KATO’S TYPE THEOREMS FOR THE CONVERGENCE OF EULER-VOIGT EQUATIONS TO EULER EQUATIONS WITH DRICHLET BOUNDARY CONDITIONS

Aibin Zang
The Center of Applied Mathematics, Yichun University
Yichun, Jiangxi 336000, China

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Abstract. After investigating existence and uniqueness of the global strong solutions for Euler-Voigt equations under Dirichlet conditions, we obtain the Kato’s type theorems for the convergence of the Euler-Voigt equations to Euler equations. More precisely, the necessary and sufficient conditions that the solution of Euler-Voigt equation converges to the one of Euler equations, as \( \alpha \to 0 \), can be obtained.

1. Introduction. Euler-Voigt model is a class of \( \alpha \)-regularization of Euler equations, which is given by

\[
\begin{align*}
\frac{\partial}{\partial t} v + (u \cdot \nabla) u + \nabla p &= 0, \quad \text{in } \Omega \times (0, T), \\
v &= u - \alpha \Delta u, \quad \text{in } \Omega \times (0, T), \\
\nabla \cdot u &= 0, \quad \text{in } \Omega \times (0, T), \\
u|_{t=0} &= u_0(x), \quad \text{in } \Omega
\end{align*}
\]

where \( \alpha > 0 \) is filtered parameter, \( u \) is velocity and \( p \) is pressure.

We consider system (1) in a simply connected smooth bounded domain \( \Omega \subset \mathbb{R}^N (N = 2, 3) \), subject to homogeneous Dirichlet boundary condition on \( \partial \Omega \), i.e.,

\[
u = 0, \quad \text{on } \partial \Omega \times (0, T).
\]

Euler-Voigt regularization belongs to the class of models known as the \( \alpha \)-models, which have substantial work (see [2, 3, 4, 5, 6, 12, 13, 16, 18, 19]). For the Voigt type regularizations in the context of various hydrodynamic models have been the focus of much recent research, see [14, 15, 18, 19] and reference therein. In [18, 19] Titi and his collaborator discuss on the higher order regularity of Voigt-regularization of MHD equations with periodic boundary conditions, specially, there let the magnetic field \( B \equiv 0 \), the model becomes Euler-Voigt equations. With aid of Galerkin method, the author in [29] has obtained the global well-posedness for Euler-Voigt equations with homogeneous Dirichlet boundary conditions (2), provided that initial velocity \( u_0 \) is regular enough.

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In this article, we focus on investigating the convergence of Euler-Voigt equations to Euler equations as filtered parameter vanishes. Formally, at the limit when \( \alpha = 0 \), the system (1) reduces to the Euler equations of ideal incompressible flow:

\[
\begin{align*}
\partial_t \bar{u} + \bar{u} \cdot \nabla \bar{u} + \nabla \bar{p} &= 0, \text{ in } \Omega \times (0, T) \\
\nabla \cdot \bar{u} &= 0, \text{ in } \Omega \times (0, T).
\end{align*}
\]

Here we consider the Euler system, (3), subject to non-penetration boundary condition,

\[
\bar{u} \cdot \hat{n} = 0, \text{ on } \partial \Omega \times (0, T),
\]

where \( \hat{n} \) denotes the exterior unit normal vector to \( \partial \Omega \).

The issue of approximation of solutions of the Euler equations has been studied by many authors. A classical challenging problem is whether the solutions of the Navier-Stokes equations converge to a corresponding solution of Euler equations when the viscosity vanishes. The answer to this question is affirmative in the absence of physical boundaries (see, e.g., [8, 9, 10, 24] and reference therein). In [7, 21, 28, 25, 27] it is shown that the answer is also affirmative when the Navier-Stokes equations are equipped with the Navier boundary condition. However, in the case when the Navier-Stokes equations are subject to the Dirichlet boundary condition the question is still open. In a remarkable work [17] Kato presents a criterion concerning the boundary layer of the solutions of the Navier-Stokes equations, with no-slip boundary conditions, that guarantees their convergence to the corresponding solutions of the Euler equations (see also [26]). For a recent survey of this subject see [1].

Other problem is concerned that the solution of Euler equations can be approximated by \( \alpha - \)regularizations models, it was shown in [20] that, in the whole space, the solutions of Euler-\( \alpha \) converge to the corresponding solutions of the Euler equations, as \( \alpha \rightarrow 0 \). The author and his collaborators have proved in [22] that the solutions of 2D Euler-\( \alpha \) equations, with Dirichlet boundary conditions, converge to the corresponding solutions of (3), (4), in \( C([0, T]; (L^2(\Omega))^2) \)-norm, as \( \alpha \rightarrow 0 \), indifference to the boundary layers. In [23], the independent limits of the solutions of second-grade fluid equations, as \( \alpha \rightarrow 0 \) and \( \nu \rightarrow 0 \), were studied for flows in a 2D bounded domain with the Dirichlet boundary conditions. In [18, 19], Larios and Titi had also proved that the convergence of Voigt regularization model to Euler equation or inviscid MHD equation with periodic boundary conditions. In this work, we try to investigate the convergence of Euler-Voigt equations to Euler equations. However, taking into account the boundary layer and the behavior of the solutions of (1) near the boundary, it is difficult to justify this convergence. Therefore, we figure out that the sufficient and necessary conditions for Euler-Voigt equations to given a smooth solution, \( \bar{u} \), of the problem (3) and (4).

The remainder of this paper is organized as follows. In section 2, we will introduce notation, present the global existence for Euler-Voigt equations. In section 3, we prove the Kato’s type theorem for the convergence of Euler-Voigt equations to Euler equations. In section 4, we describe the Kato’s conditions by tangential derivatives as in [26].

2. Preliminaries. We use the notation \( H^m(\Omega) \) for the usual \( L^2 \)-based Sobolev spaces of order \( m \), with the norm \( \| \cdot \|_m \) and the scalar product \( ( \cdot, \cdot )_m \). For the case \( m = 0 \), \( H^0(\Omega) = L^2(\Omega) \); we denote the corresponding norm by \( \| \cdot \| \) and the inner
product by $(\cdot, \cdot)$. We denote by $C^\infty_c(\Omega)$ the space of smooth functions, compactly supported in $\Omega$, and by $H^m_0(\Omega)$ the closure of $C^\infty_c(\Omega)$ under the $H^m$-norm.

We also make use of the following notations:

$$
H = \{ u \in (L^2(\Omega))^N : \nabla \cdot u = 0 \text{ in } \Omega, \ u \cdot n = 0 \text{ on } \partial \Omega \},
$$

$$
V = \{ u \in (H^1_0(\Omega))^N : \nabla \cdot u = 0 \text{ in } \Omega \},
$$

$$
\tilde{H}^1 = \{ \pi \in H^1(\Omega) : \int_{\Omega} \pi \, dx = 0 \}.
$$

Let $A = \mathbb{P}(-\Delta)$ be Stokes operator with Dirichlet boundary conditions, where $\mathbb{P}$ is Leray projector from $L^2$ to $H$. Hereafter we use $C$ for constants that, in principle, depend on $\alpha$, and $K$ for those that are independent of $\alpha$.

With these definitions and notations, we can state and prove the following theorem.

**Theorem 2.1.** Let $u_0 \in V$, then (1) has a unique solution $u \in C([0, T]; V)$ and $\frac{d}{dt} u \in C([0, T]; V)$ for arbitrary $T > 0$, in the following sense

$$
\left( \frac{d}{dt}, v \right) + \alpha (\nabla \frac{d}{dt} u, \nabla v) + (B(u), v)_{V'} = 0
$$

for all $v \in V$.

**Proof.** For complete, we can review the proof in [29], based on the following a priori estimates.

Multiplying the equations (1) by $u$, we obtain the basic energy identity

$$
\|u(t)\|^2 + \alpha \|\nabla u(t)\|^2 = \|u_0\|^2 + \alpha \|\nabla u_0\|^2.
$$

We can rewrite the equations (1) by the following evolution equation

$$
\left\{ \begin{array}{l}
(I + \alpha A) \frac{du}{dt} = -\mathbb{P}(u \cdot \nabla u), \\
\nabla \cdot u = 0, \\
u|_{t=0} = u_0,
\end{array} \right.
$$

where $I$ is the identity operator. Furthermore, by the identity (6), we obtain

$$
\|([I + \alpha A]) \frac{du}{dt}\|_{V'} \leq \|u\|^{1/2} \|\nabla u\|^{3/2} \leq K\alpha^{-3/2}.
$$

Since the operator $(I+\alpha A)^{-1}$ is bounded from $V'$ to $H$, it follows that $\|\frac{du}{dt}\|_{L^\infty(0, T; H)} \leq C$. The proof is completed. \hfill \square

3. **Kato’s type theorem.** In this section, we obtain the necessary and sufficient conditions for the convergence of the solutions to Euler-Voigt equations to the one of Euler equations, as $\alpha \to 0$.

**Theorem 3.1.** Let $u_0 \in (H^s(\Omega))^N \cap H(s \geq 3)$. Assume that

$$
\|u_0^\alpha - u_0\| \to 0, \alpha \|\nabla u_0^\alpha\|^2 \to 0 \text{ as } \alpha \to 0,
$$

the solution $u^\alpha$ of Euler-Voigt equations (1) and (2) with initial velocity $u_0^\alpha$ and the solution $\bar{u}$ of the following Euler equations with initial velocity $u_0$, then the following conditions (i) to (vi) are equivalent. (all limiting relations refer to $\alpha \to 0$)

(i) $u^\alpha(t) \to \bar{u}(t)$ in $L^2(\Omega)$, uniformly in $t \in [0, T]$;
(ii) $u^\alpha(t) \to \bar{u}(t)$ weakly in $L^2(\Omega)$, for each $t \in [0, T]$;
(iii) $\int_0^T \|\nabla u^\alpha\|^2 \, dt \to 0$;
(iv) There exists \( \delta = \delta(\alpha) \to 0 \), such that
\[
\frac{\alpha}{\delta} \to 0, \quad \alpha \int_0^T \| \nabla u^\alpha \|_{L^2(\Gamma_\delta)}^2 dt \to 0;
\]
(v) There exists \( \delta \) as in (iv), such that
\[
\frac{\alpha}{\delta} \to 0, \quad \alpha \int_0^T \|\text{curl}\, u^\alpha\|_{L^2(\Gamma_\delta)}^2 dt \to 0,
\]
where \( \text{curl}\, u \) is the vorticity of \( u \).
(vi) There exists \( \delta \) as in (iv), such that
\[
\frac{\alpha}{\delta} \to 0, \quad \alpha \int_0^T \| D(u^\alpha) \|_{L^2(\Gamma_\delta)}^2 dt \to 0;
\]
where \( D(u) = \frac{1}{2}(\partial_j u_i + \partial_i u_j)_{3 \times 3} \).
(vii) There exists \( \delta \) as in (iv), such that
\[
\frac{\alpha}{\delta} \to 0, \quad \alpha^{-1} \int_0^T \| u^\alpha \|_{L^2(\Gamma_\delta)}^2 dt \to 0.
\]

**Proof.** Since \( u^\alpha = 0 \) on the boundary \( \partial \Omega \), then it is easy to show that (iv), (v) and (vi) are equivalent, basing on the following facts, for \( \text{div} v = 0 \)
\[
\int_0^T \| \nabla u^\alpha \|^2 dt = \int_0^T \|\text{curl}\, u^\alpha\|^2 dt = \int_0^T \| D(u^\alpha) \|^2 dt
\]
\[
\int_\Omega v \cdot (u \cdot \nabla) u dx = \int_\Omega v \cdot (\text{curl} u) u dx, \quad \text{and}
\]
\[
\int_\Omega v \cdot (u \cdot \nabla) u dx = \int_\Omega v \cdot (D(u) \cdot u) dx.
\]

(a) \((i) \Rightarrow (ii) \) and \((iii) \Rightarrow (iv) \) are trivial.
(b) \((ii) \Rightarrow (iii) \): If \((ii) \) is true, energy identities i.e.
\[
\begin{align*}
\| u^\alpha(t) \|^2 + \alpha \| \nabla u^\alpha \|^2 &= \| u_0 \|^2 + \alpha \| \nabla u_0 \|^2 \\
\| \bar{u}(t) \|^2 &= \| u_0 \|^2
\end{align*}
\]
(10)
give
\[
\begin{align*}
\text{lim sup} \alpha \| \nabla u^\alpha \|^2_2 & \\
& \leq \text{lim sup} \left[ \alpha \| \nabla u_0 \|^2 - (\| u^\alpha(t) \|^2 - \| u(0) \|^2) \right] \text{ by (10)}_1 \\
& \leq 0 - (\| \bar{u}(t) \|^2 - \| \bar{u}(0) \|^2) \text{ by (10)}_2, (ii) \\
& = 0.
\end{align*}
\]
Intergrate on \((0,T)\) we obtain \((iii) \).

To prove the assertion \((iv) \Rightarrow (i) \), we need a “boundary layer” \( u_b \), which is a correction term (depending on \( \alpha \)) to be subtracted from \( \bar{u} \) to satisfy the zero boundary conditions and which has a thin support. To construct \( u_b \), we need the following lemma

**Lemma 3.2.** There exist a smooth and skew-symmetric tensor of second rank \( \bar{a} \) defined on \([0,T] \times \bar{\Omega} \), and \( \text{div} \bar{a} = \sum_k \partial_k \bar{a}_{jk} \), such that
\[
\bar{u} = \text{div} \bar{a} \text{ on } \partial \Omega, \bar{a} = 0 \text{ on } \partial \Omega.
\]

**Proof.** See[17].
Introduce a smooth cut-off function \( \xi : \mathbb{R}^+ \to \mathbb{R}^+ \) such that
\[
\xi(0) = 1, \xi(r) = 0 \text{ for } r \geq 1,
\]
and set
\[
z = z(x) = \xi(\rho) \quad \text{where } \rho = \text{dist}(x, \partial \Omega),
\]
\[
u_b = \text{div}(z\tilde{a}) = z\text{div}\tilde{a} + \tilde{a} \cdot \nabla z.
\]
(12)
where \( \tilde{a} \cdot \nabla z = \sum_k \tilde{a}_{jk} \partial_k z \). Therefore, we have the following properties by simple calculation

**Proposition 1.** Let \( u_b \) be in (12), then \( u_b \) has a thin support near \( \partial \Omega \) and satisfies
\[
u_b = \tilde{u} \text{ on } \partial \Omega, \quad \text{div}u_b = 0 \text{ in } \Omega.
\]
and the following estimates for \( u_b \) can be established
\[
\|\partial_\ell u_b\|_p \leq K\delta^k, \|\partial_\ell^\alpha \nabla^k u_b\|_p \leq K\delta^{-k+\frac{\ell}{2}},
\]
\[
\|\rho^2 \nabla u_b\|_{L^\infty} \leq K\delta, \|\rho \nabla u_b\|_p \leq K\delta^k,
\]
for any \( p \geq 1 \), \( l = 0, 1 \), \( k = 1, 2 \) and \( \|u_b\|^p = \int_\Omega |u_b|^p \text{dx} \).

We introduce a corrector with a free parameter \( \beta \) which allows us to interpolate between the laminar boundary layer theory of Prandtl and the viscous sublayer, i.e. \( \delta = 0_{\Omega^\beta}, \) here \( U = \|\tilde{u}\|_{L^\infty(\partial \Omega \times [0, T])}, \)
\[
\|u^\alpha - \tilde{u}\|^2 = \|u^\alpha\|^2 + \|\tilde{u}\|^2 - 2(u^\alpha, \tilde{u}) \\
\leq \|u^\alpha_0\|^2 + \alpha \|\nabla u^\alpha_0\|^2 + \|\tilde{u}_0\|^2 - 2(u^\alpha, \tilde{u}) \quad (14)
\]
Integrate by parts, we can calculate
\[
\text{R.H.S.} = \|u^\alpha_0\|^2 + \alpha \|\nabla u^\alpha_0\|^2 + \|\tilde{u}_0\|^2 - 2(u^\alpha, \tilde{u} - u_b) + (u^\alpha, u_b) \\
= \|u^\alpha_0\|^2 + \alpha \|\nabla u^\alpha_0\|^2 + \|\tilde{u}_0\|^2 - 2(u^\alpha, \tilde{u} - u_b) + 2(u^\alpha_0, u_b(0)) + (u^\alpha, u_b) \\
+ 2 \int_0^t (u^\alpha, \partial_t \tilde{u}) \text{ds} + 2 \int_0^t (u^\alpha \otimes u^\alpha, \nabla \tilde{u}) \text{ds} \\
- 2 \int_0^t (u^\alpha, \partial_t u_b) \text{ds} - 2 \int_0^t (u^\alpha \otimes u^\alpha, \nabla u_b) \text{ds} \\
+ 2\alpha \int_0^t (\partial_t \nabla u^\alpha, \nabla (\tilde{u} - u_b)) \text{ds} \\
:= I_1 + I_2 + I_3 + I_4. \quad (15)
\]
From (9) and \( u_b(t) \to 0 \) as \( t \to 0 \), we have \( I_1 = o(1) \). We check \( I_i (i = 2, 3, 4) \) one by one.

Since \( \tilde{u} \) is the solution of Euler equations, thus we have
\[
I_2 = 2 \int_0^t (u^\alpha, (\tilde{u} \cdot \nabla \tilde{u})) \text{ds} - 2 \int_0^t (u^\alpha \otimes u^\alpha, \nabla \tilde{u}) \text{ds}.
\]
Since \( (\tilde{u}, u^\alpha \cdot \nabla \tilde{u}) = 0 \) and \( (\tilde{u}, \tilde{u} \cdot \nabla \tilde{u}) = 0 \), then the following equality holds
\[
(u^\alpha \otimes u^\alpha, \nabla \tilde{u}) - (u^\alpha, \tilde{u} \cdot \nabla \tilde{u}) = (u^\alpha \otimes u^\alpha, \nabla \tilde{u}) - (\tilde{u}, u^\alpha \cdot \nabla \tilde{u}) + (\tilde{u}, \tilde{u} \cdot \nabla \tilde{u}) \\
- (u^\alpha, (\tilde{u} \cdot \nabla \tilde{u}) = ((u^\alpha - \tilde{u})(u^\alpha - \tilde{u}), \nabla \tilde{u}).
\]
It follows that
\[ I_2 \leq K \int_0^t \| u^\alpha - \bar{u} \|^2. \]  
(16)

By Hardy-Littlewood inequality, we obtain
\[ I_3 \leq \int_0^t \| u^\alpha \| \| \partial_s u_b \| ds + \int_0^t \| \rho^{-1} u^\alpha \|_L^2(\Gamma_\delta) \| \rho \|_\infty \| \nabla u_b \|_\infty ds \]
\[ \leq K \delta \hat{\beta} + K \frac{\alpha}{\beta} \int_0^t \| \nabla u^\alpha \|_L^2(\Gamma_\delta) ds. \]  
(17)

Now we estimate the last term \( I_4 \). Integrate by part in time, we have
\[ I_4 = 2\alpha(\nabla u^\alpha(t), \nabla (\bar{u} - u_b(t))) - 2\alpha(\nabla u^\alpha(0), \nabla (\bar{u}(0) - u_b(0))) \]
\[ - 2\alpha \int_0^t (\nabla u^\alpha, \partial_s (\bar{u} - u_b)) ds \]
\[ \leq K \delta + K \alpha \hat{\beta} \delta^{-\frac{\hat{\beta}}{2}} (\alpha \| \nabla u^\alpha \|) + o(1) + \alpha \int_0^t (\nabla u^\alpha, \partial_s u_b) ds \]
\[ \leq K \alpha + K \beta \hat{\beta} + o(1), \]

since \( \alpha \| \nabla u^\alpha \|_{L^\infty(0,T;L^2(\Omega))} \leq K \) and (9). Now we can prove that we can fix \( \delta \) and denote
\[ F_\delta = \alpha \int_0^T \| \nabla u^\alpha \|_{L^2(\Gamma_\delta)}^2 ds. \]

In fact, let \( \beta_* \) be a minimizer of \( K_1 \beta_\frac{1}{\beta} + K_2 \hat{\beta}^{-1} F_\delta \), i.e. \( \beta_* \) satisfies
\[ \beta_\frac{1}{\beta} = K_3 \hat{\beta}^{-1} F_\delta, \]
(19)

where \( K \) is independent of \( \alpha, \delta \). (19) have a solution \( \beta_* = K_3 F_\delta^{\frac{1}{2}} \).

In the case of \( \beta_* \geq \beta \),
then
\[ \delta_* = \frac{\alpha}{U \beta_*} \leq \frac{\alpha}{U \beta} = \delta, \]
and hence
\[ K_1 \beta_* \frac{1}{\beta} + K_2 \beta_*^{-1} F_\delta \leq K_1 \beta_\frac{1}{\beta} + K_2 \beta_*^{-1} F_\delta \leq K_1 K_3 \beta_*^{\frac{1}{2}} \]
(20)
since \( F_\delta \) is increasing in \( \delta \).

On the other hand \( \beta_* \leq \beta \), then \( K_3 F_\delta^{\frac{1}{2}} \leq \beta \), hence
\[ \alpha \int_0^T \| \nabla u^\alpha \|_{L^2(\Gamma_\delta)}^2 ds \leq K_4 \beta \hat{\beta} \]

Therefore,
\[ K_1 \beta \frac{1}{\beta} + K_2 \hat{\beta}^{-1} \alpha \int_0^T \| \nabla u^\alpha \|_{L^2(\Gamma_\delta)}^2 ds \leq K_5 \beta \hat{\beta}. \]

Therefore, from (15)-(18), we obtain that
\[ \| u^\alpha - \bar{u} \|^2 \leq K \int_0^t \| u^\alpha - \bar{u} \|^2 ds + K \delta \hat{\beta} + K \alpha \]
\[ + K \max \left( \beta \hat{\beta}, \left( \alpha \int_0^T \| \nabla u^\alpha \|_{L^2(\Gamma_\delta)}^2 dt \right)^{\frac{1}{2}} \right) + o(1). \]
Since $\frac{\alpha}{\delta} \to 0$ and $\delta \to 0$ as $\alpha \to 0$, from Gronwall’s inequality, it follows
\[
\sup_{t \in [0,T]} \| u^\alpha \cdot \bar{u} \| \to 0, \text{ as } \alpha \to 0.
\]

By the Poincaré inequality, (iv) implies (vii). After subtly changing the argument above, one shows that (vii) implies (i). We finish this proof.

In Theorem 3.1, we note that the width of boundary layer is bigger than $O(\alpha)$. Compared with the case of Navier-Stokes equations, the energy dissipative term is large different with the Euler-Voigt model, which depends the time derivative of dissipative term. Therefore the conditions (iv)-(vii) can be improved as in the following theorem.

**Theorem 3.3.** The following conditions are equivalent to those of Theorem 3.1:

(i')
\[
\lim_{\alpha \to 0} \sup_{t \in [0,T]} \alpha \| \nabla u^\alpha(t) \|_{L^2(\Gamma_{C,\alpha})}^2 = 0;
\]

If, in addition, the solution $u$ is in $C^1([0,T]; C^2(\Omega))$ then condition (i') and the conditions in Theorem 3.1 are equivalent to the following condition:

(vii')
\[
\lim_{\alpha \to 0} \sup_{t \in [0,T]} \alpha^{-1} \| u^\alpha(t) \|_{L^2(\Gamma_{C,\alpha})}^2 = 0,
\]

where $C$ is independent of $\alpha$.

**Proof.** From (11) of the proof in Theorem 3.1, (i) implies (iv'). The formula (17) and (18) arrive that (iv') implies (i). Thanks Poincaré inequality, it is easy to obtain that (iv') implies (vii'). Making use of the arguments in (17) and (18), one infer that (vii') implies (i). We obtain the following difference between (1) and (3)
\[
\begin{aligned}
\partial_t (u - \bar{u}) - \alpha \Delta u_t + [(u \cdot \nabla) u - (\bar{u} \cdot \nabla) \bar{u}] + \nabla (p - \bar{p}) &= 0, \quad \text{in } \Omega \times (0,T), \\
\nabla \cdot (u - \bar{u}) &= 0, \quad \text{in } \Omega \times (0,T).
\end{aligned}
\]

For any $\omega \in D(\partial \Omega \times [0,T])$, we can construct a divergence-free smooth function $\omega_{C,\alpha}$ coincides with $\omega$ on $\partial \Omega \times [0,T]$ as in Lemma 3.2 and also satisfying all estimates in (13). From the argument in in [1], it is easy to see that (i) implies (viii) or (viii'). Multiply the difference equations above, by the $u - \bar{u}$, by applying the conditions (viii) or (viii'), we obtains the (i) holds. This completes the proof. 

\[\square\]
4. Kato’s conditions by tangential derivatives. In this section, we present
the Kato’s conditions by tangential derivatives only, a small quantity in classical
boundary layer theory rather than the whole gradient as above. In this case, we
can treat more general and physically more interesting boundary condition here.

Theorem 4.1. Let \( u_0 \in (H^s(\Omega))^N \cap H(s \geq 3) \). Assume that
\[
\|u_0^\alpha - u_0\| \to 0, \alpha \|\nabla u_0^\alpha\|^2 \to 0 \quad \text{as} \quad \alpha \to 0, \tag{21}
\]
the solution \( u^\alpha \) of Euler-Voigt equation (1) with initial velocity \( u_0^\alpha \) and the solution \( \bar{u} \) of the following Euler equation with initial velocity \( u_0 \), then the following conditions
(i) to (iii) are equivalent. (all limiting relations refer to \( \alpha \to 0 \))

(i) \( u^\alpha(t) \to \bar{u}(t) \) in \( L^2(\Omega) \), uniformly in \( t \in [0,T] \);

(ii) There exists \( \delta = \delta(\alpha) \to 0 \), such that
\[
\frac{\alpha}{\delta} \to 0, \quad \alpha \int_0^T \|\nabla_\tau u^\alpha_\tau\|^2_{L^2(\Gamma_\delta)} \, dt \to 0;
\]
where \( \nabla_\tau u^\alpha_\tau \) is the tangential derivatives of tangential components of \( u^\alpha \).

(iii) There exists \( \delta = \delta(\alpha) \to 0 \), such that
\[
\frac{\alpha}{\delta} \to 0, \quad \alpha \int_0^T \|\nabla_\tau u^\alpha_n\|^2_{L^2(\Gamma_\delta)} \, dt \to 0;
\]
where \( \nabla_\tau u^\alpha_n \) is the tangential derivatives of normal component of \( u^\alpha \).

Proof. The proof follows the idea as in Theorem 3.1 or in [26].

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E-mail address: abzang@jxycu.edu.cn