“Fuzzy” stochastic resonance: robustness against noise tuning due to non Gaussian noises

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Abstract

We have analyzed the phenomenon of stochastic resonance in a system driven by non Gaussian noises. We have considered both white and colored noises. In the latter case we have obtained a consistent Markovian approximation that enables us to get quasi-analytical results for the signal-to-noise ratio. As the system departs from Gaussian behavior, our main findings are: an enhance-

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ment of the response together with a marked robustness against noise tuning.
These remarkable findings are supported by extensive numerical simulations.
We also discuss the relation with some experiments in sensory systems.

Stochastic resonance (SR) has attracted considerable interest due, among other aspects, to its potential technological applications for optimizing the response to weak external signals in nonlinear dynamical systems, as well as to its connection with some biological mechanisms. There is a wealth of papers, conference proceedings and reviews on this subject, Ref. [1] being the most recent and complete one, showing the large number of applications in science and technology, ranging from paleoclimatology [2–4], to electronic circuits [4], lasers [5], chemical systems [6], and the connection with some situations of biological interest (noise-induced information flow in sensory neurons in living systems, influence in ion-channel gating, in visual perception or in the human blood pressure regulatory system) [7]. Recent works have shown a tendency, determined by the possible technological applications, pointing towards achieving an enhancement of the system response (by means of the coupling of several SR units in what conforms an extended medium [8–10]), or analyzing the possibility of making the system response less dependent on a fine tuning of the noise intensity [11], as well as different ways to control the phenomenon [12].

A majority of studies on SR have been made on bistable one-dimensional double-well systems, and in almost all cases, with very few exceptions [13], the noises are assumed to be Gaussian. However, some experimental results in sensory systems, particularly for one kind of crayfish [14] as well as recent results for rat skin [15], offer strong indications that the noise source in these systems could be non Gaussian.

In this letter we analyze the case of SR when the noise source is non Gaussian. We start sketching the study of a particular class of Langevin equations having non Gaussian stationary distribution functions [16], a work based on the generalized thermostatistics proposed by Tsallis [17] which has been successfully applied to a wide variety of physical systems [18].
We consider the following problem

\begin{align}
\dot{x} &= f(x, t) + g(x)\eta(t) \\
\dot{\eta} &= -\frac{1}{\tau} \frac{d}{d\eta} V_q(\eta) + \frac{1}{\tau} \xi(t)
\end{align}

where \( \xi(t) \) is a Gaussian white noise of zero mean and correlation \( \langle \xi(t)\xi(t') \rangle = 2D\delta(t-t') \), and \( V_q(\eta) \) is given by \[16\]

\[ V_q(\eta) = \frac{1}{\beta(q-1)} \ln[1 + \beta(q-1)\frac{\eta^2}{2}], \]

where \( \beta = \frac{\tau}{D} \). The function \( f(x, t) \) is derived from a potential \( U(x, t) \), consisting of a double well potential and a linear term modulated by \( S(t) = F \cos(\omega t) \) (\( f(x, t) = -\frac{\partial U}{\partial x} = -U'_0 + S(t) \)). For \( F = 0 \) this problem corresponds to the case of diffusion in a potential \( U_0(x) \), induced by \( \eta \), a colored non-Gaussian noise. The stationary probability distribution for the random variable \( \eta \) is given by \( P_{st}^q(\eta) = Z_q^{-1} \left[ 1 + \beta(q-1)\frac{\eta^2}{2} \right]^{-\frac{1}{2}} \) where \( Z_q \) is the normalization factor. Clearly, when \( q \to 1 \) we recover the limit of \( \eta \) being a Gaussian colored noise (Ornstein-Uhlenbeck process). For \( q < 1 \) the above distribution cuts-off the values of the \( \eta \) process to \( |\eta| < [(1-q)\beta^2_2]^{-1/2} \). It is easy to show that the stationary mean value and variance are given, respectively, by \( \langle \eta \rangle = 0 \) and \( \langle \eta^2 \rangle = [\beta(5-3q)]^{-1} \), the latter diverging for all \( q \geq 5/3 \).

Applying the path-integral formalism to the Langevin equations given by Eqs. (1-2), and making an adiabatic-like elimination procedure [19–21] it is possible to arrive at an effective Markovian approximation. The specific details are shown elsewhere [22]. Such an approximation yields the following Fokker–Planck equation (FPE) for the evolution of the probability \( P(x, t) \)

\[ \partial_t P(x, t) = -\partial_x [A(x)P(x, t)] + \frac{1}{2} \partial_x^2[B(x)P(x, t)]. \]

Where

\[ A(x) = \frac{U'_0}{\left(1 + \frac{\tau}{2D}(q-1)U'^2_0\right) + \tau U''_0[1 + \frac{\tau}{2D}(q-1)U'^2_0]} \]
and
\[ B(x) = D \left( \frac{1 + \frac{\tau}{2D}(q-1)U'_0[1 + \frac{\tau}{2D}(q-1)U'_0^2 + [1 - \frac{\tau}{2D}(q-1)U'_0^2]]^2}{\tau U'_0[1 + \frac{\tau}{2D}(q-1)U'_0^2 + [1 - \frac{\tau}{2D}(q-1)U'_0^2]]^2} \right)^2. \] (6)

The stationary distribution of the FPE in Eq. (4) is thus
\[ P^{st}(x) = \frac{\mathcal{N}}{B(x)} \exp\left[-\Phi(x)\right] \] (7)
where \( \mathcal{N} \) is the normalization factor, and
\[ \Phi(x) = 2 \int_x^\infty \frac{A(x')}{B(x')} dx'. \] (8)

The indicated FPE and its associated stationary distribution enable us to obtain the mean-first-passage-time (MFPT) through a Kramers like approximation [23]. This quantity is the necessary ingredient to work within the two-state model (TST) [3,24]. The MFPT to reach \( x = x_0 \) starting from \( x = a \) can be obtained from
\[ T(x_0) = \int_a^{x_0} dy \int_{-\infty}^{y} dz B(z)^{-1} e^{-\Phi(z)}, \] (9)

We focus on polynomial-like forms for the potential, adopting
\[ U(x) = \frac{x^4}{4} - \frac{x^2}{2} + S(t)x, \] (10)
setting \( S(t) = F \cos(\omega t) \), and assuming that \( \omega^{-1} \) is large compared to the characteristic relaxation times in both wells.

We follow the standard TST approach [24] in order to obtain \( R \), the signal to noise ratio (SNR). In our case the result for \( R \), defined as the ratio of the strength of the output signal to the broadband noise output evaluated at the signal frequency, is
\[ R = \frac{F^2 \pi}{2 T^3} \left[ \frac{dT}{dS(t)} \right]_0^2. \] (11)
with \( S(t) \) the applied signal.

We present the results for the SNR obtained evaluating Eq. (11). In Fig. 1a we depict the SNR vs. the noise intensity \( D_c \), for a fixed value of the time correlation \( \tau \) and various \( q \),
while Fig. 1b shows the equivalent plot for a fixed value of $q$ and several values of $\tau$. However, $D_c$, the noise intensity of the non-Gaussian process is related to the Gaussian white noise $D$ through the scaling $D_c = D[5 - 3q]^{-1}$, that was the scaling we used. In the former case the general trend is that the maximum of the SNR curve increases when $q < 1$, i.e. when the system departs from the Gaussian behavior. In the latter case the general trend agrees with the results for colored Gaussian noises [25], where it was shown that the increase of the correlation time induces a decrease of the maximum of SNR as well as its shift towards larger values of the noise intensity. The latter fact is a consequence of the suppression of the switching rate with increasing $\tau$. Both qualitative trends are confirmed by extensive numerical simulations of the system in Eqs. (1-2), which has been numerically integrated using a stochastic Runge-Kutta-type method [27]. The results were obtained averaging over 2000 trajectories (5000 trajectories for $\tau = 0$). Fig. 2a shows the simulation results for the same situation and parameters indicated in Fig. 1a. Here, in addition to the increase of the maximum of the SNR curve for values of $q < 1$, we also found another remarkable aspect that is not reproduced or predicted by our effective Markovian approximation. It is the fact that the maximum of the SNR curve flattens for lower values of $q$, indicating that the system, when departing from Gaussian behavior, requires a less fine tuning of the noise intensity in order to maximize its response to a weak external signal. Fig. 2b shows the simulation results for the same situation and parameters indicated in Fig. 1b. Again we found an agreement with the general behavior found for colored Gaussian noises [25].

In order to compare with some experiments where the need to use non-Gaussian noises has been suggested, we consider the results obtained for a crayfish in Ref. [14]. A qualitative comparison of our results in Figs. 1 and 2 with those given there shows that the agreement between theory and experiment does indeed improve by the use of non-Gaussian noises corresponding to $q < 1$. However, it is known that rather than applying our simple one-variable model, to the crayfish neural system under consideration it is better to use the celebrated FitzHugh-Nagumo one. In the same way as in Ref. [14], we define the model system through the set of equations
\[
\tau_v \dot{v} = v(v - 0.5)(1 - v) - w,
\]
\[
\tau_w \dot{w} = v - w + \epsilon \cos(\Omega t) + \eta(t),
\]
(12)

where \(v(t)\) is the variable associated to the action potential (in an activator-inhibitor model, it corresponds to the activator variable), and \(w(t)\) is the recovery (inhibitor) variable. The characteristic times for such variables are, respectively, \(\tau_v\) and \(\tau_w\). The recovery variable is submitted to a periodic signal (whose (small) amplitude we denote by \(\epsilon\) and its frequency by \(\Omega\)) and a noise source \(\eta(t)\). This random variable, now chosen as white (that is with \(\tau = 0\)), is distributed accordingly to the stationary pdf of Eq. (2). We have integrated numerically Eqs. (12) using a stochastic Runge-Kutta-type method with a time step of \(\Delta t = 1\)ms.

The time series \(v(t)\) was converted into a spike train (mimicking what happens in the nervous system). One spike occurs when \(v(t)\) exceeds the threshold potential \(b_c\). Each spike was modeled as a square wave of height \(V_s = 1\)V, and duration \(t_s = 3\)ms. It is not possible to have two successive spikes with a lag smaller than \(t_s\). The SNR was evaluated following the usual procedure on this modified signal.

Along all the simulations, we fixed \(\tau_v = 10^{-5}\)s and \(\tau_w = 10^{-2}\)s. While we considered the signal as having frequency \(\Omega = 55\)Hz and amplitude \(\epsilon = 0.03\)V. All these values agree with those from Ref. [14], however their results are a particular case of ours, i.e. \(q = 1\), for the noise source.

We have compared our simulations for \(R\), the SNR, as a function of noise standard deviation for different values of \(q\) and \(b_c\), the threshold potential, with the experimental data in [14]. Here again we have scaled the noise intensity according to \(D_n = D[5 - 3q]^{-1}\), where \(D_n\) is the intensity of the Non Gaussian noise while \(D\) corresponds to the pdf parameter. Figure 3 shows the results obtained for \(b_c = 0.18\)V. We see that the (Gaussian) case \(q = 1\) underestimates the measured SNR values for the whole range of noise intensities, while \(q = 0.75\) exhibits an overestimation for large noise values. However, if we adopt \(q = 0.47\) the system exhibits a nice agreement with experimental data both near the SNR peak and for large values of noise. When the threshold value is fixed at \(b_c = 0.15\)V as in [14] (results
not shown here), the case $q = 1$ corresponds exactly to the theoretical fit in [14]. From the comparison with $q \neq 1$ it becomes apparent that, although the fit is acceptable near the $R$ maximum, for large noise intensities the $q = 1$ case underestimate the values of $R$. By setting $q = 0.75$, we find a much better fit even in the region of large noises, and when $q = 0.30$ and for large values of the noise standard deviation, the results exhibit values above the measured ones.

It is worth remarking here that the experimental behavior in the low noise limit is not reproduced by any model. This is due to the spontaneous firing of neurons in the real crayfish neural system [14].

Summarizing, motivated by some experimental results in sensory systems [14,15], we have analyzed the problem of SR when the noise source is non Gaussian. We have chosen a non Gaussian noise source (white or colored) with a probability distribution based on the generalized thermostatistics [17]. In the colored case and making use of a path integral approach, we have obtained an effective Markovian approximation that allows us to get some analytical results. In addition, we have performed exhaustive numerical simulations. Even though the agreement between theory and numerical simulations is only partial and qualitative, the effective Markovian approximation turns out to be extremely useful to predict (qualitatively) general trends in the behavior of the system under study.

Our numerical and theoretical results indicate that: (i) for a fixed value of $\tau$, the maximum value of the SNR increases with decreasing $q$; (ii) for a given value of $q$, the optimal noise intensity (that one maximizing SNR) decreases with $q$ and its value is approximately independent of $\tau$; (iii) for a fixed value of the noise intensity, the optimal value of $q$ is independent of $\tau$ and in general it turns out that $q \neq 1$.

As we depart from Gaussian behavior (with $q < 1$), the SNR shows two main aspects: firstly its maximum as a function of the noise intensity increases, secondly it becomes less dependent on the precise value of the noise intensity. Both aspects are of great relevance for technological applications [1]. However, as was indicated in Ref. [13], non Gaussian noises could be an intrinsic characteristic in biological systems, particularly in sensory systems.
In addition to the increase in the response (SNR), the reduction in the need of tuning a precise value of the noise intensity is of particular relevance both in technology and in order to understand how a biological system can exploit this phenomenon. As an example of this, simulations of a FHN model with non Gaussian noise shows a behavior in better agreement with the experimental results in a crayfish sensory system than simulations made with a Gaussian noise source. The present results indicate that the noise model used here offers an adequate framework to analyze such a problem.

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**Figure 1:** Theoretical value of the SNR, $R$, vs. the noise intensity $D$, for: (a) $\tau = 0.1$ and the following values of $q = 0.25, 0.75, 1.0, 1.25$ (continuous line, from top to bottom). (b) $q = 0.75$ and the following values of $\tau = 0.25, 0.75, 1.5$ (broken line from top to bottom). We adopted $F = 0.1$ and $\omega = 0.1$.

**Figure 2:** Simulation results of the SNR, $R$, vs. the noise intensity $D$, for (a) $\tau = 0.1$ and the following values of $q = 0.25, 0.75, 1.0, 1.25$ (from top to bottom). (b) $q = 0.75$ and the following values of $\tau = 0.25, 0.75, 1.5$ (from top to bottom). We adopted $F = 0.1$ and $\omega = 0.1$.

**Figure 3:** The SNR, $R$, for the FHN model is plotted as a function of $\sqrt{2D}$ (the noise standard deviation). The different curves show the results for different $q$-values: $q = 1$ ($\bullet$), $q = 0.75$ ($\Diamond$), $q = 0.47$ ($\nabla$). In all the cases we fixed $b_c = 0.18V$, simulation results are compared with the experimental data ($\Box$).