Path Integral Analysis of Arrival Times with a Complex Potential

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Abstract

A number of approaches to the arrival time problem employ a complex potential of a simple step function type and the arrival time distribution may then be calculated using the stationary scattering wave functions. Here, it is shown that in the Zeno limit (in which the potential becomes very large), the arrival time distribution may be obtained in a clear and simple way using a path integral representation of the propagator together with the path decomposition expansion (in which the propagator is factored across a surface of constant time). This method also shows that the same result is obtained for a wide class of complex potentials.

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I. INTRODUCTION

Some of the interesting outstanding problems in quantum theory concern situations in which time appears in a non-trivial way. One of the simplest such problems is the arrival time problem. In the one-dimensional version of this problem one considers an initial wave function concentrated in the region $x > 0$ and consisting entirely of negative momenta. The question is then to find the probability that the particle crosses $x = 0$ between time $\tau$ and $\tau + d\tau$. The classical analysis of this problem is trivial, but the quantum analysis is not so, due in part to the fact that the usual machinery of quantum measurement refers to fixed moments of time and not to measurements distributed over an interval of time.

There are many different approaches to this problem [1]. One particular approach is to include a complex potential

$$V(x) = -iV_0 \theta(-x)$$

in the Schrödinger equations, and to then compute the final state

$$|\psi(\tau)\rangle = \exp\left(-i\frac{\hbar}{\tau} H_0 \tau - \frac{V_0}{\hbar} \theta(-x) \tau\right) |\psi\rangle$$

where $H_0$ is the free Hamiltonian. The intuitive idea behind this is that the part of the wave packet that reaches the origin during the time interval $[0, \tau]$ is absorbed, so that

$$N(\tau) = \langle \psi(\tau)|\psi(\tau)\rangle$$

is the probability of not crossing during the time interval. The probability of crossing between $\tau$ and $\tau + d\tau$ is then

$$\Pi(\tau) = -\frac{dN}{d\tau}$$

Such potentials were originally considered by Allcock in his seminal work [2] and have subsequently appeared in detector models of arrival times [3, 4]. A recent interesting result of Echanobe et al. is that under certain conditions a complex potential of the form Eq.(1) is essentially the same as pulsed measurements, in which the wave function is measured at discrete time intervals [5].

The difficulty behind this approach, however, is that the wave function is not entirely absorbed by this complex potential and $N(\tau)$ includes parts of the initial wave function that have reflected off the potential. The reflection is small for small $V_0$ but then the resolution of the measurement, which is proportional to $\hbar/V_0$, is poor. On the other hand, there is a
lot of reflection for large $V_0$ and indeed the wave function is entirely reflected in the limit
$V_0 \to \infty$. This is the quantum Zeno effect [6] and plagues many different approaches to the
arrival time problem.

Echanobe et al. have made an interesting proposal which embraces the Zeno effect in
the $V_0 \to \infty$ limit yet at the same time extracts the physics hidden within it by suitable
normalization [5]. They consider the limit $V_0 \to \infty$ of the expression

$$\Pi_N(\tau) = \frac{\Pi(\tau)}{1 - N(\infty)}$$  \hspace{1cm} \text{(5)}$$

which is normalized since $\int_0^\infty d\tau \Pi(\tau) = 1 - N(\infty)$. The point is that $N(\infty)$ represents the
total amount of reflected wave function, so $1 - N(\infty)$ represents the total probability of
crossing during the time interval $[0, \infty)$ and this goes to zero as $V_0 \to \infty$. The expression
$\Pi(\tau)$ also goes to zero as $V_0 \to \infty$ but the ratio Eq.(5) is finite and independent of $V_0$, and
defines a reasonable normalized arrival time distribution function. Known results on the
stationary scattering wave functions [7] yield the result

$$\Pi(\tau) = \frac{2}{m^{3/2}V_0^{1/2}} \langle \psi_f(\tau) | \hat{p} \delta(\hat{x}) \hat{p} | \psi_f(\tau) \rangle$$  \hspace{1cm} \text{(6)}$$

from which the normalized result is

$$\Pi_N(\tau) = \frac{\hbar}{m} \langle \psi_f(\tau) | \hat{p} \delta(\hat{x}) \hat{p} | \psi_f(\tau) \rangle$$  \hspace{1cm} \text{(7)}$$

where $|\psi_f(\tau)\rangle$ is the freely evolved wave function and $\langle p \rangle$ is the average momentum in the
initial wave packet [5, 7].

An interesting question in these expressions is the origin of the form of the $\hat{p} \delta(\hat{x}) \hat{p}$ term, which is not obvious from the calculation of it in Ref.[7]. Compare this to the simplest guess
for the arrival time distribution function, the current density

$$J(t) = \frac{\hbar}{2m} \langle \psi_f(\tau) | (\hat{p} \delta(\hat{x}) + \delta(\hat{x}) \hat{p}) | \psi_f(\tau) \rangle$$  \hspace{1cm} \text{(8)}$$

which is sensible classically but is not always positive in the quantum case [8]. (Here $\delta(\hat{x}) = |0\rangle\langle 0|$, where $|0\rangle$ denotes the eigenstate $|x\rangle$ of the position operator at $x = 0$.) A simple
operator re-ordering of $J(t)$ gives the “ideal” arrival time distribution of Kijowski

$$\Pi_K(\tau) = \frac{\hbar}{m} \langle \psi_f(\tau) | \hat{p}^{1/2} \delta(\hat{x}) \hat{p}^{1/2} | \psi_f(\tau) \rangle$$  \hspace{1cm} \text{(9)}$$

which is clearly positive, but harder to relate to specific measurement schemes [9]. It would
be of value to find a derivation of Eq.(7) which gives some insight into the form of this
expression. A further question concerns the role of the complex potential. Although Eq. (7) is independent of $V_0$, it is not clear to what extent the result depends on the specific choice of complex potential, Eq. (1).

In this paper, we show that both of these questions are answered very simply using path integral methods. We show that the general form (6) is derived very easily (up to a constant, fixed by normalization). Secondly, the result (7) is seen to be true for a wide class of potentials of the form

$$V(x) = -iV_0\theta(-x)f(x)$$  \hspace{1cm} (10)

where $f(x)$ is any positive function.

With the general complex potential Eq. (10), the arrival time distribution (4) is given by

$$\Pi(\tau) = 2\langle \psi_\tau|V(\hat{x})|\psi_\tau \rangle$$

$$= 2\frac{V_0}{\hbar} \int_{-\infty}^{0} dx \, f(x) \, |\psi(x,\tau)|^2$$  \hspace{1cm} (11)

where $\psi(x,\tau)$ is defined in Eq. (2). The key is therefore to evaluate the propagator

$$g(x_1,\tau|x_0,0) = \langle x_1|\exp\left(\frac{i}{\hbar}H_0\tau - \frac{V_0}{\hbar} \theta(-\hat{x})f(\hat{x})\tau\right)|x_0\rangle$$  \hspace{1cm} (12)

for $x_1 < 0$ and $x_0 > 0$. This may be calculated using a sum over paths,

$$g(x_1,\tau|x_0,0) = \int \mathcal{D}x \exp\left(\frac{i}{\hbar}S\right)$$  \hspace{1cm} (13)

where

$$S = \int_{0}^{\tau} dt \left(\frac{1}{2}m\dot{x}^2 + iV_0\theta(-x)f(x)\right)$$  \hspace{1cm} (14)

and the sum is over all paths $x(t)$ from $x(0) = x_0$ to $x(\tau) = x_1$.

To deal with the step function form of the potential we need to split off the sections of the paths lying entirely in $x > 0$ or $x < 0$. The way to do this is to use the path decomposition expansion or PDX [10–12]. Each path from $x_0 > 0$ to $x_1 < 0$ will typically cross $x = 0$ many times, but all paths have a first crossing, at time $t_1$, say. As a consequence of this, it is possible to derive the formula,

$$g(x_1,\tau|x_0,0) = \frac{i\hbar}{2m} \int_{0}^{\tau} dt_1 \, g(x_1,\tau|0,t_1) \frac{\partial g_r}{\partial x}(x,t_1|x_0,0)|_{x=0}$$  \hspace{1cm} (15)

Here, $g_r(x,t|x_0,0)$ is the restricted propagator given by a sum over paths of the form (13) but with all paths restricted to $x(t) > 0$. It vanishes when either end point is the origin but
its derivative at \( x = 0 \) is non-zero (and in fact the derivative of \( g_r \) corresponds to a sum over all paths in \( x > 0 \) which end on \( x = 0 \) [12]). It is also useful to record a PDX formula involving the last crossing time \( t_2 \),

\[
g(x_1, \tau|x_0, 0) = -\frac{i\hbar}{2m} \int_0^\tau dt_2 \left. \frac{\partial g_r}{\partial x}(x_1, \tau|x_1, t_2) \right|_{x=0} \ g(0, t_2|0, t_1) \quad (16)
\]

These two formulae may be combined to give a first and last crossing version of the PDX,

\[
g(x_1, \tau|x_0, 0) = \frac{\hbar^2}{4m^2} \int_0^\tau dt_2 \int_0^{t_2} dt_1 \left. \frac{\partial g_r}{\partial x}(x_1, \tau|x_1, t_2) \right|_{x=0} \ g(0, t_2|0, t_1) \left. \frac{\partial g_r}{\partial x}(x_1|t_1, x_0) \right|_{x=0} \quad (17)
\]

The restricted propagators in the two regions are easier to work with than Eq.(12) since the potential is zero throughout \( x > 0 \) and \(-iV_0 f(x)\) throughout \( x < 0 \) so these may be calculated by standard methods, without the complication of the \( \theta(-x) \) term. The problem of calculating the propagator Eq.(12) therefore essentially reduces to the easier problem of calculating it between two points lying on \( x = 0 \). This can sometimes be evaluated by a mode sum calculation [13], even though the full propagator Eq.(12) is not necessarily calculable in this way.

Returning to the first crossing PDX, Eq.(15), \( g_r \) is the restricted propagator for the free particle, which is given by the method of images expression

\[
g_r(x_1, \tau|x_0, 0) = \theta(x_1)\theta(x_0) (g_f(x_1, \tau|x_0, 0) - g_f(-x_1, \tau|x_0, 0)) \quad (18)
\]

where \( g_f \) denotes the free particle propagator. It follows that

\[
\left. \frac{\partial g_r}{\partial x}(x_1|t_1, x_0) \right|_{x=0} = 2 \left. \frac{\partial g_r}{\partial x}(0, t_1|x_0, 0) \theta(x_0) \right|_{x=0} \quad (19)
\]

Inserting this in Eq.(15), and rewriting it as an operator expression, we obtain the result

\[
\langle x_1 | \exp \left( -\frac{i}{\hbar}H_0\tau - \frac{V_0}{\hbar} \theta(-\hat{x}) f(\hat{x})\tau \right) |x_0\rangle \\
= - \frac{1}{m} \int_0^\tau dt \langle x_1 | \exp \left( -\frac{i}{\hbar}H_0(\tau-t) - \frac{V_0}{\hbar} \theta(-\hat{x}) f(\hat{x})(\tau-t) \right) \langle x_0 | \delta\hat{x}\hat{p} \exp \left( -\frac{i}{\hbar}H_0t \right) \rangle \quad (20)
\]

We note the appearance of the combination \( \delta\hat{x}\hat{p} \) which clearly corresponds to a “crossing operator” (and in derivations of the PDX arises directly from a derivative of \( \theta(\hat{x}) \) [11]). Now note that the operator \( \delta\hat{x} \) has the simple property that for any operator \( A \)

\[
\delta\hat{x} A \delta\hat{x} = \delta\hat{x} \langle 0 | A | 0 \rangle \quad (21)
\]
This property together with Eq. (20) inserted in Eq. (11) yields

\[
\Pi(\tau) = \frac{2V_0}{\hbar m^2} \int_0^\tau dt' \int_0^\tau dt \int_{-\infty}^0 dx \, f(x) \times \langle 0 \rangle \exp\left(\frac{i}{\hbar} H_0^\dagger (\tau - t') \right) |x \rangle \langle x | \exp\left(-\frac{i}{\hbar} H(\tau - t) \right) |0 \rangle \\
\times \langle \psi | \exp\left(\frac{i}{\hbar} H_0 t' \right) \hat{p} \delta(\hat{x}) \hat{p} \exp\left(-\frac{i}{\hbar} H_0 t \right) |\psi \rangle
\]

where \( H = H_0 - iV_0 \theta(-x) f(x) \) is the total (non-hermitian) Hamiltonian. Now the key point is that as \( V_0 \to \infty \), the integrals over \( t \) and \( t' \) are strongly concentrated around \( \tau \), so we obtain

\[
\Pi(\tau) \approx C \langle \psi | \exp\left(\frac{i}{\hbar} H_0 \tau \right) \hat{p} \delta(\hat{x}) \hat{p} \exp\left(-\frac{i}{\hbar} H_0 \tau \right) |\psi \rangle
\]

where

\[
C = \frac{2V_0}{\hbar m^2} \int_0^\tau dt' \int_0^\tau dt \int_{-\infty}^0 dx \, f(x) \times \langle 0 \rangle \exp\left(\frac{i}{\hbar} H_0^\dagger (\tau - t') \right) |x \rangle \langle x | \exp\left(-\frac{i}{\hbar} H(\tau - t) \right) |0 \rangle
\]

Furthermore, by the change of variables \( s = \tau - t \), \( s' = \tau - t' \), it is easily seen that \( C \) is independent of \( \tau \) for large \( V_0 \), so is just a constant. It is also easily seen that the approximation (23) will hold for \( V_0 \gg k^2/2m \), where \(|k|\) is the largest momentum in the initial state.

Eq. (23) is the main result and is of precisely the form Eq. (6), up to an overall constant. Since Eq. (7) is obtained by normalization of Eq. (6), we therefore obtain Eq. (7). It holds for a general class of potentials, of the form Eq. (10), not just the simple case \( f(x) = 1 \) considered in Refs. [5, 7]. The reason for this is that for large \( V_0 \) the paths in the path integral are kept out of the region \( x < 0 \), only entering it at just before the final time, so the result has very limited dependence on the detailed form of the potential in \( x < 0 \). The result Eq. (23), and in particular the \( \hat{p} \delta(\hat{x}) \hat{p} \) form, follow as an almost immediate consequence of the PDX for large \( V_0 \), together with the simple property Eq. (21).

To fully verify the validity of this path integral approach, we now calculate the constant \( C \) in the case \( f(x) = 1 \) and look for detailed agreement with the unnormalized result Eq. (6). Eq. (24) may be written,

\[
C = \frac{2V_0}{\hbar m^2} \int_{-\infty}^0 dx \, |\phi(x)|^2
\]
where, after a change of variables $s = \tau - t$, 
\[
\phi(x) = \int_0^\tau ds \langle x| \exp \left( -\frac{i}{\hbar} H_0 s - \frac{V_0}{\hbar} \theta(-\dot{x}) s \right)|0\rangle \tag{26}
\]

The integrand may be represented as a sum over paths from $x = 0$ at time $s = 0$ to the final point $x < 0$ at time $\tau$. We may use the last crossing PDX, Eq.\((16)\), to split this into a sum over paths from $x = 0$ at $s = 0$, to the last crossing at $x = 0$, $s = s_1$, and from there propagating entirely in $x < 0$ to the final $x$ at time $s = \tau$. This reads 
\[
\phi(x) = \frac{1}{m} \int_0^\tau ds \int_0^s du \langle x| \exp \left( -\frac{i}{\hbar} H_0 (s - u) - \frac{V_0}{\hbar} (s - u) \right) \hat{p}|0\rangle \langle 0| \exp \left( -\frac{i}{\hbar} H u \right)|0\rangle \tag{27}
\]

In the second bracket expression, $H$ is the total Hamiltonian so still includes the $\theta(-x)$ potential. This is the propagator along the edge of an imaginary step potential which, fortunately was calculated in Ref.\([13]\) using a mode sum method (for the case of a real step potential, but this is readily continued to imaginary values), and we have 
\[
\langle 0| \exp \left( -\frac{i}{\hbar} H u \right)|0\rangle = \left( \frac{m}{2\pi i\hbar} \right)^{1/2} \frac{(1 - e^{-V_0 u/\hbar})}{(V_0/\hbar)u^{3/2}} \tag{28}
\]

Noting that 
\[
\int_0^\tau ds \int_0^s du = \int_0^\tau du \int_u^\tau ds \tag{29}
\]
and changing variables to $v = s - u$, we see that for large $V_0$ the dominant contribution comes from close to $u = 0$ and $v = 0$. We may therefore let $\tau \to \infty$, and the integrals then factor into a product 
\[
\phi(x) = \frac{1}{m} \int_0^\infty dv \langle x| \exp \left( -\frac{i}{\hbar} H_0 v - \frac{V_0}{\hbar} v \right) \hat{p}|0\rangle \int_0^\infty du \left( \frac{m}{2\pi i\hbar} \right)^{1/2} \frac{(1 - e^{-V_0 u/\hbar})}{(V_0/\hbar)u^{3/2}} \tag{30}
\]

The first integral may be evaluated using the familiar formula \([14]\) 
\[
\int_0^\infty dt \left( \frac{m}{2\pi i\hbar t} \right)^{1/2} \exp \left( i \frac{t}{\hbar} \left[ Et + \frac{m x^2}{2t} \right] \right) = \left( \frac{m}{2E} \right)^{1/2} \exp \left( \frac{i}{\hbar} |x| \sqrt{2mE} \right) \tag{31}
\]
with $E = iV_0$, and then applying $-i\hbar \partial / \partial x$. The second is evaluated using the formula, 
\[
\int_0^\infty dx \frac{(1 - e^{-x})}{x^{3/2}} = 2\sqrt{\pi} \tag{32}
\]

We thus obtain 
\[
\phi(x) = \left( \frac{2m}{V_0} \right)^{1/2} \exp \left( -\frac{(1 - i)}{\hbar} \sqrt{mV_0}|x| \right) \tag{33}
\]
Inserting in Eq.(25), this gives the final result

\[ C = \frac{2}{m^{3/2}V_0^{1/2}} \]  

(34)

The path integral method used here therefore gives precise agreement with the earlier result Eq.(6) obtained by scattering methods.

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