MULTIPLICATIVE PROPERTY OF LOCALIZED CHERN CHARACTERS FOR 2-PERIODIC COMPLEXES

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Abstract. In this paper, we prove the multiplicative property of localized Chern characters. As a direct consequence, a localized Chern character gives rise to a ring homomorphism from the $K$-group of periodic complexes to the bivariant Chow cohomology group. As an application, we prove the functoriality of Kiem–Li’s cosection-localized intersection homomorphisms.

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1. INTRODUCTION

Chern character is a ring homomorphism between the $K$-group and the Chow cohomology group of a scheme. It is known to be one of the most important ingredients to study Riemann–Roch theorem and may be of other interests. A localized Chern character was introduced by Baum–Fulton–MacPherson for studying Riemann–Roch for a singular scheme [BFM]. This is defined for those groups of a closed immersion rather than a scheme. (We consider an immersion of the singular scheme to a smooth space for Riemann–Roch.)

The construction was extended to a localized Chern character for an unbounded periodic complex of vector bundles (which of course may not be an element of the $K$-group) by Polishchuk–Vaintrob [PV]. They used it to define Witten’s top Chern class, which provided an algebraic definition of a virtual fundamental class of a moduli space of spin curves. Although Riemann–Roch theorem is already one very powerful application of (localized) Chern characters, this implicitly tells us that they can be used to study virtual fundamental classes as well.

Remark 1.1. The localized one is related to the usual one of the ambient space by the pushforward, see [Fu Proposition 18.1(a)] and (1.2) below.
The terminology “localized” comes from this relationship through the closed immersion.

Unlike a usual one or a localized one for a bounded complex, the multiplicative property of a localized Chern character for a periodic complex is not known in general. It is conjectured in the first remark of [PV, Section 2.3]. Meanwhile, even if a localized Chern character for a bounded complex is multiplicative [Fu, Example 18.1.5], it is mentioned that “it would be interesting to find a direct proof of this multiplicative property” there.

The purpose of this article is to prove the multiplicative property. The proof is direct enough so that it can be extended to that for Deligne–Mumford (DM for short) stacks as well. Then we would like to introduce a functorial property which we may encounter when we study virtual fundamental classes as an application of the multiplicative property.

**Multiplicative property.** We provide the precise statement of the multiplicative property. Let $Y$ be a finite type DM stack over an arbitrary field $k$. Let $E_\bullet$ be a 2-periodic complex of vector bundles on $Y$

$$E_\bullet := \cdots \overset{d}{\longrightarrow} E_+ \overset{d_+}{\longrightarrow} E_- \overset{d}{\longrightarrow} E_+ \overset{d_+}{\longrightarrow} \cdots$$

where $E_+$ sits in even degrees and $E_-$ sits in odd degrees. We say $E_\bullet$ is *strictly exact off a closed substack* $X \hookrightarrow Y$ if it satisfies the following two conditions.

- $E_\bullet$ is exact on $Y \setminus X$.
- $\ker d_+$ and $\ker d_-$ are vector bundles on $Y \setminus X$.

Throughout the paper, by a 2-periodic complex, we mean a 2-periodic complex of vector bundles.

In [PV], for a 2-periodic complex $E_\bullet$ on $Y$ strictly exact off $X$, for $i : X \hookrightarrow Y$, Polishchuk–Vaintrob constructed a localized Chern character

$$\text{ch}^Y_X(E_\bullet) : A_*(Y) \longrightarrow A_*(X)$$

satisfying

$$A_*(Y) \xrightarrow{\text{ch}^Y_X(E_\bullet)} A_*(X) \xrightarrow{i_*} A_*(Y).$$

Here $A(-)$ denotes the Chow group of DM stacks with rational coefficients, see [Gi, Vi]. Throughout the paper, every Chow group is with $\mathbb{Q}$-coefficients.

**Theorem 1.2.** Let $E_1^\bullet, E_2^\bullet$ be 2-periodic complexes on $Y$ strictly exact off $X_1 \overset{i_1}{\hookrightarrow} Y$, $X_2 \overset{i_2}{\hookrightarrow} Y$, respectively. Then the following multiplicative property holds,

$$\text{ch}^Y_{X_1 \cap X_2}(E_1^\bullet \otimes E_2^\bullet) = \text{ch}^X_{X_1 \cap X_2}(i_1^* E_2^\bullet) \circ \text{ch}^Y_{X_1}(E_1^\bullet).$$
Theorem 1.2 is proved in [PV, Proposition 2.3(vi)] for a special case when
- $Y$ is (the total space of) a vector bundle $V$ on $X_1$,
- $E_1^\star$ is the Koszul 2-periodic complex of $V^\star$ (it is then strictly exact off $X_1$), and
- $E_2^\star$ is the pullback of a 2-periodic complex on $X_1$ strictly exact off $X = X_1 \cap X_2$.

In [OS, Lemma 2.18], Sreedhar and the author proved $K$-theoretic version of Theorem 1.2.

In fact, Polishchuk–Vaintrob constructed $\text{ch}^Y_X(E_*)$ as an element of the bivariant Chow group
\[
\text{ch}^Y_X(E_*) \in A^* (X \xrightarrow{i} Y),
\]
which contains the homomorphism (1.1) as a part of the definition [Fu, Chapter 17]. Note that when $i$ is the identity morphism, the bivariant Chow group $A^* (X \xrightarrow{id} X)$ is defined to be the Chow cohomology group of $X$. We denote by $K_{0}^{Z/2, st}(Y)_X$ the Grothendieck group of an additive category of 2-periodic complexes of vector bundles on $Y$ strictly exact off $X$. As a direct consequence of Theorem 1.2 we obtain the following corollary.

**Corollary 1.3.** The additive homomorphism
\[
(1.3) \quad \text{ch}^Y_X : K_{0}^{Z/2, st}(Y)_X \longrightarrow A^* (X \xrightarrow{i} Y)
\]
is a ring homomorphism.

Note that Polishchuk–Vaintrob proved that the homomorphism (1.3) is an additive homomorphism [PV, Proposition 2.3(iv)].

**Functoriality of cosection-localized intersection homomorphisms.**
A cosection-localized intersection homomorphism constructed by Kiem–Li [KL] has become one of the most useful tools to study virtual fundamental classes of moduli spaces. It localizes the Fulton–MacPherson’s intersection homomorphism (taking a cycle in a vector bundle to the intersection with the zero section) in the following sense.

Let $M$ be a finite type DM stack and $F$ be a vector bundle on $M$. Let $\sigma : F \rightarrow \mathcal{O}_M$ be a homomorphism of sheaves (called a cosection). We denote by $w_\sigma \in \Gamma(\mathcal{O}_F)$ the function on the total space of $F$ defined by $\sigma$. Then the zero loci $Z(w_\sigma)$ of the function $w_\sigma$ and $Z(\sigma) := Z(\sigma^*)$ of the section $\sigma^*$ satisfy
\[
Z(\sigma) \subset M \subset Z(w_\sigma) \subset F,
\]
where the immersion in the middle is given by the zero section. Kiem–Li constructed an intersection homomorphism of degree $\deg = \text{rank} F$
\[
0_{F, \sigma}^\star : A_* (Z(w_\sigma)) \longrightarrow A_* - \text{rank} F (Z(\sigma))
\]
commuting the diagram

\[
\begin{array}{ccc}
A_\ast(Z(w_\sigma)) & \xrightarrow{\text{pushforward}} & A_\ast(F) \\
0'_{F,\sigma} & & 0'_{F} \\
A_\ast\text{--}\text{rank}F(Z(\sigma)) & \xrightarrow{\text{pushforward}} & A_\ast\text{--}\text{rank}F(M),
\end{array}
\]

where \(0'_{F}\) denotes the Fulton–MacPherson’s intersection homomorphism. The homomorphism \(0'_{F,\sigma}\) is a \textit{cosection-localized intersection homomorphism}.

In [KO Theorem 1.1], Kim and the author proved a cosection-localized intersection homomorphism is equivalent to a localized Chern character up to a Todd class. Using this expression and the multiplicative property of a localized Chern character, we prove the functorial property of cosection-localized intersection homomorphisms.

\textbf{Theorem 1.4.} Let \(F_1, F_2\) be vector bundles on \(M\) and

\(\sigma_1 : F_1 \rightarrow \mathcal{O}_M, \ \sigma_2 : F_2 \rightarrow \mathcal{O}_M\)

be cosections. Then the following functoriality holds true

\[
0'_{F_1 \oplus F_2, \sigma_1 \oplus \sigma_2} = 0'_{F_1, \sigma_1} \circ 0'_{F_2, \sigma_2} : A_\ast(Z(w_{\sigma_1}) \times_M Z(w_{\sigma_2})) \rightarrow A_\ast\text{--}r(Z(\sigma_1 \oplus \sigma_2)),
\]

where \(r := \text{rank}F_1 + \text{rank}F_2\) and \(0'_{F_2, \sigma_2}\) is considered to be a homomorphism

\[
A_\ast(Z(w_{\sigma_1}) \times_M Z(w_{\sigma_2})) \rightarrow A_\ast\text{--}\text{rank}F_2(Z(w_{\sigma_1}) \times_M Z(\sigma_2)).
\]

Note that the homomorphism \(0'_{F_1 \oplus F_2, \sigma_1 \oplus \sigma_2}\) is defined through

\[
Z(w_{\sigma_1}) \times_M Z(w_{\sigma_2}) \subset Z(w_{\sigma_1} \oplus \sigma_2) = Z(w_{\sigma_1} + w_{\sigma_2}).
\]

It seems interesting to think if \(0'_{F_1, \sigma_1} \circ 0'_{F_2, \sigma_2}\) can be considered to be an operator on \(A_\ast(Z(w_{\sigma_1} \oplus \sigma_2))\).

\textbf{Plan.} Here is a plan of the paper. In Section 2 we review a definition of a localized Chern character (1.1) and prove the multiplicative property (Theorem 1.2). In Section 3 we prove the functoriality of cosection-localized intersection homomorphisms (Theorem 1.4) as an application.

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2. **Multiplicative property of localized Chern characters**

2.1. **Construction of localized Chern characters** (1.1). We review the construction in [PV, Section 2] of (1.1) briefly. As we have mentioned, the idea is based on that in [BFM] for bounded complexes. It is summarized in [Fu, Chapter 18]. So we prefer to refer the results in [Fu, Chapter 18] together with those in [PV, Section 2].

**Construction.** When $X = Y$, we define

$$
(2.1) \quad \text{ch}^Y_X(E_\bullet) := \text{ch}(E_+) - \text{ch}(E_-).
$$

Now we assume that $X \neq Y$. We’d like to define $\text{ch}^Y_X(E_\bullet) \cap [V] \in A_\ast(X)$ for $V$ an integral closed substack of $Y$.

Since $X \neq Y$, $E_+$ and $E_-$ have the same rank, $\text{rank}E_+ = \text{rank}E_-$. Let $Gr := Gr(\text{rank}E_+, E_+ \oplus E_-)$ be the Grassmannian over $Y$ and $G := Gr \times_Y Gr$ be the product of the Grassmannian. Consider a locally closed immersion

$$
\phi : Y \times \mathbb{A}^1 \to G \times \mathbb{P}^1, \quad (y, \lambda) \mapsto (\Gamma_{\text{ad}_1(y)}, \Gamma_{\text{ad}_V(y)}, \lambda),
$$

where $\Gamma$ denotes the graph and $d_\pm(y) : E_\pm|_{Y_y} \to E_\mp|_{Y_y}$ is the induced homomorphism at fiber. Let $Y_\infty \subset G$ be the restriction to $\infty \in \mathbb{P}^1$ of the scheme theoretic image of $\phi$. Set-theoretically it is the closure $\overline{\phi(Y \times \mathbb{A}^1)}$.

Recall that $\text{rank}(\ker d_\pm)$ are locally constant on $Y - X$. Hence they can be extended to locally constant ones on $Y$. Let

$$
H := Gr(\text{rank}(\ker d_+), E_+) \times_Y Gr(\text{rank}(\ker d_-), E_-)
$$

be another product of Grassmannians over $Y$. A direct sum gives rise to an immersion $H \hookrightarrow Gr \to G$, where the second one is the diagonal morphism. In [Fu, Chapter 18] and [PV, Section 2], it is explained that the immersion $Y_\infty \times_Y (Y - X) \hookrightarrow G|_{Y - X}$ factors through

$$
Y_\infty \times_Y (Y - X) \hookrightarrow H|_{Y - X} \to G|_{Y - X}.
$$

This tells us that $Y_\infty \subset G|_X \cup H$.

Let $\tilde{V}$ be the scheme theoretic image of $\phi : V \times \mathbb{A}^1 \to G \times \mathbb{P}^1$. We define $[V_\infty]$ as its Gysin pullback image by the regular immersion $\{\infty\} \to \mathbb{P}^1$,

$$
[V_\infty] := \infty^\ast[\tilde{V}] \in A_{\text{dim} V}(Y_\infty).
$$

It is well-defined [Fu, Lemma 18.1(b)]. Using $Y_\infty \subset G|_X \cup H$, we can choose a class $[V_{X,\infty}] \in A_{\text{dim} V}(G|_X \cap Y_\infty)$ such that

$$
(2.2) \quad [V_\infty] - j_\ast[V_{X,\infty}] \in \text{image}(A_{\text{dim} V}(H \cap Y_\infty) \to A_{\text{dim} V}(Y_\infty)),
$$

where $j : G|_X \cap Y_\infty \to Y_\infty$ is the closed immersion. Then $\text{ch}^Y_X(E_\bullet) \cap [V]$ is defined to be

$$
(2.3) \quad \text{ch}^Y_X(E_\bullet) \cap [V] := p_X \ast((\text{ch}(\xi_+) - \text{ch}(\xi_-)) \cap [V_{X,\infty}])
$$

where $p_X : G|_X \to X$ is the projection morphism and $\xi_\pm$ are the tautological bundles on each component of $G$. 

Specialization map. Let $X_\infty := Y_\infty \times_Y X$. Unfortunately, the specialization morphism

$$A_*(Y) \rightarrow A_*(X_\infty), \quad [V] \mapsto [V_{X_\infty}],$$

constructed above is not well-defined. It is well-defined only up to a class in $H$ [Fu, Lemma 18.1 (c)]. Letting $B_*(X_\infty) := A_*(X_\infty)/A_*(X_\infty \cap H)$ (which is isomorphic to $A_*(X_\infty \setminus H)$ by [Fu, Proposition 1.8]), we obtain a well-defined specialization morphism

$$(2.4) \quad sp : A_*(Y) \rightarrow B_*(X_\infty), \quad [V] \mapsto [V_{X_\infty}].$$

Then we can rewrite (2.3) by

$$(2.5) \quad \text{ch}_Y(E_\bullet) \cap - = p_{X_\bullet}((\text{ch}(\xi_+) - \text{ch}(\xi_-)) \cap sp(-)).$$

Remark 2.1. For any morphism $Y' \rightarrow Y$, we can do the same construction using the pullback of $E_\bullet$ to $Y'$ to get a bivariant class

$$\text{ch}_X(E_\bullet) \in A^*(X \rightarrow Y).$$

2.2. Alternative description of (2.5). We keep assuming $X \neq Y$ so that we have $\text{rank} E_+ = \text{rank} E_-$. In this section, we provide an equivalent description of a localized Chern character which seems to behave better in deformations. The key point is to view $\xi_\pm$ as a part of a complex. This is also the key idea of the proof of the multiplicative property.

Periodic complex structure on $\xi_\pm$. Here we would like to explain that the tautological bundles $\xi_\pm$ form a 2-periodic complex

$$(2.6) \quad \xi_\bullet := \{ \xi_+, \frac{\delta_+}{\delta_-} \xi_- \}$$

on the scheme theoretic image of $\phi$ strictly exact off whose restriction to $X$. Precisely, letting $\tilde{Y} \subset G \times (\mathbb{P}^1 \setminus \{0\})$ be the scheme theoretic image (outside of the origin) and $\tilde{X} := \tilde{Y} \times_Y X$, it is a 2-periodic complex on $\tilde{Y}$ strictly exact off $\tilde{X}$.

We start with an exact 2-periodic complex on $G$

$$\{ E_+ \oplus E_- \xrightarrow{pr_-} E_+ \oplus E_- \}$$

where $pr_\pm$ are the projection morphisms. Let $\alpha_\pm \in \text{Hom}(\xi_\pm, (E_+ \oplus E_-)/\xi_\pm)$ be the homomorphisms defined by

$$\alpha_\pm : \xi_\pm \subset E_+ \oplus E_- \xrightarrow{pr_\pm} E_+ \oplus E_- \rightarrow (E_+ \oplus E_-)/\xi_\pm.$$ 

Then on a closed substack $(\alpha_+)^{-1}(0) \cap (\alpha_-)^{-1}(0) \subset G$, we obtain a 2-periodic complex

$$\xi_\bullet := \{ \xi_+, \frac{\delta_+}{\delta_-} \xi_- \}.$$. 

Hence the 2-periodic complex $\xi_\bullet$ is defined on $\tilde{Y}$.

The restriction of $(\tilde{Y}, \xi_\bullet)$ to each fiber $\lambda \in (A^1 \setminus \{0\})$ is $(Y, E_\bullet)$. Hence $\xi_\bullet$ is strictly exact off $\tilde{X} \cup Y_\infty$. So it remains to show that the restriction to $\infty$ is strictly exact off $\tilde{X}|_{\infty} = X_\infty \subset \tilde{Y}|_{\infty} = Y_\infty$. By the construction, $\xi_\bullet$ is strictly exact on $H$. Since $Y_\infty - X_\infty \subset H$, it is strictly exact off $X_\infty$. Note that strictness should not be checked at each fiber. However it follows from the construction.

**Alternative description.** Let

$$sp_A : A_*(Y) \longrightarrow A_*(Y_\infty), \quad [V] \longmapsto [V_\infty]$$

be the specialization morphism. Beware that it is different from $sp$ in $[24]$.

**Lemma 2.2.** We obtain an equivalence

$$ch^Y(E_\bullet) \cap - = p_{X*} \left( ch^{Y_\infty}_{X_\infty}(\xi_\bullet|_{X_\infty}) \cap sp_A(-) \right).$$

**Proof.** By $[25]$ and $[21]$, we obtain

$$ch^Y(E_\bullet) \cap - = p_{X*} \left( ch^{Y_\infty}_{X_\infty}(\xi_\bullet|_{X_\infty}) \cap sp(-) \right).$$

So it is enough to prove that

$$p_{X*} \left( ch^{Y_\infty}_{X_\infty}(\xi_\bullet|_{X_\infty}) \cap sp_A(-) \right) = p_{X*} \left( ch^{X_\infty}_{X_\infty}(\xi_\bullet|_{X_\infty}) \cap sp(-) \right).$$

For an integral closed substack $V \subset Y$, we obtain a decomposition

$$sp_A[V] = sp[V] + [V_H], \quad [V_H] \in A_{\dim V}(H \cap Y_\infty)$$

by $[22]$. By [PV] Proposition 2.3(iii), we have

$$ch^{Y_\infty}_{X_\infty}(\xi_\bullet|_{X_\infty}) \cap sp[V] = ch^{X_\infty}_{X_\infty}(\xi_\bullet|_{X_\infty}) \cap sp[V]$$

since $sp[V] \in A_{\dim V}(X_\infty)$. Let $i : H \cap X_\infty \hookrightarrow X_\infty$ be the closed immersion. Then by [PV] Proposition 2.3(i),(iii), we have

$$ch^{Y_\infty}_{X_\infty}(\xi_\bullet|_{X_\infty}) \cap [V_H] = i_* \left( ch^{H \cap Y_\infty}_{H \cap X_\infty}(\xi_\bullet|_{H \cap Y_\infty}) \cap [V_H] \right).$$

Since $\xi_\bullet$ is strictly exact on $H$, the right-hand side is zero. $\square$

**2.3. Proof of the multiplicative property (Theorem 1.2).** We follow the notations in Theorem 1.2. Let $\xi_\bullet^2$ be a 2-periodic complex $[26]$ constructed by using $E_\bullet^2$. Similarly, we define $G^2, Y_\infty^2, X_\infty^2, sp_A^2, \tilde{Y}^2, \tilde{X}^2$ using $E_\bullet^2$. Simply, we let $X := X_1 \cap X_2, X_\infty := X_\infty^2 \times X_2 X$ and $\tilde{X} := \tilde{X}^2 \times X_2 X$. 
Lemma 2.3. We have an equality of homomorphisms

\[ \operatorname{ch}^Y_X(E^1 \otimes E^2) \cap - = p_{X^*} \left( \operatorname{ch}^{Y^2}_{X^*} (E^1 \otimes \xi^2) \cap sp^2_A(-) \right). \]

Here, \( E^1 \) on the right-hand side is the pullback along \( Y^2_x \to G^2 \to Y \).

Proof. Note that the proof provides an alternative proof of Lemma 2.2 by letting

\[ E^1 := \cdots \to \mathcal{O}_Y \to 0 \to \mathcal{O}_Y \to 0 \to \cdots. \]

The main idea we use is the homotopy property. We first consider a class in

\[ p_{X^*} \left( \operatorname{ch}^{Y^2}_X (E^1 \otimes \xi^2) \cap [\tilde{V}^2] \right) \in A_{\text{dim} V + 1} \left( X \times (\mathbb{P}^1 \setminus \{0\}) \right) \]

for an integral closed substack \( V \subset Y \), where \( p_X : \tilde{X} \to X \times (\mathbb{P}^1 \setminus \{0\}) \) is the projection morphism. Homotopy property says that the Gysin map

\[ \lambda^1 : A_{\text{dim} V + 1} \left( X \times (\mathbb{P}^1 \setminus \{0\}) \right) \longrightarrow A_{\text{dim} V} (X) \]

is independent of the closed immersion \( \lambda : \{\lambda\} \to \mathbb{P}^1 \setminus \{0\} \) \cite[Corollary 6.5]{Fu}. Hence we have

\[ 1^! \left( p_{\tilde{X}^*} \left( \operatorname{ch}^{Y^2}_X (E^1 \otimes \xi^2) \cap [\tilde{V}^2] \right) \right) = \infty^! \left( p_{\tilde{X}^*} \left( \operatorname{ch}^{Y^2}_X (E^1 \otimes \xi^2) \cap [\tilde{V}^2] \right) \right). \]

Note that \( E^1 \otimes \xi^2 \) is strictly exact off \( \tilde{X} \) since \( \xi^2 \) is strictly exact off \( \tilde{X}^2_x \).

Using the commutativity of refined Gysin pullbacks and pushforwards \cite[Theorem 6.2(a)]{Fu} and the bivariant property \cite[Definition 17.1(C3)]{Fu}, the left-hand side of (2.7) becomes

\[ 1^! \left( p_{\tilde{X}^*} \left( \operatorname{ch}^{\tilde{V}^2}_X (E^1 \otimes \xi^2) \cap [\tilde{V}^2] \right) \right) = 1^! \left( \operatorname{ch}^{\tilde{V}^2}_X (E^1 \otimes \xi^2) \cap [\tilde{V}^2] \right) \]

\[ = \operatorname{ch}^{\tilde{V}^2}_X (E^1 \otimes \xi^2) \cap \lambda^1 [\tilde{V}^2] \]

\[ = \operatorname{ch}^{X^2}_X (E^1 \otimes \xi^2) \cap [V]. \]

On the other hand, the right-hand side of (2.7) becomes

\[ \infty^! \left( p_{\tilde{X}^*} \left( \operatorname{ch}^{\tilde{V}^2}_X (E^1 \otimes \xi^2) \cap [\tilde{V}^2] \right) \right) = p_{X^*} \left( \infty^! \left( \operatorname{ch}^{\tilde{V}^2}_X (E^1 \otimes \xi^2) \cap [\tilde{V}^2] \right) \right) \]

\[ = p_{X^*} \left( \operatorname{ch}^{Y^2}_{X^*} (E^1 \otimes \xi^2) \cap \infty^! [\tilde{V}^2] \right) \]

\[ = p_{X^*} \left( \operatorname{ch}^{Y^2}_{X^*} (E^1 \otimes \xi^2) \cap sp^2_A([V]) \right). \]

\[ \Box \]

Let \( sp^2 : A_*(Y) \to B_*(X^2_x) \) be the specialization morphism defined in (2.4). The proof of the following lemma is parallel to that of Lemma 2.2

Lemma 2.4. We obtain an equivalence

\[ p_{X^*} \left( \operatorname{ch}^{Y^2}_{X^*} (E^1 \otimes \xi^2) \cap sp^2_A(-) \right) = p_{X^*} \left( \operatorname{ch}^{X^2}_{X^*} (E^1 \otimes \xi^2) \cap sp^2 (-) \right). \]
Proof. This is obvious because $E_1 \otimes \xi_2^*$ is strictly exact off $X_\mathcal{Z}_\mathcal{Y} \subset X_\mathcal{Z}_\mathcal{Y}$. The rest of the proof is the same as the one of Lemma 2.2. □

Proof of Theorem 1.2 Let $p_{X_2} : X_\mathcal{Z}_\mathcal{Y} \to X_2$ be the projection morphism. Then we obtain

$$p_{X_2} \left( \text{ch}_{X_\mathcal{Z}_\mathcal{Y}}(E_1 \otimes \xi_2^*) \cap sp^2(-) \right) = p_{X_2} \left( \text{ch}_{X_\mathcal{Z}_\mathcal{Y}}(E_1^*) \cdot \text{ch}_{X_\mathcal{Z}_\mathcal{Y}}(\xi_2^*) \cap sp^2(-) \right)$$

$$= \text{ch}_{X_2}(E_1^*) \cdot p_{X_2} \left( \text{ch}_{X_\mathcal{Z}_\mathcal{Y}}(\xi_2^*) \cap sp^2(-) \right)$$

where the first equality comes from [PV] Proposition 2.3(v), the second equality is the commutativity of bivariant classes and pushforwards [Fu, Definition 17.1(C1)] and the third one comes from (2.1) and (2.5). Then the proof follows from Lemma 2.3 and 2.4.

3. Functoriality of cosection-localized intersection homomorphisms

In this section, we discuss an application to intersection theory.

3.1. Koszul 2-periodic complex. Let $Y$ be a finite type DM stack and $E$ be a vector bundle on $Y$. Let $\alpha : \mathcal{O}_Y \to E^*$ be a section of $E^*$ and $\beta : \mathcal{O}_Y \to E$ be a section of $E$. Suppose that the pairing $\langle \alpha, \beta \rangle$ is zero. The Koszul 2-periodic complex $\{\alpha, \beta\}$ is defined as

$$\{\alpha, \beta\} := \{ \oplus_k \wedge^{2k+1} E^* \circ\wedge^{2k+1} E^* \}.$$  

It is a 2-periodic complex on $Y$ strictly exact off $Z(\alpha, \beta) := Z(\alpha) \cap Z(\beta)$ [KO, Lemma 2.2(2)]. Here $\wedge_{\beta}$ denotes the contraction by $\beta$ (see [KO] Section 2.2 for the convention). We sometimes consider $\alpha$ as a cosection of $E$ without any mention when the context is clear.

Set-up. Consider the following diagram of vector bundles on $Y$

$$\begin{array}{ccc}
\mathcal{O}_Y & \xrightarrow{\beta_1} & \mathcal{O}_Y \\
\downarrow & & \downarrow \\
E_1 & \xrightarrow{f} & E \\
\downarrow & & \downarrow \\
\mathcal{O}_Y & \xrightarrow{\alpha_1} & \mathcal{O}_Y
\end{array} \quad \begin{array}{ccc}
\mathcal{O}_Y & \xrightarrow{\beta} & \mathcal{O}_Y \\
\downarrow & & \downarrow \\
E & \xrightarrow{g} & E_2 \\
\downarrow & & \downarrow \\
\mathcal{O}_Y & \xrightarrow{\alpha} & \mathcal{O}_Y \\
\downarrow & & \downarrow \\
\mathcal{O}_Y & \xrightarrow{\alpha_2} & \mathcal{O}_Y
\end{array} \quad \begin{array}{ccc}
0 & \xrightarrow{f} & 0 \\
\downarrow & & \downarrow \\
\mathcal{O}_Y & \xrightarrow{\alpha} & \mathcal{O}_Y
\end{array}$$

where the compositions of all verticals are zero and the horizontal is a short exact sequence. Suppose that the following conditions are satisfied.

1. There is a section $\beta' : \mathcal{O}_Y \to E$ such that $\beta_2 = g \circ \beta'$.
2. There is a cosection $\alpha' : E \to \mathcal{O}_Y$ such that $\alpha_1 = \alpha' \circ f$.
3. $\alpha' \circ \beta = \alpha \circ \beta' = 0$ and $\alpha - \alpha' = \alpha_2 \circ g.$
**Example 1.** The above conditions are satisfied for \( E = E_1 \oplus E_2, \alpha = \alpha_1 + \alpha_2 \) and \( \beta = (\beta_1, \beta_2) \).

With the above notations and conditions, we construct a vector bundle \( \tilde{E} \) on \( Y \times \mathbb{A}^1 \) and sections

\[
\tilde{\alpha} : \mathcal{O}_{Y \times \mathbb{A}^1} \rightarrow \tilde{E} \quad \text{and} \quad \tilde{\beta} : \mathcal{O}_{Y \times \mathbb{A}^1} \rightarrow \tilde{E}
\]

such that \( \langle \tilde{\alpha}, \tilde{\beta} \rangle = 0 \) and

\[
\{\tilde{\alpha}, \tilde{\beta}\}_\lambda = \begin{cases} 
\{\alpha, \beta\} & \text{if } \lambda = 1, \\
\{\alpha_1, \beta_1\} \otimes \{\alpha_2, \beta_2\} & \text{if } \lambda = 0
\end{cases}
\]

where \( \lambda \) is the coordinate on \( \mathbb{A}^1 \). Consider the following morphisms of vector bundles on \( Y \times \mathbb{A}^1 \),

\[
\begin{array}{ccc}
E_1 & \xrightarrow{\lambda \text{id}, f} & E_1 \oplus E \\
\downarrow & & \downarrow \\
\mathcal{O}_{Y \times \mathbb{A}^1} & \rightarrow & \mathcal{O}_{Y \times \mathbb{A}^1}
\end{array}
\]

Let \( \tilde{E} \) be the cokernel of \( (\lambda \text{id}, f) \). The section \( \tilde{\beta} : \mathcal{O}_{Y \times \mathbb{A}^1} \rightarrow \tilde{E} \) is induced by \( ((\lambda - 1)\beta_1, \lambda \beta + (1 - \lambda)\beta') \). Since we have

\[
(-\alpha_1 + (\alpha + (\lambda - 1)\alpha')) \circ (\lambda \text{id}, f) = \alpha f - \alpha' f = \alpha_2 g f = 0,
\]

we obtain an induced cosection \( \tilde{\alpha} : \tilde{E} \rightarrow \mathcal{O}_{Y \times \mathbb{A}^1} \). Since

\[
\alpha' \beta' = \alpha \beta' - \alpha_2 g \beta' = -\alpha_2 \beta_2 = 0,
\]

we have \( \langle \tilde{\alpha}, \tilde{\beta} \rangle = 0 \). The restriction of the short exact sequence

\[
0 \rightarrow E_1 \xrightarrow{\lambda \text{id}, f} E_1 \oplus E \rightarrow \tilde{E} \rightarrow 0
\]

to \( \lambda = 1 \) is

\[
0 \rightarrow E_1 \xrightarrow{\text{id}, f} E_1 \oplus E \xrightarrow{-f + \text{id}} E \rightarrow 0.
\]

Hence the restriction \( \tilde{\beta}|_{\lambda=1} \) is \(( -f + \text{id} ) \circ (0, \beta) = \beta \) and \( \tilde{\alpha}|_{\lambda=1} \) is \( \alpha \) because

\[
-\alpha_1 + \alpha - (-\alpha f + \alpha) = -\alpha_1 + \alpha f = \alpha_2 g f = 0.
\]

At \( \lambda = 0 \), the restriction is

\[
0 \rightarrow E_1 \xrightarrow{(0, f)} E_1 \oplus E \xrightarrow{-\text{id}, g} E_1 \oplus E_2 \rightarrow 0.
\]

Thus, the restriction \( \tilde{\beta}|_{\lambda=0} \) is \((- \text{id}, g) \circ (-\beta_1, \beta') = (\beta_1, \beta_2) \) and \( \tilde{\alpha}|_{\lambda=0} \) is \( \alpha_1 + \alpha_2 \).

Let us further assume that the common zero is contained in

\[
Z(\tilde{\alpha}, \tilde{\beta}) \subset X \times \mathbb{A}^1
\]
for some closed substack $X \subset Y$. For instance if $\beta' = f \circ \beta_1$, then $X$ can be taken to be $Z(\alpha, \beta)$ since it is the case of Example \cite{[1]} locally. By applying the homotopy property \cite{PV} Lemma 2.1 to $\text{ch}_X^Y \times \mathbb{A}^1(\{\tilde{\alpha}, \tilde{\beta}\})$, we have the following multiplicative formula.

**Lemma 3.1.** With the above set-up and the assumption (3.1), we obtain the following equalities

(3.2) \[ \text{ch}_X^Y(\{\alpha, \beta\}) = \text{ch}_X^Y(\{\alpha_2, \beta_2\}) \circ \text{ch}_X^Y(\{\alpha_1, \beta_1\}) = \text{ch}_X^Y(\{\alpha_2, \beta_2\}) \circ \text{ch}_X^Y(\{\alpha_1, \beta_1\}) \]

**Proof.** We have

\[
\text{ch}_X^Y(\{\alpha, \beta\}) = I^1 \text{ch}_X^Y(\{\tilde{\alpha}, \tilde{\beta}\}) = 0^1 \text{ch}_X^Y(\{\tilde{\alpha}, \tilde{\beta}\}) = \text{ch}_X^Y(\{\alpha_1, \beta_1\} \otimes \{\alpha_2, \beta_2\}) = \text{ch}_X^Y(\{\alpha_2, \beta_2\}) \circ \text{ch}_X^Y(\{\alpha_1, \beta_1\} \circ \text{ch}_X^Y(\{\alpha_1, \beta_1\})).
\]

Here, the first and third equalities are the bivariant property of localized Chern characters \cite[Definition 17.1(C3)]{Fu}, the second is the homotopy property \cite[Lemma 2.1]{PV} and the fourth equality comes from Theorem 1.2. This proves the first equality of (3.2). The proof for the second equality of (3.2) is the same. 

\[ \square \]

### 3.2. Cosection-localized intersection homomorphism.

Let $M$ be a finite type DM stack and $F$ be a vector bundle on $M$. Let $\sigma : F \to \mathcal{O}_M$ be a cosection. Recall that $w_\sigma$ is a function on $F$ induced by $\sigma$. Let $p : F \to M$ be the projection morphism and $\tau_F : \mathcal{O}_F \to p^*F$ be the tautological section induced by the diagonal morphism

\[ F \to F \times_M F. \]

Since $w_\sigma = p^* \sigma \circ \tau_F \in \Gamma(\mathcal{O}_F)$, the Koszul complex $\{p^* \sigma, \tau_F\}$ defines a 2-periodic complex on $Z(w_\sigma)$. Moreover, the zero loci are $Z(p^* \sigma) = F|_{Z(\sigma)} \subset F$ and $Z(\tau_F) = M \subset F$ where the second one is given by the zero section. Hence $Z(p^* \sigma, \tau_F) = Z(\sigma)$. Thus $\{p^* \sigma, \tau_F\}$ is strictly exact off $Z(\sigma)$.

As we have mentioned in Introduction, Kim and the author proved that the cosection-localized intersection homomorphism is expressed in terms of a localized Chern character \cite[Theorem 1.1]{KO}

(3.3) \[ 0^1_{F, \sigma} = \text{td}(F|_{Z(\sigma)}) \cdot \text{ch}_{Z(\sigma)}(\{p^* \sigma, \tau_F\}). \]

This immediately proves Theorem 1.4 using Lemma 3.1. Indeed, Theorem 1.4 can be improved to the following general situation. We provide a proof for this instead of giving that of Theorem 1.4. Consider
the following diagram of vector bundles on $M$,

$$
\begin{array}{ccc}
0 & \longrightarrow & F_1 \\
\sigma_1 & \downarrow & \sigma_1 \\
\mathcal{O}_M & \longrightarrow & \mathcal{O}_M \\
\end{array}
\begin{array}{ccc}
F & \longrightarrow & F_2 \\
\sigma & \downarrow & \sigma \\
\mathcal{O}_M & \longrightarrow & \mathcal{O}_M \\
\end{array}
\begin{array}{ccc}
0 & \longrightarrow & 0 \\
\end{array}
$$

where the horizontal is a short exact sequence. Suppose that there exists a cosection $\sigma' : F \to \mathcal{O}_M$ such that $\sigma_1 = \sigma' \circ f$ and $\sigma - \sigma' = \sigma_2 \circ g$. Letting $Y := Z(w_{\sigma_1}) \times_M Z(w_{\sigma_2})$ and $p : Y \to M$ be the projection morphism, we obtain a diagram

$$
\begin{array}{ccc}
\mathcal{O}_Y & \longrightarrow & \mathcal{O}_Y \\
\tau_{F_1} & \downarrow & \tau_{F_1} \\
0 & \longrightarrow & p^*F_1 \\
\sigma_1 & \downarrow & \sigma \\
\mathcal{O}_Y & \longrightarrow & \mathcal{O}_Y \\
\end{array}
\begin{array}{ccc}
\mathcal{O}_Y & \longrightarrow & \mathcal{O}_Y \\
\tau_F & \downarrow & \tau_F \\
p^*F & \longrightarrow & p^*F_2 \\
\sigma & \downarrow & \sigma \\
\mathcal{O}_Y & \longrightarrow & \mathcal{O}_Y \\
\end{array}
\begin{array}{ccc}
0 & \longrightarrow & 0 \\
\end{array}
$$

in the set-up of Section 3.1. By letting $\tau' := \tau_F - f \circ \tau_{F_1}$, we can check that

$$
(3.4) \quad g \circ \tau' = \tau_{F_2}, \quad \tau_F - \tau' = f \circ \tau_{F_1}, \quad p^*\sigma \circ \tau' = p^*\sigma' \circ \tau_F = 0
$$

using the local splitting of the exact sequence. The zero locus of $\sigma$ is $Z(\sigma) = Z(\sigma_1) \cap Z(\sigma_2)$. Note that we may not have the last identity in $(3.4)$ on $Z(w_{\sigma})$ (which is bigger than $Y$).

**Theorem 3.2.** Let $F, F_1, F_2, \sigma, \sigma_1, \sigma_2, f, g$ and $\sigma'$ be as above such that $\sigma_1 = \sigma' \circ f$ and $\sigma - \sigma' = \sigma_2 \circ g$. Then we have the following functorial property

$$
(3.5) \quad 0_{F,\sigma} = 0_{F_1,\sigma_1} \circ 0_{F_2,\sigma_2} = 0_{F_2,\sigma_2} \circ 0_{F_1,\sigma_1} : A_*(Y) \to A_{* - \text{rank}_F}(Z(\sigma)),
$$

where $Y := Z(w_{\sigma_1}) \times_M Z(w_{\sigma_2})$.

**Proof.** We have

$$
0_{F,\sigma} = \text{td}(F|Z(\sigma)) \cdot \text{ch}_{Z(\sigma)}^Y (\{p^*\sigma, \tau_F\})
= \text{td}(F_1|Z(\sigma)) \text{td}(F_2|Z(\sigma)) \cdot \text{ch}_{Z(\sigma)}^Y (\{p^*\sigma_1, \tau_{F_1}\}) \circ \text{ch}_{Z(\sigma)}^Y (\{p^*\sigma_2, \tau_{F_2}\})
= \text{td}(F_1|Z(\sigma)) \cdot \text{ch}_{Z(\sigma)}^Y (\{p^*\sigma_1, \tau_{F_1}\}) \circ \text{td}(F_2|Y_2) \text{ch}_{Y_2}^Y (\{p^*\sigma_2, \tau_{F_2}\})
= 0_{F_1,\sigma_1} \circ 0_{F_2,\sigma_2},
$$

where $Y_2 := Z(w_{\sigma_1}) \times_M Z(\sigma_2)$ and $F_2|Y_2$ denotes the pullback of $F_2$ to $Y_2$. Here the first and fourth equalities are from $(3.3)$, the second one comes from Lemma 3.1 and the third one is the commutativity of bivariant classes and Chern classes [Ful, Proposition 17.3.2]. This proves the first equality of $(3.5)$. The proof for the second equality of $(3.5)$ is the same. □
MULTIPLICATIVE PROPERTY

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