WEAK GEODESICS IN THE SPACE OF KÄHLER METRICS

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Abstract. Given a compact Kähler manifold \((X, \omega_0)\), according to Mabuchi, the set \(H_0\) of Kähler forms cohomologous to \(\omega_0\) has the natural structure of an infinite dimensional Riemannian manifold. We address the question whether points in \(H_0\) can be joined by a geodesic, and strengthening the finding of \([LV]\), we show that this cannot always be done even with a certain type of generalized geodesics. As in \([LV]\), the result is obtained through the analysis of a Monge–Ampère equation.

1. Introduction.

Let \(X\) be a connected compact complex manifold of dimension \(m > 0\) and \(\omega_0\) a smooth Kähler form on it. In the 1980s Mabuchi discovered that there is a natural infinite dimensional Riemannian manifold structure on the set \(H_0\) of smooth Kähler forms cohomologous to \(\omega_0\), and on the set

\[ H = \{v \in C^\infty(X) : \omega_0 + i\partial \bar{\partial}v > 0\} \]

of smooth strongly \(\omega_0\)-plurisubharmonic functions. He also showed that \(H\) is isometric to the Riemannian product \(H_0 \times \mathbb{R}\), \([M]\). In \([LV]\), answering a question posed by Donaldson, Vivas and the second author proved that in general there is no geodesic of class \(C^2\) between two points in \(H\), resp. in \(H_0\); in fact, there is not even one of Sobolev regularity \(W^{1,2}\).

Since geodesics and their generalizations, weak geodesics, potentially play an important role in the study of special Kähler metrics (for geodesics, see \([D1, M]\)), it is of interest to know whether two points in \(H\) can be connected at least by a weak geodesic. What the notion of weak geodesic should be is suggested by Sémms’ reformulation of the geodesic equation in \(H\), see \([S]\). Let \(S = \{s \in \mathbb{C} : 0 < \text{Im} s < 1\}\) and \(\omega\) the pullback of \(\omega_0\) by the projection \(\overline{S} \times X \to X\). With any \(C^2\) curve \([0, 1] \ni t \mapsto v_t \in H\) associate a function \(u : \overline{S} \times X \to \mathbb{R}\),

\[ u(s, x) = v_{\text{Im} s}(x), \]

itself a \(C^2\) function. Then \(t \mapsto v_t\) is a geodesic if and only if \(u\) satisfies the Monge–Ampère equation \((\omega + i\partial \bar{\partial}u)^{m+1} = 0\). Therefore a \(C^2\) geodesic connecting \(0, v \in H\) gives rise to a solution \(u \in C^2(\overline{S} \times X)\) of a boundary value problem for this.

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Monge–Ampère equation on $\overline{S} \times X$; furthermore $\omega + i \partial \overline{\partial} u \geq 0$. This latter is expressed by saying that $u$ is $\omega$–plurisubharmonic. By a weak, or generalized, geodesic connecting, say, $0, v \in H$ one then means an $\omega$–plurisubharmonic solution $u : \overline{S} \times X \to \mathbb{R}$ of the problem

$$
(\omega + i \partial \overline{\partial} u)^{m+1} = 0,
$$

(1.1)

$$
u(s + \sigma, x) = u(s, x), \quad \text{if } (s, x) \in \overline{S} \times X, \sigma \in \mathbb{R},
$$

$$
u(s, x) = \begin{cases} 0, & \text{if } \text{Im } s = 0 \\ v(x), & \text{if } \text{Im } s = 1. \end{cases}
$$

It has to be assumed that $u$ is sufficiently regular so that $(\omega + i \partial \overline{\partial} u)^{m+1}$ can be given sense; for example, according to [BT], the continuity of $u$ more than suffices. X.X. Chen has indeed proved that for $v \in H$ (1.1) admits a continuous $\omega$–plurisubharmonic solution whose current $\partial \overline{\partial} u$ is represented by a bounded form, see [C] and complements in [Bl]. In other words, any two points in $H$ can be connected by a weak geodesic. One should keep in mind, though, that a weak geodesic $u$ need not give rise to a curve in $H$, first because $v_t = u(t, \cdot)$ is not necessarily $C^\infty$, not even $C^2$, and second because even if $v_t$ is $C^\infty$, there is no reason why it should be strongly $\omega_0$–plurisubharmonic.

In this paper we show that the regularity that Chen obtains cannot be improved: (1.1) may have a solution with $\partial \overline{\partial} u$ bounded, but in general it will not have a solution with $\partial \overline{\partial} u$ continuous.

If $\overline{Z}$ is a complex manifold, possibly with boundary, and $Z = \text{int } \overline{Z}$, we define

$$
C^{\partial \overline{\partial}}(\overline{Z}) = 
\{ w \in C(\overline{Z}) : \text{the current } \partial \overline{\partial}(w|Z) \text{ is represented by a form continuous on } \overline{Z} \}.
$$

Given $w \in C^{\partial \overline{\partial}}(\overline{Z})$, we will simply write $\partial \overline{\partial} w$ for the continuous form on $\overline{Z}$ that represents the current $\partial \overline{\partial}(w|Z)$, and if $z_1, z_2, \ldots$ are local coordinates on $Z$, we write $w_{z_j \bar{z}_k}$ for the coefficient of $dz_j \wedge d\bar{z}_k$ in $\partial \overline{\partial} w$.

Clearly $C^2(\overline{Z}) \subset C^{\partial \overline{\partial}}(\overline{Z})$, and it is well understood in harmonic analysis that the inclusion is strict. For example, if $\overline{Z} = \{ \zeta \in \mathbb{C} : |\zeta| \leq 1/2 \}$ and $k = 2, 3, \ldots$, the function

$$
w(\zeta) = \begin{cases} \zeta^k \log \log |\zeta|^{-2}, & \text{if } 0 < |\zeta| \leq 1/2 \\ 0, & \text{if } \zeta = 0 \end{cases}
$$

is not in $C^k(\overline{Z})$, but $w_{\zeta} \in C^{k-1}(\overline{Z})$ and $w_{\zeta \zeta} \in C^{k-2}(\overline{Z})$.

**Theorem 1.1.** Suppose a connected compact Kähler manifold $(X, \omega_0)$ admits a holomorphic isometry $g : X \to X$ with an isolated fixed point, and $g^2 = \text{id}_X$. Then there is a $v \in H$ for which (1.1) has no $\omega$–plurisubharmonic solution $u \in C^{\partial \overline{\partial}}(\overline{S} \times X)$. One can choose $v$ to satisfy $g^* v = v$.

The proof will show that among symmetric potentials the $v \in H$ in Theorem 1.1 even form an open set.
Theorem 1.1 corresponds to [LV, Theorem 1.2], but the $C^3$ regularity from [LV] has been lowered. The proofs here and in [LV] are similar in that, denoting by $x_0 \in X$ an isolated fixed point of $g$, in both proofs we analyze the behavior of a regular solution $u$ in a neighborhood of $S \times \{x_0\}$. The upshot of the analysis is a condition on the Hessian of the boundary value at $x_0$, a condition that not all $v \in \mathcal{H}$ satisfy. In [LV] the analysis involved the Monge–Ampère foliation associated with a $u \in C^3(S \times X)$, and it was crucial that the foliation was of class $C^1$. The foliation method is not available when $u$ is only $C^2$, and we will have to be thriftier with our tools, but in spite of this, we will recover the same condition on the Hessian as in [LV] when $m = 1$. When $m > 1$, the present condition is slightly stronger than the one in [LV].

2. Generalities.

In this section we collect a few simple facts concerning currents and the homogeneous Monge–Ampère equation.

**Proposition 2.1.** Let $f: Y \to Z$ be a holomorphic map of complex manifolds, $\varphi$ and $\psi$ continuous forms on $Z$ satisfying $\partial \bar{\partial} \varphi = \psi$ as currents. Then $\partial \bar{\partial} f^* \varphi = f^* \psi$ as currents.

**Proof.** We can assume $Z$ is an open subset of some $\mathbb{C}^n$. Regularizing $\varphi$ and $\psi$ by convolutions gives rise to sequences of smooth forms $\varphi_k$ and $\psi_k = \partial \bar{\partial} \varphi_k$ that converge locally uniformly to $\varphi$, resp. $\psi$. Therefore $f^* \varphi_k \to f^* \varphi$ and $\partial \bar{\partial} f^* \varphi_k = f^* \partial \bar{\partial} \varphi_k \to f^* \psi$ locally uniformly, whence the claim follows from the continuity of $\partial \bar{\partial}$ in the space of currents.

Next consider a complex manifold $Z$ and a plurisubharmonic $U \in C^{0,\partial}(Z)$. Suppose $Y \subset Z$ is a one dimensional, not necessarily closed complex submanifold and $TY \subset \text{Ker} \ \partial \bar{\partial} U$. The normal bundle $NY = (T^{1,0}Z|Y)/T^{1,0}Y$ is a holomorphic vector bundle and $\partial \bar{\partial} U$ induces a possibly degenerate Hermitian metric $h$ on it. With $p: T^{1,0}Z|Y \to NY$ the canonical projection, the metric is

$$h(p\zeta) = \partial \bar{\partial} U(\zeta, \bar{\zeta}) \geq 0, \quad \zeta \in T^{1,0}Z|Y.$$ 

Thus $h$ is continuous, but can degenerate, i.e., vanish on nonzero vectors as well.

**Proposition 2.2.** The metric $h$ is seminegatively curved in the sense that $\log h \circ \sigma$ is subharmonic for any holomorphic section $\sigma$ of $NY$ over some open $Y' \subset Y$. (Here it is convenient not to exclude from subharmonic functions those that are identically $-\infty$ on some component of $Y'$.) Further, on the line bundle $\text{det} NY$ the metric induced by $h$ is also seminegatively curved.

When $h$ is smooth and nondegenerate, and moreover $(\partial \bar{\partial} U)^{\dim Z} = 0$, the seminegativity of $\text{det} NY$ was first proved by Bedford and Burns in [BB, Proposition 4.1], and [CT, Theorem 4.2.8] gives the seminegativity of $N$ itself. For possibly degenerate $h$ [BF, Lemma] represents an equivalent result, albeit without the curvature interpretation, and under the assumption that $U$ is $C^2$. Our proof is a variant of the proof in [BF].
Proof. For the first statement we only need to prove that $\log h \circ \sigma$ has the sub-mean-value property, and this at points where $h \circ \sigma \neq 0$. To do so, we can assume $Z \subset \mathbb{C}^n$ is the unit polydisc, $Y = Y' = \{ z \in Z : z_2 = \ldots = z_n = 0 \}$, and that $\sigma(z_1, 0, 0, \ldots) = p(\partial/\partial z_2)$. Thus

$$ (h \circ \sigma)(z_1, 0, \ldots) = U_{z_2\bar{z}_2}(z_1, 0, \ldots). $$

Green’s formula implies for $0 < r < 1$

$$ (2.2) \quad \frac{1}{\pi r^2} \int_0^1 \left( U(z_1, re^{2\pi it}, 0, \ldots) - U(z_1, 0, 0, \ldots) \right) dt = $$

$$ \frac{i}{\pi r^2} \int_{|z_2| \leq r} (\log r - \log |z_2|) U_{z_2\bar{z}_2}(z_1, z_2, 0, \ldots) dz_2 \wedge d\bar{z}_2, $$

certainly if $U$ is $C^2$, but then upon regularizing by convolutions, whenever $U$ and $\partial\bar{\partial}U$ are continuous—as in our case. Proposition 2.1, with $f$ the embedding $Y \to Z$, implies $\partial\bar{\partial}(U|Y) = (\partial\bar{\partial}U)|Y = 0$. Hence the left hand side of (2.2) is a subharmonic function of $z_1$, and so is the right hand side. As $r \to 0$, these functions converge locally uniformly to $U_{z_2\bar{z}_2}(z_1, 0, \ldots)$; in light of (2.1) $h \circ \sigma$ is therefore subharmonic.

If $\varphi \in \mathcal{O}(Y)$ and $\sigma$ is replaced by $e^{\varphi/2}\sigma$, we obtain that $e^{\text{Re} \varphi} h \circ \sigma$ is also subharmonic. Therefore it satisfies the maximum principle, and so does $\text{Re} \varphi + \log h \circ \sigma$; knowing this for all $\varphi \in \mathcal{O}(Y)$ is equivalent to the subharmonicity of $\log h \circ \sigma$, see e.g. [H, Theorem 1.6.3].

Now given any holomorphic vector bundle $E \to Y$ of rank $r$, endowed with a seminegatively curved, possibly degenerate continuous Hermitian metric $h$, the induced metric on the line bundle $\det E$ is also seminegatively curved. Indeed, denoting by $h(e, e')$ the inner product of $e, e' \in E_y, y \in Y$, so that $h(e) = h(e, e)$, for (local) sections $\sigma_1, \ldots, \sigma_r$ of $E$ the induced metric is given by

$$ (2.3) \quad h^\det(\sigma_1 \wedge \ldots \wedge \sigma_r) = \det(h(\sigma_j, \sigma_k)). $$

If $h$ is smooth and nondegenerate and $y \in Y$, any nonzero holomorphic section of $\det E$ in a neighborhood of $y$ can be written as $\sigma_1 \wedge \ldots \wedge \sigma_r$, where the $\sigma_j$ are holomorphic sections of $E$ near $y$, and $h(\sigma_j, \sigma_k)$ vanish to second order at $y$ for $j \neq k$. Thus $\det(h(\sigma_j, \sigma_k)) - \prod_{j=1}^r h(\sigma_j, \sigma_j)$ vanishes to fourth order at $y$. By virtue of (2.3) this implies that at $y$

$$ i\partial\bar{\partial} \log h^\det(\sigma_1 \wedge \ldots \wedge \sigma_r) = i\partial\bar{\partial} \log \prod_{j=1}^r h(\sigma_j, \sigma_j) \geq 0. $$

Therefore $h^\det$ is seminegatively curved when $h$ is smooth and nondegenerate. To prove for a general $h$ we can assume $Y \subset \mathbb{C}$ is connected, $E = Y \times \mathbb{C}^r$ is holomorphically trivial, and $h^\det$ degenerates nowhere. We can regularize $h$ by convolutions, and obtain $h^\det$ as the locally uniform limit of seminegatively curved metrics, hence itself seminegatively curved.

Lastly we record a uniqueness result and its corollary:
Proposition 2.3. Given a compact Kähler manifold $(X, \omega_0)$ and $v \in \mathcal{H}$, the equation (1.1) has at most one $\omega$–plurisubharmonic solution $u \in C^{\bar{\partial}}(\mathcal{S} \times X)$.

The result follows from [Bl, Proposition 2.2 or Theorem 2.3] or from [PS, p. 144], once one checks that for $\omega$–plurisubharmonic $u \in C^{\bar{\partial}}(\mathcal{S} \times X)$ the Monge–Ampère measure $(\omega + i\partial\bar{\partial}u)^{m+1}$, as defined e.g. in [BT], agrees with what is obtained by taking the exterior power of the continuous form $\omega + i\partial\bar{\partial}u$. Alternatively, the more elementary arguments for [D1, Lemma 6] and the first paragraph of the proof of [LV, Proposition 2.3] also give uniqueness, provided one first checks the following: if $Z$ is a complex manifold and $w \in C^{\bar{\partial}}(Z)$ is real valued, then $i\partial\bar{\partial}w \geq 0$ at any local minimum point of $w$. Because of Proposition 2.1, it suffices to verify this latter when $\dim Z = 1$, and then it is straightforward: if $i\partial\bar{\partial}w < 0$ at a point, then $i\partial\bar{\partial}w < 0$ in a neighborhood, whence $w$ is strongly superharmonic there, and has no local minimum.

Corollary 2.4. Suppose $v \in \mathcal{H}$ satisfies $g^* v = v$, and $u \in C^{\bar{\partial}}(\mathcal{S} \times X)$ is an $\omega$–plurisubharmonic solution of (1.1). Then $u(s, x) = u(s, g(x))$.

3. The proof of Theorem 1.1.

Let $X, \omega_0, \omega,$ and $g$ be as in Theorem 1.1, and let $x_0 \in X$ be an isolated fixed point of $g$. Using [LV, Proposition 2.2] we choose local coordinates $z_1, \ldots, z_m$ in a neighborhood $V \subset X$ of $x_0$ in which $g$ is expressed as $(z_j) \mapsto (-z_j)$.

Proposition 3.1. If an $\omega$–plurisubharmonic $u \in C^{\bar{\partial}}(\mathcal{S} \times V)$ solves (1.1) and $u(s, x) = u(s, g(x))$, then $u(s, x_0) = a \Im s$ for $s \in \mathcal{S}$, with some $a \in \mathbb{R}$.

Proof (essentially taken over from [BF, Proposition]). The symmetry assumption implies that $u_{s \bar{z}_j}(s, x_0) = 0$, and so

\begin{equation}
0 = (-i\omega + \partial\bar{\partial}u)^{m+1} = (-i\omega + \partial_X \bar{\partial}_X u)^m \wedge \partial_S \bar{\partial}_S u
\end{equation}

at points of $\mathcal{S} \times \{x_0\}$. Hence for any $s \in \mathcal{S}$ either $(-i\omega + \partial_X \bar{\partial}_X u)^m$ or $\partial_S \bar{\partial}_S u$ vanishes at $(s, x_0)$. The goal is to show that it is always the latter that vanishes.

We claim that on $S \times \{x_0\}$

$$
\lambda = \log (-i\omega + \partial_X \bar{\partial}_X u)^m \left( \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial \bar{z}_1} \wedge \ldots \wedge \frac{\partial}{\partial z_m} \wedge \frac{\partial}{\partial \bar{z}_m} \right)
$$

is subharmonic and not identically $-\infty$. Indeed, by Proposition 2.1 $(\partial\bar{\partial}u)|\{s\} \times X = \partial\bar{\partial}(u|\{s\} \times X)$ for $s \in S$, and by the continuity of $\partial\bar{\partial}$, also for $s \in \mathcal{S}$. But $u(0, \cdot) = 0$ is strongly $\omega_0$–plurisubharmonic, hence $\lambda(s, x_0) > -\infty$ when $s = 0$, and also when $s \in \mathcal{S}$ is near 0. As to subharmonicity, it suffices to verify it on the open set

$$
S_0 = \{ s \in S : \lambda(s, x_0) > -\infty \}.
$$

Choose a smooth $w_0$ in a neighborhood of $x_0 \in X$ such that $\omega_0 = i\partial\bar{\partial}w_0$, let $w(s, x) = w_0(x)$ and $U = u + w$. By what has been observed above, $U_{s \bar{z}_j}(s, x_0) = \cdots$
$U_{ss}(s, x_0) = 0$ if $s \in S_0$; in other words, $S_0 \times \{x_0\}$ is tangential to $\text{Ker } \partial \bar{\partial} U$. By virtue of Proposition 2.2 $\lambda$ is subharmonic on $S_0 \times \{x_0\}$, hence on $S \times \{x_0\}$, as claimed.

Once we know $\lambda|S \times \{x_0\}$ is subharmonic, it follows that $S_0$ is dense in $S$; since by (3.1) $u_{ss}$ vanishes on $S_0 \times \{x_0\}$, it vanishes on all of $S \times \{x_0\}$. The Proposition now follows, because a harmonic function on $S$ that depends only on $\text{Im } s$ must be a linear function of $\text{Im } s$.

In the proof of the next lemma we will make use of the Poisson integral representation of harmonic functions in a strip. If $\psi$ is harmonic $S$, continuous and bounded in $\mathbb{S}$, then we have the following integral representation (for more on this see [W]):

\[(3.2) \quad \psi(\xi + i\eta) = \int_{-\infty}^{\infty} P(t - \xi, \eta)\psi(t)dt + \int_{-\infty}^{\infty} P(t - \xi, 1 - \eta)\psi(t + i)dt,\]

where $P$ is the following Poisson kernel:

\[P(\xi, \eta) = \frac{\sin \pi \eta}{2(\cosh \pi \xi - \cos \pi \eta)}.\]

As expected, the above integral representation formula also gives a recipe to generate bounded continuous harmonic functions in $\mathbb{S}$ given bounded continuous boundary data on $\partial S$.

Let $\omega_0 = \sum_{j,k=1}^m \omega_{jk}dz_j \wedge d\bar{z}_k$ on $V \subset X$.

**Lemma 3.2.** Suppose $a \in \mathbb{R}$ and $u$ is a bounded continuous $\omega$-plurisubharmonic function in $\mathbb{S} \times V$ satisfying

\[u(s, x) = 0, \quad \text{if } (s, x) \in \mathbb{R} \times V,\]

\[u(s, x_0) = a \text{ Im } s, \quad \text{if } s \in \mathbb{S}.\]

If $v = u(i, \cdot)$ is twice differentiable at $x_0$ and $dv = 0$ there, then

\[(3.3) \quad \left| \sum_{j,k=1}^m v_{z_j \bar{z}_k}(x_0)\xi_j \xi_k \right| \leq \sum_{j,k=1}^m (2\omega_{jk}(x_0) + v_{z_j \bar{z}_k}(x_0))\xi_j \xi_k \quad \text{for } \xi_j \in \mathbb{C}.\]

and this estimate is sharp.

**Proof.** We will assume $a = 0$ (otherwise we replace $u(s, x)$ by $u(s, x) - a \text{ Im } s$). Thus $u(s, x_0) = v(x_0) = 0$. By passing to a slice, the proof is reduced to the case $m = 1$. We will denote the local coordinate on $V$ by $z = z_1$; it identifies $V$ and $x_0$ with a neighborhood of $0 \in \mathbb{C}$ and with $0 \in \mathbb{C}$. Since $m = 1$, we need to verify

\[(3.4) \quad |v_{zz}(0)| \leq 2\omega_{11}(0) + v_{zz}(0).\]

Suppose $f : \mathbb{S} \to \mathbb{C}$ is bounded and holomorphic with $f(\alpha) = 0$ for some $\alpha \in \mathbb{S}$. Let $q = v_{zz}(0)$, and choose real numbers $p > v_{z \bar{z}}(0)$ and $r > \omega_{11}(0)$. With a neighborhood $V' \subset V$ of $0$ we will have

\[v(z) \leq p|z|^2 + \text{Re } qz^2 \quad \text{and } \omega < i\partial \bar{\partial}r|z|^2\]
for all \( z \in V' \). Clearly, this implies that the function \( U(s, z) = r|z|^2 + u(s, z) \) is plurisubharmonic in \( S \times V' \) and if \( \zeta \) is sufficiently small then

\[
\phi(s) = U(s, \zeta f(s)) = r|\zeta f(s)|^2 + u(s, \zeta f(s))
\]

is a subharmonic function of \( s \in S \). On the boundary of \( \overline{S} \) we have the following estimates:

\[
\phi(s) \begin{cases} 
= r|\zeta f(s)|^2, & \text{if } \text{Im } s = 0 \\
\leq (p + r)|\zeta f(s)|^2 + \text{Re } q\zeta^2 f(s)^2, & \text{if } \text{Im } s = 1.
\end{cases}
\]

We take \( \zeta \) such that \( q\zeta^2 \) is nonnegative. Then \( \text{Re } q\zeta^2 f(s)^2 = |q\zeta^2|\text{Re } f(s)^2 \).

Let \( \psi_1, \psi_2 \) and \( \psi_3 \) be bounded, continuous and harmonic functions on \( \overline{S} \) defined by the following boundary data:

\[
\psi_1(s) = |f(s)|^2, \quad \text{if } s \in \partial S,
\]

\[
\psi_2(s) = 0, \quad \text{if } \text{Im } s = 0 \quad \text{and} \quad \psi_2(s) = |f(s)|^2, \quad \text{if } \text{Im } s = 1,
\]

\[
\psi_3(s) = 0, \quad \text{if } \text{Im } s = 0 \quad \text{and} \quad \psi_3(s) = \text{Re } f(s)^2, \quad \text{if } \text{Im } s = 1.
\]

Since \( \psi_1, \psi_2, \psi_3 \) and \( \phi \) are all bounded on \( \overline{S} \), by the maximum principle we obtain

\[
(3.5) \quad 0 = u(\alpha, 0) = \phi(\alpha) \leq r|\zeta|^2 \psi_1(\alpha) + p|\zeta|^2 \psi_2(\alpha) + |q||\zeta|^2 \psi_3(\alpha).
\]

We will show that \( \psi_2(\alpha)/\psi_1(\alpha) \) can be chosen arbitrarily close to \( 1/2 \) and \( \psi_3(\alpha)/\psi_1(\alpha) \) can be chosen arbitrarily close to \( -1/2 \). We can work with any \( \alpha \in S \), but if \( \alpha = \xi + i\eta = i/2 \), Poisson’s formula (3.2) simplifies and gives \( \psi_1(i/2) = (I + J)/2, \psi_2(i/2) = J/2 \) and \( \psi_3(i/2) = K/2 \), where

\[
I = I(f) = \int_{-\infty}^{+\infty} \frac{|f(t)|^2}{\cosh \pi t} dt, \quad J = J(f) = \int_{-\infty}^{+\infty} \frac{|f(t + i)|^2}{\cosh \pi t} dt,
\]

\[
K = K(f) = \int_{-\infty}^{+\infty} \frac{\text{Re } f(t + i)^2}{\cosh \pi t} dt.
\]

We need to choose \( f \) so that \( I \approx J \approx -K \). No matter what \( f \), clearly \( |K| \leq J \); to achieve \( J \approx -K \), the integrands in \( J \) and \( K \) must be negatives of each other, at least approximately and for most \( t \in \mathbb{R} \) that make the integrands large. This means that \( f(t + i) \) must be close to imaginary. If also \( |f(t)| \approx |f(t + i)| \), then \( I \approx J \). Now \( f(s) = e^{\pi s/2} - e^{\pi i/4} \) satisfies both conditions and vanishes at \( i/2 \), but it is unbounded. Instead, with a large \( \lambda \in \mathbb{R} \) we let

\[
f_\lambda(s) = \frac{e^{\pi s/2} - e^{\pi i/4}}{1 + e^{\pi(s-\lambda)/2}}.
\]

We claim that \( I(f_\lambda) \approx J(f_\lambda) \approx 2\lambda \) and \( K(f_\lambda) \approx -2\lambda \) as \( \lambda \to \infty \). This will be verified only for \( J(f_\lambda) \), the other two are treated similarly. We have

\[
J(f_\lambda) = \left( \int_{-\infty}^{0} + \int_{0}^{\lambda} + \int_{\lambda}^{+\infty} \right) \frac{|ie^{\pi t/2} - e^{\pi i/4}|^2}{|1 + ie^{\pi(t-\lambda)/2}|^2 \cosh(\pi t)} dt.
\]
Since in the first integral the numerator is bounded, and in the last it is \(O(\cosh \pi t)\), both integrals have bounds independent of \(\lambda\). After a change of variables \(\tau = t/\lambda\) in the middle integral, we obtain

\[
\int_0^\lambda \frac{|ie^{\pi t/2} - e^{\pi i/4}|^2}{(1 + ie^{\pi(t-\lambda)/2})\cosh(\pi t)} \, dt = \lambda \int_0^1 \frac{|ie^{\pi \lambda \tau/2} - e^{\pi i/4}|^2}{|1 + ie^{\pi \lambda (\tau-1)/2}|^2 \cosh(\pi \lambda \tau)} \, d\tau
\]

\[
= 2\lambda \int_0^1 \frac{|i - e^{\pi i/4 - \lambda \tau/2}|^2}{|1 + ie^{\pi \lambda (\tau-1)/2}|^2 (1 + e^{-2\pi \lambda \tau})} \, d\tau.
\]

This last expression has bounded integrand, and the dominated convergence theorem implies \(J(f_\lambda) \sim 2\lambda\), as claimed. Letting \(\lambda \to \infty\) in (3.5) (with \(\alpha = i/2\)) we obtain \(0 \leq p - |q| + 2r\), and letting \(p \to v_\xi, r \to \omega_{11}\), (3.4) follows.

To prove the sharpness of estimate (3.3), suppose that \(V \subset \mathbb{C}\) is the unit disc and \(\omega = i\partial \overline{\partial}|z|^2\). Let

\[
u(s, z) = \frac{2 \Im s}{\varepsilon + \Im s} (\Re z)^2
\]

for some \(\varepsilon > 0\). Clearly \(\nu\) is bounded and continuous on \(\overline{S} \times V\), \(\nu(s, z) = 0\) for all \((s, z) \in \mathbb{R} \times V\), and \(\nu(s, 0) = 0\) for all \(s \in \overline{S}\). One verifies that \(\nu\) is \(\omega\)-plurisubharmonic in \(S \times V\) by checking that

\[
\begin{vmatrix}
u_{ss} & \nu_{s\xi} \\
\nu_{\xi s} & 1 + \nu_{zz}
\end{vmatrix} = 0
\]

and observing that \(1 + \nu_{zz} > 0\). This confirms that the Levi form of \(|z|^2 + \nu\) is semi-positive everywhere. If \(\varepsilon \to 0\) then \(2 + v_\xi(0) = 2 + u_\xi(i, 0) \to 1\) and \(|v_{zz}(0)| = |u_{zz}(i, 0)| \to 1\); hence the estimate (3.3) is indeed sharp.

Proof of Theorem 1.1. Given a \(g\)-invariant \(v \in \mathcal{H}\), suppose (1.1) has an \(\omega\)-plurisubharmonic solution \(u \in C^{0,2}(\overline{S} \times X)\). By Corollary 2.4, \(u(s, x) = u(s, g(x))\); since \(dv(x_0) = 0\) is automatic for \(g\)-invariant \(v\), by Proposition 3.1 and by Lemma 3.2 \(v\) then satisfies (3.3). Conversely, if a \(g\)-invariant \(v \in \mathcal{H}\) does not satisfy (3.3), then (1.1) will have no \(\omega\)-plurisubharmonic solution \(u \in C^{0,2}(\overline{S} \times X)\). Such \(v\) certainly exist (and form an open set among \(g\)-invariant potentials in \(\mathcal{H}\)), because the matrices \((v_{zj}z_k(x_0)) = (p_{jk})\) and \((v_{zj}z_k(x_0)) = (q_{jk})\) can be arbitrarily prescribed for \(g\)-invariant \(v \in \mathcal{H}\), as long as \((\omega_{jk}(x_0) + p_{jk})\) is positive definite, see [LV, Lemma 3.3].

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