Delensing gravitational wave standard sirens with shear and flexion maps

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ABSTRACT
Supermassive black hole binary (SMBHB) systems are standard sirens – the gravitational wave analogue of standard candles – and if discovered by gravitational wave detectors, they could be used as precise distance indicators. Unfortunately, gravitational lensing will randomly magnify SMBHB signals, seriously degrading any distance measurements. Using a weak lensing map of the SMBHB line of sight, we can estimate its magnification and thereby remove some uncertainty in its distance, a procedure we call ‘delensing’. We find that delensing is significantly improved when galaxy shears are combined with flexion measurements, which reduce small-scale noise in reconstructed magnification maps. Under a Gaussian approximation, we estimate that delensing with a 2D mosaic image from an Extremely Large Telescope could reduce distance errors by about 25–30 per cent for an SMBHB at $z = 2$. Including an additional wide shear map from a space survey telescope could reduce distance errors by nearly a factor of 2. Such improvement would make SMBHBs considerably more valuable as cosmological distance probes or as a fully independent check on existing probes.

Key words: gravitational lensing – gravitational waves – distance scale.

1 INTRODUCTION
Recently there has been growing interest in the potential future use of coalescing compact binary systems as high-precision cosmological distance indicators. During their inspiral phase, prior to coalescence, the amplitude and frequency of the gravitational wave emission from these systems vary rapidly with time, following a characteristic ‘chirp’ waveform that is strongly dependent on the binary masses.

In a seminal paper, Schutz (1986) showed how measurement of the amplitude, frequency and frequency derivative of the inspiralling binary, using observations carried out by a network of interferometric gravitational wave detectors, could yield a precise estimate of its luminosity distance – completely independent of the traditional cosmic distance ladder. More recently, these binary systems have been labelled ‘gravitational wave standard sirens’ – by analogy with electromagnetic standard candles (although, strictly, they do not require the assumption that all sources have the same gravitational wave luminosity).

Siren candidates include neutron star–neutron star binaries, believed to be associated with short-duration gamma-ray bursts (Eichler et al. 1989). These systems are among the prime targets for detection by the next generation of ground-based interferometers, Advanced LIGO (Laser Interferometer Gravitational-wave Observatory) and Advanced Virgo, which should detect up to a few dozen such binary coalescences per year (Kopparapu et al. 2008). Even more promising siren candidates are the mergers of supermassive black hole binaries (SMBHBs). These systems are expected to be extremely luminous gravitational wave sources and will be prime observational targets for the planned Laser Interferometer Space Antenna (LISA; Bender et al. 1994). Moreover, the issues addressed in this paper are of particular importance for SMBHBs; consequently they will be the principal focus in what follows.

The expected number and redshift distribution of SMBHBs that will be observed by LISA are very uncertain, being strongly dependent on the details of our model for the history of galaxy mergers (Sesana, Volonteri & Haardt 2007; Arun et al. 2009). However, the impact of observing even a handful of SMBHBs could be very significant, as first demonstrated by Holz & Hughes (2005, hereafter HH05). Those authors performed a Monte Carlo study to calculate the fractional accuracy with which luminosity distance can be determined for sirens of a range of masses and at different redshifts, randomly distributed in orientation and sky position. Assuming that the sky position of the SMBH could be determined exactly from electromagnetic observations, HH05 found that, for e.g. the merger of a $10^5$ and $6 \times 10^5$ solar mass black hole at $z = 1$, the most probable fractional error on luminosity distance was 0.1 per cent. Moreover, the same binary system observed at $z = 3$ would yield a most probable fractional error of 0.5 per cent. HH05 then demonstrated the dramatic cosmological potential of these high-precision distance indicators: only two sirens observed at $z = 1$ and 3 could constrain cosmological parameters at a level competitive with 3000 Type Ia supernovae (SNeIa).

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As pointed out by HH05, however, there is a huge caveat: standard sirens (like standard candles) will be (de)magnified by weak lensing caused by matter fluctuations along their line of sight. The observed luminosity distance to a siren is related to its true luminosity distance by

\[ D_{\text{L}}^{\text{obs}} = D_{\text{L}} \mu^{-1/2} \approx D_{\text{L}}(1 - \delta\mu/2), \]

where \( \mu \) is the lensing magnification and \( \delta\mu \equiv \mu - 1 = 0 \) for an unlensed source. The scatter in \( \delta\mu \) is expected to be several per cent for high-redshift sirens; this adds in quadrature to the sirens’ intrinsic scatter, substantially degrading their effectiveness as cosmological probes.

Subsequent work on SMBHBs has mainly focused on better localizing and understanding their electromagnetic signatures and further reducing their intrinsic scatter in the absence of lensing (see e.g. Vecchio 2004; Lang & Hughes 2006; Kocsis & Loeb 2008; Porter & Cornish 2008). Notwithstanding this progress, however, the damaging impact of weak lensing on the performance of sirens has remained largely unaddressed – although a notable exception has been Jönsson, Goobar & Mörtsell (2007), which built on earlier work exploring weak lensing corrections to standard candles (Jönsson et al. 2005, 2006a,b). Their method proposes to model, and thus correct for, lensing magnification by using the observed photometric and spectroscopic properties of foreground galaxies along the line of sight to the siren. The authors consider the case where the lensing signal is dominated by a single dark matter halo; the magnification factor is computed from an inferred halo mass profile, which is in turn constrained by photometric and spectroscopic data using the Tully–Fisher or Faber–Jackson relations. The authors’ results are quite impressive, showing that the dispersion due to lensing for a standard candle or siren at \( z = 1.5 \) can be reduced by about a factor of 2 (Jönsson, Mörtsell & Sollerman 2009). However, their method does involve a number of specific modelling assumptions about halo mass profiles and the observable mass proxy.

In this paper, we pursue an alternative approach which makes fewer modelling assumptions: we propose calculating weak lensing corrections derived from gravitational lensing shear and flexion maps constructed in the direction of the SMBHB. In other words, we directly measure the siren magnification in order to remove it, a procedure we call ‘delensing’. Dalal et al. (2003, hereafter D03) have shown that performing such corrections for large surveys of SNeIa is unlikely to be worthwhile. Although SNeIa surveys conveniently provide galaxy shape measurements that could be used to delens each supernova, they will not have galaxy densities high enough to resolve small-scale contributions to the magnification.

We find at least three reasons to reconsider delensing in the case of SMBHBs.

(i) Since we expect SMBHB detections to be rare, we can furnish each one with a very deep pointed follow-up observation, thus obtaining the high galaxy densities needed for high-resolution shape maps.

(ii) Galaxy images from devoted observations could enable higher quality shear and flexion measurements than those available from a survey telescope. We will show that high-quality flexion measurements are a substantial advantage.

(iii) Delensing will make more of an impact on SMBHB distance measurements than on SNeIa measurements. Lensing will be by far the largest source of scatter in SMBHB distances, whereas SNeIa retain a significant intrinsic scatter even after delensing. In addition, since surveys will detect a very large number of SNeIa, the magnification noise will be partially averaged away.

In this paper, we investigate the efficacy of delensing SMBHBs with high-quality shear and flexion maps. In Section 2, we briefly review the Kaiser–Squires (KS) technique for reconstructing a magnification map from shape measurements. In Section 3, we present our methodology for estimating how precisely an SMBHB magnification can be measured from a set of shape maps. We also demonstrate the advantage of combining shear and flexion measurements to substantially reduce shape noise. In Section 4, we compute the reduction in the scatter in SMBHB distances that could be achieved by delensing with an Extremely Large Telescope (ELT). We further show that delensing can be improved by combining ELT maps with a wide space telescope survey. We conclude in Section 5 along with a discussion of topics for further investigation.

## 2 MAPMAKING

### 2.1 Basics

In order to improve our estimate of \( D_L \), the luminosity distance to a standard siren, we need to obtain an accurate estimate of \( \mu (x_{\text{BHB}}, z_{\text{BHB}}) \), the lensing magnification for the siren’s sky position, \( x_{\text{BHB}} \) and redshift, \( z_{\text{BHB}} \). If the siren’s signal has undergone weak lensing, this magnification is related to the lensing convergence \( \kappa \) by \( \mu \approx 1 + 2\kappa \). So if we can make an accurate convergence map at and around the siren’s location, then we would have an estimate of the magnification in that region.

Convergence maps can be constructed from shear measurements using the KS inversion method (Kaiser & Squires 1993; Bartelmann & Schneider 2001). Since shear and convergence are both combinations of derivatives of the lensing potential, the convergence can be estimated from the shear field, \( \gamma(x) \), by averaging shear estimators in pixels on a grid, Fourier transforming to obtain shear estimators, \( \tilde{\gamma}(l) \), and then converting the shear to convergence \( \kappa(l) \) according to simple algebraic equations. The convergence can then be inverse Fourier transformed to obtain a \( \kappa ) \) map, and hence a \( \mu \) map. Higher order shape distortions can similarly be inverted to create a \( \kappa \) map. Bacon et al. (2006) showed that flexion measurements, which characterize the slight arc of images, can be inverted in a fashion similar to KS. There are two flexion types: the 1-flexion, \( F \), represents a skew to the galaxy shape, while the 3-flexion, \( G \), represents the trefoil component of the galaxy. We review the relevant inversion equations below.

We start by introducing the complex notation for shear and flexion. For any shape, \( \Upsilon \in \{\gamma, F, G\} \),

\[ \Upsilon(x) \equiv T_1(x) + i T_2(x). \]

Subscripts on shear or flexion denote the two distinct polarizations, defined according to some fixed \( x \)-axis. Next, borrowing notation from Bartelmann & Schneider (2001), we define the following complex kernels in Fourier space:

\[ \hat{D}_\gamma(l) = \frac{\pi}{l^2} (l_1^2 + l_2^2) \]

\[ \hat{D}_F(l) = -\frac{i\pi}{l^2} (l_1 + i l_2) \]

\[ \hat{D}_G(l) = \frac{i\pi}{l^2} (l_1 + i l_2)^3, \]

where \( l \) is the Fourier conjugate to \( x \) and \( l \equiv |l| \equiv l_1^2 + l_2^2 \). Note that the subscripts on \( l \) denote the \( x \)- and \( y \)-components of the
wave-vector $\mathbf{l}$ relative to the same $x$-axis used to define polarization. Hereafter, we will not explicitly write the dependence of quantities on $\mathbf{l}$.

In Fourier space, for $\mathbf{l} \neq 0$, weak lensing distortions are related to the convergence by\(^1\) (Bacon et al. 2006)

\[
\hat{\gamma} = \pi^{-1} \kappa \hat{D}_{\gamma}
\]

(6)

\[
\hat{\mathcal{F}} = \pi^{-1} \kappa \hat{D}_{\mathcal{F}} l^2
\]

(7)

\[
\hat{\mathcal{G}} = \pi^{-1} \kappa \hat{D}_{\mathcal{G}} l^2.
\]

(8)

We can therefore use the fact that

\[
D_{\gamma}^\ast D_{\gamma} = \pi^2
\]

(9)

and

\[
D_{\gamma}^\ast D_{\mathcal{F}} = D_{\gamma}^\ast D_{\mathcal{G}} = \pi^2 l^{-2}
\]

(10)

to convert shape modes to convergence modes using the following equation for any shape:

\[
\kappa_{\text{map}} = \pi^{-1} \hat{D}_{\gamma}^\ast \hat{\gamma}.
\]

(11)

In practice, these inversions must account for the shape of the field (including masking), pixelation and smoothing of the shear and flexions. We do not study the effects of the former two in this paper.

### 2.2 Challenges

If we had perfect knowledge of the $\gamma$, $\mathcal{F}$ and/or $\mathcal{G}$ field on all scales, and at the redshift of the SMBH, this approach would yield a perfect map for $\mu$ and hence a perfect correction for $D_{\gamma}$. However, in reality our measurements of lensing distortions are noisy, exist only on lines of sight which contain a galaxy, cover a range of redshifts and are for a limited patch of sky, so our $\mu$ map will be imperfect.

First there is the impact of noisy shape measurements. We cannot measure the true shear or flexion of a galaxy; rather, we measure an estimator for these quantities, including the noise due to the galaxy’s intrinsic shape (ellipticity, skew, etc.), noise from discrete sampling (including masking), pixelation and smoothing of the shear and flexions. We do not study the effects of the former two in this paper.

In addition to these small-scale inaccuracies, a realistic survey will have a finite size in the sky. This means that large-scale matter density modes, which again contribute to the total $\mu$, will not be estimated by our survey. This issue is related to the mass-sheet degeneracy: for any constant $\beta$, the $\kappa$ estimated by KS inversion is degenerate with $(1 - \beta)\kappa + \beta$. In the weak lensing limit, we expect $\beta \ll 1$; therefore, we can ignore the factor $(1 - \beta)$ and determine the convergence modulo some constant offset, $\beta$. For large fields $\beta \to 0$, but small surveys need to take account of this effect. Our method for estimating errors in $\kappa$ must therefore deal with small-scale smoothing and large-scale cut-offs in our data. In addition, the inversion equations estimating $\kappa$ from shear and flexion should be combined in an optimal fashion, as we explain in Section 3.3.

Already it can be seen that ideally we require an ambitious lensing follow-up programme for sirens. This will include observations of the immediate region of the siren, with the best possible resolution (to minimize each $\sigma_\gamma$ and depth (to maximize $n_{\text{gal}}$); this will allow us to probe the small-scale contributions to $\mu$. In addition, we preferably require a large survey (still with considerable image resolution and depth) around the siren position, to probe the large-scale contributions to $\mu$. These requirements could be fulfilled by using an ELT (ELT Science Working Group 2006) with adaptive optics to image the central region, together with a survey space telescope to give the large-scale map. We will explore the consequences of this strategy in Section 4.2.

### 3 METHODOLOGY

#### 3.1 Estimating reduction in lensing scatter

Let $\kappa(x; z)$ denote the true value of the convergence field for an object at sky position $x$ and redshift $z$, and let $\kappa_{\text{BHB}}$ be the true convergence along the line of sight to a particular SMBH:

\[
\kappa_{\text{BHB}} \equiv \kappa(x_{\text{BHB}}; z_{\text{BHB}}).
\]

(12)

Our uncertainty in the convergence at the SMBH, $\sigma(\kappa_{\text{BHB}})$, is initially (with no lensing map) given by

\[
\sigma(\kappa_{\text{BHB}}) = \left(\kappa_{\text{BHB}}^2\right)^{1/2},
\]

(13)

where angular brackets denote an ensemble average. Suppose that we can use galaxy shape measurements to construct a map, $\kappa_{\text{map}}(x)$, which is an unbiased estimator of the projected convergence field. We can then replace our uncertainty with a smaller one, $\sigma(\kappa_{\text{BHB}})$, since the $\kappa$ map will correlate with the true convergence. Following D03 and Bower (1991), such a correlation can be used to reduce our uncertainty according to

\[
\sigma^2(\kappa_{\text{BHB}}) = (1 - r^2)\sigma(\kappa_{\text{BHB}})^2
\]

(14)

\[
r^2 \equiv \left(\kappa_{\text{BHB}}\kappa_{\text{map}}(x_{\text{BHB}})\right)^2 / \left(\kappa_{\text{BHB}}^2 / \kappa_{\text{map}}(x_{\text{BHB}})^2\right).
\]

(15)

Thus, $r^2$ can range from zero to unity, with $r^2 = 1$ when our estimator perfectly correlates with $\kappa(x; z_{\text{BHB}})$. Of course, we expect less than perfect correlation due to the noise, smoothing and broad redshift range of the shape measurements.

For our $\kappa$ map, we consider a circular field of radius $\Theta$ containing galaxies with a redshift distribution $n_{\text{gal}}(z)$. Let $\kappa_{\text{map}}(x)$ be the reconstructed convergence at sky position $x$, obtained from a KS-like inversion of shear and flexion maps. These maps will have been smoothed by a radially symmetric filter with scale radius $\theta$ (filter can be a top-hat, Gaussian, etc.). In the absence of noise, the

---

\(^1\)Note that equations (62) and (63) in Bacon et al. (2006) are missing a factor of $1/2$ on the right-hand side (cf their equation 13).
reconstructed convergence would differ from the true, smoothed convergence of the galaxies by \(K(\Theta)\), a constant determined by fluctuations larger than the field area. The mass-sheet degeneracy prevents us from obtaining this constant from shape measurements alone. With noise, we have

\[ \kappa_{\text{map}}(x) + K(\Theta) = \kappa_0(x) + v_\Theta(x; \Theta). \]  

(16)

where, \(\kappa_0\) is the true, smoothed convergence of the galaxies and \(v_\Theta\) is filtered shape noise. Thus, if we knew \(K(\Theta)\), we could add it to our map to obtain the total, smoothed convergence signal at \(x\) plus noise. Note that the noise at \(x\) depends on all galaxies in the survey area, not just those within the smoothing scale, because of the non-local properties of the KS inversion.

We will approximate the lensing fields as Gaussian random so that convergence fluctuations on different angular scales do not correlate with each other. Specifically, we assume that fluctuations within the map do not correlate with the mass-sheet degeneracy:

\[ \langle \kappa_{\text{map}}(x) K(\Theta) \rangle \approx 0. \]  

(17)

If our map is centred on the SMBHB, a simple theoretical estimate of \(K(\Theta)\) is the true convergence of the galaxies, evaluated at \(x_{\text{BHB}}\) and smoothed on the scale of the map:

\[ K(\Theta) \approx \kappa_0(x_{\text{BHB}}). \]  

(18)

where a \(\Theta\) subscript denotes top-hat smoothing. Strictly speaking then, even for a Gaussian random field, the correlation in (17) will be small but non-zero since the Fourier transform of the map area will not be a perfect top-hat in Fourier space. We will also ignore intrinsic alignments which would correlate lensing with shape noise (Crittenden et al. 2001; Heymans & Heavens 2003; Hirata & Seljak 2004), i.e. we assume

\[ \langle \kappa_0(x) v_\Theta(x; \Theta) \rangle \approx 0. \]  

(19)

Furthermore, we ignore higher order lensing effects such as reduced shear/flexion (White 2005; Dodelson, Shapiro & White 2006; Schneider & Er 2008). The simplifications implemented here should be revisited in future work.

We are now ready to express \(r^2\) in terms of expressions that we can calculate. When we square both sides of (16), take the ensemble average and include the approximations (17)–(19), we get

\[ \langle \kappa_{\text{map}}(x) + (\kappa_0(x_{\text{BHB}}))^2 \rangle = \langle \kappa_0(x)^2 \rangle + \langle v_\Theta(x; \Theta)^2 \rangle. \]  

(20)

We then plug (16) and (20) into (15) to get

\[ r^2 = \frac{\langle \kappa_0(x_{\text{BHB}}) - \kappa_{\text{BHM}} \kappa_0 \rangle^2}{\langle \kappa_{\text{BHM}} \kappa_0^2 - \kappa_0^2 \rangle}, \]  

(21)

where dependence on \(x_{\text{BHB}}\) has been omitted.

Working under the flat-sky and Limber approximations, the various correlations \(\langle \kappa(x) \kappa' \rangle\) in (21) can be calculated using the following general expression:

\[ \langle \kappa_i \kappa_j \rangle = \frac{1}{2\pi} \int_0^\infty dl \int P_s(l; i, j) \tilde{F}_l(l) \tilde{F}_l(l), \]  

(22)

for \(i, j \in \{\theta, \Theta, \text{BHB}\}\). \(P_s(l; i, j)\) is a generalized convergence power spectrum

\[ \langle \kappa_i(l) \kappa_j(l') \rangle = (2\pi)^2 \delta_0(l - l') P_s(l; i, j), \]  

(23)

where \(\delta_0\) is the Dirac delta function. \(\tilde{F}_l\) is the Fourier transform of the appropriate filter function for \(\kappa_i\); if we use Gaussian-smoothed shape maps and a top-hat-shaped survey then

\[ \tilde{F}_l(l) = e^{-l^2/2}, \]  

\[ \tilde{F}_\Theta(l) = \frac{2J_1(l\Theta)}{l\Theta}, \]  

\[ F_{\text{BHB}}(l) = 1. \]  

(24)

The generalized convergence power spectrum is given by an integral over comoving distance \(\chi\):

\[ P_s(l; i, j) \equiv \int_0^\infty \frac{d\chi}{\chi^2} W_i(\chi) W_j(\chi) P_s(k; \chi), \]  

(25)

\[ W_i(\chi) = \frac{3}{2} \Omega_m a^2(1+z)^2 \int_0^{\chi_H} dz \frac{dz}{dz_H} \frac{dz}{dz_H} \frac{dz}{dz_H}, \]  

(26)

where \(P_s(k; \chi)\) is the 3D matter power spectrum for \(k = l/\chi\) at a distance \(\chi\), accounting for the growth of structure. The scalefactor is \(a = (1+z)^{-1}\) and \(\chi_H\) is the distance to the horizon. We have assumed a flat Universe for simplicity. Here, \(p(z)\) is the redshift distribution of the sources:

\[ p_0(z) = p_{ga}(z), \]  

(27)

Using equations (22)–(27), we can calculate \(r^2\) from (21); we only need \(\langle v_\Theta(x) \rangle^2\), the variance at a point in our convergence map due to noise. In Section 3.2, we discuss the dominant sources of noise in shear and flexion maps. In Section 3.3, we calculate how this noise translates into noise in a reconstructed convergence map.

3.2 Noise contributions to shear and flexion maps

We consider two sources of noise in our shear and flexion maps. The first source, shape noise, is due to the intrinsic shapes of unlensed galaxies. We estimate that an instrument such as an ELT will provide rms intrinsic galaxy shapes of

\[ \sigma_{\gamma}^2 = 0.2, \]  

\[ \sigma_{\theta}^2 = 0.5 \text{ arcm}\min, \]  

\[ \sigma_{\Theta}^2 = 0.9 \text{ arcm}\min, \]  

(28)

where the covariance between these measures is negligible (Massey et al. 2007; Rowe, Bacon, Taylor, Massey, Heymans, Goldberg, Barden & Caldwell, in preparation).

The second source of noise stems from the finite sampling of the shear and flexion fields: we only measure these fields where we can see a galaxy. Were we to reconstruct the convergence at \(x\) using two different sets of galaxies in the neighbourhood of \(x\), the two sets would sample the shear and flexion fields in different random places. Thus, we would not expect the convergence estimates to be equal even if the galaxies were perfectly circular. Seitz & Schneider (1995) demonstrated the significance of sampling noise for shear in their reconstruction of a simulated cluster, which had a peak convergence of \(\kappa \approx 1.5\). With at most a few tens of SMBHBs available to us, we do not expect them to lie behind clusters – a random line of sight will have a \(\kappa\) of only a few per cent. We will show accordingly that sampling noise is not appreciable for shear along a random line of sight, but it is none the less significant for flexion.

Suppose we obtain continuous shape maps by smoothing our discrete shape maps with the window function \(F_\Theta(x)\). The error in each map at \(x_0\) is the error in the average of all nearby galaxy shapes, weighted by \(F_\Theta(x - x_0)\). The effective number of galaxies within the window is

\[ N_{\text{eff}} \equiv n_{\text{gal}} \int d^2x F_\Theta(x - x_0). \]  

(29)

So the expected error in \(\gamma_{\text{BHB}}(x_0)\) or \(\theta_{\text{BHB}}(x_0)\) will be \(\sigma_{\gamma} / \sqrt{N_{\text{eff}}}\), where we can define

\[ \sigma_{\gamma}^2 = \sigma_{\gamma}^2 + \sigma_{\Theta}^2. \]  

(30)

The first term is the intrinsic shape noise. The second term is sampling noise, which arises because the shear and flexion signals vary across the window.
To estimate the sampling noise, we imagine a sample of intrinsically circular galaxies within a smoothing window of scale radius \( \theta \). Then, e.g., the shears of these galaxies will only have a non-zero scatter if the cosmic shear field varies significantly on scales smaller than \( \theta \). Furthermore, for each source galaxy, there will be an angular size, \( \phi \), such that shear fluctuations on scales smaller than \( \phi \) will not influence our measurement of the galaxy’s shear.\(^2\) We expect that \( \phi \) will be of the order of a source’s half-light radius, perhaps several half-light radii, and the cut-off may not be sharp. This cut-off has not been well studied, so for simplicity, we assume a common \( \phi \) for all galaxies and experiment with different values of \( \phi \). Then, for a random line of sight, we expect that the scatter of shears in the smoothing window will be approximately given by the contribution to the cosmic shear scatter from scales that are between \( \theta \) and \( \phi \). The same will be true for flexion. Hence, we estimate the sampling noise to be

\[
\sigma^2_\theta(\theta, \phi) \approx \langle |\gamma_i|^2 \rangle - \langle |\gamma_i|^{2} \rangle, \tag{31}
\]

where, for \( i \in \{ \theta, \phi \} \),

\[
\langle |\gamma_i|^{2} \rangle = \frac{1}{2\pi} \int_{0}^{\infty} dx l^2 P_\gamma(l) \tilde{F}_i(l)^2, \tag{32}
\]

and \( P_\gamma \) is the cosmic shear or flexion power spectrum for our galaxy redshift range. The Gaussian filter \( \tilde{F}_i \) is defined in (24), and we use a top-hat filter to attenuate fluctuations smaller than galaxies:

\[
\tilde{F}_i(l) = \frac{2J_1(l\phi)}{l\phi}. \tag{33}
\]

Because the functions \( \tilde{F} \) are low-pass filters, (31) says that sampling noise contributions come from angular scales larger than \( \phi \) but not larger than \( \theta \).

Naturally, shear and flexion sampling noise will be less than but comparable to the rms of the cosmic shear or flexion signal – these are given by (31) with the function \( \tilde{F}_i \) removed. Since \( P_\gamma(\theta) = P_\gamma(l) \), \( \sigma^2_\gamma \) will be less than the rms of cosmic \( \kappa \), which is only a few per cent. Therefore, shear sampling noise will be negligible when added in quadrature to intrinsic shear. Meanwhile, the convergence/flexion relations in Section 2.1 give us \( P_\gamma(l) = P_\gamma(l) = l^2 P_\kappa(l) \) (Bacon et al. 2006). Therefore, over a single smoothing window, small-scale fluctuations in \( \kappa \) can lead to variations in \( \tilde{F} \) and \( \tilde{G} \) that are significantly larger. Fig. 1 shows that flexion sampling noise is comparable to the intrinsic flexions in (28). Note that our calculation of sampling noise is sensitive to the shape of the non-linear matter power spectrum. If matter fluctuations are suppressed on small scales, perhaps due to baryonic effects, then sampling noise will be reduced.

### 3.3 Noise in a Kaiser–Squires inversion with shear and flexion

The key to a KS-like inversion is the simple relationship, (11), between convergence and shear or flexion in Fourier space. Thus, we can easily translate uncertainty in the Fourier modes of the shape maps into uncertainty in the Fourier modes of the reconstructed

\[^2\]We find this to be a reasonable statement, but we were compelled to investigate it further. Borrowing simulations from Bacon, Amara & Read (2009), with kind permission from the authors, we verified that the shear and flexion of a source galaxy are negligibly perturbed by lensing substructures that are smaller than the solid angle of the source. Further discussion is beyond the scope of this paper.

\[\langle \tilde{\gamma}^2(l) \tilde{\gamma}^2(l') \rangle = (2\pi)^2 P_\gamma(l; \theta) \delta_0(l - l'). \tag{34}\]

Typically, we say that \( P_\gamma(l; \theta) \) contains a signal term and a term due to the intrinsic ellipticities of galaxies,

\[
P_\gamma\text{noise}(l; \theta) = \frac{F_0(l)^2 \sigma_\gamma^2}{n_{\text{gal}}} \tag{35}\]

In the limit where \( l \) is a continuous variable, the noise part of the measured power spectrum represents the error on a single mode, \( \langle \tilde{\gamma}^2(l) \rangle \). To include flexion and to account for sampling noise, we can generalize to

\[
P_\gamma\text{noise}(l; \theta) = \frac{F_0(l)^2 \sigma_\gamma^2}{n_{\text{gal}}} \tag{36}\]

with \( \sigma_\gamma \) given by (30). Using (11), measurements of \( \tilde{\gamma}^2(l) \), \( \tilde{F}(l) \), and \( \tilde{G}(l) \) provide three measurements of \( \kappa(l) \); the errors on \( \kappa(l) \) are given by \( P_\kappa\text{noise}(l) = \sqrt{P_\kappa\text{noise}(l) + P_\gamma\text{noise}(l)} \) and \( |l|^2 P_\kappa\text{noise}(l) \) respectively. We can then form an estimate of \( \kappa(l) \) from a linear combination of the shear and flexions which minimizes the error. Okura, Umetsu & Futamase (2007) show that the proper combination creates a convergence map with a noise spectrum given by

\[
P_k\text{noise}(l; \theta) = \left( \frac{1}{P_\gamma} + \frac{l^2}{P_\kappa} + \frac{l^2}{P_\gamma} \right)^{-1} \tag{37}\]

\[
= \left( \frac{F_0(l)^2}{n_{\text{gal}}} \right) \left( \frac{1}{\sigma_\gamma^2} + \frac{l^2}{\sigma_\gamma^2} + \frac{l^2}{\sigma_\gamma^2} \right)^{-1}. \tag{38}\]

For a very large map, the shape noise contribution to \( \langle \kappa_{\text{map}}^2 \rangle \), the variance at a point in our convergence map, is

\[
C_\rho(\theta) = \frac{1}{2\pi} \int_{0}^{\infty} dl l P_k\text{noise}(l; \theta). \tag{39}\]

We use the lower limit of integration, \( l = 0 \), to imply a very large map, but we will consider small maps presently. Substituting (37) into (39) and defining

\[
\sigma_{\kappa,\gamma}^2 \equiv \sigma_\gamma^2 + \sigma_\gamma^2, \tag{40}\]

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we find
\[ C_\ell(\theta) = \frac{1}{2\pi} \int_0^\infty \frac{df}{f} \left( \frac{1}{f^2 \sigma_\gamma^2} + \frac{1}{\sigma_\phi^2} \right)^{-1} (e^{-f^2 \sigma_\phi^2})^2 \]  
\[ = \frac{\sigma_\phi^2}{4\pi n_\text{gal}} \exp \left( \frac{\theta^2 \sigma_\phi^2}{4\sigma_\gamma^2} \right) \Gamma \left( 0, \frac{\theta^2 \sigma_\phi^2}{4\sigma_\gamma^2} \right), \]  
where \( \Gamma(s, x) \) is the incomplete gamma function
\[ \Gamma(s, x) = \int_x^\infty dt e^{-t} t^{s-1}. \]  

In the shear-only limit, \( \sigma_\phi \rightarrow \infty \), the property
\[ \lim_{\sigma_\phi \rightarrow \infty} \Gamma(s, x) e^{s x^{-1}} = 1 \]  
leads to \( C_\ell(\theta) \approx \sigma_\phi^2 / (\pi \theta^2 n_\text{gal}) \), as expected for a KS inversion (Seitz & Schneider 1995). Decreasing the smoothing scale, \( \theta \rightarrow \theta_0 \), we see that small-scale noise leads to a divergence in the shear-only limit; hence we must smooth to obtain finite noise. In the flexion-only limit, \( \sigma_\phi \rightarrow \infty \), \( C_\ell(\theta) \) diverges due to large-scale noise, which can be seen by ignoring the shear term in (41).

For finite maps, there are no noise contributions from modes larger than the map, so we estimate that
\[ \langle \nu_B(\mathbf{x}; \theta)^2 \rangle \approx C_\phi(\theta) - C_\ell(\theta). \]  

Although we are considering a top-hat-shaped map, and the excluded noise \( C_\ell(\theta) \) assumes a large Gaussian filter, we note that for large enough \( \theta_0 \), we recover the shear-only limit, which is the same for either filter function.

When shear and flexion are combined, they attenuate shape noise in a complementary way. We can see in (41) that the shear term prevents a divergence towards small \( l \) while the flexion term reduces the divergence towards large \( l \). Without the smoothing factor, there is an ultraviolet divergence even with flexion, but it is less severe than for shear. The benefit of a combined approach is illustrated in Fig. 2, which compares shape noise from a shear-only inversion to one that includes flexion. Since flexion measurements suppress small-scale noise in the convergence, we can use flexion to reduce our smoothing scale while maintaining the noise level of a more coarsely smoothed shear-only reconstruction. Thus, our convergence map will recover more small fluctuations, leading to a larger \( \sigma_{\text{gal}} / \sigma_{\text{BHB}} \) and therefore a larger \( r^2 \) by (15). Of course, this improvement degrades with increasing \( \sigma_\phi^2 \) and \( \sigma_\phi^2 \).

4 RESULTS

For all calculations, we adopt a flat, \( \Lambda \)CDM model with \( h = 0.7 \), \( \Omega_b = 0.27 \), \( \Omega_m = 0.045 \), \( \sigma_8 = 0.8 \) and \( n_s = 0.96 \), consistent with the Wilkinson Microwave Anisotropy Probe 5-year parameters (Dunkley et al. 2008). We calculate the linear matter power spectrum using the fitting formula of Eisenstein & Hu (1999) without baryon wiggles, and we apply to this the non-linear fitting formula of Smith et al. (2003). The weak lensing fields are assumed to be Gaussian and uncorrelated on different scales. For our cosmological model, we expect that without delensing, the uncertainty in the lensing of an SMBH, \( \sigma(\kappa) = \sigma(\mu)/2 \), will be 3.9 per cent for \( z_{\text{BHB}} = 2 \) and 5.2 per cent for \( z_{\text{BHB}} = 3 \). By (1) this uncertainty is equal to the relative distance error, \( \sigma(D_L)/D_L \).

4.1 Distance error reduction from a deep image

We first consider delensing with a narrow, deep image of a similar size and depth to the Hubble Ultra Deep Field (HUDF). The HUDF contains over 8000 galaxies detected at over 10\( \sigma \) within an area of 12 arcmin\(^2\) (Coe et al. 2006). We take our galaxy redshift distribution to be
\[ \rho_{\text{gal}}(z) \propto z^\alpha e^{-z/\theta \sigma}, \]  
with \( \alpha = 0.8, \beta = 2.0 \) and \( \theta = 1.8 \) in order to approximate the redshift distribution of the HUDF. However, we suppose that we have the resolution provided by adaptive optics with an ELT of 50 mas (ELT Science Working Group 2006). With an ELT, we can consider going deeper than the HUDF so as to obtain a denser galaxy field. We note that for \( n_{\text{gal}} = 1000 \text{arcmin}^{-2} \), the intergalaxy spacing is 1.9 arcsec, and we must therefore be wary of becoming confusion limited.

As D03 point out, there is generally an optimal smoothing angle for a shear map, obtained by maximizing \( r^2 \) with respect to \( \theta \). If the smoothing is too fine, the map will be very noisy; if smoothing is too coarse, the small-scale magnification fluctuations will be washed away. For very large \( n_{\text{gal}} \) and low enough \( \sigma_\tau \) (deep, high-resolution images), the optimal smoothing scale is found to be smaller than \( n_{\text{gal}}^{-1/2} \), the typical intergalaxy spacing. If the galaxy density field were a continuum, this would not be a problem, but of course galaxies are discrete objects, so it makes no sense to make \( \theta \) smaller than the intergalaxy spacing. We therefore enforce a conservative cut-off of \( n_{\text{gal}} \theta^2 \geq 10 \) when determining the optimal smoothing scale. For example, for a galaxy density of \( n_{\text{gal}} = 800 \text{arcmin}^{-2} \), we would use \( \theta = 0.063 \) arcmin. Fig. 2 shows that at this scale, including flexion reduces the variance in the noise by about a factor of 4 relative to shear alone.

We define the distance error remainder, \( \mathcal{R} \), to be the SMBH distance error after delensing divided by the original distance error:
\[ \mathcal{R} = \frac{\sigma(D_L)}{\sigma(D_L)} \]  

Since \( \sigma(D_L) \propto \sigma(\kappa) \), the equation for \( \mathcal{R} \) follows directly from (14):
\[ \mathcal{R} = \sqrt{1 - r^2}. \]  

Fig. 3 contains contour plots showing the \( \mathcal{R} \) we could obtain from shape maps of a given size and galaxy density. The solid contours assume an HUDF-like galaxy redshift distribution, given by (46),...
while the dashed contours assume that all galaxies lie in a plane at the redshift of the SMBHB. The latter configuration is clearly a best-case scenario and provides an upper limit on how well a map can measure the SMBHB magnification.

As an example, consider the plot with $z_{\text{SMBHB}} = 2$ in Fig. 3. Going only by the solid contours, the 'HUDF' label lies between $\mathcal{R} = 0.7$ and 0.8. Therefore, we could use an HUDF-like ELT image to reduce the SMBHB distance uncertainty to about 75 per cent of its original value (i.e. we reduce the error bar by 25 per cent); this assumes that we use no redshift information about individual galaxies and do not attempt to break the mass-sheet degeneracy. In this case, we can improve our measurement of the SMBHB magnification either by widening our map or by making it deeper. With substantial access to an ELT, we could create a mosaic to reduce our distance error to less than 70 per cent of the original value.

We reiterate that the solid contours in Fig. 3 assume that we only construct 2D weak lensing maps. If we can measure the redshifts of individual galaxies and down-weight galaxies as they deviate from $z_{\text{SMBHB}}$, we could improve on these results. The $z_{\text{SMBHB}} = 3$ plot highlights the importance of the galaxy redshift distribution. Because the galaxies have $z^\text{med} = 1.8$, most of them are closer to us than the SMBHB and are therefore lensed by different structures. Hence, there is a large discrepancy between the solid (realistic) and dashed (idealized) contours. Meanwhile, the galaxies are almost evenly distributed around an SMBHB at $z_{\text{SMBHB}} = 2$, so there is a smaller difference between the contour sets.

4.2 Distance error reduction from hybrid maps

It is apparent from the solid contours in Fig. 3 that we could significantly improve our convergence reconstruction with wider images around each SMBHB. Unfortunately, it would be impractical to use telescopes such as an ELT to create mosaics larger than a few tens of square arcminutes. A space survey telescope such as the Joint Dark Energy Mission (JDEM; http://jdem.gsfc.nasa.gov) or Euclid (Laureijs et al. 2008) would provide the required width, but it would be blind to sub-arcminute convergence fluctuations that require high galaxy densities. We therefore propose making convergence maps from a hybrid of deep, pointed observations and wider survey images. The idea here is that the deep images will be used to measure small-angle convergence modes, while the wide images will pick up modes larger than the size of the deep images.

Accordingly, we relabel $\kappa_{\text{map}}$ to $\kappa_{\text{deep}}$, and we let $\kappa_{\text{wide}}$ denote the convergence map obtained from the larger survey. We are unconcerned with small fluctuations outside of the deep map. Therefore, to minimize shape noise in the wide map, we smooth it by a top-hat filter of radius $\Theta$, the size of the deep map. Then (16) can be restated for both maps:

$$\kappa_{\text{deep}}(x) + \kappa_{\text{gal}}(x) = \kappa_0(x) + \nu_0(x; \Theta)$$

$$\kappa_{\text{wide}}(x) = \kappa_{\text{gal}}^\star(x) + \nu_0^\star(x).$$

For a sufficiently wide survey, the mass-sheet degeneracy is negligible, and the shape noise at $x$ is independent of the survey area, so we make those approximations here. Also note that because each of these terms depends on the galaxy density and redshift distributions of the corresponding images, we have introduced the * superscripts to signify a dependence on the $n_{\text{gal}}$ and $p_{\text{gal}}(z)$ of the wide survey.

We can split $\kappa_{\text{SMBHB}}$ into two terms which we will try to measure using the deep or wide map. First, we define $\kappa_L$ to be equal to the total convergence smoothed on the scale of the deep map, as in (18):

$$\kappa_L \equiv \kappa_{\text{SMBHB}, \Theta}.$$ (51)

Then, we define

$$\kappa_S \equiv \kappa_{\text{SMBHB}} - \kappa_L.$$ (52)

The subscripts are meant to imply ‘small’ and ‘large’ – each term mostly comprises Fourier modes which have wavelengths smaller or larger than the size of the deep survey, $\Theta$. Clearly, the deep map will be suited for measuring $\kappa_S$ and the wide map for $\kappa_L$. We thus define ‘small’ and ‘large’ versions of $r^2$:

$$r^2_S = \frac{\langle \kappa_{\text{deep}} \kappa_S \rangle^2}{\langle \kappa_{\text{deep}}^\star \kappa_{\text{deep}}^\star \rangle} = \frac{\langle (\kappa_0 - \kappa_S) \kappa_S \rangle^2}{\langle \kappa_0^\star \rangle \langle \kappa_0^\star - \kappa_0^\star - \nu_0^\star \rangle^2}$$ (53)

$$r^2_L = \frac{\langle \kappa_{\text{wide}} \kappa_L \rangle^2}{\langle \kappa_{\text{wide}}^\star \kappa_{\text{wide}}^\star \rangle} = \frac{\langle \kappa_{\text{gal}}^\star \kappa_L \rangle^2}{\langle \kappa_{\text{gal}}^\star \rangle \langle \kappa_{\text{gal}}^\star - \nu_{\text{gal}}^\star \rangle^2}.$$ (54)

Our final uncertainty in $\kappa_{\text{SMBHB}}$ will be given by

$$\sigma'(\kappa_{\text{SMBHB}}) = \sqrt{(1 - r^2_S) \langle \kappa_S^2 \rangle + (1 - r^2_L) \langle \kappa_L^2 \rangle}.$$ (55)

We take our wide, space-based survey to have $n_{\text{gal}} = 100$ and a redshift distribution given by (46) with $\alpha = 2$, $\beta = 1.5$ and $z_{\text{med}} = 1.5$ (Albrecht et al. 2006). We take the intrinsic shear to be $\sigma_x = 0.25$, but we do not consider flexion measurements. The intrinsic flexions will be large (±1/arcmin) for a survey instrument, and since flexion’s sensitivity to matter fluctuations decreases with angular size, it should only marginally improve a shear measurement of $\kappa_L$. The dotted contours in Fig. 3 show how including a wide shear map improves on a deep map’s ability to delens an SMBHB.
Figure 4. Uncertainty in SMBHB distance versus SMBHB redshift, with and without delensing. Solid: distance uncertainty without delensing. Dotted: reduction expected after shear-only delensing with a 30 arcmin$^2$ image from an ELT ($n_{gal} = 1000$ arcmin$^{-2}$, $\cmed = 1.8$). Dashed: shear and flexion delensing with an ELT. Dot–dashed: shear and flexion delensing with an ELT combined with a space survey telescope. 2D shear and flexion maps are assumed (no individual redshift measurements). Note that this ELT image is half as large as the one marked in Fig. 3.

For $z_{\text{BH}} = 2$, delensing with an HUDF-like image by itself can reduce the distance error by about 25 per cent, but when combined with the space survey, 45 per cent of the uncertainty can be removed. The improvement is slightly less for $z_{\text{BH}} = 3$; we should expect this since most galaxies in the space survey have a redshift lower than 3. Notice now that creating a large ELT mosaic provides a little improvement but will not be as effective as making the image deeper. Again, these results assume that we use 2D maps and do not attempt to break the mass-sheet degeneracy with information beyond shape measurements.

5 CONCLUSIONS AND FUTURE WORK

Gravitational lensing of SMBHB systems severely limits their usefulness as standard sirens – precise distance probes based on known gravitational wave signals. We suggest that ‘delensing’ each SMBHB – i.e. mapping the magnification field around it using a KS-like inversion technique – could substantially reduce distance errors due to lensing. Fig. 4 illustrates the error reduction that could be achieved with a 30 arcmin$^2$ image from an ELT combined with a space-based survey such as JDEM or Euclid. We find that combining 2D maps from these instruments could reduce SMBHB distance error from lensing by about a factor of 2 for $z_{\text{BH}} > 1.5$. The success of SMBHB delensing requires adequate flexion measurements to recover fine fluctuations in the magnification field and shear measurements to recover larger features.

It may be possible to achieve superior results by using techniques beyond the simple KS inversion considered here. For instance, incorporating a redshift for each galaxy would enable some improvement over 2D maps. Redshifts could also help break the mass-sheet degeneracy in a deep image, thereby eliminating some of the need for a wide survey. The degeneracy could also be broken by including magnification maps from quasar number counts. Furthermore, with the advent of an ELT, improved high-resolution shape measurement techniques may emerge, leading to maps with significantly lower shape noise.

We have made several simplifying assumptions that should be addressed in future work. We have extended the Smith et al. fitting formula for the matter power spectrum beyond its range of accuracy, and we have assumed that matter fluctuations remain Gaussian random. A more complete treatment of the matter power spectrum, e.g. from the results of $N$-body simulations, will be needed to refine the technique of delensing and estimate its potential success (Gair & King, in preparation). It may be possible to exploit the fact that non-linear gravitational clustering causes the true weak lensing fields to contain less information than their Gaussian random approximations (Cooray & Hu 2001; Rimes & Hamilton 2005; Doré, Lu & Pen 2009). Finally, we have ignored higher order weak lensing effects such as reduced shear/flexion and contaminating effects such as intrinsic alignments.

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