Complex Classical Mechanics of a QES Potential

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(Received March 30, 2015; revised manuscript received July 16, 2015)

Abstract We study a combined parity (P) and time reversal (T) invariant non-Hermitian quasi-exactly solvable (QES) potential, which exhibits PT phase transition, in the complex plane classically to demonstrate different quantum effects. The particle with real energy makes closed orbits around one of the periodic wells of the complex potential depending on the initial condition. However interestingly the particle escapes to an open orbit even with real energy if it is placed beyond a certain distance from the center of the well. On the other hand when the particle energy is complex the trajectory is open and the particle tunnels back and forth between two wells which are separated by a classically forbidden path. The tunneling time is calculated for different pair of wells and is shown to vary inversely with the imaginary component of energy. Our study reveals that spontaneous PT symmetry breaking does not affect the qualitative features of the particle trajectories in the analogous complex classical model.

PACS numbers: 03.65.Xp, 03.65.Ca, 11.30.Er, 02.30.Fn, 05.45.-a

Key words: PT symmetric complex systems, quasi-exactly solvable systems, tunneling in classical systems, open orbits, classical systems

1 Introduction

Consistent quantum theories with unitary time evolution and probabilistic interpretation for certain classes of non-Hermitian systems have been the subject of intensive research in the frontier physics over the last one and half decades.\textsuperscript{11–3} The huge success of complex quantum theory\textsuperscript{14–19} has lead to its extension to many other branches of physics.\textsuperscript{20–28} In particular, its application to quantum optics is the most exciting, where break down of PT-symmetry has been observed experimentally.\textsuperscript{23–24}

More recently, this formulation has been extended to the classical systems.\textsuperscript{29–33} Quantum mechanics and classical mechanics are two completely different theories and provide profoundly dissimilar description of physical systems. However, Bohr’s correspondence principle states that at very high quantum number limit quantum particle behaves like classical one. Correspondence between quantum and classical systems becomes more pronounced in the complex domain.\textsuperscript{29–30} This motivates the study of classical systems on the complex plane and/or with complex energy to grasp various quantum behaviors. Remarkably, a classical particle with complex energy exhibits tunneling like behavior which is usually realized in the quantum domain. This tunneling behavior of a classical particle with complex energy is well demonstrated in Refs. [29–30] It has been shown that a classical particle can tunnel from one classically allowed region to another allowed region separated by a classically forbidden path. Several other works in this field are devoted to study the nature of the trajectory of a classical particle with complex energy.\textsuperscript{30–34} Attempts have been made to study the effect of spontaneous PT-symmetry breaking in the particle trajectories of analogous complex classical models. However the studies of complex classical systems are only restricted to a few models.

The purpose of the present article is to extend these works further to investigate the different behaviors of a complex classical system. We have chosen a complex classical system corresponding to a QES,\textsuperscript{35–38} non-Hermitian PT-symmetric system which is described by the potential $V(x) = -(\zeta \cosh 2x - i M)^2$. By allowing the real variable $x$ to become complex, $z = x + iy$ we numerically study the dynamics of a classical particle moving in the complex $xy$-plane subjected to this potential $V(z)$ which consists of periodic wells situated to the left and right side of the imaginary axis to demonstrate the quantum tunneling effect and the trajectory of the particle at various situations. With real energy, the particle makes closed orbits around one of the wells depending on the initial condition. However, surprisingly the particle has open orbits even with real energy if it is initially placed in a certain region between the two wells on the same side of the imaginary axis. On the other hand when the particle energy is complex, the trajectory is open and the particle tunnels back and forth between two wells which are separated by a classically forbidden path. We do not observe

\textsuperscript{∗}Support from Department of Science and Technology (DST), Govt. of India under SERC Project Sanction Grant No. SR/S2/HEP-0009/2012
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any qualitative differences in the features of the particle dynamics corresponding to PT symmetry broken ($M$ even with any $\zeta$ or odd $M$ with $\zeta$ greater than a critical value ($\zeta_c$)) and unbroken ($M$ odd with $\zeta \leq \zeta_c$) situations at the quantum level. The tunneling times are calculated for different pairs of wells and is shown to vary inversely with the imaginary component of energy, similar to a situation which is usually valid in the quantum domain.

Now we present the plan of the paper. The QES potential and its solutions for broken and unbroken PT symmetry cases are reviewed in Sec. 2. We discuss the classical mechanics of this potential in a complex plane and obtain the trajectories of the classical particle in different situations in Sec. 3. Variation of tunneling time with imaginary part of the energy of the particle is discussed in Sec. 4, and Sec. 5 is kept for conclusions.

2 The QES System

The QES system is described by the Hamiltonian

$$H = p^2 - (\zeta \cosh 2x - iM)^2,$$

where $\zeta$ is an arbitrary real parameter and $M$ is an integer and we have considered $2m = h = 1$ for simplicity. This Hamiltonian is PT symmetric, where parity transformation in this particular system is taken in the general form as $x \rightarrow i(\pi/2) - x$. This system is shown to be a QES system. The first $M$ bound state energy levels along with the corresponding eigenfunctions are calculated exactly for any specific integer value of $M$. Furthermore, this QES system shows another remarkable property relevant to PT symmetric non-Hermitian system, i.e., for even values of $M$, all the eigenvalues are complex for any arbitrary value of $\zeta$. On the other hand, for odd $M$, all the eigenvalues are real if $\zeta \leq \zeta_c$. In other words, system is always in broken PT phase for even values of $M$, and shows PT phase transition when $M$ is odd. The energy eigenvalues for this system is calculated using Bender–Dunne (BD) polynomial methods. The zeros of BD polynomials give the energy eigenvalues for the QES system. Some of the low lying levels are listed as follows.

$$M = 2,$$

$$E_\pm = 3 - \zeta^2 \pm 2i\zeta, \quad \phi_+ = A \cosh x, \quad \phi_- = B \sinh x;$$

$$M = 4,$$

$$E_{\pm}^1 = 11 - \zeta^2 - 2i\zeta \pm \sqrt{1 - i\zeta - \zeta^2},$$

$$\phi_{\pm}^1 = A \sinh 3x + B \sinh x; \quad \text{with} \quad A = \frac{E - 7 + \zeta^2}{2i\zeta}.$$  

We have PT phase transition when, $M$ is odd

$$M = 1,$$

$$E = 1 - \zeta^2, \quad \phi = A \quad \text{(constant)};$$

$$M = 3,$$

$$E_{\pm} = 7 - \zeta^2 \pm \sqrt{1 - 4\zeta^2}, \quad \text{with} \quad \phi_{\pm} = A \cosh 2x + iB;$$

$$E = 5 - \zeta^2, \quad \text{with} \quad \phi = C \sinh 2x;$$

$$\text{with} \quad A = \frac{4\zeta}{E - 9 + \zeta^2}. \quad (3)$$

Note that for $M = 3$, energy eigenvalues are real subject to the condition $\zeta \leq 1/2$ i.e., $\zeta_c = 1/2$ and the wave functions are also eigenstates of the PT operator. PT phase transition occurs at $\zeta = \zeta_c$. This conclusion continues to be valid for higher odd values of $M$. \[5\]

3 Classical Mechanics in the Complex Plane

A classical particle with real energy $E$ is not allowed to travel in the region where the potential energy $V(x) > E$. However, this restriction is relaxed when we consider the particle in a complex plane with complex energy $E_1 + iE_2$ as

$$E_1 = (p_1^2 - p_2^2) + V_1, \quad E_2 = 2p_1p_2 + V_2, \quad (4)$$

where complex momentum $p = p_1 + ip_2$ and the complex potential $V = V_1 + iV_2$. Now since $p_1$ and $p_2$ can have any value from $-\infty$ to $\infty$ there is no restriction as such on the particle to be bounded in a particular region of space. The particle is allowed to move anywhere in the complex plane as long as the equations in (4) are satisfied. This is the prime reason for a classical particle with complex energy to travel through classically forbidden regions. However, even though the particle can exist anywhere in the complex plane, it prefers the region with lower value of $|E|$. The particle follows a definite trajectory depending on the initial conditions. Open trajectories with local random walk type motion have been shown for a classical particle with complex energy. On the other hand a classical particle with real energy generally has been shown to move in a closed orbit in the complex plane. \[32\]

We consider the system of a classical particle moving under the influence of the above QES potential in a complex plane. The complexified potential $V(z) = -(\zeta \cosh 2z - iM)^2$ in the complex plane $z = x + iy$ is shown in Fig. 1. The potential consists of a series of wells on both left and right side of the imaginary axis. The wells are positioned at $y = [(4n - 1)/4]\pi$ and at $y = [(4n + 1)/4]\pi$, on the left and right of the imaginary axis respectively where $n$ is an integer and serves as a label for the wells. The Hermitian counterpart of this potential $V(x) = -(\zeta \cosh 2x - M)^2$ for different values of $M$ with a fixed value of $\zeta(=0.1)$ is plotted in Fig. 2. The central barrier height increases with $M$ values. We consider a particle in this complex QES potential. The dynamics of this particle are governed by the Hamilton’s equation as

$$\ddot{x} = \frac{\partial H}{\partial p}, \quad -\dot{p} = \frac{\partial H}{\partial x}. \quad (5)$$

We find the trajectories in the complex plane for different complex energies by solving the Eq. (5) for the above QES potential numerically. For practical realization we have chosen the units of physical quantities as, energy in units of $1.0545 \times 10^{-34}$ Joules, mass in units of $10^{-3}$ gm, time in units of 1 sec and length in units of $10^{-9}$ meter in the numerical study of this QES system.
Fig. 1  (a) The magnitude of potential in the complex plane for \( \zeta = 0.1 \) and \( M = 3 \). The potential wells corresponding to the real energy orbits are distributed periodically, centered at \( x = 2.04750 \), \( y = [(4n + 1)/4]\pi \) on the right, and \( x = -2.04750 \), \( y = [(4n - 1)/4]\pi \) on the left of the imaginary axis; (b) Indicates the positions of the various wells.

Fig. 2  (Color online) The double well potential for real \( x \) and its variation with the parameter \( M \) for a fixed \( \zeta = 0.1 \).

Our numerical study reveals that the particle orbits around the position of the different wells if the energy of the particle is real and the particle is placed sufficiently close to the well. However, the particle has open orbits even with real energy if placed beyond a certain length from the center of the well. This has been demonstrated in Figs. 3–6. Figures 3 and 4 show both close and open trajectories with real energy for the PT unbroken (i.e. \( M = 1, 3, 5 \) with \( \zeta < \zeta_c \)) situation. The boundaries between open and close trajectories are also shown clearly. Similarly Figs. 5 and 6 show the trajectories of the particle with real energy for PT broken (\( M = 2, 4 \) with any \( \zeta \) and \( M = 3, 5 \) with \( \zeta > \zeta_c \)) case. It is clear from these graphs that overall nature of the trajectories are same for PT symmetry broken and unbroken situations.

Fig. 3  (Color online) Trajectories of the particle for the real energy \( E = 1 \) in the potential with \( \zeta = 0.1 \) and \( M = 1 \) (a), \( M = 3 \) (b). The trajectories start from different distances from the center of the well corresponding to \( n = 0 \) on the left side of the imaginary axis. The initial positions were picked to be at a distance \( (y) \) from the well center in the direction of the imaginary axis.

If we vary the distance of the starting point of the particle from the well along the imaginary axis \( (y\text{-axis}) \), the closed orbit opens up after a certain value. For the first well \( (n = 0) \) on the left of imaginary axis if particle is placed at a \( y \) with \( (n_0 - 0.52988875) < y < (n_0 + 0.52988875) \) we have closed orbits of the particle. \( n_0 = -\pi/4 \) is the position of first well along \( y \) direction. However if the particle is not placed sufficiently close, i.e.
$y \geq (n_0 + 0.529\;888\;75)$ or $y \leq (n_0 - 0.529\;888\;75)$ to the first well, the particle moves in an open orbit even with real energy and escapes to infinity without tunneling to any other well. We have illustrated this for the first ($n = 0$) and second well ($n = 1$) on the left side of the imaginary axis. The second well is positioned at $y = (3\pi/4)(\equiv n_1)$ along imaginary axis. The particle has open orbit to escape to infinity if it is placed at $y < (n_1 - 0.529\;888\;75)$ or $y > (n_1 + 0.529\;888\;75)$. The limiting distances ($y$) from the center of $n = 0$ well for open orbits with real energy $E = 1$ are tabulated in Table 1 for various values of the parameters $M$ and $\zeta$. We summarize the interesting observations as, (i) The particle tunnels back and forth between the wells to the left and right of the imaginary axis. (ii) When placed inside a well, the particle spirals out until it enters the classically forbidden region outside the well, then depending on the direction of its velocity, it spirals into the first well (on the other side of the imaginary axis) that it encounters in its path. The separation of the wells between which the tunneling takes place increases with $M$ values and also depends on the complex energy and on the parameter $\zeta$. This spiraling nature was also shown in Refs. [31–32].
Fig. 7 The trajectory of a particle with energy $E = 1 + i$ and for $\zeta = 0.1$. (a) When $M = 2$, the particle oscillates between the wells corresponding to $n = +3$ on the right and $n = -3$ on the left of the imaginary axis; (b) When $M = 3$, the particle oscillates between the wells corresponding to $n = +10$ on the left and $n = -10$ on the right of the imaginary axis.

Fig. 8 The trajectory of a particle with energy $E = 1 + i$ in the potential with $\zeta = 0.1$. (a) When $M = 4$, the particle oscillates between the wells corresponding to $n = +21$ on the right and $n = -21$ on the left of the imaginary axis; (b) When $M = 5$, the particle oscillates between the wells corresponding to $n = +35$ on the right and $n = -35$ on the left of the imaginary axis.

Fig. 9 The particle spiraling outwards from the well at $n = 0$ on the left side of the imaginary axis. (a) For $E_2 > 0$; (b) For $E_2 < 0$.

(iii) For $E_2 > 0$ we observe the particle to spiral in clockwise direction and spiral out in anti-clockwise direction. The situation reverses for $E_2 < 0$. The outward spiraling of the particle from the well at $n = 0$ is shown in Fig. 9 both for $E_2 > 0$ and $E_2 < 0$. (iv) The positions of the wells between which particles tunnels back and forth depend on the initial position of the particle. We have tried 3 different initial positions, (a) Placing the particle in the well on the left side, (b) placing the particle in the well on the right side, and (c) placing the particle at the
origin. All the three initial positions give qualitatively the same result. The only difference is that if the particle is placed at the origin, and it is found to tunnel between two wells \( n = -a \) on the left and \( n = +a \) on the right, then placing the particle initially in the well corresponding to \( n = -b \) on the left will make the particle tunnel between the wells \( n = -b + 2a \) on the right. Similarly, if the particle is initially placed in the well \( n = +b \) on the right, then it will tunnel between \( n = +b \) on the right and \( n = +b - 2a \) on the left. For simplicity, we have used origin as the starting point.

| Value of \( \zeta \) | Value of \( M \) | Distance (\( y \)) from the center of the \( n = 0 \) well |
|-----------------------|-----------------|-------------------------------------------------|
| \( 0.1 \)             | \( 1 \)         | \( 0.70 \)                                      |
| \( 0.1 \)             | \( 2 \)         | \( 0.62 \)                                      |
| \( 0.1 \)             | \( 3 \)         | \( 0.53 \)                                      |
| \( 0.1 \)             | \( 4 \)         | \( 0.43 \)                                      |
| \( 0.1 \)             | \( 5 \)         | \( 0.31 \)                                      |
| \( 1.0 \)             | \( 1 \)         | \( 0.78 \)                                      |
| \( 1.0 \)             | \( 2 \)         | \( 0.77 \)                                      |
| \( 1.0 \)             | \( 3 \)         | \( 0.76 \)                                      |
| \( 1.0 \)             | \( 4 \)         | \( 0.75 \)                                      |
| \( 1.0 \)             | \( 5 \)         | \( 0.74 \)                                      |

Fig. 10 The trajectory of a particle with energy \( E = 1 + i \) in the potential with \( \zeta = 1 \). (a) When \( M = 2 \), the particle oscillates between the wells corresponding to \( n = +3 \) on the right and \( n = -3 \) on the left of the imaginary axis; (b) When \( M = 3 \), the particle oscillates between the wells corresponding to \( n = +6 \) on the right and \( n = -6 \) on the left of the imaginary axis.

Fig. 11 The trajectory of a particle with energy \( E = 1 + i \) in the potential with \( \zeta = 1 \). (a) For \( M = 4 \), the particle oscillates between the wells corresponding to \( n = +11 \) on the right and \( n = -11 \) on the left of the imaginary axis; (b) For \( M = 5 \), the particle oscillates between the wells corresponding to \( n = +19 \) on the right and \( n = -19 \) on the left of the imaginary axis.

(v) We observe that particle with real energy has open orbit depending on the initial position of the particle for both broken (\( M \) even with any \( \zeta \) and \( M \) odd with \( \zeta > \zeta_c \)) and unbroken (\( M \) odd and \( \zeta < \zeta_c \)) situations. This is clearly demonstrated in Figs. 3–6. The possible reason of open orbit is explained towards the end of Sec. 4.

(vi) We numerical study both broken and unbroken PT symmetric regimes of the QES potential and find no qualitative differences in the trajectories of the particle in the broken and unbroken regimes of the potential classi-
cally unlike the analogous situations in the quantum theory. This contradicts the claims and speculations in some of the existing literature\[^{31}\] where irregular trajectories with/without crossing points and distorted behavior had been reported for the PT broken phase. However this is not very surprising as our results support the views of Ref. [34].

4 Tunneling Time

| Imaginary Component of Energy Im($E$) | Tunneling time |
|--------------------------------------|---------------|
| 0.3                                  | 54.19         |
| 0.5                                  | 32.42         |
| 0.8                                  | 20.1          |
| 1.0                                  | 15.99         |
| 1.2                                  | 13.14         |
| 1.5                                  | 10.42         |
| 1.7                                  | 9.203         |
| 2.0                                  | 7.739         |
| 2.2                                  | 7.008         |
| 2.5                                  | 6.097         |
| 2.7                                  | 5.635         |
| 3.0                                  | 5.054         |
| 3.2                                  | 4.745         |
| 3.5                                  | 4.329         |
| 3.7                                  | 4.058         |
| 4.0                                  | 3.76          |
| 4.2                                  | 3.601         |
| 4.5                                  | 3.355         |
| 4.7                                  | 3.22          |
| 5.0                                  | 3.025         |
| 5.2                                  | 2.902         |
| 5.5                                  | 2.763         |
| 5.7                                  | 2.673         |
| 6.0                                  | 2.541         |
| 6.2                                  | 2.47          |
| 6.5                                  | 2.375         |
| 6.7                                  | 2.313         |

Fig. 12 (Color online) The variation of tunneling time with the imaginary part of energy for a particle initially placed in a potential well corresponding to $n = 0$ of the PT symmetric potential with ($M = 3, \zeta = 0.1$) on the right side of the imaginary axis. The real part of energy is fixed as 1 unit.

5 Conclusion

We have studied the complex classical mechanics of a system whose non-Hermitian PT-invariant version is a QES system. We have treated this model classically on a complex plane to capture some of the strange behavior of the system. We have found that the particle tunnels back and forth between two wells (one on the left side and other on the right of the imaginary axis). Positions of the wells between which it tunnels back and forth depend on the values of the parameters $M, \zeta$ and the imaginary part of $E$. The distance between the wells in which the particle tunnels, increases with $M$ and it decreases when $\zeta$ increases. Particle never tunnels between the wells which are located on the same side of the imaginary axis. The time spent by the particle in the left well is different from the time spent in the right well. The tunneling time is inversely proportional to the imaginary part of the energy. The higher value of $E_2$ implies more tunneling effect and in that situation tunneling is between nearer wells. This further is understood by reducing $E_2$ gradually. With very less $E_2$, the particle will tunnel between wells which are very far. For real energy, the particle either stays inside the same well with a closed orbit, or tries to tunnel between two wells, which are infinitely separated in the imaginary direction, resulting in an open orbit. We have not observed any qualitative difference in the overall behavior of the particle dynamics for PT-broken and unbroken situations.

Acknowledgments

One of us (BPM) acknowledges the hospitality of the organizers of PHHPQ11 held at APC, Paris where this work was presented.
