Exotic galilean symmetry in non commutative field theory

P. A. Horváthy  
Laboratoire de Mathématiques et de Physique Théorique  
Université de Tours  
Parc de Grandmont  
F-37 200 TOURS (France)

and

L. Martina  
Dipartimento di Fisica dell’Università  
and  
Sezione INFN di Lecce. Via Arnesano, CP. 193  
I-73 100 LECCE (Italy).

December 24, 2018

Abstract

The non-relativistic version of the non commutative Field Theory, recently introduced by Lozano, Moreno and Schaposnik [1], is shown to admit the “exotic” Galilean symmetry found before for point particles.

hep-th/0207118. (with Note added)

1 Introduction

It has been known for some time that the planar Galilei group admits a two–parameter central extension [2, 3, 1, 5, 6]. The only physical examples with such an “exotic” Galilean symmetry known so far are the scalar model in [3, 7] equivalent to non-commutative quantum mechanics (NCQM) [8] and used in the context of the Hall Effect [7], and the acceleration-dependent system in [6].

Here we first revisit NCQM, viewed as a classical field theory. Then we turn to non-commutative field theory (NCFT) [9]. In these theories, which have attracted a considerable amount of recent attention, the ordinary product is replaced by the Moyal “star” product associated with the posited non-commutative structure of the plane. In [1] Lozano et al. present in particular a non-relativistic version of NCFT. Below we point out that the latter theory admits an “exotic” Galilean symmetry analogous to that of a point particle [7].

2 NC Quantum Mechanics

Let us consider a free scalar particle in the non-commutative plane, given by the standard hamiltonian \( h = \frac{p^2}{2m} \) and the fundamental commutation relations [3, 4, 5]

\[
\begin{align*}
\{x_1, x_2\} &= \theta, \\
\{x_i, p_j\} &= \delta_{ij}, \\
\{p_1, p_2\} &= 0,
\end{align*}
\]

(2.1)
where $\theta$ is the non-commutative parameter. As shown in Ref. [7], the “exotic” [meaning two-fold centrally extended] Galilei group is a symmetry for the system with associated conserved quantities $\vec{p}$, $h$, $m$, $k = -m^2 \theta$, augmented with the modified angular momentum and Galilean boosts,

$$
\begin{align*}
\hat{j} &= \vec{x} \times \vec{p} + \frac{1}{2} \theta \vec{p}^2 + s, \\
\hat{g}_i &= mx_i - p_i t + m \theta \epsilon_{ij} p_j,
\end{align*}
$$

(2.2)

where $s$ represents the anyonic spin. The commutation relations of this algebra w. r. t. the “exotic” Poisson bracket (2.1) coincide with those of the ordinary, singly-extended Galilei group except for the boosts, whose bracket yields the second, “exotic parameter”

$$
\{g_1, g_2\} = -m^2 \theta \equiv -k.
$$

(2.3)

Owing to the non-commutativity of the coordinates, the system has no position representation: $x, y$ do not form a complete commuting system. Put in another way: $\vec{x} = \text{const.}$ is not a polarization\(^1\). The momentum representation is still valid, though, and we represent therefore the wave functions by square-integrable functions of the momentum, $\phi(\vec{p})$. The representation of the “exotic” Galilei group is given as [3, 4, 5, 7]

$$
U_a \phi(\vec{p}) = \exp \left( i \left[ \frac{\vec{p}^2 e}{2m} - \vec{p} \cdot \vec{c} + s \varphi + m u \right] + im \theta \left[ \frac{1}{2} \vec{b} \times \vec{p} + m v \right] \right) \phi \left( R^{-1}(\vec{p} - m \vec{b}) \right),
$$

(2.4)

where $a$ is an element of the “exotic” Galilei group with $e$ representing a time translation, $\vec{c}$ a space translation, $\vec{b}$ the boosts, $R$ a rotation with angle $\varphi$; $u$ and $v$ represent the translations along the central directions.

The infinitesimal action of (2.4) yields the quantum operators. The momentum operator is standard, $\hat{p}_i$ is multiplication by $p_i$; the energy is $\hat{h} = \frac{\vec{p}^2}{2m}$. For the other “exotic” Galilei generators we get

$$
\begin{align*}
\hat{j} &= i \epsilon_{ij} p_i \frac{\partial}{\partial p_j} + s, \\
\hat{g}_j &= m \left( i \frac{\partial}{\partial p_j} + \frac{1}{2} \theta \epsilon_{jk} p_k \right)
\end{align*}
$$

(2.5)

while $m$ and $k = -m^2 \theta$ act trivially. The particle satisfies the usual free Schrödinger equation in the momentum space,

$$
i \partial_t \phi(\vec{p}) = \frac{\vec{p}^2}{2m} \phi(\vec{p}).
$$

(2.6)

The point is that QM can be also viewed as a classical field theory. Eq. (2.4) derives indeed from the Lagrangian

$$
L = \int \left( \frac{i}{2} (\phi \partial_t \phi - \phi \partial_t \phi) - \frac{\vec{p}^2}{2m} |\phi|^2 \right) d^2 \vec{p}.
$$

(2.7)

The theory given by (2.4) is manifestly invariant w. r. t. the “exotic” Galilei group, implemented as in (2.4). Then Noether’s theorem allows us to derive the associated conserved quantities: if $L$ changes as $\delta L = \partial_\alpha K^\alpha$ under an infinitesimal coordinate change $\delta \vec{x}$, then

\(^1\text{Quantization can be performed using canonical [Darboux] variables; but these latter will not have the physical interpretation of position, cf. the Discussion.}\)
\[ f \left( \frac{\delta L}{\delta \phi} \delta \phi + \frac{\delta L}{\delta \bar{\phi}} \delta \bar{\phi} - K \right) d^2 \vec{x} \] is a constant of the motion. Using \( (2.4) \) we get

\[
\mathcal{H} = \int \frac{\vec{p}^2}{2m} |\phi|^2 d^2 \vec{p}
\]
energy

\[
\mathcal{P}_j = \int p_j |\phi|^2 d^2 \vec{p}
\]
momenta

\[
\mathcal{J} = \int \left( \frac{1}{2i} \epsilon_{jk} (\phi \frac{\partial \bar{\phi}}{\partial p_j} - \bar{\phi} \frac{\partial \phi}{\partial p_j}) p_k + s |\phi|^2 \right) d^2 \vec{p}
\]
angular momentum

\[
\mathcal{G}_j = m \int \left( \frac{1}{2i} \left( \phi \frac{\partial \bar{\phi}}{\partial p_j} - \bar{\phi} \frac{\partial \phi}{\partial p_j} \right) + \frac{\theta}{2} \epsilon_{jk} p_k |\phi|^2 \right) d^2 \vec{p}
\]
boost

\[
\mathcal{M} = m \int |\phi|^2 d^2 \vec{p}
\]
mass

\[
\mathcal{K} = -m^2 \theta \int |\phi|^2 d^2 \vec{p}
\]
exotic charge

These quantities are the expectation values of the operators listed in \( (2.5) \) when the wave function is normalized to 1. Consistently with our previous results, these conserved quantities, with the exception of the boosts, are standard.

The free Schrödinger equation \( (2.6) \) is of the Hamiltonian form, \( \partial_t \phi = \{ \phi, \mathcal{H} \} \) with the standard Poisson bracket

\[
\{ F, G \} = \frac{1}{i} \int \left( \frac{\delta F}{\delta \phi} \frac{\delta G}{\delta \bar{\phi}} - \frac{\delta F}{\delta \bar{\phi}} \frac{\delta G}{\delta \phi} \right) d^2 \vec{p}.
\]

Then the quantities \( (2.8) \) are readily seen to close under \( (2.9) \) into the exotic Galilean relations. In particular,

\[
\{ \mathcal{G}_1, \mathcal{G}_2 \} = -\mathcal{K}, \quad \{ \mathcal{G}_1, \mathcal{G}_2 \} = -\mathcal{K},
\]

\( \text{cf. (2.8).} \)

### 3 Exotic symmetry of Moyal field theory

Much recent work has been dedicated to non-commutative field theory \[9\], where the ordinary product is replaced by the Moyal “star” product associated with the non-commutative parameter \( \theta \) \[10\],

\[
(f \star g)(x_1, x_2) = \exp \left( i \theta \left( \partial x_1 \partial y_2 - \partial x_2 \partial y_1 \right) \right) f(x_1, x_2) g(y_1, y_2) \bigg|_{x=y}
\]

associated with the non-commutative parameter \( \theta \). Lozano et al. \[11\], consider, in particular, a field theory inspired by ordinary, non-relativistic quantum mechanics. We show now that such a field theory admits the “exotic” Galilean symmetry studied above. Let us indeed consider the free non-commutative field theory with the non-local Lagrangian

\[
L_{NC} = \frac{i}{2} \left( \bar{\psi} \star \partial_t \psi - \partial_t \bar{\psi} \star \psi \right) - \frac{1}{2m} \bar{\nabla} \bar{\psi} \star \nabla \psi.
\]

The Galilean invariance of this theory is obvious from the outset since the Moyal product can be ignored under integration, \( \int f \star g d^2 \vec{x} = \int fg d^2 \vec{x} \). (Alternatively, let us observe that the equation of motion associated with \( (3.2) \) is, despite the presence of the star product in \( (3.2) \), simply the free Schrödinger equation). It is nonetheless useful to check the statement explicitly,
because the difference with the ordinary case changes to the associated conserved quantities. In fact, implementing a boost in the standard way, namely as \( \psi \rightarrow U_\beta \psi \),

\[
U_\beta \psi(x,t) = e^{im(\vec{x} \cdot \vec{b} - \frac{1}{2}b^2t)} \psi(\vec{x} - \vec{b}t, t),
\]

changes \( L \) into

\[
\frac{i}{2} \left((e^{(-)} \psi) \star (e^{(+)}) \partial_t \psi) - (e^{(-)} \partial_t \psi) \star (e^{(+)}) \psi) \right) - \frac{1}{2m} (e^{(-)} \vec{\nabla} \psi) \star (e^{(+)}) \vec{\nabla} \psi)
\]

taken at \((\vec{x}' = \vec{x} - \vec{b}t, t)\), where we used the shorthand \( e^{(\pm)} = \exp[\pm im(\vec{x} \cdot \vec{b} + \frac{1}{2}b^2t)] \). In the commutative theory, this would be simply \( \frac{1}{2} (\vec{\nabla} \psi - \vec{\nabla} \bar{\psi}) - \frac{1}{2m} |\vec{\nabla} \psi|^2 = (\vec{x} - \vec{b}t, t) = L(\vec{x} - \vec{b}t, t) \). In the noncommutative case, however, the Moyal product with the exponential factors results

\[
\text{in an additional shift of the argument,}
\]

\[
(e^{(-)} f) \star (e^{(+)}) g)(\vec{x}) = (f \star g)(\vec{x} - \frac{1}{4}m \theta \epsilon \vec{b})
\]

where \( \epsilon = (\epsilon_{ij}) \) is the totally antisymmetric matrix. The NC Lagrangian \( 3.2 \) changes according

\[
L_{NC}(x,t) \rightarrow L_{NC}(\vec{x} - \vec{b}t - \frac{1}{4}m \theta \epsilon \vec{b}, t).
\]

Boosting is hence equivalent to a “twisted shift”, so that the action \( \int L_{NC} d^2 \vec{x} dt \) is invariant, as expected. The shift yields, however, an additional term in the associated conserved quantity. The definition \( 3.1 \), together with Baker’ formula \( 10 \),

\[
(f \star g)(x) = \frac{1}{(\pi \theta)^2} \int f(\vec{x}'')g(\vec{x}'')e^{2i/\theta} \Delta d\vec{x}'d\vec{x}'', \quad \Delta = (\vec{x}' - \vec{x}) \times (\vec{x}'' - \vec{x}),
\]

allow us to establish the relation

\[
\frac{1}{2} [\vec{\psi} \star (\vec{x} \psi) + (\vec{x} \bar{\psi}) \star \psi] = \vec{x} \bar{\psi} \star \psi - \frac{\theta}{2} \epsilon \vec{j}, \quad \vec{j} = \frac{1}{2i} (\bar{\psi} \star \vec{\nabla} \psi - \vec{\nabla} \bar{\psi} \star \psi).
\]

Then for the conserved quantity associated with the boost we find

\[
G_i = \int mx_i (\bar{\psi} \star \psi) d^2 \vec{x} - tP_i + \frac{1}{4}m \theta \epsilon_{ij} P_i,
\]

where \( \vec{P} = \int \vec{j} d^2 \vec{x} = \int (1/2i)(\bar{\psi} (\vec{\nabla} \psi) - (\vec{\nabla} \bar{\psi}) \psi) d^2 \vec{x} \) is the conserved momentum, associated with the translational symmetry. Note that \( \int x_i (\bar{\psi} \star \psi) d^2 \vec{x} \neq \int x_i |\psi|^2 d^2 \vec{x} \), owing to the presence of the coordinate \( x_i \).

Similarly, the energy, associated to the time translation, is \( H = \int \frac{1}{2m} \vec{\nabla} \bar{\psi} \star \vec{\nabla} \psi d^2 \vec{x} \) is invariant, associated with the translational symmetry. A rotation by \( \varphi \) in the plane is implemented on the field according to \( U_\varphi \psi(x) = e^{i\varphi} \psi(R\vec{x}) \) where \( R = R_\varphi = e^{i\varphi} \). This leaves the free Lagrangian invariant and, using

\[
\frac{1}{4} \epsilon_{ij} [\bar{\psi} \star (\partial_i \psi x_j) - (\partial_i \bar{\psi} x_j) \star \psi] = \vec{x} \times \vec{j} - \frac{\theta}{2} \vec{\nabla} \bar{\psi} \star \vec{\nabla} \psi,
\]

we get the angular momentum involving both the exotic and the spin terms,

\[
\mathcal{J} = \int \left( \vec{x} \times \vec{j} - \frac{\theta}{2} |\vec{\nabla} \psi|^2 + s |\psi|^2 \right) d^2 \vec{x}.
\]

Note that the new term due to the noncommutativity here is separately conserved, since it is proportional to the energy.
Note that while $H, P, M = m \int |\psi|^2 \, d^2 \vec{x}$ have the standard form, the boost, $G$, the angular momentum, $J$, and the “exotic” central generator $K = -m^2 \theta \int |\psi|^2 \, d^2 \vec{x}$ involve the non-commutative parameter $\theta$. These quantities, analogous to those found for a classical particle in the non-commutative plane discussed in [2], span under the Poisson bracket, (2.9) [with the integration variable $\vec{p}$ replaced by $\vec{x}$] the “exotic” Galilei group. Bracketing the boosts yields in particular once again the “exotic” relation (2.10).

4 Discussion

For NCQM in the momentum space, Section 2, both the field-theoretical action (2.7) and the Poisson bracket, (2.9) are conventional, and the only difference with an ordinary (“non-exotic”) particle is the way boost acts on the wave function, represented by the exponential factor $\exp\left[i m(\theta/2)\vec{b} \times \vec{p}\right]$ in Eq. (2.4). This is unlike as for a classical particle, where the exotic structure appears in the Poisson bracket, (2.1), while the Galilei group acts in the usual way. Remember, however, that the exotic structure could be made disappear by redefining the position, namely introducing the “Darboux” variables $q_i = x_i + \frac{1}{2} \theta \varepsilon_{ij} p_j$ canonically conjugate to the $p_i$. In terms of the $q_i$ and the $p_i$ we would recover the standard structure of an ordinary particle – but one upon which the Galilei group acts in a non-standard way.

In NCFT considered in Section 3 instead, the action (3.2) involves the non-commutative parameter $\theta$ through the Moyal star product, while the boosts act conventionally, cf. (3.3). The action uses hence a non-local “alternative Lagrangian” for the free Schrödinger equation. It is precisely this modification that opens the way for the “exotic” Galilean symmetry.

Acknowledgement We are indebted to C. Duval and F. Schaposnik for correspondence and enlightening discussions.

5 Note added

In calculating the commutator of the boosts above, an error was committed. Straightforward calculation shows indeed that the components of the $G_i$ in (3.8) in fact commute. This can also be understood by observing that the properties of the Moyal product allows us to absorb the $\theta$-dependent term into the first integral so that (3.8) takes the traditional form

$$G_i^0 = \int m x_i \bar{\psi} \psi \, d^2 \vec{x} - t P_i.$$  

(5.1)

Yet another explanation is obtained by noting that the “Moyal stars” in (3.2) can be dropped. Owing to the “integral property” $\int f \star g = \int fg$ the Lagrangian (3.2) is in fact equivalent to the standard free expression whose associated conserved boost is (5.1).

Our theorem is, nevertheless, correct: it is in fact enough to implement the boosts by inserting a Moyal star into (3.3) i.e. to consider rather

$$U^* \psi(x, t) = e^{im(\vec{x} \cdot \vec{b} - \frac{1}{2} \theta^{2} t)} \star \psi(\vec{x} - \vec{b} t, t).$$  

(5.2)

This novel type of action is still a symmetry and yields, instead of (5.1),

$$G_i^* = m \int d^2 \vec{x} x_i \bar{\psi} \psi - t P_i - \frac{\theta}{2} \varepsilon_{ij} P_j$$  

(5.3)
which does indeed satisfy the exotic commutation relations
\[ \{ \mathcal{G}_1, \mathcal{G}_2 \} = \theta \int d^2 \vec{x} \bar{\psi} \psi. \] (5.4)

This corrects an error in [11]. See [12] for further details.

We are indebted to Professor P. Stichel for calling our attention to this point.

Similarly, the 2-dimensional version of Lévy-Leblond’s “non-relativistic Dirac equation” [13] can be considered [14, 11]. Let \( \Psi \) denote indeed a two-component spinor, and consider the Lagrange density
\[ L = \Im \left\{ \Psi^\dagger \left( \Sigma_t \partial_t - \vec{\Sigma} \cdot \vec{\nabla} + i \Sigma_s \right) \Psi \right\} \] (5.5)
where
\[ \Sigma_t = \frac{1}{2}(1 + \sigma_3), \quad \Sigma_i = \sigma_i \quad (i = 1, 2), \quad \Sigma_s = (1 - \sigma_3). \] (5.6)
Setting \( \Psi = \begin{pmatrix} \Phi \\ \chi \end{pmatrix} \) yields the planar version of the Lévy-Leblond equation [13],
\[ (\partial_1 + i \partial_2)\Phi + 2i\chi = 0 \]
\[ \partial_t - (\partial_1 - i \partial_2)\chi = 0 \] (5.7)

Implementing a boost conventionally as [13, 15]
\[ U \Psi(\vec{x}, t) = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2}(b_1 + ib_2) & 1 \end{pmatrix} e^{i(\vec{b} \cdot \vec{x} - \vec{b}^2 t/2)} \Psi(\vec{x} - \vec{b} t, t) \] (5.8)
or infinitesimally
\[ \delta^* \Phi = i\vec{b} \cdot \vec{x} \Phi - t\vec{b} \cdot \vec{\nabla} \Phi \]
\[ \delta^* \chi = i\vec{b} \cdot \vec{x} \chi - t\vec{b} \cdot \vec{\nabla} \chi \] (5.9)
we find that the LL equations (5.7) remain satisfied, establishing the Galilean symmetry.

The associated conserved boost components, again (5.1) [with the upper component, \( \Phi \), replacing the scalar field, \( \psi \)], commute, as observed by Lévy-Leblond 35 years ago and stressed recently by Hagen [14].

Inserting a Moyal “star” into the free Lagrangian (5.5) would yield an equivalent (non-local) Lagrangian for which the conventional implementation is still a symmetry; it yields the same boost generators with commuting components. (This corrects another false statement made in [11].) This is also seen by observing, as above, that formula (2.13) in that paper, although correct, can again be transformed into the conventional form using the above-mentioned properties of the Moyal product.

The statement made in [11] is still correct: it is enough to change the conventional implementation once again by inserting the Moyal product i.e. to consider rather
\[ U^* \Psi(\vec{x}, t) = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2}(b_1 + ib_2) & 1 \end{pmatrix} \Psi(\vec{x} - t\vec{b}, t) \] (5.10)
or infinitesimally
\[ \delta^* \Phi = (i\vec{b} \cdot \vec{x}) \Phi - t\vec{b} \cdot \vec{\nabla} \Phi \]
\[ \delta^* \chi = \frac{1}{2}(b_1 + ib_2) \Phi + (i\vec{b} \cdot \vec{x}) \chi - t\vec{b} \cdot \vec{\nabla} \chi, \] (5.11)
to get the “Moyal-boost” [5.3] [with \( \psi \to \Phi \)] which has noncommuting components.

Further generalization will be found elsewhere [16].
References

[1] G. S. Lozano, E. F. Moreno, F. A. Schaposnik, Self-dual Chern-Simons solitons in non-commutative space. Journ. High Energy Phys. 02 (2001) 036.

[2] J.-M. Lévy-Leblond, Galilei group and Galilean invariance. in Group Theory and Applications (Loebl Ed.), II, Acad. Press, New York, p. 222 (1972).

[3] A. Ballesteros, N. Gadella and M. del Olmo, Moyal quantization of 2+1 dimensional Galilean systems. Journ. Math. Phys. 33, 3379 (1992).

[4] Y. Brihaye, C. Gonera, S. Giller and P. Kosinski, Galilean invariance in 2 + 1 dimensions. hep-th/9503046 (unpublished).

[5] D. R. Grigore, The projective unitary irreducible representations of the Galilei group in 1+2 dimensions. Journ. Math. Phys. 37, 240 and ibid. 460 (1996).

[6] J. Lukierski, P. C. Stichel, W. J. Zakrzewski, Galilean-invariant (2+1)-dimensional models with a Chern-Simons-like term and d = 2 noncommutative geometry. Annals of Physics (N. Y.) 260, 224 (1997).

[7] C. Duval and P. A. Horváthy, The exotic Galilei group and the “Peierls substitution”, Phys. Lett. B 479, 284 (2000) [hep-th/0002233]; Exotic Galilean symmetry in the non-commutative plane, and the Hall effect. Journ. Phys. A 34, 10097 (2001).

[8] V. P. Nair and A. P. Polychronakos, Quantum mechanics on the non-commutative plane and sphere. Phys. Lett. B505, 267 (2001); J. Gamboa, M. Loewe, and J. C. Rojas, Non-commutative Quantum Mechanics. Phys. Rev. D 64, 067901 (2001); S. Bellucci, A. Nersessian, and C. Sochichiu, Two phases of the non-commutative quantum mechanics. Phys. Lett. B522, 345 (2001); R. Jackiw, Physical instances of non-commuting coordinates. Nucl. Phys. B (Proc. Suppl.) 108B, 30 (2002).

[9] see, e. g., N. Seiberg and E. Witten, String theory and non-commutative geometry. JHEP 99 09 032 (1999). For a recent review see, e. g., M. R. Douglas and N. A. Nekrasov, Noncommutative field theory. Rev. Mod. Phys. 73, 977 (2001).

[10] J. Moyal, Proc. Camb. Philos. Soc. 45, 99 (1949); G. Baker, Formulation of Quantum Mechanics. . . Phys. Rev. 109, 2198 (1958). For more recent work, see T. Curtright, D. B. Fairlie and C. K. Zachos, Features of time-independent Wigner functions. Phys. Rev. D 58, 025002 (1998); D. B. Fairlie, Moyal brackets, star products and the generalized Wigner function. hep-th/9806198.

[11] C. Duval and P. A. Horváthy, Spin and exotic Galileean symmetry. Phys. Lett. B547, 306 (2002).

[12] P. A. Horváthy, G. Martina, and P. Stichel, Galilean symmetry in noncommutative field theory, Phys. Lett. B (in press). [hep-th/0304215].

[13] J.-M. Lévy-Leblond, Comm. Math. Phys. 6, 286 (1967)
[14] C. R. Hagen, *Second central extension in Galilean covariant field theory.* Phys. Lett. **B539**, 168 (2002).

[15] C. Duval, P. A. Horváthy, L. Palla, *Spinors in non-relativistic Chern-Simons electrodynamics.* Ann. Phys. **249**, 265 (1996)

[16] P. A. Horváthy, G. Martina, and P. Stichel, work in progress.