Harmonic Oscillator Lie Bialgebras
and their Quantization

Angel Ballesteros and Francisco J. Herranz

Departamento de Física, Universidad de Burgos
Pza. Misael Bañuelos,
E-09001, Burgos, Spain

Abstract
All possible Lie bialgebra structures on the harmonic oscillator algebra are explicitly derived and it is shown that all of them are of the coboundary type. A non-standard quantum oscillator is introduced as a quantization of a triangular Lie bialgebra, and a universal $R$-matrix linked to this new quantum algebra is presented.

1 Introduction
The theory of Hopf algebra deformations of Universal Enveloping Algebras is, by construction, intimately linked to Lie bialgebras (a detailed exposition can be found in [1]). In fact, once a uniparametric deformation of $Ug$ is given, a unique Lie bialgebra structure $(g, \delta)$ on $g$ is defined via

$$\delta := (\Delta_z - \sigma \circ \Delta_z) \mod z^2,$$  \hspace{1cm} (1)

where $\sigma$ is the permutation map $\sigma (X \otimes Y) = Y \otimes X$, $\Delta_z$ is the deformed coproduct and $z$ the deformation parameter. As it was shown by Drinfel’d [2], the cocommutator $\delta$ characterizes completely the Poisson–Lie group whose quantization is given by the Hopf algebra dual to $U_zg$. Therefore, Lie bialgebra structures on $g$ are the natural candidates to explore the zoo of quantum algebra deformations of a given algebra. Note that $[1]$ means that we have extracted the first order term in $z$ by preserving such a deformation parameter as a multiplicative factor within $\delta$. The reason for this is to open the possibility of considering Lie bialgebras corresponding to multiparameter deformations of $Ug$, that will contain simultaneously various deformation parameters. In this context two natural questions arise:
1. Given a precise Lie algebra, is it possible to obtain explicitly all its Lie bialgebra structures?

2. Once \( \delta \) is given, can its quantum deformation \( U_z g \) be constructed?

Essentially, solving both problems will be equivalent to the obtention of all quantum algebra deformations of \( g \). This paper is mainly devoted to answer in the affirmative both questions for the physically meaningful case of the harmonic oscillator algebra \( h_4 \).

In particular, in Section 2 we explicitly compute all Lie bialgebra structures on \( h_4 \), the main result of the paper being the proof of the coboundary character of all of them. This fact means that the classification of the coboundary Lie bialgebras of \( h_4 \) given in [3] provides a full characterization of all possible quantum (multiparametric) deformations of this algebra. Moreover, all their corresponding coproducts were constructed in [3] by using a link between Lie bialgebras and a general method for the construction of coassociative coproducts on an arbitrary associative algebra [4] (recall that a systematic derivation of quantum oscillator groups was given in [5]).

Finally, in Section 3 we shall concentrate our attention onto a precise non-standard quantization of the oscillator algebra that can be properly called the “Jordanian q-oscillator”. Its Hopf algebra structure will be analysed, and the universal quantum \( R \)-matrix presented for this triangular quantization.

## 2 Harmonic oscillator Lie bialgebras

We begin by recalling that a Lie bialgebra \((g, \delta)\) is a Lie algebra \( g \) endowed with a map \( \delta : g \to g \otimes g \) such that

i) \( \delta \) is a 1-cocycle, i.e.,

\[
\delta([x, y]) = [\delta(x), 1 \otimes y + y \otimes 1] + [1 \otimes x + x \otimes 1, \delta(y)], \quad \forall x, y \in g. \tag{2}
\]

ii) The dual map \( \delta^* : g^* \otimes g^* \to g^* \) is a Lie bracket on \( g^* \).

A Lie bialgebra \((g, \delta)\) is called a coboundary Lie bialgebra if there exists an element \( r \in g \otimes g \) (the classical \( r \)-matrix), such that

\[
\delta(x) = [1 \otimes x + x \otimes 1, r], \quad \forall x \in g. \tag{3}
\]

When the \( r \)-matrix is such that its Schouten bracket vanishes, we shall say that \((g, \delta(r))\) is a non-standard (or triangular) Lie bialgebra. On the contrary, we shall have a standard structure. Finally, two Lie bialgebras \((g, \delta)\) and \((g, \delta')\) are said to be equivalent if there exists an automorphism \( O \) of \( g \) such that \( \delta' = (O \otimes O) \circ \delta \circ O^{-1} \).

Let us now consider the specific case of \( h_4 \), with generators \( \{N, A_+, A_-, M\} \) and commutation rules

\[
[N, A_+] = A_+, \quad [N, A_-] = -A_-, \quad [A_-, A_+] = M, \quad [M, \cdot] = 0. \tag{4}
\]

\(2\)
The most general cocommutator $\delta : h_4 \to h_4 \otimes h_4$ will be a linear combination (with real coefficients)

$$\delta(X_i) = f^{jk}_i X_j \wedge X_k,$$  \hspace{1cm} (5)

of skew-symmetric products of the generators $X_i$ of the algebra. Such a completely general cocommutator has to be computed by firstly imposing the cocycle condition (2) onto an arbitrary expression (6). This leads to the following six-parameter (pre)cocommutator:

$$\delta(N) = a_1 N \wedge A_+ + a_2 N \wedge A_+ + a_5 A_+ \wedge M + a_6 A_- \wedge M,$$

$$\delta(A_+) = a_2 N \wedge M + a_2 A_+ \wedge A_- + a_3 A_+ \wedge M,$$

$$\delta(A_-) = a_1 N \wedge M - a_1 A_+ \wedge A_- + a_4 A_- \wedge M,$$

$$\delta(M) = 0.$$  \hspace{1cm} (6)

Afterwards, Jacobi identity has to be imposed onto $\delta^* : h_4^* \otimes h_4^* \to h_4^*$ in order to guarantee that this map defines a Lie bracket. It is easy to check that this condition leads to the set of equations

$$a_1 a_3 = 0, \hspace{0.5cm} a_1 a_2 = 0, \hspace{0.5cm} a_2 a_4 = 0.$$  \hspace{1cm} (7)

The set of solutions can be splitted into three disjoint classes and they give rise to the following complete classification of Lie bialgebra structures of $h_4$:

Type A bialgebras: $a_1 \neq 0$, $a_2 = 0$ and $a_3 = 0$.

Type B bialgebras: $a_1 = 0$, $a_2 \neq 0$ and $a_4 = 0$.

Type C bialgebras: $a_1 = 0$ and $a_2 = 0$.

Note that, from (6), all these three types are four-parameter families of Lie bialgebras in which the cocommutator of the central generator $M$ always vanishes. Moreover for types A and B, respectively, $\delta(A_+) = 0$ and $\delta(A_-) = 0$.

As we have mentioned in the introduction, the classification of all coboundary Lie bialgebra structures of $h_4$ was developed in [3]. With this in mind, it is immediate to show that the classification just obtained coincides exactly with that of the coboundary structures, i.e., non-coboundary oscillator Lie bialgebras do not exist. In fact, Type A bialgebras coincide with Type $I_+$ ones in [3] under the following identification of the parameters

$$a_1 = \alpha_+ \neq 0, \hspace{0.5cm} a_4 = 2 \vartheta, \hspace{0.5cm} a_5 = \beta_+, \hspace{0.5cm} a_6 = -\beta_-.$$  \hspace{1cm} (8)

The additional condition

$$4 a_1 a_6 + a_4^2 = 0,$$  \hspace{1cm} (9)

gives rise to non-standard (or triangular) Lie bialgebras. The classical $r$-matrix for Type A structures is

$$r_A = a_1 N \wedge A_+ + a_4 (N \wedge M - A_+ \wedge A_-)/2 + a_5 A_+ \wedge M - a_6 A_- \wedge M.$$  \hspace{1cm} (10)

The connection between Type B and Type $I_-$ structures is also straightforward if we put

$$a_2 = -\alpha_- \neq 0, \hspace{0.5cm} a_3 = -2 \vartheta, \hspace{0.5cm} a_5 = \beta_+, \hspace{0.5cm} a_6 = -\beta_-.$$  \hspace{1cm} (11)
and non-standard Lie bialgebras of Type B appear when
\[ 4a_2 a_5 + a_3^2 = 0. \] (12)

The most general classical \( r \)-matrix for this kind of Lie bialgebras is
\[ r_B = -a_2 N \wedge A_- - a_3 (N \wedge M + A_+ \wedge A_-)/2 + a_5 A_+ \wedge M - a_6 A_- \wedge M. \] (13)

This Type B family is easily proven to be equivalent (as a multiparametric Lie bialgebra) to the Type A one by means of the automorphism of \( h_4 \) that interchanges the \( A_+ \) and \( A_- \) generators. Finally, if we write Type C bialgebras by using that
\[ a_3 = -\vartheta - \xi, \quad a_4 = \vartheta - \xi, \quad a_5 = \beta_+, \quad a_6 = -\beta_-, \] (14)
the result is just the cocommutator of the Type II bialgebras in \( \mathfrak{B} \). Moreover, in case that
\[ a_3 + a_4 = 0, \] (15)
we shall have recovered the non-standard cases. The classical Type C \( r \)-matrix is
\[ r_C = (a_4 - a_3) N \wedge M/2 - (a_4 + a_3) A_+ \wedge A_-/2 + a_5 A_+ \wedge M - a_6 A_- \wedge M. \] (16)

The obtention of a coassociative quantum coproduct quantizing each of these Lie bialgebras was developed in \( \mathfrak{B} \). In some cases, automorphisms of the oscillator algebra are needed to simplify the Lie bialgebras before quantizing them. For instance, by using the automorphism
\[ N' = N - (a_5/a_1) M, \] (17)
of \( h_4 \) it can be shown how \( a_5 \) is an irrelevant parameter in Type A bialgebras. Likewise \( a_6 \) plays a trivial role in the family B.

3 The Jordanian quantum oscillator

The (standard) quantum oscillator algebra obtained in \( \mathfrak{B}, \mathfrak{C} \) by a contraction method can be easily recovered from our classification. In fact, the classical \( r \)-matrix underlying that deformation is \( \mathfrak{B} \):
\[ r = -z (N \otimes M + M \otimes N) + 2z A_- \otimes A_+, \] (18)
and its skew-symmetric \( (r_-) \) part
\[ r_- = (r - \sigma \circ r)/2 = z A_- \wedge A_+ \] (19)
is, in our notation, a standard classical \( r \)-matrix of Type C with parameters \( a_5 = a_6 = 0 \) and \( a_3 = a_4 = z \). The associated oscillator bialgebra reads:
\[ \delta(N) = \delta(M) = 0, \quad \delta(A_+) = z A_+ \wedge M, \quad \delta(A_-) = z A_- \wedge M. \] (20)
At this point, it is worth recalling that this deformation was firstly obtained in \cite{[1]} by contracting the standard deformation of $sl(2, \mathbb{R})$. Therefore, it seems pertinent to wonder whether a non-standard deformation of the oscillator algebra related to the so-called Jordanian quantum $sl(2, \mathbb{R})$ \cite{[3],[7],[10]} exists. This question can be easily addressed at the Lie bialgebra level with the help of the classification here presented as follows.

The classical $r$-matrix underlying the Jordanian deformation of $sl(2, \mathbb{R})$ is

$$ r = z J_3 \wedge J_+. \tag{21} $$

Now, if we examine the non-standard oscillator Lie bialgebra of Type A with $a_4 = a_5 = a_6 = 0$ and $a_1 = z$, its classical $r$-matrix would be

$$ r = z N \wedge A+. \tag{22} $$

By taking into account that $\{N, A_+\}$ define the same Lie algebra as $\{J_3, J_+\}$, the deformation of this positive Borel subalgebra induced by both (21) and (22) will be the same. Moreover, the full non-standard deformation of $sl(2, \mathbb{R})$ can be obtained from the information contained in (21). Now it is immediate to compute the harmonic oscillator cocommutator derived from (22):

$$ \begin{align*}
\delta(A_+) &= 0, \\
\delta(M) &= 0, \\
\delta(N) &= z N \wedge A+, \\
\delta(A_-) &= z (A_- \wedge A_+ + N \wedge M). \tag{23}
\end{align*} $$

The quantum algebra obtained by quantizing this Lie bialgebra can therefore be properly called the “Jordanian q-oscillator”, and has the following coproduct and commutation rules

$$ \begin{align*}
\Delta(A_+) &= 1 \otimes A_+ + A_+ \otimes 1, \\
\Delta(M) &= 1 \otimes M + M \otimes 1, \\
\Delta(N) &= 1 \otimes N + N \otimes e^{zA_+}, \\
\Delta(A_-) &= 1 \otimes A_- + A_- \otimes e^{zA_+} + zN \otimes Me^{zA_+}; \tag{24}
\end{align*} $$

$$ \begin{align*}
[N, A_+] &= \frac{e^{zA_+} - 1}{z}, \\
[N, A_-] &= -A_-, \\
[A_-, A_+] &= Me^{zA_+}, \\
[M, \cdot] &= 0. \tag{25}
\end{align*} $$

An essential feature of this deformation is the existence of a quantum universal $R$-matrix, that can be derived by taking into account again that $N$ and $A_+$ generate a deformed Hopf subalgebra. Such a quantum Hopf subalgebra does have a universal $R$-matrix, already derived in \cite{[11]}:

$$ R = \exp\{-zA_+ \otimes N\} \exp\{zN \otimes A_+\}. \tag{26} $$

moreover, the solution (26) of the quantum Yang–Baxter equation can be shown \cite{[3]} to fulfill the relations

$$ \sigma \circ \Delta(X) = R \Delta(X) R^{-1}, \quad \text{for } X \in \{N, A_+, A_-, M\}. \tag{27} $$
Therefore, (26) is also a universal $R$-matrix for the Jordanian $q$-oscillator.

As a consequence, the construction of the corresponding quantum oscillator group is thus available by following the FRT approach [12]. For that purpose we need to recall that a matrix representation of the classical oscillator algebra is provided by

$$D(N) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad D(A_+) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$D(A_-) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad D(M) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (28)$$

A generic element $T^D$ of the oscillator group $H_4$ coming from this representation is:

$$T^D = \exp\{mD(M)\} \exp\{a_-D(A_-)\} \exp\{a_+D(A_+)\} \exp\{nD(N)\}$$
$$= \begin{pmatrix} 1 & a_-e^n & m + a_-a_+ \\ 0 & e^n & a_+ \\ 0 & 0 & 1 \end{pmatrix}. \quad (29)$$

By taking into account that the representation (28) is also a realization of the deformed algebra (25), the universal $R$-matrix (26) is represented as

$$D(R) = I \otimes I + z(D(N) \otimes D(A_+) - D(A_+) \otimes D(N)), \quad (30)$$

($I$ is the $3 \times 3$ identity matrix). Finally, by considering non-commutative entries in (29), the following quantum oscillator group is obtained (quantum coordinates are denoted by hats):

$$\Delta(\hat{n}) = 1 \otimes \hat{n} + \hat{n} \otimes 1,$$

$$\Delta(\hat{a}_+) = e^{\hat{n}} \otimes \hat{a}_+ + \hat{a}_+ \otimes 1,$$

$$\Delta(\hat{a}_-) = e^{-\hat{n}} \otimes \hat{a}_- + \hat{a}_- \otimes 1,$$

$$\Delta(\hat{m}) = 1 \otimes \hat{m} + \hat{m} \otimes 1 - e^{-\hat{n}}\hat{a}_+ \otimes \hat{a}_-. \quad (31)$$

$$[\hat{n}, \hat{a}_+] = z (e^{\hat{n}} - 1), \quad [\hat{n}, \hat{a}_-] = 0, \quad [\hat{a}_-, \hat{a}_+] = z \hat{a}_-, \quad [\hat{n}, \hat{m}] = z \hat{a}_-, \quad [\hat{a}_+, \hat{m}] = z \hat{a}_+, \quad [\hat{a}_-, \hat{m}] = -z \hat{a}_+^2. \quad (32)$$

This Jordanian quantum oscillator group is just a Weyl quantization of the Poisson–Lie bracket (expressed in local coordinates) on the classical oscillator group defined by the $r$-matrix (23) via the Sklyanin bracket.

We would like to point out some interesting features of this new Hopf algebra deformation of the oscillator algebra. Among them we emphasize that, in contradistinction to the standard deformation, exponentials of $A_+$ are the essential constituents of the deformed Hopf algebra. In the standard deformation, this role was played by the mass $M$, that now is also primitive and central. Another central element, the quantum Casimir, can be computed and reads

$$C_z = 2NM + \frac{e^{-zA_+} - 1}{z}A_- + A_- \frac{e^{-zA_+} - 1}{z}. \quad (33)$$
The eigenvalues of this Casimir can be used to label the representation theory of this non-standard $q$-oscillator, that presents stringent differences with respect to that of the standard deformation and also in relation to the well known $q$-oscillator of Biedenharn and Macfarlane [13, 14] (we recall that the latter is not a Hopf algebra deformation, therefore it cannot be recovered from our classification). For instance, the Jordanian $q$-oscillator can be realized in terms of ordinary boson operators $a_{-}$, $a_{+}$, verifying $[a_{-}, a_{+}] = 1$, in the form

$$
A_{+} = a_{+}, \quad M = \delta 1,
A_{-} = \delta e^{2a_{-}} a_{-} + 2 \beta e^{a_{+}},
N = \frac{e^{2a_{+}} - 1}{z} a_{-} + \beta \frac{e^{a_{+}} + 1}{2},
$$

(34)

where $\beta, \delta$ are related to the eigenvalue of the Casimir as $C_{z} = \delta(2\beta - 1)$. A systematic study of the representation theory of non-standard deformations, as well as a derivation of the Hopf algebra structure of the Jordanian $q$-oscillator starting from a non-standard quantum deformation of a pseudo-extended $sl(2, \mathbb{R})$ algebra can be found in [15].

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