Index definitions for nonlinear IAEs and DAEs: new classifications and numerical treatments

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Abstract

The definition of index for differential algebraic equations (DAEs) or integral algebraic equations (IAEs) in the linear case (time variable) depends only on the coefficients of integrals or differential operators and the coefficients of the unknown functions. Is this possible for the nonlinear case? In this paper we answer this question. In this paper, we generalize the index notion for the nonlinear case. One of the difficulties for nonlinear case, is its dependence on the exact solution which motivates us to give an important warning to whom want to solve DAEs using numerical methods such as Runge-Kutta, multistep or collocation methods.

keywords: Differential algebraic equation; Integral algebraic equation; Runge-Kutta methods; Collocation methods; Multistep methods.

1 Introduction

We consider DAEs and IAEs of the linear forms

\[ A(t)y'(t) + B(t)y(s)ds = f(t), \quad t \in I := [0, T] \]  \hspace{1cm} (1.1)

and

\[ A(t)y(t) + \int_0^t k(t, s)y(s)ds = f(t), \quad t \in I := [0, T] \]  \hspace{1cm} (1.2)

with \( A, B \in C(I, R^{r \times r}) \), \( f \in C(I, R^r) \), \( k \in C(\mathbb{D}, R^{r \times r}) \), and their semi-nonlinear forms

\[ A(t)y'(t) + F(t, y(t)) = f(t), \quad t \in I := [0, T], \]  \hspace{1cm} (1.3)

and

\[ A(t)y(t) + \int_0^t \kappa(t, s, y(s))ds = f(t), \quad t \in I := [0, T], \]  \hspace{1cm} (1.4)

where \( \kappa \in C(\mathbb{D} \times R^r, R^r) \) and \( F \in C(I \times R^r, R^r) \) with \( \mathbb{D} := \{(t, s) : 0 \leq s \leq t \leq T\} \). We also assume that \( A(t) \) is a singular matrix with constant rank for all \( t \in I \). Since integrating (1.1)
and (1.3) changes them to IAEs of the forms (1.2) and (1.4), we conclude that the DAEs inherit many properties of IAEs. We are interested in translating many concepts and results on DAEs to the poorly studied IAEs.

We recall the DAEs and IAEs classification by the index notions. There are many index notions by considering analytical and numerical properties of solutions. The differentiation index for DAEs goes back to the work of Campbell [1, 2] and for IAEs to the work of Gear [3]. Gear introduced the differentiation index using index reduction procedure [3] to obtain the existence and uniqueness conditions of IAEs. These definitions of index are extended in two different ways: the pioneering works of Griepentrog(1991), Reich(1991) and Rabier, Rheinbold(1991) [4, 5, 6] which are related to the differential geometric concepts of regular DAEs, and the work of Chistyakov and Bulatov (7, 8) (left index). The global index which is an extension of index notion for linear DAEs with constant coefficients using Kronecker canonical normal form, was given by Gear and Petzold [10]. In [11], Marz has shown that the global index must be replaced by tractable index. There are two reasons in [11]: 1) there are not any way for relating this index to the nonlinear case, 2) unavailability of the necessary transforming matrices, except for some interesting cases. The projection type index definition for DAEs (tractable index) which was introduced by Marz and her colleagues [11], can be extended to IAEs, which can be found at the works of Brunner and Pishbin [12, 13]. The perturbation index defined by Hairer, Lubich and Roche [14] has important role in analyzing numerical treatments of DAEs. Finally, strangeness index was introduced by Kunkel and Mehrmann [15].

This paper is organized as follows: in section 2, the preliminary definitions of index for the linear case are given, also a theorem for existence and uniqueness of solution of IAEs is proved. In section 3, after defining index for nonlinear case, the related problems are discussed. Finally, by giving numerical experiments, we observe the cautions that should be taken for solving IAEs and DAEs.

2 Index definition for the linear case

There are many definitions for index and one can refer to the reference in the introduction. Although, in this paper, we only use ‘rank degree’ index for linear case, but other index also can be used for generalization to the nonlinear case. To define ‘rank degree’ index, we need following preliminaries.

Definition 2.1 [7, 8] The matrix $A^-(t)$ is called a semi-inverse of $A(t)$, if it satisfies the equation

$$A(t)A^-(t)A(t) = A(t),$$

which can be rewritten as

$$V(t)A(t) = 0,$$

with

$$V(t) = E - A(t)A^-(t),$$ (2.1)

where $E$ is the $r \times r$ identity matrix.

The following conditions are necessary and sufficient for the existence of a semi-inverse matrix $A^-(t)$ with elements in $C^p(I, R^{r \times r})$ [8]:
1. the elements of $A(t)$ belong to $C^p(I, R^{r\times r})$;
2. rank$A(t) = \text{constant}$, $\forall t \in I$.

**Definition 2.2** [9] Suppose that $A \in C^\nu(I, R^{r\times r})$ and $k \in C^\nu(D, R^{r\times r})$. Let

$$A_0 \equiv A, \quad k_0 \equiv k,$$

$$A_i y = \frac{d}{dt} ((E - A_i(t)A_i^{-1}(t))y) + y,$$

$$A_{i+1} \equiv A_i + (E - A_i(t)A_i^{-1}(t))k_i(t, t), \quad \text{and} \quad k_{i+1} = \Lambda ik_i.$$

Then, we say the ‘rank degree’ index of $(A, k)$ is $\nu$, if

$$\text{rank}A_i(t) = \text{constant}, \quad \forall t \in I, \quad \text{for} \quad i = 0, \ldots, \nu,$$

$$\det A_i = 0, \quad \forall t \in I, \quad \text{for} \quad i = 0, \ldots, \nu - 1, \quad \text{det} A_\nu \neq 0.$$

Moreover, we say the ‘rank degree’ index of the system (1.2) is $\nu$ ($\text{ind}_r = \nu$), if in addition to the above conditions, we have $f \in C^\nu(I, R^r)$ and hence, we can define

$$F_{i+1} \equiv \Lambda_i F_i, \text{ with } F_0 \equiv f, \quad i = 0, \ldots, \nu - 1.$$

Now, we can state the following theorem of the uniqueness and existence for higher index IAEs.

**Theorem 1** [9] Suppose the following conditions are satisfied for (1.2):

1. $\text{ind}_r = \nu \geq 1$,
2. $A_i(t) \in C^1(I, R^{r\times r})$, $F_i(t) \in C^1(I, R^r)$ and $k_i \in C^1(D, R^{r\times r})$ for $i = 1, \ldots, \nu$,
3. $A_i(0)A_{i-1}^{-1}(0)F_i(0) = F_i(0)$ for $i = 0, \ldots, \nu - 1$ (consistency conditions).

Then the system (1.2) has a unique solution on $I$.

**Definition 2.3** [10] We say the ‘rank degree’ index of the system (1.1) is $\nu$, if the ‘rank degree’ index of it’s corresponding IAE

$$A(t)x(t) + \int_0^t (B(s) - A'(s))x(s)ds = \int_0^t q(s)ds. \quad (2.2)$$

be $\nu$.

### 3 index definition for the nonlinear case

Among the existing definitions for index, there are definitions which are independent of the linear case:

**Definition 3.1** [1, 2] We say the differentiation index of the system (1.3) is $\nu$ ($\text{ind}_d = \nu$), if $\nu$ is the minimum possible number of derivatives (1.3) to obtain a system of the ordinary differential equations (ODE).
Definition 3.2 [3] We say the differentiation index of the system (1.4) is $\nu$ ($\text{ind}_d = \nu$), if $\nu$ is the minimum possible number of derivatives (1.4) to obtain a system of the second kind Volterra integral equations (SVIE).

Definition 3.3 [14] The equation $F(Y', Y) = 0$ has perturbation index $\nu$ ($\text{ind}_p = \nu$) along a solution $Y$ on $I$, if $\nu$ is the smallest integer such that for all functions $\hat{Y}$ having the residual $F(\hat{Y}', \hat{Y}) = \delta(x)$, there exists an estimate

$$\|\hat{Y}(x) - Y(x)\| \leq C(\|\hat{Y}(0) - Y(0)\| + \max_{x \in I} \|\delta(x)\| + \ldots + \max_{x \in I} \|\delta^{(\nu-1)}(x)\|)$$

on $I$, whenever the expression on the right-hand side is sufficiently small. Here, $C$ denotes a constant which depends on $F$ and the length of the interval.

The perturbation index defined by Hairer, Lubich and Roche [14] has an important role in analyzing numerical treatment of DAEs, where the inequality

$$\|\hat{Y}(x) - Y(x)\| \leq C(\max_{x \in I} \|\delta(x)\| + \ldots + \max_{x \in I} \|\delta^{(\nu)}(x)\|)$$

should be considered in above definition for IAEs.

Now the question is: how can we extend the index definitions of previous sections to the nonlinear case?

Suppose $u$ denotes an approximate solution of the nonlinear equation (1.3) or (1.4), and define the residuals

$$R_1(t) := A(t)u'(t) + F(t, u(t)) - f(t) \tag{3.1}$$

or

$$R_2(t) := A(t)u(t) + \int_0^t \kappa(t, s, u(s))ds - f(t), \tag{3.2}$$

which are supposed to be very small. Subtracting (1.3) or (1.4) from (3.3) or (3.4), we obtain

$$R_1(t) = A(t)(u'(t) - y'(t)) + F(t, u(t)) - F(t, y(t)) \tag{3.3}$$

or

$$R_2(t) = A(t)(u(t) - y(t)) + \int_0^t \kappa(t, s, u(s)) - \kappa(t, s, y(s))ds. \tag{3.4}$$

Letting $e(t) := u(t) - y(t)$, and supposing that $F$ and $\kappa$ have continuous derivatives with respect to the second and third variables respectively, we have

$$R_1(t) = A(t)e'(t) + F_y(t, \eta(t))e(t) \tag{3.5}$$

or

$$R_2(t) = A(t)e(t) + \int_0^t \kappa_y(t, s, \eta(s))e(s)ds, \tag{3.6}$$

where $\eta(s)$ is an appropriate function arises from applying the mean value theorem which is in a neighborhood of the exact solution $y$. The equations (3.5) and (3.6) are of the linear form (1.1) and (1.2).

Now, the strategy for defining an index for the nonlinear case (1.3) or (1.4) is the same as in linear cases (3.5) or (3.6).
**Definition 3.4** We say that the index for the nonlinear equations (1.4) or (1.3) is $\nu$, if there exists a neighborhood of the exact solution $y(N_\epsilon(y))$, in which the index of the linear equations (3.6) or (3.5) is $\nu$.

This dependency of index definition to the exact solution (that is unknown) is the difficulty of this definition and definition 3.3. We will show by examples, how we can use this definition effectively.

We divide the nonlinear IAEs and DAEs to the following classes:

**well structure:** the IAEs or DAEs that their indices do not depend on the unknown variables (component of $y$).

**free structure:** the IAEs or DAEs that their indices depend on the unknown variables.

Fortunately, most of IAEs and DAEs in application are well structure. For example the DAEs in [14] are well structure and for every neighborhood of $y$ their indices remain constant. The IAEs of Hessenberg type

\[
\begin{bmatrix}
A_{1,1}(t) & \cdots & A_{1,\nu-1}(t) & 0 \\
0 & \ddots & \vdots & \vdots \\
A_{\nu-1,1}(t) & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0
\end{bmatrix}
\begin{bmatrix}
y_1(t) \\
y_2(t) \\
\vdots \\
y_\nu(t)
\end{bmatrix}
+ 
\int_0^t \begin{bmatrix}
k_1(t,s,y_1(s),y_2(s),\ldots,y_\nu(s)) \\
\vdots \\
k_{\nu-1}(t,s,y_1(s),y_2(s)) \\
k_\nu(t,s,y_1(s))
\end{bmatrix}
\begin{bmatrix}
f_1(t) \\
f_2(t) \\
\vdots \\
f_\nu(t)
\end{bmatrix}
ds = \begin{bmatrix}
f_1(t) \\
f_2(t) \\
\vdots \\
f_\nu(t)
\end{bmatrix},
\]

are well structure with index $\nu$, if $\prod_{i=1}^{\nu} k_i,y_{\nu+1-i}(t,t,y_1(t),\ldots,y_{\nu+1-i}(t))$ is invertible with a bounded inverse on $I$.

**Example 3.5** The DAE of the form

\[
y' = y^2 + e^y + z, \\
0 = e^y + \sin(t),
\]

is a well structure DAE of index 2, since

\[
det(F_y) = \begin{bmatrix}
2y + e^y & 1 \\
e^y & 0
\end{bmatrix} = e^y > 0, \text{ for all } y \text{ and } z.
\]

To speak about the free structure IAEs or DAEs, we divide them again to the following classes:

**the dependent form:** the IAEs or DAEs that their indices depend on the exact solution on the given interval.

**the independent form:** the IAEs or DAEs that their indices do not depend on the exact solution on the given interval.
A condition is called critical, if it changes the index in a free structure DAE or IAE. Also, a point is called critical if at which the exact solution satisfies the critical conditions of the dependent form DAE or IAE.

**Example 3.6** Let

\[
A(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
\]

\[
F(t, y) = \begin{pmatrix} -y_1^2 - e^{y_2} \\ -y_1 y_2 \end{pmatrix},
\]

and

\[
f(t) = \begin{pmatrix} \cos^2(t) - e^t - \sin(t) \\ -t \cos(t) \end{pmatrix}.
\]

Then a DAE of the form (1.3) has the exact solution

\[
y(t) = \begin{pmatrix} \cos(t) \\ t \end{pmatrix}
\]

and we have

\[
F_y(t, \eta) = \begin{bmatrix} -2\eta_1 & -e^{\eta_2} \\ -\eta_2 & -\eta_1 \end{bmatrix}.
\]

This is a free structure DAE, and it is obvious that for \(\eta_1 \neq 0\) the index is 1, and for \(\eta_2 \neq 0, \eta_1 = 0\) the index is 2. Now suppose the interval of solution is \(I_1 = [0, 1]\). In this interval, the exact solution doesn’t vanish and we can find \(\epsilon > 0\) such that \([\eta_1, \eta_2]^T \in N_\epsilon([\cos(t), t]^T)\) and \(\eta_1 \neq 0\) for all \(t \in I_1\). Hence, on \(I_1\) the index is do not change and the DAE is in the independent form. But for the interval \(I_2 = [1, 2]\), we have \(y_1(\pi/2) = 0\) and the index changes from 1 to 2 at \(t = \pi/2\). Hence, this DAE has a dependent form on \(I_2\).

**Example 3.7** Let \(A(t)\) and \(F(t, y)\) be the same matrices introduced in the previous example, and let

\[
f(t) = \begin{pmatrix} -e^{2t} \\ -te^t \end{pmatrix}.
\]

Then the exact solution is \([e^t, t]^T\). Obviously, the index of this system remain constant on \(I_1\) and \(I_2\) and on every other bounded interval and it is equal to 1. This free structure DAE has an independent form of index 1.

**Example 3.8** Consider an IAE of the form (1.4). Let

\[
A(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

and

\[
k(t, s, y(s)) = \begin{pmatrix} (y_1^2 + 2)y_2 + e^{y_2} \\ y_1^2 \end{pmatrix}
\]

and let \(f\) be a function such that the exact solution to be \(y(t) = [e^t, t]^T\). For this IAE, we have

\[
k_y(t, s, \eta(s)) = \begin{bmatrix} 2\eta_1 \eta_2 & (\eta_1^2 + 2) + e^{\eta_2} \\ 2\eta_1 & 0 \end{bmatrix}.
\]
Then the index of its corresponding linear IAE is 2, if $\eta_1 \neq 0$, and it is 0, if $\eta_1 = 0$. Hence, this IAE has a free structure. It is obvious that this IAE has an independent form of index 2.

\textbf{Example 3.9} Let $A(t)$ and $\kappa(t, s, y(s))$ be the same matrix introduced in Example 3.7 and let

$$f(t) = \begin{pmatrix} \cos(t) - \frac{\sin(2t)}{4} - e + e^t + \frac{5t^2}{4} - \frac{5}{4} \\ \frac{t^2}{2} + \frac{\sin(2t)}{4} - \frac{\sin(2)}{4} - \frac{1}{2} + \frac{\sin(1)^2}{4} + \frac{\sin(1)^2}{4} + \frac{5t^2}{4} - \frac{5}{4} \end{pmatrix}.$$ 

Then the exact solution is $[\cos(t), t]^T$. This IAE has a free structure dependent form with index changing at the point $t = \pi/2$.

\textbf{4 Numerical experiments}
Figure 2: The numerical and exact solutions of the free structure dependent form DAE in Example 4.1. This figure shows that after the point $t = \pi/2$ the numerical solution does not converge to the exact solution. This results are obtained using the command 'ode15s' with the options 'RelTol' = $1e^{-6}$ and 'AbsTol' = $[1e^{-8}, 1e^{-8}]$.

Figure 3: The numerical and exact solutions of free structure independent form DAE in Example 4.1.
Figure 4: The numerical and exact solutions of free structure independent form IAEs in Example 4.2.

We use the MATLAB codes for solving DAEs and the collocation methods introduced in [12] on the piecewise polynomial space for IAEs.

**Example 4.1** We solve the free structure DAE of Example 3.6 on $I_1$ and $I_2$ using the command ‘ode15s’ in MATLAB. The index on $I_1$ remains constant and we see that the numerical solution gives a good approximation of the exact solution (see Figure 1). But when we use ‘ode15s’ for this DAE on $I_2$, dependent form, the behaviors of the exact and approximate solutions change after $t = \frac{\pi}{2}$ and this is because of changing the index at this point (see Figure 2). This example shows that we should make caution, when we solve a free structure DAE. Figure 4 shows the numerical and exact solutions of the free structure independent form DAE in Example 3.7. This figure shows that the MATLAB code ‘ode15s’ can be used for solving free structure DAEs, although some care is needed.

**Example 4.2** We solve the free structure IAEs of Examples 3.8 and 3.9 using a continuous piecewise collocation method on the space $S_m^{(0)}$ with $c = [0, 7, 9]$, and $h = 0.025$ (see [12]). The same phenomenon of the previous Example is observed again. Figures 4 and 5 show the numerical and exact solutions of these IAEs on $[1, 2]$.

In practice, we do not have the exact solution to find the points that make a dependent form IAE or DAE. Thus the problem is that: how can one determine the interval of convergence? The answer is that, we have the data of numerical solution, using these data and critical conditions one can find the critical points and end the solver or be careful about the
Figure 5: The numerical and exact solutions of independent form free structure IAEs in Example 4.2. We can observe that numerical solution doesn’t converge to the exact solution after the point $\pi/2$. 
approximate solution after the first critical point. In Example 3.6 the condition \( y_1 = 0 \) is a critical condition. We can see from Figure 2 that the numerical values of \( y_1 \) tend to zero as \( t \) tends to \( \pi/2 \). So, we caution about the solution after this point. The same thing are observed in the numerical solution of Example 3.8 in figure 5.

5 Conclusion

The difficulty of index definitions in nonlinear DAEs and IAEs lead to dividing the IAEs and DAEs to three classes: well structure, free structure independent form and free structure dependent form. We observed that the dependency of index definition to unknown solution can not be removed, since it affects on the numerical treatments of solutions. We concluded that we can use the data of numerical solutions, but we must be careful in solving nonlinear DAEs and IAEs.

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