A proposal for a generalized canonical osp(1,2) quantization of dynamical systems with constraints

Petr M. Lavrov\textsuperscript{a}, Jorge Ananias Neto\textsuperscript{b}\textsuperscript{†} and Wilson Oliveira\textsuperscript{b}\textsuperscript{‡}

\textsuperscript{a} Tomsk State Pedagogical University, Tomsk 634041, Russia
\textsuperscript{b} Departamento de Física, ICE, Universidade Federal de Juiz de Fora, 36036-330 Juiz de Fora, MG, Brazil

Abstract

The aim of this paper is to consider a possibility of constructing for arbitrary dynamical systems with first-class constraints a generalized canonical quantization method based on the \textit{osp}(1,2) supersymmetry principle. This proposal can be considered as a counterpart to the \textit{osp}(1,2)-covariant Lagrangian quantization method introduced recently by Geyer, Lavrov and Mülsch. The gauge dependence of Green’s functions is studied. It is shown that if the parameter \( m^2 \) of the \textit{osp}(1,2) superalgebra is not equal to zero then the vacuum functional and \( S \)-matrix depend on the gauge. In the limit \( m \to 0 \) the gauge independence of vacuum functional and \( S \)-matrix are restored. The Ward identities related to the \textit{osp}(1,2) symmetry are derived.

PACS number: 11.10Ef, 11.30Ly

Keywords: constrained systems, gauge theory.

\textsuperscript{*}lavrov@tspu.edu.ru
\textsuperscript{†}jorge@fisica.ufjf.br
\textsuperscript{‡}wilson@fisica.ufjf.br
1 Introduction

The canonical version of extended BRST formalism \cite{1, 2, 3} is intended, in principle, to quantize dynamical systems with constraints. This method is based on a special type of Hamiltonian that possesses, simultaneously, the so-called BRST and anti-BRST global symmetries. These requirements can be implemented by using the global symplectic group $Sp(2)$, in which the generators of BRST and anti-BRST transformations turn out to form a doublet of the $Sp(2)$-group.

The study of \cite{4} has proposed the so-called $osp(1, 2)$-covariant Lagrangian quantization. This quantization method is based on a supergroup, which is larger than the extended BRST supergroup applied in papers \cite{5, 6, 7}, and which allows a natural procedure of including massive terms, needed to circumvent possible infrared singularities arising in the process of subtracting ultraviolet divergences, in a manner avoiding the breaking of extended BRST symmetry.

Further development of quantization procedures in the spirit of papers \cite{8, 9, 10} and \cite{1, 2, 3} makes it possible to formulate the rules of generalized canonical quantization based on the global orthosymplectic supergroup $osp(1, 2)$. Thus, the present paper should be considered as a canonical counterpart of the Lagrangian method of $osp(1, 2)$-covariant quantization \cite{4}. We will be concerned in a possibility of developing a consistent scheme of generalized canonical quantization, omitting so far all proofs of existence theorems.

In the phase space of arbitrary canonical variables $(P_A, Q^A)$ we apply the usual definition of the Poisson superbracket

$$\{G, F\} = \frac{\delta G}{\delta Q^A} \frac{\delta F}{\delta P_A} - \frac{\delta F}{\delta Q^A} \frac{\delta G}{\delta P_A} (-1)^{\epsilon(G)\epsilon(F)}, \quad (1)$$
where $\epsilon(G)$ is the Grassmann parity of a quantity $G$. The Grassmann parities of the momenta $P_A$ coincide with those of the corresponding coordinates $Q^A$: $\epsilon(P_A) = \epsilon(Q^A) = \epsilon_A$.

The Poisson superbracket (1) possesses the following properties:

$$\epsilon(\{G, F\}) = \epsilon(G) + \epsilon(F),$$

$$\{G, F\} = -(-1)^{\epsilon(G)\epsilon(F)}\{F, G\},$$

$$\{\{G, F\}, H\}(-1)^{\epsilon(G)\epsilon(H)} + \text{cycle}(G, F, H) \equiv 0.$$ \hspace{1cm} (4)

The above relation is the Jacobi identity for the Poisson superbracket. In Eq.(1) and henceforth, derivatives with respect to the coordinates are always understood as acting from the right, and those with respect to the momenta, as acting from the left.

The paper is organized as follows. In Section II we discuss the general features of canonical $osp(1,2)$ quantization, including the construction of a unitarizing Hamiltonian invariant under the extended BRST and $Sp(2)$ transformations. Section III is devoted to the question of gauge dependence and Ward identities. Conclusions are given in Section IV.

## 2 Canonical $osp(1,2)$ quantization

Let us consider a dynamical system described by a Hamiltonian $H_0 = H_0(p_i, q^i)$, $\epsilon(H_0) = 0$, in the phase space of initial canonical variables $(p_i, q^i)$, $i = 1, 2, \ldots, n$, $\epsilon(p_i) = \epsilon_i$ as well as by a set of first-class constraints $T_{a_0} = T_{a_0}(p_i, q^i)$, $a_0 = 1, 2, \ldots, m < 2n$, with Grassmann parities $\epsilon(T_{a_0}) = \epsilon_{a_0}$. The following involution relations hold true:

$$\{T_{\alpha_0}, T_{\beta_0}\} = T_{\gamma_0}U_{\alpha_0\beta_0}^{\gamma_0}.$$ \hspace{1cm} (5)
\[
\{ H_0, T_{\alpha\beta} \} = T_{\beta\alpha} V_{\alpha\beta}^{\beta_0},
\]
where the structure functions \( U_{\alpha\alpha\beta\beta}^{\gamma_0} \) possess the properties of generalized antisymmetry \( U_{\alpha\alpha\beta\beta}^{\gamma_0} = -(-1)^{\epsilon_{\alpha\beta} \epsilon_{\gamma_0}} U_{\beta\alpha\alpha\alpha}^{\gamma_0} \).

Let us introduce an extended phase space \( \Gamma \) parametrized by the following canonical variables:

\[
\Gamma = (P_A, Q_A) = (p_i, q_i; P_{\alpha a}, C_{\alpha a}, a = 1, 2; \cdots)
\]

\[
\epsilon(P_A) = \epsilon(Q_A) = \epsilon_A, \quad \epsilon(P_{\alpha a}) = \epsilon_{\alpha 0} + 1,
\]

where the dots stand for a set of possible additional auxiliary canonical variables (see [1, 2]). The explicit structure of the extended phase space \( \Gamma \) depends on the properties of the constraints \( T_{\alpha\alpha} = T_{\alpha\alpha}(p_i, q_i) \) of the initial dynamical system as well as on the existence of non-trivial solutions to the generating equations of the formalism. Here, we are not concerned in the explicit structure of \( \Gamma \), as we will discuss only the most general features of the canonical \( osp(1, 2) \) quantization.

The key role in the canonical scheme of \( osp(1, 2) \) quantization belongs to the set of generating functions \( \Omega_m^a, \Omega_\alpha \) and \( \mathcal{H} \), with \( a = 1, 2; \alpha = 0, +, -; \)
\( \epsilon(\Omega_m^a) = 1, \epsilon(\Omega_\alpha) = 0 \) and \( \epsilon(\mathcal{H}) = 0 \). The functions \( \Omega_m^a \) and \( \Omega_\alpha \) satisfy the generating equations

\[
\{ \Omega_\alpha, \Omega_\beta \} = \epsilon_{\alpha\beta}^\gamma \Omega_\gamma,
\]

\[
\{ \Omega_\alpha, \Omega_m^a \} = \Omega_m^b (\sigma_{\alpha})_{b}^a ,
\]

\[
\{ \Omega_m^a, \Omega_m^b \} = -m^2 (\sigma^a)^{ab} \Omega_\alpha ,
\]
where \( m \) is a constant (mass) parameter,

\[
(\sigma^a)^{ab} = \epsilon^{ac}(\sigma^c)^b = (\sigma^a)^a\epsilon^{cb} = \epsilon^{ac}(\sigma^a)_{cd}\epsilon^{db}, \quad (\sigma^a)^b_a = -(\sigma^a)^b_a,
\]  

while \( \sigma \) generate the group of special linear transformations with the \( sl(2) \) algebra

\[
\sigma_a\sigma_\beta = g_{a\beta} + \frac{1}{2}\epsilon_{a\gamma\beta}\sigma^\gamma, \quad \sigma^a = g^{a\beta}\sigma_\beta, \quad Tr(\sigma_a\sigma_\beta) = 2g_{a\beta},
\]

\[
g^{a\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}, \quad g^{a\gamma}g_{\gamma\beta} = \delta_\beta^a,
\]  

where \( \epsilon_{a\beta\gamma} \) is the antisymmetric tensor \( \epsilon_{0^+} = 1. \) The algebra of functions (8), (9), (10) is isomorphic to the \( osp(1,2) \)-superalgebra [11]. Here it should be noted that the right-hand side of (10) for \( m \neq 0 \) is a generalization of the conventional extended BRST relations of the \( Sp(2) \)-formalism [1, 2, 3] (for earlier discussions of the extended BRST symmetry in the generalized canonical formalism, see also [12]).

The functions \( \Omega_m^a \) and \( \Omega_\alpha \) should be considered as the generators of extended BRST and \( Sp(2) \) transformations, respectively. In its turn, the boson function \( H \) satisfies the generating equations

\[
\{ H, \Omega_\alpha \} = 0, \quad (13)
\]

\[
\{ H, \Omega_m^a \} = 0. \quad (14)
\]

These equations can be interpreted as a requirement of invariance of \( H \) under the transformations of the \( Sp(2) \) and extended BRST symmetries, respectively.
We shall now determine the unitarizing Hamiltonian $H$, in terms of $H$ and $\Omega^a_m$, by the formula

$$H = H + \frac{1}{2} \varepsilon_{ab} \{\{\Phi, \Omega^b_m\}, \Omega^a_m\} + m^2 \Phi,$$

(15)

where $\Phi$ is a boson function fixing a concrete choice of admissible gauge. In what follows we will require $\Phi$ to be an $Sp(2)$-scalar, i.e.

$$\{\Phi, \Omega_\alpha\} = 0.$$

(16)

The extended BRST and $Sp(2)$ transformations of canonical variables are given, respectively, by

$$\delta \Gamma = \{\Gamma, \Omega^a_m\} \mu_a; \quad \varepsilon(\mu_a) = 1,$$

(17)

$$\delta \Gamma = \{\Gamma, \Omega_\alpha\} \eta^\alpha; \quad \varepsilon(\eta^\alpha) = 0.$$

(18)

Here, $\mu_a$ form an $Sp(2)$-doublet of constant Grassmann parameters, and $\eta^\alpha$ are bosonic parameters. Thus, an essential property of the Hamiltonian $H$ is its invariance

$$\delta H = \{H, \Omega^a_m\} \mu_a = 0,$$

(19)

$$\delta H = \{H, \Omega_\alpha\} \eta^\alpha = 0.$$

(20)

The invariance (19) and (20) follows from Eqs. (13) and (14) as well as from the Jacobi identities for $\Omega^a_m$ and $\Omega_\alpha$. 
3 Gauge dependence and Ward identities

Let us define the vacuum functional $Z$ in terms of the following functional integral:

$$Z = \int D\Gamma \exp \left\{ -\frac{i}{\hbar} \int dt (\dot{P}_A \dot{Q}^A - H) \right\}.$$  
(21)

Making a change of the gauge $\Phi \to \Phi + \Delta \Phi$ in (21), we obtain

$$Z_{\Phi + \Delta \Phi} = \int D\Gamma \exp \left\{ -\frac{i}{\hbar} \int dt (\dot{P}_A \dot{Q}^A - H - \frac{1}{2} \epsilon_{ab} \{ \{ \Delta \Phi, \Omega^b_m \}, \Omega^a_m \} - m^2 \Delta \Phi) \right\}.$$  
(22)

The term $\frac{1}{2} \epsilon_{ab} \{ \{ \Delta \Phi, \Omega^b_m \}, \Omega^a_m \}$ in (22) can be compensated for by the change of integration variables $\Gamma \to \Gamma + \delta \Gamma$, where

$$\delta \Gamma = \frac{i}{\hbar} \{ \Gamma, \Omega^a_m \} \epsilon_{ab} \{ \Omega^b_m, \Delta \Phi \}.$$  
(23)

On the other hand, it is impossible to cancel the term $m^2 \Delta \Phi$ by using transformations of the form (18) with any functions $\eta^a$ due to the fact that matrices $\sigma_\alpha$ are traceless.

By comparison with (24), we find that the term $m^2 \Delta \Phi$ violates the independence of the vacuum functional $Z_\Phi$ from the choice of gauge. Hence, $Z_{\Phi + \Delta \Phi} \neq Z_\Phi$, and therefore in the case $m \neq 0$ the $S$ matrix within the formalism of canonical $osp(1,2)$ quantization becomes gauge-dependent. Taking the limit $m \to 0$, the gauge independence of the $S$ matrix and vacuum
functional (21) are restored. Moreover, in this limit the vacuum functional (21) is reduced to the well-known answer of the generalized canonical $Sp(2)$ formalism [1, 2].

Finally, we shall derive the Ward identities. To begin with, we assume that the $Sp(2)$-symmetries are realized on the variables $\Gamma$ as rotations of $\Gamma$ with respect to the $Sp(2)$ index only

$$\{\Gamma, \Omega_a\} = \Sigma_a \cdot \Gamma,$$

where $\Sigma_a$ is a constant matrix. In fact, this is a requirement that the transformations realized on the canonical variables form a closed algebra. Let us also consider the generating functional

$$Z(J, \Gamma^*_a, \bar{\Gamma}) = \int D\Gamma \exp \left\{ \frac{i}{\hbar} \int dt (P_A \dot{Q}_A - H + J \Gamma^*_a \{\Gamma, \Omega_m^a\}) \right\}. \tag{25}$$

Given the above generating functional, the Green functions of the theory with the Hamiltonian $H$ are calculated through taking derivatives with respect to the sources $J$ for $\Gamma^*_a = \bar{\Gamma} = J = 0$. In (25) we have introduced additional sources $\Gamma^*_a$ to the transformation $\{\Gamma, \Omega_m^a\}$ and a source $\bar{\Gamma}$ to the generator $\frac{1}{2} \epsilon_{ab} \{\{\Gamma, \Omega_m^b\}, \Omega_m^a\}$. We shall now make the change of variables (17) in the functional integral (25). Using the invariance property of $H$ (14) as well as the fact that the Berezinian of this change is equal to $\exp\{- \int dt \{\Omega_m^a, \mu_a\}\} = 1$, we obtain the following Ward identities for the generating functional $Z$ in (25):
In the functional integral (25) we shall make the change of variables (18).

With allowance for the invariance of $H$ and the fact that the Berezinian of this change is equal to unity, we obtain the following Ward identities for the generating functional $Z$ in (25):

$$J \frac{\delta Z}{\delta \Gamma^*_a} - \epsilon^{ab} \Gamma^*_b \frac{\delta Z}{\delta \Gamma} + \frac{1}{2} m^2 (\sigma^a)^{ab} \Gamma^*_b \Sigma_a \frac{\delta Z}{\delta J} + \frac{1}{2} m^2 \overline{\Gamma} \frac{\delta Z}{\delta \Gamma^*_a}$$

$$+ \frac{1}{2} m^2 (\sigma^a)^a_b \bar{\Gamma} \Sigma_a \frac{\delta Z}{\delta \Gamma^*_b} = 0. \quad (26)$$

4 Conclusion

In this paper we have considered a possibility to generalize the canonical version of extended BRST quantization [1, 2, 3], i.e. by extending it to a canonical $osp(1,2)$ quantization. This generalized canonical quantization can be considered as a canonical counterpart to the $osp(1,2)$-covariant Lagrangian quantization [4]. As has been shown above, the $osp(1,2)$ symmetry group permits introducing mass terms without breaking the extended BRST symmetry. Mass terms are important as a means applied to solve the problem of infrared divergences. Following this consistent formulation of $osp(1,2)$ approach, we have shown that the vacuum functional and $S$-matrix with massive field terms are no longer gauge-invariant. Thus, it is evident that
after performing the renormalization procedure we need to take the massless limit in order to get sensible physical answers because, in any case, physical results do not depend on the gauge.

5 Acknowledgments

The work has been supported in part by FAPEMIG, Brazilian Research Council. One of the authors (PML) appreciates partial support from FAPEMIG grant code number CEX-1308/97. The work of PML has also been supported by the grant of Ministry of General and Professional Education of Russian Federation in field of Basic Natural Sciences, as well as by the grants INTAS 96-0308 and RFBR-DFG 96-02-00180. The authors are grateful to the referee for critical remarks.

References

[1] I.A. Batalin, P.M. Lavrov and I.V. Tyutin, J.Math.Phys.31 (1990) 6.
[2] I.A. Batalin, P.M. Lavrov and I.V. Tyutin, J.Math.Phys.31 (1990) 2708.
[3] I.A. Batalin, P.M. Lavrov and I.V. Tyutin, Int.J.Mod.Phys. 6 (1990) 3599.
[4] B. Geyer, P.M. Lavrov and D. Mülsch, Acta Phys. Pol. B, 29 (1998) 2637.
[5] I.A. Batalin, P.M. Lavrov and I.V. Tyutin, J.Math.Phys.31 (1990) 1487.
[6] I.A. Batalin, P.M. Lavrov and I.V. Tyutin, J.Math.Phys.32 (1991) 532.
[7] I.A. Batalin, P.M. Lavrov and I.V. Tyutin, J.Math.Phys.32 (1991) 2513.
[8] E.S. Fradkin and G.A. Vilkovisky, Phys.Lett. B55, 224 (1975).

[9] I.A. Batalin and G.A. Vilkovisky, Phys.Lett. B69, 309 (1977).

[10] E.S. Fradkin and T.E. Fradkina, Phys.Lett. B72, 343 (1978).

[11] A. Pais and V. Rittenberg, J.Math.Phys.16 (1975) 2062;  
    W. Nahm and V. Rittenberg, J.Math.Phys.18 (1976) 146, 155;  
    F.A. Berezin and V.N. Tolstoy, Commun. Math Phys.78 (1981) 409;  
    L. Frappat, P. Sorba and A. Sciarrino, Dictionary on Lie superalgebras,  
    ENSLAPP-AL-600/96, hep-th/9607161.

[12] S. Hwang, Nucl. Phys. B231, 386 (1984);  
    H. Aratyn, R. Ingermanson and A.J. Niemi, Phys. Lett. B189, 427  
    (1987); Nucl. Phys. B307, 157 (1988);  
    V.P. Spiridonov, Nucl. Phys. B308, 527 (1988).