New constraints on the Bray conservation-of-momentum natal kick model from multiple distinct observations

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ABSTRACT
Natal supernova kicks, the linear momentum compact remnants receive during their formation, are an essential part of binary population synthesis (BPS) models. Although these kicks are well-supported by evidence, their underlying distributions and incorporation into BPS models is uncertain. In this work, we investigate the nature of natal kicks using a previously proposed analytical prescription where the strength of the kick is given by \( v_k = \alpha \frac{m_{\text{neut}}}{m_{\text{remnant}}} + \beta \) km s\(^{-1}\), for free parameters \( \alpha \) and \( \beta \). We vary the free parameters over large ranges of possible values, comparing these synthetic populations simultaneously against four constraints: the merger rate of compact binary neutron star (BNS) systems, the period-eccentricity distribution of galactic BNSs, the velocity distribution of single-star pulsars, and the likelihood for low-ejecta mass supernovae to produce low-velocity kicks. We find that different samples of the parameter space satisfy each test, and only 1 per cent of the models satisfy all four constraints simultaneously. Although we cannot identify a single best kick model, we report \( \alpha = 115^{+40}_{-55} \) km s\(^{-1}\), \( \beta = 15^{+10}_{-15} \) km s\(^{-1}\) as the center of the region of the parameter space that fulfills all of our constraints, and expect \( \beta \geq 0 \) km s\(^{-1}\) as a further constraint. We also suggest further observations that will enable future refinement of the kick model. A sensitive test for the kick model will be the redshift evolution of the BNS merger rate since this is effectively a direct measure of the delay-time distribution for mergers. For our best fitting values, we find that the peak of the BNS merger rate is the present-day.

Key words: supernovae: general – gravitational waves – methods: numerical

1 INTRODUCTION
At the terminus of a massive star’s \( (M_{\text{final}} \geq 8M_\odot) \) lifetime, it experiences a core-collapse supernova (CCSNe). This supernova imparts velocity onto the remnant it leaves behind - termed a natal supernova kick. Although evidence for a natal kick has been discussed since at least the 1960s (e.g., Blaauw 1961; Gunn & Ostriker 1970; de Loore et al. 1975; Sutantyo 1978), there are still two areas of active research. First, how the kick is caused in the core-collapse. Although it is generally understood to be the result of conservation of momentum, it is uncertain whether this is caused by an asymmetric mass ejection during the supernova or an isotropic emission of neutrinos during the cooling of the compact remnant (Wongwathanarat et al. 2013, and references therein). Secondly, and of particular interest to our work, is the distribution of velocities imparted on the remnants (e.g., Hobbs et al. 2005; Baker et al. 2008; Wongwathanarat et al. 2013; Bray & Eldridge 2018; Giacobbo & Mapelli 2020; Mandel & Müller 2020). The velocity distribution provided by Hobbs et al. (2005) is generally used, however there are conflicting accounts (cf. Fryer et al. 1998; Arzoumanian et al. 2002; Bombaci & Popov 2004; Verbunt et al. 2017) of the precise nature of the distribution.

These conflicting accounts, in part, stem from the two approaches to prescribing a kick distribution. These are based in empirically inferential methods and theoretical prescriptions. Empirically inferential methods here mean statistical fits to observed pulsar velocities, such as the fit to runaway pulsars by Hobbs et al. (2005), whereas a theoretical prescription defines the kick velocity in terms of physical parameters of the supernova or star system, such as the ejecta or remnant masses. An example of this prescription is equation 2 of Mandel & Müller (2020), which takes the neutron star kick velocity as proportional to the CO core mass and the neutron star mass. Both approaches have their merits. The empirical approach can include observational data that challenges theoretical understanding. On the other hand, theoretical prescriptions allow us to make predictions which can be validated against future observations. However, both approaches suffer from drawbacks: empirically inferential methods are subject to biases in their selection. Conversely, theoretical prescriptions are ultimately a parameterisation of complex physical phenomena. They can suffer from simplifications or assumptions that introduce difficult to quantify inaccuracies.

Two kick prescriptions we examine in detail in this work are the
theoretically prescribed kick from Bray & Eldridge (2016); Bray & Eldridge (2018) (hereinafter the ‘Bray kick’) and the empirically inferred kick from Hobbs et al. (2005) (the ‘Hobbs kick’). The reason we select these two kicks is twofold. We use the Hobbs kick as our reference kick as it has been widely traversed in literature (see, among others, Nakar 2007; Belczynski et al. 2008; Lorimer 2008; Dominik et al. 2012; Eldridge et al. 2017), and the Bray kick as its two free parameters, α and β, let us generate a population of kicks tied to the properties of the progenitors. We also consider the Bray kick and the Hobbs kick as they have differing physics underlying their selection of the kick mechanism. The Bray kick originates in an argument from conservation of momentum (as argued in Janka 2017), whereas the Hobbs kick is a random kick sampled from a Maxwell-Boltzmann distribution centered on \( \sigma = 265 \, \text{km s}^{-1} \), chosen as an empirical fit to the proper motions of 233 pulsars. Both kicks sample the direction isotropically. The Bray kick takes the kick velocity to be linear in the ejecta-remnant mass ratio,

\[
v_k = \alpha \left( \frac{M_{\text{ejecta}}}{M_{\text{remnant}}} \right) + \beta \left( \frac{M_{\text{NS}}}{M_{\text{remnant}}} \right),
\]

with the most recent fiducial parameters \( \alpha = 100^{+30}_{-20} \, \text{km s}^{-1} \) and \( \beta = -170^{+100}_{-100} \, \text{km s}^{-1} \) from Bray & Eldridge (2018), and \( M_{\text{NS}} \) the mass of a neutron star. We note that the Bray kick has an extra constant for all supernovae.

The Bray kick takesthe kick velocity to be linear in the ejecta-remnant mass ratio,

\[
P(v_k) \, dv_k = \sqrt{\frac{2}{\pi \sigma^3}} \exp\left( -\frac{v_k^2}{2\sigma^2} \right) \, dv_k,
\]

with the distribution parameter fixed at \( \sigma = 265 \, \text{km s}^{-1} \).

Some authors allow the distribution parameter of the Hobbs kick, \( \sigma \), to vary, whilst some take a bimodal approach (see, among others, Arzoumanian et al. 2002; Vigna-Gómez et al. 2018; Beniamini & Piran 2016). On the other hand, the Hobbs kick samples the velocity from a Maxwell-Boltzmann distribution, given by

\[
\text{Sources:}
\]

(i) Gravitational wave (GW) transient rates by the LIGO-Virgo-KAGRA collaboration (LIGO-Virgo (2021); LIGO-Virgo-KAGRA (2021), hereafter LVK),

(ii) Observations of galactic BNS systems (Vigna-Gómez et al. 2018, and references therein, summarised Table 2),

(iii) The velocity distributions of single-star pulsars (Willcox et al. 2021, and references therein), and

(iv) The kick velocities of USSNe.

Previous work on narrowing down the nature of the natal kick such as Bray & Eldridge (2018) has focused on a subset of these observational constraints. However, with the benefit of larger and more precise datasets, we are now in a position to apply multiple constraints to our models. Using multiple observational datasets allows us to create orthogonal, independent, constraints on our values of \( \alpha \) and \( \beta \).

It is pertinent to note that we use the terminology ‘Binary Neutron Star’ (BNS) instead of ‘Double Neutron Star’ (DNS): DNS systems include binary systems formed by, for example, the capture of a runaway star by an isolated star. On the other hand, BNS systems are the subset of DNS systems which are formed as binaries from birth. Our analysis is focussed on BNS systems.

This paper is structured as follows: in Section 2, we outline our methodology for the BPS of, and inspiral calculations for, compact BNS systems, including the BPASS project, our code base, and the statistical methods we utilise. In Section 3, we examine the results of these statistical tests, and in Section 4, we discuss our results in the broader picture of BNS evolution.

2 METHODOLOGY

2.1 The BPASS project

Our BPS is performed using the Binary Population and Spectral Synthesis (BPASS) suite of codes, described by Eldridge et al. (2017); Stanway & Eldridge (2018). We use the fiducial version 2.2.1 models, which implement the initial mass function (IMF) from Kroupa et al. (1993), take an upper mass limit of \( 300 M_\odot \), and use the initial binary parameters of Moe & Di Stefano (2017).

The BPASS models are a set of binary and single-star models generated through a modified version of Cambridge STARS code (orig-
The effect of the kick on the pre-SN orbit. The period used for this secondary binary model is given from analysing the evolution of one star at a time. The most massive (2017), pairing primary and secondary model files. Briefly, to use further extracts the velocity distribution of isolated pulsars. Our models follow the input file format described in Eldridge et al. (2017), pairing primary and secondary model files. Briefly, to use computational resources efficiently, BPASS only calculates the detailed structure and evolution of one star at a time. The most massive star, the primary, is evolved in detail first while the secondary’s evolution is approximated by using the equations of Hurley et al. (2002). These are referred to as our primary models.

After the first supernova, the evolution of the secondary is recomputed using a detailed evolution model. If accretion onto the secondary happened during the primary evolution then the maximum mass of the secondary is used. If in the first supernova the system is unbound then a single star is used. However, when the system remains bound, the secondary star is evolved in detail, modeling the compact remnant as a white dwarf, neutron star or black hole, depending on its mass. These are our secondary models. The initial period used for this secondary binary model is given from analysing the effect of the kick on the pre-SN orbit.

Then, for each kick model – either a pair of $\alpha$ and $\beta$ parameters, or the Hobbs kick – we simulate 1,000 kicks. This results in a selection of systems which have the same masses, with different eccentricities, periods, and ages, as well as kick velocities.

In this paper, we only consider solar metallicity (which we take to be $Z_\odot \equiv 0.020$), as Tang et al. (2020) showed that there is only a weak influence on BNS merger rate densities by the metallicity for BNS systems. We expect the other tests to also be only weakly dependent on metallicity, as we are using young BNS systems recently formed, and the BNS distribution is linked to the BNS merger rate, indicating weak dependence.

We have created three different sets of $\text{Tui}$ populations in this work. These are:

(i) A fiducial population based on Hobbs et al. (2005),
(ii) A wide grid of populations with $\alpha \in [0, 700]$ km s$^{-1}$ and $\beta \in [-350, 350]$ km s$^{-1}$ with a step size in both directions of 50 km s$^{-1}$ (WideGrid), and
(iii) A fine grid of populations with $\alpha \in [0, 400]$ km s$^{-1}$ and $\beta \in [-200, 200]$ km s$^{-1}$ with a step size in both directions of 10 km s$^{-1}$ (FineGrid).

Both FineGrid and WideGrid are computed as isolated binary grids and separately as distributions of isolated single-star pulsar velocities.

### 2.2 Takahe: Modelling compact remnant binary populations and their mergers

$\text{Tui}$ only provides the BNS populations at the moment after the second supernova, and thus we must compute their coalescence time. We use the equations formulated by Peters (1964) to simulate the inspiral of compact remnants. These are

\begin{equation}
\frac{da}{dt} = \frac{-\beta}{a^3 (1 - e^2)^{3/2}} \left(1 + \frac{73}{24} e^2 + \frac{37}{96} e^4\right) \tag{3}
\end{equation}

and

\begin{equation}
\frac{de}{dt} = -\frac{19}{12} \frac{\beta}{a^4 (1 - e^2)^{5/2}} \left(e + \frac{121}{304} e^3\right). \tag{4}
\end{equation}

It is pertinent to note that these equations are formulated as the equations of motion in the weak field regime. They do not account for general relativistic effects past the innermost stable circular orbit $r_{\text{ISCO}} = \frac{5GM}{c^2}$ of the primary star. However, based on prior work with the roriroo code (van Zeist et al. 2021), which accounts for up to third post-Newtonian terms, we expect that the merger time-scale will only be higher by fractions of years in this regime, and thus we do not consider effects stronger than this.

There are no closed-form solutions to these coupled, inhomogeneous, differential equations, although approximations do exist, such as Mandel (2021). It can be shown that the coalescence time of the binary system, given initial semimajor axis $a_0$ and eccentricity $e_0$ is given by

\begin{equation}
T_C(a_0, e_0) = \frac{12 \sqrt{\beta}}{19} \int_0^{e_0} \frac{e^{3/2} \left(1 + \frac{121}{304} e^3\right)}{(1 - e^2)^{5/2}} \, de, \tag{5}
\end{equation}

with $c_0$ a constant, described in Peters (1964). We therefore take the total merger time as

\begin{equation}
\tau = T_{\text{rejuv}} + T_{\text{evo}} + T_C. \tag{6}
\end{equation}

Here $T_{\text{rejuv}}$ is the evolution age – the length of time from the secondary’s zero-age main sequence until its death. $T_{\text{evo}}$ is the stellar rejuvenation age. This factor is included to account for the effects of rejuvenation of the secondary star in a binary, which is only included if during the primary model the secondary’s star’s mass increases due to mass transfer. If the companion star accretes mass, we assume it has rejuvenated to the zero-age main sequence (ZAMS). We record the time when the companion star last accreted material, and if it accreted more than 5 per cent of its initial mass, then we add this age onto the total merger time. The BPASS codes assume that the accretor is always a main sequence star, see the end of Section 7.1 in Eldridge et al. (2017) for further details on the rejuvenation age treatment.

The four parameters, $m_1, m_2, a, e$, taken at time $t = 0$, determine the future evolution of the compact binary remnant, and thus we have a four-dimensional phase space. However, as the system contains two compact objects, we treat the masses as constant throughout the evolution, and thus there are only two time-varying parameters: $a$ and $e$.

We use the Takahe library to solve the coupled equations Equations 3 and 4 for each set of parameters within the parameter spaces defined in Section 2.1. Takahe is a Python (Van Rossum & Drake 2009) library with a Julia (Bezanson et al. 2014) integrator, that allows us to efficiently solve Equations 3 and 4.

Takahe imposes both a maximum cut-off time for the integration of 10$^{11}$ years, as well as a minimum semi-major axis of 10 km. The former is chosen as a value past the age of the Universe, whilst the latter is chosen to be the average radius of a neutron star (~12 km, per Lattimer 2015). This ensures that the integrator will either evolve until coalescence, or until the system has lived too long to be
observed. TAKAHE also assumes a ΛCDM cosmology, with \( \Omega_M = 0.3, \Omega_L = 0.7, \) and \( H_0 = 70\text{\ km\ s}^{-1}\text{\ Mpc}^{-1} \) for the purposes of converting between redshift and lookback time. TAKAHE is available on GitHub, see the Data Availability Statement for details.

Our stellar formation rates are taken from Madau & Dickinson (2014),

\[
\psi(z) = 0.015 \frac{(1+z)^{2.7}}{1 + ((1+z)/(2.9))^{3.6}} \text{M}_\odot\text{yr}^{-1}\text{Mpc}^{-3}. \quad (7)
\]

The BNS merger rate density, \( \mathcal{R} \), as a function of redshift is therefore given by the convolution

\[
\mathcal{R}(z) = \psi(z) \ast m(t_z), \quad (8)
\]

where \( m(t_z) \) is the content of the delay-time distribution (DTD) at lookback time \( t_z \) corresponding to the redshift \( z \). We denote the BNS merger rate we detect today as \( \mathcal{R}_0 \equiv \mathcal{R}(0) \). Computationally, we use the binned histogram method described in Section 4 of Briel et al. (2022), with the caveat that, since our calculation is only performed for solar metallicity, our SFH does not incorporate the cosmic metallicity distribution from Langer & Norman (2006); Eldridge et al. (2018); Tang et al. (2020).

In this project, we only consider neutron stars, and assume that those have masses \( M_{\text{rem}} \leq 2.5 \text{M}_\odot \). Recent studies such as Linares et al. (2018) have discovered neutron stars with masses \( m = 2.17^{+0.17}_{-0.15} \), so our threshold admits those systems. For further information on the selection of the upper bound, the reader is referred to Section 3.2 of Stevance & Eldridge (2021).

2.3 Statistical Methods

2.3.1 BNS merger rates: The O3a Contour

Our first approach is to examine the merger rate today of BNS systems, \( \mathcal{R}_0 \). We evolve every model in our dataset, predict the BNS merger rate at \( z = 0 \), and use this to compute the DTD. We then extract all combinations of \( \alpha \) and \( \beta \) that result in a value of \( \mathcal{R}_0 \) within the contour defined by the LIGO-Virgo O3a merger rate.

2.3.2 Period-Eccentricity distributions: Bayes Factors

We quantify the goodness-of-fit of the period-eccentricity distributions by computing a probability map. Firstly, we divide the space into 2D bins, which have widths of 0.05 in eccentricity space, and 0.2 dex in \( P \) space. For each system, we trace its path through period-eccentricity space as it evolves. At each timestep, we add \( w \times dr \) to the probability map, where \( w \) is a weight term proportional to the IMF weight, quantifying the number of systems with the same parameters we would expect to occur per \( 10^6 \text{M}_\odot \) of star formation activity. The proportionality term is a weight related to the uniformly sampled kick direction.

The log-likelihood of a given model is then given by

\[
\log L = \sum_{n=0}^{14} \log e_p(\log e P_n, e_n), \quad (9)
\]

where \((P_n, e_n)\) is the period and eccentricity of the \( n \)th observed BNS system. We can then transform this log-Likelihood into a log-Bayes factor, through

\[
\log e \mathcal{K} = \log e L - \log e L_{\text{Hobbs}}. \quad (10)
\]

Table 2. The fourteen observed BNS systems we consider. J1930-1852 in particular is an important system to consider: its high period but low eccentricity places it on the periphery of the main ‘track’ of evolution. Thus, our preferred period-eccentricity distribution must be one that replicates this system. This data was originally compiled in Vigna-Gómez et al. (2018).

| Name          | log(P/days) | e   | Ref.       |
|---------------|-------------|-----|------------|
| J0453+1559    | 0.609808    | 0.113 | Martinez et al. (2015) |
| J0737-3039    | -0.991400   | 0.088 | Kramer et al. (2006)   |
| B1534+12      | -0.375718   | 0.274 | Fontese et al. (2014)  |
| J1756-2251    | -0.494850   | 0.181 | Faulkner et al. (2005) |
| B1913+16      | -0.490797   | 0.617 | Hulse & Taylor (1975)  |
| J1913+1102    | -0.686133   | 0.090 | Lazarus et al. (2016)  |
| J1757-1854    | -0.735182   | 0.606 | Cameron et al. (2018)  |
| J1518+4904    | 0.936212    | 0.249 | Janssen et al. (2008)  |
| J1811-1736    | 1.273672    | 0.828 | Corongiu et al. (2007) |
| J1829+2456    | 0.070407    | 0.139 | Champion et al. (2004) |
| J1930-1852    | 1.653791    | 0.399 | Swiggum et al. (2015)  |
| J1753-2240    | 1.134751    | 0.304 | Keith et al. (2009)    |
| J1411+2551    | 0.417638    | 0.169 | Martinez et al. (2017) |
| J1946+2052    | -1.107905   | 0.064 | Stovall et al. (2018)  |

where \( \log e L_{\text{Hobbs}} \) is the log-Likelihood of the Hobbs kick. A \( \log e \mathcal{K} > 3.4 (\sim -3.4) \) is interpreted as very strong evidence for (against) the model relative to the Hobbs kick. The reader is referred to Appendix B of Jeffreys (1961) for a comprehensive description on the precise numerical values of \( \log e \mathcal{K} \). The fourteen BNS systems we use to compute the goodness-of-fit are given in Table 2.

The catalogue of BNS systems we use do not include any Gamma Ray Bursts (GRBs). Even though a short GRB (SGRB) was discovered as the optical counterpart to a gravitational wave event in GW170817 (LIGO-Virgo 2017), giving a potential avenue of extra constraints to apply, there are still multiple unknowns which impair our ability to use these as a constraint. For example, Eldridge et al. (2018) finds that the uncertainty in the SGRB merger rate spans four orders of magnitude, whereas the BNS merger rate from LIGO-Virgo (2021) only spans 2.86 orders of magnitude. In addition, there are selection effects at play when considering SGRBs. Although the Fermi catalogue has detected > 500 SGRBs (Gruber et al. 2014; von Kienlin et al. 2014; Bhat et al. 2016; von Kienlin et al. 2020), the majority occur at low redshift, with the peak of the redshift distribution being \( z \sim 0.9 \) (Wanderman & Piran 2015). The dearth of SGRBs at higher redshifts is likely to be due to SGRBs being reasonably faint and thus more likely to be difficult to detect.

2.3.3 Single-Star Kick Velocities: Cumulative Distribution Functions

We also investigate the goodness-of-fit of the two kick models against observed pulsar velocities on a single-star model grid. We use the dataset from Willcox et al. (2021), and filter our list of pulsars in the same way: we remove millisecond pulsars, pulsars in binaries, pulsars in globular clusters, and velocities greater than 2000 km s\(^{-1}\). This leaves a selection of 81 pulsars forming a reasonably homogeneous dataset in which the velocity of our pulsars is primarily set by the natal kick and previous binary evolution.

Our single star grid spans the same range as FineGrid, and the models originate from three separate sources: BPASS primary models (models of the primary stars in binaries), BPASS secondary mod-
els (models of the secondary stars in binaries), and isolated single star models. This allows us to generate model distributions of velocities, which we can convert into probability density functions (PDFs). From here, we may compute the cumulative distribution functions (CDFs) of each model, and of the observed data set, and then perform a KS test. The KS statistic of a given model is defined as the maximum difference between its CDF and the observed CDF, and is thus a measure of the goodness-of-fit of a model to an observed CDF: a low KS statistic indicates a closer fit than a higher one.

2.3.4 Ultra-Stripped Supernovae: Kick Velocities

Our final test involves examining the results of synthetic USSNe. We use the definition of an USSNe from Tauris et al. (2015): a supernova which has an envelope mass $M_{\text{env}} \leq 0.2 M_\odot$ with a compact star companion. For every value of $\alpha$ and $\beta$, we examine the kick velocity produced by the BPASS stellar models. In particular, we only consider models which match the following criteria:

(i) $0.0 \leq M_{\text{ej}} \leq 0.3 M_\odot$, to reflect the low-ejecta mass nature of USSNe (choosing $M_{\text{ej}} \leq 0.3 M_\odot$ admits the USSNe SN2019dge from Yao et al. (2020)),
(ii) $1.1 \leq M_{\text{rem}} \leq 1.8 M_\odot$ from Tauris et al. (2015), and
(iii) $m_{\text{env}} = M - M_{\text{CO}} \leq 0.2 M_\odot$ from Tauris et al. (2015).

This leaves us with models of secondary stars which will likely undergo a USSNe. For every pair of $\alpha$ and $\beta$, we record the number of kicks between $0 \leq v \leq 30$ km s$^{-1}$ that it produces. Since the rate of Type Ic USSNe is estimated to be low – in the BPASS dataset, it is 1.8 percent of all CCSNe – we take the ‘acceptable region’ generated by this test as the region for which at least one such kick is possible.

3 RESULTS

3.1 BNS Merger Rates

We compute the BNS merger rate on a two dimensional grid of $\alpha$ and $\beta$ combinations. The result is given in Figure 1. We note that the fiducial value of the Bray kick does not lie on the Eldridge et al. (2018) contour. This is due to modifications we have made in the BPASS code around the calculation of the rejuvenation age (for details on this, the reader is referred to Ghodla et al. (2022), Section 2.1). The BNS merger rates reported by the LVK are given in Table 1, and shown against our finestgrid data in Figure 1.

We find 55 pairs of $\alpha$ and $\beta < -3$ per cent of the parameter space – that agree within the errors of the BNS merger rate from LIGO-Virgo-KAGRA (2021). These form a contour on the $\alpha - \beta$ plot in Figure 1. We term this the ‘LIGO-Virgo O3a Contour’ and discuss this below.

3.2 Period-Eccentricity Distributions

In Figure A1, we report the $\log_8 K$ of each system along the LIGO-Virgo O3a Contour alongside their period-eccentricity distributions. We also display the $\log_8 K$ values for the all of FineGrid and WideGrid in Figure 2. We see better fits towards certain pairs of $\alpha$ and $\beta$ (shown in blue), and less favorable fits towards others (shown in red). A value of $\log_8 K > 0$ for a given model implies that the model is more preferable than the Hobbs model, and the exact magnitude of $\log_8 K$ indicates how preferable it is.

Figure A1 also demonstrates that the structure in the period-eccentricity distribution is sensitive to the natal kick received. For instance, the $\alpha = 200$ km s$^{-1}$, $\beta = -50$ km s$^{-1}$ kick produces a short strand of higher-period binaries, which is not present in, for example, the $\alpha = 90$ km s$^{-1}$, $\beta = 140$ km s$^{-1}$ distribution. On the other hand, the $\alpha = 290$ km s$^{-1}$, $\beta = -180$ km s$^{-1}$ distribution has a prominent region in the lower right corner which has no systems. This region is present – though varies in size – in 42 of the distributions. The remaining distributions – for example, $\alpha = 90$ km s$^{-1}$, $\beta = 140$ km s$^{-1}$ – possess systems that are able to have high-periods and low eccentricity.

We highlight six distributions in particular. These are shown in Figure 3. Whilst the general shape of the four distributions are similar, they have some subtle differences. Firstly, and most notably, each plot has a region in the lower right corner - corresponding to low eccentricity, high-period systems - which has no mergers. This region is smallest for the $\alpha = 80$, $\beta = 160$ plot. In general, systems with high periods and low eccentricities have higher coalescence times than other systems. Given that our integrator allows systems to merge for longer than the age of the Universe – the cut-off time of our integrator is $10^{11}$ years – we can see that some kicks are able to produce systems that take longer to merge.

We note that of the six distributions plotted in Figure 3, the location of the densest region varies markedly. For the fiducial Bray kick, the densest region is in the top right region of the space. For $\alpha = 170$ km s$^{-1}$, $\beta = -10$ km s$^{-1}$, the densest region is around $(\log_{10} P, e) = (1, 0)$. We note that $\alpha = 170$ km s$^{-1}$, $\beta = -10$ km s$^{-1}$ produces a larger kick velocity than the fiducial Bray kick over all positive ejecta-remnant mass ratios. When examining this peak density over the 55 distributions in Figure A1, we find that kicks with a higher average velocity produce peaks in the lower left region of the plot – indicating lower period, circular binaries.

3.3 Single-Star Kick Velocities

Applying the methods outlined in Section 2.3.3, we see in Figure 4 that the most favorable model is $\alpha = 90$ km s$^{-1}$, $\beta = -170$ km s$^{-1}$ – almost replicating the earlier work of Bray & Eldridge (2018). This is expected, as we are effectively following in the steps of Bray & Eldridge (2018), albeit with a different compiled dataset. In general, we find a valley of well-fitting models, with the best fits in the region roughly bounded by the triangle with corners at $\alpha = 90$ km s$^{-1}$, $\beta = -150$ km s$^{-1}$, $\alpha = 90$ km s$^{-1}$, $\beta = -200$ km s$^{-1}$, and $\alpha = 120$ km s$^{-1}$, $\beta = -200$ km s$^{-1}$.

We include the PDFs of four models, shown in Figure 5. The four models we choose are the fiducial Bray kick, the kicks with the highest and lowest $\log_8 K$ values, and a kick with a low $\alpha$ and zero $\beta$. We show them relative to the observed PDF of the single-star pulsars compiled in Willcox et al. (2021). Even the model with the lowest KS statistic – $\alpha = 90$ km s$^{-1}$, $\beta = -170$ km s$^{-1}$ – has difficulty reproducing the observed PDF, though the fit to the CDF is reasonable. This is likely due to the sensitivity of the PDF to the chosen bin size, whereas the CDF is insensitive to the (essentially arbitrary) selection of bin size. Figure 5 also shows the CDF of each of the four models against the observed CDF. The model which has the highest $\log_8 K$ has a KS statistic of 0.61, which indicates it diverges quite strongly from the observed CDF. Indeed, we see that although it does produce kicks $\lesssim 500$ km s$^{-1}$, it underpredicts the probability of those kicks quite markedly.

3.4 Production of Ultra-Stripped Supernovae

We find a reasonably narrow region of the parameter space permits kick velocities consistent with USSNe. The region is broadly
consistent with the analysis that the production of USSNe requires low $\beta$ values. The region is broadest in $\beta$ towards the right edge of the plot, coinciding with high $\alpha$. This is consistent with theory: if we consider an ejecta mass of $0.1M_\odot$ resulting in a Neutron star of mass $1.4M_\odot$, both $\alpha = 30 \, \text{km s}^{-1}, \beta = 10 \, \text{km s}^{-1}$ and $\alpha = 100 \, \text{km s}^{-1}, \beta = -20 \, \text{km s}^{-1}$ would produce weak kicks of about $10 \, \text{km s}^{-1}$. To constrain the kick to have $\beta = 0$, we require the other three constraints.

### 3.5 Combining the Constraints

The four major constraints that we have established – the event rate in Figure 1, the log-Bayes factor in Figure 2, the KS statistic in Figure 4, and the space which permits USSNe – enable us to define heuristics that admit only a subset of our parameter space as acceptable.

We set the following constraints:

(i) The event rate of the model must lie along the LIGO-Virgo O3a Contour, or at worst within the uncertainty bounds,

(ii) The log-Bayes factor must be above 4.61, which represents decisive evidence for the model against the Hobbs kick,

(iii) The KS statistic must, conservatively, be below 0.5, and

(iv) The parameter space region must be capable of producing USSNe.

We select 0.5 as the point where the data fits the observed CDF better than it diverges from it. Varying the KS statistic threshold results in between 0 models (if a threshold of 0.27 or fewer is selected) and 31 models (if a threshold of 0.74 or higher is selected).

We can overplot the relevant regions from Figure 1, Figure 2, and Figure 4 into a single plot – Figure 6 – and use these to identify which regions of our parameter space fit our four constraints. Applying the four constraints to our dataset, we find 1 per cent of our parameter space fits all four constraints. These are tabulated in Table 3, along with their corresponding KS statistics, $\log_{10}K$ values, $\log_{10}R_0$ values, and redshifts that correspond to the peak in the BNS merger rate distribution.

We take the ‘best parameters’ as the center of mass of the cyan region, with uncertainty bounds covering said region. These parameters are $\alpha = 117_{-51}^{+36} \, \text{km s}^{-1}, \beta = 13_{-14}^{+8} \, \text{km s}^{-1}$, which is likely an overfit due to the finer grid used to calculate the value (see Figure 6), and as such the best parameters we report are $\alpha = 115_{-55}^{+40} \, \text{km s}^{-1}, \beta = 15^{+15}_{-10} \, \text{km s}^{-1}$. The CDF and period-eccentricity distribution for this pair of parameters are shown in Figure 7. We can see it is an adequate match to the CDF and period-eccentricity distributions by eye, and it has a merger rate of $\log_{10}R_0 = -2.84490 \pm 0.00009$, where the uncertainty is assumed to be Poissonian.

Finally, we note that the precise determination of the contour lines is accomplished via a marching squares algorithm in MATPLOTLIB. This has the added requirement of using linear interpolation to place
Constraining natal kicks

Figure 2. The log-Bayes factor of the period-eccentricity distribution for our grids, measured relative to the Hobbs kick. We note that there is a large range of log-Bayes Factors - including certain pairs of parameters which are more favourable than the Hobbs kick, and certain ones which are preferred less than the Hobbs kick. The orange contour line represents a Bayes factor confidence level: log$_e$K = 3.4.

Table 3. All 13 models that fit the three constraints in subsection 3.5 - that is, log$_e$K $\geq$ 3.4, KS statistic $\leq$ 0.5, capability of producing USSNe, and log $R_0$ on the LIGO-Virgo O3a Contour (or within its uncertainty bound). Uncertainties have been omitted for brevity, and all values have been rounded to 2 decimal places. We also report $z_{\text{peak}}$, which is the redshift $z$ at which the BNS merger rate distribution peaks, for each model.

4 DISCUSSION & CONCLUSIONS

We found a selection of 13 different pairs of parameters that fit our four constraints. For values of $\alpha \geq 100$ km s$^{-1}$, we found that low magnitude, but non-zero, values of $\beta$ replicated the BNS merger rate from LIGO-Virgo-KAGRA (2021) and replicating the observed BNS systems compiled in Vigna-Gómez et al. (2018). A true ‘best’ set of parameters likely lie within this subsample of our parameter space.

PSR J1930-1852 (Swiggum et al. 2015) is the widest known binary and, per its authors, ‘its unique spin and orbital parameters challenge models that describe BNS formation’. Whilst our work, unlike other authors such as Tauris et al. (2017), did not preferentially weight our binaries – that is, we did not prefer models that replicate extreme parameters such as PSR J1930-1852 – we do note that a number, but not all, of our period-eccentricity distributions had non-zero probability of reproducing this pulsar. Thus, certain kick prescriptions will be able to replicate its unique properties.

Whilst the current observed BNS merger rate is limited, and has wide uncertainty bands, further observations will allow the distribu-
Figure 3. Six period-eccentricity distributions from parameter pairs chosen along the O3 contour. The cyan diamonds are the galactic BNS systems as tabulated in Table 2 that are used to compute the log-Bayes factors. The white dashed line is the contour representing a constant-time coalescence of the Hubble time, and is computed using the approximation in Mandel (2021). We select $\alpha = 180$, $\beta = 10$ as the parameters with the highest log-Bayes factor from the contour, $\alpha = 170$, $\beta = -10$ as the lowest, and the remainder to illustrate structural differences. Empty regions are spaces where the number of systems per bin is zero. The colourbar has units of ‘number of systems per MWEG per 0.2 dex per 0.05 eccentricity’, as the log-Bayes factor is sensitive to the choice of bin size.

The primary limitation with our $z_{\text{peak}}$ analysis is a lack of observational data to compare against. Although factors like the galactic stellar formation rate from Madau & Dickinson (2014) are derived as fits to empirical measurements, given that a redshift of $z = 2$ (higher than the maximum of the grids) corresponds to a lookback time of $t_L \approx 10$ Gyr, we simply do not have direct observational evidence to use this as a constraint to kick models. However, indirect evidence is also found that the peak corresponds to $z \leq 0.26$, and the best model $\alpha = 115$ km s$^{-1}$, $\beta = 15$ km s$^{-1}$ peaks in the present-day, at $z = 0$. The peak redshift across both grids corresponds to $z < 1.5$, with a very clear and defined valley of the present day. Across the 13 pairs of parameters that fulfil all of our constraints, we
Figure 4. The KS statistic for each combination of $\alpha$ and $\beta$ across a single-star version of FineGrid and WideGrid. The values of $\alpha$ and $\beta$ that have the lowest KS statistic are $\alpha = 100, \beta = -180$ on FineGrid, and $\alpha = 100, \beta = -150$ on WideGrid. The orange contour line is the KS statistic for the Hobbs kick, which has a KS statistic of 0.33.

We have also highlighted the caveat that $\alpha = 115^{+20}_{-25}$ km s$^{-1}, \beta = 15^{+10}_{-15}$ km s$^{-1}$ should not be interpreted as ‘the correct choice of parameters’ but rather ‘the mean of the set of parameters which replicates the four observables we test against’. It is also pertinent to note that these values are calibrated against the BPASS version 2.2.1 models (and should not change significantly when used with future stellar models from the same stellar evolution code). Using these values with other BPS codes should be approached with caution, and readers using other codes should calibrate their choice of parameters against their BPS codes in the same way that we have.

As part of the calibration and selection of binary parameters, we simulated 1,000 kicks per binary. Testing showed a slight, but not unexpected, dependence on number of simulated kicks. We found that adjusting the number of kicks simulated by a factor of ten resulted in a shift of $\log_e K$ by approximately one.

Similar work in this field has been done for binary Black-Hole (BBH) mergers by e.g., Mapelli & Giacobbo (2018); Wiktorowicz et al. (2019); Giacobbo & Mapelli (2020); Santoliquido et al. (2021), however few authors have explored the parameter space in the way that we have. The most similar studies have varied the properties of the progenitor population – they have adjusted the IMF, or the stellar formation rate, or DTD, and assumed a kick prescription with given free parameters to be true. With this study, those population properties have been affixed, and we have varied the observed parameters we aim to replicate, to see which set of parameters replicates all four observables best.

Authors such as Fryer et al. (1998); Arzoumanian et al. (2002); Bombaci & Popov (2004) suggest that the kick distribution should ostensibly contain two peaks. In testing, we found no combination of $\alpha$ and $\beta$ for the Bray kick produced an obviously bimodal single-star velocity distribution, though we do observe interesting structure in the resultant PDFs. Figure 7 clearly shows several changes of gradient in the CDF, which can be interpreted as several ranges of kick behaviour. This behaviour arises because the Bray kick is linked to the ejecta mass distribution. Thus the behaviour of low/high kick populations could be due to the results of stellar evolution causing low/high ejecta masses for CCSNe that form neutron stars.

The reason we did not include a bimodal Hobbs kick in this study is twofold. Firstly, the unimodal Hobbs kick has been extensively discussed in literature (see, among others, Nakar 2007; Belczynski et al. 2008; Lorimer 2008; Dominik et al. 2012; Bray & Eldridge 2016; Eldridge et al. 2017; Bray & Eldridge 2018) and thus was a good starting point for our investigation. Secondly, as a point of consideration, we computed the log-Likelihood of reproducing our BNS catalog against a weighted model from Verbunt et al. (2017), taking $w = 0.42$, $\sigma_1 = 75$ km s$^{-1}$, $\sigma_2 = 316$ km s$^{-1}$. We found that $\log_e K = 0.8$. Thus, if this value were to be used as the reference model for the study, it would shift our log-Likelihoods by approximately one order of magnitude and would only admit an extra handful of systems.
Figure 5. The PDF for the single-star velocity distribution for four combinations of $\alpha$ and $\beta$. We select $\alpha = 100 \text{ km s}^{-1} - \beta = 170 \text{ km s}^{-1}$ as the fiducial Bray kick from Bray & Eldridge (2018), $\alpha = 180 \text{ km s}^{-1} - \beta = 10 \text{ km s}^{-1}$ as the combination of parameters with the highest log $\mathcal{K}$ from the LIGO-Virgo O3a contour, $\alpha = 170 \text{ km s}^{-1} - \beta = 10 \text{ km s}^{-1}$ as the combination of parameters with the lowest log $\mathcal{K}$, and $\alpha = 40 \text{ km s}^{-1} - \beta = 0 \text{ km s}^{-1}$ to illustrate a system with a low $\alpha$ and zero $\beta$. The 'empty' bins in the $\alpha = 180 \text{ km s}^{-1} - \beta = 10 \text{ km s}^{-1}$ and $\alpha = 170 \text{ km s}^{-1} - \beta = 10 \text{ km s}^{-1}$ PDFs have contents on the order of $10^3$. The dashed purple line is the PDF corresponding to the observed PDFs from Willcox et al. (2021). In the right panel, the pink dashed line is the CDF corresponding to the Hobbs kick, which has a KS statistic of 0.32. Also shown in the rightmost panel is the CDF for each model, where the colours of the lines correspond to the colours of the relevant histograms. The number in parentheses in the legend is the KS statistic of the given model, which does not apply to the observed CDF as that is the reference model we measure against.

In the future, an additional prior can be used on our dataset. The natural interpretation of the Bray kick expression is that $\beta$ is the ‘limiting value’ that the kick velocity should approach as the ejecta mass gets smaller (as in the case of USSNe and ECSNe). In events such as the direct collapse of a star to a black hole in a failed SNe, where the ejecta mass is zero, $\beta$ would identically be the kick velocity. Thus, in order for conservation of momentum to be respected for low-ejecta mass kicks, we expect $|\beta| > 0$ – i.e., we expect the magnitude of $\beta$ to be low, but most likely non-zero. This prior was not imposed on our parameter space – the log $\mathcal{K}$ and USSNe constraints independently restricted $\beta$ to be >0. It is pertinent to note, as well, that even in the limit as the ejecta mass goes to zero, a value of $\beta < 0$ would imply that the velocity vector is anti-aligned with the chosen direction of the kick. $\beta < 0$ also allows for a zero kick velocity when the ejecta mass is non-zero. At this stage, this scenario cannot be ruled out, thus we cannot yet include this constraint as a prior.

In addition, we found that the mean pair of parameters – the center of the cyan region in Figure 6 – corresponded to $\alpha = 115^{+40}_{-55} \text{ km s}^{-1}, \beta = 15^{+10}_{-15} \text{ km s}^{-1}$. Although this work does not purport to claim a single ‘best’ pair of parameters, these ones can be interpreted as the pair that optimises each of the individual tests that we have performed. We note that this value lies off-grid, and hence in Figure 7 we generated a bespoke data file for this set of two parameters.

In sum, the key conclusions we draw are:

(i) Providing multiple constraints allows for the parameter space of possible kick configurations to be drastically reduced.
(ii) Admissibility of kick velocities of $\leq 30 \text{ km s}^{-1}$ for USSNe is one of the tightest constraints we can provide.
(iii) The peak of the merger rate distribution is likely to be the present-day, and this is a further constraint we can apply to the dataset when more information is available.
(iv) The center of mass of the permitted region of the Bray parameter space is $\alpha = 115^{+40}_{-55} \text{ km s}^{-1}, \beta = 15^{+10}_{-15} \text{ km s}^{-1}$, although caution should be exercised before using this value without calibration.

The analysis presented here is an analysis pipeline to refine the
properties of kick models against four different observables. In principle, it can be applied to any kick model to validate the model. Our statistical analysis provides a simple way to validate kick models: it can be used to rule out either magnitudes of kick velocities or entire physical kick prescriptions.

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DATA AVAILABILITY

Our analysis code is available on GitHub\(^1\), and our bespoke BPASS outputs are available upon request, though we caution that the larger grids comprise some 300 GB of data when uncompressed.

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\(^1\) https://github.com/krytic/takahe
Figure 7. The CDF for a population of isolated pulsars (left) and the period-eccentricity distribution (right) corresponding to $\alpha = 115$ km s$^{-1}$, $\beta = 15$ km s$^{-1}$. The red dashed line on the CDF is the observed CDF from Willcox et al. (2021). The shaded blue region represents the CDFs within the uncertainty bands of our best model value, and were computed by computing the CDFs of every combination of $\alpha$ and $\beta$ within the range, in steps of 5 km s$^{-1}$. The shaded red region is the uncertainty bounds on the observed CDF, generated by simulating 2,000 samples of the PDF, 10,000 times, and plotting the extent of the obtained CDFs. The shaded black region is the uncertainty on the Hobbs CDF, obtained with $\sigma_{\text{max}} = 261$ km s$^{-1}$, $\sigma_{\text{min}} = 239$ km s$^{-1}$, from Verbunt et al. (2017).

Figure 8. The location in redshift $z$ at which the event rate distribution peaks, as a function of $\alpha$ and $\beta$. 
In this section, we provide plots of the 55 models that lie along the LIGO-Virgo O3a contour. The colour axis is the log of the number of systems per Milky-Way Equivalent Galaxy (MWEG), per period bin and per eccentricity bin. Bins were chosen such that $d \log P = 0.2$. Naturally, eccentricity bins span from $e = 0$ to $e = 1$, and log-period bins span from $\log P = -2$ to $\log P = 4$. Moreover, we assume that there is 1 MWEG per (4.4 Mpc)$^{-1}$, an assumption from Kasen et al. (2017).

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Figure A1. The period-eccentricity distributions for the 55 combinations of $\alpha, \beta$ along the LIGO-Virgo O3a contour. The white dashed curve is a line of constant merger time of $t = t_H \approx 13.9$ Gyr, for two 1.4 $M_\odot$ remnants. We assume one Milky-Way equivalent galaxy per (4.4 Mpc)$^3$, from Kasen et al. (2017). We use the approximation from Mandel (2021) for our calculation of the constant merger time contour. The text in the top left corner of each plot denotes the value of $\alpha$ and $\beta$ that generated that plot, alongside its $\log e K$ value. The diagrams are sorted such that the highest $\log e K$ is in the top left corner, and the lowest $\log e K$ value is in the bottom right.