On contact pseudo-metric manifolds satisfying a nullity condition

Narges Ghaffarzadeh\textsuperscript{a}, Morteza Faghfouri\textsuperscript{a,1}

\textsuperscript{a}Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran.

Abstract

In this paper, we aim to introduce and study \((\kappa, \mu)\)-contact pseudo-metric manifold and prove that if the \(\varphi\)-sectional curvature of any point of \(M\) is independent of the choice of \(\varphi\)-section at the point, then it is constant on \(M\) and accordingly the curvature tensor. Also, we introduce generalized \((\kappa, \mu)\)-contact pseudo-metric manifold and prove for \(n > 1\), that a non-Sasakian generalized \((\kappa, \mu)\)-contact pseudo-metric manifold is a \((\kappa, \mu)\)-contact pseudo-metric manifold.

Keywords: contact pseudo-metric manifold, \((\kappa, \mu)\)-contact pseudo-metric structure.

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1. Introduction

Contact pseudo-metric structures were first introduced by Takahashi \cite{1}. He defined Sasakian manifold with pseudo-metric and the classification of Sasakian manifolds of constant \(\phi\)-sectional curvatures. Next, K. L. Duggal \cite{2} and A. Bejancu \cite{3} studied contact pseudo-metric structures as a generalization of contact Lorentzian structures and contact Riemannian structures. Recently, contact pseudo-metric manifolds and curvature of \(K\)-contact pseudo-Riemannian manifolds have been studied by Calvaruso and Perrone \cite{4} and Perrone \cite{5}, respectively. Also Perrone \cite{6}, Perrone investigated contact pseudo-metric manifolds of constant curvature and CR manifolds.

In \cite{7}, D. E. Blair et al. introduced \((\kappa, \mu)\)-contact Riemannian manifold. Since then, many researchers have studied the structure \cite{8, 9, 10, 11, 12, 13}.

In this paper, we introduce and study \((\kappa, \mu)\)-contact pseudo-metric manifold. The paper is organized as follows. Section 2 contains some necessary background on contact pseudo-metric manifolds. After introducing \((\kappa, \mu)\)-contact pseudo-metric mani-
fold in section 3, we prove some relationships. In this section, we also prove if the \( \varphi \)-sectional curvature of any point of \( M \) is independent of the choice of \( \varphi \)-section at the point, then it is constant on \( M \) and we find the curvature tensor. In fact, our main purpose in this paper is to find the curvature tensor of \((\kappa, \mu)\)-contact pseudo-metric manifolds. In addition, we show that \( M \) has constant \( \varphi \)-sectional curvature if and only if \( \mu = \varepsilon \kappa + 1 \) when \( \kappa \neq \varepsilon \). In section 4, we introduce generalized \((\kappa, \mu)\)-contact pseudo-metric manifold. In this section, we also prove for \( n > 1 \), that a non-Sasakian generalized \((\kappa, \mu)\)-contact pseudo-metric manifold is a \((\kappa, \mu)\)-contact pseudo-metric manifold.

2. Preliminaries

A \((2n + 1)\)-dimensional differentiable manifold \( M \) is called an almost contact pseudo-metric manifold if there is an almost contact pseudo-metric structure \((\varphi, \xi, \eta, g)\) consisting of a \((1, 1)\) tensor field \( \varphi \), a vector field \( \xi \), a 1-form \( \eta \) and a compatible pseudo-Riemannian metric \( g \) satisfying

\[
\eta(\xi) = 1, \varphi^2(X) = -X + \eta(X)\xi, \quad (1)
\]
\[
g(\varphi X, \varphi Y) = g(X, Y) - \varepsilon \eta(X)\eta(Y), \quad (2)
\]

where \( \varepsilon = \pm 1 \) and \( X, Y \in \Gamma(TM) \). Remark that, by \(1\) and \(2\), we have

\[
\varphi \xi = 0, \quad \eta \circ \varphi = 0, \quad (3)
\]
\[
\eta(X) = \varepsilon g(\xi, X), \quad (4)
\]
\[
g(\varphi X, Y) = -g(X, \varphi Y), \quad (5)
\]

and \( \varphi \) has rank \( 2n \). In particular, \( g(\xi, \xi) = \varepsilon \) and so, the characteristic vector field \( \xi \) is either space-like or time-like, but cannot be light-like and the signature of an associated metric is either \((2p + 1, 2n - 2p)\) or \((2p, 2n - 2p - 1)\). An almost contact pseudo-metric structure becomes a contact pseudo-metric structure if \( d\eta = \Phi \), where \( \Phi(X, Y) = g(X, \varphi Y) \) is the fundamental 2-form of \( M \).

An almost contact pseudo-metric structure of \( M \) is called a normal structure if \( [\varphi, \varphi] + 2d\eta \otimes \xi = 0 \). A normal contact pseudo-metric structure is called a Sasakian structure. It can be proved that an almost contact pseudo-metric manifold is Sasakian iff

\[
(\nabla_X \varphi)Y = g(X, Y)\xi - \varepsilon \eta(Y)X, \quad (6)
\]
for any $X, Y \in \Gamma(TM)$ or equivalently, a contact pseudo-metric structure is a Sasakian structure iff $R$ satisfies

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y,$$

for $X, Y \in \Gamma(TM)$, where $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ is the curvature tensor and $\nabla$ is the Levi-Civita connection \cite{[1, 4]}. In a contact pseudo-metric manifold $M^{2n+1}(\varphi, \xi, \eta, g)$, we define the $(1, 1)$-tensor fields $\ell$ and $h$ by

$$\ell X = R(X, \xi)\xi, \quad hX = \frac{1}{2}(\mathcal{L}_\xi \varphi)(X),$$

where $\mathcal{L}$ denotes the Lie derivative. The tensors $h$ and $\ell$ are self-adjoint operators satisfying\cite{[4, 5]}

$$\text{trace}(h) = \text{trace}(h\varphi) = 0,$$

$$\eta \circ h = 0, \quad \ell \xi = 0,$$

$$h\varphi = -\varphi h,$$

$$h\xi = 0,$$

$$\nabla_X \xi = -\varepsilon\varphi X - \varphi hX,$$

$$(\nabla_X \varphi)Y = \varepsilon g(\varepsilon X + hX, Y)\xi - \eta(Y)(\varepsilon X + hX),$$

$$\nabla_\xi \varphi = 0.$$  

Due to the relation of \cite{[11]}, if $X$ is an eigenvector of $h$ corresponding to the eigenvalue $\lambda$, then $\varphi X$ is also an eigenvector of $h$ corresponding to the eigenvalue $-\lambda$.

Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a contact pseudo-metric manifold and $X \in \ker \eta$, either space-like or time-like. We put

$$K(X, \xi) = \frac{R(X, \xi, X, \xi)}{\varepsilon g(X, X)} = \frac{g(\ell X, X)}{\varepsilon g(X, X)},$$

$$K(X, \varphi X) = \frac{R(X, \varphi X, X, \varphi X)}{g(X, X)^2}.$$

We call $K(X, \xi)$ the $\xi$-sectional curvature determined by $X$, and $K(X, \varphi X)$ the $\varphi$-sectional curvature determined by $X$, where $R(X, Y, Z, W) = g(R(Z, W)Y, X)$. A Sasakian manifold with constant $\varphi$-sectional curvature $c$ is called a Sasakian space form and is denoted by $M(c)$. 

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Lemma 2.1. Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a contact pseudo-metric manifold. Then:

\[ \nabla_\xi h = \varphi - \varphi_\ell - \varphi h^2, \]

\[ \varphi_\ell \varphi - \ell = 2(\varphi^2 + h^2), \]

\[ \text{Ric}(\xi, \xi) = 2n - \text{trace}(h^2), \]

\[ \mathcal{R}(\xi, X, Y, Z) = \varepsilon(\nabla_X \Phi)(Y, Z) + g((\nabla_Y \varphi h)Z, X) - g((\nabla_Z \varphi h)Y, X). \]

3. $(\kappa, \mu)$-contact pseudo-metric manifold

A $(\kappa, \mu)$-nullity distribution of a contact pseudo-metric manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ is a distribution

\[ N_{p}(\kappa, \mu) = \{ Z \in T_{p}M : R(X, Y)Z = \kappa(g(Y, Z)X - g(X, Z)Y) \]

\[ + \mu(g(Y, Z)hX - g(X, Z)hY) \}, \]

where $(\kappa, \mu) \in \mathbb{R}^2$. Thus, the characteristic vector field $\xi$ belongs to the $(\kappa, \mu)$-distribution iff

\[ R(X, Y)\xi = \varepsilon \kappa(\eta(Y)X - \eta(X)Y) + \varepsilon \mu(\eta(Y)hX - \eta(X)hY). \]

If a contact pseudo-metric manifold satisfying $(\kappa, \mu)$, we call $(\kappa, \mu)$-contact pseudo-metric manifold. The class of $(\kappa, \mu)$-contact pseudo-metric manifold contains the class of Sasakian manifolds, which we get for $\kappa = \varepsilon$ (and hence $h = 0$, by 7).

Lemma 3.1. Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a $(\kappa, \mu)$-contact pseudo-metric manifold. Then, we have

\[ \ell \varphi - \varphi_\ell = 2\varepsilon \mu h \varphi, \]

\[ h^2 = (\varepsilon \kappa - 1) \varphi^2, \quad \varepsilon \kappa \leq 1, \quad \text{and} \quad \kappa = \varepsilon \text{ iff } M^{2n+1} \text{ is Sasakian}, \]

\[ R(\xi, X)Y = \kappa(g(X, Y)\xi - \varepsilon \eta(Y)X) + \mu(g(hX, Y)\xi - \varepsilon \eta(Y)hX), \]

\[ Q\xi = 2n\kappa \xi, \quad Q \text{ is the Ricci operator}, \]

\[ (\nabla_X h)Y - (\nabla_Y h)X = (1 - \varepsilon \kappa)\{2\varepsilon g(X, \varphi Y)\xi + \eta(X)\varphi Y - \eta(Y)\varphi X \}
\]

\[ + \varepsilon(1 - \mu)\{\eta(X)\varphi hY - \eta(Y)\varphi hX \}, \]

\[ \xi \kappa = 0, \]

where $X, Y \in \Gamma(TM)$.

Proof. Using $(\kappa, \mu)$, we obtain

\[ tX = \varepsilon \kappa(X - \eta(X)\xi) + \varepsilon \mu h X, \]
for \(X \in \Gamma(TM)\). Replacing \(X\) by \(\varphi X\) and at the same time applying \(\varphi\), we obtain

\[
\ell \varphi = \varepsilon \{\kappa \varphi + \mu h \varphi\} \quad \text{and} \quad \varphi \ell = \varepsilon \{\kappa \varphi + \mu \varphi h\}. \tag{30}
\]

Subtracting (30) and using (11), we get (24).

By using the relations (19), (11), (30), (12) and (1), we deduce the first part of (25).

Now since \(h\) is symmetric, from the second part of (1), we have \(\varepsilon \kappa \leq 1\). Moreover, \(\kappa = \varepsilon \) iff \(h = 0\). Using (23) and (7), the proof of (25) is completed.

Using (23), we get (26) and

\[
g((\nabla_X \varphi h)Y - (\nabla_Y \varphi h)X) = \varphi((\nabla_X h)Y - (\nabla_Y h)X),
\]

for any vector fields \(X, Y\) on \(M\) and hence (21) is reduced to

\[
R(Y, X)\xi = \eta(X)(Y + \varepsilon hY) - \eta(Y)(X + \varepsilon hX) + \varphi((\nabla_X h)Y - (\nabla_Y h)X).
\]

Comparing this equation with (23), we have

\[
\varphi((\nabla_X h)Y - (\nabla_Y h)X) = (\varepsilon\kappa - 1)(\eta(X)Y - \eta(Y)X) + \varepsilon(\mu - 1)(\eta(X)hY - \eta(Y)hX). \tag{31}
\]

Using (13), the symmetry of \(h\) and \(\nabla_X h\), we obtain

\[
g((\nabla_X h)Y - (\nabla_Y h)X, \xi) = 2(\varepsilon\kappa - 1)g(Y, \varphi X). \tag{32}
\]

Acting now by \(\varphi\) on (31) and using (32), we get (28).
Lemma 3.2. Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a $(\kappa, \mu)$-contact pseudo-metric manifold. Then for all $X, Y, Z \in \Gamma(TM)$, we have

$$R(X, Y)\varphi Z = \varphi R(X, Y)Z + \{ (1 - \varepsilon \kappa) \eta(X)g(\varphi Y, Z) - \eta(Y)g(\varphi X, Z) \}$$

$$+ \varepsilon(1 - \mu)[\eta(X)g(\varphi hY, Z) - \eta(Y)g(\varphi hX, Z)]\xi$$

$$- g(Y + \varepsilon hY, Z)(\varepsilon \varphi X + \varphi hX) + g(X + \varepsilon hX, Z)(\varepsilon \varphi Y + \varphi hY)$$

$$- g(\varepsilon \varphi Y + \varphi hY, Z)(X + \varepsilon hX) + g(\varepsilon \varphi X + \varphi hX, Z)(Y + \varepsilon hY)$$

$$- \eta(Z)\{ (1 - \varepsilon \kappa)[\eta(X)\varphi Y - \eta(Y)\varphi X] + \varepsilon(1 - \mu)[\eta(X)\varphi hY - \eta(Y)\varphi hX] \}.$$  

(33)

Proof. Assume that $p \in M$ and $X, Y, Z$ local vector fields on a neighborhood of $p$, such that

$$(\nabla X)_p = (\nabla Y)_p = (\nabla Z)_p = 0.$$  

The Ricci identity for $\varphi$:

$$R(X, Y)\varphi Z - \varphi R(X, Y)Z = (\nabla_X \nabla_Y \varphi)Z - \nabla_Y (\nabla_X \varphi)Z - (\nabla_{[X,Y]} \varphi)Z,$$  

(34)

at the point $p$, takes the form

$$R(X, Y)\varphi Z - \varphi R(X, Y)Z = \nabla_X (\nabla_Y \varphi)Z - \nabla_Y (\nabla_X \varphi)Z.$$  

(35)

On the other hand, combining (13) and (14), we have at $p$

$$\nabla_X (\nabla_Y \varphi)Z - \nabla_Y (\nabla_X \varphi)Z = \varepsilon g((\nabla_Y h)Y - (\nabla_X h)X, Z)\xi - g(Y + \varepsilon hY, Z)(\varepsilon \varphi X + \varphi hX)$$

$$+ \varepsilon g(\varepsilon \varphi X + \varphi hX, Z)(\varepsilon Y + hY) + g(X + \varepsilon hX, Z)(\varepsilon \varphi Y + \varphi hY)$$

$$- \varepsilon g(Z, \varepsilon \varphi Y + \varphi hY)(\varepsilon X + hX) - \eta(Z)((\nabla_X h)Y - (\nabla_Y h)X)$$

(36)

Now equation (33) is a straightforward combination of the (36), (33) and (28).

Theorem 3.1. Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a $(\kappa, \mu)$-contact pseudo-metric manifold. Then $\varepsilon \kappa \leq 1$. If $\kappa = \varepsilon$, then $h = 0$ and $M^{2n+1}$ is a Sasakian-space-form and if $\varepsilon \kappa < 1$, then $M^{2n+1}$ admits three mutually orthogonal and integrable distributions $D(0) =$

Span$\{\xi\}$, $D(\lambda)$ and $D(-\lambda)$, defined by the eigenspaces of $h$, where $\lambda = \sqrt{1 - \varepsilon \kappa}.$

Proof. By $\xi \in N(\kappa, \mu)$, we can verify Ric$(\xi, \xi) = 2\varepsilon \kappa$. Then, (20) implies $\varepsilon \kappa \leq 1$. Now, we suppose $\varepsilon \kappa < 1$. Then since $h$ is symmetric, the relations (12) and (14) imply that the restriction $h|D$ of $h$ to the contact distribution $D$ has eigenvalues $\lambda = \sqrt{1 - \varepsilon \kappa}$ and $-\lambda$. By $D(\lambda)$ and $D(-\lambda)$, we denote the distributions defined by
the eigenspaces of $h$ corresponding to $\lambda$ and $-\lambda$, respectively. By $\mathcal{D}(0)$, we denote the distribution defined by $\xi$. Then these three distributions are mutually orthogonal. Let $X \in \mathcal{D}(\lambda)$, then $hX = \lambda X$ and the relation of (11) imply $h(\varphi X) = -\lambda(\varphi X)$. Hence, we have $\varphi X \in \mathcal{D}(-\lambda).$ This means that the dimension of $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ are equal to $n$. We prove that $\mathcal{D}(\lambda)$ (resp., $\mathcal{D}(-\lambda)$) is integrable. Let $X, Y \in \mathcal{D}(\lambda)$ (resp., $\mathcal{D}(-\lambda)$). Then

$$\nabla_X \varphi = -\varphi X - \varphi h X = -(\epsilon + \lambda)\varphi X,$$

and $\nabla_Y \varphi = -(\epsilon + \lambda)\varphi Y$. So, $g(\nabla_X \varphi, Y) = g(\nabla_Y \varphi, X)$ holds. Thus, $d\eta(X, Y) = 0$ and $\eta([X, Y]) = 0$ follow. $X, Y \in \mathcal{D}(\lambda)$ and $\xi \in N(\kappa, \mu)$ imply $R(X, Y)\xi = 0$. On the other hand,

$$0 = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X, Y]}\xi$$

$$= -\epsilon \nabla_X (\varphi Y) + (\epsilon + \lambda)\nabla_Y (\varphi X) + \epsilon \varphi ([X, Y]) + \varphi h([X, Y])$$

$$= -\epsilon \nabla_X (\varphi Y) - (\nabla_Y \varphi)X \mp \lambda \varphi ([X, Y]) + \varphi h([X, Y]).$$

By (14), the first term of the last line (37) vanishes. And so, we obtain

$$\varphi h([X, Y]) = \mp \lambda \varphi ([X, Y]),$$

which together with $\eta([X, Y]) = 0$ implies $[X, Y] \in \mathcal{D}(\lambda)$ (resp., $\mathcal{D}(-\lambda)$).

**Proposition 3.1.** Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a $(\kappa, \mu)$-contact pseudo-metric manifold with $\epsilon \kappa < 1$, then

- If $X, Y \in \mathcal{D}(\lambda)$ (resp., $\mathcal{D}(-\lambda)$), then $\nabla_X Y \in \mathcal{D}(\lambda)$ (resp., $\mathcal{D}(-\lambda)$).
- If $X \in \mathcal{D}(\lambda), Y \in \mathcal{D}(-\lambda)$, then $\nabla_X Y \in \mathcal{D}(\lambda)$ (resp., $\mathcal{D}(-\lambda)$).

**Lemma 3.3.** Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a $(\kappa, \mu)$-contact pseudo-metric manifold. Then for any vector fields $X, Y$ on $M$, we have

$$(\nabla_X h)Y = (\{\epsilon - \kappa\}g(X, \varphi Y) + g(X, h\varphi Y))\xi$$

$$+ \eta(Y)[h(\epsilon \varphi X + \varphi h X)] - \epsilon \mu \eta(X) \varphi h Y.$$

**Proof.** Let $\epsilon \kappa < 1$ and $X, Y \in \mathcal{D}(\lambda)$ (resp., $\mathcal{D}(-\lambda)$). Then from Proposition 3.1, we have $\nabla_X Y \in \mathcal{D}(\lambda)$ (resp., $\mathcal{D}(-\lambda)$). Then one easily proves that

$$(\nabla_X h)Y = 0.$$
Suppose now that \( X \in \mathcal{D}(\lambda), Y \in \mathcal{D}(-\lambda) \) and \( \{E_i, \varphi E_i, \xi\} \) be a (local) \( \varphi \)-basis of vector fields on \( M \) with \( E_i \in \mathcal{D}(\lambda) \) and so \( \varphi E_i \in \mathcal{D}(-\lambda) \). For any index \( i = 1, \cdots, 2n, \) \( \{\xi, E_i\} \) spans a non-degenerate plane on the tangent space at each point where the basis is defined. Then using Proposition 3.1 and the relations (12), (13), and (14), we calculate

\[
\begin{align*}
  h \nabla_X Y &= h \left\{ \sum_{i=1}^{n} \varepsilon_i g(\nabla_X Y, \varphi E_i) \varphi E_i + \varepsilon g(\nabla_X Y, \xi) \xi \right\} \\
  &= \sum_{i=1}^{n} \varepsilon_i g(\nabla_X Y, \varphi E_i) h \varphi E_i \\
  &= \lambda \varphi \sum_{i=1}^{n} \varepsilon_i g(\varphi \nabla_X Y, E_i) E_i \\
  &= \lambda \varphi^2 (\nabla_X Y) \\
  &= \lambda (-\nabla_X Y + \varepsilon g(\nabla_X Y, \xi) \xi) \\
  &= \lambda (-\nabla_X Y - \varepsilon g(Y, \nabla_X \xi) \xi) \\
  &= \lambda (-\nabla_X Y - \varepsilon g(Y, -\varepsilon \varphi X - \varphi h X) \xi) \\
  &= \lambda (-\nabla_X Y + g(Y, \varphi X + \varepsilon \varphi h X) \xi) \\
  &= \lambda (-\nabla_X Y - g(\varphi Y, X + \varepsilon h X) \xi) \\
  &= \nabla_X h Y - \lambda (1 + \varepsilon \lambda) g(X, \varphi Y) \xi,
\end{align*}
\]

and so

\[
(\nabla_X h) Y = \lambda (1 + \varepsilon \lambda) g(X, \varphi Y) \xi, \tag{40}
\]

Similarly, we obtain

\[
(\nabla_Y h) X = \lambda (\varepsilon \lambda - 1) g(Y, \varphi X) \xi, \tag{41}
\]

Suppose now that \( X, Y \) are arbitrary vector fields on \( M \) and write

\[
X = X_\lambda + X_{-\lambda} + \eta(X) \xi,
\]

and

\[
Y = Y_\lambda + Y_{-\lambda} + \eta(Y) \xi,
\]

where \( X_\lambda \) (resp., \( X_{-\lambda} \)) is the component of \( X \) in \( \mathcal{D}(\lambda) \) (resp., \( \mathcal{D}(-\lambda) \)). Then using (39), (40), (41) and \( \nabla_\xi h = \varepsilon \mu h \varphi \), which follows from (28), we get by a direct
computation

\[(\nabla_X h)Y = \varepsilon \lambda^2 \{g(X_\lambda, \varphi Y_-\lambda) + g(X_-\lambda, \varphi Y_\lambda)\} \xi \]
\[+ \lambda \{g(X_\lambda, \varphi Y_-\lambda) - g(X_-\lambda, \varphi Y_\lambda)\} \xi \]
\[+ \eta(Y) h(\varepsilon \varphi X + \varphi hX) - \varepsilon \mu \eta(X) \varphi hY.\]  \hspace{1cm} (42)

On the other hand, we easily find that

\[ g(hX, \varphi Y) = \lambda \{g(X_\lambda, \varphi Y_-\lambda) - g(X_-\lambda, \varphi Y_\lambda)\}, \] \hspace{1cm} (43)

\[ g(hX, h\varphi Y) = \lambda^2 \{g(X_\lambda, \varphi Y_-\lambda) + g(X_-\lambda, \varphi Y_\lambda)\}. \] \hspace{1cm} (44)

The relations (43) and (44) with (42), give the required equation (38). Note that for \( \kappa = \varepsilon \) (and so \( h = 0 \)), (38) is valid identically and the proof is completed.  \( \Box \)

**Lemma 3.4.** Let \( M^{2n+1}(\varphi, \xi, \eta, g) \) be a \((\kappa, \mu)\)-contact pseudo-metric manifold. Then for any vector fields \( X, Y, Z \) on \( M \). We have

\[ R(X, Y)hZ - hR(X, Y)Z = \{\kappa[\eta(X)g(hY, Z) - \eta(Y)g(hX, Z)] \]
\[+ \mu(\varepsilon \kappa - 1)[\eta(Y)g(X, Z) - \eta(X)g(Y, Z)]\} \xi \]
\[+ \kappa \{g(Y, \varphi Z)\varphi hX - g(X, \varphi Z)\varphi hY + g(Z, \varphi hY)\varphi X \]
\[- g(Z, \varphi hX)\varphi Y + \varepsilon \eta(Z) [\eta(X)hY - \eta(Y)hX] \]
\[+ \mu \eta(Y) [(\varepsilon - \kappa)\eta(Z)X + \mu \eta(X)hZ] \]
\[- \eta(X) [(\varepsilon - \kappa)\eta(Z)Y + \mu \eta(Y)hZ] + 2\varepsilon g(X, \varphi Y)\varphi hZ}. \] \hspace{1cm} (45)

**Proof.** The Ricci identity for \( h \) is

\[ R(X, Y)hZ - hR(X, Y)Z = (\nabla_X \nabla_Y h)Z - (\nabla_Y \nabla_X h)Z - (\nabla_{[X,Y]} h)Z. \] \hspace{1cm} (46)

Using Lemma 3.3, the relations (25), (11) and the fact that \( \nabla_X \varphi \) is antisymmetric, we obtain

\[(\nabla_X \nabla_Y h)Z = \{((\varepsilon - \kappa)g(\nabla_X Y, \varphi Z) - g((\nabla_X \varphi)Y, Z)] \]
\[+ g(\nabla_X Y, h\varphi Z) + g(\nabla_X (h\varphi)Y, Z)] \xi \]
\[+ \{(\varepsilon - \kappa)g(Y, \varphi Z) + g(Y, h\varphi Z)] \nabla_X \xi \]
\[+ \varepsilon g(Z, \nabla_X \xi)[\varepsilon \varphi Y + (\varepsilon \kappa - 1)\varphi Y] \]
\[+ \eta(Z) \{[\varepsilon(\nabla_X h\varphi)Y + h\varphi(\nabla_X Y)] + (\varepsilon \kappa - 1) \{(\nabla_X \varphi)Y + \varphi(\nabla_X Y)] \}
\[- \varepsilon \mu \eta(\nabla_X Y) \varphi hZ + \varepsilon g(Y, \nabla_X \xi) \varphi hZ + \eta(Y) (\nabla_X \varphi h)Z}. \]
So, using also (38), (13), (14) and Lemma 3.3, equation (46) yields

\[
R(X, Y)hZ - hR(X, Y)Z
\]

\[
= \{(\kappa - \varepsilon)g((\nabla_X \varphi)Y - (\nabla_Y \varphi)X, Z) + g((\nabla_X h\varphi)Y - (\nabla_Y h\varphi)X, Z)\}\xi
\]

\[
+ \{(\varepsilon - \kappa)g(Y, \varphi Z) + g(Y, h\varphi Z)\}\nabla_X \xi
\]

\[
- \{(\varepsilon - \kappa)g(X, \varphi Z) + g(X, h\varphi Z)\}\nabla_Y \xi
\]

\[
+ g(Z, \nabla_X \xi)[h\varphi Y + (\kappa - \varepsilon)\varphi Y]
\]

\[
- g(Z, \nabla_Y \xi)[h\varphi X + (\kappa - \varepsilon)\varphi X]
\]

\[
+ \eta(Z)[\varepsilon(\nabla_X h\varphi)Y - (\nabla_Y h\varphi)X] + (\varepsilon\kappa - 1)[(\nabla_X \varphi)Y - (\nabla_Y \varphi)X] + \varepsilon\mu(\eta(Y)(\nabla_X \varphi)hZ - \eta(X)(\nabla_Y \varphi)hZ + 2g(X, \varphi Y)\varphi hZ).
\]

(47)

Using now (14), (12) and Lemma 3.3 we have

\[
(\nabla_X \varphi h)Y = \{g(X, hY) + (\kappa - \varepsilon)g(X, -Y + \eta(Y))\}\xi
\]

\[
+ \eta(Y)[\varepsilon hX + (\varepsilon\kappa - 1)(-X + \eta(X)\xi)] + \varepsilon\mu\eta(X)hY.
\]

Therefore, equation (47), by using (14) again, is reduced to (45) and the proof is completed.

\[\square\]

**Theorem 3.2.** Let \(M^{2n+1}(\varphi, \xi, \eta, g)\) be a \((\kappa, \mu)\)-contact pseudo-metric manifold. If \(\varepsilon\kappa < 1\), then for all \(X_\lambda, Z_\lambda, Y_\lambda \in \mathcal{D}(\lambda)\) and \(X_{-\lambda}, Z_{-\lambda}, Y_{-\lambda} \in \mathcal{D}(-\lambda)\), we have

\[
R(X_\lambda, Y_\lambda)Z_{-\lambda} = (\kappa - \varepsilon\mu)[g(\varphi Y_\lambda, Z_{-\lambda})\varphi X_\lambda - g(\varphi X_\lambda, Z_{-\lambda})\varphi Y_\lambda],
\]

(48)

\[
R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = (\kappa - \varepsilon\mu)[g(\varphi Y_{-\lambda}, Z_{-\lambda})\varphi X_{-\lambda} - g(\varphi X_{-\lambda}, Z_{-\lambda})\varphi Y_{-\lambda}],
\]

(49)

\[
R(X_{-\lambda}, Y_\lambda)Z_{-\lambda} = -\kappa g(\varphi Y_\lambda, Z_{-\lambda})\varphi X_{-\lambda} - \varepsilon\mu g(\varphi Y_\lambda, X_{-\lambda})\varphi Z_{-\lambda},
\]

(50)

\[
R(X_{-\lambda}, Y_\lambda)Z_\lambda = \kappa g(\varphi X_{-\lambda}, Z_\lambda)\varphi Y_\lambda + \varepsilon\mu g(\varphi X_{-\lambda}, Y_\lambda)\varphi Z_\lambda,
\]

(51)

\[
R(X_{-\lambda}, Y_\lambda)Z_{-\lambda} = [2(\varepsilon + \lambda) - \varepsilon\mu][g(Y_\lambda, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}],
\]

(52)

\[
R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = [2(\varepsilon - \lambda) - \varepsilon\mu][g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}],
\]

(53)

**Proof.** The first part of the Theorem follows from (25) and Lemma 3.1.

Let \(\{E_1, \cdots, E_n, E_{n+1} = \varphi E_1, \cdots, E_{2n} = \varphi E_n, E_{2n+1} = \xi\}\) be a (local) \(\varphi\)-basis of vector fields on \(M\) with \(E_i \in \mathcal{D}(\lambda)\) and so \(\varphi E_i \in \mathcal{D}(-\lambda)\). For any index \(i = 1, \cdots, 2n, \{\xi, E_i\}\) spans a non-degenerate plane on the tangent space at each point,
where the basis is defined. Then, we have

$$R(\lambda, Y, Z) = \sum_{i=1}^{n} \varepsilon_i \{ g(R(\lambda, Y, Z) \phi E_i) \phi E_i + \varepsilon g(R(\lambda, Y, Z) \phi E_i) \phi E_i \}$$

(54)

$$+ \varepsilon g(R(\lambda, Y, Z) \phi E_i) \phi E_i.$$ But since $\xi$ belonging to the $(\kappa, \mu)$-nullity distribution, using (23), we easily have

$$g(R(\lambda, Y, Z) \phi E_i) \phi E_i = -g(R(\lambda, Y, Z) \phi E_i, Z) = 0.$$ By Proposition 3.1 we get

$$g(R(\lambda, Y, Z) \phi E_i) \phi E_i = -g(R(\lambda, Y, Z) \phi E_i, Z) = 0.$$ On the other hand, if $X \in \mathcal{D}(\lambda)$ and $Y, Z \in \mathcal{D}(-\lambda)$, then applying (45), we get

$$hR(X, Y)Z + \lambda R(X, Y)Z = -2\lambda \{ \kappa g(X, \phi Z) \phi Y + \varepsilon \mu g(X, \phi Y) \phi Z \},$$ and taking the inner product with $W \in \mathcal{D}(\lambda)$, we obtain

$$g(R(X, Y)Z, W) = -\kappa g(X, \phi Z) g(\phi Y, W) - \varepsilon \mu g(X, \phi Y) g(\phi Z, W),$$ (55) for any $X, W \in \mathcal{D}(\lambda)$ and $Y, Z \in \mathcal{D}(-\lambda)$. Using (55) and the first Bianchi identity, we calculate

$$\sum_{i=1}^{n} \varepsilon_i g(R(\lambda, Y, Z) \phi E_i) \phi E_i$$

$$= -\sum_{i=1}^{n} \varepsilon_i g(R(\lambda, Y, Z) \phi E_i) \phi E_i - \sum_{i=1}^{n} \varepsilon_i g(R(\lambda, Y, Z) \phi E_i) \phi E_i$$

$$= \sum_{i=1}^{n} \varepsilon_i g(R(\lambda, Y, Z) \phi E_i, X) \phi E_i - \sum_{i=1}^{n} \varepsilon_i g(R(\lambda, Y, Z) \phi E_i, Y) \phi E_i$$

$$= \sum_{i=1}^{n} \varepsilon_i \{ -\kappa g(X, \phi^2 E_i) g(\phi Z, Y) \phi E_i - \varepsilon \mu g(X, \phi Z) g(\phi^2 E_i, Y) \phi E_i \}$$

$$- \sum_{i=1}^{n} \varepsilon_i \{ -\kappa g(X, \phi^2 E_i) g(\phi Z, Y) \phi E_i - \varepsilon \mu g(X, \phi Z) g(\phi^2 E_i, Y) \phi E_i \}$$

$$= \kappa g(\phi Z, X) \phi \sum_{i=1}^{n} \varepsilon_i g(Y, E_i) E_i + \varepsilon \mu g(Y, \phi Z) \phi \sum_{i=1}^{n} \varepsilon_i g(E_i, X)$$

$$+ \kappa g(\phi Z, Y) \phi \sum_{i=1}^{n} \varepsilon_i g(X, E_i) E_i + \varepsilon \mu g(X, \phi Z) \phi \sum_{i=1}^{n} \varepsilon_i g(E_i, Y)$$

$$= \kappa \{ g(Z, \phi Y) \phi X - g(Z, \phi X) \phi Y \}$$

$$+ \varepsilon \mu \{ g(\phi X, \phi Z) \phi Y - g(\phi Y, \phi Z) \phi X \}$$

$$= (\kappa - \varepsilon \mu) \{ g(Z, \phi Y) \phi X - g(Z, \phi X) \phi Y \}.$$
Therefore, (54) gives

$$R(X_\lambda, Y_\lambda)Z_{-\lambda} = (\kappa - \varepsilon \mu) \{ g(Z_{-\lambda}, \varphi Y_\lambda) \varphi X_\lambda - g(Z_{-\lambda}, \varphi X_\lambda) \varphi Y_\lambda \}. $$

The proof of the remaining cases are similar and will be omitted.

Then they showed the following.

**Theorem 3.3.** Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a $(\kappa, \mu)$-contact pseudo-metric manifold. If $\varepsilon \kappa < 1$, then for any $X$ orthogonal to $\xi$

(i) the $\xi$-sectional curvature $K(X, \xi)$ is given by

$$K(X, \xi) = \kappa + \mu \frac{g(hX, X)}{g(X, X)} = \begin{cases} 
\kappa + \lambda \mu, & \text{if } X \in \mathcal{D}(\lambda), \\
\kappa - \lambda \mu, & \text{if } X \in \mathcal{D}(-\lambda), 
\end{cases}$$

(ii) the sectional curvature of a plane section $(X, Y)$ normal to $\xi$ is given by

$$K(X, Y) = \begin{cases} 
2(\varepsilon + \lambda) - \varepsilon \mu, & \text{for any } X, Y \in \mathcal{D}(\lambda), n > 1, \\
-(\kappa + \varepsilon \mu) \frac{g(X, \varphi Y)^2}{g(X, X)g(Y, Y)}, & \text{for any unit vectors } X \in \mathcal{D}(\lambda), Y \in \mathcal{D}(-\lambda), \\
2(\varepsilon - \lambda) - \varepsilon \mu, & \text{for any } X, Y \in \mathcal{D}(-\lambda), n > 1, 
\end{cases}$$

(56)

(iii) The Ricci operator is given by

$$QX = \varepsilon [2(2n - 1) - n \mu] X + (2(n - 1) + \mu) hX + [2(1 - n) \varepsilon + 2n \kappa + n \varepsilon \mu] \eta(X) \xi.$$  

(57)

**Proof.**

(i) From (16), if we set $Y = \xi$ in the relation of (23), for $X$ orthogonal to $\xi$ from which, taking the inner product with $X$, we get

$$K(X, \xi) = \frac{\varepsilon \{ \kappa g(X, X) + \mu g(hX, X) \}}{\varepsilon g(X, X)}. $$

So, we have

$$K(X, \xi) = \kappa + \mu \frac{g(hX, X)}{g(X, X)} = \kappa + \mu \frac{\lambda g(hX_\lambda, X_\lambda) - \lambda g(hX_{-\lambda}, X_{-\lambda})}{g(X_\lambda, X_\lambda) + g(X_{-\lambda}, X_{-\lambda})}, $$

which is the required result.

(ii) This follows immediately from Theorem 3.2.
(iii) The first consider a \( \varphi \)-basis \( \{ E_1, \ldots, E_n, E_{n+1} = \varphi E_1, \ldots, E_{2n} = \varphi E_n, E_{2n+1} = \xi \} \) of vector fields on \( M \).

For any index \( i = 1, \ldots, 2n \), \( \{ \xi, E_i \} \) spans a non-degenerate plane on the tangent space at each point, where the basis is defined. Putting \( Y = Z = E_i \) in \( R(X, Y)Z \), adding with respect to index of \( i \) and using (1), (2) and (11), we get the following formula, for the Ricci operator, at any point of \( M \):

\[
Q X = \sum_{i=1}^{n} \varepsilon_i \{ R(X, E_i)E_i + R(X, \varphi E_i)\varphi E_i \} + \varepsilon R(X, \xi)\xi.
\]

Suppose now that \( X \) is arbitrary vector fields and write

\[
X = X_\lambda + X_{-\lambda} + \eta(X)\xi,
\]

On the other hand, from Theorem \( \ref{thm:3.2} \) we have

\[
Q X = \sum_{i=1}^{n} \varepsilon_i \{ R(X_\lambda, E_i)E_i + R(X_{-\lambda}, E_i)E_i + \eta(X) R(\xi, E_i)E_i + R(X_\lambda, \varphi E_i)\varphi E_i
\]

\[
+ \eta(X) R(\xi, \varphi E_i)\varphi E_i + R(X_{-\lambda}, \varphi E_i)\varphi E_i \} + \varepsilon R(X_\lambda, \xi)\xi + \varepsilon R(X_{-\lambda}, \xi)\xi
\]

\[
= [2(\varepsilon + \lambda) - \varepsilon \mu](n - 1)X_\lambda - (\kappa + \varepsilon \mu)X_{-\lambda} + n\kappa\eta(X)\xi - (\kappa + \varepsilon \mu)X_\lambda
\]

\[
+ [2(\varepsilon - \lambda) - \varepsilon \mu](n - 1)X_{-\lambda} + (\kappa + \varepsilon \mu)X_\lambda + n\kappa\eta(X)\xi + (\kappa + \mu h)X_{-\lambda}
\]

\[
= \varepsilon [(2 - \mu)(n - 1) - \mu](X_\lambda + X_{-\lambda})
\]

\[
+ [2(n - 1) + \mu h](X_\lambda + X_{-\lambda}) + 2n\kappa\eta(X)\xi.
\]

So, the relation of (57) is obtained.

**Theorem 3.4.** Let \( M^{2n+1}(\eta, \xi, \varphi, g) \) be a \( (\kappa, \mu) \)-contact pseudo-metric manifold and \( n > 1 \). If the \( \varphi \)-sectional curvature of any point of \( M \) is independent of the choice of \( \varphi \)-section at the point, then it is constant on \( M \) and the curvature tensor is given by

\[
R(X, Y)Z = \left( \frac{c + 3\varepsilon}{4} \right) \{ g(Y, Z)X - g(X, Z)Y \}
\]

\[
+ \left( \frac{c - \varepsilon}{4} \right) \{ 2g(X, \varphi Y)\varphi Z + g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X \}
\]

\[
+ \left( \frac{c + 3\varepsilon}{4} - \kappa \right) \{ \varepsilon \eta(X)\eta(Z)Y - \varepsilon \eta(Y)\eta(Z)X + \eta(Y)g(X, Z)\xi - \eta(X)g(Y, Z)\xi \}
\]

\[
+ \{ -g(X, Z)hY - g(hX, Z)Y + g(Y, Z)hX + g(hY, Z)X \}
\]

\[
+ \frac{\varepsilon}{2} \{ -g(hX, Z)hY + g(hY, Z)hX + \varphi hX, \varphi hY - g(\varphi hY, Z)\varphi hX \}
\]

\[
+ (1 - \mu) \{ \varepsilon \eta(X)\eta(Z)hY + \eta(Y)g(hX, Z)\xi - \varepsilon \eta(Y)\eta(Z)hX - \eta(X)g(hY, Z)\xi \},
\]

(58)

where \( c \) is the constant \( \varphi \)-sectional curvature. Moreover if \( \kappa \neq \varepsilon \), then \( \mu = \varepsilon \kappa + 1 \) and \( c = -2\kappa - \varepsilon \).
Proof. For the Sasakian case $\kappa = \varepsilon$, the proof is known (1). So, we have to prove the theorem for $\kappa \neq \varepsilon$. Let $p \in M$ and $X, Y \in T_p M$ orthogonal to $\xi$. Using the first identity of Bianchi, the basic properties of the curvature tensor, $\varphi$ is antisymmetric, $h$ is symmetric, (1) and (2), we obtain from (3), successively:

$$g(R(X, \varphi X)Y, \varphi Y) = g(R(X, \varphi Y)Y, \varphi X) + g(R(X, Y)X, Y) - \varepsilon g(X, Y)^2 - \varepsilon g(hX, Y)^2$$

$$- 2g(X, Y)g(hX, Y) + \varepsilon g(X, Y)g(Y, Y) + g(X, X)g(hY, Y) + g(Y, Y)g(hX, X) + \varepsilon g(hX, X)g(hY, Y)$$

$$+ \varepsilon g(\varphi hX, Y)g(\varphi hX, Y),$$

(59)

$$g(R(X, \varphi Y)X, \varphi Y) = g(R(X, \varphi Y)Y, \varphi X) + \varepsilon g(X, Y)^2 - \varepsilon g(hX, Y)^2$$

$$- \varepsilon g(\varphi hX, X)g(\varphi hY, Y) - \varepsilon g(X, X)g(Y, Y) - g(Y, Y)g(hX, X)$$

$$+ g(X, X)g(hY, Y) + \varepsilon g(hX, X)g(hY, Y) + \varepsilon g(\varphi X, Y)^2$$

$$+ \varepsilon g(\varphi hX, Y)^2 + 2g(\varphi X, Y)g(\varphi hX, Y),$$

(60)

$$g(R(Y, \varphi X)Y, \varphi X) = g(R(X, \varphi Y)Y, \varphi X) + \varepsilon g(X, Y)^2 - \varepsilon g(hX, Y)^2$$

$$- \varepsilon g(\varphi hX, X)g(\varphi hY, Y) + \varepsilon g(\varphi X, Y)^2 + \varepsilon g(\varphi hX, Y)^2$$

$$- 2g(\varphi X, Y)g(\varphi hX, Y) - \varepsilon g(X, X)g(Y, Y) - g(X, X)g(hY, Y)$$

$$+ g(Y, Y)g(hX, X) + \varepsilon g(hX, X)g(hY, Y),$$

(61)

$$g(R(Y, Y)\varphi X, \varphi Y) = g(R(X, Y)X, Y) - \varepsilon g(X, Y)^2 - \varepsilon g(hX, Y)^2$$

$$- 2g(X, Y)g(hX, Y) + \varepsilon g(X, X)g(Y, Y) + g(X, X)g(hY, Y)$$

$$+ g(Y, Y)g(hX, X) + \varepsilon g(hX, X)g(hY, Y) - \varepsilon g(\varphi X, Y)^2$$

$$+ \varepsilon g(\varphi hX, Y)^2 - \varepsilon g(\varphi hX, X)g(\varphi hY, Y).$$

(62)

We now suppose that the $\varphi$-sectional curvature at $p$ is independent of the $\varphi$-section at $p$, i.e. $K(X, \varphi X) = c(p)$ for any $X \in T_p M$ orthogonal to $\xi$. Let $X, Y \in T_p M$ and $X, Y$ orthogonal to $\xi$. From

$$g(R(X + Y, \varphi X + \varphi Y)(X + Y), \varphi X + \varphi Y) = -c(p)g(X + Y, X + Y)^2;$$

$$g(R(X - Y, \varphi X - \varphi Y)(X - Y), \varphi X - \varphi Y) = -c(p)g(X - Y, X - Y)^2;$$

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we get by a straightforward calculation

\[
2g(R(X, \varphi X)Y, \varphi Y) + g(R(X, \varphi Y)X, \varphi Y) + 2g(R(X, \varphi Y)Y, \varphi X) + g(R(Y, \varphi X)Y, \varphi X) = -2c(p)\{2g(X, Y)^2 + g(X, X)g(Y, Y)\}.
\]

(63)

Thus with combining (60), (61), (62) and (63), we get

\[
3g(R(X, \varphi Y)Y, \varphi X) + g(R(X, Y)X, Y) - 2\varepsilon g(hX, Y)^2 \\
- 2g(X, Y)g(hX, Y) + g(X, X)g(hY, Y) + g(Y, Y)g(hX, X) \\
+ 2\varepsilon g(hX, X)g(hY) + 2\varepsilon g(\varphi hX, Y)^2 - 2\varepsilon g(\varphi hX, X)g(\varphi hY, Y) \\
= -c(p)\{2g(X, \varphi Y)^2 + g(X, X)g(Y, Y)\}.
\]

(64)

Now, we replace \( Y \) by \( \varphi Y \) in (63), then using (63) and (11), we have

\[
-3g(R(X, Y)\varphi Y, \varphi X) + g(R(X, \varphi Y)X, \varphi Y) - 2g(\varphi hX, Y)^2 \\
+ 2g(X, \varphi Y)g(\varphi hX, Y) - g(X, X)g(hY, Y) + g(Y, Y)g(hX, X) \\
- 2g(hX, X)g(hY, Y) + 2g(hX, Y)^2 + 2g(\varphi hX, X)g(\varphi hY, Y) \\
= -c(p)\{2g(X, \varphi Y)^2 + g(X, X)g(Y, Y)\}.
\]

(65)

On the other hand, with combining the relation of (65) with (60) and (62), we get

\[
3g(R(X, Y)X, Y) + g(R(X, \varphi Y)Y, \varphi X) - 2\varepsilon g(X, Y)^2 \\
- 2\varepsilon g(hX, Y)^2 - 6g(X, Y)g(hX, Y) + 2\varepsilon g(X, X)g(Y, Y) \\
+ 3g(X, X)g(hY, Y) + 3g(Y, Y)g(hX, X) + 2\varepsilon g(hX, X)g(hY, Y) \\
- 2\varepsilon g(X, \varphi Y)^2 + 2\varepsilon g(\varphi hX, Y)^2 - 2\varepsilon g(\varphi hX, X)g(\varphi hY, Y) \\
= -c(p)\{2g(X, \varphi Y)^2 + g(X, X)g(Y, Y)\}.
\]

(66)

Now for any \( X, Y \in T_pM \) and \( X, Y \) orthogonal to \( \xi \), (66) together with (63) yield

\[
4g(R(X, Y)Y, X) = (c(p) + 3\varepsilon)\{g(X, Y)g(Y, Y) - g(X, Y)^2\} + 3(c(p) - \varepsilon)g(X, \varphi Y)^2 \\
- 2\{\varepsilon g(hX, Y)^2 + 4g(X, Y)g(hX, Y) - 2g(X, X)g(hY, Y) - 2g(Y, Y)g(hX, X) \\
- \varepsilon g(hX, X)g(hY, Y) - \varepsilon g(\varphi hX, Y)^2 + \varepsilon g(\varphi hX, X)g(\varphi hY, Y)\}.
\]

(67)

Let \( X, Y, Z \in T_pM \) and \( X, Y, Z \) orthogonal to \( \xi \). Applying (67) in

\[
g(R(X + Z, Y)Y, X + Z) = g(R(X, Y)Y, X) + g(R(Z, Y)Y, X) + 2g(R(X, Y)Y, Z).
\]
Finally, we get
\[ 4g(R(X, Y)Y, Z) = (c(p) + 3\varepsilon)(g(X, Z)g(Y, Y) - g(X, Y)g(Y, Z)) \]
\[ + 3(c(p) - \varepsilon)g(X, \varphi Y)g(\varphi Z, \varphi Y) - 2\varepsilon g(hX, Y)g(hZ, Y) + 2g(X, Y)g(hZ, Y) \]
\[ + 2g(Z, Y)g(hX, Y) - 2g(X, Z)g(hY, Y) - 2g(Y, Y)g(hX, Z) \]
\[ - \varepsilon g(hX, Z)g(hY, Y) - \varepsilon g(\varphi hX, Y)g(\varphi hZ, Y) \]
\[ + \varepsilon g(\varphi hX, Z)g(\varphi hY, Y) \].

(68)

Moreover, using (11), (23) and \( h\varphi \) is symmetric, it is easy to check that (68) is valid for any \( Z \) and for \( X, Y \) orthogonal to \( \xi \). Hence for any \( X, Y \) orthogonal to \( \xi \), the relation of (68) is reduced to
\[ 4R(X, Y)Y = (c(p) + 3\varepsilon)(g(Y, Y)X - g(X, Y)Y) + 3(c(p) - \varepsilon)g(X, \varphi Y)\varphi Y \]
\[ - 2\varepsilon g(hX, Y)hY + 2g(X, Y)hY + 2g(hX, Y)Y - 2g(hY, Y)X \]
\[ - 2g(Y, Y)hX - \varepsilon g(hY, Y)hX - \varepsilon g(\varphi hX, Y)\varphi hY + \varepsilon g(\varphi hX, Y)\varphi hY \].

(69)

Now, let \( X, Y, Z \) be orthogonal to \( \xi \). Replacing \( Y \) by \( Z \) in (69). Then from
\[ R(X, Y + Z)(Y + Z) = R(X, Y)Y + R(X, Z)Z + R(X, Y)Z + R(X, Z)Y, \]
we get
\[ 4\{R(X, Y)Z + R(X, Z)Y\} = (c(p) + 3\varepsilon)\{2g(Y, Z)X - g(X, Y)Z - g(X, Z)Y\} \]
\[ + 3(c(p) - \varepsilon)\{g(X, \varphi Y)\varphi Z + g(X, \varphi Z)\varphi Y\} \]
\[ - 2\varepsilon g(hX, Y)hY + 2g(X, Y)hY + 2g(hX, Y)Y - 4g(hY, Z)X \]
\[ + 2g(X, Z)hY + 2g(hX, Y)Z + 2g(hX, Z)Y - 4g(hY, Z)X \]
\[ - 4g(Y, Z)hX - 2\varepsilon g(hY, Z)hX - \varepsilon g(\varphi hY, Y)\varphi hY \]
\[ - \varepsilon g(\varphi hX, Z)\varphi hY + 2\varepsilon g(\varphi hX, Z)\varphi hX \].

(70)

Replacing \( X \) by \( Y \) and \( Y \) by \( -X \) in (70), we have
\[ 4\{R(X, Y)Z + R(Z, Y)X\} = (c(p) + 3\varepsilon)\{-2g(X, Z)Y + g(X, Y)Z + g(Y, Z)X\} \]
\[ + 3(c(p) - \varepsilon)\{-g(\varphi X, Y)\varphi Z - g(\varphi Z, Y)\varphi X\} \]
\[ - 2\{-\varepsilon g(hY, X)hZ - \varepsilon g(hY, Z)hX - 2g(X, Y)hZ \}
\[ - 2g(Y, Z)hX - 2g(X, hY)Z - 2g(hY, Z)X \]
\[ + 4g(hX, Z)Y + 4g(X, Z)hY + 2\varepsilon g(hX, Z)hY \]
\[ + \varepsilon g(\varphi hY, X)\varphi hZ + \varepsilon g(\varphi hY, Z)\varphi hX - 2\varepsilon g(\varphi hX, Z)\varphi hY \].

(71)
Adding (70) and (71) and using Bianchi first identity, \( \varphi \) is antisymmetric and \( \varphi h \) is symmetric, we get

\[
4R(X, Y)Z = (c(p) + 3\varepsilon)\{g(Y, Z)X - g(X, Z)Y\}
+ (c(p) - \varepsilon)\{2g(X, \varphi Y)\varphi Z + g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X\}
- 2\{\varepsilon g(hX, Z)hY + 2g(X, Z)hY + 2g(hX, Z)Y - 2g(hY, Z)X\}
- 2g(Y, Z)hX - \varepsilon g(hY, Z)hX - \varepsilon g(\varphi hX, Z)\varphi Y + \varepsilon g(\varphi hY, Z)\varphi X\},
\]

(72)

for any \( X, Y, Z \) orthogonal to \( \xi \). Moreover, using (23), (12) and the first part of (1), we conclude that (72) is valid for any \( Z \) and for \( X, Y \) orthogonal to \( \xi \). Now, let \( X, Y, Z \) be arbitrary vectors of \( T_pM \). Writing

\[
X = X_T + \eta(X)\xi, \quad Y = Y_T + \eta(Y)\xi,
\]

where \( g(X_T, \xi) = g(Y_T, \xi) = 0 \), and using (23), (26) and (12), then (72) gives (58) after a straightforward calculation.

Now, we will prove that the \( \varphi \)-sectional curvature is constant. Consider a (local) \( \varphi \)-basis \( \{E_1, \cdots, E_n, E_{n+1} = \varphi E_1, \cdots, E_{2n} = \varphi E_n, E_{2n+1} = \xi\} \) of vector fields on \( M \). For any index \( i = 1, \cdots, 2n \), \( \{\xi, E_i\} \) spans a non-degenerate plane on the tangent space at each point, where the basis is defined. Putting \( Y = Z = E_i \) in (58), adding with respect to \( i \) and using (1), (2) and (11), we get the following formula, for the Ricci operator, at any point of \( M \):

\[
2Q = \{(n+1)c + 3\varepsilon(n-1) + 2\kappa\}I
- \{(n+1)c + 3\varepsilon(n-1) - 2\kappa(2n-1)\}\eta \otimes \xi
+ 2\{2(n-1) + \mu\}h.
\]

Comparing this with (57), which is valid on any \((\kappa, \mu)\)-contact pseudo-metric manifold with \( \kappa \neq \varepsilon \), we get

\[
(n+1)c = (n-1)\varepsilon - 2n\varepsilon\mu - 2\kappa,
\]

i.e. \( c \) is constant. On the other hand, from (56), we have

\[
c = -(\kappa + \varepsilon\mu).
\]

Comparing (73) and (74), we get \( (n-1)(\varepsilon\mu - \kappa - \varepsilon) = 0 \). Moreover, since \( n > 1 \), we have \( \mu = \varepsilon\kappa + 1 \) and so \( c = -2\kappa - \varepsilon \). This completes the proof of the theorem.
Theorem 3.5. Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a $(\kappa, \mu)$-contact pseudo-metric manifold with $\varepsilon \kappa < 1$ and $n > 1$. Then $M$ has constant $\varphi$-sectional curvature if and only if $\mu = \varepsilon \kappa + 1$.

Proof. In Theorem 3.4, we proved that $\mu = \varepsilon \kappa + 1$, in the case where the non Sasakian, $(\kappa, \mu)$-contact pseudo-metric manifold has constant $\varphi$-sectional curvature. Now, we will prove the inverse, i.e. supposing $M(\varphi, \xi, \eta, g)$ is a $(2n + 1)$-dimensional $(n > 1)$, non Sasakian, $(\kappa, \mu)$-contact pseudo-metric manifold with

$$\mu = \varepsilon \kappa + 1.$$  (75)

We will prove that $M$ has constant $\varphi$-sectional curvature. Let $X \in T_pM$ be a unit vector orthogonal to $\xi$. By Theorem 3.1, we can write

$$X = X_\lambda + X_{-\lambda},$$

where $X_\lambda \in D(\lambda)$ and $X_{-\lambda} \in D(-\lambda)$.

Using Theorem 3.1, Theorem 3.2 and a long straightforward calculation, we get

$$K(X, \varphi X) = -(\kappa + \varepsilon \mu) + 4(\kappa - \varepsilon \mu + \varepsilon)(g(X_\lambda, X_\lambda)g(X_{-\lambda}, X_{-\lambda}) - g(X_\lambda, \varphi X_{-\lambda})^2)$$

and hence by (75), we have $K(X, \varphi X) = -(\kappa + \varepsilon \mu) = \text{const.}$

Example 3.1 ([15], Theorem 4.1). The tangent sphere bundle $T_\varepsilon M$ is $(\kappa, \mu)$-contact pseudo-metric manifold if and only if the base manifold $M$ is of constant sectional curvature $\varepsilon$ and $\kappa = 3\varepsilon - 2, \mu = -2\varepsilon$.

Theorem 3.6. Let $M$ be an $n$-dimensional pseudo-metric manifold, $n > 2$, of constant sectional curvature $c$. The tangent sphere bundle $T_\varepsilon M$ has constant $\varphi$-sectional curvature $(-4c(\varepsilon - 1) + c^2)$ if and only if $c = 2\varepsilon \pm \sqrt{4 + \varepsilon}$.

Example 3.2. Let $M(\varphi, \xi, \eta, g)$ be a contact pseudo-metric manifold of dimension $2n + 1$, with $g(\xi, \xi) = \varepsilon$. Then, it is easy to check that, for any real constant $a > 0$ and by choosing the tensors

$$\overline{\eta} = a\eta, \quad \overline{\xi} = \frac{1}{a}\xi, \quad \overline{\varphi} = \varphi, \quad \overline{g} = ag + \varepsilon a(a - 1)\eta \otimes \eta,$$

$M(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is a new $(\kappa, \mu)$-contact pseudo-metric manifold with

$$\kappa = \frac{\kappa + \varepsilon a^2 - \varepsilon}{a^2}, \quad \mu = \frac{\mu + 2a - 2}{a}.$$  (76)

Now, we find the value of $a$, so that $M$ has constant $\varphi$-sectional curvature. Using Theorem 3.3, we must have $\bar{\mu} = \varepsilon \bar{\kappa} + 1$. So, we get $a = \frac{(\varepsilon \kappa - 1)}{(\mu - 2)}$. In fact with choosing $a = \frac{(\varepsilon \kappa - 1)}{(\mu - 2)} > 0$, $M(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ has constant $\varphi$-sectional curvature $\overline{\kappa} = -\kappa - \varepsilon \overline{\mu} = \varepsilon (1 - 2\overline{\mu}) = \frac{(2(\mu - 2)^2 - 3(1 - \varepsilon \kappa))}{(\varepsilon - \kappa)}$. 

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4. generalized \((\kappa, \mu)\)-contact pseudo-metric manifold

In (22) and (23), if \(\kappa\) and \(\mu\) are real smooth functions on \(M\), we call generalized \((\kappa, \mu)\)-contact pseudo-metric manifold.

Example 4.1. We consider the 3-dimensional manifold \(M = \{(x, y, z) \in \mathbb{R}^3 | z \neq 0\}\), where \((x, y, z)\) are the standard coordinates in \(\mathbb{R}^3\). The vector fields
\[
e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{1}{z^2} \frac{\partial}{\partial y}, \quad e_3 = 2yz^2 \frac{\partial}{\partial x} + \frac{2x}{z} \frac{\partial}{\partial y} + \frac{1}{z^6} \frac{\partial}{\partial z}
\]
are linearly independent at each point of \(M\). Let \(g\) be the pseudo-Riemannian metric defined by \(g(e_i, e_j) = \epsilon_i \delta_{ij}\), where \(i, j = 1, 2, 3\) and \(\epsilon_1 = \epsilon, \quad \epsilon_2 = \epsilon_3 = 1\). Let \(\nabla\) be the Levi-Civita connection and \(R\) the curvature tensor of \(g\). We easily get
\[
[e_1, e_2] = 0, \quad [e_1, e_3] = \frac{2}{z^4} e_2, \quad [e_2, e_3] = 2 \left( e_1 + \frac{1}{z^4} e_2 \right)
\]
Let \(\eta\) be the 1-form defined by \(\eta(X) = \epsilon g(X, e_1)\) for \(X \in \Gamma(TM)\). \(\eta\) is a contact form. Let \(\varphi\) be the \((1, 1)\)-tensor field, defined by \(\varphi e_1 = 0, \quad \varphi e_2 = e_3, \quad \varphi e_3 = -e_2\). So, \((\varphi, \xi = e_1, \eta, g)\) defines a contact pseudo-metric structure on \(M\). Now using the Koszul formula, we calculate
\[
\nabla_{e_1} e_2 = -\left( \epsilon + \frac{1}{z^4} \right) e_3, \quad \nabla_{e_2} e_1 = -\left( \epsilon + \frac{1}{z^4} \right) e_3,
\]
\[
\nabla_{e_1} e_3 = \left( \epsilon + \frac{1}{z^4} \right) e_2, \quad \nabla_{e_3} e_1 = \left( \epsilon - \frac{1}{z^4} \right) e_2,
\]
\[
\nabla_{e_2} e_3 = \left( 1 + \frac{\epsilon}{z^4} \right) e_1 + \frac{2}{z^4} e_2, \quad \nabla_{e_3} e_2 = \left( \frac{\epsilon}{z^4} - 1 \right) e_1,
\]
\[
\nabla_{e_2} e_2 = -\frac{2}{z^4} e_3, \quad \nabla_{e_1} e_1 = \nabla_{e_3} e_3 = 0.
\]
Also Using (8), we obtain \(h e_1 = 0, \quad h e_2 = \lambda e_2, \quad h e_3 = -\lambda e_3\), where \(\lambda = \frac{1}{z^4}\). Now, putting \(\mu = 2(1 + \epsilon \lambda)\) and \(\kappa = \epsilon(1 - \lambda^2)\), we finally get
\[
R(e_2, e_1)e_1 = \epsilon (\kappa + \lambda \mu) e_2,
\]
\[
R(e_3, e_1)e_1 = \epsilon (\kappa - \lambda \mu) e_3,
\]
\[
R(e_2, e_3)e_1 = 0.
\]
These relations yield the following, by direct calculations,
\[
R(X,Y)\xi = \epsilon \kappa \{\eta(Y)X - \eta(X)Y\} + \epsilon \mu \{\eta(Y)hX - \eta(X)hY\},
\]
where \(\kappa\) and \(\mu\) are non-constant smooth functions. Hence \(M\) is a generalized \((\kappa, \mu)\)-contact pseudo-metric manifold.
Theorem 4.1. On a non Sasakian, generalized $(\kappa, \mu)$-contact pseudo-metric manifold $M^{2n+1}$ with $n > 1$, the functions $\kappa, \mu$ are constant, i.e., $M^{2n+1}$ is a $(\kappa, \mu)$-contact pseudo-metric manifold.

Theorem 4.2. Let $M$ be a non Sasakian, generalized $(\kappa, \mu)$-contact pseudo-metric manifold. If $\kappa, \mu$ satisfy the condition $a\kappa + b\mu = c$, where $a, b$ and $c$ are constant. Then $\kappa, \mu$ are constant.

The proof of the two previous theorems is the same with Theorem 3.5 and Theorem 3.6 in [10] for contact metric case. Therefore, we omit them here.

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