Integral and Absolute Hodge Classes

Ryan Keast

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Abstract

Despite the failure of the integral Hodge conjecture, we show that the rational Hodge conjecture implies an integral version (modulo torsion) of the absolute Hodge conjecture.

1 Introduction

This very brief article is based on a talk given by the author and the proceeding discussion after the lecture. The author would like to thank Donu Arapura for a very important insight.

Andre-Oort predicts the algebraic closure of CM points inside a Shimura variety is again Shimura variety. This was proven in the special case of $A_g$ by J. Tsimerman using the combined tools of o-minimality and large Galois orbits of CM abelian varieties. Effective CM Hodge structures of positive weight are always embeddable into the Hodge structure of CM abelian varieties. A natural question is if we can use this embedding to produce large Galois orbits for a whole host of varieties. The immediate road block is that it is unknown whether or not this embedding is algebraic, or even absolute.

Even when we assume the Hodge conjecture there seemed to be a gap. Let $X$ and $Y$ be smooth projective varieties with $H^n(X,\mathbb{Z})$, $H^n(Y,\mathbb{Z})$ both free. Assume there exists an integral isomorphism $j$ of Hodge structures. The absolute Hodge conjecture implies that after applying $\sigma \in Auto(\mathbb{C})$, $\sigma j$ is a rational morphism of Hodge structures. What is not clear apriori is if $\sigma j$ is still an integral isomorphism. If we are to establish a relationship between the sizes of the Galois orbits, integrality must be preserved, else we could also be counting isogenies.

Surprisingly the literature gave no answer to the question. It appeared to be silent on an even more basic question: if $\nu \in H^{n,n}(X,\mathbb{C}) \cap H^{2n}(X,\mathbb{Z})/ torsion$, is $\sigma \nu \in H^{n,n}(X^\sigma,\mathbb{C}) \cap H^{2n}(X^\sigma,\mathbb{Z})/ torsion$. If we assumed the integral Hodge conjecture, then $\nu = \sum n[Z]$ $n \in \mathbb{Z}$ and $\sigma \nu = \sum n[Z^\sigma]$. Alas, the integral Hodge conjecture is false.

Counterexamples to the integral Hodge conjecture lie in two camps: torsion and non-torsion. We will ignore the torsion counterexamples for now. A basic non-torsion counterexample to the integral Hodge conjecture was given
by Kollar(cf[V]). If we have of a very general threefold hypersurface of degree 125, then 5 divides the degree of every curve. \(H^2(X,\mathbb{Z})\cap H^{1,1}(X,\mathbb{C})\) is 1-dimensional and given by cohomology class of a hyperplane section \(h\). Assume \(H^4(X,\mathbb{Z})\cap H^{2,2}(X,\mathbb{C})\) is generated by curve \(c\).

Since the pairing of \(H^2\) and \(H^4\) is unimodular \(\int h \cap [c] = 1\). This integral also gives the intersection number, so it must be divisible by five. We arrive at a contradiction.

In this case, the rational Hodge conjecture is known for \(H^4\). Let \(c\) be a curve such that \([c]\) generates \(H^4(X,\mathbb{Q})\cap H^{2,2}(X,\mathbb{C})\).

\[\int h \cap [c] = n.\] Then \(\frac{1}{n}[c]\) is integral generates \(H^4(X,\mathbb{Z})\cap H^{2,2}(X,\mathbb{C})\). \(\int \sigma h \cap \sigma c = \int h \cap [c] = n\), so \(\frac{1}{n}[\sigma c]\) also generates \(H^4(X^\sigma,\mathbb{Z})\cap H^{2,2}(X^\sigma,\mathbb{C})\). Thus even in this counterexample of the integral Hodge conjecture, the integrality is preserved by \(\sigma\).

This example convinced the author that the preservation of integrality was at the very least not trivially false. In fact, in the proceeding section we will show that despite the failure of the integral Hodge conjecture the integrality of \(\sigma_j\) is guaranteed by the rational Hodge conjecture. In other words, the rational conjecture implies an integral version of the absolute Hodge conjecture modulo torsion.

## 2 Absolute and Integral Hodge Classes

The proof involves passing to the étale cohomology:

**Theorem 1.** Assuming the rational Hodge conjecture, if \(v \in H^{n,n}(X,\mathbb{C}) \cap H^{2n}(X,\mathbb{Z})/\text{torsion}\), then for \(\sigma \in \text{Aut}(\mathbb{C})\) \(\sigma v \in H^{n,n}(X^\sigma,\mathbb{C})\cap H^{2n}(X^\sigma,\mathbb{Z})/\text{torsion}\)

**Proof.** If we assume the rational Hodge conjecture, \(v = \sum a_i[Z_i]\) with \(a_i \in \mathbb{Q}\).

\(\sigma v = \sum a_i[Z_i^\sigma]\). By the Artin Comparison theorem[M], for each \(l\) we have a canonical isomorphism \(A_l : H^{2n}(X(\mathbb{C}),\mathbb{Z}_l) \rightarrow H^{2n}(X_{et},\mathbb{Z}_l)\). We have the embedding \(H^{2n}(X(\mathbb{C}),\mathbb{Z}) \hookrightarrow H^{2n}(X(\mathbb{C}),\mathbb{Z}_l)\). The image of \(v = \sum a_i[Z_i]\) and \(\sigma v = \sum a_i[Z_i^\sigma]\) are identical with respect to the étale cohomology, specifically they are both integral with respect to \(\mathbb{Z}_l\) in \(H^{2n}(X_{et},\mathbb{Z}_l)\). It follows that \(\sigma v = \sum a_i[Z_i^\sigma] \in H^{2n}(X^\sigma(\mathbb{C}),\mathbb{Z}_l)\).

\(\{e_n\}\) be a basis for \(H^{n,n}(X^\sigma,\mathbb{C})\cap H^{2n}(X^\sigma,\mathbb{Z})/\text{torsion}\). We rewrite \(\sigma v = \sum b_ne_n\). We will now show that \(b_n\) are integers. If \(\sum b_ne_n \in H^{2n}(X^\sigma(\mathbb{C}),\mathbb{Z})\otimes \mathbb{Z}_l\), then each \(b_n\) must be an \(l\)-adic integer. Since this is true for all \(l\), \(b_n \in \mathbb{Z}\).

**Corollary 2.** Assume the rational Hodge conjecture. If \(H^n(X,\mathbb{C})\) and \(H^n(Y,\mathbb{C})\) are free and isomorphic as integral Hodge structures, then so are \(H^n(X^\sigma,\mathbb{C})\) and \(H^n(Y^\sigma,\mathbb{C})\).

**Proof.** By assumption, there exists a integral morphism of Hodge structures \(j : H^n(X,\mathbb{C}) \rightarrow H(Y,\mathbb{C})\) with an integral inverse \(j^{-1}\). If we assume the rational Hodge conjecture \(j\) is given by an algebraic correspondence, \(\sigma j\) and \(\sigma j^{-1}\) are still inverses of each other and are still both integral.
References

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