HK MULTIPLICITY, F-THRESHOLD AND THE PALEY-WIENER THEOREM

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ABSTRACT. For a given algebraically closed field $k$ of characteristic $p > 0$ we consider the set $C_k$, of graded isomorphism classes of standard graded pairs $(R, I)$, where $R$ is a standard graded ring over the field and $I$ is a graded ideal of finite colength.

Here we give a ring homomorphism $\Pi : \mathbb{Z}[C_k] \rightarrow H(C)[X]$, where $H(C)$ denotes the ring of entire functions.

The related entire function and the homomorphism $\Pi$ keep track of the two positive characteristic invariants, $e_{HK}(R, I)$ and $c^l(m)$ of the ring: (1) composing the map $\Pi$ with the evaluation map at $z = 0$ gives a ring homomorphism $\Pi : \mathbb{Z}[C_k] \rightarrow \mathbb{R}[X]$ which sends

$$(R, I) \rightarrow e_{HK}(R^0, IR^0) + e_{HK}(R^1, IR^1)X + \cdots + e_{HK}(R^d, IR^d)X^d,$$

where $R^i$ is the union of $i$ dimensional components of $R$ and $e_{HK}(R^i, IR^i)$ is the HK multiplicity of the pair $(R^i, IR^i)$, and in particular the top coefficient is $e_{HK}(R, I)$.

(2) If, in addition, $R$ is a two dimensional ring or $\text{Proj } R$ is strongly $F$-regular, then the Fourier transform $\hat{f}_{R,I}$ belongs to the Paley-Wiener class of the real number, namely the $F$-threshold $c^l_m(R)$ of the maximal ideal $m$.

1. Introduction

Let $(R, I)$ be a standard graded pair, i.e., $R$ is a Noetherian standard graded ring over an algebraically closed field $k$ (unless otherwise stated) of characteristic $p > 0$ and $I$ is a graded ideal of finite colength. Let $m$ be the graded maximal ideal of $R$.

To study Hilbert-Kunz (HK) multiplicity $e_{HK}(R, I)$ of $R$ with respect to $I$, we had introduced in [T] a compactly supported continuous function called HK density function $f_{R,I} : [0, \infty) \rightarrow [0, \infty)$ provided $\dim \mathcal{R} \geq 2$. This function relates to the two characteristic $p$ invariants of the pair $(R, I)$.

1. $\int_0^\infty f_{R,I}(x)dx = e_{HK}(R, I)$.

2. If $R$ is strongly $F$-regular in the punctured spectrum and of dimension $\geq 2$, or a standard graded domain of dimension $= 2$ then the maximum support $\alpha(R, I)$ of $f_{R,I}$ is same as the $F$-threshold of $c^l(m)$.

The function $f_{R,I}$ was given by

$$f_{R,I}(x) = \lim_{n \rightarrow \infty} f_n(R, I)(x), \quad \text{for all } x \in \mathbb{R},$$

where $\{f_n(R, I) : [0, \infty) \rightarrow [0, \infty])\}_{n \in \mathbb{N}}$ is a sequence of compactly supported step functions which converges uniformly to $f_{R,I}$.

If $\dim \mathcal{R} = 0$ or 1 then we can still define the functions $\{f_n(R, I)\}_n$ exactly in the same way and they are compactly supported step functions.

If $\dim \mathcal{R} = 1$ then this sequence converges pointwise everywhere except at finitely many points, and in fact uniformly outside a set of arbitrarily small measure. Therefore again

$$e_{HK}(R, I) := \lim_{n \rightarrow \infty} \int_0^\infty f_n(R, I)(x)dx = \int_0^\infty f_{R,I}(x)dx.$$
However, if dim $R = 0$ then the sequence $\{f_n(R,I)\}_n$ does converge pointwise but $f_{R,I} := \lim_{n \to \infty} f_n(R,I)$ is 0 everywhere. Whereas

$$e_{HK}(R,I) = \ell(R) \neq \int_0^\infty f_{R,I}(x)dx = 0.$$ 

In this paper to every standard graded pair $(R,I)$ of dimension $d \geq 0$, we associate an entire function (that means a holomorphic function on whole of $\mathbb{C}$) $F_{R,I}$ in a uniform manner. The function $F_{R,I}$ keeps track of the invariants $e_{HK}(R,I)$ and $\alpha(R,I)$, and hence $c^I(m)$ whenever it coincides with $\alpha(R,I)$.

To do this, we use the Fourier transform, as follows.

We note that each $f_n(R,I)$ is a compactly supported step function and therefore belongs to $L^1(\mathbb{R})$. Now the Plancherel Theorem (see preliminaries) implies that the Fourier transform $\hat{f}_n(R,I)$ of $f_n(R,I)$ is a well defined entire function.

We show that $\lim_{n \to \infty} \hat{f}_n(R,I)(z)$ exists for every $z \in \mathbb{C}$ and for every $d \geq 0$. If we denote the limiting function as $F_{R,I} : \mathbb{C} \to \mathbb{C}$, i.e.,

$$F_{R,I}(z) := \lim_{n \to \infty} \hat{f}_n(R,I)(z)$$

then $F_{R,I}(0) = e_{HK}(R,I)$.

If dim $R \geq 1$ then $F_{R,I}$ is the Fourier transform of the HK density function $f_{R,I} = \lim_{n \to \infty} f_n(R,I)$. Moreover if dim $R = 1$ then there is a finite set $T_{R,I}$ of integers such that if $(S,J)$ is a standard graded pair then

$$F_{R,I} = F_{S,J} \iff f_{R,I} = f_{S,J} \text{ for all } x \in \mathbb{R} \setminus T_{R,I}.$$ 

Since $f_{R,I}$ is continuous if dim $R \geq 2$ we get $\{f_{R,I} \mid \text{dim } R \geq 2\} \hookrightarrow H(\mathbb{C})$.

One of the main result (Theorem 3.9) of this paper is to show that the correspondence $(R,I) \to F_{R,I}$ is algebraic, in the following sense.

Let $C_k$ denote the isomorphism classes of standard graded pairs $(R,I)$, where $R_0 = k$ is an algebraically closed field. Then there is a multiplicative map of monoids

$$\Phi : (C_k, \otimes) \to H(\mathbb{C}) \text{ given by } (R,I) \to F_{R,I},$$

where the identity element $(k,(0))$ of the monoid $C_k$ maps to the identity element of $H(\mathbb{C})$, namely the constant map $F_{k,(0)} : \mathbb{C} \to \mathbb{C}$ given by $z \to 1$.

Further this map extends to the ring homomorphism $\Pi : Z[C_k] \to H(\mathbb{C})[X]$ such that

$$(R,I) \to F_{R^0,IR^0} + F_{R^1,IR^1}X + \cdots + F_{R^d,IR^d}X^d,$$

where $R^i$ denotes the $i^{th}$-dimensional component of $R$ and $d = \text{dim } R$.

Moreover if $e : H(\mathbb{C})[X] \to \mathbb{C}$ denotes the evaluation map at 0, i.e., $F \to F(0)$ then the composition map $e \circ \Pi : Z[C_k] \to \mathbb{R}[X]$ is a ring homomorphism which sends

$$(R,I) \to e_{HK}(R^0,IR^0) + e_{HK}(R^1,IR^1)X + \cdots + e_{HK}(R^d,IR^d)X^d,$$

where we know by the existing theory that $e_{HK}(R^d,IR^d) = e_{HK}(R,I)$.

If $C_k^1 = \{(R,I) \in C_k \mid \text{dim } R \geq 1\}$ then $C_k^1$ is a $Z[C_k]$-module such that $\Pi|_{C_k^1} : C_k^1 \to H(\mathbb{C})[X]$ given by

$$(R,I) \to \hat{f}_{R^1,IR^1}X + \cdots + \hat{f}_{R^d,IR^d}X^d$$

is a $Z[C_k]$-linear map and therefore $(e \circ \Pi)|_{C_k^1} : C_k^1 \to \mathbb{R}[X]$ is a $Z[C_k]$-linear which sends

$$(R,I) \to e_{HK}(R^1,IR^1)X + \cdots + e_{HK}(R^d,IR^d)X^d.$$
Recall that an entire function $F : \mathbb{C} \rightarrow \mathbb{C}$ is of exponential type, if there exist constants $c_0$ and $c_1$ such that $|F(z)| \leq c_0 e^{c_1|z|}$, for all $z \in \mathbb{C}$. Here we say that the exponential index of $F$ is $(c_0, c_1)$ if $c_0$ and $c_1$ are the smallest real numbers with this property, and denote this by $\text{exp.index of } F = (c_0, c_1)$.

**Theorem 1.1.** For a standard graded pair $(R, I)$ the function $F_{R,I}$ is an entire function and $\text{exp.index of } F = (e_{HK}(R, I), \alpha(R, I))$. That means for all $z \in \mathbb{C}$,

$$|F_{R,I}(z)| \leq e_{HK}(R, I)e^{\alpha(R, I)|z|},$$

where $\alpha(R, I) = \text{maximal support of } f_{R,I}$.

Moreover $\alpha(R, I)$ is the smallest real number such that $F_{R,I}$ belongs to the Paley-Wiener class of $\alpha(R, I)$, i.e.,

$$F_{R,I} \in \text{PW}_{\alpha(R, I)} \quad \text{and} \quad F_{R,I} \notin \text{PW}_{A}, \quad \text{if } A < \alpha(R, I).$$

Further for $(R, I), (S, J) \in \mathcal{C}_k$

$$\text{exp.index of } F_{(R,I) \otimes (S,J)} = (\text{exp.index of } F_{R,I})(\text{exp.index of } F_{S,J}) = (e_{HK}(R, I) \cdot e_{HK}(S, J), \alpha(R, I) + \alpha(S, J)).$$

The above theorem gives the following

**Corollary 1.2.** If $(R, I)$ is a standard graded pair and either (1) $R$ is a two dimensional domain or (2) $\text{Proj } R$ is a strongly $F$-regular then the entire function $F_{R,I}$ has exponential index $(e_{HK}(R, I), c^l(m))$ and $F_{R,I} \in \text{PW}_{c^l(m)}$.

2. **Preliminaries**

2.1. **Some fundamental results from analysis.** In this section we recall basics from real and complex analysis. For further details reader may refer to the classical book of W. Rudin ([R]).

Here we consider $\mathbb{R}$ as an Euclidean space and with Lebesgue measure $dt$. By almost everywhere (a.e.) we mean outside a measure 0 subset of $\mathbb{R}$.

If $z = x + iy \in \mathbb{C}$, then $|z| = \sqrt{x^2 + y^2}$.

If $f : X \rightarrow \mathbb{C}$ then the supremum norm of $f$ is given as $\|f\| = \sup \{|f(z)| \mid z \in X\}$.

For an integer $1 \leq p < \infty$ and a measurable space $X$,

$$L^p(X) = \{\text{the measurable functions } f : X \rightarrow \mathbb{C} \mid \int_X |f(x)|^p dx < \infty\},$$

and the $L^p$ norm on the space $L^p(X)$ is given by $\|f\|_p = \int_X |f(x)|^p dx$.

Example: In this paper we would be dealing mainly with the following set of functions.

$$C^e_c(\mathbb{R}) = \{f : \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ is compactly supported and a.e. continuous}\}.$$

It follows easily that $f \in C^e_c(\mathbb{R})$ implies $f \in L^p(\mathbb{R})$, for every $1 \leq p < \infty$.

The binary operation $\ast : L^1(\mathbb{R}) \times L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R})$ given by

$$(f \ast g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t) dt, \quad \text{for } f, g \in L^1(\mathbb{R})$$

is called the convolution. It is easy to check that the map restricted to the space $C^e_c(\mathbb{R})$ gives the map $\ast : C^e_c(\mathbb{R}) \times C^e_c(\mathbb{R}) \rightarrow C_c(\mathbb{R})$. In fact if supp $f = [0, a_1]$ and supp $g = [0, a_2]$ then $f \ast g$ is a compactly supported continuous function with supp $f \ast g = [0, a_1 + a_2]$.

The set

$$H(\mathbb{C}) = \{f : \mathbb{C} \rightarrow \mathbb{C} \mid f \text{ is an entire function}\}$$
is a commutative ring with pointwise addition and multiplication.

The Fourier transform map, $F : C_c^{ae}(\mathbb{R}) \rightarrow H(\mathbb{C})$ is given by $f \rightarrow \hat{f}$, where
$$\hat{f}(z) = \int_{\mathbb{R}} f(t)e^{itz}dt, \quad \text{for } z \in \mathbb{C}.$$  

Moreover the map $F$ is multiplicative: it takes convolution of two functions to the pointwise multiplication of their Fourier transforms, i.e., $f * g = \hat{f} \cdot \hat{g}$ (see 9.2 (c) [R]).

A function $g : \mathbb{R} \rightarrow \mathbb{R}$ is said to vanish at infinity, if for a given $\epsilon > 0$ there is a compact set $K \subset \mathbb{R}$ such that $|g(x)| < \epsilon$, for all $x \notin K$.

$$C_0(\mathbb{R}) = \{g : \mathbb{R} \rightarrow \mathbb{R} \mid g \text{ is continuous function and vanishes at infinity}\},$$

$$C_c(\mathbb{R}) = \{g : \mathbb{R} \rightarrow \mathbb{R} \mid g \text{ is continuous function and has compact support}\}.$$  

The functions arising as the HK density functions of standard graded pairs belong to the set $C_c^{ae}(\mathbb{R})$. The relation between such functions and their Fourier transforms works very well due to the Plancherel theorem (page 404 in [R]) and Paley-Wiener theorem (Theorem 19.2 in [R]).

**The Plancherel Theorem.** If $f \in L^1(-A,A)$, for some $A \in \mathbb{R}_+$ then its Fourier transform is an entire function (denoted as $\hat{f} \in H(\mathbb{C})$) such that

1. the restriction of $\hat{f}$ to the real axis lies in $L^2(\mathbb{R})$. Moreover
2. $\hat{f}$ is of exponential type i.e. there exist positive constants $C$ and $A$ such that

$$|\hat{f}(z)| \leq Ce^{A|z|} \text{ for all } z \in \mathbb{C}, \text{ where } C = \int_{-A}^{A} |f(t)|dt.$$  

The Paley-Wiener theorem is the converse of the Plancherel Theorem.

**The Paley-Wiener theorem.** For an entire function $G : \mathbb{C} \rightarrow \mathbb{C}$,

1. if $A$ and $C$ are positive constants such that $|G(z)| \leq Ce^{A|z|}$, for all $z \in \mathbb{C}$ and
2. $G |_{\mathbb{R}} \in L^2(\mathbb{R})$

then there exists a real valued function $f \in L^2([-A,A])$ such that $G(z) = \hat{f}(z)$. Such an entire function $G$ is said to belong to the Paley-Wiener class $A$ and this is denoted by $G \in PW_A$.

**Remark 2.1.** If $f, g \in C_c^{ae}(\mathbb{R})$ then

$$\hat{f} = \hat{g} \quad \Rightarrow \quad \|\hat{f} - \hat{g}\|_2 = 0 \quad \iff \quad \|f - g\|_2 = 0 \quad \Rightarrow \quad f = g \quad \text{a.e.},$$  

where the second implication follows from the fact (Theorem 9.13 (c) in [R]) that the map $F : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ given by $f \rightarrow \hat{f}$ is an isomorphism of Hilbert-space, i.e.,

$$\|f - g\|_2 = \|\hat{f} - \hat{g}\|_2.$$  

In particular, if $f, g \in C_c(\mathbb{R})$ then $f = g$. This gives an embedding $F |_{C_c(\mathbb{R})} : C_c(\mathbb{R}) \hookrightarrow H(\mathbb{C})$.

2.2. Some relevant results from commutative algebra.

**Definition 2.2.** Let $k$ be an algebraically closed field of characteristic $p > 0$. Let $R = \oplus_{n \geq 0} R_n$ be a standard graded Noetherian ring such that $R_0 = k$. Let $m = \oplus_{m \geq 0} R_m$ denote the graded maximal ideal of $R$. Let $I \subset R$ be a graded ideal of finite colength.

Henceforth we will called such a pair an SG pair.
Given a SG pair \((R, I)\) of dimension \(d \geq 0\) we can associate a sequence of step functions \(\{f_n(R, I) : [0, \infty) \to [0, \infty)\}_n\) which are given by

\[
(2.1) \quad f_n(R, I)(x) = \frac{1}{q^{d-1}} \ell \left( \frac{R}{I^n} \right)_{[xq]}, \quad \text{where} \quad q = p^n.
\]

This is a compactly supported a.e. continuous function. In fact the support of \(f_n(R, I) \subset [0, n_0]\), where \(n_0\) is a number independent of \(n\).

**Theorem** (Theorem 1.1 of [T]). If \(R\) is a standard graded ring with \(\dim R \geq 2\) then \(\{f_n(R, I)\}\) is a uniformly convergent sequence. If \(f_{R,I} = \lim_{n \to \infty} f_n(R, I)\) then \(f_{R,I}\) is a compactly supported continuous function and

\[
e_{HK}(R, I) = \int_R f_{R,I}(x) \, dx.
\]

**Definition 2.3.** For a given SG pair \((R, I)\) the function \(f_{R,I}\) is the HK density function of \((R, I)\). Let

\[\alpha(R, I) = \sup \{x \mid f_{R,I}(x) \neq 0\}\]

the maximum support of \(f_{R,I}\).

The HK density function relates to another characteristic \(p\) invariant of the ring. We first recall the notion (introduced by [MTW]) of \(F\)-threshold of \(I\) with respect to \(J\)

\[c^J(I) := \lim_{e \to \infty} \frac{\max \{r \mid I^r \not\subseteq J^{[pe]}\}}{p^e},\]

where the existence of this limit was proved in [DsNbP].

We also recall the following definition from [HH].

**Definition 2.4.** A Noetherian domain \(R\) such that \(R \to R^{1/p}\) is module finite over \(R\), is strongly \(F\)-regular if for every nonzero \(c \in R\) there exists \(q\) such that \(R\)-linear map \(R \to R^{1/q}\) that sends 1 to \(c^{1/q}\) splits as a map of \(R\)-modules, i.e. \(Rc^{1/q} \subseteq R^{1/q}\) splits over \(R\).

Note that if Proj \(R\) is smooth then it is \(F\)-regular on the puctured spectrum.

**Theorem** (Theorem 4.9, [Tr W]). Let \((R, I)\) be a standard graded pair and \(m\) be the graded maximal ideal of \(R\). If \(R\) is strongly \(F\)-regular on the punctured spectrum (for example if Proj \(R\) is smooth) then \(\alpha(R, I) = c^J(m)\).

**Theorem** (Theorem C, [T2]). Let \((R, I)\) be a standard graded pair where \(R\) is a two dimensional domain. Then

\[c^J(m) = \alpha(R, I)\]

**Remark 2.5.** If \(\dim R = 1\) then it is obvios that

\[c^J(m) = \lim_{e \to \infty} \max \{x \mid \ell \left( \frac{R}{I^{[pe]}} \right)_{[xq]} \neq 0\} = \alpha(R, I)\]

If \(\dim R = 0\) then again \(c^J(m) = 0\) and

\[\alpha(R, I) = \sup \{x \mid \lim_{n \to \infty} f_n(R, I)(x) \neq 0\} = 0.\]
3. A MAP FROM SG PAIRS TO ENTIRE FUNCTIONS

We again recall that by a SG pair we mean $R$ is a standard graded ring over an algebraically closed field $k$ and $I \subset R$ is a graded ideal such that $\ell(R/I) < \infty$. In the rest of the paper, all SG pairs $(R, I)$ are considered over a fixed algebraically closed field $k$ of characteristic $p > 0$.

We consider a monoid $(\mathcal{C}_k, \otimes)$ generated by sg pair as follows. Let

$$\mathcal{C}_k = \{(R, I) \mid (R, I) \text{ a sg pair over } k\}/\equiv,$$

where $(R, I) \equiv (S, J)$ if there is a map $\eta : R \to S$ which is a graded isomorphism of rings of degree 0 such that $\eta(I) = J$. In other words $\mathcal{C}_k$ is the isomorphic classes of sg pairs. For $(R, I), (S, J) \in \mathcal{C}_k$, we define

$$(R, I) \otimes (S, J) = (R \otimes_k S, I \otimes_k J), \text{ where } (R \otimes S)_n = \oplus_i (R_i \otimes S_{n-i})$$

and $I \otimes_k J = \oplus_i (I_i \otimes_k J_{n-i})$.

The identity element of this monoid is $(k, (0))$, where $k = k \oplus 0 \oplus \cdots$ is the standard graded ring with $0^{th}$ component as $k$ and $n^{th}$ component $= 0$ if $n \neq 0$.

Now for a SG pair $(R, I)$ of dimension $d$, we associate an entire function. Consider $f_n(R, I)$ as in [2.1]. Since each $f_n(R, I)$ is a compactly supported a.e. continuous function, its Fourier transform is an entire function.

We know (see [T]) that if dim $R \geq 2$ then $\{f_n(R, I)\}_n$ converges uniformly to $f_{R, I}$, the HK density function of $(R, I)$. Moreover if dim $R = 1$ and $R$ is reduced then the convergence is pointwise.

**Lemma 3.1.** If $(R, I)$ is a SG pair and dim $R = 1$ then there exists a finite set of integers $T_{R, I} = \{0 = d_0 < d_1 < d_2 < \cdots < d_n\}$ and a set of constants $C_1 > C_2 > \cdots > C_{s_1} > 0$ such that for all $p^n = q \gg 0$ we have

$$f_n(R, I)(x) \leq C_{i+1} \quad \text{if } \frac{|x|}{q} \in [d_i, d_i + \frac{1}{q}, \ldots, d_i + \frac{m_i-1}{q})$$

$$= C_{i+1} \quad \text{if } \frac{|x|}{q} \in [d_i + \frac{m_i}{q}, d_i + \frac{m_i+1}{q}, \ldots, d_{i+1} - \frac{1}{q})$$

$$= 0 \quad \text{if } \frac{|x|}{q} \geq d_{s_1} + \frac{m_0}{q},$$

where $1 \leq i + 1 \leq s_1$.

In particular $f_{R, I}(x) = C_{i+1}$ if $x \in (d_i, d_{i+1})$ and the sequence $\{f_n(R, I)\}$ converges to $f_{R, I}$ uniformly outside a set of any arbitrarily small measure which contains $T_{R, I}$.

**Proof.** Let $H^0_m(R) = Q_1 \cap Q_2 \cap \cdots \cap Q_s$ be the primary decomposition of $H^0_m(R)$ in $R$. Let $\{p_1, \ldots, p_s\}$ be the corresponding prime ideals. Then dim $R/Q_i = 1$.

We note that, the canonical map $g : R \to R/Q_1 \times \cdots \times R/Q_s$ has kernel and cokernel of finite length, as the induced map by localizing at $Q_i$ is an isomorphism for all $i$.

Note that Ass $R/Q_i = \{\sqrt{Q_i} = p_i\}$ and there exists $m_0$ such that $\ell(R/Q_i)_m = e_0(R/Q_i, m)$, for $m \geq m_0$, and $\ell(R/Q_i)_m \leq e_0(R/Q_i, m)$ otherwise.

Moreover, for any element $x$ of $I$, either $x$ is nilpotent or a non zerodivisor in $R/Q_i$. Hence we can choose $q \gg 0$ such that $(I[q] + Q_i)/Q_i$ is generated by non zerodivisor of $R/Q_i$, for all $i$. Let

$$d_{Q_i} = \min\{\deg x \mid x \in I \text{ and is a non zerodivisor of } R/Q_i\}.$$

We choose $q \gg 0$ such that $qd_{Q_i} > m_0$. Now
\[ \ell \left( \frac{R}{I + Q_i} \right)_m = \ell \left( \frac{R}{Q_i} \right)_m \leq e_0(R/Q_i) \quad \text{if } m < m_0 \]
\[ = \ell \left( \frac{R}{Q_i} \right)_m = e_0(R/Q_i) \quad \text{if } m_0 \leq m < qd_i \]
\[ < \ell \left( \frac{R}{Q_i} \right)_m = e_0(R/Q_i) \quad \text{if } qd_i \leq m < m_0 + qd_i \]
\[ = 0 \quad \text{if } m_0 + qd_i \leq m, \]

where the last assertion follows by choosing a non zerodivisor \( x_i \in I \) of the ring \( R/Q_i \) and considering the exact sequence

\[ 0 \longrightarrow \left( \frac{R}{Q_i} \right)_m \overset{xq}{\longrightarrow} \left( \frac{R}{Q_i} \right)_m \longrightarrow \left( \frac{R}{x_i Q_i} \right)_m \longrightarrow 0. \]

Now we have

\[ f_n \left( \frac{R}{Q_i}, \frac{I + Q_i}{Q_i} \right)(x) \leq e_0(R/Q_i) \quad \text{if } \frac{xq_i}{q} \in \left\{ \frac{1}{q}, \ldots, \frac{(ma - 1)}{q} \right\} \]
\[ = e_0(R/Q_i) \quad \text{if } \frac{xq_i}{q} \in \left\{ \frac{ma}{q}, \ldots, \frac{ma + 1}{q} \right\} \]
\[ < e_0(R/Q_i) \quad \text{if } \frac{xq_i}{q} \in \{ d_i, d_i + \frac{1}{q}, \ldots, d_i + \frac{(ma - 1)}{q} \} \]
\[ = 0 \quad \text{if } \frac{xq_i}{q} \geq d_i + \frac{ma}{q} \]

We partition the set \( \{ Q_1, \ldots, Q_s \} = Q_1 \sqcup Q_2 \sqcup \cdots \sqcup Q_s \) such that \( Q_{j_1}, Q_{j_2} \in Q_i \) implies \( d_{Q_{j_1}} = d_{Q_{j_2}} = d_i \).

Without loss of generality we assume \( d_1 < d_2 < \cdots < d_s \).

Now we further choose \( q_0 \) such that for \( q \geq q_0 \)
\[ m_0 < qd_1 < qd_1 + m_0 < qd_2 < \cdots < qd_i < qd_i + m_0 < qd_{i+1} \cdots < qd_s. \]

Hence

\[ f_n(R, I)(x) \leq \sum_{Q \in \tilde{Q}_1 \sqcup \cdots \sqcup \tilde{Q}_{s_1}} e_0(R/Q) \quad \text{if } \frac{xq}{q} \in \left\{ \frac{1}{q}, \ldots, \frac{(ma - 1)}{q} \right\} \]
\[ = \sum_{Q \in \tilde{Q}_1 \sqcup \cdots \sqcup \tilde{Q}_{s_1}} e_0(R/Q) \quad \text{if } \frac{xq}{q} \in \left\{ \frac{ma}{q}, \frac{ma + 1}{q}, \ldots, d_1 - \frac{1}{q} \right\} \]
\[ = \sum_{Q \in \tilde{Q}_1 \sqcup \cdots \sqcup \tilde{Q}_{s_1}} e_0(R/Q) + \Delta_i \quad \text{if } \frac{xq}{q} \in \left\{ d_i, d_i + \frac{1}{q}, \ldots, d_i + \frac{(ma - 1)}{q} \right\} \]
\[ = \sum_{Q \in \tilde{Q}_1 \sqcup \cdots \sqcup \tilde{Q}_{s_1}} e_0(R/Q) \quad \text{if } \frac{xq}{q} \in \left\{ d_i + \frac{ma}{q}, \ldots, d_i + \frac{(ma - 1)}{q} \right\}, \]

where \( |\Delta_i| \leq \sum_{Q \in \tilde{Q}_1 \sqcup \cdots \sqcup \tilde{Q}_{s_1}} e_0(R/Q). \) Therefore for \( x \notin T_{R, I} = \{ 0 = d_0, d_1, d_2, \ldots, d_s \} \)

\[ \lim_{n \to \infty} f_n(R, I)(x) = \sum_{Q \in \tilde{Q}_1 \sqcup \cdots \sqcup \tilde{Q}_{s_1}} e_0(R/Q) \quad \text{if } 0 < x < d_{i_1} \]
\[ = \sum_{Q \in \tilde{Q}_2 \sqcup \cdots \sqcup \tilde{Q}_{s_1}} e_0(R/Q) \quad \text{if } d_1 < x < d_2 \]
\[ = \sum_{Q \in \tilde{Q}_1 \sqcup \cdots \sqcup \tilde{Q}_{s_1}} e_0(R/Q) \quad \text{if } d_i < x < d_{i+1} \]
\[ = 0 \quad \text{if } x \geq d_{s_1} \]

Now we take \( C_{i+1} = \sum_{Q \in \tilde{Q}_{i+1} \sqcup \cdots \sqcup \tilde{Q}_{s_1}} e_0(R/Q). \)

**Lemma 3.2.** The map \( \Phi : C_k \longrightarrow H(\mathbb{C}) \) given by

\[ (R, I) \rightarrow F_{R, I}, \quad \text{where } F_{R, I}(z) = \lim_{n \to \infty} f_n(R, I)(z) \]

is a well defined function.

Moreover, if \( \dim R \geq 1 \) then \( F_{R, I} \equiv \hat{f}_{R, I} \), where \( f_{R, I} \) is the HK density function of \( (R, I) \).
Proof. Case (1) dim $R = 0$. Then, for $q = p^n$, $f_n(R, I)(x) = q\ell(R_m)$ for $x \in [m/q, m + 1/q)$. If $m_0$ is an integer such that $R_m = 0$ for $m \geq m_0$ then $f_n(R, I)(x) = 0$ for $x \geq m_0/q$. Therefore

$$
\hat{f}_n(R, I)(z) = q \int_0^{m_0/q} f_n(R, I)(t)e^{itz}dt = q \sum_{m=0}^{m_0-1} \ell\left(\frac{R}{m/q}\right)_m \int_{m/q}^{m+1/q} e^{itz}dt.
$$

Claim. $\lim_{n \to \infty} q \int_{m/q}^{m+1/q} e^{itz}dt = 1$.

Proof of the claim: Let $c_0 = iz$. Now

$$
q \int_{m/q}^{m+1/q} e^{c_0t} = q \sum_{m=0}^{m_0-1} e^{c_0(m+1)/q} = q \sum_{m=0}^{m_0-1} \left(1 + \frac{c_0(m+1)^2}{q^2} \cdot \frac{A_m+1}{q} - \frac{c_0(m)^2}{q^2} \cdot \frac{A_m}{q} \right)
$$

where $A_1 = \frac{1}{2} + \frac{c_0}{32} + \frac{c_0^2}{4!} + \cdots$. But there exists constants $C_{m+1,z}$ and $C_{m,z}$ depending on $m$ and $z$ such that

$$
A_m+1 \leq e^{c_{0(m+1)/q}} \leq C_{m+1,z} \quad \text{and} \quad A_m \leq e^{c_{0m}/q} \leq C_{m,z}.
$$

Hence $\lim_{n \to \infty} q \int_{m/q}^{m+1/q} e^{c_0t} = 1$.

In particular

$$
\lim_{n \to \infty} \hat{f}_n(R, I)(z) = \lim_{n \to \infty} \sum_{m=0}^{m_0-1} \ell\left(\frac{R}{m/q}\right)_m \int_{m/q}^{m+1/q} e^{itz}dt = \ell(R), \text{ for all } z \in \mathbb{C}
$$

and therefore $F_{R,I}$ is a constant and hence an entire function.

Case (2) dim $R = 1$. By Lemma 3.1, there exist constants $\alpha = C_{s_1}$ and $M$ in $\mathbb{R}$ such that

$$
\cup_n (\text{Supp } f_n(R, I)) \cup (\text{Supp } f_{R,I}) \subseteq [0, \alpha] \quad \text{and} \quad \|f_n(R, I)(t) - f_{R,I}(t)\| \leq M.
$$

Fix $z = x + iy \in \mathbb{C}$ and let $\epsilon > 0$. Let $\epsilon_1 = \epsilon/(M \cdot e^{c_0y})$. By Lemma 3.1 there is an open neighborhood $T_{\epsilon_1}$ of the set $d_0, d_1, \ldots, d_{s_1}$ such that $T_{\epsilon_1}$ is of measure $< \epsilon_1$ and

$$
|f_n(R, I)(t) - f_{R,I}(t)| = 0, \quad \text{for } t \in \mathbb{R} \setminus T_{\epsilon_1}.
$$

Then

$$
|\hat{f}_n(R, I)(z) - \hat{f}_{R,I}(z)| \leq \int_{T_{\epsilon_1}} |f_n(R, I)(t) - f_{R,I}(t)|e^{-|ty|}dt < \epsilon.
$$

This proves the pointwise convergence of the sequence $\{\hat{f}_n(R, I)\}_n$ to the function $\hat{f}_{R,I}$. Since $f_{R,I} \in L^1([0, \alpha])$ its Fourier transform $\hat{f}_{R,I}$ is an entire function.

Case (2) If dim $R \geq 2$ then the assertion follows by the same argument as above. □

Lemma 3.3. The map $\Phi : (C_k, \otimes) \to H(\mathbb{C})$ given by $(R, I) \mapsto F_{R,I}$ is multiplicative, that is $F_{(R,I)\otimes(S,J)} = F_{R,I} \cdot F_{S,J}$ such that the identity element $(k, (0))$ of $(C_k, \otimes)$ maps to the identity element $F \equiv 1$ of $H(\mathbb{C})$.

Proof. Case (1) Let dim $R = 0$ and dim $S = 0$, then dim $R \otimes S = 0$. By Lemma 3.2 for all $z \in \mathbb{C}$

$$
F_{R\otimes S,I\otimes J}(z) = \ell(R \otimes S) = \ell(R)\ell(S) = F_{R,I}(z) \cdot F_{S,J}(z).
$$
Case (2) Let \( \dim R = 0 \) and \( \dim S \geq 1 \). Then \( R = R_0 \oplus R_1 \oplus \cdots \oplus R_{n_0} \), for some \( n_0 \) and \( \dim(R \otimes S) \geq 1 \). First we show that \( f_{R \otimes S, I \otimes J} = \ell(R) \cdot f_{S, J} \).

Let \( T_{S,J} \) be the finite set for the pair \((S,J)\), as in Lemma 3.1. Let us fix \( x \in \mathbb{R} \setminus T_{S,J} \). We can choose \( q \gg 0 \) such that \( \lfloor qx \rfloor > n_0 \) and \( I[q] = 0 \). Further we assume that the points \( x, x - (1/q), \ldots, x - (n_0/q) \) avoid the set \( T_{S,J} \). Then

\[
f_{R \otimes S, I \otimes J}(x) = \lim_{n \to \infty} f_n(R \otimes S, I \otimes J)(x) = \lim_{n \to \infty} \sum_{i=0}^{n_0} \ell(R_i)\ell(S/J[q])_{\lfloor qx \rfloor - i} = \sum_{i=0}^{n_0} \ell(R_i) \lim_{n \to \infty} f_n(S, J)(x - \frac{i}{q}) = \ell(R) \cdot f_{S, J}(x).
\]

Applying Fourier transform functor on both the sides we get

\[
F_{R \otimes S, I \otimes J}(z) = \hat{f}_{R \otimes S, I \otimes J}(z) = \ell(R) \cdot \hat{f}_{S, J}(z) = F_{R, I}(z) \cdot F_{S, J}(z).
\]

Case (3) Let \( \dim R \geq 1 \) and \( \dim S \geq 1 \). For the sake of abbreviation we write \( f_n(R, I) = f_n, f_n(S, J) = g_n \).

Claim. For every \( x \in \mathbb{R} \)

a) \( \lim_{n \to \infty} (f_n * g_n)(x) \rightarrow (f_{R, I} * f_{S, J})(x) \),

b) \( \lim_{n \to \infty} f_n(R \otimes S, I \otimes J)(x) \lim_{n \to \infty} (f_n * g_n)(x) \).

If we assume the proof of the claim for the moment then applying the Fourier transform we get

\[
F_{R \otimes S, I \otimes J} = \hat{f}_{R \otimes S, I \otimes J} = \hat{\lim_{n \to \infty} f_n(R \otimes S, I \otimes J)} = f_{R, I} \ast \hat{f}_{S, J} = F_{R, I} \cdot F_{S, J}.
\]

Proof of the claim [a]: By Lemma 3.1 there exist finite sets \( T_{R, I} \) and \( T_{S, J} \) (possibly empty) in \( \mathbb{R} \) such that the sequences \( \{f_n\} \) and \( \{g_n\} \) converge uniformly to \( f_{R, I} \) and \( f_{S, J} \) respectively outside a set of arbitrarily small measure containing \( T_{R, I} \) and \( T_{S, J} \).

Moreover \( f_{R, I} \) is a continuous function outside a set of measure 0 (hence so is the function \( t \to f_{R, I}(x - t) \) for a fixed \( x \)). Let

\[
M = \max \{ \|f_n\|, \|g_n\|, \|f_{R, I}\|, \|f_{S, J}\| \mid n \in \mathbb{N} \}.
\]

Fix an \( x \in \mathbb{R} \). Let \( \epsilon > 0 \) be any number. We choose a set \( U \) of the measure \( \leq \epsilon/4M^2 \) such that \( \{f_n\} \) and \( \{g_n\} \) converge outside \( U \). Further we choose \( n_0 \geq 0 \) such that for \( n \geq n_0 \)

\[
|g_n(t) - f_{S, J}(t)| < \epsilon/4|x|M \quad \text{and} \quad |f_n(x - t) - f_{R, I}(x - t)| < \epsilon/4|x|M \quad \text{for} \quad t \in \mathbb{R} \setminus U.
\]

Now

\[
|(f_n * g_n)(x) - (f_{R, I} * g_{S, J})(x)| \leq \int_0^x |f_n(x - t)||g_n(t) - f_{S, J}(t)|dt + \int_0^x |f_n(x - t)|\|f_{S, J}(t)\|dt + \int_U |f_n(x - t)||g_n(t)||dt + \int_U |f_{R, I}(x - t)||f_{S, J}(t)||dt < \epsilon.
\]

\[
< \epsilon < \epsilon/4|x|M + |x| \cdot \epsilon/4|x|M + M + M^2 \cdot \epsilon + M^2 \cdot \epsilon = \epsilon.
\]

\[
< \epsilon/4|x|M + |x| \cdot \epsilon/4|x|M + M + M^2 \cdot \epsilon + M^2 \cdot \epsilon = \epsilon.
\]
Proof of the claim \([b]\): Now \(\dim(R \otimes S) \geq 2\) implies that \(\{f_n(R \otimes S, I \otimes J)\}_n\) converges to \(f_{R \otimes S, I \otimes J}\) uniformly, where
\[
f_n(R \otimes S, I \otimes J) = \frac{1}{q^{d_1+d_2-1}} \left( \sum_{l=0}^{\frac{|xq|}{q}} \ell(\frac{R}{T[q]}_l) \cdot \ell(\frac{S}{T[q]}_{|xq|-l}) \right) = \frac{1}{q} \sum_{l=0}^{\frac{|xq|}{q}} f_n(\frac{l}{q}) g_n(\frac{|xq|-l}{q}).
\]

On the other hand, let \(T_{S, J} = \{d_0, \ldots, d_{s_1}\}\) be the finite set as in Lemma 3.1. Let \(M_{x,q} = \{l \in \mathbb{Z} \mid |\frac{xq}{q} - \frac{l+1}{q}| \leq \alpha, \ldots, d_i + \frac{ma}{q} \}\), for any common \(d_i\).

It is easy to check that the cardinality of \(M_{x,q} \leq (m_0 + 1)d_{s_1} + 1\), and for any \(l \not\in M_{x,q}\)
\[
g_n(|xq|/q - (l + 1)/q) = g_n(|xq|/q - l/q).
\]

We can write
\[
(f_n * g_n)(x) = \int_0^x f_n(t) g_n(x - t) dt = \sum_{0 \leq l \leq \frac{|xq|}{q}, l \not\in M_{x,q}} \int_{l/q}^{(l+1)/q} f_n(t) g_n(x - t) dt
\]
\[
+ \int_{\frac{|xq|}{q}}^{(l+1)/q} f_n(t) g_n(x - t) dt + \sum_{l \in M_{x,q}} \int_{l/q}^{(l+1)/q} f_n(t) g_n(x - t) dt.
\]

But
\[
\int_{l/q}^{(l+1)/q} f_n(t) g_n(x - t) dt = f_n(\frac{l}{q}) g_n(\frac{|xq|-l-1}{q}) = f_n(\frac{l}{q}) g_n(\frac{|xq|-l}{q}) \quad \text{if} \ l \not\in M_{x,q}
\]
\[
\leq M^2/q \quad \text{for any} \ l
\]

Hence,
\[
|f_n(R \otimes S, I \otimes J)(x) - (f_n * g_n)(x)| \leq ((m_0 + 1)d_{s_1} + 2) \cdot \frac{M^2}{q}.
\]

This proves the claim \([b]\) and hence the lemma.

\(\square\)

\textbf{Remark 3.4.} The case when both \((R, I)\) and \((S, J)\) are of dimension \(\geq 2\) the above lemma can also be deduced from the thesis of M. Mondal ([MM]) and Lemma 3.2.

\textbf{Remark 3.5.} \(1\) By Lemma 3.3, the map \(\Phi : C_k \rightarrow H(\mathbb{C})\) is a multiplicative map of monoids which takes the identity \((k, (0))\) of the monoid \(C_k\) to the multiplicative identity of \(H(\mathbb{C})\). If \(F : L^1(\mathbb{R}) \rightarrow H(\mathbb{C})\) denotes the Fourier transform functor then \(\text{Im}(F) \cap \text{Im}(\Phi)\) is closed under multiplication and contains whole of \(\text{Im}(\Phi)\) except \(\Phi(R, I)\), where \(\dim R = 0\).

This is because if \(f \in L^1(\mathbb{R})\) then the Fourier transform \(\hat{f} \mid_{\mathbb{R}} (x)\) tends to 0 as \(x \rightarrow \infty\) (Theorem 9.6 in [R]), whereas \(\Phi(R, I) = F_{R, I}\) is a nonzero constant function everywhere. However \(\Phi(R, I) = \lim_{n \rightarrow \infty} \hat{f}_n(R, I)\), where \(\hat{f}_n(R, I) \in \text{Im}(F)\) and \(f_n(R, I)\) are the Dirac functions.

This is analogous to the case where the set \(C_c(\mathbb{R})\) of compactly supported continuous function on \(\mathbb{R}\), by definition is the intersection of the set of compactly supported function and the set of continuous functions on \(\mathbb{R}\). This set is closed under the convolution operation. But the identity element or any nonzero constant maps is the limit of Dirac functions in \(C_c(\mathbb{R})\) does not belong to \(C_c(\mathbb{R})\).
For given SG pair \((R, I)\), the function \(F_{R,I}\) is additive for maximal dimension components of \(\text{Spec } R\): if \(\Lambda = \{ p \in \text{Spec } R \mid \dim R/p = \dim R \}\) then
\[
F_{R,I} = \sum_{p \in \Lambda} \lambda(R_p) F_{R/p,I+p/p},
\]
which follows by Lemma 3.2 (in case \(\dim R = 1\)) and by Proposition 2.14 of [T] (in case \(\dim R \geq 2\)).

However the formulation ignores lower dimensional components of \(R\). Here we extend the definition of \(\tilde{\Phi} : (\mathcal{C}_k, \otimes) \to H(\mathbb{C})\) to a multiplicative map of monoids \(\mathcal{C}_k \to H(\mathbb{C})[X]\) which keeps track of every irreducible components of \(\text{Spec } R\).

**Notations 3.6.** For a SG pair \((R, I)\) of dimension \(d\), we write
\[
\{\text{minimal primes of } R\} = \tilde{P}_1 \cup \tilde{P}_2 \cup \cdots \tilde{P}_d,
\]
where \(\tilde{P}_i = \{ p_{1i}, \ldots, p_{ij} \in \text{Spec } R \mid \dim R/p_{ij} = i \}\). Let \(Q_{ij}\) denote the \(p_{ij}\)-primary component of \(R\). Then \(R^i = R/(Q_{i1} \cap \cdots \cap Q_{ij})\) is the union of \(i\)-dimensional components of \(R\).

By Lemma 3.1
\[
F_{R^i,IR^i} = \sum_{j=1}^{ij} \ell(R_{p_{ij}}) F_{R/p_{ij},I+p_{ij}/p_{ij}}.
\]

**Proposition 3.7.** The map \(\Pi : \mathcal{C}_k \to H(\mathbb{C})[X]\) given by
\[
(R, I) \to F_{R^0,IR^0} + F_{R^1,IR^1}X + \cdots + F_{R^d,IR^d}X^d
\]
is multiplicative.

**Proof.** First we prove the following

**Claim.** Let \((R, I)\) and \((S, J)\) be in \(\mathcal{C}_k\). Then
\[
\text{min prime}(R \otimes S) = \{ p \otimes S + R \otimes q \mid p \in \text{min prime}(R), \ q \in \text{min prime}(S)\}.
\]
Further if \(p \otimes S + R \otimes q \in \text{min prime } (R \otimes S)\) then \(\ell(R_p)\ell(S_q) = \ell((R \otimes S)_P)\), where \(P = p \otimes S + R \otimes q\).

**Proof of the claim:** Let \(p \in \text{Spec } R\) and \(q \in \text{Spec } S\). Since \(k\) is algebraically closed (see Ex. 3.15 of [H])
\[
\frac{R \otimes S}{p \otimes S + R \otimes q} \cong R/p \otimes_k S/q,
\]
is an integral domain, and therefore \(p \otimes S + R \otimes q\) is a prime ideal of \(R \otimes S\).

On the other hand, if \(P \in \text{Spec } R \otimes S\) then \(\phi_R : R \to (R \otimes S)/P\) given by \(r \to r \otimes 1\) and \(\phi_S : S \to (R \otimes S)/P\) given by \(s \to 1 \otimes s\) are ring homomorphisms. Therefore \(p = \ker(\phi_R)\) and \(q = \ker(\phi_S)\) are the prime ideals of \(R\) and \(S\) respectively such that \(P \supseteq p \otimes S + R \otimes q\). In particular
\[
\text{min prime}(R \otimes S) \subseteq \{ p \otimes S + R \otimes q \mid p \in \text{Spec } R, \ q \in \text{Spec } S\} \subseteq \text{Spec } (R \otimes S)
\]
which implies the first assertion of the claim.

Now consider \(P = p \otimes R + R \otimes q \in \text{min prime } (R \otimes S)\). Then \(r \in R \setminus p\) and \(s \in S \setminus q\) implies \(r \otimes 1\) and \(1 \otimes s\) are in \((R \otimes S) \setminus P\).

So if \(T_1 = \{ r \otimes s \mid r \in R \setminus p\) and \(s \in S \setminus q\}\) and \(T = (R \otimes S) \setminus P\) then \(T^{-1}_1 \subseteq T\)
\[
R_p \otimes S_q = T^{-1}_1(R \otimes S) \quad \text{and} \quad T^{-1}(R \otimes S) = (R \otimes S)_P.
\]
Similarly, if \( k(p) \), \( k(q) \) and \( k(P) \) are the residue fields of \( p \), \( q \) and \( P \) respectively then

\[
k(p) \otimes_k k(q) = T_1^{-1} \left( \frac{R}{p} \otimes_k \frac{S}{q} \right) = T_1^{-1} \left( \frac{R \otimes_k S}{P} \right) \quad \text{and} \quad T^{-1} \left( \frac{R \otimes_k S}{P} \right) = k(P).
\]

Hence \( k(P) = T^{-1}(k(p) \otimes_k k(q)) \). By Cohen structure theorem, \( k(p) \subset R_p \), \( k(q) \subset S_q \) and \( k(P) \subset (R \otimes S) \). Now if \( \ell_{k(p)}(R_p) = m_1 \), and \( \ell_{k(q)}(S_q) = m_2 \) then we have

\[
0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{m_1m_2} = R_p \otimes_k S_q,
\]

where \( M_i/M_{i-1} \simeq k(p) \otimes_k k(q) \). But then

\[
0 = T^{-1}M_0 \subset T^{-1}M_1 \subset T^{-1}M_2 \subset \cdots \subset T^{-1}M_{m_1m_2} = T^{-1}(R_p \otimes_k S_q) = (R \otimes_k S)_P,
\]

where \( T^{-1}(M_i/M_{i-1}) = k(P) \). This proves the second part of the claim.

Let \((R, I) \) and \((S, J) \) be two SG pairs in \( \mathcal{C}_k \). Let

\[
\tilde{P}_l = \{ P_{ij} \in \text{min prime}(R \otimes_k S) \mid \text{dim}(R \otimes S)/P_{ij} = l \}.
\]

Similarly we can define the subsets \( \tilde{p}_{l_1} \subset \text{min prime}(R) \) and \( \tilde{q}_{l_2} \subset \text{min prime}(S) \). We have canonical bijection

\[
\tilde{P}_l \leftrightarrow \tilde{p}_{l_1} \times \tilde{q}_{l_2} \quad \text{where} \quad P_{ij} \rightarrow (p_{ij}, q_{ij}).
\]

if \( P_{ij} = p_{ij} \otimes S + R \otimes q_{ij} \). It is obvious that \( l_1 + l_2 = l \) as \( \text{dim} R/p_{ij} + \text{dim} S/q_{ij} = l \).

By Lemma 3.2

\[
\Phi_{R \otimes S, P_{ij}} = \Phi_{R, p_{ij}} \cdot \Phi_{S, q_{ij}}.
\]

Therefore

\[
\Phi_{(R \otimes S)^l, (I \otimes J)(R \otimes S)^l} = \sum_{P_{ij} \in \tilde{P}_l} \ell(R \otimes S) P_{ij} \Phi_{R \otimes S, P_{ij}} = \sum_{l_1 + l_2 = l} \Phi_{R^{l_1}, I^{l_1}} \cdot \Phi_{S^{l_2}, J^{l_2}}.
\]

Hence \( \Pi(R \otimes S, I \otimes J) = \Pi(R, I) \cdot \Pi(S, J) \).

We proceed to extend the monoid \((\mathcal{C}_k, \otimes)\) to a semi ring \((\{\text{graded pairs}\}/ \equiv, \otimes, \oplus)\) as follows: by a graded pair \((R, I)\) we mean \((R, I)\) is a finite sum of SG pairs, where for two SG pairs \((R, I)\) and \((S, J)\) \(\in \mathcal{C}_k\) we define the addition of pairs as (where \((R \oplus S)_n = R_n \oplus S_n\) and \((I \oplus J)_n = I_n \oplus J_n\))

\[
(R, I) \oplus (S, J) = (R \oplus S, I \oplus J),
\]

with addition and multiplication as the pointwise addition and the pointwise multiplication respaly. In other words we are considering the direct product of rings. We canonically extend this addition operation to finitely many sg pairs, and multiplication \( \otimes \) as before. Also \((R, I) \equiv (S, J)\) if there is a degree 0 graded isomorphism \( \eta : R \longrightarrow S \) of rings such that \( \eta(I) = J \).

Following lemma gives the uniqueness of the decomposition of a graded pairs into the SG pairs.

**Lemma 3.8.** Let \((R, I)\) and \((S, J)\) be graded pairs such that

\[
(R, I) = \oplus_i \oplus_m (R^i, I^i) \quad \text{and} \quad (S, J) = \oplus_j \oplus^n (S^j, J^j),
\]

where \( \{(R^i, I^i)\}_i \) and \( \{(S^j, J^j)\}_j \) are two finite sets of distinct SG pairs. If there is an isomorphism of graded pairs \( \eta : (R, I) \simeq (S, J) \) then for every \( i \) there is \( j_i \) such that \( (R^i, I^i) \simeq (S^{j_i}, J^{j_i}) \) and \( m_i = n_{j_i} \).

In particular, the decomposition of a graded pair into a finite sum of sg pairs is unique up to an isomorphism of graded pairs.
Proof The map \( \eta \) gives the isomorphism of rings \( \eta : R \to S \), which induces the homeomorphism \( \eta^* : \oplus_j \oplus^m \Spec S^j \to \oplus_i \oplus^n \Spec R^i \). Note that each \( \Spec R^i \) and \( \Spec S^j \) is a connected set (being spectrum of a standard graded ring) and hence, if \( m = \sum_i m_i \) and \( n = \sum_j n_j \) then \( m = n \).

We rewrite \( R = \oplus^m_{j=1} R^i \) and \( S = \oplus^m_{i=1} S^j \), where \( \{ R^i \} \) and \( \{ S^j \} \) need not be sets of distinct elements. Here we identify \( R^i \) as \( (0, \ldots, R^i, 0) \subset \tilde{R} \) (similarly for \( S^j \)).

Now we prove that the map \( \eta |_{P_1} : R^1 \cong S^1 \) is an isomorphism, for some \( j \).

Note that an idempotent of \( R \) is the sum of the idempotents of \( R^i \). Hence the ring \( R \) has precisely \( m \) nontrivial irreducible idempotents (i.e., the idempotent which can not be written as a sum of two nontrivial idempotents) namely

\[ \{ e_i := (a_1, \ldots, a_m) \mid a_i = 1 \text{ and } a_j = 0 \text{ for } j \neq i \}. \]

Similarly \( S \) has precisely \( m \) nontrivial irreducible idempotents say \( \{ f_i \}_i \). Since the (irreducible) idempotents map to the (irreducible) idempotents, for \( e_i \) there is \( f_j \) such that \( \eta(e_i) = f_j \), for some \( j \). This gives \( \eta(R_1) = \eta(R e_1) = S f_j = S^j \). Therefore the induced map \( R^1 \to S^j \) is an isomorphism.

Also \( \eta(I^1) \subseteq S^j \cap (J^1 \oplus \cdots \oplus J^m) = J^j \). Therefore \( \eta(R^1, I^1) \cong (S^j, J^j) \). This gives an isomorphism

\[ \eta : \oplus_{2 \leq i \leq m} (R^i, I^i) \to \oplus_{1 \leq i \leq m, i \neq j} (S^j, J^j). \]

Now the proof follows by induction on \( m \). \( \square \)

Note that the monoid \( \langle C_k, \otimes \rangle \) also gives the corresponding commutative semi ring \( (\mathbb{N}[C_k], \ast, +) \), where as a set

\[ \mathbb{N}[C_k] = \{ \sum_{P_m \in C_k} r_m P_m \mid r_m \in \mathbb{N} \text{ are all zero except for finitely many, } P_m \in C_k \}, \]

and where the addition \( + \) is a formal sum of elements of \( C_k \) given by

\[ \left( \sum_{P_m \in C_k} r_m P_m \right) + \left( \sum_{P_m \in C_k} r'_m P_m \right) = \sum_{P_m \in C_k} \left( r_m + r'_m \right) P_m, \]

and the multiplication is

\[ \left( \sum_{P_m \in C_k} r_m P_m \right) \ast \left( \sum_{P_m \in C_k} r'_m P_m \right) = \sum_{P_m \in C_k} \left( \sum_{P_n \in C_k} \sum_{P_m P_n = P_{nm}} r_m r'_n \right) P_m. \]

By the uniqueness of the decomposition of graded pairs (Lemma 3.8), there is an isomorphism

\[ (\mathbb{N}[C_k], \ast, +) \cong (\text{graded pairs}/ \equiv, \otimes, \llogical OR) \]

of semi rings given by \( \sum_m m_i(A_i, I_i) \to \oplus \oplus^m_i (A_i, I_i) \). Moreover the addition satisfies the cancellation law (i.e., \( a + c = b + c \) implies \( a = b \) hence (Theorem 20.8 of [W]) the semi ring \( \mathbb{N}[C_k] \) embeds into the ring \( \mathbb{Z}[C_k] \) = \{ \( a - b \mid a, b \in \mathbb{N}[C_k] \}\). In particular we have the embedding (respecting the binary operations)

\[ (C_k, \otimes) \hookrightarrow (\mathbb{N}[C_k], \otimes, \llogical OR) \cong (\text{graded pairs}, \equiv, \otimes, \llogical OR) \hookrightarrow (\mathbb{Z}[C_k], \otimes, \llogical OR), \]

where the first embedding is given by \( (R, I) \to 1 \cdot (R, I) \). Here the SG pairs \( (k,(0)) \) and \( (0,(0)) \) are respectively the multiplicative and the additive identity of the ring \( \mathbb{Z}[C_k] \).

In particular, if \( \phi : C_k \to (S,+,\cdot) \) is a map where \( (S,+,\cdot) \) is a ring and where

\[ \phi((R,I) \otimes (R',I')) = \phi(R,I) \cdot \phi(R',I') \quad \text{and} \quad \phi((R,I) \oplus (R',I')) = \phi(R,I) + \phi(R',I'), \]

then \( \phi \) extends uniquely to a map of rings \( \phi : (\mathbb{Z}[C_k], \otimes, \llogical OR) \to (S,+,\cdot) \).
Theorem 3.9. The map $\Pi : \mathbb{Z}[C_k] \to H(\mathbb{C})[X]$ given by

$$(R, I) \to F_{R^0,IR^0} + F_{R^1,IR^1} + \cdots + F_{R^d,IR^d}X^d$$

is a ring homomorphism, where $d = \dim R$.

Proof. We only need to check that, if $(R, I)$ and $(S, J)$ are two SG pairs then

$$F_{(R \oplus S)(I \oplus J)} = F_{R^0,IR^0} + F_{S^0,JS^0}.$$  \hfill (3.1)

It is easy to check that the set $\{P \in \text{Spec} (R \oplus S) \mid \dim(R \oplus S/P) = i\}$ is equal to

$$\{p \oplus S \mid p \in \text{Spec} R \mid \dim(R/p) = i\} \cup \{R \oplus q \mid q \in \text{Spec} S \mid \dim(S/q) = i\}.$$  \hfill \qed

Moreover $(R \oplus S)_p = R_p$, if $P = p \oplus S$. Now (3.1) follows from the additivity property of $\Phi$.

Corollary 3.10. The map $\Pi : \mathbb{Z}[C_k] \to \mathbb{R}[X]$ given by

$$(R, I) \to e_{HK}(R^0, IR^0) + e_{HK}(R^1, IR^1) + \cdots + e_{HK}(R^d, IR^d)X^d$$

is a ring homomorphism.

Also the map $\Pi : \mathbb{C}$ given by $(R, I) \to e_{HK}(R, I)$ is multiplicative, i.e.,

$e_{HK}(R \otimes S, I \otimes J) = e_{HK}(R, I)e_{HK}(S, J)$.

Proof. Consider the evaluation map $ev : H(\mathbb{C})[X] \to \mathbb{C}[X]$ given by $\sum_i f_i X^i \to \sum_i f_i(0)X^i$. Since this is a ring homomorphism the map $\Pi_e := ev \circ \Pi$ is a ring homomorphism.

Moreover the map $pr : \mathbb{C}[X] \to \mathbb{C}$ given by $a_0 + a_1 X + \cdots + a_m X^m \to a_m$ is multiplicative and hence so is $\Pi_p = pr \circ \Pi |_{\mathbb{C}[X]}$. Now the corollary follows as $e_{HK}(R, I)$ by definition is $e_{HK}(R^d, IR^d)$, where $d$ is dimension of $R$. \hfill \qed

Remark 3.11. For two SG pairs $(R, I)$ and $(S, J) \in C_k$ we define $(R, I) \cup (S, J) \in C_k$ as a standard graded ring given by $(R \cup S)_0 = k$ and $(R \cup S)_n = R_n \oplus S_n$.

Let $C_k^1 = \{(R, I) \in C_k \mid \dim R \geq 1\}$ then $C_k^1$ is a $\mathbb{Z}[C_k]$-module and it is easy to check that for $(R, I)$ and $(S, J)$ in $Ck$

$$\Pi((R, I) \oplus (S, J)) = \Pi((R, I) \cup (S, J))$$

and the map $\Pi : \mathbb{Z}[C_k] \to H(\mathbb{C})[X]$ factors through the quotient ring

$$\mathbb{Z}[C_k]/\langle (R, I) \oplus (S, J) - (R, I) \cup (S, J), (R, I), (S, J) \in C_k^1 \rangle.$$  \hfill \qed

Hence $\Pi |_{C_k^1} : C_k^1 \to H(\mathbb{C})[X]$ given by

$$(R, I) \to \hat{f}_{R^0,IR^0}X + \cdots + \hat{f}_{R^d,IR^d}X^d$$

is a $\mathbb{Z}[C_k]$-linear map and therefore $(e \circ \Pi) |_{C_k^1} : C_k^1 \to \mathbb{R}[X]$ is a $\mathbb{Z}[C_k]$-linear which sends

$$(R, I) \to e_{HK}(R^1, IR^1)X + \cdots + e_{HK}(R^d, IR^d)X^d.$$  \hfill \qed

Notations 3.12. Let $(R, I)$ be a SG pair of dimension $d$. It is easy to check that $e_{HK}(R, I) = F_{R, I}(0)$. We denote

$$\alpha(R, I) = \max\{x \mid f_{R, I}(x) \neq 0\} \quad \text{where} \quad f_{R, I} = \lim_{n \to \infty} f_n(R, I).$$

It follows that $\dim R = 0$ if and only if $\alpha(R, I) = 0$.

We note that for the Fourier transform $\hat{f}_{R, I}$ of $f_{R, I}$, we have $F_{R, I} = \hat{f}_{R, I}$ if $d \geq 1$ and $F_{R, I} \neq \hat{f}_{R, I}$, if $\dim R = 0$.  \hfill \qed
Proof of Theorem 1.1. If \((R, I)\) is a 0 dimensional pair then for all \(z\), \(F_{R,I}(z) = \ell(R) = e_{HK}(R)\). Hence the theorem holds in this case.

Henceforth we assume \(\dim R \geq 1\). We denote \(\alpha(R, I)\) by \(\alpha\). Now \(F_{R,I} = \widehat{f}_{R,I}\), where \(f_{R,I} \in C_c^{\infty}(R)\) is the HK density function of \((R, I)\).

Let \(A \in \mathbb{R}_+\) such that \(\hat{f}_{R,I} \in PW_A\). Then there is a real valued function \(g \in L^2[-A, A]\) such that

\[
\hat{f}_{R,I}(z) = \int_{-A}^{A} g(t) e^{itz} dt =: \hat{g}(z), \quad \text{for all } z \in \mathbb{C}.
\]

However the mapping \(L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})\) given by \(f \rightarrow \hat{f}\) is an isomorphism of Hilbert spaces. Hence

\[
\hat{f}_{R,I} = \hat{g} \quad \Rightarrow \quad ||\hat{f}_{R,I} - \hat{g}||_2 = 0 \quad \Rightarrow \quad \|f_{R,I} - g\|_2 = \int_{-\infty}^{\infty} |f(t) - g(t)|^2 dt = 0.
\]

Hence \(f_{R,I} = g\) a.e.. But then

\[
e_{HK}(R, I) = \hat{f}_{R,I}(0) = \int_{-A}^{A} g(t) dt = \int_{-A}^{A} f_{R,I}(t) dt = \int_{0}^{A} f_{R,I}(t) dt.
\]

If \(A < \alpha\) then we have \(f_{R,I}^\alpha(t) dt = 0\), which is a contradiction as, by Lemma 3.1 and Theorem 1.1 of [T], the function \(f_{R,I}\) is a nonnegative function which is strictly positive in a nbhd of \(\alpha\).

For \(z = x + iy \in \mathbb{C}\),

\[
|\hat{f}_{R,I}(z)| \leq \int_{0}^{\alpha} |f_{R,I}(t)e^{ixy}| dt \leq \beta e^{\alpha|z|} \int_{0}^{\alpha} f_{R,I}(t) dt = e_{HK}(R, I)e^{\alpha|z|}.
\]

Let \(C\) and \(M\) be constants such that \(|\hat{f}_{R,I}(z)| \leq Ce^{M|z|}\). Since \(\hat{f}_{R,I} \mid_{\mathbb{R}} \in L^2(\mathbb{R})\), by Paley-Wiener Theorem we have \(\hat{f}_{R,I} \in PW_M\) and hence \(\alpha \leq M\). Moreover \(e_{HK}(R, I) = \hat{f}_{R,I}(0) \leq C\).

Proof of Corollary 1.2. By Theorem 4.9 of [TrW] and Theorem C of [T2], we have \(\alpha(R, I) = e^{\ell(m)}\). Hence the corollary follows from Theorem 1.1. \(\square\)

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