The effects of cosmic microwave background (CMB) temperature uncertainties on cosmological parameter estimation

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Abstract. We estimate the effect of the experimental uncertainty in the measurement of the temperature of the cosmic microwave background (CMB) on the extraction of cosmological parameters from future CMB surveys. We find that even for an ideal experiment limited only by cosmic variance up to $\ell = 2500$ for both the temperature and polarization measurements, the projected cosmological parameter errors are remarkably robust against the uncertainty of 1 mK in the FIRAS CMB temperature monopole measurement. The maximum degradation in sensitivity is 20\%, for the baryon density estimate, relative to the case in which the monopole is known infinitely well. While this degradation is acceptable, we note that reducing the uncertainty in the current temperature measurement by a factor of five will bring it down to $\sim 1\%$. We also estimate the effect of the uncertainty in the dipole temperature measurement. Assuming the overall calibration of the data to be dominated by the dipole error of 0.2\% from FIRAS, the sensitivity degradation is insignificant and does not exceed 10\% in any parameter direction.

Keywords: CMBR theory, physics of the early universe

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1. Introduction

The precision measurement of the energy spectrum of cosmic microwave background (CMB) photons by the FIRAS instrument [1] on board the COBE satellite [2] is one of the most spectacular pieces of evidence in support of the big bang theory. Owing to the fact that cosmological expansion does not lead to spectral distortions, but merely shifts the spectrum towards longer wavelengths, today’s CMB spectrum still resembles that of a black body of temperature $T_0$ [3], even though the photons have not been in thermal equilibrium since last scattering. As it happens, $T_0$ is one of the few cosmological parameters that are accessible to direct measurement, without the need to resort to the model-dependent process of statistical inference. Combining the results of three independent estimation methods, the authors of [4] find $T_0 = 2.725 \pm 0.001$ K (at the 68% confidence level (c.l.)), an impressive accuracy $\Delta T_0/T_0$ of better than 0.04%.

Since the CMB monopole $T_0$ determines the present radiation density of the Universe, it is also a fundamental input parameter for the calculation of the temperature and polarization anisotropies of the CMB. Naturally, the experimental error in $T_0$ will also introduce a theoretical uncertainty $\Delta C_\ell/C_\ell$ in the prediction of the angular power spectra $C_\ell$. Depending on the scale, this uncertainty can reach a magnitude of order a few times 0.1% [7,8].

If one wants to use the observed $C_\ell$ data to infer constraints on the free parameters of a particular cosmological model, this temperature effect ought, in principle, to be taken into account. In a statistically stringent Bayesian analysis, one would have to treat $T_0$ as a free parameter and impose a suitable prior on its value instead of keeping it fixed. Given the statistical errors of present CMB anisotropy data [9,10], current parameter estimates are unlikely to be affected. However, in the near future, experiments such as PLANCK [11]
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or CMBPOL [12] will be able to measure the temperature and polarization angular power spectra to an accuracy that is essentially limited by cosmic variance over a wide range of multipoles up to $\ell \sim 2500$. It is therefore timely to ask whether parameter estimates using these high-quality data sets may be compromised by a possibly insufficiently accurate measurement of $T_0$.

In addition to the monopole, the analysis and interpretation of the data taken by full-sky CMB experiments also depends on the FIRAS measurement of the temperature amplitude of the CMB dipole, $\tau_{dp} = 3.381 \pm 0.007$ mK [4]. Unlike $T_0$, the dipole temperature is not important for the theoretical prediction of the power spectra, but it affects the experimental values of the $C_\ell$s. The dipole provides a convenient way to calibrate detector output with the amplitude scale of fluctuations in temperature and polarization. For WMAP [13] and the low- and high-frequency instruments of PLANCK [14,15], the error in the absolute calibration will be limited by the uncertainty in $\tau_{dp}$, inducing a normalization uncertainty in the angular power spectra data of $\Delta C_\ell^\text{exp} / C_\ell^\text{exp} = 2 \Delta \tau_{dp} / \tau_{dp} \simeq 0.4\%$ [16].

In the present work, we determine how large an effect the uncertainty in the values of $T_0$ and $\tau_{dp}$ will have on the estimates of cosmological parameters for the analysis of future CMB data. In section 2 we outline the rôle played by $T_0$ in the calculation of the anisotropies of the CMB. In section 3 we describe the technical details of our analysis, the results of which are presented in section 4. We summarize our results and conclude in section 5. A detailed account of the technicalities of generating mock CMB data is given in the appendix.

2. CMB temperature and the anisotropy spectra

The present temperature of the cosmic microwave background is one of the basic input parameters for the calculation of anisotropies, affecting the evolution of the fluctuations during various stages of the early Universe. In particular, it determines directly the current photon energy density via

$$\rho_{\gamma,0} = \frac{\pi^2}{15} T_0^4.$$  (2.1)

Let us sketch briefly how the calculation of the angular power spectra will explicitly depend on $T_0$ or $\rho_{\gamma,0}$, and to what extent these effects can mimic changes in other free parameters of the cosmological model.

Baryon-to-photon ratio. A fundamental input parameter in the Boltzmann equations for the baryon density perturbations is the baryon-to-photon density ratio,

$$R \equiv \frac{3 \rho_0}{4 \rho_{\gamma}}.$$  (2.2)

Thus, already at the level of the perturbations equations, there exists an exact degeneracy between $T_0$ and the physical baryon density $\omega_b \equiv \Omega_b h^2$. In the tight-coupling limit valid before recombination, $R$ defines the sound speed for the coupled baryon–photon fluid via $c_s^2 \equiv 1/3(1 + R)$, and enhances the compression phase (hence alternate peaks) of the acoustic oscillations [17]. The comoving sound horizon

$$r_s(\eta_*) \equiv \int_0^{\eta_*} d\eta \ c_s(\eta)$$  (2.3)
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evaluated at the time of recombination $\eta_*$ governs the spacing of the acoustic peaks in the observed CMB anisotropies. The suppression of the anisotropy spectra at high $\ell$ due to diffusion damping also depends explicitly on $R$.

Recombination. The details of the process of recombination [18,19], during which the photons decouple from the plasma, evidently have a significant influence on the eventual CMB anisotropies. Between redshifts $900 < z < 1500$, the free electron fraction $X_e$ can be approximated by [6]

$$N_e \propto \frac{h T_0^{1/2}}{z \sqrt{\Omega_m}} \exp \left( -\frac{B}{z T_0} \right),$$

(2.4)

where $h$ is the dimensionless Hubble parameter today, $\Omega_m$ is the matter density, and $B \simeq 3.9 \times 10^4$ K is a numerical constant. Because of the exponential dependence on the temperature, we can expect $\Delta N_e / N_e \gg \Delta T_0 / T_0$. From a more sophisticated calculation, it was shown in [8] that $\Delta N_e / N_e$ can be as large as 0.55% for $\Delta T_0 / T_0 \sim 0.04%$ within the standard $\Lambda$CDM model.

Matter–radiation equality. The parameter $T_0$ determines not only the photon energy density, but also, implicitly, the neutrino energy density, and thus the total radiation density before the neutrinos become non-relativistic:

$$\rho_r = \frac{\pi^2}{15} T_0^4 (1 + z)^4 \left[ 1 + \frac{7}{8} N_{\text{eff}} \left( \frac{4}{11} \right)^{4/3} \right].$$

(2.5)

Assuming the particle content of the standard model and standard neutrino decoupling, $N_{\text{eff}} \simeq 3.046$ [20]. It is apparent that any change in $\rho_r$ will shift the time of matter–radiation equality $z_{\text{eq}}$, which is manifest in an enhancement especially of the first acoustic peak relative to the low $\ell$ plateau through the early integrated Sachs–Wolfe effect. Thus, one can expect some degree of degeneracy between $T_0$ and the physical matter density $\omega_m \equiv \Omega_m h^2$ in the CMB anisotropies.

Projection. The projection of the temperature and polarization fluctuations onto the sky introduces for the observed CMB anisotropies an additional dependence on the angular diameter distance $D_*$ to the last scattering surface. For a flat geometry,

$$D_* = \int_{a_*}^{1} \frac{da}{a^2 H(a)},$$

(2.6)

where $H(a) = 100 \sqrt{h^2 + \Omega_m h^2 (a^{-3} - 1) + \Omega_r h^2 (a^{-4} - 1)}$. Evidently, projection leads to a degeneracy between $h$ and $T_0$, both directly, and indirectly through $\omega_m$’s correlation with $T_0$.

We conclude from this brief discussion that the parameter most degenerate with $T_0$ is the baryon density $\omega_b$, followed by the matter density $\omega_m$ and the Hubble parameter $h$. Combining these three effects leads to an uncertainty in the angular power spectra of $\Delta C_\ell / C_\ell \sim 0.2\%$ for $\Delta T_0 / T_0 \sim 0.04\%$. For the fiducial model of section 3, we illustrate this uncertainty in figure 1.
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Figure 1. These diagrams show the difference between the angular power spectra under a change in $T_0$ of $\pm 1$ mK (thick solid lines). We also plot the effects of changing the baryon density $\omega_b = \Omega_b h^2$ by $\Delta \omega_b / \omega_b = 4 \Delta T_0 / T_0$, holding $T_0$ fixed (thin dotted lines). Left panel: temperature autocorrelation. Right panel: polarization $E$-mode autocorrelation.

Table 1. List of the experimental parameters assumed for the PLANCK satellite [11]: $\theta_{\text{beam}}$ measures the width of the beam, $\Delta T, \Delta P$ are the sensitivities per pixel, and $\nu$ is the centre frequency of the channels.

| $\nu$ (GHz) | $\theta_{\text{beam}}$ | $\Delta T$ ($\mu$K) | $\Delta P$ ($\mu$K) |
|-------------|------------------------|---------------------|---------------------|
| 100         | 9.5′                   | 6.8                 | 10.9                |
| 143         | 7.1′                   | 6.0                 | 11.4                |
| 217         | 5.0′                   | 13.1               | 26.7                |

3. Methodology

In order to answer the question whether the standard approach of keeping $T_0$ fixed will be justified when fitting future data, we compare the results of an analysis that treats $T_0$ and $\tau_{dp}$ as free (albeit well-constrained) parameters with those of a fit where these two parameters are kept constant.

3.1. Mock data sets

Following the method outlined in [21], we generate two sets of mock CMB anisotropy data, comprising the $TT$, $TE$ and $EE$ angular power spectra for multipoles $2 \leq \ell \leq 2500$, assuming respectively the projected noise levels of the PLANCK experiment (see table 1), and an ideal, noiseless cosmic variance limited experiment (CVL). We assume a sky coverage of $f_{\text{sky}} = 0.65$ in both cases. For a more detailed discussion of the method, we refer the reader to the appendix. Our fiducial model is specified by the parameter values listed in table 2.

In the generation as well as in the subsequent analysis of the data we use the recombination code recfast [22, 23] and ignore secondary effects such as gravitational lensing or the Sunyaev–Zel’Dovich effect. Let us stress that real data would require a less...
simplistic treatment of the physics of recombination (see, e.g., [24, 25]) and the secondary effects; failure to do so can severely bias the results [26].

3.2. Parameter estimation

The multi-dimensional posterior probability distributions \( P(\theta) \) are reconstructed using a modified version of CosmoMC [27], a Markov Chain Monte Carlo (MCMC) algorithm used in conjunction with the CAMB [28] code to calculate the polarization and temperature spectra. For each analysis we generate eight Markov chains; their convergence is monitored using the Gelman and Rubin \( R \)-parameter [29]. Our convergence criterion is \( R - 1 < 0.01 \), a much stricter requirement than for instance the one used by the WMAP team [30].

We analyse two basic models: the widely used six-parameter ‘vanilla’ model, and an extended model (vanilla+Y_{\text{He}}), where in addition we vary the primordial helium fraction. With the exception of \( T_0 \) and \( \tau_{\text{dp}} \), we impose flat top-hat priors on the free parameters of the models; the limits are listed in table 2.

For each of the models we perform the analysis with the CMB temperature and dipole either kept fixed at their fiducial values, or treated as free parameters. We account for the experimental error in \( T_0 \) by imposing a Gaussian prior of the form

\[
\pi(T_0) \propto \exp \left[ -\frac{1}{2} \left( \frac{T_0 - 2.725 \text{ K}}{0.001 \text{ K}} \right)^2 \right].
\]

The dipole, in principle, has no effect on the theoretical prediction, only on the data. However, since the absolute calibration affects polarization and temperature data in the same way, on all scales, we do not need to generate new data each time \( \tau_{\text{dp}} \) changes. Instead, we shift the calibration uncertainty to the theory side, by substituting the normalization of the primordial power spectrum with

\[
A_S \rightarrow A_S \left( 1 + 2 \frac{\tau_{\text{dp}} - 3.381 \text{ mK}}{3.381 \text{ mK}} \right),
\]
with a Gaussian prior on $\tau_{dp}$,
\[ \pi(\tau_{dp}) \propto \exp \left[ -\frac{1}{2} \left( \frac{\tau_{dp} - 3.381 \text{ mK}}{0.007 \text{ mK}} \right)^2 \right], \]  
(3.3)
corresponding to the result of the FIRAS measurement.

Note that we avoid using the popular Kosowsky parameter $\theta_s$ [31], defined as the ratio of the sound horizon at recombination to the angular diameter distance to the last scattering surface, and fit instead the Hubble parameter $h$ directly. Mapping between $\theta_s$ and $h$ as implemented in CosmoMC involves the use of a fitting formula to determine the recombination redshift $z_*$, which was derived in [32] under a number of assumptions, including that of a fixed temperature, and is thus not applicable in our analysis.

Since the future data sets considered here will be able to constrain the parameters of these models extremely well, one can expect the resulting posterior distribution to be reasonably close to a multivariate Gaussian near its mode. As a consequence, adding an additional parameter, such as $T_0$, with a Gaussian posterior, is unlikely to shift the point estimates of other parameters. Also, one would not expect the errors of uncorrelated parameters to be affected; only parameters that are degenerate with the new parameter are likely to have increased error bars. Naturally, this assumes that the Gaussian is actually centred around the fixed value—using a wrong value of the temperature will of course bias the best fit.

The expected near-Gaussianity of the posterior also implies that different methods of constructing credible intervals will lead to the same results [33]. In the following, we will quote the standard deviation as a measure of uncertainty in parameter $\theta$,
\[ \sigma_\theta = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (\theta_i - \bar{\theta})^2}, \]  
(3.4)
where $i$ runs over the points of the Markov chain and $\bar{\theta}$ is the mean of the $\theta_i$. This quantity corresponds to the width of the usual minimal 68% credible interval.

4. Results

4.1. Vanilla model

The most serious potential consequence of adding extra parameters to an inference exercise is a shift in the parameter means, i.e., a bias in the point estimates. However, as expected, we find no such shift for either data set, when we compare the results from the fixed temperature analysis with the free temperature runs: the means differ by less than 0.1%.

We do find an effect on the errors though: the inferred uncertainties $\sigma_\theta/\bar{\theta}$ of the vanilla model parameters are listed in table 3. Using subsets of the full chains, we estimate the accuracy of these numbers to be of order 1%. Apart from the baryon density, the errors of the fixed temperature and free temperature analyses differ by a few per cent at most, both for PLANCK and CVL data. This corresponds roughly to the expected variance of the results for multiple runs of the same model and is consistent with a null effect. The only significant exception is the error of the baryon density, which for the CVL data set is roughly 20% when taking the temperature uncertainty into account. The reason for this
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Figure 2. This plot illustrates the correlation between the baryon density $\omega_b = \Omega_b h^2$ and the CMB temperature for a fit of the vanilla model to the CVL data set. Depicted are the joint two-dimensional 68%- and 95%-credible contours.

Table 3. In this table we list the relative uncertainties $\sigma_\theta/\bar{\theta}$ of the six parameters of the vanilla model, given in per cent.

|       | PLANCK | PLANCK fixed | CVL  | CVL fixed |
|-------|--------|--------------|------|----------|
| $\Omega_b h^2$ | 0.602  | 0.607        | 0.187| 0.149    |
| $\Omega_{dm} h^2$ | 1.14   | 1.12         | 0.587| 0.581    |
| $h$     | 0.859  | 0.848        | 0.387| 0.388    |
| $z_{re}$ | 3.43   | 3.32         | 2.00 | 2.01     |
| $\ln(10^{10} A_S)$ | 0.466  | 0.465        | 0.270| 0.274    |
| $n_S$   | 0.366  | 0.365        | 0.198| 0.200    |

increase lies in a parameter degeneracy between $T_0$ and $\omega_b$, as explained in section 2 and demonstrated in figure 2. The qualitatively similar effect of these two parameters on the anisotropy power spectra can also be seen in figure 1.

This mild degradation in the sensitivity to $\omega_b$ under an ideal situation indicates that the uncertainty in the CMB temperature measurement is, for the purpose of parameter estimation, sufficiently well controlled. Nonetheless, we find that a reduction in the error of the current temperature measurement by a factor of five will bring the degradation down to $\sim 1\%$.

It is interesting to note that had we imposed instead a temperature prior of $T_0 = 2.726 \pm 0.005$ K based on the original analysis of the FIRAS data [34], the sensitivity of CVL to the baryon density would degrade by as much as a factor of 3.6 relative to the fixed temperature case. The degeneracies between $T_0$ and the parameters $\omega_m$ and $h$ would also manifest as a 40% and a 10% degradation in their respective projected errors. Thus, through a stroke of coincidence, the current error in $T_0$ of 1 mK leads to sensitivity degradations that are large enough to still be detectable, and yet small enough not to significantly limit the constraining power of even an ideal CMB survey.
Table 4. In this table we list the relative uncertainties $\sigma_\theta/\bar{\theta}$ of the seven parameters of the extended vanilla + $Y_{\text{He}}$ model, given in per cent.

| Parameter       | CVL   | CVL-fixed |
|-----------------|-------|-----------|
| $\Omega_b h^2$  | 0.248 | 0.226     |
| $\Omega_{\text{dm}} h^2$ | 0.574 | 0.565     |
| $h$             | 0.396 | 0.392     |
| $z_{\text{re}}$ | 2.05  | 1.99      |
| $\ln(10^{10}A_S)$ | 0.339 | 0.338     |
| $n_S$           | 0.317 | 0.309     |
| $Y_{\text{He}}$ | 1.35  | 1.33      |

4.2. Extended models

While the vanilla model enjoys a large amount of popularity these days, and is generally used as the benchmark model for parameter estimates, it may be necessary in the future to consider extended models with more free parameters. One such example is the primordial Helium fraction, $Y_{\text{He}}$. While current CMB data are not very sensitive to changes in $Y_{\text{He}}$, it will be necessary, already for PLANCK data, to include it in the analysis [35]. In fact, the projected sensitivity of the CMB to $Y_{\text{He}}$ will rival that of astrophysical measurements, without being troubled by experimental systematics [36].

Generically, as pointed out above, adding extra parameters will tend to increase the uncertainties on existing parameters, provided that the data can constrain the new parameters well and barring unusual shapes of the posterior distribution. Our results for the vanilla model should thus be regarded as an estimate of the maximum possible effect. As can be seen from table 4, including $Y_{\text{He}}$ slightly weakens the bounds on the baryon density due to a degeneracy with the baryon density. As a result, the difference of the bounds of the fixed $T_0$ and free $T_0$ analyses goes down to $\sim 10\%$. The addition of other parameters degenerate with $\omega_b$ would further decrease the temperature effect.

4.3. The dipole

Equation (3.2) shows that there is a direct degeneracy between the CMB dipole and the inferred value of the normalization of the initial power spectrum. From table 3, we see that the relative error on the logarithm of the normalization, $\ln[10^{10}A_S]$, is about 0.27% even in the most optimistic case in which the dipole is infinitely well known, the data is cosmic variance limited and a minimal model is assumed. This corresponds to a relative error in $A_S$ of roughly 0.9%. Adding up this error and the dipole error of 0.4% quadratically, one expects an effect on the error of $A_S$ of less than 10%. This rough estimate is confirmed by our MCMC analysis: we find that for the minimal model and the CVL data set, fixing the dipole will lead one to underestimate the error of the normalization $A_S$ by 8%, while the other parameters are affected by less than 1%.

We can thus conclude that the FIRAS dipole measurement is sufficiently accurate for the purpose of future cosmological parameter inference.
5. Conclusion

We have shown that ignoring the uncertainty in the measurement of the present CMB temperature \([4,5]\), \(T_0 = 2.725 \pm 0.001\) K, can affect the extraction of cosmological parameters from future data. However, the magnitude of this effect is rather small. While for projected Planck data it appears to be altogether negligible, one runs the risk of underestimating the error in the baryon density by about 20\% for an ideal, cosmic variance limited experiment, assuming the current six-parameter vanilla model. For the other parameters of this model, the difference is of \(O(1)\). An improved measurement of \(T_0\), reducing the current error by a factor of five, would remedy this problem. In contrast, if one were to use the result of the original FIRAS analysis [34], \(T_0 = 2.726 \pm 0.005\) K, the effect of the temperature uncertainty would be much more dramatic: the projected error in the baryon density would increase by a factor of 3.6. Even the sensitivities to the matter density and the Hubble parameter would suffer some mild degradation.

In the same vein we have also estimated the effect of the CMB dipole uncertainty. We found that taking into account the dipole error of 0.2\% degrades the sensitivity to the normalization of the primordial power spectrum by less than 10\% for a cosmic variance limited experiment, compared to the case in which the dipole is infinitely well known.

We conclude that, at least from a parameter estimation point of view, the present precision of CMB temperature monopole and dipole measurements is ‘good enough’. However, it should be stressed that an improved measurement of the CMB spectrum would nevertheless be a worthwhile endeavour, for two reasons. Firstly, it offers the prospect for detecting possible global deviations from the blackbody spectrum, typically parameterized in terms of the Bose–Einstein and Compton distortions \(\mu\) and \(y\). As pointed out in [4], using state-of-the-art technology, the current 95\% c.l. limits of \(|\mu| < 9 \times 10^{-5}\) and \(|y| < 1.5 \times 10^{-5}\) [3] could be improved by two orders of magnitude. Secondly, an actual detection of the signatures left by the process of recombination could serve as an additional, independent probe of cosmological parameters, such as the baryon density [8], as well as testing our understanding of recombination physics.

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Appendix. Mock data generation and the likelihood function

We demonstrate in this section how to generate random realizations of future CMB data given some fiducial model, for the purpose of parameter error forecast. The method outlined below is essentially a generalization of the procedure introduced in [21], and can be applied also to forecasts for, for example, cosmic shear experiments.

Sky maps of the CMB are usually expanded in spherical harmonics, where the coefficients, or the multipole moments, \(a_{\ell m}^\mu\) in the mode \(\mu\) (\(\mu = T, E, \ldots\)) receive
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contributions from both the signal $s_{\ell m}^\mu$ and the experimental noise $n_{\ell m}^\mu$,

$$a_{\ell m}^\mu = s_{\ell m}^\mu + n_{\ell m}^\mu. \quad (A.1)$$

Assuming that the experiment has a full-sky coverage and a spatially uniform Gaussian noise spectrum, the total covariance matrix $\tilde{C}_\ell^{\mu\nu} \equiv \langle a_{\ell m}^\mu a_{\ell m}^\nu \rangle$ is diagonal in the $\ell$ basis, and can be written as a sum of the signal $C_\ell^{\mu\nu} \equiv \langle s_{\ell m}^\mu s_{\ell m}^\nu \rangle$ and noise $N_\ell^{\mu\nu} \equiv \langle n_{\ell m}^\mu n_{\ell m}^\nu \rangle$ power spectra,

$$\tilde{C}_\ell^{\mu\nu} = C_\ell^{\mu\nu} + N_\ell^{\mu\nu}. \quad (A.2)$$

Given a fiducial cosmological model $\theta_0$ and the noise specifications of the experiment of interest, one can calculate $C_\ell^{\mu\nu}|_{\theta_0}$ and hence $\tilde{C}_\ell^{\mu\nu}|_{\theta_0}$. Random realizations of the fiducial model can then be generated as follows.

(i) Generate row vectors $G_{\ell m} = \{G_{1\ell m}, G_{2\ell m}, \ldots, G_{n\ell m}\}$, each consisting of $n$ random numbers drawn from a Gaussian distribution. The number $n$ corresponds to the number of observable modes (e.g., $\mu = T, E$ makes $n = 2$).

(ii) The observables $A_{\ell m} = \{a_{1\ell m}, a_{2\ell m}, \ldots, a_{n\ell m}\}$ are defined as

$$A_{\ell m} = G_{\ell m} L^T, \quad (A.3)$$

where $L$ is a lower triangular matrix satisfying the relation $\tilde{C}_\ell|_{\theta_0} = L \cdot L^T$. The components of $L$ can be obtained from a Cholesky decomposition, so that the diagonal elements are given by

$$L_{\mu\mu} = \left( \tilde{C}_\ell^{\mu\mu}|_{\theta_0} - \sum_{\rho=1}^{\mu-1} L_{\mu\rho}^2 \right)^{1/2}, \quad (A.4)$$

and the off-diagonal elements by

$$L_{\nu\mu} = \frac{1}{L_{\mu\mu}} \left( \tilde{C}_\ell^{\mu\nu}|_{\theta_0} - \sum_{\rho=1}^{\mu-1} L_{\mu\rho} L_{\nu\rho} \right), \quad L_{\mu\nu} = 0, \quad (A.5)$$

with $\nu = \mu + 1, \mu + 2, \ldots, n$.

(iii) The mock power spectra are constructed by summing the bilinear products of $a_{\ell m}^\mu$,

$$\tilde{C}_\ell^{\mu\nu} = \frac{1}{2\ell + 1} \sum_{m=-\ell}^{\ell} a_{\ell m}^\mu a_{\ell m}^\nu. \quad (A.6)$$

To extract parameter errors from the mock data we approximate the total likelihood function $\mathcal{L}$ as a multivariate Gaussian in the mock multipole moments $a_{\ell m}^\mu$. Equivalently,

$$\chi^2_{\text{eff}} \equiv -2 \ln \mathcal{L} = \sum_\ell (2\ell + 1) \left[ \text{Tr}(\tilde{C}_\ell^{-1}\tilde{C}_\ell) + \ln \left( \frac{|\tilde{C}_\ell|}{|\tilde{C}_\ell|} \right) - n \right], \quad (A.7)$$

where we have made use of the fact that both the mock data and the noise power spectra are diagonal in the $\ell$ basis. We approximate the effect of the mandatory sky cut near the
galactic plane with a fudge factor \( f_{\text{sky}} \),

\[
\chi^2_{\text{eff}} = \sum_{\ell} (2\ell + 1) \, f_{\text{sky}} \left[ \text{Tr}(\tilde{C}_\ell^{-1} C_\ell) + \ln \frac{|\tilde{C}_\ell|}{|C_\ell|} - n \right],
\]

(A.8)

where \( f_{\text{sky}} \) stands for the actual fraction of the sky observed after the cut.

Finally, we note that it is also possible to perform a forecast using the fiducial \( \tilde{C}_\mu^{\nu} |_{\theta_0} \) instead of a random realization of the fiducial model, i.e., one can set \( C_\ell \) equal to \( \tilde{C}_\ell |_{\theta_0} \) in equation (A.8). This amounts to considering an average over an infinite number of independent realizations of the same fiducial model, and produces essentially similar error estimates as the more complicated procedure outlined above.

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