A FIRST-COUNTABLE NON-REMAINDER OF \( H \)

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Abstract. We give a (consistent) example of a first-countable continuum that is not a remainder of the real line.

Introduction

The purpose of this note is to confirm a suspicion raised in [55; 5, Question 4.2]: we show that Bell’s example, from [33, 3], of a first-countable compact space that is not an \( N^* \)-image can be adapted to produce a connected variation that is neither an \( N^* \)-image nor an \( H^* \)-image. The interest in this variation stems from the authors’ version of Parovičenko’s theorem from [99, 9]. That theorem states that every compact Hausdorff space of weight \( \aleph_1 \) or less is an \( N^* \)-image; the Continuum Hypothesis then implies that the \( N^* \)-images are exactly the compact Hausdorff spaces of weight \( 2^{\aleph_0} \) or less. We proved in [44, 4] a parallel result for \( H^* \) and continua (connected compact Hausdorff spaces). Since, by Arkhangel’skii’s theorem [11, 1], first-countable compact spaces have weight at most \( 2^{\aleph_0} \) it follows that under \( \text{CH} \) first-countable compacta/continua are \( N^* \)-images/\( H^* \)-images respectively.

Bell’s graph. A major ingredient in our construction is Bell’s graph, constructed in [22, 2]. It is a graph on the ordinal \( \omega_2 \), represented by a symmetric subset \( E \) of \( \omega_2^2 \). The crucial property of this graph is that there is no map \( \varphi : \omega_2 \to \mathcal{P}(\mathbb{N}) \) that represents this graph in the sense that \( \langle \alpha, \beta \rangle \in E \) if and only if \( \varphi(\alpha) \cap \varphi(\beta) \) is infinite.

Bell’s graph exists in any forcing extension in which \( \aleph_2 \) Cohen reals are added; for the reader’s convenience we shall describe the construction of \( E \) and adapt Bell’s proof so that it applies to continuous maps defined on \( H^* \).

A first-countable continuum

Our starting point is a connected version of the Alexandroff double of the unit interval. We topologize the unit square as follows.

1. a local base at points of the form \( \langle x, 0 \rangle \) consists of the sets
   \[
   U(x, 0, n) = (x - 2^{-n}, x + 2^{-n}) \times [0, 1] \setminus \{x\} \times [2^{-n}, 1]
   \]
2. a local base at points of the form \( \langle x, y \rangle \), with \( y > 0 \) consists of the sets
   \[
   U(x, y, n) = \{x\} \times (y - 2^{-n}, y + 2^{-n})
   \]

We call the resulting space the connected comb and denote it by \( C \). It is straightforward to verify that \( C \) is compact, Hausdorff and connected; it is first-countable by definition.

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For each \( x \in [0,1] \) and positive \( a \) we define to be the following cross-shaped closed subset of \( C^2 \):

\[
D_{x,a} = (\{x\} \times [a,1] \times C) \cup (C \times \{x\} \times [a,1])
\]

We note the following two properties of the sets \( D_{x,a} \)

1. if \( a < b \) then \( D_{x,b} \) is in the interior of \( D_{x,a} \), and
2. if \( x \neq y \) then \( D_{x,a} \cap D_{y,a} \) is the union of two squares: \( \{x\} \times [a,1] \times \{y\} \times [a,1] \)

and \( \{y\} \times [a,1] \times \{x\} \times [a,1] \)

Now take any \( \aleph_2 \)-sized subset of \([0,1]\) and index it (faithfully) as \( \{x_\alpha : \alpha < \omega_2\} \).

We use this indexing to identify \( E \) with the subset \( \{(x_\alpha, x_\beta) : (\alpha, \beta) \in E\} \) of the unit square. Next we remove from \( C^2 \) the following open set:

\[
\bigcup_{(x,y) \notin E} \bigg( (\{x\} \times \{0,1\} \times \{y\} \times \{0,1\}) \cup (\{y\} \times \{0,1\} \times \{x\} \times \{0,1\}) \bigg)
\]

The resulting compact space we denote by \( C_E \). Observe that the intersections \( D_{x_\alpha, a} \cap C_E \) are open sets in the sense that \( D_{x, a} \cap D_{y, a} \cap C_E \) is nonempty if and only if \( (\alpha, \beta) \in E \). We write \( D_{x,a} = D_{x,a} \cap C_E \).

We show that \( C_E \) is (arcwise) connected.

To begin: the square \( S \) of the base line of \( C \) is a subset of \( C_E \) and homeomorphic to the unit square so that it is (arcwise) connected.

Let \( \langle x, a, y, b \rangle \) be a point of \( C_E \) not in \( S \). If, say, \( a = 0 \) then \( \{(x,0)\} \times \{y\} \times [0,1] \) is an arc in \( C_E \) that connects \( \langle x, 0, y, b \rangle \) to the point \( \langle x, 0, y, 0 \rangle \) in \( S \). If \( a, b > 0 \) then \( \langle x, y \rangle \in E \) and the whole square \( \{x\} \times [0,1] \times \{y\} \times [0,1] \) is in \( C_E \) and it provides us with an arc in \( C_E \) from \( \langle x, a, y, b \rangle \) to \( \langle x, 0, y, 0 \rangle \).

We find that \( C_E \) is a first-countable continuum. It remains to show that it is not an \( \mathbb{H}^* \)-image.

Assume \( h : \mathbb{H}^* \to C_E \) is a continuous surjection and consider, for each \( \alpha \), the sets \( D_{x_\alpha, a}^+ \) and \( D_{x_\alpha, a}^- \).

Using standard properties of \( \beta \mathbb{H}^* \), see [2] Proposition 3.2], we find for each \( \alpha \) a sequence \( \langle a_\alpha, n \rangle : n \in \mathbb{N} \) of open intervals with rational endpoints, and with \( b_\alpha, n < a_\alpha, n+1 \) for all \( n \), such that \( h^{-1}[D_{x_\alpha, a}^+] \subseteq \text{Ex} O_\alpha \cap \mathbb{H}^*_n \subseteq h^{-1}[D_{x_\alpha, a}^-] \), where \( O_\alpha = \bigcup_n \langle a_\alpha, n, b_\alpha, n \rangle \). Because the intersections of the sets \( D_{x_\alpha, a}^\pm \) represent \( E \) the intersections of the \( O_\alpha \) will do this as well: the conditions \( 'O_\alpha \cap O_\beta \) is unbounded’ and ‘\( (\alpha, \beta) \in E \)’ are equivalent.

In the next subsection we show that for (many) \( (\alpha, \beta) \) this equivalence does not hold and that therefore \( C_E \) is not a continuous image of \( \mathbb{H}^* \).

Note also that our continuum is not an \( \mathbb{N}^* \)-image either: if \( g : \mathbb{N}^* \to C_E \) were continuous and onto we could use clopen subsets of \( \mathbb{N}^* \) and their representing infinite subsets of \( \mathbb{N} \) to contradict the unrepresentability property of \( E \).

**Destroying the equivalence.** We follow the argument from [2] and we rely on Kunen’s book [9] Chapter VII for basic facts on forcing. We let \( L = \{ (\alpha, \beta) \in \omega^2 : \alpha \leq \beta \} \) and we force with the partial order \( \text{Fn}(L, 2) \) of finite partial functions with domain in \( L \) and range in \( \{0,1\} \). If \( G \) is a generic filter on \( \text{Fn}(L, 2) \) then we let \( E = \{ (\alpha, \beta) : \bigcup G(\alpha, \beta) = 1 \text{ or } \bigcup G(\beta, \alpha) = 1 \} \).

To show that \( E \) is as required we take a nice name \( \hat{F} \) for a function from \( \omega_2 \) to \( (\mathbb{Q}^2)^\omega \) that represents a choice of open sets \( \alpha \mapsto O_\alpha \), as in above in that \( \hat{F}(\alpha) = \langle (a_\alpha, n, b_\alpha, n) : n \in \omega \rangle \) for all \( \alpha \). As a nice name \( \hat{F} \) is a subset of \( \omega_2 \times \omega \times \mathbb{Q}^2 \times \text{Fn}(L, 2) \), where for each point \( (\alpha, n, a, b) \) the set \( \{ p : (\alpha, n, a, b, p) \in \hat{F} \} \) is a maximal antichain in the set of conditions that forces the \( n \)-th term of \( \hat{F}(\alpha) \) to be \( (a, b) \).
For each \( \alpha \) we let \( I_\alpha \) be the set of ordinals that occur in the domains of the conditions that appear as a fifth coordinate in the elements of \( \hat{F} \) with first coordinate \( \alpha \). The sets \( I_\alpha \) are countable, by the ccc of \( \text{Fn}(L,2) \). We may therefore apply the Free-Set Lemma, see [8], Corollary 44.2, and find a subset \( A \) of \( \omega_2 \) of cardinality \( \aleph_2 \) such that \( \alpha \notin I_\beta \) and \( \beta \notin I_\alpha \) whenever \( \alpha, \beta \in A \) and \( \alpha \neq \beta \).

Let \( p \in \text{Fn}(L,2) \) be arbitrary and take \( \alpha \) and \( \beta \) in \( A \) with \( \alpha < \beta \) and such that \( \alpha > \eta \) whenever \( \eta \) occurs in \( p \). Consider the condition \( q = p \cup \{ \langle \alpha, \beta, 1 \rangle \} \). If \( q \) forces \( O_\alpha \cap O_\beta \) to be bounded in \([0, \infty)\) then we are done: \( q \) forces that the equivalence fails at \( \langle \alpha, \beta \rangle \).

If \( q \) does not force the intersection to be bounded we can extend \( q \) to a condition \( r \) that forces \( O_\alpha \cap O_\beta \) to be unbounded. We define an automorphism \( h \) of \( \text{Fn}(L,2) \) by changing the value of the conditions only at \( \langle \alpha, \beta \rangle \): from 0 to 1 and vice versa. The condition \( p \) as well as the names \( \hat{x}_\alpha \) and \( \hat{x}_\beta \) are invariant under \( h \). It follows that \( h(r) \) extends \( p \) and

\[
h(r) \models \bigcup \hat{G}(\alpha, \beta) = 0 \text{ and } O_\alpha \cap O_\beta \text{ is unbounded}
\]

so again the equivalence is forced to fail at \( \langle \alpha, \beta \rangle \).

Remark. The argument above goes through almost verbatim to show that Bell’s graph can also be obtained adding \( \aleph_2 \) random reals. When forcing with the random real algebra one needs only consider conditions that belong to the \( \sigma \)-algebra generated by the clopen sets of the product \( \{0,1\}^\omega \); these all have countable supports so that, again by the ccc, one can define the sets \( I_\alpha \) as before. The rest of the argument remains virtually unchanged.

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