Generalized 2-vector spaces and general linear 2-groups

Josep Elgueta
Dept. Matemàtica Aplicada II
Universitat Politècnica de Catalunya
email: Josep.Elgueta@upc.edu

Abstract
In this paper a notion of generalized 2-vector space is introduced which includes Kapranov and Voevodsky 2-vector spaces. Various kinds of generalized 2-vector spaces are considered and examples are given. The existence of non free generalized 2-vector spaces and of generalized 2-vector spaces which are non Karoubian (hence, non abelian) categories is discussed, and it is shown how any generalized 2-vector space can be identified with a full subcategory of an (abelian) functor category with values in the category \( \text{VECT}_K \) of (possibly infinite dimensional) vector spaces. The corresponding general linear 2-groups \( \text{GL} (\text{Vect}_K [C]) \) are considered. Specifically, it is shown that \( \text{GL} (\text{Vect}_K [C]) \) always contains as a (non full) sub-2-group the 2-group \( \text{Equiv}_{\text{Cat}} (C) \) (hence, for finite categories \( C \), they contain Weyl sub-2-groups analogous to the usual Weyl subgroups of the general linear groups), and \( \text{GL} (\text{Vect}_K [C]) \) is explicitly computed (up to equivalence) in a special case of generalized 2-vector spaces which include those of Kapranov and Voevodsky. Finally, other important drawbacks of the notion of generalized 2-vector space, besides the fact that it is in general a non Karoubian category, are also mentioned at the end of the paper.

1 Introduction
For the development of 1-dimensional (i.e., categorical) mathematics, where sets are systematically replaced by categories, it would be desirable to have an analog of the usual linear algebra which has proved so useful in the (0-dimensional) mathematics of sets. The first logical step in the search of such an analog is to find a good notion of categorical vector space, more often called a 2-vector space.

The notion of (finite dimensional) 2-vector space over a field \( K \) was introduced for the first time by Kapranov and Voevodsky \[12\], motivated by Segal’s definition of a conformal field theory \[16\]. The main point in their definition is to take the category \( \text{Vect}_K \) of finite dimensional vector spaces over \( K \) as analog of the field \( K \) and to define a \( \text{Vect}_K \)-module category as a symmetric monoidal category \( V \) (analog of the abelian group underlying a vector space) equipped with an action of \( \text{Vect}_K \) on it (analog of the multiplication by scalars) satisfying all the usual axioms of a \( K \)-module up to suitable coherent natural isomorphisms (cf. \[12\] for more details). Then, a 2-vector space over \( K \) is, according to these authors, a “free \( \text{Vect}_K \)-module category of finite rank”, i.e., a \( \text{Vect}_K \)-module category equivalent in the appropriate sense to \( \text{Vect}_K^n \) for some \( n \geq 0 \) (in particular, the underlying symmetric
monoidal category is a \( K \)-linear additive category equipped with the symmetric monoidal structure induced by the direct sums). When unpacked, however, this definition is quite disappointing due to the long list of required data and coherence axioms.

A simpler and essentially equivalent definition was given by Neuchl \cite{14}, who defined a (finite dimensional) 2-vector space over \( K \) as a \( K \)-linear additive category \( V \) which admits a (finite) “basis of absolutely simple objects”, i.e., a (finite) subset \( \mathbb{B} = \{X_i\}_{i \in I} \) of absolutely simple objects of \( V \) (by which I mean simple objects whose vector spaces of endomorphisms are 1-dimensional) such that, for any object \( X \) of \( V \), there exists unique natural numbers \( n_i \geq 0 \), \( i \in I \), all but a finite number of them zero, so that \( X \cong \oplus_{i \in I} X_i^{n_i} \), where \( X_i^{n_i} \) denotes any biproduct of \( n_i \) copies of \( X_i \), \( i \in I \). If such a basis \( \mathbb{B} \) exists, it is shown that \( V \) is indeed \( K \)-linear equivalent to a category \( \text{Vect}_K \) for some \( n \geq 0 \).

Another definition of 2-vector space over \( K \) was introduced almost a decade later by Baez and Crans in an attempt to define Lie 2-algebras \cite{2}. These authors defined a 2-vector space over \( K \) as a category in \( \text{Vect}_K \), and they proved that the appropriately defined 2-category of such 2-vector spaces is biequivalent to the familiar 2-category of length one complexes of vector spaces over \( K \).

The purpose of this paper is to introduce a generalized notion of 2-vector space which is in the same spirit as that of Kapranov and Voevodsky and which includes Kapranov and Voevodsky 2-vector spaces. Thus, instead of categorifying the usual notion of \( K \)-module, with \( K \) replaced by \( \text{Vect}_K \), we pay attention to the fact that any vector space is, up to isomorphism, the set \( K[X] \) of all finite formal linear combinations of elements of some set \( X \) with coefficients in \( K \), equipped with the obvious sum and multiplication by scalars, and we categorify such a constructive definition. The starting point now is going to be not a set \( X \) but a category \( C \). Then, a generalized 2-vector space over \( K \) can be defined as a \( K \)-linear additive category \( V \) which is \( K \)-linear equivalent to the free \( K \)-linear additive category generated by \( C \), for some category \( C \). By analogy with \( K[X] \), such a freely generated \( K \)-linear additive category is denoted by \( \text{Vect}_K[C] \). As shown below (Proposition \ref{proposition}), the \( K \)-linear additive categories \( \text{Vect}_K^n \) \( (n \geq 0) \) underlying Kapranov and Voevodsky 2-vector spaces are recovered (up to \( K \)-linear equivalence) as the categories \( \text{Vect}_K[C] \) for \( C \) a finite discrete category. But not all \( K \)-linear additive categories of the type \( \text{Vect}_K[C] \), for \( C \) an arbitrary category, are of this type. Thus, it is shown with examples that, in some cases, there also exists a basis whose objects, however, are non absolutely simple. Moreover, for an arbitrary category \( C \), it is likely that there exists no basis in \( \text{Vect}_K[C] \), either of absolutely simple objects or not. Arguments in favour of this possibility are discussed in the sequel.

Together with the vector space \( K[X] \), there is another vector space that can be built from an arbitrary set \( X \). Namely, the vector space \( K^X \) of all functions on \( X \) with values in \( K \). For finite sets, both vector spaces are isomorphic (actually, both are functorial and define functors \( K[-], K^{(-)} : \text{FinSets} \to \text{Vect}_K \) which are naturally isomorphic). It is then natural to consider also the analog in the category setting of this vector spaces of functions, namely, the functor categories with values in \( \text{Vect}_K \), and to compare both constructions. In contrast to what happens with vector spaces, however, they are no longer equivalent, even if we restrict to finite categories. More precisely, it will be shown that, for a finite category \( C \), \( \text{Vect}_K[C] \) is equivalent to just a certain (full) subcategory of the functor category \( \text{VECT}_K \).

My motivation for introducing this notion of generalized 2-vector space was the desire of defining an analog for 2-groups of the (Frobenius and/or Hopf) group algebras \( K[G] \). Thus, the free vector space construction \( K[X] \) is not just functorial. It actually defines a monoidal
functor $K[-] : \text{Sets} \to \text{VECT}_K$, which moreover is a left adjoint of the forgetful functor $U : \text{VECT}_K \to \text{Sets}$. The fact that $K[-]$ is monoidal implies that it indeed induces a functor $K[-] : \text{Monoids} \to \text{Alg}_K$ between the category of monoids and that of associative $K$-algebras with unit and hence, also from the category $\text{Grps}$ of groups to $\text{Alg}_K$. In a completely analogous way, if $2\text{VECT}_K$ denotes the 2-category of the above generalized 2-vector spaces, a monoidal structure on $2\text{VECT}_K$ is explicitly described in [7] and it is shown that the construction $\text{Vect}_K[\mathcal{C}]$ extends to a monoidal 2-functor $\text{Vect}_K[-] : \text{Cat} \to 2\text{VECT}_K$ which is a left 2-adjoint of the forgetful 2-functor $U : 2\text{VECT}_K \to \text{Cat}$. As in the previous setting, the fact that $\text{Vect}_K[-]$ is monoidal implies that for any 2-group $\mathcal{G}$ (more generally, for any monoidal category), the 2-vector space $\text{Vect}_K[\mathcal{G}]$ spanned by $\mathcal{G}$ inherits a 2-algebra structure. Thus, the objects $\text{Vect}_K[\mathcal{G}]$ can indeed be considered as analogs of usual group algebras.

The previous parallelism between the functor $K[-]$ and the 2-functor $\text{Vect}_K[-]$ and the view of $\text{Vect}_K[\mathcal{G}]$ as an analog of usual group algebras seems enough to make worth exploring this notion of generalized 2-vector space. It also seems worth investigating the representation theory of 2-groups on these generalized 2-vector spaces, and to compare the resulting theory with that considered in [8], where representations on Kapranov and Voevodsky 2-vector spaces are discussed. As a first step in this direction, the general linear 2-group $\text{GL}_{\text{Vect}_K[\mathcal{C}]}$ (i.e., the 2-group of $K$-linear self-equivalences of $\text{Vect}_K[\mathcal{C}]$) is completely computed (up to equivalence) in the special case where $\mathcal{C}$ is a finite (homogeneous) groupoid $\mathcal{G}$. In particular, it is shown that for any such groupoid the general linear 2-group is always split (for the definition of split 2-group, see §2.8), generalizing the situation encountered for Kapranov and Voevodsky general linear 2-groups, which correspond to the case $\mathcal{G}$ is a finite discrete category. The relation between these general linear 2-groups $\text{GL}_{\text{Vect}_K[\mathcal{C}]}$ and the 2-groups $\text{Equiv}_{\text{Cat}}(\mathcal{C})$ of self-equivalences of $\mathcal{C}$ and natural isomorphisms between them is also discussed, leading naturally to the notion of Weyl sub-2-group of $\text{GL}_{\text{Vect}_K[\mathcal{C}]}$ for a finite category $\mathcal{C}$, analogous to the Weyl subgroups of the general linear groups.

The notion of generalized 2-vector space, however, has some drawbacks with respect to Kapranov and Voevodsky 2-vector spaces. One of them is the fact that generalized 2-vector spaces are non Karoubian (hence, non abelian) categories in general. Another one is that they have no dual object in the usual sense, except when they are Kapranov and Voevodsky 2-vector spaces. Finally, the categories of morphisms in $2\text{VECT}_K$ between arbitrary generalized 2-vector spaces are not always generalized 2-vector spaces. Although on the one hand this makes the new notion a quite pathological one, it seems on the other hand the appropriate notion in order to define a natural analog of group algebras in the category setting. Another generalization of Kapranov and Voevodsky 2-vector spaces which exhibits a more pleasant behavior is given by the Karoubian completion of our generalized 2-vector spaces $\text{Vect}_K[\mathcal{C}]$. But this will be discussed elsewhere.

The outline of the paper is as follows. In Section 2, some facts concerning $K$-linear additive categories and 2-groups are reviewed (notably, the classification of 2-groups in terms of suitable 3-cocycles), and a few elementary results needed later are shown. In Section 3, the notion of generalized 2-vector space is defined as an analog for categories of the vector spaces $K[X]$, and various kinds of examples are considered. In particular, we introduce the notion of free generalized 2-vector space and discuss the possibility that there exists non free generalized 2-vector spaces. In Section 4, we consider the category analog of the vector space $K^X$ of functions on a set $X$ with values in $K$, namely the functor categories $\text{VECT}_K^{C^op}$, and
make explicit the relation with the generalized 2-vector space $\text{Vect}_K[C]$ generated by $C$. Finally, in Section 5, the general linear 2-group of the generalized 2-vector space generated by a finite homogeneous groupoid is computed, recovering Kapranov and Voevodsky general linear 2-groups as particular cases. The paper finishes with a few comments on the relation between our generalized 2-vector spaces and the notion of $\text{Vect}_K$-module category and on the above mentioned drawbacks our generalized 2-vector spaces have with respect to Kapranov and Voevodsky 2-vector spaces.

2 Preliminaries

In this section, and unless otherwise indicated, $K$ denotes an arbitrary commutative ring with unit.

§2.1. $K$-linear additive categories. Recall that a category $\mathcal{L}$ is called $K$-linear when its sets of morphisms come equipped with $K$-module structures such that all composition maps are $K$-bilinear. When $K = \mathbb{Z}$, $\mathcal{L}$ is often called a preadditive category or an $\text{Ab}$-category.

For any pair of objects $X, Y$ in a $K$-linear category $\mathcal{L}$, a biproduct (or direct sum) of $X$ and $Y$ is an object, usually denoted $X \oplus Y$, together with morphisms $\iota_X : X \to X \oplus Y$, $\iota_Y : Y \to X \oplus Y$ (called injections) and $\pi_X : X \oplus Y \to X$, $\pi_Y : X \oplus Y \to Y$ (called projections) such that

$$\pi_X \iota_X = \text{id}_X, \quad \pi_Y \iota_Y = \text{id}_Y, \quad \iota_X \pi_X + \iota_Y \pi_Y = \text{id}_X \oplus Y \tag{2.1}$$

(although the definition is usually given for preadditive categories, it actually makes sense for an arbitrary $K$, the multiplication by scalars playing no role in the definition). Any diagram in $\mathcal{L}$ of the form

$$X \xrightarrow{\iota_X} X \oplus Y \xleftarrow{\pi_X} Y$$

whose morphisms satisfy (2.1) is called a biproduct diagram. The definition extends in the obvious way to any finite set of objects $X_1, \ldots, X_n$ with $n > 2$.

A $K$-linear additive category is a $K$-linear category $\mathcal{L}$ which has a zero object and all binary biproducts (hence, all finite biproducts). A $\mathbb{Z}$-linear additive category is usually called an additive category.

For any finite biproduct $(X_1 \oplus \cdots \oplus X_n, \iota_{X_1}, \ldots, \iota_{X_n}, \pi_{X_1}, \ldots, \pi_{X_n})$ of $X_1, \ldots, X_n$, the tuples $(X_1 \oplus \cdots \oplus X_n, \iota_{X_1}, \ldots, \iota_{X_n})$ and $(X_1 \oplus \cdots \oplus X_n, \pi_{X_1}, \ldots, \pi_{X_n})$ turn out to be a coproduct and a product of $X_1, \ldots, X_n$, respectively. By the universal properties of products and coproducts, this means that the biproduct of $X_1, \ldots, X_n$ is unique up to a unique isomorphism commuting with the injections (or with the projections). Furthermore, they also make possible to describe a morphism $f : X_1 \oplus \cdots \oplus X_n \to Y_1 \oplus \cdots \oplus Y_m$ between biproduct objects in terms of an $m \times n$ matrix with entries the composite morphisms $f_{ij} = \pi_{Y_i} \circ \iota_{X_j} : X_j \to Y_i$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. Composition is then given by the formal matrix product and the composition law in $\mathcal{L}$. This notation, however, does not make explicit the injections and projections and must be used with care.
§2.2. The 2-category of $K$-linear additive categories. Given $K$-linear categories $\mathcal{L}$ and $\mathcal{L}'$, a functor $F : \mathcal{L} \to \mathcal{L}$ is called $K$-linear when it acts $K$-linearly on the $K$-modules of morphisms. If $K = \mathbb{Z}$, $F$ is called an additive functor. It is shown that $K$-linear functors $F : \mathcal{L} \to \mathcal{L}'$ map biproduct diagrams to biproduct diagrams and zero objects to zero objects.

Definition 1 Let $\text{AdCat}_K$ be the 2-category whose objects and 1- and 2-morphisms are the $K$-linear additive categories, the $K$-linear functors and all natural transformations, respectively. Composition laws and identities are the usual ones.

Observe that $\text{AdCat}_K$ is a $K$-linear 2-category, i.e., all hom-categories $\text{Hom}_{\text{AdCat}_K}(A, A')$, for $A$ and $A'$ any objects in $\text{AdCat}_K$, are $K$-linear and the composition functors are $K$-bilinear.

Among the objects in $\text{AdCat}_K$, we have the category $\text{Mod}_K$ of all $K$-modules and $K$-linear maps, and the full subcategory $\text{Mod}^1_K$ of finitely generated $K$-modules. If $K$ is a field, these categories are denoted $\text{Vect}_K$ and $\text{Vect}_K$, respectively. Objects in $\text{AdCat}_K$, for $K$ a field, further include the categories $\text{Rep}_{\text{Vect}_K}(G)$ of finite dimensional linear representations of an arbitrary group $G$.

Observe that, if $A$ and $A'$ are $K$-linear additive categories, the product $A \times A'$ inherits a $K$-linear additive structure where biproducts are given termwise. In particular, the product categories $\text{Vect}^n_K$, for any $n \geq 2$, are also objects in $\text{AdCat}_K$ for $K$ a field. Such objects play a special role in what follows.

§2.3. Krull-Schmidt $K$-linear additive categories. There is a distinguished family of objects in $\text{AdCat}_K$ characterized by the property of having a “basis”. The formal definition is as follows:

Definition 2 Let $A$ be any object in $\text{AdCat}_K$ and $\mathcal{S} = \{X_i\}_{i \in I}$ any set of objects of $A$. The $K$-linear additive subcategory of $A$ generated (or spanned) by $\mathcal{S}$, denoted by $\langle \mathcal{S} \rangle$, which contains a zero object $0$ and all biproducts $X_1 \oplus \cdots \oplus X_r$ for all objects $X_1, \ldots, X_r$ in $\mathcal{S}$ and all $r \geq 1$ (in particular, if $\mathcal{S} = \emptyset$, $\langle \mathcal{S} \rangle$ is a terminal category).

When $\langle \mathcal{S} \rangle = A$, $\mathcal{S}$ is said to be an additive generating system or to additively span $A$.

A set of objects $\mathcal{S} = \{X_i\}_{i \in I}$ of $A$ is called additively free if, whenever we have an isomorphism $X_{i_1} \oplus \cdots \oplus X_{i_r} \cong X'_{i_1} \oplus \cdots \oplus X'_{i_{r'}}$ with $X_{i_p}, X'_{i_{r'}} \in \mathcal{S}$ for all $p = 1, \ldots, r$ and $p' = 1, \ldots, r'$ ($r, r' \geq 1$), it is $r = r'$ and $X'_{i_{\sigma(p)}} = X_{i_p}$ for some permutation $\sigma \in \Sigma_r$ (in particular, the objects in $\mathcal{S}$ are non zero and pairwise nonisomorphic).

A (finite) set of objects $\mathcal{B} = \{X_i\}_{i \in I}$ of $A$ is called a (finite) basis of $A$ if it is additively free and additively spans $A$ (equivalently, if for any nonzero object $X$ there exists unique natural numbers $n_i \geq 0$, $i \in I$, all but a finite number of them zero, such that $X \cong \bigoplus_{i \in I} X_i^{n_i}$).

When all objects $X_i$ are simple (resp. simple and with 1-dimensional vector spaces of endomorphisms), $\mathcal{B}$ is said to be a basis of simple objects (resp. a basis of absolutely simple objects).

The existence of a basis in a $K$-linear additive category is related to a Krull-Schmidt type theorem. In general, such theorems have to do with the existence and uniqueness (up

---

\(^1\)If $A$ is an abelian category and $K$ is an algebraically closed field, a simple object is automatically absolutely simple by Schur’s lemma. However, this is not true in general, as it is shown below (cf. Proposition [10]).
to isomorphism and permutations) of a decomposition as a "product" of certain "indecomposable" objects of some of the objects in certain categories (mostly additive categories, but not necessarily). The concrete notions of product and indecomposable object depend on the particular version of the theorem. Thus, there is a Krull-Schmidt theorem for the category of groups in which the product is taken to be the usual direct product of groups, and where the indecomposable objects are are the groups $G \neq 1$ such that $G \cong H \times K$ implies that $H \cong 1$ or $K \cong 1$. The theorem then states that any group $G$ satisfying either the ascending or descending chain condition on normal subgroups is isomorphic to a direct product of a finite number of indecomposable groups and that, when it satisfies both conditions, this decomposition is unique up to isomorphism and permutation of the factors (see for ex. [11]).

For $K$-linear additive categories, one usually takes the biproduct as the appropriate notion of product, and the indecomposable objects are the objects $X \not\cong 0$ such that $X \cong X' \oplus X''$ implies $X' \cong 0$ or $X'' \cong 0$. Clearly, if a basis $B = \{X_i\}_{i \in I}$ indeed exists in such a category, the objects $X_i$ are necessarily indecomposable in this sense (otherwise, it will be $X_i \cong X \oplus X'$ for some $X, X' \not\cong 0$ and, hence, $X_i \cong X_{i_1} \oplus \cdots X_{i_k} \oplus X'_{i_1} \oplus \cdots X'_{i'_k}$ for some $k, k' \geq 1$, in contradiction with the additive freeness of $B$). This suggests introducing the following terminology:

**Definition 3** A Krull-Schmidt $K$-linear additive category is a $K$-linear additive category which has a basis.

Notice that any basis $B$ in a Krull-Schmidt $K$-linear additive category $A$ necessarily includes one (and only one) representative from each isomorphism class of indecomposable objects (otherwise, $B$ will not span $A$ additively). Hence, in contrast to what happens with vector spaces, the basis in a Krull-Schmidt $K$-linear additive category is unique up to isomorphism. More precisely, we have the following

**Proposition 4** If $A$ is a Krull-Schmidt $K$-linear additive category, there exists a unique basis up to isomorphism, given by one representative in each isomorphism class of indecomposable objects. In particular, all bases of $A$ have the same cardinal (called the rank of $A$ and denoted $\text{rk}(A)$).

**Example 5** For any field $K$ and $n \geq 1$, $\text{Vect}^n_K$ is a Krull-Schmidt $K$-linear additive category, a basis being given by $B = \{K(i, n), \ i = 1, \ldots, n\}$, with $K(i, n) = (0, \ldots, \delta^n, \ldots, 0)$ for all $i = 1, \ldots, n$.

**Example 6** If $K$ is an algebraically closed field and $G$ a finite group, $\text{Rep}_{\text{Vect}_K}(G)$ is also a Krull-Schmidt $K$-linear additive category, a basis being given by one representative in each isomorphism class of simple objects, usually called the irreducible representations (there are as many such isomorphism classes as conjugacy classes in $G$; see, for ex., Fulton-Harris [10]).

Let us finally point out that Krull-Schmidt $K$-linear additive categories can also be characterized in terms of the commutative (additive) monoid with the isomorphism classes of objects as elements and with the sum induced by the biproduct of corresponding representative objects. Specifically, if we denote by $M(A)$ this monoid, for any $K$-linear additive category $A$, we have the following obvious result:
Proposition 7 A $K$-linear additive category $A$ is of the Krull-Schmidt type if and only if $\text{M}(A)$ is free.

For example, for $n \geq 1$ it is $\text{M}(\text{Vect}_K^n) \cong \mathbb{N}^n$, while $\text{M}(\text{Rep}_{\text{Vect}_K}(G)) \cong \mathbb{N}^r$ with $r$ the number of conjugacy classes in $G$.

§2.4. Free $K$-linear categories. For any category $C$, the free $K$-linear category (or free preadditive category when $K = \mathbb{Z}$) generated by $C$ is the $K$-linear category $K[C]$ with the same objects as $C$, with vector spaces of morphisms

$$\text{Hom}_{K[C]}(X, X') := K[\text{Hom}_C(X, X')]$$

and with composition law given by the $K$-bilinear extension of the composition law in $C$ (identities are the obvious ones).

There is an obvious inclusion functor $k_C : C \to K[C]$, and the pair $(K[C], k_C)$ has the following universal property, which follows from the universal property of free $K$-modules: for any $K$-linear category $L$ and any functor $F : C \to L$, there exists a unique $K$-linear functor $\overline{F} : K[C] \to L$, called the $K$-linear extension of $F$, such that $F = \overline{F} k_C$. Furthermore, any natural transformation $\tau : F \Rightarrow G : C \to L$ defines a natural transformation between the $K$-linear extensions $\overline{\tau} : \overline{F} \Rightarrow \overline{G}$.

Note also that the construction $K[C]$ preserves coproducts, i.e., for an arbitrary family of categories $\{C_i\}_{i \in I}$ it is

$$K[\bigsqcup_{i \in I} C_i] \cong K \bigsqcup_{i \in I} K[C_i]$$ (2.2)

where $\cong_K$ denotes $K$-linear equivalence and $\bigsqcup$ denotes the coproduct of $K$-linear categories, given by the usual disjoint union of categories except that for pairs of objects in different categories the corresponding hom-set in the coproduct is the zero vector space, instead of the empty set.

In general, it is possible that non isomorphic objects in $C$ become isomorphic in $K[C]$. This suggests introducing the following

Definition 8 A category $C$ is called $K$-stable if isomorphic objects in $K[C]$ are also isomorphic in $C$.

Examples of categories which are $K$-stable for any $K$ include all groupoids and all free categories. Another example which will be needed later (see Lemma 44) is provided by the following

Proposition 9 Let $C$ be a category such that, for any object $X$ of $C$, an endomorphism $f : X \to X$ is an isomorphism if and only if it is a monomorphism. Then, $C$ is $K$-stable for any $K$.

Proof. Suppose $X, Y$ are isomorphic objects in $K[C]$, and let $\sum_i \lambda_i f_i : X \to Y$ be an isomorphism, with inverse $\sum_j \mu_j g_j : Y \to X$. In particular, it is

$$\sum_{i,j} \lambda_i \mu_j g_j f_i = \text{id}_X, \quad \sum_{i,j} \lambda_i \mu_j f_i g_j = \text{id}_Y$$
Since the hom-sets in \( \mathcal{C} \) constitute linear bases for the corresponding vector spaces of morphisms in \( K[\mathcal{C}] \), it follows that there exists pairs \((i_0, j_0)\) and \((i_1, j_1)\) such that \( g_{j_0}f_{i_0} = \text{id}_X \) and \( f_{i_1}g_{j_1} = \text{id}_Y \). In particular, both \( f_{i_0} \) and \( g_{j_1} \) are sections (hence, monomorphisms) and consequently, the composite \( f_{i_0}g_{j_1} : X \to X \) is a monomorphism. By hypothesis, \( f_{i_0}g_{j_1} \) is then an isomorphism, from which we conclude that \( f_{i_0} \) is an epimorphism. But a section which is at the same time an epimorphism is necessarily an isomorphism. Therefore, \( X \cong Y \) already in \( \mathcal{C} \).

Let us finally remark that, when the category \( \mathcal{C} \) is already \( K \)-linear, the \( K \)-linear structure on \( K[\mathcal{C}] \) has nothing to do with that on \( \mathcal{C} \). Thus, it is a priori possible that the biproduct of two objects \( X, Y \) exists in \( \mathcal{C} \) while it does not exist in \( K[\mathcal{C}] \), and conversely. Similary, there is no zero object in \( K[\mathcal{C}] \) although it may exists one in \( \mathcal{C} \).

### §2.5. Free additive categories.

Suppose we are now given a \( K \)-linear (a preadditive when \( K = \mathbb{Z} \)) category \( \mathcal{L} \). The free additive category generated by \( \mathcal{L} \) is the category \( \text{Add}(\mathcal{L}) \) with objects all finite (possibly empty) ordered sequences of objects in \( \mathcal{L} \) and with arrows the matrices of arrows in \( \mathcal{L} \). More explicitly, a morphism in \( \text{Add}(\mathcal{L}) \) between two nonempty sequences \((X_1, \ldots, X_n)\) and \((X'_1, \ldots, X'_{n'})\) is an \( n' \times n \) matrix \( A \) whose \((i', j)\)-entry is \( A_{i'i} \in \text{Hom}_{\mathcal{L}}(X_i, X'_j) \) (if one or both sequences are empty, the corresponding hom-set is a singleton, whose element is generically denoted by 0 and called a zero morphism). Composition is given by the formal matrix product and the composition law in \( \mathcal{L} \) when all involved objects are nonempty, it is equal to the appropriate zero matrix when only the middle object is empty and it is the corresponding zero morphism otherwise.

\( \text{Add}(\mathcal{L}) \) has the obvious \( K \)-linear structure inherited from \( \mathcal{L} \) and the empty sequence as a zero object, and it is an additive category, with biproducts given, for example, by concatenation of sequences. There may exists, however, other zero objects (for instance, if \( \mathcal{L} \) has a zero object \( 0 \), any sequence \((0, \ldots, 0)\) is also a zero object for \( \text{Add}(\mathcal{L}) \)), and other biproducts (for instance, if \( X, X' \) already have a biproduct \( X \oplus X' \) in \( \mathcal{L} \), a biproduct of \((X)\) and \((X')\) is also given by the length one sequence \((X \oplus X')\); see Proposition 10). Note also that any sequence \((X_1, \ldots, X_k)\) of length \( n \geq 2 \) can be thought of as the object part of a biproduct of the length one sequences \( S_i = (X_i), \ldots, S_k = (X_k) \), and that the matrix \( A = (A_{i'i}) \) giving a morphism \((X_1, \ldots, X_n) \to (X'_1, \ldots, X'_{n'})\) coincides with the matrix representation of \( A \) with respect to the corresponding projections and injections.

Two easy facts concerning free additive categories and which will be useful in the sequel are the following:

**Proposition 10** Let \( \mathcal{L}, \mathcal{L}' \) be \( K \)-linear categories. Then:

1. There is a \( K \)-linear equivalence \( \text{Add}(\mathcal{L}) \times \text{Add}(\mathcal{L}) \cong K \text{Add}(\mathcal{L}) \uplus \mathcal{L}' \), where \( \uplus \) denotes the coproduct of \( K \)-linear categories (see §2.4).

2. For any objects \( X, X_1, \ldots, X_n \) in \( \mathcal{L} \), the following statements are equivalent:
   
   (a) \( X \cong (X_1, \ldots, X_n) \) in \( \text{Add}(\mathcal{L}) \).

   (b) The biproduct of \( X_1, \ldots, X_n \) exists in \( \mathcal{L} \) and \( X \cong X_1 \oplus \cdots \oplus X_n \).
A $K$-linear equivalence $E : \text{Add}(\mathcal{L}) \times \text{Add}(\mathcal{L}') \to \text{Add}(\mathcal{L} \sqcup \mathcal{L}')$ is the functor which maps the object $((X_1, \ldots, X_n), (X'_1, \ldots, X'_{n'}))$ to $(X_1, \ldots, X_n, X'_1, \ldots, X'_{n'})$ and a morphism $(A, A')$ to the morphism $A \oplus A'$, the usual direct sum of matrices. Such a functor is indeed essentially surjective because any object $(Y_1, \ldots, Y_m)$ in $\text{Add}(\mathcal{L} \sqcup \mathcal{L}')$ is isomorphic to any of its permuted sequences. The proof of (ii) is an easy check left to the reader. \hfill \Box

A feature worth emphasizing of the free additive categories $\text{Add}(\mathcal{L})$ is that the associated monoid $M(\text{Add}(\mathcal{L}))$ is not necessarily equal to the free commutative monoid generated by the isomorphism classes of objects in $\mathcal{L}$. Thus, for an arbitrary $K$-linear category $\mathcal{L}$, it may happen that two objects $(X_1, \ldots, X_n)$ and $(X'_1, \ldots, X'_{n'})$ in $\text{Add}(\mathcal{L})$ are isomorphic even when $n \neq n'$. Examples of this naturally arise when $\mathcal{L}$ is already an additive category, as shown by the previous Proposition. Actually, it seems to be false that $\text{Add}(\mathcal{L})$ is always a Krull-Schmidt $K$-linear additive category, in spite that it is freely generated as an additive category.

Finally, let $a_\mathcal{L} : \mathcal{L} \to \text{Add}(\mathcal{L})$ be the $K$-linear embedding mapping $\mathcal{L}$ into the full subcategory with objects the length one sequences. Then, the pair $(\text{Add}(\mathcal{L}), a_\mathcal{L})$ has the following universal property, which justifies the name given to $\text{Add}(\mathcal{L})$:

**Proposition 11** For any $K$-linear additive category $\mathcal{A}$ and any $K$-linear functor $F : \mathcal{L} \to \mathcal{A}$, there exists a $K$-linear functor $\tilde{F} : \text{Add}(\mathcal{L}) \to \mathcal{A}$, unique up to isomorphism, such that $F = \tilde{F} \circ a_\mathcal{L}$ (we shall call $\tilde{F}$ a $K$-linear extension of $F$). Furthermore, given a second $K$-linear functor $F' : \mathcal{L} \to \mathcal{A}$, a natural transformation $\tau : F \Rightarrow F'$ and any $K$-linear extensions $\tilde{F}$ and $\tilde{F}'$ of $F$ and $F'$, respectively, there exists a unique natural transformation $\tilde{\tau} : \tilde{F} \Rightarrow \tilde{F}'$ extending $\tau$ (i.e., such that $\tau = \tilde{\tau} \circ 1_{a_\mathcal{L}}$), and $\tilde{\tau}$ is an isomorphism if $\tau$ is an isomorphism.

§2.6. Notion of 2-group and the 2-category $\text{2Grp}$ of 2-groups. There are various definitions of (weak) 2-group, depending on the amount of structure assumed on it. In this paper, by a 2-group (also called a categorical group) we shall mean a monoidal category $(\mathcal{G}, \otimes, I, a, l, r)$ satisfying the following additional conditions: (1) $\mathcal{G}$ is a groupoid, and (2) any object $A$ of $\mathcal{G}$ is invertible, in the sense that the functors $- \otimes A, A \otimes - : \mathcal{G} \to \mathcal{G}$ are equivalences. When the monoidal category is strict (i.e., $a, l$ and $r$ are identities) and any object $A$ is strictly invertible, in the sense that the functors $- \otimes A, A \otimes -$ are not only equivalences but isomorphisms, the 2-group is said to be strict. For example, if $\mathcal{C}$ is any bicategory and $X$ any object of $\mathcal{C}$, the category $\text{Equiv}_{\mathcal{C}}(X)$ with objects the autoequivalences $f : X \to X$ and morphisms all 2-isomorphisms between these is a 2-group, the composition and the tensor product being respectively given by the vertical composition of 2-morphisms and the composition of 1-arrows and horizontal composition of 2-arrows, and with $\text{id}_X$ as unit object (actually, any 2-group is of this type for some bicategory $\mathcal{C}$ and some object $X$ of $\mathcal{C}$). In case $\mathcal{C}$ is a strict 2-category, the full subcategory $\text{Aut}_{\mathcal{C}}(X)$ of $\text{Equiv}_{\mathcal{C}}(X)$ with objects only the strict invertible endomorphisms of $X$ is a strict 2-group.

2-groups are the objects of a 2-category $\text{2Grp}$, whose 1-arrows are the monoidal functors between the underlying monoidal categories and whose 2-arrows are the monoidal natural transformations between these (for the precise definitions, see for instance [S]). It is shown that any 2-group is equivalent (in $\text{2Grp}$) to a strict 2-group.

§2.7. Classification of 2-groups up to equivalence. It is a fundamental result in the theory of 2-groups, first proved apparently by Sinh [L], that these are completely classified
(up to the corresponding notion of equivalence in \(2\text{Grp}\)) by triples \((G, M, [\alpha])\), with \(G\) a group, \(M\) a \(G\)-module and \([\alpha] \in H^3(G, M)\). For a given 2-group \(\mathbb{G}\), the corresponding group \(G\) and \(G\)-module \(M\) are usually denoted by \(\pi_0(\mathbb{G})\) and \(\pi_1(\mathbb{G})\) and called the homotopy groups of \(\mathbb{G}\). They are respectively equal to the group of isomorphism classes of objects of \(\mathbb{G}\) (thought of as abelian groups when \(\mathbb{G}\) is a strict 2-group) and the group \(\text{Aut}_\mathbb{G}(1)\) of automorphisms of the unit object (it is an abelian group with the product given by the composition of automorphisms). A basic feature of 2-groups is that they are “homogeneous”, in the sense that \(\pi_1(\mathbb{G}) \cong \text{Aut}_\mathbb{G}(A)\) for any object \(A\) of \(\mathbb{G}\). There are two particularly important such isomorphisms of groups, denoted \(\delta_A\) and \(\gamma_A\) and defined by

\[
\delta_A(u) = r_A \circ (\text{id}_A \otimes u) \circ r_A^{-1}, \quad \gamma_A(u) = l_A \circ (u \otimes \text{id}_A) \circ l_A^{-1}
\]  

(2.3)

for all \(u \in \pi_1(\mathbb{G})\). In terms of these isomorphisms, the action of \(\pi_0(\mathbb{G})\) on \(\pi_1(\mathbb{G})\) is given by

\[
[A] \cdot u = \gamma_A^{-1}(\delta_A(u))
\]  

(2.4)

for any representative \(A \in [A]\). Finally, the cohomology class \([\alpha] \in H^3(\pi_0(\mathbb{G}), \pi_1(\mathbb{G}))\) classifying \(\mathbb{G}\) (also called the Postnikov invariant of \(\mathbb{G}\); see [3]) is basically determined by the associator \(\alpha\) of the underlying monoidal category. More explicitly, let us choose a representative \(A_g\) for each class \(g = [A_g] \in \pi_0(\mathbb{G})\), and for any other object \(A' \in g\), choose an isomorphism \(\iota_{A'} : A' \to A_g\), with \(\iota_{A_g} = \text{id}_{A_g}\). Then, a classifying 3-cocycle \(\alpha \in Z^3(\pi_0(\mathbb{G}), \pi_1(\mathbb{G}))\) is

\[
\alpha(g_1, g_2, g_3) = \gamma_{A_{g_1}g_2g_3}^{-1}(\alpha_{g_1, g_2, g_3}) \in \pi_1(\mathbb{G})
\]  

(2.5)

with \(\alpha_{g_1, g_2, g_3} \in \text{Aut}_\mathbb{G}(A_{g_1}g_2g_3)\) defined by

\[
\alpha_{g_1, g_2, g_3} = \iota_{A_{g_1}g_2g_3} \circ (\text{id}_{A_{g_1}} \otimes \iota_{A_{g_2}g_3}) \circ a_{A_{g_1}g_2, A_{g_2}g_3} \circ \iota_{A_{g_1}g_2}^{-1} \otimes \iota_{A_{g_2}g_3}^{-1} \circ \text{id}_{A_{g_1}g_2g_3}
\]  

(2.6)

As a consequence of the pentagon axiom on \(\alpha\), it is seen that the map \(\alpha : \pi_0(\mathbb{G})^3 \to \pi_1(\mathbb{G})\) defined in this way is indeed a (normalized) 3-cocycle, whose cohomology class turns out to be independent of the chosen representatives \(A_g\) and isomorphisms \(\iota_{A'}\).

§2.8. Split 2-groups. A particularly simple type of 2-groups are those for which the Postnikov invariant is trivial, i.e. \([\alpha] = 0\). It is easily seen (cf. [3]) that these are exactly the 2-groups equivalent (in \(2\text{Grp}\)) to skeletal strict 2-groups, i.e., to strict 2-groups whose underlying categories are skeletal (isomorphic objects are equal). These 2-groups are called split because a strict 2-group is of this kind when a certain exact sequence of 2-groups splits. Specifically, for any group \(G\) and any abelian group \(A\), let \(G[0]\) be the group \(G\) thought of as a discrete 2-group, and let \(A[1]\) be the group \(A\) thought of as a 2-group with only one object and \(A\) as group of automorphisms of the unique object. Then, for any 2-group \(\mathbb{G}\) there is an inclusion of 2-groups \(\pi_1(\mathbb{G})[1] \to \mathbb{G}\) and a “projection” morphism \(p : \mathbb{G} \to \pi_0(\mathbb{G})[0]\), the last one mapping each object \(A\) of \(\mathbb{G}\) to the corresponding isomorphism class \([A]\). Together, they define a sequence of 2-group morphisms

\[
1 \to \pi_1(\mathbb{G})[1] \to \mathbb{G} \to \pi_0(\mathbb{G})[0] \to 1
\]  

(2.7)

which is exact in the sense that \(\pi_1(\mathbb{G})[1]\) is equivalent to the kernel of \(p\) (i.e., the homotopy fiber of \(p\) over the unit object \([I]\) of \(\pi_0(\mathbb{G})[0]\)). We have then the following:
Proposition 12 Suppose $G$ is a strict 2-group (as pointed out before, this implies no loss of generality). Then, if there exists a strict section for the exact sequence (2.7) (i.e., a strict monoidal functor $S : \pi_0(G)[0] \to G$ such that $pS = \text{id}_{\pi_0(G)[0]}$), $G$ is split.

Proof. The existence of such a strict monoidal functor $S$ amounts to the existence of a choice of representatives $A_g$ compatible with the tensor product, i.e., such that $A_{g_1} \otimes A_{g_2} = A_{g_1 g_2}$ and $A_e = I$. In this case, it readily follows from (2.3), (2.5), (2.6) and the fact that $G$ is strict that $\alpha$ maps each triple $(g_1, g_2, g_3)$ to the identity of $\pi_1(G)$ and hence, $[\alpha] = 0$. $\square$

Note that for strict 2-groups $G$, not only $\pi_0(G)$ but also the set $|G|$ of objects of $G$ inherits a group structure given by the tensor product. In these cases, the group $\pi_0(G)$ is nothing but the quotient of $|G|$ modulo the normal subgroup of objects isomorphic to the unit object $I$ of $G$ (see §3.7). The existence of the above strict section $S : \pi_0(G)[0] \to G$ then corresponds to the existence of a group morphism section $s : \pi_0(G) \to |G|$.

3 The 2-category $2GVECT_K$

From now on, $K$ denotes an arbitrary field.

§3.1. Notion of generalized 2-vector space, universal property and stability under categorical products. For any nonempty category $C$, let

$$\text{Vect}_K[C] := \text{Add}(K[C])$$

Thus, an object in $\text{Vect}_K[C]$ is any finite (possibly empty) ordered sequence $(X_1, \ldots, X_n)$ ($(X_i)_n$ for short) of objects of $C$, with $n \geq 0$, and a morphism between two nonempty objects $(X_i)_n$ and $(X'_i)_n'$ is an $n' \times n$ matrix $A = (A_{i,i'})$ with $(i', i)$-entry $A_{i,i'} \in K[\text{Hom}_C(X_i, X'_i)]$, i.e., of the form

$$A_{i,i'} = \sum_{\alpha=1}^{d_{i,i'}} \lambda(i', i)_{\alpha} f^\alpha_{i,i'}$$

where $f^\alpha_{i,i'} : X_i \to X'_i$ is a morphism in $C$ for all $\alpha$. Composition is given by the composition law in $C$ (extended $K$-bilinearly) and the formal matrix product. Observe that the empty sequence is the unique zero object of $\text{Vect}_K[C]$, because $K[C]$ has no zero object. Finally, we shall agree that $\text{Vect}_K[\emptyset]$ is the terminal category 1.

$\text{Vect}_K[C]$ may be thought of as an analog of the vector spaces $K[X]$ constructed from arbitrary sets $X$ (more properly, it is an analog of $\mathbb{N}[X]$). The fact that any vector space is of this kind up to isomorphism suggests the following definition of generalized 2-vector space:

Definition 13 A generalized 2-vector space over $K$ is a $K$-linear additive category $\mathbb{V}$ which is $K$-linear equivalent to $\text{Vect}_K[C]$ for some category $C$. When $C$ is finite, $\text{Vect}_K[C]$ is called a finitely generated generalized 2-vector space. In particular, terminal categories (isomorphic to $\text{Vect}_K[\emptyset]$) are called zero 2-vector spaces.
Generalized 2-vector spaces have the following universal property, which is an immediate consequence of the universal properties of the pairs \((K'[C], k_C)\) and \((\text{Add}(K'[C]), a_{K'[C]})\) and the analog of the fact that a \(K\)-linear map between vector spaces is uniquely determined by the image of a basis:

**Proposition 14** For any category \(C\), let \(\beta_C : C \to \text{Vect}_K[C]\) denote the canonical embedding given by the composite \(C \xrightarrow{k} K[C] \xrightarrow{a_K[C]} \text{Vect}_K[C]\). Then, for any \(K\)-linear additive category \(A\) and any functor \(F : C \to A\), there exists a \(K\)-linear functor \(\tilde{F} : \text{Vect}_K[C] \to A\) (called a \(K\)-linear extension of \(F\)), unique up to isomorphism, such that \(F = \tilde{F} \beta_C\). Furthermore, given a second functor \(F' : C \to A\), a natural transformation (resp. isomorphism) \(\tau : F \Rightarrow F'\) and any \(K\)-linear extensions \(\tilde{F}\) and \(\tilde{F}'\) of \(F\) and \(F'\), respectively, there exists a unique natural transformation (resp. isomorphism) \(\bar{\tau} : \tilde{F} \Rightarrow \tilde{F}'\) such that \(\tau = \bar{\tau} \circ 1_{\beta_C}\).

It readily follows from this universal property that, for any category \(C\) and any \(K\)-linear additive category \(A\), there exists a \(K\)-linear equivalence

\[
\text{Hom}_{\text{AdCat}_K}(\text{Vect}_K[C], A) \simeq_K \text{Hom}_{\text{Cat}}(C, A)
\]

Furthermore, it is easily seen that the construction \(\text{Vect}_K[C]\) extends to a 2-functor \(\text{Vect}_K[-] : \text{Cat} \to \text{AdCat}_K\) (for details, see [7]). Thus, if \(C\) and \(C'\) are equivalent categories, the corresponding generalized 2-vector spaces \(\text{Vect}_K[C]\) and \(\text{Vect}_K[C']\) are also equivalent objects in \(\text{AdCat}_K\). Hence, it can be assumed without loss of generality that all involved categories are skeletal.

Note also that generalized 2-vector spaces are stable under finite products. More explicitly, we have the following

**Proposition 15** For any categories \(C_1, \ldots, C_n\) there is a \(K\)-linear equivalence

\[
\text{Vect}_K[C_1] \times \cdots \times \text{Vect}_K[C_n] \simeq_K \text{Vect}_K[C_1 \sqcup \cdots \sqcup C_n]
\]

In particular, the cartesian product of a finite number of generalized 2-vector spaces is a generalized 2-vector space.

*Proof.* It is a direct consequence of (3.2) and Proposition 14(i). \(\square\)

### §3.2. Kapranov and Voevodsky 2-vector spaces.

The simplest examples of generalized 2-vector spaces are those generated by finite discrete categories. These turn out to be the \(K\)-linear additive categories \(\text{Vect}_K^n\) (\(n \geq 0\)) underlying usual Kapranov and Voevodsky 2-vector spaces. More generally, for any (non necessarily finite) set \(X\) and any \(K\)-linear additive category \(A\), let \(A^X\) be the full subcategory of \(\prod_{x \in X} A\) with objects the ordered sequences \((A_x)_{x \in X}\) of objects in \(A\) such that \(A_x = 0\) (the zero object) for all but a finite number of \(x \in X\). Then, we have the following

**Proposition 16** For any set \(X\), let \((X[0])\) denote the corresponding discrete category. Then, \(\text{Vect}_K[X[0]] \simeq_K \text{Vect}_K^{\mathbb{K}^X}\), where \(\simeq_K\) means \(K\)-linear equivalence. In particular, for a finite set \(X\) of cardinal \(n \geq 1\), it is \(\text{Vect}_K[X[0]] \simeq_K \text{Vect}_K^n\).
Proof. Let $\text{Mat}_K$ be the category with objects the natural numbers and morphisms between non zero objects $n \to m$ the $m \times n$ matrices with entries in $K$. $\text{Mat}_K$ is $K$-linear equivalent to $\text{Vect}_K$, so that it is enough to see that $\text{Mat}_K^{\oplus X} \simeq_K \text{Vect}_K[X[0]]$. Such an equivalence is defined as follows:

- map the object $(k_x)_{x \in X}$ of $\text{Mat}_K^{\oplus X}$ to the finite sequence $(x, k_x, x)_{x \in X}$ (in particular, the zero object $(\ldots, 0, \ldots)$ is mapped to the empty sequence);
- map the morphism $(A_x)_{x \in X} : (k_x)_{x \in X} \to (k'_x)_{x \in X}$, with $A_x$ a $k_x \times k'_x$ matrix, to the morphism $A : (x, k_x, x)_{x \in X} \to (x, k'_x, x)_{x \in X}$ given by

$$A = \begin{pmatrix} A_{x_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_{x_n} \end{pmatrix}$$

where $x_1, \ldots, x_n$ are the elements $x \in X$ for which $k_x, k'_x \neq 0$ (here, the entries in $A_x$ have to be thought of as the corresponding scalar multiples of $\text{id}_{x_i}$, for each $i = 1, \ldots, n$).

The functor so defined is clearly fully faithful and it is essentially surjective because any nonempty sequence $(x_1, \ldots, x_n)$ is isomorphic in $\text{Vect}_K[X[0]]$ to any one of its permuted sequences. \hfill \Box

A fundamental feature of the finitely generated 2-vector spaces $\text{Vect}_K^n (n \geq 1)$ is that they have a finite basis of absolutely simple objects (see Definition 2), given by $B = \{K(i, n)\}_{i=1}^n$, with $K(i, n) = (0, \ldots, \underbrace{i}_{i}, \ldots, 0)$, $i = 1, \ldots, n$ (see Example 1). In fact, this property characterizes them up to $K$-linear equivalence. Indeed, any $K$-linear additive category $A$ having a finite (possibly empty) basis of absolutely simple objects turns out to be $K$-linear equivalent to $\text{Vect}_K^n$ for some $n \geq 0$. This suggests introducing the following

Definition 17 A generalized 2-vector space $\mathbb{V}$ is called an absolutely simple free 2-vector space if it has a basis of absolutely simple objects. If, moreover, the basis (which is unique up to isomorphism by Proposition 4) is finite, $\mathbb{V}$ is called of finite rank or a Kapranov and Voevodsky 2-vector space.

Example 18 More examples of generalized 2-vector spaces of the Kapranov and Voevodsky type are given by the above mentioned representation categories of finite groups; see Example 2.

§3.3. Generalized free 2-vector spaces. Not every generalized 2-vector space is a Kapranov and Voevodsky 2-vector space. For example, there exists generalized 2-vector spaces which have a basis but whose basic objects are non absolutely simple. Among such generalized 2-vector spaces, we have those generated by non trivial monoids. Specifically, let $M[1]$, for any monoid $M$, be the category with only one object $*$ and the elements of $M$ as morphisms. Then, we have the following

Proposition 19 For any non trivial monoid $M$, $\mathbb{B} = \{(*)\}$ is a basis of non absolutely simple objects for $\text{Vect}_K[M[1]]$. 

13
Proof. \( S = \{ (\ast) \} \) clearly spans additively the category \( \text{Vect}_K[M[1]] \), so that we only need to see that it is additively free, which in this case means to see that \( \text{Vect}_K[M[1]] \) is skeletal, i.e., that \((\ast, k), (\ast, k') \in S \) implies \( k = k' \). To see this, observe that a morphism \( A = (m_{i,j}) : (\ast, k), (\ast, k') \rightarrow (\ast, k''), (\ast) \) can be thought of as a \( K[M]\)-linear map between the free \( K[M]\)-modules \( K[M]^k \) and \( K[M]^{k'} \), with the composition in \( \text{Vect}_K[M[1]] \) corresponding to the composition of linear maps. Hence, the condition \((\ast, k), (\ast) \in S \) is equivalent to the condition \( K[M]^k \cong K[M]^{k'} \) as free \( K[M]\)-modules. The result follows then from the fact that the ring \( K[M] \) has the invariant dimension property \(^2\), and this in turn follows from the fact that \( K[M] \) has a homomorphic image, namely \( K \), which is a division ring \((\text{see Hungerford [11], Ch. IV, \S 2})\). Furthermore, \((\ast) \) is clearly a non absolutely simple object because it has \( K[M] \) as vector space of endomorphisms, which is of dimension \( > 1 \) for any non trivial monoid \( M \) \((\text{if } M \text{ is trivial, we recover Kapranov and Voevodsky 2-vector space } \text{Vect}_K)\). In fact, \((\ast) \) is neither a simple object, because any element \( a \in K[M] \) which is left cancellable but not a unit defines a monomorphism \( a : (\ast) \rightarrow (\ast) \) which is not an isomorphism. \( \square \)

This suggests introducing the following more general notion of a free 2-vector space:

**Definition 20** A generalized 2-vector space \( \mathbb{V} \) is called free when the underlying \( K\)-linear additive category is of the Krull-Schmidt type \((i.e., it has a basis, possibly of non absolutely simple objects)\). If it has a finite basis, it is called of finite rank \((equal to the cardinal of any basis)\). Otherwise, it is called of infinite rank.

Notice that, defined in this way, a generalized free 2-vector space may simultaneously be non finitely generated and of finite rank. For instance, if \( M \) is a non finite monoid, it follows from the previous result that \( \text{Vect}_K[M[1]] \) is a non finitely generated free 2-vector space of rank one.

Generalized free 2-vector spaces, as well as generalized free 2-vector spaces of finite rank, constitute a subclass of the class of all generalized 2-vector spaces which remains stable under finite products, i.e.

**Proposition 21** If \( \mathbb{V}, \mathbb{V}' \) are both \((finite rank)\) free 2-vector spaces, with bases \( \mathbb{B} = \{ X_i \}_{i \in I} \) and \( \mathbb{B}' = \{ X'_i \}_{i' \in I'} \), respectively, then \( \mathbb{V} \times \mathbb{V}' \) is also a \((finite rank)\) free 2-vector space, with basis \( \mathbb{B} \cup \mathbb{B}' = \{ (X_i, 0') \}_{i \in I} \cup \{ (0, X'_i) \}_{i' \in I'} \) \((0 \text{ and } 0' \text{ stand for zero objects in } \mathbb{V} \text{ and } \mathbb{V}', \text{ respectively})\).

*Proof*. It is easy check left to the reader. \( \square \)

The above examples \( \text{Vect}_K[M[1]] \) of generalized finite rank free 2-vector spaces which are not of the Kapranov and Voevodsky type are special cases of the following

**Proposition 22** For any category \( \mathcal{C} \) with finitely many isomorphism classes of objects and such that \( \text{Hom}_\mathcal{C}(X, X') = \emptyset \) when \( X \ncong X' \), \( \text{Vect}_K[\mathcal{C}] \) is a generalized finite rank free 2-vector space, a basis being given by any family of length one sequences \( \{ (X_1), \ldots, (X_n) \} \) with \( \{ X_1, \ldots, X_n \} \) a set of representative objects of \( \mathcal{C} \).

\(^2\)Recall that a ring \( R \) is said to have the invariant dimension property if for any free \( R\)-module \( F \), any two bases of \( F \) have the same cardinality.
Proof. Let \( \{X_1, \ldots, X_n\} \) be any set of representative objects in \( \mathcal{C} \). Since \( \text{Hom}_\mathcal{C}(X, X') = \emptyset \) when \( X \not\cong X' \), we have \( \mathcal{C} \simeq \biguplus_{i=1}^n M_i[1] \), where \( M_i = \text{End}_\mathcal{C}(X_i) \) \( (i = 1, \ldots, n) \). Then, the statement readily follows from Proposition 15, 19 and 21. ✷

Finally, observe that, besides the universal property in Proposition 14, generalized free 2-vector spaces satisfy the following additional universal property:

**Proposition 23** Let \( \mathcal{V} \) be a generalized free 2-vector space, with basis \( \mathbb{B} \), and let \( \mathcal{V}^\mathbb{B} \) be the full subcategory of \( \mathcal{V} \) with \( \mathbb{B} \) as set of objects. Then, for any \( K \)-linear additive category \( \mathcal{A} \) and any \( K \)-linear functor \( F : \mathcal{V} \to \mathcal{A} \), there exists a \( K \)-linear extension \( \tilde{F} : \text{Vect}_K[\mathbb{C}] \to \mathcal{A} \) (called also a \( K \)-linear extension of \( F \)), unique up to isomorphism, such that \( F = \tilde{F} \beta_c \). Furthermore, given a second \( K \)-linear functor \( F' : \mathcal{V} \to \mathcal{A} \), a natural transformation \( \tau : F \Rightarrow F' \) and any \( K \)-linear extensions \( \tilde{F} \) and \( \tilde{F}' \) of \( F \) and \( F' \), respectively, \( \tau \) extends uniquely to a natural transformation \( \tilde{\tau} : \tilde{F} \Rightarrow \tilde{F}' \).

Proof. It is left to the reader. □

§3.4. On the existence of generalized non-free 2-vector spaces. At this point, the question naturally arises whether any generalized 2-vector space is free. \(^3\)

\( \text{Vect}_K[\mathbb{C}] \) always has an additive generating system, whatever the category \( \mathbb{C} \) is. For instance, the set of all length one sequences. Moreover, since the empty sequence is the unique zero object in \( \text{Vect}_K[\mathbb{C}] \), all indecomposable sequences \( S \) are of length one. Hence, determining if it indeed exists a basis in \( \text{Vect}_K[\mathbb{C}] \) and finding it explicitly involves in the following two steps: (1) to determine which length one sequences are indecomposable, and (2) to see that the indecomposable length one sequences are indeed additively free.

As regards the first step, note that in all examples of generalized free 2-vector spaces considered until now (as well as in those considered in §3.5), all length one sequences turn out to be indecomposable, so that a basis is given by any representative set of objects of \( \mathbb{C} \) (see Proposition 4).

The following result gives sufficient conditions on \( \mathbb{C} \), different from those in Proposition 22 and Proposition 26 below, which ensure that all length one sequences are also indecomposable:

**Proposition 24** Let \( \mathbb{C} \) be a category all whose hom-sets are finite (in particular, \( \mathbb{C} \) may be a finite category, but not necessarily) and such that all monomorphisms \( f : X \to X \) are isomorphisms for any object \( X \) in \( \mathbb{C} \). Then, all length one sequences \( (X) \) are indecomposable objects of \( \text{Vect}_K[\mathbb{C}] \). Consequently, if \( \text{Vect}_K[\mathbb{C}] \) has a basis for such a category \( \mathbb{C} \), it is necessarily given by any family of length one sequences \( \mathbb{B} = \{(X_i)\}_{i \in I} \) with \( \{X_i\}_{i \in I} \) a set of representative objects of \( \mathbb{C} \).

Proof. Let us first see that, for any object \( X \) of \( \mathbb{C} \), \( X \) is not a biproduct in \( K[\mathbb{C}] \) of objects all of them nonisomorphic to \( X \) (either in \( K[\mathbb{C}] \) or in \( \mathbb{C} \), because the isomorphism classes of objects are the same in both categories by Proposition 9). Indeed, suppose \( X \) is a biproduct (in \( K[\mathbb{C}] \)) of \( X_1, \ldots, X_n \), and let \( \iota_k : X_k \to X \) and \( \pi_k : X \to X_k \) be the corresponding

\(^3\)Conversely, we can ask whether any Krull-Schmidt \( K \)-linear additive category is a generalized 2-vector space.
injections and projections, with $t_k = \sum_i \lambda_{ki} f_{ki}$ and $\pi_k = \sum_j \mu_{kj} g_{kj}$, where $f_{ki} : X_k \to X$ and $g_{kj} : X \to X_k$ are morphisms in $\mathcal{C}$ and $\lambda_{ki}, \mu_{kj} \in K$. They are such that
\begin{equation}
\pi_k t_k = \text{id}_{X_k}, \quad k = 1, \ldots, n
\end{equation}
\begin{equation}
\sum_{k=1}^n t_k \pi_k = \text{id}_X
\end{equation}
Then, it follows from (3.3) that $\sum_{i,j} \lambda_{ki} \mu_{kj} (g_{kj} f_{ki}) = \text{id}_{X_k}$ for all $k = 1, \ldots, n$. But $\text{End}_{K[C]}(X_k)$ is the vector space with basis $\text{End}_C(X_k)$ and hence, for each $k = 1, \ldots, n$, there exists at least one pair $(i_k, j_k)$ such that $g_{kj_k} f_{ki_k} = \text{id}_{X_k}$. In particular, $f_{ki_k}$ is a section. Similarly, it follows from (3.4) that $\sum_{i,j,k} \lambda_{ki} \mu_{kj} (f_{ki} g_{kj}) = \text{id}_X$ and hence, there exists, for at least one value of $k$, at least one pair $(i'_k, j'_k)$ such that $f_{ki'_k} g_{kj'_k} = \text{id}_X$. In particular, $g_{kj'_k}$ is also a section (hence, a monomorphism). The argument is now the same we have made to prove Proposition 0. Namely, the composite $f_{ki_k} g_{kj'_k} : X \to X$ is a monomorphism and consequently, an isomorphism by hypothesis. This implies that $f_{ki_k} : X_k \to X$, which is a section, is also an epimorphism and hence, an isomorphism. Therefore, at least one factor $X_k$ is isomorphic to $X$ in $\mathcal{C}$.

Suppose now that $(X)$ is decomposable in $\text{Vect}_K[\mathcal{C}]$, i.e., $(X) = S \oplus S'$ for some sequences $S, S'$ in $\text{Vect}_K[\mathcal{C}]$ both of length $\geq 1$. This means that there exists objects $X_0, \ldots, X_k$ in $\mathcal{C}$, with $k \geq 1$, such that $(X) \cong (X_0, X_1, \ldots, X_k)$ in $\text{Vect}_K[\mathcal{C}]$. According to Proposition 10 (ii), however, this holds if and only if the biproduct of $X_0, X_1, \ldots, X_k$ exists in $K[\mathcal{C}]$ and $X_0 \oplus X_1 \oplus \cdots \oplus X_k \cong X$ (in $K[\mathcal{C}]$). It follows then from the previous observation that $X \cong X_i$ for at least one $i \in \{0, 1, \ldots, k\}$. Let us assume that $X = X_0$. Then, for any other object $Y$ in $\mathcal{C}$, we have a linear isomorphism
\begin{equation}
\text{Hom}_{K[\mathcal{C}]}(X, Y) \cong \text{Hom}_{K[\mathcal{C}]}(X \oplus X_1 \oplus \cdots \oplus X_k, Y)
\end{equation}
i.e.,
\begin{equation}
K[\text{Hom}_\mathcal{C}(X, Y)] \cong K[\text{Hom}_\mathcal{C}(X, Y)] \oplus \left( \bigoplus_{i=1}^k K[\text{Hom}_\mathcal{C}(X_i, Y)] \right)
\end{equation}
Since all involved hom-sets are finite, this is an isomorphism of finite dimensional vector spaces. Hence
\begin{equation}
\dim_K \left( \bigoplus_{i=1}^k K[\text{Hom}_\mathcal{C}(X_i, Y)] \right) = 0
\end{equation}
for any object $Y$, which implies $k = 0$, in contradiction with the fact that $k \geq 1$. \hfill \square

**Example 25** Take $\mathcal{C} = \text{Mat}_{F_q}$, the category of matrices with entries in the finite field $F_q$ of $q$ elements. This category satisfies none of the conditions stated in Propositions 22 or 24. However, all its hom-sets are finite, and an endomorphism $A : n \to n$ is a monomorphism if and only if it is an isomorphism. Hence, by Proposition 21 all length one sequences $(n)$, with $n \geq 1$, are indecomposable objects in $\text{Vect}_K[\text{Mat}_{F_q}]$. Note that this is not in contradiction with the fact that $n = 1 \oplus \cdots \oplus 1$ in $\text{Mat}_{F_q}$, because $(n) \cong (1, n, 1)$ is equivalent to this equality in $K[\text{Mat}_{F_q}]$, not in $\text{Mat}_{F_q}$.
In general, however, it is false that all length one sequences are indecomposable. For instance, this is not the case if $C$ is already additive (see Proposition 10(ii)). Furthermore, we have already pointed out in §2.5 that, for an arbitrary $K$-linear category $L$, the monoid $M(Add(L))$ is not always isomorphic to the free commutative monoid generated by the isomorphism classes of objects in $L$.

As regards the second step above, notice that it is equivalent to the essential uniqueness part in a Krull-Schmidt theorem for these kind of $K$-linear additive categories.

There are several versions of this theorem, concerning various types of $K$-linear additive categories. The classical version, which goes back to Schmidt (1913) and Krull (1925), refers to the abelian categories of modules over a commutative ring with unit $K$, and it states that any $K$-module of finite length (more generally, any $K$-module which is a direct sum of $K$-modules with local endomorphism rings) decomposes as a direct sum of finitely many indecomposables and that the decomposition is unique up to isomorphism and permutation of the direct summands (see for ex. [18] or [15]). The result was later shown for the categories of sheaves by Atiyah [1]. More generally, he proved that in any exact category satisfying a suitable finiteness condition (called the “bichain condition”), each object has an essentially unique decomposition as a finite direct sum of indecomposables. A third version can be found in [3] (p. 20), where the essential uniqueness of the decomposition for objects analogous to the above is demonstrated for any Karoubian additive category (i.e., any additive category where all idempotents split).

Although our categories $Vect_K[C]$ are $K$-linear additive, in general they are neither Karoubian nor exact (see §3.6), so that none of the above three versions applies. Moreover, the proofs of the essential uniqueness in these three versions make essential use of the fact that the endomorphism rings of the involved indecomposable objects are local, while our endomorphism rings $End_{Vect_K[C]}(X) = K[End_C(X)]$ need not be local, even for finite categories $C$.

This suggests that our generalized 2-vector spaces $Vect_K[C]$ will have no basis in general, even if we restrict to finite categories $C$ (with more than one object, of course), and that two notions of freeness should be distinguished in the category setting. A notion of “external” freeness, when the category is a free object in the appropriate 2-category, and a notion of “internal” freeness, when there exists in the category a family of basic objects from which all objects can be generated in an essentially unique way with the help of some given associative operation. An externally free category may be internally non free. In the context of additive categories, this corresponds to free additive categories $Add(L)$ whose monoid $M(Add(L))$ is non free. The above mentioned existence of non finitely generated free generalized 2-vector spaces of finite rank also fits naturally with this situation.

§3.5. More examples of generalized free 2-vector spaces of infinite rank. The simplest examples of generalized free 2-vector spaces of infinite rank are those generated by non finite discrete categories (see Proposition 10). These examples are a special case of a more general situation.

Indeed, according to Proposition 22 for any category $C$ equivalent to a finite disjoint union of monoids (viewed as one object categories), $Vect_K[C]$ is a generalized finite rank free 2-vector. It turns out that, when the finite hypothesis is replaced by the assumption that all involved monoids are isomorphic to the same monoid $M$, the result remains true.
except that the generalized 2-vector space is now of a possibly infinite rank. Explicitly, let us call a category $C$ homogeneous when it is a disjoint union of copies (possibly infinite in number) of the same monoid $M$, and call the monoid $M$ the underlying monoid of $C$. Then, we have the following

**Proposition 26** For any homogeneous category $C$, $\text{Vect}_K[C]$ is a generalized free 2-vector space, a basis being given by any family of length one sequences $B = \{(X_i)_{i \in I}\}$ with $\{X_i\}_{i \in I}$ a set of representative objects of $C$.

**Proof.** We only need to see that $B$ is additively free. Let $M$ be the underlying monoid of $C$. Then, the same argument used to prove Proposition 19 can now be used to show that if $(X_{i_1}, \ldots, X_{i_k})$ and $(X'_{i_1}, \ldots, X'_{i_k})$ are any isomorphic objects in $\text{Vect}_K[C]$, then $k = k'$ (any morphism $A$ between them is thought of as a morphism of free $K[M]$-modules $K[M]^k \to K[M]^{k'}$). It remains to see that both sequences are the same up to isomorphism and permutation of the objects. For any $j = 1, \ldots, k$, let $r_j$ and $r_j'$ be the number of copies of $X_{i_j}$ present (up to isomorphism) in the first and second sequences, respectively. By definition, it is $r_j \geq 1$, and since $A$ is invertible, it is also $r_j' \geq 1$ (otherwise, the corresponding matrix column in $A$ will be entirely made of zeros and hence, non invertible). By symmetry, we conclude that both sequences, if isomorphic, necessarily contain the same objects (up to isomorphism), but probably with different multiplicities. Let $X_1, \ldots, X_s$ be the pairwise non isomorphic objects present in both sequences, and let $p_1, \ldots, p_s$ and $p'_1, \ldots, p'_s$ the respective multiplicities, so that

$$(X_{i_1}, \ldots, X_{i_k}) \cong (X_1^{p_1}, X_1, \ldots, X_s^{p_s}, X_s)$$

and

$$(X'_{i_1}, \ldots, X'_{i_k}) \cong (X_1^{p'_1}, X_1, \ldots, X_s^{p'_s}, X_s)$$

Then, the isomorphism $A$ is necessarily of the form

$$A = \begin{pmatrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_s \end{pmatrix}$$

with $A_l$ a $p'_l \times p_l$ matrix with entries in $K[M]$, $l = 1, \ldots, s$. Furthermore, since $A$ is an isomorphism, there exists a second matrix

$$A' = \begin{pmatrix} A'_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A'_s \end{pmatrix}$$

with $A'_l$ a $p_l \times p'_l$ matrix with entries also in $K[M]$, for all $l = 1, \ldots, s$, such that

$$A_lA'_l = \text{Id}_{p'_l}, \quad A'_lA_l = \text{Id}_{p_l}, \quad l = 1, \ldots, s$$

Using again the invariance dimension property of $K[M]$, we conclude that $p_l = p'_l$ for all $l = 1, \ldots, s$. $\square$

---

[4]See footnote [2].
Corollary 27  For any homogeneous groupoid \( \mathcal{G} \) (in particular, for any discrete category or any 2-group), finite or not, \( \text{Vect}_K[\mathcal{G}] \) is a generalized free 2-vector space, a basis being given by any family of length one sequences \( \mathbb{B} = \{(X_i)_{i \in I}\} \) with \( \{X_i\}_{i \in I} \) a set of representative objects of \( \mathcal{G} \).

§3.6. Existence of generalized 2-vector spaces which are non abelian (even non Karoubian) categories. Kapranov and Voevodsky 2-vector spaces are abelian categories (in particular, Karoubian), but this is no longer true for generalized 2-vector spaces.

To see this, let us consider the case of the generalized 2-vector spaces \( \text{Vect}_K[M[1]] \) generated by non trivial monoids (see §3.3 for the notation). Given a non zero endomorphism \( a : (\ast) \to (\ast) \), with \( a \in K[M] \), we want to know if \( a \) has a kernel or not. Note first the following

Lemma 28  Let \( K \) be of characteristic \( \neq 2 \). Then, any monomorphism \( S \to (\ast) \) in \( \text{Vect}_K[M[1]] \), with \( S \neq \emptyset \), is necessarily an endomorphism \( b : (\ast) \to (\ast) \) for some \( b \in K[M] \).

Proof. Let us suppose that there exists a monomorphism \( A : (\ast, ?, \ast) \to (\ast) \) with \( r > 1 \), given by a row matrix \( A = (a_1 \cdots a_r) \), \( a_i \in K[M] \). At least one of the entries is non zero (otherwise, it will not be monic). Let \( a_1 \neq 0 \), so that \( a_1 \neq -a_1 \) (\( K \) is of characteristic \( \neq 2 \)). Then, the morphisms \( B, B' : (\ast) \to (\ast, ?, \ast) \) given by

\[
B = \begin{pmatrix}
  a_2 \\
  -a_1 \\
  0 \\
  \vdots \\
  0
\end{pmatrix} \quad B' = \begin{pmatrix}
  -a_2 \\
  a_1 \\
  0 \\
  \vdots \\
  0
\end{pmatrix}
\]

clearly satisfy \( B \neq B' \) and \( AB = AB' \), in contradiction with the hypothesis that \( A \) is monic. \( \square \)

The answer to the above question reads then as follows:

Proposition 29  Let \( M \) be a non trivial monoid and let \( a \in K[M] \). Then, as a morphism \( a : (\ast) \to (\ast) \) in \( \text{Vect}_K[M[1]] \), it has a kernel if and only if \( a \) is not a right zero divisor of \( K[M] \), in which case the kernel is the morphism \( \emptyset \to (\ast) \).

Proof. If \( a \) is not a right zero divisor, the only morphisms \( B : S \to (\ast) \) such that \( AB = 0 \) are the zero morphisms, so that \( \emptyset \to (\ast) \) is clearly a kernel of \( a \).

Suppose now that \( a \) is a right zero divisor. In this case, the zero morphism \( \emptyset \to (\ast) \) can no longer be a kernel of \( a \), because there are non zero morphisms \( B : S \to (\ast) \) such that \( AB = 0 \). For example, all morphisms \( b : (\ast) \to (\ast) \) with \( b \in K[M] \) such that \( ab = 0 \). By the previous Lemma, if a kernel exists, being monic, it is necessarily an endomorphism \( b : (\ast) \to (\ast) \) for some \( b \in K[M] \) such that \( ab = 0 \). But the universal property of the kernel further requires \( b \) to be left cancellable, in contradiction with the fact that it is a left zero divisor. \( \square \)

For example, if \( M = \mathbb{Z}_2 = \{\pm\} \), it is easily checked that \( (+) - (-) \) is a zero divisor in \( K[\mathbb{Z}_2] \) and hence it has no kernel as endomorphism of \( (\ast) \) in \( \text{Vect}_K[\mathbb{Z}_2[1]] \).
It might be thought that the categories $\text{Vect}_K[C]$, although non abelian in general, they are at least Karoubian. But this is also false, as Example 33 below shows.

§3.7. The 2-category of generalized 2-vector spaces and some full sub-2-categories. Let the 2-category of generalized 2-vector spaces over $K$, denoted by $\text{2GVECT}_K$, be the full sub-2-category of $\text{AdCat}_K$ with objects all generalized 2-vector spaces over $K$. In particular, 1-morphisms in $\text{2GVECT}_K$ are $K$-linear functors and 2-morphisms arbitrary natural transformations between these. As full sub-2-category of $\text{AdCat}_K$, observe that $\text{2GVECT}_K$ is a $K$-linear 2-category (see §2.2).

There are various full sub-2-categories of $\text{2GVECT}_K$ that can be distinguished, according to the various types of generalized 2-vector spaces considered before. Thus, let

- $\text{2GVECT}_K^f$ be the full sub-2-category of $\text{2GVECT}_K$ with objects only the generalized free 2-vector spaces;
- $\text{2GVECT}_K^{ff}$ be the full sub-2-category of $\text{2GVECT}_K$ with objects only the generalized finite rank free 2-vector spaces;
- $\text{2GVect}_K$ be the full sub-2-category of $\text{2GVECT}_K$ with objects only the finitely generated generalized 2-vector spaces;
- $\text{2GVect}_K^f$ be the full sub-2-category of $\text{2GVECT}_K$ with objects only the finitely generated free 2-vector spaces (hence, of a necessarily finite rank), and
- $\text{2Vect}_K$ be the full sub-2-category of $\text{2GVECT}_K$ with objects only the Kapranov and Voevodsky 2-vector spaces.

All these sub-2-categories fit into the following diagram of inclusion 2-functors, where the label ($\star$) denotes a strict inclusion while (id?) means that it could be an identity:

Thus, an example of an object in $\text{2GVECT}_K^f$ which is not in $\text{2GVECT}_K^{ff}$ is $\text{Vect}_K[X[0]]$ for any infinite set, an example of an object in $\text{2GVECT}_K^{ff}$ which is not in $\text{2GVECT}_K^f$ is $\text{Vect}_K[M[1]]$ for any infinite monoid (or $\text{Vect}_K[G]$ for any 2-group $G$ such that $\pi_0(G)$ is finite and $\pi_1(G)$ is infinite), and an example of an object in $\text{2GVECT}_K^f$ which is not in $\text{2Vect}_K$ is $\text{Vect}_K[M[1]]$ for any finite monoid (or $\text{Vect}_K[G]$ for any finite 2-group).
§3.8. Finite rank free generalized 2-vector spaces up to equivalence. An arbitrary free generalized 2-vector space of finite rank encodes a more involved structure than a Kapranov and Voevodsky 2-vector space. Thus, these 2-vector spaces are completely characterized (up to equivalence) by their rank (as it occurs for finite dimensional vector spaces). However, characterizing an arbitrary free generalized 2-vector space of finite rank (up to equivalence) generally requires a whole set of structure constants in the field $K$, taking account of the nontrivial composition law for morphisms between basic objects. More explicitly, if $V$ is a free generalized 2-vector space over $K$ of finite rank $r$ and $B = \{X_1, \ldots, X_r\}$ is a basis of $V$, we may choose for each pair of basic objects $X_i, X_j \in B$ a linear basis $B(\alpha, \beta, \gamma) = \{f(i, j)_{\alpha, \beta}, \alpha \in \Lambda(i, j), \beta \in \Lambda(j, k), \gamma \in \Lambda(i, k)\}$ defined by

$$f(j, k)_{\beta, \gamma} f(i, j)_{\alpha, \beta} = \sum_{\gamma \in \Lambda(i, k)} c(i, j, k)_{\alpha, \beta, \gamma} f(i, k)_{\alpha, \beta}$$

These constants satisfy the following associativity and unit equations coming from the corresponding axioms on the composition law:

- **(associativity)** For all $i, j, k, l \in \{1, \ldots, r\}$, $\alpha \in \Lambda(i, j)$, $\beta \in \Lambda(j, k)$, $\gamma \in \Lambda(k, l)$ and $\delta \in \Lambda(i, l)$ it is

$$\sum_{\mu \in \Lambda(i, k)} c(i, j, k)_{\mu, \beta} c(i, k, l)_{\mu, \gamma} = \sum_{\nu \in \Lambda(j, l)} c(i, j, l)_{\alpha, \nu} c(j, k, l)_{\beta, \gamma}$$

- **(unit conditions)** For all $i, j \in \{1, \ldots, r\}$ and $\alpha, \beta \in \Lambda(i, j)$ it is

$$c(i, i, j)_{\beta, \alpha} = \delta_{\alpha, \beta}, \quad c(i, j, j)_{\beta, \alpha} = \delta_{\alpha, \beta}$$

For free generalized 2-vector spaces of rank one, these are nothing but the equations satisfied by the structure constants of an associative $K$-algebra with unit.

Although the above constants depend on the basis $B$ of $V$ and on the chosen linear basis of morphisms between basic objects, they serve to completely determine $V$ in the following sense:

**Proposition 30** Two finite rank free generalized 2-vector spaces $V$ and $V'$ are equivalent (as objects in $2\text{VECT}_{K}$) if and only if they have the same structure constants for suitably chosen bases of objects and linear bases of morphisms between basic objects.

**Proof.** Left to the reader. \hfill \Box

It is worth pointing out, however, that not all sets of constants satisfying the above associativity and unit conditions are the structure constants of a finite rank free 2-vector space. For instance, in the rank one case, it should further exist a linear basis of endomorphisms of the basic object for which the constants are given by $c(1, 1, 1)_{\alpha, \beta} = \delta_{\gamma, m(\alpha, \beta)}$. In other words, among all possible associative algebras with unit, only the algebras of a monoid correspond to free generalized 2-vector spaces of rank one.
4 Vect$_K[C]$ versus the functor category $\text{VECT}_K^{op}$.

Together with $K[X]$, there is one more vector space which can be built from a set $X$. Namely, the vector space $K^X$ of all functions on $X$ with values in $K$. This construction is also functorial. In fact, if restricted to finite sets, both functors $K[-], K(-): \text{FinSets} \to \text{Vect}_K$ are naturally isomorphic.

The purpose of this section is to consider the analog for categories of the vector spaces $K^X$ and to show that the corresponding construction is no longer equivalent to $\text{Vect}_K[C]$, even if we restrict to finite categories.

§4.1. The functor categories $\text{VECT}_K^{op}$. To define the analog of $K^X$, we follow again Kapranov and Voevodsky insight of replacing $K$ by the category of vector spaces, except that we shall consider the category $\text{VECT}_K$ of all vector spaces, finite dimensional or not. If we further replace the set $X$ by a category $C$, we are led to the category $\text{VECT}_K$ with objects all functors $F: C \to \text{VECT}_K$ and the natural transformations between these as morphisms. For various reasons, however, it is more convenient to consider the category $\text{VECT}_K^{op}$ of contravariant functors.

Note that $\text{VECT}_K^{op}$ is a $K$-linear additive category for any $C$. A zero object is given by the constant functor $F_0$ mapping each object of $C$ to the zero vector space, and a biproduct of two functors $F,G: C^{op} \to \text{VECT}_K$ is given by the composite functor $C^{op} \xrightarrow{\Delta} C^{op} \times C^{op} \xrightarrow{F \times G} \text{VECT}_K \times \text{VECT}_K \xrightarrow{\Delta} \text{VECT}_K$ and the obvious injections and projections (here, $\Delta$ denotes the diagonal functor and $\oplus$ the usual direct sum functor on $\text{VECT}_K$). Such a biproduct is denoted by $F \oplus G$.

Actually, in contrast to the 2-vector spaces $\text{Vect}_K[C]$, the categories $\text{VECT}_K^{op}$ are always abelian, for any category $C$, because the target category $\text{VECT}_K$ is already abelian.

Example 31 If $C = \mathbb{N}[1]$, with $\mathbb{N}$ the additive monoid of natural numbers, a functor $F: \mathbb{N}[1]^{op} \to \text{VECT}_K$ is completely given by a vector space $V$ (the image of $*$) together with a $K$-linear map $f: V \to V$ (the image of the morphism $1: * \to *$), and both can be chosen arbitrarily because $\mathbb{N}$ is free. By identifying $f$ with the action of an indeterminate $T$ on $V$ and extending this action in the obvious way to the whole polynomial algebra $K[T]$, objects of $\text{VECT}_K^{[\mathbb{N}[1]]^{op}}$ are naturally identified with modules over the polynomial algebra $K[T]$. These identifications extend to a $K$-linear equivalence $\text{VECT}_K^{[\mathbb{N}[1]]^{op}} \simeq_K K[T]-\text{Mod}$, where $K[T]-\text{Mod}$ denotes the $K$-linear abelian category of $K[T]$-modules.

§4.2. Relation between both constructions. As mentioned before, even for finite categories $C$, $\text{VECT}_K^{op}$ and $\text{Vect}_K[C]$ are generally non equivalent. This is easily understood because $\text{VECT}_K^{op}$ is always abelian, while $\text{Vect}_K[C]$ is not (see §3.6).

To make precise the relation between both constructions, observe that, among the objects in $\text{VECT}_K^{op}$, we have the representable functors, isomorphic to the functors $K[\text{Hom}_C(-, X)]: C^{op} \to \text{VECT}_K$ for some object $X$ in $C$ (note that, if the hom-sets of $C$ are finite, such functors actually take values in $\text{Vect}_K$). Then, we have the following:

Theorem 32 For any category $C$ (resp. category $C$ whose hom-sets are finite), $\text{Vect}_K[C]$ is $K$-linear equivalent to the $K$-linear additive subcategory of $\text{VECT}_K^{op}$ (resp. of $\text{Vect}_K^{op}$).
generated by the representable functors (see Definition 2).

Proof. For short, let \( F_X \) stand for the functor \( K[\text{Hom}_C(-, X)] \). Then, define a \( K \)-linear functor \( E : \text{Vect}_K[\mathcal{C}] \to \text{VECT}_K \) as follows:

- on objects: \( E(X_1, \ldots, X_r) = F_{X_1} \oplus \cdots \oplus F_{X_r} \) for \( r \geq 1 \) and \( E(\emptyset) = F_0 \), the constant zero functor.
- on morphisms: for any \( f_{i'i} \in \text{Hom}_C(X_i, X_{i'}) \), let \( A(f_{i'i}) : (X_1, \ldots, X_r) \to (X'_1, \ldots, X'_{r'}) \) be the morphism with all entries equal to zero except the \((i', i)\)-entry, which is equal to \( f_{i'i} \). Then, define \( E(A) : F_{X_1} \oplus \cdots \oplus F_{X_r} \to F_{X'_1} \oplus \cdots \oplus F_{X'_{r'}} \) as the natural transformation whose \( Y \)-component \( E(A)_Y : F_{X_1}(Y) \oplus \cdots \oplus F_{X_r}(Y) \to F_{X'_1}(Y) \oplus \cdots \oplus F_{X'_{r'}}(Y) \) is the linear map described by the \( r' \times r \) matrix with entries \( (E(A)_Y)_{j'j} : K[\text{Hom}_C(Y, X_j)] \to K[\text{Hom}_C(Y, X'_j)] \) given by

\[
(E(A)_Y)_{j'j} = \begin{cases} f_{i'i} \circ - & \text{if } j = i \text{ and } j' = i' \\ 0 & \text{otherwise} \end{cases}
\] (4.1)

The morphisms \( A(f_{i'i}) \), for all \( f_{i'i} \in \text{Hom}_C(X_i, X_{i'}) \) and all \( (i', i) \in \{1, \ldots, r'\} \times \{1, \ldots, r\} \), constitute a linear basis of \( \text{Hom}_{\text{Vect}_K[\mathcal{C}]}((X_1, \ldots, X_r), (X'_1, \ldots, X'_{r'})) \) and this action of \( E \) is extended \( K \)-linearly to arbitrary morphisms between both sequences.

It is easily checked that these assignments are functorial. We only need to prove that it is a fully faithful functor.

Let us first see that the linear map \( E(\cdot, X') : K[\text{Hom}_C(X, X')] \to \text{Nat}(F_X, F_{X'}) \) defined by (4.1) is an isomorphism for any length one sequences \((X), (X')\) of \( \text{Vect}_K[\mathcal{C}] \). On the one hand, we have a set bijection

\[
\text{Yon} : \text{Hom}_C(X, X') \to \text{Nat}(\text{Hom}_C(-, X), \text{Hom}_C(-, X'))
\]

mapping \( f : X \to X' \) to the natural transformation \( \sigma(f) \) with \( Y \)-component \( \sigma(f)_Y = f \circ - \) (Yoneda lemma). On the other hand, we have a linear map

\[
\Phi : K[\text{Nat}(\text{Hom}_C(-, X), \text{Hom}_C(-, X'))] \to \text{Nat}(F_X, F_{X'})
\]
given by \( \Phi(\sigma) = \sigma^K \), with \( \sigma^K = 1_{K[\cdot]} \circ \sigma \), for all \( \sigma \in \text{Nat}(\text{Hom}_C(-, X), \text{Hom}_C(-, X')) \). The images \( \{\sigma^K\}_\sigma \) are linearly independent vectors of \( \text{Nat}(F_X, F_{X'}) \) (and hence, \( \Phi \) is injective). Indeed, let \( \sigma_1, \ldots, \sigma_n \) be arbitrary natural transformations from \( \text{Hom}_C(-, X) \) to \( \text{Hom}_C(-, X') \), with \( \sigma_i \neq \sigma_j \) if \( i \neq j \). Note that \( \sigma_i \) is completely given by the morphism \( (\sigma_i)_X(\text{id}_X) \) (Yoneda lemma once more), so that the maps \( (\sigma_1)_X(\text{id}_X), \ldots, (\sigma_n)_X(\text{id}_X) : X \to X' \) are pairwise different. Then, if \( \sum_{i=1}^n \lambda_i \sigma^K_i : F_X \Rightarrow F_{X'} \) is the zero natural transformation, we have in particular that

\[
\sum_{i=1}^n \lambda_i (\sigma^K_i)_X(\text{id}_X) = \sum_{i=1}^n \lambda_i (\sigma_i)_X(\text{id}_X) = 0,
\]
in \( K[\text{Hom}_C(X, X')] \). Hence, \( \lambda_i = 0 \) for all \( i = 1, \ldots, n \). Furthermore, given any natural transformation \( \tau : F_X \Rightarrow F_{X'} \), suppose that

\[
\tau_X(\text{id}_X) = \sum_{i=1}^n \lambda_i f_i
\]
where \( f_i \in \text{Hom}_C(X, X') \), and define \( \sigma_i : \text{Hom}_C(-, X) \Rightarrow \text{Hom}_C(-, X') \) by \( (\sigma_i)_X(id_X) = f_i \) for all \( i = 1, \ldots, n \). Then, it is easily checked that \( \tau = \sum_{i=1}^n \alpha_i \sigma_i^K \), so that \( \Phi \) is also surjective. Therefore, \( \Phi \) is an isomorphism of vector spaces and it is immediately seen that the composite

\[
K[\text{Hom}_C(X, X')] \xrightarrow{K[\text{Yon}]} K[\text{Nat}(\text{Hom}_C(-, X), \text{Hom}_C(-, X'))]\n\xrightarrow{\Phi} \text{Nat}(K[\text{Hom}_C(-, X)], K[\text{Hom}_C(-, X')])
\]

coincides with the linear map \( E(X, X') \).

More generally, for nonzero objects \((X_1, \ldots, X_r)\) and \((X'_1, \ldots, X'_{r'})\) of arbitrary lengths, it is

\[
\text{Hom}_{\text{Vect}_K}\left((X_1, \ldots, X_r), (X'_1, \ldots, X'_{r'})\right) \cong \prod_{(i, i')} K[\text{Hom}_C(X_i, X'_{i'})]
\]

while

\[
\text{Nat}(F_{X_1} \oplus \cdots \oplus F_{X_r}, F_{X'_1} \oplus \cdots \oplus F_{X'_{r'}}) \cong \prod_{(i, i')} \text{Nat}(F_{X_i}, F_{X'_{i'}}),
\]

Under these identifications, it follows from the definition of \( E \) that

\[
E_{(X_1, \ldots, X_r), (X'_1, \ldots, X'_{r'})} = \prod_{(i, i')} E(X_i, X'_{i'})
\]

Hence, the linear maps \( E_{(X_1, \ldots, X_r), (X'_1, \ldots, X'_{r'})} \), for any \( r, r' \geq 1 \), are also isomorphisms and \( E \) is indeed fully faithful. \( \square \)

**Example 33** If \( C = \mathbb{N}[1] \), there is a unique representable functor up to isomorphism. Namely, \( F_* = K[\text{Hom}_{\mathbb{N}[1]}(-, \ast)] \). Under the identification \( \text{VECT}_K^{[\mathbb{N}[1]]^{op}} \simeq_K K[T]\text{-Mod} \) (see Example 31), this functor corresponds to \( K[T] \) as module over itself. Hence, \( \text{Vect}_K[\mathbb{N}[1]] \) can be identified with the full subcategory \( K[T]\text{-Mod}_f \) of \( K[T]\text{-Mod} \) with objects the free \( K[T] \)-modules. Note that this subcategory, and hence \( \text{Vect}_K[\mathbb{N}[1]] \), is non Karoubian. Thus, if \( P \) is any projective non free \( K[T] \)-module and \( F \) is the free \( K[T] \)-module of which \( P \) is a direct summand, so that \( F \cong P \oplus M \) for some \( K[T] \)-submodule \( M \) of \( F \), the projection \( p : F \to F \text{ onto } P \) is a non split idempotent in \( K[T]\text{-Mod}_f \) (an idempotent \( e : X \to X \) in an additive category splits if and only if \( id_X - e \) has kernel, and \( id_X - p \) has no kernel in \( K[T]\text{-Mod}_f \)).

In some special cases, both categories \( \text{Vect}_K[C] \) and \( \text{VECT}_K^{cpc} \) may in fact be equivalent, mimicking the situation for vector spaces. For instance, this is clearly the case if \( C \) is a finite discrete category and hence, for the Kapranov and Voevodsky 2-vector spaces. As shown by the previous example, however, this is not true in general.

## 5 General linear 2-groups \( G\text{LL}(\text{Vect}_K[C]) \)

Recall that for any bicategory \( C \) (non necessarily a strict one) and any object \( X \) of \( C \), the category \( \text{Equiv}_C(X) \) with objects the autoequivalences \( f : X \to X \) and with morphisms all 2-isomorphisms between these is a 2-group (see §2.6).
We are interested in the case \( \mathcal{C} = 2\text{GVECT}_K \). By analogy with the case of vector spaces, let us denote by \( \text{GL}(\mathcal{V}) \) the 2-group \( \text{Equiv}_{2\text{GVECT}_K}(\mathcal{V}) \) corresponding to a generalized 2-vector space \( \mathcal{V} \) and call it the \textit{general linear 2-group} of \( \mathcal{V} \). The purpose of this section is to compute \( \text{GL}(\mathcal{V}) \) (up to equivalence) for a special type of generalized 2-vector spaces which include Kapranov and Voevodsky 2-vector spaces.

§5.1. \( \text{GL}(\text{Vect}_K[\mathcal{C}]) \) versus \( \text{Equiv}_{\text{Cat}_K}(K[\mathcal{C}]) \) and \( \text{Equiv}_{\text{Cat}}(\mathcal{C}) \). Computing \( \text{GL}(\mathcal{V}) \) for an arbitrary generalized 2-vector space seems to be difficult. There are, however, general results relating \( \text{GL}(\text{Vect}_K[\mathcal{C}]) \) to the 2-groups \( \text{Equiv}_{\text{Cat}_K}(K[\mathcal{C}]) \) and \( \text{Equiv}_{\text{Cat}}(\mathcal{C}) \) which we want to discuss first, before considering any particular case.

Let \( \mathcal{V} = \text{Vect}_K[\mathcal{C}] \), with \( \mathcal{C} \) an arbitrary category, and let \( H_C : \text{End}_{\text{Cat}_K}(K[\mathcal{C}]) \to \text{End}_{2\text{GVECT}_K}(\text{Vect}_K[\mathcal{C}]) \) be a functor mapping a \( K \)-linear functor \( \overline{F} : K[\mathcal{C}] \to K[\mathcal{C}] \) to some \( K \)-linear extension of the composite \( K[\mathcal{C}] \to K[\mathcal{C}] \xrightarrow{\text{Vect}_K} \text{Vect}_K[\mathcal{C}] \), and a natural transformation \( \tau : \overline{F} \Rightarrow \overline{F}' : K[\mathcal{C}] \to K[\mathcal{C}] \) to the unique natural transformation \( H_C(\tau) \) such that \( 1_{\text{Vect}_K[\mathcal{C}]} \circ \tau = H_C(\tau) \circ 1_{\text{Vect}_K[\mathcal{C}]} \) (cf. Proposition 11). There are various such functors \( H_C \), but all of them are isomorphic because \( K \)-linear extensions are unique up to isomorphism. They are clearly injective on objects. In general, however, they are non essentially surjective because a \( K \)-linear endomorphism of \( \text{Vect}_K[\mathcal{C}] \) can apply length one sequences to sequences of length greater than one. For instance, if \( \mathcal{C} = 1 \) (the terminal category), \( K[1] \) is isomorphic to the one object \( K \)-linear category \( K[1] \) with \( K \) as vector space of endomorphisms, while \( \text{Vect}_K[1] \simeq K \text{ Vect}_K \). Then, \( \text{End}_{\text{Cat}_K}(K[1]) \) is a one object category, because a \( K \)-linear functor \( F : K[1] \to K[1] \) is nothing but a \( K \)-linear map \( f : K \to K \) and the condition of preservation of identities implies that \( f = \text{id}_K \) necessarily. In contrast, the set of isomorphism classes of objects in \( \text{End}_{2\text{GVECT}_K}(\text{Vect}_K) \) is in bijection with the set \( \mathbb{N} \) of natural numbers (see for ex. [9]).

However, the following general result holds:

**Theorem 34** Let \( \mathcal{C} \) be an arbitrary category. Then, any functor \( H_C : \text{End}_{\text{Cat}_K}(K[\mathcal{C}]) \to \text{End}_{2\text{GVECT}_K}(\text{Vect}_K[\mathcal{C}]) \) as above is a full monoidal embedding. Moreover, if \( \mathcal{C} \) is finite, it restricts to an equivalence of monoidal categories (hence, an equivalence of 2-groups)

\[
\text{Equiv}_{\text{Cat}_K}(K[\mathcal{C}]) \cong \text{GL}(\text{Vect}_K[\mathcal{C}])
\]

**Proof.** For any natural transformation \( \tau_1, \tau_2 : \overline{F} \Rightarrow \overline{F}' \) between \( K \)-linear functors \( \overline{F}, \overline{F}' : K[\mathcal{C}] \to K[\mathcal{C}] \), \( H_C(\tau_1) = H_C(\tau_2) \) and the definition of \( H_C \) on morphisms implies that \( 1_{\text{Vect}_K[\mathcal{C}]} \circ \tau_1 = 1_{\text{Vect}_K[\mathcal{C}]} \circ \tau_2 \) and hence, \( \tau_1 = \tau_2 \). Thus, \( H_C \) is always faithful. It is also full because given \( \sigma : H_C(\overline{F}) \Rightarrow H_C(\overline{F}') \), the morphisms

\[
\tau_X = \sigma_X, \quad X \in \text{Obj}(\mathcal{C})
\]

define a natural transformation \( \tau : \overline{F} \Rightarrow \overline{F}' \) such that \( 1_{\text{Vect}_K[\mathcal{C}]} \circ \tau = \sigma \circ 1_{\text{Vect}_K[\mathcal{C}]} \) and hence, we have \( \sigma = H_C(\tau) \). Furthermore, the functors \( H_C(\overline{F}\overline{F}') \) and \( H_C(\overline{F})H_C(\overline{F}') \) both are \( K \)-linear extensions of the functor \( a_{\text{Vect}_K[\mathcal{C}]} : K[\mathcal{C}] \to \text{Vect}_K[\mathcal{C}] \), and \( H_C(\text{id}_{\text{Vect}_K[\mathcal{C}]}) \) and \( \text{id}_{\text{Vect}_K[\mathcal{C}]} \) both are \( K \)-linear extensions of \( a_{\text{Vect}_K[\mathcal{C}]} : K[\mathcal{C}] \to \text{Vect}_K[\mathcal{C}] \). Consequently, there are isomorphisms \( H_C(\overline{F}\overline{F}') \cong H_C(\overline{F})H_C(\overline{F}') \) and \( H_C(\text{id}_{\text{Vect}_K[\mathcal{C}]}) \cong \text{id}_{\text{Vect}_K[\mathcal{C}]} \). Let \( H_{C,2}(\overline{F}, \overline{F}') : H_C(\overline{F}\overline{F}') \Rightarrow \).
$H_C(F)H_C(F')$ and $H_{C,0} : H_C(id_{K[C]}) \Rightarrow id_{\text{Vect}_K[C]}$ be the unique natural isomorphisms such that $1_{a_{K[C]}} \circ H_{C,2}(F, F') = H_{C,2}(F, F') \circ 1_{a_{K[C]}}$, and $1_{a_{K[C]}} = H_{C,0} \circ 1_{a_{K[C]}}$ (cf. Proposition 11). Then, it is easy to check that these isomorphisms define a monoidal structure on $H_C$. For instance, given any $K$-linear endomorphisms $F, F', F''$ of $K[C]$, the following coherence condition needs to be checked:

$\left( H_{C,2}(F, F') \circ 1_{H_{C}(F'')} \right) \cdot H_{C,2}(F F', F'') = \left( 1_{H_{C}(F)} \circ H_{C,2}(F', F'') \right) \cdot H_{C,2}(F, F' F'')$

Now, this equality holds if and only if the horizontal precomposites with $1_{a_{K[C]}}$ of both members are equal. But

$\left[ \left( H_{C,2}(F, F') \circ 1_{H_{C}(F'')} \right) \cdot H_{C,2}(F F', F'') \right] \circ 1_{a_{K[C]}} = \left( H_{C,2}(F, F') \circ 1_{H_{C}(F'')} \cdot 1_{a_{K[C]}} \right) \cdot 1_{a_{K[C]}} \cdot F F' F''$

$= \left( H_{C,2}(F, F') \circ 1_{a_{K[C]}} \cdot F F' F'' \right)$

$= 1_{a_{K[C]}} \cdot F F' F''$

and similarly

$\left( \left( 1_{H_{C}(F)} \circ H_{C,2}(F', F'') \right) \cdot H_{C,2}(F, F' F'') \right) \circ 1_{a_{K[C]}} = \left( 1_{H_{C}(F)} \circ H_{C,2}(F', F'') \cdot 1_{a_{K[C]}} \right) \cdot 1_{a_{K[C]}} \cdot F F' F''$

$= 1_{H_{C}(F)} \circ 1_{a_{K[C]}} \cdot F F' F''$

$= 1_{a_{K[C]}} \cdot F F' F''$

We leave to the reader checking the remaining coherence conditions and the naturality of $H_{C,2}(F, F')$ in $(F, F')$.

Suppose now that $C$ is finite. Let $H^0_C$ be the restriction of $H_C$ to $\text{Equiv}_{\text{Cat}_K}(K[C])$. It is a fully faithful monoidal functor $H^0_C : \text{Equiv}_{\text{Cat}_K}(K[C]) \rightarrow GL(\text{Vect}_K[C])$. For any $K$-linear functor $\bar{F} : \text{Vect}_K[C] \rightarrow \text{Vect}_K[C]$, let $F = \bar{F} \circ_C : C \rightarrow \text{Vect}_K[C]$ be its restriction to $C$. $\bar{F}$ is obtained from $F$ by extending it with the help of some biproduct functors and zero object in $\text{Vect}_K[C]$. Consequently, if $\bar{F}$ is an equivalence, the set of image objects $\{F(X)\}_{X \in \text{Obj}(C)}$ generates $\text{Vect}_K[C]$ additively. In particular, this set necessarily contains (up to isomorphism) all indecomposable objects in $\text{Vect}_K[C]$. But for a finite category $C$, all length one sequences are indecomposable (see Proposition 24). It follows that there exists a (unique) functor $\bar{F} : C \rightarrow K[C]$, which uniquely extends to a $K$-linear functor $\bar{F} : K[C] \rightarrow K[C]$, making commutative the diagram
Similarly, for any category $\mathcal{C}$, the functor $\text{End}_{\text{Cat}}(\mathcal{C}) \rightarrow \text{End}_{\text{Cat}_K}(K[\mathcal{C}])$ mapping $F : \mathcal{C} \rightarrow \mathcal{C}$ to the unique $K$-linear extension $\overline{F}$ of the composite $\mathcal{C} \xrightarrow{F} \mathcal{C} \xrightarrow{k_\mathcal{C}} K[\mathcal{C}]$ and $\tau : F \Rightarrow F'$ to the unique natural transformation $\overline{\tau} : \overline{F} \Rightarrow \overline{F}'$ such that $\tau = \tau \circ 1_{k_\mathcal{C}}$ is a monoidal non essentially surjective embedding. But it is a non necessarily full functor now. Thus, in the simplest case $\mathcal{C} = 1$, it is $\text{End}_{\text{Cat}}(1) \cong 1$ while $\text{End}_{\text{Cat}_K}(K[1]) \cong K[1]$ (any scalar $\lambda \in K$ defines a natural endomorphism of $\text{id}_{K[1]}$). Furthermore, the restriction $\text{Equiv}_{\text{Cat}}(\mathcal{C}) \rightarrow \text{Equiv}_{\text{Cat}_K}(K[\mathcal{C}])$ continues to be neither full ($\text{Equiv}_{\text{Cat}_K}(K[1]) \cong K^*[1]$ is not a terminal category) nor essentially surjective (basically, because an arbitrary $K$-linear equivalence $K[\mathcal{C}] \rightarrow K[\mathcal{C}]$ need not map morphisms of $\mathcal{C}$ to morphisms also in $\mathcal{C}$).

**Example 35** Let $\mathcal{C}$ be a group $G$ thought of as a category with only one object. Then, if the restriction $\text{Equiv}_{\text{Cat}}(\mathcal{C}) \rightarrow \text{Equiv}_{\text{Cat}_K}(K[\mathcal{C}])$ really gives an essentially surjective functor, any $K$-linear equivalence $K[\mathcal{C}] \rightarrow K[\mathcal{C}]$ should be isomorphic to the $K$-linear extension of some equivalence $\mathcal{C} \rightarrow \mathcal{C}$. Now, a $K$-linear equivalence $K[\mathcal{C}] \rightarrow K[\mathcal{C}]$ is nothing but a (unit preserving) algebra automorphism of $K[G]$, while an equivalence $\mathcal{C} \rightarrow \mathcal{C}$ is just a group automorphism of $G$. Furthermore, two algebra automorphisms $\phi, \phi' : K[G] \rightarrow K[G]$ define isomorphic $K$-linear equivalences if and only if there exists a unit $u \in K[G]^*$ such that $\phi'(x) = u^{-1}\phi(x)u$ for all $x \in K[G]$. In particular, if $G$ is abelian, they must be equal. But for an abelian group $G$, an arbitrary automorphism of $K[G]$ does not restrict to an automorphism of $G$. For ex., if $G = \mathbb{Z}_2 = \{\pm\}$, it is $\text{Aut}_{\text{Grp}}(\mathbb{Z}_2) = 1$, while $\text{Aut}_{\text{Alg}}(K[\mathbb{Z}_2]) \cong \Sigma_2$, the non trivial automorphism being that which maps $(-)$ to $-(-)^5$.

Therefore, the most general statement as regards the relation between $\text{GL}(\text{Vect}_K[\mathcal{C}])$ and $\text{Equiv}_{\text{Cat}}(\mathcal{C})$ reads as follows:

**Theorem 36** For any category $\mathcal{C}$, the composite functor $\text{End}_{\text{Cat}}(\mathcal{C}) \hookrightarrow \text{End}_{\text{Cat}_K}(K[\mathcal{C}]) \rightarrow \text{End}_{\text{2GE}\text{VCT}_K}(\text{Vect}_K[\mathcal{C}])$ restricts to a monoidal embedding $\text{Equiv}_{\text{Cat}}(\mathcal{C}) \hookrightarrow \text{GL}(\text{Vect}_K[\mathcal{C}])$. In particular, $\text{Equiv}_{\text{Cat}}(\mathcal{C})$ is equivalent to a (non full) sub-2-group of $\text{GL}(\text{Vect}_K[\mathcal{C}])$.

For a finite category $\mathcal{C}$, this is to be thought of as an analog of the fact that the group $\text{Aut}(X) \cong \Sigma_n$ of automorphisms of a finite set $X$ of cardinal $n$ is isomorphic to a subgroup (usually called the Weyl subgroup) of the general linear group $\text{GL}(K[X]) \cong \text{GL}(n,K)$. This suggests introducing the following

**Definition 37** For any finite category $\mathcal{C}$, the Weyl sub-2-group of $\text{GL}(\text{Vect}_K[\mathcal{C}])$ is the image of the previous monoidal embedding $\text{Equiv}_{\text{Cat}}(\mathcal{C}) \hookrightarrow \text{GL}(\text{Vect}_K[\mathcal{C}])$.

It is not clear at all, however, that there exists some sort of analog for $\text{GL}(\text{Vect}_K[\mathcal{C}])$ of the Bruhat decomposition of the general linear groups $\text{GL}(n,K)$.

---

5The group $\text{Aut}_{\text{Alg}}(K[G])$ is computed in the next paragraph for an arbitrary finite $G$ and an algebraically closed field $K$ whose characteristic does not dived the order of $G$; see proof of Lemma 15.
§5.2. General linear 2-group of the generalized 2-vector space generated by a finite homogeneous groupoid. Recall that a groupoid $\mathcal{G}$ is a category equivalent to a disjoint union of groups which are viewed as one object categories, i.e., $\mathcal{G} \simeq \bigsqcup_{i \in I} G_i[1]$ for some groups $G_i$. Let us call the cardinal of $I$ the coarse size of $\mathcal{G}$. We shall say that $\mathcal{G}$ is homogeneous when all groups $G_i$ are isomorphic to a given group $G$, called the underlying group of $\mathcal{G}$.

Suppose $\mathcal{G}$ is a finite homogeneous groupoid (i.e., finite coarse size and finite underlying group). Examples include all finite discrete categories $X[0]$ and all finite 2-groups $\mathcal{G}$, the first of coarse size equal to the cardinal of $X$ and underlying group $G = 1$, and the second of coarse size equal to the cardinal of $\pi_0(\mathcal{G})$ and underlying group $G = \pi_1(\mathcal{G})$.

To simplify notation, we shall denote by $\mathbb{GL}(\mathcal{G})$ the general linear 2-group $\mathbb{GL}(\mathbf{Vect}_K[\mathcal{G}])$. The purpose of this paragraph is to prove the following

**Theorem 38** Let $K$ be an algebraically closed field and $\mathcal{G}$ a finite homogeneous groupoid of coarse size $n$ and underlying group $G$. Suppose that the order of $G$ is not divisible by the characteristic of $K$ (in particular, this is the case if $\text{char}(K) = 0$). Then, $\mathbb{GL}(\mathcal{G})$ is a split 2-group with

$$\pi_0(\mathbb{GL}(\mathcal{G})) \cong \Sigma_n \times (\Sigma_{k_1} \times \cdots \times \Sigma_{k_s})^n \quad (5.1)$$

$$\pi_1(\mathbb{GL}(\mathcal{G})) \cong (K^*)^n \quad (5.2)$$

where $\Sigma_p$ denotes the symmetric group on $p$ elements ($p \geq 1$), $r$ is the number of conjugacy classes of $G$ and $k_i \geq 1$, for all $i = 1, \ldots, s$, is the number of non equivalent irreducible representations of $G$ of a given dimension $d_i$ (in particular, $k_1 + \cdots + k_s = r$). Furthermore, under these identifications, the action of $\pi_0(\mathbb{GL}(\mathcal{G}))$ on $\pi_1(\mathbb{GL}(\mathcal{G}))$ is given by

$$\begin{pmatrix}
\lambda_{11}^{(1)} & \cdots & \lambda_{1n}^{(1)} \\
\vdots & & \vdots \\
\lambda_{k_1}^{(1)} & \cdots & \lambda_{k_n}^{(1)} \\
\vdots & & \vdots \\
\lambda_{11}^{(s)} & \cdots & \lambda_{1n}^{(s)} \\
\vdots & & \vdots \\
\lambda_{k_1}^{(s)} & \cdots & \lambda_{k_n}^{(s)}
\end{pmatrix}
= \begin{pmatrix}
\lambda_{11}^{(1)} & \cdots & \lambda_{1n}^{(1)} \\
\lambda_{11}^{(s)} & \cdots & \lambda_{1n}^{(s)} \\
\vdots & & \vdots \\
\lambda_{k_1}^{(1)} & \cdots & \lambda_{k_n}^{(1)} \\
\vdots & & \vdots \\
\lambda_{k_1}^{(s)} & \cdots & \lambda_{k_n}^{(s)}
\end{pmatrix}
\begin{pmatrix}
\lambda_{11}^{(1)} & \cdots & \lambda_{1n}^{(1)} \\
\lambda_{11}^{(s)} & \cdots & \lambda_{1n}^{(s)} \\
\vdots & & \vdots \\
\lambda_{k_1}^{(1)} & \cdots & \lambda_{k_n}^{(1)} \\
\vdots & & \vdots \\
\lambda_{k_1}^{(s)} & \cdots & \lambda_{k_n}^{(s)}
\end{pmatrix}
\quad (5.3)
$$

for any $\sigma \in \Sigma_n$ and $\sigma_{i_1} \in \Sigma_{k_1}$ for all $i = 1, \ldots, s$ and $q = 1, \ldots, n$, and where we have identified the elements $\Lambda \in (K^*)^n$ with $r \times n$ matrices with entries $\lambda_{i_1}^{(i)} \in K^*$.

Notice that for $G = 1$ (hence, $r = 1$) we indeed recover the general linear 2-groups of Kapranov and Voevodsky 2-vector spaces $\mathbf{Vect}_K^n$, for which $\pi_0 \cong \Sigma_n$, $\pi_1 \cong (K^*)^n$ and with $\Sigma_n$ acting on $(K^*)^n$ in the usual way, i.e.

$$\sigma \cdot (\lambda_1, \ldots, \lambda_n) = (\lambda_{\sigma^{-1}(1)}, \ldots, \lambda_{\sigma^{-1}(n)})$$
(cf. [3], Proposition 6.3).

To prove the theorem, we shall first compute the homotopy groups \( \pi_0 \) and \( \pi_1 \) of the groupoid \( G \), next we shall determine the action of the first onto the second and finally, we shall see that its classifying 3-cocycle \( \alpha \) is cohomologically trivial.

Recall that, for any \( A \)-algebra \( G \), its group \( \text{Out}_{\text{Alg}}(A) \) of outer automorphisms is the quotient of the group \( \text{Aut}_{\text{Alg}}(A) \) of all its (unit preserving) algebra automorphisms modulo the normal subgroup \( \text{Inn}_{\text{Alg}}(A) \) of the inner ones, i.e., of those of the form \( \phi_u(x) = u^{-1} xu \) for some unit \( u \in A^* \).

**Lemma 39** For any finite homogeneous groupoid \( G \) of coarse size \( n \) and underlying group \( G \), there is a group isomorphism
\[
\pi_0(\text{GL}(G)) \cong \Sigma_n \times (\text{Out}_{\text{Alg}}(K[G]))^n
\] (5.4)
In particular, \( \pi_0(\text{GL}(G)) \) is a finite group.

**Proof.** Note first that for a groupoid \( G \) of the above kind it is \( K[G] \cong \bigcup K[G][1] \), where \( \bigcup K \) denotes the coproduct in \( \text{Cat}_K \). Hence, a \( K \)-linear equivalence of \( K[G] \) is completely determined by a permutation \( \sigma \in \Sigma_n \) giving the action on objects together with a collection of \( K \)-algebra automorphisms \( \phi_1, \ldots, \phi_n : K[G] \to K[G] \) giving the action on the vector spaces of morphisms, and any such data \( (\sigma, \phi_1, \ldots, \phi_n) \) defines a \( K \)-linear equivalence \( F(\sigma, \phi_1, \ldots, \phi_n) : K[G] \to K[G] \). Moreover, it is immediate to check that the equivalences \( F(\sigma, \phi_1, \ldots, \phi_n) \) and \( F(\sigma', \phi'_1, \ldots, \phi'_n) \) are isomorphic if and only if \( \sigma = \sigma' \) and there exists units \( u_1, \ldots, u_n \in K[G]^* \) (the components of an isomorphism) such that \( \phi'_i(x) = u_i^{-1} \phi_i(x) u_i \) for all \( x \in K[G] \). The isomorphism (5.4) is then a consequence of Theorem 5.3. As regards the last assertion, it follows from a result due to Karpilovsky (see [13], Theorem 8.5.2), according to which the group \( \text{Out}_{\text{Alg}}(K[G]) \), for any \( G \), is in bijection with the isomorphism classes of \( K[G \times G] \)-modules whose underlying additive group is \( K[G] \) and with \( K[G \times G] \) acting on it by
\[
\left( \sum_i \lambda_i (g_i, g'_i) \right) x = \sum_i \lambda_i x f ((g'_i)^{-1}), \quad x \in K[G]
\]
for some \( f \in \text{Aut}_{\text{Alg}}(K[G]) \), and from the known fact (see [13], Theorem 79.13) that for a finite group \( G \), there are only finitely many isomorphism classes of such modules. \( \square \)

**Lemma 40** Let \( K \) be an algebraically closed field and \( G \) a finite group whose order is not divisible by the characteristic of \( K \). Then, there is an isomorphism of groups
\[
\text{Out}_{\text{Alg}}(K[G]) \cong \Sigma_{k_1} \times \cdots \times \Sigma_{k_s}
\]
where \( k_i \geq 1 \) is the number of non equivalent irreducible representations of \( G \) of a given dimension \( d_i \), for all \( i = 1, \ldots, s \) (in particular, \( k_1 + \cdots + k_s \) is the number of conjugacy classes of \( G \)).

**Proof.** Under the assumptions on \( K \) and on the order of \( G \), it is well known (see for ex. [15]) that there exists an algebra isomorphism \( K[G] \cong M_{n_1}(K) \times \cdots \times M_{n_r}(K) \), where \( r \) is the number of conjugacy classes of \( G \) and \( n_1, \ldots, n_r \) are the dimensions of the non equivalent
irreducible representations of $G$. Furthermore, it follows from Skolem-Noether theorem (see [2], Corollary 4.4.3) that all automorphisms of the algebra $M_n(K)$ are inner. Hence

$$\text{Inn}_{\text{Alg}}(M_{n_1}(K) \times \cdots \times M_{n_r}(K)) = \text{Inn}_{\text{Alg}}(M_{n_1}(K)) \times \cdots \times \text{Inn}_{\text{Alg}}(M_{n_r}(K)) = \text{Aut}_{\text{Alg}}(M_{n_1}(K)) \times \cdots \times \text{Aut}_{\text{Alg}}(M_{n_r}(K))$$

In general, however, the obvious embedding of $\text{Aut}_{\text{Alg}}(M_{n_1}(K)) \times \cdots \times \text{Aut}_{\text{Alg}}(M_{n_r}(K))$ into $\text{Aut}_{\text{Alg}}(M_{n_1}(K) \times \cdots \times M_{n_r}(K))$ is non surjective, so that the quotient $\text{Out}_{\text{Alg}}(K[G])$ is non trivial. To compute this quotient, let us denote by $I_n$ the identity $n \times n$ matrix and by $0$ any zero matrix. Then, if $e_j = (0, \ldots, 1_{n_j}, \ldots, 0)$ ($j = 1, \ldots, r$), the elements $e_1, \ldots, e_r$ are pairwise orthogonal central idempotents of the product algebra $A = M_{n_1}(K) \times \cdots \times M_{n_r}(K)$.

Hence, any algebra automorphism $\phi : A \to A$ necessarily maps them to pairwise orthogonal central idempotents of $A$. Since the center of $M_n(K)$ is $Z(M_n(K)) = K I_n$, this means that $\phi(e_j) = \sum_{i=1}^r \lambda_{ij} e_i$ for some scalars $\lambda_{ij} \in \{0, 1\}$, $i, j = 1, \ldots, r$ (note that $\phi(e_j)$ idempotent implies that $\lambda_{jj}^2 = \lambda_{jj}$). Moreover, since $\phi$ preserves the identity of $A$, we also have $\phi(e_1 + \cdots + e_r) = e_1 + \cdots + e_r$, from which it follows that $\lambda_{ij} = \delta_{ij}$, i.e., $\phi(e_j) = e_j$, for some $j'$ which depends on $j$. Together with the fact that, for any $N_j \in M_{n_j}(K)$, it is

$$\phi(0, \ldots, N_j, \ldots, 0) = \phi((0, \ldots, N_j, \ldots, 0)(0, \ldots, I_{n_j}, \ldots, 0)) = \phi(0, \ldots, N_j, \ldots, 0) \phi(0, \ldots, I_{n_j}, \ldots, 0)$$

it follows that any automorphism $\phi$ of $A$ necessarily maps each factor $M_{n_j}(K)$ isomorphically onto some other factor $M_{n_j}(K)$. In particular, the subscript $j'$ for which $\lambda_{jj} = \delta_{jj}$ must be such that $n_{j'} = n_j$. Inner or decomposable automorphisms $\phi \in \text{Aut}_{\text{Alg}}(M_{n_1}(K)) \times \cdots \times \text{Aut}_{\text{Alg}}(M_{n_r}(K))$ correspond to the case $j' = j$ for all $j = 1, \ldots, r$. These will be the unique possible automorphisms of $A$ when the positive integers $n_1, \ldots, n_r$ are pairwise different. In general, however, $G$ may have non equivalent irreducible representations of the same dimension. Specifically, suppose we have $k_i$ non equivalent irreducible representations of dimension $d_i$ for $i = 1, \ldots, s$ (for example, suppose that $n_1 + \cdots + n_{k_1} = d_1$, $n_{k_1+1} + \cdots + n_{k_1+k_2} = d_2$, etc.). In particular, we have $k_1 + \cdots + k_s = r$. In this case, a generic automorphism of $A = M_{d_1}(K) \times \cdots \times M_{d_s}(K) \times \cdots \times M_{d_1}(K) \times \cdots \times M_{d_s}(K)$ will decompose in a unique way as the composite of a permutation automorphism $\phi_{\sigma_1, \ldots, \sigma_s}$ given by

$$\phi_{\sigma_1, \ldots, \sigma_s}(N_{11}, \ldots, N_{k_1}, \ldots, N_{1s}, \ldots, N_{ks}, s) = (N_{\sigma_{1}(1)}, \ldots, N_{\sigma_{1}(k_1)}, \ldots, N_{\sigma_{s}(1)}, \ldots, N_{\sigma_{s}(k_s)}, s) \quad (5.5)$$

for some $(\sigma_1, \ldots, \sigma_s) \in \Sigma_{k_1} \times \cdots \times \Sigma_{k_s}$, followed by a decomposable automorphism $\phi_1 \times \cdots \times \phi_s$. In other words, by identifying $\Sigma_{k_1} \times \cdots \times \Sigma_{k_s}$ with the above subgroup of permutation automorphisms of $A$, we conclude that $\text{Aut}_{\text{Alg}}(A)$ is the semidirect product of the (normal) subgroup of inner automorphisms and $\Sigma_{k_1} \times \cdots \times \Sigma_{k_s}$ and therefore, $\text{Out}_{\text{Alg}}(K[G]) \cong \Sigma_{k_1} \times \cdots \times \Sigma_{k_s}$ as claimed. \hfill \Box

Isomorphism (5.4) of Theorem 38 is now an immediate consequence of Lemmas 39 and 40.

Let us now compute $\pi_1(\mathbb{G}_L(\mathbb{G}))$.
Lemma 41 For any finite homogeneous groupoid $G$ of coarse size $n$ and underlying group $G$, there is an isomorphism of abelian groups

$$
\pi_1(\text{GL}(G)) \cong Z(K[G]^*)^n
$$

(5.6)

where $Z(K[G]^*)$ denotes the center of $K[G]^*$.

Proof. By Theorem 34, $\pi_1(\text{GL}(G)) \cong \pi_1(\text{Equiv}_{\text{Cat}_K}(K[G])) = \text{Aut}(\text{id}_{K[G]})$. Now, a natural automorphism of $\text{id}_{K[G]}$ (actually, of any $F : K[G] \to K[G]$) is given by invertible elements $u_1, \ldots, u_n$ (its components) in $K[G]$, and naturality further requires that the $u_q$ belong to the center of $K[G]^*$. Moreover, composition of automorphisms clearly corresponds to the product in $Z(K[G]^*)^n$. □

Lemma 42 Let $K$ be an algebraically closed field and $G$ a finite group whose order is not divisible by the characteristic of $K$. Then

$$
Z(K[G]^*) \cong (K^*)^r
$$

where $r$ is the number of conjugacy classes of $G$.

Proof. The result is a direct consequence of the isomorphism of algebras $K[G] \cong M_{n_1}(K) \times \cdots \times M_{n_r}(K)$ and the fact that $Z(M_n(K)) = K I_n$. □

Combining Lemmas 41 and 42 we readily get isomorphism (5.2) of Theorem 38. Let us now prove that, with the above identifications, the action is indeed given by (5.3).

Lemma 43 For any finite homogeneous groupoid $G$ of coarse size $n$ and underlying group $G$, the action of $\pi_0(\text{GL}(G)) \cong \Sigma_n \times (\text{Out}_{\text{Alg}_K}(K[G]))^n$ on $\pi_1(\text{GL}(G)) \cong Z(K[G]^*)^n$ is given by

$$(\sigma, [\phi_1], \ldots, [\phi_n]) \cdot (u_1, \ldots, u_n) = (\phi_{\sigma^{-1}(1)}(u_{\sigma^{-1}(1)}), \ldots, \phi_{\sigma^{-1}(n)}(u_{\sigma^{-1}(n)}))$$

(5.7)

for any representatives $\phi_1, \ldots, \phi_n$ of $[\phi_1], \ldots, [\phi_n] \in \text{Out}_{\text{Alg}_K}(K[G])$.

Proof. Let us identify $\text{GL}(G)$ with $\text{Equiv}_{\text{Cat}_K}(K[G])$. Then, by definition, given $[F] \in \pi_0(\text{GL}(G))$ and $\tau \in \pi_1(\text{GL}(G))$, it is

$$
[F] \cdot \tau = \gamma_F^{-1}(\delta_F(\tau))
$$

for any representative $F : K[G] \to K[G]$ of $[F]$ (see §2.7). Now, identifying $\text{Aut}(F)$ with $Z(K[G]^*)$ as above, it is easy to see that

$$
\delta_F(\sigma, \phi_1, \ldots, \phi_n)(u_1, \ldots, u_n) = (\phi_1(u_1), \ldots, \phi_n(u_n))
$$

$$
\gamma_F(\sigma, \phi_1, \ldots, \phi_n)(u_1, \ldots, u_n) = (u_{\sigma(1)}, \ldots, u_{\sigma(n)})
$$

from which (5.3) readily follows (note that the action so defined is indeed independent of the representatives $\phi_i$ because the $u_i$ are central). □

Equation (5.3) follows now from (5.7) by making the appropriate identifications. Thus, as discussed above, when $K[G]$ is identified with the corresponding product algebra $A = M_{n_1}(K) \times \cdots \times M_{n_r}(K)$, each equivalence class $[\phi_q] \in \text{Out}_{\text{Alg}_K}(A)$ in (5.7) can be identified...
with an element \((\sigma_q, \ldots, \sigma_{qs}) \in \Sigma_{k_1} \times \cdots \times \Sigma_{k_n}\), a representative of \([\phi_q]\) being then the permutation automorphism \(\phi_{\sigma_1, \ldots, \sigma_{qs}}\) defined by (5.3). At the same time, each \(u_q \in Z(A^*)\), \(q = 1, \ldots, n\), can be identified with an element \((\lambda^{(1)}_{1q}, \ldots, \lambda^{(1)}_{k_1q}, \ldots, \lambda^{(s)}_{1q}, \ldots, \lambda^{(s)}_{k_nq}) \in (K^*)^r\), corresponding in fact to \((\lambda^{(1)}_{1q}I_{d_1}, \ldots, \lambda^{(1)}_{k_1q}I_{d_1}, \ldots, \lambda^{(s)}_{1q}I_{d_s}, \ldots, \lambda^{(s)}_{k_nq}I_{d_s}) \in Z(A^*)\). With these identifications, it is straightforward checking that (5.7) translates into (5.8).

Let us finally see that \(\GL(G)\) is split. Note first the following general result:

**Lemma 44** For any finite category \(C\), there is an equivalence of 2-groups \(\text{Equiv}_{\text{Cat}_K}(K'[C]) \simeq \text{Aut}_{\text{Cat}_K}(K'[C])\) and hence, an equivalence of 2-groups \(\GL(Vect_K[C]) \simeq \text{Aut}_{\text{Cat}_K}(K'[C])\).

**Proof.** To prove the first assertion, it is enough to see that \(\text{Equiv}_{\text{Cat}_K}(K'[C]) = \text{Aut}_{\text{Cat}_K}(K'[C])\) for any finite skeletal category \(C\). The claimed equivalence follows then from the fact that any category is equivalent to a skeletal one. Let \(C\) be skeletal and let \(E : K[C] \to K[C]\) be any \(K\)-linear equivalence. In particular, there exists a \(K\)-linear functor \(\overline{E} : K[C] \to K[C]\) and a natural isomorphism \(\tau : \overline{E}E \Rightarrow \text{id}_{K[C]}\). Since \(C\) is finite, it follows from Proposition 12 that \(K'[C]\) is also skeletal. Hence, \(\overline{E}E(X) \cong X\) implies \(\overline{E}E(X) = X\). Then, if \(\overline{E} : K[C] \to K'[C]\) is the \(K\)-linear functor uniquely defined by \(\overline{E}(X) = X\) for any object \(X\) of \(C\) and \(\overline{E}(f) = \tau_Y f \tau_X^{-1}\) for any morphism \(f : X \to Y\) in \(C\), it is easily checked that \(\overline{E}E\) is a strict inverse of \(E\). The last assertion follows from Theorem 13. \(\square\)

Let us now consider the case \(C\) is a finite homogeneous groupoid \(G\), which we may assume it is skeletal. By the previous Lemma, to prove that \(\GL(G)\) is split it is enough to see that \(\text{Aut}_{\text{Cat}_K}(K'[G])\) is split. But this is a strict 2-group and hence, Proposition 12 and the subsequent remark can be applied. Now, if \(G\) is skeletal, we have an strict equality \(K'[G] = \sqcup^n K'[G][1]\) and hence

\[
|\text{Aut}_{\text{Cat}_K}(\sqcup^n K'[G][1])| = \Sigma_n \times (\text{Aut}_{\text{Alg}_K}(K'[G]))^n,
\]

while

\[
\pi_0(\text{Aut}_{\text{Cat}_K}(\sqcup^n K'[G][1])) = \Sigma_n \times (\text{Out}_{\text{Alg}_K}(K'[G]))^n.
\]

Therefore, the split character of \(\text{Aut}_{\text{Cat}_K}(K'[G])\) readily follows from the next result:

**Lemma 45** Let \(K\) be an algebraically closed field and \(G\) a finite group whose order is not divisible by the characteristic of \(K\). Then, the exact sequence of groups \(1 \to \text{Inn}_{\text{Alg}_K}(K[G]) \to \text{Aut}_{\text{Alg}_K}(K[G]) \to \text{Out}_{\text{Alg}_K}(K[G]) \to 1\) splits.

**Proof.** It has already been shown that, under the above hypothesis on \(K\) and \(G\), the group \(\text{Aut}_{\text{Alg}_K}(K[G])\) is the semidirect product of \(\text{Inn}_{\text{Alg}_K}(K[G])\) and the subgroup of the so called permutation automorphisms, which is isomorphic to a direct product of symmetric groups (see proof of Lemma 18). \(\square\)

**Corollary 46** For any finite 2-group \(G\), the general linear 2-group of the 2-vector space it generates is split and with homotopy groups

\[
\pi_0(\GL(G)) \cong \Sigma_n \times \Sigma_p^n \\
\pi_1(\GL(G)) \cong (K^*)^{pn}
\]

with \(n\) and \(p\) the cardinals of \(\pi_0(G)\) and \(\pi_1(G)\), respectively.
Proof. \( G \) is a finite homogeneous groupoid of coarse size \( n \) and underlying group \( \pi_1(\mathbb{G}) \), which is abelian.

6 Final comments

The notion of generalized 2-vector space introduced in this work includes Kapranov and Voevodsky 2-vector spaces, defined as a special kind of \( \text{Vect}_K \)-module category (see [12]). It is then worth comparing the notion of generalized 2-vector space with the general notion of \( \text{Vect}_K \)-module category. In this sense, it is tedious but not difficult to see that any generalized 2-vector space \( \text{Vect}_K[C] \) has a “canonical” \( \text{Vect}_K \)-module category structure, with \( \text{Vect}_K \) acting on \( \text{Vect}_K[C] \) by

\[
V \otimes S = \begin{cases} (S, n, n), S & \text{if } \dim V = n \\ \emptyset & \text{if } \dim V = 0 \end{cases}
\]

for any vector space \( V \) and any object \( S \) of \( \text{Vect}_K[C] \), and

\[
f \otimes A = \begin{pmatrix} \alpha_1A & \cdots & \alpha_nA \\ \vdots & \ddots & \vdots \\ \alpha_{n'}A & \cdots & \alpha_{n'n}A \end{pmatrix}
\]

for any linear map \( f : V \to V' \) and morphism \( A : S \to S' \), where \( (\alpha_{i'j}) \) is the matrix of \( f \) in previously chosen linear bases of \( V \) and \( V' \) \footnote{This is a special case of the “canonical” \( \text{Vect}_K \)-module category structure that can be defined on any \( K \)-linear additive category \( \mathcal{A} \) once we choose particular biproduct functors of all orders and a zero object in \( \mathcal{A} \) as well as a linear basis in each vector space.}. This can be seen as the analog of the canonical \( K \)-linear structure on the sets \( K[X] \). However, it is unlikely that an arbitrary \( \text{Vect}_K \)-module category is equivalent to a generalized 2-vector space equipped with such a \( \text{Vect}_K \)-module category structure, as it happens with vector spaces. Consequently, our notion of generalized 2-vector space should still be thought of as a particular kind of \( \text{Vect}_K \)-module category, although of a less restrictive kind than Kapranov and Voevodsky 2-vector spaces.

To finish this paper, it seems worth pointing out the drawbacks our notion of generalized 2-vector space has with respect to Kapranov and Voevodsky 2-vector spaces. In particular, let us mention some nice properties Kapranov and Voevodsky 2-vector spaces have that are lost when moving to generalized 2-vector spaces. One such lost property has already been mentioned before. Namely, an arbitrary generalized 2-vector space is not always a Karoubian category (see §3.6). Another important drawback concerns the property of having a dual object. Thus, an arbitrary generalized 2-vector space has no dual object, at least in the usual sense of the term, except when it is a Kapranov and Voevodsky 2-vector. Indeed, if for a given \( K \)-linear additive category \( \mathcal{A} \) there exists a \( K \)-linear additive category \( \mathcal{A}^* \) and \( K \)-linear functors \( \text{ev} : \mathcal{A}^* \boxtimes \mathcal{A} \to \text{Vect}_K \) and \( \text{coev} : \text{Vect}_K \to \mathcal{A} \boxtimes \mathcal{A}^* \) such that \( (\text{id}_A \boxtimes \text{ev})(\text{coev} \boxtimes \text{id}_A) \cong \text{id}_A \) and \( (\text{ev} \boxtimes \text{id}_A^*)(\text{id}_A \boxtimes \text{coev}) \cong \text{id}_A^* \) \footnote{Here, \( \boxtimes \) denotes the tensor product pseudofunctor for the 2-category \( \text{AdCat}_K \) of \( K \)-linear additive categories; see [12].}, it may be shown that \( \mathcal{A} \) is necessarily a Kapranov and Voevodsky 2-vector space (this result seems to be due to P. Schauenburg; a proof can
be found in Neuchl’s thesis [14]). If \( \mathcal{A} \simeq_{K} \text{Vect}^\ast_K \), such a dual object indeed exists and it is given by the natural candidate, i.e., the hom-category \( \text{Hom}_{2\text{Vect}_K}(\mathcal{A}, \text{Vect}_K) \), which is again a Kapranov and Voevodsky 2-vector space of rank \( n \), in complete analogy with the situation for vector spaces. Such a drawback may be serious when trying to define a Frobenius structure on the 2-algebra \( \text{Vect}_K[\mathcal{G}] \) generated by a 2-group \( \mathcal{G} \), because in the zero dimensional setting such a structure makes explicit use of duals. But it is also possible that a definition of Frobenius structure on a 2-algebra (more generally, on any pseudomonoid in a monoidal 2-category) may exists which makes no use of duals (actually, such a definition where duals do not appear already exists in the context of algebras or, more generally, monoids in a monoidal category). Finally, another important drawback of generalized 2-vector spaces is that, for arbitrary categories \( \mathcal{C} \) and \( \mathcal{D} \), the category of morphisms between the corresponding generalized 2-vector spaces \( \text{Hom}_{2\text{VECT}_K}(\text{Vect}_K[\mathcal{C}], \text{Vect}_K[\mathcal{D}]) \) may no longer be a generalized 2-vector space. For instance, if \( \mathcal{C} = M[1] \), for some monoid \( M \), and \( \mathcal{D} = 1 \), this category of morphisms is nothing but the category of finite dimensional linear representations of \( M \), which is not a generalized 2-vector space for an arbitrary \( M \).

**Acknowledgments.** I would like to acknowledge B. Toen for his many valuable comments, in particular, for pointing out to me the relation discussed in Section 4 between the constructions \( \text{Vect}_K[\mathcal{C}] \) and \( \text{VECT}^{op}_K \).

**References**

[1] M. Atiyah. On the krull-schmidt theorem with application to sheaves. *Bull. Soc. Math. France*, 84:307–317, 1956.

[2] J. Baez and A. Crans. Higher-dimensional algebra vi: Lie 2-algebras. *Preprint (arXiv: math.QA/0307263)*, 2003.

[3] H. Bass. *Algebraic K-theory*. W. Benjamin Inc., 1968.

[4] L. Breen. Théorie de schreier supérieure. *Ann. Scient. École Norm. Sup.*, 25:465–514, 1992.

[5] C.W. Curtis and I. Reiner. *Representation theory of finite groups and associative algebras*. Interscience, 1962.

[6] Y.A. Drozd and V.V. Kirichencko. *Finite dimensional algebras*. Springer-Verlag, 1994.

[7] J. Elgueta. An analog of group algebras in the category setting. *(in preparation)*.

[8] J. Elgueta. Representation theory of 2-groups on kapranov and voevodsky 2-vector spaces. *arxiv.org: math.CT/0408...*.

[9] J. Elgueta. A strict totally coordinatized version of kapranov and voevodsky 2-category \( 2\text{vect}_k \). to appear in *Math. Proc. Cambridge Phil. Soc.*

[10] W. Fulton and J. Harris. *Representation theory. A first course*, volume 129 of *GTM*. Springer Verlag, 1991.
[11] T.W. Hungerford. *Algebra*, volume 73 of *GTM*. Springer Verlag, 1974.

[12] M. Kapranov and V. Voevodsky. 2-categories and zamolodchikov tetrahedra equations. In *Proc. Sympos. Pure Math.*, volume 56(2), pages 177–260. American Mathematical Society, 1994.

[13] G. Karpilovsky. *Unit groups of classical rings*. Oxford University Press, 1988.

[14] M. Neuchl. *Representation theory of Hopf categories*. PhD Dissertation. University of Munich, 1997.

[15] L. Rowen. *Ring Theory, vol. 1*, volume 127 of *Pure and Applied Mathematics*. Academic Press, 1988.

[16] G. Segal. Two-dimensional conformal field theory and modular functors. In *Proc. IXth Intern. Congress on Mathematical Physics (Swansea, 1988)*, pages 22–37. Hilger, 1989.

[17] Hoang Xuan Sinh. *Gr-catégories*. Thèse de doctorat. Université Paris-VII, 1975.

[18] S.H. Weintraub. *Representation theory of finite groups: Algebra and Arithmetic*, volume 59 of *Graduate Studies in Mathematics*. AMS, 2003.