Alternative to Morse-Novikov Theory for a closed 1-form (II)

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Abstract

This paper is a continuation of [1] and establishes:

a) a refinement of Poincaré duality to an equality between the configurations $BM_{\delta^\omega}$ and $\delta^\omega$ resp. $BM_{\gamma^\omega}$ and $\gamma^\omega$ in complementary dimensions,

b) the stability property for the configurations $\delta^\omega_{r^r}$,

c) a result needed for the proof of Theorems 1.2 and 1.3 in [1].

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1 Introduction

This paper is a continuation of [1] and establishes:

a) a refinement of Poincaré duality to an equality between the configurations $BM_{\delta^\omega}$ and $\delta^\omega$ resp. $BM_{\gamma^\omega}$ and $\gamma^\omega$ in complementary dimensions,

b) the stability property for the configurations $\delta^\omega_{r^r}$,

c) a result needed for the proof of Theorems 1.2 and 1.3 in [1].

Since we are interested in Poincaré duality which has to be considered for non compact manifolds, it is necessary to involve the functor $BM_{H_r}$, the Borel-Moore homology with coefficients in a fixed field $\kappa$, abbreviated BM-homology, and the natural transformation $\theta_r : H_r \rightarrow BM_{H_r}$. We suggest [7] chapter 5 as reference to Borel-Moore homology.

Recall that Borel-Moore homology with coefficients in a field $\kappa$ is a collection of $\kappa$—vector space-valued functors $BM_{H_r}$ defined on the category of pairs $(X, Y)$, $Y \subseteq X$, $X$ locally compact space, with $Y$ closed subset of $X$, and proper continuous maps of pairs, which are homotopic functors (i.e. proper homotopic maps induce equal linear maps) with the excision and the long exact sequence property for any such pair.

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(X,Y). When restricted to the subcategory of pairs of compact spaces they coincide with the standard homology. The reader should be aware that for a pair (M,N) of f.d. manifolds, each interiors of compact manifold with boundary, with N a closed subset, $BMH_r(M,N) = H_r(\tilde{M} , \tilde{N})$ and $BMH_r(M) = H_r(\tilde{M} , *)$ where $\tilde{X}$ denotes the one point compactification of the locally compact space $X$. This partially explains the natural maps $H_r(\cdot , \cdot ) \rightarrow BM H_r(\cdot , \cdot )$ from the standard homology (= singular homology) to the Borel-Moore homology as well as the contravariant behavior of Borel-Moore homology when restricted to open sets of a locally compact space; both properties hold for any $X,Y$ locally compact spaces not necessary manifolds.

The Borel-Moore homology is not a priory defined for a pair (X,Y) when both X and Y are locally compact but Y is not closed, however a vector space $BMH^f_\omega(X,Y)$ can be introduced in the case Y is open in X and appears as an open sub-level of a continuous real-valued map $f$, as described in Section 3.

Note that all definitions and concepts derived in [1] using standard homology can be ”word by word” repeated for Borel-Moore homology. In particular one can produce the Borel-Moore version of the vector spaces $1^f_\omega$, $g^f_\omega$, $f^f_\omega$ and ultimately of the numbers $\delta^f_\omega(a,b)$, $\gamma^f_\omega(t)$, $\lambda^f_\omega(a)$ denoted by the same symbols with the addition of the left-side exponent $BM$. The vector spaces $\hat{\lambda}^f_\omega(a)$ / numbers $\lambda^f_\omega(a)$ have not been described in [1] ; they are defined here in Section 3. For f a lift of a tame map they are finite dimensional/ finite and vanish for $a$ regular value. We believe they vanish for any a being actually irrelevant intermediate objects. In general these spaces / numbers defined using Borel-Moore homology are different from the ones defined using standard homology. This is reviewed in . The natural transformations $\theta_r : H_r(\cdot , \cdot ) \rightarrow BM H_r(\cdot , \cdot )$ induce the natural linear maps $\theta_r(a) : \hat{H}_r(\tilde{X}_a, \tilde{X} , <a) \rightarrow BM H_r(\tilde{X}_a, \tilde{X} , <a)$, $\theta^f_r(a,b) : \delta^f_\omega(a,b) \rightarrow BM \delta^f_\omega(a,b)$, $\theta^f_r(a,b) : \gamma^f_\omega(a,b) \rightarrow BM \gamma^f_\omega(a,b)$ and $\theta^\omega_r(a) : \lambda^f_\omega(a) \rightarrow BM \lambda^f_\omega(a)$. As already noticed in [1] if $\delta^f_\omega(a,b) \neq 0$ resp $\gamma^f_\omega(a,b) \neq 0$ implies that both a and b are homological critical values w.r. to the homology under consideration. It is a consequence of Proposition 1.4 below that t is a critical value w.r. to standard homology iff is a critical value w.r. to Borel-Moore homology.

Recall from [1] that $Z^1(X;\xi)$ denotes the subspace of tame topological closed one forms with the topology induced from the compact open topology of $Z^1(X;\xi)$, the space of topological closed one forms of cohomology class $\xi$, as defined in Section 2 of [1], and $Conf_{BM}(X;\xi)(\mathbb{R})$ denotes the space of configurations of points with multiplicity in $\mathbb{R}$ of total cardinality $\beta^N_r(X;\xi)$ equipped with the collision topology. Here $\beta^N_r(X;\xi)$ denotes the Novikov-Betti number of $(X,\xi)$, $\xi \in H^1(X;\kappa)$. Recall that the space of configurations of points in Y of total cardinality N with the collision topology can be identified to the quotient space $Y^N/\Sigma_N$, where $\Sigma_N$ denotes the group of permutations of N elements acting on $Y^N$ by permutations.

The main results verified in this paper are the following:

**Theorem 1.1**

For M is a closed topological manifold and $\omega \in Z^1_\omega(X;\xi)$ the following holds true.

1. $BM \delta^\omega_\omega(t) = \delta^\omega_{n-r}(-t)$,
2. $BM \gamma^\omega_\omega(t) = \gamma^\omega_{n-r-1}(t)$.

**Theorem 1.2**

The assignments $\omega \mapsto \delta^\omega_\omega$ from $Z^1_\omega(X;\xi)$ to $Conf_{BM}(X;\xi)(\mathbb{R})$ is continuous when the source $Z^1_\omega(X;\xi)$ is equipped with the compact-open topology and the target $Conf_{BM}(X;\xi)(\mathbb{R})$ with the collision topology.

The assignment can be extended continuously to the entire $Z^1_\omega(X;\xi)$. If $f : \tilde{X} \rightarrow \mathbb{R}$ is a lift of a tame TC1-form $\omega$, on a compact ANR $X$, cf. [1] or Section 3 for definition, one has
Proposition 1.3

1. \( \dim(H_r(\tilde{X}_a, \tilde{X}_{<a})) = \sum_{t \in \mathbb{R}} \delta^f_r(a, t) + \sum_{t>a} \gamma^f_r(a, t) + \sum \gamma^f_{r-1}(t, a) + \dim \tilde{\chi}^f_{r-1}(a) \).

2. \( \dim(BM H_r(\tilde{X}_a, \tilde{X}_{<a})) = \sum_{t \in \mathbb{R}} BM \delta^f_r(a, t) + \sum_{t>a} BM \gamma^f_r(a, t) + \sum_{t<a} BM \gamma^f_{r-1}(t, a) + \dim BM \tilde{\chi}^f_{r-1}(a) \).

Formula 1 was established in [1]. We will also establish the following results essential for the proofs in Theorems 1.2 and 1.3 in [1].

Proposition 1.4

The linear maps \( \theta_r(a) : H_r(\tilde{X}_a, \tilde{X}_{<a}) \rightarrow BM H_r(\tilde{X}_a, \tilde{X}_{<a}) \) are isomorphisms.

One expects that both \( \theta^f_r(a, b) : \delta^f_r(a, b) \rightarrow BM \delta^f_r(a, b) \) and \( \lambda^f_r(a, b) : \gamma^f_r(a, b) \rightarrow BM \gamma^f_r(a, b) \) are isomorphisms. This is indeed the case when \( \omega \) is of degree of irrationality 1.

Theorem [1.2] verifies Theorem 3 item 1 in [1]. Item 2 will be treated in part 3 of this work.

Theorem [1.1] verifies Theorem T2 announced in [1] provided \( \theta^f_r \) and \( \lambda^f_r \) are isomorphisms, fact not yet established in full generality and possibly not always true. Proposition [1.4] shows however that \( \omega \) is homological critical value for standard homology iff \( \omega \) is homological critical value for Borel-Moore homology a first step towards verifying \( \theta^f_r \), \( \theta^f_{r+1} \), \( \theta^f_{r+2} \) and \( \lambda^f_r \) are isomorphisms if the case.

The paper requires some basic linear algebra, most likely familiar to most of the readers but for convenience of the reader reviewed in Section 2. The reader can skip this section at the first reading and return to it when necessary.

2 Linear algebra preliminary

2.1 Linear algebra

In this section ”=” designates equality or canonical isomorphism and \( \simeq \) indicates the existence of an isomorphism.

Let \( \kappa \) be a fixed field. All vector spaces are \( \kappa \)-vector spaces and the linear maps are \( \kappa \)-linear. Recall that for a linear map \( \alpha : A \rightarrow B \) the canonical exact sequence associated to \( \alpha \) is

\[
0 \rightarrow \ker \alpha \xrightarrow{i_\alpha} A \xrightarrow{\alpha} B \xrightarrow{\pi_\alpha} \operatorname{coker} \alpha \rightarrow 0.
\]

By passing to ”duals” one obtains the exact sequence

\[
0 \xleftarrow{\ker(\alpha^*)^*} (\ker(\alpha^*))^* \xrightarrow{i_{\alpha^*}} A^* \xrightarrow{\alpha^*} B^* \xrightarrow{\pi_{\alpha^*}} (\operatorname{coker} \alpha^*)^* \xrightarrow{\theta} 0,
\]

canonically isomorphic to the canonical exact sequence of \( \alpha^* \).

\[
0 \rightarrow \ker(\alpha^*) \xrightarrow{i_{\alpha^*}} B^* \xrightarrow{\alpha^*} A^* \xrightarrow{\pi_{\alpha^*}} (\operatorname{coker} \alpha^*) \xrightarrow{\theta} 0. \tag{1}
\]

For a diagram

\[
\begin{array}{ccc}
D := & A & B_1 \\
\alpha_1 & \downarrow \alpha_2 & \downarrow \beta_1 \\
B_2 & \beta_2 & C
\end{array}
\]
let \( \alpha : A \to \alpha_2 \cup_A \alpha_1 \) be the push-forward of \( \alpha_2 \) and \( \alpha_1 \), and \( \beta : \beta_2 \times_C \beta_1 \) be the pullback of \( \beta_2 \) and \( \beta_1 \), with
\[
\alpha_2 \cup_A \alpha_1 := B_2 \oplus B_1 / \{ \alpha_2(a) - \alpha_1(a) \mid a \in A \},
\]
\[
\beta(a) := (\alpha_2(a) + \alpha_1(a)),
\]
\[
\beta_1 \times_C \beta_2 := \{ (b_1, b_2) \in B_1 \times B_2 \mid \beta_1(b_1) = \beta_2(b_2) \}.
\]
where \( (b_2 \oplus b_1) \) denotes the image of \( b_1 \oplus b_2 \) in \( \alpha_2 \cup_A \alpha_1 \).

Let \( \alpha : A \to \beta_2 \times_C \beta_1 \) and \( \beta : \alpha_2 \cup A \alpha_1 \to C \) be given by
\[
\alpha(a) = (\alpha_2(a), \alpha_1(a))
\]
\[
\beta((b_2 \oplus b_1)) = \beta_2(b_1) + \beta_2(b_2).
\]
Define
\[
\ker(D) := \ker \alpha
\]
\[
coker(D) := \coker \beta.
\]

**Observation 2.1** The canonical isomorphisms \( \theta \) extend to the canonical isomorphisms
\[
\theta(D) : (\coker D)^* \to \ker(D^*)
\]
and
\[
\theta(D) : (\ker D)^* \to \coker(D^*).
\]

**Proof**: To check the statements observe that the diagram \( D \) can be completed to the diagrams \( \bar{D} \) and \( \overline{D} \)

and notice that \((D)^*\) identifies to \((\overline{D}^*)\) and \((\bar{D})^*\) identifies to \((D^*)\) which imply the statements.

q.e.d.

Let \( \alpha : A \to B, \beta : B \to C, \gamma : C \to D \) be linear maps. To these three maps we associate the diagrams, \( \hat{D} \) and \( \bar{D} \):

\[
\hat{D}(\alpha, \beta, \gamma) \equiv \begin{cases} 
& \ker(\gamma \beta \alpha) \xrightarrow{j_2} \ker(\gamma \beta) \xrightarrow{i_1} \ker(\beta) \\
& \ker(\beta \alpha) \xrightarrow{j_1} \ker(\beta)
\end{cases}
\]
\[
\bar{D}(\alpha, \beta, \gamma) \equiv \begin{cases} 
& \text{coker}(\gamma \beta \alpha) \xrightarrow{j_2'} \text{coker}(\gamma \beta) \xrightarrow{i'_1} \text{coker}(\beta) \\
& \text{coker}(\beta \alpha) \xrightarrow{j_1'} \text{coker}(\beta)
\end{cases}
\]

In the diagram \( \hat{D} \)

(a) \( i_1 \) and \( i_2 \) are injective,

(b) \( i_1 : \ker j_1 \to \ker j_2 \) is an isomorphism,
and in the diagram $D$

(a) $j'_1$ and $j'_2$ are surjective,

(b) $\text{coker}i'_1 \to \text{coker}i'_2$ is an isomorphism,

(c) $\text{ker}(j'_2 \cdot i'_1 = i'_2 \cdot j'1) = \text{ker}i'_1 + \text{ker}j'_1$.

In view of Observation 2.1 one has

**Observation 2.2** $(\hat{D}(\alpha, \beta, \gamma))^* = D(\gamma^*, \beta^*, \alpha^*)$ and $(D(\alpha, \beta, \gamma))^* = \hat{D}(\gamma^*, \beta^*, \alpha^*)$.

One defines the vector space

$$\hat{\omega}(\alpha, \beta, \gamma) = \text{coker} \hat{D}(\alpha, \beta, \gamma) := \text{coker}(j_1 \cup_{\ker(\beta \alpha)} i_1 \to \ker(\gamma \beta))$$

$$= \ker(\gamma \beta)/(j_2(\ker(\gamma \beta \alpha)) + i_2(\ker \beta)).$$

(2)

a quotient space of $\ker(\gamma \beta)$, and the vector space

$$\omega(\alpha, \beta, \gamma) = \ker D(\alpha, \beta, \gamma) := \ker(coker(\beta \alpha) \to i'_2 \times coker(\gamma \beta) j'_2)$$

a subspace of $\text{coker}(\beta \alpha)$.

Note that the assignments $(\alpha, \beta, \gamma) \mapsto \hat{\omega}(\alpha, \beta, \gamma)$ and $(\alpha, \beta, \gamma) \mapsto \omega(\alpha, \beta, \gamma)$ are functorial and in view of the definitions above if $\alpha$ is surjective or if $\gamma$ is injective then $\hat{\omega}(\alpha, \beta, \gamma) = 0$.

**Theorem 2.3**

1. For $\alpha, \beta, \gamma$ linear maps the isomorphisms $\theta$ extend to the canonical isomorphisms

$$\theta : \hat{\omega}(\alpha, \beta, \gamma)^* \to \omega(\gamma^*, \beta^*, \alpha^*)$$

$$\theta : \omega(\alpha, \beta, \gamma)^* \to \hat{\omega}(\gamma^*, \beta^*, \alpha^*)$$.

2. For $\alpha, \beta, \gamma$ and $\alpha', \beta', \gamma'$ linear maps consider the diagram

$$\begin{array}{ccc}
M & \overset{\lambda_A}{\longrightarrow} & M \\
\lambda_B \downarrow & & \lambda_C \downarrow \\
A & \overset{\alpha}{\longrightarrow} & B \\
\downarrow d & & \downarrow c \\
C & \overset{\beta}{\longrightarrow} & D \\
\downarrow b & & \downarrow d \\
B' & \overset{\alpha'}{\longrightarrow} & C' \\
\downarrow \theta_A & & \downarrow \theta_B \\
N & \overset{\theta_C}{\longrightarrow} & N \\
\downarrow \theta_D & & \downarrow \theta_D \\
D' & \overset{\gamma'}{\longrightarrow} & N.
\end{array}$$

(3)

If the columns are exact sequences then the linear maps

$$\hat{\omega}(\alpha', \beta', \gamma') \to \hat{\omega}(\alpha, \beta, \gamma)$$

and

$$\omega(\alpha', \beta', \gamma') \to \omega(\alpha, \beta, \gamma)$$

are isomorphisms.

3. For $\alpha, \beta, \gamma$ linear maps the following holds true.
(i) A factorization \( \alpha = \alpha_2 \cdot \alpha_1 \), with \( \alpha_1 : A \to A' \) and \( \alpha_2 : A' \to B \) linear maps, induces the short exact sequences
\[
0 \to \tilde{\omega}(\alpha_1, \beta \alpha_2, \gamma) \to \tilde{\omega}(\alpha, \beta, \gamma) \to \tilde{\omega}(\alpha_2, \beta, \gamma) \to 0, \\
0 \to \omega(\alpha_1, \beta \alpha_2, \gamma) \to \omega(\alpha, \beta, \gamma) \to \omega(\alpha_2, \beta, \gamma) \to 0. 
\]

(5)

(ii) A factorization \( \beta = \beta_2 \cdot \beta_1 \), with \( \beta_1 : B \to B' \) and \( \beta_2 : B' \to B \) linear maps, induces the short exact sequences
\[
0 \to \tilde{\omega}(\alpha, \beta_1, \beta_2) \to \tilde{\omega}(\alpha, \beta_1, \gamma \beta_2) \to \tilde{\omega}(\alpha, \beta, \gamma) \to 0, \\
0 \to \omega(\alpha, \beta_1, \beta_2) \to \omega(\alpha, \beta_1, \gamma \beta_2) \to \omega(\alpha, \beta, \gamma) \to 0. 
\]

(iii) A factorization \( \gamma = \gamma_2 \cdot \gamma_1 \), with \( \gamma_1 : C \to C' \) and \( \gamma_2 : C' \to D \) maps, induces the short exact sequences
\[
0 \to \tilde{\omega}(\alpha, \beta, \gamma_1) \to \tilde{\omega}(\alpha, \beta, \gamma) \to \tilde{\omega}(\alpha, \gamma_1 \beta, \gamma_2) \to 0, \\
0 \to \omega(\alpha, \beta, \gamma_1) \to \omega(\alpha, \beta, \gamma) \to \omega(\alpha, \gamma_1 \beta, \gamma_2) \to 0. 
\]

\[ \text{Proof:} \] Item 1. follows from Observations (2.1) and (2.2). To prove Item 2. notice the following.

(a) If \( N = 0 \), then the induced maps
\[
\ker(\beta') \to \ker(\beta), \\
\ker(\gamma' \beta') \to \ker(\gamma \beta) \\
\ker(\beta' \alpha') \to \ker(\beta \alpha), \\
\ker(\gamma' \beta' \alpha') \to \ker(\gamma \alpha \beta).
\]
are isomorphisms hence \( \hat{D}(\alpha', \beta', \gamma') = \hat{D}(\alpha, \beta, \gamma) \) and then \( \hat{\omega}(\alpha', \beta', \gamma') = \hat{\omega}(\alpha, \beta, \gamma) \).

(b) If \( M = 0 \) then the obvious induced maps
\[
\coker(\beta' \alpha') \to \coker(\beta \alpha) \\
\coker(\gamma' \beta' \alpha') \to \coker(\gamma \beta \alpha) \\
\coker(\gamma' \beta') \to \coker(\gamma \beta) \\
\coker(\beta') \to \coker(\beta)
\]
are isomorphisms consequently, the induced map \( \hat{\omega}(\alpha', \beta', \gamma') \to \hat{\omega}(\alpha, \beta, \gamma) \) is an isomorphism.

(c) To prove the result for \( M \) and \( N \) arbitrary consider the diagram
\[
\begin{array}{c}
A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & D \\
\text{img a} & \xrightarrow{\alpha''} & \text{img b} & \xrightarrow{\beta''} & \text{img c} & \xrightarrow{\gamma''} & \text{img d} \\
A' & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C' & \xrightarrow{\gamma'} & D'
\end{array}
\]

and the linear maps
\[
\hat{\omega}(\alpha', \beta', \gamma') \to \hat{\omega}(\alpha'', \beta'', \gamma'') \to \hat{\omega}(\alpha, \beta, \gamma).
\]
The first arrow is an isomorphism by (b) above and the second by (ii) above. This establishes Item 2.
To prove Item 3 consider the diagram

\[ \begin{array}{ccc}
A_2 & \xrightarrow{i_2} & B_2 \\
| & \downarrow{j_A} & | \\
A_1 & \xrightarrow{i_1} & B_1
\end{array} \quad \begin{array}{ccc}
B_2 & \xrightarrow{j_C} & C_2 \\
| & \downarrow{j_B} & | \\
B_1 & \xrightarrow{j_1} & C_1
\end{array} \]

and make the following observation.

**Observation 2.4** Suppose that each of the three diagrams \( \mathbb{B}_1, \mathbb{B}_2, \mathbb{B} \), associated with (7),
\( \mathbb{B}_1 \) with vertices \( A_1, A_2, B_1, B_2 \),
\( \mathbb{B}_2 \) with vertices \( B_1, B_2, C_1, C_2 \) and
\( \mathbb{B} \) with vertices \( A_1, A_2, C_1, C_2 \)
satisfy the properties (a) (b) (c) of the diagram \( \hat{D} \). Then (7) induces the exact sequence

\[
\begin{array}{ccc}
B_2/(i_2^A(A_2) + j_B(B_1)) & \xrightarrow{i} & 0 \\
0 & \xrightarrow{p} & C_2/(i_2^C(A_2) + j_C(C_1)) \\
& \xrightarrow{\cdot j_1} & C_2/(i_2^B(B_2) + j_C(C_1))
\end{array}
\]

with \( i \) induced by \( i^B_2 \), well defined because \( \text{img}(i^B_2 \cdot j_B) \subseteq \text{img}j_C \), and \( p \) the projection induced by the inclusion \( (i_2(A_2) + j_C(C_1)) \subseteq (i_2^B(B_2) + j_C(C_1)) \).

Clearly \( p \) is surjective and \( p \cdot i = 0 \). Property (c) implies that \( i \) is injective. Properties (a), (b) (c) imply that the sequence is exact.

A similar observation holds for the diagram

\[ \begin{array}{ccc}
A_3 & \xrightarrow{i_3} & B_3 \\
| & \downarrow{j_A} & | \\
A_2 & \xrightarrow{i_2} & B_2 \\
| & \downarrow{j_B} & | \\
A_1 & \xrightarrow{i_1} & B_1
\end{array} \]

**Observation 2.5** Suppose that each of the three diagrams \( \mathbb{B}_1, \mathbb{B}_2, \mathbb{B} \), associated with (8),
\( \mathbb{B}_1 \) with vertices \( A_2, A_3, B_2, B_3 \),
\( \mathbb{B}_2 \) with vertices \( A_1, A_2, B_1, B_2 \) and
\( \mathbb{B} \) with vertices \( A_1, A_3, B_1, B_2 \)
satisfy the properties (a) (b) (c) of the diagram $\hat{D}$ Then (8) induces the exact sequence

$$\begin{array}{ccc}
B_2/(i_2(A_2) + j^B_1(B_1)) & \xrightarrow{j} & B_3/(i_3(A_3) + j^B_1(B_1)) \\
\downarrow & & \downarrow \\
0 & & 0 \\
\downarrow & & \downarrow \\
B_3/(i_3(A_3) + j^B_2(B_2)) & \xrightarrow{p} & B_3/(i_3(A_3) + j^B_2(B_2))
\end{array}$$

Observation 2.4 applied to diagram (7) with

$A_1 = \ker(\beta \alpha)$, \hspace{1em} $A_2 = \ker(\gamma \beta \alpha)$, \\
$B_1 = \ker(\beta \alpha_2)$, \hspace{1em} $B_2 = \ker(\gamma \beta \alpha_2)$, \\
$C_1 = \ker(\beta)$, \hspace{1em} $C_2 = \ker(\gamma \beta)$

verifies Item 3. (i).

Observation 2.5 applied to diagram (8) with

$A_1 = \ker(\beta_1 \alpha)$, \hspace{1em} $B_1 = \ker(\beta_1)$, \\
$A_2 = \ker(\beta \alpha)$, \hspace{1em} $B_2 = \ker(\beta)$, \\
$A_3 = \ker(\gamma \beta \alpha)$, \hspace{1em} $B_3 = \ker(\gamma \beta)$

verifies Item 3. (ii) and applied to diagram (8) with

$A_1 = \ker(\beta \alpha)$, \hspace{1em} $B_1 = \ker(\beta)$, \\
$A_2 = \ker(\gamma_1 \beta \alpha)$, \hspace{1em} $B_2 = \ker(\gamma_1 \beta)$, \\
$A_3 = \ker(\gamma \beta \alpha)$, \hspace{1em} $B_3 = \ker(\gamma \beta)$

verifies Item 3. (iii).

The above considerations/proofs were already contained in [2] but under the additional hypothesis that all the linear maps were Fredholm.

**Direct and inverse limits**

A totally order set $\mathcal{I} = (I, \leq)$ can be viewed as a category whose objects are its elements and for any two objects $x, y \in I$ there is only one morphism if $x \leq y$ and no morphism otherwise. A system $\mathcal{A}$ indexed by $I$ is a covariant functor from $\mathcal{I}$ to the category of $\kappa$–vector spaces, i.e the collection of vector spaces $\{A_t, t \in I\}$ and linear maps $i_{ts}^t : A_t \to A_s, t \leq s$. Each such system has a direct limit $\lim_{t \in I} A_t$ and inverse limit $\lim_{t \in I} A_t$ both vector spaces. A priori there is also a derived direct limit but always identical to zero and a derived inverse limit $\lim_{t \in I} A_t$ a vector space not always trivial, as well as the obvious linear maps $\pi_t, i_t, i$ making the diagram below commutative for any $t \leq s$.

$$\begin{array}{ccc}
\lim_{t \in I} A_t & \xrightarrow{i} & \lim_{t \in I} A_t \\
\downarrow & & \downarrow \\
A_t & \xrightarrow{i_t} & A_s
\end{array}$$
Note that a subset $I'$ of $I$ it is called cofinal w.r.to $I$ to the left resp. to the right if for any $t \in I'$ there exists $s \in I$ s.t. $s \leq t$ resp. $s \geq t$ and if so $\lim_{t \in I'} A_t = \lim_{t \in I} A_t$ resp. $\lim_{t \in I'} A_t = \lim_{t \in I} A_t$.

Note that $\mathbb{Z} \subseteq \mathbb{N}$ is cofinal to $\mathbb{R}$ to the left (as well to $\mathbb{Z}$), $\mathbb{Z} \supseteq \mathbb{N}$ is cofinal to $\mathbb{R}$ to the right (as well to $\mathbb{Z}$) and $\mathbb{Z}$ is cofinal to $\mathbb{R}$ both to the left and right.

**Observation 2.6**

1. An exact sequence of systems indexed by the same $I$, $\cdots A \rightarrow B \rightarrow C \rightarrow D \rightarrow \cdots$ induces by passing to direct limits an exact sequence

\[ \cdots \lim A_t \rightarrow \lim B_t \rightarrow \lim C_t \rightarrow \lim D_t \rightarrow \cdots. \]

This is not true for inverse limits however one has.

2. A short exact sequence of systems indexed by the same $I$, $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ induces by passing to the inverse limits the exact sequence

\[ 0 \rightarrow \lim A_t \rightarrow \lim B_t \rightarrow \lim C_t \rightarrow \lim' A_t \rightarrow 0. \]

As indicated in [8]

1. For the set $\mathbb{Z} \subseteq \mathbb{N}$ consider the linear map

\[ \mathcal{I} : \bigoplus_{k \geq N} A_k \rightarrow \bigoplus_{k \geq N} \text{ defined by } \]

\[ \mathcal{I}(a_N, a_{N+1}, a_{N+2}, \cdots) = (a_N, a_{N+1} - i_{N+1}^N(a_N), a_{N+2} - i_{N+2}^N(a_{N+1}), \cdots). \]

Then

\[ \lim A_k = \text{coker } \mathcal{I} \]

2. For the set $\mathbb{Z} \supseteq \mathbb{N}$ consider the linear map $\mathcal{P} : \prod_{k \leq N} A_k \rightarrow \prod_{k \leq N}$ defined by

\[ \mathcal{P}(a_{-N}, a_{-N-1}, a_{-N-2}, \cdots) = (a_{-N}, a_{-N-1} - i_{N-1}^{-N}(a_{-N-1}), a_{-N-2} - i_{N-2}^{-N}(a_{-N-2}), \cdots). \]

Then

\[ \lim A_k = \ker \mathcal{P}, \quad \lim' A_k = \text{coker } \mathcal{P} \]

and note that $\lim' A_k = 0$ if there exists $l$ s.t for $j$ large enough $i_{j-l}^j$ is surjective, i.e Mittag- Leffler condition is satisfied.

For the proof of Theorem [11] one needs some additional definitions.

**Definition 2.7**

A sub-surjection $\tilde{\pi} : A \rightarrow A'$ is a pair $\tilde{\pi} := \{ \pi : A \rightarrow P, P \supseteq A' \}$ consisting of a surjective linear map $\pi$ and a subspace $A'$ of $P$.

To a directed system of sub-surjections

\[ A_1 \xrightarrow{\pi_1} A_2 \xrightarrow{\pi_2} A_3 \xrightarrow{\pi_3} \cdots \xrightarrow{\pi_{k-1}} A_k \xrightarrow{\pi_k} A_{k+1} \xrightarrow{\pi_{k+1}} \cdots \]

one provides a unique maximal directed system of surjections

\[ \begin{array}{cccccccc}
A_1 & A_2 & A_3 & \cdots & A_k & A_{k+1} \\
\subseteq & \subseteq & \subseteq & \cdots & \subseteq & \subseteq \\
A_1^\infty & A_2^\infty & A_3^\infty & \cdots & A_k^\infty & A_{k+1}^\infty \\
\end{array} \]
The construction of this sequence is rather straightforward and it goes as follows:

Starting with the right side of the Diagram (10) below (i.e. the collections of linear maps \( \{ \pi_i, \supseteq \} \), one inductively, from right to left and from up to down (i.e from the lower index \( i \) to lower index \( (i - 1) \) and from the upper index \( k \) to the upper index \( (k + 1) \)) one produces the subspace \( A^{k+1}_i \subset A^k_i \) and the surjective linear maps \( \pi^k_i : A^k_i \to A^{k+1}_i \) defined by:

- \( A^{k+1}_i = (\pi^k_i)^{-1}(A_{k+1}) \),
- \( A^k_i = (\pi^k_i)^{-1}(A_{k+1}) \) and
- \( \pi^k_i \) = restriction of \( \pi^k_i \).

Take \( A^\infty_k = \cap_{i > k} A^k_i \) and \( \pi^\infty_i \) the restriction of \( \pi^\infty_i \) and define

\[
\lim_{i \to \infty} \pi_i : = \lim_{i \to \infty} \pi^\infty_i.
\]

\[
\text{(10)}
\]

3 Notations and definitions

Most of the definitions and notations below are the same as in [1] where they were considered for the standard (=singular) homology only. However, they can be considered for any homology theory with coefficients in a fixed field \( \kappa \), in particular for Borel-Moore homology of interest in this paper. In this paper a homology theory is a collection of \( \chi \)-vector space-valued covariant homotopy functors denoted by \( H_r(\cdots) \) defined on the category of pairs of locally compact Hausdorff spaces \((X, Y)\), \( Y \) closed subset of \( X \), and of proper continuous maps which satisfy the Eilenberg-Steenrod axioms. Standard homology is defined for any pair \((X, Y)\) and in addition satisfies Milnor continuity axiom. Recall that the continuity axiom states that if \( X(0) \subset \cdots \subset X(i) \subset X(i + 1) \subset \cdots \subset X \) is a filtration of an ANR \( X \) by the ANRs \( X(i) \) with \( X = \cup_i X(i) \), then

\[
H_r(X) = \lim_{i \to \infty} H_r(X_i).
\]

10
In this section any homology theory, in particular the standard or the Borel-Moore homology, will be denoted by $H_r$. In the next sections the notation $H_r$ will be reserved exclusively for the standard homology and the Borel-Moore homology will acquire the left-side exponent "BM" (e.g. $BMH_r$).

As in [1], for $X$ a compact ANR, a cohomology class $[\omega] \in H^1(X; \mathbb{Z})$ determines the group $\Gamma := \text{img}(\omega) : H_1(X; \mathbb{Z}) \to \mathbb{R}$ and the principal $\Gamma$–covering $\tilde{X} \to X$ with $\tilde{X}$ a locally compact ANR. As indicated in [1] the TC1-form $\omega$ representing $[\omega]$ determines and is determined by a $\Gamma$– continuous equivariant map $f : \tilde{X} \to \mathbb{R}$, unique up to an additive constant, referred to as a lift of $\omega$. Such a map up to an equivalence (two such maps are equivalent iff their difference is a locally constant map) can be taken as an alternative definition for a TC-1 form.

In consistency with the notation in [1], for a continuous map $f : Y \to \mathbb{R}$, one denotes by $Y_t := f^{-1}((\minus\infty, t])$, $Y^t := f^{-1}([t, \infty))$, $Y(t) := f^{-1}((\minus\infty, t))$, $Y^t := f^{-1}((t, \infty))$, which are closed resp. open subsets of $Y$.

Note that if one specifies $f$ in the notation above, precisely if one writes $Y_a^f, Y_{<a}^f, Y_a^g, Y_{>a}^g$ instead of $Y_a, Y_{<a}, Y_a, Y_{>a}$, then

$$Y_a^f = Y_{<a}^f \text{ and } Y_{<a}^f = Y_{>a}^f.$$  

Since $\tilde{X}_{<a}$ is an open set, $H_r(\tilde{X}_a, \tilde{X}_{<a})$ is not a priory defined [1], but we introduce the vector space

$$H_r^f(\tilde{X}_a, \tilde{X}_{<a}) := \lim_{\epsilon \to 0} H_r(\tilde{X}_a, \tilde{X}_{a-\epsilon}).$$

By passing to direct limit when $\epsilon \to 0$ the long exact sequence for the pair $(\tilde{X}_a, \tilde{X}_{a-\epsilon})$ leads to the long exact sequence

$$\cdots \longrightarrow \lim_{\epsilon \to 0} H_r(\tilde{X}_{a-\epsilon}) \longrightarrow H_r(\tilde{X}_a) \longrightarrow H_r^f(\tilde{X}_a, \tilde{X}_{<a}) \longrightarrow \lim_{\epsilon \to 0} H_{r-1}(\tilde{X}_{a-\epsilon}) \longrightarrow \cdots.$$  

Similarly one introduces the vector space

$$H_r^f(\tilde{X}_a, \tilde{X}_{>a}) := \lim_{\epsilon \to 0} H_r(\tilde{X}_a, \tilde{X}_{a+\epsilon})$$

and by passing to direct limit when $\epsilon \to 0$ the long exact sequence for the pair $(\tilde{X}_a, \tilde{X}_{a+\epsilon})$ leads to the long exact sequence

$$\cdots \longrightarrow \lim_{\epsilon \to 0} H_r(\tilde{X}_{a+\epsilon}) \longrightarrow H_r(\tilde{X}_a) \longrightarrow H_r^f(\tilde{X}_a, \tilde{X}_{>a}) \longrightarrow \lim_{\epsilon \to 0} H_{r-1}(\tilde{X}_{a+\epsilon}) \longrightarrow \cdots.$$  

For $f : Y \to \mathbb{R}$ a continuous map

(a) $t \in \mathbb{R}$ is a regular value iff for any $r \geq 0$ and any open set $U \subseteq Y$ one has $h_r(U_t, U_{<t}) = h_r(U^t, U^{>t}) = 0$ with $h_r$ denoting the singular homology. If $Y$ is a smooth manifold, possibly with boundary $\partial Y$, and $f$ is a smooth map with $t$ a regular value in the sense of differential calculus for both $Y$ and $\partial Y$, then $t$ is a regular value in the sense mentioned above.

(b) $t \in \mathbb{R}$ is a critical value if not regular. Denote by $CR(f) \subset \mathbb{R}$ the set of all critical values.

(c) $x \in X$ is a critical point if for some small enough neighborhood $U$ of $x$, $f(x)$ is a critical value for the restriction of $f$ to $U$. Denote by $Cr(f) \subset X$ the set of critical points.

\footnote{for example for Borel Moore homology}

\footnote{for standard homology $\lim_{\epsilon \to 0} H_r(\tilde{X}_{a-\epsilon}) \simeq H_r(X_{<a})$ and $H_r^f(\tilde{X}_a, \tilde{X}_{>a}) \simeq H_r^f(\tilde{X}_a, \tilde{X}_{>a})$}
The continuous map $f : Y \to \mathbb{R}$ is *tame* if the following holds true:

(i) $Y$ is a locally compact ANR and $f^{-1}(I)$ is an ANR for any closed interval $I \subset \mathbb{R}$,

(ii) $CR(f)$ is countable, hence the set of regular values is dense in $\mathbb{R}$,

(iii) for any $t \in CR(f)$ the set $f^{-1}(t) \cap CR(f)$ is a compact ANR.

Note that $CR(f) = C_r(-f)$ and $CR(f) = -CR(-f)$ for $f$ a lift of the tame TC1-form $\omega$ resp. $-f$ lift of the tame TC1-form $-\omega$.

A TC1-form $\omega$ on the compact ANR $X$ is *tame* if one lift $f : \tilde{X} \to \mathbb{R}$, and then any other lift, is a tame map and in addition the set of orbits of the free action of $\Gamma$ on $CR(f)$ is finite.

As in [1], for $f : \tilde{X} \to \mathbb{R}$ and $a \in \mathbb{R}$ one defines

• $\mathbb{I}_a^f(r) := \text{img}(H_r(\tilde{X}_a) \to H_r(\tilde{X}))$,

• $\mathbb{I}_{<a}^f(r) := \lim_{t \to 0} \text{img}(H_r(\tilde{X}_{a-t}) \to H_r(\tilde{X}))$,

• $\mathbb{I}_f^a(r) := \text{img}(H_r(\tilde{X}_a) \to H_r(\tilde{X}))$,

• $\mathbb{I}_f^{a+\epsilon}(r) := \lim_{t \to 0} \text{img}(H_r(\tilde{X}_{a+\epsilon}) \to H_r(\tilde{X}))$,

• $\mathbb{I}_f^a(r) := \frac{\text{img}(H_r(\tilde{X}_a) \to H_r(\tilde{X}))}{\text{img}(H_r(\tilde{X}_{a-t}) \to H_r(\tilde{X}))}$ for $a' < a$,

• $\mathbb{I}_f^a(r) := \frac{\text{img}(H_r(\tilde{X}_a) \to H_r(\tilde{X}))}{\text{img}(H_r(\tilde{X}_{a-t}) \to H_r(\tilde{X}))}$ for $a' \leq a$,

• $\mathbb{I}_f^a(r) := \frac{\text{img}(H_r(\tilde{X}_a) \to H_r(\tilde{X}))}{\text{img}(H_r(\tilde{X}_{a-t}) \to H_r(\tilde{X}))}$,

• $\mathbb{I}_f^{a+\epsilon}(r) := \lim_{t \to 0} \text{img}(H_r(\tilde{X}_{a+\epsilon}) \to H_r(\tilde{X}))$,

In order to lighten the notation, when implicit from the context, $f$ might be dropped off the notation.

For a box $B \subset \mathbb{R}^2$, $B = (a', a] \times [b, b')$, $a' < a, b < b'$, consider the diagrams $\mathcal{F}_r(B)$ and $\mathcal{G}_r(B)$

$$
\mathcal{F}_r(B) := \left\{ \begin{array}{c}
\mathbb{F}_r(a', b') \subseteq \mathbb{F}_r(a, b) \\
\mathbb{F}_r(a', b) \subseteq \mathbb{F}_r(a, b)
\end{array} \right\}, \\
\mathcal{G}_r(B) := \left\{ \begin{array}{c}
\mathbb{G}_r(a', b') \subseteq \mathbb{G}_r(a, b) \\
\mathbb{G}_r(a', b) \subseteq \mathbb{G}_r(a, b)
\end{array} \right\}
$$

whose arrows are the obviously induced linear maps. In the diagram $\mathcal{F}_r(B)$ all these induced maps are injective and in the diagram $\mathcal{G}_r(B)$ all are surjective.

As in [1] one defines

• $\mathbb{F}_r(B) := \text{coker}(\mathcal{F}(B)) = \frac{\mathbb{F}_r(a, b)}{\mathbb{F}_r(a', b) + \mathbb{F}_r(a, b')}$

• $\mathbb{G}_r(B) := \ker\mathcal{G}_r(B) = \ker(G_r(\alpha, \beta) \to G_r(\alpha, \beta') \times_{G_r(\alpha', \beta')} G_r(\alpha', \beta))$.

The inclusion $\mathbb{I}_a(r) \cap \mathbb{I}_b(r) \subset (\mathbb{I}_a'(r) + \mathbb{I}_b'(r)) \cap (\mathbb{I}_a'(r) + \mathbb{I}_b'(r))$ induces a canonical isomorphism

$$
\theta_r(B) : \mathbb{F}_r(B) \to \mathbb{G}_r(B)
$$

(13)

\(^3\text{for standard homology } \mathbb{I}^f_{<a}(r) \simeq \text{img}(H_r(\tilde{X}_{<a}) \to H_r(\tilde{X})) \text{ and } \mathbb{I}^f_{>a}(r) \simeq \text{img}(H_r(\tilde{X}_{>a}) \to H_r(\tilde{X}))\)
For $a'' < a' < a$ and $b < b' < b''$ consider the boxes $B_1, B_2, B$ with $B = (a'', a] \times [b, b'')$ and either $B_1 = (a'', a'] \times [b, b''), B_2 = (a', a] \times [b, b'')$ or $B_1 = (a'', a] \times [b', b''), B_2 = (a'', a] \times [b, b')$. In both cases $B = B_1 \sqcup B_2$. As in [1] or [2] one has the following.

**Proposition 3.1** (cf. [1]) The boxes $B_1, B_2, B$ described above induce the commutative diagram whose rows are exact sequences and vertical arrows are isomorphisms.

$$
\begin{array}{cccccc}
0 & \rightarrow & \mathbb{F}_r(B_1) & \overset{i_{B_1}^{B_2}}{\rightarrow} & \mathbb{F}_r(B) & \overset{\pi_{B_1}^{B_2}}{\rightarrow} & \mathbb{F}_r(B_2) & \rightarrow & 0 \\
\downarrow & & \theta_1(B_1) & & \theta_1(B) & & \theta_1(B_2) & & \\
0 & \rightarrow & \mathbb{G}_r(B_1) & \overset{j_{B_1}^{B_2}}{\rightarrow} & \mathbb{G}_r(B) & \overset{\pi_{B_1}^{B_2}}{\rightarrow} & \mathbb{G}_r(B_2) & \rightarrow & 0.
\end{array}
$$

\[13\]

**Observation 3.2** As a consequence if $B_1 \subset B$ are boxes, with $B_1$ located in the upper-left corner of $B$, then the induced linear map $i_{B_1}^{B_2} : \mathbb{F}_r(B_1) \rightarrow \mathbb{F}_r(B)$ is injective and, if $B_2$ is located in the down-right corner of $B$, then the induced linear map $\pi_{B_1}^{B_2} : \mathbb{F}_r(B) \rightarrow \mathbb{F}_r(B_2)$ is surjective.

In Figure 1 below $B_{11} = (a'', a'] \times [b, b'')$ is located in the upper-left corner of $B = (a'', a] \times [b, b'')$ and $B_{22} = (a', a] \times [b, b')$ in the lower-right corner of $B$.

![Figure 1](image_url)

Define

$$F \hat{\delta}_r(a, b) := \lim_{\epsilon, \epsilon' \to 0} \mathbb{F}_r((a - \epsilon, a] \times [b, b + \epsilon')) \quad \text{and} \quad G \hat{\delta}_r(a, b) := \lim_{\epsilon, \epsilon' \to 0} \mathbb{G}_r((a - \epsilon, a] \times [b, b + \epsilon')).$$

In view of (13) one has

$$F \hat{\delta}_r(a, b) = G \hat{\delta}_r(a, b),$$

and then simplify the notation to

$$\hat{\delta}_r(a, b) := F \hat{\delta}_r(a, b) = G \hat{\delta}_r(a, b)$$

and denote

$$\delta_r(a, b) := \dim \hat{\delta}_r(a, b).$$
Note also that
\[
\delta_f^s (a, b) = \frac{\mathbb{F}_r^f (a, b)}{\mathbb{F}_r^f (a, b) + \mathbb{F}_r^f (a, > b)}
\]  \hspace{1cm} (14)
and in view of (14) one has the obvious surjective linear map
\[
\pi_{a, b}^f : \mathbb{F}_r^f (a, b) \to \delta_f^s (a, b).
\]

As in [1], for \( f : \tilde{X} \to \mathbb{R} \) a tame map and \( a, b \in \mathbb{R} \cup \{\infty\} \) with \( a < b \leq \infty \), let \( i_a^b (r) : H_r (\tilde{X}_a) \to H_r (\tilde{X}_b) \), with \( \tilde{X}_\infty := \tilde{X} \), be the inclusion induced linear map and define

- \( \mathcal{T}_r (a, b) := \ker (i_a^b (r) : H_r (\tilde{X}_a) \to H_r (\tilde{X}_b)) \),
- \( \mathcal{C}_r (a, b) := \text{coker} (i_a^b (r) : H_r (\tilde{X}_a) \to H_r (\tilde{X}_b)) \).

and then

\[
\begin{align*}
\mathcal{T}_r (a, < b) := & \lim_{\varepsilon \to 0} \mathcal{T}_r (a - \varepsilon, b), \\
\mathcal{T}_r (a, < b) := & \lim_{\varepsilon \to 0} \mathcal{T}_r (a, b - \varepsilon), \hspace{0.5cm} 0 < \varepsilon < b - a \\
\mathcal{C}_r (a, < b) := & \lim_{\varepsilon \to 0} \mathcal{C}_r (a - \varepsilon, b).
\end{align*}
\]

For \( a' < b' \leq b' < b \) denote by \( i_{a' b'}^a (r) : \mathcal{T}_r (a', b') \to \mathcal{T}_r (a, b) \) the induced linear map.

For a box above diagonal \( B = (a', a] \times (b', b] \) with \( a' < a \leq b' < b \leq \infty \) observe that \( \mathcal{T}_r (a, b') \subseteq \mathcal{T}_r (a, b) \) and define
\[
\mathcal{T}_r (B) := \frac{\mathcal{T}_r (a, b)}{i_{a' b'}^a (r) (\mathcal{T}_r (a', b')) + \mathcal{T}_r (a', b')}, \hspace{1cm} (15)
\]

In view of formula \( (2) \) (in subsection 2.1)\n\[
\mathcal{T}_r (B) = \tilde{\omega} (i_a^a (r), i_a^{b'} (r), i_b^b (r)) \hspace{1cm} (16)
\]
with \( i_a^a (r), i_a^{b'} (r), i_b^b (r) \) the linear maps in the sequence
\[
\begin{array}{c}
H_r (\tilde{X}_a) \xrightarrow{i_a^a (r)} H_r (\tilde{X}_a) \xrightarrow{i_a^{b'} (r)} H_r (\tilde{X}_b) \xrightarrow{i_b^b (r)} H_r (\tilde{X}_b).
\end{array}
\]

For \( a'' < a' < a \) and \( b > b' > b'' \) consider boxes above diagonal \( B_1, B_2, B \) with \( B = (a'', a] \times (b'', b] \) and either \( B_1 = (a'', a'] \times (b'', b], B_2 = (a', a] \times (b'', b] \) or \( B_1 = (a'', a] \times (b'', b'], B_2 = (a'', a] \times (b', b) \). In both cases \( B = B_1 \sqcup B_2 \). As in [1] or [2] one has the following proposition.

**Proposition 3.3** (cf. [1]) The boxes \( B_1, B_2, B \) as above induce the linear maps \( i_{B_1}^{B_2} (r) \) and \( \pi_{B_1}^{B_2} (r) \) which make the following sequence exact.
\[
\begin{array}{c}
0 \longrightarrow \mathcal{T}_r (B_1) \xrightarrow{i_{B_1}^{B_2} (r)} \mathcal{T}_r (B) \xrightarrow{\pi_{B_1}^{B_2} (r)} \mathcal{T}_r (B_2) \longrightarrow 0
\end{array}
\]

**Observation 3.4** As a consequence, if \( B_1 \subseteq B \) is located in the down-left corner of \( B \) then the induced linear map \( i_{B_1}^{B_2} (r) : \mathcal{T}_r (B_1) \to \mathcal{T}_r (B) \) is injective and, if \( B_2 \subseteq B \) is located in the upper-right corner of \( B \) then the induced linear map \( \pi_{B_1}^{B_2} (r) : \mathcal{T}_r (B) \to \mathcal{T}_r (B_2) \) is surjective.
In Figure 2 below \( B_{11} = (a'', a'] \times (b'', b'] \) is located in the down-left corner of \( B = (a'', a] \times (b'', b] \) and \( B_{22} = (a', a] \times (b', b] \) in the upper-right corner of \( B \).

In view of (15) there is the obvious surjective linear map

\[
\pi_{a,b} : T_f^f(a, b) \rightarrow \hat{\gamma}_f^f(a, b)
\]

Suppose that \( f : \tilde{X} \rightarrow \mathbb{R} \) is a lift of a tame TC1-form \( \omega \). Suppose \( H_f(\ldots) \) is a homology theory s.t. \( \mathcal{H}_f(a, \tilde{X}, \tilde{X}) \) and \( H_f(\tilde{X}^a, \tilde{X}^{>a}) \) are finite dimensional for any \( a \in \mathbb{R} \), hypotheses satisfied for both standard and Borel-Moore homologies.

As in section 5 of [1], where only the standard homology is considered, one has the following result, valid for both standard and Borel-Moore homology theory.

**Proposition 3.5**

*Under the above hypotheses the supports of \( \delta_f^f \) and \( \gamma_f^f \) are subsets of \( CR(f) \times CR(f) \) with the following properties:

1. If \( (a, b) \in \text{supp } \delta_f^f \) resp. \( (a, b) \in \text{supp } \gamma_f^f \) then for any \( g \in \Gamma \) one has \( (a + g, b + g) \in \text{supp } \delta_f^f \) resp. \( (a + g, b + g) \in \text{supp } \gamma_f^f \), hence \( \text{supp } \delta_f^f \) and \( \text{supp } \gamma_f^f \) are \( \Gamma \)-invariant w.r. to the action \( \mu(g, (x, y)) = (g + x, g + y) \).

2. For any \( a \in \mathbb{R} \) \( \text{supp } \delta_f^f \cap \mathbb{R} \times a \), \( \text{supp } \delta_f^f \cap a \times \mathbb{R} \), \( \text{supp } \gamma_f^f \cap \mathbb{R} \times a \), \( \text{supp } \gamma_f^f \cap a \times \mathbb{R} \) are finite sets, empty if \( a \in \mathbb{R} \setminus CR(f) \).

4In Figure 2 all boxes are supposed to be above diagonal and not necessary in the first quadrant.
3. There exists a finite set of lines in the plane $\mathbb{R}^2$, $\Delta^\delta_i(r)$ resp. $\Delta^\gamma_i(r)$, given by the equations $y = x + t^1_i$, $i = 1, 2, \cdots N^\delta_i$ resp. $y = x + t^2_i$, $i = 1, 2, \cdots N^\gamma_i$. s.t. $\text{supp } \delta^f_i \subset \bigcup_{i=1,\ldots,N^\delta_i} \Delta^\gamma_i(r)$.

Proof: (sketch)
Item (1) follows from the $\Gamma$-equivariance of the lift $f$ and the definitions, cf. (14) and (15).
To verify item (2) proceed as in (1).
Introduce:

- for $a' < a, b < b'$
  \[ F^f_r((a', a] \times b) := \lim_{\epsilon \to 0} F^f_r((a', a] \times [b, b + \epsilon)) \]
  \[ F^f_r(a \times [b, b')) := \lim_{\epsilon \to 0} F^f_r((a - \epsilon, a] \times [b, b')) \]

and note that $F^f_r(a \times [b, b')) = F^{-f}_r((-b', -b] \times -a)$.\(^5\)

- for $a' < a \leq b' < b \leq \infty$
  \[ T^f_r((a', a] \times b) := \lim_{\epsilon \to 0} T^f_r((a', a] \times (b - \epsilon, b)), \quad 0 < \epsilon < b - a \]
  \[ T^f_r((a', b) \times b) := \lim_{b > a \to b} T^f_r((a', a] \times b) \]
  \[ T^f_r(a \times (b', b]) := \lim_{\epsilon \to 0} T^f_r((a - \epsilon, a] \times (b', b)) \].

Clearly
\[
\hat{\delta}^f_r(a, b) = \lim_{\epsilon, \epsilon' \to 0} F^f_r((a - \epsilon, a] \times [b, b + \epsilon')) = \lim_{\epsilon \to 0} F^f_r((a - \epsilon, a] \times [b, b + \epsilon)) = \lim_{\epsilon \to 0} F^f_r(a \times [b, b + \epsilon') \\
\hat{\gamma}^f_r(a, b) = \lim_{\epsilon, \epsilon' \to 0} T^f_r((a - \epsilon, a] \times (b - \epsilon', b)) = \lim_{\epsilon \to 0} T^f_r((a - \epsilon, a] \times (b - \epsilon', b)) = \lim_{\epsilon \to 0} T^f_r(a \times (b - \epsilon', b)),
\]
(18)
\[ \hat{\delta}^f_r(a, b) = \hat{\delta}^{-f}_r(-b', -b). \]

Observe that because $\mathcal{H}_r(X_a, \bar{X}_{<a})$ is a finite dimensional vector space, the exact sequence (11) implies $\dim \mathbb{T}_{r-1}(<a, a) < \infty$ and $\dim \mathbb{C}_r(<a, a) < \infty$ for any lift $f$ of the tame $\omega$.

In view of Proposition 3.3 and the exact sequence of the triple $(X_a \subseteq \bar{X}_{b-\epsilon} \subseteq \bar{X}_b)$ on has
\[ \dim \mathbb{T}_r((a', a] \times b) \leq \dim \mathbb{T}_r((a', b) \times b) \leq \dim \mathbb{T}^{-f}_r(<b, b) \leq \dim \mathcal{H}_{r+1}(\bar{X}_b, \bar{X}_{<b}) < \infty \]
(19)
and in view of definitions
\[ \dim(\mathbb{T}^{-f}_r(a, b)/\mathbb{T}^{-f}_r(<a, b)) \leq \dim \mathbb{C}_r(<a, a) \leq \dim \mathcal{H}_r(\bar{X}_a, \bar{X}_{<a}) < \infty \]
(20)
\[ \dim(\mathbb{I}^{-f}_a(r)/\mathbb{I}^{-f}_{<a}(r)) \leq \dim \mathbb{C}_r(<a, a) \leq \dim \mathcal{H}_r(\bar{X}_a, \bar{X}_{<a}) < \infty \]
(21)

Note that if a lift of the tame TCI-form $\omega$ makes $-f$ a lift of the tame TC-1 form $-\omega$ and one has
\[ \mathbb{I}^{-f}_a(r) = \mathbb{I}^{-f}_b(r), \quad \mathbb{I}^{-f}_{<a}(r) = \mathbb{I}^{-f}_{<b}(r), \]
\(^5\)in view of the fact that $\bar{X}_a = \bar{X}_{-a}$ and $\bar{X}_{<a} = \bar{X}_{-a}$.
hence

$$\dim(\overline{I}_f^b(r)/\overline{I}_f^b(r)) = \dim(\overline{I}_b^f(r)/\overline{I}_b^f(r)) < \dim \mathcal{C}_r^{-f}(< -b, -b) < \dim \mathcal{H}_r(\overline{X}_f^b, \overline{X}_f^b) < \infty. \quad (22)$$

One can extend $F_r(a \times [\alpha, \beta])$, and $F_r((\alpha, \beta) \times c)$ with $\alpha < \beta$ to $\alpha = -\infty$ or $\beta = \infty$ by defining

$$F_r(a \times [\alpha, \infty)) = \lim_{\beta \rightarrow \alpha \rightarrow \infty} F_r(a \times [\alpha, \beta])$$

$$F_r(a \times (-\infty, \beta)) = \lim_{\beta \rightarrow \alpha \rightarrow -\infty} F_r(a \times [\alpha, \beta])$$

$$F_r(a \times (-\infty, \infty)) = \lim_{\alpha \rightarrow \infty} F_r(a \times [\alpha, \infty)) = F_r((\alpha, \infty) \times c) = \lim_{\beta \rightarrow \alpha \rightarrow -\infty} F_r((\alpha, \beta) \times c)$$

$$F_r((\alpha, \infty) \times c) = \lim_{\alpha \rightarrow \beta \rightarrow \infty} F_r((\alpha, \beta) \times c)$$

$$F_r((-\infty, \infty) \times c) = \lim_{\beta \rightarrow \infty} F_r((\alpha, \beta) \times c)$$

Similarly one can extend $T_r(a \times (\alpha, \beta)), a \leq \alpha < \beta < \infty$ to $\beta = \infty$

$$T_r(a \times (\alpha, \infty)) = \lim_{\alpha \rightarrow \beta \rightarrow \infty} T_r(a \times (\alpha, \beta))$$

and $T_r((\alpha, \beta) \times c), -\infty < \alpha < \beta < c$ to the case $\alpha = -\infty$ or $c$ and $\beta = c$ by defining

$$T_r((-\infty, \beta) \times c) = \lim_{\beta \rightarrow \alpha \rightarrow -\infty} T_r((\alpha, \beta) \times c)$$

$$T_r((\alpha, c) \times c) = \lim_{\alpha \rightarrow \beta \rightarrow c} T_r((\alpha, \beta) \times c)$$

$$T_r((-\infty, c) \times c) = \lim_{\beta \rightarrow \infty} T_r((\alpha, \beta) \times c)$$

In view of (21) resp. (19), resp. (20), resp. (22) and of Propositions 3.1 and 3.3 the assignments

1. $(-\infty, b) \ni t \rightsquigarrow \dim F_r(a \times [t, b]), -\infty < t \leq b$
2. $(a', b) \ni t \rightsquigarrow \dim T_r((a', t] \times b), -\infty \leq a' < t < b$
3. $(b', \infty) \ni t \rightsquigarrow \dim T_r(a \times (b', t]), a \leq b' < t < \infty$
4. $(a', \infty) \ni t \rightsquigarrow \dim F_r((a', t] \times b), a' \leq t < \infty$

are bounded $\mathbb{Z}_{\geq 0}$-valued functions with (1) decreasing and (2), (3) and (4) increasing in $t$, hence step functions with finitely many jumps at

1. $-\infty < t_1^a < t_2^a < \cdots < t_{N_a}^a < \infty$
2. $-\infty < t_1^b < t_2^b < \cdots < t_{M_b}^b < b$
3. $a < t_1^a < t_2^a < \cdots < t_{M_a}^a < \infty$
4. $-\infty < t_1^b < t_2^b < \cdots < t_{M_b}^b < \infty$
Proposition 3.1 for assignment (4), Proposition 3.3 for assignment (2), Proposition 3.1 for assignment (1), Proposition 3.3 for assignment (3),

1. Suppose \( \dim \) is given, in view of (18), by
\[
\lim_{\epsilon \to 0} \dim(F_r((a \times [t, b]) \cap F_r((a \times [t + \epsilon, b]))) = \lim_{\epsilon \to 0} \dim(F_r((a \times [t, t + \epsilon))) = \delta_r(a, t) \text{ in view of Proposition 3.1 for assignment (1),}
\]
\[
\lim_{\epsilon \to 0} \dim(T_r((a', t) \times b) / T_r((a', t - \epsilon) \times b)) = \lim_{\epsilon \to 0} \dim T_r((t - \epsilon, t) \times b) = \gamma_r(t, b) \text{ in view of Proposition 3.3 for assignment (2),}
\]
\[
\lim_{\epsilon \to 0} \dim(T_r((a \times (b', t]) \cap T_r((a \times (b', t - \epsilon)) = \lim_{\epsilon \to 0} \dim T_r((a \times (t - \epsilon, t) = \gamma_r(a, t) \text{ in view of Proposition 3.3 for assignment (3),}
\]
\[
\lim_{\epsilon \to 0} \dim(F_r((a', t) \times b) / F_r((a', t - \epsilon) \times b)) = \lim_{\epsilon \to 0} \dim F_r((t - \epsilon, t) \times b) = \delta_r(t, b) \text{ in view of Proposition 3.1 for assignment (4),}
\]

In particular we have the following:

1. Suppose \( \dim H_r(X_a, X_{<a}) < \infty \). Then
   \[
   \hat{\delta}^f_r(a, t) = 0 \text{ of } t \neq \{t^1_b, \ldots, t^{N_b}_a\}, \text{ hence the map } \hat{\delta}^f : a \times \mathbb{R} \to \mathbb{Z}_{\geq 0} \text{ is a configuration of points,}
   \]
   \[
   \text{(b) for any } x, y \text{ with } -\infty \leq x \leq y \leq \infty \text{ one has}
   \]
   \[
   \mathbb{F}_r((a \times [x, y])) \simeq \bigoplus_{x \leq t < y} \hat{\delta}^f_r(a, t).
   \]

2. Suppose \( \dim H_r(X_b, X_{<b}) < \infty \). Then
   \[
   \hat{\gamma}^f_r(t, b) = 0 \text{ of } t \neq \{t^1_b, \ldots, t^{M_b}_b\} \text{ hence the map } \gamma^f_r : (-\infty, b) \to \mathbb{Z}_{\geq 0} \text{ is a configuration of points,}
   \]
   \[
   \text{(b) for any } x, y \text{ with } -\infty \leq x < y < b \text{ one has}
   \]
   \[
   \mathbb{T}_r((x, y] \times b) \simeq \bigoplus_{x < t \leq y} \hat{\gamma}^f_r(t, b)
   \]
   \[
   \mathbb{T}_r((x, b) \times b) \simeq \bigoplus_{x < t < b} \hat{\gamma}^f_r(t, b).
   \]

3. Suppose \( \dim H_r(X_a, X_{<a}) < \infty \). Then
   \[
   \hat{\gamma}^f_r(a, t) = 0 \text{ of } t \neq \{t^1_b, \ldots, t^{M_b}_b\} \text{ hence the map } \gamma^f_r : (a, \infty) \to \mathbb{Z}_{\geq 0} \text{ is a configuration of points,}
   \]
   \[
   \text{(b) for any } x, y \text{ with } a \leq x < y \leq \infty \text{ one has}
   \]
   \[
   \mathbb{T}_r((a \times (x, y]) \simeq \bigoplus_{x < t \leq y} \hat{\gamma}^f_r(a, t)
   \]

4. Suppose \( \dim H_r(X^b, X^{>b}) < \infty \). Then
   \[
   \hat{\delta}^f_r(t, b) = 0 \text{ of } t \neq \{t^1_b, t^2_b, \ldots, t^{N_b}_b\} \text{ hence the map } \delta^f_r : \mathbb{R} \times b \to \mathbb{Z}_{\geq 0} \text{ is a configuration of points,}
   \]
   \[
   \text{(b) for any } -\infty \leq x \leq y \leq \infty \text{ one has}
   \]
   \[
   \mathbb{F}_r((x, y] \times b) \simeq \bigoplus_{x < t \leq y} \hat{\delta}^f_r(t, b)
   \]
Part (a) of (1), (2), (3) and (4) establish item 2. Part (b) provide calculations used below.

To prove Item 3 first observe that both \( \text{supp}\delta^f \) and \( \text{supp}\gamma^f \) are \( \Gamma \)-invariant w.r. to the diagonal action of \( \Gamma \) on \( CR(f) \times CR(f) \subset \mathbb{R} \times \mathbb{R}, \mu(g, (x, y)) \sim (g + x, g + y) \).

A collection \( \mathcal{B} \) of elements of \( \text{supp}\delta^f \) or \( \text{supp}\gamma^f \) is called a base if :

1. for any \( v \in \text{supp}\delta^f \) resp. \( v \in \text{supp}\gamma^f \) there exists \( u \in \mathcal{B} \) and \( g \in \Gamma \) s.t. \( v = \mu(g, u) \)

2. If \( u_1, u_2 \in \mathcal{B} \) and \( u_2 = \mu(g, u_1) \) then \( g = 0 \).

For a base \( \mathcal{B} \) define

\[
\mathcal{B}^\delta(t) := \sum_{(a, b) \in \mathcal{B} : b - a = t} \delta^f(a, b), \quad \delta(a, b) = b - a
\]

resp.

\[
\mathcal{B}^\gamma(t) := \sum_{(a, b) \in \mathcal{B} : b - a = t} \gamma^f(a, b), \quad \gamma(a, b) = b - a
\]

which, when \( \mathcal{B} \) is finite, are clearly a \( \mathbb{Z}_{\geq 0} \)-valued configurations on \( \mathbb{R} \) resp. \( \mathbb{R}_{>0} \) and in view of the properties of the base are independent of \( \mathcal{B} \).

Each finite base \( \mathcal{B} \) provides a finite collection of real numbers \( t = b - a, (a, b) \in \mathcal{B} \), which in view of the properties of the base is independent of \( \mathcal{B} \) and of the lift \( f \) hence define the configurations \( \delta^\omega \) and \( \gamma^\omega \).

To finalize the proof one needs to check the existence of bases \( \mathcal{B} \). In the case \( \delta \) we proceed as follows. For any \( o \in CR(f)/\Gamma \) one chooses \( a^o \in CR(f) \) \( a^o \in o \); then the collection

\[
\mathbb{B} := \bigcup_{o \in CR(f)/\Gamma} \{ (a^o, t_1^o), \cdots (a^o, t_{N_o}^o) \}
\]

is a base. One can also one chooses \( b_o \in CR(f) \) \( b_o \in o \); then the collection

\[
\mathbb{B} := \bigcup_{o \in CR(f)/\Gamma} \{ (b_o, t_1^o), \cdots (b_o, t_{N_o}^o) \}
\]

is a base. The case of \( \gamma \) is entirely similar.

\[\square\]

Define

\[
\mathbb{T}_r(\alpha, a) := \lim_{a \to \alpha} \mathbb{T}_r(a, a)
\]

\[
\mathbb{T}_r(a, \infty) := \lim_{a \to \infty} \mathbb{T}_r(< a, x)
\]

and denote by

\[
\hat{\lambda}_r(a) := \text{img}(\mathbb{T}_r(\alpha, a) \to \mathbb{T}_r(< a, x))
\]

which is a f.d. vector space when \( H_r(X_a, X_{<a}) \) is finite dimensional.

**Observation 3.6**

1. \( \mathbb{T}_r(a \times (a, \infty)) = \mathbb{T}_r(a, \infty) / \mathbb{T}_r(< a, \infty) \)
Note that we have the following filtrations:

\[
\mathbb{T}_r((-\infty, a) \times a) = \mathbb{T}_r(<a,a)/i\mathbb{T}_r(-\infty, a) = \mathbb{T}_r(<a,a)/i\mathbb{T}_r(-\infty, a) = \mathbb{T}_r(<a,a)/(\hat{\lambda}_r(a))
\]

**Proof:** Apply the definitions.

**Theorem 3.7**

Suppose \( \dim H_r(X_a, X_{<a}) < \infty \). Then:

1. \( T_r(<a,a) = \hat{\lambda}_r(a) \oplus T_r((-\infty, a) \times a) = \hat{\lambda}_r(a) \oplus \bigoplus_{x < a} \hat{\gamma}_r(x,a) \)
2. \( \lim_{\gamma \to 0} \text{coker}(H_r(X_{a-\gamma}) \to H_r(X_a)) = \mathbb{I}_a(r)/\mathbb{I}_{<a}(r) \oplus \bigoplus_{x < a} \hat{\gamma}_r(a,x) \)
3. \( \mathcal{H}_r(X_a, X_{<a}) = \mathbb{I}_a(r)/\mathbb{I}_{<a}(r) \oplus \bigoplus_{x > a} \hat{\gamma}_r(a,x) \oplus \hat{\lambda}_{r-1}(a) \oplus \bigoplus_{x < a} \hat{\gamma}_{r-1}(x,a) \)

**Proof:** Item 1 follows from Observation 3.6 item 2 and (2).

To check items 2 and 3 some additional notations, not existing in (1), are of help.

\[
\mathbb{T}_r(a', a : t) := \ker(i_{a',t}^a), \quad a' < a < t \leq \infty,
\]

\[
\mathbb{C}_r(a', a : t) := \text{coker}(i_{a',t}^a), \quad a' < a < t \leq \infty,
\]

\[
\mathbb{I}_r(a', a : t) := (\mathbb{I}_a^f(r) \cap \mathbb{I}_r^f(r)) / (\mathbb{I}_a^f(r) \cap \mathbb{I}_r^f(r)) \subseteq \mathbb{I}_a^f(r) / (\mathbb{I}_a^f(r), a' < a,
\]

where \( i_{a',t}^a : \mathbb{T}_r(a', t) \to \mathbb{T}_r(a, t) \) is the obviously induced map considered in formula (15). With these notations one has the diagram (31) below whose rows and columns are exact sequences.

\[
\begin{array}{c}
\mathbb{C}_r(a - \epsilon, a; \infty) \\
\downarrow \\
\mathbb{C}_r(a - \epsilon, a) \\
\downarrow \\
\mathbb{I}_r(a - \epsilon, a) \\
\downarrow \\
0
\end{array}
\]

\[
\begin{array}{c}
\mathbb{C}_r(a - \epsilon, a) \\
\downarrow \\
H_r(X_a, X_{a-\epsilon}) \\
\downarrow \\
\mathbb{T}_{r-1}(a - \epsilon, a) \\
\downarrow \\
0
\end{array}
\]

Note that we have the following filtrations:

1. \( \mathbb{I}_r(a - \epsilon, a) \cdots \supseteq \mathbb{I}_r(a - \epsilon, a ; t) \supseteq \mathbb{I}_r(a - \epsilon, a ; t + \epsilon) \supseteq \cdots \) indexed by \( t \in \mathbb{R} \) with the property

\[
\mathbb{I}_r(a - \epsilon, a ; t) / \mathbb{I}_r(a - \epsilon, a ; t + \epsilon) \supseteq \mathbb{I}_r((a - \epsilon, a) \times [t, t + \epsilon]).
\]

2. \( \mathbb{C}_r(a - \epsilon, a) \cdots \supseteq \mathbb{C}_r(a - \epsilon, a ; t) \supseteq \mathbb{C}_r(a - \epsilon, a ; t - \epsilon) \supseteq \cdots \), \( t - \epsilon > a \) indexed by \( t \in (a, \infty) \),

which in view of (11) satisfies

\[
\mathbb{C}_r(a - \epsilon, a ; t) / \mathbb{C}_r(a - \epsilon, a ; t - \epsilon) = \mathbb{T}_r(a - \epsilon, a) \times (t - \epsilon, t]).
\]

Recall / observe that:
(i) $\mathcal{H}_r(\tilde{X}_a, \tilde{X}_{<a}) = \lim_{\epsilon \to 0} H_r(\tilde{X}_a, \tilde{X}_{a-\epsilon}),$

$T_r(<a, a) := \lim_{\epsilon \to 0} T_r(a - \epsilon, a),$

$C_r(<a, a) := \lim_{\epsilon \to 0} C_r(a - \epsilon, a),$

$I_r(<a, a) := \lim_{\epsilon \to 0} I_r(a - \epsilon, a) = I_r(a)(r)/I_{<a}(r).$

(ii) $I_r(<a, a; t) := \lim_{\epsilon \to 0} I_r(a - \epsilon, a; t),$ and

$\hat{\delta}_f(a, t) := \lim_{\epsilon \to 0} I_r(<a, a; t)/I_r(<a, a; t + \epsilon)$

(iii) $C_r(<a, a; t) := \lim_{\epsilon \to 0} C_r(a - \epsilon, a; t), a < t$ and

$\hat{\gamma}_f(a, t) := \lim_{\epsilon \to 0} C_r(<a, a; t)/C_r(<a, a; t - \epsilon), a < t - \epsilon.$

(iv) $T_r(<a, a; t) := \lim_{\epsilon \to 0} T_r(a - \epsilon, a; t), a < t.$

By passing to limit when $\epsilon \to 0$ the diagram (31) becomes

\[\begin{array}{ccc}
0 & \downarrow & 0 \\
\downarrow & & \downarrow \\
C_r(<a, a; \infty) & \downarrow & C_r(<a, a) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & H_r(\tilde{X}_a, \tilde{X}_{<a}) \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
I_r(<a, a) & \longrightarrow & T_r(<a, a) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & 0.
\end{array}\]

The exactness of the columns and rows in the diagram

\[\begin{array}{ccc}
0 & \downarrow & 0 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & T_r(a', a; t) \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
0 & \longrightarrow & H_r(\tilde{X}_{a'}) \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \tilde{H}_r(\tilde{X}_{a'}) \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
H_r(\tilde{X}_t) & \longrightarrow & H_r(\tilde{X}_{t'})
\end{array}\]

for $a' < a < t$ and the commutativity of the diagram below for for $a' < a < t < t'$
0 \rightarrow T_r(a', a; t) \xrightarrow{i_{a', a}^0} T_r(a', t) \xrightarrow{i_{a', a}^0} T_r(a', a; t) \rightarrow 0

0 \rightarrow T_r(a', a; t') \xrightarrow{i_{a', a}^0} T_r(a', t') \xrightarrow{i_{a', a}^0} T_r(a', a; t') \rightarrow 0

0 \rightarrow T_r(a', a) \xrightarrow{H_r(\tilde{X}_a')} \xrightarrow{i_{a', a}^0} H_r(\tilde{X}_a) \xrightarrow{H_r(\tilde{X}_a')} \xrightarrow{i_{a', a}^0} C_r(a', a) \rightarrow 0

imply the injectivity of all arrows $T_r(a', a; t) \rightarrow T_r(a', a; t') \rightarrow T_r(a', a)$ and $C_r(a', a; t) \rightarrow C_r(a', a; t') \rightarrow C_r(a', a)$.

As already noticed, in view of the exactness of the row in (32), if $\dim H_r(\tilde{X}_a, \tilde{X}_{<a})$ is finite then so are $\dim T_r(\alpha, a, \infty), \dim C_r(\alpha, a, \infty), \dim I_a(r)/I_{<a}(r)$ and then so are $\dim T_r(\alpha, a, a; t), \dim C_r(\alpha, a, a; t)$ and $\dim I_r(\alpha, a, a; t)$ are integer valued with finitely many jumps located at $t$ where $\gamma_{\alpha}^r(a, t) \neq 0$ resp. $\delta_{\alpha}^r(t, a) \neq 0$. They are also continuous from the right. In view of (ii), (iii), (iv) above one has

(a) $I_r(\alpha, a, a; t) = \oplus_{t<a} \delta_r(t, a)$,
(b) $C_r(\alpha, a, a; t) = \oplus_{t>a} \gamma_r(t, a)$,

and combined with item 1 one has

$$\dim H_r(\tilde{X}_a, \tilde{X}_{<a}) = \sum_{t<a} \delta_r(t, a) + \sum_{t>a} \gamma_r(t, a) + \sum_{t<a} \delta_{r-1}(t, a) + \lambda_{r-1}(a).$$

which finalizes the proof of Theorem 3.7.

In case $H_r$ is the standard homology this is Corollary 4 in [1].

3.1 Splittings

Recall from [1] section 4:

A splitting $i_{a,b}^{\delta} : \delta_{\alpha}^r(a, b) \rightarrow F_r(a, b)$, for $(a, b) \in \text{supp} \delta_{\alpha}^r \subset CR_r(f) \times CR_r(f)$, is a right inverse of the canonical projection $\pi_{a,b}^{\delta}(r) : F_r(a, b) \rightarrow \delta_{\alpha}^r(a, b)$ i.e. $\pi_{a,b}^{\delta}(r) \cdot i_{a,b}^{\delta}(r) = \text{id}.$

A splitting $i_{a,b}^{\gamma} : \gamma_{\alpha}^r(a, b) \rightarrow T_r(a, b)$, for $(a, b) \in \text{supp} \gamma_{\alpha}^r$, is a right inverse of the canonical projection $\pi_{a,b}^{\gamma}(r) : T_r(a, b) \rightarrow \gamma_{\alpha}^r(a, b)$.

For a box $B = (a', a] \times [b', b)$, $a' < a, b < b'$, resp. for a box above diagonal $B = (a', a] \times (b', b)$, $a' < a \leq b' < b$ one writes $i_{a,b}^{B}(r)$ for the composition of $i_{a,b}^{\delta}(r)$ resp. $i_{a,b}^{\gamma}(r)$ with the projection $F_r(a, b) \rightarrow F_r(B)$ resp $T_r(a, b) \rightarrow T_r(B)$. Clearly the linear maps $i_{a,b}^{B}$ remain splittings of the canonical projections $F_r(B) \rightarrow \delta_{\alpha}^r(a, b)$, resp. $T_r(B) \rightarrow \gamma_{\alpha}^r(a, b)$.

One extends these splittings to any $(a, b) \in B$ as being the linear injective maps $i_{a,b}^{B}(r) : \delta_{\alpha}^r(a, b) \rightarrow F_r(B)$ resp. $i_{a,b}^{B}(r) : \gamma_{\alpha}^r(a, b) \rightarrow T_r(B)$ defined by the composition $i_{a,b}^{B}(r) \cdot i_{a,b}^{\delta}(r)$ where $B' = (a' \alpha] \times [\beta, b']$, $a' < a \alpha, b < \beta < b'$ resp. $B' = (a' \alpha] \times (b', \beta]$, $a' < a, b' < \beta < b$.

For $f : \tilde{X} \rightarrow \mathbb{R}$ a lift of a tame TC1-form $\omega$, a collection $S^6$ resp. $S^7$ of splittings $S^6 = \{i_{a,b}^{\delta}(r)\}$ resp. $S^7 = \{i_{a,b}^{\gamma}(r)\}$ is called $\Gamma-$compatible if for any $(a, b) \in \text{supp} \delta_{\alpha}^r$ resp. $(a, b) \in \text{supp} \gamma_{\alpha}^r$ the diagrams...
\[
\begin{align*}
\mathbb{R}^f(a, b) & \longrightarrow \mathbb{R}^e(g + a, g + b) & \text{resp.} & \mathbb{R}^f(a, b) & \longrightarrow \mathbb{R}^e(g + a, g + b) \\
\tilde{i}_{a,b}(r) & \longrightarrow \tilde{i}_{a+b}(r) & \tilde{i}_{a,b}(r) & \longrightarrow \tilde{i}_{a+b}(r) \\
\delta_{a,b}(r) & \longrightarrow \delta_{a+b}(r) & \delta_{a,b}(r) & \longrightarrow \delta_{a+b}(r) \\
\tilde{\gamma}_{a,b}(r) & \longrightarrow \tilde{\gamma}_{a+b}(r) & \tilde{\gamma}_{a,b}(r) & \longrightarrow \tilde{\gamma}_{a+b}(r) \\
\delta_{a,b}(r) & \longrightarrow \delta_{a+b}(r) & \delta_{a,b}(r) & \longrightarrow \delta_{a+b}(r)
\end{align*}
\]

remain commutative. Note that collections of \( \Gamma \)–compatible splittings \( S^6 \) resp. \( S^7 \) exist.

Indeed, if one chooses one point \((a_i,b_i)\) in each \( \Delta_{r}^\delta \backslash \text{supp} \delta_{r}^f \) resp. \( \Delta_{r}^\gamma \backslash \text{supp} \gamma_{r}^f \) and a splitting \( \tilde{i}_{a_i,b_i}(r) \) resp. \( \tilde{\gamma}_{a_i,b_i}(r) \) and one defines

\[
i_{a_i}^{\gamma}(r) = (g) \cdot \tilde{i}_{a_i,b_i}(r) \cdot (g)^{-1}
\]

then, since \((a_i + b_i, g)\) exhaust all points in the support of \( \delta_{r}^f \) resp. \( \gamma_{r}^f \), one obtains a collection of \( \Gamma \)–compatible splittings \( S^6 \) resp. \( S^7 \).

Given a collection of splittings \( S^6 \) and a set \( A \subset \text{supp} \delta_{r}^f \), as in [1] Proposition 4, one shows that the linear map

\[
S^6 I_A(r) = (\oplus_{(\alpha, \beta) \in A} i_{\alpha, \beta}^\delta(r)) : \oplus \delta_{r}^f(\alpha, \beta) \rightarrow H_r(\hat{X})
\]

is injective, has the image contained in \( \mathbb{R}^e(I) \) provided \( A \subset (\infty, a] \times [b, \infty) \), and remains injective when composed by \( \pi_{a,b} : \mathbb{R}^e(I) \rightarrow \mathbb{R}^e(B) \) resp. \( \pi_{a,b} : \mathbb{R}^e(I) \rightarrow \mathbb{R}^e(I) \) where \( B = (a', a] \times [b', b') \) resp. \( I = a \times [b, b'] \) or \( I = a \times [b', b'], a' < a, b < b' \). Moreover, if one write \( A = A(I) := \text{supp} \delta_{r}^f \cap I \) the composition

\[
S^6 I_{A(I)}(r) = (\oplus_{(\alpha, \beta) \in A(I)} i_{\alpha, \beta}^\delta(r)) : \oplus \delta_{r}^f(\alpha, \beta) \rightarrow \mathbb{R}^e(I)
\]

is an isomorphism.

The same remains true for a collection of splittings \( S^7 \), \( A \subset \{\alpha, \beta \in \text{supp} \gamma_{r}^f \mid \alpha \leq a\} \) and the map

\[
S^7 I_A(r) = (\oplus_{(\alpha, \beta) \in A} i_{\alpha, \beta}^\gamma(r)) : \oplus \gamma_{r}^f(\alpha, \beta) \rightarrow T_r(a, \infty).
\]

and

\[
S^7 I_{A(B)}(r) : \oplus (\alpha, \beta) \in A(B) \gamma_{r}^f(\alpha, \beta) \rightarrow T_r(B)
\]

resp.

\[
S^7 I_{A(I)}(r) : \oplus (\alpha, \beta) \in A(I) \gamma_{r}^f(\alpha, \beta) \rightarrow T_r(B)
\]

for \( B = (a', a] \times (b', b] \) resp. \( I = (a', a] \times b \) or \( a \times (b', b], a' < a \leq b' < b \). Precisely \( S^7 I_A(r) \) is injective and \( S^7 I_{A(I)}(r) \) with target \( T_r(I) \) is isomorphism.

4 Proof of Poincaré duality, Theorem 1.1

Suppose \( a, b \) regular values for \( f : \tilde{M} \rightarrow \mathbb{R} \) a lift of a locally smooth \( \tilde{f} \) hence tame, TC1-form \( \omega \) on a topological closed manifold \( M \), hence the levels \( \tilde{M}(a), \tilde{M}(b) \) are codimension one submanifolds of \( \tilde{M} \). Poincaré duality for the topological manifolds with boundary \( \tilde{M}_a \) and \( \tilde{M}_b \) combined with the excision property for standard cohomology provide the canonical isomorphisms “PD” from the Borel-Moore homology \( BM H_r \) to the standard cohomology \( H^{n-r} \) and then the commutative diagrams below.

\footnote{in the neighborhood of any point there exists coordinates in which \( \omega \) is a differential closed one form whose components are given by polynomial maps}
The first equality holds by exactness of the first row in these diagrams, the second by the equality of the top-
is an isomorphism. Suppose $a < b < c < d$ with all $a, b, c, d$ regular values and consider the box above diagonal $B = (a, b] \times (c, d]$. Poincaré duality and excision property provide the following commutative diagram
In view of Theorem 2.3 item 2 this diagram implies that \( \omega(\alpha, \beta, \gamma) = \omega(\gamma^*, \beta^*, \alpha^*) \) (cf subsection 2.1 for definitions) which by Theorem 2.3 item 1 is canonically isomorphic to \( \omega(\gamma', \beta', \alpha')^* \). In particular one has the isomorphism

\[
BM \xrightarrow{PD_r(B)} BM \xrightarrow{T^f_r(a, b) \times (c, d)} BM \xrightarrow{T^{-f}_{n-r-1}((-d, -c) \times (-b, -a))} BM
\]

For \( a, b \in \mathbb{R} \) and \( \epsilon_1 > \epsilon_2 \) one considers two sub-surjections

\[
1\pi : A = BM \xrightarrow{BM} BM \xrightarrow{BM} BM \xrightarrow{BM} BM \xrightarrow{BM} BM = \pi
\]

and

\[
2\pi : A = (G_n^{-r}(B_1))^* \xrightarrow{BM} (G_n^{-r}(B_2))^* = A
\]
defined based on the boxes

(a) \( B_1 = (a - \epsilon_1, a + \epsilon_1) \times [b - \epsilon_1, b + \epsilon_1] \)

\( B_2 = (a - \epsilon_2, a + \epsilon_2) \times [b - \epsilon_2, b + \epsilon_2] \)

\( C = (a - \epsilon_2, a + \epsilon_1) \times [b - \epsilon_1, b + \epsilon_2] \)

with \( C \) as a lower-right corner of \( B_1, B_2 \) as an upper-left corner of \( C \)

and on the boxes

(b) \( \underline{B}_1 = (b - \epsilon_1, b + \epsilon_1) \times [a - \epsilon_1, a + \epsilon_1] \)

\( \underline{B}_2 = (b - \epsilon_2, b + \epsilon_2) \times [a - \epsilon_2, a + \epsilon_2] \)

\( \underline{C} = (b - \epsilon_1, b + \epsilon_2) \times [a - \epsilon_2, a + \epsilon_1] \)

with \( \underline{C} \) as a upper left corner of \( \underline{B}_1, \underline{B}_2 \) as an down right corner of \( \underline{C} \).

The boxes \( B_1, C, B_2 \) are the symmetric of the boxes \( B_1, C, B_2 \) w.r. to the first diagonal \( \Delta_1 := \{(x, y) \in \mathbb{R}^2 \mid x - y = 0\} \). See Figure 3 below.
In view of Observation 3.2
\( \pi_{B_1} : ^{BM} \mathbb{F}_r(B_1) \to ^{BM} \mathbb{F}_r(C) \) is surjective and
\( i_{B_2}^* : ^{BM} \mathbb{F}_r(B_2) \to ^{BM} \mathbb{F}_r(C) \) is injective and identifies \(^{BM} \mathbb{F}_r(B_2)\) to a subspace of \(^{BM} \mathbb{F}_r(C)\).

Define the sub-surjection \(^1\pi\) by: \( A := ^{BM} \mathbb{F}_r(B_1), P = ^{BM} \mathbb{F}_r(C), A' = ^{BM} \mathbb{F}_r(B_2) \) and \( \pi := \pi_{B_1}^C \).

In view of Observation 3.2
\( (i_{B_2}^C)^*: (G_{n-r}(B_1))^* \to (G_{n-r}(C))^* \) is surjective and
\( (\pi_{B_1}^C)^*: (G_{n-r}(B_2))^* \to (G_{n-r}(C))^* \) is injective.

Define the second sub-surjection \(^2\pi\) by: \( A := (G_{n-r}(B_1))^*, P = (G_{n-r}(C))^*, A' = (G_{n-r}(B_2))^* \) and \( \pi := (i_{B_2}^C)^* \). In view of (36) Poincaré duality identifies the two sub-surjections.

For \( a < b, a, b \in \mathbb{R} \) and \( \epsilon_1 > \epsilon_2 \) with \( a < 2\epsilon_1 < b \) one considers two sub-surjections \(^3\pi : A = ^{BM} \mathbb{T}_r^f(B_1) \sim ^{BMT} \mathbb{T}_r^f(B_2) \) and \(^4\pi : (\mathbb{T}_r^{r-1}(B_1))^* \sim (\mathbb{T}_r^{r-1}(B_2))^* \) based on the boxes above diagonal.

(a) \( B_1 = (a + \epsilon_1, a + \epsilon_1] \times (b - \epsilon_1, b + \epsilon_1] \)
\( B_2 = (a - \epsilon_2, a + \epsilon_2] \times (b - \epsilon_2, b + \epsilon_2] \)
\( C = (a - \epsilon_2, a + \epsilon_1] \times (b - \epsilon_2, b + \epsilon_1] \)
with \( C \) as a upper-right corner of \( B_1 \) and \( B_2 \) as the lower-left corner of \( C \) and on the boxes above diagonal.

(b) \( B_1 = (-b - \epsilon_1, -b + \epsilon_1] \times (-a - \epsilon_1, -a + \epsilon_1] \)
\( B_2 = (-b - \epsilon_2, -b + \epsilon_2] \times (-a - \epsilon_2, -a + \epsilon_2] \)
\( C = (-b - \epsilon_1, -b + \epsilon_2] \times (-a - \epsilon_1, -a + \epsilon_2] \)
with \( C \) as a lower left corner of \( B_1, B_2 \) as an upper-right corner of \( C \)

\(^7\)Since \( i_{B_2}^C : G(C) \to G(B_2) \) is injective and \( \pi_{B_2}^C : G(C) \to G(B_2) \) is surjective
In view of Observation 3.4 \[
\pi^C_{B_1} : BM \mathbb{T}^f(B_1) \to BM \mathbb{T}^f(C)
\] is surjective and \[
i^C_{B_2} : BM \mathbb{T}^f(B_2) \to BM \mathbb{T}^f(C)
\] is injective identifying \(BM \mathbb{T}^f(B_2)\) to a subspace of \(BM \mathbb{T}^f(C)\).

Define the sub-surjection \(3\tilde{\pi}\) with \(A := BM \mathbb{T}^f(B_1), P = BM \mathbb{T}^f(C), A' = BM \mathbb{T}^f(B_2)\) and \(\pi := \pi^C_{B_1}\).

In view of Observation 3.4 \[
(\tau^C_{B_1})^* : (\mathbb{T}^f_{n-r-1}(B_1))^* \to (\mathbb{T}^f_{n-r-1}(C))^*
\] is surjective. \[
(\tau^C_{B_2})^* : (\mathbb{T}^f_{n-r-1}(B_2))^* \to (\mathbb{T}^f_{n-r-1}(C))^*
\] is injective.

Define the sub-surjection \(4\tilde{\pi}\) with \(A := (\mathbb{T}^f_{n-r-1}(B_1))^*, P = (\mathbb{T}^f_{n-r-1}(C))^*, A' = (\mathbb{T}^f_{n-r-1}(B_2))^*\) and \(\pi := (\tau^C_{B_1})^*\).

In view of (39) Poincaré duality identifies the two quasi surjections.

Given \(a, b \in CR(f)\) choose a sequence \(\epsilon_1 > \epsilon_2 > \epsilon_3 > \cdots \epsilon_k > \cdots > 0\) with \(\lim_{i \to \infty} \epsilon_i = 0\) s.t. \(a \pm \epsilon_i\) are regular values for any \(i\). This is possible since the set of regular values of \(f\) is everywhere dense in \(\mathbb{R}\). In case \(a < b\) assume in addition that \(a < 2\epsilon_1 < b\).

First consider the collection of sub-surjections \(BM^F\tilde{\pi}_i : A_i \to A_{i+1}\) with \(A_i := BM \mathbb{F}_r(B_i), B_i\) the box \(B_i := (a - \epsilon_i, a + \epsilon_i) \times [b - \epsilon_i, b + \epsilon_i]\) which identifies by Poincaré duality to the collection of sub-surjections \(G^F\tilde{\pi}_i : A_i \to A_{i+1}\) with \(A_i := (G_{n-r}(B_i))^*\) with \(B_i\) with the box \(B_i = (b - \epsilon_i, b + \epsilon_i) \times [a - \epsilon_i, a + \epsilon_i]\). Since in view of (36) \(\lim_{i \to \infty} \tilde{\pi}_i = \lim_{i \to \infty} 2\tilde{\pi}_i\), Proposition 4.1 below implies

\[
BM^F\delta^f_{n-r}(a, b) = (G^F\delta^f_{n-r}(b, a))^*.
\]

In view of the finite dimensionality of \(\delta^f_{n-r}(b, a) = F\delta^f_{n-r}(b, a) = G\delta^f_{n-r}(b, a)\) one has \((G^F\delta^f_{n-r}(b, a))^* = \]
\[\text{since } \tau^C_{B_1} : \mathbb{T}^f(C) \to \mathbb{T}^f(B_1) \text{ is surjective and } \tau^C_{B_2} : \mathbb{T}^f(C) \to \mathbb{T}^f(B_2) \text{ is injective}\]

\[\text{Figure 4}\]
\begin{align*}
\dim G\delta^f_{n-r}(b,a), \text{ hence } BM\delta^f_r(a,b) = \delta^f_{n-r}(b,a) \text{ hence }
BM\delta^\omega_r(t) &= \delta^\omega_{n-r}(-t).
\end{align*}

This establishes Item 1 in Theorem [1.1].

Similarly, given \(a < b\) consider the collection of sub-surjections \(3\pi^f_i : A_i \rightrightarrows A_{i+1}\) with \(A_i := BM \mathbb{T}_{i}^f(B_i), B_i\) the box above diagonal \(B_i := (a - \epsilon_i, a + \epsilon_i] \times (b - \epsilon_i, b + \epsilon_i]\) which identifies by Poincaré duality to the collection of sub-surjections \(4\pi^f_i : A_i \rightrightarrows A_{i+1}\) with \(A_i := (\mathbb{T}_{n-r-1}^{-f}(B_i))^*\) with \(B_i\) the box above diagonal \(B_i = (-b - \epsilon_i, -b + \epsilon_i] \times (-a - \epsilon_i, a + \epsilon_i].\)

In view of (39) one has \(\lim_{i \to \infty} 3\pi^f_i = \lim_{i \to \infty} 4\pi^f_i\).

Proposition 4.1 below implies
\[BM\tilde{\gamma}^f_r(a,b) = (\tilde{\gamma}_{n-r-1}^{-f}(-b,-a))^*\]
and in view of the finite dimensionality of \(\tilde{\gamma}_{n-r-1}^{-f}(-b,-a)\) one has
\[\dim \tilde{\gamma}_{n-r-1}^{-f}(-b,-a) = (\dim \tilde{\gamma}_{n-r-1}^{-f}(-b,-a))^*\]

hence \(BM\gamma^\omega_r(a,b) = \gamma^\omega_{n-r-1}(-b,-a)\) hence
\[BM\gamma^\omega_r(t) = \gamma^\omega_{n-r-1}(t).\]

This establishes Item 2 in Theorem [1.1]

\textbf{Proposition 4.1}

1. \(\lim_{i \to \infty} 1\pi_i = BM\tilde{\delta}^f_r(a,b)\)
2. \(\lim_{i \to \infty} 2\pi_i = (\tilde{\delta}^f_{n-r}(b,a))^*\)
3. \(\lim_{i \to \infty} 3\pi^f_i = BM\tilde{\gamma}^f_r(a,b)\)
4. \(\lim_{i \to \infty} 4\pi^f_i = (\tilde{\gamma}_{n-r-1}^{-f}(-b,-a))^*\)

\textbf{Proof:}

Using the description of \(A_i^\infty\) (at the end of section 1) Observations [3.2] and Observation [3.4] one concludes that for each directed system \(1\pi_i, 2\pi_i, 3\pi_i, 4\pi_i\) the corresponding \(A_i^\infty\) are given by

\[A_i^\infty = \begin{cases}
BM\mathbb{T}_{r}((a - \epsilon_i, a] \times [b, b + \epsilon_i])
\mathbb{G}_{n-r}((-b - \epsilon_i, b] \times [a, a + \epsilon_i])^*
BM\mathbb{T}_{r}^f((a - \epsilon_i, a] \times (-b - \epsilon_i, b])
(\mathbb{T}_{n-r-1}^{-f}((-b - \epsilon_i, -b] \times (-a - \epsilon_i, -a]))^*.
\end{cases}\]

In view of the Definitions [3.1] and of the definitions of \(BM\tilde{\delta}^f_r, G\tilde{\delta}^f_r, BM\gamma^f_r, \) one has

\[\lim_{i \to \infty} (\pi^f_i : A_i^\infty \to A_{i+1}^\infty) = \begin{cases}
BM\tilde{\delta}^f_r(a,b)
(G\tilde{\delta}^f_{n-r}(b,a))^*
BM\tilde{\gamma}^f_r(a,b)
(\tilde{\gamma}_{n-r-1}^{-f}(-b,-a))^*.
\end{cases}\]

\textbf{q.e.d}
5 Proof of stability property, Theorem 1.2

In view of Proposition 3.5 Note that for each diagonal $\Delta_t := \{(x, y) \in \mathbb{R}^2, y - x = t\}$ the set $\Delta_t \cap \text{supp} \delta^f_t$ if not empty is $\kappa$—invariant w.r. to the diagonal action and the set $(\Delta_t \cap \text{supp} \delta^f_t)/\Gamma$ consists of finitely many orbits say $o_1, o_2, \cdots o_k$. Choose in any such orbit one point $(a_i, b_i)$, and let $\delta^f_t(a_i, b_i) = \sum_{i=1,2,\cdots k} \delta^f_t(a_i, b_i)$. Clearly $\delta^f_t(a, b)$ does not depend on the choice of $(a_i, b_i)$. Recall that $\delta^f_t(a, b) \neq 0$ implies that both $a$ and $b$ are critical values.

Denote by

1. $\tilde{\mathbb{F}}^f_t(\{a, b\}) := \bigoplus_{g \in \Gamma} \tilde{\delta}^f_t(a + g, b + g),$
2. $\tilde{\mathbb{F}}^f_t(t) := \bigoplus_{\{(a, b) \in \mathbb{R}^2, b-a=t\}} \tilde{\delta}^f_t(a, b) = \bigoplus_{\{(a, b) \in \text{supp} \delta^f_t, b-a=t\}} \tilde{\delta}^f_t(a, b) = \bigoplus_{i(a, b) \in \Delta_t} \tilde{\mathbb{F}}^f_t(\{a, b\}),$
3. $\tilde{\mathbb{F}}^f_t(\leq t) := \bigoplus_{\{(a, b) \in \mathbb{R}^2, b-a \leq t\}} \tilde{\delta}^f_t(a, b) = \bigoplus_{\{(a, b) \in \text{supp} \delta^f_t, b-a \leq t\}} \tilde{\delta}^f_t(a, b) = \bigoplus_{s \leq t} \tilde{\mathbb{F}}^f_t(s),$
4. $\tilde{\mathbb{F}}^f_t(< t) := \bigoplus_{\{(a, b) \in \mathbb{R}^2, b-a < t\}} \tilde{\delta}^f_t(a, b) = \bigoplus_{\{(a, b) \in \text{supp} \delta^f_t, b-a < t\}} \tilde{\delta}^f_t(a, b) = \bigoplus_{s < t} \tilde{\mathbb{F}}^f_t(s),$
5. $\tilde{\mathbb{F}}^f_r := \bigoplus_{\{(a, b) \in \mathbb{R}^2\}} \tilde{\delta}^f_r(a, b) = \bigoplus_{\{(a, b) \in \text{supp} \delta^f_r\}} \tilde{\delta}^f_r(a, b) = \bigoplus_{s \in \mathbb{R}} \tilde{\mathbb{F}}^f_r(s).$

Let $\langle g \rangle_{a,b} : \delta^f_t(a, b) \to \tilde{\delta}^f_t(a + g, b + g)$ be the linear isomorphism derived from the linear isomorphism $\langle g \rangle : H_r(X) \to H_r(X)$ end equip $\tilde{\mathbb{F}}^f_r$ with a structure of $\kappa[\Gamma]$— module given by

$$\langle g \rangle := \bigoplus_{\{(a, b) \in \mathbb{R}^2\}} \langle g \rangle_{a,b} \tilde{\delta}^f_t(a, b) \to \tilde{\delta}^f_t(a + g, b + g).$$

The $\kappa[\Gamma]$—modules $\tilde{\mathbb{F}}^f_t(\{a, b\}), \tilde{\mathbb{F}}^f_t(t), \tilde{\mathbb{F}}^f_t(\leq t), \tilde{\mathbb{F}}^f_t(< t)$ and $\tilde{\mathbb{F}}^f_r$, are all free f.g. $\kappa[\Gamma]$— modules and in view of Proposition 3.5 of finite rank

1. $\delta^f_t(a, b),$
2. $\delta^f_r(t) = \sum_{i(a, b) \in \Delta_t} \delta^f_t(a, b),$
3. $\sum_{s \in \text{supp} \delta^f_r, s \leq t} \delta^f_r(s),$
4. $\sum_{s \in \text{supp} \delta^f_r, s < t} \delta^f_r(s),$
5. $\sum_{s \in \text{supp} \delta^f_r} \delta^f_r(s).$

Note that $\text{Tor} H_r(X) = \bigcap_{l} \tilde{\mathbb{F}}^f_l = \bigcap_{l} \tilde{\mathbb{F}}^f_l(r)$ which implies that $\mathbb{F}_r(a, b) \supseteq \text{Tor} H_r(H_r(X)) \subseteq H_r(X)/\text{Tor}(H_r(X))$

Denote by

$$\mathbb{F}^f_t := H_r(X)/\text{Tor} H_r(X),$$
$$\mathbb{F}^f_t(\{a, b\}) := \bigoplus_{g \in \Gamma} \mathbb{F}^f_t(a + g, b + g)/\text{Tor} H_r(X) \subseteq \mathbb{F}^f_r,$$
$$\mathbb{F}^f_t(t) := \bigoplus_{\{(a, b) \in \text{supp} \delta^f_t, b-a=t\}} (\mathbb{F}^f_t(a, b)/\text{Tor} H_r(X)) \subseteq \mathbb{F}^f_r,$$
$$\mathbb{F}^f_t(\leq t) := \bigoplus_{\{(a, b) \in \text{supp} \delta^f_t, b-a \leq t\}} (\mathbb{F}^f_t(a, b)/\text{Tor} H_r(X)) \subseteq \mathbb{F}^f_r,$$
$$\mathbb{F}^f_t(< t) := \bigoplus_{\{(a, b) \in \text{supp} \delta^f_t, b-a < t\}} (\mathbb{F}^f_t(a, b)/\text{Tor} H_r(X)) \subseteq \mathbb{F}^f_r.$$
2. \( \mathbb{P}^f_r(\leq t) = \mathbb{P}^\omega_r(\leq t_i) \) for \( t_i \leq t < t_{i+1} \),
3. \( \mathbb{P}^f_r(\leq t) = H_r(\tilde{X})/Tor H_r(\tilde{X}) \) for \( t_k \leq t \).
4. \( \mathbb{P}^f_r(\leq t_i)/\mathbb{P}^\omega_r(\leq t_{i-1}) = \mathbb{P}^\omega_r(t_i) \).

Choose a collection of \( \Gamma \) compatible splittings \( S^\delta := \{i_{a,b}(r) : \delta^f_r(a,b) \rightarrow \mathbb{P}^f_r(a,b) \mid (a,b) \in \text{supp} \, \delta^f_r \} \) (like in subsection 2.4) and consider
\[
S I(r) : \mathbb{P}^\omega_r \rightarrow \mathbb{P}^f_r = H_r(\tilde{X})/Tor H_r(\tilde{X}).
\]

The map \( S I(r) \) is \( \kappa[\Gamma] \)-linear, in view of Proposition 4 in [1] is injective and sends \( \mathbb{P}^\omega_r \) into \( \mathbb{P}^f_r \), and restricts to the maps
\[
S I(a,b)(r) : \tilde{\mathbb{P}}^f_r(\{a,b\}) \rightarrow \mathbb{P}^f_r(\{a,b\}),
S I(r)_t : \tilde{\mathbb{P}}^f_r(t) \rightarrow \mathbb{P}^\omega_r(t),
S I(r)_{\leq} : \tilde{\mathbb{P}}^f_r(\leq t) \rightarrow \mathbb{P}^f_r(\leq t),
S I(r)_{<} : \tilde{\mathbb{P}}^f_r(< t) \rightarrow \mathbb{P}^f_r(< t)
\]
all injective \( \kappa[\Gamma] \)-linear maps.

**Proposition 5.1** The above maps are all bijective, hence \( \mathbb{P}^f_r(t), \mathbb{P}^\omega_r(\leq t), \mathbb{P}^\omega_r(< t), \mathbb{P}^f_r \) are all f.g. free modules of rank

1. \( \delta^\omega_r(t) \),
2. \( \sum_{s \in \text{supp} \, \delta^\omega_r | s \leq t} \delta^\omega_r(s) \),
3. \( \sum_{s \in \text{supp} \, \delta^\omega_r | s < t} \delta^\omega_r(s) \),
4. \( \sum_{s \in \text{supp} \, \delta^\omega_r} \delta^\omega_r(s) \).

In view of the above it suffices to check the surjectivity of \( S I(r) \).

Indeed for \( x \in H_r(\tilde{X}) \) define:

- \( \alpha(x) := \inf \{a \mid x \in \mathbb{P}^f_r(r)\} \), \( \alpha(x) \in [-\infty, \infty) \)
  
  with \( \alpha(x) = -\infty \) if \( x \in \cap_{a \in \mathbb{R}} I^f_a(r) \),
- \( \beta(x) := \sup \{b \mid x \in \mathbb{P}^f_r(r)\} \), \( \beta(x) \in (-\infty, \infty] \)
  
  with \( \beta(x) = \infty \) if \( x \in \cap_{b \in \mathbb{R}} I^f_b(r) \),
- \( t^\delta(x) = \beta(x) - \alpha(x), t^\delta(x) \in (-\infty, \infty] \).

In view of Proposition 3.1 in [1] \( x \in Tor H_k(\tilde{X}) \) iff \( t(x) = \infty \). Observe that if \( x \in H_r(\tilde{X}) \) with \( \alpha(x) = a, \beta(x) = b, t = t(x) = b - a \), hence \( x \notin Tor H_r(\tilde{X}) \), then it can be written (in the presence of a collection of splittings) as \( x = i_{a,b}(\hat{x}) + x' \) with \( \hat{x} = \pi_{a,b}(x) \in \delta_r(a,b) \) and \( x' \in H_r(\tilde{X}) \) with \( t(x') < t \). This implies that in view of finite generation of \( H_r(\tilde{X}) \) there exists a finite sequence \( t_0 > t_1 > t_2 \cdots > t_k \) and \( x_i \) with \( \alpha(x_i) = a_i, \beta(x_i) = b_i, t(x_i) = t_i \) s.t. \( x = \sum_{0 \leq i \leq k} i_{a_i,b_i}(\hat{x}_i), \hat{x}_i \neq 0 \); otherwise \( H_r(\tilde{X}) \) contains an infinite sequence of submodules \( \cdots \subset \mathbb{P}^f_r(t_i) \supset \mathbb{P}^f_r(t_{i+1}) \supset \cdots \) which contradicts the f.g property of \( H_r(\tilde{X}) \).

q.e.d.

To finalize the proof of Theorem [12] one follows the arguments in [4] or in [2] section 5. Observe that for any two TC 1-forms \( \omega_1, \omega_2 \) in the same cohomology class and two lifts \( f_1 : \tilde{X} \rightarrow \mathbb{R} \) of \( \omega_1 \) and \( f_2 : \tilde{X} \rightarrow \mathbb{R} \) of \( \omega_2 \) with \( ||f_1 - f_2||_{\infty} < \epsilon \) and \( a, b \in \mathbb{R} \) one has:
pairs \( \theta \). Commutative. When \( \theta \) standard homology versus Borel-Moore homology (Proposition 1.4)

\[ f_{a+\epsilon}(r) \subseteq f_{a}(r) \subseteq f_{a+\epsilon}(r) \]

and therefore

\[ f_{a}(a - \epsilon, b + \epsilon) \subseteq f_{a}(a, b) \subseteq f_{a}(a + \epsilon, b - \epsilon) \]

and for any \( t \in supp \delta^\omega_r \)

\[ f_{\omega_r}(t - 2\epsilon) \subseteq f_{\omega_r}(t) \subseteq f_{\omega_r}(t + 2\epsilon) \]

\[ f_{\omega_r}(t < (t - 2\epsilon)) \subseteq f_{\omega_r}(t < t) \subseteq f_{\omega_r}(t < (t + 2\epsilon)) \]

Clearly, if \( f_{\omega_r}(t - \epsilon) = f_{\omega_r}(t + \epsilon) \) resp. \( f_{\omega_r}(t < (t - \epsilon)) = f_{\omega_r}(t < (t + \epsilon)) \) then \( f_{\omega_r}(t) = f_{\omega_r}(t - \epsilon) = f_{\omega_r}(t < t) = f_{\omega_r}(t < t) = f_{\omega_r}(t < t) \).

For a tame TC1-form \( \omega \) denote by \( \sigma_f(\omega) := \inf |t_i - t_j|, t_i \neq t_j \) with \( t_i, t_j \in supp \delta^\omega_r \). As an immediate consequence of Proposition (5.1) one has Proposition (5.2) which, as in [4] or [2] section 5, implies the continuity of the assignment \( \omega \rightsquigarrow \delta^\omega_r \).

**Proposition 5.2** For any \( \epsilon < \sigma_f(\omega), \omega, \omega' \) two tame TC1-forms in the same cohomology class with \( |\omega - \omega'| < \epsilon/3 \) and \( t_i \in supp \delta^\omega_r \) one has

\[ \delta^\omega_r(t_i) = \sum_{t_i - \epsilon < s < t_i + \epsilon} \delta^\omega_r(s) \]

\[ supp \delta^\omega_r \subset \bigcup_{t_i \in supp \delta^\omega_r} \delta^\omega_r(t_i - \epsilon, t_i + \epsilon) \]

In fact

\[ \hat{\delta}^\omega_r(t_i) \simeq \bigoplus_{t_i - \epsilon < s < t_i + \epsilon} \hat{\delta}^\omega_r(s) \]

6 Standard homology versus Borel-Moore homology (Proposition 1.4)

In this section \( H_r \) denotes the standard (singular) homology, \( B^mH_r \) the Borel-Moore homology and the linear maps \( \theta^Y_r : H_r(X,Y) \rightarrow B^mH_r(X,Y) \) define the natural transformation from the standard to the Borel-Moore homology. In general the linear maps \( \theta^Y_r : H_r(X,Y) \rightarrow B^mH_r(X,Y) \) are not isomorphisms.

Consider pairs \( (U, V) \) of locally compact ANRs, \( V \) closed subset of \( U \). An inclusion of such pairs \( (U_1, V_1) \subseteq (U_2, V_2) \), s.t. \( U_1 \) open subset in \( U_2 \) and \( V_1 \) open set of \( V_2 \) induces the linear maps \( i_{U_1,V_1}^U : H_r(U_1, V_1) \rightarrow H_r(U_2, V_2) \) and \( p_{U_2,V_2}^{U_1,V_1} : B^mH_r(U_2, V_2) \rightarrow B^mH_r(U_1, V_1) \) which together with \( \theta^V_r, U_1, V_1 \) and \( \theta^V_r, U_2, V_2 \) make the diagram

\[ H_r(U_1, V_1) \rightarrow \theta^V_r, U_1, V_1, B^mH_r(U_1, V_1) \]

\[ i_{U_2,V_2}^{U_1,V_1} \]

\[ H_r(U_2, V_2) \rightarrow \theta^V_r, U_2, V_2, B^mH_r(U_2, V_2) \]

commutative. When \( V_1 \) and \( V_2 \) are the empty sets one has the commutative diagram

\[ H_r(U_1) \rightarrow \theta^V_r, U_1, B^mH_r(U_1) \]

\[ i_{U_2} \]

\[ H_r(U_2) \rightarrow \theta^V_r, U_2, B^mH_r(U_2) \]

\[ 31 \]
By similar arguments as in [3], when regarded as a contravariant functor on the category whose objects are open subsets $U$ of a locally compact ANR $X$ and morphisms are the inclusions, the assignment $U \mapsto H_r(U)$ induces the exact sequence

$$0 \xrightarrow{\lim'_{i \to \infty}} \lim'_{i \to \infty} B^M H_{r-1}(U(i)) \xrightarrow{BM} H_r(X) \xrightarrow{\lim'_{i \to \infty}} \lim'_{i \to \infty} B^M H_r(U(i)) \xrightarrow{BM} 0$$  \hspace{1cm} (43)

for any open filtration of $X$ by open sets $U(i)$,

$$U(1) \subset U(2) \subset \cdots \subset U(i) \subset U(i + 1) \subset \cdots \subset X,$$

$X = \bigcup_i U(i)$. Here $\lim$ and $\lim'$ denotes inverse limit and the derived inverse limit of the system of vector spaces and linear maps

$$p_{U_{i+1}}^{U_i} : B^M H_r(U_{i+1}) \to B^M H_r(U_i).$$

For a pair $(X, Y)$ as above equipped with a filtration by open pairs,

$$(U(1), V(1)) \subset (U(2), V(2)) \subset \cdots \subset (U(i), V(i)) \subset (U(i + 1), V(i + 1)) \subset (X, Y),$$

i.e. $U(i)$ open subset of $X$, $V(i)$ open subset of $Y$, $X = \bigcup_i (U(i), Y = \bigcup_i V(i)$, the exact sequence (43) implies the short exact sequence

$$0 \xrightarrow{\lim'_{i \to \infty}} \lim'_{i \to \infty} B^M H_{r-1}(U(i), V(i)) \xrightarrow{BM} H_r(X, Y) \xrightarrow{\lim'_{i \to \infty}} \lim'_{i \to \infty} B^M H_r(U(i), V(i)) \xrightarrow{BM} 0$$ \hspace{1cm} (44)

with the derived inverse limit and the inverse limit considered for of the system of vector spaces and linear maps

$$p_{U_{i+1}, V_{i+1}}^{U_i, V_i} : B^M H_r(U(i + 1), V(i + 1)) \to B^M H_r(U(i), V(i)).$$

Consider $f : \tilde{X} \to \mathbb{R}$ a lift of a tame TC1-form $\omega$ and

$$U(1) \subset U(2) \subset \cdots \subset U(i) \subset U(i + 1) \subset \cdots \subset U(\infty) = \tilde{X}$$

a filtration by open sets of $\tilde{X}$. By applying the exact sequence (44) to $X = \tilde{X}_a$, $Y = \tilde{X}_{a-\epsilon}$ and the filtration by the pairs $(U(i)_a, U(i)_a)$ of the pair $(\tilde{X}_a, \tilde{X}_{a-\epsilon})$ and by passing to direct limit when $\epsilon \to 0$ one obtains the exact sequence

$$0 \xrightarrow{\lim'_{i \to \infty}} \lim'_{i \to \infty} B^M H_{r-1}(U(i)_a, U(i)_a) \xrightarrow{BM} H_r(\tilde{X}_a, \tilde{X}_{a-\epsilon}) \xrightarrow{\lim'_{i \to \infty}} \lim'_{i \to \infty} B^M H_r(U(i)_a, U(i)_a) \xrightarrow{BM} 0$$ \hspace{1cm} (45)

For the proof of Proposition 1.4, we will also need the following Proposition 6.1.

**Proposition 6.1** Suppose $M$ is a compact manifold with boundary $\partial M$ and interior $\text{Int} M = U$, and $f : M \to \mathbb{R}$ a tame map with with $C^r(f) \subset U$. Suppose $t$ is a regular value for $f|_{\partial M}$ the restriction of $f$ to the boundary, but possibly critical value for $f$. Then for $\epsilon > 0$ small enough the inclusion induced linear map $i_\epsilon(t) := i_{U_t, U_{t-\epsilon}}$ and the $\theta_\epsilon(t) := \theta_{U_t, U_{t-\epsilon}}$ in the sequence (1) below are isomorphisms.

$$H_r(M_t, M_{t-\epsilon}) \xrightarrow{i_{\epsilon}(t)} H_r(U_t, U_{t-\epsilon}) \xrightarrow{\theta_{\epsilon}(t)} B^M H_r(U_t, U_{t-\epsilon}) \hspace{1cm} (46)$$
Proof: We leave to the reader to check that the inclusion $(U_t, U_{t-\epsilon}) \subset (M_t, M_{t-\epsilon})$ is a homotopy equivalence of pairs which implies $i_r(t)$ is an isomorphism. Concerning $\theta_r(t)$ observe that since $t$ is regular value for $f_{\partial M}$ there exists $\epsilon > 0$ small enough s.t. the inclusion $\partial M(t-\epsilon) \subset \partial M[t_{t-\epsilon},t]$ is a homotopy equivalence, hence the inclusion $(M_t, M_{t-\epsilon}) \subset (M_t, M_{t-\epsilon} \cup \partial M[t_{t-\epsilon},t])$ is a homotopy equivalence of compact pairs. Hence

$$H_r(M_t, M_{t-\epsilon}) = H_r(M_t, M_{t-\epsilon} \cup \partial M[t_{t-\epsilon},t]) =_{BM} H_r(M_t, M_{t-\epsilon} \cup \partial M[t_{t-\epsilon},t]).$$

(47)

Since $M_t \setminus (M_{t-\epsilon} \cup \partial M[t_{t-\epsilon},t]) = (U_t \setminus U_{t-\epsilon})$ one has

$$BM H_r(M_t, M_{t-\epsilon} \cup \partial M[t_{t-\epsilon},t]) =_{BM} H_r(U_t, U_{t-\epsilon}).$$

(48)

Combining the two isomorphisms one concludes that $\theta_r(U_t, U_{t-\epsilon}(t) : i_r(t)^{-1}$ is an isomorphism, hence so is

$$\theta_r(U_t, U_{t-\epsilon}(t).$$

q.e.d.

**Proof of Proposition (1.4)**

For a fix $t \in \mathbb{R}$ choose a filtration of $\tilde{X}$ by open sets

$$U(1) \subset U(2) \subset \cdots \subset U(k) \subset U(k+1) \subset \cdots \subset \tilde{X},$$

$\bigcup_i U(i) = \tilde{X}$, with the properties:

(a) $\overline{U}(i)$ compact submanifold with boundary s.t. $\overline{U}(i) \subset U(i + 1)$,

(b) the restrictions of $f$ to $\overline{U}(i)$ are tame maps,

(c) $t \in CR(f)$ is a regular value for the restriction of $f$ to $\partial(\overline{U}(i))$

In view of tameness of $f$

(d): for any $t \in CR(f)$ there exists $N(t) \in \mathbb{Z}_{\geq 0}$ s.t. the compact set $f^{-1}(t) \cap CR(f)$ is contained in $U(i)$ for $i > N(t)$.

Clearly such filtrations exists. One can choose $\overline{U}(i)$ to be the closure of a finite union of translates of a properly chosen fundamental domain of the free action of $\Gamma$ on $\tilde{X}$ (i.e. whose interior is a manifold). It satisfies (a) and (b). One can increase (arbitrary little $\overline{U}(i)$ inside $U(i+1)$) to make (c) also satisfied.

Observe that diagram $[42]$ implies the commutative diagram $[11]$
s.t. $i_j^{j+k+r} = i_j^{j+k+r} + i_j^{j+k}$ and $p_{t+i+k}^j = p_{j+k}^j + p_{j+k+r}^j$. Inspection of this diagram combined with Propositions 6.1 and the exact sequence (45) lead to the result as follows.

F1. In view of Property (d) of the filtration, Proposition 6.1 implies that for $i > N(t)$ and $\epsilon$ small enough $\theta_{r(U(i))} : H_r(U(i)_t, U(i)_{t-\epsilon}) \to^BM H_r(U(i)_t, U(i)_{t-\epsilon})$ is an isomorphism. Passing to direct limit when $\epsilon \to 0$ one obtains

$$\theta^i_r(t) : H_r(U(i)_t, U(i)_{<t}) \to^BM H_r(U(i)_t, U(i)_{<t})$$

is an isomorphism.

F2. For $j > N(t)$ the linear map $i_j^{j+k}$ is an isomorphism by excision property of the standard homology. Combined with the commutativity of the diagram (49) and F1. one obtains $p_{j+k}^j$ is an isomorphism for any $j > N(t)$ and $k$.

F.3 The isomorphisms $i_j^{j+k}$ stated in F2. imply $i_{\infty}^j$ is isomorphism for $j > N(t)$, and the isomorphisms $p_{j+k}^j$ for $j > N(t)$ stated in F2., in view of the exact sequence (E45) imply $p_{\infty}^j$ is isomorphism for $j > N(t)$. Hence in view of diagram (49), $\theta_r(t) = \theta_{\infty}^i_r(t)$ is an isomorphism. This finalizes the proof of Proposition 1.4.

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