The Kudla-Millson form via the Mathai-Quillen formalism

Romain Branchereau

November 21, 2022

Abstract

In [6], Kudla and Millson constructed a $q$-form $\varphi_{KM}$ on an orthogonal symmetric space using Howe’s differential operators. It is a crucial ingredient in their theory of theta lifting. This form can be seen as a Thom form of a real oriented vector bundle. In [9] Mathai and Quillen constructed a canonical Thom form and we show how to recover the Kudla-Millson form via their construction. A similar result was obtained by [3] for signature $(2, q)$ in case the symmetric space is hermitian and we extend it to an arbitrary signature.

1 Introduction

Let $(V, Q)$ be a quadratic space over $\mathbb{Q}$ of signature $(p, q)$ and let $G$ be its orthogonal group. Let $D$ be the space of oriented negative $q$-planes in $V(\mathbb{R})$ and $D^+$ one of its connected components. It is a Riemannian manifold of dimension $pq$ and an open subset of the Grassmannian. The Lie group $G(\mathbb{R})^+$ is the connected component of the identity and acts transitively on $D$. Hence we can identify $D$ with $G(\mathbb{R})^+/K$, where $K$ is a compact subgroup of $G(\mathbb{R})^+$ and is isomorphic to $SO(p) \times SO(q)$. Moreover, let $L$ be a lattice in $V(\mathbb{Q})$ and $\Gamma$ be a torsion free subgroup of $G(\mathbb{R})^+$ preserving $L$.

For every vector $v$ in $V(\mathbb{R})$ there is a totally geodesic submanifold $D^+_v$ of codimension $q$ consisting of all the negative $q$-planes that are orthogonal to $v$. Let $\Gamma_v$ denote the stabilizer of $v$ in $\Gamma$. We can view $\Gamma_v \setminus D^+_v$ as a rank $q$ vector bundle over $\Gamma_v \setminus D^+_v$, so that the natural embedding $\Gamma_v \setminus D^+_v$ in $\Gamma_v \setminus D^+_v$ is the zero section. In [6], Kudla and Millson constructed a closed $G(\mathbb{R})^+$-invariant differential form

$$\varphi_{KM} \in \left[\Omega^q(D^+) \otimes \mathcal{S}(V(\mathbb{R}))\right]^{G(\mathbb{R})^+},$$

where $G(\mathbb{R})^+$ acts on the Schwartz space $\mathcal{S}(V(\mathbb{R}))$ from the left by $(gf)(v) := f(g^{-1}v)$ and on $\Omega^q(D^+) \otimes \mathcal{S}(V(\mathbb{R}))$ from the right by $g \cdot (\omega \otimes f) := g^* \omega \otimes (g^{-1}f)$. In particular $\varphi_{KM}(v)$ is a $\Gamma_v$-invariant form on $D^+$. The main property of the Kudla-Millson form is its Thom form property: if $\omega$ in $\Omega^q(\Gamma_v \setminus D^+_v)$ is a compactly supported form, then

$$\int_{\Gamma_v \setminus D^+_v} \varphi_{KM}(v) \wedge \omega = 2^{-\frac{q}{2}} e^{-\pi Q(v, v)} \int_{\Gamma_v \setminus D^+_v} \omega.$$  (1.2)

Another way to state it is to say that in cohomology we have

$$[\varphi_{KM}(v)] = 2^{-\frac{q}{2}} e^{-\pi Q(v, v)} \text{PD}(\Gamma_v \setminus D^+_v) \in H^q(\Gamma_v \setminus D^+_v),$$

where PD$(\Gamma_v \setminus D^+_v)$ denotes the Poincaré dual class to $\Gamma_v \setminus D^+_v$.  


Kudla-Millson theta lift. In order to motivate the interest in the Kudla-Millson form, let us briefly recall how it is used to construct a theta correspondence between certain cohomology classes and modular forms. Let $\omega$ be the Weil representation of $SL_2(\mathbb{R})$ in $\mathcal{S}(V(\mathbb{R}))$. We extend it to a representation in $\Omega^q(\mathbb{D}^+) \otimes \mathcal{S}(V(\mathbb{R}))$ by acting in the second factor of the tensor product. Building on the work of [10], Kudla and Millson [7, 8] used their differential form to construct the theta series

$$\Theta_{KM}(\tau) := y^{-\frac{3q+1}{2}} \sum_{v \in \mathcal{L}} (\omega(g_v) \varphi_{KM})(v) \in \Omega^q(\mathbb{D}^+), \tag{1.4}$$

where $\tau = x + iy$ is in $\mathbb{H}$ and $g_v$ is the matrix $\left( \begin{array}{cc} \sqrt{y} & x \sqrt{y}^{-1} \\ 0 & \sqrt{y}^{-1} \end{array} \right)$ in $SL_2(\mathbb{R})$ that sends $i$ to $\tau$ by Möbius transformation. This form is $\Gamma$-invariant, closed and holomorphic in cohomology. Kudla and Millson showed that if we integrate this closed form on a compact $q$-cycle $C$ in $Z_q(\Gamma \backslash \mathbb{D}^+)$, then

$$\int_C \Theta_{KM}(\tau) = c_0(C) + \sum_{n=1}^\infty \langle C, C_{2n} \rangle e^{2i\pi n \tau} \tag{1.5}$$

is a modular form of weight $\frac{q+q}{2}$, where

$$C_n := \sum_{v \in \mathcal{L} \cap Q(v,v)=n} C_v \tag{1.6}$$

and the special cycles $C_v$ are the images of the composition

$$\Gamma_v \backslash \mathbb{D}^+ \longrightarrow \Gamma_v \backslash E \longrightarrow \Gamma \backslash E. \tag{1.7}$$

Thus, the Kudla-Millson theta series realizes a lift between the (co)-homology of $\Gamma \backslash \mathbb{D}^+$ and the space of weight $\frac{q+q}{2}$ modular forms.

The result. Let $E$ be a $G(\mathbb{R})$-equivariant vector bundle of rank $q$ over $\mathbb{D}^+$ and $E_0$ the image of the zero section. By the equivariance we also have a vector bundle $\Gamma_v \backslash E$ over $\Gamma_v \backslash \mathbb{D}^+$. The Thom class of the vector bundle is a characteristic class $\text{Th}(\Gamma_v \backslash E)$ in $H^q(\Gamma_v \backslash E, \Gamma_v \backslash (E - E_0))$ defined by the Thom isomorphism; see Subsection 3.6. A Thom form is a form representing the Thom class. It can be shown that the Thom class is also the Poincaré dual class to $\Gamma_v \backslash E_0$. Let $s_v : \Gamma_v \backslash \mathbb{D}^+ \longrightarrow \Gamma_v \backslash E$ be a section whose zero locus is $\Gamma_v \backslash \mathbb{D}_{v}^+$, then

$$s_v^* \text{Th}(\Gamma_v \backslash E) \in H^q(\Gamma_v \backslash \mathbb{D}^+, \Gamma_v \backslash (\mathbb{D}^+ - \mathbb{D}_{v}^+)). \tag{1.8}$$

Viewing it as a class in $H^q(\Gamma_v \backslash \mathbb{D}^+)$ it is the Poincaré dual class of $\Gamma_v \backslash \mathbb{D}_{v}^+$. Since the Poincaré dual class is unique, property (1.3) implies that

$$[\varphi_{KM}(v)] = 2^{-\frac{q}{2}} e^{-\pi Q(v,v)} s_v^* \text{Th}(\Gamma_v \backslash E) \in H^q\left( \Gamma_v \backslash \mathbb{D}^+ \right), \tag{1.9}$$

on the level of cohomology.

For arbitrary oriented real metric vector bundles, Mathai and Quillen used the Chern-Weil theory to construct in [9] a canonical Thom forms on $E$. We denote by $U_{MQ}$ the canonical Thom form in $\Omega^q(E)$ of Mathai and Quillen. Since $U_{MQ}$ is $\Gamma$-invariant, it is also a Thom form for the bundle $\Gamma_v \backslash E$ for every vector $v$. The main result is the following.

2
Theorem. (Theorem 4.5) We have \( \varphi_{KM}(v) = 2 - 2e^{-\pi Q(v,v)}s_\nu U_{Mq} \) in \( \Omega^2(\Gamma_v \backslash \mathbb{D}^+) \).

For signature \((2, q)\), the spaces are hermitian and the result was obtained by a similar method in [3] using the work of Bismut-Gillet-Soulé.

Acknowledgements

This project is part of my thesis and I thank my advisors Nicolas Bergeron and Luis Garcia for suggesting me this topic and for their support. I was funded from the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement N\textsuperscript{3}o754362.

2 The Kudla-Millson form

2.1 The symmetric space \( \mathbb{D} \)

Let \((V, Q)\) be a rational quadratic space and let \((p, q)\) be the signature of \(V(\mathbb{R})\). Let \(e_1, \ldots, e_{p+q}\) be an orthogonal basis of \(V(\mathbb{R})\) such that

\[
Q(e_\alpha, e_\alpha) = 1 \quad \text{for} \quad 1 \leq \alpha \leq p,
Q(e_\mu, e_\mu) = -1 \quad \text{for} \quad p + 1 \leq \mu \leq p + q. \tag{2.1}
\]

Note that we will always use letters \(\alpha\) and \(\beta\) for indices between 1 and \(p\), and letters \(\mu\) and \(\nu\) for indices between \(p + 1\) and \(p + q\). A plane \(z\) in \(V(\mathbb{R})\) is a negative plane if \(Q|_z\) is negative definite. Let

\[
\mathbb{D} := \{ z \subset V(\mathbb{R}) \mid z \text{ is an oriented negative plane of dimension } q \} \tag{2.2}
\]

be the set of negative oriented \(q\)-planes in \(V(\mathbb{R})\). For each negative plane there are two possible orientations, yielding two connected components \(\mathbb{D}^+\) and \(\mathbb{D}^-\) of \(\mathbb{D}\). Let \(z_0\) in \(\mathbb{D}^+\) be the negative plane spanned by the vectors \(e_{p+1}, \ldots, e_{p+q}\) together with a fixed orientation. The group \(G(\mathbb{R})^+\) acts transitively on \(\mathbb{D}^+\) by sending \(z_0\) to \(g z_0\). Let \(K\) be the stabilizer of \(z_0\), which is isomorphic to \(SO(p) \times SO(q)\). Thus we have an identification

\[
G(\mathbb{R})^+/K \longrightarrow \mathbb{D}^+
\]

\[
g K \mapsto g z_0. \tag{2.3}
\]

For \(z\) in \(\mathbb{D}^+\) we denote by \(g_z\) any element of \(G(\mathbb{R})^+\) sending \(z_0\) to \(z\).

For a positive vector \(v\) in \(V(\mathbb{R})\) we define

\[
\mathbb{D}_v := \{ z \in \mathbb{D} \mid z \subset v^\perp \}. \tag{2.4}
\]

It is a totally geodesic submanifold of \(\mathbb{D}\) of codimension \(q\). Let \(\mathbb{D}_v^+\) be the intersection of \(\mathbb{D}_v\) with \(\mathbb{D}^+\).

Let \(z\) in \(\mathbb{D}^+\) be a negative plane. With respect to the orthogonal splitting of \(V(\mathbb{R})\) as \(z^\perp \oplus z\) the quadratic form splits as

\[
Q(v, v) = Q|_{z^\perp}(v, v) + Q|_z(v, v). \tag{2.5}
\]

We define the Siegel majorant at \(z\) to be the positive definite quadratic form

\[
Q^+_z(v, v) := Q|_{z^\perp}(v, v) - Q|_z(v, v). \tag{2.6}
\]
The Lie algebras $g$ and $k$

Let
\[
g := \left\{ \begin{pmatrix} A & x \\ t^x & B \end{pmatrix} \middle| A \in \mathfrak{so}(z_0^+), B \in \mathfrak{so}(z_0), x \in \text{Hom}(z_0, z_0^+) \right\}, \tag{2.7}\]
\[
f := \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \middle| A \in \mathfrak{so}(z_0^+), B \in \mathfrak{so}(z_0) \right\}. \tag{2.8}\]

be the Lie algebras of $G(\mathbb{R})^+$ and $K$ where $\mathfrak{so}(z_0)$ is equal to $\mathfrak{so}(q)$. The latter is the space of skew-symmetric $q$ by $q$ matrices. Similarly we have $\mathfrak{so}(z_0^+)$ equals $\mathfrak{so}(p)$. Hence we have a decomposition of $\mathfrak{f}$ as $\mathfrak{so}(z_0^+) \oplus \mathfrak{so}(z_0)$ that is orthogonal with respect to the Killing form. Let $\epsilon$ be the Lie algebra involution of $g$ mapping $X$ to $-tX$. The $+1$-eigenspace of $\epsilon$ is $k$ and the $-1$-eigenspace is $p$:
\[
p := \left\{ \begin{pmatrix} 0 & x \\ t^x & 0 \end{pmatrix} \middle| x \in \text{Hom}(z_0, z_0^+) \right\}. \tag{2.9}\]

We have a decomposition of $g$ as $\mathfrak{f} \oplus p$ and it is orthogonal with respect to the Killing form. We can identify $p$ with $g/k$. Since $\epsilon$ is a Lie algebra automorphism we have that
\[
[p, p] \subset \mathfrak{f}, \quad [\mathfrak{f}, p] \subset p. \tag{2.10}\]

We identify the tangent space of $\mathbb{D}^+$ at $eK$ with $p$ and the tangent bundle $T\mathbb{D}^+$ with $G(\mathbb{R})^+ \times_K p$ where $K$ acts on $p$ by the Ad-representation. We have an isomorphism
\[
T : \wedge^2 V(\mathbb{R}) \rightarrow g, \\
e_i \wedge e_j \mapsto T(e_i \wedge e_j)e_k := Q(e_i, e_k)e_j - Q(e_j, e_k)e_i. \tag{2.11}\]

A basis of $g$ is given by the set of matrices
\[
\{ X_{ij} := T(e_i \wedge e_j) \in g | 1 < i < j < p + q \} \tag{2.12}\]
and we denote by $\omega_{ij}$ its dual basis in the dual space $g^*$. Let $E_{ij}$ be the elementary matrix sending $e_i$ to $e_j$ and the other $e_k$'s to 0. Then $p$ is spanned by the matrices
\[
X_{\alpha\mu} = E_{\alpha\mu} + E_{\mu\alpha}, \tag{2.13}\]
and $\mathfrak{f}$ is spanned by the matrices
\[
X_{\alpha\beta} = E_{\alpha\beta} - E_{\beta\alpha}, \quad X_{\nu\mu} = -E_{\nu\mu} + E_{\mu\nu}. \tag{2.14}\]

Poincaré duals

Let $M$ be an arbitrary $m$-dimensional real orientable manifold without boundary. The integration map yields a non-degenerate pairing [2, Theorem. 5.11]
\[
H^q(M) \otimes_{\mathbb{R}} H^m-q_c(M) \rightarrow \mathbb{R}, \\
[\omega] \otimes [\eta] \mapsto \int_M \omega \wedge \eta, \tag{2.15}\]

4
where $H_c(M)$ denotes the cohomology of compactly supported forms on $M$. This yields an isomorphism between $H^q(M)$ and the dual $H^{m-q}_c(M)^* = \text{Hom}(H^{m-q}_c(M), \mathbb{R})$. If $C$ is an immersed submanifold of codimension $q$ in $M$ then $C$ defines a linear functional on $H^{m-q}_c(M)$ by

$$
\omega \mapsto \int_C \omega.
$$

Since we have an isomorphism between $H^{m-q}_c(M)^*$ and $H^q(M)$ there is a unique cohomology class $\text{PD}(C)$ in $H^q(M)$ representing this functional, i.e.

$$
\int_M \omega \wedge \text{PD}(C) = \int_C \omega
$$

for every class $[\omega]$ in $H^{m-q}_c(M)$. We call $\text{PD}(C)$ the Poincaré dual class to $C$, and any differential form representing the cohomology class $\text{PD}(C)$ a Poincaré dual form to $C$.

### 2.4 The Kudla-Millson form

The tangent plane at the identity $T_eK\mathbb{D}^+$ can be identified with $p$ and the cotangent bundle $(T\mathbb{D}^+)^*$ with $G(\mathbb{R})^+ \times_K p^*$, where $K$ acts on $p^*$ by the dual of the Ad-representation. The basis $e_1, \ldots, e_{p+q}$ identifies $V(\mathbb{R})$ with $\mathbb{R}^{p+q}$. With respect to this basis the Siegel majorant at $z_0$ is given by

$$
Q_{z_0}^+(v, v) := \sum_{i=1}^{p+q} x_i^2. \tag{2.18}
$$

Recall that $G(\mathbb{R})^+$ acts on $\mathcal{S}(\mathbb{R}^{p+q})$ from the left by $(g \cdot f)(v) = f(g^{-1}v)$ and on $\Omega^q(\mathbb{D}^+) \otimes \mathcal{S}(\mathbb{R}^{p+q})$ from the right by $g \cdot (\omega \otimes f) := g^* \omega \otimes (g^{-1}f)$. We have an isomorphism

$$
[\Omega^q(\mathbb{D}^+) \otimes \mathcal{S}(\mathbb{R}^{p+q})]^{G(\mathbb{R})^+} \xrightarrow{\varphi} \left[\bigwedge^q p^* \otimes \mathcal{S}(\mathbb{R}^{p+q})\right]^K
$$

by evaluating $\varphi$ at the basepoint $eK$ in $G(\mathbb{R})^+/K$, corresponding to the point $z_0$ in $\mathbb{D}^+$. We define the **Howe operator**

$$
D: \bigwedge^* p^* \otimes \mathcal{S}(\mathbb{R}^{p+q}) \rightarrow \bigwedge^{*+q} p^* \otimes \mathcal{S}(\mathbb{R}^{p+q}) \tag{2.20}
$$

by

$$
D := \frac{1}{2q} \prod_{\mu=p+1}^{p+q} \sum_{\alpha=1}^{p} A_{\alpha \mu} \otimes \left(x_\alpha - \frac{1}{2\pi} \frac{\partial}{\partial x_\alpha}\right) \tag{2.21}
$$

where $A_{\alpha \mu}$ denotes left multiplication by $\omega_{\alpha \mu}$. The Kudla-Millson form is defined by applying $D$ to the Gaussian:

$$
\varphi_{KM}(v)_e := D \exp \left(-\pi Q_{z_0}^+(v, v)\right) \in \bigwedge^q p^* \otimes \mathcal{S}(\mathbb{R}^{p+q}) \tag{2.22}
$$

Kudla and Millson showed that this form is $K$-invariant. Hence by the isomorphism (2.19) we get a form

$$
\varphi_{KM} \in \left[\Omega^q(\mathbb{D}^+) \otimes \mathcal{S}(\mathbb{R}^{p+q})\right]^{G(\mathbb{R})^+}. \tag{2.23}
$$
In particular it is $\Gamma_v$-invariant and defines a form on $\Gamma_v\backslash \mathbb{D}^+$. It is also closed and satisfies the Thom form property: for every compactly supported form $\omega$ in $\Omega^{p-q}_c(\Gamma_v\backslash \mathbb{D}^+)$ we have

$$
\int_{\Gamma_v\backslash \mathbb{D}^+} \omega \land \varphi_{KM}(v) = 2^{\frac{q}{2}} e^{-\pi Q(v,v)} \int_{\Gamma_v\backslash \mathbb{D}^+} \omega.
$$

(2.24)

3 The Mathai-Quillen formalism

We begin by recalling a few facts about principal bundles, connections and associated vector bundles. For more details we refer to [1] and [5]. The Mathai-Quillen form is defined in Subsection 3.7 following [1]; see also [4].

3.1 $K$-principal bundles and principal connections

Let $K$ be $\text{SO}(p) \times \text{SO}(q)$ as before and $P$ be a smooth principal $K$-bundle. Let

$$
R: K \times P \to P
$$

(3.1)

be the smooth right action of $K$ on $P$ and

$$
\pi: P \to P/K
$$

(3.2)

the projection map. For a fixed $p$ in $P$ consider the map

$$
R_p: K \to P
$$

(3.3)

$$
 k \mapsto R_k(p).
$$

Let $V_p P$ be the image of the derivative at the identity

$$
d_e R_p: \mathfrak{k} \to T_p P,
$$

(3.4)

which is injective. It coincides with the kernel of the differential $d_p \pi$. A vector in $V_p P$ is called a vertical vector. Using this map we can view a vector $X$ in $\mathfrak{k}$ as a vertical vector field on $P$. The space $P$ can a priori be arbitrary, but in our case we will consider either

1. $P$ is $G(\mathbb{R})^+$ and $R_k$ the natural right action sending $g$ to $gk$. Then $P/K$ can be identified with $\mathbb{D}^+$,

2. $P$ is $G(\mathbb{R})^+ \times z_0$ and the action $R_k$ maps $(g, w)$ to $(gk, k^{-1}w)$. In this case $P/K$ can be identified with $G(\mathbb{R})^+ \times_K z_0$. It is the vector bundle associated to the principal bundle $G(\mathbb{R})^+$ as defined below.

A principal $K$-connection on $P$ is a 1-form $\theta_P$ in $\Omega^1(P, \mathfrak{k})$ such that

- $\iota_X \theta_P = X$ for any $X$ in $\mathfrak{k}$,

- $R_k^* \theta_P = \text{Ad}(k^{-1}) \theta_P$ for any $k$ in $K$,

where $\iota_X$ is the interior product

$$
\iota_X: \Omega^k(P) \to \Omega^k(P)
$$

$$
\omega \mapsto (\iota_X \omega)(X_1, \ldots, X_{p-1}) := \omega(X, X_1, \ldots, X_{p-1}).
$$

(3.5)
and we view $X$ as a vector field on $P$. Geometrically these conditions imply that the kernel of $\theta_p$ defines a horizontal subspace of $TP$ that we denote by $HP$. It is a complement to the vertical subspace \textit{i.e.} we get a splitting of $T_pP$ as $V_pP \oplus H_pP$.

Let $\mathfrak{g}$ be the Lie algebra of $G(\mathbb{R})^+$ and let $p$ be the orthogonal projection from $\mathfrak{g}$ on $\mathfrak{k}$. After identifying $\mathfrak{g}^*$ with the space $\Omega^1(G(\mathbb{R})^+)G(\mathbb{R})^+$ of $G(\mathbb{R})^+$-invariant forms we define a natural 1-form

$$
\sum_{1 \leq i<j \leq p+q} \omega_{ij} \otimes X_{ij} \in \Omega^1(G(\mathbb{R})^+) \otimes \mathfrak{g}
$$

(3.6)
called the \textit{Maurer-Cartan form}, where $X_{ij}$ is the basis of $\mathfrak{g}$ defined earlier and $\omega_{ij}$ its dual in $\mathfrak{g}^*$. After projection onto $\mathfrak{k}$ we get a form

$$
\theta := p \left( \sum_{1 \leq i<j \leq p+q} \omega_{ij} \otimes X_{ij} \right) \in \Omega^1(G(\mathbb{R})^+) \otimes \mathfrak{k}
$$

(3.7)
where we identify $\Omega^1(G(\mathbb{R})^+,\mathfrak{k})$ with $\Omega^1(G(\mathbb{R})^+) \otimes \mathfrak{k}$. A direct computation shows that it is a principal $K$-connection on $P$ when $P$ is $G(\mathbb{R})^+$.

If $P$ is $G(\mathbb{R})^+ \times z_0$ then the projection

$$
\pi: G(\mathbb{R})^+ \times z_0 \rightarrow G(\mathbb{R})^+
$$

(3.8)
induces a pullback map

$$
\pi^* : \Omega^1(G(\mathbb{R})^+) \rightarrow \Omega^1(G(\mathbb{R})^+ \times z_0).
$$

(3.9)
The form

$$
\tilde{\theta} := \pi^* \theta \in \Omega^1(G(\mathbb{R})^+ \times z_0) \otimes \mathfrak{k}
$$

(3.10)
is a principal connection on $G(\mathbb{R})^+ \times z_0$.

### 3.2 The associated vector bundles

Since $z_0$ is preserved by $K$ we have an orthogonal $K$-representation

$$
\rho: K \rightarrow SO(z_0)
$$

$$
k \mapsto \rho(k)w := k|_{z_0}w,
$$

(3.11)
where we will usually simply write $kw$ instead of $k|_{z_0}w$. We can consider the \textit{associated vector bundle} $P \times_K z_0$ which is the quotient of $P \times z_0$ by $K$, where $K$ acts by sending $(p,w)$ to $(R_k(p),\rho(k)^{-1}w)$. Hence an element $[p,w]$ of $P \times_K z_0$ is an equivalence class where the equivalence relation identifies $(p,w)$ with $(R_k(p),\rho(k)^{-1}w)$. This is a vector bundle over $P/K$ with projection map sending $[p,w]$ to $\pi(p)$. Let $\Omega^i(P/K, P \times_K z_0)$ be the space of $i$-forms valued in $P \times_K z_0$, when $i$ is zero it is the space of smooth sections of the associated bundle.

In the two cases of interest to us we define

$$
E := G(\mathbb{R})^+ \times_K z_0,
$$

$$
\tilde{E} := (G(\mathbb{R})^+ \times z_0) \times_K z_0.
$$

(3.12)
Note that in both cases $P$ admits a left action of $G(\mathbb{R})^+$ and that the associated vector bundles are $G(\mathbb{R})^+$-equivariant. Moreover it is a Euclidean bundle, equipped with the inner product
\[ \langle v, w \rangle := -Q \big|_{z_0}(v, w) \] (3.13)
on the fiber. Let $\Omega^i(P, z_0)$ be the space of $z_0$-valued differential $i$-forms on $P$. A differential form $\alpha$ in $\Omega^i(P, z_0)$ is said to be horizontal if $\iota_X \alpha$ vanishes for all vertical vector fields $X$. There is a left action of $K$ on a differential form $\alpha$ in $\Omega^i(P, z_0)$ defined by
\[ k \cdot \alpha := \rho(k)(R^*_k \alpha), \] (3.14)
and $\alpha$ is $K$-invariant if it satisfies $k \cdot \alpha = \alpha$ for any $k$ in $K$, i.e. we have $R^*_k \alpha = \rho(k^{-1})\alpha$. We write $\Omega^i(P, z_0)^K$ for the space of $K$-invariant $z_0$-valued forms on $P$. Finally a form that is horizontal and $K$-invariant is called a basic form and the space of such forms is denoted by $\Omega^i(P, z_0)_{bas}$.

Let $X_1, \ldots, X_N$ be tangent vectors of $P/K$ at $\pi(p)$ and $\tilde{X}_i$ be tangent vectors of $P$ at $p$ that satisfy $d_p \pi(\tilde{X}_i) = X_i$. There is a map
\[ \Omega^i(P, z_0)_{bas} \to \Omega^i(P/K, P \times_K z_0) \]
\[ \alpha \mapsto \omega_\alpha \] (3.15)
defined by
\[ \omega_\alpha|_{\pi(p)}(X_1 \wedge \cdots \wedge X_N) = \alpha|_{p}(\tilde{X}_1 \wedge \cdots \wedge \tilde{X}_N). \] (3.16)

**Proposition 3.1.** The map is well-defined and yields an isomorphism between $\Omega^i(P/K, P \times_K z_0)$ and $\Omega^i(P, z_0)_{bas}$. In particular if $z_0$ is 1-dimensional then $\Omega^i(P/K)$ is isomorphic to $\Omega^i(P)_{bas}$.

**Proof.** In the case where $i$ is zero the horizontally condition is vacuous and the isomorphism simply identifies $\Omega^0(P/K, P \times_K z_0)$ with $\Omega^0(P, z_0)^K$. We have a map
\[ \Omega^0(P, z_0)^K \to \Omega^0(P/K, P \times_K z_0) \]
\[ f \mapsto s_f(\pi(p)) := [p, f(p)], \] (3.17)
which is well defined since
\[ f(R_k(p)) = \rho(k)^{-1}f(p). \] (3.18)
Conversely every smooth section $s$ in $\Omega^0(P/K, P \times_K z_0)$ is given by
\[ s(\pi(p)) = [p, f_s(p)] \] (3.19)
for some smooth function $f_s$ in $\Omega^0(P, z_0)^K$. The map sending $s$ to $f_s$ is inverse to the previous one. The proof is similar for positive $i$. \qed

### 3.3 Covariant derivatives

A covariant derivative on the vector bundle $P \times_K z_0$ is a differential operator
\[ \nabla_P : \Omega^0(P/K, P \times_K z_0) \to \Omega^1(P/K, P \times_K z_0) \] (3.20)
such that for every smooth function $f$ in $C^\infty(P/K)$ we have
\[ \nabla_P(fs) = df \otimes s + f \nabla_P(s). \] (3.21)

The inner product on $P \times_K z_0$ defines a pairing
\[ \Omega^i(P/K, P \times_K z_0) \times \Omega^j(P/K, P \times_K z_0) \to \Omega^{i+j}(P/K) \]
\[ (\omega_1 \otimes s_1, \omega_2 \otimes s_2) \mapsto \langle \omega_1 \otimes s_1, \omega_2 \otimes s_2 \rangle = \omega_1 \wedge \omega_2(s_1, s_2), \] (3.22)
and we say that the derivative is compatible with the metric if
\[ d(s_1, s_2) = \langle \nabla_P s_1, s_2 \rangle + \langle s_1, \nabla_P s_2 \rangle \] (3.23)
for any two sections $s_1$ and $s_2$ in $\Omega^0(P/K, P \times_K z_0)$. There is a covariant derivative that is induced by a principal connection $\theta_P$ in $\Omega^1(P) \otimes \mathfrak{k}$ as follows. The derivative of the representation gives a map
\[ d\rho: \mathfrak{k} \to \mathfrak{so}(z_0) \subset \text{End}(z_0), \] (3.24)
which we also denote by $\rho$ by abuse of notation. Note that for the representation (3.11) this is simply the map
\[ \rho: \mathfrak{k} \to \mathfrak{so}(z_0) \]
\[ X \mapsto X|_{z_0} \] (3.25)
since $\mathfrak{k}$ splits as $\mathfrak{so}(z_0) = \mathfrak{so}(z_0)$. Composing the principal connection with $\rho$ defines an element
\[ \rho(\theta_P) \in \Omega^1(P, \mathfrak{so}(z_0)). \] (3.26)

In particular, if $s$ is a section of $P \times_K z_0$ then we can identify it with a $K$-invariant smooth map $f_s$ in $\Omega^0(P, z_0)^K$. Since $\rho(\theta_P)$ is a $\mathfrak{so}(z_0)$-valued form and $\mathfrak{so}(z_0)$ is a subspace of $\text{End}(z_0)$ we can define
\[ df_s + \rho(\theta_P) \cdot f_s \in \Omega^1(P, z_0). \] (3.27)

**Lemma 3.2.** The form $df_s + \rho(\theta_P) \cdot f_s$ is basic, hence gives a $P \times_K z_0$-valued form on $P/K$. Thus $d + \rho(\theta_P)$ defines a covariant derivative on $P \times_K z_0$. Moreover, it is compatible with the metric.

**Proof.** See [1, p. 24]. For the compatibility with the metric, it follows from the fact that the connection $\rho(\theta_P)$ is valued in $\mathfrak{so}(z_0)$ that
\[ \langle \rho(\theta_P)f_{s_1}, f_{s_2} \rangle + \langle f_{s_1}, \rho(\theta_P)f_{s_2} \rangle = 0. \] (3.28)
Hence if we denote by $\nabla_P$ is the covariant derivative defined by $d + \rho(\theta_P)$ then
\[ \langle \nabla_P s_1, s_2 \rangle + \langle s_1, \nabla_P s_2 \rangle = \langle df_{s_1}, f_{s_2} \rangle + \langle f_{s_1}, df_{s_2} \rangle = d(f_{s_1}, f_{s_2}) = d\langle s_1, s_2 \rangle. \] (3.29)
\[ \square \]

Let us denote by $\nabla_P$ the covariant derivative $d + \rho(\theta_P)$. It can be extended to a map
\[ \nabla_P: \Omega^i(P/K, P \times_K z_0) \to \Omega^{i+1}(P/K, P \times_K z_0) \] (3.30)
by setting
\[ \nabla_P(\omega \otimes s) := d\omega \otimes s + (-1)^i \omega \wedge \nabla_P(s) \] (3.31)
where
\[ \omega \otimes s \in \Omega^i(P/K) \otimes \Omega^0(P/K, P \times_K z_0) \simeq \Omega^i(P/K, P \times_K z_0). \] (3.32)

We define the curvature \( R_P \) in \( \Omega^2(P, k) \) by
\[ R_P(X,Y) := [\theta_P(X), \theta_P(Y)] - \theta_P([X,Y]) \] (3.33)
for two vector fields \( X \) and \( Y \) on \( P \). It is basic by [1, Proposition. 1.13] and composing with \( \rho \) gives an element
\[ \rho(R_P) \in \Omega^2(P, \mathfrak{so}(z_0))_{bas}, \] (3.34)
so that we can view it as an element in \( \Omega^2(P/K, P \times_K \mathfrak{so}(z_0)) \) where \( K \) acts on \( \mathfrak{so}(z_0) \) by the \text{Ad}-representation. For a section \( s \) in \( \Omega^0(P/K, P \times_K z_0) \) we have [1, Proposition. 1.15]
\[ \nabla^2_P s = \rho(R_p)s \in \Omega^2(P/K, P \times_K z_0). \] (3.35)

From now on we denote by \( \nabla \) and \( \tilde{\nabla} \) the covariant derivatives on \( E \) and \( \tilde{E} \) associated to \( \theta \) and \( \tilde{\theta} \) defined in (3.7) and (3.10). Let \( \tilde{R} \) and \( \tilde{R} \) be their respective curvatures.

### 3.4 Pullback of bundles

The pullback of \( E \) by the projection map gives a canonical bundle
\[ \pi^*E := \{(e, e') \in E \times E \mid \pi(e) = \pi(e')\} \] (3.36)
over \( E \). We have the following diagram

\[ \begin{array}{ccc}
\pi^*E & \longrightarrow & E \\
\downarrow & & \downarrow \pi \\
E & \longrightarrow & \mathbb{D}^+. 
\end{array} \] (3.37)

The projection induces a pullback of the sections
\[ \pi^* : \Omega^i(\mathbb{D}, E) \longrightarrow \Omega^i(E, \tilde{E}). \] (3.38)

We can also pullback the covariant derivative \( \nabla \) to a covariant derivative
\[ \pi^*\nabla : \Omega^0(E, \pi^*E) \longrightarrow \Omega^1(E, \pi^*E) \] (3.39)
on \( \pi^*E \). It is characterized by the property
\[ (\pi^*\nabla)(\pi^*s) = \pi^*(\nabla s). \] (3.40)

**Proposition 3.3.** The bundles \( \tilde{E} \) and \( \pi^*E \) are isomorphic, and this isomorphism identifies \( \tilde{\nabla} \) and \( \pi^*\nabla \).
Proof. By definition \([(g_1, w_1), [g_2, w_2])\] are elements of \(\pi^* E\) if and only if \(g_1^{-1}g_2\) is in \(K\). We have a \(G(\mathbb{R})^+\)-equivariant morphism

\[
\pi^* E \rightarrow \tilde{E}
\]

\[
([g_1, w_1], [g_2, w_2]) \mapsto ([g_1, g_1^{-1}g_2w_2], w_1).
\]  

(3.41)

This map is well defined and has as inverse

\[
\tilde{E} \rightarrow \pi^* E
\]

\[
([g, w_1], [g, w_2]) \mapsto ([g, w_2], [g, w_1]).
\]  

(3.42)

The second statement follows from the fact that \(\tilde{\theta}\) is \(\pi^* \theta\).

3.5 A few operations on the vector bundles

We extend the \(K\)-representation \(z_0\) to \(\wedge^j z_0\) by

\[
k(w_1 \wedge \cdots \wedge w_j) = (kw_1) \wedge \cdots \wedge (kw_j).
\]  

(3.43)

We consider the bundles \(P \times_K \wedge^j z_0\) and \(P \times_K \wedge z_0\) over \(P/K\), where \(\wedge z_0\) is defined as \(\bigoplus_i \wedge^i z_0\). Denote the space of differential forms valued in \(P \times_K \wedge^j z_0\) by

\[
\Omega_{\wedge^j} := \bigoplus_{i,j} \Omega_{P}^i(P/K, P \times_K \wedge^j z_0) = \Omega_{P}^i(P/K) \otimes \Omega^0(P/K, P \times_K \wedge^j z_0).
\]  

(3.44)

The total space of differential forms

\[
\Omega(P/K, P \times_K \wedge z_0) = \bigoplus_{i,j} \Omega_{P}^{i,j}
\]  

(3.45)

is an (associative) bigraded \(C^\infty(P/K)\)-algebra where the product is defined by

\[
\wedge: \Omega_{P}^{i,j} \times \Omega_{P}^{k,l} \rightarrow \Omega_{P}^{i+k,j+l}
\]

\[
(\omega \otimes s, \eta \otimes t) \mapsto (\omega \otimes s) \wedge (\eta \otimes t) := (-1)^{jl}(\omega \wedge \eta) \otimes (s \wedge t).
\]  

(3.46)

This algebra structure allows us to define an exponential map by

\[
\exp: \Omega(P/K, P \times_K \wedge z_0) \rightarrow \Omega(P/K, P \times_K \wedge z_0)
\]

\[
\omega \mapsto \exp(\omega) := \sum_{k \geq 0} \frac{\omega^k}{k!}
\]  

(3.47)

where \(\omega^k\) is the \(k\)-fold wedge product \(\omega \wedge \cdots \wedge \omega\).

Remark 3.1. Suppose that \(\omega\) and \(\eta\) commute. Then the binomial formula

\[
(\omega + \eta)^k = \sum_{l=0}^{k} \binom{k}{l} \omega^l \eta^{k-l}
\]  

(3.48)

holds and one can show that \(\exp(\omega + \eta) = \exp(\omega) + \exp(\eta)\) in the same way as for the real exponential map. In particular the diagonal subalgebra \(\bigoplus \Omega_{i}^{i}\) is a commutative since for two forms \(\omega\) and \(\eta\) in \(\Omega_{P}\) we have

\[
\omega \wedge \eta = (-1)^{\deg(\omega) + \deg(\eta)} \eta \wedge \omega
\]  

(3.49)

and similarly for two sections \(s\) and \(t\) in \(\Omega^0(P/K, P \times_K z_0)\).
The inner product $\langle -, - \rangle$ on $z_0$ can be extended to an inner product on $\wedge z_0$ by

$$
\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_l \rangle := \begin{cases} 
0 & \text{if } k \neq l, \\
\det \langle v_i, w_j \rangle_{i,j} & \text{if } k = l.
\end{cases}
$$

If $e_1, \ldots, e_q$ is an orthonormal basis of $z_0$, then the set

$$
\{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \leq k \leq q, \ i_1 < i_2 < \cdots < i_k \}
$$

is an orthonormal basis of $\wedge z_0$. We define the Berezin integral $\int^B$ to be the orthogonal projection onto the top dimensional component, that is the map

$$
\int^B : \wedge z_0 \rightarrow \mathbb{R}
$$

$$
w \mapsto \langle w, e_1 \wedge \cdots \wedge e_q \rangle.
$$

The Berezin integral can then be extended to

$$
\int^B : \Omega(P/K, P \times_K \wedge z_0) \rightarrow \Omega(P/K)
$$

$$
\omega \otimes s \mapsto \omega \int^B s
$$

where $\int^B s$ in $C^\infty(P/K)$ is the composition of the section with the Berezinian in every fiber. Let $s_1, \ldots, s_q$ be a local orthonormal frame of $P \times_K z_0$. Then $s_1 \wedge \cdots \wedge s_q$ is in $\Omega^0(P/K, \wedge^q P \times_K z_0)$ and defines a global section. Hence for $\alpha$ in $\Omega(P/K, P \times_K \wedge z_0)$ we have

$$
\int^B \alpha = \langle \alpha, s_1 \wedge \cdots \wedge s_q \rangle.
$$

Finally, for every section $s$ in $\Omega^{0,1}$ we can define the contraction

$$
i(s) : \Omega^{i,j}_p \rightarrow \Omega^{i,j-1}_p
$$

$$
\omega \otimes s_1 \wedge \cdots \wedge s_j \mapsto \sum_{k=1}^j (-1)^{i+k-1} \langle s_k, \omega \rangle \otimes s_1 \wedge \cdots \wedge \hat{s}_k \wedge \cdots \wedge s_j
$$

and extended by linearity, where the symbol $\hat{}$ means that we remove it from the product. Note that when $j$ is zero then $i(s)$ is defined to be zero. The contraction $i(s)$ defines a derivation on $\oplus \Omega^{i,j}$ that satisfies

$$
i(s)(\alpha \wedge \alpha') = (i(s)\alpha) \wedge \alpha' + (-1)^{i+j} \alpha \wedge (i(s)\alpha')
$$

for $\alpha$ in $\Omega^{i,j}$ and $\alpha'$ in $\Omega^{k,l}$.

### 3.6 Thom forms

We denote by $E$ the bundle $G(\mathbb{R})^+ \times_K z_0$. On the fibers of the bundle we have the inner product given by $\langle w, w' \rangle := -Q(w, w')$. Let $v$ be arbitrary vector in $L$ and $\Gamma_v$ its stabilizer. Since the bundle is $G(\mathbb{R})^+$-equivariant we have a bundle

$$
\Gamma_v \backslash E \rightarrow \Gamma_v \backslash D^+,
$$
and let $D(\Gamma_v \setminus E)$ be the closed disk bundle. If we have a closed $(q + i)$-form on $\Gamma_v \setminus E$ whose support is contained in $D(\Gamma_v \setminus E)$, then it has compact support in the fiber and represents a class in $H^{q+i}(\Gamma_v \setminus E, \Gamma_v \setminus E - D(\Gamma_v \setminus E))$. The cohomology group $H^\bullet(\Gamma_v \setminus E, \Gamma_v \setminus E - D(\Gamma_v \setminus E))$ is equal to the cohomology group $H^\bullet(\Gamma_v \setminus E, \Gamma_v \setminus (E - E_0))$ that we used in the introduction, where $E_0$ is the zero section. Fiber integration induces an isomorphism on the level of cohomology

\[
\text{Th}: H^{q+i}(\Gamma_v \setminus E, \Gamma_v \setminus E - D(\Gamma_v \setminus E)) \longrightarrow H^i(\Gamma_v \setminus \mathbb{D}^+)
\]

\[\omega \mapsto \int_{\text{fiber}} \omega \]  

(3.58)

known as the Thom isomorphism [2, Theorem. 6.17]. When $i$ is zero then $H^i(\Gamma_v \setminus \mathbb{D}^+)$ is $\mathbb{R}$ and we call the preimage of 1

\[
\text{Th}(\Gamma_v \setminus E) := \text{Th}^{-1}(1) \in H^0(\Gamma_v \setminus E, \Gamma_v \setminus E - D(\Gamma_v \setminus E))
\]  

(3.59)

the Thom class. Any differential form representing this class is called a Thom form, in particular every closed $q$-form on $\Gamma_v \setminus E$ that has compact support in every fiber and whose integral along every fiber is 1 is a Thom form. One can also view the Thom class as the Poincaré dual class of the zero section $E_0$ in $E$, in the same sense as for (2.24).

Let $\omega$ in $\Omega(\mathbb{E})$ be a form on the bundle and let $\omega_z$ be its restriction to a fiber $E_z = \pi^{-1}(z)$ for some $z$ in $\mathbb{D}^+$. After identifying $z_0$ with $\mathbb{R}^q$ we see $\omega_z$ as an element of $C^\infty(\mathbb{R}^q) \otimes \wedge(\mathbb{R}^q)^*$. We say that $\omega$ is rapidly decreasing in the fiber if $\omega_z$ lies in $\mathcal{S}(\mathbb{R}^q) \otimes \wedge(\mathbb{R}^q)^*$ for every $z$ in $\mathbb{D}^+$. We write $\Omega_{rd}^q(E)$ for the space of such forms.

Let $\Omega_{rd}^q(\Gamma_v \setminus E)$ be the complex of rapidly decreasing forms in the fiber. It is isomorphic to the complex $\Omega_{rd}^q(E)^{\Gamma_v}$ of rapidly decreasing $\Gamma_v$-invariant forms on $E$. Let $H_{rd}(\Gamma_v \setminus E)$ the cohomology of this complex. The map

\[
h: \Gamma_v \setminus E \longrightarrow \Gamma_v \setminus E
\]

\[
w \longrightarrow \frac{w}{\sqrt{1 - \|w\|^2}}
\]  

(3.60)

is a diffeomorphism from the open disk bundle $D(\Gamma_v \setminus E)^{\circ}$ onto $\Gamma_v \setminus E$. It induces an isomorphism by pullback

\[
h^*: H_{rd}(\Gamma_v \setminus E) \longrightarrow H(\Gamma_v \setminus E, \Gamma_v \setminus E - D(\Gamma_v \setminus E)),
\]  

(3.61)

which commutes with the fiber integration. Hence we have the following version of the Thom isomorphism

\[
H_{rd}^{q+i}(\Gamma_v \setminus E) \longrightarrow H^i(\Gamma_v \setminus \mathbb{D}^+).
\]  

(3.62)

The construction of Mathai and Quillen produces a Thom form

\[
U_{MQ} \in \Omega_{rd}^q(E)
\]  

(3.63)

which is $G(\mathbb{R})^+$-invariant (hence $\Gamma_v$-invariant) and closed. We will recall their construction in the next section.

13
3.7 The Mathai-Quillen construction

As earlier let \( \tilde{E} \) be the bundle \( (G(\mathbb{R})^+ \times z_0) \times_K z_0 \). Let \( \wedge^j \tilde{E} \) be the bundle \( (G(\mathbb{R})^+ \times z_0) \times_K \wedge^j z_0 \) and

\[
\Omega^{i,j} := \Omega^i(\mathbb{D}^+, \wedge^j \tilde{E})
\]

\[
\tilde{\Omega}^{i,j} := \Omega^i(\tilde{E}, \wedge^j \tilde{E}).
\]

(3.64)

First consider the tautological section \( s \) of \( E \) defined by

\[
s[g, w] := [(g, w), w] \in \tilde{E}.
\]

(3.65)

This gives a canonical element \( s \) of \( \tilde{\Omega}^{i,0} \). Composing with the norm induced from the inner product we get an element \( \|s\|^2 \) in \( \tilde{\Omega}^{0,0} \).

The representation \( \rho \) on \( z_0 \) induces a representation on \( \wedge^i z_0 \) that we also denote by \( \rho \). The derivative at the identity gives a map

\[
\rho : \mathfrak{e} \longrightarrow \mathfrak{so}(\wedge^i z_0).
\]

(3.66)

The connection form \( \rho(\tilde{\theta}) \) in \( \Omega^j(G(\mathbb{R})^+ \times z_0, \wedge^j z_0) \) defines a covariant derivative

\[
\tilde{\nabla} : \tilde{\Omega}^{0,j} \longrightarrow \tilde{\Omega}^{1,j}
\]

(3.67)

on \( \wedge^j \tilde{E} \). We can extend it to a map

\[
\tilde{\nabla} : \tilde{\Omega}^{i,j} \longrightarrow \tilde{\Omega}^{i+1,j}
\]

(3.68)

by setting

\[
\tilde{\nabla}(\omega \otimes s) := d\omega \otimes s + (-1)^i \omega \wedge \tilde{\nabla}(s),
\]

(3.69)

as in (3.30). The connection on \( \tilde{\Omega}^{i,j} \) is compatible with the metric. Finally, the covariant derivative \( \tilde{\nabla} \) defines a derivation on \( \oplus \tilde{\Omega}^{i,j} \) that satisfies

\[
\tilde{\nabla}(\alpha \wedge \alpha') = (\tilde{\nabla}\alpha) \wedge \alpha' + (-1)^{i+j} \alpha \wedge (\tilde{\nabla}\alpha')
\]

(3.70)

for any \( \alpha \) in \( \tilde{\Omega}^{i,j} \) and \( \alpha' \) in \( \tilde{\Omega}^{k,l} \).

Taking the derivative of the tautological section gives an element

\[
\tilde{\nabla}s = ds + \rho(\tilde{\theta})s \in \tilde{\Omega}^{1,1}.
\]

(3.71)

Let \( \mathfrak{so}(\tilde{E}) \) denote the bundle \( (G(\mathbb{R})^+ \times z_0) \times_K \mathfrak{so}(z_0) \) and consider the curvature \( \rho(\tilde{R}) \) in \( \Omega^2(\tilde{E}, \mathfrak{so}(\tilde{E})) \). We have an isomorphism

\[
T^{-1}\big|_{z_0} : \mathfrak{so}(z_0) \longrightarrow \wedge^2 z_0
\]

\[
A \mapsto \sum_{i<j} \langle Ae_i, e_j \rangle e_i \wedge e_j.
\]

(3.72)

The inverse sends \( v \wedge w \) to the endomorphism \( u \mapsto \langle v, u \rangle w - \langle w, u \rangle v \), and is the isomorphism from (2.11) restricted to \( z_0 \). Note that we have

\[
T(v \wedge w)u = \iota(u)v \wedge w.
\]

(3.73)

Using this isomorphism we can also identify \( \mathfrak{so}(\tilde{E}) \) and \( \wedge^2 \tilde{E} \) so that we can view the curvature as an element

\[
\rho(\tilde{R}) \in \tilde{\Omega}^{2,2}.
\]

(3.74)
Lemma 3.4. The form $\omega := 2\pi ||s||^2 + 2\sqrt{\pi} \nabla s - \rho(\bar{R})$ lying in $\bar{\Omega}^{0,0} \oplus \bar{\Omega}^{1,1} \oplus \bar{\Omega}^{2,2}$ is annihilated by $\tilde{\nabla} + 2\sqrt{\pi} i(s)$. Moreover

$$d \int^{B} \alpha = \int^{B} \tilde{\nabla} \alpha,$$

(3.75)

for every form $\alpha$ in $\bar{\Omega}^{i,j}$. Hence $\int^{B} \exp(-\omega)$ is a closed form.

Proof. We have

$$\left( \tilde{\nabla} + 2\sqrt{\pi} i(s) \right) \left( 2\pi ||s||^2 + 2\sqrt{\pi} \nabla s - \rho(\bar{R}) \right) = 2\pi \tilde{\nabla} ||s||^2 + 4\pi \frac{\sqrt{\pi}}{2} i(s) ||s||^2 + 2\sqrt{\pi} \tilde{\nabla}^2 s + 4\pi i(x) \tilde{\nabla} s - \tilde{\nabla} \rho(\bar{R}) - 2\sqrt{\pi} i(s) \rho(\bar{R}).$$

(3.76)

It vanishes because we have the following:

- $i(s) ||s||^2 = 0$ since $||s||$ is in $\bar{\Omega}^{0,0}$,
- $\tilde{\nabla} \rho(\bar{R}) = 0$ by Bianchi’s identity,
- $\tilde{\nabla} ||s||^2 = 2\langle \nabla s, s \rangle = -2i(s) \tilde{\nabla} s$,
- $\tilde{\nabla}^2 s = \rho(\bar{R}) s = i(s) \rho(\bar{R})$.

For the last point we used (3.73) where we view $\rho(\bar{R})$ as an element of $\Omega^2(E, \mathfrak{so}(\bar{E}))$, respectively of $\Omega^2(E, \wedge^2 \bar{E})$.

Let $s_1 \wedge \cdots \wedge s_q$ in $\bar{\Omega}^q(E, \wedge^q \bar{E})$ be a global section where $s_1, \ldots, s_q$ is a local orthonormal frame for $\bar{E}$. Then for any $\alpha$ in $\bar{\Omega}^{i,j}$ we have

$$\int^{B} \alpha = \langle \alpha, s_1 \wedge \cdots \wedge s_q \rangle.$$  

(3.77)

This vanishes if $j$ is different from $q$, hence we can assume $\alpha$ is in $\bar{\Omega}^q$. If we write $\alpha$ as $\beta s_1 \wedge \cdots \wedge s_q$ for some $\beta$ in $\Omega^q(E)$ then

$$\int^{B} \alpha = \beta.$$  

(3.78)

On the other hand, since the connection on $\bar{\Omega}^{i,q}$ is compatible with the metric, we have

$$0 = d(s_1 \wedge \cdots \wedge s_q, s_1 \wedge \cdots \wedge s_q) = 2\langle \nabla (s_1 \wedge \cdots \wedge s_q), s_1 \wedge \cdots \wedge s_q \rangle.$$  

(3.79)

Then we have

$$\int^{B} \tilde{\nabla} \alpha = \langle \tilde{\nabla} \alpha, s_1 \wedge \cdots \wedge s_q \rangle$$

$$= \langle d\beta \otimes s_1 \wedge \cdots \wedge s_q + (-1)^j \beta \wedge \tilde{\nabla} (s_1 \wedge \cdots \wedge s_q), s_1 \wedge \cdots \wedge s_q \rangle$$

$$= d\beta$$

$$= d \int^{B} \alpha.$$  

(3.80)
Since \( \nabla + 2\sqrt{\pi i}(s) \) is a derivation that annihilates \( \omega \) we have
\[
\left( \nabla + 2\sqrt{\pi i}(s) \right) \omega^k = 0 \tag{3.81}
\]
for positive \( k \). Hence it follows that
\[
d \int^B \exp(-\omega) = \int^B \nabla \exp(-\omega) \\
= \int^B \left( \nabla + 2\sqrt{\pi i}(s) \right) \exp(-\omega) \\
= 0. \tag{3.82}
\]

In [9] Mathai and Quillen define the following form
\[
U_{MQ} := (-1)^{\frac{q(q+1)}{2}} (2\pi)^{-\frac{q}{2}} \int^B \exp \left( -2\pi \|s\|^2 - 2\sqrt{\pi} \nabla s + \rho(\tilde{R}) \right) \in \Omega_{q-1}(E), \tag{3.83}
\]
We call it the Mathai-Quillen form.

**Proposition 3.5.** The Mathai-Quillen form is a Thom form.

*Proof.* From the previous lemma it follows that the form is closed. It remains to show that its integral along the fibers is 1. The restriction of the form \( U_{MQ} \) along the fiber \( \pi^{-1}(eK) \) is given by
\[
U_{MQ} = (-1)^{\frac{q(q+1)}{2}} (2\pi)^{-\frac{q}{2}} e^{-2\pi \|s\|^2} \int^B \exp(-2\sqrt{\pi} ds) \\
= (-1)^{\frac{q(q+1)}{2}} 2^q e^{-2\pi \|s\|^2} (-1)^q \int^B (dx_1 \otimes e_1) \wedge \cdots \wedge (dx_q \otimes e_q) \\
= 2^q e^{-2\pi \|s\|^2} dx_1 \wedge \cdots \wedge dx_q, \tag{3.84}
\]
and its integral over the fiber \( \pi^{-1}(eK) \) is equal to 1. \( \square \)

3.8 Transgression form

For \( t > 0 \) consider the map \( t: E \to E \) given by multiplication by \( t \) in the fibers. Consider the \( K \)-invariant vector field
\[
X := \sum_{i=1}^q x_i \frac{\partial}{\partial x_i} \tag{3.85}
\]
on \( G(\mathbb{R})^+ \times \mathbb{R}^q \). Since it is \( K \)-invariant it also induces a vector field on \( E \). We define the transgression form \( \psi \) in \( \Omega^{q-1}(E) \) to be \( t_X U_{MQ} \), where \( t_X \) is the interior product.

**Proposition 3.6** (Transgression formula). The transgression satisfies:
\[
\left( \frac{d}{dt} t^* U_{MQ} \right)_{t=t_0} = -\frac{1}{t_0} d(t_0^* \psi). \tag{3.86}
\]
Proof. This is due to Mathai and Quillen. Let us view the multiplication map by \( t \) as a map

\[
m: E \times \mathbb{R}_{>0} \rightarrow E
\]

\[
(e, t) \mapsto et.
\]

(3.87)

The differential \( \tilde{d} \) on \( E \times \mathbb{R}_{>0} \) splits as \( d + d_{\mathbb{R}_{>0}} \). Since \( U_{MQ} \) is closed (hence its pullback) we have

\[
0 = \tilde{d}(m^*U_{MQ}) = d(m^*U_{MQ}) + \frac{d}{dt}(m^*U_{MQ})dt. \tag{3.88}
\]

Moreover the pushforward of the vector field \( t \frac{\partial}{\partial t} \) by \( m \) is \( X \), hence for the contraction we have

\[
\iota_{\frac{\partial}{\partial t}} m^*U_{MQ} = \frac{1}{t} m^* \iota_X U_{MQ}. \tag{3.89}
\]

Since the differential \( d \) is independent of \( t \) it commutes with the contraction \( \iota_{\frac{\partial}{\partial t}} \). Combining with (3.88) yields

\[
\frac{d}{dt}(m^*U_{MQ}) = -\frac{1}{t} d(m^*\psi). \tag{3.90}
\]

Finally, pulling back by the section

\[
t_0: E \rightarrow E \times \mathbb{R}_{>0}
\]

\[
e \mapsto (e, t_0) \tag{3.91}
\]

gives the desired formula. \( \square \)

Let \( \Gamma_v \) be the stabilizer of \( v \) in \( \Gamma \), which acts on the left on \( E \). By the \( G(\mathbb{R})^+ \)-invariance (hence \( \Gamma_v \)-invariance) of \( U_{MQ} \), it is also a form in \( \Omega^q(\Gamma_v \backslash E) \). Let \( S_0 \) denote the image \( \Gamma_v \backslash E_0 \) of the zero section in \( \Gamma_v \backslash E \).

**Proposition 3.7.** The form \( U_{MQ} \) represents the Poincaré dual of \( S_0 \) in \( \Gamma_v \backslash E \).

**Sketch of proof.** For \( 0 < t_1 < t_2 \) we have

\[
t^*_2 U_{MQ} - t^*_1 U_{MQ} = \int_{t_1}^{t_2} \left( \frac{d}{dt} t^* U_{MQ} \right) dt
\]

\[
= -\int_{t_1}^{t_2} d(t^*\psi) \frac{dt}{t}
\]

\[
= -d \int_{t_1}^{t_2} t^*\psi \frac{dt}{t} \tag{3.92}
\]

so that \( t^*_2 U_{MQ} \) and \( t^*_1 U_{MQ} \) represent the same cohomology class in \( H^q(\Gamma_v \backslash E) \). Then, one can show that

\[
\lim_{t \to \infty} t^* U_{MQ} = \delta_{S_0} \tag{3.93}
\]

17
where $\delta_{S_0}$ is the current of integration along $S_0$. Hence if $\omega$ is a form in $\Omega^m_c(E)$, where $m$ is the dimension of $\mathbb{D}^+$, then
\[
\int_{\Gamma_v \setminus E} U_{MQ} \wedge \omega = \lim_{t \to \infty} \int_{\Gamma_v \setminus E} t^* U_{MQ} \wedge \omega = \int_{S_0} \omega.
\] (3.94)

4 Computation of the Mathai-Quillen form

4.1 The section $s_v$

Let $\text{pr}$ denote the orthogonal projection of $V(\mathbb{R})$ on the plane $z_0$. Consider the section
\[
s_v : \mathbb{D}^+ \longrightarrow E
\]
\[
z \mapsto [g_z, \text{pr}(g^{-1}_z v)],
\] (4.1)
where $g_z$ is any element of $G(\mathbb{R})^+$ sending $z_0$ to $z$. Let us denote by $L_g$ the left action of an element $g$ in $G(\mathbb{R})^+$ on $\mathbb{D}^+$. We also denote by $L_g$ the action on $E$ given by $L_g[g_z, v] = [g g_z, v]$. The bundle is $G(\mathbb{R})^+$-equivariant with respect to these actions.

**Proposition 4.1.** The section $s_v$ is well-defined and $\Gamma_v$-equivariant. Moreover its zero locus is precisely $\mathbb{D}_v^+$. 

**Proof.** The section is well-defined, since replacing $g_z$ by $g_z k$ gives
\[
s_v(z) = [g_z k, \text{pr}(g^{-1}_z v)] = [g_z k, k^{-1} \text{pr}(g^{-1}_z v)] = [g_z, \text{pr}(g^{-1}_z v)] = s_v(z).
\] (4.2)
Suppose that $z$ is in the zero locus of $s_v$, that is to say $\text{pr}(g^{-1}_z v)$ vanishes. Then $g^{-1}_z v$ is in $z_0^\perp$. It is equivalent to the fact that $z = g_z z_0$ is a subspace of $v_0^\perp$, which means that $z$ is in $\mathbb{D}_v^+$. Hence the zero locus of $s_v$ is exactly $\mathbb{D}_v^+$. For the equivariance, note that we have
\[
s_v \circ L_g(z) = [g g_z, \text{pr}(g^{-1}_z g^{-1}_v)] = L_g \circ s_{g^{-1}_v}(z).
\] (4.3)
Hence if $\gamma$ is an element of $\Gamma_v$ we have
\[
s_v \circ L_\gamma = L_\gamma \circ s_v.
\] (4.4)

We define the pullback $\varphi^0(v) := s_v^* U_{MQ}$ of the Mathai-Quillen form by $s_v$. It defines a form
\[
\varphi^0 \in C^\infty(\mathbb{R}^{p+q}) \otimes \Omega^q(\mathbb{D})^+.
\] (4.5)
It is only rapidly decreasing on $\mathbb{R}^q$, and in order to make it rapidly decreasing everywhere we set
\[
\varphi(v) := e^{-\pi Q(v,v)} \varphi^0(v).
\] (4.6)
It defines a form $\varphi \in \mathcal{S}(\mathbb{R}^{p+q}) \otimes \Omega^q(\mathbb{D})^+$. 

18
**Proposition 4.2.** 1. For fixed $v$ in $V(\mathbb{R})$ the form $\varphi^0(v)$ in $\Omega^q(\mathbb{D}^+)$ is given by

$$
\varphi^0(v) = (-1)^{\frac{q(q+1)}{2}}(2\pi)^{-\frac{q}{2}} \exp \left(2\pi Q|_{z_0}(v, v)\right) \int \exp \left(-2\sqrt{\pi} \nabla s_v + \rho(R)\right). \tag{4.7}
$$

2. It satisfies $L^*_g \varphi^0(v) = \varphi^0(g^{-1}v)$, hence

$$
\varphi^0 \in \left[\Omega^q(\mathbb{D}^+) \otimes C^\infty(\mathbb{R}^{p+q})\right]^{G(\mathbb{R})^+}. \tag{4.8}
$$

3. It is a Poincaré dual of $\Gamma_v \backslash \mathbb{D}^+$ in $\Gamma_v \backslash \mathbb{D}^+$.

**Proof.** 1. Recall that $\tilde{\nabla} = \pi^* \nabla$ and $\tilde{R} = \pi^* R$. We pullback by $s_v$

$$
\begin{array}{ccc}
E \cong s_v^* \tilde{E} & \longrightarrow & \tilde{E} \\
\downarrow & & \downarrow \pi \\
\mathbb{D}^+ & \longrightarrow & E.
\end{array}
$$

Since $\pi \circ s_v$ is the identity we have

$$
s_v^* \tilde{\nabla} = s_v^* \pi^* \nabla = \nabla. \tag{4.9}
$$

Hence, the pullback connection $s_v^* \tilde{\nabla}$ satisfies

$$
s_v^*(\tilde{\nabla}s) = (s_v^* \tilde{\nabla})(s_v^* s) = \nabla s_v \tag{4.10}
$$

since $s_v^* s = s_v$. We also have $s_v^* \tilde{R} = R$ and

$$
s_v^* ||s||^2 = ||s_v||^2 = \langle s_v, s_v \rangle = -Q|_{z_0}(v, v). \tag{4.11}
$$

The expression for $\varphi^0$ then follows from the fact that $\exp$ and $s_v^*$ commute.

2. The bundle $E$ is $G(\mathbb{R})^+$ equivariant. By construction the Mathai-Quillen is $G(\mathbb{R})^+$-invariant, so $L^*_g U_{MQ} = U_{MQ}$. On the other hand we also have

$$
s_v \circ L_g(z) = L_g \circ s_{g^{-1}v}(z), \tag{4.12}
$$

and thus

$$
L^*_g \varphi^0(v) = L^*_g s_v^* U_{MQ} = \varphi^0(g^{-1}v). \tag{4.13}
$$

3. Since $s_v$ is $\Gamma_v$-equivariant we view it as a section

$$
s_v : \Gamma_v \backslash \mathbb{D}^+ \longrightarrow \Gamma_v \backslash E, \tag{4.14}
$$

whose zero locus is precisely $\Gamma_v \backslash \mathbb{D}^+$. Let $S_0$ (respectively $S_v$) be the image in $\Gamma_v \backslash E$ of the section $s_v$ (respectively the zero section). By Proposition 3.7 the Thom form $U_{MQ}$ is a Poincaré dual of
For a form $\omega$ in $\Omega_m^{-q}(\Gamma_v \setminus \mathbb{D}^+)$ we have
\[
\int_{\Gamma_v \setminus \mathbb{D}^+} \varphi^0(v) \wedge \omega = \int_{S_v} U_{MQ} \wedge \pi^* \omega = \int_{S_v \cap S_0} \pi^* \omega = \int_{\Gamma_v \setminus \mathbb{D}^+} \omega.
\] (4.15)

The last step follows from the fact $\pi^{-1}(S_v \cap S_0)$ equals $\Gamma_v \setminus \mathbb{D}^+$. □

As in (2.19) we have an isomorphism
\[
[\Omega^q(\mathbb{D}^+) \otimes C^\infty(\mathbb{R}^{p+q})]^G(\mathbb{R}) \longrightarrow [\bigwedge^q p^* \otimes C^\infty(\mathbb{R}^{p+q})]^K
\] (4.16)

by evaluating at the basepoint $eK$ of $G(\mathbb{R})/K$ that corresponds to $z_0$ in $\mathbb{D}^+$. We will now compute $\varphi^0|_{eK}$.

### 4.2 The Mathai-Quillen form at the identity

From now on we identify $\mathbb{R}^{p+q}$ with $V(\mathbb{R})$ by the orthonormal basis of (2.1), and let $z_0$ be the negative spanned by the vectors $e_{p+1}, \ldots, e_{p+q}$. Hence we identify $z_0$ with $\mathbb{R}^q$ and the quadratic form is
\[
Q|_{z_0}(v, v) = -\sum_{\mu=p+1}^{p+q} x^2_{\mu}
\] (4.17)

where $x_{p+1}, \ldots, x_{p+q}$ are the coordinates of the vector $v$.

Let $f_v$ in $\Omega^0(G(\mathbb{R})^+, z_0)^K$ be the map associated to the section $s_v$, as in Proposition 3.1. It is defined by
\[
f_v(g) = \text{pr}(g^{-1} v).
\] (4.18)

Then $df_v + \rho(\theta)f_v$ is the horizontal lift of $\nabla s_v$, as discussed in Section 3.1. Let $X$ be a vector in $\mathfrak{g}$ and let $X_p$ and $X_t$ be its components with respect to the splitting of $\mathfrak{g}$ as $p \oplus \mathfrak{t}$. We have
\[
(df_v + \rho(\theta)f_v)_e(X) = df_v(X_p).
\] (4.19)

In particular we can evaluate on the basis $X_{\alpha\mu}$ and get:
\[
df_v(X_{\alpha\mu}) = \frac{d}{dt} \bigg|_{t=0} f_v(\exp t X_{\alpha\mu}) = -\text{pr}(X_{\alpha\mu} v) = -\text{pr}(x_{\mu} e_{\alpha} + x_{\alpha} e_{\mu}) = -x_{\alpha} e_{\mu}.
\] (4.20)
So as an element of $p^* \otimes z_0$ we can write

$$d_e f_v = - \sum_{\mu=p+1}^{p+q} \left( \sum_{\alpha=1}^{p} x_\alpha \omega_{\alpha \mu} \right) \otimes e_\mu = - \sum_{\alpha=1}^{p} x_\alpha \eta_\alpha,$$

with

$$\eta_\alpha := \sum_{\mu=p+1}^{p+q} \omega_{\alpha \mu} \otimes e_\mu \in \Omega^{1,1}. \quad (4.22)$$

**Proposition 4.3.** Let $\rho(R_e)$ in $\wedge^2 p^* \otimes \mathfrak{so}(z_0)$ be the curvature at the identity. Then after identifying $\mathfrak{so}(z_0)$ with $\wedge^2 z_0$ we have

$$\rho(R_e) = - \frac{1}{2} \sum_{\alpha=1}^{p} \eta_\alpha^2 \in \wedge^2 p^* \otimes \wedge^2 z_0, \quad (4.23)$$

where $\eta_\alpha^2 = \eta_\alpha \wedge \eta_\alpha$.

**Proof.** Using the relation $E_{ij} E_{kl} = \delta_{il} E_{kj}$ one can show that

$$[X_{\alpha \nu}, X_{\beta \mu}] = \delta_{\mu \nu} X_{\alpha \beta} + \delta_{\alpha \beta} X_{\mu \nu} \quad (4.24)$$

for two vectors $X_{\alpha \nu}$ and $X_{\beta \mu}$ in $p$. Hence we have

$$Re(X_{\alpha \nu} \wedge X_{\beta \mu}) = \theta([X_{\alpha \nu}, X_{\beta \mu}]) = -\theta([X_{\alpha \nu}, X_{\beta \mu}]) = -p (\delta_{\alpha \beta} X_{\nu \mu} + \delta_{\nu \mu} X_{\alpha \beta}) = -\delta_{\alpha \beta} X_{\nu \mu}. \quad (4.25)$$

On the other hand, since $\eta_i(X_{jr}) = \delta_{ij} e_r$, we also have

$$\sum_{i=1}^{p} \eta_i^2 (X_{\alpha \nu} \wedge X_{\beta \mu}) = \sum_{i=1}^{p} \eta_i (X_{\alpha \nu} \wedge \eta_i (X_{\beta \mu} - \eta_i (X_{\beta \mu} \wedge \eta_i (X_{\alpha \nu})

$$= 2\delta_{\alpha \beta} e_\nu \wedge e_\mu. \quad (4.26)$$

The lemma follows since $\rho(X_{\nu \mu}) = T(e_\nu \wedge e_\mu)$ in $\mathfrak{so}(z_0)$, because

$$Q(\rho(X_{\nu \mu}) e_\nu, e_\mu) e_\nu \wedge e_\mu = -Q(e_\mu, e_\mu) e_\nu \wedge e_\mu = e_\nu \wedge e_\mu. \quad (4.27)$$

Using the fact that the exponential satisfies $\exp(\omega + \eta) = \exp(\omega) \exp(\eta)$ on the subalgebra $\bigoplus \Omega^{1,i}$ - see Remark 3.1 - we can write

$$\varphi^0|_e(v) = (-1)^{\frac{(p+1)}{2}(2\pi)^{-\frac{q}{2}}} \exp \left( 2\pi Q|_{z_0} (v,v) \right) \prod_{\alpha=1}^{p} \exp \left( 2\sqrt{\pi} x_\alpha \eta_\alpha - \frac{1}{2} \eta_\alpha^2 \right). \quad (4.28)$$
We define the $n$-th Hermite polynomial by

$$H_n(x) := \left( 2x - \frac{d}{dx} \right) \cdot 1 \in \mathbb{R}[x]. \quad (4.29)$$

The first three Hermite polynomials are $H_0(x) = 1$, $H_1(x) = 2x$ and $H_2(x) = 4x^2 - 2$.

**Lemma 4.4.** Let $\eta$ be a form in $\bigoplus \Omega^{i,i}$. Then

$$\exp(2x\eta - \eta^2) = \sum_{n \geq 0} \frac{1}{n!} H_n(x) \eta^n, \quad (4.30)$$

where $H_n$ is the $n$-th Hermite polynomial.

**Proof.** Since $\eta$ and $\eta^2$ are in $\bigoplus \Omega^{i,i}$, they commute and we can use the binomial formula:

$$\exp(2x\eta - \eta^2) = \sum_{k \geq 0} \frac{1}{k!} (2x\eta - \eta^2)^k$$

$$= \sum_{k \geq 0} \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (2x\eta)^{k-l} (-\eta^2)^l$$

$$= \sum_{k \geq 0} \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (2x)^{k-l}(1)^l \eta^{l+k}$$

$$= \sum_{n \geq 0} P_n(x) \eta^n, \quad (4.31)$$

where

$$P_n(x) := \sum_{0 \leq l \leq k \leq n \atop k + l = n} \frac{(-1)^l}{l!(k-l)!} (2x)^{k-l}. \quad (4.32)$$

The conditions on $k$ and $l$ imply that $n$ is less than or equal to $2k$. First suppose that $n$ is even. Then we have that $k$ is between $\frac{n}{2}$ and $n$, so that the sum above can be written

$$\sum_{k=\frac{n}{2}}^{n} \frac{(-1)^{n-k}}{(n-k)!(2k-n)!} (2x)^{2k-n} = \sum_{m=0}^{\frac{n}{2}} \frac{(-1)^{\frac{n}{2} - m}}{(\frac{n}{2} - m)!(2m)!} (2x)^{2m} = \frac{1}{n!} H_n(x), \quad (4.33)$$

where in the second step we let $m$ be $k - \frac{n}{2}$. If $n$ is odd then $k$ is between $\frac{n+1}{2}$ and $n$, so that the sum can be written

$$\sum_{k=\frac{n-1}{2}}^{n} \frac{(-1)^{n-k}}{(n-k)!(2k-n)!} (2x)^{2k-n} = \sum_{m=0}^{\frac{n-1}{2}} \frac{(-1)^{\frac{n-1}{2} - m}}{(\frac{n-1}{2} - m)!(2m+1)!} (2x)^{2m+1} = \frac{1}{n!} H_n(x). \quad (4.34)$$
Applying the lemma to (4.28) we get

\[ \int_B \prod_{\alpha=1}^p \exp \left( 2\sqrt{\pi} x_{\alpha} \eta_{\alpha} - \frac{1}{2} \eta_{\alpha}^2 \right) \]
\[ = \int_B \prod_{\alpha=1}^p \exp \left( 2\sqrt{2\pi} x_{\alpha} \left( \sqrt{\frac{\eta_{\alpha}}{2}} - \frac{(\eta_{\alpha}/\sqrt{2})^2}{2} \right) \right) \]
\[ = \int_B \prod_{\alpha=1}^p \sum_{n=0}^{2-n/2} \frac{1}{n!} H_n \left( \sqrt{2\pi} x_{\alpha} \right) \eta_{\alpha}^n \]
\[ = \sum_{n_1, \ldots, n_p} \frac{2^{-\frac{n_1+\cdots+n_p}{2}+1}}{n_1! \cdots n_p!} H_{n_1} \left( \sqrt{2\pi} x_1 \right) \cdots H_{n_p} \left( \sqrt{2\pi} x_p \right) \int_B \eta_{n_1}^{n_1} \cdots \eta_{n_p}^{n_p}. \quad (4.35) \]

If \( n_1 + \cdots + n_p \) is different from \( q \), then the Berezinian of \( \eta_{n_1}^{n_1} \cdots \eta_{n_p}^{n_p} \) vanishes and we get

\[ \sum_{n_1, \ldots, n_p} \frac{2^{-\frac{n_1+\cdots+n_p}{2}+1}}{n_1! \cdots n_p!} H_{n_1} \left( \sqrt{2\pi} x_1 \right) \cdots H_{n_p} \left( \sqrt{2\pi} x_p \right) \int_B \eta_{n_1}^{n_1} \cdots \eta_{n_p}^{n_p} = 2^{-\frac{q}{2}} \sum_{n_1+\cdots+n_p=q} \frac{1}{n_1! \cdots n_p!} H_{n_1} \left( \sqrt{2\pi} x_1 \right) \cdots H_{n_p} \left( \sqrt{2\pi} x_p \right) \int_B \eta_{n_1}^{n_1} \cdots \eta_{n_p}^{n_p}. \quad (4.36) \]

Note that

\[ \eta_{\alpha}^n = \left( \sum_{\mu=p+1}^{p+q} \omega_{\alpha \mu} \otimes e_\mu \right)^n = \sum_{\mu_1, \ldots, \mu_n} (\omega_{\alpha \mu_1} \otimes e_{\mu_1}) \wedge \cdots \wedge (\omega_{\alpha \mu_n} \otimes e_{\mu_n}) = n! \sum_{\mu_1 < \cdots < \mu_n} (\omega_{\alpha \mu_1} \otimes e_{\mu_1}) \wedge \cdots \wedge (\omega_{\alpha \mu_n} \otimes e_{\mu_n}), \quad (4.37) \]

where the sums are over all \( \mu_i \)'s between \( p+1 \) and \( p+q \). If \( n_1 + \cdots + n_p \) is equal to \( q \) we have

\[ \int_B \eta_{n_1}^{n_1} \cdots \eta_{n_p}^{n_p} = \int_B \prod_{\alpha=1}^p \left( \sum_{\mu=p+1}^{p+q} \omega_{\alpha \mu} \otimes e_\mu \right)^{n_\alpha} \]
\[ = \int_B \prod_{\alpha=1}^p n_{\alpha}! \sum_{\mu_1 < \cdots < \mu_n} (\omega_{\alpha \mu_1} \otimes e_{\mu_1}) \wedge \cdots \wedge (\omega_{\alpha \mu_n} \otimes e_{\mu_n}) \]
\[ = n_1! \cdots n_p! \int_B (\omega_{\alpha (p+1)} \otimes e_1) \wedge \cdots \wedge (\omega_{\alpha (p+q)} \otimes e_q) \]
\[ = (-1)^{q+1} n_1! \cdots n_p! \int_B \omega_{\alpha (p+1)} \wedge \cdots \wedge \omega_{\alpha (p+q)}, \quad (4.38) \]

where the sums in the last two lines go over all tuples \( \alpha = (\alpha_1, \ldots, \alpha_q) \) with \( \alpha \) between 1 and \( p \), and
the value $\alpha$ appears exactly $n_\alpha$-times in $\alpha$. Hence
\[
\varphi^0 \big|_e (v) = 2^{-q/2} \pi^{-\frac{q}{2}} \sum \omega_{\alpha_1(p+1)} \land \cdots \land \omega_{\alpha_q(p+q)} \otimes H_{n_1} \left( \sqrt{2\pi x_1} \right) \cdots H_{n_p} \left( \sqrt{2\pi x_p} \right) \exp \left( 2\pi Q \big|_{z_0} (v, v) \right).
\]
(4.39)

After multiplying by $\exp \left( -\pi Q(v, v) \right)$ we get
\[
\varphi \big|_e (v) = 2^{-q/2} \pi^{-\frac{q}{2}} \sum \omega_{\alpha_1(p+1)} \land \cdots \land \omega_{\alpha_q(p+q)} \otimes H_{n_1} \left( \sqrt{2\pi x_1} \right) \cdots H_{n_p} \left( \sqrt{2\pi x_p} \right) \exp \left( -\pi Q^+_{z_0} (v, v) \right).
\]
(4.40)

The form is now rapidly decreasing in $v$, since the Siegel majorant is positive definite. We have
\[
\varphi \big|_e \in \left[ \bigwedge^q p^* \otimes \mathcal{S}(\mathbb{R}^{p+q}) \right]^K.
\]
(4.41)

**Theorem 4.5.** We have $2^{-\frac{q}{2}} \varphi(v) = \varphi_{KM}(v)$.

**Proof.** It is a straightforward computation to show that
\[
(2\pi)^{-n_\alpha/2} H_{n_\alpha} \left( \sqrt{2\pi x_\alpha} \right) \exp(-\pi x_\alpha^2) = \left( x_\alpha - \frac{1}{2\pi} \frac{\partial}{\partial x_\alpha} \right)^{n_\alpha} \exp(-\pi x_\alpha^2).
\]
(4.42)

Hence applying this we find that the Kudla-Millson form, defined by the Howe operators in (2.22), is
\[
\varphi_{KM} \big|_e (v) = 2^{-q/2} \pi^{-\frac{q}{2}} \sum \omega_{\alpha_1(p+1)} \land \cdots \land \omega_{\alpha_q(p+q)} \otimes H_{n_1} \left( \sqrt{2\pi x_1} \right) \cdots H_{n_p} \left( \sqrt{2\pi x_p} \right) \exp \left( -\pi Q \big|_{z_0} (v, v) \right)
\]
\[= 2^{-\frac{q}{2}} e^{-\pi Q(v,v)} \varphi^0 \big|_e (v).
\]
(4.43)

\[\square\]

5 Examples

1. Let us compute the Kudla-Millson as above in the simplest setting of signature $(1, 1)$. Let $V(\mathbb{R})$ be the quadratic space $\mathbb{R}^2$ with the quadratic form $Q(v, w) = x'y + xy'$ where $x$ and $x'$ (respectively $y$ and $y'$) are the components of $v$ (respectively of $w$). Let $e_1 = \frac{1}{\sqrt{2}} (1, 1)$ and $e_2 = \frac{1}{\sqrt{2}} (1, -1)$. The 1-dimensional negative plane $z_0$ is $\mathbb{R} e_2$. If $r$ denotes the variable on $z_0$ then the quadratic form is $Q \big|_{z_0} (r) = -r^2$. The projection map is given by
\[
\text{pr}: V(\mathbb{R}) \longrightarrow z_0
\]
\[v = (x, x') \mapsto \frac{x - x'}{\sqrt{2}}.
\]
(5.1)

The orthogonal group of $V(\mathbb{R})$ is
\[
G(\mathbb{R})^+ = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, t > 0 \right\},
\]
(5.2)
and \( \mathbb{D}^+ \) can be identified with \( \mathbb{R}_{>0} \). The associated bundle \( E \) is \( \mathbb{R}_{>0} \times \mathbb{R} \) and the connection \( \nabla \) is simply \( d \) since the bundle is trivial. Hence the Mathai-Quillen form is

\[
U_{MQ} = \sqrt{2} e^{-2\pi^2 t} dt \in \Omega^1(E),
\]

as in the proof of Proposition 3.5. The section \( s_v : \mathbb{R}_{>0} \to E \) is given by

\[
s_v(t) = \left( t, \frac{-t^{-1}x - tx'}{\sqrt{2}} \right),
\]

where \( x, x' \) are the components of \( v \). We obtain

\[
s_v^* U_{MQ} = e^{-\pi \left( \frac{x}{t} + tx' \right)^2} \left( \frac{x}{t} + tx' \right) \frac{dt}{t}.
\]

Hence after multiplication by \( 2^{-\frac{1}{2}} e^{-\pi Q(v,v)} \) we get

\[
\varphi_{KM}(x, x') = 2^{-\frac{1}{2}} e^{-\pi \left( \frac{x}{t} + tx' \right)^2} \left( \frac{x}{t} + tx' \right) \frac{dt}{t}.
\]

2. The second example illustrates the functorial properties of the Mathai-Quillen form. Suppose that we have an orthogonal splitting of \( V(\mathbb{R}) \) as \( \bigoplus_i V_i(\mathbb{R}) \). Let \((p_i, q_i)\) be the signature of \( V_i(\mathbb{R}) \). We have

\[
\mathbb{D}_1 \times \cdots \times \mathbb{D}_r \simeq \left\{ z \in \mathbb{D} \mid z = \bigoplus_{i=1}^r z \cap V_i(\mathbb{R}) \right\}.
\]

Suppose we fix \( z_0 = z_0^1 \oplus \cdots \oplus z_0^r \) in \( \mathbb{D}_1^+ \times \cdots \times \mathbb{D}_r^+ \subset \mathbb{D} \), where \( z_0^i \) is a negative \( q_i \)-plane in \( V_i(\mathbb{R}) \). Let \( G_i(\mathbb{R}) \) be the subgroup preserving \( V_i(\mathbb{R}) \), let \( K_i \) the stabilizer of \( z_0^i \) and \( \mathbb{D}_i \) be the symmetric space associated to \( V_i(\mathbb{R}) \).

Over \( \mathbb{D}_1^+ \times \cdots \times \mathbb{D}_r^+ \) the bundle \( E \) splits as an orthogonal sum \( E_1 \oplus \cdots \oplus E_r \), where \( E_i \) is the bundle \( G_i(\mathbb{R})^+ \times K_i z_0^i \). Moreover the restriction of the Mathai-Quillen form to this subbundle is

\[
U_{MQ} \big|_{E_1 \oplus \cdots \oplus E_r} = U_{MQ}^1 \wedge \cdots \wedge U_{MQ}^r,
\]

where \( U_{MQ}^i \) is the Mathai-Quillen form on \( E_i \). The section \( s_v \) also splits as a direct sum \( \oplus s_{v_i} \) where \( v_i \) is the projection of \( v \) onto \( v_i \). In summary the following diagram commutes

\[
\begin{array}{ccc}
E_1 \oplus \cdots \oplus E_r & \longrightarrow & E \\
\oplus s_{v_i} & \uparrow & s_v \\
\mathbb{D}_1^+ \times \cdots \times \mathbb{D}_r^+ & \longrightarrow & \mathbb{D}^+
\end{array}
\]

and we can conclude that

\[
\varphi_{KM}(v) \big|_{\mathbb{D}_1^+ \times \cdots \times \mathbb{D}_r^+} = \varphi_{KM}(v_1) \wedge \cdots \wedge \varphi_{KM}(v_r)
\]

where \( \varphi_{KM}^i \) is the Kudla-Millson form on \( \mathbb{D}_i^+ \).
References

[1] N. Berline, E. Getzler, and M. Vergne. *Heat Kernels and Dirac Operators*. Grundlehren Text Editions. Springer Berlin Heidelberg, 2003. ISBN: 9783540200628.

[2] Raoul Bott and Loring W Tu. *Differential forms in algebraic topology*. 1st ed. Vol. 82. Springer-Verlag New York, 1982. DOI: 10.1007/978-1-4757-3951-0.

[3] Luis E. Garcia. “Superconnections, theta series, and period domains”. In: *Advances in Mathematics* 329 (2018), pp. 555–589. ISSN: 0001-8708. DOI: https://doi.org/10.1016/j.aim.2017.12.021. URL: http://www.sciencedirect.com/science/article/pii/S0001870816305564.

[4] Ezra Getzler. “The Thom class of Mathai and Quillen and probability theory”. In: *Stochastic analysis and applications (Lisbon, 1989)*. Vol. 26. Progr. Probab. Birkhäuser Boston, Boston, MA, 1991, pp. 111–122. DOI: 10.1007/978-1-4612-0447-3_8.

[5] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry*. Ed. by Academic Press. Vol. Volume 1. Interscience publisher, 1963.

[6] Stephen S. Kudla and John J. Millson. “The theta correspondence and harmonic forms. I”. In: *Mathematische Annalen* 274.3 (1986), pp. 353–378. DOI: 10.1007/BF01457221.

[7] Stephen S. Kudla and John J. Millson. “The theta correspondence and harmonic forms. II”. In: *Mathematische Annalen* 277.2 (1987), pp. 267–314. DOI: 10.1007/BF01457364.

[8] Stephen S. Kudla and John J. Millson. “Intersection numbers of cycles on locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several complex variables”. en. In: *Publications Mathématiques de l’IHÉS* 71 (1990), pp. 121–172. URL: http://www.numdam.org/item/PMIHES_1990__71__121_0/.

[9] Varghese Mathai and Daniel Quillen. “Superconnections, thom classes, and equivariant differential forms”. In: *Topology* 25.1 (1986), pp. 85–110. ISSN: 0040-9383. DOI: 10.1016/0040-9383(86)90007-8.

[10] André Weil. “Sur certains groupes d’opérateurs unitaires”. In: *Acta Mathematica* 111.none (1964), pp. 143–211. DOI: 10.1007/BF02391012.