INTEGRATION OF THE GHP EQUATIONS
IN SPACETIMES ADMITTING
A GEODESIC SHEAR-FREE
EXPANDING NULL CONGRUENCE

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Abstract

We perform a complete \( \rho \)-integration of the GHP equations for all spacetimes that admit a geodesic shear-free expanding null congruence, whose Ricci spinor is aligned to the congruence and whose Ricci scalar is constant. We also deduce the system of GHP equations after the integration is completed, and discuss a few applications.

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1 Introduction

1.1 Conventions

We will use spacetime definitions and conventions from [16]. In particular this means that the metric $g_{ab}$ is assumed to have signature $(+ - - -)$. We will use spinors for our calculations, but as all results are local in nature there is no need to postulate the existence of a global spinor structure on spacetime. All spinor dyads $(\nu^A, \nu^A')$ will be assumed to be normalized i.e., $\nu^A \nu^A' = 1$. $\nabla_{AA'}$ denotes the Levi-Civita connection i.e., the uniquely defined metric and torsion-free (symmetric) connection on spacetime.

1.2 Outline

When looking for exact solutions of Einstein’s field equations in NP- or GHP-formalism, a common approach is to assume the existence of a geodesic shear-free expanding null congruence. In vacuum spacetimes, the Goldberg-Sachs theorem tells us that this condition is necessary and sufficient for the Weyl curvature spinor to be algebraically special. In this paper we will relax the vacuum condition and instead assume that the GHP-components of the Ricci spinor satisfy

$$\Phi_{00} = \Phi_{01} = \Phi_{02} = 0$$

or equivalently

$$\Phi_{ABA'B'O^{A'}O^{B'}} = 0$$

where $l^a = \nu^A \nu^A'$ denotes the geodesic shear-free expanding congruence. This assumption is still sufficient to ensure algebraic speciality of the Weyl spinor. For technical reasons we will also assume

$$\Lambda = \text{constant}.$$ 

We then show that all GHP equations containing $\Phi$ can be integrated explicitly, something which is done in [10] for the vacuum case. We also show that the remaining GHP equations take on a very simple appearance after the integration procedure is completed and discuss redundancy in the final system of equations.

It is worth noting that a similar integration in NP-formalism for a non-vacuum subclass of these spacetimes is performed in [14] and [15] and, under even more restrictive assumptions, this NP integration have been performed in e.g., [17], [13], [18] and [19].

Kerr [12] has also integrated the GHP equations using this method for the case $\Phi_{ABA'B'} = 0$ i.e., Einstein spacetimes.
2 Spacetimes admitting a geodesic shear-free expanding null congruence

2.1 The general case

We will consider spacetimes admitting a geodesic, shear-free null congruence $l^a = o^A o^{A'}$ and whose Ricci spinor satisfies the condition

$$\Phi_{ABA'B'} o^A o^{B'} = 0.$$  \hspace{1cm} (1)

Take $o^A$ as the first spinor of a spinor dyad. In GHP-formalism the above conditions are equivalent to

$$\Phi_{00} = \Phi_{01} = \Phi_{02} = 0, \quad \kappa = \sigma = 0$$ \hspace{1cm} (2)

By the Goldberg-Sachs theorem we obtain

$$\Psi_0 = \Psi_1 = 0$$ \hspace{1cm} (3)

so the spacetime is algebraically special.

In addition we will assume that $\rho \neq 0$ so that $l^a$ is expanding. Then we can use a null rotation about $o^A$ to achieve $\tau = 0$, and the Ricci equations \[16\] then imply that also

$$\tau' = \sigma' = 0.$$ \hspace{1cm} (4)

We introduce Held’s \[10\] modified operators which can be written

$$\tilde{\partial} = \frac{1}{\rho} \partial, \quad \tilde{\partial}' = \frac{1}{\rho'} \partial', \quad \tilde{P}' = P' + \frac{p}{2\rho} (\Psi_2 + 2\Lambda) + \frac{q}{2\rho} (\bar{\Psi}_2 + 2\Lambda) \hspace{1cm} (5)$$

in this dyad. Note that the definition of $\tilde{P}'$ is slightly modified from Held’s in our non-vacuum case. The purpose of using Held’s modified operators is simply to reduce the length of calculations; in particular the new operators have the nice properties

$$\left[ \tilde{P}, \tilde{\partial} \right] = \left[ \tilde{P}, \tilde{\partial}' \right] = 0$$ \hspace{1cm} (6)

and

$$\left[ \tilde{P}, \tilde{P}' \right] \eta = \left[ -\frac{1}{2\rho} (\Psi_2 + 2\Lambda) - \frac{1}{2\rho} (\bar{\Psi}_2 + 2\Lambda) \right] \tilde{P} \eta \hspace{1cm} (7)$$

so if $\eta^o$ satisfies $\tilde{P} \eta^o = 0$ (a degree sign will, throughout the paper, be used to denote a quantity that is killed by $\tilde{P}$) then

$$\tilde{P} \tilde{\partial}' \eta^o = \left[ \tilde{P}, \tilde{\partial}' \right] \eta^o = 0$$

and the same result is true if $\tilde{\partial}'$ is replaced with $\tilde{\partial}$ or $\tilde{P}'$.  

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From the Ricci equations [16] we obtain the equations
\[ \pi \rho = \rho^2, \quad \ddot{\rho} = 0 \] (8)

The first of these equations can be used to ‘\( \rho \)-integrate’ some of the Ricci- and Bianchi equations. However, we first need to calculate \( \tilde{\partial} \rho \). Therefore, define the twist of the congruence \( \Omega^o \) as
\[ \Omega^o = \frac{1}{\rho} - \frac{1}{\bar{\rho}} \] (9)

We then obtain
\[ \pi \Omega^o = -\frac{\bar{\rho}^2}{\rho^2} + \frac{\rho^2}{\bar{\rho}^2} = 0 \]
so the notation is consistent. We also obtain
\[ \tilde{\partial} \rho = \rho^2 \tilde{\partial} \Omega^o \]. (10)

The Ricci- and Bianchi equations [16] involving \( \pi \), but not \( \Phi_{22} \) are
\[ \pi \rho' = \bar{\rho}' \rho - \Psi_2 - 2\Lambda \]
\[ \pi \kappa' = -\Psi_3 - \Phi_{21} \]
\[ \pi (\Psi_2 + 2\Lambda) = 3\rho \Psi_2 + 2\rho \Phi_{11} \]
\[ \pi \Psi_3 - \bar{\pi} \Phi_{21} - \rho \ddot{\rho} (\Psi_2 + 2\Lambda) = 2\rho \Psi_3 - 2\rho \Phi_{21} \]
\[ \pi \Psi_4 - \rho \ddot{\rho} \Psi_3 - \rho \ddot{\rho} \Phi_{21} = \rho \Psi_4 \]
\[ \pi (\Phi_{11} + 3\Lambda) = 2(\rho + \bar{\rho}) \Phi_{11} \]
\[ \pi \Phi_{21} - \rho \ddot{\rho} (\Phi_{11} - 3\Lambda) = (\rho + 2\bar{\rho}) \Phi_{21} \] (11)

In order to integrate these with respect to \( \pi \) we need to know the dependence of the Ricci scalar \( \Lambda \) on \( \pi \). Therefore we will make the additional assumption that \( \Lambda \) is constant\(^2\). Then these equations reduce to
\[ \pi \rho' = \bar{\rho}' \rho - \Psi_2 - 2\Lambda \]
\[ \pi \kappa' = -\Psi_3 - \Phi_{21} \]
\[ \pi \Psi_2 = 3\rho \Psi_2 + 2\rho \Phi_{11} \]
\[ \pi \Psi_3 - \bar{\pi} \Phi_{21} - \rho \ddot{\rho} \Psi_2 = 2\rho \Psi_3 - 2\rho \Phi_{21} \]
\[ \pi \Psi_4 - \rho \ddot{\rho} \Psi_3 - \rho \ddot{\rho} \Phi_{21} = \rho \Psi_4 \]
\[ \pi \Phi_{11} = 2(\rho + \bar{\rho}) \Phi_{11} \]
\[ \pi \Phi_{21} - \rho \ddot{\rho} \Phi_{11} = (\rho + 2\bar{\rho}) \Phi_{21} \] (12)

\(^2\)It would in fact be sufficient to assume that \( \Lambda \) is killed by \( \pi \), \( \partial \) and \( \partial' \), but unless \( \rho = \bar{\rho} \) the commutators would imply that also \( \pi' \Lambda = 0 \), so we choose to ignore this possibility. It may also be sufficient to assume some other explicit \( \pi \)-, \( \partial \)- and \( \partial' \)-dependence of \( \Lambda \), \( \Psi_2 \), or \( \Phi_{11} \).
The sixth of these equations is equivalent to
\[
0 = \frac{1}{\rho^2 \phi^2} \text{tr} \Phi_{11} - \left( \frac{2}{\rho \phi} + \frac{2}{\rho^2 \phi} \right) \Phi_{11} = \text{tr} \left( \frac{\Phi_{11}}{\rho^2 \phi^2} \right).
\]
Thus, we obtain
\[
\Phi_{11} = \rho^2 \phi^2 \Phi_{11}^0
\]
where, as usual \( \text{tr} \Phi_{11}^0 = 0 \). Then the third equation can be written
\[
0 = \frac{1}{\rho^3} \text{tr} \Psi_2 - \frac{3}{\rho^2} \Psi_2 - 2 \phi \Phi_{11}^0 = \text{tr} \left( \frac{\Psi_2}{\rho^3} - 2 \phi \Phi_{11}^0 \right).
\]
This gives us
\[
\Psi_2 = \rho^3 \Phi_{11}^0 + 2 \rho^3 \phi \Phi_{11}^0.
\]
In a similar way, the remaining of the above equations can be integrated, to give
\[
\begin{align*}
\rho' &= \rho \rho - \frac{1}{2} (\rho^2 + \phi \rho) \Psi_2 - \phi \rho \Phi_{11} + \frac{1}{\phi} \\
\kappa' &= \kappa - \rho \Psi_3 - \frac{1}{2} \rho^2 \Psi_2 - \frac{1}{2} \rho \phi \Psi_2 + \rho \phi \Phi_{11} - \rho \phi \Phi_{11} + \phi \rho \Phi_{11}^0 \\
\Phi_{21} &= \rho \phi \Phi_{21} + \rho \phi \Phi_{20} + \phi \rho \Phi_{21} + \phi \rho \Phi_{21} \\
\Psi_3 &= \rho \Psi_3 + \rho \phi \Psi_2 + \frac{3}{2} \rho \phi \Psi_2 + \phi \rho \Phi_{11} + \phi \rho \Phi_{11} + 3 \phi \rho \Phi_{11}^0 \\
\Psi_4 &= \rho \Psi_4 + \rho \phi \Psi_2 + \left( \frac{3}{2} \rho \phi \Psi_2 + \phi \rho \Phi_{11} + \phi \rho \Phi_{11}^0 + 3 \phi \rho \Phi_{11}^0 \Psi_2 \right) \\
&+ \frac{3}{2} \rho \phi \Psi_2 + \phi \rho \Phi_{11} + \phi \rho \Phi_{11}^0 + 3 \phi \rho \Phi_{11}^0 + 3 \phi \rho \Phi_{11}^0 \Psi_2 \\
&+ \rho \phi \Phi_{11}^0 + \rho \phi \Phi_{11}^0 + 3 \phi \rho \Phi_{11}^0 \Psi_2 + \phi \rho \Phi_{11}^0 \Psi_2 + \phi \rho \Phi_{11}^0 \Psi_2.
\end{align*}
\]
The remaining Ricci- and Bianchi equations are
\[
\begin{align*}
\text{tr} \rho' &= \text{tr} \rho - \text{tr} \Psi_2 - 2 \Lambda \\
\text{tr} \rho' &= \partial \kappa' = \rho^2 + \Phi_{22} \\
\partial \kappa' &= -\Psi_4 \\
\partial \rho' &= (\rho - \rho) \kappa' - \Psi_3 + \Phi_{21} \\
\partial \Psi_2 &= 2 \rho \Phi_{12} \\
\text{tr} \Psi_2 &= \partial \Psi_3 - \partial \Phi_{21} + \partial \Phi_{22} = 3 \rho \Psi_2 + \phi \Phi_{22} + 2 \rho \Phi_{11} \\
\text{tr} \Psi_4 &= \partial \Psi_4 + \partial \Phi_{21} + \partial \Phi_{22} = 4 \rho \Psi_4 - 3 \kappa' \Psi_2 - 2 \rho \Phi_{21} + 2 \kappa' \Phi_{11} \\
\text{tr} \Phi_{22} &= \partial \Phi_{11} - \partial \Phi_{12} - \partial \Phi_{21} = (\rho + \rho) \Phi_{22} + 2 (\rho + \rho) \Phi_{11}
\end{align*}
\]
We will now use the expressions (15), to rewrite these equations. Starting with the first one, we obtain
\[
\text{tr} \rho' = \rho^2 \rho - \frac{1}{2} \rho^2 \rho \Psi_2 - \frac{1}{2} \rho \phi \Psi_2 - \rho \phi \Phi_{11} + \frac{\rho}{\phi} \Lambda.
\]
From this equation we obtain the relation
\[ \tilde{\mathbf{P}}' \Omega^o = \tilde{\rho}^o - \rho^o, \]
and by applying the \([\partial, \tilde{\partial}']\)-commutator to \(\rho\) we obtain the useful relation
\[ \tilde{\partial} \tilde{\partial}' \Omega^o = 2 \Omega^o \tilde{\rho}^o + \Psi_2^o - \Psi_2. \]
Similarly, the substitution of known expressions into the third, fourth and fifth equation of (16) easily give us
\[ \tilde{\partial} \rho^o \Omega^o = -\Omega^o k^o - \Psi_3^o \]
As we have already calculated the derivative operators action on \(\Omega^o\) we can replace their action on \(\rho\) with their action on the real \((-1, -1)\)-quantity \(\frac{1}{\rho} + \frac{1}{\rho} \rho \)
\[ \tilde{\mathbf{P}} \left( \frac{1}{\rho} + \frac{1}{\rho} \right) = -2 \]
\[ \tilde{\partial} \left( \frac{1}{\rho} + \frac{1}{\rho} \right) = \tilde{\partial} \Omega^o \]
\[ \tilde{\partial}' \left( \frac{1}{\rho} + \frac{1}{\rho} \right) = -\tilde{\partial}' \Omega^o \]
\[ \tilde{\mathbf{P}}' \left( \frac{1}{\rho} + \frac{1}{\rho} \right) = -\left( \rho^o + \tilde{\rho}^o \right) + \rho \Psi_2^o + \tilde{\rho} \Psi_2^o + 2 \rho \tilde{\rho} \Phi_{11}^o - \frac{2 \Lambda}{\rho \tilde{\rho}}. \]  
(19)
We will next use the last equation of (16) to \(\rho\)-integrate for \(\Phi_{22}\), but before that we give two of the commutators that will be used in the integration
\[ \left[ \tilde{\mathbf{P}}, \tilde{\partial} \right] = \left( -\frac{k^o}{\rho} + \Psi_3^o + \frac{1}{2} \rho^o \Psi_2^o + \frac{1}{2} \rho^2 \Psi_2^o \Omega^o + \tilde{\rho} \Phi_{21}^o + 2 \rho \tilde{\rho} \Phi_{11}^o \right) \tilde{\mathbf{P}} + p \left( \frac{1}{\rho} + \frac{1}{\rho} \right) \]
\[ \left[ \tilde{\partial}, \tilde{\partial}' \right] = \left( \frac{\rho^o}{\rho} + \frac{1}{\rho} \right) \Psi_2^o + \frac{1}{2} \rho^2 \Psi_2^o \Omega^o + \Omega^o \right) \tilde{\mathbf{P}} + p \left( \frac{1}{\rho} + \frac{1}{\rho} \right) \]
\[ + \Omega^o \tilde{\mathbf{P}}' + p \left( \rho^o + \Omega^o \right) - q \left( \rho^o + \Omega^o \right) \]  
(20)
The last equation of (16) requires considerably more work to \(\rho\)-integrate than the ones integrated previously. However, after a very long calculation involving these commutators, we obtain
\[ \Phi_{22} = \rho \tilde{\rho} \Phi_{22}^o + \rho^2 \rho \left( \tilde{\partial} \Phi_{21}^o \right) - \rho^2 \left( \tilde{\partial} \Phi_{11}^o \right) + \rho^2 \left( \tilde{\partial} \Phi_{11}^o \right) - \frac{1}{2} \tilde{\partial} \Phi_{11}^o 
\]
\[ + \rho^3 \rho \tilde{\rho} \Phi_{21}^o \tilde{\partial} \Phi_{11}^o - \rho^2 \rho^2 \rho \left( \tilde{\partial} \tilde{\partial}' \Phi_{11}^o \right) - \rho^3 \rho \tilde{\rho} \Phi_{21}^o \tilde{\partial} \Omega^o \]
\[ + \rho^3 \rho^2 \rho \tilde{\partial} \Omega^o \tilde{\partial} \Phi_{11}^o + \rho^2 \rho^2 \rho \tilde{\partial} \tilde{\partial}' \Phi_{11}^o \tilde{\partial} \Phi_{11}^o - \rho^3 \rho^3 \Phi_{11}^o \tilde{\partial} \Phi_{11}^o - \rho^3 \rho^3 \Phi_{11}^o \tilde{\partial} \Omega^o \tilde{\partial} \Omega^o \]  
(21)
Next we will look at the sixth equation of (16). We start by subtracting it from the last equation of (16) to get rid of the $I^\Phi_{22}$-term

$$\dot{\mathbf{p}} \Phi_{11} - \dot{\mathbf{p}} \Phi_{12} - \dot{\mathbf{p}} \Psi_2 + \partial \Psi_3 = -3\rho^3 \Psi_2 + 2\rho^2 \Phi_{11} + \rho \Phi_{22}. \quad (22)$$

By using the previous equations along with the $[\tilde{\partial}, \tilde{\partial}']$-commutator it becomes

$$\tilde{\partial} \Psi_3 - \dot{\mathbf{p}} \Psi_2 = \Phi_{22} \quad (23)$$

It remains to check the second and seventh equation of (16). Using the equations obtained so far, the second gives us

$$\dot{\mathbf{p}} \rho^\circ - \tilde{\partial} \kappa^\circ = \Lambda(2\Omega^\circ \rho^\circ + \Psi_2^\circ - \Psi_2). \quad (24)$$

Checking the seventh equation of (16) is a very long and tedious calculation where both of the commutators (20) are used. The end result is that

$$\tilde{\partial} \Psi_4 - \dot{\mathbf{p}} \Psi_3 = \Lambda(\tilde{\partial'} \Psi_2^\circ - 2\Phi_{21}^\circ - 2\Omega^\circ \Psi_3^\circ) \quad (25)$$

As could be expected, the system of equations that we have obtained contains considerable redundancy. As an example of this we note that the imaginary part of equation (23) and the whole of equation (25) are actually consequences of the other equations.

### 3 Summary and conclusions

#### 3.1 Summary

In this section we will collect the resulting equations in one place.

First of all, the equations for the GHP-operators acting on $\rho$

$$\begin{align*}
\tilde{\mathbf{p}} \rho &= \rho^2 \\
\tilde{\partial} \rho &= 0 \\
\tilde{\partial'} \rho &= \rho^2 \tilde{\partial'} \Omega^\circ \\
\dot{\mathbf{p}}' \rho &= \rho^2 \rho^\circ - \frac{1}{2} \rho^2 \rho \overline{\Psi_2} - \frac{1}{2} \rho^3 \Psi_2^\circ - \rho \Phi_{11}^\circ + \frac{\rho}{\tilde{\rho}} \Lambda
\end{align*} \quad (26)$$

can be split into the relations

$$\begin{align*}
\tilde{\mathbf{p}}\left(\frac{1}{\rho} + \frac{1}{\tilde{\rho}}\right) &= -2 \\
\tilde{\partial}\left(\frac{1}{\rho} + \frac{1}{\tilde{\rho}}\right) &= \tilde{\partial} \Omega^\circ
\end{align*}$$
\[ \bar{d}'\left(\frac{1}{\rho} + \frac{1}{\rho}\right) = -\bar{d}'\Omega^0 \]

\[ \bar{p}'\left(\frac{1}{\rho} + \frac{1}{\rho}\right) = -(\rho^\omega + \bar{p}'^\omega) + \rho\Psi_2^\circ + \bar{\rho}\Psi_{11}^\circ - \frac{2\Lambda}{\rho\rho} \]

\[ \bar{p}\Omega^0 = 0 \]

\[ \bar{p}'\Omega^0 = \bar{p}'^\omega - \rho^\omega \]

(27)

From the commutators on \( \rho \) we have the useful relation

\[ \bar{d}\bar{d}'\Omega^0 = 2\Omega^\omega \rho^\omega + \Psi_2^\circ - \bar{\Psi}_2. \]

(28)

The curvature scalars and the spin coefficients are

\[ \rho' = \bar{\rho}\rho'^\circ - \frac{1}{2}(\rho^2 + \rho\bar{\rho})\Psi_3^\circ - \rho^2\rho\Psi_{11}^\circ + \frac{1}{2}\Lambda \]

\[ \kappa' = \kappa'^\circ - \rho\Psi_3^\circ - \frac{1}{2}\rho^2\bar{\Psi}_2^\circ - \frac{1}{2}\rho^2\bar{\Psi}_2^\circ \Omega^0 - \rho\bar{\rho}\Psi_{11}^\circ - \rho^2\bar{\rho}\bar{d}'\Phi_{11}^\circ - \rho^2\rho\Phi_{11}^\circ \Omega^0 \]

\[ \Psi_2 = \rho^3\Psi_3^\circ + 2\rho^3\Phi_{11}^\circ \]

\[ \Psi_3 = \rho^2\Psi_3^\circ + \rho^3\bar{\rho}^\circ\Psi_2^\circ + \frac{3}{2}\rho^4\Psi_2^\circ\bar{\rho}\Omega^0 + \rho^2\bar{\rho}\Phi_{21}^\circ + 2\rho^3\bar{\rho}\Phi_{11}^\circ + 3\rho^4\rho\Phi_{11}^\circ \bar{d}'\Omega^0 \]

\[ \Psi_4 = \rho^3\Psi_4^\circ + \rho^3\bar{\rho}^\circ\Psi_3^\circ + \frac{1}{2}\rho^3(\bar{d}'\Omega^0 - \Psi_2^\circ\Phi_{11}^\circ + \bar{\Psi}_2^\circ\Phi_{11}^\circ) + \frac{1}{2}\rho^4(\Psi_2^\circ\Phi_{11}^\circ + \bar{\Psi}_2^\circ\Phi_{11}^\circ) \]

\[ + \rho^4\bar{\rho}(\bar{d}'\Omega^0 + 3\bar{\rho}'\bar{\rho}'\Phi_{11}^\circ + 3\rho\Phi_{11}^\circ(\bar{d}'\Omega^0)^2 \]

\[ \Phi_{11} = \rho^2\rho\Phi_{11}^\circ \]

\[ \Phi_{21} = \rho^3\Phi_{21}^\circ + \rho^3\bar{\rho}^2\bar{\Phi}_{11}^\circ + \rho^3\rho^2\bar{\Phi}_{11}^\circ \bar{d}'\Omega^0 \]

\[ \Phi_{22} = \rho^2\Phi_{22}^\circ + \rho^2\rho\bar{\Phi}_{21}^\circ - \frac{1}{2}\bar{\Phi}_{11}^\circ + \rho^2(\bar{d}'\Phi_{21}^\circ - \frac{1}{2}\Phi_{11}^\circ) \]

\[ + \rho^3\rho\Phi_{21}^\circ \bar{d}'\Omega^0 + \frac{1}{2}(\bar{d}'\Phi_{11}^\circ + \bar{d}'\Phi_{11}^\circ) - \rho^3\Phi_{11}^\circ \bar{d}'\Omega^0 \]

\[ + \rho^2\rho\bar{d}'\Omega^0 \bar{d}'\Phi_{11}^\circ - \rho^2\rho^2\bar{d}'\Omega^0 \Phi_{11}^\circ + \rho^2\rho^2\bar{d}'\Omega^0 \Phi_{11}^\circ - \rho^3\rho\Phi_{11}^\circ \bar{d}'\Omega^0. \]

(29)

The remaining Ricci and Bianchi equations are

\[ \bar{d}'\rho'^\circ - \bar{d}\kappa'^\circ = (2\Omega^\omega \rho'^\circ + \Psi_2^\circ - \bar{\Psi}_2)\Lambda \]

\[ \bar{d}'\kappa'^\circ = -\Psi_4^\circ \]

\[ \bar{d}'\rho'^\circ = -\Omega^\omega \rho'^\circ - \Psi_3^\circ \]

\[ \partial\Psi_3^\circ = \bar{d}'\left(\Psi_2^\circ + \bar{\Psi}_2\right) = 2\Phi_{22}^\circ. \]

(30)
The commutators become

\[
\begin{align*}
\left[ \hat{p}, \partial' \right] &= 0 \\
\left[ \hat{p}, \partial \right] &= 0 \\
\left[ \hat{p}, \hat{p}' \right] &= -\left( \frac{1}{2} \rho^2 \Psi_2^o + \frac{1}{2} \rho^2 \tilde{\Psi}_2 \right) + \rho^2 \tilde{\rho} \Phi_{11}^o + \rho \tilde{\rho}^2 \Phi_{11}^o + \Lambda \left( \frac{1}{\rho} + \frac{1}{\tilde{\rho}} \right) \tilde{\Phi} \\
\left[ \hat{p}', \partial \right] &= \left( -\frac{k_0}{\rho} + \tilde{\Psi}_3^o + \frac{1}{2} \tilde{\rho} \tilde{\rho} \tilde{\phi}_2 - \frac{1}{2} \tilde{\rho} \tilde{\phi}_2 \tilde{\phi}_2 \Omega^o + \rho \tilde{\phi}_2^o + \rho \tilde{\phi}_2 \tilde{\phi}_2 \\
&\quad - \rho \tilde{\phi}_2 \tilde{\phi}_2 \tilde{\phi}_2 \Omega^o \right) \hat{p} + q \left( k_0 - \Lambda \tilde{\phi}_2 \right) \\
\left[ \hat{p}', \partial' \right] &= \left( -\frac{k_0}{\rho} + \Psi_3^o + \frac{1}{2} \rho^2 \tilde{\rho} \tilde{\phi}_2 + \frac{1}{2} \rho^2 \tilde{\rho} \tilde{\phi}_2 \tilde{\phi}_2 \Omega^o + \rho \tilde{\phi}_2^o + \rho \tilde{\phi}_2 \tilde{\phi}_2 \\
&\quad + \rho^2 \Phi_{11}^o \tilde{\phi}_2 \Omega^o \right) \hat{p} + p \left( k_0 + \Lambda \tilde{\phi}_2 \Omega^o \right) \\
\left[ \partial, \partial' \right] &= \left( -\frac{\rho'}{\rho} - \frac{\rho^2}{2} \left( \frac{1}{\rho} + \frac{1}{\tilde{\rho}} \right) \tilde{\phi}_2 - \frac{\tilde{\rho}}{2} \left( \frac{1}{\rho} + \frac{1}{\tilde{\rho}} \right) \tilde{\phi}_2 - \Omega^o \right) \hat{p} + \Omega^o \tilde{\phi}_2 + p \left( \rho^o + \Omega^o \Lambda \right) - q \left( \rho^o - \Omega^o \Lambda \right)
\end{align*}
\]

\[ (31) \]

3.2 Conclusions

The most obvious application of these results is when searching for exact non-vacuum solutions of Einstein’s field equations using the GHP integration procedure suggested by Held [10], [11] and developed by Edgar and Ludwig [6], [7], [8]. It is worth noting that among the remaining Ricci- and Bianchi equations (30) we can view the second one as the definition of $\Psi_4^o$, the third as the definition of $\Phi_2^o$, the fourth as the definition of $\Phi_{21}$ and the fifth as the definition of $\Phi_{22}$. This leaves only a system of equations consisting of the equations (27), (31) and the first equation of (30) for the unknown functions $\frac{1}{\rho} + \frac{1}{\tilde{\rho}}$, $\Omega^o$, $k_0^o$, $\rho^o$, $\Psi_2^o$, $\Phi_{11}^o$ and the constant $\Lambda$. Of these, $\frac{1}{\rho} + \frac{1}{\tilde{\rho}}$, $\Phi_{11}^o$ and $\Lambda$ are real while $\Omega^o$ is purely imaginary and the others are complex.

There are however two important points to keep in mind.

- The GHP integration program requires that the commutators are applied to four real $(0, 0)$-weighted quantities and one non-trivially weighted complex quantity. Only then is all the information extracted from the commutators.

- We have not yet imposed Einstein’s field equations. When we do this some of the equations taken as definitions of the Ricci components will actually become constraints e.g., if we impose the condition that the spacetime be vacuum we obtain the extra conditions

\[ \tilde{\partial} \Psi_2^o = 0 \]
\[ \tilde{\partial} \Psi_3^\circ - \tilde{\partial}' \Psi_2^\circ = 0. \]

Also, impositions of other features such as symmetries, special Petrov types etc., may result in extra constraint equations.

Another important application is to quasi-local momentum in these space-times. Recall that a symmetric spinor \( L_{ABCA'} \) is said to be a Lanczos potential of \( \Psi_{ABCD} \) if it satisfies the Weyl-Lanczos equation i.e.,

\[ \Psi_{ABCD} = 2 \nabla_{(A} A' L_{BCD)} \]

This equation can be translated into GHP-formalism (see e.g., [1]) and if we look for solutions satisfying

\[ L_{ABCA'} o^{A'} = 0 \]

the GHP Weyl-Lanczos equations are easy to \( \rho \)-integrate. Such Lanczos potentials turn out to be very useful when looking for metric asymmetric curvature-free connections in these spacetimes. Such a connection have been used in the Kerr spacetime to construct quasi-local momentum. For details of this construction and some generalizations, see [3], [4], [5], [1], and [2]. This application is further developed in [2].

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