A universal threshold for primordial black hole formation

Albert Escrivà,1, 2 Cristiano Germani,1 and Ravi K. Sheth3

1 Institut de Ciències del Cosmos, Universitat de Barcelona, Martí i Franquès 1, 08028 Barcelona, Spain 2 Departament de Física Quàntica i Astrofísica, Facultat de Física, Universitat de Barcelona, Martí i Franquès 1, 08028 Barcelona, Spain 3 Center for Particle Cosmology, University of Pennsylvania, Philadelphia, PA 19104, USA

In this letter, we show numerically that the threshold to form primordial black holes from an initial spherically symmetric perturbation can be written as a universal function of the compaction function averaged over a sphere of radius \( r_m \), where \( r_m \) is the scale on which the compaction function is maximum. This can be understood as the requirement that, for a black hole to form, each shell of the averaged compaction function should have an amplitude larger than the Harada limit. In passing, we show that the minimal threshold for black hole formation in a radiation dominated universe is \( \delta_c = 0.4 \).

I. INTRODUCTION

In a Friedmann-Robertson-Walker (FRW) universe filled with a single fluid component having equation of state \( p = \omega \rho \), a spherically symmetric local perturbation can be approximated, to leading order in gradient expansion (super-horizon scales), as

\[
\text{ds}^2 \simeq -dt^2 + a(t)^2 \left[ \frac{dr^2}{1-K(r)r^2} + r^2 d\Omega^2 \right].
\]

Here, the local “gravitational potential” \( K(r)r^2 \) parameterizes the initial curvature perturbation.

In Ref. 3, numerical simulations were used to argue that the threshold for the amplitude of an over-density peak forming a spherically symmetric black hole in a FRW universe, only depends upon two master parameters: the integral of the initial \( K(r) \) and the edge of the over-density distribution. Musco [4] recently refined the arguments of Ref. 3 by showing that the threshold may be more conveniently given in terms of the amplitude of the gravitational potential at its maximum \( (r = r_m) \). More precisely, in Ref. 3, the threshold was given in terms of the “compaction function” \( C(r) \) at super-horizon scales (here and after we shall simply call it compaction function): The compaction function, which closely resembles the Schwarzschild gravitational potential, is defined as twice the local excess-mass over the co-moving areal radius. Indeed, at super-horizon scales, it is (in units \( G_N = 1) \)

\[
C(r) = f(\omega)K(r)r^2,
\]

where \( f(\omega) \equiv 3(1+\omega)/(5+3\omega) \). From this, one finds \( r_m \) as the first root of \( C'(r) = 0 \).

Regularity – the gravitational potential within a vanishingly small volume must be zero – ensures that \( K(r)r^2 \to 0 \) for \( r \to 0 \). Thus, the behavior of \( K(r) \) around the origin plays little role in black hole formation.

In addition, the threshold for primordial black hole formation should be quite insensitive to the curvature beyond \( r_m \). The reason is simple: the threshold is the amplitude above which a “virtual” black hole of zero mass is formed. Therefore, all the over-density beyond \( r_m \) will be diffused away while that just in the vicinity of \( r_m \) will hinder collapse. Therefore, we also expect the threshold to be very weakly dependent on the exact location of \( r_m \). However, above threshold a larger part of the initial profile would be involved in the collapse. This is why the mass of the black hole with peak density above threshold also depends non-trivially on the scale of the peak.

Summarizing, we expect the threshold to be only related to the curvature of the compaction function in the vicinity of \( r_m \). What we will then find in the following is an appropriate expansion of \( C \) around its peak.

II. APPROXIMATING THE CURVATURE

We have checked numerically that the family of centrally peaked exponentials used in Ref. 4 is an efficient basis with which to approximate the compaction function around its maximum. By defining the parameter

\[
q \equiv -\frac{C''(r_m)r_m^2}{4C(r_m)},
\]

we consider the basis

\[
K_b(r) = \frac{C(r_m)}{f(\omega)r_m^2}e^\frac{q}{2}(1-(\frac{r}{r_m})^{4}) .
\]

The threshold will be given in terms of \( C(r_m) \).

The first thing we note is that, in our basis, \( r_m \) only defines the units of length for the scaling of \( K_b \). Thus, as already argued before, the threshold does not depend on it.
We have tested our basis by considering a representative class of curvatures for the case of a radiation-dominated universe $\omega = \frac{1}{3}$. We have found that the threshold for black hole formation, obtained by the use of our basis, only differs by (1 $\pm$ 2)% from the one obtained by the exact curvature profiles considered.

In the next section we use the basis (4) to provide an analytical formula for the thresholds. We demonstrate that our formula accurately reproduces the numerical results obtained from the publicly available code for black hole formation of [5]. In turn, this will also show numerically our claim that the basis (4) well-approximates any realistic desired curvature for the calculation of the threshold.

III. UNIVERSAL THRESHOLD

As noticed by [4], the threshold for $C(r_m)$ is not universal: it depends upon the shape of the curvature profile. This implies that, if initial conditions for primordial black hole (PBH) formation are generated during inflation (see e.g. [6, 7]), then the threshold for PBH formation strongly depends on the form of the inflationary power spectrum [8]. What we show below is the remarkable fact that, nevertheless, the threshold for the average compaction function is, within 2% with respect to the simulations, universal.

Let us define

$$\bar{C}_c \equiv \frac{3}{r_m^2} \int_0^{r_m} C_c(x)x^2 dx ,$$  

where $C_c(r)$ is the critical compaction function for generating a black hole with zero mass. By using the basis [4], we have

$$\bar{C}_c = \frac{3}{2} e^{\frac{4}{3}q-1+\frac{2}{5q}} \left[ \Gamma \left( \frac{5}{2q} \right) - \Gamma \left( \frac{5}{2q}, \frac{1}{q} \right) \right] \delta_c ,$$

where $\Gamma(x)$ is the gamma function and $\Gamma(x, y)$ the incomplete gamma function and we have defined $\delta_c \equiv C_c(r_m)$.

Following [4], if the initial perturbation is not already a black hole, the compaction function is bounded by the threshold whenever $q \to \infty$, as shown numerically in [4]. At large $q$ we have

$$\bar{C}_c \sim \frac{3}{5} \delta_c .$$

Thus, at least in that limit, and in radiation, $\bar{C}_c = \frac{2}{5}$. Our assumption, that we will prove both numerically and argue in the following, is that

$$\bar{C}_c = \frac{2}{5}$$

for any value of $q$. In other words, we find that for radiation, the threshold for different curvature profiles is given by

$$\delta_c = \frac{4}{15} e^{-\frac{4}{3}q} \left[ \frac{q^{1-\frac{2}{5q}}}{\Gamma \left( \frac{5}{2q} \right) - \Gamma \left( \frac{5}{2q}, \frac{1}{q} \right) } \right] .$$  

(9)

A. Numerical checks

In this section, by using the publicly available code developed in [5], we show several numerical checks of (8).

In addition to the basis $K_b$, we have also considered the following families of curvatures:

$$K_1 = \frac{A_1}{1 + \frac{2\alpha}{p-2} \left( \frac{r}{r_m} \right)^{\alpha}} ,$$

$$K_2 = A_2 \left[ \frac{2(\lambda + 1)}{\alpha} \left( \frac{r}{r_m} \right)^{2\lambda} e^{-\frac{(\lambda+1)}{\alpha} \left( \frac{r}{r_m} \right)^{2\alpha}} \right] ,$$

$$K_3 = A_3 \left[ \frac{3n}{2k_p r} \left\{ -k_p r \left( E_{3+n}(-ik_p r) + E_{3+n}(ik_p r) \right) \right. \right.
\left. + i \left( E_{4+n}(ik_p r) - E_{4+n}(-ik_p r) \right) \right] ,$$

where

$$E_n(x) = \int_1^{\infty} e^{-xt} \frac{dt}{t^n} .$$

(13)

The constants $A_i$ parameterize the amplitudes of the different curvatures. The last (oscillating) curvature profile is related to specific templates for inflationary power spectrum [6], as explained in [5]. In Fig. 1 we compare these different profiles by fixing the same threshold.

In Figs. 1, 4, 5 and 2 we test the validity of (8) by comparing the analytical values of $\delta^A$ obtained from (9) with the numerical ones $\delta^N$. The subplots show the relative errors $d \equiv |\delta^N - \delta^A|/\delta^A$: the agreement is better than $\sim 98\%$.

We also give numerical evidence that $\delta_c$ reaches values below the one quoted in [10], as can be seen in the upper inner plot Fig. 5 for the value $q = 0.05$.

B. Analytical argument

The threshold for the average compaction function ($\bar{C}_c = 2/5 = 0.4$) is very close to the so-called Harada limit which was analytically found to be $\sim 0.41$. The value of the second significant digit is related to assumptions about the Jeans length of the perturbation [10]. As already mentioned, in Fig. 3 we provide evidence that this limit is actually closer to our theoretical value of 0.40. Nevertheless, we shall still call the minimal threshold the Harada limit, as the interpretation of it will not change.
FIG. 1. Comparison between the profiles listed in eqs. 10 and 12 and \( K_b \) in the case of the same threshold (proportional to the curvature value at \( r = r_m \)). For illustration, we have chosen the case \( q = 1.3 \) leading to \( \delta_c \approx 0.5035 \). For \( K_1 \), this translates to \( p = 4.6 \); for \( K_2 \) one has \( \alpha = 1 \) and \( \lambda = 0.3 \); for \( K_3 \), \( n \approx 6.67 \).

FIG. 2. Thresholds for the exponential basis profile \( K_b \) for different \( q \). Black points show \( \delta^A_c \) (eq. (9)) and reds show \( \delta^N_c \).

The Harada limit is the threshold for which a very sharply peaked over-density profile would collapse into a zero mass black hole, as discussed in [4]. Let us then approximate an initial over-density to be a Dirac delta function
\[
\frac{\delta \rho}{\rho} \propto \delta_D\left(\frac{r}{r_m} - 1\right).
\]

One can always find an initial time where the linear approximation is good enough and find [8] (see also [9])
\[
C(r) \propto \frac{1}{r} \int_0^r \frac{\delta \rho(x)}{\rho} x^2 dx
\]
and therefore
\[
C(r) = \frac{r_m}{r} C(r_m) \theta(r - r_m).
\]

Because, as discussed above, what happens at \( r > r_m \) is not crucial for the calculation of the threshold, we can approximate (16) as a very thin shell with finite amplitude \( C(r_m) \) positioned at \( r = r_m \). In other words we shall cut-off the tail in (16). The Harada limit indicates that such a shell would collapse and form a zero mass black hole if \( C(r_m) \sim 0.4 \).

Now suppose we have a continuum of concentric shells forming a homogeneous ball. This ball would then collapse to a black hole of zero mass if each shell had the same amplitude equal to the Harada threshold. Our averaging relates the problem of a generic compaction function shape to this homogeneous one.

We then conjecture that the same would happen for any \( \omega \) and so, for a generic fluid matter, the threshold
FIG. 5. Thresholds for the profiles. Black points show $\delta_A^c$ (eq. (9)) and reds show $\delta_N^c$. The subplot shows the difference $d$ between these two values.

would be obtained for

$$\bar{C}^c_c = C^\text{hom}_c,$$

where $C^\text{hom}_c$ is the threshold for a homogeneous ball. As a first approximation, one then may consider the functional form [10]

$$\bar{C}^c_c \sim f(\omega) \sin^2 \left[ \pi \sqrt{\omega \left( 1 + 3\omega \right)} \right].$$

While for radiation we could exactly fix $\bar{C}^c_c$ by using the limit of very peaked compaction function, we cannot do the same for other $\omega$. We then leave for future work the extensive proof of our conjecture [18].

IV. CONCLUSIONS

Primordial black holes can account for the majority of dark matter if they are in the range of $[10^{-16}, 10^{-12}] \ M_{\odot}$ (see e.g. [11]). The seeds for primordial black hole formation might be generated by large statistical fluctuations during inflation. The abundance of these statistical fluctuation, and in turn of the generated PBHs, is extremely sensitive to the threshold required to form a PBH [11,12]. To date, analytical estimates of this threshold (see for example [10,13]) are insufficiently accurate, so numerical analyses have been employed (for the latest results see [14,15]).

In this paper, we have shown that although the threshold to form a PBH is initial curvature profile dependent, as noticed by [4], the threshold for the mean (i.e. volume averaged) compaction function is universal and equal to the Harada limit. We used this remarkable result to provide an analytical formula for the threshold that only depends upon the normalized second derivative of the compaction function at its maximum. Specifically, for a radiation dominated universe, the threshold for a compaction function $C(r)$ is

$$\delta_c \equiv C_{\text{thr}}(r_m) = \frac{4}{15} e^{-\frac{1}{7}} \frac{q^{1 - \frac{2}{7}}}{\Gamma \left( \frac{5}{27} \right) - \Gamma \left( \frac{5}{27}, \frac{1}{7} \right)},$$

where

$$q \equiv -\frac{C''(r_m)r_m^2}{4C(r_m)},$$

and $r_m$ is the first root of $C'(r) = 0$.

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