General limitations on trajectories suitable for super-Penrose process

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Abstract – Collisions of particles near a rotating black hole can lead to unbound energies \( E_{\text{c.m.}} \) in their centre-of-mass frame. There are indications that the Killing energy of debris at infinity can also be unbound for some scenarios of collisions near the extremal black-hole horizon (the so-called super-Penrose process). They include the participation of a particle that i) has generic (not fine-tuned) parameters and ii) moves away from a black hole before collision. We show that for any finite particle’s mass, such a particle cannot be obtained as a result of the preceding collision. However, this can be done if one of the initial infalling particles has the mass of the order \( N^{-2} \) that generalizes previous observations made in the literature for radial infall in the Kerr background.

Introduction. – Nowadays, high-energy particles collisions near black holes attract much attention. In doing so, there are two related but different issues. The first one consists in the question as to under what conditions high (even unbound) energies \( E_{\text{c.m.}} \) in the centre-of-mass frame can be obtained. Let the parameters of one of the particles be fine-tuned (case i)). Then, this can give rise to the so-called Bañados-Silk-West (BSW) effect [1]. In case ii), no fine-tuning is required [2–4]. There is also one more case iii), when fine-tuning of parameters itself is the consequence of the fact that a particle moved on a circular orbit, provided this orbit is situated near the horizon of a near-extremal black hole [5,6].

The second issue concerns the properties of debris after collision. The central question here is whether and under what conditions one can detect at infinity high Killing energy \( E \). The main difficulty here is that a strong redshift can “eat” even a significant gain of the energy obtained in the collision. It turned out that the type of scenario between colliding particles is important here as well. For case i), when the BSW effect occurs, it was shown that there exist upper limits on \( E \) [7–9]. Meanwhile, quite recently, a series of works have appeared in which case ii) was investigated. It implies that the main effect comes from collision between an ingoing and outgoing particles near the horizon (head-on collision, as far as the radial motion is concerned). As was shown in [10] numerically, this type of scenario can lead to a significant increase of \( E \). In doing so, the outgoing particle was obtained as a result of reflection of another ingoing particle from the potential barrier. Then, the outgoing particle under discussion turns out to be close to the fine-tuned one, provided reflection occurred near the horizon. In [11], an outgoing generic particle was taken as an initial condition \( \text{per se} \). As a result, the authors obtained numerically unbound \( E \). The latter property is called in [11] super-Penrose process. It was pointed out in [12] that this leads to some limitations on the mass \( m_\star \) of the ingoing particle that should be very large (formally diverging) when the point of collision approaches the horizon (see also the reply in the revised preprint version of [11]). Head-on collision with participation of a generic (not fine-tuned) particle of a finite arbitrary mass was considered analytically in [13]. It was shown there that unbound \( E_{\text{c.m.}} \) are indeed possible in such a process.

As the outcome of collision is very sensitive to the type of scenario and properties of particles before collision, it is important to understand, what kinds of particles are
suitable for the super-Penrose process in general. The presence of a so-called usual (not fine-tuned) outgoing particle near the horizon is an essential ingredient for collisions considered in [11–13]. In principle, one can impose this initial condition by hand. However, if a particle moves away from the horizon of a black hole, this looks quite unusual. Therefore, it is desirable to find physical justification for such trajectories and learn, what such trajectory can come from. More precisely, we have to elucidate, whether or not they can be obtained from preceding particle collisions. We consider this issue and show that this is impossible. The result is valid for any number of particles participating in the collision near the horizon of the extremal black hole. It is obtained in a model-independent way, so it does not depend on the particular form of the metric. Additionally, we generalize the observation made in [12] and obtain the value of $m_*$ for a quite generic metric and without additional assumption about radial motion of infalling particles.

It is also worth mentioning that for the extremal Reissner-Nordström black hole, unbound $E_{c.m.}$ can be obtained easily even in the standard BSW picture, without initially outgoing particles [14, 15]. However, for astrophysical purpose, this metric is irrelevant. Throughout the paper, the fundamental constants $G = c = 1$.

**Basic formulas.** We consider the axially symmetric metric of the form

$$ds^2 = -N^2 dt^2 + g_\phi(d\phi - \omega dt)^2 + \frac{dr^2}{A} + g_\theta d\theta^2,$$  \hspace{1cm} (1)

where the metric coefficients do not depend on $t$ and $\phi$. Correspondingly, the energy $E = -mu_0$ and angular momentum $L = mu_\phi$ are conserved. Here, $m$ is a particle’s mass, $u^a = dx^a/d\tau$, being the four-velocity, $\tau$ the proper time along a trajectory. We restrict ourselves to motion within the equatorial plane $\theta = \frac{\pi}{2}$. Then, without loss of generality, we can redefine the radial coordinate in such a way that $A = N$. We do not restrict ourselves to the Kerr or another concrete form of the metric, so the results are quite general. The equations of motion for a geodesic particle read

$$m\ddot{t} = \frac{X}{N^2},$$  \hspace{1cm} (2)

$$m\ddot{\phi} = \frac{L}{g_\phi} + \omega X \frac{1}{N^2},$$  \hspace{1cm} (3)

$$m\dot{\tau} = \sigma Z,$$  \hspace{1cm} (4)

where

$$X = E - \omega L,$$  \hspace{1cm} (5)

$$Z = \sqrt{X^2 - N^2 \left( \frac{m^2 + L^2}{g_\phi} \right)},$$  \hspace{1cm} (6)

and the dot denotes differentiation with respect to $\tau$, the factor $\sigma = +1$ or $-1$ depending on the direction of motion.

As usual, we assume the forward-in-time condition $\dot{t} > 0$, whence

$$X \geq 0.$$  \hspace{1cm} (7)

The equality can be achieved on the horizon only where $N = 0$. Particles with $X_H = 0$ separated from zero are called usual, particles with $X_H = 0$ are called critical. If $X_H \neq 0$ but is small we call a particle near-critical (see next section for more detailed explanations). Subscript $H$ means that the quantity is calculated on the horizon.

**Properties of particle’s motion near the horizon.** In what follows, we consider the extremal horizon. In combination with formulas from the previous section, this gives rise to some quite definite properties of motion near the horizon. Let us consider first usual particles. In the vicinity of the horizon, there is no turning point since $N \to 0$, $X > 0$, so $Z^2 > 0$. If the impact parameter $b \equiv \frac{L}{E} < \omega_H^{-1}$, the particle can reach the horizon. If $b > \omega_H^{-1}$, this is impossible since it would contradict condition (7). This means that a usual particle can reach the turning point, where $Z = 0$, and bounce back. We are interested in collisions with small $N$ only, so we take into account the situation when the turning point lies close to the horizon. Then, $b$ must be close to the “critical” value $\omega_H^{-1}$. Near the extremal horizon, the requirement of regularity gives rise to the expansion [16]

$$\omega = \omega_H - B_1 N + O(N^2),$$  \hspace{1cm} (8)

where $B_1$ is some model-dependent coefficient. Correspondingly,

$$X = X_H + B_1 LN + O(N^2).$$  \hspace{1cm} (9)

There are two clear definitions of usual ($X_H = 0$) and critical ($X_H \neq 0$) particles. For near-critical particles, we required $X_H$ to be small (see above) but have not yet specified more precisely how small $X_H$ should be. Now, this can be done on the basis of the expansion (9). It is natural to assume that $X_H$ (the value of $X$ on the horizon) has the same order as the second term taken in the point of collision, where $N = N_c$ (hereafter, subscript $c$ indicates the point of collision). Otherwise, we return to previous definitions. Indeed, if $X_H \ll B_1 LN$, the contribution from $X_H$ is negligible, and the particle behaves practically like the critical one. The opposite case, $X_H \gg B_1 LN$, is typical of a usual particle. Only the intermediate case when

$$X_H = DN_c$$  \hspace{1cm} (10)

deserves special attention. Here, $D$ is some constant. Thus, we specify our definition of near-critical particles requiring that (10) is valid. In eq. (10) $N_c$ is not a free variable since it is taken only in the point of collision. This is in contrast with eqs. (8), (9) which are valid for arbitrary $N$, small enough. Then,

$$N_c = N_c(D + B_1 LN) + O(N^2).$$  \hspace{1cm} (11)

If the particle is exactly critical, $D = 0$. According to our definitions, in the point of collision both critical and
near-critical particles are similar in the sense that in both cases
\[ X_c = O(N_c). \] (12)
It is seen from (6) that
\[ Z(N_c) = O(N_c). \] (13)
As a result, for all cases we can write
\[ \lim_{N_c \to 0} Z = X. \] (14)
In particular, for critical particles both sides of (14) are equal to zero. If a particle is near-critical, sending \( N_c \to 0 \) we also send \( X_H \) to zero since it is adjusted to \( N_c \) according to (10). In doing so, a near-critical particle becomes more and more similar to the critical one and, in the limit, turns into it.

**Type of outgoing particles from collisions near horizon.** – In this section we extend the approach of [8] (see also sect. V of [9]) to the case of multiple collision. Let several particles initially move toward the horizon and collide in some point near the horizon. We assume that in the point of collision, the total energy and angular momentum are conserved. As a consequence, in this point we have
\[ X_{\text{fin}} = X_{\text{ini}}, \] (15)
where \( X_{\text{ini}} \) is the total contribution of initial particles and \( X_{\text{fin}} \) is the total contribution of the final outcome. Also, we assume the conservation of the radial momentum. If there are \( p \) particles before collision and \( q \) particles after it, this entails
\[ \sum_{i=1}^{p} \sigma_i Z_i = \sum_{k=1}^{q} \sigma_k Z_k. \] (16)
Let us consider the limit in which the point of collision approaches the horizon. We want to elucidate, whether or not a usual outgoing particle (that was absent initially) can arise as a result of such a collision. In doing so, all masses \( m_i \) and \( m_k \) are considered to be finite.

**Statement.** If in the initial configuration usual outgoing particles are absent, they cannot appear after collision.

**Proof.** It follows from (16) that in the limit under discussion,
\[ \sum_{i=1}^{n} \sigma_i (X_i)_H = \sum_{k=1}^{m} \sigma_k (X_k)_H. \] (17)
By assumption, all initial particles are ingoing. Therefore, in the left-hand side of (17), only terms with \( \sigma_i = -1 \) are present. Then, we have
\[ -X_{\text{ini}} = X_{\text{fin}}^{(+)} - X_{\text{fin}}^{(-)}. \] (18)
Here, \( X_{\text{fin}}^{(+)} \) and \( X_{\text{fin}}^{(-)} \) are, correspondingly, contributions into \( X_{\text{fin}} \) coming from outgoing and ingoing particles,
\[ X_{\text{fin}} = X_{\text{fin}}^{(+)} + X_{\text{fin}}^{(-)}. \] (19)
Now, comparing (15), (18) and (19), we arrive at the result
\[ X_{\text{fin}}^{(+)} = 0. \] (20)
However, according to (7), each term in this sum is non-negative. Therefore, the equality (20) is possible only in the case in which each of the terms is equal to zero. It means that none of the outgoing particles after collision can be usual, which completes the proof. If there are several critical or near-critical particles initially, the statement still holds true since such particles do not contribute to \( X_{\text{fin}} \) in the limit under discussion.

If, instead of taking the exact limit, we consider small but nonzero \( N_c \), we should use eq. (13) instead of eq. (14). Then, instead of (20) we obtain that
\[ X_{\text{fin}}^{(+)} = O(N_c). \] (21)
Thus, outgoing particles can be near-critical but, again, they cannot be usual.

The results can be generalized for nonequatorial motion, provided that \( \theta \)-components of velocities are finite. They will enter \( Z \), being multiplied by \( N^2 \) [17] and will not change the conclusion.

**Case of supermassive particle.** – If some of the particles are so heavy that, formally, \( m = O(N^{-1}) \) or higher, eq. (14) becomes invalid and our proof fails. This is just the situation discussed in [12] (see their eq. (4)) and [11]. The authors of ref. [12] considered, under which conditions an outgoing particle can be obtained near the horizon as a result of collisions of two particles falling from infinity. It was shown in [12] that for the Kerr metric the corresponding mass value of one of the particles has the order \( N^{-2} \). Now, we generalize this observation for an arbitrary metric of the form (1). What is even more important, this enables us to elucidate whether or not the main restriction on the mass of the infalling particle found in [12] depends on the details of the process.

Let two particles, 2 and 3 (we use the same notations as in [12]), move from infinity (or some finite distance) towards the horizon. After collision, particle 3 falls in a black hole and particle 2 escapes to infinity. Then, the conservation of the radial momentum reads
\[ p_2^r + p_3^r = p_2^\phi + p_3^\phi. \] (22)
Here, \( p^r \) for each particle is taken from (4), so
\[ p_2^r = -\sqrt{X_2^2 - N^2 \left( m_2^2 + \frac{L_2^2}{g_\phi} \right)}, \] (23)
\[ p_3^r = -\sqrt{(m_3 - \omega H L_2)^2 - N^2 \left( m_3^2 + \frac{L_2^2}{g_\phi} \right)}, \]
\[ p_X^r = \sqrt{X_X^2 - N^2 \left( m_X^2 + \frac{L_2^2}{g_\phi} \right)}, \]
\[ p_3^\phi = -\sqrt{(X_2 + X_3 - X_X)^2 - N^2 \left( \frac{L_2^2}{g_\phi} + m_3^2 \right)}. \] (24)
where we assume for simplicity that \( E_* = m_* \) and took into account the conservation of \( X \) (15). All particles are taken to be usual. Then, near the horizon, for small \( N \), one can write for any particle that

\[
Z \approx X - \frac{N^2}{2X} \left( m^2 + \frac{L^2}{g^2} \right). \tag{25}
\]

Then, after some algebra, we obtain in the main approximation that \( m_* \) should be large,

\[
m_* = \frac{4X}{N^2} + O \left( \frac{1}{N} \right). \tag{26}
\]

For the extremal Kerr metric, \( N^2 \approx \frac{(r-M)^2}{4M^2} \), where \( M \) is the black-hole mass, \( r \) is the Boyer-Lindquist coordinate. Then, eq. (26) corresponds just to eq. (4) of [12]. In [11], the large value of \( m_* \) was related to the simplifying assumption \( L_2 = L_* = 0 \) taken in [12]. However, it follows from our consideration that for any finite angular momenta the result \( m_* = O(N^{-2}) \) is qualitatively the same and, in the main approximation, depends on \( X \) only.

**Discussion and conclusions.** – Thus we showed that the most simple option for arranging the super-Penrose process near black holes is closed. However, there are several other options. Let us enumerate them. Actually, the present result in combination with some previous works gives a complete classification of potentially suitable states.

**Collision with a supermassive particle.** This includes a very massive particle falling in a black hole in the first collision to obtain a usual escaping particle in the near-horizon region. This is the inevitable price for it.

**Collision near past horizons (white holes).** A particle that moves near the horizon in the outward direction can correspond to white holes rather than to black ones, which suggests one more type of high-energy collisions. This circumstance was briefly mentioned in [18]. Detailed consideration of such a process was done in [19]. Now, the particle crosses the horizon with \( X_H > 0 \) by assumption, so we have a usual particle moving in the outward direction. If one bears in mind the potential astrophysical consequences, there is difficulty here since white holes are probably unstable (see sect. 15.2 of [20]). Nonetheless, a white hole (past horizon) is an unavoidable part of the whole picture of an eternal black-white hole. Therefore, the complete and coherent consideration of all possible scenarios should include this case as well.

**Collisions with (near-)critical particles.** There is a scenario in which one of the ingoing near-critical particles bounces back near the horizon thus turning into an outgoing one and collides afterwards with another ingoing particle. Then, relatively high energies were obtained numerically in [10]. It remains to be seen analytically, when this type of scenario indeed leads to the super-Penrose process.

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