Scaling in erosion of landscapes: renormalization group analysis of a model with turbulent mixing

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Abstract

The model of landscape erosion, introduced in (1998 Phys. Rev. Lett. 80 4349, 1998 J. Stat. Phys. 93 477) and modified in (2016 Theor. Math. Phys. in press (arXiv:1602.00432)), is advected by anisotropic velocity field. The field is Gaussian with vanishing correlation time and the pair correlation function of the form $\rho(t-t')/k^4\ln^{4-1+\xi}$, where $k = |\mathbf{k}|$ and $k_\perp$ is the component of the wave vector, perpendicular to a certain preferred direction—the $d$-dimensional generalization of the ensemble introduced by Avellaneda and Majda (1990 Commun. Math. Phys. 131 381). Analogous to the case without advection, the model is multiplicatively renormalizable and has infinitely many coupling constants. The one-loop counterterm is derived in a closed form in terms of the certain function $V(h)$, entering the original stochastic equation, and its derivatives with respect to the height field $h(t,\mathbf{x})$. The full infinite set of the one-loop renormalization constants, $\beta$-functions and anomalous dimensions is obtained from the Taylor expansion of the counter-term. Instead of a two-dimensional surface of fixed points there are two such surfaces; they are likely to contain infrared attractive region(s). If that is the case, the model exhibits scaling behaviour in the infrared range. The corresponding critical exponents are non-universal because they depend on the coordinates of the fixed points on the surface; they also satisfy certain universal exact relation.

Keywords: renormalization group, turbulent mixing, landscape erosion, scaling
1. Introduction and description of the model

The problem of landscape erosion due to the flow of air or water over it, and related problems (e.g. granular flows), have attracted constant interest over the past few decades: see [1–18] and the literature cited therein. A plethora of widely varied physical phenomena is related to these issues, which makes a construction of the underlying dynamical model a complicated task. These models have been a source of much controversy [4–17]. However, landscape erosion has some universal aspects (such as the exponents in scaling laws) which, in analogy with critical phenomena, can be described within the framework of relatively simple semi-phenomenological models.

A similar situation takes place in the related problem of kinetic roughening of surfaces or interfaces, as described by the well known Kardar–Parisi–Zhang (KPZ) stochastic model [19] and its descendants [20–22]. Another example is provided by the problem of self-organized criticality, which is described in the continuum limit by the Hwa–Kardar stochastic model [23] and its modifications [24, 25].

A semiphenomenological model of landscape erosion was suggested by the authors of [14, 15]. In contrast with existing isotropic continuum models of erosion [10–12] based on a diffusion equation with added noise, this model is built to be nonlinear and anisotropic—the latter choice was motivated by the features of experimental data on erosion. Indeed, experimental measurements of the ‘roughness’ exponent (exponent in the scaling law) indicate that the exponent is either small (0.30–0.55) [26, 27] or large (0.70–0.85) [27–29], while other measurements display a crossover from large to small values that happens when length scales exceed the mark of approximately 1 km [27]. The authors of [14, 15] proposed that the large values of the exponent at small length scales may be due to the influence of a preferred direction for the transport of eroded material—the downhill direction.

Dimensionality and symmetry considerations gave further basis to the anisotropic model [14, 15]. In particular, one of the symmetries—with respect to the sign of the space coordinate in the direction parallel to the slope—made the model different from the models of self-organized criticality [23–25]. The dynamical renormalisation group analysis of the anisotropic model yielded a first-order estimate of the roughness exponents. This estimate was in good agreement with the sea floor measurements completed by the authors of [14, 15]; desert environment measurements did not yield any conclusive power-law but confirmed that the topography in the perpendicular direction to the slope is rougher than the topography in the parallel direction. The latter is the main result of [14, 15] as it applies to different types of landscapes. The anisotropic model also predicts that correlations in different directions decay quantitatively differently than they do for isotropic topography.

Since the model applies to a surface with the fixed tilt of the slope, it applies only locally to the relatively small scales where the preferred direction of the flux is approximately constant. As the anisotropy vanishes at large length scales, an isotropic model may be better fitted to describe the large scale features. The KPZ model [19–22] can be such an isotropic theory. The authors of [14, 15] suggested that the difference between KPZ predictions and those of the anisotropic model might be used to distinguish statistically between features of the landscape due to erosion and features due to larger-scale processes.

It is also worth noticing that the anisotropic model has a fixed tilt of the slope, and it would be interesting to consider the model with random tilt, described by various statistics.

Standard field-theoretic renormalization group analysis was applied to the anisotropic model in [30]. It was shown that the the model is not suitable for renormalization analysis because it gives rise to an infinite amount of counterterms which are not present in the original stochastic equation, truncated as it is on the leading $h^3$ term ($h$ being the height of the
landscape profile). This consideration also changes the upper critical value of space dimension \(d\) from supposed \(d = 4\) to \(d = 2\).

Because of this, the anisotropic model was modified in [30] to include the whole series in the powers of \(h\) which made it renormalizable. The modified model has an infinite amount of coupling constants; such models are generally considered to lack predictive power. However, in the case of the modified model, it is possible to calculate the one-loop counterterm and find universal relations for critical exponents (i.e. relations which do not depend on the coupling constants). A two-dimensional surface of fixed points which is likely to contain IR attractive region(s) was found. Varying experimental results [26–29] indicate two wide ranges of roughening exponent value, which may be explained by the existence of two different IR attractive regions.

Let us describe the modified model.

A unit constant vector \(\mathbf{n}\) determines a certain preferred direction (tilt of the slope) and, therefore, establishes an intrinsic anisotropy of the model. Any vector can be decomposed into the components perpendicular and parallel to \(\mathbf{n}\). For the \(d\)-dimensional horizontal position \(\mathbf{x}\) one has \(\mathbf{x} = \mathbf{x}_n + \mathbf{x}_\perp\) with \(\mathbf{x}_n \cdot \mathbf{n} = 0\). In the following, we denote the derivative in the full \(d\)-dimensional \(\mathbf{x}\) space by \(\partial = \partial \partial x_i\) with \(i = 1 \ldots d\), and the derivative in the subspace orthogonal to \(\mathbf{n}\) by \(\partial_\perp = \partial \partial x_i\) with \(i = 1 \ldots d - 1\). Then the derivative in the parallel direction is written as \(\partial_\parallel = \mathbf{n} \cdot \partial\).

The stochastic differential equation for the height of the profile, i.e. for the height field \(h(x) = h(t, \mathbf{x})\), proposed in [14, 15] and modified in [30], is taken in the form

\[
\partial_t h = \nu_{x_0} \partial_\parallel^2 h + \nu_{\perp 0} \partial_\parallel^2 h + \partial_\perp^2 V(h) + f. \tag{1.1}
\]

Here \(\partial_t = \partial \partial t\), \(\nu_{x_0}\) and \(\nu_{\perp 0}\) are topographic diffusion coefficients, \(V(h)\) is some function that depends only on the field \(h(x)\) (and not on its derivatives) and \(f(x)\) is a Gaussian random noise with zero mean and prescribed pair correlation function

\[
\langle f(x) f(x') \rangle = D_0 \delta(t - t') \delta^{(d)}(\mathbf{x} - \mathbf{x}') \tag{1.2}
\]

with some positive amplitude \(D_0\).

Here and below the subscript ‘0’ means that the parameters in (1.1) are bare, i.e. not yet renormalized.

The function \(V(h)\) is a series in powers of \(h(x)\). In [14, 15] is was taken to be odd in \(h\) which was explained by the symmetry \(h, f \rightarrow -h, -f\) (another symmetry of the model is \(x_i \rightarrow -x_i\)). The authors of [14, 15] also truncated the Taylor expansion of \(V(h)\) on the leading \(h^3\) term, but the whole series in \(h\) should be considered instead [30].

When added to the problem, the various kinds of deterministic or chaotic flows change the behaviour of the critical systems (such as liquid crystals or binary mixtures near their consolution points). Indeed, the flow can destroy the usual critical behaviour, change it to the mean-field behaviour, or give rise to a new non-equilibrium universality class [31–36]. That is why it is vital to study the influence of turbulent mixing on critical behavior.

Gaining a clear quantitative picture of the erosion under influence of the turbulent motion is the main goal of this paper. That is why the velocity field will be modelled by a relatively simple synthetic ensemble which, nonetheless, is quite powerful as a tool of quantitative analysis. It is a strongly anisotropic Gaussian ensemble, with vanishing correlation time and prescribed power-like pair correlation function—a \(d\)-dimensional generalization of the ensemble introduced by Avellaneda and Majda in [37]. At the same time, the ensemble is an anisotropic modification of the popular Kraichnan’s rapid-change model; see [38] for the review and references. Anisotropic flow is a natural consideration, because the modified model of erosion already involves intrinsic anisotropy, related to the overall tilt of the landscape.
Coupling with the velocity field \( v_i(x) \) is introduced by the replacement

\[
\partial_t \to \nabla_i \equiv \partial_t + v_i \partial_i
\]

(1.3)

where \( \nabla_i \) is the Lagrangian (Galilean covariant) derivative.

The velocity field will be taken in the form

\[
v = n v(t, x_i),
\]

(1.4)

where \( v(t, x_i) \) is a scalar function independent of \( x_k \). Then the incompressibility condition is automatically satisfied:

\[
\partial_i v_i = \partial_i n v(t, x_i) = 0.
\]

(1.5)

We assume that \( v(t, x_i) \) has a Gaussian distribution with zero mean and the pair correlation function of the form

\[
\langle v(t, x_i) v(t', x'_i) \rangle = \delta(t - t') \int \frac{dk}{(2\pi)^d} \exp\{i k \cdot (x - x')\} D_r(k)
\]

\[
= \delta(t - t') \int \frac{dk_\perp}{(2\pi)^{d-1}} \exp\{i k_\perp \cdot (x_i - x'_i)\} \tilde{D}_r(k_\perp),
\]

(1.6)

with \( k_\perp = |k_\perp| \) and the scalar coefficient function \( D_r \):

\[
D_r(k) = 2\pi \delta(k_\parallel) \tilde{D}_r(k_\perp), \quad \tilde{D}_r(k_\perp) = B_0 k_\perp^{-d+1-\xi}.
\]

(1.7)

Here \( B_0 > 0 \) is a constant amplitude factor; \( \xi \) is an arbitrary exponent, which will play the role of a formal RG expansion parameter (along with \( \epsilon = 2 - d \)). The cutoff \( k_\perp > m \) provides the infrared (IR) regularization in (1.6). The precise form of the cutoff is unimportant; the sharp cutoff is the most convenient choice from the calculational viewpoint.

We apply the standard field theoretic renormalization group (RG) to the modified model of erosion with turbulent mixing and arrive at the results similar to those presented in [30].

The plan of the paper and the main results are as follows.

In section 2 the field theoretic formulation of the stochastic problem (1.1) and (1.2) for the arbitrary (not necessarily odd) full-scale (not truncated) function \( V(h) \) is presented.

In section 3 ultraviolet (UV) divergences and renormalization procedure of the resulting field theory are discussed. The upper critical dimension is established to be \( d = 2 \); this leads to the emergence of infinitely many coupling constants in renormalized model, and, subsequently, to the emergence of infinitely many \( \beta \)-functions in the corresponding RG equations.

We write down the corresponding renormalized action functional, renormalization relations for the fields and parameters, RG equations and RG functions (\( \beta \)-functions and anomalous dimensions).

In section 4 the renormalization procedure is performed in the leading one-loop order. Despite the fact that the model involves infinitely many couplings, the one-loop counterterm is derived in a closed form in terms of the function \( V(h) \) and its derivatives with respect to the variable \( h(x) \). Its Taylor expansion gives rise to the full infinite set of one-loop renormalization constants, and, therefore, to all \( \beta \)-functions and anomalous dimensions.

In this derivation, we adopt the functional method applied earlier by Vasil’ev and one of the authors [40] to an isotropic model of surface roughening, proposed in [39] as a possible modification of the Kardar–Parisi–Zhang equation; see also [41, 42]. This method was also applied in [30].

In section 5 attractors of the obtained RG equations are analyzed in the infinite-dimensional space of coupling constants. Instead of a set of fixed points (as for most
multicoupling models), there are two two-dimensional surfaces of fixed points — one of them corresponds to the IR asymptotic regime, where turbulent mixing is irrelevant, and coincides with the surface obtained in \[30\]. These surfaces are likely to contain IR attractive region(s). If so, the model exhibits scaling behaviour in the IR range. The corresponding critical exponents are nonuniversal because they depend on the coordinates of the specific fixed points on the surface, but satisfy certain exact relations.

The remaining problems are briefly discussed in section 6.

2. Field theoretic formulation of the model

According to the general statement (see, e.g. the books \[43, 44\] and the references therein), the stochastic problem (1.1) and (1.2) is equivalent to the field theoretic model of the set of fields \( \Phi = \{h, h', v\} \) with the action functional

\[
S(\Phi) = h'h' + h' \left\{ -\partial_t h + \nu_{\perp,0} \partial^2 h + \nu_{\parallel,0} \partial^2 h + \partial^2 \sum_{n=2}^{\infty} \frac{\lambda_{n}h^n}{n!} \right\} + S_v \tag{2.1}
\]

(we have scaled out \( D_0 \) and other factors of \( h'h' \) by adjusting the values of \( \lambda_{00} \)).

The last term in (2.1) corresponds to the Gaussian averaging over \( v \) with correlator (1.6) and has the form

\[
S_v = \frac{1}{2} \int dt \int dx dx' v(t,x) \tilde{D}_v (x-x') v(t,x'), \tag{2.2}
\]

where

\[
\tilde{D}_v^{-1}(r_{\perp}) \propto B_0^{-1} r_{\perp}^{2(1-d)-\xi} \tag{2.3}
\]

is the kernel of the inverse linear operation \( D_v^{-1} \) for the correlation function \( D_v \) in (1.7).

Here and below, all the required integrations over \( x = (t,x) \) are always implied, e.g.

\[
h'h' = \int dt \int dx h'(t,x) h'(t,x). \tag{2.4}
\]

The field theoretic formulation of the stochastic problem identifies various correlation and response functions of the stochastic problem (1.1) and (1.2) with various Green’s functions of the field theoretic model with the action (2.1). In other words, the correlation functions are now represented by functional averages over the full set of fields \( \Phi = \{h, h', v\} \) with the weight \( \exp(S(\Phi)) \).

A standard Feynman diagrammatic technique applies to the model (2.1). There are three bare propagators (lines in the diagrams): \( \langle vv \rangle_0 \) (given by (1.6) and (1.7)), and the propagators of the scalar fields (in the frequency–momentum and time–momentum representations):

\[
\langle hh' \rangle_0 = \langle h'h' \rangle_0^{-1}, \quad \langle hh \rangle_0 = 2 \left( \omega^2 + \varepsilon^2(k) \right)^{-1}, \tag{2.5}
\]

where \( \varepsilon(k) = \nu_{\perp,0} k_{\perp}^2 + \nu_{\parallel,0} k_{\parallel}^2 \). The propagator \( \langle h'h' \rangle_0 \) vanishes identically for any field theory of the type (2.1). The interaction terms \( -h'\partial_t V(h) \) and \( -h'(v\partial_0)h \) give rise to the vertices with bare coupling constants \( g_{n0} \) \( (n = 2, 3, \ldots) \) and \( w_0 \):

\[
\lambda_{n0} = 8g_{n0}^{(n+3)/4} \nu_{\perp,0}^{(n-1)/4} \nu_{\parallel,0}^{(n+1)/4}, \quad B_0 = w_0\nu_{\parallel,0}. \tag{2.6}
\]

so that by dimension \( g_{n0} \sim \ell^{-c(n-1)/2} \) and \( w_0 \sim \ell^{-\xi} \), where \( \ell \) has the order of the smallest length scale in our problem.
3. UV divergences and renormalization

To analyze the UV divergences the analysis of canonical dimensions is used; see, e.g. [43, 44]. Dynamic models of the type (2.1) usually have two scales, i.e. their dimensions are described by the two numbers — the frequency dimension \( \omega \) and the momentum dimension \( k \). These two numbers completely define the canonical dimension of a quantity \( F \) (a field or a parameter): \[ \sim - \omega F T L d F k \] (\( L \) is the typical length scale and \( T \) is the time scale); see, e.g. chapter 5 in [44]. In the present case, there are two independent momentum scales because of the anisotropy of the model. Namely, two independent momentum canonical dimensions \( \perp d F k \) and \( / \perp un 2225 d F k \) has to be introduced so that \[ \sim - \omega \perp F T L d d F F \] where \( \perp L \) and \( / \perp un 2225 L \) are (independent) length scales in the corresponding subspaces. The obvious normalization conditions are \[ d 1 d k x, / \perp un 2225 d 0 d k x, \] etc; moreover, each term of the action functional (2.1) is assumed to be dimensionless with respect to all the three independent dimensions separately. The original momentum dimension can be found from the relation \[ = + / \perp un 2225 d F k F F, \] Then, the total canonical dimension is \[ = + = = + + + \] The factor 2 in the last term comes from the consideration that in the free theory \[ \propto / \perp un 2225 d F k \].

The canonical dimensions for the model (2.1) are presented in table 1. The renormalized parameters (without the subscript \( ' \)) and the renormalization mass \( \mu \) will be introduced later.

As can be seen from table 1, all the coupling constants \( g_{n0} \) and \( w_0 \) become simultaneously dimensionless at \( d = 2 \), which makes \( d = 2 \) the upper critical dimension of the model. It should be noted that the total canonical dimension of the field \( h \) vanishes for this value of \( d \).

The UV divergences in the Green’s functions of the full-scale model manifest themselves as poles in \( \varepsilon = - d - 2 \), and that is why \( \varepsilon \) plays the role of the expansion parameter in the RG expansions.

The total canonical dimension of an arbitrary 1-irreducible Green’s function \( \Gamma = \langle \Phi \cdots \Phi \rangle_{1-irr} \) with \( \Phi = \{ h, h', v \} \) in the frequency–momentum representation is given by the relation:

\[ d_{\Gamma} = d + 2 - d_{h} N_{h} - d_{h'} N_{h'} - N_{v}, \] (3.1)

where \( N_{h}, N_{h'}, N_{v} \) are the numbers of the corresponding fields entering into the function \( \Gamma \); see, e.g. [44].

The total dimension \( d_{\Gamma} \) in the logarithmic theory (i.e. at \( \varepsilon = 0 \)) is, in fact, the formal index of the UV divergence: \( \delta_{\Gamma} = d_{\Gamma} \big|_{\varepsilon=0} \). The superficial UV divergences, whose removal requires counterterms, can be present only in those functions \( \Gamma \) for which \( \delta_{\Gamma} \) is a non-negative integer. The counterterm is a polynomial in frequencies and momenta of degree \( \delta_{\Gamma} \) (given that \( \omega \propto k^2 \) is implied).
If a number of external momenta occurs as an overall factor in all diagrams of a certain Green’s function, the real index of divergence \( \delta' \) will be smaller than \( \delta \) by the corresponding number. This happens in our model: the derivative at the vertex \( \mathcal{H}'(\phi) \) can be moved onto the field \( h' \) via integration by parts. The derivative in the vertex \( -h'(\partial_0 h) \) can be placed, at will, on \( h \) or on \( h' \). This means that any appearance of \( h' \) in some function \( \Gamma \) gives either an external momentum or a square of it, and \( \delta' \) is either equal to \( \delta - N h' \) or \( \delta - 2N h' \). Moreover, \( h' \) can appear in the corresponding counterterm only in the form of derivative \( \frac{\partial}{\partial h} \).

From table 1 and the expression (3.1) one obtains:

\[
\delta' = \delta - N h' = 4 - 4N h' - N_x,
\]

or

\[
\delta' = \delta - N h' = 4 - 3N h' - N_x.
\]

As all the 1-irreducible Green’s functions without the response fields vanish identically in dynamical models (their diagrams always involve closed circuits of retarded lines; see, e.g. [44]), it is sufficient to consider only the case \( N h' > 0 \).

Straightforward analysis of the expression (3.2) and (3.3) shows that superficial UV divergences can be present only in the 1-irreducible functions of the form \( (h' h\ldots h)_{i=ir} \) with the counter-term \( (\partial_0^2 h') h^n \) (for any \( n \geq 1 \)). Indeed, all the other counter-terms (e.g. \( h'h', h'\partial_0^2 h, h'\partial_0 h^n \), and hence from Galileyan symmetry \( h'(\partial_0 h) h^n \) are not needed as the corresponding 1-irreducible functions are finite.

The model is multiplicatively renormalizable because all the terms \( (\partial_0^2 h') h^n \) are present in the action (2.1). The renormalized action can be written in the form:

\[
S_{\text{ren}}(\Phi) = h'h' + h\left\{-\frac{\partial_0^2 h}{2} - \nu_0 \partial_0^2 h + Z g_0 \partial_0^2 h + \partial_0^2 \sum_{n=2}^{\infty} \frac{Z_0 Z_n h^n}{n!}\right\} + S_v.
\]

(3.4)

Here \( \nu_0 \) and \( \lambda_n \) are renormalized analogs of the bare parameters (those with subscript ‘0’). \( S_v \) does not require renormalization—there is no corresponding counter-term—and the same is true for \( \nu_0 \), i.e. \( \nu_0 = \nu_{0,0} \).

The renormalization constants \( Z_{\parallel}, Z_{\perp}, \) and \( Z_n \) depend only on the completely dimensionless parameters \( g_0 \) and \( w \) and absorb the poles in \( \varepsilon \) and \( \xi \). The bare charges \( w_0, g_0 = \{g_n\} \) and bare parameters \( \lambda_{0,0} \) completely dimensionless renormalized charges \( w, g = \{g_n\} \) (\( n = 2, 3, \ldots \)) and renormalized parameters \( \nu_0, B, \lambda_n \) are related as follows:

\[
\lambda_{0,0} = g_0 \nu_0^\parallel \nu_0^{n-1/4} \nu_0^{1/4} \mu^{1/2}, \quad \lambda_n = g_n \nu_0^\parallel \nu_0^{n-1/4} \nu_0^{1/4} \mu^{1/2} \quad (3.5)
\]

\[
B_0 = \nu_0 w_0 \quad B = \nu_0 w \mu^{1/2} \quad (3.6)
\]

Here the renormalization mass \( \mu \) is an additional parameter of the renormalized theory; its canonical dimension is shown in table 1.

The renormalized action (3.4) is obtained from the original (2.1) by the renormalization of the parameters (the renormalization of the fields \( h, h', v \) and parameter \( \nu_{1,1} \) is not required):

\[
\nu_{0,0} = \nu_0 \nu_{0,0} \quad g_{0,0} = \mu^{1/2} g_0 Z_{\parallel} \quad \lambda_{0,0} = \lambda_{0,0} \quad w_0 = w Z_{\parallel} \mu^{1/2} \quad (3.7)
\]

The renormalization constants in equations (3.4) and (3.7) are related as follows:

\[
Z_{0,0} = Z_{\parallel} \nu_0^\parallel \nu_0^{1/4}, \quad Z_{\parallel} Z_{\perp} = 1 \quad (3.8)
\]
Let us consider an elementary derivation of the RG equations \[43, 44\]. The RG equations are written for the renormalized Green’s functions \( G_R = \langle \Phi \cdots \Phi \rangle_R \). In the present case, however, the original (unrenormalized) Green’s functions \( G \) could be considered instead—the fields are not renormalized and, therefore, \( G(e_0, \ldots) = G_R(e, \mu, \ldots) \). Here, \( e_0 = \{ w_0, g_{00}, \nu_{00}, \nu_{01}, \ldots \} \) is a full set of bare parameters and \( e = \{ w, g_n, \nu_n, \nu_n, \ldots \} \) are their renormalized counterparts; the ellipsis stands for the other arguments (times, coordinates, momenta etc).

We use \( \mathcal{D}_e \) to denote the differential operation \( \mu \partial, |_{\nu} \). When expressed in the renormalized variables it looks as follows:

\[
\mathcal{D}_{RG} \equiv \mathcal{D}_e + \sum_{n=2}^{\infty} \beta_n \partial g_n + \beta_w \partial w - \gamma \mathcal{D}_{\nu} \tag{3.9}
\]

where \( \mathcal{D}_x \equiv x \partial_x \) for any variable \( x \). The anomalous dimensions \( \gamma \) are defined as

\[
\gamma \equiv \mathcal{D}_e \ln Z_F \quad \text{for any quantity } F,
\]

and the \( \beta \) functions for the dimensionless coupling constants \( g_n \) and \( w \) are

\[
\beta_n \equiv \mathcal{D}_e g_n = g_n [1 - \varepsilon(n - 1)/2 - \gamma], \quad \beta_w \equiv \mathcal{D}_e w = w [-\xi - \gamma_w]. \tag{3.11}
\]

### 4. One-loop expressions for the counterterm, renormalization constants and RG functions

The model involves infinitely many coupling constants. Despite that, the one-loop counterterm can be obtained—in an explicit closed form in terms of the function \( V(h) \). Let us follow the calculation process.

The 1-irreducible Green’s functions of our model correspond to the generating functional \( \Gamma_R(\Phi) \). Its expansion in the number \( p \) of loops looks as follows:

\[
\Gamma_R(\Phi) \equiv \sum_{p=0}^{\infty} \Gamma^{(p)}(\Phi), \quad \Gamma^{(0)}(\Phi) = S_0(\Phi). \tag{4.1}
\]

The loopless (tree-like) contribution is simply the action; the one-loop contribution can be calculated via the following relation, see, e.g. [45]:

\[
\Gamma^{(1)}(\Phi) = -(1/2) \text{Tr} \ln(W/W_0), \tag{4.2}
\]

where \( W \) is a linear operation with the kernel

\[
W(x, y) = -\delta^2 S_0(\Phi)/\delta \Phi(x) \delta \Phi(y), \tag{4.3}
\]

and \( W_0 \) is the similar expression for the free parts of the action. Both \( W \) and \( W_0 \) are 3 × 3-matrices in the set of the fields \( \Phi = \{ h, h', v \} \).

By removing UV divergences in (4.1) and using the minimal subtraction scheme, we can find the uniquely determined values for constants \( Z \). We put \( Z = 1 \) in (4.2) in the one-loop approximation. In the loopless contribution we keep leading-order terms in the coupling constants \( g_n, w \) in the constants \( Z \). For internal consistency we suppose that \( g_n \simeq g_2^{n-2} \).

The Taylor expansion of the function \( V(h) \) is

\[
V(h) = \sum_{n=2}^{\infty} \lambda_n h^n(x)/n!, \quad V_0(h) = \sum_{n=2}^{\infty} Z_n \lambda_n h^n(x)/n!, \tag{4.4}
\]
In the following, we interpret similar objects as functions of a single variable \( h(x) \), and \( V', V'' \), etc, as the corresponding derivatives with respect to this variable. Thus, the matrix \( W \) (under the condition that \( Z = 1, \nu = 0 \)) can be symbolically represented as

\[
W = \begin{pmatrix}
-\partial_\parallel h' \cdot V'' & L^T & -\partial h' \\
L & -2 & \partial h \\
\partial_\parallel h & \partial_\parallel h & D_v
\end{pmatrix}
\]  \hspace{1cm} \text{(4.5)}

where \( D_v(k) = 2\pi \delta(k_0)B_0 k^{-d+1-\xi} \) from (1.7); \( L \equiv -\partial_t -\nu \partial_\perp^2 - \nu_\perp \partial_\perp^2 \) \( \partial_\perp \) \( V' \), and \( L^T \equiv -\partial_t -\nu \partial_\perp^2 - \nu_\perp \partial_\perp^2 - V' \partial_\perp^2 \) is the transposed operation.

Only the divergent part of expression (4.1) is required to calculate the constants \( Z \); this part was previously established to have the form

\[
\int dx \partial_\parallel h(x) R(h(x))
\]

with a function \( R(h) \) similar to \( V(h) \). How can one extract this part? Let us recall the well-known formula: \( \delta(\ln K) = \ln(K^\nu) \) for any variation \( \delta K \). By varying the matrix \( W \) by \( h' \) we obtain

\[
\int dx \partial_\parallel^2 h(x) R(h(x)) \\
= -\frac{1}{2} \left( -D^{hh}V''\partial_\parallel^2 h' + D^{hh'}\partial h'h' + D^{hh'}\partial h'h' \right) \\
\equiv -\frac{1}{2} \int dx \left( -D^{hh}V''(h(x))\partial_\parallel^2 h'(x) + D^{hh'}\partial h'(x) + D^{hh'}\partial h'(x) \right),
\]  \hspace{1cm} \text{(4.6)}

where \( D^{hh'} = (W^{-1})_{hh'} \) at \( h', \nu = 0 \) (the fields are kept equal to zero in \( W^{-1} \) because we do not need the terms with them to extract the divergent part). Due to the way it was constructed, \( D^{hh'} \) is the ordinary propagator \( \langle hh' \rangle \) of the model (3.4) with \( Z = 1 \) and with \( \nu \partial_\parallel^2 + \nu_\perp \partial_\perp^2 + \partial_\parallel^2 V' \) substituted for \( \nu \partial_\parallel^2 + \nu_\perp \partial_\perp^2 \).

Another consideration should be taken into account. After \( \partial_\parallel^2 \) is moved to the external factor \( h' \), only a logarithmically divergent expression remains in the counterterm. This means that during calculation of the divergent part of a given diagram all the external momenta can be put to zero (IR regularization is ensured by the cutoff). Moreover, we can ignore the inhomogeneity of \( \partial_\parallel h'(x) \) and \( h(x) \), assuming them to be constants in (4.6) while we select the poles in \( \varepsilon \) and \( \xi \). Then \( D^{hh'}(x, x), D^{hh'}(x, x), \) and \( D^{hh'}(x, x) \) can be calculated by going over to the momentum-frequency representation:

\[
D^{hh'}(x, x) = \int \frac{d\omega dk}{(2\pi)^{d+1}} \frac{2}{\omega^2 + [\nu k_\parallel^2 + \nu_\perp k_\perp^2 + k_\perp^2 V']^2} \\
= \frac{S_d}{(2\pi)^d} \frac{\mu^{-\varepsilon}}{\varepsilon \sqrt{\nu \partial_\parallel^2 + \nu_\perp \partial_\perp^2 + \partial_\parallel^2 V'}} + \ldots,
\]

\[
D^{hh'}(x, x) = \partial h \int \frac{d\omega dk}{(2\pi)^{d+1}} \frac{B_0 \delta(k_0)}{k_\perp^{d-1+\xi}(i\omega + \nu k_\parallel^2 + \nu_\perp k_\perp^2 + k_\perp^2 V')}.
\]

\[
D^{hh'}(x, x) = \partial h \int \frac{d\omega dk}{(2\pi)^{d+1}} \frac{B_0 \delta(k_0)}{k_\perp^{d-1+\xi}(-i\omega + \nu k_\parallel^2 + \nu_\perp k_\perp^2 + k_\perp^2 V')}.
\]

\[
D^{hh'}(x, x) + D^{hh'}(x, x) = \partial h \frac{S_d}{(2\pi)^{d-1}} \frac{\mu^{-\varepsilon}}{\xi B_0} + \ldots \hspace{1cm} \text{(4.7)}
\]
where the ellipsis stands for the UV-finite part; $S_d$ is the surface area of the unit $d$-dimensional sphere: 

$$S_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}.$$ 

Substituting (4.6) and (4.7) into (4.2), we obtain the following expression for the divergent part of $\Gamma_1(\Phi)$:

$$\Gamma_1(\Phi) \sim \frac{S_d}{2(2\pi)^d} \frac{\mu^{-\varepsilon}}{\varepsilon} \int dx \frac{V''(h(x))}{\sqrt{\nu_1(\nu_1 + V'(h(x)))}} \partial_\varepsilon^2 h(x) - \frac{S_{d-1}}{2(2\pi)^{d-1}} B_0 \frac{\mu^{-\xi}}{\xi} \int dx \partial_\xi^2 h(x) h.$$ 

(4.8)

The sum of (4.8) and the loopless contribution in (4.2) has no poles in $\varepsilon$, $\xi$, or their linear combination (they cancel out). This allows us to find the one-loop contributions of order $1/\varepsilon$ and $1/\xi$ in all constants $Z$.

Let us introduce the representation

$$\frac{V''(h(x))}{\sqrt{\nu_1(\nu_1 + V'(h(x)))}} = \sum_{n=0}^\infty \frac{\mu^\varepsilon \nu_1^{n+1}}{\nu_1^{n+1}} \nu_1^{-1/4} \nu_1^{-1/4} \nu_1^{-1/4} \frac{r_n \partial h^n}{n!}$$

(4.9)

for the Taylor expansion of the integrand in (4.8).

Then $r_n$ are completely dimensionless coefficients—polynomials in the charges $g_n$. Combining the above condition for the canceling out of poles in $\varepsilon$, $\xi$, and (3.5)–(3.8), we obtain

$$Z_\parallel = 1 - \frac{r_0 S_d}{2(2\pi)^d \varepsilon} + \frac{w S_{d-1}}{2(2\pi)^{d-1} \xi} + \ldots, \quad Z_\parallel = 1 - \frac{r_n}{g_n} \frac{S_d}{2(2\pi)^d \varepsilon} + \ldots.$$ 

(4.10)

The operation $\bar{D}_\parallel$ in (3.11) assumes the form

$$\bar{D}_\parallel = \sum_n \left( \bar{D}_\parallel g_n \partial_{\varepsilon_n} + \beta_n \partial_w \right) = \sum_n \beta_n \partial_{\varepsilon_n} + \beta_w \partial_w,$$

which means that in order to achieve the required accuracy it is sufficient to use only the first terms in the $\beta$-functions (3.11). This yields

$$\bar{D}_\parallel \equiv -\frac{\varepsilon}{2} \bar{D}_\parallel - \xi \bar{D}_w, \quad \bar{D}_\parallel = \sum_{n=2}^\infty (n-1) \bar{D}_\parallel.$$ 

(4.11)

Applying this to (3.10), (3.11) and (4.10) we obtain the following expressions for the one-loop RG-functions:

$$\gamma_\parallel = a \bar{D}_\parallel r_1/2 - bw, \quad a \equiv \frac{S_d}{2(2\pi)^d}, \quad b \equiv \frac{S_{d-1}}{2(2\pi)^{d-1}};$$

(4.12)

$$\beta_w = -\xi w + w \gamma_\parallel;$$

(4.13)

$$\beta_n = -\varepsilon - \frac{n - 1}{2} g_n + \frac{n + 3}{4} g_n \gamma_\parallel - \frac{a}{2} (\bar{D}_\parallel - n + 1) r_n.$$ 

(4.14)

Let us consider the explicit expressions for the first four coefficients $r_n$ (the first term with $r_0$ in (4.9) contributes nothing to (4.8)); they could be found from the definitions (3.5), (4.4), (4.9):

$$r_1 = g_3 - \frac{1}{2} g_5^2, \quad r_2 = g_4 - \frac{3}{2} g_2 g_3 + \frac{3}{4} g_5^2,$$

$$r_3 = g_5 - 2 g_2 g_4 - \frac{3}{2} g_3^2 + \frac{9}{2} g_5^2 g_3 - \frac{15}{8} g_5^4.$$


When substituted into (4.14) they yield:

\[
\gamma_\parallel = \frac{d}{2} (2g_3 - g_2^2) - bw, 
\]

(4.15)

\[
\beta_w = -\xi w + w \frac{a}{2} (2g_3 - g_2^2) - bw^2, 
\]

(4.16)

\[
\beta_2 = \left( -\frac{\varepsilon}{2} - \frac{5}{4} bw \right) g_2 + a \left( -g_4 + \frac{11}{4} g_2 g_3 - \frac{11}{8} g_3^2 \right), 
\]

(4.17)

\[
\beta_3 = \left( -\varepsilon - \frac{3}{2} bw \right) g_3 + a \left( -g_5 + 2g_2 g_4 + 3g_2^2 - \frac{21}{4} g_2^2 g_3 + \frac{15}{8} g_3^2 \right). 
\]

(4.18)

(We recall that we have to admit \( g_n \sim g_2^{n-1} \) for the sake of consistency of the approximation). These two examples—\( \beta_2 \) and \( \beta_3 \)—give us the general form of the functions (4.14).

5. Attractors and critical exponents

Let us turn to the complete system (4.13) and (4.14) of the \( \beta \)-functions. The fixed points of RG equations can be found from the requirement that \( \beta_w(w_n, g_n) = 0, \beta_k(w_n, g_n) = 0, n = 2, 3, \ldots \). The first equation \( \beta_w(w_n, g_n) = 0 \) has two solutions: \( w_n^{(1)} = 0 \) and \( w_n^{(2)} = (-\xi + a(2g_3 - g_2^2)/2)/b \). The explicit form of the \( \beta \)-functions (4.16)–(4.18) shows that we can choose the coordinates \( g_2 \) and \( g_3 \), arbitrarily, while all the other \( g_n \) with \( n \geq 4 \) are then uniquely determined from the equations \( \beta_k(g_n) = 0, k \geq 3 \). Instead of a set of a fixed points in the infinite-dimensional space of the couplings \( \{w, g\} \equiv \{w_n, g_n\} \), the RG-equation (3.9) has two two-dimensional surfaces of fixed points, parametrized by the values of \( g_2 \) and \( g_3 \), with either \( w_n^{(1)} = 0 \) or \( w_n^{(2)} = (-\xi + a(2g_3 - g_2^2)/2)/b \). The former surface corresponds to IR asymptotic regime, where turbulent mixing is irrelevant; it coincides with the surface obtained in [30].

In general, it is difficult to establish the character of these fixed points. According to the general rule [43], a point \( w^*, g^* \equiv \{w^*, g^*_n\} \) is IR stable if the real parts of all the eigen-numbers of the matrix \( \omega_{ij} = \partial \beta_k / \partial w_i \bigg|_{w^*,g^*} \) (where \( \omega_{11} = \partial \beta_{w} / \partial w \bigg|_{w^*,g^*} \) are strictly positive. The requirement that all the diagonal elements \( \omega_{kk} \) be positive is the necessary condition for IR-stability. Equations (4.13), (4.14) yield these elements for all values of \( k \):

\[
\omega_{11} = -\xi + \frac{a}{2} \left[ 2g_3 - g_2^2 \right] - 2bw, 
\]

\[
\omega_{22} = -\xi + a \left[ \frac{11}{4} g_3 - \frac{33}{8} g_2^2 \right] - \frac{5}{4} bw, 
\]

\[
\omega_{33} = -\xi + a \left[ 6g_3 - \frac{21}{4} g_2^2 \right] - \frac{3}{2} bw, 
\]

and for \( n \geq 4 \) we have...
\[
\omega_{\text{in}} = -\varepsilon \frac{n-1}{2} + a \frac{(n+1)^2 + 2}{4} \delta_{3z} - a \frac{n(3n+4)}{8} \delta_{2z} - \frac{n+3}{4} b_{\text{water}}.
\]

For the most realistic values of \(\varepsilon\) and \(\xi\) (0 and 2), regions where all these quantities are positive exist. However, this is just a necessary condition; still, we can assume that the surfaces of fixed points \(u, g\) contain regions of IR stability. If this is indeed so, the model exhibits IR scaling with nonuniversal critical dimensions, (i.e., they depend on the the parameters \(g_{2,\ast}\) and \(g_{3,\ast}\)).

In dynamic models of the type (2.1) the critical exponents \(\Delta_F\) of an arbitrary quantity \(F\) (a field or a parameter) are given by the following expression (for detailed explanation see, e.g., [35]):

\[
\Delta_F = d^F + d^\bot \Delta_{\text{IR}} + d^\ast \Delta_{\ast} + \gamma^\ast_F, \quad \Delta_{\ast} = 2 - \gamma_{\bot}^\ast, \quad \Delta_{\ast} = 1 + \gamma^\ast_2/2. \quad (5.1)
\]

For \(F = h\) we have \(\gamma^\ast_h = 0\) and \(\gamma_{\bot}^\ast = 0\) (the fields and the parameter \(\nu_\bot\) are not renormalized). Relations (5.1) together with the table 1 yield the exact result \(2\Delta_{\ast} = d - 1 + \Delta_{\text{IR}} - \Delta_\ast\); from (4.15) we find that \(\Delta_{\text{IR}} = 1 + (a(2\delta_{3,\ast} - \delta_{2,\ast})b_{\text{water}})/4 - b_{\text{water}}/2\), \(\Delta_{\ast} = a(2\delta_{3,\ast} - \delta_{2,\ast})b_{\text{water}})/8 - b_{\text{water}}/4\) in the one-loop approximation.

6. Conclusion

We applied the standard field theoretic RG to the model of landscape erosion (proposed in [14, 15] and modified in [30]) subjected to advection by anisotropic velocity ensemble [37]. It turned out that the model could be reformulated as a multiplicatively renormalizable field theoretic model with an infinite set of independent renormalization constants (thus, infinite set of coupling constants). Usually, this means that the theory has no predictive power; however, it turns out that the one-loop counterterm can still be derived by employing the method earlier proposed in [30, 40] for an isotropic model of surface roughening. The method yields two two-dimensional surfaces of fixed points; one of them corresponds to IR asymptotic regime, where turbulent mixing is irrelevant, and coincides with the surface obtained in [30]. This means that the turbulence influence leads to the appearance of two distinct modes of behavior: a mode where the velocity field is IR irrelevant (in the sense of Wilson) and the mode when it is IR relevant.

These surfaces of fixed points are likely to contain IR attractive region(s). If that is the case, then the model exhibits scaling behavior. Indeed, experimental results [26–29] indicate two wide ranges of roughening exponent value which might be explained by the existence of two different IR attractive regions. The corresponding scaling exponents turn out to be nonuniversal because of their dependence on the coordinates of specific fixed point on the surfaces. Nonetheless, they satisfy a certain exact universal relation, that, in principle, can be tested experimentally. This result is similar to the one in [30], meaning that addition of the velocity field, while giving rise to a new type of behavior, preserved the lack of universality of scaling exponents and existence of universal relations for them. This non-universality of exponents is the distinct difference between other models of erosion [5–18] and the one proposed here.

As the modified anisotropic model of erosion has a fixed tilt of the slope it would be interesting to consider the model with random tilt, described by various statistics. From a more theoretical point of view, it is desirable to write down the RG equations and find the attractors directly in terms of the function \(V(h)\), so that, instead of infinitely many \(\beta\) functions for the couplings, we would have the only functional \(\beta(V)\) with the only functional argument \(V(h)\); see the discussion in [46] for a general case. This work remains to be completed in the future and is partly in progress.
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