Quasiparticle spectrum in the vortex state of $d-$wave superconductors

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Quasiparticle spectrum associated with the nodal structure in $d$-wave superconductors is of great interest. We study theoretically the quasi-particle spectrum in a planar magnetic field, where the effect of the magnetic field is treated in terms of the Doppler shift. We obtain the angular dependent specific heat in the presence of a planar magnetic field and impurities, both in the superclean limit ($\frac{H}{\Delta} \ll 1$) and in the clean limit ($\frac{H}{\Gamma} \ll \frac{\Delta}{\Gamma} \ll 1$). Also a similar analysis is used for the thermal conductivity tensor within the $a-b$ plane. In particular, in contrast to the earlier works, we find a fourfold symmetry term in $\kappa_\parallel$ and $\kappa_\perp \sim -H \sin(2\theta)$ where $\kappa_\parallel$ and $\kappa_\perp$ are the diagonal- and the off-diagonal components of the thermal conductivity tensor and $\theta$ is the angle between the heat current and the magnetic field.

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I. INTRODUCTION

In the last few years $d$-wave superconductivity in hole-doped high-$T_c$ cuprates is established. More recently, Tsuei and Kirtley and other groups have shown by means of the phase sensitive tri-crystal geometry and the magnetic penetration depth measurement that the superconductivity in the electron doped high-$T_c$ cuprates NCCO and PCCO is also of $d$-wave. This suggests strongly the universality of $d$-wave superconductivity for all high-$T_c$ cuprate superconductors. Further, recent measurements of the magnetic penetration depth in organic superconductor $\kappa$-(BEDT-TTF)$_2$Cu(N(CN)$_2$)Br indicates this system has also $d$-wave superconductivity, although the phase sensitive experiment is not available for this system. Also uncertain is the nodal directions of the order parameters in $\kappa$-(ET)$_2$ salts in spite of the theoretical suggestion that it belongs to $d_{xy}$-symmetry. In this circumstance, the experimental probe which indicates the nodal directions in $d$-wave superconductors is of prime importance.

We recall earlier study of the thermal conductivity tensor in the optimally doped YBCO in a planar magnetic field indicates clearly the nodal directions of $d$-wave superconductors consistent with $d_{x^2-y^2}$ symmetry. In the meanwhile Volovik$^{13}$ made the crucial observation that the thermodynamics of the vortex state in $d$-wave superconductors is dominated by the extended quasi-particle states associated with the nodal structure and the effect of the magnetic field is incorporated by the Doppler shift in the energy spectrum. This approach is further developed by Barash et al$^{14}$ and Kübert et al$^{15}$.

In particular, those authors consider the supercurrent associated with individual vortices explicitly and then made spatial average of individual vortex contribution over the unit cell of the vortex lattice. Indeed, Kübert et al have succeeded not only to describe the early specific heat data by K. Moler et al and B. Revaz et al$^{16}$, but also predicted the thermal conductivity in a magnetic field parallel to the $c$-axis in the low temperature limit, which is confirmed by the thermal conductivity measurement by Chiao et al$^{17}$. More recently, this method has been used to calculate a variety of quantities, which includes the angular dependence of the specific heat of $d$-wave superconductivity in a planar magnetic field$^{18,19}$.

The object of this paper is to study both the thermodynamics and the thermal conductivity tensor in a planar magnetic field. We find the analytic expressions of the density of states and the thermal conductivity tensor in the presence of a magnetic field and impurity. Also, the Doppler shift is handled consistently with the layered structure common to both high-$T_c$ cuprates and $\kappa$-(ET)$_2$ salts. This enables us to analyze these quantities both in the superclean limit ($\frac{\Delta}{\Gamma} \ll \frac{H}{\Gamma} \ll 1$) and in the clean limit ($\frac{H}{\Gamma} \ll \frac{H}{\Delta} \ll 1$), while in [16-18] they are called the clean limit and the dirty limit. We think the word ”dirty limit” should not be used to the system where $\frac{\Delta}{\Gamma} \ll 1$, in [16-18] they are treated in the unitarity limit. Therefore, our result is similar to [18] in the clean limit, though it is different from [18] in the superclean limit. So the thermal conductivity increases with $H$ as in [18] but in contradiction to [13]. Further the thermal conductivity exhibits a similar $\theta$-dependence as observed in [13], but again it has the opposite sign; $\kappa_\parallel$ takes the minimum value for $\theta = \pm \frac{\pi}{4}$ contrary to [13]. Perhaps this is due to the fact there is more quasi-particle for $\theta = 0$ than for $\theta = \pm \frac{\pi}{4}$. On the other hand, the transverse thermal conductivity is described by $\kappa_\perp \sim -H \sin(2\theta)$ in the both limits. This dependence appears to describe quite well the data reported in [11,12]. For simplicity, we limit our analysis to $T \ll T_c$, and $H \ll H_2$ and the impurity scattering is treated in the unitarity limit$^{20}$.
II. DENSITY OF STATES, SPECIFIC HEAT, AND SUPERFLUID DENSITY

Following Barash et al. the impurity-renormalized quasiparticle energy $\tilde{\omega}$ is given by

$$\tilde{\omega} = \omega + i \frac{\Gamma}{g(\omega)}$$  \hspace{1cm} (1)

and

$$g(\omega) = \frac{\tilde{\omega} - \mathbf{v} \cdot \mathbf{q}}{\sqrt{(\tilde{\omega} - \mathbf{v} \cdot \mathbf{q})^2 - \Delta^2 \cos^2(2\phi)}}$$  \hspace{1cm} (2)

where $|\mathbf{v} \cdot \mathbf{q}| = \frac{1}{2\pi} \sqrt{v\nu'(\sin^2 \chi + \sin^2(\phi - \theta))}$, the Doppler-shifted energy due to the circulating supercurrent around the vortex, $r$ the distance from the vortex, $v$ and $v'$ are the Fermi velocity within the $a$-$b$ plane and the one parallel to the $c$-axis, respectively, where the effect of the layered structure is explicitly considered. Here $\chi = \theta_3$, $\phi$ is the angle the quasi-particle moment in the $a$-$b$ plane makes from the $a$-axis, $\theta$ is the angle the magnetic field makes from the $a$-axis, and $\ldots \ldots = \frac{1}{2\pi^2} \int dx \int d\phi \ldots$ the angular average. We assume the unitarity limit of the impurity scattering. The low temperature limit of both the specific heat and the superfluid density is easily obtained from the residual density of states (i.e. the density of states on the Fermi surface or at $\omega = 0$). The quasiparticle density of states at $\omega = 0$ in the presence of both impurities and a magnetic field is given by

$$\frac{N(\omega = 0)}{N_0} = \text{Re} \ g(\omega = 0)$$

$$= \frac{2}{\pi} \left[ C_0 \ln \left( \frac{4}{\sqrt{C_0^2 + x^2}} \right) + x \tan^{-1} \left( \frac{x}{C_0} \right) \right]$$  \hspace{1cm} (3)

where $x = |\mathbf{v} \cdot \mathbf{q}|/\Delta$ and we used

$$\tilde{\omega}_\Delta = iC_0 = \frac{i\Gamma}{\Delta g(\omega = 0)}$$  \hspace{1cm} (4)

This set of equation is solved in the following:

For $C_0 \ll \langle x \rangle \ll 1$ (i.e. $\frac{\Gamma}{\Delta} \ll \frac{H}{\hbar c} \ll 1$),

$$\frac{N(\omega = 0)}{N_0} \equiv \frac{N(H, \theta)}{N_0} = \langle x \rangle + \frac{2 \Gamma}{\pi \Delta \langle x \rangle} \left( \ln \left( \frac{4}{\langle x \rangle} \right) - 1 \right)$$  \hspace{1cm} (5)

and

$$C_0 = \frac{\Gamma}{\Delta} \langle x \rangle + \ldots$$

For $\langle x \rangle \ll C_0 \ll 1$ (i.e. $\frac{\hbar c}{\pi \Delta} \ll \frac{\Gamma}{\Delta} \ll 1$),

$$\frac{N(\omega = 0)}{N_0} \equiv \frac{N(H, \theta)}{N_0} = \frac{N_{\text{imp}}(0)}{N_0} \left( 1 + \frac{1}{2} \frac{\Delta}{\Gamma} \langle x^2 \rangle \right)$$  \hspace{1cm} (6)

and

$$C_0 \approx \sqrt{\frac{\pi \Gamma}{2 \Delta \ln(4 \frac{2 \Delta}{\pi \Gamma})}}$$

where $N_{\text{imp}}(0)$ is the density of states in the $H = 0$ case with the unitarity impurity scatterers and is given

$$\frac{N_{\text{imp}}(0)}{N_0} = \frac{2}{\pi} C_0 \ln(\frac{4}{C_0}) \approx \frac{2 \Delta}{\pi \Gamma} \ln(4 \frac{2 \Gamma}{\pi \Delta})$$  \hspace{1cm} (7)

We call the former the superclean limit while the latter the clean limit. These have been obtained essentially in [16] except for a few typos. We used the same angular and the spatial average as in Barash et al and K"ubert et al. Finally, the density of states in the both limits is the following:

For the superclean limit ($C_0 \ll \langle x \rangle \ll 1$);

$$\frac{N(H, \theta)}{N_0} \approx \frac{\sqrt{\nu' eH}}{\Delta} I(\theta) + \frac{\Gamma}{\pi \nu' eH} \left( \ln \left( \frac{8\Delta}{\nu' eH} \right) - \frac{3}{2} J(\theta) \right)$$  \hspace{1cm} (8)

where

$$I(\theta) = \frac{1}{2} \Sigma_{\pm} \left( \sqrt{\sin^2 \chi + \sin^2(\pm \frac{\pi}{4} - \theta)} \right)$$

$$= \frac{1}{2} \Sigma_{\pm} \sqrt{\frac{3 \pm s}{2}} E(\sqrt{\frac{2}{3 \pm s}})$$

$$\approx 0.0285 \cos(4\theta) + 0.955$$  \hspace{1cm} (9)

and

$$J(\theta) = \frac{1}{2} \Sigma_{\pm} \left( \ln \sqrt{\sin^2 \chi + \sin^2(\pm \frac{\pi}{4} - \theta)} \right)$$

$$= \frac{1}{4} \Sigma_{\pm} \ln \left( \frac{2 \pm s + \sqrt{(3 \pm s)(1 \pm s)}}{4} \right)$$

$$\approx -0.778 + 0.744 \text{Max}\{ |\cos \theta|, |\sin \theta| \}$$  \hspace{1cm} (10)

where $s = \sin(2\theta)$ and $E(k)$ is the complete elliptic integral. Here, instead of average over $\phi$, we average over $\phi = \pm \frac{\phi}{2}$ (i.e. over the two nodal directions $\underline{\underline{E}}$).

The $\theta$-dependence of $I(\theta)$ and $J(\theta)$ are shown in Fig.1 and Fig.2. In Fig.1 we compared also the angular dependent obtained in [23] where the layered structure is ignored. In the present calculation, the angular dependent term is about the 4% of the coefficient $\sim \hbar v$, while in [23] it is about 20%. A recent specific data from stoichiometric YBa$_2$Cu$_3$O$_{7\pm\delta}$ crystals appear to be more consistent with the present analysis. Perhaps we have to point out the angular dependence of $J(\theta)$ appeared since we cut off the the short-range logarithmic divergence at $x = \frac{1}{2}$ where the uniform order parameter started to be greatly disturbed, though the exact value $\frac{1}{2}$ in the present case is not so important.
For the clean limit \((x \ll C_0 \ll 1)\):

\[
\frac{N(H, \theta)}{N_0} = \frac{N_{\text{imp}}(0)}{N_0} \left[ 1 + \frac{1}{4} \frac{\nu v e H}{\Gamma \Delta} \ln(\sqrt{\Delta / \nu v e H} - F(\theta)) \right] \tag{11}
\]

where

\[
F(\theta) = \frac{1}{2} \sum_k \langle \sin^2 \chi + \sin^2(\pm \frac{\pi}{4} - \theta) \rangle \times \\
\ln \sqrt{\sin^2 \chi + \sin^2(\pm \frac{\pi}{4} - \theta)} \\
= \frac{1}{4} \sum_\pm \left[ (2 \pm s) \ln \left( \frac{2 \pm s + \sqrt{(3 \pm s)(1 \pm s)}}{4} \right) \\
+ 4 - \sqrt{(3 \pm s)(1 \pm s)} \right] \\
\simeq 0.147 - 0.082 \cos(4\theta) \tag{12}
\]

The function \(F(\theta)\) is shown in Fig.3.

The residual density of states is most readily accessible to the low temperature limit of the spin susceptibility as in NMR, the specific heat and the superfluid density.

In particular the low temperature specific heat and the superfluid density are given by

\[
C_s(H, \theta) = \frac{2\pi^2 T \Delta}{3} N(H, \theta) \tag{13}
\]

and

\[
\rho_s(H, \theta) = 1 - N(H, \theta)/N_0 \tag{14}
\]

Also, in the superclean limit and \(T/\Delta > \langle x \rangle\), we obtain

\[
C_s = 18\zeta(3) \frac{T^2}{\Delta} N_0 \tag{15}
\]

and

\[
\rho_s(T)/\rho(0) = 1 - 2(\ln 2) \frac{T}{\Delta} \tag{16}
\]

To conclude this section, we have considered the residual density of states in a planar magnetic field both in the superclean and in the clean limit. In both limits the residual density of states exhibits the fourfold symmetry, though magnitude of this terms is roughly 4 times smaller than the earlier result.

Similarly

\[
\kappa_\perp/\kappa_0 = \frac{\pi}{2} \left[ \sin(2\phi) \frac{1}{2} \frac{C_0^2 + x^2 - \cos^2(2\phi)}{\Re \sqrt{(C_0 + i x)^2 + \cos^2(2\phi)}} \right] \tag{17}
\]

where \(\kappa_0 = \kappa_\parallel\) \((H = 0)\)

First, for the superclean limit \((C_0 \ll \langle x \rangle \ll 1)\) Eq.(17) reduce to

\[
\kappa_\parallel/\kappa_n \simeq \frac{2}{\pi} \langle x \rangle^2 \\
= \frac{2}{\pi} \frac{\nu v e H}{\Delta^2} (I(\theta))^2 \tag{19}
\]

and

\[
\kappa_\perp/\kappa_n = - \frac{2}{\pi} \frac{\nu v e H}{\Delta^2} L(\theta) \tag{20}
\]

where \(\kappa_n = \frac{\pi^2 T N_0}{6\Gamma m}\), the thermal conductivity in the normal state, and

\[
L(\theta) = \frac{1}{\pi} \left( \sqrt{\frac{3 + s}{2}} E\left(\sqrt{\frac{3 + s}{2}}\right) - \sqrt{\frac{3 + s}{2}} E\left(\sqrt{\frac{3 + s}{2}}\right) \right) \\
\simeq 0.29 \sin(2\theta) \tag{21}
\]

The function \(L(\theta)\) is shown in Fig.4 together with the approximate form. We note also that \(\kappa_\parallel\) has the same angular dependence as \([N(H, \theta)]^2\). Also \(\kappa_\perp\) is proportional to \(\sin(2\theta)\) in a good approximation (see Fig.4 and Eq.(9)).

On the other hand, in the clean limit \((x \ll C_0 \ll 1)\), we obtain

\[
\kappa_\parallel/\kappa_n = 1 + \frac{1}{3} \frac{\langle x \rangle^2}{C_0^2} \\
= 1 + \frac{1}{3\pi} \frac{\nu v e H}{\Gamma \Delta} \ln(4\sqrt{\frac{2\Delta}{\pi \Gamma}} \left[ \ln(\frac{2\Delta}{\sqrt{\nu v e H}}) \\
- F(\theta) \right] \tag{22}
\]

where \(F(\theta)\) is given in Eq.(12). Note that \(F(\theta)\) appears from the lower cut-off of \(r\) as explained after Eq.(10).

And

\[
\kappa_\perp/\kappa_n = - \frac{1}{3\pi} \frac{\nu v e H}{\Gamma \Delta} \left[ \ln(4\sqrt{\frac{2\Delta}{\pi \Gamma}} \sin(2\theta) \ln(\frac{2\Delta}{\sqrt{\nu v e H}}) \\
- G(\theta) \right] \tag{23}
\]

where

\[
G(\theta) = \frac{1}{4} \left[ (2 + s) \ln\left( \frac{2 + s + \sqrt{(3 + s)(1 + s)}}{4} \right) \\
- (2 - s) \ln\left( \frac{2 - s + \sqrt{(3 - s)(1 - s)}}{4} \right) + 2 \sin(2\theta) \\
- \sqrt{(3 + s)(1 + s)} + \sqrt{(3 - s)(1 - s)} \right] \\
\simeq 0.422 \sin(2\theta) \tag{24}
\]
The function $G(\theta)$ is shown in Fig.5 with the approximate form. Here, $\kappa_0 = \frac{\pi T n}{3 \Delta m}$, $n$ is the quasiparticle density and $m$ the quasiparticle mass. This form of thermal conductivity in the absence of magnetic field is derived first by Lee. In general, however, $\Delta$ in $\kappa_0$ depends on both on $\Gamma$ and $H$. In the superclean limit, $\Delta(H)$ may be approximately given by

$$\Delta(H)/\Delta_0 = 1 - \frac{1}{3} (x^3), \quad \text{for} \quad C_0 \ll \langle x \rangle \ll 1$$

$$\simeq 1 - \frac{1}{6} \frac{v' e H}{\Delta^2}$$

(25)

In the clean limit $\langle x \rangle \ll C_0 \ll 1$,

$$\Delta(\Gamma)/\Delta_0 \approx 1 - \frac{\pi}{4} \frac{\Gamma}{\Delta}$$

(26)

In both limits, the longitudinal conductivity increases with $H$, linearly in superclean limit and $H \ln(Hc_2/H)$ in clean limit, respectively. (Our result agrees with [18] only in the clean limit, $\frac{\kappa_0}{\kappa_{c2}} \ll \Delta \ll 1$.) Further it exhibits the $\theta$ dependence in both the superclean and the clean limit. In the superclean limit the $\theta$-dependence comes from that of the density of states, while in the clean limit this arises from the short range cut-off we have introduced after Eq.(10). Also this angular dependence is very similar to the one reported in [13], but of opposite sign. But perhaps of particular interest is the transverse thermal conductivity. The dominant terms in Eq.(20) and Eq.(23), exhibit $\sin(2\theta)$ dependence, which is fully consistent with the early experiment. Also this $\sin(2\theta)$ dependence is appreciable even when $H \sim Hc_2$.

IV. CONCLUDING REMARKS

We have extended earlier analysis of the thermodynamics and the transport properties in d-wave superconductors in 2 directions. First, we take account of the layered structure of the underlying superconductors explicitly. Second, we focused on the angular dependence of thermal conductivity tensor, which exhibits clear signs of the nodal structures in d-wave superconductors. Indeed the diagonal thermal conductivity exhibits the fourfold symmetry as observed in [13], but of opposite sign. In particular, the present result describes the transverse thermal conductivity (or Righi-Leduc effect) observed in YBCO. Indeed, we have found recently a similar transverse thermal conductivity in p-wave superconductors as Sr$_2$RuO$_4$ and f-wave superconductors as UPt$_3$. Therefore, this $\sin(2\theta)$ dependence is rather common to most of unconventional superconductors.

In summary, the exploration of the nodal structure in unconventional superconductors will provide useful insight in the quasi-particle spectrum in the vortex state.

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FIG. 1. $I(\theta)$ and the approximate form are shown as function of $\theta$. $I_1(\theta)$ is the angular dependence in [23] where the layer structure is ignored.

FIG. 2. $J(\theta)$ is shown as a function of $\theta$. It has cusps at $\pi/4$ and $3\pi/4$ together with the approximate form.

FIG. 3. $F(\theta)$ is shown as a function of $\theta$ together with the approximate form. This angular dependence appears in the specific heat $C_v$, the superfluid density $\rho_s$, and thermal conductivity tensor $\kappa_\parallel$ in clean limit ($\frac{H}{T_c^2} \ll \frac{1}{\Delta} \ll 1$).

FIG. 4. $L(\theta)$ is shown as a function of $\theta$ to together with the approximate form.

FIG. 5. $G(\theta)$ is shown as a function of $\theta$ together with the approximate form.
0.0285 \cos(4\theta) + 0.955

H. Won and K. Maki, Fig.1
\[ J(\theta) = 0.744 \max\{ |\cos \theta|, |\sin \theta| \} - 0.778 \]

H. Won and K. Maki, Fig. 2
H. Won and K. Maki, Fig. 3

\[ F(\theta) = -0.082 \cos(4\theta) + 0.147 \]
H. Won and K. Maki, Fig. 4
H. Won and K. Maki, Fig.5