \) is either symmetric or de Jonquières.

The next, classical, lemma is very important for our purposes, and we give here a proof only for the reader’s convenience.

**Lemma 13** (cf. [26, p. 72]) If \( d \geq 3 \), the maximal number of base points of a homaloidal net \( L_\gamma \) is \( \sum_{i=1}^{d-1} v_i = 2d - 1 \) and, in that case, \( \gamma \) is a de Jonquières transformation.

**Proof** In (3), multiply the second equation by \( d \) and then subtract the first equation. Thus,

\[
(d - 1)(2d - 1) = \sum_{i=1}^{d-1} i(d - i)v_i.
\]

On the real interval \([1, d - 1]\), the function \( g(x) = x(d - x) \) has a maximum at \( x = d/2 \) and a minimum at \( x = 1 \) and at \( x = d - 1 \), so \( 1(d - 1) \leq g(x) \leq d^2/4 \). Hence, (6) implies \( 2d - 1 \geq \sum_{i=1}^{d-1} v_i \). Equality holds if and only if \( v_1 + v_{d-1} = 2d - 1 \) and \( v_i = 0, i = 2, \ldots, d - 2 \). Remark 7, formulas (3), and the hypothesis \( d \geq 3 \) force \( v_{d-1} = 1 \).

**Remark 14** In algebraic geometry, when a family of objects \( \{ X_\eta \}_{\eta \in \Sigma} \) like varieties, maps, etc., is parameterized by the points of an irreducible algebraic variety \( \Sigma \), one
usually says that the general object $X$ has a certain property $P$ if the subset of points $q \in \Sigma$, such that $X_q$ has the property $P$, contains a Zariski open dense subset of $\Sigma$.

Later we will use finite sets of points in $\mathbb{P}^2$ and in its symmetric products.

**Definition 15** (cf. [29]) The $r$-th symmetric product of $\mathbb{P}^2$, denoted by $\text{Sym}^r(\mathbb{P}^2)$, is the quotient $(\mathbb{P}^2)^r/\mathcal{S}_r$ of the Cartesian product $(\mathbb{P}^2)^r$ under the action of the symmetric group $\mathcal{S}_r$ permuting the factors. One has that $\text{Sym}^1(\mathbb{P}^2) = \mathbb{P}^2$, if $r = 1$; otherwise, $\text{Sym}^r(\mathbb{P}^2)$ is a rational, singular, irreducible, projective variety of dimension $2r$. If $p_1, \ldots, p_r \in \mathbb{P}^2$ are distinct points, then $\text{Sym}^r(\mathbb{P}^2)$ is smooth at the class of $(p_1, \ldots, p_r)$.

A useful tool to take care of infinitely near points is the Hilbert scheme.

**Definition 16** (cf. [19, 20, 27]) The Hilbert scheme $\text{Hilb}^r(\mathbb{P}^2)$ parameterizes zero-dimensional subschemes of $\mathbb{P}^2$ with length $r$, i.e., whose Hilbert polynomial has degree 0 and is equal to $r$. It is a rational, smooth, irreducible, projective variety of dimension $2r$ and is a desingularization of $\text{Sym}^r(\mathbb{P}^2)$.

Recall that $r$ points in general position in $\mathbb{P}^2$ determine an open dense subset of either $\text{Hilb}^r(\mathbb{P}^2)$ or $\text{Sym}^r(\mathbb{P}^2)$.

**Definition 17** Let $d$ be a positive integer. A multi-index $v_I := (v_1, v_2, \ldots, v_{d-2}, v_{d-1})$ with length $\rho$ and reduced length $r$ is a $(d - 1)$-tuple of non-negative integers with $r = \sum_{i=1}^{d-1} v_i$ and $\rho = \sum_{i=1}^{d-1} i(i + 1) v_i/2$. Setting $I^* = \{i \mid v_i > 0\}$, let us define

$$\text{Hilb}^v(\mathbb{P}^2) = \prod_{i \in I^*} \text{Hilb}^{v_i}(\mathbb{P}^2).$$

Thus, a point $Z \in \text{Hilb}^v(\mathbb{P}^2)$ is $Z = ([Z_i])_{i \in I^*}$ with $[Z_i] \in \text{Hilb}^{v_i}(\mathbb{P}^2)$, for each $i \in I^*$.

Let us denote by $\text{Hilb}^v_*(\mathbb{P}^2)$ the dense open subset of $\text{Hilb}^v(\mathbb{P}^2)$ whose elements are $Z = ([Z_i])_{i \in I^*}$ such that the zero-dimensional scheme $Z' = \bigcup_{i \in I^*} Z_i$ is supported on $r$ distinct points in $\mathbb{P}^2$; in particular, $Z_i$ is a collection of points $p_{i,1}, \ldots, p_{i,v_i}$ in $\mathbb{P}^2$. To a point $Z \in \text{Hilb}^v_*(\mathbb{P}^2)$, we associate the zero scheme $[Z_{v_I}]$ in $\text{Hilb}^v(\mathbb{P}^2)$ with $(Z_{v_I})_{\text{red}} = Z'$ and $Z_{v_I}$ given by the union of the points $p_{i,j}$ with multiplicity $i$, for each $i \in I^*$ and $j = 1, \ldots, v_i$.

**Remark 18** Note that $[Z_{v_I}]$ is not a general point of $\text{Hilb}^v(\mathbb{P}^2)$. This is the main motivation for introducing both length and reduced length. Equations (3) force the length $\rho$ of $v_I$ to be uniquely determined by $d$, namely, $\rho = (d + 4)(d - 1)/2$.

**Definition 19** We say that a multi-index $v_I = (v_1, v_2, \ldots, v_{d-2}, v_{d-1})$ is admissible if there is an element $Z \in \text{Hilb}^v_*(\mathbb{P}^2)$ such that the linear system $\Lambda_Z := [I_{Z_{v_I}}(d)]$ is non-empty, of the expected dimension

$$\dim \Lambda_Z = \left( \frac{d + 2}{2} \right) - \sum_{i=1}^{d-1} \frac{i(i + 1)v_i}{2},$$
and the general element of $\Lambda_Z$ is an irreducible curve. In such a case we say that $Z$ is an admissible cycle associated with $\nu_I$.

The following theorem is classical; cf., e.g., [18, V.II.20] and [1, Theorem 5.1.1].

**Theorem 20** Fix a positive integer $d$ and a multi-index $\nu_I$. Assume that $\nu_I$ satisfies (3). Let $Z \in \Hilb_{\nu_I}(\mathbb{P}^2)$ be a point. Then $\nu_I$ and $Z$ are admissible if and only if $\Lambda_Z$ is a homaloidal net.

**Remark 21** There are finitely many multi-indexes $\nu_I$ satisfying (3), when $d$ is fixed, but not all such multi-indexes are H-types, i.e., do give rise to homaloidal nets in $\text{Bir}^o_d$. The first, classical, example is $(6, 0, 2, 0)$, $d = 5$. In this case the line through the two triple points is a fixed component of the linear system of 5-ics with the assigned multiplicities. The movable part of the net is given by elliptic quartic curves. In particular, the point representing it in $\mathbb{P}^62$ is not in $\text{Bir}_5$.

To get rid of this behavior we proceed as follows; cf. Hudson’s test in [26] and in [1].

**Definition 22** Let $\nu_I = (\nu_1, \ldots, \nu_{d-1})$, $d \geq 2$, be a multi-index satisfying (3). When $d = 2$, we say that $\nu_I$ is (3) is 1-irreducible. Suppose then that $d \geq 3$, and let $m_1 \geq m_2 \geq m_3$ be the maximal multiplicities associated with $\nu_I$; cf. Definition 9.

We say that $\nu_I$ is 1-irreducible if $m_1 + m_2 \leq d$. Let $d' = 2d - m_1 - m_2 - m_3$. By (3) and $d \geq 3$, the same proof of Noether’s inequality (5) in [1] shows that $d > d' \geq 2$.

Now define a new multi-index $q(\nu_I) = (\nu_1', \nu_2', \ldots, \nu_{d-2}', \nu_{d-1}')$ by the following steps:

- for each $j = 1, 2, 3$, decrease $\nu_{m_j}$ by 1 (if $m_3 = m_2 = m_1$, this means decrease $\nu_{m_1}$ by 3);
- set $\epsilon = d - m_1 - m_2 - m_3$;
- for each $j = 1, 2, 3$, set $k = m_j + \epsilon$ and, if $k > 0$, increase $\nu_k$ by 1;
- finally, for each $i = 1, \ldots, d' - 1$, set $\nu_i' = \nu_i$.

We say that $\nu_I$ is irreducible if $\nu_I$ is 1-irreducible, $q(\nu_I)$ is 1-irreducible, $q(q(\nu_I))$ is 1-irreducible, and so on, for all new multi-indexes until one stops, when $d$ becomes 2. A script, which runs this irreducibility test, is listed in the Appendix.

**Remark 23** If $\nu_I$ is the H-type of a Cremona transformation $\gamma \in \text{Bir}^o_d$, and the maximal multiplicities $m_1 \geq m_2 \geq m_3$ of the base points of the homaloidal net $\mathcal{L}_\gamma$ occur at three points $p_1, p_2, p_3$ such that a quadratic transformation $\omega$ centered at $p_1, p_2, p_3$ is well defined, then $q(\nu_I)$ is just the H-type of $\omega \circ \gamma \in \text{Bir}^o_d$.

Setting $p_4, \ldots, p_r$ the other base points of $\mathcal{L}_\gamma$, with respective multiplicities $m_4 \geq \cdots \geq m_r$, one has (cf., e.g., Corollary 4.2.6 in [1]) that the multiplicities of the homaloidal net of $\omega \circ \gamma$ at the points corresponding via $\omega$ to $p_1, \ldots, p_r$ are respectively $m_1', m_2', \ldots, m_r'$, where

\[ m_i' = m_i - \epsilon, \quad i = 1, 2, 3, \quad \epsilon = m_1 + m_2 + m_3 - d, \quad m_j' = m_j, \quad j \geq 4. \]
Note that \(d' = d - \varepsilon, m_1' = d - m_2 - m_3, m_2' = d - m_1 - m_3, \) and \(m_3' = d - m_1 - m_2.\)

The same formulas hold even if \(m_1, m_2, m_3\) are not the maximal multiplicities of the base points of \(L_\gamma.\) Note that \(\varepsilon > 0\) if and only if \(m_1 + m_2 + m_3 > d.\)

The next theorem appears to be classical, and it has been implicitly used by Hudson, but it had probably fallen into oblivion; cf. historical remark 5.3.6 in [1] and the references therein. For a modern proof, see Theorems 5.2.19 and 5.3.4 in [1].

**Theorem 24** Fix an integer \(d\) and a multi-index \(\nu_I\) which satisfies (3) and is irreducible, according to Definition 22. Setting \(r = \sum_{i=1}^{d-1} \nu_i\), let \(m_1 \geq m_2 \geq \cdots \geq m_r\) be the multiplicities associated with \(\nu_I;\) cf. Definition 9.

Then \(\nu_I\) is admissible, and there exists a non-empty Zariski open subset \(U\) of \((\mathbb{P}^2)^r\) (that is, \(\mathbb{P}^2 \times \cdots \times \mathbb{P}^2, r\) times) such that for each \((p_1, \ldots, p_r) \in U\) there exists a Cremona transformation of degree \(d\) which has \(p_i\) as the base point of multiplicity \(m_i, i = 1, \ldots, r,\) and has no other base points.

In our notation, this translates as follows.

**Corollary 25** In the hypothesis of the previous theorem, there exists an open dense subset \(U_{\nu_I} \subset \text{Hilb}^{\nu_I}(\mathbb{P}^2)\) such that for any point \(Z \in U_{\nu_I}, Z\) is associated with \(\nu_I\) and the linear system \(\Lambda_Z\) is a homaloidal net.

The algebraic structure of \(\text{Bir}_d\) and \(\text{Bir}_d^o\) is already known; cf. [6, Lemma 2.4].

**Lemma 26** The subsets \(\text{Bir}_d\) and \(\text{Bir}_d^o\) in \(\mathbb{P}^{3N-1}\) are quasi-projective varieties.

We will use also planes, i.e., linear subspaces of dimension 2, in \(\mathbb{P}^{N-1}\), or equivalently three-dimensional vector subspaces of \(\mathbb{C}^N.\) A convenient setting is Grassmannians.

**Definition 27** Denote by \(\text{Gr}(3, N)\) the Grassmannian variety parameterizing three-dimensional vector subspaces of \(\mathbb{C}^N\), i.e., planes in \(\mathbb{P}^{N-1}.\) It is a smooth, irreducible, rational, projective variety of dimension \(3(N - 3).\)

### 3 Cremona Transformations of Fixed Degree

In this section we use notation introduced in Sect. 2. Fix \(d\) a positive integer.

**Remark 28** For each \(d \geq 2,\) there is a one-to-one map
\[
\text{Bir}_d \setminus \text{Bir}_d^o \rightarrow \prod_{a=1}^{d-1} \left(\mathbb{P}(\mathbb{C}[x, y, z]_a) \times \text{Bir}_{d-a}^o\right),
\]
\[
[f_1 : f_2 : f_3] \mapsto \left(h, \left[\frac{f_1}{h} : \frac{f_2}{h} : \frac{f_3}{h}\right]\right).
\]
where \( h = \gcd(f_1, f_2, f_3) \). The inverse is the collection of the maps
\[
\tau_a : \mathbb{P}(\mathbb{C}[x, y, z]_a) \times \text{Bir}_{d-a}^\circ \to \text{Bir}_d,
\]
\[
\tau_a(h, [f_1 : f_2 : f_3]) = [hf_1 : hf_2 : hf_3],
\]
which are injective, for each \( a = 1, \ldots, d - 1 \).

Let us focus on \( \gamma = [f_1 : f_2 : f_3] \in \text{Bir}_d^\circ \); in particular, \( \gcd(f_1, f_2, f_3) = 1 \). Let \( \mathcal{L}_\gamma = \mathbb{P}((f_1, f_2, f_3)) \) be the homaloidal net associated with \( \gamma \), recalled in Sect. 2.

**Definition 29** We denote by “Bs” the map which sends a Cremona transformation \( \gamma \in \text{Bir}_d^\circ \) to its base locus:
\[
\text{Bs} : \text{Bir}_d^\circ \to \text{Hilb}(\mathbb{P}^2), \quad \gamma = [f_1 : f_2 : f_3] \mapsto \text{Bs}(\gamma) = Z(f_1, f_2, f_3), \quad (8)
\]
where \( \text{Bs}(\gamma) = Z(f_1, f_2, f_3) \) is a 0-dimensional subscheme of \( \mathbb{P}^2 \).

**Remark 30** The map \( \text{Bs} \) is algebraic. To see this, fix an irreducible component \( A \subset \text{Bir}_d^\circ \); take
\[
I_c = \{([f_1 : f_2 : f_3], [Z]) \mid Z \subset Z(f_1, f_2, f_3) \subset A \times \text{Hilb}^c(\mathbb{P}^2),
\]
and let \( p_1, p_2 \) be the projections on the two factors. By (3), each homaloidal net in \( \text{Bir}_d^\circ \) has the expected dimension. Hence, for each \( A \) there is a unique \( c \) such that
\[
p_1^{-1}([f_1 : f_2 : f_3]) \cap I_c \text{ is a point. In this setting, } \text{Bs} = p_2 \circ p_1^{-1}.
\]

There is no inverse map to \( \text{Bs} \) in (8), because the map \( \text{Bs} \) is not injective. The definition of \( \text{Bir}_d^\circ \) is such that two different bases of the same homaloidal net gives different elements in \( \text{Bir}_d^\circ \) while having the same base locus. This is quite awkward, at least from the algebraic geometry point of view, but can be, somehow, settled as follows.

**Lemma 31** Let \( \gamma = [f_1 : f_2 : f_3] \) and \( \delta = [g_1 : g_2 : g_3] \) be in \( \text{Bir}_d^\circ \) such that \( \text{Bs}(\gamma) = \text{Bs}(\delta) \). Then \( \langle f_1, f_2, f_3 \rangle = \langle g_1, g_2, g_3 \rangle = W \subset \mathbb{C}^{d(d+3)/2+1} \), and there exists a unique change of basis matrix \( \omega \in \text{PGL}(3) \) such that the triplet \( \langle g_1, g_2, g_3 \rangle = (f_1, f_2, f_3) \omega \) in \( W \cong \mathbb{C}^3 \).

**Proof** Let \( Z_\gamma = (\text{Bs}(\gamma))_{\text{red}} = (\text{Bs}(\delta))_{\text{red}} \) be the reduced base locus of \( \gamma \) and \( \delta \). Theorem 20 says that \( \dim A_{Z_\gamma} = 2 \) and thus \( W = \langle f_1, f_2, f_3 \rangle \ni g_i, i = 1, 2, 3 \). Hence, by definition, the \( f_i \)'s and the \( g_i \)'s are two bases of the 3-dimensional vector space \( W \subset \mathbb{C}^{d(d+3)+1} \), and there is a unique change of basis \( \omega \) sending one to the other. To conclude, observe that \( [f_1 : f_2 : f_3] \) and \( [\lambda f_1 : \lambda f_2 : \lambda f_3], \lambda \in \mathbb{C}^*, \) represent the same element in \( \text{Bir}_d^\circ \). \( \square \)

The previous lemma suggests how to change the target space in (8) in order to get a birational map. For this purpose, we construct suitable morphisms.

Fix a positive integer \( d \) and an irreducible (according to Definition 22) multi-index \( \nu \), satisfying (3). By Corollary 25 there is a dense open subset \( U_\nu \subset \text{Hilb}^\nu(\mathbb{P}^2) \)
made of admissible cycles associated with \( v_I \). Recall that \( Z = ([Z_i])_{i \in I^*} \in U_{v_I} \) is such that \([Z_i] \in \text{Hilb}^{v_i}(\mathbb{P}^2)\) and \( Z_i \) is a collection of \( v_i \) points \( p_{i,1}, \ldots, p_{i,v_i} \) in \( \mathbb{P}^2 \).

Let us define the map

\[
\beta_{v_I} : U_{v_I} \to \text{Gr}(3, N), \quad \beta_{v_I}(Z) = [H^0(I_{Z_{v_I}}(d))],
\]

which sends \( Z \) to the homaloidal net of plane curves of degree \( d \) with multiplicity \( i \) at the points \( p_{i,j} \), for each \( i \in I^* \) and \( j = 1, \ldots, v_i \); cf. Definition 17.

**Remark 32** It is important to stress that the base locus in \( \text{Hilb}^{\rho}(\mathbb{P}^2) \) of a homaloidal net associated with a multi-index \( \nu_I \) and supported on \( r = \sum_{i=1}^{d-1} v_i \) distinct points determines a unique admissible cycle in \( \text{Hilb}^{\nu_I}(\mathbb{P}^2) \) associated with \( \nu_I \). In that case, we say that such a base locus in \( \text{Hilb}^{\rho}(\mathbb{P}^2) \) is admissible. Vice versa, an admissible cycle determines uniquely the homaloidal net.

This, together with Lemma 31, yields that the morphism \( \beta_{v_I} \) is a birational map onto its image in the Grassmannian \( \text{Gr}(3, N) \) of planes in \( \mathbb{P}^{N-1} = \mathbb{P}(\mathbb{C}[x, y, z]_d) \).

Next we want to go from the Grassmannian to \( \text{Bir}^0_d \). This is done by distinguishing a basis in the general point of the image of \( \beta_{v_I} \). To do this, choose three general \( N - 3 \) planes \( H_1, H_2, H_3 \) in \( \mathbb{P}(\mathbb{C}[x, y, z]_d) = \mathbb{P}^{N-1} \), e.g., we may choose

\[
H_1 = \langle y^d, y^{d-1}z \rangle, \quad H_2 = \langle z^d, z^{d-1}x \rangle, \quad H_3 = \langle x^d, x^{d-1}y \rangle.
\]

This allows us to universally choose the basis \( W \cap H_1, W \cap H_2, W \cap H_3 \) for a general 3-dimensional linear vector subspace \( W \subset \mathbb{C}^N \). In other words, we have chosen three sections \( \sigma_i : \text{Gr}(3, N) \to U \) of the universal bundle over the Grassmannian \( \text{Gr}(3, N) \).

**Definition 33** In the above setting, define the rational map

\[
\alpha_{v_I} : \text{PGL}(3) \times U_{v_I} \dashrightarrow \text{Bir}^0_d, \\
(\omega, Z) \mapsto [\omega(\sigma_1(\beta_{v_I}(Z))) : \omega(\sigma_2(\beta_{v_I}(Z))) : \omega(\sigma_3(\beta_{v_I}(Z)))],
\]

where \( \omega \in \text{PGL}(3) \) is acting on the 3-dimensional vector subspace \( W \subset \mathbb{C}^N \) as described in Lemma 31. The map \( \alpha_{v_I} \) is well defined; in fact, \( \alpha_{v_I}(\lambda \omega, Z) = \alpha_{v_I}(\omega, Z) \), for any \( \omega \in \text{PGL}(3) \) and \( \lambda \in \mathbb{C}^* \).

**Lemma 34** The map \( \alpha_{v_I} \) is birational onto its image.

**Proof** It is enough to prove that \( \alpha_{v_I} \) is generically injective. But this follows immediately by Lemma 31 and Remark 32. \( \square \)

**Lemma 35** Let \( v_I = (v_1, \ldots, v_{d-1}) \) and \( \mu_I = (\mu_1, \ldots, \mu_{d-1}) \) be two distinct admissible multi-indexes. Then \( \text{Im}(\alpha_{v_I}) \) and \( \text{Im}(\alpha_{\mu_I}) \) lie in two different components of \( \text{Bir}^0_d \).

**Proof** Let \( r \) and \( m \) be the two reduced lengths of \( v_I \) and \( \mu_I \), respectively. If \( r = m \) we conclude again by Remark 32. Assume that \( r > m \). Then we have to prove
that $\text{Im}(\alpha_{\nu_I}) \not\supset \text{Im}(\alpha_{\mu_I})$. This is equivalent to saying that the base locus of a general element in $\text{Im}(\alpha_{\mu_I})$ cannot be obtained as the limit of base loci of general elements in $\text{Im}(\alpha_{\nu_I})$.

Assume that this is not the case. Then in $\text{Hilb}^\rho(\mathbb{P}^2)$ there is a curve whose general point represents the base locus of an element in $\text{Im}(\alpha_{\nu_I})$ and with a special point associated with $Z_{\mu_I}$. In other words, we are saying that a bunch of points of multiplicity $m_{i,1}, \ldots, m_{i,h_1}$, with ordinary singularities, limits to a point of some multiplicity $m_i$, with ordinary singularity, and this is done in such a way that Noether’s equations (3) are always satisfied.

Fix one point in the limit, say $p_1$ of multiplicity $m_1$, and assume that $p_1$ is the limit of $\{q_1, \ldots, q_{h_1}\}$ of respective multiplicity $m_{1,j}$. The existence of the limit forces
\[
\frac{m_1(m_1 + 1)}{2} = \sum_j \frac{m_{1,j}(m_{1,j} + 1)}{2}; \text{ in particular, } m_1 \leq \sum_j m_{1,j},
\]
with strict inequality if $h_1 > 1$. This, together with (3) yields
\[
d^2 - 1 = \sum_i m_i^2 = \sum_i \left( \sum_j (m_{i,j}^2 + m_{i,j}) - m_i \right) = d^2 - 1 + \sum_i (\sum_j m_{i,j}) - m_i.
\]

Hence we have the contradiction $\sum_i (\sum_j m_{i,j}) - m_i = 0$. □

In the previous lemmas, we found an irreducible component of $\text{Bir}_d^{\circ}$ for each admissible multi-index $\nu_I$. In the next lemma, we show that each Cremona transformation $\gamma \in \text{Bir}_d^{\circ}$ belongs to one of these irreducible components.

**Lemma 36** Let $\gamma \in \text{Bir}_d^{\circ}$ be a birational transformation, $Z' = \text{Bs}(\gamma)$ its base locus, and $\nu_I = (\nu_1, \ldots, \nu_{d-1})$ the corresponding $H$-type. Setting $\rho$ to be the length of $\nu_I$ and $r$ its reduced length, there is a curve $C \in \text{Hilb}^\rho(\mathbb{P}^2)$ such that

(i) $[Z'] \in C$,
(ii) the general point $[Z_t] \in C$ is a zero-dimensional scheme supported on $r$ distinct points, with ordinary singularities, in $\mathbb{P}^2$,
(iii) $Z_t$ is admissible.

**Proof** Let $\mathcal{L}_\gamma \subset |\mathcal{O}(d)|$ be the homaloidal net associated with $Z'$. Consider the index $i(\gamma)$ introduced in Definition 6. If $i(\gamma) = 0$, the assertion is immediate, for any degree $d$. To conclude, we argue by induction on $i(\gamma)$. Assume that $i(\gamma) = M > 0$, and let $p \in Z_{\text{red}}$ be a point of multiplicity $m$ with infinitely near other base points. Let $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a quadratic transformation centered in $p$ and two general points $q_1$ and $q_2$, and $\mathcal{L}' = \omega_* \mathcal{L}_\gamma$ the strict transform linear system. Then $\mathcal{L}' \subset |\mathcal{O}(2d - m)|$ is a homaloidal net defining $\gamma' = \omega \circ \gamma$. By construction, $i(\gamma') \leq i(\gamma) - 1$ and by inductive hypothesis we may describe its base locus $Z' = \text{Bs} \mathcal{L}'$ as the limit of admissible cycles with two ordinary points of multiplicity $d$. Then applying $\omega^{-1}$ we get the desired curve in $\text{Hilb}^\rho(\mathbb{P}^2)$. □
**Definition 37** Let \( \nu_I = (v_1, \ldots, v_{d-1}) \) be an admissible multi-index. We denote by \( \operatorname{Bir}_{\nu_I}^o \) the irreducible component of \( \operatorname{Bir}_d^o \) whose general element is defined by a homaloidal net of H-type \( \nu_I \). Moreover, we denote by \( \operatorname{Bir}_{v_I} \) the intersection of the Zariski closure of \( \operatorname{Bir}_{\nu_I}^o \) in \( \mathbb{P}^{3N-1} \) with \( \operatorname{Bir}_d \).

**Remark 38** It is important to stress that a homaloidal net may degenerate to a linear system with a fixed component in such a way that the residual part is not a homaloidal net. In particular, the Zariski closure of \( \operatorname{Bir}_{\nu_I}^o \) in \( \mathbb{P}^{3N-1} \) is not contained in \( \operatorname{Bir}_d \). The easiest occurrences of this behavior are as follows.

Let \( \mathcal{L} \) be a general homaloidal net in \( \operatorname{Bir}_{(4,1)}^o \) given by plane cubics with a double point \( p_0 \) and four simple base points \( p_1, \ldots, p_4 \). If we let \( p_1, \ldots, p_4 \) become aligned, then \( \mathcal{L} \) degenerates to a net with a fixed line and the residual part is composed with the pencil of lines through \( p_0 \).

Let \( \mathcal{L} \) be a general homaloidal net in \( \operatorname{Bir}_{(3,3,0)}^o \) given by plane quartics with three double points \( p_1, p_2, p_3 \) and three simple base points. If we let \( p_1, p_2, p_3 \) become aligned, then \( \mathcal{L} \) degenerates to a linear system with a fixed line and the residual part is a 3-dimensional linear system of cubics with six simple base points.

The next lemma is classical; see, e.g., [26, p. 73], but the proof therein is not complete.

**Lemma 39** If \( d \geq 4 \) and \( \gamma \in \operatorname{Bir}_d^o \) is not a de Jonquières transformation, then the number of base points of the homaloidal net \( \mathcal{L}_\gamma \) is at most \( d + 2 \).

**Proof** By Lemma 36 (cf. also Theorems 20 and 24), the H-type \( \nu_I \) of \( \gamma \), and hence the number \( r = \sum_{i=1}^{d-1} v_i \) of base points \( \mathcal{L}_\gamma \), does not depend on the position of the points in \( \mathbb{P}^2 \). Therefore, we may and will assume that these base points are in general position, in such a way that \( \gamma \) can be factored as a composition of quadratic transformations, centered at base points of \( \mathcal{L} \) only, each one decreasing the degree of \( \gamma \).

Now we proceed by induction on the degree \( d \) of \( \gamma \).

The base of induction is \( d = 4 \). In this case there is only one possible H-type, which is \((3,3,0)\), i.e., six base points and the assertion is trivially true.

Suppose then that \( d > 4 \). We set \( m_1, m_2, m_3, \tilde{d}, \epsilon, \mu_i \) according to Definition 22; in particular, \( \tilde{d} = 2d - m_1 - m_2 - m_3 = d + \epsilon \). Recall that \( r = \sum_{i=1}^{d-1} v_i \) and set \( \tilde{r} = \sum_{i=1}^{d-1} \mu_i \).

Assume first that \( \mu_I \) is the H-type of a de Jonquières transformation, namely, \( \mu_I = (2\tilde{d} - 2, 0, \ldots, 0, 1) \) and \( \tilde{r} = 2\tilde{d} - 1 \). Since \( \nu_I \) is not the H-type of a de Jonquières transformation and \( \tilde{d} < d \), it follows that \( \nu_I \) is obtained from \( \mu_I \) by performing a quadratic transformation centered at points of multiplicity \( e_I \leq 1 \), say \( 1 \geq e_1 \geq e_2 \geq e_3 \geq 0 \). On the other hand, the point with multiplicity \( e_3 \) for \( \mu_I \) would have multiplicity \( m_3 = e_3 + c = \tilde{d} - e_1 - e_2 \) for \( \nu_I \), that still have the point of multiplicity \( \tilde{d} - 1 \) corresponding to the highest multiplicity point of \( \mu_I \). Since \( m_1, m_2, m_3 \) are chosen to be the highest multiplicities, this is possible only if \( e_2 = e_3 = 0 \). It follows that \( d = 2\tilde{d} - e_1, m_1 = \tilde{d} \), and \( m_2 = m_3 = \tilde{d} - e_1 \); hence, \( \tilde{r} = 2\tilde{d} - 1 = d + e_1 - 1 \).
and \( r = \tilde{r} + 3 - e_1 = d + 2 \), where \( r \) is the number of base points of \( \mathcal{L}_\gamma \), which is the assertion.

Now we may and will assume that \( \mu_I \) is not the H-type of a de Jonquières transformation, so the inductive hypothesis says that \( \tilde{r} \leq \tilde{d} + 2 \).

The construction of the \( \mu_i \) implies that \( r \leq \tilde{r} + 3 \), and one sees that equality holds if and only if \( m_1 = m_2 = m_3 = -\varepsilon = d/2 = \tilde{d} \). Since \( d > 4 \), one has \( \tilde{r} \leq \tilde{d} + 2 = d/2 + 2 \leq d - 1 \) and therefore \( r = \tilde{r} + 3 \leq d + 2 \), which is the assertion.

Assume then \( r \leq \tilde{r} + 2 \); One sees that equality holds if and only if \( m_1 = \tilde{d} > m_2 = m_3 = -\varepsilon \) and \( d = m_1 + m_2 \). Since \( \mu_I \) is not a de Jonquières transformation, one has \( \tilde{d} = m_1 \leq d - 2 \) and therefore \( r \leq \tilde{r} + 2 \leq \tilde{d} + 4 \leq d + 2 \), which is the assertion.

Finally, the remaining case is \( r \leq \tilde{r} + 1 \). Since \( \tilde{d} \leq d - 1 \), one has

\[
r \leq \tilde{r} + 1 \leq \tilde{d} + 3 \leq d + 2,
\]

which concludes the proof of this lemma. \( \Box \)

We are finally ready for the proof of Theorem 1 in the Introduction.

**Proof of Theorem 1** Fix an integer \( d \geq 2 \). By Lemma 26, \( \text{Bir}_d^{a} \) is a quasi-projective variety. By Lemma 36, each irreducible component \( \text{Bir}_{\nu_I}^{o} \) of \( \text{Bir}_d^{a} \) is determined by an admissible and irreducible multi-index \( \nu_I = (\nu_1, \ldots, \nu_{d-1}) \) satisfying (3).

By Lemma 34, \( \text{Bir}_{\nu_I}^{o} \) is rational and has dimension \( 8 + \dim U_{\nu_I} = 8 + 2r \), where \( r = \sum_{i=1}^{d-1} \nu_i \) is the reduced length of \( \nu_I \). By Lemma 13, the maximum \( r \) is 2\( d - 1 \) and occurs for \( \nu_I = (2d - 2, 0, \ldots, 0, 1) \), which gives an irreducible component of dimension \( 8 + 2(2d - 1) = 4d + 6 \), whose elements are de Jonquières transformations. By Lemma 39, when \( d \geq 4 \), the other irreducible components have dimension at most \( 8 + 2(d + 2) = 2d + 12 \). \( \square \)

**Definition 40** For any fixed positive integers \( d \) and \( a < d \) let

\[
\text{Bir}_d^{a} = \mathbb{P}^{(\binom{a+2}{2})-1} \times \text{Bir}_{d-a}.
\]

As already observed in Remark 28, there is a natural inclusion of \( \text{Bir}_d^{a} \) into \( \text{Bir}_d \). For this we often identify \( \text{Bir}_d^{a} \) with its image in \( \text{Bir}_d \).

**Remark 41** For integers \( a < b < d \) we have, with natural identifications,

\[
\text{Bir}_d^{a} \cap \text{Bir}_d^{b} = \mathbb{P}^{(\binom{a+2}{2})-1} \times \text{Bir}_{d-a}.
\]

**Remark 42** When \( 0 < a < b < d \), the general element \( [f_1 : f_2 : f_3] \) of \( \tau_b(\text{Bir}_{d-b}^{a}) \subset \text{Bir}_d \) cannot be the limit of elements in \( \tau_a(\text{Bir}_{d-a}^{b}) \subset \text{Bir}_d \) because \( \gcd(f_1, f_2, f_3) \) is an irreducible polynomial of degree \( b > a \).

We are able to completely describe the behavior of these varieties in low degrees.

**Example 43** By Theorem 1, \( \text{Bir}_2^{0} = \text{Bir}_2^{(3)} \) is irreducible of dimension 14. By Remark 28, \( \text{Bir}_2 = \text{Bir}_2^{0} \cup \text{Bir}_2^{1} \). A general element \( \gamma \) in \( \text{Bir}_2^{(3)} \) is given by a homaloidal
net of conics with three distinct base points \( p_1, p_2, p_3 \). If we let \( p_3 \) move to a general point of the line through \( p_1 \) and \( p_2 \), the line splits from the net and we get a degeneration of \( \gamma \) to an element in \( \text{Bir}_1^1 \). Since each element in \( \text{Bir}_1^1 \) can be obtained in this way, \( \text{Bir}_2 \) is irreducible.

**Example 44** Again by Theorem 1 and Remark 28, \( \text{Bir}_3^2 = \text{Bir}_{(4,1)}^2 \) is irreducible of dimension 18 and \( \text{Bir}_3 = \text{Bir}_{(4,1)} \cup \text{Bir}_1^1 \cup \text{Bir}_2^2 \), where \( \text{Bir}_3^2 = \tau_2(\text{Bir}_4^2) \) has dimension 13 and \( \text{Bir}_1^1 \) contains \( \tau_1(\text{Bir}_2^2) \), which has dimension 16.

We claim that \( \text{Bir}_3 \) has three irreducible components and is connected.

By Remark 41, \( \text{Bir}_1^1 \cap \text{Bir}_2^2 = \mathbb{P}^2 \times \text{Bir}_1^1 \) has dimension 12. On the other hand, a general element \( \gamma \) in \( \text{Bir}_{(4,1)}^2 \) is given by a homaloidal net of cubics with a double point \( p \) and four simple base points \( q_1, \ldots, q_4 \). If we let \( q_2 \) move to a general point of the line through \( p \) and \( q_1 \), the line splits from the net and we get a degeneration of \( \gamma \) to an element in \( \text{Bir}_1^1 \). This shows that \( \text{Bir}_3 \) is connected.

Note that any degeneration of an element in \( \text{Bir}_3^2 \) has to contain a double point. Linear systems of conics with a double point are homaloidal only in the presence of a fixed component. Then the only possible degenerations of \( \text{Bir}_3^2 \) are either a pair of fixed lines together with the linear system of lines or a fixed line, say \( l \), and a linear system of conics with a base point on \( l \). However, the general element in either \( \text{Bir}_1^1 \) or \( \text{Bir}_2^2 \) is such that there is no base point, of the mobile part, lying on the fixed component, and therefore it cannot be obtained as a limit of elements in \( \text{Bir}_3^2 \). This means that \( \text{Bir}_1^1 \) and \( \text{Bir}_2^2 \) give two further irreducible components, other than \( \text{Bir}_{(4,1)} \). In [10], it seems that the authors missed the component \( \text{Bir}_3^2 \) in \( \text{Bir}_3 \).

**Example 45** By Theorem 1, \( \text{Bir}_4^2 \) has two irreducible components \( \text{Bir}_{(6,0,1)}^2 \) and \( \text{Bir}_{(3,3,0)}^2 \), having respective dimensions 22 and 20.

Reasoning as in the previous examples, one may check that \( \text{Bir}_4 \) is connected (we will prove it in Theorem 55 later) and its decomposition into irreducible components is

\[
\text{Bir}_4 = \text{Bir}_{(3,3,0)} \cup \text{Bir}_{(6,0,1)} \cup \text{Bir}_1^1 \cup \text{Bir}_2^2 \cup \text{Bir}_3^3,
\]

where the last three components have respective dimensions 17, 19, and 20.

To study the connectedness of \( \text{Bir}_4^2 \), we need to understand degenerations of base loci of homaloidal systems. This is a hard task, and almost nothing is known. The only example we are aware of is the one stated in [18, V.III.25, p. 231] of a quartic curve with three double points degenerating to a quartic with a triple point. This suggests that the component of \( \text{Bir}_4^2 \) associated with the multi-index \((3,3,0)\) intersects the component of a de Jonquières transformation.

**Example 46** Take the linear system of quartic curves with three double base points at \( p_1 = [0, 0, 1], p_2 = [t, 0, 1], p_3 = [0, t, 1], t \neq 0 \). Its affine equation is

\[
a_0x^4 + a_1x^3y + a_2x^2y^2 + a_4xy^3 + a_5y^4 - 2a_0tx^3 + a_3x^2y + (-a_4t + a_1t + a_3)xy^2 - 2a_5ty^3 + a_0t^2x^2 + (-a_4t^2 - ta_3)xy + a_5t^2y^2 = 0.
\]
For $t = 0$, this is a linear system of quartics, whose general member is irreducible, with a triple base point at $[0, 0, 1]$ and three infinitely near simple base points in the direction of the lines $x = 0$, $y = 0$, and $x + y = 0$, which are the limits of the lines $\overline{p_1p_3} : x = 0$, $\overline{p_1p_2} : y = 0$, and $\overline{p_2p_3} : x + y - tz = 0$.

By imposing further three simple base points $p_4, p_5, p_6$ in general position, one gets a homaloidal net of type $(3, 3, 0)$ for a general $t \neq 0$ and a homaloidal net of type $(6, 0, 1)$ for $t = 0$. E.g., we choose $p_4 = [1, -2, 1]$, $p_5 = [-2, 1, 1]$, and $p_6 = [2, 3, 1]$ and we get the following Cremona transformation, for $t = 1$,

$$
\begin{bmatrix}
3x^3 - 6x^2z + 80xy^2 - 107xyz + 3xz^2 - 9y^3 - 98y^2z + 107yz^2
\end{bmatrix}x
$$

$$
-3\left(-x^2 + 10xy - 12xz - y^2 - 12yz + 13z^2\right)xy
$$

$$
3\left(-y + z\right)\left(12x^2 - 3xy - 12xz - y^2 + yz\right)y.
$$

which has the inverse map

$$
\begin{bmatrix}
-36x^2 - 243xy + 42xz - 396y^2 + 116yz
\end{bmatrix}
$$

$$
\begin{bmatrix}
-36xy + 39xz - 99y^2 + 107yz
\end{bmatrix}
$$

$$
\begin{bmatrix}
-36xy + 39xz - 99y^2 + 107yz
\end{bmatrix}
$$

$$
\begin{bmatrix}
36xy - 42xz + 99y^2 - 125yz + 10z^2
\end{bmatrix}
$$

$$
\begin{bmatrix}
-108x^3z - 1296x^2y^2 + 1539x^2yz - 1152x^2z^2 - 7128xy^3 + 10809xyz
$$

$$
-6195xyz^2 - 30xz^3 - 9801y^4 + 15840y^3z - 8317y^2z^2 - 90yz^3
$$

while, for $t = 0$, we get the de Jonquières transformation

$$
\begin{bmatrix}
12x^3 - 217xy^2 + 308xyz - 30y^3 + 308y^2z
\end{bmatrix}
$$

$$
6\left(-2x^2 - 19xy + 28xz
$$

$$
-2y^2 + 28yz\right)xy : 6\left(-23x^2y + 42x^2z - 5xy^2 + 42xyz - 2y^3\right)y,
$$

which has the inverse map

$$
\begin{bmatrix}
-7\left(6x + 11y\right)^2\left(-3y + 2z\right)\left(-6x - 17y + 4z\right)
\end{bmatrix}
$$

$$
\begin{bmatrix}
14\left(-6x - 17y + 4z\right)\left(6x + 11y\right)\left(-3y + 2z\right)^2
\end{bmatrix}
$$

$$
\begin{bmatrix}
216x^3z - 2484x^2y^2 + 4896x^2yz - 1368x^2z^2 - 9648xy^3
$$

$$
+ 16710xy^2z - 5544xyz^2 + 96xz^3 - 9555y^4
$$

$$
+ 15942y^3z - 5854y^2z^2 + 240yz^3
$$

These computations have been performed by using Maple.

**Proposition 47** $\text{Bir}_{3}^{2} = \text{Bir}_{(6,0,1)}^{2} \cup \text{Bir}_{(3,3,0)}^{2}$ is connected, $\dim(\text{Bir}_{(6,0,1)}^{2} \cap \text{Bir}_{(3,3,0)}^{2}) = 19$. 
**Proof** The connectedness follows from Example 46, which shows that the general element in Bir$_{3}^{o}(3,3,0)$ may degenerate to a special element in the component Bir$_{6}^{o}(6,0,1)$ of a de Jonquières transformation. The general choice of the base points of such de Jonquières transformations is one triple point in the plane, three infinitely near simple points, and further three simple points in the plane. Thus it lies in an open subset of $\mathbb{P}^2 \times \text{Sym}^3 \mathbb{P}^1 \times \text{Sym}^3 \mathbb{P}^2$, which has dimension 11, plus 8 dimensions for the action of PGL(3) on the homaloidal net; cf. Lemma 31.

$\Box$

**Proposition 48** Bir$_5^{o} = \text{Bir}_{8}^{o}(8,0,0,1) \cup \text{Bir}_{3}^{o}(3,3,1,0) \cup \text{Bir}_{6}^{o}(0,6,0,0)$ is connected.

**Proof** The decomposition of Bir$_5^{o}$ into three irreducible components Bir$_{8}^{o}(8,0,0,1)$, Bir$_{3}^{o}(3,3,1,0)$, and Bir$_{6}^{o}(0,6,0,0)$, having respective dimensions 26, 22, and 20, follows from Theorem 1.

One has Bir$_{3}^{o}(3,3,1,0) \cap \text{Bir}_{6}^{o}(0,6,0,0) \neq \emptyset$ for the same reason of the previous proposition, namely, that a linear system of quintics with three double base points may degenerate to a linear system of quintics with a triple base point and three infinitely near simple base points. Using the notation of Example 46, one gets such a degeneration just by applying a quadratic transformation centered at $p_4$, $p_5$, $p_6$ to the degeneration of the linear system of quartics.

In order to prove the connectedness of Bir$_5^{o}$, it is enough to show, with an example, that Bir$_{8}^{o}(8,0,0,1) \cap \text{Bir}_{3}^{o}(3,3,1,0) \neq \emptyset$. We remark that, if we make collide a triple point and three double points, in general we get a quintuple point, not a quadruple one. Thus we perform a special degeneration: We take the linear system $L$ of quintics with a triple base point and three infinitely near simple base points, and further three simple points in the plane. Thus it lies in an open subset of $\mathbb{P}^2 \times \text{Sym}^3 \mathbb{P}^1 \times \text{Sym}^3 \mathbb{P}^2$, which has dimension 11, plus 8 dimensions for the action of PGL(3) on the homaloidal net; cf. Lemma 31.

$\Box$

$Bir_5^{o} = \text{Bir}_{8}^{o}(8,0,0,1) \cup \text{Bir}_{3}^{o}(3,3,1,0) \cup \text{Bir}_{6}^{o}(0,6,0,0)$ is connected.

**Proof** The decomposition of Bir$_5^{o}$ into three irreducible components Bir$_{8}^{o}(8,0,0,1)$, Bir$_{3}^{o}(3,3,1,0)$, and Bir$_{6}^{o}(0,6,0,0)$, having respective dimensions 26, 22, and 20, follows from Theorem 1.

One has Bir$_{3}^{o}(3,3,1,0) \cap \text{Bir}_{6}^{o}(0,6,0,0) \neq \emptyset$ for the same reason of the previous proposition, namely, that a linear system of quintics with three double base points may degenerate to a linear system of quintics with a triple base point and three infinitely near simple base points. Using the notation of Example 46, one gets such a degeneration just by applying a quadratic transformation centered at $p_4$, $p_5$, $p_6$ to the degeneration of the linear system of quartics.

In order to prove the connectedness of Bir$_5^{o}$, it is enough to show, with an example, that Bir$_{8}^{o}(8,0,0,1) \cap \text{Bir}_{3}^{o}(3,3,1,0) \neq \emptyset$. We remark that, if we make collide a triple point and three double points, in general we get a quintuple point, not a quadruple one. Thus we perform a special degeneration: We take the linear system $L$ of quintics with an oscnode at $[0,0,1]$, along the direction of the conic $xz + y^2 = 0$, and a triple point at $[t,0,1]$, when $t \neq 0$. The affine equation of $L$ is

$$a_0 x^5 + a_1 x^4 y + a_2 x^3 y^2 + a_3 x^2 y^3 + a_4 x y^4 + a_5 y^5$$

$$- 3 a_0 t^4 x^4 + 2 a_1 t x^3 y + (2 a_0 t^2 - a_2 t) x^2 y^2 + (a_1 t^2 + a_5) x y^3$$

$$- a_0 t^3 y^4 + 3 a_0 t^2 x^3 + a_1 t^2 x^2 y - 2 a_0 t^3 x y^2 - a_0 t^3 x^2 = 0.$$
while, for $t = 0$, we get the de Jonquières transformation
\[
[ - 60 y^3 x^2 - 5 y^4 x + 30 y^5 + 5 x^5 + 30 x y^3 z : 12 y^5 - 29 y^3 x^2 + 5 y x^4 + 12 x y^3 z \\
: 6 y^5 - 5 y^4 x - 12 x^3 y^2 + 5 y^2 x^3 + 6 x y^3 z ].
\]

We are able to check the properties of these maps and to find their inverse maps by using Maple, as we did in Example 46. \hfill \square

**Proposition 49** $\text{Bir}_6^o = \text{Bir}_6(10,0,0,0,1) \cup \text{Bir}_6(3,4,0,1,0) \cup \text{Bir}_6(4,1,3,0,0) \cup \text{Bir}_6(1,4,2,0,0)$ is connected.

*Proof* The decomposition of $\text{Bir}_6^o$ into four irreducible components follows from Theorem 1. The usual degeneration of three double base points to one triple point with infinitely near three simple base points implies that $\text{Bir}_6(1,3,0,0) \cap \text{Bir}_6(1,4,2,0,0) \neq \emptyset$. It can be obtained by that of Example 46 by applying a quadratic transformation centered at $p_4$, $p_5$, and at a general point in the plane.

To conclude we show, with two examples, that

\[
\text{Bir}_{(3,4,0,1,0)}^o \cap \text{Bir}_{(1,4,2,0,0)}^o \neq \emptyset \quad \text{and} \quad \text{Bir}_{(3,4,0,1,0)}^o \cap \text{Bir}_{(10,0,0,0,1)}^o \neq \emptyset.
\]

First example: Take the linear system of sextics with a triple base point at $[0, 0, 1]$, with an infinitely near double base point in the direction of the line $x = 0$, and another triple base point at $[t, 0, 1]$, $t \neq 0$. When $t = 0$, we get a linear system of sextics with a quadruple base point with an infinitely near double base point. By imposing further three double base points and a simple base point, e.g., we choose $[1, 1, 1]$, $[-1, 1, 1]$, $[2, 1, 1]$, and $[2, -3, 1]$, respectively, we get a homaloidal net of type $(1, 4, 2, 0, 0)$ for a general $t \neq 0$ and a homaloidal net of type $(3, 4, 0, 1, 0)$ for $t = 0$. In particular, for $t = 1$, we get the map

\[
\left[ 27 x^6 - 216 y^6 - 81 x^2 y z^3 + 135 x^3 y z^2 - 108 x^2 y^2 z^2 \\
- 368 x y z^2 - 27 x^4 y z + +108 x^3 y z^2 + 520 x y^4 z \\
- 27 x^3 z^3 + 81 x^4 z^2 - 81 x^5 z + 324 y^5 z - 27 x^5 y - 260 x y^5 \\
: 3 x y (-z + y) \left( -4 y^3 + 22 y^2 z - 18 x y^2 + 9 x z^2 - 18 x z^2 + 9 x^3 \right) \\
: 9 y^2 (-z + y) \left( 6 y^3 + 4 x y^2 - 7 y x z - 3 x z^2 y + 3 x^3 - 3 x z^2 \right) \right]
\]

and, for $t = 0$, we get the map

\[
\left[ 27 x^6 - 540 y^6 - 430 x y^3 z^2 + 54 x^4 y z + 216 x^3 y^2 z - 702 x^2 y^3 z + 644 x y^4 z \\
+ 648 y^5 z - 108 x^5 y + 513 x^2 y^4 - 322 x y^5 \\
: 3 y (-z + y) \left( 36 y^4 + 20 x y^3 - 20 y^2 x z - 45 x^2 y^2 + 9 x^4 \right) \\
: 9 y^2 (-z + y) \left( 6 y^3 + 2 x y^2 - 5 y x z - 6 x^2 y + 3 x^3 \right) \right].
\]
Second example: Take the linear system of sextics with a double base point at \([0, 0, 1]\) with three other infinitely near double base points, each one infinitely near to the previous one, along the conic \(xz + y^2 = 0\), and a quadruple base point at \([t, 0, 1]\), \(t \neq 0\). When \(t = 0\), we get a linear system of sextics with a base point of multiplicity 5. By imposing further three simple base points, e.g., we choose \([1, 1, 1]\), \([-1, 1, 1]\), and \([2, 1, 1]\), we get a homaloidal net of type \((3, 4, 0, 1, 0)\) for a general \(t \neq 0\) which degenerates to a homaloidal net of de Jonquières type \((10, 0, 0, 0, 1)\). In particular, for \(t = 1\), we get the map

\[
\begin{align*}
-2x^6 - 8y^6 - 4xy^2z^3 + 8x^2y^2z^2 - 4x^3y^2z \\
+ 5x^2y^3z - 8xy^4z - 2x^2z^4 + 8x^3z^3 - 12x^4z^2 \\
- 2y^4z^2 + 8x^5z - 5x^3y^3 + 13x^2y^4 + 5xy^5
\end{align*}
\]

and, for \(t = 0\), we get the de Jonquières transformation

\[
\begin{align*}
-3x^6 - 20y^6 - 20xy^4z + 20x^3y^3 + 23x^2y^4 : \\
&- y\left(8y^5 + 8y^3xz + 3x^5 - 11x^3y^2 - 8x^2y^3\right) \\
&: -y^2\left(4y^4 + 4xyz^2 + 3x^4 - 4x^3y - 7x^2y^2\right).
\end{align*}
\]

We again checked the properties of these maps and found their inverse maps by using Maple, as we did in Example 46.

\[\square\]

**Remark 50** We want to stress a difference between Bir\(^n\) and Bir\(^o\). It is not difficult to prove that any pair of components in Bir\(^n\) intersects. The situation for Bir\(^o\) is, quite unexpectedly, different. We claim that Bir\(^o\)(4,1,3,0,0) \(\cap\) Bir\(^o\)(10,0,0,0,1) = \(\emptyset\). Let \(\mathbb{P}^2 \times \Delta \to \Delta\) be a degeneration, over a complex disk \(\Delta\), with a linear system \(\mathcal{L}\) such that the map induced by \(\mathcal{L}_0\) is in Bir\(^o\)(10,0,0,0,1) and the map induced by \(\mathcal{L}_1\) is in Bir\(^o\)(4,1,3,0,0), for \(t \in \Delta \setminus \{0\}\). Let \(\mu : Y \to \mathbb{P}^2 \times \Delta\) be the blow up of the unique singular point \(p\) in \(\mathcal{L}_0\), with exceptional divisor \(E \simeq \mathbb{P}^2\). Note that the sections associated with the singular points of \(\mathcal{L}_0\) have to intersect in \(p\), because \(p\) is the unique singular point in \(\mathcal{L}_0\). Hence, the strict transform \(\mathcal{L}_0^Y\) is such that \(\mathcal{L}_0^Y_{0,E}\) is a curve with degree equal to the multiplicity of \(\mathcal{L}_0\) at \(p\), which is 5, and with three triple points and a double point. This forces, by a direct computation, \(\mathcal{L}_0^Y_{0,E}\) to be non-reduced and therefore introduces a fixed component in \(\mathcal{L}_0\). Hence, the map induced by \(\mathcal{L}_0\) is in Bir\(^o\)(10,0,0,0,1) \(\setminus\) Bir\(^o\)(10,0,0,0,1) and Bir\(^o\)(4,1,3,0,0) \(\cap\) Bir\(^o\)(10,0,0,0,1) = \(\emptyset\).

The connectedness of Bir\(_d\) is considerably simpler even if not all irreducible components intersect each other.
Remark 51 If \( \nu_I = (2d - 2, 0, \ldots, 0, 1) \), i.e., if \( \nu_I \) is the H-type of de Jonquières transformations, then it is easy to check that \( \text{Bir}_{\nu_I} \) meets \( \text{Bir}_{d}^{a} \), for each \( a = 1, \ldots, d - 1 \).

However, the same statement does not hold for each admissible multi-index \( \nu_I \).

For example, one may check that the minimum \( d \) such that there exists an admissible \( \nu_I = (\nu_1, \ldots, \nu_{d-1}) \) with \( \text{Bir}_{\nu_I} \cap \text{Bir}_{d}^{a} = \emptyset \) is \( d = 10 \). Moreover, there are exactly two such admissible \( \nu_I \) of degree \( d = 10 \), namely, \((0, 0, 7, 0, 0, 1, 0, 0, 0, 0)\) and \((3, 0, 0, 6, 0, 0, 0, 0, 0)\).

We already recognized that \( \text{Bir}_{d}^{a} \cap \text{Bir}_{d}^{b} \neq \emptyset \). Hence, to conclude the connectedness it is enough to show that for any admissible index \( \nu_I \) there is a degeneration with a fixed component. The following lemmas allow us to produce these degenerations.

Lemma 52 Let \( \chi : \mathbb{P}^2 \times \Delta \rightarrow \Delta \) be a one-dimensional family over a complex disk \( \Delta \). Let \( F_t \) be the fiber over the point \( t \in \Delta \). Let \( C_1, C_2, \text{ and } C_3 \) be three disjoint sections of \( \chi \). Let \( p_i^t := C_i \cap F_t \) be the intersection of the \( i \)-th section with the fiber \( F_t \), \( i = 1, 2, 3 \). Assume that the points \( p_i^t \) are in general position for any \( t \). Then there is a birational modification \( \Omega : \mathbb{P}^2 \times \Delta \dashrightarrow \mathbb{P}^2 \times \Delta \) such that \( \Omega_t := \Omega |_{F_t} \) is the quadratic Cremona transformation centered at the points \( \{ p_i^t \} \). Assume that there is a linear system \( \mathcal{H} \in \text{Pic}(\mathbb{P}^2 \times \Delta) \) such that \( \mathcal{H}_t \) is a homaloidal net associated with a multi-index \( \nu_I \) and \( \mathcal{H}_0 \) is a homaloidal net associated with a multi-index \( \mu_I \) and

\[
\sum_i \text{mult}_{Z_{\nu_I}} p_i^1 < \sum_i \text{mult}_{Z_{\mu_I}} p_0^i.
\]

Let \( \mathcal{H}' := \Omega_\ast \mathcal{H} \) be the transformed linear system. Then \( \mathcal{H}'_{|F_0} \) has a fixed component.

Proof Let \( D_{ij} \) be the divisor covered by lines spanning the points \( p_i^t \) and \( p_j^t \) inside \( F_t \). The general position assumption ensures that \( D_{ij} \) is a smooth minimally ruled surface. Let \( \phi : Y \rightarrow \mathbb{P}^2 \times \Delta \) be the blow up of \( \mathbb{P}^2 \times \Delta \) along the disjoint sections \( C_i \) with exceptional divisors \( E_i \). Let \( D_{ij}^Y \) be the strict transform of \( D_{ij} \) on \( Y \) and \( l_t^{ij} \) the strict transform of the line \( \langle p_i^t, p_j^t \rangle \subset F_t \), and \( F_t^Y \) the strict transform of the fiber \( F_t \). Then we have the following intersection numbers.

\[
K_Y \cdot l_t^{ij} = K_{F_t^Y} \cdot l_t^{ij} = -1
\]

and

\[
D_{ij}^Y \cdot l_t^{ij} = D_{ij}^Y \cdot D_{ij}^Y \cdot F_t^Y = (l_t^{ij} \cdot l_t^{ij}) p_t^Y = -1.
\]

Moreover, \( D_{ij}^Y \) is ruled by \( l_t^{ij} \) and all fibers are irreducible and reduced. This shows, by Mori theory (see, for instance, [2, Theorem 4.1.2]), that \( l_t^{ij} \) spans an extremal ray and the extremal ray can be contracted to a smooth curve \( Z_{ij} \) in a smooth 3-fold. Let \( \psi \) be the blow down of the three disjoint divisors \( D_{ij} \). Then \( \psi \) is a morphism from \( Y \) to \( \mathbb{P}^2 \times \Delta \). The required map \( \Omega \) is just \( \psi \circ \phi^{-1} \). To conclude, observe that for the general fiber \( \Omega_t(\mathcal{H}_t) = \mathcal{H}'_{|F_t} \) and the deg \( \Omega_t(\mathcal{H}_t) = \deg \mathcal{H}' = 2d - \sum \text{mult}_{Z_{\nu_I}} p_i^1 \).
The numerical assumption on the multiplicities forces \( \deg \Omega_0(H_0) < \deg H' \). This yields a fixed component in \( H'|_{F_0} \).

**Remark 53** The usage of the above lemma is to produce degenerations with fixed components starting from a known degeneration in a different pure degree.

To apply the above lemma we have to construct degenerations. This is the aim of the next lemma.

**Lemma 54** Let \( \nu_I \) be an admissible multi-index in degree \( d \). Let \( Z_{\nu_I} \) be a base locus of a general homaloidal net associated with the multi-index \( \nu_I \), and \( p_1, p_2 \in Z_{\nu_I} \) two points. Assume that

\[
m_1 := \text{mult}_{Z_{\nu_I}} p_1 \leq \text{mult}_{Z_{\nu_I}} p_2 =: m_2.
\]

Then there is a degeneration \( \chi : \mathbb{P}^2 \times \Delta \rightarrow \Delta \) and a base scheme \( Z \) such that \( Z_t := Z|_{F_t} \) is associated with the multi-index \( \nu_I \) and \( Z_0 := Z|_{F_0} \) has the point \( p_1 \) infinitely near to \( p_2 \).

**Proof** The multiplicity of \( p_2 \) is at least the one of \( p_1 \) and we may degenerate \( p_1 \) into \( p_2 \). Assume that \( \text{mult}_{p_1} Z_{\nu_I} = m_1 \) and \( \text{mult}_{p_2} Z_{\nu_I} = m_2 \). Then a local equation of such a degeneration can be

\[
t x_0^{d-\nu_1} p + x_1^{d-\nu_2} (x_2^m h + t g) + x_2^d = 0,
\]

where \( g \in \mathbb{C}[x_0, x_1, x_2] \) is such that for \( t \neq 0 \) the points \([1, 0, 0]\) and \([0, 1, 0]\) are ordinary points of multiplicities \( m_1 \) and \( m_2 \), respectively, and for \( t = 0 \) the point \([0, 1, 0]\) is of multiplicity \( m_2 \) with an infinitely near point of multiplicity \( m_1 \).

**Theorem 55** The quasi-projective variety \( \text{Bir}_d \) is connected.

**Proof** As already observed, we have only to prove that for any admissible index \( \nu_I \) the general element admits a degeneration with a fixed component. Let \( L \) be a general homaloidal net associated with \( \nu_I \) and \( \omega \) a standard Cremona transformation centered in three points of \( \text{Bs} L \) that lowers the degree. Let \( L' \) be the transformed homaloidal net and \( q_1, q_2, q_3 \) the three points of indeterminacy of \( \omega^{-1} \). Then by the Noether–Castelnuovo theorem the \( q_i \)'s are not of maximal multiplicity. That is, we may assume that there is a point \( x \in \text{Bs} L' \) with \( \text{mult}_x L' > \text{mult}_{q_i} L' \). Then by Lemma 54 there is a degeneration \( \chi : \mathbb{P}^2 \times \Delta \rightarrow \Delta \) and a base scheme \( Z \) such that \( Z_t := Z|_{F_t} \) is associated with the multi-index \( \nu_I \) and \( Z_0 := Z|_{F_0} \) has the point \( q_1 \) infinitely near to \( x \).

Let \( C_i \) be the section of \( \chi \) associated with the point \( q_i \) and \( \Omega, H' \) the birational modification and linear system on \( \mathbb{P}^2 \times \Delta \) as in Lemma 52. Then we may apply Lemma 52 to produce a Cremona transformation \( \Omega^{-1} : \mathbb{P}^2 \times \Delta \rightarrow \mathbb{P}^2 \times \Delta \) that induces \( \omega^{-1} \) on the general fiber and produces a fixed component in the special linear system \( H|_{F_0} \). In particular, this produces a degeneration of \( L \) to a homaloidal net with a fixed component.
We are now ready to complete the proof of Theorem 3.

**Proof of Theorem 3** By Remark 28, Bir$_d$ is the union of Bir$_0^d$ and Bir$_a^d$, for each $a = 1, \ldots, d - 1$. Note that Bir$_{d-1}^d$, that is, $\tau_{d-1}(\text{Bir}_0^d)$, has dimension $8 + \binom{d+1}{2} - 1 = d(d + 1)/2 + 7$, for $d \geq 2$. By Theorem 1, irreducible components of Bir$_d$ coming from irreducible components of Bir$_{d-a}$, $a \leq d - 2$, have dimension at most

$$4(d - a) + 6 + a(a + 3)/2 = 4d + 6 + a(a - 5)/2 \leq d(d - 1)/2 + 13.$$ 

When $d \geq 7$, this implies that Bir$_a^d$, $a \leq d - 2$, has components of smaller dimension. □

**Acknowledgements** It is a pleasure to thank Ciro Ciliberto for the reference in Enriques–Chisini. Furthermore, we thank the anonymous referee for a thorough reading, suggestions, and remarks that improved the paper, in particular for pointing out a mistake in Definition 17.

**Appendix: Test if a Multi-Index is Irreducible, i.e., Admissible**

The following script defines a function “adm” in PARI/GP that checks if a multi-index $\nu_I$ is irreducible, or equivalently admissible; cf. Definition 22 and Theorem 24.

The input is a vector $\nu_I = [\nu_1, \nu_2, \ldots, \nu_{d-1}]$, where the square brackets are used in GP for denoting row vectors. The output of adm($\nu_I$) is either 1 = true, i.e., $\nu_I$ is admissible, or 0 = false, i.e., $\nu_I$ is not admissible.

```plaintext
adm(v) = { local(d=1+#v, s=1-(#v+1)^2, m=vector(3), t=0, e=0);
    for( i=1,#v, s = s+i^2*v[i] );
    if( s, print("ERROR: the self-intersection is not 1"); return(0) );
    s = -#v*3;
    for( i=1,#v, s = s+i*v[i] );
    if( s, print("ERROR: the genus is not 0"); return(0) );
    while( d>2, s = 0; t = d; e = d;
        for( i=1,3, while( !s, t=t-1; s=s+v[t] );
            m[i]=t; e=e-t; v[t]=v[t]-1 ; s=s-1 );
        if( m[1]+m[2]>d, print("The net is reducible"); return(0) );
        for( j=1,3, t=m[j]+e; if( t, v[t] = v[t] +1 );
            d = d+e );
    1)
```

For example, after defining this function “adm” in GP, the command “adm([0, 6, 0, 0])” returns “1”, while the command “adm([6, 0, 2, 0])” prints “The net is reducible” and returns “0”; cf. Remark 21.
References

1. Alberich-Carramiñana, M.: Geometry of the Plane Cremona Maps. Lecture Notes in Mathematics, vol. 1769. Springer, Berlin (2002)
2. Andreotti, A., Mella, M.: Morphisms of projective varieties from the viewpoint of minimal model theory. Warsawa: Polska Akademia Nauk. Instytut matematyczny, 72 pp. Dissertationes Mathematicae 413 (2003)
3. Bisi, C.: On commuting polynomial automorphisms of \( \mathbb{C}^2 \). Publ. Mat. 48(1), 227–239 (2004)
4. Bisi, C., Calabri, A., Mella, M.: On dynamical behaviour of plane Cremona transformations with applications. Preprint
5. Blanc, J.: Elements and cyclic subgroups of finite order of the Cremona group. Comment. Math. Helv. 86(2), 469–497 (2011)
6. Blanc, J., Furter, J.-P.: Topologies and structures of the Cremona groups. Ann. Math. 178(3), 1173–1198 (2013)
7. Campillo, A., Olivares, J.: A plane foliation of degree different from 1 is determined by its singular scheme. C. R. Acad. Sci. Paris 328, 877–882 (1999)
8. Campillo, A., Olivares, J.: Special subschemes of the scheme of singularities of a plane foliation. C. R. Acad. Sci. Paris, Ser. I 344, 581–585 (2007)
9. Cantat, S., Lamy, S.: Normal subgroups in the Cremona group. Acta Math. 210(1), 31–94 (2013). With an appendix by Yves de Cornulier.
10. Cerveau, D., Déserti, J.: Transformations birationnelles de petit degré. Cours Spécialisés, Société Mathématique de France, Volume 19, to appear. arXiv:0811.2325
11. Cerveau, D., Déserti, J., Garba Belko, D., Meziani, R.: Géométrie classique de certains feuilletages de degré deux. Bull. Braz. Math. Soc., New Ser. 41(2), 161–198 (2010)
12. Cerveau, D., Lins Neto, A.: Irreducible components of the space of holomorphic foliations of degree 2 in \( \mathbb{C}^n \), \( n \geq 3 \). Ann. Math. (2) 143(3), 577–612 (1996)
13. Cukierman, F., Pereira, J.V., Vainsencher, I.: Stability of foliations induced by rational maps. Ann. Fac. Sci. Toulouse Math. (6) 18(4), 685–715 (2009)
14. Déserti, J.: Some properties of the Cremona group. In: Ensaios Matemáticos, vol. 21. Sociedade Brasileira de Matemática, Rio de Janeiro (2012)
15. Dolgachev, I.: Classical Algebraic Geometry: a modern view. Cambridge University Press, Cambridge (2012)
16. Dolgachev, I., Iskovskikh, V.A.: Finite subgroups of the plane Cremona group. In: Algebra, Arithmetic, and Geometry. Progress in Math., vol. I, vol. 269, pp. 443–548. Birkhäuser, Basel (2009). In honor of Y.I. Manin
17. Edo, E., Furter, J.-P.: Some families of polynomial automorphisms. J. Pure Appl. Algebra 194, 263–271 (2004)
18. Enriques, F., Chisini, O.: Lezioni Sulla Teoria Geometrica Delle Equazioni e Delle Funzioni Algebriche, 4 vols. Zanichelli, Bologna (1985)
19. Fogarty, J.: Algebraic families on an algebraic surface. Am. J. Math. 90, 511–521 (1968)
20. Fogarty, J.: Algebraic families on an algebraic surface. II. The Picard scheme of the punctual Hilbert scheme. Am. J. Math. 95, 660–687 (1973)
21. Friedland, S., Milnor, J.: Dynamical properties of plane polynomial automorphisms. Ergod. Theory Dyn. Syst. 9, 67–99 (1989)
22. Furter, J.-P.: On the variety of automorphisms of the affine plane. J. Algebra 195, 604–623 (1997)
23. Furter, J.-P.: On the length of polynomial automorphisms of the affine plane. Math. Ann. 322, 401–411 (2002)
24. Furter, J.-P.: Plane polynomial automorphisms of fixed multidegree. Math. Ann. 343, 901–920 (2009)
25. Gizatullin, M.: Defining relations for the Cremona group of the plane. Math. USSR Izv. 21(2), 211–268 (1983)
26. Hudson, H.P.: Cremona Transformations in Plane and Space. Cambridge University Press, Cambridge (1927)
27. Iarrobino, A.: Punctual Hilbert schemes. Bull. Am. Math. Soc. 78, 819–823 (1972)
28. Jung, H.W.E.: Ober ganze birationale Transformationen der Ebene. J. Reine Angew. Math. 184, 161–174 (1942)
29. Mattuck, A.: On the symmetric product of a rational surface. Proc. Am. Math. Soc. 21, 683–688 (1969)
30. Nguyen Dat, D.: Groupe de Cremona. PhD thesis, Université de Nice-Sophia Antipolis (2009)
31. Shafarevich, I.R.: On some infinite dimensional groups. Rend. Mat. Appl. (Roma) 25, 208–212 (1966)