ON DISTRIBUTIONAL SYMMETRIES ON THE CAR ALGEBRA

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Abstract. Spreadable, exchangeable, and rotatable states on the CAR algebra are shown to be the same.

Mathematics Subject Classification: 46L55, 46L53, 60G09, 81V74.
Key words: CAR algebra, exchangeable states, spreadable states, rotatable states, non-commutative ergodic theory.

1. Introduction

A (discrete) stochastic process enjoys a distributional symmetry when its finite-dimensional joint distributions are invariant under the action of remarkable groups or semigroups of measurable transformations. Among these, a notable role is played by the group of translations, the group of finite permutations, the group of rotations, and the semigroup of increasing monotone maps. The corresponding distributional symmetries are known as stationarity, exchangeability, rotatability and spreadability, respectively. Exchangeability is settled by de Finetti’s theorem that a sequence of random variables is exchangeable if and only if its joint distribution is a mixture of measures each of which is the joint distribution of an independent and identically distributed process, see [13, 16]. Rotatability is a stronger symmetry since permutations can be seen as orthogonal transformations. As a consequence, for rotatable sequences the common distribution of the independent and identically distributed processes above cannot be arbitrary. In fact, it is forced to be Gaussian, which is the content of Freedman’s theorem, [15, 17]. Lastly, spreadability should in principle be a weaker symmetry than exchangeability. However, it turns out to be the same as exchangeability in light of a classical result of Ryll-Nardzewski, [20]. Distributional symmetries for quantum stochastic processes can be considered as well and have recently been focused on in a number of papers, see e.g. [7, 8, 10, 5, 18]. In this paper, we mainly deal with processes based on the CAR algebra CAR(\mathbb{Z}). These are sequences of quantum
random variables with sample algebra \( M_2(\mathbb{C}) \) and which are subject to the canonical anticommutation relations. As explained in e.g. \([8, 11]\), processes of this type are in a one-to-one correspondence with states on \( \text{CAR}(\mathbb{Z}) \). Furthermore, the distributional symmetry of such a process amounts to the invariance of the corresponding state.

The set \( S^p(\text{CAR}(\mathbb{Z})) \) of exchangeable states of the CAR algebra has been addressed in \([6]\), where it is shown to be a Choquet simplex whose extremes are Araki-Moriya product states \([1]\) of a single even state of \( M_2(\mathbb{C}) \).

In this paper, we show that on the CAR algebra spreadable states are the same as exchangeable states, Theorem 3.4, thus establishing a fermionic version of the Ryll-Nardzewski theorem \([20]\). This fixes a mistake contained in (both statement and proof of) \([11, \text{Theorem 4.2}]\) from previous work by two of the authors of the present paper. Moreover, in Theorem 3.5 we show that the set of spreadable/exchangeable is somewhat small in the larger set of all stationary states being a proper face. Theorem 3.4 allows us to come to an accomplished characterization of symmetric states, which is the fermionic version of the so-called extended de Finetti theorem. Precisely, combining our result with those in \([6, 7]\), we have that spreadability and exchangeability for processes on the CAR algebra are both the same as conditional independence and identical distribution with respect to the tail algebra. Interestingly, the techniques we employ to prove our result are not limited to the CAR algebra. In fact, they can be used to handle tensor products of \( C^*\)-algebras, thus including the formalism established by Størmer in \([22]\). Among other things, applying the general result to commutative sample \( C^*\)-algebras yields a proof of the classical Ryll-Nardzewski theorem for essentially bounded random variables.

The strategy to arrive at our results has an interest in its own right not least because non-commutative ergodic theory for automorphic actions of groups on \( C^*\)-algebras does not apply as spreadability is implemented by the action of the semigroup \( \mathbb{J}_\mathbb{Z} \) of all monotone increasing functions of \( \mathbb{Z} \) to itself with cofinite range. This obstacle, however, can be circumvented by using a geometric construction naturally arising from the symmetry under consideration which allows one to realize the set of all spreadable states on the CAR algebra as the set of all states on a larger CAR algebra that are invariant under the action of a group, Theorem 3.1. In more detail, the larger algebra is \( \text{CAR}(\mathbb{Z}[\frac{1}{2}]) \), i.e. the CAR algebra on the set \( \mathbb{Z}[\frac{1}{2}] \) of dyadic numbers, obtained as inductive limit of the increasing sequence \( \{\text{CAR}(\mathbb{Z}[\frac{n}{2}]) : n \in \mathbb{N}\} \) of CAR algebras. Notably, this construction features the property that
any spreadable state on CAR $\left(\frac{\mathbb{Z}}{2n}\right)$ uniquely extends to a spreadable state on CAR $\left(\frac{\mathbb{Z}}{2n+1}\right)$, and the inductive limit state one ends up with on CAR($\mathbb{Z} \left[\frac{1}{2}\right]$) is invariant under the action of a group $G$ of continuous piecewise linear bijections of $\mathbb{Z} \left[\frac{1}{2}\right]$. The group $G$ is in turn an inductive limit of a sequence of groups $G_n$.

CAR($\mathbb{Z}$) is also acted on by orthogonal transformations in a natural way through Bogolubov automorphisms. This enables us to consider rotatability for states on the CAR algebra as well. Contrary to the classical case, rotatable states, $S^O_{\mathbb{Z}}(\text{CAR}(\mathbb{Z}))$, are exactly the same as exchangeable states, Proposition 4.1. The overall picture of invariant states on CAR($\mathbb{Z}$) is then the following

$$S^O_{\mathbb{Z}}(\text{CAR}(\mathbb{Z})) = S^P_{\mathbb{Z}}(\text{CAR}(\mathbb{Z})) = S^J_{\mathbb{Z}}(\text{CAR}(\mathbb{Z})) \subset S_{\mathbb{Z}}(\text{CAR}(\mathbb{Z})),$$

where $S^J_{\mathbb{Z}}(\text{CAR}(\mathbb{Z}))$ denotes the stationary states and the last strict inclusion is shown in e.g. [11].

Finally, the so-called self-adjoint subalgebra $\mathcal{C}(\mathbb{Z})$ of CAR($\mathbb{Z}$) is also addressed, since its generators are the Fermi analogues of Bernoulli random variables. As proved in [11], the vacuum state is the only exchangeable state on $\mathcal{C}(\mathbb{Z})$; therefore, it is also its only rotatable state. Moreover, we prove in Corollary 3.7 that the vacuum state is the only spreadable state on $\mathcal{C}(\mathbb{Z})$. The picture of distributional symmetries on $\mathcal{C}(\mathbb{Z})$ is the same as what we saw above for the CAR algebra. Phrased differently, the sets of spreadable, symmetric, and rotatable states on $\mathcal{C}(\mathbb{Z})$ all coincide with the singleton made up of the vacuum state, whereas there exist stationary states other than it, cf. [11][Proposition 4.1].

2. Preliminaries

By a realization of a quantum stochastic process labelled by the index set $J$ we mean a quadruple $(\mathfrak{A}, \mathcal{H}, \{\iota_j\}_{j \in J}, \xi)$, where $\mathfrak{A}$ is a (unital) $C^*$-algebra, referred to as the sample algebra of the process, $\mathcal{H}$ is a Hilbert space, whose inner product is denoted by $\langle \cdot , \cdot \rangle$, the maps $\iota_j$ are (unital) $*$-representation of $\mathfrak{A}$ on $\mathcal{H}$, and $\xi \in \mathcal{H}$ is a unit vector, which is cyclic for the von Neumann algebra $\bigvee_{j \in J} \iota_j(\mathfrak{A})$.

The assignment of such a quadruple amounts to a state $\varphi$ on the free product $C^*$-algebra $*_{j} \mathfrak{A}$. For free products of $C^*$-algebras the reader is referred to [2]. For completeness, we next quickly recall how the correspondence between stochastic processes and states works. If one starts with a stochastic process, then a state $\varphi$ on the free product $*_{j} \mathfrak{A}$ can be defined by setting

$$\varphi(i_{j_1}(a_1)i_{j_2}(a_2)\cdots i_{j_n}(a_n)) := \langle (\iota_{j_1}(a_1)\iota_{j_2}(a_2)\cdots \iota_{j_n}(a_n)\xi, \xi \rangle$$

(2.1)
for all $n \in \mathbb{N}$, $j_1 \neq j_2 \neq \cdots \neq j_n \in J$ and $a_1, a_2, \ldots, a_n \in \mathfrak{A}$, where $i_j : \mathfrak{A} \to *_J \mathfrak{A}$ is the $j$-th embedding of $\mathfrak{A}$ into $*_J \mathfrak{A}$. The values of $\varphi$ on monomials of the type above are often referred to as finite-dimensional distributions of the process itself.

Conversely, all states on the free product $*_J \mathfrak{A}$ arise in this way, see [7, Theorem 3.4]. Phrased differently, starting now with a state $\varphi \in \mathcal{S}(*_J \mathfrak{A})$, a stochastic process can be defined by using the GNS representation $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ of $\varphi$. Indeed, for every $j \in J$ we can set $\iota_j(a) := \pi_\varphi(i_j(a))$, $a \in \mathfrak{A}$, so as to get the quadruple $(\mathfrak{A}, \mathcal{H}_\varphi, \{\iota_j\}_{j \in J}, \xi_\varphi)$.

The Canonical Anticommutation Relations (CAR for short) algebra over $\mathbb{Z}$ is the universal unital $C^*$-algebra $\text{CAR}(\mathbb{Z})$, with unit $I$, generated by the set $\{a_j, a_j^\dagger : j \in \mathbb{Z}\}$ (i.e. the Fermi annihilators and creators respectively), satisfying the relations

$$(2.2) \quad (a_j)^* = a_j^\dagger, \quad \{a_j^\dagger, a_k\} = \delta_{j,k}I, \quad \{a_j, a_k\} = \{a_j^\dagger, a_k^\dagger\} = 0, \quad j, k \in \mathbb{Z},$$

where $\{\cdot, \cdot\}$ is the anticommutator and $\delta_{j,k}$ is the Kronecker symbol.

The CAR algebra is the completion of the subalgebra of so-called localized elements. More precisely,

$$\text{CAR}(\mathbb{Z}) = \overline{\text{CAR}_0(\mathbb{Z})},$$

where

$$\text{CAR}_0(\mathbb{Z}) := \bigcup \{\text{CAR}(F) : F \subset \mathbb{Z} \text{ finite} \}$$

and $\text{CAR}(F)$ is the $C^*$-subalgebra generated by the finite set $\{a_j, a_j^\dagger : j \in F\}$. We will sometimes say that an element $a$ of $\text{CAR}(\mathbb{Z})$ is localized in $F$ if $a \in \text{CAR}(F)$.

The parity automorphism $\theta$ acting on the generators as

$$\theta(a_j) = -a_j, \quad \theta(a_j^\dagger) = -a_j^\dagger, \quad j \in \mathbb{Z},$$

makes the CAR algebra into a $\mathbb{Z}_2$-graded algebra. Clearly, $\text{CAR}(\mathbb{Z})$ decomposes as $\text{CAR}(\mathbb{Z}) = \text{CAR}(\mathbb{Z})_+ \oplus \text{CAR}(\mathbb{Z})_-$, where

$$\text{CAR}(\mathbb{Z})_+ := \{a \in \text{CAR}(\mathbb{Z}) | \theta(a) = a\},$$

$$\text{CAR}(\mathbb{Z})_- := \{a \in \text{CAR}(\mathbb{Z}) | \theta(a) = -a\}.$$

Elements in $\text{CAR}(\mathbb{Z})_+$ and in $\text{CAR}(\mathbb{Z})_-$ are called even and odd, respectively.

A state $\varphi$ on $\text{CAR}(\mathbb{Z})$ is said to be even if $\varphi \circ \theta = \varphi$, which is the same as $\varphi|_{\text{CAR}(\mathbb{Z})_-} = 0$, where $\varphi|_{\text{CAR}(\mathbb{Z})_-}$ is the restriction to $\text{CAR}(\mathbb{Z})_-$ of $\varphi$. 
The CAR algebra can be presented in several equivalent ways. Below we recollect two presentations that are particularly suited to the purposes of the paper. First, the CAR algebra is isomorphic with the infinite graded tensor product of the \( \mathbb{Z}_2 \)-graded \( \mathbb{C}^* \)-algebra \((\mathbb{M}_2(\mathbb{C}), \text{ad}(U))\) with itself, where \( \text{ad}(U)(\cdot) := U(\cdot)U \) is the grading induced by the self-adjoint unitary (Pauli) matrix \( U := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \), see Example 3.2 of [12].

Second, the CAR algebra can also be seen as a quotient of the free product \( \ast_{\mathbb{Z}} \mathbb{M}_2(\mathbb{C}) \). Indeed, if we define \( A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \), it is easy to see that the quotient of \( \ast_{\mathbb{Z}} \mathbb{M}_2(\mathbb{C}) \) modulo the relations \( \{i_j(A^*), i_k(A)\} = \delta_{j,k}I \) and \( \{i_j(A), i_k(A)\} = \{i_j(A^*), i_k(A^*)\} = 0 \), for all \( j, k \in \mathbb{Z} \), is isomorphic with the CAR algebra by (2.2). It is this very presentation of the CAR algebra that allows one to interpret its states as particular stochastic processes defined on the sample algebra \( \mathbb{M}_2(\mathbb{C}) \).

By a slight abuse of notation, for every \( k \in \mathbb{Z} \), we continue to denote by \( i_k : \mathbb{M}_2(\mathbb{C}) \to \text{CAR}(\mathbb{Z}) \) the composition \( \pi \circ i_k \), where \( \pi : \ast_{\mathbb{Z}} \mathbb{M}_2(\mathbb{C}) \to \text{CAR}(\mathbb{Z}) \) is the quotient map.

For completeness’ sake, we rather quickly recall that \( \text{CAR}(\mathbb{Z}) \) has a distinguished (faithful) irreducible representation on the Fermi Fock space \( \mathcal{F}_-(\ell^2(\mathbb{Z})) \), where for every \( j \in \mathbb{Z} \), the operator \( a_j^\dagger \) (or \( a_j \)) acts as the Fermi creator (or annihilator) of a particle in the state \( e_j \), \( \{e_j : j \in \mathbb{Z}\} \) being the canonical orthonormal basis of \( \ell^2(\mathbb{Z}) \). For an exhaustive account of the Fermi Fock space the reader is referred to Chapter 5.2 of [4]. The vector state associated with the Fock vacuum vector \( \Omega \in \mathcal{F}_-(\ell^2(\mathbb{Z})) \) \( \text{i.e.} \) the one corresponding to the state with no particles at all) is called the vacuum state and denoted by \( \omega_\Omega \).

Along with \( \text{CAR}(\mathbb{Z}) \), we will also need to consider the \( \mathbb{C}^* \)-subalgebra \( \mathcal{C}(\mathbb{Z}) \subset \text{CAR}(\mathbb{Z}) \) generated by the so-called position operators \( x_i := a_i + a_i^\dagger \), \( i \in \mathbb{Z} \). In the literature, the \( \mathbb{C}^* \)-algebra \( \mathcal{C}(\mathbb{Z}) \) is often referred to as the self-adjoint subalgebra of the CAR algebra, and we will stick to this terminology as well.

Comparing the distributional symmetries involved in the present paper requires considering some algebraic structures on \( \mathbb{Z} \). We denote by \( \mathbb{P}_\mathbb{Z} \) the group of finite permutations of the set \( \mathbb{Z} \). Its elements are bijective maps of \( \mathbb{Z} \) which only moves finitely many integers. The group operation is given by the map composition.
We denote by \( L_Z \) the unital semigroup of all strictly increasing maps of \( Z \) to itself. As is known from Classical Probability, the importance of the semigroup \( L_Z \) lies in the role it plays in relation with spreadability. However, this distributional symmetry is equivalently implemented by the action of a smaller semigroup, \( J_Z \), which was introduced in [10]. This is by definition the set of all strictly increasing maps \( f \) of \( Z \) to itself whose range is cofinite, that is \( |Z \setminus f(Z)| < \infty \), where for any set \( E \), \(|E|\) denotes the cardinality of \( E \). We next describe a useful set of generators of \( J_Z \). For every fixed \( h \in Z \), define the \( h \)-right hand-side partial shift as the element \( \theta_h \) of \( J_Z \) given by

\[
\theta_h(k) := \begin{cases}
    k & \text{if } k < h , \\
    k + 1 & \text{if } k \geq h .
\end{cases}
\]

In [9, 10] the set \( \{ \theta_h, \tau, \tau^{-1} : h \in Z \} \) has been shown to generate \( J_Z \) as a unital semigroup, where \( \tau \) is given by \( \tau(k) = k + 1, k \in Z \).

The semigroup \( J_Z \) is known to be left but not right amenable, see [11]. Despite failing to be right amenable, \( J_Z \) does have a right Følner sequence. Precisely, the sequence of finite sets \( \{ F_n : n \in \mathbb{N} \} \subset J_Z \), with

\[
F_n := \left\{ \theta_{-n}^h \cdots \theta_0^h \cdots \theta_{-n+1}^h \theta_0^h \cdots \theta_n^h : \sum_{i=-n}^{n} h_i \leq n^2 \right\}
\]

is proved to be a right Følner sequence in [11, Proposition 3.1]. Explicitly, this means that for any fixed \( h \in J_Z \), one has

\[
\lim_{n \to \infty} \frac{|F_n \Delta F_n h|}{|F_n|} = 0 ,
\]

where \( \Delta \) denotes the symmetric difference.

By universality of the CAR algebra, the natural action on \( Z \) of both \( \mathbb{P}_Z \) and \( J_Z \) can be lifted to the CAR algebra by making them act directly on the indices of the generators \( a_i \). More precisely, if \( h \) is in either \( \mathbb{P}_Z \) or \( J_Z \), there exists a unique endomorphism \( \alpha_h \) of \( \text{CAR}(Z) \) uniquely determined by

\[
\alpha_h(a_i) = a_{h(i)} , \; i \in Z .
\]

Accordingly, one can consider the invariant states for these actions. In light of Equality (2.1), the associated processes will enjoy the relative distributional symmetry. In particular, invariant states for the action of \( \mathbb{P}_Z \) are known as symmetric or exchangeable states and the corresponding processes are often referred to as exchangeable processes. We will denote by \( \mathcal{S}^{\mathbb{P}_Z}(\text{CAR}(Z)) \) the compact convex set of all symmetric states on the CAR algebra.

Invariant states under the action of \( J_Z \) are known as spreadable states.
and the corresponding processes are referred to as spreadable processes as well. We will denote by $S^{JZ}(\text{CAR}(\mathbb{Z}))$ the compact convex set of all spreadable states on the CAR algebra.

Stationary states, which we will denote by $S^{Z}(\text{CAR}(\mathbb{Z}))$, are those invariant under the action of the automorphism $\alpha_\tau$, where $\tau$ is the right shift map, that is $\tau(k) = k + 1$, $k \in \mathbb{Z}$. Note that by definition there holds the inclusion $S^{JZ}(\text{CAR}(\mathbb{Z})) \subseteq S^{Z}(\text{CAR}(\mathbb{Z}))$.

Denote by $\mathbb{O}_Z$ the group of (infinite) orthogonal matrices $O := [O_{i,j}]_{i,j \in \mathbb{Z}}$ such that there are only finitely many entries $O_{i,j}$ not equal to $\delta_{i,j}$, the Kronecker symbol. We briefly recall how $\mathbb{O}_Z$ acts on $\text{CAR}(\mathbb{Z})$ in a natural way. Given a matrix $O = [O_{i,j}] \in \mathbb{O}_Z$, set

$$\rho_O(a_i) := \sum_{k \in \mathbb{Z}} O_{k,i}a_k \quad i \in \mathbb{Z}.$$ 

In order to verify that each $\rho_O$ defines an automorphism of $\text{CAR}(\mathbb{Z})$, by universality we only need to check that \{\rho_O(a_i) : i \in \mathbb{Z}\} still satisfies the defining relations of the CAR algebra. One has \{\rho_O(a_i), \rho_O(a_j)\} = 0 and \{\rho_O(a_i), \rho_O(a_j^\dagger)\} = \delta_{i,j}I for any $i, j \in \mathbb{Z}$. The first equality is trivially verified. As for the second, we have:

$$\{\rho_O(a_i), \rho_O(a_j^\dagger)\} = \sum_{k,l \in \mathbb{Z}} O_{k,i}O_{l,j}\{a_k, a_l^\dagger\} = \sum_{k,l \in \mathbb{Z}} O_{k,i}O_{l,j}\delta_{k,l}I$$

$$= \sum_{k \in \mathbb{Z}} O_{k,i}O_{k,j}I = \delta_{i,j}I$$

where in the last equality we have exploited the orthogonality conditions. Finally, the equality $\rho_{O_1}\rho_{O_2} = \rho_{O_1O_2}$ can be immediately verified as well for any $O_1, O_2 \in \mathbb{O}_Z$.

The automorphisms $\rho_O$ are known as Bogolubov automorphisms. Actually, in the second-quantization formalism they are obtained by functoriality in a way that we next sketch for completeness’ sake. First, it is convenient to think of $\text{CAR}(\mathbb{Z})$ as the concrete $C^*$-algebra $\text{CAR}(\ell^2(\mathbb{Z}))$. Associated with any unitary $U$ acting on $\ell^2(\mathbb{Z})$ there is a unitary $\mathcal{F}(U)$ acting on the Fermi Fock space $\mathcal{F}_-(\ell^2(\mathbb{Z}))$. Now the inner automorphism $\text{ad}(\mathcal{F}(U))$ restricts to an automorphism of $\text{CAR}(\mathbb{Z})$, see e.g. [4]. The automorphisms $\rho_O$ defined above are nothing but the restriction of $\text{ad}(\mathcal{F}(O))$ to the CAR algebra, where by a minor abuse of notation $O$ denotes the unitary operator of $\ell^2(\mathbb{Z})$ defined as

$$Oe_i := \sum_{k \in \mathbb{Z}} O_{k,i}e_k \quad , i \in \mathbb{Z},$$
\{e_i : i \in \mathbb{Z}\} being the canonical orthonormal basis of \(l^2(\mathbb{Z})\).

We will say that a state \(\omega\) on CAR(\(\mathbb{Z}\)) is \textit{rotatable} if it is invariant under the above action of \(O_{\mathbb{Z}}\), namely if \(\omega \circ \rho_O = \omega\) for every \(O \in O_{\mathbb{Z}}\). The set of all rotatable states will be denoted by \(S^{O_{\mathbb{Z}}}(\text{CAR}(\mathbb{Z}))\). The terminology comes from Classical Probability in that the finite-dimensional distributions of the process corresponding to such a state will be invariant under orthogonal transformations thanks to (2.1).

At this point it is also worth noting that the states we are going to consider in this paper are all automatically even because stationarity implies evenness, \(cf.\) Example 5.2.21 in [4].

The actions of \(P_{\mathbb{Z}}\) and \(O_{\mathbb{Z}}\) on the CAR algebra are examples of a \(C^*\)-dynamical system, by which we mean a triple \((A, H, \beta)\), where \(A\) is a \(C^*\)-algebra, \(H\) is a group, and \(\beta : H \rightarrow \text{Aut}(A)\) is a group homomorphism from \(H\) to Aut(\(A\)), the group of all \(*\)-automorphisms of \(A\). Systems of this type can be seen as the non-commutative counterpart of topological dynamics and feature a much developed theory, for which the reader is referred to \(e.g.\) [21]. In an effort to keep the present paper as self-consistent as we possibly can, however, we need to recall a few notions. First, a state \(\omega\) on \(A\) is said to be invariant if \(\omega \circ \beta_h = \omega\) for all \(h \in H\). On the GNS representation \((\mathcal{H}_\omega, \pi_\omega, \xi_\omega)\) of any such state \(\omega\) the action \(\beta\) of \(H\) can be implemented unitarily. More precisely, for each \(h \in H\), \(U_h^\omega \pi_\omega(a)\xi_\omega := \pi_\omega(\beta_h(a))\xi_\omega\), defines a unitary on \(\mathcal{H}_\omega\) such that \(U_h^\omega \pi_\omega(a)U_{h^{-1}}^\omega = \pi_\omega(\beta_h(a))\) for all \(a \in A\). Lastly, a \(C^*\)-dynamical system \((A, H, \beta)\) is said to be \(H\)-abelian if, for any \(H\)-invariant state \(\omega\), the family \(E_\omega \pi_\omega(A)E_\omega\) is abelian, where \(E_\omega\) is the orthogonal projection onto \(\mathcal{H}_\omega^H := \{\xi \in \mathcal{H}_\omega : U_h^\omega \xi = \xi, \text{ for all } h \in H\}\). \(H\)-abelianness is often inferred from asymptotical abelianness, \(see \ e.g.\) [21, Proposition 3.1.16]. A dynamical system \((A, H, \beta)\) is called asymptotically abelian if there exists a sequence \(\{h_n : n \in \mathbb{N}\} \subset H\) such that for all \(a, b \in A\) one has

\[
\lim_{n \to \infty} ||[\beta_{h_n}(a), b]|| = 0,
\]

where \([\cdot, \cdot]\) denotes the commutator.

3. Spreadable states

Let \(\omega\) be a spreadable state on CAR(\(\mathbb{Z}\)). Associated with any \(h \in \mathbb{J}_\mathbb{Z}\) there is an isometry acting on the Hilbert space \(\mathcal{H}_\omega\) of the GNS representation of \(\omega\) as

\[
T_h^\omega \pi_\omega(a)\xi_\omega := \pi_\omega(\alpha_h(a))\xi_\omega, \ a \in \text{CAR}(\mathbb{Z}).
\]
We denote by $E_\omega$ the orthogonal projection onto the closed subspace $\mathcal{H}_\omega := \{ \xi \in \mathcal{H}_\omega : T_h^* \xi = \xi, \text{ for all } h \in \mathbb{J}_z \}$. Notice that $\mathbb{C} \xi_\omega \subset \mathcal{H}_\omega$, where $\xi_\omega$ is the GNS vector of $\omega$.

Let $\mathbb{Z}[\frac{1}{2}]$ be the (additive) group of dyadic numbers. For every fixed $n \in \mathbb{N}$, we denote by $\mathbb{Z}_n$ the set of rational numbers $\{ \frac{k}{2^n} : k \in \mathbb{Z} \}$.

In order to perform the geometric construction alluded to in the introduction, we need to think of the generators of $\mathbb{J}_z$ as functions acting on the whole set of dyadic numbers. We do so by suitably extending $\theta_0$ to a bijection $\tilde{\theta}_0$ of $\mathbb{Z}[\frac{1}{2}]$, which is defined below

$$\tilde{\theta}_0(d) := \begin{cases} 
  d & \text{if } d \leq -1, \\
  2d + 1 & \text{if } -1 \leq d \leq 0 \\
  d + 1 & \text{if } d \geq 0.
\end{cases}$$

For each natural $n$, we consider the dilation $\delta_n$ by $2^n$ acting on $\mathbb{Z}[\frac{1}{2}]$, that is

$$\delta_n(d) = 2^n d, \ d \in \mathbb{Z}[\frac{1}{2}].$$

Let us define $\tilde{\theta}_n := \delta_n^{-1} \circ \tilde{\theta}_0 \circ \delta_n$ and $\tau_{k,n}(r) = r + \frac{k}{2^n}, \ r \in \mathbb{Z}[\frac{1}{2}]$, for $n \in \mathbb{N}$ and $k \in \mathbb{Z}$.

For each natural $n$, we then consider the group $G_n$ generated by $\tilde{\theta}_k$ and $\tau_{l,n}$ with $k = 1, \ldots, n$ and $l$ varies in $\mathbb{Z}$. Note that $G_n \subset G_{n+1}$ for each $n$. Therefore, we can consider the group $G$ which is by definition the inductive limit of the sequence $\{ G_n : n \in \mathbb{N} \}$ w.r.t. the inclusions, namely

$$G := \bigcup_n G_n.$$

Notably, $G$ acts through automorphisms on $\text{CAR}(\mathbb{Z}[\frac{1}{2}])$ by moving the corresponding indices, namely for every $g \in G$ a $*$-automorphism $\alpha_g$ of $\text{CAR}(\mathbb{Z}[\frac{1}{2}])$ is uniquely defined by

$$\alpha_g(a_d) := a_{g(d)}, \ d \in \mathbb{Z}[\frac{1}{2}].$$

As usual, we denote by $S^G(\text{CAR}(\mathbb{Z}[\frac{1}{2}]))$ the set of all invariant states on $\text{CAR}(\mathbb{Z}[\frac{1}{2}])$ under the action of $G$.

The next proposition allows us to realize any spreadable state on $\text{CAR}(\mathbb{Z})$ as a unique $G$-invariant state on $\text{CAR}(\mathbb{Z}[\frac{1}{2}])$. 
Proposition 3.1. The map \( T : S^G(\text{CAR}(\mathbb{Z}[\frac{1}{2}]))) \rightarrow S^G(\text{CAR}(\mathbb{Z}[\frac{1}{2}])) \)
\[
T(\omega) := \omega[\text{CAR}(\mathbb{Z}[\frac{1}{2}])], \ \omega \in S^G(\text{CAR}(\mathbb{Z}[\frac{1}{2}]))
\]
establishes an affine homeomorphism of compact convex sets.

Proof. We start by observing that the map is well defined, in that it sends a \( G \)-invariant state on \( \text{CAR}(\mathbb{Z}[\frac{1}{2}]) \) to a spreadable state on \( \text{CAR}(\mathbb{Z}) \). The statement amounts to proving that, conversely, any spreadable state on \( \text{CAR}(\mathbb{Z}) \) can uniquely be lifted to a \( G \)-invariant state on \( \text{CAR}(\mathbb{Z}[\frac{1}{2}]) \). To do so, for each \( n \geq 1 \), we consider the \( C^* \)-algebra \( A_n := \text{CAR}(\mathbb{Z}[\frac{1}{2}]) \). Since \( \mathbb{Z}[\frac{1}{2}] \subset \mathbb{Z}[\frac{1}{2}]+1 \), we also have \( A_n \subset A_{n+1} \).

Moreover, the map \( \Psi_n : A_{n+1} \rightarrow A_n \) given by \( \Psi_n(a_r) := a_{2r} \) for every \( r \in \mathbb{Z}[\frac{1}{2}]+1 \) is actually a \( * \)-isomorphism by universality of the \( \text{CAR} \) algebras.

Let now \( \varphi_0 \) be any spreadable state on \( \text{CAR}(\mathbb{Z}) \). We first extend \( \varphi_0 \) to \( A_n \) inductively by setting \( \varphi_{n+1} := \varphi_n \circ \Psi_n \), for \( n \geq 0 \). By definition, each \( \varphi_n \) is a spreadable state on \( \text{CAR}(\mathbb{Z}[\frac{1}{2}]) \), by which we mean that \( \varphi_n \) is invariant under the natural action of all strictly monotone maps on \( \mathbb{Z}[\frac{1}{2}] \) with cofinite range. Now \( \bigcup_{n=1}^{\infty} A_n \) is dense in \( \text{CAR}(\mathbb{Z}[\frac{1}{2}]) \), which means a state \( \varphi \) can be defined on the whole \( \text{CAR}(\mathbb{Z}[\frac{1}{2}]) \) as inductive limit of the sequence of states \( \{ \varphi_n : n \in \mathbb{N} \} \). All is left to do is ascertain that the state thus obtained is \( G \)-invariant. To this end, it is enough to note that, for each \( n \), the state \( \varphi_n \) is invariant under the action of \( \tilde{\theta}_k \) and \( \tilde{\tau}_{l,n} \), for \( k = 1, \ldots, n \) and \( l \) in \( \mathbb{Z} \). At this point, \( G \)-invariance of \( \varphi \) follows immediately by density of \( \bigcup_n A_n \) in \( \text{CAR}(\mathbb{Z}[\frac{1}{2}]) \) and the fact that \( G \) is generated by \( \tilde{\theta}_k \) and \( \tilde{\tau}_{l,n} \), as \( n \) varies in \( \mathbb{N} \) and \( k, l \) vary in \( \mathbb{Z} \). The argument above also shows that the extension obtained is in fact the only \( G \)-invariant extension, because \( \varphi_n \) is the only spreadable extension of \( \varphi_0 \) to \( A_n \).

The continuity of the restriction map is entirely obvious. Finally, its inverse is continuous as well by compactness. □

For any given \( G \)-invariant state \( \omega \) on \( \text{CAR}(\mathbb{Z}[\frac{1}{2}]) \) and for each \( g \in G \), we can define a unitary \( T_g^{\omega} \) acting on \( \mathcal{H}_\omega \) as
\[
T_g^{\omega} \pi_\omega(a) \xi_\omega = \pi_\omega(\alpha_g(a)) \xi_\omega, \ \ a \in \text{CAR}(\mathbb{Z}[\frac{1}{2}]).
\]
We then consider the closed subspace
\[
\mathcal{H}_G^\omega := \{ \xi \in \mathcal{H}_\omega : T_g^{\omega} \xi = \xi, \text{ for all } g \in G \}.
\]
of invariant vectors, and denote by $E_\omega$ the corresponding orthogonal projection. Finally, we denote by $\omega_0$ the restriction of $\omega$ to CAR($\mathbb{Z}$).

We next need to consider the unital semigroup $S \subset G$ generated by $\tilde{\theta}_n$ and $\tau_{k,n}$ as $n$ varies in $\mathbb{Z}$ and $k$ in $\mathbb{N}$. Note that for any element $h \in S$ one has $h(q) \geq q$ for all $q$ in $\mathbb{Z}[\frac{1}{2}]$.

Now, for any $G$-invariant state $\omega$ on CAR($\mathbb{Z}[\frac{1}{2}]$) recall that $\mathcal{H}^G_\omega := \{ \xi \in \mathcal{H}_\omega : T_h^\omega \xi = \xi, h \in G \}$ and define $\mathcal{H}^S_\omega := \{ \xi \in \mathcal{H}_\omega : T_h^\omega \xi = \xi, h \in S \}$.

**Lemma 3.2.** For any $G$-invariant state $\omega$ on CAR($\mathbb{Z}[\frac{1}{2}]$), one has then $\mathcal{H}^G_\omega = \mathcal{H}^S_\omega$.

**Proof.** The inclusion $\mathcal{H}^G_\omega \subseteq \mathcal{H}^S_\omega$ is trivial since $S \subset G$. As for the inclusion $\mathcal{H}^S_\omega \subseteq \mathcal{H}^G_\omega$, let $\xi$ be a vector sitting in $\mathcal{H}^S_\omega$. In order to prove that $\xi$ lies in $\mathcal{H}^G_\omega$, it is enough to observe that $S$ generates $G$ as a group and that a vector that is invariant for a unitary is also invariant for its adjoint. \hfill $\Box$

As an application of the above lemma, we find that the projections $E^G_\omega$ and $E^S_\omega$ onto $\mathcal{H}^G_\omega$ and $\mathcal{H}^S_\omega$, respectively, are actually the same.

**Lemma 3.3.** The dynamical system $(\text{CAR}(\mathbb{Z}[\frac{1}{2}]), G, \alpha)$ is $G$-abelian.

**Proof.** Set $\mathfrak{A} := \text{CAR}(\mathbb{Z}[\frac{1}{2}])$ for brevity. We have to prove that for any $G$-invariant state $\omega$, the operators of the set $E^G_\omega \pi_\omega(\mathfrak{A}) E^G_\omega$ commute with one another, where $E^G_\omega$ is the projection onto the subspace $\mathcal{H}^G_\omega = \{ \xi \in \mathcal{H}_\omega : T_g^\omega \xi = \xi, \text{ for all } g \in G \}$ of all invariant vectors.

We shall accomplish this task in two steps. The next step is to prove that $E^G_\omega \pi_\omega(\mathfrak{A}_+) E^G_\omega$ is abelian, where $\mathfrak{A}_+$ is the even subalgebra of $\mathfrak{A}$. The second step is to prove that $E^G_\omega \pi_\omega(\alpha)(a) E^G_\omega = 0$ for any odd element $a \in \mathfrak{A}$.

In order to take the first step, we start by observing that $\mathfrak{A}_+$ is invariant under the action of $G$ as $\alpha_g \circ \theta = \theta \circ \alpha_g$ for all $g \in G$, where $\theta$ is the grading of CAR($\mathbb{Z}[\frac{1}{2}]$). The dynamical system $(\mathfrak{A}_+, \alpha, G)$ obtained by restricting the dynamics to $\mathfrak{A}_+$ is asimptotically abelian. Indeed, if we consider the sequence $g_n := \tau_{n,0}$, that is $g_n(r) = r + n$ for every $r \in \mathbb{Z}[\frac{1}{2}]$, one can verify that for all localized $a, b \in \mathfrak{A}_+$ one has $[\alpha_{g_n}(a), b] = 0$ for $n$ big enough. But then from a standard density arguments it follows that

$$\lim_{n \to \infty} \| [\alpha_{g_n}(a), b] \| = 0,$$

for all $a, b \in \mathfrak{A}_+$. By applying Proposition 3.1.16 in [21] we finally see that $(\mathfrak{A}_+, \alpha, G)$ is $G$-abelian.

In order to take the second step, we closely follow the argument in
Example 5.2.21 in [4]. As above, it is straightforward to verify that for all odd \(a\) in \(\mathfrak{A}\), we have
\[
\lim_{n \to \infty} \|\{\alpha_{g_n}(a), a^*\}\| = 0.
\]
Now denote by \(F\) the orthogonal projection onto the closed subspace \(\{\xi \in \mathcal{H}_\omega : T_{g_n}\xi = \xi, \text{ for all } n \in \mathbb{N}\}\). Note that \(E^G_\omega \leq F\). We aim to show that \(F\) is an infinite product of a single even state \(\rho\) on \(M_2(\mathbb{C})\).

As an application of the von Neumann ergodic theorem to the projection \(F\), one sees that for any such \(a\) it holds
\[
\{F \pi_\omega(a) F, F \pi_\omega(a^*) F\} = \lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^n F \pi_\omega(\{\alpha_{g_k}(a), a^*\}) F = 0.
\]

But then \(F \pi_\omega(a) F = 0\), hence \(E^G_\omega \pi_\omega(a) E^G_\omega = E^G_\omega F \pi_\omega(a) F E^G_\omega = 0\).

As an application of the previous result, we find that \(E^S_\omega\) is a rank-one projection if \(\omega\) is an extreme \(G\)-invariant state, as follows from Proposition 3.1.12 in [21]. We are now ready to state and prove our main result.

**Theorem 3.4.** On \(\text{CAR}(\mathbb{Z})\) any spreadable state is also exchangeable, that is
\[
S^g_z(\text{CAR}(\mathbb{Z})) = S^p_z(\text{CAR}(\mathbb{Z})).
\]

**Proof.** We need only prove the inclusion \(S^g_z(\text{CAR}(\mathbb{Z})) \subseteq S^p_z(\text{CAR}(\mathbb{Z}))\), for the reverse inclusion is trivial. By the Krein-Milman theorem, it is enough to handle extreme spreadable state. Precisely, our strategy is to prove that any extreme spreadable state \(\omega\) is an infinite product of a single even state \(\rho\) on \(M_2(\mathbb{C})\).

In the light of Proposition 3.1, we are reconducted to showing this: the restriction of any extreme state \(\omega\) in \(S^G(\text{CAR}(\mathbb{Z}[\frac{1}{2}])))\) to \(\text{CAR}(\mathbb{Z})\) is an infinite product of a single even state \(\rho\) on \(M_2(\mathbb{C})\).

Now, if \(\omega\) is an extreme \(G\)-invariant state on \(\text{CAR}(\mathbb{Z}[\frac{1}{2}])))\), then \(E^G_\omega\) is the rank-one projection onto \(\mathbb{C}\xi_\omega\), thanks Lemma 3.3 and Proposition 3.1.12 in [21]. By virtue of Lemma 3.2, \(E^S_\omega\) is the projection onto \(\mathbb{C}\xi_\omega\) as well.

The next thing to do is to apply the Alalgolu-Birkhoff theorem, see e.g. Proposition 4.3.4 in [3], to the family of unitaries \(\{T^\omega_h : h \in S\} \subseteq \mathcal{B}(\mathcal{H}_\omega)\). This yields a net \(\{S_{\gamma} : \gamma \in I\}\), whose terms are finite convex combinations of the form \(S_{\gamma} = \sum_{k=1}^{n_{\gamma}} \lambda_k T^\omega_{k}\), for some \(\lambda_k \geq 0\) with \(\sum_{k=1}^{n_{\gamma}} \lambda_k = 1\), which is strongly convergent to \(E^S_\omega\). Let now \(a, b\) be fixed.
elements of CAR \((\mathbb{Z}[\frac{1}{2}])\). For each \(\gamma\) in \(I\), define \(b_\gamma := \sum_{k=1}^{n_\gamma} \lambda_k^\gamma \alpha_{h_k^\gamma}(b)\).

We have

\[
\lim_{\gamma} \omega(ab_\gamma) = \lim_{\gamma} \langle \pi_\omega(ab_\gamma) \xi_\omega, \xi_\omega \rangle \\
= \lim_{\gamma} \langle \pi_\omega(a) \sum_{k=1}^{n_\gamma} \lambda_k^\gamma T_{h_k^\gamma} \pi_\omega(b) \xi_\omega, \xi_\omega \rangle \\
= \langle \pi_\omega(a) E_{C,\xi_\omega} \pi_\omega(b) \xi_\omega, \xi_\omega \rangle = \omega(a)\omega(b).
\]

We are now ready to prove our claim. Let \(\rho\) be the even state on \(M_2(\mathbb{C})\) defined by \(\rho(A) := \omega(i(A))\), \(A\) in \(M_2(\mathbb{C})\), where \(i\) is any of the embeddings of \(M_2(\mathbb{C})\) into its infinite \(\mathbb{Z}_2\)-graded tensor product. Note that \(\rho\) is well defined, i.e. the definition does not depend on \(l\), because \(\omega\) is spreadable. In order to show that \(\omega_0\) (that is the restriction of \(\omega\) to \(\text{CAR}(\mathbb{Z})\)) coincides with the infinite product of \(\rho\) with itself, we need to prove the equality

\[
\omega_0(i_{j_1}(A_1)i_{j_2}(A_2)\cdots i_{j_n}(A_n)) = \rho(A_1)\rho(A_2)\cdots \rho(A_n)
\]

for every \(n \in \mathbb{N}\), for every \(j_1 < j_2 < \ldots < j_n \in \mathbb{Z}\), and for every \(A_1, A_2, \ldots, A_n\) in \(M_2(\mathbb{C})\). This task can be accomplished by induction on \(n\). For \(n = 1\), there is nothing to prove. Let us move on to take the inductive step. We will show that the equality holds with \(n + 1\) if it holds with \(n\). To this end, we start by observing that, for all \(n \in \mathbb{N}\), \(j_1 < j_2 < \ldots < j_{n+1} \in \mathbb{Z}\), and \(A_1, \ldots, A_n, B \in M_2(\mathbb{C})\), spreadability of \(\omega_0\) gives

\[
\omega_0(i_{j_1}(A_1)\cdots i_{j_n}(A_n)i_{j_{n+1}}(B)) = \omega_0(i_{j_1}(A_1)\cdots i_{j_n}(A_n)i_{h_{k}^\gamma(j_{n+1})}(B)),
\]

where, for each \(\gamma\) in \(I\), the \(h_k^\gamma\)'s are the monotone functions in \(S\) (in particular, \(h_k^\gamma(m) \geq m\) for all \(m\) in \(\mathbb{Z}\)) appearing in the definition of the net \(S_\gamma\) we introduced above. By summing the equalities above on all \(k\)'s between 1 and \(n_\gamma\), we find

\[
\omega_0(i_{j_1}(A_1)\cdots i_{j_n}(A_n)i_{j_{n+1}}(B)) \\
= \sum_{k=1}^{n_\gamma} \lambda_k^\gamma \omega_0(i_{j_1}(A_1)\cdots i_{j_n}(A_n)i_{h_k^\gamma(j_{n+1})}(B)) \\
= \omega_0(i_{j_1}(A_1)\cdots i_{j_n}(A_n)b_\gamma)
\]

where \(b_\gamma := \sum_{k=1}^{n_\gamma} \lambda_k^\gamma i_{h_k^\gamma(j_{n+1})}(B) = \sum_{k=1}^{n_\gamma} \lambda_k^\gamma \alpha_{h_k^\gamma(i_{j_{n+1}}(B))}\).

Now thanks to 3.1 (applied to \(b = i_{j_{n+1}}(B)\)) we have
\[
\lim_{\gamma} \omega_0(i_{j_1}(A_1) \cdots i_{j_n}(A_n)b_\gamma) = \omega_0(i_{j_1}(A_1) \cdots i_{j_n}(A_n))\rho(B),
\]
and we are done because by our inductive hypothesis we have
\[
\omega_0(i_{j_1}(A_1) \cdots i_{j_n}(A_n))\rho(B) = \rho(A_1) \cdots \rho(A_n)\rho(B).
\]

Furthermore, exchangeable (or spreadable) states make up a face of the larger convex of all stationary states, as we next show.

**Theorem 3.5.** The convex subset \( S_{\pi}(\mathrm{CAR}(\mathbb{Z})) = S_{\lambda_\pi}(\mathrm{CAR}(\mathbb{Z})) \) of exchangeable/spreadable states of \( \mathrm{CAR}(\mathbb{Z}) \) is a proper face of the convex set of all stationary states \( S_{\pi}(\mathrm{CAR}(\mathbb{Z})) \).

**Proof.** We start by showing that the extreme states of \( S_{\pi}(\mathrm{CAR}(\mathbb{Z})) \) are extreme in \( S_{\pi}(\mathrm{CAR}(\mathbb{Z})) \) as well.

Let \( \omega \) be any such state, which by [6, Theorem 5.3] is the infinite product of a single even state \( \rho \) on \( \mathbb{M}_2(\mathbb{C}) \). In particular, its GNS representation \( \pi_\omega \) factors as the infinite \( \mathbb{Z}_2 \)-graded product \( \hat{\otimes}_{\mathbb{Z}} \pi \rho \), see [12].

We need to prove that \( E^\tau_\omega \) is the orthogonal projection onto the one-dimensional subspace \( \mathbb{C}\xi_\omega \). By the von Neumann ergodic theorem, \( E^\tau_\omega \) can be obtained as the limit \( \lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} (U^\tau_\omega)^k \) in the strong operator topology. Therefore, all we have to do is show that
\[
\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} \langle (U^\tau_\omega)^k \xi, \eta \rangle = 0,
\]
for every \( \xi \in (\mathbb{C}\xi_\omega)^\perp \) and every \( \eta \in \mathcal{H}_\omega \).

Since the sequence \( \left\{ \frac{1}{n+1} \sum_{k=0}^{n} (U^\tau_\omega)^k : n \in \mathbb{N} \right\} \) is uniformly bounded, it is enough to verify the sought equality on a dense subspace. By cyclicity of \( \xi_\omega \) and density of the *-subalgebra of localized elements of CAR(\( \mathbb{Z} \)), we can assume that \( \eta \) has the form below
\[
\eta = \cdots \hat{\otimes}_{\mathbb{Z}} \xi_{\rho_1} \hat{\otimes} \cdots \xi_{\rho_k} \hat{\otimes} \xi_{\rho_l} \hat{\otimes} \cdots \pi_{\rho_k} (A_{-k}) \xi_{\rho_k} \hat{\otimes} \cdots \pi_{\rho_k} (A_k) \xi_{\rho_k} \hat{\otimes} \cdots \pi_{\rho_l} (B_l) \xi_{\rho_l} \hat{\otimes} \cdots - \lambda \xi_\omega,
\]
where \( k \geq 1 \) is a fixed integer and \( A_j \in \mathbb{M}_2(\mathbb{C}) \) for all \( j \in \{-k, \ldots, k\} \).

Now a moment’s reflection shows that vectors of the form
\[
\xi = (\cdots \otimes \xi_{\rho_l} \hat{\otimes} \cdots \pi_{\rho_l} (B_l) \xi_{\rho_l} \hat{\otimes} \cdots \pi_{\rho_l} (B_l) \xi_{\rho_l} \hat{\otimes} \cdots) - \lambda \xi_\omega,
\]
with \( \lambda := \prod_{j=-l}^{l} (\pi_{\rho_l} (B_j) \xi_{\rho_l}, \xi_{\rho_l}) \), where \( l \) is any natural number and \( B_j \in \mathbb{M}_2(\mathbb{C}) \) for all \( l = -j, \ldots, j \), are dense in \( (\mathbb{C}\xi_\omega)^\perp \). We are thus led
to verify that
\[
\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} \langle (U_{\omega}^\tau)^{k} \xi', \eta \rangle = \lambda \langle \xi_\omega, \eta \rangle
\]
for
\[
\eta = \cdots \otimes \xi_\rho \otimes \pi_\rho(A_{-k}) \xi_\rho \otimes \cdots \otimes \pi_\rho(A_1) \xi_\rho \otimes \cdots \otimes \pi_\rho(B_k) \xi_\rho \otimes \xi_\rho \otimes \cdots
\]
and
\[
\xi' = \cdots \otimes \xi_\rho \otimes \pi_\rho(B_{-l}) \xi_\rho \otimes \cdots \otimes \pi_\rho(B_1) \xi_\rho \otimes \cdots \otimes \pi_\rho(B_l) \xi_\rho \otimes \xi_\rho \otimes \cdots
\]
where \(\lambda\) is the product
\[
\prod_{j=-l}^{l} \langle \pi_\rho(B_j) \xi_\rho, \xi_\rho \rangle.
\]
But the desired limit equality follows from the stronger equality below
\[
\lim_{n \to \infty} \langle (U_{\omega}^\tau)^{n} \xi', \eta \rangle = \lambda \langle \xi_\omega, \eta \rangle,
\]
which can be verified by easy direct computation.

We are now ready to reach the conclusion. Let \(\varphi\) be a symmetric state which is a convex combination of the type \(\varphi = t\varphi_1 + (1-t)\varphi_2\), \(t \in (0,1)\), where \(\varphi_1\) and \(\varphi_2\) are stationary states. We need to make sure that both \(\varphi_1\) and \(\varphi_2\) are actually symmetric. Denote by \(\mathcal{K} \subset S_{\mathbb{Z}}(\text{CAR}(\mathbb{Z}))\) the compact set of extreme symmetric states. Since \(S_{\mathbb{Z}}(\text{CAR}(\mathbb{Z}))\) is a Choquet simplex, there exists a unique measure \(\mu\), which is supported on \(\mathcal{K}\), such that \(\varphi = \int_{\mathcal{K}} \varphi_\mu d\mu(\mu)\). Since \(S_{\mathbb{Z}}(\text{CAR}(\mathbb{Z}))\) is a Choquet simplex as well, see Example 5.2.21 in [4], the decomposition above is also the (unique) decomposition of \(\varphi\) as a point of \(S_{\mathbb{Z}}(\text{CAR}(\mathbb{Z}))\).

Denote now by \(\mathcal{E}\) the set of all extreme stationary states. Again, since \(S_{\mathbb{Z}}(\text{CAR}(\mathbb{Z}))\) is a Choquet simplex, there exist unique measures \(\nu_1, \nu_2\) supported on \(\mathcal{E}\) such that
\[
\varphi_i = \int_{\mathcal{E}} \omega d\nu_i(\omega), \ i = 1, 2.
\]
By uniqueness of the decomposition, the equality \(\nu = t\nu_1 + (1-t)\nu_2\) must hold. In particular, the support of \(t\nu_1 + (1-t)\nu_2\) is the same as the support of \(\nu\), which is contained in \(\mathcal{K}\). But then the supports of both \(\nu_1\) and \(\nu_2\) are contained in \(\mathcal{K}\) as well, which means \(\varphi_1\) and \(\varphi_2\) are symmetric. \(\square\)

**Remark 3.6.** Despite \(S_{\mathbb{Z}}(\text{CAR}(\mathbb{Z}))\) being a face of \(S_{\mathbb{Z}}(\text{CAR}(\mathbb{Z}))\), the so-called Olshen theorem [19] fails on the CAR algebra. More explicitly, there exist exchangeable states on CAR(\(\mathbb{Z}\)) whose exchangeable algebra is strictly contained in the stationary algebra. The vacuum state itself provides the most elementary counterexample. Indeed, its

\[\text{\footnotesize By this we mean that } \nu_i(\mathcal{E}) = 1, \ i = 1, 2, \text{ and this equality makes perfect sense as } \mathcal{E} \text{ is a } G_\delta \text{-set thanks to the general lemma of Choquet, see e.g. [21, 3.4.1].}\]
exchangeable algebra $\pi''_{\omega}(\text{CAR}(\mathbb{Z}))_{\mathbb{P}_Z}$ is trivial as follows from [12, Proposition 2.9, Proposition 2.10]. However, its stationary algebra $\pi''_{\omega}(\text{CAR}(\mathbb{Z}))_{\mathbb{Z}} = \{U^r_{\omega}\}'$ is not trivial.

Theorem 3.4 allows us to determine the structure of the set of all spreadable states on the so-called self-adjoint subalgebra $\mathfrak{C}(\mathbb{Z})$ of $\text{CAR}(\mathbb{Z})$. This is by definition the $C^*$-subalgebra generated by the set $\{x_i : i \in \mathbb{Z}\}$, with $x_i := a_i + a_i^\dagger$, $i \in \mathbb{Z}$. It turns out that this set only consists of the vacuum state.

**Corollary 3.7.** The vacuum state is the only spreadable state on $\mathfrak{C}(\mathbb{Z})$.

**Proof.** Let $\varphi$ be a spreadable state on $\mathfrak{C}(\mathbb{Z})$. Consider any extension $\tilde{\varphi}$ of $\varphi$ to $\text{CAR}(\mathbb{Z})$. For each $n \in \mathbb{N}$, define $\tilde{\varphi}_n := \frac{1}{|F_n|} \sum_{h \in F_n} \tilde{\varphi} \circ \alpha_h$, where $\{F_n : n \in \mathbb{N}\}$ is the Følner sequence of $J_{\mathbb{Z}}$. By weak* compacteness, up to extracting a subsequence, we can suppose that the sequence $\tilde{\varphi}_n$ weakly* converges to a state $\omega$. By construction, $\omega$ is a spreadable state on $\text{CAR}(\mathbb{Z})$ and its restriction to $\mathfrak{C}(\mathbb{Z})$ is the same as $\varphi$. Now by Theorem 3.4 $\omega$ is exchangeable, and thus $\varphi$ is exchangeable as well. The thesis is finally reached as an application of Proposition 4.3 in [11], where the vacuum state is shown to be the unique exchangeable state on $\mathfrak{C}(\mathbb{Z})$. \hfill \Box

Before ending the section, we would like to point out that our techniques can also be made use of to address infinite tensor products of a (nuclear) sample $C^*$-algebra $\mathfrak{A}$. Denote by $\bigotimes_{\mathbb{Z}} \mathfrak{A}$ the infinite tensor product of $\mathfrak{A}$ with itself. Both $P_{\mathbb{Z}}$ and $J_{\mathbb{Z}}$ act naturally on $\bigotimes_{\mathbb{Z}} \mathfrak{A}$. Invariant states under the action of permutations, often referred to as symmetric states, make up a Choquet simplex whose extreme points are infinite product of a single state on $\mathfrak{A}$, see Theorem 2.7 in [22].

As with the CAR algebra, the action of $J_{\mathbb{Z}}$ on $\bigotimes_{\mathbb{Z}} \mathfrak{A}$ can be extended to an action of $G$ on $\bigotimes_{[\frac{1}{2}]} \mathfrak{A} \supseteq \bigotimes_{\mathbb{Z}} \mathfrak{A}$. Again, the $C^*$-dynamical system thus obtained is $G$-abelian. As a consequence, the projection $E_\omega$ associated with any extreme spreadable state $\omega$ is one-dimensional. This allows one to exploit the arguments employed in the proof of Theorem 3.4. Therefore, one has that the Choquet simplex of spreadable states on $\bigotimes_{\mathbb{Z}} \mathfrak{A}$ is the same as the simplex of symmetric states. This applies in particular to commutative sample $C^*$-algebras. In this way one also finds an independent proof of the classical Ryll-Nardzweski theorem (for bounded random variables).
4. Rotatable states

In this section we show that the set of rotatable states on the CAR algebra agrees with the set of exchangeable states. This result can in a sense be regarded as a version of Freedman’s theorem for the CAR algebra, in that it can be combined with Theorem 5.3 and Theorem 5.5 in [6] to provide an explicit description of what rotatable states look like.

**Theorem 4.1.** Let $\omega$ be a state on $\text{CAR}(\mathbb{Z})$. The following are equivalent:

(i) $\omega$ is exchangeable;
(ii) $\omega$ is spreadable;
(iii) $\omega$ is rotatable.

**Proof.** The equivalence between (i) and (ii) has been established in Theorem 3.4. As for the equivalence between (i) and (iii), we need only prove $S_{\mathbb{Z}}^e(\text{CAR}(\mathbb{Z})) \subseteq S_{\mathbb{Z}}^o(\text{CAR}(\mathbb{Z}))$. To this end, it is enough to ascertain that the extreme states in $S_{\mathbb{Z}}^e(\text{CAR}(\mathbb{Z}))$ are rotatable thanks to the Krein-Milman theorem as $S_{\mathbb{Z}}^e(\text{CAR}(\mathbb{Z}))$ and $S_{\mathbb{Z}}^o(\text{CAR}(\mathbb{Z}))$ are convex and weakly* compact.

As proved in [6, Theorem 5.3], the extreme states of $S_{\mathbb{Z}}^e(\text{CAR}(\mathbb{Z}))$ are precisely the Araki-Moriya product states $\varphi_{\mu}$, $\mu \in [0,1]$, with $\varphi_{\mu} = \times \rho_{\mu}$ and $\rho_{\mu}$ is the state on $\mathbb{M}_2(\mathbb{C})$ given by

\[\rho_{\mu} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \mu \alpha + (1 - \mu) \delta.\]

Note that in particular $\varphi_{\mu}(a_i^\dagger a_i) = 1 - \mu$ for all $i \in \mathbb{Z}$.

In order to check the involved equalities, thanks to the CAR relations it suffices to work with Wick-ordered words (words where all creators come before all annihilators), namely of the form

\[X = a_{j_1}^\dagger a_{j_2}^\dagger \cdots a_{j_l}^\dagger a_{k_1} a_{k_2} \cdots a_{k_m}.\]

Furthermore, we may assume that the indices of the set $\{j_1, j_2, \ldots, j_l\}$ are different from one another as are those of the set $\{k_1, k_2, \ldots, k_m\}$. In particular, for any given $j_i$, $i = 1, 2, \ldots, l$ there can exist at most one $k_h$, $h = 1, 2, \ldots, m$ such that $j_i = k_h$. In this case, by using the CAR relations, the word can be re-ordered in such a way that two matched $a_{j_i}^\dagger$ and $a_{k_h}$ with $j_i = k_h$ are placed next to each other as $a_s^\dagger a_s$ with $s$ being the common value of the two indices. The overall re-ordering procedure, though, can change the sign of the original word depending on how many exchanges are necessary. By virtue of (4.1) any product state $\varphi_{\mu}$ vanishes on a word of the type above as soon as
a singleton appears in the word itself. By singleton we mean either a creator or an annihilator corresponding to an index that appears only once in \(\{j_1, j_2, \ldots, j_l\} \cup \{k_1, k_2, \ldots, k_m\}\). Because \(\varphi_\mu\) vanishes on all words with \(l \neq m\), there is no loss of generality in assuming \(l = m = n\), that is \(X = a_{j_1}^1 a_{j_2}^1 \cdots a_{j_n}^1 a_{k_1} a_{k_2} \cdots a_{k_n}\). For every \(O \in \mathcal{O}_Z\) we have

\[
\varphi_\mu(\rho_O(X)) = \sum_{p_1, \ldots, p_n, q_1, \ldots, q_n} O_{p_1,j_1} \cdots O_{p_n,j_n} O_{q_1,k_1} \cdots O_{q_n,k_n} \varphi_\mu(a_{p_1}^1 \cdots a_{p_n}^1 a_{p_\sigma(1)} \cdots a_{p_\sigma(n)}) .
\]

In the sum above only the summands with \(p_i \neq p_j\) for all \(i \neq j\) and \(q_i = p_{\sigma(i)}, i = 1, 2, \ldots, n\), for some permutation \(\sigma\), survive. If we denote by \(D_n \subset \mathbb{Z}^n\) the subset of \(n\)-tuple \(p = (p_1, p_2, \ldots, p_n)\) of integers all different from one another, the right-hand side of the previous equality rewrites as

\[
\sum_{\sigma \in \mathcal{F}_n} \sum_{p \in D_n} O_{p_1,j_1} \cdots O_{p_n,j_n} O_{p_\sigma(1),k_1} \cdots O_{p_\sigma(n),k_n} \varphi_\mu(a_{p_1}^1 \cdots a_{p_n}^1 a_{p_\sigma(1)} \cdots a_{p_\sigma(n)})
\]

\[
= \sum_{\sigma \in \mathcal{F}_n} \sum_{p \in \mathbb{Z}^n} O_{p_1,j_1} \cdots O_{p_n,j_n} O_{p_\sigma(1),k_1} \cdots O_{p_\sigma(n),k_n}(1 - \mu)^n(-1)^{f(\sigma)}
\]

\[
= \sum_{\sigma \in \mathcal{F}_n} \sum_{p \in \mathbb{Z}^n} \sum_{\sigma \in \mathcal{F}_n} O_{p_1,j_1} \cdots O_{p_n,j_n} O_{p_\sigma(1),k_1} \cdots O_{p_\sigma(n),k_n}(1 - \mu)^n(-1)^{f(\sigma)}
\]

\[- \sum_{\sigma \in \mathcal{F}_n} \sum_{p \in D_n} O_{p_1,j_1} \cdots O_{p_n,j_n} O_{p_\sigma(1),k_1} \cdots O_{p_\sigma(n),k_n}(1 - \mu)^n(-1)^{f(\sigma)},
\]

where \((-1)^{f(\sigma)}\) is the sign of \(\varphi_\mu(a_{p_1}^1 \cdots a_{p_n}^1 a_{p_\sigma(1)} \cdots a_{p_\sigma(n)})\). As suggested by the notation, \((-1)^{f(\sigma)}\) depends on the permutation \(\sigma\).

We treat the two summands separately. Thanks to the orthogonality relations, the first is equal to

\[
\sum_{\sigma \in \mathcal{F}_n} \sum_{p \in \mathbb{Z}^n} O_{p_1,j_1} \cdots O_{p_n,j_n} O_{p_\sigma(1),k_1} \cdots O_{p_\sigma(n),k_n}(1 - \mu)^n(-1)^{f(\sigma)}
\]

\[
= \sum_{\sigma \in \mathcal{F}_n} \delta_{j_1,k_{\sigma - 1(1)}} \cdots \delta_{j_n,k_{\sigma - 1(n)}}(1 - \mu)^n(-1)^{f(\sigma)}
\]

\[
= \varphi_\mu(X).
\]

To prove the last equality, we consider two cases depending on whether \(X\) has a singleton or not. If \(X\) has a singleton, then \(\varphi_\mu(X) = 0\). But the sum \(\sum_{\sigma \in \mathcal{F}_n} \delta_{j_1,k_{\sigma - 1(1)}} \cdots \delta_{j_n,k_{\sigma - 1(n)}}(1 - \mu)^n(-1)^{f(\sigma)}\) is zero as well because each term vanishes separately. If no singleton shows up, then \(\varphi_\mu(X) = \pm(1 - \mu)^n\). In \(\sum_{\sigma \in \mathcal{F}_n} \delta_{j_1,k_{\sigma - 1(1)}} \cdots \delta_{j_n,k_{\sigma - 1(n)}}(1 - \mu)^n(-1)^{f(\sigma)}\) only the term corresponding to the permutation \(\sigma\) s.t. \(j_l = k_{\sigma - 1(l)}\) for
all \( l = 1, \ldots, n \) survives, and \((-1)^{f(\sigma)}\) is just the sign of \( \varphi_\mu(X) \).
The conclusion will be reached once we show
\[
\sum_{\sigma \in \Pi_n} \sum_{p \in D_n^\mu} O_{p_1,j_1} \cdots O_{p_n,j_n} O_{p_{\sigma(1)},k_1} \cdots O_{p_{\sigma(n)},k_n} (1 - \mu)^n (-1)^{f(\sigma)} = 0.
\]
Denote by \( \mathcal{P}_n^2 \) the collection of all partitions of \( \{1, 2, \ldots, n\} \) containing
at least one subset of \( \{1, 2, \ldots, n\} \) with cardinality at least two. Note that \( \mathcal{P}_n^2 \)
contains all partitions except for \( \{\{1\}, \{2\}, \ldots, \{n\}\} \).
For any partition \( \pi = \{S_1, S_2, \ldots, S_l\} \) in \( \mathcal{P}_n^2 \), define \( A_\pi \subset \mathbb{Z}^n \) the set of
all \( n \)-tuples \( \mathbf{p} = (p_1, p_2, \ldots, p_n) \) such that there exist \( s_1, s_2, \ldots, s_l \in \mathbb{Z} \)
with \( s_i \neq s_j \) for all \( i \neq j \) such that \( p_h = s_j \) if \( h \in S_j \), \( j = 1, \ldots, l \).
The sum that we want to prove zero can be rewritten as
\[
\sum_{\sigma \in \Pi_n} \sum_{\pi \in \mathcal{P}_n^2} \sum_{p \in A_\pi} O_{p_1,j_1} \cdots O_{p_n,j_n} O_{p_{\sigma(1)},k_1} \cdots O_{p_{\sigma(n)},k_n} (1 - \mu)^n (-1)^{f(\sigma)}.
\]
The conclusion will then follow if we can show that for every fixed partition \( \pi \in \mathcal{P}_n^2 \), the partial sum
\[
\sum_{\sigma \in \Pi_n} \sum_{p \in A_\pi} O_{p_1,j_1} \cdots O_{p_n,j_n} O_{p_{\sigma(1)},k_1} \cdots O_{p_{\sigma(n)},k_n} (1 - \mu)^n (-1)^{f(\sigma)}
\]
vanishes. Now, by definition of \( \mathcal{P}_n^2 \) there exist at least two elements \( i, j \in \{1, 2, \ldots, n\} \) such that \( p_i = p_j \). Denote by \( \bar{\sigma} \) the permutation
that only switches \( i \) and \( j \) and leaves everything else fixed. From the equality
\[
\sum_{p \in A_\pi} O_{p_1,j_1} \cdots O_{p_n,j_n} O_{p_{\sigma(1)},k_1} \cdots O_{p_{\sigma(n)},k_n} (1 - \mu)^n (-1)^{f(\sigma)}
\]
\[
= - \sum_{p \in A_\pi} O_{p_1,j_1} \cdots O_{p_n,j_n} O_{p_{\bar{\sigma}(1)},k_1} \cdots O_{p_{\bar{\sigma}(n)},k_n} (1 - \mu)^n (-1)^{f(\bar{\sigma}\sigma)}
\]
which holds for every given \( \sigma \), by summing over all permutations we finally see that
\[
\sum_{\sigma \in \Pi_n} \sum_{p \in A_\pi} O_{p_1,j_1} \cdots O_{p_n,j_n} O_{p_{\sigma(1)},k_1} \cdots O_{p_{\sigma(n)},k_n} (1 - \mu)^n (-1)^{f(\sigma)} = 0,
\]
and the proof is complete. \( \square \)

**Acknowledgments**

All authors acknowledge the support of the Italian INDAM-GNAMPA
Project Code CUP_E55F2233270001 and the Italian PNRR MUR
project PE0000023-NQSTI.
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