NP IS NOT AL AND P IS NOT NC IS NOT NL IS NOT L

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1. Overview

This paper talk about that NP is not AL and P is not NC, NC is not NL, and NL is not L. The point about this paper is the depend relation of the problem that need other problem’s result to compute it. I show the structure of depend relation that could divide each complexity classes.

2. The condition to emulate the TM by using UTM

We will begin by considering the important nature of Turing Machine (TM) in this paper.

Definition 1. I will use the term “Action Configuration” to the part of the computation configuration that decide next transition. Action configuration include the information of state, transition function, position (head), and read memory (tape alphabet at head). I will use the term “Origin Configuration”, “Moving Configuration”, “Target Configuration”, “Affirm Configuration”, “Negate Configuration”, and “Computation Progress” to the action configuration of start configuration, computing configuration, halting configuration, accepting configuration, rejecting configuration.

Action configuration have all information to yield the next configuration, therefore we can make the universal turing machine (UTM) that emulate TM by using the information include the action configuration.

Definition 2. I will use the term “VTM” as “Virtual Turing Machine” to the TM that UTM emulate.

Theorem 3. Log space is necessary and sufficient to record action configuration.

Proof. The memory space that is required with the action configuration would be as follows: state and transition function is determined for each TM, and TM can record in the constant space. And position can record with log space. And read memory can record with constant space. Therefore, UTM can record TM’s action configuration into log space. □

Next, I talk about the sharing of the information of VTM.

Theorem 4. In NTM, VTM that execute nondeterministic branches does not share information and result each other. If VTM share the information and result, VTM must be executed in same branch.

Proof. VTM that execute non deterministic branches converges to single VTM. The VTM is one of these branches. Another branches do not exist, and these VTM can not affects the single VTM. Therefore, VTM that execute non deterministic branches does not share information and result each other. □
Theorem 5. The VTM’s moving configuration that execute in parallel must be recorded in different space. VTM that need each other’s information and result needs to execute in parallel.

Proof. If VTM is recorded in the other VTM space whether deterministic or non-deterministic, UTM have to overwrite with the last VTM. So the VTM that was overwritten another VTM can not keep the moving configuration (especially head position) and can not continue computation. If UTM emulates some VTM in same space, the predecessor VTM can not use successor VTM information (like tail-recursive.) So the VTM’s moving configuration that execute in parallel must be recorded in different space. □

I will define TM in this paper as follows;

Definition 6. I will use the term “NTM” to the Nondeterministic Turing Machine that can compute NP problems. I will use the term “LATM” to the Logarithmic space Alternating Turing Machine that can compute AL problems. I will use the term “LNTM” to the Logarithmic space Nondeterministic Turing Machine that can compute NL problems. I will use the term “LDTM” to the Logarithmic space Deterministic Turing Machine that can compute L problems.

To simplify, I will define UTM and VTM in this paper as follows;

Definition 7. Tape alphabet of TM is \{0, 1\}. Input data of TM is \(w\). TM treats \(w\) with special tape and head, and TM does not write \(w\). Length of \(w\) is \(O(n)\). UTM write I will use the term “Working Memory” to the memory that TM can read and write. TM write number of the steps, therefore computation history is acyclic. TM treats decide problems and TM must halt.

Theorem 8. We think about the set that’s elements are target configurations. DTM’s computation history is singleton, NTM’s computation history is set, ATM’s computation history is family. And structure of TM is well-founded set.

Proof. To think about the relation TM’s computation history and result. Because Computation history have no cyclic path in this paper, computation history become directed acyclic graph (DAG). This DAG have root as origin configuration, trunk as moving configuration, leaf as target configuration. We can characterize each TM by using the DAG of the computation history. Therefore, we can associate TM with set that correspond with the DAG of TM’s computation history. And the set is well-founded set which minimal elements are target configurations because DAG have no cyclic part.

DTM’s computation history is only one path and have only one target configuration. Therefore, DTM’s computation history correspond with singleton of the target configuration.

NTM’s computation history is DAG. But target configuration that included DAG affect to the NTM’s result, and DAG structure does not affect to the NTM’s result. Therefore, NTM’s computation history correspond with set of the target configuration.

ATM’s computation history is DAG. And DAG correspond with hypergraph that edge correspond with universal state and existential state. Therefore, ATM’s computation history correspond with family of the target configuration’s set. □
3. The depend relation between some problems

Think the situation that some VTM is sharing the results. The problem that describe incomplete and need the another problem’s result to complete meets the condition.

**Definition 9.** The problem $P_i, P_j$, if $P_i$ value does not confirmed until $P_j$ value is determining, I will use the term “Variable Problem” to $P_i$, and “Blocking Problem” to the $P_j$. And I will use “$P_i P_j$” to the problem that compute $P_i$ after computed $P_j$. The value or some condition of $P_i P_j$ is “$P_i P_j$”.

If I assume a certain value or some condition of $P_j$, I will use “$P_j$” to the some blocking problem of $P_i$, and “$P_i P_j$” to the problem that compute $P_i$ after computed $[P_j]$. Furthermore, $[P_i]$ may be variable problem. the case that $[P_i]$ is variable problem, $[P_i] P_i$ is also variable problem. The blocking problem of $[P_i] P_i$ is $[[P_i]] = [P_i]^2$.

I will use the term “Combined Problem” and “$CP$” to the issues covered in the following discussion. Combined problem is the problem that combines some variable problems in a complexity class. I will use the term “Element Problem” and “$CP = \{P_0, P_1, \ldots, P_{k-1}\}$” to the variable class. I will use “$k$” to the total number of element problems. Satisfiability of $P$ decide the value of $CP$. The combined problem’s value is the satisfaction of the element problems. I will use the term “$V$” to the truth value assignment of $CP$. And I add number to each truth value assignment like $VT = \{V^0, V^1, \ldots, V^{2k-1}\}$.

**Example 10.** Parity problem of Blocking problems’ true or false is variable problem. These are four type, true is even, true is odd, false is even, false is odd.

**Definition 11.** I will use the term “Depend Relation” and “$[P_i] \rightarrow P_j$” to the relation of $[P_i] P_j$. And I will use the term “Depend Path” and “$[P_i]^n \rightarrow P_j$” to the transitive depend relation $[P_i]^n \rightarrow [P_i]^{n-1} \rightarrow \cdots \rightarrow P_j$, and “$\{[P_i]^n \rightarrow P_j\}$” to the set of the problems that include $[P_i]^n \rightarrow P_j$. For simplicity, the depend path is partial order.

I will use the term “Rotate Path” to $P_i \sim P_j$. And I will use “$[P_i]^n ? \{[P_i]^n \rightarrow P_j\}$” to the computation that assume $[P_i]^n$ and compute $[P_i]^n \rightarrow P_j$ and $P_j$!

I will use the term “Depend Path Length” and “$L ([P_i]^n \rightarrow P_j)$” to the maximum number of the depend relations in the single chain of $[P_i]^n \rightarrow P_j$.

**Theorem 12.** VTM that compute $P_i$! share the result of the VTM that compute $[P_i]$!. If UTM can not record value of $[P_i]$! UTM must execute $[P_i]$! VTM and $P_i$! VTM in pararrel. And UTM can not record $[P_i]$! VTM and $P_i$! VTM into the same space.

**Proof.** If UTM can not record all $[P_i]$!, VTM must compute $[P_i]$! when compute $P_i$! $P_i$! need $[P_i]$! and $[P_i]$! need the timing to compute $[P_i]$!. Therefore, $[P_i]$! and $P_i$! is necessary to share the information each other.

**Theorem 13.** We can treat $CP$ as the family of the family $P$ of the set $V$ of the $P$ that value is true. And $CP$ is not well-formed set because of cyclic of transitive relation.

**Proof.** If we decide $[P_i]$? to some $V$, $V$? is $P_i$? = $P_i$! or $P_i$? $\neq$ $P_i$!. Therefore, $P_i$ classify $V$ into $P_i$? = $P_i$? $P_i$! or $P_i$? $\neq$ $P_i$? $P_i$!. If we define $V$ as the set that include $P_i$? = $\top$ of $[P_i]$?, and $P_i$ as the set that include $V$ of $P_i$? = $P_i$? $P_i$!, $CP$ is the family of $P_i$. 

□
For simplification, I will define $CP$ as follows. $P$ is the part of rotate path. $CP$ is efficient and do not have redundant. Therefore, all $P$ has $V$ that only $P$ is conflict. And $CP$ like $P_i \in CP \not\subset P_j \in \{P_i\}$ exist. Such $CP$ have no limitation with $P_j$, therefore $CP$ can take $P_j! = \top$ or $P_j! = \bot$.

4. $NP \supseteq AL = P$

Using the problem that’s all part depends on whole, I show $NP \supseteq AL = P$.

**Definition 14.** I will use the term “CHAOS” to the combined problem that made the following element problems.

$P_i \in \text{ClassNP}$

$[P_i] = CP$

I prove $NP \supseteq AL$ by using CHAOS with $NP \ni \text{CHAOS}$ and $AL \not\ni \text{CHAOS}$.

**Theorem 15.** $NP \ni \text{CHAOS}$

**Proof.** NTM can compute CHAOS to choose $P_i?$ in nondeterministic and check $\forall i ([P_i]?P_i! = P_i?)$. And UTM use $O(n)$ time to compute the choose of $P_i?$ and $P_i$, and compute $P_i!$ and compare $P_i?$ and $P_i!$. So $NP \ni \text{CHAOS}$. □

I extend CHAOS and prove $AL \not\ni \text{CHAOS}$.

**Theorem 16.** If we treat $CP$ as mentioned above $\text{CHAOS}$, CHAOS is the problem that decide $\bigcap CP = P_0 \cap P_1 \cap \cdots \cap P_{k-1} \neq \emptyset \leftrightarrow CP \in \text{CHAOS}$. And CHAOS is not well-formed set.

**Proof.** If we treat $CP$ as set, $V$ that include $P$ means consistent value $P$?. And if all $P$ include same $V$, $CP$ consistent at $V$. Therefore, $CP$ satisfy $\bigcap CP = P_0 \cap P_1 \cap \cdots \cap P_{k-1} \neq \emptyset$. And condition of CHAOS can not remove the cyclic of $CP$, therefore CHAOS is not well-formed set. □

**Theorem 17.** $AL \not\ni \text{CHAOS}$

**Proof.** We assume that LATM can compute the CHAOS. But the assumption contradict with CHAOS and we can see $AL \not\ni \text{CHAOS}$.

From assumptions, there is a mapping from CHAOS to LATM. But this mapping must relate CHAOS and LATM by using LDTM. Therefore, composition of LDTM and LATM (of computation history) must make CHAOS structure. And as mentioned above, LDTM and LATM is well-formed, therefore CHAOS structure made from LDTM and LATM must be well-formed.

But as mentioned above, CHAOS is not well-formed. If we want to treat CHAOS as well-formed structure, we must treat some elements as minimal element and remove the cyclic of transitive relation. And CHAOS does not include minimal elements, LDTM must create minimal elements and LDTM or LATM must record these elements. To remove the cyclic of transitive relation, we can use two ways a) all $P$ include $V$ change to $P?$, and b) all $V$ include $P$ change to $V$?. a) need space as $P$ cardinality $k = \sqrt{n} > \lg(n)$. b) need space as power set of $P$ cardinality $2^{\sqrt{n}} > \lg(n)$. LDTM and LATM does not have a) or b) space and can not remove the cyclic. Therefore, we can not make CHAOS structure by using composition of LDTM and LATM.

From the above, the assumption that LATM can compute CHAOS contradict with LATM and LDTM condition. Therefore, we can say from the reductio ad absurdum that LATM can not compute CHAOS, and $AL \not\ni \text{CHAOS}$. □
Theorem 18. \( NP \supseteq AL \)

Proof. \( NP \ni CHAOS \), \( AL \not\ni CHAOS \), and \( NP \supset P = AL \), thus we see \( NP \supseteq AL = P \). \( \square \)

5. \( AL = P \supseteq NC \)

Using the problem that’s linear order structure, I show \( NP \supseteq AL \).

Definition 19. I will use the term “ORDER” to the CHAOS that made the following element problems.

\[ P \in ClassP \]
\[ [P_{i=0}] = \{ P_j : j < i \} \]

I prove \( P \supseteq NC \) by using CHAOS with \( P \ni ORDER \) and \( NC \not\ni ORDER \).

Theorem 20. \( P \ni ORDER \)

Proof. UTM can compute ORDER by using this operation; both case of \( P_0? = 1 \) and \( P_0? = 0 \), UTM compute \([P_i]P_i! \) from smaller number, and check \( P_0? \{ P_0 \leadsto P_0 \}! = P_0? \). And UTM use \( O(n) \) time and space to compute all \([P_i]P_i! \). So \( P \ni ORDER \). \( \square \)

Theorem 21. \( NC \not\ni ORDER \)

Proof. If \([P_i]! \) is variable, \([P_i]P_i! \) is also variable problem and \([P_i]P_i! \) is variable. If UTM compute each \( P_i! \) in parallel, UTM must assume the combination of \([P_i]!? \). But \([P_i]!? \) is reached to \( O(2^n) \) and UTM can not record into \( O(n) \) space. And UTM must compute \([P_i]! \) to save the computing space whenever \( P_i! \) need \([P_i]! \). But UTM must compute \( P_i! \) sequentially from smaller numbers. So UTM can not compute \( P_i! \) in parallel.

From the above, \( NC \not\ni ORDER \). \( \square \)

Theorem 22. \( P \supseteq NC \)

Proof. \( P \ni ORDER \), \( NC \not\ni ORDER \), and \( P \supset NC \), thus we see \( P \supseteq NC \). \( \square \)

6. \( NC \supseteq NL \)

Using the problem that’s partial order structure, I show \( NC \supseteq NL \).

Definition 23. I will use the term “LAYER” to the ORDER that made the following element problems.

\[ P_i \in ClassNC \]
\[ m > 1 \text{, } length = (\log(n))^m \text{, } width = \frac{n}{\text{length}} \]
\[ \{ P \}_p = \{ P_q : q \leq width \times p \} \]
\[ [P_0] = \{ P \}_{j=0}, [P_{i=0}] = \{ P \}_{j < \lfloor \frac{\text{width}}{n} \rfloor} \]

I prove \( NC \supseteq NL \) by using CHAOS with \( NC \ni LAYER \) and \( NL \not\ni LAYER \).

Theorem 24. \( NC \ni LAYER \)

Proof. LAYER is the problem that have \( width = O \left( \frac{n}{\log(n)^m} \right) \) size anti chain of variable problem, and have \( length = O((\log(n))^m) \) length rotate path. Each variable problem in anti chain is independent each other and UTM can compute these
problems in parallel. Therefore UTM that have $O\left(\frac{n}{\log(n)}\right) < O(n)$ TM can compute LAYER in $O((\log(n))^m)$ time.

From the above, $NC \ni LAYER$. \hfill \Box

**Theorem 25.** $NL \not\ni LAYER$

*Proof.* We assume that LNTM can compute the LAYER. But the assumption contradict with LAYER and we can see $NL \not\ni LAYER$.

In LAYER, LNTM must use $[P_i]? = \{P\}_j^{<\left\lfloor \frac{\log(n)}{m} \right\rfloor}$ to compute $P_i!$. But LNTM can not record all $[P_i]?$ into $O(\log(n))$ space. Therefore, LNTM must divide $[P_i]?$ to fit $O(\log(n))$ space.

But LNTM must need the information of divided $[P_i]?$ combination because $P_i!$ is changed by the $[P_i]?$ combination. LNTM can not use universal state, therefore LNTM must record the information of each $[P_i]?$ combination. And $[P_i]^2, [P_i]^3, [P_i]^4, \cdots$ will also like $[P_i]$ and LNTM can not stop until round rotate path. Therefore, LNTM must record at least $length = O((\log(n))^m)$ space.

From the above, the assumption that LNTM can compute LAYER contradict with LNTM’s condition. Therefore, we can say from the reductio ad absurdum that LNTM can not compute LAYER, and $NL \not\ni LAYER$. \hfill \Box

**Theorem 26.** $NC \supseteq NL$

*Proof.* $NC \ni LAYER$, $NL \not\ni LAYER$, and $NC \ni NL$, thus we see $NC \supseteq NL$. \hfill \Box

7. $NL \supseteq L$

Using the problem that relation spread to whole, I show $NC \supseteq NL$.

**Definition 27.** I will use the term “TWINE” to the LAYER that made the following element problems.

$P_i \in ClassNL$

$[P_0] \subseteq \{P\}_{j \neq 0}, [P_{1\neq0}] \subseteq \{P\}_{j < \left\lfloor \frac{\log(n)}{m} \right\rfloor}, |[P_i]| = O(\log(n))$

$O(L (P_0 \sim P_0)) > O(1)$

I prove $NL \supseteq L$ by using CHAOS with $NL \ni TWINE$ and $L \not\ni TWINE$.

**Theorem 28.** $NL \ni TWINE$

*Proof.* LNTM can compute TWINE following procedure.

First, LNTM choose $[P_0]$ by nondeterministic that satisfies $[P_0] P_0? = 1$. If $[P_0]$! is not exist, LNTM choose $[P_0] P_0? = 0$ by nondeterministic. If $[P_0]$! is not also exist, LNTM accept input. If $[P_0]$! is exist, LNTM choose $P_i \in [P_0]$ and choose $[P_i]$! by nondeterministic that satisfies previous $[P_0]$! condition. If $[P_i]$! is not exist, LNTM choose $[P_0] P_0? = 0$ by nondeterministic. If $[P_i]$! is not also exist, LNTM accept input. If $[P_i]$! is exist, LNTM repeat same procedure to $P_0$. If LNTM reach to $P_0$, LNTM check $P_0? = P_0!$. If $P_0? = P_0!$ then LNTM accept input, $P_0? \neq P_0!$ in case $P_0? = 1$ and $P_0? = 0$, LNTM reject input.

Such procedure, LNTM can verify all possible combinations of $P_i!$. Because LNTM can verify whether all blocking problem of $P_0$? . The case of $P_i$ is three case, a) $P_i!$ is the value that never possible value of $P_i$, b) all $P_i!$ of any depend path is same value, c) some $P_i!$ of depend path is different values each other. In case a),
the depend path is never exist and LNTM can accept the branch. In case b), the depend path is correct constraint and LNTM can continue computing. In case c), the same depend path take true and false because the different $P_i!$ leads different $[P_0!]$, and rotate path will contradict at $P_0!$ or never possible value that refer a). Therefore LNTM can compute correctly in a)b)c).

And this procedure use $O(\log(n))$ space because LNTM use one $P_i!$ nondeterministic and compare $P_0!? = P_0!$. From the above, $NL \not\supset TWINE$.

I prove following lemma, and $L \not\in TWINE$.

**Theorem 29.** If Combined Problem is true, all rotate path is symmetric about satisfiability. In other words, Decision of the Combined Problem is true, include the decision of these rotate path is symmetric about satisfiability.

**Proof.** If Combined Problem is true, all rotate path is satisfied and symmetric about satisfiability. Therefore, it is possible to determine whether these rotate path is symmetric about satisfiability by determine the true that Combined Problem. □

**Theorem 30.** The rotate path of Combined Problem is not necessarily symmetric about satisfiability.

**Proof.** As you can see easily that is possible to create rotate path with true and false result at same problem. Therefore, it is possible to create rotate path that is asymmetry each other, and the rotate path of Combined Problem is not necessarily symmetric about satisfiability. □

**Theorem 31.** LDTM can handled elements atmost $O(n)$. Therefore, LDTM can check elements symmetry or asymmetry atmost $O(n)$.

**Proof.** In order to tell apart each element, LDTM need the information. LDTM can tell apart each element by using the pointer. But LDTM can use atmost $O(\log(n))$ space, LDTM can tell apart atmost $O(n)$ elements. Therefore, LDTM can handled elements atmost $O(n)$.

And to check the symmetry of two elements, it’s necessary to tell apart these elements. Therefore, LDTM can check elements symmetry or asymmetry atmost $O(n)$. □

**Theorem 32.** When dealing with a Combined Problem, NTM can deal with the symmetry of the elements in same step. But DTM can not deal with the symmetry of the elements in same step.

**Proof.** When computing a Combined Problem, DTM have at most one computation history that is one way from starting configuration to halting configuration. DTM’s computation configuration can not replace another. And DTM can not deal some elements symmetry at each step. But NTM have branching computation history that is Directed Acyclic Graph which root is starting configuration. Therefore, some branches that have same trunk is symmetry and can replace each other. And NTM can deal some element symmetry by dealing these element as branches. □

**Theorem 33.** In TWINE, number of different sequences of values in a rotate path is $O(n^{L(P_0\Rightarrow P_0)}) > O(n^c)$.
Proof. In TWINE, number of different sequences of values \([P_i]\) is atmost \(O(n)\), because \(|[P_i]| = \lg(n)\). And because of TWINE’s structure, length of rotate path is atmost \(L(P_0 \leadsto P_0) > O(1)\). Therefore, number of different sequences of values in a rotate path is \(O\left(\prod_{i=0}^{L(P_0 \leadsto P_0)} [P_i]\right) = O\left(n^{L(P_0 \leadsto P_0)}\right) > O(n^c)\). □

Theorem 34. \(L \not\ni \text{TWINE}\)

Proof. We assume that LDTM can compute the TWINE. But the assumption contradict with CHAOS and we can see \(L \not\ni \text{TWINE}\).

First, We think that compute rotate path. Proof. As mentioned above, all rotate path symmetry in satisfiability if TWINE is true. Thus computing that TWINE is true include that all rotate path is symmetry. And as mentioned above, the rotate path of TWINE is not necessarily symmetric about satisfiability, LDTM must compute to compare their satisfiability. And as mentioned above, DTM can not deal some symmetry, DTM must deal these rotate path separately.

As mentioned above, number of rotate path is \(O\left(n^{L(P_0 \leadsto P_0)}\right) > O(n^c)\). As mentioned above, LDTM can check rotate path symmetry or asymmetry atmost \(O(n)\), and can not check all rotate path. Therefore, LDTM must use multiple LDTM to check all rotate path symmetry.

For checking the symmetry of rotate path, LDTM must tell apart each rotate path. LDTM can handle each element atmost \(O(n)\). Therefore, LDTM must split all rotate path to fit \(O(n)\). The number of the rotate path pack are \(O\left(\frac{n^{L(P_0 \leadsto P_0)}}{n}\right) = O\left(n^{L(P_0 \leadsto P_0)-1}\right)\). LDTM can check symmetry all rotate path to check these pack. But LDTM can not tell apart each rotate path pack, LDTM must repeat thus splitting \(O(L(P_0 \leadsto P_0))\) times.

We think the number of required LDTM to split rotate path. LDTM must split rotate path and execute sub LDTM to check symmetry, and finally check each sub LDTM’s result and each symmetry. I will use the term “Caller LDTM” to the LDTM that split rotate path and execute sub LDTM, and “Callee LDTM” to the LDTM that called by Caller LDTM. Callee LDTM must get the rotate path pack information to check the symmetry from Caller LDTM. Caller LDTM must get the result information from Callee LDTM. Therefore, as mentioned above, Caller LDTM and Callee LDTM must execute in parallel and must use different space.

Thus chain from Caller LDTM to Callee LDTM exist \(O(L(P_0 \leadsto P_0)) > O(1)\). Constant LDTM can not compute these chain. That is inconsistent with assumptions and thus can not compute with LDTM.

From the above, \(L \not\ni \text{TWINE}\). □

Theorem 35. \(NL \ni L\)

Proof. \(NL \ni \text{TWINE}, L \not\ni \text{TWINE}, \) and \(NL \ni L\), thus we see \(NL \ni L\). □

8. Conclusion

These results lead to the conclusion.

Theorem 36. \(NP \ni AL = P \ni NC \ni NL \ni L\)
References

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