New lower bounds for the minimum M-eigenvalue of elasticity M-tensors and applications

Haitao Che¹, Haibin Chen²*, Na Xu³ and Qingni Zhu⁴

¹Correspondence: chenhabin508@qfnu.edu.cn
²School of Management Science, Qufu Normal University, Rizhao, Shandong 276800, China
Full list of author information is available at the end of the article

Abstract

M-eigenvalues of elasticity M-tensors play an important role in nonlinear elasticity and materials. In this paper, we present several new lower bounds for the minimum M-eigenvalue of elasticity M-tensors and propose numerical examples to illustrate the efficiency of the obtained results. As applications, we provide several checkable sufficient conditions for the strong ellipticity and positive definiteness of irreducible elasticity M-tensors.

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1 Introduction

A tensor \( A = (a_{ijk}) \in E_{4n} \) is called a fourth-order real partially symmetric tensor if

\[
a_{ijkl} = a_{jikl} = a_{ijlk}, \quad i, j, l, k \in [n],
\]

where \([n] = \{1, 2, \ldots, n\}\). The tensor of elastic moduli for a linearly anisotropic elastic solid is a fourth-order real partially symmetric tensor [1], and the components of such a tensor are considered as the coefficients of the following optimization problem:

\[
\begin{align*}
\min \quad & f(x, y) = Axyx = \sum_{i,j,k,l \in [n]} a_{ijkl} x_i y_j x_k y_l, \\
\text{s.t.} \quad & x^T x = 1, \quad y^T y = 1, \\
& x, y \in \mathbb{R}^n.
\end{align*}
\]

Problem (1.1) has applications in the ordinary ellipticity and strong ellipticity and nonlinear elastic materials analysis [2–28]. The strong ellipticity condition is stated as \( f(x, y) > 0 \) for all nonzero vectors \( x, y \in \mathbb{R}^n \), which guarantees the existence of solutions of basic boundary-value problems of elastostatics and ensures an elastic material to satisfy some mechanical properties [29]. In fact, the KKT condition of (1.1) can be regarded as the following definition of M-eigenvalues.

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Definition 1.1 ([1]) Let $A \in \mathbb{E}_{4,n}$. If there are $\lambda \in \mathbb{R}$ and $x, y \in \mathbb{R}^n \setminus \{0\}$ such that

\[
\begin{align*}
Axy^2 &= \lambda x, \\
Ax^2y &= \lambda y, \\
x^Tx &= 1, \\
y^Ty &= 1,
\end{align*}
\]

(1.2)

where $(Axy^2)_i = \sum_{j,k,l \in [n]} a_{ijkl} x_j y_k$, and $(Ax^2y)_l = \sum_{i,j,k \in [n]} a_{ijkl} x_i y_j$, then the scalar $\lambda$ is called an M-eigenvalue of $A$, and $x, y$ are called the corresponding left and right M-eigenvectors of $A$, respectively.

Furthermore, Han et al. revealed that the strong ellipticity condition holds if and only if the smallest M-eigenvalue is positive [1]. Recently, Ding et al. [30] investigated a fourth-order structured partially symmetric tensors named elasticity M-tensors, and some sufficient conditions for the strong ellipticity were provided. Since the strong ellipticity condition and M-positive definiteness can be identified by the smallest M-eigenvalue, He et al. [31] proposed some lower bounds for the minimum M-eigenvalue of elasticity M-tensors.

In this paper, we present several new bounds for the minimum M-eigenvalue of elasticity M-tensors. We prove that the bounds are tighter than those proposed in [31]. Numerical examples illustrate the efficiency of the obtained results. As applications, we give some checkable sufficient conditions for the strong ellipticity and positive definiteness of elasticity tensors.

2 Main results

For an elasticity tensor $A \in \mathbb{E}_{4,n}$, its M-spectral radius is denoted by

\[
\rho(A) = \max \{|\lambda| : \lambda \text{ is an M-eigenvalue of } A\}.
\]

The identity tensor $I = (e_{ijkl}) \in \mathbb{E}_{4,n}$ is defined by

\[
e_{ijkl} = \begin{cases} 
0 & \text{if } i = j, k = l, \\
1 & \text{otherwise.}
\end{cases}
\]

Let $\alpha_i = \max_{k \in [n]} |a_{ilk}|$, $\beta_i = \max_{k \in [n]} |a_{ilk}|$, and

\[
\begin{align*}
\gamma_i &= \sum_{j \in [n], j \neq i} \max_{k \in [n]} |a_{ijkl}|, \\
r_i(A) &= \sum_{j,k \in [n], k \neq l} |a_{ijkl}|, \\
r_i'(A) &= \sum_{k,l \in [n], k \neq l} |a_{ikll}|, \\
c_i(A) &= \sum_{i,j \in [n], i \neq j} |a_{ijkl}|, \\
c_i'(A) &= \sum_{i,k \in [n], i \neq k} |a_{ijkl}|,
\end{align*}
\]

To continue, we need the following definitions and technical results.
**Definition 2.1** ([30]) A tensor $A \in \mathbb{E}_{m,n}$ is called an elasticity M-tensor if there exist a nonnegative tensor $B \in \mathbb{E}_{m,n}$ and a real number $s \geq \rho(B)$ such that $A = sI - B$, where $\rho(B)$ is the M-spectral radius of $B$. Furthermore, if $s > \rho(B)$, then $A$ is called a nonsingular elasticity M-tensor.

**Definition 2.2** ([32]) A tensor $A = (a_{i_1i_2...i_m})$ of order $m$ and dimension $n$ is called reducible if there exists a nonempty proper index subset $J = \{1, 2, \ldots, n\} \subset [n]$ such that

$$a_{i_1i_2...i_m} = 0, \quad \forall i_1 \in J, \forall i_2 \ldots i_m \in [n] \setminus J.$$ 

If $A$ is not reducible, then we say that $A$ is irreducible.

**Theorem 2.1** ([31]) Let $A = (a_{ijkl}) \in \mathbb{E}_{m,n}$ be an irreducible and nonnegative partially symmetric tensor, and let $\tau(A)$ be the minimal M-eigenvalue of $A$. Then $\tau(A) \geq 0$ is an M-eigenvalue of $A$ with positive eigenvectors. Moreover, there exist a nonnegative tensor $B$ and a real number $c \geq \rho(B)$ such that $A = cI - B$.

**Theorem 2.2** ([31]) Let $A = (a_{ijkl}) \in \mathbb{E}_{m,n}$ be an irreducible elasticity M-tensor. Then

$$\tau(A) \leq \min_{i \in [n]} \{a_{iij} \}.$$ 

**Theorem 2.3** ([31]) Let $A = (a_{ijkl}) \in \mathbb{E}_{m,n}$ be an irreducible elasticity M-tensor. Then

$$\tau(A) \geq \max \left\{ \min_{i \in [n]} \{a_{ijl} - R_i(A)\}, \min_{i \in [n]} \{\beta_i - C_i(A)\} \right\}.$$ 

Now we are in a position to propose some lower bounds for $\tau(A)$.

**Theorem 2.4** Let $A = (a_{ijkl}) \in \mathbb{E}_{m,n}$ be an irreducible elasticity M-tensor. Then the minimum M-eigenvalue satisfies

$$\tau(A) \geq \max \left\{ \min_{i,j \in [n], i \neq j} \{\eta_1(A)\}, \min_{k,l \in [n], k \neq l} \{\eta_2(A)\} \right\},$$

where $\eta_1(A) = \frac{a_{ij} - R_i(A)}{2} - \frac{\Delta_{ij}}{2}$, $\eta_2(A) = \beta_i - \frac{c_i(A)}{2}$, and

$$\Delta_{ij} = (\alpha_i - R_i(A) - \alpha_j)^2 + 4(\alpha_i - R_i(A) - \alpha_j)(\gamma_j)R_i(A),$$

$$\delta_i = (\beta_k - c_i(A) - \beta_l)^2 + 4(\beta_k - c_i(A) - \beta_l)(\delta_i)C_i(A).$$

**Proof** By Theorem 2.1 suppose that $x = (x_i)_{i=1}^n > 0 \in \mathbb{R}^n$ and $y = (y_i)_{i=1}^n > 0 \in \mathbb{R}^n$ are the corresponding left and right M-eigenvectors, respectively. Let $x_p \geq x_q \geq \max_{i \in [n], j \in [p]} \{x_i\}.$
From the $p$th equation of $Ax^2 = \tau(A)x$ in (1.2) we obtain

\[
\tau(A)x_p = \sum_{j,k,l \in [n]} a_{pjk}x_jy_l
\]

\[
= \sum_{k,l \in [n], k \neq l} a_{pkl}x_ky_l + \sum_{j,k,l \in [n], j \neq k, k \neq l} a_{pjk}x_jy_l
\]

\[
+ \sum_{j,k,l \in [n], j \neq p} a_{pjl}y_j^2 + \sum_{l \in [n]} a_{plp}y_l^2,
\]

that is,

\[
\sum_{l \in [n]} a_{plp}y_l^2 - \tau(A)x_p
\]

\[
= - \sum_{k,l \in [n], k \neq l} a_{pkl}x_ky_l - \sum_{j,k,l \in [n], j \neq k, k \neq l} a_{pjk}x_jy_l - \sum_{j,l \in [n], j \neq p} a_{pjl}y_j^2.
\]

Let $\alpha_p = \min_{l \in [n]} \{a_{plp}\}$. It follows from Theorem 2.2 that

\[
0 \leq (\alpha_p - \tau(A))x_p \leq \left( \sum_{l \in [n]} a_{plp}y_l^2 - \tau(A) \right)x_p
\]

\[
\leq \sum_{k,l \in [n], k \neq l} |a_{pkl}|x_k + \sum_{j,k,l \in [n], j \neq k, k \neq l} |a_{pjk}|x_j + \sum_{j,l \in [n], j \neq p} |a_{pjl}|x_j^2.
\]

Note that

\[
\sum_{l \in [n]} |a_{plp}|y_l^2 = \sum_{l \in [n]} \left( \sum_{l \in [n]} |a_{plp}|y_l^2 \right)x_l
\]

\[
\leq \sum_{l \in [n]} \max_{j \in [n]} |a_{plp}| \left( \sum_{l \in [n]} y_l^2 \right)x_l
\]

\[
= \sum_{l \in [n]} \max_{j \in [n]} \{a_{plp}\}x_l.
\]

Furthermore,

\[
(\alpha_p - \tau(A) - r_p^p(A))x_p \leq (r_p(A) - r_p^p(A) + y_p)x_p.
\]  

(2.1)

From the $s$th equation of $Ax^2 = \tau(A)x$ in (1.2) we have

\[
\tau(A)x_s = \sum_{j,k,l \in [n]} a_{sjk}x_jy_l
\]

\[
= \sum_{j,k,l \in [n], k \neq l} a_{skl}x_ky_l + \sum_{j,l \in [n], j \neq s} a_{sjl}y_j^2 + \sum_{l \in [n]} a_{ssl}y_l^2x_s.
\]

Let $\alpha_s = \min_{l \in [n]} \{a_{ssl}\}$. It follows from Theorem 2.2 that

\[
(\alpha_s - \tau(A))x_s \leq R_s(A)x_p.
\]  

(2.2)
Multiplying (2.1) and (2.2), we have
\[
(\alpha_p - \tau(A) - r_p^p(A)) (\alpha_s - \tau(A)) \leq (r_p(A) - r_p^p(A) + \gamma_p) R_s(A),
\]
which means that
\[
\tau(A) \geq \frac{\alpha_p - r_p^p(A) + \alpha_s - \Delta_{\rho,s}}{2}, \tag{2.3}
\]
where \(\Delta_{\rho,s} = (\alpha_p - r_p^p(A) - \alpha_s)^2 + 4(r_p(A) - r_p^p(A) + \gamma_p) R_s(A)\).

On the other hand, let \(|y_q| \geq |y_1| \geq \max_{i \in [n], j \neq i} |y_j|\). From the qth equation of \(Ax^2y = \tau(A)y\) in (1.2) it follows that
\[
\tau(A)y_q = \sum_{i,j,k \in [n]} a_{ijk} x_i x_j y_k
\]
\[
= \sum_{i,j,k \in [n], j \neq q} a_{ijk} x_i x_j y_k + \sum_{i,k \in [n], j \neq q} a_{ikq} x_i y_k + \sum_{k \in [n], j \neq q} a_{ikq} y_k + \sum_{i \in [n]} a_{iiq} y_q x_i^2.
\]
Let \(\beta_q = \min_{i \in [n]} \{|a_{iiq}|\}\). It follows from Theorem 2.2 that
\[
0 \leq (\beta_q - \tau(A))y_q \leq \left(\sum_{i \in [n]} a_{iiq} y_q x_i^2 - \tau(A)\right)y_q
\]
\[
= - \sum_{i \in [n], j \neq q} a_{ijq} x_i y_q - \sum_{i,k \in [n], j \neq q} a_{ijk} x_i y_k - \sum_{k \in [n], j \neq q} a_{ikq} y_k
\]
\[
\leq \sum_{i \in [n], j \neq q} |a_{ijq}| y_q + \sum_{i,k \in [n], j \neq q} |a_{ijk}| y_k + \sum_{k \in [n], j \neq q} |a_{ikq}| y_k,
\]
that is,
\[
(\beta_q - \tau(A) - c_q^2(A))y_q \leq (c_q(A) - c_q^2(A) + \delta_q) y_1. \tag{2.4}
\]
From the tth equation of \(Ax^2y = \tau(A)y\) in (1.2) we obtain
\[
\tau(A)y_t = \sum_{i,j,k \in [n]} a_{ijk} x_i x_j y_k
\]
\[
= \sum_{i,j,k \in [n], j \neq t} a_{ijk} x_i x_j y_k + \sum_{i,k \in [n], j \neq t} a_{ikt} x_i y_k + \sum_{k \in [n], j \neq t} a_{ikt} y_k + \sum_{i \in [n]} a_{iit} x_i^2 y_t.
\]
Let \(\beta_t = \min_{i \in [n]} \{|a_{iit}|\}\). This yields
\[
(\beta_t - \tau(A))y_t \leq C_t(A) y_q. \tag{2.5}
\]
Multiplying (2.4) and (2.5), we have
\[
(\beta_q - \tau(A) - c_q^2(A)) (\beta_t - \tau(A)) \leq (c_q(A) - c_q^2(A) + \delta_q) C_t(A), \tag{2.6}
\]
which means that
\[ \tau(A) \geq \frac{\beta_q - c^0_p(A) + \beta_1 - \Theta_q^2}{2}, \]  
(2.7)
where \( \Theta_q^2 = (\beta_q - c^0_p(A) - \beta_1)^2 + 4(c_p(A) - c^0_p(A) + \delta_q)C(A). \) Then the conclusion follows. \( \square \)

Next, we compare the bound in Theorem 2.3 with that in Theorem 2.4 and obtain the following conclusion.

**Theorem 2.5** Let \( A = (a_{ijk}) \in \mathbb{R}_{kn} \) be an irreducible elasticity M-tensor. Then
\[ \tau(A) \geq \max \left\{ \min_{i,j \in [n], i \neq j} \{ \eta_1(A), \eta_2(A) \} \right\} \]
\[ \geq \max \left\{ \min_{i \in [n]} \{ \alpha_i - R(A), \beta_i - C(A) \} \right\}. \]

**Proof** We first show that \( \min_{i,j \in [n], i \neq j} \{ \eta_1(A) \} \geq \min_{i \in [n]} \{ \alpha_i - R(A) \} \) and divide the argument into two cases.

Case 1. For any \( i, j \in [n], i \neq j \), if \( \alpha_i - R_i(A) \leq \alpha_j - R_j(A) \), then
\[ \alpha_i - \alpha_j + R_i(A) \geq R_j(A) \geq 0. \] 
(2.8)

From (2.8) we deduce
\[ (\alpha_i - r_i(A) - \alpha_j)^2 + 4(r_i(A) - r_j(A) + \gamma_i)R_j(A) \]
\[ \leq (\alpha_i - r_i(A) - \alpha_j)^2 + 4(r_i(A) - r_j(A) + \gamma_i)(\alpha_j - \alpha_i + R_j(A)) \]
\[ = (\alpha_i - r_i(A) - \alpha_j)^2 + 4(R_i(A) - r_j(A))(\alpha_j - \alpha_i + R_j(A)) \]
\[ = (\alpha_i - r_i(A) - \alpha_j)^2 + 4(R_i(A) - r_j(A))(\alpha_j - \alpha_i + r_j(A) - r_j(A) + R_j(A)) \]
\[ = (\alpha_i - r_i(A) - \alpha_j)^2 + 4(R_i(A) - r_j(A))(\alpha_j - \alpha_i + r_j(A)) + 4(R_i(A) - r_j(A))^2 \]
\[ = (\alpha_j - \alpha_i - r_j(A) + 2R_i(A))^2. \]

Thus
\[ \frac{1}{2} \left( \alpha_i - r_i(A) + \alpha_j - \sqrt{(\alpha_i - r_i(A) - \alpha_j)^2 + 4(r_i(A) - r_j(A) + \gamma_i)R_j(A)} \right) \]
\[ \geq \frac{1}{2} \left( \alpha_i - r_i(A) + \alpha_j - (\alpha_j - \alpha_i - r_j(A) + 2R_i(A)) \right) \]
\[ = \alpha_i - R_i(A), \]
which means that
\[ \frac{1}{2} \min_{i,j \in [n], i \neq j} \left\{ \alpha_i - r_i(A) + \alpha_j - \sqrt{(\alpha_i - r_i(A) - \alpha_j)^2 + 4(r_i(A) - r_j(A) + \gamma_i)R_j(A)} \right\} \]
\[ \geq \min_{i \in [n]} \{ \alpha_i - R_i(A) \}. \]
Case 2. For any \( i, j \in [n], i \neq j \), if \( \alpha_i - R_i(A) \geq \alpha_j - R_j(A) \), then

\[
\alpha_i - r_i(A) - \alpha_j + R_j(A) \geq \gamma_i. \tag{2.9}
\]

From (2.9) we have

\[
\begin{align*}
&\left( \alpha_i - r_i(A) - \alpha_j \right)^2 + 4 \left( r_i(A) - r_i(A) + \gamma_i \right) R_i(A) \\
\leq &\left( \alpha_i - r_i(A) - \alpha_j \right)^2 + 4 \left( r_i(A) - r_i(A) + \alpha_i - r_i(A) - \alpha_j + R_j(A) \right) R_j(A) \\
= &\left( \alpha_i - r_i(A) - \alpha_j \right)^2 + 4 \left( \alpha_i - r_i(A) - \alpha_j + R_j(A) \right) R_j(A) \\
= &\left( \alpha_i - r_i(A) - \alpha_j + 2 R_j(A) \right)^2.
\end{align*}
\]

Then

\[
\begin{align*}
&\frac{1}{2} \left( \alpha_i - r_i(A) + \alpha_j - \sqrt{\left( \alpha_i - r_i(A) - \alpha_j \right)^2 + 4 \left( r_i(A) - r_i(A) + \gamma_i \right) R_i(A)} \right) \\
\geq &\frac{1}{2} \left( \alpha_i - r_i(A) + \alpha_j - \left( \alpha_i - r_i(A) - \alpha_j + 2 R_j(A) \right) \right) \\
= &\alpha_j - R_j(A),
\end{align*}
\]

which implies

\[
\frac{1}{2} \min_{i,j \in [n], i \neq j} \left\{ \alpha_i - r_i(A) + \alpha_j - \sqrt{\left( \alpha_i - r_i(A) - \alpha_j \right)^2 + 4 \left( r_i(A) - r_i(A) + \gamma_i \right) R_i(A)} \right\} \\
\geq &\min_{j \in [n]} \{ \alpha_j - R_j(A) \}.
\]

Therefore we obtain \( \min_{j \in [n], i \neq j} \{ \eta_i(A) \} \geq \min_{i \in [n]} \{ \alpha_i - R_i(A) \} \).

Similarly, we have \( \min_{i,j \in [n], i \neq j} \{ \eta_j(A) \} \geq \min_{l \in [n]} \{ \beta_l - C_l(A) \} \). Thus we deduce

\[
\max \left\{ \min_{i,j \in [n], i \neq j} \{ \eta_i(A) \}, \min_{k,l \in [n], k \neq l} \{ \eta_l(A) \} \right\} \geq \max \left\{ \min_{i \in [n]} \{ \alpha_i - R_i(A) \}, \min_{l \in [n]} \{ \beta_l - C_l(A) \} \right\},
\]

and the desired result follows. \( \square \)

In what follows, we propose another lower bound for \( \tau(A) \).

**Theorem 2.6** Let \( A = (a_{i,j}) \in \mathbb{E}_{k,n} \) be an irreducible elasticity \( M \)-tensor. Then

\[
\tau(A) \geq \max \left\{ \min_{i,j \in [n], i \neq j} \{ \theta_1(A), \alpha_i - r_i(A), \alpha_j - r_j(A) \}, \right.
\]

\[
\left. \min_{k,l \in [n], k \neq l} \{ \theta_2(A), \beta_k - c_k^I(A), \beta_l - c_l^I(A) \} \right\},
\]

where

\[
\theta_1(A) = \frac{(\alpha_i - r_i(A)) + (\alpha_j - r_j(A)) - \Omega_i^j}{2},
\]

\[
\theta_2(A) = \frac{(\beta_k - c_k^I(A)) + (\beta_l - c_l^I(A)) - \Phi_i^j}{2},
\]
\[
\Omega_{ij} = (\alpha_i - \tau_i(A) - (\alpha_j - \tau_j(A)))^2 + 4(R_i(A) - \tau_i(A))(R_j(A) - \tau_j(A)),
\]

and

\[
\Phi_{kl} = (\beta_k - c_i^k(A) - (\beta_l - c_i^l(A)))^2 + 4(C_k(A) - c_i^k(A))(C_l(A) - c_i^l(A)).
\]

**Proof.** Let \( \tau(A) \) be the minimal M-eigenvalue of tensor \( A \). From Theorem 2.1 we suppose that \( x = \{x_i\}_{i=1}^n > 0 \in \mathbb{R}^n \) and \( y = \{\gamma_i\}_{i=1}^n > 0 \in \mathbb{R}^n \) are the corresponding left and right M-eigenvectors, respectively. Let \( x_p \geq x_i \geq \max_{e \in [n], i \neq p} \{x_i\} \). From the \( s \)th equation of \( Ax^2 = \tau(A)x \) in (1.2) we have

\[
\tau(A)x_i = \sum_{j,k \in [n]} a_{ijk}x_jy_kx_i = \left( \sum_{l \in [n]} a_{ijkl}x_jy_k \right)x_i + \sum_{s} \sum_{j,k,l \in [n]} a_{ijkl}x_jy_k + \sum_{s} \sum_{j,k,l \in [n]} a_{ijkl}y_kx_i + \sum_{l \in [n]} a_{ijkl}y_l^2x_i.
\]

Let \( \alpha_s = \min_{e \in [n]} \{a_{ijkl}\} \). It follows from Theorem 2.2 that

\[
0 \leq (\alpha_s - \tau(A))x_i \leq \left( \sum_{l \in [n]} a_{ijkl}x_jy_k \right)x_p + \sum_{s} \sum_{j,k,l \in [n]} a_{ijkl}x_jy_k + \sum_{s} \sum_{j,k,l \in [n]} a_{ijkl}y_kx_i + \sum_{l \in [n]} a_{ijkl}y_l^2x_i.
\]

Moreover,

\[
(\alpha_s - \tau(A) - r_s^p(A))x_i \leq (r_s(A) - r_s^p(A) + \gamma_s)x_p,
\]

(2.10)

When \( \alpha_s - r_s^p(A) > \tau(A) \) or \( \alpha_p - r_s^p(A) > \tau(A) \), multiplying (2.1) and (2.10), we have

\[
(\alpha_p - \tau(A) - r_s^p(A))(\alpha_s - \tau(A) - r_s^p(A)) \leq (r_p(A) - r_s^p(A) + \gamma_p)(r_s(A) - r_s^p(A) + \gamma_s),
\]

(2.11)

that is,

\[
\tau(A) \geq \frac{(\alpha_p - r_s^p(A)) + (\alpha_s - r_s^p(A)) - \Omega_{ps}^{1/2}}{2},
\]

(2.12)

where \( \Omega_{ps} = (\alpha_p - r_s^p(A) - (\alpha_s - r_s^p(A)))^2 + 4(R_p(A) - r_s^p(A))(R_s(A) - r_s^p(A)). \)

On the other hand, let \( |y_q| \geq |y_l| \geq \max_{e \in [n], i \neq q, l} |y_l| \). From the \( t \)th equation of \( Ax^2y = \tau(A)y \) in (1.2) we obtain

\[
\tau(A)y_t = \sum_{j,k \in [n]} a_{ijk}x_jy_kx_i = \sum_{l \in [n]} a_{ijkl}x_jy_k + \sum_{l \in [n]} a_{ijkl}y_kx_i + \sum_{l \in [n]} a_{ijkl}y_l^2x_i.
\]
Let $\beta_t = \min_{i \in [n]} |a_{i,ij}|$. It follows from Theorem 2.2 that

$$0 \leq (\beta_t - \tau(A))y_t \leq \left( \sum_{i \in [n]} a_{i,ij}x_iy_t - \tau(A)y_t \right) = -\sum_{i,j \in [n], j \neq i} a_{i,j}x_i y_j - \sum_{i \in [n], j \neq i} a_{i,ij}x_i y_t \leq \sum_{i,j \in [n], j \neq i} |a_{i,j}|y_t + \sum_{i \in [n], j \neq i} |a_{i,ij}|y_t + \sum_{i \in [n], j \neq i} |a_{i,ij}|y_t,$$

that is,

$$(\beta_t - \tau(A) - c_j^1(A))y_t \leq (c_t(A) - c_j^1(A) + \delta_t)y_t, \quad (2.13)$$

When $\beta_t - c_j^1(A) > \tau(A)$ or $\beta_q - c_q^0(A) > \tau(A)$, multiplying (2.6) and (2.13), we have

$$(\beta_q - \tau(A) - c_q^0(A))(\beta_t - \tau(A) - c_j^1(A)) \leq (c_q(A) - c_q^0(A) + \gamma_q)(c_t(A) - c_j^1(A) + \gamma_t), \quad (2.14)$$

which means that

$$\tau(A) \geq \frac{(\beta_q - c_q^0(A)) + (\beta_t - c_j^1(A)) - \Phi_{q,t}^1}{2}, \quad (2.15)$$

where $\Phi_{q,t} = (\beta_q - c_q^0(A) - (\beta_t - c_j^1(A)))^2 + 4(C_q(A) - c_q^0(A))(C_t(A) - c_j^1(A))$. \hfill \Box

Next, we compare the bound in Theorem 2.3 with that in Theorem 2.6 and obtain the following result.

**Theorem 2.7** Let $A = (a_{ijk}) \in \mathbb{E}_{k,n}$ be an irreducible elasticity $M$-tensor. Then

$$\tau(A) \geq \max \left\{ \min_{i,j \in [n], j \neq i} \{ \theta_1(A), \alpha_i - r_i(A), \alpha_j - r_j(A) \}, \min_{k,l \in [n], k \neq l} \{ \theta_2(A), \beta_k - c_k^1(A), \beta_l - c_l^1(A) \} \right\} \geq \max \left\{ \min_{i \in [n]} \{ \alpha_i - R_i(A) \}, \min_{l \in [n]} \{ \beta_l - C_l(A) \} \right\}.$$

**Proof** We will show $\min_{i,j \in [n], j \neq i} \{ \theta_1(A), \alpha_i - r_i(A), \alpha_j - r_j(A) \} \geq \min_{i \in [n]} \{ \alpha_i - R_i(A) \}$ and divide the argument into two cases.

Case 1. For any $i,j \in [n]$, $i \neq j$, if $\alpha_i - R_i(A) \leq \alpha_j - R_j(A)$, then from (2.8) we have

$$(\alpha_i - r_i(A) - (\alpha_j - r_j(A)))^2 + 4(R_i(A) - r_i(A))(R_j(A) - r_j(A)) \leq (\alpha_i - r_i(A) - (\alpha_j - r_j(A)))^2 + 4(R_i(A) - r_i(A))(\alpha_j - r_j(A) + R_i(A) - r_i(A))$$

$$= (\alpha_j - \alpha_i + r_i(A) - r_j(A)) + 2R_i(A) - 2r_i(A)).$$
Since
\[ \alpha_j - \alpha_i + r_j^i(A) - r_i^j(A) + 2R_i(A) - 2r_j^i(A) \]
\[ = \alpha_j - \alpha_i + R_i(A) - R_j(A) + R_j(A) - r_j^i(A) + R_i(A) - r_j^i(A) \geq 0, \]
we have
\[ \theta_1(A) = \frac{(\alpha_i - r_j^i(A)) + (\alpha_j - r_i^j(A)) - \Omega_{ij}^\frac{1}{2}}{2} \]
\[ \geq \frac{1}{2} (\alpha_i - r_j^i(A)) + (\alpha_j - r_i^j(A)) - (\alpha_i - \alpha_i + r_j^i(A) - r_i^j(A) + 2R_i(A) - 2r_j^i(A)) \]
\[ = \alpha_i - R_i(A), \]
which means that
\[ \min_{i,j \in [n], i \neq j} \{ \theta_1(A), \alpha_i - r_j^i(A), \alpha_j - r_i^j(A) \} \geq \min_{i \in [n]} [\alpha_i - R_i(A)]. \]

Case 2. For any \( i, j \in [n] \), \( i \neq j \), if \( \alpha_i - R_i(A) \geq \alpha_j - R_j(A) \), then
\[ \alpha_i - \alpha_j + R_j(A) \geq R_i(A). \] (2.16)

From (2.16) we have
\[ (\alpha_i - r_j^i(A) - (\alpha_j - r_i^j(A)))^2 + 4(R_i(A) - r_j^i(A))(R_j(A) - r_i^j(A)) \]
\[ \leq (\alpha_i - r_j^i(A) - (\alpha_j - r_i^j(A)))^2 + 4(\alpha_i - \alpha_j + R_i(A) - r_i^j(A))(R_j(A) - r_i^j(A)) \]
\[ = (\alpha_i - r_j^i(A) - (\alpha_j - r_i^j(A)))^2 \]
\[ + 4(\alpha_i - \alpha_j + r_i^j(A) - r_j^i(A) + R_i(A) - r_j^i(A))(R_j(A) - r_i^j(A)) \]
\[ = (\alpha_i - r_j^i(A) - (\alpha_j - r_i^j(A)) + 2R_j(A) - 2r_j^i(A))^2. \]

Since
\[ \alpha_i - r_j^i(A) - (\alpha_j - r_i^j(A)) + 2R_j(A) - 2r_j^i(A) \]
\[ = \alpha_i - \alpha_j + R_i(A) - R_j(A) + R_i(A) - r_j^i(A) + R_i(A) - r_j^i(A) \geq 0, \]
we have
\[ \theta_1(A) = \frac{(\alpha_i - r_j^i(A)) + (\alpha_j - r_i^j(A)) - \Omega_{ij}^\frac{1}{2}}{2} \]
\[ \geq \frac{1}{2} (\alpha_i - r_j^i(A)) + (\alpha_j - r_i^j(A)) - (\alpha_i - r_j^i(A) - (\alpha_j - r_i^j(A)) + 2R_i(A) - 2r_j^i(A)) \]
\[ = \alpha_j - R_j(A), \]
which means that
\[ \min_{i,j \in [n], i \neq j} \{ \theta_1(A), \alpha_i - r_j^i(A), \alpha_j - r_i^j(A) \} \geq \min_{i \in [n]} [\alpha_i - R_i(A)]. \]
Similarly, we have \( \min_{k, l \in [n], k \neq l} \{ \theta_2(A), \beta_k - c_k^1(A), \beta_l - c_l^1(A) \} \geq \min_{l \in [n]} \{ \beta_l - C_l(A) \} \). Thus we deduce

\[
\tau(A) \geq \max \left\{ \min_{i, j \in [n], i \neq j} \{ \theta_1(A), \alpha_i - \gamma_i(A), \alpha_j - \gamma_j(A) \}, \right.
\]

\[
\left. \min_{k, l \in [n], k \neq l} \{ \theta_2(A), \beta_k - c_k^1(A), \beta_l - c_l^1(A) \} \right\}
\]

\[
\geq \max \left\{ \min_{i \in [n]} \{ \alpha_i - R_i(A) \}, \min_{l \in [n]} \{ \beta_l - C_l(A) \} \right\},
\]

and the desired result follows.

The following example shows the superiority of the conclusions obtained in Theorems 2.4 and 2.6.

**Example 2.1 ([30])** Let \( A = (a_{ijkl}) \in \mathbb{E}_{4,2} \) be an elasticity M-tensor defined by

\[
a_{ijkl} = \begin{cases} 
 a_{1111} = 13, & a_{2222} = 12, & a_{1122} = 2, & a_{2211} = 2, \\
 a_{1112} = a_{1121} = -2, & a_{2212} = a_{2221} = -1, \\
 a_{2111} = a_{1211} = -2, & a_{1222} = a_{2122} = -1, \\
 a_{1212} = a_{2112} = a_{1221} = a_{2121} = -4.
\end{cases}
\]

From the matrices

\[
A(f_1) = \begin{bmatrix} a_{1111} & a_{1121} \\
 a_{2111} & a_{2211} \end{bmatrix} = \begin{bmatrix} 13 & -2 \\
 -2 & 2 \end{bmatrix}, \quad A(f_2) = \begin{bmatrix} a_{1122} & a_{1222} \\
 a_{2122} & a_{2222} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\
 -4 & 12 \end{bmatrix},
\]

\[
A(g_1) = \begin{bmatrix} a_{1111} & a_{1112} \\
 a_{1121} & a_{1122} \end{bmatrix} = \begin{bmatrix} 13 & -2 \\
 -2 & 2 \end{bmatrix}, \quad A(g_2) = \begin{bmatrix} a_{2211} & a_{2212} \\
 a_{2221} & a_{2222} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\
 -1 & 12 \end{bmatrix},
\]

we know that \( A \) is irreducible. By simple computation, \( A \) has six M-eigenvalues: 13.4163, 12.1118, 11.2036, 6.1778, 0.2442, and 0.1964. The minimal M-eigenvalue of \( A \) is 0.1964. Furthermore, we obtain

\[
\alpha_1 = 13, \quad \alpha_2 = 12, \quad \beta_1 = 13, \quad \beta_2 = 12, \\
\rho_1(A) = 12, \quad \rho_2(A) = 10, \quad C_1(A) = 12, \quad C_2(A) = 10, \\
\gamma_1 = 2, \quad \gamma_2 = 2, \quad \delta_1 = 2, \quad \delta_2 = 2, \\
\rho_1^2(A) = 4, \quad \rho_2^2(A) = 2, \quad C_1^2(A) = 4, \quad C_2^2(A) = 2, \\
R_1(A) = 14, \quad R_2(A) = 12, \quad C_1(A) = 14, \quad C_2(A) = 12.
\]

From Theorems 3.1 and 3.2 in [31] we have \( \tau(A) \geq -1 \) and \( \tau(A) \geq -0.8655 \), respectively. By Theorem 2.4 we have \( \tau(A) \geq -0.5567 \), and by Theorem 2.6 we have \( \tau(A) \geq -0.5125 \). Their comparison is drawn in Fig. 1, which reveals that our bounds are tighter than those of [31].
3 Strong ellipticity and positive definiteness

In this section, based on the results in Theorems 2.4 and 2.6, we present some sufficient conditions for the strong ellipticity and positive definiteness.

**Theorem 3.1** Let $A = (a_{ijk}) \in \mathbb{E}_{4,n}$ be an irreducible elasticity M-tensor. If

$$\max \left\{ \min_{i,j \in [n], i \neq j} \left\{ \eta_1(A) \right\}, \min_{k,l \in [n], k \neq l} \left\{ \eta_2(A) \right\} \right\} > 0,$$

then $A$ is positive definite, and the strong ellipticity condition holds.

**Proof** From Theorem 2.4 we have

$$\tau(A) \geq \max \left\{ \min_{i,j \in [n], i \neq j} \left\{ \eta_1(A) \right\}, \min_{k,l \in [n], k \neq l} \left\{ \eta_2(A) \right\} \right\} > 0.$$

Hence $A$ is positive definite, and the strong ellipticity condition holds. $\square$

**Theorem 3.2** Let $A = (a_{ijk}) \in \mathbb{E}_{4,n}$ be an irreducible elasticity M-tensor. If

$$\max \left\{ \min_{i,j \in [n], i \neq j} \left\{ \theta_1(A), \alpha_i - r_i(A), \alpha_j - r_j(A) \right\}, \min_{k,l \in [n], k \neq l} \left\{ \theta_2(A), \beta_k - c_k(A), \beta_l - c_l(A) \right\} \right\} > 0,$$

then $A$ is positive definite, and the strong ellipticity condition holds.

**Proof** From Theorem 2.6 we have

$$\tau(A) \geq \max \left\{ \min_{i,j \in [n], i \neq j} \left\{ \theta_1(A), \alpha_i - r_i(A), \alpha_j - r_j(A) \right\}, \min_{k,l \in [n], k \neq l} \left\{ \theta_2(A), \beta_k - c_k(A), \beta_l - c_l(A) \right\} \right\} > 0.$$

Hence $A$ is positive definite, and the strong ellipticity condition holds. $\square$

The following example reveals that Theorems 3.1 and 3.2 can identify the positive definiteness of elasticity M-tensors.
Example 3.1 Let \( A = (a_{ijkl}) \in \mathbb{E}_{4,2} \) be an elasticity M-tensor such that

\[
a_{ijkl} = \begin{cases} 
a_{1111} = 7, & a_{2222} = 8, & a_{1122} = 7, & a_{2211} = 8, \\
a_{1112} = a_{1121} = -1, & a_{2212} = a_{2221} = -1, \\
a_{2111} = a_{1211} = -1, & a_{1212} = a_{2121} = -1, \\
a_{1212} = a_{2112} = a_{1221} = a_{2121} = -0.5. 
\end{cases}
\]

By a direct computation we have

\[
A(\cdot, f_1) = \begin{bmatrix} 7 & -1 \\
-1 & 8 \end{bmatrix}, \quad A(\cdot, f_2) = \begin{bmatrix} 7 & -1 \\
-1 & 8 \end{bmatrix},
\]

\[
A(g_1, \cdot) = \begin{bmatrix} 7 & -1 \\
-1 & 7 \end{bmatrix}, \quad A(g_2, \cdot) = \begin{bmatrix} 8 & -1 \\
-1 & 8 \end{bmatrix}.
\]

Then \( A \) is irreducible. Furthermore, by simple computation we obtain

\[
\alpha_1 = 7, \quad \alpha_2 = 8, \quad \beta_1 = 7, \quad \beta_2 = 7, \\
\gamma_1 = 3, \quad \gamma_2 = 3, \quad \delta_1 = 3, \quad \delta_2 = 3, \\
r_1(A) = 2, \quad r_2(A) = 2, \quad c_1(A) = 2, \quad c_2(A) = 2, \\
R_1(A) = 4, \quad R_2(A) = 4, \quad C_1(A) = 4, \quad C_2(A) = 4.
\]

From Theorem 3.1 we have

\[
\tau(A) \geq \max \left\{ \min_{i \in \{1, \ldots, n\}, j \neq i} \left\{ \eta_1(A) \right\}, \quad \min_{k \in \{1, \ldots, n\}, l \neq k} \left\{ \eta_2(A) \right\} \right\} = 3.2984 > 0.
\]

From Theorem 3.2 we have

\[
\tau(A) \geq \max \left\{ \min_{i \in \{1, \ldots, n\}, j \neq i} \left\{ \theta_1(A), \alpha_i - r_1^i(A), \alpha_j - r_1^j(A) \right\}, \quad \min_{k \in \{1, \ldots, n\}, l \neq k} \left\{ \theta_2(A), \beta_k - c_1^k(A), \beta_l - c_1^l(A) \right\} \right\} = 3.4384 > 0.
\]

Thus from Theorems 3.1 and 3.2 we obtain that \( A \) is positive definite.

4 Conclusion

In this paper, we present some bounds for the minimum M-eigenvalue of elasticity M-tensors, which are tighter than some existing results. We propose numerical examples that illustrate the efficiency of the obtained results. As applications, we provide some checkable sufficient conditions for the strong ellipticity and positive definiteness.

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Author details
1 School of Mathematics and Information Science, Weifang University, Weifang, Shandong 261061, China. 2 School of Management Science, Qufu Normal University, Rizhao, Shandong 276800, China. 3 Department of College English Teaching, Qufu Normal University, Rizhao, Shandong 276800, China. 4 Department of Basic Teaching, Shandong Water Conservancy Vocational College, Rizhao, Shandong 276800, China.

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