Disorder, Path Integrals and Localization
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Anderson localization is derived directly from the path integral representation of quantum mechanics in the presence of a random potential energy function. The probability distribution of the potential energy is taken to be a Gaussian in function space with a given autocorrelation function. Averaging the path integral itself we find that the localization length, in one-dimension, is given by \( \langle E_{\xi}/\sigma \rangle (K E_{\text{cl}}/\sigma) \xi \) where \( E_{\xi} \) is the "correlation energy", \( KE_{\text{cl}} \) the average classical kinetic energy, \( \sigma \) the root-mean-square variation of the potential energy and \( \xi \) the autocorrelation length. Averaging the square of the path integral shows explicitly that closed loops in the path when traversed forward and backward in time lead to exponential decay, and hence localization. We also show how, using Schwinger proper time, the path integral result can be directly related to the Greens function commonly used to study localization.

I. INTRODUCTION

The fact that the wave function in a random or disordered potential is localized in one and two dimensions and does not diffuse was predicted by Anderson in the late 1950’s [1]. Since then much work has been done on ”Anderson Localization”, both theoretically and experimentally, in a wide range of physical systems [2]. From the theoretical perspective, localization is generally studied by considering the Greens function \( G(\vec{x}, \vec{x}') = \langle \vec{x} | (\vec{z} - H)^{-1} | \vec{x}' \rangle \), or products of the Greens function with itself, where \( H \) is a Hamiltonian and \( z \) is a complex number. In the simplest case \( H = \vec{p}^2/2m + V(\vec{x}) \) where the potential \( V(\vec{x}) \) is treated as disordered, i.e., as a random function of position. Computing the average of \( G(\vec{x}, \vec{x}') \) over some distribution of \( V(\vec{x}) \), or powers of \( G(\vec{x}, \vec{x}') \), with \( V(\vec{x}) \) in the denominator is rather complicated and as pointed out in "Quantum Field Theory in a Nutshell" by A. Zee [3] that has led to two main approaches for studying localization using field theory techniques, the replica approach pioneered by Parisi and others [4] and the supersymmetry approach pioneered by Efetov [5]. But other approaches have also been used, see for example the self-consistent approach developed by Wölffe and Vollhardt, Chapter 4 in [2].

Here we show how localization emerges directly from the path integral for nonrelativistic quantum mechanics and also show how the path integral result can be related to the above Greens function. Averaging the path integral over a Gaussian probability distribution, in function space, for the potential energy \( V(\vec{x}) \) with a Hamiltonian of the form \( H = \vec{p}^2/2m + V(\vec{x}) \) is straightforward. This same approach was used by Dashen in [6]. There he considered the propagation of light in a medium with a randomly varying index of refraction in the paraxial approximation. His results therefore are equivalent to the 2D Schrodinger equation with \( (x, y) \) being position and the \( z \) direction being time. Since he treats the randomness in the index of refraction as spatially isotropic his analysis doesn’t allow for the randomness to be independent of \( z \) which is required to get localization. Hence, although localization is implicit in his analysis, he doesn’t identify it as such. But, since he does include real temporal variation in the index of refraction he does find some very interesting "speckle" and "twinkle" effects. Chakravarty and Schmid [7] studied localization by considering the path integral from a semiclassical point of view which illustrated explicitly how closed loops in the path contribute to localization. Here the path integral for a random potential is analyzed from the fully quantum mechanical viewpoint. The average of the square of the path integral shows explicitly the contribution of closed loops.

Using the Schwinger proper time representation of the inverse of a time independent operator [8] we can write, for \( \text{Im} \, [z] > 0 \),

\[
G(\vec{x}, \vec{x}') = \frac{1}{i\hbar} \int_{0}^{\infty} dT e^{izT/\hbar} \left\langle \vec{x} \left| \exp \left[ -i \frac{HT}{\hbar} \right] \right| \vec{x}' \right\rangle = \frac{1}{i\hbar} \int_{0}^{\infty} dT e^{izT/\hbar} K(\vec{x}, T, \vec{x}', 0)
\]

(1)

The inclusion of \( \hbar \) is not necessary but it does allow for the interpretation of \( T \) as actual time. This can be rewritten in terms of a path integral [3, 9] as

\[
K(\vec{x}, T, \vec{x}', 0) = \int_{\vec{x}, 0}^{\vec{x}, T} \delta(\vec{x}(t)) \exp \left[ \frac{i}{\hbar} \int_{0}^{T} dt \left( \frac{m}{2} (\partial_{t} \vec{x}(t))^2 - V(\vec{x}(t)) \right) \right]
\]

(2)
where \( \int_{\vec{x}',0}^{\vec{x},T} \delta \vec{\xi} (t) \) indicates integration over all paths \( \vec{\xi} (t) \) starting at \( \vec{\xi} \) at \( t = 0 \) and ending at \( \vec{\xi}' \) at \( t = T \). As is well known \( K (\vec{\xi}, T, \vec{\xi}', 0) \) propagates a wave function, \( \psi (\vec{\xi}, t) \), in time, i.e.,

\[
\psi (\vec{\xi}, T) = \int d^N x' K (\vec{\xi}, T, \vec{\xi}', 0) \psi (\vec{\xi}', 0)
\]

(3)

where \( N \) is the space dimension. Here we consider localization from the point of view of the path integral itself. The point of (1) is to show how results obtained using the path integral, which can easily be averaged over \( V (\vec{\xi}) \) with a Gaussian probability distribution, can be directly related to the more standard Greens function analysis of localization. That relationship will not be explored in detail here.

Take \( V (\vec{\xi}) \) to be a random function with a Gaussian probability distribution \( P [V (\vec{\xi})] \) in function space given by

\[
P [V (\vec{\xi})] = \frac{1}{Z_0} \exp \left( -\frac{1}{2} \int d^N x d^N x' V (\vec{\xi}) C^{-1} (\vec{\xi}, \vec{\xi}') V (\vec{\xi}') \right)
\]

(4)

Here \( N \) is the number of space dimensions and \( Z_0 \) is a normalization factor which we won’t need. We will generally use the notation \( \int dx \) instead of \( \int d^N x \).

Expectation values with respect to \( P [V (\vec{\xi})] \) are defined by

\[
\langle F (V (\vec{\xi})) \rangle_p = \int \delta V (\vec{\xi}) F [V (\vec{\xi})] P [V (\vec{\xi})]
\]

(5)

where \( \int \delta V (\vec{\xi}) \) indicates functional integration over \( V (\vec{\xi}) \). We find via standard procedures \cite{3},

\[
\langle V (\vec{\xi}) \rangle_p = 0
\]

\[
\langle V (\vec{\xi}) V (\vec{\xi}') \rangle_p = C (\vec{\xi}, \vec{\xi}')
\]

(6)

where \( C (\vec{\xi}, \vec{\xi}') \) is the inverse of \( C^{-1} (\vec{\xi}, \vec{\xi}') \), i.e.,

\[
\int d\vec{\xi}'' C^{-1} (\vec{\xi}, \vec{\xi}'') C (\vec{\xi}'', \vec{\xi}') = \delta (\vec{\xi} - \vec{\xi}')
\]

(7)

Although not necessary, we will choose \( C (\vec{\xi}, \vec{\xi}') \) to be isotropic and homogeneous,

\[
C (\vec{\xi}, \vec{\xi}') = C (|\vec{\xi} - \vec{\xi}'|)
\]

(8)

and we will use the exponential form for the autocorrelation function

\[
C (x, \vec{x}') = \sigma^2 \exp \left( -\frac{|x - |\vec{x}'|}{\xi} \right)
\]

(9)

where \( \sigma^2 = \langle V (\vec{\xi})^2 \rangle_p \) is the mean square variation in the potential energy with \( \xi \) the autocorrelation length of \( V (\vec{\xi}) \).

Again via thoroughly standard procedures \cite{3} we have

\[
\left\langle \exp \left[ i \int d\vec{\xi} J (\vec{\xi}) V (\vec{\xi}) \right] \right\rangle_p = \exp \left[ -\frac{1}{2} \int d\vec{\xi} d\vec{\xi}' J (\vec{\xi}) C (\vec{\xi}, \vec{\xi}') J (\vec{\xi}') \right]
\]

(10)

Note that this is properly normalized since

\[
\langle 1 \rangle_p = \exp \left[ -\frac{1}{2} \int d\vec{\xi} d\vec{\xi}' J (\vec{\xi}) C (\vec{\xi}, \vec{\xi}') J (\vec{\xi}') \right]_{J(\vec{\xi})=0} = 1
\]

(11)

so an explicit form for \( Z_0 \) is not needed.

Averaging the product of the wave function at one position with its conjugate at another position,

\[
\langle \sum_i \delta (E - E_i) \psi_i (\vec{\xi})^* \psi_i (\vec{\xi}') \rangle_p
\]

(1)

shows how the wave function, at energy \( E \), correlates with itself as a function of \( |\vec{\xi} - \vec{\xi}'| \). Here the \( E_i \) are the energy eigenvalues and \( \psi_i (\vec{\xi}) \) the eigenfunctions, respectively, of \( H \). But, as pointed out in \cite{3}, since \( \psi (\vec{\xi}) \) has a phase, this average potentially could yield zero. Instead it is suggested in \cite{3} to average the square of the wave function at two positions, i.e.,

\[
\left\langle \sum_i \delta (E - E_i) |\psi_i (\vec{\xi})|^2 |\psi_i (\vec{\xi}')|^2 \right\rangle_p
\]

(12)
Here we consider more generally the behavior of both the average of the product of the wave function at different times and positions which leads to averaging the path integral itself, and the behavior of the average of the product of the square of the wave functions at two different times and positions which leads to averaging the square of the path integral.

As pointed out above, the path integral evolves the wave function in time. Consider a wave function initially localized on a scale much smaller than $\xi$, e.g., $\psi(\vec{x}, 0) = \exp \left[ -\frac{\pi^2}{4w^4} \right] / (2\pi w^2)^{1/4}$ with $w \ll \xi$ so that $|\psi(\vec{x}, 0)|^2 \sim \delta(\vec{x})$ on the scale of $\xi$, and evaluate how it correlates with itself at later times and positions. Using the chosen $\psi(\vec{x}, 0)$, setting $\vec{x}_e = \vec{x}$ and $\vec{x}_a = \vec{x}'$ we have

$$
\langle \psi(\vec{x}, T) \psi(0, 0)^* \rangle_p = \int d^N x' \langle K(\vec{x}, T, \vec{x}', 0) \rangle_p \psi(\vec{x}', 0) \psi(0, 0)^* \\
\sim \langle K(\vec{x}, T, 0, 0) \rangle_p
$$

which is the average of the path integral itself. As shown below this does not yield zero and in fact shows evidence of localization, at least when evaluated for the classical trajectory. In line with evaluating the average of the product of the square of the wave function at two positions, we also consider

$$
\langle |\psi(\vec{x}, T)|^2 |\psi(0, 0)|^2 \rangle_p = \int d^N x' d^N x'' \langle K(\vec{x}, T, \vec{x}', 0) K(\vec{x}, T, \vec{x}'', 0) \rangle_p \psi(\vec{x}', 0) \psi(\vec{x}'', 0) |\psi(0, 0)|^2
$$

$$
\sim \langle |K(\vec{x}, T, 0, 0)|^2 \rangle_p
$$

which, for the chosen $\psi(\vec{x}, 0)$, we see is approximately the average of the square of the path integral.

II. AVERAGING THE PATH INTEGRAL

Letting $J(\vec{x}) = -\frac{i}{\hbar} \int_0^T dt \delta(\vec{x} - \vec{x}(t))$ we have

$$
\langle \exp \left[ -\frac{i}{\hbar} \int_0^T dt V(\vec{x}) \right] \rangle_p = \langle \exp \left[ i \int d^N x J(\vec{x}) V(\vec{x}) \right] \rangle_p
$$

$$
= \exp \left[ -\frac{1}{2\hbar^2} \int_0^T dt dt' C(|\vec{x}(t) - \vec{x}(t')|) \right]
$$

and so

$$
\langle K(\vec{x}, T, \vec{x}', 0) \rangle_p = \int_{\vec{x}', \vec{x}} \delta(\vec{x}(t)) \exp \left[ i \int_0^T dt \frac{m}{2} \left( \partial_t \vec{x}(t) \right)^2 - \frac{1}{2\hbar^2} \int_0^T dt dt' C(|\vec{x}(t) - \vec{x}(t')|) \right]
$$

Thus averaging over $V(\vec{x})$ has replaced the phase factor $\exp \left[ -\frac{i}{\hbar} \int_0^T dt V(\vec{x}(t)) \right]$ with the exponential factor $\exp \left[ -\frac{1}{2\hbar^2} \int_0^T dt dt' C(|\vec{x}(t) - \vec{x}(t')|) \right]$. After accounting for differences in notation this is the same as equation (2.5) in Dashen [6]. He considers this result to be exponentially small and "therefore not particularly interesting." We now show that this result indicates the existence of localization. We work in one dimension for simplicity.

Let $x' = 0$ and $x = L$. The phase factor $\exp \left[ i \int_0^T dt \frac{m}{2} \left( \partial_t^2 x(t) \right)^2 \right]$ is stationary when $x(t)$ is the classical solution $x_{cl}(t) = \frac{L}{2} t$. Also the exponential factor $\exp \left[ -\frac{1}{2\hbar^2} \int_0^T dt dt' C(|x(t) - x(t')|) \right]$ is smaller the more times $x(t)$ intersects itself, i.e., the more times for which $|x(t) - x(t')| \ll \xi$. Both these facts indicate that the dominant contribution to the path integral comes from the straightest or classical path $x_{cl}(t) = \frac{L}{2} t$. Substituting this into exponential factor and using the explicit form for $C(|\vec{x} - \vec{x}'|)$ we find

$$
\langle K(L, T, 0, 0) \rangle_p \sim \sqrt{\frac{m}{2\pi i \hbar T}} \exp \left[ i \frac{mL^2}{2\hbar T} - \frac{\sigma^2 \xi^2 T^2}{h^2 T^2} \left( \frac{|L|}{\xi} + e^{-|L|/\xi} - 1 \right) \right]
$$

$$
\sim \sqrt{\frac{m}{2\pi i \hbar T}} \exp \left[ i \frac{mL^2}{2\hbar T} - \frac{\sigma^2 \xi}{h^2 (L/T)^2} |L| \right] \text{ for } L \gg \xi
$$

(17)
where we have included the standard factor $\sqrt{|\mathcal{M}_{\text{dir}}|}$. The exponential decay with increasing $|L|$ indicates localization with the localization length $\ell$ proportional to the factor $\hbar^2 (L/T)^2 /\sigma^2 \xi$. Multiplying and dividing by $\xi m/2$ and rearranging we have

$$\frac{\hbar^2 (L/T)^2}{\sigma^2 \xi} = 2 \left( \frac{\hbar^2}{m^2} \right) \left( \frac{m}{T} \left( \frac{t}{\xi} \right)^2 \right) \xi = 2 \frac{E_L KE_{cl}}{\sigma} \xi \equiv 2 \ell \quad (18)$$

where $E_\xi = \hbar^2 /m\xi^2$ is the so called "correlation energy" [10-12]. It is proportional to $(\lambda_{dB}/\xi)^2$ where $\lambda_{dB}$ is the de Broglie wavelength. Hence $E_\xi$ is large when $\lambda_{dB} \gg \xi$. In this regime the wave function can spread over long distances via tunneling [10-12] and leads to larger values of $\ell$. $KE_{cl} = m/2 (L/T)^2$ is the classical kinetic energy for the path $x(t) = Lt/T$. Since $\langle V(x) \rangle_p = 0$ we have the classical kinetic energy is equal to the average total energy, $KE_{cl} = \langle E_{\text{total}} \rangle_p$. The Gaussian form for $C(|\vec{x} - \vec{x}'|)$ yields the same result for $L \gg \xi$ but without the factor of 2.

The justification for replacing $(L/T)^2$ with $(2/m) KE_{cl}$ is because the path integral is a point to point propagator, hence it includes all possible energies. Localization on the other hand is usually studied at specific energy scales, e.g., the Fermi energy. We can filter the path integral for specific average kinetic energies, $KE_{ave}$ by inserting a Fadeev-Popov type factor [3]

$$1 = \int_0^\infty d(KE_{ave}) \delta \left( KE_{ave} - \frac{1}{T} \int_0^T dt \frac{m}{2} (\partial_t x(t))^2 \right)$$

in the path integral. This is similar to the approach used by Chakravarty and Schmid [7].

III. AVERAGING THE \text{[Path Integral]}^2

Consider the average of the product of the square of the wave function at two times, 0 and $T$, and positions 0 and $\vec{L}$, which, as shown above, for the chosen initial wave function reduces to

$$\left\langle \left| \psi \left( \vec{L}, T \right) \right|^2 \left| \psi \left( 0, 0 \right) \right|^2 \right\rangle_p \sim \left\langle K \left( \vec{L}, T, 0, 0 \right)^* K \left( \vec{L}, T, 0, 0 \right) \right\rangle_p \quad (20)$$

Expressing one propagator as a path integral over $\vec{x}_1(t)$ and the other over $\vec{x}_2(t)$ where both $\vec{x}_1(t)$ and $\vec{x}_2(t)$ go from $\vec{x}_s = 0$ to $\vec{x}_c = \vec{x}$, and letting $J(t) = -\hbar \int_0^T dt (\delta(\vec{x} - \vec{x}_1(t)) - \delta(\vec{x} - \vec{x}_2(t)))$ we have

$$\left\langle K \left( \vec{L}, T, 0, 0 \right)^* K \left( \vec{L}, T, 0, 0 \right) \right\rangle_p = \int_0^L \delta \vec{x}_1(t) \delta \vec{x}_2(t) \left\{ \exp \left[ \frac{i}{\hbar} \int_0^T dt \frac{m}{2} \left( \partial_t \vec{x}_2(t) \right)^2 - \left( \partial_t \vec{x}_1(t) \right)^2 \right] \right\}$$

$$\times \exp \left[ - \frac{1}{2\hbar^2} \int_0^T dt dt' \left( \begin{array}{c} C \left( |\vec{x}_1(t') - \vec{x}_1(t)\right| \\ +C\left( |\vec{x}_2(t') - \vec{x}_2(t)| \right) \\ -2C\left( |\vec{x}_2(t') - \vec{x}_1(t)| \right) \end{array} \right) \right] \quad (21)$$

After accounting for differences in notation the is the same as equation (2.9) in Dashen [6]. Dashen notes that a dominant contribution to the path integral comes from paths for which $\vec{x}_1(t) \approx \vec{x}_2(t)$, and he uses that in his evaluation. But he does not note that paths with closed loops traversed in opposite directions, equivalent to forward and backward in time around the loop, also make a dominant contribution to the path integral. For such paths $|\vec{x}_2(t) - \vec{x}_1(t)|$ can be arbitrarily large. Consider identical $\vec{x}_1(t)$ and $\vec{x}_2(t)$ with one or more closed loops. If both loops are traversed in the same direction this is still simply $\vec{x}_1(t) = \vec{x}_2(t)$. But for any of the loops traversed in the opposite direction then, with the loop starting at $t = t_a$ and ending at $t = t_b$, we have $\vec{x}_2(t') = \vec{x}_1(t_a + t_b - t')$ for $t_a \leq t' \leq t_b$. Changing the integration variable from $t'$ to $t'' = t_a + t_b - t'$ for $t_a \leq t' \leq t_b$ gives the same result as $\vec{x}_1(t) = \vec{x}_2(t)$ in both exponential factors in the path integral. Hence even though for oppositely traversed loops $|\vec{x}_2(t) - \vec{x}_1(t)|$ can be arbitrarily large, we still get a unity contribution to the path integral. The fact that forward and backward loops contribute equally to the path integral is simply due to the fact that the accrued phase along the paths in the forward and backward directions around each loop is the same and so the two directions add coherently [2, 5, 7]. This fact is captured in the very form of the second exponential.
It is not possible to evaluate the double path integral in an exact analytical way. So we need to consider the character of the paths that make dominant contributions.

The first exponential is unity for any $\vec{x}_2 (t) \neq \vec{x}_1 (t)$ if the two paths have identical time average kinetic energy,

$$\frac{m}{2} \int_0^T dt \left( \partial_t \vec{x}_2 (t) \right)^2 = \frac{m}{2} \int_0^T dt \left( \partial_t \vec{x}_1 (t) \right)^2$$

(22)

Hence, from a stationary phase point of view, since $\langle V (\vec{x}) \rangle_p = 0$, the first exponential “filters out” paths with very different average energies or, equivalently, very different average de Broglie wavelengths.

To see how localization emerges, we must show that, in the second exponential the first two terms, on average, outweigh the third term. First note that for any paths whatsoever, the first two terms in the integrand are maximal for all $t' = t$ and so their integrals will scale with $T$. In the third term this can only happen if $|\vec{x}_2 (t') - \vec{x}_1 (t)| \ll \xi$ for all $t' \approx t$ or if both paths contain the same closed loops which again may be traversed in either direction. But for very different paths, i.e., paths for which $|\vec{x}_2 (t') - \vec{x}_1 (t)| \gg \xi$ for almost all $t'$ and $t$, then, after accounting for closed loops, the third term will generally only be nonzero for specific pairs of range of $t$ and $t'$ where the paths might cross or get close to one another, hence, on average, the first two terms dominate the path integral and lead to localization.

Consider 1D. Both paths must cover the distance from 0 to $L$ in time $T$. If both paths remain between 0 and $L$ for $0 \leq t \leq T$, then the third term and first two terms scale the same since there are multiple ranges of times for which $x_2 (t') = x_1 (t)$ with $t' \neq t$. But for paths where, say, $x_2 (t)$ spends a significant portion of time at positions $x > L$ while $x_1 (t)$ spends a significant portion of time at positions $x < L$, or vice versa, then, in this case the contribution from the third term will be much less than that from the first two terms simply because there are now far fewer ranges of time for which $x_2 (t') = x_1 (t)$.

Dimensionality plays a role in localization in the following way. Localization occurs for any energy in one and two dimensions. In three dimensions there is a “mobility edge” in terms of energy which separates extended from localized wave functions [2, 3, 5, 13, 14]. This is encapsulated in the double path integral above as follows. The path integral integrates over all continuous random walks. But random walks in 1D and 2D are recurrent, i.e., every random walk in 1D and 2D will eventually cross itself somewhere, creating closed loops. This was proven by Pólya in 1921 [15]. In 1D this is because any change in direction makes the path recurrent and creates a closed loop with a range of times, $t$ and $t'$, for which $x_2 (t') = x_1 (t)$ with $t' \neq t$. In 2D recurrence occurs because the path itself forms a boundary which, when crossed, again creates a closed loop. In this case generally you have individual values of $t$ and $t'$, for which $\vec{x}_2 (t') = \vec{x}_1 (t)$ with $t' \neq t$. Effectively, paths with closed loops are “dense” in path space in 1D and 2D. The probability of a path being recurrent in 3D is less than unity and paths with closed loops are, effectively, not “dense” in path space. The Pólya result is the flip side of the Poincare-Bendixon theorem [16, 17] which states that an autonomous dynamical system in continuous time needs a phase space of at least 3 dimensions to have chaotic trajectories, i.e., trajectories that never intersect themselves while remaining in a finite volume of the phase space.

Finally note that if we let $V (\vec{x})$ be random in time as well as space, i.e., $V (\vec{x}) \rightarrow V (\vec{x}, t)$, then the autocorrelation function becomes $C (|\vec{x} - \vec{x}'|, |t - t'|)$. Given a finite autocorrelation time $\tau$, if $T \ll \tau$ we have $C (|\vec{x} - \vec{x}'|, |t - t'|) \simeq C (|\vec{x} - \vec{x}'|, 0)$ and we have the same results as above, i.e., localization. But for $T \gg \tau$, peaks in $V (\vec{x}, t)$ evolve into valleys and vice versa which leads to diffusion. From the point of view of the time integrals over the autocorrelation function, the temporal variation of $V (\vec{x}, t)$ destroys the phase coherence of oppositely traversed closed loops for times $T \gg \tau$. Effectively the relevant time scale for these integrals switches from $T$ to $\tau$, causing localization to degenerate into diffusion [2, 7].

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