NORMAL EQUIVARIANT COMPACTIFICATIONS OF $\mathbb{G}_a^2$ WITH PICARD NUMBER ONE

PINAKI MONDAL

Abstract. We classify all normal $\mathbb{G}_a^2$-surfaces with Picard number one, and characterize which of these surfaces have at worst log canonical, and which have at worst log terminal singularities, answering a question of Hassett and Tschinkel [HT99]. We also find all $\mathbb{G}_a^2$-structures on these surfaces and show that these surfaces and their minimal desingularizations have the same $\mathbb{G}_a^2$-structures (modulo equivalence of $\mathbb{G}_a^2$-actions). In particular, we show that some of these surfaces admit one dimensional moduli of $\mathbb{G}_a^2$-structures, answering another question of Hassett and Tschinkel [HT99].

1. Introduction

Hassett and Tschinkel [HT99] started the study of $\mathbb{G}_a^n$-varieties; these are equivariant compactifications of $\mathbb{G}_a^n$, i.e. $\mathbb{C}^n$ with the additive group structure. They classified $\mathbb{G}_a^n$-structures on projective spaces and Hirzebruch surfaces, and showed that in dimension $\geq 6$ projective spaces admit moduli of $\mathbb{G}_a^n$-structures. In particular, they asked the following questions regarding the $n = 2$ case:

Problem 1.1 ([HT99, Section 5.2]).

(1) Can the $\mathbb{G}_a^2$-structures on a given (smooth) surface have moduli?
(2) Classify $\mathbb{G}_a^2$-structures on projective surfaces with log terminal singularities and Picard number one.

Motivated by these questions, in this article we undertake a study of normal $\mathbb{G}_a^2$-surfaces with Picard rank one. In particular, we answer both these questions.

Indeed, every $\mathbb{G}_a^2$-surface of Picard rank one is trivially a primitive compactification of $\mathbb{C}^2$, i.e. a compact complex analytic surface containing $\mathbb{C}^2$ such that the curve at infinity is irreducible. In [Mon13] we gave an explicit description of automorphisms of normal primitive compactifications of $\mathbb{C}^2$. Using that description, in this article we classify all normal surfaces with Picard rank one which have $\mathbb{G}_a^2$-structures (theorem 4.2). Moreover, we give an explicit description of all $\mathbb{G}_a^2$-structures on these surfaces (theorem 4.2) and of the space of $\mathbb{G}_a^2$-structures modulo equivalence (theorem 4.3). In particular, it turns out that some of these spaces admit one dimensional moduli of $\mathbb{G}_a^2$-structures. On the other hand, we show that normal primitive compactifications of $\mathbb{C}^2$ have the special property that all of their automorphisms lift to automorphisms of their minimal desingularizations (theorem B.4), which implies that the spaces of $\mathbb{G}_a^2$-structures modulo equivalence on normal $\mathbb{G}_a^2$-surfaces of Picard rank one and on their minimal desingularizations are isomorphic (Corollary 5.3). In particular, it follows that some of these minimal desingularizations also admit one dimensional moduli of $\mathbb{G}_a^2$-structures, thereby answering question (1).

In [Mon16] we gave an explicit description of minimal desingularizations of normal primitive compactifications of $\mathbb{C}^2$. Combining this with Kawamata’s [Kaw88] classification of log canonical surface singularities (we follow the description of Alexeev [Ale92]) and our classification of $\mathbb{G}_a^2$-structures on normal surfaces of Picard rank one (theorems 4.2 and 4.3), we immediately obtain a classification of $\mathbb{G}_a^2$-structures on projective surfaces with log terminal or log canonical singularities and Picard number one (theorem 6.7), which answers question (2).
Some (more precisely, four, up to isomorphism,) of the $\mathbb{G}_a^2$-surfaces of Picard rank one are also singular del Pezzo surfaces in the sense of Derenthal and Loughran [DL10] corresponding to dual graphs of type $A_1, A_2 + A_1, A_4$ and $D_5$ (Corollary 4.5). In particular, the first two are respectively weighted projective spaces $\mathbb{P}^2(1, 1, 2)$ and $\mathbb{P}^2(1, 3, 2)$, and have precisely two $\mathbb{G}_a^2$-structures modulo equivalence. The third one admits a one dimensional moduli of $\mathbb{G}_a^2$-structures (modulo equivalence) - it is described in section 2. The other one admits only one $\mathbb{G}_a^2$-structure modulo equivalence.

2. A SIMPLE NON-SINGULAR SURFACE WITH ONE DIMENSIONAL MODULI OF $\mathbb{G}_a^2$-STRUCTURES.

Let $\bar{X} := \mathbb{P}^2$ and $L$ be a line on $\bar{X}$. Blow up a point $P$ on $L$, then blow up the point where the strict transform of $L$ intersects the exceptional curve, and then blow up again the point of intersection of the strict transform of $L$ and the new exceptional curve. Finally blow up a point on the newest exceptional curve which is not on the strict transform of either $L$ or any of the older exceptional curves. Let $X'$ be the resulting surface. Identifying $X := \bar{X} \setminus L$ with $\mathbb{C}^2$, we see that $\bar{X}'$ is a non-singular compactification of $\mathbb{C}^2$ and the ‘weighted dual graph’ of the curve at infinity on $\bar{X}'$ is as in fig. 1.

![Figure 1. Weighted dual graph of the curve at infinity on $\bar{X}'$](image)

Let $\bar{X}''$ be the surface formed by contracting (the strict transforms of) $L$, $E_1$, $E_2$ and $E_3$. In the notation introduced in section 3, $\bar{X}''$ is the normal primitive compactification of $\mathbb{C}^2$ corresponding to key sequence $\bar{\omega} := (3, 2, 5)$, and $\bar{X}'$ is the minimal desingularization of $\bar{X}''$. Choose homogeneous coordinates $[u : v : w]$ on $X$ such that $L = \{w = 0\}$ and $P$ has coordinates $[1 : 0 : 0]$. Then $(x, y) := (u/w, v/w)$ are coordinates on $X$. It follows from theorem 4.3 and Corollary 5.3 that the moduli (up to equivalence) of $\mathbb{G}_a^2$-structures on $\bar{X}'$ and $\bar{X}''$ consists of $\mathbb{G}_a^2$-actions $\tau_\lambda$, $\lambda \in \mathbb{C}$ defined as follows:

\[(t_1, t_2) \cdot \tau_\lambda (x, y) = \left(x + \lambda \left(\frac{(t_1)^2}{2} + t_1y\right) + t_2, y + t_1\right)\]

3. PRELIMINARIES ON NORMAL PRIMITIVE COMPACTIFICATIONS OF $\mathbb{C}^2$

A primitive compactification of $\mathbb{C}^2$ is an analytic surface containing $\mathbb{C}^2$ such that the curve at infinity is irreducible. In this section we recall some properties of normal primitive compactifications of $\mathbb{C}^2$ from [Mon13].

**Definition 3.1 (Key sequences).** A sequence $\bar{\omega} := (\omega_0, \ldots, \omega_{n+1}), n \in \mathbb{Z}_{\geq 0}$, of integers is called a key sequence if it has the following properties:

1. $\omega_0 \geq 1$.
2. Let $e_k := \gcd(|\omega_0|, \ldots, |\omega_k|), 0 \leq k \leq n + 1$ and $\alpha_k := e_{k-1}/e_k, 1 \leq k \leq n + 1$. Then $e_{n+1} = 1$, and $\omega_{k+1} < \alpha_k \omega_k, 1 \leq k \leq n$.
3. Moreover, $\bar{\omega}$ is called primitive if $\omega_{n+1} > 0$ (or equivalently, $\omega_k > 0$ for all $k, 0 \leq k \leq n + 1$), and it is called algebraic if $\omega_k \in \mathbb{Z}_{\geq 0}(\omega_0, \ldots, \omega_{k-1}), 1 \leq k \leq n$.
4. Finally, $\bar{\omega}$ is called essential if $\alpha_k \geq 2$ for $1 \leq k \leq n$. Note that $\bar{\omega}$ is called algebraic if $\alpha_k \geq 2$ for $1 \leq k \leq n$.

(a) Given an arbitrary key sequence $(\omega_0, \ldots, \omega_{n+1})$, it has an associated essential subsequence $(\omega_0, \omega_i, \ldots, \omega_i, \omega_{n+1})$ where $\{i\}$ is the collection of all $k, 1 \leq k \leq n$, such that $\alpha_k \geq 2$.
(b) If $\bar{\omega}$ is an algebraic key sequence, then its essential subsequence is also algebraic.
Remark 3.2. Let $\bar{\omega} := (\omega_0, \ldots, \omega_{n+1})$ be a key sequence. It is straightforward to see that property 3 implies the following: for each $k, 1 \leq k \leq n$, $\alpha_k \omega_k$ can be uniquely expressed in the form $\alpha_k \omega_k = \beta_k,0 \omega_0 + \beta_k,1 \omega_1 + \cdots + \beta_k,k-1 \omega_{k-1}$, where $\beta_k',s$ are integers such that $0 \leq \beta_k,j < \alpha_j$ for all $j \geq 1$. If $\bar{\omega}$ is in additional algebraic, then $\beta_k,0$’s of the preceding sentence are non-negative.

Definition 3.3. Let $\bar{\omega} := (\omega_0, \ldots, \omega_{n+1})$ be a key sequence and $\bar{\theta} \in (\mathbb{C}^*)^n$. Let $\bar{W}\bar{P}$ be the weighted projective space $\mathbb{P}^{n+2}(1, \omega_0, \omega_1, \ldots, \omega_{n+1})$ with (weighted) homogeneous coordinates $[w : y_0 : \cdots : y_{n+1}]$. We write $\bar{X}_{\bar{\omega},\bar{\theta}}$ for the subvariety of $\bar{W}\bar{P}$ defined by weighted homogeneous polynomials $G_k, 1 \leq k \leq n$, given by

$$G_k := w^{\alpha_k \omega_k - \omega_{k+1}} y_{k+1} - \left( y_k^{\alpha_k} - \theta_k \prod_{j=0}^{k-1} y_j^{\beta_k,j} \right)$$

where $\alpha_k$’s and $\beta_k,j$’s are as in Remark 3.2.

Proposition 3.4 ([Mon13, Proposition 3.4]). If $\bar{\omega}$ is primitive and algebraic, then $\bar{X}_{\bar{\omega},\bar{\theta}}$ is a normal primitive algebraic compactification of $\mathbb{C}^2$. Conversely, every normal primitive algebraic compactification of $\mathbb{C}^2$ is isomorphic to $\bar{X}_{\bar{\omega},\bar{\theta}}$ for some primitive algebraic key sequence $\bar{\omega} := (\omega_0, \ldots, \omega_{n+1})$ and $\bar{\theta} \in (\mathbb{C}^*)^n$ for some $n \geq 0$.

Let $\bar{\omega} := (\omega_0, \ldots, \omega_{n+1})$ be a key sequence, and $\bar{\omega}_0 := (\omega_{i_0}, \ldots, \omega_{i_{l+1}})$, where $0 = i_0 < i_1 < \cdots < i_{l+1} = n+1$, be the essential subsequence of $\bar{\omega}$. Define

$$\chi_j := \frac{1}{\omega_0} (\omega_i - \sum_{k=1}^{j-1} (\alpha_k - 1) \omega_i), \quad 1 \leq j \leq l + 1.$$

where $\alpha_1, \ldots, \alpha_{n+1}$ are as in Definition 3.1. Let

$$\mathcal{E}_{\bar{\omega}} := \begin{cases} \{k \omega_0 \omega_1 - 1 : k \in \mathbb{Z}, \max\{0, (\chi_{l+1} + 1) \omega_0 \omega_1 < k < \omega_0 \omega_1 + 1\} \cup \{0\} & \text{if } \omega_0 > 0, \\
\{k \omega_0 \omega_1 - 1 : k \in \mathbb{Z}, 0 < k < (\chi_{l+1} + 1) \omega_0 \omega_1\} & \text{if } \omega_0 < 0. \end{cases}$$

Let $\beta \in \mathbb{Q}$. Let

$$\hat{k}(\beta) := \begin{cases} 0 & \text{if } \beta \geq \chi_1, \\
\max\{k : 1 \leq k \leq l + 1, \beta < \chi_k\} & \text{otherwise}. \end{cases}$$

$$\hat{\omega}_\beta := \omega_0 \beta + \sum_{j=1}^{\hat{k}(\beta)} (\alpha_{i_j} - 1) \omega_{i_j}$$

$$\hat{I}_\beta = \begin{cases} \{i : i_{k(\beta)} < i < i_{k(\beta)+1}\} & \text{if } \hat{k}(\beta) \leq l \\
\emptyset & \text{if } \hat{k}(\beta) = l + 1. \end{cases}$$

Note that $\hat{\omega}_{\chi_j} = \omega_{i_j}, 1 \leq j \leq l + 1$.

Definition 3.5. We say that a key sequence $\bar{\omega} := (\omega_0, \ldots, \omega_{n+1})$ is in the normal form if it satisfies one of the following (mutually exclusive) conditions:

(N0) (a) $n = 0$.
(b) $\omega_0 \geq \omega_1$.

(N1) (a) $n \geq 1$.
(b) $\omega_0 > \omega_1$.
(c) $\frac{\omega_k}{\omega_0} \notin \{\frac{k}{n} : \omega_0 \in \mathbb{Z}, k \geq 1\} \cup \{0\}$.
(d) For each $\beta \in \mathcal{E}_{\bar{\omega}}$, there does not exist $i \in \hat{I}_\beta$ such that $\omega_i = \hat{\omega}_\beta$.

Proposition 3.6 ([Mon13, Theorems 4.6 and 6.3]). Let $\bar{X}$ be a primitive normal algebraic compactification of $\mathbb{C}^2$.

(1) There exist a unique $n \geq 0$ and a unique primitive algebraic key sequence $\bar{\omega} := (\omega_0, \ldots, \omega_{n+1})$ in normal form such that $\bar{X} \cong \bar{X}_{\bar{\omega},\bar{\theta}}$ for some $\bar{\theta} \in (\mathbb{C}^*)^n$. 
(2) Let $\alpha_i$’s and $\beta_{i,j}$’s be as in Remark 3.2. Moreover, set $\alpha_0 := 1$. Let $\bar{\omega}_e := (\omega_{i_0}, \ldots, \omega_{i_{k+1}})$ be the essential subsequence of $\bar{\omega}$. Define $\mu_1, \ldots, \mu_n \in \mathbb{Z}$ as follows: for each $i$, $1 \leq i \leq n$, pick the unique $k$ such that $i_k \leq i < i_{k+1}$, and set

$$
\mu_i := \alpha_{i_0} \cdots \alpha_{i_k} - \sum_{j=1}^{k} \alpha_{i_0} \cdots \alpha_{i_{j-1}} \beta_{i,j}
$$

If $\bar{\theta} \in (\mathbb{C}^*)^n$ is such that $\bar{X} \cong \bar{X}_{\bar{\omega}, \bar{\theta}}$ as well, then there exist $\lambda_1, \lambda_2 \in \mathbb{C}^*$ such that

$$
\theta_i' = \lambda_1^{-\beta_{i,0}} \lambda_2^{\mu_i} \theta_i, \quad i = 1, \ldots, n.
$$

**Theorem 3.7** ([Mon13, Theorem 5.2]). Fix a system of coordinates $(x, y)$ on $X := \mathbb{C}^2$. Let $\bar{\omega} := (\omega_0, \ldots, \omega_{n+1})$ be a primitive key sequence in normal form, $\bar{\theta} := (\theta_1, \ldots, \theta_n) \in (\mathbb{C}^*)^n$, and $\bar{X} := \bar{X}_{\bar{\omega}, \bar{\theta}}$ be the corresponding primitive compactification of $X$. Let $\mathcal{G}$ be the group of automorphisms of $\bar{X}$.

1. If (N0) holds, then $\bar{X} \cong \mathbb{P}^2(1, \omega_0, \omega_1)$. Fix (weighted) homogeneous coordinates $[z : x : y]$ on $\bar{X}$.
   (a) If $\omega_0 = \omega_1 = 1$, then $\bar{X} \cong \mathbb{P}^2$ and $\mathcal{G} \cong \text{PGL}(3, \mathbb{C})$.
   (b) If $\omega_0 > \omega_1 = 1$, then $\mathcal{G} = \{[z : x : y] \mapsto [az + by : cx + f(y, z) : dz + ey] : a, b, d, e \in \mathbb{C}, \ ad - be \neq 0, \ c \in \mathbb{C}^*, \ f \text{ is a homogeneous polynomial in } (y, z) \text{ of degree } \omega_0\}$.\]
   (c) If $\omega_0 > \omega_1 > 1$, then $\mathcal{G} = \{[z : x : y] \mapsto [z : ax + f(y, z) : by + c\bar{\omega}_1] : a, b \in \mathbb{C}^*, \ c \in \mathbb{C}, \ f \text{ is a weighted homogeneous polynomial in } (y, z) \text{ of weighted degree } \omega_0\}$.\]

2. If (N1) holds, define $\bar{\omega}_k := \omega_k/\alpha_{n+1}$, $0 \leq k \leq n$, and $\bar{\omega}_k^* := \alpha_1 \bar{\omega}_1 + \sum_{j=2}^{k} (\alpha_j - 1) \bar{\omega}_j - \bar{\omega}_k$, $2 \leq k \leq n$, where $\alpha_1, \ldots, \alpha_{n+1}$ are as in Definition 3.1. Set $\bar{\omega}^* := \gcd(\bar{\omega}_2^*, \ldots, \bar{\omega}_n^*)$ (note that $\bar{\omega}^*$ is defined only if $n \geq 2$) and

$$
(8) \quad k_\bar{X} := \left(\omega_0 + \omega_{n+1} + 1 - \sum_{k=1}^{n} (\alpha_k - 1) \omega_k\right)
$$

where $\alpha_1, \ldots, \alpha_{n+1}$ are as in Definition 3.1. Then $\mathcal{G}$ consists of all $F : \bar{X} \rightarrow \bar{X}$ such that $F|_{X} : (x, y) \mapsto (a \omega_0 x + f(y), a \bar{\omega}_1 y + c)$, where

$$
a = \begin{cases} 
\text{an arbitrary element of } \mathbb{C}^* & \text{if } n = 1, \\
\text{an } \bar{\omega}^*-\text{th root of unity} & \text{if } n \geq 2.
\end{cases}
$$

$$
c = \begin{cases} 
0 & \text{if } 0 \leq k_\bar{X} < 0, \\
\text{an arbitrary element in } \mathbb{C} & \text{otherwise}.
\end{cases}
$$

and $f(y) \in \mathbb{C}[y]$ is a polynomial such that $\deg(f) \leq -(k_\bar{X} + \omega_1 + 1)/\omega_1$. \hfill \Box

4. $G^2_a$-structures on normal surfaces with Picard rank one

Let $G$ be a connected linear algebraic group. A $G$-variety is a variety $Y$ with a fixed left action of $G$ such that the stabilizer of a generic point is trivial and the orbit of a generic point is dense. A $G^2_a$-surface is a $G$-variety for $G = G^2_a$. Two left actions $\sigma_1, \sigma_2$ of $G$ on $Y$ are equivalent if there is a commutative diagram as follows:

$$
\begin{array}{ccc}
G \times Y & \xrightarrow{(\alpha, j)} & G \times Y \\
\downarrow \sigma_1 & & \downarrow \sigma_2 \\
Y & \xrightarrow{j} & Y
\end{array}
$$

where $\alpha$ (resp. $j$) is an automorphism of $G$ (resp. $Y$).

Lemma 4.1 below studies a class of $G^2_a$-actions on $\mathbb{C}^2$ of a very special form. It will be used in the classification of $G^2_a$-surfaces (theorem 4.2).
Lemma 4.1. Let $\Phi$ be the morphism $\mathbb{G}_a^2 \times \mathbb{C}^2 \to \mathbb{C}^2$ given by

\[
(t_1, t_2) \cdot (x, y) = \left( a(t_1, t_2)x + \sum_{i=0}^{m} b_i(t_1, t_2)y^i, b(t_1, t_2)y + c(t_1, t_2) \right),
\]

where $(t_1, t_2)$ (resp. $(x, y)$) are coordinates on $\mathbb{G}_a^2$ (resp. $\mathbb{C}^2$), $m \geq 0$ and $a, b, c, b_1, \ldots, b_m \in \mathbb{C}[t_1, t_2]$. Then $\Phi$ defines a $\mathbb{G}_a^2$-action on $\mathbb{C}^2$ iff each of the following conditions holds:

1. $a(t_1, t_2) = b(t_1, t_2) = 1$ for all $(t_1, t_2) \in \mathbb{G}_a^2$.
2. There are $c_1, c_2 \in \mathbb{C}$ such that $c(t_1, t_2) = c_1t_1 + c_2t_2$ for all $t_1, t_2 \in \mathbb{G}_a^2$.
3. If $(c_1, c_2) = (0, 0)$, then $b_i$ is linear in $(t_1, t_2)$ for each $i$.
4. If $(c_1, c_2) \neq (0, 0)$, then there exists $\lambda_0, \ldots, \lambda_m, \mu_0 \in \mathbb{C}$ such that

\[
b_i(t_1, t_2) := \begin{cases} g_0(c_1t_1 + c_2t_2) + \mu_0(\bar{c}_2t_1 - \bar{c}_1t_2) & \text{if } i = 0, \\ g_i(c_1t_1 + c_2t_2) & \text{if } i = 1, \ldots, m, \end{cases}
\]

where for each $i = 0, \ldots, m$,

\[
g_i(r) := \lambda_ir + \frac{\lambda_{i+1}}{2}(i+1)r^2 + \cdots + \frac{\lambda_m}{m-i+1}(\frac{m}{m-i})r^{m-i+1}
\]

Proof. See appendix A. \(\square\)

Let $\bar{\omega} := (\omega_0, \ldots, \omega_{n+1})$ be a primitive key sequence in the normal form, $\bar{\theta} := (\theta_1, \ldots, \theta_n) \in (\mathbb{C}^*)^n$ and $X := \bar{X}_{\bar{\omega}, \bar{\theta}}$ be the corresponding primitive compactification of $\mathbb{C}^2$. Define $k_X$ as in (8).

Theorem 4.2.

1. The following are equivalent:
   a. $X$ admits the structure of a $\mathbb{G}_a^2$-surface,
   b. $\omega_0 + k_X < 0$.

2. Assume $X \not\cong \mathbb{P}^2$ and that there is a $\mathbb{G}_a^2$-action $\sigma$ on $\bar{X}$ which makes it a $\mathbb{G}_a^2$-surface. Then there is an automorphism $F$ of $\bar{X}$ such that $F$ such that $X$ is invariant under $\sigma \circ (1, F)$, where $1$ is the identity map of $\mathbb{G}_a^2$, and $(\sigma \circ (1, F))|_X$ is of the form

\[
(t_1, t_2) \cdot (x, y) = \left( x + \sum_{i=0}^{m_2} g_i(c_1t_1 + c_2t_2)y^i + \mu(\bar{c}_2t_1 - \bar{c}_1t_2), y + c_1t_1 + c_2t_2 \right),
\]

where

\[
m_2 := [-k_X + \omega_0 + 1]/\omega_1,
\]

$(c_1, c_2) \in \mathbb{C}^2 \setminus \{(0, 0)\}$, $\mu \in \mathbb{C} \setminus \{0\}$, $\bar{c}_j$ are complex conjugates of $c_j$, and

\[
g_i(r) := \lambda_i r + \frac{\lambda_{i+1}}{2}(i+1)r^2 + \cdots + \frac{\lambda_{m_2}}{m_2-i+1}(\frac{m_2}{m_2-i})r^{m_2-i+1}
\]

for some $\lambda_0, \ldots, \lambda_{m_2} \in \mathbb{C}$.

3. Conversely, for every $g_0, \ldots, g_{m_2}, c_1, c_2$ as above, identity (12) defines a $\mathbb{G}_a^2$-structure on $X$.

Proof. Theorem 3.7 implies that given any two copies of $\mathbb{C}^2$ in $\bar{X}$, there is an automorphism of $\bar{X}$ that takes one to the other. Let $F$ be any automorphism of $\bar{X}$ which maps a $\sigma$-invariant copy of $\mathbb{C}^2$ to $\bar{X}$; set $\tau := \sigma \circ (1, F)$. Theorem 3.7 implies that $\tau|_X$ is given by

\[
(t_1, t_2) \cdot (x, y) = \left( a(t_1, t_2)x + \sum_{i=0}^{m_2} b_i(t_1, t_2)y^i, b(t_1, t_2)y + c(t_1, t_2) \right),
\]

where $m_2$ is from (13). Now the result follows from Lemma 4.1. \(\square\)
We continue with the notation from theorem 4.2. If $\omega_0 + k_\bar{X} < 0$, then theorem 4.2 implies that the following equation defines a $G_2^0$-structure $\tau_\lambda$ on $\bar{X}_{\omega,\bar{\theta}}$ for each $\lambda \in \mathbb{C}$.

\begin{equation}
(t_1, t_2) \circ \tau_\lambda (x, y) = \left( x + \lambda \sum_{i=0}^{m} \frac{1}{m - i + 1} \left( \sum_{j=0}^{i-1} h_i \binom{i}{j} t_1^{i-j} y^j \right) + t_2, y + t_1 \right)
\end{equation}

where $m := m_\omega$ from (13). Note that $\tau_0$ is simply the action $(t_1, t_2) \cdot (x, y) = (x + t_2, y + t_1)$.

**Theorem 4.3.** Let $\bar{\omega}_0, \ldots, \bar{\omega}_n$ be as in assertion (2) of theorem 3.7. Assume $\bar{X} \not\equiv \mathbb{P}^2$ and $\omega_0 + k_\bar{X} < 0$.

1. If $m = 0$, then every $G_2^0$-structure on $\bar{X}$ is equivalent to $\tau_0$.
2. If $n = 0$ (or equivalently, $\bar{X}$ is isomorphic to a weighted projective variety) and $m > 0$, then up to equivalence there are precisely two $G_2^0$-structures on $\bar{X}$, namely $\tau_0$ and $\tau_1$.
3. If $n \geq 1$ and $m > 0$, then
   a. every $G_2^0$-structure on $\bar{X}$ is equivalent to $\tau_\lambda$ for some $\lambda \in \mathbb{C}$;
   b. if $n = 1$, then $\tau_\lambda$ is equivalent to $\tau_{\lambda'}$ if $\bar{X} = \bar{\omega}_1$ for some $\omega_1$-th root $\zeta$ of identity;
   c. if $n > 1$, then $\tau_\lambda$ is equivalent to $\tau_{\lambda'}$ if $\bar{X} = \bar{\omega}_1$ for some $\omega_i$-th root $\zeta_i$ of identity, where $\omega_i := \gcd(\bar{\omega}_1, \ldots, \bar{\omega}_n)$.

**Proof.** The $m = 0$ case follows immediately from theorem 4.2. So assume $m \geq 1$. Let $X$ be a copy of $\mathbb{C}^2$ in $\bar{X}$ with coordinates $(x, y)$. Let $\sigma$ be an arbitrary $G_2^0$-action on $\bar{X}$. Theorem 3.7 implies that replacing $\sigma$ by a $G_2^0$-equivalent action if necessary, we may assume that $X$ is invariant under $\sigma$. From (12) it is straightforward to see that after a change of coordinates on $G_2^0$ we may assume that the action of $\sigma$ on $X$ has the following form:

\begin{equation}
(t_1, t_2) \circ \sigma (x, y) = \left( x + \sum_{i=0}^{m} g_i(t_1) y^i + t_2, y + t_1 \right),
\end{equation}

where $g_i$'s are defined as in (14). We would like to understand when there are automorphisms $F$ of $\bar{X}$ and $\phi$ of $G_2^0$ which induce an $G_2^0$-equivalence of $\sigma$ and $\tau_\lambda$. Lemma 5.1 below implies that $F(X) = X$. Therefore theorem 3.7 implies that $F|_X$ is of the form

\begin{equation}
(x, y) \mapsto (ax + h(y), dy + e)
\end{equation}

for some $h \in \mathbb{C}[y]$ such that $\deg(h) \leq m$. The $G_2^0$-equivalence of $\sigma$ and $\tau_\lambda$ is equivalent to the identity

\begin{equation}
F((t_1, t_2) \circ \sigma (x, y)) = \phi(t_1, t_2) \circ \tau_\lambda F(x, y)
\end{equation}

Writing $\phi(t_1, t_2) = (s_1, s_2)$ identity (19) becomes

\[ F \left( x + \sum_{i=0}^{m} g_i(t_1) y^i + t_2, y + t_1 \right) = (s_1, s_2) \circ \tau_\lambda (ax + h(y), dy + e) \]

which is equivalent to

\begin{equation}
\left( ax + \sum_{i=0}^{m} g_i(t_1) y^i + t_2 \right) + h(y + t_1), dy + dt_1 + e
\end{equation}

\[ = \left( ax + h(y) + \lambda \sum_{i=0}^{m} \frac{1}{m - i + 1} \binom{m}{m - i} s_1^{m-i+1}(dy + e)^i + s_2, dy + e + s_1 \right) \]

Comparing the $y$-coordinates of both sides of (20) gives that $s_1 = dt_1$. Write $h(y) = \sum_{i=0}^{m} h_i y^i$, $h_i \in \mathbb{C}$, $i = 0, \ldots, m$. Then a comparison of the $x$-coordinates of both sides of (20) implies that

\[ a \sum_{i=0}^{m} g_i(t_1) y^i + at_2 + \sum_{i=0}^{m} \sum_{j=0}^{i-1} h_i \binom{i}{j} t_1^{i-j} y^j = \lambda \sum_{i=0}^{m} \frac{1}{m - i + 1} \binom{m}{m - i} t_1^{m-i+1} s_2 + s_2 \]
or equivalently,

\[ a g_m(t_1) y^m + \sum_{j=0}^{m-1} \left( a g_j(t_1) + \sum_{i=j+1}^{m} h_i \binom{i}{j} t_1^{i-j} \right) y^j + a t_2 \]

\[ = \lambda d^m y^m + \lambda \sum_{j=0}^{m-1} \sum_{i=j}^{m} \frac{1}{m-i+1} \binom{m-i-j}{j} e^{i-j} t_1^{m-i+1} d^j y^j + s_2 \]

Write

\[ g_j(t_1) = \sum_{k=j}^{m} \frac{\lambda_k}{k-j+1} \binom{k}{k-j} t_1^{k-j+1}, \]

Comparing coefficients of \( y^m \) gives

\[ a \lambda_m = d^m \lambda \]

For each \( j = 1, \ldots, m-1 \), comparing coefficients of \( y^j \) gives

\[ \frac{a \lambda_m}{m-j+1} \binom{m}{j} t_1^{m-j+1} = \sum_{l=1}^{m-j} \left( \frac{a \lambda_{l+j-1}}{l} \binom{l+j-1}{j} + h_{l+j} \binom{l+j}{j} \right) t_1^l \]

\[ = \frac{d^j \lambda}{m-j+1} \binom{m}{j} t_1^{m-j+1} + \sum_{l=1}^{m-j} \frac{\lambda d^j}{l} \binom{m-l-1}{j} t_1^l \]

and similarly, for \( j = 0 \), we have

\[ \frac{a \lambda_m}{m+1} t_1^{m+1} + \sum_{l=1}^{m} \left( \frac{a \lambda_{l-1}}{l} + h_l \right) t_1^l + a t_2 = \frac{\lambda}{m+1} t_1^{m+1} + \sum_{l=1}^{m} \frac{\lambda}{l} \binom{m}{l-1} t_1^l + s_2 \]

Identity (22) and a comparison of the coefficients of \( t_1^j \) from (24) for \( j = m-1 \) imply that \( d = 1 \).

But then it is straightforward to check that

\[ h_k = \frac{a \lambda_{k-1}}{k} + \frac{1}{m+1} \binom{m+1}{k} e^{m+1-k}, \quad k = 1, \ldots, m \]

\[ s_2 = a t_2 \]

is a solution to equations (24) and (25). It follows that \( \sigma \) is equivalent to \( \tau_3 \) if and only if it can be arranged that \( a \lambda_m = \lambda \) and \( d = 1 \). The theorem now follows from theorem 3.7. \( \square \)

**Definition 4.4.** A singular del Pezzo surface is a singular normal projective surface \( Y \) with only ADE-singularities, i.e., singularities for which the dual graphs of minimal resolutions are Dynkin diagrams of type \( A_k, D_l, \) or \( E_m \) for some \( k, l \) or \( m \), and each of the irreducible exceptional curve are rational curves with self intersection number \(-2\). The type of a singular point on \( Y \) is the type (as a Dynkin diagram) of the weighted dual graph of its minimal resolution.

**Corollary 4.5.** Adopt the notation of theorem 4.2.

1. The following are equivalent:
   (a) \( X_{\omega, \delta} \) is a singular del Pezzo surface which admits a \( G_2^a \)-structure.
   (b) \( \omega \) is either (2, 1), (3, 2), (3, 2, 5) or (3, 2, 4).

2. If \( \omega = (2, 1) \), then \( X_{\omega, \delta} \cong \mathbb{P}^2(1, 1, 2) \); \( X_{\omega, \delta} \) has only one singular point and it is of type \( A_1 \). Up to equivalence \( X_{\omega, \delta} \) has precisely two \( G_2^a \)-structures.

3. If \( \omega = (3, 2) \), then \( X_{\omega, \delta} \cong \mathbb{P}^2(1, 3, 2) \); \( X_{\omega, \delta} \) has two singular points - one of type \( A_2 \) and the other of type \( A_1 \). Up to equivalence \( X_{\omega, \delta} \) has precisely two \( G_2^a \)-structures.

4. If \( \omega = (3, 2, 5) \), then \( X_{\omega, \delta} \) is isomorphic to the hypersurface in \( \mathbb{P}^2(1, 3, 2, 5) \) (with weighted homogeneous coordinates \( [w : x : y : z] \)) defined by \( wz - (y^3 + x^2) = 0 \); \( X_{\omega, \delta} \) has only one singular point and it is of type \( A_4 \). Up to equivalence \( X_{\omega, \delta} \) has a one dimensional moduli of \( G_2^a \)-structures.
(5) If \( \bar{\omega} = (3, 2, 4) \), then \( \bar{X}_{\bar{\omega}, \bar{g}} \) is isomorphic to the hypersurface in \( \mathbb{P}^2(1, 3, 2, 4) \) (with weighted homogeneous coordinates \([w : x : y : z]\)) defined by \( w^2z - (y^3 + x^2) = 0 \); \( \bar{X}_{\bar{\omega}, \bar{g}} \) has only one singular point and it is of type \( D_5 \). Up to equivalence \( \bar{X}_{\bar{\omega}, \bar{g}} \) has a one dimensional moduli of \( \mathbb{G}^2_a \)-structures.

Proof. \([\text{Mon13}, \text{Corollary 7.9}]\) implies that \( \bar{X}_{\bar{\omega}, \bar{g}} \) is a singular del Pezzo surface iff \( \bar{\omega} = (2, 1), (3, 2), (3, 2, 5, 1) \) or \( (3, 2, 6 - r), r = 1, \ldots, 5 \). Assertion (1) then follows from theorem 4.2 and the observation that \( \omega_0 + k_\bar{X} < 0 \) for precisely those \( \bar{\omega} \) listed in assertion (1b). The other assertions follow from theorem 4.3, \([\text{Mon13, corollary 7.9}]\) and the description in \([\text{Mon16, theorem 4.5}]\) of weighted dual graphs of minimal resolutions of singularities of primitive compactifications of \( \mathbb{C}^2 \).

5. \( \mathbb{G}^2_a \)-structures on minimal desingularizations of \( \mathbb{G}^2_a \)-surfaces of Picard rank one

Lemma 5.1. Let \( Y \) be an irreducible variety, \( G \) be a group, \( U \) be an open subset of \( Y \), and \( \sigma_1, \sigma_2 \) be two actions of \( G \) on \( Y \) such that \( U \) is an orbit of \( G \) under both \( \sigma_1 \) and \( \sigma_2 \). Assume there are automorphisms \( \alpha \) of \( G \) and \( j \) of \( Y \) such that \( \sigma_{\alpha}(a(g), j(y)) = j(\sigma_1(g, y)) \) for all \( g \in G \), \( y \in Y \).

Then \( j(U) = U \).

Proof. Indeed, since \( U \) is open in \( Y \), there exist \( U \cap j^{-1}(U) \neq \emptyset \). Pick \( y \in U \cap j^{-1}(U) \). Then

\[
U = G\text{-orbit of } j(y) \text{ under } \sigma_2 = \{ \sigma_2(a(g), j(y)) : g \in G \} = \{ j(\sigma_1(g, y)) : g \in G \} = j(G\text{-orbit of } y \text{ under } \sigma_1) = j(U)
\]
as required. \(\square\)

Let \( \bar{\omega} \) be a key sequence in normal form, \( \bar{X} := \bar{X}_{\bar{\omega}, \bar{g}} \) be a \( \mathbb{G}^2_a \)-surface of Picard rank 1 containing \( X \cong \mathbb{C}^2 \), and \( \pi : X^{\min} \to \bar{X} \) be the minimal desingularization. Let \( \sigma_1 \) and \( \sigma_2 \) be actions of \( \mathbb{G}^2_a \) on \( \bar{X} \) which fix \( X \). Assertion (1) of theorem B.4 implies that there are group actions \( \sigma'_j \), \( 1 \leq j \leq 2 \) of \( \mathbb{G}^2_a \) on \( X^{\min} \) which are compatible with \( \sigma \) and \( \pi \), i.e. \( \pi(\sigma'_j(a, x')) = \sigma_j(a, \pi(x')) \) for each \( j \) and \( a \in \mathbb{G}^2_a \), \( x' \in X^{\min} \).

Lemma 5.2. If \( \sigma'_1 \) is equivalent to \( \sigma'_2 \), then \( \sigma_1 \) is equivalent to \( \sigma_2 \).

Proof. Assume \( \sigma'_1 \) is equivalent to \( \sigma'_2 \) via automorphisms \( \alpha \) of \( \mathbb{G}^2_a \) and \( j \) of \( \bar{X}^{\min} \). Assertion (2) of theorem B.4 implies that \( j \) induces an automorphism \( j \) of \( \bar{X} \). It is straightforward to see that \( \sigma'_1 \) is equivalent to \( \sigma'_2 \) via \( (\alpha, j) \).

Corollary 5.3. The space of \( \mathbb{G}^2_a \)-structures modulo equivalence on \( X^{\min} \) is isomorphic to the space of \( \mathbb{G}^2_a \)-structures modulo equivalence on \( \bar{X} \).

Proof. It follows immediately from combining theorem 4.3 and lemma 5.2 once we observe that every \( \mathbb{G}^2_a \)-action on \( X^{\min} \) is equivalent to an action which fixes \( X \).

Corollary 5.4. If \( \omega_0 + k_\bar{X} < 0 \) and \( m_{\bar{\omega}} \geq 1 \), then the minimal resolution of singularities of \( \bar{X} \) admits a moduli of \( \mathbb{G}^2_a \)-structures.

6. Log terminal and log canonical and \( \mathbb{G}^2_a \)-surfaces with Picard rank one

In section 6.1 we recall following Alexeev [Ale92] a part of Kawamata’s [Kaw88] classification of two dimensional log terminal and log canonical singularities in terms of dual graphs of their resolutions of singularities. In section 6.2 we recall from [Mon16] the description of dual graphs of resolution of singularities of primitive normal compactifications of \( \mathbb{C}^2 \). In section 6.3 we combine these results to classify log terminal and log canonical primitive normal compactifications of \( \mathbb{C}^2 \), and among these characterize those which admit \( \mathbb{G}^2_a \)-structures.
6.1. Two dimensional log terminal and log canonical singularities.

Definition 6.1. Let $E_1, \ldots, E_k$ be non-singular curves on a (non-singular) surface such that for each $i \neq j$, either $E_i \cap E_j = \emptyset$, or $E_i$ and $E_j$ intersect transversally at a single point. Then $E = E_1 \cup \cdots \cup E_k$ is called a simple normal crossing curve. The (weighted) dual graph of $E$ is a weighted graph with $k$ vertices $V_1, \ldots, V_k$ such that

- there is an edge between $V_i$ and $V_j$ if $E_i \cap E_j \neq \emptyset$,
- the weight of $V_i$ is the self intersection number of $E_i$.

Usually we will abuse the notation, and label $V_i$’s also by $E_i$. If $\pi : Y' \to Y$ is the resolution of singularities of a surface, then the weighted dual graph of $\pi$ is the weighted dual graph of the union of ‘exceptional curves’ (i.e. the curves that contract to points under $\pi$) of $\pi$.

Definition 6.2. Let $(Y, P)$ be a germ of a normal surface, and $\pi : Y' \to Y$ be a resolution of the singularity of $(Y, P)$ such that the inverse image of $P$ is a normal crossing curve $E$. If $K_Y, K_{Y'}$ are respectively canonical divisors of $Y$, then

$$K_{Y'} = \pi^*(K_Y) + \sum_j a_j E_j$$

where the sum is over irreducible components $E_j$ of $E$ and $a_j$ are rational numbers. The singularity $(Y, P)$ is called log canonical (resp. log terminal) if $a_j \geq -1$ (resp. $a_j > -1$) for all $j$.

Definition 6.3. Given a simple normal crossing curve $E$ on a surface, we denote by $\Delta(E)$ the absolute value of the determinant of the matrix of intersection numbers of components of $E$. If $\Gamma$ is the weighted dual graph of $E$, then we also write $\Delta(\Gamma)$ for $\Delta(E)$.

The result below is a special case of Kawamata’s classification of log terminal and log canonical singularities. We follow the notation of Alexeev. The dual graphs of possible resolutions are listed in figs. 2 and 3. The notation used in these figures is as follows:

- each dot denotes a vertex;
- a number next to a dot represents the weight of the vertex;
- each empty oval denotes a chain, i.e. a tree such that every vertex has at most two edges;
- an oval with a symbol $\Delta$ in the interior denotes a chain $\Gamma$ with $\Delta(\Gamma) = \Delta$.

Theorem 6.4 (Kawamata [Kaw88], Alexeev [Ale92]). Let $\pi : Y' \to Y$ be a resolution of singularities of a germ $(Y, P)$ of normal surface. Assume that the exceptional curve $E$ of $\pi$ satisfies the following properties: $E$ is a simple normal crossing curve, each irreducible component of $E$ is a rational curve, and the dual graph of $E$ is a tree.

1. The singularity of $(Y, P)$ is log terminal iff the dual graph $\Gamma$ of $E$ is one of the graphs listed in fig. 2.

(2) The singularity of $(Y, P)$ is log canonical but not log terminal iff the dual graph $\Gamma$ of $E$ is one of the graphs listed in fig. 3.
6.2. Dual graphs of resolution of singularities of normal primitive compactifications of \(\mathbb{C}^2\). Let \(n \geq 0\), \(\bar{\omega} = (\omega_0, \ldots, \omega_{n+1})\) be a primitive key sequence in normal form, \(\bar{\theta} \in (\mathbb{C}^*)^n\), and \(\bar{X} \cong \tilde{X}_{\bar{\omega}, \bar{\theta}}\) be the corresponding primitive normal compactification of \(\mathbb{C}^2\). Let \(C_{\infty}\) be the curve at infinity on \(\bar{X}\). It turns out that one can associate a formal descending Puiseux series, i.e. a formal sum
\[
\hat{\phi}(x, \xi) = a_1x^{\beta_1} + \ldots + a_sx^{\beta_s} + \xi^{\beta_{s+1}}
\]
where
- \(s \geq 0\),
- \(a_1, \ldots, a_s \in \mathbb{C}^*\),
- \(\beta_1 > \cdots > \beta_{s+1}\) are rational numbers,
- \(\xi\) is an indeterminate,

such that for each \(f \in \mathbb{C}[x, y]\),
\[
\text{pole}_{C_{\infty}}(f) = \omega_0 \deg_x \left(f|_{y=\hat{\phi}(x, \xi)}\right)
\]

Note that the normality of \(\bar{\omega}\) implies that either \(s = 0\), or \(\beta_1\) is a positive rational number between 0 and 1 such that neither \(\beta_1\) nor \(1/\beta_1\) is an integer.

Let \(d_j\) be the lowest common denominator of (reduced forms of) \(\beta_1, \ldots, \beta_j, 1 \leq j \leq s + 1\). The sequence of formal characteristic exponents of \(\hat{\phi}\) is the sequence \(\beta_1 = \beta_{j_1} > \cdots > \beta_{j_{s+1}} = \beta_{s+1}\) of exponents which satisfy the following property:
- \(d_j = d_{j_k}\) for each \(j = j_k, \ldots, j_{k+1} - 1\),
- \(d_{j_{k+1}} > d_{j_k}\) for each \(k = 1, \ldots, l - 1\).

The formal Newton pairs of \(\hat{\phi}\) are \((q'_{1,1}, p_{1,1}), \ldots, (q'_{l+1,1}, p_{l+1,1})\), where
\[
p_k = \begin{cases} 
  d_{j_1} = d_1 & \text{if } k = 1, \\
  d_{j_k}/d_{j_{k-1}} & \text{if } k = 2, \ldots, l + 1. 
\end{cases}
\]
\[
q'_k = \begin{cases} 
  d_{j_1}\beta_1 & \text{if } k = 1, \\
  d_{j_k}(\beta_{j_k} - \beta_{j_{k-1}}) & \text{if } k = 2, \ldots, l + 1. 
\end{cases}
\]

The formal Newton pairs are completely determined by (and also completely determine) the essential subsequence (Definition 3.1) of the key sequence \(\bar{\omega}\). The relation among them is as follows:

**Proposition 6.5** ([Mon13, Propositoin A.1]). Let \(\bar{\omega}_e = (\omega_{i_0}, \ldots, \omega_{i_{l'+1}})\) be the essential subsequence of \(\bar{\omega}\) and \(\alpha_1, \ldots, \alpha_{n+1}\) be as in Definition 3.1. Then \(l' = l\), and for each \(j, 1 \leq j \leq l + 1\),
\[
p_j = \alpha_{i_j},
\]
\[
\beta_{i_j} := \frac{1}{\omega_0}(\omega_{i_j} - \sum_{k=1}^{j-1}(\alpha_{i_k} - 1)\omega_{i_k}), \quad 1 \leq j \leq l + 1.
\]

**Theorem 6.6** ([Mon16, Proposition 4.2]).

1. If \(p_{l+1} = 1\), then there is a resolution of singularities of \(\bar{X}\) such that the dual graph is of the form displayed in fig. 4.
(2) If $p_{l+1} > 1$, then there is a resolution of singularities of $\bar{X}$ such that the dual graph is a disjoint union of a chain $\Gamma$ with $\Delta(\Gamma) = p_{l+1}$ and a graph of the form displayed in fig. 4.

\[ \begin{array}{c}
|q'_1| & \cdots & |q'_{l+1}| \\
\bullet & \cdots & \bullet \\
p_1 & \cdots & p_l
\end{array} \]

**Figure 4.** Dual graphs in theorem 6.6

6.3. **Primitive compactifications with log terminal and log canonical singularities.** In this section we combine the results of preceding two sections to classify primitive normal compactifications of $\mathbb{C}^2$, including all Picard rank on $G^2_n$-surfaces, with log terminal and log canonical singularities.

We continue to use the notation from section 6.2. In particular, $\bar{\omega} = (\omega_0, \ldots, \omega_{n+1})$ is a primitive key sequence in normal form, $\bar{\theta} \in (\mathbb{C}^*)^n$, and $\tilde{X} := \bar{X}_{\bar{\omega}, \bar{\theta}}$ is the corresponding primitive normal compactification of $\mathbb{C}^2$. The following are straightforward consequences of the normality of $\bar{\omega}$ and proposition 6.5:

(i) $\gcd(q'_l, p_1) = 1$;
(ii) if $l \geq 1$, then $p_1 > 2$, $p_1 > q_1$, and neither $q'_1/p_1$ nor $p_1/q'_1$ is an integer.

**Theorem 6.7.** Let $\bar{\omega}_e$ be the essential subsequence (Definition 3.1) of $\bar{\omega}$.

1. $\bar{X}$ has only log terminal singularities iff $\bar{\omega}_e$ is one of the key sequences in the first column of table 1. $\bar{X}$ is in addition a $G^2_n$-surface iff it satisfies the conditions from the 4th column of table 1.
2. $\bar{X}$ has a log canonical singularity which is not log terminal iff $\bar{\omega}_e$ is the key sequence from table 2. $\bar{X}$ is in addition a $G^2_n$-surface iff it satisfies the condition from the 4th column of table 2.

**Proof.** The first statements of assertions (1) and (2) follow from observations (i), (ii), proposition 6.5 and theorems 6.4 and 6.6. For the criteria for having $G^2_n$-surface structures, we use theorem 4.2. If $n = 0$, then $k_{\tilde{X}} + \omega_0 = -\omega_1 - 1 < 0$, so that $\tilde{X}$ is a $G^2_0$-surface; this explains the first two rows of table 1. The key sequences in table 2 and the remaining cases of table 1 are of the form $(p_1 p_2, q_1 p_2, q_1 p_2 - r)$, with $\alpha_1 = p_1$ and $\alpha_2 = p_2$. It follows that

$$k_{\tilde{X}} + \omega_0 = -q_1 p_1 p_2 + r - 1 + (p_1 - 1) q_1 p_2 = r - 1 - q_1 p_2$$

Therefore $k_{\tilde{X}} + \omega_0 < 0$ iff $q_1 p_2 \geq r$. This explains the criteria for $G^2_n$-surface structures in the 4th columns of tables 1 and 2. □

**Appendix A. Proof of Lemma 4.1**

At first we prove the ($\Rightarrow$) implication. The compatibility of the action implies that

\begin{align*}
(32a) \quad a(t_1 + t'_1, t_2 + t'_2) &= a(t_1, t_2) a(t'_1, t'_2) \\
(32b) \quad b(t_1 + t'_1, t_2 + t'_2) &= b(t_1, t_2) b(t'_1, t'_2), \\
(32c) \quad c(t_1 + t'_1, t_2 + t'_2) &= b(t_1, t_2) c(t'_1, t'_2) + c(t_1, t_2), \\
(32d) \quad f(t_1 + t'_1, t_2 + t'_2, y) &= a(t_1, t_2) f(t'_1, t'_2, y) + (t_1, t_2) b(t'_1, t'_2) y + c(t'_1, t'_2), \\
(32e) \quad f(t_1, t_2, y) := \sum_{i=0}^{m} b_i(t_1, t_2) y^i
\end{align*}
| $\tilde{\omega}_e$ | Formal Newton pairs | Dual graph of resolution of singularities | $G_2^3$-surface iff |
|-----------------|---------------------|----------------------------------------|------------------|
| $(1, 1)$        | $(1, 1)$            | -                                      | always           |
| $(p, q)$        | $(q, p)$            | if $q = 1$                             | always           |
| $p > q \geq 1$, $\text{gcd}(p, q) = 1$ |                             | $q \cup p$                           |                  |
| $(p_1, p_2, q_1, p_2 - 1)$ | $(q_1, p_1, -1, p_2)$ | if $p_2 = 1$                          | always           |
| $p_1 > q_1 > 1$, $\text{gcd}(p_1, q_1) = 1$, $p_2 \geq 1$ |                             | $q_1 \cup p_2$                     |                  |
| $(p_1, p_2, 2p_1, 2p_1 - 2)$ | $(2, p_1, -2, p_2)$ | if $p_2 = 1$                          | always           |
| $p_1, p_2 \text{ odd, } p_1 > 2$, $p_2 \geq 1$ |                             | $2 \cup p_2$                        |                  |
| $(p_1, p_2, 2p_1, 2p_1 - r)$ | $(2, p_1, -r, p_2)$ | if $p_2 = 1$                          | never            |
| $(p_1, r) \in \{(3, 3), (3, 4), (3, 5), (5, 3)\}$, $p_2 \geq 1$, $\text{gcd}(p_2, r) = 1$ |                             | $2 \cup p_2$                        | $p_2 \geq 3$ if $(p_1, r) = (3, 5)$, $p_2 \geq 2$ otherwise |
| $(p_1, 3p_2, 3p_2 - 2)$ | $(3, p_1, -2, p_2)$ | if $p_2 = 1$                          | always           |
| $p_1 = 4, 5$, $p_2 \geq 1$, $\text{gcd}(p_2, 2) = 1$ |                             | $3 \cup p_2$                        |                  |

**Table 1.** Log terminal primitive compactifications

| $\tilde{\omega}_e$ | Formal Newton pairs | Dual graph of resolution of singularities | $G_2^3$-surface iff |
|-----------------|---------------------|----------------------------------------|------------------|
| $(3p, 2p, 6p - 6)$ | $(2, 3, -6, p)$     | $2 \cup \{p\}$                        | $p \geq 3$       |
| $p \geq 2$, $\text{gcd}(p, 6) = 1$ |                             |                                        |                  |

**Table 2.** Primitive compactifications which are log canonical but not log terminal
Since \(a, b\) are non-zero polynomials in \((t_1, t_2)\), identities (32a) and (32b) imply that \(a(t_1, t_2) = b(t_1, t_2) = 1\) for all \((t_1, t_2) \in \mathbb{C}^2\). Identity (32c) then implies that \(c\) is a linear function in \((t_1, t_2)\), i.e. \(c(t_1, t_2) = c_1 t_1 + c_2 t_2\) for some \(c_1, c_2 \in \mathbb{C}\). Consequently identity (32d) implies that

\[
\sum_{i=0}^{m} (b_i(t_1 + t'_1, t_2 + t'_2) - b_i(t_1', t_2')) = \sum_{i=0}^{m} b_i(t_1, t_2)(y + c_1 t'_1 + c_2 t'_2)
\]

If \(c_1 = c_2 = 0\), then (33) implies that each \(b_i\) is linear and the action is given by

\[
(t_1, t_2) \cdot (x, y) = \left( x + \sum_{i=0}^{m} b_i(t_1, t_2)y^i, y \right)
\]

Now assume \((c_1, c_2) \neq (0, 0)\). For all \(\lambda \in \mathbb{C}\), plugging in \((t'_1, t'_2) = (\lambda c_2, -\lambda c_1)\) in (33) gives that

\[
\sum_{i=0}^{m} (b_i(t_1 + \lambda c_2, t_2 - \lambda c_1) - b_i(\lambda c_2, -\lambda c_1) - b_i(t_1, t_2))y^i = 0
\]

so that

\[
b_i(t_1 + \lambda c_2, t_2 - \lambda c_1) - b_i(\lambda c_2, -\lambda c_1) - b_i(t_1, t_2) = 0
\]

for each \(i = 0, \ldots, m\). Let \(\sigma : \mathbb{C}^2 \to \mathbb{C}^2\) be the map defined by

\[
(t_1, t_2) \mapsto (t_1', t_2') = \left( \tilde{c}_2, \frac{c_1}{c_2} \right)
\]

where \(\tilde{c}_i\) denotes the conjugate of \(c_i\), \(i = 1, 2\). Let \(\tilde{b}_i := b_i \circ \sigma^{-1}\). Identity (35) implies that

\[
\tilde{b}_i(s_1 + \lambda |c|^2, s_2) - \tilde{b}_i(\lambda |c|^2, 0) - \tilde{b}_i(s_1, s_2) = 0, \quad i = 0, \ldots, m
\]

where we wrote \((s_1, s_2)\) for \(\sigma(t_1, t_2)\) and \(|c|^2\) for \(|c_1|^2 + |c_2|^2\). It follows from (36) in a straightforward manner that for each \(i = 0, \ldots, m\),

\[
\tilde{b}_i(s_1, s_2) = g_i(s_2) + \mu_is_1
\]

for some \(\mu_i \in \mathbb{C}\) and \(g_i \in \mathbb{C}[s_2]\) such that \(g_i(0) = 0\). Plugging \(b_i(t_1, t_2) = \tilde{b}_i \circ \sigma(t_1, t_2) = g_i(c_1 t_1 + c_2 t_2) + \mu_i(c_2 t_1 - c_1 t_2)\) into (33) gives

\[
\sum_{i=0}^{m} (g_i(r + r') - g_i(r') + \mu_is)x^i = \sum_{i=0}^{m} (g_i(r) + \mu_is)(x + r')^i
\]

where \(r := c_1 t_1 + c_2 t_2\), \(r' := c_1 t_1 + c_2 t_2\), and \(s := c_2 t_1 - c_1 t_2\), \(i = 0, \ldots, s\). Substituting \(r = 0\) in (38) and using \(g_i(0) = 0\) yields that

\[
\sum_{i=0}^{m} \mu_ix^i = \sum_{i=0}^{m} \mu_i(x + r')^i
\]

which implies \(\mu_i = 0\) for \(i = 1, \ldots, m\). On the other hand, differentiating (38) with respect to \(r\) and then substituting \(r = 0\) yields

\[
\sum_{i=0}^{m} g'_i(r')x^i = \sum_{i=0}^{m} g'_i(0)(x + r')^i
\]

A comparison of coefficients of \(x^i\) from both sides of (40) gives

\[
g'_i(r') = g'_i(0) + g'_{i+1}(0) \binom{i + 1}{1} r' + \cdots + g'_m(0) \binom{m}{m-i} r'^{m-i}, \quad i = 0, \ldots, m.
\]

Since \(g_i(0) = 0\) for each \(i\), this implies that \(g_i\)'s are as in (11) with \(\lambda_i := g'_i(0), i = 0, \ldots, m\), and completes the proof of \((\Rightarrow)\) direction.
Now we prove the \((\Leftarrow)\) implication. It suffices to show that if \((c_1, c_2) \neq (0, 0)\), then identity (33) holds with \(b_j\)'s defined in (10). But then with \(r, r'\) defined as in (38), identity (33) is equivalent to the following identity:

\[
\sum_{i=0}^{m} (g_i(r + r') - g_i(r'))x^i = \sum_{i=0}^{m} g_i(r)(x + r')^i,
\]

which is in turn equivalent to identities below:

\[
g_i(r + r') - g_i(r') = \sum_{j=i}^{m} \binom{j}{j-i+1} (r + r')^{j-i+1} - r'^{j-i+1} \tag{42}
\]

Now (11) implies that for each \(i = 0, \ldots, m\),

\[
g_i(r + r') - g_i(r') = \sum_{j=i}^{m} \lambda_j \binom{j}{j-i+1} ((r + r')^{j-i+1} - r'^{j-i+1})
\]

\[
= \sum_{j=i}^{m} \lambda_j \frac{j!}{(j-k-i+1)!} \left( \frac{k}{j-k-i} \right) (r + r')^{j-i+1} - r'^{j-i+1}
\]

\[
= \sum_{k=0}^{m-i} \frac{(k+i)!}{k!} r^k \sum_{j=k+i}^{m} \frac{\lambda_j}{(j-k-i+1)!} (j-k-i)! (j-k-i+1)
\]

\[
= \sum_{k=0}^{m-i} \frac{(k+i)!}{k!} r^k g_{k+i}(r)
\]

\[
= \sum_{j=i}^{m} \binom{j}{j-i+1} (r')^{j-i} g_j(r),
\]

as required. \(\square\)

**Appendix B. Automorphisms of minimal desingularizations of primitive compactifications of \(\mathbb{C}^2\)**

In this section we show that every automorphism of a primitive compactification \(\bar{X}\) of \(X := \mathbb{C}^2\) lifts to an automorphism of the minimal desingularization \(X^{\text{min}}\) of \(\bar{X}\). Conversely, we also show that every automorphism of \(X^{\text{min}}\) which fixes \(X\) descends to an automorphism of \(\bar{X}\).

Let \((Y, P)\) be a germ of a non-singular analytic surface. Choose analytical coordinates \((u, v)\) on \(Y\) such that \(P = \{u = v = 0\}\). Let \(\tilde{\rho}, \tilde{q}\) be relatively prime positive integers such that \(\tilde{\rho} > \tilde{q} \geq 1\), and let \(\pi' : Y' \rightarrow Y\) be the minimal resolution of the singularity of the curve \(C := \{u(v^\tilde{\rho} - u^\tilde{q}) = 0\}\) at \(P\) i.e.

(i) \(\pi'\) is an isomorphism outside the inverse image of \(P\);
(ii) the strict transform of \(u(v^\tilde{\rho} - u^\tilde{q}) = 0\) on \(Y'\) intersects the union of the exceptional curves of \(\pi'\) transversally;
(iii) every \(\pi'' : Y'' \rightarrow Y\) satisfying the above two properties factors through \(\pi'\).
The morphism \( \pi' \) can be expressed as a sequence of blow ups. Let \( E_j, j = 1, 2, \ldots \), be the strict transform of the \( j \)'th blow up on \( Y' \). Denote by \( E_0 \) the strict transform of \( u = 0 \) on \( Y' \). Given a germ \( C \) of a curve at \( P \), we say that \( C \) is an \( E_j \)-curvette if the strict transform of \( C \) on \( Y' \) intersects \( E_j \) transversally. The following lemma follows from standard theory of resolution of curve singularities.

**Lemma B.1.** Express \( \tilde{p}/\tilde{q} \) as a continued fraction in the following way:

\[
\frac{\tilde{p}}{\tilde{q}} = m_1 + \frac{1}{m_2 + \frac{1}{\ddots + \frac{1}{m_N}}}
\]

where \( m_j \geq 2, j = 1, \ldots, N \). Then

1. The dual graph of \( E_0 \cup E_1 \cup \cdots \) is as in fig. 5.

![Figure 5. Dual graph for the minimal resolution of singularities of monomial curve singularities](image)

(2) The self intersection number of \( E_0 \) is \( 1 - \left\lfloor \frac{\tilde{p}}{\tilde{q}} \right\rfloor \).

(3) Set \( M_j := \sum_{i=1}^{j} m_j, j = 0, \ldots, N \). For each \( j = 0, \ldots, N - 1 \) and each \( k = 1, \ldots, m_j + 1 \), let \( \check{p}_{M_j+k}, \check{q}_{M_j+k} \) be the positive relatively prime integers such that

\[
\frac{\check{p}_{M_j+k}}{\check{q}_{M_j+k}} = m_1 + \frac{1}{m_2 + \frac{1}{\ddots + \frac{1}{m_j + \frac{1}{k}}}}
\]

Then for generic \( \xi' \in \mathbb{C} \), the germ of \( v^{\check{p}_{M_j+k}} - \xi' u^{\check{q}_{M_j+k}} = 0 \) is an \( E_{M_j+k} \)-curvette. \( \square \)

**Claim B.2.** Adopt the notation of lemma B.1. Fix \( j, 0 \leq j \leq N - 1 \).

1. Assume \( j \) is even. Then

\[
\frac{(\check{p}_{M_j+k} - \check{q}_{M_j+k})/\check{p}_{M_j+k} < (\tilde{p} - \tilde{q})/\tilde{p}}
\]

(46)

\[
\left[ (\check{p}_{M_j+k} - \check{q}_{M_j+k})/\check{p}_{M_j+k} \right] = \left[ (\tilde{p} - \tilde{q})/\tilde{p} \right] = 0
\]

2. Assume \( j \) is odd. Then

\[
(\check{p}_{M_j+k} - \check{q}_{M_j+k})/\check{p}_{M_j+k} > (\tilde{p} - \tilde{q})/\tilde{p}
\]
Let $\Gamma$ be the weighted chain (where the weight of a vertex is the self intersection number of the corresponding curve) connecting $E_0$ to $E_{M_1+k}$. If $\Gamma$ is not as in fig. 6, then

$$[\tilde{p}_{M_1+k}/(\tilde{p}_{M_1+k} - \tilde{q}_{M_1+k})] \geq [\tilde{p}/(\tilde{p} - \tilde{q})]$$

(48)

\[ E_0 \quad -1 \quad E_{m_1+1} \quad -2 \quad E_{M_1+k} \]

\[ E_0 \quad -1 \quad E_2 \quad -2 \quad E_1 \]

\textbf{Figure 6.} ‘Irrelevant’ weighted chain

\textbf{Figure 7.} Case $\tilde{p}/\tilde{q} = 2$

Proof. Inequalities (45) and (47) follows immediately from assertion (3) of lemma B.1. Since $\tilde{q} < \tilde{p}$, inequality (46) follows from (45). We now prove (48). If $\tilde{p}/\tilde{q} > 2$ then

$$[\tilde{p}_{M_1+k}/(\tilde{p}_{M_1+k} - \tilde{q}_{M_1+k})] \geq 1 = [\tilde{p}/(\tilde{p} - \tilde{q})]$$

If $\tilde{p}/\tilde{q} = 2$, then the dual graph from fig. 5 is as follows, where we also list the weights (i.e. self intersection number of the corresponding curves): Therefore (48) is vacuously true. Now assume

$\tilde{p}/\tilde{q} < 2$. Then $m_1 = 1$ and $N \geq 2$ in identity (43). In particular, identities (43) and (44) imply that

$$[\tilde{p}_{1+m_2}/(\tilde{p}_{1+m_2} - \tilde{q}_{1+m_2})] = [\tilde{p}/(\tilde{p} - \tilde{q})] = 1 + m_2$$

(49)

It is straightforward to see that the weighted chain consisting of $E_0, E_2, E_3, \ldots, E_{m_2}$ is as in fig. 6, which proves (48).

Now we apply the preceding observations to minimal resolution of a primitive compactification $\tilde{X}$ of $\mathbb{C}^2$. Pick the (unique) primitive key sequence $\vec{\omega} = (\omega_0, \ldots, \omega_{n+1})$ in normal form and $\vec{\beta} \in (\mathbb{C}^*)^n$ such that $\tilde{X} \cong \tilde{X}_{\vec{\omega}, \vec{\beta}}$. As in section 6.2 let $\tilde{\phi}(x, \xi) = \sum_{j=1}^s a_j x^{\beta_j} + \xi x^{\beta_{s+1}}$ be the formal descending Puiseux series associated to $\tilde{X}$. Let $\beta_1 = \beta_{j_1} > \cdots > \beta_{j_{s+1}} = \beta_{s+1}$ be the formal characteristic exponents, and $(q_{j_1}', p_{j_1}), \ldots, (q_{j_{s+1}}', p_{j_{s+1}})$ be the formal Newton pairs of $\vec{\omega}$. Embed $X := \mathbb{C}^2$ into $\mathbb{P}^2$ via $(x, y) \mapsto [x : y : 1]$. Then $(u, v) := (1/x, y/x)$ are analytic coordinates near $P := [1 : 0 : 0] \in \mathbb{P}^2$; note that $u = 0$ is the equation of the line at infinity on $\mathbb{P}^2$. Pick a generic $\xi' \in \mathbb{C}$. Then

$$\tilde{\psi}(u, \xi') := u \tilde{\phi}(1/u, \xi)|_{\xi = \xi'} = \sum_{j=1}^s a_j u^{1-\beta_j} + \xi u^{1-\beta_{s+1}}$$

is a (finite) Puiseux series in $u$. Let $C$ be the germ at $P$ of the (reduced) union of the line at infinity and the irreducible analytic curve with Puiseux expansion $v = \tilde{\psi}(u, \xi')$. It turns out (see e.g. [Mon16, Proposition 4.2]) that

(iv) If $\pi' : X' \to \mathbb{P}^2$ is the minimal resolution (in the sense of properties (i)-(iii)) of the singularity at $P$ of $C$, then $X'$ is also a resolution of singularities of $\tilde{X}$.

(v) The dual graph of the resolution $\sigma' : X' \to X$ is of the form described in theorem 6.6.

More precisely, in fig. 4

(1) the strict transform $E_0$ of the line at infinity on $\mathbb{P}^2$ is the ‘left end’ of the leftmost chain (with $\Delta$-value $|q_1'| = q_1'$).
Let $m(i)$ from the proof of Theorem 5.2 imply that $E$ be the integer associated to (the key sequence associated to) exceptional curves of $\pi$, and has an associated formal descending Puiseux series $\tilde{x}$, and possibly also the strict transform of the line at infinity. The latter gets contracted if and only if $q_i \leq p_i/2$, where $(q_1,p_1), \ldots, (q_{l+1},p_{l+1})$ are formal Newton pairs of $\phi$.

Let $E$ be an exceptional curve of $\pi'$. Then $E$ defines a divisorial valuation centered at infinity on $C[x,y]$, and has an associated formal descending Puiseux series $\phi_E(x,\xi)$. Moreover,

(vi) The minimal resolution $\bar{X}_{\text{min}}$ of singularities of $X$ is formed by contracting some of the exceptional curves of $\pi'$, and possibly also the strict transform of the line at infinity. The latter gets contracted if and only if $q_i \leq p_i/2$, where $(q_1,p_1), \ldots, (q_{l+1},p_{l+1})$ are formal Newton pairs of $\phi$.

Let $m_E$ be the integer associated to (the key sequence associated to) $E$ defined as in (13), with $\bar{\omega}$ replaced by the key sequence associated to $E$. Observations (v.2), (v.3) and [Mon13, Observation (i) from the proof of Theorem 5.2] imply that

\[
\begin{align*}
(54) & \quad m_{\omega} = \frac{\text{ord}_x(\phi) + 1}{\deg_x(\phi)} - 1 = \left[ \frac{p_1}{q_1} (\beta_{s+1} + 1) - 1 \right] \\
(55) & \quad m_E = \frac{\text{ord}_x(\phi_E) + 1}{\deg_x(\phi_E)} - 1 \begin{cases} 
\frac{\tilde{p}_k}{\tilde{p}_k - \tilde{q}_k} & \text{in the scenario of (v.3.b) with } i = 1, \\
\frac{p_1}{q_1} \left( \beta_{j_i-1} - \frac{\tilde{q}_k}{p_1 \cdots p_{i-1} \tilde{p}_k} + 1 \right) - 1 & \text{in the scenario of (v.3.b) with } i > 1, \\
\frac{p_1}{q_1} (\beta_{j_i} + 1) - 1 & \text{in the scenario of (v.3.a),} \\
\frac{p_1}{q_1} (\beta_{j_i} + 1) - 1 & \text{in the scenario of (v.2).}
\end{cases}
\end{align*}
\]
Lemma B.3. Let $E$ be an exceptional curve of the minimal resolution $\sigma : \bar{X}^{\min} \to \bar{X}$ of singularities of $\bar{X}$. Then $m_E \geq m_\varnothing$, where $m_\varnothing$ is as in (13).

Proof. Observation (vi) implies that either

(a) $E$ comes from either an exceptional curve of $\pi' : X' \to \mathbb{P}^2$,

(b) or $E$ is the strict transformation of the line at infinity on $\mathbb{P}^2$.

At first consider case (a). In the scenario of observations (v.2) or (v.3.a) Identities (54) and (55) immediately imply that $m_E \geq m_\varnothing$. Now note that

\[(56) \quad m_\varnothing \leq \left\lfloor \frac{p_1}{q_1}(\beta_1 + 1) - 1 \right\rfloor = \left\lfloor \frac{p_1}{q_1} \right\rfloor \]

If (v.3.b) holds with $i = 1$, then (45), (48) and (56) imply that $m_E \geq m_\varnothing$. On the other hand, if (v.3.b) holds with $i > 1$, then (46) and (47) imply that

\[|\tilde{p}_k - q_k|/\tilde{p}_k \geq |(p_i - |q_i'|)/p_i| \]

Since $p_1 \cdots p_{i-1} \beta_{j_i-1}$ is an integer, it follows that

\[
\begin{align*}
(p_1 \cdots p_{i-1} \beta_{j_i-1} - \tilde{q}_k)/\tilde{p}_k &\geq |p_1 \cdots p_{i-1} \beta_{j_i-1} - |q_i'|/p_i| = |p_1 \cdots p_{i-1} \beta_{j_i}| \\
\Rightarrow p_1 \cdots p_{i-1} (\beta_{j_i-1} - q_k/(p_1 \cdots p_{i-1} \tilde{p}_k) + 1) &\geq |p_1 \cdots p_{i-1} (\beta_{j_i} + 1)| \\
\Rightarrow (\beta_{j_i-1} - q_k/(p_1 \cdots p_{i-1} \tilde{p}_k) + 1)p_1/q_1' &\geq |\beta_{j_i} + 1)p_1/q_1'| \geq m_\varnothing 
\end{align*}
\]

as required.

Now consider Case (b). Since $E_0$ does not get contracted, it follows from the arguments in the proof of Claim B.2 that $p_1/|q_i'| > 2$. Identity (56) then implies that $m_\varnothing \leq 1 = m_E$. \qed

Adopt the notation of lemma B.3. Let Aut$_X(\bar{X})$ (resp. Aut$_X(\bar{X}^{\min})$ be the set of automorphisms of $\bar{X}$ (resp. $\bar{X}^{\min}$) that fix $X$.

Theorem B.4.

1. Every automorphism $F$ of $\bar{X}$ lifts to an automorphism of $\bar{X}^{\min}$ and $F(E) = E$ for every exceptional curve of $E$ of $\sigma$.

2. Every automorphism of $\bar{X}^{\min}$ that fixes $X$ descends to an automorphism of $\bar{X}$.

3. $\sigma$ induces an isomorphism Aut$_X(\bar{X}) \cong$ Aut$_X(\bar{X}^{\min})$.

Proof. Let $F$ be an automorphism of $\bar{X}$. If $F(X) = X$, then lemma B.3 and [Mon13, Theorem 4.9] imply that assertion (1) holds for $F$. If $\bar{X}$ is not isomorphic to a weighted projective space of the form $\mathbb{P}^2(1,1,p)$, then [Mon13, Proposition 5.1] implies that every automorphism of $\bar{X}$ fixes $X$, so that assertion (1) holds for $\bar{X}$. Now assume $\bar{X} \cong \mathbb{P}^2(1,1,p)$. Since $\bar{X}$ is in the normal form, this is due to [Mon13, theorem 5.2] that $\bar{X} = (p,1)$. It is then straightforward to see that there is only one irreducible exceptional curve $E$ of $\sigma$ and the order of pole of a polynomial $f$ along $E$ is precisely $\deg_y(f)$. [Mon13, Theorems 4.9 and 5.2] then imply that assertion (1) holds for $\bar{X}$.

Since $\bar{X}$ is in the normal form, observation (vii) implies that for every irreducible exceptional curve of $E$ of $\sigma$, the ‘key sequence’ of the pole along $E$ is in the ‘normal form’ (in the sense of [Mon13, section 4]) with respect to $(x,y)$-coordinates; moreover, the key sequences are distinct for distinct (irreducible) exceptional curve. [Mon13, Theorem 4.6] then implies that $F(E) = E$ for every irreducible exceptional curve of $E$ of $\sigma$. Assertion (2) then follows from [Mon13, theorems 4.9 and 5.2]. Assertion (3) is a consequence of the assertions (1) and (2). \qed

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