THE SPECTRA OF GENERALIZED PALEY GRAPHS AND OF THEIR ASSOCIATED IRREDUCIBLE CYCLIC CODES

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Abstract. For $p = p^m$ with $p$ prime and $k | q - 1$, we consider the generalized Paley graph $\Gamma(k, q) = Cay(\mathbb{F}_q, R_k)$, with $R_k = \{x^k : x \in \mathbb{F}_q^*\}$, and the irreducible $p$-ary cyclic code $C(k, q) = \langle (\text{Tr}_{q/p}(\gamma^{jk} \cdot 1))_{\gamma \in \mathbb{F}_q^*} \rangle$, with $\omega$ a primitive element of $\mathbb{F}_q$ and $n = \frac{q - 1}{k}$. We compute the spectra of $\Gamma(k, q)$ in terms of Gaussian periods and give $Spec(\Gamma(k, q))$ explicitly in the semiprimitive case. We then show that the spectra of $\Gamma(k, q)$ and $C(k, q)$ are mutually determined by each other if further $k | \frac{q - 1}{k}$. Also, we use known characterizations of generalized Paley graphs which are cartesian decomposable to explicitly compute the spectra of the corresponding associated irreducible cyclic codes. As applications, we give reduction formulas for the number of rational points in Artin-Schreier curves and to the computation of Gaussian periods.

1. Introduction

We will assume henceforth that $q = p^m$ for some natural $m$ with $p$ prime and $k$ a positive integer such that $k | q - 1$.

Generalized Paley graphs. If $G$ is a group and $S$ is a subset of $G$ not containing $0$, the associated Cayley graph $\Gamma = X(G, S)$ is the digraph with vertex set $G$ and where two vertices $u, v$ form a directed edge from $u$ to $v$ in $\Gamma$ if and only if $v - u \in S$. If $S$ is symmetric ($S = -S$), then $X(G, S)$ is a simple (undirected) graph. The generalized Paley graph is the Cayley graph

$$\Gamma(k, q) = X(\mathbb{F}_q, R_k) \text{ with } R_k = \{x^k : x \in \mathbb{F}_q^*\}$$

($GP$-graph for short). That is, $\Gamma(k, q)$ is the graph with vertex set $\mathbb{F}_q$ and two vertices $u, v \in \mathbb{F}_q$ are neighbors (directed edge) if and only if $v - u = x^k$ for some $x \in \mathbb{F}_q^*$.

These graphs are denoted $GP(q, \frac{q - 1}{k})$ in [11]. Notice that if $\omega$ is a primitive element of $\mathbb{F}_q$, then $R_k = \langle \omega^k \rangle$, and this implies that $\Gamma(k, q)$ is a $\frac{q - 1}{k}$-regular graph. The graph $\Gamma(k, q)$ is simple if $q$ is even or if $k | \frac{q - 1}{k}$ for $p$ odd, and it is connected if $\frac{q - 1}{k}$ is a primitive divisor of $q - 1$ (i.e. $\frac{q - 1}{k}$ does not divide $p^a - 1$ for any $a < m$).

For $k = 1, 2$ we get the complete graph $\Gamma(1, q) = K_q$ and the classic Paley graph $\Gamma(2, q) = P(q)$. We will also consider the complementary graph $\hat{\Gamma}(k, q) = X(\mathbb{F}_q, R_k \setminus \{0\})$.

Let $\alpha$ be a primitive element of $\mathbb{F}_q^m$ and consider the cosets $R^{(j)}_k = \alpha^j R_k$ for $0 \leq j \leq n$. If we put $\Gamma^{(j)}(k, q) = \Gamma(\mathbb{F}_q^m, R^{(j)}_k)$ then we have the disjoint union $\hat{\Gamma}(k, q) = \Gamma^{(1)}(k, q) \cup \cdots \cup \Gamma^{(n-1)}(k, q)$ where $\Gamma^{(j)}(k, q) \simeq \hat{\Gamma}(k, q)$ for every $1 \leq j \leq n - 1$ (same proof as in Lemma 4.2 in [14]).

The spectrum of a graph $\Gamma$, denoted $Spec(\Gamma)$, is the spectrum of its adjacency matrix $A$ (i.e. the set of eigenvalues of $A$ counted with multiplicities). If $\Gamma$ has distinct eigenvalues $\lambda_0, \ldots, \lambda_t$ with multiplicities $m_0, \ldots, m_t$, we write as usual $Spec(\Gamma) = \{[\lambda_0]^{m_0}, \ldots, [\lambda_t]^{m_t}\}$. It is well-known that an $n$-regular graph $\Gamma$ has $n$ as one of its eigenvalues, with multiplicity equal to the number of connected components of $\Gamma$. There are few cases of known spectrum of $GP$-graphs. For instance unitary Cayley graphs over rings $X(R, R^*)$, where $R$ is a finite abelian ring and $R^*$ is the group of units (see [1], this includes the cases $X(\mathbb{Z}_n, \mathbb{Z}_n^*)$ and $X(\mathbb{F}_q^m, S\ell)$ with $S\ell = \{x^{q^d+1} : x \in \mathbb{F}_q^*\}$.)
where \( \ell \mid m \) (see [14], this includes the classical Paley graphs \( P(q) \)). We will compute \( \text{Spec}(\Gamma(k, q)) \), which includes \( X(F_{q^m}, S_\ell) \) since \( S_\ell = R_{q^\ell+1} \).

**Irreducible cyclic codes.** A linear code of length \( n \) over \( \mathbb{F}_q \) is a vector subspace \( \mathcal{C} \) of \( \mathbb{F}_q^n \). The weight of a codeword \( c = (c_0, \ldots, c_{n-1}) \) is the number \( w(c) \) of its nonzero coordinates. The spectrum of \( \mathcal{C} \), denoted \( \text{Spec}(\mathcal{C}) = (A_0, \ldots, A_n) \), is the sequence of frequencies \( A_i = \# \{ c \in \mathcal{C} : w(c) = i \} \). A linear code \( \mathcal{C} \) is cyclic if for every \( (c_0, \ldots, c_{n-1}) \) in \( \mathcal{C} \) the shifted codeword \( (c_1, \ldots, c_{n-1}, c_0) \) is also in \( \mathcal{C} \). An important subfamily of cyclic codes is given by the irreducible cyclic codes. For \( k \mid q - 1 \) we will be concerned with the weight distribution of the irreducible cyclic codes

\[
\mathcal{C}(k, q) = \{ c_{\gamma} = (\text{Tr}_{q/p}(\gamma \omega^{ki}))_{i=0}^{n-1} : \gamma \in \mathbb{F}_q \}
\]

where \( \omega \) is a primitive element of \( \mathbb{F}_q \) and \( n = \frac{q-1}{k} \). \( \mathcal{C}(k, q) \) is the set of codewords of length \( n \). \( \omega \) is a primitive element of \( \mathbb{F}_q \) and \( \mathcal{C}(k, q) \) is the coset in \( \mathbb{F}_q \) of the subgroup \( \langle \omega^N \rangle \) of \( \mathbb{F}_q^* \). From Theorem 14 in [9], we have the following integrality results:

\[
\eta_i^{(N, q)} \in \mathbb{Z} \quad \text{and} \quad N\eta_i^{(N, q)} + 1 \equiv 0 \pmod{p}.
\]

The spectra of irreducible cyclic codes with few weights are known. In 2002, Schmidt and White conjectured a characterization of 2-weights codes. In 2007, Wolfmann and Vega characterized all 1-weight irreducible cyclic codes ([20]) and found a characterization of all projective 2-weights codes (not necessarily irreducible). We now list several cases where the spectra of irreducible cyclic codes are known. The survey of Ding and Yang ([9]) summarizes all the results on spectra of irreducible cyclic codes until 2013. If \( q = p^m \), \( s = q^j \) and \( N \) as in (1.3), then we have:

- **Conditions on \( k \) and \( n \):**
  - (a) the semiprimitive case: \( k \mid q^j + 1 \) and \( j \mid \frac{1}{2} \) ([2], [6], [12]);
  - (b) \( n \) is a prime power ([17]);
  - (c) \( k \) is a prime with \( k \equiv 3 \pmod{4} \) and \( \text{ord}_q(k) = \frac{k-1}{2} \) ([3]).

- **Small values of \( N \):** \( k \mid s - 1 \) and
  - (a) \( N = 1 \) for all \( q \) ([7], [9]);
  - (b) \( N = 2 \) for all \( q \) ([2], [9]);
  - (c) \( N = 3 \) for all \( q \), with \( p \equiv 1 \pmod{3} \) ([7], [8], [9]) or \( p \equiv 2 \pmod{3} \) and \( mt \) even ([9]);
  - (d) \( N = 4 \) and \( p \equiv 1 \pmod{4} \) ([7], [8], [9]).

- **Few weights:**
  - (a) 1-weight codes ([20], [9]);
  - (b) 2-weight codes ([19], [9]);
  - (c) some 3-weight codes ([8]).

**Outline and results.** We now give a brief summary of the results of the paper. In Section 2 we compute the spectrum of \( \Gamma(k, q) \) and of its complement \( \overline{\Gamma}(k, q) \) in terms of Gaussian periods (Theorem 2.1). If further \( k \mid \frac{q-1}{p-1} \), both such spectra are integral. In the next two sections we consider the case of (see Definition 3.1) a semiprimitive pair \( (k, q) \). In Theorem 3.3 we deduce explicit expressions for the spectra of \( \Gamma(k, q) \) and \( \overline{\Gamma}(k, q) \). From this, we obtain that these graphs are strongly regular, so we give their srg-parameters and their intersection arrays as distance
regular graphs, and we also prove that they are Latin square graphs in half of the cases (see Proposition 3.5). In Section 4 we study the property of being Ramanujan. In Proposition 4.1, we give a partial characterization of all semiprimitive pairs \((k, q)\) such that \(\Gamma(k, q)\) is Ramanujan, showing that \(k\) can only take the values \(2, 3, 4\) or \(5\).

In the following section, we show that the spectrum of the graph \(\Gamma(k, q)\) is closely related with the corresponding one for the code \(C(k, q)\). In Theorem 5.1, we show that the eigenvalue \(\lambda_\gamma\) of \(\Gamma(k, q)\) and the weight \(c_\gamma\) the code \(C(k, q)\) satisfy the expression

\[
\lambda_\gamma = \frac{q^1 - 1}{k} - \frac{p}{p-1} w(c_\gamma).
\]

As a consequence, in Corollary 5.4 we get a lower bound for the minimum distance \(d\) of \(C(k, q)\) in the case that \(\Gamma(k, q)\) is Ramanujan. In Section 6, using this result and the known weight distribution of the codes \(C(3, q)\) and \(C(4, q)\), we compute the spectra of \(\Gamma(3, q)\) and \(\Gamma(4, q)\).

Next, in Section 7, we consider the weight distribution of \(C(k, q)\) associated to graphs \(\Gamma(k, q)\) which are cartesian decomposable. More precisely, if \(\Gamma = \square\Gamma_0\), in Theorem 7.2 we show that the weight distribution of \(\Gamma\) can be computed from the weight distribution of the codes \(C_0\) associated to \(\Gamma_0\). We then obtain two applications of this reduction result: in the first one we express the number of rational points of an Artin-Schreier curve over a field \(\mathbb{F}_{p^n}\) in terms of linear combinations of the number of rational points of similar curves over a subfield \(\mathbb{F}_{p^r}\) (Corollary 7.4). The second application is given in the next section, where we present an expression for Gaussian periods in terms of Gaussian periods of smaller parameters (Corollary 8.1). In the particular case that the smaller pair is semiprimitive, we get the simple explicit expression of Proposition 8.2.

In Section 9, we obtain the weight distribution of cyclic codes constructed from \(1\) and \(2\)-weight irreducible cyclic codes. In the case of \(2\)-weight codes, they are of three different kind: subfield, semiprimitive and exceptional. We leave out the subfield subcodes since they do not satisfy the required conditions. The semiprimitive case is considered in Proposition 9.1. The exceptional case is treated separately in the last section, where we compute the spectrum of \(\Gamma(k, q)\) and \(C(k, q)\) for the eleven exceptional pairs (Theorem 10.1). As in Section 9, one can use this to compute the weight distributions of irreducible cyclic codes constructed from these exceptional ones.

2. The spectrum of generalized Paley graphs and Gaussian periods

Here, we compute the spectrum of \(\Gamma(k, q)\) and of its complement \(\bar{\Gamma}(k, q) = X(\mathbb{F}_q, R_k^r \setminus \{0\})\), in terms of Gaussian periods. Let \(n = \frac{q^1 - 1}{k}\) and \(\eta_0 = \eta_0^{(N,q)}, \ldots, \eta_{k-1} = \eta_{k-1}^{(N,q)}\) be the Gaussian periods as in (1.4). Also, let \(\eta_1, \ldots, \eta_s\) denote the different Gaussian periods non equal to \(n\) and, for \(0 \leq i \leq k-1\), define the following numbers

\[
\mu = \# \{0 \leq i \leq k-1 : \eta_i = n\} \geq 0 \quad \text{and} \quad \mu_i = \# \{0 \leq j \leq k-1 : \eta_j = \eta_i\} \geq 1.
\]

We now show that, under mild conditions, both \(GP\)-graphs as well as their complements are what we call \(GP\)-spectral, that is their spectra are determined by Gaussian periods.

**Theorem 2.1.** Let \(q = p^m\) with \(p\) prime and \(k \in \mathbb{N}\) such that \(k \mid q-1\) and also \(k \mid \frac{q^1 - 1}{2}\) if \(p\) is odd. If we put \(n = \frac{q^1 - 1}{k}\) then, in the previous notations, we have

\[
\text{Spec}(\Gamma(k, q)) = \{[n]^{1+\mu n}, [\eta_1]^{\mu_1 n}, \ldots, [\eta_s]^{\mu_s n}\}
\]

and \(\text{Spec}(\bar{\Gamma}(k, q)) = \{[(k-1)n]^{1+\mu n}, [-1-\eta_1]^{\mu_1 n}, \ldots, [-1-\eta_s]^{\mu_s n}\}\), where \(\eta_{k-q}^{(k,q)}\) are the Gaussian periods as in (1.4). Moreover, we have:

(a) \(\Gamma(k, q)\) and \(\bar{\Gamma}(k, q)\) are connected if and only if \(\mu = 0\).

(b) If \(k \mid \frac{q^1 - 1}{p}\) then \(\text{Spec}(\Gamma(k, q))\) and \(\text{Spec}(\bar{\Gamma}(k, q))\) are integral.
Proof. We first compute the eigenvalues of $\Gamma(k, q)$. It is well-known that the spectrum of a Cayley graph $X(G, S)$ is determined by the irreducible characters of $G$. If $G$ is abelian, each irreducible character $\chi$ of $G$ induces an eigenvalue of $X(G, S)$ by the expression

\begin{equation}
\chi(S) = \sum_{g \in S} \chi(g)
\end{equation}

with eigenvector $v_\chi = (\chi(g))_{g \in G}$.

For $\Gamma(k, q)$ we have $G = \mathbb{F}_q$ and $S = R_k$. The irreducible characters of $\mathbb{F}_q$ are $\{\chi_\gamma\}_{\gamma \in \mathbb{F}_q}$ where

\begin{equation}
\chi_\gamma(y) = \zeta_p^{Tr_{q/p}(\gamma y)}
\end{equation}

for $y \in \mathbb{F}_q$. Thus, since $R_k = \langle \omega^k \rangle = C_{0,q}^k$, the eigenvalues of $\Gamma(k, q)$ are

\begin{equation}
\lambda_\gamma = \chi_\gamma(R_k) = \sum_{y \in R_k} \chi_\gamma(y) = \sum_{y \in C_{0,q}^k} \zeta_p^{Tr_{q/p}(\gamma y)}.
\end{equation}

We have $\mathbb{F}_q = \{0\} \cup C_{0,q}^1 \cup \cdots \cup C_{0,q}^{k-1}$, a disjoint union, and $\#C_{i,q}^{(k,q)} = \#(\omega^k) = \frac{q^i - 1}{k}$ for every $i = 0, \ldots, k - 1$. For $\gamma = 0$ we have

\[ \lambda_0 = \chi_0(R_k) = |R_k| = n, \]

since $\chi_0$ is the principal character. This is in accordance with the fact that since $\Gamma(k, q)$ is $n$- regular with $n = \frac{q^i - 1}{k}$, then $n$ is an eigenvalue of $\Gamma(k, q)$. If $\gamma \in C_{i,q}^{(k,q)}$ then $\gamma y$ runs over $C_{i,q}^{(k,q)}$ when $y$ runs over $C_{0,q}^k$ and thus, by (2.5), we have

\[ \lambda_\gamma = \sum_{x \in C_{i,q}^{(k,q)}} \zeta_p^{Tr_{q/p}(x)} = \eta_i^{(k,q)} \]

which does not depend on $\gamma$.

Let $\eta_1, \ldots, \eta_s$ be the different Gaussian periods. Notice that each $\gamma \in C_{i,q}^{(k,q)}$ gives the same $\lambda_\gamma$ and $|C_{i,q}^{(k,q)}| = |C_{0,q}^{(k,q)}| = n$. Thus, it is clear that the multiplicity of $\lambda_\gamma$ is

\[ m(\lambda_0) = 1 + \sum_{0 \leq j \leq k-1} |C_j^{(k,q)}| \quad \text{and} \quad m(\lambda_\gamma) = \sum_{0 \leq j \leq k-1} |C_j^{(k,q)}| \] (for $\gamma \neq 0$),

that is $m(n) = 1 + \mu n$ and $m(\eta_i) = \mu_i n$ for $1 \leq l \leq s$.

If $A$ is the adjacency matrix of $\Gamma(k, q)$ then $J - A - I$ is the adjacency matrix of $\tilde{\Gamma}(k, q)$, where $J$ stands for the all $1$’s matrix. Since $\Gamma(k, q)$ is $n$-regular with $q$ vertices, then $\tilde{\Gamma}(k, q)$ is $(q - n - 1)$-regular, that is $\tilde{\lambda}_0 = q - n - 1 = (k - 1)n$. The remaining eigenvalues of $\tilde{\Gamma}(k, q)$ are $-1 - \lambda$ where $\lambda$ are the non-trivial eigenvalues, and hence the result follows by (2.2).

It remains to show items (a) and (b).

(a) Being $n$-regular, $\Gamma(k, q)$ is connected if and only if the multiplicity of $n$ is $1$, i.e. if $\mu = 0$.

(b) Expression (2.2) gives the spectra of $\Gamma(k, q)$ and $\tilde{\Gamma}(k, q)$ in terms of the Gaussian periods $\eta_i^{(N,q)}$ for $k \mid \frac{q^i - 1}{p - 1}$. If $k$ satisfies $k \mid \frac{q^i - 1}{p - 1}$ then $k = N$, by (1.3), and hence all the Gaussian periods $\eta_i^{(k,q)}$ are integers, by (1.5), which clearly implies that $Spec(\Gamma(k, q))$ is integral. The same happens for $Spec(\tilde{\Gamma}(k, q))$, and the result follows. \(\square\)

Remark 2.2. If all the Gaussian periods are different, $\eta_i \neq \eta_j$ for $0 \leq i < j \leq k - 1$, then $Spec(\Gamma(k, q)) = \{[\eta_1]^n, [\eta_0]^n, [\eta_1]^n, \ldots, [\eta_{k-1}]^n\}$. This holds, for instance, for Paley graphs $P(q) = \Gamma(2, q)$, as one can see in (2.8) in Example 2.4, and also for $\Gamma(3, q)$ and $\Gamma(4, q)$ in the non-semiprimitive case (see Theorems 6.1 and 6.3 and (iii) in Remark 6.5).
The period polynomial is defined by \( \Psi_{(k,q)}(X) = \prod_{i=0}^{k-1} (X - \eta_{i}^{(k,q)}) \) where \( \eta_{i}^{(k,q)} \) are the Gaussian periods. In the previous notations we have the following direct consequence of Theorem 2.1.

**Corollary 2.3.** Let \( q = p^m \) with \( p \) prime and let \( k \in \mathbb{N} \) such that \( k \mid q - 1 \) and \( k \mid \frac{q-1}{p} \) if \( p \) is odd. Then, the period polynomial satisfies

\[
\Psi_{(k,q)}(X) = \frac{P_{\Gamma(k,q)}(X)}{X - n}
\]

where \( n = \frac{q-1}{k} \) and \( P_{\Gamma(k,q)}(X) \) denotes the characteristic polynomial of the graph \( \Gamma(k,q) \).

The Ihara zeta function \( \zeta_{\Gamma}(u) \) for a regular graph \( \Gamma \) has a determinantal expression in spectral terms. For a GP-graph \( \Gamma(k,q) \), we have (see (8.3) in [14] for details)

\[
\zeta_{\Gamma}(u) = \frac{(1 - u^{2})^{\eta} - \frac{p}{4}}{\prod_{i=1}^{\eta}(1 - \lambda_{i}u - (n - 1)u^{2})^{m_{i}}},
\]

where \( \{\lambda_{1}^{m_{1}}, \ldots, \lambda_{\eta}^{m_{\eta}}\} \) is the spectrum of \( \Gamma(k,q) \).

In the next example we will obtain the already known spectrum of classical Paley graphs \( P(q) \) throughout Gaussian periods.

**Example 2.4** (Paley graphs). We will use that the periods for \( k = 2 \) are known and that \( \Gamma = \Gamma(2,q) = P(q) \) for \( q \) odd. Let \( q = p^m \) with \( p \) an odd prime and \( m = 2t \) and let \( n = \frac{q-1}{2} \). We will show that \( \Gamma(2,q) \) is a connected strongly regular graph having integral spectrum

\[
\text{Spec}(\Gamma(2,q)) = \{[1]^{1}, [\eta]^{n}, [-1 - \eta]^{n}\}
\]

where \( \eta = \eta_{0}^{(2,q)} = \frac{-1 - \sqrt{\sigma}}{2} \) where \( \sigma = 1 \) if \( p \equiv 1 \mod 4 \) and \( \sigma = (-1)^t \) if \( p \equiv 3 \mod 4 \).

In fact, \( 2 \mid \frac{q-1}{2} \) since \( p \) is odd and \( m \) is even, and hence \( \text{Spec}(\Gamma) = \{[1]^{1+m_{n}}, [\eta]^{m_{n}}, [\eta]^{m_{1}}\} \) by Theorem 2.1, where \( \eta_{i} = \eta_{i}^{(2,q)} \) are the Gaussian periods for \( i = 1, 2 \). The spectrum is integral since 2 also divides \( \frac{m_{1}}{p-1}, m \) being even.

The above periods are given in Lemma 12 in [9]. In our notations, we have \( \eta_{1} = -1 - \eta_{0} \) and \( \eta_{0} = \frac{-1 + (-1)^{m-1} - \sqrt{\sigma}}{2} \) if \( p \equiv 1 \mod 4 \) and \( \eta_{0} = \frac{-1 + (-1)^{m-1} + \sqrt{\sigma}}{2} \) if \( p \equiv 3 \mod 4 \). Using that \( m = 2t \) we get that

\[
\eta_{0} = \begin{cases} 
-\frac{1 - p}{2} & \text{if } p \equiv 1 \mod 4, \\
-\frac{1 - (-1)^{t} p}{2} & \text{if } p \equiv 3 \mod 4.
\end{cases}
\]

Now, it is clear that \( \eta_{0} \neq \eta_{1} \) and \( \eta_{i} \neq n, i = 0, 1, \) and hence \( \mu = 0 \) and \( \mu_{0} = \mu_{1} = 1 \). In this way, the spectrum is as given above. This coincides with the known spectrum

\[
\text{Spec}(P(q)) = \{[\frac{q-1}{2}]^{1}, [\frac{\sqrt{q} - 1}{2}]^{n}, [-\frac{\sqrt{q} - 1}{2}]^{n}\}.
\]

From this and using (2.8) one can get the Ihara zeta function of \( P(q) \). The period polynomial is

\[
\Psi_{(2,q)}(X) = (x - \eta)^{n}(x + 1 + \eta)^{n} = (x^{2} + x - \eta(1 + \eta))^{n} = (x^{2} + x + \frac{q - \sqrt{q}}{2})^{n}.
\]

The graph \( \Gamma(2,q) \) is connected since the multiplicity of the regularity degree is 1. Since \( p \) is odd and \( m \) is even it is clear that the eigenvalues are integers. Finally, since \( \Gamma(2,q) \) is regular and connected and has exactly 3 eigenvalues it is a strongly regular graph. All these facts are of course well-known.

3. The spectrum of semiprimitive GP-graphs \( \Gamma(k,q) \)

Using that the Gaussian periods in the semiprimitive case are known, we give the spectrum of the corresponding GP-graphs \( \Gamma(k,q) \) and of their complements \( \bar{\Gamma}(k,q) \). We then use these spectra to give structural properties of the graphs \( \Gamma(k,q) \) in this case.
We recall that in the semiprimitive case one has: $k \geq 2$, $q = p^m$ with $p$ prime and
\begin{equation}
  k \mid p^t + 1 \quad \text{for some} \quad t \mid \frac{m}{2}.
\end{equation}
If $k = p^t + 1$ then the graph $\Gamma(k, q)$ is not connected (see Proposition 4.6 in [14]).

**Definition 3.1.** We say that $(k, q)$ satisfying (3.1) and $k \neq p^t + 1$ is a semiprimitive pair of integers. We will refer to $\Gamma(k, q)$ as a semiprimitive GP-graph (hence $\Gamma(k, q)$ is connected).

**Example 3.2.** (i) For instance, if $p = 3$ and $m = 4$, to find the semiprimitive pairs $(k, 81)$ we take $k \mid 3^2 + 1 = 2 \cdot 5$ and $k \mid 3^3 + 1 = 4$. Hence $k = 2, 4$ or $5$, while $k = 10$ is not allowed since $10 = 3^2 + 1$.

(ii) Three infinite families of semiprimitive pairs, for $p$ prime and $m = 2t \geq 2$, are:
- $(2, p^{2t})$ with $p$ odd,
- $(3, p^{2t})$ with $p \equiv 2 \mod 3$ and $t \geq 1$ (where $t \geq 2$ if $p = 2$),
- $(4, p^{2t})$ with $p \equiv 3 \mod 4$ and $t \geq 1$ (where $t \geq 2$ if $p = 3$).

The first of the three families of pairs give rise to the classical Paley graphs $\Gamma(2, p^{2t})$.

(iii) Another infinite family of semiprimitive pairs is given by
\begin{equation}
  (p^\ell + 1, p^m)
\end{equation}
with $p$ prime, $m \geq 2$, $\ell \mid m$ and $\frac{m}{2}$ even. They give the GP-graphs $\Gamma(q^\ell + 1, q^m)$, with $q = p$, considered in [14] for $q$ a power of $p$.

(iv) Using the previous definition and items (ii) and (iii), we give a list of the smallest semiprimitive pairs $(k, q)$ with $q = p^m$ for $p = 2, 3, 5, 7$ and $m = 2, 4, 6, 8$.

| $m = 2$ | $m = 4$ | $m = 6$ | $m = 8$ |
|---------|---------|---------|---------|
| $p = 2$ | 3       | 3       | 5       |
| $p = 3$ | 2, 4, 5 | 2, 4, 7, 14 | 2, 4, 5, 10, 41 |
| $p = 5$ | 2, 3, 6, 13 | 2, 3, 6, 7, 9, 14, 18, 21, 42, 63 |
| $p = 7$ | 2, 4, 5, 8, 10, 25 | 2, 4, 5, 8, 10, 25, 43, 50, 86, 172 | 2, 4, 5, 8, 10, 25, 50, 1201 |

Here we have marked in bold those $k$ which are different from 2 and not of the type $p^\ell + 1$ for some $p$ and $\ell$, showing that in general there are much more semiprimitive graphs $\Gamma(k, q)$ than Paley graphs or the GP-graphs of the form $\Gamma(p^\ell + 1, p^m)$.

We now give the spectrum of semiprimitive GP-graphs $\Gamma(k, q)$ explicitly. We will need the following notation. Define the sign
\begin{equation}
  \sigma = (-1)^{s + 1}
\end{equation}
where $s = \frac{m}{2}$ and $t$ is the least integer $j$ such that $k \mid p^j + 1$ (hence $s \geq 1$).

**Theorem 3.3.** Let $(k, q)$ be a semiprimitive pair with $q = p^m$, $m$ even, and put $n = \frac{2 - 1}{k}$. Then, the spectrum of $\Gamma = \Gamma(k, q)$ and $\bar{\Gamma} = \bar{\Gamma}(k, q)$ are respectively given by
\[\text{Spec}(\Gamma) = \{\left|m\right|, \left|\lambda_1\right|^n, \left|\lambda_2\right|(k-1)n\} \quad \text{and} \quad \text{Spec}(\bar{\Gamma}) = \{\left|(k - 1)n\right|, \left|\lambda_2\right|, \left|\lambda_2\right|(k-1)n\},\]
where
\begin{equation}
  \lambda_1 = \frac{\sigma(k-1)p^m - 1}{k} \quad \text{and} \quad \lambda_2 = \frac{\sigma p^m + 1}{k}
\end{equation}
with $\sigma$ as given in (3.2).
Proof. We first compute the spectrum of $\Gamma = \Gamma(k, q)$, which by Theorem 2.1 is given in terms of Gaussian periods. From Lemma 13 in [9] the Gaussian periods $\eta_j^{(k, q)}$, for $j = 0, \ldots, k - 1$, are given by:

(a) If $p, \alpha = \frac{p}{k} + 1$ and $s$ are all odd then

\begin{equation}
\eta_j^{(k, q)} = \begin{cases} 
\frac{(k-1)\sqrt{-1}}{k} & \text{if } j = \frac{k}{2}, \\
\frac{-\sqrt{-1}+1}{k} & \text{if } j \neq \frac{k}{2}.
\end{cases}
\end{equation}

(b) In any other case we have $\sigma = (-1)^{s+1}$ and

\begin{equation}
\eta_j^{(k, q)} = \begin{cases} 
\frac{\sigma(k-1)\sqrt{-1}}{k} & \text{if } j = 0, \\
\frac{-\sqrt{-1}+1}{k} & \text{if } j \neq 0.
\end{cases}
\end{equation}

Thus, by Theorem 2.1, the spectrum of $\Gamma(k, q)$ is $Spec(\Gamma(k, q)) = \{[n]^1, [\eta_j/(k-1)n]\}$ if $p, \alpha, s$ are odd or $Spec(\Gamma(k, q)) = \{[n]^1, [\eta_j]^n, [\eta_j/(k-1)n]\}$ otherwise.

Suppose we are in case (a), i.e. $p, \alpha$ and $s$ are odd. Then we have

$$\lambda_1 = \eta_{k/2} = \frac{(k-1)p\sqrt{-1}}{k}$$

and

$$\lambda_2 = \eta_j = \eta_0 = -\frac{\sqrt{-1}+1}{2}(j \neq \frac{k}{2}).$$

It is clear that $\lambda_2 \neq n$ and $\lambda_2 \neq \lambda_1$. Also, $n \neq \lambda_1$ since $k \neq p\sqrt{-1} + 1$. Thus, all three eigenvalues are different and their corresponding multiplicities are as given in the statement.

In case (b), we have $\eta_0 = \frac{\sigma(k-1)p\sqrt{-1}}{k}$ and $\eta_j = -\sigma \frac{\sqrt{-1}+1}{k}$ for $j \neq 0$. Again, one checks that $\eta_0 \neq \eta_j$, $\eta_0 \neq n$ and $\eta_j \neq n$ for every $j \neq \frac{k}{2}$. Thus, the corresponding multiplicities are as stated in the proposition.

Combining cases (a) and (b) we get (3.3), as we wanted to show. Finally, the spectrum of $\Gamma(k, q)$ follows by Theorem 2.1.

\[ \square \]

Note. Since $\lambda_1, \lambda_2 \in \mathbb{Z}$, we have that $\sigma = \pm 1$ if and only if $k \mid p\sqrt{-1} \pm 1$, respectively.

Remark 3.4. Recently, we computed the spectrum of the GP-graphs $\Gamma_{q,m}(\ell) = \Gamma(q^\ell+1, q^m)$ and $\Gamma_{q,m}(\ell)$, with $\ell \mid m$ and $p\sqrt{-1}$ even (see Theorem 3.5 and Proposition 4.3 in [14]), by using certain trigonometric sums associated to the quadratic forms

$$Q_{\gamma, \ell}(x) = \text{Tr}_{p^m/p}(\gamma x^{p^\ell+1})$$

with $\gamma \in \mathbb{F}_{p^m}^*$. By (ii) in Example 3.2, the graph $\Gamma(p^\ell+1, p^m)$, i.e. with $q = p$ prime, is semiprimitive and hence its spectrum is given by Theorem 2.1. Indeed, $Spec(\Gamma(p^\ell+1, p^m)) = \{[n]^1, [\lambda_1]^n, [\lambda_2]^m\}$ where

\begin{equation}
\begin{align*}
n &= \frac{p^m - 1}{p^\ell + 1}, \\
\lambda_1 &= \frac{2p^{2\ell}+1}{p^\ell + 1}, \\
\lambda_2 &= -\frac{2p^{2\ell}+1}{p^\ell + 1},
\end{align*}
\end{equation}

with $\sigma = (-1)^{2\ell+1}$. It is reassuring that both computations of the spectrum coincide after using these two different methods. The same happens for the complementary graphs.

Some graph invariants are given directly in terms of the spectrum $\lambda_0, \lambda_1, \ldots, \lambda_{q-1}$ of $\Gamma$. For instance, the energy $E(\Gamma) = \sum_i |\lambda_i|$ of $\Gamma$ and the number of walks $w_r(\Gamma) = \sum_i \lambda_i^r$ of length $r$ in $\Gamma$. Also, if $\Gamma$ is connected and $n$-regular, the number of spanning trees of $\Gamma$ is given by $t(\Gamma) = \frac{1}{q} \eta_1 \cdots \eta_{q-1}$, where $\eta_i = n - \lambda_i$ are the Laplace eigenvalues for $i = 1, \ldots, q - 1$. In our case, after some straightforward calculations we have

\begin{equation}
\begin{align*}
E(\Gamma(k, q)) &= \{2(p\sqrt{-1} + \sigma \lambda_2) + 1 + \sigma\}, \\
w_r(\Gamma(k, q)) &= n \{n^{r-1} + \sigma \{p\sqrt{-1} + \sigma \lambda_2\}^r + (k-1)(\sigma \lambda_2)^r\}, \\
t(\Gamma(k, q)) &= (-\sigma)^{q-1} p^{m(q-3)}(\lambda_2 + 1)^n \lambda_2^{(k-1)n}.
\end{align*}
\end{equation}
Now, we will give some structural properties of the graphs \( \Gamma(k, q) \) throughout the spectrum.

**Proposition 3.5.** Let \((k, q)\) be a semiprimitive pair with \( q = p^m \), \( m = 2t \)s where \( t \) is the least integer satisfying \( k \mid p^t + 1 \) and put \( n = \frac{q - 1}{k} \). Then we have:

(a) \( \Gamma(k, q) \) and \( \bar{\Gamma}(k, q) \) are primitive, non-bipartite, integral, strongly regular graphs with corresponding parameters \( \text{sr}(q, n, e, d) \) and \( \text{sr}(q, (k - 1)n, e', d') \) given by

\[
e = d + (\sigma p^m + 2\lambda_2), \quad d = n + (p^m + \lambda_2)\lambda_2, \quad e' = q - 2 - 2n + d, \quad d' = q - 2n + e.
\]

(b) \( \Gamma(k, q) \) and \( \bar{\Gamma}(k, q) \) are distance regular graphs of diameter 2 with intersection arrays

\[
\mathcal{A} = \{n, n - e - 1; 1, d\} \quad \text{and} \quad \bar{\mathcal{A}} = \{(k - 1)q, n - d; 1, q - 2n + e\}.
\]

(c) If \( s \) is odd then \( \Gamma(k, q) \) and \( \bar{\Gamma}(k, q) \) are Latin square graphs with parameters

\[
L_s(w) = \text{sr}(w^2, \delta(w - 1), \delta^2 - 3\delta + w, \delta(\delta - 1)),
\]

where \( w = f - g \), \( \delta = -g \) and \( f > 0 > g \) are the non-trivial eigenvalues of \( \Gamma(k, q) \) or \( \bar{\Gamma}(k, q) \).

**Proof.** We will use the spectral information from Proposition 3.3. We prove first the results for \( \Gamma(k, q) \).

(a) Since the multiplicity of the degree of regularity \( n \) is 1, the graph is connected. Also, one can check that \( -n \) is not an eigenvalue of \( \Gamma(k, q) \) and hence the graph is non-bipartite. Now, since \( k \mid p^t + 1 \) then \( k \mid p^t + 1 \) if \( s \) is odd and \( k \mid p^t - 1 \) if \( s \) is even, hence \( \beta = \frac{p^t + \sigma}{k} \) is an integer. Hence, since \( \lambda_1 = \sigma(p^{ts} - \beta) \) and \( \lambda_2 = -\sigma\beta \), by (3.3), the eigenvalues are all integers (we also know this from Theorem 2.1).

Finally, since the graph is connected, \( n \)-regular with \( q \)-vertices and has exactly 3 eigenvalues, it is a strongly regular graph with parameters \( \text{sr}(q, n, e, d) \). We now compute \( e \) and \( d \). It is known that the non-trivial eigenvalues of an \( \text{sr} \) graph are of the form \( \lambda^\pm = \frac{1}{2} \{(e - d) \pm \Delta\} \) where \( \Delta = \sqrt{(e - d)^2 + 4(n - d)} \). Thus, \( d = n + \lambda^+\lambda^- \) and \( e = d + \lambda^+\lambda^- \). From this and (3.3) the result follows.

(b) Similar to the proof of Corollary 5.3 in [14].

(c) Note that the regularity degree of \( \Gamma = \Gamma(k, q) \) equals the multiplicity of a non-trivial eigenvalue by Theorem 3.3. Thus, \( \Gamma \) is of pseudo-Latin square type graph (PL), of negative Latin square type (NL) or a conference graph (see Proposition 8.14 in [5]). By definition, a conference graph satisfy \( 2n + (q - 1)(e - d) = 0 \). It is easy to check that this condition holds for \( \Gamma(k, q) \) if and only if \( \Gamma(1, 4) = K_4 \), and hence \( \Gamma \) is not a conference graph. Put \( w = f - g \), where \( f, g \) are the non-trivial eigenvalues with \( f > 0 > g \), and \( \delta = -g \). Then, \( \Gamma \) is a pseudo-Latin square graph with parameters

\[
\text{PL}_{\delta}(w) = \text{sr}(w^2, \delta(w - 1), \delta^2 - 3\delta + w, \delta(\delta - 1))
\]

or a negative Latin square graph with parameters

\[
\text{NL}_{\delta}(w) = \text{sr}(w^2, \delta(w + 1), \delta^2 + 3\delta - w, \delta(\delta + 1)).
\]

It is clear that \( n = \delta(w - 1) \) if and only if \( s \) is odd and that for \( s \) even \( k \neq \delta(w + 1) \).

Thus, we are only left to prove that in the case of \( s \) odd, \( \Gamma \) is actually a Latin square graph. Hence, it is enough to show that \( \Gamma \) is geometric, that is, it is the point graph of a partial geometry \( pg(a, b, \alpha) \) with parameters \( \text{sr}(a + 1)(ab + \alpha)\alpha^{-1}, (b + 1)a, a - 1 + b(\alpha - 1), (b + 1)\alpha) \). To see this, we use a result of Neumaier (Theorem 7.12 in [5]) asserting that if \( g < -1 \) is integer and

\[
f + 1 \leq \frac{1}{2}g(g + 1)(d + 1)
\]

then \( \Gamma \) is the point graph of a partial geometry \( pg(a, b, \alpha) \) with \( \alpha = b \) or \( b + 1 \). We know that for \( s \) odd we have that \( g < -1 \) is integer. Expression (3.10) takes the form

\[
k - 1 \leq -\frac{1}{2}(\sigma\lambda_2 + 1)(n + 1 + (\sigma p^m + \lambda_2)\lambda_2).
\]
By straightforward calculations, this inequality is equivalent to
\[
(3.11) \quad 2k^3(k - 1) + k(2p^n + 4p\overline{p} - 2kp\overline{p} + 2 - 2k + k^2) \leq p^\frac{3n}{2} + 3p^n + 3p\overline{p} + 1.
\]
Notice that the l.h.s. of (3.11) increases as k does. By hypothesis \( s > 1 \) is odd, thus \( k \leq K := p\overline{p} + 1 \) since \( s \) satisfies \( k \mid p\overline{p} + 1 \) and \( s \geq 3 \). Therefore, the l.h.s. of (3.11) is less than
\[
C_K := 2K^4 - K^3 + 2Kp^n + (2 - K)2Kp\overline{p} + 2K(1 - K).
\]
It is not difficult to show that \( C_K \) is less than the right hand side of (3.11). Therefore, (3.11) is true in this case, and this implies that \( \Gamma \) satisfies (3.10) as required.

Now, it is easy to see that \( \bar{\Gamma}(k, q) \) is also a primitive non-bipartite integral strongly regular graph with parameters and intersection array as stated. The proof that \( \bar{\Gamma}(k, q) \) is a Latin square if \( s \) is odd is analogous to the previous one for \( \Gamma(k, q) \) and we omit the details. Since \( \bar{\Gamma}(k, q) \) is a pseudo-Latin graph \( PL_5(\delta) \) then \( \bar{\Gamma}(k, q) \) is a pseudo-Latin graph \( PL_5(\delta) \) with \( \delta' = u + 1 - \delta \). Thus, it is enough to check that \( \bar{\Gamma}(k, q) \) is geometric, that is, that inequality (3.10) holds in this case also with \( g' \) in place of \( g \). This completes the proof.

\textbf{Example 3.6.} From Theorem 3.3 and Proposition 3.5 we obtain Table 2. Here \( s = \frac{m}{27} \) where \( t \) is the least integer such that \( k \mid p^t + 1 \). We mark in bold those graphs with \( k \neq p^t + 1 \) for \( p^m \).

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
graph & srg parameters & spectrum & \( t \) & \( s \) & latin square \\
\hline
\( \Gamma(3, 2^4) \) & (16, 5, 0, 2) & \( \{5\}^1, \{1\}^{10}, \{3\}^5 \) & 1 & 2 & no \\
\( \Gamma(3, 2^4) \) & (16, 10, 6, 6) & \( \{10\}^1, \{2\}^5, \{3\}^{10} \) & 1 & 2 & no \\
\( \Gamma(3, 2^5) \) & (64, 21, 8, 6) & \( \{21\}^1, \{5\}^{21}, \{3\}^{12} \) & 1 & 3 & \( L_3(8) \) \\
\( \Gamma(3, 2^5) \) & (64, 42, 26, 30) & \( \{42\}^1, \{2\}^{42}, \{6\}^{21} \) & 1 & 3 & \( L_6(8) \) \\
\( \Gamma(3, 5^2) \) & (25, 8, 3, 2) & \( \{8\}^1, \{3\}^6, \{2\}^{16} \) & 1 & 1 & \( L_5(5) \) \\
\( \Gamma(3, 5^2) \) & (25, 16, 9, 12) & \( \{16\}^1, \{3\}^{16}, \{4\}^{8} \) & 1 & 1 & \( L_4(5) \) \\
\( \Gamma(3, 5^4) \) & (625, 208, 63, 72) & \( \{208\}^1, \{8\}^{116}, \{17\}^{208} \) & 1 & 2 & no \\
\( \Gamma(3, 5^4) \) & (625, 416, 279, 272) & \( \{416\}^1, \{16\}^{208}, \{9\}^{416} \) & 1 & 2 & no \\
\( \Gamma(4, 3^4) \) & (81, 20, 1, 6) & \( \{20\}^1, \{2\}^{60}, \{7\}^{20} \) & 1 & 2 & no \\
\( \Gamma(4, 3^4) \) & (81, 60, 45, 42) & \( \{60\}^1, \{6\}^{20}, \{3\}^{60} \) & 1 & 2 & no \\
\( \Gamma(4, 3^5) \) & (729, 182, 55, 42) & \( \{182\}^1, \{20\}^{182}, \{7\}^{540} \) & 1 & 3 & \( L_7(27) \) \\
\( \Gamma(4, 3^5) \) & (729, 546, 105, 420) & \( \{546\}^1, \{6\}^{54}, \{21\}^{182} \) & 1 & 3 & \( L_3(21) \) \\
\( \Gamma(4, 7^2) \) & (49, 12, 5, 2) & \( \{12\}^1, \{5\}^{12}, \{2\}^{16} \) & 1 & 1 & \( L_2(7) \) \\
\( \Gamma(4, 7^2) \) & (49, 36, 25, 30) & \( \{36\}^1, \{1\}^{36}, \{6\}^{12} \) & 1 & 1 & \( L_6(7) \) \\
\( \Gamma(4, 7^4) \) & (2401, 600, 131, 156) & \( \{600\}^1, \{12\}^{600}, \{37\}^{1560} \) & 1 & 2 & no \\
\( \Gamma(4, 7^4) \) & (2401, 1800, 1332, 1555) & \( \{1800\}^1, \{36\}^{1800}, \{13\}^{1555} \) & 1 & 2 & no \\
\( \Gamma(5, 3^4) \) & (81, 16, 7, 2) & \( \{16\}^1, \{7\}^{16}, \{2\}^{20} \) & 2 & 1 & \( L_2(9) \) \\
\( \Gamma(5, 3^4) \) & (81, 64, 49, 56) & \( \{64\}^1, \{1\}^{64}, \{8\}^{16} \) & 2 & 1 & \( L_8(9) \) \\
\( \Gamma(5, 7^4) \) & (2401, 480, 119, 90) & \( \{480\}^1, \{39\}^{480}, \{10\}^{1920} \) & 2 & 1 & \( L_{10}(9) \) \\
\( \Gamma(5, 7^4) \) & (2401, 1920, 1560, 1529) & \( \{1920\}^1, \{9\}^{1920}, \{40\}^{480} \) & 2 & 1 & \( L_{40}(9) \) \\
\hline
\end{tabular}
\end{table}

We point out that, for instance, the graphs with \( q = 7^4 \) do not appear in the Brouwer’s lists ([4]) of strongly regular graphs.

\textbf{Remark 3.7.} (i) Notice that if we take \( h = \min\{\vert f\vert, \vert g\vert\} \), then for \( s \) even (in the previous notations), the graph \( \Gamma(k, q) \) satisfy the same parameters as in (3.9) with \( \delta \) replaced by \( h \), that is \( \Gamma(k, q) \) is a strongly regular graph with parameters, in terms of the eigenvalues, given by
\[\overline{NL} = \text{srg}(w^2, h(w + 1), h^2 + 3h - w, h(h + 1))\].

(ii) Since the complement of \( \Gamma(k, q) \) is a Latin square \( L_\delta(w) \) and its complement \( \bar{\Gamma}(k, q) \) is also a Latin square \( L_\delta(w) \) with \( \delta' = u + 1 - \delta \), by Proposition 3.5, the graphs can be constructed from
a complete set of mutually orthogonal Latin squares (MOLS). This is equivalent to the existence of an affine plane of order $w$. Since $w = f - g = p^{\frac{m}{2}}$ with $m$ even, the semiprimitive GP-graphs give a standard way to construct complete sets of MOLS of order $w$ and affine planes of order $w$, with $w$ a power of a prime.

4. RAMANUJAN SEMIPRIMITIVE GP-GRAPHS

Here we will give a partial characterization of the semiprimitive generalized Paley graphs which are Ramanujan.

If $\Gamma$ is an $n$-regular graph, then $n$ is the greatest eigenvalue of $\Gamma$. A connected $n$-regular graph is called Ramanujan if

$$\lambda(\Gamma) := \max_{\lambda \in \text{Spec}(\Gamma)} \{|\lambda| : |\lambda| \neq n\} \leq 2\sqrt{n - 1}.$$  

**Proposition 4.1.** Let $q = p^m$ with $p$ prime and let $(k, q)$ be a semiprimitive pair. If $\Gamma(k, q)$ is Ramanujan then $k \in \{2, 3, 4, 5\}$. Conversely, if $k \in \{2, 3, 4, 5\}$ then $\Gamma(k, q)$ is Ramanujan if $m$ is even and $k, p, m$ are as given in one of the following cases:

(a) $k = 2$, $p \neq 2$ and $m \geq 2$.
(b) $k = 3$, $p = 2$ and $m \geq 4$.
(c) $k = 3$, $p \neq 2$ with $p \equiv 2 \pmod{3}$ and $m \geq 2$.
(d) $k = 4$, $p = 3$ and $m \geq 4$.
(e) $k = 4$, $p \neq 3$ with $p \equiv 3 \pmod{4}$ and $m \geq 2$.
(f) $k = 5$, $p = 2$ and $m \geq 8$ with $4 \mid m$.
(g) $k = 5$, $p \neq 2$ with $p \equiv 2, 3 \pmod{5}$ and $m \geq 4$ with $4 \mid m$.
(h) $k = 5$, $p \equiv 4 \pmod{5}$ and $m \geq 2$.

Moreover, $\Gamma(k, q)$ is Ramanujan for every semiprimitive pair $(k, q)$.

**Proof.** First, notice that $\lambda(\Gamma) = |\lambda_1|$. Suppose that $\sigma = -1$. In this case, if $\Gamma$ is Ramanujan we have that

$$\frac{(k-1)p_{\frac{m}{2}}}{k} + 1 \leq 2\sqrt{n - 1} = 2\sqrt{\frac{p^m - (k + 1)^2}{k}}.$$  

This inequality is equivalent to $(k - 1)^2 p^m + 2(k - 1)p_{\frac{m}{2}} + 1 \leq 4k(p^m - (k + 1))$ and, then, we have that

$$2(k - 1)p_{\frac{m}{2}} + 4k(k + 1) + 1 \leq p^m(4k - (k - 1)^2).$$  

Since the left hand side of this inequality is positive, we have $4k - (k - 1)^2 > 0$, and this can only happen if $k \in \{1, 2, 3, 4, 5\}$. The case $k = 1$ is excluded because $(1, q)$ is not a semiprimitive pair. The case $\sigma = 1$ can be proved in a similar way, taking into account that in general the equation (4.2) takes the form

$$4k(k + 1) + 1 \leq p^m(4k - (k - 1)^2) + 2(k - 1)\sigma p_{\frac{m}{2}}.$$  

Now, assume that $k \in \{2, 3, 4, 5\}$. Notice that $k = 2$ corresponds to the classic Paley graph which is well-known to be Ramanujan. In this case $p$ is neccessarily odd since the pair $(2, 2^m)$ is not semiprimitive for any $m$. If $k = 3$, necessarily $p \equiv 2 \pmod{3}$ and $m$ is even, for if not the pair $(k, p^m)$ is not semiprimitive. In this case, (4.3) is given by

$$49 \leq 8p^m + 4\sigma p_{\frac{m}{2}}.$$  

The worst case is when $\sigma = -1$, and in this case the inequality is equivalent to

$$12 + \frac{1}{k} \leq p_{\frac{m}{2}}(2p_{\frac{m}{2}} - 1).$$  

This clearly holds for $p$ odd and $m \geq 2$ or $p = 2$ and $m \geq 4$. The remaining cases can be proved it in the same manner.
For $\Gamma(k, q)$, we have that $\lambda(\Gamma) = |\lambda_1| = (k - 1)\frac{p^m + m}{k}$ and the regularity degree of $\Gamma$ is $n(k - 1)$. Without loss of generality we can assume that $\sigma = 1$. Inequality (4.1) becomes

$$(k - 1)\frac{p^m + 1}{k} \leq 2\sqrt{\frac{p^m - 1}{k} - (k - 1)}$$

which is equivalent to $(k - 1)^2p^m + 2(k - 1)p^m + (k - 1)^2 \leq 4k(p^m(k - 1) - (2k - 1))$ and therefore we have

$$2(k - 1)p^m + (k - 1)^2 + 4k(2k - 1) \leq p^m(4k(k - 1) - (k - 1)^2).$$

Since $k \mid p^\ell + 1$ for some $\ell$ and $k \neq \frac{p^m}{2} + 1$, if the term on the right hand side between parenthesis is $> 0$ then this inequality is satisfied for any semiprimitive pair $(k, q)$. Since $4k(k - 1) - (k - 1)^2 > 0$ for all $k > 1$, then $\Gamma$ is Ramanujan for all semiprimitive pair, as desired. The case $\sigma = -1$ can be proved analogously. \hfill \Box

The previous result gives the following 8 infinite families of Ramanujan semiprimitive GP-graphs:

- (a) $\{\Gamma(2, p^2)\}_{\ell \geq 1}$ with $p$ odd (Paley graphs),
- (b) $\{\Gamma(3, 4^\ell)\}_{\ell \geq 2}$,
- (c) $\{\Gamma(3, p^2)\}_{\ell \geq 1}$ with $p \equiv 2 \mod 3$ and $p \neq 2$ (e.g. 5, 11, 17),
- (d) $\{\Gamma(4, 9^\ell)\}_{\ell \geq 2}$,
- (e) $\{\Gamma(4, p^2)\}_{\ell \geq 1}$ with $p \equiv 3 \mod 4$ and $p \neq 3$ (e.g. 7, 11, 19),
- (f) $\{\Gamma(5, 16^\ell)\}_{\ell \geq 2}$,
- (g) $\{\Gamma(5, p^4)\}_{\ell \geq 1}$, with $p \equiv 2, 3 \mod 5$ and $p \neq 2$,
- (h) $\{\Gamma(5, p^2)\}_{\ell \geq 1}$ with $p \equiv 4 \mod 5$.

Notice that five of them are valid for an infinite number of primes.

**Remark 4.2.** (i) The Ramanujan GP-graphs $\Gamma(k, q)$ with $k = p^\ell + 1$ are characterized in Theorem 8.1 in [14]. There, we proved that $\Gamma_{q,m}(\ell) = \Gamma(p^\ell + 1, p^m)$, with $\ell \mid m$ such that $m_k$ even and $\ell \neq \frac{m_k}{2}$, is Ramanujan if and only if $q = 2, 3, 4$ with $\ell = 1$ and $m \geq 4$ even. This says that $\Gamma(p^\ell + 1, p^m)$ is Ramanujan only in the cases (b), (d) and (f), giving the infinite families

$$\{\Gamma(3, 4^\ell)\}_{\ell \geq 2}, \quad \{\Gamma(4, 9^\ell)\}_{\ell \geq 2} \quad \text{and} \quad \{\Gamma(5, 16^\ell)\}_{\ell \geq 2}$$

of Ramanujan graphs. The first two families coincide with those in (b) and (d), while the third one gives just half the graphs in (f), precisely those with $t$ even in (f). Thus, the last proposition extends this characterization of Ramanujan GP-graphs $\Gamma(q^\ell + 1, q^m)$ to all semiprimitive pairs $(k, p^m)$, that is in the case $q = p$.

(ii) When $p = 2$, the last proposition does not give any novelty, since the possible values of $k \in \{2, 3, 4, 5\}$ such that $(k, 2^m)$ is a semiprimitive pair reduces to $k = 3, 5$, which correspond to the cases $p = 2$ with $\ell = 1, 2$ in (i) above.

**Example 4.3.** From Proposition 4.1, the following GP-graphs are Ramanujan:

| $p$  | $\Gamma(3, 16)$, $\Gamma(3, 64)$, $\Gamma(3, 256)$, $\Gamma(5, 256)$, $\Gamma(2, 6.561)$, $\Gamma(4, 6.561)$, $\Gamma(5, 6.561)$ |
|------|--------------------------------------------------------------------------------------------------|
| $p$  | $\Gamma(2, 81)$, $\Gamma(4, 81)$, $\Gamma(5, 81)$, $\Gamma(2, 729)$, $\Gamma(4, 729)$, $\Gamma(2, 6.561)$, $\Gamma(4, 6.561)$, $\Gamma(5, 6.561)$ |
| $p$  | $\Gamma(2, 25)$, $\Gamma(3, 25)$, $\Gamma(2, 625)$, $\Gamma(3, 625)$, $\Gamma(2, 15.625)$, $\Gamma(3, 15.625)$, $\Gamma(2, 390.625)$, $\Gamma(3, 390.625)$ |

where $6.561 = 3^8$, $15.625 = 5^8$ and $390.625 = 5^8$. In bold those graphs with $k \neq p^\ell + 1$ for $p^m$. \hfill \n
5. The relation between the spectra of $\Gamma(k, q)$ and $C(k, q)$

In this section we relate the spectrum of the graph $\Gamma(k, q)$ given in (1.1) with the weight distribution of the code $C(k, q)$ defined in (1.2). We will see that $Spec(C(k, q))$ determines $Spec(\Gamma(k, q))$. 

and conversely. To this end we will have to assume further that $k \mid \frac{q-1}{p-1}$. Under this hypothesis, the numbers in (1.3) become $N = k$ and $n = \frac{q-1}{k}$.

**Theorem 5.1.** Let $q = p^m$ with $p$ prime, $k \in \mathbb{N}$ such that $k \mid \frac{q-1}{p-1}$ and put $n = \frac{q-1}{k}$. Let $\Gamma(k, q)$ and $C(k, q)$ be as in (1.1) and (1.2). Then, we have:

(a) The eigenvalue $\lambda_\gamma$ of $\Gamma(k, q)$ and the weight of $c_\gamma \in C(k, q)$ are related by the expression

$$\lambda_\gamma = n - \frac{p}{p-1} w(c_\gamma).$$

(b) If $\Gamma(k, q)$ is connected the multiplicity of $\lambda_\gamma$ is $A_{w(c_\gamma)}$ for all $\gamma \in \mathbb{F}_q$. In particular, the multiplicity of $\lambda_0 = n$ is $A_0 = 1$.

**Proof.** (a) Let $c_\gamma \in C(k, q)$. Thus, $w(c_0) = w(0) = 0$, by (1.2). Also, if $\gamma \in C_i^{(k,q)}$ (see (1.4)), from equation (12) in [9] we have

$$w(c_\gamma) = \frac{p-1}{pk}(q-1 - k\eta_i^{(k,q)}).$$

By Proposition 2.1, the eigenvalues of $\Gamma(k, q)$ are $\lambda = n$ or $\lambda_\gamma = \eta_i^{(k,q)}$ if $\gamma \in C_i^{(k,q)}$. Putting this in (5.2) we get (5.1) for $\gamma \neq 0$. But (5.1) also holds for $\lambda = n$ and $w = 0$, as we wanted.

(b) The assertion about the multiplicities of the eigenvalues $\lambda_\gamma$ with $\gamma \neq 0$ is clear. For $\gamma = 0$, we have $c_\gamma = 0$, so $\lambda_0 = n$. Every $n$-regular graph $\Gamma$ has $n$ as one of its eigenvalues, with multiplicity equal to the number of connected components of the graph. Therefore, since $\Gamma$ is connected, the multiplicity of $\lambda_0$ equals $A_{w(0)} = A_0 = 1$. \qed

Note that from (5.1), $\Gamma(k, q)$ is integral if and only if $C(k, q)$ is $(p-1)$-divisible. Actually, it is known that if $k \mid \frac{q-1}{p-1}$ then $C(k, q)$ is a $(p-1)$-divisible code ([19]) which is in accordance with part (b) of Theorem 2.1.

**Remark 5.2.** Combining Theorem 5.1 with Theorem 24 in [9], which gives the spectrum of $C(k, q)$ in the semiprimitive case, we obtain Theorem 3.3 in another way.

**Corollary 5.3.** Let $q = p^m$ with $p$ prime, $k \in \mathbb{N}$ such that $k \mid \frac{q-1}{p-1}$ and $n = \frac{q-1}{k}$ is a primitive divisor of $q-1$. Then, $\Gamma(k, q)$ is a strongly regular graph if and only if $C(k, q)$ is a 2-weight code.

**Proof.** We use the well-known fact that a simple connected graph is strongly regular if and only if it has 3 different eigenvalues. Since the graph $\Gamma(k, q)$ is simple and connected by hypothesis (see the comments after (1.1)), the remaining assertion follows by (a) in Theorem 5.1. \qed

A lower bound for the minimum distance of $C(k, q)$. Note that if $\Gamma = \Gamma(k, q)$ with $k$ and $q$ satisfying $k \mid \frac{q-1}{p-1}$ and $n = \frac{q-1}{k}$ is a primitive divisor of $q-1$ then, by (5.1), we have (see (4.1))

$$\lambda(\Gamma) = \max \left\{ n - (\frac{p-1}{p}) d, (\frac{p-1}{p}) d' - n \right\}$$

where $d = w_{\min}$ is the minimum distance of $C(k, q)$ and $d' = w_{\max}$ is the maximum weight of it.

We now show that if $\Gamma(k, q)$ is Ramanujan, under the non-restrictive additional assumption $d \leq \left( \frac{p-1}{p} \right) n$ we get a lower bound for the minimum distance of the associated code $C(k, q)$.

**Corollary 5.4.** In the previous notations and under the same hypothesis of Theorem 5.1, if $\Gamma(k, q)$ is Ramanujan and the minimum distance of $C(k, q)$ satisfies $d \leq \left( \frac{p-1}{p} \right) n$ then

$$d \geq \left( \frac{p-1}{p} \right) (n - 2\sqrt{n} - 1).$$

More precisely, we have:

(a) If $\lambda(\Gamma) = n - (\frac{p-1}{p}) d$, then $\Gamma(k, q)$ is Ramanujan if and only if (5.4) holds.

(b) Suppose $\lambda(\Gamma) = (\frac{p-1}{p}) d' - n$ and $d \leq \left( \frac{p-1}{p} \right) n$. If $\Gamma(k, q)$ is Ramanujan then (5.4) holds.
Proof. The first assertion is a consequence of (a) and (b) in the statement. If \( \lambda(\Gamma) = n - (\frac{p}{p-1})d \), the equivalence in (a) follows immediately from (4.1). To show (b), by hypothesis we have
\[
0 \leq n - (\frac{p}{p-1})d = |n - (\frac{p}{p-1})d| \leq \lambda(\Gamma) \leq 2\sqrt{n-1}
\]
since \( \Gamma \) is Ramanujan, and the proof is thus complete. \( \Box \)

Example 5.5. Fix \( m \) and suppose we take \( q = p^m \) with \( p \) prime and \( k = p^\ell + 1 \) with \( 1 \leq \ell \leq m \). Consider the graph and the code
\[
\Gamma_{p,m}(\ell) = \Gamma(p^\ell + 1, p^m), \quad C_{p,m}(\ell) = C(p^\ell + 1, p^m),
\]
as in (1.1) and (1.2), respectively. Both \( \Gamma_{p,m}(\ell) \) and \( C_{p,m}(\ell) \), and their corresponding spectra, were respectively studied in \([14]\) and \([15]\). In the definition of \( C_{p,m}(\ell) \) we need to assume that \( p^\ell + 1 \mid p^m - 1 \). This is equivalent to the conditions \( \ell \mid m \) and \( \frac{m}{\ell} \) even. But this is just the condition for \( \Gamma_{p,m}(\ell) \) to be a truly generalized Paley graph, i.e. not a Paley graph (see \([14]\), Proposition 2.5). Also, take \( \ell \neq \frac{m}{2} \) since for \( \ell = \frac{m}{2} \) the graph is not connected.

Taking \( q = p \) in Theorem 4.1 and Table 5 in \([15]\) we know that \( C_\ell \) is a 2-weight code with weights and frequencies given by
\[
\begin{align*}
w_1 &= \frac{(p-1)(p^m-1+\varepsilon_\ell p^{\frac{m-1}{2}})}{p^\ell+1}, & A_{w_1} &= n, \\
w_2 &= \frac{(p-1)(p^m-1-\varepsilon_\ell p^{\frac{m-1}{2}})}{p^\ell+1}, & A_{w_2} &= np^\ell,
\end{align*}
\]
where \( n = \frac{p^m-1}{p^\ell+1} \) and \( \varepsilon_\ell = (-1)^{\frac{m}{2}} \). On the other hand, for \( \ell \mid m \) with \( \frac{m}{\ell} \) even, we have that \( \Gamma_\ell \) has three eigenvalues as given in (3.6) –see also Theorem 3.5 in \([14]\) with \( q = p \). From (3.6) and (5.5), it is easy to check that both (a) and (b) in Theorem 5.1 hold. In particular, we have
\[
w_1 = (k_\ell - \mu_\ell)\frac{(p-1)}{p} \quad \text{and} \quad w_2 = (k_\ell - \nu_\ell)\frac{(p-1)}{p}.
\]
Note that this implies that \( k_\ell \equiv \nu_\ell \equiv \mu_\ell \mod p \) since \( w_1, w_2 \) are integers, a fact that we do not know from \([14]\).

From Theorem 8.1 in \([14]\), the graphs \( \Gamma_{q,m}(\ell) \) are Ramanujan if and only if \( \ell = 1, q \in \{2, 3, 4\} \) and \( m = 2t \) with \( t \geq 2 \). Thus, by Remark 2.10 in \([14]\), we have that \( \Gamma_{2,2t}(1) \) and \( \Gamma_{2,2t}(2) \) are Ramanujan graphs for every \( t \geq 2 \). From Theorem 4.1 in \([15]\), the parameters of \( C_{p,m}(1) \) and \( C_{p,m}(2) \) are
\[
n = \frac{2^{2t} - 1}{2^t + 1}, \quad k = m, \quad d = \begin{cases} 2^{t-1}\frac{2^{2t-1}}{2^t + 1} & \text{if } t \text{ is even}, \\
2^{t-1}\frac{2^{2t-1} - 2}{2^t + 1} & \text{if } t \text{ is odd}, \end{cases}
\]
for \( j = 1, 2 \). From here one can check that (5.4) in Theorem 5.4 holds. \( \Box \)

Remark 5.6. Using Theorem 5.1, one can recover the weights of the irreducible semiprimitive codes (given in \([19]\)) from the spectrum –obtained in Proposition 3.3– of the generalized Paley graphs \( \Gamma(k, q) \) where \( (k, q) \) is a semiprimitive pair (i.e. \( k \mid p^\ell + 1 \) for some \( t \neq \frac{m}{2} \) such that \( t \mid m \) and \( m_\ell = \frac{m}{2} \) even). If \( t \) is minimal satisfying these conditions we have that \( \sigma = (-1)^{\frac{m}{2^{t+1}}} \) (see (3.2)). In fact, the nonzero weights of \( \mathcal{C}(k, q) \) are
\[
w_1 = \frac{(p-1)p^{\frac{m-1}{2}}}{m}(p^{\frac{m}{2}} - \sigma(k-1)) \quad \text{and} \quad w_2 = \frac{(p-1)p^{\frac{m-1}{2}}}{k}(p^{\frac{m}{2}} + \sigma)
\]
with multiplicities \( n \) and \( (k-1)n \), respectively. Notice that \( \sigma = -\varepsilon_\ell \), where \( t = \ell \) and \( \varepsilon_\ell \) as in the previous example.

6. The spectrum of the graphs \( \Gamma(3, q) \) and \( \Gamma(4, q) \)

We have given the spectrum of \( \Gamma(k, q) \) for \( (k, q) \) a semiprimitive pair. Without this assumption, we know the spectrum in the cases \( k = 1, 2 \) since \( \Gamma(1, q) = K_q \) and \( \Gamma(2, q^2) = P(q^2) \) are the complete and the classical Paley graphs. Now, using the main result of the previous section and
the known spectra of irreducible cyclic codes $C(3,q)$ and $C(4,q)$, we compute the spectra of the associated GP-graphs $\Gamma(3,q)$ and $\Gamma(4,q)$, under certain conditions on $q$.

**Theorem 6.1.** Let $q = p^m$ with $p$ prime such that $3 \nmid \frac{q-1}{p-1}$ and $q \geq 5$. Thus, the graph $\Gamma(3,q)$ is connected with integral spectrum given as follows:

(a) If $p \equiv 1 \pmod{3}$ then $m = 3t$ for some $t \in \mathbb{N}$ and

$$\text{Spec}(\Gamma(3,q)) = \{ [n]^1, [\frac{a}{3}(q-1)]^n, [\frac{1}{3}(a+9bq-1)]^n, [\frac{1}{3}(a-9bq-1)]^n \}$$

where $a, b$ are integers uniquely determined by

$$4\sqrt{q} = a^2 + 27b^2, \ a \equiv 1 \pmod{3} \quad \text{and} \quad (a,p) = 1.$$  

(b) If $p \equiv 2 \pmod{3}$ then $m = 2t$ for some $t \in \mathbb{N}$ and

$$\text{Spec}(\Gamma(3,q)) = \begin{cases} 
\{ [n]^1, [\frac{a}{3}(q-1)]^n, [\frac{1}{3}(a+9bq-1)]^n \} & \text{for } m \equiv 0 \pmod{4}, \\
\{ [n]^1, [\frac{a}{3}(q-1)]^n, [\frac{1}{3}(a-9bq-1)]^n \} & \text{for } m \equiv 2 \pmod{4}.
\end{cases}$$

In particular, $\Gamma(3,q)$ is a strongly regular graph.

**Proof.** Let $q = p^m$. First note that condition $3 \nmid \frac{q-1}{p-1} = p^m - 1 + \cdots + p + 1$ implies that $m = 3t$ if $p \equiv 1 \pmod{3}$ and $m = 2t$ if $p \equiv 2 \pmod{3}$. We will apply Theorem 5.1 to the code $C(3,q)$. The spectrum of $C(3,q)$ is given in Theorems 19 and 20 in [9], with different notations ($r$ for our $q$, $N$ for our $k$, etc).

By (a) in (5.1), the eigenvalues of $\Gamma(3,q)$ are given by

$$\lambda_i = \frac{q-1}{3} - \frac{p}{p-1}w_i$$

where $w_i$ are the weights of $C(3,q)$.

If $p \equiv 1 \pmod{3}$, by Theorem 19 in [9], the four weights of $C(3,q)$ are $w_0 = 0$,

$$w_1 = \frac{(p-1)(q-a\sqrt{q})}{3p}, \quad w_2 = \frac{(p-1)(q+a\sqrt{q})}{3p}, \quad w_3 = \frac{(p-1)(q+9b\sqrt{q})}{3p},$$

with frequencies $A_0 = 1$ and $A_1 = A_2 = A_3 = \frac{q-1}{3}$; where $a$ and $b$ are the only integers satisfying $4\sqrt{q} = a^2 + 27b^2$, $a \equiv 1 \pmod{3}$ and $(a,p) = 1$.

On the other hand, if $p \equiv 2 \pmod{3}$, by Theorem 20 in [9], the three weights of $C(3,q)$ are

$$w_0 = 0, \quad w_1 = \frac{(p-1)(q-2\sqrt{q})}{3p}, \quad w_2 = \frac{(p-1)(q+2\sqrt{q})}{3p},$$

with frequencies $A_0 = 1, A_1 = \frac{2(q+1)}{3}$ and $A_2 = \frac{q-1}{3}$ if $m \equiv 0 \pmod{4}$ and

$$w_0 = 0, \quad w_1 = \frac{(p-1)(q-2\sqrt{q})}{3p}, \quad w_2 = \frac{(p-1)(q+\sqrt{q})}{3p},$$

with frequencies $A_0 = 1, A_1 = \frac{q-1}{3}$ and $A_2 = \frac{2(q-1)}{3}$ if $m \equiv 2 \pmod{4}$. By introducing (6.2), (6.3) and (6.4) in (6.1), we get the eigenvalues in (a) and (b) of the statement.

To compute the multiplicities, we use (b) of Theorem 5.1. The hypothesis that $\Gamma(3,q)$ be connected is equivalent to the fact that $n = \frac{q-1}{3}$ is a primitive divisor of $q - 1$ (see Introduction, after (1.1)). We now show that this is always the case for $q \geq 5$.

Suppose that $p \geq 5$. Then, we have that $p^{m-1} - 1 \leq \frac{q-1}{3}$, since this inequality is equivalent to $3p^{m-1} - 3 \leq p^m - 1$ which holds for $p \geq 5$. This implies that $n$ is greater that $p^{m-1}$ for all $a < m$ and, hence, $n$ is a primitive divisor of $q - 1$. Now, if $p = 2$ we only have to check that $n$ does not divide $2^{m-1} - 1$, since $n > 2^{m-2} - 1$. Notice that $n \mid 2^n - 1$ if and only if $2^n - 1 \mid 3(2^{m-1} - 1)$, which can only happen if $m = 2$. If $m > 2$ we have $3(2^{m-1} - 1) \equiv 2^{m-1} - 2 \not\equiv 0 \pmod{2^{m-1} - 1}$ as we wanted. The prime $p = 3$ is excluded by hypothesis.
Thus, by part (b) of Theorem 5.1, the multiplicities of the eigenvalues of $\Gamma(3, q)$ are the frequencies of the weights of $C(3, q)$, and we are done.

Example 6.2. Let $p = 7$ and $m = 3$, hence $q = 7^3 = 343$. Since $p \equiv 1 \mod 3$, we have to find integers $a, b$ such that $28 = a^2 + 27b^2, a \equiv 1 \mod 3$ and $(a, 7) = 1$. Clearly $a = b = 1$ satisfy these conditions. By (i) in Theorem 6.1 we have $Spec(\Gamma(3, 7^3)) = \{[114]^1, [9]^114, [2]^114, [-12]^114\}$. 

Theorem 6.3. Let $q = p^m$ with $p$ prime such that $4 \mid \frac{q - 1}{p - 1}$ and $q \geq 5$ with $q \neq 9$. Thus, the graph $\Gamma(4, q)$ is integral and connected and its spectrum is given as follows:

(a) If $p \equiv 1 \mod 4$ then $m = 4t$ for some $t \in \mathbb{N}$ and

$$Spec(\Gamma(4, q)) = \{[n]^1, [\sqrt[4]{\frac{q}{4}}]^n, [-\sqrt[4]{\frac{q}{4}}]^n, [\frac{a}{n} - 2\sqrt[4]{\frac{q}{4}}]^n\}$$

where $c, d$ are integers uniquely determined by

$$\sqrt[4]{q} = c^2 + 4d^2, \quad c \equiv 1 \mod 4 \quad \text{and} \quad (c, p) = 1.$$

(b) If $p \equiv 3 \mod 4$ then $m = 2t$ for some $t \in \mathbb{N}$ and

$$Spec(\Gamma(4, q)) = \{[n]^1, [\sqrt[4]{\frac{q}{4}}]^3n, [-\sqrt[4]{\frac{q}{4}}]^n\}.$$

In particular, $\Gamma(4, q)$ is a strongly regular graph.

Proof. The proof is similar to the one of Theorem 6.1. We apply Theorem 5.1 to the code $C(4, q)$ since the spectrum of this code is given in Theorem 21 in [9]. Thus, we leave out the details and only show that if $q \geq 5$ with $q \neq 9$ then $\frac{q - 1}{4}$ is a primitive divisor of $q - 1$ and hence $\Gamma(4, q)$ is connected.

Suppose that $p \geq 5$. Then, we have that $p^{m-1} - 1 \leq \frac{q - 1}{4}$ since this inequality is equivalent to $4p^{m-1} - 4 \leq p^{m} - 1$ which is true because $p \geq 5$. This implies that $n$ is greater that $p^6 - 1$ for all $a < m$ and hence $n$ is a primitive divisor of $q - 1$. Now, if $p = 3$ we only have to check that $n$ does not divide $3^{m-1} - 1$, since $n > 3^{m-2} - 1$. Notice that $n \mid 3^{m-1} - 1$ if and only if $3^{m-1} \mid 4(3^{m-1} - 1)$, which can only happen if $m = 2$. If $m > 2$ in this case $4(3^{m-1} - 1) \equiv 3^{m-1} - 3 \not\equiv 0 \mod 3^{m-1}$ as we wanted. The prime $p = 2$ is excluded by hypothesis.

Example 6.4. Let $q = 5^4 = 625$, that is $p = 5$ and $m = 4$. Since $p \equiv 1 \mod 4$, we have to find integers $c, d$ such that $25 = c^2 + 4d^2, c \equiv 1 \mod 4$ and $(c, 5) = 1$. One can check that $(c, d) = (-3, 2)$ satisfy these conditions and hence by (i) in Theorem 6.3, the spectrum of $\Gamma(4, 625)$ is given by $Spec(\Gamma(4, 625)) = \{[156]^1, [16]^156, [1]^156, [-4]^156, [-14]^156\}$. 

We end this section with some remarks.

Remark 6.5. (i) The integers $b$ and $d$ in Theorems 6.1 and 6.3 are determined up to sign. However, by symmetry, the eigenvalues in these theorems are not affected by these choices of sign.

(ii) The spectrum of $C(k, q)$ with $k = 3t$ is computed in Theorems 19 and 20 in [9], since it requires $N = (\frac{q - 1}{2})$, and the spectrum of $C(4t, q)$ is provided by Theorems 21 in [9]. However, to apply Theorem 5.1 to compute the spectrum of $\Gamma(k, q)$ with $k = 3t$ (resp. $k = 4t$), one needs the extra assumption $k \mid \frac{q - 1}{p - 1}$, which forces $N = k = 3$ (resp. 4). Hence, another method must be used to compute the spectra of $\Gamma(3t, q)$ and $\Gamma(4t, q)$ for $t > 1$.

(iii) In the cases (b) in Theorems 6.1 and 6.3, the graphs $\Gamma(3, q)$ and $\Gamma(4, q)$ are semiprimitive by (ii) in Example 3.2. Thus, their spectra is already given by Theorem 3.3 (without assuming $n$ to be a primitive divisor of $q - 1$). However, the cases in (a) are new.
7. Spectrum of cyclic codes associated to decomposable GP-graphs

The cartesian product of the graphs $\Gamma_1, \ldots, \Gamma_t$, denoted by $\Gamma_1 \square \cdots \square \Gamma_t$, is the graph $\Gamma$ with vertex set $V(\Gamma) = V(\Gamma_1) \times \cdots \times V(\Gamma_t)$, such that $(v_1, \ldots, v_t)$ and $(w_1, \ldots, w_t)$ form an edge if and only if there is one $j \in \{1, \ldots, t\}$ such that $v_j, w_j$ is an edge in $\Gamma_j$ and $v_i = w_i$ for all $i \neq j$. A graph $\Gamma$ is (cartesian) decomposable if it can be written as a product of smaller graphs $\Gamma = \Gamma_1 \square \cdots \square \Gamma_t$ with $t > 1$.

Generalized Paley graphs which are cartesian decomposable are characterized in [13]. It is proved there that, in this case, the graph is a product of copies of a single graph, which is necessarily another GP-graph. More precisely, if $\Gamma = \Gamma(k, p^m)$ is simple and connected (equivalently, if $k$ divides $\frac{p^m - 1}{2}$ when $p$ is odd and $n = \frac{p^m - 1}{k}$ is a primitive divisor of $p^m - 1$), the following conditions are equivalent:

(a) $\Gamma = \Gamma(k, p^m)$ is cartesian decomposable.

(b) $\Gamma = \square k \Gamma_0$, where $\Gamma_0 = \Gamma(u, p^m)$ with $u = \frac{p^m - 1}{c}$ for some $b, c$.

(c) $n = bc$ with $b > 1$, $b \mid m$ and $c$ is a primitive divisor of $p^m - 1$.

Remark 7.1. To check that $n = bc$ as in (7.1) is a primitive divisor of $q - 1$ can be somewhat difficult. However, taking $b$ a prime different from $p$ such that $bc \equiv 1 \pmod{b}$ then $n = bc$ is a primitive divisor of $q - 1$. Conversely, if $n = bc$ is a primitive divisor of $q - 1$ and $b \neq p$ is a prime not dividing $u$ then $p^m \equiv 1 \pmod{b}$.

We now show that if $\Gamma$ is decomposable, say $\Gamma = \square b \Gamma_0$, then the computation of the spectrum of the cyclic code $C$ associated to $\Gamma$ reduces to the one of the smaller code $C_0$ associated to $\Gamma_0$.

Theorem 7.2. Let $p$ be a prime and let $k, m, b, c, n, u, b$ be positive integers as in (7.1) such that $n = \frac{p - 1}{k}$ is a primitive divisor of $q - 1$ and $u \mid \frac{p - 1}{2}$ if $p$ is odd. Consider the irreducible cyclic codes $C = C(k, p^m)$ and $C_0 = C(u, p^m)$. If $k \mid \frac{p - 1}{c}$ and $p - 1 \mid c$ then $\text{Spec}(C)$ is determined by $\text{Spec}(C_0)$. More precisely, if the weights of $C_0$ are $w_i$ with frequencies $A_{w_i} = m_i$ for $i = 1, \ldots, s$, then the weights of $C$ are given by

$$w_{\ell_1, \ldots, \ell_s} = \sum_{i=1}^{s} \ell_i w_i$$

with frequency

$$A_{\ell_1, \ldots, \ell_s} = \binom{b}{\ell_1, \ldots, \ell_s} m_{\ell_1}^{1} \cdots m_{\ell_s}^{s}$$

where $(\ell_1, \ldots, \ell_s) \in \mathbb{N}_0^s$ such that $\ell_1 + \cdots + \ell_s = b$.

Proof. We have $k \mid \frac{p^m - 1}{p - 1}$ by hypothesis and $u \mid \frac{p^m - 1}{p - 1}$ since $p - 1 \mid c$. Also, the graphs $\Gamma = \Gamma(k, p^m)$ and $\Gamma_0 = \Gamma(u, p^m)$ are connected because of the primitiveness of $n$ and $c$, respectively. Thus, we can apply Theorem 5.1 to the codes $C$ and $C_0$.

By hypothesis, we have $\Gamma = \square b \Gamma_0$ and therefore $\text{Spec}(\Gamma) = \text{Spec}(\square b \Gamma_0)$. Now, since the eigenvalues of the cartesian product of graphs is the sum of the eigenvalues of its factors (see [10]), if $\text{Spec}(\Gamma_0) = \{\lambda_1^{m_1}, \ldots, \lambda_s^{m_s}\}$, then the eigenvalues of $\Gamma$ are

$$\lambda_{\ell_1, \ldots, \ell_s} = \ell_1 \lambda_1 + \cdots + \ell_s \lambda_s$$

with multiplicity $\binom{b}{\ell_1, \ldots, \ell_s} m_{\ell_1}^{1} \cdots m_{\ell_s}^{s}$

where the $r$-tuple of integers $(\ell_1, \ldots, \ell_s)$ satisfies that $\ell_i \geq 0$ and $\ell_1 + \cdots + \ell_s = b$ and $\binom{b}{\ell_1, \ldots, \ell_s}$ denotes the multinomial coefficient. By Theorem 5.1, $\lambda_i = c - \frac{w_i}{p - 1}$ for each $i = 1, \ldots, s$ and hence

$$\lambda_{\ell_1, \ldots, \ell_s} = \sum_{i=1}^{s} \ell_i (c - \frac{w_i}{p - 1}) = n - \frac{w}{p - 1} \sum_{i=1}^{s} \ell_i w_i,$$

since $c (\ell_1 + \cdots + \ell_s) = bc = n$. Also, the frequency of $w_i$ in $C_0$ is $m_i$ for all $i = 1, \ldots, s$. By Theorem 5.1 again, we have that the weights of $C$ are

$$w_{\ell_1, \ldots, \ell_s} = \frac{w}{p - 1} (n - \lambda_{\ell_1, \ldots, \ell_s}) = \ell_1 w_1 + \cdots + \ell_s w_s$$
with frequency \( \sum_{i=1}^{b} m_{i}^{2} \), as desired.

\[ \text{Remark 7.3.} \] Suppose that \( \Gamma \simeq \square \Gamma_{0} \). Then \( \Gamma \) and \( \Gamma_{0} \) have associated irreducible cyclic codes \( C \) and \( C_{0} \). Under the hypothesis of the theorem, we can only assure that the spectrum of \( C \) equals the spectrum of the direct sum \( C_{0} \oplus \cdots \oplus C_{0} \), with \( C_{0} \) repeated \( b \)-times, which is not cyclic in general. Thus, one may wonder if there is some code operation \( \ast \) such that \( C = C_{0} \ast \cdots \ast C_{0} \), with \( C_{0} \) repeated \( b \)-times.

**Number of rational points of Artin-Schreier curves in extensions.** As an application of the previous result, we obtain a relationship between the rational points of some Artin-Schreier curves defined over two different fields. Let \( C \) be a prime and \( m \) be an integer. Under the hypothesis of Theorem 7.2, let \( C_{k,\beta}(m) \) be the Artin-Schreier curve with affine equations

\[ C_{k,\beta}(m) : \quad y^p - y = \beta x^k, \quad \beta \in \mathbb{F}_{p^m}. \]

We will use the following notation

\[ \Psi_b(x) = \frac{x^{b-1}}{x-1} = x^{b-1} + \cdots + x^2 + x + 1. \]

**Corollary 7.4.** Let \( p \) be a prime and \( m = br \). For each \( \beta \in \mathbb{F}_{p^m} \) there are \( \alpha_1, \ldots, \alpha_b \in \mathbb{F}_{p^r} \) such that

\[ \text{number of rational points of Artin-Schreier curves in extensions.} \]

\[ \text{Corollary 7.4.} \]

**Proof.** The code \( C_k = \{ c_k(\beta) = (\text{Tr}_{p^m/p}(\beta x^k))_{x \in \mathbb{F}_{p^m}} : \beta \in \mathbb{F}_{p^m} \} \) is obtained from \( k \)-copies of \( C(k, p^m) \). This implies that

\[ w(c_k(\beta)) = k w(c(\beta)) \quad \text{where} \quad c(\beta) = (\text{Tr}_{p^m/p}(\beta \omega^j))_{j=1}^n. \]

On the other hand, the weight of the codeword \( c_k(\beta) \) is related to the number of \( \mathbb{F}_{p^m} \)-rational points of the curve \( C_{k,\beta} \). In fact, by Theorem 90 of Hilbert we have

\[ \text{Tr}_{p^m/p}(\beta x^k) = 0 \iff y^p - y = \beta x^k \quad \text{for some} \quad y \in \mathbb{F}_{p^m}. \]

Since \( C_{k,\beta} \) is a \( p \)-covering of \( \mathbb{P}^1 \), considering the point at infinity, we get

\[ \#C_{k,\beta}(\mathbb{F}_{p^m}) = 1 + p \# \{ x \in \mathbb{F}_{p^m} : \text{Tr}_{p^m/p}(\beta x^k) = 0 \} = p^{m+1} - p w(c_k(\beta)) + 1. \]

In the same way, we have that

\[ w(c_u(\alpha)) = u w(c(\alpha)) \quad \text{and} \quad \#C_{u,\alpha}(\mathbb{F}_{p^r}) = 1 + p \# \{ x \in \mathbb{F}_{p^r} : \text{Tr}_{p^r/p}(\alpha x^u) = 0 \} = p^{r+1} - p w(c_u(\alpha)) + 1. \]

First notice that \( \mathbb{F}_{p^r} \subset \mathbb{F}_{p^m} \). Now, by Theorem 7.2, for each element \( \beta \in \mathbb{F}_{p^m} \) there exist elements \( \alpha_1, \ldots, \alpha_b \in \mathbb{F}_{p^r} \) such that \( w(c(\beta)) = w(c(\alpha_1)) + \cdots + w(c(\alpha_b)) \). Moreover, given \( \alpha_1, \ldots, \alpha_b \in \mathbb{F}_{p^r} \), \( w(c(\alpha_1)) + \cdots + w(c(\alpha_b)) \) defines a weight in \( C(k, p^m) \), i.e. there must be some \( \beta \in \mathbb{F}_{p^m} \) such that \( w(c(\beta)) = w(c(\alpha_1)) + \cdots + w(c(\alpha_b)) \). Therefore, the number \( \#C_{k,\beta}(\mathbb{F}_{p^m}) \) equals

\[ p^{m+1} + 1 - pk \sum_{i=1}^{b} w(c(\alpha_i)) = p^{m+1} + 1 - \frac{k}{b} \sum_{i=1}^{b} (p^{r+1} + 1 - \#C_{u,\alpha_i}(\mathbb{F}_{p^r})). \]

Since \( \frac{k}{b} = \frac{m^m - 1}{m^r - 1} = \frac{1}{b} \Psi_b(p^r) \), after straightforward calculations we get (7.5) as desired.

In particular, we have

\[ \#C_{k,\beta}(\mathbb{F}_{p^m}) \equiv \frac{1}{b} \Psi_b(p^r) \sum_{i=1}^{b} \#C_{u,\alpha_i}(\mathbb{F}_{p^r}) \pmod{M}. \]
with $M = p + 1$ or $M = p^r$. Since $\Psi_{\ell+1}(x) = x^\ell + \Psi_{\ell}(x)$, we also have

$$b \cdot \#C_{k,\beta}(\mathbb{F}_{p^m}) = \begin{cases} 
\Psi_b(p^r) \sum_{i=1}^{b} \#C_{u,\alpha_i}(\mathbb{F}_{p^r}) \pmod{p^r}, & \text{if } b \mid p^r, \\
\sum_{i=1}^{b} \#C_{u,\alpha_i}(\mathbb{F}_{p^r}) \pmod{\Psi_b(p^r)}, & \text{if } b \not\mid p^r.
\end{cases}$$

**Example 7.5.** In the notations of Theorem 7.2, take $p = 2$ and $u = 1$. Hence, $c = 2^r - 1$, $n = b(2^r - 1)$ and $m = br$. Obviously $2^r - 1$ is a primitive divisor of itself and it can be shown that if $b$ is odd and $x = 2^r \equiv 1 \pmod{b}$ then $n$ is a primitive divisor of $2^m - 1$. If $k = \Psi_b(x)$, by the last corollary the $\mathbb{F}_{p^m}$-rational points of the curve $C_{k,\beta} : y^2 + y = \beta x^k$ with $\beta \in \mathbb{F}_{2^r}$ can be calculated in terms of the $\mathbb{F}_{2^r}$-rational points of the curves $C_{1,\alpha} : y^2 + y = \alpha x$ for some $\alpha_1, \ldots, \alpha_b \in \mathbb{F}_{2^r}$.

The simplex code $C(1,2^r)$ has only one nonzero weight, which is $2^r - 1$. Taking into account that $\#C_{1,\alpha}(\mathbb{F}_{2^r}) = 2^r + 1$ with $w(c_1(\alpha)) = 1$ we have that $\#C_{1,\alpha}(\mathbb{F}_{2^r}) = 2^r + 1$ for all $\alpha \in \mathbb{F}_{2^r}$. By (7.6), we have that the number of $\mathbb{F}_{q^m}$-rational points of the curves in the family $\{C_{k,\beta}(\mathbb{F}_{2^m})\}_{\beta \in \mathbb{F}_{2^m}}$ is given by $\{2^m + 1 - k\ell 2^r : 0 \leq \ell \leq b\}$. \hfill $\Box$

### 8. A reduction formula for Gaussian periods

It is known that the Gaussian periods satisfy the relations

$$\sum_{i=0}^{k-1} \eta_i^{(k,q)} = -1 \quad \text{and} \quad \sum_{i=0}^{k-1} \eta_i^{(k,q)} \eta_{i+j}^{(k,q)} = q \theta_j - n \quad (0 \leq j \leq k-1)$$

where $n = \frac{2^r - 1}{q}$ and $\theta_j = 1$ if and only if $-1 \in C_j^{(k,q)}$ and $\theta_j = 0$ otherwise (see [18]). Equivalently, $\theta_j = 1$ if and only if $n$ is even and $j = 0$ or $n$ is odd and $j = \frac{r}{2}$.

Apart from (1.5) and (8.1), there are not many known relations for Gaussian periods. Next, as another application of Theorem 7.2, we give a relation between Gaussian periods defined over different fields ($\mathbb{F}_{p^r} \subset \mathbb{F}_{p^m}$), showing that one can reduce the computation of $\eta_i^{(k,q)}$ to linear combinations of Gaussian periods $\eta_j^{(u,r)}$ with smaller parameters, namely $u \mid k$ and $r \mid q$.

**Corollary 8.1.** Let $p$ be a prime and $k,m,n,b,c,u$ integers as in Theorem 7.2. Then, for each $i = 0, \ldots, k - 1$ there exists some $s$ and $\ell_0, \ell_1, \ldots, \ell_{s-1} \in \mathbb{N}$ such that

$$\eta_i^{(k,p^m)} = c\ell_0 + \sum_{j=1}^{s-1} \ell_j \eta_j^{(u,p^m)}$$

where the non-negative integers $\ell_i$ run over all possible $s$-tuples $(\ell_0, \ell_1, \ldots, \ell_{s-1}) \in \mathbb{N}_0^s$ such that $\ell_0 + \cdots + \ell_{s-1} = b$ different from $(b,0,\ldots,0)$.

**Proof.** By hypothesis, $\Gamma = \Gamma(k,p^m)$ decomposes as $\Gamma = \square^b \Gamma_0$ where $\Gamma_0 = \Gamma(u,p^m)$. We know by Proposition 2.1 that the spectra of $\Gamma$ and $\Gamma_0$ are given in terms of Gaussian periods. Equation (7.2) in the proof of Theorem 7.2 gives a relation between the eigenvalues of $\Gamma$ and $\Gamma_0$, where $s$ is the number of distinct eigenvalues of $\Gamma$ and $\ell_0, \ldots, \ell_{s-1}$ are integers satisfying $\ell_0 + \cdots + \ell_{s-1} = b$. Also, $\lambda_0 = c$. Finally, we have to rule out all the cases giving

$$\eta_i^{(k,q)} = n = bc.$$

Since $\ell_0 + \cdots + \ell_{s-1} = b$, the only possibility to have $\eta_i^{(k,q)} = n$ is given by $(\ell_0, \ell_1, \ldots, \ell_{s-1}) = (b,0,\ldots,0)$, as we wanted to show. \hfill $\square$

Suppose we take $(k,q)$ a semiprimitive pair, with $q = p^m$. Then, the Gaussian periods $\eta_i^{(k,q)}$ are known (see (3.4) and (3.5)). On the other hand, one can ask if there exists $\ell_0 = \Gamma(u,p^r)$ such that $\Gamma = \square^b \Gamma_0$ with $r = \frac{m}{b}$ and $k,m,b,c,n,u$ as in Theorem 7.2. Notice that necessarily $b = 2$
and hence \( \Gamma_0 \) must have only one non-trivial eigenvalue since \( \Gamma \) have two non-trivial eigenvalues. This implies that \( \Gamma_0 \) must be the complete graph, i.e. \( u = 1 \) and \( r = \frac{m}{2} \) and therefore \( k = \frac{p^r - 1}{2} \).

Suppose \( \Gamma = \Gamma(k, p^m) \) is decomposable \( \Gamma \simeq \Box \Gamma_0 \) with \( \Gamma_0 = \Gamma_0(u, p^r) \) a semiprimitive graph. Then we can explicitly compute the Gaussian periods \( \eta_i^{(k,q)} \).

**Proposition 8.2.** Let \( q = p^m \) with \( p \) prime and \( k | q - 1 \) such that \( n = \frac{q-1}{k} = bc \) where \( m = br \), \( u = \frac{p^r - 1}{2} \) and \( (u, p^r) \) is a semiprimitive pair. Then, the different Gaussian periods modulo \( q \) are given by

\[
\eta_i^{(k,q)} = \ell_0 c + \ell_1 \frac{(u-1)\sqrt{p^r-1}}{u} - \ell_2 \frac{2\sqrt{p^r+1}}{u}
\]

where the non-negative integers \( \ell_0, \ell_1, \ell_2 \) run in the set

\[
\{ L_i = (\ell_0, \ell_1, \ell_2) : \ell_0 + \ell_1 + \ell_2 = b \} \setminus \{(b, 0, 0)\}
\]

and \( \sigma = (-1)^{\frac{m}{2}} \) with \( t \) the least integer such that \( u | p^t + 1 \).

**Proof.** By Corollary 8.1 we have an expression for \( \eta_i^{(k,q)} \) in terms of \( \eta_i^{(u,p^r)} \). Since \( (u, p^r) \) is a semiprimitive pair, there are only two different such periods, given by (3.4) or (3.5), depending the case. In case (a), that is \( p, \alpha \) and \( s \) odd, we have \( \eta_0^{(u,p^r)} = \frac{(u-1)\sqrt{p^r-1}}{u} \) and \( \eta_1^{(u,p^r)} = -\frac{2\sqrt{p^r+1}}{u} \) while in case (b) we have \( \eta_0^{(u,p^r)} = -\frac{(u-1)\sqrt{p^r-1}}{u} \) and \( \eta_1^{(u,p^r)} = \frac{\sigma(u-1)\sqrt{p^r-1}}{u} \). Now, by (8.2) we have

\[
\eta_i = \ell_0 c + \ell_1 \eta_0^{(u,p^r)} + \ell_2 \eta_1^{(u,p^r)}
\]

Since the triples \( (\ell_0, \ell_1, \ell_2) \) satisfying \( \ell_0 + \ell_1 + \ell_2 = b \) are symmetric, the above expression is the same no matter if we are in case of (a) or (b), or if \( \sigma = 1 \) or \( \sigma = -1 \), and hence we get (8.3). \( \square \)

**Example 8.3.** Take \( u = 2, r = 2, b = 3 \) and \( p = 5 \). Then \( (u, p^r) = (2, 5^2) \) is a semiprimitive pair (and \( \Gamma_0 = \Gamma(2, 5^2) = P(25) \)). Thus \( m = br = 6, q = 5^6 = 15625 \), \( c = \frac{p^r - 1}{u} = \frac{5^2 - 1}{2} = 12 \) and \( n = bc = 36 \); hence \( k = \frac{q-1}{n} = 434 \).

By (8.3), the Gaussian periods for \((k, q) = (434, 15625)\) are given by

\[
\eta_i^{(434,15625)} = 12\ell_0 + 2\ell_1 - 3\ell_2
\]

where \( \ell_0 + \ell_1 + \ell_2 = 3 \); compare with (1.4). There are 9 such triples \((\ell_0, \ell_1, \ell_2) \neq (3, 0, 0)\), namely \( L_1 = (2, 1, 0), L_2 = (2, 0, 1), L_3 = (1, 2, 0), L_4 = (1, 1, 1), L_5 = (1, 0, 2), L_6 = (0, 3, 0), L_7 = (0, 2, 1), L_8 = (0, 1, 2) \) and \( L_9 = (0, 0, 3) \). Thus, we have that

\[
\eta_1 = 26, \quad \eta_2 = 21, \quad \eta_3 = 16, \quad \eta_4 = 11, \quad \eta_5 = 6 = \eta_6, \quad \eta_7 = 1, \quad \eta_8 = -4, \quad \eta_9 = -9.
\]

Note that \( \eta_i \equiv 1 \mod 5 \) for \( i = 1, \ldots, 9 \) as it should be, by (1.5).

We now check the expressions in (8.1). If \( \eta_i^{(k,q)} \) is associated with \((\ell_0, \ell_1, \ell_2)\), then its frequency is given by \( \mu_i = \frac{1}{n} A_i \), where \( A_i = A_{\ell_0, \ell_1, \ell_2} = \left( \frac{3}{3} \right) m_{\ell_0}^0 m_{\ell_1}^1 m_{\ell_2}^2 \), with \( m_0, m_1, m_2 \) the multiplicities of the Paley graph \( P(25) \). Hence, \( m_0 = 1, m_1 = m_2 = 12 \) (see (2.8)). Thus, we have \( A_{2,1,0} = A_{2,0,1} = 3 \cdot 12 = 36, A_{2,1,0} = A_{1,0,2} = 3 \cdot 12^2 = 432, A_{1,1,1} = 6 \cdot 12^2 = 684, A_{0,2,1} = A_{0,1,2} = 3 \cdot 12^3 = 5184 \) and \( A_{0,0,3} = A_{0,0,3} = 1 \cdot 12^3 = 1728, \) and hence

\[
\mu_1 = \mu_2 = 1, \quad \mu_3 = \mu_5 = 12, \quad \mu_4 = 24, \quad \mu_6 = \mu_9 = 48, \quad \mu_7 = \mu_8 = 144.
\]

In this way we have

\[
\sum_{i=0}^{433} \eta_i^{(434,5^6)} = \sum_{i=1}^{9} \mu_i \eta_i = \mu_1 (\eta_1 + \eta_2) + \mu_3 (\eta_3 + \eta_5) + \eta_4 \mu_4 + \mu_6 (\eta_6 + \eta_9) + \mu_7 (\eta_7 + \eta_8)
\]

\[
= (26 + 21) + 12(16 + 6) + 24 \cdot 11 + 48(6 - 9) + 144(1 - 4) = -1.
\]

One can also check that

\[
\sum_{i=1}^{9} \mu_i \eta_i^2 = 15.589 = q - n \quad \text{and} \quad \sum_{i=1}^{9} \mu_i \mu_i+j \eta_i \eta_{i+j} = -36 = -n
\]
for \( j = 1, \ldots, 9 \), and hence the second identity of (8.1) holds.

\[ \odot \]

9. Spectra of irreducible cyclic codes constructed from 2-weight codes

Here, we will compute the spectra of irreducible cyclic codes constructed from irreducible 2-weight codes.

In [19], Schmidt and White conjectured that all two-weight irreducible cyclic codes over \( \mathbb{F}_p \), with \( p - 1 \mid n \), belong to one of the following disjoint families:

- The subfield subcodes, corresponding to \( C(u, p^r) \) where \( u = \frac{p^r - 1}{p - 1} \) with \( a < r \).
- The semiprimitive codes, which are those \( C(u, p^r) \) such that \(-1\) is a power of \( p \) modulo \( u \). Equivalently, \( (k, q) \) is a semiprimitive pair as in (3.1).
- The exceptional codes, i.e. irreducible 2-weight cyclic codes which are neither subfield subcodes nor semiprimitive codes.

If one does not require the condition \( p - 1 \mid n \), Rao and Pinnawala ([16]) has given a family of 2-weight irreducible cyclic codes which are not of the previous kind.

Notice that in the subfield subcode case, the graph \( \Gamma(u, p^r) \) is not connected since \( c = \frac{p^r - 1}{c} \) is not a primitive divisor of \( p^r - 1 \); and thus we cannot apply Theorem 7.2. Hence, we are only interested in the other two cases.

We now compute the spectrum of the code \( C \) associated to the decomposable graph \( \Gamma \simeq \square^b \Gamma_0 \), where \( \Gamma_0 \) is a semiprimitive GP-graph.

**Proposition 9.1** (semiprimitive case). Let \( p \) be a prime and let \( k, b, m, u, c, n, r \) be positive integers such that \( p - 1 \mid c, b > 1, u = \frac{p^r - 1}{c}, m = br \) such that \( (u, p^r) \) is a semiprimitive pair. If \( n = \frac{p^m - 1}{k} = bc \) is a primitive divisor of \( p^m - 1 \) then the code \( C = C(k, p^m) \) has weights

\[
\begin{align*}
    w_{\ell_1, \ell_2} &= \frac{(p-1)p^{\ell_2-1}}{u} \left\{ \ell_1(p^{\ell_1} - \sigma(u - 1)) + \ell_2(p^{\ell_2} + \sigma) \right\} \\
\end{align*}
\]

for every pair of non-negative integers \( \ell_1, \ell_2 \) such that \( 0 \leq \ell_1 + \ell_2 \leq b \), with frequencies

\[
\begin{align*}
    A_{\ell_1, \ell_2} &= \left( \frac{b}{\ell_1} \right) \left( \frac{b+1}{\ell_2} \right) c^{\ell_1+\ell_2}(u - 1)^{\ell_1} \\
\end{align*}
\]

where we put \( \sigma = \pm 1 \) if \( u \mid p^\ell + 1 \).

**Proof.** Consider the semiprimitive irreducible cyclic code \( C_0 = C(u, p^r) \). Thus, \( u \mid p^\ell + 1 \) for some \( \ell \mid r \) with \( r \) even and \( C_0 \) is a two weight irreducible code. By Remark 5.6 its weights are

\[
\begin{align*}
    w_1 &= \frac{(p-1)p^{\ell-1}}{u}(p^{\ell} - \sigma(u - 1)) \quad \text{and} \quad w_2 = \frac{(p-1)p^{\ell-1}}{u}(p^{\ell} + \sigma) \\
\end{align*}
\]

where \( \sigma = (-1)^{\frac{m}{2}k} \) with \( k \) the minimal positive integer such that \( u \mid p^\ell + 1 \) and \( m_\ell = \frac{m}{p^\ell} \). By hypothesis \( p - 1 \mid c \), this implies that \( u \mid \frac{p^m - 1}{p-1} \) and \( k \mid \frac{m}{p-1} \). By Remark 5.6 the frequencies of \( w_1, w_2 \) are \( m_1 = c \) and \( m_2 = c(u - 1) \), respectively.

Finally, by hypothesis we have that \( m = br \) and \( n = bc \) is a primitive divisor of \( p^m - 1 \). Hence, by Theorem 7.2 the weights of \( C(k, p^m) \) are given by

\[
\begin{align*}
    w_{\ell_1, \ell_2} &= \ell_1 w_1 + \ell_2 w_2 = \frac{(p-1)p^{\ell_2-1}}{u} \left\{ \ell_1(p^{\ell_1} - \sigma(u - 1)) + \ell_2(p^{\ell_2} + \sigma) \right\} \\
\end{align*}
\]

with corresponding multiplicities

\[
\begin{align*}
    \frac{b!}{\ell_1!\ell_2!(b - (\ell_1 + \ell_2))} m_1^{\ell_1} m_2^{\ell_2} = \left( \frac{b}{\ell_1} \right) \left( \frac{b+1}{\ell_2} \right) c^{\ell_1+\ell_2}(u - 1)^{\ell_1} \\
\end{align*}
\]

where \( (\ell_1, \ell_2) \) runs over all integer 2-tuples such that \( \ell_i \geq 0 \) for \( i = 1, 2 \) and \( \ell_1 + \ell_2 \leq b \) and therefore we obtain (9.1) and (9.2), as we wanted. \( \square \)
By Example 3.2 (ii) we can consider the semiprimitive pair \((u, p^r)\) as \((2, p^2)\) with \(p \equiv 2 \mod 3\) or \((4, p^2)\) with \(p \equiv 3 \mod 4\). The spectrum of \(\mathcal{C}(u, p^2)\) with \(u = 2, 3, 4\) are known in these cases by Example 2.4 and Theorems 6.1 and 6.3. Thus, to apply Theorem 7.2 in these cases we can use Remark 7.1 to ensure that \(n = bc\) is a primitive divisor of \(q - 1\). This can be done for instance if we take \(b\) a prime different from \(p\) such that \(b \nmid u\) and \(t \in o_b(p)\mathbb{Z}\), where \(o_b(p)\) is the order of \(p\) mod \(b\). We next give an example with \(u = 2\).

Example 9.2. Let \(p\) be an odd prime, \(b = 3, r = 2\) and \(u = 2\). In this case, the graph \(\Gamma_0 = \Gamma(2, p^2)\) is the classic Paley graph over \(\mathbb{F}_{p^2}\) with \(\text{Spec}(\Gamma_0) = \{\{\frac{p^2-1}{2}\}, \{\frac{p^2-1}{2}, \{-\frac{p+1}{2}\}\}, \{-\frac{p+1}{2}\}\}\) by (2.8) and then the two nonzero weights of the code \(\mathcal{C}_0 = \mathcal{C}(2, p^2)\) have multiplicity \(\frac{p^2-1}{2}\). We have \(m = br = 6\) and \(c = \frac{p^2-1}{2}\) and thus

\[n = bc = 3\left(\frac{p^2-1}{2}\right)\]

Clearly, \(c\) is a primitive divisor of \(p^2 - 1\). Notice that if \(p \neq 3\), then \(9 \mid n\). In particular, if \(p \equiv 2, 5, 7\mod 9\), then \(n\) is a primitive divisor of \(p^6 - 1\), since in these cases the order of \(p\) modulo 9 is 6 and then 9 does not divide \(p^a - 1\) when \(1 \leq a < 6\), this implies that \(n\) does not divide \(p^a - 1\), either.

It can be seen that in this case we can choose \(\sigma = 1\) or \(-1\) indistinctly in the formula (9.1). Therefore the code

\[C = \mathcal{C}\left(\frac{3(p^6-1)}{2}, p^6\right) = \mathcal{C}\left(\frac{3}{2}(p^4 + p^2 + 1), p^6\right)\]

has weights

\[w_{\ell_1, \ell_2} = \frac{(p-1)}{2}\{\ell_1(p-1) + \ell_2(p+1)\} = \frac{(p-1)}{2}\ell_1 + c\ell_2\]

for every pair \(0 \leq \ell_1 + \ell_2 \leq 3\), with frequency

\[A_{\ell_1, \ell_2} = \left(\frac{3}{\ell_1}\right)\left(\frac{1}{\ell_2}\right)\left(\frac{p^2-1}{2}\right)\ell_1 + \ell_2\]

If \(\ell_2 = 0\), then \(w_{1,0} = \frac{(p-1)^2}{2}\), \(w_{2,0} = (p-1)^2\), and \(w_{3,0} = \frac{3(p-1)^2}{2}\). If \(\ell_1 = 0\), then \(w_{0,1} = \frac{p^2-1}{2}\), \(w_{0,2} = p^2 - 1\), and \(w_{0,3} = \frac{3(p-1)}{2}\). Also, if \(\ell_1\) and \(\ell_2\) are nonzero, then \(w_{1,1} = p(p-1)\), \(w_{2,1} = \frac{(p-1)}{2}(3p-1)\) and \(w_{1,2} = \frac{(p-1)}{2}(3p + 1)\). One can check that if \(p \neq 5\), all these weights are different and hence the spectrum of \(C\) is given by Table 3.

**Table 3. Weight distribution of \(C\) with \(p = 2, 5, 7\mod 9\) and \(p > 5\).**

| weight       | frequency     |
|--------------|---------------|
| 0            | \(A_{0,0} = 1\) |
| \(w_{1,0} = \frac{(p-1)^2}{2}\) | \(A_{1,0} = 3\left(\frac{p^2-1}{2}\right)\) |
| \(w_{2,0} = (p-1)^2\) | \(A_{2,0} = 3\left(\frac{p^2-1}{2}\right)^2\) |
| \(w_{3,0} = \frac{3(p-1)^2}{2}\) | \(A_{3,0} = \left(\frac{p^2-1}{2}\right)^3\) |
| \(w_{0,1} = \frac{p^2-1}{2}\) | \(A_{0,1} = 3\left(\frac{p^2-1}{2}\right)\) |

| weight       | frequency     |
|--------------|---------------|
| \(w_{0,2} = p^2 - 1\) | \(A_{0,2} = 3\left(\frac{p^2-1}{2}\right)^2\) |
| \(w_{0,3} = \frac{3(p^2-1)}{2}\) | \(A_{0,3} = \left(\frac{p^2-1}{2}\right)^3\) |
| \(w_{1,1} = p(p-1)\) | \(A_{1,1} = 6\left(\frac{p^2-1}{2}\right)^2\) |
| \(w_{2,1} = \frac{p^2-1}{2}(3p-1)\) | \(A_{2,1} = 3\left(\frac{p^2-1}{2}\right)^3\) |
| \(w_{1,2} = \frac{p^2-1}{2}(3p + 1)\) | \(A_{1,2} = 3\left(\frac{p^2-1}{2}\right)^3\) |

Notice that adding all the frequencies we get

\[\sum_{0 \leq i+j \leq 3} A_{i,j} = 1 + 6c + 12c^2 + 8c^3 = p^6\]

and therefore the code \(C\) has dimension 6 and minimum distance \(\frac{(p-1)^2}{2}\). That is, \(C\) has parameters \(\left[\frac{3(p^2-1)}{2}, 6, \frac{(p-1)^2}{2}\right]\).

If \(p = 5\), the weights \(w_{3,0} = w_{0,2}\) coincide, hence the weights are given by \(w_{1,0} = 8, w_{2,0} = 16, w_{3,0} = 24 = w_{0,2}, w_{0,1} = 12, w_{0,3} = 36, w_{1,1} = 20, w_{2,1} = 28\) and \(w_{1,2} = 32\) with frequencies

\[A_8 = A_{12} = 3c, A_{16} = 3c^2, A_{20} = 6c^2, A_{24} = (c + 3)c^2, A_{28} = A_{32} = 3c^3, A_{36} = c^3\]

where \(c = 12\).
10. The exceptional case

In this final section we compute the spectrum of the graphs associated to the exceptional 2-weight irreducible cyclic codes.

Schmidt and White gave the following list of pairs \((k, q)\), with \(k\) in ascending order,
\[
(11, 3^5), \ (19, 5^9), \ (35, 3^{12}), \ (37, 7^9) \ (43, 11^7), \ (67, 17^{33}),
\]
\[
(107, 3^{53}), \ (133, 5^{18}), \ (163, 41^{81}), \ (323, 3^{141}), \ (499, 5^{249}),
\]
(10.1)
such that \(C(k, q)\) is an exceptional 2-weight irreducible cyclic code and they conjectured that these are all such codes ([19]). We refer to them as exceptional pairs.

For all these pairs \((k, q)\), the number \(n = \frac{q-1}{k}\) is always a primitive divisor of \(q - 1\). For instance, for \((11, 3^5)\), we see that \(n = \frac{3^5-1}{11} = 22\) is a primitive divisor of \(3^5 - 1 = 242\) since \(22 \mid 3^5 - 1\) for \(a = 1, 2, 3, 4\). In fact, if \(n\) is not a primitive divisor of \(q - 1\), then the code \(C(k, q)\) would be a subfield subcode, which is not the case. Therefore, all these graphs \(\Gamma(k, q)\) are connected.

Now, we give the spectra of \(C(k, q)\) and \(\Gamma(k, q)\) for the exceptional pairs given above.

**Theorem 10.1.** Consider the eleven exceptional pairs \((k, q)\) from (10.1). The spectra of \(\Gamma(k, q)\) and \(C(k, q)\) can be explicitly computed. In Tables 4–6 we give these spectra, as well as the parameters as strongly regular graphs, for the first eight such pairs.

**Proof.** From Corollary 3.2 and Table 1 in [19], the nonzero weights of \(C(k, q)\) are given by
\[
w_1 = \frac{1}{2}(p-1)p^{\theta-1}(p^{m-\theta}-e^t) \quad \text{and} \quad w_2 = w_1 + \epsilon(p-1)p^{\theta-1},
\]
where, in our notations \((u = k, f = m, k = t)\),
\[
\begin{array}{cccccc}
 k & p & m & \theta & t & \epsilon \\
 11 & 3 & 5 & 2 & 5 & 1 \\
 19 & 5 & 9 & 4 & 9 & 1 \\
 35 & 3 & 12 & 5 & 17 & 1 \\
 37 & 7 & 9 & 4 & 9 & 1 \\
 43 & 11 & 7 & 3 & 21 & 1 \\
 67 & 17 & 33 & 16 & 33 & 1 \\
 107 & 3 & 53 & 25 & 53 & 1 \\
 133 & 5 & 18 & 8 & 33 & -1 \\
 163 & 41 & 81 & 40 & 81 & 1 \\
 323 & 3 & 144 & 70 & 161 & 1 \\
 499 & 5 & 249 & 123 & 249 & 1 \\
\end{array}
\]
(10.3)
From this, by Theorem 5.1, putting \(n = \frac{q-1}{k}\), the eigenvalues of \(\Gamma(k, q)\) are
\[
\lambda_1 = n - \frac{p}{p-1}w_1 \quad \text{and} \quad \lambda_2 = n - \frac{p}{p-1}w_2.
\]

To compute the multiplicities, notice that since \(\Gamma(k, q)\) is connected and \(C(k, q)\) is a 2-weight code, then \(\Gamma(k, q)\) is an strongly regular graph with parameters \(srg(q, n, e, d)\). It is known that the eigenvalues and their multiplicities of such a graph are respectively given by
\[
\lambda \pm = \frac{(e-d) \pm \Delta}{2} \quad \text{and} \quad m(\lambda \pm) = \frac{1}{2}(q-1) + \frac{2n+(q-1)(n-d)}{2}\Delta
\]
(10.5)
where
\[
\Delta = \sqrt{(e-d)^2 + 4(n-d)}.
\]
Thus \(\lambda_1 = \lambda^+\) and \(\lambda_2 = \lambda^-\). Since \(\lambda_1 + \lambda_2 = e - d\) and \(\lambda_1 - \lambda_2 = \Delta\), we obtain
\[
d = n - \frac{(\lambda_1-\lambda_2)^2-(\lambda_1+\lambda_2)^2}{4} \quad \text{and} \quad e = d + \lambda_1 + \lambda_2,
\]
(10.6)
and hence the multiplicities can be calculated from (10.5). Since \(\Gamma(k, q)\) is connected, the multiplicities \(m_i\) are also the frequencies \(A_{w_i}\) of the weights \(w_i\), \(i = 1, 2\), by Theorem 5.1.

By computing (10.2), using (10.3) and performing all the calculations in (10.4), (10.5) and (10.6), one can get the spectrum of \(\Gamma(k, q)\) and \(C(k, q)\) for all the exceptional pairs \((k, q)\). The spectra of the first eight pairs are given in Tables 4–6. The remaining pairs \((163, 41^{81}), (323, 3^{144})\) and \((499, 5^{249})\) are quite unmanageable, and we do not give them for readability. □
The frequencies $A_{w_j}$ of the weights $w_j$ of $C(k,q)$ in the table below are the multiplicities $m_i$ of the corresponding eigenvalues $\lambda_i$ of $\Gamma(k,q)$.

**Table 5.** Spectra of $C(k,q)$ for the first 5 exceptional pairs

| weights | $C(11,3^5)$ | $C(19,5^9)$ | $C(35,3^{12})$ | $C(37,7^9)$ | $C(43,11^7)$ |
|---------|-------------|-------------|----------------|-------------|-------------|
| $w_1$   | 22          | 82.000      | 10.044         | 934.332     | 411.400     |
| $w_2$   | 18          | 82.500      | 10.026         | 936.390     | 412.610     |

The pairs $(67,17^{33})$, $(107,3^{53})$ and $(133,5^{18})$ have intermediate complexity and are given separately in Table 6.

**Table 6.** Spectra of $\Gamma(k,q)$ and $C(k,q)$ for the 6th exceptional pair

| Pair $(67,17^{33})$ | $q$ | 40.254.497.110.927.943.179.349.807.054.456.171.205.137 |
|---------------------|-----|--------------------------------------------------------|
|                     | $n$ | 600.813.389.715.342.435.512.683.687.379.942.853.808 |
|                     | $e$ | 8.967.364.025.602.125.902.458.937.044.032.599.119 |
|                     | $d$ | 8.967.364.025.602.125.903.185.223.489.938.034.768 |
| $\lambda_1$        |     | 23.967.452.714.880.696.416                              |
| $m_1$               | 20.427.655.250.321.642.807.431.245.370.918.057.029.472 |
| $\lambda_2$        |     | -24.693.739.160.786.172.065                               |
| $m_2$               | 19.826.841.860.606.300.371.918.561.683.538.114.175.664 |
| $w_1$               | 565.471.425.614.439.939.283.497.632.625.940.854.016    |
| $w_2$               | 565.471.425.614.439.939.329.296.401.450.097.906.704    |

**Remark 10.2.** We can use the exceptional 2-weight irreducible cyclic codes to calculate the spectra of new irreducible cyclic codes constructed from them, in the same way as we did in the previous section for the semiprimitive 2-weight irreducible cyclic codes.

**Remark 10.3.** In [19], Schmidt and White gave an expression for the weights of the (conjecturally) all 2-weight irreducible cyclic codes. From Theorems 3.3, 5.1 and 10.1, we provide the frequencies of these weights in the semiprimitive and exceptional cases, thus completing the computation of the weight distributions of the connected 2-weight irreducible cyclic codes which are non subfield subcodes; i.e. those associated to connected GP-graphs.
Table 7. Spectra of $\Gamma(k, q)$ and $C(k, q)$ for the 7th and 8th exceptional pairs.

| Pair $(107, 3^{51})$ | Pair $(133, 5^{18})$ |
|----------------------|----------------------|
| $q = 19.383.245.667.680.019.896.796.723$ | $q = 3.814.697.265.625$ |
| $n = 181.151.828.669.906.728.007.446$ | $n = 28.681.934.328$ |
| $e = 360.610.649.595.226.895.872.817$ | $e = 215.848.943$ |
| $d = 360.610.649.595.234.814.457.952$ | $d = 215.652.162$ |
| $\lambda_1 = 419.685.012.154$ | $\lambda_1 = -96.922$ |
| $m_1 = 9.782.198.748.174.963.312.402.084$ | $m_1 = 2.868.193.432.800$ |
| $\lambda_2 = -427.603.597.289$ | $\lambda_2 = 293.703$ |
| $m_2 = 9.601.046.919.505.056.584.394.638$ | $m_2 = 946.503.832.824$ |
| $w_1 = 120.767.885.779.658.028.663.528$ | $w_1 = 22.945.625.000$ |
| $w_2 = 120.767.885.780.222.887.736.490$ | $w_2 = 22.945.312.500$ |

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