The Poincaré mass operator in terms of a hyperbolic algebra

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Abstract
The Poincaré mass operator can be represented in terms of a $Cl(3,0)$ Clifford algebra. With this representation the quadratic Dirac equation and the Maxwell equations can be derived from the same mathematical structure.

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1 Introduction

The complex numbers are naturally related to rotations and dilatations in the plane, whereas the so-called hyperbolic numbers can be related to Lorentz transformations and dilatations in the two-dimensional Minkowski space-time. The hyperbolic numbers are also known as perplex, unipodal, duplex or split-complex numbers.

The hyperbolic numbers are the universal one-dimensional Clifford algebra. They have been applied by Reany to 2nd-order linear differential equations. A function theory for hyperbolic numbers has been presented by Motter and Rosa. Extensions to an n-dimensional space have been given by several authors.

It was shown by Hucks that a four-component Dirac spinor is equivalent to a two-component hyperbolic complex spinor, and that the Lorentz group is equivalent to a generalized SU(2). In this work Hucks also found that the operations of C, P, and T on Dirac spinors are closely related to the three types of complex conjugation that exist when both hyperbolic and ordinary imaginary units are present. Xuegang et al. investigated in this context the Dirac wave equation, Clifford algebraic spinors, a hyperbolic Hilbert space, and the hyperbolic spherical harmonics in hyperbolic spherical polar coordinates.

It has been shown by Baylis and Jones that a $Cl(3,0)$ Clifford algebra has enough structure to describe relativity as well as the more usual $Cl(1,3)$ or $Cl(3,1)$. Baylis represents relativistic space-time points as paravectors. He applied these paravectors to Electrodynamics (for another approach to relativity see Hestenes). The paravectors are also used in this work, where
Hyberbolic numbers are taken to explicitly represent the $Cl(3,0)$ Clifford algebra. Therefore, this algebra is denoted here as hyperbolic algebra.

The Poincaré mass operator is arising in a natural way as an operator which is invariant under rotations and translations in four-dimensional space-time. In the following it is shown that the Poincaré mass operator can be represented with the $Cl(3,0)$ hyperbolic algebra and it is then possible to derive the quadratic Dirac equation and the Maxwell equations directly from this operator.

2 Hyperbolic algebra

In this work the hyperbolic numbers are used for the algebraic representation of relativistic vectors. The hyperbolic numbers $z \in \mathbb{H}$ are defined as

$$z = x + jy, \quad x, y \in \mathbb{C},$$

where the hyperbolic unit $j$ has the property

$$j^2 = 1.$$

In addition to the complex conjugation, a hyperbolic conjugation can be defined which changes the sign of the hyperbolic unit

$$\bar{z} = x - jy.$$

The hyperbolic numbers are a commutative extension of the complex numbers to include new roots of the polynomial equation $z^2 - 1 = 0$. In the terminology of Clifford algebras they are represented by $Cl(1,0)$, i.e. they correspond to a one-dimensional Clifford algebra.

Using the hyperbolic unit an algebra for the description of relativistic vectors can be introduced. A contravariant Lorentz vector with the coordinates $p^\mu = (p^0, p^i)$ is represented as

$$P = p^\mu \sigma_\mu.$$

The basis vectors $\sigma_\mu$ include the unity and the elements of the Pauli algebra multiplied by the hyperbolic unit $\sigma_\mu = (1, j\sigma_i)$. Note that this algebra is congruent to the $Cl(3,0)$ paravector algebra of Baylis [14, 15].

3 Poincaré mass operator

With the above vector representation the Poincaré mass operator can be introduced as a product of a momentum vector and its hyperbolic conjugated counterpart

$$M^2 = P \bar{P}.$$

$\bar{P}$ is a realization of the so-called Clifford conjugation, the involution of spatial reversal [15].

The explicit form of the mass operator is obtained by a multiplication of the basis matrices. The mass operator can be separated into a spin dependent and a spin independent contribution

$$P \bar{P} = p_\mu p^\mu - i\sigma_\mu p^\mu p^\nu.$$
where the spin operator is given by
\[
\sigma_{\mu\nu} = \begin{pmatrix}
0 & -i j \sigma_1 & -i j \sigma_2 & -i j \sigma_3 \\
-i j \sigma_1 & 0 & \sigma_3 & -\sigma_2 \\
i j \sigma_2 & -\sigma_3 & 0 & \sigma_1 \\
i j \sigma_3 & \sigma_2 & -\sigma_1 & 0
\end{pmatrix}.
\] (7)

Since the spin operator is anti-symmetric, the last term in Eq. (6) is in this case zero. However, the spin term becomes important when interactions are considered.

A basic fermion equation can be introduced as an eigenvalue equation of the mass operator. With the hyperbolic algebra defined above the equation can be written as
\[
M^2 \psi(x) = m^2 \psi(x),
\] (8)
The wave function \(\psi(x)\) has the general structure
\[
\psi(x) = \varphi(x) + j \chi(x),
\] (9)
where \(\varphi(x)\) and \(\chi(x)\) can be represented as two-component spinor functions [10]. They depend on the four space-time coordinates \(x^\mu\). Equation (8) is a second order differential equation in coordinate space. Therefore, the momentum vectors included in \(M^2\) are replaced by the operators \(p^\mu = i \partial^\mu\).

### 4 Quadratic Dirac equation

The following considerations focus on the description of electrons and photons. Electromagnetic interactions can be introduced with the minimal substitution of the momentum operator. The mass operator of Eq. (5) transforms into
\[
M^2 = (p - eA(x)) (\bar{p} - e\bar{A}(x)),
\] (10)
and is now invariant under local gauge transformations. This mass operator can be inserted into Eq. (8). If Pauli matrices and electromagnetic fields are expressed with the anti-symmetric tensor \(\sigma_{\mu\nu}\) given in Eq. (7) and \(F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu\) one finds
\[
\left( (p - eA)_{\mu} (p - eA)^{\mu} - \frac{e}{2} \sigma_{\mu\nu} F^{\mu\nu} - m^2 \right) \psi(x) = 0.
\] (11)

This is the quadratic Dirac equation. Using the hyperbolic algebra it can be represented as a \(2 \times 2\) matrix equation, whereas conventionally the quadratic Dirac equation is a \(4 \times 4\) matrix equation. One finds in both cases the same two coupled differential equations. In the hyperbolic formalism the terms proportional to the hyperbolic unit include the differential equation of the lower component, the other terms describe the differential equation of the upper component. Conventionally, the coupled differential equations for upper and lower components are separated by the matrix structure.

### 5 Maxwell equations

The Maxwell equations can be derived from an eigenvalue equation of the mass operator, where the mass operator is now acting on a vector field
\[
M^2 A(x) = 0.
\] (12)
The equation can be expressed with the electromagnetic fields according to

\[ P \bar{P} A = -\nabla \cdot E - \partial^\mu C + ij \nabla \cdot B - j(\nabla \times B - \partial^\mu E - \nabla C) - i(\nabla \times E + \partial^\mu B) = 0. \]  

(13)

This expression is obtained if one evaluates \( \bar{P}A(x) \), inserts the usual definitions for the electromagnetic fields, and then multiplies the resulting terms by the operator \( P \). If \( P \bar{P} \) is calculated first, Eq. (12) reduces to the wave operator acting on the vector potential giving zero. Both forms are equivalent in the Lorentz gauge. Note, that the Pauli algebra is implicitly part of the 3-dimensional vectors, e.g. \( E(x) = E^i(x)\sigma_i \).

In Eq. (13) the four homogeneous Maxwell equations are included. The calculation provides two additional terms depending on

\[ C(x) = \partial_\mu A^\mu(x). \]  

(14)

These terms disappear in the Lorentz gauge.

6 Summary

The representation of the Poincaré mass operator in terms of a \( Cl(3,0) \) Clifford algebra can be used to construct equations of motion of relativistic quantum fields. In general the representation has the form

\[ M^2 = P \bar{P}, \]  

(15)

where \( P \) is the \( Cl(3,0) \) momentum vector and \( \bar{P} \) denotes the Clifford conjugation of \( P \). If more complex symmetry transformations like local gauge transformations are included, the mass operator is modified (see Eq. (10)). The mass operator is then a Casimir operator of the extended symmetry group, i.e. it is invariant also under the additional symmetry transformations.

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