Cooling in a parametrically driven optomechanical cavity

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We obtain a master equation for a parametrically driven optomechanical cavity. We use a more correct dissipation model that accounts for the modification of the quasi-energy spectrum caused by the driving. When the natural frequency of the mechanical object oscillates periodically around its mean value, the master equation with the improved dissipation model is expressed using Floquet operators. We apply the corresponding master equation to model the laser cooling of the mechanical object. Using an adiabatic approximation, an analytical expression for the number of excitations of the mechanical oscillator can be obtained. We find that the number of excitations can be lower than in the non-time dependent case. Our results raise the possibility of achieving lower temperatures for the mechanical object if its natural frequency can be controlled as a function of time.

I. INTRODUCTION

Quantum cavity optomechanics studies systems composed of macroscopic mechanical objects, such as mirrors, and an optical cavity’s quantized light field, coupled via radiation pressure. In a common scheme one of the end-mirrors of a Fabry-Perot cavity is suspended while able to freely oscillate. When photons are reflected by the mirror, there is a momentum transfer between the light field and the mirror; as the cavity’s resonance depends on its length, the mechanical displacement in turn affects the light field inside the cavity. Some of the first theoretical work predicting this sort of coupling, between light and mechanical object, is described in [1]. This interaction between the macroscopic mechanical object and the light field leads to several interesting effects such as optomechanically induced transparency [2], the optical spring effect [3] or, most relevant to this study, optomechanical cooling [4–7], which was first proposed by Mancini, et al [8].

Optomechanical cooling consists of the damping of the end mirror’s mechanical motion due to the radiative coupling to the cavity field. Sideband cooling takes place when the cavity’s resonance is much narrower than the mechanical frequency. It can be understood as Raman scattering of incident photons [9] which are red-detuned from the cavity resonance. When the parameters are chosen appropriately, incident photons absorb a phonon from the mechanical oscillator in order to scatter into the cavity’s resonance mode, resulting in cooling of the resonator. For coherent quantum control over a mechanical object, it must be close to a pure quantum mechanical state [10], so effective methods of cooling macroscopic objects to low temperatures are highly desirable.

One possible avenue for manipulating the mechanical object and improving cooling lies in controlling the mechanical resonator’s frequency as a function of time [11]. There have been other studies that include the modulation of optomechanical parameters. Some of these include modulating the spring constant and the interaction strength to achieve a splitting of the cavity sidebands [12] reach a non-linear quantum regime [13], or achieve controllable quantum squeezing [14]. Other studies cover the periodic Langevin equations that arise in a bi-chromatically driven optical cavity [15] and modulating the amplitude of the driving field [16] in order to achieve squeezing of the mechanical resonator. The effect of a modulated spring constant on the mechanical object’s final temperature was studied in [17]. In that study it was found that the final temperature of the parametrically driven harmonic oscillator was larger than the non-driven case. However, the master equation from which the cooling rates were derived accounted for the natural frequency of the mechanical resonator via ad-hoc time-dependent coefficients that were introduced after performing the Markov approximation. In this paper we extend the study of the cooling dynamics to the case when the time-dependence of the system is taken into account in the derivation of the master equation.

The formalism we apply (section [11] is based on Floquet theory and was demonstrated to be a more accurate treatment [18]. For the case where the drive consists of a small periodic oscillation with respect to the central frequency of the mechanical oscillator, the Floquet operators can be given explicitly (section [11]). Under the adiabatic approximation, we derive an approximated expression for the mean mechanical excitation number in the final stages of optomechanical cooling and compare this prediction to the time-independent case (section [11]). From the treatment presented here, it follows that lower temperatures can be obtained if the mechanical object is parametrically driven. Our result suggests that the details of the theoretical dissipation model for the mechanical oscillator can have a significant influence on the resulting temperature (section [11]).
II. OPTOMECHANICAL HAMILTONIAN

A. Hamiltonian with Floquet Operators

The Hamiltonian for a parametrically driven optomechanical system \[\Sigma\], in a reference system that rotates with the same frequency as a laser that continuously pumps photons into the cavity is

\[ H(t) = H_{\text{cav}} + H_{\text{mec}}(t) + H_{\text{int}} + H_{\text{pump}}, \] (1)

where

\[ H_{\text{cav}} = -\hbar \delta a^\dagger a, \] (2)
\[ H_{\text{mec}}(t) = \frac{p^2}{2M} + \frac{1}{2}M\nu^2(t)x^2, \] (3)
\[ H_{\text{int}} = -\hbar g_c a^\dagger ax, \] (4)
\[ H_{\text{pump}} = \hbar \frac{\Omega}{2} (a^\dagger + a), \] (5)

where \( \delta = \omega_{\text{laser}} - \omega_{\text{cav}} \) is the detuning between laser and cavity. The mechanical oscillator’s mass is denoted by \( M \), \( p \) and \( x \) are its momentum and position operator, and \( \nu(t) \) the modulated mechanical frequency. The term \( H_{\text{int}} \) models the interaction between the field and the cavity mirror where \( g_c \) sets the strength of the coupling \[\Omega\]. Finally, \( H_{\text{pump}} \) describes the pumping of the cavity by a field with strength proportional to \( \Omega \). The mechanical oscillator Hamiltonian has an explicit time dependence given by \( \nu(t) \). We assume here a periodic function of time, which allows us to employ the Floquet formalism.

The Floquet operators are analogous to the usual creation and annihilation operators for the standard harmonic oscillator and can be expressed in terms of the mechanical oscillator’s position and momentum operators \[\hat{p}, \hat{x}\]. These operators are

\[ \Gamma(t) = \frac{1}{2i} \left[ \hat{x} \sqrt{\frac{2M}{\hbar}} \dot{f}(t) - \hat{p} \sqrt{\frac{2}{M\hbar}} f(t) \right], \] (6)

as well as its Hermitian conjugate. \( f(t) \) is the solution to the classical time-dependent harmonic oscillator equation of motion in one dimension

\[ \ddot{f} + \nu(t)^2 f = 0, \] (7)

and is generally a complex function \[\phi(t)\]. This equation has two solutions \[\phi(t) = e^{i\alpha t}\phi(t)\], (8)
and its complex conjugate, where \( \phi(t) \) is a periodic function of time with the same period as \( \nu(t) \). \( \eta \) is, in general, a complex number \[\eta\]. The Floquet operators follow the usual commutation relations for creation and annihilation operators

\[ [\Gamma^\dagger(t), \Gamma(t)] = 1. \] (9)

Using these operators, \( H_{\text{mec}}(t) \) (see Eq. (3)), can be written in the same form as the non time-dependent harmonic oscillator with the Floquet operators playing the role of the creation and annihilation operators, with the exception of a global time-dependent scalar coefficient \[\Omega\]

\[ H_{\text{mec}}(t) = \hbar \frac{W}{|f(t)|^2} \left[ \Gamma^\dagger(t) \Gamma(t) + \frac{1}{2} \right] \] (10)

where \( W \) is the Wronskian for the differential equation (7). Using Eqs. (9), (10) we can define the number of excitations of a parametrically driven oscillator in a manner analogous to the quantum harmonic oscillator: as the expectation value of the number operator \((n) = \langle \Gamma^\dagger \Gamma \rangle\).

The explicit time dependence of the Floquet operators will not be noted from now on for the sake of brevity. Equation (6) can be inverted and solved for the harmonic oscillator’s position operator

\[ \hat{x} = \gamma^*_\nu(t) \Gamma + \gamma^\nu(t) \Gamma^\dagger, \] (11)

this expression can be substituted into the interaction Hamiltonian, getting

\[ H_{\text{int}}(t) = g_c \sqrt{\frac{\hbar}{2M}} a^\dagger a [\gamma^*_\nu(t) \Gamma + \gamma^\nu(t) \Gamma^\dagger] \] (12)

The explicit expressions for \( \gamma^\nu(t) \) are obtained when explicit expressions for the solutions \( f(t) \) are available. The Hamiltonian (1) contains two separate harmonic oscillator-like terms, \( H_{\text{cav}} \) and \( H_{\text{mec}}, \) that commute. This allow us to derive a master equation following the same procedure depicted in [18] for the mechanical oscillator, and the standard procedure for the cavity. This derivation involves the Markov approximation; in previous attempts to study a parametrically driven oscillator, the time dependence of the frequency was included after the Markov approximation had been performed, via time-dependent ad-hoc coefficients for the damping [17]. Under the formalism developed in [18] the frequency’s time dependence is accounted for when the Markov approximation is performed, via the Floquet operators. As demonstrated in [18], the method employed here is a more complete, and thus accurate, treatment.

The optomechanical master equation with improved dissipation model is

\[ \dot{\rho} = \frac{1}{i\hbar} [H, \rho] + L_a \rho + L_T \rho, \] (13)

where

\[ L_a \rho = -\frac{\kappa}{2} (n_\rho + 1) [a^\dagger a \rho + \rho a^\dagger a - 2a^\dagger a \rho] \] (14)
\[ - \kappa (n_\rho) [aa^\dagger \rho + \rho a a^\dagger - 2a^\dagger \rho a], \]
\[ L_{\Gamma} \rho = -\frac{\gamma}{2}(n_m + 1)[\Gamma^\dagger \Gamma \rho + \rho \Gamma^\dagger \Gamma - 2 \Gamma \rho \Gamma^\dagger] \]  
\[ -\frac{\gamma}{2}(n_m)[\Gamma^\dagger \rho + \rho \Gamma^\dagger - 2 \Gamma^\dagger \rho \Gamma], \]

\( \kappa \) is the energy decay rate for the cavity, \( \gamma \) is the decay rate for the mechanical oscillator, \( n_p \) is the number of thermal excitations of the bath at the frequency resonant with the cavity \( \omega_{\text{cav}} \), and \( n_m \) is known as the effective thermal-bath occupation number \[18\]. In the undriven case, \( n_m \) reduces to the number of thermal excitation of the bath at the natural frequency of the mechanical oscillator \( \nu \). In the absence of interaction between the cavity and the mechanical oscillator, the stationary state for the cavity is a thermal state with mean photon number \( n_p \), and for the oscillator it is a thermal state with mean number \( n_m \) of mechanical excitations.

The superoperators \( L_{\Gamma} \) and \( L_{\alpha} \) model the energy exchanges between the environment, and the cavity and the mechanical resonator respectively. Note that the time dependence of the Floquet operators implies that the dissipation of the mechanical oscillator, given by \[15\], is time-modulated. Equation \[13\] is the master equation for a parametrically driven optomechanical system with time-modulated operators. The primes in the operators will be omitted in order to calculate explicit expressions for several of the coefficients and to deal with the \( \dot{\Gamma} \) and \( \dot{\Gamma}^\dagger \) fulfill the differential equations

\[ \dot{\alpha} = \alpha \left(-\frac{\kappa}{2} + i(\delta + g_c \sqrt{\frac{\hbar}{2M}}(\gamma^-(t) \beta^* + \gamma^+(t) \beta)) - \frac{i}{2} \Omega, \right. \]  
\[ \dot{\beta} = \beta \left(-\frac{\gamma}{2} - i \frac{W}{|f(t)|^2} \right) + i g_c \sqrt{\frac{\hbar}{2M}} |\alpha|^2 \gamma^-(t). \]  

Proceeding further requires an explicit solution for equation \[7\] in order to calculate explicit expressions for several of the coefficients and to deal with the \( \Gamma \) and \( \Gamma^\dagger \) operators. The primes in the operators will be omitted from now on as all calculations will be carried out in the displaced frame.

### B. Displaced Frame

In order to eliminate the pump term and find useful approximations, we employ a unitary transformation to shift equation \[13\] into a displaced reference frame. This transformation depends on two time-dependent coefficients, \( \alpha(t) \) and \( \beta(t) \), which are chosen in a convenient manner to simplify the Hamiltonian. The transformation is given by the operator

\[ U_{\alpha,\Gamma} = e^{(\alpha(t) a^\dagger - \alpha^*(t) a)} e^{(\beta(t) \Gamma^\dagger - \beta^*(t) \Gamma)}, \]  

and results in a displaced master equation for the time evolution of the density operator \( \rho'(t) = U \rho U^\dagger \)

\[ \dot{\rho}' = \frac{1}{i\hbar}[H', \rho'] + L_{\alpha} \rho' + L_{\Gamma} \rho' + C(t) \rho', \]  

where

\[ C(t) = -((\beta^2 - |\beta|^2) [\Gamma^\dagger, \Gamma^\dagger]) - ((|\beta|^2)^2 - |\beta|^2) [\Gamma, \Gamma]. \]  

This term arises due to the explicit time dependence in the Floquet operators as they do not, in general, commute with their own time derivatives. The primes indicate that the transformation has been applied. The displaced Hamiltonian, which includes a pump-like term that appears due to the transformation being applied to the time derivative term, is

\[ H' = U H U^\dagger \]

\[ = -\hbar \delta' a^\dagger a + \hbar \frac{W}{|f(t)|^2} \Gamma^\dagger \Gamma \]

\[ -\hbar g_c \sqrt{\frac{\hbar}{2M}} [\alpha^\dagger a + \alpha a^\dagger + \alpha^* a] (\gamma^-(t) \Gamma^\dagger + \gamma^+(t) \Gamma) + i\hbar(\beta^* \Gamma - \beta \Gamma^\dagger), \]  

with \( \delta' = \delta + g_c \sqrt{\frac{\hbar}{2M}} (\gamma^-(t) \beta + \gamma^+(t) \beta^*) \). This Hamiltonian is valid as long as the coefficients \( \alpha(t) \) and \( \beta(t) \) fulfill the differential equations

\[ \dot{\alpha} = \alpha \left(-\frac{\kappa}{2} + i(\delta + g_c \sqrt{\frac{\hbar}{2M}}(\gamma^-(t) \beta^* + \gamma^+(t) \beta)) - \frac{i}{2} \Omega, \right. \]  
\[ \dot{\beta} = \beta \left(-\frac{\gamma}{2} - i \frac{W}{|f(t)|^2} \right) + i g_c \sqrt{\frac{\hbar}{2M}} |\alpha|^2 \gamma^-(t). \]  

### III. Solution for Small Oscillations

In order to obtain an explicit form of the Floquet operators we focus on the case of small oscillations around a central frequency, specifically

\[ \nu(t) = \nu_0 + \epsilon' \cos(2\omega t), \]

with \( \epsilon' \ll \nu_0 \), where \( \nu_0 \) is the mean frequency. This leads to the time-dependent harmonic oscillator equation

\[ \ddot{f} + \nu_0^2 f + 2\epsilon' \nu_0 \cos(2\omega t) f = 0, \]  

when neglecting terms of order 2 or higher in \( \epsilon \) and represents a particular case of the Mathieu equation \[21\]. In order to guarantee stable solutions with the required periodicity \[20\] we require the scattering relation

\[ \frac{\nu_0^2}{\omega^2} = n^2, \]  

with \( n \in \mathbb{Z}^+ \). The solutions for equation \[23\] are, to first order in \( \epsilon \approx 2\nu_0^2 \omega^2 \),

\[ f(t) = \frac{1}{\sqrt{n\omega}} (e^{i\omega t} + \frac{1}{8(n+1)} e^{i(n+2)\omega t} - \frac{1}{8(n-1)} e^{i(n-2)\omega t}), \]  

(25)
and its complex conjugate. To simplify the comparison with the non-parametrically driven case we define

\[ \tilde{\Gamma}(t) = e^{-in\omega t} \Gamma(t), \]
\[ \tilde{\Gamma}(t) = e^{in\omega t} \Gamma(t). \]

These operators retain the same commutation relations as the original \( \Gamma \) operators. In general, any term involving the same number of \( \Gamma \) and \( \Gamma^\dagger \) operators is unchanged. All calculations will use the \( \tilde{\Gamma} \) operators, so the tilde will be omitted from this point. The operators can be written as

\[ \Gamma(t) = \frac{1}{2i} \left[ \frac{2M}{\hbar} h(t) - \dot{\rho} \left( \frac{2}{M\hbar} g(t) \right) \right], \]
\[ \Gamma^\dagger(t) = -\frac{1}{2i} \left[ \frac{2M}{\hbar} h^*(t) - \dot{\rho} \left( \frac{2}{M\hbar} g^*(t) \right) \right] \]

with

\[ g(t) = \frac{1}{\sqrt{n\omega}} (1 + \frac{\epsilon}{8(n+1)} e^{2i\omega t} - \frac{\epsilon}{8(n-1)} e^{-2i\omega t}), \]
\[ h(t) = \frac{1}{\sqrt{n\omega}} (in\omega + \frac{\epsilon i\omega(n+2)}{8(n+1)} e^{2i\omega t} - \frac{\epsilon i\omega(n-2)}{8(n-1)} e^{-2i\omega t}). \]

We can then calculate all of the time dependent terms that require specific solutions for \( f(t) \), which can be just as easily obtained in terms of \( g(t) \) and \( h(t) \). We have, neglecting terms of order \( \frac{\epsilon^2}{n\omega} \)

\[ \frac{W}{|f(t)|} \approx \nu_0, \]  

(30)

and

\[ \gamma_+(t) = g^*(t), \]
\[ \gamma_-(t) = g(t). \]

With these coefficients we can solve equations (20). We are interested in case where the stationary case is reached on a small time scale. In that case we can assume that \( \dot{\alpha} = \dot{\beta} = 0 \). We also assume that the coupling is weak enough to be neglected at first order. The solutions are then

\[ \alpha_0 = \frac{\Omega}{2\delta + i\kappa}, \]
\[ \beta_0 = 0. \]

(33)

(34)

The subscript indicates that these solutions are valid up to order 0 in the coupling parameter. The Hamiltonian is now

\[ H = -\hbar \delta a^\dagger a + \hbar \nu_0 \Gamma^\dagger \Gamma - H_{int}, \]

(35)

with

\[ H_{int}(t) = g_0 \sqrt{\frac{\hbar}{2M}} (\alpha_0^* a + \alpha_0 a^\dagger) (g^*(t) \Gamma(t) + g(t) \Gamma^\dagger(t)). \]

Setting \( \chi_0 = g_0 \sqrt{\frac{\hbar}{\alpha_0^2\delta}} \) we write

\[ H_{int}(t) = H_{int}^0 + H_{int}^f(t), \]

with

\[ H_{int}^0(t) = \chi_0 (\alpha_0^* a + \alpha_0 a^\dagger) (\Gamma(t) + \Gamma^\dagger(t)), \]
\[ H_{int}^f(t) = \chi_0 (\alpha_0^* a + \alpha_0 a^\dagger) \times \]
\[ \frac{1}{8(n+1)} \left( e^{-2i\omega t} \Gamma(t) + e^{2i\omega t} \Gamma^\dagger(t) \right) \]
\[ - \frac{1}{8(n-1)} \left( e^{2i\omega t} \Gamma(t) + e^{-2i\omega t} \Gamma^\dagger(t) \right). \]

Due to (33), the terms in the master equation involving the time derivatives of the Floquet operators, both the commutator terms and the pump-like term, vanish. Then, the master equation (17) can be written as

\[ \dot{\rho} = \frac{1}{i\hbar} [H, \rho] + L_\alpha \rho + L_\gamma \rho = \mathcal{L} \rho. \]

(36)

This last equation is a model for a mechanical oscillator with time dependent frequency, interacting with an electromagnetic field, and with a dissipation model that takes into account that the mechanical object’s frequency depends on time. It looks similar to the standard optomechanical master equation but it has Floquet operators instead of creation and annihilation operators for the mechanical oscillator and an explicit time dependence in the interaction Hamiltonian. It is one of the main results of this paper, it gives the evolution of the parametrically driven optomechanical system with an improved dissipative model. In the next sections we will focus on calculating the number of excitations of the mechanical object \( \langle n \rangle = \langle \Gamma^\dagger \Gamma \rangle \).

IV. LASER COOLING

We use the master equation (36) to study laser cooling of the parametrically driven mechanical object. Our goal is to minimize the temperature of the mechanical object. In the displaced frame, where the master equation is written, this is equivalent to minimizing the number of mechanical excitations. Our focus is on the parameter regime where the coupling is weak enough that we may take \( \alpha_0 \) and \( \beta_0 \) to be the solutions to (20). After projecting into the subspace corresponding to the slowly evolving time scale and tracing over the cavity degrees of freedom, we arrive at the following master equation for the density operator \( \mu(t) = Tr_c[P \rho(t)], \)

\[ \dot{\mu} = (A_-(t) + \gamma_0 (n_m + 1)) D[\Gamma^\dagger] \mu + (A_+(t) + \gamma_0 n_m) D[\Gamma] \mu, \]

(37)
with $D[\Gamma] = 2\Gamma_\mu\Gamma^\dagger - \{\Gamma^\dagger\Gamma, \mu\}$. $A_-(t)$ and $A_+(t)$ are known as the cooling and heating rates, respectively. This equation is obtained in Appendix [B] The coefficients

$$A_\pm(t) = A^0_\pm + \epsilon \sin(2\omega t) A^0_\pm, \quad (38)$$

can be written as the usual rates for the non driven case

$$A^0_\pm = \frac{\chi^2_0|\alpha_0|^2}{2} \frac{\kappa}{(\delta + \nu_0)^2 + \frac{\omega^2}{\Gamma}}.$$  

plus a correction proportional to $\epsilon$

$$A^\epsilon_\pm = \frac{\chi^2_0|\alpha_0|^2}{2} \frac{(\delta + \nu_0)}{n_\pm \left(\frac{\omega}{\Gamma} + (\delta + \nu_0)^2\right)}.$$  

We wish to obtain an expression for the mean number of mechanical excitations $\langle m \rangle$, which is a measure for the system’s temperature. We use the system’s covariance matrix [22] to do that. Defining

$$X = \sqrt{\frac{n_\omega}{2}} (g(t)^\dagger \Gamma + g(t)\Gamma^\dagger), \quad (39)$$

$$P = \frac{1}{\sqrt{2n_\omega}} (h(t)^\dagger \Gamma + h(t)\Gamma^\dagger), \quad (40)$$

and

$$\mathcal{R} = [X, P]^T,$$

the expectation value of the covariance matrix is then expressed as

$$\overline{\gamma}_{i,j} = \frac{1}{2} \langle \mathcal{R}_i \mathcal{R}_j + \mathcal{R}_j \mathcal{R}_i \rangle - \langle \mathcal{R}_i \rangle \langle \mathcal{R}_j \rangle. \quad (42)$$

The calculations to obtain an expression for the covariance matrix $\gamma(t)$ are performed in Appendix [C]. The mean number

$$\langle m \rangle = \frac{1}{2} (\text{Tr}[\gamma] - 1), \quad (43)$$

of mechanical excitations, can be calculated as a function of the trace of $\gamma$ [17]. Defining

$$\tilde{A}^0 = A^0 + \frac{\gamma}{2}(n_m + 1),$$

$$\tilde{A}^0_+ = A^0_+ + \frac{\gamma}{2}n_m,$$

we obtain that

$$\text{Tr}[\gamma(t)] = \frac{\tilde{A}^0 + \tilde{A}^0_+}{\tilde{A}^0_+ - \tilde{A}^0_+} + \frac{\epsilon (A^0_+ + A^0_-)}{\tilde{A}^0_+ - \tilde{A}^0_+} \sin(2\omega t)$$

$$- \epsilon (A^0_+ + A^0_-) A^{0+}_- A^{0+}_- \cos(2\omega t) + \frac{\epsilon (A^0_+ - A^0_-)}{\tilde{A}^0_+ - \tilde{A}^0_+} A^{0+}_+ A^{0+}_+$$

$$- \epsilon (A^0_+ - A^0_-) \frac{\omega \sin(2\omega t)}{\tilde{A}^0_+ - \tilde{A}^0_+}.$$  

These results are valid when $\delta < 0$, if $\delta > 0$ we have heating and the number of mechanical excitations diverges in the framework of our theory. The number of mechanical excitations, Eq. (13), has four correction terms proportional to $\epsilon$, two are time independent and the other two oscillate with the frequency of the drive.

As expected, when $\omega \rightarrow 0$, $\langle m \rangle$ becomes the number of mechanical excitations for the non parametrically driven case. To first order in $\omega$, the time independent corrections vanish, and the number of mechanical excitations oscillates around the non parametrically driven case with a frequency given by $2\omega$. If we take the time average of $\langle m \rangle$ over one period, there will be a time independent correction of order $\omega^2$. We now analyze the effects of the correction terms on the number of mechanical excitations. In the unresolved sideband regime ($\kappa >> \nu_0$), the constant correction terms tend towards zero (as in this regime $A^0_+ \approx A^0_-$). The time dependent corrections remain, but these are zero when time averaged. Note that $A^0_+ \sim \frac{1}{\kappa}$ and $A^0_- \sim \frac{1}{\kappa^2}$ so as $\kappa$ increases, the correction terms become irrelevant.

We will focus in the resolved sideband regime $\kappa < \nu_0$.

When $\omega^2 \ll (A^0_+ - A^0_-)^2$, we can approximate the average, over one period of time, of the number of mechanical excitations

$$\langle m \rangle = \frac{\pi}{\omega} \int_0^{\pi/\omega} \langle m \rangle \, dt,$$

as

$$\langle m \rangle \approx \langle m \rangle_{n_m=0} + \frac{\gamma n_m}{\Gamma_{\text{cool}} + \gamma/2} + \epsilon \omega \frac{\gamma n_m}{2(\Gamma_{\text{cool}} + \gamma/2)^3} (A^0_+ - A^0_-). \quad (45)$$

$$\langle m \rangle_{n_m=0} \approx \frac{A^0_+}{\Gamma_{\text{cool}} + \gamma/2} + \epsilon \omega \frac{A^0_+ + A^0_- + \gamma/2}{2(\Gamma_{\text{cool}} + \gamma/2)^3} (A^0_+ - A^0_-). \quad (46)$$

The second and third term in Eq. (45) are the contribution, to the mean number of mechanical excitations, when the temperature of the mechanical bath is not zero. When $\Gamma_{\text{cool}} \gg \gamma n_m$ this contribution is negligible and we obtain $\langle m \rangle \approx \langle m \rangle_{n_m=0}$.

Sideband cooling is used as a final cooling stage [23]. When sideband cooling is begun, $n_m$ can be in the range of 1000 excitations and $\Gamma_{\text{cool}} \gg \gamma n_m$ [24]. Under these conditions we have that $A^0_+ \approx A^0_-$ and we get that
\begin{equation}
\langle m \rangle \approx -\frac{(\nu_0 + \delta)^2 + \kappa^2/4}{4\delta \nu_0} + \frac{\epsilon \omega}{32\delta^3 \nu_0^3 \chi_0^2 |\alpha_0|^2} \left[ (\nu_0^2 - \delta^2 + \kappa^2/4)(\nu_0^2 + \delta^2 + \kappa^2/4)/(\nu_0^2 + \delta^2 + \kappa^2/4)(\nu_0 - \delta^2 + \kappa^2/4) \right].
\end{equation}

When \( \nu_0^2 - \delta^2 + \kappa^2/4 = 0 \) there is no difference in the number of mechanical excitations between the parametrically driven and the non parametrically driven cases. Note that when \( \epsilon > 0 \) and \( \delta^2 < \nu_0^2 + \kappa^2/4 \), or \( \epsilon < 0 \) and \( \delta^2 > \nu_0^2 + \kappa^2/4 \), the mean number of mechanical excitations in the parametrically driven case is smaller than in the non parametrically driven case. An example of this is shown in figure 1. The detuning is chosen for the case where \( \langle m \rangle \) is minimal at \( t = 0 \) and the number of mechanical excitations \( \langle m \rangle \) is plotted, as a function of time, for the non parametrically driven and the parametrically driven cases. The number of mechanical excitations, for the parametrically driven case, is lower than the non parametrically driven case for most of the time period.

In some cases \( \langle m \rangle \) can be smaller than the smallest achievable temperature in the non parametrically driven case. To show this we calculate \( \langle m \rangle \), over the range \( \delta/\nu_0 = [-1.2, -0.8] \), for the parametrically and non parametrically driven case. The result can be seen in figure 2. We can see that the minimum number of mechanical excitations can be lower than in the non parametrically driven case. The value of \( \delta \), where the minimum is achieved, depends on the sign of \( \epsilon \), as predicted by Eq. (47). In Fig. 3 we compare the ratio of \( \langle m \rangle \) between the parametrically and non parametrically driven cases. For the parameters in the figure, which are consistent with the approximations used in the calculations, the difference can be up to 10%.

\section{V. CONCLUSIONS}

Using an improved theoretical description of the dissipation of a parametrically driven mechanical object in an optomechanical setup, we found that the temperature can be lower than in the non-driven case. Moreover, the usage of a consistent dissipation model affects the predictions for the cooling dynamics. The results of this paper allow for the analysis of the discrepancy when compared to conventional approaches.
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Appendix A: The Damping Basis

Master equations of the type

\[ \dot{\rho} = \mathcal{L}_{\text{cav}} \rho = \frac{1}{i\hbar} [H, \rho] + L_a \rho, \quad (A1) \]

with

\[ L_a \rho = -\kappa (n_p + 1)[a^\dagger a \rho + \rho a^\dagger a - 2a^\dagger \rho a] \]
\[ - \kappa (n_p)[aa^\dagger \rho + \rho aa^\dagger - 2a^\dagger \rho a], \quad (A2) \]

and

\[ H = \hbar \omega_c a^\dagger a, \quad (A3) \]

model the behavior of a bosonic field inside a one mode leaky cavity with frequency \( \omega_c \); the cavity is in contact with a thermal bath characterized by \( n_p \) thermal photons, the cavity damping is given by \( \kappa \) and \( a^\dagger \) and \( a \) are cavity photons creation and annihilation operators. The density operator can be expressed in a basis given by the right Lindblad superoperator's eigenstates, \( \hat{\rho}_n \), \( n = 0, 1, 2, \ldots \) \( j = 0, \pm 1, \pm 2, \ldots \), where

\[ L_a \hat{\rho}_n = \lambda_n \hat{\rho}_n, \quad (A4) \]

with

\[ \lambda_n = ij \omega_c - \kappa [n + |j|/2]. \quad (A5) \]

The real part of these eigenvalues corresponds to the eigenvalues of the operator \( L_a \). This basis is known as the damping basis \([25]\) and is given by

\[ \hat{\rho}_n = \frac{(-1)^n}{(n_p + 1)!} \left[ \frac{a^\dagger a}{n_p + 1} \right] [e^{-\frac{a^\dagger a}{n_p + 1}} : j \geq 0, \quad (A6) \]

\[ \hat{\rho}_n = \frac{(-1)^n}{(n_p + 1)!} \left[ \frac{a^\dagger a}{n_p + 1} \right] [e^{-\frac{a^\dagger a}{n_p + 1}} : a^\dagger |j| \quad (A7) \]

The Lindblad operator is not hermitian and the left eigenstates must be considered to find the coefficients of the density operator expansion in the damping basis. These are the eigenstates of the equation \( \hat{\lambda}_n L = \lambda_n \hat{\rho}_n \) they have the same eigenvalues and are given by

\[ \hat{\rho}_n = \frac{(-1)^n}{(n_p + 1)!} \left[ \frac{a^\dagger a}{n_p + 1} \right] [e^{-\frac{a^\dagger a}{n_p + 1}} : j \geq 0, \quad (A8) \]

\[ \hat{\rho}_n = \frac{(-1)^n}{(n_p + 1)!} \left[ \frac{a^\dagger a}{n_p + 1} \right] [e^{-\frac{a^\dagger a}{n_p + 1}} : a^\dagger |j| \quad (A9) \]

The left and right eigenstates are orthogonal under the product

\[ (\hat{\rho}_n^l, \hat{\rho}_n^r) = Tr[\hat{\rho}_n^l \hat{\rho}_n^r] = \delta_{n,n'} \delta_{j,j'}, \quad (A10) \]

and fulfill

\[ \sum_{\lambda} \hat{\rho}_\lambda \otimes \hat{\rho}_\lambda = \mathbb{I}, \quad (A11) \]

where the sum is over all possible eigenvalues. An important case is a cavity at zero temperature, in this case the right states are \([25]\)

\[ \hat{\rho}_n^l = \frac{n!}{(n + j)!} \left[ \frac{a^\dagger a}{n} \right] ^j \quad j \geq 0, \quad (A12) \]
\[ \hat{\rho}_n^l = \frac{n!}{(n + |j|)!} \left[ \frac{a^\dagger a}{n} \right] ^{|j|} \quad j < 0. \quad (A13) \]

These states play an important part in the derivation of the master equation of the cavity state in the adiabatic approximation.

In Appendix B we consider a harmonic oscillator with no damping, so the left and right eigenstates of the damping basis reduce to

\[ \hat{\rho}_n = |n + l\rangle \langle n| = \hat{\rho}_n^l, \quad (A14) \]

with eigenvalues

\[ \lambda_l = il \nu_0, \quad (A17) \]

|\( n \rangle \) is the number state of the harmonic oscillator, \( l \) is an integer satisfying \( n + l > 0 \).

Appendix B: Laser Cooling and Projection Operators

In order to find the master equation \([37]\) we begin with the equation based on the Hamiltonian \([35]\)

\[ \dot{\rho} = (\mathcal{L}_0 + \mathcal{L}_1) \rho, \quad (B1) \]

where

\[ \mathcal{L}_0 = \mathcal{L}_{\text{cav}} + \mathcal{L}_{\text{mec}} = \left( \frac{1}{\hbar} \mathcal{H}_{\text{cav}} \right) + \mathcal{L}_a + \left( \frac{1}{\hbar} \mathcal{H}_{\text{mec}} \right), \quad (B2) \]
gives the free dynamics and
\[ \mathcal{L}_1 = \mathcal{L}_0 + \mathcal{L}_1' = \frac{1}{\hbar} [H\text{_{int}} + H\text{_{int}}, \bullet], \tag{B3} \]
gives the field-mechanic oscillator interaction.

Equation (B1) is the same as equation [17] without the mechanical damping, which occurs on a slower time scale than the other processes and can be incorporated after the adiabatic approximation. We employ projection operators, like those in [26], to separate the evolution into different time scales and perform an adiabatic approximation. The projection operator \( P \) projects the state into a slow-decaying evolution space whereas the projection operator \( Q \) projects the system into a fast-decaying evolution space, the projection operators fulfill the completeness relation
\[ 1 = P + Q, \tag{B4} \]
and have the properties

1. \( P\mathcal{L}_0 = \mathcal{L}_0 P = 0 \) as \( P \) projects the state to the stationary subspace

2. \( P\mathcal{L}_1 P = 0 \) as the interaction does not couple states in \( P \)

3. \( P^2 = P \) \( Q^2 = Q \) as \( P \) and \( Q \) are projectors.

In the decay picture the master equation is
\[ \dot{\rho}' = \mathcal{L}_1' \rho', \tag{B5} \]
where
\[ \rho' = e^{\int_0^t \mathcal{L}_0 dt'} \rho, \]
\[ \mathcal{L}_1' = e^{-\int_0^t \mathcal{L}_0 dt'} \mathcal{L}_1 e^{\int_0^t \mathcal{L}_0 dt'}. \tag{B7} \]

or more explicitly
\[ \dot{\rho}' = e^{-\mathcal{L}_0 t} \rho, \tag{B6} \]
\[ \mathcal{L}_1' = e^{-\mathcal{L}_0 t} \mathcal{L}_1' e^{\mathcal{L}_0 t}. \tag{B7} \]

We project the master equation (B5) into both \( P \) and \( Q \) to obtain the equations
\[ P\dot{\rho}' = P\mathcal{L}_1' Q \rho', \]
\[ Q\dot{\rho}' = Q\mathcal{L}_1' P \rho' + Q\mathcal{L}_1' Q \rho'. \]

The equation for \( Q \) can be formally integrated
\[ Q\rho = Q\rho'(t_0) + \int_{t_0}^t dt' Q\mathcal{L}_1'(t') P \rho'(t') \]
\[ + \int_{t_0}^t dt' Q\mathcal{L}_1'(t') Q \rho'(t'), \]
and then the Markov approximation is performed, approximating \( \rho(t') \) by \( \rho(t_0) \)
\[ Q\rho \approx Q\rho'(t_0) + \int_{t_0}^t dt' Q\mathcal{L}_1'(t') P \rho'(t_0) \]
\[ + \int_{t_0}^t dt' Q\mathcal{L}_1'(t') Q \rho'(t_0), \]
and this is substituted into the \( P \) equation
\[ P\dot{\rho}'(t) = P\mathcal{L}_1 Q \rho'(t_0) \]
\[ + P\mathcal{L}_1 \int_{t_0}^t dt' Q\mathcal{L}_1(t') P \rho'(t_0) \]
\[ + P\mathcal{L}_1 \int_{t_0}^t dt' Q\mathcal{L}_1(t') Q \rho'(t_0), \tag{B8} \]
where only the second term is non zero as we can choose the initial condition to have no part in \( Q \). We focus on this term and transform back from the decay picture
\[ P\dot{\rho}'(t) = Pe^{-\mathcal{L}_0 t} \mathcal{L}_1 e^{-\mathcal{L}_0 t^2} \]
\[ \int_{t_0}^t dt' Qe^{-\mathcal{L}_0 t'} \mathcal{L}_1 e^{-\mathcal{L}_0 t'} Pe^{-\mathcal{L}_0 t_0} \rho(t_0). \tag{B9} \]

We write the projectors as
\[ P = \sum_{\lambda} (\hat{\rho}_\lambda^{cav} \otimes \hat{\rho}_\lambda^{mec}) \otimes (\hat{\rho}_\lambda^{cav} \otimes \hat{\rho}_\lambda^{mec}), \tag{B10} \]
\[ = \sum_{\lambda} P_\lambda, \]
\[ Q = \sum_{\lambda'} (\hat{\rho}_{\lambda'}^{cav} \otimes \hat{\rho}_{\lambda'}^{mec}) \otimes (\hat{\rho}_{\lambda'}^{cav} \otimes \hat{\rho}_{\lambda'}^{mec}), \tag{B11} \]
\[ = \sum_{\lambda} Q_{\lambda'}, \]
the projectors with the \( \lambda \) label project the state into the slow-decaying time-scale subspace, they are eigenstates of \( \mathcal{L}_0 \) with only eigenvalues equal to zero. The projectors with the \( \lambda' \) label corresponds to the fast-decaying time-scale, they are eigenstates of \( \mathcal{L}_0 \) with eigenvalues with a non-zero real part, those states decay quickly. The projectors are applied via the product
\[ PX = \sum_{\lambda} \hat{\rho}_\lambda Tr[\hat{\rho}_\lambda X], \tag{B12} \]
with
\[ \hat{\rho}_\lambda = \hat{\rho}_\lambda^{mec} \otimes \hat{\rho}_\lambda^{cav}. \tag{B13} \]
We employ the states defined in Appendix A for both the cavity and the mechanical resonator.
Using equations (B10) and (B11) in (B9) and applying the operator $\mathcal{L}_0$ we obtain

$$P\dot{\rho}(t) = P e^{-\mathcal{L}_0 t} \mathcal{L}_1 \left( \sum_{\lambda',\lambda} \int_{t_0}^t dt' e^{i\lambda t' - i\lambda_0 t} \hat{\rho}_{\lambda'} \otimes \hat{\rho}_{\lambda} e^{-i\lambda t'} \mathcal{L}_1 \right)$$  \hspace{1em} (B14)

$\mathcal{L}_1$ is time-independent and the integration can be easily performed. Returning $P$ and $Q$ to their original notation we may write

$$P\dot{\rho}(t) = P e^{-\mathcal{L}_0 t} \mathcal{L}_1 \left( \sum_{\lambda',\lambda} \frac{1}{\lambda - \lambda'} e^{i\lambda t - i\lambda_0 t} \mathcal{L}_1 \mathcal{P}_{\lambda' \rho(t_0)} \right).$$  \hspace{1em} (B15)

The integration is straightforward and we obtain

$$P\dot{\rho}(t) = P e^{-\mathcal{L}_0 t} \mathcal{L}_1 \left( \sum_{\lambda',\lambda} \frac{1}{\lambda - \lambda'} (e^{i\lambda(t - t_0)} - e^{i\lambda(t - t_0)}) \mathcal{Q}_{\lambda'} \mathcal{P}_{\rho(t_0)} \right).$$  \hspace{1em} (B16)

We neglect terms proportional to $e^{\lambda t}$ because for the slow time scale these terms tend to zero. Using that $\lambda = 0$, we can write

$$P\dot{\rho}(t) = \sum_{\lambda'} \left( -\frac{1}{\lambda} P \mathcal{L}_1^0(t) \mathcal{Q}_{\lambda'} \mathcal{L}_1^0(t) \mathcal{P}_{\rho(t_0)} \right)$$  \hspace{1em} (B17)

$$- P\mathcal{L}_1^0(t) \mathcal{Q}_{\lambda'} \mathcal{L}_1^0(t) \mathcal{P}_{\rho(t_0)}.$$  \hspace{1em} (B18)

Now, we trace over all of the cavity degrees of freedom as we are interested only in the mechanical degrees of freedom. Defining $\mu(t) = \text{Tr}_{c}[P \rho(t)]$ we have

$$\mu(t) = \sum_{\lambda'} \frac{1}{\hbar^2} \left( \text{Tr}_{c} \left[ \frac{1}{\lambda} P[H_{\text{int}}^0(t), \bullet] \mathcal{Q}_{\lambda'} [H_{\text{int}}^0(t), \bullet] P \rho(0) \right] \right. \hspace{1em} (B20)$$

$$+ \text{Tr}_{c} \left[ \frac{1}{\lambda} P[H_{\text{int}}^0(t), \bullet] \mathcal{Q}_{\lambda'} [H_{\text{int}}^0(t), \bullet] P \rho(0) \right) \left. \right) + \text{Tr}_{c} \left[ \frac{1}{\lambda} P[H_{\text{int}}^0(t), \bullet] \mathcal{Q}_{\lambda'} [H_{\text{int}}^0(t), \bullet] P \rho(0) \right].$$

The first term yields the usual master equation for the non-driven case and the other two terms yield correction terms proportional to $\epsilon$. We may calculate term by term, using the notation

$$F_a = (\alpha_0 a + \alpha_0 a^\dagger),$$  \hspace{1em} (B21)

$$F_{\Gamma} = (\Gamma + \Gamma^\dagger),$$  \hspace{1em} (B22)

$$F_{\Gamma}^+ = \frac{\epsilon}{8(n + 1)} (e^{4i\omega t} \Gamma + e^{-4i\omega t} \Gamma^\dagger),$$  \hspace{1em} (B23)

$$F_{\Gamma}^- = \frac{\epsilon}{8(n - 1)} (e^{-4i\omega t} \Gamma + e^{4i\omega t} \Gamma^\dagger),$$  \hspace{1em} (B24)

$$F_{\Gamma}^i = F_{\Gamma}^+ - F_{\Gamma}^-,$$  \hspace{1em} (B25)

$$H_{\text{int}}^0 = \chi_0 F_a F_{\Gamma},$$  \hspace{1em} (B26)

$$H_{\text{int}}^1 = \chi_0 F_a F_{\Gamma}^i.$$  \hspace{1em} (B27)

With this, in the case of the first term of equation (B20)

$$\chi_0^2 \sum_{\lambda'} \frac{1}{\hbar^2} \text{Tr}_{c} \left[ P F_a F_{\Gamma} \mathcal{Q}_{\lambda'} [F_a F_{\Gamma}, \mu_{\text{st}}] \right]$$  \hspace{1em} (B28)

$$= \chi_0^2 \left( \frac{1}{\lambda} \left( \text{Tr}_{c} [P F_a F_{\Gamma} \mathcal{Q}_{\lambda'} (F_a F_{\Gamma}, \mu_{\text{st}})] \right) \right.$$  \hspace{1em} (B29)

$$- \text{Tr}_{c} [P \mathcal{Q}_{\lambda'} (F_a F_{\Gamma}, \mu_{\text{st}}) F_a F_{\Gamma}]$$  \hspace{1em} (B30)

$$- \text{Tr}_{c} [P \mathcal{Q}_{\lambda'} (\mu_{\text{st}} F_a F_{\Gamma}, F_a F_{\Gamma})]$$  \hspace{1em} (B31)

$$+ \text{Tr}_{c} [P \mathcal{Q}_{\lambda'} (\mu_{\text{st}} F_a F_{\Gamma}, F_a F_{\Gamma})].$$

Here we have assumed that $\rho(0) = \rho_{\text{mech}} \otimes \rho_{\text{st}}$, that is that the initial condition is separable. We separate the projection operators into the mechanical and cavity parts, indicated by the appropriate sub-index

$$P = P_a P_{\Gamma},$$  \hspace{1em} (B29)

$$\mathcal{Q}_{\lambda'} = \mathcal{Q}_{\lambda'} \mathcal{Q}_{\lambda'}.$$  \hspace{1em} (B30)

The mechanical parts can then be taken out of the trace and we have
\[ \chi^2 \sum_{\lambda'} \frac{1}{\hbar^2} \frac{1}{X'} Tr_c[P[F_{\lambda'} F_R, Q_{\lambda'} [F_{\lambda'} F_R, \mu_{\rho_{st}}]]] \]  
\[ = \chi^2 \sum_{\lambda'} \frac{1}{\hbar^2} \frac{1}{X'} \left( \frac{1}{X'(Tr_c[P_{\lambda} Q_{\lambda'} (F_{\rho_{st}}) F_{\lambda}])P_{\lambda'} F_{\lambda'} Q_{\lambda'} (F_R \mu) - Tr_c[P_{\lambda} Q_{\lambda'} (F_{\rho_{st}}) F_{\lambda}])P_{\lambda'} Q_{\lambda'} (F_R \mu) F_R - Tr_c[P_{\lambda} Q_{\lambda'} (\rho_{st} F_{\lambda}) F_{\lambda}] P_{\lambda'} Q_{\lambda'} (\mu F_R) + Tr_c[P_{\lambda} Q_{\lambda'} (\rho_{st} F_{\lambda}) F_{\lambda}] P_{\lambda'} Q_{\lambda'} (\mu F_R) F_R) \right). \]

This can be written as
\[ \chi^2 \sum_{\lambda'} \frac{1}{\hbar^2} \frac{1}{X'} Tr_c[P[F_{\lambda'} F_R, Q_{\lambda'} [F_{\lambda'} F_R, \mu_{\rho_{st}}]]] \]  
\[ = \chi^2 \sum_{\alpha_0} \frac{1}{\hbar^2} \frac{1}{X'} \left( T_{1c} P_{\lambda'} F_R, Q_{\lambda'} F_R \mu \right] - T_{2c} P_{\lambda'} F_R, Q_{\lambda'} \mu F_R] \right), \]

with
\[ T_{1c} = Tr_c[F_{\lambda'} Q_{\lambda'} F_{\lambda'} F_{\lambda} \rho_{st}] = \hbar^2 |\alpha_0|^2 \delta_{\lambda,1} \delta_{n,0}, \]  
\[ T_{2c} = Tr_c[F_{\lambda'} Q_{\lambda'} \rho_{st} F_{\lambda}] = \hbar^2 |\alpha_0|^2 \delta_{\lambda,-1} \delta_{n,0}, \]

and
\[ P_{\lambda'}[F_{\lambda'} Q_{\lambda'} F_R \mu] = ((\Gamma^\dagger \mu - \Gamma^\dagger \mu \Gamma) \delta_{\lambda,-1} + (\Gamma^\dagger \mu - \Gamma^\dagger \mu \Gamma) \delta_{\lambda,1}), \]  
\[ = T_{1c} P_{\lambda'} F_R, Q_{\lambda'} (F_R \mu) \]  
\[ + (\Gamma^\dagger \mu - \Gamma^\dagger \mu \Gamma) \delta_{\lambda,-1} \]  
\[ + (\Gamma^\dagger \mu - \Gamma^\dagger \mu \Gamma) \delta_{\lambda,1}. \]

The other two terms in equation (B20) are handled in the exact same manner. Both terms yield the exact same result which then acquires a factor of 2 and we have
\[ Tr_c[\frac{1}{X'} P[H_{int}(t), \bullet] Q_{\lambda'} [H_{int}^\dagger(t), \bullet] P_\rho(0)] = \chi^2 \sum_{\lambda'} \frac{1}{\hbar^2} \frac{2}{X'} \left( T_{1c} P_{\lambda'} F_R, Q_{\lambda'} (F_R^\dagger \mu) \right] - T_{2c} P_{\lambda'} F_R, Q_{\lambda'} (\mu F_R^\dagger) \right), \]

with
\[ P_{\lambda'}[F_{\lambda'} Q_{\lambda'} (F_R^\dagger \mu)] = \epsilon(T(-\omega) \Gamma^\dagger \mu \delta_{\lambda,-1} - T(-\omega) \Gamma^\dagger \mu \Gamma \delta_{\lambda,-1} - T(\omega) \Gamma^\dagger \mu \delta_{\lambda,1} - T(\omega) \mu \Gamma^\dagger \delta_{\lambda,1}), \]

and
\[ P_{\lambda'}[F_{\lambda'} Q_{\lambda'} (\mu F_R^\dagger)] = \epsilon(T(-\omega) \Gamma^\dagger \mu \delta_{\lambda,-1} - T(-\omega) \mu \Gamma \Gamma^\dagger \delta_{\lambda,-1} + T(\omega) \Gamma^\dagger \mu \delta_{\lambda,1} - T(\omega) \mu \Gamma \Gamma^\dagger \delta_{\lambda,1}). \]

where we have defined
\[ T(\omega) = \frac{1}{8(n+1)} e^{2i\omega t} - \frac{1}{8(n-1)} e^{-2i\omega t}. \]

We approximate this function, neglecting terms of order \( \frac{1}{\pi^2} \) as
\[ T(\omega) = \frac{i \sin(2\omega t)}{4n}. \]

The Kronecker delta functions apply to the eigenvalues \( \lambda' \) which are, adding together the values for the cavity and the mechanical resonator
\[ \chi' = i(\delta j + \nu_0 l) - \kappa(n + \frac{|l|}{2}), \]

and equation (B20) can be re-arranged as
\[ \dot{\mu}(t) = A_- D[\Gamma] \mu + A_+ D[\Gamma^\dagger] \mu, \]

if we neglect a small term proportional to \( \Gamma^\dagger \Gamma \), with
\[ A_\pm = A_\pm^0 + \epsilon \sin(2\omega t) A_\pm^0, \]
\[ A_\pm^0 = \frac{\chi^2 |\alpha_0|^2}{2} \frac{\kappa}{\left( \delta + \nu_0 \right)^2 + \frac{\kappa^2}{4}}, \]
\[ A_\pm^0 = \frac{\chi^2 |\alpha_0|^2}{2} \frac{\left( \delta + \nu_0 \right)}{n \left( \frac{\kappa^2}{4} + \left( \delta + \nu_0 \right)^2 \right)}. \]

In the adiabatic approximation, as presented here, the mechanical dissipation can be incorporated into the master equation later, as it occurs on a much longer timescale than other processes (\( \gamma \ll \kappa, \chi_0 |\alpha_0| \)). By adding [13] to [B43] we obtain
\[ \dot{\mu} = (A_- (t) + \gamma \frac{2}{n} (n_0 + 1)) D[\Gamma] \mu + (A_+ (t) + \gamma \frac{2}{n_m} D[\Gamma^\dagger] \mu, \]

which is the desired result.

Appendix C: Calculation of the Covariance Matrix

We follow [17, 22] to calculate the covariance matrix. It is useful to first change to dimensionless position and momentum operators
The expectation value of the covariance matrix is then
\[ X = \sqrt{\frac{n \omega}{2}} (g(t)^\dagger \Gamma + g(t) \Gamma^\dagger), \tag{C1} \]
\[ P = \frac{1}{\sqrt{2n \omega}} (h(t)^\dagger \Gamma + h(t) \Gamma^\dagger), \tag{C2} \]
and define the vector
\[ \mathcal{R} = [X, P]^T. \tag{C3} \]

The expectation value of the covariance matrix is then expressed as
\[ \mathcal{F}_{i,j} = \frac{1}{2} \langle \mathcal{R}_i \mathcal{R}_j + \mathcal{R}_j \mathcal{R}_i \rangle - \langle \mathcal{R}_i \rangle \langle \mathcal{R}_j \rangle. \tag{C4} \]

If the master equation (36) can be expressed in the form
\[ \frac{d\mu}{dt} = \sum_k \gamma_k D[L_k \mathcal{R}] \mu, \tag{C5} \]
for a pair of vectors \( L_1 \) and \( L_2 \), a differential equation for the matrix \( \mathcal{F} \) can then be found. The differential equation for the covariance matrix is
\[ \frac{d\mathcal{F}}{dt} = \mathcal{H}_{eff} \mathcal{F} + \mathcal{F} \mathcal{H}_{eff}^T + \mathcal{J}, \tag{C6} \]
with
\[ \mathcal{F}_{i,j} = \frac{1}{i} \langle \mathcal{R}_i \mathcal{R}_j \rangle, \tag{C7} \]
\[ G_{i,j} = \sum_k \gamma_k (L_k)_i (L_k)_j, \tag{C8} \]
\[ \mathcal{H}_{eff} = 2 \mathcal{F} (Im(\mathcal{G})) \tag{C9} \]
\[ \mathcal{J} = 2 \mathcal{F} (Re(\mathcal{G})) \mathcal{F}^T. \tag{C10} \]

The equation can be integrated as
\[ \mathcal{F}(t) = e^{\frac{f_0}{t} dt} \mathcal{H}_{eff}(t') \mathcal{F}(0) e^{\frac{-f_0}{t} dt} \mathcal{H}_{eff}^T(t') \]
\[ + \int_0^t dt e^{\frac{f_0}{t} - dt} \mathcal{H}_{eff}(t') \mathcal{J}(t) e^{\frac{f_0}{t} - dt} \mathcal{H}_{eff}^T(t'). \tag{C11} \]

In order to express the master equation (36) in the form (C5), we require two vectors \( L \) such that
\[ L_1 \mathcal{R} = \Gamma, \tag{C12} \]
\[ L_2 \mathcal{R} = \Gamma^\dagger. \tag{C13} \]

This is simple given the form of the \( \Gamma \) operators
\[ \Gamma = \sqrt{\frac{2}{n \omega} \frac{h(t)}{2i}} X - \sqrt{2n \omega} \frac{g(t)}{2i} P, \tag{C15} \]
\[ \Gamma^\dagger = -\sqrt{\frac{2}{n \omega} \frac{h^*(t)}{2i}} X + \sqrt{2n \omega} \frac{g(t)^*}{2i} P. \tag{C16} \]

We can then easily write
\[ T_1 = \frac{1}{2i} (\sqrt{\frac{2}{n \omega}} h(t), -\sqrt{2n \omega} g(t)), \tag{C17} \]
\[ T_2 = -\frac{1}{2i} (\sqrt{\frac{2}{n \omega}} h^*(t), -\sqrt{2n \omega} g^*(t)). \tag{C18} \]

Given the form of equation (C5) we can see that
\[ \gamma_1 = A-(t), \tag{C19} \]
\[ \gamma_2 = A+(t), \tag{C20} \]
We can then calculate all of the matrices in (C7). The commutator matrix is
\[ \mathcal{F}_{i,j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{C21} \]
and so
\[ \mathcal{G}_{1,1} = A_-(\overline{L}_1)_1 (\overline{L}_1)_1 + A_+(\overline{L}_2)_1 (\overline{L}_2)_1, \tag{C22} \]
\[ \mathcal{G}_{1,2} = A_-(\overline{L}_1)_1 (\overline{L}_1)_2 + A_+(\overline{L}_2)_1 (\overline{L}_2)_2, \tag{C23} \]
\[ \mathcal{G}_{2,1} = A_-(\overline{L}_1)_2 (\overline{L}_1)_1 + A_+(\overline{L}_2)_2 (\overline{L}_2)_1, \tag{C24} \]
\[ \mathcal{G}_{2,2} = A_-(\overline{L}_1)_2 (\overline{L}_1)_2 + A_+(\overline{L}_2)_2 (\overline{L}_2)_2. \tag{C25} \]

This allows us to write expressions for \( \mathcal{H}_{eff} \) and \( \mathcal{J} \)
\[ \mathcal{H}_{eff} = (A_+ - A_-) \mathcal{I}, \tag{C26} \]
\[ \mathcal{J} = (A_+ + A_-) \mathcal{I}, \tag{C27} \]
with
\[ \mathcal{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{C28} \]

After separating the coefficients \( A_+ \) as in equation (38), we begin the integration process. We begin with the integrals appearing in the exponentials in equation (C11).
\[ \int_0^t dt' \bar{H}_{eff}(t') = (A_0^0 - A_0^0) \mathcal{T} \]

After a long enough time, if the parameters are chosen to favor cooling, where \( A_- > A_+ \), all of the exponential terms in the effective Hamiltonian will be small.
terms proportional to $\epsilon^2(A_0^+ - A_0^-)^2$. including the initial condition, drop out and equation (C11) simplifies to

$$\gamma(t) = \frac{1}{2} A_0^+ + A_0^- T$$

$$- \frac{1}{2} \epsilon (A_0^+ - A_0^-) (A_0^+ - A_0^-)^{2} + \omega^2$$

$$\frac{\epsilon}{2\omega} (A_0^+ - A_0^-)^{2} + \omega^2$$

$$- \frac{\epsilon}{2\omega} (A_0^+ - A_0^-) (A_0^+ + A_0^-) T.$$ (C44)

To obtain $\langle n \rangle$ we must then simply take the trace

$$\text{Tr}[\gamma(t)] = \frac{A_0^+ + A_0^-}{A_0^+ - A_0^-}$$

$$- \frac{\epsilon (A_0^+ + A_0^-) (A_0^+ - A_0^-) \sin(2\omega t)}{(A_0^+ - A_0^-)^2 + \omega^2}$$

$$- \frac{\epsilon (A_0^+ + A_0^-) \omega \cos(2\omega t)}{(A_0^+ - A_0^-)^2 + \omega^2}$$

$$+ \frac{\epsilon (A_0^+ - A_0^-) (A_0^+ + A_0^-)}{\omega}$$

$$- \frac{\epsilon (A_0^+ - A_0^-) (A_0^+ - A_0^-)}{(A_0^+ - A_0^-)^2 + \omega^2}.$$ (C45)

This is the desired result.

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