**I. INTRODUCTION**

Modified gravity \[ f(R) \] is a crucial direction that one can follow in order to explain the early- and late-time phases of accelerated expansion of the Universe, instead of the introduction of dark energy \[ \Phi, \chi \] and the inflaton \[ \phi \]. Most theories that are modifications of General Relativity (GR) are based on various extensions of the standard Einstein-Hilbert action, and thus lie within the curvature-based gravitational formulation, namely \( f(R) \) gravity \[ 6 \], \( f(G) \) gravity \[ 5 \], etc. Alternatively, one can construct gravity theories starting from the torsion based formulation, and in particular from the Teleparallel Equivalent of General Relativity (TEGR) \[ 8,13 \].

In this formulation the Lagrangian is the torsion scalar \( T \), which is obtained by the contraction of the torsion tensor, and thus one can use it to construct torsional extended theories, namely \( f(T) \) gravity \[ 14,15 \], \( f(T_G) \) gravity \[ 16,17 \], etc (see \[ 18 \] for a comprehensive review). Although TEGR and GR are equivalent at the level of equations, their modifications correspond to different theoretical developments, and therefore in recent years theories of torsional modified gravity have attracted the interest of physicists in the literature \[ 19,32 \].

To examine which modified gravitational theories amongst the huge zoo of proposals are good candidates for the description of Nature, we resort to comparison with observations. Apart from the standard observational data that one can use, including Supernovae (SN), Cosmic Microwave Background (CMB), Baryonic Acoustic Oscillations (BAO), the growth rate, the Hub-

**II. \( f(T) \) GRAVITY**

Let us briefly review the \( f(T) \) gravitational theory \[ 18 \]. In torsional and teleparallel gravity one uses the tetrad fields \( e^a_i \) as the dynamical variables, which are defined at each point of the manifold as a base of orthonormal
vectors $e^A_{\mu}$, where $A, B, C... = 0, 1, 2, 3$ label the tangent spacetime coordinates, while $\mu, \nu, \rho... = 0, 1, 2, 3$ are the spacetime coordinates. Furthermore, a co-tetrad $e^A_{\mu}$ is defined through $e^\mu_A e^A_\nu = \delta^\mu_\nu$ and $e^\mu_A e^B_\mu = \delta^A_B$.

In order to describe the orthogonality and normalization of tetrad fields one introduces the tetrad metric $\eta_{AB} = \delta^B_A$ and thus the space-time metric can be reconstructed as

$$g_{\mu\nu} = \eta_{AB} e^A_\mu e^B_\nu .$$  \hspace{1cm} (1)

In teleparallel gravity, one uses the Weitzenböck connection $\hat{\Gamma}^{\lambda}_{\mu\nu} \equiv \epsilon^A_\lambda \partial_\mu e^A_\nu - \epsilon^A_\mu \partial_\mu e^A_\nu$, which is a connection leading to zero curvature but non-zero torsion. The resulting torsion tensor is

$$T^\lambda_{\mu\nu} \equiv \hat{\Gamma}^{\lambda}_{\nu\mu} - \hat{\Gamma}^{\lambda}_{\mu\nu} = \epsilon^A_\lambda (\partial_\mu e^A_\nu - \partial_\nu e^A_\mu) ,$$  \hspace{1cm} (2)

and therefore one can construct the torsion scalar through its contractions, namely

$$T \equiv \frac{1}{4} T^{\rho\mu\nu} T_{\rho\mu\nu} + \frac{1}{2} T^{\rho\mu\nu} T_{\rho\nu\mu} - T_{\rho\mu} T^{\rho\nu\mu} .$$  \hspace{1cm} (3)

As is well known, the Levi-Civita connection $\Gamma^\sigma_{\mu\nu}$ is related to any other connection, and thus to Weitzenböck connection too, through

$$\hat{\Gamma}^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} + K^\rho_{\mu\nu} ,$$  \hspace{1cm} (4)

where the contorsion tensor writes as

$$K^\rho_{\mu\nu} \equiv \frac{1}{2} (T^\rho_{\mu\nu} + T^\rho_{\nu\mu} - T^\rho_{\mu\nu}) .$$  \hspace{1cm} (5)

Similarly, the covariant derivative of a quantity $A_{\mu}$ with respect to the Levi-Civita connection $\nabla_{\mu}$ is related to its covariant derivative with respect to the Weitzenböck connection $\hat{\nabla}_{\mu}$ through

$$\nabla_{\mu} A_{\nu} = \hat{\nabla}_{\mu} A_{\nu} - K^\rho_{\mu\nu} A_{\rho} .$$  \hspace{1cm} (6)

Hence, since the curvature (Riemann) tensor corresponding to the Levi-Civita connection is

$$R^\rho_{\lambda\mu\nu} = \partial_\mu \Gamma^\rho_{\lambda\nu} + \Gamma^\sigma_{\mu\lambda} \Gamma^\rho_{\sigma\nu} - \partial_\nu \Gamma^\rho_{\lambda\mu} - \Gamma^\rho_{\nu\lambda} \Gamma^\sigma_{\mu\nu} ,$$  \hspace{1cm} (7)

one can straightforward derive the relation

$$R = -T + 2 \nabla_{\mu} T^\mu .$$  \hspace{1cm} (8)

Here, $R$ is the Ricci scalar corresponding to the Levi-Civita connection, $T$ is the torsion scalar \[8\] corresponding to the Weitzenböck connection, and $T^\mu$ is the contraction of the torsion tensor, defined as $T^\mu \equiv T^\nu_{\nu\mu}$.

In teleparallel gravity one uses the above torsion scalar as the Lagrangian of the theory, in a similar way to the use of the Ricci scalar as the Lagrangian of general relativity. Due to relation \[8\] one can immediately see that the two theories will be completely equivalent at the level of equations, and that is why this theory is called TTEGR. Nevertheless, one can use TTEGR as a base of extended gravity. Inspired by the $f(R)$ extensions of GR, one can generalize $T$ to a function $f(T)$, resulting to $f(T)$ gravity, which is characterized by the action

$$S = \int d^4x \frac{M_P^2}{2} f(T) ,$$  \hspace{1cm} (9)

where $e = \det(e^A_\mu)$ is the Ricci scalar as the action of general relativity, which satisfies the condition $T_{\rho\mu} T^{\rho\mu} = \frac{e}{e^A_\mu}$. Additionally, one can use TTEGR as a base of extended gravity. Inspired by the $f(R)$ extensions of GR, one can generalize $T$ to a function $f(T)$, resulting to $f(T)$ gravity, which is characterized by the action

$$S = \int d^4x \frac{M_P^2}{2} f(T) ,$$  \hspace{1cm} (9)

where $e = \det(e^A_\mu) = \sqrt{-\gamma}$ and with $M_P$ the Planck mass in units where the light speed is set to $c = 1$. Varying the above action with respect to the tetrads we extract the field equations as

$$e^{-1} \partial_\mu (e e^A_\sigma S^\sigma_{\rho\mu}) f_T + e^A_\delta S^\rho_{\mu\sigma} \partial_\sigma (f_T)$$

$$- f_T e^A_\mu T^\rho_{\sigma\mu} S^\sigma_{\rho\sigma} + \frac{1}{4} e^A_\delta f(T) = 4 \pi G e^A_\alpha \Theta^\alpha_{\rho\nu} ,$$  \hspace{1cm} (10)

where $f_T = \partial f / \partial T$, $f_T = \partial^2 f / \partial T^2$, and with $\Theta^\alpha_{\rho\nu}$ denoting the matter energy-momentum tensor. For convenience, in the above equation we have introduced the “super-potential”

$$S^\rho_{\mu\nu} = \frac{1}{2} (K^\mu_{\rho\nu} + \delta^\mu_\rho T^\alpha_{\nu\alpha} - \delta^\nu_\rho T^\alpha_{\mu\alpha}) .$$  \hspace{1cm} (11)

III. GWS IN f(T) GRAVITY

In this section we investigate cosmological GWs generated in $f(T)$ gravity. Since the dynamical variables are the four vector tetrad fields, instead of the symmetric metric field, we need to consider all the 16 components of the tetrads instead of the 10 components of the metric tensor. Definitely, comparing with the metric tensor, which has only coordinate indices, we should note that the tetrad $e^A_\mu$ has additional tangent space-time indices, and therefore the local Lorentz invariance will release 6 extra degrees of freedom \[58, 59\].

Since in the present work we are interested in the gravitational waves, which are detected through the change of line element, we only need to focus on the components of tetrad corresponding to the components of metric. In particular, decomposing the tetrad as \[58\]

$$e^A_\mu(x) = e^A_\mu(x) + \epsilon^A_\mu(x) ,$$  \hspace{1cm} (12)

which satisfies the condition

$$g_{\mu\nu}(x) = \eta_{AB} e^A_\mu e^B_\nu = \eta_{AB} e^A_\mu e^B_\nu ,$$  \hspace{1cm} (13)

where $e^A_\mu$ illustrates the part of tetrad corresponding to metric components and $\epsilon^A_\mu$ represents the degrees of freedom released from the local Lorentz transformation.
where (0) expansion (from now on the superscript a from the beginning of the alphabet spanning the spatial 2 torsion scalar from (3) reads as ϵ(whose number is thus six), we only need to focus on the e4·a part.

As usual, we perturb the tetrad fields eμ a around a flat Friedmann-Robertson-Walker (FRW) background as follows,

\[ e_0^μ = \delta_0^μ (1 + \psi) + a \delta_i^μ (G_i + \partial_i F) , \]
\[ e_a^μ = a \left[ \delta_a^μ (1 - \phi) + \delta_i^μ \Theta_3 a \left( \frac{1}{2} h_{ij} + \partial_i \partial_j B + \partial_i C_j + \partial_j C_i \right) \right] , \]
\[ e_0^μ = \delta_0^μ (1 - \psi) - \frac{1}{a} \delta^μ i (G_i + \partial_i F) , \]
\[ e_a^μ = \frac{1}{a} \left[ \delta_a^μ (1 + \phi) - \delta^μ i \delta_a \left( \frac{1}{2} h_{ij} + \partial_i \partial_j B + \partial_i C_j + \partial_j C_i \right) \right] , \tag{14} \]

where a(t) is the scale factor, and with small Latin indices from the beginning of the alphabet spanning the spatial part of the tangent space. In the above expressions we have introduced the scalar modes φ and ψ, the transverse vector modes C_i and G_i, and the transverse traceless tensor mode h_{ij}.

The above perturbed tetrad gives rise to the standard perturbed FRW metric

\[ g_{00} = -1 - 2\psi , \]
\[ g_{0i} = -a [\partial_i F + G_i] , \]
\[ g_{ij} = a^2 [1 - 2\phi] \delta_{ij} + h_{ij} + \partial_i \partial_j B + \partial_i C_j + \partial_j C_i . \tag{15} \]

We mention that the above perturbation expansions are slightly different from those provided in [20], since there are more than one forms of tetrads that correspond to the same metric.

In the rest of the manuscript we focus on the GWs, i.e. the tensor perturbations. Correspondingly, from now on, we set the scalar and vector perturbations to zero for convenience. Inserting [13] into [2] we obtain the perturbed torsion tensor as

\[ T^i_{0j} = H \delta_{ij} + \frac{1}{2} h_{ij} \]
\[ T^i_{jk} = \frac{1}{2} (\delta_j h_{ik} - \delta_k h_{ij}) , \tag{16} \]

where H = \dot{a}/a is the Hubble function. As a result, the torsion scalar from [23] reads as 3

\[ T = T^{(0)} + O(h^2) = 6H^2 + O(h^2) , \tag{17} \]

where T^{(0)} is the zeroth-order part of the torsion scalar expansion (from now on the superscript (0) marks the zeroth-order part of an expanded quantity). Thus, relation [17] implies that the torsion scalar is not affected by the tensor fluctuations at linear expansion. Additionally, from [11] we acquire the perturbed super-potential as

\[ S_i^{0j} = H \delta_{ij} - \frac{1}{4} h_{ij} \]
\[ S_i^{jk} = \frac{1}{4a^2} (\partial_j h_{ik} - \partial_k h_{ij}) . \tag{18} \]

Inserting the above perturbations in the field equations [10] we can obtain

\[ 4f_T \left[ (H + 3H^2) \delta_{ij} + \frac{1}{4} \left( -\dot{h}_{ij} + \nabla^2 h_{ij} - 3H h_{ij} \right) \right] + 4f_T (H \delta_{ij} - \frac{\dot{h}_{ij}}{4}) - f \delta_{ij} = 16\pi G \Theta^4 j , \tag{19} \]

where the derivative f_T is calculated at \( T = T^{(0)} = 6H^2 \). The perturbation part of the above equation leads to the equation of motion for the GWs, namely

\[ \ddot{h}_{ij} + (3H + \frac{f}{f_T}) \dot{h}_{ij} - \nabla^2 h_{ij} = 0 . \tag{20} \]

IV. GRAVITATIONAL WAVES IN \( f(T) \) GRAVITY VIA THE EFT APPROACH

In this section we will investigate the GWs in \( f(T) \) gravity through the EFT approach [40]. Inspired by the theory of spontaneous symmetry breaking of the SU(2) × U(1) gauge theory of the Standard model, one can apply the EFT to cosmological perturbations of modified gravity theories, by treating them as the Goldstone boson of spontaneously broken time-translations. Similarly to the gauge field theory with spontaneous symmetry breaking, one can also choose the unitary gauge, “eating” the would-be Goldstone bosons and making the theory to display only metric degrees of freedom. A significant advantage is that this process can organize the terms of the action as number of perturbations, allowing us to deal with the background and perturbations separately [41]. The EFT approach has been applied to the inflationary context [42,43] or to the dark energy paradigm [44,45] (see also [46] for an EFT analysis of dark energy models in the light of GW170817).

A. The description of EFT approach

Let us first describe the EFT approach. For simplicity we use the curvature-based formulation of gravity, and

2 Mind the sign difference comparing to the majority of \( f(T) \) works, due to the mostly-plus signature we use in this manuscript.
we start by considering a general action of the form [43]

\[
S = \int \! d^4x \left\{ \frac{M_p^2}{2} \sqrt{-g} \left( \Omega(t) R - \Lambda(t) - b(t) g^{00} \right) + M_p^2 (\delta g^{00})^2 - \hat{\delta} g^{00} \delta K - \tilde{M}_p^2 \delta K^2 - \tilde{M}_p^2 \delta K_{\mu}^{\nu} \delta K^\nu_{\mu} + \mu_1^2 \delta g^{00} \delta R \right. \\
+ \left. \frac{1}{3} \mu_2^2 \delta g^{00} \delta g^{00} + \lambda_1 \delta R^2 + \lambda_2 \delta R_{\mu\nu} \delta R^{\mu\nu} + \mu_1^2 \delta g^{00} \delta R \right\} + \sqrt{-g} \left[ \frac{M_p^2}{3} (\delta g^{00})^3 - \hat{\delta} g^{00} (\delta g^{00})^2 \delta K + \ldots \right],
\]

(21)

where \( C^{\mu\nu\rho\sigma} \) is the Weyl tensor, \( \delta K_{\mu}^{\nu} \) is the perturbation of the extrinsic curvature and \( R \) is the Ricci scalar corresponding to the Levi-Civita connection. Additionally, we have included the time-dependent functions \( \Omega(t) \), \( \Lambda(t) \), \( b(t) \) which are determined by the background evolution, and finally we have allowed for various time-dependent coefficients in front of the various terms. We mention here that the first line of the action corresponds to the background evolution, the lines from second to fourth are quadratic in perturbations, while the fifth line is cubic in perturbations.

**B. The EFT of teleparallel gravity**

Now we proceed to the application of the EFT approach in TEGR and the modified theory based on this framework [33], an application which is facilitated by the fact that teleparallel gravity can be seen as a translational gauge theory of gravity [12]. As we will see, in order to do this we need to add some extra terms to the above action, both at the background and perturbation parts.

We begin by referring to the unitary gauge. In a general perturbed FRW geometry, a scalar degree of freedom is decomposed as

\[
\phi(t, \vec{x}) = \phi_0(t) + \delta \phi(t, \vec{x}).
\]

(22)

The unitary gauge is to choose the coordinate \( t \) to be a function of \( \phi \), namely \( t = t(\phi) \), thus \( \delta \phi = 0 \) and the action displays only metric degrees of freedom.

The unitary gauge action must be invariant under the unbroken symmetries. This implies that the action should leave spatial diffeomorphisms unbroken. Thus, having relations [34,35] and [33] in mind, it is reasonable to include both curvature and torsion terms in the action. In summary, the action of EFT can contain [50]:

- i) Terms that are invariant under all diffeomorphisms. These are four-dimensional diffeomorphisms invariant scalars such as \( R \) and \( T \), which are in general multiplied by functions of time.
- ii) Terms that are invariant only under spatial diffeomorphisms. Firstly, these can be scalars that are constructed by spatial tensors such as the spatial Riemann tensor \( \mathcal{R}_{\mu\nu\rho\sigma} \), the extrinsic curvature \( K_{\mu\nu} \), as well as the the spatial torsion tensor \( T^\nu_{\mu\nu} \) and the “extrinsic torsion” \( \hat{K}_{\mu\nu} \). We will explain the latter two terms more clearly below.

Secondly, these can be four-dimensional covariant tensors with upper 0 indices such as \( g^{00} \), \( R^{00} \) and \( T^0 (T^0 \text{ is the 0-index component of the contracted torsion tensor } T^\nu) \).

The terms of type ii) arise from the definition of a preferred time slicing by the scalar field \( \phi \), namely

\[
n_{\mu} = \frac{\partial_{\mu} \phi(t)}{\sqrt{-(\partial \phi)^2}} = \frac{\delta_{\mu}^0}{\sqrt{-g^{00}}}. \quad (23)
\]

Since the time-translation is broken, we can contract covariant tensors with this unitary vector orthogonal to the \( t = \text{const} \) surfaces, and this is where the terms with upper 0 indices arise from. Then we consider the Weitzenböck covariant derivative of \( n_{\mu} \), projected on the surface of constant \( t \), i.e.

\[
h_{\mu} \nabla_{n} n_{\nu} \equiv \hat{K}_{\mu\nu}. \quad (24)
\]

where \( h_{\mu\nu} \equiv g_{\mu\nu} + n_{\mu} n_{\nu} \) is the induced metric of this surface. It is easy to verify that this quantity is a spatial tensor, therefore we refer to it as “extrinsic torsion”. Given the relation (24) between the Weitzenböck covariant derivative and the ordinary covariant derivative, we can extract the following relations between extrinsic curvature and extrinsic torsion:

\[
\hat{K}_{\mu\nu} = h_{\mu} \nabla_{n} n_{\nu} = K_{\mu\nu} - K_{\mu}^{\lambda} n_{\lambda} + n_{\mu} \frac{1}{g^{00}} T^{00}_{\nu}. \quad (25)
\]

After contracting its two indices, we have

\[
K = \nabla_{\mu} n^{\mu} = \nabla_{n} n_{\mu} = K_{\mu}^{\lambda} n_{\lambda} = K + T^{00}_{\nu} = \hat{K} + \left( -g^{00} \right)^{-1/2} T^0. \quad (26)
\]

Accordingly, having in mind (25) and (26) and observing the action (21), we can deduce that, if we allow \( T^{00}_{\nu} \) and \( T^0 \) to be present in our action, we can avoid the use of \( \hat{K} \) and \( K_{\mu\nu} \).

We proceed by considering the covariant derivative of \( n_{\mu} \) perpendicular to the time slicing, which gives

\[
n^{\sigma} \nabla_{n} n_{\nu} = n^{\sigma} \nabla_{n} n_{\nu} + \frac{1}{g^{00}} T^{00}_{\nu}. \quad (27)
\]

As illustrated in [43], the first term on the right-hand side leads to a term that contains \( g^{00} \) and \( h_{\mu}^{n} \), which has already been included in action (21). Therefore, we can also avoid \( n^{\sigma} \nabla_{n} n_{\nu} \) if we allow for \( T^{00}_{\nu} \) in the action.

Using the unit normal vector \( n_{\mu} \) (or the projection operator \( h_{\mu}^{n} \)), we can construct three-dimensional spatial tensors, whose contractions provide the spatial diffeomorphisms invariant scalar, which can then be used in the action. On the other hand, for convenience we only use four-dimensional tensors, since their three-dimensional
Finally, by observing relation (8), one deduces that the integration of the boundary term with a time-dependence coefficient takes the form

$$S = \int d^4x \sqrt{-g} f(t) \nabla_\mu T^\mu = -\int d^4x \sqrt{-\hat{g}f(t)} T^0 .$$

(34)

Having all the above discussions in mind, we can now write down the EFT action of teleparallel gravity. As we showed, the remaining background terms are $R$ and $T^0$, and hence we have

$$S = \int d^4x \sqrt{-g} \left[ \frac{M^2}{2} \Omega(t) R - \Lambda(t) - b(t) g^{00} + \frac{M^2}{2} d(t) T^0 \right]$$

$$+ S^{(2)},$$

(35)

with $d(t)$ a time-dependent function. In the second line of the above action, $S^{(2)}$ has been introduced to include all the terms that start explicitly quadratic in the perturbations, and hence its presence does not affect the background dynamics. In addition to the terms shown in action (21), $S^{(2)}$ can also include: i) Pure torsion terms such as $\delta T^2$, $\delta T^0\delta T^0$ and $\delta T^{\mu\nu}\delta T_{\rho\nu}$ (note that since we include $T^0$ in the action, according to (20) we can avoid the presence of $\dot{K}$); ii) Terms mixing torsion and curvature such as $\delta g^{00}\delta T^0$, $\delta g^{00}\delta T^0$, $\delta T\delta R$ and $\delta K\delta T^0$.

C. The EFT of $f(T)$ gravity

In the previous subsection we applied the EFT approach to teleparallel equivalent of general relativity. Thus, we have all the machinery to proceed to the EFT approach of $f(T)$ gravity.

A first complication that arises from such project is the incorporation of the time slicing. Similarly to the discussion of $f(R)$ theory within EFT formalism [15], we firstly expand the action (40) with respect to the background as

$$S = \frac{M^2}{2} \int d^4x \sqrt{-g} \left[ f_T T + f(T^{(0)}) - f_T T^{(0)} \right]$$

$$+ \frac{1}{2} f_{TT} \delta T^2 + \ldots .$$

(36)

Afterwards, we can fix the time slicing in a way that it coincides with uniform $T$ hypersurfaces. This treatment will make the terms beyond the linear order in the above expansion to vanish, since their contribution to the equations of motion will always include at least one power of $\delta T$. Thus, we obtain the unitary-gauge action as follows

$$S = \frac{M^2}{2} \int d^4x \sqrt{-g} \left[ -f_T R - 2f_T T^{(0)} - f_T T + f(T^{(0)}) \right],$$

(37)

which comparing with (36) then implies:

$$\Omega(t) = -f_T(T^{(0)}) , \quad d(t) = -2f_T(T^{(0)}) ,$$

$$\Lambda(t) = \frac{M^2}{2} \left[ T^{(0)} f_T(T^{(0)}) - f(T^{(0)}) \right] , \quad b(t) = 0 .$$

(38)
Note that from Eq. (37) and (34) we also deduce that $S^{(2)} = 0$, and thus one only needs to deal with the background part. Lastly, if the additional $T^0$ term vanishes, then the above action will reproduce the EFT form of $f(R)$ gravity 45.

D. Propagations of GWs in $f(T)$ gravity from EFT

We have now all the tools in order to proceed to the investigation of GWs in the EFT approach. As we mentioned earlier, in order to study the GWs we only need to focus on the $h_{ij}$ component of the tetrad. Additionally, we mention that the quantities in action 37 must be expanded to quadratic order in perturbations in order to obtain the dynamical behavior. Thus, the perturbative components of the metric, up to second order in perturbations, read

$$g_{00} = -1 \ , \ g_{0\mu} = 0 \ ,$$
$$g_{ij} = a^2 (\delta_{ij} + h_{ij} + \frac{1}{2} \delta_{ik} h_{kj}) \ ,$$  \hspace{1cm} (39)

which can be derived from the perturbative tetrads (up to second order in perturbations as well):

$$e^0_{\mu} = \delta^0_{\mu} \ ,$$
$$e^\mu_a = a \delta^\mu_a + \frac{a}{2} \delta^\mu_i h_{ij} + \frac{a}{8} \delta^\mu_i \delta^a_k h_{kj} \ ,$$
$$e^\mu_0 = \delta^\mu_0 \ ,$$
$$e^a_i = \frac{1}{a} \delta^a_i - \frac{1}{2a} \delta^\mu_i \delta^a_j h_{ij} + \frac{1}{8a} \delta^i_\mu \delta^a_j h_{kj} \ .$$  \hspace{1cm} (43)

Accordingly, we can calculate $T^0$ and find that its perturbation part vanishes up to second order, which is given by

$$T^0 = -T_0 = e^A_\Lambda \partial_\nu e_\Lambda^A - e^A_\Lambda \partial_\Lambda e_0^A = 3H \ .$$  \hspace{1cm} (44)

As a result, the $T^0$ term in action 37 does not lead to a new kinetic term. This feature lies beyond the main result of the present work.

With the metric provided in 39, we calculate $R$ as follows

$$R = (3) R + K_{\mu
u} K^{\mu\nu} - K^2 + 2 T_{\nu} (Kn^\nu - n^\nu T_{\nu}) \ ,$$  \hspace{1cm} (45)

and up to second order we have the following expressions:

$$(3) R \approx - \frac{1}{4} a^{-2} (\partial_i h_{ki} \partial_j h_{kj}) \ ,$$
$$K^{ij} K_{ij} \approx 3H^2 + \frac{1}{4} h_{ij} h_{ij} \ ,$$
$$K \approx 3H \ ,$$  \hspace{1cm} (47)

with $R$ being the spatial curvature scalar. We mention that when a scalar field $\phi$ is non-minimally coupled to $R$, the total derivative in 45 does not vanish. Thus, we should consider its integral with a time-dependent coefficient, which is given by

$$\int d^4 x \sqrt{-g} f(t) 2 \nabla_{\nu} (Kn^\nu - n^\nu \nabla_{\nu} n^\nu)$$
$$= \int d^4 x \sqrt{-g} (-2fKn^\nu) = \int d^4 x \sqrt{-g} (6H \dot{f}) \ .$$  \hspace{1cm} (49)

Hence, we deduce that this term does not contribute to tensor perturbations up to second order.

As a result, we obtain the final form of the action 37 for the linearized GWs within a cosmological background as follows:

$$S = \frac{M_p^2}{2} \int d^4 x \sqrt{-g} \left[ \frac{f_T}{4} (a^{-2} T_{ij} \nabla_i h_{jj} - h_{ij} h_{ij}) \right.$$
$$+ 6H^2 f_T - 12H \dot{f}_T - T^{(0)} f_T + f(T^{(0)}) \] \ .$$  \hspace{1cm} (50)

The above action is exactly the EFT action of cosmological GWs within the $f(T)$ gravity up to second order. Varying this action with respect to $h_{ij}$, we can again yield the equation of motion 20. Then, performing the Fourier transformation and tracing the evolution of a fixed Fourier mode of GWs we obtain

$$\ddot{h}_{ij} + 3H (1 - \beta_T) h_{ij} + \frac{k^2}{a^2} h_{ij} = 0 \ ,$$  \hspace{1cm} (51)

where we have introduced the dimensionless parameter

$$\beta_T = - \frac{T'}{3H f_T} \ .$$  \hspace{1cm} (52)

Observing Eq. (51), and comparing it with the general evolution equation of linear, transverse-traceless perturbations over an FRW background, we can immediately deduce that the speed of GWs is equal to one, i.e. equal to the speed of light. As a result, one can see that the experimental constraint of GW170817 on the GW speed in $f(T)$ gravity is trivially satisfied.

We can proceed with our analysis by referring to the dispersion relation and the frequency of cosmological GWs in $f(T)$ gravity. Taking the ansatz of the Fourier transformation of cosmological GWs as

$$h_{ij} = \int d^3 k e^{i\vec{k} \cdot \vec{x}} [A_{ij} e^{i\omega t} + B_{ij} e^{-i\omega t}] \ ,$$  \hspace{1cm} (53)

and inserting it into 20, we obtain

$$\left( \frac{k^2}{a^2} - \omega^2 \right) \pm 3iH \omega \left( 1 + \frac{f_T}{3H f_T} \right) = 0 \ .$$  \hspace{1cm} (54)

The solution of the above equation leads to the dispersion relation, which can be expressed as

$$\left| \frac{d\omega}{dk} \right| = \frac{1}{a} \left[ 1 + \frac{9a^2}{4k^2} H^2 (1 - \beta_T)^2 \right]^{-\frac{1}{2}} \ .$$  \hspace{1cm} (55)

Note that the dimensionless coefficient $\beta_T$ from 52, using that $T = 6H^2$ can be further written as

$$\beta_T = \frac{d ln f_T}{d ln T} (1 + \omega_{tot}) \ ,$$  \hspace{1cm} (56)
with $w_{\rm tot} \equiv 1 - \frac{2\beta}{\beta T}$ the total equation-of-state parameter of the universe ($w_{\rm tot} = p_{\rm tot}/\rho_{\rm tot}$ with $p_{\rm tot}$ and $\rho_{\rm tot}$ the total pressure and energy density of the universe respectively).

It is straightforward to notice that if the gravitational theory is GR or TEGR, then $\beta_T = 0$. However, for a general case of $f(T)$ gravity, the form of $\beta_T$ deviates from zero. Therefore, even if the propagation of GWs in $f(T)$ gravity remains at the speed of light, similarly to general relativity, a high precision measurement of the dispersion relation of cosmological GWs can impose observational constraints upon the parameter $\beta_T$, and hence reveal the effect of $f(T)$ gravity. Definitely, since in viable $f(T)$ models $f_{\beta T} \ll 1$ [51, 54], we deduce that it is quite difficult to utilize the present GWs experiments to probe such a deviation in the dispersion relation. However, a future measurement of the value of the $\beta_T$ parameter may become possible under the great development of GW astronomy [55, 56] (note that $\beta_T$ is related to the running of the effective Planck mass [54, 57] but it does not coincide with it). If a non-zero $\beta_T$ is measured in future observations, it could be the smoking gun of modified gravity, and the significance of this signature would become even greater if one has in mind that probably there is no impact of $f(T)$ gravity upon the polarization modes of GWs when comparing to the case of GR [57, 58].

V. CONCLUSIONS

In this work we performed a detailed analysis of the GWs in $f(T)$ gravity and cosmology. Taking advantage of the fact that teleparallel gravity can be seen as a translational gauge theory of gravity, we applied the EFT approach, which allows to analyze the perturbations in a systematic way and separately from the background evolution. Constructing all terms in the perturbative action up to second order, we extracted the propagation equation for the GWs. For completeness, we alternatively extracted the same equation through the standard scalar tensor vector and tensor perturbation analysis around a cosmological background.

From the GW propagation equation we deduced that the speed of GWs in $f(T)$ gravity is equal to the light speed. This is the main result of the present work and it is very important since it shows that $f(T)$ gravity can trivially satisfy the combined constraints of GW170817 and GRB170817A. Note that this is not guaranteed in a general theory of modified gravity, in which GWs propagate with a speed that may be different from the speed of light. Therefore, it is necessary to examine the observational constraints upon the propagation of cosmological GWs in modified gravities. The above result offers an additional advantage for $f(T)$ gravity.

Finally, examining the dispersion relation and the frequency of the GWs in $f(T)$ gravity within an FRW background, we found that there is a deviation from the result of GR, which is quantified by a new parameter $\beta_T$, due to a modification of the friction term in the perturbation equation of cosmological GWs. Although for $f(T)$ models that are allowed by present observations the value of $\beta_T$ is typically small and is difficult to be tested in the current GW data, a possible future measurement in advancing GW astronomy would be the smoking gun of testing this type of modified gravity.

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3 Using the best-fit values for the parameters of specific $f(T)$ models [54, 55] we obtain $\beta_T \sim 10^{-2} - 10^{-1}$ in the low-redshift region.

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