3D Cartesian vertically boundary-adaptive grids construction for coastal hydrophysics problems

V V Sidoryakina
Taganrog Institute of A.P. Chekhov (branch) RGEU (RINE), Taganrog, Russia
*e-mail: cvv9@mail.ru

Abstract. Coastal systems are characterized by coastline complex shape, the presence of far protruding spits, bays of complex shape. The rectangular grids using for modeling hydrodynamic processes in the coastal zone when constructing discrete models leads to significant errors. In this regard, when considering such processes, the initial geometry of the region is often specified in the surface grid form, and disturbances in the system in the form of boundary conditions. Particular attention is paid to the construction of a high-quality computational grid near the boundaries. The paper presents a technology for constructing 3D Cartesian vertically boundary-adaptive grids based on surface 2D grids, which are created using the procedure of minimizing the generalized Dirichlet functional. Previously, this approach has shown its effectiveness in constructing 2D non-degenerate regular grids containing the minimum number of cells (convex quadrangles) for test areas of the Z-shape, such as the «Maltese cross» and others, as well as in the numerical solution of 2D coastal hydrophysics problems.

1. Introduction
One of the essential features of coastal systems is the complex shape of the coastline, which greatly complicates their mathematical modeling. The use of rectangular, in general, irregular grids does not allow taking into account with due accuracy the presence of long coastal streamers, narrow winding water bodies such as bays, estuaries, straits, etc. [1 - 5].

For good accuracy of numerical calculations near the boundary, the grid data should have very small steps in spatial variables. Therefore, their use would lead to unacceptable computations in the 3D case. The advantage of curvilinear boundary-adaptive grids is flexible grid structure that allows more accurate mapping of the computational domain geometry compared to rectangular grids and generating grid at a lower cost for areas of complex geometry, mainly spatial configurations [6 - 10]. The algorithms were developed and implemented for constructing 3D Cartesian vertically boundary-adaptive grids based on surface 2D grids associated with solving the system of Euler-Lagrange equations for the Dirichlet functional (or energy functional) and the procedure for its minimization. The calculation of the initial data and the solution of these equations is performed in the right-hand Cartesian coordinate system. Previously, this approach has shown its effectiveness in constructing minimal 2D non-degenerate grids for test regions of the Z-shape, such as the «Maltese cross» and others, as well as in the numerical solution of 2D problems of hydrodynamics of coastal systems [11 - 14].
2. Technology for constructing 3D boundary-adaptive grids description

Let use a rectangular Cartesian coordinate system \( Oxyz \), where \( Ox \) is the axis passes along the surface of an undisturbed water surface and directed to the north, the axis \( Oy \) oriented to the east, and the axis \( Oz \) down. Coordinate system \( Oxyz \) forms the right triple of vectors.

Let be \( h=H+\eta \) is the total water depth; \( H \) is the depth at undisturbed water surface; \( \eta \) is the elevation of the free surface relative to the geoid (sea level).

Let us present a mathematical model for constructing 3D boundary-adaptive grids for a certain water volume \( V \), the vertical edges of which consist of segments parallel to the vertical axis. \( Oz \).

On the free surface of the reservoir, we build an optimal boundary-adaptive 2D grid (surface grid) \( \Omega_{ij} \). To number the cells of this grid, we will use a pair of indices \((i,j)\) for the lower left node of the grid cell. The cells of this grid after optimization are a set of convex quadrangular cells built on the undisturbed surface of the reservoir.

Next, find the Cartesian product of the set of these cells and the set of 1D cells (grids) \( \Omega_{ij} \), having uniform steps within the area, the number of which will, in the general case, depend on the bottom topography (depth) function \( H(x,y) \) at the location of the cell \((x_i,y_j)\) and from the function of raising the water level \( \eta(x,y) = \eta(x_i,y_j) \); the grid steps in the near-surface and bottom cells also depend on the functions \( H(x,y), \eta(x,y) \) and may not match the vertical grid spacing within the grid. The construction of such grids is carried out based on the requirements for the connectivity of the resulting 3D grid with respect to the templates used by the difference (finite-volume) operators involved in the approximations of the equations of hydrophysics.

In this case, for the node \((x_i,y_j)\) grid \( \Omega_{ij} \), grid \( \Omega_{ij} \) is constructed as follows

\[
\Omega_{ij} = \{z_{ij,k} = z_{ij,k} = h_{ij,k} \mid k = \lfloor N_{ijy} / 2 \rfloor \},
\]

where \( h_{ij,k} \) are grid steps in direction \( Oz \), \( N_{ijy} \) is the number of grid cells along the direction \( Oz \), depending on the depth of the reservoir at the location of the 2D grid cell with the number \((i,j)\) (Fig.1).

![Figure 1. Grid building along the vertical direction](image-url)
Note that the step \( h_{i,j} \), adjacent to the free surface and step \( h_{i,j,N} \), adjacent to the bottom surface, grid \( \Omega_{xy} \) are determined from the relations
\[
h_{i,j} = h_y - z_{i,j}^0,
\]
\[
h_{i,j,N} = H_y - z_{i,j,N}^0.
\]

The desired 3D grid is defined as a set that is the union of all indices \((i,j)\) Cartesian products of a 2D grid \( \Omega_{xy} \) and 1D grid \( \Omega_{zg} \), i.e.
\[
\Omega_n = \bigcup \Omega_{xy} \times \Omega_{zg}.
\]

Let dwell in more detail on the construction of 2D boundary-adaptive planar grids \( \Omega_{xy} \).

3. 2D boundary-adaptive grids construction

The curved grid in the «physical» area is obtained as a result of one-to-one mapping of the reference grid of the «computational» area.

Let \( x' = x, x'' = y \). In the new notation, the «physical» area is in the space of variables \((x',x'')\), and the «calculated» area is set in the space of variables \((\xi',\xi'')\). The process of constructing boundary-adaptive grid is reduced to establishing a one-to-one correspondence between the «physical» and «computational» areas, i.e. in getting dependencies:
\[
x' = x'(\xi',\xi''), x'' = x''(\xi',\xi'').
\]

Let \( D \) is curved rectangle, \( D \subset V \), to which unit square is assigned \( \Xi(0 \leq \xi' \leq 1, 0 \leq \xi'' \leq 1) \).

We build in the area \( D \) Cartesian rectangular computational grid
\[
\Omega_{xy} = \{(x'_i,x'_j) \mid x'_i = x'_i, x'_j = x'_j, i = 1,N_x, j = 1,N_y\}.
\]

On surface \((\xi',\xi'')\) a reference grid is constructed
\[
\Omega_{z=0} = \{(\xi'_i,\xi'_j) \mid \xi'_i = \xi'_i, \xi'_j = \xi'_j, i = 1,N_x, j = 1,N_y\}.
\]

Suppose there is some function \( \phi(\xi',\xi'') \), translating unit square \( \Xi \) to the region \( D \):
\[
\phi(\xi',\xi'') : \Xi \rightarrow D.
\]

Function \( \phi(\xi',\xi'') \) is smooth at every point of the domain \( D \).

Positiveness of the Jacobian transformation (5)
\[
J = J(\xi',\xi'') = \det \begin{vmatrix} \frac{\partial x'_i}{\partial \xi'} & \frac{\partial x'_i}{\partial \xi''} \\ \frac{\partial x''_i}{\partial \xi'} & \frac{\partial x''_i}{\partial \xi''} \end{vmatrix} i,j = 1,2
\]
introduces restrictions when choosing the shape of the cell and ensures that each of the corners of the cell is less \( \pi \).

Continuous transformation (5) is determined at each specific cell by the formulas
\[
x' = x'(\xi',\xi'') = (1-\xi')\phi(0,\xi'') + \xi'\phi(1,\xi''),
\]
\[
x'' = x''(\xi',\xi'') = (1-\xi'')\phi(\xi',0) + \xi''\phi(\xi',1).
\]
Display \( x'(\xi, \bar{\xi}) \) converts opposite sides \( \xi = 0 \) and \( \xi = 1 \) square \( \Xi \) in opposite directions \( \phi(0, \xi) \) and \( \phi(1, \xi) \) areas \( D \); display \( x'(\xi, \bar{\xi}) \) converts opposite sides \( \xi = 0 \) and \( \xi = 1 \) in opposite directions \( \phi(\xi, 0) \) and \( \phi(\xi, 1) \).

Algebraic methods for constructing grids are quite simple, but they have such disadvantages as lack of orthogonality, as well as a strong difference in the size of neighboring cells. All this leads to a decrease in the accuracy of the solution. Differential methods associated with the solution of a system of differential equations, which were used in this work to build the grid, are to some extent devoid of these shortcomings.

Boundary-responsive grid in area \( D \) is carried out first on its selected edges, then inside this area. Therefore, at each step, the intermediate transformation \( \phi(\xi, \bar{\xi}) \) known at the border of the region \( \Xi \), and this boundary transformation continues from the border to the interior of the region.

Desired coordinate transformation \( (x') \rightarrow (\xi') \) in the region of \( D \), the inverse of the intermediate conversion \( \phi(\xi, \bar{\xi}) \) is determined from the solution of the Dirichlet problem for the system of Euler – Lagrange equations for the energy functional

\[
I = \iint (g^{11} + g^{22}) dx' dx'' = \iint g^{11} + g^{22} d\xi d\bar{\xi}, \quad (7)
\]

\[
g = g_{11} + g_{22}, \quad g_{ij} = \left( \frac{\partial x'}{\partial \xi} \right) ^2 + \left( \frac{\partial x''}{\partial \bar{\xi}} \right) ^2 \quad \text{and} \quad g_{22} = \left( \frac{\partial x'}{\partial \xi} \right) ^2 + \left( \frac{\partial x''}{\partial \bar{\xi}} \right) ^2 \quad \text{are components of the metric coordinate transformation tensor.}
\]

For functional (7), we can write a system of nonlinear equations for the sought functions \( x' = x'(\xi, \bar{\xi}), x'' = x'(\xi, \bar{\xi}) \).

Discrete analog of the functional \( I \) is the functional obtained by summing over all four corners (triangles formed when a quadrangular cell is divided by its diagonals) of all cells. Nonlinear programming methods are used to directly minimize the value of the discrete Dirichlet functional on the set of convex grids by means of a quasi-Newton procedure in the «computational» domain \((\xi, \bar{\xi})\). To do this, it is advisable to use the well-proven gradient methods tested and described in [15 - 16] in relation to the water area of the Azov Sea and the Taganrog Bay.

Sign change in the denominator of the integrand of functional (7) of the Jacobian \( J \), leads to discontinuities of the second kind, which turns out to be critical for methods of the second order (quasi-Newtonian procedure). To get out of this situation near the degeneracy boundary \((J=0)\) coordinate transformation is performed \( J \) on the

\[
J' = \max(J, \varepsilon) = (J - \varepsilon)^+ \quad (8)
\]

\( \varepsilon \) is rather small number, \( \varepsilon > 0 \).

4. Software implementation results

The described algorithms for constructing 3D boundary-adaptive grids were implemented in software and tested on a number of test areas.

The areas under consideration are \( 6 \times 15 \times 6 \) (area 1) and \( 9 \times 12 \times 6 \) (area 2).

In fig. 2 (a) and 2 (b) show the initial approximations for mappings to the 3D Z-shaped regions 1 and 2, respectively, with the same sets of boundary, but different choices of corner nodes. In fig. 2 (c) and 2 (d) show the results of grid building 1, 2, after reaching the convexity of all cells of the 2D grid. Figures 2 (e) and 2 (f) show the results of grid generation for regions 1 and 2, where the 2D grid is constructed based on the minimization of the Dirichlet functional using a quasi-Newtonian procedure.
Figure 2. Results of building grids for test areas 1 and 2

5. Conclusions
The paper presents a technology for constructing a boundary-adaptive 3D curvilinear grid and the results of numerical modeling using this technology. On the example of specific areas, tests were carried out to demonstrate the effectiveness of the methods used. In the future, this work will serve as the basis for constructing boundary-adaptive 3D-grids of water areas with a complex coastline shape (Taganrog Bay, Azov Sea, etc.), to improve the accuracy of numerical modeling of real problems of hydrophysics and hydrobiology.

Acknowledgments
This paper was supported by the Russian Foundation for Basic Research (RFBR) grant No. 19-01-00701.

References
[1] Chetverushkin B N 2013 Resolution limits of continuous media models and their mathematical formulations (Math. Models Comput. Simul., 5:3) pp. 266–279
[2] Liu X, Qi S, Huang Y, Chen Y, Du P 2015 Predictive modeling in sediment transportation across multiple spatial scales in the Jialing River Basin of China (International Journal of Sediment Research, 30:3) pp. 250–255
[3] Dou H S 2006 Mechanism of flow instability and transition to turbulence Internatioinal (Journal Non-Linear Mechanics, 41) pp.512-517
[4] Belotserkovskii O M, Gushchin V A, Shchennikov V V 1975 *Use of the splitting method to solve problems of the dynamics of a viscous incompressible fluid* (Computational Mathematics and Mathematical Physics, 15:1) pp. 190-200

[5] Tao J J, Chen S Y and Su W D 2013 *Local Reynolds number and thresholds of transition in shear flows* Science China Physics (Mechanics & Astronomy, 56) pp. 263-269

[6] Sukhinov A I, Chistyakov A E, Protosenko E A, Sidoryakina V V, Protosenko S V 2020 *Accounting Method of Filling Cells for the Solution of Hydrodynamics Problems with a Complex Geometry of the Computational Domain* (Mathematical Models and Computer Simulations 12(2)) pp. 232-245

[7] Sukhinov A I, Chistyakov A E, Protosenko E A, Sidoryakina V V, Protosenko S V 2019 *Accounting method of filling cells for the hydrodynamics problems solution with complex geometry of the computational domain* (Matem. Mod., 31:8), pp. 79–100

[8] Sidoryakina V V, Sukhinov A I 2017 *Well-posedness analysis and numerical implementation of a linearized two-dimensional bottom sediment transport problem* (Comput. Math. Math. Phys., 57(6)), pp. 978–994

[9] Sukhinov A I, Sukhinov A A 2005 *3D Model of Diffusion-Advection-Aggregation Suspensions in Water Basins and Its Parallel Realization* (Parallel Computational Fluid Dynamics, Multidisciplinary Applications, Proceedings of Parallel CFD 2004 Conference, Las Palmas de Gran Canaria, Spain, ELSEVIER, Amsterdam-Berlin-London-New York-Tokyo), pp.223-230.

[10] Sukhinov A I, Sidoryakina V V, Sukhinov A A 2017 *Sufficient conditions for convergence of positive solutions to linearized two-dimensional sediment transport problem* (Vestnik of Don State Technical University 17(1)) pp. 5-17

[11] Ivaneiko S A 1988 *Construction of nondegenerate grids* (Zh. Vychisl. Math. and Math. fiz., 28:10) pp. 1498-1506

[12] Noelle S, Rosenbaum W and Rumpf M 2001 *An adaptive staggered grid scheme for conservation laws* (Int. Ser. Numer. Math., 141) pp. 775–784

[13] Arminjon P, St-Cyr A and Madrane A 2002 *New two- and three-dimensional non-oscillatory central finite volume methods on staggered cartesian grids* (Appl. Numer. Math., 40) pp. 367–390

[14] Berger M, Helzel C, and LeVeque R 2003 *H-box methods for the approximation of hyperbolic conservation laws on irregular grids* (SIAM J. Numer. Anal. 41:3) pp. 893-918

[15] Sukhinov A Chistyakov A, Sidoryakina V 2017 *Investigation of Nonlinear 2D Bottom Transportation Dynamics in Coastal Zone on Optimal Curvilinear Boundary Adaptive Grids* (XIII International Scientific-Technical Conference Dynamic of Technical Systems (DTS-2017), Matec Web of Conferences, 132, ed. Najafabadi T, Sevostianov I, Yeghiazaryan K. Dong A, Mladenovic V., E D P Sciences, 2017, UNSP 04003)

[16] Vasiliev V S, Sukhinov A I 2003 *Precision two-dimensional models of shallow water bodies* (Matem. Modeling, 15:10) pp. 17-34