DYNAMICS OF A DELAYED LOTKA-VOLTERRA MODEL WITH 
TWO PREDATORS COMPETING FOR ONE PREY

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Abstract. In this paper, we study the local dynamics of a class of 3-dimensional
Lotka-Volterra systems with a discrete delay. This system describes two preda-
tors competing for one prey. Firstly, linear stability and Hopf bifurcation are
investigated. Then some regions of attraction for the positive steady state are
obtained by means of Liapunov functional in a restricted region. Finally, suf-
ficient and necessary conditions for the principle of competitive exclusion are
obtained.

1. Introduction. Lotka [21] and Volterra [29] considered a model (now known
as the standard Lotka-Volterra model) for one predator and one single prey in a
constant and uniform environment. In this model the populations of predator and
prey permanently oscillate for almost all positive initial conditions. Recently, there
have been some excellent works on global dynamics of resource competitive models
(see [4, 5, 11, 16, 17, 19, 24, 22, 25, 30, 31, 32, 33, 34]), which are important in
understanding of the mechanism of natural selection: the principle of competitive
exclusion (see [1, 3, 10, 26, 27, 28, 35, 36]) or the coexistence of competing species.
For example, Volterra [29] observed that the coexistence of two or more predators
competing for fewer prey resources is impossible. To describe two predators com-
peting for one prey, Llibre and Xiao [20] obtained sufficient and necessary conditions
for the principle of competitive exclusion to hold in the following system (1) and

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described the global dynamical behavior of the three species in the first octant:

\[
\begin{aligned}
\frac{dS(t)}{dt} &= S(t) \left[r_3 - \frac{S(t)}{K} - b_1 x_1(t) - b_2 x_2(t)\right], \\
\frac{dx_1(t)}{dt} &= x_1(t) \left[-r_1 + a_1 S(t)\right], \\
\frac{dx_2(t)}{dt} &= x_2(t) \left[-r_2 + a_2 S(t)\right].
\end{aligned}
\]  

(1)

In the real ecological predation environment, time lag often occurs, which is due to the time required for the predator to catch up with the prey or the gestation period for the predator to reproduce [6, 8, 7, 9, 18, 23]. Kuang and his collaborators had a deep study of the effect of delay on the Lotka-Volterra model, such as [2, 12, 13, 14, 37]. In particular, Beretta and Kuang [2] provided a detailed and explicit procedure of obtaining some regions of attraction for the positive steady state of a well-known Lotka-Volterra type predator-prey system by constructing a proper Liapunov functional in a restricted region.

Motivated by Llibre and Xiao [20] and Beretta and Kuang [2], we aim to obtain some regions of attraction for the positive steady state of the following 3-dimensional predator-prey system:

\[
\begin{aligned}
\frac{dS(t)}{dt} &= S(t) \left[r_3 - \frac{S(t)}{K} - b_1 x_1(t) - b_2 x_2(t)\right], \\
\frac{dx_1(t)}{dt} &= x_1(t) \left[-r_1 + a_1 S(t - \tau)\right], \\
\frac{dx_2(t)}{dt} &= x_2(t) \left[-r_2 + a_2 S(t - \tau)\right],
\end{aligned}
\]  

(2)

where \(x_i(t) (i = 1, 2)\) represents the population density of the \(i\)-th predator at time \(t\), \(S(t)\) represents the population density of the prey at time \(t\), \(r_3 > 0\) is the intrinsic rate of growth of the prey, and \(K > 0\) is the carrying capacity of the prey, which describes the richness of resources for prey. \(b_i > 0\) is the effect of the \(i\)-th predation on the prey, \(r_i > 0\) is the natural death rate of the \(i\)-th predator in the absence of prey, and \(a_i > 0\) is the efficiency and propagation rate of the \(i\)-th predation in the presence of prey, the delay \(\tau > 0\) can be regarded as a gestation period or reaction time of the predators. For relevant background of this kind of systems, see [2, 12]. It is clear that \(x_i(t) \geq 0\) and \(S(t) \geq 0\). Hence, system (2) is considered only in the closed positive octant \(\mathbb{R}_+^3\), here of course \(\mathbb{R}_+ = [0, +\infty)\). For simplicity, we denote the open positive octant by \(\text{Int}(\mathbb{R}_+^3)\).

The only difference between systems (1) and (2) is that a delay appears in system (2), which, as we shall see, makes system (2) have much richer dynamics than system (1). For example, Llibre and Xiao [20] found that system (1) with \(r_3 K > \frac{a_2}{a_1} = \frac{r_1}{a_2}\) has infinitely many positive equilibria \(E_+(S, x_1, x_2)\) filling up a segment

\[
L = \left\{(S, x_1, x_2) : S = \frac{r_1}{a_1}, b_1 x_1 + b_2 x_2 = r_3 - \frac{r_1}{K a_1}, x_1 \geq 0, x_2 \geq 0\right\},
\]

which attract all solutions satisfying positive initial conditions, the end of \(L\) attracts all solutions satisfying nonnegative initial conditions \((S, x_1, 0)\) or \((S, 0, x_2)\), respectively, and the origin \((0, 0, 0)\) attracts all solutions satisfying nonnegative initial conditions \((0, x_1, x_2)\). In system (2), however, the presence of time delay \(\tau\) may cause some nonlinear oscillations and lead to a completely different computational performance (see Theorem 3.4). Thus, even though the time delay \(\tau\) is small enough,
we cannot claim that the segment $L$ attracts all solutions of system (2) satisfying positive initial conditions. We shall see that every positive equilibrium $E_+(S, x_1, x_2)$ just attracts solutions of system (2) with initial values close to $E_+(S, x_1, x_2)$ when $\tau$ is small enough (see Theorem 4.2). For this purpose, in section 4 we shall provide a detailed and explicit procedure of obtaining some regions of attraction for the positive equilibrium $E_+(S, x_1, x_2)$ by constructing a proper Liapunov functional. In section 5, we analyze the asymptotical behavior of the boundary equilibrium in the absence of positive equilibria. In particular, sufficient and necessary conditions for the principle of competitive exclusion are obtained. Our theoretical results are illustrated by an example in section 6. Finally, some discussions about generalization to predator-prey models with $n$ predators competing for one prey are given in section 7.

2. Preliminaries. We first consider the boundedness of solutions of system (2).

**Lemma 2.1.** There exists a positive constant $M$ such that every positive solution $(\tilde{S}(t), x_1(t), x_2(t))$ of system (2) with nonnegative initial condition satisfies

$$\limsup_{t \to +\infty} \tilde{S}(t) < M, \quad \limsup_{t \to +\infty} x_1(t) < M, \quad \limsup_{t \to +\infty} x_2(t) < M.$$

**Proof.** Consider the following two systems:

$$\begin{align*}
\frac{d\tilde{S}(t)}{dt} &= \tilde{S}(t)(r_3 - \tilde{S}(t) - b_1 \hat{x}_1(t)), \\
\frac{d\hat{x}_1(t)}{dt} &= \hat{x}_1(t)(-r_1 + a_1 \tilde{S}(t - \tau)), \\
\frac{d\hat{x}_2(t)}{dt} &= \hat{x}_2(t)(-r_2 + a_2 \tilde{S}(t - \tau)),
\end{align*}$$

(3)

and

$$\begin{align*}
\frac{d\hat{S}(t)}{dt} &= \hat{S}(t)(r_3 - \hat{S}(t) - b_2 \hat{x}_2(t)), \\
\frac{d\hat{x}_1(t)}{dt} &= \hat{x}_1(t)(-r_1 + a_1 \hat{S}(t - \tau)), \\
\frac{d\hat{x}_2(t)}{dt} &= \hat{x}_2(t)(-r_2 + a_2 \hat{S}(t - \tau)).
\end{align*}$$

(4)

Denote by $(\tilde{S}(t), \hat{x}_1(t), \hat{x}_2(t))$ and $(\hat{S}(t), \hat{x}_1(t), \hat{x}_2(t))$ the solutions to systems (3) and (4), respectively. In view of [12, page 247], we know that there exist $M_1 > 0$ and $M_2 > 0$ such that $\limsup_{t \to +\infty} \hat{S}(t) < M_1$, $\limsup_{t \to +\infty} \hat{x}_1(t) < M_1$, $\limsup_{t \to +\infty} \hat{S}(t) < M_2$, and $\limsup_{t \to +\infty} \hat{x}_2(t) < M_2$. It follows from the principle of comparison that the solution $(\tilde{S}(t), x_1(t), x_2(t))$ of system (2) is less than each of the solutions of systems (3) and (4). So we can take $M = \max\{M_1, M_2\}$ such that

$$\limsup_{t \to +\infty} S(t) < M, \quad \limsup_{t \to +\infty} x_1(t) < M, \quad \limsup_{t \to +\infty} x_2(t) < M.$$

This completes the proof. \hfill \square

Consider the following autonomous system of delay differential equation

$$\hat{x}(t) = F(x_1)$$

(5)

where $F: C([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n$ is Lipschitzian and satisfies $F(0) = 0$, $\tau > 0$, $C([-\tau, 0], \mathbb{R}^n)$ is the Banach space of continuous functions defined on $[-\tau, 0]$ equipped with the norm $\|\phi\| = \max_{\theta \in [-\tau, 0]} |\phi(\theta)|$, and $|\cdot|$ is any norm in $\mathbb{R}^n$. 

**Lemma 2.2** (Lemma 2.1 in [2]). Let $w_1(\cdot)$ and $w_2(\cdot)$ be nonnegative continuous scalar functions such that $w_i(0) = 0$, $i = 1, 2$; $w_2(r) > 0$ for $r > 0$, $\lim_{r \to \infty} w_1(r) = +\infty$ and $V : C([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}$ be a continuously differentiable scalar functional such that for a special set $S$ of solutions of (5), the following are satisfied
\[
V(\phi) \geq w_1(\|\phi(0)\|), \quad \dot{V}(\phi) \leq -w_2(\|\phi(0)\|).
\]
Then $x = 0$ is asymptotically stable with respect to the set $S$. That is, solutions that stay in $S$ converge to $x = 0$.

In the process of the construction of a key Liapunov functional in the next section, we will need the solution of the following optimization problem. Let $m, n, p, q, h, k$ and $\sigma$ be positive constants and $\gamma, \varsigma$ be two positive variables satisfying
\[
\gamma > \sigma \quad \text{and} \quad \varsigma = \sigma/(\gamma - \sigma).
\]
Find
\[
\xi = \max_{\gamma} \left\{ \min \left\{ \frac{\varsigma}{q + p\gamma}, \frac{1}{n + m\gamma}, \frac{1}{k + h\varsigma} \right\} \right\}.
\]

**Lemma 2.3.** Let $\mu_1 = n + (m + p)\sigma + [(n + 2q + (m + p)\sigma)^2 - 4(n + q)(q + p\sigma)]^{1/2}$, and $\mu_2 = k + (h + p)\sigma + [(k + 2q + (h + p)\sigma)^2 - 4(k + q)(q + p\sigma)]^{1/2}$, then the solution to the above optimization problem is $\xi = \min\{2\mu_1^{-1}, 2\mu_2^{-1}\}$.

**Proof.** Substituting $\varsigma = \sigma/(\gamma - \sigma)$ into (8) yields
\[
\xi = \max_{\gamma} \left\{ \min \left\{ \frac{\sigma}{(q + ps)\gamma - q\sigma}, \frac{\gamma - \sigma}{(n + m\sigma)\gamma - n\sigma}, \frac{\gamma - \sigma}{(k + h\sigma)\gamma - k\sigma} \right\} \right\}.
\]
Let
\[
f(\gamma) = \frac{\sigma}{(q + p\sigma)\gamma - q\sigma}, \quad g(\gamma) = \frac{\gamma - \sigma}{(n + m\sigma)\gamma - n\sigma}, \quad l(\gamma) = \frac{\gamma - \sigma}{(k + h\sigma)\gamma - k\sigma}.
\]

Note that
\[
f'(\gamma) = \frac{-\sigma(q + p\sigma)}{[(q + p\sigma)\gamma - q\sigma]^2}, \quad g'(\gamma) = \frac{m\sigma^2}{[(n + m\sigma)\gamma - n\sigma]^2}, \quad l'(\gamma) = \frac{h\sigma^2}{[(k + h\sigma)\gamma - k\sigma]^2}.
\]
It is easy to see that for $\gamma \geq \sigma$, $f$ is strictly decreasing while both $g$ and $l$ are strictly increasing, and that $f(\sigma) = \frac{1}{p\sigma} > g(\sigma) = l(\sigma) = 0$. Hence the solution of the optimization problem is unique if it exists. In what follows, we prove the existence of the unique solution. Solving the equation $f(\gamma) = g(\gamma)$ for $\gamma > \sigma$, and denoting the root by $\gamma_1$, we have
\[
\gamma_1 = \frac{\sigma(\mu_1 + 2q)}{2(p\sigma + q)}, \quad f(\gamma_1) = \frac{2}{\mu_1}.
\]
Solving the equation $f(\gamma) = l(\gamma)$ for $\gamma > \sigma$, and denoting the root by $\gamma_2$, we have
\[
\gamma_2 = \frac{\sigma(\mu_2 + 2q)}{2(p\sigma + q)}, \quad f(\gamma_2) = \frac{2}{\mu_2}.
\]
It follows from the monotonicity of functions $f$, $g$ and $l$ that $\xi = \min\{2\mu_1^{-1}, 2\mu_2^{-1}\}$. This completes the proof. \qed
3. Local stability. We first consider the existence and types of equilibria of (2). Obviously, system (2) always has the boundary equilibria $O(0, 0, 0)$ and $E_0(r_3 K, 0, 0)$. Moreover, we have the following observations, which are similar to proposition 1 in [20],

(i): In $\mathbb{R}^3$, system (2) has an additional boundary equilibrium $E_1 \left( \frac{r_3}{a_1}, \frac{r_3}{b_1} (r_3 K - \frac{r_3}{a_1}), 0 \right)$ (respectively, $E_2 \left( \frac{r_2}{a_2}, 0, \frac{1}{a_2} (r_3 K - \frac{r_3}{a_2}) \right)$) when $r_3 K > \frac{r_3}{a_1}$ (respectively, $r_3 K > \frac{r_3}{a_2}$).

(ii): System (2) has a positive equilibrium $E_+ (S, x_1, x_2)$ in $\text{Int} (\mathbb{R}^3)$ if and only if $\frac{r_2}{a_1} = \frac{r_2}{a_2}$ and $r_3 K > \frac{r_2}{a_1}$. Moreover, if system (2) has a positive equilibrium in $\text{Int} (\mathbb{R}^3)$, then system (2) has infinitely many positive equilibria $E_+(S, x_1, x_2)$ in $\text{Int} (\mathbb{R}^3)$ which fill a segment $L$ with endpoints at the boundary equilibrium $E_1$ and $E_2$.

In what follows, we investigate the linear stability of these equilibria. Note that the characteristic equation of system (2) at the equilibrium $E = (S, x_1, x_2)$ is

$$J_r (E) = \begin{bmatrix} r_3 - \frac{2}{S} S - b_1 x_1 - b_2 x_2 & -b_1 S & -b_2 S \\ a_1 x_1 e^{-\lambda \tau} & -r_1 + a_1 S & 0 \\ a_2 x_2 e^{-\lambda \tau} & 0 & -r_2 + a_2 S \end{bmatrix}.$$  \hspace{1cm} (9)

We start with equilibria $O(0, 0, 0)$ and $E_0(r_3 K, 0, 0)$ and observe that $J_r (O)$ has three eigenvalues: $r_3$, $-r_1$, and $-r_2$, and $J_r (E_0)$ has three eigenvalues: $-r_3$, $-r_1 + a_1 r_3 K$, $-r_2 + a_2 r_3 K$. Thus, we have the following results.

**Theorem 3.1.** For system (2) with $\tau \neq 0$ the following statements hold:

(i): $O(0, 0, 0)$ is a saddle with a 2-dimensional stable manifold and a 1-dimensional unstable manifold for all the values of the parameters.

(ii): $E_0(r_3 K, 0, 0)$ has the following local phase portraits depending on the values of the parameters.

(a): $E_0$ is a stable node if $r_3 K < \min \{ \frac{r_2}{a_1}, \frac{r_2}{a_2} \}$;

(b): $E_0$ is a saddle with a 2-dimensional stable manifold and a 1-dimensional unstable manifold if $\min \{ \frac{r_2}{a_1}, \frac{r_2}{a_2} \} < r_3 K < \max \{ \frac{r_2}{a_1}, \frac{r_2}{a_2} \}$;

(c): $E_0$ is a saddle with a 1-dimensional stable manifold and a 2-dimensional unstable manifold if $\max \{ \frac{r_2}{a_1}, \frac{r_2}{a_2} \} < r_3 K$;

(d): $E_0$ is a degenerate equilibrium if either $r_3 K = \frac{r_2}{a_1}$ or $r_3 K = \frac{r_2}{a_2}$. Here a degenerate equilibrium is an equilibrium having at least one zero eigenvalue.

Theorem 3.1 coincides with Proposition 2 in [20], which presents the linear stability of equilibria $O(0, 0, 0)$ and $E_0(r_3 K, 0, 0)$ of system (1).
It follows from $r_3 K > \frac{r_1}{a_1}$ that $\frac{r_1}{K}(r_3 K - \frac{r_1}{a_1}) e^{-\lambda \tau} > 0$ for all nonnegative real $\lambda$ and $\tau \geq 0$, and hence that $P_\tau(\lambda) > 0$ for all $\lambda \geq 0$ and $\tau \geq 0$. Therefore, all real solutions of equation (11) are negative. Now consider a complex solution $\lambda = i \omega$ with $\omega > 0$ of $P_\tau(\lambda) = 0$. Substituting $\lambda = i \omega$ into $P_\tau(\lambda) = 0$ and then separating real and imaginary parts of the equation, we obtain

$$\omega^4 + \frac{r_1^2 \omega^2}{a_1^2 K^2} - \left( r_1 r_3 - \frac{r_1^2}{a_1 K} \right)^2 = 0. \quad (12)$$

Solving equation (12) for $\omega$ yields $\omega = \omega_1$, where

$$\omega_1 = \frac{\sqrt{2}}{2} \left\{ - \frac{r_1^2}{a_1^2 K^2} + \left[ \left( \frac{r_1}{a_1 K} \right)^4 + 4 \left( r_1 r_3 - \frac{r_1^2}{a_1 K} \right)^2 \right]^{\frac{1}{2}} \right\}^\frac{1}{2} > 0.$$ 

Hence, equation (11) may have a pair of pure imaginary roots $\pm i \omega_1$. Moreover, solving $P_\tau(\omega_1) = 0$ for $\tau$ yields

$$\tau = \tau_1^{(k)} := \frac{1}{\omega_1} \arctan \frac{r_1}{a_1 K \omega_1} + k \pi \omega_1 \quad (k = 0, 1, 2, \ldots). \quad (13)$$

Differentiating the equation (11) with respect to $\tau$ yields

$$\frac{d\lambda}{d\tau} = \frac{a \lambda e^{-\lambda \tau}}{b + 2 \lambda - a \tau e^{-\lambda \tau}}$$

where $a = r_1 r_3 - \frac{r_1^2}{a_1 K}$ and $b = \frac{r_1}{a_1 K}$. Therefore,

$$\text{Re} \left\{ \frac{d\lambda}{d\tau} \bigg|_{\lambda = i \omega_1, \tau = \tau_1^{(k)}} \right\} = \text{Re} \left\{ \frac{-i a \omega_1 e^{-i \omega_1 \tau_1^{(k)}}}{b + 2 i \omega_1 - a \tau_1^{(k)} e^{-i \omega_1 \tau_1^{(k)}}} \right\} = \frac{b^2 \omega_1^2 + 2 \omega_1^4}{|b - a \tau_1^{(k)} \cos(\omega_1 \tau_1^{(k)})|^2 + [2 \omega_1 + a \tau_1^{(k)} \sin(\omega_1 \tau_1^{(k)})]^2} > 0.$$ 

In particular, for each fixed $\tau \in [0, \tau_1^{(0)})$, all solutions to the equation $P_\tau(\cdot) = 0$ have negative real parts and hence that we have the following results.

**Theorem 3.2.** When $r_3 K > \frac{r_1}{a_1}$,

(a): $E_1$ is stable if $\frac{r_1}{a_1} < \frac{r_2}{a_2}$ and $\tau \in [0, \tau_1^{(0)})$; And is unstable if $\frac{r_1}{a_1} < \frac{r_2}{a_2}$ and $\tau > \tau_1^{(0)}$.

(b): $E_1$ has a 1-dimensional unstable manifold if $\frac{r_1}{a_1} > \frac{r_2}{a_2}$ and $\tau \in [0, \tau_1^{(0)})$.

(c): $E_1$ has an unstable manifold of dimension at least 3 if $\frac{r_1}{a_1} > \frac{r_2}{a_2}$ and $\tau > \tau_1^{(0)}$.

(d): $E_1$ is a degenerate equilibrium if $\frac{r_1}{a_1} = \frac{r_2}{a_2}$.

(e): There will be a periodic solution branching from the equilibrium $E_1$ when $\tau = \tau_1^{(k)}$. The periodic solution is unstable if $\frac{r_1}{a_1} > \frac{r_2}{a_2}$; it is not certain whether the periodic solution is stable or not if $\frac{r_1}{a_1} < \frac{r_2}{a_2}$.

When $\tau = 0$, system (2) reduces to (1). Llibre and Xiao [20] observed that $E_1$ is a stable node of (1) if

$$\frac{r_1}{a_1} < r_3 K \leq \frac{r_1 + \sqrt{r_1^2 + r_1 r_3}}{2a_1} \quad \text{and} \quad \frac{r_1}{a_1} < \frac{r_2}{a_2}.$$
$E_1$ is a saddle of (1) with a 2-dimensional stable manifold and a 1-dimensional unstable manifold if

$$\frac{r_1}{a_1} < r_3 K \leq \frac{r_1 + \sqrt{r_1^2 + r_1 r_3}}{2 a_1} \text{ and } \frac{r_1}{a_1} > \frac{r_2}{a_2},$$

$E_1$ is a stable focus of (1) if

$$r_3 K > \frac{r_1 + \sqrt{r_1^2 + r_1 r_3}}{2 a_1} \text{ and } \frac{r_1}{a_1} < \frac{r_2}{a_2},$$

$E_1$ is a saddle-focus of (1) with a 2-dimensional stable manifold and a 1-dimensional unstable manifold if

$$r_3 K > \frac{r_1 + \sqrt{r_1^2 + r_1 r_3}}{2 a_1} \text{ and } \frac{r_1}{a_1} > \frac{r_2}{a_2},$$

and $E_1$ is a degenerate equilibrium of (1) if

$$\frac{r_1}{a_1} < r_3 K \text{ and } \frac{r_1}{a_1} = \frac{r_2}{a_2}.$$

Obviously, this observation coincides with Theorem 3.2.

Next, we consider the boundary equilibrium $E_2 (\frac{r_2}{a_2}, 0, \frac{1}{a_2} (r_3 K - \frac{r_2}{a_2}))$ under the assumption that $r_3 K > \frac{r_2}{a_2}$. Note that the characteristic equation $\det(\lambda I_3 - J_{\tau}(E_2)) = 0$ can be written as

$$\left(\lambda + r_1 - a_1 \frac{r_2}{a_2}\right) \left[\lambda^2 + \frac{r_2 \lambda}{a_2 K} + \frac{r_2}{K} \left(r_3 K - \frac{r_2}{a_2}\right) e^{-\lambda \tau}\right] = 0. \quad (14)$$

It is obvious that equation (14) has a solution $\lambda = -r_1 + a_1 \frac{r_2}{a_2}$ and the other solutions $\lambda$ of equation (14) satisfy

$$\tilde{P}_\tau(\lambda) := \lambda^2 + \frac{r_2 \lambda}{a_2 K} + \frac{r_2}{K} \left(r_3 K - \frac{r_2}{a_2}\right) e^{-\lambda \tau} = 0. \quad (15)$$

It follows from $r_3 K > \frac{r_2}{a_2}$ that all real solutions of equation (15) are negative. Substituting $\lambda = i \omega$ with $\omega > 0$ into $\tilde{P}_\tau(\lambda) = 0$ and then separating real and imaginary parts of the equation, we have

$$\omega^4 + \frac{r_2^2 \omega^2}{a_2^2 K^2} - \left(r_2 r_3 - \frac{r_2^2}{a_2 K}\right)^2 = 0. \quad (16)$$

Solving equation (16) for $\omega$ yields $\omega = \omega_2$, where

$$\omega_2 = \frac{\sqrt{2}}{2} \left\{-\frac{r_2^2}{a_2^2 K^2} + \left[\left(\frac{r_2}{a_2 K}\right)^4 + 4 \left(r_2 r_3 - \frac{r_2^2}{a_2 K}\right)^2\right]^\frac{1}{2}\right\}^\frac{1}{2} > 0.$$

Hence, equation (15) may have a pair of pure imaginary roots $\pm i \omega_2$. Moreover, solving $\tilde{P}_\tau(i \omega_2) = 0$ for $\tau$ yields

$$\tau = r_2^{(k)} := \frac{1}{\omega_2} \arctan \frac{r_2}{a_2 K \omega_2} + \frac{k \pi}{\omega_2} \quad (k = 0, 1, 2, \ldots). \quad (17)$$

On differentiating the equation (15) with respect to $\tau$, we have

$$\frac{d\lambda}{d\tau} = \frac{a_1 \lambda e^{-\lambda \tau}}{b + 2 \lambda - a \tau e^{-\lambda \tau}}.$$
where \( a = r_2 r_3 - \frac{r_2^2}{a_2 K} \) and \( b = \frac{r_2}{a_2 K} \). Therefore
\[
\text{Re} \left\{ \frac{d\lambda}{d\tau} \right\}_{\lambda = i\omega_2, \tau = \tau_2^{(k)}} = \text{Re} \left\{ \frac{i a \omega_2 e^{-i \omega_2 \tau_2^{(k)}}}{b + 2i \omega_2 - a \tau_2^{(k)} e^{-i \omega_2 \tau_2^{(k)}}} \right\}_{\lambda = i\omega_2, \tau = \tau_2^{(k)}} = \frac{b^2 \omega_1^2 + 2 \omega_1^4}{|b - a \tau_2^{(k)} e^{-i \omega_2 \tau_2^{(k)}}|^2 + [2 \omega_2 + a \tau_2^{(k)} \sin(\omega_2 \tau_2^{(k)})]^2} > 0.
\]

In particular, for each fixed \( \tau \in [0, \tau_2^{(0)}) \), all solutions to the equation \( \tilde{P}_r(\cdot) = 0 \) have negative real parts and hence we have the following results.

**Theorem 3.3.** When \( r_3 K > \frac{r_2}{a_2} \)

(a): \( E_2 \) is stable if \( \frac{r_1}{a_1} > \frac{r_2}{a_2} \) and \( \tau \in [0, \tau_2^{(0)}) \); and is unstable if \( \frac{r_1}{a_1} > \frac{r_2}{a_2} \) and \( \tau > \tau_2^{(0)} \).

(b): \( E_2 \) has a 1-dimensional unstable manifold if \( \frac{r_1}{a_1} < \frac{r_2}{a_2} \) and \( \tau \in [0, \tau_2^{(0)}) \);

(c): \( E_2 \) has an unstable manifold of dimension at least 3 if \( \frac{r_1}{a_1} < \frac{r_2}{a_2} \) and \( \tau > \tau_2^{(0)} \);

(d): \( E_2 \) is a degenerate equilibrium if \( \frac{r_1}{a_1} = \frac{r_2}{a_2} \) and \( \tau = \tau_2^{(k)} \).

(e): There will be a periodic solution branching from the equilibrium \( E_2 \) when \( \tau = \tau_2^{(k)} \). The periodic solution is unstable if \( \frac{r_1}{a_1} < \frac{r_2}{a_2} \); it’s not certain whether the periodic solution is stable or not if \( \frac{r_1}{a_1} > \frac{r_2}{a_2} \).

Llibre and Xiao [20] observed that \( E_2 \) is a stable node of system (1) if

\[
\frac{r_2}{a_2} < r_3 K \leq \frac{r_2 + \sqrt{r_2^2 + r_2 r_3}}{2 a_2} \quad \text{and} \quad \frac{r_1}{a_1} > \frac{r_2}{a_2},
\]

\( E_2 \) is a saddle of system (1) with a 2-dimensional stable manifold and a 1-dimensional unstable manifold if

\[
\frac{r_2}{a_2} < r_3 K \leq \frac{r_2 + \sqrt{r_2^2 + r_2 r_3}}{2 a_2} \quad \text{and} \quad \frac{r_1}{a_1} < \frac{r_2}{a_2},
\]

\( E_2 \) is a stable focus of system (1) if

\[
r_3 K > \frac{r_2 + \sqrt{r_2^2 + r_2 r_3}}{2 a_2} \quad \text{and} \quad \frac{r_1}{a_1} > \frac{r_2}{a_2},
\]

\( E_2 \) is a saddle-focus of system (1) with a 2-dimensional stable manifold and a 1-dimensional unstable manifold if

\[
r_3 K > \frac{r_2 + \sqrt{r_2^2 + r_2 r_3}}{2 a_2} \quad \text{and} \quad \frac{r_1}{a_1} < \frac{r_2}{a_2},
\]

and \( E_2 \) is a degenerate equilibrium if

\[
\frac{r_2}{a_2} < r_3 K \quad \text{and} \quad \frac{r_1}{a_1} = \frac{r_2}{a_2},
\]

similarly, this observation coincides with Theorem 3.3.

Finally, we consider the positive equilibrium \( E_+(S, x_1, x_2) \) under the assumption that \( \frac{r_1}{a_1} = \frac{r_2}{a_2} \) and \( r_3 K > \frac{r_1}{a_1} \). Note that the characteristic equation \( \det(\lambda M_{d_3} - J_r(E_+)) = 0 \) is

\[
\lambda \left[ \lambda^2 + \frac{r_1}{a_1 K} \lambda + \left( r_1 b_1 \bar{x}_1 + r_2 b_2 \bar{x}_2 \right) e^{-\lambda r} \right] = 0. \tag{18}
\]
It is obvious that characteristic equation (18) has a solution λ = 0 and the other solutions of (18) satisfy
\[ \hat{P}_r(\lambda) := \lambda^2 + \frac{r_1}{a_1 K} \lambda + (r_1 b_1 \bar{x}_1 + r_2 b_2 \bar{x}_2) e^{-\lambda \tau} = 0. \] (19)
Note that \( r_1 b_1 \bar{x}_1 + r_2 b_2 \bar{x}_2 > 0 \), then we conclude that all real solutions of equation (19) are negative. Substituting \( \lambda = i \omega \) with \( \omega > 0 \) into \( \hat{P}_r(\lambda) = 0 \) yields
\[ -\omega^2 + \frac{i \omega r_1}{a_1 K} + (r_1 b_1 \bar{x}_1 + r_2 b_2 \bar{x}_2) e^{-i \omega \tau} = 0, \]
and hence
\[ \omega^4 + \frac{r_1^2 \omega^2}{a_1^2 K^2} - (r_1 b_1 \bar{x}_1 + r_2 b_2 \bar{x}_2)^2 = 0. \] (20)
Solving equation (20) for \( \omega \) yields \( \omega = \omega_+ \), where
\[ \omega_+ = \frac{\sqrt{2}}{2} \left\{ - \frac{r_1^2}{a_1^2 K^2} + \left[ \left( \frac{r_1}{a_1 K} \right)^4 + 4 (r_1 b_1 \bar{x}_1 + r_2 b_2 \bar{x}_2)^2 \right] \right\}^{\frac{1}{2}} > 0. \]
Hence, equation (19) may have a pair of pure imaginary roots \( \pm i \omega_+ \). Moreover, solving \( \hat{P}_r(i \omega_+) = 0 \) for \( \tau \) yields
\[ \tau = \tau_+^{(k)} = \frac{1}{\omega_+} \arctan \frac{r_1}{a_1 K \omega_+} + \frac{k \pi}{\omega_+} \quad (k = 0, 1, 2, \ldots). \] (21)
On differentiating the equation (19) with respect to \( \tau \), we have
\[ \frac{d\lambda}{d\tau} = \frac{a \lambda e^{-\lambda \tau}}{b + 2 \lambda - a \tau e^{-\lambda \tau}}, \]
where \( a = r_1 b_1 \bar{x}_1 + r_2 b_2 \bar{x}_2 \) and \( b = \frac{r_1}{a_1 K} \). Therefore
\[ \text{Re} \left\{ \frac{d\lambda}{d\tau} \bigg|_{\lambda = i \omega_+, \tau = \tau_+^{(k)}} \right\} = \text{Re} \left\{ \frac{i a \omega_+ e^{-i \omega_+ \tau_+^{(k)}}}{b + 2 \omega_+ - a \tau_+ e^{-i \omega_+ \tau_+^{(k)}}} \right\} \]
\[ = \frac{b^2 \omega_+^2 + 2 \omega_+^4}{[b - a \tau_+ \cos(\omega_+ \tau_+)]^2 + [2 \omega_+ + a \tau_+ \sin(\omega_+ \tau_+)]^2} > 0. \]
So the positive equilibrium of system (2) undergoes a Hopf bifurcation at the critical value \( \tau = \tau_+^{(k)} \). In particular, for each fixed \( \tau \in [0, \tau_+^{(0)}) \), all solutions to the equation \( \hat{P}_r(\cdot) = 0 \) have negative real parts and hence that we have the following results.

**Theorem 3.4.** Each positive equilibrium \( E_+ \) of system (2) is a degenerate equilibrium with a 1-dimensional center manifold filled by equilibria. Moreover, \( E_+ \) has a co-dimensional one stable manifold when \( \tau \in [0, \tau_+^{(0)}) \), and has an unstable manifold of dimension at least 2 when \( \tau > \tau_+^{(0)} \).

Llibre and Xiao [20] observed that each positive equilibrium \( E_+ \) of system (2) with \( \tau = 0 \) is a degenerate equilibrium with a 2-dimensional stable manifold and a 1-dimensional center manifold filled by equilibria. This coincides with Theorem 3.4. Moreover, Theorem 3.4 means that the presence of time delay \( \tau \) may cause some nonlinear oscillations and lead to a completely different computational performance. Thus, the time delay \( \tau \) can be regarded as a source of instability and oscillatory response of system (2).
4. Regions of attraction. This section is devoted to seeking for regions of attraction for system (2). Llibre and Xiao [20] observed that

(i): If \( r_3 K > \frac{r_1}{a_1} = \frac{r_2}{a_2} \), then system (2) with \( \tau = 0 \) has infinitely many positive equilibria \( E_+(S, x_1, x_2) \) filling up a segment \( L \) which attract all solutions of system (2) with \( \tau = 0 \) and positive initial conditions, the end of \( L \) attracts all solutions of system (2) with \( \tau = 0 \) and nonnegative initial conditions \((S, x_1, 0)\) or \((S, 0, x_2)\), respectively, and the origin \((0, 0, 0)\) attracts all solutions of system (2) with \( \tau = 0 \) and nonnegative initial conditions \((0, x_1, x_2)\).

(ii): If \( r_3 K \leq \min\{\frac{r_1}{a_1}, \frac{r_2}{a_2}\} \), then system (2) with \( \tau = 0 \) has only two equilibria \((0, 0, 0)\) and \((r_3 K - \frac{r_1}{a_1}, 0, 0)\), the equilibrium \((r_3 K - \frac{r_1}{a_1}, 0, 0)\) attracts all solutions of system (2) with \( \tau = 0 \) expect the orbits in the \( x_1 x_2 \)-plane, and the equilibrium \((0, 0, 0)\) attracts all solutions of system (2) with \( \tau = 0 \) in the \( x_1 x_2 \)-plane.

(iii): If \( \frac{r_1}{a_1} > \frac{r_2}{a_2} \), then all orbits of system (2) with \( \tau = 0 \) in \( \text{Int}(\mathbb{R}_+^3) \) are asymptotic to the orbits on the \( S x_2 \)-plane in forward time.

(iv): If \( \frac{r_1}{a_1} < \frac{r_2}{a_2} \), then all orbits of system (2) with \( \tau = 0 \) in \( \text{Int}(\mathbb{R}_+^3) \) are asymptotic to the orbits on the \( S x_1 \)-plane in forward time.

If \( r_3 K > \frac{r_1}{a_1} = \frac{r_2}{a_2} \), then system (2) has infinitely many positive equilibria \( E_+(S, x_1, x_2) \) which fill up the segment \( L \). And system (2) can be transformed into the following system

\[
\begin{align*}
\frac{dS(t)}{dt} &= S(t) \left[ r_3 - \frac{S(t)}{K} - b_1 x_1(t) - b_2 c x_1(t)^{a_2/a_1} \right], \\
\frac{dx_1(t)}{dt} &= x_1(t) [-r_1 + a_1 S(t - \tau)],
\end{align*}
\]

where \( c = x_2(0)x_1(0)^{-a_2/a_1} \) depends on the initial values of \( x_1 \) and \( x_2 \). Hence the study of the dynamics of system (2) in \( \text{Int}(\mathbb{R}_+^3) \) is equivalent to that of system (22) restricted on each foliation \( c = x_2x_1^{-a_2/a_1} \) of \( \text{Int}(\mathbb{R}_+^3) \), where \( c > 0 \).

In what follows, we consider the restricted system (22). It is easy to see that system (22) has only one positive equilibrium \( E^*(S^*, x^*_1) \), where \( S^* = \frac{r_2}{a_2} \) and \( x^*_1 \) satisfies

\[
b_1 x^*_1 + b_2 c x^*_1^{a_2/a_1} = r_3 - \frac{r_1}{a_1 K}.
\]

If we define

\[
u_1 = \frac{S - S^*}{S^*}, \quad u_2 = \frac{x_1 - x^*_1}{x^*_1},
\]

then system (22) can be rewritten as

\[
\begin{align*}
\frac{du_1(t)}{dt} &= -(1 + u_1(t)) \left[ S^* u_1(t) + b_1 x^*_1 u_2(t) + b_2 c x^*_1^{a_2/a_1} ((u_2(t) + 1)^{a_2/a_1} - 1) \right], \\
\frac{du_2(t)}{dt} &= a_1 S^* (1 + u_2(t)) u_1(t - \tau).
\end{align*}
\]

Thus, the positive equilibrium \( E^*(s^*, x^*_1) \) of system (22) corresponds to the trivial solution of system (24). Let \( u_3(t) = (u_2(t) + 1)^{a_2/a_1} - 1 \), then we have

\[
\frac{du_3(t)}{dt} = a_2 S^* (1 + u_3(t)) u_1(t - \tau).
\]
Combining (24) and (25), we get
\[
\begin{align*}
\frac{du_1(t)}{dt} &= -(1 + u_1(t))(Au_1(t) + Bu_2(t) + Cu_3(t)), \\
\frac{du_2(t)}{dt} &= D(1 + u_2(t))u_1(t - \tau), \\
\frac{du_3(t)}{dt} &= E(1 + u_3(t))u_1(t - \tau),
\end{align*}
\]
(26)
where \( A = \frac{S^*}{K}, \ B = b_1x_1, \ C = b_2cx_1^{\frac{p-2}{p}}, \ D = a_1S^*, \) and \( E = a_2S^*. \) Consider the following scalar function \( V_0(t), \)
\[
V_0(t) = \ln(1 + u_1(t)) + \alpha \ln(1 + u_2(t)) + \beta \ln(1 + u_3(t)),
\]
(27)
where \( \alpha \) and \( \beta \) are positive constants whose values are to be determined later. For convenience, let
\[
z_i \equiv z_i(u_i(t)) = \ln(1 + u_i(t)), \quad i = 1, 2, 3.
\]
(28)
It is easy to see that
\[
\frac{d}{dt} \left( \frac{1}{2} V_0^2(t) \right) = (z_1 + \alpha z_2 + \beta z_3)(-Au_1 - Bu_2 - Cu_3 + \alpha Du_1(t - \tau) + \beta Eu_1(t - \tau)).
\]
Note that
\[
u_1(t - \tau) = u_1(t) - \int_{t-\tau}^{t} u_1'(s)ds,
\]
then we have
\[
\frac{d}{dt} \left( \frac{1}{2} V_0^2(t) \right) = -Az_1u_1 - Bz_1u_2 + \alpha Du_1 z_1 + \beta Eu_1 z_1 - Cz_1 u_3 - \alpha A z_2 u_1 - \alpha B z_2 u_2 - \alpha C z_2 u_3 + \alpha^2 D u_1 z_2 + \alpha \beta Eu_1 z_2 - \beta A z_3 u_1 - \beta B z_3 u_2 - \beta C z_3 u_3 + \beta^2 E u_1 z_3 - (z_1 + \alpha z_2 + \beta z_3)(\alpha D + \beta E) \int_{t-\tau}^{t} u_1'(s)ds.
\]
Choosing \( \alpha D + \beta E = A, \) we have
\[
\frac{d}{dt} \left( \frac{1}{2} V_0^2(t) \right) = -Bz_1 u_2 - Cz_1 u_3 - \alpha B z_2 u_2 - \alpha C z_2 u_3 - \beta B z_3 u_2 - \beta C z_3 u_3 - A(z_1 + \alpha z_2 + \beta z_3) \int_{t-\tau}^{t} u_1'(s)ds.
\]
(29)
Next, we construct
\[
V_1(t) = u_1(t) - z_1 + \gamma(u_2 - z_2) + \eta(u_3 - z_3),
\]
(30)
where \( \gamma \) and \( \eta \) are positive constant to be determined later. Here we state \( V_1 \) is a positive define function of \((u_1, u_2, u_3)\). According to the previous symbol denotation \( u_1 = \frac{S-S^*}{S} \) and \( z_i = \ln(1 + u_i), \ i = 1, 2, 3, \) it is clear that \( u_1 > -1 \) (because of \((S, x_1, x_2)\) in \( \text{Int}(\mathbb{R}^+))\) and \( V_1(u_1, u_2, u_3) = u_1 - \ln(1 + u_1) + \gamma(u_2 - \ln(1 + u_2)) + \eta(u_3 - \ln(1 + u_3)). \) Let \( v_1(u_1) = u_1 - \ln(1 + u_1), \) then \( v_1'(u_1) = 1 + u_1 \) and \( v_1''(u_1) = \frac{1}{(1 + u_1)^2} > 0, \) so \( u_1 = 0 \) is the minimum point. Hence \( v_1'(u_1) \geq v_1'(0) = 0, \) the equal is hold only if \( u_1 = 0, \) it means \( u_1 - \ln(1 + u_1) \) is positive definite. For the second and third terms of the function \( V_1, \) there have the same explanation.
Therefore, the function $V_1$ is positive definite.

The derivative of the function $V_1$ with respect to $t$ is

$$V_1'(t) = -Au_1^2 - Bu_2 - Cu_1 u_3 + \gamma Du_2 u_1 + \eta E u_3 u_1$$

$$- (\gamma Du_2 + \eta E u_3) \int_{t-\tau}^{t} u_1'(s)ds.$$  

Hence we have

$$\begin{align*}
\frac{d}{dt} \left( \frac{1}{2} V_0^2(t) + \varsigma V_1(t) \right) \\
= -\alpha Bz_2 u_2 - \beta C z_3 u_3 - \varsigma Au_1^2 + [\varsigma(\gamma D - B)u_1 - Bz_1 - \beta B z_3] u_2 \\
+ [\varsigma(\eta E - C)u_1 - Cz_1 - \alpha C z_2] u_3 \\
+ \frac{1}{S^*} A(z_1 + \alpha z_2 + \beta z_3) \int_{t-\tau}^{t} S(s)[Au_1(s) + Bu_2(s) + Cu_3(s)]ds \\
+ \frac{1}{S^*} \varsigma(\gamma Du_2 + \eta E u_3) \int_{t-\tau}^{t} S(s)[Au_1(s) + Bu_2(s) + Cu_3(s)]ds.
\end{align*}$$  

(31)

Observe that

$$z_1 \int_{t-\tau}^{t} S(s)u_1(s)ds \leq \frac{1}{2} z_1^2 \tau + \frac{1}{2} \int_{t-\tau}^{t} S^2(s)u_1^2(s)ds,$$

then by similar manipulations for other integral terms, we obtain

$$\begin{align*}
\frac{1}{S^*} A(z_1 + \alpha z_2 + \beta z_3) \int_{t-\tau}^{t} S(s)[Au_1(s) + Bu_2(s) + Cu_3(s)]ds \\
+ \frac{1}{S^*} \varsigma(\gamma Du_2 + \eta E u_3) \int_{t-\tau}^{t} S(s)[Au_1(s) + Bu_2(s) + Cu_3(s)]ds \\
\leq \frac{1}{2S^*}[A(A + B + C)z_1^2 \tau + \alpha A(A + B + C)z_2^2 \tau + \beta A(A + B + C)z_3^2 \tau \\
+ \varsigma \gamma D(A + B + C)u_2^2 \tau + \varsigma \eta E(A + B + C)u_3^2 \tau] + P \int_{t-\tau}^{t} S^2(s)u_1^2(s)ds \\
+ Q \int_{t-\tau}^{t} S^2(s)u_2^2(s)ds + R \int_{t-\tau}^{t} S^2(s)u_3^2(s)ds,
\end{align*}$$

where

$$\begin{align*}
P & \equiv \frac{1}{2S^*} A(A + \alpha A + \beta A + \varsigma \gamma D + \varsigma \eta E), \\
Q & \equiv \frac{1}{2S^*} B(A + \alpha A + \beta A + \varsigma \gamma D + \varsigma \eta E), \\
R & \equiv \frac{1}{2S^*} C(A + \alpha A + \beta A + \varsigma \gamma D + \varsigma \eta E).
\end{align*}$$  

(32)
Therefore

\[
\frac{d}{dt} \left( \frac{1}{2} V_2^2(t) + \varsigma V_1(t) \right) \\
\leq -\alpha Bz u_2 - \beta Cz u_3 - \varsigma Au^2 + \varsigma (\gamma D - B)u_1 - Bz_1 - \beta Bz_3 u_2 \\
+ [\varsigma (\eta E - C)u_1 - Cz_1 - \alpha Cz_2]u_3 + \frac{1}{2S^*} A(A + B + C)z_1^2\tau \\
+ \frac{1}{2S^*} \alpha A(A + B + C)z_2^2\tau + \frac{1}{2S^*} \beta A(A + B + C)z_3^2\tau \\
+ \frac{1}{2S^*} \gamma D(A + B + C)u_2^2\tau + \frac{1}{2S^*} \varsigma \eta E(A + B + C)u_3^2\tau \\
+ P \int_{t-\tau}^t S^2(s)u_1^2(s)ds + Q \int_{t-\tau}^t S^2(s)u_2^2(s)ds + R \int_{t-\tau}^t S^2(s)u_3^2(s)ds.
\]

(33)

In order to have some negative definite expression in the right-hand side of the above inequality, we need to find ways to control the last three integral terms. Consider

\[
V_2(t) \equiv P \int_{t-\tau}^t \int_{s-\tau}^s S^2(v)u_1^2(v)dvds + Q \int_{t-\tau}^t \int_{s-\tau}^s S^2(v)u_2^2(v)dvds \\
+ R \int_{t-\tau}^t \int_{s-\tau}^s S^2(v)u_3^2(v)dvds.
\]

(34)

We have

\[
V_2'(t) = PS^2(t)u_1^2(t)\tau - P \int_{t-\tau}^t S^2(s)u_1^2(s)ds + QS^2(t)u_2^2(t)\tau \\
- Q \int_{t-\tau}^t S^2(s)u_2^2(s)ds + RS^2(t)u_3^2(t)\tau - R \int_{t-\tau}^t S^2(s)u_3^2(s)ds.
\]

(35)

We now define a Liapunov functional \( V \) as

\[
V(t) = \frac{1}{2} V_2^2(t) + \varsigma V_1(t) + V_2(t).
\]

(36)

Then, by combining (33) and (35), we obtain

\[
V'(t) \leq -\alpha Bz u_2 - \beta Cz u_3 - \varsigma Au^2 + [\varsigma (\gamma D - B)u_1 - Bz_1 - \beta Bz_3] u_2 \\
+ [\varsigma (\eta E - C)u_1 - Cz_1 - \alpha Cz_2]u_3 + \frac{1}{2S^*} A(A + B + C)z_1^2\tau \\
+ \frac{1}{2S^*} \alpha A(A + B + C)z_2^2\tau + \frac{1}{2S^*} \beta A(A + B + C)z_3^2\tau \\
+ \frac{1}{2S^*} \gamma D(A + B + C)u_2^2\tau + \frac{1}{2S^*} \varsigma \eta E(A + B + C)u_3^2\tau \\
+ PS^2(t)u_1^2(t)\tau + QS^2(t)u_2^2(t)\tau + RS^2(t)u_3^2(t)\tau.
\]

(37)
Recall that $z_i = \ln(1 + u_i)$ for $i = 1, 2, 3$. We define $\varepsilon_i u_i \equiv \varepsilon_i (u_i) u_i \equiv z_i - u_i$ for $i = 1, 2, 3$. Substituting $z_i = u_i + \varepsilon_i u_i$ and $S = S'(1 + u_1)$ into (37), we obtain

$$V'(t) \leq -\alpha Bu_2(u_2 + \varepsilon_2 u_2) - \beta Cu_3(u_3 + \varepsilon_3 u_3) - \zeta Au_1^2$$

$$+ \zeta(\gamma D - B)u_1 u_2 - Bu_2(u_1 + \varepsilon_1 u_1) - \beta Bu_2(u_3 + \varepsilon_3 u_3)$$

$$+ \zeta(\eta E - C)u_1 u_3 - Cu_3(u_1 + \varepsilon_1 u_1) - \alpha Cu_3(u_2 + \varepsilon_2 u_2)$$

$$+ \frac{1}{2S^*} A(A + B + C)((u_1 + \varepsilon_1 u_1)^2 \tau + \alpha(u_2 + \varepsilon_2 u_2)^2 \tau + \beta(u_3 + \varepsilon_3 u_3)^2 \tau]$$

$$+ \frac{1}{2S^*} \zeta(A + B + C)[\gamma Du_2^2 \tau + \eta Eu_3^2 \tau]$$

$$+ S^*[P u_1^2 \tau + Q u_2^2 \tau + R u_3^2 \tau](1 + u_1)^2,$$

that is,

$$V'(t) \leq -\left\{ \zeta A - \frac{1}{2S^*} \left[ A(A + B + C) + 2P S^* \right] \tau \right\} u_1^2$$

$$- \left\{ \alpha B - \frac{1}{2S^*} \left[ \alpha A(A + B + C) + \zeta \gamma D(A + B + C) + 2QS^* \right] \tau \right\} u_2^2$$

$$- \left\{ \beta C - \frac{1}{2S^*} \left[ \beta A(A + B + C) + \zeta \eta E(A + B + C) + 2RS^* \right] \tau \right\} u_3^2$$

$$+ [\zeta(\gamma D - B) - B]u_1 u_2 + [\zeta(\eta E - C) - C]u_1 u_3 - (\alpha C + \beta B)u_2 u_3$$

$$- B\varepsilon_1 u_1 u_2 - C\varepsilon_1 u_1 u_3 - \alpha C\varepsilon_2 u_2 u_3 - \beta B\varepsilon_3 u_2 u_3 - \alpha B\varepsilon_2 u_2^2 - \beta C\varepsilon_3 u_3^2$$

$$+ \frac{1}{2S^*} A(A + B + C)(2\varepsilon_1 + \varepsilon_2^2)\tau u_1^2 + \frac{1}{2S^*} \alpha A(A + B + C)(2\varepsilon_2 + \varepsilon_3^2)\tau u_2^2$$

$$+ \frac{1}{2S^*} \beta A(A + B + C)(2\varepsilon_3 + \varepsilon_3^2)\tau u_3^2$$

$$+ PS^*(u_1 + u_1^2)\tau u_1^2 + QS^*(u_1 + u_1^2)\tau u_2^2 + RS^*(u_1 + u_1^2)\tau u_3^2.$$

Notice that the first three terms of (38) have nothing to do with $\varepsilon_1$, $\varepsilon_2$, and $\varepsilon_3$ and that when $u_1$, $u_2$, $u_3$ are very small, $\varepsilon_1$, $\varepsilon_2$, and $\varepsilon_3$ are also very small. So, in order to have the above expression negative definite for at least small values of $u_1$, $u_2$ and $u_3$, it is highly desirable that the coefficients of the first three terms be as negative as possible.

Recall that $\alpha D + \beta E = A$, $\gamma$, $\eta$ and $\zeta$ are yet to be determined. An obvious choice is to eliminate the coefficients of $u_1 u_2$ and $u_1 u_3$ by choosing suitable values of $\gamma$, $\eta$, and $\zeta$. This can be done easily by using values

$$\frac{\gamma}{\eta} = \frac{BE}{CD}, \quad \zeta = \frac{B}{\gamma D - B},$$

where $\gamma > \frac{B}{D}$ and $\eta > \frac{C}{D}$.

In order to have negative coefficients for the first three terms of (38), we must have $\tau$ smaller than a threshold value $\bar{\tau}$ which is defined as

$$\bar{\tau} = \min \left\{ \frac{2\zeta S^* A}{A(A + B + C) + 2P S^*}, \frac{2\zeta S^* B}{(\alpha A + \zeta \gamma D)(A + B + C) + 2Q S^*}, \frac{2\beta S^* C}{(\beta A + \zeta \eta E)(A + B + C) + 2R S^*} \right\}.$$
Substituting the expressions for $P$, $Q$ and $R$ in (32) into the above equation, we have

$$
\bar{\tau} = \min \left\{ \frac{2\varsigma S^* A}{A(A + B + C) + A(A + \alpha A + \beta A + \varsigma \gamma D + \varsigma \eta E)S^*^2}, \frac{2\alpha S^* B}{(\alpha A + \varsigma \gamma D)(A + B + C) + B(A + \alpha A + \beta A + \varsigma \gamma D + \varsigma \eta E)S^*^2}, \frac{2\beta S^* C}{(\beta A + \varsigma \eta E)(A + B + C) + C(A + \alpha A + \beta A + \varsigma \gamma D + \varsigma \eta E)S^*^2} \right\}.
$$

Clearly, the ideal choice of $\alpha$ and $\gamma$ is to maximize the value of $\bar{\tau}$, provided that $0 \leq \alpha \leq \frac{A}{B}$ and $\gamma > \frac{B}{A}$. First, we fix $\alpha$ as a constant, consider $\gamma$ as the only one variable. Then Lemma 2.3 can be applied here and an explicit and unique choice of $\gamma^*$ can be made to realize the maximum value of $\bar{\tau}$, which we denote by $\hat{\tau}$. Notice that $\hat{\tau}$ depends on the value of $\alpha$, then we regard $\hat{\tau}$ as a function of $\alpha$, denoted by $\hat{\tau}(\alpha)$. Obviously, $\hat{\tau}(\alpha)$ is a continuous function. Moreover $\hat{\tau}(\alpha)$ for $\alpha \in [0, \frac{A}{B}]$ can take the maximum $\tau^*$. In the rest of this paper, we assume $\alpha = \alpha^*$, $\beta = \beta^*$, $\gamma = \gamma^*$, $\eta = \eta^*$, and $\varsigma = \varsigma^*$ in the expression of $V$ in (32) and (36).

By taking advantage of the value $\tau^*$, we have from (38)

$$
V'(t) \leq -\varsigma A(1 - \frac{\tau^*}{\tau})u_1^2 - \alpha^* B(1 - \frac{\tau^*}{\tau})u_2^2 - \varsigma^* C(1 - \frac{\tau^*}{\tau})u_3^2 - (\alpha C + \beta B)u_2u_3 - B\varepsilon_1 u_1 u_2 - C\varepsilon_1 u_1 u_3 - \alpha^* B\varepsilon_2 u_2^2 - \varsigma^* C\varepsilon_2 u_2 u_3 - \beta B\varepsilon_3 u_3^2 - \beta C\varepsilon_3 u_3^2 + \frac{1}{2S^*} A(A + C + B + C)(2\varepsilon_1 + \varepsilon_2^2 + \varepsilon_2^3 - \varepsilon_1^2)\tau u_2^2 + \frac{1}{2S^*} \beta^* A(A + B + C)(2\varepsilon_3 + \varepsilon_2^3 - \varepsilon_1^2)\tau u_3^2 + PS^*^2(2u_1 + u_1^2)\tau u_1^2 + QS^*^2(2u_1 + u_1^2)\tau u_2^2 + RS^*^2(2u_1 + u_1^2)\tau u_3^2 + PS^*^2(2u_1 + u_1^2)\tau u_1^2 + QS^*^2(2u_1 + u_1^2)\tau u_2^2 + RS^*^2(2u_1 + u_1^2)\tau u_3^2.
$$

We are now ready to state and prove our main result in a general form. As usual, $\|u_0\| \equiv \max\{|u_1(\cdot)|, |u_2(0)|, |u_3(0)|\}$.

**Theorem 4.1.** Assume that $\tau < \tau^*$ then there is an explicitly expressible positive constant $\delta$ such that the solution of $(u_1(t), u_2(t), u_3(t))$ of (26) with initial value $u_0$ satisfying $\|u_0\| < \delta$ tends to $(0, 0, 0)$ as $t \to \infty$. Equivalently, the solution $(S(t), x_1(t))$ of the original system (22) tends to $(S^*, x_1^*)$ as $t \to \infty$.

The following result is essentially a corollary of Theorem 4.1.

**Theorem 4.2.** Assume that $\tau < \tau^*$ in system (22). Let $\delta_0$ be defined as in (42),

$$
\Delta = \frac{5}{25} \varsigma^* \min \{1, \gamma^*, \eta^*\}
$$

and

$$
\delta = \min \left\{ \Delta^{\frac{3}{2}} \delta_0, \frac{3}{5} \right\}
$$

Then $\| S_0 - S^* \| < S^* \delta$ and $| x_1(0) - x_1^* | < x_1^* \delta$ implies that $\lim_{t \to +\infty} (S(t), x_1(t)) = (S^*, x_1^*)$. 

Now, we first prove Theorem 4.1. Using the inequality $2u_1u_2 \leq u_1^2 + u_2^2$ and (40), we obtain

$$V'(t) \leq \left[-c^*A(1 - \frac{\tau}{\tau^*}) + \frac{1}{2S^*} A(A + B + C)(2\varepsilon_1 + \varepsilon_1^2)\tau \right] u_1^2$$

$$+ \left[\frac{B + C}{2} \varepsilon_1 + PS^{\tau^*^2}(2u_1 + u_1^2)\tau \right] u_1^2$$

$$+ \left[-\alpha^* B(1 - \frac{\tau}{\tau^*}) + \frac{B}{2} \varepsilon_1 + \alpha^* B\varepsilon_2 + \frac{\alpha^* C}{2} \varepsilon_2 + \frac{\beta^* B}{2} \varepsilon_3 \right] u_1^2$$

$$+ \left[\frac{1}{2S^*} \alpha^* A(A + B + C)(2\varepsilon_2 + \varepsilon_2^2)\tau + QS^{\tau^*^2}(2u_1 + u_1^2)\tau \right] u_2^2$$

$$+ \left[-\beta^* C(1 - \frac{\tau}{\tau^*}) \varepsilon_1 + \frac{C}{2} \varepsilon_1 + \frac{\alpha^* C}{2} \varepsilon_2 + \frac{\beta^* B}{2} \varepsilon_3 + \beta^* C\varepsilon_3 \right] u_2^2$$

$$+ \left[\frac{1}{2S^*} \beta^* A(A + B + C)(2\varepsilon_3 + \varepsilon_3^2)\tau + RS^{\tau^*^2}(2u_1 + u_1^2)\tau \right] u_3^2.$$

It is easy to show that

$$|\ln(1 + u) - u| \leq |u|^2$$ if $$|u| < \frac{3}{5}.$$ Assume below that $|u_i| < \frac{3}{5},$ then $|\varepsilon_1| \leq |u_i|,$ $|\varepsilon_2| \leq |u_i|,$ and $u_i^2 \leq |u_i|.$ Let $||u|| = \max\{|u_1|, |u_2|, |u_3|\},$ then we obtain

$$V'(t) \leq -\Delta_1 u_1^2 - \Delta_2 u_2^2 - \Delta_3 u_3^2,$$ (41)

where

$$\Delta_1 = c^* A(1 - \frac{\tau}{\tau^*}) - \left[\frac{B + C}{2} + \frac{3}{2S^*} A(A + B + C)\tau + 3PS^{\tau^*^2}\tau \right] ||u||,$$

$$\Delta_2 = \alpha^* B(1 - \frac{\tau}{\tau^*})$$

$$- \left[\frac{B + \alpha^* C + \beta^* B + 2\alpha^* B}{2} + \frac{3}{2S^*} \alpha^* A(A + B + C)\tau + 3QS^{\tau^*^2}\tau \right] ||u||,$$

$$\Delta_3 = \beta^* C(1 - \frac{\tau}{\tau^*})$$

$$- \left[\frac{C + \alpha^* C + \beta^* B + 2\beta^* C}{2} + \frac{3}{2S^*} \beta^* A(A + B + C)\tau + 3RS^{\tau^*^2}\tau \right] ||u||.$$  

Let

$$\delta_0 = \min \left\{ \frac{2c^* S^* A(1 - \tau/\tau^*)}{S^*(B + C) + 3A(A + B + C) + 6PS^{\tau^*^2}\tau}, \right.$$

$$\frac{2\alpha^* S^* B(1 - \tau/\tau^*)}{S^*(B + \alpha^* C + \beta^* B + 2\alpha^* B) + 3A(A + B + C)\tau + 6QS^{\tau^*^2}\tau},$$

$$\frac{2\beta^* S^* C(1 - \tau/\tau^*)}{S^*(\alpha^* C + \beta^* B + C + 2\beta^* C) + 3\beta^* A(A + B + C)\tau + 6RS^{\tau^*^2}\tau} \right\}. \quad (42)$$

Then we see that $|u(t)| < \delta_0$ for $t \geq 0$ implies that $\Delta_1 > 0,$ $\Delta_2 > 0,$ and $\Delta_3 > 0,$ and hence that $-\Delta_1 u_1^2 - \Delta_2 u_2^2 - \Delta_3 u_3^2$ is negative definite. Lemma 2.2 ensures that $\lim_{t \to +\infty} u_i(t) = 0,$ $i = 1, 2, 3,$ which implies that $\lim_{t \to +\infty} S(t) = S^*$ and $\lim_{t \to +\infty} x_i(t) = x_i.$ Therefore, to complete the proof, we need to find $\delta$ such that
\[ \| u(0) \| < \delta \] then implies that \( \| u(t) \| < \delta \) for all \( t \geq 0 \). For this purpose, we define
\[ L = \min \{ \frac{1}{2} V_0^2 + \zeta^* V_1 : \| u \| = \delta \}, \]
and set
\[ \mathcal{S} = \left\{ \begin{array}{l}
\phi_1 \in C([\tau, 0], \mathbb{R}), \phi_2(\theta) \equiv \phi_2(0), \phi_3(\theta) \equiv \phi_3(0) \\
\phi = (\phi_1, \phi_2, \phi_3) : \theta \in [\tau, 0], \max\{|\phi_1|, |\phi_2(0)|, |\phi_3(0)|\} < \delta, \\
\text{and } V(\phi_1, \phi_2, \phi_3) < L \end{array} \right\}, \]
where \( |\phi_1| = \max_{\theta \in [\tau, 0]} |\phi_1(\theta)| \). We claim that for initial value chosen from \( \mathcal{S} \), we have \( \| u(t) \| < \delta \) for all \( t \geq 0 \). Otherwise, there is a \( t_0 > 0 \), such that \( \| u(t_0) \| = \delta \) and \( \| u(t) < \delta \| \) for \( t \in [0, t_0) \). Obviously,
\[ V(u(t_0)) \geq \frac{1}{2} V_0^2 + \zeta^* V_1 \geq L. \]
However, for \( t \in [0, t_0) \), we have \( \| u(t) \| < \delta \), and hence \( V'(u(t)) \leq 0 \), which implies that for \( t \in [0, t_0) \) we must have
\[ V(u(t)) \leq V(u(0)) < L. \]
By the continuity of \( V \), we can get
\[ V(u(t_0)) \leq V(u(0)) < L, \]
which contradicts \( V(u(t_0)) \geq \frac{1}{2} V_0^2 + \zeta^* V_1 \geq L \). This proves our claim. Since \( V \) is continuous, clearly there is a \( 0 < \delta < \delta_0 \) such that
\[ \mathcal{S}_\delta = \left\{ \phi = (\phi_1, \phi_2, \phi_3) : \phi_1 \in C([\tau, 0], \mathbb{R}), \phi_2(\theta) \equiv \phi_2(0), \phi_3(\theta) \equiv \phi_3(0), \theta \in [\tau, 0], \max\{|\phi_1|, |\phi_2(0)|, |\phi_3(0)|\} < \delta \right\} \subset \mathcal{S}. \]
This is a desired value for \( \delta \) in our theorem, and hence completes the proof of Theorem 4.1.

Finally, we prove Theorem 4.2. Since \( \delta \leq \frac{3}{5} \), for every \( \phi \) satisfying \( \| \phi \| \leq \frac{3}{5} \), we have
\[ |\ln(1 + \phi_i(0))| \leq |\phi_i(0)| \leq \frac{8}{5}|\phi_i(0)|, \quad i = 1, 2, 3, \]
and
\[ \frac{5}{16}|\phi_i(0)|^2 \leq |\phi_i(0)| - \ln(1 + \phi_i(0)) \leq \frac{5}{4}|\phi_i(0)|^2, \quad i = 1, 2, 3. \]
Also \( |S(0)| = S^*|1 + \phi_1(0)| < \frac{5}{2} S^* \). Hence, we obtain
\[ V(\phi_1, \phi_2, \phi_3) \leq \frac{1}{2} \left[ \frac{8}{5} (1 + \alpha^* + \beta^*) \| \phi \|^2 + \frac{5}{4} \zeta^* (1 + \gamma^* + \eta^*) \| \phi \|^2 \right] + \frac{32}{25} (P + Q + R) \tau^2 S^* \| \phi \|^2, \]
and the value \( L \) satisfies
\[ L > \frac{5}{16} \zeta^* \min\{1, \gamma^*, \eta^*\} \delta_0^2. \]
Hence if
\[ \| \phi \|^2 < \frac{\frac{5}{16} \zeta^* \min\{1, \gamma^*, \eta^*\} \delta_0^2}{\left( \frac{32}{25} (1 + \alpha^* + \beta^*)^2 + (P + Q + R) \tau^2 S^* \right) + \frac{5}{4} \zeta^* (1 + \gamma^* + \eta^*)} = \Delta \delta_0^2, \]
then \( \phi \in \mathcal{S} \). Clearly, if we define \( \delta = \min\{\Delta \delta_0^2, \frac{3}{2}\} \), then \( \mathcal{S}_\delta \subset \mathcal{S} \). The proof of Theorem 4.2 is completed.
5. Asymptotic behavior in the absence of positive equilibria. In this section, we analyze the asymptotical behavior of the boundary equilibrium when $\frac{r_1}{a_1} \neq \frac{r_2}{a_2}$.

**Theorem 5.1.** If $r_3 K < \min\{\frac{r_1}{a_1}, \frac{r_2}{a_2}\}$, then the equilibrium $E_0$ of system (2) is globally asymptotically stable.

**Proof.** Constructing a Liapunov functional:

$$V(t) = r_3 K H \left( \frac{S(t)}{r_3 K} \right) + \frac{b_1}{a_1} x_1(t) + \frac{b_2}{a_2} x_2(t) + V_+,$$

where $H(x) = x - 1 - \ln x$, $V_+ = \int_{-\tau}^{0} b_1 x_1(t) S(t + \alpha) d\alpha + \int_{-\tau}^{0} b_2 x_2(t) S(t + \alpha) d\alpha$.

Then $V(t)$ is positive definite, and $V(T) \geq 0$ with equality if and only if $S(t) = S^*$, $x_1(t) = x_2(t) = 0$. The derivative of $V$ along system (2) is given as

$$\frac{dV(t)}{dt} \bigg|_{(2)} = \left( 1 - \frac{r_3 K}{S(t)} \right) S(t) \left[ r_3 - \frac{S(t)}{K} - b_1 x_1(t) - b_2 x_2(t) \right]$$

\[+ \frac{b_1}{a_1} x_1(t) [-r_1 + a_1 S(t - \tau)] + \frac{b_2}{a_2} x_2(t) [-r_2 + a_2 S(t - \tau)]
\[+ b_1 x_1(t) S(t) + b_2 x_2(t) S(t) - b_1 x_1(t) S(t - \tau) - b_2 x_2(t) S(t - \tau)
\[= - \frac{1}{K} (S(t) - r_3 K)^2 - b_1 \left( \frac{r_1}{a_1} - r_3 K \right) x_1(t) - b_2 \left( \frac{r_2}{a_2} - r_3 K \right) x_2(t).

Thus $r_3 K < \min\{\frac{r_1}{a_1}, \frac{r_2}{a_2}\}$ ensures that $\frac{dV(t)}{dt} \bigg|_{(2)} \leq 0$ for all $S > 0$, $X_1 > 0$, $X_2 > 0$. Let $\frac{dV(t)}{dt} \bigg|_{(2)} = 0$ if and only if $(S(t), x_1(t), x_2(t)) = (r_3 K, 0, 0)$. By the LaSalle’s invariance principle [15], we conclude that $E_0$ is indeed globally attractive. In addition, by Theorem 3.1, we know that $E_0$ is locally asymptotically stable. Then $E_0$ is globally asymptotically stable. This proof is completed. \qed

**Theorem 5.2.** (i): If $\frac{r_1}{a_1} > \frac{r_2}{a_2}$, then all orbits of system (2) in $\text{Int}(\mathbb{R}_+^2)$ are asymptotic to the orbits on the $Sx_2$-plane in forward time.

(ii): If $\frac{r_1}{a_1} < \frac{r_2}{a_2}$, then all orbits of system (2) in $\text{Int}(\mathbb{R}_+^2)$ are asymptotic to the orbits on the $Sx_1$-plane in forward time.

**Proof.** It follows from system (2) that

$$x_1(t) = x_1(0) e^{-r_1 t} e^{\int_0^t S(t - \tau) d\tau},$$

and

$$x_2(t) = x_2(0) e^{-r_2 t} e^{\int_0^t S(t - \tau) d\tau}.$$  

Combining (43) and (44), we obtain

$$\frac{x_2(0)}{x_1(0)} = \frac{x_2(0)}{x_1(0)} e^{(a_2 r_1 - a_1 r_2) t}.$$

It is obvious that if $\frac{r_1}{a_1} < \frac{r_2}{a_2}$, then $\lim_{t \to +\infty} \frac{x_2(t)}{x_1(t)} = 0$. Due to the fact that $x_1(t)$ is bounded, we have $\lim_{t \to +\infty} |x_2(t)|^{a_1} = 0$, which means that $\lim_{t \to +\infty} X_2(t) = 0$. Similarly, if $\frac{r_2}{a_2} < \frac{r_1}{a_1}$, then $\lim_{t \to +\infty} x_1(t) = 0$. The proof is completed. \qed

**Remark 1.** In view of Theorem 5.2, we see that one of the two predators will become extinct when $\frac{r_1}{a_1} \neq \frac{r_2}{a_2}$. Thus, system (2) becomes system (1.1) in [2].
Consider the following 2-dimensional system
\[
\begin{align*}
\frac{dS(t)}{dt} &= S(t) \left[ r_3 - \frac{S(t)}{K} - bx(t) \right], \\
\frac{dx(t)}{dt} &= x(t) \left[-r + aS(t - \tau)\right].
\end{align*}
\]  
(45)

System (45) always has the boundary equilibria \(O(0,0)\) and \(E_0(r_3K,0)\). If system (45) has no positive equilibria, then the equilibrium \(E_0\) attracts all solutions of system (45) expect the orbits in the S-axis, and the equilibrium \(O\) attracts all solutions of system (45) in the S-axis. If \(r_3K > \frac{5}{\alpha}\), system (45) has positive equilibrium \(E_+((\frac{5}{\alpha}, \frac{1}{\alpha}r_3K - \frac{5}{\alpha}))\). Beretta and Kuang [2] provided a detailed and explicit procedure of obtaining some regions of attraction for the positive steady state \(E_+\) of system (45). This coincides with Theorem 4.1.

6. An illustrating example. We present below a simple example to illustrate the procedures of applying our results and to gain a better understanding of the magnitude of \(\delta\) in Theorem 4.1. Consider a special case of (2) with \(r_3 = 5, K = 1, b_1 = b_2 = 1, r_1 = 2, a_1 = 1, r_2 = 4, a_2 = 2\). Namely, we consider the following system
\[
\begin{align*}
\frac{dS(t)}{dt} &= S(t) \left[ 5 - S(t) - x_1(t) - x_2(t) \right], \\
\frac{dx_1(t)}{dt} &= x_1(t) \left[-2 + S(t - \tau)\right], \\
\frac{dx_2(t)}{dt} &= x_2(t) \left[-4 + 2S(t - \tau)\right],
\end{align*}
\]  
(46)

which has a positive equilibrium \(E_+(S^*, x_1^*, x_2^*) = (2, 1, 2)\). After making the change of variable \(S = 2(1 + u_1), x_1 = 1 + u_2, x_2 = 2(1 + u_3)\), we obtain a special case of (26) with \(A = C = D = 2, B = 1, E = 4\). This, together with (32) and (39), implies that \(\alpha + 2\beta = 1, \gamma = \eta\) and \(Q = \frac{1}{2}(1 + \alpha + \beta + \gamma + 2\gamma), P = R = 2Q\), and
\[
\bar{\tau} = \min \left\{ \frac{4\gamma}{5 + 8(1 + \alpha + \beta + \gamma + 2\gamma)}, \frac{2\alpha}{5(\alpha + \gamma) + 4(1 + \alpha + \beta + \gamma + 2\gamma)} \right\},
\]
and \(\gamma > \frac{1}{2}, \zeta = \frac{1}{\sqrt{\tau}}\). Following the proof of Lemma 2.3, \(\bar{\tau}\) can attain its maximum value \(\tau^* = 0.0035\) when \(\alpha^* = 0.345, \beta^* = 0.3275, \gamma^* = \eta^* = 2.1623, \zeta^* = 0.3008\). This in turn leads to \(Q = 1.8118, P = R = 3.6237\). Hence the \(\delta_0\) defined by (42) is
\[
\delta_0 = \min \left\{ \frac{2.4063(1 - \tau/\tau^*)}{36 + 173.9377\tau}, \frac{1.38(1 - \tau/\tau^*)}{5.415 + 97.3185\tau}, \frac{2.62(1 - \tau/\tau^*)}{8.655 + 183.762\tau} \right\}.
\]
If \(\tau\) is very small, then the first number is smaller and hence will be taken as the value of \(\delta_0\). After some simple computations, we have the value of \(\Delta\) as follows
\[
\Delta = \frac{0.0939975}{5.58246819 + 46.3832\tau^2}.
\]
We assume that \(\tau = \tau^*/2\), in which case we have \(\delta_0 = 0.0331\), and \(\Delta = 0.1683\). This will give an approximate value of \(\delta\) is 0.0136.

Next, we present some numerical simulations of the reduced system of system (46) to support and supplement the above analytic results.
In Figure 1, we selected two groups initial value that conforms with the stability condition in Theorem 4.2. This result of numerical simulation accords with Theorems 4.1 and 4.2. In Figure 2, the two groups of initial values do not meet the stability condition in Theorems 4.1 and 4.2, however, we can see the convergence is still established in the larger range, that is, δ have a lot of room to improve. In the later study, we will explore a better method to expand δ.

In section 3, we know the positive equilibrium of system (2) undergoes Hopf bifurcation at the critical value $\tau = \tau^{(k)}_+$. For system (46), the calculation results of the value $\tau_+^{(0)}$ in equation (21) is 0.212. Under the Matlab to draw the waveform plots of the reduced system of (46) at different value of $\tau$, see Figure 3, which supports some part of the results in section 3.
7. Discussion. In this paper, we consider just two predators competing for one prey. It is natural to ask what happens to the case where there are three predators. Namely, it is interesting to consider the following system

\[
\begin{align*}
\frac{dS(t)}{dt} &= S(t) \left[ r - \frac{S(t)}{K} - b_1 x_1(t) - b_2 x_2(t) - b_3 x_3(t) \right], \\
\frac{dx_1(t)}{dt} &= x_1(t) [-r_1 + a_1 S(t - \tau)], \\
\frac{dx_2(t)}{dt} &= x_2(t) [-r_2 + a_2 S(t - \tau)], \\
\frac{dx_3(t)}{dt} &= x_3(t) [-r_3 + a_3 S(t - \tau)].
\end{align*}
\]

(47)

Obviously, system (47) has always has two equilibria \( O(0, 0, 0) \) and \( E_0(rK, 0, 0, 0) \). However, the existence of the other equilibria is conditional. In fact, it is not difficult to obtain the following result.

**Theorem 7.1.** For system (47), the following statements hold.

(i): \( E_1 \left( \frac{r_1}{a_1}, 0, 1, 0 \right) \) exists if \( rK > \frac{r_1}{a_1} \).

(ii): \( E_2 \left( \frac{r_2}{a_2}, 0, 0, 1 \right) \) exists if \( rK > \frac{r_2}{a_2} \).

(iii): \( E_3 \left( \frac{r_3}{a_3}, 0, 0, 1 \right) \) exists if \( rK > \frac{r_3}{a_3} \).

(iv): \( E_4 \left( \frac{r_4}{a_4}, x_1, x_2, 0 \right) \) exists if \( rK > \frac{r_4}{a_4} = \frac{a_2}{a_1} \), where \( b_1 x_1 + b_2 x_2 = r - \frac{r_1}{K a_1} \).

(v): \( E_5 \left( \frac{r_5}{a_5}, x_1, 0, x_3 \right) \) exists if \( rK > \frac{r_5}{a_5} = \frac{a_3}{a_1} \), where \( b_1 x_1 + b_3 x_3 = r - \frac{r_1}{K a_1} \).

(vi): \( E_6 \left( \frac{r_6}{a_6}, 0, x_2, x_3 \right) \) exists if \( rK > \frac{r_6}{a_6} = \frac{a_3}{a_2} \), where \( b_2 x_2 + b_3 x_3 = r - \frac{r_2}{K a_2} \).

(vii): \( E_7 \left( \frac{r_7}{a_7}, x_1, x_2, x_3 \right) \) exists if \( rK > \frac{r_7}{a_7} = \frac{a_3}{a_1} = \frac{a_2}{a_3} = \frac{a_1}{a_4} \), where \( b_1 x_1 + b_2 x_2 + b_3 x_3 = r - \frac{r_3}{K a_3} \).

Both Lemma 2.1 and Theorem 5.2 are also applicable to system (47). So, there is either a definite function relationship or a competitive exclusion principle among these predators.
Furthermore, imagine there are $n$ predators competing for one prey.

\[
\begin{aligned}
\frac{dS(t)}{dt} &= S(t) \left[ r - \frac{S(t)}{K} - \sum_{i=1}^{n} b_i x_i(t) \right], \\
\frac{dx_i(t)}{dt} &= x_i(t) \left[ -r_i + a_i S(t - \tau) \right], \quad i = 1, 2, ..., n.
\end{aligned}
\] (48)

Sorting $\frac{r_1}{a_1}, \frac{r_2}{a_2}, ..., \frac{r_n}{a_n}$ from small to large, and writing them as:

\[
\frac{r(1)}{a(1)} \leq \frac{r(2)}{a(2)} \leq \cdots \leq \frac{r(n)}{a(n)}.
\]

Assuming the former $m$ terms equal, we have

\[
\frac{r(1)}{a(1)} = \cdots = \frac{r(m)}{a(m)} < \frac{r(m+1)}{a(m+1)} \leq \cdots \leq \frac{r(n)}{a(n)},
\]

and hence, as time goes on, the predators $x(m+1), ..., x(n)$ will become extinct while predators $x(1), x(2), ..., x(m)$ live together with a certain function relation.

In order to study the final state of the solutions of system (48), we can reduce system (48) to the following 2-dimensional system

\[
\begin{aligned}
\frac{dS(t)}{dt} &= S(t) \left[ r - \frac{S(t)}{K} - b_{(1)} x_{(1)}(t) - F(x_{(1)}(t)) \right], \\
\frac{dx_{(1)}(t)}{dt} &= x_{(1)}(t) \left[ -r_{(1)} + a_{(1)} S(t - \tau) \right],
\end{aligned}
\] (49)

where $F(x_{(1)}(t))$ is determined by $a_{(1)}$ and the initial value of $x_{(1)}(0)$ ($i = 1, 2, \ldots, m$). As for the detailed investigation of system (49), we don’t study in-depth here. It’s left to be revealed in the future.

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