The Minimal Coloring Number Of Any Non-splittable Z-colorable Link Is Four

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Abstract

K. Ichihara and E. Matsudo introduced the notions of Z-colorable links and the minimal coloring number for Z-colorable links, which is one of invariants for links. They proved that the lower bound of minimal coloring number of a non-splittable Z-colorable link is 4. In this paper, we show the minimal coloring number of any non-splittable Z-colorable link is exactly 4.

Keywords: Z-colorable links; minimal coloring number; equivalent local moves.

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1. Introduction

Imitating Fox-coloring [1] and the minimal coloring number [2] for links with Fox colorings, in order to deal with links of determinant 0, in [3], K. Ichihara and E. Matsudo introduced the notions of Z-coloring and minimal coloring number, denoted by mincol_Z(L), for Z-colorable links, which is one of invariants for links.

Definition 1.1. Let L be a link and D a diagram of L. We consider a map γ : {arcs of D} → Z. If γ satisfies the condition 2γ(a) = γ(b) + γ(c) at each crossing of D with the over arc a and the under arcs b and c, then γ is called a Z-coloring on D. A Z-coloring which assigns the same color to all arcs of the diagram is called the trivial Z-coloring. A link is called Z-colorable if it has a diagram admitting a non-trivial Z-coloring.
Definition 1.2. Let us consider the cardinality of the image of a non-trivial \( \mathbb{Z} \)-coloring on a diagram of a \( \mathbb{Z} \)-colorable link \( L \). We call the minimum of such cardinalities among all non-trivial \( \mathbb{Z} \)-colorings on all diagrams of \( L \) the minimal coloring number of \( L \), and denote it by \( \text{mincol}_\mathbb{Z}(L) \).

Definition 1.3. Let \( L \) be a \( \mathbb{Z} \)-colorable link, and \( \gamma \) a non-trivial \( \mathbb{Z} \)-coloring on a diagram \( D \) of \( L \). Suppose that there exists a positive integer \( d \) such that, at all the crossings in \( D \), the differences between the colors of the over arcs and the under arcs are \( d \) or 0. Then we call \( \gamma \) a simple \( \mathbb{Z} \)-coloring.

Then they proved:

Theorem 1.4. 1. Let \( L \) be a non-splittable \( \mathbb{Z} \)-colorable link. Then \( \text{mincol}_\mathbb{Z}(L) \geq 4 \).
2. Let \( L \) be a non-splittable \( \mathbb{Z} \)-colorable link. If there exists a simple \( \mathbb{Z} \)-coloring on a diagram of \( L \), then \( \text{mincol}_\mathbb{Z}(L) = 4 \).
3. If a non-splittable link \( L \) admits a \( \mathbb{Z} \)-coloring with five colors, then \( \text{mincol}_\mathbb{Z}(L) = 4 \).

In the end of [3], they posed two questions:

Question 1.5.
1. Does \( \text{mincol}_\mathbb{Z}(L) = 4 \) always hold for any non-splittable \( \mathbb{Z} \)-colorable link \( L \)?
2. Does every non-splittable \( \mathbb{Z} \)-colorable link admit a simple \( \mathbb{Z} \)-coloring?

In this paper, we give a positive answer to Question 1.5 (2), and hence a positive answer to Question 1.5 (1) by Theorem 1.4 (2).

2. Main result and its proof

For convenience, we denote the crossing with over arc \( b \) and under arcs \( a, c \) by \( a|b|c \). Let \( d \) be a nonnegative integer, we call a crossing a \( d \)-diff one if the difference between the colors of the over arc and the under arcs is \( d \).

In the figures through the rest of this paper, we don’t distinguish the over arc and under arcs of the uni-colored crossings. Moreover, a uni-colored \( b|b|b \) crossing often represents a finite number, including 0, of \( b|b|b \) crossings.

Definition 2.1. Let \( L \) be a \( \mathbb{Z} \)-colorable link, and \( \gamma \) a non-trivial \( \mathbb{Z} \)-coloring on a diagram \( D \) of \( L \). Two non-0-diff crossings are adjacent if there are only a finite number of 0-diff crossings between them, as shown in the Figure 1.1 (containing 4 cases and 10 subcases altogether).
In this section we shall prove:

**Theorem 2.2.** Any non-splittable ℤ-colorable link admits a simple ℤ-coloring.

To prove Theorem 2.2, we need two lemmas.

**Lemma 2.3.** Let \( L \) be a ℤ-colorable link, and \( \gamma \) a ℤ-coloring on a diagram \( D \) of \( L \). If there exists a pair of adjacent \( n \)-diff crossing and \( qn \)-diff crossing (\( q \geq 2, q \in \mathbb{N}^+ \)), then the \( qn \)-diff crossing can be eliminated by equivalent local moves. Moreover, any newly created crossing in the process of elimination is either a 0-diff crossing or an \( n \)-diff crossing.

**Proof.** We prove it case by case. Take \( m = qn \) in Figure 1.1.

**Case 1:** See Figure 1.1 (1). In this case we prove Lemma 2.3 by induction on \( q \). When \( q = 2 \), we can eliminate \( b - 2n|b||b|b + 2n \) as shown in Figure 1.1.1, and any newly created crossing in the process of elimination is either a 0-diff crossing or an \( n \)-diff crossing. When \( q = 3 \), we can eliminate \( b - 3n|b||b|b + 3n \) as shown in Figure 1.1.2, and any newly created crossing in the process of elimination is either a 0-diff crossing or an \( n \)-diff crossing. Now we assume that when \( q \leq k - 1, \), the lemma holds, and shall prove when \( q = k \), the lemma also holds. When \( q = k \), we can reduce it to \( q = k - 2 \) as shown in Figure 1.1.3.

**Case 2:** See Figure 1.1 (2). When \( q = 2 \), we can eliminate \( b - 2n|b||b|b + 2n \) as shown in Figure 1.2.1, and any newly created crossing in the process of elimination is either a 0-diff crossing or an \( n \)-diff crossing. When \( q = k \),
Figure 1.1.1
Figure 1.1.2
Figure 1.1.3: The structure with newly created $b - (k - 2)n | b | b + (k - 2)n$ contained in the dashed box is the Case 1 with $q = k - 2$. We can eliminate it by induction.
$k \geq 3$, we can operate as shown in Figure 1.2.2. Although there is newly created $b-(k-2)n|b|b+(k-2)n$ crossing, we can use the structure contained in the dashed box to eliminate it since it is the Case 1 with $q = k-2$.

![Diagram](image)

**Figure 1.2.2**

**Case 3:** See Figure 1.1 (3). When $q = 2$, we can eliminate $b|b+2n|b+4n$ and $b|b-2n|b-4n$ as shown in Figure 1.3.1, and any newly created crossing in the process of elimination is either a 0-diff crossing or an $n$-diff crossing. When $q = k$, $k \geq 3$, we can operate as shown in Figure 1.3.2. Let $b' = b + kn$, $b'' = b - kn$. Although there is newly created $b + n|b + kn|b + (2k-1)n$ crossing, that is $b' - (k-1)n|b'|b' + (k-1)n$ in the figure (above) and newly created $b - n|b - kn|b - (2k-1)n$, that is $b'' + (k-1)n|b''|b'' - (k-1)n$ in the figure (below), both the structures contained in the dashed boxes are the Case 2 with $q = k-1$. We can eliminate them as done in Case 2.

**Case 4:** See Figure 1.1 (4). There are 4 subcases.
Figure 1.3.1
Figure 1.3.2
For the first and second subcases, we can eliminate \( b|b + kn|b + 2kn \) and \( b|b - kn|b - 2kn \) as shown in Figure 1.4.1. Let \( b' = b + kn, \ b'' = b - kn \). The newly created \( b - n|b + kn|b + (2k + 1)n \), that is \( b' - (k + 1)n|b'|b' + (k + 1)n \) in the figure (above) and the newly created \( b + n|b - kn|b - (2k + 1)n \), that is \( b'' + (k + 1)n|b''|b'' - (k + 1)n \) in the figure (below), contained in the dashed boxes are the Case 2 with \( q = 2 \). We can eliminate them as done in Case 2, and any newly created crossing in the process of elimination is either a 0-diff crossing or an \( n \)-diff crossing.

For the third and fourth subcases. When \( q = 2 \), we can eliminate \( b|b - 2n|b - 4n \) and \( b|b + 2n|b + 4n \) as shown in Figure 1.4.2, and any newly created crossing in the process of elimination is either a 0-diff crossing or an \( n \)-diff crossing. When \( q = k \), \( k \geq 3 \), we can eliminate \( b|b - kn|b - 2kn \) and \( b|b + kn|b + 2kn \) as shown in Figure 1.4.3. Let \( b' = b - kn, \ b'' = b + kn \). The newly created \( b - n|b - kn|b - (2k - 1)n \), that is \( b' + (k - 1)n|b'|b' - (k - 1)n \) in the figure (above) and newly created \( b + n|b + kn|b + (2k - 1)n \), that is \( b'' - (k - 1)n|b''|b'' + (k - 1)n \) in the figure (below), contained in the dashed boxes are the Case 2 with \( q = k - 1 \). We can eliminate them as done in Case 2, and any newly created crossing in the process of elimination is either a 0-diff crossing or an \( n \)-diff crossing.

\[ \square \]

Lemma 2.4. Let \( L \) be a \( \mathbb{Z} \)-colorable link, and \( \gamma \) a \( \mathbb{Z} \)-coloring on a diagram \( D \) of \( L \). If there exists a pair of adjacent \( n \)-diff crossing and \( m \)-diff crossing, we can convert this local structure to a new one by equivalent local moves, containing only 0 or \( d \)-diff crossings, where \( d = \gcd(m, n) \).

Proof. Without loss of generality, we assume \( m > n \). We shall prove Lemma 2.4 by induction on \( n \). If \( n = 1 \), then \( d = 1 \). Applying Lemma 2.3, we can convert the local structure and obtain a new local structure containing only 0 or 1-diff crossings, thus the lemma holds. Next, assuming that when \( n \leq z - 1, \ z \geq 2 \), the lemma holds, we shall prove when \( n = z \), the lemma also holds.

If \( m = qz \ (q \geq 1, q \in \mathbb{N}^+) \), then \( d = z \). Applying Lemma 2.3, we can obtain a new structure, containing only 0 or \( d \)-diff crossings, which means the lemma holds. If \( m = qz + r \ (q \geq 1, 0 < r < z) \), we shall further prove, for each case, we can create a \( r \)-diff crossing which is adjacent to a \( z \)-diff crossing and the difference of any newly created crossing in the operation is \( kd \) for some nonnegative integer \( k \). Note that \( n = z, m = qz + r \) and \( d = \gcd(m, n) = \gcd(n, r) \).
Figure 1.4.1
Figure 1.4.2
For (1) in Figure 1.1 with $n = z$, $m = qz + r$, we can operate as shown in Figure 2.1.

For (2) in Figure 1.1 with $n = z$, $m = qz + r$, we can operate as shown in Figure 2.2.1, 2.2.2.

For (3) in Figure 1.1 with $n = z$, $m = qz + r$, we can operate as shown in Figure 2.3.

For (4) in Figure 1.1 with $n = z$, $m = qz + r$, we can operate as shown in Figure 2.4.1, 2.4.2.

In the figures above, there always exists a $r$-diff crossing which is adjacent to a $z$-diff crossing. See dashed boxes. After proving that, we can turn the adjacent $z$-diff crossing and $r$-diff crossing to a new local structure, containing only 0 or $d$-diff crossings (and a $d$-differ crossing is always created) by hypothesis induction, because $0 < r < z$. Moreover, because the difference of any newly created crossing in the operation of creating a $r$-diff crossing is $kd$ for some nonnegative integer $k$, the great common divisor of the $d$-diff crossing and its adjacent crossings is also $d$. Applying Lemma 2.3, we can turn these kinds of adjacent crossings to new structures containing only 0 or $d$-diff crossings, and continuing this process repeatedly, the lemma will hold.

Proof of Theorem 2.2. Let $L$ be a non-splittable $\mathbb{Z}$-colorable link, and $\gamma$ a $\mathbb{Z}$-coloring on a diagram $D$ of $L$. If this coloring is not simple, then there exists a pair of adjacent $m$-diff and $n$-diff crossings, where $m, n \in \mathbb{N}^+$ and $m \neq n$. Applying the Lemma 2.4 to this pair of crossings, we can obtain a new equivalent diagram by converting the local structure containing $m$-diff crossing, $n$-diff crossing and 0-diff crossings between them to a new local structure containing only 0 or $d$-diff crossings, where $d = \gcd(m, n)$. Since $L$ is non-splittable, continue the above process repeatedly, we can obtain a simple coloring.

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Figure 2.1
Figure 2.2.1

Figure 2.2.2
Figure 2.3
Figure 2.4.1
Figure 2.4.2
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