Some Poisson-Lie sigma models

ARM Boris
arm.boris@gmail.com

Abstract
We calculate the Poisson-Lie sigma model for every 4-dimensional Manin triples (function of its structure constant) and we give the 6-dimensional models for the Manin triples
\((\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})^*, \mathfrak{sl}(2, \mathbb{C}), \mathfrak{sl}(2, \mathbb{C})^*)\),
\((\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})^*, \mathfrak{sl}(2, \mathbb{C})^*), \mathfrak{sl}(2, \mathbb{C}), \mathfrak{sl}(2, \mathbb{C})^*)\),
\((\mathfrak{sl}(2, \mathbb{C}), \mathfrak{su}(2, \mathbb{C}), \mathfrak{sb}(2, \mathbb{C}))\) and
\((\mathfrak{sl}(2, \mathbb{C}), \mathfrak{sb}(2, \mathbb{C}), \mathfrak{su}(2, \mathbb{C}))\)
1 Introduction

A Manin triples \((\mathcal{D}, \mathfrak{g}, \tilde{\mathfrak{g}})\) is a bialgebra \((\mathfrak{g}, \tilde{\mathfrak{g}})\) which don’t intersect each others and a direct sum of this bialgebra \(\mathcal{D} = \mathfrak{g} \oplus \tilde{\mathfrak{g}}\). If the corresponding Lie groups have a Poisson structure, they are called Poisson-Lie groups. A Poisson-Lie sigma models is an action \((3.13)\) calculated by a Poisson vector field matrix. \[3\] have deduced the extremal field which minimize the action of this models, which gives the motion equation \((3.19)\). We calculate here the action and the equations of motion for some 6-dimensionals Manin triples and we give a general formula for each 4-dimensional Manin triples. The 6-dimensional Manin triples are \((\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})^*), \mathfrak{sl}(2, \mathbb{C}), \mathfrak{sl}(2, \mathbb{C})^*), (\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})^*), \mathfrak{sl}(2, \mathbb{C}), (\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{su}(2, \mathbb{C}))\) and \((\mathfrak{sl}(2, \mathbb{C}), \mathfrak{sb}(2, \mathbb{C}), \mathfrak{su}(2, \mathbb{C}))\).
2 Some Manin triples

The Drinfeld double $D$ is defined as a Lie group such that its Lie algebra $\mathfrak{D}$ equipped by a symmetric ad-invariant nondegenerate bilinear form $\langle ., . \rangle$ can be decomposed into a pair of maximally isotropic subalgebras $\mathfrak{g}, \tilde{\mathfrak{g}}$ such that $\mathfrak{D}$ as a vector space is the direct sum of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$. Any such decomposition written as an ordered set $(\mathfrak{D}, \mathfrak{g}, \tilde{\mathfrak{g}})$ is called a Manin triples $(\mathfrak{D}, \mathfrak{g}, \tilde{\mathfrak{g}})$.

One can see that the dimensions of the subalgebras are equal and that bases $\{T_i\}, \{\tilde{T}^i\}$ in the subalgebras can be chosen so that

$$\langle T_i, T_j \rangle = 0, \quad \langle T_i, \tilde{T}^j \rangle = \langle \tilde{T}^j, T_i \rangle = \delta^j_i, \quad \langle \tilde{T}^i, \tilde{T}^j \rangle = 0 \quad (2.1)$$

This canonical form of the bracket is invariant with respect to the transformations

$$T'_i = T_k A^k_i, \quad \tilde{T}'^j = (A^{-1})^j_k \tilde{T}^k \quad (2.2)$$

Due to the ad-invariance of $\langle ., . \rangle$ the algebraic structure of $\mathfrak{D}$ is

$$[T_i, T_j] = c_{ij}^k T_k, \quad [\tilde{T}^i, \tilde{T}^j] = f^{ij}_k \tilde{T}^k$$

There are just four types of nonisomorphic four-dimensional Manin triples.

**Abelian Manin triples**:

$$[T_i, T_j] = 0, \quad [\tilde{T}^i, \tilde{T}^j] = 0, \quad [T_i, \tilde{T}^j] = 0, \quad i, j = 1, 2 \quad (2.3)$$

**Semi-Abelian Manin triples (only non trivial brackets are displayed)**:

$$[\tilde{T}^1, \tilde{T}^2] = \tilde{T}^2, \quad [T_2, \tilde{T}^1] = T_2, \quad [T_2, \tilde{T}^2] = -T_1 \quad (2.4)$$

**Type A non-Abelian Manin triples ($\beta \neq 0$)**:

$$[T_1, T_2] = T_2, \quad [\tilde{T}^1, \tilde{T}^2] = \beta \tilde{T}^2$$

$$[T_1, \tilde{T}^2] = -T^2, \quad [T_2, \tilde{T}^1] = \beta T_2, \quad [T_2, \tilde{T}^2] = -\beta T_1 + \tilde{T}^1 \quad (2.5)$$

**Type B non-Abelian Manin triples**:

$$[T_1, T_2] = T_2, \quad [\tilde{T}^1, \tilde{T}^2] = \tilde{T}^1$$

$$[T_1, \tilde{T}^1] = T_2, \quad [T_1, \tilde{T}^2] = -T_1 - \tilde{T}^2, \quad [T_2, \tilde{T}^2] = \tilde{T}^1 \quad (2.5)$$

Now we focus some six dimensional Manin triples. We recall that the commutation relations of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ of the Lie group $SL(2, \mathbb{C})$:

$$[T_1, T_2] = 2T_2, \quad [T_1, T_3] = -2T_3, \quad [T_2, T_3] = T_1 \quad (2.5)$$

The dual Lie algebra $\mathfrak{sl}(2, \mathbb{C})^*$ of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ has the commutation relations :

$$[\tilde{T}^1, \tilde{T}^2] = \frac{1}{4} \tilde{T}^2, \quad [\tilde{T}^1, \tilde{T}^3] = \frac{1}{4} \tilde{T}^3, \quad [\tilde{T}^2, \tilde{T}^3] = 0 \quad (2.5)$$
There is a scalar product on $(\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})^*)$ such that (see [2]):

$$(T_i, \bar{T}^j) = \delta^j_i$$

(2.7)

Finally, we have that $(\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})^*, \mathfrak{sl}(2, \mathbb{C}), \mathfrak{sl}(2, \mathbb{C})^*)$ with this scalar product is a Manin triple. We note that $(\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})^*, \mathfrak{sl}(2, \mathbb{C})^*$, $\mathfrak{sl}(2, \mathbb{C})$) with this scalar product is also a Manin triples.

The Iwasawa decomposition allows us to decompose:

$$\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{su}(2, \mathbb{C}) \oplus \mathfrak{sb}(2, \mathbb{C})$$

(2.8)

where $\mathfrak{su}(2, \mathbb{C})$ is the Lie algebra of the Lie group $SU(2)$ with commutation relations:

$$[T_1, T_2] = T_3, \quad [T_2, T_3] = T_1, \quad [T_3, T_1] = T_2$$

(2.9)

$\mathfrak{sb}(2, \mathbb{C})$ is the Lie algebra of the Borel subgroup $SB(2, \mathbb{C})$ with commutation relations:

$$[\bar{T}^1, \bar{T}^2] = \bar{T}^3, \quad [\bar{T}^1, \bar{T}^3] = \bar{T}^2, \quad [\bar{T}^2, \bar{T}^3] = 0$$

(2.10)

Here we can see in comparing (2.10) and (2.6) that $\mathfrak{sb}(2, \mathbb{C}) \simeq \mathfrak{sl}(2, \mathbb{C})^*$.

The Iwasawa decomposition (2.8) allows us to identify $\mathfrak{sb}(2, \mathbb{C}) \simeq \mathfrak{su}(2, \mathbb{C})^*$. We define a scalar product on $(\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})^*)$ such that $(x, y) = \text{Im}(\text{Tr}(x|y))$. With this scalar product we have (see [2]):

$$(T_i, \bar{T}^j) = \delta^j_i$$

(2.11)

Finally we have that $(\mathfrak{sl}(2, \mathbb{C}), \mathfrak{su}(2, \mathbb{C}), \mathfrak{sb}(2, \mathbb{C}))$ with this scalar product is a Manin triple. We note that $(\mathfrak{sl}(2, \mathbb{C}), \mathfrak{sb}(2, \mathbb{C}), \mathfrak{su}(2, \mathbb{C}))$ with this scalar product is also a Manin triples.

### 3 Poisson-Lie sigma models

Given a Lie group $M$ and a Poisson structure on it. We define the action of this model (see [1]) as:

$$S_1 = \int_{\Sigma} (< dgg^{-1}, A > - \frac{1}{2} < A, (r - Ad_g r Ad_g)A >)$$

(3.12)

where $g \in G, A = A_\alpha^i d\xi^\alpha X_i$ and $r \in g \otimes g$ is a classical $r$ matrix with $g$ as the Lie algebra of $G$ and $\{X_i\}$ as a basis of $g$. Note that the above action can be applied for simple or nonsemisimple Lie group $G$ with ad-invariant symmetric bilinear nondegenerate form $< X_i, X_j > = G_{ij}$ on the Lie algebra $g$. When the metric $G_{ij}$ of Lie algebra is degenerate then the above action is not good. Here we use the following action instead of the above one:

$$S_2 = \int_{\Sigma} (dX_i \wedge A_i - \frac{1}{2} P^{ij} A_i \wedge A_j)$$

(3.13)

where $x$ are Lie group parameters with parametrization (e.g.)

$$\forall g \in G, g = e^{X_1 T_1} e^{X_2 T_2} ...$$

(3.14)
where $P^{ij}$ is the Poisson structure on the Lie group which for coboundary Poisson Lie groups it is obtained from

$$ (\mathcal{P}(g))_\chi = b(g)a(g)^{-1} $$

(3.15)

We can obtain $a(g)^{-1}$ and $b(g)$ in computing :

$$ (Ad_{g^{-1}})_\chi = \begin{pmatrix} a(g)^T & b(g)^T \\ 0 & d(g)^T \end{pmatrix} $$

(3.16)

$$ (Ad_g)_\chi = \begin{pmatrix} a(g)^{-T} & -a(g)^{-T}b(g)^T d(g)^{-T} \\ 0 & d(g)^{-T} \end{pmatrix} $$

(3.17)

where $^T$ denotes the transpose matrix.

The extremal fields $(X, A)$ which minimize the action (3.13) have to satisfy the equation written locally (see [3]) as :

$$ dX_i + P^{ij}(X)A_j = 0 $$

(3.18)

$$ dA_k + \frac{1}{2} P^{ij}_{,k}(X)A_i \wedge A_j = 0 $$

(3.19)

where $P^{ij}_{,k} = \partial_k P^{ij}|_{X_k=0}$.

### 4 Poisson-Lie sigma model of any 4-dimensional Manin triple

We first calculate the matrix of the adjoint actions function of structure constant :

$$ ad_{T_1} = \begin{pmatrix} 0 & c_{12}^1 & 0 & -f_{12}^1 \\ 0 & c_{12}^2 & f_{12}^1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -c_{12}^1 & -c_{12}^2 \end{pmatrix} $$

$$ ad_{T_2} = \begin{pmatrix} -c_{12}^1 & 0 & 0 & -f_{12}^2 \\ -c_{12}^2 & 0 & f_{12}^2 & 0 \\ 0 & 0 & c_{12}^1 & c_{12}^2 \\ 0 & 0 & 0 & 0 \end{pmatrix} $$

To obtain the matrix $\mathcal{P}$, we calculate the adjoint action matrix of a general element $g = \prod_{i=1}^{2} e^{\alpha_i T_i}$ by the formula :

$$ (Ad_{\prod_{i=1}^{2} e^{X_i T_i}})\chi = \prod_{i=1}^{2} e^{X_i (ad_{T_i})_\chi} $$

(4.20)

Similarly, we have :

$$ (Ad_{\prod_{i=1}^{2} e^{X_i T_i} - 1})\chi = \prod_{i=1}^{2} e^{-X_{3-i} (ad_{T_{3-i}})_\chi} $$

(4.21)

We can deduce the matrix $P^{ij}$ :

$$ P^{ij} = \begin{pmatrix} 0 & -p^{21} \\ p^{21} & 0 \end{pmatrix} $$
where
\[ P^{21} = c_{12}^1(-1 + e^{c_{12}^2 X_1})f^{12}_1 + c_{12}^2 e^{c_{12}^2 X_1} - c_{12}^1 X_2 (-1 + e^{c_{12}^1 X_2})f^{12}_2 \] (4.22)

Now, we can calculate the action \[ S_2 = \int_\Sigma \sum_{i=1}^2 dX_i \wedge A_i - P^{21} A_2 \wedge A_1 \] (4.23)
and the equations of motion : \[ \begin{align*}
  dX_1 - P^{21} A_2 &= 0 \\
  dX_2 + P^{21} A_1 &= 0 \\
  dA_1 - \frac{c_{12}^1 c_{12}^2 f^{12}_1 + c_{12}^2 c_{12}^2 (-e^{-c_{12}^1 X_2} + 1)f^{12}_2}{c_{12}^1 c_{12}^2} A_2 \wedge A_1 &= 0 \\
  dA_2 + \frac{c_{12}^2 e^{c_{12}^2 X_1} c_{12}^1 f^{12}_2}{c_{12}^1 c_{12}^2} A_2 \wedge A_1 &= 0
\end{align*} \] (4.24)

5 Poisson-Lie sigma model of \( \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})^*, \mathfrak{sl}(2, \mathbb{C}), \mathfrak{sl}(2, \mathbb{C})^* \)

We first calculate the matrix of the adjoint actions :
\[
\begin{align*}
  ad_{T_1} &= \begin{pmatrix}
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 2 & 0 & 0 & 0 & 0 \\
  0 & 0 & -2 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & -2 & 0 \\
  0 & 0 & 0 & 0 & 0 & 2
\end{pmatrix} \\
  ad_{T_2} &= \frac{1}{4} \begin{pmatrix}
  0 & 0 & 4 & 0 & -1 & 0 \\
  -8 & 0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 8 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & -4 & 0 & 0
\end{pmatrix} \\
  ad_{T_3} &= \frac{1}{4} \begin{pmatrix}
  0 & -4 & 0 & 0 & 0 & -1 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  8 & 0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & -8 & 0 \\
  0 & 0 & 0 & 4 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{align*}
\]
To obtain the matrix \( \mathcal{P} \), we calculate the adjoint action matrix of a general element \( g = \prod_{i=1}^{3} e^{\alpha_i T_i} \) by the formula:

\[
(Ad_{\prod_{i=1}^{3} e^{x_i T_i}}) = \prod_{i=1}^{3} e^{X_i (adr_{x_i})}
\]  

(5.25)

Similarly, we have:

\[
(Ad_{\prod_{i=1}^{3} e^{-x_i T_i}}) = \prod_{i=1}^{3} e^{-X_i (adr_{X_i})}
\]  

(5.26)

We can deduce the matrix \( \mathcal{P}_{ij} \):

\[
\mathcal{P}_{ij} = \begin{pmatrix}
0 & -\frac{X_2}{4}(1 + X_2 X_3)e^{2X_1} & -\frac{X_3}{2}e^{-2X_1} \\
\frac{X_2}{4}(1 + X_2 X_3)e^{2X_1} & 0 & \frac{X_2 X_3}{2} \\
\frac{X_3}{2}e^{-2X_1} & -\frac{X_2 X_3}{2} & 0
\end{pmatrix}
\]

Now, we can calculate the action (3.13) of the model

\[
S_2 = \int_{\Sigma} \sum_{i=1}^{3} dX_i \wedge A_i + \left( \frac{X_2}{4}(1 + X_2 X_3)e^{2X_1} \right) A_1 \wedge A_2 + \frac{X_3}{2}e^{-2X_1} A_1 \wedge A_3 - \frac{X_2 X_3}{2} A_2 \wedge A_3
\]  

(5.27)

and the equations of motion (3.19):

\[
\begin{align*}
    dX_1 - \left( \frac{X_2}{4}(1 + X_2 X_3)e^{2X_1} \right) A_2 - \frac{X_3}{4}e^{-2X_1} A_3 &= 0 \\
    dX_2 + \left( \frac{X_2}{4}(1 + X_2 X_3)e^{2X_1} \right) A_1 + \frac{X_2 X_3}{2} A_2 &= 0 \\
    dX_3 + \frac{X_3}{2}e^{-2X_1} A_1 - \frac{X_2 X_3}{2} A_2 &= 0 \\
    dA_1 - \frac{X_2}{2}(1 + X_2 X_3) A_1 \wedge A_2 + \frac{X_3}{2} A_1 \wedge A_2 &= 0 \\
    dA_2 - \frac{e^{2X_1}}{4} A_1 \wedge A_2 + \frac{X_3}{2} A_2 \wedge A_3 &= 0 \\
    dA_3 - \frac{X_2}{4}e^{2X_1} A_1 \wedge A_2 - \frac{e^{-2X_1}}{4} A_1 \wedge A_3 + X_2 A_2 \wedge A_3 &= 0
\end{align*}
\]

6 Poisson-Lie sigma model of \((\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}))^*, \mathfrak{sl}(2, \mathbb{C}), \mathfrak{sl}(2, \mathbb{C})^*\)

Now to obtain this Poisson Lie sigma model, we have to change \( T_i \rightarrow \tilde{T}_i \) and \( \tilde{T}_i \rightarrow T_i \) of the previous model. And we can calculate the matrix of the adjoint actions as we do previously. With this we can deduce the matrix \( \mathcal{P}_{ij} \) for this model:

\[
\mathcal{P}_{ij} = \begin{pmatrix}
0 & -2e^{\frac{X_3}{2} X_2} & 2e^{X_3} \\
2e^{\frac{X_3}{2} X_2} & 0 & 2 - 2e^{\frac{X_3}{2} (4 + X_2 X_3)} \\
-2e^{\frac{X_3}{2} X_2} & -2 + \frac{X_3}{2} e^{\frac{X_3}{2} (4 + X_2 X_3)} & 0
\end{pmatrix}
\]
Now, we can calculate the action \((3.13)\) of the model

\[
S_2 = \int \sum_{i=1}^{3} dX_i \wedge A_i + 2e^{\frac{\Sigma}{2}} X_2 A_1 \wedge A_2 + 2e^{\frac{\Sigma}{2}} X_3 A_1 \wedge A_3 + (-2 + \frac{1}{2} e^{\frac{\Sigma}{2}} (4 + X_2 X_3)) A_2 \wedge A_3
\]

and the equations of motion \((3.19)\) :

\[
\begin{align*}
    dX_1 & - 2e^{\frac{\Sigma}{2}} X_2 A_2 + 2e^{\frac{\Sigma}{2}} X_3 A_3 = 0 \\
    dX_2 & + 2e^{\frac{\Sigma}{2}} X_2 A_1 + (2 - \frac{1}{2} e^{\frac{\Sigma}{2}} (4 + X_2 X_3)) A_3 = 0 \\
    dX_3 & - 2e^{\frac{\Sigma}{2}} X_3 A_1 - (2 - \frac{1}{2} e^{\frac{\Sigma}{2}} (4 + X_2 X_3)) A_2 = 0 \\
    dA_1 & - \frac{X_2}{2} A_1 \wedge A_2 + \frac{X_3}{4} A_1 \wedge A_3 - \frac{1}{4} (4 + X_2 X_3) A_2 \wedge A_3 = 0 \\
    dA_2 & - 2e^{\frac{\Sigma}{2}} A_1 \wedge A_2 - \frac{1}{2} e^{\frac{\Sigma}{2}} A_2 \wedge A_3 = 0 \\
    dA_3 & + 2e^{\frac{\Sigma}{2}} A_1 \wedge A_3 - \frac{e^{\frac{\Sigma}{2}} X_2}{2} A_2 \wedge A_3 = 0
\end{align*}
\]

7 Poisson-Lie sigma model of \((\text{sl}(2, \mathbb{C}), \text{su}(2, \mathbb{C}), \text{so}(2, \mathbb{C}))\)

We can calculate the matrix of the adjoint actions as we do previously. With this we can deduce the matrix \(P^{ij}\) for this model :

\[
P^{ij} = \begin{pmatrix}
0 & -\cos X_1 \cos X_3 \sin X_2 + \sin X_1 \sin X_3 & -\cos X_3 \sin X_1 \sin X_2 - \cos X_1 \sin X_3 \\
\cos X_1 \cos X_3 \sin X_2 - \sin X_1 \sin X_3 & 0 & -1 + \cos X_2 \cos X_3 \\
\cos X_3 \sin X_1 \sin X_2 + \cos X_1 \sin X_3 & 1 - \cos X_2 \cos X_3 & 0
\end{pmatrix}
\]

Now, we can calculate the action \((3.13)\) of the model

\[
S_2 = \int \sum_{i=1}^{3} dX_i \wedge A_i - (\cos X_1 \cos X_3 \sin X_2 + \sin X_1 \sin X_3) A_1 \wedge A_2 - (\cos X_3 \sin X_1 \sin X_2 - \cos X_1 \sin X_3) A_1 \wedge A_3 - (-1 + \cos X_2 \cos X_3) A_2 \wedge A_3
\]

and the equations of motion \((3.19)\) :

\[
\begin{align*}
    dX_1 + (\cos X_1 \cos X_3 \sin X_2 + \sin X_1 \sin X_3) A_2 + (\cos X_3 \sin X_1 \sin X_2 - \cos X_1 \sin X_3) A_3 = 0 \\
    dX_2 + (\cos X_1 \cos X_3 \sin X_2 - \sin X_1 \sin X_3) A_1 + (-1 + \cos X_2 \cos X_3) A_3 = 0 \\
    dX_3 + (\cos X_3 \sin X_1 \sin X_2 + \cos X_1 \sin X_3) A_1 (1 - \cos X_2 \cos X_3) A_2 = 0 \\
    dA_1 + \sin X_3 A_1 \wedge A_2 - \cos X_3 \sin X_2 A_1 \wedge A_3 = 0 \\
    dA_2 - \cos X_1 \cos X_3 A_1 \wedge A_2 - \cos X_2 \sin X_1 A_1 \wedge A_3 = 0 \\
    dA_3 + \sin X_1 A_1 \wedge A_2 - \cos X_1 A_1 \wedge A_3 = 0
\end{align*}
\]
8 Poisson-Lie sigma model of \((\text{sl}(2, \mathbb{C}), \text{sb}(2, \mathbb{C}), \text{su}(2, \mathbb{C}))\)

We can calculate the matrix of the adjoint actions as we do previously. With this we can deduce the matrix \(\mathcal{P}^{ij}\) for this model:

\[
\mathcal{P}^{ij} = \begin{pmatrix}
0 & -e^{X_1}X_3 & -e^{X_1}X_2 \\
e^{X_1}X_3 & 0 & \frac{1}{2}(1 - e^{2X_1}(1 + X_2^2 + X_3^2)) \\
e^{X_1}X_2 & \frac{1}{2}(1 - e^{2X_1}(1 + X_2^2 + X_3^2)) & 0
\end{pmatrix}
\]

Now, we can calculate the action (3.13) of the model

\[
S_2 = \int \Sigma \sum_{i=1}^{3} dX_i \wedge A_i + e^{X_1}X_3A_1 \wedge A_2 + e^{X_1}X_2A_1 \wedge A_3 - \frac{1}{2}(1 - e^{2X_1}(1 + 2X_2^2 + 2X_3^2))A_2 \wedge A_3
\]

and the equations of motion (3.19):

\[
\begin{align*}
&dX_1 - e^{X_1}X_3A_2 - e^{X_1}X_2A_3 = 0 \\
dX_2 + e^{X_1}X_3A_1 + \frac{1}{2}(1 - e^{2X_1}(1 + X_2^2 + X_3^2))A_3 = 0 \\
dX_3 + e^{X_1}X_2A_1 - \frac{1}{2}(1 - e^{2X_1}(1 + X_2^2 + X_3^2))A_2 = 0 \\
dA_1 - X_3A_1 \wedge A_2 - X_2A_1 \wedge A_3 - (1 + X_2^2 + X_3^2)A_2 \wedge A_3 = 0 \\
dA_2 - e^{X_1}A_1 \wedge A_3 = 0 \\
dA_3 - e^{X_1}A_1 \wedge A_2 = 0
\end{align*}
\]

9 Discussion

We gives here the Poisson-Lie sigma models of some Manin triples. Concerning the general formula (3.13), we have to say that this is no problem when \(c_{12}^2\) and \(c_{12}^2\) is zero because

\[
\mathcal{P}^{21} = \frac{(-1 + e^{c_{12}^2X_1})f^{12}_1}{c_{12}^2} + \frac{e^{c_{12}^2X_1} - c_{12}^1X_2(-1 + e^{c_{12}^1X_2})f^{12}_2}{c_{12}^2}
\]

which can be approximate by

\[
\mathcal{P}^{21} = (X_1 + \frac{c_{12}^2}{2}X_1^2 + ... )f^{12}_1 + e^{c_{12}^2X_1} - c_{12}^1X_2X_2 + \frac{c_{12}^1}{2}X_2^2 + ... )f^{12}_2
\]

We tried to obtain the equivalent formula for \(n = 3\) but the calculus was too hard.
Références

[1] Hajizadeh S., Rezaei-Aghdam A., Poisson-Lie Sigma models over low dimensional real Poisson-Lie groups
[2] Kosmann-Schwarzbach Y., Lie bialgebras, Poisson Lie Groups and Dressing Transformation
[3] Vysoký J., Hlavatý, Poisson Lie Sigma Models on Drinfeld double