RICCI TENSOR IN GRADED GEOMETRY

FRIDRICH VALACH

Abstract. We define the notion of the Ricci tensor for NQ symplectic manifolds of degree 2 and show that it corresponds to the standard generalized Ricci tensor on Courant algebroids. We use an appropriate notion of connections compatible with the generalized metric on the graded manifold.

1. Introduction

It has been known for some time that Courant algebroids [10] provide a natural framework for the study of several aspects of string theory. First, the string sigma models can be seen [14] as particular cases of the Courant sigma models [7, 12] on manifolds with boundary. Furthermore, the low-energy dynamics of string theory can be described via a suitable generalization of the Ricci tensor and scalar curvature, defined for Courant algebroids [5, 2, 4, 8, 16].

While for some purposes it is enough to consider only a special class of the so-called exact Courant algebroids [13], for the study of dualities (such as the Poisson-Lie T-duality [9]), it is necessary to understand the constructions in the general setup [13]. Such generalized Ricci tensor was introduced (by several different means) and studied in [5, 4, 8, 16]. In particular, its properties were used to provide a proof of the compatibility of the Poisson-Lie T-duality with the RG flow and with the string background equations, as well as to find new solutions to generalized supergravity equations.

On the other hand, it is known [13, 11] that graded geometry offers a much more conceptual viewpoint of Courant algebroids, simplifying the formulas and providing various new insights. It is therefore desirable to also find a formulation of the curvature tensors in the graded language.

Motivated by the recent work [1], where the graded analogues of connections, curvature and torsion were introduced and studied, we propose a simple definition of the generalized Ricci tensor in the graded setup. The purpose of this is twofold – it provides a more conceptual viewpoint of the generalized Ricci tensor and at the same time opens a very concrete door towards generalizations, for example in the context of U-duality, following [3]. This will be explored in a future work.

This note is structured as follows. We start by reviewing the notion of generalized metric, from the perspective of graded geometry. We then recall the necessary definitions from [1], introduce the notion of Levi-Civita connections and present the Ricci tensor. Finally, we explore the relation to the more standard Courant algebroid connections [5] and discuss the exact case, where this Ricci tensor produces the usual one from Riemannian geometry.

Acknowledgements. The author would like to thank Andreas Deser and Pavol Ševera for helpful discussions, suggestions, and comments on the preliminary version of the paper. The author would also like to acknowledge the COST action MP 1405 Quantum structure of spacetime and the Corfu Summer Institute 2019 at EISA, where the main idea of the paper was conceived.

This work was supported by the GAČR Grant EXPRO 19-28628X.
2. Generalized metric

In what follows, we shall use $\mathcal{E}$ to denote an NQ symplectic manifold of degree 2 [15]. This means that $\mathcal{E}$ is an $\mathbb{N}$-graded manifold, equipped with a symplectic form of degree 2, and a degree 1 symplectic vector field $Q_{\mathcal{E}}$, satisfying $Q_{\mathcal{E}}^2 = 0$. There is an associated sequence of fibrations

$$\mathcal{E} = \mathcal{E}_2 \to \mathcal{E}_1 \to \mathcal{E}_0,$$

which corresponds to the subsheafs generated by coordinates of degrees up to 2, up to 1, and up to 0, respectively. In particular, the last arrow gives a vector bundle.

A generalized metric is, in the graded language, a symplectic involution $\iota$ on $\mathcal{E}$, which preserves the basis $\mathcal{E}_0$, i.e. it is a diffeomorphism of $\mathcal{E}$ satisfying

$$\iota^* = \text{id}_{\mathcal{E}}, \quad \iota^* \omega = \omega, \quad \iota|_{\mathcal{E}_0} = \text{id}_{\mathcal{E}_0}.$$

Given a generalized metric, it is always possible to locally choose the coordinates $x^i$, $e^α$, $e^β$, $p_i$ on $\mathcal{E}$ of degrees 0, 1, 1, 2, respectively, such that

$$\omega = dp_i dx^i + de_a dx^a + de_e dx^e,$$

$$\iota^* x^i = x^i, \quad \iota^* p_i = p_i, \quad \iota^* e^α = e^α, \quad \iota^* e^β = -e^β,$$

where $e_a := g_{ab} e^b$, $e_e := g_{ab} e^b$ for some constants $g_{ab}$, $g_{ab}$. We will sometimes denote the coordinates $e^α$, $e^β$ collectively as $e^{α, β}$.

It is easy to see that $Q_{\mathcal{E}}$ is always Hamiltonian, i.e. it comes from a degree 3 function $H$. The most general such function has the form

$$H = \rho_{αβ}(x) p_α e^α - \frac{1}{6} c_{αβγ}(x) e^α e^β e^γ,$$

giving

$$Q_{\mathcal{E}} = \rho_{αβ} e^α \partial_α + (\rho^{i}_{α} p_i - \frac{1}{6} c_{αβγ} e^β e^γ) \partial_α + (\frac{1}{6} c_{αβγ} e^α e^β e^γ - \rho^{i}_{αβ} e^α p_i) \partial_i,$$

where, in setting $e_a := g_{αβ} e^β$, we extended $g_{ab}$, $g_{ab}$ by adding $g_{ab} = g_{ab} = 0$. The condition $Q_{\mathcal{E}}^2 = 0$ translates to the classical master equation $\{H, H\} = 0$.

Since $\iota$ preserves the degree, it induces an involution on $\mathcal{E}_1$. Because of the base-fixing condition, $\iota$ is in turn fully determined by the corresponding fixed point set $V_1 \subset \mathcal{E}_1$. Seeing $V_1$ as a vector bundle over $\mathcal{E}_0$, we can pull it back along $\mathcal{E} \to \mathcal{E}_0$ to obtain a bundle $V \to \mathcal{E}$. The latter bundle is locally described by coordinates $x^i$, $e^α$, $ξ^α$, $e^β$, $p_i$, with one copy of $ξ^α$, deg $ξ^α = 1$, corresponding to each $e^α$. Finally, notice that $\mathcal{E}_1, V_1$, and $V$ are all symplectic vector bundles.

**Remark.** The relation to Courant algebroids is as follows [11, 13]. The Courant algebroid is given by the ordinary vector bundle $E = V_1[-1]$ over $\mathcal{E}_0$. (In other words, $x^i$ are coordinates on the base and $e^α$ correspond to the linear coordinates on the fibers of the algebroid.) The coefficients $c$ and $ρ$ give the structure functions of the bracket and the anchor, respectively, while $g$ encodes the fiberwise inner product. Finally, the involution $\iota$ corresponds to the usual viewpoint of generalized metric as a fiberwise reflection on the Courant algebroid, or equivalently, as a subbundle $V_+ = V_1[-1] \subset E$.

---

1. i.e. an ordinary manifold $\mathcal{E}_0$ together with a sheaf of $\mathbb{N}$-graded commutative algebras, locally of the form $C^∞(U) \otimes S(V)$, with $S(V)$ the free graded commutative algebra generated by a finite-dimensional vector space $V = \bigoplus_{i=1}^m V_m$, and $U$ an open subset of $\mathcal{E}_0$

2. The proof goes as follows: Using the (graded) Darboux theorem we first find coordinates $x^i$, $e^α$, $p_i$ such that $ω = dp_i dx^i + g_{αβ} de^α dx^β$, for $g_{αβ}$ a diagonal matrix with only 1 and $-1$ on the diagonal. It follows that the (x-dependent) matrix $R_{αβ}^i$ defined by $\iota^* e^α = R_{αβ}^i(x) e^β$ is idempotent and orthogonal w.r.t. $g$, and thus can be made into the form diag(1,...,1,−1,...,−1) using $e^{α, β} = O_{β}^α(x) e^β$, for $O$ orthogonal. Making an appropriate shift of $p_i$, the form of $ω$ is preserved in the new coordinates.
3. Tautological section and contraction

Let us now denote the vector bundle morphism $\mathcal{V} \to \mathcal{V}_1$ by $\varphi$. There is a unique section $\iota : \mathcal{E} \to \mathcal{V}$ such that $\varphi \circ \iota$ coincides with the map $\mathcal{E} \to \mathcal{E}_1 \to \mathcal{V}_1$ (the last arrow is the orthogonal projection). Since the bundle $\mathcal{V}$ is symplectic, we get an induced section $\tau$ on the dual bundle $\mathcal{V}^* \to \mathcal{E}$. We call $\tau$ the tautological section [1]. More concretely, identifying sections of $\mathcal{V}^*$ with functions on $\mathcal{V}$ which are linear in the fiber coordinates, we get

$$\tau = c_a \xi^a.$$

The involution on $C^\infty(\mathcal{V})$ (induced by $\iota$) allows us to split any vector field on $\mathcal{V}$ into the sum of its self-dual and anti-self-dual part. We will denote the anti-self-dual part of a vector field $D$ by $\pi D$. Let us now consider a special subspace $\text{End}_2$ of the space of vector fields, given by degree 2 bundle endomorphisms of $\mathcal{V}^*$. Locally we have

$$D = D^a_{\beta}(x)p_{\xi}^b \xi^b \xi^c \partial_c + \frac{1}{2} D^a_{\alpha\beta}(x) \epsilon^{\alpha} \epsilon^{\beta} \partial_c \xi^c \mapsto D^a_{\alpha\beta}(x) \epsilon^{\alpha} \epsilon^{\beta} \partial_c \xi^c.$$

Writing $\mathcal{V}^*_1 \subset \mathcal{E}_1$ for the subbundle (over $\mathcal{E}_0$) perpendicular to $\mathcal{V}_1 \subset \mathcal{E}_1$, we have an identification (we use $\Gamma$ for the space of sections)

$$\pi(\text{End}_2) \cong \Gamma(\mathcal{V}^*_1 \otimes \mathcal{V}^*_1 \otimes \text{End}(\mathcal{V}^*_1)) \cong \Gamma(\mathcal{V}^*_1 \otimes \mathcal{V}^*_1 \otimes \mathcal{V}_1 \otimes \mathcal{V}^*_1).$$

We define the contraction map $C : \text{End}_2 \to \Gamma(\mathcal{V}^*_1 \otimes \mathcal{V}^*_1)$ as the projection $\pi$ followed by the contraction of the first and third factor in the last expression. Explicitly,

$$C : D \mapsto \partial_{\psi}(\pi D) \xi^a.$$

4. Connections, torsion and curvature

Following [1], a connection on a $\mathcal{V}^* \to \mathcal{E}$ is a degree 1 vector field $Q$ on $\mathcal{V}$, which projects to $Q_\mathcal{E}$, and which preserves the space of sections $\Gamma(\mathcal{V}^*) \subset C^\infty(\mathcal{V})$. The torsion is then a particular section of $\mathcal{V}^*$, defined as $Q \tau$. The curvature of $Q$ is the vector field $Q^2 \equiv \tfrac{1}{2}[Q, Q]$ on $\mathcal{V}$. One easily sees that $Q^2 \in \text{End}_2$.

We will say that a connection $Q$ on $\mathcal{V}$ is Levi-Civita if its torsion is invariant under the induced involution on $\Gamma(\mathcal{V}^*)$. Finally, we define the Ricci tensor Ric by

$$\text{Ric} := CQ^2 \in \Gamma(\mathcal{V}^*_1 \otimes \mathcal{V}^*_1).$$

**Remark.** The contraction corresponds to the usual procedure for obtaining the Ricci tensor from the Riemann tensor. The insertion of the projection $\pi$ keeps only the part of the tensor which can be identified, via $\Gamma(\mathcal{V}^*_1 \otimes \mathcal{V}^*_1) \cong \text{Hom}_{\xi_0}(\mathcal{V}_1, \mathcal{V}_1^*)$, with an infinitesimal deformation of the generalized metric.\(^3\) This is the generalized Ricci flow [17].

Passing again to a coordinate description, a general connection on $\mathcal{V}^*$ has the form

$$Q = Q_\mathcal{E} + \psi^a_{\lambda\beta}(x) \epsilon^{\alpha} \xi^b \partial_c \xi^c.$$

Its torsion is

$$Q \tau = \rho^a_{\lambda\beta} p_{\xi}^c \epsilon^{\alpha} \epsilon^{\beta} \epsilon^{\gamma} \xi^b \epsilon^{\lambda} \psi^a_{\gamma\beta} \epsilon^{\alpha} \xi^b + \psi^a_{\lambda\beta} \epsilon^{\alpha} \epsilon^{\beta} \epsilon^{\gamma} \xi^b \epsilon^{\lambda} \psi^a_{\gamma\beta} \epsilon^{\alpha} \xi^b.$$

Since the involution preserves the coordinates $\xi^a$, the invariance of torsion is equivalent to the constraint

$$\psi^a_{\lambda\beta} = \epsilon^{a}_{\lambda\beta}.$$

In particular, the coefficients $\psi^a_{\lambda\beta}$ are left unrestricted. However, as we will see, in the Levi-Civita case the curvature only depends on $\psi^a_{\lambda\beta}$ through the trace

$$\lambda^a_{\beta} := \psi^a_{\lambda\beta}.$$\(^3\) It also allows us to define the contraction.
Concretely, a short calculation (see Appendix) reveals that for a Levi-Civita connection \( Q \),
\[
\text{Ric} = (e^c_{ab} \lambda_c - \rho^a_b e^a_{ba,i} + \rho^a_b \lambda_{b,i} + c_{ca} e^a_{b} e^b) \xi^c e^a.
\]

This recovers exactly the formula for the generalized Ricci tensor from [16].

**Remark.** More precisely, using the notation of [16] we have
\[
\text{Ric} = GRic_{\circ} \text{div}(e_a, e_a) \xi^a e^a,
\]
where the divergence operator is given by \( \text{div}(e_a) = -\lambda_a \).

5. **Connection with Courant algebroid connections**

From the definition it follows that connections are in one-to-one correspondence with linear maps\(^4\) \( Q : \Gamma(V^\ast) \to \Gamma_2(V^\ast) \), satisfying \( Q(fu) = f(Qu) + (Q_e f)u \), for \( f \in C^\infty_0(\mathcal{E}) \), \( u \in \Gamma(V^\ast) \). Since \( C^\infty_0(\mathcal{E}) \cong \Gamma(E^\ast) \), \( C^\infty_1(\mathcal{E}) \cong \Gamma((E^\ast)^\ast) \), \( \Gamma_1(V^\ast) \cong \Gamma(V^\ast_+) \), \( \Gamma_2(V^\ast) \cong \Gamma(E^\ast \otimes V^\ast_+) \), we can understand \( Q \) as a map
\[
\nabla : \Gamma(V^\ast_+) \to \Gamma(E^\ast \otimes V^\ast_+), \quad \text{such that} \quad \nabla(fu) = f\nabla u + (Q_e f)u.
\]
Dually, we have a map \( \nabla : \Gamma(V_+) \to \Gamma(E^\ast \otimes V^\ast) \) (satisfying the same Leibniz identity). This is known in the literature as the **Courant algebroid (or generalized) connection** [5].

If \( Q \) is Levi-Civita, then it is uniquely determined (c.f. (1)) by the restriction of \( \nabla \) to a map \( \Gamma(V_+) \to \Gamma(V^\ast_+ \otimes V^\ast_+) \).

6. **The exact case**

Let us now consider the following example [13, 14].\(^5\) First,
\[
\mathcal{E} = T^*[2]T[1]E_0, \quad H = d + \eta, \quad \text{for} \ \eta \in \Omega^2_\text{closed}(E_0),
\]
with the standard symplectic form on the cotangent bundle. Here \( d \) is understood as a vector field on \( T[1]E_0 \) (thus a function on \( T^*[2]T[1]E_0 \)), and \( \eta \in \Omega(M) \cong C^\infty(T[1]M) \) pulled back to \( T^*[2]T[1]E_0 \). (Such \( \mathcal{E} \) is called exact.) We fix the generalized metric by the requirement that the submanifold \( V_1 \subset \mathcal{E}_1 \cong (T \oplus T^\ast)[1]E_0 \) corresponds to the graph of a (pseudo Riemannian) metric \( \langle \cdot, \cdot \rangle \) on \( E_0 \).

In particular, we have a vector-bundle isomorphism \( T^\ast E_0 \cong V_+ \). Combining this with the result of the previous section, a Levi-Civita connection on \( V^\ast \) is now given by an ordinary connection on \( T^\ast E_0 \). Let us now take \( Q \) given by the Levi-Civita connection (in the usual sense) on \( T^\ast E_0 \) w.r.t. the metric on \( E_0 \).

Choose a local frame \( E_a \) on \( T^\ast E_0 \) satisfying \( \langle E_a, E_b \rangle = \pm \delta_{ab} \). We define the frame \( F_a \) on \( T^\ast E_0 \) by \( F_a := \langle E_a, \cdot \rangle \) and we denote by \( E^a, F^a \) the induced fiber coordinates on \( \mathcal{E}_1 \cong (T \oplus T^\ast)[1]E_0 \). Finally, setting \( e^a := \frac{1}{2}(E^a + F^a) \), \( e^b := \frac{1}{2}(E^b - F^b) \), we have
\[
\text{Ric} = \text{Ric}_\eta(E_a, E_b) e^a e^b,
\]
where \( \text{Ric}_\eta \) is the (usual) Ricci tensor on \( T^\ast E_0 \) for the metric connection (w.r.t. \( \langle \cdot, \cdot \rangle \)) with torsion given by \( \eta \). For the proof of this fact we refer the reader to [16]. It can also be verified by a direct calculation.

\(^4\)We use subscript (for \( C^\infty \) and \( \Gamma \)) to denote the degree.

\(^5\)Every \( \mathcal{E} \) with vanishing tangential cohomology and a generic generalized metric can be put into this form [13, 6].
Using the fact that $Q^2 = 0$, we have (for a general $Q$)

$$
Q^2 = (Q \xi \psi^a ba e^a \xi - \psi^b c d e^b \xi \psi^a ba e^a) \partial \xi^a
$$

$$
= (\rho^a_{c d} \psi^a ba, i e^a \xi - Q \xi \psi^b a \xi \xi^b - \psi^b c d e^b \xi \psi^a ba e^a) \partial \xi^a
$$

$$
\pi Q^2 = (\rho^a_{c d} \psi^a ba, i e^a \xi + \rho^a_{c d} \psi^a ba, i e^a \xi - c_{a d a} \psi^a a \xi \xi^b + \psi^b c d \psi^a ba e^a \xi)
$$

$$
CQ^2 = (\rho^a_{c d} \psi^a ba, i - \rho^a_{c d} \psi^a ba, i - c_{a d a} \psi^a a \xi \xi^b + \psi^b c d \psi^a ba e^a \xi)
$$

Using $\psi^a ba = c^a ba$ and $\psi^a ba = \lambda_b$ we recover (2).

REFERENCES

[1] P. Aschieri, F. Bonechi, A. Deser, On Curvature and Torsion in Courant Algebroids, arXiv:1910.11273 [math.DG]

[2] A. Coimbra, C. Strickland-Constable, D. Waldram, Supergravity as generalised geometry I: type II theories, J. High Energ. Phys. (2011) 091.

[3] A. Coimbra, C. Strickland-Constable, D. Waldram, $E_{d(d)} \times \mathbb{R}^+$ generalized geometry, connections and M-theory, JHEP 1402 (2014) 054.

[4] M. García-Fernandez, Torsion-free generalized connections and Heterotic Supergravity, (2014). Communications in Mathematical Physics 332 (1), 89-115.

[5] M. Gualtieri, Branes on Poisson Varieties. A Tribute to Nigel Hitchin. (2007). Preprint arXiv:0710.2719 [math.DG]

[6] M. Gualtieri, Generalized Kähler geometry, Commun. Math. Phys. 331 (2014) no.1, 297-331.

[7] N. Iekeda, Chern-Simons Gauge Theory coupled with BF Theory, Int.J.Mod.Phys. A18 (2003) 2689-2702.

[8] B. Jurco, J. Vysoky, Courant Algebroid Connections and String Effective Actions, in: T. Voronov, ed., Quantization, Poisson Brackets and Beyond (Manchester, 2001), Contemporary Mathematics, vol. 315, American Mathematical Society, Providence, RI (2002), pp. 169-185

[9] D. Roytenberg, On the structure of graded symplectic supermanifolds and Courant algebroids, in T. Voronov, ed., Quantization, Poisson Brackets and Beyond (Manchester, 2001), Contemporary Mathematics, vol. 315, American Mathematical Society, Providence, RI (2002), pp. 169-185

[10] D. Roytenberg, AKSZ–BV Formalism and Courant Algebroid-Induced Topological Field Theories, Lett.Math.Phys. 79 (2007) 143-159.

[11] P. Severa, Letters to Alan Weinstein about Courant algebroids, 1998-2000, arXiv:1707.00265

[12] P. Severa, Poisson-Lie T-duality as a boundary phenomenon of Chern-Simons theory, JHEP 1605 (2016) 044

[13] P. Severa, Some title containing the words “homotopy” and “symplectic”, e.g. this one, Travaux Mathématiques, Fasc. XVI (2005), pp. 121-137, also math.SG/0105080.

[14] P. Severa, F. Valach, Courant algebroids, Poisson-Lie T-duality, and type II supergravities, arXiv:1810.07763.

[15] P. Severa, F. Valach, Ricci flow, Courant algebroids, and renormalization of Poisson-Lie T-duality, Letters in Mathematical Physics, 1-13, (2017).