On rates in Euler’s formula for $C_0$-semigroups

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To Fritz Gesztesy on the occasion of his sixtieth anniversary, with admiration

ABSTRACT. By functional calculus methods, we obtain optimal convergence rates in Euler’s approximation formula for $C_0$-semigroups restricted to ranges of generalized Stieltjes functions. Our results include a number of partial cases studied in the literature and cannot essentially be improved.

1. Introduction

Let $-A$ be the generator of a bounded $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on a Banach space $X$. Then the abstract Cauchy problem

\begin{align}
\begin{cases}
x'(t) = -Ax(t), & t \geq 0, \\
x(0) = x_0, & x_0 \in X,
\end{cases}
\end{align}

(1.1)

is well-posed and all its mild solutions are given by the formula

$$x(t) = e^{-tA}x_0, \quad t \geq 0, \quad x_0 \in X.$$  

However, even if $A$ is bounded, the exponential function $e^{-tA}$ can hardly be given in an explicit form. Thus it is of importance for applications to find approximation formulas for $e^{-tA}$ amenable for the purposes of numerical analysis, e.g. formulas involving rational functions of $A$. Starting from the pioneering works of Hersh and Kato [10] and P. Brenner and V. Thomée [2], the methods of Hille-Phillips functional calculus have played an important role in the theory of rational approximations of $C_0$-semigroups, see e.g. [15] Introduction and Chapter 1 for a survey.

In this paper, we extend further the functional calculus approach by replacing the “conventional” Hille-Phillips functional calculus by the extended Hille-Phillips functional calculus and then restricting ourselves to the important part of the extended Hille-Phillips calculus given by generalized Stieltjes functions. This approach proved to be quite successful in dealing with rates in mean ergodic theorems for continuous and discrete operator semigroups, see [6], [7] and [8]. To demonstrate the power of our approach we consider the simplest semigroup approximation...
known as Euler’s exponential formula or the Post-Widder inversion formula. The approximation arises when the abstract Cauchy problem \((1.1)\) is time-discretized by the so-called Euler backward method and it can be defined as

\[
E_{n,t}(A)x := \left(1 + \frac{t}{n}A\right)^{-n} x, \quad x \in X, \quad n \in \mathbb{N}, \quad t > 0.
\]

It is well known that for every \(x \in X\),

\[
e^{-tA}x = \lim_{n \to \infty} E_{n,t}(A)x\tag{1.2}
\]
uniformly for \(t\) in compact intervals of positive semi-axis. Thus a natural question is whether it is possible to quantify the convergence in \((1.2)\). It is easy to show that, in general, there is no rate of convergence in \((1.2)\) uniform with respect to all elements \(x \in X\). However such a rate does exist when the elements are taken from the domain of an appropriate function of \(A\), e.g. a power function. The next theorem surveys known results on the rates of convergence of Euler’s formula in this case. Denote

\[
\Delta_{n,t}(A) := E_{n,t}(A) - e^{-tA}, \quad n \in \mathbb{N}, \quad t > 0.
\]

**Theorem 1.1.** Let \(-A\) be the generator of a bounded \(C_0\)-semigroup \((e^{-tA})_{t \geq 0}\) on a Banach space \(X\).

(i) [2] Theorem 4] There exists \(c > 0\) such that for any \(n \in \mathbb{N}\) and \(t > 0\),

\[
\|\Delta_{n,t}(A)x\| \leq c \left(\frac{t}{\sqrt{n}}\right)^2 \|A^2x\|, \quad x \in \text{dom}(A^2);
\]

(ii) [4] Theorem 1.7] There exists \(c > 0\) such that for any \(n \in \mathbb{N}\) and \(t > 0\),

\[
\|\Delta_{n,t}(A)x\| \leq c \left(\frac{t}{\sqrt{n}}\right) \|Ax\|, \quad x \in \text{dom}(A);
\]

(iii) [16] Corollary 4.4] There exists \(c > 0\) such that for any \(n \in \mathbb{N}\), \(t > 0\) and \(0 < \alpha \leq 2\),

\[
\|\Delta_{n,t}(A)x\| \leq c \left(\frac{t}{\sqrt{n}}\right)^\alpha \|x\|_{\alpha,2,\infty}, \quad t \geq 0, \quad n \in \mathbb{N}, \quad x \in X_{\alpha,2,\infty},\tag{1.3}
\]

where the Banach space \(X_{\alpha,2,\infty}\) (called a Favard space) is defined as

\[
X_{\alpha,2,\infty} := \left\{ x \in X : \|x\|_{\alpha,2,\infty} := \|x\| + \sup_{t > 0} \frac{\|(e^{-tA} - I)x\|}{t^\alpha} < \infty \right\}.
\]

Some comments concerning the last result are in order. Note that by [14] Theorem 4.3] (see also [18] Theorem 11.3.5]) if \(\alpha \in (0, 2)\) then \(X_{\alpha,2,\infty}\) coincides with Komatsu’s (Banach) space \(D_\infty^\alpha = D_{\infty,2}^\alpha\),

\[
D_\infty^\alpha := \left\{ x \in X : \|x\|_{D_\infty^\alpha} := \|x\| + \sup_{\lambda > 0} \lambda^\alpha \|A(A + \lambda)^{-1}\|^2x\| < \infty \right\},\tag{1.4}
\]
in the sense that \(X_{\alpha,2,\infty} = D_\infty^\alpha\) as sets and the norms are equivalent. On the other hand, \(\text{dom}(A^2) \subset X_{2,2,\infty}\) and, by [14] Proposition 2.8], \(\text{dom}(A^\alpha)\) is embedded continuously in \(D_\infty^\alpha\), \(\alpha \in (0, 2)\). However, there are examples (see e.g. [13] p. 340]) showing the inclusion \(\text{dom}(A^\alpha) \subset D_\infty^\alpha\) is in general strict.

Thus, \((1.3)\) implies that there exists \(c > 0\) such that for any \(\alpha \in (0, 2]\), \(n \in \mathbb{N}\), and \(t > 0\),

\[
\|\Delta_{n,t}(A)x\| \leq c \left(\frac{t}{\sqrt{n}}\right)^\alpha \|x\|_{\text{dom}(A^\alpha)}, \quad x \in \text{dom}(A^\alpha),\tag{1.5}
\]
where
\[ \|x\|_{\text{dom}(A^\alpha)} := \|x\| + \|A^\alpha x\|, \quad x \in \text{dom}(A^\alpha). \]

(Remark that [16] Corollary 4.4 only states that (1.3) implies (1.5) for \( \alpha = 1, 2 \).)

We should also emphasize that the results mentioned in Theorem 1.1 are in fact partial cases of more general statements on convergence rates for \( A^\alpha \)-stable and stable rational approximations of \( e^{-At} \) obtained in [2, 4] and [16]. (For similar results see also [5].) In this paper we consider a very particular case of Euler’s approximation but our results are more general and complete (see also a remark at the end of this section). The main problem addressed in this paper is the characterization of decay rates for \( \Delta_{n,t}(A)x, x \in \text{ran}(f(A)), \) where \( f \) is a generalized Stieltjes function of the class \( S_2 \).

In particular, we extend Theorem [16] substantially by replacing power functions with reciprocals of generalized Stieltjes functions. As a corollary, we are also able to improve Theorem [1.1] by showing that there exists \( c > 0 \) such that for any \( n \in \mathbb{N}, \ t > 0, \) and \( \alpha \in (0, 2], \)

\[ \|\Delta_{n,t}(A)x\| \leq c \left( \frac{t}{\sqrt{n}} \right)^\alpha \|A^\alpha x\|, \quad x \in \text{dom}(A^\alpha). \]

This result does not hold for \( \alpha > 2 \) as it is explained in Remark 6.7. We also show that (1.5) is an easy consequence of our main result and so it is possible to avoid interpolation theory used in [16]. Moreover, we prove that our estimates of convergence rates are optimal.

We believe that our method will be fruitful for more general rational approximations as well. However, being confined by space limits, we present only its sample which nevertheless is significant enough to be of value as for semigroup theory so for numerical analysis.

2. Preliminaries and notations

The following elementary integrals will be used frequently throughout the paper:

\[
\begin{align*}
(2.1) \quad \frac{1}{(n-1)!} \int_0^\infty s^{n-1} e^{-s} \, ds = 1, \\
(2.2) \quad \int_0^\infty s^{n-1} e^{-s(1-s/n)} \, ds = 0, \\
(2.3) \quad \frac{1}{(n-1)!} \int_0^\infty s^{n-1} e^{-s(1-s/n)^2} \, ds = \frac{1}{n},
\end{align*}
\]

where \( n \in \mathbb{N} \).

To simplify our presentation we introduce the next notation:

\[ e_t(z) := e^{-tz}, \quad r(z) := \frac{1}{1+z}, \quad r_n,t(z) := r^n(zt/n), \]
\[ \Delta_{n,t}(z) := r_n,t(z) - e_t(z), \quad n \in \mathbb{N}, \ t > 0, \ h := \frac{t}{n}. \]

Thus, in particular, by (2.1) we have for \( z \in \mathbb{C}_+ \):

\[
(2.4) \quad (n-1)! r_n,t(z) = \frac{1}{h^n} \int_0^\infty s^{n-1} e^{-hs} e^{-zst/n} \, ds = \int_0^\infty s^{n-1} e^{-s} e^{-zst/n} \, ds.
\]
For a closed linear operator $A$ on a complex Banach space $X$ we denote by $\text{dom}(A)$, $\text{ran}(A)$ and $\sigma(A)$ the domain, the range, and the spectrum of $A$, respectively, and let $\overline{\text{ran}}(A)$ stand for the norm-closure of the range. The space of bounded linear operators on $X$ is denoted by $L(X)$. Finally, we let

$$C_+ = \{z \in \mathbb{C} : \Re z > 0\}, \quad \mathbb{R}_+ = [0, \infty).$$

3. Functional calculus

In this subsection we recall definition and basic properties of functional calculus useful for the sequel.

Let $M(\mathbb{R}_+)$ be a Banach algebra of bounded Radon measures on $\mathbb{R}_+$. Define the Laplace transform of $\mu \in M(\mathbb{R}_+)$ as

$$(L\mu)(z) := \int_0^\infty e^{-sz} \mu(ds), \quad z \in \mathbb{C}_+.$$ 

Note that the space

$$A_1^+(\mathbb{C}_+) := \{L\mu : \mu \in M(\mathbb{R}_+)\}$$

is a commutative Banach algebra with pointwise multiplication and with the norm

$$||L\mu||_{A_1^+(\mathbb{C}_+)} := ||\mu||_{M(\mathbb{R}_+)} = ||\mu||(\mathbb{R}_+),$$

where $||\mu||(\mathbb{R}_+)$ stands for the total variation of $\mu$ on $\mathbb{R}_+$. Moreover, the Laplace transform

$$L : M(\mathbb{R}_+) \mapsto A_1^+(\mathbb{C}_+)$$

is an isometric isomorphism.

Let $-A$ be the generator of a bounded $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on a Banach space $X$. Then the mapping

$$A_1^+(\mathbb{C}_+) \mapsto L(X),$$

$$H(L\mu)x := \int_0^\infty e^{-sA}x \mu(ds), \quad x \in X,$$

defines a continuous algebra homomorphism such that

$$||H(L\mu)|| \leq \sup_{t \geq 0} ||e^{-tA}|| ||\mu||(\mathbb{R}_+).$$

The homomorphism is called the Hille-Phillips (HP-) functional calculus for $A$, and we set

$$g(A) = H(L\mu) \quad \text{if} \quad g = L\mu.$$ 

Basic properties of the Hille-Phillips functional calculus can be found in [11, Chapter XV].

The HP-calculus has an automatic extension to a function class much larger then $A_1^+(\mathbb{C}_+)$. Let us recall how this extension is constructed: if $f : \mathbb{C}_+ \rightarrow \mathbb{C}$ is holomorphic such that there exists $e \in A_1^+(\mathbb{C}_+)$ with $ef \in A_1^+(\mathbb{C}_+)$ and the operator $e(A)$ is injective, then one defines

$$\text{dom}(f(A)) := \{x \in X : (ef)(A)x \in \text{ran}(e(A))\}$$

$$f(A) := e(A)^{-1}(ef)(A).$$

Such $f$ is called regularizable, and $e$ is called a regularizer for $f$. This definition of $f(A)$ does not depend on the choice of $e$ and $f(A)$ is a closed operator on $X$. The
set of all regularizable functions \( f \) constitute an algebra \( A \) depending on \( A \) (see e.g. [9] p. 4-5 and [3] p. 246-249). We call the mapping

\[
A \ni f \mapsto f(A)
\]

the extended Hille-Phillips calculus for \( A \). The next product rule of the extended Hille-Phillips calculus (see e.g. [9] Chapter 1) will be crucial for the sequel: if \( f \) is regularizable and \( g \in A_1^1(\mathbb{C}_+) \), then

\[
g(A)f(A) \subset f(A)g(A) = (fg)(A),
\]

where products of operators have their natural domains. From (3.3) it follows that if \( f \) is regularizable and \( e \) is a regularizer, then

\[
\text{ran}(e(A)) \subset \text{dom}(f(A)).
\]

4. Generalized Stieltjes functions

Our considerations will rely on the notion of generalized Stieltjes function. We say that a function \( f : (0, \infty) \mapsto [0, \infty) \) is generalized Stieltjes of order \( \alpha > 0 \) if it can be written as

\[
f(z) = a + \int_0^\infty \frac{\mu(d\tau)}{(z + \tau)^\alpha}, \quad z > 0,
\]

where \( a \geq 0 \) and \( \mu \) is a positive Radon measure on \([0, \infty)\) satisfying

\[
\int_0^\infty \frac{\mu(d\tau)}{(1 + \tau)^\alpha} < \infty.
\]

Observe that if \( f \) is generalized Stieltjes (of any positive order), then \( f \) admits an (unique) analytic extension to \( \mathbb{C} \setminus (-\infty, 0] \) which will be identified with \( f \) and denoted by the same symbol. The class of generalized Stieltjes functions of order \( \alpha \) will be denoted by \( S_\alpha \). In this terminology, Stieltjes functions constitute precisely the class \( S_1 \) of generalized Stieltjes functions of order 1, and we will write \( S \) in place of \( S_1 \) to denote the class of Stieltjes functions thus using an established notation. Note that \( S \subset S_2 \), and moreover \( S \cdot S \subset S_2 \). Since for every \( \alpha \in (0, 2] \) one has \( z^{-\alpha} \in S_\alpha \) and \( S_\alpha \subset S_2 \), it clearly follows that \( z^{-\alpha} \in S_2 \) for every \( \alpha \in (0, 2] \). For these as well as many other properties of generalized Stieltjes functions see [12]. A very informative discussion of Stieltjes functions is contained in [20] Chapter 2.

We will also need a subclass \( \tilde{S}_2 \) of \( S_2 \) consisting of products of Stieltjes functions:

\[
\tilde{S}_2 := \{ f = f_1 \cdot f_2 : f_1, f_2 \in S \}.
\]

Observe that the implication \( f_1, f_2 \in S \Rightarrow f_1^{1/2} \cdot f_2^{1/2} \in S \) (see [20] Proposition 7.10) yields

\[
\tilde{S}_2 = \{ f = f_0^2 : f_0 \in S \}.
\]

We can define the class of complete Bernstein functions \( CBF \) as \( CBF := \{ zf : f \in S \} \). An important link between the classes of Stieltjes and complete Bernstein functions is provided by the fact that \( f \in CBF, f \neq 0, \) if and only if \( 1/f \in S \), see e.g. [20] Theorem 7.3].

Let now \(-A\) be the generator of a bounded \( C_0\)-semigroup on a Banach space \( X \). By [7] Lemma 2.5 any complete Bernstein function is regularizable by \( 1/(1 + z) \in A_1^1(\mathbb{C}_+) \). Thus if \( A \) is injective then every \( f \in S \) is regularizable by \( z/(1 + z) \in A_1^1(\mathbb{C}_+) \). The next proposition shows, in particular, that functions from \( S_2 \) (and then from \( \tilde{S}_2 \)) are regularizable as well and identifies cores of the corresponding
operators. To deal with densely defined operators we assume below that the range of $A$ dense. Note that under this condition, for every $x \in X$,
\[ \lambda (\lambda + A)^{-1} x \to 0, \quad \lambda \to 0+, \]
and therefore $A$ is also injective (see e.g. [1] p. 261).

**Proposition 4.1.** Let $-A$ be the generator of a bounded $C_0$-semigroup on a Banach space $X$, and $\text{ran}(A) = X$.

(i) If $f \in S_2$, then $f$ is regularizable by $e(z) = z^2/(1 + z)^2 \in A_1^1(C_+)$, and thus belongs to the extended Hille-Phillips calculus. Moreover, $\text{ran}(A^2)$ is a core for $f(A)$.

(ii) If $f \in S_2$, $f \neq 0$, then $1/f$ is regularizable by $e(z) = 1/(1 + z)^2 \in A_1^1(C_+)$. Hence $1/f$ belongs to the extended Hille-Phillips calculus and, moreover, $\text{dom}(A^2)$ is a core for $(1/f)(A)$.

**Proof.** To prove (i) note that, since $A^2(I + A)^{-2}$ is injective, a holomorphic function $f : C_+ \to \mathbb{C}$ is regularizable by $e$ if and only if $ef \in A_1^1(C_+)$. Without loss of generality, we can assume that $f \in S_2$ is of the form
\[ f(z) = \int_0^\infty \frac{\mu(d\tau)}{(z + \tau)^2}, \quad z \in C_+. \]
Then
\[ f(z) = \int_0^\infty e^{-zs} \int_0^\infty e^{-st} \mu(d\tau) ds, \quad z \in C_. \]
Since
\[ f(z) = \int_0^\infty e^{-zs} \int_0^\infty e^{-st} \mu(dt) ds \]
\[ = \int_0^1 e^{-zs} \int_0^\infty e^{-st} \mu(dt) ds + \int_1^\infty e^{-zs} \int_0^\infty e^{-st} \mu(dt) ds \]
it is enough to prove that the second term above is regularizable by $e$.

To this aim note that
\[ \int_1^\infty e^{-zs} \int_0^\infty e^{-st} \mu(dt) ds \]
\[ = \frac{e^{-z}}{2} \int_0^\infty e^{-t} \mu(dt) + \frac{1}{z} \int_1^\infty e^{-zs} \int_0^\infty (1 - st)e^{-st} \mu(dt) ds \]
\[ = \frac{e^{-z}}{2} \int_0^\infty e^{-t} \mu(dt) + \frac{e^{-z}}{z^2} \int_0^\infty (1 - t)e^{-t} \mu(dt) \]
\[ + \frac{1}{z^2} \int_1^\infty e^{-zs} \int_0^\infty (t(st - 2))e^{-st} \mu(dt) ds. \]
The first two terms in the last display are clearly regularizable by $e$. Let us show that the third term is regularizable by $e$ too. Since
\[ \int_1^\infty \left| \int_0^\infty t(st - 2)e^{-st} \mu(dt) \right| ds \leq \int_0^\infty \int_1^\infty t(ts + 2)e^{-st} ds \mu(dt), \]
and
\[ \int_1^\infty t(ts + 2)e^{-st} ds = (t + 3)e^{-t}, \]
it follows that there exists \( c > 0 \) such that
\[
\int_1^\infty \left| \int_0^\infty t(ts - 2)e^{-st} \mu(dt) \right| ds \leq \int_0^\infty (t + 3)e^{-t} \mu(dt) \\
\leq c \int_0^\infty \frac{\mu(dt)}{(1 + t)^2} < \infty.
\]
Thus, the function
\[
s \mapsto \int_0^\infty t(st - 2)e^{-st} \mu(dt)
\]
is integrable on \([1, \infty)\). This shows that
\[
z \mapsto \int_1^\infty e^{-zs} s \int_0^\infty e^{-st} \mu(dt) ds
\]
is regularizable by \( e \), and yields \( ef \in A_1^+(\mathbb{C}_+)\).
Moreover, by (3.4) we have
\[
\text{ran } (A^2) = \text{ran } (A^2(I + A)^{-2}) \subset \text{dom } (f(A)).
\]
To prove that \( \text{ran } (A^2) \) is a a core for \( f(A) \) note that if \( e_\epsilon(A) = A^2(\epsilon + A)^{-2}, \epsilon > 0 \), then \( e_\epsilon(A)x \to x \) for every \( x \in X \) as \( \epsilon \to 0 \). Since \( e_\epsilon(z) \in A_1^+(\mathbb{C}_+) \) for each \( \epsilon > 0 \), the product rule (3.3) implies that if \( x \in \text{dom } (f(A)) \) and \( f(A)x = y \) then \( f(A)e_\epsilon(A)x = e_\epsilon(A)y \). As \( \text{ran } (e_\epsilon(A)) = \text{ran } (A^2), \epsilon > 0 \), the statement follows.
Let us prove (ii) now. Set \( g(z) = 1/f(z), z > 0 \). As the reciprocal of a nonzero complete Bernstein function is a Stieltjes function, \( g \) is a product of two complete Bernstein functions. Then \((1 + z)^{-z}g \in A_1^+(\mathbb{C}_+)\) and, since \((1 + A)^2\) is injective, the function \( g \) is regularizable by \( 1/(1 + z)^2 \). Hence (3.4) yields
\[
\text{dom } (A^2) = \text{ran } ((1 + A)^{-2}) \subset \text{dom } (g(A)).
\]
Furthermore, if \( e_\epsilon(A) = (1 + \epsilon A)^{-2}, \epsilon > 0 \), then \( e_\epsilon(A)x \to x \) for every \( x \in X \) as \( \epsilon \to 0 \). Arguing as in the proof of (i), we infer that \( \text{dom } (A^2) \) is a core for \( g(A) \). \( \square \)

**Remark 4.2.** Let \( f \in S_2 \) be of the form (4.1). Using (4.2) and
\[
\left( \frac{z}{1 + z} \right)^2 = 1 + \int_0^\infty (t - 2)e^{-t}e^{-zt} dt,
\]
by simple transformations, we obtain
\[
\frac{z^2}{(1 + z)^2} f(z) = \int_0^\infty e^{-zt} r(t) dt, \quad z \in \mathbb{C}_+
\]
where
\[
r(t) := \int_0^\infty r_0(t, \tau) \mu(d\tau), \quad t \geq 0,
\]
\[
r_0(t, \tau) := te^{-t\tau} + e^{-t} \int_0^t e^{-s\tau}e^{s(t - s - 2)} ds
\]
\[
= \begin{cases} 
\frac{(-2 + (t - \tau)t)e^{-t\tau} + (t + (2 - t)\tau)e^{-t}}{(1 - \tau)^2}, & \tau \neq 1, \\
te^{-t(t^2/6 - t + 1)}, & \tau = 1.
\end{cases}
\]
Moreover, it is not to hard to show that
\[
\int_0^\infty |r(t)|\,dt \leq \int_0^\infty \int_0^\infty |r_0(t, \tau)|\,dt\,d\mu(\tau)
\leq c \int_0^\infty \frac{\mu(\tau)}{(1 + \tau)^2} < \infty,
\]
for some constant \( c > 0 \). This leads to an alternative proof of Proposition 4.1 (i).

For \( n \in \mathbb{N} \) and \( t > 0 \), denote
\[
(n - 1)!L_{n,t}[m] := \int_0^\infty s^{n-1}e^{-s} \int_0^{[1-s/n]} m(v)\,dv\,ds
+ \int_0^\infty \left| \int_0^{n(v/t+1)} s^{n-1}e^{-s}[m(v + t/n) - m(v)]\,dv \right| \,ds.
\]

**Lemma 4.3.** Let \( m \) be a positive measurable function on \([0, \infty)\) such that its Laplace transform \((Lm)(z)\) exists for every \( z \in \mathbb{C}_+ \). Then for any \( n \in \mathbb{N} \) and \( t > 0 \),
\[
\Delta_{n,t}(z)(Lm)(z) = \int_0^\infty e^{-sz}q_{n,t}(s)\,ds, \quad z \in \mathbb{C}_+,
\]
where \( q_{n,t}(s) \) is a measurable on \([0, \infty)\) and
\[
\int_0^\infty |q_{n,t}(s)|\,ds \leq L_{n,t}[m].
\]

**Proof.** By (2.4) we have for every \( z \in \mathbb{C}_+ \)
\[
r_{n,t}(z)(Lm)(z) = \frac{1}{(n-1)!} \int_0^\infty \left| \int_0^{nu/t} \int_0^{s^{n-1}e^{-s}} m(u - s)\,du \right| \,ds.
\]
Hence after a change of variable \( s \mapsto st/n \)
\[
r_{n,t}(z)(Lm)(z) = \frac{1}{(n-1)!} \int_0^\infty \left| \int_0^{nu/t} \int_0^{s^{n-1}e^{-s}} (u - st/n)\,du \right| \,ds.
\]
On the other hand,
\[
e^{-tz}(Lm)(z) = \int_0^\infty e^{-(u+t)z} m(u)\,du = \int_t^\infty e^{-uz} m(u - t)\,du, \quad z \in \mathbb{C}_+.
\]
Then the above two formulas yield (4.4) with
\[
q_{n,t}(u) := \frac{1}{(n-1)!} \int_0^{nu/t} s^{n-1}e^{-s} m(u - st/n)\,ds - \chi(u - t)m(u - t),
\]
where \( \chi(\cdot) \) is the characteristic function of \([0, \infty)\).

Taking into account (2.1) we transform (4.5) further to the form
\[
q_{n,t}(u) = \frac{\chi(t - u)}{(n - 1)!} \int_0^{nu/t} s^{n-1}e^{-s} m(u - st/n)\,ds
- \frac{\chi(u - t)}{(n - 1)!} \int_0^\infty s^{n-1}e^{-s} m(u - t)\,ds
+ \frac{\chi(u - t)}{(n - 1)!} \int_0^{nu/t} s^{n-1}e^{-s} [m(u - st/n) - m(u - t)]\,ds.
\]
Hence for any \( n \in \mathbb{N} \) and \( t > 0 \),

\[
\int_0^\infty |q_{n,t}(u)| \, du \leq \frac{1}{(n-1)!} \int_0^t \int_0^{nu/t} s^{n-1} e^{-s} m(u-st/n) \, ds \, du \\
+ \frac{1}{(n-1)!} \int_0^t \int_0^{nu/t} s^{n-1} e^{-s} m(u-t) \, ds \, du \\
+ \frac{1}{(n-1)!} \int_0^t \int_0^{nu/t} s^{n-1} e^{-s} [m(u-st/n) - m(u-t)] \, ds \, du \\
= \frac{1}{(n-1)!} \int_0^\infty s^{n-1} e^{-s} \left( \int_0^{t[1-s/n]} m(v) \, dv \right) ds \\
+ \frac{1}{(n-1)!} \int_0^\infty s^{n-1} e^{-s} \left( \int_0^{n(v/t+1)} s^{n-1} e^{-s} [m(v+t-st/n) - m(v)] \, ds \right) \, dv.
\]

The proof is complete. \( \square \)

Let us illustrate Lemma 4.3 with several examples.

**Example 4.4.** a) Let

\[
f_1(z) := \frac{1}{z} = \int_0^\infty e^{-zv} \, dv, \quad z \in \mathbb{C}_+.
\]

Then \( f_1 = \mathcal{L}m \) with \( m(v) \equiv 1 \) for \( v \geq 0 \), and using (2.1) and (2.3), we have

\[
L_{n,t}[m] = \frac{1}{(n-1)!} \int_0^\infty s^{n-1} e^{-st} [1 - s/n] \, ds \\
\leq \frac{t}{(n-1)!} \left( \int_0^\infty s^{n-1} e^{-s} \, ds \right)^{1/2} \left( \int_0^\infty s^{n-1} e^{-s} (1 - s/n)^2 \, ds \right)^{1/2} \\
= \frac{t}{\sqrt{n}}, \quad n \in \mathbb{N}, \ t > 0.
\]

Hence by Lemma 4.3

\[
\|\Delta_{n,t}f_1\|_{A_n^1(\mathbb{C}_+)} \leq \frac{t}{\sqrt{n}}, \quad n \in \mathbb{N}, \ t > 0.
\]

b) Let

\[
f_2(z) := \frac{1}{z^2} = \int_0^\infty e^{-zv} v \, dv, \quad z \in \mathbb{C}_+,
\]

so that \( f_2 = \mathcal{L}m \) with \( m(v) = v \) for \( v \geq 0 \). By (2.2) we have

\[
(4.6) \quad \left| \int_0^{n(v/t+1)} s^{n-1} e^{-s} (1 - s/n) \, ds \right| = \int_0^{n(v/t+1)} s^{n-1} e^{-s} (s/n - 1) \, ds,
\]
and then (see (2.3))
\[ L_{n,t}[m] = \frac{t^2}{2(n-1)!} \int_0^\infty s^{n-1} e^{-s} (1 - s/n)^2 \, ds \]
\[ + \frac{t}{(n-1)!} \int_0^\infty \int_{n(v/t+1)}^\infty s^{n-1} e^{-s} \, ds \, dv \]
\[ \leq \frac{3t^2}{2(n-1)!} \int_0^\infty s^{n-1} e^{-s} (1 - s/n)^2 \, ds \, dv \]
\[ = \frac{3t^2}{2n}, \quad n \in \mathbb{N}, \quad t > 0. \]

So, in this case, by Lemma 4.3
\[ \|\Delta_{n,t} f\|_{A_1^+(C_+)} \leq \frac{3t^2}{2n}, \quad n \in \mathbb{N}, \quad t > 0. \]

c) Let
\[ f_3(z) := \left(1 + \frac{\lambda}{z}\right)^2, \quad \lambda > 0, \quad z \in \mathbb{C}_+. \]

Instead of identifying \( m \) here we use the previous two examples. Observe that by (2.4) we have \( \Delta_{n,t} \in A_1^+(C_+) \), and
\[ (4.7) \quad \|\Delta_{n,t}\|_{A_1^+(C_+)} \leq 2, \quad n \in \mathbb{N}, \quad t > 0. \]

By (4.7) and Examples 4.4 a), b),
\[ \|\Delta_{n,t} f_3\|_{A_1^+(C_+)} \leq 2 + 2\lambda \|\Delta_{n,t} f_1\|_{A_1^+(C_+)} + \lambda^2 \|\Delta_{n,t} f_2\|_{A_1^+(C_+)} \]
\[ \leq 2 + \frac{2\lambda t}{\sqrt{n}} + \frac{3\lambda^2 t^2}{2n} \leq 2 \left(1 + \frac{\lambda t}{\sqrt{n}}\right)^2, \quad n \in \mathbb{N}, \quad t > 0. \]

The following technical lemma is crucial in the proof of the main result of this section, Theorem 4.6. We shift its proof to Appendix to clarify our presentation.

**Lemma 4.5.** Let \( \tau \geq 0, \quad t > 0, \quad n \in \mathbb{N} \) be fixed. If
\[ (4.8) \quad Q_{n,t}^{(1)}(\tau) := \frac{1}{\tau^2} \int_0^\infty s^{n-1} e^{-s} \left(1 - (1 + \tau |1 - s/n|)e^{-\tau (1 - s/n)}\right) \, ds, \]
and
\[ (4.9) \quad Q_{n,t}^{(2)}(\tau) := \int_0^\infty e^{-\tau v} \psi_{n,t}(v) \, dv, \]
\[ \psi_{n,t}(v) := \int_0^{n(v/t+1)} s^{n-1} e^{-s} \left((v + t - st/n)e^{-\tau (1 - s/n)} - v\right) \, ds, \]
then
\[ (4.10) \quad Q_{n,t}(\tau) := \frac{Q_{n,t}^{(1)}(\tau) + Q_{n,t}^{(2)}(\tau)}{(n-1)!} \leq \frac{12}{(\sqrt{n}/t + \tau)^2}. \]

Lemma 4.5 implies the following key estimate for \( A_1^+(C_+) \) norms of \( \Delta_{n,t} f \) when \( f \in S_2 \).

**Theorem 4.6.** Let \( f \in S_2 \). Then \( \Delta_{n,t} f \in A_1^+(C_+) \), and
\[ (4.11) \quad \|\Delta_{n,t} f\|_{A_1^+(C_+)} \leq 12 f(\sqrt{n}/t), \quad n \in \mathbb{N}, \quad t > 0. \]
PROOF. According to (4.7) it suffices to consider \( f \) of the form (4.1). Then
\[
(4.12) \quad \Delta_{n,t}(z)f(z) = \int_0^\infty \frac{\Delta_{n,t}(z)}{(z + \tau)^2} \mu(d\tau), \quad z \in \mathbb{C}_+.
\]
For fixed \( \tau \geq 0 \) define
\[
\varphi_{-}(z) := \frac{1}{(z + \tau)^2} = \int_0^\infty e^{-zw}m(v, \tau) \, dv, \quad m(v, \tau) := ve^{tv}, \quad z \in \mathbb{C}_+.
\]
Using Lemma 4.3 with \( m(v, \tau) = ve^{tv} \) and noting that
\[
\int_0^w ve^{-v}dv = 1 - (1 + w)e^{-w},
\]
we obtain
\[
L_{n,t}[m(\cdot, \tau)] = Q_{n,t}(\tau), \quad n \in \mathbb{N}, \quad t > 0, \quad \tau \geq 0,
\]
where \( L_{n,t}[m(\cdot, \tau)] \) is defined by (4.3) and \( Q_{n,t}(\tau) \) is given by (4.10). So, by Lemmas 4.3 and 4.5 we have
\[
(4.13) \quad \|\Delta_{n,t}\varphi_{-}\|_{\mathcal{A}_1^+(\mathbb{C}_+)} \leq Q_{n,t}(\tau) \leq \frac{12}{(\sqrt{n}/t + \tau)^2}.
\]
Observe further that, in view of (4.7), \( \tau \mapsto \Delta_{n,t}\varphi_{-} \) is a continuous, \( \mathcal{A}_1^+(\mathbb{C}_+) \)-valued function on \((0, \infty)\). Moreover, by (4.13), \( \tau \mapsto \|\Delta_{n,t}\varphi_{-}\|_{\mathcal{A}_1^+(\mathbb{C}_+)} \) is Lebesgue integrable on \([0, \infty)\) for any \( t > 0 \) and \( n \in \mathbb{N} \). Thus the \( \mathcal{A}_1^+(\mathbb{C}_+) \)-valued Bochner integral
\[
\int_0^\infty \Delta_{n,t}\varphi_{-} \mu(d\tau)
\]
is well-defined. Since point evaluations are bounded functionals on \( \mathcal{A}_1^+(\mathbb{C}_+) \) and separate elements of \( \mathcal{A}_1^+(\mathbb{C}_+) \), (4.12) implies that the integral coincides with \( \Delta_{n,t}f \).

Then by (4.12), (4.13), and a standard inequality for Bochner integrals (see e.g. [11, Theorem 3.7.6]), we obtain for any \( n \in \mathbb{N} \) and \( t > 0 \):
\[
\|\Delta_{n,t}f\|_{\mathcal{A}_1^+(\mathbb{C}_+)} \leq \int_0^\infty \|\Delta_{n,t}\varphi_{-}\|_{\mathcal{A}_1^+(\mathbb{C}_+)} \mu(d\tau) \leq 12f(\sqrt{n}/t).
\]

\[\square\]

5. Main results

Theorem 4.6 and (3.2) imply immediately the following statement.

THEOREM 5.1. Let \(-A\) be the generator of a \( C_0\)-semigroup \( (e^{-tA})_{t \geq 0} \) on a Banach space \( X \), and let \( \text{ran} (A) = X \). Assume that
\[
M := \sup_{t \geq 0} \|e^{-tA}\| < \infty.
\]
If \( f \in \mathcal{S}_2 \), then for any \( x \in X \),
\[
(5.1) \quad \|f(A)\Delta_{n,t}(A)x\| \leq 12M\|x\|f(\sqrt{n}/t), \quad n \in \mathbb{N}, \quad t > 0,
\]
and for any \( x = f(A)y, \quad y \in \text{dom} (f(A)) \),
\[
(5.2) \quad \|\Delta_{n,t}(A)x\| \leq 12M\|y\|f(\sqrt{n}/t), \quad n \in \mathbb{N}, \quad t > 0.
\]
In particular, if \( f(z) = z^{-\alpha}, \quad \alpha \in (0, 2) \), then for every \( x \in \text{dom} (A^\alpha) \),
\[
(5.3) \quad \|\Delta_{n,t}(A)x\| \leq 12M\|A^\alpha x\| \left( \frac{t}{\sqrt{n}} \right)^\alpha, \quad n \in \mathbb{N}, \quad t > 0.
\]
Let us consider now the case when \( -A \) generates a bounded \( C_0 \)-semigroup but the range of \( A \) may not be dense, so that \( A \) may not be injective. We will need the next approximation result.

**Theorem 5.2.** Let \( -A \) be the generator of a bounded \( C_0 \)-semigroup on a Banach space \( X \), and let \( f \in \hat{S}_2, f \neq 0 \). If \( g = 1/f \), then for every \( x \in \text{dom}(A^2) \),
\[
\lim_{\delta \to 0^+} g(A + \delta)x = g(A)x.
\]

**Proof.** Recall that by Proposition 4.1 \( \text{dom}(A^2) \) is a core for \( g(A + \delta) \), \( \delta \geq 0 \). Let \( x \in \text{dom}(A^2) \). Then for every \( \delta > 0 \) there exists \( y_\delta \in X \) such that \( x = (A + 1 + \delta)^{-2}y_\delta \). Note that
\[
g(A + \delta)x = g(A + \delta)(A + 1 + \delta)^{-2}y_\delta = [g(z + \delta) \cdot (z + 1 + \delta)^{-2}](A)y_\delta.
\]
Moreover \( g(z)/(z+1)^2 \in \mathcal{L}_1^1(\mathbb{C}_+) \) and hence
\[
\lim_{\delta \to 0^+} \frac{g(\cdot + \delta)}{(\cdot + 1 + \delta)^2} = \frac{g(\cdot)}{(\cdot + 1)^2}
\]
in the Banach algebra \( \mathcal{L}_1^1(\mathbb{C}_+) \). Since
\[
\lim_{\delta \to 0^+} y_\delta = \lim_{\delta \to 0^+} (A + 1 + \delta)^2x = (A + 1)^2x,
\]
we have
\[
\lim_{\delta \to 0^+} g(A + \delta)x = [g(z) \cdot (z+1)^{-2}](A)(A + 1)^2x
\]
\[= g(A)(A + 1)^{-2}(A + 1)^2x = g(A)x.
\]
\[\Box\]

Theorem 5.2 allows us to adopt Theorem 5.1 to the case when the range of the generator may not be dense.

**Corollary 5.3.** Let \( -A \) be the generator of a \( C_0 \)-semigroup \( (e^{-tA})_{t \geq 0} \) on a Banach space \( X \). Assume that
\[
M := \sup_{t \geq 0} \|e^{-tA}\| < \infty.
\]
If \( g = 1/f \), where \( f \in \hat{S}_2, f \neq 0 \), then for every \( x \in \text{dom}(g(A)) \),
\[
\|\Delta_{n,t}(A)x\| \leq 12M \frac{\|g(A)x\|}{g(\sqrt{n}/t)}, \quad n \in \mathbb{N}, \quad t > 0.
\]
In particular, if \( g(z) = z^\alpha, \alpha \in (0,2] \), then (5.3) holds.

**Proof.** Note that for any \( \delta > 0 \) one has \( \text{ran}(A + \delta) = X \). Thus \( f(A + \delta) \) is well defined and bounded on \( X \), moreover
\[
f(\delta + A) = f_\delta(A),
\]
where \( f_\delta(z) := f(z + \delta), z > 0 \). For \( f \in \hat{S}_2, f \neq 0 \), define
\[
g(z) := \frac{1}{f(z)}, \quad g_\delta(z) := \frac{1}{f_\delta(z)}, \quad \delta > 0.
\]
Then, by the product rule (4.3),
\[
f_\delta(A)g_\delta(A)x = g_\delta(A)f_\delta(A)x = x, \quad x \in \text{dom}(A^2).
\]
From Theorem 5.1 and (5.5) it follows that if $\delta > 0$ and $x \in \dom(A^2)$ then for any $n \in \mathbb{N}$ and $t > 0$,

$$\|\Delta_{n,t}(A)x\| \leq 12Mf(\sqrt{n}/t)\|g_h(A)x\| \leq 12Mf(\sqrt{n}/t)\|g_h(A)x\|.$$

Let $\delta \to 0+$ in the above inequality. Since $\dom(A^2)$ is core for $g(A)$, (5.4) implies the statement.

We finish this section with the estimate of convergence rate in Euler’s formula for Komatsu’s spaces $D^\alpha_\infty$ (defined in Introduction).

**Theorem 5.4.** Let $-A$ be the generator of a $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on a Banach space $X$. Suppose that $M := \sup_{t \geq 0} \|e^{-tA}\| < \infty$.

For any $\alpha \in (0, 2]$ and $x \in D^\alpha_\infty$,

(5.6) $$\|\Delta_{n,t}(A)x\| \leq 8M \left(\frac{t}{\sqrt{n}}\right)^\alpha \|x\|_{D^\alpha_\infty}, \quad n \in \mathbb{N}, \quad t > 0.$$

**Proof.** For fixed $\lambda > 0$ define

$$f_\lambda(z) := \left(1 + \frac{\lambda}{z}\right)^2, \quad g_\lambda(z) := 1/f_\lambda(z) = \left(\frac{z}{\lambda + z}\right)^2, \quad z > 0.$$

Note that $f_\lambda \in \tilde{S}_2$. Using Corollary 5.3 with $g = g_\lambda$ and taking into account Example 4.4 c), we have for any $x \in X$ and $n \in \mathbb{N}$

(5.7) $$\|\Delta_{n,t}(A)x\| \leq 2M \left(1 + \frac{\lambda t}{\sqrt{n}}\right)^2 \|[A(A + \lambda)^{-1}]^2x\|$$

$$= 2MX^\alpha\|[A(A + \lambda)^{-1}]^2x\| \frac{(\lambda t/\sqrt{n} + 1)^2}{\lambda^\alpha}, \quad t > 0.$$

Setting now $\lambda = \sqrt{n}/t$ in (5.7) it follows that

$$\|\Delta_{n,t}(A)x\| \leq 8M \left(\frac{t}{\sqrt{n}}\right)^\alpha \sup_{s > 0} (s^\alpha\|[A(A + s)^{-1}]^2x\|), \quad x \in D^\alpha_\infty,$$

and (5.6) holds.

$$□$$

### 6. Optimality of rates

In this section we show that our estimates for convergence rates in Euler’s approximation formula are in a sense optimal. We will need the next estimate for Stieltjes functions proved independently in many papers. It seems the earliest reference is [19, Lemma 2].

**Lemma 6.1.** If $f \in S$, then

(6.1) $$f(s) \leq \sqrt{2}|f(\pm is)|, \quad s > 0.$$

It will also be convenient to single out an auxiliary inequality involving $\Delta_{n,t}$.

**Lemma 6.2.** For any $n \in \mathbb{N}$ and $t > 0$,

(6.2) $$|\Delta_{n,t}(\pm i\sqrt{n}/t)| \geq 1 - \frac{1}{\sqrt{2}}.$$
Proof. For any \( n \in \mathbb{N} \) and \( t > 0 \),
\[
|\Delta_{n,t}(\pm i\sqrt{n}/t)| = \left| \frac{1}{(1 + i/\sqrt{n})^n} - e^{-i\sqrt{n}} \right| \geq 1 - |1 + i/\sqrt{n}|^{-n}.
\]
Since \( \log(1 + t) \geq t \log 2 \), \( t \in [0,1] \), it follows that
\[
|1 + i/\sqrt{n}| = e^{(n/2) \log(1 + 1/n)} \geq e^{(1/2) \log 2} = \sqrt{2}, \quad n \in \mathbb{N},
\]
and this yields (6.2).

The result below shows that Theorem 5.1 and Corollary 5.3 are sharp if the spectrum of the generator is large enough.

**Theorem 6.3.** Let \(-A\) be the generator of a bounded \( C_0 \)-semigroup \( (e^{-tA})_{t \geq 0} \) on a Banach space \( X \). Suppose that \( \text{ran}(A) = X \) and
\[
\{ \{s\} : s \in \mathbb{R}, is \in \sigma(A) \} = \mathbb{R}^+.
\]
If \( f \in \tilde{S}_2 \), then
\[
\|f(A)\Delta_{n,t}(A)\| \geq cf(\sqrt{n}/t), \quad n \in \mathbb{N}, \quad t > 0, \quad c = \frac{1}{2} \left( 1 - \frac{1}{\sqrt{2}} \right).
\]
In particular,
\[
\|A^\alpha \Delta_{n,t}(A)\| \geq c(t/\sqrt{n})^\alpha, \quad n \in \mathbb{N}, \quad t > 0,
\]
for any \( \alpha \in (0,2] \).

Proof. Let \( n \in \mathbb{N} \) and \( t > 0 \) be such that \( i\sqrt{n}/t \in \sigma(A) \). By the spectral inclusion theorem for the Hille-Phillips functional calculus (see e.g. [11, Theorem 16.3.5] or [7, Theorem 2.2]) we obtain for every \( t > 0 \):
\[
\|f(A)\Delta_{n,t}(A)\| = \sup_{\lambda \in \sigma(A)} |(f \cdot \Delta_{n,t})(\lambda)| \geq |f(i\sqrt{n}/t)| = |\Delta_{n,t}(i\sqrt{n}/t)| \cdot |f(i\sqrt{n}/t)|.
\]
Then, by Lemmas 6.1 and 6.2, (6.5) implies (6.4).

If \(-i\sqrt{n}/t \in \sigma(A)\), then, by Lemmas 6.1 and 6.2, the argument completely analogous to the above gives the same estimate (6.4).

The assumptions of Theorem 6.3 can trivially be satisfied as the next simple example shows.

**Example 6.4.** Let \( X = L^2(\mathbb{R}^+) \). Define
\[
(Au)(s) := isu(s), \quad u \in L^2(\mathbb{R}^+),
\]
with the maximal domain. Then \(-A\) generates a \( C_0 \)-semigroup \( (e^{-At})_{t \geq 0} \) given by \( (e^{-At}u)(s) = e^{-ist}u(s), t \geq 0, \) on \( X \), and \( \sigma(A) = i\mathbb{R}^+ \). Thus, \( A \) satisfies the conditions of Theorem 6.3.

The following statement complementing Theorem 6.3 can be proved in the same way as Theorem 6.3.
Theorem 6.5. Let \(-A\) be the generator of a bounded \(C_0\)-semigroup \((e^{-tA})_{t \geq 0}\) on a Banach space \(X\). Suppose that \(\text{ran}(A) = X\) and \(\sigma(A) \cap i\mathbb{R}\) has an accumulation point at infinity. If \(f \in \tilde{S}_2\), then

\[
\limsup_{\sqrt{n/t} \to \infty} \frac{\|f(A)\Delta_{n,t}(A)\|}{f(\sqrt{n/t})} > 0.
\]

Finally, we show that Theorem 5.1 and Corollary 5.3 are sharp in a slightly weaker sense than in Theorem 6.3 but with no restriction on the spectrum of the generator.

Corollary 6.6. Let \(A\) and \(f\) satisfy the assumptions Theorem 6.5. Suppose in addition that

\[
\tau^2 f(\tau) \to \infty,
\]

Then, whenever \(\epsilon : (0, \infty) \mapsto (0, \infty)\) is a decreasing function with \(\lim_{\tau \to \infty} \epsilon(\tau) = 0\), there exists \(y \in \text{ran}(f(A))\) such that

\[
\limsup_{\sqrt{n/t} \to \infty} \frac{\|\Delta_{n,t}(A)y\|}{\epsilon(\sqrt{n/t})f(\sqrt{n/t})} = \infty.
\]

Proof. By the Theorem 6.5 we have

\[
\limsup_{\sqrt{n/t} \to \infty} \frac{\|f(A)\Delta_{n,t}(A)\|}{\epsilon(\sqrt{n/t})f(\sqrt{n/t})} = \infty.
\]

Since \(\text{ran}(A^2) \subset \text{dom}(f(A))\), the product rule \((3.3)\) implies that \(f(A)\Delta_{n,t}(A)\) is similar to its restriction to \(\text{dom}(A^{-2}) = \text{ran}(A^2)\) by means of the isomorphism \(A^2(I + A)^{-2} : X \mapsto \text{ran}(A^2)\). Then the uniform boundedness principle yields \(x \in \text{ran}(A^2) \subset \text{dom}(f(A))\) such that

\[
\limsup_{\sqrt{n/t} \to \infty} \frac{\|f(A)\Delta_{n,t}(A)x\|_{\text{dom}(A^{-2})}}{\epsilon(\sqrt{n/t})f(\sqrt{n/t})} = \infty.
\]

On the other hand, setting \(y = f(A)x\), and using Example 4.4b), we obtain

\[
\|f(A)\Delta_{n,t}(A)x\|_{\text{dom}(A^{-2})} = \|\Delta_{n,t}(A)f(A)x\| + \|A^{-2}\Delta_{n,t}(A)f(A)x\| = \|\Delta_{n,t}(A)y\| + \|A^{-2}\Delta_{n,t}(A)y\| \\
\leq \|\Delta_{n,t}(A)y\| + \frac{3M}{2} \left(\frac{t}{\sqrt{n}}\right)^2 \|y\|.
\]

Since \(\tau^2 f(\tau) \to \infty\), \(\tau \to \infty\), we may replace \(\epsilon(\tau)\) by \(\max\{\epsilon(\tau), (\tau^2 f(\tau))^{-1}\}\), \(\tau \geq 1\), and suppose without loss of generality that

\[
\beta := \inf_{\tau \geq 1} \epsilon(\tau)\tau^2 f(\tau) > 0.
\]

Hence, in view of

\[
\sup_{\sqrt{n/t} \geq 1} \frac{1}{\epsilon(\sqrt{n/t})f(\sqrt{n/t})^2} \leq \frac{1}{\beta} < \infty,
\]

the statement follows from (6.7). \(\square\)
Remark 6.7. Note that Theorem 4.6 does not hold for wider classes $S_{\alpha}$, $\alpha > 2$, of generalized Stieltjes functions. Indeed, note that

$$\lim_{z \to 0^+} \frac{\Delta n,t(z)}{z^2} = \lim_{z \to 0^+} \frac{r_n,t(z) - e_t(z)}{z^2} = \frac{t^2}{2n} \neq 0, \text{ if } t > 0.$$ 

Hence if $f \in S_{\alpha}$, $\alpha > 2$, is such that $\lim_{z \to 0^+} z^2 f(z) = \infty$, and $\sigma(A)$ has accumulation point at zero, then $f(A)\Delta n,t(A)$ is not bounded. (Otherwise, the inequality

$$\|f(A)\Delta n,t(A)\| \geq \sup_{\lambda \in \sigma(A)} |(f \cdot \Delta n,t)(\lambda)|,$$

leads to a contradiction.) Thus, (5.1) and (5.3) are not true in this case.

7. Appendix

Proof of Lemma 4.5.

It will be convenient to denote

$$(7.1) \quad w := w_n,t(s,\tau) = \tau t|1 - s/n|.$$ 

Using $(4.9)$ we obtain

$$Q^{(2)}_{n,t}(\tau) \leq \int_0^\infty e^{-\tau v} \int_0^{n/v(t+1)} s^{n-1} e^{-s} \left[(v + t - st/n)e^{-\tau(1-s/n)} - v\right] ds dv$$

$$= \frac{1}{\tau^2} \int_0^n s^{n-1} e^{-s} \int_0^\infty e^{-v} \left|(v + w)e^{-w} - v\right| dv ds$$

$$+ \frac{1}{\tau^2} \int_n^\infty s^{n-1} e^{-s} \int_w^\infty e^{-v} \left|(v - w)e^w - v\right| dv ds$$

$$= \frac{1}{\tau^2} \int_0^\infty s^{n-1} e^{-s} \int_0^\infty e^{-v} \left|(1 - e^{-w})v + we^{-w}\right| dv ds,$$

and therefore

$$(7.2) \quad Q^{(2)}_{n,t}(\tau) \leq \frac{1}{\tau^2} \int_0^\infty s^{n-1} e^{-s} \left[(1 - e^{-w}) + we^{-w}\right] ds.$$ 

Then, by $(1.8)$ and $(7.2)$,

$$(7.3) \quad Q_{n,t}(\tau) = \frac{Q^{(1)}_{n,t}(\tau) + Q^{(2)}_{n,t}(\tau)}{(n - 1)!} \leq \frac{2}{(n - 1)!\tau^2} \int_0^\infty s^{n-1} e^{-s} \left[1 - e^{-w}\right] ds \leq \frac{2}{\tau^2}.$$ 

Now let us prove that

$$(7.4) \quad Q_{n,t}(\tau) \leq \frac{3t^2}{n}, \text{ if } t \geq 0, \text{ } n \in \mathbb{N}, \text{ } \tau > 0.$$ 

Define

$$q_{n,t}(v, s, \tau) := \left|v[e^{-\tau(1-s/n)} - 1 + \tau t(1 - s/n)] + t(1 - s/n)(e^{-\tau(1-s/n)} - 1)\right|.$$
Then using (4.6) we have
\[
Q_{n,t}^{(2)}(\tau) \leq \int_0^\infty e^{-\tau v} \int_0^{n(v/t+1)} s^{n-1} e^{-s} q_{n,t}(v, s, \tau) \, ds \, dv
+ t \int_0^\infty e^{-\tau v} \int_0^{n(v/t+1)} s^{n-1} e^{-s} (s/n - 1) \, ds \, dv
= \frac{1}{\tau^2} \int_0^n s^{n-1} e^{-s} \int_0^\infty e^{-v} \tau q_{n,t}(v/\tau, s, \tau) \, dv \, ds
+ \frac{1}{\tau^2} \int_n^\infty s^{n-1} e^{-s} \int_w^\infty e^{-v} \tau q_{n,t}(v/\tau, s, \tau) \, dv \, ds
+ \frac{1}{\tau^2} \int_n^\infty s^{n-1} e^{-s} w u(w) \, ds,
\]
where
\[
u(w) := \int_0^w |1 - v| e^{-v} \, dv \leq 1 - e^{-w}.
\]

For \(s \leq n\) we have
\[
\tau q_{n,t}(v/\tau, s, \tau) = |v[e^{-w} - 1 + w] + w(e^{-w} - 1)|
\leq v(e^{-w} - 1 + w) + w(1 - e^{-w}),
\]
and similarly if \(s \geq n\) then
\[
\tau q_{n,t}(v/\tau, s, \tau) = |v[e^{-w} - 1 - w] - w(e^w - 1)|
\leq v(e^w - 1 - w) + w(e^w - 1).
\]

So,
\[
\tau^2 Q_{n,t}^{(2)}(\tau)
\leq \int_0^n s^{n-1} e^{-s} \int_0^\infty e^{-v} [v[e^{-w} - 1 + w] + w(1 - e^{-w})] \, dv \, ds
+ \int_n^\infty s^{n-1} e^{-s} \int_w^\infty e^{-v} [v[e^{-w} - 1 - w] + w(e^w - 1)] \, dv \, ds
+ \int_n^\infty s^{n-1} e^{-s} w(1 - e^{-w}) \, ds
= \int_0^n s^{n-1} e^{-s} [2w - 1 + (1 - w)e^{-w}] \, ds
+ \int_n^\infty s^{n-1} e^{-s} [1 + 3w - (1 + 4w + w^2)e^{-w}] \, ds.
\]

Then, by (4.8) and (7.5), we infer that
\[
\tau^2 (Q_{n,t}^{(1)}(\tau) + Q_{n,t}^{(2)}(\tau)) \leq 2 \int_0^n s^{n-1} e^{-s} w(1 - e^{-w}) \, ds
+ \int_n^\infty s^{n-1} e^{-s} [2 + 3w - (w^2 + 5w + 2)e^{-w}] \, ds.
\]

Using now elementary inequalities
\[
w(1 - e^{-w}) \leq w^2, \quad 2 + 3w - (w^2 + 5w + 2)e^{-w} \leq 3w^2, \quad w \geq 0,
\]
where \( w^2 = \tau^2 t^2 (1 - s/n)^2 \) (see (7.1)), we obtain that

\[
Q_{n,t}(\tau) \leq \frac{3t^2}{(n-1)!} \int_0^\infty s^{n-1} (1 - s/n)^2 \, ds = \frac{3t^2}{n},
\]

i.e. (7.4) holds.

Hence, from (7.3), (7.6) and the inequality

\[
\min \left\{ \frac{1}{a^2}, \frac{1}{b^2} \right\} \leq \frac{4}{(a+b)^2}, \quad a, b > 0,
\]

it follows that

\[
Q_{n,t}(\tau) \leq 3 \min \left\{ \frac{1}{\tau^2}, \frac{t^2}{n} \right\} \leq \frac{12}{\sqrt{n/t + \tau^2}}, \quad n \in \mathbb{N}, \quad \tau \geq 0, \quad t > 0.
\]

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