Coulomb Blockade and Super Universality of the \( \theta \) Angle

I.S. Burmistrov\textsuperscript{1,2} and A.M.M. Pruisken\textsuperscript{3}

\textsuperscript{1}L.D. Landau Institute for Theoretical Physics RAS, Kosygina street 2, 119334 Moscow, Russia
\textsuperscript{2}Department of Theoretical Physics, Moscow Institute of Physics and Technology, 141700 Moscow, Russia
\textsuperscript{3}Institute for Theoretical Physics, University of Amsterdam, Valckenierstraat 65, 1018XE Amsterdam, The Netherlands

Based on the Ambegaokar-Eckern-Schön (AES) model\textsuperscript{1} that has attracted a considerable amount of interest over the years. The standard experimental set-up is the single electron transistor (SET) which is a mesoscopic metallic island coupled to a gate and connected to two metallic reservoirs by means of tunnelling contacts with a total conductance \( g \). Even though the physical conditions of the AES model are limited and well known,\textsuperscript{2,3,4} the theory nevertheless displays richly complex and fundamentally new behavior, much of which has not been understood to date. To explain the observed tunnelling phenomena with varying temperature \( T \) and gate voltage \( V_g \) one usually considers an isolated island or a single electron box obtained by putting the tunnelling conductance \( g \) equal to zero. The AES model then leads to the standard semiclassical or electrostatic picture of the Coulomb blockade which says that at \( T = 0 \) the average charge \( \langle Q \rangle \) on the island is robustly quantized except for very special values of the gate voltage \( V^{(k)} = e(k + 1/2)/C_g \) with integer \( k \) and \( C_g \) denoting the gate capacitance. At these very special values a quantum phase transition occurs where the average charge \( Q \) on the island changes from \( Q = k \) to \( Q = (k + 1) \) in units of \( e \).

The experiments on the SET always involve finite values of the tunnelling conductance \( g \), however, and this dramatically complicates the semiclassical picture of the Coulomb blockade. Despite ample theoretical work on both the strong and weak coupling side of the problem, the matter still lacks basic physical clarity since the averaged charge \( Q \) is known to be un-quantized for any finite value of \( g \), no matter how small.\textsuperscript{5} This raises fundamental questions about the exact meaning of the experiments and the physical quantities in which the Coulomb blockade is usually expressed.

In this Letter we present a complete quantum theory of the SET that is motivated by the formal analogies that exist between the AES theory on the one hand, and the theory of the quantum Hall effect\textsuperscript{6} on the other. Each of these theories describe an interesting experimental realization of the topological issue of a \( \theta \) vacuum that originally arose in QCD.\textsuperscript{7} In each case one deals with different physical phenomena and therefore different quantities of physical interest. What has remarkably emerged over the years is that the basic scaling behavior is always the same, independent of the specific application of the \( \theta \) angle that one is interested in.\textsuperscript{8} Within the Grassmannian \( U(m + n)/U(m) \times U(n) \) non-linear \( \sigma \) model, for example, one finds that quantum Hall physics, in fact, a super universal topological feature of the theory for all values of \( m \) and \( n \). It is therefore of interest to know whether super universality is retained in the AES theory where physical concepts such as the “Hall conductance” and \( \theta \) renormalization have not been recognized.

In direct analogy with the theory of the quantum Hall effect we develop, in the first part of this Letter, a quantum theory of observable parameters \( q' \), \( E_c' \) and \( q' \) for the AES model obtained by studying the response of the system to changes in the boundary conditions. Here, \( q' \) is identified as the SET conductance, \( E_c' \) is the charging energy whereas the \( q' \) is a novel physical quantity that is fundamentally related to the current noise in the SET. The \( q' \) is in all respects the same as the Hall conductance in the quantum Hall effect and, unlike the averaged charge \( Q \) on the island, it is robustly quantized in the limit \( T \rightarrow 0 \). The crux of this Letter is the unifying scaling diagram of Fig. 1 indicating that \( q' \) and \( q' \) are the appropriate renormalization group parameters of the AES theory. These scaling results which are the main objective of the remaining part of this Letter, provide the complete conceptual framework in which the various disconnected pieces of existing computational knowledge of the AES theory can in general be understood.

**AES model.** The action involves a single abelian phase \( \phi(\tau) \) describing the potential fluctuations on the island \( V(\tau) = i\phi(\tau) \) with \( \tau \) denoting the imaginary time.\textsuperscript{11}
The theory is defined by
\[ Z = \int D[\phi] e^{-S[\phi]}, \quad S[\phi] = S_d + S_t + S_c. \] (1)

The action \( S_d \) describes the tunneling between the island and the leads
\[ S_d[\phi] = \frac{g}{4} \int_0^\beta d\tau_1 d\tau_2 \alpha(\tau_{12}) e^{-i[\phi(\tau_1) - \phi(\tau_2)]} \] (2)
where \( \beta = 1/T, \tau_{12} = \tau_1 - \tau_2 \) and the kernel \( \alpha(\tau) \) is usually expressed as \( \alpha(\tau) = (T/\pi) \sum_n |\omega_n| e^{-i\omega_n \tau} \) with \( \omega_n = 2\pi T n \). The part \( S_t \) describes the coupling between the island and the gate and \( S_c \) is the effect of the Coulomb interaction between the electrons
\[ S_c[\phi] = -2\pi i q \mathcal{C}[\phi], \quad S_t[\phi] = \frac{1}{4E_c} \int_0^\beta d\tau \phi^2. \] (3)
Here, \( q = C_g V_g/e \) is the external charge and \( \mathcal{C}[\phi] = 1/(2\pi \int_0^\beta d\phi) \) is the winding number or topological charge of the \( \phi \) field. For the system in equilibrium the winding number is strictly an integer \([3]\) which means that Eq. (3) is only sensitive to the fractional part \(-\frac{1}{2} < q < \frac{1}{2}\) of the external charge \( q \). The main effect of \( S_c \) in Eq. (3) is to provide a cut-off for large frequencies. Eq. (2) has classical finite action solutions \( \phi_0(\tau) \) with a non-zero winding number that are completely analogous to Yang-Mills instantons. The general expression for winding number \( W \) is given by \([10, 11]\)
\[ e^{i\phi_0(\tau)} = e^{-i2\pi W T} \sum_{\lambda=1}^{\lvert W \rvert} e^{i2\pi \lambda T \tau - \frac{2}{3}i \lambda}, \] (4)
For instantons \( W > 0 \) the complex parameters \( z_\lambda \) are all inside the unit circle and for anti-instantons \( W < 0 \) they are outside. Considering \( W = \pm 1 \) which is of interest to us, one identifies \( \arg z/2\pi T \) as the position (in time) of the single instanton whereas \( \lambda = (1 - \lvert z \rvert^2) \beta \) is the scale size or the duration of the potential pulse \( i\phi_0(\tau) \). The semiclassical instanton expression for the thermodynamic potential \( \Omega = -T \ln Z \) can be written in the standard form \([12]\)
\[ \Omega_{\text{inst}} = -gD T^3 \left[ \frac{\lambda}{\beta^2} - \frac{\pi \lambda}{2} \right] \cos 2\pi q. \] (5)
Here, \( D = 2e^{-\gamma E_0} \) with \( \gamma_E \approx 0.577 \) denoting the Euler constant. Introducing a frequency scale \( \nu_0 = gE_c/(\pi^2 D) \) then \( g(\lambda) \) and \( E_c(\lambda) \) are given by \([13]\)
\[ g(\lambda) = g - 2\ln \lambda \nu_0, \quad E_c(\lambda) = E_c \left[ 1 - \frac{2}{g} \ln \lambda \nu_0 \right]. \] (6)

The logarithmic corrections are the same as those computed in ordinary perturbation theory in \( 1/g \). Based on Eq. (3) alone one expects that the SET always scales from a good conductor at high \( T \) or short times \( \lambda \nu_0 \ll 1 \) to an insulator at low \( T \) or long times \( \lambda \nu_0 \gg 1 \).

**Kubo formulae.** To develop a theory of observable parameters of the SET \([3]\) we employ the background field \( \phi(\tau) = \omega_n \tau \) that satisfies the classical equation of motion of Eq. (1). Write
\[ e^{-S'[\tilde{\phi}]} = Z^{-1} \int D[\phi] e^{-S[\phi + \tilde{\phi}]} \] (7)
then a detailed knowledge of \( S'[\tilde{\phi}] \) generally provides complete information on the low energy dynamics of the system. The effective action \( S'[\tilde{\phi}] \) is properly defined in terms of a series expansion in powers of \( \omega_n \). Retaining only the lowest order terms in the series we can write
\[ S'[\tilde{\phi}] = \beta \left( \frac{g'}{4\pi} \omega_n - iq' \omega_n + \frac{\omega_n^2}{4E_c'} \right). \] (8)
The quantities of physical interest are \( g', q' \) with \( q' < 1/2 \) and \( E_c' \) that are formally given in terms of Kubo-like expressions \([14]\)
\[ g' = 4\pi \text{Im} \left. \langle K(\eta) \rangle \right|_{\eta \to 0}, \quad q' = q + \frac{i \langle \phi \rangle}{2E_c} + \text{Re} \left. \langle K(\eta) \rangle \right|_{\eta \to 0}, \]
\[ \frac{1}{E_c'} = E_c \left( 1 + \int_0^\beta d\tau e^{i\eta \tau} \left. \langle \phi(\tau) K(\eta) \rangle \right|_{\eta \to 0} \right) \] (9)
where the expectation is with respect to the theory of Eq. (1). Here, \( K(\eta) \) is obtained from the expression
\[ K(i\omega_n) = \frac{g}{4\beta} \int_0^\beta d\tau_1 d\tau_2 \left[ e^{i\omega_n \tau_{12}} - \frac{1}{i\omega_n} \right] \alpha(\tau_{12}) e^{i[\phi(\tau_2) - \phi(\tau_1)]} \] (10)
followed by the analytic continuation \( i\omega_n \to \eta + i0^+ \) which is standard. As we shall point out in what follows, the main advantage of the background field formalism of Eqs (7)-(10) is that it unequivocally determines the
The renormalization of the AES model while retaining the close contact with the physics of the SET. To see this we notice first that by expanding the effective action of Eq. (9) in powers of $\omega_n$ we essentially treat the discrete variable $\omega_n$ as a continuous one. This means that the quantities $q'$, $q''$ with $|q'| < 1/2$ and $1/E_c'$ in Eqs (11) - (10) are, by construction, a measure for the response of the system to infinitesimal changes in the boundary conditions. This observation immediately leads to a general criterion for strong coupling $\textit{Coulomb blockade phase}$ of the SET that the perturbative results of Eq. (10) could not give. More specifically, the general statement which says that the SET scales toward an $\textit{insulator}$ as $T \to 0$ implies that the response quantity $q'$, the fractional part of $q'$ as well as the dimensionless quantity $1/\beta E_c'$ all render equal to zero except for corrections that are exponentially small in $\beta$. Since the expressions of Eqs (11) - (10) are all invariant under the shift $q \to q + k$ and $q' \to q' + k$ for integer $k$, we conclude that the AES theory on the strong coupling side generally displays the $\textit{Coulomb blockade}$ with the novel quantity $q'$, unlike the averaged charge $Q$ on the island, now identified as the $\textit{robustly quantized}$ quasi particle charge of the SET. This quantization phenomenon which is depicted in Fig. 1 by the infrared stable fixed points located at integer values $q' = k$, is fundamentally different from semiclassical picture of the Coulomb blockade since it elucidates the discrete nature of the electronic charge which is independent of tunneling.

Before embarking on the details of scaling it is important to emphasize that Eqs (8) - (10) are precisely the same quantities that one normally would obtain in ordinary linear response theory. For example, $q'$ is exactly same as the Kubo formula [13] relating a small potential difference $V$ between the leads to the current $I$ across the island according to $I = e^2 G V / h$ where $G = g_l g_r g'/(g_l + g_r)^2$. Here, $h$ is Planck’s constant and $g_{l,r}$ are the bare tunneling conductances across the different leads. To understand the new quantity $q'$ we notice that the first two terms in Eq. (10) are equal to the average charge $Q$ on the island, $Q = q - (2E_c')^{-1}\partial I/\partial q$. The last piece in $q'$ is related to the current noise [14] and a more transparent expression is obtained by writing

$$q' = Q + \frac{(g_l + g_r)^2}{2g_l g_r} \frac{i}{2} \frac{\partial}{\partial V} \int_0^\infty dt \langle [I(t), I(0)] \rangle \bigg|_{V=0}.$$ (11)

It can be shown that the last term in Eq. (11) is the result of an inductive coupling between tunneling current of the SET and the external current in the circuit. Similarly, it can be shown that $E_c'$ describes the frequency dependence of the tunneling current ($I_e$) and the potential difference ($V_e$) according to $\partial (I_e/V_e)/\partial \omega = i e^2/(2E_c')$.

**Weak coupling phase.** By evaluating Eqs (11) - (10) in a series expansion in powers of $1/g$ one obtains the same lowest order results as in Eq. (10) but with $q' = 0$. The quantity $q'$ is generally unaffected by the quantum fluctuations and to establish the renormalization of $q'$ it is necessary to include instantons. Following the detailed methodology of Ref. [8] we express the observable theory in terms of renormalization group $\beta = \beta(\eta', q')$ and $\gamma = \gamma(\eta', q')$ functions which are universal and given by

$$\beta_\eta = \frac{d\eta'}{d \ln \lambda} = -2 - \frac{4}{\eta'} - \frac{D q'^2 e^{-\eta'/\lambda^2}}{2q'} \cos 2\pi q' \quad (12)$$

$$\beta_q = \frac{d\eta'}{d \ln \lambda} = -\frac{D}{4\pi} q'^2 e^{-\eta'/\lambda^2} \sin 2\pi q' \quad (13)$$

$$\gamma = \frac{d\ln E_c'}{d \ln \lambda} = -2 q' + \frac{D}{2} q'^2 e^{-\eta'/\lambda^2} \cos 2\pi q'. \quad (14)$$

Here, $D$ is the same as in Eq. (5) and we have included in Eq. (12) the perturbative contribution of order $1/g$.

The results indicate that instantons are the fundamental objects of the theory that facilitate the $\textit{cross-over}$ between the metallic phase with $g' \gg 1$ at high $T$ and the Coulomb blockade phase with $g' \lesssim 1$ that generally appears at a much lower $T$ only.

**Strong coupling phase.** We next evaluate Eqs (9) - (10) in terms of a strong coupling expansion about the theory with $g = 0$. [2, 17] Remarkably, this expansion is in many respects the same as the one recently reported for the two dimensional $CP^{N-1}$ model with large values of $N$. [15] The results for small values of $g'$ and $u' = q' - k - 1/2$ can be written as follows

$$\beta_\eta = -\frac{g'^2}{\pi^2}, \quad \beta_q = u' \left(1 - \frac{g'}{\pi^2}\right), \quad \gamma = O(g'^2) \quad (15)$$

indicating that $u' = g' = 0$ is the $\textit{critical}$ fixed point of the AES theory with $g'$ a marginally irrelevant scaling variable. Eq. (15), together with the weak coupling results of Eqs (12) - (14), are the main justification of the unifying scaling theory illustrated in Fig. 1. To make contact with the existing strong coupling analysis of $g'$ [4, 2, 19] we employ the basic principles of the renormalization group and obtain the general scaling results $g' = g'(X, Y)$ and $q' = q'(X, Y)$ where

$$X = \tilde{E}_c \tilde{u}/(\tilde{T} \tilde{g}), \quad Y = (T/\tilde{E}_c) e^{-1/\tilde{g}} \quad (16)$$

with $\tilde{u}$, $\tilde{g}$ and $\tilde{E}_c$ denoting the renormalization group starting point (which, by the way, is slightly different from the bare theory $u$, $g$ and $E_c$). An explicit computation gives $g'(0, Y) = \ln Y^{-1}$ indicating that the maximum of $g'$ decreases with $T$ like $\ln T^{-1}$. Similarly we find $q'(X, Y) = k + 1/2 - X \ln Y^{-1}$ indicating that the width $\Delta V_q$ of the transition with varying $V_q \propto q$ vanishes with $T$ according to $\Delta V_q \propto T \ln T$. [19]

**Critical correlations.** Of general interest are the critical correlations of the AES theory with $u, g \approx 0$. These are most elegantly described by the fermionic effective action [18]

$$S = \int \bar{\psi}(\partial_\tau + b \alpha_\tau) \psi + \frac{g}{4} \int_{1,2} S_-(\tau_1) \alpha(\tau_1) S_+(\tau_2). \quad (17)$$
Here, \( b = E_x u \approx 0 \), \( \sigma_{x,y,z} \) are the Pauli matrices and \( \psi \) and \( \bar{\psi} \) denote two component fermion fields. The operators \( S_±(\tau) = \frac{i}{2} \bar{\psi}(\tau)(\sigma_± + i\sigma_y)\psi(\tau) \) in Eq. (17) are identified with the AES operators \( e^{i\phi(\tau)} \) in Eqs (11) - (13) that create (annihilate) a unit charge on (from) the island at time \( \tau \). In the absence of tunneling \( (g = 0) \) one has \( \langle \psi \sigma_± \psi \rangle = b/|b| \) indicating that the transition at \( b = 0 \) is a first order one. Furthermore,

\[
\langle e^{i(\phi(0)-\phi(\tau))} \rangle = \langle S_+(0) S_- (\tau) \rangle = \vartheta(b\varpi/|b|) e^{-2|b|\tau}
\]

where \( \vartheta \) is the Heaviside step function. These results are precisely in accordance with the semiclassical picture of the SET where \( 2|b| \) denotes the continuously vanishing energy gap between the states \( q' = Q = k \) and \( k + 1 \) of the island as one approaches the critical point. In the presence of tunneling \( g \neq 0 \), however, the correlations get more complicated and the main technical problem is to find the modified operators \( S_± \) that change the quasi particle charge \( q' \) of the SET rather than the averaged charge \( Q \) of the island. \[14\]

In summary, based on the new concept of \( \theta \) or \( q' \) renormalization we assign a universal significance to the Coulomb blockade in the SET that previously did not exist beyond the semiclassical picture. We have shown that the AES model is, in fact, an extremely interesting and exactly solvable example of a \( \theta \) vacuum that displays all the super universal topological features that have arisen before in the context of the quantum Hall liquids \[8\] as well as quantum spin liquids. \[20\] These include not only the existence of gapless or critical excitations at \( q' = k + 1/2 \) (or \( \theta = \pi \)) but also the robust topological quantum numbers that explain quantization of the electronic charge in the SET at finite values of \( g \). Unlike the conventional theories of the \( \theta \) angle, however, the strong coupling behavior of the AES model can be studied analytically and our results for the novel quantity \( q' \) should in general be taken as an experimental challenge. Following Eq. (11) it involves the anti-symmetric part of the current noise which can be experimentally observed by a coupling of the SET to an \( LC \) component. \[21\] for example, see also Ref. \[22\].

Notice that the critical behavior of the SET is likely to change when the number of channels in the tunneling contacts are taken to be finite rather than infinite. \[7\] Under these circumstances one expects a second order transition at \( q' = k + 1/2 \) \[7\] with a finite value of the SET conductance \( g' \) which closely resembles the more complicated physics of the quantum Hall effect. \[8\] Finally, the AES theory is known to map onto the “circular brane” model \[23\] such that the findings of this Letter apply to the latter theory as well. It should be mentioned that physical objectives similar to ours have recently been pursued in Ref. \[24\] using otherwise heuristic arguments.

The reported ideas and conjectures, however, are in many ways in conflict with the present theory.

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