Lattice Topological Field Theory in Two Dimensions

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Abstract

The lattice definition of a two-dimensional topological field theory (TFT) is given generically, and the exact solution is obtained explicitly. In particular, the set of all lattice topological field theories is shown to be in one-to-one correspondence with the set of all associative algebras $R$, and the physical Hilbert space is identified with the center $Z(R)$ of the associative algebra $R$. Perturbations of TFT's are also considered in this approach, showing that the form of topological perturbations is automatically determined, and that all TFT’s are obtained from one TFT by such perturbations. Several examples are presented, including twisted $N = 2$ minimal topological matter and the case where $R$ is a group ring.

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1 Introduction

Any consistent quantum field theory is expected to be realized as a continuum limit of a lattice model. Furthermore, the lattice definition is the only known method to investigate the non-perturbative structure of quantum field theories.

In this paper, we show that 2D topological field theories (TFT’s), especially topological matter systems, can also be realized as lattice models, which will be called lattice topological field theories (LTFT’s). The advantage of this approach to TFT over the conventional continuum field theoretic one [1] is in that this lattice definition makes much easier the understanding of geometric and algebraic structure of TFT. Moreover, since there should not be any dimensionful parameters in TFT or in LTFT, we do not need to take a continuum limit in our lattice model. This fact allows easy calculation of various quantities.

We first recall the basic axiom of TFT. Let \( \hat{g}_{\mu\nu} \) be a background metric on a surface, on which matter field \( X_{\text{matter}} \) lives. The partition function \( Z[\hat{g}_{\mu\nu}] \) is defined by

\[
Z[\hat{g}] \equiv \int \mathcal{D} \hat{g} X_{\text{matter}} \exp \left( -S[X_{\text{matter}}, \hat{g}] \right), \quad (1.1)
\]

with \( S[X_{\text{matter}}, \hat{g}] \) the action. We also assume the existence of fermionic conserved quantity \( Q_{\text{BRST}} \) which generates all the symmetry of the theory and satisfies the nilpotency condition: \( Q_{\text{BRST}}^2 = 0 \). The theory is called a TFT if the energy-momentum tensor \( T_{\mu\nu} \) is BRST-exact:

\[
T_{\mu\nu} = \{ Q_{\text{BRST}}, \ast \}. \quad (1.2)
\]

Recall here that the energy-momentum tensor is defined by

\[
T_{\mu\nu}(x) \equiv \frac{-4\pi}{\sqrt{\hat{g}(x)}} \frac{\delta}{\delta \hat{g}^{\mu\nu}(x)} (-\ln Z[\hat{g}]). \quad (1.3)
\]

Therefore, the condition (1.2) implies that our partition function \( Z[\hat{g}] \) is invariant under local changes of background metric if we restrict ourselves to the “physical” Hilbert space \( \mathcal{H}_{\text{phys}} \equiv \{ |\text{phys}\rangle \mid Q_{\text{BRST}} |\text{phys}\rangle = 0 \} \);

\[
\frac{\delta Z[\hat{g}]}{\delta \hat{g}^{\mu\nu}(x)} \sim 0, \quad (1.4)
\]
and thus any BRST-invariant quantity calculated over this physical space is topological.

How do we translate this property of TFT into lattice language? Intuitive consideration tells us that each background metric $g_{\mu\nu}$ in continuous theory should correspond to a triangulation $T$ in our lattice framework. In fact, in 2D quantum gravity, the summation over all quantum fluctuations of metric can be replaced by the summation over all triangulations $\sum T$:

$$\int Dg_{\mu\nu} \leftrightarrow \sum_{T:\text{triangulation}}.$$  

(1.5)

Thus, we might wish to characterize our LTFT by the condition that the partition function of the lattice model is independent of triangulations. However, since the condition (1.4) is local, we should further require our LTFT to have the following property:

**Ansatz 1** The partition function of LTFT, which is first constructed with a given triangulation, should be invariant under any local changes of the triangulation.

The present paper is organized as follows. In section 2, we rewrite the above ansatz for LTFT into more concrete form by introducing 2D version of Matveev move. The general solutions to this ansatz are obtained explicitly in section 3, where we find that there is a one-to-one correspondence between the set of all TFT’s and the set of all associative algebras $R$. In section 4, we then define physical operators and investigate the structure of their correlation functions. There we see that each physical operator in a given TFT has a one-to-one correspondence to an element of the center $Z(R)$ of the associative algebra $R$ associated with the LTFT. We further show that these operators actually satisfy all the properties known in conventional TFT. The results of these two sections can be summarized schematically as follows:

$$\text{LTFT} \rightarrow \text{TFT}$$

$$\downarrow \quad \downarrow$$

$$R \rightarrow Z(R)$$

(1.6)

In section 5, as an example, we consider the LTFT that corresponds to a group ring $R = \mathbb{C}[G]$ with $G$ a group. The physical operators in this case
have one-to-one correspondence to the conjugacy classes of $G$, and their correlation functions are calculated explicitly, showing the coincidence with Witten’s result obtained in continuum approach [7]. In section 6, we also study the perturbation of TFT’s with introducing the concept of the moduli of TFT’s. We there show that the form of perturbation is automatically determined in our lattice formulation upon requiring its locality and topological property to be preserved under the perturbation. We further show that every TFT can be obtained by perturbation from what will be called the standard topological field theory (STFT). As a simple example, the twisted $N = 2$ minimal topological matters [8] are considered, and shown to live on the boundary of the moduli space of TFT’s. Section 7 is devoted to the discussion about how to incorporate gravity into our lattice formulation.

This paper was inspired by the work by Turaev and Viro [10] who constructed a series of three-dimensional topological invariants by using lattice approach (see also [11]).

2 Definition of LTFT

Let $\Sigma_g$ be a closed oriented surface of genus $g$, $T_g$ a triangulation of $\Sigma_g$. Then, the partition function of the lattice model associated with $T_g$ is defined as follows: First, for an oriented triangle in $T_g$ we make a coloring as preserving its orientation. That is, we give a set of color indices running from 0 through $A$ to three edges of the triangle (see fig. 1). We then assign a complex value $C_{ijk}$ to a triangle with ordered color indices $i, j, k$. We here assume that $C_{ijk}$ is symmetric under cyclic permutations of the indices:

$$C_{ijk} = C_{jki} = C_{kij}. \quad (2.1)$$

Note, however, that $C_{ijk}$ is not necessarily totally symmetric. Next, we glue these triangles by contracting their indices with $g_{ij} = g_{ji}$ (see fig. 2). We further assume that $g^{ij}$ has its inverse $g_{ij}$; $g_{ij} g^{kj} = \delta_i^j$, and raise or lower indices with these matrices. Thus, we have a complex-valued function of $C_{ijk}$ and $g^{ij}$ for each triangulation $T_g$, and we will interpret it as the partition function of our lattice model, denoting it by $Z(T_g)$. For example, the partition function for the triangulation of $\Sigma_0 = S^2$ depicted in fig. 3 is expressed as

$$Z(T_0) = g^{ii'} g^{jj'} g^{kk'} g^{ll'} g^{mm'} g^{nn'} C_{ijk} C_{klm} C_{mnp} C_{j'n'l'}. \quad (2.2)$$
Note that in dual diagrams, $g^{ij}$, $C_{ijk}$ and $g_{ij}$ are interpreted as propagator, 3-point vertex and 2-point vertex, respectively (see fig. 4 and fig. 5).

Now we consider LTFT and, by following the argument given in the previous section, we require that the partition function is invariant under local changes of triangulation, which will set some conditions on $C_{ijk}$ and $g^{ij}$. There have been known several systematic methods to deal with these local moves, which can also generate all the triangulations with fixed topology. Among the best known are the so-called bond-flip method and the Alexander-move method, but it is difficult to find solutions when we require the invariance of our partition function $Z(T_g)$ under these moves. In this paper, we adopt instead the other one, 2D version of Matveev-move method, which can be expressed only in dual diagrams and consists of a sequence of two fundamental local moves; fusion transformation and bubble transformation, as shown in fig. 6. The reason why we adopt this is that we can easily find the general solutions when we require the invariance of the partition function under these 2D Matveev moves.

We conclude this section by summarizing our ansatz for the partition function of LTFT:

**Ansatz 2** Let $Z(T_g)$ be the partition function of LTFT associated with a triangulation $T_g$ of genus-$g$ closed surface. Then it should be invariant under any 2D Matveev moves acting on the triangulation $T_g$.

### 3 General solutions

In this section, we obtain general solutions of our ansatz, and show that our LTFT has a one-to-one correspondence to a (generally noncommutative) associative algebra.

First, the invariance of the partition function $Z$ under fusion transformations is expressed in terms of $g_{ij}$ and $C_{ijk}$ as (see fig. 7)

$$C'_{ij}pC'_{pk} = C'_{jk}pC'_{ip}.$$  \[(3.1)\]

\[1\] Although, as can be easily proven, 2D Matveev moves actually generate other kinds of local moves such as bond-flips or Alexander moves, 2D Matveev move cannot be obtained from these moves since 2D Matveev move can only be represented in dual diagrams.
This equation allows us to introduce an associative algebra \( R \) with a basis \( \{ \phi_i \} \) \((i = 0, 1, 2, \ldots, A)\) and the structure constant \( C_{ij}^k \); \( \phi_i \phi_j = C_{ij}^k \phi_k \). It is easy to see that eq. (3.1) ensures the associativity of our algebra \( R \): \( \phi_i \phi_j \phi_k = \phi_i (\phi_j \phi_k) \). Then the invariance of the partition function \( Z \) under bubble transformations is now expressed as (see fig. 8)

\[
g_{ij} = C_{ik}^l C_{jl}^k. \tag{3.2}
\]

We thus have obtained the map from LTFT to an associative algebra \( R = \bigoplus_{i=0}^A \mathbb{C} \phi_i \) with the product \( \phi_i \phi_j = C_{ij}^k \phi_k \) and the metric \( g_{ij} = C_{ik}^l C_{jl}^k \). We can further show that this map is bijective. In fact, by introducing a metric by eq. (3.2), we can define 3-point vertex \( C_{ijk} \equiv C_{ij}^l g_{lk} \) and propagator \( (g^{ij}) = (g_{ij})^{-1} \), which both satisfy the invariance conditions. Therefore, we have proven the following theorem:

**Theorem 3** The set of all LTFT’s defined above has a one-to-one correspondence to the set of semi-simple associative algebras \( R \).

A few remarks are now in order:

1. The condition that the metric \( g_{ij} \) in (3.2) has its inverse is stated in the word *semi-simple* above. In fact, the necessary and sufficient condition for the metric to have its inverse turns out that the algebra is semi-simple. The sufficiency of the condition follows simply from Wedderburn’s theorem \([12]\), applied to the algebra \( R \) over \( \mathbb{C} \), which says that the algebra is isomorphic to direct products of matrix rings over \( \mathbb{C} \). On the other hand, to prove the necessity of the condition we note that Maschke’s theorem \([12]\) for finite group \( G \) extends naturally to our case of \( R \). Maschke’s theorem says that if an arbitrary finite dimensional \( G \)-module \( V \) over \( \mathbb{C} \) contains \( W \) as its submodule then \( V \) decomposes into a direct sum as \( G \)-module; \( V = W \oplus W' \) where \( W' \) is complement to \( W \). The proof is essentially based on taking average of the projection map \( P_W : V \to W \) over \( G \):

\[
\bar{P}_W = \frac{1}{|G|} \sum_{g \in G} \pi_W(g^{-1}) P_W \pi_V(g). \tag{3.3}
\]

The average (3.3) naturally extends to our case of \( R \) as

\[
\bar{P}_W = \sum_{ij} \pi_W(\phi_i) g^{ij} P_W \pi_V(\phi_j), \tag{3.4}
\]
if the metric has its inverse. Taking $V = R$ as left $R$-module we can
conclude that the ring $R$ is semi-simple. We, however, show in section 6 that we
can still define LTFT’s which do not necessarily correspond to semi-simple
associative algebras by using topological perturbations.

(2) It is easy to show that $C_{ijk}$ is totally symmetric if and only if $R$ is com-
mutative. In such a case, however, the partition function $Z$ has a value which
is independent of topology, so that the model only has trivial structure. \[2\]

(3) If we introduce the regular representation $(\pi, V)$ of algebra $R$ by $(\pi(\phi_i))^k_j = C_{ij}^k$, then $g_{ij}$ and $C_{ijk}$ can be simply expressed as follows:

\[
\begin{align*}
  g_{ij} &= \text{tr}_V \pi(\phi_i)\pi(\phi_j) & (3.5) \\
  C_{ijk} &= \text{tr}_V \pi(\phi_i)\pi(\phi_j)\pi(\phi_k). & (3.6)
\end{align*}
\]

This representation is useful in constructing LTFT directly from a given
algebra (see section 5).

(4) If we set $\phi_0 = 1$ (unit element of $R$), then we have

\[
C_{0i}^j = C_{i0}^j = \delta_i^j,
\]

since $\phi_0\phi_i = \phi_i\phi_0 = \phi_i$.

4 Physical observables and their correlation functions

In the previous section, we found that our LTFT has a one-to-one corre-
spondence with an associative algebra $R$. In this section, we investigate the
structure of physical observables, and show that all information we need can
be reduced to the center $Z(R)$ of the algebra $R$, and further show that our
method actually reproduces the well-known results of continuous TFT.

We first define operators $\mathcal{O}_i$ ($i = 0, 1, 2, \ldots, A$) by interpreting the inser-
tion of $\mathcal{O}_i$ into correlation functions as creating a loop boundary with color
index $i$, and we denote the correlation function of $\mathcal{O}_{i_1}, \ldots, \mathcal{O}_{i_n}$ on genus-$g$
closed surface by $\langle \mathcal{O}_{i_1} \ldots \mathcal{O}_{i_n} \rangle_g$.

\[2\] We, however, can construct nontrivial theories by perturbation from such a trivial
LTFT that corresponds to a commutative algebra.
Let us consider 2-point function of $O_i$ and $O_j$ on sphere; $\eta_{ij} \equiv \langle O_i O_j \rangle_0$, and investigate the property of physical operators. The simplest triangulation for $\eta_{ij}$ is depicted in fig. 9, and written as

$$\eta_{ij} = C_{ik}^l C_{lj}^k. \quad (4.1)$$

Furthermore, due to its independence of triangulation, $\eta_{ij}$ can also be calculated from another graph shown in fig. 10, which yields an important identity; $\eta_{ij} = \eta_{ik}^k \eta_{kj}$, or

$$\eta_{ji} = \eta_{ik}^k \eta_{jk}^i. \quad (4.2)$$

Thus, we know that the operator $\eta = (\eta_{ji}^j)$ acting on $R$ is idempotent; $\eta^2 = \eta$, and so expect that $\eta$ is a kind of projection map. In fact, we can prove the following theorem:

**Theorem 4** $\eta = (\eta_{ji}^j)$ is the surjective projector from $R$ to its center $Z(R) = \{ \tilde{\phi} \in R | \phi \tilde{\phi} = \tilde{\phi} \phi \forall \phi \in R \}$.\n
To prove this, we first show that $\eta$ is a map from $R$ into its center $Z(R)$. We only have to show that $\tilde{\phi}_i \phi_k = \phi_k \tilde{\phi}_i$ (\forall i, k) with $\tilde{\phi}_i \equiv \eta_{ji}^j \phi_j$, and this is easily seen from the relation $\eta_{ij}^j C_{jk}^l = \eta_{ik}^i C_{kj}^l$ as depicted in fig. 11. Moreover, we can also show that

$$\eta \tilde{\phi} = \tilde{\phi} \text{ for } \forall \tilde{\phi} \in Z(R), \quad (4.3)$$

which asserts that this map $\eta : R \to Z(R)$ is surjective. We thus proved that $\eta = (\eta_{ji}^j)$ is the surjective projector from $R$ to its center $Z(R)$.

**proof of eq. (4.3)**

For $\phi = c^j \phi_i \in Z(R)$, we have a relation $c^j C_{ik}^l = c^i C_{ki}^l$ since $\tilde{\phi} \phi_k = \phi_k \tilde{\phi}$. Thus, we have

$$\eta \tilde{\phi} = \eta_{ji}^j c^i \phi_j$$

$$= C_{ik}^l C_{jk}^i c^i \phi_j$$

$$= C_{ki}^l C_{ij}^k c^i \phi_j$$

$$= c^i \phi_j$$

$$= \tilde{\phi}. \quad \text{[Q.E.D.]}$$
Next, we study 3-point function on sphere, \( N_{ijk} \equiv \langle O_i O_j O_k \rangle_0 \). The simplest triangulation is shown in fig. 12, and evaluated as

\[
N_{ijk} \equiv \eta^i_{j'} \eta^j_{k'} \eta^k_{i'} C_{i'j'k'}.
\]  

(4.5)

Note that the indices \( i, j \) and \( k \) in \( C_{ijk} \) are subject to the projection of \( \eta \), and so we know that \( N_{ijk} \) is now totally symmetric even though \( C_{ijk} \) is not so.

Such a graphical consideration can be easily generalized to the case of other multi-point functions and of higher genuses, and we see that every insertion of operator \( O_i \) is necessarily subject to the projection of \( \eta \). Thus, we obtain the following theorem:

**Theorem 5** The set of physical operators is in one-to-one correspondence with the center \( Z(R) \) of the associative algebra \( R \) associated with the LTFT we consider. In particular, the number of independent physical operators is equal to the dimension of \( Z(R) \).

To get correlation functions, we only have to combine \( N_{ijk} \)'s by contracting their indices with \( \eta_{ij} \), as exemplified in fig. 13. In the following, we relabel the indices of basis \( \{ \phi_i \} (i = 0, 1, \ldots, A) \) of \( R \) in such a way that the first \( (K + 1) \) indices represent a basis of \( Z(R) \):

\[
R = \bigoplus_{i=0}^{A} C\phi_i = Z(R) \bigoplus Z^C(R) = \left( \bigoplus_{\alpha=0}^{K} C\phi_{\alpha} \right) \bigoplus \left( \bigoplus_{p=K+1}^{A} C\phi_{p} \right).
\]

(4.6)

Since \( \eta = (\eta_{ij}) (i, j = 0, 1, \ldots, A) \) is the projector onto \( Z(R) \) and the relation (4.3) holds, \( \eta \) has the following form under the above decomposition (4.6):

\[
\eta_{ij} = \begin{bmatrix}
\eta_{\alpha\beta} = g_{\alpha\beta} & 0 \\
0 & 0
\end{bmatrix}, \\
\eta^i_{j} = \begin{bmatrix}
\eta^i_{\alpha} = \delta^i_{\alpha} & 0 \\
0 & 0
\end{bmatrix}, \\
\eta^{ij} = \begin{bmatrix}
\eta^{ij}_{\alpha\beta} = g^{ij}_{\alpha\beta} & 0 \\
0 & 0
\end{bmatrix}.
\]

(4.7)
and the relation $\eta_{ik} \eta_{kj} = \eta_{ji}$ implies that $(\eta^{\alpha\beta})$ is the inverse to $(\eta_{\alpha\beta})$ if we restrict their defining region to $Z(R)$:

$$\eta_{\alpha\gamma} \eta^{\gamma\beta} = \delta_\alpha^\beta.$$  

(4.8)

Note also that

$$N_{\alpha\beta\gamma} = C_{\alpha\beta\gamma}.$$  

(4.9)

Equations (4.7)-(4.9) simplify the calculation of correlation functions since we only have to sum over indices $\alpha = 0, 1, \ldots, K$ of $Z(R)$ in glueing. In summary,

**Theorem 6** *All correlation functions are obtained by connecting cylinder $\eta^{\alpha\beta}$ and diaper $N_{\alpha\beta\gamma}$ (see fig. 14).*

In the rest of this section, we show that our LTFT actually satisfies all the known properties in continuous TFT. Recall that due to our redefinition of indices (4.6), physical operators $O_\alpha (\alpha = 0, 1, \ldots, K)$ correspond to a basis $\phi_\alpha$ of the center $Z(R)$.

Let $A(O)$ be a function of physical operators (e.g. $A(O) = O_{\alpha_1} O_{\alpha_2} \ldots O_{\alpha_n}$). Then we have the following theorem:

**Theorem 7** *Calculation of correlation functions with genus $g$ can always be reduced to that with genus 0 by using the handle operator $H$:

$$\langle A(O) \rangle_g = \langle A(O) H^g \rangle_0$$  

(4.10)

with

$$H \equiv w^\alpha O_\alpha, \quad w^\alpha \equiv N^{\alpha\beta}_{\beta}.$$  

(4.11)

**proof**

Correlation function with genus $g$ is calculated

$$\langle A(O) \rangle_g = \langle A(O) O_{\alpha_1} O_{\alpha_2} \ldots O_{\alpha_s} \rangle_0 w^{\alpha_1} w^{\alpha_2} \ldots w^{\alpha_s}$$  

(4.12)

as shown in fig. 15. Thus, if we introduce $H$ as in eq. (4.11), then we have

$$\langle A(O) \rangle_g = \langle A(O) H^g \rangle_0.$$  

[Q.E.D.]

Furthermore, we can show
Theorem 8 Operators $O_{\alpha}$ satisfy the following OPE:

$$O_{\alpha}O_{\beta} = N_{\alpha\beta}^{\gamma}O_{\gamma}, \quad (N_{\alpha\beta}^{\gamma} \equiv N_{\alpha\beta\gamma}\eta^{\gamma\gamma})$$ (4.13)

as an identity in any correlation functions.

proof If we introduce the regular representation $(\rho, W)$ of commutative algebra $Z(R)$ as $\rho(\phi_{\alpha})^{\gamma}_{\beta} \equiv N_{\alpha\beta}^{\gamma} = C_{\alpha\beta}^{\gamma}$, then $\rho(\phi_{\alpha})$ satisfies the product law: $\rho(\phi_{\alpha})\rho(\phi_{\beta}) = N_{\alpha\beta}^{\gamma}\rho(\phi_{\gamma})$. On the other hand, as can be seen graphically, the expectation value of $A(O)$ with $g = 1$ (torus) is represented as a trace over this representation space $W$: $\langle A(O) \rangle_{g=1} = \text{tr}_{W} A(\rho(\phi))$. We thus have the following relation:

$$\langle O_{\alpha}O_{\beta}A(O) \rangle_{g} = \langle O_{\alpha}O_{\beta}A(O)H^{g-1} \rangle_{1} = \text{tr}_{W} \rho(\phi_{\alpha})\rho(\phi_{\beta})A(\rho(\phi))\rho(H)^{g-1} = N_{\alpha\beta}^{\gamma}\text{tr}_{W} \rho(\phi_{\gamma})A(\rho(\phi))\rho(H)^{g-1} = N_{\alpha\beta}^{\gamma}\langle O_{\gamma}A(O) \rangle_{g}.$$ (4.14)

By using this OPE, we can further show that our model has a strong factorization property\[3\] (see fig. 16):

$$\langle A(O) \rangle_{g} = \eta^{\alpha\beta}\langle A(O)O_{\alpha}O_{\beta} \rangle_{g-1}$$ (4.15)

since $H = w^{\alpha}O_{\alpha} = \eta^{\alpha\beta}O_{\alpha}O_{\beta}$.

5 Example: $R = C[G]$  

In this section, we deal with the special case where $R$ is a group ring:

$$R = C[G] = \bigoplus_{x \in G} C\phi_{x},$$ (5.1)

with the product induced from the group multiplication; $\phi_{x}\phi_{y} = \phi_{xy}$. Here we assume that $G = \{x, y, z, \ldots, g, h, \ldots\}$ is a finite group, for simplicity.

\[3\] For the reason why we call eq. (4.14) strong factorization, see section 7.
Extension to continuous group is straightforward, but yields more fruitful structure in the obtained theory, as will be reported elsewhere.

In order to calculate 2- and 3-point vertices, it is useful to use the regular representation \((\pi, V)\) of \(R = \mathbb{C}[G]\):

\[
\pi(\phi_x)^z_y = C_{xy}^z = \delta(xy, z),
\]

where

\[
\delta(x, y) \equiv \begin{cases} 
1 & (x = y) \\
0 & \text{(otherwise)}.
\end{cases}
\]

Thus, if we use eqs. (3.3) and (3.6) together with the following formula:

\[
\text{tr}_V \pi(\phi_x) = |G| \delta(x, 1),
\]

we have

\[
\begin{align*}
g_{xy} &= |G| \delta(xy, 1) \\
C_{xyz} &= |G| \delta(xyz, 1).
\end{align*}
\]

By using these equations, we easily obtain

\[
\eta_{xy} = \langle O_x O_y \rangle_0 = \frac{|G|}{h[x]} \delta_{[x],[y^{-1}]}.
\]

Here \([x]\) denotes the conjugacy class of \(x\); \([x] \equiv \{y \in G \mid y = gxg^{-1}, \exists g \in G\}\), and \(h[x]\) is the number of the elements of \([x]\); \(h[x] = \#([x])\). In the following, we denote \([x^{-1}]\) by \(\hat{x}\), and label conjugacy classes by Greek letters. Note that \(h_{\hat{\alpha}} = h_{\alpha}\).

Let us investigate the property of the projection operator \(\eta = (\eta_{xy})\):

\[
\eta_{xy} = \eta_{xz} g_{yz} = \frac{1}{h[x]} \delta_{[y]}^{[y^{-1}]}.
\]

By operating \(\eta\) on \(R\), we obtain

\[
\tilde{\phi}_x = \sum_{y \in G} \eta_{xy} \phi_y = \frac{1}{h[x]} \sum_{y \in [x]} \phi_y,
\]
and thus know that $Z(R = C[G])$ is spanned by the orbits of conjugacy classes:

$$Z(R) = \bigoplus_{\alpha} CC_{\alpha}, \quad C_{\alpha} \equiv \frac{1}{\sqrt{h_{\alpha}}} \sum_{x \in \alpha} \phi_{x}. \quad (5.9)$$

We here normalize the basis $\{C_{\alpha}\}$ by the factor $\sqrt{h_{\alpha}}$ for later convenience. In this basis, $\eta_{\alpha\beta}$ is represented by

$$\eta_{\alpha\beta} = \frac{1}{\sqrt{h_{\alpha}h_{\beta}}} \sum_{x \in \alpha} \sum_{y \in \beta} |G| \delta_{[x], [y^{-1}]} \delta_{\alpha} = |G| \delta_{\beta} = |G| \delta_{\beta}, \quad (5.10)$$

$N_{\alpha\beta\gamma}$ is now easily read out from the following relation:

$$C_{\alpha}C_{\beta} = N_{\alpha\beta\gamma}C_{\gamma}, \quad (5.11)$$

and found to be

$$N_{\alpha\beta\gamma} = \frac{1}{\sqrt{h_{\alpha}h_{\beta}h_{\gamma}}} \sum_{x \in \alpha} \sum_{y \in \beta} \delta_{\gamma}^{[xy]}. \quad (5.12)$$

This expression in turn gives us 3-point function on sphere:

$$N_{\alpha\beta\gamma} = \langle O_{\alpha}O_{\beta}O_{\gamma} \rangle_0 = N_{\alpha\beta\gamma} \eta_{\gamma}^{\gamma} = \frac{|G|}{\sqrt{h_{\alpha}h_{\beta}h_{\gamma}}} \sum_{x \in \alpha} \sum_{y \in \beta} \delta_{\gamma}^{[xy]} \delta_{\gamma}, \quad (5.13)$$

These forms of $\eta_{\alpha\beta}$ and $N_{\alpha\beta\gamma}$, however, are not so useful for direct calculation, and so in the following we rewrite them into more convenient form. Since $\eta_{\alpha\beta}$ and $N_{\alpha\beta\gamma}$ both are functions of conjugacy classes, these must be expanded with respect to irreducible characters. In fact, short algebraic calculation shows that

$$\eta_{\alpha\beta} = \sum_{j} \chi_{j}(C_{\alpha})\chi_{j}(C_{\beta}),$$

$$N_{\alpha\beta\gamma} = \sum_{j} \chi_{j}(C_{\alpha})\chi_{j}(C_{\beta})\chi_{j}(C_{\gamma}) \frac{d_{j}}{d_{j}}. \quad (5.14)$$

---

4 In the previous section, $C_{\alpha}$ was written as $\phi_{\alpha}$. We, however, use different symbol here in order to avoid confusion.
Here \( \chi_j \) is the character of an irreducible representation \( j \), and its defining region is extended to \( \mathbb{C}[G] \) by linearity. Furthermore, \( d_j \) stands for the dimension of the representation \( j \). Recall that \( d_j = \chi_j(1) \).

It is further convenient to introduce the following symbol:

\[
\chi_j \leftrightarrow \langle \chi_j | \rangle
\]

Then, the first, and the second orthogonality relation of irreducible characters are expressed in the following form:

\[
\langle \alpha | \beta \rangle = |G| \delta^\alpha_\beta, \quad \langle \chi_j | \chi_k \rangle = \delta^j_k.
\]

(5.15)

\[
\frac{1}{|G|} \sum_{\alpha} |\alpha \rangle \langle \alpha| = 1, \quad \sum_j |\chi_j \rangle \langle \chi_j| = 1.
\]

(5.16)

Note that

\[
\langle \chi_j | \hat{\alpha} \rangle = \langle \alpha | \chi_j \rangle,
\]

(5.17)

since \( \langle \chi_j | \hat{\alpha} \rangle = \chi_j(C_\alpha) = \chi_j(C_\alpha^*) = \langle \alpha | \chi_j \rangle \). Thus, we have the following expression for \( \eta_{\alpha\beta} \), \( N_{\alpha\beta\gamma} \) and \( N_{\alpha\beta}^\gamma \):

\[
\eta_{\alpha\beta} = \sum_j \langle \chi_j | \alpha \rangle \langle \chi_j | \beta \rangle = \langle \hat{\beta} | \alpha \rangle = \langle \hat{\alpha} | \beta \rangle
\]

\[
N_{\alpha\beta\gamma} = \sum_j \frac{\langle \chi_j | \alpha \rangle \langle \chi_j | \beta \rangle \langle \chi_j | \gamma \rangle}{\langle \chi_j | 0 \rangle}
\]

\[
N_{\alpha\beta}^\gamma = \frac{1}{|G|} \sum_j \frac{\langle \chi_j | \alpha \rangle \langle \gamma | \chi_j \rangle \langle \chi_j | \beta \rangle}{\langle \chi_j | 0 \rangle}.
\]

(5.18)

Here we denote the conjugacy class of identity by 0; \( C_0 = 1 \), and so we have \( d_j = \langle \chi_j | 0 \rangle \).

We now can calculate all correlation functions explicitly. Following the prescription given in the previous section, we first introduce the regular representation \( (\rho, W) \) of \( Z(\tilde{R}) = \bigoplus_{\alpha=1}^K \mathbb{C} C_{\alpha} \):

\[
\rho(C_\alpha)^\gamma_\beta \equiv N_{\alpha\beta}^\gamma.
\]

(5.19)

\(^5\) \( (K+1) \) is the dimension of center, and equal to the number of conjugacy classes, which is also equal to the number of irreducible representations, as is clear from the orthogonality relations of irreducible characters.\(^{14} \)

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We then get the following formula:

$$\langle O_{\alpha_1} \ldots O_{\alpha_n} \rangle_g = \langle O_{\alpha_1} \ldots O_{\alpha_n} H^{g-1} \rangle_1 = \text{tr}_W \rho(C_{\alpha_1}) \ldots \rho(C_{\alpha_n}) \rho(H)^{g-1}. \quad (5.20)$$

Here, $\rho(H) = w^\alpha \rho(C_\alpha)$ is calculated as

$$\rho(H)^\gamma_\beta = N^\alpha_\delta N^\gamma_{\alpha \beta} \rho(H) = \frac{1}{|G|} \sum_j \frac{\langle \gamma | \chi_j \rangle \langle \chi_j | \beta \rangle}{\langle \chi_j | 0 \rangle^2}, \quad (5.21)$$

and, by substituting this equation into eq. (5.20) and using eq. (5.16), we finally obtain

$$\langle O_{\alpha_1} \ldots O_{\alpha_n} \rangle_g = \sum_j \frac{\langle \chi_j | a_1 \rangle \ldots \langle \chi_j | a_n \rangle}{\langle \chi_j | 0 \rangle^{2g-2+n}} = \sum_j \frac{\chi_j(C_{\alpha_1}) \ldots \chi_j(C_{\alpha_n})}{d_j^{2g-2+n}}. \quad (5.22)$$

This has the same form as Witten’s result calculated by using continuous TFT [7].

6 Moduli of TFT’s and their perturbation

In this section, we investigate the moduli space of TFT’s. In particular, we show that every TFT can be obtained by perturbation from the standard topological field theory (STFT) to be defined below. The following discussions are inspired by ref. [13].

6.1 Standard basis and standard topological field theory

As has been shown in preceding sections, a TFT with $(K+1)$ independent physical operators has a one-to-one correspondence to a commutative algebra $\tilde{R}$ of dimension $(K+1)$, which can be regarded as the center of an associative algebra $R$ in our lattice language; $\tilde{R} = Z(R)$. In particular, the physical
operators $\mathcal{O}_\alpha$ correspond to a basis $\phi_\alpha$ of $\tilde{R}$. In the following, we further investigate these correspondences in order to introduce the concept of the moduli of TFT’s.

We again consider the regular representation $(\rho, W)$ of $\tilde{R}$: $\rho(\phi_\alpha)^\gamma_\beta = N_{\alpha\beta}^\gamma$ with $\rho(\phi_\alpha)\rho(\phi_\beta) = N_{\alpha\beta}^\gamma \rho(\phi_\gamma)$. Since $\tilde{R}$ is commutative, the following relation holds:

$$\rho(\phi_\alpha)\rho(\phi_\beta) = \rho(\phi_\beta)\rho(\phi_\alpha), \quad (6.1)$$

from which we know that the $\rho(\phi_\alpha)$’s ($\alpha = 0, 1, \ldots, K$) are simultaneously diagonalizable;

$$\rho(\phi_\alpha) = \begin{bmatrix} \lambda^{(\alpha)}_0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \lambda^{(\alpha)}_K \end{bmatrix}, \quad (6.2)$$

that is, $N_{\alpha\beta}^\gamma = \lambda^{(\alpha)}_\beta \delta_\beta^\gamma$. Moreover, since $N_{\alpha\beta}^\gamma = N_{\beta\alpha}^\gamma$, we can further make a transformation of the basis in such a way that $N_{\alpha\beta}^\gamma$ has the following form:

$$N_{\alpha\beta}^\gamma = \lambda_\alpha \delta_\alpha^\beta \delta_\beta^\gamma. \quad (6.3)$$

Thus, the physical operators $\{\mathcal{O}_\alpha\}$ now have the following OPE [13]:

$$\mathcal{O}_\alpha \mathcal{O}_\beta = \lambda_\alpha \delta_\alpha^\beta \mathcal{O}_\alpha. \quad (6.4)$$

Let $\mathcal{M}_{\text{TFT}}$ be the moduli space of TFT’s, which is nothing but the set of all commutative algebras. For the physical operators of almost all TFT’s in $\mathcal{M}_{\text{TFT}}$, all the $\lambda_\alpha$’s in eq. (6.4) have nonvanishing values. Thus, by properly normalizing $\mathcal{O}_\alpha$, we have the following OPE:

$$\mathcal{O}_\alpha \mathcal{O}_\beta = \delta_\alpha^\beta \mathcal{O}_\alpha. \quad (6.5)$$

We will call $\{\mathcal{O}_\alpha\}$ with this OPE the standard basis of the TFT we consider. Since this form of OPE completely determines the basis $\{\phi_\alpha\}$ up to their permutation, and any correlation functions are uniquely calculated from their one-point functions on sphere;

$$v_\alpha \equiv \langle \mathcal{O}_\alpha \rangle_0, \quad (6.6)$$
we now know [13] that \( \mathcal{M}_{\text{TFT}} \) is parametrized by the number \((K+1)\) of physical operators (the dimension of the algebra \( \hat{R} \)) and their one-point functions \( \{v_\alpha\} (\alpha = 0, 1, \ldots, K) \). Note that for this standard basis, the handle operator \( H \) [eq. (4.11)] is expressed as

\[
H = \sum_\alpha \frac{O_\alpha}{v_\alpha}.
\]  

(6.7)

We, in particular, call the TFT where \( v_\alpha \equiv 1 \) (\( \alpha = 0, 1, \ldots, K \)) the \( K \)-th standard topological field theory (STFT).

**Example.** \( R = \mathbb{C}[G] \)

We follow the notation in section 5: \( G \) is a finite group, and \( \alpha \) (resp. \( j \)) labels conjugacy classes (resp. irreducible representations) of \( G \). We can always construct the standard basis in the LTFT corresponding to \( R = \mathbb{C}[G] \), group ring of \( G \). In fact, if we make a transformation of basis as

\[
\bar{O}_j \equiv \frac{d_j}{|G|} \sum_\alpha \langle \alpha | \chi_j \rangle \ O_\alpha
\]

\[
= \frac{d_j}{|G|} \sum_\alpha \chi_j(C_\alpha)^* O_\alpha,
\]

then \( \{\bar{O}_j\} \) satisfies the following OPE:

\[
\bar{O}_j \bar{O}_k = \delta_{jk} \bar{O}_j.
\]

(6.9)

The one-point functions are easily calculated to be found

\[
v_j \equiv \langle \bar{O}_j \rangle_0 = d_j^2.
\]

(6.10)

Recall here that \( d_j \equiv 1 \) (\( \forall j \)) for commutative groups. Thus, \( K \)-th STFT can be realized by the LTFT that corresponds to a group ring \( R = \mathbb{C}[G] \) of commutative group \( G \) with order \( |G| = K + 1 \).

**6.2 Perturbation of TFT**

In this and the next subsections, we show that every TFT can be obtained form STFT by perturbation. In particular, we see that the TFT’s which have vanishing \( \lambda_\alpha \) for some \( \alpha \) can also be expressed by this perturbation.
Suppose that we have chosen a TFT, and let us perturb it by adding \( \delta S \) to the original action. Perturbed correlation functions to be denoted with prime are thus calculated by inserting the operator \( \exp(-\delta S) \) into the original correlation function:

\[
\langle \ldots \rangle' \equiv \langle \ldots \exp(-\delta S) \rangle.
\] (6.11)

In the following, we show that the possible form of \( \delta S \) can be determined automatically if we require its locality and topological property. We first fix a triangulation \( T_g \). Then locality condition leads to the following form of \( \delta S \):

\[
\delta S = \sum_\alpha \sum_x f_\alpha(n_x) \mathcal{O}_\alpha,
\] (6.12)

where \( x \) parametrizes vertices in the triangulation, and \( n_x \) stands for the number of triangles around the vertex \( x \) (see fig. 17). Then, by the invariance of \( \exp(-\delta S) \) under the fusion and bubble transformations, \( f_\alpha(n_x) \) is determined to have the form

\[
f_\alpha(n_x) = A_\alpha (n_x - 6) \quad (A_\alpha : \text{constant}),
\] (6.13)

which implies that \( f_\alpha(n_x) \) is proportional to the deficit angle around the vertex \( x \).

**proof**

First, the invariance under the fusion transformation (fig. 18) yields the following identity:

\[
f_\alpha(n_x) + f_\alpha(n_y) + f_\alpha(n_z) + f_\alpha(n_w) = f_\alpha(n_x - 1) + f_\alpha(n_y + 1) + f_\alpha(n_z - 1) + f_\alpha(n_w + 1),
\] (6.14)

from which \( f_\alpha(n_x) \) is known to be a linear function of \( n_x; f_\alpha(n_x) = A_\alpha n_x + B_\alpha \). Next, from the invariance under the bubble transformation, we have the relation (see fig. 19)

\[
f_\alpha(n_x) + f_\alpha(n_y) = f_\alpha(n_x + 2) + f_\alpha(n_y + 2) + f_\alpha(2),
\] (6.15)

\[\text{In the language of continuum theory, this ansatz corresponds to requiring that } \delta S \text{ has the following form:}
\]

\[
\sum_\alpha \int d^2x \sqrt{g} f_\alpha(R) \mathcal{O}_\alpha(x)
\]

with \( R \) scalar curvature.
which gives \( B_\alpha = -6A_\alpha \). [Q.E.D.]

Thus, by setting \( A_\alpha = (1/12) \mu_\alpha \), we have:

\[
e^{-\delta S} = \exp \left( -\frac{1}{12} \sum_\alpha \mu_\alpha \sum_x (n_x - 6) \mathcal{O}_\alpha \right).
\]

If we insert this operator into genus-\( g \) correlation function \( \langle \ldots \rangle_g \), we then obtain

\[
e^{-\delta S} = \exp \left\{ (1-g) \sum_\alpha \mu_\alpha \mathcal{O}_\alpha \right\},
\]

since \( \mathcal{O}_\alpha \), which corresponds to an element of commutative algebra, is independent of its location. We here also used the Gauss-Bonnet theorem: \( \sum_x (n_x - 6) = -12(1-g) \). In particular, if \( \{\mathcal{O}_\alpha\} \) is the standard basis of the TFT we consider, then we have

\[
e^{-\delta S} = 1 + \sum_\alpha \left( e^{(1-g)\mu_\alpha} - 1 \right) \mathcal{O}_\alpha
\]

in genus-\( g \) correlation functions.

Now we have the general form of the perturbation operator \( \exp(-\delta S) \), it is easy to see that every TFT can be obtained from STFT by perturbation. In fact, we have the following formula for the standard basis of STFT \[13\]:

\[
v'_\alpha \equiv \langle \mathcal{O}_\alpha \rangle_0' = \langle \mathcal{O}_\alpha e^{-\delta S} \rangle_0 = e^{\mu_\alpha} \langle \mathcal{O}_\alpha \rangle_0 = e^{\mu_\alpha} v_\alpha.
\]

Thus, if we, in particular, start from STFT where \( v_\alpha \equiv 1 \), we then have \( v'_\alpha = e^{\mu_\alpha} \), and so can obtain all values of \( v'_\alpha \) by adjusting the parameters \( \mu_\alpha \). On the other hand, the form of OPE is preserved under perturbation. Therefore, \[7\] This corresponds to

\[
\delta S = -\frac{1}{2} \sum_\alpha \mu_\alpha \int d^2x \sqrt{g} R \mathcal{O}_\alpha(x),
\]

the form of which is the same with the one given in ref. \[13\].
we know that every TFT which can have standard basis is obtained from STFT by perturbation. Moreover, as will be shown in the following examples, TFT's which do not have the standard basis are also obtained from STFT by perturbations in a suitable limit of the perturbation parameters \( \mu_\alpha \) and with infinite renormalization of physical operators. In this sense, such TFT's live on the boundary of \( \mathcal{M}_{\text{TFT}} \).

6.3 Examples

In the following, we consider some examples, and explain how to obtain those TFT's by perturbations which do not necessarily have the standard basis.

**Example 1.** TFT associated with \( A^{(1)}_K \) WZW of level 1

Let \( \omega \) be the primitive \((K + 1)\)-th root of unity, and \( \{\mathcal{O}_\alpha\} \) the standard basis of \( K \)-th STFT. If we make a transformation of the basis into the following form:

\[
A_j \equiv \sum_{\alpha=0}^{K} \omega^{j\alpha} \mathcal{O}_\alpha \quad (j = 0, 1, \ldots, K),
\]

then it is easy to see that the following OPE holds:

\[
A_j A_k = A_{[j+k]} \tag{6.21}
\]

with the one-point function

\[
\langle A_j \rangle_0 = \sum_{\alpha=0}^{K} \omega^{j\alpha} = (K + 1) \delta_{j,0}. \tag{6.22}
\]

Here \( [l] \) stands for \( l \) modulo \((K + 1)\). Thus, we now know that the \( K \)-th STFT is nothing but the TFT associated with \( A^{(1)}_K \) WZW of level 1.

**Example 2.** twisted \( N = 2 \) minimal topological matter of level \( K \)

This theory is characterized by the following OPE and the vacuum expectation value of physical operators \( \sigma_j \) \((j = 0, 1, \ldots, K)\):

\[
\sigma_j \sigma_k = \theta (j + k \leq K) \sigma_{j+k} \tag{6.23}
\]

\[
\langle \sigma_j \rangle_0 = \delta_{j,K}. \tag{6.24}
\]
What is special in this case is that we cannot introduce the standard basis in the commutative algebra corresponding to this theory, since eq. (6.23) means that the matrix $\rho(\sigma_j)$ in the regular representation has some vanishing eigenvalues. However, we can realize the theory as a limit of perturbed theory. In fact, if we define the operators $\sigma_j^{(\epsilon)}$ from the operators $A_j$ in example 1 as

$$\sigma_j^{(\epsilon)} \equiv \epsilon^j A_j = \epsilon^j \sum_{\alpha=0}^{K} \omega^{j\alpha} O_\alpha,$$

and set the perturbation parameters $\mu_\alpha$ as

$$e^{\mu_\alpha} = \epsilon^{-K} \frac{1}{K+1} \omega^\alpha,$$

then we have the desired form of OPE and vacuum expectation values in the limit of $\epsilon \to 0$:

$$\sigma_j^{(\epsilon)} \sigma_k^{(\epsilon)} = \theta (j + k \leq K) \sigma_{j+k}^{(\epsilon)} + O(\epsilon)$$

$$\langle \sigma_j^{(\epsilon)} \rangle'_0 = \delta_{j,K}.$$

We thus know that the twisted $N = 2$ minimal topological matter is obtained at the boundary of the moduli space $\mathcal{M}_{\text{TFT}}$.

## 7 Discussion

In this paper, we give the lattice definition of topological matter system, find its explicit solution and investigate physical consequences, with emphasis on the algebraic structure of lattice topological field theory.

What still remains to be investigated is how to incorporate gravity, especially topological gravity, in our formalism. There seem to be the following two possibilities:

1. “Topological gravity can also be treated within our framework without any essential modification.” In fact, gravity can also be regarded as a matter field if we expand metric $g_{\mu\nu}$ around a background metric $\hat{g}_{\mu\nu}$:

$$g_{\mu\nu} = \hat{g}_{\mu\nu} + \delta g_{\mu\nu},$$

under some proper gauge condition on $\delta g_{\mu\nu}$. For example, in the conformal gauge gravitational quantum fluctuations are represented by the Liouville
field, which is in turn regarded as a conformal matter on a Riemann surface with fixed background metric $\tilde{g}_{\mu\nu}$ \(^4\). However, to go ahead in this direction, we need more machinery than we now have. In fact, we should set the dimension of $Z(R)$ to infinity ($K \to \infty$), since there are infinitely many physical observables in topological gravity \(^5\). Moreover, the Schwinger-Dyson equation of gravity \(^6\) shows that its quantum theory only has the weak factorization property. That is, factorization of a surface along trivial cycles is necessarily accompanied by factorization along nontrivial cycles, while the topological matter system has the strong factorization property in the sense that the geometry can be factorized along any cycles independently. However, the limiting procedure of $K \to \infty$ requires some regularization, which might reduce the strong factorization property to the weak one.

(2) “Quantum fluctuations of gravity can only be described by summing over different geometries.” If this is the case, the results obtained in this paper will not work directly for any quantum gravity. However, it will then be interesting to incorporate our lattice model in the Kontsevich model \(^{14}\), and to investigate whether the resulting model is equivalent to the so-called generalized Kontsevich model given in ref. \(^{15}\) (see also \(^{16}\)). Investigations along these two lines would be interesting.

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References

[1] E. Witten, Comm. Math. Phys. 117 (1988) 353, Int. J. Mod. Phys. A6 (1991) 2775.

[2] V. Kazakov, Phys. Lett. B150 (1985) 282;
   F. David, Nucl. Phys. B257 (1985) 45, 543;
   J. Ambjørn, B. Durhuus and J. Fröhlich, Nucl. Phys. B257 (1985) 433;
D. Boulatov, V. Kazakov, I. Kostov and A. Migdal, Nucl. Phys. B275 (1986) 641.

[3] A. Polyakov, Mod. Phys. Lett. A2 (1987) 899;
V. Knizhnik, A. Polyakov and A. Zamolodchikov, Mod. Phys. Lett. A3 (1988) 819.

[4] J. Distler and H. Kawai, Nucl. Phys. B321 (1989) 509;
F. David, Mod. Phys. Lett. A3 (1988) 165.

[5] E. Brézin and V. Kazakov, Phys. Lett. B236 (1990) 144;
M. Douglas and S. Shenker, Nucl. Phys. B335 (1990) 635;
D. Gross and A. Migdal, Phys. Rev. 64 (1990) 127.

[6] M. Fukuma, H. Kawai and R. Nakayama, Int. J. Mod. Phys. A6 (1991) 1385, Comm. Math. Phys. 143 (1992) 371, Comm. Math. Phys. 148 (1992) 101;
E. Verlinde and H. Verlinde, Nucl. Phys. B348 (1991) 457;
R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. B348 (1991) 435.

[7] E. Witten, Comm. Math. Phys. 141 (1991) 153, preprint, IASSNS-HEP-92/15 (1992).

[8] T. Eguchi and S.K. Yang, Mod. Phys. Lett. A5 (1991) 1693.

[9] W. Lerche, C. Vafa and N.P. Warner, Nucl. Phys. /bf B324 (1989) 427.

[10] V.G. Turaev and O.Y. Viro, “State Sum Invariants of 3-Manifolds and Quantum 6j-Symbols,” preprint (1990).

[11] H. Ooguri and N. Sasakura, Mod. Phys. Lett. A6 (1991) 3591;
S. Mizoguchi and T. Tada, Phys. Rev. Lett. 68 (1992) 1795;
F. Archer and R. M. Williams, Phys. Lett. B273 (1991) 438;
B. Durhuus, H. Jacobsen and R. Nest, Nucl. Phys. (Proc. Suppl.) 25A (1992) 109.

[12] T.W. Hungerford, Algebra, Springer-Verlag, New York (1974).

[13] S. Elitzur, A. Forge and E. Rabinovici, preprint, CERN-TH.6326 (1991).

[14] M. Kontsevich, Funk. Anal. Priloz. 25 (1991) 50.
[15] S. Kharchev, A. Marshakov, A. Mironov, A. Morozov and A. Zabrodin, preprint, FIAN-TD-9-91, FIAN-TD-10-91 (1991).

[16] K. Li, Nucl. Phys. B354 (1991) 711, 725.
Figure Captions

fig. 1  Colored triangle with a complex value $C_{ijk}$.

fig. 2  Gluing two triangles $C_{ijm}$ and $C_{nkl}$ with a propagator $g^{mn}$.

fig. 3  A triangulation $T_0$ of sphere $\Sigma_0 = S^2$.

fig. 4  Propagator $g^{ij}$ and three-point vertex $C_{ijk}$ in dual diagrams. Cutting lines represent the truncation of external lines.

fig. 5  Two-point vertex $g_{ij}$. Crossed lines represent that two external lines are truncated.

fig. 6  Fusion transformation and bubble transformation in dual diagrams.

fig. 7  Diagramatic representation of the invariance under fusion transformation [eq. (3.1)].

fig. 8  Diagramatic representation of the invariance under bubble transformation [eq. (3.2)].

fig. 9  A triangulation of $\eta_{ij}$.

fig. 10  Another triangulation of $\eta_{ij}$.

fig. 11  Graphical proof of $\eta_i^j C_{jk}^l = \eta_i^j C_{kj}^l$.

fig. 12  A triangulation of the three-point function $N_{ijk}$.

fig. 13  One-point function $\langle O_i \rangle_{g=1}$ on torus.

fig. 14  (a) Cylinder $\eta^\alpha\beta$ and (b) diaper $N_{\alpha\beta\gamma}$.

fig. 15  Calculation of correlation functions with genus $g$ is reduced to that with genus 0.
fig. 16
Strong factorization property.

fig. 17
There are five triangles around a vertex $x$, $n_x = 5$.

fig. 18
Invariance under fusion transformation.

fig. 19
Invariance under bubble transformation.
\[ C_{ijk} = \begin{array}{c} i \ \ \ j \ \ \ k \end{array} \quad i, j, k \in \{0, 1, \ldots, A\} \]

\[ \begin{array}{c} \text{fig. 1} \end{array} \]

\[ \begin{array}{c} i \ \ \ j \ \ \ l \ \ \ k \end{array} = g_{mn} \begin{array}{c} i \ \ \ m \ \ \ n \ \ \ k \end{array} \]

\[ = C_{ijm} g_{mn} C_{nkl} \]

\[ = C_{ijn} C_{nkl} \]

\[ \begin{array}{c} \text{fig. 2} \end{array} \]

\[ T_0 = \begin{array}{c} \text{fig. 3} \end{array} \]
\[ g_{ij} = i \quad \text{---} \quad j \]

\[ C_{ijk} = \]

\[ g_{ij} = i \quad \times \quad j \]

fig. 4

fig. 5

fig. 6

bubble

fusion
\[ \langle O_i \rangle_{g=1} = \begin{array}{c}
\text{(fig. 13)}
\end{array} \]

\[ \begin{array}{c}
\text{fig. 14 (a)}
\end{array} \]

\[ \begin{array}{c}
\text{fig. 14 (b)}
\end{array} \]
\[ \mathcal{W}^\alpha = \alpha \]

\[ = \alpha \]

\[ = \alpha \]

\[ = \gamma^{\alpha\alpha'} N_{\alpha'\beta\delta} \gamma^{\beta\delta} \]

\[ = N_{\alpha\beta} \]

fig. 15 (a)

fig. 15 (b)
\[ \begin{align*}
\text{fig. 16} \\
\text{fig. 17} \\
\text{fig. 18} \\
\text{fig. 19}
\end{align*} \]