1. A Markov chain on the symmetric group

Let $S_n$, $n \geq 3$ denote the symmetric group on $n$ letters and let $(i, j)$ denote the transposition which swaps $i$ and $j$. We use conventions so that left multiplication acts on values and right multiplication acts on positions.

Define a matrix $P = (p_{w,v})_{w,v \in S_n}$

$$p_{w,v} = \begin{cases} x_{w^{-1}(i+1)} & \text{if } v = (i, i+1)w < w. \\ x_{w^{-1}(1)} & \text{if } v = (1, n)w > w. \\ * & \text{if } w = v. \\ 0 & \text{otherwise.} \end{cases}$$

where $*$ is chosen so that $\sum_{v \in S_n} p_{w,v} = 1$ for each $w \in S_n$. If the $x_i$’s are non-negative real numbers summing to at most 1, then we can think of $P$ as defining a Markov chain on $S_n$. When we set $x_i = 1/n$, we obtain the Markov chain defined in [Lam] Section 3.

**Proposition 1.** The matrix $P^T - I$ has a one-dimensional nullspace for generic values of $x$. In particular, when the $x_i$’s are nonnegative real numbers summing to at most 1, the Markov chain defined by $P$ has a unique stationary distribution.

**Proof.** When all $x_i$ are positive and sum to at most 1, then it follows from [Lam] Proposition 1 that we have an irreducible and aperiodic Markov chain on $S_n$, and thus we have a unique invariant distribution. If we treat $x_1, \ldots, x_{n-2}, x_{n-1}$ as variables, then a basis of the nullspace of $P^T - I$ can be written as a rational function in the $x_i$. This nullspace must be one-dimensional. \hfill \Box

Let $\{\zeta(w) \in \mathbb{Q}(x_1, x_2, \ldots, x_{n-1}) \mid w \in W\}$ denote a vector spanning the nullspace of Proposition 1 which we normalize by setting

$$\zeta(w_0) = x_1^{1+2+\cdots+n-2}x_2^{1+2+\cdots+n-3} \cdots x_{n-2}.$$
Suppose \( w = w_1 w_2 \cdots w_n \in S_n \). Let \( \chi(w) = (w_1 + 1)(w_2 + 1) \cdots (w_n + 1) \in S_n \) be the cyclic shift of \( w \), where the letters of \( \chi(w) \) are interpreted modulo \( n \). The following follows immediately from the definitions.

**Proposition 2.** For each \( w \in W \), we have \( \zeta(\chi(w)) = \zeta(w) \).

### 2. Schubert polynomials

We fix notations concerning Schubert polynomials. Let \( \partial_i \) denote the divided difference operator on polynomials in \( x_1, x_2, \ldots \), defined by

\[
\partial_i f(x_1, x_2, \ldots) = \frac{f(x_1, \ldots, x_i, x_{i+1}, \ldots) - f(x_1, \ldots, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}}.
\]

For the longest permutation \( w_0 \in S_n \), we first define

\[
\mathcal{S}_{w_0}(x_1, x_2, \ldots) := x_1^{n-1} x_2^{n-2} \cdots x_{n-1}.
\]

Next for \( w \in S_n \), we let \( w^{-1} w_0 = s_{i_1} s_{i_2} \cdots s_{i_k} \) be a reduced expression. Then

\[
\mathcal{S}_w := \partial_{i_1} \partial_{i_2} \cdots \partial_{i_k} \mathcal{S}_{w_0}.
\]

The polynomial \( \mathcal{S}_w \) does not depend on the choice of reduced expression. Furthermore, \( \mathcal{S}_w \) does not depend on which symmetric group \( w \) is considered an element of.

### 3. Conjectures

Our main conjecture is

**Conjecture 1.** In increasing strength:

1. Each \( \zeta(w) \) is a polynomial.
2. Each \( \zeta(w) \) is a polynomial with nonnegative integer coefficients.
3. Each \( \zeta(w) \) is a nonnegative integral sum of Schubert polynomials \( \mathcal{S}_u(x_1, x_2, \ldots) \).

Let \( \eta(w) \) denote the largest monomial that can be factored out of \( \zeta(w) \). By Proposition \( \mathcal{S}_w \), \( \eta(w) = \eta(\chi(w)) \). Write \( [m] \) to denote \( \{0, 1, 2, \ldots, m\} \).

**Conjecture 2** (Monomial factor). Assume Conjecture \( \mathcal{S}_w \). The map \( w \mapsto \eta(w) \) is an \( n \)-to-1 map from \( S_n \) to

\[
\left\{ x_1^{a_1} x_2^{a_2 + \cdots + a_{n-2}} \cdots x_n^{a_{n-2}} \mid (a_1, a_2, \ldots, a_{n-2}) \in [n-2] \times [n-3] \times \cdots \times [1] \right\}.
\]

Moreover, \( \eta(w) = x_1^{a_1} x_2^{a_2 + \cdots + a_{n-2}} \cdots x_n^{a_{n-2}} \) is given by

\[
a_i = \# \{ k \in [i+2, n] \mid w_k \in [w_i, w_{i+1}] \},
\]

where \([w_i, w_{i+1}]\) denotes a cyclic subinterval of \([n]\).

**Conjecture 3** (Special value).

\[
\zeta(\text{id}) = \mathcal{S}_{123\cdots n} \mathcal{S}_{1n23\cdots(n-1)} \mathcal{S}_{1(n-1)n23\cdots(n-2)} \cdots \mathcal{S}_{134\cdots n2}.
\]

**Conjecture 4** (Special Schubert factors). Consider the letters of \( w \in S_n \) in (cyclic) order. If there is an adjacent string of letters \( 1, 2 \), then \( \zeta(w) \) is a multiple of the Schubert polynomial \( \mathcal{S}_{134\cdots n2} \). More generally, if there is an adjacent string of letters \( 1, 2, 3, \ldots, k \), then \( \zeta(w) \) is a multiple of the Schubert polynomial \( \mathcal{S}_{1(k+1)(k+2)\cdots n23\cdots k} \).
4. Data

We provide experimental data supporting these conjectures.

4.1. \( n = 3 \). See Figure 1

4.2. \( n = 4 \). Using Proposition 2, we need only provide data for permutations \( w \) where \( w_1 = n \). In the following we use \( a = x_1 \), \( b = x_2 \), and \( c = x_3 \). We also write the answers as products of Schubert polynomials. Since a product of Schubert polynomials is also a nonnegative linear combination of Schubert polynomials this supports Conjecture 1(3).

\[
\begin{array}{c|c|c}
\text{\( w \)} & \zeta(w) & \text{\( \eta(w) \)} \\
4123 & (a^2 + ab + b^2)(ab + ac + bc) & S_{1423}S_{1342} \\
4132 & (a^2 + ab + b^2)ab & S_{1423}S_{231} \\
4213 & (a + b + c)a^2b & S_{1243}S_{321} \\
4231 & (a^2b + a^2c + ab^2 + abc + b^2c)a & S_{1432}S_{21} \\
4312 & (ab + ac + bc)a^2 & S_{1342}S_{312} \\
4321 & a^3b & S_{4213}
\end{array}
\]

Note that \( a^2b + a^2c + ab^2 + abc + b^2c \) is the only non-trivial factor which is not a symmetric polynomial.

4.3. \( n = 5 \). For \( n = 5 \) we write our answers as products and sums of Schubert polynomials, multiplied by the monomial factor \( \eta(w) \).
| $w$  | $\zeta(w)$                                                                 |
|------|---------------------------------------------------------------------------|
| 51234| $\mathcal{S}_{15234} \mathcal{S}_{14523} \mathcal{S}_{13452}$           |
| 51243| $\mathcal{S}_{15234} \mathcal{S}_{14523} abc$                          |
| 51324| $\mathcal{S}_{15234} \mathcal{S}_{12453} a^2 b^2 c$                     |
| 51342| $\mathcal{S}_{15234} \mathcal{S}_{14532} ab$                           |
| 51423| $\mathcal{S}_{15234} \mathcal{S}_{13452} a^2 b^2$                       |
| 51432| $\mathcal{S}_{15234} \mathcal{S}_{1234} a^2 b^2 c$                      |
| 52134| $\mathcal{S}_{12534} \mathcal{S}_{13452} a^3 b^2$                      |
| 52143| $\mathcal{S}_{12534} a^3 b^2 c$                                         |
| 52314| $(\mathcal{S}_{15432} + \mathcal{S}_{164235}) a^2 b c$                  |
| 52413| $(\mathcal{S}_{1753246} + \mathcal{S}_{265314} + \mathcal{S}_{2743156} + \mathcal{S}_{356214} + \mathcal{S}_{364215} + \mathcal{S}_{365124}) a$ |
| 52431| $(\mathcal{S}_{164325} + \mathcal{S}_{25431}) a^2 b$                    |
| 53124| $\mathcal{S}_{15243} a^3 b c$                                           |
| 53142| $\mathcal{S}_{12543} a^3 b c$                                           |
| 53214| $\mathcal{S}_{12354} a^3 b c$                                           |
| 53241| $\mathcal{S}_{13542} a^3 b c$                                           |
| 53412| $\mathcal{S}_{15423} \mathcal{S}_{13452} a^2$                          |
| 53421| $\mathcal{S}_{15423} a^3 b c$                                           |
| 54123| $\mathcal{S}_{14523} \mathcal{S}_{13452} a^3$                          |
| 54132| $\mathcal{S}_{14523} a^3 b c$                                           |
| 54213| $\mathcal{S}_{12453} a^3 b^2 c$                                         |
| 54231| $\mathcal{S}_{14532} a^3 b$                                             |
| 54312| $\mathcal{S}_{13452} a^3 b^2$                                           |
| 54321| $a^3 b^3 c$                                                             |

**REFERENCES**

[Lam] T. Lam, The shape of a random affine Weyl group element and random core partitions, preprint, 2011.

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109 USA
E-mail address: tfylam@umich.edu

Department of Mathematics, University of California, Berkeley, CA 94705 USA
E-mail address: williams@math.berkeley.edu