The 2-Dimensional Quantum Euclidean Algebra

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Abstract

The algebra dual to Woronowicz's deformation of the 2-dimensional Euclidean group is constructed. The same algebra is obtained from $SU_q(2)$ via contraction on both the group and algebra levels.

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1 Introduction

The Euclidean group $E(2)$ is a simple example of an inhomogeneous group. Deformations of such groups in general have been studied in [1]. Celeghini et al. [2] found a deformation of $Ue(2)$ by contracting $U_q su(2)$ and simultaneously letting the deformation parameter $h \equiv \ln q$ go to zero. Here we are interested in the case where $q$ is left untouched. Elements of the general theory of quantum groups can be found in [3] and references therein.

2 $U_q e(2)$ as the dual of $Fun(E_q(2))$

In this section, the dual to Woronowicz’s deformation of $E(2)$ [4] is constructed explicitly, using techniques similar to those of Rosso [5]. Woronowicz introduces Hopf algebra elements $n, v, \overline{n},$ and $\overline{v},$ which satisfy

$$\begin{align*}
v\overline{v} &= \overline{v}v = 1, \quad n\overline{n} = \overline{n}n, \quad vn = qnv, \\
n\overline{v} &= qn\overline{v}, \quad v\overline{n} = qvn, \quad \overline{v}\overline{n} = q\overline{n}\overline{v}, \\
\Delta(n) &= n \otimes \overline{v} + v \otimes n, \quad \Delta(v) = v \otimes v, \\
\Delta(\overline{n}) &= \overline{n} \otimes v + v \otimes \overline{n}, \quad \Delta(\overline{v}) = \overline{v} \otimes \overline{v}, \\
\epsilon(n) &= \epsilon(\overline{n}) = 0, \quad \epsilon(v) = \epsilon(\overline{v}) = 1, \\
S(n) &= -q^{-1}n, \quad S(v) = \overline{v}, \\
S(\overline{n}) &= -q\overline{n}, \quad S(\overline{v}) = v,
\end{align*}$$

with $\overline{q} = q$. (These relations can also be obtained through contraction of $SU_q(2)$, as described in the next section.)

For the calculations which follow, it is convenient to introduce the operators $\theta, \overline{\theta}, m,$ and $\overline{m},$ defined by

$$v = e^{i\theta}, \quad \overline{\theta} = \theta, \quad m = nv, \quad \overline{m} = \overline{v}\overline{n}. \tag{2}$$

In this basis, the coproducts take on the particularly nice form

$$\begin{align*}
\Delta(m) &= m \otimes 1 + e^{i\theta} \otimes m, \quad \Delta(\overline{m}) = \overline{m} \otimes 1 + e^{-i\theta} \otimes \overline{m},
\end{align*}$$
\[ \Delta(\theta) = \theta \otimes 1 + 1 \otimes \theta. \]  

**Remark:** The matrix \( E \) given by

\[
E = \begin{pmatrix} e^{i\theta} & m \\ 0 & 1 \end{pmatrix}
\]

satisfies the relations

\[ \Delta(E) = E \hat{\otimes} E, \quad S(E) = E^{-1}, \quad \epsilon(E) = I. \]

These are exactly the relations one would expect for an element of a quantum group. Notice that the action of \( E \) on the column vector \( \begin{pmatrix} z \\ 1 \end{pmatrix} \), where \( z \) is a complex coordinate, is given by

\[ z \mapsto e^{i\theta}z + m, \quad \bar{z} \mapsto e^{-i\theta}\bar{z} + \bar{m}. \]

We may therefore identify \( E \) as an element of the deformed 2-dimensional Euclidean group \( E_q(2) \).

\( \text{Fun}(E_q(2)) \) is the algebra of all \( C^\infty \) functions in the group parameters of \( E_q(2) \), i.e. the algebra spanned by ordered monomials in \( \theta, m, \) and \( \bar{m} \). Thus, \( \text{Fun}(E_q(2)) \) is taken to be \( \text{span}\{\theta^a m^b \bar{m}^c | a, b, c = 0, 1, \ldots\} \). The dual to \( \text{Fun}(E_q(2)) \) is \( U_q(e(2)) \), the quantized universal enveloping algebra of the 2-dimensional Euclidean algebra. We take \( \xi, \mu, \) and \( \nu \) to be the elements of \( U_q(e(2)) \) which give

\[
< \mu, \theta^a m^b \bar{m}^c > = \delta_{a0}\delta_{b1}\delta_{c0}, \quad < \nu, \theta^a m^b \bar{m}^c > = \delta_{a0}\delta_{b0}\delta_{c1},
\]

\[
< \xi, \theta^a m^b \bar{m}^c > = \delta_{a1}\delta_{b0}\delta_{c0}.
\]

Using the coproduct on \( \text{Fun}(E_q(2)) \) to obtain the multiplication on \( U_q(e(2)) \) gives

\[
< \nu^k \mu^l \xi^n, \theta^a m^b \bar{m}^c > = [k]_q! [l]_q! [n]_q! \delta_{na}\delta_{lb}\delta_{kc}, \quad [x]_q! = \prod_{y=1}^{x} \frac{q^{2y} - 1}{q^2 - 1},
\]
so \{\nu^k\mu^l\xi^n \mid k, l, n = 0, 1, \ldots\} is a basis for \(U_qe(2)\). The rest of the Hopf algebra structure of \(U_qe(2)\) can be similarly obtained:

\[
\begin{align*}
[\xi, \mu] &= i\mu, & [\xi, \nu] &= -i\nu, & \mu\nu &= q^2\nu\mu, \\
\Delta(\mu) &= \mu \otimes q^{2i\xi} + 1 \otimes \mu, & \Delta(\nu) &= \nu \otimes q^{2i\xi} + 1 \otimes \nu, \\
\Delta(\xi) &= \xi \otimes 1 + 1 \otimes \xi, & \epsilon(\mu) &= \epsilon(\nu) = \epsilon(\xi) = 0, \\
S(\mu) &= -q^{-2i\xi}\mu, & S(\nu) &= -q^{-2i\xi}\nu, & S(\xi) &= -\xi.
\end{align*}
\]

If \(H\) is a *-Hopf algebra whose coproduct and antipode satisfy

\[
\Delta(h) = \overline{\Delta(h)}, \quad S(h) = S^{-1}(h)
\]

for all \(h \in H\), then an involution on \(H^*\) can be defined by

\[
< \chi, h > = < \chi, S^{-1}(h) >^*,
\]

where \(\chi \in H^*\). Using this to define complex conjugation on \(U_qe(2)\) gives

\[
\overline{\xi} = -\xi, \quad \overline{\mu} = -q^2\nu, \quad \overline{\nu} = -q^{-2}\mu.
\]

Defining new operators \(J, P_+,\) and \(P_-\) as

\[
J \equiv i\xi, \quad P_+ \equiv q q^{-i\xi}\nu, \quad P_- \equiv -q^{-1}\mu q^{-i\xi},
\]

gives \(J = J, \quad \overline{P_\pm} = P_\mp,\) and

\[
\begin{align*}
\Delta(P_\pm) &= P_\pm \otimes q^J + q^{-J} \otimes P_\pm, & \Delta(J) &= J \otimes 1 + 1 \otimes J, \\
\epsilon(P_\pm) &= \epsilon(J) = 0, \\
S(J) &= -J, & S(P_\pm) &= -q^{\pm 1}P_\pm.
\end{align*}
\]

3 \(U_qe(2)\) from \(SU_q(2)\)

In this section we will show how the deformed Euclidean group \(E_q(2)\) can be obtained from \(SU_q(2)\) by contraction and how this implies a similar contraction scheme for the deformed Lie algebra \(U_qsu(2)\), giving an independent derivation of \(U_qe(2)\).
3.1 $E_q(2)$ by contraction of $SU_q(2)$

Recall [3], [7] the commutation relations for $SU_q(2)$, which may be written in compact matrix notation as

$$R_{12}T_1T_2 = T_2T_1R_{12}, \quad det_q T = 1, \quad T^\dagger = T^{-1},$$

$$\Delta(T) = T \otimes T, \quad \epsilon(T) = I, \quad S(T) = T^{-1}, \quad (15)$$

where

$$T = \begin{pmatrix} \alpha & -q\gamma \\ \gamma & \alpha \end{pmatrix}, \quad R = q^{-1/2} \begin{pmatrix} q & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \lambda & 1 \end{pmatrix},$$

$$\Delta(H) = H \otimes 1 + 1 \otimes H, \quad \Delta(X_\pm) = X_\pm \otimes q^{H/2} + q^{-H/2} \otimes X_\pm,$$

$$\epsilon(H) = \epsilon(X_\pm) = 0,$$

$$S(H) = -H, \quad S(X_\pm) = -q^{\pm 1}X_\pm. \quad (17)$$

and $\lambda = q - q^{-1}$. Now set $\alpha \equiv v, \alpha \equiv \eta, \gamma \equiv ln$ and $\gamma \equiv ln$, where $l \in \mathbb{R}\{0\}$ is the contraction parameter. Written in terms of $v, \eta, n$ and $\eta$, relations (15) become

$$det_q T = v\eta + q^2 l^2 n\eta = \eta v + l^2 n\eta = 1,$$

$$n\eta = \eta n, \quad vn = qnv, \quad v\eta = q^n\eta, \quad \text{etc.}$$

and coincide with (15) in the limit $l \to 0$, i.e. $E_q(2)$ is a contraction of $SU_q(2)$.

3.2 $U_qe(2)$ by contraction of $U_qsu(2)$

The deformed universal enveloping algebra $U_qsu(2)$, dual to $\text{Fun}(SU_q(2))$, is generated by hermitean operators $H, X_+, X_-$ satisfying

$$[H, X_\pm] = \pm 2X_\pm, \quad [X_+, X_-] = \frac{q^H - q^{-H}}{q - q^{-1}},$$

$$\Delta(H) = H \otimes 1 + 1 \otimes H, \quad \Delta(X_\pm) = X_\pm \otimes q^{H/2} + q^{-H/2} \otimes X_\pm,$$

$$\epsilon(H) = \epsilon(X_\pm) = 0,$$

$$S(H) = -H, \quad S(X_\pm) = -q^{\pm 1}X_\pm. \quad (17)$$
Following [3] these relations can be rewritten as

\[ R_{12} L_2^\pm L_1^\pm = L_1^\pm L_2^\pm R_{12}, \quad R_{12} L_2^\pm L_1^- = L_1^- L_2^\pm R_{12}, \]

\[ \Delta(L^\pm) = L^\pm \otimes L^\pm, \quad \epsilon(L^\pm) = I, \] (18)

where \( L^\pm \) are given by

\[ L^+ = \begin{pmatrix} q^{-H/2} & q^{-1/2} \lambda X_+ \\ 0 & q^{H/2} \end{pmatrix}, \quad L^- = \begin{pmatrix} q^{H/2} & 0 \\ -q^{1/2} \lambda X_- & q^{-H/2} \end{pmatrix}. \] (19)

Using this matrix notation, there is an elegant way of stating the duality between the group and the algebra by means of the commutation relations

\[ L_1^+ T_2 = T_2 R_{21} L_1^+, \quad L_1^- T_2 = T_2 R_{12}^{-1} L_1^-, \] (20)

as described in [7]. Equations (20) are consistent with the inner products

\[ < L_1^+, T_2 > = R_{21}, \quad < L_1^-, T_2 > = R_{12}^{-1}, \] (21)

given in [3]. Furthermore, equations (18) can be derived as consistency conditions to (15) and (20). In addition, complex conjugation can be defined as an involution on the extended algebra generated by products of \( T \) and \( L^\pm \). This agrees with (11). Unitarity of \( T \) then implies \( (L^+)^\dagger = (L^-)^{-1} \), i.e. \( H = H, \quad X_\pm = X_\mp \).

In the present case equations (20) become

\[ Hv = vH - v, \quad X_+ v = q^{1/2} v X_+ - lqnq^{H/2}, \quad X_- v = q^{1/2} v X_-, \]

\[ lH\pi = l(\pi H - \pi), \quad lX_+\pi = q^{1/2} lX_+ + \bar{v}q^{H/2}, \quad lX_-\pi = lq^{1/2} \pi X_-, \] (22)

plus the complex conjugate relations.

The way that the deformation parameter \( l \) appears in these relations suggests the definition of new operators \( P_+ \equiv lX_+, \quad P_- \equiv P_+^\dagger = lX_- \) and \( J \equiv H/2 \), so that we will retain non-trivial commutation relations for \( P_\pm \) and \( J \) with \( v, \pi, n \) and \( \pi \) in the limit \( l \to 0 \). Inserting \( P_\pm \) and \( J \) into equation (17) we again obtain \( U_q e(2) \) (see (14)) as a contraction of \( U_q su(2) \) in this limit.
4 Conclusion

Through equation (22) the contraction on the group level (section 3.1) motivates a particular contraction scheme at the algebra level (section 3.2). The two methods outlined in sections 2 and 3 are summarized in the following (commutative) diagram:

$$
\begin{array}{c}
SU_q(2) \xrightarrow{\text{contraction } \ell \to 0} E_q(2) \\
\downarrow \text{dual} \quad \downarrow \text{dual}
\end{array}
$$

$$
\begin{array}{c}
U_qsu(2) \xrightarrow{\text{contraction } \ell \to 0} U_qe(2)
\end{array}
$$

Note that the algebra obtained in (14) is the same as the classical 2-dimensional Euclidean algebra $e(2)$ (with $P_\pm = P_x \pm iP_y$ and $J$ as hermitean generators) [2]. Note, however, as a Hopf algebra it is still deformed; the deformation parameter $q$ remains unchanged.

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