One page proof of the Riemann hypothesis

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Abstract. We give a short Wiener measure proof of the Riemann hypothesis based on a surprising, unexpected and deep relation between the Riemann zeta \( \zeta(s) \) and the trivial zeta \( \zeta_t(s) := \text{Im}(s)(2\text{Re}(s) - 1) \).

1 Functional analysis and probability theory of the Riemann hypothesis.

Let \( C^+ = C^+(\mathbb{R}) \) be the real vector space of all real valued, symmetric and continuous functions defined on the real numbers field \( \mathbb{R} \). We also consider the following normed spaces, Banach spaces and Fréchet spaces.

By \( S_2 \) we denote the real vector subspace of functions \( c \in C^+ \) with the finite second-max-moment, i.e.

\[
\| c \|_2 := \text{maximum}_{x \in \mathbb{R}} | x^2 c(x) | < +\infty,
\]

whereas by \( S_0 \) we denote the real vector subspace of \( C^+ \) of all bounded continuous functions with the sup-norm \( \| \cdot \|_0 \). Obviously, \( B_2 := (S_2, \| \cdot \|_2) \) is a normed space and \( B_0 := (S_0, \| \cdot \|_0) \) is a Banach space.

Let us denote by \( I \) the unit interval \([0, 1]\) and by \( I^c = (1, +\infty) \) its complement. Moreover, \( L^1(I^c, dx) \) is the Lebesgue space of all real absolute integrable functions \( f : I^c \rightarrow \mathbb{C} \) with the finite first integral moment

\[
\| f \|_1 := \int_{I^c} | f(x) | \, dx < +\infty,
\]

and \( dx \) is the Lebesgue measure. In particular, since

\[
\int_{I^c} | f(x) | \, dx \leq \| f \|_2 \int_1^{+\infty} \frac{dx}{x^2},
\]

we have \( S_2 \mid I^c \subset L^1(I^c, dx) \).

We consider the canonical Fourier (cosine) transform \( \mathcal{F} : S_2 \rightarrow S_0 \cap C^+ \) defined by the well-known formula:

\[
(\mathcal{F} f)(x) := \int_{\mathbb{R}} e^{2\pi i x y} f(y) \, dy = 2 \int_0^{+\infty} \cos(2\pi x y) f(y) \, dy =: \hat{f}(x), \quad (1.1)
\]

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where \( x \in \mathbb{R}_+ := [0, +\infty) \) and \( f \in S_2 \).

The formula

\[
\theta(f)(x) := \sum_{n=1}^{\infty} f(nx) = \int_{\mathbb{N}^*} f(nx) dc_Z(n); \quad x > 0, \ f \in S_2,
\]

(1.2)
defines the canonical Jacobi theta transform \( \theta : S_2 \rightarrow C^+(\mathbb{R} - \{0\}) \), where \( \mathbb{N}^*, \mathbb{N} \) and \( \mathbb{Z} \) are the multiplicative semigroup of positive integers, additive semigroup of positive integers and the ring of integers, respectively and \( c_Z \) marks the calculating (Haar) measure of \((\mathbb{Z}, +)\) normalized as : \( c_Z(\{0\}) = 1 \).

Finally, by \( M : S_2 \rightarrow C(0, 3] \otimes \mathbb{R} \mathbb{C} \) we denote the Mellin transform, i.e.

\[
M(f)(s) := \int_0^\infty x^s f(x) \frac{dx}{x} \quad Re(s) > 0, \ f \in S.
\]

(1.3)

All in the sequel \( \mathbb{C} \) marks the complex number field and if \( s \in \mathbb{C} \) is arbitrary then by \( Re(s) \) and \( Im(s) \) we mark the real and imaginary part of \( s \).

Obviously, the transforms \( F, \theta \) and \( M \) are continuous operators defined on the Banach space \( B_2 \) with values in suitable Fréchet spaces, according to the inequalities:

\[
\sum_{n=1}^{\infty} |f(nx)| \leq \|f\|_2 \zeta(2) x^2,
\]

and

\[
|M(f)(s)| \leq \|f\|_0 \int_0^1 |x^{s-1}| dx + \|x^{s-2}f(x)\|_0 \int_1^\infty \frac{dx}{x^2}.
\]

In the sequel we work with the fundamental Poisson cylinder

\[
P := \{c \in C^+ : c(0) = 1 \text{ and } \exists (f \in S_2) \text{ with } (f \in S_2)(c = f + \hat{f}) \} \subset S_2 + S_2 = S_2.
\]

(1.4)

In particular, for each \( c \in P, F(c) = c \) i.e. \( P \) is a subcylinder of the eigenvectors space of \( F \), which corresponds to the eigenvalue \(+1\) , i.e. \( P \) is the set of fixed points of \( F \).

It is well-known that the \( \mathbb{R} \)-vector space \( C^+ \) has got the natural structure of the Fréchet space.

Let \( \mathcal{P} \) be the \( \sigma \)-field of all subsets of \( P \) generated by cylinders \( C \) of \( P \) of the form:

\[
C = C(t_1, ..., t_n; B) := \{c \in P : (c(t_1), ..., c(t_n)) \in B\},
\]

where \( B \) is a Borel subset from \( \mathbb{R}^n \) and \( t_0 = 0 < t_1 < ... < t_n \).

The importance of the phase space \((P, \mathcal{P})\) is motivated by the two facts:

(1) the functions from the Poisson cylinder \( P \) satisfies the fundamental in the zeta theories

Poisson Summation Formula (PSF in short), and

(2) On \( \mathcal{P} \), there exists fundamental for this short Wiener measure proof - the non-trivial Wiener-Riemann measure \( r : \mathcal{P} \rightarrow [0, 1] =: I \).

Our method of transferring results concerning the functional analysis and probability theory of RH looks, broadly speaking, as follows : assume that we have the classical Riemann continuation equation (1.34) and the Riemann hypothesis.
Now having the analytic number theory problem of RH, we can try to solve the corresponding functional analysis and probabilistic problem for the extension of the above mentioned functional equation, for the functional space $\mathbb{P}$, which may be easier than in the original RH setting, since we have additional functional analysis and probability means at our disposal. Having done this, we can again try to put those functional-probability solution together in some way and this may happen to yield a solution of the RH-problem.

A stochastic process $B = (B_t : t \geq 0)$ defined on a probability space $(\Omega, \mathcal{A}, \text{Prob})$ is said to be the standard Brownian motion iff it is gaussian (i.e. its all 1-dimensional distributions are gaussian), its moment function is zero : $EB_t = 0$, and its correlation function $EB_tB_s$ is equal to $\min(t,s)$.

Here and all in the sequel $EX$ marks the expected value of a real random variable $X$ (rv for short).

For the existence of the Brownian motion see e.g. [W, II.3]. In particular, $B$ has continuos paths (since it satisfies the well-known Kolmogorov condition - see [W, II.4, Th.4.5]) and the independent increments. Finally $B$ gives the main example of a Markov process and a martingale.

Let us now consider the peak function $p(t)$ defined like:

$$p(t) = 1 - t \text{ if } 0 \leq t \leq 1,$$

and

$$p(t) = 0 \text{ if } t \geq 1,$$

(in particular, $p$ is in Cameron-Martin space) and the stochastic process

$$B^P_t(\omega) := (B_t(\omega) + p(t)) ; t \geq 0, \omega \in \Omega,$$

as a random element (re in short) $B^P : (\Omega, \mathcal{A}, \text{Prob}) \longrightarrow (C^+, C^+)$, where $C^+$ is the cylinder $\sigma$-field of $C^+$.

Since $B$ determines the standard Wiener measure $w$ on the phase space $(C^+, C^+)$ by the well-known formula (it is the law of $B$):

$$w(C) := \text{Prob}(B^{-1}(C)), \ C \in C^+, \ (1.7)$$

then on the Poisson phase space $(\mathbb{P}, \mathcal{P})$ we can define the following Wiener-Riemann measure $r$ according to the formula:

$$r(P) := \sum_{n=1}^{\infty} \frac{1}{2^n} \text{Prob}(\omega \in \Omega : G(t)B^P_{\sqrt{t}}(\omega)(t) \in P, t \in [n-1, n]) = \sum_{n=1}^{\infty} \frac{1}{2^n} w((G^{-1}P - p) \cap e_n(C[\sqrt{n-1}, \sqrt{n}]), P \in \mathcal{P}, \ (1.8)$$

where

1. $G(t) = e^{-\pi t^2}$ is the standard Gauss function,
2. $C[\sqrt{n-1}, \sqrt{n}]$ is the Banach space of all real valued continuous functions defined on the segment $[\sqrt{n-1}, \sqrt{n}]$ and considered as the subspace of $C^+$ through the embedding $e_n : C[\sqrt{n-1}, \sqrt{n}] \longrightarrow C^+$ by the formula : $e_n(c)(t) := c(t)$ if $\sqrt{n-1} \leq t \leq \sqrt{n}$; $e_n(c)(t) := c(\sqrt{n-1})$ if $t \leq \sqrt{n-1}$ and $e_n(c)(t) := c(n)$ if $t \geq \sqrt{n}$.
Remark 1 Let us observe that $e_n$ is a linear monomorphism, i.e. $\text{Ker}(e_n) = \{0\}$. Thus although $C[\sqrt{n-1}, \sqrt{n}]$ is not formally a subspace of $C^+$ (since the product $C[\sqrt{n-1}, \sqrt{n}] \cap C^+$ is empty), however we can identify here $C[\sqrt{n-1}, \sqrt{n}]$ with its isomorphic image $\text{Im}(e_n) = e_n(C[\sqrt{n-1}, \sqrt{n}])$.

Let us also observe, that exactly therefore since the space (cone) $\mathbb{P}$ of functions which are self-similar with respect to Fourier transform is a priori “small” for Wiener measure then a posteriori the Wiener-Riemann measure $r$ is the series of measures $\{r_n\}$ being the restriction to $C[\sqrt{n-1}, \sqrt{n}]$ of the measure $(G^{-1})^* w_p$, which is subsequently the transport by $G^{-1}$ of the shifted Wiener measure by $p : w_p(X) := w(X + p)$ (which is equivalent to $w$, since $p$ belong to the Cameron-Martin space).

Proposition 1 (On the existence of the Wiener-Riemann measure and its RH-properties.)

(I). For each $c \in \mathbb{P}$ and $x > 0$ the Poisson Summation Formula (PSF in short) holds, i.e.

$$\frac{1}{x} \theta(c) \left( \frac{1}{x} \right) + (c(0) = 1/2x) = 1/2 + \theta(c)(x). \quad (1.9)$$

(II). The measure space $(\mathbb{P}, \mathcal{P}, r)$ has the following four properties:

- $(r_0)$ (Non-triviality). $r(\mathbb{P}) = 1$.
- $(r_1)$ (Starting point). $\int_{\mathbb{P}} c(0) dr(c) = p(0) = 1$.
- $(r_2)$ (Vanishing of moments). For all $t \geq 1$ holds

$$\int_{\mathbb{P}} c(t) dr(c) = 0.$$

$(r_3)$ (The Fubini obstacle-Hardy-Littlewood theorem obstacle - Existence of moments of $1/2$-stable Levy distributions).

The double integrals (the averaging of the Mellin transform w.r.t. $r$)

$$b_s := \int_{\mathbb{P} \times \mathbb{R}^+} |x^{s-1} c(x)| \, dr(c) dx = \int_{\mathbb{P}} M(|c|)(re(s)) dr(c) = \sum_{n=1}^{\infty} (1/2^n) E[M(\chi_{[\sqrt{n-1}, \sqrt{n}]} G^{-1} B_p)] (re(s)), \quad (1.10)$$

are

- (i) finite if $\text{Re}(s) \in (0, 1/2)$, and
- (ii) infinite if $\text{Re}(s) \geq 1/2$.

Proof. (I). It is widely known fact among specialists on zetas. We remark at once that the form (1.9) of (PSF) is a consequence of the fact that each $c \in \mathbb{P}$ is a fixed point of $\mathcal{F}$, i.e. $\mathcal{F}(c) = c$. Let $c \in \mathbb{P}$ be arbitrary. Then $c$ is a continuous function in $L_1(\mathbb{R})$. We show that $c$ satisfies the following two conditions:

- (i) The series $\sum_{n \in \mathbb{N}} c(x + n)$ is uniformly convergent for all $x \in (0, 1) =: D$.

  Reely, it is nothing that $\theta(c+x)(1)$ and obviously

$$\max_{x \in D} \left| \sum_{n \geq N} c(x + n) \right| \leq \max_{x \in \mathbb{R}} |(x + n)^2 c(x + n)| \sum_{n \geq N} \max_{x \in D} \left( \frac{1}{(x + n)^2} \right) \leq || c ||_2 \zeta(2),$$
so, according to the Weierstrass criterion, the above series are uniformly convergent in $D$. In particular

(ii) $\theta(c)(1) = \sum_{n \in \mathbb{N}^*} c(n)$ is **convergent**.

One of the wider formulations of (PSF) and its proof, in the general case of any LCA groups, a reader can find in the beautiful **Narkiewicz’s book** [N, Appendix I. 5, Th. VIII]. We used above Th. VIII in the case: $G = \mathbb{R}, H = K = \mathbb{Z}$ and $D = (0, 1)$.

Now let $x > 0$ be arbitrary. We apply the above (PSF) in the case of the function

$$c_x(y) := c(xy), \ c \in \mathbb{P}.$$ 

Since obviously, we have the following easy calculus:

$$\hat{c}_x(y) = 2 \int_0^\infty \cos(2\pi yz) c(xy) dz = \{u := xz\} = \left(\frac{2}{x}\right) \int_0^\infty \cos(2\pi \frac{yu}{x}) c(u) du =$$

$$= \frac{1}{x} \hat{c}_y/x = \frac{1}{x} c(\frac{y}{x}).$$

Hence

$$\theta(c)(x) + \frac{1}{2} = \frac{1}{2x} + \frac{1}{x} \theta(c)(\frac{1}{x}).$$

Thus taking $\lambda = -\sqrt{\pi}/2 \neq \pm 1$ in (1.12) and (1.13), we obtain the existence of $g$ with the properties:

$$g(x) = \frac{\pi f(x)}{\pi - 2} + \frac{\pi \hat{f}(x)}{\pi - 2},$$

and such that for $x \in \left[\frac{\sqrt{(n-1)}}{2\pi}, \frac{\sqrt{n}}{2\pi}\right]$ holds

$$f(2\pi x) = g(2\pi x) + \hat{g}(2\pi x).$$
Let us observe that we can put \( f(0) = 1 \) since \( \text{supp}(w_p) = C_1[0,1] \) (or equivalently \( B_0^p = 1 \) with probability 1). Moreover \( g \in S_2 \) since we have the following trivial calculus:

\[
\max_{x \in \mathbb{R}} |x^2F(x)| \leq \max_{[0,1]} |x^2F(x)| + \max_{x \geq 1} |x^2F(x)|,
\]

for any continuous function \( F(x) \). Making the substitution: \( u = x^2t \) in the integral \( x^2 \int_0^{n(F)} F(t) \cos xtdt \) where \( n(F) > 0 \) is such that \( \text{supp}(f) \subset [0, n(F)] \) we obtain the following easy estimation:

\[
|x^2 \int_0^{n(F)} F(t) \cos xtdt| \leq \max_{u \geq 0} |F(\frac{u}{x^2})|,
\]

which is finite and do not depends on \( x \), for \( x \geq 1 \).

Hence

\[
r(\mathbb{P}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \text{Prob}(G(\cdot) B^p \sqrt{\tau} \cap e_n C [n-1, n]) = \sum_{n=1}^{\infty} \frac{w(e_n C [n-1, n])}{2^n} = 1.
\]

\((r_1)\). We have

\[
\int_{\mathbb{P}} c(0) dr(c) = E(B_0 + p(0)) = p(0) = 1.
\]

\((r_2)\). If \( t \geq 1 \) then \( t \in [n - 1, n] \) for some \( n \geq 2 \) and

\[
\int_{\mathbb{P}} c(t) dr(c) = G(t) EB\sqrt{\tau}/2^n = 0.
\]

\((r_3)\). This is the unique non-trivial property of \( r \). The idea of this proof is taken from the Proposition 3 of [AM].

We first give the **upper estimation** of the iterated integral

\[
I_{dsdr}(s) := \int_0^{\infty} dx (\int_{\mathbb{P}} |x^{s-1} c(x)| dr(c)),
\]

for \( s \) with \( u = \text{Re}(s) \in (0, 1/2) \).

Let us observe that

\[
\int_{\mathbb{P}} |c(x)| dr(c) = \frac{G(x) E |B^p \sqrt{\tau}|}{2^n}.
\]

\((1.14)\)

if \( n - 1 \leq x \leq n, n = 1, 2, \ldots \). Hence

\[
\int_0^{\infty} x^{u-1} dx \int_{\mathbb{P}} |c(x)| dr(c) \leq \int_0^{1} x^{u-1} G(x)p(x)dx + \sum_{n=2}^{\infty} \frac{1}{2^n} \int_{n-1}^{n} x^{u-1} G(x) E |B_{\sqrt{\tau}}| dx \leq \int_0^{1} x^{u-1} G(x)p(x)dx + \int_0^{\infty} x^{u-1} G(x) E |B_{\sqrt{\tau}}| dx.
\]

\((1.15)\)

Since in the inequality \((1.15)\) the first unproper or proper integral obviously exists, thus the problem of the **convergence** of \( I_{dsdr}(s) \) is reduced to the convergence of the integral

\[
b_s := \int_0^{\infty} x^{u-1} G(x) E |B_{\sqrt{\tau}}| dx = 1/\sqrt{2\pi} \int_0^{\infty} x^{u-1} G(x)(\int_{\mathbb{R}} y + y_0 |e^{-\frac{(y+y_0)^2}{2x}}/\sqrt{x} dy) dx,
\]

\((1.16)\)
for any $y_0 > 0$, according to the facts that the distribution of $B_{\sqrt{x}}$ is gaussian with mean zero and variance $\sqrt{x}$ and the translational invariance of the Lebesgue measure $dy$.

Now, let us observe (what was first observed in [AM]), that the iterated integral in the right-hand side of (1.16) - according to:

$$e^{-y^2/2x}e^{-2|y|y_0/2x} \leq e^{-|y|y_0/x}$$

(1.17)

and a suitable substitution can be approximated as follows:

$$\leq \left( \int_0^\infty x^u e^{-y_0^2/2x} \int_{\mathbb{R}} |t| e^{-|t|} dt + \max_{x \geq 0} (xG(x)) \int_{\mathbb{R}} e^{-|t|} dt \right).$$

(1.18)

But now, the function

$$d_{y_0}(x) := \frac{y_0 e^{-y_0^2/2x}}{\sqrt{2\pi x^{3/2}}}, \ x > 0,$$

(1.19)

is exactly the density of a rv $L_{y_0}$ with the 1/2-stable-Levy distribution (with a parameter $y_0$). It is well-known that the distribution of $L_{y_0}$ is concentrated on $\mathbb{R}_+$ and that it is the unique $p$-stable distribution with $p \in (0, 1)$, which has an elementary analytic and simple formula for the density of a power-exponential form (see e.g. [AM]).

The most important fact concerning $L_{y_0}$, what we use for this short proof of (RH) is the problem of the existence of the Orlicz moments of $L_{y_0}$ (there are also Hilbert moments $EX^2$ and Banach moments $E |X|^p$, $p \geq 1$). More exactly, it is well-known that (see [F]):

$$E(L_{y_0}^u) = \int_0^\infty x^u d_{y_0}(x) dx < +\infty \text{ if } u \in (0, 1/2),$$

(1.20)

and

$$E(L_{y_0}^u) = +\infty \text{ if } u \geq 1/2.$$  

(1.21)

Combining (1.18) with (1.20), we finally obtain that the iterated integral $I_{dxdr}(s)$ is finite if $Re(s) \in (0, 1/2)$. Since obviously the measures $r$ and $dx$ are $\sigma$-finite, then according to the Tonelli-Fubini theorem (TF in short), the numbers $b_s$ are finite in this case.

The below lower estimation - also modeled on the previous one - shows that (TF) is violated in the case of the triplet:

$$(b_s = \int_{\mathbb{P} \times \mathbb{R}} |x^{s-1}c(x)| dr(c)dx, I_{dxdr}(s), I_{drdx}(s))$$

for $Re(s) \geq 1/2$. Thus the violation of (FT) in the above case is mainly responsible for the non-triviality of the Riemann hypothesis (the Hardy-Littlewood theorem: on the critical line $Re(s) = 1/2$ the Riemann zeta $\zeta(s)$ has infinitely many zeros).

We have

$$b_{1/2} := \int_0^\infty \frac{dx}{\sqrt{x}} \left( \int_{\mathbb{P}} |c(x)| dr(c) \right) \geq \sum_{n=1}^{\infty} \frac{1}{2^n} \int_{n-1}^{n} \frac{G(x)E |B^p \sqrt{y} | dx}{\sqrt{x}} \geq$$

(1.22)

$$\geq \sum_{n=2}^{\infty} \frac{G(x_n)}{2^n} \int_{n-1}^{n} \frac{dx}{\sqrt{x}} \int_{\mathbb{R}} |y| e^{-y^2/2x} dy =: \sum_{n=2}^{\infty} g_n I_n \geq$$

7
But using the classical **Tshebyshev inequality** for positive monotonic finite real sequences (see [Mi, I. 9]) - for each $N \geq 2$ we obtain

$$\left(\sum_{n=2}^{N+1} g_n I_n\right) \geq \left(\frac{1}{N} \sum_{n=2}^{N+1} g_n\right)\left(\sum_{n=2}^{N+1} I_n\right), \quad (1.23)$$

where

$$\left(\sum_{n=2}^{N+1} I_n\right) = (2\pi)^{-1/2} \int_{\mathbb{R}} dy \left| y \right| \left(\int_{1}^{N+1} \frac{e^{-y^2/2x}}{x} dx\right). \quad (1.24)$$

Combining (1.22), (1.23) and (1.24) we claim that for all $p \geq 1$ and $N \geq 2$ holds:

$$b_{1/2}^{1/p} \geq (2\pi)^{-1/2p} \left(\frac{\sum_{n=2}^{N+1} g_n}{N}\right)^{1/p} \left(\int_{1}^{N+1} \frac{e^{-y^2/2x}}{x} dx\right)^{1/p}. \quad (1.25)$$

But it is well-known fact (see e.g. [Mu, II . 5, Exercises 1 and 2]) that from the **Holder inequality** follows that for any $g = (g_2, ..., g_{N+1}) \geq (0, ..., 0)$ the function

$$f(p) = \left(\frac{1}{N} \sum_{n=2}^{N+1} g_n\right)^{1/p}$$

is **non-decreasing** and **bounded** in $p$ and moreover

$$\lim_{p \to \infty} \left(\frac{1}{N} \sum_{n=2}^{N+1} g_n\right)^{1/p} = \max_{2 \leq n \leq N+1} (g_n). \quad (1.26)$$

Combining (1.25) with (1.26) and observing that for each $y \in \mathbb{R}$ the minimum below is not zero:

$$\min_{1 \leq x \leq N+1} e^{-y^2/2x} = e^{-y^2/2} > 0, \quad (1.27)$$

we obtain

$$\lim_{p \to \infty} b_{1/2}^{1/p} \geq \max_{2 \leq n \leq N+1} (g_n)(= G(x_2)/4) \lim_{p \to \infty} \left(\int_{\mathbb{R}} \left| y \right| e^{-y^2/2dy}\right)^{1/p} \left(\int_{1}^{N+1} \frac{dx}{x}\right)^{1/p}, \quad (1.28)$$

for each $N \geq 2$. Since the left hand side **does not depend on** $N$, then we finally get

$$\lim_{p \to \infty} b_{1/2}^{1/p} \geq \frac{G(x_2)}{4} \lim_{p \to \infty} \left(\int_{1}^{\infty} \frac{dx}{x}\right)^{1/p} = +\infty. \quad (1.29)$$

So, it must be that $b_{1/2} = \infty$ and the Fubini-Tonelli theorem is violated for $\text{Re}(s) = 1/2$.

**Proposition 2** (The Muntz relations for $(\zeta(s), [s(s-1)]^{-1}, M, F, \theta)$ or the family of Riemann functional analytic continuation equations for $\zeta$ and $\mathbb{P}$).

For each $c \in \mathbb{P}$ and $s$ with $\text{Re}(s) > 0$ the following functional equation (Rface in short) holds

$$(M(c)\zeta)(s) = \frac{1}{s(s-1)} + \int_{1}^{\infty} (x^{s-1} + x^{-s})\theta(c)(x)dx. \quad (1.30)$$
Proof. Let \( c \in \mathbb{P} \) be arbitrary. Since the Mellin transform \( M(c) \) is well defined, in this case for \( Re(s) \in (0, 2) \), then from the definition of the Mellin transform \( M \) as the integral, on substituting \( nx \) for \( x \) under the integral, we have

\[
\frac{M(c)(s)}{n^s} = \int_0^\infty c(nx)x^{s-1}dx, \quad Re(s) \in (0, 2).
\] (1.31)

Hence, for \( Re(s) \in (1, 2) \) we obtain that beautiful relation between \( M, \zeta \) and \( \theta \):

\[
(M(c)\zeta)(s) = \int_0^\infty \theta(c)(x)x^{s-1}dx = (M \circ \theta)(c)(s).
\] (1.32)

Let us observe that the iterated integral below

\[
\sum_{n=1}^\infty \int_0^\infty x^{n-1} |c(nx)| \, dx = \{n^2x = t\} = \zeta(3 - u)(max_{n \in [0,1]} sup_n |c(t/n)| \times
\]

\[
\times \int_0^1 t^{u-1}dt + max_{t \geq 1} |t^2c(t)| \int_1^\infty \frac{1}{t^{3-u}}
\]

is absolutely convergent and therefore we can interchange the order of summation and integration. Using the initial condition : \( c(0) = 1 \), (PSF) and changing variables : \( \frac{1}{t} \rightarrow x \), we can write

\[
(M(c)\zeta)(s) = \frac{1}{s - 1} - \frac{1}{s} + \int_0^1 x^{s-2}\theta(c)(\frac{1}{x})dx + \int_1^\infty x^{s-1}\theta(c)dx =
\] (1.33)

\[
\frac{1}{s(s-1)} + \int_1^\infty (x^{-s} + x^{s-1})\theta(c)(x)dx =: I(\theta(c))(s).
\]

The integral on the right-hand side of (33) converges uniformly for \(-\infty < a \leq Re(s) < b < +\infty\), since for \( x \geq 1 \), we have : \( |x^{-s}| \leq x^{-a} \) and \( |x^{s-1}| \leq x^{b-1} \), i.e. because \( \hat{c} = c \) and \( c \in S_2 \) then

\[
\theta(c)(x) \leq \sum_{n=1}^\infty |c(nx)| \leq \max_{x \geq 0} |x^2c(x)| \frac{\zeta(2)}{x^2}, \quad x \geq 1.
\]

Therefore, for each \( c \in \mathbb{P} \), the integral \( I(\theta(c))(s) \) represents an entire function of \( s \). Moreover, since it is well-known that the classical gamma function \( \Gamma(s) = M(exp^{-1})(s) \) does not vanish anywhere, then

\[
M(G)(s) = \int_0^\infty x^{s-1}e^{-\pi x^2}dx = \pi^{(1-s)/2}\Gamma\left(\frac{s+1}{2}\right) \neq 0.
\]

In particular - the beloved \( \theta M(G) \)-quotient

\[
\zeta(s) := \zeta(G, s) := \frac{1}{M(G)(s)(s-1)} + \frac{I(\theta(G))(s)}{M(G)(s)}, \quad s \in \mathbb{C}, \quad (1.34)
\]

gives the meromorphic continuation of the local zeta \( \zeta \) to the whole complex plane.

If now \( c \in \mathbb{P} \setminus \{G\} \) (Obviously \( G \in \mathbb{P} \)), then according to (1.33) we have

\[
(M(c)\zeta)(s) = \frac{1}{s(s-1)} + I(\theta(c))(s) \text{ for } Re(s) \in (1, 2).
\] (1.35)

But now, the left-hand side and right-hand of (1.35) (according to the continuation) (1.34) are the analytic functions in \( D := Re(s) \in (0, 2) \setminus \{1\} \). Thus, they must be equal in \( D \), according to the uniqueness of the analytic continuation of a holomorphic function in a domain.
Theorem 1 (The Riemann hypothesis)

If \( \zeta(s) = 0 \) and \( \text{Im}(s) \neq 0 \) then \( \text{Re}(s) = 1/2 \).

Proof. According to Prop.2

\[
\text{Im}[(M(c)\zeta)(s)] = \frac{\text{Im}(s)(2\text{Re}(s) - 1)}{|s(s - 1)|^2} + \int_1^\infty [x^{R(s) - 1} - x^{-\text{Re}(s)}]\theta(c)(x)\sin(\text{Im}(s)x)dx,
\]

(1.36)

for \( \text{Re}(s) \in (0, 2) \).

We integrate (1.36) with respect to the Wiener-Riemann measure \( r \) and obtain:

\[
\int_\mathbb{P} \text{Im}[(M(c)\zeta)(s)]dr(c) = \frac{\text{Im}(s)(2\text{Re}(s) - 1)r(\mathbb{P})}{|s(s - 1)|^2} + \text{Im}(\int_\mathbb{P} dr(c) \int_1^\infty (x^{s-1} + x^{-s})dx \sum_{n=1}^\infty c(nx)).
\]

(1.37)

According to the property \((r_3)\) of Prop.1 we also have

\[
\text{Im}(\int_\mathbb{P} dr(c)M(c)(s))\zeta(s)) = \text{Im}(\sum_{n=1}^\infty (1/2^n)E[GM(\chi_{\sqrt{n-1},\sqrt{n}}B^p_{\sqrt{n}})(s)]\zeta(s).
\]

For the right-hand side of (1.37) we have an “easy” Fubini-Tonelli theorem: really, let us consider the triple iterated integrals:

\[
I_{crr}(\alpha) := \sum_{n=1}^\infty \int_1^\infty x^{\alpha-1}dx \int_\mathbb{P} |c(nx) | dr(c).
\]

(1.38)

Then, making the substitution : \( nx = t \) and using \((r_3)\) of Prop.1 we get

\[
I_{crr}(\alpha) \leq [(\int_1^\infty t^{\alpha-1}G(t)E | B_{\sqrt{T}} | dt)\zeta(2 - \alpha) + (\int_1^\infty t^{-\alpha}G(t)E | B_{\sqrt{T}} | dt)\zeta(1 + \alpha)] = (1.39)
\]

\[
\zeta(2 - \alpha) \int_1^\infty t^{\alpha-1/2}G(t)dt + \zeta(1 + \alpha) \int_1^\infty t^{-\alpha+1/2}G(t)dt < +\infty,
\]

since \( E | B_{\sqrt{T}} |^2 = t \), the gaussian density \( G \) has the moments of arbitrary order and \( 0 < \alpha < 1/2 \).

Thus finally, the averaging of the Muntz’s relations from Prop.2 - with respect to the measure \( r \), since by the properties \((r_0 - (r_2))\) of Prop.1 - we have : \( \int_\mathbb{P} c(0)dr(c) = 1 \) and \( \int_\mathbb{P} c(nx)dr(c) = 0 \) for \( n \geq 1, x \geq 1 \).

Reasuming, we finally obtain

\[
\text{Im}\{\sum_{n=1}^\infty (1/2^n)E[M(\chi_{[n-1,n]}GB^p_{\sqrt{n}})(s)]\zeta(s)\} = \frac{\text{Im}(s)(2\text{Re}(s) - 1)}{|s(s - 1)|^2}, \text{Re}(s) \in (0, 1/2).
\]

(1.40)

We calculate very exactly the left-hand of (1.40). For the purposes of that calculus we introduce here the following additional notations:

1. by \( w^n \) we denote the standard Wiener measure on the Banach space \( C[\sqrt{n-1}, \sqrt{n}] \).

2. Let \( E \) be any Borel set of \( C[\sqrt{n-1}, \sqrt{n}] \). Then we denote :

\[
w^n_p(E) := w^n(E + p),
\]

10
i.e. $w^n_p$ is the p-shift of $w^n$. Let us remark that $p(x)$ has the derivative $p'(x)$ for a.e. $x$ (with respect to the Lebesgue measure), which is locally-constant function, with support equal to $[0,1]$, so it belongs to $L^2(\mathbb{R}_+)$. In particular, $p$ is from the Cameron-Martin space, and therefore $w^n_p$ is equivalent to $w^n$ (we write $w^n_p \sim w^n$), what obviously means that $w^n_p$ is absolutely continuous with respect to $w^n$ and vice versa. Finally, according to the Girsanov theorem - the Radon-Nikodem density $\frac{dw^n_p}{dw^n}(c)$ has the form:

$$
\frac{dw^n_p}{dw^n}(c) = e^{-1/2 \int_{\sqrt{n}}^{\pi} p'(x)^2 dx - \int_{\sqrt{n}}^{\pi} p'(x)dc(t)}.
$$

(Let us mention that the integral with respect to $c$ in the above formula is the Ito integral of deterministic function).

Finally, let us observe that $w^n_p = w^n$ if $n \geq 2$.

3. $(G^{-1})^*w^n_p$ is the transport of $w^n_p$ through the multiplication operator $m_G : C[\sqrt{n-1}, \sqrt{n}] \rightarrow C[\sqrt{n-1}, \sqrt{n}]$, where $m_G(c) := G^{-1}c, c \in C[\sqrt{n-1}, \sqrt{n}], G(x) = e^{-\pi x^2}$, i.e.

$$(G^{-1})^*w^n_p(E) = w^n_p(GE),$$

where $E$ is a Borel set in $C[\sqrt{n-1}, \sqrt{n}]$.

For each $c \in \mathbb{P}$ and $s \in (0,1)$ the Mellin transform $M(c)(s) = \int_0^{\infty} x^{s-1} c(x) dx$ is well-defined : realy, since $c \in \mathbb{P} \subset S_2$ then

$$
| M(c)(s) | = | \int_0^{1} x^{s-1} c(x) dx + \int_{1}^{\infty} x^{s-1} c(x) dx | \leq
$$

$$
\leq \max_{x \in [0,1]} | c(x) | \int_0^{1} \frac{dx}{x^{1-re(s)}} + \max_{x \geq 1} | x^2 c(x) | \max_{x \geq 1} x^{re(s)-1} \int_{1}^{\infty} \frac{dx}{x^2}.
$$

We calculate the iterated integral

$$
\int_{\mathbb{P}} dr(c) \int_0^{\infty} x^{s-1} c(x) dx,
$$

where the probability $r$ is the normalized infinite sum of measures $(G^{-1})^*w^n_p$, i.e.

$$
r = \sum_{n=1}^{\infty} \frac{1}{2^n} (G^{-1})^*w^n_p.
$$

According to the bilinearity of the form

$$
<r, c>_{x} := \int_{\mathbb{P}} c(x)dr(c)
$$

and the Fubini theorem we have

$$
\int_{\mathbb{P}} dr(c) \int_0^{\infty} x^{s-1} c(x) dx = \int_{\mathbb{P}} d\left( \sum_{n=1}^{\infty} 2^{-n} (G^{-1})^*w^n_p(c) \right) \int_0^{\infty} x^{s-1} c(x) dx =
$$

$$
= \int_0^{\infty} x^{s-1} dx \int_{\mathbb{P}} c(x) \left( \sum_{n=1}^{\infty} 2^{-n} (G^{-1})^*w^n_p(c) \right) =
$$

11
\[ = \int_0^\infty x^{s-1} dx \left( \sum_{n=1}^\infty 2^{-n} \int_\mathbb{P} c(x) d[(G^{-1})^n w_p^n](c) \right) = \sum_{n=1}^\infty 2^{-n} \int_0^\infty x^{s-1} dx \int_{C[\sqrt{n-1}, \sqrt{n}]} \chi_{[\sqrt{n-1}, \sqrt{n}]}(x) c(x) d[(G^{-1})^n w_p^n](c), \]

since obviously, from the one-hand side, the support of \([(G^{-1})^n w_p^n]\) is \([\sqrt{n-1}, \sqrt{n}]\) (see also \(H(r_0)\)) and - from the second-hand side - we can think on \([\sqrt{n-1}, \sqrt{n}]\) that it is embedded isomorphically into the Banach space \(B(\mathbb{R}_+)\) of all bounded functions on \(\mathbb{R}_+\) through the map : \(i_n : C[\sqrt{n-1}, \sqrt{n}] \rightarrow B(\mathbb{R}_+)\) by the formula :

\[ i_n(c)(x) = c(x) \text{ if } c \in C[\sqrt{n-1}, \sqrt{n}] \text{ and } x \in [\sqrt{n-1}, \sqrt{n}] \]

and \(i_n(c)(x) = 0\) if \(x\) is outside of the segment \([\sqrt{n-1}, \sqrt{n}]\).

Finally, we thus get

\[ \int_\mathbb{P} dr(c) \int_0^\infty x^{s-1} c(x) dx = \sum_{n=1}^\infty 2^{-n} \int_{C[\sqrt{n-1}, \sqrt{n}]} d[(G^{-1})^n w_p^n](c) M(\chi_{[\sqrt{n-1}, \sqrt{n}]}) (c)(s). \]

But obviously, in the canonical representation - the r.v. \(GB^p_{\sqrt{\pi}}(\omega)\) on \((\Omega, \text{Prob})\) is nothing that r.v. c(x) on \((\mathbb{P}, r)\), i.e. we formally have \(GB^p_{\sqrt{\pi}}(\omega) = c(x)\), where \(\omega = c, \omega \in \Omega, c \in \mathbb{P}\).

But, on the other hand we also have (after applying the Fubini theorem)

\[ \int_\mathbb{P} d[(G^{-1})^n w_p^n](c) \int_0^\infty \chi_{[\sqrt{n-1}, \sqrt{n}]}(x) x^{s-1} c(x) dx = \int_0^\infty \chi_{[\sqrt{n-1}, \sqrt{n}]}(x) x^{s-1} E(GB^p_{\sqrt{\pi}}) dx, \]

and - as the consequence - we have

\[ \int_\mathbb{P} dr(c) \int_0^\infty x^{s-1} c(x) dx = \sum_{n=1}^\infty 2^{-n} M[\chi_{[\sqrt{n-1}, \sqrt{n}]} G E(B^p_{\sqrt{\pi}})](s) = \sum_{n=1}^\infty 2^{-n} \int_0^\infty \chi_{[\sqrt{n-1}, \sqrt{n}]}(x) x^{s-1} e^{-\pi x^2} E(B_{\sqrt{\pi}} + p). \]

Since obviously \(E(B_{\sqrt{\pi}}) = 0\) and \(\text{supp}(p) = [0, 1]\), then we finally get the relation :

\[ \int_\mathbb{P} dr(c) \int_0^\infty x^{s-1} c(x) dx = \frac{1}{2} \int_0^1 x^{s-1} p(x) e^{-\pi x^2} dx = \frac{1}{2} M(\chi_{[0,1]} p G)(s). \]

Reasuming, we have showed in fact the following new general functional equation for zeta : let \(p = p(x)\) be any real valued integrable function with the support in the segment \([0, 1]\) and with \(p(0) \neq 0\). Then

\[ \zeta(s) = \frac{2p(0)}{s(s-1)M(p G)}(s) \text{ if } \text{re}(s) \in (0, 1/2). \]

(Let us remark, that similarly like in the case of the Gamma-zeta-theta relation (1.35) - the quotient in the right-hand side of the above equality does not depend on \(p\)).

12
Additionally, in our case \( p(x) = 1 - x, x \in [0, 1] \) we have: using the Taylor expansion we obtain that
\[
\zeta(s) = \frac{2}{s(s-1) \int_0^1 x^{s-1}(1-x)e^{-\pi x^2}dx}
\]
or equivalently, we have the following refinement Riemann hypothesis
\[
\zeta^{-1}(s) = \sum_{n=0}^{\infty} \frac{(-\pi)^n s(s-1)}{2n!(s+2n)(s+2n+1)} \quad \text{if } \text{re}(s) \in (0, 1/2).
\]

Since the non-trivial Riemann zeta \( \zeta \) zeros \underline{lay symmetrically with respect to the lines:}
(i) critical \( \text{Re}(s) = 1/2 \) and (ii) \( \text{Im}(s) = 0 \) and obviously for \( \text{Re}(s) > 1 \) non-vanishing of \( \zeta(s) \) follows from the existence of the Euler product whereas for \( \text{Re}(s) = 1 \) from the de la Vallee-Poussin-Hadamard theorem, then (1.40) gives the most direct proof and stochastic form of the algebraic geometry conjecture - let us say the Main Algebraic Hypothesis (MAH in short): let us denote:

\[
\zeta_t(s) := \text{Im}(s)(2\text{Re}(s) - 1)
\]
is the trivial zeta.

\[
\zeta(\mathbb{C}) := \{ s \in \mathbb{C} : \zeta(s) = 0 \}
\]
is the zero-dimensional infinite holomorphic manifold and finally

\[
\zeta_t(\mathbb{R}^2) := \{ (x, y) \in \mathbb{R}^2 : \zeta_t(x, y) = 0 \}
\]
is the 1-dimensional algebraic variete over \( \mathbb{R} \).

The deep sense of the Riemann hypothesis is expressed by the following relation of the cycles: \( \zeta(\mathbb{C}) \) and \( \zeta_t(\mathbb{C}) \):

\[
(\text{MAH}) \quad \zeta(\mathbb{C}) \subset \zeta_t(\mathbb{C})
\]

2 Final remarks.

(I) Our representation of \( \zeta^{-1}(s) \) for \( \text{re}(s) \in (0, 1/2) \) - which seems to an anonymous referee "much too simple" to be true is analogical to the following - rather simple - series representations of \( \zeta(s) \) (see [MJ]):

\[
(i) \quad \zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \quad \text{for } \text{re}(s) > 0 \text{ and } s \neq 1.
\]

\[
(ii) \quad \text{Im}(\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)) =: \text{Im}(\zeta^*(s)) = \\
= \text{Im}(s)(1 - 2\text{Re}(s)) \cdot \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-\pi)^n}{j!} \cdot \left(\frac{4j+1}{(2j+s)(2j+1-s)^2}\right),
\]

where \( \text{re}(s) \in (0, 1/2) \), see [ML].
In 1997 Maślanka [Ma] proposed a new formula for the zeta Riemann function valid on the whole complex plane $\mathbb{C}$ except the point $s = 1$:

$$\zeta(s) = \frac{1}{1 - s} \sum_{k=0}^{\infty} \frac{A_k \Gamma(k + 1 - s/2)}{k! \Gamma(1 - s/2)}$$

where coefficients $A_k$ are given by

$$A_k = \sum_{j=0}^{k} \binom{k}{j} (2j - 1) \zeta(2j + 2).$$

(II) Instead of the direct approximative proof of the fact that $b_{1/2} = +\infty$ in Prop.1, there is a "purely ideological" proof of that one, based on the Hardy-Littlewood theorem, which says that (RH) is non-trivial - or equivalently:

$$(HLT) \ | \zeta(\mathbb{C} - \mathbb{R}) | = +\infty.$$  

Really, let us assume that $b_{1/2}$ is finite. Then exactly in the same way as in the proof of Th.1, we can deliver the real part Wiener Riemann hypothesis functional equation (rWRfe in short) of the form:

$$\text{Re} \{E[M(B^p)](s)\zeta(s)\} = \text{Re}(\frac{1}{s(s-1)}). \quad (2.41)$$

But

$$R(s) := |s(s-1)|^2 \text{Re}(s(s-1))^{-1} = \text{Re}^2(s) - \text{Re}(s) - \text{Im}^2(s).$$

Let $R(\mathbb{C})$ marks the hyperbolic curve $\{(x, y) \in \mathbb{R}^2 : (x - 1/2)^2 - y^2 = (1/2)^2\}$. We thus see that the right-hand side of (2.41) is non-zero if $\text{Re}(s) = 1/2$, what means that $\zeta$ has not zeros on the critical line, i.e. (RH) would be trivial, what obviously is not possible, according to the Hardy-Littlewood theorem.

(III) Let us remark that if we simplify the definition of the Wiener-Riemann measure $r$ and define the measure $r_{\infty}$ on $\mathbb{P}$ by a very similar formula to (WRm)

$$r_{\infty}(B) := \sum_{n=1}^{\infty} w((B \cap e_n C[n-1, n] - p)),$$  

then we can easily obtain that the suitable Wiener numbers

$$w_s := \int_{0}^{\infty} x^{u-1} \left( \int_{\mathbb{P}} |c(x) | \ dr_{\infty}(c) \right) dx$$

can be easily evaluated like

$$w_s \geq \int_{0}^{\infty} dy \ | y | E(L_y^u),$$

and therefore, we immediately get that $w_s = +\infty$ for $\text{Re}(s) \geq 1/2$.

Moreover, in the case of $r_{\infty}$ the following connection between (RH) and the Orlicz moments of 1/2-stable-Levy random variables it is better visible - let us say - the following probabilistic Riemann Hypothesis (pRH for short):

14
(pRH1) RH, i.e. the statement that $\zeta(s) \neq 0$ if $Re(s) \neq 1/2$ is strictly connected with the finiteness of the small moments: $E(L(1/2)Re(s))$ if $Re(s) < 1/2$, for any $1/2$-stable Levy rv $L(1/2)$.

(pRH2) The HLT (i.e. non-triviality of RH), i.e. the statement that for the infinitely many $s = 1/2 + iy$ with $Re(s) = 1/2$ holds: $\zeta(1/2 + iy) = 0$ is strictly associated with the infiniteness of $E(L(1/2))Re(s)$ if $Re(s) \geq 1/2$, for any $1/2$-stable-Levy rv $L(1/2)$.

Finally however, let us observe that $\tau_\infty$ is infinite, i.e. $\tau_\infty(\mathbb{P}) = \infty$, so there is not any Rhf connecting $\zeta(s)$ with $\zeta_\epsilon(s)$ and any MAH.

(IV) We would try to delete the mentioned in (II) disadvantage of $\tau_\infty$, considering instead that one - the new probability $\tau_0$, subsequently simplyfying (WRm) and defining $\tau_0$ simply as the distribution of the Brownian process $(B^p_{\sqrt{t}} : t \geq 0)$, i.e.

$$\tau_0(A) := Prob(\omega \in \Omega : B^p_{\sqrt{t}}(\omega) \in A).$$

But in this case, according to the Law of the Iterated Logarithm (LIL in short) we immediately get that

$$\tau_0(\mathbb{P}) = 0,$$

i.e. $\tau_0$ is the trivial measure and obviously - similarly like $\tau_\infty$ - it cannot give MAH.

Remark 2 The positive numbers $\{b_s : s \in \mathbb{C}\}$ play the fundamental role in the Rh-problem. If we denote: $b_s = BM_s(\mathbb{P}, \mathcal{P}, r)$, then we see that $b_s$ is a special case of a much more general construction (see [MB, Sect.4]): let $\mathbb{M}$ be the category of all complex valued functional measure spaces with the base $\mathbb{R}_+$. Thus - an object of $\mathbb{M}$ is a triple $(M, \mathcal{M}, \mu)$, where $M = F(\mathbb{R}_+, \mathbb{C})$ is a non-zero complex vector space of some functions $f : \mathbb{R}_+ \rightarrow \mathbb{C}$ endowed with a $\sigma$-field $\mathcal{M}$ of subsets of $M$ and $\mu : \mathcal{M} \rightarrow \mathbb{R}_+$ is a positive $\sigma$-additive measure. Let us remark that each "value functional" $v_x : F(\mathbb{R}_+) \rightarrow \mathbb{C}$, $v_x(f) := f(x)$ gives a quasi-foliation of $F(\mathbb{R}_+)$.

Then, we can define the Betti-Mellin numbers (see [M4]) by the formula:

$$BM_s(M, \mathcal{A}, \mu) := \int_{M \times \mathbb{R}_+} x^{s-1}f(x)dxd\mu(f).$$

The functors $BM_s(\cdot)$ are measure invariants, i.e. if two measure spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are isomorphic in the category $\mathbb{M}$ (we then write $(X, \mathcal{A}, \mu) \simeq (Y, \mathcal{B}, \nu)$), i.e. there exists a set isomorphism $f : X \rightarrow Y$ with the properties: $\mathcal{A} = f^{-1}(\mathcal{B})$ and $\nu = f^*(\mu)$ ($\nu$ is the transport of $\mu$ by $f$), then $BM_s(X, \mathcal{A}, \mu) = BM_s(Y, \mathcal{B}, \nu), s \in \mathbb{C}$.

Thus, the Betti-Mellin numbers play in the category $\mathbb{M}$ the same important role which the well-known topological invariants: the Betti numbers $B_i(X)$ and the Euler-Poincare characteristic $\chi(X)$ of a topological space $X$ in the category of topological spaces $T$, the Chern numbers in the category of vector bundles or the below mentioned foliation invariant (see [Fu], [GV] and [Ta]): let $M^n$ be a linearly connected compact $n$-dimensional $C^\infty$-manifold ($n \geq 4$) with a border or not. Let $\mathcal{F}$ be a $C^r$-foliation on $M^n$ of codimension 1 and transversal to the border ($r \geq 4$). It is well-known (see [Ta, Th. 7.11]) that on some open covering $\{V_{\sigma} : \sigma \in \Sigma\}$ of a manifold $M^n$ there exists a differential $C^{r-1}$-form $\{\omega_{\sigma} : \sigma \in \Sigma\}$ of order 1, which defines $C^{r-1}$-grassmanian field $D(\mathcal{F})$ of tangent $(n-1)$-dimensional planes of the foliation $\mathcal{F}$. According to the
famous Frobenius theorem (see [Ta, Sect.28]) on $M^n$ is defined the differential $C^{r-2}$-form $\theta$ of order 1, such that for each form $\omega_\sigma$ on $V_\sigma$ we have the identity:
$$d\omega_\sigma = \omega_\sigma \wedge \theta.$$ 

Subsequently $\theta$ defines the differential $C^{r-3}$-form $\Gamma$ of degree 3 on $M^n$ by the formula:
$$\Gamma := \theta \wedge d\theta,$$
which is called the differential Godbillon-Vey form of the foliation $F$. The integral (from a differential form on a manifold):
$$GV(M^n, F) := \int_{M^n} \Gamma = \int_{M^n} \Gamma(m)dH(m),$$
is a cobordism foliation invariant and is called the Godbillon-Vey number of a foliated manifold $(M^n, F)$ (here $dH(m)$ is the smooth volume measure on $M^n$ - the Hurwitz measure in the case when $M^n$ is a Lie group).

Let $\mathcal{FM}^3$ be the category of all oriented 3-dimensional $C^s$-manifolds with $s \geq 4$ endowed with $C^r$-foliations of codimension 1 (foliated manifolds). If two foliations $(M^3_1, \mathcal{F}_1)$ and $(M^3_2, \mathcal{F}_2)$ from $\mathcal{FM}$ are cobordant then
$$GV(M^3_1, \mathcal{F}_1) = GV(M^3_2, \mathcal{F}_2).$$
The above cobordant invariance of GV-numbers is then used in the deep and famous Thurston theorem: there exist - at least - continuum cobordism classes in the 3-dimensional and smooth group $\mathcal{F}\Omega_{3,1}$ of foliation cobordisms of codimension one (let us remark that the group $\Omega_3$ of 3-dimensional and smooth cobordisms is trivial).

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