ON THE CLOSURE OF THE POSITIVE HODGE LOCUS

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Abstract. Given a polarized variation of \( \mathbb{Z} \)-Hodge structures on a smooth quasi-projective variety \( S \), the Hodge locus for \( \mathbb{V} \otimes \) is the set of points \( s \) of \( S \) where the fiber \( \mathbb{V}_s \) has more Hodge tensors than the very general one. A classical result of Cattani, Deligne and Kaplan states that the Hodge locus is a countable union of irreducible algebraic subvarieties of \( S \), called the special subvarieties of \( S \) for \( \mathbb{V} \).

We show that the union of the positive special subvarieties for \( \mathbb{V} \) is either an algebraic subvariety of \( S \) or is Zariski-dense in \( S \) (under the assumption that the adjoint group of the generic Mumford-Tate group of \( \mathbb{V} \) is simple). Here a special subvariety \( Y \) is called positive if the local system \( \mathbb{V} \mid Y \) is not constant. This implies for instance the following typical intersection statement: given an irreducible Hodge-generic subvariety \( S \) of the moduli space \( \mathcal{A}_g \) of principally polarized Abelian varieties of dimension \( g \), \( g \geq 2 \), the union of the positive dimensional irreducible components of the intersection of \( S \) with the strict special subvarieties of \( \mathcal{A}_g \) is either an algebraic subvariety of \( S \) or is Zariski-dense in \( S \).

1. Introduction

1.1. Motivation: Hodge loci and the theorem of Cattani, Deligne and Kaplan.

Let \( f : X \rightarrow S \) be a smooth projective morphism of smooth complex connected quasi-projective varieties and \( s_0 \) a closed point of \( S \). Given a non-zero class \( \lambda_{s_0} \) in the Betti cohomology \( H^2(X_{an}^{s_0}, \mathbb{Z}(i)) \) of some fiber \( X_{s_0}^{an} \) (where \( X_{s_0}^{an} \) denotes the complex analytic manifold associated to \( X_{s_0} \)), Weil asked in [Weil79] whether or not the locus of closed points \( s \in S \) where some determination of the flat transport \( \lambda_s \in H^2(X_{s}^{an}, \mathbb{Z}(i)) \) of \( \lambda_{s_0} \) is a Hodge class is an algebraic subvariety of \( S \) (recall that a Hodge class in a pure \( \mathbb{Z} \)-Hodge structure \( H \) with Hodge filtration \( F^\bullet \) is a class in \( H^{\mathbb{Z}} \) whose image in \( H^C \) lies in \( F^0 H^C \), or equivalently a morphism of Hodge structures \( \mathbb{Z}(0) \rightarrow H \)). A positive answer to Weil’s question would easily follow from the rational Hodge conjecture.

More generally let \( p : \mathbb{V} \rightarrow S^{an} \) be a polarized \( \mathbb{Z} \)-variation of Hodge structure (abbreviated ZVHS) on \( S \), assumed to be without torsion. Thus \( \mathbb{V} = \mathbb{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}^{an}} \mathbb{C}^{S^{an}} \) where \( \mathbb{V}_{\mathbb{Z}} \) is a local system of free \( \mathbb{Z} \)-modules of finite rank over \( S^{an} \), with associated holomorphic connection \( (\nabla^{an} : = \nabla \otimes_{\mathbb{C}^{S^{an}}} \mathcal{O}_{S^{an}}, \nabla^{an}) \) on \( S^{an} \) and Hodge filtration \( F^\bullet \) on the holomorphic vector bundle \( \mathbb{V}^{an} \). Weil’s case corresponds to \( \nabla_{\mathbb{Z}} = (R^{2i}f_! \mathbb{Z}(\chi^{an}(i))/\text{torsion}, (\mathbb{V}, \nabla) \) being the Gauß-Manin connection associated to \( f : X \rightarrow S \). Following Griffiths (see [Sc73, (4.13)]) the holomorphic bundle \( \mathbb{V}^{an} \) admits a unique algebraic structure \( \mathbb{V} \) such that the holomorphic connection \( \nabla^{an} \) is the analytification of an algebraic connection \( \nabla \) on \( \mathbb{V} \) which is regular, and the filtration \( F^\bullet \mathbb{V}^{an} \) is the analytification of an algebraic filtration \( F^\bullet \mathbb{V} \). Thus from now on our notations will not distinguish between algebraic objects and their analytifications, the meaning being clear from the context.

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The local system $V_Z$ is uniquely written as $\tilde{S} \times_{\rho} V_Z$, where $\pi : \tilde{S} \to S$ denotes the (complex analytic) universal cover of $S$ associated to the choice of a point $s_0$ in $S$, $V_Z := H^0(\tilde{S}, \pi^*V_Z) \simeq V_{s_0, Z}$ is a free $\mathbb{Z}$-module of finite rank and $\rho : \pi_1(S, s_0) \to \text{GL}(V_Z)$ denotes the monodromy representation of the local system $V_Z$. This corresponds to a complex analytic trivialization of $\hat{V} := V \times S \tilde{S}$ as a product $\tilde{S} \times V$, where $V := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} C$. We still let $\pi : \tilde{S} \times V \to V$ denote the natural projection.

Given $\lambda \in V_Z$ the locus $\text{Hdg}(\lambda) := \mathcal{V}(\lambda) \cap F^0\mathcal{V} \subset F^0\mathcal{V}$ intersection of the flat leaf $\mathcal{V}(\lambda) := \pi(\tilde{S} \times \{\lambda\})$ of the flat connection $\nabla$ through $\lambda$ with the subbundle $F^0\mathcal{V}$ is the locus of $V$ where the flat transport of $\lambda$ becomes a Hodge class; and $\text{HL}(S, \lambda) := \rho(\text{Hdg}(\lambda)) \subset S$ is the Hodge locus of $\lambda$ considered by Weil, namely the locus of points of $S$ where some determination of the flat transport of $\lambda$ becomes a Hodge class. Cattani, Deligne and Kaplan proved in [CDK95] the following celebrated result, answering positively and unconditionally a vast extension of Weil’s question (we also refer to [BKT18] for an alternative proof):

**Theorem 1.1.** (Cattani-Deligne-Kaplan) Let $S$ be a smooth complex quasi-projective algebraic variety and $V$ be a polarized $\mathbb{Z}$VHS over $S$. Then for any $\lambda \in V_Z$ the locus $\text{Hdg}(\lambda)$ is an algebraic subvariety of $V$, finite over the algebraic subvariety $\text{HL}(S, \lambda)$ of $S$.

Notice that the sets $\text{Hdg}(\lambda)$ and $\text{HL}(S, \lambda)$ are of interest only if $\lambda$ belongs to the complement $V^\text{nc}$ of the vector subspace $V^c \subset V$ of vectors fixed under $\rho(\pi_1(S, s_0))$. Indeed it follows from the theorem of the fixed part (see [Sc73, Cor. 7.23]) that $V^c$ is naturally defined over $\mathbb{Q}$ and carries a polarized $\mathbb{Q}$-Hodge structure making the corresponding constant local system $V^c_Q$ a direct factor of the $\mathbb{Q}$VHS $V_Q$; hence for $\lambda \in V_Z \cap V^c$ the locus $\text{Hdg}(\lambda)$ (resp. $\text{HL}(S, \lambda)$), is either $\mathcal{V}(\lambda)$ (resp. $S$) if $\lambda$ belongs to $F^0V^c$ or empty otherwise. We define $V^\text{nc}_Z := V_Z \cap V^\text{nc}$.

**Definition 1.2.** Let $\text{Hdg}(V)$ be the locus of non-trivial Hodge classes $\bigcup_{\lambda \in V^\text{nc}_Z} \text{Hdg}(\lambda)$ and $\text{HL}(S, V) := \bigcup_{\lambda \in V^\text{nc}_Z} \text{HL}(S, \lambda)$ the Hodge locus of closed points $s \in S$ for which the fiber $\mathcal{V}_s$ contains more Hodge classes than the very general fiber.

**Corollary 1.3.** (Cattani-Deligne-Kaplan) The subsets $\text{Hdg}(V) \subset V$ and $\text{HL}(S, V) \subset S$ are countable unions of strict algebraic subvarieties.

### 1.2. Main result.

Our goal in this paper is to investigate the geometry of the Zariski-closure of the Hodge locus $\text{HL}(S, \mathcal{V})$.

#### 1.2.1. Tannakian context and special subvarieties.

First, it will be crucial to consider the countable direct sum of polarized $\mathbb{Z}$VHS

$$\mathcal{V}^\oplus := \bigoplus_{a, b \in \mathbb{N}} \mathcal{V}^a \otimes (\mathcal{V}^\vee)^b$$

(where $\mathcal{V}^\vee$ denotes the dual of $\mathcal{V}$) rather than $\mathcal{V}$ itself. The notion of locus of Hodge classes and of Hodge locus generalize immediately to this setting. In particular $\text{HL}(S, \mathcal{V}^\oplus)$ is the set of closed points $s \in S$ for which exceptional Hodge tensors for $\mathcal{V}_s$ do occur. It obviously contains $\text{HL}(S, \mathcal{V})$, usually strictly. As Theorem 1.1 extends immediately to
countable direct sums of polarized ZVHS, the Hodge locus $HL(S, V^\otimes)$ is still a countable union of algebraic subvarieties of $S$.

The Hodge locus $HL(S, V^\otimes)$ has the advantage over $HL(S, V)$ of being group-theoretic. Recall that the Mumford-Tate group $MT(H) \subset GL(H)$ of a $\mathbb{Q}$-Hodge structure $H$ is the Tannakian group of the Tannakian category $(H^\otimes)$ of $\mathbb{Q}$-Hodge structures tensorially generated by $H$ and its dual $H^\vee$. Equivalently, the group $MT(H)$ is the fixator of the Hodge tensors for $H$. This is a connected $\mathbb{Q}$-algebraic group, which is reductive if $H$ is polarized. Given a polarized ZVHS $V$ on $S$ as above and $W \hookrightarrow S$ an irreducible algebraic subvariety, a point $s$ of the smooth locus $W^0$ of $W$ is said to be Hodge-generic for $V|_{W^0}$ if $MT(V_{s, \mathbb{Q}})$ is maximal when $s$ ranges through $W^0$. Two Hodge-generic points of $W^0$ have the same Mumford-Tate group, called the generic Mumford-Tate group $MT(W, V)$ of $(W, V|_W)$. Hence the Hodge locus $HL(S, V^\otimes)$ is also the subset of points of $S$ which are not Hodge-generic.

**Definition 1.4.** A special subvariety of $S$ for $V$ is an irreducible algebraic subvariety of $S$ maximal among the irreducible algebraic subvarieties of $S$ with a given generic Mumford-Tate group.

Hence any special subvariety of $S$ for $V$ is either contained in $HL(S, V^\otimes)$ (in which case we say that $S$ is strict), or $S$ itself.

**1.2.2. Positive Hodge locus.** The geometric tools used in this paper only detect the special subvarieties of $S$ for $V$ which are positive in the following sense:

**Definition 1.5.** An irreducible subvariety $Y$ of $S$ is said to be positive if the local system $V|_{Y}$ is not constant.

Equivalently, $Y$ is positive if and only if the period map $\Phi_S : S \rightarrow \text{Hod}^0(S, V) := \Gamma\backslash D^+$ describing $V$ (see [K17, Def. 3.18] or Section 2 below) does not contract $Y$ to a point in the connected Hodge variety $\text{Hod}^0(S, V)$. Thus, when $V$ satisfy the infinitesimal Torelli condition (i.e. the period map $\Phi_S$ is an immersion), an irreducible subvariety $Y$ of $S$ is positive if and only if it is positive dimensional.

**Definition 1.6.** We define the positive Hodge locus $HL(S, V^\otimes)_{\text{pos}}$ as the union of the positive strict special subvarieties of $S$ for $V$.

**1.2.3. Statement of the main result.** Our main result describes the Zariski-closure of the positive Hodge locus $HL(S, V^\otimes)_{\text{pos}}$:

**Theorem 1.7.** Let $V$ be a polarized ZVHS on a smooth quasi-projective variety $S$. Suppose that adjoint group of the generic Mumford-Tate group $MT(S, V)$ is simple (we will say that $MT(S, V)$ is non-product). Then either $HL(S, V^\otimes)_{\text{pos}}$ is a finite union of strict positive special subvarieties of $S$, hence algebraic; or it is Zariski-dense in $S$.

In other words: either the set of strict positive special subvarieties of $S$ has finitely many maximal elements (for the inclusion) or their union is Zariski-dense in $S$.

We will illustrate Theorem 1.7 in the cases where the ZVHS $V$ is of small level. We warn the reader that although this situation is much simpler to describe it is not representative of the general case: in higher level we expect $HL(S, V^\otimes)_{\text{pos}}$ to be algebraic in general.
1.2.4. **Example:** $\mathbb{V}$ of weight 1. Let us first illustrate *Theorem 1.7* in the case where $\mathbb{V}$ is (effective) of weight 1 or, a bit more generally, is parametrized by a Shimura variety.

Let $Sh^0_K(G,X)$ be a connected Shimura variety associated to a Shimura datum $(G,X)$, with $G$ non-product, and a level $K$ chosen to be neat so that $Sh^0_K(G,X)$ is smooth. For $(G,X) = (\text{GSp}(2g), H_g)$, $g \geq 1$, the Shimura variety $Sh^0_K(G,X)$ is the moduli space $A_g$ of principally polarized Abelian varieties of dimension $g$ (endowed with some additional level structure in order to have a fine moduli space).

Let $\mathbb{V}$ be the ZVHS on $Sh^0_K(G,X)$ associated to a given faithful representation $\mathbb{V}$ of $G$. In the example of $A_g$ one can take $\mathbb{V} = R^1f_*\mathbb{Z}$, the relative cohomology of the universal Abelian variety $f: A_g \to A_g$ over $A_g$. The Hodge locus $HL(Sh^0_K(G,X)) := HL(Sh^0_K(G,X), \mathbb{V}^\otimes)$ is well-known to be independent of the choice of the faithful representation $\mathbb{V}$. The special subvarieties of $Sh^0_K(G,X)$ are also called the subvarieties of Hodge type of $Sh^0_K(G,X)$, see [Moo98]. The special points of $Sh^0_K(G,X)$ are the CM-points, i.e. the points of $Sh^0_K(G,X)$ whose Mumford-Tate group is commutative. In $A_g$ special points corresponds to Abelian varieties with complex multiplication.

Any connected Shimura variety contains an analytically dense set of special points (see [Mi05, Lemma 3.3 and 3.5]), in particular $HL(Sh^0_K(G,X))$ is analytically dense in $Sh^0_K(G,X)$. The same proof shows that $HL(Sh^0_K(G,X))_{\text{pos}}$ is analytically dense in $Sh^0_K(G,X)$ as soon as it is not empty. Examples of Shimura varieties with empty positive Hodge locus are the Kottwitz arithmetic varieties (see [Cle]), obtained by taking for $G$ the group of invertible elements of a division algebra of prime degree endowed with an involution of the second kind. Ball quotients of Kottwitz type are the simplest examples.

If $S \subset Sh^0_K(G,X)$ is a smooth irreducible subvariety the special subvarieties of $S$ for $\mathbb{V}|_S$ are precisely the irreducible components of the intersection of $S$ with the special subvarieties of $Sh^0_K(G,X)$. *Theorem 1.7* applied to subvarieties of $Sh^0_K(G,X)$ thus gives immediately:

**Corollary 1.8.** Let $Sh^0_K(G,X)$ be a smooth connected Shimura variety associated to a Shimura datum $(G,X)$ with $G$ non-product. Let $S \subset Sh^0_K(G,X)$ be a smooth irreducible subvariety such that the generic Mumford-Tate group $\text{MT}(S, \mathbb{V}|_S)$ is non-product (for instance choose $S$ Hodge-generic in $Sh^0_K(G,X)$). Either the positive dimensional components of the intersection of $S$ with the strict special subvarieties of $Sh^0_K(G,X)$ form a set with finitely many maximal elements, or their union is Zariski-dense in $S$.

In particular:

**Corollary 1.9.** Any smooth connected Hodge-generic subvariety $S \subset A_g$ has either finitely many maximal positive dimensional subvarieties contained in strict special subvarieties of $A_g$, or the union of such subvarieties is Zariski-dense in $S$.

**Remark 1.10.** *Corollary 1.8* should be compared with the classical André-Oort conjecture (now a theorem when $Sh^0_K(G,X)$ is of abelian type, for instance for $Sh^0_K(G,X) = A_g$). In the same situation, the André-Oort conjecture says that there are finitely many maximal special subvarieties of $Sh^0_K(G,X)$ contained in $S$; while *Corollary 1.8* describes the positive dimensional intersections of $S$ with the special subvarieties of $Sh^0_K(G,X)$. Thus the André-Oort conjecture is an “atypical intersection” statement in the sense of [Za12], while *Theorem 1.7* is a “typical intersection” statement (in particular both
statement seem completely independent). More generally the full Theorem 1.7 seems to be the “typical intersection” counterpart to the “atypical intersection” conjecture for ZVHS proposed in [K17, Conj. 1.9], generalizing the Zilber-Pink conjectures for Shimura varieties. In particular it provides an answer to the geometric part of the naïve [K17, Question 1.2] (we warn the reader that our HL(S, V) is denoted by HL(S, V) in [K17]).

Remark 1.11. Various examples of S ⊂ Sh^0_K(G, X) for which HL(S, V^\circ)_{|S} is Zariski-dense in S have been constructed.

For Sh^0_K(G, X) = A_g, Izadi [Iz98], following ideas of [CiPi90], proved that HL(S, V^\circ)_{|S} is analytically (hence Zariski-) dense in S for any irreducible S ⊂ A_g of codimension at most g. Her proof adapts immediately to show that HL(S, V^\circ)_{|S} is analytically dense in S if S has codimension at most g – 1.

Generalizing the results of [Iz98] to a general connected Shimura variety Sh^0_K(G, X), Chai (see [Chai98]) showed the following. Let H ⊂ G be a Hodge subgroup. Let HL(S, V^\circ, H) ⊂ HL(S, V^\circ) denote the subset of points s ∈ S whose Mumford-Tate group MT^s(V) is G(Q)-conjugated to H. Then there exists an explicit constant c(H, G, X, H) ∈ N, whose value is g in the example above, which has the property that HL(S, V^\circ, H) is analytically dense in S as soon as S has codimension at most c(H, G, X, H) in Sh^0_K(G, X). Once more it follows from the analysis of the proof of [Chai98] that HL(S, V^\circ)_{pos} is analytically dense in S as soon as S has codimension at most c(H, G, X, H) – 1.

1.2.5. Example: V of weight 2; Noether-Lefschetz locus. Let B ⊂ P^H^0(P^3, O(d)) be the open subvariety parametrizing the smooth surfaces of degree d in P^3. From now on we suppose d > 3. The classical Noether theorem states that any surface Y ⊂ P^3 corresponding to a very general point [Y] ∈ B has Picard group Z: every curve on Y is a complete intersection of Y with another surface in P^3. The countable union NL(B) of algebraic subvarieties of B corresponding to surfaces with bigger Picard group is called the Noether-Lefschetz locus of B. Let V → B be the ZVHS R^f_J, where f : Y → B denotes the universal family. Clearly NL(B) ⊂ HL(B, V^\circ). Green (see [Vo02, Prop.5.20]) proved that NL(B) is analytically dense in B (see also [CHM88] for a weaker result). In particular HL(B, V^\circ) is dense in B. Once more the analysis of Green’s proof shows that in fact HL(B, V^\circ)_{pos} is dense in B. Now Theorem 1.7 states the following:

Corollary 1.12. Let S ⊂ B be any smooth irreducible Hodge-generic subvariety. Either S ∩ HL(B, V^\circ)_{pos} contains only finitely many maximal positive dimensional subvarieties of S, or the union of such subvarieties is Zariski-dense in S.

Remark 1.13. We don’t know if Corollary 1.12 remains true if we replace HL(B, V^\circ)_{pos} by NL(B).

1.3. Main ingredients.

1.3.1. Locus of F^i-type. The main idea for proving Theorem 1.7 consist in studying the locus of F^i-type for a general (i.e. not necessarily integral) class λ ∈ V and a general index i. Given λ ∈ V and i ∈ Z we define the set

F^i(λ) := V(λ) ∩ F^iV ⊂ F^iV
intersection of the flat leaf \( V(\lambda) := \pi(\tilde{S} \times \{ \lambda \}) \) of the flat connection \( \nabla \) through \( \lambda \) with the subbundle \( F^i V \); and its projection
\[
S^i(\lambda) := p(V^i(\lambda)) \subset S .
\]

For \( \lambda \in V^*_E \) and \( i = 0 \) the equalities \( \text{Hdg}(\lambda) = V^0(\lambda) \) and \( \text{HL}(S, \lambda) := S^0(\lambda) \) hold. Notice also that \( V^i(z \cdot \lambda) = z V^i(\lambda) \) and \( S^i(z \cdot \lambda) = S^i(\lambda) \) for any \( z \in \mathbb{C}^* \). Hence \( V^i(\lambda) \subset F^i V \) is the cone over a subset \( V^i([\lambda]) \subset \mathbb{P}F^i V \); and \( S^i(\lambda) \) depends only on \([\lambda] \in \mathbb{P} V\).

It is an immediate corollary of their definitions that for any \( \lambda \in V^* \), the sets \( V(\lambda) \) and \( V^i(\lambda), \ i \in \mathbb{Z}, \) are naturally complex analytic subspaces of the \( \acute{e}tale \) space of the complex local system \( V \). We will always endow \( V^i(\lambda) \) with its reduced analytic structure.

When \( \lambda \in V \) is not a complex multiple of an element of \( V_E \), the orbit of \( \lambda \) in \( V \) under the monodromy group \( \rho(\pi_1(S, s_0)) \subset \text{GL}(V) \) has usually accumulation points, in which case \( V(\lambda) \) is not even an analytic subvariety of \( V \) in this case. There is no obvious reason why \( V^i(\lambda) \) should behave better. A fortiori its projection \( S^i(\lambda) \subset S \) is a priori not a complex analytic subvariety of \( S \).

**Definition 1.14.**

(a) A component of \( V^i(\lambda) \) is an irreducible component of the complex analytic subvariety \( V^i(\lambda) \) of the \( \acute{e}tale \) space of the complex local system \( V \).

(b) A component of \( S^i(\lambda) \) is the image under \( p : V \rightarrow S \) of a component of \( V^i(\lambda) \).

(c) For \( \lambda \in V^*, \ i \in \mathbb{Z} \) and \( d \in \mathbb{N} \) let \( V^i(\lambda)_{\geq d} \subset F^i V \), respectively \( S^i(\lambda)_{\geq d} \subset S \), be the union of components of \( V^i(\lambda) \), resp. \( S^i(\lambda) \), of dimension at least \( d \). We define
\[
V^i_{\geq d} := \bigcup_{\lambda \in V^*} V^i(\lambda)_{\geq d} \subset F^i V \quad \text{and} \quad S^i(V)_{\geq d} := p(V^i_{\geq d}) \subset S .
\]

1.3.2. A global algebraicity result. The first ingredient of independent interest in the proof of Theorem 1.7 is a global algebraicity statement for \( V^i_{\geq d} \):

**Theorem 1.15.** Let \( V \) be a polarized \( \mathbb{Z} \)-VHS on a smooth quasi-projective variety \( S \). For any \( i \in \mathbb{Z} \) and any \( d \in \mathbb{N} \), the set \( V^i_{\geq d} := \bigcup_{\lambda \in V^*} V^i(\lambda)_{\geq d} \subset F^i V \) (respectively its projection \( S^i(V)_{\geq d} \)) is an algebraic subvariety of \( V \) (resp. of \( S \)).

In fact Theorem 1.15 is a special case of a general result on algebraic flat connections, see Theorem 5.1.

1.3.3. On the Zariski-closure of the components of \( S^i(\lambda) \). The second ingredient in the proof of Theorem 1.7 describes the Zariski-closure of any component of \( S^i(\lambda) \) for general \( i \in \mathbb{Z} \) and \( \lambda \in V^* \). The notion of special subvariety for \( V \) can be weakened to the notion of weakly special subvariety for \( V \), whose precise definition is given in Definition 2.5. For the convenience of the reader let us give the following provisional definition, which is shown in Corollary 2.14 to be equivalent to Definition 2.5 (notice that this provisional definition does not obviously imply that a special subvariety of \( S \) for \( V \) is weakly special). Given an irreducible algebraic subvariety \( i : W \hookrightarrow S \) let \( n : W^{\text{nor}} \rightarrow W \) be its normalization and \( W^{\text{nor}, 0} \subset W^{\text{nor}} \) the smooth locus of \( W^{\text{nor}} \). We define the algebraic monodromy group of \( W \) for \( V \) as the identity component of the Zariski-closure in \( \text{GL}(V) \) of the monodromy of the restriction to \( W^{\text{nor}, 0} \) of the local system \( n^* V \).
Definition 1.16. Let $\mathcal{V}$ be a polarized ZVHS on a smooth complex quasi-projective variety $S$. A weakly special subvariety $W \subset S$ for $\mathcal{V}$ is an irreducible algebraic subvariety $W$ of $S$ maximal among the irreducible algebraic subvarieties of $S$ having the same algebraic monodromy group as $W$ for $\mathcal{V}$.

From now on we will write (weakly) special subvariety for (weakly) special subvariety for $\mathcal{V}$ when the reference to $\mathcal{V}$ is clear.

Theorem 1.17. For any $i \in \mathbb{Z}$ and any $\lambda \in \mathcal{V}$, the Zariski-closure of any of the (possibly infinitely many) components of $S^i(\lambda)$ is a weakly special subvariety of $S$.

1.3.4. A converse to Theorem 1.1. As a preliminary to Theorem 1.17, Theorem 1.15 and Theorem 1.7, we also provide for the convenience of the reader the following converse to Theorem 1.1, which is probably well-known to the experts but which does not seem to have appeared before.

Proposition 1.18. Let $\lambda \in \mathcal{V}$ and $i \in \mathbb{Z}$ be such that $\mathcal{V}^i(\lambda)$ is an algebraic subvariety of $\mathcal{V}$. Then the projection $S^i(\lambda)$ of $\mathcal{V}^i(\lambda)$ is a finite union of special subvarieties of $S$. Moreover, $\mathcal{V}^i(\lambda)$ is finite over $S^i(\lambda)$.

1.4. Organization of the paper. The paper is organized as follows. Section 2 establishes the basic properties of the weakly special subvarieties of $S$ for $\mathcal{V}$ needed in the following sections. In particular we prove that they are algebraic and coincide in fact with the bi-algebraic subvarieties of $S$ for the natural bi-algebraic structure on $S$ defined by $\mathcal{V}$ (a result stated in [K17, Prop.7.4] without proof). The following sections provide the proofs of Proposition 1.18, Theorem 1.17, Theorem 1.15 and Theorem 1.7 successively.

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2. Weakly special subvarieties and bi-algebraic geometry for $(S, \mathcal{V})$

In this section we recall the definition of the weakly special subvarieties of $S$ for $\mathcal{V}$ given in [K17], study their geometry and prove that their bi-algebraic characterisation stated in [K17] without proof. We use the definitions of Hodge theory introduced in [K17] (themselves inspired by [Pink89] and [Pink05]), to which we refer for more details.

2.1. Weakly special subvarieties.

2.1.1. Weakly special subvarieties of Hodge varieties. In this paper all connected Hodge data (see [K17, Def. 3.3]) and connected Hodge varieties (see [K17, Def. 3.8]) are pure. Let $(G, \mathcal{D}^+)$ be a connected Hodge datum and $Y = \Gamma \backslash \mathcal{D}^+$ an associated connected Hodge variety. Hence $Y$ is an arithmetic quotient in the sense of [BKT18, Section 1] endowed with a natural complex analytic structure (which is not algebraic in general). Recall that a Hodge morphism between connected Hodge varieties is the complex analytic map deduced from a morphism of the corresponding Hodge data (see [K17, Lemma 3.9]). The special and weakly special subvarieties of $Y$ are irreducible analytic subvarieties of $Y$ defined as follows (see [K17, Def.7.1]):

Definition 2.1. Let $Y$ be a connected Hodge variety.
The image of any Hodge morphism \( T \rightarrow Y \) between connected Hodge varieties is called a special subvariety of \( Y \).

Consider any Hodge morphism \( \varphi : T_1 \times T_2 \rightarrow Y \) between connected Hodge varieties and any point \( t_2 \in T_2 \). Then the image \( \varphi(T_1 \times \{t_2\}) \) is called a weakly special subvariety of \( Y \). It is said to be strict if it is distinct from \( Y \).

Remark 2.2. [K17, Def. 7.1], valid more generally for \( Y \) a mixed Hodge variety and generalizing [Pink05, Def. 4.1] to this context, gives the following apparently more general definition of a weakly special subvariety. Consider any Hodge morphisms \( R \xleftarrow{i} T \xrightarrow{\pi} Y \) between (possibly mixed) connected Hodge varieties and any point \( r \in R \). Then any irreducible component of \( i(\pi^{-1}(r)) \) is called a weakly special subvariety of \( Y \). When \( Y \) is pure, i.e. \( G \) is a reductive group, one easily checks that this definition reduces to Definition 2.1(2) above.

Remark 2.3. Considering the connected Hodge variety \( T_2 = \{t_2\} \) associated to the trivial algebraic group, any special subvariety of \( Y \) is a weakly special subvariety of \( Y \).

Remark 2.4. As noticed in [Pink05, Rem. 4.8] in the case of Shimura varieties, any irreducible component of an intersection of special (resp. weakly special) subvarieties of the Hodge variety \( Y \) is a special (resp. a weakly special) subvariety of \( Y \). The proof is easy and the details are left to the reader.

2.1.2. Weakly special subvarieties for \( V \). Let \( V \) be a polarized \( \mathbb{Z} \) VHS on \( S \). Let \( (G, \mathcal{D}) \) be the generic Hodge datum for \( (S, V) \) (see [K17, Section 2.6]), \( \text{Hod}^0(S, V) := \Gamma \backslash \mathcal{D}^+ \) the associated connected Hodge variety and \( \Phi_S : S \rightarrow \text{Hod}^0(S, V) := \Gamma \backslash \mathcal{D}^+ \) the associated period map describing \( V \), with lifted period map \( \tilde{\Phi}_S : \tilde{S} \rightarrow \mathcal{D}^+ \) (see [K17, Def. 3.18 and below]). As in [K17, Prop. 3.20 and Def. 7.1] we define:

Definition 2.5. Let \( p : V \rightarrow S \) be a polarized \( \mathbb{Z} \) VHS over a quasi-projective complex manifold \( S \) with associated period map \( \Phi_S : S \rightarrow \text{Hod}^0(S, V) \).

Any irreducible complex analytic component of \( \Phi_S^{-1}(Y) \), where \( Y \) is a special (resp. weakly special) subvariety of the connected mixed Hodge variety \( \text{Hod}^0(S, V) \), is called a special (resp. weakly special) subvariety of \( S \) for \( V \). It is said to be strict if it is distinct from \( S \).

Notice that an irreducible component of an intersection of special (resp. weakly special) subvarieties of \( S \) for \( V \) is not anymore necessarily a special (resp. a weakly special) subvariety of \( S \). The following follows immediately from Remark 2.4:

Lemma 2.7. An irreducible component of an intersection of special (resp. weakly special) intersections for \( V \) is a special (resp. weakly special) intersection for \( V \).
2.1.3. Algebraicity of weakly special subvarieties of \( S \). The very definition of the Hodge locus \( \text{HL}(S, V^\otimes) \) imply that special subvarieties of \( S \) for \( V \) in the sense of Definition 2.5 coincide with the ones defined in Definition 1.4. In particular, in view of Theorem 1.1, any special subvariety of \( S \) (hence any special intersection in \( S \)) is an algebraic subvariety of \( S \). An alternative proof of Theorem 1.1 using o-minimal geometry was provided in [BKT18, Theor. 1.6]. The approach of [BKT18] gives immediately the following more general algebraicity result, which is implicit in the discussion of [K17, Section 7]:

**Proposition 2.8.** Any weakly special subvariety \( Z \) for \( V \) (hence also any weakly special intersection for \( V \)) is an algebraic subvariety of \( S \).

**Proof.** The proof is stricly analogous to the proof of [BKT18, Theor. 1.6]. By [BKT18, Theor. 1.1(1)] the Hodge variety \( \text{Hod}^0(S, V) = \Gamma\backslash D^+ \) is an arithmetic quotient endowed with a natural structure of \( \mathbb{R}_{\text{alg}} \)-definable manifold. By [BKT18, Theor. 1.3] the period map \( \Phi_S \) is \( \mathbb{R}_{\text{an}, \exp} \)-definable with respect to the natural \( \mathbb{R}_{\text{alg}} \)-structures on \( S \) and \( \text{Hod}^0(S, V) \). Let \( Y \) be the unique weakly special subvariety of \( \text{Hod}^0(S, V) \) such that \( Z \) is an irreducible component of \( \Phi_S^{-1}(Y) \). By [BKT18, Theor. 1.1(2)] \( Y \) is an \( \mathbb{R}_{\text{alg}} \)-definable subvariety of \( \text{Hod}^0(S, V) \); hence its preimage \( \Phi_S^{-1}(Y) \) is an \( \mathbb{R}_{\text{an}, \exp} \)-definable subvariety of \( S \). By the definable Chow theorem of Peterzil and Starchenko [PS09, Theor. 4.4 and Corollary 4.5], the complex analytic \( \mathbb{R}_{\text{an}, \exp} \)-definable subvariety of the complex algebraic variety \( S \) is necessarily an algebraic subvariety of \( S \). Hence its irreducible complex analytic component \( Z \) too. \( \square \)

2.1.4. Special and weakly special closure. One deduces immediately from Lemma 2.7 the following

**Corollary 2.9.** Any irreducible algebraic subvariety \( i: W \to S \) is contained in a smallest weakly special (resp. special) intersection \( \langle W \rangle_{\text{ws}} \) (resp. \( \langle W \rangle_s \)) of \( S \) for \( V \), called the weakly special (resp. special) closure of \( W \) in \( S \) for \( V \).

**Remark 2.10.** Obviously \( W \subset \langle W \rangle_{\text{ws}} \subset \langle W \rangle_s \).

The geometric description of \( \langle W \rangle_s \) is easy. Let \( (G_W, D_W) \subset (G, D) \) be the generic Hodge datum of the restriction of \( V \) to the smooth locus of \( W \). This induces a Hodge morphism of connected Hodge varieties \( \varphi: \Gamma_W\backslash D_W^+ \to \Gamma\backslash D^+ \), where \( \Gamma_W := \Gamma \cap G_W(\mathbb{Q}) \). The restriction of the period map \( \Phi_S \) to the smooth locus of \( W \) factorizes through the special subvariety \( \varphi(\Gamma_W\backslash D_W^+) \) of \( \Gamma\backslash D^+ \) and we obtain:

**Lemma 2.11.** The special closure \( \langle W \rangle_s \) is the unique irreducible component of intersections of components of \( \Phi_S^{-1}(\varphi(\Gamma_W\backslash D_W^+)) \) containing \( W \).

The description of the weakly special closure \( \langle W \rangle_{\text{ws}} \) is a bit more involved but similar to the one obtained by Moonen [Moo98, Section 3] in the case of Shimura varieties. Let \( n: W^{\text{nor}} \to W \) be the normalization of \( W \). The restriction to the smooth locus \( W^{\text{nor}, 0} \) of \( W^{\text{nor}} \) of the local system \( n^* V \) is a ZVHS with generic Mumford-Tate group \( G_W \). Let \( \Phi_{W^{\text{nor}, 0}}: W^{\text{nor}, 0} \to \Gamma_W\backslash D_W^+ \).
be its period map, hence we have a commutative diagram

\[
\begin{array}{ccc}
W^{\text{nor},0} & \xrightarrow{\Phi_{W^{\text{nor},0}}} & \Gamma_W \backslash D^+_W \\
\downarrow n & & \downarrow \varphi \\
W & \xrightarrow{\Phi_{S|W}} & \Gamma \backslash D^+ \\
\end{array}
\]

Let \( H_W \) be the algebraic monodromy group of the local system \( n^*V|_{W^{\text{nor},0}} \). As \( W^{\text{nor}} \) is irreducible normal, the fundamental group \( \pi_1(W^{\text{nor},0}) \) surjects onto \( \pi_1(W^{\text{nor}}) \). Hence \( H_W \) is also the identity component of the Zariski-closure of \((\Phi_S \circ n)_*(\pi_1(W^{\text{nor}})) \subset \Gamma \) in \( \text{GL}(V) \). We know from [An92, Th.1] that \( H_W \) is a normal subgroup of the derived group \( G_W^{\text{der}} \). As \( G_W \) is reductive there exists a normal subgroup \( G'_W \subset G_W \) such that \( G_W \) is an almost direct product of \( H_W \) and \( G'_W \). In this way we obtain a decomposition of the adjoint Hodge datum \((G_W^{\text{ad}}, D^+_W)\) into a product

\[
(G_W^{\text{ad}}, D^+_W) = (H_W^{\text{ad}}, D^+_{H_W}) \times (G'_W^{\text{ad}}, D^+_{G'_W}) ,
\]

inducing a decomposition of connected Hodge varieties

\[
\Gamma_W \backslash D^+_W = \Gamma_{H_W} \backslash D^+_{H_W} \times \Gamma_{G'_W} \backslash D^+_{G'_W} .
\]

**Lemma 2.12.** The projection of \( \Phi_{W^{\text{nor},0}}(W^{\text{nor},0}) \subset \Gamma_W \backslash D^+_W \) on \( \Gamma_{G'_W} \backslash D^+_{G'_W} \) is a single point \( \{t'\} \).

**Proof.** When \( \Gamma \backslash D^+ \) is a connected Shimura variety this is proven in [Moo98, Prop. 3.7]. Moonen’s argument does not extend to our more general situation: he uses that \( D^+ \) is a bounded domain in some \( \mathbb{C}^N \) in the Shimura case, which is not true for a general flag domain \( D^+ \). Instead we argue as follows. Choose any faithful linear representation \( \rho : G_W^{\text{ad}} \rightarrow \text{GL}(H) \) and a \( \mathbb{Z} \)-structure \( H_Z \) on the \( \mathbb{Q} \)-vector space \( H \) such that \( \rho(G_W^{\text{der}}) \subset \text{GL}(H_Z) \). The \( \mathbb{Z} \)-local system on \( W^{\text{nor},0} \) with monodromy representation

\[
\lambda : \pi_1(W^{\text{nor},0}) \xrightarrow{\Phi_{W^{\text{nor},0}}} \Gamma_W \xrightarrow{p_2} \Gamma_{G'_W} \xrightarrow{\rho} \text{GL}(H_Z)
\]

is a ZVHS with period map

\[
W^{\text{nor},0} \xrightarrow{\Phi_{W^{\text{nor},0}}} \Gamma_W \backslash D^+_W \xrightarrow{p_2} \Gamma_{G'_W} \backslash D^+_{G'_W} .
\]

By the very definition of the algebraic monodromy group \( H_W \) the group \( \lambda(\pi_1(W^{\text{nor},0})) \subset \text{GL}(H_Z) \) is finite. Applying the theorem of the fixed part (see [Sc73, Cor. 7.23]) to the corresponding étale cover of \( W^{\text{nor},0} \) we deduce that the period map \( p_2 \circ \Phi_{W^{\text{nor},0}} \) is constant. \( \Box \)

**Lemma 2.12** implies that \( W \) is contained in \( \Phi_S^{-1}(\varphi((\Gamma_{H_W} \backslash D^+_{H_W}) \times \{t'\})) \). Conversely, as any irreducible component of an intersection of weakly special subvarieties of \( \Gamma_W \backslash D^+_W \) is still weakly special, one easily checks that any weakly special subvariety \( Y := \psi(T_1 \times \{t_2\}) \subset \Gamma_W \backslash D^+_W \) containing \( \Phi_{W^{\text{nor},0}}(W^{\text{nor},0}) \) has to contain \( (\Gamma_{H_W} \backslash D^+_{H_W}) \times \{t'\} \). Thus:
Proposition 2.13. The weakly special closure \( ⟨W⟩_{ws} \) of \( W \) is the unique irreducible component of the intersection of components of \( \Phi_S^{-1}(\varphi((\Gamma_{H_W} \setminus D^+_{H_W}) \times \{t′\})) \) containing \( W \).

It then follows immediately:

Corollary 2.14. Weakly special subvarieties of \( S \) for \( V \) in the sense of Definition 2.5 coincide with the ones defined in Definition 1.16.

2.2. Bi-algebraic geometry for \((S,V)\). Let us start by recalling the general functional transcendence context of “bi-algebraic geometry” (see [KUY18], [K17, Section 7]):

Definition 2.15. A bi-algebraic structure on a connected complex algebraic variety \( S \) is a pair \((D : \tilde{S} \rightarrow X, \rho : \pi_1(S) \rightarrow \text{Aut}(X))\) where \( \pi : \tilde{S} \rightarrow S \) denotes the universal cover of \( S \), \( X \) is a complex algebraic variety, \( \text{Aut}(X) \) its group of algebraic automorphisms, \( \rho : \pi_1(S) \rightarrow \text{Aut}(X) \) is a group morphism (called the holonomy representation) and \( D \) is a \( \rho \)-equivariant holomorphic map (called the developing map).

The datum of a bi-algebraic structure on \( S \) tries to emulate an algebraic structure on the universal cover \( \tilde{S} \) of \( S \):

Definition 2.16. Let \( S \) be a connected complex algebraic variety endowed with a bi-algebraic structure \((D, \rho)\).

(i) An irreducible analytic subvariety \( Z \subset \tilde{S} \) is said to be an irreducible algebraic subvariety of \( \tilde{S} \) if \( Z \) is an analytic irreducible component of \( D^{-1}(\overline{D(Z)^{\text{Zar}}}) \) (where \( \overline{D(Z)^{\text{Zar}}} \) denotes the Zariski-closure of \( D(Z) \) in \( X \)).

(ii) An irreducible algebraic subvariety \( Z \subset \tilde{S} \), resp. \( W \subset S \), is said to be bi-algebraic if \( \pi(Z) \) is an algebraic subvariety of \( S \), resp. any (equivalently one) analytic irreducible component of \( \pi^{-1}(W) \) is an irreducible algebraic subvariety of \( \tilde{S} \).

Remark 2.17. As in Section 2.1.2 an irreducible component of an intersection of algebraic subvarieties of \( \tilde{S} \) is not necessarily algebraic in the sense above, as the map \( D \) is not assumed to be injective. Let us call such an irreducible component an algebraic intersection in \( \tilde{S} \). An algebraic intersection \( Z \subset \tilde{S} \), resp. an irreducible algebraic subvariety \( W \subset S \), is called a bi-algebraic intersection if \( \pi(Z) \) is an algebraic subvariety of \( S \), resp. any (equivalently one) analytic irreducible component of \( \pi^{-1}(W) \) is an algebraic intersection in \( \tilde{S} \).

Let \( V \) be a polarized \( \mathbb{Z} \)VHS on \( S \). It canonically defines a bi-algebraic structure on \( S \) as follows. Let \( \Phi_S : \tilde{S} \rightarrow \hat{D} \) be the composite \( j \circ \tilde{\Phi}_S \) where \( j : D \hookrightarrow \hat{D} \) denotes the open embedding of the Mumford-Tate domain \( D \) in its compact dual \( \hat{D} \), which is an algebraic flag variety for \( G(\mathbb{C}) \) (see [K17, Section 3.1]).
Definition 2.18. Let \( p : \mathcal{V} \rightarrow S \) be a polarized \( \mathcal{Z} \) VHS on a quasi-projective complex manifold \( S \). The bi-algebraic structure on \( S \) defined by \( \mathcal{V} \) is the pair \((\Phi_S : \hat{S} \rightarrow \hat{D}, \rho_S := (\Phi_S)_* : \pi_1(S) \rightarrow \Gamma \subset G(\mathbb{C}))\).

The following proposition, stated in [K17, Prop. 7.4] without proof, characterizes the weekly special subvarieties of \( S \) for \( \mathcal{V} \) in bi-algebraic terms. It was proven by Ullmo-Yafaev [UY11] in the case where \( S \) is a Shimura variety, and in some special cases by Friedman and Laza [FL15].

Proposition 2.19. Let \((S, \mathcal{V})\) be a \( \mathcal{Z} \) VHS. The weakly special subvarieties (resp. the weakly special intersections) of \( S \) for \( \mathcal{V} \) are the bi-algebraic subvarieties (resp. the bi-algebraic intersections) of \( S \) for the bi-algebraic structure on \( S \) defined by \( \mathcal{V} \).

Proof. The proof is similar to the proof of [UY11, Theor. 4.1], we provide it for completeness.

Notice that the statement for the weakly special intersections follows immediately from the statement for the weakly special subvarieties. Hence we are reduced to prove that the weakly special subvarieties of \( S \) coincide with the bi-algebraic subvarieties of \( S \).

That a weakly special subvariety of \( S \) is bi-algebraic follows from the fact that a Hodge morphism of Hodge varieties \( \varphi : T \rightarrow Y \) is defined at the level of the universal cover by a closed analytic embedding \( \hat{D}_T^+ \hookrightarrow \hat{D}_Y^+ \) restriction of a closed algebraic immersion \( \hat{D}_T \hookrightarrow \hat{D}_Y \).

Conversely let \( W \) be a bi-algebraic subvariety of \( S \). With the notations of Proposition 2.13 the period map \( \Phi_S|_W : W \rightarrow \text{Hod}^0(S, \mathcal{V}) \) factorises trough the weakly special subvariety \( \varphi((\Gamma_{\mathcal{H}_W} \backslash D^+_{\mathcal{H}_W}) \times \{t'\}) \) of \( \text{Hod}^0(S, \mathcal{V}) \). Let \( Z \) be an irreducible component of the preimage of \( W \) in \( \hat{S} \) and consider the lifting \( \tilde{\Phi}|_Z : Z \rightarrow D^+_{\mathcal{H}_W} \) of \( \Phi_S|_W \) to \( Z \). As \( W \) is bi-algebraic the Zariski-closure of \( \tilde{\Phi}|_Z(Z) \) in \( D^+_{\mathcal{H}_W} \) has to be stable under the monodromy group \( \mathcal{H}_W(\mathbb{C}) \), hence equal to \( D^+_{\mathcal{H}_W} \). Thus \( Z = (\tilde{\Phi}|_Z)^{-1}(D^+_{\mathcal{H}_W}) \) and \( W \) is weakly special. \( \square \)

We will need the following result, proven for Shimura varieties in [KUY16], conjectured in general in [K17, Conj. 7.6] as a special case of [K17, Conj. 7.5], and proven by Bakker-Tsimerman [BT17, Theor. 1.1]:

Theorem 2.20. (Ax-Lindemann for \( \mathcal{Z} \) VMHS) Let \((S, \mathcal{V})\) be a \( \mathcal{Z} \) VMHS. Let \( Y \subset \hat{S} \) be an algebraic subvariety for the bi-algebraic structure defined by \( \mathcal{V} \). Then \( \pi(Y)_{\text{Zar}} \) is a bi-algebraic subvariety of \( S \), i.e. a weakly special subvariety of \( S \) for \( \mathcal{V} \).

3. A converse to Theorem 1.1: proof of Proposition 1.18

Let \( f : \mathcal{V}' \rightarrow \mathcal{V} \) be a finite étale cover and let \( \mathcal{V}' := f^*\mathcal{V} \). By abuse of notation let \( f \) still denote the natural map \( \mathcal{V}' \rightarrow \mathcal{V} \). The reader will immediately check the following (where, with the notations of Section 1, we naturally identify \( V' \) with \( V \)):

Lemma 3.1. (a)
\[ \forall \lambda \in V'^*, \forall i \in \mathbb{Z}, \quad \mathcal{V}^{ri}(\lambda) = f^{-1}\mathcal{V}^i(\lambda) \quad \text{and} \quad f(\mathcal{V}^{ri}(\lambda)) = \mathcal{V}^i(\lambda). \]
(b) the \( f \)-image of a special subvariety of \( S' \) for \( \forall' \) is a special subvariety of \( S \) for \( \forall \); conversely the \( f \)-preimage of a special subvariety of \( S \) for \( \forall \) is a finite union of special subvarieties of \( S' \) for \( \forall' \).

Hence proving Proposition 1.18 for \( \forall \) is equivalent to proving it for \( \forall' \). As any finitely generated linear group admits a torsion-free finite index subgroup (Selberg’s lemma) we can thus assume without loss of generality by replacing \( S \) by a finite étale cover if necessary that the monodromy \( \rho(\pi_1(S, s_0)) \subset GL(V) \) is torsion-free.

Let \( \lambda \in V^* \) be such that \( V^i(\lambda) \) is an algebraic subvariety of \( V \). Hence \( V^i([\lambda]) \subset \mathbb{P}V \) is also algebraic. As the projection \( p : PV \to S \) is a proper morphism, it follows that the set \( S^i(\lambda) := p(V^i(\lambda)) \) is an algebraic subvariety of \( S \).

Let \( n : S' \to S^i(\lambda) \) be the smooth locus of the normalisation of one irreducible component of \( S^i(\lambda) \). Hence \( S' \) is connected. Let \( \pi' : \tilde{S}' \to S' \) be its universal cover and by \( \rho' : \pi_1(S', s_0') \to GL(V) \) the monodromy of the local system \( \forall' := i^{-1}V \) on \( S' \). Let

\[
\widetilde{V^i}(\lambda) := \pi'^{-1}(V^i(\lambda)) \subset \widetilde{V}(\lambda) := \pi'^{-1}(V'(\lambda)) \cong \tilde{S}' \times \{ \lambda \} \subset \tilde{V}' \cong \tilde{S}' \times V .
\]

As \( V^i(\lambda) \subset \tilde{S}' \times \{ \lambda \} \) and \( p : \widetilde{V^i}(\lambda) \to \tilde{S}' \) is surjective, it follows that \( \forall^i(\lambda) = \tilde{S}' \times \{ \lambda \} \), hence \( \forall^i(\lambda) = V^i(\lambda) \).

In particular

\[
V^i(\lambda) \cap V = V^i(\lambda) \cap V = \rho(\pi_1(S', s_0')) \cdot \lambda \subset V ,
\]

where \( V \) is identified with \( V_{s_0}' \). As \( \forall^i(\lambda) \subset \forall' \) is an algebraic subvariety, its fiber \( \forall^i(\lambda) \cap V \) is an algebraic subvariety of \( V \). On the other hand the set \( \rho(\pi_1(S', s_0')) \cdot \lambda \) is countable. Thus \( \forall^i(\lambda) \cap V \) is a finite set of points, in particular \( p : \forall^i(\lambda) \to S' \) is finite étale.

It follows that the smallest \( \mathbb{Q} \)-sub-local system \( \forall^i_{\mathbb{Q}} \subset \forall' \) whose complexification \( \forall^i_{\mathbb{Q}} \subset \forall' \) contains \( \forall^i(\lambda) \) has finite monodromy. As the monodromy \( \rho(\pi_1(S')) \) is a subgroup of \( \rho(\pi_1(S)) \) which is assumed to be torsion-free, it follows that the local system \( \forall^i_{\mathbb{Q}} \) is trivial. By the theorem of the fixed part (see [Sc73, Cor. 7.23]) \( \forall^i_{\mathbb{Q}} \) is a constant sub-\( \mathbb{Q} \)VHS of \( \forall' \). It follows easily that \( n(S') \) is the smooth locus of an irreducible component of the Hodge locus in \( S \) defined by the fiber \( W_{\mathbb{Q}} \subset V_{\mathbb{Q}} \) of \( \forall^i_{\mathbb{Q}} \).

This finishes the proof that \( S^i(\lambda) \) is a union of special subvarieties of \( S \) and that \( p : \forall^i(\lambda) \to S^i(\lambda) \) is finite.

\[\square\]

4. Proof of Theorem 1.17

Theorem 1.17 follows from Theorem 2.20 and the following

Proposition 4.1. Any component of \( S^i(\lambda) \) is the image under \( \pi \) of an algebraic subvariety of \( \tilde{S} \) (for the bi-algebraic structure on \( S \) defined by \( \forall \)).

Proof of Proposition 4.1. The quadruple \((\forall, \nabla, F^\bullet)\) defining the ZVHS \( \forall \) is the pull-back under \( \Phi_S \) of a similar quadruple \((\forall_{\mathbb{Z}, \Gamma \setminus D^+, \nabla_{\Gamma \setminus D^+}, F^*_{\Gamma \setminus D^+})\) on the connected Hodge variety \( \Gamma \setminus D^+ \), which however does not satisfy Griffiths transversality. This quadruple itself comes, by restriction to \( D^+ \) and descent to \( \Gamma \setminus D^+ \), from a \( G(\mathbb{C}) \)-equivariant
quadriples \((V_{\hat{D}}, V_D, \nabla_D, F_D^*)\) on \(\hat{D}\). As \(\hat{D}\) is simply connected the algebraic flat connection \(\nabla_D\) induces a canonical algebraic trivialization \(V_{\hat{D}} \cong V_D \times V\). Hence we have a commutative diagram

\[
\begin{array}{c}
\mathcal{V} \xleftarrow{\pi} \tilde{\mathcal{V}} \cong \tilde{S} \times V \xrightarrow{\Phi_S \times \text{Id}} \hat{D}_S \times V \cong V_{\hat{D}} \\
p \downarrow \quad p \downarrow \quad p_1 \\
S \xleftarrow{\pi} \tilde{S} \xrightarrow{\Phi_S} \hat{D}_S.
\end{array}
\]

Let \(N\) be a component of \(S^i(\lambda)\). Hence \(N = \pi(Y)\), where \(Y = p(W)\) for \(W \subset \tilde{V} = \tilde{S} \times V\) an analytic irreducible component of the complex analytic subvariety

\[
\tilde{V}(\lambda) := (\Phi_S \times \text{Id})^{-1}(\{\lambda\} \cap F^iV_D)
\]

of \(\tilde{V}\). Let us define \(\hat{D}(\lambda) \subset \hat{D}\) as the projection \(p_1((\hat{D} \times \{\lambda\}) \cap F^iV_D)\). In particular, corresponding to the diagram (4.1), we have a commutative diagram

\[
\begin{array}{c}
\pi(W) \xleftarrow{\pi} W \xrightarrow{\Phi_S \times \text{Id}} (\hat{D} \times \{\lambda\}) \cap F^iV_D \\
p \downarrow \quad p \downarrow \quad p_1 \\
N \xleftarrow{\pi} Y \xrightarrow{\Phi_S} \hat{D}(\lambda)
\end{array}
\]

Notice that both \((\hat{D} \times \{\lambda\})\) and \(F^iV_D\), hence also their intersection \((\hat{D} \times \{\lambda\}) \cap F^iV_D\), are algebraic subvarieties of \(\hat{D} \times V\). Thus their projection \(\hat{D}(\lambda)\) is an algebraic subvariety of \(\hat{D}\). It follows that the irreducible component \(Y\) of \(\Phi_S^{-1}(\hat{D}(\lambda))\) is an algebraic subvariety of \(\tilde{S}\) for the bi-algebraic structure on \(S\) for \((S, V)\).

\[\square\]

5. Algebraicity of the Positive Dimensional \(F^i\)-Locus and Relation with the \(\mathbb{Q}\)-Structure

5.1. An algebraicity result for flat complex connections.

**Theorem 5.1.** Let \(d\) be a positive integer. Let \(p : (V, \nabla) \rightarrow S\) be an algebraic flat connection on a smooth quasi-projective complex variety \(S\) and \(V \subset \nabla\) the associated complex local system. Let \(F \subset V\) be an algebraic subvariety. For \(x \in F\) let \(N_{F,x}\) denote the union of irreducible components containing \(x\) of the complex analytic subvariety \((V(x) \cap F)^{\text{red}}\) of the étale space of \(V\).

The locus \(A_{F \geq d}\) of closed points \(x \in F\) such that \(N_{F,x}\) has dimension at least \(d\) at \(x\), is an algebraic subvariety of \(V\).

**Proof.** Let \(T_hV \subset TV\) denote the horizontal algebraic subbundle of the tangent bundle \(TV\) defined by the flat connection \(\nabla\). We write \(q : \mathbb{P}(TV) \rightarrow V\) for the (proper) natural projection. Let \(T_hF := T_hV \times_{TV} TF\). We define inductively reduced algebraic varieties \((A_{F \geq d,n})_{n \in \mathbb{N}} \subset V\) by
- \( A_{F, \geq d, 0} := F, \)
- \( A_{F, \geq d, n+1} := \{ x \in A_{F, \geq d, n} \mid \dim((T_h A_{F, \geq d, n})_x) \geq d \} \).

Let \( A_{F, \geq d, \infty} := \bigcap_{n \in \mathbb{N}} A_{F, \geq d, n} \). As the \( A_{F, \geq d, n} \) are algebraic subvarieties of \( \mathcal{V} \), so is \( A_{F, \geq d, \infty} \).

From now on we write for simplicity \( A_d := A_{F, \geq d} \), \( A_{d, n} := A_{F, \geq d, n} \), \( A_{d, \infty} := A_{F, \geq d, \infty} \), and \( N_x := N_{F, x} \). The result then follows from Lemma 5.2 below.

**Lemma 5.2.** The equality \( A_d = A_{d, \infty} \) holds.

**Proof.** The inclusion \( A_d \subset A_{d, \infty} \) is equivalent to the inclusions \( A_d \subset A_{d, n} \) for all \( n \in \mathbb{N} \), which we show by induction on \( n \). By definition \( A_d \subset F = A_{d, 0} \). Assume that \( A_d \subset A_{d, n} \) for some \( n \in \mathbb{N} \). By definition of \( A_d \), for any \( x \in A_d \) the variety \( A_d \) contains an irreducible component \( h \) of \( N_{F, x} \) through \( x \) of dimension at least \( d \). Hence

\[
d \leq \dim(N) \leq \dim(T_x N) \leq \dim((T_h A_{d, n})_x),
\]

hence \( x \in A_{d, n+1} \). This shows \( A_d \subset A_{d, n+1} \) and finishes the proof by induction that \( A_d \subset A_{d, \infty} \).

Conversely let us prove that \( A_{d, \infty} \subset A_d \). Let \( h : \tilde{\mathcal{V}} \rightarrow \mathcal{V} \) denote the composition

\[
h : \tilde{\mathcal{V}} \simeq \tilde{S} \times X \overset{p_2}{\rightarrow} \mathcal{V}
\]

(where the first isomorphism is provided by the flat trivialisation). For \( x \in \mathcal{V} \) and \( \tilde{x} \in \pi^{-1}(x) \subset \tilde{\mathcal{V}} \simeq \tilde{S} \times X \) let \( N_{\tilde{x}} \) be the union of the irreducible components passing through \( \tilde{x} \) of the complex analytic subvariety \( h^{-1}(h(\tilde{x})) \cap \pi^{-1}(F) \) of \( \mathcal{V} \). Thus the local biholomorphism \( \pi : \tilde{\mathcal{V}} \rightarrow \mathcal{V} \) identifies \( N_{\tilde{x}} \) locally at \( \tilde{x} \) with \( N_x \) locally at \( x \).

By noetherianity there exists an \( n \in \mathbb{N} \) such that \( A_{d, n} = A_{d, n+1} = A_{d, \infty} \). Hence for any \( x \in A_{d, \infty} \) we have \( \dim((T_h A_{d, \infty})_x) \geq d \). Let us consider the restriction

\[
h_{|A_{d, \infty}} : A_{d, \infty} \rightarrow \mathcal{V}
\]

of \( h \) to \( A_{d, \infty} \). Let \( U_{d, \infty} \subset A_{d, \infty} \) be the Zariski-dense open subset of smooth points \( x \in A_{d, \infty} \) such that the complex analytic map \( h_{|A_{d, \infty}} \) is smooth and locally submersive onto its image at any \( \tilde{x} \in \{ x \} \times S \tilde{S} \). Hence, for \( x \in U_{d, \infty} \),

\[
\dim_{\tilde{x}}(A_{d, \infty} \times_X h(\tilde{x})) = \dim((T_h A_{d, \infty})_x) \geq d.
\]

Since \( U_{d, \infty} \) is Zariski-dense in \( A_{d, \infty} \), the inequality \( \dim_{\tilde{x}}(A_{d, \infty} \times_X h(\tilde{x})) \geq d \) holds for any \( \tilde{x} \) in the preimage \( A_{d, \infty} \) of \( A_{d, \infty} \) in \( \tilde{\mathcal{V}} \). As \( A_{d, \infty} \subset F \), any analytic irreducible component of \( A_{d, \infty} \times_X h(\tilde{x}) \) containing \( x \) is contained in \( N_{\tilde{x}} \). Thus Equation (5.1) implies that for any \( x \in A_{d, \infty} \) we have \( \dim_{\tilde{x}}(N_{\tilde{x}}) \geq d \), i.e. \( x \in A_d \). \( \square \)

### 5.2. Applications to \( \mathbb{Q} \)-local systems

The following saturation result will be crucial in the proof of Theorem 1.7:

**Proposition 5.3.** In the situation of Theorem 5.1 suppose moreover that the local system \( \mathcal{V} = \mathcal{V}_Q \otimes_{\mathbb{Q}} \mathbb{C} \) is defined over \( \mathbb{Q} \). Let \( A_{F, \geq d, Q} := A_{F, \geq d} \cap \mathcal{V}_Q \) be the locus of rational classes whose flat transport meets \( F \) in dimension \( \geq d \) and let \( A'_{F, \geq d} := A_{F, \geq d, Q}^{\text{zar}} \) be its Zariski-closure in \( \mathcal{V} \). There is a Zariski-open dense subset \( U \) of \( A'_{F, \geq d} \) such that \( U \subset \bigcup_{x \in U} N_{F, x} \subset A'_{F, \geq d} \).
Notice first that \( A_{F_i \geq d,Q} \subseteq A_{F \geq d} \) hence \( A'_{F \geq d} \subseteq A_{F \geq d} \) as \( A_{F \geq d} \) is algebraic by Theorem 5.1. As in the proof of Theorem 5.1 we remove from now on the reference to \( F \) in our notations. For each irreducible component \( W \) of \( A'_{\geq d} \), let \( E \) be the Zariski-open dense subset of all \( x \in W \) such that the variety \( W \) is smooth at \( x \) and the morphism \( h_{|W}: \tilde{W} \to V \) is locally surjective onto its image at any \( \tilde{x} \in \{x\} \times_S \tilde{S} \). The fibers of the morphism \( h_{|E}: \tilde{E} \to V \) are smooth of constant dimension \( D \geq d \) and for any \( \tilde{x} \in \tilde{E} \) we have

\[
D = \dim_{\mathbb{C}} \left( \tilde{W} \times_{H_x} \tilde{x} \right) = \dim_{\mathbb{C}} (N_{\tilde{x}} \cap \tilde{W}) \leq \dim_{\mathbb{C}} (N_{\tilde{x}}).
\]

Let us define \( U := E \setminus (E \cap A_{\geq D+1}) \). For any \( x \in U \) the variety \( N_x \) is smooth at \( x \), hence it is irreducible. Moreover, we have \( D = \dim_x (N_x \cap U) \leq \dim_x (N_x) = D \), hence \( N_x \cap U \) is open and dense in \( N_x \). Hence \( U \) is dense in \( \bigcup_{x \in U} N_x \) (for the usual topology) and \( \bigcup_{x \in U} N_x \subseteq W \).

Since \( A_{\geq D+1} \) is Zariski-closed in \( V \), \( U \) is Zariski-open in \( E \), hence in \( W \). Moreover, for any \( x \in A_{\geq d,Q} \), we have \( N_x \subseteq A_{\geq d,Q} \), hence for any \( x \in A_{\geq d,Q} \cap E \), the locus \( N_x \cap E \) is an open subset of \( N_x \). Hence \( \dim_x (N_x) = \dim_x (N_x \cap E) = D \) and \( x \in U \). Since \( A_{\geq d,Q} \) is Zariski-dense in \( W \), \( A_{\geq d,Q} \cap E \) is not empty, hence \( U \) is not empty.

\[\square\]

5.3. Application to ZVHS: proof of Theorem 1.15. Theorem 1.15 is the special case of Theorem 5.1 where \( V \) is a ZVHS and \( F = F^n \). In this case \( A_{F_i \geq d} = V_i \geq d \) and \( A_{F_i \geq d,Q} = V_i \geq d \cap V_Q \). Proposition 5.3 reads in this case:

**Proposition 5.4.** Let \( S \) be a smooth complex quasi-projective algebraic variety and \( V \) be a polarized ZVHS over \( S \). Let \( i \in \mathbb{Z} \) and \( d \in \mathbb{N} \). There is a Zariski-open dense subset \( U \) of \( V_i \geq d \cap V_Q \) such that \( U \subseteq \bigcup_{x \in U} N_{F_i,x} \subseteq V_i \geq d \).

For \( x \in F \) corresponding by flat transport to a class \( \lambda \in V^* \) the analytic variety \( N_{F,x} \) is the union of components of \( V^i(\lambda) \) passing through \( x \). Hence the projection \( p(N_{F,x}) \) is a union of components of \( S^i(\lambda) \). By Theorem 1.17 the Zariski-closure of any such components is a weakly special subvariety of \( S \). We thus obtain

**Corollary 5.5.** Let \( S \) be a smooth complex quasi-projective algebraic variety and \( V \) be a polarized ZVHS over \( S \). Let \( d \in \mathbb{N} \). Then \( S^i(\bigcap V_i)_{\geq d} \cap \operatorname{HL}(S, V)_{\Zar} \) contains a Zariski-open dense set \( U \) with the following property: for each point \( x \in U \) there exists a weakly special subvariety \( Y_x \subseteq S^i(\bigcap V_i)_{\geq d} \cap \operatorname{HL}(S, V)_{\Zar} \) of dimension at least \( d \) passing through \( x \).

6. Proof of Theorem 1.7

The following result of Deligne (see [Voi13, Theor. 4.10]) will be important for us: there exists a bound on the tensors one has to consider for defining \( \operatorname{HL}(S, V^\otimes) \). Thus \( \operatorname{HL}(S, V^\otimes) = \bigcup_{i=1}^n \operatorname{HL}(S, V_i) \) for finitely many irreducible ZVHS \( V_i \subseteq V^\otimes \). It follows that \( \operatorname{HL}(S, V^\otimes)_{\pos} = \bigcup_{i=1}^n \operatorname{HL}(S, V_i)_{\pos} \).

To make the proof of Theorem 1.7 more transparent we deal first with special cases.

**Case 1:** the period map \( \Phi_S \) is an immersion. In that case

\[
\operatorname{HL}(S, V^\otimes)_{\pos} = S^0(V^\otimes)_{\geq 1} \cap \operatorname{HL}(S, V^\otimes) = \bigcup_{i=1}^n S^0(V_i)_{\geq 1} \cap \operatorname{HL}(S, V_i) .
\]
Applying Corollary 5.5 to each \( V_i \), \( 1 \leq i \leq n \), it follows that \( \text{HL}(S, V^\otimes_{\text{pos}})_{\text{Zar}} \) contains a Zariski-open dense subset \( U \) with the following property: for each point \( x \in U \) there exists a positive dimensional weakly special subvariety \( W_x \) for \( (S, V) \) passing through \( x \) and contained in \( \text{HL}(S, V^\otimes_{\text{pos}})_{\text{Zar}} \).

Either there exists \( x \in U \) such that \( W_x = S \), in which case \( \text{HL}(S, V^\otimes_{\text{pos}})_{\text{Zar}} = S \). Or for all \( x \in U \) the weakly special subvariety \( W_x \) of \( S \) is strict. In this case the assumption that \( \text{MT}(S, V) \) is non-product and the description of weakly special subvarieties given in Section 2.1 implies that each \( W_x \) is contained in a unique strict positive dimensional special subvariety \( S_x \) of \( S \). As \( S_x \) belongs by definition to \( \text{HL}(S, V^\otimes_{\text{pos}}) \), it follows in this case that \( \text{HL}(S, V^\otimes)_{\text{pos}} = \text{HL}(S, V^\otimes)_{\text{pos}} \) is a finite union of strict special subvarieties of \( S \), hence the result.

\textbf{Case 2:} the period map \( \Phi_S \) has constant relative dimension \( d \). The proof is the same as in the first case, replacing \( S^0(V^\otimes_{\geq 1}), S^0(V^\otimes_{\geq 1}) \), and “positive dimensional” by \( S^0(V^\otimes_{\geq d+1}), S^0(V^\otimes)_{\geq d+1} \), and “at least \((d+1)\)-dimensional”.

\textbf{General case:} As the period map \( \Phi_S \) is definable in the o-minimal structure \( \mathbb{R}_{\text{an,exp}} \) (see [BKT18]), it follows from the trivialization [VDD98, Theor. 1.2] that the locus \( S_d \subset S \) where the fibers of \( \Phi_S \) are of complex dimension at least \( d \) is an \( \mathbb{R}_{\text{an,exp}} \)-definable subset of \( S \). As \( S_d \) is also a closed complex analytic subset of \( S \), if follows from the o-minimal Chow theorem [PS09, Theor.4.4 and Cor. 4.5] of Peterzil-Starchenko that \( S_d \) is a closed algebraic subvariety of \( S \). Finally we obtain an algebraic filtration \( S = S_{d_0} \supseteq S_{d_1} \supseteq \cdots \supseteq S_{d_k} \supseteq S_{d_{k+1}} = \emptyset \).

Suppose that \( \text{HL}(S, V^\otimes_{\text{pos}}) \) is not algebraic. Let \( 0 \leq i \leq k \) be the smallest integer such that \( (S_{d_i} - S_{d_{i+1}}) \cap \text{HL}(S, V^\otimes_{\text{pos}}) \) is not an algebraic subvariety of \( S_{d_i} - S_{d_{i+1}} \). As \( \text{HL}(S - S_{d_{i+1}}, V^\otimes_{S - S_{d_{i+1}}}) = \text{HL}(S, V^\otimes_{\text{pos}}) \cap (S - S_{d_{i+1}}) \), to prove that \( \text{HL}(S, V^\otimes_{\text{pos}}) \) is Zariski-dense in \( S \) we can and will assume without loss of generality that \( i = k \) (replacing \( S \) by \( S - S_{d_{i+1}} \) if necessary).

Without loss of generality we can assume that \( \text{HL}(S, V^\otimes_{\text{pos}}) \) is contained in \( S_{d_i} \); this is clear if \( i = 0 \), as \( S = S_{d_i} \) in this case; if \( i > 0 \) there are only finitely many maximal positive special subvarieties \( Z_1, \ldots, Z_m \) of \( S \) for \( V \) intersecting \( S_{d_{i-1}} - S_{d_i} \), and we can without loss of generality replace \( S \) by \( S - (Z_1 \cup \cdots \cup Z_m) \).

Thus \( \text{HL}(S, V^\otimes_{\text{pos}}) \) coincide with \( S^0(V^\otimes_{\geq d_{i+1}}) \cap \text{HL}(S, V^\otimes) \). Applying Corollary 5.5 with \( d = d_{i+1} \), it follows that the union \( Z \) of irreducible components of \( \text{HL}(S, V^\otimes_{\text{pos}})_{\text{Zar}} \) contains a Zariski-open dense set \( U \) such that for every point \( x \in U \) there exists a weakly special subvariety \( W_x \) of \( S \) for \( V \) of dimension at least \( d_i + 1 \) passing through \( x \) and contained in \( Z \).

If \( i > 0 \) the weakly special subvariety \( W_x \subset Z \subset S_{d_i} \) is strict, and we conclude as above: each \( W_x \) is contained in a unique strict positive special subvariety \( S_x \) for \( V \), thus \( Z = \text{HL}(S, V^\otimes_{\text{pos}}) \), which contradicts the assumption that \( \text{HL}(S, V^\otimes_{\text{pos}}) \) is not an algebraic subvariety of \( S \).

Thus \( i = 0 \). Hence we are in Case 2 above and we conclude that \( \text{HL}(S, V^\otimes) \) is Zariski-dense in \( S_{d_0} = S \). This finishes the proof of Theorem 1.7. 
\( \square \)
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