Geometry of Batalin-Vilkovisky quantization.

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Abstract

A very general and powerful approach to quantization of gauge theories was proposed by Batalin and Vilkovisky [1],[2]. The present paper is devoted to the study of geometry of this quantization procedure. The main mathematical objects under consideration are $P$-manifolds and $SP$-manifolds (supermanifolds provided with an odd symplectic structure and, in the case of $SP$-manifolds, with a volume element). The Batalin-Vilkovisky procedure leads to consideration of integrals of the form $\int L H d\lambda$ where $L$ is a Lagrangian submanifold of an $SP$-manifold $M$ and $H$ satisfies the equation $\Delta H = 0$ where $\Delta$ is an odd analog of Laplacian. The choice of $L$ can be interpreted as a choice of gauge condition; Batalin and Vilkovisky proved that in some sense their procedure is gauge independent. Namely they proved that $\int_{L_0} H d\lambda_0 = \int_{L_1} H d\lambda_1$ if Lagrangian submanifolds $L_0$ and $L_1$ are connected by a continuous family $L_t$ of Lagrangian submanifolds. We will prove that the same conclusion can be made in the much more general case when the bodies $m(L_0)$ and $m(L_1)$ of submanifolds $L_0$ and $L_1$ are homologous in the body $m(M)$ of $M$. This theorem leads to a conjecture that one can modify the quantization procedure in such a way as to avoid the use of the notion of Lagrangian submanifold. In the next paper we will show that this is really so at least in the semiclassical approximation. Namely if $H$ is written in the form $\exp \hbar^{-1} S$ where $S = S_0 + \hbar S_1 + \ldots$ we will find the asymptotics of $\int L H d\lambda$ as an integral over some set of critical points of $S_0$ with the integrand expressed in terms of Reidemeister torsion. This leads to a simplification of quantization procedure and to the possibility to get rigorous results also in the infinite-dimensional case, using the results of [4]. (We are talking about the semiclassical approximation.)

The present paper contains also a compete classification of $P$-manifolds and $SP$-manifolds. The classification is interesting by itself, but in this paper it plays also a role of an important tool in the proof of other results.
Let us consider a domain $U$ in a superspace $R^{n|n}$ with coordinates $(x^1, \ldots, x^n, \xi_1, \ldots, \xi_n)$. An odd Poisson bracket (antibracket) of functions $F$ and $G$ on $U$ can be defined by the formula

$$\{F, G\} = \frac{\partial_r F}{\partial x^a} \frac{\partial G}{\partial \xi_a} - \frac{\partial_r F}{\partial \xi_a} \frac{\partial G}{\partial x^a}$$ \hspace{1cm} (1)$$

where $\partial_r$ and $\partial_l$ denote the right derivative and the left derivative correspondingly. (We suppose usually that $x^1, \ldots, x^n$ are even and $\xi_1, \ldots, \xi_n$ are odd. However one can weaken this assumption by requiring only that $x^a$ and $\xi_a$ have opposite parity.) The transformations of $U$ preserving the bracket (1) will be called $P$-transformations (or odd symplectic transformations). Volume preserving $P$-transformations (i.e. $P$-transformations having unimodular Jacobian) will be called $SP$-transformations. (Of course we have in mind the supervolume. Unimodularity of the Jacobian matrix means that the Berezinian of this matrix is equal to 1.) $P$-manifold (or odd symplectic manifold) is by definition a manifold pasted together from $(n|n)$-dimensional superdomains by means of $P$-transformations. Replacing in this definition $P$-transformations by $SP$-transformations we get the definition of $SP$-manifold. \[1\] In a general local coordinate system $(z^1, \ldots, z^{2n})$ one can write the Poisson bracket (1) in the form

$$\{F, G\} = \frac{\partial_r F}{\partial z^i} \omega^{ij}(z) \frac{\partial G}{\partial z^j}$$ \hspace{1cm} (2)$$

where $\omega^{ij}(z)$ is an invertible matrix. Its inverse matrix $\omega_{ij}(z)$ determines a differential form

$$\omega = dz^i \omega_{ij} dz^j$$ \hspace{1cm} (3)$$

It is easy to check that this form is closed ($d\omega = 0$). As in standard symplectic geometry one can construct a vector field $K_H$ corresponding to a function $H$ on a $P$-manifold $M$ by the formula

$$K_H(z) = \omega^{ij}(z) \frac{\partial H}{\partial z^j}$$ \hspace{1cm} (4)$$

By definition $K_H$ is a Hamiltonian vector field with Hamiltonian $H$. If the function $H$ is odd then $K_H$ is even and vice versa. The bracket (2) determines the structure of a Lie superalgebra on the linear (super)space $F$ of (super)functions on $M$. It is easy to check that the map $H \rightarrow K_H$ is a homomorphism of $F$ into the Lie superalgebra $diffM$ of vector fields on $M$. A submanifold $L \subset M$ is called isotropic if $\omega$ vanishes on $L$ (i.e. $t^a \omega_{ab}(x) \delta^b = 0$ for every pair $t, \bar{t}$ of tangent vectors to $L$ at the point $x \in L$). A Lagrangian manifold $L$ is by definition an isotropic manifold of dimension $(|k|n - k)$, $0 \leq k \leq n$.

\[1\] Using the language of $G$-structures we can say that $P$-manifold is a supermanifold provided with locally flat $P$-structure where $P$ is a group consisting of linear transformations of $R^{n|n}$ preserving the bilinear form $x^a \xi_a$. To get the definition of $SP$-manifold we have to replace here the group $P$ by its subgroup $SP = P \cap SL(n|n)$. We will not use the language of $G$-structures; speaking about $P$-structure or $SP$-structure we will have in mind the structure of $P$-manifold or $SP$-manifold.
One can give an invariant definition of $P$-manifold. Namely such a manifold can be defined as an $(n|n)$-dimensional supermanifold provided with a non-degenerate closed odd 2-form $\omega$. This definition is equivalent to a previous one because one can prove an analog of Darboux theorem: a non-degenerate closed odd 2-form $\omega$ locally can be written as $dx^a d\xi_a$ by an appropriate choice of coordinates $(x^1, ..., x^n, \xi_1, ..., \xi_n)$—Darboux coordinates. Moreover if $L$ is a Lagrangian submanifold of $M$ one can choose Darboux coordinates in the neighborhood of a point $a \in L$ in such a way that in this neighborhood $L$ is singled out by the equations $x^{k+1} = ... = x^n = 0, \xi_1 = ... = \xi_k = 0$. If we don’t require that $x^i$ are even, $\xi_i$ are odd, then we always can define a Lagrangian submanifold locally by the equations $\xi_1 = ... = \xi_n = 0$. The volume element in arbitrary coordinates $(z^1, ..., z^{2n})$ can be specified by means of the density function $\rho(z)$. In such a way $SP$-structure on $M$ is determined by non-degenerate closed odd 2-form $\omega$ and by density $\rho$. (It is necessary to emphasize that $\rho$ is not arbitrary; one has to require that in the neighborhood of every point in $M$ one can make $\rho \equiv 1$ by means of appropriate choice of Darboux coordinates.) As usual the volume element in $M$ determines the divergence of vector field $K^a$ by the formula

$$\text{div} K = \rho^{-1} \frac{\partial}{\partial z^a} (\rho K^a) = \frac{\partial}{\partial z^a} K^a + \frac{\partial}{\partial z^a} \ln \rho K^a$$

Therefore one can define an operator $\Delta$ on the space $F$ of functions on $SP$-manifold $M$ by the formula

$$\Delta H = \frac{1}{2} \text{div} K_H$$

One can check that $\Delta^2 = 0$ using the existence of local coordinates with $\omega = dx^a d\xi_a, \rho = 1$. In these coordinates

$$\Delta = \frac{\partial}{\partial x^a} \frac{\partial}{\partial \xi_a}$$

and the relation $\Delta^2 = 0$ is evident. In a general coordinate system the relation $\Delta^2 = 0$ leads to conditions on $\rho$. One can prove that these conditions are sufficient to assert that the non-degenerate closed odd 2-form $\omega$ and the density function $\rho(z)$ determine an $SP$-structure; see Theorem 5 below. In the formula (7) we suppose that the variables $x^a, \xi_a$ have opposite parity; if $x^1, ..., x^n$ are even and $\xi_1, ..., \xi_n$ are odd as we assume usually the right derivative with respect to $x^a$ in (7) is of course the standard derivative. If $L$ is a Lagrangian submanifold of $SP$-manifold $M$ one can define a volume element in $L$ (up to a sign). Namely, if in Darboux coordinates $x^1, ..., x^n, \xi_1, ..., \xi_n$ the manifold $L$ is singled out by the equations $x^{k+1} = ... = x^n = 0, \xi_1 = ... = \xi_k = 0$ then the volume element in $L$ can be defined as $dx^1 ... dx^k d\xi_{k+1}...d\xi_n$. It is easy to check that this volume element is well defined up to a sign. (This fact follows immediately from another
description of the volume element in \( L \) given in the proof of the Lemma 4.) One has to impose some global conditions to define the volume element globally. Namely, we will prove that it suffices to require that \( M \) be orientable, i.e. that \( m(M) \) be orientable. We denote by \( m(M) \) the body of \( M \), i.e. the bosonic part of \( M \).

The Batalin-Vilkovisky approach to quantization is based on the following theorem: if \( L_0 \) and \( L_1 \) are closed oriented Lagrangian submanifolds connected with a smooth family of closed oriented Lagrangian submanifolds \( L_t \) and an even function \( H \) on \( M \) satisfies the condition \( \Delta H = 0 \) then \( \int_{L_0} H d\lambda_0 = \int_{L_1} H d\lambda_1 \). For completeness we will sketch a proof of this statement. As usual it is sufficient to consider an infinitesimal deformation of the Lagrangian manifold \( L \); moreover one can assume that \( L \) is deformed only in a domain where (after appropriate change of coordinates) it is singled out by equations \( \xi_1 = ... = \xi_n = 0 \) and where \( \rho = 1 \). Then the deformed manifold can be specified by means of an odd function \( \Psi(x^1, ..., x^n) \) that vanishes outside of this domain. Namely, the deformed manifold can be defined by the equations \( \xi_j = \frac{\partial \Psi}{\partial x_j} \). The variation of the integral \( \int_L H d\lambda \) by this deformation can be written as

\[
\int L \frac{\partial_r H}{\partial x^j} \frac{\partial \Psi}{\partial \xi_j} dx^1 ... dx^n.
\]

Integrating by parts and using \( \Delta H = 0 \) we obtain that this variation is equal to zero.

In the formulation of Batalin-Vilkovisky theorem we assume that the volume elements \( d\lambda_0 \) and \( d\lambda_1 \) in \( L_0 \) are choosen in an appropriate way: namely we require the existence of volume elements \( d\lambda_t \) in \( L_t \) depending continuously on \( t \) and connecting \( d\lambda_0 \) and \( d\lambda_1 \). A similar assumption must be made about the orientation of \( L_0 \) and \( L_1 \). Our aim is to prove a generalization of this theorem. Namely, we will prove the following:

**Theorem 1.** Let \( L_0 \) and \( L_1 \) be closed oriented Lagrangian submanifolds of an orientable \( SP \)-manifold \( M \). If the cycles \( m(L_0) \) and \( m(L_1) \) are homologous in \( m(M) \) over \( R \) (i.e. \( m(L_0) \) and \( m(L_1) \) determine the same element of \( H_k(m(M), R) \)) then

\[
\int_{L_0} H d\lambda_0 = \int_{L_1} H d\lambda_1 \tag{8}
\]

for every function \( H \) satisfying \( \Delta H = 0 \).

We will prove also

**Theorem 2.** If \( H = \Delta K \) then for every closed Lagrangian manifold \( L \)

\[
\int_L H d\lambda = 0. \tag{9}
\]

The proof of these theorems will be based on an explicit description of \( P \)-manifolds and their Lagrangian submanifolds. I don’t know any direct proof of these theorems.
We begin with the remark that every transformation $\tilde{x} = f(x)$ of an $n$-dimensional domain with coordinates $x^1, ..., x^n$ can be extended to a $P$-transformation $(x^1, ..., x^n, \xi_1, ..., \xi_n) \rightarrow (\tilde{x}^1, ..., \tilde{x}^n, \tilde{\xi}_1, ..., \tilde{\xi}_n)$ by means of the formula
\begin{equation}
\tilde{\xi}_i = \frac{\partial x^j}{\partial \tilde{x}^i} \xi_j.
\end{equation}

This means that a cotangent bundle $T^*N$ to an $n$-dimensional manifold $N$ has a natural structure of $P$-manifold (the formula (10) coincides with transformation law of covectors). We will prove

**Theorem 3.** Every $(n|n)$-dimensional $P$-manifold $M$ is equivalent to a $P$-manifold of the form $T^*N$. Namely, one can take $N = m(M)$.

Let us begin with a remark that for every $m$-dimensional vector bundle $\alpha$ over an $n$-dimensional manifold $N$ one can construct an $(m|n)$-dimensional supermanifold considering the fibres as odd linear spaces. More precisely, if a vector bundle over $N$ has transition functions $\hat{x}^i = f^i(x^1, ..., x^n)$, $\hat{\eta}^j = \alpha^j_k(x^1, ..., x^n) \eta^k$, where $x^i$ are coordinates in the base, $\eta^k$ are coordinates in the fibre, one can construct a supermanifold pasted together by means of the same formulas where $\eta^k$ are considered as odd coordinates. It is well known that every real $m|n$-dimensional supermanifold can be obtained by means of this construction [5]; therefore we can assume that $P$-manifold $M$ is a bundle $\alpha$ over $N = m(M)$.

(The bundle $\alpha$ has an invariant description as so called conormal bundle [5], we will not use this description.) Sometimes we will use the notation $N_\alpha$ for the supermanifold corresponding to the bundle $\alpha$ over $N$. Let us restrict the form $\omega$ specifying the $P$-structure in $M = N_\alpha$ to $N \subset M$ (i.e. we take $\eta = 0$). The expression
\begin{equation}
\omega|_{\eta=0} = \omega_{ij}(x)dx^i d\eta^j
\end{equation}
determines a non-degenerate pairing between fibres of $\alpha$ and tangent spaces to $N$. The existence of this pairing permits us to identify $\alpha$ with cotangent bundle and $M$ with $T^*N$. However it is possible a priori that the $P$-structure on $T^*N$ arising from this identification and the standard $P$-structure on $T^*N$ are different. To show the equivalence of these $P$-structures we note that corresponding forms $\omega$ and $\omega_0$ can be connected by a smooth family $\omega_t = (1-t)\omega_0 + t \omega$ of closed non-degenerate odd forms. (To check that the forms $\omega_t$ are non-degenerate we use the fact that $\omega$ and $\omega_0$ coincide on $N$ imbedded in standard way into $T^*N$. Non-degeneracy of $\omega_t$ on $T^*N$ follows from non-degeneracy on $N \subset T^*N$.) To finish the proof we utilize the following:

**Lemma 1.** If $\omega$ is a non-degenerate closed odd 2-form and $\sigma$ is a closed odd 2-form then one can find a vector field $V$ in such a way that $\sigma = L_V \omega$ where $L_V \omega$ is the Lie derivative of $\omega$ with respect to $\omega$ (the change of $\omega$ by the infinitesimal transformation $V$).

To prove this lemma we note that $L_V \omega$ can be represented as
\begin{equation}
L_V \omega = (d\omega)_V - d\omega_V
\end{equation}
where for every $k$-form $\sigma$ we denote by $\sigma_V$ the $(k-1)$-form obtained from $\sigma$ by contraction with the vector field $V$. For example, if $\omega = dz^i \omega_{ij} dz^j$ then $\omega_V = V^i \omega_{ij} dz^j$. If $\omega$ is closed then $L_V \omega = -d\omega_V$. Every closed odd 2-form $\sigma$ is exact: $\sigma = d\lambda$, where $\lambda = \lambda_j dz^j$. It remains to say that $V^i$ can be found from the equation

$$\lambda_j = -V^i \omega_{ij} \tag{13}$$

This equation always has a solution because $\omega_{ij}$ is non-degenerate. Moreover this solution is unique.

The proof of the lemma repeats the standard proof of the fact that an even symplectic structure on closed manifold does not change if the 2-form defining it changes, but corresponding cohomology class remains intact.

If $\omega_t$ is a smooth family of non-degenerate closed odd 2-forms on $M = T^*N$ coinciding on $N \subset M$ it follows immediately from the lemma that all these forms determine equivalent $P$-structures. The lemma shows that an infinitesimal variation of form $\omega$ gives an equivalent $P$-structure. The study of a smooth deformation of $\omega$ can be reduced to the study of infinitesimal variation. To construct the transformations proving the equivalence we have to solve differential equation

$$\dot{z}(t) = V(t) z(t) \tag{14}$$

where the vector field $V(t)$ satisfies

$$\dot{\omega}_t = L_{V(t)} \omega_t \tag{15}$$

It follows from the proof of the lemma that one can find $V(t)$ is such a way that it will be differentiable with respect to $t$. This assumption guarantees the existence of solution to (14). (In the case of even symplectic structure it is necessary to assume compactness of symplectic manifold to guarantee the existence of solution to the analog of (14). In the case at hand we don’t need this assumption because $V^i$ generates a zero vector field on the body $N$ of $M = T^*N$.)

In what follows we restrict ourselves by the case when the $P$-manifold $M$ is realized as $T^*N$ with standard $P$-structure; as we proved this can be made without loss of generality. Let us define standard Lagrangian submanifolds of $T^*N$ in the following way. Let us suppose that $K$ is a $k$-dimensional submanifold of $N$. Then we can construct an $(n-k)$-dimensional bundle $\lambda$ over $K$ consisting of covectors orthogonal to $K$. Supermanifold $L_K$ corresponding to this bundle is naturally imbedded into $T^*N$ and can be considered as $(k|n-k)$-dimensional Lagrangian submanifold of $T^*N$.

**Theorem 4.** For every Lagrangian submanifold of $T^*N$ one can find a smooth deformation of this submanifold into a standard Lagrangian submanifold (i.e. into a submanifold of the form $L_K$). To prove this theorem we consider at first the group $G_M$ of all transformations of arbitrary supermanifold $M$. Without loss of generality we assume
that $M = N_\sigma$ where $\sigma$ is a vector bundle over a manifold $N$. Let us denote by $G_\sigma$ the group of automorphisms of the bundle $\sigma$. In local coordinates these automorphisms are given by formulas

$$\tilde{x}^i = f^i(x^1, \ldots, x^n) + \sum_{k=1}^{j_1} \sum_{j_{2k}} f_{j_1 \ldots j_{2k}}^i (x^1, \ldots, x^n) \eta^{j_1} \cdots \eta^{j_{2k}},$$

and

$$\tilde{\eta}^i = a_i^j(x^1, \ldots, x^n) \eta^i + \sum_{k=1}^{j_1} \sum_{j_{2k+1}} a^i_{j_1 \ldots j_{2k+1}} (x^1, \ldots, x^n) \eta^{j_1} \cdots \eta^{j_{2k+1}}.$$  \hfill (16), (17)

Leaving only the first term in (16), (17) we get an automorphism of $\sigma$. (In more invariant words one can say that the transformation of a supermanifold generates naturally an automorphism of corresponding conormal bundle). It is easy check that for $i = 16, 17$ determine a transformation of $\sigma$ generating an automorphism of $\sigma$. Therefore we can simply multiply all these functions by $\tau$, $0 \leq \tau \leq 1$, to obtain a family $Q_\tau$, $0 \leq \tau \leq 1$, of transformations of $M$ obeying $Q_1 = i\alpha$, $Q_0 = i\pi$, $\pi Q_\tau = \pi$.

If $M$ is a $P$-manifold, we will denote by $S_M$ the group of all $P$-transformations of $M$ (transformations preserving the $P$-structure in $M$). In this case $\sigma$ is a cotangent bundle and $G_\sigma$ is imbedded in $S_M$. One can prove the following lemma which is interesting by itself.

**Lemma 2.** The imbedding $i$ of $G_\sigma$ into $S_M$ and the natural map $\pi$ of $S_M$ onto $G_\sigma$ determine a homotopy equivalence between $G_\sigma$ and $S_M$.

To prove this statement we use the deformation $Q_\tau$ constructed above and the arguments used in the proof of Lemma 1. Namely, we will define the deformation $Q_\tau$ as $R_\tau Q_\tau$ where $R_\tau$ is a transformation of $M = T^*N$ satisfying $(R_\tau Q_\tau)^* \omega = \omega$. Such a transformation $R_\tau$ can be found by solving the equations (14), (15). To guarantee the continuity of $R_\tau$ with respect to $\tau$ we have to eliminate the freedom in the construction of $R_\tau$. This can be made if we construct $R_\tau$ by means of (15) with $V_i$ found as a solution of the equation $\lambda^l_i = -V^l_i \cdot \omega_{ij}$, where $\lambda^l = \lambda^l_j dz^j$ is specified by the formula $\lambda^l = (Q_t)^{-1} \lambda$, $d\lambda = \omega$.

Now we are able to prove the Theorem 4. It is easy to check that for arbitrary Lagrangian submanifold $L$ of $P$-manifold $M$ one can find a map $\varphi$ of $T^*L$ into $M$ preserving $P$-structure. Representing $L$ as $K_\beta$ where $K = m(L)$, $\beta$ is a vector bundle over $K$ we can construct a map of $L$ onto a standard Lagrangian submanifold of $M = T^*N$ and extend this map to a map $\tilde{\varphi}$ of $T^*L$ into $M$ preserving $P$-structure. Using the Lemma 2 we can deform $\varphi$
Let us consider a manifold \(N\) provided with a volume element \(\alpha\) (one can consider \(\alpha\) as an \(n\)-form \(\alpha(x^1, ..., x^n)dx^1 \wedge ... \wedge dx^n\) where \(\alpha(x^1, ..., x^n)\) is a non-vanishing function on \(N\)). Differential forms on \(N\) can be considered as functions on a supermanifold \(T^*N\), corresponding to the tangent bundle over \(N\). If \(x^1, ..., x^n\) are local coordinates in \(N\) then \(x^1, ..., x^n, \eta^1 = dx^1, ..., \eta^n = dx^n\) can be considered as coordinates in \(T^*N\). If \(\omega\) is a differential form on \(N\) (a function on \(T^*N\)) one can define a function \(\tilde{\omega} = F\omega\) on \(T^*N\) by the formula

\[
\tilde{\omega}(x, \xi) = \int e^{\xi_i \eta^i} \omega(x, \eta) \alpha(x) d^n\eta
\]

(18)

In other words \(\tilde{\omega}\) is a Fourier transform of \(\omega\) with respect to odd variables. (The functions on \(T^*N\) can be identified with antisymmetric polyvector fields. In this interpretation Fourier transformation is simply the duality transformation, i.e. contraction of covector field \(\omega\) with the universal antisymmetric tensor \(\alpha\).) It is easy to check that

\[
\frac{\partial}{\partial \xi_i} (F\omega) = F(\eta^i \omega), \quad \frac{\partial}{\partial x^i} (F\omega) = F(\frac{\partial \omega}{\partial x^i}) + \alpha^{-1} \frac{\partial \alpha}{\partial x^i} F(\omega).
\]

(19)

Using these formulas we obtain that

\[
F(d\omega) = \Delta F(\omega)
\]

(20)

where \(d\) denotes the exterior differential of \(\omega\) (in the language of functions on \(T^*N\) we have \(d = \eta^i \frac{\partial}{\partial x^i}\) and the operator \(\Delta\) is constructed by means of \(SP\)-structure on \(P\)-manifold \(T^*N\), specified by the volume element

\[
\rho_0(x, \xi) d^n x d^n \xi = \alpha^{-2}(x) d^n x d^n \xi.
\]

(21)

(This connection between \(d\) and \(\Delta\) was used in [3]).

Now we able to prove Theorems 1 and 2 for the case when the manifold \(M = T^*N\) is provided with standard \(P\)-structure and with the volume element (21). We will use the following statement that can be easily proved in this case.

**Lemma 3.** If \(\omega\) is a form on \(N\) and \(K\) is a closed oriented submanifold of \(N\) then

\[
\int_K \omega = \int_{L_K} F(\omega) d\lambda
\]

(22)

where \(L_K\) denotes the Lagrangian submanifold of \(T^*N\) corresponding to \(K\).

The proof of the Lemma 3 in the case when \(\omega\) has a support in a domain where \(K\) in an appropriate coordinate system can be singled out by equations \(x^{k+1} = 0, ..., x^n = 0\) is immediate. Without loss of generality one can assume that \(\omega\) is a monomial with respect to \(\eta^1, ..., \eta^n\). Only the monomial
\( \omega = \gamma(x)\eta^1...\eta^k \) gives a non-zero contribution to the integrals in (22). For this monomial we have

\[ F(\omega) = \gamma(x)\alpha(x)\xi_{k+1}...\xi_n. \]

The volume element \( d\lambda \) on \( L_K \) induced by (21) can be written in the form

\[ d\lambda = \alpha(x)^{-1}dx^1...dx^k d\xi_{k+1}...d\xi_n \]

(we omit the proof of this assertion because a more general fact will be proven later; see Lemma 4). Using the expressions for \( F(\omega) \) and \( d\lambda \) we obtain (22) in the case at hand. The general case can be reduced to this simplest case by means of standard technique (one has to use the partition of unity).

The statements of Theorems 1 and 2 follow immediately from (22) and (20) when the Lagrangian submanifolds are standard. The case of general Lagrangian submanifolds of \( T^*N \) can be reduced to this simplest case by means of Theorem 3. Therefore we can say that Theorems 1 and 2 are proved in the case when \( SP \)-structure in \( T^*N \) is determined by the density that does not depend on \( \xi \).

(The volume element corresponding to such a density can be represented up to a sign in the form (21).)

In the consideration above we did not pay sufficient attention to the choice of the sign of the volume element in Lagrangian submanifold. It suffices to analyze this question for standard Lagrangian submanifold \( L_K \). Let us introduce the notation \( \Lambda(E) \) for the one-dimensional linear space of translationally invariant real measures in the linear superspace \( E \). In other words \( \Lambda(E) \) consists of functions of bases in \( E \) having degree 1 (i.e. to specify an element \( \alpha \in \Lambda(E) \) we have to assign to every basis \( e \in E \) a real number \( \alpha(e) \) in such a way that \( \alpha(Ae) = \det A \cdot \alpha(e) \) where \( \tilde{e} = Ae \) denote a basis obtained from \( e \) by means of a linear transformation: \( \tilde{e}_i = A_i^j e_j \).) To specify the volume element in \( L_K \) one has to single out a non-zero element of \( \Lambda(TL_K(z)) \) a non-zero measure in tangent space \( TL_K(z) \) to \( L_K \) at every point \( z \in L_K \). One can identify \( \Lambda(TL_K(z)) \) with \( \Lambda(TK(m(z))) \otimes \Pi \Lambda(TN(m(z))/TK(m(z)))^* = \Lambda(TN(m(z))) \). (Here \( \Pi \) denotes the parity reversion. We used that \( \Lambda(\Pi E) = \Lambda(E)^* \), \( \Lambda(E^*) = \Lambda(E)^* \), \( \Lambda(E_1) = \Lambda(E_2) \otimes \Lambda(E_1/E_2) \) if \( E_2 \subset E_1 \).) The spaces \( \Lambda(TL_K(z)) \) can be considered as fibres of a line bundle over \( L_K \). If this bundle is trivial and locally the volume element is defined up to a sign then the volume element can be defined globally. Conversely if the volume element is defined globally it can be considered as a non-zero section of this bundle and this bundle is trivial. Using the identification \( \Lambda TL_K(z) = \Lambda TL_K(m(z)) = \Lambda(TN(m(z))) \) we conclude that in the case when \( N \) is orientable (i.e. the bundle over \( N \) with the fibres \( \Lambda(TN(x)) \) is trivial) the volume element on every Lagrangian submanifold can be defined globally. In the general case Lagrangian submanifold \( L \) of \( SP \)-manifold \( M \) can be provided with global volume element if and only if its body \( m(L) \) can be imbedded in an orientable submanifold of \( m(M) \).

Now we have to give a proof of Theorems 1 and 2 for general \( SP \)-manifold. The proof is based on the following
Lemma 4. Let us suppose that $SP$-structure in a $SP$-manifold $M$ is specified by the density $\rho$. If the density $\tilde{\rho} = \rho e^\sigma$ in $M$ also determines a $SP$-structure in $M$ then

$$\Delta_\rho \sigma + \frac{1}{4} \{\sigma, \sigma\} = 0$$

(23)

where $\Delta_\rho$ denotes the operator $\Delta$ corresponding to the $SP$-structure determined by the density $\rho$. The operator $\Delta$ corresponding to the density $\tilde{\rho}$ can be written in the form

$$\Delta_{\tilde{\rho}} = e^{-\sigma/2} \Delta_\rho (e^{\sigma/2}H).$$

(24)

Volume elements $d\tilde{\lambda}$ and $d\lambda$ in the Lagrangian submanifold $L \subset M$ corresponding to $SP$-structures at hand are connected by the formula

$$d\tilde{\lambda} = e^{\sigma/2} d\lambda.$$ 

(25)

The statement of the Lemma 4 is local and therefore we can simplify the proof using Darboux coordinates and assuming that $\rho = 1$. We can write in these coordinates

$$\Delta_{\tilde{\rho}} H = \Delta_\rho H + \frac{1}{2} \{\sigma, H\} = \frac{\partial}{\partial x^a} \frac{\partial}{\partial \xi^a} H + \frac{1}{2} \{\sigma, H\}.$$ 

(26)

Calculating $\Delta_{\tilde{\rho}}^2$ we get

$$\Delta_{\tilde{\rho}}^2 H = \{\Delta_\rho \sigma + \frac{1}{4} \{\sigma, \sigma\}, H\} = 0$$

(27)

This equation shows that in the case when the density $\tilde{\rho}$ determines an $SP$-structure, $\Delta_\rho \sigma + \{\sigma, \sigma\}/4 = \text{const}$. This equation can be written also in the form

$$\Delta_{\tilde{\rho}} e^{\sigma/2} = \text{const} \cdot e^{\sigma/2}.$$ 

(28)

Applying $\Delta_{\tilde{\rho}}$ to (28) we obtain from $\Delta_{\tilde{\rho}}^2 = 0$ that the constant in this equation is equal to 0. In such a way

$$\Delta_{\tilde{\rho}} e^{\sigma/2} = 0.$$ 

(29)

Using (29) one can check that (24) follows from (23). To prove (25) we will give another description of volume element in Lagrangian submanifold $L$. Let us fix a basis $(e_1, ..., e_n)$ in the tangent space $TL(z)$ to $L$ at the point $z \in L$. Then one can find a basis $(e_1, ..., e_n, f^1, ..., f^n)$ in the tangent space $TM(z)$ to $M$ satisfying $\omega(e_i, f^j) = \delta_i^j$ ($P$-structure in $M$ determines an odd bilinear form $\omega$ on $TM(z)$). The volume element $\lambda$ in $L$ can be defined by the formula

$$\lambda(e_1, ..., e_n) = \mu(e_1, ..., e_n, f^1, ..., f^n)^{1/2}$$

(30)

where $\mu$ denotes the volume element determined by $SP$-structure in $M$. The equation (25) follows immediately from (30).
Using the Lemma 4 we can reduce the study of $SP$-structure with density function $\tilde{\rho} = pe^\sigma$ to the study of $SP$-structure with density function $\rho$. In particular we are able now to prove the Theorems 1 and 2 for all $SP$-manifolds. As we mentioned already it is sufficient to consider manifolds of the form $M = T^*N$ with standard $P$-structure. If $SP$-structure in $M$ is specified by the density

$$\rho(x, \xi) = \rho_0(x) + \sum_{k>1} \rho^{i_1 \ldots i_k}(x)\xi_{i_1} \ldots \xi_{i_k}$$

(31)

we can consider another $SP$-structure in $M$ determined by the density $\rho_0(x)$. It is clear that $\rho(x, \xi) = \rho_0(x)e^\sigma(x, \xi)$ where $\sigma(x, \xi) = 0$ for $\xi_1 = \ldots = \xi_n = 0$. It follows from (25) that for every Lagrangian submanifold $L \subset M$ we have

$$\int_L H d\lambda = \int_L H e^{\sigma(x, \xi)} d\lambda_0.$$  

(32)

Further if $\Delta H = 0$ we obtain from (24) that $\Delta_0(H e^{\sigma/2}) = 0$ and if $H = \Delta K$ we get that $H e^{\sigma/2} = \Delta_0(K e^{\sigma/2})$. (We use the notations $d\lambda$ and $d\lambda_0$ for volume elements in $L$ determined by the densities $\rho$ and $\rho_0$; the notations $\Delta$ and $\Delta_0$ have similar meaning.) Using these remarks we reduce the proof of Theorems 1 and 2 for the density $\rho$ to the case of density $\rho_0$. This case was analyzed already.

The consideration above permits us to construct one-to-one correspondence between $SP$-structures in connected $P$-manifold $M$ and cohomology classes $s \in H(m(M), R)$ satisfying $s^n \neq 0$. (Recall that by definition $H(N, R)$ is the direct sum of $k$-dimensional cohomology groups $H^k(N, R)$; we represent $s \in H(m(M), R)$ as $s^0 + s^1 + \ldots + s^n$ where $s^k \in H^k(m(M), R)$.) We suppose without loss of generality that $M$ coincides with $T^*N$ provided with standard $P$-structure. Let us fix a volume element $\alpha$ in $N$. If $\omega$ is a differential form $\omega = \sum_{k=0}^n \omega^k$ in $N$ (i.e. a function $\omega(x, \eta)$ on $TN$) we define a function $\tilde{\omega}(x, \xi)$ on $T^*N$ by means of Fourier transformation (18). Let us assume that the $n$-dimensional component $\omega^n$ of the form $\omega$ does not vanish (i.e. $\omega^n = \beta(x)dx^1 \wedge \ldots \wedge dx^n$ where $\beta(x) \neq 0$). We will define the density function $\rho_\omega(x, \xi)$ on $T^*N$ by the formula

$$\rho_\omega(x, \xi) = \alpha^{-2}(x)\tilde{\omega}(x, \xi)^2.$$ 

(33)

It is easy to check that $\rho_\omega(x, \xi)$ does not depend on the choice of $\alpha$:

$$\rho_\omega(x, \xi) = (\int e^{\xi_i \eta_j} \omega(x, \eta)d^n \eta)^2.$$  

(34)

**Theorem 5.** The density function $\rho_\omega(x, \xi) = \alpha^{-2}(x)\tilde{\omega}(x, \xi)^2$ determines an $SP$-structure in $T^*N$ if and only if the form $\omega$ is closed and its $n$-dimensional component $\omega^n$ does not vanish. Every $SP$-structure in $T^*N$ can be described by means of density function of such a kind. If $\rho_\omega$ and $\rho_{\omega'}$ are density functions
corresponding to closed forms $\omega$ and $\omega'$ then corresponding $SP$-structures are equivalent only in the case when the form $\omega' - \omega$ is exact.

We say here that two $SP$-structures on $T^*N$ are equivalent if there exists a $P$-transformation connecting these $SP$-structures and homotopic to the identity mapping. (As we have seen the transformation of the supermanifold $T^*N$ is homotopic to identity if and only if corresponding transformation of $N$ is homotopic to identity.)

Let us begin the proof with the remark that the application of Lemma 4 to $\tilde{\rho} = \rho_\omega$, $\rho_0 = \alpha^{-2}$, $\tilde{\omega}(x, \xi) = e^{\sigma/2}$ shows that the operator $\Delta$ corresponding to the density $\rho_\omega$ satisfies $\Delta^2 = 0$ if and only if the form $\omega$ is closed. Therefore if $\rho_\omega$ determines an $SP$-structure then $\omega$ is closed. To prove that in the case of closed $\omega$ the density $\rho_\omega$ determines an $SP$-structure we will construct a family $\omega_t$ of closed forms: $\omega_t = (1 - t)\omega^n + t\omega$, corresponding densities $\rho_t = \rho_{\omega_t}$ and operators $\Delta_t$ defined by the formula

$$\Delta_t H = e^{-\sigma_t/2}\Delta_0(e^{\sigma_t/2}H) = \tilde{\omega}_t^{-1} \cdot \Delta_0(\tilde{\omega}_t H).$$ (35)

Here $\Delta_0$ is constructed by means of the density $\rho_0$ corresponding to the form $\omega_t|_{t=0} = \omega^n$. It is clear that the density $\rho_0$ determines an $SP$-structure. To prove that $\rho_\omega$ also determines an $SP$-structure it is sufficient to check that at least locally we can transform $\rho_\omega$ into $\rho_0$ by means of $P$-transformation. To find such a $P$-transformation we construct at first an infinitesimal $P$-transformation (Hamiltonian vector field) transforming $\rho_t$ into $\rho_{t+dt}$. To verify the existence of such a field we note that the change of density $\rho_t$ by the infinitesimal transformation generated by the vector field $K_t$ can be written as

$$\frac{\partial}{\partial z^a}(\rho_t K_t^a) = \rho_t \text{div} K_t$$ (36)

If $K_t$ is a Hamiltonian vector field with Hamiltonian $H_t$ we obtain

$$\dot{\rho}_t = 2\rho_t \Delta_t H$$ (37)

or equivalently

$$\dot{\sigma}_t = 2\Delta_t H_t = 2e^{-\sigma_t/2}\Delta_0(e^{\sigma_t/2}H_t).$$ (38)

From the other side we obtain from Lemma 4 that $\Delta_0 e^{\sigma_t/2} = 0$ Differentiating this equation with respect to $t$ we get

$$\Delta_0(\dot{\sigma}_t e^{\sigma_t/2}) = 0$$ (39)

It follows from (39) that (38) considered as an equation for $H_t$ can be solved at least locally. (One should make the Fourier transformation (18) and use the Poincare lemma). As usual to find the transformation connecting $\rho_\omega$ and $\rho_0$ we have to integrate the equation $\dot{z} = K_t(z)$.

In such a way we proved that the density (33) determines an $SP$-structure if $\omega$ is closed. The same arguments can be used to check that the $\rho_{\omega'}$ and
If $\omega'$ is exact (it follows from the exactness of $\omega'$ that the equation (38) for $H_t$ can be solved globally). To finish the proof of Theorem 5 we have to check that in the case when $\omega'$ is not exact the densities $\rho_{\omega'}$ and $\rho_{\omega}$ cannot determine equivalent $SP$-structures.

Let us suppose that there exists a $P$-transformation $Q$ connecting $\rho_{\omega'}$ and $\rho_{\omega}$ and homotopic to identity. One can conclude from Lemma 2 that in this case we can find a smooth family $Q_t$ of $P$-transformations connecting $Q = Q_1$ with the identity map $Q_0$. Let us denote by $\rho_t$ the density obtained from $\rho_{\omega}$ by means of $Q_t$; corresponding form will be denote by $\omega_t$. The density $\rho_{t+dt}$ can be obtained from the density $\rho_t$ by means of infinitesimal $P$-transformation (Hamiltonian vector field $K_t = Q_tQ_t^{-1}$) and we can apply (38). It follows from (38) that the form $\omega_t$ is exact, therefore the form $\omega' - \omega = \int_0^1 \omega_t dt$ is exact too.

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