AIRY FUNCTIONS
IN THE THERMODYNAMIC BETHE ANSATZ

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Abstract. Thermodynamic Bethe ansatz equations are coupled non-linear integral equations which appear frequently when solving integrable models. Those associated with models with $N=2$ supersymmetry can be related to differential equations, among them Painlevé III and the Toda hierarchy. In the simplest such case the massless limit of these non-linear integral equations can be solved in terms of the Airy function. This is the only known closed-form solution of thermodynamic Bethe ansatz equations, outside of free or classical models. This turns out to give the spectral determinant of the Schrodinger equation in a linear potential.

A great deal of interesting mathematical physics has arisen from the study of integrable models of statistical mechanics and field theory. One interesting area is known as the thermodynamic Bethe ansatz (TBA), which has proven a useful tool for computing the free energy of an integrable 1+1 dimensional system. One ends up with a set of coupled non-linear integral equations, the “TBA equations”. One completely-unexpected result was a correspondence between a limit of these integral equations and some very well-studied non-linear differential equations, namely the Toda hierarchy. The purpose of this paper is to extend these results further, and show that in at least one case there is a closed-form but non-trivial solution of the integral equations. Not only is it interesting that such complicated equations have a simple solution in terms of the Airy function, but proving it requires some utilizing some very intricate results involving the Painlevé III differential equation. Moreover, it turns out to be related to the spectral determinant of the Schrodinger equation in a linear potential.

The TBA integral equations are generically of the form

$$\epsilon_a(\theta) = m_a \cosh \theta - \sum_b \int \frac{d\theta'}{2\pi} \phi_{ab}(\theta - \theta') \ln(1 + e^{\mu_b - \epsilon_b(\theta')}).$$

(1)

Physically, $T\epsilon_a(\theta)$ is the energy for creating a particle of type $a$ and rapidity $\theta$ in a thermal bath at temperature $T$. The $m_a$ are the particle masses over temperature, while the $\mu_a$ are their chemical potentials over temperature. The kernels $\phi_{ab}$ are a result of the interactions between particles. This and all unlabelled integrals in this paper run from $-\infty$ to $\infty$. The free energy per unit length is

$$F = -T^2 \sum_a \int \frac{d\theta}{2\pi} m_a \cosh \theta \ln(1 + e^{\mu_a - \epsilon_a(\theta)}).$$

(2)

Here we study TBA equations where the underlying physical system has $N=2$ supersymmetry. The amazing thing is that solutions of a particular limit of such TBA equations are simply related to solutions of a non-linear differential equation.
Particles in an $N=2$ theory all have a charge, $f_a$, known as their fermion number. When the chemical potentials are $\mu_a = i \pi f_a$, a consequence of supersymmetry is that the $\epsilon_a$ in (3) are all constants and the free energy is $F = 0$. This is known as the Witten index (the usual integer contributions to the index do not contribute to the free energy per unit length) (3). The result of (3) is that for chemical potentials $\mu_a = i(\pi - h) f_a$, one can derive a differential equation for the order $h$ piece of the free energy. The simplest cases give the Painlevé III differential equation and the Toda hierarchy.

The TBA equations for the case at hand were derived in (2). They have $\phi_{12} = \phi_{13} = 1 / \cosh \theta$ with $\phi_{ab} = \phi_{ba}$ and other $\phi_{ab} = 0$, while $m_2 = m_3 = 0$, and $f_1 = 0$, $f_2 = -1$. For small positive $h$, the functions $A$ and $B$ are defined by $\epsilon_1(\theta) = A(\theta) - \ln h$, and $\epsilon_2(\theta) = \epsilon_3(\theta) = - h B(\theta)$. The order $h$ TBA equations are

\begin{align}
A(\theta) &= 2u(\theta) - \int \frac{d\theta'}{2\pi \cosh(\theta - \theta')} \ln(1 + B^2(\theta')) \\
B(\theta) &= \int \frac{d\theta'}{2\pi \cosh(\theta - \theta')} e^{-A(\theta')}
\end{align}

where $2u(\theta) = m_1 \cosh \theta$ here. In (2), a physics proof was given that the resulting free energy is simply related to a solution of the Painlevé III differential equation with variable $m_1$. This is a physics proof because one method of computation gives integral equations, the other the differential equations. This result was extended considerably in (3). Subsequently, the equivalence was proven directly and rigorously in (4).

A particularly interesting situation is the “massless” limit, where $m_1$ is very small. Then $2u(\theta) = e^\theta$, because $m_1$ can be removed by redefining $\theta$ by a shift. The result of this paper is $A$ and $B$ in (3-4) can be found in closed form in this massless limit.

Result:
When $u(\theta) = e^\theta/2$, the solution of (3-4) is

\begin{align}
e^{-A(\theta)} &= -2\pi \frac{d}{dz} (Ai(z))^2 \\
B(\theta) &= -2\pi \frac{d}{dz} \left[ Ai(ze^{i\pi/3})Ai(ze^{-i\pi/3}) \right]
\end{align}

where $z = (3e^\theta/4)^{2/3}$ and $Ai(z)$ is the Airy function.

To check the normalization, note that (3-4) imply that $e^{-A(\theta)} \to 2/\sqrt{3}$ and $B(\theta) \to 1/\sqrt{3}$ as $\theta \to -\infty$, in agreement with the limits of the appropriate Airy functions as $z \to 0$. $Ai(z)$ is a solution of the differential equation $w'' = zw$, while $e^{-A(\theta)}$ and $B(\theta)$ solve $w'' - 4zw' = 6w$.

As far as I can tell, it is difficult to prove the result by direct substitution into the integrals, but requires utilizing some additional structure. First, define the integral operator $K$ which maps functions to functions with the kernel

$$K(\theta, \theta') = \frac{2 E(\theta)E(\theta')}{e^\theta + e^{\theta'}},$$

where

$$E(\theta) = e^{\theta/2} e^{-u(\theta)}.$$
Solutions of Painlevé III can be expressed in terms of this operator $K$ when $m = m_1 \cosh \theta$. The functions $Z_+$ and $Z_-$ are defined as

\begin{equation}
Z_+ = e^{-2\theta/3}(I + K)^{-1}E \quad Z_- = e^{-\theta/3}(I - K)^{-1}E
\end{equation}

These $Z_\pm$ are simply related to the $Q \mp P = (I \pm K)^{-1}E$ used in $\textbf{[4]}$. It was proven in $\textbf{[2, 3, 4]}$ that functions $A$ and $B$ solve $\textbf{[3, 4]}$ if

\begin{align}
e^{-A(\theta)} &= 4\pi Z_+ (\theta)Z_- (\theta) \\
B(\theta) &= 4\pi e^{i\theta/3}e^{u(\theta+i\pi/2)+u(\theta-i\pi/2)}Z_+ (\theta-i\pi/2)Z_- (\theta+i\pi/2) - i
\end{align}

The functions $\textbf{[8, 9]}$ obey $\textbf{[3, 4]}$ for any entire $u(\theta)$ obeying $u(\theta) = u(\theta + 2\pi i)$ $\textbf{[3, 4]}$, not only the special case $u(\theta) = e^\theta/2$ analyzed in this paper.

With the similarity between $\textbf{[8, 9]}$ and $\textbf{[3, 4]}$, proving the result is equivalent to proving that

\begin{equation}
Z_+ (\theta) = Ai (z) \quad Z_- (\theta) = -Ai' (z)
\end{equation}

when $u(\theta) = e^\theta/2$. The expression $\textbf{[8]}$ for $B(\theta)$ follows by using a few Airy-function identities. The expressions for $Z_\pm$ in $\textbf{[10]}$ can be proven by directly evaluating the integral in their definition. Specifically, $\textbf{[10]}$ requires that

\begin{equation}
E(\theta) = e^{2\theta/3}Z_+(\theta) + 2E(\theta) \int d\theta'Z_+(\theta')e^{-e^{\theta'}/2} \frac{e^{7\theta'/6}}{e^{\theta} + e^{\theta'}}
\end{equation}

Defining $x \equiv e^\theta/2$ and using the expression of an Airy function in terms of a Bessel function, the integral is proportional to

\[
\int_0^\infty dx \ e^{-x} \frac{\sqrt{2x}}{e^{\theta} + 2x} K_{1/3}(x).
\]

This can be looked up in $\textbf{[13]}$, or evaluated by using Mellin transforms, under which the Bessel function $K_\nu$ has nice properties. One indeed finds that $\textbf{[11]}$ and the analogous relation for $Z_-$ are true when $Z_\pm$ are given by $\textbf{[10]}$. Since the solution with the appropriate analyticity properties (no zeroes and bounded in the strip $|\text{Im} \theta| < \pi/2$) is unique $\textbf{[4]}$, the result is proven.

The functions $Z_+(\theta)$ and $Z_-(\theta)$ are interesting in their own right. They arose as a sort of partition function of the boundary-sine Gordon problem $\textbf{[12]}$ at coupling $g = 2/3$, and equivalently as the continuum version of the Baxter $Q$-operator $\textbf{[4]}$ in studies of conformal field theory $\textbf{[14]}$. The $Q$ operator gives the generating function of the conserved quantities (local and non-local) of the theory. Up to normalization, the functions $Z_\pm$ are $Z_{BSG}(z, \pm 1/3)$ in $\textbf{[12]}$, and $\langle Q (z) \rangle$ at $p = \pm 1/6$ in $\textbf{[14]}$. Thus the result here provides the only case (other than where the system is free or classical) where these quantities can be computed explicitly. In fact, it provides strong evidence that the results of $\textbf{[12, 14]}$ are applicable in the repulsive regime $g > 1/2$ as conjectured. For example, it shows that the zeroes of the eigenvalues of the $Q$-operator obey the pattern conjectured in $\textbf{[14]}$. The “quantum Wronskian” of $\textbf{[14]}$ becomes equivalent to the Proposition of $\textbf{[4]}$, and both are easily verified by using the (ordinary) Wronskian of the Airy functions. Also, by using the series expansion of $\textbf{[14]}$, it gives a strange sequence of identities of sums of products of gamma functions.

Recently, the work of $\textbf{[14]}$ also arose in a completely new context. Namely, it was observed in $\textbf{[3]}$ that the spectral determinant for the Schrodinger equation in a
potential $|\alpha|^\alpha$ for any $\alpha$ is given precisely by the vacuum eigenvalue of this Baxter $Q$-operator. A non-zero angular momentum can be added, and this correspondence still holds. This is because the quantum Wronskian is equivalent to a set of functional relations for the spectral determinant derived in [3, 4]. The result above then means that Airy function is the spectral determinant for the Schrodinger equation in a linear potential, a result shown directly in [4]. It would be most interesting to understand how to extend this result.

Since this system discussed in this paper provides the simplest example of the differential equation/TBA correspondence [2], it seems likely that there is a simple solution of the massless limit of any TBA equations of this type. What is not yet known outside of this case is the detailed analysis of the differential equation of [10], which was vital to the results of [4]. Given how beautifully the Airy function solved the problem here, it would be quite interesting to see how this result is generalized.

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