Meta-conformal invariance and the boundedness of two-point correlation functions

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Abstract

The covariant two-point functions, derived from Ward identities in direct space, can be affected by consistency problems and can become unbounded for large time- or space-separations. This difficulty arises for several extensions of dynamical scaling, for example Schrödinger-invariance, conformal Galilei invariance or meta-conformal invariance, but not for standard ortho-conformal invariance. For meta-conformal invariance in $(1 + 1)$ dimensions, which acts as a dynamical symmetry of a simple advection equation, these difficulties can be cured by going over to a dual space and an extension of these dynamical symmetries through the construction of a new generator in the Cartan sub-algebra. This provides a canonical interpretation of meta-conformally covariant two-point functions as correlators. Galilei-conformal correlators can be obtained from meta-conformal invariance through a simple contraction. In contrast, by an analogous construction, Schrödinger-covariant two-point functions are causal response functions. All these two-point functions are bounded at large separations, for sufficiently positive values of the scaling exponents.

Keywords: conformal invariance, conformal Galilean invariance, Ward identity, two-point function, dynamical scaling, Schrödinger invariance

1. Dynamical symmetries are playing an useful rôle in the elucidation of the properties of many complex systems. One of the best-known examples is conformal invariance in...
equilibrium phase transitions [8, 10]. When turning to dynamics, an often-studied case is the one of Schrödinger-invariance [20], which arises either from the free Schrödinger equation or else from the free diffusion equation, whose many applications to non-equilibrium dynamics include for example phase-ordering kinetics. One of the most elementary predictions of dynamical symmetries (such as conformal or Schrödinger-invariance) concerns the prediction of the form of the co-variant two-point functions, to be derived form the (e.g. conformal or Schrödinger) Ward identities [8, 9]. These are built from quasi-primary scaling operators \( \phi_i(t_i, r) \), depending locally on a ‘time’ coordinate \( t_i \in \mathbb{R} \) and a ‘space’ coordinate \( r \in \mathbb{R}^d \). Since both the conformal and the Schrödinger group contain time- and space-translations, and also spatial rotations, we can restrict to the difference \( t = t_1 - t_2 \) and the absolute value \( r = |r| = |r_1 - r_2| \). Then, for conformal [30] and Schrödinger-invariance [16], respectively, the covariant two-point functions of scalar quasi-primary operators read, up to a global normalisation constant

\[
\mathcal{G}_{12,\text{conf}}(t; r) = \langle \phi_1(t, r) \phi_2(0, 0) \rangle = \delta_{\text{n.a.}} [t^2 + r^2]^{-x_i}, \tag{1a}
\]

\[
\mathcal{R}_{12,\text{Schr}}(t; r) = \langle \phi_1(t, r) \tilde{\phi}_2(0, 0) \rangle = \delta_{\text{n.a.}} \delta(M_1 + \tilde{M_2}) r^{-x_i} \exp \left[-\frac{M_1 r^2}{2} - \frac{\tilde{M}_2}{t}\right]. \tag{1b}
\]

A few more comments are required:

1. For conformal invariance, the properties of the two-point function, built from the scaling operators \( \phi_i \), are described by the scaling dimensions \( x_i \). The two-point function is a correlator and is symmetric under permutation of the two scaling operators, viz. \( \mathcal{G}_{12}(t; r) = \mathcal{G}_{21}(-t; -r) \).

2. For Schrödinger-invariance, the two-point function is a (linear) response function—it is cast here formally in the form of a correlator by appealing to Janssen–de Dominicis theory [32] where one introduces a response operator \( \phi_i \) conjugate to the scaling operator \( \phi_j \). The two-point function is now characterised by the pair \( (x_i, M_i) \) of a scaling dimension and a mass \( M_i \) associated to each scaling operator \( \phi_i \). For ‘usual’ scaling operators, masses are positive by convention, whereas response operators \( \phi_i \) have formally negative masses, viz. \( \tilde{M}_i = -M_i < 0 \). Because of causality, a response function \( \mathcal{R}_{12}(t; r) \) is maximally asymmetric under permutation of the scaling operators, in the sense that for \( t > 0 \), one has \( \mathcal{R}_{12} = 0 \), whereas as it must vanish, viz. \( \mathcal{R}_{12} = 0 \), for \( t < 0 \), because of causality.

In this Letter, we wish to point that a straightforwardish application of the Ward identities of time–space symmetries, often schematically reproduced in the literature to derive results such as equations (1), implicitly assumes analyticity in the time–space arguments. If the two-point functions are not analytic, such an approach leads to inconsistencies\(^4\). As we shall show, in order to avoid this, a certain extension of the dynamical symmetry is required and we shall indicate under which circumstances Ward identities can indeed be written down straightforwardly, such that the results are physically consistent. Such co-variant \( n \)-point functions arise for example in several distinct non-relativistic versions of the AdS/CFT correspondence, see e.g. [1, 3–7, 11–14, 22, 27, 28, 33] and references therein.

2. Clearly, the result (1a) of conformal invariance [30] is physically reasonable for a correlator and if \( x_i > 0 \) it decays to zero for large time- or space separations, viz. \( |t| \to \infty \) or

\(^4\) An explicit example, leading to (8), will be given below: a special case is conformal galilean invariance.
$r \to \infty$. On the other hand, the result (1b) of Schrödinger-invariance [16] does not explicitly contain the constraint $t > 0$, physically required by causality. In addition, it is not obvious why the response should vanish for large separations, even if $x_i > 0$ is admitted. Of course, in the special case of Schrödinger-invariance, one might simply put in these features by hand. However, it is better to use the following, algebraically sound, procedure [19]:

(i) Consider the mass $\mathcal{M}$ as an additional coordinate.
(ii) Dualise by Fourier-transforming with respect to $\mathcal{M}_i$, which introduces dual coordinates $\zeta_i$. The terminology is borrowed from non-relativistic versions of the AdS/CFT correspondence.
(iii) Construct an extension of the Schrödinger Lie algebra $\widehat{\mathfrak{sch}}(d) := \mathfrak{sch}(d) \oplus \mathbb{CN}$, where the new generator $N$ is in the Cartan sub-algebra of $\widehat{\mathfrak{sch}}(d)$.
(iv) Use the extended Schrödinger Ward identities, in the dual coordinates, to find the covariant two-point function $\mathcal{R}(\zeta_1 - \zeta_2, t, r)$.
(v) Finally, transform back to the fixed masses $\mathcal{M}_i$. The result is [19, 21]

$$\mathcal{R}_{12, \text{Sch}}(t; r) = \langle \phi_1 (t, r) \bar{\phi}_2 (0, 0) \rangle = \delta_{x_1, x_2} \delta(\mathcal{M}_1 + \widehat{\mathcal{M}}_2) \times \Theta(\mathcal{M}_i t) t^{-\eta} \exp \left[ -\frac{\mathcal{M}_i r^2}{2t} \right]. \quad (2)$$

If one uses the convention $\mathcal{M}_i > 0$, the Heaviside function $\Theta$ expresses the causality condition $t > 0$. In addition, if $x_i > 0$, the response function decays to zero for large time- or space separations, as physically expected.

Comparing (2) with (1b), we see the importance of formulating the extended Schrödinger Ward identities in dual space, where it is legitimate to treat $\mathcal{R}$ as an analytic function of several variables. The extra generator $N$ provides an important restriction such that in direct space, with the $\mathcal{M}_i$ fixed, it becomes explicit that $\mathcal{R}$ rather is a distribution.

This comes about since the Ward identities lead to systems of first-order differential equations whose coefficients are holomorphic functions of time and space. Hence its solution, the sought two-point function, will be holomorphic as well [24]. A contrario, if the physically required form of the two-point function is not holomorphic, as in (2), a different form of the Ward identities must be sought.

Here, we shall describe how to implement this procedure for a non-standard representation of conformal invariance which leads to a two-point correlation function distinct from (1a). To be precise, we shall distinguish between ortho- and meta-conformal transformations$^5$.

**Definition 1.** (i) **Meta-conformal transformations** are maps $(t, r) \mapsto (t', r') = \mathcal{H}(t, r)$, depending analytically on several parameters, such that they form a Lie group. The associated Lie algebra is isomorphic to the conformal Lie algebra $\mathfrak{conf}(d)$. A physical system is meta-conformally invariant if its $n$-point functions transform covariantly under meta-conformal transformations. (ii) **Ortho-conformal transformations** (called ‘conformal transformations’ for

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$^5$ For the non-semi-simple Schrödinger algebra, the mathematical context are parabolic sub-algebras and their representations [25]. Parabolic sub-algebras are made up from the Cartan sub-algebra $\mathfrak{h}$ and only ‘positive’ generators. Our first construction of this kind still used the complete extension $\mathfrak{sch}(d) \subseteq (\mathfrak{conf}(d + 2))\subset$ into a conformal algebra in $d + 2$ dimensions [19]. We realised later that the extension by a single generator $N$ is sufficient [21], such that $\mathfrak{sch}(d)$ is a non-trivial parabolic sub-algebra. If the extension by $N$ is not made, the physical consistency problems remain [21].

$^6$ From the Greek prefixes ortho: right, standard; and meta: of secondary rank.
brevity) are those meta-conformal transformations \((t, r) \mapsto (t', r') = \mathcal{O}(t, r)\) which keep the angles in the time–space of points \((t, r) \in \mathbb{R}^{1+2}\) invariant.

In this Letter, we study the meta-conformal transformations, in \((1 + 1)\) time and space dimensions, with the following infinitesimal generators \([18, 20]\):

\[
X_n = -t^{n+1} \partial_t - \mu^{-1}[(t + \mu r)^n + t^n] \partial_r - (n + 1) \frac{\gamma}{\mu}[(t + \mu r)^n - t^n] - (n + 1)x t^n
\]

\[
Y_n = -(t + \mu r)^{n+1} \partial_r - (n + 1)(t + \mu r)^n
\]

\[(3)\]

such that \(\mu^{-1}\) can be interpreted as a velocity (‘speed of light or sound’) and where \(x, \gamma\) are constants (‘scaling dimension’ and ‘rapidity’). The generators obey the Lie algebra, for \(n, m \in \mathbb{Z}\)

\[
[X_n, X_m] = (n - m)X_{n+m}, \quad [X_n, Y_m] = (n - m)Y_{n+m}, \quad [Y_n, Y_m] = \mu(n - m)Y_{n+m}
\]

\[(4)\]

The isomorphism of \((4)\) with the conformal Lie algebra \(\text{conf}(2)\) is seen as follows \([18, 23]\): write \(X_0 = \ell_\alpha + \ell_\beta\) and \(Y_0 = \mu \ell_\alpha\), where the generators \(\langle \ell_\alpha, \ell_\beta \rangle_{n+m}\) satisfy \([\ell_\alpha, \ell_\beta] = (n - m)\ell_{n+m}\), \([\ell_\alpha, \ell_\beta] = (n - m)\ell_{n+m}\) and \([\ell_\alpha, \ell_\beta] = 0\). If \(\mu = 0\), the above Lie algebra \((4)\) is reproduced and hence is isomorphic to a pair of Virasoro algebras \(\text{vec}(S^1) \oplus \text{vec}(S^1) \cong \text{conf}(2)\) with a vanishing central charge.

The meta-conformal Lie algebra \((4)\) acts as a dynamical symmetry on the linear advection equation \([26]\)

\[
S\phi(t, r) = (-\mu \partial_t + \partial_r)\phi(t, r) = 0
\]

\[(5)\]

in the sense that a solution \(\phi\) of \(S\phi = 0\), with scaling dimension \(x_0 = x = \gamma/\mu\), is mapped onto another solution of the same equation. Hence the space of solutions of \(S\phi = 0\) is meta-conformal invariant \([18]\) (extended to Jeans–Poisson systems in \([31]\)). This follows from, with \(n \in \mathbb{Z}\)

\[
[S, Y_n] = 0, \quad [S, X_n] = -(n + 1)t^n \hat{S} + n(n + 1)(\mu x - \gamma) t^{n-1}
\]

\[(6)\]

Furthermore, the meta-conformal generators \((3)\) make up one of only four closed Lie algebras of local time–space scaling transformations, constructed by requiring the presence of time-translations with generator \(X_{-1}\), dilatations with dynamical exponent \(z\) and generator \(X_0\) and spatial translations generated by \(Y_{-1}\). The only three other closed Lie algebras which obey these requirements are ortho-conformal, conformal galilean and Schrödinger transformations \([18]\).

Now, quasi-primary scaling operators \([8]\) are characterised by co-variance under the maximal finite-dimensional sub-algebra \(\langle X_{\pm 1, 0}, Y_{\pm 1, 0} \rangle \cong \mathfrak{s}(2, \mathbb{R}) \oplus \mathfrak{s}(2, \mathbb{R})\) for \(\mu \neq 0\). Explicitly

\[
X_{-1} = -\partial_r, \quad X_0 = -t \partial_t - r \partial_r - x, \quad X_1 = -t^2 \partial_t - 2t r \partial_r - \mu r^2 \partial_r - 2x t - 2 \gamma r,
\]

\[
Y_{-1} = -\partial_r, \quad Y_0 = -t \partial_t - \mu r \partial_r - \gamma, \quad Y_1 = -t^2 \partial_t - 2 \mu t r \partial_r - \mu^2 r^2 \partial_r - 2 \gamma t - 2 \gamma \mu r.
\]

\[(7)\]

Here, the generators \(X_{-1}, Y_{-1}\) describe time- and space-translations, \(Y_0\) is a (conformal) Galilei transformation, \(X_0\) gives the dynamical scaling \(t \mapsto \lambda t, r \mapsto \lambda r\) (with \(\lambda \in \mathbb{R}\)) such that the so-called ‘dynamical exponent’ \(z = 1\) since both time and space are re-scaled in the same way and finally \(X_{+1}, Y_{+1}\) give ‘special’ meta-conformal transformations. For a meta-conformally invariant system, quasi-primary operators \(\phi_i\) are characterised by the parameters \((x_i, \gamma_i)\), while \(\mu\) is simply
a global dimensionful scale. Using the generators (7) and extending to two-body generators $X_n^{[2]}, Y_n^{[2]}$ in order to construct the meta-conformal Ward identities $X_n^{[2]} \langle \phi_1 \phi_2 \rangle = Y_n^{[2]} \langle \phi_1 \phi_2 \rangle = 0$, one finds the co-variant two-point function, up to normalisation [18, 20]

$$
\langle \phi_1(t_1, r_1) \phi_2(t_2, r_2) \rangle = \delta_{t_1, t_2} \delta_{r_1, r_2} (t_1 - t_2)^{-2\gamma} \left(1 + \mu \frac{r_1 - r_2}{t_1 - t_2}\right)^{-2\gamma/\mu}
$$

clearly distinct from the result (1a) of ortho-conformal invariance. However, the result (8) raises immediately the following questions:

1. Is $\langle \phi_1 \phi_2 \rangle$ a correlator or rather a response, since neither of the symmetry or causality conditions are obeyed?
2. Even if $x_i > 0$ and $\gamma_i/\mu > 0$, Why does $\langle \phi_1 \phi_2 \rangle$ not always decay to zero for large separations $|t_1 - t_2| \to \infty$ or $|r_1 - r_2| \to \infty$?
3. Why is there a singularity at $\mu(r_1 - r_2) = -(h - t_2)$?

These difficulties, which cannot be fixed ad hoc, become even more pronounced in the ‘non-relativistic limit’ $\mu \to 0$. From (3) and (4), one sees that the Lie algebra (4) contracts to the non-semi-simple ‘altern-Virasoro algebra’ alt(1) (but without central charges). Its maximal finite-dimensional sub-algebra is the conformal galilean algebra

$$
\mathfrak{alt}(1) \cong \mathfrak{CGA}(1) \cong \mathfrak{bms} [4, 5, 15, 17, 19, 29].
$$

The co-variant two-point function would become $\langle \phi_1 \phi_2 \rangle \sim \exp(-2\gamma r/\mu)$ [3, 18, 27] and does suffer from an analogous difficulty with boundedness. However, for the CGA algebra, it has been shown recently that a procedure analogous to the one of the Schrödinger algebra, as outlined above, can be applied. This finally leads to $\mathcal{C} = \langle \phi_1 \phi_2 \rangle \sim \exp(-2\gamma r/1)$. Then boundedness is satisfied and $\mathcal{C}$ does obey the symmetry relations of a correlator [23].

These two known examples might suggest that the correct identification of the Ward identities might only be possible for non-semi-simple Lie algebras, such as $\mathfrak{sch}(d)$ or CGA($d$). This is not true, as we shall now show. Our results will be summarised in propositions 2 and 3.

3. Our construction follows the same steps as outlined above which have already been used to recast the co-variant two-point functions of Schrödinger- and conformal Galilean invariance into a physically reasonable form, see [19, 21, 23].

First, we consider the ‘rapidity’ $\gamma$ as a new variable. Second, it is dualised through a Fourier transformation, which gives the quasi-primary scaling operator

$$
\hat{\phi}(\zeta, t, r) = \frac{1}{\sqrt{2\pi}} \int \mathbb{R} d\gamma \, e^{i\gamma \zeta} \, \phi_\gamma(t, r).
$$

The representation (3) of the meta-conformal algebra becomes

$$
X_n = \frac{i (n + 1)}{\mu} \left[ (t + \mu r)^n - t^n \right] \partial_t - t^{n+1} \partial_r - \frac{1}{\mu} \left[ (t + \mu r)^{n+1} - t^{n+1} \right] \partial_r - (n + 1) x^n,
$$

$$
Y_n = i (n + 1) (t + \mu r)^n \partial_t - (t + \mu r)^{n+1} \partial_r.
$$

Third, we seek an extension of the Cartan sub-algebra $\mathfrak{h}$ by looking for a new generator $N$ such that

$^7$ CGA($d$) is non-isomorphic to either the standard Galilei algebra or else the Schrödinger algebra $\mathfrak{sch}(d)$. 

5
where the constants $ab, nm$ are to be determined. To find $N$, we start from the ansatz

$$N = A(\zeta, t, r, \mu) \partial_t + B(\zeta, t, r, \mu) \partial_r + C(\zeta, t, r, \mu) \partial_\zeta + D(\zeta, t, r, \mu) \partial_\mu + E(\zeta, t, r, \mu)$$

which is slightly more general than the one used for the CGA(1) [23] and impose the commutators (11). The result is

$$N = a_{\alpha} \partial_t + b_r \partial_r + (\beta - \alpha)((\zeta + c) \partial_\zeta - \mu \partial_\mu) + \nu,$$

where $\alpha = a, \beta = b, c, \nu$ are arbitrary constants. Since we seek an independent generator in $h$, we can subtract the contribution proportional to $X_0$, retain as our definition of $N$

$$N := -r \partial_\zeta - (\zeta + c) \partial_\zeta + \mu \partial_\mu - \nu$$

and then have, for $n \in \mathbb{Z}$

$$[X_n, N] = 0, \quad [Y_n, N] = -Y_n. \quad (14)$$

$N$ is a dynamical symmetry of (5), since $[S, N] = -S$. This achieves the construction of the extended meta-conformal algebra $\mathfrak{conf}(2) := \mathfrak{conf}(2) \oplus \mathbb{C} N$, with commutators (4) and (14).

4. Co-variant two-point functions of quasi-primary scaling operators are found from the Ward identities $X_1 \phi \phi' = Y_1 \phi \phi' = N \phi \phi' = 0$, with $X_m, Y_m, N \in \mathfrak{conf}(2)$ and $n = \pm 1, 0$ [20]. Given the form of $N$, we also consider $\mu$ to be a further variable and set

$$\langle \tilde{\phi}(\zeta_1, t_1, r_1; \mu_1) \tilde{\phi}(\zeta_2, t_2, r_2; \mu_2) \rangle = \hat{F}(\zeta_1, \zeta_2, t_1, t_2, r_1, r_2; \mu_1, \mu_2). \quad (15)$$

Clearly, co-variability under $X_1$ and $Y_1$ implements time- and space-translation-invariance, such that $\hat{F} = \hat{F}(\zeta_1, \zeta_2, t, r; \mu_1, \mu_2)$, with $t = t_1 - t_2$ and $r = r_1 - r_2$. Next, co-variability under $X_0$ and $Y_0$ produces

$$(t \partial_t + r \partial_r + x_1 + x_2)\hat{F}(\zeta_1, \zeta_2, t, r; \mu_1, \mu_2) = 0, \quad (16)$$

$$(t \partial_t + \mu_1 r \partial_r - i(\partial_\zeta + \partial_{\zeta_1}) + (\mu_1 - \mu_2)r_2 \partial_\zeta)\hat{F}(\zeta_1, \zeta_2, t, r; \mu_1, \mu_2) = 0. \quad (17)$$

Since $\hat{F}$ must not any longer depend explicitly on $r_2$, (17) shows first that $\mu_1 = \mu_2 = \mu$ and

$$(t \partial_t + \mu r \partial_r - i(\partial_\zeta + \partial_{\zeta_1}) + (\mu_1 - \mu_2)r_2 \partial_\zeta)\hat{F}(\zeta_1, \zeta_2, t, r; \mu) = 0. \quad (18)$$

Similarly, co-variance under the special meta-conformal transformation $X_1, Y_1$ leads to

$$(r \partial_r - \partial_{\zeta_1}) - t(x_1 - x_2)\hat{F}(\zeta_1, \zeta_2, t, r; \mu) = 0, \quad (19)$$

$$(t + \mu r)(\partial_\zeta - \partial_{\zeta_1})\hat{F}(\zeta_1, \zeta_2, t, r; \mu) = 0. \quad (20)$$

Equation (20) states that $(\partial_\zeta - \partial_{\zeta_1})\hat{F} = 0$ such that $\hat{F} = \hat{F}(\zeta_1, t, r, \mu)$, with $\zeta_1 := \frac{1}{2}(\zeta + \zeta_2)$. Then equation (19) produces the constraint $x_1 = x_2$.

Finally, the required condition on the causality comes from co-variance under $N$, which gives (we shall absorb from now on $c$ into a translation of $\zeta_1$)

$$(r \partial_r + (\zeta_1 + c) \partial_{\zeta_1} - \mu \partial_\mu + \nu_1 + \nu_2)\hat{F}(\zeta_1, t, r; \mu) = 0, \quad (21)$$
5. The three conditions (16), (18) and (21) fix the function \( \hat{F}(\zeta_1, t, r; \mu) \) which depends on three variables and the constant \( \mu \), and also on the pairs of constants \( (x_1, \nu_1) \) and \( (x_2, \nu_2) \) which characterise the two quasi-primary scaling operators \( \hat{\phi}_{1,2} \). Solving equation (16), it follows that

\[
\hat{F}(\zeta_1, t, r; \mu) = t^{-2x_1} \mathcal{F}(u, \zeta_1, \mu), \quad \text{with } u = r/t \text{ and } x = x_1 = x_2. \tag{22}
\]

Changing variables according to \( v = \zeta_1 + iu \) and \( f(u, \zeta_1, \mu) = \mathcal{F}(u, v, \mu) \), equations (18) and (21) become

\[
\left( \frac{1 + i\mu u}{\mu} \partial_u + i\partial_v \right) \mathcal{F}(u, v, \mu) = 0 \quad \text{and} \quad (u\partial_u + v\partial_v - \mu\partial_{\mu} + \nu_1 + \nu_2)\mathcal{F}(u, v, \mu) = 0. \tag{23}
\]

If \( 1 + \mu u \neq 0 \), the first equation (23) gives \( \mathcal{F}(u, v, \mu) = \hat{G}(w, \mu) \), where \( w \) is obtained from

\[
w = \int du \frac{\mu u}{1 + \mu u} + i \int dv = u - \frac{\ln(1 + \mu u)}{\mu} + iv. \tag{24}
\]

Inserted in the other equation (23), this leads to \( (w\partial_w + \nu_1 + \nu_2)\hat{G}(w, \mu) = 0 \), hence

\[
\hat{G}(w, \mu) = \hat{G}_0(\mu)w^{-\nu_1-\nu_2} = \hat{G}_1(\mu)\left( \zeta_1 + i\frac{\ln(1 + \mu u)}{\mu} \right)^{-\nu_1-\nu_2}. \tag{25}
\]

Since \( \mu \) is merely a parameter, \( \hat{G}_1(\mu) \) is just a normalisation constant. We have proven

**Proposition 1.** The dual two-point function, covariant under the generators \( X_{1,0}^\pm, Y_{1,0}^\pm, N \) of the dual representation (10) and (13) of the meta-conformal algebra \( \text{conf}(2) \), is up to normalisation

\[
\hat{F}(\zeta_1, \zeta_2, t, r) = \langle \hat{\phi}_1(t, r, \zeta_1) \hat{\phi}_2(0, 0, \zeta_2) \rangle = \delta_{\nu_1, \nu_2} |t|^{-2x} \left( \frac{\zeta_1 + \zeta_2}{2} + i\frac{\ln(1 + \mu r/t)}{\mu} \right)^{-\nu_1-\nu_2}. \tag{26}
\]

6. To un-dualise, that is to carry out the the inverse Fourier transform on the dual two-point function (26), a precise mathematical fact is required. We write \( \mathbb{H}_+ (\mathbb{H}_-) \) for the upper (lower) complex half-plane \( w = u + iv \) with \( v > 0 \) (\( v < 0 \)). Recall from [2, chapter 11]:

**Definition 2.** A holomorphic function \( g: \mathbb{H}_+ \to \mathbb{C} \) is in the Hardy class \( \mathcal{H}_+^2 \) if the bound

\[
M^2 = \sup_{v > 0} \int_{\mathbb{R}} du |g(u + iv)|^2 < \infty, \tag{27}
\]

holds true. Analogously, a holomorphic function \( g: \mathbb{H}_- \to \mathbb{C} \) is in the Hardy class \( \mathcal{H}_-^2 \), when the supremum in (27) is taken over \( v < 0 \).
Lemma 1. [2] If \( g \in L^2(H^\mathbb{R}_+^\nu) \), then there are square-integrable functions \( G_\pm \in L^2(\mathbb{R}_+) \), such that for \( v > 0 \), one has the integral representation
\[
G(v) = G(u \pm iv) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} d\gamma e^{\pm iv\gamma} G_\pm(\gamma).
\]
(28)

To use this lemma, we fix
\[
\lambda := \frac{\ln(1 + \mu r/t)}{\mu}
\]
(29)
and recalling (26), we write \( F = \delta_{x,t_2} |t|^{-2\mu} \hat{f}(\zeta_+ + i\lambda) \) such that
\[
f_\pm(\zeta_+) := \hat{f}(\zeta_+ + i\lambda) = (\zeta_+ + i\lambda)^{-\nu_1 - \nu_2}.
\]
(30)

Lemma 2. [23] Let \( 2\nu := \nu_1 + \nu_2 > \frac{1}{2} \). Then \( f_\pm \in H^\mathbb{R}_+^\nu \) for \( \lambda > 0 \) and \( f_\pm \in H^\mathbb{R}_-^\nu \) for \( \lambda < 0 \).

Proof. Since \( f_\pm \) is holomorphic where it is defined, it remains to check the bound (27). Clearly, \(|f_\pm(u + iv)| = |(u + i(\nu + \lambda))^2| = (u^2 + (\nu + \lambda)^2)^{-\nu} \). For \( \lambda > 0 \), by explicit computation
\[
M^2 = \sup_{v > 0} \int_\mathbb{R} |f_\pm(u + iv)|^2 = \frac{\sqrt{\pi} \Gamma(2\nu - \frac{1}{2})}{\Gamma(2\nu)} \sup_{v > 0} (\nu + \lambda)^{1 - 4\nu} < \infty
\]
since the integral converges for \( \nu > \frac{1}{4} \). For \( \lambda < 0 \), the argument is similar. q.e.d.

Therefore, we have from lemma 1, equation (28), where \( \Theta \) is the Heaviside function
\[
\sqrt{2\pi} \hat{f}(\zeta_+ + i\lambda) = \Theta(\lambda) \int_0^{\infty} d\gamma_+ e^{i(\zeta_+ + i\lambda)\gamma_+} \hat{F}_+ (\gamma_+) + \Theta(-\lambda) \int_{-\infty}^{0} d\gamma_+ e^{-i(\zeta_+ + i\lambda)\gamma_+} \hat{F}_+ (\gamma_+)
\]
(31)
To carry out the back-transformation, we distinguish the cases \( \lambda > 0 \) and \( \lambda < 0 \). If one has \( \lambda > 0 \), we have from (29) and (31), with \( \zeta_\pm = \frac{1}{2}(\zeta_1 \pm \zeta_2) \)
\[
F = \frac{|t|^{-2\nu}}{\sqrt{2\pi}} \hat{G}_1(\mu) \int_\mathbb{R} d\gamma_+ \Theta(\gamma_+) d\gamma_- e^{-i(\gamma_+ + \gamma_-)\zeta_+} e^{-i(\gamma_+ + \gamma_-)\zeta_-} \int_\mathbb{R} d\gamma_+ \Theta(\gamma_+) \hat{F}_+ (\gamma_+) e^{-\gamma_+ \lambda} \int_\mathbb{R} d\gamma_- e^{-\gamma_- \lambda} \int_\mathbb{R} d\zeta_+ e^{i(\gamma_+ - \gamma_-)\zeta_+} \int_\mathbb{R} d\zeta_- e^{i(\gamma_- - \gamma_+)\zeta_-} \\
= \frac{|t|^{-2\nu}}{\sqrt{2\pi}} \hat{G}_1(\mu) \int_\mathbb{R} d\gamma_+ \Theta(\gamma_+) \hat{F}_+ (\gamma_+) e^{-\gamma_+ \lambda} \int_\mathbb{R} d\zeta_+ e^{-\gamma_- \lambda} \int_\mathbb{R} d\zeta_- e^{i(\gamma_+ - \gamma_-)\zeta_+} \\
= \delta_{x,t_2} \delta(\gamma_1 - \gamma_2) \Theta(\gamma_1) |t|^{-2\nu} f_1(\mu) f_2(\gamma_1) \exp(-2\gamma_1 \ln(1 + \mu r/t)/\mu) \\
= \delta_{x,t_2} \delta(\gamma_1 - \gamma_2) \Theta(\gamma_1) f_1(\mu) f_2(\gamma_1) |t|^{-2\nu}(1 + \mu r/t)^{-2\nu}/\mu,
\]
(32)
where in the third line two delta functions where recognised, and \( f_1, f_2 \) contain unspecified dependencies on \( \mu \) and \( \gamma_1 \), respectively. Analogously, for \( \lambda < 0 \), we have from (29) and (31)

An eventual shift \( \zeta_+ \to \zeta_+ + c \), see (21), can be absorbed into the re-definition \( \hat{F}(\gamma_+) e^{-\gamma_+ c} \to \hat{F}(\gamma_+) \).
Now, we discuss the meaning of the signs of $\lambda$, using (29). We adopt the convention that the mass $\mu > 0$. For $\lambda > 0$, one must have that $\ln(1 + \mu r/t) > 0$, hence $r/t > 0$. From the explicit expression (32), we also have $\gamma_1 > 0$. Conversely, for $\lambda < 0$, one must have $\ln(1 + \mu r/t) < 0$, hence $r/t < 0$ and from (33), we also have $\gamma_1 < 0$. Hence we have always $\gamma_1 r/t = |\gamma_1 r/t| > 0$, independently of the sign of $\lambda$. Therefore, we can always write for the time–space argument

$$\frac{\mu r}{t} = \frac{\mu}{\gamma_1} \frac{\gamma_1 r}{t} = \frac{\mu}{\gamma_1} \left| \frac{\gamma_1 r}{t} \right|$$

(if $\gamma_1 \neq 0$) and we have identified the source of the non-analyticity in the two-point function, which rendered the naïve use of the meta-conformal Ward identities (which would have led to (8)) inapplicable. Now, equations (32) and (33) combine as follows, which is or main result.

**Proposition 2.** With the convention that $\mu = \mu_1 = \mu_2 > 0$, and if $\nu_1 + \nu_2 > \frac{1}{2}$, the two-point correlation function, co-variant under the representation (7), enhanced by (13), of the extended meta-conformal algebra $\tilde{\text{con}}[2]$, reads up to normalisation

$$\mathcal{G}_{12}(t, r) = \langle \phi_1(t, r) \phi_2(0, 0) \rangle = \delta_{\nu_1 \nu_2} \delta_{\gamma_1 \gamma_2} \left| t \right|^{-2\nu_1} \left( 1 + \frac{\mu}{\gamma_1} \left| \frac{\gamma_1 r}{t} \right| \right)^{-2\gamma_1/\mu}. \quad (34)$$

This form has the correct symmetry $\mathcal{G}_{12}(t, r) = \mathcal{G}_{21}(-t, -r)$ under permutation of the scaling operators of a correlator. For $\gamma_1 > 0$ and $x_1 > 0$, the correlator decays to zero for $t \to \pm \infty$ or $r \to \pm \infty$. This is analogous to ortho-conformal invariance (1a), but distinct from a maximally asymmetric response function (2) as one finds for a Schrödinger-invariant system.

In the limit $\mu \to 0$, the extended meta-conformal algebra (4) and (14) contracts to the extended altern-Virasoro algebra $\tilde{\alpha}(1)$, whose maximal finite-dimensional sub-algebra is the extended conformal Galilean algebra $\tilde{\text{CGAL}}(1) = \text{CGA}(1) \oplus \mathbb{C}$. We recover as a special limit case:

**Proposition 3.** [23] If $\nu_1 + \nu_2 > \frac{1}{2}$, the two-point correlation function, co-variant under the extended conformal Galilean algebra $\tilde{\text{CGAL}}(1)$, reads up to normalisation

$$\mathcal{G}_{12}(t, r) = \langle \phi_1(t, r) \phi_2(0, 0) \rangle = \delta_{\nu_1 \nu_2} \delta_{\gamma_1 \gamma_2} \left| t \right|^{-2\nu_1} \exp \left( - \frac{2\gamma_1 r}{t} \right) \quad (35)$$

Summarising, we have shown that for time–space meta-conformal invariance, as well as for conformal galilean invariance, its $\mu \to 0$ limit, the co-variant two-point correlators are given by equations (34) and (35) and are explicitly non-analytic in the temporal-spatial variables. This implies that any form of the Ward identities which implicitly assumes such an
analyticity cannot be correct. In our construction of physically sensible Ward identities, it was necessary to extend the Cartan sub-algebra to a higher rank. This points towards the possibility for an extension of the meta-conformal and conformal galilean symmetries, in analogy to the embedding of the Schrödinger algebra $\mathfrak{sch}(1) \subset \mathfrak{conf}(3)$ into a conformal algebra in three-dimensions [19]. These extensions and their physical consequences remain to be found.

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References

[1] Aizawa N and Dobrev V K 2012 Schrödinger algebra and non-relativistic holography J. Phys.: Conf. Ser. 343 012007
Aizawa N and Dobrev V K 2010 Intertwining operator realization of non-relativistic holography Nucl. Phys. B 828 581
[2] Akhiezer N I 1988 Lectures on Integral Transforms (Translation of Mathematical Monographs) vol 70 (Providence, RI: Charkiv University 1984/American Mathematical Society)
[3] Bagchi A and Gopakumar R 2009 Galilean conformal algebras and AdS/CFT J. High Energy Phys. JHEP07(2009)037
[4] Bagchi A, Gopakumar R, Mandal I and Miwa A 2010 CGA in 2d J. High Energy Phys. JHEP08(2010)004
[5] Barnich G and Compère G 2007 Classical central extension for asymptotic symmetries at null infinity in three spacetime dimensions Class. Quantum Grav. 24 F15
Barnich G and Compère G 2007 Class. Quantum Grav. 24 3139
[6] Barnich G and Troessaert C 2011 BMS charge algebra J. High Energy Phys. JHEP12(2011)105
[7] Barnich G and Oblak B 2014 Notes on the BMS group in three dimensions J. High Energy Phys. JHEP06(2014)129
[8] Belavin A A, Polyakov A M and Zamolodchikov A B 1984 Infinite conformal symmetry in two-dimensional quantum field theory Nucl. Phys. B 241 333
[9] Boyer T H 1967 Conserved currents, renormalization and the Ward identity Ann. Phys 44 1
[10] di Francesco P, Mathieu P and Sénéchal D 1997 Conformal Field-Theory (Heidelberg: Springer)
[11] Duval C and Horváthy P A 2009 Non-relativistic conformal symmetries and Newton–Cartan structures J. Phys. A: Math. Theor. 42 465206
[12] Duval C, Hassaine M and Horváthy P A 2009 The geometry of Schrödinger symmetry in non-relativistic CFT Ann. Phys. 324 1158
[13] Fuertes C A and Moroz S 2009 Correlation functions in the non-relativistic AdS/CFT correspondence Phys. Rev. D 79 106004
[14] Gray N, Minic D and Pleimling M 2013 On non-equilibrium physics and string theory Int. J. Mod. Phys. A 28 1330009
[15] Havas P and Plebanski J 1978 Conformal extensions of the Galilei group and their relation to the Schrödinger group J. Math. Phys. 19 482
[16] Henkel M 1994 Schrödinger-invariance and strongly anisotropic critical systems J. Stat. Phys. 75 1023
[17] Henkel M 1997 Extended scale-invariance in strongly anisotropic equilibrium critical systems Phys. Rev. Lett. 78 1940
[18] Henkel M 2002 Phenomenology of local scale-invariance: from conformal invariance to dynamical scaling Nucl. Phys. B 641 405
[19] Henkel M and Unterberger J 2003 Schrödinger invariance and space–time symmetries Nucl. Phys. B 660 407
[20] Henkel M and Pleimling M 2010 Non-Equilibrium Phase Transitions vol. 2: Ageing and Dynamical Scaling far from Equilibrium (Heidelberg: Springer)
[21] Henkel M 2014 Causality from dynamical symmetry: an example from local scale-invariance
Algebra, Geometry and Mathematical Physics (Springer Proceedings in Mathematics &
Statistics) (Heidelberg: Springer) ed A Makhlouf et al vol 85, p 511 (arxiv:1205.5901)
[22] Henkel M, Hosseiny A and Rouhani S 2014 Logarithmic exotic conformal Galilean algebras Nucl.
Phys. B 879 292
[23] Henkel M 2015 Dynamical symmetries and causality in non-equilibrium phase transitions
Symmetry 7 2108
[24] Hille E 1997 Ordinary Differential equations in the Complex Domain (New York: Dover)
[25] Knapp A W 1986 Representation Theory of Semi-Simple Groups: An Overview Based on
Examples (Princeton, NJ: Princeton University Press)
[26] LeVeque R J 1999 Numerical Methods for Conservation Laws 2nd edn (Basel: Birkhäuser)
[27] Martelli D and Tachikawa Y 2010 Comments on Galilean conformal field theories and their
geometric realization J. High Energy Phys. JHEP05(2010)091
[28] Minic D, Vaman D and Wu C 2012 Three-point functions of ageing dynamics and the AdS-CFT
 correspondence Phys. Rev. Lett. 109 131601
[29] Negro J, del Olmo M A and Rodríguez-Marco A 1997 Nonrelativistic conformal groups J. Math.
Phys. 38 3786
  Negro J, del Olmo M A and Rodríguez-Marco A 1997 Nonrelativistic conformal groups II
J. Math. Phys. 38 3810
[30] Polyakov A M 1970 Conformal symmetry of critical fluctuations Sov. Phys.—JETP Lett. 12 381
[31] Stoimenov S and Henkel M 2015 From conformal invariance towards dynamical symmetries of
the collisionless Boltzmann equation Symmetry 7 1595
[32] Täuber U C 2014 Critical Dynamics (Cambridge: Cambridge University Press)
[33] Zhang P-M and Horváthy P A 2010 Non-relativistic conformal symmetries in fluid mechanics Eur.
Phys. J. C 65 607