MODULAR PERIODICITY OF THE EULER NUMBERS
AND A SEQUENCE BY ARNOLD

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Abstract. For any positive integer $q$, the sequence of the Euler up/down numbers reduced modulo $q$ was proved to be ultimately periodic by Knuth and Buckholtz. Based on computer simulations, we state for each value of $q$ precise conjectures for the minimal period and for the position at which the sequence starts being periodic. When $q$ is a power of 2, a sequence defined by Arnold appears, and we formulate a conjecture for a simple computation of this sequence.

1. Introduction

The sequence of Euler up/down numbers $(E_n)_{n \geq 0}$ is the sequence with exponential generating series

$$\sum_{n=0}^{\infty} \frac{E_n}{n!} x^n = \sec x + \tan x.$$  

It is referenced as sequence A000111 in Slo17 and its first terms are $1, 1, 1, 2, 5, 16, 61, 272, 1385, 7936, 50521, 353792, 2702765, \ldots$

The numbers $E_n$ were shown by André And79 to count up/down permutations on $n$ elements (see Section 3).

Knuth and Buckholtz KB67 proved that for any integer $q \geq 1$, the sequence $(E_n \mod q)_{n \geq 0}$ is ultimately periodic. For any $q \geq 1$ we define:

- $s(q)$ to be the minimum number of terms one needs to delete from the sequence $(E_n \mod q)_{n \geq 0}$ to make it periodic;
- $d(q)$ to be the smallest period of the sequence $(E_n \mod q)_{n \geq s(q)}$.

For example, the sequence $(E_n \mod 3)$ starts with

$1, 1, 1, 2, 2, 1, 1, 2, 2, 1, 1, 2, 2, \ldots$

so one might expect to have $s(3) = 1$ and $d(3) = 4$. Clearly $s(1) = 0$ and $d(1) = 1$. In the remainder of this paper, we formulate precise conjectures for the values of $s(q)$ and $d(q)$ for any $q \geq 2$. 


Organisation of the paper. In Section 2 we reduce the problem to the case when \( q \) is a prime power and we conjecture the values of \( s(q) \) and \( d(q) \) when \( q \) is an odd prime power. In Section 3 we conjecture the values of \( s(q) \) and \( d(q) \) when \( q \) is a power of 2, after having introduced the Entringer numbers and a sequence defined by Arnold describing the 2-adic valuation of the Entringer numbers. In Section 4, we provide a simple construction which conjecturally yields the Arnold sequence.

2. Case when \( q \) is not a power of 2

The following lemma implies that it suffices to know the values of \( s(q) \) and \( d(q) \) when \( q \) is a prime power in order to know the values of \( s(q) \) and \( d(q) \) for any \( q \geq 2 \).

**Lemma 1.** Fix \( q \geq 2 \) and write its prime number decomposition as

\[
q = \prod_{i=1}^{k} p_i^{\alpha_i},
\]

where \( k \geq 1, p_1, \ldots, p_k \) are distinct prime numbers and \( \alpha_1, \ldots, \alpha_k \) are positive integers. Then

\[
s(q) = \max_{1 \leq i \leq k} s(p_i^{\alpha_i})
\]

\[
d(q) = \text{lcm}(d(p_1^{\alpha_1}), \ldots, d(p_k^{\alpha_k})).
\]

The proof is elementary and uses the Chinese remainder theorem.

When \( q \) is an odd prime power, Knuth and Buckholtz [KB67] found the following:

**Theorem 2 ([KB67]).** Let \( p \) be an odd prime number.

1. If \( p \equiv 1 \mod 4 \), then

\[
d(p) = p - 1.
\]

2. If \( p \equiv 3 \mod 4 \), then

\[
d(p) = 2p - 2.
\]

3. For any \( k \geq 1 \),

\[
s(p^k) \leq k.
\]

4. For any \( k \geq 2 \),

\[
d(p^k) \mid p^{k-1}d(p).
\]

We conjecture the following for the exact values of \( s(q) \) and \( d(q) \) when \( q \) is an odd prime power:

**Conjecture 1.** Let \( p \) be an odd prime number.
(1) For any $k \geq 1$,
\[ s(p^k) = k. \]

(2) For any $k \geq 2$,
\[ d(p^k) = p^{k-1}d(p). \]

Conjecture 1 is supported by Mathematica simulations done for all odd prime powers $q < 1000$.

3. ENTRINGER NUMBERS AND CASE WHEN $q$ IS A POWER OF 2

Formulating a conjecture analogous to Conjecture 1 for powers of 2 requires to define, following Arnold [Arn91], a sequence describing the behavior of the 2-adic valuation of the Entringer numbers.

3.1. The Seidel-Entringer-Arnold triangle. The Entringer numbers are a refined version of the Euler numbers, enumerating some subsets of up/down permutations. For any $n \geq 0$, a permutation $\sigma \in S_n$ is called up/down if for any $2 \leq i \leq n$, we have $\sigma(i-1) < \sigma(i)$ (resp. $\sigma(i-1) > \sigma(i)$) if $i$ is even (resp. $i$ is odd). André [And79] showed that the number of up/down permutations on $n$ elements is $E_n$. For any $1 \leq i \leq n$, the Entringer number $e_{n,i}$ is defined to be the number of up/down permutations $\sigma \in S_n$ such that $\sigma(n) = i$. The Entringer numbers are usually displayed in a triangular array called the Seidel-Entringer-Arnold triangle, where the numbers $(e_{n,i})_{1 \leq i \leq n}$ appear from left to right on the $n$-th line (see Figure 1).

\[
\begin{array}{cccccc}
1 \\
0 & 1 \\
1 & 1 & 0 \\
0 & 1 & 2 & 2 \\
5 & 5 & 4 & 2 & 0 \\
\end{array}
\]

**Figure 1.** First five lines of the Seidel-Entringer-Arnold triangle.

The Entringer numbers can be computed using the following recurrence formula (see for example [Sta97]). For any $n \geq 2$ and for any
\[1 \leq i \leq n, \text{ we have}\]
\[
e_{n,i} = \begin{cases} 
\sum_{j<i} e_{n-1,j} & \text{if } n \text{ is even} \\
\sum_{j\geq i} e_{n-1,j} & \text{if } n \text{ is odd}
\end{cases}
\]

3.2. **Arnold’s sequence.** Replacing each entry of the Seidel-Entringer-Arnold triangle by its 2-adic valuation, we obtain an infinite triangle denoted by \( T \) (see Figure 2).

\[
\begin{array}{ccccccc}
0 & \infty & 0 \\
0 & 0 & \infty \\
\infty & 0 & 1 & 1 \\
0 & 0 & 2 & 1 & \infty
\end{array}
\]

**Figure 2.** First five lines of the triangle \( T \) of 2-adic valuations of the Entringer numbers.

We read this triangle \( T \) diagonal by diagonal, with diagonals parallel to the left boundary. For any \( i \geq 1 \), denote by \( D_i \) the \( i \)-th diagonal of the triangle \( T \) parallel to the left boundary. For example \( D_1 \) starts with \( 0, \infty, 0, \infty, 0, \ldots \). For any \( i \geq 1 \), denote by \( m_i \) the minimum entry of diagonal \( D_i \). Arnold [Arn91] observed that the further away one moves from the left boundary, the higher the 2-adic valuation of the Entringer numbers becomes. In particular, he observed (without proof) that the sequence \( (m_i)_{i \geq 1} \) was weakly increasing to infinity. He defined the following sequence: for any \( k \geq 1 \),
\[
u_k := \max \{i \geq 1 | m_i < k\}.
\]

In other words, \( u_k \) is the number of diagonals containing at least one entry that is not zero modulo \( 2^k \). The sequence \( (u_k)_{k \geq 1} \) is referenced as the sequence A108039 in OEIS [Slo17] and its first few terms are given in Table 1.

\[
\begin{array}{cccccccccccccccccccc}
& k & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
u_k & 2 & 4 & 4 & 8 & 8 & 8 & 8 & 8 & 10 & 12 & 12 & 16 & 16 & 16 & 16 & 16 & 16 & 18 & 18 & 20
\end{array}
\]

**Table 1.** The first few values of \( u_k \).
Note that the first few terms given by Arnold were incorrect, because the entry 4 appeared four times, whereas it should be appearing only three times. We also remark that we cannot define any sequence analogous to \((u_k)\) when studying the \(p\)-adic valuations of the Entringer numbers for odd primes \(p\). Indeed, the \(p\)-adic valuation 0 seems to appear in diagonals of arbitrarily high index.

3.3. Case when \(q\) is a power of 2. Using the sequence \((u_k)_{k \geq 1}\), we formulate the following conjecture for \(s(q)\) and \(d(q)\) when \(q\) is a power of 2:

**Conjecture 2.** For any \(k \geq 1\), we have
\[
s(2^k) = u_k.
\]
Furthermore, if \(k \geq 1\) and \(k \neq 2\), we have
\[
d(2^k) = 2^k.
\]
Finally, we have \(d(4) = 2\).

Numerical simulations performed on Mathematica for \(k \leq 12\) support Conjecture 2.

4. Construction of Arnold’s sequence

In this section we provide a construction which conjecturally yields Arnold’s sequence \((u_k)_{k \geq 1}\).

We denote by \(\mathbb{Z}_+\) the set of nonnegative integers and we denote by
\[
S := \bigcup_{d \geq 1} \mathbb{Z}_+^d
\]
the set of all finite sequences of nonnegative integers. We define a map \(f : S \to S\), which maps each \(\mathbb{Z}_+^d\) to \(\mathbb{Z}_+^{2d}\), as follows. Fix \(x = (x_1, \ldots, x_d) \in S\). If all the \(x_i\)'s are equal to \(x_d\), we set
\[
f(x) = (x_d, \ldots, x_d, 2x_d, \ldots, 2x_d),
\]
where \(x_d\) and \(2x_d\) both appear \(d\) times on the right-hand side. Otherwise, define
\[
s := \max\{1 \leq i \leq d - 1 | x_i \neq x_d\}
\]
and set
\[
f(x) = (x_1, \ldots, x_d, x_1 + x_d, \ldots, x_{s-1} + x_d, 2x_d, \ldots, 2x_d),
\]
where \(2x_d\) appears \(d-s+1\) times on the right-hand side. For example, we have
\[
f((2, 4, 4, 4)) = (2, 4, 4, 4, 8, 8, 8, 8)
\]
and

\( f(2, 4, 4, 4, 8, 8, 8, 8) = (2, 4, 4, 4, 8, 8, 8, 8, 10, 12, 12, 16, 16, 16, 16, 16, 16) \).

By iterating this function \( f \) indefinitely, one produces an infinite sequence:

**Lemma 3.** Fix \( d \geq 1 \) and \( x \in \mathbb{Z}_+^d \). There exists a unique (infinite) sequence \((X_k)_{k \geq 1}\) such that for any \( k \geq 1 \) and for any \( n \geq \log_2(k/d) \), \( X_k \) is the \( k \)-th term of the finite sequence \( f^n(x) \).

This infinite sequence is called the \( f \)-transform of \( x \). The lemma follows from the observation that for any \( \ell \geq 1 \) and for any \( y \in \mathbb{Z}_+^\ell \), \( y \) and \( f(y) \) have the same first \( \ell \) terms.

We can now formulate a conjecture about the construction of the sequence \((u_k)_{k \geq 1}\):

**Conjecture 3.** Arnold’s sequence \((u_k)_{k \geq 1}\) is the \( f \)-transform of the quadruple \((2, 4, 4, 4)\).

Conjecture 3 is supported by the estimation on Mathematica of \( u_k \) for every \( k \leq 512 \).

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