THE NUMBER OF ROOTS OF A RANDOM POLYNOMIAL OVER THE FIELD OF $p$-ADIC NUMBERS

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Abstract. We study the roots of a random polynomial over the field of $p$-adic numbers. For a random monic polynomial with coefficients in $\mathbb{Z}_p$, we obtain an asymptotic formula for the factorial moments of the number of roots of this polynomial. In addition, we show the probability that a random polynomial of degree $n$ has more than $\log n$ roots is $O(n^{-K})$ for some $K > 0$.

1. Introduction

Consider the random polynomial

$$f(X) = \xi_0 + \xi_1 X + \cdots + \xi_n X^n$$

where $\xi_0, \ldots, \xi_n$ are independent random variables taking values in some field $F$. For a subset, $E$, of the algebraic closure of $F$, we define $R_E(f)$ to be the number of distinct roots in $E$ of the polynomial $f$, i.e.

$$R_E(f) = \# \{ x \in E : f(x) = 0 \}.$$

The study of the distribution of $R_E(f)$ when $F = \mathbb{R}$, has a long history. It goes back to Bloch and Pólya [BP31] who showed that $\mathbb{E}[R_E(f)] = O(\sqrt{n})$ as $n \to \infty$ when the coefficients of $f$ take the values of $\pm 1$ both with equal probability. Later, their results were improved and generalized on many occasions, see [LO38, LO39, Kac43, EO56, IM71, NNV16, Söz17a, Söz17b]. In particular, Maslova [Mas74a, Mas74b] determined asymptotically all higher moments of $R_E(f)$ as $n \to \infty$ when $f$ is a general random real polynomial.

Evans [Eva06] studied $R_{\mathbb{Q}_p}(f)$ in the $p$-adic setting, i.e., $E = F = \mathbb{Q}_p$ and in fact a multivariate version. In his model, the coefficients are not independent and the randomness comes from the Haar measure (cf. [KL21, Theorem 5] for generalizations). Buhler, Goldstein, Moews, and Rosenberg [BGMR06] worked in our setting where the coefficients are independent and distributed according to the Haar measure on $\mathbb{Z}_p$. They computed the probability that $R_{\mathbb{Q}_p}(f) = n$, i.e. $f$ totally splits, conditioned on $\xi_n = 1$. Recently, Caruso [Car21] generalized their result and obtained an integral formula for $R_E(f)$, for any open subset $E$ of finite field extension $K/\mathbb{Q}_p$.

In [Shm21], the author estimated $\mathbb{E}[R_{\mathbb{Q}_p}(f)]$ for general distributions of $f$. For example, if $p \neq 2$ and $\xi_0, \ldots, \xi_{n-1}$ are taking values of $\pm 1$ both with equal distribution then for any $\varepsilon > 0$

$$\mathbb{E}[R_{\mathbb{Q}_p}(f)] = \frac{p-1}{p+1} + O(n^{-1/4+\varepsilon})$$

as $n \to \infty$.

In this paper, we will consider the case where $f$ is a monic polynomial with $p$-adic integer coefficients, i.e., $\xi_n = 1$ and $\xi_0, \ldots, \xi_{n-1} \in \mathbb{Z}_p$. In this case, we have that $R_{\mathbb{Q}_p}(f) = R_{\mathbb{Z}_p}(f)$, so we abbreviate and write $R(f) = R_{\mathbb{Z}_p}(f)$.

Our study is focused on the factorial moments of $R(f)$ which is defined as follows. We call a set of exactly $d$ elements a $d$-set, and we denote $R^{(d)}(f)$ to be the number of $d$-sets of roots of $f$ i.e.

$$R^{(d)}(f) = \binom{R(f)}{d}.$$
The expected value of \( R^{(d)}(f) \) is called the \( d \)-th factorial moment of \( R(f) \). Computing the factorial moments of \( R(f) \) is equivalent to computing its distribution and its standard moments, using:

\[
\mathbb{E}[R(f)^m] = \sum_{d=1}^{m} \binom{m}{d} d! \mathbb{E}[R^{(d)}(f)],
\]

where the curly braces denote Stirling numbers of the second kind.

Bhargava, Cremona, Fisher and Gajović in [BCFG22], computed the \( d \)-factorial moments \( R(f) \) when \( \xi \) is distributed according to Haar measure normalized on \( \mathbb{Z}_p \) and \( p\mathbb{Z}_p \). They showed that for \( n \geq 2d \) the expectation \( \mathbb{E}[R^{(d)}(f)] \) is independent of \( n \). Moreover, if we denote \( \alpha(d) \) (respectively \( \beta(d) \)) to be \( \mathbb{E}[R^{(d)}(f)] \) when \( n \geq 2d \) and \( \xi \) is distributed according to Haar measure normalized on \( \mathbb{Z}_p \) (respectively \( p\mathbb{Z}_p \)), then we have the following power series relation between them:

\[
\sum_{d=0}^{\infty} \alpha(d)t^d = \left( \sum_{d=0}^{\infty} \beta(d)t^d \right)^p.
\]

Our result deals with a rather general distribution for the coefficients. Using a similar power series, we define \( \gamma(d) \) by

\[
\sum_{d=0}^{\infty} \gamma(d)t^d = \left( \sum_{d=0}^{\infty} \beta(d)t^d \right)^{p-1}.
\]

The values of \( \gamma(d) \) satisfy the following theorem.

**Theorem 1.** Let \( f(X) = \xi_0 + \xi_1X + \cdots + \xi_{n-1}X^{n-1} + X^n \) where \( \xi_0, \ldots, \xi_{n-1} \) are i.i.d. random variables taking values in \( \mathbb{Z}_p \) such that \( \xi_i \mod p \) is non-constant for all \( i = 0, \ldots, n-1 \). Then for any \( d = d(n) = o\left( \log^{1/2} n \right) \) and any \( \varepsilon > 0 \) we have

\[
\mathbb{E}\left[R^{(d)}(f) \mid p \mid \xi_0 \right] = \gamma(d) + O\left( n^{-1/4+\varepsilon} \right)
\]

as \( n \to \infty \). Here the implied constant depends only on \( p, \varepsilon \) and the distribution of \( \xi_0 \).

For larger values of \( d \), the main term of Theorem 3 is smaller than the error term since \( \gamma(d) \leq e^{-cd^2} \) for some \( c > 0 \), see Lemma 16. For larger \( d \), we take a wider point of view.

**Assumption 2.** Assume \( \xi_0, \ldots, \xi_{n-1} \) are independent random variables taking values in \( \mathbb{Z}_p \) such that there exists \( 0 < \tau < 1 \) independent of \( n \) which satisfies that

\[
\sum_{x \in \mathbb{Z}_p \setminus \{0\}} \mathbb{P}(\xi_i \equiv \bar{x} \pmod{p})^2 < 1 - \tau
\]

for each \( i = 1, \ldots, n-1 \).

Note that Assumption 2 does not give any requirements on \( \xi_0 \) beside being independent of the other \( \xi_i \) and taking values in \( \mathbb{Z}_p \).

**Theorem 3.** Let \( f(X) = \xi_0 + \xi_1X + \cdots + \xi_{n-1}X^{n-1} + X^n \) where \( \xi_0, \ldots, \xi_{n-1} \) satisfy Assumption 2. Then for any \( d = d(n) \) such that \( \limsup_{n \to \infty} d/\log n < (16 \log p)^{-1} \) there exists an explicit constant \( C > 0 \) such that for any \( \varepsilon > 0 \)

\[
\mathbb{E}[R^{(d)}(f) \mid p \mid \xi_0] = \gamma(d) + O\left( n^{-C+\varepsilon} \right)
\]

as \( n \to \infty \). Here the implied constant depends only on \( p, \varepsilon \) and \( \tau \) from Assumption 2.

The constant \( C \) is determined by the following

\[
C = \frac{1}{4} - \frac{1}{4} H_p \left( 4 \log p \cdot \limsup_{n \to \infty} \frac{d}{\log n} \right),
\]

where \( H_p \) is the binary entropy function with base \( p \), defined by

\[
H_p(x) = x \log_p \frac{1}{x} + (1-x) \log_p \frac{1}{1-x}.
\]
If the random variables $\xi_0, \ldots, \xi_{n-1}$ are i.i.d. and $\xi_i \mod p$ is non-constant, then they satisfy Assumption 2. Thus, Theorem 1 follows from Theorem 3 and (4).

Some special cases that are studied frequently when the coefficients are i.i.d. distributed uniformly in one of the sets $\{-1, 0, 1\}$, $\{0, 1\}$ or $\{0, \ldots, p-1\}$. In those cases, we can estimate $\mathbb{E}[R^{(d)}(f)]$ without conditioning on $\xi_0$ using the following:

**Theorem 4.** Let $f(X) = \xi_0 + \xi_1 X + \cdots + \xi_{n-1} X^{n-1} + X^n$ be a random polynomial where $\xi_0, \ldots, \xi_{n-1}$ are random variables taking values in $\mathbb{Z}_p^\infty \cup \{0\}$ and satisfying Assumption 2. Then for any $d = d(n)$ such that $\limsup_{n \to \infty} d/\log n < (16 \log p)^{-1}$ there exists an explicit constant $C > 0$, defined in (4), such that for any $\varepsilon > 0$

$$\mathbb{E}[R^{(d)}(f)] = \gamma(d) + \mathbb{P}(\xi_0 = 0)\gamma(d - 1) + O(n^{-C + \varepsilon})$$

as $n \to \infty$. Here the implied constant depends only on $p$ and $\varepsilon$.

Theorem 3 combined with (1) gives us a way to estimate any fixed moment of $R(f)$. In particular, we can compute the expected value and the variance of $R(f)$ as follows:

**Corollary 5.** Let $f(X) = \xi_0 + \xi_1 X + \cdots + \xi_{n-1} X^{n-1} + X^n$ where $\xi_0, \ldots, \xi_{n-1}$ satisfy Assumption 2 then for any $\varepsilon > 0$

$$\mathbb{E}[R(f) \mid p \nmid \xi_0] = \frac{p - 1}{p + 1} + O\left(n^{-1/4 + \varepsilon}\right)$$

and

$$\text{Var}[R(f) \mid p \nmid \xi_0] = \frac{(p^2 + 1)^2(p - 1)}{(p^4 + p^3 + p^2 + p + 1)(p + 1)} + O\left(n^{-1/4 + \varepsilon}\right)$$

as $n \to \infty$. Here the implied constant depends only on $p$, $\varepsilon$ and $\tau$ from Assumption 2.

We can also use the results with Markov’s inequality to obtain a bound on the probability that a random polynomial has a large number of roots.

**Corollary 6.** Let $f(X) = \xi_0 + \xi_1 X + \cdots + \xi_{n-1} X^{n-1} + X^n$ where $\xi_0, \ldots, \xi_{n-1}$ satisfy Assumption 2. Then

(a) There exists a constant $K > 0$ such that

$$\mathbb{P}(R(f) \geq \log n \mid p \nmid \xi_0) = O\left(n^{-K}\right)$$

as $n \to \infty$.

(b) For any $0 < \lambda \leq 1$ there exists a constant $K > 0$ such that

$$\mathbb{P}(R(f) \text{ when } \geq n^\lambda \mid p \nmid \xi_0) = O\left(\exp\left(-K \log^2 n\right)\right)$$

as $n \to \infty$. In particular, if $\lambda = 1$ then

$$\mathbb{P}(f \text{ totally split } \mid p \nmid \xi_0) = O\left(\exp\left(-K \log^2 n\right)\right)$$

as $n \to \infty$.

1.1. **Structure of the paper.** Section 2 contains generalizations and variants of known facts regarding $p$-adic number, uniform random $p$-adic polynomials, and random walks. In Section 3, we study the distribution of $R(g)$ for specific family of polynomials $g$ and in Section 4 we study the distribution of the Hasse derivatives of $f$. We prove Theorem 3 and Theorem 4 in Section 5, mainly using Lemma 29 which is described and proved in the same section. Finally, in Section 6 we bound the probability of $f$ having a large number of roots, see Corollary 6.

1.2. **Basic Notations and conventions.** In this paper, we will assume that $p$ is a fixed prime and all the implied constants of the big O notation may depend on $p$.

For a ring $A$, we denote the ring of polynomials over $A$ with $A[X]$. For $n \geq 0$ we denote $A[X]_n$ to be the subset of $A[X]$ contains all polynomials of degree $n$ and denote $A[X]_n^1$ to be the subset of $A[X]$ contains all monic polynomials of degree $n$. 

Theorem 7 (Newton-Raphson method). If \( f \in \mathbb{Z}_p[X] \) and \( \bar{x} \in \mathbb{Z}/p^{k-1}\mathbb{Z} \) satisfies
\[
f(\bar{x}) \equiv 0 \pmod{p^{k-1}} \quad \text{and} \quad f'(\bar{x}) \not\equiv 0 \pmod{p^k},
\]
then \( \bar{x} \) can be lifted uniquely from \( \mathbb{Z}/p^k\mathbb{Z} \) to a root of \( f \) in \( \mathbb{Z}_p \), i.e., there is a unique \( x \in \mathbb{Z}_p \) such that \( f(x) = 0 \) and \( x \equiv \bar{x} \pmod{p^k} \).

The other generalization is used to factor polynomial in \( p \)-adic fields, see [FJ05, Proposition 3.5.2] or [Neu99, Lemma II.4.6].

Theorem 8 (Hensel’s Lemma). Let \( f \in \mathbb{Z}_p[X] \) be a polynomial and \( \bar{g}, \bar{h} \in \mathbb{Z}/p\mathbb{Z}[X] \) coprime polynomials satisfying \( f \equiv \bar{g}h \pmod{p} \). Then \( \bar{g}, \bar{h} \) can be lifted uniquely to polynomials \( g, h \in \mathbb{Z}_p[X] \) such that \( f = gh \). Moreover, if \( f \) is monic polynomial then also \( g \) and \( h \) are also monic polynomials.

We also note another lemma regarding random polynomials over \( \mathbb{Q}_p \).

Lemma 9. Let \( f(X) = \xi_0 + \xi_1 X + \cdots + \xi_{n-1} X^{n-1} + X^n \) be a random polynomial where \( \xi_0, \ldots, \xi_{n-1} \) are random variables taking values in \( \mathbb{Z}_p^\times \cup \{0\} \). Let \( f_0(X) = f(pX) \), then \( f_0 \) has no non-zero roots in \( \mathbb{Z}_p \) almost surely.

Proof. The proof of [Shm21, Lemma 20] gives the stronger statement of our lemma.
2.2. Uniform random $p$-adic polynomial. The $p$-adic absolute value induces a metric on $\mathbb{Q}_p$ defined by $d(x, y) = |x - y|_p$. The open balls of this metric are of the form $x + p^n \mathbb{Z}_p$ for some $x \in \mathbb{Q}_p$ and $k \in \mathbb{Z}$. Since the $p$-adic absolute value is discrete, every open ball is also closed and compact. By Haar’s theorem (see [Hal50, Chapter XI]), there exists up to a positive multiplicative constant, a unique regular non-trivial measure $\mu$ on Borel subsets of $\mathbb{Q}_p$, such that for any Borel set $E \subseteq \mathbb{Q}_p$, $x \in \mathbb{Q}_p$,

$$
\mu(x + E) = \mu(E) \quad \text{and} \quad \mu(xE) = |x|_p \mu(E).
$$

This measure is called a Haar measure on $\mathbb{Q}_p$.

For a compact set $K \subseteq \mathbb{Q}_p$, we call $\mu$ the Haar measure normalized on $K$ if $\mu$ is a Haar measure on $\mathbb{Q}_p$ and $\mu(K) = 1$. In the vector space $\mathbb{Q}_p^n$ for a compact set $K \subseteq \mathbb{Q}_p^n$ we say $\mu$ is the Haar measure normalized on $K$ if it is a product of some Haar measures on $\mathbb{Q}_p$ and $\mu(K) = 1$. The Haar measure normalized on $K$ is unique and always exists.

We use the embedding of $\mathbb{Q}_p[X]^1_n$ in $\mathbb{Q}_p^n$:

$$
a_0 + a_1 X + \cdots + a_{n-1}X^{n-1} + X^n \mapsto (a_0, \ldots, a_{n-1}),
$$

to define a topology in $\mathbb{Q}_p[X]^1_n$ and equivalent definition of Haar measure on normalized compact subsets. Moreover, for a compact set $P \subseteq \mathbb{Q}_p[X]^1_n$ we say that a random polynomial $h \in \mathbb{Q}_p[X]^1_n$ is distributed uniformly on $P$ if $h$ is taking values in $P$ and distributed according to the Haar measure normalized on $P$ and restricted to $P$.

For integers $m \leq n$ we define the set $P_{m,n} \subseteq \mathbb{Z}_p[X]^1_n$ to be the set of all monic polynomials $h(X) = a_0 + a_1 X + \cdots + a_{n-1}X^{n-1} + X^n$ such that $a_m$ is the first coefficient not divisible by $p$. We note two special cases of this set when $m = n$ and when $m = 0$. The set $P_{n,n}$ is the set of all monic polynomials $f \in \mathbb{Z}_p[X]^1_n$ which their reduction modulo $p$ is $X^n$, i.e.

$$
P_{n,n} = \left\{ f \in \mathbb{Z}_p[X]^1_n : f \equiv X^n \pmod{p} \right\}.
$$

And $P_{0,n}$ is the set of all monic polynomials $g \in \mathbb{Z}_p[X]^1_n$ such that the free coefficient $g(0)$ is not divisible by $p$, i.e.

$$
P_{0,n} = \left\{ g \in \mathbb{Z}_p[X]^1_n : g(0) \not\equiv 0 \pmod{p} \right\}.
$$

Lemma 10. For any integers $m < n$, a random polynomial $h$ is distributed uniformly in $P_{m,n}$ if and only if there exists random independent polynomials $f$ and $g$ distributed uniformly in $P_{m,m}$ and $P_{0,n-m}$, respectively, such that $h = fg$.

Proof. Let $f \in P_{m,m}$ and $g \in P_{0,n-m}$. From the definition $f$ and $g$ are monic and their reductions modulo $p$ are coprime. Therefore, we use [BCFG22, Corollary 2.5] to infer that the resultant $\text{Res}(f, g)$ is a unit in $\mathbb{Z}_p$ for all $f \in P_{m,m}$ and $g \in P_{n,n}$. From [BCFG22, Corollary 2.7], we conclude that the multiplication map $P_{m,m} \times P_{0,n-m} \rightarrow P_{m,m}P_{0,n-m}$ is measure preserving.

Thus, it is sufficient to show that $P_{m,m}P_{0,n-m} = P_{n,n}$.

Let $h(X) = a_0 + a_1 X + \cdots + a_{n-1}X^{n-1} + X^n \in P_{m,n}$, then taking reduction modulo $p$ gives

$$
h(X) \equiv X^m(a_m + \cdots + a_{n-1}X^{n-m-1} + X^{n-m}) \pmod{p}.
$$

Since $a_m \not\equiv 0 \pmod{p}$, the polynomials $X^m$ and $a_m + \cdots + a_{n-1}X^{n-m-1} + X^{n-m}$ are coprime modulo $p$. By Hensel’s Lemma (Theorem 8), there exists a lift $f$ of $X^m$ and a lift $g$ of $a_m + \cdots + a_{n-1}X^{n-m-1} + X^{n-m} \pmod{p}$ such that $h = fg$. The polynomial $f$ is a lift of $X^m$ hence $f \in P_{m,m}$. Also, $g(0) \equiv \xi_m \not\equiv 0 \pmod{p}$ hence $g \in P_{0,n-m}$. So we got that $h = fg \in P_{m,m}P_{0,n-m}$.

Therefore, $P_{m,n} \subseteq P_{m,m}P_{0,n-m}$ and the other is direction is trivial when reducing the product modulo $p$. \hfill \qed

We set $\alpha(n, d) = \mathbb{E}[R(d)(f)]$ (respectively $\beta(n, d)$) where $f$ is a random polynomial distributed uniformly on $\mathbb{Z}_p[X]^1_n$ (respectively $P_{n,n}$). We have the following lemmas regarding the values of $\alpha(n, d)$ and $\beta(n, d)$ which are taken from [BCFG22].
Lemma 11. The values of $\alpha(n,d)$ and $\beta(n,d)$ can be computed using the following recurrence relation. First, for all $n \geq d \geq 0$ we have

\begin{equation}
\alpha(n,d) = p^{-n} \sum_{f \in \mathbb{F}_p[X]} \sum_{d_0 + \cdots + d_{p-1} = d} \prod_{r=0}^{p-1} \beta(n_r, d_r)
\end{equation}

where the inner sum runs over all non-negative integers $d_0, \ldots, d_{p-1}$ such that $d_0 + \cdots + d_{p-1} = d$ and $n_r$ is the multiplicity of $r$ as a root of $f$ over $\mathbb{F}_p$, i.e.,

\[ n_i = \max \left\{ k \geq 0 : (X - i)^k \mid f \right\} . \]

Second, for all $n \geq d \geq 0$ we have

\begin{equation}
\beta(n,d) = p^{-\binom{d}{2}} \alpha(n,d) + (p-1) \sum_{d \leq s < r < n} p^{-\binom{s+1}{2}} p^s \alpha(s,d).
\end{equation}

And the initial conditions are

\begin{equation}
\alpha(n,d) = \beta(n,d) = 0, \quad \alpha(n,0) = \beta(n,0) = 1 \quad \text{and} \quad \alpha(1,1) = \beta(1,1) = 1
\end{equation}

for all $0 \leq n < d$.

Proof. We start with proving the initial condition, (7). Those identities are true because if $0 \leq n < d$ then $R^{(d)}(f) = 0$ and $R^{(0)}(f) = 1$. The last identity of (7) holds since all linear polynomial has exactly one root.

Next we prove (5). This equation is a consequence of [BCFG22, eq. (30)] after settings $N_\sigma = \sum_{f \in \mathbb{F}_p[X]} \sigma(f) = \sigma$ and changing the order of summations. Also note that in [BCFG22, eq. (30)] increasing $k$ by adding $n_r = 0$ does not change the inner sum since $\beta(0,0) = 1$ and $\beta(0,0) = 0$ when $d_r > 0$.

Finally, we get (6) by plugging $\alpha(s,d) = 0$ for $s < d$ into [BCFG22, eq. (33)].}

\[ \square \]

Lemma 12. The expectations $\alpha(n,d)$ and $\beta(n,d)$ are rational functions in $p$ and are independent of $n$ for $n \geq 2d$. Moreover, we have the following equality of power series

\[ \sum_{d=0}^{\infty} \alpha(2d,d)t^d = \left( \sum_{d=0}^{\infty} \beta(2d,d)t^d \right)^p . \]

Proof. By [BCFG22, Theorem 1.(a) and Theorem 1.(c)] $\alpha(n,d)$ and $\beta(n,d)$ are rational functions in $p$ and are independent of $n$ for $n \geq 2d$. We use the notation of $A_d$ and $B_d$ as defined in [BCFG22]. According to [BCFG22, Theorem 1.(c)] and the identity $A_d(1) = A_d(p)$ (see paragraph after [BCFG22, eq. (38)]) we have

\begin{equation}
\alpha(2d,d) = A_d(1) = A_d(p) \quad \text{and} \quad \beta(2d,d) = B_d(1).
\end{equation}

The equality of power series is proved by setting $t = 1$ in [BCFG22, eq. (5)] and then plugging (8).

\[ \square \]

Due to Lemma 12 we can abbreviate and write $\alpha(d) = \alpha(n,d)$ and $\beta(d) = \beta(n,d)$ for some $n \geq 2d$. Using the shorthanded notation, (2) is immediate result of Lemma 12.

Lemma 13. For all $n > 2d$,

\begin{align*}
\alpha(d) &= (1 - p) \sum_{m=0}^{n-1} \alpha(m,d)p^m + \alpha(n,d)p^n \quad \text{and} \\
\beta(d) &= (1 - p^{-1}) \sum_{m=0}^{n-1} \beta(m,d)p^{-m} + \beta(n,d)p^{-n}.
\end{align*}

Proof. By setting $t = p$ in [BCFG22, eq. (38)] and writing $n - 1$ instead of $n$, we obtain

\[ A_d(p) = (1 - p) \sum_{m=0}^{n-1} \alpha(m,d)p^m + \alpha(n,d)p^n . \]
Plugging (8) into the last equation gives the equality for \( \alpha(d) \). And the other equality is obtained by applying the inversion \( p \leftrightarrow 1/p \) and [BCFG22, Theorem 1.(a)].

Next, we prove two lemmas regarding the values of \( \alpha(n,d) \), \( \beta(n,d) \) that are not covered in [BCFG22].

**Lemma 14.** Let \( f \) be a random polynomial distributed uniformly in \( \mathbb{Z}_p[X]_n^1 \). Then for any \( 0 < d \leq n/2 \) we have that
\[
\mathbb{E}[R^d(f_0)] = \beta(d),
\]
where \( f_0(X) = f(pX) \).

**Proof.** Let \( m \leq n \) be a non-negative integer and assume that \( h \in P_{m,n} \) occurs. So \( h \) is distributed uniformly in \( P_{m,n} \) and by Lemma 10 we get that there exists two random polynomials \( f \) and \( g \) distributed uniformly in \( P_{m,n} \) and \( P_{0,n-m} \), respectively, such that \( h = fg \).

The map \( x \mapsto px \) is a bijection from integer roots of \( h_0 \) to integer roots of \( f \). Indeed, if \( x \) is an integer root of \( h_0 \) then \( f(px)g(px) \equiv h_0(x) = 0 \) and since \( g(px) \equiv g(0) \not\equiv 0 \pmod{p} \) we get that \( f(px) = 0 \). For the other direction, if \( y \in \mathbb{Z}_p \) is a root of \( f \) then \( y^m \equiv 0 \pmod{p} \) and then \( p \mid y \).

So there exists \( x \in \mathbb{Z}_p \) such that \( y = px \) and \( h_0(x) = 0 \) follows immediately. Therefore, we have that \( R^d(h_0) = R^d(f) \).

Since \( \mathbb{Z}_p[X]_n^1 = \bigsqcup_{m=0}^n P_{m,n} \), we use the law of total expectation to get
\[
\mathbb{E}[R^d(h_0)] = \sum_{m=0}^n \mathbb{E}[R^d(f_0) \mid f \in P_{m,n}] \mathbb{P}(f \in P_{m,n})
\]
\[
= \sum_{m=0}^n \mathbb{E}[R^d(f) \mid f \in P_{m,n}] \mathbb{P}(f \in P_{m,n}).
\]

We recall that \( f \) is distributed uniformly on \( P_{m,n} \) when \( h \in P_{m,n} \), hence \( \mathbb{E}[R^d(f) \mid h \in P_{m,n}] = \beta(n,d) \). Moreover, the probability \( \mathbb{P}(f \in P_{m,n}) \) equals \((p-1)/p^{m+1}\) when \( m < n \) and \( 1/p^n \) when \( m = n \). Therefore,
\[
\mathbb{E}[R^d(h_0)] = \sum_{m=0}^{n-1} \beta(m,d) \cdot \frac{p-1}{p^{m+1}} + \beta(n,d) p^{-n}
\]
\[
= (1 - p^{-1}) \sum_{m=0}^n \beta(m,d) p^{-m} + \beta(n,d) p^{-n}.
\]

And using Lemma 13 finish the proof.

**Lemma 15.** For any integers \( 0 \leq d \leq n \),
\[
\log_p \alpha(n,d) = -\frac{d^2}{2(p-1)} + O(d \log d) \quad \text{and}
\]
\[
\log_p \beta(n,d) = -\frac{pd^2}{2(p-1)} + O(d \log d)
\]
as \( d \to \infty \).

**Proof.** Let \( f \) be a random polynomial distributed uniformly on \( \mathbb{Z}_p[X]_n^1 \), so \( \alpha(n,d) = \mathbb{E}[R^d(f)] \).

We take a look at the values of \( \alpha(n,d) \) when \( n = d \). In this case we have at most \( d \) roots, hence \( R^d(f) \) is an indicator function of the event that \( f \) has exactly \( d \) roots i.e. \( f \) totally splits. So \( \alpha(d,d) = \mathbb{P}(f \text{ totally splits}) \) and this probability has an asymptotic formula in [BGMR06, Theorem 5.1] which implies
\[
\log_p \alpha(d,d) = -\frac{d^2}{2(p-1)} + O(d \log d).
\]

Setting \( n = d \) in (6) gives
\[
\beta(d,d) = p^{-\binom{d}{2}} \alpha(d,d).
\]
Since \((d \choose 2) = d^2/2 + O(d)\) we obtain
\[
\log_p \beta(d, d) = - \frac{pd^2}{2(p-1)} + O(d \log d).
\]

From the definition of big-O notations we that there exists a constant \(C_0 > 0\) such that for any \(d \geq 0\)
\[
\frac{d^2}{2(p-1)} + C_0 \log d < 0
\]
(9) \[
\log_p \alpha(d, d) + \frac{d^2}{2(p-1)} < C_0 d \log d \quad \text{and}
\]
(10) \[
\log_p \beta(d, d) + \frac{pd^2}{2(p-1)} < C_0 d \log d.
\]

We continue our proof by bounding \(\log_p \alpha(n, d)\) and \(\log_p \beta(n, d)\) from both sides.

**Upper bound:** We prove by induction on \(n\) the following inequalities
\[
\log_p \alpha(n, d) < - \frac{d^2}{2(p-1)} + C_0 d \log d \quad \text{and}
\]
\[
\log_p \beta(n, d) < - \frac{pd^2}{2(p-1)} + C_0 d \log d.
\]

For the base of the induction we take \(n = d\). In this case, (11) and (12) are immediate implication of (9) and (10) respectively.

For \(n > d\), consider the inner sum \(\sum_{d_0 + \cdots + d_{p-1} = d} \prod_{i=0}^{p-1} \beta(n_i, d_i)\), of (5) in the case that \(\tilde{f} = (X - r)^n\) for some \(r \in \mathbb{F}_p\). If there exists \(i \neq r\) such that \(d_i \neq 0\) then \(\beta(n_i, d_i) = \beta(0, d_i) = 0\) by (7) and the product eliminates. Therefore, this sum has only one non-zero summand which obtained when \(d_r = d\) and \(d_i = 0\) for \(i \neq r\). Hence, when \(f = (X - r)^n\)
\[
\sum_{d_0 + \cdots + d_{p-1} = d} \prod_{i=0}^{p-1} \beta(n_i, d_i) = \beta(n, d).
\]

We plug this in (5) and get
\[
\alpha(n, d) = p^{-n} \sum_{f \in \mathbb{F}_p[X]} \sum_{n_i < n} \prod_{i=0}^{p-1} \beta(n_i, d_i) + p^{-n+1} \beta(n, d).
\]

We change the order of summation:
\[
\alpha(n, d) = p^{-n} \sum_{d_0 + \cdots + d_{p-1} = d} \sum_{f \in \mathbb{F}_p[X]} \prod_{i=0}^{p-1} \beta(n_i, d_i) + p^{-n+1} \beta(n, d).
\]

We note that if \(n_i < d_i\) for some \(i\) then \(\beta(n_i, d_i) = 0\) and the product eliminates. Hence,
\[
\alpha(n, d) = p^{-n} \sum_{d_0 + \cdots + d_{p-1} = d} \sum_{f \in \mathbb{F}_p[X]} \prod_{i=0}^{p-1} \beta(n_i, d_i) + p^{-n+1} \beta(n, d).
\]

We take a look at the product of (13) for some non-negative integers \(d_0, \ldots, d_{p-1}\) such that \(d_0 + \cdots + d_{p-1} = d\) and polynomial \(f \in \mathbb{F}_p[X]\) such that \(d_i \leq n_i < n\) for all \(i\). Using (12) from the induction’s assumption we get that
\[
\log_p \left( \prod_{i=0}^{p-1} \beta(n_i, d_i) \right) < \sum_{i=1}^{p-1} \left( - \frac{pd^2}{2(p-1)} + C_0 d_i \log d_i \right)
\]
\[
\leq - \frac{p}{2(p-1)} \sum_{i=0}^{p-1} d_i^2 + C_0 d \log d.
\]
By Cauchy-Schwarz inequality, 
\[ d^2 = \left( \sum_{i=0}^{p-1} d_i \right)^2 \leq p \sum_{i=0}^{p-1} d_i^2. \]
Hence,
\[ \log_p \left( \prod_{i=0}^{p-1} \beta(n_i, d_i) \right) < -\frac{d^2}{2(p-1)} + C_0 d \log d. \]  
(14)

Define the constant \( A_d \) by
\[ \log_p A_d = -\frac{d^2}{2(p-1)} + C_0 d \log d. \]
So we can write (14) as
\[ \prod_{i=0}^{p-1} \beta(n_i, d_i) < A_d. \]

We put this in the outer sum of (13) and get
\[ \sum_{d_0 + \cdots + d_{p-1} = d} \sum_{f \in \mathbb{F}_p[X]^1_n} \prod_{i=0}^{p-1} \beta(n_i, d_i) \]
(15)
\[ = A_d \sum_{d_0 + \cdots + d_{p-1} = d} \sum_{f \in \mathbb{F}_p[X]^1_n} 1. \]

We look at the sums in right most side of (15), and change the order of summation so
\[ \sum_{d_0 + \cdots + d_{p-1} = d} \sum_{f \in \mathbb{F}_p[X]^1_n} 1 = \sum_{f \in \mathbb{F}_p[X]^1_n} \sum_{d_0 + \cdots + d_{p-1} = d} 1. \]
We add to the outer sum the \( p \) polynomials of the form \((X - r)^d\). For each of those polynomials the inner sum is equal 1 since all but one \( n_i \) is equal 0. Therefore,
\[ \sum_{d_0 + \cdots + d_{p-1} = d} \sum_{f \in \mathbb{F}_p[X]^1_n} 1 = \sum_{f \in \mathbb{F}_p[X]^1_n} \sum_{d_0 + \cdots + d_{p-1} = d} 1 - p. \]

Changing order of summation in the right most side gives
\[ \sum_{d_0 + \cdots + d_{p-1} = d} \sum_{f \in \mathbb{F}_p[X]^1_n} 1 = \sum_{d_0 + \cdots + d_{p-1} = d} \# \left\{ \hat{f} \in \mathbb{F}_p[X]^1_n : d_i \leq n_i \right\} - p. \]  
(16)

We consider the set \( \left\{ f \in \mathbb{F}_p[X]^1_n : n_r \geq d_r \right\}. \) This set is the set of all polynomials \( \hat{f} \in \mathbb{F}_p[X]^1_n \) such that \( \prod_{r=0}^{p-1} (X - r)^{d_r} \mid \hat{f} \). Hence,
\[ \# \left\{ \hat{f} \in \mathbb{F}_p[X]^1_n : n_r \geq d_r \right\} = p^{n-d}. \]  
(17)
Plugging (17) into (16) gives
\[ \sum_{d_0 + \cdots + d_{p-1} = d} \sum_{f \in \mathbb{F}_p[X]^1_n} 1 = \sum_{d_0 + \cdots + d_{p-1} = d} p^{n-d} - p \leq p^n - p, \]
where the inequality is true since the number of summands in the sum is at most \( p^d \). We plug the last inequality into (15) to get
\[ \sum_{d_0 + \cdots + d_{p-1} = d} \sum_{f \in \mathbb{F}_p[X]^1_n} \prod_{i=0}^{p-1} \beta(n_i, d_i) < A_d(p^n - p). \]

We put the last inequality in (13) and apply simple calculation we get that
\[ p^{n-1} \alpha(n, d) - \beta(n, d) < (p^{n-1} - 1) A_d. \]  
(18)
Next, we use (11) from the induction’s assumption to infer
\[ \alpha(s, d) < A_d \]
for \( d \leq s < n \). We plug this into (6):
\[ \beta(n, d) < p^{-\binom{d}{2}} \alpha(n, d) + (p - 1) \sum_{d \leq s < r < n} p^{-\binom{r+1}{2}} p^s A_d. \]

We split the sum in the left most side into two sums and apply some basic computations
\[
\sum_{d \leq s < r < n} p^{-\binom{r+1}{2}} p^s = \sum_{r=d+1}^{n-1} p^{-\binom{r+1}{2}} \sum_{s=d}^{r-1} p^s
= \sum_{r=d+1}^{n-1} p^{-\binom{r+1}{2}} \cdot \frac{p^r - p^d}{p - 1}
= \frac{1}{p - 1} \sum_{r=d+1}^{n-1} \left( p^{-\binom{r}{2}} - p^{-\binom{r+1}{2}} \right)
\leq \frac{1}{p - 1} \sum_{r=d+1}^{n-1} \left( p^{-\binom{r}{2}} - p^{-\binom{r+1}{2}} \right)
= \frac{1}{p - 1} \left( p^{-\binom{d+1}{2}} - p^{-\binom{d}{2}} \right).
\]

We combine this with (19) to create the inequality
\[ \beta(n, d) - p^{-\binom{d}{2}} \alpha(n, d) < \left( p^{-\binom{d+1}{2}} - p^{-\binom{d}{2}} \right) A_d. \]

Adding (18) and (20) gives
\[ \left( p^{n-1} - p^{-\binom{d}{2}} \right) \alpha(n, d) < \left( p^{n-1} + p^{-\binom{d+1}{2}} - 1 - p^{-\binom{d}{2}} \right) A_d. \]

We have that \( p^{-\binom{d+1}{2}} - 1 \leq 0 \) so
\[ \alpha(n, d) < \frac{p^{n-1} + p^{-\binom{d+1}{2}} - 1 - p^{-\binom{d}{2}}}{p^{n-1} - p^{-\binom{d}{2}}} \cdot A_d \leq A_d. \]

Taking \( \log_p \) on both side of the inequality gives (11).

We use (21) and (20) to get the bound
\[ \beta(n, d) < p^{-\binom{d}{2}} \alpha(n, d) + \left( p^{-\binom{d+1}{2}} - p^{-\binom{d}{2}} \right) A_d \leq p^{-\binom{d+1}{2}} A_d. \]

Taking \( \log_p \) on both side of the inequality gives
\[ \log_p \beta(n, d) < -\binom{d+1}{2} - \frac{d^2}{2(p - 1)} + C_0 d \log d. \]

Since \( \binom{d+1}{2} > d^2/2 \), we get that
\[ \log_p \beta(n, d) < -\frac{pd^2}{2(p - 1)} + C_0 d \log d. \]

The inequality (21) after taking \( \log_p \) and (22) finish the induction.
**Lower Bound:** We start with bounding \( \beta(n, d) \). For \( n = d \), by (10) we have

\[
\log_p \beta(d, d) > -\frac{pd^2}{2(p-1)} - C_0 d \log d.
\]

For \( n = d + 1 \), let \( g \) be a random polynomial distributed uniformly on \( P_{n,n} \), so that \( \beta(n, d) = \mathbb{E}[R^{(d)}(g)] \). The polynomial \( g \) cannot miss exactly one root, so \( R(g) = d + 1 \) or \( R(g) < d \). In the later case, we have that \( R^{(d)}(g) = 0 \). And when \( R(g) = d + 1 \), i.e. \( g \) totally split, we have that \( R^{(d)}(g) = p \). Also the probability that \( g \) totally split is \( \beta(d + 1, d + 1) \). Therefore, we use (10) to get that

\[
\log_p \beta(d + 1, d + 1) = \log_p (p \beta(d + 1, d + 1))
\]

\[
> 1 - \frac{pd + 1}{2(p-1)} - C_0 (d + 1) \log(d + 1).
\]

So there exists a constant \( C_1 > 0 \) such that

\[
\log_p \beta(d + 1, d + 1) > -\frac{pd^2}{2(p-1)} - C_1 d \log d.
\]

Next, let \( n > d + 1 \). We look at (6) and omit all terms except the summand where \( s = d \) and \( r = d + 1 \):

\[
\beta(n, d) > (p - 1)p^{-(d+2)}p^2 \alpha(d, d).
\]

Taking \( \log_p \) in both sides and using (9) gives

\[
\log_p \beta(n, d) > \log_p (p - 1) - \left( \frac{d + 2}{2} \right) + d - \frac{d^2}{2(p-1)} - C_0 d \log d.
\]

Since \( \left( \frac{d + 2}{2} \right) = \frac{d^2}{2} + O(d) \), then

\[
\log_p \beta(n, d) > -\frac{pd^2}{2(p-1)} - C_2 d \log d,
\]

for some constant \( C_2 > 0 \). From (23), (24) and (25), there exists \( C_3 > 0 \) such that for all \( n \geq d \)

\[
\log_p \beta(n, d) > -\frac{pd^2}{2(p-1)} - C_3 d \log d.
\]

We also define the constant \( B_d \) by

\[
\log_p \beta(n, d) > -\frac{pd^2}{2(p-1)} - C_3 d \log d.
\]

So applying exponent with base \( p \) on both sides of (26) gives

\[
\beta(n, d) > B_d.
\]

Next we bound the values of \( \alpha(n, d) \). We change the order of summation in (5), so

\[
\alpha(n, d) = p^{-n} \sum_{d_0 + \ldots + d_{p-1} = d} \sum_{f \in \mathbb{F}_p[X]_n^1} \prod_{r=0}^{p-1} \beta(n_r, d_r)
\]

As before, if \( n_r < d_r \) for some \( r \) then \( \beta(n_r, d_r) = 0 \) and the product eliminates. Therefore,

\[
\alpha(n, d) = p^{-n} \sum_{d_0 + \ldots + d_{p-1} = d} \sum_{n_r \geq d_r} \prod_{r=0}^{p-1} \beta(n_r, d_r).
\]
Using (27) gives

\[ \alpha(n, d) > p^{-n} \sum_{d_0 + \cdots + d_{p-1} = d} \sum_{d_r \geq d_r}^{p-1} \prod_{r=0}^{p-1} B_{d_r} \]

\[ = p^{-n} \sum_{d_0 + \cdots + d_{p-1} = d} \# \{ f \in \mathbb{F}_p[X]^1_n : n_r \geq d_r \} \prod_{r=0}^{p-1} B_{d_r}. \]

We consider the set \( \{ f \in \mathbb{F}_p[X]^1_n : n_r \geq d_r \} \). This set is the set of all polynomials \( f \in \mathbb{F}_p[X]^1_n \) such that \( \prod_{r=0}^{p-1} (X - r)^{d_r} \mid f \). Hence, its size is \( p^{-n} \) and the last equation becomes

\[ (28) \quad \alpha(n, d) > p^{-d} \sum_{d_0 + \cdots + d_{p-1} = d} \prod_{r=0}^{p-1} B_{d_r}. \]

Note that \( \sum_{r=0}^{p-1} [(d + r)/p] = d \). So we omit all terms in the sum of (28) except when \( d_r = [(d + r)/p] \) to get

\[ \alpha(n, d) > p^{-d} \prod_{i=0}^{p-1} B_{[(d + i)/p]}. \]

Taking \( \log_p \) on both sides gives

\[ \log_p \alpha(n, d) > -d - \sum_{i=0}^{p-1} \left( \frac{p}{2(p-1)} \left( \frac{d + i}{p} \right)^2 + C_3 \left( \frac{d + i}{p} \right) \log \left( \frac{d + i}{p} \right) \right) \]

\[ \geq -d - \sum_{i=0}^{p-1} \frac{p}{2(p-1)} \left( \frac{d + i}{p} \right)^2 - C_3 \sum_{i=0}^{p-1} \left( \frac{d + i}{p} \right) \log \left( \frac{d + i}{p} \right) \]

\[ \geq - \frac{p^2}{2(p-1)} \left( \frac{d + p}{p} \right)^2 - C_4 d \log d. \]

for some constants \( C_4 > 0 \). Since \( \left( \frac{d + p}{p} \right)^2 = d^2/p^2 + O(d) \), there exists a constant \( C_5 > 0 \) such that

\[ (29) \quad \log_p \alpha(n, d) \geq - \frac{d^2}{2(p-1)} - C_5 d \log d. \]

The inequalities (29) and (26) gives the required lower bounds for the proof. \( \square \)

We define the values \( \gamma(d) \) using the power series equality (3). From this definition we have that:

\[ (30) \quad \gamma(d) = \sum_{d_1 + \cdots + d_{p-1} = d} \prod_{r=1}^{p-1} \beta(d_r), \]

where the sum runs on all non-negative integers \( d_1, \ldots, d_{p-1} \) such that \( d_1 + \cdots + d_{p-1} = d \). Finally, we give an asymptotic estimate for the values of \( \gamma(d) \).

**Lemma 16.** We have that

\[ \log_p \gamma(d) = - \frac{pd^2}{2(p-1)^2} + O(d \log d), \]

when \( d \to \infty \).

**Proof.** By Lemma 15, there exists \( C_0 > 0 \) such that for all \( d \geq 0 \) we have

\[ (31) \quad \left| \log_p \beta(d) + \frac{pd^2}{2(p-1)} \right| < C_0 d \log d. \]

We continue our proof by bounding \( \log_p \gamma(d) \) from both sides.
Upper Bound: We look at the product of (30) for some non-negative integers \( d_1, \ldots, d_{p-1} \) such that \( d_1 + \cdots + d_{p-1} = d \), so by (31) we infer
\[
\log p \prod_{r=1}^{p-1} \beta(d_r) < \sum_{r=1}^{p-1} \left( -\frac{pd_r^2}{2(p-1)} + C_0 d_r \log d_r \right)
\leq -\frac{p}{2(p-1)} \sum_{r=1}^{p-1} d_r^2 + C_0 \sum_{r=1}^{p-1} d_r \log d.
\]
By Cauchy-Schwarz inequality, \( d^2 = \left( \sum_{r=1}^{p-1} d_r \right)^2 \leq (p-1) \sum_{r=0}^{p-1} d_r^2 \). Hence,
\[
\log p \prod_{r=1}^{p-1} \beta(d_r) < -\frac{pd^2}{2(p-1)} + C_0 d \log d.
\]
Define the constant \( G_d \) by
\[
\log p G_d = -\frac{pd^2}{2(p-1)} + C_0 d \log d.
\]
We plug (32) into (30) to obtain
\[
\gamma(d) < \sum_{d_1 + \cdots + d_{p-1} = d} G_d = \left( \frac{d+p-1}{p-1} \right)^{p-1} G_d \leq p^d G_d.
\]
We take \( \log p \) on both sides of the inequality, so
\[
\log p \gamma(d) < d - \frac{pd^2}{2(p-1)^2} + C_0 d \log d
\leq -\frac{pd^2}{2(p-1)^2} + C_1 d \log d,
\]
for some \( C_1 > 0 \). And the last inequality gives the required upper bound.

Lower Bound: We take a look on (30). Note that \( \sum_{r=1}^{p-1} \left( \frac{d+r-1}{p-1} \right) = d \), hence
\[
\gamma(d) \geq \prod_{r=1}^{p-1} \beta \left( \left\lceil \frac{d+r-1}{p-1} \right\rceil \right).
\]
Taking \( \log p \) on both sides gives
\[
\log_p \gamma(d) \geq \sum_{r=1}^{p-1} \log_p \beta \left( \left\lceil \frac{d+r-1}{p-1} \right\rceil \right).
\]
We use (31) in (33) to get
\[
\log_p \gamma(d) \geq -\sum_{r=1}^{p-1} \left( \frac{p}{2(p-1)} \left\lceil \frac{d+r-1}{p-1} \right\rceil \right)^2 + C_0 \left\lceil \frac{d+r-1}{p-1} \right\rceil \log \left\lceil \frac{d+r-1}{p-1} \right\rceil
\geq -\sum_{r=1}^{p-1} \left( \frac{p}{2(p-1)} \left( \frac{d+r-1}{p-1} \right) ^2 + C_0 \left( \frac{d+r-1}{p-1} \right) \log \left( \frac{d+r-1}{p-1} \right) \right)
\geq -\sum_{r=1}^{p-1} \left( \frac{p}{2(p-1)} \left( \frac{d+p}{p-1} \right)^2 + C_2 d \log d \right.
\geq -\frac{p}{2(p-1)^2} d^2 + C_3 d \log d,
\]
for some constants \( C_2, C_3 > 0 \). And the lower bound is shown as needed. \( \square \)
2.3. Random walks. Let $k, m$ be positive integers and let $\xi_0, \ldots, \xi_{n-1}$ be random variables satisfying Assumption 2. Set $V = (\mathbb{Z}/p^k\mathbb{Z})^m$, for some vectors $\vec{v}_0, \vec{v}_1, \ldots, \vec{v}_n$ in $V$, we construct the random walk over the additive group $(V, +)$ whose $n$-th step is $\sum_{i=0}^n \xi_i \vec{v}_i$ (we set $\xi_n = 1$ for convenience).

For two vectors $\vec{u}, \vec{w} \in V$, we denote by $\langle \vec{u}, \vec{w} \rangle$ the formal dot product i.e.

$$\langle \vec{u}, \vec{w} \rangle = u_1w_1 + \cdots + u_mw_m.$$ 

For a non-zero vector $\vec{u} \in V$, we call the number of vectors in $\vec{v}_0, \ldots, \vec{v}_n$ such that $\langle \vec{u}, \vec{v}_i \rangle \neq 0$, the $\vec{u}$-weight of $\vec{v}_0, \ldots, \vec{v}_n$, and we denote it by $\text{Weight}_\vec{u}(\vec{v}_0, \ldots, \vec{v}_n)$. We define the minimal weight of $\vec{v}_0, \ldots, \vec{v}_n$ to be

$$\sigma(\vec{v}_0, \ldots, \vec{v}_n) = \min_{\vec{w} \in V \setminus \{\vec{0}\}} \text{Weight}_\vec{w}(\vec{v}_0, \ldots, \vec{v}_n).$$

The relation between $\tau$ from Assumption 2, $\sigma$ and the $n$-step of the random walk is:

**Proposition 17.** For any $S \subseteq V$. We have:

$$\mathbb{P}\left( \sum_{i=0}^n \xi_i \vec{v}_i \in S \right) = \frac{\# S}{\# V} + O\left( \frac{\# S \exp\left( -\tau \frac{\sigma(\vec{v}_0, \ldots, \vec{v}_n)}{p^k} \right) }{\# V} \right),$$

as $n \to \infty$.

To prove this proposition we follow the proof of [Shm21, Proposition 12]. The proof of both propositions is mostly the same except for few adjustments allowing $\xi_0, \ldots, \xi_{n-1}$ to have different laws from each other.

Let $\mu_0, \ldots, \mu_n$ be the laws of $\xi_0, \ldots, \xi_n$ and $\nu$ be the law of $\sum_{i=0}^n \xi_i \vec{v}_i$. For each index $i$ and positive integer $k$, let $\mu_i^{(k)}$ be the pushforward of $\mu_i$ to $\mathbb{Z}/p^k\mathbb{Z}$ and for $k = 1$ we also denote $\mu_i' = \mu_i^{(1)}$. Those measures satisfy the following

$$\mu_i^{(k)}(\vec{x}) = \mu_i(\vec{x} + p^k\mathbb{Z}_p) \quad \text{and} \quad \mu_i'(\vec{x}) = \mu_i(\vec{x} + p\mathbb{Z}_p).$$

From Assumption 2, we have that $1 - \sum_{\vec{x} \in \mathbb{Z}/p\mathbb{Z}} \mu_i'(\vec{x})^2 > \tau$ for all $0 < i < n$.

Let $\delta_\vec{w}$ be the Dirac measure on $V$, i.e.

$$\delta_\vec{w}(\vec{u}) = \begin{cases} 1, & \vec{u} = \vec{w}, \\ 0, & \vec{u} \neq \vec{w}. \end{cases}$$

We write $\mu_i \ast \delta_\vec{w}$ for the following probability measure on $V$:

$$\mu_i \ast \delta_\vec{w}(\cdot) = \sum_{\vec{x} \in \mathbb{Z}/p^k\mathbb{Z}} \mu_i^{(k)}(\vec{x}) \delta_{\vec{x} \ast \vec{w}}(\cdot).$$

With this notation, we can write:

$$\nu = \mu_0 \ast \delta_{\vec{v}_0} \ast \mu_1 \ast \delta_{\vec{v}_1} \ast \cdots \ast \mu_n \ast \delta_{\vec{v}_n}.$$ 

where $\ast$ is the convolution operator.

In this section we denote the Fourier transform by $\hat{\cdot}$ and we let $\zeta$ be a primitive $p^k$-th root of unity. So for any function $f: V \to \mathbb{C}$ we have the following relations

$$\hat{f}(\vec{u}) = \sum_{\vec{w} \in V} f(\vec{w}) \zeta^{-\langle \vec{u}, \vec{w} \rangle} \quad \text{and} \quad f(\vec{u}) = \frac{1}{\# V} \sum_{\vec{w} \in V} \hat{f}(\vec{w}) \zeta^{\langle \vec{u}, \vec{w} \rangle}.$$ 

The following lemma and its proof are based on [Shm21, Lemma 13] and [BV19, lemma 31].

**Lemma 18.** Let $\vec{u}, \vec{w} \in V$ and integer $0 < i < n$. If $\langle \vec{u}, \vec{w} \rangle \neq 0$ then

$$|\mu_i \ast \delta_\vec{w}(\vec{u})| < \exp\left( -\frac{\tau}{p^{2k}} \right).$$
Proof. By direct computation using (37) and (35) we get
\[
\left| \mu_i \delta_w(\vec{u}) \right|^2 = \sum_{x, y \in V} \mu_i(x) \mu_i(y) \zeta^{(x-y, \vec{u})}
\]  
which then from (34)
\[
\left| \mu_i \delta_w(\vec{u}) \right|^2 = \sum_{x, y \in Z/p^kZ} \mu_i(x) \mu_i(y) \zeta^{(x-y, \vec{u})}.
\]
Then from (34)
\[
\left| \mu_i \delta_w(\vec{u}) \right|^2 = \sum_{x, y \in Z/p^kZ} \mu_i(x) \mu_i(y) \zeta^{(x-y, \vec{u})}.
\]
We denote by \(L(\vec{t})\) the lift of \(\vec{t} \in \mathbb{Z}/p^k\mathbb{Z}\) to the interval \(-\frac{q^k}{2}, \frac{q^k}{2}\) \(\cap \mathbb{Z}\). Since \(\left| \mu_i \delta_w(\vec{u}) \right|^2 \in \mathbb{R}\) and \(\Re(\zeta^t) \leq 1 - 2L(\vec{t})^2/p^{2k}\), we get
\[
\left| \mu_i \delta_w(\vec{u}) \right|^2 \leq \sum_{x, y \in Z/p^kZ} \mu_i(x) \mu_i(y) \Re\left(\zeta^{(x-y, \vec{u})}\right)
\]
\[
\leq \sum_{x, y \in Z/p^kZ} \mu_i(x) \mu_i(y) \left(1 - \frac{2L((x-y, \vec{u})^2}{p^{2k}}\right)
\]
\[
= 1 - \frac{2}{p^{2k}} \sum_{x, y \in Z/p^kZ} \mu_i(x) \mu_i(y) L((x-y, \vec{u})^2.
\]
If \(p \nmid x - y\) then \((x-y, \vec{u})\) is non-zero. So
\[
L((x-y, \vec{u})^2 \geq 1,
\]
hence
\[
(39) \quad \left| \mu_i \delta_w(\vec{u}) \right|^2 \leq 1 - \frac{2}{q^2} \sum_{x, y \in Z/p^kZ} \mu_i(x) \mu_i(y).
\]
Since \(\mu'_i\) is also the pushforward measure of \(\mu_i\), we have
\[
\mu_i'(x') = \sum_{x \in Z/p^kZ} \mu_i(x).
\]
Hence
\[
\sum_{x, y \in Z/p^kZ} \mu_i(x) \mu_i(y) = \sum_{x', y' \in Z/p^kZ} \mu_i(x') \mu_i(y').
\]
By direct computation
\[
\sum_{x, y \in Z/p^kZ, x \neq y} \mu_i(x) \mu_i(y) = \sum_{x', y' \in Z/p^kZ} \mu_i(x') \mu_i(y') - \sum_{x' \in Z/p^kZ} \mu_i(x')^2
\]
\[
= \left( \sum_{x' \in Z/p^kZ} \mu_i(x') \right)^2 - \sum_{x' \in Z/p^kZ} \mu_i(x')^2
\]
\[
= 1 - \sum_{x' \in Z/p^kZ} \mu_i(x')^2 > \tau.
\]
Plugging this into (39) and using the inequality \(1 - t \leq \exp(-t)\), we get
\[
\left| \mu_i \delta_w(\vec{u}) \right|^2 < 1 - \frac{2\tau}{q^2} \leq \exp\left(-\frac{2\tau}{q^2}\right).
\]
We finish the proof by taking square root on both sides of the inequality. \(\square\)
Lemma 19. Let \( \bar{u} \in V \setminus \{ \bar{0} \} \). Then there exists \( C_0 > 0 \) such that
\[
|\hat{\nu}(\bar{u})| < C_0 \exp\left( -\frac{\tau \text{Weight}_{\bar{u}}(\bar{v}_1, \ldots, \bar{v}_n)}{p^{2k}} \right).
\]

Proof. We define the following set \( I(\bar{u}) = \{0 < i < n : \langle \bar{u}, \bar{v}_i \rangle \neq 0 \} \), so that
\[
\text{Weight}_{\bar{u}}(\bar{v}_1, \ldots, \bar{v}_{n-1}) = \#I(\bar{u})
\]
by definition. For \( i \in I(\bar{u}) \), Lemma 18 infers that
\[
(40) \quad |\hat{\nu}_i(\bar{u})| \leq \exp\left( -\frac{\tau}{p^{2k}} \right).
\]
Otherwise, for \( i \notin I(\bar{u}) \) we have that
\[
(41) \quad \left| \mu_i \cdot \delta_{\bar{v}_i}(\bar{u}) \right| \leq \sum_{\bar{w} \in \bar{V}} \left| \mu_i \cdot \delta_{\bar{v}_i}(\bar{w}) \right| \zeta^{-\langle \bar{u}, \bar{w} \rangle} = 1.
\]
By (36), (40), (41) and since the Fourier transform maps convolutions to products we get
\[
(42) \quad |\hat{\nu}(\bar{u})| = \prod_{i=0}^{n} \left| \hat{\nu}_i(\bar{u}) \right| 
\leq \prod_{i \in I(\bar{u})} \exp\left( -\frac{\tau}{p^{2k}} \right) \prod_{i \notin I(\bar{u})} 1 
= \exp\left( -\frac{\tau \#I(\bar{u})}{p^{2k}} \right) 
= \exp\left( -\frac{\tau \text{Weight}_{\bar{u}}(\bar{v}_1, \ldots, \bar{v}_{n-1})}{p^{2k}} \right).
\]
From the definition of \( \bar{u} \)-weight we get the following inequality
\[
\text{Weight}_{\bar{u}}(\bar{v}_1, \ldots, \bar{v}_{n-1}) \geq \text{Weight}_{\bar{u}}(\bar{v}_0, \ldots, \bar{v}_n) - 2.
\]
Plugging this inequality into (42) gives
\[
|\hat{\nu}(\bar{u})| \leq \exp\left( \frac{2\tau}{p^{2k}} \right) \exp\left( -\frac{\tau \text{Weight}_{\bar{u}}(\bar{v}_0, \ldots, \bar{v}_n)}{p^{2k}} \right).
\]
Since \( \tau < 1, p \geq 2 \) and \( k \geq 1 \) we have that \( \exp\left( 2\tau/p^{2k} \right) < e \) which finish the proof. \( \square \)

Proof of Proposition 17. We have that \( \nu \) is the law of \( \sum_{i=0}^{n} \xi_i \bar{v}_i \), hence
\[
P\left( \sum_{i=0}^{n} \xi_i \bar{v}_i \in S \right) = \sum_{\bar{u} \in S} \nu(\bar{u}).
\]
Using the triangle inequality gives
\[
(43) \quad \left| P\left( \sum_{i=0}^{n} \xi_i \bar{v}_i \in S \right) - \frac{\#S}{\#V} \right| \leq \sum_{\bar{u} \in S} |\nu(\bar{u}) - \frac{1}{\#V}|.
\]
Since \( \nu \) is a probability measure on \( V \), we have by (37) that
\[
\hat{\nu}(\bar{0}) = \sum_{\bar{u} \in V} \nu(\bar{u}) = 1.
\]
Hence, by (38)
\[
(44) \quad \nu(\bar{u}) = \frac{1}{\#V} \sum_{\bar{w} \in V} \hat{\nu}(\bar{u}) \zeta(\bar{u}, \bar{w}) = \frac{1}{\#V} + \frac{1}{\#V} \sum_{\bar{w} \in V \setminus \{\bar{0}\}} \hat{\nu}(\bar{u}) \zeta(\bar{u}, \bar{w}).
\]
Therefore, by the triangle inequality and Lemma 19
\[
\left| \nu(\bar{a}) - \frac{1}{\#V} \right| \leq \frac{1}{\#V} \sum_{\bar{u} \in V \setminus \{\bar{a}\}} |\hat{\nu}(\bar{u})|
\leq \frac{1}{\#V} \sum_{\bar{u} \in V \setminus \{\bar{a}\}} C_0 \exp \left( -\frac{\tau \sigma(\bar{v}_0, \ldots, \bar{v}_n)}{p^{2k}} \right)
\leq C_0 \exp \left( -\frac{\tau \sigma(\bar{v}_0, \ldots, \bar{v}_n)}{p^{2k}} \right).
\]

We finish the proof by using this bound in (43). \(\square\)

We write the following lemma to give a lower bound for the minimal weight.

Lemma 20. Let \(\bar{v}_0, \ldots, \bar{v}_n\) be vectors such that \(\bar{v}_0 \mod p, \ldots, \bar{v}_n \mod p\) have a cycle \(t < n\) then
\[
\left( \frac{n}{t} - 1 \right) \sigma(\bar{v}_0 \mod p, \ldots, \bar{v}_{t-1} \mod p) \leq \sigma(\bar{v}_0, \ldots, \bar{v}_n).
\]
Also, if \(\bar{v}_0 \mod p, \ldots, \bar{v}_n \mod p\) contains a basis for \(\mathbb{F}_p^n\), then
\[
\frac{n}{t} - 1 \leq \sigma(\bar{v}_0, \ldots, \bar{v}_n).
\]

Proof. We have that
\[
\text{Weight}_{\bar{u}}(\bar{v}_0, \ldots, \bar{v}_n) \geq \text{Weight}_{\bar{u} \mod p}(\bar{v}_0 \mod p, \ldots, \bar{v}_n \mod p),
\]
for all non-zero \(\bar{u} \in V\). So it suffices to prove the lemma for \(k = 1\) and rest will follow.

The first part is clear since,
\[
\text{Weight}_{\bar{u}}(\bar{v}_0, \ldots, \bar{v}_n) \geq \sum_{i=0}^{\lfloor n/t \rfloor - 1} \text{Weight}_{\bar{u}}(\bar{v}_{it}, \ldots, \bar{v}_{it+t-1}) > \left( \frac{n}{t} - 1 \right) \text{Weight}_{\bar{u}}(\bar{v}_0, \ldots, \bar{v}_{t-1}).
\]

For the second part, let \(i_1, \ldots, i_m\) such that \(\bar{v}_{i_1}, \ldots, \bar{v}_{i_m}\) is basis of \(\mathbb{F}_p^n\). We can assume without loss of generality that \(i_j < t\) for all \(j = 1, \ldots, m\). Hence
\[
\sigma(\bar{v}_0, \ldots, \bar{v}_{t-1}) \geq \sigma(\bar{v}_{i_1}, \ldots, \bar{v}_{i_m}),
\]
and it suffices to show that
\[
\sigma(\bar{v}_{i_1}, \ldots, \bar{v}_{i_m}) \geq 1.
\]
Assume otherwise, so there is a non-zero vector \(\bar{u}\) such that \(\langle \bar{u}, \bar{v}_{i_j} \rangle = 0\) for all \(j\). Hence, \(\bar{u}\) is a non-trivial solution of the following linear equation:
\[
\begin{pmatrix}
- \bar{v}_{i_1} \\
- \bar{v}_{i_2} \\
\vdots \\
- \bar{v}_{i_m}
\end{pmatrix}
\bar{u} \equiv \bar{0} \pmod{p}.
\]
But the matrix is invertible since its rows form a basis of \(\mathbb{F}_p^m\). So we got a contradiction and the proof is finished. \(\square\)

3. A SPACE OF POLYNOMIALS MODULO \(p^k\)

Define the following subset of \(\mathbb{Z}/p^k\mathbb{Z}[X]\)
\[
\Upsilon_k = \left\{ \sum_{i=0}^{k-1} \bar{a}_i p^i X^i \left| \forall i, \bar{a}_i \in \mathbb{Z}/p^{k-1} \mathbb{Z} \right. \right\}.
\]
We have a natural bijection \(\mathbb{Z}/p^k \mathbb{Z} \times \cdots \times \mathbb{Z}/p \mathbb{Z} \to \Upsilon_k\), so
\[
\# \Upsilon_k = \prod_{i=0}^{k-1} p^{k-1} = p^{k(k+1)/2}.
\]
Proposition 21. For any positive integer $n$, let $d = d(n)$ and $k = k(n)$ be positive integers such that $\limsup_{n \to \infty} d \log n/k^2 < \log p/8$. Assume $g \in \mathbb{Z}_p[X]_n$ is a random polynomial such that $g \mod p^k$ is distributed uniformly in $\Upsilon_k$. Then
\[
E[R^{(d)}(g)] = \beta(d) + O\left(p\left(-\frac{1}{2} + \frac{1}{4} \eta_p(\Psi)\right)\right)
\] as $n \to \infty$.

For $\bar{x} \in \mathbb{Z}/p^k\mathbb{Z}$ we say that $\bar{x}$ is simple root of $g$ modulo $p^k$ if $g(\bar{x}) \equiv 0 \pmod{p^k}$ and $g'(\bar{x}) \not\equiv 0 \pmod{p^k}$. And, we say $\bar{x} \in \mathbb{Z}/p^k\mathbb{Z}$ is non-simple root of $g$ modulo $p^k$ if $g(\bar{x}) \equiv g'(\bar{x}) \equiv 0 \pmod{p^k}$.

Lemma 22. For any $k > 0$, let $g \in \mathbb{Z}_p[X]$ be a random polynomial such that $g \mod p^k$ is distributed uniformly in $\Upsilon_k$. Then we have that
\[
\mathbb{P}(g \text{ has a non-simple root modulo } p^k) = O(p^{-k})
\] as $k \to \infty$.

Proof. Let $\mathcal{M}$ be the set of all non-simple roots of $g$ modulo $p^k$. We write $g(X) = \eta_0 + \eta_1 pX + \eta_2 p^2 X^2 + \ldots$ and then for a fixed $\bar{x} \in \mathbb{Z}/p^k\mathbb{Z}$ we have
\[
P(\bar{x} \in \mathcal{M}) = \mathbb{P}\left\{ \begin{array}{l}
g(\bar{x}) \equiv 0 \pmod{p^k} \\
g'(\bar{x}) \equiv 0 \pmod{p^k}
\end{array} \right\} = \mathbb{P}\left\{ \begin{array}{l}
\eta_0 \equiv -\eta_1 p \bar{x} + \ldots \pmod{p^k} \\
\eta_1 p \equiv -2\eta_2 p^2 \bar{x}^2 + \ldots \pmod{p^k}
\end{array} \right\} = \mathbb{P}\left\{ \begin{array}{l}
\eta_0 \equiv -\eta_1 p \bar{x} + \ldots \pmod{p^k} \\
\eta_1 \equiv -2\eta_2 p^2 \bar{x}^2 + \ldots \pmod{p^{k-1}}
\end{array} \right\} = p^{-2k+1}.
\]

The last equality holds true because $(\eta_0 \mod p^k, \eta_1 \mod p^{k-1})$ is distributed uniformly in $\mathbb{Z}/p^k\mathbb{Z} \times \mathbb{Z}/p^{k-1}\mathbb{Z}$.

We finish the proof by using union bound and plugging (46)
\[
\mathbb{P}(g \text{ has a non-simple root modulo } p^k) = \sum_{\bar{x} \in \mathbb{Z}/p^k\mathbb{Z}} \mathbb{P}(\bar{x} \in \mathcal{M}) = p^{-k+1}.
\]

Definition 23. For a polynomial $g \in \mathbb{Z}_p[X]$ we say that $\bar{x} \in \mathbb{Z}/p^k\mathbb{Z}$ is a $k$-Henselian root of $g$ if $g'(\bar{x}) \not\equiv 0 \pmod{p^k}$ and there is a lift $\bar{y}$ of $\bar{x}$ in $\mathbb{Z}/p^{2k-1}\mathbb{Z}$ such that $g(\bar{y}) \equiv 0 \pmod{p^{2k-1}}$.

We denote by $H_k^{(d)}(g)$ the number of $d$-sets of $k$-Henselian roots of $g$. By Newton-Raphson method (Theorem 7) every $k$-Henselian root can be lifted uniquely to a root of $g$ in $\mathbb{Z}_p$, hence
\[
R_k^{(d)}(g) \geq H_k^{(d)}(g).
\]

Equality holds as follows

Lemma 24. If a polynomial $g \in \mathbb{Z}_p[X]$ has only simple roots modulo $p^k$, then $R_k^{(d)}(g) = H_k^{(d)}(g)$.

Proof. It suffices to show that $R(g) = H_k^{(1)}(g)$. So, we consider the map $x \mapsto x \mod p^k$ and prove that this map is a bijection from integer roots of $g$ to $k$-Henselian roots of $g$. Indeed, if $x$ is a root of $g$ then $x \mod p^k$ is root of $g$ modulo $p^k$. Since $g$ has only simple roots modulo $p^k$, we obtain $g'(\bar{x}) \not\equiv 0 \pmod{p^k}$. Also, we have that $x \mod p^{2k-1}$ is a lift of $x \mod p^k$ and $g(x \mod p^{2k-1}) \equiv 0 \pmod{p^{2k-1}}$. Finally, Newton-Raphson method (Theorem 7) gives that the map is invertible.

Lemma 25. For any positive integer $n$, let $d = d(n)$ and $k = k(n)$ be positive integers such that $\limsup_{n \to \infty} d \log n/k^2 < \log p/2$. Then
\[
E[R^{(d)}(g)] = E\left[\left. H_k^{(d)}(g) \right| \right] + O\left(p\left(-\frac{1}{2} + \frac{1}{4} \eta_p(\Psi)\right)\right)
\]
as \( n \to \infty \).

**Proof.** We define the following events:

(a) \( S \) be the event that \( g \) has only simple roots modulo \( p^k \);
(b) \( M \) be the event that \( g \) has a non-simple root modulo \( p^k \) and \( g \not\equiv 0 \pmod{p^k} \) and
(c) \( Z \) be the event that \( g \equiv 0 \pmod{p^k} \).

Clearly, the events are disjoints and \( \mathbb{P}(S \sqcup M \sqcup Z) = 1 \). From the total law of expectation we get that

\[
\mathbb{E}[R^{(d)}(g)] = \mathbb{E}[R^{(d)}(g) \mid S] \mathbb{P}(S) + \mathbb{E}[R^{(d)}(g) \mid M] \mathbb{P}(M) + \mathbb{E}[R^{(d)}(g) \mid Z] \mathbb{P}(Z).
\]

We know that when \( S \) occurs then \( R^{(d)}(g) = H^d_k(g) \) by Lemma 24, so

\[
\mathbb{E}[R^{(d)}(g) \mid S] \mathbb{P}(S) = \mathbb{E}[H^d_k(g) \mid S] \mathbb{P}(S) \leq \mathbb{E}[H^d_k(g)].
\]

Recall that \( R^{(d)}(g) \geq H^d_k(g) \), and with (47) we obtain

\[
\mathbb{E}[R^{(d)}(g)] = \mathbb{E}[H^d_k(g)] + O(\mathbb{E}[R^{(d)}(g) \mid M] \mathbb{P}(M) + \mathbb{E}[R^{(d)}(g) \mid Z] \mathbb{P}(Z)).
\]

To bound the error term we first consider \( \mathbb{E}[R^{(d)}(g) \mid M] \mathbb{P}(M) \). If \( g \not\equiv 0 \pmod{p^k} \) then by [Shm21, Proposition 5] we get that \( R(g) < k \). Hence,

\[
\mathbb{E}[R^{(d)}(g) \mid M] = O\left( p^{H_k^+(\frac{d}{2})} \right)
\]

Using binomial coefficient’s approximation [CT06, Example 11.1.3] we get

\[
\mathbb{E}[R^{(d)}(g) \mid M] = O\left( p^{H_k^+(\frac{d}{2})} \right).
\]

Moreover, from Lemma 22 we have \( \mathbb{P}(M) = O(p^{-k}) \), so

\[
\mathbb{E}[R^{(d)}(g) \mid M] \mathbb{P}(M) = O\left( p^{H_k^+(\frac{d}{2})} p^{-k} \right) = O\left( p^{-1} p^{H_k^+(\frac{d}{2})} \right).
\]

Next we bound \( \mathbb{E}[R^{(d)}(g) \mid Z] \mathbb{P}(Z) \). Since \( R(g) \leq n \),

\[
\mathbb{E}[R^{(d)}(g) \mid Z] = \mathbb{E}\left[\left(\frac{R(g)}{d}\right) \mid Z\right] \leq \left(\frac{n}{d}\right) \leq n^d.
\]

From (45) we have that \( \mathbb{P}(Z) = p^{-k(k+1)/2} = O\left( p^{-k^2/2} \right) \) so

\[
\mathbb{E}[R^{(d)}(g) \mid Z] \mathbb{P}(Z) = O\left( n^d p^{-k^2/2} \right) = O\left( \exp\left( d \log n - \frac{1}{2} k^2 \log p \right) \right).
\]

Since \( \limsup_{n \to \infty} d \log n / k^2 < \log p / 2 \) there exists a constant \( c > 0 \) such that \( d \log n - k^2 \log p / 2 < -ck^2 \), hence

\[
\mathbb{E}[R^{(d)}(g) \mid Z] \mathbb{P}(Z) = O(\exp(-ck^2)).
\]

Finally, plugging (49) and (50) into (48) finish the proof.

**Proof of Proposition 21.** Let \( h \in \mathbb{Z}_p[X] \) be a random polynomial distributed uniformly in \( \mathbb{Z}_p[X] \). We write \( h(X) = \eta_0 + \eta_1 X + \cdots + \eta_{k-1} X^{k-1} + X^k \), and set

\[
h_0(X) = h(pX) = \eta_0 + \eta_1 pX + \cdots + \eta_{k-1} p^{k-1} X^{k-1} + p^k X^k.
\]

The random variable \( \eta_i \) is distributed according to the normalized Haar measure on \( \mathbb{Z}_p \) hence \( \eta_i \mod p^k \) is distributed uniformly in \( \mathbb{Z}/p^k\mathbb{Z} \).

Therefore, \( h_0 \mod p^k \) and \( g \mod p^k \) has the same distribution which is the uniform distribution on \( T \). Also \( h_0 \mod p^{k/2} \) and \( g \mod p^{k/2} \) are both uniform in \( T_{[k/2]} \), so Lemma 25 gives

\[
\mathbb{E}[R^{(d)}(h_0)] = \mathbb{E}[H^{(d)}_k(h_0)] + O\left( p^{-1} p^{H_k^+(\frac{d}{2})} \right),
\]

\[
\mathbb{E}[R^{(d)}(g)] = \mathbb{E}[H^{(d)}_k(g)] + O\left( p^{-1} p^{H_k^+(\frac{d}{2})} \right).
\]


Also, $\mathbb{E}\left[H_{k/2}^{(d)}(g)\right]$ depends only on the distribution of $g \mod p^k$, thus

$$\mathbb{E}\left[H_{k/2}^{(d)}(g)\right] = \mathbb{E}\left[H_{k/2}^{(d)}(h_0)\right]$$

and by subtracting (52) from (51) we get

$$\mathbb{E}\left[R^{(d)}(g)\right] = \mathbb{E}\left[R^{(d)}(h_0)\right] + O\left(p^{\left(-\frac{1}{2} + \frac{1}{2} \mathbb{H}_p(\frac{d}{4})\right)}\right).$$

Using Lemma 14 in the last equation finish the proof. \( \square \)

4. The distribution of Hasse derivatives

In this section we explore properties of the distribution of the Hasse derivatives of $f$, $D^{(j)} f$, modulo powers of $p$. We recall that the Hasse derivatives are given by the following

$$(53) \quad D^{(j)} f(r) = \frac{1}{j!} f^{(j)}(r) = \sum_{i=1}^{n} \xi_i {\binom{i}{j}} r^{i-j}.$$  

Also note that if $\xi_0, \ldots, \xi_{n-1}, r \in \mathbb{Z}_p$ then also $D^{(j)} f(r) \in \mathbb{Z}_p$, so it is possible to talk about its reduction modulo $p$.

**Proposition 26.** Let $f$ be random polynomial with coefficients satisfying Assumption 2 and let $m = m(n) < n/p^2$ be a positive integer. Assume $k_0, \ldots, k_{m-1}$ are non-negative integers and $a_r^{(j)} \in \mathbb{Z}_p$ is $p$-adic integer for all $j \in \{0, \ldots, m-1\}$ and $r \in \{1, \ldots, p-1\}$. Then

$$\mathbb{P}\left( \bigwedge_{0 \leq j < m} D^{(j)} f(r) \equiv a_r^{(j)} \pmod{p^{k_j}} \right) = p^{-N(p-1)} + O\left(\exp\left(-\frac{\tau n}{p^{2k_r+r_m}} + Kmp\log p\right)\right)$$

as $n \to \infty$ where $K = \max_{0 \leq j < m} k_j$, $N = \sum_{j=0}^{m-1} k_j$ and $\tau$ is taken from Assumption 2.

Set $V = (\mathbb{Z}/p^E \mathbb{Z})^{m(p-1)}$ and construct a random walk over the additive group $(V, +)$ as in Subsection 2.3 with the vectors $\vec{v}_0, \ldots, \vec{v}_n$ defined by

$$\vec{v}_j = \left( \binom{i}{j} r^{i-j} \right)_{0 \leq j < m, 1 \leq r < p}, \quad (54)$$

From (53) the $n$-step of the random walk is

$$\sum_{i=0}^{n} \xi_i \vec{v}_i = \left( D^{(j)} f(r) \right)_{0 \leq j < m, 1 \leq r < p} \quad (55)$$

**Lemma 27.** Let $\vec{v}_0, \ldots, \vec{v}_n$ be vectors as in (54). Then

$$\frac{n}{p^{2m}} - 1 < \sigma(\vec{v}_0, \ldots, \vec{v}_n).$$

**Proof.** Let $\ell$ be the integer such that $p^\ell - 1 < m \leq p^\ell$. First, we show that the vectors $\vec{v}_0, \ldots, \vec{v}_n$ has cycle of $(p-1)p^\ell$ modulo $p$. By Lucas’s Theorem (see [Fin47]) for any $j < m \leq p^\ell$

$$\binom{i + (p-1)p^\ell}{j} \equiv \binom{i}{j} \pmod{p}.$$  

Also, by Fermat’s Little Theorem, $r^{p-1} \equiv 1 \pmod{p}$. So

$$\vec{v}_{j+(p-1)p^\ell} \equiv \left( \binom{i + (p-1)p^\ell}{j} r^{i+(p-1)p^\ell-j} \right)_{0 \leq j < m, 1 \leq r < p} \equiv \left( \binom{i}{j} r^{i-j} \right)_{0 \leq j < m, 1 \leq r < p} \equiv \vec{v}_i \pmod{p}.$$  

Next, we show that $\vec{v}_0 \mod p, \ldots, \vec{v}_{n-1} \mod p$ contains a basis of $\mathbb{F}_p^{m(p-1)}$. We take a look on two vectors sequences $\vec{u}_0, \ldots, \vec{u}_{n-1} \in \mathbb{F}_p^m$ and $\vec{w}_1, \ldots, \vec{w}_{p-1} \in \mathbb{F}_p^{p-1}$ defined by

$$\vec{u}_i = \left( \binom{i}{0}, \ldots, \binom{i}{m-1} \right), \quad \text{and} \quad \vec{w}_i = \left( 1, 2^{i-1}, \ldots, (p-1)^{i-1} \right).$$
The sequence \( \tilde{u}_0, \ldots, \tilde{u}_{m-1} \) is a basis of \( \mathbb{F}_p^m \) since its vectors form a triangular matrix

\[
\begin{pmatrix}
- & \tilde{u}_0 \\
- & \tilde{u}_1 \\
\vdots \\
- & \tilde{u}_{m-1}
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
1 & 1 & 0 & \cdots & 0 \\
1 & 2 & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \ddots & \cdots \\
1 & m-1 & (m-1) & \cdots & 1
\end{pmatrix}.
\]

The sequence \( \tilde{w}_1, \ldots, \tilde{w}_{p-1} \) is a basis of \( \mathbb{F}_p^{p-1} \) since its vectors are columns of Vandermonde matrix.

Since both sequences are basis of their vector spaces then the set \( \{ \tilde{u}_t \otimes \tilde{w}_s : 0 \leq t < m, 1 \leq s < p \} \) is basis of the tensor product \( \mathbb{F}_p^m \otimes \mathbb{F}_p^{p-1} \cong \mathbb{F}_p^{m(p-1)} \).

Define \( T : \mathbb{F}_p^{m(p-1)} \rightarrow \mathbb{F}_p^{m(p-1)} \) to be the linear map which multiplies the element in the \((j, r)\) coordinates by \( r^{-j} \) i.e.

\[
T \left( \begin{pmatrix}
x_{r}^{(j)} \\
i \leq j < m \\
1 \leq r < p
\end{pmatrix} \right) = \begin{pmatrix}
r^{-j}x_{r}^{(j)} \\
i \leq j < m \\
1 \leq r < p
\end{pmatrix}.
\]

The map \( T \) is an invertible, so the set \( B = \{ T(\tilde{u}_t \otimes \tilde{w}_s) : 0 \leq t < m, 1 \leq s < p \} \) is also a basis of \( \mathbb{F}_p^{m(p-1)} \).

Moreover, the basis \( B \) is contained in \( \tilde{v}_0 \mod p, \ldots, \tilde{v}_{n-1} \mod p \). Indeed, for every \( 0 \leq t < m \) and \( 1 \leq s < p \) we set \( i_s^{(t)} \) to be the positive integer such that \( i_s^{(t)} \equiv t \pmod{p^k} \) and \( i_s^{(t)} \equiv s - 1 \pmod{p - 1} \). By the Chinese Remainder Theorem there exists such \( i_s^{(t)} \) which also satisfy \( i_s^{(t)} < p^k(p - 1) < mp^2 \leq n \). So for any \( T(\tilde{u}_t \otimes \tilde{w}_s) \in B \)

\[
T(\tilde{u}_t \otimes \tilde{w}_s) = T \left( \begin{pmatrix}
x_{r}^{(j)} \\
i \leq j < m \\
1 \leq r < p
\end{pmatrix} \right) = \begin{pmatrix}
r^{-j}x_{r}^{(j)} \\
i \leq j < m \\
1 \leq r < p
\end{pmatrix} = \tilde{v}_s^{i_s^{(t)}}.
\]

We proved the requirements of Lemma 20, so

\[
\sigma(\tilde{v}_0, \ldots, \tilde{v}_{n-1}) \geq \frac{n}{(p-1)p^2} - 1 > \frac{n}{p^2m} - 1.
\]

**Proof of Proposition 26.** Define

\[
S = \left\{ \begin{pmatrix} x_{r}^{(j)} \end{pmatrix}_{0 \leq j < m} \in V \mid \forall j, r, \ x_{r}^{(j)} \equiv a_r^{(j)} \pmod{p^k} \right\} \subseteq V,
\]

so by (55) we get

\[
P \left( \bigwedge_{0 \leq j < m} \bigwedge_{1 \leq r < p} D^{(i)} f(r) \equiv a_r^{(j)} \pmod{p^k} \right) = P \left( \sum_{i=0}^{n} \xi_i \tilde{v}_i \in S \right).
\]
Proposition 17 gives
\[
P\left( \sum_{i=0}^{n} \xi_i \bar{v}_i \in S \right) = \frac{\#S}{\#V} + O\left( \frac{\tau \sigma(\bar{v}_0, \ldots, \bar{v}_n)}{p^2K} \right).
\]
Since \(\#S = \prod_{i=0}^{n-1} p^{(K-k)(p-1)} = p^{Km(p-1)}p^{-N(p-1)} \leq p^{Kmp}, \#V = p^{Km(p-1)},\) and by Lemma 27
\[
\mathbb{P}\left( \sum_{i=0}^{n} \xi_i \bar{v}_i \in S \right) = p^{-N(p-1)} + O\left( \exp\left( -\frac{\tau n}{p^{2K+2m}} + Kmp \log p \right) \right).
\]

5. Proofs of the main theorems

For a polynomial \(f \in \mathbb{Z}_p[X]\) and \(r \in \{0, \ldots, p-1\},\) we define \(f_r(X)\) to be the polynomial \(f(r + pX).\) Note that by Taylor expansion we have that
\[
f_r(X) = D^{(0)} f(r) + D^{(1)} f(r) pX + D^{(2)} f(r) p^2 X^2 + \ldots.
\]
Each integer root of \(f_r\) correspond to an integer root of \(f\) which congruent to \(r\) modulo \(p.\) Hence, we divide the roots of \(f\) into \(p\) sets by their values modulo \(p\) and get
\[
R^{(d)}(f) = \sum_{d_0+\ldots+d_{p-1}=d} \prod_{r=0}^{p-1} R^{(d_r)}(f_r).
\]
Denote the sequence of \(p-1\) polynomials \((f_1, f_2, \ldots, f_{p-1})\) with \(\hat{f}.\) We stress that \(\hat{f}_0\) does not appear in this sequence.

Lemma 28. Let \(f\) be a random polynomial with coefficients satisfying Assumption 2 and let \(0 < \varepsilon < 1.\) Then there exists \(C_0 > 0\) such that for any positive integer \(k \leq \frac{(1-\varepsilon) \log n}{2 \log p}\) and a sequence of polynomials \(\bar{h} = (h_1, \ldots, h_{p-1}) \in \mathcal{T}_k^{p-1},\) we have
\[
\mathbb{P}\left( \hat{f} \equiv \bar{h} \pmod{p^k} \right) = \left( \frac{1}{\# \mathcal{T}_k} \right)^{p-1} + O(\exp(-C_0 n^\varepsilon))
\]
as \(n \to \infty.\) Moreover, \(C_0\) is dependent only on \(\varepsilon\) and \(\tau\) from Assumption 2.

Proof. As each \(h_r \in \mathcal{T}_k,\) it is of the form \(h_r(X) = \bar{a}_r^{(0)} + \bar{a}_r^{(1)} pX + \ldots + \bar{a}_r^{(k-1)} p^{k-1} X^{k-1}.\) So by (56) we have
\[
f_r \equiv h_r \pmod{p^k} \iff D^{(j)} f(r) \equiv \bar{a}_r^{(j)} \pmod{p^{k-j}}, \quad j = 0, \ldots, k-1.
\]
We apply Proposition 26 with \(m = k, k_i = k - i\) and \(\bar{a}_r^{(i)}\) be some lift of \(\bar{a}_r^{(i)},\) so \(K = k, N = k(k+1)/2\) and
\[
\mathbb{P}\left( \hat{f} \equiv \bar{h} \pmod{p^k} \right) = P\left( \bigwedge_{r=1}^{p-1} f_r \equiv h_r \pmod{p^k} \right)
\]
\[
= P\left( \bigwedge_{0 \leq j < k} D^{(j)} f(r) \equiv \bar{a}_r^{(j)} \pmod{p^{k-j}} \right)
\]
\[
= p^{-k(k+1)(p-1)/2} + O(\exp\left( -\frac{\tau n}{p^{2k+2m}} + k^2 p \log p \right))
\]
where \(\tau\) is taken from Assumption 2. This finish the proof. Indeed, the main term is \((1/\# \mathcal{T}_k)^{p-1}\) by (45) and by the assumption on \(k,\) the error term is \(O(\exp(-C_0 n^\varepsilon))\) as needed. \(\square\)
Lemma 29. Let \( f \) be a random polynomial with coefficients satisfying Assumption 2. Then for any \( d = d(n) \) such that \( \lim \sup_{n \to \infty} d! \log n < (16 \log p)^{-1} \) and for any \( \varepsilon > 0 \) there exists a constant \( C > 0 \), as defined in (4), satisfying
\[
E \left[ \sum_{d_1 + \cdots + d_{p-1} = d} \prod_{r=1}^{p-1} R^{(d_r)}(f_r) \right] = \gamma(d) + O(n^{-C + \varepsilon})
\]
as \( n \to \infty \).

Proof. Set
\[
(58) \quad \tilde{R}^{(d)}(f) = \sum_{d_1 + \cdots + d_{p-1} = d} \prod_{r=1}^{p-1} R^{(d_r)}(f_r).
\]

Let \( k = \lfloor \frac{(1-\varepsilon_1) \log n}{2 \log p} \rfloor \) where \( \varepsilon_1 \) is a positive real to be defined later. So we apply the law of total expectation and Lemma 28 to get \( C_0 > 0 \) such that
\[
(59) \quad E \left[ \tilde{R}^{(d)}(f) \right] = \sum_{\tilde{h} \in \tilde{Y}_k^{p-1}} E \left[ \tilde{R}^{(d)}(f) \right| \tilde{f} \equiv \tilde{h} \pmod{p^k} \] \[
\times \Pr \left( \tilde{f} \equiv \tilde{h} \pmod{p^k} \right) \right] \left( \frac{1}{\# \tilde{Y}_k} \right)^{p-1} + O(\exp(-C_0 n^{\varepsilon_1})).
\]

Since \( \tilde{R}^{(d)}(f) \leq R^{(d)}(f) \leq n^d \) (see (57) and (58)) and \( \# \tilde{Y}_k = p^{k(k+1)/2} \) (see (45)) then we can bound the error term in (59)
\[
\sum_{\tilde{h} \in \tilde{Y}_k^{p-1}} E \left[ \tilde{R}^{(d)}(f) \right| \tilde{f} \equiv \tilde{h} \pmod{p^k} \] \[
\times \exp(-C_0 n^{\varepsilon_1}) = O \left( n^d p^{k(k+1)(p-1)/2} \exp(-C_0 n^{\varepsilon_1}) \right) = O(\exp(d \log n + k^2 \log p - C_0 n^{\varepsilon_1})) = O(\exp(C_1 \log^2 n - C_0 n^{\varepsilon_1})) = O(\exp(-C_2 n^{\varepsilon_1}))
\]
for some constants \( C_1, C_2 > 0 \). So (59) becomes
\[
(60) \quad E \left[ \tilde{R}^{(d)}(f) \right] = \left( \frac{1}{\# \tilde{Y}_k} \right)^{p-1} \sum_{\tilde{h} \in \tilde{Y}_k^{p-1}} E \left[ \tilde{R}^{(d)}(f) \right| \tilde{f} \equiv \tilde{h} \pmod{p^k} \] \[
+ O(\exp(-C_2 n^{\varepsilon_1})).
\]

To evaluate the main term, we use linearity of expectation and change the order of summation to get
\[
(61) \quad \left( \frac{1}{\# \tilde{Y}_k} \right)^{p-1} \sum_{\tilde{h} \in \tilde{Y}_k^{p-1}} E \left[ \tilde{R}^{(d)}(f) \right| \tilde{f} \equiv \tilde{h} \pmod{p^k} \] \[
= \left( \frac{1}{\# \tilde{Y}_k} \right)^{p-1} \sum_{\tilde{h} \in \tilde{Y}_k^{p-1}} \sum_{d_1 + \cdots + d_{p-1} = d} E \left[ \prod_{r=1}^{p-1} R^{(d_r)}(f_r) \right| \tilde{f} \equiv \tilde{h} \pmod{p^k} \] \[
= \sum_{d_1 + \cdots + d_{p-1} = d} \left( \frac{1}{\# \tilde{Y}_k} \right)^{p-1} \sum_{\tilde{h} \in \tilde{Y}_k^{p-1}} E \left[ \prod_{r=1}^{p-1} R^{(d_r)}(f_r) \right| \tilde{f} \equiv \tilde{h} \pmod{p^k} \]
Next we evaluate the summands of the outer sum. Let \( \vec{g} = (g_1, \ldots, g_{p-1}) \) be a sequence of random polynomials in \( \mathbb{Z}_p[X]_n \) distributed according to the law

\[
\mathbb{P}(\vec{g} \in E) = \left( \frac{1}{\#Y_k} \right)^{p-1} \sum_{\vec{h} \in Y_k^{p-1}} \mathbb{P}\left( \vec{f} \in E \mid \vec{f} \equiv \vec{h} \pmod{p^k} \right), \quad E \subseteq (\mathbb{Z}_p[X]_n)^{p-1}.
\]

This distribution is well-defined for \( n \) sufficiently large, since \( \mathbb{P}\left( \vec{f} \equiv \vec{h} \pmod{p^k} \right) \) is bounded away from zero by Lemma 28.

On the one hand, we have

\[
\mathbb{E} \left[ \prod_{r=1}^{p-1} R^{(d_r)}(g_r) \right] = \prod_{r=1}^{p-1} \mathbb{E} \left[ R^{(d_r)}(g_r) \right] = \prod_{r=1}^{p-1} \left( \beta(d_r) + O\left(p^{-\frac{1}{2} + \frac{1}{2} \log_p(2d^2)}\right) \right).
\]

By Lemma 15 we have that \( \beta(d_r) = O(1) \) so expanding the left most side of (63) gives

\[
\mathbb{E} \left[ \prod_{r=1}^{p-1} R^{(d_r)}(g_r) \right] = \prod_{r=1}^{p-1} \beta(d_r) + O\left(p^{-\frac{1}{2} + \frac{1}{2} \log_p(2d^2)}\right).
\]

We get expression for the summands of the outer sum of (61) by combining (62) and (64):

\[
\left( \frac{1}{\#Y_k} \right)^{p-1} \sum_{\vec{h} \in Y_k^{p-1}} \mathbb{E} \left[ \prod_{r=1}^{p-1} R^{(d_r)}(f_r) \mid \vec{f} \equiv \vec{h} \pmod{p^k} \right] = \prod_{r=1}^{p-1} \beta(d_r) + O\left(p^{-\frac{1}{2} + \frac{1}{2} \log_p(2d^2)}\right).
\]

We return to (61) and set the summands of the outer sum:

\[
\sum_{d_1, \ldots, d_{p-1} = d} \left( \prod_{r=1}^{p-1} \beta(d_r) + O\left(p^{-\frac{1}{2} + \frac{1}{2} \log_p(2d^2)}\right) \right) = \gamma(d) + O\left(p^{-\frac{1}{2} + \frac{1}{2} \log_p(2d^2)}\right).
\]
Consider the error term, so for $n$ sufficiently large we have
\[
\binom{p + d - 2}{p - 1} p^{-\frac{1}{4} + \frac{1}{2} H_p(\frac{2d}{k})} = O\left(d^p p^{-\frac{1}{4} + \frac{1}{2} H_p(\frac{2d}{k})}\right)
\]
\[
= O\left(\exp\left(p \log d + \left(-\frac{1}{2} + \frac{1}{2} H_p\left(\frac{2d}{k}\right)\right) k \log p\right)\right)
\]
\[
= O\left(\exp\left(p \log(L \log n) + (1 - \varepsilon_1)\left(-\frac{1}{4} + \frac{1}{4} H_p\left(\frac{2d}{k}\right)\right) \log n\right)\right)
\]
\[
= O\left(\exp\left((1 - \varepsilon_1)\left(-\frac{1}{4} + \frac{1}{4} H_p\left(\frac{2d}{k}\right) + \varepsilon_1\right) \log n\right)\right).
\]
Note that since $L < (16 \log p)^{-1}$ we have that $\limsup_{n \to \infty} 2d/k < 4L \log p < 1/4$. Moreover, since $H_p$ is increasing in $(0, 1/2)$, for $n$ sufficiently large
\[
\left(-\frac{1}{4} + \frac{1}{4} H_p\left(\frac{2d}{k}\right)\right) \leq \left(-\frac{1}{4} - \frac{1}{4} H_p(4L \log p) + \varepsilon_1 = -C + \varepsilon_1,
\]
where $C$ is defined as in (4). We return to (66) and choose $\varepsilon_1 < \min(\varepsilon/(2 + C), 1 - \sqrt[4]{4L \log p})$ to get that
\[
\left(\binom{p + d - 2}{p - 1} p^{-\frac{1}{4} + \frac{1}{2} H_p(\frac{1}{k})}\right) = O(\exp((1 - \varepsilon_1)(-C + 2\varepsilon_1)) \log n) = O(n^{-C + \varepsilon}).
\]
And (65) becomes
\[
\left(\frac{1}{\#\mathcal{Y}_k}\right)^{p-1} \sum_{\tilde{h} \in \mathcal{Y}_k^{-1}} E[R^{(d)}(f) \mid h \equiv \tilde{h} \quad \text{(mod } p^k\text{)}] = \gamma(d) + O(n^{-C + \varepsilon}).
\]
Plugging the last equation into (60) finish the proof. \qed

**Proof of Theorem 3.** When conditioning on $p \nmid \xi_0$ no root is divided by $p$ and then $R(f_0) = 0$. Hence $R^{(d_0)}(f_0) = 1$ and $R^{(d_0)}(f_0) = 0$ for $d_0 \geq 1$. Plugging that in (57) and adding expectation gives
\[
E[R^{(d_0)}(f)] = E\left[\sum_{d_1 + \ldots + d_{p-1} = d} \prod_{r=1}^{p-1} R^{(d_r)}(f_r) \mid p \nmid \xi_0\right].
\]
Note that Assumption 2 still holds after conditioning on $p \mid \xi_0$, so we can Lemma 29 to finish the proof. \qed

**Proof of Theorem 4.** By Lemma 9 we have that almost surely
\[
R^{(d_0)}(f_0) = \begin{cases} 1, & d_0 = 0, \\ \mathbb{I}_{\xi_0 = 0}, & d_0 = 1, \\ 0, & d_0 > 1. \end{cases}
\]
Using (57) gives
\[
E[R^{(d)}(f)] = E\left[\sum_{d_1 + \ldots + d_{p-1} = d} \prod_{r=1}^{p-1} R^{(d_r)}(f_r)\right] + E\left[\mathbb{I}_{\xi_0 = 0} \sum_{d_1 + \ldots + d_{p-1} = d-1} \prod_{r=1}^{p-1} R^{(d_r)}(f_r)\right].
\]
By total law of expectation
\[
E[R^{(d)}(f)] = E\left[\sum_{d_1 + \ldots + d_{p-1} = d} \prod_{r=1}^{p-1} R^{(d_r)}(f_r)\right]
\]
\[
+ E\left[\sum_{d_1 + \ldots + d_{p-1} = d-1} \prod_{r=1}^{p-1} R^{(d_r)}(f_r) \mid \xi_0 = 0\right] \mathbb{P}(\xi_0 = 0).
\]
And the proof is finished by using Lemma 29 on each expectation on the right side of (68). Note that this can be done since Assumption 2 still holds after conditioning on $\xi_0$. \qed

6. LARGE NUMBER OF ROOTS

In this section, we consider the probability that $f$ has a large number of roots relatively to $n$.

Proof of Corollary 6. Let $r = r(n)$ be positive integer and set $d = \left\lfloor \frac{\log n}{17 \log p} \right\rfloor$. We define the following function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\phi(x) = \begin{cases} 0, & x \leq d, \\ \frac{x}{d!} x(x-1) \cdots (x-d+1), & x > d. \end{cases}$$

(69)

Note that $\phi$ is non-decreasing function hence Markov inequality gives

$$P(R(f) \geq r \mid p \nmid \xi_0) \leq \frac{E[\phi(R(f)) \mid p \nmid \xi_0]}{\phi(r)},$$

(70)

when $r > d$.

Assume that

$$\limsup_{n \to \infty} \frac{d}{r} < 1,$$

(71)

then for $n$ sufficiently large we have that $r > d$ and (70) holds. For any non-negative integer $k$, we have that $\phi(k) = \binom{k}{d}$. In particular, we have that $\phi(R(f)) = R(d)(f)$. Therefore, (70) becomes

$$P(R(f) \geq r \mid p \nmid \xi_0) \leq \frac{E[R(d)(f) \mid p \nmid \xi_0]}{\phi(r)},$$

From Theorem 3, we have that $E[R(d)(f) \mid p \nmid \xi_0] = \gamma(d) + O(n^{-C_0})$ for some $C_0 > 0$. Also from Lemma 16 we have that

$$\log_p \gamma(d) \leq -C_1 d^2 \leq -C_2 \log^2 n.$$

Therefore,

$$E[R(d)(f) \mid p \nmid \xi_0] = O(n^{-C_0}).$$

(72)

Next we bound the term $1/\phi(r)$ from (70). We have that for $r > d$

$$\phi(r) = \frac{r \cdot (r-1) \cdots (r-d+1)}{d!} \geq \left( \frac{r}{d} \right)^d.$$

Hence,

$$\frac{1}{\phi(r)} = O\left( \left( \frac{d}{r} \right)^d \right).$$

(73)

We plug (72) and (73) into (70) to get

$$P(R(f) \geq r \mid p \nmid \xi_0) = O\left( \left( \frac{d}{r} \right)^d \cdot n^{-C_0} \right)$$

$$= O\left( \exp\left( d \log \frac{d}{r} - C_0 \log n \right) \right)$$

$$= O\left( \exp\left( \frac{\log n}{17 \log p} \log \frac{d}{r} - C_0 \log n \right) \right).$$

From (71), for $n$ sufficiently large $\log(r/d) < 0$, hence

$$P(R(f) \geq r \mid p \nmid \xi_0) = O\left( \exp\left( C_3 \log n \log \frac{d}{r} \right) \right)$$

(74)

for some $C_3 > 0$. 


We consider the case where \( r = \log n \). So since \( p \geq 2 \) we have
\[
\limsup_{n \to \infty} \frac{d}{r} = \frac{1}{17 \log p} < \frac{1}{10} < 1.
\]
Hence (71) holds, and from (74) we get
\[
P(R(f) \geq \log n \mid p \nmid \xi_0) = O(\exp(-C_3 \log 10 \log n)).
\]
Choosing \( K = C_3 \log 10 \) finish the proof of Corollary 6.(a).

Next, consider the case where \( r = n^\lambda \) for some \( 0 < \lambda \leq 1 \). Since \( \limsup_{n \to \infty} \frac{d}{r} = 0 \), the assumption (71) holds. Thus (74) also holds. Moreover, because
\[
\log \frac{d}{r} \leq \log \frac{\log n}{17 \log p} - \lambda \log n,
\]
there exists \( K > 0 \) such that
\[
P(R(f) \geq n^\lambda \mid p \nmid \xi_0) = O(\exp(-K \log^2 n)).
\]
And the last equation finish the proof of Corollary 6.(b).

\[\square\]

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