The Black–Scholes equation in the presence of arbitrage

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We apply Geometric Arbitrage Theory to obtain results in Mathematical Finance, which do not need stochastic differential geometry in their formulation. First, for generic market dynamics given by a subclass of multidimensional Itô processes we specify and prove the equivalence between No-Free-Lunch-with-Vanishing-Risk (NFLVR) and expected utility maximization. As a by-product, we provide a geometric characterization of the No-Unbounded-Profit-with-Bounded-Risk (NUPBR) condition given by the zero curvature (ZC) condition for this subclass of Itô processes. Finally, we extend the Black–Scholes partial differential equation to markets allowing arbitrage.

Keywords: Geometric Arbitrage Theory; Gauge transform; Curvature; Non linear Black Scholes PDE

JEL Classification: C00

1. Introduction

This paper provides applications of a conceptual structure—called Geometric Arbitrage Theory (GAT in short)—to prove results in financial mathematics which are comprehensible without the use of stochastic differential geometry and extend well-known classical facts. We expect therefore to make GAT accessible to a wider public in the mathematical finance community.

GAT rephrases classical stochastic finance in stochastic differential geometric terms to characterize arbitrage. The main idea of the GAT approach consists of modelling markets made of basic financial instruments together with their term structures as principal fibre bundles. Financial features of this market—like no arbitrage and equilibrium—are then characterized in terms of standard differential geometric constructions—like curvature—associated to a natural connection in this fibre bundle. Principal fibre bundle theory has been heavily exploited in theoretical physics as the language in which laws of nature can be best formulated by providing an invariant framework to describe physical systems and their dynamics. These ideas can be carried over to mathematical finance and economics. A market is a financial-economic system that can be described by an appropriate principle fibre bundle. A principle like the invariance of market laws under change of numéraire can be seen then as gauge invariance.

The fact that gauge theories are the natural language to describe economics was first proposed by Malaney and Weinstein in the context of the economic index problem (Malaney 1996, Weinstein 2006). Ilinski (see Ilinski 2000, 2001) and Young (1999) proposed to view arbitrage as the curvature of a gauge connection, in analogy to some physical theories. Independently, Smith and Speed (1998) further developed (Flesaker and Hughston 1996) seminal work and utilized techniques from differential geometry to reduce the complexity of asset models before stochastic modelling. Moreover, a numéraire independent framework with assets interpreted as sections of fibre bundles, as well as an application of group theoretical methods to the formulation of time evolution and no-arbitrage condition for self-financing strategies, has been suggested in Heymann (2018), Pelts (2012), Pelts (2018a) and Pelts (2018b).

Why is arbitrage modelling important? The no arbitrage condition is only an approximation and it is not fulfilled when we consider real markets. This is the case for non-traded assets, traded assets when the frequency of the trades falls below 2 minutes (cf. Farinelli and Vazquez 2012) or electricity markets, where we do not have the possibility of completely liquidating the portfolio at any given time, as we implicitly assume in mathematical finance. This has been recognized for a long time and in recent years the modelling of markets allowing for arbitrage beyond pathological cases has made a relevant progress (see f.i. Ruf 2013, Hugonnier and Prieto 2015). The benchmark approach to mathematical finance by
Heath and Platen (2006) allows for arbitrage even if this fact is not explicitly mentioned.

This paper is structured as follows. Section 2 reviews classical stochastic finance and GAT, summarizing (Farinelli 2015), where GAT has been given a rigorous mathematical foundation utilizing the formal background of stochastic differential geometry as in Schwartz (1980), Elworthy (1982), Emery (1989), Hackenbroch and Thalmair (1994), Stroock (2000) and Hsu (2002). We state the main results from previous publications needed for the following sections omitting the proofs. The potential arbitrage strategies are quantified by means of the curvature of a principal fibre bundle representing the market. The zero curvature condition is a weaker condition than No-Free-Lunch-with-Vanishing-Risk (NFLVR). It becomes equivalent under additional assumptions introduced for a guiding example, a market whose asset prices are Itô processes. This treatment is novel. In general, the zero curvature condition follows from the No-Unbounded-Profit-with-Bounded-Risk (NUPBR) condition, as we prove in section 3, where we analyse the relationship between arbitrage and expected utility maximization. The equivalence is proved for a certain subclass of Itô processes. This is the first new contribution of this paper. In section 4, GAT is applied to prove an extension of the Black–Scholes PDE in the case of markets allowing for arbitrage. This is the second new contribution of this publication. Section 5 concludes, and appendix A reviews Nelson’s stochastic derivatives.

2. Geometric arbitrage theory background

In this section, we explain the main concepts of Geometric Arbitrage Theory introduced in Farinelli (2015), to which we refer for proofs and additional examples. Since the differential geometric thinking is not so widespread in the mathematical finance community, we explain in detail the reformulation of the asset model as principal fibre bundle with a connection, whose curvature can be seen as a measure of arbitrage. New results and more pedagogical results in comparison to Farinelli (2015) are provided.

2.1. The classical market model

In this subsection, we will summarize the classical set up, which will be rephrased in section 2.4 in differential geometric terms. We basically follow Hunt and Kennedy (2004) and the ultimate reference Delbaen and Schachermayer (2008).

We assume continuous time trading and that the set of trading dates is \([0, +\infty]\). This assumption is general enough to embed the cases of finite and infinite discrete times as well as the one with a finite horizon in continuous time. Note that while it is true that in the real-world trading occurs at discrete times only, these are not known a priori and can be virtually any points in the time continuum. This motivates the technical effort of continuous time stochastic finance.

The uncertainty is modelled by a filtered probability space \((\Omega, \mathcal{A}, \mathbb{P})\), where \(\mathbb{P}\) is the statistical (physical) probability measure, \(\mathcal{A} = \{\mathcal{A}_t\}_{t \in [0, +\infty]}\) an increasing family of \(\sigma\)-algebras of \(\mathcal{A}_\infty\) and \((\Omega, \mathcal{A}_\infty, \mathbb{P})\) is a probability space. The filtration \(\mathcal{A}\) is assumed to satisfy the usual conditions, that is

- right continuity: \(\mathcal{A}_t = \bigcap_{t' > t} \mathcal{A}_{t'}\) for all \(t \in [0, +\infty]\).
- \(\mathcal{A}_0\) contains all null sets of \(\mathcal{A}_\infty\).

The market consists of finitely many assets indexed by \(j = 1, \ldots, N\), whose nominal prices are given by the vector valued semimartingale \(S : [0, +\infty] \times \Omega \to \mathbb{R}^N\) denoted by \((S_j)_{t \in [0, +\infty]}\) adapted to the filtration \(\mathcal{A}\). The stochastic process \((S_j^0)_{t \in [0, +\infty]}\) describes the price at time \(t\) of the \(j\)th asset in terms of unit of cash at time \(t = 0\). More precisely, we assume the existence of a 0th asset, the cash, a strictly positive semimartingale, which evolves according to \(S_0^0 = \exp(\int_0^t \mu^0 \, dt)\), where the integrable semimartingale \((\mu^j)_{t \in [0, +\infty]}\) represents the continuous interest rate provided by the cash account: one always knows in advance what the interest rate on the own bank account is, but this can change from time to time. The cash account is therefore considered the locally riskless asset, in contrast to the other assets, the risky ones. We remark that the notion of being ‘riskless’ is the subject of the numéraire choice. In the following, we will mainly utilize discounted prices, defined as \(S_j^1 := S_j^0 / S_0^0\), representing the asset prices in terms of current unit of cash.

We remark that there is no need to assume that asset prices are positive. But, there must be at least one strictly positive asset, in our case the cash. If we want to renormalize the prices by choosing another asset instead of the cash as reference, i.e. by making it to our numéraire, then this asset must have a strictly positive price process. More precisely, a generic numéraire is an asset, whose nominal price is represented by a strictly positive stochastic process \((B_t)_{t \in [0, +\infty]}\), and which is a portfolio of the original assets \(j = 0, 1, 2, \ldots, N\). The discounted prices of the original assets are then represented in terms of the numéraire by the semimartingales \(S_j^1 := S_j^0 / B_t\).

We assume that there are no transaction costs and that short sales are allowed. Remark that the absence of transaction costs can be a serious limitation for a realistic model. The filtration \(\mathcal{A}\) is not necessarily generated by the price process \((S_j)_{t \in [0, +\infty]}\); other sources of information than prices are allowed. All agents have access to the same information structure, that is to the filtration \(\mathcal{A}\).

Let \(v\) be a positive real number. A \(\mathcal{A}\)-admissible strategy \(x^v = (x_t)_{t \in [0, +\infty]}\) is an \(\mathcal{S}\)-integrable predictable process for which the Itô integral \(\int_0^t x \cdot dS^v \geq -v\) a.s. for all \(t \geq 0\) with \(x_0 = 0\). A strategy is admissible if it is \(\mathcal{A}\)-admissible for some \(v \geq 0\).

**Definition 1** (Arbitrage, cf. Delbaen and Schachermayer 2008) Let the process \((S_j)_{t \in [0, +\infty]}\) be a semimartingale and \((x_t)_{t \in [0, +\infty]}\) be admissible self-financing strategy. Let us consider trading up to time \(T \leq \infty\). The portfolio wealth at time \(t\) is given by \(V(x) := V_0 + \int_0^t x_t \cdot dS_\tau\), and we denote by \(K_0\) the subset of \(L^1(\Omega, \mathcal{A}_T, \mathbb{P})\), the set of \(\mathcal{A}_T\)-measurable random variables, containing all such \(V_T(x)\), where \(x\) is any admissible self-financing strategy. Furthermore, let \(L^\infty(\Omega, \mathcal{A}_T, \mathbb{P})\) be the set of bounded random variables, and \(L^0\) and \(L^\infty\) denote the subset of positive random variables in \(L^0\) and \(L^\infty\) respectively.
We define
- $C_0 := K_0 - L_0^0(\Omega, A_T, P)$
- $C := C_0 \cap L^\infty(\Omega, A_T, P)$
- $\overline{C}$: the closure of $C$ in $L^\infty$ with respect to the norm topology
- $Y^0 := \{ (V_t)_{t \in [0, +\infty]} : V_t = V_0(x), \text{ where } x \in V_0 \}$
- $Y^{V_0} := \{ V_t : (V_t)_{t \in [0, +\infty]} \in Y^0_k \}$: terminal wealth for $V_0$-admissible self-financing strategies.

We say that $S$ satisfies
- (NA), no arbitrage, if and only if $C \cap L^\infty(\Omega, A_T, P) = \{0\}$.
- (NFLVR), no-free-lunch-with-vanishing-risk, if and only if $C \cap L^\infty(\Omega, A_T, P) = \{0\}$.
- (NUPBR), no-unbounded-profit-with-bounded-risk, if and only if $Y^{V_0}$ is bounded in $L^0$ for some $V_0 > 0$.

The relationship between these three different types of arbitrage has been elucidated in Delbaen and Schachermayer (1994) and in Kabanov (1997) with the proof of the following result.

**Theorem 2.1**

\[(NFLVR) \iff (NA) + (NUPBR).\]

**Remark 1** We recall that, as shown in Delbaen and Schachermayer (1994), Kabanov (1997), Kabanov and Kramkov (1994) and Karatzas and Kardaras (2007), (NUPBR) is equivalent to (NAA1), i.e. no asymptotic arbitrage of the first kind, and equivalent to (NA1), i.e. no arbitrage of the first kind.

### 2.2. Geometric reformulation of the market model: primitives

We are going to introduce a more general representation of the market model introduced in section 2.1, which better suits to the arbitrage modelling task.

The rationale behind this model is to identify every asset with its cash flow stream. Every asset has its term structure allowing to compute its value up to the division by a numéraire, value that we term deflator following terminology outside the context of fixed income.

**Remark 2** This gauge transform constructs a bond from the original gauge. The cashflow intensity $\pi$ specifies the bond cashflow structure. The bond value at time $t$ expressed in terms of the market model numéraire is given by $D^T_t$. The term structure of forward prices for the bond future expressed in terms of the bond current value is given by $P^T_{t,s}$.

**Proposition 2.2** Gauge transforms induced by cashflow vectors have the following property:

\[((D, P)\pi)^\nu = ((D, P)^\nu)^\pi = (D, P)^{\pi \ast \nu},\]

where $\ast$ denotes the convolution product of two cashflow vectors or intensities respectively:

\[\ast := \int_0^t dh \pi_{h\nu_{t-h}}.\]
We can observe that
\[
(D_f^*)^\pi = D_f^\pi \int_0^{+\infty} dv \int_0^{+\infty} du \pi_u P_{t+h+u}.
\]
By changing variables \(v = h + u\), one has
\[
(D_f^*)^\pi = D_f \int_0^{+\infty} dv \left( \int_0^{v} dv \pi_v P_{t+h} \right) P_{t+h+v} = (D_f)^{\pi+v}
\]
and this coincide with \((D^\pi)^\tau\), proving the first component of (1). The second component can be derived similarly. ■

The group \(G\) acts freely and differentiably on \(B\) to the right.

The market fibre bundle repackages all the information concerning market dynamics of the asset futures and their underlyings. The principal bundle structure reflects the portfolio construction possibilities at a fixed time, as well as the synthetic bond construction possibilities for given cash flow patterns specified by the gauge transforms.

### 2.4.1. Nelson weak \(\mathcal{D}\)-Differentiable market model.

We continue to reformulate the classic asset model introduced in section 2.1 in terms of stochastic differential geometry. We refer to appendix for the definition of the different stochastic derivatives.

**Definition 5** A Nelson weak \(\mathcal{D}\)-differentiable market model for \(N\) assets is described by \(N\) gauges which are Nelson weak \(\mathcal{D}\)-differentiable with respect to the time variable. More exactly, for all \(t \in [0, +\infty[\) and \(s \geq t\) there is an open time interval \(I \ni t\) such that for the deflators \(D_{i} := [D_{i}^{1}, \ldots, D_{i}^{N}]\) and the term structures \(P_{t,s} := [P_{t,s}^{1}, \ldots, P_{t,s}^{N}]\), the latter seen as processes in \(t\) and parameter \(s\), there exist a weak \(\mathcal{D}\)-derivative with respect to the time variable \(t\) (see appendix). The short rates are defined by \(r_{t} := \lim_{s \to t} \frac{1}{s} \log P_{t,s}\).

A strategy is a curve \(\gamma : I \to X\) in the portfolio space parameterized by the time. This means that the allocation at time \(t\) is given by the vector of nominals \(x_{t} := \gamma(t)\). We denote by \(\gamma\) the lift of \(\gamma\) to \(M\), that is \(\gamma(t) := (\gamma(t), t)\). A strategy is said to be closed if it represented by a closed curve. A weak \(\mathcal{D}\)-admissible strategy is predictable and weak \(\mathcal{D}\)-differentiable.

**Remark 5** We require weak \(\mathcal{D}\)-differentiability and not strong \(\mathcal{D}\)-differentiability because imposing \(\alpha\)-priori regularity properties on the trading strategies corresponds to restricting the class of admissible strategies with respect to the classical notion of Delbaen and Schachermayer. Every (no-)arbitrage consideration depends crucially on the chosen definition of admissibility. Therefore, restricting the class of admissible strategies may lead to the automatic exclusion of potential arbitrage opportunities, leading to vacuous statements for kinds of Fundamental Theorem of Asset Pricing. An admissible strategy in the classical sense (see section 2) is weak \(\mathcal{D}\)-differentiable.

In general, the allocation can depend on the state of the nature \(i.e.\ x_{t} = x_{t}(\omega)\) for \(\omega \in \Omega\).

**Proposition 2.4** A weak \(\mathcal{D}\)-admissible strategy is self-financing if and only if
\[
\mathcal{D}(x_{t} \cdot D_{t}) = x_{t} \cdot DD_{t} - \frac{1}{2} \mathcal{D} \mathcal{D}_{+}(x, D)_{t} \cdot D_{t} = -\frac{1}{2} \mathcal{D} \mathcal{D}_{+}(x, D)_{t} \cdot D_{t} = 0, \tag{4}
\]
almost surely. The bracket $\langle \cdot, \cdot \rangle$ denotes the continuous part of
the quadratic covariation.

**Proof** The strategy is self-financing if and only if

$$x_t \cdot D_t = x_0 \cdot D_0 + \int_0^t x_u \cdot dD_u.$$ 

Since on the r.h.s we have an Itô’s integral, this is equivalent to

$$\mathcal{D}(x_t \cdot D_t) = x_t \cdot \mathcal{DD}_t,$$

or equivalent to

$$\mathcal{D}x_t \cdot D_t = 0.$$ (6)

The self-financing condition can be expressed by means of the anticipative differential $d_t$ as

$$x_t \cdot D_t = x_0 \cdot D_0 + \int_0^t x_u \cdot dD_u - \int_0^t d\langle x, D\rangle_u,$$

which is equivalent to

$$\mathcal{D}_*(x_t \cdot D_t) = x_t \cdot \mathcal{D}_*D_t - \mathcal{D}_* \langle x, D\rangle_t.$$ (7)

By summing equations (5) and (7), we obtain

$$\mathcal{D}(x_t \cdot D_t) = \frac{1}{2} (\mathcal{D} + \mathcal{D}_*)(x_t \cdot D_t) = x_t \cdot \mathcal{DD}_t - \frac{1}{2} \mathcal{D}_* \langle x, D\rangle_t.$$ (8)

To prove the second statement in expression (4), we consider the integration by parts formula for Itô’s integral

$$\int_0^t x_u \cdot dD_u + \int_0^t D_u \cdot dx_u = x_t \cdot D_t - x_0 \cdot D_0 - \langle x, D\rangle_t,$$

which, expressed in terms of Stratonovich’s integral, leads to

$$\int_0^t x_u \circ dD_u - \frac{1}{2} \langle x, D\rangle_t + \int_0^t D_u \circ dx_u - \frac{1}{2} \langle x, D\rangle_t
= x_t \cdot D_t - x_0 \cdot D_0 - \langle x, D\rangle_t.$$ (9)

By taking Stratonovich’s derivative $\mathcal{D}$ on both sides, we get

$$\mathcal{D}(x_t \cdot D_t) = \mathcal{D}x_t \cdot D_t + x_t \cdot \mathcal{DD}_t,$$

which together with the first statement in expression (4) proves the second one. ■

For the remainder of this paper unless otherwise stated we will deal only with weak $\mathcal{D}$-differentiable market models, weak $\mathcal{D}$-differentiable strategies, and, when necessary, with weak $\mathcal{D}$-differentiable state price deflators. All Itô processes are weak $\mathcal{D}$-differentiable, so that the class of considered admissible strategies is very large.

### 2.4.2. Arbitrage as curvature

The Lie algebra of $G$ is the function space of all real-valued functions on $[0, +\infty[$ denoted by

$$\mathfrak{g} = \mathbb{R}^{[0, +\infty[}$$

and therefore commutative. Following Ilinski’s idea proposed in Ilinski (2001), we motivate the choice of a particular $\mathfrak{g}$-valued connection 1-form by the fact that it allows to encode portfolio rebalance (or foreign exchange) and discounting as parallel transport.

**Theorem 2.5** With the choice of connection

$$\chi(x, t, g).(\delta x, \delta t) := \left(D^g_x \frac{dx}{D^g_t} - r_t \delta t \right)g,$$ (10)

the stochastic parallel transport in $\mathcal{B}$ has the following financial interpretations:

- Parallel transport along the nominal directions $(x$-lines) corresponds to a multiplication by an exchange rate.
- Parallel transport along the time direction $(t$-line) corresponds to a division by a stochastic discount factor.

**Proof** We refer to theorem 28 in Farinelli (2015). ■

Recall that time derivatives needed to define the parallel transport along the time lines have to be understood in Stratonovich’s sense. We see that the bundle is trivial, because it has a global trivialization, but the connection is not trivial. The connection $\chi$ writes as a linear combination of basis differential forms as

$$\chi(x, t, g) = \left(\frac{1}{D^g_t} \sum_{j=1}^N D^j_x \, dx_j - r_t \, dt \right)g.$$ (11)

The $\mathfrak{g}$-valued curvature 2-form is defined as

$$R := d\chi + [\chi, \chi],$$ (12)

meaning by this, that for all $(x, t, g) \in \mathcal{B}$ and for all $\xi, \eta \in T_{(x,t)}\mathcal{M}$

$$R(x, t, g)(\xi, \eta) := d\chi(x, t, g)(\xi, \eta) + [\chi(x, t, g)(\xi), \chi(x, t, g)(\eta)].$$ (13)

Remark that, being the Lie algebra commutative, the Lie bracket $[\cdot, \cdot]$ vanishes. After some calculations, we obtain

$$R(x, t, g) = \frac{g}{D^g_t} \sum_{j=1}^N D^j_t \left(r_t^j + \mathcal{D} \log(D^j_t) - r_t^j - \mathcal{D} \log(D^j_t) \right) dx_j \wedge dr,$$ (14)

summarized as the following.
PROPOSITION 2.6 Curvature Formula Let $R$ be the curvature. Then, the following equality holds:

$$R(x, t, g) = g \ dr \wedge d_s \left[ D \log(D^j_t) + r_t^j \right].$$  \hspace{1cm} (13)$$

The curvature represents the capacity of instantaneous arbitrage allowed by the market.

Proof See proposition 38 in Farinelli (2015).

We can prove following results which characterizes arbitrage as curvature.

THEOREM 2.7 (No Arbitrage) The following assertions are equivalent:

(i) The market model (consisting base assets and futures with discounted prices $D$ and $P$) satisfies the no-free-lunch-with-vanishing-risk condition.

(ii) There exists a positive local martingale $\beta = (\beta_t)_{t \geq 0}$ such that deflators and short rates satisfy for all portfolio nominals and all times the condition

$$r_t^i = -D \log(\beta_tD^i_t).$$  \hspace{1cm} (14)$$

(iii) There exists a positive local martingale $\beta = (\beta_t)_{t \geq 0}$ such that deflators and term structures satisfy for all portfolio nominals and all times the condition

$$P_t^i = \frac{E_t[\beta_tD^i_t]}{\beta_tD^i_t}.$$  \hspace{1cm} (15)$$

Proof We refer to theorem 33 in Farinelli (2015).

This motivates the following definition.

DEFINITION 6 The market model satisfies the zero curvature (ZC) if and only if the curvature vanishes a.s.

Therefore, we have following implication relying two different definitions of no-arbitrage:

COROLLARY 2.8

$$(NFLVR) \Rightarrow (ZC).$$

As an example to demonstrate how the most important geometric concepts of section 2 can be applied we consider an asset model whose dynamics is given by a multidimensional Itô process. Let us consider a market consisting of $N + 1$ assets labelled by $j = 0, 1, \ldots, N$, where the 0th asset is the cash account utilized as a numéraire. Therefore, as explained in the introductory section 2.1, it suffices to model the price dynamics of the other assets $j = 1, \ldots, N$ expressed in terms of the 0th asset. As vector valued semimartingales for the discounted price process $\hat{S} : [0, +\infty[ \times \Omega \to \mathbb{R}^N$ and the short rate $r : [0, +\infty[ \times \Omega \to \mathbb{R}^N$, we chose the multidimensional Itô processes given by

$$d\hat{S}_t = \hat{S}_t (\alpha_t \ dt + \sigma_t \ dW_t), \hspace{1cm} dr_t = \alpha_t \ dt + b_t \ dW_t,$$  \hspace{1cm} (16)$$

where

- $(W_t)_{t \in [0, +\infty]}$ is a standard $\mathbb{P}$-Brownian motion in $\mathbb{R}^K$, for some $K \in \mathbb{N}$,
- $(\sigma_t)_{t \in [0, +\infty]}$, $(\alpha_t)_{t \in [0, +\infty]}$ are $\mathbb{R}^{N \times K}$-valued stochastic processes, $\sigma_t$ has maximal rank, i.e. $\text{rank}(\sigma_t) = K$, and
- $(b_t)_{t \in [0, +\infty]}$, $(\alpha_t)_{t \in [0, +\infty]}$ are $\mathbb{R}^{N \times K}$-valued stochastic processes.

PROPOSITION 2.9 Let the dynamics of a market model be specified by following Itô processes as in (16), where we additionally assume that the coefficients

- $(\alpha_t)$, $(\sigma_t)$, and $(r_t)$ satisfy

$$\lim_{s \to t} E_t[\alpha_t] = \alpha_t, \hspace{1cm} \lim_{s \to t} E_t[r_t] = r_t,$$

- $(\sigma_t)$ is an Itô process,
- $(\alpha_t)$ and $(W_t)$ are independent processes.

Then, the market model satisfies the (ZC) condition if and only if

$$\alpha_t + r_t \in \text{Range}(\sigma_t).$$  \hspace{1cm} (17)$$

Remark 6 In the case of the classical model, where there are no term structures (i.e. $r \equiv 0$), the condition (17) reads as $\alpha_t \in \text{Range}(\sigma_t)$.

Proof Let us consider the expression for Itô’s integral with respect to Stratonovich’s

$$\int_0^t \sigma_u \ dW_u = \int_0^t \sigma_u \circ dW_u - \frac{1}{2} \int_0^t d (\sigma, W)_u,$$

and take Nelson’s derivative corresponding to the Stratonovich’s integral:

$$\mathcal{D} \int_0^t \sigma_u \ dW_u = \sigma_t DW_t - \frac{1}{2} \mathcal{D} (\sigma, W)_t.$$  \hspace{1cm} (18)$$

Since

$$DW_t = \frac{W_t}{2t},$$  \hspace{1cm} (19)$$

and, because of the independence assumption for the two Itô processes $(\sigma_t)$ and $(W_t)$, \hspace{1cm} (19)

$$\langle \sigma, W \rangle_t \equiv 0,$$

we obtain

$$\mathcal{D} \int_0^t \sigma_u \ dW_u = \sigma_t \frac{W_t}{2t},$$

which, inserted into the asset dynamics

$$\hat{S}_t = \hat{S}_0 \exp \left( \int_0^t (\alpha_u - \frac{1}{2} \text{diag}(\sigma_u \sigma_u^t)) \ du + \int_0^t \sigma_u \ dW_u \right),$$

leads to

$$\mathcal{D} \log \hat{S}_t = \alpha_t - \frac{1}{2} \text{diag}(\sigma_t \sigma_t^t) + \sigma_t \frac{W_t}{2t}.$$
By proposition 2.6, the curvature vanishes if and only if for all \( x \in \mathbb{R}^N \)
\[
D \log \tilde{S}_t^i + r_t^i = C_i,
\]
for a real-valued stochastic process \((C_t)_t \geq 0\), or, equivalently
\[
D \log \tilde{S}_t = C_t e,
\]
where \( e := [1, \ldots, 1]^\top \) or
\[
\alpha_t + r_t = \frac{1}{2} \text{diag}(\sigma \sigma_t^\top) + \frac{1}{2} \text{diag}(\sigma_t \sigma) = C_t e, \tag{20}
\]
Equation (20) is the formulation of the (ZC) condition for the market model (16). By taking on both sides of (20) \( \lim_{h \to 0^+} E_{t-h}[\cdot] \), and utilizing the independence assumption, from which
\[
\mathbb{E}_{t-h} \left[ \sigma_t \frac{W_t}{2t} \right] = \mathbb{E}_{t-h} \left[ \sigma_t \right] \mathbb{E}_{t-h} \left[ \frac{W_t}{2t} \right] = 0
\]
follows, we obtain, using the continuity assumption for \((\alpha_t)_t\), \((\sigma_t)_t\), and \((r_t)_t\),
\[
\alpha_t + r_t - \frac{1}{2} \text{diag}(\sigma \sigma_t^\top) = \beta_t e,
\]
where \( \beta_t := \lim_{h \to 0^+} E_{t-h}[C_t] \) is a predictable process. Therefore, equation (20) becomes
\[
\sigma_t \frac{W_t}{2t} = (C_t - \beta_t) e, \tag{21}
\]
and, thus
\[
e \in \text{Range}(\sigma_t), \tag{22}
\]
the space spanned by the column vectors of \( \sigma_t \). Since \( \sigma_t \) has maximal rank, the \( K \) column vectors of \( \sigma_t \) are linearly independent and \( C_t - \beta_t \neq 0 \).

Let \( P_{\sigma_t} \) denote the orthogonal projections onto Range(\( \sigma_t \)) and its orthogonal complement \( \text{Range}(\sigma_t)^\perp \), respectively. Then, we can decompose
\[
\alpha_t + r_t = P_{\sigma_t}(\alpha_t + r_t) + P_{\sigma_t}^\perp(\alpha_t + r_t), \tag{23}
\]
and
\[
P_{\sigma_t}^\perp(\alpha_t + r_t)
\]
\[
= P_{\sigma_t}^\perp(C_t e) - P_{\sigma_t}^\perp \left( \sigma_t \frac{W_t}{2t} + \frac{1}{2} \text{diag}(\sigma_t \sigma) \right).
\tag{24}
\]
Since \( e \) and \( \sigma_t W_t \) lie in Range(\( \sigma_t \)), the first two addenda on the right-hand side of (24) vanish. By lemmata 2.10 and 2.11, the third one vanishes as well, so that \( P_{\sigma_t}^\perp(\alpha_t + r_t) = 0 \), i.e. \( \alpha_t + r_t \in \text{Range}(\sigma_t) \). Conversely, if \( \alpha_t + r_t \in \text{Range}(\sigma_t) \), then equation (20) holds true, and the proof of the equivalence between the (ZC) condition and (17) is completed.

**Lemma 2.10** Let \( A \) be a linear operator on the Euclidean \( \mathbb{R}^N \). The vector
\[
\text{diag}(A) := \sum_{j=1}^{N} (\sigma_j e_j) e_j
\]
does not depend on the choice of the orthonormal basis \( \{e_1, \ldots, e_N\} \) of \( \mathbb{R}^N \) and defines the diagonal of \( A \).

**Proof** The coordinates of \( \text{diag}(A) \) with respect to the orthonormal basis \( \{e_1, \ldots, e_N\} \) can be written as
\[
[\text{diag}(A)]_i = \sum_{j=1}^{N} (\sigma_j e_j) [A]_{ij} (e_j e_i)_j
\]
Let us consider another orthonormal basis \( \{f_1, \ldots, f_N\} \) of \( \mathbb{R}^N \). This means that there exists an orthogonal linear operator \( U \) on \( \mathbb{R}^N \) such that \( U e_j = f_j \) for all \( j = 1, \ldots, N \). Therefore we can write
\[
[\text{diag}(A)]_i = \sum_{j=1}^{N} (U f_j)_i (U f_j)_i
\]
Therefore, the coordinates of the diagonal transform like a vector during a change of basis, and hence the diagonal is well defined. ■

**Lemma 2.11** Let \( \sigma \) be an \( \mathbb{R}^{N \times K} \) real matrix of rank \( K \) and \( P \) the orthogonal projection onto the orthogonal complement to the subspace generated by the column vectors of \( \sigma \). Then,
\[
P \text{diag}(\sigma) = 0 \in \mathbb{R}^N.
\]

**Proof** The real symmetric matrix \( \sigma \sigma^\top \in \mathbb{R}^{N \times N} \) induces via standard orthonormal basis a self-adjoint linear operator on \( \mathbb{R}^N \), which by lemma 2.10 has a well-defined diagonal. Let us enlarge \( \sigma \) to an \( \mathbb{R}^{N \times N} \) matrix, by adding \( N - K \) zero column vectors. The matrix \( \sigma \sigma^\top \in \mathbb{R}^{N \times N} \) remains the same. Let us consider an orthonormal basis of \( \mathbb{R}^N \), \( \{f_1, \ldots, f_N\} \), where \( \{f_1, \ldots, f_N\} \) is a basis of \( \text{Range}(\sigma) \) and \( \{f_{k+1}, \ldots, f_N\} \) is a basis of its orthogonal complement, \( \text{Range}(\sigma)^\perp \). The diagonal of \( \sigma \sigma^\top \) reads
\[
\text{diag}(\sigma) = \sum_{j=1}^{N} (\sigma f_j, \sigma f_j) f_j = \sum_{j=1}^{N} (\sigma^\top f_j, \sigma^\top f_j) f_j
\]
\[
= \sum_{j=1}^{K} (\sigma^\top f_j, \sigma^\top f_j) f_j, \tag{27}
\]
Since \( \sigma \) is a matrix, the equation (27) holds true for any \( \sigma \) and completes the proof.
because \(\sigma^j f_j = 0\) for \(j = K + 1, \ldots, N\), being \(f_j\) in the orthogonal complement of Range(\(\sigma\)). Therefore,

\[
P \text{diag}(\sigma^j) = \sum_{j=1}^{K} (\sigma^j f_j \cdot \sigma^j) P f_j = 0,
\]

because \(f_j\) is in Range(\(\sigma\)) for \(j = 1, \ldots, K\) and \(P\) is the projection onto Range(\(\sigma\))\(^\perp\).

Next, we show the equivalence of the (ZC) condition with (NFLVR) in the case of Itô’s dynamics.

**Proposition 2.12** Under the same assumptions as proposition 2.9, the zero curvature condition for the market model specified by (16), that is

\[
\mathcal{D} \log \tilde{S}_t + r_i = C_i e,
\]

is equivalent to the no-free-lunch-with-vanishing-risk condition if the positive stochastic process \((\beta_t)_{t \geq 0}\), defined as

\[
\beta_t := \exp \left( - \int_0^t C_u \, du \right)
\]

is a local martingale.

**Proof** By proposition 2.6, the zero curvature (ZC) condition \(R = 0\) is equivalent with the existence of a stochastic process \((C_t)_{t \geq 0}\) such that for all \(i = 1, \ldots, N\) the equation

\[
\mathcal{D} \log \tilde{S}_t + r_i = C_i
gives

holds. This means that

\[
\mathcal{D} \log \tilde{S}_t = C_i - r_i
\]

\[
\log \frac{\tilde{S}_t}{\tilde{S}_0} = \int_0^t (C_u - r_u^i) \, du
\]

\[
\tilde{S}_t = \tilde{S}_0 \exp \left( \int_0^t C_u \, du \right) \exp \left( - \int_0^t r_u^i \, du \right).
\]

Therefore,

\[
\mathcal{D} \log(\tilde{\beta} D^i) + r_i^i = 0
\]

for all \(i = 1, \ldots, N\). By theorem 2.7, if \((\tilde{\beta}_t)_{t \geq 0}\) is a martingale, then we have proved (NFLVR).

We can reformulate the result of proposition 2.9 as follows.

**Corollary 2.13** Let \([J^1, \ldots, J^K]\) be an orthonormal basis of Ker(\(\sigma_i\)) \(\subseteq \mathbb{R}^N\). Under the same assumptions as proposition 2.9 the (ZC) condition for the market model (16), which is equivalent to (NFLVR), is equivalent to

\[
\rho_t := J^i (\alpha_t + r^i) \equiv 0 \in \mathbb{R}^K,
\]

where \(J_i := [J^1_i, \ldots, J^K_i]\).

**Remark 7** (Counterexamples) Let us consider a financial market with a cash account with \(r_i^0 = 0\) and a single risky asset with (discounted) price given by

\[
S_t = e^{X_t}, \quad \text{where } X_t := \int_0^t \frac{W_u}{u} + W_t,
\]

for a standard univariate Brownian motion \((W_t)\). By Itô’s formula, it follows that

\[
dS_t = \frac{1}{2} \frac{W_t^2}{t} + S_t \, dW_t \quad \text{and} \quad S_0 = 1.
\]

In the notation of proposition 2.9, this corresponds to \(\alpha_t = \frac{1}{2} \frac{W_t^2}{t}\) and \(r_t = 0\). The coefficient \(\alpha_t\) does not satisfy the assumption \(\lim_{s \to t} E_s[\alpha_t]^\ast \neq \alpha_t\), because

\[
\lim_{s \to t} E_s \left[ \frac{1}{2} \frac{W_t^2}{t} \right] = \frac{1}{2} \frac{1}{2} \lim_{s \to t} E_s [W_t]^2 \neq \alpha_t.
\]

The process \((S_t)\) does not satisfy (NFLVR), since \(\int_0^t \frac{W_u^2}{u} \, du \geq 0\) for all \(t > 0\) as a consequence of corollary 3.2 of Jeulin and Yor (1979). In the terminology of Delbaen & Schachermayer, model (30) generates immediate arbitrage opportunities. Other simple counterexamples can be constructed from Brownian bridges, which provide well-known examples of models admitting arbitrage (see Loewenstein and Willard 2000).

Moreover, Fontana and Runggaldier present asset models in Fontana (2015) (example 7.5) and Fontana and Runggaldier (2013) (page 59) based on Bessel processes, which do not fulfill the assumptions of propositions 2.9 and 2.12. They are an example of dynamics satisfying (NUPBR) but not (NFLVR). The proof of the (NUPBR) property is based on its equivalence with the non-existence of arbitrage possibilities of the first kind.

The measure of arbitrage \(\rho_t\), defined in (29) does not depend on the model parametrization. More exactly, it is invariant under an invertible linear transform of the base assets.

**Proposition 2.14** Let \(\Phi : (x, D, r) \mapsto (\tilde{x}, \tilde{D}, \tilde{r})\) be a transform corresponding to a change of basis for the deflators preserving the portfolio values, that is:

(i) There exists a \(\psi \in \text{GL}(\mathbb{R}^N)\) such that \(\Phi(x, D, r) = (\psi^{-1}(x), \psi(D), \tilde{r})\) for all \(x, D\).

(ii) \(D_i = D_i^\ast\) for all \(x, D\).

Under the same assumptions and notation as proposition 2.9, if the dynamics of the asset model satisfies

\[
dD_t = D_t (\alpha_t \, dt + \sigma_t \, dW_t)
\]

with arbitrage measure \(\rho_t\), then, the transformed asset model fulfills

\[
d\tilde{D}_t = \tilde{D}_t (\tilde{\alpha}_t \, dt + \tilde{\sigma}_t \, dW_t)
\]

for appropriate vector and matrix valued \(\tilde{\alpha}_t\) and \(\tilde{\sigma}_t\), and the arbitrage measure satisfies

\[
\tilde{\rho}_t = \rho_t.
\]
Remark 8 The invariance of the arbitrage measure described in (33) can be seen as a special case of the invariance of the curvature differential 2-form $R$ representation (12) in local coordinates, being $\Phi$ a change of (global) coordinates on the market fibre bundle.

Proof We utilize the notation $ab$ and $alb$ for component-wise multiplication and division of vectors $a, b \in \mathbb{R}^n$. From property (ii) we infer

$$x \cdot D_t = \varphi^{-1}(\tilde{x}) \cdot \varphi(D_t) = \varphi^\dagger \varphi^{-1}(\tilde{x}) \cdot \tilde{D}_t = \tilde{x} \cdot \tilde{D}_t$$

(34)

for all $\tilde{x}, \tilde{D}_t$, and hence

$$\varphi^\dagger \varphi^{-1} = 1_{\mathbb{R}^n}, \text{i.e. } \varphi^\dagger = \varphi.$$

(35)

The invariance of the connection differential 1-form $\chi$ representation (9) in local coordinates reads for all $x, t, g$

$$\chi(x, t, g) = \chi^\Phi (\varphi^{-1}(x), t, g),$$

(36)

that is

$$\left(\frac{D \cdot dx}{D \cdot \bar{x}} - r^x\right) g = \left(\frac{\tilde{D} \cdot d\tilde{x}}{\tilde{D} \cdot \tilde{x}} - \tilde{r}^\tilde{x}\right) g,$$

(37)

which, by (i), leads to

$$\tilde{r}^\tilde{x} = r^x,$$

(38)

that is

$$\frac{\tilde{x} \tilde{D} \cdot \tilde{r}}{\tilde{x} \cdot \tilde{D}} = \frac{x D \cdot r}{x \cdot D},$$

(39)

and hence for all $x$

$$xD \cdot r = \tilde{x} \tilde{D} \cdot \tilde{r} = \varphi^{-1}(x) \cdot \tilde{D} \tilde{r} = x \cdot (\varphi^{-1})^\dagger (\tilde{D} \tilde{r}).$$

(40)

Therefore,

$$Dr = (\varphi^{-1})^\dagger (\tilde{D} \tilde{r})$$

$$\varphi^\dagger (Dr) = \tilde{D} \tilde{r},$$

(41)

and, finally

$$\tilde{r} = \varphi^\dagger (Dr)^\dagger (Dr) / \varphi(D) = \varphi(Dr) / \varphi(D) = (\varphi(Dr)) / \varphi(D) = r,$$

(42)

where we have utilized the fact that, for any self-adjoint $\varphi$, the equality

$$\varphi(Da) = a \varphi(D)$$

(43)

holds for all vectors $D, a$.

Now, if the base assets follow an Itô dynamics

$$dD_t = D_t (\alpha_t dt + \sigma_t dW_t),$$

(44)

we have

$$D_t = D_0 \exp\left(\int_0^t (\alpha_u - \frac{1}{2} \text{diag}(\sigma_u \sigma_u^\dagger)) + \int_0^t \sigma_u dW_u\right)$$

$$\mathcal{D}D_t = \left(\int_0^t (\alpha_u - \frac{1}{2} \text{diag}(\sigma_u \sigma_u^\dagger)) + \int_0^t \sigma_u dW_u\right) D_t,$$

(45)

The transformed base assets follow an Itô dynamics, too, as

$$d\tilde{D}_t = \tilde{D}_t (\tilde{\alpha}_t dt + \tilde{\sigma}_t dW_t),$$

(46)

and, hence, being $\tilde{D} = \varphi(d)$ and $\phi$ linear,

$$\mathcal{D} \varphi(D_t) = \left(\alpha_t - \frac{1}{2} \text{diag}(\sigma_t \sigma_t^\dagger) + \frac{\sigma_t W_t}{2}\right) \varphi(D_t),$$

(47)

where we have utilized property (43) on the second equation of (45). By comparing the two equations in (47) we obtain

$$\tilde{\alpha}_t = \alpha_t \quad \text{and} \quad \sigma_t = \sigma_t,$$

(48)

and finally

$$\tilde{\rho}_t = \rho_t.$$  

(49)  

3. Arbitrage and utility

Let us now consider a utility function, that is a real $C^2$-function of a real variable, which is strictly monotone increasing (i.e. $u' > 0$) and concave (i.e. $u'' < 0$). Typically, a market participant would like to maximize the expected utility of its wealth at some time horizon. Let us assume that he (or she) holds a portfolio of synthetic zero bonds delivering at maturity base assets and that the time horizon is infinitesimally near, that is that the utility of the instantaneous total return has to be maximized. The portfolio values read as:

- At time $t-h$: $D_{t-h}^r P_{t-h,t+h}^r$
- At time $t$: $D_t^r P_{t,t+h}^r$
- At time $t+h$: $D_{t+h}^r$

From now on we make the following Assumptions:

(A1): The market filtration $(\mathcal{A}_t)_{t \geq 0}$ is the coarsest filtration for which $(D_t)_{t \geq 0}$ is adapted.

(A2): The process $(D_t)_{t \geq 0}$ is Markov with respect to the filtration $(\mathcal{A}_t)_{t \geq 0}$.

Proposition 3.1 Under the assumptions (A1) and (A2) the synthetic bond portfolio instantaneous return can be computed as

$$\text{Ret}_t := \lim_{h \to 0^-} \mathbb{E}_t \left[ \frac{D_t^r - D_{t-h}^r P_{t-h,t+h}^r}{2hD_{t-h}^r P_{t-h,t+h}^r} \right] = \mathcal{D} \log(D_t^r) + r_t^\dagger.$$
Proof Under the assumptions (A1) and (A2) the conditional expectations with respect to the market filtration \((\mathcal{A}_t)_{t \geq 0}\) are the same as those computed with respect to the present \((\mathcal{N}_t)_{t \geq 0}\), past \((\mathcal{P}_t)_{t \geq 0}\) and future \((\mathcal{F}_t)_{t \geq 0}\) filtrations (see Appendix). Therefore, we can develop the instantaneous return as

\[
\lim_{h \to 0^+} E_t \left[ \frac{D^{x^h}_{t+h} - D^{x^h}_{t-h} - h P_{t-r}^{x^h}}{2hD_{t-r}^{x^h} P_{t-r}^{x^h}} \right] = \lim_{h \to 0^+} E_t \left[ \frac{D^{x^h}_{t+h} - D^{x^h}_{t-h} + 1 - P_{t-r}^{x^h}}{2hP_{t-r}^{x^h}} \right] = \frac{1}{D_{t}^{r}} D_{s}^{D_{t}^{r}} + \lim_{h \to 0^+} \exp \left( \int_{t-h}^{t+h} ds f^{x^h}_{s-h,x} \right) - 1 = D_{t}^{D_{t}^{r}} + r_{t}.
\]

REMARK 9 This portfolio of synthetic zero bonds in the theory corresponds to a portfolio of futures in practice. If the short rate vanishes, then the future corresponds to the original asset.

**Definition 7** (Expected Utility of Synthetic Bond Portfolio Return) Let \(t \geq s \) be fixed times. The expected utility maximization problem at time \(s\) for the horizon \(T\) for initial capital \(x\) writes

\[
\sup_{x \in [x_0, h], D_{t}^{r} = \xi} E_s \left[ u \left( \exp \left( \int_{s}^{T} dt \left( D_{t}^{r} \log(D_{t}^{r}) + r_{t}^{h} \right) \right) D_{s}^{r} P_{s,T}^{x} \right) \right],
\]

where the supremum is taken over all weak \(\mathcal{D}\)-differentiable self-financing admissible strategies \(x = [x_0]_{t \geq 0}\).

Now we can formulate the first result of this section.

**Theorem 3.2** Let us assume (A1) and (A2), and that \((D_t^{r})_{t \geq 0}\) are weakly \(\mathcal{D}\)-differentiable semimartingales. The market curvature vanishes if and only if the expected utility maximization problem can be solved for all times and horizons for a chosen utility function.

This result can be seen as the natural generalization of the corresponding result in discrete time, as theorem 3.5 in Föllmer and Schied (2004), see also Rogers (1994). Compare with Bellini’s, Frittelli’s and Schachermayer’s results for infinite dimensional optimization problems in continuous time, see theorem 22 in Bellini and Frittelli (2002) and theorem 2.2 in Schachermayer (2001). Nothing is said about the fulfilment of the no-free-lunch-with-vanishing-risk condition: only the weaker zero curvature condition is equivalent to the maximization of the expected utility at all times for all horizons.

Proof The optimization problem (50) into a standard problem of stochastic optimal theory in continuous time which can be solved by means of a fundamental solution of the Hamilton–Jacobi–Bellman partial differential equation.

However, there is a direct method, using Lagrange multipliers for Banach spaces (see Luenberger David 1969 pp. 239–270 and Zeidler 1995 section 4.14, pp. 270–271). First, remark that problem (7) is a concave optimization problem with convex domain and concave utility function and has therefore a unique solution corresponding to a global maximum. The Lagrange principal function corresponding to the maximum problem

\[
\Phi(x, \lambda, \mu) := E_s \left[ u \left( \exp \left( \int_{s}^{T} dt \left( D_{t}^{r} \log(D_{t}^{r}) + r_{t}^{h} \right) \right) D_{s}^{r} P_{s,T}^{x} \right) \right] - \int_{s}^{T} dt \lambda_{t} \mathcal{D} x_{t} \cdot D_{t} \right) - \mu(D_{s}^{r} - \xi).
\]

Note that the Lagrange multiplier \(\lambda\) corresponding to the self-financing condition (6), expressed in terms of Nelson’s derivative corresponding to Itô’s differential, is a stochastic process \((\lambda_{t})_{t \geq 0}\). This Lagrange multiplier is weak \(\mathcal{D}\)-differentiable as all process involved so far are. The Lagrange multiplier \(\mu\) corresponding to the initial wealth is a real number. To solve the maximization problem for \(\Phi\) with respect to the processes \((x_{t})\) and \((\lambda_{t})\) we embed the optimal solution into a one parameter family as

\[
x_{t}(\epsilon) := x_{t} + \epsilon \delta x_{t}, \quad \lambda_{t}(\eta) := \lambda_{t} + \eta \delta \lambda_{t}, \quad \mu(\nu) := \mu + \nu \delta \mu,
\]

where \(\epsilon, \eta \) and \(\nu\) are real parameters defined in a neighbourhood of 0, and \(\delta x_{t}, \delta \lambda_{t}\) and \(\delta \nu\) are arbitrary variations such that the boundary conditions

\[
x_{t}(\epsilon) \equiv x_{t}, \quad x_{T}(\epsilon) \equiv x_{T},
\]

are satisfied. The Lagrange principal equations associated to this maximization problem read

\[
\begin{aligned}
\frac{\partial \Phi}{\partial \epsilon} \bigg|_{\epsilon = \eta = \nu = 0} &= \mathbb{E}_{s} \left[ u \left( \exp \left( \int_{s}^{T} dt \left( D_{t}^{r} \log(D_{t}^{r}) + r_{t}^{h} \right) \right) D_{s}^{r} P_{s,T}^{x} \right) \right] \cdot \exp \left( \int_{s}^{T} dt \left( D_{t}^{r} \log(D_{t}^{r}) + r_{t}^{h} \right) \right) D_{s}^{r} P_{s,T}^{x} \right] \cdot \int_{s}^{T} dt \frac{\partial}{\partial x_{t}} \left( D_{t}^{r} \log(D_{t}^{r}) + r_{t}^{h} \right) \bigg|_{x_{t} = x_{t}} = 0, \\
\frac{\partial \Phi}{\partial \eta} \bigg|_{\epsilon = \eta = \nu = 0} &= \int_{s}^{T} dt \delta \lambda_{t} \mathcal{D} x_{t} \cdot D_{t} = 0, \\
\frac{\partial \Phi}{\partial \nu} \bigg|_{\epsilon = \eta = \nu = 0} &= -\delta \mu(D_{s}^{r} - \xi) = 0,
\end{aligned}
\]

where, by Leibniz’s theorem, we have interchanged differentiation with respect to \(\epsilon\) or \(\eta\) with the integration with respect to \(t\) and the conditional expectation. The boundary condition implies \(\delta x_{t} = 0\), and hence \(\mu D_{s}^{\lambda_{t}} = 0\).
Integration by parts with respect to the time variable shows that

\[- \int_s^T dt \lambda_t \mathcal{D} \delta x_t \cdot D_t = \int_s^T dt (\mathcal{D} (\lambda_s D_s) \cdot \delta x_t),\]

which, inserted into the first equation of (53) leads to

\[
\mathbb{E}_x \left[ \int_s^T dt \left( M \frac{\partial}{\partial x} \left( \mathcal{D} \log(D_t^x) + r_t^x \right) \right) C_{s,T} \right] = 0, \tag{54}
\]

where

\[
M := u' \left( \exp \left( \int_s^T dt (\mathcal{D} \log(D_t^x) + r_t^x) \right) D_s^x P_{s,T} \right)
\cdot \left( \exp \left( \int_s^T dt (\mathcal{D} \log(D_t^x) + r_t^x) \right) D_s^x P_{s,T} \right)
\]

is a strictly positive random variable. Since the variation $\delta x_t$ is arbitrary we infer from (54)

\[
M \frac{\partial}{\partial x} \left( \mathcal{D} \log(D_t^x) + r_t^x \right) \bigg|_{x=x_t} + \mathcal{D} (\lambda_s D_s)
= 0 \quad \text{for any } t \in [s, T],
\]

and, thus, for the choice $t := s$, it follows, being the initial condition $x_s \in \mathbb{R}^N$ arbitrary

\[
\mathcal{D} \log(D_t^x) + r_t^x = - \frac{1}{M} \mathcal{D} (\lambda_s D_s) x_t + C_t^i \quad \text{for all } j = 1, \ldots, N,
\]

for a stochastic process $(C_t^i)_{t \geq 0}$. Therefore

\[
- \frac{1}{M} \mathcal{D} (\lambda_s D_s) x_t + C_t^i = - \frac{1}{M} \mathcal{D} (\lambda_s D_s) x_t + C_t^i \quad \text{for all } j \neq i,
\]

which can hold true if and only if

\[
\begin{cases}
\mathcal{D} (\lambda_s D_s) = 0 \\
C_t^i = C_t
\end{cases} \tag{55}
\]

for all $j = 1, \ldots, N$. Hence, for the optimal Lagrange multiplier,

\[
\mathcal{D} (\lambda_s D_s) = 0 \in \mathbb{R}^N
\]

and

\[
\mathcal{D} \log(D_t^x) + r_t^x = C_t \quad \text{for all } j = 1, \ldots, N. \tag{56}
\]

Therefore, by proposition 2.6, the curvature must vanish, which means that the existence of a solution to the maximization problem implies the vanishing of the curvature. The converse is also true, as it can be seen by following back the steps in this proof from (56) to (51). Hence, the equivalence between (ZC) and (50) holds true.

It turns out that the two weaker notions of arbitrage, the zero curvature and the no-unbounded-profit-with-bounded-risk are equivalent.

\[\text{Theorem 3.3} \quad \text{Let us assume (A1) and (A2), and that } (D_t)_t, (r_t)_t \text{ are semimartingales. Then,}\]

\[(\text{NUPBR}) \Rightarrow (\text{ZC}).\]

\[\text{Remark 10} \quad \text{Counterexample The model given by (30) satisfies (ZC), because it is one dimensional in the risky assets, but does not fulfill (NUPBR), because it allows for immediate arbitrage opportunities as shown in remark 7.}\]

\[\text{Proof of theorem 3.3} \quad \text{By proposition 2.1 (4) in Hulley and Schweizer (2010), the (NUPBR) is equivalent with the existence of a growth optimal portfolio. We apply the classic set up of portfolio optimization to the portfolio of futures under consideration, (which covers as a special case the portfolio of base assets). Since the value of the portfolio at time } t \text{ is}\]

\[D_t^x P_{t,T}^x,\]

and the growth factor from $s$ to $T$ is

\[
\exp \left( \int_s^T dt (\mathcal{D} \log(D_t^x) + r_t^x) \right),
\]

the solution of the expected utility maximization for $s = 0$ and arbitrary $T$ with utility function $u := \log$ must be equal to the optimal growth portfolio. Therefore, by theorem 3.2 (ZC) follows.}

Under what conditions is the converse of theorem 3.3 true? The equivalence of expected utility maximization and (NFLVR) can be proved for a particular choice of a Markov dynamics. Namely, if the asset dynamics follows an Itô process, proposition 2.12 and theorem 3.2 lead to

\[\text{Proposition 3.4} \quad \text{Let the dynamics of a market model be specified by following Itô processes as in (16), where we additionally assume that the coefficients}\]

\[\begin{align*}
&\alpha_t, \sigma_t, \text{ and } (r_t), \\
&\lim_{s \to t^-} \mathbb{E}_s [\alpha_t] = \alpha_t, \quad \lim_{s \to t^-} \mathbb{E}_s [r_t] = r_t, \\
&\lim_{s \to t^-} \mathbb{E}_s [\sigma_t] = \sigma_t,
\end{align*}\]

\[\begin{itemize}
\item (\sigma_t)_t, \text{ and } (W_t)_t \text{ are independent processes.}
\end{itemize}\]

Then, the (NFLVR) condition holds true if and only if the expected utility maximization problem can be solved for all times and horizons for a chosen utility function.

\[\text{Corollary 3.5} \quad \text{Under the same assumptions of proposition 3.4,}\]

\[(\text{ZC}) \Rightarrow (\text{NFLVR}).\]

These last two results are in line with the well-known results of Becherer (2001), Christensen and Larsen (2007), Karatzas and Kardaras (2007) and Hulley and Schweizer (2010).
4. Arbitrage and derivative pricing

The (NFLVR) is an equilibrium condition for financial markets, and the Black–Scholes PDE allows for a unique pricing of derivatives of the base assets of those financial markets. Even if the (ZC) is not fulfilled, the market forces determine an asset dynamics minimizing the total quantity of arbitrage allowed by the market, as it was shown in Farinelli (2015, 2021). The minimal arbitrage is an equilibrium condition as well, which generalizes the benchmark approach (e.g. Heath and Platen 2006) leading to a probability measure equivalent to the statistical one, which is the best possible approximation for the risk neutral measure (cf. Farinelli and Takada 2021). In this case too, a (non-linear) PDE allows for a unique pricing of derivatives of the base assets, in which the arbitrage measure explicitly appears.

We remark that market models allowing for bid and ask spreads have been modelled in Madan Dilip (2017) and Eberlein et al. (2014) by means of non-linear expectations constructed with probability distortions. Interestingly, bid and ask stock prices turn out to be the solutions of non-linear partial integro-differential equations.

4.1. The Black–Scholes PDE in the presence of arbitrage

For markets allowing for arbitrage we are in the position to derive the price dynamics of derivatives whose underlying following an Itô process. It is a nonlinear partial differential equation which coincides with the linear Black–Scholes partial differential equation as soon as the arbitrage vanishes.

**Theorem 4.1** Let us consider a market consisting in a bank account, an asset and a derivative whose discounted prices $X_t$ and $Φ(t, Χ_t)$ follow an Itô’s process. In particular

$$dΧ_t = Χ_t(α_t dt + σ_t dW_t),$$

where $(α_t)_{t∈[0, +∞]}$ and $(σ_t)_{t∈[0, +∞]}$ are real-valued adapted processes, the latter with finite variation. Assuming that the pay-off function $Φ = Φ(t, Χ_t) ∈ C^{1,2}$, the derivative discounted price solves the PDE

$$\frac{∂Φ}{∂t} + \frac{σ_t^2}{2} \frac{∂^2Φ}{∂Χ_t^2} = ρ_tΦ \left(1 + \left(\frac{1}{2} \frac{∂Φ}{∂Χ_t} Χ_t \right)^2\right),$$

where $ρ_t$, defined in (29) measures the arbitrage allowed by the market.

**Proof** We prove the theorem in the context of corollary 2.13 with vanishing short rate $r_t$. By assumption, choosing $N = 2$ and $B = 1$, the market dynamics reads

$$d\hat{S}_t = \hat{S}_t(\hat{α}_t dt + \hat{σ}_t dW_t),$$

where

$$\hat{S}_t := \begin{bmatrix} X_t \\ Φ(t, X_t) \end{bmatrix}, \quad \hat{α}_t := \begin{bmatrix} α_t \\ β_t \end{bmatrix}, \quad \hat{σ}_t := \begin{bmatrix} σ_t \\ τ_t \end{bmatrix},$$

for appropriate real valued predictable processes $(β_t)_{t∈[0, +∞]}$ and $(τ_t)_{t∈[0, +∞]}$ characterizing the dynamics of the derivative.

We apply Itô’s Lemma to the second component of (58). By comparing deterministic and stochastic terms we obtain

$$\begin{cases}
\frac{∂Φ}{∂t} + \frac{σ_t^2}{2} \frac{∂^2Φ}{∂Χ_t^2} = \frac{σ_t^2}{2} \frac{∂^2Φ}{∂Χ_t^2} = β_tΦ \\
\frac{∂Φ}{∂Χ_t} Χ_tσ_t = τ_tΦ.
\end{cases}$$

The one-dimensional $\ker(σ_t)$ is spanned by

$$J_t := (σ_t^2 + τ_t^2)^{-\frac{1}{2}} \begin{bmatrix} -τ_t \\ +σ_t \end{bmatrix},$$

and the vector $\tilde{α}_t$ admits the decomposition

$$\tilde{α}_t = λ_t \tilde{α}_t + ρ_t J_t,$$

for reals $λ_t$ and $ρ_t = \tilde{α}_t^T J_t$. Now we can insert (61) into (59) and eliminate $λ_t$, since the $λ_j$ terms cancel out. The first equation of (59) becomes

$$\frac{∂Φ}{∂t} + \frac{σ_t^2}{2} \frac{∂^2Φ}{∂Χ_t^2} = ρ_t Φ \left(\frac{σ_t^2 + τ_t^2}{τ_t^2}\right)^{\frac{1}{2}}.$$  

The second equation of (59) can be written as

$$\frac{σ_t}{τ_t} = \frac{Φ}{Χ_t^{\frac{σ_t^2}{τ_t}}},$$

which, inserted into (62) gives (57).

**Remark 11** In Farinelli and Vazquez (2012), utilizing another measure of arbitrage $\bar{ρ}_t$, the PDE

$$\frac{∂Φ}{∂t} + \frac{σ_t^2}{2} \frac{∂^2Φ}{∂Χ_t^2} = -\sqrt{2} \bar{ρ}_t Φ \left[1 + \frac{Χ_t^2}{Φ^2} \left(\frac{∂Φ}{∂Χ_t} \right)^2 - \frac{∂Φ}{∂Χ_t} \right],$$

was derived. After some computations, it turns out that

$$\bar{ρ}_t = \frac{1}{\sqrt{2}} \left(1 + \frac{Χ_t^2}{Φ^2} \left(\frac{∂Φ}{∂Χ_t} \right)^2 - \frac{∂Φ}{∂Χ_t} \right)^{\frac{1}{2}} ρ_t,$$

thus guaranteeing that both (59) and (63) are two representations of the same nonlinear Black–Scholes PDE for the price of a derivative in the presence of arbitrage.

**Remark 12** If arbitrage possibilities are allowed, there is no risk neutral probability measure. Asset pricing can nevertheless be obtained as (conditional) expectation of discounted asset’s cash flows with respect to the minimal arbitrage probability measure, as explained in Farinelli and Takada (2021).

**Remark 13** Theorem 4.1 determines the pricing of a derivative in terms of the base assets, if arbitrage opportunities among the base assets are allowed. It does not cover the more general case, where arbitrage opportunities among derivative and base assets are allowed.

It is possible to reformulate theorem 4.1 directly in terms of prices and not discounted prices.
Corollary 4.2 Let us consider a market consisting in a bank account with constant instantaneous risk free rate \( r \), an asset and a derivative whose prices \( S_t \) and \( \Psi(t,S_t) \) follow an \( \mathcal{H} \) process. In particular

\[
dS_t = S_t(\alpha_t dt + \sigma_t dW_t),
\]

where \( (\alpha_t)_{t \in [0, T]} \) and \( (\sigma_t)_{t \in [0, T]} \) are real-valued adapted processes, the latter with finite variation. Assuming that the pay-off function \( \Psi = \Psi(t,s) \in C^{1,2} \), the derivative price solves the PDE

\[
\frac{\partial \Psi}{\partial t} + r \frac{\partial \Psi}{\partial s} + \frac{\sigma_t^2}{2} \frac{\partial^2 \Psi}{\partial s^2} - r \Psi = \rho_t \Psi \left( 1 + \left( \frac{1}{ \Psi } \frac{\partial \Psi}{\partial s} \right) \right)^{\frac{1}{2}}, \tag{64}
\]

where \( \rho_t \), defined in (29) measures the arbitrage allowed by the market.

Note that in the (ZC) case (64) becomes the celebrated linear Black–Scholes PDE well-known from textbooks.

Proof In equation (57), we insert

\[
\Phi(t,x) = \Psi(t,s)e^{-rt}
\]

and, taking into account that

\[
\frac{\partial}{\partial x} = e^{rt} \frac{\partial}{\partial s}, \quad \frac{\partial^2}{\partial x^2} = e^{2rt} \frac{\partial^2}{\partial s^2}, \quad \frac{\partial}{\partial t} = rs,
\]

we obtain, after some algebra equation (64). \( \blacksquare \)

### 4.2. Approximate solution of the modified Black–Scholes PDE

In this section, we derive a dependence relation between a call option price, the price of its underlying and the arbitrage measure \( \rho \) in an implicit form. For this purpose, we assume that the arbitrage measure \( \rho_t \equiv \rho \) is constant during the period considered, typically between 0 and the derivative maturity \( T \). As Farinelli and Vazquez (2012) discussed empirically, arbitrage measure is relatively small so we consider perturbations with respect to \( \rho \) and seek an approximate solution of the modified Black–Scholes PDE (57). We note that the nonlinear term of the modified Black–Scholes PDE (57) is multiplied by \( \rho \) linearly.

Theorem 4.3 For sufficiently small \( \rho > 0 \), an approximated solution of the modified Black–Scholes PDE (57) under the terminal condition \( \Phi(T,X_T) = (X_T - K)^+ \), where \( K \) is the strike price at time \( T \) on the discounted value of the underlying with constant volatility \( \sigma \), is given by

\[
\Phi(t,X_t) = Ke^{\frac{\rho}{2} \log \frac{X_t}{K} - \frac{1}{2} \sigma^2 (T-t)} u \left( \frac{1}{2} \sigma^2 (T-t), \log \frac{X_t}{K} \right). \tag{65}
\]

where

\[
\begin{align*}
\Phi(t,y) &= u(0,\tau) + \rho U_1(\tau, y) + \rho^2 U_2(\tau, y) + O(\rho^3) \quad (\rho \to 0)
\end{align*}
\]

and \( u(0,\tau) \) is the solution of \( (\partial_t - \frac{1}{2} \sigma^2)u(0,\tau) = 0 \) with the initial condition \( u(0,y) = \max\{e^y - e^{-\frac{1}{2} \tau}, 0\} \), and

\[
\Phi(\tau, x) = \frac{2K}{\sigma^2} \sqrt{\frac{\tau}{4 \pi}} e^{-\frac{x^2}{4 \tau}} + \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{(y-z)^2}{4(\tau-s)} \right)
\]

\[
U_1(\tau, y) = \int_0^\tau \int_{-\infty}^{\infty} dz G(\tau, y, s, z) (u_0(s,z), u_0' (s,z))
\]

\[
U_2(\tau, y) = \int_0^\tau \int_{-\infty}^{\infty} dz G(\tau, y, s, z)
\]

\[
\times \left[ f_{11}(u_0(s,z), u_0'(s,z)) U_1(s,z)
\right.
\]

\[
\left. + f_{12}(u_0(s,z), u_0'(s,z)) U_1'(s,z) \right]. \tag{66}
\]

The prime \( \prime \) denotes the derivative with respect to the second argument and \( f_{ij} \) is the derivative of the function \( f \) with respect to the \( j \)th variable.

Proof By means of the change of variables as \( x = Ke^\nu, t = T - 2\tau/\sigma^2 \) and

\[
\frac{\partial}{\partial t} = -\frac{\sigma^2}{2} \frac{\partial}{\partial \nu}, \quad \frac{\partial}{\partial x} = \frac{1}{2} \frac{\partial}{\partial \nu},
\]

the modified Black–Scholes PDE (57) and the terminal condition \( \Phi(T,X_T) = (X_T - K)^+ \) are rewritten for the unknown function \( v(\tau, y) := K^{-\frac{1}{2}} \Phi(t,x) \) as

\[
\frac{\partial v(\tau, y)}{\partial \tau} = \frac{\partial^2 v(\tau, y)}{\partial \nu^2} - \frac{\partial v(\tau, y)}{\partial \nu} + \frac{2\rho K}{\sigma^2} \sqrt{v(\tau, y)^2 + \left( \frac{\partial v(\tau, y)}{\partial \nu} \right)^2}
\]

\[
v(0, y) = \max \{e^y - 1, 0\}.
\]

By introducing the new unknown function \( u = u(\tau, y) \) defined as \( v(\tau, y) = e^{\frac{1}{2} \nu - \frac{1}{2} \nu^2} u(\tau, y) \), we obtain the canonical form of diffusion equation

\[
\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \nu^2} + \rho f(\tau, y, u(\tau, y)).
\]

Here the terminal condition is changed to \( u(0,y) = \max\{e^y - e^{-\frac{1}{2} \tau}, 0\} \). By introducing an unknown function \( B(k, \tau) \), suppose
that the solution of (67) has the form
\[ u(\tau, y) = u_0(\tau, y) + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} B(k, \tau) e^{iky} \, dk, \]  
(67)
where \( u_0(\tau, y) \) is the solution for the case \( \rho = 0 \), i.e. \((\partial_\tau - \partial_y^2)u_0(\tau, y) = 0\). Thus
\[ u_0(\tau, y) = \int_{-\infty}^{\infty} G(\tau, y; 0, z) \max\{e^{iz} - e^{-iz}, 0\} \, dz. \]
Inserting the representation of \( u_0(\tau, y) \) into (67) gives
\[ (\partial_\tau - \partial_y^2)u = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \left( \frac{\partial B(k, \tau)}{\partial \tau} + k^2 B(k, \tau) \right) e^{iky} \, dk, \]
which is equivalent to
\[ \left( \partial_\tau + \partial_y^2 \right) u = \rho f(u, u'). \]
via Fourier transform,
\[ \frac{\partial B(k, \tau)}{\partial \tau} = -k^2 B(k, \tau) + \rho \tilde{f}(\tau, k), \]
(68)
where
\[ \tilde{f}(\tau, k) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{d}{dy} \left( u(\tau, y), u'(\tau, y) \right) e^{-iky} \, dy. \]
We solve (68) via variation of parameters. By introducing new function \( B(k, \tau) \), we assume that the solution has the form
\[ B(k, \tau) = e^{-iz\tau} \tilde{B}(k, \tau). \]
(69)
Inserting this into (68) gives
\[ e^{-iz\tau} \frac{\partial \tilde{B}(k, \tau)}{\partial \tau} = \rho \tilde{f}(\tau, k), \]
which is equivalent to
\[ \tilde{B}(k, \tau) = \rho \int_{0}^{\tau} e^{iz\tau} \tilde{f}(t, k) \, dt. \]
Consequently, the difference between the arbitrage solution \( u \) and the no arbitrage solution \( u_0 \) is
\[ u(\tau, y) - u_0(\tau, y) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{iky} B(k, \tau) \, dk \]
\[ = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{iky} e^{-iz\tau} \left( \rho \int_{0}^{\tau} e^{iz\tau} \tilde{f}(t, k) \, dt \right) \, dk \]
\[ = \rho \frac{1}{\sqrt{2\pi}} \int_{0}^{\tau} \left( \int_{-\infty}^{\infty} e^{-i(z(\tau-x)) + ik(y-z)} f(u(s, z), u'(s, z)) \, dz \right) \, ds \]
\[ = \rho F[u](\tau, y). \]
(70)

The nonlinear Black–Scholes PDE (57) with the terminal condition is therefore equivalent to the functional equation
\[ G[u] := u - u_0 - \rho F[u] = 0, \]
(71)
which can be solved by a Newton’s approximation scheme. The first element of the approximation sequence of the solution \( u \) is \( u_0 \). The second, \( u_1 \) is the solution of the linearization of (71) at \( u_0 \)
\[ G[u_0] + G'[u_0] \cdot (u_1 - u_0) = 0, \]
(72)
where the star denotes the Gâteaux derivative. The solution reads
\[ u_1 = u_0 + \rho (1 - \rho F^*[u_0])^{-1} F[u_0] \]
\[ = u_0 + \rho (1 + \rho F^*[u_0]) F[u_0] + O(\rho^3) \]
\[ = u_0 + \rho U_1 + \rho^2 F^*[u_0] U_1 + O(\rho^3) \quad (\rho \to 0), \]
where \( U_1 := F[u_0] \) corresponds to (66). We now compute the Gâteaux derivative of \( F \) at \( u_0 \) as
\[ F^*[u], U_1(\tau, y) \]
\[ = \int_{0}^{\tau} ds \int_{-\infty}^{\infty} dz G(\tau, y; s, z) \left[ f_{-1}(u_0(s, z), u'_0(s, z)) U_1(s, z) \right. \]
\[ + f_{-2}(u_0(s, z), u'_0(s, z)) U'_1(s, z) \].

We can now derive the second-order approximate solution for \( u \) as
\[ u(\tau, y) = u_0(\tau, y) \]
\[ + \rho \int_{0}^{\tau} ds \int_{-\infty}^{\infty} dz G(\tau, y; s, z) f(u_0(s, z), u'_0(s, z)) \]
\[ + \rho^2 \int_{0}^{\tau} ds \int_{-\infty}^{\infty} dz G(\tau, y; s, z) \]
\[ \times \left[ u_1(s, z) f_{-1}(u_0(s, z), u'_0(s, z)) \right. \]
\[ + u'_1(s, z) f_{-2}(u_0(s, z), u'_0(s, z)) \]
\[ + O(\rho^3) \quad (\rho \to 0). \]

By tracing back of the change of variables in (57) we can obtain the solution \( \Phi(\tau, X) \) as in (65).

5. Conclusion
We apply Geometric Arbitrage Theory to obtain results in Mathematical Finance, which do not need stochastic differential geometry in their formulation. First, we utilize the equivalence for a certain subclass of Itô processes between the no-unbounded-profit-with-bounded-risk condition and the expected utility maximization to prove the equivalence between the NUPBR condition with the ZC condition. Then, we generalize the Black–Scholes PDE to markets allowing arbitrage, computing an approximated solution for the nonlinear PDE for a call option with arbitrage.
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No potential conflict of interest was reported by the authors.

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References

Becherer, D., The numéraire portfolio for unbounded semimartingales. Finance Stoch., 2001, 5(3), 327–341.
Bellini, F. and Frittelli, M., On the existence of minimax martingale measures. Math. Finance, 2002, 12(1), 1–21.
Christensen, M.M. and Larsen, K., No arbitrage and growth optimal portfolio. Stoch. Anal. Appl., 2007, 25(1), 255–280.
Delbaen, F. and Schachermayer, W., A general version of the fundamental theorem of asset pricing. Math. Ann., 1994, 300(1), 463–520.
Delbaen, F. and Schachermayer, W., The Mathematics of Arbitrage, 2008 (Springer-Verlag Berlin Heidelberg: New York).
Eberlein, E., Madan, D.B., Pistorius, M. and Yor, M., Ask prices as non-linear continuous time G-Expectations based on distortions. Math. Financ. Econom., 2014, 8(3), 265–289.
Elworthy, K.D., Stochastic Differential Equations on Manifolds. London Mathematical Society Lecture Notes Series, 1982.
Emery, M., Stochastic Calculus on Manifolds- With an Appendix by P. A. Meyer, 1989 (Springer-Verlag Berlin Heidelberg: New York).
Farinelli, S., Geometric arbitrage theory and market dynamics. J. Geom. Mech., 2015, 7, 431–471.
Farinelli, S., Geometric arbitrage theory and market dynamics reloaded. preprint, arXiv, 2021.
Farinelli, S. and Takada, H., Can you hear the shape of market? geometric arbitrage and spectral theory. Axioms, 2021, 10(4), 242–266.
Fellesaker, B. and Hughston, L., Positive interest. Risk Mag., 1996, 9, 46–49.
Föllmer, H. and Schied, A., Stochastic Finance: An Introduction In Discrete Time, 2nd ed., 2004 (De Gruyter Studies in Mathematics: Boston).
Fontana, C., Weak and strong no-arbitrage conditions for continuous financial markets. Int. J. Theor. Appl. Finance, 2015, 18(01), 1–34.
Fontana, C. and Runggaldier, W.J., Diffusion-based models for financial markets without martingale measures. Chapter 4. In Risk Measures and Attitudes, edited by F. Biagini, A. Richter and H. Schlesinger, pp. 45–91, 2013 (Springer-Verlag, London).
Gliklikh, Y.E., Global and Stochastic Analysis with Applications to Mathematical Physics. Theoretical and Mathematical Physics, 2010 (Springer-Verlag London).
Hackenbroch, W. and Thalmaier, A., Stochastische Analyse. Eine Einführung in die Theorie der stetigen Semimartingale, 1994 (Teubner Verlag: Wiesbaden).
Heath, D. and Platen, E., A Benchmark Approach to Quantitative Finance, 2006 (Springer: New York).
Heymann, M., Adaptive Curve Evolution Model for Interest Rates, 2018 (CreateSpace Independent Publishing Platform: Scotts Valley).
Hörmander, L., The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis, 2003 (Springer-Verlag Berlin Heidelberg: New York).
Hugonnier, J. and Prieto, R., Asset pricing with arbitrage activity. J. Financ. Econ., 2015, 115, 411–428.
Hsu, E.P., Stochastic Analysis on Manifolds. Graduate Studies in Mathematics, Vol. 38, 2002 (AMS).
Hulley, H. and Schweizer, M., M*-on minimal market models and minimal martingale measures. In Contemporary Quantitative Finance. Essays in Honour of Eckhard Platen, edited by C. Chiarella and A. Novikov, pp. 35–51, 2010 (Springer-Verlag Berlin Heidelberg).
Hunt, P.J. and Kennedy, J.E., Financial Derivatives in Theory and Practice. Wiley Series in Probability and Statistics, 2004 (John Wiley & Sons).
Ilinski, K., Gauge geometry of financial markets. J. Phys. A: Math. Gen., 2000, 33(1), 5–14.
Ilinski, K., Physics of Finance: Gauge Modelling in Non-Equilibrium Pricing, 2001 (Wiley: New York).
Jleuni, T. and Yor, M., Inégalité de Hardy, semimartingales, et faux-amaris. Séminaire de Probabilités, Vol. XIII, pp. 332–359,1979.
Kabanov, Y.M., On the FTAP of Kreps-Delbaen-Schachermayer. In Statistics and Control of Stochastic Processes, The Liptser Festschrift Proceedings of Steklov Mathematical Institute Seminar, Moscow, Russia, edited by Y. M. Kabanov, pp. 191–203, 1997 (World Scientific, Singapore).
Karatzas, I. and Kardaras, C., The numéraire portfolio in semimartingale financial models. Finance Stoch., 2007, 11(4), 447–493.
Kabanov, Y.M. and Kramkov, D.O., Large financial markets: asymptotic arbitrage and contiguity. Probab. Theory Appl., 1994, 39, 222–229.
Loewenstein, M. and Willard, G.A., Local martingales, arbitrage, and viability. Econom. Theory, 2000, 16(1), 135–161.
Luenberger David, G., Local Theory of Constrained Optimization: Optimization by Vector Space Methods, 1969 (New York John Wiley & Sons: New York).
Malaney, P.N., The index number problem: A differential geometric approach. Ph.D thesis, Harvard University Economics Department, 1996.
Madan Dilip, B., Financial Equilibrium with Non-Linear Valuations, 2017, www.papers.ssrn.com.
Pelts, G., Unspanned Volatility in Non-Affine Short Rate Models and Conformal Symmetry, 2012, www.papers.ssrn.com.
Pelts, G., Quantum Pricing, 2018a, www.papers.papers.ssrn.com.
Pelts, G., Modeling Interest Rate and FX Derivatives with Division Algebras, 2018b, www.researchgate.net.
Rogers, L.C.G., Equivalent martingale measures and no-arbitrage. Stoch. Stoch. Rep., 1994, 51(1–2), 41–49.
Ruf, J., Hedging under arbitrage. Math. Finance, 2013, 23(2), 297–317.
Schachermayer, W., Optimal investment in incomplete markets when wealth may become negative. Ann. Appl. Probab., 2001, 11(3), 694–734.
Appendix. Generalized derivatives of stochastic processes

In stochastic differential geometry, one would like to lift the constructions of stochastic analysis from open subsets of $\mathbb{R}^N$ to $N$-dimensional differentiable manifolds. To that aim, chart invariant definitions are needed and hence a stochastic calculus satisfying the usual chain rule and not Itô’s integral is required, (cf. Hackenbroch 1999). That is why the papers about geometric derivatives can be substituted by the sigma algebra $\mathcal{P}_t$ (‘past’) and $\mathcal{F}_t$ (‘future’) in the definitions of forward and backward derivatives can be substituted by the sigma algebra $\mathcal{N}_t$ (‘present’), see Chapter 6.1 and 8.1 in Gliklikh (2010).

Stochastic derivatives can be defined pointwise in $\omega \in \Omega$ outside the class $\mathcal{C}^1$ in terms of generalized functions.

Definition 9 Let $Q : I \times \Omega \rightarrow \mathbb{R}^N$ be a continuous linear functional in the test processes $\psi : I \times \Omega \rightarrow \mathbb{R}^N$ for $\psi(\cdot, \omega) \in C^\infty_c(I, \mathbb{R}^N)$. We mean by this that for a fixed $\omega \in \Omega$ the functional $Q(\cdot, \omega) \in \mathcal{D}(I, \mathbb{R}^N)$, the topological vector space of continuous distributions. We can then define Nelson’s generalized stochastic derivatives:

$\mathcal{D}Q(\psi) := -Q(\mathcal{D}\psi)$: forward generalized derivative,

$\mathcal{D}_+ Q(\psi) := -Q(\mathcal{D}_+ \psi)$: backward generalized derivative,  \(\mathcal{D}_- Q(\psi) := -Q(\mathcal{D}_- \psi)$: mean generalized derivative.

If the generalized derivative is regular, then the process has a derivative in the classic sense. This construction is nothing else than a straightforward pathwise lift of the theory of generalized functions to a wider class of stochastic processes which do not a priori allow for Nelson’s derivatives in the strong sense.