On the deformation groupoid of the inhomogeneous pseudo-differential Calculus

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Abstract

Recently, Van-Erp and Yuncken and independently Choi and Ponge defined an inhomogeneous deformation groupoid. As shown by Van-Erp and Yuncken, this deformation groupoid allows to fully recover the general inhomogeneous pseudo-differential calculus.

In this article we simplify and generalise this construction using a double (multiple) deformation to the normal cone.

Introduction

In order to construct a parametrix for Hörmander’s subelliptic operators on a contact manifold, Folland and Stein defined a noncommutative pseudo-differential calculus where the principal cosymbol is a function on a bundle of Heisenberg groups. A fundamental characteristic of this pseudo-differential calculus is that a vector field defines a differential operator of order 1 if it is everywhere tangent to the contact subbundle and of order 2 if not. Later on, this was generalised to an arbitrary subbundle of the tangent bundle, and even further to a filtration of the tangent bundle under conditions on the Lie bracket (see Folland, Stein, Connes, Moscovici). To such a structure one associates a bundle of graded nilpotent Lie groups over which the cosymbols are functions. Let us remark that the general situation is more involved because the bundle of graded nilpotent Lie groups doesn’t need to be locally trivial and the analogue of the theorem of Darboux doesn’t hold in general.

This calculus was later used by many authors in index theory and $C^*$-algebras, for instance by Connes and Moscovici to define a transversal signature operator on foliated manifolds and do computations in cyclic cohomology, following a construction of Hilsum and Skandalis, and by Julg and Kasparov to compute the $SU(n, 1)$ equivariant $KK$-theory following the work of Rumin.

In Debord and Skandalis gave a global definition of the classical pseudo-differential calculus thanks to the tangent groupoid of Connes. This definition was adapted to the case of inhomogeneous calculus by Van-Erp and Yuncken. To this end they used an inhomogeneous deformation groupoid instead of Connes’s tangent groupoid. This inhomogeneous groupoid was constructed in the case of contact manifolds by Ponge and van-Erp independently, and in the general case of filtrations by Choi and Ponge and van Erp and Yuncken following work by Julg and van Erp.

This groupoid was also used by van Erp and later (with Baum) to formulate and prove an index formula in the same spirit as that of Atiyah-Singer. Their index theorem is for differential operators whose cosymbol is invertible in the above calculus associated to a contact structure. These operators are necessarily hypoelliptic, hence their analytic index is well defined but they are rarely elliptic. It was also used by van Erp to formulate and prove an index theorem for hypoelliptic operators on foliated manifolds.

In the present article, we give an elementary construction of this Carnot groupoid using the deformation to the normal cone construction. Our approach gives rise to noncommutative Lie groupoids/symbols precisely because we deform Lie groupoids with respect to subgroupoids and not with respect to spaces, and contrary to the methods used in Folland, Stein, Connes, Moscovici. No analysis on local coordinates is needed to construct the Lie groupoid, only functoriality of

Following Ponge’s recommendation, the inhomogeneous deformation groupoid will be called the Carnot groupoid.
the DNC construction, furthermore no analysis on higher jets is needed to construct the Lie groupoid.

We now briefly describe our construction.

Let us first recall the deformation to the normal cone construction. If $V \subseteq M$ is submanifold, then the set $\text{DNC}(M, V) = M \times \mathbb{R}^* \cup N^0_V \times \{0\}$ admits a natural smooth structure, where $N^0_V$ is the normal bundle. It then follows from the functoriality of the construction that if $H \subseteq H^0$ is a Lie subgroupoid of $G \subseteq G^0$, then $\text{DNC}(G, H)$ is naturally a Lie groupoid over $\text{DNC}(G^0, H^0)$. Connes's tangent groupoid is then the groupoid $\text{DNC}(M \times M, M) = M \times \mathbb{R}$.

Our construction of the inhomogeneous deformation groupoid is as follows. Let $H \subseteq TM$ be a vector bundle. Recall the tangent groupoid

$$\text{DNC}(M \times M, M) = M \times M \times \mathbb{R}^* \cup TM \times \{0\} \supset M \times \mathbb{R}$$

defined by Connes. The space $H \times \{0\} \subseteq TM \times \{0\} \subseteq \text{DNC}(M \times M, M)$ is a Lie subgroupoid. Hence by the Functoriality of the DNC construction, the space

$$\text{DNC}(\text{DNC}(M \times M, M), H \times \{0\}) \supset \text{DNC}(M \times \mathbb{R}, M \times \{0\}) = M \times \mathbb{R}^2$$

is a Lie groupoid. We prove that the fiber over $M \times \{1\} \times \mathbb{R}$ is the Carnot Lie groupoid. Furthermore the groupoid $\text{DNC}(\text{DNC}(M \times M, M), H \times \{0\}) \supset M \times \mathbb{R}^2$ is a quite natural object to study because it contains ‘the deformations in all the directions’. In the case of a 2-step filtration $H^1 \subseteq H^2 \subseteq TM$ with the hypothesis $[H^1, H^1] \subseteq H^2$, we construct a Lie groupoid as follows: by the previous construction with $H = H^1$, we have a Lie groupoid

$$M \times M \times \mathbb{R}^* \cup H^1 \oplus TM/H^1 \times \{0\},$$

with a nilpotent group structure on $H^1 \oplus TM/H^1$. The condition $[H^1, H^1] \subseteq H^2$ is then precisely the condition needed for

$$H^1 \oplus H^2/H^1 \times \{0\} \subseteq M \times M \times \mathbb{R}^* \cup H^1 \oplus TM/H^1 \times \{0\}$$

to be a Lie subgroupoid. Hence by functoriality of DNC the space

$$\text{DNC}(M \times M \times \mathbb{R}^* \cup H^1 \oplus TM/H^1 \times \{0\}, H^1 \oplus H^2/H^1 \times \{0\}) \supset M \times \mathbb{R}^2$$

is a Lie groupoid. We restrict to $M \times \{1\} \times \mathbb{R}$ to obtain the deformation groupoid associated to the filtration. The general case is then treated in section 3 by induction.

The methods developed here can be used to give a variety of examples of Lie groupoids which can be used to define an associated inhomogenous pseudo-differential calculi in a variety of geometric situations. In particular we extend the Carnot Lie groupoid to cover the case of transverse (to a foliation) hypoelliptic pseudo-differential calculus without any difficulty (examples 2.9 and 3.3).

This article is organised as follows.

In Section 1 some preliminaries are recalled.

In Section 1.1 we recall the notion of the deformation to the normal cone following [39, 16].

In Section 1.2 the iterated deformation to the normal cone construction is introduced.

In Section 1.3 a proposition is proved which will be used in Section 2.1 in order to give us the algebraic structure of the symbol part of the Carnot groupoid.

In Section 2 the case of a single bundle is treated thoroughly.

In Section 2.1 we define the Carnot groupoid and we calculate the Lie groupoid structure of the symbol part of Carnot groupoid.

In Section 2.2 as the construction in [39, 7, 8, 35] is based on local charts, we describe our construction locally, and show that the two construction agree.

In Section 3 we generalize the construction given in Section 2.1 but for a filtration of the tangent bundle proving that iterated deformation to the normal cone gives rise to the Carnot groupoid in the general case. This section is independent of Section 2.1 and provides another proof of Theorem 2.1.

The paper ends with a paragraph on a related construction of Sadegh and Higson [32] seen here as the quotient of a Lie groupoid by a Lie subgroupoid.
1 Preliminaries

1.1 Deformation to the normal cone construction

In this section, we recall the deformation to the normal cone construction following [39, 16]. The deformation to the normal cone of a manifold $M$ along an immersed submanifold $V$ is a manifold whose underlying set is

$$\text{DNC}(M, V) := M \times \mathbb{R}^2 \sqcup N_V^M \times \{0\},$$

where $N_V^M$ is the normal bundle of $V$ inside $M$. The smooth structure is defined by covering $\text{DNC}(M, V)$ with two sets. The first is $M \times \mathbb{R}^2$. The second is $\phi(N_V^M) \times \mathbb{R}^2 \sqcup N_V^M \times \{0\}$ where $\phi: N_V^M \to M$ is a tubular embedding. The smooth structure on $\phi(N_V^M) \times \mathbb{R}^2 \sqcup N_V^M \times \{0\}$ is given by declaring the following bijection a diffeomorphism

$$\tilde{\phi}: N_V^M \times \mathbb{R} \to \phi(N_V^M) \times \mathbb{R}^2 \sqcup N_V^M \times \{0\}$$

$$\tilde{\phi}(x, X, t) = (\phi(x, tX), t) \in M \times \mathbb{R}^2, \quad t \neq 0$$

$$\tilde{\phi}(x, X, 0) = (x, X, 0) \in N_V^M \times \{0\}.$$

This smooth structure is independent of $\phi$. This follows by noticing that the following functions are smooth functions that generate the smooth structure:

1. the function

$$(\pi_M, \pi_\mathbb{R}): \text{DNC}(M, V) \to M \times \mathbb{R}$$

$$(x, t) \to (x, t), \quad t \neq 0$$

$$(x, X, 0) \to (x, 0)$$

2. Let $f \in C^\infty(M)$ be a smooth function which vanishes on $V$. Therefore $df: N_V^M \to \mathbb{R}$ is well defined. The following function is smooth

$$\text{DNC}(f): \text{DNC}(M, V) \to \mathbb{R}$$

$$(x, t) \to \frac{f(x)}{t}, \quad t \neq 0$$

$$(x, X, 0) \to df_x(X)$$

The group $\mathbb{R}^2$ acts smoothly on $\text{DNC}(M, V)$. The action is given by $\lambda_u(x, t) = (x, ut)$ and $\lambda_u(x, X, 0) = (x, \frac{X}{u}, 0)$ for $u \in \mathbb{R}^2$.

**Proposition 1.1** (Functoriality of DNC). Let $M, M'$ be smooth manifolds, $V \subseteq M$, $V' \subseteq M'$ submanifolds, $f: M \to M'$ a smooth map such that $f(V) \subseteq V'$. Then the map defined by

$$\text{DNC}(M, V) \to \text{DNC}(M', V')$$

$$(x, t) \to (f(x), t), \quad t \neq 0$$

$$(x, X, 0) \to (f(x), df_x(X), 0)$$

is a smooth map, that will be denoted by $\text{DNC}(f)$. Furthermore the map $\text{DNC}(f)$ is

- a submersion if and only if $f$ is a submersion and $f|_V: V \to V'$ is also a submersion.

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2To simplify the exposition, we will always assume that tubular neighbourhoods are diffeomorphisms on $N_V^M$. In the case of immersed manifolds, the tubular neighbourhoods are only local in $V$.

3In the case where $V'$ is an immersed submanifold, one must also suppose that $f|_V: V \to V'$ is continuous.
• an immersion if and only if \( f \) is an immersion and for every \( v \in V \), \( T_v V = df_v^{-1}(TV') \).

Proof. Smoothness of DNC(\( f \)) follows from the description of smooth maps given above. For statements concerning submersions and immersions. Let \( U \subseteq DNC(M, V) \) be the set where the differential of DNC(\( f \)) is onto (respectively injective). It is clear that \( U \) is an open set that is invariant under the \( \mathbb{R}^n \) action and contains \( M \times \mathbb{R}^n \). To prove that \( U = DNC(M, V) \), it suffices to prove that \( V \times \{0\} \subseteq U \). If \( v \in V \), then one sees directly that

\[
T_{(v,0)} DNC(M, V) = \mathbb{R} \oplus T_v V \oplus T_v M/T_v V
\]

The differential of DNC(\( f \)) is then \( dDNC(f)_{(v,0)}(t, X, Y) = (t, df_v(X), df_v(Y)) \). The proposition is then clear.

The map

\[
N_{V_1}^M \to N_{V_2}^M, \quad (x, X) \to (f(x), df_x(X))
\]

will be denoted by \( Nf \).

Remark 1.2. More generally if \( V \) is a smooth manifold, \( i : V \to M \) is an immersion but not necessarily injective, then the manifold DNC(\( M, V \)) can still be defined using the same charts as above. The main difference is that the manifold DNC(\( M, V \)) is no longer Hausdorff.

Remark 1.3. It follows from Proposition 1.1 that if \( G \) is a Lie group acting smoothly on a manifold \( M \) that leaves a submanifold \( V \) invariant, then \( G \) acts smoothly on DNC(\( M, V \)). This action commutes with the \( \mathbb{R}^n \) action \( \lambda \). In particular the group \( G \times \mathbb{R}^n \) acts on DNC(\( M, V \)).

Proposition 1.4. Let \( M_1, M_2, M \) be manifolds, \( V_1 \subseteq M_1, V \subseteq M \) submanifolds, \( f_i : M_i \to M \) smooth maps such that

1. \( f_i(V_i) \subseteq V \) for \( i \in \{1, 2\} \)
2. the maps \( f_i \) are transverse
3. the maps \( f_i|_{V_i} : V_i \to V \) are transverse

Then

1. (a) the maps \( Nf_i : N_{V_1}^{M_i} \to N_{V}^{M} \) are transverse.
   (b) the natural map
   \[
   N_{V_1}^{M_1} \times N_{V_2}^{M_2} \to N_{V_1}^{M_1} \times N_{V_2}^{M_2}
   \]
   is a diffeomorphism.

   Similarly for DNC, we have

2. (a) the maps DNC(\( f_i \)) : DNC(\( M_i, V_i \)) \to DNC(\( M, V \)) are transverse.
   (b) the natural map
   \[
   DNC(M_1 \times M_2, V_1 \times V_2) \to DNC(M_1, V_1) \times DNC(M_2, V_2)
   \]
   is a diffeomorphism.

Proof. Let us prove 1. (a). The group \( \mathbb{R}^n \) actson \( N_{V_1}^{M_1} \) and \( N_{V_2}^{M_2} \). Both \( Nf_i \) are \( \mathbb{R}^n \) equivariant. Since transversality is an open condition it follows that it suffices to check transversality at the origin. In that case for \( x_1 \in V_1, x_2 \in V_2 \) such that \( f_1(x_1) = f_2(x_2) \), one has

\[
T_{(x_1,0)} N_{V_1}^M = T_{x_1} V_1 \oplus N_{V_2,x_2}^M.
\]

Transversality of \( f_i \) at \( (x_1,0) \) follows from condition 2 and 3.

Statement 1. (b) and bijectivity of the natural map

\[
DNC(M_1 \times M_2, V_1 \times V_2) \to DNC(M_1, V_1) \times DNC(M_2, V_2)
\]

directly follow from statement 1. (a). To prove that it is a diffeomorphism and that the maps DNC(\( f_i \)) are transverse, we use the same argument as in Proposition 1.1. The two conditions are open conditions which are \( \mathbb{R}^n \)-invariant. Hence it suffices to check that them at \( V_1 \times V_2 \) which follows directly from 1. (a) and 1. (b).

Proposition 1.5. Let \( V \subseteq M \) a submanifold, then
1. \(TN^M = N_T^M V\)

2. If \(\pi_\mathcal{N} : \text{DNC}(M, V) \to \mathbb{R}\) is the natural projection, then \(\ker(\pi_\mathcal{N}) = \text{DNC}(TM, TV)\).

Proof. We will define a diffeomorphism \(\phi : TN^M \to N_T^M V\). Let \(c(t, s) : \mathbb{R}^2 \to M\) be a smooth map such that \(c(t, 0) \in V\) for all \(t\). It follows that for each fixed \(t\), \(\partial_t c(t, 0) \in N^M_V\), and hence \(\partial_t \partial_t c(0, 0) \in TN^M\). Conversely for each fixed \(s\), \(\partial_s c(0, s) \in TM\) and its value at \(s = 0\) is in \(TV\), hence \(\partial_t \partial_t c(0, 0) \in N^M_V\). The map \(\phi\) is the map sending \(\partial_t \partial_t c(0, 0)\) to \(\partial_t \partial_t c(0, 0)\), for each path \(c\). One easily checks in local coordinates this \(\phi\) is indeed a diffeomorphism.

For the deformation to the normal cone, by definition \(\ker(\pi_\mathcal{N}) = TM \times \mathbb{R}^\ast \sqcup TN^M\). Using the map \(\phi\), one defines a map \(\ker(\pi_\mathcal{N}) \to \text{DNC}(TM, TV)\). It is straightforward to check that the resulting map is a diffeomorphism by checking so in local coordinates. \(\square\)

Let us recall the notion of a \(\mathcal{VB}\)-groupoid from [20][26].

**Definition 1.6.** Let \(H\) be a Lie groupoid. A \(\mathcal{VB}\)-groupoid over \(H\) is given by

- a vector bundle \(G\) over \(H\)
- a vector bundle \(G^0\) over \(H^0\)
- a Lie groupoid structure on \(G\) whose space of objects is \(G^0\), such that the map range map \(r : G \to G^0\), the inverse map \(i : G \to G\), the multiplication map \(m : G \times_{s,r} G \to G\) are respectively bundle maps over the range map \(r : H \to H^0\), the inverse map \(i : H \to H\) and the multiplication map \(m : H \times_{s,r} H \to H\).

By abuse of notation we will call \(G \cong G^0\) a \(\mathcal{VB}\)-groupoid over \(H\).

A \(\mathcal{VB}\)-subgroupoid of \(G\) is a Lie subgroupoid \(K \subseteq K^0\) of \(G\) such that \(K\) is a subbundle of \(G\) and \(K^0\) is a subbundle of \(G^0\).

**Theorem 1.7.** Let \(G\) be a Lie groupoid, \(H\) a Lie subgroupoid. Then

1. the space \(N^G_H \supseteq N^G_H^0\) is a Lie groupoid whose structure maps are \(Ns, Nr\) and whose Lie algebroid is equal to \(N^{\mathcal{A}G}_H\). Furthermore, \(N^G_H\) is a \(\mathcal{VB}\)-groupoid over \(H\).

2. the manifold \(\text{DNC}(G, H) \supseteq \text{DNC}(G^0, H^0)\) is a Lie groupoid whose structure maps are \(\text{DNC}(s), \text{DNC}(r)\) and Lie algebroid is equal to \(\text{DNC}(\mathcal{A}G, \mathcal{A}H)\).

3. if \(K \subseteq H\) is a Lie subgroupoid, then the restriction of the normal bundle \(N^G_H\mid_K \supseteq N^G_H^0\mid_K\) is a Lie subgroupoid of \(N^G_H \supseteq N^G_H^0\) whose Lie algebroid is \(N^{\mathcal{A}G}_H\mid_K\). Furthermore \(N^G_H\mid_K\) is a \(\mathcal{VB}\)-groupoid over \(K\).

Proof. The fact that \(N^G_H \supseteq N^G_H^0\) and \(\text{DNC}(G, H)\) are Lie groupoids is a direct consequence of propositions [1][4] and [3][4]. For example the product map of \(N^G_H \supseteq N^G_H^0\) is defined using proposition [4] If \(M : G^\ast \to G\) denotes the product map then

\[NM : (N^G_H)^{(2)} \to N^G_H\]

is well defined. Now using proposition [3][4] it follows that

\[N^{(2)}_H = N^G_H \times_{ds,dr} N^G_H\]

Hence \(NM\) can be identified with a map

\[N^G_H \times_{ds,dr} N^G_H \to N^G_H\]

This is by definition the product map of \(N^G_H\). Axioms like associativity and the identity all follow from the fact that the corresponding axioms hold for \(G\) and the functoriality of the constructions \(N\) and \(\text{DNC}\).

For the Lie algebroid computations, since \(N^G_H\) is a Lie groupoid, its Lie algebroid can be identified with kernel of the source map. The source map is \(Ns : N^G_H \to N^G_H^0\). Hence using proposition [3][4] \(dNs\) can be identified with \(Nds : N^G_H \to N^G_H^0\). The kernel under this identification becomes \(N^{\mathcal{A}G}_H\). Here we similarly identified \(\mathcal{A}G, \mathcal{A}H\) with the kernel of the source map.
For the Lie algebroid of the deformation to the normal cone, one proceeds similarly. The Lie algebroid of DNC\((G, H)\) is the kernel of \(d\text{DNC}(s) : T\text{DNC}(G, H) \rightarrow T\text{DNC}(G^0, H^0)\). The kernel of such map has to lie in \(\ker(d\pi_R)\). Hence one can instead only consider
\[
d\text{DNC}(s) : \ker(d\pi_R) \subseteq T\text{DNC}(G, H) \rightarrow \ker(d\pi_R) \subseteq T\text{DNC}(G^0, H^0).
\]
One then proceeds exactly the same as for \(N^G_H\).

The third statement follows from the first and because the projection map onto the base
\[
\begin{array}{ccc}
N^G_H & \longrightarrow & H \\
\downarrow & & \downarrow \\
N^G_{H^0} & \longrightarrow & H^0
\end{array}
\]
is a submersive morphism of groupoids, hence the inverse image of the Lie subgroupoid \(K\) is a Lie groupoid.

From now on, for a Lie groupoid \(G\) and a Lie subgroupoid \(H\), we will use \(N^G_H\) to denote the space \(N^G_H\) equipped with the structure of a Lie groupoid given by Theorem 1.7.

**Remarks 1.8.**

1. Let \(E \rightarrow M\) be a vector bundle, \(V \subseteq M\) a submanifold, \(F \rightarrow V\) a subbundle of the restriction of \(E\) to \(V\). By Theorem 1.7, the space \(\text{DNC}(E, F)\) is a vector bundle over \(\text{DNC}(M, V)\). Since a section of \(\text{DNC}(E, F)\) is determined by its values on the dense set \(M \times \mathbb{R}^n\), it follows that
\[
\Gamma(\text{DNC}(E, F)) = \{X \in \Gamma(E \times \mathbb{R}) : X_{|V \times \{0\}} \in \Gamma(F)\},
\]
where \(\Gamma\) denotes the set of global sections (continuous or smooth).

2. Let \(V = V_0 + a \subseteq \mathbb{R}^n\) be an affine subspace where \(V_0\) is the underlying vector space, \(a \in \mathbb{R}^n\). Let \(L\) be the orthogonal of \(V_0, \pi_{V_0}, \pi_L\) the orthogonal projections. The space \(\text{DNC}(\mathbb{R}^n, V)\) will be identified with \(\mathbb{R}^{n+1}\) by the following map
\[
\begin{array}{l}
\text{DNC}(\mathbb{R}^n, V) \rightarrow \mathbb{R}^{n+1} \\
(x, t) \rightarrow (a + \pi_{V_0}(x-a) + \pi_L(x-a)/t, t), \quad t \neq 0 \\
(x, x, 0) \rightarrow (x + X, 0),
\end{array}
\]
where in the last identity we identified \(N^\mathbb{R}_{V}^n\) with \(L\).

**Examples 1.9.**

1. If \(M\) is a smooth manifold, then
\[
\text{DNC}(M \times M, M) = M \times M \times \mathbb{R}^n \cup TM \times \{0\} \cong M \times \mathbb{R}
\]
is the tangent groupoid of Connes. He used it to give a short elegant proof of Atiyah Singer index theorem \([12]\). The product law is given by
\[
(x, y, t) \cdot (y, z, t) = (x, y, 0) \cdot (y, z, 0) = (x, y + z),
\]
where \((x, y, 0) = (x, y, 0)\).

2. Let \(L \subseteq G^0\) be a submanifold. Here we will calculate \(N^L_G\). Notice that \(N^L_G\) is equal to \(\ker(ds)|_L \oplus N^L_L^0\). If \((Y, Z) \in \ker(ds)|_L \oplus N^L_L^0\), then \(s_{X^L_L}(Y, Z) = ds(Y) + ds(Z) = Z\) by assumption on \(Y\). Also \(r_{X^L_L}(Y, Z) = dr(Y) + dr(Z) = z(Y) + Z\). Here we used the definition of the anchor map \(z := dr - ds\). It follows that the groupoid \(N^L_G \cong N^L_G^0\) is equal to
\[
\{(X, Y, Z) : l \in L, X, Z \in \text{N}^L_L^0, Y \in \mathfrak{g}G, X = Y + z(Z)\},
\]
with the structural maps
\[
r(X, Y, Z) = X, s(X, Y, Z) = Z.
\]
Finally the product is given by
\[
(A, B, C)(C, D, E) = (A, B + D, E).
\]
To see this notice that one has a natural map \(N^L_G \rightarrow N^G_{H^0}\). This is simply the quotient map. In the above identification this map sends \((X, Y, Z) \rightarrow Y\). By functoriality of the map, it has to be a morphism of groupoids, hence the product has to agree with the product of the adiabatic groupoid \(N^G_{H^0}\).
1.2 DNC iterated

Let $M$ be a smooth manifold, $V_0 \subseteq M$ a submanifold, $V_1 \subseteq \text{DNC}(M, V_0)$ a submanifold. One defines

$$\text{DNC}^2(M, V_0, V_1) := \text{DNC}(\text{DNC}(M, V_0), V_1).$$

This space being a deformation space admits an $\mathbb{R}^*$-action that will be denoted by $\lambda^{(1)}$, and a projection map $\pi_2^{(1)} : \text{DNC}^2(M, V_0, V_1) \to \mathbb{R}$.

If $V_1$ is $\mathbb{R}^*$-invariant, then by Remark 1.3, the group $\mathbb{R}^*$ acts on $\text{DNC}^2(M, V_0, V_1)$. This action will be denoted by $\lambda^{(0)}$, furthermore the group $(\mathbb{R}^*)^2$ acts on $\text{DNC}^2(M, V_0, V_1)$ by $\lambda^{(0)} \times \lambda^{(1)}$.

Let $\pi : \text{DNC}(M, V) \to \mathbb{R}$ the projection constructed in Section 1.1. If $\pi(V_1)$ is a point of $\mathbb{R}$, then the map

$$\pi^{(0,1)} := \text{DNC}(\pi) : \text{DNC}^2(M, V_0, V_1) \to \text{DNC}(\mathbb{R}, \pi(V_1)) = \mathbb{R}^2$$

is a smooth submersion, where we identified $\text{DNC}(\mathbb{R}, \pi(V_1))$ with $\mathbb{R}^2$ using remarks 1.8.

If $V_1$ is furthermore $\mathbb{R}^*$-invariant (hence $\pi(V_1) = \{0\}$), then one has for all $u, t \in \mathbb{R}^*$

$$\pi^{(0,1)} \lambda^{(1)} = \left( \frac{\pi^{(0)}}{u}, u \pi^{(1)} \right), \pi^{(0,1)} \lambda^{(0)} = \left( u \pi^{(0)}, \pi^{(1)} \right),$$

where $\pi^{(0,1)} = (\pi^{(0)}, \pi^{(1)})$.

By induction, given a sequence of submanifolds

$$V_0 \subseteq M, V_1 \subseteq \text{DNC}(M, V_0), V_2 \subseteq \text{DNC}^2(M, V_0, V_1), \cdots, V_k \subseteq \text{DNC}^k(M, V_0, \cdots, V_{k-1}).$$

We define the space

$$\text{DNC}^{k+1}(M, V_0, \cdots, V_k) := \text{DNC}(\text{DNC}^k(M, V_0, \cdots, V_{k-1}), V_k).$$

If for each $1 \leq i \leq k$, $\pi^{(0,\cdots,i-1)}(V_i)$ is an affine subspace of $\mathbb{R}^i$ and $\pi^{(0,\cdots,i-1)} : V_i \to \pi^{(0,\cdots,i-1)}(V_i)$ is a submersion, then by Proposition 1.1 the map

$$\pi^{(0,\cdots,k)} := \text{DNC}(\pi^{(0,\cdots,k-1)}) : \text{DNC}^{k+1}(M, V_0, \cdots, V_k) \to \text{DNC}(\mathbb{R}^k, \pi^{(0,\cdots,k-1)}(V_k)) = \mathbb{R}^{k+1}$$

is a smooth submersion, where we identified $\text{DNC}(\mathbb{R}^k, \pi^{(0,\cdots,k-1)}(V_k))$ with $\mathbb{R}^{k+1}$ using remarks 1.8.

If each $V_i$ is $(\mathbb{R}^*)^i$-invariant, then the space $\text{DNC}^{k+1}(M, V_0, \cdots, V_k)$ admits $k + 1$ pairwise commuting actions of $\mathbb{R}^*$-denoted $\lambda^{(k)}, \cdots, \lambda^{(0)}$.

Propositions 1.1 1.4 and Theorem 1.7 have obvious extensions to $\text{DNC}^k$.

**Corollary 1.10.** If $G \rightrightarrows G^0$ is a Lie groupoid, $H_0 \subseteq G$, $H_1 \subseteq \text{DNC}(G, H_0)$, $\cdots$, $H_k \subseteq \text{DNC}^k(G, H_0, \cdots, H_{k-1})$. are Lie subgroupoids, then

$$\text{DNC}^{k+1}(G, H_0, H_1, \cdots, H_k) \rightrightarrows \text{DNC}^{k+1}(G^0, H_0^0, \cdots, H_k^0)$$

is a Lie groupoid.

1.3 Description of the symbol part

The Carnot groupoid is a groupoid of the form

$$M \times M \times \mathbb{R}^* \sqcup Q \times \{0\} \rightrightarrows M \times \mathbb{R}.$$
Proposition 1.11. Let $G \cong G^0$ be a Lie groupoid, $H \subseteq G$ a Lie subgroupoid which is a bundle of connected Lie groups such that
\[(dr - ds)(T_h G) \subseteq T_{s(h)} H^0, \quad \forall h \in H.\]

Then
1. the Lie groupoid $N^G_H \rightrightarrows N^G_{H^0}$ is a bundle of Lie groups.
2. the Lie groupoid $N^G_H \times H^0 \rightrightarrows N^G_{H^0}$ is a bundle of abelian Lie groups which is isomorphic (as a bundle of Lie groups over $N^G_{H^0}$) to $\mathfrak{A}G/\mathfrak{A}H \times H^0 N^G_{H^0}$.
3. the Lie groupoid $N^G_H \rightrightarrows N^G_{H^0}$ sits in an exact sequence of bundles of Lie groups over $N^G_{H^0}$ whose fiber at $(x_0, X_0) \in N^G_{H^0}$ is
\[1 \to \mathfrak{A}G_{x_0}/\mathfrak{A}H_{x_0} \to \left(\frac{N^G_H}{(x_0, X_0)}\right) \to H_{x_0} \to 1.\]

Furthermore the action associated to this exact sequence of the Lie algebra $\mathfrak{A}H_{x_0}$ on the abelian group $\mathfrak{A}G_{x_0}/\mathfrak{A}H_{x_0}$ is as follows; if $X, Y \in \Gamma^\infty(\mathfrak{A}G)$ such that $X_{|H^0} \in \Gamma^\infty(\mathfrak{A}H)$, then by our assumption,
\[\{X, Y\}|_{(x_0)} \mod \mathfrak{A}H_{x_0}\]
only depends on $X(x_0) \in \mathfrak{A}H_{x_0}$ and $Y(x_0) \mod \mathfrak{A}H_{x_0} \in \mathfrak{A}G_{x_0}/\mathfrak{A}H_{x_0}$. In particular the above exact sequence is central if and only if this action is trivial.

Proof. 1. The condition $(dr - ds)(T_h G) \subseteq T_{s(h)} H^0$ can be restated as the equality of the maps $Ns, Nr : T_hG/T_hH \to T_{s(h)}G/T_{s(h)}H^0$. Those two maps are the source and the target maps of the Lie groupoid $N^G_H = \sqcup_{h \in H} T_hG/T_hH \cong N^G_{H^0}$. By assumption, they coincide which means that $N^G_H \rightrightarrows N^G_{H^0}$ is a bundle of Lie groups.

2. If $X \subseteq Y \subseteq Z$ are manifolds, then $N^G_H \rightrightarrows N^G_{H^0}$ is the surjective image by a groupoid morphism of the Lie groupoid $N^G_H \times H^0$ with kernel $N^G_{H^0}$. One has
\[N^G_H = \{(X, Y, Z) : X, Z \in N^G_{H^0}, Y \in \mathfrak{A}G/\mathfrak{A}H, Z = X + \gamma(Y)\} \subseteq N^G_{H^0}.\]

By assumption, the map $\gamma : \mathfrak{A}G/\mathfrak{A}H \to N^G_{H^0}$ is the zero map. Hence $N^G_H \rightrightarrows N^G_{H^0}$ is a bundle of abelian Lie groups, hence $N^G_H \rightrightarrows N^G_{H^0}$ as well.

3. the exact sequence is the natural sequence
\[N^G_H \to N^G_{H^0} \to H.\]

(a) exactness at $\mathfrak{A}G_{x_0}/\mathfrak{A}H_{x_0}$ is clear, because $N^G_H \rightrightarrows N^G_{H^0}$ is a subgroupoid of $N^G_H$.

(b) exactness at $\left(\frac{N^G_H}{(x_0, X_0)}\right)$ follows directly from the definitions.

(c) the map $s : G \to G^0$ is a submersion, hence exactness at $H_{x_0}$.

Let us prove that $\{X, Y\}$ only depends on $X(x_0)$ and $Y(x_0)$, where $X, Y \in \Gamma^\infty(\mathfrak{A}G)$ such that $X_{|H^0} \in \Gamma^\infty(\mathfrak{A}H)$.

• If $Y$ vanishes at $x_0$, then locally it can be written as the sum of sections of the form $fZ$, where $f : M \to \mathbb{R}$ vanishes at $x_0$ and $Z \in \Gamma^\infty(\mathfrak{A}G)$. One has
\[\{X, fZ\} = f(x_0)\{X, Z\}(x_0) + df_{x_0}(\gamma(X(x_0)))Z(x_0) = 0,\]
because $X(x_0) \in \mathfrak{A}H_{x_0}$ and $H$ is a bundle of Lie groups, hence $\gamma(X(x_0)) = 0$.

• If $Y_{|H^0} \in \Gamma^\infty(\mathfrak{A}H)$, then $\{X, Y\}(x_0) \in \mathfrak{A}H_{x_0}$ because the Lie bracket computation could be carried out inside $\mathfrak{A}H$.

• If $X$ vanishes at $x_0$, then $dX_{x_0} : T_{x_0} G^0 \to \mathfrak{A}x_0 G$ is well defined. It is well known that $\{X, Y\}(x_0) = -dX_{x_0}(\gamma(Y(x_0)))$. This formula can be proved locally by writing $X$ as sum of $FZ$. The condition $X_{|H^0} \in \Gamma^\infty(\mathfrak{A}H)$ implies that $dX_{x_0}(T_{x_0} H^0) \subseteq \mathfrak{A}H^0$. The assumption on $dr - ds$ implies that $\gamma(Y(x_0)) \in T_{x_0} H^0$, hence $\{X, Y\}(x_0) \in \mathfrak{A}H_{x_0}$.

That is the action associated to the abelian extension of $\left(\frac{N^G_H}{(x_0, X_0)}\right)$ is then clear. \(\blacksquare\)
2 The case of a single subbundle

2.1 The product structure of the symbol part

Let $M$ be a smooth manifold, $H \subseteq \mathcal{N}_{\mathcal{M}}^{M \times M} = TM$ a subbundle. In this section we prove Proposition 1.11 which proves the claim made in the introduction (at least on the algebraic level) that the fiber of the groupoid $\text{DNC}^2(M \times M, H \times \{0\}) \cong \text{DNC}^2(M, M, M \times \{0\}) = M \times \mathbb{R}^2$ over $M \times \{1\} \times \mathbb{R}$ is equal to the groupoid constructed in \([6, 8, 7, 38]\). In Section 2.2 we will write local charts which will prove that in fact the fiber is equal as a smooth manifold to one constructed in \([6, 8, 7, 38]\).

Before stating the theorem, let us recall the construction of the Levi form $\mathcal{L} :$ the map

$$\Gamma^\infty(H) \times \Gamma^\infty(H) \rightarrow \Gamma^\infty(TM/H), \quad (X, Y) \rightarrow [X, Y] \quad \text{mod } H \tag{2}$$

is $C^\infty(M)$-linear because

$$[fX, Y] = f[X, Y] - XF = f[X, Y] \mod H.$$ 

Hence it comes from an anti symmetric bilinear bundle map $\mathcal{L} : H \times H \rightarrow TM/H.$

**Theorem 2.1.** The groupoid $\mathcal{N}_{H \times \{0\}}^{\text{DNC}(M \times M, M)} \cong \mathcal{N}_{M \times \{0\}}^{M \times M} = M \times \mathbb{R}$ is a bundle of Lie groups. It is isomorphic to the bundle of Lie groups $H \oplus TM/H \times \mathbb{R}$ equipped with the group law

$$(h, n, t) \cdot (h', n', t') = \left(h + h', n + n' + \frac{t}{2} \mathcal{L}(h, h'), t\right).$$

**Proof.** First we apply Proposition \([1,1]\) to $\text{DNC}(M \times M, M) \cong M \times \mathbb{R}$ and $H \times \{0\} \cong M \times \{0\}.$ Let us check the condition of Proposition \([1,1]\) and the triviality of the action.

- Since $\pi_2 \circ r = \pi_2 \circ s,$ the condition of Proposition \([1,1]\) is satisfied.

- the triviality of the action is immediate to check. If $X$ is a section of $TM$ over $M \times \mathbb{R}$ which vanishes on $M \times \{0\},$ $Y$ is a section of $TM$ over $M \times \mathbb{R}$ which vanishes on $M \times \{0\}$ and whose $\partial_t$-derivative on $M \times \{0\}$ is in $H,$ then the vector field $[X, Y]$ vanishes over $M \times \{0\}.$

The central exact sequence of bundles of Lie groups over $(x_0, t_0) \in \mathcal{N}_{M \times \{0\}}^M = M \times \mathbb{R}$ given by Proposition \([1,1]\) is then equal to

$$1 \rightarrow T_{x_0}M/H_{x_0} \rightarrow \left(\mathcal{N}_{H \times \{0\}}^{\text{DNC}(M \times M, M)}\right)_{(x_0, t_0)} \rightarrow H_{x_0} \rightarrow 1.$$

There exists a quite natural section of this exact sequence: let $h \in H_{x_0},$ $f : \mathbb{R} \rightarrow M$ any smooth function such that $f(0) = x_0,$ $f'(0) = h$ and $f'(t) \in H_{f(t)} \forall t,$

$$\begin{align*}
\sigma_{x_0, t_0}(h, \cdot) & : \mathbb{R} \rightarrow \text{DNC}(M \times M, M) \\
\sigma_{x_0, t_0}(h, u) & = (f(t_0u), x_0, t_0u) \quad \text{if } t_0u \neq 0 \\
\sigma_{x_0, t_0}(h, 0) & = (x_0, h, 0) \quad \text{if } t_0u = 0
\end{align*}$$

One then sees immediately that the map

$$\mathcal{S}_{x_0, t_0} : H_{x_0} \rightarrow \left(\mathcal{N}_{H \times \{0\}}^{\text{DNC}(M \times M, M)}\right)_{(x_0, t_0)}$$

$$h \rightarrow \left(\frac{\partial}{\partial u}|_{u=0}\sigma_{x_0, t_0}(h, u) \mod T_{(x_0, h, 0)}(H \times \{0\})\right)$$

is well defined (i.e., doesn’t depend on the choice of $f$) and is a section of the above exact sequence.

The map $\mathcal{S}_{x_0, t_0}$ is not a group homomorphism. For $h_1, h_2 \in H_{x_0},$ we have

$$\mathcal{S}_{x_0, t_0}(h_1)\mathcal{S}_{x_0, t_0}(h_2)\mathcal{S}_{x_0, t_0}(-h_1 - h_2) = \frac{t_0}{2} \mathcal{L}(h_1, h_2) \in T_{x_0}M/H_{x_0}.$$ 

This follows from the definition of $\mathcal{L}.$ See eq. (2).
Corollary 2.2. The fiber of the groupoid

$$\text{DNC}^2(M \times M, M, H \times \{0\}) \Rightarrow M \times \mathbb{R}^2$$

over $M \times \{1\} \times \mathbb{R}$ is equal to (as an algebraic groupoid) to

$$M \times M \times \mathbb{R}^* \sqcup H \oplus TM/H \times \{0\} \Rightarrow M \times \mathbb{R},$$

where the groupoid structure on $M \times M \times \mathbb{R}^*$ is the pair groupoid, and on $H \oplus TM/H$ is the bundle of nilpotent Lie groups

$$(h,n) \cdot (h',n') = \left( h + h', n + n' + \frac{1}{2}L(h,h') \right).$$

Since $H$ is $\mathbb{R}^*$ invariant, by Section [1] we have two group actions $\lambda^1, \lambda^0$ of $\mathbb{R}^*$ on $\mathcal{N}^\text{DNC}(M, V)_{(0)}$. Under the above identification the two actions $\lambda^1$ and $\lambda^0$ become

$$\lambda^0(h,n,t) = \left( \frac{h}{u}, \frac{n}{u}, ut \right), \quad \lambda^1(h,n,t) = \left( h, \frac{n}{u}, t \right).$$

2.2 Local charts for $\mathcal{N}^\text{DNC}(M,V)_{H \times \{0\}}$

In this section the development done in section 2.1 at the level of Lie algebroids is done in parallel at the level of local charts. This is more general as it applies to $\mathcal{N}^\text{DNC}(M, V)_{H \times \{0\}}$ which is in general only a smooth manifold.

Let $M$ be a smooth manifold, $V$ a submanifold, $H \subseteq N^M_V$ a smooth subbundle, $\mathcal{H}$ the lift of $H$ to $TM$. In other words $\mathcal{H}$ is a subbundle of the restriction of $TM$ to $V$ such that $TV \subseteq \mathcal{H}$ and $H = \mathcal{H}/TV$. In this section we give an alternate description of the fiber $\mathcal{N}^\text{DNC}(M, V)_{H \times \{0\}}$.

Definition 2.3. Let $\tilde{N}^M_{V,H}$ the set of smooth functions $f : \mathbb{R} \to M$ such that $f(0) \in V$ and $f'(0) \in \mathcal{H}$. Let $N^M_{V,H}$ be the quotient of $\tilde{N}^M_{V,H}$ by the equivalence relation where $f, g \in \tilde{N}^M_{V,H}$ are equivalent if and only if

1. $f(0) = g(0)$
2. $f'(0) - g'(0) \in T_{f(0)}V$.
3. for every smooth function $l : M \to \mathbb{R}$ which vanishes on $V$ and whose derivative $dl$ vanishes on $\mathcal{H}$, one has $(l \circ f)'(0) = (l \circ g)'(0)$.

Let $\pi : \text{DNC}(M, V) \to \mathbb{R}$ be the projection. Since $\pi_{(0)}(H) = 0$, the map $N_{\pi_{(0)}} : N^\text{DNC}(M, V)_{H \times \{0\}} \to N^\pi_{(0)} = \mathbb{R}$ is well defined. We claim that the set $N^M_{V,H}$ is in a natural bijection with $(N_{\pi_{(0)}})^{-1}(1)$.

To see this let $f \in \tilde{N}^M_{V,H}$. Since $f(0) \in V$, the function

$$\text{DNC}(f) : \mathbb{R} \to \text{DNC}(M, V), \quad t \to (f(t), t), \text{if } t \neq 0, 0 \to (f'(0), 0)$$

is smooth. In the previous formula instead of the domain being $\text{DNC}(\mathbb{R}, 0)$, we replace the domain with $\mathbb{R}$ using the inclusion

$$\mathbb{R} \to \text{DNC}(\mathbb{R}, 0)$$

$$t \to (t, t)$$

$$0 \to (1,0).$$

Since $f'(0) \in H$ it follows that $\text{DNC}^2(f) : \mathbb{R} \to \text{DNC}^2(M, V, H)$ is a well defined smooth map. Its value at zero is an element in $N^\text{DNC}(M, V)_{H \times \{0\}}$ which is clearly in $(N_{\pi_{(0)}})^{-1}(1)$.

Proposition 2.4. the map

$$\beta : N^M_{V,H} \to (N_{\pi_{(0)}})^{-1}(1), \quad [f] \to [\text{DNC}(f)]$$

is a well defined bijection.
Let us remark that the map \( \beta \) is not a linear map and in fact the space \( N_{V,H}^M \) is not a vector bundle.

**Proof.** In Section 1.1 two types of functions on \( \text{DNC}(M,V) \) were described which generate the ring of smooth functions on \( \text{DNC}(M,V) \). By regarding each type we see that for two functions \( f, g \in \tilde{N}_{V,H}^M \), the classes in \( N_{\text{DNC}(M,V)}^2 \) of \( \text{DNC}(f) \) and \( \text{DNC}(g) \) are equal if and only if the classes of \( f \) and \( g \) are equal in \( N_{V,H}^M \). Hence \( \beta \) is well defined and injective. Surjectivity follows by looking at a local chart as described below. \( \square \)

Let \( \psi : N_{V}^M \to M \) be a tubular neighbourhood embedding, \( L : H \oplus N_{V}^M / H \to N_{V}^M \) a linear isomorphism given by the choice of a complementary subbundle of \( H \) inside \( N_{V}^M, \phi = \psi \circ L \).

By the local charts described in Section 1.1 the following is a local chart for \( \text{DNC}(M,V) \):

\[
\phi : H \oplus N_{V}^M / H \times \mathbb{R} \to \text{DNC}(M,V)
\]

\[
(h, n, t, u) \mapsto (\phi(un, u^2 n), \phi(L(h, u, n, t))\).
\]

Therefore the following is a local chart for \( \text{DNC}(M,V,H \times \{0\}) \):

\[
H \oplus N_{V}^M / H \times \mathbb{R} \times \mathbb{R} \to \text{DNC}(M,V,H \times \{0\})
\]

\[
(h, n, t, u, u) \mapsto (\phi(un, u^2 n), \phi(L(h, u, n, t))\).
\]

Let \( \text{DNC}_H(M,V) := M \times \mathbb{R}^* \sqcup N_{V,H}^M \times \{0\} \).

We equip \( \text{DNC}_H(M,V) \) with a smooth structure by identifying it with \( (\pi_0^{(0,1)})^{-1}(\{1\} \times \mathbb{R}) \) using the map \( \beta \). Its local charts are hence given by

\[
H \oplus N_{V}^M / H \times \mathbb{R} \to \text{DNC}_H(M,V)
\]

\[
(h, n, u) \mapsto (\phi(un, u^2 n), u),
\]

\[
(h, n, 0) \mapsto (L(h, u, n, t)\).
\]

The space \( \text{DNC}_H(M,V) \) is called the deformation to the normal cone of \( M \) along \( V \) with weight \( H \).

**Remark 2.5.** All the other fibers \( (\pi_0^{(0,1)})^{-1}(\{1\} \times \mathbb{R}) \) for \( t \neq 0 \) are isomorphic to \( (\pi_0^{(0,1)})^{-1}(\{1\} \times \mathbb{R}) \) by a rescaling in the \( u \)-variable. The fiber \( (\pi_0^{(0,1)})^{-1}(\{0\} \times \mathbb{R}) \) is equal to \( \text{DNC}(N_{V}^M, H) \). In particular the space \( \text{DNC}^2(M, V, H) \) should be seen as a deformation of the space \( \text{DNC}_H(M,V) \) to the simpler space \( \text{DNC}(N_{V}^M, H) \).

Since \( H \) is \( \mathbb{R}^* \)-invariant, by Section 1.2 it follows that there is an \( (\mathbb{R}^*)^2 \) action on \( \text{DNC}^2(M, V, H \times \{0\}) \). It follows from Equation (1) in Section 1.2 that \( (\pi_0^{(0,1)})^{-1}(\{1\} \times \mathbb{R}) \) is invariant under the diagonal \( \lambda_H^{(1)} \lambda_H^{(0)} \). This action is described by \( u \cdot (x,t) = (x, tu) \) and \( u \cdot ([f,0]) = ([f(\tilde{\psi})],0) \) for \( f \in \tilde{N}_{V,H}^M \).

**Corollary 2.6.** Let \( (M,V), (M',V') \) be smooth manifold pairs, \( H \subseteq N_{V}^M, H' \subseteq N_{V'}^{M'} \) subbundles, \( g : M \to M' \) a smooth map such that \( g(V) \subseteq V' \) and \( dg(H) \subseteq H' \). Then the maps

- \( Ng : N_{V,H}^M \to N_{V',H'}^{M'} \) \( [f] \to [g \circ f] \)
- \( \text{DNC}(g) : \text{DNC}_H(M,V) \to \text{DNC}_{H'}(M',V') \)

\[
(x,t) \mapsto (g(x),t)
\]

\[
([f],0) \to ([g \circ f],0)
\]
are well defined and smooth.

Proof. This is a corollary of Proposition 2.6 applied twice and the identification of $\text{DNC}_H(M, V)$ with $\left(\nu^{(0,1)}_\mathbb{R}\right)^{-1}(\{1\} \times \mathbb{R}) \subseteq \text{DNC}^2(M, V, H \times \{0\})$.

Proposition 2.7. Let $M_1, M_2, M$ be manifolds, $V_i \subseteq M_i, V \subseteq M$ submanifolds, $H_i \subseteq N^M_i, H \subseteq N^M$ vector subbundles, $f_i : M_i \to M$ smooth maps such that

1. $f_i(V_i) \subseteq V$
2. the maps $f_i : M_i \to M$ are transverse
3. the maps $f_i|V : V_i \to V$ are transverse
4. $H = df_1(H_1) + df_2(H_2)$,

then

1. the maps $\text{DNC}(f_i) : \text{DNC}_H(M_i, V_i) \to \text{DNC}_H(M, V)$ are transverse.
2. the natural map

\[ \text{DNC}_{H_1 \times H_2}((M_1 \times M_2, V_1 \times V_2) \to \text{DNC}_{H_1}(M_1, V_1) \times_{\text{DNC}_H(M,V)} \text{DNC}_{H_2}(M_2, V_2) \]

is a diffeomorphism.

Proof. This is a corollary of Proposition 2.6 applied twice and the identification of $\text{DNC}_H(M, V)$ with $\left(\nu^{(0,1)}_\mathbb{R}\right)^{-1}(\{1\} \times \mathbb{R}) \subseteq \text{DNC}^2(M, V, H \times \{0\})$.

Theorem 2.8. Let $G \rightrightarrows G^0$ be a groupoid, $G' \rightrightarrows G^0$ a subgroupoid, $\mathcal{H} \subseteq N^G_{G'}$ a VB-subgroupoid $\Surd$ $\Surd$. Then

1. the space $N^G_{G', \mathcal{H}} \subseteq N^{G,G'}_{G', \mathcal{H}}$ is a Lie groupoid whose algebroid is equal to $N^{\mathcal{A}G}_{\mathcal{A}G', \mathcal{A}G'}$.
2. the space $\text{DNC}_H(G, G') \rightrightarrows \text{DNC}_{H^0}(G^0, G^0')$ is a Lie groupoid whose Lie algebroid is equal to $\text{DNC}_{H^0}(\mathcal{A}G, \mathcal{A}G')$.

Proof. This is a corollary of Corollary 2.6 and Proposition 2.7.

Example 2.9. Let $F \subseteq TM$ be an integrable subbundle. We regard the foliation groupoid $\mathcal{G}(M, F) \rightrightarrows M$ as an immersed subgroupoid of $M \times M \rightrightarrows M$ by the map

\[ (x, [\gamma], y) \to (x, y). \]

This map is not injective but the Lie groupoid $\text{DNC}(M \times M, \mathcal{G}(M, F)) \rightrightarrows M \times \mathbb{R}$ is still well defined by remark 1.2. Its underlying manifold is a second countable locally Hausdorff manifold.

The vector bundle $TM/F$ will be denoted by $\nu(F)$. If $\gamma : [0, 1] \to M$ is path tangent to the leaves, then its holonomy defines a map $\nu(\gamma) : \nu(F)_{\gamma(0)} \to \nu(F)_{\gamma(1)}$. One then sees that the groupoid

\[ N^M(M, F) = \{(x, [\gamma], y, X) : (x, [\gamma], y) \in \mathcal{G}(M, F), X \in \nu(F)y \} \rightrightarrows M. \]

The product is then given by

\[ (x, [\gamma], y, X) \cdot (y, [\gamma'], z, Y) = (x, [\gamma \gamma'], z, d\gamma'(X) + Z). \]

Let $H \subseteq \nu(F)$ be a holonomy invariant subbundle, i.e. such that for any leafwise path $\gamma : [0, 1] \to M$, one has $\gamma(0) = H_0$. Then $\gamma(1)$. It follows that

\[ L := \{(x, [\gamma], y, X) \in N^M(M, F) : X \in H_y \} \subseteq N^M(M, F) \]

is a Lie subgroupoid. The groupoid

\[ N^M(M, F, L) = \{(x, [\gamma], y, X, Y) : X \in H_y, Y \in \nu(F)y \} \rightrightarrows M \]

has then the groupoid law

\[ (x, [\gamma], y, X, Y) \cdot (y, [\gamma'], z, X', Y') = (x, [\gamma \gamma'], z, d\gamma'(X) + X', d\gamma'(Y) + Y' + \frac{1}{2} \mathcal{L}(d\gamma'(X), X')), \]

where $\mathcal{L} : H \times H \to \nu(F)/H$ is a Levi form defined similarly to the one defined in Section 2.4.
3 Carnot Groupoid

A more general groupoid will be constructed starting from the following data: Let $M$ be a smooth manifold, $0 = H^0 \subseteq H^1 \subseteq \cdots \subseteq H^{k+1} = TM$ be vector bundles such that

$$[\Gamma^\infty(H^i), \Gamma^\infty(H^j)] \subseteq \Gamma^\infty(H^{i+j}),$$

where $H^i = TM$ for $i > k$. We will calculate the Lie algebroid of this groupoid and hence show that it is equal to the groupoid constructed in \([28, 6, 8, 7, 38]\). See remark 3.2.3 for more details.

Since $[\Gamma^\infty(H^i), \Gamma^\infty(H^j)] \subseteq \Gamma^\infty(H^{i+j})$, it follows that the map

$$\Gamma^\infty(H^i/H^{i-1}) \times \Gamma^\infty(H^j/H^{j-1}) \to \Gamma^\infty(H^{i+j}/H^{i+j-1})$$

$$(X, Y) \to [X, Y] \mod \Gamma^\infty(H^{i+j-1})$$

is a $C^\infty(M)$-bilinear map, hence it comes from an antisymmetric bilinear map

$$\mathcal{L} : H^i/H^{i-1} \times H^j/H^{j-1} \to H^{i+j}/H^{i+j-1}.$$ 

For each $a \in M$, the map $\mathcal{L}$ defines the structure of a Lie algebra on $\mathcal{G}(H)_a := \oplus H^i_a/H^{i-1}_a$ by

$$[X, Y] = \mathcal{L}(X, Y), \quad X \in H^i_a/H^{i-1}_a, Y \in H^j_a/H^{j-1}_a.$$ 

By Baker–Campbell–Hausdorff formula, the vector space $\mathcal{G}(H)_a$ admits the structure of a nilpotent Lie group. It is clear that the structure of group is $C^\infty$ in $a$, hence $\mathcal{G}(H)$ is a bundle of nilpotent Lie groups. We will define a Lie groupoid denoted by $\text{DNC}_H(M \times M, M)$ by induction on $k$ whose underlying set is equal to

$$M \times M \times \mathbb{R}^* \cup \mathcal{G}(H) \times \{0\},$$

and whose Lie algebroid is equal to

$$\Gamma^\infty(\mathfrak{A}_H) = \{X \in \Gamma^\infty(TM \times \mathbb{R}) : \partial^i_t X|_{t=0} \in \Gamma^\infty(H^i) \forall i \geq 0\}$$

For $k = 1$, this is just $\text{DNC}_{H^1}(M \times M, M) \cong M \times \mathbb{R}$ defined in Section 2.2. By induction assuming it is defined for $k - 1$, that is the Lie groupoid

$$\text{DNC}_{H^1, \ldots, H^{k-1}}(M \times M, M) = M \times M \times \mathbb{R}^* \cup \mathcal{G}(H^1, \ldots, H^{k-1}) \times \{0\} = M \times M \times \mathbb{R}^* \cup H^1 \oplus H^2/H^1 \oplus \cdots \oplus TM/H^{k-1} \times \{0\}$$

is well defined. The subset $H^1 \oplus H^2/H^1 \oplus \cdots \oplus H^{k-1}/H^{k-1}$ is a Lie subgroupoid of $\mathcal{G}(H^1, \ldots, H^{k-1})$ precisely because

$$[\Gamma^\infty(H^i), \Gamma^\infty(H^j)] \subseteq \Gamma^\infty(H^k), \quad i + j = k.$$ 

Therefore the space

$$\text{DNC}(\text{DNC}_{H^1, \ldots, H^{k-1}}(M \times M, M), H^1 \oplus H^2/H^1 \oplus \cdots \oplus H^{k}/H^{k-1} \times \{0\})$$

is a Lie groupoid, where we used Remarks\[LS\]. The Lie algebroid of this groupoid is then

$$\text{DNC}(\mathfrak{A}_{H^1, \ldots, H^{k-1}}, H^1 \oplus \cdots \oplus H^k/H^{k-1})$$

Using Remarks\[LS\] we get that the space of sections of this algebroid is then equal to

$$\Gamma^\infty(\text{DNC}(\mathfrak{A}_{H^1, \ldots, H^{k-1}}, H^1 \oplus \cdots \oplus H^k/H^{k-1}))$$

$$= \{X \in \Gamma^\infty(TM \times \mathbb{R} \times \mathbb{R}) : \partial^i_t(X)(0, u) \in \Gamma^\infty(H^i) \forall 0 \leq i \leq k - 1, \ u \in \mathbb{R} \land \partial^i_t(X)(0, 0) \in \Gamma^\infty(H^k)\}.$$ 

We define $\text{DNC}_{H^1, \ldots, H^k}(M \times M, M)$ as the fiber of $\text{DNC}(\text{DNC}_{H^1, \ldots, H^{k-1}}(M \times M, M), H^1 \oplus H^2/H^1 \oplus \cdots \oplus H^k/H^{k-1} \times \{0\})$ over $M \times \{1\} \times \mathbb{R}$. This is clearly a Lie groupoid.
It follows from the above description of $\Gamma^\infty(\mathcal{DNC}(\mathfrak{A}_H^{1,\ldots,H^{k-1}},H^1 \oplus \cdots \oplus H^k/H^{k-1}))$ by restricting to the diagonal we get that if

$$X \in \Gamma^\infty(\mathcal{DNC}(\mathfrak{A}_H^{1,\ldots,H^{k-1}},H^1 \oplus \cdots \oplus H^k/H^{k-1}))$$

then $\partial_i X(0,0) \in \Gamma^\infty(H^i)$ for all $0 \leq i \leq k$, where we used that $X(0,u) = 0$. This finishes the induction, and proves that Lie algebroid of $\mathcal{DNC}_H(M \times M, M)$ is equal to $\mathfrak{A}_H$. Hence we proved the following

**Theorem 3.1.** The Lie groupoid $\mathcal{DNC}_H(M \times M, M)$ is the same as the groupoid constructed in [2, 3, 4, 38].

**Remarks 3.2.** 1. In [38], a more general case is regarded where starting from a groupoid $G$, subbundles $H^1 \subseteq \cdots \subseteq H^r = \mathfrak{f}G$ such that $[\Gamma^\infty(H^1),\Gamma^\infty(H^2)] \subseteq \Gamma^\infty(H^{r+1})$ they construct a groupoid $\mathcal{DNC}_H(G,G^0)$. It is clear that the above construction works equally well for this case with only notational changes. The advantage of our approach is that we can do the more general case of a groupoid inside another without any extra difficulty.

2. The groupoid

$$\mathcal{DNC}^{k+1}(M \times M, M, H^1 \times \{0\}, \ldots, H^1 \oplus \cdots \oplus H^k/H^{k-1} \times \{0\})$$

is a Lie groupoid which contains the ‘deformations in all the directions’. This groupoid admits an $(\mathbb{R}^*)^{k+1}$ action as in Section 1.2. The fiber over $(1,1,1,1,0)$ is then equal to $\mathcal{DNC}_{H^1,\ldots,H^k}(M \times M, M)$. The action $\mathbb{R}^*$ defined on $\mathcal{DNC}_{H^1,\ldots,H^k}(M \times M, M)$ defined in [38] is then just the diagonal action of $\mathbb{R}^{k+1}$ which by induction is easily seen to preserve the fiber $(1,1,1,1,0)$.

For example, in the case $k = 2$, this gives

$$\mathcal{DNC}^3(M \times M, H^1 \times \{0\}, H^1 \oplus H^2/H^1 \times \mathbb{R}) = M \times M \times \mathbb{R}^* \times \mathbb{R}^* \times \mathbb{R}^*$$

$$\cup TM \times \{0\} \times \mathbb{R}^* \times \mathbb{R}^* \cup H^1 \oplus TM/H^1 \times \mathbb{R} \times \{0\} \times \mathbb{R}^*$$

$$\cup H^1 \oplus H^2/H^1 \oplus TM/H^2 \times \mathbb{R} \times \{0\}$$

Let us remark that the subgroupoid $H^1 \oplus H^2/H^1 \oplus TM/H^2 \times \mathbb{R} \times \{0\}$ is not trivial as a groupoid, it has a structure

$$(h_1, h_2, h_3, t, u, 0) \cdot (k_1, k_2, k_3, t, u, 0) = (h_1 + k_1, h_2 + k_2 + \frac{t^2}{2} h_1, k_3)$$

$$h_3 + k_3 + \frac{tu}{2} (\{h_1, k_2\} + [h_2, k_1]) + \frac{t^2}{12} ([h_1, [h_1, k_1]] + [k_1, [k_1, h_1]], t, u, 0)$$

Similarly for $M \times \{0\} \times \mathbb{R}^* \times \mathbb{R}^*$ and $H^1 \oplus TM/H^1 \times \mathbb{R} \times \{0\} \times \mathbb{R}^*$.

3. The existence of the Lie groupoid $\mathcal{DNC}_H(M \times M, M)$ follows from Debord’s result on integrability of Lie algebroids [14]. Debord’s result applies to the Lie algebroid of $\mathcal{DNC}_H(M \times M, M)$. It shows that there exists a unique minimal Lie groupoid integrating the Lie algebroid of $\mathcal{DNC}_H(M \times M, M)$. Minimal in the sense that any other Lie groupoid projects by a submersion morphism of groupoids into it.

4. The groupoid $\mathcal{DNC}_H(M \times M, M)$ constructed above is the minimal groupoid integrating its Lie algebroid. The maximal Lie groupoid is the groupoid

$$\tilde{M} \times_{\pi_1(M)} \tilde{M} \times \mathbb{R}^* \cup H^1 \oplus \cdots \oplus H^k/H^{k-1} \times \{0\}$$

where $\tilde{M} \times_{\pi_1(M)} \tilde{M}$ is the Poincaré groupoid.

**Example 3.3.** Following the notation of Example 2.9. Let $F$ be a foliation, $H^1 \subseteq \cdots \subseteq H^{k+1} = \nu(F)$ subbundles such that if $X \in \Gamma^\infty(H^1)$ and $Y \in \Gamma^\infty(H^2)$, then

$$[X, Y] \in \Gamma^\infty(H^{i+j})$$

with the convention $H^s = \nu(F)$ for $s > k$ and such that if $i \in \{1,\ldots,k\}$, $\gamma:\{0,1\} \to M$ a path tangent to the leaves, then $d\gamma^* H^{i}(0) = H^{i}(1)$. In Example 2.9 we defined the groupoid $\mathcal{DNC}_{\nu}(M \times M, g(M,F))$. We can by an induction, similar to the above, construct the groupoid $\mathcal{DNC}_H(M \times M, g(M,F))$. 

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**Quotient of Lie groupoids** Let $G \rightrightarrows G^0$ be a Lie groupoid, $H \subseteq G$ a Lie subgroupoid. The Lie groupoid $H$ acts on the smooth manifold $G_{H^0}$ by right translation. This action is clearly free. The action is proper if $H$ is closed in the pullback of $G$ by $H^0 \subseteq G^0$. In this case, by [3, section 5.9.5], the quotient space $G_{H^0}/H$ is a smooth manifold, that will be denoted by $G/H$.

**Example 3.4.**

1. If $V$ is a submanifold of $M$, then $\text{DNC}(V \times V, V)$ is a Lie subgroupoid of $\text{DNC}(M \times M, M)$. It is clear that the quotient space is equal to $\text{DNC}(M \times M, M)/\text{DNC}(V \times V, V) = \text{DNC}(M, V)$.

2. Let $V \subseteq M$ a smooth submanifold such that $H^1 \cap TV$ is of locally of finite rank. It is then clear that $[\Gamma^\infty(H^1 \cap TV), \Gamma^\infty(H^j \cap TV)] \subseteq \Gamma^\infty(H^{j+1} \cap TV)$. Let $G(H^j)$ the bundle of nilpotent Lie groups $\oplus_i H^i/H^{i-1}$, $G(H \cap TV)$ be the bundle of nilpotent Lie groups $\oplus(H^i \cap TV)/(H^{i+1} \cap TV)$. In [42], the authors define a smooth manifold whose underlying set is equal to $M \times \mathbb{R}^* \cup G(H \cap TV)$, where $G(H \cap TV)$ is the restriction of $G(H)$ to $V$.

Similarly to the description of the classical deformation to the normal as a quotient space, the space defined in [32] can also be written as $\text{DNC}(H \cdot (M \times M, M))/\text{DNC}(H \cap TV (V \times V, V))$.

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