Space-time fractional diffusion equations and asymptotic behaviors of a coupled continuous time random walk model

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Abstract

In this paper, we consider a type of continuous time random walk model where the jump length is correlated with the waiting time. The asymptotic behaviors of the coupled jump probability density function in the Fourier-Laplace domain are discussed. The corresponding fractional diffusion equations are derived from the given asymptotic behaviors. Corresponding to the asymptotic behaviors of the joint probability density function in the Fourier-Laplace space, the asymptotic behaviors of the waiting time probability density and the conditional probability density for jump length are also discussed.

Keywords: Space-time fractional diffusion equation, Caputo fractional derivative, Riesz fractional derivative, coupled continuous time random walk, asymptotic behavior

1 Introduction

The continuous time random walk (CTRW) theory, which was introduced in the 1960s by Montroll and Weiss to describe a walker hopping randomly on a periodic lattice with the steps occurring at random time intervals [1], has been applied successfully in many fields (e.g. the reviews [2-4] and references therein).

In a continuum one-dimensional space, the CTRW scheme is characterised by a jump probability density function (PDF) $\psi(x, t)$, which is the probability density that the walker makes a jump of length $x$ after some waiting time $t$. Let $P(x, t)$ be the PDF of finding the walker at a given place $x$ and at time $t$ with the initial condition $P(x, 0) = \delta(x)$. A CTRW process can be described by the following integral equation [3]:

$$P(x, t) = \int_{-\infty}^{+\infty} dx' \int_0^t \psi(x - x', t - t')P(x', t')dt' + \delta(x)\Phi(t),$$

where $\Phi(t) = 1 - \int_0^t \varphi(\tau)d\tau$ is the probability of not having made a jump until time $t$ and $\varphi(t) = \int_{-\infty}^{+\infty} \psi(x, t)dx$ is the waiting time PDF.

Fractional diffusion equations (FDEs) arise quite naturally as the limiting dynamic equations of the CTRW models with temporal and/or space memories [5]. The asymptotic relation between the CTRW models and fractional diffusion processes was studied firstly by Balakrishnan in 1985, dealing with the anomalous diffusion in one dimension [6]. Later, many authors discussed the relation between CTRW and FDEs [3-5,7-19]. However, the usual assumption in most of these
works is that the CTRW is decoupled, which means that the jump lengths and the waiting times are independent. Recently the coupled CTRW models have attracted more attention [20-25]. Here we focus on the coupled CTRW models with the jump length correlated with the waiting time [25], i.e. \( \psi(x,t) = \varphi(t) \lambda(x|t) \), and derive the corresponding FDEs from the asymptotic behaviors of the waiting time PDF \( \varphi(t) \) and the jump PDF \( \psi(x,t) \) in the Fourier-Laplace space.

This paper is organized as follows. In section 2, we introduce a space-time fractional diffusion equation which can be obtained from the standard diffusion equation by replacing the first-order time derivative and/or the second-order space derivative by a Caputo derivative of order \( \alpha \in (0,2] \) and/or a Riesz derivative of order \( \beta \in (0,2] \), respectively. In section 3, the asymptotic behaviors of the jump PDF \( \psi(x,t) \) in the Fourier-Laplace domain are given and the corresponding FDEs are derived. In section 4, corresponding to the asymptotic behaviors of the jump PDF \( \psi(x,t) \) in the Fourier-Laplace domain, the asymptotic behaviors of the waiting time PDF \( \varphi(t) \) and the conditional PDF of jump length \( \lambda(x|t) \) are discussed. In section 5, some conclusions are presented.

2 The space-time fractional diffusion equation

We consider a space-time FDE [10]

\[
\left( C_0^\alpha D_t^\alpha \right) u(x,t) = K \frac{\partial \psi(x,t)}{\partial |x|^\beta}, \quad x \in \mathbb{R}, t > 0,
\]

where \( u(x,t) \) is the field variable, \( K \) is the generalized diffusion constant and the real parameters \( \alpha, \beta \) are restricted to the range \( 0 < \alpha \leq 2, \ 0 < \beta \leq 2 \).

In Eq. (2), the time derivative is the Caputo fractional derivative of order \( \alpha \), defined as [26]

\[
\left( C_0^\alpha D_t^\alpha \right) g(t) = \left\{ \begin{array}{ll}
\frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{g^{(n)}(\tau)}{(t-\tau)^{\alpha+n}} d\tau, & n-1 < \alpha < n, \\
g^{(n)}(t), & \alpha = n \in \mathbb{N},
\end{array} \right.
\]

and the space derivative is the Riesz fractional derivative of order \( \beta \), defined as [27]

\[
\frac{d^\beta}{d|x|^{\beta}} f(x) = \left\{ \begin{array}{ll}
\Gamma(1+\beta) \frac{\sin(\beta \pi/2)}{\pi} \int_0^{+\infty} \frac{f(x+\xi)-2f(x)+f(x-\xi)}{\xi^{1+\beta}} d\xi, & 0 < \beta < 2, \\
\frac{d^2 f(x)}{d x^2}, & \beta = 2.
\end{array} \right.
\]

Let

\[
\hat{f}(k) = \mathcal{F}\{f(x)\} = \int_{-\infty}^{+\infty} f(x) e^{ikx} dx
\]

be the Fourier transform of \( f(x) \) and

\[
\tilde{g}(s) = \mathcal{L}\{g(t)\} = \int_0^{+\infty} g(t) e^{-st} dt
\]

be the Laplace transform of \( g(t) \).

Now, let us recall the following fundamental formulas about the Laplace transform of the Caputo fractional derivative of order \( \alpha \) and the Fourier transform of the Riesz fractional derivative of order \( \beta \):

\[
\mathcal{L}\{C_0^\alpha D_t^\alpha g(t)\} = s^\alpha \tilde{g}(s) - \sum_{m=0}^{n-1} s^{\alpha-1-m} g^{(m)}(0), \quad n-1 < \alpha \leq n,
\]

\[
\mathcal{F}\left\{ \frac{d^\beta}{d|x|^{\beta}} f(x) \right\} = -|k|^\beta \hat{f}(k).
\]
After applying the formula (7), in the Laplace space, the space-time FDE (2) appears in the form
\[ s^\alpha \tilde{u}(x, s) - s^{\alpha-1} u(x, 0) = K \frac{\partial^\beta \tilde{u}(x, s)}{\partial |x|^\beta} \] (9)
for \( 0 < \alpha \leq 1 \) and in the form
\[ s^\alpha \tilde{u}(x, s) - s^{\alpha-1} u(x, 0) - s^{\alpha-2} u_t(x, 0) = K \frac{\partial^\beta \tilde{u}(x, s)}{\partial |x|^\beta} \] (10)
for \( 1 < \alpha \leq 2 \).

Taking the Fourier transform of Eq.(9) with initial condition \( u(x, 0) = \delta(x) \), or of Eq.(10) with initial conditions \( u(x, 0) = \delta(x), \ u_t(x, 0) = 0 \), we get
\[ s^\alpha \hat{\tilde{u}}(k, s) - s^{\alpha-1} = -K |k|^\beta \hat{\tilde{u}}(k, s), \] (11)
and obtain immediately
\[ \hat{\tilde{u}}(k, s) = \frac{s^{\alpha-1}}{s^\alpha + K |k|^\beta}, \ 0 < \alpha \leq 2, 0 < \beta \leq 2. \] (12)

**Remark 1**: The fundamental solutions and the asymptotic solutions of Eq.(2) (containing its special cases) have been considered in many previous works [10,16,28-34]. In the following section we will consider two types of asymptotic behaviors of the jump PDF \( \psi(x, t) \) in the Fourier-Laplace space and derive the corresponding space-time FDEs.

3 From the coupled CTRW models to FDEs

After taking the Laplace transform in the variable \( t \) and the Fourier transform in the variable \( x \) of Eq.(1), we get the following well-known relation [3]
\[ \hat{P}(k, s) = \frac{1 - \hat{\varphi}(s)}{s} \cdot \frac{1}{1 - \hat{\psi}(k, s)}, \] (13)
which is called the Montroll-Weiss equation.

Different types of CTRW processes can be categorised by the existence or non-existence of the characteristic waiting time [3]
\[ T = \int_0^{+\infty} dt \int_{-\infty}^{+\infty} t\psi(x, t)dx, \] (14)
and the second moment of the jump length
\[ \sigma^2 = \int_{-\infty}^{+\infty} dx \int_0^{+\infty} x^2\psi(x, t)dt. \] (15)

For finite \( T \) and \( \sigma^2 \), the Laplace transform of the waiting time PDF \( \varphi(t) \) and the Fourier transform of the jump length PDF \( \lambda(x) \) are of the forms
\[ \hat{\varphi}(s) = 1 - sT + o(s), \ s \to 0, \] (16)
\[ \hat{\lambda}(k) = 1 - \sigma^2 k^2 + o(k^2), \ k \to 0. \] (17)

In many applications, one needs to consider long waiting time and/or long jump length, meaning that the characteristic waiting time and/or the second moment of the jump length are infinite. It is natural to generalize Eq. (16) and Eq. (17) to the following forms [3]:
\[ \hat{\varphi}(s) = 1 - A_\alpha s^\alpha + o(s^\alpha), \ s \to 0, 0 < \alpha \leq 1, \] (18)
which implies that \( \hat{\psi}(k) = 1 - A_\beta |k|^{\beta} + o(|k|^{\beta}), \quad k \to 0, 0 < \beta \leq 2, \) (19)

where \( A_\alpha \) and \( A_\beta \) are two positive normal constants.

Therefore, for the decoupled case, in the limit \((k, s) \to (0, 0)\), one has

\[
\hat{\psi}(k, s) = (1 - A_\alpha s^\alpha + o(s^\alpha))(1 - A_\beta |k|^{\beta} + o(|k|^{\beta}))
\]

\[
= 1 - A_\alpha s^\alpha - A_\beta |k|^{\beta} + O(s^\alpha |k|^{\beta}).
\]

In Eq. (20), the term \( 1 - A_\alpha s^\alpha - A_\beta |k|^{\beta} \) has main influence on \( \hat{\psi}(k, s) \) in the limit \((k, s) \to (0, 0)\). We can weaken the independent condition \( \psi(x, t) = \lambda(x) \varphi(t) \) and assume \( \psi(x, t) \) has the following form in the Fourier-Laplace domain:

\[
\hat{\psi}(k, s) = 1 - A_\alpha s^\alpha - A_\beta |k|^{\beta} + o(s^\alpha, |k|^{\beta}),
\]

which implies that \( \psi(x, t) \) is coupled. If \( o(s^\alpha, |k|^{\beta}) = O(s^\alpha |k|^{\beta}) \), Eq. (21) reduces to the decoupled case.

Inserting Eq. (18) and Eq. (21) into Eq. (13), in the limit \((k, s) \to (0, 0)\), we obtain

\[
\hat{P}(k, s) = \frac{s^{\alpha-1}}{s^\alpha + K |k|^{\beta}}, \quad 0 < \alpha \leq 1, 0 < \beta \leq 2,
\]

where \( K = \frac{A_\beta}{A_\alpha} \).

By comparing Eq. (22) with Eq. (12), with the initial condition \( P(x, 0) = \delta(x) \), the following space-time fractional equation is derived immediately:

\[
_0^\alpha D_t^\beta P(x, t) = K \frac{\partial^\beta P(x, t)}{\partial |x|^{\beta}}, \quad 0 < \alpha \leq 1, 0 < \beta \leq 2.
\]

**Remark 2:** We derived Eq. (23) by using the coupled CTRW model with the asymptotic relations Eqs. (18) and (21). The same space-time FDE has been also derived using the decoupled CTRW models in Refs. [14,18,35], where the distribution of the waiting times and that of the jump lengths are required to be independent of each other. In Refs. [18,35], the authors showed how the integral equation for the CTRW reduces to the space-time fractional diffusion equation by a properly scaled passage to the limit of compressed waiting times and jump lengths. Here we extend their consideration to the coupled case. In Ref. [14], the authors noted that the same result can be derived by weakening the independent hypothesis and replacing it with \( \hat{\psi}(k, s) \sim 1 - s^\gamma - |k|^{\beta} \). But they did not discuss under what conditions one has the above limiting behavior for the joint distribution \( \psi(x, t) \). In the following section, we will explore the problem and consider a specific case.

Next, let us extend the asymptotic relation (21) further and suppose \( \psi(x, t) \) has the following form of in the Fourier-Laplace space:

\[
\hat{\psi}(k, s) \sim 1 - A_\alpha s^\alpha - A_\beta |k|^{\beta} \frac{|k|^{\beta}}{s^{\alpha-\gamma}}, \quad 0 < \alpha \leq 1, \alpha < \gamma \leq 2, 0 < \beta \leq 2,
\]

which implies that \( \psi(x, t) \) cannot be decoupled in any event. When \( \beta = 2 \) the asymptotic behavior of \( \hat{\psi}(k, s) \) in (24) has been discussed in Ref. [36].

Inserting Eq. (18) and Eq. (24) into Eq. (13), in the limit \((k, s) \to (0, 0)\), we obtain

\[
\hat{P}(k, s) = \frac{A_\alpha s^{\alpha-1}}{A_\alpha s^\alpha + A_\beta |k|^{\beta} s^{\alpha-\gamma}}
\]

\[
= \frac{s^{\gamma-1}}{s^\gamma + K |k|^{\beta}}, \quad 0 < \alpha \leq 1, \alpha < \gamma \leq 2,
\]
where \( K = \frac{A_\beta}{A_\alpha} \).

By comparing Eq.(25) with Eq.(12), we obtain the following space-time FDE:

\[
\partial^\beta_t P(x,t) = K \frac{\partial P(x,t)}{\partial |x|^\beta}, \quad 0 < \alpha \leq 1, \alpha < \gamma \leq 2, 0 < \beta \leq 2,
\]

with the initial condition \( P(x,t=0) = \delta(x) \) for \( \alpha < \gamma \leq 1 \) or the initial conditions \( P(x,t=0) = \delta(x), P_t(x,t=0) = 0 \) for \( 1 < \gamma \leq 2 \).

4 The derivation of the asymptotic behaviors of the jump PDF \( \psi(x,t) \)

In this work, we focus on the coupled CTRW model where the jump PDF \( \psi(x,t) \) has the form \( \psi(x,t) = \varphi(t) \lambda(x|t) \), meaning that the jump length is correlated with the waiting time [25]. In the following, in the Fourier-Laplace domain, we derive the asymptotic behaviors of \( \hat{\psi}(k,s) \) in the limit \( (k,s) \to (0,0) \) which are introduced in the previous section.

For the waiting time PDF \( \varphi(t) \), we assume in the Laplace space

\[
\hat{\varphi}(s) \sim 1 - A_\alpha s^\alpha, \quad s \to 0, 0 < \alpha \leq 1.
\]

For the conditional PDF \( \lambda(x|t) \), we assume

\[
\lambda(x|t) = \begin{cases} 
\frac{1}{\sqrt{4\pi g(t)}} \exp(-\frac{x^2}{4g(t)}), & \text{if } \beta = 2, \\
\frac{1}{(g(t))^{\beta/2}} L_\beta\left(\frac{x}{(g(t))^{\beta/2}}\right), & \text{if } 0 < \beta < 2,
\end{cases}
\]

where \( L_\beta(x) \) is two-sided Lévy stable probability density, defined in Ref. [37]

\[
L_\beta(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(-|k|^{\beta})e^{-ikx}dk
\]

and \( g(t) > 0 \) is an auxiliary function. In the Fourier space, we obtain

\[
\hat{\lambda}(k|t) = \exp(-g(t)|k|^\beta), \quad 0 < \beta \leq 2.
\]

Then, in the limit \( k \to 0 \), we have the asymptotic relation

\[
\hat{\lambda}(k|t) \sim 1 - g(t)|k|^\beta, \quad k \to 0, 0 < \beta \leq 2.
\]

In the Fourier-Laplace space, in the limit \( (k,s) \to (0,0) \) we have

\[
\hat{\psi}(k,s) - \hat{\varphi}(s) = \int_0^{+\infty} dt \int_{-\infty}^{+\infty} \psi(x,t) \exp(ikx-st)dx - \int_0^{+\infty} \varphi(t) \exp(-st)dt
\]

\[
= \int_0^{+\infty} \hat{\lambda}(k|t) - 1 |\varphi(t) \exp(-st)dt
\]

\[
\sim -|k|^\beta \int_0^{+\infty} g(t)\varphi(t) \exp(-st)dt
\]

\[
= -|k|^\beta L\{g(t)\varphi(t)\}.
\]

So

\[
\hat{\psi}(k,s) \sim 1 - A_\alpha s^\alpha - |k|^\beta L\{g(t)\varphi(t)\}, \quad 0 < \alpha \leq 1, 0 < \beta \leq 2.
\]
If
\[ \mathcal{L}\{g(t)\varphi(t)\} \sim 1 - s^\mu, \quad \mu > 0, s \to 0, \] (34)
we can obtain the asymptotic relation
\[ \tilde{\psi}(k, s) \sim 1 - A_\alpha s^\alpha - |k|^\beta, \quad 0 < \alpha \leq 1, 0 < \beta \leq 2, \] (35)
which is the same as Eq. (21).

If
\[ \mathcal{L}\{g(t)\varphi(t)\} = \frac{\Gamma(\gamma - \alpha)}{s^{\gamma-\alpha}}, \quad 0 < \alpha < \gamma, s \to 0, \] (36)
we have
\[ \tilde{\psi}(k, s) \sim 1 - A_\alpha s^\alpha - A_\beta \frac{|k|^\beta}{s^{\gamma-\alpha}}, \quad 0 < \alpha \leq 1, \alpha < \gamma, 0 < \beta \leq 2. \] (37)
which is the same as Eq. (24).

Now we consider the specific case
\[ \varphi(t) \sim t^{-1-\alpha}, \quad 0 < \alpha < 1, \] (38)
and
\[ g(t) = t^\gamma, \quad 0 < \gamma \leq 2. \] (39)
Then
\[ g(t)\varphi(t) \sim t^{-1-\alpha+\gamma}, \quad 0 < \alpha < 1, 0 < \gamma \leq 2. \] (40)
If \( \gamma < \alpha \), according to the Tauberian theorem \cite{38}, we have
\[ \mathcal{L}\{g(t)\varphi(t)\} \sim 1 - s^{\alpha-\gamma}, \quad 0 < \gamma < \alpha < 1. \] (41)
After taking \( \alpha - \gamma = \mu \), Eq. (41) satisfies the condition Eq. (34). We then obtain the asymptotic relation Eq. (35), and the corresponding space-time FDE is
\[ \partial_t^\gamma P(x, t) = K \frac{\partial^\beta P(x, t)}{\partial |x|^\beta}, \quad 0 < \gamma < \alpha < 1, 0 < \beta \leq 2, \] (42)
which implies that the order of time fractional derivative in Eq. (42) is determined by the parameter \( \alpha \) of the waiting time PDF \( \varphi(t) \).

If \( \gamma > \alpha \), then \(-1 - \alpha + \gamma > -1\). Using the Laplace transform formula of power function, we have
\[ \mathcal{L}\{g(t)\varphi(t)\} = \frac{\Gamma(\gamma - \alpha)}{s^{\gamma-\alpha}}, \quad 0 < \alpha < 1, \alpha < \gamma \leq 2. \] (43)
It satisfies the condition Eq. (36). So we obtain the asymptotic relation (37), and the corresponding space-time FDE is
\[ \partial_t^\gamma P(x, t) = K \frac{\partial^\beta P(x, t)}{\partial |x|^\beta}, \quad 0 < \alpha < 1, \alpha < \gamma \leq 2, 0 < \beta \leq 2, \] (44)
which implies that the order of time fractional derivative in Eq. (44) is determined by the parameter \( \gamma \) of the auxiliary function \( g(t) \).

According to above discussions, we find that for long waiting time, i.e. \( 0 < \alpha < 1 \), there exists a competition between the waiting time PDF \( \varphi(t) \) and the auxiliary function \( g(t) \) to decide the order of the time fractional derivative in the space-time FDEs.
5 Conclusions

In this work, we discuss the asymptotic behaviors of the jump PDF $\psi(x,t)$ in the Fourier-Laplace space in the coupled CTRW model with $\psi(x,t) = \varphi(t)\lambda(x|t)$. The corresponding space-time FDEs are derived from the asymptotic behaviors of the jump PDF $\psi(x,t)$ in the Fourier-Laplace space and the waiting time PDF $\varphi(t)$ in the Laplace space. We also discuss the asymptotic behaviors of the conditional PDF of jump length $\lambda(x|t)$ and show that there exists a competition between the waiting time PDF $\varphi(t)$ and an auxiliary function $g(t)$ of the conditional PDF of jump length $\lambda(x|t)$ to determine the order of the time derivative in the space-time FDE. We also conclude that when $\beta = 2$, the derived FDE Eq.(42) from the given coupled CTRW model yields subdiffusion. Moreover, FDE (44) yields subdiffusion for the case of $0 < \alpha < \gamma < 1$, normal diffusion for the case of $0 < \alpha < \gamma = 1$, superdiffusion for the case of $0 < \alpha < 1 < \gamma \leq 2$.

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