Calabi-Yau compactifications of toric Landau-Ginzburg models for smooth Fano threefolds

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Abstract. We prove that smooth Fano threefolds have toric Landau-Ginzburg models. More precisely, we prove that their Landau-Ginzburg models, represented as Laurent polynomials, admit compactifications to families of K3 surfaces, and we describe their fibres over infinity. We also give an explicit construction of Landau-Ginzburg models for del Pezzo surfaces and any divisors on them.

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§ 1. Introduction

The Mirror Symmetry conjecture, which is one of the deepest recent ideas in mathematics, connects varieties or certain families of varieties, the so-called Landau-Ginzburg models. It states that every variety has a dual object whose symplectic properties correspond to algebraic properties for the original variety and, conversely, whose algebraic properties correspond to symplectic properties for the original variety.

The Mirror Symmetry conjecture has several levels. In this paper we study the Mirror Symmetry Conjecture for variations of Hodge structures. It connects the (symplectic) Gromov-Witten invariants of Fano varieties (which count the expected numbers of (rational) curves lying on the varieties) with the periods of the dual algebraic families. Givental suggested dual Landau-Ginzburg models and proved the Mirror Symmetry Conjecture for variations of Hodge structures for smooth toric varieties and Fano complete intersections in them, see [1]. Another description of Mirror Symmetry for toric varieties treats it as a duality between toric varieties (or the polytopes which define them), see [2], for instance. From this point of view Laurent polynomials appear naturally as anticanonical sections of toric varieties.

Based on these considerations the idea of a toric Landau-Ginzburg model was proposed in [3]. A toric Landau-Ginzburg model for a smooth Fano variety \(X\) of dimension \(n\) and a chosen divisor \(D\) on it is a Laurent polynomial \(f_X\) in \(n\) variables satisfying three conditions. The period condition relates the periods of the family of

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fibres of the map \( f_X : (\mathbb{C}^*)^n \to \mathbb{C} \) to the generating series of one-pointed Gromov-Witten invariants of \( X \) depending on \( D \). The second, Calabi-Yau condition, says that fibres of \( f_X \) can be compactified to Calabi-Yau varieties. This condition is motivated by the so called Compactification Principle (see [3], Principle 32) stating that a fibrewise compactification of the family of fibres of a toric Landau-Ginzburg model is a Landau-Ginzburg model from the point of view of Homological Mirror Symmetry. Finally, the toric condition states that there is a degeneration of \( X \) to a toric variety \( T \) whose fan polytope is the Newton polytope of \( f_X \). This condition is motivated by the treatment of Mirror Symmetry for toric varieties as a duality between polytopes and by the deformation invariance of Gromov-Witten invariants.

The existence of toric Landau-Ginzburg models has been shown for Fano threefolds of Picard rank one and for complete intersections in projective spaces (see [3] and [4]). Other than this only partial results are known. In particular, the existence of Laurent polynomials satisfying the period condition for all smooth Fano threefolds (and trivial, that is, equal to zero, divisors on them) with very ample anticanonical classes was shown in [5], and the toric condition for them is checked in the forthcoming papers [6] and [7].

The main goal of the paper is to complete the proof of the existence of toric Landau-Ginzburg models for smooth Fano threefolds, in other words, to check the Calabi-Yau condition for them. There are 105 families of smooth Fano threefolds; the anticanonical classes of 98 of them are very ample. We consider ‘good’ toric degenerations of these, that is, degenerations to Gorenstein toric varieties. Given a Fano threefold \( X \) and its Gorenstein toric degeneration \( T \), we denote its fan polytope by \( \Delta \). It is reflexive, which means that its dual polytope \( \nabla \) is integral. We assume that a Laurent polynomial \( f_X \), whose Newton polytope is \( \Delta \), is of Minkowski type, that is, its coefficients correspond to the decompositions of facets of \( \Delta \) into Minkowski sums of elementary polygons (see Definition 2.2). Assume further that \( f_X \) satisfies the period condition for \( X \) and a trivial divisor. (According to [5] and [8], such polynomials exist for all 98 smooth Fano threefolds with very ample anticanonical classes.) The family of fibres of the map given by \( f_X \) is a one-dimensional linear subsystem in the anticanonical linear system of a toric variety \( T^\vee \) whose fan polytope is \( \nabla \). Since \( \Delta \) is three-dimensional and reflexive, \( T^\vee \) has a crepant resolution. One of the members of the family given by \( f_X \) is a boundary divisor of \( T^\vee \). The base locus of the family is a union of smooth rational curves due to the special choice of the coefficients of \( f_X \). This enables us to resolve the base locus, and, cutting out the boundary divisor, get a Calabi-Yau compactification. Moreover, this gives a log-Calabi-Yau compactification, that is, a compactification to a compact variety over \( \mathbb{P}^1 \) whose anticanonical divisor is a fibre. In addition, one gets a description of the ‘fibre over infinity’ of the compactified Landau-Ginzburg model.

**Theorem 1.1.** Any Minkowski Laurent polynomial in three variables admits a log-Calabi-Yau compactification.

We also construct Calabi-Yau compactifications ‘by hand’ for Laurent polynomials corresponding to the seven Fano threefolds whose anticanonical classes are not very ample. More precisely, we compactify the families to families of (singular) quartics in \( \mathbb{P}^3 \) or to hypersurfaces of bidegree \((2, 3)\) in \( \mathbb{P}^1 \times \mathbb{P}^2 \), which have crepant
resolutions. In the proof of Theorem 1.1 we do not use the fact that the variety $T$ is a toric degeneration of $X$. This fact is proved in the forthcoming papers [6] and [7] (see also [9]). This, together with the results discussed above, gives the following result.

**Corollary 1.2.** A pair, consisting of a smooth Fano threefold $X$ and a trivial divisor on it, has a toric Landau-Ginzburg model. Moreover, if $-K_X$ is very ample, then each Minkowski Laurent polynomial satisfying the period condition for $(X, 0)$ is a toric Landau-Ginzburg model.

We can get log-Calabi-Yau compactifications for del Pezzo surfaces (of degree greater than 2) in a similar way. In this case the base locus is just a set of points (possibly with multiplicities), and the resolution of the base locus is given by just a number of blow-ups of these points. Landau-Ginzburg models for del Pezzo surfaces equipped with general divisors were constructed in [10]; see also [11] for another description for arbitrary divisors. Our compactification procedure gives a precise way to write down a Laurent polynomial (and to describe the singularities of its fibres) for any chosen divisor.

The paper is organized as follows. In §2 we recall some preliminaries and give the main notation and definitions. In particular, we define toric Landau-Ginzburg models and give the definition of Minkowski Laurent polynomials. In §3 we consider the two-dimensional case. We present an explicit way to write down a toric Landau-Ginzburg model for a del Pezzo surface with an arbitrary divisor on it. In §4 we study the case of Fano threefolds and trivial divisors on them. We describe the structure of the base loci of the families we are interested in. Using this we prove the existence of Calabi-Yau compactifications for Minkowski Laurent polynomials (Theorem 1.1). Then we construct Calabi-Yau compactifications for the remaining cases. We summarize our arguments in Corollary 1.2, where we show that smooth Fano threefolds have toric Landau-Ginzburg models. We also make some remarks and state some questions relating to fibres over infinity of their compactifications.

**Notation and conventions.** All varieties are defined over the field $\mathbb{C}$ of complex numbers. We use the standard notation for multidegrees: given an element $a = (a_1, \ldots, a_k) \in \mathbb{Z}^k$ and a multivariable $x = (x_1, \ldots, x_k)$, we denote $x_1^{a_1} \cdots x_k^{a_k}$ by $x^a$. For a variety $X$, we denote $\text{Pic}(X) \otimes \mathbb{C}$ by $\text{Pic}(X)_C$. For a Laurent polynomial $f \in \mathbb{C}[\mathbb{Z}^n]$ we denote its Newton polytope, that is, the convex hull in $\mathbb{Z}^n \otimes \mathbb{R}$ of the exponents of nonzero monomials of $f$, by $N(f)$. The smooth del Pezzo surface of degree $d$ (excluding the case of a quadric surface) is denoted by $S_d$. We let $X_{k-m}$ denote the smooth Fano variety (regarded as an element of the family of Fano varieties of this type) of Picard rank $k$ and number $m$ in the lists from [12]. We denote the missing variety of Picard rank 4 and degree 26 by $X_{4-2}$ and shift subscripts for varieties of the same Picard rank and greater degree. We denote the weighted projective space with weights $a_0, \ldots, a_n$ by $\mathbb{P}(a_0, \ldots, a_n)$. The (weighted) projective space with coordinates $x_0, \ldots, x_n$ is denoted by $\mathbb{P}[x_0: \ldots: x_n]$. The affine space with coordinates $x_0, \ldots, x_n$ we denote by $\mathbb{A}[x_1, \ldots, x_n]$. The ring $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is denoted by $\mathbb{T}[x_1, \ldots, x_n]$. A polytope with integral vertices is said to be integral. We take a pencil in the birational sense: a pencil for us is a family birational to a family of fibres of a morphism to $\mathbb{P}^1$. 


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§ 2. Preliminaries

2.1. Toric geometry. Consider a toric variety $T$. A fan (or spanning) polytope $F(T)$ is the convex hull of the integral generators of the fan’s rays for $T$. Let $\Delta = F(T) \subset N_\mathbb{R} = \mathbb{Z}^n \otimes \mathbb{R}$. Let

$$\nabla = \{ x \mid \langle x, y \rangle \geq -1 \text{ for all } y \in \Delta \} \subset M_\mathbb{R} = N^\vee \otimes \mathbb{R}$$

be the dual polytope.

With an integral polytope $\Delta$ we associate a (singular) toric Fano variety $T_\Delta$ defined by the fan whose cones are cones over the faces of $\Delta$. We also associate a toric variety $eT_\Delta$ with $F(eT_\Delta) = \Delta$ (not uniquely defined) such that for any toric variety $T'$ with $F(T') = \Delta$ and any morphism $T' \to eT_\Delta$ we have $T' \cong eT_\Delta$. In other words, $eT_\Delta$ is given by a ‘maximal triangulation’ of $\Delta$.

In the rest of the paper we shall assume that $\nabla$ is integral, in other words, that $\Delta$ (or $\nabla$, or $T$, or $T_\nabla$, or $eT_\nabla$) is reflexive. In particular, this means that integral points of both $\Delta$ and $\nabla$ are either the origin or lie on the boundary. We denote $T_\nabla$ by $T^\vee$ and $eT_\nabla$ by $eT^\vee$.

Lemma 2.1. Let $T$ be a threefold reflexive toric variety. Then $eT^\vee$ is smooth.

Proof. Let $C$ be a two-dimensional cone of the fan of $eT^\vee$. It is a cone over an integral triangle $R$ without strictly internal integral points, lying in the affine plane $L = \{ x \mid \langle x, y \rangle \geq -1 \}$ for some $y \in N$. This means that in some basis $e_1$, $e_2$ and $e_3$ in $M$ we have $L = \{ a_1e_1 + a_2e_2 + e_3 \}$. Let $P$ be a pyramid over $R$ whose vertex is the origin. Then by Pick’s formula $\text{vol } R = 1/2$ and $\text{vol } P = 1/6$, which means that the vertices of $R$ form a basis in $M$, so that $eT^\vee$ is smooth.

An analogue of Lemma 2.1 holds for ‘nice’ toric degenerations of Fano complete intersections [13], so that log-Calabi-Yau compactifications can be constructed for their Landau-Ginzburg models. Unfortunately, Lemma 2.1 does not hold for higher dimensions in general, because there are $n$-dimensional simplices whose only integral points are vertices, and whose volume is greater than $1/n!$.

Definition 2.2 (see [8]). An integral polygon is said to be of type $A_n$, $n \geq 0$, if it is a triangle such that two of its edges have integral length 1 and the third one has integral length $n$. (In other words, its integral points are the three vertices and $n - 1$ points lying on the same edge.) In particular, $A_0$ is a segment of integral length 1.

An integral polygon $P$ is said to be Minkowski, or of Minkowski type, if it is a Minkowski sum of several polygons of type $A_n$, that is,

$$P = \{ p_1 + \cdots + p_k \mid p_i \in P_i \}$$

for some polygons $P_i$ of type $A_{n_i}$, and if the affine lattice generated by $P \cap \mathbb{Z}^2$ is the sum of the affine lattices generated by $P_i \cap \mathbb{Z}^2$. Such a decomposition is called an admissible lattice Minkowski decomposition and denoted by $P = P_1 + \cdots + P_k$. 
An integral three-dimensional polytope is said to be Minkowski if it is reflexive and all its facets are Minkowski polygons.

Finally we summarize some facts relating to toric varieties and their anticanonical sections. (See, for instance, [14] for details.) To do this it is more convenient to start from the toric variety $T^\vee$.

**Fact 1.** Let the anticanonical class $-K_{T^\vee}$ be very ample (in particular, this holds in the reflexive threefold case; see [15] and [16]). We can embed $T^\vee$ in a projective space in the following way. Consider the set $A \subset M$ of integral points in the polytope $\Delta$ dual to $F(T^\vee)$. Consider a projective space $\mathbb{P}$ whose coordinates $x_i$ correspond to elements $a_i$ in $A$. Associate the homogenous equation $\prod x_i^{\alpha_i} = \prod x_j^{\beta_j}$ with any homogenous relation $\sum \alpha_i a_i = \sum \beta_j a_j$, $\alpha_i, \beta_j \in \mathbb{Z}_+$. The variety $T^\vee$ is given in $\mathbb{P}$ by the equations associated with all homogenous relations on the $a_i$.

**Fact 2.** The anticanonical linear system of $T^\vee$ is $O_{\mathbb{P}}(1)$. In particular, it can be described as the projectivisation of a linear system of Laurent polynomials whose Newton polytopes lie in $\Delta$.

**Fact 3.** Toric strata of $T^\vee$ of dimension $k$ correspond to $k$-dimensional faces of $\Delta$. Let $R_f$ denote the anticanonical section corresponding to the Laurent polynomial $f \in \mathbb{C}[N]$ and $F_Q$, the stratum corresponding to the face $Q$ of $\Delta$. Denote the sum of those monomials of $f$ whose supports lie in $Q$ by $f|_Q$. Let $\mathbb{P}_Q$ be a projective space whose coordinates correspond to $\mathbb{P}_Q \cap N$. (In particular, $Q$ is given in $\mathbb{P}_Q$ by homogenous relations on the integral points of $Q \cap N$.) Then

$$R_{Q,f} = R_f|_{F_Q} = \{f|_Q = 0\} \subset \mathbb{P}_Q.$$ 

**Fact 4.** It follows from Fact 3 that $R_f$ does not pass through a toric point corresponding to a vertex of $\Delta$ if and only if its coefficient at this vertex is nonzero. The constant Laurent polynomial corresponds to the boundary divisor of $T^\vee$.

**2.2. Laurent polynomials and toric Landau-Ginzburg models.** Different polynomials with the same Newton polytope $\Delta$ give the same toric variety $T_\Delta$. However, the choice of the particular coefficients of a polynomial is important from the Mirror Symmetry point of view.

We briefly recall the notion of a toric Landau-Ginzburg model. (For more details see [3], for instance.)

Let $X$ be a smooth Fano variety of dimension $n$ and Picard number $\rho$. Let the number

$$\langle \tau_{a_1}, \gamma_1, \ldots, \tau_{a_k}, \gamma_n \rangle_\beta, \quad a_i \in \mathbb{Z}_{\geq 0}, \quad \gamma_i \in H^*(X, \mathbb{C}), \quad \beta \in H_2(X, \mathbb{Z}),$$

be the $k$-pointed genus-0 Gromov-Witten invariant with descendants for $X$ (see [17], VI-2.1).

Choose $\mathbb{C}$-divisors $L$ and $D$ on $X$. We can view $L$ as a direction (one-dimensional subtorus) on $\mathbb{T} = \text{Spec } \mathbb{C}[H_2(X, \mathbb{Z})]$ and $D$ as a point $p_D$ on $\mathbb{T}$. Let $1$ be the fundamental class of $X$. Let $R$ be a set of classes of effective curves. The series

$$\widetilde{I}_0^{X,L,D}(t) = 1 + \sum_{\beta \in R, \ a \in \mathbb{Z}_{\geq 0}} (\beta \cdot L)! \langle \tau_a 1 \rangle_\beta \cdot e^{-\beta \cdot D} t^{\beta \cdot L}$$
is called the constant term of the regularized $I$-series for $X$. It is a flat section for the restriction of the second Dubrovin’s connection on $\mathbb{T}$ to the direction of $D$ passing through $p_D$.

Now, we are interested in the anticanonical direction, so $L = -K_X$. We denote $\overline{I}_0^{X,-K_X,D}$ by $\overline{I}_0^{X,D}$. The latter series is the main object we associate to the pair $(X,D)$. Moreover, we often consider the case $D = 0$; in this case we use the notation $\overline{I}_0^{X}$ for $\overline{I}_0^{X,0}$.

Now consider the other side of Mirror Symmetry.

Let $\phi[f]$ be the constant term of the Laurent polynomial $f$. Set

$$I_f(t) = \sum \phi[f^j] t^j.$$ 

(The analogue of the series $I_f(t)$ for iterated Laurent series over rings was studied by algebraic methods in \[18\]–\[20\], for instance.)

The following theorem (which is mathematical folklore; see \[21\], Proposition 2.3, for the proof) justifies this definition.

**Theorem 2.3.** Let $f$ be a Laurent polynomial in $n$ variables. Let $P$ be a Picard-Fuchs differential operator for the pencil of hypersurfaces in a torus which is provided by $f$. Then $P[I_f(t)] = 0$.

**Definition 2.4** (see \[3\], §6). A toric Landau-Ginzburg model for the pair of a smooth Fano variety $X$ of dimension $n$ and a divisor $D$ on it is a Laurent polynomial $f \in \mathbb{T}[x_1, \ldots, x_n]$ which satisfies the following.

**Period condition.** The series $I_f = \overline{I}_0^{X,D}$.

**Calabi-Yau condition.** There exists a relative compactification of the family $f: (\mathbb{C}^*)^n \to \mathbb{C}$, whose total space is a (noncompact) smooth Calabi-Yau variety $Y$. Such a compactification is called a Calabi-Yau compactification.

**Toric condition.** There is a degeneration $X \sim T_X$ to a toric variety $T_X$ such that $F(T_X) = N(f)$.

A Laurent polynomial satisfying the period condition is called a weak Landau-Ginzburg model.

**Definition 2.5.** A compactification of the family $f: (\mathbb{C}^*)^n \to \mathbb{C}$ to a family $f: Z \to \mathbb{P}^1$, where $Z$ is smooth and $-K_Z = f^{-1}(\infty)$, is called a log-Calabi-Yau compactification (cf. the notion of tame compactified Landau-Ginzburg model in \[22\]).

We consider Mirror Symmetry as a correspondence between Fano varieties and Laurent polynomials. That is, the strong version of the Mirror Symmetry conjecture for variations of Hodge structures states the following.

**Conjecture 2.6** (see \[3\], Conjecture 38). Any pair of a smooth Fano variety and a divisor on it has a toric Landau-Ginzburg model.

In addition to the threefold case discussed in §1, the existence of Laurent polynomials satisfying the period condition was proved in \[1\] for smooth toric varieties;
in [23]–[25] for complete intersections in Grassmannians; and in [26] and [9] for certain complete intersections in some toric varieties. In [9] the toric condition was also checked for certain complete intersections in toric varieties and partial flag varieties.

In a lot of cases polynomials satisfying the period and toric conditions satisfy the Calabi-Yau condition as well. However, it is not easy to check this condition: unlike for the other two conditions there are no sufficiently general approaches; usually the Calabi-Yau condition has to be checked ‘by hand’. The natural idea is to compactify the fibres of the map $f: (\mathbb{C}^*)^n \to \mathbb{C}$ using the embedding

$$(\mathbb{C}^*)^n \hookrightarrow T,$$

where the toric variety $T = T_\Delta$ corresponds to $\Delta = N(f)$. Fibres compactify to anticanonical sections in $T$, which have trivial canonical classes. However, first, $T$ is usually singular, and even if we resolve it (provided it has a crepant resolution!), we can only conclude that its general anticanonical section is a smooth Calabi-Yau variety, and it is hard to say anything about the particular sections we need. Second, the family of anticanonical sections we are interested in has the base locus which we need to blow up to construct a Calabi-Yau compactification, and this blow up is not necessarily crepant.

The coefficients of the polynomials that correspond to trivial divisors tend to be very special, at least for the simplest toric degenerations. In this case the base loci are simpler and enable us to construct Calabi-Yau compactifications.

Consider a Laurent polynomial $f$. Recall that we can construct a Fano toric variety $T = T_\Delta$, where $\Delta = N(f)$, the dual toric Fano variety $T = T_\nabla$, where $\nabla$ is the dual polytope for $\Delta$, and the maximally triangulated toric variety $eT = eT_\nabla$. We make two assumptions about $f$. First, we assume that $\nabla$ is integral, in other words, $\Delta$ is reflexive. In particular, this means that integral points in both $\Delta$ and $\nabla$ either coincide with the origin or lie on the boundary. The other assumption is related to the special ‘symmetric’ choice of the coefficients of $f$.

**Definition 2.7** (see [8]). Let $P \in \mathbb{Z}^2 \otimes \mathbb{R}$ be an integral polygon of type $A_n$. Let $v_0, \ldots, v_n$ be consecutive integral points on the edge of $P$ with integral length $n$ and let $u$ be the remaining integral point of $P$. Let $x = (x_1, x_2)$ be a multivariable that corresponds to the integral lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$. Put

$$f_P = x^u + \sum \binom{n}{k} x^{v_k}.$$  

(In particular, $f_P = x^u + x^{v_0}$ for $n = 0$.)

Let $Q = Q_1 + \cdots + Q_s$ be an admissible lattice Minkowski decomposition of an integral polygon $Q \subset \mathbb{R}^2$. Put

$$f_{Q_1, \ldots, Q_s} = f_{Q_1} \cdots f_{Q_s}.$$  

A Laurent polynomial $f \subset \mathbb{T}[x_1, x_2, x_3]$ is called *Minkowski* if $N(f)$ is Minkowski and for each facet $Q \subset N(f)$ as an integral polygon, there exist an admissible lattice Minkowski decomposition $Q = Q_1 + \cdots + Q_s$ such that $f|_Q = f_{Q_1, \ldots, Q_s}$.
Remark 2.8 (cf. [8]). There are 105 families of smooth Fano threefolds (see [27]–[29]). Among these, 98 have weak Landau-Ginzburg models of Minkowski type, and seven varieties, namely $X_{1-1}$, $X_{1-11}$, $X_{2-1}$, $X_{2-2}$, $X_{2-3}$, $X_{9-1}$ and $X_{10-1}$, do not have reflexive toric degenerations.

§ 3. Del Pezzo surfaces

We start the section by recalling well-known facts about del Pezzo surfaces. There is a vast literature on del Pezzo surfaces. We refer the reader to [30].

The initial definition of del Pezzo surface is the following one given by del Pezzo himself.

Definition 3.1 (see [31]). A del Pezzo surface is a nondegenerate irreducible linear normal (that is, not a projection of a degree-d surface in $\mathbb{P}^{d+1}$) surface in $\mathbb{P}^d$ of degree $d$ that is not a cone.

In modern terms this means that a del Pezzo surface is an (anticanonically embedded) surface with very ample anticanonical class and canonical singularities (the same as du Val, simple surface, Kleinian, or rational double points). Classes of canonical and Gorenstein singularities for surfaces coincide, so we use the following more general definition.

Definition 3.2. A del Pezzo surface is a complete surface with very ample anticanonical class and canonical singularities. A weak del Pezzo surface is a complete surface with nef and big anticanonical class and canonical singularities.

Remark 3.3. Weak del Pezzo surfaces are (partial) minimal resolutions of singularities of del Pezzo surfaces. The exceptional divisors of the resolutions are $(-2)$-curves.

The degree of a del Pezzo surface $S$ is the number $d = (-K_S)^2$. We have $1 \leq d \leq 9$. If $d > 2$, then the anticanonical class of $S$ is very ample and it gives the embedding $S \hookrightarrow \mathbb{P}^d$, so both the definitions coincide. From now on we assume that $d > 2$.

Obviously, projecting a degree-$d$ surface in $\mathbb{P}^d$ from a point on it gives a degree-$(d-1)$ surface in $\mathbb{P}^{d-1}$. This projection is nothing but the blow up of the centre of the projection and the blow down of all lines passing through this point. (By the adjunction formula these lines are $(-2)$-curves.) If we choose general centres of projections (that do not lie on lines, say), then we get the classical description of a smooth del Pezzo surface of degree $d$ as a quadric surface (with $d = 8$) or a blow-up of $\mathbb{P}^2$ in $9 - d$ points. They degenerate to singular surfaces which are projections from nongeneral points (including infinitely close points). Moreover, all del Pezzo surfaces of given degree lie in the same irreducible deformation space, with the exception of degree 8, when there are two components (one for a quadric surface and one for the blow up $F_1$ of $\mathbb{P}^2$ in a point). General elements of the families are smooth, and all singular del Pezzo surfaces are degenerations of smooth surfaces in these families. This description enables us to construct toric degenerations of del Pezzo surfaces.

Remark 3.4. Del Pezzo surfaces of degree 1 or 2 also have toric degenerations. Indeed, these surfaces can be described as hypersurfaces in weighted projective
spaces, that is, ones of degree 4 in $\mathbb{P}(1,1,1,2)$ and of degree 6 in $\mathbb{P}(1,1,2,3)$ respectively, so they can degenerate to binomial hypersurfaces. However their singularities are worse than canonical.

Let $T_S$ be a Gorenstein toric degeneration of a del Pezzo surface $S$ of degree $d$. Let $\Delta = F(T_S) \subset N_\mathbb{R} = \mathbb{Z}^2 \otimes \mathbb{R}$ be the fan polygon of $T_S$. Let $f$ be a Laurent polynomial such that $N(f) = \Delta$.

Our goal now is to describe in detail a way to construct a Calabi-Yau compactification for $f$. More precisely, we will construct a commutative diagram

$$
\begin{array}{ccc}
(C^*)^2 & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
\mathbb{A}^1 & \xrightarrow{f} & \mathbb{P}^1
\end{array}
$$

where $Y$ and $Z$ are smooth, the fibres of the maps $Y \to \mathbb{A}^1$ and $Z \to \mathbb{P}^1$ are compact and $-K_Z = f^{-1}(\infty)$; we denote all ‘vertical’ maps in the diagram by $f$ for simplicity.

The strategy is the following. First we consider a natural compactification of the pencil $\{f = \lambda\}$ to an elliptic pencil in a toric del Pezzo surface $T^\vee$. Then we resolve the singularities of $T^\vee$ and get a pencil in a smooth toric weak del Pezzo surface $\widetilde{T}^\vee$. Finally, we resolve the base locus of the pencil to get $Z$. We get $Y$ by cutting out the strict transform of the boundary divisor of $\widetilde{T}^\vee$.

The polygon $\Delta$ has integral vertices in $N_\mathbb{R}$ and has the origin as a unique strictly internal integral point. The dual polygon $\nabla = \Delta^\vee \subset M = N^\vee$ has integral vertices and a unique strictly internal integral point too. Geometrically this means that the singularities of $T$ and $T^\vee$ are canonical.

Remark 3.5. The normalized volume of $\nabla$ is given by

$$\text{vol } \nabla = |\text{integral points in } \nabla| - 1 = (-K_S)^2 = d.$$

It is easy to see that

$$|\text{integral points in } \Delta| + |\text{integral points in } \nabla| = 12.$$

In particular, $\text{vol } \Delta = 12 - d$.

Compactification construction 3.6. By Fact 2, the anticanonical linear system on $T^\vee$ can be described as the projectivisation of the linear space of Laurent polynomials whose Newton polygons are contained in $\nabla^\vee = \Delta$. Thus the natural way to compactify this family is to use the embedding $(\mathbb{C}^*)^2 \hookrightarrow T^\vee$. Fibres of the family are anticanonical divisors in this (possibly singular) toric variety. Two anticanonical sections intersect in $(-K_{T^\vee})^2 = \text{vol } \Delta = 12 - d$ points (counted with multiplicities), so the compactification of the pencil in $T^\vee$ has $12 - d$ base points (possibly with multiplicities). The pencil $\{\lambda_0 f = \lambda_1\}$, $(\lambda_0 : \lambda_1) \in \mathbb{P}$, is generated by its general member and the divisor corresponding to a constant Laurent polynomial, that is, the boundary divisor of $T^\vee$. We note that, by Fact 4, the torus invariant points of $T^\vee$ do not lie in the base locus of the family.
Let $\tilde{T}^\vee \to T^\vee$ be a minimal resolution of singularities of $T^\vee$. Pull back the pencil under consideration. We get an elliptic pencil with $12 - d$ base points (with multiplicities), which are smooth points of the boundary divisor $D$ of the toric surface $\tilde{T}^\vee$; this divisor is a wheel of $d$ smooth rational curves. Blowing up these base points we get an elliptic surface $Z$. Let $E_1, \ldots, E_{12 - d}$ be the exceptional curves of the blow-up $\pi : Z \to \tilde{T}^\vee$; in particular, $Z$ is not toric. Denote the strict transform of $D$ by $D$ for simplicity. Then

$$-K_Z = \pi^*(-K_{\tilde{T}^\vee}) - \sum E_i = D + \sum E_i - \sum E_i = D.$$  

Thus the anticanonical class $-K_Z$ contains $D$ and consists of fibres of $Z$. In particular, this means that the open variety $Y = Z \setminus D$ is a Calabi-Yau compactification of the pencil given by $f$. This variety has $e > 0$ sections, where $e$ is the number of base points of the pencil in $\tilde{T}^\vee$ counted without multiplicities.

Summarizing, we obtain an elliptic surface $f : Z \to \mathbb{P}^1$ with smooth total space $Z$ and a wheel $D$ of $d$ smooth rational curves over $\infty$.

**Remark 3.7.** Let the polynomial $f$ be general among Laurent polynomials with a given Newton polygon. Then singular fibres of $Z \to \mathbb{P}^1$ are either curves with a single node or a wheel of $d$ rational curves over $\infty$. By Noether’s formula

$$12 \chi(O_Z) = (-K_Z)^2 + e(Z) = e(Z),$$

where $e(Z)$ is the topological Euler characteristic. Thus, singular fibres of $Z \to \mathbb{P}^1$ are $d$-curves with one node and a wheel of $d$-curves over $\infty$. This description was given in [10].

**Remark 3.8.** All toric Landau-Ginzburg models for all del Pezzo surfaces of degree at least 3 can be compactified simultaneously. That is, all reflexive polygons are contained in the biggest polygon $B$, which has vertices $(2, -1), (-1, 2)$ and $(-1, -1)$. Thus, fibres of all toric Landau-Ginzburg models can be simultaneously compactified to (possibly singular) anticanonical curves on $T_{B^\vee} = \mathbb{P}^2$. Blowing up the base locus gives a base point free family. However, in this case a general member of the family can pass through toric points as it may be that $N(f) \not\subseteq B$. This means that some exceptional divisors of the minimal resolution are extra curves in the wheel over infinity.

In other words, consider a triangle of lines on $\mathbb{P}^2$. A general member of the pencil given by $f$ is an elliptic curve on $\mathbb{P}^2$. The total space of the log-Calabi-Yau compactification is a blow up of nine points of intersection (counted with multiplicities) of the elliptic curve and the triangle of lines. Exceptional divisors for points lying over vertices of the triangle are components of the wheel over infinity for the log-Calabi-Yau compactification; the others are either sections of the pencil or components of fibres over finite points.

Now we describe toric Landau-Ginzburg models for del Pezzo surfaces and toric weak del Pezzo surfaces. That is, for a del Pezzo surface $S$, its Gorenstein toric degeneration $T$ with a fan polygon $\Delta$, its crepant resolution $\tilde{T}$ with the same fan polygon and a divisor $D \in \text{Pic}(S)_C \cong \text{Pic}(\tilde{T})_C$, we construct by induction two Laurent polynomials $f_{S,D}$ and $f_{\tilde{T},D}$ that are toric Landau-Ginzburg models for $S$.
and $\widetilde{T}$, respectively. To do this, in particular, we use Givental’s construction of
Landau-Ginzburg models for smooth toric varieties, see [1].

Let $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ be a quadric surface, and let $D_S$ be an $(a, b)$-divisor on it.
Let $T_1 = S$, and let $T_2$ be a quadratic cone; $T_1$ and $T_2$ are the only Gorenstein toric
degenerations of $S$. The crepant resolution of $T_2$ is a Hirzebruch surface $F_2$, so let
$D_{F_2} = a_s + b_\alpha + \beta f$, where $s$ is the class of a section of $F_2$, so that $s^2 = -2$, and let $f$
be a fibre of the map $F_2 \rightarrow \mathbb{P}^1$. Define

$$f_{S, D_S} = f_{T_1, D_S} = x + \frac{e^{-a}}{x} + y + \frac{e^{-b}}{y}$$

for the first toric degeneration and

$$f_{S, D_S} = y + \left(1 - \frac{e^{-a} + e^{-b}}{y}\right) x + \frac{e^{-b}}{y}, \quad f_{T_2, D_{F_2}} = y + \frac{e^{-\beta}}{xy} + \frac{e^{-a}}{y} + \frac{x}{y}$$

for the second.

Now assume that $S$ is a blow-up of $\mathbb{P}^2$. First let $S = T = \mathbb{P}^2$, let $l$ be the
class of a line on $S$, and let $D = a_0 l$. Then up to a toric change of variables

$$f_{\mathbb{P}^2, D} = x + y + \frac{e^{-a_0}}{xy}.$$

Now let $S'$ be a blow-up of $\mathbb{P}^2$ in $k$ points with exceptional divisors $e_1, \ldots, e_k$, let $S$ be a blow-up of $S'$ in a point, and let $e_{k+1}$ be the exceptional divisor for this
blow-up. We identify divisors on $S'$ and their strict transforms on $S$, so

$$\text{Pic}(S') = \text{Pic}(\mathbb{P}^2) = Zl + Ze_1 + \cdots + Ze_k$$

and

$$\text{Pic}(S) = Zl + Ze_1 + \cdots + Ze_k + Ze_{k+1}.$$  

Let

$$D' = a_0 l + a_1 e_1 + \cdots + a_k e_k \in \text{Pic}(S')_C$$

and

$$D = D' + a_{k+1} e_{k+1} \in \text{Pic}(S)_C.$$  

First we describe the polynomial $f_{\mathbb{T}, D}$. Combinatorially, $\Delta = F(\mathbb{T})$ is obtained
from the polygon $\Delta' = F(\mathbb{T}')$ by adding one integral point $K$ that corresponds
to the exceptional divisor $e_{k+1}$ and taking the convex hull. Let $L$ and $R$
be boundary points of $\Delta$ adjacent to $K$ on the left and right with clockwise order-
ing. Let $c_L$ and $c_R$ be coefficients in $f_{\mathbb{T}, D'}$ of the monomials corresponding to $L$ and $R$. Let $M \in \mathbb{T}[x, y]$ be a monomial corresponding to $K$. Then from Givental’s
description of Landau-Ginzburg models for toric varieties (see [1]) we get

$$f_{\mathbb{T}, D} = f_{\mathbb{T}', D'} + c_L c_R e^{-a_{k+1}} M.$$  

The polynomial $f_{S, D}$ differs from $f_{\mathbb{T}, D}$ by its coefficients at nonvertex boundary
points. For any boundary point $K \subset \Delta$ define the mark $m_K$ as the coefficient of
Consider a facet of $\Delta$ and let $K_0, \ldots, K_r$ be the integral points of this facet going clockwise. Then the coefficient of $f_{S,D}$ at $K_i$ is the coefficient of $s^i$ in the polynomial

$$m_{K_0} \left(1 + \frac{m_{K_1}}{m_{K_0}} s\right) \cdots \left(1 + \frac{m_{K_r}}{m_{K_{r-1}}} s\right).$$

Remark 3.9. Note that $\text{Pic}(S) \cong \text{Pic}(\tilde{T})$. That is, if $S$ is not a quadric, then both $S$ and $\tilde{T}$ are obtained by a sequence of blow-ups in points (the only difference is that points for $\tilde{T}$ can lie on exceptional divisors of the previous blow-ups). Thus in both cases Picard groups are generated by the class of a line on $\mathbb{P}^2$ and the exceptional divisors $e_1, \ldots, e_k$. However the image of $e_i$ under the map of Picard groups given by the degeneration of $S$ to $\tilde{T}$ can equal not $e_i$ itself but some linear combination of exceptional divisors. In other words these bases do not agree with the degeneration map.

Remark 3.10. The spaces parametrizing toric Landau-Ginzburg models for $S$ and for $\tilde{T}$ are the same — they are the spaces of Laurent polynomials with Newton polygon $\Delta$ modulo toric rescaling. Thus any Laurent polynomial corresponds to different elements of $\text{Pic}(S)_C \cong \text{Pic}(\tilde{T})_C$. This gives a map $\text{Pic}(S)_C \to \text{Pic}(\tilde{T})_C$. However, this map is transcendental because of the exponential nature of the parametrization.

**Proposition 3.11.** The Laurent polynomial $f_{S,D}$ is a toric Landau-Ginzburg model for $(S,D)$.

**Proof.** It is well known that a del Pezzo surface $S$ is either a smooth toric variety or a complete intersection in a smooth toric variety. This allows us to compute the series $\tilde{I}^S$ and hence $\tilde{I}^{S,D}$ following [1]. Using this it is straightforward to check that the period condition for $f_{S,D}$ holds. The Calabi-Yau condition holds by Compactification Construction 3.6. Finally, the toric condition holds by the construction of the Laurent polynomial.

**Proposition 3.12.** Consider two distinct Gorenstein toric degenerations $T_1$ and $T_2$ of a del Pezzo surface $S$. Let $\Delta_1 = F(T_1)$ and $\Delta_2 = F(T_2)$. Consider families of Calabi-Yau compactifications of Laurent polynomials with Newton polygons $\Delta_1$ and $\Delta_2$. Then there is a birational isomorphism between these families. In other words, there is a birational isomorphism between the affine spaces of Laurent polynomials with supports in $\Delta_1$ and $\Delta_2$ modulo a toric change of variables that preserves Calabi-Yau compactifications.

**Proof.** We can see that polygons $\Delta_1$ and $\Delta_2$ differ by a sequence of mutations (see [32], for example). These mutations agree with fibrewise birational isomorphisms of toric Landau-Ginzburg models modulo a change of basis in $H^2(S, \mathbb{Z})$ by construction. The statement follows from the fact that birational elliptic curves are isomorphic.

**Remark 3.13.** Let $D = 0$. Then the polynomial $f_{S,0}$ has coefficients 1 at vertices of its Newton polygon and $\binom{n}{k}$ at the $k$th integral point of an edge of integral length $n$. In other words, $f_{S,0}$ is binomial.

**Example 3.14.** Let $S = S_7$. This surface has two Gorenstein toric degenerations: it is toric itself, and it also can be degenerated to a singular surface which is obtained
by a blow up of a point in \( \mathbb{P}^2 \), a blow up of a point on the exceptional curve, and a blow down of the first exceptional curve to a point of type \( A_2 \).

Let \( \Delta_1 \) be the polygon with vertices \((1, 0), (1, 1), (0, 1), (-1, -1) \) and \((0, -1)\), and let \( D = a_0l + a_1e_1 + a_2e_2 \). Then

\[
f_{\bar{T}_{\Delta_1},D} = f_{S,D} = x + y + e^{-a_0} \frac{1}{xy} + e^{-(a_0+a_1)} \frac{1}{y} + e^{-a_2xy}.
\]

Let \( \Delta_2 \) be the polygon with vertices \((1, 0), (0, 1), (-1, -1) \) and \((1, -1)\), and let \( D = a_0l + a_1e_1 + a_2e_2 \). Then

\[
f_{\bar{T}_{\Delta_2},D} = x + y + e^{-a_0} \frac{1}{xy} + e^{-(a_0+a_1)} \frac{1}{y} + e^{-(a_0+a_1+a_2)} \frac{x}{y}
\]

and

\[
f_{S,D}' = x + y + e^{-a_0} \frac{1}{xy} + (e^{-(a_0+a_1)} + e^{-(a_0+a_2)}) \frac{1}{y} + e^{-(a_0+a_1+a_2)} \frac{x}{y}.
\]

(Here \( f_{S,D} \) and \( f_{S,D}' \) are toric Landau-Ginzburg models for \( (S, D) \) in different bases in \( (\mathbb{C}^*)^2 \).) It is easy to check that the mutation

\[
x \rightarrow x, \quad y \rightarrow \frac{y}{1 + e^{-a_2x}}
\]

sends \( f_{S,D} \) to \( f_{S,D}' \).

The surface \( S \) is toric, so by Givental

\[
\widetilde{I}_0^{S,D} = \sum_{k,l,m} \frac{(2k + 3l + 2m)!}{(k + l)! (l + m)! k! l! m!} e^{-a_0(k+l+m)-a_1k-a_2m} t^{2k+3l+2m}
\]

(see [5]). Note that \( \widetilde{I}_0^{S,D} = I_{f_{S,D}} = I_{f_{S,D}'} \).

§ 4. Minkowski toric Landau-Ginzburg models

**Lemma 4.1.** Let \( f \) be a Minkowski Laurent polynomial. Then for any face \( Q \) of \( \Delta \) the curve \( R_{Q,f} \) is a union of (transversally intersecting) smooth rational curves (possibly with multiplicities).

**Proof.** Let \( Q \) be of type \( A_k, k > 0 \). In appropriate basis \( Q \) has vertices \( u = (0, 1), v_0 = (0, 0) \) and \( v_k = (k, 0) \), and integral points \( v_i = (i, 0) \). Let \( x, x_0, \ldots, x_k \) be coordinates corresponding to \( u, v_0, \ldots, v_k \). Then, according to Fact 1, \( F_Q \) is given by the relations \( x_i x_j = x_i x_s \), where \( i + j = r + s \), in \( \mathbb{P}[x : x_0 : \ldots : x_k] \). This means that \( F_Q = v_k(\mathbb{P}(1, 1, k)) \) is the image of the \( k \)th Veronese map of \( \mathbb{P}(1, 1, k) \). Let \( y_0, y_1 \) and \( y_2 \) be coordinates on \( \mathbb{P}(1, 1, k) \), where \( y_2 \) has weight \( k \). Then

\[
R_{Q,f} = \sum x_i \binom{n}{i} + x = 0 \cap F_Q \subset \mathbb{P}^Q.
\]

Hence

\[
R_{Q,f} = \{(y_0 + y_1)^k + y_2 = 0 \} \subset \mathbb{P}(1, 1, k),
\]
so \( R_{Q,f} \) projects isomorphically to \( \mathbb{P}^1 \) under projection of \( \mathbb{P}(1,1,k) \) on \( \mathbb{P}^1 \) along the third coordinate. Thus \( R_{Q,f} \) is a smooth rational curve with multiplicity 1.

Now let \( Q = Q_1 + \cdots + Q_n \) be an admissible lattice Minkowski decomposition, where \( Q_i \) is of type \( A_{k_i} \), such that \( f|_Q = f_{Q_1} \cdots f_{Q_n} \). Then, as above, there are Veronese embeddings \( v_{k_i} : \mathbb{P}(1,1,k_i) \to \mathbb{P}^{k_i+1} \), where \( \mathbb{P}^{k_i+1} \) are different projective spaces. Let \( \Pi \) be a product of these projective spaces over all \( Q_i \), so that coordinates in \( \Pi \) can be described as collections of integral points in \( (Q_1, \ldots, Q_n) \). Denote the map \( F_{Q_1} \times \cdots \times F_{Q_n} \to \Pi \) by \( \varphi \). Let \( \psi : \Pi \to \mathbb{P}S \) be a Segre embedding. Let \( \mathbb{P} \) be the projective space whose coordinates correspond to integral points in \( Q \). Let \( x_{b_1}, \ldots, x_{b_n} \) be natural coordinates in \( \mathbb{P}S \). The space \( \mathbb{P} \) can be described as a linear section of \( \mathbb{P}S \) defined by the linear space

\[
L = \{ x_{b_1}, \ldots, x_{b_n} = x_{b_1}' , \ldots, x_{b_n}' \mid b_1 + \cdots + b_n = b_1' + \cdots + b_n' \},
\]

and \( F_Q = \psi \varphi(F_{Q_1} \times \cdots \times F_{Q_n}) \cap L \) in \( \mathbb{P}S \). This gives birational isomorphisms \( F_{Q_i} \to F_Q \) for \( k_i > 0 \) and \( \mathbb{P}^1 \)-bundles for \( k_i = 0 \). (In other words, coordinates on \( F_{Q_i} \) correspond to points of type \( a+b_1+\cdots+b_{n-1} \) on \( F_Q \), where \( a \in Q_i \) and \( b_j \) are some fixed points on \( Q_j \), \( j \neq i \).) In these coordinates the function \( f|_Q \) splits into \( n \) functions \( f_{Q_1}, \ldots, f_{Q_n} \), such that \( f_{Q_i} = f_{Q_j} \) for \( Q_i \neq Q_j \). This gives the required splitting \( R_{Q,f} = B_1 \cup \cdots \cup B_n \), where \( B_i \) is isomorphic to \( R_{Q_i,f} \) for \( k_i > 0 \) and a standard linear section \( f_i \) on \( \mathbb{P}(1,1,a_i) \) as above, \( B_j \) is a line (fibre) for \( k_j = 0 \), and \( B_r = B_s \) for \( Q_r = Q_s \).

**Proposition 4.2.** Let \( W \) be a smooth threefold and let \( F \) be a one-dimensional anticanonical linear system on \( W \) with reduced fibre \( D = F_\infty \). Let the base locus \( B \subset D \) be a union of smooth curves (possibly with multiplicities) such that any two components \( D_1 \) and \( D_2 \) of \( D \) satisfy \( D_1 \cap D_2 \not\subset B \). Then there is a resolution of the base locus \( f : Z \to \mathbb{P}^1 \) with a smooth total space \( Z \) such that \( -K_Z = f^{-1}(\infty) \).

**Proof** (cf. Compactification Construction 3.6). Let \( \pi : W' \to W \) be a blow-up of one component \( C \) of \( B \) on \( W \). Since \( \pi \) is a blow-up of a smooth curve on a smooth variety, \( W' \) is smooth. Let \( E \) be an exceptional divisor of the blow-up. Let \( D' = \bigcup D_i' \) be the strict transform of \( D = \bigcup D_i \). Since the multiplicity of \( C \) in \( D \) is 1, we have

\[
-K_{W'} = \pi^*(-K_W) - E = D' + E - E = D'.
\]

Moreover, the base locus of the family on \( W' \) is the same as \( B \) or \( B \setminus C \), possibly together with a smooth curve \( C' \) which is isomorphic to \( E \cap D_i' \); in particular, \( C \) is isomorphic to \( \mathbb{P}^1 \). (There are no isolated base points as the base locus is the intersection of two divisors on a smooth variety.) Thus all the conditions of the proposition hold for \( W' \). Since \( (W,F) \) is a canonical pair, the base locus \( B \) can be resolved in finite number of blow ups. This gives the required resolution.

**Proof of Theorem 1.1.** Let \( f \) be the Minkowski Laurent polynomial. Recall that the Newton polytope \( \Delta \) of \( f \) is reflexive, and the (singular Fano) toric variety whose fan polytope is \( \nabla = \Delta^\vee \) is denoted by \( T^\vee \). The family of fibres of the map given by \( f \) is a pencil \( \{ f = \lambda \} \), \( \lambda \in \mathbb{C} \). Its members have natural compactifications to anticanonical sections of \( T^\vee \). This family (more precisely, its compactification to a family \( \{ \lambda_0 f = \lambda_1 \} \) over \( \mathbb{P}[\lambda_0 : \lambda_1] \)) is generated by a general member and the
member that corresponds to the constant Laurent polynomial. The latter is nothing but the boundary divisor $D$ of $T^\vee$. Denote the pencil obtained on $T^\vee$ by $f: Z_{T^\vee} \to \mathbb{P}^1$ (we are using the same notation $f$ for the Laurent polynomial, the corresponding pencil and resolutions of this pencil for simplicity). By Lemma 4.1 the base locus of $f$ on $Z_{T^\vee}$ is a union of smooth (rational) curves (possibly with multiplicities). By Lemma 2.1 the variety $\tilde{T}^\vee$ is a crepant resolution of $T^\vee$. By the definition of a Newton polytope, the coefficients of the Minkowski Laurent polynomial at the vertices of $\Delta$ are nonzero. This means that the base locus does not contain any torus invariant strata of $T^\vee$ since it does not contain torus invariant points by Fact 4. Thus we get a pencil $f: Z \to \mathbb{P}^1$ of the base locus on $Z_{\tilde{T}^\vee}$ such that $Z$ is smooth and $-K_Z = f^{-1}(\infty)$. Thus $Z$ is the required log-Calabi-Yau compactification and $Y = Z \setminus f^{-1}(\infty)$ is a Calabi-Yau compactification.

Remark 4.3. The construction of a Calabi-Yau compactification is not canonical: it depends on the order of blow-ups of base locus components. However all log-Calabi-Yau compactifications are isomorphic in codimension 1.

There are 105 families of smooth Fano threefolds. By Remark 2.8 and [8], there are 98 families among them that have degenerations to toric varieties whose fan polytopes coincide with Newton polytopes of Minkowski Laurent polynomials satisfying the period condition (for trivial divisors). Thus they have toric Landau-Ginzburg models by Theorem 1.1. Two other varieties, $X_{1-1}$ and $X_{1-11}$, have toric Landau-Ginzburg models by [3]. The following proposition, in the spirit of [3], proves the existence of (log-)Calabi-Yau compactifications for some degenerations of the other threefolds.

Proposition 4.4. The Fano threefolds $X_{2-1}$, $X_{2-2}$, $X_{2-3}$, $X_{9-1}$ and $X_{10-1}$ have toric Landau-Ginzburg models.

Proof. The Fano variety $X_{2-1}$ is a hypersurface section of type $(1,1)$ in $\mathbb{P}^1 \times X_{1-11}$ in an anticanonical embedding; in other words, it is a complete intersection of hypersurfaces of types $(1,1)$ and $(0,6)$ in $\mathbb{P}^1 \times \mathbb{P}(1,1,1,2,3)$. The Fano variety $X_{2-2}$ is a hypersurface in a certain toric variety; see [8]. The Fano variety $X_{2-3}$ is a hyperplane section of type $(1,1)$ in $\mathbb{P}^1 \times X_{1-12}$ in an anticanonical embedding; in other words, it is a complete intersection of hypersurfaces of types $(1,1)$ and $(0,4)$ in $\mathbb{P}^1 \times \mathbb{P}(1,1,1,1,2)$. Finally, $X_{9-1} = \mathbb{P}^1 \times S_2$ and $X_{10-1} = \mathbb{P}^1 \times S_1$.

We now construct the Givental-type Landau-Ginzburg model for a variety $X_{i-j}$; see [1], for a concise explanation see [23]. Then we will represent it by a Laurent polynomial $f_{i-j}$ (see [33] and [3], for instance). We see that it satisfies the period condition using [5], and the toric condition using [6] and [7]. In the spirit of [3] we compactify the family given by $f_{i-j}$ to a family of (singular) anticanonical sections in $\mathbb{P}^1 \times \mathbb{P}^2$ or $\mathbb{P}^3$ and then crepantly resolve the singularities of the total space of the family. Consider these cases one by one.

Givental’s Landau-Ginzburg model for $X_{2-1}$ is the complete intersection

$$\begin{cases} u + v_0 = 0, \\ v_1 + v_2 + v_3 = 0 \end{cases}$$
in $\text{Spec } T[u, v_0, v_1, v_2, v_3]$ with the function

$$u + \frac{1}{u} + v_0 + v_1 + v_2 + v_3 + \frac{1}{v_1 v_2 v_3^3}.$$ 

Making the birational change of variables (see [33])

$$v_1 = \frac{x}{x + y + 1}, \quad v_2 = \frac{y}{x + y + 1}, \quad v_3 = \frac{1}{x + y + 1}, \quad u = \frac{z}{z + 1}, \quad v_0 = \frac{1}{z + 1}$$

up to an additive shift we get the function

$$f_{2-1} = \frac{(x + y + 1)^6 (z + 1)}{xy^2} + \frac{1}{z}$$

on the torus $\text{Spec } T[x, y, z]$.

Consider the family $\{f_{2-1} = \lambda\}, \lambda \in \mathbb{C}$. Make the birational change of variables (cf. the proof of [3], Theorem 18)

$$x = \frac{1}{b_1} - \frac{1}{b_1^2 b_2} - 1, \quad y = \frac{1}{b_1^2 b_2} \quad \text{and} \quad z = \frac{1}{a_1} - 1,$$

and multiply the expression thus obtained by the denominator. We see that the family is birational to

$$\{(1 - a_1)b_2^3 = ((1 - a_1)\lambda - a_1)(b_1 b_2 - b_1^2 b_2 - 1)\} \subset \mathbb{A}[a_1, b_1, b_2] \times \mathbb{A}[\lambda].$$

Now this family can be compactified to a family of hypersurfaces of bidegree $(2, 3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$ using the embedding $T[a_1, b_1, b_2] \hookrightarrow \mathbb{P}[a_0 : a_1] \times \mathbb{P}[b_0 : b_1 : b_2]$. The (noncompact) total space of the family has trivial canonical class and its singularities are a union of (possibly) ordinary double points and rational curves which are du Val along a line at general points. If we blow up any of these curves we get singularities of similar type again. After several crepant blow-ups we obtain a threefold with just ordinary double points; these points admit algebraic small resolution. This resolution completes the construction of the Calabi-Yau compactification. Note that the total space $(\mathbb{C}^*)^3$ of the initial family is embedded in the resolution.

We obtain Calabi-Yau compactifications for the other varieties similarly. We have

$$f_{2-2} = \frac{(x + y + z + 1)^2}{x} + \frac{(x + y + z + 1)^4}{yz}.$$ 

Applying the change of variables

$$x = ab, \quad y = bc \quad \text{and} \quad z = c - ab - bc - 1,$$

to the family $\{f_{2-2} = \lambda\}$ and multiplying by the denominator gives the family of quartics

$$ac^3 = (c - ab - bc - 1)(\lambda ab - c^2).$$
The embedding $\text{Spec} \mathbb{T}[a, b, c] \hookrightarrow \mathbb{P}[a : b : c : d]$ gives a compactification to a family of quartics over $\mathbb{A}^1$.

We have

$$f_{2-3} = \frac{(x + y + 1)^4(z + 1)}{xyz} + z + 1.$$ 

Applying the change of variables

$$x = ac, \quad y = a - ac - 1 \quad \text{and} \quad z = \frac{b}{c} - 1,$$

to the family $\{f_{2-3} = \lambda\}$ and multiplying by the denominator gives the family

$$a^3b = (\lambda c - b)(b - c)(a - ac - 1).$$

The embedding $\text{Spec} \mathbb{T}[a, b, c] \hookrightarrow \mathbb{P}[a : b : c : d]$ gives a compactification to a family of quartics over $\mathbb{A}^1$.

One has

$$f_{9-1} = x + \frac{1}{x} + \frac{(y + z + 1)^4}{yz}.$$ 

Applying the change of variables

$$x = \frac{c}{b}, \quad y = ac \quad \text{and} \quad z = a - ac - 1,$$

to the family $\{f_{9-1} = \lambda\}$ and multiplying by the denominator gives the family

$$a^3b = (\lambda bc - b^2 - c^2)(a - ac - 1).$$

The embedding $\text{Spec} \mathbb{T}[a, b, c] \hookrightarrow \mathbb{P}[a : b : c : d]$ gives a compactification to a family of quartics over $\mathbb{A}^1$.

Next,

$$f_{10-1} = \frac{(x + y + 1)^6}{xy^2} + z + \frac{1}{z}.$$ 

Applying the change of variables

$$x = \frac{1}{b_1} - \frac{1}{b_1^2b_2} - 1, \quad y = \frac{1}{b_1^2b_2}, \quad z = a_1,$$

to the family $\{f_{10-1} = \lambda\}$ and multiplying by the denominator gives the family

$$a_1b_2^3 = (\lambda a_1 - a_1^2 - 1)(b_1b_2 - b_1^2b_2 - 1).$$

The embedding $\text{Spec} \mathbb{T}[a_1, b_1, b_2] \hookrightarrow \mathbb{P}[a_0 : a_1] \times \mathbb{P}[b_0 : b_1 : b_2]$ gives a compactification to a family of singular hypersurfaces of bidegree $(2, 3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$ over $\mathbb{A}^1$.

In all cases the total spaces of the families have crepant resolutions. Proposition 4.4 is proved.

In some cases a Calabi-Yau compactification can be constructed in another way, using the multipotential technique (see [34], for example) and elliptic fibrations.
Proposition 4.5 (Harder). The polynomial $f_{10-1}$ satisfies the Calabi-Yau condition.

Proof. Consider the surface $B = \mathbb{A}[w] \times \mathbb{P}[s_0 : s_1]$. Compactify the family given by $f_{10-1}$ to a family of elliptic curves over $B$ so that the projection onto $\mathbb{A}^1$ gives a partial compactification of the initial family. This Weierstrass fibration can be given by the equation

$$Y^2 = X^3 + f_2X + f_3$$

with

$$f_2 = -\frac{1}{3}(2w^2s_1^2 - 3ws_0s_1 + s_0^2 + s_1^2)^4,$$

$$f_3 = \frac{2}{27}(2w^2s_1^2 - 3ws_0s_1 - 864ws_1^2 + s_0^2 + 864s_0s_1 + s_1^2)$$

$$\times (2w^2s_1^2 - 3ws_0s_1 + s_0^2 + s_1^2)^5.$$

This fibration is singular and its degeneracy locus over $B$ is given by the equation

$$s_1(ws_1 - s_0)(2w^2s_1^2 - 3ws_0s_1 - 432ws_1^2 + s_0^2 + 432s_0s_1 + s_1^2)$$

$$\times (2w^2s_1^2 - 3ws_0s_1 + s_0^2 + s_1^2)^{10} = 0.$$

Each component of this singular locus is a smooth curve over $B$. The singularities in the total space of this fibration are in the fibres over the curve given by

$$2w^2s_1^2 - 3ws_0s_1 + s_0^2 + s_1^2 = 0.$$

Above this curve, we get a curve of du Val singularities of type $E_8$. These singularities can be resolved by blowing up 8 times. This gives a smooth variety $Y$ which is relatively compact and fibred over $\mathbb{A}^1$. To see that this resolution is actually a Calabi-Yau variety, we can use the canonical bundle formula (see [35], p. 132). The equation is basically $K_Y = g^*(K_B + L)$, where $g$ is a map $Y \to B$, and $L$ is a divisor on the base of the fibration, which in this case is the pullback from $\mathbb{P}^1$ to $B$ of a section of $\mathcal{O}_{\mathbb{P}^1}(2)$. Therefore, $K_Y$ is the pullback of the trivial divisor on $B$, and hence is itself trivial. Thus $Y$ is a Calabi-Yau compactification of the family given by $f_{10-1}$.

Using [3], Theorem 1.1, Proposition 4.4 and the forthcoming papers [7] and [6] we obtain Corollary 1.2.

Remark 4.6. Recall that $\tilde{T}$ is a smooth toric variety with $F(\tilde{T}) = \Delta$. Let $f$ be a general Laurent polynomial with $N(f) = \Delta$. The Laurent polynomial $f$ is a toric Landau-Ginzburg model for a pair $(\tilde{T}, D)$, where $D$ is a general $\mathbb{C}$-divisor on $\tilde{T}$. Using [1], we see that the period condition for it is satisfied. Following the compactification procedure described in the proof of Theorem 1.1, we can see that the base locus $B$ is a union of smooth transversally intersecting (not necessary rational) curves. This means that in the same way as above the statement of Theorem 1.1 holds for $f$, so that $f$ satisfies the Calabi-Yau condition (cf. [36]). Finally, the toric condition holds for $f$ tautologically. Thus $f$ is a toric Landau-Ginzburg model for $(\tilde{T}, D)$. 
Problem 4.7. Prove an analogue of Theorem 1.1 for smooth Fano threefolds and any divisor. A description of Laurent polynomials for all Fano threefolds is contained in [7].

Question 4.8. Is it true that the Calabi-Yau condition follows from the period and toric conditions? If not, what conditions should be put on a Laurent polynomial to ensure that it does?

Another advantage of the compactification procedure described in Theorem 1.1 is that it enables us to describe ‘the fibres of compactified toric Landau-Ginzburg models over infinity’. These fibres play an important role, for example, in computing the Landau-Ginzburg Hodge numbers (see [22] and [37] for a thorough investigation of the del Pezzo case). We summarize these observations in the following assertion.

Corollary 4.9 (cf. [36], Conjecture 2.3.13). Suppose that \( f \) is a Minkowski Laurent polynomial. Let \( \tilde{T}^{\vee} \) be a (smooth) maximally triangulated toric variety such that \( F(\tilde{T}^{\vee}) = N(f) \), and let \( D \) be the boundary divisor of \( \tilde{T}^{\vee} \). Then there exists a log-Calabi-Yau compactification \( f: Z \to \mathbb{P}^1 \) with \( -K_Z = f^{-1}(\infty) = D \). In particular, \( D \) consists of \((−K_{T_{N(f)}})^3/2 + 2\) components given combinatorially by a triangulation of a sphere. (This means that vertices of the triangulation correspond to components of \( D \), edges correspond to intersections of the components, and triangles correspond to triple intersection points of the components.)

Proof. Let \( v \) be the number of vertices in the triangulation of \( \nabla \); in other words, \( v \) is the number of integral points on the boundary of \( \nabla \) or, equivalently, the number of components of \( D \). Let \( e \) be the number of edges in the triangulation of \( \nabla \), and let \( f \) be a number of triangles in the triangulation. As we are dealing with the triangulation of a sphere, we have \( v - e + f = 2 \). On the other hand \( 2e = 3f \). This means that \( v = f/2 + 2 \). The assertion of the corollary follows from the fact that both \((−K_{T_{N(f)}})^3 \) and \( f \) are equal to the normalized volume of \( \nabla \).

Remark 4.10. Let \( g = (−K_X)^3/2 + 1 \) be the genus of the Fano threefold \( X \); in particular, \( D \) consists of \( g + 1 \) components. Then \( g + 1 = \dim |−K_X| \).

Remark 4.11. The description of the fibres of Landau-Ginzburg models over infinity fits well with Mirror Symmetry considerations from the point of view of [38] and [6]. In these papers Fano varieties and their Landau-Ginzburg models are connected, via their toric degenerations by means of elementary transformations called basic links. From our point of view these are given by elementary subtriangulations of the sphere of boundary divisors.

General fibres of compactified toric Landau-Ginzburg models for Fano threefolds are smooth K3 surfaces. However, some of them can be singular and even reducible. Our lines of reasoning give almost no information about them; however, singular fibres of Landau-Ginzburg models are of special interest: they contain information about derived categories of singularities. There are few examples where these categories are computed. An invariant which is simpler to calculate is the number of components of reducible fibres.

Conjecture 4.12 (see [39], Conjecture 1.1, and also [40]). Let \( X \) be a smooth Fano variety of dimension \( n \). Let \( f_X \) be its toric Landau-Ginzburg model corresponding
to the trivial divisor on $X$. Let $k_{f_X}$ be the number of all the components of all reducible fibres (without multiplicities) of a Calabi-Yau compactification for $f_X$ minus the number of reducible fibres. Then

$$h^{1,n-1}(X) = k_{f_X}.$$ 

This conjecture has been established for Fano threefolds of rank one (see [3]) and for complete intersections (see [39]).

**Problem 4.13.** Prove Conjecture 4.12 for all Fano threefolds.

**Remark 4.14.** Most Fano threefolds have ‘simple’ toric degenerations, say, degenerations to toric varieties with cDV singularities (combinatorially this means that, apart from the origin, their fan polytopes only have integral points on edges). In these special cases we can keep track of the exceptional divisors appearing in the resolution procedure described in Proposition 4.2 and Theorem 1.1. That is, we can compute the multiplicities of the base curves (each multiplicity greater than 1 gives exceptional divisors in the fibres) and the local behaviour of their intersections. Then, proceeding as in Resolution Procedure 4.4 in [39], we can compute the required number of components.

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