THRESHOLD ASYMPTOTIC BEHAVIORS FOR A DELAYED NONLOCAL REACTION-DIFFUSION MODEL OF MISTLETOES AND BIRDS IN A 2D STRIP

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Abstract. A time-delayed reaction-diffusion system of mistletoes and birds with nonlocal effect in a two-dimensional strip is considered in this paper. By the background of model deriving, the bird diffuses with a Neumann boundary value condition, and the mistletoes does not diffuse and thus without boundary value condition. Making use of the theory of monotone semiflows and Kuratowski measure of non-compactness, we discuss the existence of spreading speed $c^*$. The value of $c^*$ is evaluated by using two auxiliary linear systems accompanied with approximate process.

1. Introduction. Mistletoes are typical aerial stem-parasites plants (Kuijt [5]), whose seeds are mainly transmitted by fruit-eating birds highly specialized to consume their berries. Mistletoes produce flowers and fruits. Birds in some trees search and encounter mistletoe fruits, then handle the fruits and deposit their droppings with seeds after they ate the fruits. Once a mistletoe seed is deposited by a bird on an appropriate host plant, it sticks to a branch, germinates and forms a haustorium that taps into the xylem of the host to absorb water and minerals. Mistletoe was often regarded as a pest because it kills trees and devalues natural habitats, while it was recently recognized as an ecological keystone species, an organism that has a disproportionately pervasive influence over its community (Watson [17]).

In the mutually beneficial relationships between mistletoes and birds species that disperse mistletoe seeds, recently, Wang et al. [15] derived and proposed a reaction-diffusion system with proper initial and boundary value conditions:

\[
\begin{aligned}
\frac{\partial u_1}{\partial t} &= -d_m u_1 + \alpha e^{-d_i \tau} \int_{\Omega} k(x, z) \frac{u_1(t-\tau, z)}{u_1(t-\tau, z)} u_2(t-\tau, z) dz, \quad t > 0, x \in \bar{\Omega}, \\
\frac{\partial u_2}{\partial t} &= D \Delta u_2 - \gamma \nabla (u_2 \nabla u_1) + u_2(1 - u_2) + d \int_{\Omega} k(x, z) \frac{u_1(t, z)}{u_1(t, z)} u_2(t, z) dz, \quad t > 0, x \in \bar{\Omega}, \\
(D \nabla u_2 &- \gamma u_2 \nabla u_1) \cdot n(x) = 0, \quad t > 0, x \in \partial \Omega, \\
u_1(t, x) &= u_1^{(0)}(t, x), \quad u_2(t, x) = u_2^{(0)}(t, x), \quad t \in [-\tau, 0], \quad x \in \Omega,
\end{aligned}
\]  

(1)
where \( u_1(t, x) \) and \( u_2(t, x) \) denote the densities of adult mistletoes and birds under consideration at time \( t \geq -\tau \) and location \( x \in \Omega \), respectively. \( \Omega \) is a bounded connected open domain in \( \mathbb{R}^n \) (\( n \geq 1 \)) with smooth boundary \( \partial \Omega \). In the deriving procedure, Fickian diffusion and chemotaxis are regarded as the random movement and the aggregation of birds. A convolution integral with a kernel function \( k(x, z_x) \) expresses the spread of mistletoes by birds from location \( z_x \) to location \( x \). A Holling type II functional response is used to model the fruits removal process by birds. The time delay \( \tau \) models the maturation time of mistletoes. \( d_i \) and \( d_m \) present respectively, the mortality rates of immature and mature mistletoes. \( \alpha \) expresses the hanging rate of mistletoes fruits on trees. \( D \) models the diffusion rate of birds. \( \gamma \) is a chemotactic term that models birds are attracted by trees with more mistletoes. \( d \) is the rate of the conversion from mistletoe fruits into bird population. \( \omega \) presents the fact that birds may perch on some other trees without mistletoes and structures independent of the dynamic course of mistletoes. A no-flux boundary condition here imposed for birds’ population, where \( n(x) \) is the outer normal vector at \( x \in \partial \Omega \), means that birds cannot go outside the spatial habitat \( \Omega \). In [15], Wang, Liu, Shi & Martinez discussed the existence, bounds and uniqueness of solutions of (1) with \( \gamma = 0 \), as well as linearized stability of the model with a Dirac delta kernel function and bifurcation analysis of the model with chemotactic and nonlocal effect for \( \gamma \geq 0 \).

The spreading speed (or, asymptotic speed of spread) is one of the key elements to the developmental process of population dynamics. The research for spreading speed can be traceable from Aronson & Weinberger[1] for reaction-diffusion systems and Weinberger[18] for recursion systems, and also can be found in references for many types of evolutionary systems (e.g. see [3, 8, 12, 13, 14, 20, 21, 22, 23, 24, 26]). Spreading speed \( c^* \) is in fact a threshold value, the existence of which reveals the different asymptotic patterns as time \( t \to \infty \), while the wave speed parameter \( c \) crosses \( c^* \), generating two unbounded sub-domain for solution spreading (see details in Theorem 5.3). General speaking, the study of spreading speed is carried out in an unbounded domain, which can be \( \mathbb{R}^n \) or \( \mathbb{R}^{n-1} \times [-L, L] \). Although the living planet for the mankind is earth, and the area of which is in fact finite, one can still take the extending space and infinite time to see the asymptotic behaviors and patterns in unbounded domain as \( t \to \infty \). In [16], Wang, Liu, Shi & Martinez discussed the existence of an asymptotic spreading speed and traveling wave solutions for the model

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= -d_m u_1 + \alpha e^{-d_i \tau} \int_{\mathbb{R}} k(x, z_x) \frac{u_1(t-\tau, z_x)}{u_1(t-\tau, z_x) + u_2(t-\tau, z_x)} u_2(t-\tau, z_x) dz_x, \quad t > 0, x \in \mathbb{R}, \\
\frac{\partial u_2}{\partial t} &= D \Delta u_2 + u_2(1-u_2) + d \int_{\mathbb{R}} k(x, z_x) \frac{u_1(t, z_x)}{u_1(t, z_x) + \omega} u_2(t, z_x) dz_x, \quad t > 0, x \in \mathbb{R}.
\end{align*}
\]

The purpose of this article is to study the dynamics of the model in a two-dimensional strip because the species may be live in a two-dimensional strip rather than in a one-dimension space. Thus, in this article, we shall consider the reaction-diffusion model of mistletoes and birds with the chemotactic term \( \gamma = 0 \) in \( \Omega = \mathbb{R}^n \).
$\mathbb{R} \times (-L, L)$, then the system (1) without initial value conditions turns into:

$$
\begin{cases}
\frac{\partial u_1}{\partial t} = -d_m u_1 + ae^{-d \tau} \int_{-L}^{L} k(x, y, z, z_y) \frac{u_1(t - \tau, z, z_y)}{u_1(t - \tau, z, z_y) + \omega} u_2(t - \tau, z, z_y) dz_y d\tau \\
\frac{\partial u_2}{\partial t} = D(\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2}) + u_2(1 - u_2) + d \int_{-L}^{L} k(x, y, z, z_y) dz_y d\tau \\
\frac{\partial u_2}{\partial y} = 0,
\end{cases}
$$

(2)

where the parameters $d_m, a, d, D, \omega$ are positive constant, the time delay $\tau$ and parameter $d$ are nonnegative, the boundary condition here expresses the requirement that the population size $u_i$ are confined in $\Omega$ which means that no migration occurs through the boundary $\partial \Omega$ and $k(x, y, z, z_y)$ is a kernel function with separated variables, that is,

$$
k(x, y, z, z_y) = k(x - z, y, z_y) = k_1(x - z)k_2(y, z_y),
$$

depends only on the distance between $x$ and $z$, and satisfies the following assumptions throughout this paper.

(K1) $k_1$ is a nonnegative Lebesgue measurable function, $k_1(\xi) = k_1(-\xi)$, $\forall \xi \in \mathbb{R}$, $\int_{\mathbb{R}} k_1(\xi) d\xi = 1$, and $K_1(\varsigma) := \int_{\mathbb{R}} k_1(\xi) e^{\varsigma \xi} d\xi < \infty, \forall \varsigma \in [0, \infty).

(K2) $k_2(y, z_y) \in C([-L, L] \times [-L, L], \mathbb{R}_+), k_2(y, z_y) = k_2(\varsigma, y, z_y) > 0$ for $\forall y, z_y \in [-L, L]$, and $\int_{-L}^{L} k_2(y, z_y) dz_y = 1$ for every $y \in [-L, L].$

The main results for (2) in this article are about the spreading speed. Similar to [16], we use the theory of monotone semiflow developed by Liang and Zhao in [9]. However, there are some difficulties and challenging because of the region $\Omega$ has been changed into a two-dimensional strip now. Firstly, we must obtain the Green function by solving the following initial boundary value problem:

$$
\begin{cases}
\frac{\partial w}{\partial t} = D(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}), & t > 0, (x, y) \in \mathbb{R} \times (-L, L), \\
\frac{\partial w}{\partial y} = 0, & t \geq 0, (x, y) \in \mathbb{R} \times (-L, L), \\
w(0, x, y) = \psi(x, y), & (x, y) \in \mathbb{R} \times (-L, L).
\end{cases}
$$

Secondly, to estimate the upper and lower bounds of the spreading speed $c^*$, we need more complex and elaborate techniques.

The organization of the article is as follows. We give preliminaries in Section 2. In Section 3, we first translate (2) to an equivalent system (5), and then obtain the existence and uniqueness of solutions, the comparison principle and the strong positivity for system (5). In Section 4 and 5, by the usage of the theory of monotone semiflows developed in Liang & Zhao [9, 27], we obtain the existence of the spreading speed $c^*$. We give an estimation of $c^*$ in Section 6 in view of two linear systems accompanied with an approximate process. An appendix is added at the end to illustrate the eigenvalues and eigenfunctions for a specify eigenvalue problem. We hope that the results in this article could be significative for understanding about the spreading patterns and interaction rule among mistletoes and birds.
2. Preliminaries.

Let
\[ C([-L, L]) = \{ u(y) : u \in C([-L, L], \mathbb{R}) \}, \]
\[ BC(\mathbb{R} \times [-L, L]) = \{ u(x, y) : u \in C(\mathbb{R} \times [-L, L], \mathbb{R}), \]
\[ u \text{ is bounded and uniformly continuous on } \mathbb{R} \times [-L, L] \}\]
and the norm of which be expressed as \( ||u||_{C([-L, L])} \) and \( ||u||_{BC(\mathbb{R} \times [-L, L])} \), respectively. Here \( ||u||_{C([-L, L])} \) is the supremum norm, and \( ||u||_{BC(\mathbb{R} \times [-L, L])} \) could be a norm induced from the supremum or the compact open topology. Define two linear operators as follows:
\[ K_2[u](y) := \int_{-L}^{L} k_2(y, z_y)u(z_y)dz_y \text{ for } u \in C([-L, L]), \]
\[ K[u](x, y) := \int_{-L}^{L} \int_{-L}^{L} k(x, y, z_x, z_y)u(z_x, z_y)dz_xdz_y \text{ for } u \in BC(\mathbb{R} \times [-L, L]). \]
It is obviously that \( K \) and \( K_2 \) satisfy
\[ (1) ||K[u]||_{BC(\mathbb{R} \times [-L, L])} \leq A_1 ||u||_{BC(\mathbb{R} \times [-L, L])} \text{ for } A_1 = 1; \]
\[ (2) K \text{ and } K_2 \text{ are strongly positive, order preserved and compact}. \]
Also see the statements after the assumption (K3) and the equation (49) in [15].

From [6, 7], we know that there is no stable non-constant equilibrium of a cooperative system with homogeneous Neumann boundary value condition in a convex domain. Therefore, in this paper, we just consider constant equilibria of (2). It is easy to see that the system (2) has the following two boundary equilibria: \( E_0 := (0, 0), E_1 := (0, 1) \). Note that the positive constant equilibria of (2) are determined by the following system:
\[ \begin{cases} 
\frac{\alpha e^{-d_1 t} u_2}{u_1 + \omega} - d_m = 0, \\
1 - u_2 + \frac{d u_1}{u_1 + \omega} = 0.
\end{cases} \]
(3)
Substitute the second equation into the first one, then we have
\[ \omega u_1^2 + (2\omega - 1 - d)u_1 + (\omega^2 - \omega) = 0, \]
(4)
where \( \iota := \frac{d_m}{\alpha e^{-d_1 t}} \).
Now, let
\[ \Delta := (2\omega - 1 - d)^2 - 4\omega(\omega^2 - \omega). \]
Note that if \( \Delta < 0 \), which is equivalent to \( d > \frac{\alpha e^{-d_1 t}}{4\omega(\omega^2 - \omega)} \), then there is no other constant equilibrium. In the following, we further consider the situation \( \Delta \geq 0 \), and obtain the following lemma by summarizing.

**Lemma 2.1.** The following statements are valid.

(i) Suppose \( \Delta < 0 \), then there are exact two constant equilibria of (2): \( E_0 = (0, 0), E_1 = (0, 1) \), that is, there is no positive constant equilibrium of (2).

(ii) Suppose \( \Delta = 0 \). If \( d > 1 \), then there is a positive constant equilibria of (2), which is a double root of (4), otherwise, there is no positive constant equilibrium of (2).

(iii) Suppose \( d_m < d_m := \frac{\alpha e^{-d_1 t}}{\omega} \), then \( \Delta > 0 \) and there are two different interior equilibria of (2), one is positive and the other is negative.

(iv) Suppose \( d_m = d_m \) and \( d > 1 \), then \( \Delta > 0 \) and there exist only one positive constant equilibrium of (2).
Suppose \( d_m > d'_m \) and \( d > 1 \). If \( \triangle > 0 \), then there are two positive constant equilibria of (2).

The proof is similar to the proof in [16], we omit it here.

3. Existence, uniqueness and comparison principle. Throughout this article, we suppose that the following assumption holds.

\[
E \quad \text{(E)} \quad \therefore \quad \text{we know that (5) has an unique positive equilibrium}
\]

\[
\text{where}\ u_1^+ := \frac{1 + d + \sqrt{(1 + d)^2 - 4d\omega}}{2\omega} - \omega > 0, \quad u_2^+ := 1 + \frac{du_1^+}{u_1^+ + \omega} \in (1, 1 + d), \quad u_1^- < 0.
\]

Let \( \bar{u}_1 = u_1, \bar{u}_2 = u_2 - 1 \). Thus we transform the system (2) into the following equivalent system (for the sake of convenience, we drop out “\( u \)”):

\[
\begin{aligned}
\frac{\partial u_1}{\partial t} &= -d_m u_1 + \alpha e^{-d_1} \int_{-L}^{L} k(x, y, z_x, z_y) \frac{u_1(t - \tau, z_x, z_y)}{u_1(t - \tau, z_x, z_y) + \omega} [u_2(t - \tau, z_x, z_y) + 1]dz_ydz_x, \\
\frac{\partial u_2}{\partial t} &= D(D^2 u_2 + \frac{\partial^2 u_2}{\partial y^2}) - u_2(2u_2 + 1) + d \int_{-L}^{L} k(x, y, z_x, z_y) \frac{u_1(t, z_x, z_y)}{u_1(t, z_x, z_y) + \omega} [u_2(t, z_x, z_y) + 1]dz_ydz_x, \\
\frac{\partial u_1}{\partial y} &= 0,
\end{aligned}
\]

Therefore, we know that (5) has an unique positive equilibrium

\[
\mathcal{E}_+ := (\bar{u}_1^+, \bar{u}_2^+) = (u_1^+, u_2^+ - 1)
\]

and other three equilibria: \( \mathcal{E}_0 = (0, 0) \), \( \mathcal{E}_- = (0, -1) \) and \( \mathcal{E}_- = (\bar{u}_1^-, \bar{u}_2^-) = (u_1^-, u_2^- - 1) \). In other words, for system (5), in the rectangle with vertices \( \mathcal{E}_0 \), \( \mathcal{E}_- \), there are no other equilibria of (5).

We first give some notations. Let \( \mathbb{R}^+ = [0, \infty) \) and

\[
\begin{aligned}
\mathcal{X}_1 &:= \{ \phi_1 : \phi_1 \in BC(\mathbb{R} \times [-L, L], \mathbb{R}) \}, \\
\mathcal{X}_2 &:= \{ \phi_2 : \phi_2 \in BC(\mathbb{R} \times [-L, L], \mathbb{R}) \text{ with } \frac{\partial \phi_2}{\partial y}(x, y) \in \mathbb{R} \times [-L, L] = 0 \}, \\
\mathcal{X} &:= \mathcal{X}_1 \times \mathcal{X}_2, \\
\mathcal{X}_+ &:= \{ (\phi_1, \phi_2) \in \mathcal{X} : 0 \leq \phi_i(x, y), (x, y) \in \mathbb{R} \times [-L, L], i = 1, 2 \}, \\
\mathcal{X}_- &:= \{ (\phi_1, \phi_2) \in \mathcal{X} : 0 \leq \phi_i(x, y) \leq \beta_i, (x, y) \in \mathbb{R} \times [-L, L], i = 1, 2 \}.
\end{aligned}
\]

Here, \( BC(\mathbb{R} \times [-L, L], \mathbb{R}) \) denotes the space with all of bounded and uniformly continuous functions from \( \mathbb{R} \times [-L, L] \) to \( \mathbb{R} \), \( \beta = (\beta_1, \beta_2) \) with \( \beta_i \geq 0 \). Note that any vector \( \beta \in \mathbb{R}^2 \) is a constant function \( \beta \) in \( \mathcal{X} \). Obviously, \( \mathcal{X}_+ \) is a positive cone in \( \mathcal{X} \) and \( (\mathcal{X}, \mathcal{X}_+) \) is an ordered Banach space. For \( \phi = (\phi_1, \phi_2), \varphi = (\varphi_1, \varphi_2) \in \mathcal{X} \), we write \( \phi \leq \varphi \) provided \( \phi_i(x, y) \leq \varphi_i(x, y) \), \( \forall (x, y) \in \mathbb{R} \times [-L, L], i = 1, 2 \); \( \phi < \varphi \) provided \( \phi < \varphi \) and \( \phi \neq \varphi \); \( \varphi \) \( \varphi \) provided \( \phi_i(x, y) < \varphi_i(x, y) \), \( \forall (x, y) \in \mathbb{R} \times [-L, L], i = 1, 2 \). Equip \( \mathcal{X} \) with a compact open topology: \( \phi_n \to \phi (n \to \infty) \) in \( \mathcal{X} \) which is equivalent to \( \phi_n(x, y) \to \phi(x, y) (n \to \infty) \) uniformly for \( (x, y) \) in any compact set on \( \mathbb{R} \times [-L, L] \). Note that \( \mathcal{X} \) is a Banach lattice.
Define $C := C([-\tau, 0], \mathcal{X})$ with all of continuous functions mapping from $[-\tau, 0]$ into $\mathcal{X}$, $C_+ := C([-\tau, 0], \mathcal{X}_+)$ a closed positive cone of $C$, and $C_\beta := \{ \phi \in C : \phi(\theta) \in \mathcal{X}_\beta, \theta \in [-\tau, 0] \}$ for $\forall \beta = (\beta_1, \beta_2), \beta_i \geq 0, i = 1, 2$. Note that any function in $\mathcal{X}$ is an element of $C$. We identify any member $\phi \in C$ with a function mapping from $[-\tau, 0] \times \mathbb{R} \times [-L, L]$ into $\mathbb{R}^2$: $\phi(\theta, x, y) = \phi(\theta)(x, y) = (\phi_1(\theta), \phi_2(\theta))(x, y)$ for any $\theta \in [-\tau, 0]$ and $(x, y) \in \mathbb{R} \times [-L, L]$. Similarly, we can define $\phi \leq \varphi$, $\phi \leq \varphi$, and $\phi \leq \varphi$ for $\phi, \varphi \in C$, and write $u_t \in C : u_t(\theta) = u(t + \theta)$ for $\theta \in [-\tau, 0]$ with $t \in [0, b]$, if the function $u(\cdot) : [-\tau, b] \rightarrow \mathcal{X}$ with $b > 0$. Let the compact open topology in $C$ is introduced by a norm which is denoted as $\| \cdot \|_C$. We shall consider the initial value problem of system (5) corresponding to an initial value condition $\phi \in C_+$ in the present article, from the biological consideration.

Let $(\mu_n, \Psi_n(y))$, $n = 0, 1, \ldots$ be the eigenvalues and their corresponding normalized eigenfunctions of operator $-\frac{\partial^2}{\partial y^2}$ on $[-L, L]$ with respect to homogeneous Neumann boundary condition. Let $\Gamma(t, x, y, z_x, z_y)$ be the Green function of initial-boundary value problem

\[
\begin{align*}
\frac{\partial w}{\partial t} &= D \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right), \quad t > 0, (x, y) \in \mathbb{R} \times (-L, L), \\
\frac{\partial w}{\partial y} &= 0, \quad \quad \quad t \geq 0, (x, y) \in \mathbb{R} \times \{-L, L\}, \\
w(0, x, y) &= \psi(x, y), \quad \quad \quad (x, y) \in \mathbb{R} \times [-L, L],
\end{align*}
\]

(6)

where

\[
\Gamma(t, x, y, z_x, z_y) := \frac{1}{\sqrt{4\pi D t}} e^{-\frac{(x-x_0)^2}{4Dt}} \sum_{k=1}^{\infty} \frac{1}{M_k} \Psi_k(y) \Psi_k(z_y) e^{-\mu_k D t},
\]

$M_k = \int_{-L}^{L} \Psi_k^2(y) dy$ satisfying

\[
\int_{-L}^{L} \int_{-L}^{L} \Gamma(t, x, y, z_x, z_y) d z_y d z_x = 1, \quad t \geq 0, (x, y) \in \mathbb{R} \times [-L, L]
\]

(see more details in the Appendix).

Therefore, for any given $\psi \in \mathcal{X}_2$, the solution of (6) can be written as

\[
(T_2(t)[\psi])(x, y) = \begin{cases} \int_{-L}^{L} \int_{-L}^{L} \Gamma(t, x, y, z_x, z_y) \psi(z_x, z_y) d z_y d z_x, & t > 0, \\
\psi(x, y), & t = 0,
\end{cases}
\]

for $(x, y) \in \mathbb{R} \times (-L, L)$. Note that $T_2(t)$ is compact for $t > 0$ (see [25]).

Let $T_1(t) := I$ be the identify operator defined on $\mathcal{X}_1$. Define a family of bounded linear operators $T(t) = (T_1(t), T_2(t)), t \geq 0$ on $\mathcal{X}$ by

\[
T(t)[\phi] := \begin{pmatrix} T_1(t)[\phi_1] \\ T_2(t)[\phi_2] \end{pmatrix} = \begin{pmatrix} \phi_1 \\ T_2(t)[\phi_2] \end{pmatrix}, \quad \forall \phi = (\phi_1, \phi_2) \in \mathcal{X}.
\]

Then, $T(t) : \mathcal{X} \rightarrow \mathcal{X}$ is a $C_0$-semigroup with $T(t)\mathcal{X}_+ \subseteq \mathcal{X}_+$ for any $t \geq 0$. For any $\phi = (\phi_1, \phi_2) \in \mathcal{X}_+$, define a map $f = (f_1, f_2) : \mathcal{X}_+ \rightarrow \mathcal{X}$ as follows:

\[
f_1(\phi)(x, y) := -d_{\alpha} \phi_1(0, x, y)
\]

\[
\alpha e^{-d_{\alpha} \tau} \int_{-L}^{L} k(x, y, z_x, z_y) \frac{\phi_1(-\tau, z_x, z_y)}{\phi_1(-\tau, z_x, z_y) + \omega}[\phi_2(-\tau, z_x, z_y) + 1] d z_y d z_x,
\]

\[
f_2(\phi)(x, y) := -\phi_2(0, x, y)(\phi_2(0, x, y) + 1)
\]
For any given \( t \) then, as \( T \) for any

\[
\text{Theorem 3.2.}
\]

supersolution (subsolution) of (5) on \([0, b)\).

Definition 3.1. If \( w \) in \((\cdot, \cdot, \cdot) \) has an equivalent integral form

\[
\text{Remark 1.}\quad \text{Now, We define a pair of supersolution and subsolution for (5).}
\]

\[(7)\] has an equivalent integral form

\[
\text{whose solution } u(t, x, y; \phi) \text{ of (8) is called a mild solution of (5). Here, } \phi = (\phi_1, \phi_2) \in \mathcal{C}_+. \]

Now, We define a pair of supersolution and subsolution for (5).

**Definition 3.1.** A continuous function \( w : [-\tau, b) \rightarrow \mathcal{X}_+ \) is said a supersolution (subsolution) of (5), if it satisfies

\[
w(t) \geq (\leq) T(t) f(\phi(0)) + \int_0^t T(t-s) f(w_s) ds, \quad \forall t \in [0, b), \phi \in \mathcal{C}_+. \tag{9}
\]

**Remark 1.** If \( w = (w_1, w_2) \in C([-\tau, b) \times \mathbb{R} \times [-L, L], \mathbb{R}^2_+) \) with \( b > 0 \) such that \( w_1 \in C^2 \) in \((x, y) \in \mathbb{R} \times [-L, L], w_1, w_2 \in C^1 \) in \( t \in [0, b) \) and satisfies

\[
\text{Then, as } T(t)\mathcal{X}_+ \subset \mathcal{X}_+ \text{ for all } t \geq 0, \text{ it follows that (9) holds, and hence, } w \text{ is a supersolution (subsolution) of (5) on } [0, b). \tag{10}
\]

**Theorem 3.2.** For any \( \phi \in \mathcal{C}_{\mathcal{E}_+} \), (5) has a unique non-negative mild solution \( u(t, x, y; \phi) \) with \( u_0(\cdot, \cdot, \cdot; \phi) = \phi \) defined on \([-\tau, \infty) \) such that \( u_1(\cdot, \cdot, \cdot; \phi) \in \mathcal{C}_{\mathcal{E}_+} \) for \( t \geq 0 \), which means that \( \mathcal{C}_{\mathcal{E}_+} \) is a positive invariant subset in \( \mathcal{C}_+ \).

**Proof.** For any given \( \phi \in \mathcal{C}_{\mathcal{E}_+} \), when \( h > 0 \) is small enough such that \( \min\{1 - d_m h, 1 - h, 1 - h \alpha^2 \} > 0 \), we have

\[
\phi_1(0, x, y) + h f_1(\phi)(x, y) = \phi_1(0, x, y) - d_m h \phi_1(0, x, y)
\]

\[
+ \alpha e^{-d_1 h} \int_0^T \int_{[-L, L]} k(x, y, z_x, z_y) \frac{\phi_1(-\tau, z_x, z_y)}{\phi_1(-\tau, z_x, z_y) + \omega} [\phi_2(-\tau, z_x, z_y) + 1] dz_x dz_y \geq (1 - d_m h) \phi_1(0, x, y) \geq 0
\]

and

\[
\phi_2(0, x, y) + h f_2(\phi)(x, y) = \phi_2(0, x, y) - h \phi_2(0, x, y)(\phi_2(0, x, y) + 1)
\]

Therefore, (5) with the initial value \( \phi \in \mathcal{C}_+ \) can be written as an abstract system

\[
\begin{aligned}
\frac{\partial u_1}{\partial t} &= f_1(u_1), & t > 0, (x, y) \in \mathbb{R} \times [-L, L], \\
\frac{\partial u_2}{\partial t} &= D \Delta u_2 + f_2(u_t), & t > 0, (x, y) \in \mathbb{R} \times (-L, L), \\
\frac{\partial u_2}{\partial y} &= 0, & t \geq -\tau, (x, y) \in \mathbb{R} \times \{-L, L\}, \\
u_i(\theta, x, y) &= \phi_i(\theta, x, y) & \theta \in [-\tau, 0], (x, y) \in \mathbb{R} \times (-L, L), i = 1, 2.
\end{aligned}
\tag{7}
\]
where

By Theorem 1 in Martin & Smith [10] (taking $\supseteq \leq <$)

Denote $\hat{0} := (0,\phi_0)$ for $\phi_0 \in \mathbb{R}$. Moreover, when $h$ is so small that $\min\{ae^{-d_1\tau}h(\bar{u}_2^+ + 1), dh(\bar{u}_2^+ + 1)\} < \epsilon$, we have

and

Therefore, if $h$ is sufficiently small, we have $\phi(0) + hf(\phi) \in \mathcal{C}_{\mathcal{E}_+} := \{\phi \in \mathbb{C} : 0 \leq \phi(\theta, x, y) \leq \mathcal{E}_+ + \epsilon, (\theta, x, y) \in [-\tau, 0] \times \mathbb{R} \times [-L, L], \ i = 1, 2\}$, and thus

For the arbitrariness of $\epsilon$, it follows that

By Theorem 1 in Martin & Smith [10] (taking $S(t,s) = T(t-s)$ and $B(t,\phi) = f(\phi)$), it follows that (5) has a unique non-negative mild solution $u(t, x, y; \phi)$ defined on $[-\tau, \infty)$ such that $u_{\epsilon}(\cdot, \cdot, \cdot; \phi) \in \mathcal{C}_{\mathcal{E}_+}$ for $t \geq 0, \phi \in \mathcal{C}_{\mathcal{E}_+}$. Moreover, by Corollary 2.2.5 in Wu [25], we know that $u(t, x, y; \phi)$ is a classical solution of (5) with the initial value condition $\phi \in \mathcal{C}_{\mathcal{E}_+}$ for $t > \tau$.

Denote $\hat{0} := (0,0)$. Let $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2)$ and $\alpha_i, \beta_i \in \mathbb{R}, i = 1, 2$. Define an vector interval $[\alpha, \beta] := \{\gamma = (\gamma_1, \gamma_2) \in \mathbb{R}^2 : \alpha \leq \gamma \leq \beta, i = 1, 2\}, \forall \alpha, \beta \in \mathbb{R}^2$. Note that $I(r, s) := \frac{r}{r+s+1}(s+1)$ satisfies

where $L_{I,1} = \frac{d+1}{\omega}$ and $L_{I,2} = 1$. Moreover, $J_1(r, s) := -d_2m r + ae^{-d_1\tau} s$ and $J_2(r, s) := -r(r+1) + ds$ are global Lipschitz continuous, that is, there exist $L_{J,1, j} > 0, i, j = 1, 2$ such that

$\{I(r_1, s_1) - J_1(r_2, s_2)\} \leq L_{I,1}|r_1 - r_2| + L_{I,2}|s_1 - s_2|$, $\forall r_1, r_2 \in [0, \bar{u}_1^+], s_1, s_2 \in [0, \bar{u}_2^+]$.

Moreover, $J_1(r_1, s_1) - J_1(r_2, s_2) \leq L_{J,1,1}|r_1 - r_2| + L_{J,1,2}|s_1 - s_2|$, $\forall r_1 \in [0, \bar{u}_1^+], s_1 \in [0, \bar{u}_2^+]$.

Lemma 3.3. Suppose that $\bar{u}(t, x, y) \in [0, \mathcal{E}_+], (t, x, y) \in [-\tau, \infty) \times \mathbb{R} \times [-L, L]$, are a pair of supersolution and subsolution of (5) with $\bar{u}(\theta, x, y) \leq \bar{u}(\theta, x, y) \leq \bar{u}(\theta, x, y) \leq \bar{u}(\theta, x, y) \leq \bar{u}(\theta, x, y)$ for $(t, x, y) \in \mathbb{R} \times \mathbb{R} \times [-L, L]$. Moreover, when $\bar{u}(\theta, x, y) \leq \bar{u}(\theta, x, y), \theta \in [-\tau, 0]$ with
Thus, $u(t, x, y) < \hat{u}(0, x, y)$ for $(x, y) \in \mathbb{R} \times [-L, L]$, it holds that $u(t, x, y) < \bar{u}(t, x, y)$ for $(t, x, y) \in (0, \infty) \times \mathbb{R} \times [-L, L]$.

Proof. We have from the statements ahead this lemma that $f$ is global Lipschitz in $C_{E_+}$. We now prove that $f$ is quasi-monotone in $C_{E_+}$, that is,

$$\lim_{h \to 0^+} \frac{1}{h} d(\varphi(0) - \psi(0) + h[f(\varphi) - f(\psi)]; \mathcal{X}_+) = 0$$

for $\varphi, \psi \in C_{E_+}$ with $\varphi \geq \psi$ and $h > 0$ small.

Note that $I(r, s) := \frac{1}{r + \omega}(s + 1)$ satisfies that

$$I(r_1, s_1) - I(r_2, s_2) \leq \frac{d + 1}{\omega}(r_1 - r_2) + (s_1 - s_2), \forall (r_1, r_2), (s_1, s_2) \in [0, \mathcal{E}_+].$$

Then,

$$f_1(\psi)(x, y) - f_1(\varphi)(x, y)$$

$$= -d_m[\psi_1(0, x, y) - \varphi_1(0, x, y)] + \alpha e^{-d_1 \tau} \int_{R_L}^{L} k(x, y, z_x, z_y) \cdot \left\{ \psi_1(-\tau, z_x, z_y) - \varphi_1(-\tau, z_x, z_y) + \psi_2(-\tau, z_x, z_y) + \varphi_2(-\tau, z_x, z_y) \right\} dz_y dz_x$$

$$\leq -d_m[\psi_1(0, x, y) - \varphi_1(0, x, y)] + \alpha e^{-d_1 \tau} \int_{R_L}^{L} k(x, y, z_x, z_y) \cdot \left\{ \psi_1(-\tau, z_x, z_y) - \varphi_1(-\tau, z_x, z_y) + \psi_2(-\tau, z_x, z_y) - \varphi_2(-\tau, z_x, z_y) \right\} dz_y dz_x.$$

Similarly,

$$f_2(\psi)(x, y) - f_2(\varphi)(x, y)$$

$$= -[\psi_2(0, x, y) - \varphi_2(0, x, y)] - \psi_2(0, x, y) + \varphi_2(0, x, y) + d \int_{R_L}^{L} k(x, y, z_x, z_y) \cdot \left\{ \psi_1(0, z_x, z_y) - \varphi_1(0, z_x, z_y) + \psi_2(0, z_x, z_y) + \varphi_2(0, z_x, z_y) \right\} dz_y dz_x$$

$$\leq -[\psi_2(0, x, y) - \varphi_2(0, x, y)] + [\psi_2(0, x, y) + \varphi_2(0, x, y)] + \alpha e^{-d_1 \tau} \int_{R_L}^{L} k(x, y, z_x, z_y) \cdot \left\{ \psi_1(0, z_x, z_y) - \varphi_1(0, z_x, z_y) + \psi_2(0, z_x, z_y) - \varphi_2(0, z_x, z_y) \right\} dz_y dz_x$$

$$\leq -[\psi_2(0, x, y) - \varphi_2(0, x, y)] + \alpha e^{-d_1 \tau} \int_{R_L}^{L} k(x, y, z_x, z_y) \cdot \left\{ \psi_1(0, z_x, z_y) - \varphi_1(0, z_x, z_y) + \psi_2(0, z_x, z_y) - \varphi_2(0, z_x, z_y) \right\} dz_y dz_x.$$

Thus, $f$ is quasi-monotone in $C_{E_+}$. Note that $E_0 = 0$ is a subsolution and $E_+$ is a supersolution of system (5), respectively. Taking the initial function with $0 \leq u_0(0, x, y) \leq \hat{u}_0(0, x, y) \leq E_+, (x, y) \in \mathbb{R} \times [-L, L]$, then we have from Corollary 1 in Martin & Smith [10] that

$$0 \leq u(t, \cdot, \cdot; u_0) \leq u(t, \cdot, \cdot; \hat{u}_0) \leq E_+, t \geq 0.$$
By the fact that $\bar{u}(t, x, y)$, $\underline{u}(t, x, y)$ are a pair of supersolution and subsolution of (5), respectively, again applying Corollary 1 in Martin & Smith [10], we have

$$
\underline{u}(t, \cdot, \cdot) \leq u(t, \cdot, \cdot; \underline{u}_0) \leq \mathcal{E}_t, \ t \geq 0,
$$

$$
\hat{0} \leq u(t, \cdot, \cdot; \bar{u}_0) \leq \bar{u}(t, \cdot, \cdot), \ t \geq 0,
$$
which means that

$$
\underline{u}(t, x, y) \leq \bar{u}(t, x, y), \quad \forall (t, x, y) \in \mathbb{R}_+ \times \mathbb{R} \times [-L, L].
$$

Define $w(t, x, y) = \bar{u}(t, x, y) - \underline{u}(t, x, y)$, then $w(t, x, y) \geq 0$ for all $(t, x, y) \in \mathbb{R}_+ \times \mathbb{R} \times [-L, L]$. Similarly to the above arguments, we can obtain

$$
f_1(\bar{u}_t)(x, y) - f_1(\underline{u}_t)(x, y) \geq -d_m[\bar{u}_1(t, x, y) - \underline{u}_1(t, x, y)],
$$

$$
f_2(\bar{u}_t)(x, y) - f_2(\underline{u}_t)(x, y) \geq -[\bar{u}_2(t, x, y) - \underline{u}_2(t, x, y)][\bar{u}_2(t, x, y) + \underline{w}_2(t, x, y) + 1]
$$

$$
\geq -(2\bar{u}_2^+ + 1)[\bar{u}_2(t, x, y) - \underline{u}_2(t, x, y)],
$$

which together with the definition of supersolution and subsolution, and the fact that $T(t)\mathcal{X}_+ \subset \mathcal{X}_+$ for all $t \geq 0$, leads to

$$
w_1(t) \geq T_1(t)w_1(0) + \int_0^t T_1(t-s)[f_1(\bar{u}_s) - f_1(\underline{u}_s)]ds
$$

$$
\geq T_1(t)w_1(0) - d_m \int_0^t T_1(t-s)w_1(s)ds, \quad t \geq 0,
$$

and

$$
w_2(t) \geq T_2(t)w_2(0) + \int_0^t T_2(t-s)[f_2(\bar{u}_s) - f_2(\underline{u}_s)]ds
$$

$$
\geq T_2(t)w_2(0) - (2\bar{u}_2^+ + 1) \int_0^t T_2(t-s)w_2(s)ds, \ t \geq 0.
$$

Let $v_1(t) = e^{-d_m t}T_1(t)w_1(0)$ and $v_2(t) = e^{-(2\bar{u}_2^+ + 1)t}T_2(t)w_2(0)$. Then it is easy to know that $v_1(t)$ is a solution of

$$
v(t) = T_1(t)v(0) - d_m \int_0^t T_1(t-s)v(s)ds, \quad t \geq 0,
$$

and $v_2(t)$ is a solution of

$$
v(t) = T_2(t)v(0) - (2\bar{u}_2^+ + 1) \int_0^t T_2(t-s)v(s)ds, \ t \geq 0.
$$

According to Proposition 3 in Martin & Smith [10] with $v^-(t) = v(t)$, $v^+(t) = \infty$, $S(t, s) = T_1(t-s)$, $B(t, \phi) = B^-(t, \phi) = -d_m \phi(0)$, we obtain that $w_1(t) \geq v_1(t)$ for $t \geq 0$, which is equivalent to

$$
w_1(t) \geq e^{-d_m t}T_1(t)w_1(0) = e^{-d_m t}w_1(0) \quad \text{for} \ t \in \mathbb{R}_+.
$$

We can also get a similar result for $w_2(t)$. Thus, we have that $w(t) = (w_1(t), w_2(t)) \gg \hat{0}$ for $t > 0$, if $w(0, x, y) > 0$ for $(x, y) \in \mathbb{R} \times [-L, L]$.

\qed
4. Monotone semiflow. Recall a family of operators \( \{ \Pi_t \}_{t \geq 0} \) which is called a semiflow on a metric space \( (Z, \| \cdot \|_Z) \) with the norm \( \| \cdot \|_Z \); if \( \{ \Pi_t \}_{t \geq 0} \) satisfies the following properties:

(a) \( \Pi_0(v) = v \) for \( v \in Z \);
(b) \( \Pi_s(\Pi_t(v)) = \Pi_{t+s}(v) \) for \( t, s \in \mathbb{R}_+, v \in Z \);
(c) \( \Pi(t, v) := \Pi_t(v) \) is continuous in \( (t, v) \in \mathbb{R}_+ \times Z \).

It is easy to see that property (c) holds if \( \Pi(\cdot, v) \) is continuous on \( \mathbb{R}_+ \) for each \( v \in Z \), and \( \Pi(t, \cdot) \) is uniformly continuous for \( t \) in any bounded intervals in the sense that for any \( v_0 \in Z \), bounded interval \( I \) and \( \epsilon > 0 \), there exists \( \delta = \delta(v_0, I, \epsilon) > 0 \) such that if \( \| v - v_0 \|_Z < \delta \), then \( \| \Pi_t(v) - \Pi_t(v_0) \|_Z < \epsilon \) for all \( t \in I \). Let \( u(t, x, y; \phi) \) be the mild solution of \( (5) \) with initial condition \( u_0 = \phi \in C Z \). Define a family of operators \( \{ Q_t \}_{t \geq 0} \) on \( C Z \) by \( Q_t(\phi) := (Q_{1t}(\phi), Q_{2t}(\phi)) \) and

\[
Q_t(\phi)(\theta, x, y) = u_t(\theta, x, y; \phi) = u(t + \theta, x, y; \phi)
\]

for any \( t \geq 0 \), \( (\theta, x, y) \in [-\tau, 0] \times \mathbb{R} \times [-L, L] \), \( \phi \in C Z \). For \( C Z \) is a positive invariant set for \( (5) \), it follows that \( Q_t : C Z \rightarrow C Z \) is the solution operator of \( (5) \) defined on \( C Z \).

**Theorem 4.1.** \( \{ Q_t \}_{t \geq 0} \) is a monotone semiflow on \( C Z \). Here ‘monotone’ means that \( \{ Q_t \}_{t \geq 0} \) satisfies \( Q_t(\phi) \leq Q_t(\psi) \) whenever \( \phi \leq \psi \) and \( t \geq 0 \).

**Proof.** From the integral form \( (8) \) of system \( (5) \) and the property of \( \Pi(\cdot, v) \), it follows that \( Q_t(\phi) \) is continuous in \( t \) in any bounded intervals in the sense that for any \( v_0 \in Z \), bounded interval \( I \) and \( \epsilon > 0 \), there exists \( \delta = \delta(v_0, I, \epsilon) > 0 \) such that if \( \| v - v_0 \|_Z < \delta \), then \( \| \Pi_{1t}(v) - \Pi_{1t}(v_0) \|_Z < \epsilon \) for all \( t \in I \). Let \( u(t, x, y; \phi) \) be the mild solution of \( (5) \) with initial condition \( u_0 = \phi \in C Z \). Define a family of operators \( \{ Q_t \}_{t \geq 0} \) on \( C Z \) by \( Q_t(\phi) := (Q_{1t}(\phi), Q_{2t}(\phi)) \) and

\[
Q_t(\phi)(\theta, x, y) = u_t(\theta, x, y; \phi) = u(t + \theta, x, y; \phi)
\]

for any \( t \geq 0 \), \( (\theta, x, y) \in [-\tau, 0] \times \mathbb{R} \times [-L, L] \), \( \phi \in C Z \). For \( C Z \) is a positive invariant set for \( (5) \), it follows that \( Q_t : C Z \rightarrow C Z \) is the solution operator of \( (5) \) defined on \( C Z \).

In the following, we claim that for \( \forall t_0 > 0 \), \( Q_t(\phi) \) is continuous in \( \phi \) with respect to the compact open topology uniformly for \( t \in [0, t_0] \). For the sake of convenience, we first make some denotations. Let \( t \in [0, t_0] \), \( \phi, \tilde{\phi} \in C Z \). Denote

\[
v(t, x, y) := \| u(t, x, y; \phi) - u(t, x, y; \tilde{\phi}) \|, \quad (x, y) \in \mathbb{R} \times [-L, L],
\]

\[
U_+ := \sqrt{(\tilde{u}_1^+)^2 + (\tilde{u}_2^+)^2},
\]

\[
\mathcal{H}_t := \sup_{s \in [-\tau, t], x \in \mathbb{R}, y \in [-L, L]} v(s, x, y) \leq 2U_+,
\]

\[
\Upsilon_x(x) := [-\tau, 0] \times [-\rho + x, x + \rho] \times [-L, L],
\]

\[
|\phi|_{\Upsilon_x(x)} := \sup_{(\theta, z, y) \in \Upsilon_x(x)} \| \phi(\theta, z, y) \|.
\]

From the definition of supremum, it follows that for \( \forall \epsilon > 0 \), there exists \( (\tilde{t}, \tilde{x}, \tilde{y}) \in [-\tau, t] \times \mathbb{R} \times [-L, L] \) such that \( v_\epsilon(\theta, x, y) \leq \mathcal{H}_t \leq v(\tilde{t}, \tilde{x}, \tilde{y}) + \epsilon, \quad s \in [0, t], \theta \in [-\tau, 0], (x, y) \in \mathbb{R} \times [-L, L] \). Moreover, there exists \( M_1 > 0 \) and \( M_2 > 0 \) such that

\[
\int_{|z - \tilde{z}| > M} \int_{-L}^{L} k(\tilde{x}, \tilde{y}, z_x, z_y)dz_ydz_x < \frac{\epsilon}{2U_+} \quad \text{for any } M > M_1,
\]

\[
\int_{|z - \tilde{z}| > M} \int_{-L}^{L} \Gamma(\tilde{t}, \tilde{x}, \tilde{y}, z_x, z_y)dz_ydz_x < \frac{\epsilon}{2U_+} \quad \text{for any } M > M_2.
\]
There exists a $M > \max \{M_1, M_2\}$ large enough, such that
\[
\int_R \int_{-L}^{L} k(\tilde{x}, \tilde{y}, z_x, z_y) u_s(-\tau, z_x, z_y) dz_y dz_x \\
= \int_{|z_x - \tilde{z}| > M} \int_{-L}^{L} k(\tilde{x}, \tilde{y}, z_x, z_y) u_s(-\tau, z_x, z_y) dz_y dz_x \\
+ \int_{|z_x - \tilde{z}| \leq M} \int_{-L}^{L} k(\tilde{x}, \tilde{y}, z_x, z_y) u_s(-\tau, z_x, z_y) dz_y dz_x \\
< |v_s|_{\Upsilon_M(\tilde{x})} + 2U + \frac{\epsilon}{2U_+} \\
= |v_s|_{\Upsilon_M(\tilde{x})} + \epsilon.
\]
Similarly, we have
\[
\int_R \int_{-L}^{L} \Gamma(\tilde{t}, \tilde{x}, \tilde{y}, z_x, z_y) |\phi(0, z_x, z_y) - \phi(0, z_x, z_y)| dz_y dz_x < |\phi - \bar{\phi}|_{\Upsilon_M(\tilde{x})} + \epsilon,
\]
and
\[
\int_R \int_{-L}^{L} \Gamma(\tilde{t}, \tilde{x}, \tilde{y}, z_x, z_y) \int_R \int_{-L}^{L} k(z_x, z_y, \xi, \eta)|u_s(0, \xi, \eta; \phi) - u_s(0, \xi, \eta; \bar{\phi})| dz_y dz_x < |v_s|_{\Upsilon_M(\tilde{x})} + \epsilon.
\]
Thus, there exists $\delta < \epsilon$, for $\phi, \bar{\phi} \in C_{\mathcal{E}}$, $|\phi - \bar{\phi}|_{\Upsilon_M(\tilde{x})} < \delta$, such that
\[
|u_1(\tilde{t}, \tilde{x}, \tilde{y}; \phi) - u_1(\tilde{t}, \tilde{x}, \tilde{y}; \bar{\phi})| \\
\leq T_1(\tilde{t})|\phi(0, \cdot)(\tilde{x}, \tilde{y}) - \bar{\phi}(0, \cdot)(\tilde{x}, \tilde{y})| \\
+ \int_0^{\tilde{t}} T_1(\tilde{t} - s)|f_1(u_s(\phi))(\tilde{x}, \tilde{y})) - f_1(u_s(\bar{\phi}))(\tilde{x}, \tilde{y}))| ds \\
\leq |\phi(0, \tilde{x}, \tilde{y}) - \bar{\phi}(0, \tilde{x}, \tilde{y})| + L_{J_1,1} \int_0^{\tilde{t}} v_s(0, \tilde{x}, \tilde{y}) ds \\
+ L_{J_1,2}(L_{I,1} + L_{I,2}) \int_0^{\tilde{t}} \int_R \int_{-L}^{L} k(\tilde{x}, \tilde{y}, z_x, z_y) u_s(-\tau, z_x, z_y) dz_y dz_x ds \\
\leq |\phi - \bar{\phi}|_{\Upsilon_M(\tilde{x})} + [L_{J_1,1} + L_{J_1,2}(L_{I,1} + L_{I,2})] \int_0^{\tilde{t}} |v_s|_{\Upsilon_M(\tilde{x})} ds + L_{J_1,2}(L_{I,1} + L_{I,2})\tilde{t}\epsilon \\
< \epsilon + L_1 t_0 \epsilon + L_1 \int_0^{\tilde{t}} |v_s|_{\Upsilon_M(\tilde{x})} ds = (1 + L_1 t_0) \epsilon + L_1 \int_0^{\tilde{t}} |v_s|_{\Upsilon_M(\tilde{x})} ds,
\]
where $L_1 := L_{J_1,1} + L_{J_1,2}(L_{I,1} + L_{I,2})$, and
\[
|u_2(\tilde{t}, \tilde{x}, \tilde{y}; \phi) - u_2(\tilde{t}, \tilde{x}, \tilde{y}; \bar{\phi})| \\
\leq T_2(\tilde{t})|\phi(0, \cdot)(\tilde{x}, \tilde{y}) - \bar{\phi}(0, \cdot)(\tilde{x}, \tilde{y})| \\
+ \int_0^{\tilde{t}} T_2(\tilde{t} - s)|f_2(u_s(\phi))(\tilde{x}, \tilde{y})) - f_2(u_s(\bar{\phi}))(\tilde{x}, \tilde{y}))| ds \\
\leq \int_R \int_{-L}^{L} \Gamma(\tilde{t}, \tilde{x}, \tilde{y}, z_x, z_y)|\phi(0, z_x, z_y) - \bar{\phi}(0, z_x, z_y)| dz_y dz_x.
\]
Using Gronwall inequality, we can obtain

\[ + L_{J_2,1} \int_0^t \int_{\mathbb{R}} \int_{-L}^L \Gamma(t, \tilde{x}, \tilde{y}, z_x, z_y)v_s(0, z_x, z_y)dz_ydz_xds \]

\[ + L_{J_2,2}(L_{I_1,1} + L_{I,2}) \int_0^t \int_{\mathbb{R}} \int_{-L}^L \Gamma(t, \tilde{x}, \tilde{y}, z_x, z_y) \]

\[ \int_{\mathbb{R}} \int_{-L}^L k(z_x, z_y, \xi, \eta)v_s(0, \xi, \eta)d\xi d\eta dz_ydz_xds \]

\[ \leq |\phi - \tilde{\phi}|_{\mathcal{M}(\tilde{x})} + \epsilon + [L_{J_2,1} + L_{J_2,2}(L_{I_1,1} + L_{I,2})] \]

\[ \int_0^t |v_s|_{\mathcal{M}(\tilde{x})} ds + [L_{J_2,1} + L_{J_2,2}(L_{I_1,1} + L_{I,2})] t \epsilon \]

\[ < \epsilon + \epsilon + L_2 \int_0^t |v_s|_{\mathcal{M}(\tilde{x})} ds + L_2 t_0 \epsilon = (2 + L_2 t_0) \epsilon + L_2 \int_0^t |v_s|_{\mathcal{M}(\tilde{x})} ds, \]

where \( L_2 := L_{J_2,1} + L_{J_2,2}(L_{I_1,1} + L_{I,2}) \).

Denote \( \mathcal{L} := \max\{L_1, L_2\} \), then

\[ |u_i(\tilde{t}, \tilde{x}, \tilde{y}; \phi) - u_i(\tilde{t}, \tilde{x}, \tilde{y}; \tilde{\phi})| < (2 + \mathcal{L} t_0) \epsilon + \mathcal{L} \int_0^t |v_s|_{\mathcal{M}(\tilde{x})} ds, \quad i = 1, 2. \]

Therefore,

\[ |v_t|_{\mathcal{M}(\tilde{x})} \leq v(\tilde{t}, \tilde{x}, \tilde{y}) + \epsilon = (2 + \mathcal{L} t_0) \epsilon + \mathcal{L} \int_0^t |v_s|_{\mathcal{M}(\tilde{x})} ds \]

Using Gronwall inequality, we can obtain

\[ |v_t|_{\mathcal{M}(\tilde{x})} \leq (1 + 2\sqrt{2} + \sqrt{2\mathcal{L} t_0}) \epsilon + \sqrt{2\mathcal{L} t_0} \int_0^t |v_s|_{\mathcal{M}(\tilde{x})} ds. \]

Thus, we can conclude that for \( \forall \epsilon > 0, t_0 > 0 \), and compact subset \( K \subseteq [-\tau, 0] \times \mathbb{R} \times [-L, L] \), there exist \( \delta < \epsilon \) and a compact set \( \mathcal{M}(\tilde{x}) \) such that \( K \subseteq \mathcal{M}(\tilde{x}) \) (e.g. let \( M \) large enough.) and

\[ |v_t|_K \leq |v_t|_{\mathcal{M}(\tilde{x})} < (1 + 2\sqrt{2} + \sqrt{2\mathcal{L} t_0}) \epsilon + \sqrt{2\mathcal{L} t_0} \int_0^t |v_s|_{\mathcal{M}(\tilde{x})} ds \]

Thus, \( Q_t(\cdot) \) is uniformly continuous for \( t \) in any bounded interval. That is, \( Q_t(\cdot) \) satisfies (c) in the definition of semiflow with respect to the compact open topology.

5. Asymptotic speed of spread. In this section, we will apply the theorems in Liang & Zhao [9] and Zhao [27] to study the asymptotic speed of spread of system (5), and then state a corresponding result for (2).

Let \( \mathcal{C} = \mathcal{C}(\mathbb{R}^2) \) with the maximum norm \( \| \cdot \|_\mathcal{C} \) such that it is a Banach space. We can also regard every member of \( \mathcal{C} \) as a function in \( \mathcal{C} \).

Define the reflection operator \( \mathcal{R} \) on \( \mathcal{C} \) by \( \mathcal{R}(\phi)(\theta, x, y) := \phi(\theta, -x, y) \) for \( \theta \in [-\tau, 0], (x, y) \in \mathbb{R} \times [-L, L] \) and the translation operator \( \mathcal{T}_z \) by \( \mathcal{T}_z(\phi)(\theta, x, y) := \phi(\theta, x - z, y) \) for \( \theta \in [-\tau, 0], x, z \in \mathbb{R}, y \in [-L, L] \). A set \( D \subseteq \mathcal{C} \) is said to be \( \mathcal{T} \)-invariant if \( \mathcal{T}_z D = D \) for any \( z \in \mathbb{R} \). Let \( Q : \mathcal{C}_\beta \to \mathcal{C}_\beta \) be a map, where \( \beta \in \mathcal{C} \) and \( \beta \gg 0 \). Now we introduce the following assumptions on \( Q \).
(A1) \( Q[R[u]] = R[Q[u]], Q[T_z[u]] = T_z Q[u] \) for any \( z \in \mathbb{R} \).

(A2) \( Q : \mathcal{C}_\beta \to \mathcal{C}_\beta \) is continuous with respect to the compact open topology.

(A3) \( Q : \mathcal{C}_\beta \to \mathcal{C}_\beta \) is monotone(order-preserving) in the sense that \( Q[u] \geq Q[v] \) whenever \( u \geq v \) in \( \mathcal{C}_\beta \).

Note that from (A3) we have \( Q[\mathcal{C}_\beta] \subset \mathcal{C}_\beta \).

(A4) \( Q : \mathcal{C}_\beta \to \mathcal{C}_\beta \) admits exactly two fixed points 0 and \( \beta \), and for any positive number \( \epsilon \), there is \( \alpha \in \mathcal{C}_\beta \) with \( ||\alpha|| < \epsilon \) such that \( Q[\alpha] \gg \alpha \).

(A5) There is a number \( l \in [0, 1) \) such that for any \( A \subset \mathcal{C}_\beta \) and \( x \in \mathbb{R} \), \( \alpha((Q[u](\cdot, x, \cdot) : u \in A)) \leq l\alpha(\{u(\cdot, x, \cdot) : u \in A\}) \), where \( \alpha \) is the Kuratowski measure of non-compactness on the Banach space \( C \).

For a given \( \phi \in \mathcal{C}_\beta \) and a compact interval \( I = [a, b] \subset \mathbb{R} \), define a function \( \phi_I \in C([-\tau, 0] \times I \times [-L, L], \mathbb{R}^2) \) by \( \phi_I(\theta, x, y) = \phi(\theta, x, y) |_{x \in I} \). Moreover, for any subset \( D \) of \( \mathcal{C}_\beta \), define \( D_I := \{ \phi_I \in C([-\tau, 0] \times I \times [-L, L], \mathbb{R}^2) : \phi \in D \} \).

(A6) For any number \( \delta > 0 \), there exists \( l = l(\delta) \in [0, 1] \) such that for any \( D \subset \mathcal{C}_\beta \) and any interval \( I = [a, b] \) of the length \( \delta \), we have \( \alpha(Q[D_I]) \leq l\alpha(D_I) \), where \( \alpha \) is the Kuratowski measure of non-compactness on the Banach space \( C([-\tau, 0] \times I \times [-L, L], \mathbb{R}^2) \).

**Lemma 5.1.** \( Q_t \) satisfies (A1)-(A4) with \( \beta = \mathcal{E}_+ \) for \( t > 0 \).

**Proof.** It is easy to see that \( Q_t \) satisfies (A1) - (A3) with \( \beta = \mathcal{E}_+ \) for \( t > 0 \). Let \( \tilde{Q}_t := Q_{t|\mathcal{E}} \) be the restriction of \( Q_t \) to \( \mathcal{E} \), then \( \tilde{Q}_t \) is a semiflow generated from the initial boundary value problem:

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= -d_mu_1 + \alpha e^{-d_1 t} \int_{-L}^L k_2(y, z_y) \frac{u_1(t-\tau, z_y)}{u_1(t-\tau, z_y) + \omega}[u_2(t-\tau, z_y) + 1]dz_y, \\
\frac{\partial u_2}{\partial t} &= D \frac{\partial^2 u_2}{\partial y^2} - u_2(1 + u_2) + d \int_{-L}^L k_2(y, z_y) \frac{u_1(t, z_y)}{u_1(t, z_y) + \omega}u_2(t, z_y)dz_y, \\
\frac{\partial u_2}{\partial y} &= 0, \\
u_1(\theta, y) &= u_{1,0}(\theta, y), \quad (\theta, y) \in [-\tau, 0] \times (-L, L), \\
u_2(\theta, y) &= u_{2,0}(\theta, y), \quad (\theta, y) \in [-\tau, 0] \times (-L, L).
\end{align*}
\]

Note that the above problem has the same equilibria as (5) and \( \mathcal{E}_0 \) is unstable.

For \( \tilde{Q}_t \) is a strongly monotone semiflow in \( [0, \mathcal{E}_+] \), appealing to Dancer-Hess connecting orbit lemma (see Chapter 2 of Zhao [28]), the semiflow \( \tilde{Q}_t \) admits a strongly monotone full orbit connecting \( \mathcal{E}_0 \) and \( \mathcal{E}_+ \). Thus, (A4) holds for \( t > 0 \). \( \square \)

**Lemma 5.2.** \( Q_t \) satisfies (A5) with \( \beta = \mathcal{E}_+ \) for \( t > \tau \).

**Proof.** We note that (5) has another integral form

\[
\begin{align*}
u(t, x, y) &= \tilde{T}(t) \phi(0, \cdot, \cdot)(x, y) + \int_0^t \tilde{T}(t-s) f(u_s)(x, y) ds, \quad t \geq 0, \\
u(\theta, x, y) &= \phi(\theta, x, y), \quad \theta \in (0, \tau], (x, y) \in \mathbb{R} \times (-L, L),
\end{align*}
\]

where \( \tilde{T}(t) = (T_1(t), T_2(t))^T \) defined on \( \mathcal{X} \) with

\[
\tilde{T}(t) \phi := \begin{pmatrix} T_1(t) \phi_1 \\ T_2(t) \phi_2 \end{pmatrix} = \begin{pmatrix} e^{-d_{1,1} t} \phi_1 \\ e^{-d_{2,2} t} \phi_2 \end{pmatrix}, \quad \forall \phi = (\phi_1, \phi_2) \in \mathcal{X},
\]
and \( \tilde{f} = (\tilde{f}_1, f_2) : C_+ \to \mathcal{X} \). Here \( f_2 \) is defined in Section 3 and \( \tilde{f}_1 \) is defined as

\[
\tilde{f}_1(\phi)(x, y) := \alpha e^{-d_1\tau} \int_{\mathbf{R}} \int_L^{-L} k(x, y, z_x, z_y) \frac{\phi_1(-\tau, z_x, z_y)}{\phi_1(-\tau, z_x, z_y) + \omega} \left[ \phi_2(-\tau, z_x, z_y) + 1 \right] dz_y dz_x.
\]

Indeed, \( \tilde{T}(t) : \mathcal{X} \to \mathcal{X} \) is a \( C_0 \)-semigroup with \( \tilde{T}(t)\mathcal{X}^+ \subseteq \mathcal{X}^+ \) for any \( t \geq 0 \).

We define a linear operator \( L(t)[\phi] := (\tilde{T}_1(t)\phi_1(0)(\cdot, \cdot), 0) \) for all \( \phi \in C_+ \) and a nonlinear map

\[
S(t)[\phi] := (\int_0^t \tilde{T}_1(t-s)\tilde{f}_1(u_s)(\cdot, \cdot)ds, \tilde{u}_2(t, \cdot; \phi)), \quad \forall \phi \in C_{\mathcal{E}_+}.
\]

Note that \( Q_t = L(t) + S(t) \) for \( t \geq 0 \) and for each \( \phi \in C_{\mathcal{E}_+} \), we have

\[
\|L(t)[\phi]\| \leq e^{-d_1t}\|\phi\|, \quad t > 0,
\]

then it follows that \( \|L(t)\| \leq e^{-d_1t} \) for all \( t > 0 \). For the expression of (11) and the compactness of \( T_2(t) \), we have that \( S(t) \) is compact for each \( t > \tau \). Therefore, for any subset \( A \subseteq C_{\mathcal{E}_+} \) and \( x \in \mathbb{R} \), it follows that

\[
\alpha((Q_t[A])(\cdot, \cdot, \cdot)) \leq \alpha((L(t)[A])(\cdot, \cdot, \cdot)) + \alpha((S(t)[A])(\cdot, \cdot, \cdot)) \leq e^{-d_1t}\alpha([A](\cdot, \cdot, \cdot)).
\]

Thus, \( Q_t \) satisfies (A5) for \( t > \tau \).

**Remark 2.** We have from Lemma 5.1 that \( Q_1 \) satisfies (A5) if \( \tau \in (0, 1) \).

Moreover, for any given \( \tau > 0 \), there exists a constant \( v \in (0, 1) \) such that \( \bar{v} := \tau v < 1 \). Now we non-dimensionalize system (5) with the following scaling:

\[
\bar{u}_1(\bar{t}, x, y) = u_1(\frac{\bar{t}}{v}, x, y), \quad \bar{u}_2(\bar{t}, x, y) = u_2(\frac{\bar{t}}{v}, x, y), \quad \bar{t} = tv, \quad \bar{\tau} = \tau v,
\]

then (5) turns into:

\[
\begin{align*}
\frac{\partial \bar{u}_1}{\partial \bar{t}} &= -\bar{d}_m \bar{u}_1 + \bar{\alpha} e^{-d_1\bar{t}} \int_R \int_L^{L} k(x, y, z_x, z_y) \frac{\bar{u}_1(\bar{t} - \bar{\tau}, z_x, z_y)}{\bar{u}_1(\bar{t} - \bar{\tau}, z_x, z_y) + \bar{\omega}} [\bar{u}_2(\bar{t} - \bar{\tau}, z_x, z_y) + 1] dz_y dz_x, \\
&\quad \bar{t} > 0, (x, y) \in \mathbb{R} \times [-L, L], \\
\frac{\partial \bar{u}_2}{\partial \bar{t}} &= \bar{D} \left( \frac{\partial^2 \bar{u}_2}{\partial x^2} + \frac{\partial^2 \bar{u}_2}{\partial y^2} \right) - \frac{1}{\bar{v}} \bar{u}_2(\bar{t} + 1) + \bar{d} \int_R \int_L^{L} k(x, y, z_x, z_y) \frac{\bar{u}_1(\bar{t}, z_x, z_y)}{\bar{u}_1(\bar{t}, z_x, z_y) + \bar{\omega}} [\bar{u}_2(\bar{t}, z_x, z_y) + 1] dz_y dz_x, \\
&\quad \bar{t} > 0, (x, y) \in \mathbb{R} \times (-L, L), \\
\frac{\partial \bar{u}_2}{\partial \bar{y}} &= 0,
\end{align*}
\]

Note

\[
d_m < \bar{d}_m = \frac{\alpha e^{-d_1\tau}}{\bar{\omega}} \iff \bar{d}_m < \frac{\alpha e^{-d_1\tau}}{\bar{\omega}},
\]

and \( \mathcal{E}_+ \) is still the positive equilibrium of (12). Similarly, denote \( \bar{u}(\bar{t}, x, y; \phi) \) as the mild solution of (12) with initial condition \( \bar{u}_0 = \phi \in C_{\mathcal{E}_+}^\mathbb{R} := \{BC([-\bar{\tau}, 0] \times \mathbb{R} \times \mathbb{R})\}

Let \( \tau > 0 \). Theorem 5.4. Asymptotic patterns of the mistletoe-bird system (2). Let \( (12) \) be expressed as
\[
\begin{align*}
&\tau > 0, \\
&\text{(12) is expressed as } \quad Q_t \in \mathbb{R}^2 : \phi(s) \in X_{\mathcal{E}_+} \text{ for } s \in [-\bar{\tau}, 0]. 
\end{align*}
\]
The solution operator generalized from (12) is expressed as \( \{Q_t\}_{t \geq 0} \) with
\[
Q_t(\phi)(s, x, y) = \tilde{u}_t(s, x, y; \phi) = \tilde{u}(t + s, x, y; \phi).
\]
It follows that \( \{Q_t\}_{t \geq 0} : C^0_{\mathcal{E}_+} \to C^0_{\mathcal{E}_+} \) is a monotone semiflow, and have property (A1)-(A4). Similar arguments leads to that \( Q_t \) satisfies (A5) for \( \bar{t} > \bar{\tau} \), and so is \( \tilde{Q}_t \).

Therefore, from [9, Theorem 2.17] and [27, Theorem 2.1], \( Q_t \) has an asymptotic speed of spread \( c^* > 0 \) and we have the following conclusions for (5).

**Theorem 5.3.** Assume (K1), (K2), (D) hold and \( \tau \geq 0 \), then the following statements for (5) are valid.

1. If \( \phi \in \mathcal{E}_+ \) with \( 0 \leq \phi \ll \mathcal{E}_+ \), and \( \phi(\cdot, x, y) = 0 \) for \( |x| > N \) (\( N > 0 \) is some constant), \( y \in [-L, L] \), then
\[
\lim_{t \to \infty, |x| \geq ct} u(t, x, y; \phi) = 0 \quad \text{for } c > c^*
\]
uniformly for \( y \in [-L, L] \);

2. For any \( \varpi \in \mathcal{C} \), with \( \varpi(\theta, y) \gg 0 \) for \( \theta \in [-\tau, 0], y \in [-L, L] \), there is a positive number \( r_{\varpi} \) such that if \( \phi \in \mathcal{E}_+ \) with \( \phi(\cdot, x, y) \gg \varpi(\cdot, y) \) for \( x \) on an interval of length \( 2r_{\varpi} \) and \( y \in [-L, L] \), then
\[
\lim_{t \to \infty, |x| \leq ct} u(t, x, y; \phi) = \mathcal{E}_+ \quad \text{for } 0 < c < c^*,
\]
uniformly for \( y \in [-L, L] \).

**Proof.** If \( \tau \in (0, 1) \), then the conclusion is obtained directly from our Lemma 5.1, Remark 2, [9, Theorem 2.17] and [27, Theorem 2.1]. If \( \tau \geq 1 \), then we obtain same results for system (12) with \( \bar{\tau} < 1 \). Let the spreading speed for system (12) is expressed as \( c^* \), and \( c^* = \bar{v}c^* \), where \( v \) is the constant in \( \bar{\tau} = v\tau \). Then we can obtain conclusion (1) and (2) in this theorem from
\[
\lim_{t \to \infty, |x| \geq ct} \tilde{u}(t, x, y; \phi) = 0 \quad \text{for } c > \bar{c}^*,
\]
and
\[
\lim_{t \to \infty, |x| \leq ct} \tilde{u}(t, x, y; \phi) = \mathcal{E}_+ \quad \text{for } 0 < c < \bar{c}^*.
\]

\( \blacksquare \)

Now, summarizing the above results for (5), we obtain our main result for asymptotic patterns of the mistletoe-bird system (2).

**Theorem 5.4.** Let \( \tau \geq 0 \), and (K1), (K2), (D) are satisfied. Assume that \( u_2^{(0)}(\theta, x, y) \) satisfies the homogeneous Neumann condition, then the following statements for the model (2) are valid.

1. If \( 0 \leq u_1^{(0)}(\theta, x, y) < u_1^* \), \( 1 \leq u_2^{(0)}(\theta, x, y) < u_2^* \) for \( \theta \in [-\tau, 0], (x, y) \in \mathbb{R} \times [-L, L] \), and \( u_1^{(0)}(\cdot, x, y) = 0, u_2^{(0)}(\cdot, x, y) = 1 \) for \( |x| > N \) (\( N > 0 \) is some constant), \( y \in [-L, L] \), then
\[
\lim_{t \to \infty, |x| \geq ct} (u_1(t, x, y; \phi), u_2(t, x, y; \phi)) = (0, 1) \quad \text{for } c > c^*
\]
uniformly for \( y \in [-L, L] \);
(2) For any \( \varphi \in \tilde{C}_{\mathcal{E}} \) with \( \varphi(\theta, y) = (\varphi_1(\theta, y), \varphi_2(\theta, y)) \gg 0 \) for \( \theta \in [-\tau, 0], y \in [-L, L] \), there is a positive number \( r_\varphi \) such that if \( \varphi_1(t, y) < u_1^{(0)}(\cdot, x, y) \leq u_1^+ \), \( 1 + \varphi_2(t, y) < u_2^{(0)}(\theta, x, y) \leq u_2^+ \) for \( x \) on an interval of length \( 2r_\varphi \) and \( y \in [-L, L] \), then

\[
\lim_{t \to \infty, |x| \leq ct}(u_1(t, x, y; \phi), u_2(t, x, y; \phi)) = (u_1^+, u_2^+) \quad \text{for} \quad 0 < c < c^*,
\]

uniformly for \( y \in [-L, L] \).

6. Estimation of spreading speed \( c^* \). Consider the following linear boundary value system (noting that \( \tilde{u}_2^++1 = u_2^+ > 1 \)):

\[
\begin{align*}
\partial u_1 \partial t &= -d_m u_1 + \frac{\alpha (\tilde{u}_2^+ + 1)e^{-d_y t}}{\omega} \int_{R}^{L} k(x, y, z_x, z_y) u_1(t - \tau, z_x, z_y) dz_y dz_x, \\
\partial u_2 \partial t &= D((\partial^2 u_2 \partial x^2 + \partial^2 u_2 \partial y^2) - u_2 \bigg) + \frac{d(\tilde{u}_2^+ + 1)}{\omega} \int_{R}^{L} k(x, y, z_x, z_y) u_1(t, z_x, z_y) dz_y dz_x, \\
\partial u_2 \partial y &= 0,
\end{align*}
\]

(13)

Denote the solution map of (13) as \( \tilde{M}_t \). Then \( \tilde{M}_t \) satisfies the properties (C1)-(C5) in [9] and we have

\[ Q_\tau[u] \leq \tilde{M}_t[u] \quad \text{for all} \quad u \in C_{\mathcal{E}}, t \geq 0. \]

Let \( u(t, x, y) = w(t, y)e^{-c_\tau x} \), then we obtain

\[
\begin{align*}
\partial w_1 \partial t &= -d_m w_1 + \frac{\alpha (\tilde{u}_2^+ + 1)e^{-d_y t}}{\omega} K_1(\varsigma) \int_{-L}^{L} k_2(y, z_y) w_1(t - \tau, z_y) dz_y, \\
\partial w_2 \partial t &= D(\partial^2 w_2 \partial y^2) + (Dc^2 - 1)w_2 + \frac{d(\tilde{u}_2^+ + 1)}{\omega} K_1(\varsigma) \int_{-L}^{L} k_2(y, z_y) w_1(t, z_y) dz_y, \\
\partial w_2 \partial y &= 0,
\end{align*}
\]

(14)

where \( K_1(\varsigma) := \int_R k_1(\varsigma)e^{c_\varsigma d_\varsigma} d\varsigma \). Furthermore, we claim that

\[ K_1(\varsigma) = \int_0^\infty k_1(\xi)[e^{c_\xi} + e^{-c_\xi}] d\xi \geq 1 \quad \text{for any} \quad \varsigma \geq 0. \]

Note that the integral operator \( K_2[u](y) := \int_{-L}^{L} k_2(y, z_y) u(z_y) dz_y \) defined on \( C([-L, L]) \) is compact and strongly positive, i.e. for each \( u \in C([-L, L]) \) and \( u > 0, K_2[u] \gg 0 \). Thus, it follows that \( K_2 \) possesses a sequence of eigenvalues \( \ell_j \) such that \( \ell_j \in \mathbb{R} \),

\[ 0 \leq \cdots \leq |\ell_2| \leq |\ell_1| \leq |\ell_0|, \]

(15)

and the only possible limit point of \( \{\ell_j\} \) is zero (see [2, 15]). What’s more, as \( K_2 \) is strongly positive, from Krein-Rutman theorem (see [11]), the principal eigenvalue \( \ell_0 \) of \( K_2 \) is real satisfying \( \ell_0 > 0 \) and corresponding to a positive eigenfunction \( \Phi_0 = \Phi_0(y) > 0 \). In fact, we can obtain directly from the definition of \( K_2 \) that
\[ \ell_0 = 1 \text{ and } \Phi_0(y) \equiv 1. \] Here, we denote \( \Phi_j(y) \) as the corresponding eigenfunction of eigenvalue \( \ell_j \) for \( j \geq 0. \)

Substitute \( w(t, y) = e^{\ell_j(y)}(\Phi_j(y), \Psi_n(y))^T \) with \( j, n = 0, 1, \ldots \) into (14), where \( \Psi_n \) is defined in Appendix. Recall that \( K_2[\Phi_j(y) = \ell_j(\Phi_j(y)) \) and \( \Psi_n''(y) = -\mu_n \Psi_n(y) \), then we have

\[
\begin{align*}
\lambda \Phi_j(y) &= -d_m \Phi_j(y) + \frac{\alpha(\bar{u}_2^+ + 1)e^{-d_j \tau}}{\omega} K_1(\zeta) e^{-\lambda \tau} \ell_j(y), \quad y \in [-L, L], \\
\lambda \Psi_n(y) &= (D\zeta^2 - D\mu_n - 1) \Psi_n(y) + \frac{d(\bar{u}_2^+ + 1)}{\omega} K_1(\zeta) \ell_j(y), \quad y \in (-L, L), \\
\Psi_n''(y) &= 0, \quad y \in \{-L, L\},
\end{align*}
\]

for any \( \lambda \). Noting \( \Psi_n(0) = 1, \) we claim the following conclusion (C).

(i) For \( n = 0, 1, 2, \ldots \), every \( \lambda_2^{(n)}(\zeta) \) is a real and simple root of the second equation in (16), if \( \zeta \geq 0 \). Furthermore, \( \lambda_2^{(n)}(\zeta) \) is nonincreasing on \( \zeta \) for \( \zeta \geq 0 \), namely, \( \lambda_2^{(n)}(\zeta) \leq \lambda_2^{(n-1)}(\zeta) \) for \( n \geq 1 \).

(ii) \( \lambda_1^{(j)}(\zeta) \) is a real and simple root of \( \lambda = -d_m + \frac{\alpha(\bar{u}_2^+ + 1)e^{-d_j \tau}}{\omega} K_1(\zeta) e^{-\lambda \tau} \) for \( j = 0, 1, 2, \ldots, \) if \( \zeta \geq 0 \). Furthermore, for any \( j \geq 1 \), we always have \( \lambda_1^{(j)}(\zeta) \leq \max\{0, \lambda_1^{(0)}(\zeta)\} \).

(iii) Let \( \lambda(\zeta) = \max_{n \geq 0, j \geq 0} \{|\lambda_1^{(j)}(\zeta), \lambda_2^{(n)}(\zeta)|\} = \max\{|\lambda_1^{(0)}(\zeta), \lambda_2^{(0)}(\zeta)|\} \), then \( \lambda(\zeta) > 0 \) for \( \zeta \geq 0 \).

Note that \( \mu_n = (\frac{\mu_n}{2})^2 \) increases in \( n \) (see Appendix). Then

\[
\lambda_2^{(n)}(\zeta) = D\zeta^2 - D\mu_n - 1 \leq \lambda_2^{(n-1)}(\zeta) = D\zeta^2 - D\mu_{n-1} - 1 \leq D\zeta^2 - D\mu_0 - 1 = D\zeta^2 - 1 = \lambda_2^{(0)}(\zeta),
\]

for any \( n = 0, 1, 2, \ldots \) and \( \zeta \geq 0 \). Thus, (i) holds.
Let $j = 0$, for any $\varsigma \geq 0$, we know that
\[
\Lambda = -d_m + \frac{\alpha(\bar{u}_2^+ + 1)e^{-d_1\tau}}{\omega} K_1(\varsigma)\ell_0 e^{-\Lambda \tau}
\quad (17)
\]
has a real zero $\Lambda_1(\varsigma)$ and complex conjugate pair of zeros $\lambda_2(\varsigma)$, $\overline{\lambda}_2(\varsigma)$, $\lambda_3(\varsigma)$, $\overline{\lambda}_3(\varsigma)$, \ldots such that $\Lambda_1(\varsigma) > \text{Re}\lambda_2(\varsigma) > \text{Re}\lambda_3(\varsigma) > \cdots$ (noting $\ell_0 = 1$). In fact, let $E(\Lambda) := \Lambda + d_m - \frac{\alpha(\bar{u}_2^+ + 1)e^{-d_1\tau}}{\omega} K_1(\varsigma)e^{-\Lambda \tau}$, then we have
\[
E(-\infty) = -\infty, \quad E(0) = d_m - \frac{\alpha(\bar{u}_2^+ + 1)e^{-d_1\tau}}{\omega} K_1(\varsigma), \quad E(\infty) = \infty,
\quad \frac{dE}{d\Lambda} = 1 + \frac{\tau \alpha(\bar{u}_2^+ + 1)e^{-d_1\tau}}{\omega} K_1(\varsigma)e^{-\Lambda \tau} > 0 \quad \text{for any } \Lambda \in \mathbb{R}.
\]
That is, $E(\Lambda) = 0$ has a unique real root $\Lambda_1(\varsigma)$ which is simple. Furthermore, substituting $\Lambda = p + iq$ into $E(\Lambda) = 0$ and separating the real part and image part, one can see that $p = p(\varsigma) < \Lambda_1(\varsigma)$. Therefore, $\lambda_1^{(0)}(\varsigma) = \Lambda_1(\varsigma)$. The argument for the conclusion of $j \geq 1$ is similar, we omit the details. Moreover, from the first equation of (16), we have that
\[
|\lambda_1^{(j)}(\varsigma) + d_m|e^{\lambda_1^{(j)}(\varsigma)\tau} = \tilde{d}_m K_1(\varsigma)\ell_j.
\]
Thus, if $\ell_j < 0$, then $\lambda_1^{(j)}(\varsigma) \leq 0$; if $\ell_j > 0$, then we have from $\ell_j \leq \ell_0$ that
\[
|\lambda_1^{(j)}(\varsigma) + d_m|e^{\lambda_1^{(j)}(\varsigma)\tau} = \tilde{d}_m K_1(\varsigma)\ell_j \leq \tilde{d}_m K_1(\varsigma)\ell_0 = |\lambda_1^{(0)}(\varsigma) + d_m|e^{\lambda_1^{(0)}(\varsigma)\tau}.
\]
It follows from the increasing property of $A(x) = (x + d_m)e^{x\tau}$ for $x \in \mathbb{R}$ that $\lambda_1^{(j)}(\varsigma) \leq \lambda_1^{(0)}(\varsigma)$. Thus, we obtain conclusion (ii) of (C).

Note that (17) can be rewritten as $(\xi + A)e^{\xi} + B = 0$, where
\[
\xi = \lambda \tau, \quad A = \tau d_m, \quad B = -\frac{\alpha(\bar{u}_2^+ + 1)e^{-d_1\tau}}{\omega} K_1(\varsigma)\ell_0 = -\tau \tilde{d}_m (\bar{u}_2^+ + 1) K_1(\varsigma).
\quad (18)
\]
We have from the assumption (D) and the fact: $K_1(\varsigma) \geq 1$, that $A + B < 0$. By using Lemma 6.1 that there exists a root $\hat{\lambda}$ such that $\text{Re}\hat{\lambda} \geq 0$. Since $\lambda_1^{(0)}(\varsigma)$ is the principal eigenvalue which is real and simple, we have $\lambda_1^{(0)}(\varsigma) \geq 0$. Moreover, $\lambda = 0$ is not the root of $(\xi + A)e^{\xi} + B = 0$, therefore $\lambda_1^{(0)}(\varsigma) > 0$ for $\varsigma \geq 0$. The definition of $\hat{\lambda}(\varsigma)$ leads to $\hat{\lambda}(\varsigma) > 0$ for $\varsigma \geq 0$. The conclusion (iii) is true.

Let $\tilde{B}_t^x$ be the solution operator associated with (14), then it follows that
\[
\tilde{B}_t^x[\varphi](\theta, y) = \tilde{M}_t[\varphi e^{-c_t\tau}](\theta, 0, y) = e^{\lambda(t)\tau} \varphi(\theta, y), \quad t > 0, \quad \varphi \in \tilde{C}_e^x,
\]
where $e^{\lambda(t)\tau}$ is any eigenvalue of $\tilde{B}_t^x$ and $\lambda(\varsigma)$ is the eigenvalue of the infinitesimal generator of $\tilde{B}_t^x$. In fact, $\lambda(\varsigma)$ takes any root of (16).

Define $\tilde{\Phi}(\varsigma) := \frac{\hat{\lambda}(\varsigma)}{\varsigma}$, then $\tilde{\Phi}(\varsigma) > 0$ for $\varsigma \geq 0$. By [9, Lemma 3.8], we then have the following lemma.

Lemma 6.2. The following statements are valid:

(i) $\tilde{\Phi}(\varsigma) \to \infty$ as $\varsigma \downarrow 0$;
(ii) $\tilde{\Phi}(\varsigma)$ is decreasing near $0$;
(iii) $\tilde{\Phi}(\varsigma)$ changes sign at most once on $(0, \infty)$;
(iv) $\lim_{\varsigma \to \infty} \tilde{\Phi}(\varsigma)$ exists, where the limits may be infinite.
Let $t = 1$, $\zeta_0$ be the principal eigenvector of $\tilde{B}_0 := \tilde{B}_0^1$, $\mathcal{M}[u] := \min\{\zeta_0, \tilde{M}_1[u]\}$. Choose $\zeta_0$ such that $\zeta_0 \gg \varepsilon_1$. Note that $\zeta_0$ is the fixed point of $\mathcal{M}$ (see [9, Lemma 3.3]). Then we have from [9, Proposition 3.9, Theorem 3.10] that $\bar{c}^\ast = \inf_{\zeta_0} \Phi(\zeta)$ is the spreading speed of $\mathcal{M}$ (also called the spreading speed of the linear system (13)) and $c^\ast \leq \bar{c}^\ast$.

For any $\epsilon > 0$, now we consider another linear system:

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= -d_m u_1 + \frac{\alpha \epsilon^{-d_\tau}}{\epsilon + \omega} \int_R \int_{-L}^L k(x, y, z_x, z_y) u_1(t - \tau, z_x, z_y) dz_y dz_x, \\
\frac{\partial u_2}{\partial t} &= D_1 \left( \frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} \right) - (\epsilon + 1) u_2 \\
&\quad + \frac{d}{\epsilon + \omega} \int_R \int_{-L}^L k(x, y, z_x, z_y) u_1(t, z_x, z_y) dz_y dz_x, \\
\frac{\partial u_2}{\partial y} &= 0,
\end{align*}
\]  

with $t > 0, (x, y) \in \mathbb{R} \times [-L, L]$, $t \geq 0, (x, y) \in \mathbb{R} \times \{-L, L\}$.

Denote $M_1^\ast$ be the solution map of (19). Then we have

$Q_0[u] \geq M_1^\ast[u]$ for all $u \in \mathcal{C}_0, t \geq 0$.

Let $u(t, x, y) = w(t, y) = e^{\lambda t}(\Phi_j(y), \Psi_n(y))^T$ with $j, n = 0, 1, \ldots$. Then we have

\[
\begin{align*}
\lambda \Phi_j(y) &= -d_m \Phi_j(y) + \frac{\alpha \epsilon^{-d_\tau}}{\omega + \epsilon} K_1(\zeta)e^{-\lambda \tau} \ell_j \Phi_j(y), \quad y \in [-L, L], \\
\lambda \Psi_n(y) &= (D_1 \zeta^2 - D_\mu_n - 1 - \epsilon) \Psi_n(y) + \frac{d}{\omega + \epsilon} K_1(\zeta) \ell_j \Phi_j(y), \quad y \in (-L, L), \\
\Psi_n''(y) &= 0, \quad y \in \{-L, L\}.
\end{align*}
\]

Let

\[
\begin{vmatrix}
\lambda + d_m - \frac{\alpha \epsilon^{-d_\tau}}{\omega + \epsilon} K_1(\zeta) e^{-\lambda \tau} \ell_j & 0 \\
- \frac{d}{\omega + \epsilon} K_1(\zeta) \ell_j & \lambda - \zeta^2 + D_\mu_n + 1 + \epsilon
\end{vmatrix} = 0,
\]

which is equivalent to

\[
\begin{align*}
\lambda &= -d_m + \frac{\alpha \epsilon^{-d_\tau}}{\omega + \epsilon} K_1(\zeta)e^{-\lambda \tau} \ell_j, \\
\lambda &= D_1 \zeta^2 - D_\mu_n - 1 - \epsilon, \quad \text{where } \zeta \geq 0, \quad j, n = 0, 1, 2, \ldots.
\end{align*}
\]  

Let $\lambda_{1,\epsilon,n}(\zeta) (j = 0, 1, 2, \ldots)$ denote the principal eigenvalue of the first equation in (20), and the second equation has roots $\lambda_{2,\epsilon,n}(\zeta) = D_1 \zeta^2 - D_\mu_n - 1 - \epsilon$ ($n = 0, 1, 2, \ldots$) in all. Similar to the discussion for the system (16) as above, we have the following conclusion ($C^\epsilon$).

(i) For $n = 0, 1, 2, \ldots$, every $\lambda_{2,\epsilon,n}(\zeta)$ is a real and simple root of the second equation in (20), if $\zeta \geq 0$. Furthermore, $\lambda_{2,\epsilon,n}(\zeta)$ is nonincreasing on $n$ for any $\zeta \geq 0$, namely, $\lambda_{2,\epsilon,n}(\zeta) \leq \lambda_{2,\epsilon,n+1}(\zeta)$ for $n \geq 1$.

(ii) $\lambda_{1,\epsilon,j}(\zeta)$ is a real and simple root of $\lambda = -d_m + \frac{\alpha \epsilon^{-d_\tau}}{\omega + \epsilon} K_1(\zeta) \ell_j e^{-\lambda \tau}$ for $j = 0, 1, 2, \ldots$, if $\zeta \geq 0$. Furthermore, $\lambda_{1,\epsilon,j}(\zeta) \leq \max\{0, \lambda_{1,\epsilon,j-1}(\zeta)\}$ for $j \geq 1$. 


We can only obtain the result between the solution operators of (5) and (13): here \( \bar{\lambda} \) from [9, Proposition 3.9, Theorem 3.10], that \( c \) the result:

\[
\text{Let the solution operator of (21) is (21) the system of } M \text{ such as mentioned in [19]. Therefore, The conclusion in Theorem 6.3 could not lead to the linear determinacy as mentioned in [19].}
\]

\[
\text{(iii) Let } \tilde{\lambda}^\epsilon(\varsigma) = \max\{\lambda_1(\epsilon, i)(\varsigma), \lambda_2(\epsilon, n)(\varsigma)\} = \max\{\lambda_{1,0}(\epsilon, 0)(\varsigma), \lambda_{2,0}(\epsilon, 0)(\varsigma)\}, \text{ then } \tilde{\lambda}^\epsilon(\varsigma) > 0 \text{ for } \varsigma > 0.
\]

Define \( B_{\epsilon}^{st} \) be the solution operator associated with (20), it follows that

\[
B_{\epsilon}^{st}[\varphi](\theta, y) = M_{\epsilon}^{t}[\varphi e^{-ct}] (\theta, 0, y) = e^{\lambda^\epsilon(\varsigma) t} \varphi(\theta, y), \ t > 0, \ \varphi \in \mathcal{C}_{\epsilon+},
\]

with \( e^{\lambda^\epsilon(\varsigma)t} \) is any eigenvalue of \( B_{\epsilon}^{st} \) and \( \lambda^\epsilon(\varsigma) \) is the eigenvalue of the infinitesimal generator of \( B_{\epsilon}^{st} \). In fact, \( \lambda^\epsilon(\varsigma) \) takes any root of (20).

Define \( \Phi^\epsilon(\varsigma) := \frac{\tilde{\lambda}^\epsilon(\varsigma)}{\varsigma} \), then \( \Phi^\epsilon(\varsigma) \geq 0 \) for \( \varsigma \geq 0 \). Let \( t = 1, \varsigma_0^\epsilon \) be the principal eigenvector of \( B_0^\epsilon := B_0^\epsilon \), \( M^\epsilon[u] := \min\{\varsigma_0^\epsilon, M_{t}[u]\} \). Note that \( \varsigma_0^\epsilon \) is the fixed point of \( M^\epsilon \) (see [9, Lemma 3.3]). Choose \( \varsigma_0^\epsilon \) such that \( \varsigma_0^\epsilon \ll \min\{\epsilon, \mathcal{E}_+\} \). Then we have from [9, Proposition 3.9, Theorem 3.10], that \( \varsigma^\epsilon = \inf_{\varsigma > 0} \Phi^\epsilon(\varsigma) \) is the spreading speed of \( M^\epsilon \), and \( c^\epsilon \geq \varsigma^\epsilon \). Let \( \epsilon \to 0 \), then we obtain

\[
c^\epsilon \geq c^\epsilon = \inf_{\varsigma > 0} \Phi^0(\varsigma).
\]

Let \( \epsilon \to 0 \) in (19), we obtain the linearized system of (5) at \( \mathcal{E}_0^\epsilon \):

\[
\begin{cases}
\frac{\partial u_1}{\partial t} = -d_m u_1 + \frac{\alpha e^{-ct} \tau}{\omega} \int_{-L}^{L} k(x, y, z_x, z_y) u_1(t - \tau, z_x, z_y) dz_x dz_y, \\
\frac{\partial u_2}{\partial t} = D \left( \frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} \right) - u_2 + \frac{d}{\omega} \int_{-L}^{L} k(x, y, z_x, z_y) u_1(t, z_x, z_y) dz_x dz_y, \\
\frac{\partial u_2}{\partial y} = 0,
\end{cases}
\]

That is, \( c^\epsilon \) is the spreading speed of the system (21).

Summarizing the above discussion, we obtain the estimation of \( c^\epsilon \).

**Theorem 6.3.** The following estimation for the spreading speed \( c^\epsilon \) of system (5) holds.

\[
\varsigma^\epsilon := \inf_{\varsigma > 0} \Phi^0(\varsigma) \leq c^\epsilon \leq \inf_{\varsigma > 0} \Phi(\varsigma) = \bar{c}^\epsilon.
\]

Here \( \bar{c}^\epsilon \) is the spreading speed of the system (13), and \( c^\epsilon \) is the spreading speed of the system (21).

**Remark 3.** Let the solution operator of (21) is \( M_{t} \). We are not able to have the the result:

\[
Q_t[u] \leq M_{t}[u] \text{ for all } u \in \mathcal{C}_{\epsilon+}, t \geq 0.
\]

We can only obtain the result between the solution operators of (5) and (13):

\[
Q_t[u] \leq \bar{M}_{t}[u] \text{ for all } u \in \mathcal{C}_{\epsilon+}, t \geq 0.
\]

Therefore, The conclusion in Theorem 6.3 could not lead to the linear determinacy as mentioned in [19].
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Figure 1. (1) \( \lambda_1^{(0)}(\varsigma) = -\frac{d_m}{\varsigma} + \frac{ae^{-d_1\tau}}{u_2' \varsigma} + K_1(\varsigma) e^{-\lambda_1^{(0)}(\varsigma)\tau} \);
(2) \( \lambda^{(e,0)}(\varsigma) = -\frac{d_m}{\varsigma} + \frac{ae^{-d_1\tau}}{\omega \varsigma} K_1(\varsigma) e^{-\lambda^{(e,0)}(\varsigma)\tau} \);
(3) \( \frac{\varsigma^2 - 1}{\varsigma} \).

7. Numerical simulations and concluding discussions. In the following, we make some numerical simulation. From the conclusion \((C)(iii)\) and \((C')\)(iii), we know that \( \bar{\lambda}(\varsigma) \) and \( \tilde{\lambda}(\varsigma) \) are independing on \( \ell_j \) and \( \mu_n \) for \( j, n = 1, 2, \ldots \). Recall that \( \ell_0 = 1 \) and \( \mu_0 = 0 \). Let \( k_1(x) \) be the Gaussian Function and the parameter values given in Table 1, namely, \( k_1(x) = \frac{1}{\sqrt{4\pi}} e^{-\frac{x^2}{4}} \). Then,

\[
K_1(\varsigma) = c_\varsigma^2, \quad \varsigma \geq 0.
\]

From Table 1, it follows that \( \bar{d}_m = \frac{ae^{-d_1\tau}}{u_2'} = e^{-0.1} \approx 0.9048 > d_m = 0.8 \) and \( u_2' = \bar{u}_2' + 1 \approx 1.5318 \). Thus (D) holds. By Figure 1, it follows that \( \bar{c}^* \approx 0.7572 \) and \( c^* \approx c_\varsigma^* \approx 0.3234 \) (noting that the value of \( \epsilon \) in Table 1 is very small). Thus by Theorem 6.3, we have that \( c_\varsigma \in (0.3234, 0.7572) \). Let

\[
\bar{c}^* = \bar{\Phi}(c^*) = \min_{\varsigma \geq 0} \{ \bar{\Phi}(\varsigma) \}, \quad c_\varsigma^* = \Phi^e(\varsigma^*) = \min_{\varsigma \geq 0} \{ \Phi^e(\varsigma) \}.
\]

We see from Figure 1 that \( \bar{\Phi}(\varsigma) = \frac{\lambda_1^{(0)}(\varsigma)}{\varsigma} \) for \( \varsigma \in (0, \varsigma^*) \) and \( \Phi^e(\varsigma) = \frac{\lambda_e^{(0)}(\varsigma)}{\varsigma} \) for \( \varsigma \in (0, \varsigma^*) \) in this example.

In this article, we commit ourself to obtain the threshold asymptotic patterns related to spreading speed of model (2). We want to mention here that the model (2) is an unusual system, which has several characteristics. Firstly, the system is defined on a strip domain, and it is a partially degenerate system. Secondly, the boundary value of which is defined only for the second species. Thirdly, the model...
is both spatial nonlocal and with time delay. Moreover, as stated in Remark 3, the solution operator of the system could not compare with the solution operator of its linearized system at zero equilibrium. Therefore, the linear determinacy is still an open problem.

We also mention here, that if we take \( k_1(x) \) to be the Dirac-delta function, we believe that the discussions for spreading speed can be proceeded.

8. Appendix. Consider the following initial boundary value problem with \( \psi \in \mathcal{X}_2 \):

\[
\begin{aligned}
\frac{\partial w}{\partial t} &= D \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right), \quad t > 0, (x, y) \in \mathbb{R} \times (-L, L), \\
\frac{\partial w}{\partial y} &= 0, \quad t \geq 0, (x, y) \in \mathbb{R} \times \{-L, L\}, \\
w(0, x, y) &= \psi(x, y), \quad (x, y) \in \mathbb{R} \times (-L, L). \\
\end{aligned}
\tag{22}
\]

Using the separation of variables: \( w(t, x, y) = V(t, x)Y(y) \), and \( V(t, x), Y(y) \neq 0 \), then we have

\[
\frac{\partial V}{\partial t} - D \frac{\partial^2 V}{\partial x^2} = Y''(y) / Y(y), \quad t > 0, (x, y) \in \mathbb{R} \times (-L, L).
\]

For the left-hand side of the above is the function of \((t, x)\) and the right-hand side is the function of \(y\), there must exist a constant \( \mu \) such that

\[
\begin{aligned}
\frac{\partial V}{\partial t} &= D \frac{\partial^2 w}{\partial x^2} - \mu DV(t, x), \quad t > 0, x \in \mathbb{R}, \\
\frac{\partial w}{\partial y} &= 0,
\end{aligned}
\tag{23}
\]

Note that the eigenvalues and their corresponding eigenfunctions of (24) are

\[
\mu_k = \left( \frac{k\pi}{2L} \right)^2, \quad k = 0, 1, \ldots \quad \text{and} \quad \Psi_k(y) = \begin{cases} 
\sin \left( \frac{k\pi y}{2L} \right), & y \in [-L, L], \quad k = 1, 3, \ldots, \\
\cos \left( \frac{k\pi y}{2L} \right), & y \in [-L, L], \quad k = 0, 2, \ldots,
\end{cases}
\]

respectively, which satisfy \( \int_{-L}^{L} \Psi_n(y) \Psi_m(y) dy = 0, n, m = 0, 1, \ldots, n \neq m \).

We have from (23) that

\[
V_k(t, x) = \frac{1}{\sqrt{4D\pi t}} e^{-\mu_kDt} \int_{-L}^{L} e^{-\left( x - \frac{\xi}{2t} \right)^2} V_k(0, \xi) d\xi, \quad t > 0, x \in \mathbb{R}, k = 0, 1, \ldots
\]

Thus, there are a series of solutions of (22):

\[
w_k(t, x, y) = V_k(t, x)Y_k(y) = \frac{1}{\sqrt{4D\pi t}} B_k \Psi_k(y) e^{-\mu_kDt} \int_{-L}^{L} e^{-\left( x - \frac{\xi}{2t} \right)^2} V_k(0, \xi) d\xi,
\]

\( t > 0, (x, y) \in \mathbb{R} \times [-L, L] \), \( B_k \) is any constant, \( k = 0, 1, \ldots \) Therefore, the solution of (22) can be written as

\[
w(t, x, y) = \sum_{k=0}^{\infty} w_k(t, x, y) = \sum_{k=0}^{\infty} \frac{1}{\sqrt{4D\pi t}} B_k \Psi_k(y) e^{-\mu_kDt} \int_{-L}^{L} e^{-\left( x - \frac{\xi}{2t} \right)^2} V_k(0, \xi) d\xi.
\]

We have from \( w(0, x, y) = \psi(x, y) = \sum_{k=1}^{\infty} V_k(0, x)B_k \Psi_k(y) \) that

\[
\int_{-L}^{L} \psi(x, \eta) \Psi_n(\eta) d\eta = \int_{-L}^{L} \sum_{k=0}^{\infty} V_k(0, x)B_k \Psi_k(\eta) \Psi_n(\eta) d\eta = \sum_{k=0}^{\infty} V_k(0, x)B_k \int_{-L}^{L} \Psi_k(\eta) \Psi_n(\eta) d\eta = V_n(0, x)B_n M_n.
\]
where $M_n = \int_{-L}^{L} \Psi_n^2(y)dy$, $n = 0, 1, 2, \ldots$ Thus $B_k V_k(0, x) = \frac{1}{M_k} \int_{-L}^{L} \psi(x, \eta) \Psi_k(\eta) d\eta$ and

$$w(t, x, y) = \int_{-L}^{L} \int_{-L}^{L} \frac{1}{\sqrt{4 Do t}} e^{-\frac{(x-\xi)^2}{4 Do t}} \sum_{k=0}^{\infty} \frac{1}{M_k} \Psi_k(\eta) \Psi_k(y) e^{-\mu_k D t} \psi(\xi, \eta) d\eta d\xi$$

is a classical solution of (22). Therefore, we can conclude that the Green function of (22) is

$$\Gamma(t, x, y, \xi, \eta) = \frac{1}{\sqrt{4 D t}} e^{-\frac{(x-\xi)^2}{4 D t}} \sum_{k=0}^{\infty} \frac{1}{M_k} \Psi_k(\eta) \Psi_k(y) e^{-\mu_k D t}.$$ 

Let $\psi(x, y) \equiv 1$ for $(x, y) \in \mathbb{R} \times [-L, L]$, then we have from (22) that $w(t, x, y; \psi) \equiv 1$ is the solution of (22) with regard to the initial function $\psi$. By the expression of $w(t, x, y)$ above, we see

$$\int_{-L}^{L} \int_{-L}^{L} \Gamma(t, x, y, \xi, \eta) d\eta d\xi = 1, \ t \geq 0, (x, y) \in \mathbb{R} \times [-L, L].$$

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