HITTING TIMES FOR GAUSSIAN PROCESSES

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We establish a general formula for the Laplace transform of the hitting times of a Gaussian process. Some consequences are derived, and particular cases like the fractional Brownian motion are discussed.

1. Introduction. Consider a zero mean continuous Gaussian process \((X_t, t \geq 0)\), and for any \(a > 0\), we denote by \(\tau_a\) the hitting time of the level \(a\) defined by

\[
\tau_a = \inf \{t \geq 0 : X_t = a\} = \inf \{t \geq 0 : X_t \geq a\}.
\]

Thus, the map \((a \mapsto \tau_a)\) is left-continuous and increasing, hence, with right limits. The map \((a \mapsto \tau_{a+})\) is right continuous where

\[
\tau_{a+} = \lim_{b \downarrow a, b > a} \tau_b = \inf \{t \geq 0 : X_t > a\}.
\]

Little is known about the distribution of \(\tau_a\). It is explicitly known in particular cases like the Brownian motion. If \(X\) is a fractional Brownian motion with Hurst parameter \(H\), there is a result by Molchan [5] which stands that

\[
P(\tau_a > t) = t^{-(1-H)+o(1)}
\]

as \(t\) goes to infinity.

When \(X\) is a standard Brownian motion, it is well known that

\[
E(\exp(-\alpha \tau_a)) = \exp(-a \sqrt{2\alpha})
\]

for all \(\alpha > 0\). This result is easily proved using the exponential martingale

\[
M_t = \exp(\lambda B_t - \frac{1}{2} \lambda^2 t).
\]
By Doob’s optional stopping theorem applied at time $t \wedge \tau_a$ and letting $t \to \infty$, one gets $1 = E(M_{\tau_a}) = E(\exp(\lambda B_{\tau_a} - \lambda^2 \tau_a/2))$. Since $B_{\tau_a} = a$, we thus obtain (1.2). If we consider a general Gaussian process $X_t$, the exponential process

$$M_t = \exp(\lambda X_t - \frac{1}{2} \lambda^2 V_t),$$

where $V_t = E(X_t^2)$ is no longer a martingale. However, it is equal to 1 plus a divergence integral in the sense of Malliavin calculus. The aim of this paper is to take advantage of this fact in order to derive a formula for $E(\exp(-\frac{1}{2} \lambda^2 V_{\tau_a}))$. We derive an equation involving this expectation in Theorem 3.4, under rather general conditions on the covariance of the process.

As a consequence, we show that if the partial derivative of the covariance is nonnegative, then $E(\exp(-\frac{1}{2} \lambda^2 V_{\tau_a})) \leq 1$, which implies that $V_{\tau_a}$ has infinite moments of order $p$ for all $p \geq \frac{1}{2}$ and finite negative moments of all orders. In particular, for the fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$, we have the inequality

$$E(\exp(-\alpha \tau_a^{2H})) \leq \exp(-a \sqrt{2 \alpha})$$

for all $\alpha, a > 0$.

The paper is organized as follows. In Section 2 we present some preliminaries on Malliavin calculus, and the main results are proved in Section 3.

2. Preliminaries on Malliavin calculus. Let $(X_t, t \geq 0)$ be a zero mean Gaussian process such that $X_0 = 0$ and with covariance function

$$R(s,t) = E(X_t X_s).$$

We denote by $\mathcal{E}$ the set of step functions on $[0, +\infty)$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$(1_{[0,t]}, 1_{[0,s]})_\mathcal{H} = R(t,s).$$

The mapping $1_{[0,t]} \to X_t$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space $H_1(X)$ associated with $X$. We will denote this isometry by $\varphi \mapsto X(\varphi)$.

Let $\mathcal{S}$ be the set of smooth and cylindrical random variables of the form

$$(2.1) \quad F = f(X(\phi_1), \ldots, X(\phi_n)),$$

where $n \geq 1$, $f \in C_b^\infty(\mathbb{R}^n)$ ($f$ and all its partial derivatives are bounded), and $\phi_i \in \mathcal{H}$.

The derivative operator $D$ of a smooth and cylindrical random variable $F$ of the form (2.1) is defined as the $\mathcal{H}$-valued random variable

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(X(\phi_1), \ldots, X(\phi_n)) \phi_i.$$
The derivative operator $D$ is then a closable operator from $L^2(\Omega)$ into $L^2(\Omega; \mathcal{H})$. The Sobolev space $D^{1,2}$ is the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,2}^2 = E(F^2) + E(\|DF\|_{\mathcal{H}}^2).$$

The divergence operator $\delta$ is the adjoint of the derivative operator. We say that a random variable $u$ in $L^2(\Omega; \mathcal{H})$ belongs to the domain of the divergence operator, denoted by $\text{Dom} \delta$, if

$$|E((DF, u)_{\mathcal{H}})| \leq c_u \|F\|_{L^2(\Omega)}$$

for any $F \in \mathcal{S}$. In this case $\delta(u)$ is defined by the duality relationship

$$E(F \delta(u)) = E((DF, u)_{\mathcal{H}}),$$

for any $F \in D^{1,2}$.

Set $V_t = R(t, t)$. For any $\lambda > 0$, we define

$$M_t = \exp(\lambda X_t - \frac{1}{2} \lambda^2 V_t).$$

Formally, the Itô formula for the divergence integral, proved, for instance, in [1], implies that

$$(2.3) \quad M_t = 1 + \lambda \delta(M_{1[0,t]}),$$

where $M_{1[0,t]}$ represents the process $(s \mapsto M_s1_{[0,t]}(s), s \geq 0)$. However, the process $M_{1[0,t]}$ does not belong, in general, to the domain of the divergence operator. This happens, for instance, in the following basic example.

**Example 1.** Fractional Brownian motion with Hurst parameter $H \in (0, 1)$ is a zero mean Gaussian process $(B^H_t, t \geq 0)$ with the covariance

$$(2.4) \quad R_H(t, s) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).$$

In this case, the processes $(B^H_t1_{[0,t]}(s), s \geq 0)$ and $(\exp(\lambda B^H_s - \frac{1}{2} \lambda^2 s^{2H})1_{[0,t]}(s), s \geq 0)$ do not belong to $L^2(\Omega; \mathcal{H})$ if $H \leq \frac{1}{4}$ (see [2]).

In order to define the divergence of $M_{1[0,t]}$ and to establish formula (2.3), we introduce the following additional property on the covariance function of the process $X$.

**$(H0)$** The covariance function $R(t, s)$ is continuous, the partial derivative $\frac{\partial R}{\partial s}(s, t)$ exists in the region $\{0 < s, t, s \neq t\}$, and for all $T > 0$,

$$\sup_{t \in [0, T]} \int_0^T \left| \frac{\partial R}{\partial s}(s, t) \right| ds < \infty.$$

Notice that this property is satisfied by the covariance (2.4) for all $H \in (0, 1)$. 
Define
\[ \delta_t M = \frac{1}{\lambda} (M_t - 1). \]

The following proposition asserts that \( \delta_t M \) satisfies an integration by parts formula, and in this sense, it coincides with an extension of the divergence of \( M_{1[0,t]} \).

**Proposition 2.1.** Suppose that (H0) holds. Then, for any \( t > 0 \), and for any smooth and cylindrical random variable of the form \( F = f(X_{t_1}, \ldots, X_{t_n}) \), we have

\[ E(F \delta_t M) = E \left( \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X_{t_1}, \ldots, X_{t_n}) \int_0^t M_s \frac{\partial R}{\partial s}(s, t_i) \, ds \right). \]

**Proof.** First notice that condition (H0) implies that the right-hand side of equation (2.6) is well defined. Then, it suffices to show equation (2.6) for a function of the form

\[ f(x_1, \ldots, x_n) = \exp \left( \sum_{i=1}^{n} \lambda_i x_i \right), \]

where \( \lambda_i \in \mathbb{R} \). In this case we have, for all \( 0 < t_1 < \cdots < t_n \),

\[
\begin{align*}
\frac{1}{\lambda} E(F(M_t - 1)) &= \frac{1}{\lambda} \exp \left\{ \frac{1}{2} \sum_{i=1}^{n} \lambda_i \lambda_j R(t_i, t_j) \right\} \left( \exp \left\{ \sum_{i=1}^{n} \lambda_i R(t, t_i) \right\} - 1 \right) \\
&= \sum_{i=1}^{n} \int_0^t \exp \left\{ \frac{1}{2} \sum_{i=1}^{n} \lambda_i \lambda_j R(t_i, t_j) + \lambda \sum_{i=1}^{n} \lambda_i R(s, t_i) \right\} \lambda_i \frac{\partial R}{\partial s}(s, t_i) \, ds \\
&= \int_0^t E \left( \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X_{t_1}, \ldots, X_{t_n}) M_s \frac{\partial R}{\partial s}(s, t_i) \right) \, ds,
\end{align*}
\]

which completes the proof of the proposition. \( \square \)

In many cases like in Example 1 with \( H > \frac{1}{4} \), the process \( M_{1[0,t]} \) belongs to the space \( L^2(\Omega; \mathcal{H}) \), and then the right-hand side of equation (2.6) equals

\[ E\langle DF, M_{1[0,t]} \rangle_{\mathcal{H}}. \]

In this situation, taking into account the duality formula (2.2), equation (2.6) says that \( M_{1[0,t]} \) belongs to the domain of the divergence and \( \delta(M_{1[0,t]}) = \delta_t M \).
3. **Hitting times.** In this section we will assume the following conditions:

(H1) The partial derivative \( \frac{\partial R}{\partial s}(s, t) \) exists and it is continuous in \([0, +\infty)^2\).

(H2) \( \limsup_{t \to \infty} X_t = +\infty \) almost surely.

(H3) For any \( 0 \leq s < t \), we have \( E(|X_t - X_s|^2) > 0 \).

Under these conditions, the process \( X \) has a continuous version because

\[
E(|X_t - X_s|^2) = R(t, t) + R(s, s) - 2R(s, t)
\]

\[
= \int_s^t \left[ \frac{\partial R}{\partial u}(u, t) - \frac{\partial R}{\partial u}(u, s) \right] du
\]

\[
\leq 2|t - s| \sup_{s \leq u \leq t} \left| \frac{\partial R}{\partial u}(u, t) \right|.
\]

For any \( a > 0 \), we define the hitting time \( \tau_a \) by (1.1). We know that

\[
P(\tau_a < \infty) = 1 \text{ by condition (H2)}.\]

Set

\[
S_t = \sup_{s \in [0, t]} X_s.
\]

From the results of [6], it follows that, for all \( t > 0 \), the random variable \( S_t \) belongs to the space \( D^{1,2} \). Furthermore, condition (H3) allows us to compute the derivative of this random variable.

**Lemma 3.1.** For all \( t > 0 \), with probability one, the maximum of the process \( X \) in the interval \([0, t] \) is attained in a unique point, that is, \( \tau_{S_t} = \tau_{S_t^+} \) and \( DS_t = 1_{[0, \tau_{S_t}]} \).

**Proof.** The fact that the maximum is attained in a unique point follows from condition (H3) and Lemma 2.6 in Kim and Pollard [4]. The formula for the derivative of \( S_t \) follows easily by an approximation argument. \( \square \)

We need the following regularization of the stopping time \( \tau_a \). Suppose that \( \varphi \) is a nonnegative smooth function with compact support in \((0, +\infty)\) and define for any \( T > 0 \)

\[
Y = \int_0^\infty \varphi(a)(\tau_a \wedge T) \, da.
\]

The next result states the differentiability of the random variable \( Y \) in the sense of Malliavin calculus and provides an explicit formula for its derivative.

**Lemma 3.2.** The random variable \( Y \) defined in (3.2) belongs to the space \( D^{1,2} \), and

\[
D_r Y = - \int_0^{\tau_r} \varphi(y) 1_{[0, \tau_y]}(r) \, d\tau_y.
\]
Proof. Clearly, $Y$ is bounded. On the other hand, for any $r > 0$, we have
\[ \{ \tau_a > r \} = \{ S_r < a \}. \]
Therefore, we can write using Fubini’s theorem
\[ Y = \int_0^\infty \varphi(a) \left( \int_0^{\tau_a \wedge T} d\theta \right) da = \int_0^T \left( \int_0^\infty \varphi(a) da \right) d\theta, \]
which implies that $Y \in D_{1,2}$ because $S_{\theta} \in D_{1,2}$, and
\[ D_r Y = -\int_0^T \varphi(S_{\theta}) D_r S_{\theta} d\theta = -\int_0^T \varphi(S_{\theta}) 1_{[0,\tau_{S_{\theta}}]}(r) d\theta. \]
Finally, making the change of variable $S_{\theta} = y$ yields
\[ D_r Y = -\int_0^{S_T} \varphi(y) 1_{[0,\tau_y]}(r) d\tau_y. \]

Notice that $M_Y \equiv \exp(AX_Y - \frac{1}{2} \lambda^2 V_Y)$. Hence, letting $t = Y$ in equation (2.5) and taking the mathematical expectation of both members of the equality yields
\[ (3.4) \quad E(M_Y) = 1 + \lambda E(\delta_t M_{|t=Y}). \]
We are going to show the following result which provides a formula for the left-hand side of equation (3.4).

**Lemma 3.3.** Assume conditions (H1), (H2) and (H3). Then, we have
\[ (3.5) \quad E(M_Y) = 1 - \lambda E \left( M_Y \int_0^{S_T} \varphi(y) \frac{\partial R}{\partial s}(Y, \tau_y) d\tau_y \right). \]

**Proof.** The proof will be done in two steps.

**Step 1.** We claim that for any function $p(x)$ in $C_0^\infty(\mathbb{R})$ we have
\[ (3.6) \quad E(\delta_t M p(Y)) = -E \left( \int_0^t M_s p'(Y) \int_0^{S_T} \varphi(y) \frac{\partial R}{\partial s}(s, \tau_y) d\tau_y ds \right). \]

We can write $Y = \int_0^T \psi(S_{\theta}) d\theta$, where $\psi(x) = \int_x^\infty \varphi(a) da$. Consider an increasing sequence $D_n$ of finite subsets of $[0, T]$ such that their union is dense in $[0, T]$. Set $Y_n = \int_0^T \psi(S_{\theta}^n) d\theta$, and $S_{\theta}^n = \max\{X_t, t \in D_n \cap [0, \theta]\}$. Then, $Y_n$ is a Lipschitz function of $\{X_t, t \in D_n\}$. Hence, formula (2.6), which holds for Lipschitz functions, implies that
\[ E(\delta_t M p(Y_n)) = -E \left( p'(Y_n) \int_0^T \varphi(S_{\theta}^n) \left( \int_0^t M_s \frac{\partial R}{\partial s}(s, \tau_{S_{\theta}^n}) ds \right) d\theta \right). \]
The function $r \mapsto \int_0^t M_s \frac{\partial R}{\partial u}(s, r) \, ds$ is continuous and bounded by condition (H1). As a consequence, we can take the limit of the above expression as $n$ tends to infinity and we get

$$E(\delta_t M p(Y)) = -E \left( p'(Y) \int_0^T \varphi(S_\theta) \left( \int_0^t M_s \frac{\partial R}{\partial s}(s, \tau_{S_y}) \, ds \right) \, d\theta \right).$$

Finally, making the change of variable $S_\theta = y$ yields (3.6).

**Step 2.** We write

$$E(\delta_t M | _{t=Y}) = E \left( \lim_{\varepsilon \to 0} \int_{-\infty}^\infty \delta_t M p_\varepsilon(Y - t) \, dt \right),$$

where $p_\varepsilon(x)$ is an approximation of the identity, and by convention, we assume that $\delta_t M = 0$ if $t$ is negative. We can commute the expectation with the above limit by the dominated convergence theorem because

$$\int_{-\infty}^\infty |\delta_t M| p_\varepsilon(Y - t) \, dt = \int_{-\infty}^\infty \frac{1}{\lambda} |M_t - 1| p_\varepsilon(Y - t) \, dt \leq \frac{1}{\lambda} \sup_{0 \leq t \leq T+1} (|M_t| + 1),$$

if the support of $p_\varepsilon(x)$ is included in $[-\varepsilon, \varepsilon]$, and $\varepsilon \leq 1$. Hence,

$$E(\delta_t M | _{t=Y}) = \lim_{\varepsilon \to 0} \int_{-\infty}^\infty E(\delta_t M p_\varepsilon(Y - t)) \, dt. \quad (3.7)$$

Using formula (3.6) yields

$$E(\delta_t M p_\varepsilon(Y - t)) \quad (3.8)$$

$$= -\int_0^t E \left( p'_\varepsilon(Y - t) M_s \left( \int_0^{S_T} \varphi(y) \frac{\partial R}{\partial s}(s, \tau_y) \, ds \right) \right) \, ds.$$

Hence, substituting (3.8) into (3.7) and integrating by parts, we obtain

$$E(\delta_t M | _{t=Y})$$

$$= -\lim_{\varepsilon \to 0} \int_{-\infty}^\infty p'_\varepsilon(Y - t) \left( \int_0^t M_s \left( \int_0^{S_T} \varphi(y) \frac{\partial R}{\partial s}(s, \tau_y) \, ds \right) \, dt \right) \, ds.$$

Notice that

$$\left| \int_0^{S_T} \varphi(y) \frac{\partial R}{\partial s}(s, \tau_y) \, ds \right| \leq T \sup_{0 \leq s, u \leq T} \left| \frac{\partial R}{\partial s}(s, u) \right| ||\varphi||_\infty.$$
Hence, applying the dominated convergence theorem, we get
\[
E(M_Y) = 1 + \lambda E(\delta_t M_{t=Y}) = 1 - \lambda \lim_{\varepsilon \to 0} E\left(\int_{-\infty}^{\infty} p_{\varepsilon}(Y-t) \left( M_t \int_0^{S_T} \varphi(y) \frac{\partial R}{\partial s}(t, \tau_y) d\tau_y \right) dt \right)
\]
\[
= 1 - \lambda E\left( M_Y \int_0^{S_T} \varphi(y) \frac{\partial R}{\partial s}(Y, \tau_y) d\tau_y \right).
\]
□

The next step will be to replace the function \( \varphi(x) \) by an approximation of the identity and let \( T \) tend to infinity. Notice that (3.5) still holds for \( \varphi(x) = 1_{[0,b]}(x) \) for any \( b \geq 0 \). In this way we can establish the following result.

**Theorem 3.4.** Assume conditions (H1), (H2) and (H3). For any \( a > 0 \) and \( \lambda \in \mathbb{R} \), we have
\[
\int_0^a E(M_{\tau_y}) dy = a - \lambda E\left( \int_0^a \int_0^1 M_{\tau_{y+b}(1-z)\tau_y} \varphi(y) \frac{\partial R}{\partial s}(y, \tau_y) d\tau_y dy \right).
\]

Notice that we are not able to differentiate with respect to \( a \), the integral in the rightmost expectation of (3.9), because the (random) measure \( d\tau_y \), in general, is not absolutely continuous with respect to the Lebesgue measure.

**Proof of Theorem 3.4.** Fix \( a > 0 \). We first replace the function \( \varphi(x) \) by an approximation of the identity of the form \( \varphi_{\varepsilon}(x) = \varepsilon^{-1} 1_{[0,1]}(x/\varepsilon) \) in formula (3.5). We will make use of the following notation:
\[
Y_{\varepsilon,a} = \int_0^\infty \varphi_{\varepsilon}(x-a)(\tau_x \wedge T) dx.
\]
At the same time we fix a nonnegative smooth function \( \psi(x) \) with compact support such that \( \int_{\mathbb{R}} \psi(a) da = c \) and we set
\[
\int_{\mathbb{R}} E(M_{Y_{\varepsilon,a}}) \psi(a) da = c - \lambda \int_{\mathbb{R}} E\left( M_{Y_{\varepsilon,a}} \int_0^{S_T} \varphi_{\varepsilon}(y-a) \frac{\partial R}{\partial s}(Y_{\varepsilon,a}, \tau_y) d\tau_y \right) \psi(a) da.
\]
The increasing property of the function \( x \to \tau_x \) implies that \( \tau_{a+} \wedge T \leq Y_{\varepsilon,a} \leq \tau_{a+} \wedge T \). Hence, \( Y \) converges to \( \tau_{a+} \wedge T \) as \( \varepsilon \) tends to zero. Thus, almost surely, we have
\[
\lim_{\varepsilon \to 0} M_{Y_{\varepsilon,a}} = \exp(\lambda X_{\tau_{a+} \wedge T} - \frac{1}{2} X_{\tau_{a+} \wedge T}).
\]
By the dominated convergence theorem,
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}} E(M_{Y, a}) \psi(a) \, da = \int_{\mathbb{R}} E(\exp(\lambda X_{\tau_{a+} \wedge T} - \frac{1}{2} \lambda^2 V_{\tau_{a+} \wedge T})) \psi(a) \, da.
\]

Now, set \( F(t) = M_t \frac{\partial R}{\partial s}(t, \tau_y) \). Then, assuming that \( \varphi_\varepsilon(x) = \varepsilon^{-1} 1_{[0,1]}(x/\varepsilon) \), we have
\[
\int_{y-\varepsilon}^y \varphi_\varepsilon(y - a) M_{Y, a} \frac{\partial R}{\partial s}(Y_{\varepsilon,a}, \tau_y) \psi(a) \, da
\]
\[
= \frac{1}{\varepsilon^2} \int_{y-\varepsilon}^y 1_{[0,1]} \left( \frac{y - a}{\varepsilon} \right) F \left( \int_0^{a+\varepsilon} \frac{1}{0,1} \left( \frac{x - a}{\varepsilon} \right) (\tau_x \wedge T) \, dx \right) \psi(a) \, da
\]
\[
= \int_0^1 F \left( \int_0^{\tau_y + \varepsilon \xi - \varepsilon \eta \wedge T} \, dx \right) \psi(y - \varepsilon \eta) \, d\eta
\]
\[
= \int_0^1 F \left( \int_0^{\eta} \left( \tau_y + \varepsilon \xi - \varepsilon \eta \wedge T \right) \, d\xi \right) \psi(y - \varepsilon \eta) \, d\eta.
\]

As \( \varepsilon \) tends to zero, this expression clearly converges to
\[
\psi(y) \int_0^1 F(\eta(\tau_y \wedge T) + (1 - \eta)(\tau_y + \wedge T)) \, d\eta.
\]

So, we have proved that
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}} M_{Y, a} \varphi_\varepsilon(y - a) \frac{\partial R}{\partial s}(Y_{\varepsilon,a}, \tau_y) \psi(a) \, da
\]
\[
= \psi(y) \int_0^1 M_{z\tau_y + (1-z)\tau_y} \frac{\partial R}{\partial s}(z\tau_y + (1-z)\tau_y, \tau_y) \, dz.
\]

In order to complete the proof of the theorem, we will apply the dominated convergence theorem. We have the following estimate for \( y \leq S_T \):
\[
\left| \int_{\mathbb{R}} M_{Y, a} \varphi_\varepsilon(y - a) \frac{\partial R}{\partial s}(Y_{\varepsilon,a}, \tau_y) \psi(a) \, da \right| \leq \| \psi \|_{\infty} \sup_{s,t \leq T} \left| \frac{\partial R}{\partial s}(s,t) \right| \sup_{t \leq T} |M_t|,
\]
which allows us to commute the limit (3.10) with the integral with respect to the measure \( P \times d\tau_y \) on the set \{ (\omega, y) : y \leq S_T(\omega) \}. In this way we get
\[
\int_{\mathbb{R}} E(M_{\tau_y}) \psi(y) \, dy
\]
\[
= \int_{\mathbb{R}} \psi(y) \, dy
\]
\[
- \lambda \mathbb{E} \left( \int_0^{S_T} \psi(y) \int_0^1 M_{z\tau_y + (1-z)\tau_y} \frac{\partial R}{\partial s}(z\tau_y + (1-z)\tau_y, \tau_y) \, dz \, d\tau_y \right).
\]

Approximating \( 1_{[0,a]} \) by a sequence of smooth functions \( \psi_n, n \geq 1 \) and letting \( T \) tend to infinity completes the proof. \( \square \)
If we assume that the partial derivative $\frac{\partial R}{\partial t}(t,s)$ is nonnegative, then we can derive the following result.

**Proposition 3.5.** Assume that $X$ satisfies hypotheses (H1), (H2) and (H3). If $\frac{\partial R}{\partial s}(s,t) \geq 0$, then, for all $\alpha,a > 0$, we have

$$E(\exp(-\alpha V_{\tau_a})) \leq e^{-a\sqrt{2}\alpha}. \quad (3.11)$$

**Proof.** Since $\frac{\partial R}{\partial t}(t,s) \geq 0$, we obtain

$$E(M_{\tau_a}) \leq 1,$$

that is,

$$E(\exp(\lambda a - \frac{1}{2}\lambda^2 V_{\tau_a})) \leq 1,$$

or

$$E(\exp(-\alpha V_{\tau_a})) \leq e^{-a\sqrt{2}\alpha}.$$

The result follows. \qed

The above proposition means that the Laplace transform of the random variable $V_{\tau_a}$ is dominated by the Laplace transform of $\tau_a$, where $\tau_a$ is the hitting time of the level $a$ for the ordinary Brownian motion. This domination implies some consequences on the moments of $V_{\tau_a}$. In fact, for any $r > 0$, we have, multiplying (3.11) by $\alpha^r$,

$$E(V_{\tau_a}^{-r}) = \frac{1}{\Gamma(r)} \int_0^{\infty} E(\exp(-\alpha V_{\tau_a})) \alpha^{-1} d\alpha$$

$$\leq \frac{1}{\Gamma(r)} \int_0^{\infty} e^{-a\sqrt{2}\alpha} \alpha^{-1} d\alpha$$

$$= 2r \Gamma(r + 1/2) a^{-2r}. \quad (3.12)$$

On the other hand, for $0 < r < 1$,

$$E(V_{\tau_a}^r) = \frac{r}{\Gamma(1-r)} \int_0^{\infty} (1 - E(\exp(-\alpha V_{\tau_a}))) \alpha^{-r} d\alpha$$

$$\geq \frac{r}{\Gamma(1-r)} \int_0^{\infty} (1 - e^{-a\sqrt{2}\alpha}) \alpha^{-r} d\alpha. \quad (3.13)$$

In particular, for $r \in [1/2, 1)$, $E(V_{\tau_a}^r) = +\infty$.

**Remark 3.6.** If $X$ is the standard Brownian motion, its covariance $s \land t$ does not satisfy condition (H1), but we still can apply our approach.
It is known from [3] that \( d\tau_a \) has no absolutely continuous part and that \( \{a, \tau_a = \tau_a^+\} \) is a Cantor set, hence, of zero Lebesgue measure. It follows from this observation and from (3.10) that

\[
\int E(M_{\tau_y})\psi(y)\,dy = \int \psi(y)\,dy.
\]

Choosing \( \psi = 1_{[0,a]} \) yields to the expected result:

\[
E\left(\int_0^a e^{\lambda y - \left(\lambda^2/2\right)V(\tau_y)}\,dy\right) = a.
\]

If \( X \) has independent increments and satisfies (H3), then

\[
E(e^{-(\lambda^2/2)V(\tau_a)}) = e^{-\lambda a}.
\]

This follows easily from the fact that \( X \) can be written as a time-changed Brownian motion.

**Remark 3.7.** Consider that \( X \) is a fractional Brownian motion of Hurst index \( H = 1 \). Then \( R(s,t) = st \), and consequently, \( X_t = Y_t \), where \( Y \) is a one-dimensional standard Gaussian random variable. Then, \( \tau_a = \tau_{a^+} = a/Y^+ \). It is then easy to compute the Laplace transform of \( \tau_a \) and we obtain

\[
(3.14) \quad E(\exp(-\alpha\tau_a^2)) = \frac{1}{2} e^{-a\sqrt{2\alpha}}.
\]

We show now that our formula also yields to the right answer. We just note that \( (y \mapsto \tau_y) \) is continuous. This entails that

\[
\frac{\partial R}{\partial s}(z\tau_y^+ + (1-z)\tau_y, \tau_y) = \frac{\partial R}{\partial s}(\tau_y, \tau_y) = \frac{1}{2} V'(\tau_y)
\]

and

\[
\int_0^a E\left(\exp\left(\frac{\lambda y - \lambda^2}{2}V(\tau_y)\right)\right)\,dy
\]

\[
= a - \frac{\lambda}{2} E\left(\int_0^a \exp\left(\frac{\lambda y - \lambda^2}{2}V(\tau_y)\right)V'(\tau_y)\,d\tau_y\right).
\]

Set

\[
\Psi(a, \lambda) = E\left(\exp\left(\frac{\lambda a - \lambda^2}{2}V(\tau_a)\right)\right),
\]

then

\[
(3.15) \quad \frac{\partial\Psi}{\partial a}(a, \lambda) = \lambda \Psi(a, \lambda) - \frac{\lambda^2}{2} E\left(M_{\tau_a} \frac{\partial V(\tau_a)}{\partial a}\right),
\]
Substitute (3.15) into (3.16) to obtain
\[ \frac{\partial \Psi}{\partial a} = 2\lambda \Psi - \lambda. \]

Then, there exists a function \( C(\lambda) \) such that
\[ \Psi(a, \lambda) = \frac{1}{2} + C(\lambda) e^{2\lambda a} \]
so that \( E\left( \exp\left( -\frac{\lambda^2}{2} \tau_a^2 \right) \right) = \frac{1}{2} e^{-\lambda a} + C(\lambda) e^{\lambda a}. \)

By dominated convergence, it is clear that, for any \( \lambda \),
\[ E\left( \exp\left( -\frac{\lambda^2}{2} \tau_a^2 \right) \right)^{a \to \infty} 0, \]
thus, \( C(\lambda) = 0 \) and
\[ E\left( \exp\left( -\frac{\lambda^2}{2} \tau_a^2 \right) \right) = \frac{1}{2} e^{-\lambda a}. \]

Changing \( \lambda^2/2 \) into \( \alpha \) gives (3.14).

**Remark 3.8.** Consider the case of a fractional Brownian motion with Hurst parameter \( H > \frac{1}{2} \). Conditions (H1), (H2) and (H3) are satisfied and we obtain
\[
\int_0^a E(M_y) \, dy = a - \lambda HE\left( \int_0^a \int_0^1 M_{z \tau_y + (1-z)\tau_y} \left[ (z \tau_y + (1-z)\tau_y)^{2H-1} \right. \right.
\]
\[ \left. - |z(\tau_y - \tau_y)|^{2H-1} \right) \, dz \, d\tau_y. \]
Moreover, \( E(e^{-\alpha \tau_a^{2H}}) \leq e^{-a \sqrt{2\alpha}} \), and therefore, \( E(\tau_a^p) < \infty \) if \( p < H \). According to (3.13), \( E(\tau_a^p) \) is infinite if \( pH > 1/4 \) and (3.12) entails that \( \tau_a \) has finite negative moments of all orders.

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**REFERENCES**

[1] ALÔS, E., MAZET, O. and NUALART, D. (2001). Stochastic calculus with respect to Gaussian processes. *Ann. Probab.* 29 766–801. MR1849177

[2] CHERIDITO, P. and NUALART, D. (2005). Stochastic integral of divergence type with respect to fractional Brownian motion with Hurst parameter \( H \in (0,1/2) \). *Ann. Inst. H. Poincaré Probab. Statist.* 41 1049–1081. MR2172209
[3] Itô, K. and McKean, H. P., Jr (1974). *Diffusion Processes and Their Sample Paths*. Springer, Berlin. MR0345224

[4] Kim, J. and Pollard, D. (1990). Cube root asymptotics. *Ann. Probab.* 18 191–219. MR1041391

[5] Molchan, G. M. (2000). On the maximum of fractional Brownian motion. *Theory Probab. Appl.* 44 97–102. MR1751192

[6] Nualart, D. and Vives, J. (1988). Continuité absolue de la loi du maximum d’un processus continu. *C. R. Acad. Sci. Paris Sér. I Math.* 307 349–354. MR0958796