OPERATIONS ON INTEGRAL LIFTS OF $K(n)$

JACK MORAVA

Abstract. This very rough sketch is a sequel to [16]; it presents evidence that the 'motivic' group of multiplicative automorphisms of lifts of the functors $K(n)$ to cohomology theories over valuation rings $\mathcal{O}_L$ of local number fields, indexed by Lubin-Tate groups of such fields, are extensions

$$1 \to \text{Spec } E^*(\nu_F) \to \text{Aut}_{\otimes}(K(L)) \to \text{Aut}(F) \cong \mathcal{O}_L^\times \to 1,$$

where $E^*(\nu_F)$ is the exterior algebra on the normal bundle to the orbit of $F$ in the Lubin-Tate space of lifts.

Conventions and inadequacies Among other things, a left group action $(g, x) \to gx$ may be confused, e.g. for symmetry in displays, with the corresponding right action $(x, g) \to g^{-1}x$. Similarly, homology may mean cohomology, represented may mean corepresented, and gradings may be suppressed for notational convenience. Things are sometimes claimed for general local number fields but are (sketchily) discussed only in the unramified case. Boldface type will be inconsistently used to distinguish spectra from the (co)homology theories they (co)represent.

Acknowledgements and thanks to Mona Merling, Xiyuan Wang, Tobias Barthel, Vitaly Lorman, and Apurv Nakade for their interest and encouragement!

§ 1 : Constructions involving formal groups

1.1 Let $\Phi$ be a one-dimensional formal group law of height $n < \infty$ over a finite field $k$ of characteristic $p$. In 1966 Lubin and Tate constructed a formal affine scheme

$$\text{Spf } E_{\Phi} = \text{Spf } W(k)[[u_1, \ldots, u_{n-1}]] = \text{Spf } W(k)[[u_*]]$$

parametrizing lifts of $\Phi$ (up to a certain notion of equivalence) to Artin local algebras with residue field $k$, and they showed that the pro-etale
groupscheme $\text{Aut}_{k}(\Phi)$ of automorphisms of $\Phi$ acts naturally and continuously (but not smoothly) on $\text{Spf} E_{\Phi}$, in the etale topology.

More recently Hopkins, Goerss, Miller, Rezk and others [9,19,...] have constructed a periodic $E_{\infty}$ ring spectrum $E_{\Phi}$ with homotopy groups

$$E_{\Phi*} = E_{\Phi}[v,v^{-1}] \in (W(k) - \text{Mod})$$

(with $v$ of degree two, corresponding to $v_{n} = v^{q - 1}$, see below) admitting $\text{Aut}_{k}(\Phi)$ as group of multiplicative automorphisms\footnote{That is, the profinite group $\text{Aut}_{k}(\Phi)$ of automorphisms of $\Phi \times_{k} \overline{k}$ defined over a separable closure $\overline{k}$ of $k$ acts $\text{Gal}(\overline{k}/k)$-equivariantly on $W(\overline{k}) \otimes_{W(k)} E_{\Phi}$. This and related issues will usually be backgrounded in this account.}, with $E_{\Phi}^{0}(\mathbb{C}P^{\infty})$ realizing the modular lift of $\Phi$ to $E_{\Phi}$.

Let $F$ be a lift of $\Phi$ to a Lubin-Tate group (i.e. a formal $W(k)$-module) over $W(k)$, thus classified by a homomorphism

$$F : E_{\Phi} = W(k)[[u_*]] \to W(k)$$

of $W(k)$-algebras: the constructions below can be extended to the valuation rings of more general local number fields, but the unramified case illustrates the argument well enough. Note that the values $F(u_i)$ lie in the maximal ideal of $W(k)$, and so vanish mod $p$. The parameters $u_i$ are not canonical, and we can equally well describe $E_{\Phi}$ as the algebra $W(k)[[v_1, \ldots, v_{n - 1}]]$ with parameters $v_i = u_i - F(u_i)$.

[Lemma: If $f(x) = \sum_{i \geq 0} f_i x^i \in W(k)[[x]]$ and $w \in (p) \subset W(k)$, then

$$f_w(x) := \sum_{k \geq 0} \left(\sum_{r \geq 0} \binom{k + r}{k} f_{k+r} w^r\right) x^k := f(x + w) \in W(k)[[x]] .$$

In this new system of coordinates, the classifying map $F$ sends the parameters $v_i$ to zero.

Note that Lubin and Tate show [13 Theorem 3.1] that the action of $\text{Aut}_{k}(\Phi)$ sends the closed ideal $m_{\Phi} = [[v_*]] = [[u_*]]$ of $E_{\Phi}$ generated by the parameters to itself.

**Example** If $F$ is Honda’s (Lubin-Tate) group for $W(F_q) \otimes \mathbb{Q}$, i.e. with logarithm

$$\sum_{i \geq 0} p^{-i} T^{q^i} \in W(F_p)[[T]]$$

($q = p^n$), then we can identify $\text{Aut}_{F_p}(\Phi)$ (as a groupscheme over $W(F_p)$) with the group $\mathcal{O}_{\mathfrak{D}}^{\times}$ of units in the domain

$$\mathcal{O}_{\mathfrak{D}} = W(F_q) \langle F \rangle / (F^n = p, \ F a = a^{q} F)$$
where \( a \mapsto a^g \) is the automorphism of \( W(\mathbb{F}_q) \) induced by the Frobenius map \( x \to x^p \) of \( \mathbb{F}_q \), with the action of a generator of \( \text{Gal}(\mathbb{F}_p/\mathbb{F}_p) \) on \( \mathcal{O}_D^\times \) defined to be conjugation by \( F \).

On the other hand, the automorphism group \( \text{Aut}_\mathcal{O}(G) \) of a formal group \( G \) over the valuation ring \( \mathcal{O} \) of a local number field is always commutative: for if

\[
\alpha(T) = \sum_{i \geq 0} \alpha_i T^{i+1}
\]

is an automorphism of \( G \) with coefficients in the valuation ring \( \overline{\mathcal{O}} \) (of an algebraic closure of the quotient field of \( \mathcal{O} \)) then the tangent map \( \alpha \mapsto \alpha_0 \in \overline{\mathcal{O}}^\times \) must be injective, for otherwise \( \alpha \) isn’t invertible. In Honda’s example

\[
\text{Aut}_{W(\mathbb{F}_p)}(F) = W(\mathbb{F}_q)^\times \subset \mathcal{O}_D^\times,
\]

with the induced action of \( \text{Gal}(\mathbb{F}_p/\mathbb{F}_p) \).

Aside: When \( L \) is a ramified local field (\( i.e. \) properly containing a maximal unramified subfield \( W(k) \otimes \mathbb{Q} \)) the image \( \Phi \) (over the residue field \( k \), of a Lubin-Tate group \( F \) for \( L \)) has a modular lift defined over \( \mathcal{O} \otimes_{W(k)} E_\Phi \), which is realized by the homotopy groups of a spectrum\(^2\)

\[
E_F := \mathcal{O}_L \otimes_{W(k)} E_\Phi,
\]

a free \( A_\infty \)-algebra of finite rank over \( E_\Phi \). I believe the constructions below work equally well using \( E_F \) where appropriate; but because \( E_\infty \) structures do not play well with ramification (\( e.g. \) \( p \)-adic \( K \)-theory with a primitive \( p \)th root of unity adjoined to its ground ring does not take values in the the category of \( \lambda \)-rings) we can not expect \( E_F \) to be an \( E_\infty \) ring spectrum.

1.2.1 The category of (one-dimensional) formal group laws and isomorphisms can be regarded as the simplicial affine scheme

\[
[FG/\Gamma] : \text{Spec } \mathbb{Z} \longrightarrow \text{FG} \longrightarrow \text{FG} \times \Gamma \longrightarrow \cdots
\]

defined by the action of a group object \( \Gamma \) on an object \( \text{FG} \), interpreted as the functor

\[
\text{Spec } \mathbb{L} : (\text{Comm}) \ni A \mapsto \text{Hom}_{\text{Comm}}(\mathbb{L}, A) := \text{FG}(A)
\]

\(^2\)regarded as an object with \( \text{Gal}(\mathbb{L}/L) \)-action
of points defined by the Lazard ring \( \mathbb{L} \), with \( \Gamma := \text{Spec } S \) the proalgebraic groupscheme of formal diffeomorphisms of the line at the origin, defined similarly by the commutative Hopf algebra

\[
S = \mathbb{Z}[t_i \mid i \geq 0][t_0^{-1}]
\]

with coproduct \( \Delta t(T) = (t \otimes 1)((1 \otimes t)(T)) \), \( t(T) = \sum_{i \geq 0} t_i T_i \). The simplicial object \([FG/\Gamma]\) is the spectrum (in the algebraic sense) defined by the homotopy groups of the cosimplicial ring spectrum

\[
\cdots \rightarrow \text{MU} \rightarrow \text{MU} \wedge_{S_0} \text{MU} \rightarrow \cdots
\]

(in the homotopy-theoretic sense): an Adams or Mahowald or (co)bar construction... There is a similar interpretation of the (co)operations on BP in terms of the groupoid-scheme of \( p \)-typical formal group laws.

For Lubin-Tate theories there is an analogous groupoid-valued formal scheme \([\text{Spf } E_F/\text{Spf } H_S(\Phi)]\), where \( H_S(\Phi) \) is a completed Hopf algebra corepresenting \( \text{Aut}_k(\Phi) \), i.e. an algebra of \( \text{Gal}(\mathbb{T}/L) \)-equivariant continuous functions from the group \( O_{\mathbb{D}}^x \) of units of a division algebra to \( O_{\mathbb{T}} \); the co-operations for \( E_F \) can thus be corepresented by a Hopf algebroid

\[
\eta_{L,R} : E_{F,*} E_F \cong E_{F,*} \otimes_{O_L} H_S(\Phi)
\]

(or, dually, by thinking of \( E_{F,*} \) as taking values in a category of modules over a twisted group ring \( E_{F,*}(\text{Aut}(\Phi)) \), as in [6,11]).

Note that both maps send the ideal \( m_F \) to \( m_F \otimes_{W} H_S(\Phi) \).

1.2.2 In the category of sets, an action \( \mu \) of a group \( G \) on a set \( X \) defines a similar simplicial object, and a map \( i : A \rightarrow X \) defines a pulled-back isotropy groupoid

\[
[S(A)] : A \rightarrow \text{iso}(A) \rightarrow \text{iso}(A) \times_A \text{iso}(A) \cdots
\]

\[
[X/G] : X \rightarrow X \times G \rightarrow X \times G \times G \cdots
\]

with \( \text{iso}(A) \rightarrow A \) defined by the fiber product

\[
A \rightarrow G \times A \rightarrow G \times X \rightarrow X ;
\]
the two projections to $A$ define an equivalence relation $\sim_G$ on $A$, defining a groupoid with skeleton

$$\coprod_{[a] \in A/\sim_G} \ast/\iso([a])$$

where $a \to A/\sim_G \to X/G$ and

$$\iso([a]) = \{ g \in G \mid g \cdot i(a) = i(a) \}.$$

Thus we can write (a little imprecisely)

$$\iso(A) := A \times_X (G \times A) \cong A \times_X (G \times X) \times_X A.$$

1.2.3 This construction can be extended to categories of (affine, formal, ...) schemes by Yoneda’s lemma, and because fiber products of affine schemes are represented by tensor products of their algebras of functions, the stabilizer groupoid-scheme is representable as a cosimplicial algebra. For example in this context the groupoid defined by a homomorphism $F : \mathbb{L} \to \mathcal{O}_L$ is (co)represented by the Hopf algebroid

$$\mathcal{O} \longrightarrow \mathcal{O} \otimes_F (\mathbb{L} \otimes_{\mathbb{Z}} S) \otimes_F \mathcal{O}$$

(see [18 §5.1.13] for $\Phi : \mathbb{L} \to k$, which defines ‘classical’ $K(n)$). Similarly, in the Lubin-Tate context the stabilizer groupoid $[S(\mathcal{O}_L)] := \text{Aut}(F)$ of a morphism $F : E_F \to \mathcal{O}_L$ classifying a lift of $\Phi$ to $\mathcal{O}_L$ is corepresented by

$$\mathcal{O}_L \longrightarrow \mathcal{O}_L \otimes_{E_F} (E_F \otimes_{\mathcal{O}_L} H_{S(\Phi)}) \otimes_{E_F} \mathcal{O}_L \cong H_{S(\Phi)} \otimes_{E_F} \mathcal{O}_L.$$

Regarded as an object over the Zariski spectrum of $\mathcal{O}_L$, this has as fiber a pro-etale groupscheme $\mathcal{O}_D^\times$ over the closed point, with the maximal torus $\mathcal{O}_L^\times \subset \mathcal{O}_D^\times$ as fiber of over the generic point.

When $L$ is unramified [16], we can take $F$ to be the formal group law with

$$[p]_L(T) = pT + F(v^{q-1}T^q),$$

i.e. with Araki generators $v_i = 0$ if $i > 0$ and $i \neq n$, while $v_n = v^{q-1}$, e.g. with $v = 1$ if we disregard the grading.

§II Constructions involving ring spectra

This section sketches a construction of a periodic spectrum

$$K(L) := |\mathbb{K}_{E_8}(\mathcal{O}_L)|$$

(in notation detailed below), which defines an $A_\infty$ complex oriented spectrum with homotopy groups in the category of modules over $\mathcal{O}_L$; but for simplicity I will usually take $L$ to be unramified. The orientation
defines a formal $O_L$-module structure on $K(L)^0(\mathbb{C}P^\infty)$, with the group law defined by the Chern class being a Lubin-Tate formal group for $L$.

This construction is classical, going back to work of Sullivan and Baas [2,15,22,23 ...], and is by now standard, cf e.g. [10 §3.3]. Indeed, this whole section may be overkill, but one of its points is to acknowledge great advances in our understanding of symmetric monoidal structures on categories of spectra (cf e.g. [8,14, ...]). A key tool is the analog for spectra of the spectral sequence for the (co)homology of the totalization of a simplicial space; another is the construction of the homotopy colimit of a diagram as the totalization of a simplicial object [7 §14.15]. The goal of this section is to frame the necessary constructions in terms of Koszul homology in commutative algebra.

2.1 Following §1.1, let $F/O_L$ be a Lubin-Tate group law, defined by the quotient homomorphism

$$E_F \to E_F/m_F = O_L;$$

recall that the ideal $m_F = [[v_1, \ldots, v_{n-1}]]$ is generated by the regular sequence $v_*$ of parameters. Let $v_i : S^{2(p^i - 1)} \to E_F$ be the choice of a morphism representing $v_i$, and write

$$v_i : S^{2(p^i - 1)}E_F \to E_F$$

for the endomorphism of $E_F$ defined by the product with $v_i$.

At this point I will simplify notation by dropping the subscript $F$, so $E_F := E$, with homotopy groups $E_*$. Similarly, $S^0$ is better read as something like $S^0 \otimes O_L$, with a Gal($\overline{L}/L$) - action as icing on the cake ...

2.2 The classical Koszul complex

$$K_{A*}(a_*):= \otimes_{1 \leq i \leq n-1} K_A(a_i)$$

[24 §15.28] associated to a commutative $k$-algebra $A$ (with $k$ a commutative ring, not necessarily a field) and a sequence $a_* = (a_1, \ldots, a_{n-1})$ of elements of $A$, is the graded commutative DGA defined as the product of elementary Koszul complexes

$$K_{A*}(a) := A \otimes_k E_k(e) : 0 \to A[1] \xrightarrow{a} A \to 0,$$

with multiplication by $a$ as differential; $E_k(e)$ is the exterior algebra over $k$ with one generator. This extends to graded commutative rings, assuming that the elements $a_i$ are homogeneous.
When $A$ is concentrated in even degree and the sequence $a_*$ is regular, i.e. $a_i$ is not a zero-divisor in the quotient $A/(a_1, \ldots, a_{i-1})$ (e.g. $v_* \in E_*$), the homology of $\mathbb{K}_A(a_*)$ vanishes in positive grading, with zero-dimensional homology $A/(a_*)$; it is thus a resolution of this quotient by a commutative $A$-free DGA. In a nice symmetric monoidal category, e.g. of spectra, a sequence of ‘elements’ such as $v_*$ defines an analog

$$\wedge_{E, 1 \leq i \leq n-1}(\text{Cofiber } v_i)$$

of the Koszul complex [8 V §2.6, 3.2; note that some restrictions apply]. Suppressing gradings for readability, the functor $D(v)$ from the diagram category $[0 \to 1]$ sending the arrow to the endomorphism of $E$ defined by $v$-multiplication has a commutative, associative multiplication

$$D(v) \wedge S^0 D(v) \to D(v)$$

in the category of diagrams, i.e.

$$\begin{array}{ccc}
E \wedge E & \xrightarrow{1 \wedge v} & E \wedge E \\
\downarrow \mu(v \wedge 1) & & \downarrow \mu(1 \wedge v) \\
E \wedge E & \xleftarrow{v \wedge 1} & E \wedge E \\
\downarrow \mu \wedge 1 & & \downarrow \mu \\
E & \xrightarrow{v} & E,
\end{array}$$

which totalizes to define a morphism $|D(v)| \wedge S^0 |D(v)| \to |D(v)|$ of homotopy colimits, making $|D(v)| = \text{Cofiber } v$ an $A_\infty$ algebra.$^3$

Similarly, interpreting $E$ as a monoid in a suitable category of spectra, the diagram category

$$D(v_*) := \wedge_{E, 1 \leq i \leq n-1} D(v_i)$$

defines a simplicial spectrum $\mathbb{K}_E(v_*)$ which totalizes to a homotopy colimit $K(L) := |\mathbb{K}_E(v_*)|$ of the diagram, with an $A_\infty$ multiplication

$$K(L) \wedge S^0 K(L) \to K(L).$$

Totalizations are canonically filtered, defining spectral sequences [20 Prop 5.1, 3] which can sometimes be used to calculate (co)homological invariants. [Note that the associated filtration is defined by a diagram with vertices indexed by \textit{proper} subsets of $\{1, \ldots, n-1\}$ [15, p 185].]

$^3$This diagram is meant to be interpreted as a morphism from the diagram defined by the top face of the prism to the diagram defined by its bottom line.
In the case at hand, the homotopy groups of $K(L) = |K_E(v_*)|$ are the abutment of a spectral sequence with the algebraic Koszul complex

$$E^1 = K_{E_*}(v_*)$$

as first page: but since $v_*$ is regular, this is a resolution, and the spectral sequence collapses to an isomorphism

$$\pi_*|K_E(v_*)| : = K(L)_* \cong \mathcal{O}_L[v, v^{-1}] ;$$

subject, however, to the ambiguities (e.g. with respect to choice of representatives) noted above.

Recent work [10] of Hopkins and Lurie on Azumaya objects in spectra seems to have changed the geography of this subject.

§III Cohomology automorphisms of $K(L)$

3.1 The spectral sequence used above to calculate the homotopy groups of $|K_E(v_*)|$ can similarly be used to calculate its $E_*$-homology, which can be identified with $K(L)_*(E)$. This spectral sequence has the algebraic Koszul complex

$$K_E\cdot E(v_*)$$

(with the $v_i$ acting as left multiplication) as its $E^1$ page; but $E_*E \cong E_* \otimes_{\mathcal{O}_L} H_{S(\phi)}$ is left-free over $E_*$, so this again collapses to an isomorphism

$$K(L)_*(E) \cong K(L)_* \otimes_{\mathcal{O}_L} H_{S(\phi)} \cong H_{S(\phi)}[v, v^{-1}] .$$

[For example, the co-operation ring $K_*K$ for classical $p$-adic topological $K$-theory [5,11] is an algebra of continuous functions from $\mathbb{Z}_p^\times$ to $\mathbb{Z}_p$ . . .]

3.2 We can now compute $K(L)_*K(L)$ using the equivalence

$$K(L) \wedge_{S^0} K(L) \simeq (K(L) \wedge_{S^0} E) \wedge_E K(L)$$

together with the spectral sequence for smash products of module-spectra [8 Theorem 6.4], which in our case has

$$\text{Tor}^{E_*}_*(K(L), E, K(L)_*) .$$

as $E^2$ page. Using $K_{E_*}(v_*)$ as an $E_*$-module resolution for $K(L)_*$, we thus need to calculate the homology of the Koszul complex

$$K_K(L) \otimes_{\mathcal{O}_L} H_{S(\phi)}(\eta(v_*)) ,$$

where $\eta(v_i) \in K(L) \otimes_{\mathcal{O}_L} H_{S(\phi)}$ is the image, under the (right unit) coaction map

$$E_* \xrightarrow{\eta_R} E_* \otimes_{\mathcal{O}_L} H_{S(\phi)} \rightarrow K(L)_* \otimes_{\mathcal{O}_L} H_{S(\phi)}$$
of $v_i$. But, following the remark in §1.1, the coefficients $v_{i, \alpha}$ of

$$
\eta_R(v_i) := \sum_{\alpha} v_{i, \alpha} \otimes g_{i, \alpha} \in E_* E
$$

lie in the ideal $m_\Phi = (v_1, \ldots, v_{n-1})$, and so vanish when mapped to $K(L)_\ast$. Following §1.2.3, this identifies the $E^2$ page of the Tor spectral sequence as the homology of the complex $K(K(L) \otimes_{\mathcal{O}_L} H_{S(\Phi)}(0))$: where $0 = (0, \ldots, 0)$.

This is easy to calculate: it collapses to the tensor product of the Hopf algebroid

$$
K(L)_\ast \longrightarrow K(L)_\ast \otimes_{\mathcal{O}_L} H_{S(\Phi)} \otimes_{\mathcal{O}_L} K(L)_\ast
$$

representing the stabilizer groupoid of a Lubin-Tate group $F_L$, with an exterior (Hopf) algebra on the free module

$$
\nu_F := K(L)_\ast \otimes E_\ast m_\Phi / m_\Phi^2
$$

of rank $n - 1$ representing (normal) deformations, i.e. transverse to the orbit under the stabilizer action, of Spec $\mathcal{O}_L$ in Spec $E$. With mod $p$ coefficients

$$
K(L)^\ast (M(\mathbb{Z}/p\mathbb{Z}, 0) \wedge (-) \cong K(n)^\ast (-) \otimes_{\mathcal{O}_L} k
$$

we recover the usual $K(n)$ associated to the formal group law $\Phi$, together with one additional generator for the exterior algebra of normal deformations, corresponding to the classical mod $p$ Bockstein.

### 3.3

I believe that the $A_\infty$ structure on $K(L)$ defined in §2 implies that this is a spectral sequence of Hopf algebroids which collapses to an isomorphism of $K(L)_\ast K(L)$ with the ‘derived’ Hopf algebroid defined by an extension of the Hopf algebra representing the Lubin-Tate stabilizer groupoid of $F_L$ by the exterior Hopf $\mathcal{O}_L$-algebra of normal deformations. Alternately, writing $\text{Aut}_{\otimes}((K(L))$ for the groupscheme of automorphisms of the multiplicative cohomology theory $K(L)_\ast$, we have an extension

$$
1 \to \text{Spec } E^\ast(\nu_F) \to \text{Aut}_{\otimes}(K(L)) \to \text{Aut}(F) \to 1,
$$

with Spec $E^\ast$ interpreted as a (super) group object associated to a $\mathbb{Z}/2\mathbb{Z}$-graded exterior algebra. The periodic localization of the trace morphism

$$
k(L)^\ast \to \text{THH}^\ast(\mathcal{O}_{L_\infty})
$$

of multiplicative cohomology theories [17] would then factor through a morphism

$$
K(L)^\ast \to \text{THH}^\ast(\mathcal{O}_{L_\infty}) \otimes \mathbb{Q}
$$
compatible with the identification of \( \text{Aut}(F_L) \) with \( \text{Gal}(L^\infty/L) \) defined by Artin reciprocity.

In particular, the conjecture is that there are moreover no extension problems in this collapsed spectral sequence. When \( n = 1 \) or 2 this is pretty obvious, but it is evidently beyond my ability to prove it in general.

REFERENCES

1. V Angeltveit, Uniqueness of Morava K-theory, 
   https://arxiv.org/abs/0810.5032
2. N A Baas, On bordism theory of manifolds with singularities, Math. Scand. 33 (1973), 279 - 302 (1974)
3. A K Bousfield, D Kan, Homotopy limits, completions and localizations, Springer LNM 304 (1972)
4. Ching-Li Chai, The group action on the closed fiber of the Lubin-Tate moduli space, Duke Math. J. 82 (1996) 725 - 754
5. F Clarke, p-adic analysis and operations in \( K \)-theory, Groupe de travail d'analyse ultrametrique, tome 14 (1986-1987) no 15, p 1 – 12, available at www.numdam.org/article/GAU_1986-198_14_A7_0.pdf
6. E Devinatz, M Hopkins, The action of the Morava stabilizer group on the Lubin-Tate moduli space of lifts, Am. J. Math. 117 (1995) 669 - 710
7. D Dugger, A primer on homotopy colimits, available at pages.uoregon.edu/ddugger/hocolim.pdf
8. A Elmendorf, I Kriz, M Mandell, JP May, Rings, modules, and algebras in stable homotopy theory . . ., Mathematical Surveys and Monographs 47, AMS (1997)
9. P Goerss, M Hopkins, Moduli spaces of commutative ring spectra, in Structured ring spectra 151 - 200, LMS Lecture Notes 315, Cambridge (2004)
10. M Hopkins, J Lurie, On Brauer groups of Lubin-Tate spectra I www.math.harvard.edu/~lurie/papers/Brauer.pdf
11. M Hovey, Operations and co-operations in Morava \( E \)-theory, Homology, Homotopy and Applications, 6 (2004) 201 - 236
12. J Lubin, J Tate, Formal complex multiplication in local fields, Ann. Math. 81 (1965) 380 - 387
13. ——, —— Formal moduli for one-parameter formal Lie groups, Bull. Soc. Math. France 94 (1966) 49 - 59
14. M Mandell, JP May, Orthogonal spectra and S-modules, www.math.uchicago.edu/~may/PAPERS/mmLMSdec30.pdf
15. J Morava, A product for the odd-primary bordism of manifolds with singularities, Topology 18 (1979) 177 - 186
16. ——, Local fields and extraordinary K-theory, 
   https://arxiv.org/abs/1207.4011
17. ——, Complex orientations for THH of some perfectoid fields, 
   https://arxiv.org/abs/1608.04702
18. D Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics 121, Academic Press (1986)
19. C Rezk, Notes on the Hopkins-Miller theorem, in *Homotopy theory via algebraic geometry and group representations* 313 - 366, Contemp. Math. 220, AMS (1998)
20. G Segal, Classifying spaces and spectral sequences, IHES Publ. Math. 34 (1968) 105 - 112
21. JP Serre, Local class field theory, in *Algebraic Number Theory*, Brighton (1965) 128 - 161, Thompson (1967)
22. N Shimada, N Yagita, Multiplications in the complex bordism theory with singularities. Publ. Res. Inst. Math. Sci. 12 (1976/77) 259 - 293
23. D Sullivan, Singularities in spaces, in *Proceedings of Liverpool Singularities Symposium II* (1969/1970) 196 - 206; Springer LNM 209 (1971)
24. The Stacks Project, [https://stacks.math.columbia.edu/tag/0621](https://stacks.math.columbia.edu/tag/0621)

Department of Mathematics, The Johns Hopkins University, Baltimore, Maryland 21218

E-mail address: jack@math.jhu.edu