AN EFFECTIVE VERSION OF BELYI’S THEOREM IN POSITIVE CHARACTERISTIC

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Abstract. The purpose of the present paper is to give an effective version of the noncritical $p$-tame Belyi theorem. That is to say, we compute explicitly an upper bound of the minimal degree of tamely ramified Belyi maps in positive characteristic which are unramified at a prescribed finite set of points.

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1. Introduction: Effective noncritical $p$-tame Belyi theorem

Let $X$ be a curve over a field $k$, which means, in the present paper, a geometrically connected proper smooth scheme over $k$ of dimension 1. A Belyi map on $X$ is a dominant $k$-morphism $f : X \to \mathbb{P}^1_k$ with $\text{Br}(f) \subseteq \{0, 1, \infty\}$, where $\mathbb{P}^1_k$ denotes the projective line over $k$ and $\text{Br}(f)$ denotes the set of branch points of $f$. A celebrated theorem of Belyi (cf. [2], [3]) asserts that if $k = \mathbb{Q}$ (i.e., an algebraic closure of the field of rational numbers), then $X$ always admits at least one Belyi map. This result has attracted much attention ever since Grothendieck noticed in his article (cf. [6]) that it implies surprising correspondences between curves defined over number fields and a certain class of bipartite graphs embedded in a topological surface called dessins d’enfants.

There are several variations and enhancements of Belyi’s theorem. For example, Mochizuki (cf. [14]) and Scherr-Zieve (cf. [16]) proved the noncritical enhancement, asserting the existence of Belyi maps which are unramified at a prescribed finite set of points. Khadjavi (cf. [11]) and Litcanu (cf. [12]) considered the effective versions, which give explicit upper bounds for the minimal degree of such morphisms in terms of height. The bounds can be interpreted, for example, bounding the number of edges of a dessin d’enfant given by the corresponding Belyi map (cf. [6]). On the other hand, we can find analogues of Belyi’s theorem in positive characteristic...
characteristic, i.e., the $p$-tame Belyǐ theorem (asserting the existence of a tamely ramified Belyǐ map) and the $p$-wild Belyǐ theorem (asserting the existence of a Belyǐ map which admits at most one branch point). See [11, 14, 10, 13, 17], and [18].

The purpose of the present paper is to give an effective version of the noncritical $p$-tame Belyǐ theorem in positive characteristic. (The $p$-wild case can be obtained immediately from the previous works, see Proposition 4.1) A point of our study is that although the effective bounds of Belyǐ maps obtained so far have been given only for $X = \mathbb{P}_k^1$, we compute, in our situation (i.e., the case of positive characteristic), an upper bound for an arbitrary curve. We shall state the main theorems. Denote by $\mathbb{F}_q$ the finite field with $q$ elements, where $q$ is a power of an odd prime $p$, and by $\overline{\mathbb{F}}_q$ its algebraic closure. Let $X$ be a curve over $\mathbb{F}_q$ and let $S, T$ be (possibly empty) finite sets of $\mathbb{F}_q$-rational points of $X$ with $S \cap T = \emptyset$, where $s := \sharp S, t := \sharp T$. For each field $k$ over $\mathbb{F}_q$, we write $X_k$, $S_k$, and $T_k$ for the base-changes over $k$ of $X$, $S$, and $T$ respectively. Here, by a $p$-tame Belyǐ map on $(X, S, T)$ over $k$, we shall mean a tamely ramified $k$-dominant morphism $f : X_k \to \mathbb{P}_k^1$ satisfying the following conditions:

$$f(S_k) \cup \text{Br}(f) \subseteq \{0, 1, \infty\}, \quad \{0, 1, \infty\} \cap f(T_k) = \emptyset.$$  

The $p$-tame Belyǐ degree of $(X, S, T)$ is defined as

$$(2) \quad \mathcal{B}(X, S, T) := \min \{\text{deg}(\phi) \mid \phi \text{ is a } p\text{-tame Belyǐ map } \phi \text{ on } (X, S, T) \text{ over } \overline{\mathbb{F}}_q\}$$

(where $\mathcal{B}(X, S, T) := \infty$ if there is no $p$-tame Belyǐ map on $(X, S, T)$ over $\overline{\mathbb{F}}_q$). The value $\mathcal{B}(X) := \mathcal{B}(X, \emptyset, \emptyset)$ (i.e., the minimal degree of tamely ramified Belyǐ maps on $X$) is simply referred as the $p$-tame Belyǐ degree of $X$. Then, the main assertion of the present paper is as follows. (Note that the upper bound of the degree asserted in the theorem is somewhat rough; by treating strictly various inequalities at each step in our proof, e.g., the inequalities appearing in Proposition 2.1 we can obtain a sharper bound.)

**Theorem A** (Effective version of the noncritical $p$-tame Belyǐ theorem).

There exists at least one $p$-tame Belyǐ map on $(X, S, T)$ over $\overline{\mathbb{F}}_q$, and moreover, the following inequality holds:

$$(3) \quad \mathcal{B}(X, S, T) \leq (2g + t + 1) \cdot (q^{5\log_q(10^2(2g+t+1)!\cdot(2g+t+s+1)^2\cdot(2g+s+1)^{2g+s+1})}\cdot L(6g+2t) - 1)^{6g+s+2t+1}.$$  

Here, $\lceil \cdot \rceil$ denotes the ceiling function and, for each nonnegative integer $m$, $L(m)$ denotes the least common multiple of $1, 2, \ldots, m$ (where $L(0) := 1$ if $m = 0$). In particular, if $S = T = \emptyset$, then we have

$$(4) \quad \mathcal{B}(X) \leq (2g + 1) \cdot (q^{5\log_q(10^2(2g+1)!\cdot(2g+1)^2\cdot(2g+1)^{2g+1})}\cdot L(6g) - 1)^{6g+1}.$$  

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2. First step of the proof

In this section, we prove Proposition 2.1 described below, which is a slightly strengthened version of [4], Proposition 8.1, and also is the main body of Theorem A. (The statement of Proposition 2.1 include the case where the base field is algebraically closed, but this will not be used in our proof of Theorem A.) Here, recall that a dominant morphism \( f : Y \to \mathbb{P}^1_k \) from a curve \( Y \) over a field \( k \) onto \( \mathbb{P}^1_k \) is called a simple covering if the discriminant \( \delta(f) \) of \( f \) is a simple divisor, i.e., has no multiple components after base-change over an algebraically closed field \( \mathbb{F} \) over \( k \). (If \( f \) is as above, then \( \deg(\delta(f)) = 2g + 2\deg(f) - 2 \), where \( g \) denotes the genus of \( Y \).) In particular, if \( k \) has characteristic \( \neq 2 \), then any simple covering is tamely ramified (cf. [4], Theorem 5.6).

Proposition 2.1.
Let \( k \) be a field of characteristic \( \neq 2 \), \( X \) a curve over \( k \) of genus \( g \) (\( \geq 0 \)), and \( S, T \) (possibly empty) finite sets of \( k \)-rational points of \( X \) with \( S \cap T = \emptyset \). Write \( s = \sharp S \) and \( t := \sharp T \). Also, let \( n \) be a positive integer with \( n \geq g + \max\{t, g\} \), and suppose that one of the following conditions (\(*\), (\(*\)*)) is satisfied:

(\(*\)) \( k \) is algebraically closed;
(\(*\)*) \( k = \mathbb{F}_q \) for a power \( q \) of a prime \( p \) such that there exists an integer \( A \) with \( A \geq 3 \) satisfying the inequality:

\[
q \geq \max \left\{ A^2 \cdot g^2, 10^2 \cdot n! \cdot (n^2 + s) \cdot \left( \frac{5A + 4}{9A - 6} \right)^n \right\}.
\]

Then, there exists a simple covering \( \zeta : X \to \mathbb{P}^1_k \) of degree \( n \) such that \( \zeta(S) \cap \zeta(T) = \emptyset \) and \( \zeta(T) \) consists of one point (if \( T \neq \emptyset \)) over which \( \zeta \) is unramified. In particular, if \( k = \mathbb{F}_q \) with \( q \geq 10^2 \cdot (2g + t + 1)! \cdot (2g + t + s + 1)^2 \cdot (5/6)^{2g+t+1} \), then (by considering the case where \( n = 2g + t + 1 \) and \( A = 3 \), we see that) there exists a simple covering \( \zeta : X \to \mathbb{P}^1_{\mathbb{F}_q} \) of degree \( 2g + t + 1 \) satisfying the above requirements.

Proof. For each positive integer \( r \), we shall denote by \( X^{(r)} \) the \( r \)-th symmetric product of \( X \) over \( k \); it is a geometrically connected proper smooth scheme over \( k \) of dimension \( r \), each of whose \( k \)-rational point corresponds to an effective divisor of degree \( r \) on \( X \). If, moreover, \( X^r \) denotes the product of \( r \) copies of \( X \) over \( k \), then we have the natural projection

\[
\pi_r : X^r \to X^{(r)}.
\]

Let us first consider the following lemma.

Lemma 2.2.
Suppose that (\(*\)*) is satisfied.

(i) There exists a subset \( R \) of \( X(\mathbb{F}_p) \setminus (S \cup T) \) with \( \sharp R = n - t - g \).

(ii) For each positive integer \( r \), the following inequalities hold:

\[
\frac{1}{7 \cdot r!} \cdot \left( 3 - \frac{2}{A} \right)^r \cdot q^r < \sharp X^{(r)}(\mathbb{F}_q) < \left( \frac{5}{3} + \frac{4}{3A} \right)^r \cdot q^r.
\]
Proof. Recall the Hasse-Weil theorem, asserting the following equality for each positive integer \( m \):

\[
\left| \#X(\mathbb{F}_{q^m}) - (q^m + 1) \right| \leq 2g\sqrt{q^m}.
\]

Then, assertion (i) follows from the above inequality for \( m = 1 \) and the inequality \( q+1-2g\sqrt{q} \geq n+s-g \) (= \((n-t-g)+s+t\)) arising immediately from the assumption (5).

In what follows, let us prove assertion (ii). The assumption \( q \geq A^2 \cdot g^2 \) \((\Rightarrow 2A \cdot q^m \geq 2g\sqrt{q^m} \text{ for any } m \geq 1)\) and (8) imply the inequalities:

\[
\left(1 - \frac{2}{A}\right) \cdot q^m + 1 \leq \#X(\mathbb{F}_{q^m}) \leq \left(1 + \frac{2}{A}\right) \cdot q^m + 1.
\]

Since \( X(r)(\mathbb{F}_q) \) is the set of effective divisors of degree \( r \) on \( X \), its number can be calculated by

\[
\#X(r)(\mathbb{F}_q) = \sum_{i=1}^{r} \frac{1}{i!} \sum_{a_1, \ldots, a_i \geq 1} \prod_{j=1}^{i} \frac{\#X(\mathbb{F}_{q^{a_j}})}{a_j}.
\]

By means of (10) and the first inequality of (9), we have the following sequence of inequalities:

\[
\#X(r)(\mathbb{F}_q) \geq \sum_{i=1}^{r} \frac{1}{i!} \sum_{a_1, \ldots, a_i \geq 1} \prod_{j=1}^{i} \frac{1}{a_j} \cdot \left(\left(1 - \frac{2}{A}\right) \cdot q^{a_j} + 1\right)
\]

\[
> \sum_{i=1}^{r} \frac{1}{i!} \sum_{a_1, \ldots, a_i \geq 1} \left(\frac{r}{\sum_{j=1}^{i} a_j}\right) \prod_{j=1}^{i} \left(1 - \frac{2}{A}\right) \cdot q^{a_j}
\]

\[
= q^r \cdot \sum_{i=1}^{r} \frac{1}{i!} \sum_{a_1, \ldots, a_i \geq 1} \left(1 - \frac{2}{A}\right)^i
\]

\[
= q^r \cdot \left(1 - \frac{2}{A}\right) \cdot \sum_{i=1}^{r} \frac{1}{i!} \cdot \left(1 - \frac{2}{A}\right)^{i-1} \cdot \left(\frac{r-1}{i-1}\right)
\]

\[
\geq q^r \cdot \left(1 - \frac{2}{A}\right) \cdot \sum_{i=1}^{r} \frac{2^{(r-1)-(i-1)}}{r!} \cdot \left(1 - \frac{2}{A}\right)^{i-1} \cdot \left(\frac{r-1}{i-1}\right)
\]

\[
\geq q^r \cdot \frac{1}{r!} \cdot \left(1 - \frac{2}{A}\right) \cdot \left(3 - \frac{2}{A}\right)^{r-1}
\]

\[
\geq q^r \cdot \frac{1}{7} \cdot \left(3 - \frac{2}{A}\right)^{r},
\]

where the second inequality follows from the geometric-harmonic inequality \( \left(\prod_{j=1}^{i} \frac{1}{a_j}\right)^{1/r} \geq \frac{r}{\sum_{j=1}^{r} a_j} \). On the other hand, let \( a := \frac{3}{5} \) if \( A = 3 \) and \( a := \frac{2}{3} \) if \( A \geq 4 \). Then, by (10) and the
second inequality of (9), we have

\begin{equation}
\#X^{(r)}(F_q) \leq \sum_{i=1}^{r} \frac{1}{i!} \sum_{a_1, \ldots, a_i \geq 1 \atop \sum_j a_j = r} \prod_{j=1}^{i} \frac{1}{a_j} \cdot \left(1 + \frac{2}{A}\right) \cdot q^{a_j} + 1
\end{equation}

\begin{equation}
\leq \sum_{i=1}^{r} \frac{1}{i!} \sum_{a_1, \ldots, a_i \geq 1 \atop \sum_j a_j = r} \prod_{j=1}^{i} 2a \cdot \left(1 + \frac{2}{A}\right) \cdot q^{a_j}
\end{equation}

\begin{equation}
= q^r \cdot \sum_{i=1}^{r} \frac{1}{i!} \cdot (2a)^i \cdot \left(1 + \frac{2}{A}\right)^i
\end{equation}

\begin{equation}
= q^r \cdot \left(2a + \frac{4a}{A}\right) \cdot \sum_{i=1}^{r} \frac{2^{i-1}}{i!} \cdot \left(r - 1\right) \cdot \left(a + \frac{2a}{A}\right)^{i-1}
\end{equation}

\begin{equation}
\leq q^r \cdot \left(2a + \frac{4a}{A}\right) \cdot \sum_{i=1}^{r} \left(r - 1\right) \cdot \left(a + \frac{2a}{A}\right)^{i-1}
\end{equation}

\begin{equation}
= q^r \cdot \left(2a + \frac{4a}{A}\right) \cdot \left(a + 1 + \frac{2a}{A}\right)^{r-1}
\end{equation}

\begin{equation}
< q^r \cdot \left(\frac{5}{3} + \frac{4}{3A}\right)^r
\end{equation}

This completes the proof of the lemma. □

Next, denote by \( J \) the Jacobian variety of \( X \), and let us fix a \( k \)-rational point \( P \) of \( X \). (If the condition (**)) is satisfied, then it suffices to choose \( P \) from \( R \cup S \cup T \). In fact, the assumptions \( n \geq 1 \) and \( n \geq g + \max\{t, g\} \) implies \( \#(R \cup S \cup T) = n + s - g \geq n - g \geq 1 \). In what follows, if \( L \) is a line bundle of degree 0 on \( X \), we abuse notation and also write \( L \) for the corresponding point of \( J \). For each positive integer \( r \), we denote by

\begin{equation}
\varphi_r : X^{(r)} \to J
\end{equation}

the morphism giving by sending each divisor \( D \) to the line bundle \( O(-rP + D) \). It follows from [13], § 5, Theorem 5.1 (a), that \( \varphi_m \) is surjective (resp., birational onto its image; resp., a birational morphism) if \( m > g \) (resp., \( m < g \); resp., \( m = g \)). The fiber \( \varphi_r^{-1} \) of this morphism over \( L \in J(k) \) can be identified with the complete linear system \(|L(rP)|\) of \( L(rP) \). In particular, if \( k = \mathbb{F}_q \) (i.e., the assumption (**)) holds), then the number of \( \mathbb{F}_q \)-rational points of \( \varphi_r^{-1}(L) \) is given by

\begin{equation}
\#\varphi_r^{-1}(L)(\mathbb{F}_q) = \varpi|L(rP)|(\mathbb{F}_q) = \sum_{j=0}^{L} q^j = q^L + \frac{q^L - 1}{q - 1} < q^L + \frac{q^L}{2} = \frac{3}{2} \cdot q^L,
\end{equation}

where \( L := \dim|L(rP)| \).

Now, we shall denote by

\begin{equation}
\iota : X^{(g)} \to X^{(n)}
\end{equation}
the closed immersion given by sending each divisor $D$ to $D + \sum_{Q \in R \cup T} Q$. The composite $\varphi_n \circ \iota : X^{(g)} \to J$ coincides with the composite of $\varphi_g$ and the translation of $J$ by the line bundle $O(-(n - g)P + \sum_{Q \in R \cup T} Q)$. It follows that $\varphi_n \circ \iota$ is a birational morphism. More precisely, (according to [13], Lemma 5.2 (b), and the comment following that lemma) we can find a dense open subset $U$ of $X^{(g)}$ such that $h^0(O(D)) = 1$ (or equivalently, $h^0(\Omega_{X/k}(-D)) = 0$ by the Riemann-Roch theorem) for all $D$’s in $U$; the restriction of $\varphi_n \circ \iota$ to $U$ is an open immersion. We shall write

$$J' := (\varphi_n \circ \iota)(U), \quad E_1 := J \setminus J', \quad \text{and} \quad E_1 := \varphi_n^{-1}(E_1),$$

where $E_1$ and $E_1$ are considered as reduced closed subschemes of $J$ and $X^{(n)}$ respectively. Since the restriction $\varphi_n|_{(U)} : \iota(U) \to J'$ of $\varphi_n$ to $\iota(U)$ is an isomorphism, there exists its inverse morphism

$$\psi : J' \xrightarrow{\sim} \iota(U) \quad (\subseteq X^{(n)}).$$

Also, for each $k$-rational point $Q$ of $X$, denote by

$$\gamma_Q : X^{(g-1)} \to X^{(g)}$$

the morphism given by $D \mapsto D + Q$. Then, we have a reduced closed subscheme

$$E_Q := \varphi_n^{-1}(\text{Im}(\varphi_n \circ \iota \circ \gamma_Q))$$

of $X^{(n)}$.

**Lemma 2.3.**

*Let $Q$ be an $k$-rational point of $X$. Then, both $E_1$ and $E_Q$ are of dimension $\leq n - 1$. If, moreover, the condition (**) is satisfied, then the following inequalities hold:*

$$\dim(E_1(\mathbb{F}_q), \dim(E_Q(\mathbb{F}_q) \leq \frac{3}{2} \cdot \left(\frac{5}{3} + \frac{4}{3A}\right)^{g-1} \cdot q^{n-1}. $$

**Proof.** We will only consider the case of $E_1$ because the proof of the other is similar. Denote by $\Theta'$ the reduced closed subscheme of $X^{(g)}$ classifying divisors $D$ with $h^0(\Omega_{X/k}(D + P)) > 0$. Since $\deg(\Omega_{X/k}(D + P)) = g - 1$ for any such divisor $D$, it follows from a well-known fact that the image $(\varphi_n \circ \iota)(\Theta')$ coincides with the theta divisor $\Theta := \text{Im}(\varphi_{g-1}) (\subseteq J)$ up to translation. The obvious inequality $h^0(\Omega_{X/k}(D + P)) \geq h^0(\Omega_{X/k}(-D))$ implies that $(\varphi_n \circ \iota)(\Theta')$ contains $E_1$, and hence,

$$\dim(E_1) \leq \dim((\varphi_n \circ \iota)(\Theta')) \leq \dim(\Theta') = g - 1.$$

Let us take a line bundle $L$ classified by $E_1 (\subseteq J)$. Since $\deg(\Omega_{X/k} \otimes L(nP)) = 2g - 2 - n < 0 \quad (\implies h^1(L(nP)) = 0$ by assumption, the Riemann-Roch theorem gives

$$h^0(L(nP)) = h^0(L(nP)) - h^1(L(nP)) = n + 1 - g.$$

Thus, the equality $\dim(|LP|) = n - g$ holds, and

$$\dim(E_1) \leq \dim(|LP|) \leq \dim(E_1) \leq \frac{3}{2} \cdot \left(\frac{5}{3} + \frac{4}{3A}\right)^{g-1} \cdot q^{n-1}.$$
Next, we shall estimate the number $\#E_1(\mathbb{F}_q)$ when the condition (**) holds. By the comment preceding (21), we have
\begin{equation}
\#E_1(\mathbb{F}_q) \leq \#(\varphi_n \circ i)(\Theta)(\mathbb{F}_q) = \#(\Theta)(\mathbb{F}_q).
\end{equation}
Moreover, the fiber of $\varphi_{g-1}: X^{(g-1)} \to (\Theta \subseteq J)$ over each $\mathbb{F}_q$-rational point of $\Theta$ is isomorphic to a projective space (which, in particular, contains at least one $\mathbb{F}_q$-rational point). This implies that $\varphi_{g-1}$ induces a surjective map $X^{(g-1)}(\mathbb{F}_q) \twoheadrightarrow \Theta(\mathbb{F}_q)$, and hence,
\begin{equation}
\#\Theta(\mathbb{F}_q) \leq \#X^{(g-1)}(\mathbb{F}_q).
\end{equation}
By (24), (25), and Lemma 2.2 (ii), the following sequence of inequalities holds:
\begin{equation}
\#E_1(\mathbb{F}_q) \leq \#\Theta(\mathbb{F}_q) \leq \#X^{(g-1)}(\mathbb{F}_q) \leq \left(\frac{5}{3} + \frac{4}{3A}\right)^{g-1} \cdot q^{g-1}.
\end{equation}
On the other hand, for each line bundle $\mathcal{L}$ classified by $E_1$, the result of (14) reads
\begin{equation}
\#\varphi_{n-1}(\mathcal{L})(\mathbb{F}_q) < \frac{3}{2} \cdot q^{n-g}.
\end{equation}
Thus, by (26) and (27), we have
\begin{equation}
\#E_1(\mathbb{F}_q) \leq \sum_{\mathcal{L} \in E_1(\mathbb{F}_q)} \#\varphi_{n-1}(\mathcal{L})(\mathbb{F}_q)
\leq \left(\frac{5}{3} + \frac{4}{3A}\right)^{g-1} \cdot q^{g-1} \cdot \frac{3}{2} \cdot q^{n-g}
= \frac{3}{2} \cdot \left(\frac{5}{3} + \frac{4}{3A}\right)^{g-1} \cdot q^{n-1}.
\end{equation}
This completes the proof of the lemma. \hfill \Box

Next, let us consider the reduced closed subscheme
\begin{equation}
E_2 := \varphi_{n-1}(\varphi_n(\text{Im}(\alpha^{(n)}) \cap \text{Im}(\iota)))
\end{equation}
of $X^{(n)}$, where, for each positive integer $r$, $\alpha^{(r)}$ denotes the morphism $X^{r-1} \to X^{(r)}$ given by
\begin{equation}
\alpha^{(r)}(Q_1, \cdots, Q_{r-1}) := 2Q_1 + \sum_{i=2}^{r-1} Q_i.
\end{equation}

**Lemma 2.4.**
$E_2$ is of dimension $\leq n - 1$. If, moreover, the condition (**) is satisfied, then the following inequality holds:
\begin{equation}
\#E_2(\mathbb{F}_q) < \left(\frac{3n}{2} + \frac{3g}{8}\right) \cdot \left(\frac{5}{3} + \frac{4}{3A}\right)^{g-1} \cdot q^{n-1}.
\end{equation}
Proof. Each divisor classified by a point of \( \text{Im}(\alpha^{(n)}) \cap \text{Im}(\iota)) \) may be expressed as \( D + \sum_{Q \in R \cup T} Q \), where \( D \) is a divisor in either \( \text{Im}(\iota \circ \alpha^{(g)}) \) or \( \text{Im}(\iota \circ \gamma_Q) \) for some \( Q \in R \cup T \). This implies that 

\[
\dim(\varphi_n(\text{Im}(\alpha^{(n)}) \cap \text{Im}(\iota))) \leq \dim(\text{Im}(\alpha^{(n)}) \cap \text{Im}(\iota)) \\
\leq \max \{ \dim(\text{Im}(\iota \circ \alpha^{(g)})), \dim(\text{Im}(\iota \circ \gamma_Q)) \} \\
\leq \max \{ \dim(\text{Im}(\alpha^{(g)})) \} \cup \{ \dim(\text{Im}(\gamma_Q)) \}_{Q \in R \cup T} \\
\leq \max \{ \dim(X^{g-1}), \dim(X^{(g-1)}) \} \\
= g - 1.
\]

Hence, by the same argument as the inequalities (22), we have 

\[
\dim(E_2) \leq (n - g) + \dim(\varphi_n(\text{Im}(\alpha^{(n)}) \cap \text{Im}(\iota)))) = n - 1,
\]

which completes the proof of the former assertion.

Next, let us consider the latter assertion. Suppose that the condition (**) is satisfied. Then, the above discussion implies that 

\[
\sharp \text{Im}(\alpha^{(n)}) \cap \text{Im}(\iota))(\mathbb{F}_q) \leq \sharp \text{Im}(\iota \circ \alpha^{(g)})(\mathbb{F}_q) + \sum_{Q \in R \cup T} \sharp \text{Im}(\iota \circ \gamma_Q)(\mathbb{F}_q) \\
\leq \sharp \text{Im}(\alpha^{(g)}) + \sum_{Q \in R \cup T} \sharp X^{(g-1)}(\mathbb{F}_q) \\
\leq \sharp \text{Im}(\alpha^{(g)}) + (n - g) \cdot \left(\frac{5}{3} + \frac{4}{3A}\right)^{g-1} \cdot q^{q-1},
\]

where the last inequality follows from Lemma 22 (ii).

In what follows, let us consider an upper bound for \( \sharp \text{Im}(\alpha^{(g)}) \). For each \( \mathbb{F}_q \)-rational point \( Q \) of \( X \), we shall denote by \( Q_{\text{div}} \) the reduced effective divisor determined by its image. The assignment \( Q \mapsto Q_{\text{div}} \) can be extended naturally to a map \( D \mapsto D_{\text{div}} \) from \( X^{(r)}(\mathbb{F}_q) \) (for each \( r \)) to the set of divisors on \( X \). Now, let us write 

\[
\hat{X}^{(m:l)} := \{(D, D') \in X^{(m)}(\mathbb{F}_q) \times X^{(l)}(\mathbb{F}_q) \mid D \geq D'_{\text{div}} \}
\]

\((m \geq l \geq 1)\). Since the fiber of the first projection \( \hat{X}^{(m:l)} \to X^{(m)}(\mathbb{F}_q) \) over each element of \( X^{(m)}(\mathbb{F}_q) \) has at most \( \binom{m}{l} \) elements. It follows that 

\[
\hat{X}^{(m:l)} \leq \binom{m}{l} \cdot X^{(m)}(\mathbb{F}_q) \leq \binom{m}{l} \cdot \left(\frac{5}{3} + \frac{4}{3A}\right)^m \cdot q^m.
\]

Now, let us consider the set 

\[
\hat{X}^{(g):2} := \{(D, D') \in \text{Im}(\alpha^{(g)})(\mathbb{F}_q) \times X(\overline{\mathbb{F}_q}) \mid D \geq 2Q_{\text{div}} \}.
\]

The projection to the first factor \( \hat{X}^{(g):2} \to \text{Im}(\alpha^{(g)})(\mathbb{F}_q) \) is surjective, and hence we have 

\[
\sharp \text{Im}(\alpha^{(g)})(\mathbb{F}_q) \leq \sharp \hat{X}^{(g):2}.
\]

Moreover, notice that the assignment \( (D, D') \mapsto D - D'_{\text{div}} \) give an injection 

\[
\hat{X}^{(g):2} \to \prod_{i=1}^{g-1} \hat{X}^{(i:1)}.
\]
Hence, by (36), we have
\[ \tilde{X}_2^{(g)} \leq \sum_{i=1}^{g-1} \tilde{X}^{(i;1)} \]
\[ \leq \sum_{i=1}^{g-1} i \cdot \left( \frac{5}{3} + \frac{4}{3A} \right)^i \cdot q^i \]
\[ < g \cdot \left( \frac{5}{3} + \frac{4}{3A} \right)^{g-1} \cdot q^{g-1} + g \cdot \left( \frac{5}{3} + \frac{4}{3A} \right)^{g-1} \cdot q \cdot \frac{(5/3 + 4/3A) \cdot q^{g-2} - 1}{(5/3 + 4/3A) \cdot q - 1} \]
\[ < g \cdot \left( \frac{5}{3} + \frac{4}{3A} \right)^{g-1} \cdot q^{g-1} + g \cdot \left( \frac{5}{3} + \frac{4}{3A} \right)^{g-1} \cdot q \cdot \frac{1}{4} \cdot \left( \frac{5}{3} + \frac{4}{3A} \right)^{g-2} \cdot q^{g-2} \]
\[ = g \cdot \frac{5}{4} \cdot \left( \frac{5}{3} + \frac{4}{3A} \right)^{g-1} \cdot q^{g-1}. \]

By combining (34), (38), and (40), we obtain
\[ \# \text{Im}(\alpha \cap \text{Im}(\iota))(F_q) < \frac{g \cdot 5}{4} \cdot \left( \frac{5}{3} + \frac{4}{3A} \right)^{g-1} \cdot q^{g-1} + \left( n - g \right) \cdot \left( \frac{5}{3} + \frac{4}{3A} \right)^{g-1} \cdot q^{g-1} \]
\[ = \left( n + \frac{g}{4} \right) \cdot \left( \frac{5}{3} + \frac{4}{3A} \right)^{g-1} \cdot q^{g-1}. \]

Finally, since each fiber of \( \varphi_n \) is isomorphic to a projective space, the following inequalities hold:
\[ \# E_2(F_q) \leq \sum_{\mathcal{L} \in \varphi_n(\text{Im}(\alpha(n) \cap \text{Im}(\iota)))(F_q)} \# \varphi_n^{-1}(\mathcal{L})(F_q) \]
\[ \leq \sum_{D \in \text{Im}(\alpha(n) \cap \text{Im}(\iota)))(F_q)} \# \varphi_n^{-1}(\varphi_n(D))(F_q) \]
\[ \leq \sum_{D \in \text{Im}(\alpha(n) \cap \text{Im}(\iota)))(F_q)} \frac{3}{2} \cdot q^{n-q} \]
\[ < \left( \frac{n + \frac{g}{4}}{2} \right) \cdot \left( \frac{5}{3} + \frac{4}{3A} \right)^{g-1} \cdot q^{g-1} \cdot \frac{3}{2} \cdot q^{n-q} \]
\[ < \left( \frac{3n + 3g}{8} \right) \cdot \left( \frac{5}{3} + \frac{4}{3A} \right)^{g-1} \cdot q^{n-1}, \]
where the third inequality follows from (14) and the last inequality follows from (41). This completes the proof of the lemma. \( \square \)

Let \( \beta : X^{n-2} \rightarrow X^{(n)} \) and \( \gamma : X^{n-2} \rightarrow X^{(n)} \) be the morphisms given by
\[ \beta(Q_1, \cdots, Q_{n-2}) := 2Q_1 + 2Q_2 + \sum_{l=3}^{n-2} Q_l, \quad \gamma(Q_1, \cdots, Q_{n-2}) := 3Q_1 + \sum_{l=2}^{n-2} Q_l. \]

Write
\[ \delta : ((\text{Im}(\beta) \cup \text{Im}(\gamma)) \setminus E_1) \times \mathbb{P}^1_k \rightarrow X^{(n)} \]

given by \((D, t) \mapsto \psi \circ \varphi_n(D) + t(D - \psi \circ \varphi_n(D))\), and write

\[(45) \quad E_3 := \text{Im}(\delta)\]

(considered as a reduced subscheme of \(X^{(n)}\)).

**Lemma 2.5.**

\(E_3\) is of dimension \(\leq n - 1\). If, moreover, the condition \((**)\) is satisfied, then the following inequality holds:

\[(46) \quad \#E_3(\mathbb{F}_q) < \frac{5n^2}{2} \cdot \left(\frac{5}{3} + \frac{4}{3A}\right)^{n-2} \cdot q^{n-1}.\]

**Proof.** Since \(\dim(\text{Im}(\beta)), \dim(\text{Im}(\gamma)) \leq \dim(X^{n-2}) = n - 2\), the following inequalities hold:

\[(47) \quad \dim(E_3) \leq \dim(\mathbb{P}^1_k) + \max\{\dim(\text{Im}(\beta)), \dim(\text{Im}(\gamma))\} \leq 1 + (n - 2) = n - 1.\]

Next, we shall prove the latter assertion under the assumption that \((**)\) is satisfied. Let us write

\[(48) \quad \hat{X}^{(n)}_\beta := \{(D, D') \in \text{Im}(\beta)(\mathbb{F}_q) \times X^{(2)}(\overline{\mathbb{F}}_q) | D \geq 2Q_{\text{div}}\},\]

\[(49) \quad \hat{X}^{(n)}_\gamma := \{(D, D') \in \text{Im}(\gamma)(\mathbb{F}_q) \times X(\overline{\mathbb{F}}_q) | D \geq 3Q_{\text{div}}\}.\]

The projections to the first factors \(\hat{X}^{(n)}_\beta \to \text{Im}(\beta)(\mathbb{F}_q)\) and \(\hat{X}^{(n)}_\gamma \to \text{Im}(\gamma)(\mathbb{F}_q)\) are surjective, and we have

\[(49) \quad \#\text{Im}(\beta)(\mathbb{F}_q) \leq \#\hat{X}^{(n)}_\beta, \quad \#\text{Im}(\gamma)(\mathbb{F}_q) \leq \hat{X}^{(n)}_\gamma.\]

Moreover, notice that the assignments \((D, D') \mapsto D - D'_{\text{div}}\) and \((D, D') \mapsto D - 2D'_{\text{div}}\) give injections

\[(50) \quad \hat{X}^{(n)}_\beta \hookrightarrow \prod_{i=2}^{n-2} \hat{X}^{(i;2)}, \quad \hat{X}^{(n)}_\gamma \hookrightarrow \prod_{i=1}^{n-2} \hat{X}^{(i;1)}\]
respectively, where \( \hat{X}^{(m,l)} \)'s are as in the proof of Lemma 2.4. Hence, by (49), (50), and (36), the following inequalities hold:

\[
\sum_{i=2}^{n-2} \hat{X}^{(i;2)} \leq \left( \begin{array}{c} n \end{array} \right) \cdot \left( \frac{5}{3} + \frac{4}{3A} \right)^i \cdot q^i \leq \frac{n^2}{2} \cdot \left( \frac{5}{3} + \frac{4}{3A} \right)^{n-2} \cdot q^n + \frac{n^2}{2} \cdot \left( \frac{5}{3} + \frac{4}{3A} \right)^2 \cdot q^2 \cdot \frac{((5/3 + 4/3A)q)^{n-4} - 1}{(5/3 + 4/3A)q - 1} < n \cdot \left( \frac{5}{3} + \frac{4}{3A} \right)^{n-2} \cdot q^n + n \cdot \frac{1}{4} \cdot \left( \frac{5}{3} + \frac{4}{3A} \right)^{n-2} \cdot q^n \leq \frac{5n^2}{8} \cdot \left( \frac{5}{3} + \frac{4}{3A} \right)^{n-2} \cdot q^n. 
\]

\[
\sum_{i=1}^{n-2} \hat{X}^{(i;1)} \leq \left( \begin{array}{c} n \end{array} \right) \cdot \left( \frac{5}{3} + \frac{4}{3A} \right)^i \cdot q^i \leq \frac{n^2}{2} \cdot \left( \frac{5}{3} + \frac{4}{3A} \right)^{n-2} \cdot q^n + \frac{n^2}{2} \cdot \left( \frac{5}{3} + \frac{4}{3A} \right)^2 \cdot q^2 \cdot \frac{((5/3 + 4/3A)q)^{n-3} - 1}{(5/3 + 4/3A)q - 1} < n \cdot \left( \frac{5}{3} + \frac{4}{3A} \right)^{n-2} \cdot q^n + n \cdot \frac{1}{4} \cdot \left( \frac{5}{3} + \frac{4}{3A} \right)^{n-2} \cdot q^n \leq \frac{5n}{4} \cdot \left( \frac{5}{3} + \frac{4}{3A} \right)^{n-2} \cdot q^n. 
\]

Thus, we have

\[
\#E_3(F_q) \leq (\#\text{Im}(\beta) + \#\text{Im}(\gamma)) \cdot \#P^1(F_q) < \left( \frac{5n^2}{8} \cdot \left( \frac{5}{3} + \frac{4}{3A} \right)^{n-2} \cdot q^n + \frac{5n}{4} \cdot \left( \frac{5}{3} + \frac{4}{3A} \right)^{n-2} \cdot q^n \right) \cdot (q + 1) = \frac{5n(n+2)}{8} \cdot \left( \frac{5}{3} + \frac{4}{3A} \right)^{n-2} \cdot q^n \cdot (q + 1) < \frac{5n^2}{2} \cdot \left( \frac{5}{3} + \frac{4}{3A} \right)^{n-2} \cdot q^n.
\]

This completes the proof of the lemma.

Denote by

\[
E'_4
\]
the closed subscheme of $X^g \times_{\varphi_n \circ \eta_{g,J},\varphi_n} X^{(n)}$ consisting of points $((Q_i)_i, D)$ such that $Q \in \text{Supp}(D)$ for some $Q \in R \cup T$ or $Q = Q_i (i = 1, \cdots, g)$. Denote by

\begin{equation}
E_4
\end{equation}

the scheme-theoretic image of the second projection $E'_4 \to X^{(n)}$.

**Lemma 2.6.**

$E_4$ is of dimension $\leq n - 1$. If, moreover, the condition (**) is satisfied, then the following inequality holds:

\begin{equation}
\sharp E_4(\mathbb{F}_q) < \frac{3}{2} \left(\frac{5}{3} + \frac{4}{3A}\right)^g \cdot q^{n-1}.
\end{equation}

*Proof.* Let $D \in X^{(n)}$. Since $\deg(D) = n \geq 2g$, the linear system $|D|$ has no base points (cf. [7, Chap. IV, §3, Corollary 3.2]). This implies that $|D(-Q)| \neq |D|$ (i.e., $\dim(|D(-Q)|) \leq n - g - 1$) for each point $Q$ of $X$. Hence, each fiber of the morphism $\varphi_n|_{E_4} : E_4 \to J$ induced by $\varphi_n$ is contained in the union of $n$ hyperplanes embedded in the $n - g$ dimensional projective space.

By the same argument as the inequalities (23), we have

\begin{equation}
\text{dim}(E_4) \leq \text{dim}(E'_4) \leq \text{dim}(J) + (n - g - 1) = n - 1,
\end{equation}

which completes the former assertion.

Also, if the condition (**) is satisfied, then the above discussion shows that $\sharp (\varphi_n|_{E_4})^{-1}(\mathcal{L}) \leq n \cdot \frac{3}{2} \cdot q^{n-g-1}$ for each $\mathcal{L} \in J(\mathbb{F}_q)$ (cf. (14)). Thus,

\begin{equation}
\sharp E_4(\mathbb{F}_q) \leq \sum_{\mathcal{L} \in J(\mathbb{F}_q)} \sharp (\varphi_n|_{E_4})^{-1}(\mathcal{L})
\end{equation}

\begin{align*}
&\leq \sharp X^g(\mathbb{F}_q) \cdot \left(n \cdot \frac{3}{2} \cdot q^{n-g-1}\right) \\
&< \left(\frac{5}{3} + \frac{4}{3A}\right)^g \cdot q^g \cdot \left(n \cdot \frac{3}{2} \cdot q^{n-g-1}\right) \\
&\leq \frac{3}{2} \cdot \left(\frac{5}{3} + \frac{4}{3A}\right)^g \cdot q^{n-1}.
\end{align*}

This completes the proof of the lemma. $\square$

Now, let us complete the proof of Proposition 2.1. Write $E := E_1 \cup E_2 \cup E_3 \cup E_4 \cup \bigcup_{Q \in S} E_Q$. If the condition (**) is satisfied, then the lemmas proved so far and the assumption (5) imply
the following sequence of inequalities:

\[
\begin{align*}
(59) \quad \#(X^{(n)} \setminus E)(\mathbb{F}_q) \\
&\geq \#X^{(n)}(\mathbb{F}_q) - \#E_1(\mathbb{F}_q) - \#E_2(\mathbb{F}_q) - \#E_3(\mathbb{F}_q) - \#E_4(\mathbb{F}_q) - \sum_{Q \in S} \#E_Q(\mathbb{F}_q) \\
&> \frac{1}{7 \cdot n!} \left( \frac{3 - 2}{A} \right)^n \cdot q^n - \frac{3}{2} \cdot \left( \frac{5}{3} + \frac{4}{3A} \right)^{g-1} \cdot q^{n-1} - \left( \frac{3n}{2} + \frac{3g}{8} \right) \cdot \left( \frac{5}{3} + \frac{4}{3A} \right)^{g-1} \cdot q^{n-1} \\
&\quad - \frac{5n^2}{2} \cdot \left( \frac{5}{3} + \frac{4}{3A} \right)^{n-2} \cdot q^{n-2} - \left( \frac{3}{2} \cdot \left( \frac{5}{3} + \frac{4}{3A} \right)^g \cdot q^{n-1} - s \cdot \frac{3}{2} \cdot \left( \frac{5}{3} + \frac{4}{3A} \right)^{g-1} \cdot q^{n-1} \\
&\geq \frac{1}{7 \cdot n!} \left( \frac{3A - 2}{A} \right)^n \cdot q^{n-1} \left( q - 10^2 \cdot n! \cdot (n^2 + s) \cdot \left( \frac{5A + 4}{9A - 6} \right)^n \right) \\
&> 0.
\end{align*}
\]

Hence, there exists a \( k \)-rational point \( Q \) of \( X^{(n)} \) which is not contained in \( E \). We can also verify this claim in the case of \((*)\) because of the dimension estimates proved in the lemmas, which implies that \( \dim(E) \leq n - 1 \). Denote by \( l \) the line in the linear system \(|\mathcal{O}_X(D)|\) passing through \( Q \) and \((\psi \circ \varphi_n)(Q)\). (Notice that \((\psi \circ \varphi_n)(Q)\) can be defined since \( Q \) is not contained in \( E_1 \).) Choose \( \zeta_0, \zeta_1 \in H^0(X, \mathcal{O}_X(D)) \) such that \( l \) consists of all divisors \( \text{div}(\lambda_1 \zeta_0 - \lambda_0 \zeta_1) \), where \( \lambda_0, \lambda_1 \in k \) are not both zero. Since \( D \) is not contained in \( E_4 \), two divisors corresponding to \( Q \) and \((\psi \circ \varphi_n)(Q)\) have no common support. Hence, the morphism \( \zeta : X \to \mathbb{P}^1_k \) given by \( \zeta(P) = [\zeta_0(P) : \zeta_1(P)] \) is everywhere-defined and of degree \( n \). In particular, \( \zeta^{-1}((\lambda_0, \lambda_1)) = \text{div}(\lambda_1 \zeta_0 - \lambda_0 \zeta_1) \). Since \( l \cap \bigcup_{Q \in S} E_Q = \emptyset \) (resp., \( l \cap E_2 = \emptyset \); resp., \( l \cap E_3 = \emptyset \)), we see that \( \zeta(T) \cap \zeta(S) = \emptyset \) (resp., \( \zeta \) is unramified over \( \zeta(T) \); resp., \( \zeta \) is a simple covering by \( \bar{H} \), Theorem 5.6). Consequently, \( \zeta \) specifies the desired morphism. This completes the proof of Proposition 2.1. \( \square \)

3. Second step of the proof

Next, we prove the following effective version of [16], Proposition 4.1.

Proposition 3.1.
Let \( q \) be a power of an odd prime, and let \( S \) be a (possibly empty) finite set of \( \mathbb{F}_q \)-rational points of \( \mathbb{P}^1_{\mathbb{F}_q} \). Write \( s := \#S \).

(i) There exists a \( p \)-tame Belyi map on \((\mathbb{P}^1_{\mathbb{F}_q}, \emptyset, S)\) over \( \mathbb{F}_q \) of degree \( q - 1 \).

(ii) Let \( \tau \) be an \( \mathbb{F}_q \)-rational point of \( \mathbb{P}^1_{\mathbb{F}_q} \). If \( s \leq 3 \) (resp., \( s > 3 \)), then there exists a \( p \)-tame Belyi map on \((X, S, \{\tau\})\) over \( \mathbb{F}_q \) of degree 1 (resp., \( (q - 1)^{s-3} \)).

Proof. Assertion (i) follows from the fact that the \( \mathbb{F}_q \)-endomorphism of the field \( \mathbb{F}_q(x) \) (= the rational function field of one variable \( x \) over \( \mathbb{F}_q \)) given by \( x \mapsto x^{q-1} - 1 \) specifies the desired endomorphism of \( \mathbb{P}^1_{\mathbb{F}_q} \) (since any element \( v \) of \( \mathbb{F}_q = \mathbb{P}^1_{\mathbb{F}_q}(\mathbb{F}_q) \setminus \{\infty\} \) satisfies the equality \( v^{q-1} = 1 \)).

Next, we shall consider assertion (ii). The non-resp’d portion is immediately verified by taking a suitable linear transformation on \( \mathbb{P}^1_{\mathbb{F}_q} \). In what follows, let us prove the resp’d portion
One verifies that the composite over \( F \) of a pull-back of \( \zeta \) for any \( \xi \) results in a \( \{ \infty \} \subseteq S \). Let us fix \( \alpha \in S \setminus \{0, \infty \} \), and denote by \( \xi_1 \) the \( \mathbb{F}_q \)-endomorphism of \( \mathbb{P}^1_{\mathbb{F}_q} \) (of degree \( q-1 \)) corresponding to the \( \mathbb{F}_q \)-endomorphism of \( \mathbb{F}_q(x) \) given by \( x \mapsto -x^{q-1}+\alpha^{-1}x \). Then, \( \xi_1(0) = \xi_1(\alpha) = 0 \), and \( \xi_1 \) is unramified away from \( \infty \) (since \( (2, q) = 1 \)). Since \( \xi_1(\beta) = \alpha^{-1}\beta-1 \) for any \( \beta \in \mathbb{F}_q \), the map of sets \( \mathbb{F}_q \setminus \{0, \alpha\} \to \mathbb{F}_q \setminus \{0\} \) induced by \( \xi_1 \) is injective. Hence, \( \sharp \xi_1(S) = s-1 \) and \( \xi_1(\tau) \notin \xi_1(S) \). By applying the inductive assumption (or the non-resp’d portion already proved), we obtain a \( p \)-tame Belyï map \( \xi_2 : \mathbb{P}^1_{\mathbb{F}_q} \to \mathbb{P}^1_{\mathbb{F}_q} \) on \( (\mathbb{P}^1_{\mathbb{F}_q}, \psi_1(S), \{\xi_1(\tau)\}) \) over \( \mathbb{F}_q \) of degree \( (q-1)^{s-4} \). The composite \( \xi := \xi_2 \circ \xi_1 \) has degree \( \deg(\xi_2) \cdot \deg(\xi_1) = (q-1)^{s-4} \cdot (q-1) = (q-1)^{s-3} \) and gives the desired morphism. This completes the proof of the proposition. \( \square \)

**Proof of Theorem A.** By the last assertion of Proposition 2.1, we see that if \( m \) is a positive integer with \( q^m \geq 10^2 \cdot (2g+t+1)! \cdot (2g+t+s+1)^2 \cdot (5/6)^{2g+t+1} \), then there exists a simple covering \( \zeta : X_{\mathbb{F}_{q^m}} \to \mathbb{F}_{q^m} \) over \( \mathbb{F}_{q^m} \) of degree \( 2g+t+1 \) such that \( \zeta(S_{\mathbb{F}_{q^m}}) \cap \zeta(T_{\mathbb{F}_{q^m}}) = \emptyset \) and \( \zeta(T_{\mathbb{F}_{q^m}}) \) consists of one point \( \tau_0 \) (if \( T \neq \emptyset \)). Indeed, we can choose \( m \) as

\[
(60) \quad m := \lceil \log_q(10^2 \cdot (2g+t+1)! \cdot (2g+t+s+1)^2 \cdot (5/6)^{2g+t+1}) \rceil.
\]

Since the discriminant \( \delta(\zeta) \) of \( \zeta \) has degree \( 2g+2(2g+t+1)-2 = 6g+2t \), all the points of \( \text{Br}(\zeta) \) are \( \mathbb{F}_{q^m-L(6g+2t)} \)-rational. Hence, if \( S' \) denotes the set of closed points of \( X_{\mathbb{F}_{q^m}} \) defined as the pull-back of \( \zeta(S) \cup \text{Br}(\zeta) \), then all elements of \( S' \) are \( \mathbb{F}_{q^m-L(6g+2t)} \)-rational and \( \sharp S' \leq 6g+s+2t \). Let us choose a \( p \)-tame Belyï map \( \zeta \) resulting from Proposition 3.1 (i) or (ii), where the triple \( (q, S, \tau) \) is taken to be \( (q^m-L(6g+2t), S', \tau_0) \). Then, the following inequality holds:

\[
(61) \quad \deg(\zeta) \leq (q^{m-L(6g+2t)}-1)^{6g+s+2t+1}.
\]

One verifies that the composite \( f := \xi \circ \zeta \) has the degree

\[
(62) \quad \deg(f) = \deg(\zeta) \cdot \deg(\xi) = (2g+t+1) \cdot (q^{m-L(6g+2t)}-1)^{6g+s+2t+1},
\]

and moreover, satisfies the required conditions. This completes the proof of Theorem A. \( \square \)

### 4. Appendix: Effective noncritical \( p \)-wild Belyï theorem

The \( p \)-wild version of our main theorem follows from an argument in the previous works, as we will discuss in the proof of the theorem below. Let us keep the notation preceding Theorem A. By a **\( p \)-wild Belyï map** on \((X, S, T)\) over a field \( k \) (where \( k \) denotes a field over \( \mathbb{F}_q \)), we shall mean a \( k \)-dominant morphism \( f : X_k \to \mathbb{P}^1_k \) satisfying the following conditions:

\[
(63) \quad f(S_k) \cup \text{Br}(f) \subseteq \{\infty\}, \quad \{\infty\} \cap f(T_k) = \emptyset.
\]

The **\( p \)-wild Belyï degree** of \((X, S, T)\) is defined as

\[
(64) \quad wB(X, S, T) := \min \{ \deg(\phi) \mid \phi \text{ is a } p\text{-wild Belyï map } \phi \text{ on } (X, S, T) \text{ over } \mathbb{F}_q \}.
\]
Theorem 4.1.
Suppose that the following inequality holds:
(65) \[ q + 1 - 2g \sqrt{q} \geq N + s, \]
where \( N := \max\{2g - 1 + t, t, 2\} \). Then, there exists a \( p \)-wild Belyï map on \( (X, S, T) \) of degree \( < N \cdot p^{s+2(g+N)} \). In particular, we have
(66) \[ wB(X, S, T) < N \cdot p^{s+2(g+N)}. \]

Proof. By the Hasse-Weil theorem (8) and the assumption (63), we can find distinct \( \mathbb{F}_q \)-rational points \( P_1, \cdots, P_{N-t} \) with \( \{P_1, \cdots, P_{N-t}\} \cap (S \cup T) = \emptyset \). Also, it follows from the discussion in [16], Lemma 2.2, that there exists an element
(67) \[ v \in \Gamma(X, \mathcal{O}_X(\sum_{i=1}^{N-t} P_i + \sum_{Q \in T} Q))) \setminus \bigcup_{Q \in T} \Gamma(X, \mathcal{O}_X(\sum_{i=1}^{N-t} P_i + \sum_{Q' \in T \setminus \{Q\}} Q')), \]
and the morphism \( \phi : X \to \mathbb{P}_{\mathbb{F}_q}^1 \) determined by \( v^{-1} \) is of degree \( N \) and satisfies that \( \phi(T) \subseteq \{0\} \) and \( \{0\} \cap (\phi(S) \cup \text{Br}(\phi)) = \emptyset \). Let \( \mathcal{B} \) be the set of \( \text{Gal}([\mathbb{F}_q/\mathbb{F}_q]) \)-conjugates of elements in \( (\phi(S) \cup \text{Br}(\phi)) \setminus \{\infty\} \), and let \( V \) denote the \( \mathbb{F}_p \)-span of \( \mathcal{B} \subseteq \mathbb{P}_{\mathbb{F}_q}^1 (\mathbb{F}_q) \setminus \{\infty\} = \mathbb{F}_q \). Also, set
(68) \[ h_1 := x^p + \frac{x^p}{x^p - h_0(x) + h_0(x)^p}, \quad h_2 := h_1^p + h_1, \]
where \( h_0(x) := \prod_{\alpha \in V} (x - \alpha) \in \mathbb{F}_q(x) \) (hence \( x|h_0(x) \)). That is to say, \( h_1 \) and \( h_2 \) are the rational functions “\( q \)” and “\( f \)”, respectively, in the proof of [16], Proposition 4.1, where \( B \) is taken to be \( \phi(S) \cup \text{Br}(\phi) \). According to the discussion in loc. cit., the endomorphism \( \psi \) of \( \mathbb{P}_{\mathbb{F}_q}^1 \) determined by \( h_2 \) satisfies that \( \text{Br}(\psi) = \{\infty\} \) and \( \psi(0) \neq \infty \). Therefore, the composite
(69) \[ f := \psi \circ \phi : X \to \mathbb{P}_{\mathbb{F}_q}^1 \]
forms a \( p \)-wild Belyï map on \( (X, S, T) \). Finally, we shall compute its degree. By the Riemann-Hurwitz theorem, the discriminant of \( \psi \) turns out to have degree \( (g + N - 1) > 1 \), which implies the inequalities \( 1 \leq \sharp(\mathcal{B}) \leq s + 2(g + N - 1) \). Hence, \( 2 < \sharp V \) \((= \deg(h_0)) \leq p^{s+2(g+N-1)} \), and
(70) \[ \deg(h_2) = p \cdot \deg(h_1) = p^2 \cdot (\deg(h_0) - 1) < p^{s+2(g+N)}. \]
Consequently,
(71) \[ \deg(f) = \deg(\psi) \cdot \deg(\phi) < N \cdot p^{s+2(g+N)}. \]
This completes the proof of the proposition. \( \square \)

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