KAHLER, POISSON GEOMETRY OF CR LIE GROUPS.

"Las frutas de la honestidad se recogen
en muy poco tempo y duran para siempre."
Colombian Proverb.

Abstract.

A Cauchy Riemann (CR) Lie group is a Lie group \( G \) which Lie algebra \( \mathfrak{g} \) has a vector subspace \( \mathcal{H} \) endowed with an endomorphism \( j \) such that \( j^2 = -\text{Id} \), and for each elements \( x, y \) in \( \mathcal{H} \), we have \( [j(x), j(y)] - [x, y] \) is an element of \( \mathcal{H} \), and \( [j(x), j(y)] - [x, y] = j([j(x), y] + [x, j(y)]) \). In this paper we study the geometry of a CR Lie groups \( G \) when its Lie algebra \( \mathfrak{g} \) is endowed with more geometric structures compatible with \( j \), as kahler, and poisson type structures.

0. Introduction.

Let \((M, j)\) be a complex manifold, and \( H \) an hypersurface of \( M \), for each element \( x \) of \( H \) the tangent space \( TH_x \) of \( H \) at \( x \) is endowed with a maximal complex vector space \( E_x = TH_x \cap j(TH_x) \). The collection of vector spaces \( E_x \) defines a vector bundle \( E \) such that for sections \( X \) and \( Y \) of \( E \), we have:

\[
[j(X), j(Y)] - [X, Y] \in E
\]

since \((M, j)\) is a complex structure, we also have:

\[
[j(X), j(Y)] - [X, Y] = j([j(X), Y] + [X, j(Y)]).
\]

More generally, a Cauchy Riemann (CR) structure on a manifold \( M \), is defined by a subbundle \( H \) of \( TM \) endowed with an endomorphism \( j \) such that:

\[
j^2 = -\text{Id}_H
\]

For each sections \( X \) and \( Y \) of \( H \), we have:

\[
[j(X), j(Y)] - [X, Y] \in H
\]

\[
[j(X), j(Y)] - [X, Y] = j([j(X), Y] + [X, j(Y)]),
\]
The notion of CR manifolds is studied by many authors see[3]
In this paper we study Left-invariant CR−structures on Lie groups compatible with geometric properties as poisson, and kahler type properties. More precisely:

**Definition 0.1.**
A kahlerian complex real Lie group $(G,\mathcal{H},j,<,>)$, is a Lie group $G$ endowed with a CR structure defined by the vector subspace $\mathcal{H}$, $j$ is an endomorphism of $\mathcal{G}$ which image is $\mathcal{H}$. We suppose that the following properties are verified:
1. $j$ preserves $\mathcal{H}$, the restriction of $j^2$ to $\mathcal{H}$ is $-Id$.
2. $[X,Y]−[jX,jY] \in \mathcal{H}$ if $X,Y \in \mathcal{H}$
3. $[j(X),j(Y)] = [X,Y] + j([X,j(Y)] + [j(X),Y]) if X,Y \in \mathcal{H}$
Moreover we will suppose that there exists a left-invariant riemannian metric on $G$, defined by a scalar product on $G$ such that $\omega =<,j>$ is closed. This means that $\omega$ is antisymmetric and
$$\omega([x,y],z) + \omega([z,x],y) + \omega([y,z],x) = 0$$
We remark that the restriction of $\omega$ to $\mathcal{H}$ is not degenerated.

We show results related to those structures, for example, we show that a semi-simple kahlerian CR Lie group such that the codimension of $\mathcal{H}$ is 1 is locally isomorphic to $so(3)$ or $sl(2)$.

1. **The structure of kahlerian Lie groups.**

Let $G$ be Lie group. A left symmetric structure on $G$, is defined by a product on its Lie algebra $(\mathcal{G},[.,.])$

$$\mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$$

$$(x,y) \longrightarrow xy$$

which verified

$$xy - yx = [x,y]$$

and

$$x(yz) - (xy)z = y(xz) - (yx)z.$$  

This is equivalent to endows $G$ with a left invariant connection which curvature and torsion forms vanish identically.

In this part we consider a CR kahlerian Lie group $(G,\mathcal{H},j,<,>)$. The Lie algebra $\mathcal{G}$ of $G$, will be call a CR kahlerian Lie algebra.
We will define the following product on $\mathcal{H}$:
For $x, y, z \in \mathcal{H}$, we set

$$\omega(xy,z) = -\omega(y,[x,z]) = \omega([x,z],y) = <[x,z],j(y)>$$
Proposition 1.1.
For every \( x, y, u \) and \( z \) in \( H \) we have

\[
\omega(xy - yx, u) = \omega([x, y], u)
\]

and if the bracket \([x, y]' = xy - yx\) satisfies the Jacobi identity

\[
x(yz) - (xy)z = y(xz) - (yx)z
\]

Proof.
The proof almost copy the one of left-invariant symplectic structures on Lie groups.
Let \( u \) be an element of \( H \), for \( x, y, z \) in \( H \), we have \( \omega(xy - yx, u) = -\omega(y, [x, u]) + \omega(x, [y, u]) \) then (1) follows from the definition of \( \omega \) (the closed property).
Now we prove the second assertion:

\[
\omega(x(yz), u) = -\omega(yz, [x, u]) = \omega(z, [y, [x, u]])
\]

We also have:

\[
\omega((xy)z, u) = -\omega(z, [xy, u]) = \omega(z, [u, xy]) = -\omega(uz, xy)
\]

This implies that

\[
\omega(x(yz) - (xy)z, u) - \omega(y(xz) - (yx)z, u) =
\]

\[
\omega(z, [y, [x, u]] - [x, [y, u]]) + \omega(uz, xy - yx)
\]

The property (1) implies that:

\[
\omega(uz, xy - yx) = \omega(uz, [x, y]') = -\omega(z, [u, [x, y]]')
\]

We deduce that

\[
\omega(x(yz) - (xy)z, u) - \omega(y(xz) - (yx)z, u) = \omega(z, [y, [x, u]]' + [x, [u, y]]' + [u, [y, x]]') = 0
\]

Corollary 1.2.
Let \( G, H, j \) be a Kahlerian cr-algebra, then the product \( (x, y) \to xy - yx = [x, y]' \) defined on \( H \) a structure of a Kahlerian Lie algebra if it satisfies the Jacobi identity.

Proof.
We deduce from the property (2) that the product defined on \( H \), \( (x, y) \to xy \) endows \( H \) with a structure of Left symmetric algebra which underlying Lie algebra is \([,]'\), the morphism \( j \) defines also a complex structure on \( H \), and the scalar product \(<, >\) a Kahlerian structure.
Proposition 1.3.
Consider the vector subspace \( L \) such that for each \( x \in L \), we have \( \omega(x, G) = 0 \). \( L \) is a Lie subalgebra of \( G \) and is the \((<,>)\) orthogonal vector space of \( H \).

Proof.
Let \( x \) and \( y \) two elements of \( L \), for every element \( z \) of \( G \) we have:
\[
\omega([x, y], z) = \omega(y, [z, x]) + \omega(x, [y, z])
\]
since \( x \) and \( y \) are elements of \( L \), we deduce that \( \omega([x, y], z) = 0 \), and that \( [x, y] \in L \).

For \( x \in L \), and \( y \in H \), we have:
\[
\omega(x, y) = [x, j(y)] = 0,
\]
Since the restriction of \( j \) to \( H \) is an automorphism, we deduce that \( L \) and \( H \) are \((<,>)\) orthogonal each other.

Proposition 1.4.
Let \( L \) be the Lie subgroup which Lie algebra is \( L \), and \( M \) the right quotient \( G/L \), then \( M \) is a kahlerian manifold.

Proof.
The fact that \( H \) and \( L \) are orthogonal each other implies that the riemannian metric \((<,>)\) gives rise to a metric \((<,>)'\) of \( M \), the morphism \( j \) also gives rise to a complex structure \( j' \) of \( M \). Denote by \( p \) the projection \( p : G \rightarrow G/L \), we have \( \omega = p^* <, j'> \). This implies that \( \omega' = <, j'>' \) is a symplectic form defined on \( M' \), thus \((M, \omega', j')\) is a kahlerian manifold.

Let \((\mathcal{H}, [., .]', j', <, >')\) be a kahlerian Lie algebra, and \( V \) a vector space. Suppose defined a Lie algebra structure on \( G = H + V \) such that there exists a map:
\[
\alpha : \mathcal{H} \times \mathcal{H} \rightarrow V
\]
such that for \( x, y \in \mathcal{H} \) we have \([x, y] = [x, y]' + \alpha(x, y)\), suppose that \( \alpha(j(x), j(y)) = \alpha(x, y) \).

We suppose also that there exists a scalar product \(<, >\) on \( G \) which extends \((<, >')\) such that \( H \) and \( V \) are orthogonal, we also extend \( j' \) to an endomorphism \( j \) of \( G \) such that \( j(V) = 0 \). We suppose that the form \( \omega = <, j > \) is closed, then \((G, j, <, >)\) is a \( CR \)–kahlerian algebra.

Remark that the fact that \( G \) is a kahlerian \( CR \)–Lie algebra implies the following property:
\[
\oint \alpha([x, y]', z) + [\alpha(x, y), z] = 0
\]
Remark that if \( H \) is a sub Lie algebra of \( G \), then its symplectic structure defined on \( G \) a left invariant Poisson structure.

Examples of kahlerian \( CR \)–structures.
1. Consider the $n-$dimensional commutative Lie algebra $\mathbb{R}^n$ endowed with its flat riemannian metric $\langle , \rangle$, and $V$ an even dimension subspace of $\mathbb{R}^n$ endowed with a linear map $j$ such that $j^2 = -id$, then $(\mathbb{R}^n, V, \langle , \rangle)$ is a Lie kahlerian $CR-$algebra.

2. Consider the semi-simple algebra $so(3)$, and $(e_1, e_2, e_3)$ its basis in which its Lie structure is defined by $[e_1, e_2] = e_3, [e_1, e_3] = -e_2, [e_2, e_3] = e_1$

On $\text{Vect}(e_1, e_2)$ the subspace generated by $e_1$ and $e_2$, we consider the linear map $j$ defined by $j(e_1) = e_2$ and $j(e_2) = -e_1$.

Let $\langle , \rangle$ be the scalar product defined on $so(3)$ by $\langle e_i, e_j \rangle = \delta_{ij}$, The family $(so(3), V, j, \langle , \rangle)$ is a $cr-$kahlerian Lie algebra.

**Proposition 1.6.**

Let $Z(G)$ be the center of $G$, then $U = (Z(G) \cap H) + j(Z(G) \cap H)$ is a Lie commutative algebra and for every element $z$ of $U$, $\text{ad}(z)H \subset H$.

**Proof.**

Let $z$ and $z'$ be elements of $Z(G) \cap H$, we have $[j(z), j(z')] = [z, z'] + j([z, j(z')] + [j(z), z'])$. Since $z$ and $z'$ are in the center of $G$, we deduce that $[j(z), j(z')] = 0$.

The fact that $[z, H] \subset H$ follows from the fact that for $x, y$ in $H$, we have $[x, j(y)] + [j(x), y]$ is an element of $H$.

**Proposition 1.7.**

Let $\mathcal{G}$ be a $CR-$algebra, suppose that there exists an ideal $I$ supplementary to $H$, then $H$ is endowed with a complex Lie structure.

**Proof.**

The projection $p : \mathcal{G} \rightarrow H$ parallel to $I$ defines on $H$ a Lie-complex structure.

Conversely suppose given an extension

$$0 \rightarrow I \rightarrow \mathcal{G} \rightarrow U \rightarrow 0$$

where $U$ is a Lie algebra endowed with a complex structure, then a supplementary space $H$ of $I$ defines a $CR-$structure on $\mathcal{G}$ if and only if for every $x, y$ in $H$, $[jx, jy] - [x, y]$ is an element of $H$ and $[j(x), j(y)] - [x, y] = j([j(x), y] + [x, j(y)])$, where $j$ is the pulls-back of the complex structure of $U$.

**Proposition 1.8.**

Let $(G, H, \langle , \rangle, j)$ be a $CR$ kahlerian Lie group. Suppose that $H^\perp = \ker j$ is an ideal, then $\mathcal{G}$ is not semi-simple.

**Proof.**

We have seen that if $H^\perp$ then $H$ is endowed with a structure of a kahlerian Lie algebra which is known not be semi-simple. Since the quotient of $\mathcal{G}$ by $H^\perp$ is isomorphic as a Lie algebra to $H$, we deduce that $\mathcal{G}$ is not semi-simple.
Proposition 1.9.
Suppose that $\mathcal{H}$ is an ideal, then $G$ is the product of a family of groups $G_i$, where $G_0$ is flat, and for $i \geq 1$, the holonomy of the riemannian structure of $G_i$ is irreductible. Each group $G_i$ is endowed with a Kahlerian $CR$-structure defined by the subspace $\mathcal{H}_i$ of the Lie algebra $\mathfrak{G}_i$ of $G_i$ such that the sum of the dimension of $\mathcal{H}_i$ is $\mathcal{H}$.

Proof.
Suppose that $\mathcal{H}$ is an ideal, then the quotient $L$ of $G$ by $\mathcal{H}$ is a Kahlerian Lie algebra. The theorem (Lichnerowicz Medina [12]), implies that $L = \sum L_i$ where each $L_i$ is a Kahlerian Lie algebra. Now consider the De Rham decomposition of $G$ as a product of groups $G_i$. The projection $G \to L$ respects this decomposition.

Proposition 1.10.
The algebra $(C^\infty(G_H), \{,\})$ is a Poisson algebra, i.e $\{,\}$ verifies the Jacobi identity.

2. Homogeneous Kahler $CR$ manifold.

Let $(G, \mathcal{H}, j, <, >)$ be a Kahler $CR$ Lie group and $\Gamma$ a cocompact discrete subgroup of $G$, the manifold $M = G/\Gamma$ inherits a $CR$ structure from $G$.

The orbits of the left action of the group $L$ on $G$ defines a foliation $\mathcal{F}_L$ on $M$. (The Lie algebra of $L$ is $\mathcal{H} <, >$ orthogonal.

Proposition 2.1.
The orbits of the foliation $\mathcal{F}_L$ are closed if and only if the group $\Gamma L$ is closed in $G$, in this case $M$ is the total space of a fibration over a Kahler manifold.

Proof.
Suppose that the group $\Gamma L$ is closed, then the quotient $G/\Gamma L$ is a Kahlerian manifold $N$, the projection map $M \to N$ induced by the identity map of $G$ which is the given fibration.

Conversely suppose that the orbits of the foliation $\mathcal{F}_L$ are closed. Consider a sequence $g_n$ of $\Gamma L$ which converges towards the element $g$ of $G$. Consider a neighbourhood $U$ of $g$ such that the restriction of the projection $p : G \to M$ to $U$ is injective, then $p(g)$ is an element of the adherence of $p((\Gamma L)e) = \mathcal{F}_{p(e)}$, where $e$ is the neutral element of $G$. Since we have supposed that the leaves of $\mathcal{F}_L$ are closed, $p(g)$ is an element of $\mathcal{F}_{p(e)}$ which means that $\Gamma L$ is closed in $G$. We can thus define the Kahler manifold $G/\Gamma L$. 
Deformation of CR kahlerian structures of homogeneous manifolds.

Let $(G, \mathcal{H}, j, <, >)$ be a Lie group endowed with a CR kahlerian structure. Consider a cocompact subgroup $\Gamma$ of $G$ the manifold $M = G/\Gamma$ inherits from $G$ a CR structure. In this section, we will define the deformation of those structures from two points of view.

Supposed fixed the CR kahlerian structure of $G$, and a compact manifold $M$, Let $\Gamma$ be a group we consider the set of representations $R(\Gamma, G)$ such that for each $u \in R(\Gamma, G)$, $u$ is injective and $G/u(\Gamma)$ is a compact manifold.

To elements $u$ and $u'$ of $R(\Gamma, G)$ will be said equivalent if and only there exists an element $g$ of $G$ such that $u' = gug^{-1}$. We denote by $Def_1(\Gamma, G, H, j, <, >)$ the space of equivalence classes of those CR kahlerian structures.

Now consider $RCK(G)$, the set of real complex kahlerian structures of $G$, then for a cocompact subgroup $\Gamma$ of $G$ $M = G/\Gamma$ inherits a CR kahlerian structure for each element $u$ of $RCK(G)$ denotes by $(M, u)$. We will say that $(M, u_1)$ is equivalent to $(M, u_2)$ if and only if there exists an isomorphism of CR kahlerian complex manifolds between $(M, u_1)$ and $(M, u_2)$. and denote by $Def_2(M, G)$ the set of those CR structures.

3. Kahlerian codimension 1 CR–structures.

Theorem 3.1.

Suppose that the codimension of $\mathcal{H}$ is $l$, and $G$ is semi-simple then $G$ is a Lie group of rank $\leq l$.

Proof.

Let $(\mathcal{G}, \mathcal{H}, j, <, >)$ be a Lie semi-simple algebra endowed with a codimension $l$ CR–kahlerian structure. This means that the codimension of $\mathcal{H}$ in $\mathcal{G}$ is $l$.

The map $\mathcal{G} \to \mathcal{G}^*$,

$$X \mapsto \omega(X, .)$$

is a 1–cocycle for the coadjoint representation. Since $\mathcal{G}$ is semi-simple, this cocycle is trivial. There exists an element $\alpha$ of $\mathcal{G}^*$ such that for $x, y \in \mathcal{G}$ we have:

$$\omega(x, y) = \alpha([x, y])$$

Let $K$ be the Killing form of $\mathcal{G}$, there exists $X \in \mathcal{G}$ such that for each $Y \in \mathcal{G}$ we have:

$$K(X, Y) = \alpha(Y)$$

The Lie algebra $\mathcal{L} = \{x : \omega(x, \mathcal{G}) = 0\}$ is a dimension $l$ subalgebra of $\mathcal{G}$ since the codimension of $\mathcal{H}$ is $l$. The Lie algebra $\mathcal{L}$ is the Lie algebra of the subgroup $L$ of $G$ which preserves $\alpha$, $L$ is also the subgroup which preserves $X$ since $K$ is invariant by the adjoint representation. We deduce that the rank of $G$ is less or equal than $l$, and then that $G$ is isomorphic to $sl(2)$ or $so(3)$ if the codimension of $\mathcal{H}$ is 1.
Corollary 3.2.
Suppose that $G, H, <, >, j$ is a semi-simple codimension 1 $cr$–structure (the codimension of $H$ is 1), then $G$ is $so(3)$ or $sl(2)$.

4. Poisson $CR$–structures.
Let $M$ be a manifold, and $TH$ and $TU$ two supplementary subbundles of its tangent bundle $TM$.

Definition 4.1.
An $(TH, TU)$–pseudo-Poisson structure on $M$ is defined by a bivector $\Lambda \in \Lambda^2 TM$ such that
$$[\Lambda, \Lambda] \in TUA^2 TM$$
where $[\Lambda, \Lambda]$ is the schouten product of $\Lambda$ by $\Lambda$. The bivector $\Lambda$ defined on $C^\infty(M)$ the bracket $\{,\}$ by the formula
$$\{f, g\} = \Lambda(df, dg)$$
A morphism $f : (M, \Lambda) \to (M', \Lambda')$ is a differentiable map which commutes with $\{,\}$.
Suppose that the distribution $TH$ defines on $M$ a $cr$–structure, we will say that $(M, TH, TU, j)$ defines a pseudo-Poisson $cr$–structure on $M$, if $j$ preserves $\Lambda$.

Let $G$ be a Lie group which Lie algebra is $G$. Consider a subspace $H$ of $G$ and $U$ a supplementary space to $H$. The vector spaces $H$ and $U$ define on $G$ right invariant distributions $TH$ and $TU$.

Definition 4.2.
A pseudo-Lie Poisson structure on $G$ is a bivector $\Lambda \in \Lambda^2 TG$ such that
$$[\Lambda, \Lambda] \in TUA^2 TG.$$
Moreover we suppose that $\Lambda$ is multiplicative i.e that the product $G \times G \to G$ is a morphism of pseudo-Poisson structures.

The bracket $\{,\}$ defined by $\Lambda$ satisfies the following properties:
$$\{f, f'\}(xy) = \{f \circ L_x, f' \circ L_x\}(y) + \{f \circ R_y, f' \circ R_y\}(x).$$
If we denote by $T_u L_x$ the differential of $L_x$ in $u$, we have
$$\Lambda(xy) = T_y L_x \Lambda_y + T_x R_y \Lambda_x$$
Consider the tensors $\Lambda_R(x) = T_x R_x^{-1} \Lambda_x$ and $\Lambda_L(x) = T_x L_x^{-1} \Lambda_x$

Proposition 4.3.
The fact that $\pi$ is multiplicative is equivalent to
$1. \pi_R$ is a 1–cocycle for the adjoint representation $G \to \Lambda^2 G$, i.e $\pi_R(xy) = \pi_R(x) + Ad_x(\pi_R(y))$. 

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2. \( \pi_L \) is a 1-cocycle for the adjoint action of the opposite group of \( G \) in \( \Lambda^2 G \), i.e. \( \pi_L(xy) = \pi_L(y) + \text{Ad}_y^{-1}(\pi_L(x)) \).

Let \( r \) be an element of \( \Lambda^2 G \), \( r_+ \), and \( r_- \) the left and right invariant tensors defined by \( r \). We denote by \( \pi \) the tensor \( r_+ - r_- \).

The tensor \( \pi \) defines a pseudo-Poisson structure if and only if \( [\pi, \pi] = [r, r]_+ - [r, r]_- \in TU\Lambda^2 TG \), or equivalently if

\[
\text{Ad}_x[r, r] - [r, r] \in U\Lambda^2 G.
\]

Suppose that \( H \) defines on \( G \) a \( cr \)-structure. We will say that \( (G, H, \Lambda, j) \) is a \( cr \)-Poisson structure if \( j \) preserves \( \Lambda \).

Moreover we assume that \( \Lambda \) is invariant by \( j \).

**Remark.**

Suppose defined the \( cr \)-Poisson structure \( (G_1, H_1, j_1, \Lambda_1) \) and \( (G_2, H_2, j_2, \Lambda_2) \), then the tensor \( \Lambda_1 \times \Lambda_2 \) defines a \( cr \)-structure \( (G_1 \times G_2, H_1 \times H_2, \Lambda \times \Lambda_2, j_1 \times j_2) \) called the product of the Poisson \( cr \)-structures.

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