THE CENTER MANIFOLD THEOREM
FOR CENTER EIGENVALUES
WITH NON-ZERO REAL PARTS

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Abstract
We define center manifold as usual as an invariant manifold, tangent
to the invariant subspace of the linearization of the mapping defining a
continuous dynamical system, but the center subspace that we consider
is associated with eigenvalues with small but not necessarily zero real
parts. We prove existence and smoothness of such center manifold assuming
that certain inequalities between the center eigenvalues and the
rest of the spectrum hold. The theorem is valid for finite-dimensional
systems, as well as for infinite-dimensional systems provided they sat-
ify an additional condition. We show that the condition holds for the
Navier-Stokes equation subject to appropriate boundary conditions.

Key words: center manifold theorem, center manifold reduction,
Navier-Stokes equation
Introduction

Investigation of bifurcations in complex dynamical systems, e.g., hydrodynamic or magnetohydrodynamic ones, can be simplified by reducing dimension of the state space. This can be done by the center manifold (CM) [8] or Lyapunov-Schmidt [6] reductions. CM is an invariant manifold, tangent to an invariant subspace of the linearization of the mapping defining the continuous dynamical system. We will refer to the eigenvalues associated with the invariant subspace as center eigenvalues. In conventional definitions of CM employed in applications (e.g., [1, 3, 14]) imaginary center eigenvalues were assumed [4, 8, 16, 17]. Here we consider expanded CM, allowing center eigenvalues with small but not necessarily zero real parts.

Our interest in such CM stems from the works [12, 13], where they were applied for investigation of bifurcations in an ABC forced hydrodynamic system. While the 6-dimensional reduced system, obtained by the conventional CM reduction, reproduced only the first bifurcation of the trivial steady state [2], the 8-dimensional reduced system constructed with the use of an expanded CM reproduced well the complex sequence of bifurcations of the original hydrodynamic system [12, 13].

To the best of our knowledge, the variants of definitions of CM, where center eigenvalues with real parts unequal to one were allowed, were introduced before only for discrete finite-dimensional dynamical systems [5, 15]. Nontrivial problems in the theory of CM are the questions of their existence and smoothness. Theorems, guaranteeing existence and smoothness of CM for the discrete finite-dimensional dynamical systems, where real parts of center eigenvalues are close to one, are available [5, 15], but they cannot be generalized by the standard technique [9] or other simple arguments to cover the continuous infinite-dimensional case.

Our goal is to present a strict mathematical proof of the expanded CM (for the sake of simplicity, we will henceforth refer to them without the qualifier “expanded”) theorem, which is applicable for hydrodynamic system. First, the theorem is proved for finite-dimensional systems. Second, we introduce a class of infinite-dimensional systems, for which the theorem remains valid. Finally, we show that the Navier-Stokes equation belongs to this class, if it is considered for appropriate boundary conditions and provided certain inequalities hold for eigenvalues of the linearization of the equation near the trivial steady state.

The theory which we develop here involves modifications of the proof of the CM theorem for finite-dimensional systems [16] (pp. 91-123), and of generalization of this theorem for infinite-dimensional systems [17] (pp. 126-160). We use a similar notation and follow the presentation of the papers. If a theorem or a lemma proved in these papers is applied here in its original form, we present only its statement. Our presentation is otherwise complete.

\footnote{If a continuous system is transformed into a discrete one by time discretization [9], eigenvalues of linearization increase by 1, and thus in discrete dynamical systems center eigenvalues have real parts close to 1.}
1. The center manifold theorem for center eigenvalues with non-vanishing real parts. Finite-dimensional systems

1.1. The global CM theorem

We consider differential equations of the form

\[ \dot{x} = f(x) \equiv Ax + \tilde{f}(x), \]  

where \( x \in \mathbb{R}^n \), \( f : \mathbb{R}^n \to \mathbb{R}^n \) is a \( C^k \) vector field, \( k \geq 1 \), \( f(0) = 0 \), \( A = Df(0) \in \mathcal{L}(\mathbb{R}^n) \) and hence \( \tilde{f}(0) = 0 \), \( D\tilde{f}(0) = 0 \). For each \( x \in \mathbb{R}^n \) we denote by \( t \to \tilde{x}(t, x) \) the unique solution to (1), satisfying \( x(0) = x \); the maximal interval of its existence is denoted by \( J(x) \). For an open \( \Omega \subset \mathbb{R}^n \) and \( x \in \Omega \) denote by \( J_0(x) \) the maximal interval of \( t \) such that \( \tilde{x}(t, x) \in \Omega \).

Let the spectrum of the operator \( A, \sigma(A) \subset \mathbb{C} \), be decomposed as a disjoint union of the stable spectrum \( \sigma_s \), the center spectrum \( \sigma_c \) and the unstable spectrum \( \sigma_u \), where

\[
\begin{align*}
\sigma_s &= \{ \lambda \in \sigma | \operatorname{Re}\lambda < -\Lambda^- \}, \\
\sigma_c &= \{ \lambda \in \sigma | -\Lambda^- \leq \operatorname{Re}\lambda \leq \Lambda^+ \}, \\
\sigma_u &= \{ \lambda \in \sigma | \operatorname{Re}\lambda > \Lambda^+ \}
\end{align*}
\]

and \( \Lambda^\pm \geq 0 \). Denote by \( X_s, X_c \) and \( X_u \) (the stable, the center and the unstable subspaces) the subspaces of \( \mathbb{R}^n \) spanned by the generalized eigenvectors of \( A \) associated with the respective sets of eigenvalues; thus \( \mathbb{R}^n = X_s \oplus X_c \oplus X_u \). We call \( X_h = X_s \oplus X_u \) the hyperbolic subspace. Denote by \( \pi \) projections onto corresponding subspaces:

\[
\pi_s : \mathbb{R}^n \to X_s, \quad \pi_c : \mathbb{R}^n \to X_c, \quad \pi_u : \mathbb{R}^n \to X_u
\]

and \( \pi_h = \pi_s + \pi_u \).

Denote

\[
\begin{align*}
\beta_+ &= \min\{ \operatorname{Re}\lambda \mid \lambda \in \sigma_u \} \\
\alpha_+ &= \max\{ \operatorname{Re}\lambda \mid \lambda \in \sigma_c \} \\
\alpha_- &= -\min\{ \operatorname{Re}\lambda \mid \lambda \in \sigma_c \} \\
\beta_- &= -\max\{ \operatorname{Re}\lambda \mid \lambda \in \sigma_s \}
\end{align*}
\]

(\( \beta_+ = +\infty \) if \( \sigma_u = \emptyset \), and \( \beta_- = +\infty \) if \( \sigma_s = \emptyset \)). From (2), \( \beta_+ > \alpha_+ \geq 0 \) and \( \beta_- > \alpha_- \geq 0 \).

**Lemma 1.** For any \( \epsilon > 0 \) there exists a constant \( M(\epsilon) \) such that the following inequalities hold:

\[
\begin{align*}
&\| e^{At\pi_c} \| \leq M(\epsilon)e^{(\alpha_+ + \epsilon)t}, \quad \forall t \geq 0, \\
&\| e^{At\pi_c} \| \leq M(\epsilon)e^{-(\alpha_- + \epsilon)t}, \quad \forall t \leq 0, \\
&\| e^{At\pi_u} \| \leq M(\epsilon)e^{(\beta_+ - \epsilon)t}, \quad \forall t \geq 0, \\
&\| e^{At\pi_u} \| \leq M(\epsilon)e^{-(\beta_- - \epsilon)t}, \quad \forall t \leq 0.
\end{align*}
\]
The proof is identical to the proof of Lemma 1.1 in [16] and it is omitted here.

Denote by $C^k_b(X;Y)$ the set of all bounded mappings from a Banach space $X$ to a Banach space $Y$ with the norm

$$
\|w\|_{C^k_b} = \max_{0 \leq j \leq k} |w|_j
$$

where

$$
|w|_j = \sup_{x \in X} \|D^j w(x)\|,
$$

and $C^k_b(X;X)$ is denoted by $C^k_b(X)$.

Consider a system

$$
\dot{x} = Ax + g(x),
$$

where $x \in \mathbb{R}^n$, $A \in \mathcal{L}(\mathbb{R}^n)$ and $g \in C^k_b(\mathbb{R}^n)$ for some $k \geq 1$. Denote by $\tilde{x}_g(t,x)$ the solution to (5), satisfying $x(0) = x$. Since $g$ is bounded, it is defined for all $t$.

**Theorem 1.** There exists $\delta_0 > 0$ (depending on $A \in \mathcal{L}(\mathbb{R}^n)$) such that for each $g \in C^1_b(\mathbb{R}^n)$ with $|g|_1 < \delta_0$ the following holds:

(i) Existence and invariance: the set

$$
M_c = \{ x \in \mathbb{R}^n \mid \sup_{t \in \mathbb{R}} \|\pi_h \tilde{x}_g(t,x)\| < \infty \}
$$

(which is called global CM) is invariant for (5). It is also a $C^0$-submanifold in $\mathbb{R}^n$. More precisely, there exists $\psi \in C^0_b(X_c;X_h)$ such that

$$
M_c = \{ x_c + \psi(x_c) \mid x_c \in X_c \};
$$

(ii) Uniqueness: if $\phi \in C^0_b(X_c;X_h)$ is such that a manifold

$$
W_\phi = \{ x_c + \phi(x_c) \mid x_c \in X_c \}
$$

is invariant under (5), then $W_\phi = M_c$ and $\phi = \psi$.

The proof of invariance and uniqueness of $M_c$ is the same as in the proof of Theorem 2.1 in [16], and we do not present it. The proof of existence of $M_c$ follows.

**Lemma 2.** Suppose $g \in C^1_b(\mathbb{R}^n)$, $\eta_+ \in (\alpha_+, \beta_+)$ and $\eta_- \in (\alpha_-, \beta_-)$. Then

$$
M_c = \{ x \in \mathbb{R}^n \mid \max_{t>0} (\sup_{t>0} e^{-\eta_+ t} \|\tilde{x}_g(t,x)\|, \sup_{t<0} e^{\eta_- t} \|\tilde{x}_g(t,x)\|) < \infty \}. \tag{8}
$$

**Proof.** The proof is based on the variation-of-constants formula

$$
\tilde{x}_g(t,x) = e^{A(t-t_0)} \tilde{x}_g(t_0,x) + \int_{t_0}^t e^{A(t-\tau)} g(\tilde{x}_g(\tau,x)) d\tau, \tag{9}
$$

which holds for all $t, t_0 \in \mathbb{R}$. 

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First, we show that (6) is a subset of (8). Since \( \eta_+ > \alpha_+ \geq 0 \) and \( \eta_- > \alpha_- \geq 0 \), for \( x \) from the set (6)
\[
\sup_{t>0} e^{-\eta_+ t} \| \pi_h \bar{x}_g(t, x) \| < \infty, \quad \text{and} \quad \sup_{t<0} e^{\eta_- t} \| \pi_h \bar{x}_g(t, x) \| < \infty. \tag{10}
\]
Application of \( \pi_c \) to (9) with \( t_0 = 0 \) yields
\[
\pi_c \bar{x}_g(t, x) = e^{A t} \pi_c x + \int_0^t e^{A(t-\tau)} \pi_c g(\bar{x}_g(\tau, x)) d\tau. \tag{11}
\]
Lemma 1 implies that for \( t > 0 \)
\[
\| \pi_c \bar{x}_g(t, x) \| \leq M(\eta_+ - \alpha_+) e^{\eta_+ t} \| x \| + M(\eta_+ - \alpha_+) \| g \|_0 \int_0^t e^{\eta_+(t-\tau)} d\tau
\leq M(\eta_+ - \alpha_+) e^{\eta_+ t} (\| x \| + \eta_+^{-1} \| g \|_0)
\]
and hence
\[
\sup_{t>0} e^{-\eta_+ t} \| \pi_c \bar{x}_g(t, x) \| < \infty. \tag{12}
\]
It can be shown similarly that
\[
\sup_{t<0} e^{\eta_- t} \| \pi_c \bar{x}_g(t, x) \| < \infty,
\]
which together with (10) and (12) yields
\[
\max(\sup_{t>0} e^{-\eta_+ t} \| \bar{x}_g(t, x) \|, \sup_{t<0} e^{\eta_- t} \| \bar{x}_g(t, x) \|) < \infty.
\]

Conversely, assume that \( x \in \mathbb{R}^n \) is from the set (8). Project (9) onto \( X_u \) to obtain
\[
\pi_u \bar{x}_g(t, x) = e^{A(t-t_0)} \pi_u \bar{x}_g(t_0, x) + \int_{t_0}^t e^{A(t-\tau)} \pi_u g(\bar{x}_g(\tau, x)) d\tau. \tag{13}
\]
For a fixed \( t \in \mathbb{R} \), \( t_0 \geq \max(0, t) \) and \( \epsilon \in (0, \beta_+ - \eta_+) \) Lemma 1 and (8) imply
\[
\| e^{A(t-t_0)} \pi_u \bar{x}_g(t_0, x) \| \leq M(\epsilon) e^{(\beta_+ - \epsilon)(t-t_0)} C e^{\eta_+ t_0}
= M(\epsilon) C e^{(\beta_+ - \epsilon)t} e^{-(\beta_+ - \eta_+)t_0}. \tag{14}
\]
The r.h.s. of (14) tends to zero when \( t_0 \to \infty \). Consequently, in the limit \( t_0 \to \infty \) (13) takes the form
\[
\pi_u \bar{x}_g(t, x) = -\int_t^\infty e^{A(t-\tau)} \pi_u g(\bar{x}_g(\tau, x)) d\tau, \quad \forall t \in \mathbb{R}. \tag{15}
\]
Thus, for any \( \epsilon \in (0, \beta_+) \) and any \( t \in \mathbb{R} \)
\[
\| \pi_u \bar{x}_g(t, x) \| \leq M(\epsilon) \| g \|_0 \int_t^\infty e^{(\beta_+-\epsilon)(t-\tau)} d\tau = (\beta_+ - \epsilon)^{-1} M(\epsilon) \| g \|_0. \tag{16}
\]
Similarly, for any $\epsilon \in (0, \beta_-)$ and any $t \in \mathbb{R}$

$$\pi_s \tilde{x}_g(t, x) = \int_{-\infty}^{t} e^{A(t-\tau)} \pi_s g(\tilde{x}_g(\tau, x)) d\tau$$  \hspace{1cm} (17)

and

$$\|\pi_s \tilde{x}_g(t, x)\| \leq (\beta_- - \epsilon)^{-1} M(\epsilon) \|g\|_0.$$  \hspace{1cm} (18)

Together, (16) and (18) imply (6). The proof of Lemma 2 is completed.

**Definition 1.** For a vector $\eta = (\eta_+, \eta_-)$, where $\eta_+, \eta_- \geq 0$, $Y_\eta$ is the Banach space

$$Y_\eta = \{y \in C^0(\mathbb{R}; \mathbb{R}^n) \mid \|y\|_\eta = \sup_{t \in \mathbb{R}} e^{-\eta(t)} \|y(t)\| < \infty\},$$  \hspace{1cm} (19)

where

$$\eta(t) = \begin{cases} \eta_+ t & \text{if } t \geq 0, \\ -\eta_- t & \text{if } t < 0. \end{cases}$$  \hspace{1cm} (20)

The inequality $\zeta \geq \eta$ means that $\zeta_+ \geq \eta_+$ and $\zeta_- \geq \eta_-$, and $\zeta > \eta$ – that $\zeta_+ > \eta_+$ and $\zeta_- > \eta_-$. $Y_\eta$ are a scale of Banach spaces: if $\zeta \geq \eta$, then $Y_\eta \subset Y_\zeta$, and the embedding is continuous

$$\|y\|_\zeta \leq \|y\|_\eta, \quad \forall y \in Y_\eta.$$  

In this notation, the manifold (8) can be expressed as

$$M_c = \{x \in \mathbb{R}^n \mid \tilde{x}_g(\cdot, x) \in Y_\eta\}$$

$$= \{y(0) \mid y \in Y_\eta \quad \text{and } y \text{ solves (5)}\}$$  \hspace{1cm} (21)

for some

$$\eta \in (\alpha_+, \beta_+) \times (\alpha_-, \beta_-).$$

The scale of Banach spaces $Y_\eta$, $\eta > 0$, employed in the proof of the conventional CM theorem [16], coincides with the scale (19), where $\eta = \eta_+ = \eta_+$; the spaces for $0 < \eta < \beta$ are employed, where $\beta = \min(\beta_+, \beta_-)$ (cf. (19) for $\alpha_+ = \alpha_- = 0$).

As it was shown in the proof of Lemma 2, (11), (15) and (17) hold for $\tilde{x}_g(t, x)$ on the CM. Summing up these equations we find that $x \in \mathbb{R}^n$ belongs to $M_c$ if and only if $\forall t \in \mathbb{R}$

$$\tilde{x}_g(t, x) = e^{At} \pi_c x + \int_0^t e^{A(t-\tau)} \pi_c g(\tilde{x}_g(\tau, x)) d\tau + \int_{-\infty}^{+\infty} B(t-\tau) g(\tilde{x}_g(\tau, x)) d\tau,$$

where $B : \mathbb{R} \rightarrow \mathcal{L}(\mathbb{R}^n)$ is

$$B(t) = \begin{cases} -e^{At} \pi_u, & \text{if } t < 0, \\ e^{At} \pi_s, & \text{if } t \geq 0. \end{cases}$$  \hspace{1cm} (22)
Lemma 1 implies that for any $\epsilon > 0$

$$
\|B(t)\| < \begin{cases} 
M(\epsilon)e^{(\beta_+ - \epsilon)t}, & \forall t < 0, \\
M(\epsilon)e^{-(\beta_- - \epsilon)t}, & \forall t > 0.
\end{cases}
$$  \quad (23)

**Lemma 3.** Suppose $g \in C^1_b(R^n)$, $\eta \in (\alpha_+, \beta_+) \times (\alpha_-, \beta_-)$ and $y \in Y_{\eta}$. Then $y$ is a solution to (5) if and only if there exists $x_c \in X_c$, such that for any $t \in R$

$$
y(t) = e^{At}x_c + \int_0^t e^{A(t-\tau)}\pi_c g(y(\tau))d\tau + \int_{-\infty}^{+\infty} B(t-\tau) g(y(\tau))d\tau.
$$  \quad (24)

The proof is identical to the proof of Lemma 2.8 in [16] and it is omitted here.

Let $\Sigma$ be the set of all $(x_c, y) \in X_c \times Y_{\eta}$ such that (24) holds; (21) implies

$$
M_c = \{y(0) \mid (x_c, y) \in \Sigma\} = \{x_c + \pi_h y(0) \mid (x_c, y) \in \Sigma\},
$$  \quad (25)

since $\pi_c y(0) = x_c$ for any $(x_c, y) \in \Sigma$. To determine the set $\Sigma$, rewrite (24) in the form

$$
y = Sx_c + KG(y)
$$  \quad (26)

where the following notation is used:

$$
Sx_c : R \to R^n, \quad (Sx_c)(t) = e^{At}x_c \quad \forall x_c \in X_c;
$$

$$
G(y) : R \to R^n, \quad G(y)(t) = g(y(t)) \quad \text{for each function } y : R \to R^n;
$$

$$
Ky : R \to R^n, \quad Ky(t) = \int_0^t e^{A(t-\tau)}\pi_c y(\tau)d\tau + \int_{-\infty}^{+\infty} B(t-\tau) y(\tau)d\tau
$$  \quad (27)

for such functions $y : R \to R^n$ that the integrals are defined.

**Lemma 4.** $S$ is a bounded operator from $X_c$ to $Y_{\eta}$ for any $\eta_+ > \alpha_+$ and $\eta_- > \alpha_-$. 

**Proof.** Lemma 1 implies that for any $\eta_+ > \alpha_+$

$$
\|e^{At}x_c\| \leq M(\eta_+ - \alpha_+)^t\|x_c\|, \quad \forall t > 0,
$$

and for any $\eta_- > \alpha_-$

$$
\|e^{At}x_c\| \leq M(\eta_- - \alpha_-)^{-t}\|x_c\|, \quad \forall t < 0.
$$

Hence

$$
\|Sx_c\|_{\eta} \leq \max(M(\eta_+ - \alpha_+), M(\eta_- - \alpha_-))\|x_c\|, \quad \forall x_c \in X_c.
$$

**Lemma 5.** If $g \in C^0_b(R^n)$, then $G$ maps $C^0(R; R^n)$ into $C^0_b(R; R^n)$, and $G$ maps each $Y_{\eta}$, $\eta \geq 0$, into itself. If $g \in C^1_b(R^n)$, then for any $\eta > 0$

$$
\|G(y_1) - G(y_2)\|_{\eta} \leq |g|_1\|y_1 - y_2\|_{\eta}, \quad \forall y_1, y_2 \in Y_{\eta}.
$$
Proof. The first part is obvious. If \( g \in C_b^1(\mathbb{R}^n) \), \( y_1, y_2 \in Y_\eta \), then
\[
\sup_{t > 0} e^{-\eta_+ t} \| G(y_1) - G(y_2) \| = \sup_{t > 0} e^{-\eta_+ t} \| g(y_1(t)) - g(y_2(t)) \|
\]
\[
\leq \sup_{t > 0} e^{-\eta_+ t} \| g \|_1 \| y_1(t) - y_2(t) \| \leq \| g \|_1 \| y_1(t) - y_2(t) \| \eta.
\]
A similar inequality holds for negative \( t \). Thus, by virtue of (19) and (20), the proof is complete.

Lemma 6. For any \( \eta \in (\alpha_+, \beta_+) \times (\alpha_-, \beta_-) \) the operator \( K : Y_\eta \to Y_\eta \) is bounded; there exists a continuous function \( \gamma : (\alpha_+, \beta_+) \times (\alpha_-, \beta_-) \to \mathbb{R} \) such that
\[
\| K \eta \| \leq \gamma(\eta), \quad \forall \eta \in (\alpha_+, \beta_+) \times (\alpha_-, \beta_-).
\] (28)

Proof. Suppose \( \eta_+ \in (\alpha_+, \beta_+) \), \( \eta_- \in (\alpha_-, \beta_-) \), \( y \in Y_\eta \) and \( t > 0 \). The definition of \( K \) (27) and bounds (23) imply
\[
e^{-\eta_+ t} \| Ky(t) \| \leq \| y \| \eta \sup_{t > 0} e^{-\eta_+ t} \left[ \int_0^t \| e^{A(t-\tau)} \pi_c \| e^{\eta_+ \tau} d\tau + \int_0^0 \| B(t-\tau) \| e^{-\eta_+ \tau} d\tau \right] + \int_0^t \| B(t-\tau) \| e^{-\eta_+ \tau} d\tau + \int_{t}^{+\infty} \| B(t-\tau) \| e^{-\eta_+ \tau} d\tau
\]
\[
\leq \| y \| \eta \sup_{t > 0} \left[ \int_0^+ \| e^{A\tau} \pi_c \| e^{-\eta_+ \tau} d\tau + \int_0^0 \| B(\tau) \| e^{-\eta_+ \tau} d\tau \right] + \int_0^t \| B(t-\tau) \| e^{-\eta_+ \tau} d\tau + \int_{t}^{+\infty} \| B(t-\tau) \| e^{-\eta_+ \tau} d\tau
\]
\[
\leq \| y \| \eta \left[ M(\epsilon_1)(\eta_+ - \alpha_+ - \epsilon_1)^{-1} + M(\epsilon_2)(\beta_- - \eta_- - \epsilon_2)^{-1} + M(\epsilon_3)(\beta_+ + \eta_+ - \epsilon_3)^{-1} + M(\epsilon_4)(\beta_+ - \eta_+ - \epsilon_4)^{-1} \right],
\]
if \( \epsilon_i \) satisfy \( \eta_+ - \alpha_+ - \epsilon_1 > 0 \), \( \beta_- - \eta_- - \epsilon_2 > 0 \), \( \beta_+ + \eta_+ - \epsilon_3 > 0 \) and \( \beta_+ - \eta_+ - \epsilon_4 > 0 \). Similarly, for \( t < 0 \)
\[
e^{\eta_- t} \| Ky(t) \| \leq \| y \| \eta \left[ \int_{-\infty}^0 \| e^{A\tau} \pi_c \| e^{\eta_- \tau} d\tau + \int_{-\infty}^0 \| B(\tau) \| e^{-\eta_- \tau} d\tau \right] + \int_{-\infty}^0 \| B(\tau) \| e^{-\eta_- \tau} d\tau + \int_{0}^{+\infty} \| B(\tau) \| e^{\eta_- \tau} d\tau
\] (30)
Thus \( K \in \mathcal{L}(Y_\eta) \). The norm of \( K \) is bounded by the function \( \gamma(\eta_+, \eta_-) \), defined as the maximum of the sums (29) and (30); this is a continuous function of the two arguments. The proof of Lemma 6 is complete.
Lemma 7. If $\eta \in (\alpha_+, \beta_+) \times (\alpha_-, \beta_-)$ and $g \in C_b^1(\mathbb{R}^n)$ is such that

$$\kappa = \|K\| |g|_1 < 1$$

(31)

then $(I - K \circ G)$ is a homeomorphism on $Y_{\eta}$, whose inverse $\Psi : Y_{\eta} \to Y_{\eta}$ is Lipschitzian with the Lipschitz constant $\kappa$, and

$$\Sigma = \{ (x_c, \Psi(Sx_c)) \mid x_c \in X_c \}.$$  

(32)

The proof is identical to the proof of Lemma 2.12 in [16].

We finish now the proof of Theorem 1. For a $\gamma(\eta)$ satisfying (28), denote

$$\delta_0 = \sup_{\eta \in (\alpha_+, \beta_+) \times (\alpha_-, \beta_-)} \gamma(\eta)^{-1}.$$  

If $g \in C_b^1(\mathbb{R}^n)$ and $|g|_1 < \delta_0$, there exists $\eta \in (\alpha_+, \beta_+) \times (\alpha_-, \beta_-)$ such that $|g|_1 \gamma(\eta) < 1$. By (28) this implies (31) and therefore (32) holds. Combining it with (25), obtain (7) with $\psi : X_c \to X_h$ defined by

$$\psi(x_c) = \pi_h \Psi(Sx_c)(0), \quad \forall x_c \in X_c.$$  

(33)

Since $\Psi$ is continuous, $\psi$ is also continuous. Moreover, since $\Psi = (I - K \circ G)^{-1}$ by definition,

$$\Psi(Sx_c) = Sx_c + KG(\Psi(Sx_c)).$$

From the definitions of $S$, $G$ and $K$ it follows that

$$\psi(x_c) = \int_{-\infty}^{+\infty} B(-\tau) g(Sx_c)(\tau) d\tau,$$

Thus, the bounds (23) imply

$$\|\psi(x_c)\| < (M(\epsilon_+)(\beta_+ - \epsilon_+)^{-1} + M(\epsilon_-)(\beta_- - \epsilon_-)^{-1}) |g|_0$$

$$\forall x_c \in X_c, \forall \epsilon_+ \in (\alpha_+, \beta_+), \forall \epsilon_- \in (\alpha_-, \beta_-).$$

Finally, note that $\psi \in C_b^0(\mathbb{R}^n)$ is globally Lipschitzian, because $\Psi$ is globally Lipschitzian and (33) holds. The Theorem is proved.

1.2. Smoothness of CM

The Theorem can be applied to study bifurcations, if a CM is sufficiently smooth. In the sequel we prove smoothness of the manifold under certain additional assumptions.

Theorem 2. Let the spectrum of $A \in \mathcal{L}(\mathbb{R}^n)$ in (5) be split as $\sigma(A) = \sigma_u \cup \sigma_c \cup \sigma_s$ in accordance with (2), with $\alpha_\pm$ and $\beta_\pm$ (see (3) ) satisfying

$$\alpha_+ < \beta_+ / l \quad \text{and} \quad \alpha_- < \beta_- / l$$

\[9\]
for some \( l \geq 1 \). Then for each \( k, \ 1 \leq k \leq l \), there exists \( \delta_k \in (0, \delta_0] \) such that if \( g \in C^k_b(\mathbb{R}^n) \) and \( |g|_1 < \delta_k \), then the unique global center manifold \( M_c \) of (5) is \( C^k \). More precisely, the mapping \( \psi \) constructed in Theorem 1 belongs to \( C^k_b(X_c; X_h) \).

Since \( \psi(x_c) = \pi_h \Psi(Sx_c)(0) \), it is sufficient to show that the mapping \( \Psi \) constructed in Lemma 7 is \( C^k \). Then the smoothness might be established by application of the implicit function theorem to the equation (26), if the operator \( G \) were \( C^k \). The difficulty is that as a mapping from \( Y_\eta \) into itself \( G \) is not in general differentiable. But \( G \in C^k(Y_\eta, Y_\zeta) \), if \( g \in C^k_b(\mathbb{R}^n) \) and \( \zeta > k\eta \). The proof of Theorem 2 employs the following Lemma.

**Lemma 8.** Suppose \( g \in C^k_b(\mathbb{R}^n) \) for some \( k \geq 1 \). Let \( \eta, \zeta \in (\alpha_+, \beta_+) \times (\alpha_-, \beta_-) \) be such that \( \zeta > k\eta \). Suppose

\[
\kappa = \sup_{\xi \in [\eta, \zeta]} \|K\|_1 |g|_1 < 1. \tag{34}
\]

Then the mapping \( \Psi : Y_\eta \rightarrow Y_\eta \) constructed in Lemma 7 belongs to \( C^k(Y_\eta, Y_\zeta) \). More precisely,

\[
\Psi - J_{\eta, \zeta} \in C^k_b(Y_\eta; Y_\zeta),
\]

where \( J_{\eta, \zeta} \) is the embedding of \( Y_\eta \) into \( Y_\zeta \).

The proof of this Lemma coincides with the proof of Lemma 3.2 of [16] (pp. 104-115) after replacement of \( \eta, \zeta \) and \( \xi \) by \( \eta, \zeta \) and \( \xi \), respectively. We do not repeat it here.

**Proof of Theorem 2.** For each \( k \geq 1 \) denote

\[
\delta_k = \sup_{\eta \in (\alpha, \beta/k)} \inf_{\xi \in [\eta, \kappa\eta]} \gamma(\xi)^{-1},
\]

where \( \gamma \) is the function constructed in Lemma 6. If \( g \in C^k_b(\mathbb{R}^n) \) and \( |g|_1 < \delta_k \), then there exists \( \eta \in (\alpha, \beta/k) \) such that \( |g|_1 < \inf\{\gamma(\xi)^{-1} \mid \xi \in [\eta, \kappa\eta]\} \). Since \( \gamma \) is continuous, this implies existence of \( \zeta \in (k\eta, \beta) \) such that \( |g|_1 < \inf\{\gamma(\xi)^{-1} \mid \xi \in [\eta, \zeta]\} \). By (28) this implies (34). It follows from Lemma 8 that \( \Psi \in C^k(Y_\eta, Y_\zeta) \), all its derivatives being globally bounded. Since \( S : X_c \rightarrow Y_\eta \) is a bounded linear operator (Lemma 4), the mapping \( x_c \rightarrow \Psi(Sx_c) \) is also \( C^k(X_c, Y_\zeta) \) with all its derivatives globally bounded. Hence (33) implies that \( \psi \in C^k_b(X_c; X_h) \). The proof is complete.

### 1.3. The local CM theorem

Theorems 1 and 2 hold for all functions \( g \), bounded by certain constants. Now let us return to the equation (1), where \( f \) does not satisfy this condition.

**Theorem 3.** Suppose \( f \in C^k(\mathbb{R}^n), \ k \geq 1 \), and \( f(0) = 0 \). Split the set of eigenvalues of \( A = Df(0) \) in agreement with (2): \( \sigma(A) = \sigma_u \cup \sigma_c \cup \sigma_s \). Assume

\[
\alpha_+ < \beta_+/k \quad \text{and} \quad \alpha_- < \beta_-/k \tag{35}
\]
\( \alpha_\pm \) and \( \beta_\pm \) are defined by (3). Then there exists \( \psi \in C^k_b(X_c, X_h) \) \((X_c \text{ and } X_h \text{ denote the respective center and hyperbolic subspaces})\) and an open neighborhood \( \Omega \) of the origin in \( \mathbb{R}^n \) such that

(i) \( \psi(0) = 0 \text{ and } D\psi(0) = 0; \)
(ii) the manifold

\[ W_\psi = \{ x_c + \psi(x_c) \mid x_c \in X_c \} \]

is locally invariant for (1), i.e.

\[ \tilde{x}(t, x) \in W_\psi, \quad \forall x \in W_\psi \cap \Omega, \ \forall t \in J_\Omega(x) \]

(iii) if \( x \in \Omega \text{ and } J_\Omega(x) = \mathbb{R}, \) then \( x \in W_\psi. \)

To prove Theorem 3, apply Theorems 1 and 2 to the system

\[ \dot{x} = Ax + \tilde{f}_\rho(x), \quad (36) \]

where

\[ \tilde{f}_\rho(x) = \tilde{f}(x)\chi(\rho^{-1}x), \quad \forall x \in \mathbb{R}^n, \]

and \( \chi \) is a smooth cut-off function \( \chi : \mathbb{R}^n \rightarrow \mathbb{R} \) with the following properties:

(i) \( 0 \leq \chi \leq 1, \forall x \in \mathbb{R}^n; \)
(ii) \( \chi(x) = 1, \text{ if } \|x\| \leq 1; \)
(iii) \( \chi(x) = 0, \text{ if } \|x\| \geq 2. \)

The constant \( \rho \) can be chosen small enough, so that the system (36) satisfies conditions of the Theorems. Equations (1) and (36) coincide in \( \Omega = \{ x \in \mathbb{R}^n \mid \|x\| \leq \rho \}. \)

If \( x \in \Omega \text{ and } J_\Omega(x) = \mathbb{R}, \) then \( \tilde{x}(\cdot, x) = \tilde{x}_\rho(\cdot, x) \) is a bounded solution to (36) and, according to (6), belongs to its global CM, thus implying \( x \in W_\psi. \)

If the unstable spectrum is empty, the CM is attracting. The proof is the same as in the case of a conventional CM theorem.

2. CM theorem for infinite-dimensional systems

Let \( X, Y \) and \( Z \) be Banach spaces with \( X \) continuously embedded in \( Y, \) and \( Y \) continuously embedded in \( Z. \) Consider a differential equation

\[ \dot{x} = Ax + g(x), \quad (37) \]

where \( A \in \mathcal{L}(X, Z) \) and \( g \in C^k(X, Y), \ k \geq 1. \)

Definition 2. For a vector \( \eta = (\eta_+, \eta_-) \), where \( \eta_\pm \geq 0, \) and a Banach space \( E, \) define a Banach space \( BC^\eta(\mathbb{R}, E): \)

\[ BC^\eta(\mathbb{R}, E) = \{ w \in C^0(\mathbb{R}; E) \mid \|w\|_E = \sup_{t \in \mathbb{R}} e^{-\eta(t)} \|w(t)\|_E < \infty \}, \quad (38) \]

where \( \eta(t) \) is defined by (20).
Assume the operator $A$ satisfies the following hypothesis (H):

There exists a continuous projection $\pi_c \in \mathcal{L}(Z, X)$ onto a finite-dimensional subspace $Z_c = X_c \subset X$, such that

$$A\pi_c x = \pi_c Ax, \quad \forall x \in X,$$

and such that for

$$Z_h = (I - \pi_c)(Z), \quad X_h = (I - \pi_c)(X), \quad Y_h = (I - \pi_c)(Y),$$

$$A_c = A|_{X_c} \in \mathcal{L}(X_c), \quad A_h = A|_{X_h} \in \mathcal{L}(X_h, Z_h),$$

the following statements hold:

(i) there exist $\alpha_+ \geq 0$ and $\alpha_- \geq 0$ such that

$$-\alpha_- \leq \Re \lambda \leq \alpha_+ \quad \forall \lambda \in \sigma(A_c);$$

(ii) there exist $\beta_-$ and $\beta_+, \beta_\pm > k\alpha_\pm$, such that for any $\eta = (\eta_-, \eta_+), \eta_\pm \in [0, \beta_\pm)$, and for any $f \in BC^\eta(R, Y_h)$ the linear problem

$$\dot{x}_h = A_h x_h + f(t), \quad x_h \in BC^\eta(R, X_h)$$

has a unique solution $x_h = K_h f$, where $K_h \in \mathcal{L}(BC^\eta(R, Y_h), BC^\eta(R, X_h))$ and

$$\|K_h\|_\eta \leq \gamma(\eta)$$

for a continuous function $\gamma: [0, \beta_-) \times [0, \beta_+) \to \mathbb{R}$. 

**Lemma 8.** Assume (H) and $g \in C^0_b(X, Y)$. Let $\tilde{x} : R \to X$ be a solution of (37), and let $\eta = (\eta_-, \eta_+) \in (\alpha_-, \beta_-) \times (\alpha_+, \beta_+)$. Then the following statements are equivalent:

(i) $\tilde{x} \in BC^\eta(R, X)$;

(ii) $\tilde{x} \in BC^\xi(R, X), \quad \forall \xi = (\xi_-, \xi_+), \xi_\pm > \alpha_\pm$;

(iii) $\pi_h \tilde{x} \in C^\eta_b(R, X_h)$.

The proof is identical to that of Lemma 1 in [17].

**Lemma 9.** Assume (H) and $g \in C^0_b(X, Y)$. Let $\tilde{x} \in BC^\eta(R, X)$ for some $\eta = (\eta_-, \eta_+) \in (\alpha_-, \beta_-) \times (\alpha_+, \beta_+)$. Then $\tilde{x}$ is a solution of (37) if and only if

$$\tilde{x}(t) = e^{At} \pi_c \tilde{x}(0) + \int_0^t e^{A_c(t-s)} \pi_c g(\tilde{x}(s)) ds + K_h(\pi_h g(\tilde{x}))(t), \quad \forall t \in R.$$ 

The Lemma is identical to Lemma 2 of [17].

**Theorem 4.** Assume (H). Then there exist $\delta_0 > 0$ such that for all $g \in C^0_b(X, Y)$, which are globally Lipschitz with the Lipschitz constant $|g|_{\text{Lip}}$, satisfying

$$|g|_{\text{Lip}} < \delta_0,$$

(39)
there exist a unique \( \psi \in C^0_b(X_c, X_h) \) possessing the property that for all \( \tilde{x} : \mathbb{R} \to X \) the following statements are equivalent:

(i) \( \tilde{x} \) is a solution of (37) and \( \tilde{x} \in BC^\eta(\mathbb{R}, X) \) for some \( \eta = (\eta_-, \eta_+) \in (\alpha_-, \beta_-) \times (\alpha_+, \beta_+) \);

(ii) \( \pi_h \tilde{x}(t) = \psi(\pi_c \tilde{x}(t)) \) for all \( t \in \mathbb{R} \) and \( \pi_c \tilde{x} : \mathbb{R} \to X_c \) is a solution of the equation

\[
\dot{x}_c = A_c x_c + \pi_c g(x_c \psi(x_c)).
\]

As pointed out in [17], the proof is similar to the proof of Theorem 1 in [16] and is the same as the proof of Theorem 1 in the present paper.

Theorem 5. Assume \( (H) \) and \( g \in C^0_b(X, Y) \) satisfying (39), the problem

\[
\begin{cases}
\dot{x} = Ax + g(x) \\
\pi_c x(0) = x_c, \ x \in BC^\eta(\mathbb{R}, X)
\end{cases}
\]

with \( \eta = (\eta_-, \eta_+) \in (\alpha_-, \beta_-) \times (\alpha_+, \beta_+) \) has for each \( x_c \in X_c \) a unique solution

\[
\tilde{x}(t, x_c) = \tilde{x}_c(t, x_c) + \psi(\tilde{x}_c(t, x_c)),
\]

where \( \tilde{x}_c(t, x_c) \) is the unique solution of (40) satisfying \( x_c(0) = x_c \).

As in the finite-dimensional case (Section 1), the set

\[
M_c = \{ x_c + \psi(x_c) | x_c \in X_c \} \subset X
\]

is called the global center manifold of (37).

Theorem 5. Assume \( (H) \). Then for any \( l \leq k \) there exist \( \delta_l > 0 \), such that if \( g \in C^0_b(X, Y) \cap C^l_b(V_\rho, Y) \), with \( V_\rho = \{ x \in X | \| \pi_h x \| < \rho \} \) and \( \rho > \| K_h \|_0 \| \pi_h g \|_0 \),

\[
|g|_{\text{Lip}} < \delta_l
\]

the mapping \( \psi \) given by Theorem 1 belongs to the space \( C^l_b(X_c, X_h) \).

Similarly to Theorem 4, the proof follows the proof of Theorem 2 for finite-dimensional systems.

Theorem 6. Assume \( (H) \), \( g \in C^k_b(X, Y) \) for \( k \geq 1 \), \( g(0) = 0 \) and \( Dg(0) = 0 \). Then there exist a neighborhood \( \Omega \) of the origin in \( X \) and a mapping \( \psi \in C^k_b(X_c, X_h) \) with \( \psi(0) = 0 \) and \( D\psi(0) = 0 \) such that the following statements hold:

(i) if \( \tilde{x} : I \to X_c \) is a solution of (40) such that \( \tilde{x}(t) = \tilde{x}_c(t) + \psi(\tilde{x}_c(t)) \in \Omega \) for all \( t \in I \), then \( \tilde{x} : I \to X \) is a solution of (37);

(ii) if \( \tilde{x} : \mathbb{R} \to X \) is a solution of (37) such that \( \tilde{x}(t) \in \Omega \) for all \( t \in \mathbb{R} \), then

\[
\pi_h \tilde{x}(t) = \psi(\pi_c \tilde{x}(t)), \ \forall t \in \mathbb{R},
\]
and \( \pi_c \tilde{x} : \mathbb{R} \to X_c \) is a solution of (40).

Unlike in the cases of Theorems 4 and 5, the proof is different from the one for finite-dimensional systems, since the cut-off function \( \chi \in C^k_c(X, \mathbb{R}) \) used in the proof of Theorem 3 does not always exist for a general Banach space \( X \). The proof for infinite-dimensional systems given in [17] involves construction of a cut-off function from the finite-dimensional \( X_c \) to \( \mathbb{R} \).

3. The Navier-Stokes equation

Consider the Navier-Stokes equation

\[
\frac{\partial \mathbf{v}}{\partial t} = \mathbf{v} \times (\nabla \times \mathbf{v}) - \nabla p + \nu \Delta \mathbf{v} + \mathbf{f}
\]  

subject to the incompressibility condition

\[
\nabla \cdot \mathbf{v} = 0,
\]

where the force \( \mathbf{f} \) is a smooth bounded function, defined in a bounded domain \( \Omega \subset \mathbb{R}^3 \) with a smooth boundary \( \partial \Omega \).

We assume one of the following boundary conditions:

space-periodic:

\[
\mathbf{v}(\mathbf{x}) = \mathbf{v}(\mathbf{x} + \mathbf{T}), \quad \mathbf{T} \in \mathbb{R}^3;
\]

no-slip:

\[
\mathbf{v}|_{\partial \Omega} = 0.
\]

Our theory is equally applicable to other commonly used boundary conditions, e.g. stress-free and periodicity in one (the Taylor-Couette problem) or two directions (in a layer).

Denote by \( \mathcal{F} \) the space of functions, satisfying the boundary conditions (44) or (45).

Let \( \mathbf{v}_0 \) be a steady solution of (42), (43) with (44) or (45). For \( \mathbf{v} = \mathbf{v}_0 + \mathbf{w} \), (42) reduces to

\[
\frac{\partial \mathbf{w}}{\partial t} = A\mathbf{w} + N(\mathbf{w}),
\]

where

\[
A\mathbf{w} = \pi_0(\nu \Delta \mathbf{w} + \mathbf{v}_0 \times (\nabla \times \mathbf{w}) + \mathbf{w} \times (\nabla \times \mathbf{v}_0)),
\]

\[
N(\mathbf{w}) = \pi_0(\mathbf{w} \times (\nabla \times \mathbf{w})).
\]

We set

\[
Z = \{ \mathbf{w} \in \mathcal{F} \cap (L_2(\Omega))^3 | \nabla \cdot \mathbf{w} = 0 \},
\]

denote by \( \pi_0 \) the orthogonal projection of \( (L_2(\Omega))^3 \) onto \( Z \), and define

\[
X = Z \cap (H_2(\Omega))^3, \quad Y = Z \cap (H_1(\Omega))^3.
\]
It is shown in [17] that \( A \in \mathcal{L}(X, Z) \) and \( N \in C^\infty(X, Y) \). Since \( A \) is an elliptic operator, for any constant \( C \) it has a finite number of eigenvalues with \( \text{Re} \lambda > C \) (counting with multiplicities).

Theorem 6 is applicable to the Navier-Stokes equation, if the equation satisfies the hypothesis \((H)\). Decompose the spectrum of the operator \( A \), \( \sigma(A) \subset \mathbb{C} \), into a disjoint union of the stable spectrum \( \sigma_s \), the center spectrum \( \sigma_c \) and the unstable spectrum \( \sigma_u \), where

\[
\sigma_s = \{ \lambda \in \sigma \mid \text{Re} \lambda < -\beta_- \}, \\
\sigma_c = \{ \lambda \in \sigma \mid -\alpha_- \leq \text{Re} \lambda \leq \alpha_+ \}, \\
\sigma_u = \{ \lambda \in \sigma \mid \text{Re} \lambda > \beta_+ \}
\]

with \( \beta_+ > k \alpha_+ \geq 0 \). Due to the properties of the operator \( A \) stated above, it can be easily examined for a particular bifurcation by computing several eigenvalues with the largest real parts for the system linearized in the vicinity of the steady state, whether for a given \( k \) such constants \( \alpha_\pm \) and \( \beta_\pm \) can be found that (47) can be constructed.

If the decomposition (47) can be constructed, the Navier-Stokes equation satisfies the following hypothesis \((\Sigma)\):

There exist \( \alpha'_\pm \geq 0 \) and \( \beta'_\pm > k \alpha'_\pm \) such that

(i) \( \sigma(A) \cap [a'_-, a'_+] \times i\mathbb{R} \) consists of a finite number of isolated eigenvalues, each associated with a finite-dimensional generalized eigenspace;

(ii) \( ([\beta'_-, a'_-] \cup [a'_+, \beta'_+]) \times i\mathbb{R} \subset \rho(A) \);

(iii) there exist constants \( \omega_0 > 0 \), \( C > 0 \) and \( \alpha \in [0, 1) \) such that for all \( \omega \in \mathbb{R} \) with \( |\omega| \geq \omega_0 \) we have \( i\omega \in \rho(A) \),

\[
\| (i\omega - A)^{-1} \|_{\mathcal{L}(Z)} \leq \frac{C}{|\omega|} \quad \text{and} \quad \| (i\omega - A)^{-1} \|_{\mathcal{L}(X,Y)} \leq \frac{C}{|\omega|^{1-\alpha}},
\]

where \( \rho(A) \) is the resolvent set of \( A \).

In [17] the hypothesis \((\Sigma)\) with \( \alpha'_\pm = 0 \) is employed, and it is shown that it holds for the Navier-Stokes equation. Their proof can be easily extended for our case (the condition (47) is required to allow for non-vanishing \( \alpha'_\pm \)). A trivial modification of arguments of [17] proves \((\Sigma) \Rightarrow (H)\). Thus Theorem 6 is applicable for the Navier-Stokes equation under the condition (47).

The equation (42) involves the parameter \( \nu \) (and possibly others, e.g. included into the force \( f \)). Denote by \( \mu \) all parameters of the system. CM can be made parameters dependent by the standard [8] extension of the system by considering the parameters as variables and setting \( \dot{\mu} = 0 \). Evidently, Theorem 6 is applicable to the extended system, if it is applicable to the original one.
Conclusion

We have proved CM theorems, including the one for infinite-dimensional systems, under less restrictive assumptions than those required by existing theorems. Although the proof is just a modification of the existing proofs [16, 17], the new variant of the theorem (Theorem 6) is important for applications, providing a more powerful tool for investigation of bifurcations in dynamical systems of infinite dimensions. Its advantage was demonstrated by applying our theorem to the ABC-forced Navier-Stokes equation [12, 13].

The demonstration that the theorem is applicable for the Navier-Stokes equation (if additional inequalities for eigenvalues of the linearization in a vicinity of a steady state hold) relies only on the fact that the linearization is an elliptic operator. Thus, it can be easily extended to accommodate other boundary conditions, the Rayleigh-Bénard convection, magnetohydrodynamic and other systems.

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