ALTERNATIVE PROOF FOR THE EXISTENCE OF GREEN’S FUNCTION

Sungwon Cho

Department of Mathematics Education, Gwangju National University of Education, 93 Pilmunlo Bugku, Gwangju 500-703, Republic of Korea

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Abstract. We present a new method for the existence of a Green’s function of nod-divergence form parabolic operator with H"older continuous coefficients. We also derive a Gaussian estimate. Main ideas involve only basic estimates and known results without a potential approach, which is used by E.E. Levi.

1. Introduction. We consider the following non-divergence form parabolic operator:

\[ L := \frac{\partial}{\partial t} - \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial^2}{\partial x_i \partial x_j}, \]

where its coefficients satisfy symmetric and ellipticity condition. Namely,

\[ a_{ij} = a_{ji}, \quad \lambda |\xi|^2 \leq a_{ij}(x,t) \xi_i \xi_j \leq \Lambda |\xi|^2 \]

holds for some positive constants \( \lambda \) and \( \Lambda \) in a given domain \( Q := \Omega \times (0,T), \Omega \) is an open connected bounded set in \( \mathbb{R}^n \) for \( n \geq 1 \). We study the Green’s function (of the first kind) for the operator \( L \) in a smooth domain. Following [12], a function \( G \) defined in \( Q \times Q \), is called the Green’s function if, for any fixed \( Y = (y,s) \in Q \),

\[ \text{LG}(X,Y) = 0, \quad t > s, X = (x,t) \in Q, \]

\[ G(X,Y) = 0 \quad \text{for } X \in \partial_p Q, \]

\[ \lim_{t \to s^+} G(X,Y) = \delta_y(x), \]

where \( \partial_p Q \) is the parabolic boundary of \( Q \) (see the definition below), \( \delta_y \) is the Dirac mass at \( y \), and the last expression is understood in a distributional sense. For its fundamental feature, this has been a topic of many publications. See [13, 9, 12, 15] and references therein.

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Throughout this paper, we assume $a_{ij}$ to be a $\alpha$-Hölder continuous with its norm of $A_\alpha$. Namely,

$$A_\alpha := \sup_{X, Y \in Q, X \neq Y} \frac{|a_{ij}(X) - a_{ij}(Y)|}{|X - Y|^\alpha} + \sup_{X \in Q} |a_{ij}(X)| < \infty,$$

where $|X - Y|^\alpha := \max(|x - y|^\alpha, (t - s)^{\alpha/2})$,

for some fixed $\alpha \in (0, 1)$. In this case, the fundamental solution was constructed from the parametrix method of E. E. Levi [16, 17], where volume potentials were use of essential. For more details and history, see [12, 7, 24, 14].

In this paper, we present an alternative method for the existence of the Green’s function along with its pointwise estimation. Our method does not involve any potential theory, but basic estimates including maximum principle and classical Schauder theory. Now we state our main result along with some remarks:

**Theorem 1.1.** Let $L$ be a non-divergence parabolic operator of the form (1) satisfying (2). Assume its coefficients $a_{ij}$ are $\alpha$-Hölder continuous, and $\Omega$ be a bounded open connected set in $\mathbb{R}^n$ with $C^{2,\alpha}$ boundary. Then the Green’s function $G$ of $L$ in $Q := \Omega \times (0, T)$ exists and satisfies the following Gaussian estimate: for some positive constants $C := C(n, \alpha, \lambda, A_\alpha, \text{diam}(\Omega), T)$ and $c := c(n, \alpha, \lambda, A_\alpha, \text{diam}(\Omega))$,

$$G(x, t; y, s) \leq C(t - s)^{-\frac{n}{2}} \exp \left(- \frac{c|x - y|^2}{(t - s)}\right) \quad \text{in} \quad Q.$$

**Remark 1.** In the nondivergence elliptic case, the continuity of the Green’s function and classical Harnack principle away from the pole $(x = y)$ are proved in [23] and [1], respectively. With only continuous assumption on coefficients $a_{ij}$, the Green function is not even bounded. See [4].

**Remark 2.** For the measurable coefficients case, two-sided pointwise estimations of fundamental solution (the Green function when $\Omega = \mathbb{R}^n$) become available by Escauriaza [10], using a certain adjoint solution for the bound functions. In the proof, he used the concept of normalized adjoint solution, which was first introduced by Bauman [5] in elliptic, and Fabes, Garofalo, Salsa [11] in parabolic case, respectively.

**Remark 3.** More extensive results are available in divergence case. See Aronson [2], and [3, 19, 25, 21, 20, 8, 22].

Our main estimation (6) is not optimal if we consider $G$ is vanishing near its parabolic boundary. See [6] for two sided estimations including its boundary in the divergence case.

We conclude this section with a sketch of the paper and some notations. In section 2, we solve the Poisson problem with weakly singular right hand side along with pointwise estimate of the solution. In section 3, using Theorem 2.1, we prove our main result, Theorem 1.1.

Now, we enlist some standard notations: $D_{ij} := D_iD_j$, $D_i := \frac{\partial}{\partial x_i}$ for $1 \leq i, j \leq n$. $D_t := \frac{\partial}{\partial t}$. The expression $a_{ij} \in C^\alpha(Q)$ means $A_\alpha < \infty$, where $Q := \Omega \times (0, T)$. For $X = (x, t)$, $C_r(X)$ denote the standard parabolic cylinder of radius $r$ centered at $X$. Namely, $C_r(X) := B_r(x) \times (t - r^2, t + r^2)$, where $B_r(x) := \{y \in \mathbb{R}^n \mid |y - x| < r\}$. The parabolic boundary of $Q$ is denoted by $\partial_p Q$. Namely, $\partial_p Q := \{(x, 0) \mid x \in \Omega\} \cup \{(x, t) \mid x \in \partial \Omega, t \in [0, T)\}$. The diameter of $\Omega$, $\sup_{x, y \in \Omega} \{|x - y|\}$, will be denoted.
by \( \text{diam}(\Omega) \). We write \( u \in C^{2,1} \) if \( \sup |u| \) and \( D_i u, D_{ij} u, i, j = 1, \ldots, n \) exist and are bounded in a given domain. Also \( u \in C^{2,1,\alpha} \) if \( u \in C^{2,1} \) and

\[
\sup_{X,Y \in Q, X \neq Y} \frac{|D_{ij} u(X) - D_{ij} u(Y)|}{|X - Y|^\alpha} < \infty, \\
\sup_{X,Y \in Q, X \neq Y} \frac{|D_i u(X) - D_i u(Y)|}{|X - Y|^\alpha} < \infty,
\]

for \( i, j = 1, \ldots, n \). The expression \( N(\cdots) \) denotes the various constants \( N \) which will be determined by the quantities described in the parentheses.

2. The Poisson problem with weakly singular right hand side. Let \( \Omega \) be a bounded domain with \( C^{2,\alpha} \)-boundary, and \( Q = \Omega \times (0, T) \) for \( T > 0 \). Fix \( Y = (y, s) \in Q \). Since the matrix \( A := A(Y) = (a_{ij}(Y))_{i,j=1,\ldots,n} \) is strictly positive and symmetric, we have a symmetric, positive, and invertible matrix \( S := S^2 \). Let \( f \) be a function such that \( f \in C^\alpha(Q \setminus C_d(Y)) \) for all \( \delta > 0 \) and verifies

\[
|f(X)| \leq N_1 |X - Y|^\alpha \left( \frac{1}{t - s} + \frac{|x - y|^2}{(t - s)^2} \right) |t - s|^{-\frac{\alpha}{2}} \exp \left( -\frac{|S^{-1}(x - y)|^2}{4(t - s)} \right) \tag{7}
\]

for \( t > s, X = (x, t) \in Q \), and identically zero for \( t \leq s \). Here, \( S^{-1} \) denotes the inverse matrix of \( S \), and \( N_1 \) is a fixed positive constant. Consider the following Poisson problem with weakly singular right hand side:

\[
\begin{align*}
Lu &= f & \text{in } Q \setminus \{Y\}, \\
u &= 0 & \text{on } \partial_p Q. \\
\end{align*}
\tag{PS}
\]

The following theorem gives us the existence and an estimate of the solution \( u \):

**Theorem 2.1.** There exists a solution \( u \) of the problem (PS) in the class \( C^{2,1,\alpha}(Q \setminus C_d(Y)) \) for any \( \delta > 0 \), and satisfying

\[
|u(X)| \leq N_1 \cdot N_2 \cdot (t - s)^{-\frac{\alpha}{2}} \exp \left( -\frac{|x - y|^2}{N_3(t - s)} \right) \text{ in } \{t > s\} \cap Q, \tag{8}
\]

where \( N_2 = N_2(n, \alpha, \lambda, A_\alpha, \text{diam}(\Omega), T), N_3 := N_3(n, \alpha, \lambda, A_\alpha, \text{diam}(\Omega)). \)

**Proof.** Dividing \( N_1 \), using \( \alpha \equiv 0 \) for \( t < s \), we may assume \( N_1 = 1 \), \( y = 0 \), \( s = 0 \). Furthermore, assume temporarily \( f \in C^\alpha(Q) \), then the existence of solution \( u \in C^{2,1,\alpha}(Q) \cap C(Q) \) of the Poisson problem (PS) will be guaranteed by [18, Proposition 4.25]. We will show that the estimate (8) holds. Define

\[
h(X) := N_4 t^{-\frac{n-\alpha}{2}} \exp \left( -\frac{|S^{-1}x|^2}{4N_5 t} \right)
\]

for \( t > 0, X = (x, t) \), and identically zero for \( t \leq 0 \), where \( S^{-1} := (s_{ij}^{-1}) \) is the inverse matrix of \( S \), and \( N_4 \) and \( N_5 \) are positive constants which will be fixed later. Also, let

\[
L^Y u := D_t u - \sum_{i,j=1}^n a_{ij}(Y)D_{ij} u.
\]

It is easy to check the following by direct computations:

\[
D_t h = \left( -\frac{n - \alpha}{2t} + \frac{|S^{-1}x|^2}{4N_5 t^2} \right) h, \\
D_j |S^{-1}x|^2 = D_j \sum_{k=1}^n \left( \sum_{l=1}^n s_{kl}^{-1} x_l \right)^2 = 2 \sum_{k=1}^n \left( \sum_{l=1}^n s_{kl}^{-1} x_l \right) s_{kj}^{-1}.
\]
$$D_{ij} h = \left( -\frac{\sum_{k=1}^{n} s_{ki}^{-1} s_{kj}^{-1}}{2N_{5}t} + \frac{\sum_{k,l=1}^{n} s_{ki}^{-1} s_{kj}^{-1} x_{l}}{4N_{5}^{2}t^{2}} \right) h,$$

$$\sum_{i,j=1}^{n} a_{ij} \sum_{k,l=1}^{n} s_{ki}^{-1} s_{kj}^{-1} x_{l} \sum_{k,l=1}^{n} s_{ki}^{-1} s_{kj}^{-1} x_{l} = \sum_{i=1}^{n} x_{i} \sum_{k,l=1}^{n} s_{ki}^{-1} s_{kj}^{-1} x_{l} = |S^{-1}x|^2,$$

$$L^Y h = \left( \frac{\alpha - n + \frac{n}{N_{5}}}{2t} + \frac{|S^{-1}x|^2}{4t^{2}} \left( \frac{1}{N_{5}^2} - \frac{1}{N_{5}^{2}} \right) \right) h$$

$$\geq \min \left\{ \frac{\alpha - n + \frac{n}{N_{5}}}{2}, \frac{1}{4\Lambda \left( \frac{1}{N_{5}^2} - \frac{1}{N_{5}^{2}} \right)} \right\} \left( 1 + \frac{|x|^2}{t^2} \right) h$$

for any $N_{5} \in \left( 1, \frac{n}{n - \alpha} \right)$. Fix $N_{5} := \frac{1 + \frac{n}{2}}{2} = \frac{2n - \alpha}{n - \alpha}$, then we have

$$L^Y h \geq 2f$$

for some $N_{4} := N_{4}(n, \alpha, \Lambda)$. Here, we used the fact that

$$\left( \frac{|x|}{\sqrt{t}} \right)^{\alpha} \exp \left( -\frac{|S^{-1}x|^2}{4t} + \frac{|S^{-1}x|^2}{4N_{5}t} \right) \leq \left( \frac{|x|}{\sqrt{t}} \right)^{\alpha} \exp \left( \frac{1}{4N_{5}^2 \Lambda} - \frac{1}{4\Lambda} \right) \frac{|x|^2}{t^2} \leq N$$

for some constant $N$ depending on $N_{5}, \Lambda, \alpha$. For these fixed constants $N_{1}$ and $N_{5},$

$$|(L - L^Y)h| \leq N_{6}|X|^\alpha \left( \frac{1}{t} + \frac{|x|^2}{t^2} \right) h$$

for some $N_{6} := N_{6}(n, \alpha, \Lambda, \Lambda, A_{0})$. We can choose small $\epsilon_{0}$ such that

$$N_{6}|X|^\alpha \leq \frac{1}{2} \min \left\{ \frac{\alpha - n + \frac{n}{N_{5}}}{2}, \frac{1}{4\Lambda \left( \frac{1}{N_{5}^2} - \frac{1}{N_{5}^{2}} \right)} \right\}$$

for $|X| < \epsilon_{0}$, where $\epsilon_{0}$ depends on $N_{5}, N_{6}, n, \alpha, \Lambda$. From (9), we have

$$|(L - L^Y)h| \leq \frac{L^Y h}{2}$$

for $|X| < \epsilon_{0}$. Let

$$v(t) := N_{7}t^{\frac{\alpha}{2}} \exp \left( -\frac{\epsilon_{0}^2}{4N_{5}^2 \Lambda t} \right)$$

for $t > 0$, and identically zero for $t \leq 0$. Here the positive constant $N_{7}$ will be fixed later. Note

$$Lv = v_{t} = \left( -\frac{n - \alpha}{2t} + \frac{\epsilon_{0}^2}{4N_{5}^2 \Lambda t^2} \right) v > 0, \quad 0 < t < t_{0}$$

for small $t_{0}$ depending on $n, \alpha, \Lambda, \epsilon_{0}, N_{5}$, and $v \geq h$ for $|x| \geq \epsilon_{0}$ for any $N_{7} \geq N_{4}$. Here we may assume $t_{0} \leq \sqrt{\epsilon_{0}}$ without loss of generality. Let $w := h + v$, then, for $|X| < \epsilon_{0}$ and $0 < t < t_{0},$

$$Lw \geq L^Y h + (L - L^Y)h \geq \frac{L^Y h}{2} \geq f$$

(12)

For $|x| \geq \epsilon_{0}$ and $t < t_{0}$, from (10) and (11), we have

$$Lv - |(L - L^Y)h| > 0$$

(13)

for large $N_{7}$ depending on $N_{6}, diam(\Omega), t_{0}$. Thus in all, by (12) and (13),

$$Lw \geq f$$
for $t \in (0, t_0)$. By Lemma 2.2 below, we have $|u| \leq w$ in $\Omega \times (0, t_0)$. Fix $N_3$, depending on $N_5, \lambda, \epsilon_0, \text{diam}(\Omega)$, which satisfies

$$\frac{|x|^2}{N_3 t} \leq \min \left\{ \frac{|S^{-1}x|^2}{4N_5 t}, \frac{c_0^2}{4N_5 t} \right\},$$

and this implies the inequality (8) in $\Omega \times (0, t_0)$ for any $N_2$ satisfying $N_2 \geq N_4 + N_7$. Now we consider $t \geq t_0$. Applying the well known comparison principle to $\pm u$ and $t \sup_{\Omega \times (t_0, T)} f + \sup_{\Omega \times \{t_0\}} w$, it is easy to see

$$|u| \leq t \sup_{\Omega \times (t_0, T)} f + \sup_{\Omega \times \{t_0\}} w$$

for $t \geq t_0$. It is enough to choose $N_2$ such that

$$t \sup_{\Omega \times (t_0, T)} f + \sup_{\Omega \times \{t_0\}} w \leq N_2 t^{-\frac{n - \alpha}{2}} \exp \left( - \frac{|x|^2}{N_3 t} \right) \text{ when } x \in \Omega \text{ and } t \geq t_0.$$

For this, note that

$$T^{-\frac{n - \alpha}{2}} \exp \left( \frac{(\text{diam}(\Omega))^2}{N_3 t_0} \right) \leq t^{-\frac{n - \alpha}{2}} \exp \left( - \frac{|x|^2}{N_3 t} \right), \quad t \in [t_0, T], x \in \Omega,$$

$$\sup_{\Omega \times (t_0, T)} f \leq \max \{ \text{diam}(\Omega), T^{1/2} \} \alpha \left( \frac{1}{t_0} + \frac{\text{diam}(\Omega)^2}{t_0^2} \right) f_0^{-\frac{1}{2}},$$

$$\sup_{\Omega \times \{t_0\}} w \leq N_4 t_0^{-\frac{n - \alpha}{2}} + N_7 t_0^{-\frac{n - \alpha}{2}}.$$

Thus we can choose $N_2$ depending on $n, \alpha, \lambda, A_\alpha, \text{diam}(\Omega), N_1, T$ such that

$$T^{-\frac{n - \alpha}{2}} \exp \left( \frac{(\text{diam}(\Omega))^2}{N_3 t_0} \right) \left( \sup_{\Omega \times (t_0, T)} f \cdot T + \sup_{\Omega \times \{t_0\}} w \right) \leq N_2,$$

which leads to the inequality (8).

For the general case, choose $f_m \in C^\alpha(Q)$ for any positive integer $m$, such that $f_m = f$ in $C(Q \setminus \frac{1}{m} \Omega(Y))$ and $f_m$ also satisfies the given singular estimate (7). We showed that there exists $u_m$ such that

$$(P_m) \begin{cases} Lu_m = f_m & \text{in } Q, \\ u_m = 0 & \text{on } \partial_p Q, \end{cases}$$

and verifying (8) independent of $m$. We claim that there exists $u \in C^{2,1,\alpha}(Q \setminus C_\delta(Y))$ such that $u_m$ has a subsequence converging to $u$ in $C^{2,1}(Q \setminus C_\delta(Y))$ for any $\delta > 0$. For this, note that $u_m$ solves the following:

$$(P_{m-\delta}) \begin{cases} Lu = f_m & \text{in } Q \setminus C_\delta(Y), \\ u = u_m & \text{on } \partial_p(Q \setminus C_\delta(Y)). \end{cases}$$

Thus $\|u_m\|_{C^{2,1,\alpha}(Q \setminus C_\delta(Y))} \leq N\|f_m\|_{C^{\alpha}(Q \setminus C_\delta(Y))} + \sup_{Q \setminus C_\delta(Y)} u_m$ by [18, Proposition 4.25] again. By Arzela-Ascoli theorem, there exists $u \in C^{2,1}(Q \setminus C_\delta(Y))$ such that $u_m$ has a subsequence converging to $u$ and $Lu = f$ in $Q \setminus C_\delta(Y)$. By the standard Schauder theory and the uniqueness of the solution, $u \in C^{2,1,\alpha}(Q \setminus C_\delta(Y))$. Since $\delta$ was arbitrary, by the standard diagonal process, we have $u \in C^{2,1,\alpha}(Q \setminus C_\delta(Y))$ for any $\delta > 0$. \hfill \Box

Lemma 2.2. Let $u \in C^{2,1,\alpha}_{loc}(Q) \cap C(\overline{Q})$ and $w \in C^{2,1}_{loc}(Q) \cap C(\overline{Q} \setminus \{0\})$. Also, $Lu \leq Lw$, $w \geq 0$ in $Q$, $u = 0$ on $\partial_p Q$. Then $u \leq w$ in $Q$. 


Proof. Fix $\epsilon > 0$. Since $u$ is uniformly continuous in $Q$, we can find $\delta > 0$ such that $u \leq \epsilon \leq w + \epsilon$ on $\partial_{p}Q_{3}$, where $Q_{3} := \Omega \times (\delta, T)$. By the comparison principle, we have $u \leq w + \epsilon$ in $Q_{5}$. Observing that $\epsilon$ was arbitrary and $\delta \to 0^{+}$ as $\epsilon \to 0^{+}$, we have the desired result.

3. Existence and estimates of the Green’s function. In this section, we will prove Theorem 1.1. Fix $Y \in Q$. Similar to the proof of Theorem 2.1, for the strictly positive, symmetric matrix $A := A(Y) = (a_{ij}(Y))_{i,j=1,\ldots,n}$, we have a symmetric, positive, and invertible matrix $S := (s_{ij})_{i,j=1,\ldots,n}$ such that $A = S^{2}$. Define

$$H(X,Y) := \det(S^{-1}) (4\pi(t-s))^{-\frac{n}{2}} \exp\left(-\frac{|S^{-1}(x-y)|^{2}}{4(t-s)}\right),$$

for $t > s, X = (x,t), Y = (y,s)$, and identically zero for $t \leq s$, where $S^{-1}$ is the inverse matrix of $S$. By [12], $H$ is the fundamental solution of $L^{Y}$, where

$$L^{Y}u := D_{t}u - \sum_{i,j=1}^{n} a_{ij}(Y)D_{ij}u.$$  

We will construct $G$ in the form

$$G(X,Y) := H(X,Y) + G_{1}(X,Y) + G_{2}(X,Y),$$

where

$$\begin{cases}LG_{1} = 0 & \text{in } Q, \\ G_{1} = -H & \text{on } \partial_{p}Q,\end{cases}$$

and

$$\begin{cases}LG_{2} = f & \text{in } Q, \\ G_{2} = 0 & \text{on } \partial_{p}Q,\end{cases}$$

for $f := (L^{Y} - L)H$. The existence of $G_{1}$ is well known. For example, see [18]. For $G_{2}$, note that

$$f := (L^{Y} - L)H = \sum_{i,j=1}^{n} (a_{ij} - a_{ij}(Y))D_{ij}H,$$

$$D_{j}|S^{-1}(x-y)|^{2} = D_{j} \sum_{k=1}^{n} \left(\sum_{l=1}^{n} s_{kl}^{-1}(x_{l} - y_{l})\right)^{2} = 2 \sum_{k=1}^{n} \left(\sum_{l=1}^{n} s_{kl}^{-1}(x_{l} - y_{l})\right) s_{kj}^{-1},$$

$$D_{ij}H = \left(-\sum_{k=1}^{n} s_{kl}^{-1}s_{kj}^{-1} + \sum_{k,l=1}^{n} s_{kl}^{-1}s_{kj}^{-1}(x_{l} - y_{l}) \sum_{k,l=1}^{n} s_{kl}^{-1}s_{kj}^{-1}(x_{l} - y_{l})\right) \frac{4(t-s)^{2}}{2(t-s)},$$

Using a Hölder continuity of $a_{ij}$, the function $f$ satisfies (7) with $y = 0, s = 0, N_{1} = N_{1}(n, \lambda, \Lambda, A_{0})$. Thus, the existence of $G_{2}$ is proved by Theorem 2.1. Furthermore, it is immediate to see that $G$ satisfies the definition of Green’s function, (3)–(5).

Lastly, we will prove (6). For this, note that $G_{1} \leq 0$ by the maximum principle, and the estimate holds for $H$ and $G_{2}$.

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E-mail address: scho@gnue.ac.kr