Mathematical analysis of an epidemic through qualitative analysis with a third order model

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Abstract. The present article exposes the analysis of the dynamic model of an epidemic, which from certain considerations pretends to present the way in which for a given population the dispersion of a virus can occur, initially the preliminary concepts of the theory are exposed to then incur the qualitative solution forms of greatest use in the specialized literature, finally the third order dynamic model of the epidemic is exposed, where the considerations and analysis procedures to obtain the final results are shown.

1. Introduction
The analysis of formal dynamic systems begins with the study of linear systems, since the behavior of these systems can be easily predicted analytically, however few physical systems are really linear, so the study should be focused on non-linear systems, a traditional way of solving these systems is linearizing around a point, although this procedure is not inappropriate, it biases the knowledge of the behavior of the non-linear model.

There are several analytical methods to solve the problem, but they really have high difficulty even for the simplest nonlinear systems, even so, there are easy application techniques to solve nonlinear systems in a simple way, these techniques contain a geometric analysis in which from the given system the trajectories are drawn and information is extracted from the solutions, in most cases this geometric reasoning allows to trace the trajectories in the phase plane without solving the system. This (qualitative) analysis that this document focuses on, aims to show how, from observing the first-order differential equation plotted on the plane, it shows the trajectories it contains, from the observation of critical points. In addition, it will be explained how the variation of the parameters involved in the model makes stable and unstable critical points change position or even stability (bifurcation). This analysis is made from the technique known as flows in the lines [1].

2. Dynamical systems
The notion of a dynamic system is the mathematical formalization of the scientific concept of “a deterministic process”. The future and past state of many physical, biological, and economic chemical systems can be predicted to some extent by knowing their present states and the laws that govern their evolution [2]. The traditional way of writing a dynamic system is Equation (1):
2.1. Non-linear models

The analysis that works for linear systems is based on the principle of superposition, however, as mentioned above, most of the models that exist in nature have a non-linear structure which complicates the analysis a bit because the principle of overlap does not work anymore and it begins to require more extended mathematical tools. One way to solve non-linear systems is to linearize around a nominal operating point and start working from there, and although this practice is correct, it has the disadvantage that only the local behavior of the system can be predicted [3].

A clear example of this linearization procedure is when the simple pendulum, Figure 1, is modeled by a second order system (PS) and linearized (by Taylor series of first degree) considering small amplitude oscillations (small angle $\theta$), and in this way you can have clarity in the behavior of the pendulum with a more simple expression, however, you will not have knowledge for large oscillations Equation (2).

$$\ddot{\theta} + \frac{g}{L} \sin(\theta) = 0 \rightarrow \ddot{\theta} + \frac{g}{L} \theta = 0$$

Figure 1. Simple pendulum.

It is usual to perform the process described above, but in many nonlinear models it is not possible to predict their behavior through linearization’s, the custom of performing these procedures is that the linear models are easy to solve because the problem can be divided into more parts. small, and each of these parts is even easier to solve than the original problem and later these answers can be combined to obtain the general solution of the problem. The difficulty of non-linear systems, as has been indicated, lies in the fact that it is not possible to apply the principle of superposition. Fortunately, there are techniques to predict behavior outside the operating points of the system, as will be seen below [4].

2.2. Qualitative analysis

Qualitative analysis is a way of solving non-linear systems such as those mentioned in Equation (1), solutions of this type of systems can be visualized as trajectories that ow through a phase space of dimension $n$ with $(x_1, x_2, ..., x_n)$ coordinates. However, the purpose of this document is to focus on solving a first-order system Equation (3):

$$\dot{x} = f(x),$$

where $x(t)$ is a function of real value of time $t$ and $f(x)$ is a smooth function of real values of $x$. The methodology to analyze this system is the observation of graphics, or said in a more formal way; a differential equation is interpreted as a vector field. For example, for the system Equation (4) [5].

$$\dot{x} = \sin (x)$$
Whose analytical solution for the initial value $x = x_0$, $t = 0$, is Equation (5):

$$t = \ln \left| \frac{\csc x_0 + \cot x_0}{\csc x + \cot x} \right|$$

Even after having the analytical solution to the equation, it is difficult to know its behavior over time. Through qualitative analysis, we have $t$ is time, $x$ is the position of an imaginary particle moving along a real line, and $\dot{x}$ as the velocity of that particle [6]. It will be then that the differential Equation (4) represents a vector field in the line, which indicates the velocity vector for each position $x$.

As seen in Figure 2, the vector field indicates the direction of $\dot{x}$, through the arrows drawn on the $x$ axis it is possible to see that they are to the right when $\dot{x} > 0$ and to the left when $\dot{x} < 0$, in this way you can see also if the critical points $\dot{x} = 0$ are stable or unstable (attractors or repulsors) [7].

2.3. **Bifurcations**

The bifurcations indicate a change in the critical points of the dynamic system, the change that can occur is in the existence of the critical points or in the stability of these, that is, in a moment the number of critical points of the system can change. those that were stable can be unstable, the moment in which this happens is defined by the variation of the system parameter Equation (6) [8].

$$\dot{x} = \mu + x^2$$

For the system Equation (6) the qualitative analysis of Figure 3 to Figure 5 is carried out, in which the parameter $\mu$ is varied.

It is observed that for values of $\mu > 0$ there are no critical points for the system, when $\mu < 0$ there are two critical points, one stable and one unstable, and when $\mu = 0$ there is a critical point “stable medium”, which is where the bifurcation of the system occurs, since, if the $\mu$ parameter increases or decreases a little, the critical points disappear or appear. This is the qualitative change called bifurcation and it is important from the scientific point of view since it provides a transition and instability model for some control parameters that are varied in a system [9,10].
3. Model of an epidemic

The population to analyze can be divided into three classes: \( x(t) \) number of healthy people; \( y(t) \): number of sick people; \( z(t) \): number of people killed. Assuming that the total population remains constant in size including deaths, it is possible to ignore the slower changes in populations due to births, emigration or deaths from other causes. Therefore, the resulting Equation (7).

\[
\begin{align*}
\dot{x} &= -kxy \\
\dot{y} &= kxy - ly \\
\dot{z} &= ly
\end{align*}
\]

Where \( k \) and \( l \) are positive constants. The equations are based on that first; healthy people get sick at a rate proportional to the product of \( x \) and \( y \). This would be true if healthy and sick people are at a rate proportional to their number, and if there is a constant rate \( k \) that the meeting leads to the transmission of the disease. Second; sick people die at a constant rate of \( l \). Because the sum \( \frac{d}{dt}(x + y + z) = \dot{x} + \dot{y} + \dot{z} = 0 \), then indicates that the population remains constant \( x + y + z = N \).

It is also possible to solve \( x \) in terms of \( z \) Equation (8) to Equation (10):

\[
\frac{dx}{dz} = \frac{k}{l} x \rightarrow x(z) = x_0 e^{-kz/l} \tag{8}
\]

Next; \( \dot{z} = ly = l(N - x - z) \rightarrow z = l(N - z - x_0 e^{-kz/l}) \). Now, leaving \( u = kz/l \), we obtain:

\[
\frac{du}{dt} = \frac{k}{l} \frac{dz}{dt} \tag{9}
\]

\[
\frac{1}{k} \frac{du}{dt} = l \left( N - \frac{1}{k} u - x_0 e^{-u} \right) \rightarrow \frac{du}{dt} = kN - lu - kx_0 e^{-u} \tag{10}
\]

Then \( \tau = kx_0 t \) Equation (11):

\[
\begin{align*}
\frac{du}{d\tau} &= \frac{N}{x_0} - \frac{1}{kx_0} u - e^{-u} \\
\frac{du}{d\tau} &= a - bu - e^{-u} \tag{11}
\end{align*}
\]

Now, by definition we have Equation (12):

\[
a = N/x_0 > 1 \quad b = 1/kx_0 > 0 \tag{12}
\]

The obtaining of the fixed points is done according to the method of the flows in the lines Equation (13).
\[ \dot{u} = 0 \rightarrow 0 = a - bu^* - e^{-u^*} \rightarrow e^{-u^*} = a - bu^* \]  

(13)

Figure 6. Fixed points.

From Figure 6 it is possible to observe that there are two fixed points, \( u^+ \) and \( u^* \), of which only one of them makes sense \( u^* \), this due to the direct relationship that exists between the variable \( z \) and \( u \). According to the above then the variable will indicate in some way the number of deaths and if at any time an epidemic occurs. Also, with the qualitative analysis can be seen in Figure 6 that there is a bifurcation when the parameters \( a = b = 1 \), however, this bifurcation does not provide more information on the behavior of the population. The analysis of real interest is observed when the parameter values are modified [11].

Figure 7 and Figure 8 it can be verified that at the moment when the parameter \( b < 1 \) the value of \( u^* \) is greater than the value of parameter \( a \), which indicates the size of the population, \( x = 1 \), this means that a from \( b = 1 \) the epidemic occurs, since for values of \( b > 1 \) the fixed point \( u^* \) is smaller than the parameter \( a \), thus tending a stable number of deaths without an epidemic occurring [12].

4. Conclusion

The model of an epidemic of Kermack and McKendrick is very useful for studying the behavior of an epidemic in an ideal environment and under certain assumptions. The qualitative techniques of dynamic systems analysis prove that it is not necessary to obtain a formalized function to be able to carry out an in-depth analysis of a deterministic model, in addition to the fact that the analysis of a third-order model could be simplified to a first-order one and this The critical point of the model was identified. The analysis made in this document also allows us to give an outline of future studies of dynamic population models in which other important social phenomena are studied.

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