Gauge fields and interactions in matrix string theory

Thomas Wynter\textsuperscript{o}\textsuperscript{†}

Service de Physique Théorique, C.E.A. - Saclay, F-91191 Gif-Sur-Yvette, France

It is shown that all possible $N$ sheeted coverings of the cylinder are contained in type IIA matrix string theory as non-trivial gauge field configurations. Using these gauge field configurations as backgrounds the large $N$ limit is shown to lead to the type IIA conformal field theory defined on the corresponding Riemann surfaces. The sum over string diagrams is identified as the sum over non-trivial gauge backgrounds of the SYM theory.

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\textsuperscript{o} wynter@wasa.saclay.cea.fr

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1. Introduction

Matrix theory\[1\] is a concrete proposal for a non-perturbative description of M theory\[2\], the 11 dimensional theory postulated as the strong coupling limit of type IIA string theory. Formulated in the infinite momentum frame it describes M-theory by the $N \to \infty$ limit of the supersymmetric quantum mechanics of D0 branes\[3\]\[4\]\[5\].

Type IIA string theory is recovered from M theory by compactifying M theory on a circle and identifying the string coupling constant $g_s$ with the radius of compactification. One is thus naturally led to study matrix theory compactified on a circle \[6\]\[7\]\[8\]. The 2d SYM theory resulting from this compactification should give a non-perturbative description of type IIA string theory. This was studied in the most depth in the elegant paper by Dijkgraaf, Verlinde and Verlinde \[8\]. The string coupling constant in this model is related to the SYM coupling by $g_s = 1/\sqrt{\alpha'} g_{QCD}$. By looking at the strong coupling limit of the SYM they argued that perturbative string theory could indeed be recovered. The deep infra-red (strong coupling) limit of the theory was identified as an $S_N$ orbifold conformal theory describing freely propagating strings. From this infra-red limit the original SYM theory is seen as an irrelevant perturbation of the conformal field theory. They further argued that this irrelevant perturbation corresponded to the splitting and joining of strings and constructed a local operator to describe such an interaction. By dimensional analysis they showed that it would, as required, be associated with a linear power of the string coupling constant $g_s$.

In this paper we aim to understand directly from the SYM theory how string interactions emerge. We start in section 2 by recalling the model. In section 3 we study the classical solutions of the theory and identify not only the solutions corresponding to multiply wound (long) strings but also those corresponding to arbitrary string interactions. They are characterized by particular non-trivial gauge field configurations which we show how construct explicitly. In section 4 we turn to the effective action for the low energy excitations of the theory in the limit $N \to \infty$. We show that they are described by the orbifold conformal field theory defined on the multisheeted Riemann surface specified by the background gauge field configuration, irrespective of the value of $g_s$. The sum over string diagrams is identified as the sum over non-trivial gauge configurations of the SYM theory.
2. $N = 8$ two dimensional gauge theory as non-perturbative string theory

In this section we briefly summarize the principle ideas of matrix string theory as first proposed in [6] and [7] and developed most fully in [8]. The starting point is 10-d supersymmetric Yang-Mills theory dimensionally reduced to two dimensions:

$$S = \int d\tau d\sigma \text{Tr}[\frac{1}{2}(D_\alpha X^I)^2 + \frac{i}{2} \Theta^T D \Theta - \frac{1}{4} F_{\alpha\beta}^2 + \frac{1}{4 g_s^2} [X^I, X^J]^2 + \frac{1}{2 g_s} \Theta^T \gamma_i [X^I, \Theta]].$$

(2.1)

This can be obtained by compactifying matrix theory on a circle (see [1][9]). All fields are $N \times N$ hermitean matrices. The index $I$ runs from 1 to 8 and the 16 component fermion fields split into the $8_s$ and $8_c$ representations of $SO(8)$. $g_s$ is the string coupling constant and the coordinate $\sigma$ runs from 0 to $2\pi$.

Weakly coupled strings are to be recovered by considering the limit $g_s \rightarrow 0$ or equivalently the infra-red limit of the theory. It was argued in the above papers that the limiting point $g_s = 0$ would be described by a conformal theory in which all matrices commute. The matrix coordinates can then be written as:

$$X^I = U x^I U^\dagger \quad \text{and} \quad \Theta = U \theta U^\dagger,$$

(2.2)

where $x^I$ and $\theta$ are diagonal matrices and $U$ is a unitary matrix. The abelian remnant of the gauge fields would decouple leaving $N$ copies of the light cone Green Schwarz action. We will discuss this reduction in more detail in section 4.

As was first realised in [3] it is possible to find field configurations (2.2) corresponding to strings wound multiply around the compact direction $\sigma$. Specifically we can choose the matrix $U$ in (2.2) such that

$$U(\sigma + 2\pi) = U(\sigma) g \quad \Rightarrow \quad x(\sigma + 2\pi) = g x(\sigma) g^\dagger$$

(2.3)

where $g$ is an element of the Weyl group of $U(N)$. The result being that in going around the compact direction the eigenvalues are interchanged and thus form cycles of varying length. For a cycle of length $n$ consisting of the eigenvalues $x_1, \ldots, x_n$ one has

$$x_i(\sigma + 2\pi) = x_{i+1}(\sigma) \quad \text{with} \quad x_{n+1} = x_1.$$

(2.4)

In the infra red limit we thus would expect a conformal field theory whose Hilbert space decomposes into “twisted sectors” given by the different ways of partitioning the $N$ eigenvalues into cycles:

$$N = \sum_N n N_n.$$

(2.5)
with \( N_n \) the number of cycles of length \( n \). For each of these twisted sectors the original non-abelian gauge symmetry is broken down to a discrete \( S_N \) symmetry which acts by (a) permuting cycles of identical length, and by (b) cycling through the eigenvalues in a given cycle, i.e. \( x_i \rightarrow x_{i+1}, \quad i = 1, \ldots, n \) for the cycle described above.

The outcome is that the conformal field theory in the infra-red (\( g_s = 0 \)) limit is identified as the \( N = 8 \) supersymmetric sigma model defined on the orbifold target space

\[
S^N \mathbb{R}^8 = (\mathbb{R}^8)^N / S_N.
\]  

(2.6)

The final step in identifying freely propagating strings is to send \( N \rightarrow \infty \) and to keep only the lowest energy excitations i.e. those corresponding to strings of length \( \mathcal{O}(N) \). The length of an individual string divided by the total length of all the strings is then identified with the light cone \( p^+ \) momenta and the discrete remnant gauge symmetry (b), described above, becomes the symmetry under translation \( \sigma \rightarrow \sigma + \text{const} \) of the string world sheet.

String interactions arise when two eigenvalues approach each other and interchange. This is illustrated in Fig. 1 where for clarity we have chosen \( N = 2 \). In this case the interchange of eigenvalues describes the transition between an incoming state consisting of a single string and an outgoing state consisting of two strings. At the point where the eigenvalues meet (marked by the cross in the diagram) a \( U(2) \) subgroup of the otherwise completely broken \( U(N) \) symmetry is restored. The authors of \cite{8} argued that near the infra-red fixed point (where the non-abelian part of the SYM theory is partially restored) the theory would be described by the conformal field theory perturbed by an irrelevant operator corresponding to such an interaction. Using twist operators for the bosonic and fermionic fields they constructed a Lorentz invariant supersymmetric operator and showed that it was the unique least irrelevant operator of this type. Furthermore its dimension is such that its contribution to the action is naturally associated with a linear power of the string coupling constant \( g_s \).

3. Gauge field configurations and string interactions

The objective of this paper is to understand directly from the SYM theory how string interactions arise. We begin by investigating the non-trivial gauge configurations of the theory. We will identify them by studying the classical equations of motion of (2.1). In particular we will construct field configurations corresponding to type IIA string theory defined on arbitrary \( N \) sheeted Riemann surfaces, where \( N \) is the size of the matrix fields.
We restrict our attention to field configurations in which all bosonic and fermionic matrices commute.

Let us start with the free strings wrapped multiply around the \( \sigma \) direction and focus on one particular block corresponding to a single cycle of length \( n \). It is trivial to find such configurations which also satisfy the classical equations of motion. We will do this just for the bosonic fields. The generalization to the fermions is obvious. We start from the configuration

\[
A_\alpha = 0 \quad \text{and} \quad X(\sigma, \tau) = \text{diag}(x_1(\sigma, \tau), \ldots, x_n(\sigma, \tau)) \quad \text{with} \quad x_i(\sigma + 2\pi, \tau) = x_{i+1}(\sigma, \tau).
\]  

which is multivalued and with \( X \) satisfying the equations of motion, \( \partial_\alpha \partial^\alpha X = 0 \). Using the matrix \( U(\sigma) \), which satisfies

\[
U(\sigma + 2\pi) = U(\sigma)g \quad \text{with} \quad g = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
1 & 0 & \ldots & 0 & 0
\end{pmatrix},
\]

we gauge transform (3.1) to arrive at the singlevalued configuration

\[
X = U(\sigma)\text{diag}(x_1(\sigma, \tau), \ldots, x_n(\sigma, \tau))U^\dagger(\sigma), \quad A_\sigma = igU^\dagger(\partial_\sigma U) \quad \text{and} \quad A_\tau = 0, \quad (3.3)
\]

which satisfies the non-abelian equations of motion \( D_\alpha D^\alpha X = 0 \). We see that multiply wound strings are associated with a non-zero pure gauge field \( A_\alpha \) (3.3) which cannot be gauge transformed away by a single valued gauge transformation.

For convenience we now make the standard change of coordinates from cylindrical coordinates \( \sigma, \tau \) to \( z \) and \( \bar{z} \) defined in the complex plane by \( z = e^{\tau+i\sigma} \). The \( N \) eigenvalues will then form an \( N \) sheeted covering of the complex plane. The winding sectors are characterized by having branch points of different orders which are placed at the origin and which connect together the sheets into cycles. The order of the branch cut determines the length of the corresponding cycle.

In this picture string interactions are trivial, they would correspond to having branch cuts occurring away from the origin (see fig. 1). What is not immediately obvious, however, is whether there exist singlevalued matrix configurations whose multivalued eigenvalues correspond to such a Riemann surface.
Below we demonstrate that this is indeed the case. We will show that any $N$ sheeted Riemann surface connected together by an arbitrary set of branch points can be described by the multivalued eigenvalues of a single-valued hermitean matrix. We will achieve this by construction. Specifically we will search for single-valued matrices, $X$, whose eigenvalues are multivalued holomorphic functions of $z$ and which furthermore are hermitean in the sense that $X(z, \bar{z})^\dagger = X(\bar{z}, z)$.

The starting point is the fact that an arbitrary $N$ sheeted Riemann surface can always be generated by a polynomial equation of degree $N$ whose coefficients are single-valued functions of $z$. We thus search for single valued matrix solutions of the equation

$$\sum_{n=0}^{N} a_n X^n = 0,$$

with $a_n(z)$ single valued and $a_N = 1$. If such a solution, $X$, exists its $N$ eigenvalues $x_j$, $(j = 1, \cdots, N)$ will correspond to the $N$ solutions of (3.4) and together will describe the $N$ sheeted Riemann surface. Furthermore since they are holomorphic functions of $z$ they will trivially satisfy the equations of motion: $\partial \bar{\partial} x(z) = 0$. Thus, in complete analogy with the winding sectors discussed above, the non-abelian equations of motion will be satisfied by the field configuration

$$X = U \text{diag}(x_1, \cdots, x_N) U^\dagger, \quad A_\alpha = igU^\dagger (\partial_\alpha U).$$

We now demonstrate that such a solution exists. We start by solving (3.4) for complex matrices $M$. It is trivial to see that the matrix

$$M = \begin{pmatrix}
-a_{N-1} & -a_{N-2} & \cdots & -a_1 & -a_0 \\
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0
\end{pmatrix}$$

Fig. 1. (a) A simple string interaction and (b) its representation in the complex plane.
satisfies equation (3.4). The matrix $M$ can be diagonalized by the complex invertible matrix $S$:

$$M = S \text{diag}(x_1, \cdots, x_N) S^{-1} \quad \text{with} \quad S = \begin{pmatrix}
  x_1^{N-1} & x_2^{N-1} & \cdots & x_N^{N-1} \\
  x_1^{N-2} & x_2^{N-2} & \cdots & x_N^{N-2} \\
  \vdots & \vdots & \ddots & \vdots \\
  1 & 1 & \cdots & 1
\end{pmatrix} \quad (3.7)$$

where the $x_i, i = 1, \cdots, N$ are the $N$ eigenvalues of the matrix $M$, i.e. the $N$ scalar solutions of the polynomial equation (3.4). Note that strictly this matrix is only invertible away from the branch points where two or more eigenvalues coincide. Now let us examine what happens about the branch points of the Riemann surface. In circling a branch point some of the eigenvalues, $x_i$ are interchanged

$$\text{diag}(x_1, \cdots, x_N) \rightarrow g \text{diag}(x_1, \cdots, x_N) g^\dagger \quad \text{and} \quad S \rightarrow Sg, \quad (3.8)$$

where $g$ is, as before, an element of the Weyl group of $U(N)$. We now decompose $S$ into a hermitean matrix (positive definite) times a unitary matrix

$$S(z) = H(z, \bar{z})U(z, \bar{z}). \quad (3.9)$$

Since this is a unique decomposition it follows from (3.8) (and the fact that $g$ is an element of $U(N)$) that in circling a branch point

$$H(z, \bar{z}) \rightarrow H(z, \bar{z}) \quad \text{and} \quad U(z, \bar{z}) \rightarrow U(z, \bar{z})g. \quad (3.10)$$

From the matrix $U(z, \bar{z})$ (3.9) and the matrix $\text{diag}(x_1(z), \cdots, x_N(z))$ we then construct the single valued solution (3.5). We illustrate this construction in the appendix with a simple example, that of $Z_N$ Riemann surfaces.

The most important result of this construction is not the particular matrix $X$ (3.5) but rather the matrix $U$ (3.9). We have shown that for every $N$ sheeted Riemann surface there exists a unitary matrix $U$ which generates the correct monodromies around the branch points. Thus an arbitrary function defined on an $N$ sheeted Riemann surface, covering the complex plane, can always be described by a single valued $N \times N$ matrix. Furthermore each Riemann surface is characterized by a particular (up to gauge transformations) non-trivial gauge field configuration $A_\alpha = igU^\dagger(\partial_\alpha U)$, which interpolates between the winding sector in the infinite past ($z = 0$) and the winding sector in the infinite future ($z = \infty$). These field configurations are singular with the gauge fields diverging at the branch points. Since these are non-trivial gauge configurations they will necessarily play a crucial role in the effective action for the low energy excitations.

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3 I thank Michel Bauer for suggesting this matrix.
4. The effective action

The key idea is that string scattering amplitudes are calculated as SYM correlation functions connecting together different winding sectors. The sum over all the possible string interactions is reproduced by the sum over all non-trivial gauge configurations connecting the winding sector in the infinite past with the winding sector in the infinite future.

Let us therefore study the effective action calculated about one of the non-trivial gauge field configurations of the previous section. We focus on a Riemann surface \( \Sigma \) corresponding to a few strings wound multiply around the \( \sigma \) direction which interact at one or two points. Denoting by \( U_\Sigma \) its associated gauge field configuration we define field variables, \( X_\Sigma^I, \Theta^\Sigma \) and \( A^\Sigma_\alpha \) for which the structure of the Riemann surface is manifest:

\[
\begin{align*}
X^I &= U_\Sigma X^I_\Sigma U_\Sigma^\dagger \\
\Theta &= U_\Sigma \Theta^\Sigma U_\Sigma^\dagger \\
A_\alpha &= U_\Sigma A^\Sigma_\alpha U_\Sigma^\dagger - i g_s (\partial_\alpha U_\Sigma) U_\Sigma^\dagger.
\end{align*}
\]

These are well defined everywhere except at the interaction points, where the transformation is singular. Their diagonal elements, by definition, live on the \( N \) sheeted Riemann surface \( \Sigma \). Their off-diagonal elements connect together the different sheets (diagonal elements). We illustrate this in Fig. 2 where for clarity we have unwound the multiply wound long strings and have illustrated the part of the resulting Riemann surface surrounding an interaction point. The vertical wall represents a square root cut.

Fig. 2 Diagonal \((a_{ij})\) and off diagonal \((w_{ij})\) elements defined on the Riemann surface \( \Sigma \).
The strips are of width $2\pi$ in the $\sigma$ direction with each strip labeled by an index $i$ or $j$ specifying the diagonal element $a_i$ defined on it. The “strings”\(^1\) connecting together the different strips represent the off diagonal elements $w_{ij}$ with indices given by the strips on which they end. In terms of the Riemann surface in which the long strings have been unwound they are bilocal fields and can connect together a long string to itself or to another long string.

Ultimately we are interested in the $N \to \infty$ limit where non-compact ten-dimensional space is recovered. The limit is taken while rescaling the coordinates $(\sigma, \tau)$ by $1/N$ so that only states invariant under change of $N$ are left. This also allows one to equate the string length with the $p^+$ momentum carried by the string as required by light cone string theory. The large $N$ limit is thus the infra-red fixed point of the theory with invariance under change of $N$ equivalent to conformal invariance\(^2\). There is of course a subtlety in that the dimension of the matrix fields is increasing as $N$ and could compensate for the renormalization group flow.

In the light of the above comment let us now turn to the determination of the effective action paying particular attention to the large $N$ limit. Roughly speaking we would like to integrate out the off-diagonal elements of the matrices to arrive at an effective action for the diagonal elements. This will only in fact be possible for configurations in which the diagonal elements do not touch.

For compactness we write the action (2.1) in it’s ten-dimensional form (we drop from here on the superscript $\Sigma$):

$$S = \int d^2 \sigma \text{Tr} \left[ -\frac{1}{4} F_{\mu\nu}^2 + \frac{i}{2} \Theta^T \partial \Theta \right], \quad (4.2)$$

where $\mu, \nu = 0, \cdots, 9$. The indices split into $\alpha, \beta = 0, 9$ and $I, J = 1 \cdots 8$. The fields depend upon the coordinates $\sigma_\alpha = (\tau, \sigma)$ and the gauge fields $A_I$ are the bosonic coordinates $X_I$ of (2.1). We then decompose the fields into their diagonal components ($a$ and $\theta$) and their off diagonal components ($w$ and $\eta$):

$$A_\mu = a_\mu + w_\mu,$$

$$\Theta = \theta + \eta, \quad (4.3)$$

\(^1\) The action (2.1) can be thought of as the action for D-strings. The “strings” are then the fundamental type IIB strings stretched between the D-strings

\(^2\) See [10] where sigma model versions of matrix theory are studied and the Ricci flatness condition is recovered as a fixed point of the large $N$ renormalization group of the theory.
and choose an abelian gauge to privilege the diagonal elements. Specifically we add the gauge fixing term

\[ \frac{1}{2} (\partial_\mu a^\mu)^2 + (D_\mu w_\mu)^2 \] 

where \( D_\mu = \partial_\mu - \frac{i}{g} [a_\mu, \cdot] \) \( (4.4) \)

We then write the action as an integral over the Riemann surface \( \Sigma \) and replace, for \( N \) large, the indices of the off diagonal elements by continuous coordinates corresponding to having a “string” connecting the arbitrary points \( \sigma \) and \( \sigma' \) of \( \Sigma \) (at equal \( \tau \)). The sum over indices is then replaced by an integral over all such positions with coordinates rescaled by \( 1/N \). For example

\[ \int d\tau d\sigma \text{Tr} \left[ \frac{1}{g^2} [a_\mu, w_\nu]^2 \right] \to N^3 \int d\tau d\sigma d\sigma' \frac{1}{g^2} (a_\mu(\sigma, \tau) - a_\mu(\sigma', \tau))^2 |w_\nu(\sigma, \sigma', \tau)|^2, \] 

where the factor of \( N^3 \) comes from rescaling the three coordinates. Focusing on the purely bosonic part of the action we find that for non-coinciding \( a^\mu \) there is a single term (the above mass term) in the integral over \( w^\mu \) that dominates all the others. This is made clear by rescaling the \( w^\mu \) field by \( 1/N^{3/2} \) (see appendix). The bosonic part of the action then takes the form

\[ S_b = \int \left[ -\frac{1}{2} (\partial_\mu a_\nu)^2 - \frac{1}{2} (\nabla_\mu w_\nu)^2 + \mathcal{O} \left( \frac{1}{\sqrt{N}} \right) \right], \quad \text{where} \quad \nabla_\mu = \frac{1}{N} \partial_\mu - \frac{i}{g} [a_\mu, \cdot] \] \( (4.6) \)

For the complete action (see Appendix) one finds

\[ S = S_{G.S.} + S_{g.f.} + S_0 + \mathcal{O}\left( \frac{1}{\sqrt{N}} \right), \] 

where \( S_{G.S.} \) is the Green Schwarz action defined on the Riemann surface \( \Sigma \), \( S_{g.f.} \) is the gauge fixed action for the abelian gauge fields which decouple and \( S_0 \) is given by

\[ S_0 = \int d^2 \sigma \text{Tr} \left[ -\frac{1}{2} (\nabla_\mu w_\nu)^2 - \frac{i}{2} \eta \gamma_\mu \nabla^\mu + \omega^* (\nabla_\mu)^2 \omega \right] \] \( (4.8) \)

with \( \omega \) off diagonal ghost fields from the Fadeev-Popov determinant. Providing \( |a^\mu_i - a^\mu_j|^2 \neq 0 \quad \forall i \neq j \) we can drop the \( \mathcal{O}\left( \frac{1}{\sqrt{N}} \right) \) terms in (4.7) and the integrals over \( w^\mu, \eta \) and \( \omega \) can be performed with the determinants canceling due to supersymmetry in the usual way.

The conclusion is that away from the interaction points, where diagonal elements meet, the effective action is the Green Schwarz action defined on the Riemann surface \( \Sigma \), irrespective of the value of \( g_s \).
At this level there is no way of determining what weight to associate with a given Riemann surface and it might even appear that Riemann surfaces for freely propagating strings should be treated on an equal footing with those corresponding to interactions. What distinguishes them of course is the singularities. In light cone string theory the amplitudes have to be carefully calculated around the interaction points, a cut off length $\epsilon$ being introduced and an operator inserted at the interaction point to preserve supersymmetry and Lorentz invariance\cite{12} \cite{13}. From the point of view of the SYM theory we know that the Riemann surface interpretation breaks down around the branch point and thus we can expect the SYM to provide a regularization of the singularity. To explicitly calculate this from the SYM theory would involve including all the terms of $O(1/\sqrt{N})$ of (4.7) (see the appendix). It is not yet known how to do this. In \cite{8} however it was argued on dimensional grounds that interactions would be associated with a factor of $g_s$, which is the only constant in the SYM and which has the dimension of length. This was achieved by constructing a supersymmetric twist operator associated with the branch point and demonstrating that it has the correct dimension to appear with a factor of $g_s$. What is not clear from this line of reasoning however is what power of $N$ should be associated with the string coupling constant. Indeed the large $N$ limit has been argued above to correspond to the infra-red fixed point. The string interaction operator (including the integrals over its position), has dimension 1, and thus might simply be scaled away with a factor of $1/N$.

In terms of the unwound long strings of Fig. 2 the interaction weight comes from integrating out the off diagonal elements (“strings”) in a small region around the branch point. As well as the “strings” that connect the two sheets at points directly over each other there are also the “strings” that connect points shifted with respect to each other by an integer number of strips. As $N \to \infty$ the density of such “strings” that contribute around the interaction point increases with a factor of $N$. This could lead to a canceling of the factor of $1/N$ coming from scaling the coordinates. A further factor that could be important is the measure associated with the integration over the position of the branch points. It should be determined by restricting the gauge field integration to the field configurations associated with a string interaction. Clearly more work needs to be done in these directions to resolve this question satisfactorily.
5. Conclusions

We have set out to understand in more detail how string theory emerges from matrix theory compactified on a circle. The principle result of this paper is that all the possible Riemann surfaces associated with light cone string interactions are contained in the SYM as non-trivial gauge field configurations. Treating these configurations as backgrounds we have demonstrated that in the large $N$ limit the effective action is described by the conformal field theory defined on the associated Riemann surface. The weight to be associated with a branch point can be deduced, up to its dependence on $N$, on dimensional grounds by following the reasoning in [8]. The crucial issue of what power of $N$, if any, is associated with the branch points remains to be addressed.

Finally it is interesting to note that matrix description of string theory appears to be a concrete realization of a proposal for the sum over all string diagrams suggested ten years ago by Knizhnik[14]. He considered the $N \to \infty$ limit of string theory defined on an $N$ sheeted covering of the complex plane and speculated on whether it would contain non-perturbative phenomena.

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7. Appendix

7.1. $Z_N$ Riemann surfaces

We illustrate the construction of section 3 on the simplest subset of general $N$ sheeted Riemann surfaces, those with a $Z_N$ symmetry, i.e. Riemann surfaces where all branch cuts are of order $N$. In terms of the polynomial equation (3.4) they are described by the single valued matrix solution to the equation

$$X^N = a(z),$$  \hspace{1cm} (7.1)

4 I thank Ivan Kostov for drawing my attention to this paper
with the position of the branch points given by the zeros of \( a(z) \). Defining the coordinate \( \phi \) as the argument of \( a(z) \),
\[
a(z) = |a(z)|e^{i\phi},
\]
we have for the matrix \( S \)
\[
S_{kl} = |a|^{(1 - \frac{k}{N})} e^{i(1 - \frac{k}{N})\phi} \omega^{kl},
\]
where \( \omega \) is the \( N \)th root of unity. Using equations (3.3) and (3.2) we find
\[
U_{kl} = \frac{1}{\sqrt{N}} e^{i(1 - \frac{k}{N})\phi} \omega^{kl} \quad \text{and} \quad (A_\phi)_{kl} = \begin{cases} -\frac{2}{N} \left( 1 - \frac{1}{N} \right), & \text{for } k = l; \\ \frac{g}{2N} \left( 1 - i \cot \left[ \frac{\pi}{N} (k - l) \right] \right), & \text{for } k \neq l. \end{cases}
\]
For the special case of a winding sector consisting of several cycles the unitary matrix \( U \) is composed of the sub-blocks for the individual cycles placed down the diagonal. Each sub-block is given by (7.4) with \( N \) the length of the cycle and \( \phi \) the original cylindrical coordinate \( \sigma \).

More general cases involve solving polynomial equations of degree 3 or higher.

7.2. The gauge fixed Lagrangian

The total Lagrangian is given by
\[
S = S_b + S_f + S_{gh},
\]
where the bosonic and fermionic parts are given by
\[
S_b = \int \text{Tr} \left[ -\frac{1}{2} (\partial_\mu a_\nu)^2 - \frac{1}{2} (D_\mu w_\nu)^2 + \frac{i}{g} [w_\mu, w_\nu] \partial^\mu a^\nu + \frac{i}{g} [w_\mu, w_\nu] D^\mu w^\nu + \frac{1}{4g^2} [w_\mu, w_\nu]^2 \right]
\]
\[
S_f = \int \text{Tr} \left[ \frac{i}{2} \theta^T \gamma_\mu \partial^\mu \theta + \frac{i}{2} \eta^T \gamma_\mu D^\mu \eta + \frac{1}{g} \eta^T \gamma_\mu [w^\mu, \theta] + \frac{1}{2g} \eta^T \gamma_\mu [w^\mu, \eta] \right]
\]
\[
S_{gh} = \int \text{Tr} \left[ \zeta^* (\partial_\mu)^2 \zeta + \omega^* (D_\mu)^2 \omega - \frac{i}{g} \omega^* [D_\mu w^\mu, \zeta] \right]
\]
and the ghost field contribution is
\[
S_{gh} = \int \text{Tr} \left[ \zeta^* (\partial_\mu)^2 \zeta + \omega^* (D_\mu)^2 \omega - \frac{i}{g} \omega^* [D_\mu w^\mu, \zeta] \right]
\]
\[
= \frac{1}{N^3} \left[ -\frac{i}{g} \zeta^* \partial_\mu [w^\mu, \omega] - \frac{i}{g} \omega^* D_\mu [w^\mu, \omega] + \frac{1}{g^2} \omega^* [w_\mu, [w^\mu, \omega]|_{\text{diag}}] \right]
\]
\[
N^2 \quad N^4 \quad N^4
\]
The fields $\zeta$ and $\omega$ are respectively the diagonal and off diagonal ghost fields. Below each term is its corresponding factor of $N$ coming from replacing the sums over indices by integrals and rescaling the coordinates $(\tau, \sigma)$ to be independent of $N$. The factors of $N$ associated with the covariant derivatives $D_\mu = \partial_\mu - i/g [a_\mu,]$ come from the commutator with $a_\mu$. Rescaling the fields by

$$w^\mu \to \frac{1}{N^{3/2}} w^\mu, \quad \eta \to \frac{1}{N} \eta, \quad \omega \to \frac{1}{N^{3/2}} \omega,$$

we arrive at equations (4.6), (4.7) and (4.8). There is one subtlety: after rescaling the third term in the ghost action (7.7) is of order $N^0$. It does not contribute at order $N^0$ however since it must always be accompanied by the 4th term which (after rescaling) is of order $1/N$ and which is the only term with the fields $\zeta^*$ and $\omega$. 

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