FLUCTUATIONS OF THE RETARDED VAN DER WAALS FORCE

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Abstract

The retarded Van der Waals force between a polarizable particle and a perfectly conducting plate is re-examined. The expression for this force given by Casimir and Polder represents a mean force, but there are large fluctuations around this mean value on short time scales which are of the same order of magnitude as the mean force itself. However, these fluctuations occur on time scales which are typically of the order of the light travel time between the atom and the plate. As a consequence, they will not be observed in an experiment which measures the force averaged over a much longer time. In the large time limit, the magnitude of the mean squared velocity of a test particle due to this fluctuating Van der Waals force approaches a constant, and is similar to a Brownian motion of a test particle in an thermal bath with an effective temperature. However the fluctuations are not isotropic in this case, and the shift in the mean square velocity components can even be negative. We interpret this negative shift to correspond to a reduction in the velocity spread of a wavepacket. The force fluctuations discussed in this paper are special case of the more general problem of stress tensor fluctuations.

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These are of interest in a variety of areas of physics, including gravity theory. Thus the effects of Van der Waals force fluctuations serve as a useful model for better understanding quantum effects in gravity theory.

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I. INTRODUCTION

The retarded Van der Waals forces between pairs of atoms and between an atom and a perfectly conducting plate were first calculated by Casimir and Polder [1]. In the long distance limit, where the atoms may be described by a static polarizability, these forces may be interpreted as due to shifts in the vacuum energy of the quantized electromagnetic field. This is most clearly illustrated by the Casimir effect [2], which may be viewed either as the retarded Van der Waals force between a pair of perfectly conducting plates, or as the shift in the vacuum energy due to the plates. It has recently been measured accurately [4–6]. Similarly, the Casimir-Polder force between a plate and an atom has been confirmed by an experiment by Sukenik et al [3]. Note that the large distance limit of the theory can be applied to any polarizable particle, but not just an atom.

Because these forces have their origin in the vacuum fluctuations of the electromagnetic field, it is perhaps not surprising that the forces themselves are fluctuating forces. The first discussion of the force fluctuations was given by Barton [7,8], who considered fluctuations of the Casimir force between plates. In this approach, one considers a spatial and/or time average of the force. It is found that the fluctuations diverge in the limit that the averaging time goes to zero. Further work along the same lines was done by Eberlein [9]. Jaekel and Reynaud [10] have also discussed Casimir force fluctuations, especially for accelerating mirrors, using an approach based upon fluctuation-dissipation theorems. In this paper, we will consider the fluctuations of the force between an atom and a perfectly conducting plate from an approach somewhat different from that adopted by either of the above sets of authors. Our approach is based upon the Langevin equation. The solution of this equation to find the mean squared velocity of the particle involves a time integration which introduces a natural averaging scale. We will show that this averaging is sufficient to yield finite results.

The problem addressed in the present paper can be viewed as a special case of the larger problem of understanding the quantum fluctuations of the stress tensor [13–15]. This problem is of interest for a variety of reasons, ranging from radiation pressure noise in an interferometer [17], to quantum fluctuations of spacetime geometry driven by stress tensor fluctuations [13,16].

This paper is organized as follows: The Van der Waals force is reviewed in Sec. II and then the force-force correlation function will be calculated in Sec. III. In Sec. IV, we use this correlation function to study the velocity fluctuations of a test particle. Here it will be useful to use a decomposition of the correlation function into three parts, and to study the effect of each part individually. Our results will be summarized and discussed in Sect. V.

II. THE VAN DER WAALS FORCE

First, let us recall the result for the mean force. We assume that the atom can be described as a point particle with a static polarizability $\alpha$. Its interaction energy with a classical electromagnetic field, $E$, is

$$U = -\frac{\alpha}{2} E^2. \quad (1)$$

We will use Lorentz-Heaviside units with $\hbar = c = 1$, but will restore factors of $\hbar$ and $c$ in key results. We now assume that the electromagnetic field is quantized, and that its quantum state is such that $\langle E \rangle = 0$. However, $\langle E^2 \rangle \neq 0$, and there is a mean force given by

$$\langle F \rangle = \frac{\alpha}{2} \nabla \langle E^2 \rangle. \quad (2)$$
Quantities such as $\langle \mathbf{E}^2 \rangle$ in the presence of a plate may be calculated from the photon Hadamard function:

$$G_{\mu\nu}(x, x') \equiv \frac{1}{2} \langle A_\mu(x) A_\nu(x') + A_\nu(x') A_\mu(x) \rangle = G^{(0)}_{\mu\nu} + \tilde{G}_{\mu\nu},$$

(3)

where

$$G^{(0)}_{\mu\nu} = \frac{\delta_{\mu\nu}}{4\pi^2 D(x, x')}$$

(4)

is the Hadamard function for empty space, and

$$\tilde{G}_{\mu\nu} = -\frac{\delta_{\mu\nu} - 2 \hat{z}_\mu \hat{z}_\nu}{4\pi^2 D(x, \tilde{x}')}$$

(5)

is an “image” term due to the presence of the conducting boundary \[11,12\]. Here $\delta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric, and $\hat{z}$ is the unit vector in the $z$ direction

$$\hat{z}_\mu = (0, 0, 0, 1).$$

Furthermore, $D(x, x')$ is the squared geodesic distance between $x$ and $x'$,

$$D(x, x') = -(t - t')^2 + (x - x')^2 + (y - y')^2 + (z - z')^2,$$

(7)

and $\tilde{D}(x, \tilde{x}')$ is the corresponding distance between $x$ and the image point $\tilde{x}' = (t', x', y', -z'),$

$$\tilde{D}(x, \tilde{x}') = -(t - t')^2 + (x - x')^2 + (y - y')^2 + (z + z')^2.$$  

(8)

The vacuum expectation value of a product of electric fields in the presence of a conducting plate is given by

$$\langle E_i E_{j'} \rangle = \partial_0 \partial_{0'} G_{ij' j'} + \partial_i \partial_{j'} G_{00'} - \partial_0 \partial_{j'} G_{i0'} - \partial_i \partial_{0'} G_{0j'}. $$

(9)

We are using a notation in which unprimed indices refer to the spacetime point $x$, and primed indices to the point $x'$. Thus, $\partial_0 = \partial / \partial t$, $\partial_{j'} = \partial / \partial x_{j'}$, etc. The quantity $\langle E_i E_{j'} \rangle$ is divergent in the coincidence limit, $x' \to x$, but the divergent part does not contribute to the force in Eq. (2). For the calculation of the mean force, the only part which is of interest is the renormalized expectation value, which is obtained when $\tilde{G}_{\mu\nu}$ rather than $G_{\mu\nu}$ is used in Eq. (3). This is simply subtracting out the pure vacuum contribution, and is same as normal ordering with respect to the Minkowski vacuum. Equation (4) can be rewritten as

$$\langle E_i E_{j'} \rangle = \langle : E_i E_{j'} : \rangle + \langle E_i E_{j'} \rangle_0,$$  

(10)

where the normal-ordered term is

$$\langle : E_i E_{j'} : \rangle = \partial_0 \partial_{0'} \tilde{G}_{ij' j'} + \partial_i \partial_{j'} \tilde{G}_{i0' 0'} - \partial_0 \partial_{j'} \tilde{G}_{i0' j'} - \partial_i \partial_{0'} \tilde{G}_{0j' j'}$$

(11)

and the vacuum term is

$$\langle E_i E_{j'} \rangle_0 = \partial_0 \partial_{0'} G^{(0)}_{ij' j'} + \partial_i \partial_{j'} G^{(0)}_{i0' 0'} - \partial_0 \partial_{j'} G^{(0)}_{i0' j'} - \partial_i \partial_{0'} G^{(0)}_{0j' j'}.$$  

(12)

If we combine Eqs. (2), (3), and (11), we obtain the Casimir-Polder result for the mean force:
\[ \langle F \rangle = -\frac{3\alpha \hbar c}{8\pi^2 z^5} \hat{z}. \]

This is an attractive force in the direction perpendicular to the conducting plate.

Recall that Eq. (13) strictly holds only when the particle is described by a static (frequency-independent) polarizability. For the case of a one electron atom in its ground state, Casimir and Polder gave a more complicated expression which reduces to Eq. (13) in the large \( z \) limit. In the case of a macroscopic particle with nontrivial dispersive properties, there is the possibility of having a force which is either attractive or repulsive, and larger in magnitude than that given by the above expression [17]. In the present paper, we will deal only with the case of a frequency-independent polarizability.

### III. THE FORCE-FORCE CORRELATION FUNCTION

Now we wish to study the fluctuations in this force. This may be done by examining the correlation function \( \langle F(x) : F(x') : \rangle \) and the expectation value of the squared force \( \langle F : F^2 \rangle \). However, we will encounter the quantity \( \langle E^2(x) : E^2(x') : \rangle \), which is formally divergent in the coincident limit \( x' \to x \). Unlike the quadratic expectation values encountered in the case of the mean force, we cannot simply render this quantity finite by subtracting its expectation value in the Minkowski vacuum state. Following the method used in our previous works [14,15], this two point function can be decomposed into three different terms by using Wick’s theorem

\[
\langle E^2(x) : E^2(x') : \rangle = \langle E_i E_i E_j E_{j'} : \rangle + \langle E^2(x) : E^2(x') : \rangle_{\text{cross}} + \langle E^2(x) : E^2(x') : \rangle_0,
\]

which are the fully normal-ordered term, the cross term and the pure vacuum term, respectively. In the coincidence limit \( x \to x' \), the fully normal-ordered term is a well-defined local quantity. The cross term contains a state-dependent divergence, but can be made finite with careful regularization in the integral. The pure vacuum term is also divergent in the coincidence limit, but it is state-independent and cancels when we measure the difference due to a change of the boundary condition. The fully normal-ordered term can be expressed explicitly as

\[
\langle E_i E_i E_j E_{j'} : \rangle = \langle E_i E_{j'} : \rangle \langle E_i E_j : \rangle + \langle E_i E_j : \rangle \langle E_i E_{j'} : \rangle + \langle E_i E_{j'} : \rangle \langle E_j E_{j'} : \rangle \]

and the cross term is

\[
\langle E^2(x) : E^2(x') : \rangle_{\text{cross}} = 4 \langle E_i E_{j'} : \rangle_{\text{cross}} = \frac{\alpha^2}{4} \partial_k \partial_{k'} \langle E_i E_{j'} : \rangle_{\text{cross}} \bigg|_{x=x'}. \]

The physical content of both of these terms has been discussed by us [14,15] in other contexts. In general, both terms can contribute to the fluctuations of the stress tensor, or other quadratic operators.

The force-force correlation function \( \langle F_i : F_{j'} : \rangle \), evaluated at \( \mathbf{x} = \mathbf{x}' \) but at different times, can be obtained by the formula

\[
\langle F_k(t, z) : F_k(t', z) : \rangle = \frac{\alpha^2}{4} \partial_k \partial_{k'} \langle E_i E_{j'} : \rangle \bigg|_{x=x'}. \]
Again, this correlation function contains two parts we are interested in, namely the fully normal-ordered term and the cross term, and the cross term is divergent in the coincident limit, \( t \to t' \). The contribution from these two terms will be examined separately in the following section. The idea is to investigate the velocity dispersion of a test particle due to this fluctuating Van der Waals force.

IV. VELOCITY FLUCTUATIONS OF A TEST PARTICLE

We can better understand the effects of these fluctuations by studying the motion of particles subjected to the fluctuating force, which will be described by a Langevin equation. Consider particles which start at rest at time \( t = 0 \). The mean velocity at a later time \( t \) is given by

\[
\langle v_k(t) \rangle = \frac{1}{m} \int_0^t dt_1 \int_0^t dt_2 \langle F_k(t_1, z) : F_k(t_2, z) \rangle.
\]

where \( k = x, y, z \). To simplify the analysis, we assume that the distance of the particle from the plate does not change significantly in a time \( t \), so that \( z \) is approximately constant. Then the dispersion around the mean velocity in the \( k \)-direction at a later time is given by

\[
\langle \Delta v_k^2(t) \rangle = \frac{1}{m^2} \int_0^t dt_1 \int_0^t dt_2 \left[ \langle F_k(t_1, z) : F_k(t_2, z) \rangle - \langle F_k(t_1, z) \rangle \langle F_k(t_2, z) \rangle \right].
\]

This can be decomposed into two terms

\[
\langle \Delta v_k^2 \rangle = \langle \Delta v_k^2 \rangle + \langle \Delta v_k^2 \rangle_{\text{cross}},
\]

which are the fully normal-ordered term

\[
\langle \Delta v_k^2(t) \rangle = \frac{1}{m^2} \int_0^t dt_1 \int_0^t dt_2 \left[ \langle F_k(t_1, z) F_k(t_2, z) \rangle - \langle F_k(t_1, z) \rangle \langle F_k(t_2, z) \rangle \right]
\]

and the cross term

\[
\langle \Delta v_k^2(t) \rangle_{\text{cross}} = \frac{1}{m^2} \int_0^t dt_1 \int_0^t dt_2 \langle F_k(t_1, z) : F_k(t_2, z) \rangle_{\text{cross}}.
\]

Here the pure vacuum term is dropped because we are only interested in the difference due to a change of boundary conditions, which is the change caused by adding a plate. The fully normal-ordered term and the cross term will now be discussed in turn.

1. The fully normal-ordered term

Consider the force fluctuations due to the fully normal-ordered term, Eq. (23). The diagonal components of the force-force correlation functions are

\[
\langle F_k(t, z) F_k(t', z) \rangle = \frac{\alpha^2}{4} \left[ \partial_k \partial_{k'} \langle E_i E^i E_j E^{j'} \rangle \right]_{x=x'}.
\]

Note that the off-diagonal terms will be zero in the limit, \( x \to x' \). Use Eq. (3) and Eq. (11), and we find that the electric field two point functions can be expressed as
\[ \langle E_x E_{x'} \rangle = \frac{1}{2\pi^2} \left[ \frac{2}{D^2} - \frac{4(x - x')^2}{D^3} + \frac{4(t - t')^2}{D^3} \right] \]  
\[ \langle E_y E_{y'} \rangle = \frac{1}{2\pi^2} \left[ \frac{2}{D^2} - \frac{4(y - y')^2}{D^3} + \frac{4(t - t')^2}{D^3} \right] \]  

and

\[ \langle E_z E_{z'} \rangle = \frac{-1}{2\pi^2} \left[ \frac{2}{D^2} - \frac{4(z + z')^2}{D^3} + \frac{4(t - t')^2}{D^3} \right]. \]

Plug Eqs. (24), (25) and (26) into Eqs. (13) and (23). We find

\[ \langle F_x(t, z) F_x(t', z) \rangle = \langle F_y(t, z) F_y(t', z) \rangle = -\frac{4\alpha^2 (5T^4 + 16T^2z^2 + 48z^4)}{\pi^4 (T^2 - 4z^2)^7} \]  

and

\[ \langle F_z(t, z) F_z(t', z) \rangle - \langle F_z(t, z) \rangle \langle F_z(t', z) \rangle = \frac{4\alpha^2 (5T^6 + 308T^4z^2 + 944T^2z^4 + 1728z^6)}{\pi^4 (T^2 - 4z^2)^8}, \]

where \( T = t - t' \) and the product of the mean force is

\[ \langle F_z(t, z) : F_x(t', z) \rangle = \frac{\alpha^2}{4} \left[ \partial_z \langle E_x E_x' \rangle \partial_{z'} \langle E_y E_y' \rangle \right]_{x=x'}. \]

All of these results are independent of \( x \) and \( y \), and are Lorentz invariant under boosts in the directions parallel to the plate. In the limit \( t' \to t \), these fluctuations become

\[ \langle \Delta F_x^2(z, t) \rangle = \langle \Delta F_y^2(z, t) \rangle = \frac{3\alpha^2}{256 \pi^4 z^{10}} \]  

and

\[ \langle \Delta F_z^2(z, t) \rangle = \frac{27\alpha^2}{256 \pi^4 z^{10}}. \]

Here the \( z \) component is about 5 times the \( x \) and \( y \) components. If we compare the expectation value of the squared force from Eq. (31) with the square of the expectation value from Eq. (13), we obtain a measure of the force fluctuations:

\[ \Delta = \left| \frac{\langle F_z(t, z) \rangle^2 - \langle F_z(t, z)^2 \rangle}{\langle F_z(t, z)^2 \rangle} \right| = \frac{3}{4}, \]

which is of order of unity and shows that the force is fluctuating considerably. Note that even though there are no mean forces in \( x \) and \( y \) directions, the deviation of the force in these directions are still non-zero. Furthermore, the correlation function, Eq. (27) and Eq. (28), becomes small if \( T \gg z \), i.e. for time separations large compared to the distance of the atom from the plate. This shows that the characteristic fluctuation time is of the order of \( z \).

However, the behavior at time scales larger than the characteristic fluctuation time \( z \) is also important, and is needed to find the velocity fluctuations. Note that the force correlation
functions, Eqs. (27) and (28), are singular at $T = 2z$, a time separation equal to the round-trip time light travel between the particle and the plate. This singularity is presumably an artifact of our assumption of a perfectly reflecting plate, and would hence be smeared out in a more realistic treatment. However, we will see that the integrals can be made well-defined even with this singular integrand.

Let us now change integration variables to $T = t_1 - t_2$ and $\tau = t_1 + t_2$. If $F(T)$ is an even function, then

$$\int_0^t dt_1 \int_0^t dt_2 F(T) = \frac{1}{2} \int_{-t}^t dT \int_{|T|}^{2t-T} d\tau F(T) = 2 \int_0^t dT (t - T) F(T).$$

(33)

With Eqs. (13), (27) and (28), we may write Eq. (21) as

$$\langle : \Delta v_k^2(t) : \rangle = \frac{1}{m^2} \int_0^t dT \frac{f_k(T)}{(T - 2z)^8},$$

(34)

where

$$f_x(T) = f_y(T) = -\frac{8 \alpha^2 (5 T^4 + 16 T^2 z^2 + 48 z^4) (t - T)}{\pi^4 (T + 2 z)^7}$$

(35)

and

$$f_z(T) = \frac{8 \alpha^2 (5 T^6 + 308 T^4 z^2 + 944 T^2 z^4 + 1728 z^6) (t - T)}{\pi^4 (T + 2 z)^8}.$$  

(36)

The dispersion $\langle : \Delta v_k^2(t) : \rangle$ in Eq. (34) will be defined as a generalized principle value integral \[18\]. Such integrals involving higher-order poles may be evaluated by successive integrations by parts, which remove the divergence at the point $T = 2z$ and lead to the formula

$$\wp \int_a^b dx \frac{f(x)}{(x-c)^n} = -\frac{1}{(n-1)!} \left[ \sum_{i=0}^{n-2} (n - 2 - i)! f^{(i)}(x) (x-c)^{-n+1+i} \right]_a^b + \frac{1}{(n-1)!} \wp \int_a^b dx f^{(n-1)}(x) (x-c)^{-1}.$$  

(37)

We may now apply this formula to evaluate $\langle : \Delta v_k^2(t) : \rangle$. The result simplifies considerably if we assume that $t \gg 2z$, which is the limit of greatest physical interest. In this case, after some calculation one finds that

$$\langle : \Delta v_x^2(t, z) : \rangle = \langle : \Delta v_y^2(t, z) : \rangle = \frac{1}{m^2} \left( \frac{9}{1280 \pi^4} \right) \frac{\alpha^2 h^2}{z^8} + O \left( \frac{z}{t} \right)$$

(38)

and

$$\langle : \Delta v_z^2(t, z) : \rangle = \frac{1}{m^2} \left( \frac{63}{1280 \pi^4} \right) \frac{\alpha^2 h^2}{z^8} + O \left( \frac{z}{t} \right).$$

(39)

Note that even though there is no mean force in the $x$ and $y$ directions, the dispersion of the velocity in the direction parallel to the plate is still nonzero.
2. The cross term

The other intriguing part of the quantum fluctuation of the Van der Waals force is the cross term. Its contribution to the velocity fluctuation, Eq. (22), is formally divergent. However, it can be made finite by an integration by parts procedure analogous to that used in the previous section. The key assumption which we need to introduce is one of *adiabatic switching*. This means that the effect of the plates is smoothly switched on in the past and then off in the future. Physically, this might be achieved by means of a plate whose reflectivity could be controlled. This switching will allow us to drop surface terms which would otherwise be divergent. An analogous switching was assumed in a treatment of the quantum fluctuations of radiation pressure [15]. There it was shown that the cross term plays a central role, and in fact gives the sole contribution when a laser beam in a coherent state is shined on a mirror. In this case, it was necessary to assume that the laser beam is switched on in the past and then off in the future in order to obtain finite velocity fluctuations for the mirror.

In analogy to Eq. (23), the cross term of these force-force two point functions is defined by

$$\langle F_k(t, z) F_k(t', z) \rangle_{\text{cross}} = \frac{\alpha^2}{4} \partial_k \partial_{k'} \langle : E_i E^i : \rangle_{\text{cross}} \bigg|_{x=x'} .$$

(40)

Due to Eq. (16), we need to know the vacuum two point function $\langle E_i E_{j'} \rangle_0$ as well as the normal-ordered two point function $\langle : E_i E_{j'} : \rangle$ to compute the cross term. Use the equations (14) and (17) to compute these vacuum two-point functions Eq. (12), and find

$$\langle E_x E_{x'} \rangle_0 = -\frac{1}{2\pi^2} \left[ \frac{2}{D^2} - \frac{4(x - x')^2}{D^3} + \frac{4(t - t')^2}{D^3} \right]$$

(41)

$$\langle E_y E_{y'} \rangle_0 = -\frac{1}{2\pi^2} \left[ \frac{2}{D^2} - \frac{4(y - y')^2}{D^3} + \frac{4(t - t')^2}{D^3} \right]$$

(42)

and

$$\langle E_z E_{z'} \rangle_0 = -\frac{1}{2\pi^2} \left[ \frac{2}{D^2} - \frac{4(z - z')^2}{D^3} + \frac{4(t - t')^2}{D^3} \right].$$

(43)

Plug these vacuum two point functions, along with the normal-ordered two point functions Eqs. (24), (25) and (26) into Eq. (10) and compute the derivatives in Eq. (10). The force-force two point functions then becomes

$$\langle F_k(t, z) F_k(t', z) \rangle_{\text{cross}} = \frac{\alpha^2}{\pi^2} \left[ \frac{f_{k,4}}{(t_1 - t_2)^4} - \frac{4 f_{k,6}}{(t_1 - t_2)^6} \right],$$

(44)

Here $f_{k,4}$ and $f_{k,6}$ are formed from the normal-ordered two point functions and their derivatives, and can be expressed as

$$f_{k,4} = \left[ \partial_{k,k'} \langle : E_x(x) E_{x'}(x') : \rangle + \partial_{k,k'} \langle : E_y(x) E_{y'}(x') : \rangle + \partial_{k,k'} \langle : E_z(x) E_{z'}(x') : \rangle \right]_{x=x'}$$

(45)

and as

$$f_{x,6} = \left[ 2 \langle : E_x(x) E_{x'}(x') : \rangle + 2 \langle : E_y(x) E_{y'}(x') : \rangle + 2 \langle : E_z(x) E_{z'}(x') : \rangle \right]_{x=x'},$$

(46)
\[ f_{y,6} = \left[ 2\langle E_x(x)E_{x'}(x') \rangle + \langle E_y(x)E_{y'}(x') \rangle + 2\langle E_z(x)E_{z'}(x') \rangle \right]_{x=x'} \]  
and
\[ f_{z,6} = \left[ 2\langle E_x(x)E_{x'}(x') \rangle + 2\langle E_y(x)E_{y'}(x') \rangle + \langle E_z(x)E_{z'}(x') \rangle \right]_{x=x'} . \]

The singular parts, \(1/(t_1 - t_2)^4\) and \(1/(t_1 - t_2)^6\), in Eq. (14) are caused by the vacuum two point functions. Use Eq. (14) and change the variables \((t_1,t_2)\) to dimensionless ones \((s_1 = t_1/2z,s_2 = t_2/2z)\). The velocity fluctuation Eq. (22) becomes

\[ \langle \Delta v_k^2(t) \rangle_{\text{cross}} = \frac{\alpha^2}{\pi^2 m^2} \int_0^{t/2z} \int_0^{t/2z} \frac{f_{k,4}}{(s_1 - s_2)^4} - \frac{4 f_{k,6}}{(2z)^2(s_1 - s_2)^6} \, ds_1 ds_2 . \]

Because of the adiabatic switching assumption discussed above, we can now integrate by parts and drop the surface terms, using the relations

\[ \int \int \frac{f_{k,4}}{(s_1 - s_2)^4} \, ds_1 ds_2 = -\frac{1}{12} \int \int \left[ (\partial_{s_1})^2(\partial_{s_2})^2 f_{k,4} \right] \ln(s_1 - s_2)^2 \, ds_1 ds_2 \]
and
\[ \int \int \frac{f_{k,6}}{(s_1 - s_2)^6} \, ds_1 ds_2 = \frac{1}{240} \int \int \left[ (\partial_{s_1})^3(\partial_{s_2})^3 f_{k,6} \right] \ln(s_1 - s_2)^2 \, ds_1 ds_2 . \]

Plug these re-defined integrals into Eq. (49) and change variables to \((u = s_1 - s_2, v = s_1 + s_2)\). Use of the relation

\[ \int_0^a \int_0^a \, ds_1 ds_2 = \frac{1}{2} \left( \int_0^a \, du \int_{-u}^{u+2a} \, dv + \int_0^a \, du \int_{-a}^{a-u} \, dv \right) , \]
leads to

\[ \langle \Delta v_k^2(t) \rangle_{\text{cross}} = \frac{\alpha^2}{\pi^2 m^2} \int_0^{t/2z} \int_0^{t/2z} g_k(s_1, s_2) \ln(s_1 - s_2)^2 \, ds_1 ds_2 \]

\[ = \frac{\alpha^2}{\pi^2 m^2} \left( 2t \int_0^{t/2z} g_k(u^2) \ln u^2 \, du - 2 \int_0^{t/2z} u g_k(u^2) \ln u^2 \, du \right) , \]

where

\[ g_x = g_y = -\frac{3(5 + 275u^2 + 1325u^4 + 1041u^6 + 42u^8)}{16\pi^2(u^2 - 1)^9z^8} \]
and
\[ g_z = \frac{3(23 + 663u^2 + 1573u^4 + 429u^6)}{8\pi^2(u^2 - 1)^9z^8} . \]

When \(t \gg 2z\), the first term in Eq. (53) goes to zero for all the components \(k = x, y, z\), which leads to

\[ \langle \Delta v_k^2(z) \rangle_{\text{cross}} = \langle \Delta v_x^2(z) \rangle_{\text{cross}} \rightarrow \frac{13}{240 \frac{\hbar^2 \alpha^2}{\pi^4 m^2 z^8}} . \]
The magnitude of the velocity fluctuation of the $z$ component is about 19 times of that of $x$ and $y$ components. In all cases, the contributions of the cross terms are larger than those of the fully normal-ordered terms, Eqs. (38) and (39)

\[ \langle \Delta v_x^2 \rangle_{\text{cross}} = \langle \Delta v_y^2 \rangle_{\text{cross}} \approx 7.7 \langle : \Delta v_x^2 : \rangle \]

\[ \langle \Delta v_z^2 \rangle_{\text{cross}} \approx -21 \langle : \Delta v_z^2 : \rangle. \]  

The most surprising result is the negative $z$-component due to the cross term. The total velocity fluctuations are

\[ \langle \Delta v_y^2 \rangle = \langle \Delta v_x^2 \rangle = \langle : \Delta v_x^2 : \rangle + \langle \Delta v_x^2 \rangle_{\text{cross}} = \frac{47}{768} \frac{\hbar^2 \alpha^2}{\pi^4 m^2 z^8}, \]

\[ \langle \Delta v_z^2 \rangle = \langle : \Delta v_z^2 : \rangle + \langle \Delta v_z^2 \rangle_{\text{cross}} = \frac{-3787}{3840} \frac{\hbar^2 \alpha^2}{\pi^4 m^2 z^8}. \]

The results are independent of time and the $z$ component is still negative after the fully normal-ordered term is added to the cross term. The time-independent result shows a behavior similar to the case of Brownian motion in thermal equilibrium system. However we should also note that the velocity dispersion is not isotropic. The $x$ and $y$ components are much smaller than the magnitude of the $z$ component

\[ \langle \Delta v_z^2 \rangle = \langle \Delta v_y^2 \rangle \approx 0.06 |\langle \Delta v_z^2 \rangle|. \]

The non-isotropic behavior is also reflected in the fact that the mean force is zero in the parallel direction, but non-zero in the perpendicular direction. That the $\langle \Delta v_i^2 \rangle$ approach constant values, as opposed to growing in time, can be understood on the basis of energy conservation.

Of particular interest is the fact that $\langle \Delta v_z^2 \rangle < 0$. Recall that this quantity is a difference between a mean squared velocity with the plate and one without it, hence it is possible for this difference to be negative. (Similarly, the negative energy density in the Casimir effect arises from energy density being defined as a difference.) However, this negative value requires a physical interpretation. The most plausible explanation is that one cannot ignore the quantum nature of the test particles we have been discussing. The particle must have both a position uncertainty $\Delta z$ and a momentum uncertainty $\Delta p_z$, obeying the uncertainty principle. Furthermore, because the particle is massive, there will be wavepacket spreading in which $\Delta z$ is an increasing function of time. Thus, even if the particle is initially in a minimum uncertainty wavepacket, at later time it will satisfy the uncertainty principle by a wide margin. Our interpretation of the negative $\langle \Delta v_z^2 \rangle$ is that the electromagnetic vacuum fluctuations cause a small reduction in the velocity spread of the wavepacket compare to what it would have been without the plate present. Imagine that we initially prepare the particle in a minimum uncertainty state, and then allow it to evolve for a time $\tau >> 2z$). During this time $\Delta z$ will increase, but $\Delta v_z$ will decrease slightly.

In any case, the magnitude of the velocity changes due to electromagnetic vacuum fluctuations is always very small compared to the velocity spread due to quantum uncertainty. Let the latter be
\[ \Delta v_q = \frac{\Delta p_z}{m} > \frac{\Delta z}{m}. \]  

(62)

Compare this to the spread due to vacuum fluctuations,

\[ \Delta v_f = \max(\sqrt{\langle (\Delta v_i^2) \rangle}) \approx \frac{\alpha}{\pi^2 m z^3}. \]  

(63)

Their ratio satisfies

\[ \frac{\Delta v_f}{\Delta v_q} < \left(\frac{\Delta z}{z}\right) \left(\frac{\alpha}{\pi^2 z^3}\right). \]  

(64)

However, both factors on the right-hand-side of the above expression are small compared to one. The particle must be localized in a region small compared to the distance to the plate, so \( \Delta z \ll z \). The size of the particle must also be small compared to \( z \), and because the polarizability \( \alpha \) is at most of the order of the volume of the particle, \( \alpha \ll z^3 \). Thus

\[ \frac{\Delta v_f}{\Delta v_q} \ll 1. \]  

(65)

V. DISCUSSION

Equations (59) and (60) tell us that the effect of the fluctuations of the retarded Van der Waals force is to generate a random motion around that described by the classical trajectory. Of course, if a particle is released in the vicinity of a conducting plate, it tends to fall toward the plate under the influence of the mean force, Eq. (2). However, we could apply a compensating classical force \( F_{cl} = -\langle F \rangle \), so that the classical trajectory is that of a particle at fixed \( z \). Nonetheless, it will still develop a mean squared velocity given by Eqs. (38) and (39). If we look at the \( x \)-direction (or \( y \)-direction), this is equivalent to thermal motion at an effective temperature of

\[ T_{eff} = \left(\frac{47}{768\pi^4}\right) \frac{\alpha^2 h^2}{k_B m z^8}, \]  

(66)

where \( k_B \) is Boltzmann’s constant. Equation (66) can be written as

\[ T_{eff} \approx 10^{-1} K \left(\frac{m_H}{m}\right) \left(\frac{1\text{Å}}{z}\right)^8 \left(\frac{\alpha}{\alpha_H}\right)^2, \]  

(67)

where \( m_H \) and \( \alpha_H \) are the mass and static polarizability of atomic hydrogen, respectively. This effective temperature is essentially the temperature below which the system must be cooled so that the quantum fluctuation effects are not masked by ordinary thermal fluctuations. The effect in the \( z \)-direction is different from that in the transverse directions in that the mean squared velocity in that direction is reduced. Nonetheless, Eq. (66) gives an estimate of the magnitude even in this case. The magnitude of the effect depends crucially upon how small \( z \) can be. For atoms near a metal plate, both the assumptions of perfect conductivity and of using the static (as opposed to dynamic polarizability) break down for sufficiently small \( z \), typically for \( z \lesssim 10^3 \text{Å} \). Thus the effect of the fluctuations will be very small in the range that both of these assumptions hold well. However, there is likely to be
some effect even at much smaller values of \( z \). A metal surface acts as a partial reflector of electromagnetic waves even up into the x-ray range, where Bragg scattering can produce reflectivities close to 100% at special angles \([21]\). Thus although Eq. (66) is strictly valid only for \( z > 10^3 \AA \), it may produce crudely correct answers for \( z \) as small as as a few \( \AA \). If so, the fluctuation effects could conceivably approach observable levels. This conjecture needs to be confirmed by more detailed treatments.

The appearance of nonzero values for \( \langle \Delta v_x^2 \rangle \) and \( \langle \Delta v_y^2 \rangle \) requires some comment. By symmetry, a particle is equally likely to be deflected by the electromagnetic field in the \(+x\) or \(-x\) directions, and hence \( \langle v_x \rangle = \langle v_y \rangle = 0 \). However, the history of an individual particle does not have to respect the symmetry of the problem. Some particles acquire nonzero transverse components of velocity, leading to \( \langle \Delta v_x^2 \rangle \neq 0 \). A similar situation arises in lightcone fluctuations due to quantum gravity effects in a compact space \([20]\). Here Lorentz invariance holds on the average, but not for the history of an individual test particle.

The time scale of the fluctuations due to the fully normal-ordered term are of the order of \( z \), the light travel time between the particle and the plate, as may be seen from the fact that the correlation functions, Eqs. (27) and (28), vanish for \( T \gg z \). The time scale associated with the fluctuations arising from the cross term is of the same order. The short distance singularity of the cross term indicates that it contains fluctuations on arbitrarily short scales. However, these very rapid fluctuations are averaged out by the time integrations. The final integral for \( \langle \Delta v_x^2(t) \rangle_{\text{cross}}, \) Eq. (53), again contains an integrand which vanishes rapidly for \( u = T/(2z) \ll 1 \).

In summary, the Casimir-Polder result is a mean force, whereas the actual force is rapidly fluctuating. The typical magnitude of the fluctuations is of the same order as the mean force itself, but the time scale of the fluctuations is of the order of the light travel time between the atom and the plate. For most purposes, such as the Sukenik et al \([3]\) experiment, the fluctuations average to zero and are not seen. In principle, it is possible to detect the fluctuations through the random motions which they will induce in test particles. For ordinary atomic systems, this effect is very small.

The effect discussed in this paper is also of interest in gravity theory. When quantum matter fields act as the source of gravity, fluctuations of the stress tensor will lead to “passive” fluctuations of the spacetime geometry. These fluctuations are one of the physical phenomena to be expected in any quantum theory of gravity. Quantum fluctuations of the spacetime metric imply Brownian motion of the test particles which probe the fluctuating metric \([13,14]\). Thus the Brownian motion due to electromagnetic vacuum fluctuations treated here is a useful analogy for understanding the quantum nature of gravity.

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