Analysis and Control of Aircraft Longitudinal Dynamics with Large Flight Envelopes

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Abstract—The paper contributes towards the development of a unified control approach for longitudinal aircraft dynamics with large flight envelopes. Prior to the control design, we analyze the existence and the uniqueness of the equilibrium orientation along a reference velocity. We show that shape symmetries and aerodynamic stall phenomena imply the existence of the equilibrium orientation irrespective of the reference velocity. The equilibrium orientation, however, is not in general unique, and this may trigger an aircraft loss-of-control for specific reference velocities. Conditions that ensure the local and the global uniqueness of the equilibrium orientation are stated. We show that the uniqueness of the equilibrium orientation is intimately related to the so-called spherical equivalency, i.e., the existence of a thrust change of variable rendering the direction of the transformed external force independent of the vehicle’s orientation, as in the case of spherical shapes. Once this transformation is applied, control laws for reference velocities can be designed. These laws extend the so-called vectored-thrust control paradigm to the case of generic body shapes subject to steady aerodynamics.

Index Terms—Aerodynamic Modelling and Stall, Flight Equilibrium Analysis, Flight Control, Hover-to-cruise flight.

I. INTRODUCTION

The profound human curiosity about nature’s flight systems and the dream of flying have prompted the long, irregular, and faltering understanding of basic aerodynamic phenomena. Yet, despite decades of research on aerodynamics and flight control, flying machines’ dynamics are a far cry from being fully understood. This paper contributes towards the comprehension and the control of the aircraft longitudinal dynamics subject to steady aerodynamic forces.

Flight control makes extensive use of linear control techniques [1]. One reason why is the existence of numerous tools to assess the robustness properties of a linear feedback controller [2]. Another reason is that flight control techniques have been developed primarily for commercial airplanes, which are designed and optimized to fly along very specific trajectories. Control design is then typically achieved from the linearized equations of motion along desired trajectories that often represent steady-state conditions. Clearly, the hidden assumption behind linearization techniques is the existence of equilibrium conditions that — to the best of the author’s knowledge — has never been investigated before.

Some aerial vehicles, however, are required to fly in very diverse conditions that involve large variations of the angle of attack, i.e., large flight envelopes. Fighter aircraft, convertible Vertical Take-Off and Landing (VTOL) aircraft, and small Unmanned Aerial vehicles (UAVs) in windy conditions are few examples of aircraft with large flight envelopes. It then matters to ensure large stability domains that are achievable via the use of nonlinear feedback designs.

The nonlinear effects of large flight envelopes on aircraft dynamics are of potentially fatal importance in practice. These effects, in fact, can give rise to an aircraft loss-of-control (LOC), which remains one of the most important contributors to fatal accidents [3], [4]. A number of different types of LOCs in longitudinal and lateral/directional motion is related to stall phenomena [5], s. 2.2]. Among these forms, one can mention a stable flight at high angle of attack without rotation, also called a deepstall condition. Other forms of LOCs are due to the roll- and the inertia-coupled problems [6], [7], [8].

Many types of aircraft LOCs are related to the equilibria pattern variations depending on the vehicle control settings. In fact, aircraft dynamics may have more than one equilibrium point associated with a given control setting [9, p. 728]: when this setting varies, the aircraft equilibrium may jump from one stable configuration to another, which may cause abrupt aircraft responses and, eventually, an aircraft LOC.

The qualitative behavior of aircraft dynamics in relation to their equilibria pattern can be obtained from the bifurcation analysis and catastrophe theory methodology [10]. For an introduction to bifurcation analysis of aircraft dynamics, the reader is referred to [10]. In essence, the nonlinear problem is formulated in the form of a set of ordinary differential equations depending on parameters [5], which often represent the control surface deflections. Yet, the analysis of how the equilibria pattern varies versus the reference trajectory is still missing. We shall see in this paper that this analysis characterizes a set of non-trackable transition maneuvers between hovering and high velocity cruising.

Once the bifurcations are identified, one may apply the so-called bifurcation control to stabilize the system around the interested bifurcation. In particular, one can modify the equilibria characteristics via a designed control input [13]. Applications of bifurcation control to aircraft dynamics are [14], [15]. The assumption that the system is autonomous, however, clearly impairs the proposed control approach when the error dynamics is time dependent, as in the case of a time-varying reference. Also, the effectiveness of the bifurcation control is related to the model of the chaotic region where bifurcations occur — often the stall region — which is very difficult to model accurately.

Avoiding the conditions that may eventually trigger a LOC is then fundamental for any nonlinear feedback control. Following [16], control laws based on the dynamic inversion technique have been proposed to extend the flight envelope of military aircraft (see, e.g., [17] and the references therein). The control design strongly relies on tabulated models of
aerodynamic forces and moments, like the High-Incidence Research Model [18]. Compared to linear techniques, this type of approach extends the flight domain without involving gain scheduling strategies, but the angle of attack is assumed to remain away from the stall zone. However, should this assumption be violated the system’s behavior is unpredictable.

Compared to commercial airplane control, nonlinear control of VTOL vehicles is more recent, and it has been addressed with a larger variety of techniques, such as dynamic inversion [19], Lyapunov-based design [20], [21], Backstepping [22], Sliding modes [22], [23], and Predictive control [24], [25] (see [26] for more details). Most of these studies address the stabilization of hover flight or low-velocity trajectories; little attention has been paid to aerodynamic effects, which are typically either ignored or modeled as additive perturbations.

When considering the class of convertibles vehicles, one of the major control problems is related to the transition maneuvers between hovering and high-velocity cruising. During this transition, the aerodynamic effects become from negligible to preponderant, and the issues related to the LOCs can be carefully dealt with. Several studies have been dedicated to the control of transition maneuvers for convertibles [27], [28], [29], [30], [31], [32], the common denominator of which is a “switching” policy between a hover and a cruise control depending on the actual flight state. A technical difficulty is ensuring the stability of the closed-loop system along the transition, which is very sensitive to the “switching” policy usually tuned for specific classes of reference trajectories. To the author’s knowledge, a unified approach for the control of the transition maneuvers not based on a “switching” policy and allowing for large flight envelopes is missing.

This paper extends and encompasses the contributions [33], [34], [35] on the longitudinal dynamics control of aerial vehicles subject to steady aerodynamic forces. Given a reference velocity, we first investigate the existence of an equilibrium orientation about which the vehicle must be stabilized: the existence of an equilibrium orientation along any reference velocities can be designed. These laws extend the vectored–thrust control paradigm [38], [39], [40], developed for systems with orientation-dependent external forces to orientation-dependent external forces.

The paper is organized as follows. Section II provides the notation, the background, and some definitions. In Section III we address the problem of the existence of the equilibrium orientation, and in Section IV we study the multiplicity of this orientation in the case of NACA airfoils. The spherical equivalency and its relation to the uniqueness of an equilibrium orientation is presented in Section V. In this section, we also present the application of the spherical equivalency to NACA airfoils. The spherical equivalency is used in Section VI to propose a feedback control design method applicable to several vehicles. Simulation results for the airfoil NACA 0021 performing a transition maneuver between low velocity hovering and high velocity cruising are reported in Section VII. Remarks and perspectives conclude the paper.

II. BACKGROUND

A. Notation

- The $i_{th}$ component of a vector $x \in \mathbb{R}^n$ is denoted as $x_i$.
- $\mathcal{I} = \{O; \hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\}$ is a fixed inertial frame with respect to (w.r.t) which the vehicle’s absolute pose is measured.
- $\mathcal{B} = \{G; \bar{i}, \bar{j}\}$ is a frame attached to the body, and $\bar{i}$ is parallel to the thrust force $\bar{T}$. This leaves two possible and opposite directions for $\bar{i}$. The direction chosen here is consistent with the convention used for VTOL vehicles.
- For the sake of brevity, $(x_1 \bar{i} + x_2 \bar{j})$ is written as $(\bar{i}, \bar{j})x$.
- $\{e_1, e_2\}$ is the canonical basis in $\mathbb{R}^2$, and $I$ is the $(2 \times 2)$ identity matrix.
- $\bar{x} \cdot \bar{y}$ denotes the scalar product of two vectors $\bar{x}, \bar{y}$.

Given a function of time $f : \mathbb{R} \rightarrow \mathbb{R}^n$, its first time derivative is denoted as $\frac{df}{dt}$. Given a function $f$ of several variables, its partial derivative w.r.t. some of them, say $x$, is denoted as $\partial_x f = \partial f / \partial x$. Given a function $f(x) : \mathbb{R} \rightarrow \mathbb{R}$, the first and second order derivative w.r.t. $x$ can be denoted by $f'$ and $f''$, respectively.

- $G$ is the body’s center of mass and $m$ is the (constant) mass of the vehicle.
- $\bar{p} := O\bar{G} = (\bar{i}_0, \bar{j}_0)x$ denotes the body’s position. $\bar{v} = \frac{d\bar{p}}{dt} = (\bar{i}_0, \bar{j}_0)\dot{x} = (\bar{i}, \bar{j})\dot{v}$ denotes the body’s linear velocity, and $\bar{a} = \frac{d\bar{v}}{dt}$ the linear acceleration.

The vehicle orientation is given by the angle $\theta$ between $\bar{i}_0$ and $\bar{i}$. The rotation matrix of $\theta$ is $R(\theta)$. The column vectors of $R$ are the vectors of coordinates of $\bar{i}, \bar{j}$ in $\mathcal{I}$. The matrix $S = R(\pi/2)$ is a unitary skew-symmetric matrix. The body’s angular velocity is $\omega := \dot{\theta}$. 

![Fig. 1. Propelled vehicle subject to aerodynamic forces.](image-url)
B. Equations of motion

We consider two control inputs to derive the equations of motion: a thrust force $T$ along the body fixed direction $\hat{c}(\hat{T} = \hat{c})$, whose main role is to produce longitudinal motions, and a torque actuation, typically created via secondary propellers, rudders or flaps, etc. We assume that any desired torque can be produced so that the vehicle’s angular velocity $\omega$ can be used as a control variable. In the language of Automatic Control, this is a backstepping assumption, and producing the angular velocity can be achieved via classical nonlinear techniques \cite{41} p. 589). In the language of Aircraft Flight Dynamics, instead, this is the assumption of the guidance loop, which focuses on the problem of determining the thrust intensity and the vehicle’s orientation to track a desired reference position/velocity.

The external forces acting on the body are assumed to be composed of the gravity $mg\hat{g}_0$ and the aerodynamic forces $\vec{F}_a$. Applying Newton’s law yields:

$$
\begin{align*}
ma &= mg\hat{g}_0 + \vec{F}_a - T\hat{c}, \\
\dot{\theta} &= \omega,
\end{align*}
$$

with $g \in \mathbb{R}$ the gravitational acceleration.

C. Aerodynamic forces

Steady aerodynamic forces at constant Reynolds and Mach numbers can be written as follows \cite{42} p. 34

$$
\vec{F}_a = k_a|\vec{v}_a| \left[ c_L(\alpha)\vec{v}_a^\perp - c_D(\alpha)\vec{v}_a \right],
$$

with $k_a := \frac{\beta}{\Sigma}$, $\rho$ the free stream air density, $\Sigma$ the characteristic surface of the vehicle’s body, $c_L(\cdot)$ the lift coefficient, $c_D(\cdot) > 0$ the drag coefficient ($c_L$ and $c_D$ are called aerodynamic characteristics), $\vec{v}_a = \vec{v} - \vec{v}_w$ the air velocity, $\vec{v}_w$ the wind’s velocity, $\vec{v}_a^\perp$ obtained by rotating the vector $\vec{v}_a$ by $90^\circ$ anticlockwise, i.e. $\vec{v}_a^\perp = v_{a,1}\hat{c} - v_{a,2}\hat{h}$, and $\alpha$ the angle of attack. This latter variable is here defined as the angle between the body-fixed zero-lift direction $\vec{Z}_L$, along which the airspeed does not produce lift forces, and the airspeed vector $\vec{v}_a$, i.e.

$$
\alpha := \text{angle}(\vec{v}_a, \vec{Z}_L).
$$

The model \cite{2} neglects the so-called unsteady aerodynamics, e.g., the flow pattern effects induced by fast angular velocity motions \cite{9} p.199]. This assumption is commonly accepted in the literature dealing with large flight-envelope aircraft control \cite{43, 44, 45, 46, 47, 48, 49, 50, 49, 51, 52, 53}: in fact, building global, unsteady aerodynamic models employable for control design purposes is still a challenge for the specialized aerodynamic literature \cite{54, 55}.

Now, denote the constant angle between the zero-lift direction $\vec{Z}_L$ and the thrust $\hat{T}$ as $\delta$, i.e. $\delta := \text{angle}(\vec{Z}_L, \hat{T})$, and the angle between the gravity $g\hat{g}_0$ and $\vec{v}_a$ as $\gamma$, i.e. $\gamma := \text{angle}(\hat{g}_0, \vec{v}_a)$. Then (see Figure 1):

$$
\begin{align*}
\alpha &= \theta - \gamma + (\pi - \delta), \\
v_{a1} &= -|\vec{v}_a| \cos(\alpha + \delta), \\
v_{a2} &= |\vec{v}_a| \sin(\alpha + \delta).
\end{align*}
$$

1) Symmetric shapes: To characterize two kinds of shape symmetries and their properties, let $B_z = \{Z, \vec{Z}_L, \vec{Z}_Z\}$ be an orthonormal frame, and $P$ a point of the body surface $S$ – see Figure 2. Consider the vector $Z\vec{P}$ and its expression w.r.t. the frame $B_z$, i.e. $Z\vec{P} := x\hat{Z}_z + y\hat{Z}_y$, with $x, y \in \mathbb{R}$. Then, symmetric and bisymmetric shapes satisfy what follows.

Assumption 1 (Symmetry). There exists a choice for the frame $B_z$ such that the point $P$ defined by the vector $Z\vec{P} = x\hat{Z}_z - y\hat{Z}_y$ belongs to $S$ for any point $P$ of the surface $S$. Then, the shape is said to be symmetric, with axis of symmetry given by $\{Z, \vec{Z}_z\}$.

Assumption 2 (Bisymmetry). There exists a choice for the frame $B_z$ such that the point $P_{bs}$ defined by the vector $Z\vec{P}_{bs} = -x\hat{Z}_z - y\hat{Z}_y$ belongs to $S$ for any point $P$ of the surface $S$. Then, the shape is said to be bisymmetric, with axes of symmetry given by $\{Z, \vec{Z}_Z\}$ and $\{Z, \vec{Z}_Z\}$.

We assume that an axis of symmetry identifies two zero-lift-directions. Then, we choose the zero-lift-direction $\vec{Z}_L$ in parallel to an axis of symmetry, which implies that $c_L(0) = c_L(\pi) = 0$. Note that this choice still leaves two possible and opposite directions for the definition of the vector $\vec{Z}_L$, which in turn may reflect in two possible values of the angle $\delta$. Without loss of generality, the direction here chosen is that minimizing the angle $\delta$.

In light of the above choice, a symmetric shape induces aerodynamic characteristics $c_D(\alpha)$ and $c_L(\alpha)$ that are even and odd functions, respectively.

Property 1. If the body shape $S$ is symmetric and the zero-lift-direction $\vec{Z}_L$ is parallel to the axis of symmetry, then hold:

$$
\begin{align*}
c_D(\alpha) &= c_D(-\alpha), \\
c_L(\alpha) &= -c_L(-\alpha), \quad \forall \alpha,
\end{align*}
$$

(B) $c_L(0) = c_L(\pi) = 0$.

Bisymmetric shapes have an additional symmetry about the axis $\vec{Z}_Z$, thus implying the invariance of the aerodynamic forces w.r.t. body rotations of $\pm \pi$. Then, the aerodynamic characteristics of bisymmetric shapes are $\pi$–periodic functions versus the angle $\alpha$.

Property 2. If the body shape $S$ is bisymmetric and the zero-lift-direction $\vec{Z}_L$ is parallel to an axis of symmetry, then the aerodynamic coefficients satisfy (6) and

$$
\begin{align*}
c_D(\alpha) &= c_D(\alpha \pm \pi), \\
c_L(\alpha) &= c_L(\alpha \pm \pi), \quad \forall \alpha.
\end{align*}
$$
D. Problem statement and preliminary definitions

The control objective is the asymptotic stabilization of a reference velocity. Let \( \vec{v}_r(t) \) denote the differentiable reference velocity, and \( \vec{a}_r(t) \) its time derivative, i.e. \( \vec{a}_r(t) = \dot{\vec{v}}_r(t) \).

Now, define the velocity error as follows

\[
\vec{e}_v := \vec{v} - \vec{v}_r
\]  

(8)

Using System (1) one obtains the following error model

\[
m\dot{\vec{e}}_v = \vec{F} - T\vec{I}, \quad \dot{\theta} = \omega,
\]

(9)

with \( \vec{F} \) the apparent external force defined by

\[
\vec{F} := m\vec{g} + \vec{F}_a - m\vec{a}_r.
\]

(10)

Eq. (9) indicates that the equilibrium condition \( \vec{e}_v = 0 \) requires \( T\vec{I}(\theta) = \vec{F}(\vec{v}_r(t), \theta, t), \forall t \), which in turn implies

\[
T = \vec{F}(\vec{v}_r(t), \theta, t) \cdot \vec{I}(\theta),
\]

(11a)

\[
0 = \vec{F}(\vec{v}_r(t), \theta, t) \cdot \vec{J}(\theta) \quad \forall t.
\]

(11b)

The existence of an orientation \( \theta \) such that Eq. (11b) is satisfied cannot be ensured a priori. In fact, the apparent external force \( \vec{F} \) depends on the vehicle’s orientation, and any change of this orientation affects both vectors \( \vec{F} \) and \( \vec{J} \). The dependence of \( \vec{F} \) on the orientation \( \theta \) comes from the dependence of the aerodynamic force \( \vec{F}_a \) upon \( \alpha \) (see Eqs. (2) and (4)). In view of Eq. (11b), we state the definition next.

**Definition 1.** An equilibrium orientation \( \theta_e(t) \) is a time function such that Eq. (11b) is satisfied with \( \theta = \theta_e(t) \).

The existence of an equilibrium orientation is a necessary condition for the asymptotic stabilization of a reference velocity. In general, a reference velocity \( \vec{v}_r(t) \) may induce several equilibrium orientations. To classify the number of these equilibrium orientations, define the set \( \Theta_{\vec{v}_r}(t) \) as

\[
\Theta_{\vec{v}_r}(t) := \{ \theta_e(t) \in \mathbb{S}^1 : \vec{F}(\vec{v}_r(t), \theta_e(t), t) \cdot \vec{J}(\theta_e(t)) = 0 \}.
\]

(12)

We further introduce a terminology when there exist only two opposite equilibrium orientations over large domains of the reference velocity \( \vec{v}_r \).

**Definition 2.** We say that System (9) possesses a generically-unique equilibrium orientation if and only if there exists \( \theta_e(t) \) such that

\[
\Theta_{\vec{v}_r}(t) \cap \{ \theta_e(t) \} = \mathbb{S}^1
\]

for any reference velocity \( \vec{v}_r(t) \) except for a unique continuous velocity \( \vec{v}_b(t) \) such that \( \Theta_{\vec{v}_b}(t) = \mathbb{S}^1 \) \( \forall t \).

For the systems possessing a generically-unique equilibrium orientation, the reference velocity can be tracked with only two, opposite vehicle orientations at any time \( t \). This holds for any reference velocity except for a unique bad reference velocity \( \vec{v}_b \). At this bad reference velocity, any orientation is of equilibrium, i.e. (11b) is satisfied for any orientation \( \theta \).

Remark that given an equilibrium orientation \( \theta_e(t) \), the thrust intensity \( T \) at the equilibrium configuration is given by Eq. (11a) with \( \theta = \theta_e(t) \). The existence of an equilibrium orientation ensuring a positive thrust is of particular importance, since positive-thrust limitations represent a common constraint when considering aerial vehicles. To characterize this existence, define

\[
\Theta^+(\vec{v}_r)(t) := \{ \theta_e(t) \in \Theta_{\vec{v}_r}(t) : \vec{F}(\vec{v}_r, \theta_e, t) \cdot \vec{n}(\theta_e) \geq 0 \}.
\]

(13)

III. Existence of the equilibrium orientation

We know from experience that airplanes do fly, so the equilibrium orientation must exist in most cases. One may conjecture that the existence of an equilibrium orientation follows from aerodynamic properties that hold independently of the body’s shape, alike the passivity of aerodynamic forces. The next lemma, however, points out that the aerodynamic force passivity is not sufficient to assert the existence of an equilibrium orientation.

**Lemma 1.** The passivity of the aerodynamic force, i.e.

\[
\vec{a}_n \cdot \vec{F}_a(\vec{v}_n, \alpha) \leq 0, \quad \forall (\vec{v}_n, \alpha),
\]

(14)

is not a sufficient condition for the existence of an equilibrium orientation.

The proof is given in the Appendix. Another route that we may follow to conclude about the existence of an equilibrium orientation is by considering specific classes of body’s shapes.

**Theorem 1.** Assume that the aerodynamic coefficients \( c_{il}(\alpha) \) and \( c_{ip}(\alpha) \) are continuous functions, and that the reference velocity is differentiable, i.e. \( \vec{v}_r(t) \in \mathbb{C}^1 \).

i) If the body shape is symmetric and the thrust is parallel to the its axis of symmetry, then there exist at least two equilibrium orientations for any reference velocity, i.e.

\[
\text{cardinality}(\Theta_{\vec{v}_r}(t)) \geq 2 \quad \forall t, \quad \forall \vec{v}_r(t) \in \mathbb{C}^1.
\]

ii) If the body’s shape is bisymmetric, then there exists at least one equilibrium orientation ensuring a positive-semidefinite thrust for any reference velocity, i.e.

\[
\text{cardinality}(\Theta^+(\vec{v}_r)(t)) \geq 1 \quad \forall t, \quad \forall \vec{v}_r(t) \in \mathbb{C}^1,
\]

whatever the (constant) angle \( \delta \).

The proof is given in the Appendix. Item i) asserts that for symmetric body’s shapes powered by a thrust force parallel to their axis of symmetry, e.g. \( \delta = 0 \), the existence of (at least) two equilibrium orientations is guaranteed for any reference velocity. Item ii) states that the bisymmetry of the shape implies the existence of an equilibrium orientation independently of the thrust direction with respect to the body, i.e. the angle \( \delta \). Of most importance, this item points out that the shape’s bisymmetry implies the existence of an equilibrium orientation inducing a positive-semidefinite thrust intensity independently of reference trajectories.

Now, assume that the body’s shape is symmetric and not bisymmetric. If the thrust force is not parallel to the shape’s axis of symmetry, the assumptions of Theorem 1 are not satisfied and the existence of an equilibrium orientation cannot be asserted. Yet, common sense makes us think that an equilibrium orientation still exists. By considering symmetric shapes,
the next theorem states conditions ensuring the existence of an equilibrium orientation independently of reference velocities and thrust directions w.r.t. the body’s zero-lift direction.

**Theorem 2.** Consider symmetric shapes. Assume that the aerodynamic coefficients \( c_L(\alpha) \) and \( c_D(\alpha) \) are continuous functions, and that \( c_D(\pi) > c_D(0) \). If there exists an angle \( \alpha_s \in (0, \pi/2) \) such that \( c_L(\alpha_s) > 0 \) and

\[
\tan(\alpha_s) \leq \frac{c_D(\alpha_s) - c_D(\pi)}{c_L(\alpha_s)},
\]

then there exists at least one equilibrium orientation for any reference velocity, i.e.

\[
\text{cardinality}(\Theta_{\vec{v}_r}(t)) \geq 1 \quad \forall t, \quad \forall \vec{v}_r(t) \in \mathbb{C}^1,
\]

whatever the (constant) angle \( \delta \) between the zero-lift direction and the thrust force.

The proof is given in the Appendix. The key hypothesis in Theorem 2 is the existence of an angle \( \alpha_s \) such that the condition \( 15 \) is satisfied. Seeking for this angle requires some aerodynamic data, and it may be airfoil and flow regime specific. Recall, however, that stall phenomena (see, e.g., Figure 3b) involve rapid, usually important, lift decreases and drag increases. Then the likelihood of satisfying the condition \( 15 \) with \( \alpha_s \) belonging to the stall region is very high. In fact, we verified that Theorem 2 applies with \( \alpha_s \) belonging to the stall region for the NACA airfoils 0012, 0015, 0018, and 0021 at \( M = 0.3 \) and several Reynolds numbers (data taken from \[56\]).

In light of the above, an equilibrium orientation exists in most cases if the airfoil is quasi-symmetric, and this existence is independent from the thrust direction relative to the body. From a control perspective, however, one is also interested in the number of possible vehicle orientation equilibria along the reference velocity.

**IV. A CASE STUDY: MULTIPLICITY OF THE EQUILIBRIUM ORIENTATION IN THE CASE OF NACA AIRFOILS**

This section studies the multiplicity of the equilibrium orientation and its control consequences by considering the experimental aerodynamic coefficients of the symmetric airfoil NACA 0021 in steady, horizontal flight. Recall that this multiplicity equals the cardinality of the set \( \Theta_{\vec{v}_r}(t) \) given by \( 12 \). For the sake of simplicity, we assume no wind, i.e. \( \vec{v}_w = 0 \), a thrust force aligned with the zero-lift direction, i.e. \( \delta = 0 \), and a desired steady-horizontal flight, i.e. \( \dot{x}_r = \nu \vec{e}_2 \), where \( \dot{x}_r \) is the vector of coordinates of \( \vec{v}_r \), expressed in the inertial frame. For an analysis of the equilibrium orientation multiplicity along other flight directions see [36] p. 76.

Since the airfoil NACA 0021 is symmetric, then Theorem 1 with \( \delta = 0 \) ensures the existence of at least one equilibrium orientation along any reference velocity. Now, from the definition of the set \( \Theta_{\vec{v}_r} \) given by \( 12 \), the cardinality of this set equals the number of solutions \( \theta_e \) to the following equation

\[
F(\dot{x}_r, \theta_e)^T R(\theta_e) \vec{e}_2 = 0,
\]

where \( F \) is the vector of coordinates of \( \vec{F} \), given by \( 10 \), expressed in the inertial frame, i.e.

\[
F = mgc_1 + k_a|\dot{x}_r| [c_L(\alpha_e)S - c_D(\alpha_e)I] \dot{x}_r,
\]

with

\[
\alpha_e = \theta_e - \gamma_r + \pi,
\]

and \( \gamma_r = \text{angle}(\nu, \dot{x}_r) = \pi/2 \). By replacing \( \theta_e = \alpha_e + \gamma_r - \pi \) in Eq. \( 16 \), one gets

\[
[1-a_\nu c_L(\alpha_e)] \cos(\alpha_e) - a_\nu c_D(\alpha_e) \sin(\alpha_e) = 0,
\]

where \( a_\nu \) is a dimensionless number defined by

\[
a_\nu := \frac{k_a \nu^2}{mg}.
\]

Therefore, the problem of seeking for the equilibrium orientations \( \theta_e \) is equivalent to the problem of finding the equilibrium angles of attack \( \alpha_e \) that satisfy \( 17 \). Observe that from \( 17 \), we can find the explicit expression of the parameter \( a_\nu \) in function of the equilibrium angles \( \alpha_e \).

\[
a_\nu(\alpha_e) = \frac{\cot(\alpha_e)}{c_D(\alpha_e) + c_L(\alpha_e) \cot(\alpha_e)}.
\]

A picture of the equilibrium angle \( \alpha_e \) as a function of the cruise velocity \( \nu \) is then obtained by plotting Eq. \( 18 \) along the angle of attack \( \alpha_e \). Figure 5a depicts the function \( 18 \) evaluated with the experimental aerodynamic characteristics shown in Figure 5b. From this figure we see that the equilibrium angle of attack is unique as long as the parameter \( a_\nu < 1.35 \). At \( a_\nu = 1.35 \), the equilibrium angle of attack bifurcates in multiple points. The local bifurcation of \( \alpha_e \) is a saddle-node kind since a couple of equilibrium angles collide and annihilate each other \[57\], \[58\] when crossing the bifurcation values \( a_\nu = 1.35 \) and \( a_\nu = 1.45 \). When \( a_\nu \) belongs to a neighborhood of 1.4, three equilibrium angles of attack arise, and a steady-horizontal flight may theoretically be performed with three different vehicle’s orientations.

The bifurcation analysis of the equilibrium orientations is beyond the scope of the present paper, all the more so because these local phenomena occur principally on the highly nonlinear and chaotic stall region. Let us just remark that stall phenomena – intended as a lift decrease when \( \alpha \in (0, 45^\circ) \) – do not always imply a bifurcation of the equilibrium orientation \[36 \] Lemma 7.5, p. 74).

**A. Ill-conditioning of the control problem for constant height transition maneuvers**

A consequence of the existence of several equilibria is that given a continuous reference velocity, the associated equilibrium orientation \( \theta_e(t) \) may be discontinuous. In this case, the reference velocity cannot be perfectly tracked. Also, the continuity of the equilibrium orientation \( \theta_e(t) \) is a necessary condition for the asymptotic stabilization of the equilibrium \( \vec{v}_r = 0 \) associated with System 12. In fact, the control input \( \omega \) at the equilibrium, i.e. \( \omega = \dot{\theta}_e(t) \), must be defined for any \( t \), and this is not the case when \( \theta_e(t) \) is discontinuous.
calls for specific studies on the planning of desired reference trajectories representing transition maneuvers between hovering and high-velocity cruising.

\[ \dot{x}_r(t) = \nu t (0, 1)^T, \]  

(19)

with \( \nu \) a (small) positive number. On the time interval \( t \in (0, t_1) \) (see Figure 4), one has \( \alpha_e \approx 90^\circ \) because the horizontal reference velocity is of low intensity (the thrust opposes the weight). As time goes by, the intensity of the reference velocity increases, and this in turn implies smaller values of the angle of attack at the equilibrium configuration. At \( t = \tilde{t} \), the equilibrium attitude \( \alpha_e(t) \) instantaneously goes from \( 19^\circ \) to \( 8^\circ \), thus making the equilibrium orientations \( \theta_e(t) \) discontinuous. Such discontinuities destroy the well-posedness of the asymptotic stabilization problem related to the transition maneuver given by (19), and may cause a LOC if not carefully dealt with. Then, the reference velocity (19) cannot be perfectly tracked by any aircraft whose aerodynamic characteristics are similar to those shown in Figure 3. This calls for specific studies on the planning of desired reference trajectories representing transition maneuvers between hovering and high-velocity cruising flight, which are, however, beyond the scope of the present paper.

V. Uniqueness of the equilibrium orientation and spherical equivalency

In the previous section, we have seen that a reference velocity \( \vec{v}_r \) may be perfectly tracked with several equilibrium orientations. This poses two interesting problems.

Global uniqueness: can we find aerodynamic models inducing a unique equilibrium orientation for any given reference velocity?

Local uniqueness: can we find conditions ensuring that the equilibrium orientation is isolated for a given reference velocity and aerodynamic model?

We shall see that the uniqueness of the equilibrium orientation is related to the possibility of transforming system (9) – either globally or locally – into an equivalent meaningful system. This transformed system is of the same form of (9), but the transformed apparent external force has a direction independent of the vehicle’s orientation, as in the case of spherical shapes. The transformation is called spherical equivalency.

A. Global spherical equivalency and generically-unique equilibrium orientation

To introduce the spherical equivalency, define

\[ \bar{c}_L(\alpha, \lambda) := c_L(\alpha) - \lambda \sin(\alpha + \delta), \]  

(20a)

\[ \bar{c}_D(\alpha, \lambda) := c_D(\alpha) + \lambda \cos(\alpha + \delta), \]  

(20b)

with \( \lambda \in \mathbb{R} \). In view of \( \vec{v}_a \perp \vec{v}_s \) and (5), one can verify that \( \vec{F}_a \) given by (2) can be decomposed as follows

\[ \vec{F}_a = k_a |\vec{v}_a|^2 \left[ \bar{c}_L \vec{v}_a^\perp - \bar{c}_D \vec{v}_a \right] - \lambda k_a |\vec{v}_a|^2 \vec{r}. \]

Then, the dynamics of the velocity errors (9) become

\[ m \ddot{x}_e = \vec{F}_p - T_p \vec{r}, \]  

(20c)

\[ \dot{\theta} = \omega, \]  

(20d)

with

\[ \vec{F}_p := mg\bar{v}_0 + \vec{f}_p - m \ddot{x}_e, \]  

(21a)

\[ \vec{f}_p := k_a |\vec{v}_a|^2 \left[ \bar{c}_L(\alpha, \lambda) \vec{v}_a^\perp - \bar{c}_D(\alpha, \lambda) \vec{v}_a \right], \]  

(21b)

\[ T_p := T + k_a \lambda |\vec{v}_a|^2. \]  

(21c)
In light of the above, one has the following result.

**Lemma 2.** Assume that the aerodynamic coefficients are twice differentiable functions.

i) System (9) can be transformed into the form (21) with \( \vec{F}_p \) independent of \( \theta \), if and only if
\[
(c_D'' - 2c_D' \sin(\alpha + \delta) + c_D' + 2c_D' \cos(\alpha + \delta)) = 0. \tag{22}
\]
In this case, the function \( \lambda \) in (20) is given by
\[
\lambda(\alpha) = c_D'(\alpha) \cos(\alpha + \delta) + c_D'(\alpha) \sin(\alpha + \delta). \tag{23}
\]
ii) Assume that the thrust force is parallel to the zero-lift direction \( \vec{z}_L \) so as \( \delta = 0 \). If
\[
\vec{c}_D = c_D'(\alpha) + c_D'(\alpha) \cot(\alpha) \tag{24}
\]
is a constant number, then
\[
\lambda(\alpha) = \frac{c_D'(\alpha)}{\sin(\alpha)} \tag{25}
\]
allows to transform System (9) into the form (21) with \( \vec{F}_p \) independent of \( \theta \).

The proof is given in the Appendix. Item i) states a necessary and sufficient condition on the aerodynamic coefficients that defines the cases when the vector \( \vec{F}_p \), evaluated with \( \lambda \) as (23), is independent of the vehicle’s orientation. A simpler, only sufficient condition that allows for the aforementioned transformation is stated in item ii) when the thrust force is parallel to the zero lift direction. More precisely, if the condition (24) is satisfied, one has \( \vec{c}_L = 0 \) in (20). Then, the equivalent aerodynamic force \( \vec{F}_p \) in (21) is reduced to drag forces, i.e.
\[
\vec{f}_p = -k_a|\vec{v}_a|\vec{c}_D\vec{v}_a.
\]
This means that the shapes whose aerodynamic coefficients satisfy (24) with \( \vec{c}_D > 0 \) can be viewed as spheres once the variable change (21c) in the thrust is applied.

When the conditions in Lemma 2 are satisfied, one can inherit the equilibria analysis and the control design developed for spherical shapes but applied to the transformed system (21). In fact, the following result holds.

**Lemma 3.** Assume that the aerodynamic coefficients satisfy the condition (22), and that \( \vec{c}_D \), given by (20b),(23), is a positive-constant. Then System (9) possesses a generically-unique equilibrium orientation given by
\[
\theta_e(t) = \text{angle} \left( \vec{r}_0, \vec{F}_p(\vec{v}_r(t), t) \right), \tag{26}
\]
The proof is given in the Appendix. The above lemma highlights that the aerodynamic models satisfying the condition (22) induce an explicit expression for the equilibrium orientation \( \theta_e \). Furthermore, this equilibrium orientation is unique in the sense of definition 2.

**B. Spherical equivalent shape of NACA airfoils**

We show in this section that there exist aerodynamic models capable of approximating experimental data taken for NACA airfoils and satisfying the conditions (22)–(24).

**Proposition 1.** The following results hold.

i) The modeling functions
\[
\begin{align*}
\vec{c}_L(\alpha) &= c_1 \sin(2\alpha) \\
\vec{c}_D(\alpha) &= c_0 + 2c_1 \sin^2(\alpha),
\end{align*}
\tag{27}
\]
satisfy the condition (22) and yield
\[
\begin{align*}
\vec{c}_L &= -c_1 \sin(2\delta), \\
\lambda &= 2c_1 \cos(\alpha - \delta), \\
\vec{c}_D &= c_0 + 2c_1 \cos^2(\delta), \\
T_p &= T + 2c_1k_a|\vec{v}_a|^2 \cos(\alpha - \delta).
\end{align*}
\tag{28}
\]

ii) If the thrust force \( \vec{T} \) is parallel to the zero-lift direction \( \vec{z}_L \) so as \( \delta = 0 \), then the modeling functions
\[
\begin{align*}
\vec{c}_L(\alpha) &= \frac{0.5c^2_2}{c_2 - c_3} \cos^2(\alpha) + c_3 \\
\vec{c}_D(\alpha) &= c_0 + \frac{c_2c_3}{(c_2 - c_3) \cos^2(\alpha) + c_3} \sin^2(\alpha),
\end{align*}
\tag{29}
\]
satisfy the condition (24) and yield
\[
\begin{align*}
\vec{c}_L &= 0, \\
\lambda &= \frac{c^2_2 \cos(\alpha)}{(c_2 - c_3) \cos^2(\alpha) + c_3}, \\
\vec{c}_D &= c_0 + c_2, \\
T_p &= T + \frac{c_2k_a|\vec{v}_a|^2 \cos(\alpha)}{(c_2 - c_3) \cos^2(\alpha) + c_3}.
\end{align*}
\]
Concerning the functions (27), we show the following result.

**Lemma 4.** Consider symmetric shapes. The model (27) is the only family of aerodynamic coefficients independent of \( \delta \) that yield a vector \( \vec{F}_p \) independent of \( \theta \) whatever the angle \( \delta \).

The proof is in the Appendix. The process of approximating experimental data with the functions (27) is shown in Figure 5 (NACA 0021 with Mach and Reynolds numbers equal to \( (R_e, M) \approx (60 \cdot 10^3, 0.3) \)). For this example, the identified coefficients are \( c_0 = 0.0139 \) and \( c_1 = 0.9430 \). The approximation result, although not perfect, should be sufficient for control design purposes at small Reynolds numbers \( R_e \) – e.g. small-chord-length airfoils – at which stall phenomena are less pronounced. In this respect, small vehicles are advantaged over large ones. The model (27), in fact, is reminiscent of the aerodynamic coefficients of a flat plate when setting \( c_0 = 0 = 0 \). When \( \delta = 0 \), we then speculate that a flat plate is equivalent to a sphere once the variable change \( T \rightarrow T_p \) in the thrust input is applied.

Figure 5 shows that the experimental data are basically independent of the Reynolds number when the angle of attack increases beyond the stall region. Then, the higher the Reynolds number, the worse the approximation result at small \( \alpha \) only. In contrast, Figure 6 shows that the modeling functions (28) yield better approximations at small angles of attack independently of the Reynolds number. In fact, the second order Taylor expansions of the functions (28) at \( \alpha = 0 \) is \( c_L(\alpha) = c_2\alpha, c_D(\alpha) = c_0 + c_2\alpha^2 \), which are the classical modeling functions used to approximate steady aerodynamic characteristics at low angles of attack. The quality of the approximations provided by (28), however, worsens when the angle of attack gets close to the stall region.

Therefore, we combine the models (28) and (27) to approximate the experimental data taken at large domains of \( (R_e, \alpha) \).

Consider, for instance, the following smooth-rectangular function \( \sigma(\cdot) \) defined by
\[
\sigma(\bar{\alpha}, \bar{k}, \alpha) = \frac{1+\tanh(k_2\bar{\alpha}^2 - k_2\alpha^2)}{1+\tanh(k_2\bar{\alpha}^2)}, \quad \alpha \in [-\pi, \pi], \tag{29}
\]
with $\bar{k}, \bar{\alpha} \in \mathbb{R}$. This function is chosen so as to have $\sigma \approx 1$ at small angles of attack, and $\sigma \approx 0$ at large angles of attack. Let $(c_{L_0}, c_{D_0})$ and $(c_{L_2}, c_{D_2})$ denote the modeling functions given by Eqs. (27) and (28), respectively. A combined model is then given by

$$c_L(\alpha) = c_{L_0}(\alpha) \sigma(\alpha, \bar{k}_L, \alpha) + c_{L_2}(\alpha) [1 - \sigma(\alpha, \bar{k}_L, \alpha)]$$  \hspace{1cm} (30a)$$
$$c_D(\alpha) = c_{D_0}(\alpha) \sigma(\alpha, \bar{k}_D, \alpha) + c_{D_2}(\alpha) [1 - \sigma(\alpha, \bar{k}_D, \alpha)]$$  \hspace{1cm} (30b)$$

Figure 7 depicts typical approximations given by the functions (30). The estimated parameters at $R_e = 160 \cdot 10^3$ are

$$\begin{cases}
(c_0, c_1, c_2, c_3) = (14 \cdot 10^{-3}, 0.95, 5.5, 0.3) \\
(\bar{\alpha}, k_L, k_D) = (11^\circ, 28, 167),
\end{cases}$$  \hspace{1cm} (31)$$

while at $R_e = 5 \cdot 10^6$ they are $c_0 = 0.0078$, $c_1 = 0.9430$, $c_2 = 6.3025$, $c_3 = 0.1378$, $\bar{\alpha} = 18^\circ$, $k_L = 12$, and $k_D = 86$. Figure 7 also shows that (30) can approximate the main aerodynamic coefficient variations including stall phenomena.

C. Local spherical equivalency and local uniqueness of the equilibrium orientation

By construction of the model (30), the force $\vec{F}_p$ in (21a)-(23) is almost independent of the vehicle’s orientation if the angle of attack is away from the stall region, and $\delta = 0$. The control problem remains open when the reference velocity requires crossing this region and, more generally, when the aerodynamic model does not satisfy the condition (22). This section shows that the transformed dynamics (21)- (23) encompasses meaningful properties, which are instrumental for control design purposes, independently of the aerodynamic forces.

**Theorem 3.** Assume that there exists an equilibrium orientation $\theta_e(t)$ for the reference velocity $\vec{v}_r(t)$, and that the aerodynamic coefficients are twice differentiable. Consider System (21) with $\lambda$ given by (23). If the vector $\vec{F}_p$ given by (21a) is different from zero at the equilibrium point, i.e.

$$|\vec{F}_p(\vec{v}_r(t), \theta_e(t), t)| > 0,$$  \hspace{1cm} (32)$$

then

i) the direction of $\vec{F}_p$ is locally constant w.r.t. the vehicle’s orientation at the equilibrium point, i.e.

$$\partial_{\theta} \left[ \frac{\vec{F}_p}{|\vec{F}_p|} \right]_{(\vec{v}_r, \theta)=(0, \theta_e(t))} = 0;$$  \hspace{1cm} (33)$$

ii) the equilibrium orientation $\theta_e(t)$ is isolated and differentiable at the time $t$.

iii) the linearization of (21) at $(\vec{v}_r, \theta) = (0, \theta_e(t))$ is controllable with $(T_p, \omega)$ taken as control inputs.

The proof is in the Appendix. The result i) asserts that in a neighborhood of the equilibrium configuration, varying the thrust direction $\vec{v}$ does not perturb the direction of the vector $\vec{F}_p$; this is why we say that system (9) is locally equivalent to a spherical shape system. The condition $|\vec{F}_p| \neq 0$, in fact, implies that the linearization of System (21) at $(\vec{v}_r, \theta) = (0, \theta_e(t))$ is controllable (see the result iii)) as in the cases where the external apparent force does not depend upon the vehicle orientation, e.g. spherical shape case.
VI. CONTROL DESIGN

This section presents control laws for the asymptotic stabilization of the reference velocity \( \bar{c}_v(t) \).

Theorem 3 points out that the direction of \( \bar{F}_p \) in (21) is almost constant close to the equilibrium configuration \((\bar{c}_v, \theta) = (0, \theta)\). A local control strategy then consists in aligning the thrust direction \( \vec{v} \) with the direction of \( \bar{F}_p \) and in opposing the magnitude \( T_p \) to the intensity of \( \bar{F}_p \).

A. Equilibria of interest

The transformed system (21) shows that \( \bar{c}_v = 0 \) implies

\[
T_p = \bar{F}_p(\bar{c}_v, \theta, t) \cdot \vec{v}(\theta), \quad 0 = \bar{F}_p(\bar{c}_v, \theta, t) \cdot \vec{p}(\theta), \quad \forall t.
\]

Let \( \tilde{\theta} \in (0, \pi) \) denote the angle between \( \vec{v} \) and \( \bar{F}_p \). The control objective is then equivalent to the asymptotic stabilization of either \( \tilde{\theta} = 0 \) or \( \tilde{\theta} = \pi \), depending on the equilibrium orientation \( \theta_e \). These two equilibria correspond to either \( T_p = |\bar{F}_p| \) or \( T_p = -|\bar{F}_p| \), respectively. However, we derive control laws stabilizing \( (T_p, \tilde{\theta}) = (|\bar{F}_p|, 0) \) only. Let us justify this choice. Assume a thrust parallel to the zero-lift direction so as \( \delta = 0 \).

Now assume also an equilibrium configuration at small velocities, i.e. the second term on the right hand side of the above equation is negligible. Then, stabilizing this reference velocity with a positive thrust \( T_p \) requires a positive \( T_p \), and consequently \( \tilde{\theta} = 0 \). Analogously, if one assumes an equilibrium configuration at high velocities, which is typically associated with small, positive angles of attack, the second term on the right hand side of \( T_p \) is likely to be positive, thus implying a positive \( T_p \) and \( \tilde{\theta} = 0 \) for positive thrust forces \( T \). Also, simulations that we have performed tend to show that the equilibrium configurations associated with \( \tilde{\theta} = \pi \) are those requiring an angle of attack belonging to the stall region.

Then, stabilizing \( \tilde{\theta} = 0 \) may ensure that the equilibrium angle of attack does not belong to the stall region.

We have also observed that large-constant reference velocities representing a descending phase may be associated with \( \tilde{\theta} = 0 \), but a negative thrust intensity. Hence, to comply with the additional constraint \( T > 0 \), one must stabilize \( \tilde{\theta} = \pi \) in these cases. Although this kind of reference velocities are seldom used in practice, remark that the choice of stabilizing either \( \tilde{\theta} = 0 \) or \( \tilde{\theta} = \pi \) requires in general close attention: stabilizing the former equilibrium does not always ensure a positive thrust at the equilibrium configuration \( \bar{c}_v \equiv 0 \).

B. Assumptions

Assumption 3. There exists a continuous equilibrium orientation \( \theta_e(t) \) such that \( \hat{\theta} = 0 \) and

\[
|\bar{F}_p(\bar{c}_v(t), \theta_e(t), t)| > \eta, \quad \forall t \in \mathbb{R}^+, \quad \eta \in \mathbb{R}^+.
\]

The above assumption ensures that the control problem is well-posed. In particular, the continuity of the equilibrium orientation ensures that no jump of the equilibrium can occur, while the satisfaction of the condition (34) ensures that the equilibrium orientation is differentiable (see Theorem 3, so the angular velocity along the reference velocity is defined \( \forall t \). To avoid non-essential complications, we make the following additional assumption.

Assumption 4. The aerodynamic coefficients \( c_L(\alpha) \) and \( c_D(\alpha) \) are twice differentiable functions, and their derivatives are bounded \( \forall \alpha \in \mathbb{S}^1 \). The wind velocity \( \vec{v}_w \) and the reference velocity \( \bar{c}_v \), along with their first and second order derivatives, are bounded in norm on \( \mathbb{R}^+ \).

C. Velocity control

Proposition 2. Assume that Assumptions 3 and 4 are satisfied. Let \( k_i > 0 \), \( i = \{1, 2, 3\} \) and apply the control

\[
T = \bar{F}_1 + k_1 |\bar{F}_p| \bar{c}_1, \quad \omega = k \left[ k_2 |\bar{F}_p| \bar{c}_2 + k_3 |\bar{F}_p| \bar{c}_3 + \frac{\bar{F}_p^T S R T \bar{F}_3}{|\bar{F}_p|^2} \right],
\]

\[
to System (5) with \( \bar{F}_p = R^T \bar{F}_p, \bar{F} = R^T F \),
\]

\[
F = m \bar{c}_1 + F_\perp - m \bar{c}_2, \quad F_p = m \bar{c}_1 + F_\perp - m \bar{c}_2, \quad F_k := \partial_{\vec{x}_a} \bar{F}_p \bar{x}_a - \partial_{\alpha} \bar{F}_p \gamma - m \bar{c}_2(t),
\]

\[
\begin{align*}
F_a &= k_a |\bar{c}_1| c_L(\alpha) S - c_D(\alpha) \bar{c}_1, \\
F_p &= k_a |\bar{c}_1| c_L(\alpha) S + c_D(\alpha) \bar{c}_1,
\end{align*}
\]

and \( k \) given by:

\[
k := \left( 1 + \frac{\bar{F}_p^T S R T}{|\bar{F}_p|^2} \partial_{\theta} \left[ \bar{F}_3 \right] \right)^{-1}.
\]

Then,

i) the control laws (35) are well defined in a neighborhood of the equilibrium point \( (\bar{c}_v, \tilde{\theta}) = (0, 0) \);

ii) \( (\bar{c}_v, \tilde{\theta}) = (0, 0) \) is a locally asymptotically stable equilibrium point of System (5).

The proof can be found in the Appendix. The interest of System (21) lies precisely in the expression \( k \) of \( k \). More precisely, in view of the result i) of Theorem 3 the direction of \( \bar{F}_p \) is locally independent of the vehicle’s orientation and the nonlinear gain \( k \) is equal to one at the equilibrium configuration. Then, by continuity, the control law (35) is well-defined in a neighborhood of the equilibrium point, and stability can be easily proven.

The expression (38) points out that if the vector \( \bar{F}_p \) does not depend on the vehicle’s orientation \( \theta \), e.g. the conditions in Lemma 2 are satisfied, then \( k \equiv 1 \). In this case, the control design for System (21) can be addressed by adapting the methods developed for the class of systems subject to an orientation-independent external force. For example, proposes globally stabilizing controllers for this kind of system. One can then verify that the velocity control derived by the application of (35) to System (21) coincides with that given by (35) with \( k = 1 \) and \( \bar{F}_p \) independent of the orientation \( \theta \).

1An implementable expression of \( k \) is given by

\[
k = \left( 1 + k_a \bar{c}_1^2 \bar{F}_2 \left( \cos(\alpha + \delta) \bar{c}_L - \sin(\alpha + \delta) \bar{c}_L \right) / |\bar{F}_p|^2 \right)^{-1}.
\]
Also, the control laws \(35\) yield a large domain of attraction in this case. These facts are stated in the following proposition.

**Proposition 3.** If the aerodynamic coefficients satisfy the condition \(22\), then the control laws \(35\) coincide with the velocity control proposed by \(63\) when applied to system \(21\). Consequently, if \(|F_p| > \rho\ \forall t, \ \rho > 0, \text{and the Assumption}\) holds, then the application of the controls \(35\) to System \(21\) renders \((\tilde{v}, \tilde{\theta}) = (0, 0)\) an asymptotically stable equilibrium point with domain of attraction equal to \(R^2 \times (-\pi, \pi)\).

A consequence of the above proposition is that when the aerodynamic characteristics are given by \(27\) any reference velocity is quasi-globally asymptotically stable if \(|F_p| > \rho\ \forall t\). This latter condition characterizes the set of reference velocities for which the control is not defined. For example, among constant reference velocities and no wind, the unique reference velocities are designed, it is straightforward to add integral correction terms for stabilizing reference trajectories. Consequently, if \(\rho > 2\pi^2\), then the control laws \(35\) yield a large domain of attraction equal to \(R^2 \times (-\pi, \pi)\).

**Remark** Once control laws for the asymptotic stabilization of reference velocities are designed, it is straightforward to add integral correction terms for stabilizing reference trajectories. Such an extension can be found at \[50\] p. 83, Sec. 8.3]

**D. Control robustification**

The control laws \(35\) use terms that involve singularities for specific situations. To obtain control laws that are well-defined everywhere, we first set the nonlinear coefficient \(k \equiv 1\) so that we do not destroy the local stability property of the above control laws \((k \approx 1)\) near the reference trajectory since \(F_2 \approx 0\). Secondly, we multiply the terms \(1/(|F_p| + F_p)\) and \(|F_p|\) by the function \(\mu_T \in C^1: [0, +\infty) \to [0, 1]\), i.e.: \(\mu_T(s) = \begin{cases} \sin \left( \frac{\pi^2 s}{2\pi^2} \right), & \text{if } s \leq \tau \\ 1, & \text{otherwise} \end{cases}\) \(39\)

with \(\tau > 0\). This yields the well-defined expression given by \(T = kF_p|\tilde{v}| + \tilde{F}_1, \ \omega = k_2|F_p|\tilde{v} + \mu_\tau(|F_p| + \tilde{F}_1)\frac{k_3|F_p|}{|F_p| + \tilde{F}_1}\) \(40a\)

\(- \mu_\tau(|F_p|)\frac{F_pT}{|F_p|}\). \(40b\)

The property: \(\lim_{s \to 0} \frac{\mu_T(s)}{s} = \lim_{s \to 0} \sin \left( \frac{\pi^2 s}{2\pi^2} \right) s^{-2} = \frac{\pi^2}{4\pi^2}\), implies that the modified control is well-defined everywhere.

**Remark** The control laws \(35\) use the feedforward terms \(\tilde{a}_a, \ \gamma\) that are not always available in practice. Simulations with wing models have shown that neglecting this term when its actual value is not too large does not much affect the control performance in terms of ultimate tracking errors.

**Remark** In practice, the control law \(40\) shall ensure some performance criterion: this may be achieved by relating the control law gains to the chosen criterion, e.g. the system settling time. Along this direction, the control gain values in the next section have been chosen using a linearization of the closed loop system at constant velocity, and then by applying an eigenvalue placement technique. The actual value of the eigenvalues may be related to the linearized system settling time. For additional details on this method see [64] p. 54.

**VII. Simulations**

In this section, we illustrate through a simulation the performance of the proposed approach for the airfoil NACA 0021 with the thrust force parallel to the zero-lift-line, e.g. \(\delta = 0\). The equations of motion are defined by Eqs. \(1\) \(2\) and the aerodynamic coefficients are given by \(50\) \(51\). The other physical parameters are: \(m = 10\) \([Kg]\), \(\rho = 1.292\) \([Kg/m^3]\), \(\Sigma = 1\) \([m^2]\), \(k_a = \frac{\Sigma}{2} = 0.6460\) \([Kg/m]\).

We assume here that the control objective is the asymptotic stabilization of a reference velocity and we apply the control laws given by \(40\). Other values are used for the calculation of the control laws in order to test the robustness w.r.t. parametric errors. They are chosen as follows: \(\hat{m} = 9\) \([Kg]\), \(\bar{k}_a = 0.51\) \([Kg/m]\), \((c_0, c_1, c_2, c_3) = (20 \cdot 10^{-3}, 0.9, 5, 0.5)\), \(\bar{\sigma} = 10^\circ\). The feedforward term \(F_k\) in \(40\) is kept equal to zero, thus providing another element to test the robustness of the controller. The parameters of the control laws are \(k_1 = 0.1529, k_2 = 0.0234, k_3 = 6, \tau = 80\).

**A. From hovering to cruising flight**

The chosen reference velocity represents an hover-to-cruise flight. It is composed of: \(i\) an horizontal velocity ramp on the time interval \([0, 10]\) \([sec]\); \(ii\) cruising with constant horizontal velocity of \(20\) \([m/sec]\) for \(t \geq 10\) \([sec]\). Hence, it is given by: \(\dot{x}_r(t) = \begin{cases} (0, 2t)^T & 0 \leq t < 10, \\ (0, 20)^T & t \geq 10. \end{cases}\) \(41\)

Remark that perfect tracking of this reference velocity is not possible because it involves discontinuous variations of the vehicle orientation (see Section IV-A), using \(41\) constitutes another robustness test of the proposed control strategy. The vehicle’s initial velocity and attitude are \(\dot{x}(0) = (0, 0)^T\) and \(\theta(0) = 0\) respectively. No wind is assumed.

Figure 8 depicts the evolution of the desired reference velocity, the velocity errors, the angle of attack, the angular velocity, the thrust-to-weight ratio, and the vehicle orientation.

At \(t = 0\), the vehicle attitude is zero (vertical configuration), and the thrust tends to oppose the body’s weight. Because of modeling errors and a nonzero reference acceleration, the thrust-to-weight ratio is different from one. In the interval \((0, 10)\) \([sec]\), the horizontal velocity of the vehicle increases, the angle of attack decreases, and the vehicle’s orientation tends towards \(-90^\circ\) (horizontal configuration). At \(t = 8\), the equilibrium orientation associated with the reference velocity \(41\) jumps, and perfect tracking of this reference is not feasible. At this time instant, the norm of the vector \(\tilde{F}_p\) may cross zero. This generates abrupt variations of the thrust intensity and of the (desired) angular velocity. Note that the control value just after the jump depends sensitively upon the constant \(\tau\). The jump of the equilibrium orientation forbids perfect tracking of the reference velocity: the velocity errors significantly increase right after the discontinuity occurrence.

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VIII. Conclusions

Extensions of the vectored–thrust control paradigm [65] to the case of aerial vehicles subject to steady aerodynamic forces has been addressed. Before tackling the control design, we have studied the existence and the uniqueness of the vehicle equilibrium orientation at a reference velocity. We have shown that in the case of symmetric shapes, the existence of a vehicle equilibrium orientation can be asserted independently of the vehicle’s shape and reference velocity. Concerning the uniqueness, we have presented conditions that ensure either the local or the global uniqueness of the equilibrium orientation. The uniqueness of the equilibrium orientation allows us to view the body shape as a sphere after the application of a specific thrust input. This is what we call the spherical equivalency, and applies to various cases, e.g., flat plates and NACA airfoils flying at low Reynolds numbers: in these cases, the application of the proposed thrust input allows one to view the body with a spherical shape. Once the transformation is applied, the control design is simplified and laws for reference velocities and positions can be designed.

Leaving aside the adaptations of this work before it is implemented on a device, the next step is to extend the analysis presented here to the three-dimensional case, eventually to the case of articulated aircraft [69]. Preliminary achievements in this direction have already been published. For instance, one can show that both global spherical equivalency and the global control design presented in Sec. V and VI can be extended to the case of 3D axis-symmetric aerial vehicles [37]. Furthermore, one can use this result for attempting the control of non-symmetric 3D aircraft [45]. Forthcoming studies may then focus on extending the in-depth equilibrium analysis presented in the paper to the three-dimensional case.

REFERENCES

[1] B. L. Stevens and F. L. Lewis, Aircraft Control and Simulation, 2nd ed. Wiley-Interscience, 2003.
[2] C. Roos, C. Döll, and J.-M. Biannic, “Flight control laws: recent advances in the evaluation of their robustness properties,” Aerospace Lab., vol. 4, p., 2012.
[3] H. Kwatny, J.-E. Dongmo, B.-C. Chang, G. Bajpai, C. Belcastro, and M. Yasar, “Aircraft accident prevention: Loss-of-control analysis,” in AIAA Guidance, Navigation, and Control Conference and Exhibit, 2009.
[4] C. Belcastro and J. Foster, Aircraft Loss-of-Control Accident Analysis, American Institute of Aeronautics and Astronautics, 7 2010. [Online]. Available: http://dx.doi.org/10.2514/6.2010-8004
[5] M. Goman, G. Zagainov, and A. Khramtsovsky, “Application of bifurcation methods to nonlinear flight dynamics problems,” Progress in Aerospace Sciences, vol. 33, pp. 539–586, 1997.
[6] W. Phillips, “Effect of steady rolling on longitudinal and directional stability,” NACA, Tech. Rep. 1627, 1948.
[7] T. Hacker and C. Opirsiu, “A discussion of the roll-coupling problem,” Progress in Aerospace Sciences, vol. 15, no. 5, pp. 151–180, 1974.
[8] C. C. Jahnke, “On the roll-coupling instabilities of high-performance aircraft,” Philosophical Transactions of the Royal Society of London, vol. 356, no. 1745, pp. 2223–2239, 1998, serie A.
[9] R. F. Stengel, Flight Dynamics. Princeton University Press, 2004.
[10] J. V. Carroll and R. K. Mehra, “Bifurcation control of nonlinear aircraft dynamics,” Journal of Guidance, Control, and Dynamics, vol. 5, no. 5, pp. 529–536, 1982.
[11] P. A. Cummings, “Continuation methods for qualitative analysis of aircraft dynamics,” NASA, Tech. Rep. CR-2004-213035, 2004.
[12] M. Lowenberg, Bifurcation and Continuation Method. Berlin, Heidelberg: Springer Berlin Heidelberg, 2002, pp. 89–106. [Online]. Available: http://dx.doi.org/10.1007/3-540-45864-8_6
[13] G. Chen, J. L. Moiola, and H. O. Wang, “Bifurcation control: theories, methods, and applications,” International Journal of Bifurcation and Chaos, vol. 10, no. 3, pp. 511–548, 2000.
[14] H. G. Kwatny, W. H. Bennett, and J. Berg, “Regulation of relaxed static stability aircraft,” IEEE Trans. on Automatic Control, vol. 36, no. 11, pp. 1315 – 1323, 1991.
[15] E. H. Abed and H. C. Lee, “Nonlinear stabilization of high angle-of-attack flight dynamics using bifurcation control,” in American Control Conference, 1990, pp. 2235 – 2238.
[16] S. N. Singh and A. Schy, “Output feedback nonlinear decoupled control synthesis and observer design for maneuvering aircraft,” International Journal of Control, vol. 31, no. 31, pp. 781–806, 1980.
[17] Q. Wang and R. Stengel, “Robust nonlinear flight control of high-performance aircraft,” IEEE Transactions on Control Systems Technology, vol. 13, no. 1, pp. 15–26, 2005.
[18] E. Muir, “Robust flight control design challenge problem formulation and manual: The high incidence research model (HIRM),” in Robust Flight Control, A Design Challenge (GARTEUR), ser. Lecture Notes in Control and Information Sciences. Springer Verlag, 1997, vol. 224, pp. 419–443.
[19] J. Hauser, S. Sastry, and G. Meyer, “Nonlinear control design for slightly non-minimum phase systems: Application to V/STOL,” Automatica, vol. 28, pp. 651–670, 1992.
[20] L. Marconi, A. Isidori, and A. Serrani, “Autonomous vertical landing on an oscillating platform: an internal-model based approach,” Automatica, vol. 38, pp. 21–32, 2002.
[21] A. Isidori, L. Marconi, and A. Serrani, Robust autonomous guidance: an internal-model based approach. Springer Verlag, 2003.
[22] S. Bouabdallah and R. Siegwart, “Backstepping and sliding-mode techniques applied to an indoor micro quadrotor,” in IEEE International Conference on Robotics and Automation, 2005.
[23] R. Xu and U. Ozguner, “Sliding mode control of a class of underactuated systems,” Automatica, vol. 44, pp. 233–241, 2008.
[24] H. J. Kim, D. H. Shim, and S. Sastry, “Nonlinear model predictive track-
ing control for rotorcraft-based unmanned aerial vehicles,” in American Control Conference, 2002, pp. 3576–3581.

[25] S. Bertrand, H. Piet-Lahanier, and T. Hamel, “Contractive model predictive control of an unmanned aerial vehicle model,” in 17th IFAC Symp. on Automatic Control in Aerospace, vol. 17, 2007.

[26] M.-D. Hua, T. Hamel, P. Morin, and C. Samson, “Introduction to feedback control of underactuated vtol vehicles,” IEEE Control Systems Magazine, pp. 61–75, 2013.

[27] M. Benosman and K. Lum, “Output trajectory tracking for a switched nonlinear non-minimum phase system: The vtol aircraft,” in IEEE Int. Conf. on Control Applications, 2007, pp. 262–269.

[28] A. Frank, J. S. McGrew, M. Valenti, D. Levine, and J. P. How, “Hover, transition, and level flight control design for a single-propeller indoor airplane,” in Guidance, Navigation and Control Conference and Exhibit (AIAA), 2007, pp. 6318–6336.

[29] M. Oishi and C. Tomlin, “Switched nonlinear control of a vtol aircraft,” in IEEE Conf. on Decision and Control (CDC), 1999, pp. 2685–2690.

[30] A. L. Deshenski, A. Asbeck, and M. Cutkosky, “Hybrid aerial and scarrional robotics,” in IEEE Conf. on Robotics and Automation, 2010, pp. 1–6.

[31] R. Naldi and L. Marconi, “Optimal transition maneuvers for a class of v/stol aircraft,” Automatica, vol. 47, no. 5, pp. 870–879, 2011.

[32] P. Casau, D. Cabecinhas, and C. Silvestre, “Hybrid control strategy for the autonomous transition flight of a fixed-wing aircraft,” IEEE Transactions on control systems technology, vol. PP, pp. 1–18, 2011.

[33] D. Pucci, T. Hamel, P. Morin, and C. Samson, “Nonlinear control of PVTOL Vehicles subjected to Drag and Lift,” in IEEE Conf. on Decision and Control, 2011, pp. 6177–6183.

[34] D. Pucci, “Flight dynamics and control in relation to stall,” in American Control Conf. (ACC), 2012, pp. 118–124.

[35] D. Pucci, T. Hamel, P. Morin, and C. Samson, “Nonlinear control of aerial vehicles subjected to aerodynamic forces,” in IEEE Conf. on Decision and Control (CDC), 2013, pp. 4839 – 4846.

[36] D. Pucci, “Towards a unified approach for the control of aerial vehicles,” Ph.D. dissertation, Université de Nice-Sophia Antipolis and “Sapienza” Università di Roma, 2013.

[37] D. Pucci, T. Hamel, P. Morin, and C. Samson, “Nonlinear feedback control of axisymmetric aerial vehicles,” Automatica, vol. 31, no. 31, pp. 781–806, 2015.

[38] N. Guenard, T. Hamel, and V. Moreau, “Dynamic modeling and intuitive control strategy for an “x4-flyer”,” in International Conference on Control and Automation (ICCA2005), 2005, pp. 141–146.

[39] M.-D. Hua, T. Hamel, P. Morin, and C. Samson, “A control approach for thrust-propelled underactuated vehicles and its application to VTOL drones,” IEEE Trans. on Automatic Control, vol. 54(8), pp. 1837–1853, 2009.

[40] R. Naldi, M. Furci, R. G. Sanfelice, and L. Marconi, “Robust global trajectory tracking for underactuated vtol aerial vehicles using inner-loop control paradigms,” IEEE Transactions on Automatic Control, vol. 62, no. 1, pp. 97–112, 2017.

[41] H. Khalil, Nonlinear systems, 3rd ed. Prentice Hall, 2002.

[42] J. Anderson, Fundamentals of Aerodynamics, 5th ed. McGraw Hill Series in Aeronautical and Aerospace Engineering, 2010.

[43] R. Naldi and L. Marconi, “Optimal transition maneuvers for a class of v/stol aircraft,” Automatica, vol. 47, no. 5, pp. 870–879, 2011.

[44] P. Casau, D. Cabecinhas, and C. Silvestre, “Hybrid control strategy for the autonomous transition flight of a fixed-wing aircraft,” IEEE Transactions on control systems technology, vol. 21, no. 6, pp. 2194–2211, 2013.

[45] M. D. Hua, D. Pucci, T. Hamel, P. Morin, and C. Samson, “A novel approach to the automatic control of scale model airplanes,” in 53rd IEEE Conf. on Decision and Control, Dec 2014, pp. 805–812.

[46] Y. Jung and D. H. Shim, “Development and application of controller for transition flight of tail-sitter uav,” Journal of Intelligent and Robotic Systems, vol. 65, no. 1, pp. 137–152, 2012.

[47] W. E. Green and P. Y. Oh, “A hybrid mav for ingress and egress of urban environments,” IEEE Transactions on Robotics, vol. 25, no. 2, pp. 253–263, April 2009.

[48] E. Çetinsoy, S. Dikyar, C. Hanger, K. Onen, E. Sirimoglu, M. Unel, and M. Akşit, “Design and construction of a novel quad tilt-wing uav,” Mechatronics, vol. 22, no. 6, pp. 723–745, 2012.

[49] K. Muraoaka, N. Okada, and D. Kubo, Quad Tilt Wing VTOL UAV: Aerodynamic Characteristics and Prototype Flight. American Institute of Aeronautics and Astronautics, 2017/05/24 2009.

[50] G. Flores and R. Lozano, “Transition flight control of the quad-tilting rotor convertible mav,” in Unmanned Aircraft Systems (ICUAS), 2013 International Conference on. IEEE, 2013, pp. 789–794.
In coordinates, the aerodynamic force passivity (14) writes
\[ \dot{x}_a^T F_a \leq 0 \quad \forall (\dot{x}_a, \alpha). \] (44)

To show that (44) does not in general imply the existence of an equilibrium orientation, it suffices to find an aerodynamic force satisfying (44) and such that the function given by (42) never crosses zero for some reference and wind velocities at a some time instant. Hence, choose
\[
\begin{align*}
    c_L(\alpha) &= \sin(\alpha) \\
    c_D(\alpha) &= c_0 + 1 - \cos(\alpha) > 0, \quad \forall \alpha,
\end{align*}
\] (45)
with \(c_0 > 0\). It is then straightforward to verify that the aerodynamic force given by (43c) with the coefficients (45) satisfies (44); in addition, note also that \(c_L(0) = c_L(\pi) = 0\). Since the vector \(F\) on the right hand side of (42) is evaluated at the reference velocity, we have to evaluate the quantities (43) at \(\dot{x}\). Let us assume that

**A1:** the thrust force is perpendicular to the zero lift direction so that \(\delta = \pi/2\);

**A2:** there exists a time \(t\) such that

i) the reference and wind velocities imply
\[ \gamma(\dot{x}_r(t) - \dot{x}_w(t)) = \pi/2 \quad \text{and} \quad k_a|\dot{x}_r(t) - \dot{x}_w(t)|^2 = 1; \]

ii) the reference acceleration \(\ddot{x}_r(t)\) implies
\[ F_{gr1}(t) = 0 \quad \text{and} \quad F_{gr2}(t) = c_0. \]

By evaluating the angle of attack (43c) at the reference velocity with the assumption A1 and A2, one verifies that \(\alpha(t) = 0\). Then, (42) at \(t = \ell\) becomes
\[ f_\ell(\theta) = [F_{gr2}(t) - c_D(\theta)]\cos(\theta) + [c_L(\theta) - F_{gr1}(t)]\sin(\theta). \]
In view of the aerodynamic coefficients (45) and the assumption Aii, one has \(f_\ell(\theta) = 1 \neq 0\). Hence, there exists an aerodynamic force that satisfies (44), but for which there does not exist an equilibrium orientation for some reference and wind velocities at a fixed time instant.

**PROOF OF THEOREM 1**

Recall that the existence of an equilibrium orientation such that (11b) holds is equivalent to the existence, at any time \(t\), of one zero of the function \(f_\ell(\theta)\) in (42).

**Proof of the item i)**

Assume that the thrust force is parallel to the zero-lift-line so that \(\delta = 0\); the existence of the equilibrium orientation in the case \(\delta = \pi\) can be proven using the same arguments as those below. Now, in view of Eqs. 5. \(\dot{x}_a = R(\theta)v_a, S = R^T(\theta)SR(\theta)\), and of \(\delta = 0\), one verifies that the function \(f_\ell(\theta)\) given by (42) becomes
\[ f_\ell(\theta) = F_{gr1}(t)R(\theta)e_2 - k_a|\dot{x}_{rw}(t)|^2[c_L(\alpha_r)\cos(\alpha_r) + c_D(\alpha_r)\sin(\alpha_r)], \]
where \(F_{gr1}(t)\) is given by (43b) and
\[ \begin{align*}
\alpha_r(\theta, t) &= \theta - \gamma_r(t) + \pi, \\
\gamma_r(t) &= \tau(2\dot{x}_{rw}, \dot{x}_{rw}, \dot{x}_{rw}).
\end{align*} \] (46a)
\[ \dot{x}_{rw}(t) := \dot{x}_r(t) - \dot{x}_w(t). \] (47b)

It follows from (46b) that at any time \(t\) there exists an orientation \(\theta_0(t)\) such that \(\theta = \theta_0(t)\) yields \(\alpha_r(t) = 0\), i.e.,\[ \theta = \theta_0(t) = \gamma_r(t) - \pi \quad \Rightarrow \quad \alpha_r(t) = 0. \]

Consequently, \(\theta = \theta_0(t) + \pi\) yields \(\alpha_r(t) = \pi\) and \(\theta = \theta_0(t) - \pi\) yields \(\alpha_r(t) = -\pi\). Since it is assumed that the body shape is symmetric, then (46b) holds. Thus, Eq. (46) yields
\[ f_\ell(\theta_0(t) + \pi) = f_\ell(\theta_0(t) - \pi) = -f_\ell(\theta_0(t)). \] (48)

Since \(c_L(\theta) + c_D\) are continuous and defined \((\dot{x}, \theta)\) is differentiable). These two zeros, denoted by \(\theta_{e1}(t)\) and \(\theta_{e2}(t)\), belong to \(\theta_{e1}(t) \in [\theta_0(t) - \pi, \theta_0(t)]\) and \(\theta_{e2}(t) \in [\theta_0(t), \theta_0(t) + \pi]. \)

**Remark** By looking at the proof of item i), remark that the key assumption is that \(c_L(\theta) = c_D(\alpha) = 0\), which does not depend on the drag coefficient. Hence, drag forces have no role in the existence an equilibrium orientation when considering symmetric shapes powered by a thrust force parallel to their axis of symmetry. If the thrust force is not parallel to the shape’s axis of symmetry, one easily shows that the condition \(c_L(\theta) = c_L(\pi) = 0\) is no longer sufficient to ensure the equilibrium orientation existence for any reference velocity.

**Proof of the item ii)**

Under the assumption that the body’s shape is bisymmetric, Eqs. (7) hold, i.e., \(c_l(\alpha) = c_l(\pi/2)\) \(\forall \alpha\), \(c_D(\alpha) = c_L(\pi/2)\) \(\forall \alpha\). This property of the aerodynamic coefficients, in view of (43c), implies \(F_a(\dot{x}_a, \alpha) = F_a(\dot{x}_0, \alpha)\). Consequently, using the expression of the angle of attack (43c), one verifies that the apparent external force given by (43a) satisfies
\[ F(\dot{x}, \theta, \pi, t) = F(\dot{x}, \theta, \pi, t) \quad \forall (\dot{x}, \theta, t). \] (49)

In turn, it is straightforward to verify that the function \(f_\ell(\theta)\) given by (42) satisfies, at any time \(t\), the following
\[ f_\ell(\theta + \pi) = f_\ell(\theta - \pi) = -f_\ell(\theta) \quad \forall \theta. \] (50)

Then, analogously to the proof of the Item 1), the existence of at least two equilibrium orientations \(\theta_{e1}(t)\) and \(\theta_{e2}(t)\) such that \(f_\ell(\theta_{e1}(t)) = f_\ell(\theta_{e2}(t)) = 0\) can be shown by applying the intermediate value theorem.

Observe that Eqs. (50) imply that if \(\theta_{e1}(t)\) is an equilibrium orientation, i.e., \(f_\ell(\theta_{e1}(t)) = 0\) \(\forall t\), then another equilibrium orientation is given by \(\theta_{e2}(t) = \theta_{e1}(t) + \pi\). Now, to show that there always exists an equilibrium orientation ensuring a positive-semi definite thrust intensity, from Eq. (11a) observe that the thrust intensity at the equilibrium point is given by \(T_e = F^T(\dot{x}_r(t), \theta_{e1}(t), t)R(\theta_{e1}(t))e_1\). Then, it follows from (49) that if the thrust intensity is negative-semi definite at \(t\) along an equilibrium orientation, i.e., \(T_e(\dot{x}_r(t), \theta_{e1}(t), t) \leq 0\), then it is positive-semi definite at the the equilibrium orientation given by \(\theta_{e2}(t) = \theta_{e1}(t) + \pi\), i.e., \(T_e(\dot{x}_r(t), \theta_{e1}(t) + \pi, t) \geq 0\). Hence, one can always build up an equilibrium orientation \(\theta_{e1}(t)\) associated with a positive-semi definite thrust intensity.
Proof of Theorem 2

First, observe that if \( \sin(\delta) = 0 \), then the equilibrium orientation existence follows from Theorem 1 since the thrust force is parallel to the zero-lift-direction. Hence assume that
\[
\sin(\delta) \neq 0. \tag{51}
\]
Recall that the equilibrium orientation existence is equivalent to the existence, at any fixed time \( t \), of one zero of the function \( f_1(\theta) \) given by (42). In view of (5), \( \dot{x}_a = R(\theta) v_a \), and of \( S = R^T(\theta)SR(\theta) \), one can verify that (42) becomes
\[
f_t(\theta) = F_{gr}^T(\theta) R(\theta) c_2 - k_a |\dot{x}_w(t)|^2 [c_L(\alpha_r) \cos(\alpha_r + \bar{\delta}) + c_D(\alpha_r) \sin(\alpha_r + \bar{\delta})], \tag{52}
\]
where \( F_{gr} \) is given by (43b),
\[
\alpha_r = \alpha_r(\theta, t) = \theta - \gamma_r(t) + \pi - \bar{\delta}, \tag{53}
\]
\( \gamma_r \) by (17a), and \( \dot{x}_w \) by (17b). From Eq. (52) note that if \( |\dot{x}_w(t)| = 0 \), then there exist at least two zeros for the function \( f_t(\theta) \), i.e. at least two equilibrium orientations at the time \( t \). Thus, let us focus on the following case
\[
|\dot{x}_w(t)| \neq 0. \tag{54}
\]
It follows from (53) that at any fixed time \( t \), there exists an orientation \( \theta_0(t) = \gamma_r(t) + \pi - \bar{\delta} \) such that \( \theta = \theta_0(t) \) yields \( \alpha_r = 0 \), so \( \theta = \theta_0(t) + \pi \) yields \( \alpha_r = \pi \). Now, if \( f_t(\theta_0(t)) f_t(\theta_0(t) + \pi) \leq 0 \), then there exists a zero for the function \( f_t(\theta) \), and this zero belongs to \( [\theta_0(t), \theta_0(t) + \pi] \); in fact, the function \( f_t(\theta) \) changes sign on this domain and is continuous versus \( \theta \). We are thus interested in the case when the above inequality is not satisfied. Hence, assume also that
\[
f_t(\theta_0(t)) f_t(\theta_0(t) + \pi) > 0. \tag{55}
\]
Given the assumption that the body’s shape is symmetric, one has \( c_L(0) = c_L(\pi) = 0 \). So, in view of (52), imposing (55) divided by \( k_a^2 |\dot{x}_w(t)|^4 \sin(\delta)^2 \), which we recall to be assumed different from zero, yields
\[
|a_t - c_D(0)| |c_D(\pi) - a_t| > 0, \tag{56}
\]
where \( a_t := \frac{F_{gr}^T(\theta(t)) R(\theta(t)) c_2}{k_a \sin(\delta)|\dot{x}_w(t)|^2} \). Under the assumption that \( c_D(0) < c_D(\pi) \), the inequality (56) implies that
\[
c_D(0) < a_t < c_D(\pi). \tag{57}
\]
When the constraint (57) is satisfied, the inequality (55) holds and we cannot (yet) claim the existence of an equilibrium orientation at the time instant \( t \). The following shows that when the inequality (57) is satisfied, the existence of an equilibrium orientation at the time instant \( t \) follows from the symmetry of the body’s shape provided that the conditions of Theorem 2 hold true. Recall that when the inequality (57) is not satisfied, the existence of an equilibrium orientation at the time \( t \) follows from the fact that \( f_t(\theta_0(t)) f_t(\theta_0(t) + \pi) \leq 0 \).

Now, under the assumption that the body’s shape is symmetric, we have that Eqs. 6 hold. Let \( \alpha \in \mathbb{R}^+ \); then, by using \( c_D(\alpha) = c_D(-\alpha), c_L(\alpha) = -c_L(-\alpha) \), and (52), one verifies that (recall that \( \theta = \theta_0(t) \Rightarrow \alpha_r(t) = 0 \), so \( \theta = \theta_0(t) \pm \bar{\delta} \Rightarrow \alpha_r(t) = \pm \bar{\delta} \))
\[
\begin{align*}
f_t(\theta_0(t) - \bar{\delta}) f_t(\theta_0(t) + \bar{\delta}) &= \Delta_a^2 \sin^2(\delta) - \Delta_p^2, \tag{58} \\
\Delta_p &= \Delta_a(\bar{\delta}) := [a_t - c_D(\bar{\delta})] \cos(\bar{\delta}) + c_L(\bar{\delta}) \sin(\bar{\delta}), \tag{59} \\
\Delta_b(\bar{\delta}) &= [b_t + c_D(\bar{\delta}) \cos(\bar{\delta})] \sin(\bar{\delta}) + c_L(\bar{\delta}) \cos(\bar{\delta}), \tag{60}
\end{align*}
\]
\( \Delta_b(\bar{\delta}) \) equals zero everywhere. Differentiating (20), one has
\[
\Delta_a(\alpha) \Delta_a(\alpha) = \Delta_a(\alpha) \Delta_a(\alpha) \tag{61}
\]
which implies (50) with \( \alpha \in [\alpha_0, \alpha_s] \) since \( \Delta_a(\bar{\delta}) \) is continuous versus \( \bar{\delta} \). Now, in view of (59) note that \( \forall \alpha : c_D(0) < a_t < c_D(\pi) \) one has \( \Delta_a(\alpha) > 0 \). From still (59), note also that \( \forall \alpha : c_D(0) < a_t < c_D(\pi) \) one has
\[
\Delta_a(\bar{\delta}) \leq c_D(\pi) \cos(\bar{\delta}) - c_D(\bar{\delta}) \cos(\bar{\delta}) + c_L(\bar{\delta}) \sin(\bar{\delta}). \tag{62}
\]
If there exists \( \alpha_s \in (0, 90^\circ) \) such that \( c_L(\alpha_s) > 0 \) and
\[
c_L(\alpha_s) \sin(\alpha_s) - [c_D(\alpha_s) - c_D(\pi)] \cos(\alpha_s) = 0 \Leftrightarrow \text{Cond. (15)}
\]
then \( \Delta_a(\alpha_s) \leq 0 \) and (61) holds with \( \alpha_0 = 0 \). Consequently, there exists an angle \( \alpha_0 \) such that (61) is satisfied and, subsequently, an equilibrium orientation \( \theta_0(t) \).

Proof of the item i)

The vector \( \vec{F}_p \) in (21a) is independent of the vehicle’s orientation \( \vec{v}_a \). If and only if the coefficients \( c_L \) and \( c_D \) in (21b) are independent of \( \theta \). A necessary and sufficient condition for this independence is that the derivative of \( c_L \) and \( c_D \) w.r.t. \( \alpha \) equals zero everywhere. Differentiating (20) w.r.t. \( \alpha \) yields
\[
\begin{align*}
0 &= c_L'(\alpha) - \lambda' \sin(\alpha + \delta) - \lambda \cos(\alpha + \delta), \tag{63a} \\
0 &= c_D'(\alpha) + \lambda' \cos(\alpha + \delta) - \lambda \sin(\alpha + \delta). \tag{63b}
\end{align*}
\]
Multiply (62a) by \( \cos(\alpha + \delta) \) and (62b) by \( \sin(\alpha + \delta) \). Summing up the obtained relationships yields (24). Also, multiply (62a) by \( \sin(\alpha + \delta) \) and (62b) by \( -\cos(\alpha + \delta) \). Summing up the obtained relationships yields \( \lambda' \). Hence:
\[
\begin{align*}
\lambda &= c_L'(\alpha) \cos(\alpha + \delta) + c_D'(\alpha) \sin(\alpha + \delta), \tag{63a} \\
\lambda' &= c_L'(\alpha) \sin(\alpha + \delta) - c_D'(\alpha) \cos(\alpha + \delta). \tag{63b}
\end{align*}
\]
Therefore, the vector \( \vec{F}_p \) is independent of \( \theta \) \( \forall \vec{v}_a \) if and only if \( \lambda \) and \( \lambda' \) are given by (63). The function given by (63b), however, is not always equal to the derivative of (63a). By differentiating (63a) and by imposing the outcome equal to (63b), one yields the condition (22).
Proof of the item ii)

One verifies that if the condition (24) is satisfied and \( \delta = 0 \), substituting (25) in (20) yields \( \bar{c}_D(\alpha, \lambda) = \bar{c}_D \), i.e. a constant number, and \( \bar{c}_b = 0 \). Thus, \( \bar{F}_p \) in Eq. (21b) and \( \bar{F}_p \) in Eq. (21a), are independent of the orientation \( \theta \).

Proof of Lemma 3

Once the velocity errors dynamics are transformed into the form (21), i.e. \( m\ddot{\mathbf{v}} = \bar{F}_p - T_\mathbf{p} x \), with \( \bar{F}_p \) independent of \( \theta \), one verifies that \( \bar{c}_v \equiv 0 \) implies

\[
\begin{align*}
T_{p} &= \left| \bar{F}_p(v, t), \right| \\
\bar{v}(\theta) &= \bar{F}_p(v, t) \Rightarrow \bar{v} = \xi_p(t),
\end{align*}
\]

(64)

\[
\begin{align*}
T_{p} &= -\left| \bar{F}_p(v, t), \right| \\
\bar{v}(\theta) &= \bar{F}_p(v, t) \Rightarrow \bar{v} = \xi_p(t) + \pi,
\end{align*}
\]

(65)

where \( \xi_p \) denotes the angle between the vertical direction \( \bar{v}_0 \) and \( \bar{F}_p(v, t) \), i.e. \( \xi_p = \angle(\bar{v}_0, \bar{F}_p(v, t)) \). Consequently, at the time instant \( t \) one has \( \Theta_{\bar{v}}(t) = \{\bar{v}(t), \bar{v} + \bar{F}_p(v, t)\} \), if \( \bar{F}_p(v, t) \neq 0 \) and \( \Theta_{\bar{v}}(t) = S^1 \), if \( \bar{F}_p(v, t) = 0 \). Then, system (21) has a generically-unique equilibrium orientation (see the definition 2) if and only if there exists a unique, continuous reference velocity \( \bar{v}(t) \) such that \( |\bar{F}_p(\bar{v}(t), t)| = 0 \) \( \forall t \). Now, from Eq. (21a) observe that

\[
\bar{x}_b = f(\bar{x}_b, t),
\]

(66)

with \( f(\bar{x}_b, t) := g e_1 - \bar{c}\bar{x}_b - \bar{x}_w |\bar{c}| \bar{v} \bar{F} S - \bar{c} D I |\bar{x}_b - \bar{x}_w|\). Then, one verifies that \( f(\bar{x}_b, t) \) is uniformly locally Lipschitz \([67] \text{ p. 90 Lemma 3.2}]\) on any compact, convex \( D_c \subset \mathbb{R}^2 \) since \( \exists \delta \in \mathbb{R}^+ : |\bar{x}_w| < \bar{c}, \forall |\bar{x}_w| < \bar{c} \). Then, one verifies that \( f(\bar{x}_b, t) \) is uniformly locally Lipschitz \([67] \text{ p. 93 Example 3.3}]\). By considering the derivative of the positive-definite function given by \( V = \frac{1}{2}|\bar{x}_b - \bar{x}_w|^2 \) with \( \bar{c} D > 0 \) one shows that the solutions to the differential system \( (66) \) in \( D_c \). However, we cannot claim that the solution \( \bar{x}_b(t) \) is unique in \( \mathbb{R}^2 \) since this solution may leave any compact set – it may tend to infinity in finite time \([67] \text{ p. 93 Example 3.3}]\). By considering the derivative of the positive-definite function given by \( V = \frac{1}{2}|\bar{x}_b - \bar{x}_w|^2 \) with \( \bar{c} D > 0 \) one shows that the solutions to the differential system \( (66) \) are bounded, so there exists a convex, compact \( D_c \) that contains any solution starting at \( \bar{x}_b(0) \in \mathbb{R}^2 \). Therefore, we deduce that there exists a unique, continuous solution to System \( (66) \forall \bar{x}_b(0) \in \mathbb{R}^2 \) and, consequently, to \( F(\bar{v}(t), t) = 0 \) \( \forall t \).

Proof of Lemma 4

Assume that the aerodynamic coefficients are independent of the angle \( \delta \). Then, the condition \( (22) \) is satisfied for any value of \( \delta \) only if \( c_D^p = 2 c_L^p = 0 \forall \alpha \), \( c_D^p + 2 c_L^p = \forall \alpha \), whose general solution is given by

\[
\begin{align*}
c_D(\alpha) &= b_0 + b_1 \sin(2\alpha) - b_2 \cos(2\alpha), \\
c_L(\alpha) &= b_3 + b_4 \cos(2\alpha) + b_5 \sin(2\alpha),
\end{align*}
\]

where \( b_j \) denoting constants number. When the shape of the body is symmetric, the above functions must also satisfy the conditions \( (6) \). This implies that \( b_1 \) and \( b_3 \) are equal to zero. Using the fact that \( \cos(2\alpha) = 1 - 2 \sin^2(\alpha) \), one has \( (27) \) with \( c_0 = b_0 - b_2 \) and \( b_2 = c_1 \).

Proof of Theorem 5

Proof of the item i): To show that the direction of the vector \( \bar{F}_p \) is constant w.r.t. the vehicle’s orientation at the equilibrium configuration, we equivalently show that

\[
\partial_\theta \xi_p |(\bar{v}, \theta) = (0, \theta_i(t)) = 0,
\]

(67)

where \( \xi_p := \angle(\bar{v}_i, \bar{F}_p) \), and \( \bar{F}_p = (\bar{v}_i, \bar{v}_b) \bar{F}_p \). From relations \( (20) \), \( (21) \), and \( (22) \) one has

\[
\begin{align*}
F_p &= m g e_1 + f_p - m \bar{x}_b(t), \\
f_p &= k_s |\bar{v}_a| \bar{v} S - D I \bar{x}_a,
\end{align*}
\]

(68a)

\[
\begin{align*}
\{ c_L = c_L - [c_L^p \cos(\alpha + \delta) + c_D^p \sin(\alpha + \delta)] \sin(\alpha + \delta), \\
\{ c_D = c_D + [c_L^p \cos(\alpha + \delta) + c_D^p \sin(\alpha + \delta)] \cos(\alpha + \delta),
\end{align*}
\]

(68c)

where \( \bar{a}_i = (\bar{v}_i, \bar{v}_b) \bar{a}_i \). By using \( (68a) \) and \( (3) \), computing the partial derivative of \( \xi_p \) w.r.t. \( \bar{v}_a \) yields

\[
\begin{align*}
\partial_\theta \xi_p &= -\frac{\bar{F}_p^{T} \bar{v}_a}{|\bar{F}_p|^2}, \\
\partial_\theta f_p &= \bar{k}_s |\bar{v}_a| \bar{v} S - D I \bar{x}_a.
\end{align*}
\]

(69a)

Given \( (69b), (68c), \bar{a}_i = R \bar{v}_a \), and \( (58) \), one verifies that

\[
e_2^T R^T \partial_\theta f_p = 0.
\]

(70)

Then, from Eq. \( (69a) \) one obtains

\[
\partial_\theta \xi_p = -\frac{\bar{F}_p^{T} \partial_\theta f_p}{|\bar{F}_p|^2} e_2^T R^T F_p.
\]

(71)

Now, the transformed System \( (21) \) points out that the equilibrium condition \( \bar{v}_b \equiv 0 \) implies that \( \bar{F}_p \cdot \bar{v}_b \equiv 0 \) \( \forall t \). This latter condition writes in terms of coordinates

\[
e_2^T R^T (\theta_i(t)) F_p (\bar{v}_i, \theta_i(t), t) = 0 \quad \forall t,
\]

(72)

where \( \bar{v}_i = (\bar{v}_i, \bar{v}_b) \bar{v}_i \). By combining \( (72) \) and \( (71) \), one shows \( (67) \) when the condition \( (32) \) is satisfied.

Proof of the item ii): Local uniqueness of an equilibrium orientation is related to the equilibrium condition \( (42) \). In particular, if

\[
\partial_\theta f_1(\theta) |_{\theta = \theta_i(t)} = \partial_\theta F^T R^T e_2 |_{\theta = \theta_i(t)} \neq 0,
\]

(73)

then the implicit function theorem ensures the existence of a unique differentiable function \( \theta = \phi(t) \) satisfying \( f_1(t) = 0 \) when \( t \) belongs to a neighborhood \( \mathcal{I}_t \) of \( t \), and such that \( \theta_i(t) = \phi(t) \), i.e. \( f_1(\phi(t)) = 0 \) \( \forall t \in \mathcal{I}_t \). Hence, the condition \( (73) \) ensures that the equilibrium orientation \( \theta_i(t) \) is isolated and differentiable at the time instant \( t \).

Using \( (5), \partial_\theta F = \partial_\theta \bar{F}_p = \partial_\theta \bar{F}_p \), \( (42a), \) and \( (68a) \), one gets

\[
\partial_\theta e_2^T R^T F = -e_2^T R^T F_p.
\]

(74)

Now, it is a simple matter to verify that

\[
e_2^T R^T F = e_2^T R^T F_p.
\]

(75)
Also, recall that at the equilibrium one has:
\[ e_2^T R^T F_p |_{(x, \theta)\equiv (\bar{x}, \theta_0)} = 0, \]
therefore, in view of (75), at the equilibrium one has
\[ e_2^T R^T F_p |_{(x, \theta)\equiv (\bar{x}, \theta_0)} = 0, \] (76)
i.e. the second component of the vector \( R^T F_p \) is equal to zero at the equilibrium point. Now, in view of \( |F_p| = |R^T F_p| \), and of the assumption that the vector \( F_p \) is different from zero at the equilibrium point, one has that the first component of the vector \( R^T F_p \) is necessarily different from zero at the equilibrium point, i.e. \( e_1^T R^T F_p |_{(x, \theta)\equiv (\bar{x}, \theta_0)} \neq 0 \). This in turn implies, via (74), that the condition (73) is satisfied. Consequently, the equilibrium orientation \( \theta_0(t) \) is isolated and differentiable at the time instant \( t \).

**Proof of the item iii):** Using \( e_v := (\bar{v}_1, \bar{v}_2, \bar{v}_3) \) and System (21) yields the following tracking error dynamics
\[ m \ddot{x} = F_p(x, \theta(t) - T_p R(\theta)) e_1, \] (77a)
\[ \ddot{\theta} = \omega - \dot{\theta}_e(t), \] (77b)
where \( \dot{\theta} := \theta - \theta_0(t) \). In view of the item ii), the equilibrium orientation \( \theta_0(t) \) is differentiable, so the Eq. (77) is well-conditioned. Now, take \( (T_p, \omega) \) as control inputs and \( (m \ddot{x}, \dot{\theta}) \) as state variables. Observe that \( \partial_\theta F_p = \frac{\partial F_p}{\partial \theta} |_{F_p} \partial_\theta |_{F_p} + |F_p| \partial_\theta \right[ \frac{F_p}{|F_p|} \right], \) and that \( F_p / |F_p| = \pm R e_1 \) and \( T_p = \pm |F_p| \) at the equilibrium \((m \ddot{x}, \dot{\theta}) = (0, 0)\). In view of the item i), one shows that the state and control matrices associated with the linearization of System (77) are given by
\[ A = \begin{pmatrix} \frac{\partial F_p}{\partial \theta} |_{F_p} & (a(t) R(\theta)) e_1 + b(t) R(\theta) e_2 \\ 0 & 0 \end{pmatrix}, \]
\[ B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \]
where \( 0_{n \times m} \in \mathbb{R}^{n \times m} \) denotes a matrix of zeros, \( a(t) := \pm \partial_\theta |F_p| |_{F_p} |_{F_p} e_1(t), \theta(t), \dot{\theta}(t) \), and \( b(t) := \pm |F_p| (\dot{x}(t), \theta(t), \dot{\theta}(t), t) \). When the condition (52) holds, one has \( b(t) \neq 0 \) \( \forall t \) and it is a simple matter to verify that the matrix \((B \ A B - B)\) is of full rank, which implies the controllability of (21).

**Proof of Proposition 2**

By decomposing the velocity errors in the body frame basis, i.e. \( e_v := (\bar{v}_1, \bar{v}_2, \bar{v}_3) \) from System (21) one obtains
\[ m \ddot{v} = -m \omega S \bar{v} - T_p e_1 + F_p, \quad \theta = \omega, \] (78)
with \( F_p = R^T F_p \), and \( F_p \) given by (68). Recall that \( \dot{\theta} \in (-\pi, \pi) \) denotes the angle between the two vectors \( e_1 \) and \( F_p \) so that \( |F_p| \cos(\dot{\theta}) = F_{p1} \), and that the control objective is the asymptotic stabilization of \( \dot{\theta} \) to zero. Then, we consider the candidate Lyapunov function defined by:
\[ V = \frac{m}{2} |\dot{\bar{v}}|^2 + \frac{1}{k_2} \left( 1 - \frac{\bar{F}_{p}}{\bar{F}_{p}} \right). \] (79)
Now, to compute \( \dot{V} \), first verify that for any \( x \in \mathbb{R}^2 \) one has
\[ |x|^2 I + S x x^T S = x x^T. \]
By using \( \dot{\bar{R}} = \omega SR, \quad RS = SR, \quad S^2 = -I \), and the above identity, one can verify that
\[ \frac{d}{dt} \left( 1 - \frac{\bar{F}_{p}}{\bar{F}_{p}} \right) = \frac{\dot{\bar{F}}_{p}}{\bar{F}_{p}} \left( -\omega S \bar{F}_{p} + R^T \left( I - \frac{F_p}{F_p} \right) \bar{F}_{p} \right) \]
\[ = -\frac{\bar{F}_{p}}{\bar{F}_{p}} \left( \omega + \frac{F_p S \bar{F}_{p}}{\bar{F}_{p}} \right). \] (80)
In view of (68), \( F_p \) as (65), \( \partial_\theta F_p = \partial_\theta F_p = \partial_\theta F_p \), and \( \alpha = \theta - \gamma + \pi - \delta \), the term \( \bar{F}_{p} \) in the right hand side of the above equation becomes
\[ \bar{F}_{p} = \partial_\theta F_p \bar{x}_a + \partial_\theta F_p \bar{a} - m \bar{x}_r \]
\[ = \partial_\theta F_p \bar{x}_a + \partial_\theta F_p \bar{a} - m \bar{x}_r. \] (81)
Observe also that
\[ \partial_\theta F_p = \partial_\theta \left( |F_p| \frac{F_p}{|F_p|} \right) = \frac{F_p}{|F_p|} \partial_\theta |F_p| + |F_p| \partial_\theta \left( \frac{F_p}{|F_p|} \right) \] (82)
In light of (80), (81), and (82) the function \( V \) along the solutions of System (78) becomes
\[ \dot{V} = -k_1 |F_p| \bar{v}_2 - \frac{k_2}{k_2} \frac{n \cos^2(\theta/2)}{k_2} \] (83)
because \( \frac{n \cos^2(\theta/2)}{k_2} = \frac{F_p}{|F_p|} + \bar{F}_{p} \). Since \( \bar{V} \) is negative semi-definite, the velocity error term \( \bar{v} \) is locally bounded. The next step of the proof consists in showing the uniform continuity of \( \bar{V} \) along every system solution and, using Barbalat’s lemma, one deduces the convergence of \( \bar{v} \) and \( \bar{v} \) to zero. In order to prove the mentioned uniform continuity of \( \bar{V} \), it suffices to show that \( \bar{V} \) is bounded. Note that in view of Assumption 3 the vector \( F_p \) is different from zero in an open neighborhood of \((\bar{v}, \bar{v}) = (0, 0)\). Consequently, it is a simple matter to verify that there exists an open neighborhood of \((\bar{v}, \bar{v}) = (0, 0)\) in which \( \bar{V} \) is bounded. As a consequence, in view of (83), there exists an open neighborhood of \((\bar{v}, \bar{v}) = (0, 0)\) such that for any initial condition in it, \( \bar{v} \) and \( \bar{v} \) converge to zero. Now, in order to show that \( \bar{v} \) tends to zero, observe that
\[ \frac{d}{dt} \frac{\bar{F}_{p2}}{\bar{F}_{p1}} = -\frac{k_2}{k_2} \bar{v}_2 + \frac{k_3}{k_3} \frac{\bar{F}_{p1}}{|F_p| + \bar{F}_{p1}^2} \] (85)
where the control inputs \((T, \omega)\) were chosen as (35). By applying Barbalat’s Lemma, one verifies the uniform continuity of (85) in a neighborhood of \((\bar{v}, \bar{v}) = (0, 0)\). Then, the right hand side of (85) tends to zero. Since
\[ \bar{F}_{p1} \to 0 \quad (|F_p| + \bar{F}_{p1})^2 > 0, \quad \bar{F}_{p1} \to |F_p| > 0 \]
in a neighborhood of \((\bar{v}, \bar{v}) = (0, 0)\), then there exists an open neighborhood of \((\bar{v}, \bar{v}) = (0, 0)\) such that for any initial condition in it, \( \bar{v} \) necessarily tends to zero. As for the stability of the equilibrium \((\bar{v}, \bar{v}) = (0, 0)\), it is a consequence of relations (79) and (84).