The Scalar Magnetic Potential in Magnetoencephalography

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Abstract.
Two results on Magnetoencephalography (MEG) are reported in this presentation. First, we present an integral formula connecting the scalar magnetic potential with the values of the electric potential on the boundary of a conductive region. This formula provides the magnetic potential analogue of the well known Geselowitz formula. Second, we construct the scalar magnetic potential for the realistic ellipsoidal model of the brain, as an eigenfunction expansion in terms of surface ellipsoidal harmonics.

1. Magnetoencephalography
The electromagnetic activity of the brain is governed by the quasi-static approximation of Maxwell’s equations [8]

\[ \nabla \times \mathbf{E} = 0 \] (1)
\[ \nabla \times \mathbf{B} = \mu_0 (\mathbf{J}^p + \sigma \mathbf{E}) \] (2)
\[ \nabla \cdot \mathbf{B} = 0 \] (3)

where \( \mathbf{E} \) and \( \mathbf{B} \) are the electric and magnetic induction fields respectively, \( \mathbf{J}^p \) is the neuronal (primary) current, \( \mu_0 \) is the magnetic permeability and \( \sigma \) is the conductivity of the brain tissue.

Outside the head, there is no neuronal current and the conductivity is zero. Hence, the right hand side of equation (2) vanishes in the exterior of the head. This means that the magnetic induction field in the exterior of the head is irrotational and solenoidal. Consequently, it can be represented as the gradient of a harmonic function \( U \) which is known as the scalar magnetic potential, i.e.

\[ \mathbf{B} = \frac{\mu_0}{4\pi} \nabla U. \] (4)

In the case of a dipolar current-source at the point \( \mathbf{\tau} \) with moment \( \mathbf{Q} \), which is located inside a homogeneous conducting sphere the magnetic potential can be calculated in closed form [1,9]

\[ U(\mathbf{r}) = \frac{\mathbf{Q} \times \mathbf{\tau} \cdot \mathbf{r}}{F(\mathbf{r}, \mathbf{\tau})} \] (5)

1 On leave from the University of Patras
where

\[ F(r, \tau) = |r - \tau| |r - \tau| + r \cdot (r - \tau). \tag{6} \]

Similarly, equation (1) implies that

\[ E = -\nabla u \tag{7} \]

where \( u \) is the electric potential.

Taking the divergence of (2) and using (7) we obtain

\[ \sigma \nabla u(r) = \nabla \cdot J^p(r), \ r \in \Omega \tag{8} \]

where \( \Omega \) denotes the domain where the conductivity is not zero. If we demand from \( u \) to satisfy the standard boundary condition for the conductor problem

\[ \partial_n u(r) = 0, \ r \in \partial \Omega \tag{9} \]

then \( u \) is uniquely determined up to an additive constant (as it is the case with any potential problem).

Once we solve the Neumann problem (8), (9) we construct the dyadic field

\[ \tilde{D}(r) = \int_{\Omega} J^p(\tau) \otimes \frac{r - \tau}{|r - \tau|^3} dv(\tau) - \sigma \int_{\partial \Omega} u(r') \hat{n}(r') \otimes \frac{r - r'}{|r - r'|^3} ds(r'). \tag{10} \]

The dyadic \( \tilde{D} \) incorporates both the electric and the magnetic activity of the brain, in the sense that its scalar invariant provides the Geselowitz formula for Electroencephalography (EEG)

\[ 4\pi \sigma u(r) = \int_{\Omega} J^p(\tau) \cdot \frac{r - \tau}{|r - \tau|^3} dv(\tau) - \sigma \int_{\partial \Omega} u(r') \hat{n}(r') \cdot \frac{r - r'}{|r - r'|^3} ds(r') \tag{11} \]

and its vector invariant provides the Geselowitz formula for Magnetoencephalography (MEG)

\[ \frac{4\pi}{\mu_0} B(r) = \int_{\Omega} J^p(\tau) \times \frac{r - \tau}{|r - \tau|^3} dv(\tau) - \sigma \int_{\partial \Omega} u(r') \hat{n}(r') \times \frac{r - r'}{|r - r'|^3} ds(r'). \tag{12} \]

In fact the information carried by \( u \) is complementary to the information carried by \( B \). Therefore, it is the dyadic field \( \tilde{D} \) that tells the whole story about the neuronal activity of the brain.

2. The equation of the magnetic potential.

In view of (4) and (12) the magnetic potential \( U \) can be calculated as follows

\[ U(r) = -\int_{-\infty}^{+\infty} \frac{\partial}{\partial t} U(t \hat{r}) dt = \frac{4\pi}{\mu_0} \int_{-\infty}^{+\infty} \hat{r} \cdot B(t \hat{r}) dt \]

\[ = \int_{\Omega} J^p(\tau') \cdot (r' \times \hat{r}) \left[ \int_{-\infty}^{+\infty} \frac{dt}{|t \hat{r} - r'|^3} \right] dv(\tau') \]

\[ - \sigma \int_{\partial \Omega} u(r') \hat{n}(r') \cdot (r' \times \hat{r}) \left[ \int_{-\infty}^{+\infty} \frac{dt}{|t \hat{r} - r'|^3} \right] ds(r') \tag{13} \]
If we define the magnetic kernel

\[ M(r; r') = \frac{r}{F(r; r')} = \hat{r} \int_r^{+\infty} \frac{dt}{|\hat{r} - r'|^3} \tag{14} \]

then (13) provides the representation

\[
U(r) = \int_{\Omega} J^p(r') \times r' \cdot M(r; r') dv(r') \\
- \sigma \oint_{\partial \Omega} u(r') \hat{n}(r') \times r' \cdot M(r; r') ds(r') \tag{15}
\]

which is the analogue of the Geselowitz formula for the magnetic potential in terms of the trace of the electric potential over the boundary of the conductor.

A second view on the above calculations reveals that a knowledge of the radial component of \( B \) is enough to recover its tangential component as well. Indeed, equation (13) maps the radial component of \( B \) to the magnetic potential \( U \), then (4) maps \( U \) to \( B \) and finally by subtracting from \( B \) the radial component we obtain the tangential part of \( B \).

For the case of a spherical conductor the unit normal is parallel to the position vector and therefore the surface integral on the right hand side of (15) vanishes. This means that when the conducting medium is a sphere the magnetic potential is independent of the conductivity. Furthermore, for a dipolar source, the magnetic potential is also independent of the radius of the conductive sphere. As a consequence, shells of different conductivity, such as the cerebrospinal fluid, the skull and the scalp that surround the brain tissue, become invisible when the spherical model is adapted. This extreme simplicity reflects the high symmetry of the one dimensional spherical geometry, and in fact it is not realistic, since the brain is a triaxial ellipsoid which is a genuine three dimensional object.

3. The ellipsoidal system.

The human brain is shaped in the form of an ellipsoid with average semi-axes 6cm, 6.5cm and 9cm [10]. Its boundary is given by the equation

\[
\frac{x_1^2}{\alpha_1^2} + \frac{x_2^2}{\alpha_2^2} + \frac{x_3^2}{\alpha_3^2} = 1, \quad 0 < \alpha_3 < \alpha_2 < \alpha_1 < +\infty \tag{16}
\]

where

\[
h_1^2 = \alpha_2^2 - \alpha_3^2 \\
h_2^2 = \alpha_1^2 - \alpha_3^2 \\
h_3^2 = \alpha_1^2 - \alpha_2^2 \tag{17}
\]

are the squares of the semi-axes of the ellipsoid.

The ellipsoidal coordinates \((\rho, \mu, \nu)\) [7] are connected to the Cartesian ones by

\[
x_1 = \frac{1}{h_2 h_3} \rho \mu \nu \\
x_2 = \frac{1}{h_1 h_3} \sqrt{\rho^2 - h_3^2} \sqrt{\mu^2 - h_3^2} \sqrt{\nu^2 - h_3^2} \\
x_3 = \frac{1}{h_1 h_2} \sqrt{\rho^2 - h_2^2} \sqrt{\mu^2 - h_2^2} \sqrt{\nu^2 - h_2^2} \tag{18}
\]
where

$$0 \leq \nu^2 \leq h_3^2 \leq \mu^2 \leq h_2^2 \leq \rho^2 < +\infty.$$  

(19)

The coordinate $\rho$ describes a family of confocal ellipsoids and corresponds to the radial spherical coordinate $r$, the coordinates $\mu$ and $\nu$ describe two families of confocal hyperboloids of one and of two sheets respectively, and correspond to the angular spherical coordinates $\theta$ and $\phi$.

The ellipsoidal system has the metric coefficients

$$h_{\rho}^2 = \frac{(\rho^2 - \mu^2)(\rho^2 - \nu^2)}{(\rho^2 - h_3^2)(\rho^2 - h_2^2)}$$

(20)

$$h_{\mu}^2 = \frac{(\mu^2 - \nu^2)(\mu^2 - \rho^2)}{(\mu^2 - h_3^2)(\mu^2 - h_2^2)}$$

(21)

$$h_{\nu}^2 = \frac{(\nu^2 - \rho^2)(\nu^2 - \mu^2)}{(\nu^2 - h_3^2)(\nu^2 - h_2^2)}$$

(22)

and the outward unit normal on the ellipsoid $\rho$ is given by

$$\hat{\rho} = \frac{\rho}{h_{\rho}} \sum_{i=1}^{3} \frac{x_i}{\rho^2 - \alpha_i^2 + \alpha_i^2 \hat{x}_i}.$$  

(23)

Laplace’s operator in ellipsoidal coordinates reads

$$\Delta = \frac{1}{(\rho^2 - \mu^2)(\rho^2 - \nu^2)} \left[ (\rho^2 - h_3^2)(\rho^2 - h_2^2) \frac{\partial^2}{\partial \rho^2} + \rho(2\rho^2 - h_3^2 - h_2^2) \frac{\partial}{\partial \rho} \right]$$

$$+ \frac{1}{(\mu^2 - \rho^2)(\mu^2 - \nu^2)} \left[ (\mu^2 - h_3^2)(\mu^2 - h_2^2) \frac{\partial^2}{\partial \mu^2} + \mu(2\mu^2 - h_3^2 - h_2^2) \frac{\partial}{\partial \mu} \right]$$

$$+ \frac{1}{(\nu^2 - \rho^2)(\nu^2 - \mu^2)} \left[ (\nu^2 - h_3^2)(\nu^2 - h_2^2) \frac{\partial^2}{\partial \nu^2} + \nu(2\nu^2 - h_3^2 - h_2^2) \frac{\partial}{\partial \nu} \right].$$

(24)

The interior ellipsoidal eigenvalues of Laplace’s equation, known as interior ellipsoidal harmonics[7], are denoted by

$$E_n^m(\rho, \mu, \nu) = E_n^m(\rho)E_n^m(\mu)E_n^m(\nu)$$

(25)

and the corresponding exterior ellipsoidal harmonics are denoted by

$$F_n^m(\rho, \mu, \nu) = F_n^m(\rho)E_n^m(\mu)E_n^m(\nu)$$

(26)

where $n = 0, 1, 2, \ldots$ and $m = 1, 2, \ldots, 2n + 1$. The functions $F_n^m(\rho)$ are given by

$$F_n^m(\rho) = (2n + 1)E_n^m(\rho) \int_{\rho}^{+\infty} dx \frac{dx}{[E_n^m(x)]^2 \sqrt{x^2 - h_3^2} \sqrt{x^2 - h_2^2}}$$

(27)

and they satisfy the asymptotic behaviour

$$F_n^m(\rho) = O \left( \frac{1}{\rho^{n+1}} \right), \rho \to \infty.$$  

(28)
The surface ellipsoidal harmonics \{E_m^m(\mu)E_n^m(\nu)\}_{n,m} form a complete orthogonal system over the surface of every ellipsoid \( \rho = \rho_0 \) with respect to the inner product

\[
(f(\mu, \nu), g(\mu, \nu)) = \int_{\rho=\rho_0} f(\mu, \nu)g(\mu, \nu) \frac{ds(\mu, \nu)}{\sqrt{\rho_0^2 - \mu^2} \sqrt{\rho_0^2 - \nu^2}}.
\]

That is for \( n = 0, 1, \ldots \) and \( m = 1, 2, \ldots, 2n + 1 \)

\[
\int_{\rho=\rho_0} E_n^m(\rho)E_n^m(\nu)E_n^m(\mu)E_n^m(\nu) \frac{ds(\mu, \nu)}{\sqrt{\rho_0^2 - \mu^2} \sqrt{\rho_0^2 - \nu^2}} = \gamma_n^m \delta_{nn'} \delta_{mm'}
\]

where \( \gamma_n^m \) are the normalization constants.

4. The magnetic potential for the ellipsoid.

It is obvious that for the ellipsoid \( \rho = \alpha_1 \), given by (16), the surface integral on the right hand side of (12), describing the contribution of the conductive medium to the magnetic induction field, does not vanish. The evaluation of this integral offers the main difficulty in the solution of the forward problem of Magnetoencephalography. In this section, we use eigenfunction expansions in terms of surface ellipsoidal harmonics to evaluate this integral for the case of a single dipole, i.e. we basically evaluate the Green’s function for Magnetoencephalography in ellipsoidal geometry.

For a dipolar current located at the point \( \tau \), with a dipole moment \( Q \) we have

\[
J^p(\rho) = Q \delta(\rho - \tau)
\]

where \( \delta \) denotes the Dirac measure. For this current, equation (12) can be written as

\[
\frac{4\pi}{\mu_0} B(\rho) = \nabla \times \left( \frac{Q}{|\rho - \tau|} \right) - \sigma \nabla \times \int_{\rho'=\alpha_1} u(\rho') \frac{\hat{\rho} \times (\rho - \rho')}{|\rho - \rho'|} ds(\rho').
\]

The fundamental solution provides the expansion [2]

\[
\nabla_{\rho'} \frac{1}{|\rho - \rho'|} = \nabla \rho \sum_{n=1}^{2n+1} \sum_{m=1}^{2n+1} \frac{4\pi}{2n+1} \frac{1}{\gamma_n^m} E_n^m(\rho')E_n^m(\rho).
\]

The outward unit normal is expressed as [3]

\[
\hat{\rho} = \frac{\alpha_1 \alpha_2 \alpha_3}{h_1 h_2 h_3 \sqrt{\alpha_1^2 - \mu^2} \sqrt{\alpha_1^2 - \nu^2}} \sum_{m=1}^{3} \frac{h_m}{\alpha_m} E_1^m(\mu)E_1^m(\nu) \hat{e}_m.
\]

Finally, the boundary values of the electric potential can be expanded as [3]

\[
\sigma u(\alpha_1, \mu, \nu) = Q \sum_{n=1}^{2n+1} \sum_{m=1}^{3} \nabla_{\rho} E_n^m(\tau) \frac{\alpha_2 \alpha_3 \gamma_n^m E_n^m(\alpha_1)}{\alpha_1^2 - \mu^2} E_n^m(\mu)E_n^m(\nu).
\]

Inserting (33)-(35) in (32) and using orthogonality we arrive at the expansion of \( B \) in terms of ellipsoidal harmonics. Nevertheless, we can avoid this complicated calculations if we follow the corresponding steps for the sphere [9] using the characteristics of the ellipsoidal system. More precisely, in the case of a sphere, we calculated the magnetic potential by integrating the radial
component of \( \mathbf{B} \) along a ray from the observation point \( \mathbf{r} = (r, \theta, \phi) \) to infinity. This ray is defined as the intersection of the coordinate surfaces corresponding to the angular coordinates \( \theta \) and \( \phi \). In the case of the ellipsoid, the corresponding “ray” is the curve defined by the intersection of the coordinate surfaces associated with the “angular” coordinates \( \mu \) and \( \nu \), where the observation point is now given by \( \mathbf{r} = (\rho, \mu, \nu) \). Therefore, all we need to do is to integrate the \( \rho \)-component of \( \mathbf{B} \) from the point \((\rho, \mu, \nu)\) to infinity along the \((\mu, \nu)\)-coordinate curve

\[
C(\rho) = \frac{1}{h_1h_2h_3} \left[ h_1\rho\mu\mathbf{x}_1 + h_2\sqrt{\rho^2 - h_3^2}\sqrt{\mu^2 - h_3^2}\sqrt{\nu^2 - h_3^2} \right.
+ \left. h_3\sqrt{\rho^2 - h_3^2}\sqrt{\mu^2 - h_3^2}\sqrt{\nu^2 - h_3^2} \right]
\]  

(36)

In view of the identities

\[
\nabla = \frac{\hat{\rho}}{\rho^2} \frac{\partial}{\partial \rho} + \frac{\hat{\mu}}{\mu^2} \frac{\partial}{\partial \mu} + \frac{\hat{\nu}}{\nu^2} \frac{\partial}{\partial \nu}
\]  

(37)

and

\[
\frac{\partial}{\partial \rho} = h_\rho \hat{\rho} \cdot \nabla
\]  

(38)

we obtain

\[
U(\rho, \mu, \nu) = -\int_\rho^{+\infty} \frac{\partial}{\partial \rho} U(\rho', \mu, \nu) d\rho'
\]  

\[
= -\int_\rho^{+\infty} h_\rho \hat{\rho}' \cdot \nabla U(\rho', \mu, \nu) d\rho'
\]  

(39)

from which we see that it is only the \( \rho \)-component of \( \mathbf{B} \) that is needed in order to calculate \( \mathbf{B} \).

A series of long calculations now leads to the following expression for the magnetic potential in ellipsoidal form

\[
U(\rho, \mu, \nu) = -4\pi \sum_{n=1}^{2n+1} \sum_{m=1}^{3} \sum_{i=1}^{3} f_{nm}^i(\mu, \nu)(\mathbf{Q} \cdot \mathbf{D}_n^m(\tau) \cdot \hat{\mathbf{x}}_i) \int_\rho^{+\infty} \frac{E_n^m(\rho')E_n^i(\rho')}{\sqrt{\rho^2 - h_3^2}\sqrt{\rho'^2 - h_3^2}} d\rho'
\]  

(40)

where

\[
f_{nm}^i(\mu, \nu) = \frac{h_1h_2h_3(\mu^2 - \nu^2)}{h_1h_2h_3(\mu^2 - \nu^2)} \left[ E_1^i(\mu)E_1^i(\nu) E_m^i(\mu)E_m^i(\nu) - E_1^i(\mu)E_1^i(\nu) E_m^i(\mu)E_m^i(\nu) \right]
\]  

(41)

\[
\mathbf{D}_n^m(\tau) = \frac{1}{(2n+1)\gamma_n^m} \left[ \mathbf{Q} \cdot \mathbf{E}_n^m(\tau) \mathbf{I} + \sum_{\alpha_2=1}^{n+1} \sum_{\alpha_1=1}^{2\kappa+1} \sum_{\lambda_1=1}^{\kappa} \frac{E_\kappa^\lambda(\alpha_1)}{E_\kappa^\lambda(\alpha_1)} (\mathbf{\nabla} \mathbf{E}_n^m(\tau)) \otimes C_{\alpha_1\alpha_2}^{m\lambda} \right]
\]  

(42)

with \( \mathbf{I} \) denoting the identity dyadic, and

\[
C_{\alpha_1\alpha_2}^{m\lambda} = \int_{\rho=\alpha_1} \frac{E_n^m(\mu)E_n^m(\nu)E_\kappa^\lambda(\mu)E_\kappa^\lambda(\nu)\hat{\rho}(\mu, \nu)}{\sqrt{\alpha_1^2 - \mu^2}\sqrt{\alpha_1^2 - \nu^2}} ds(\mu, \nu)
\]

\[
= \alpha_1\alpha_2\alpha_3 \sum_{i=1}^{3} \frac{h_i}{h_1h_2h_3} \int_{\rho=\alpha_1} \frac{E_n^m(\mu)E_n^m(\nu)E_\kappa^\lambda(\mu)E_\kappa^\lambda(\nu)\hat{\rho}(\mu, \nu)}{\sqrt{\alpha_1^2 - \mu^2}\sqrt{\alpha_1^2 - \nu^2}} ds(\mu, \nu).
\]  

(43)
The first few multipole terms in the expansion (40) can be calculated analytically by using orthogonality properties [2,3]. Nevertheless, all integrals are continuous functions over compact sets and therefore they can be evaluated numerically.

There are both quantitative and qualitative differences between the spherical model of the brain and the realistic ellipsoidal one. In the ellipsoidal model, confocal shells of different conductivity are visible [5,6] and silent sources are not radial [4]. Nevertheless, the most important point is related to the error that one makes when he collects measurements from a realistic ellipsoidal conductive tissue and uses the direct solution of the spherical problem to implement an inversion algorithm in order to identify any neuronal activity. This inconsistency is almost universal today in magnetoencephalographic practice.

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