TOPOLOGICAL INVARIANTS OF SOME CHEMICAL REACTION NETWORKS

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ABSTRACT. This is a revision and extension of earlier joint work [25] with N Kitchloo, tentatively attempting to apply ideas of Baker and Richter [7,8] to the cobordism theory of (quasi-Hamiltonian symplectic) toric manifolds [1,2]. Some possible applications to the statistical mechanics [16, 37] of chemical reaction networks [12,18,21,...] and their associated toric varieties are proposed.

Organization

The monographs [1,2] (of Guillemin, Ginzburg, and Karshon on Hamiltonian group actions, and of Bukhstaber and Panov on toric topology), serve as foundations for this paper; sections 1.1 and 1.2 summarize some of this background material. Its motivation, however (which comes from classical Hamiltonian mechanics of quasi-periodic systems) is deferred to an afterword.

We apply these techniques to the modern theory [12,18,19,...] of chemical reaction networks, some of which is summarized in §1.3. The point is that these formal models for chemical interactions have interesting associated toric varieties, with rich topological structures. A suitable notion of (equivariant) cobordism for such geometric objects would provide a natural context for their study, but the relevant technical details seem to be just outside the available literature, briefly reviewed in §2.

The third section of the paper is consequently concerned with characteristic numbers for the noncommutative cobordism spectrum \(M_\xi\) (roughly, of ‘omnioriented’ manifolds) defined by Baker, Richter, and Kitchloo [7,8,25]; these are perhaps to the classical characteristic numbers of Thom and others as the classical symmetric functions are to quasi- and non-commutative symmetric functions. The complication of this technology justifies working over \(\mathbb{Q}\) for simplicity, which allows us to make some connections with statistical mechanics and the theory of free probability, extending work of Friedrich and McKay [15,16,17] and Marcolli [29,30]. The paper ends with a conjecture [§3.2.3] relating the Hopf algebroid of (co)operations on \(M_\xi\) to the (neither commutative nor cocommutative) Hopf algebra of formal diffeomorphisms at the origin of the noncommutative line [9].

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I Some background and preliminaries

1.1 Some toric geometry

For our purposes a $2n$-dimensional complex-oriented toric manifold $M$ will be a closed compact smooth complex-oriented [34] manifold with a compatible [2 B.6.1 p 434] action of an $n$-dimensional torus $T \cong (S^1)^n$, with Lie algebra $\text{Lie } T \cong \mathbb{Z}^n \otimes \mathbb{R}$ (i.e. given a preferred integral basis) acting effectively on the tangent bundle.

A quasi-Hamiltonian (complex-oriented) toric manifold will have additional structure, defined by an equivariant differential form $\bar{\omega} = \omega - \Phi$ representing a class $[\bar{\omega}] \in H^2_T(M; \mathbb{R})$, where $\omega$ is a $T$-invariant symplectic 2-form (i.e. $\omega^n$ is nowhere zero, with $d\omega = 0$), and

\[
\begin{array}{c}
M \\
\Phi \\
\Phi \rightarrow (\text{Lie } T)^* \\
M/T := P
\end{array}
\]

is a diagram of $T$-equivariant maps factoring through a simple (i.e. exactly $n$ faces meet at each vertex [2 §1.7 p 46]) polyhedron, with an ordered set of $m$ codimension one faces. Such an object thus has an underlying symplectic manifold $(M, \omega)$, as well as an underlying stably almost complex manifold with a compatible $T$-action [1 D §1.3 p 231], which is Hamiltonian in the sense of [1 §2.15 p 25, Th E.37 p 266]. The correspondence between isotropy groups of faces and one-parameter subgroups of $T$ defines a surjective (omni-orientation) homomorphism

$\Lambda : \mathbb{Z}^m \rightarrow \mathbb{Z}^n$

[2 §7.3.9 p 247] such that if the vertex $v$ is the intersection of faces $\{i_v\} \subset [m] := \{1, \ldots, m\}$, and $\Lambda_{\{i_v\}}$ is the minor of $\Lambda$ defined by the columns indexed by $\{i_v\}$, then $\det \Lambda_{\{i_v\}} = \pm 1$.

Example A projective toric manifold $M$ (defined by a smooth complex toric variety) inherits a Kähler class $\omega$ from its projective embedding [2 §5.5.4 p 197, §7.3 p 243], as well as a moment map $\Phi$ defining a polyhedron in $(\text{Lie } T)^*$ bounded by $m$ half-planes $\{v \mid v \cdot e_i \geq \lambda_i\}$, such that [20]

$[\omega] = -2\pi \sum \lambda_i[v_i] \in H^2(M, \mathbb{R})$

for certain classes $[v_i] \in H^2(M, \mathbb{Z})$ defined in §1.2.2 below.
1.2 Some simplicial topology

1.2.1 Let $K$ be a finite simplicial complex with vertices $[m]$, and let $(X_i, A_i)$ be a family of reasonable pairs of spaces, also indexed by $[m]$; then Browder’s polyhedral product

$$K \mapsto (X_*, A_*)_K$$

([2 §4.2.3 p 137], functorial under maps of families $(X_*, A_*)$ and inclusions of complexes) associates to $K$, the moment-angle complex [2 §4.1.1 p 131]

$$Z_K := (D^2, S^1)^K \in (\text{Spaces with } \mathbb{T} - \text{action})$$

[2 §4.1.1 p 131], where now $(X_i, A_i) = (D^2, S^1)$ is a pair with an action of the circle $S^1$, and $\mathbb{T} = (S^1)^m$.

Remarks

1) $K \mapsto Z_K$ is monoidal, taking joins of simplicial complexes to Cartesian products of spaces [2 §4.5.5], and

2) If $|K| \cong S^{n-1}$ then $Z_K$ is a closed topological manifold of dimension $n + m$ [2 §4.1.4].

1.2.2 Following [2 §3.1.1 p 93], let $\mathbb{Z}[K]$ denote the Stanley-Reisner face ring of the simplicial complex $K$: this is the quotient of the augmented polynomial algebra $\mathbb{Z}_{1 \leq i \leq m}[v_i] = \mathbb{Z}[v_*]$ generated by the vertices of $K$, modulo the ideal generated by relations $v_I = \prod_{k \in I} v_k$, where the subset $I \subset K$ is not a simplex of $K$. Evidently $\mathbb{Z}[K]$ is a $\mathbb{Z}[v_*]$-algebra.

There has been a great deal of interesting recent work relating combinatorics (formulated for example in terms of face rings of simplicial complexes) and the equivariant topology of their associated moment-angle complexes. Some of this work will be summarized below; [2] contains further references to the original literature. Thus [2 Prop 4.5.5 p 147]:

$$\text{Tor}_{\mathbb{Z}[v_*]}^\mathbb{Z}(\mathbb{Z}[K], \mathbb{Z}) \cong H^*(\mathbb{Z}_K; \mathbb{Z}),$$

$$H^*_\mathbb{T}(\mathbb{Z}_K; \mathbb{Z}) \cong \mathbb{Z}[K]$$

(as graded rings). Moreover, if $P$ is an omnioriented simple polytope, and we let $T_\Lambda$ denote the kernel of

$$\Lambda \otimes \mathbb{R}/\mathbb{Z} : T^m \rightarrow T^n,$$

then $T^m \cong \mathbb{T}/T_\Lambda$ acts on $\mathbb{Z}_P/T_\Lambda := M$, which carries the structure of a projective toric manifold with $T^m$-action [2 §7.3.8 - 13 p 248] so

$$H^*_T(M; \mathbb{Z}) \cong \mathbb{Z}[K]$$

while $H^*(M; \mathbb{Z})$ is isomorphic to the quotient of $\mathbb{Z}[K]$ by the ideal generated by the image of $\Lambda$ [2 Th 5.3.1 p 186]. These isomorphisms have natural interpretations in terms of Eilenberg-Moore spectral sequences.
Remarks

1) Writing $T$ as a product $\prod_{1 \leq i \leq m} T_i$ of $m$ circles, the projection $T \to T_i$ defines a representation $C(i)$ of $T$ and a complex line bundle

$$L_i := Z_P \times_T C(i) \to M$$

whose Chern classes can naturally be identified with the images of the classes $v_i$ in $H^*(Z_P; \mathbb{Z})$. The images $[v_i]$ of these classes in $H^*(M; \mathbb{Z})$ are those cited above in Guillemin’s formula for the Kähler class of $M$ [2 §7.3.15 p 249].

2) When $P$ is a Gorenstein complex [2 Th 3.4.2 p 112], $M$ is a manifold with fundamental class

$$H^{2n}_{T \lambda}(Z_P; \mathbb{Z}) \cong H^{2n}(M; \mathbb{Z})$$

of the tangent bundle of $M$ [6, 2 Th 7.3.15 p 249].

1.3 Some chemical dynamical systems

1.3.1 A chemical reaction network $A$ (for alembic) is defined by a triple

$$A = \{G, \kappa, Y \} \in M^G_n(\mathbb{R}), Y \in M^s_n(\mathbb{Z})$$

where

*) $G$ is a graph with a set $V$, $|V| = n$ of vertices, called ‘reactions’, with edge set $E \subset \{(i, j) \in V \times V \mid i \neq j\}$ (i.e. no closed loops),

*) $\kappa = [\kappa^j_k]$ is a matrix of ‘reaction rates’ satisfying $\kappa^j_k > 0$ if $j \neq k$,

$$\kappa^j_k = -\sum_{k \neq j} \kappa^j_k$$

(i.e. zero row sums), and

*) $Y = [Y^m_j] \in M^s_n(\mathbb{Z})$ is a non-negative integer matrix, with $s$ the number of chemical ‘species’ involved in the network.

[1] [https://reaction-networks.net/wiki/Mathematics_of_Reaction_Networks](https://reaction-networks.net/wiki/Mathematics_of_Reaction_Networks)
Following [12,18,21 Def 2.1,22,...], a chemical reaction network is a constant positive solution \( c : \mathbb{R} \rightarrow \mathbb{R}^s \) (a vector of species concentrations) for the toric dynamical system
\[
\dot{c} = Y \cdot \kappa \cdot c^Y;
\]
that is, the system
\[
\frac{dc_i}{dt} = \sum_{1 \leq j,k \leq n} Y^k_i A^j_k \prod_{1 \leq m \leq s} c^m_{Y^m_i} = 0
\]
of ODEs associated to \( A \).

1.3.2 It will simplify this summary to assume that \( G \) is strongly connected, in the sense that for any pair \( i,j \) of vertices, there is a directed path from \( i \) to \( j \); the definitions below generalize [21 §4.1, 38 Th 2.2.9 p 21] to a disjoint union of strongly connected graphs. In this strongly connected case the matrix \( \kappa \) then has rank \( n - 1 \); let \((-1)^{n-1}K_i\) denote the determinant of a minor of \( \kappa \) defined by deleting the \( i \)th row and any column. A toric variety \( V(\mathbf{A}) \) is then defined by the ideal of the polynomial algebra \( \mathbb{Q}[K_1, \ldots, K_n] \) [12 Th 9 p 8, 38 §2.2.1 p 16] generated by elements
\[
\prod K^u_0_i - \prod K^u_1_i
\]
indexed by vectors \( u_0, u_1 \in \mathbb{N}^n \) satisfying \( \kappa \cdot (u_0 - u_1) = 0 \). The interior of the convex hull of the columns of \( \kappa \) (a polytope of dimension one less than the rank of \( \kappa \)) can be identified with the positive vectors in the image of the moment map of \( V(\mathbf{A}) \). When \( G \) is the disjoint union of \( l \) strongly connected subgraphs, the matrix \( \kappa \) needs to be replaced by a more general \((s+l) \times n\) Cayley matrix \( \text{Cay}_G(\kappa) \) [12 p 7] in the description of the moment polytope; this is probably best displayed in terms of the explicit matrices in the original sources. Further developments involving Hopf bifurcations are considered in [19].

Note that the real positive points on such a complex toric variety are suggested as asymptotic limits of states of dissipative biochemical systems. Following [13], the short term mechanics of living organisms are mostly metabolic, but in the long run (in terms of eggs rather than chickens) they are low-energy information management processes. The complex points of these varieties are thus not assigned any immediate physical or biological interpretation.
§II Cobordism theory

2.1 Characteristic numbers

The Thom spectrum

$$MU : S^2MU(k) \to MU(k + 1)$$

(for the stable unitary group $U = \cup_{k \geq 1} U(k)$) is defined by the Thom spaces $MU(k)$ of the canonical $\mathbb{C}^k$-bundles $\xi_k$ over the classifying spaces $BU(k)$. The classifying map

$$\nu : M \to BU(N - n), \ n \gg 0$$

of the stable normal bundle of a complex-oriented $2n$-dimensional manifold $M \subset \mathbb{C}^N$ defines a class

$$[\mathbb{C}^N \to M' \to MU(N - n)] \in \pi_{2n}MU := MU_{2n}$$

in the complex bordism ring $MU_*$ (with $MU_{odd} = 0$). The Thom isomorphism

$$H_*(MU; \mathbb{Z}) \to H_*(BU; \mathbb{Z})$$

(over comodules) identifies Hurewicz’s ring monomorphism

$$\pi_*MU \to \text{Hom}^{-*}(H^*(MU), H^*(S^0)) \cong H_*(BU)$$

with the characteristic number homomorphism which sends $M$ to the evaluation

$$c^I(\nu)[M] := \prod_{1 \leq k \leq n} c^i_k(\nu)[M]$$

of a degree $2n$ word in the Chern classes of the stable normal bundle of $M$ on the fundamental class $[M] \in H_{2n}(M; \mathbb{Z})$. This map becomes an isomorphism after tensoring with $\mathbb{Q}$.

The group completion

$$\coprod_{k \geq 0} BU(k) \to \Omega B(\coprod_{k \geq 0} BU(k)) \simeq \mathbb{Z} \times BU \supset 0 \times BU$$

defines an $H$-space structure on the classifying space for stable complex vector bundles. Following Borel and Hirzebruch, we identify its cohomology

$$H^*(BU; \mathbb{Z}) \cong \mathbb{Z}_{k \geq 1}[e_k] \subset \mathbb{Z}_{k \geq 1}[x_k] \cong \lim_{r \to \infty} H^*(B(T^r); \mathbb{Z}) := S_*$$

($|x_k| = 2, |e_k| = 2k$), with the graded polynomial algebra of symmetric functions [28 I §2.7 p 22] generated by elementary symmetric functions, identifying $e_k$ with the Chern class $c_k$. In terms of generating functions,

$$E(t) := \prod_{i \geq 1} (1 + x_i t) = \sum_{j \geq 0} e_j t^j,$$

and the \textbf{complete} symmetric functions $h_j$, $H(t) = \sum_{j \geq 0} h_j t^j$ are defined by $H(t) = E(-t)^{-1}$. Thus, for example, $e_0 = h_0 = 1$, $e_1 = h_1$.

\footnote{Such words $(I = 1^{i_1} 2^{i_2} \ldots, |I| = \sum k_i k)$ are partitions of $n$.}
The ring of symmetric functions has a canonical pair of involutions generating an action of the Klein four-group: the first is defined by \( x_i \mapsto -x_i \), sending \( e_k \) to \((-1)^k e_k\), while the second maps \( e_k \) to \( \bar{\omega}(e_k) = -h_k \) (so \( \bar{\omega}(p_k) = -p_k \)). These correspond to the operations which send a stable complex vector \( V \) respectively to its complex conjugate \( V^* \) and to its \( H \)-space inverse \(-V\). It follows that the \( i \)th Chern class of the stable normal bundle \( \nu \) of a stably almost-complex manifold corresponds to \(-h_i\) of its stably almost-complex tangent bundle \( T_M \) [11].

2.2 Symplectic cobordism

VL Ginzburg’s definition [1 Thm H.10 p 309]

\[
(W, \omega) : (V_0, \omega_0) \to (V_1, \omega_1)
\]

of a cobordism \( \partial W = V^\text{op}_0 \coprod V_1 \) between symplectic manifolds involves a closed two-form \( \omega \) on a \((2n + 1)\)-dimensional \( W \), having maximal rank (i.e. \( \ker \omega^n \) is a real line bundle supported on the interior of \( W \)). Symplectic manifolds \((V, \omega)\) admit homotopically well-defined stably almost complex structures, defining characteristic number homomorphisms indexed by unordered partitions \( I \) of \( i \leq n \), sending \( V \) to

\[
(-1)^{\sum k_i} \left( \prod h_i^{1/k} (T^C_V) : [\omega]^{n-i} |V] \in \mathbb{R}. \right.
\]

If we furthermore require the class \((2\pi)^{-1}[\omega] \in H^2_{dR}(V)\) to be integral, the resulting symplectic cobordism ring \( \mathcal{B}_* \) can be naturally identified with \( MU_* CP^\infty \), and the characteristic number homomorphism defined above can be identified with the (injective) Hurewicz homomorphism

\[
MU_* CP^\infty \to H_*(BU; \mathbb{Z}) \otimes H_*(CP^\infty; \mathbb{Z}) \cong S_* \otimes \mathbb{Z}_{i \geq 1}[b_{(i)}]
\]

(where \( b_{(i)} = b^i/i! \) is a divided power). These groups are torsion-free, and it will be convenient below to work with their characteristic zero localizations, defined by tensoring with \( \mathbb{Q} \). The completed localization \( \mathcal{B}((b^{-1})) \) (with \( b = b_1 \) [31]) can be regarded as a \( \mathbb{Z}/2\mathbb{Z} \)-graded filtered version of \( \mathcal{B}_* \).
§III The Baker-Richter spectrum

This section summarizes some of the work of Baker and Richter on quasi- and noncommutative symmetric functions, and their role in algebraic topology.

3.1 More characteristic numbers

3.1.1 The $2^{a-1}$ ordered partitions

$a := a_1 + \cdots + a_k$
of $a = \sum a_i$ into nonempty parts define the quasisymmetric function

$\langle a \rangle(x^*):= \sum_{0<i_1<\cdots<i_k} \prod x_{i_j}^{a_j} \in \mathbb{Z}[x^*]$

(denoted $[a_1, \ldots, a_k]$ in [7], cf. [22 §4]). The subring generated by such sums is, by Baker and Richter or Hazwinkel’s proof of Ditters’ conjecture, an evenly graded (‘quasi-shuffle’) Hopf algebra $Q\text{Symm}^*$ over $\mathbb{Z}$, generated over $\mathbb{Q}$ by certain Lyndon partitions [7 §2.2]; for brevity it will here be denoted $Q^*$. The dual Hopf algebra of noncommutative symmetric functions

$N^* = N\text{Symm}^* = \mathbb{Z}_{i\geq 1} \langle Z_i \rangle$
is free associative, with coproduct $\Delta Z_i = \sum_{i=j+k} Z_j \otimes Z_k$; the word $Z_a := \prod_{1\leq i\leq k} Z_{a_i} \in N_{2a}$ is dual to $\langle a \rangle \in Q^{2a}$. We will identify the generators $Z_i$ with Cartier’s elements $\Lambda_i$, cf. [10 §4.1F eq 155]. Abelianization defines dual maps

$N^* \to S^*$, $S^* \to Q^*$
of Hopf algebras, the second sends $h_k \mapsto \langle 1^k \rangle$. [The classical ring of symmetric functions is canonically self-dual; see further [32 §2].]

James’ construction provides a stable splitting

$$\Omega \Sigma BU(1) \sim \bigvee_{n\geq 0} \mathbb{B}U(1)^{\wedge n}$$

and thus Hopf algebra isomorphisms

$$H_*(\Omega \Sigma CP^\infty; \mathbb{Z}) \cong N_*, \quad H^*(\Omega \Sigma CP^\infty; \mathbb{Z}) \cong Q^* .$$

3.1.2 Regarding a complex line bundle as a complex vector bundle defines a map from $BU(1)$ to $BU$; composing with the stable inverse map of §2.1.1 above defines a morphism

$$\Omega \Sigma BU(1) \longrightarrow BU \xrightarrow{V} BU$$
of loop-spaces. Pulling back the canonical stable bundle over $BU$ back defines the $A_\infty$ spectrum $M_\xi$ of [8]. A class in $\pi_{2n}M_\xi$ can then be interpreted as the cobordism class of a complex-oriented $2n$-manifold $M$, together with a preferred isomorphism

$$T_M \cong \oplus L_i$$

Convention: $- \otimes \mathbb{Q}$ sends the graded module $M_* = M^{-*}$ to $M^*_Q = M_Q^{-*}$. 
of its stable tangent bundle as a sum of complex line bundles; quasitoric manifolds provide examples. Forgetting the splitting defines (some kind of) abelianization homomorphism $M_{\xi,*} \to MU_*$; note that expressing this in terms of characteristic numbers will involve the involution $\bar{\omega}$.

The Hurewicz map

$$M_{\xi,*} := \pi_* M_{\xi} \to H_*(M_{\xi}; \mathbb{Z}) \cong H_*(\Omega \Sigma \mathbb{C}P^\infty; \mathbb{Z}) \cong \mathbb{N}_*$$

is injective, and becomes an isomorphism after tensoring with $\mathbb{Q}$ (see §3.2.1 below); it sends $M$ to

$$(\langle n \rangle([v_*]) [M]) := \sum_{0 < i_1 < \cdots < i_k} (\prod c_1(L_{ij})^{n_j}) [M] : \mathbb{Q}^{2n} \to \mathbb{Z},$$

taking the quasisymmetric function $\langle n \rangle$ (of the Chern classes $[v_i]$) to its evaluation on the fundamental class of $M$.

More generally, a symplectic manifold $(V, \omega)$ with a compatible quasitoric structure (for example, a quasi-Hamiltonian manifold as in §1.1) has characteristic numbers $(i \leq n)$

$$(V, \omega) \mapsto [i \mapsto (\langle i(T_M) \rangle [[\omega]^{n-i}] [M] \cdot b_{(i)}) \in \text{Hom}(\mathbb{Q}_*, \mathbb{R}[b]) \cong \mathbb{N}_* \otimes \mathbb{R}[b].$$

If $(V, \omega)$ is projective toric, Guillemin’s theorem [§1.1] expresses this invariant in terms of the face parameters $\lambda_i$.

Remarks:

1) By the construction in §1.2.2, we can associate to quasisymmetric functions, characteristic classes in the face rings of simplicial spheres, and evaluate them to define characteristic numbers for such spheres, which are essentially the same as the characteristic numbers for their associated (toric) moment-angle manifolds. Since joins map to products under that correspondence, this defines a homomorphism from the noncommutative graded ring (with respect to joins) of isomorphism classes of simplicial spheres, to $\mathbb{N}_*$. For example,

$$\partial \Delta^n \mapsto \sum_{|a|=n} Z_a.$$

2) The collection $\mathcal{A}$ defining a chemical reaction network, as in §1.3, has an associated projective toric variety $V(\mathcal{A})$; when smooth, this is a quasi-Hamiltonian toric manifold, which defines an element in $M_{\xi,*} \mathbb{C}P^\infty \otimes \mathbb{R}$, with a symplectic cobordism class as image in $\mathcal{B}_* \otimes \mathbb{R}$. It seems reasonable to hope that this construction will extended to quasi-Hamiltonian orbifolds as well.

3) A good geometric definition of cobordism between quasi-Hamiltonian manifolds $(V, \bar{\omega})$ would imply a homomorphism from the resulting cobordism ring to $\mathbb{N}_* \otimes \mathbb{R}[b]$ with the extra characteristic numbers defined as above by the pullback $[\omega]$ of $[\bar{\omega}] \in H^2_\mathbb{F}(M; \mathbb{R})$. 

3.2 Formal diffeomorphisms of the noncommutative line

3.2.1 Although composition of formal power series with noncommuting coefficients is not associative, Baker and Richter [8 Prop 3.1] show that, with suitable care, $M\xi^\ast\mathbb{C}P^\infty$ can be understood as having a formal one-dimensional Lie group structure defined by a coproduct

$$\Delta_\xi x_\xi = \sum a_{i,j}x_{\xi i}^{i+1} \otimes x_{\xi j}^{j+1}$$

where, however, $x_\xi \in M\xi^2\mathbb{C}P^\infty$ is not a central element. Nevertheless, their Prop 2.3 uses the Hurewicz homomorphism to define an algebra monomorphism

$$x_\xi \mapsto \Theta(x_\xi) := \sum_{i\geq 0} Z_i x_{H i}^{i+1} = Z(x_H) : M\xi^\ast\mathbb{C}P^\infty \to N_\ast[[x_H]]$$

with $x_H$ central. Moreover, the rationalization $\Theta \otimes \mathbb{Q}$ is an isomorphism, and we have a commutative diagram

$$\begin{array}{c}
M\xi^\ast\mathbb{C}P^\infty \xrightarrow{\Delta_\xi} M\xi^\ast\mathbb{C}P^\infty \times \mathbb{C}P^\infty \\
\cong \Theta_\ast \quad \cong \Theta_\ast \otimes \Theta_\ast \\
\mathbb{N}_\ast[[x_H]] \xrightarrow{\Delta_N} \mathbb{N}_\ast[[x_H \otimes 1, 1 \otimes x_H]]
\end{array}$$

defining a formal group law over $\mathbb{N}_\ast \otimes \mathbb{Q}$ with central coordinate $x_H$.

Similarly, $M\xi^\ast\mathbb{C}P^\infty \cong \mathbb{N}_\ast[b]$ is a binomial Hopf algebra [with

$$b^H(T) = \sum_{i\geq 0} b_i^H T_i, \quad \Delta b^H(T) = b^H(T) \otimes \mathbb{N} b^H(T),$$

where $b_i^H \in \mathbb{N}_\ast[b]$ is a polynomial Kronecker dual to $x_i^H$, e.g. $b_1^H = b$.]

Note that $M\xi$ is complex-orientable (i.e., possesses a Thom isomorphism for complex vector bundles) even though it is not an $MU$-algebra [7 §7].

3.2.2 The Landweber-Novikov Hopf algebra $S_\ast = \mathbb{Z}_{i\geq 0}[t_i]$ of formal diffeomorphisms of the line at the origin (with $|t_i| = 2i$ and $t(T) = \sum_{i\geq 0} t_i T^{i+1}$) is defined by the coproduct

$$(\Delta S t)(T) = (t \otimes 1)((1 \otimes t)(T)) \in (S \otimes S)_\ast[[T]]$$

$$(|T| = 2).$$

Miščenko’s logarithm

$$\log_{MU}(T) = \sum_{n\geq 1} \frac{\mathbb{C}P_{n-1}}{n} T^n \in MU^\ast_\mathbb{Q}[[T]]$$

for the formal group law on $MU^\ast(\mathbb{C}P^\infty) \cong MU^\ast[[c]]$ defines a coaction

$$\psi_{MU}(log_{MU}(T)) = log_{MU}(t(T))$$

of $S_\ast$ on $MU_\ast$, yielding an isomorphism

$$(MU_\ast, MU_\ast \otimes S_\ast) \cong (MU_\ast, MU_\ast MU)$$
of Hopf algebroids \[35\]. The generators \( b_i \in MU_* (\mathbb{C}P^\infty) \) (Kronecker dual to \( c^i \in MU_*[[c]] \)) satisfy
\[
 b(T) = \sum_{i \geq 0} b_i T^i = \exp(b \log MU(T)) \in MU_* Q \mathbb{C}P^\infty \]
\[31\], defining a Hopf coproduct \( \Delta b(T) = b(T) \otimes MU b(T) \) making \( MU_* \mathbb{C}P^\infty \) an \( MU_* MU \)-comodule coalgebra with\[
 \psi_{MU} b(T) = b(t(T)) .
\]
I am indebted to Michiel Hazewinkel for drawing attention to work \[9\] of Brouder, Frabetti, and Krattenthaler on a (neither commutative nor co-commutative) generalization \((N_* , \Delta_N)\) of \((S_* , \Delta_S)\), defined on generators in terms of a formal residue homomorphism
\[
 \text{res}_{T=0} (\prod_{i \in \mathbb{Z}} a_i T^i) := a_{-1} ,
\]
so that
\[
 \Delta_N Z(T) = \text{res}_{U=0} (U^{-1} Z(U) \otimes (1 - U^{-1} Z(T))^{-1}) \in (N \otimes N)_r [[T]]
\]
(where \( T \) and \( U \) are central formal indeterminants of degree -2); their Theorem 2.14 provides an explicit formula for an antipode \( S \) on the generators \( Z_* \). It is an exercise, using the formal analog of Cauchy’s theorem, to see that the BFK Hopf algebra abelianizes to the Landweber-Novikov algebra.

It may be helpful to provide a concordance of the notation of BFK with ours. They work with a complexification \( H_{\text{dif}} := C_{i \geq 1} \langle a_i \rangle \) of \((N_* , \Delta_N)\), with their \( a_i \) corresponding to our \( Z_i \). They also consider a cocommutative Hopf algebra \( H_{\text{inv}} := C_{i \geq 1} \langle b_i \rangle \) with binomial coproduct \( \Delta b(T) = b(T) \otimes_C b(T) \); for example, Cartier’s primitive generators \( \Psi_k \in N_*^Q \) define such elements by \( b(T) := \exp(\Psi(T)) \). The algebra homomorphism
\[
 \psi_{BFK} : H_{\text{inv}} \to H_{\text{inv}} \otimes_C H_{\text{dif}}
\]
defined on generators by
\[
 \psi_{BFK} b(T) = \text{res}_{U=0} (U^{-1} b(U) \otimes (1 - U^{-1} a(T))^{-1})
\]
is, by their lemma 3.3, a morphism of coalgebras, defining (a family of) Hopf algebroid(s) \( (H_{\text{inv}} , H_{\text{inv}} \otimes_C H_{\text{dif}}) \) depending on a choice of \( b \)’s. The choice above defines a Hopf algebroid \((N_*^Q , N_*^Q \otimes (N_* , \Delta_N))\).

**3.2.3 Conjecture** Setting \( b^H(T) = \exp(b \Psi(T)) \) in \(3.2.1\) defines an isomorphism
\[
 (N_*^Q \langle b \rangle , N_*^Q \langle b \rangle \otimes (N_* , \Delta_N)) \to (M_{\xi_*^Q} , M_{\xi_*^Q} \mathbb{C}P^\infty \wedge M_{\xi})
\]
of Hopf algebroids, which specializes at \( b = 1 \) to an isomorphism
\[
 (N_*^Q , N_*^Q \otimes (N_* , \Delta_N)) \to (M_{\xi_*^Q} , M_{\xi_*^Q} M_{\xi}) .
\]
Remarks
1) Under this correspondence, abelianization $Mξ → MU$ sends
\[ \Psi_k → k^{-1} CP_{k-1}. \]

2) Remark 3.1.2.3 suggests hoping for a reasonable map from a cobordism theory of quasi-Hamiltonian toric manifolds (or varieties) to $E_∞ \otimes R$ induced by a factorization through an $A_∞$ morphism
\[ Mξ \otimes R → (MU ∧ CP^∞) \otimes R \]
of ring spectra; for the possible significance of $R$ versus $Q$ cf. \[10 § 4.4 II\]. . .

3.2.4 I am also indebted to Masoud Khalkhali for drawing my attention to the relevance of Voiculescu’s theory of free probability\[5\] to these matters. Indeed, McKay and Friedrich \[15,16,17 § 6.1\] show its relevance to cobordism theory by demonstrating that Miščenko’s logarithm corresponds naturally to the cumulant generating function of statistical mechanics.

This is closely related to the Helmholtz free energy (roughly, the Legendre transform [30 §2.1,37] of the cumulant generating function) of physical chemistry, which Schrödinger [36 Ch 6 p 74] identified with negative entropy. In [24 Th 3.1] an analog of the cumulant generating function is lifted to the noncommutative Hopf algebra of formal diffeomorphisms as the series
\[ S(x_H Θ(xξ)^{-1}) = S((\sum_{i≥0} Z_i x_H^i)^{-1}) ∈ N_∗[[x_H]], \]
with $S$ the antipode of $N_∗$. This might have interesting connections with [29] . . .

Afterword

This note is motivated by the idea that the work [4,26,33] of Kolmogorov, Arnol’d, Moser and others on the stability of quasi-periodic solutions of certain Hamiltonian mechanical systems (originating in celestial mechanics, but since generalized to hydrodynamics [3,5,14,27]) might have applications to the analysis of self-replicating systems in biology. This is suggested by D’Arcy Thompson’s ‘simple and most beautiful [inkdrop] experiment’ (On falling drops, \textit{On Growth and Form} p 395-6).
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