A FORMULA FOR THE $R$-MATRIX USING A SYSTEM OF WEIGHT PRESERVING ENDOmorphisms

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Abstract. We give a formula for the universal $R$-matrix of the quantized universal enveloping algebra $U_q(g)$. This is similar to a previous formula due to Kirillov-Reshetikhin and Levendorskii-Soibelman, except that where they use the action of the braid group element $T_{w_0}$ on each representation $V$, we show that one can instead use a system of weight preserving endomorphisms. One advantage of our construction is that it is well defined for all symmetrizable Kac-Moody algebras. However we have only established that the result is equal to the universal $R$-matrix in finite type.

1. Introduction

Let $g$ be a finite type complex simple Lie algebra and $U_q(g)$ the corresponding quantized universal enveloping algebra. In [KR] and [LS], Kirillov-Reshetikhin and Levendorskii-Soibelman developed a formula for the universal $R$-matrix

$$R = (X^{-1} \otimes X^{-1})\Delta(X),$$

where $X$ belongs to a completion of $U_q(g)$. The element $X$ is constructed using the braid group element $T_{w_0}$ corresponding to the longest word of the braid group, and as such only makes sense when $g$ is of finite type.

The element $X$ in (1) defines a vector space endomorphism $X_V$ on each representation $V$ of $U_q(g)$, and in fact $X$ is defined by the system of endomorphisms $\{X_V\}$. Furthermore, any natural system of vector space endomorphisms $\{E_V\}$ can be represented as an element $E$ in a certain completion of $U_q(g)$ (see [KT]). The action of the coproduct $\Delta(E)$ on a tensor product $V \otimes W$ is then simply $E_V \otimes W$. Thus the right side of (1) is well defined if $X$ is replaced by $E = \{E_V\}$.

In this note we consider the case where $g$ is a symmetrizable Kac-Moody algebra. We define a system of weight preserving endomorphisms $\Theta = \{\Theta_V\}$ of all integrable highest weight representations $V$ of $U_q(g)$. When $g$ is of finite type, we show that

$$R = (\Theta^{-1} \otimes \Theta^{-1})\Delta(\Theta),$$

where the equality means that, for any type $1$ finite dimensional modules $V$ and $W$, the actions of the two sides of (2) on $V \otimes W$ agree. We expect this remains true in other cases, although this has not been proven.

Our endomorphisms $\Theta_V$ are not linear over the field $\mathbb{C}(q)$, but are instead compatible with the automorphism which inverts $q$. For this reason, $\Theta$ cannot be realized using an element in a completion of $U_q(g)$, and it is crucial to work with
systems of endomorphisms. There is a further technically in that $\Theta_V$ actually depends on a choice of global basis for $V$. Nonetheless, we give a precise meaning to (2).

This note is organized as follows. In Section 2 we fix notation and conventions. In Section 3 we review the universal $R$-matrix. In Section 4 we review a method developed by Henriques and Kamnitzer [HK] to construct isomorphisms $V \otimes W \rightarrow W \otimes V$. In Section 5 we state some background results on crystal bases and global bases. In Section 6 we construct our endomorphism $\Theta$. In Section 7 we prove our main theorem (Theorem 7.11), which establishes (2) when $g$ is of finite type. In Section 8 we briefly discuss future directions for this work.

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2. Conventions

We must first fix some notation. For the most part we follow [CP].

- $g$ is a symmetrizable Kac-Moody algebra with Cartan matrix $A = (a_{ij})_{i,j \in I}$ and Cartan subalgebra $\mathfrak{h}$.
- $\langle \cdot, \cdot \rangle$ denotes the paring between $\mathfrak{h}$ and $\mathfrak{h}^*$ and $(\cdot, \cdot)$ denotes the usual symmetric bilinear form on either $\mathfrak{h}$ or $\mathfrak{h}^*$. Fix the usual elements $\alpha_i \in \mathfrak{h}^*$ and $H_i \in \mathfrak{h}$, and recall that $\langle H_i, \alpha_j \rangle = a_{ij}$.
- $d_i = (\alpha_i, \alpha_i)/2$, so that $(H_i, H_j) = d_j^{-1}a_{ij}$ and, for all $\lambda \in \mathfrak{h}^*$, $(\alpha_i, \lambda) = d_i(H_i, \lambda)$.
- $B$ is the symmetric matrix $(d_j^{-1}a_{ij})$.
- $\rho \in \mathfrak{h}^*$ satisfies $(H_i, \rho) = 1$ for all $i$. Note that this implies $(\alpha_i, \rho) = d_i$. If $A$ is not invertible this condition does not uniquely determine $\rho$, and we simply choose any one solution.
- $H_\rho$ is the element of $\mathfrak{h}$ such that, for any $\lambda \in \mathfrak{h}^*$, $\langle H_\rho, \lambda \rangle = (\rho, \lambda)$. In particular, $\langle H_\rho, \alpha_i \rangle = d_i$ for all $i$.
- $U_q(g)$ is the quantized universal enveloping algebra associated to $g$, generated over $\mathbb{C}(q)$ by $E_i, F_i$ for all $i \in I$, and $K_H$ for $H$ in the coweight lattice of $g$. As usual, let $K_i = K_{d_i H_i}$. For convenience, we recall the exact formula for the coproduct:

$$\Delta E_i = E_i \otimes K_i + 1 \otimes E_i
\Delta F_i = F_i \otimes 1 + K_i^{-1} \otimes F_i
\Delta K_H = K_H \otimes K_H$$

and the following commutation relations

$$K_H E_i K_H^{-1} = q^{\langle H, \alpha_i \rangle} E_i \quad \text{and} \quad K_H F_i K_H^{-1} = q^{-\langle H, \alpha_i \rangle} F_i.$$

At times it will be necessary to adjoin a fixed $k$-th root of $q$ to the base field $\mathbb{C}(q)$, where $k$ is twice the dual Coxeter number of $g$.

- $[n] = q^{n-2}/q^{-n}$, and $X(n) = X^n_{[n][n-1] \cdots [2]}$.
- Fix a representation $V$ of $U_q(g)$ and $\lambda \in \mathfrak{h}^*$. We say $v \in V$ is a weight vector of weight $\lambda$ if, for all $H \in \mathfrak{h}$, $K_H(v) = q^{\langle H, \lambda \rangle} v$.
- $\lambda \in \mathfrak{h}^*$ is called a dominant integral weight if $\langle H_i, \lambda \rangle \in \mathbb{Z}_{\geq 0}$ for all $i$.
- For each dominant integral weight $\lambda$, $V_\lambda$ is the type 1 irreducible integrable representation of $U_q(g)$ with highest weight $\lambda$. 
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$B_\lambda$ is a fixed global basis for $V_\lambda$, in the sense of Kashiwara (see [K]). $b_\lambda$ and $b^\text{low}_\lambda$ are the highest weight and lowest weight elements of $B_\lambda$ respectively.

3. The $R$-matrix

We briefly recall the definition of a universal $R$-matrix, and the related notion of a braiding.

**Definition 3.1.** A braided monoidal category is a monoidal category $\mathcal{C}$, along with a natural system of isomorphisms $\sigma_{V,W}^{br}$ : $V \otimes W \rightarrow W \otimes V$ for each pair $V, W \in \mathcal{C}$, such that, for any $U, V, W \in \mathcal{C}$, the following two equalities hold:

\[
\begin{align*}
(\sigma_{U,W}^{br} \otimes \text{Id}) \circ (\text{Id} \otimes \sigma_{V,W}^{br}) &= \sigma_{U \otimes V,W}^{br} \\
(\text{Id} \otimes \sigma_{U,V}^{br}) \circ (\sigma_{U,V}^{br} \otimes \text{Id}) &= \sigma_{U,V \otimes W}^{br}.
\end{align*}
\]

The system $\sigma^{br} := \{\sigma_{V,W}^{br}\}$ is called a braiding on $\mathcal{C}$.

Let $\widetilde{U_q(\mathfrak{g})} \otimes \widetilde{U_q(\mathfrak{g})}$ be the completion of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ in the weak topology defined by all matrix elements of representations $V_\lambda \otimes V_\mu$, for all ordered pairs of dominant integral weights $(\lambda, \mu)$.

**Definition 3.2.** A universal $R$-matrix is an element $R$ of $\widetilde{U_q(\mathfrak{g})} \otimes \widetilde{U_q(\mathfrak{g})}$ such that $\sigma^{br}_{V,W} := \text{Flip} \circ R$ is a braiding on the category of $U_q(\mathfrak{g})$ representations.

Note in particular that, since the braiding is an isomorphism, $R$ must be invertible. It is central to the theory of quantized universal enveloping algebras that, for any symmetrizable Kac-Moody algebra $\mathfrak{g}$, $U_q(\mathfrak{g})$ has a universal $R$-matrix. The universal $R$-matrix is not truly unique, but there is a well-studied standard choice. See [CP] for a thorough discussion when $\mathfrak{g}$ is of finite type, and [L] for the general case.

When $\mathfrak{g}$ is of finite type, the $R$-matrix can be described explicitly as follows. Note that the expression below is presented in the $h$-adic completion of $U_h(\mathfrak{g})$, whereas here we are working in $U_q(\mathfrak{g})$. However, it is straightforward to check that this gives a well defined endomorphism of $V \otimes W$ for any integrable highest weight $U_q(\mathfrak{g})$-representations $V$ and $W$, with the only difficulty being that certain fractional powers of $q$ can appear.

**Theorem 3.3.** (see [CP, Theorem 8.3.9]) Assume $\mathfrak{g}$ is of finite type. Then the standard universal $R$-matrix for $U_q(\mathfrak{g})$ is given by the expression

\[
(6) \quad R_h = \exp \left( h \sum_{i,j} (B^{-1})_{ij} H_i \otimes H_j \right) \prod_{\beta} \exp \left[ (1 - q_{\beta}^2) E_{\beta} \otimes F_{\beta} \right],
\]

where the product is over all the positive roots of $\mathfrak{g}$, and the order of the terms is such that $\beta_r$ appears to the left of $\beta_s$ if $r > s$. \hfill \Box

We will not explain all the notation in (6), since the only thing we use is the fact that $E_{\beta}$ acts as 0 on any highest weight vector, and so the product in the expression acts as the identity on $b_\lambda \otimes c \in V_\lambda \otimes V_\mu$. 
4. Constructing isomorphisms using systems of endomorphisms

Here and throughout this note a representation of $U_q(\mathfrak{g})$ will mean a direct sum of possibly infinitely many of the irreducible integrable type 1 representations $V_\lambda$. We note that the category of such representations is closed under tensor product. When $\mathfrak{g}$ is of finite type, we can restrict to finite direct sums, or equivalently finite dimensional type 1 modules, since this category is already closed under tensor product.

In this section we review a method for constructing natural systems of isomorphisms $\sigma_{V,W} : V \otimes W \to W \otimes V$. This idea was used by Henriques and Kamnitzer in [HK], and was further developed in [KT]. The data needed to construct such a system is:

(i) An algebra automorphism $C_\xi$ of $U_q(\mathfrak{g})$ which is also a coalgebra anti-automorphism.

(ii) A natural system of invertible vector space endomorphisms $\xi_V$ of each representation $V$ of $U_q(\mathfrak{g})$ which is compatible with $C_\xi$ in the sense that the following diagram commutes for all $V$:

\[
\begin{array}{ccc}
V & \xrightarrow{\xi_V} & V \\
\downarrow & & \downarrow \\
U_q(\mathfrak{g}) & \xrightarrow{C_\xi} & U_q(\mathfrak{g}).
\end{array}
\]

It follows immediately from the definition of coalgebra anti-automorphism that

\[
\sigma_{V,W}^\xi := \text{Flip} \circ (\xi_V^{-1} \otimes \xi_W^{-1}) \circ \xi_{V \otimes W}
\]

is an isomorphism of $U_q(\mathfrak{g})$ representations from $V \otimes W$ to $W \otimes V$, where Flip is the map from $V \otimes W$ to $W \otimes V$ defined by Flip$(v \otimes w) = w \otimes v$.

We will normally denote the system $\{\xi_V\}$ simply by $\xi$, and will denote the action of $\xi$ on the tensor product of two representations by $\Delta(\xi)$. This is justified since, as explained in [KT], $\xi$ in fact belongs to a completion of $U_q(\mathfrak{g})$, and the action of $\xi$ on $V \otimes W$ is calculated using the coproduct. With this notation $\sigma^\xi := \{\sigma_{V,W}^\xi\}$ can be expressed as

\[
\sigma^\xi = \text{Flip} \circ (\xi^{-1} \otimes \xi^{-1}) \circ \Delta(\xi).
\]

In the current work we require a little more freedom: we will sometimes use automorphisms $C_\xi$ of $U_q(\mathfrak{g})$ which are not linear over $\mathbb{C}(q)$, but instead are bar-linear (i.e. invert $q$). This causes some technical difficulties, which we deal with in Section 6. Once we make this precise, we will use all the same notation for a bar-linear $C_\xi$ and compatible system of $\mathbb{C}$ vector space automorphisms $\xi$ as we do in the linear case, including using $\Delta(\xi)$ to denote $\xi$ acting on a tensor product.

Comment 4.1. Since the representations we are considering are all completely reducible, to describe the data $(C_\xi, \xi)$ it is sufficient to describe $C_\xi$ and to give the action of $\xi_V\lambda$ on any one vector $v$ in each irreducible representation $V_\lambda$. This is usually more convenient then describing $\xi_{V_\lambda}(v)$ explicitly. Of course, the choice of $C_\xi$ imposes a restriction on $\xi_{V_\lambda}(v)$, so when we give such a description of $\xi$, we must check that the action on our chosen vector in each $V_\lambda$ is compatible with $C_\xi$. 
Comment 4.2. If $C_\xi$ is an coalgebra automorphism as opposed to a coalgebra anti-automorphism, the same arguments show that $(\xi^{-1}_V \otimes \xi^{-1}_W) \circ \xi_{V \otimes W} : V \otimes W \to V \otimes W$ is an isomorphism.

5. Crystal bases and Global bases

In order to extend the construction described in the Section 4 to include bar linear $\xi$, we will need to use some results concerning crystal bases and global bases. We state only what is relevant to us, and refer the reader to [K] for a more complete exposition. Unfortunately, the conventions in [K] and [CP] do not quite agree. In particular, the theorems from [K] that we will need are stated in terms of a different coproduct, so we have modified them to match our conventions.

Definition 5.1. Fix an integrable highest weight representation $V$ of $U_q(\mathfrak{g})$. Define the Kashiwara operators $\tilde{F}_i, \tilde{E}_i : V \to V$ by linearly extending

$$(9) \quad \begin{cases} \tilde{F}_i(F_i^{(n)}(v)) = F_i^{(n+1)}(v) \\ \tilde{E}_i(F_i^{(n)}(v)) = F_i^{(n-1)}(v). \end{cases}$$

for all $v \in V$ such that $E_i(v) = 0$.

Definition 5.2. Let $A_\infty = \mathbb{C}[q^{-1}]$ be the algebra of rational functions in $q^{-1}$ over $\mathbb{C}$ whose denominators are not divisible by $q^{-1}$.

Definition 5.3. A crystal basis of a representation $V$ (at $q = \infty$) is a pair $(\mathcal{L}, \tilde{B})$, where $\mathcal{L}$ is an $A_\infty$-lattice of $V$ and $\tilde{B}$ is a basis for $\mathcal{L} \otimes q^{-1} \mathcal{L}$, such that

(i) $\mathcal{L}$ and $\tilde{B}$ are compatible with the weight decomposition of $V$.

(ii) $\mathcal{L}$ is invariant under the Kashiwara operators and $\tilde{B} \cup 0$ is invariant under their residues $e_i := \tilde{E}_i(-\text{mod } q^{-1} \mathcal{L})$, $f_i := \tilde{F}_i(-\text{mod } q^{-1} \mathcal{L}) : \mathcal{L}/q^{-1} \mathcal{L} \to \mathcal{L}/q^{-1} \mathcal{L}$.

(iii) For any $b, b' \in \tilde{B}$, we have $e_i b = b'$ if and only if $f_i b' = b$.

Definition 5.4. Let $(\mathcal{L}, \tilde{B})$ be a crystal basis for $V$. The highest weight elements of $\tilde{B}$ are those $b \in \tilde{B}$ such that, for all $i$, $e_i(b) = 0$.

Proposition 5.5. (see [K]) Each $V_\lambda$ has a crystal basis $(\mathcal{L}_\lambda, \tilde{B}_\lambda)$. Furthermore, $(\mathcal{L}_\lambda, \tilde{B}_\lambda)$ has a unique highest weight element, and this occurs in the $\lambda$ weight space.

Theorem 5.6. [K, Thoerem 1] Let $V, W$ be representations with crystal bases $(\mathcal{L}, \tilde{A})$ and $(\mathcal{M}, \tilde{B})$ respectively. Then $(\mathcal{L} \otimes \mathcal{M}, \tilde{A} \otimes \tilde{B})$ is a crystal basis of $V \otimes W$. Furthermore, the highest weight elements of $\tilde{A} \otimes \tilde{B}$ are all of the form $a^\text{high} \otimes b$, where $a^\text{high}$ is a highest weight element of $\tilde{A}$.

Definition 5.7. Let $(\mathcal{L}_\lambda, \tilde{B}_\lambda)$ and $(\mathcal{L}_\mu, \tilde{B}_\mu)$ be crystal bases for $V_\lambda$ and $V_\mu$. Set

$$S_{\lambda, \mu}^\nu := \{ b \in \tilde{B}_\mu : b_\lambda \otimes b \text{ is a highest weight element of } \tilde{B}_\lambda \otimes \tilde{B}_\mu \text{ of weight } \nu \}.$$ 

For any $V_\lambda$, and any choice of highest weight vector $b_\lambda \in V_\lambda$, there is a canonical choice of basis $B_\lambda$ for $V_\lambda$, which contains $b_\lambda$, and such that $(B_\lambda + q\mathcal{L}, \mathcal{L})$ is a crystal basis for $V$, where $\mathcal{L}$ is the $A_\infty$-span of $B_\lambda$. That is not to say there is a unique basis for $V_\lambda$ satisfying these two conditions, only that one can find a canonical “good” choice. This is known as the global basis for $V_\lambda$. A complete construction can be found in [K], although here we more closely follow the presentation from
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In the present work we simply use the fact that the global basis exists, and state the properties of $B_\lambda$ that we need.

**Definition 5.8.** $\overline{C} : U_q(g) \to U_q(g)$ is the $\mathbb{C}$-algebra involution defined by

\[
\begin{align*}
C_{\overline{\cdot}}(E_i) &= E_i \\
C_{\overline{\cdot}}(F_i) &= F_i \\
C_{\overline{\cdot}}(K_i) &= K_i^{-1} \\
C_{\overline{\cdot}}(q) &= q^{-1}.
\end{align*}
\]

**Theorem 5.9.** (Kashiwara [K]) Fix a highest weight vector $b_\lambda \in V_\lambda$. There is a canonical choice of a “global” basis $B_\lambda$ of $V_\lambda$. This has the properties (although is not defined by these alone) that:

(i) $b_\lambda \in B_\lambda$.

(ii) $B_\lambda$ is a weight basis for $V_\lambda$.

(iii) Let $L$ be the $A_{\infty}$ span of $B_\lambda$. Then $(B_\lambda + q^{-1}L, L)$ is a crystal basis for $V_\lambda$.

(iv) Define the involution $\overline{b_{\lambda, B_\lambda}}$ of $V_\lambda$ by $\overline{b_{\lambda, B_\lambda}}(f(q)b) = f(q^{-1})b$ for all $f(q) \in \mathbb{C}(q)$ and $b \in B_\lambda$. Then $\overline{b_{\lambda, B_\lambda}}$ is compatible with $C_{\overline{\cdot}}$, in the sense discussed in Section 4.

Furthermore, if a different highest weight vector is chosen, $B_\lambda$ is multiplied by an overall scalar. □

**Definition 5.10.** If $V$ is any (possibly reducible) representation of $U_q(g)$, we say a basis $B$ of $V$ is a global basis if there is a decomposition of $V$ into irreducible components such that $B$ is a union of global bases for the irreducible pieces.

6. THE SYSTEM OF ENDOMORPHISMS $\Theta$

We now introduce a $\mathbb{C}$-algebra automorphism $C_\Theta$ of $U_q(g)$. Notice that this inverts $q$, so it is not a $\mathbb{C}(q)$ algebra automorphism, but is instead bar linear:

\[
\begin{align*}
C_\Theta(E_i) &= E_iK_i^{-1} \\
C_\Theta(F_i) &= K_iF_i \\
C_\Theta(K_i) &= K_i^{-1} \\
C_\Theta(q) &= q^{-1}.
\end{align*}
\]

One can check that $C_\Theta$ is a well defined algebra involution and a coalgebra anti-involution. In order to use the methods of section 4, we must define a $\mathbb{C}$-vector space automorphism $\Theta_{V_\lambda}$ of each $V_\lambda$ which is compatible with $C_\Theta$. This is complicated by the fact that $C_\Theta$ does not preserve the $\mathbb{C}(q)$ algebra structure, but instead inverts $q$. We must actually work in the category of representations with chosen global bases. An element of this category will be denoted $(V, B)$, where $B$ is the chosen global basis of $V$.

**Definition 6.1.** Fix a global basis $B_\lambda$ for $V_\lambda$. The action of $\Theta_{(V_\lambda, B_\lambda)}$ on $V_\lambda$ is defined by requiring that it be compatible with $C_\Theta$, and that $\Theta_{(V_\lambda, B_\lambda)}(b_\lambda) = q^{-(\lambda, \lambda)/2 + (\lambda, \rho)}b_\lambda$. This is extended by naturality to define $\Theta_{(V, B)}$ for any (possibly reducible) $V$. 

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[CP, Chapter 14.1C].
Comment 6.2. To ensure that Definition 6.1 makes sense, one must check that there is a map which sends $b_\lambda$ to $q^{-(\lambda,\lambda)/2+2\lambda,q}b_\lambda$ and is compatible with $C_\Theta$. This amounts to checking that $b_\lambda$ is still a highest weight vector if the action of $U_q(\mathfrak{g})$ is twisted by the automorphism $C_\Theta$, and is not difficult.

Comment 6.3. In some cases $\Theta$ acts on a weight vector as multiplication by a fractional power of $q$. To be completely precise we should adjoin a fixed $k^{th}$ root of unity to the base field $\mathbb{C}(q)$, where $k$ is twice the dual Coxeter number of $\mathfrak{g}$. This causes no significant difficulties.

The construction described in Section 4 uses the action of $\xi_{V \otimes W}$ on $V \otimes W$. Thus we will need to define how $\Theta$ acts on a tensor product. In particular, we need a well defined notion of tensor product in the category of representations with chosen global bases.

Definition 6.4. Let $V^\nu_{\lambda,\mu}$ denote the isotypic component of $V_\lambda \otimes V_\mu$ with highest weight $\nu$. Let $V_{\lambda,\mu}^{>\nu} := \bigcup_{\gamma > \nu} V_{\lambda,\mu}^\gamma$, $V_{\lambda,\mu}^{\geq \nu} := \bigcup_{\gamma \geq \nu} V_{\lambda,\mu}^\gamma$, and $Q_{\lambda,\mu}^{\nu} := V_{\lambda,\mu}^{>\nu} / V_{\lambda,\mu}^{\geq \nu}$. Here we use the partial order of the weight lattice where $\gamma > \nu$ iff $\gamma - \nu$ is a non-negative linear combination of the $\alpha_i$.

Comment 6.5. It is clear that the inclusion $V_{\lambda,\mu}^{\nu} \hookrightarrow V_{\lambda,\mu}^{\geq \nu}$ descends to an isomorphism from $V_{\lambda,\mu}^{\nu}$ to $Q_{\lambda,\mu}^{\nu}$.

Definition 6.6. The tensor product $(V_\lambda, B_\lambda) \otimes (V_\mu, B_\mu)$ is defined to be $(V_\lambda \otimes V_\mu, A)$, where $A$ is the unique global basis of $V \otimes W$ such that the projections of the highest weight elements of $A$ of weight $\nu$ in $Q_{\lambda,\mu}^{\nu}$ are equal to the projections of $b_\lambda \otimes b$ for those $b \in S_{\lambda,\mu}^\nu$. This is well defined by Comment 6.5. Extend by naturality to can a tensor product $(V, B) \otimes (W, C)$ for possibly reducible $V$ and $W$.

7. Proof that we obtain the $R$-matrix when $\mathfrak{g}$ is of finite type

The proof of our main theorem uses a relationship between the $R$-matrix and the braid group element $T_{w_0}$ first observed in [KR] and [LS]. Thus for this section we must restrict to finite type. We hope the result will prove to be true in greater generality, but establishing this would certainly require a different approach. We start by introducing a few more automorphisms of $U_q(\mathfrak{g})$ and of its representations.

Definition 7.1. Let $\theta$ to be the diagram automorphism such that $w_0(\alpha_i) = -\alpha_{\theta(i)}$, where $w_0$ is the longest element in the Weyl group.

Definition 7.2. $C_\Gamma$ is the $\mathbb{C}$-Hopf algebra automorphism of $U_q(\mathfrak{g})$ defined by

\begin{align*}
C_\Gamma(E_i) &= -K_{\theta(i)} F_{\theta(i)} \\
C_\Gamma(F_i) &= -E_{\theta(i)} K^{-1}_{\theta(i)} \\
C_\Gamma(K_i) &= K_{\theta(i)} \\
C_\Gamma(q) &= q^{-1}.
\end{align*}

Define the action of $\Gamma_{(V_\lambda, B_\lambda)}$ on $V_\lambda$ to be the unique $\mathbb{C}$-linear endomorphism of each $V_\lambda$ which is compatible with $C_\Gamma$, and which is normalized so that $\Gamma(b_\lambda) = b_\lambda^{w_0}$. Extend this by naturality to get the action of $\Gamma_{(V, B)}$ on any (possible reducible) representation $V$ with chosen global basis $B$. 
**Comment 7.3.** It is a simple exercise to check that $C_T$ is in fact a Hopf algebra automorphism, and is compatible with a $\mathbb{C}$-vector space automorphism of $V_\lambda$, which takes $b_\lambda$ to $b^\text{low}_\lambda$.

**Definition 7.4.** $C_{T_{w_0}}$ and $C_J$ are the $\mathbb{C}(q)$-algebra automorphisms of $U_q(\mathfrak{g})$ defined by

$$
\begin{aligned}
C_{T_{w_0}}(E_i) & = -F_{\theta(i)} K_{\theta(i)} \\
C_{T_{w_0}}(F_i) & = -K_{\theta(i)}^{-1} E_{\theta(i)} \\
C_{T_{w_0}}(K_H) & = K_{w_0(H)}, 	ext{ so that } C_{T_{w_0}}(K_i) = K_{\theta(i)}^{-1} \\
C_J(E_i) & = K_i E_i \\
C_J(F_i) & = F_i K_i^{-1} \\
C_J(K_H) & = K_H.
\end{aligned}
$$

The systems of $\mathbb{C}(q)$-vector space automorphisms $T_{w_0}$ and $J$ of each $V_\lambda$ are the unique automorphisms which are compatible with $C_{T_{w_0}}$ and $C_J$ respectively, and such that $T_{w_0}(b^\text{low}_\lambda) = b_\lambda$ and $J(b_\lambda) = q^{(\lambda,\lambda)/2 + (\lambda,\rho)} b_\lambda$, where $b_\lambda$ and $b^\text{low}_\lambda$ are the highest and lowest weight elements in some global basis $B_\lambda$.

**Comment 7.5.** It is a straightforward exercise to show that the formulas in Definition 7.4 do define algebra automorphisms of $U_q(\mathfrak{g})$ and compatible vector space automorphisms of each $V_\lambda$. There is an action of the braid group on each $V_\lambda$, and $T_{w_0}$ is in fact the action of the longest element (for an appropriate choice of conventions). Note also that $J$ and $T_{w_0}$ do not depend on the choice of global basis as they are stable under simultaneously rescaling $b_\lambda$ and $b^\text{low}_\lambda$. All of this is discussed in [KT].

**Lemma 7.6.** The following identities hold:

1. $\Gamma_{(V,B)} = \text{bar}_B(V,B) \circ T_{w_0}^{-1}$,
2. $\Theta_{(V,B)} = K_{2H_\rho} \circ \text{bar}_B(V,B) \circ J$,
3. For any weight vector $v \in V$ with $\text{wt}(v) = \mu$, $\text{J}(v) = q^{(\mu,\rho)/2 + (\mu,\rho)} v$,
4. For any $b \in B$ with $\text{wt}(b) = \mu$, $\Theta_{(V,B)}(b) = q^{-(\mu,\rho)/2 + (\mu,\rho)} b$,
5. $\Gamma^{-1}_{(V,B)} \circ \Theta_{(V,B)} = JT_{w_0}$.

Here $\text{bar}_B(V,B)$ is the involution defined in Theorem 5.9, part (iv).

**Proof.** Let $C_{K_{2H_\rho}}$ be the algebra automorphism of $U_q(\mathfrak{g})$ defined by $C_{K_{2H_\rho}}(X) = K_{2H_\rho} X K_{2H_\rho}^{-1}$. It follows directly from (4) that

$$
C_{K_{2H_\rho}}(K_i^{-1} E_i) = E_i K_i^{-1} \quad \text{and} \quad C_{K_{2H_\rho}}(F_i K_i) = K_i F_i.
$$

Using (15) and the relevant definitions, a simple check on generators shows that

$$
C_T = C_{\text{bar}} \circ C_{T_{w_0}}^{-1}, \quad \Theta = C_{K_{2H_\rho}} \circ \text{bar} \circ C_J, \quad \text{and} \quad C_{\Gamma^{-1}} \circ \Theta = C_J \circ C_{T_{w_0}}.
$$

Thus, to prove (i), (ii) and (v), it suffices to check each identity when each side acts on any one chosen vector $b$ in each $V_\lambda$. For parts (i) and (ii), choose $b = b_\lambda$ and the identity is immediate from definitions.

For part (iii), it is sufficient to consider $V = V_\lambda$. By Definition 7.4, (iii) holds for $b = b_\lambda$. Furthermore, vectors of the form $F_{i_k} \cdots F_i b_\lambda$ generate $V_\lambda$ as a $\mathbb{C}(q)$
module. Assume that $v$ is a weight vector of weight $\mu$, and $J(v) = q^{(\mu, \rho)/2 + (\mu, \rho)}$. Fix $i \in I$. Then
\begin{equation}
J(F_i v) = C_i(J(F_i)v) = F_i K_i^{-1} q^{(\mu, \rho)/2 + (\mu, \rho)} v = F_i q^{-d(v, w)} q^{(\mu, \rho)/2 + (\mu, \rho)} v
\end{equation}
(17)
\[= q^{-(\alpha_i, \mu)} q^{(\mu, \rho)/2 + (\mu, \rho)} v = q^{(\mu - \alpha_i, \mu - \alpha_i)/2 + (\mu - \alpha_i, \rho)} v.\]

The claim now follows by induction on $k$.

Part (iv) follows by directly calculating the action of the right side of (ii) on $b$ and using Part (iii) to evaluation the action of $J$.

The definitions of $\Theta_{(V, B)}$ and $\Gamma_{(V, B)}$, along with parts (iii) and (iv), now immediately imply that $\Gamma_{(V_\lambda, B_\lambda)}^{-1} \circ \Theta_{(V_\lambda, B_\lambda)}(b_{\lambda}^{\text{low}}) = J T_{w_0}(b_{\lambda}^{\text{low}}) = q^{(\lambda, \lambda)/2 + (\lambda, \rho)} b_{\lambda}$, completing the proof of (v). $\square$

We also need the following construction of the $R$ matrix due to Kirillov-Reshetikhin and Levendorskii-Soibelman. Due to a different choice of conventions, our $T_{w_0}$ is $K_{H_0} T_{w_0}^{-1}$ in those papers, so we have modified the statement accordingly. As with Theorem 7.7, this expression is written using the $h$-adic completion of $U_h(\mathfrak{g})$, but gives a well defined action on $V \otimes W$ for any finite dimensional type $1 U_q(\mathfrak{g})$-module.

**Theorem 7.7.** [KR, Theorem 3], [LS, Theorem 1] The standard universal $R$-matrix can be realized as
\begin{equation}
R = \exp \left( h \sum_{i,j \in I} (B^{-1})_{ij} H_i \otimes H_j \right) \left( T_{w_0}^{-1} \otimes T_{w_0}^{-1} \right) \Delta(T_{w_0}).
\end{equation}
(18)

\[\square\]

**Corollary 7.8.** $(T_{w_0}^{-1} \otimes T_{w_0}^{-1}) \Delta(T_{w_0}) = \prod_{\beta} \exp_{q_{\beta}} \left[ (1 - q_{\beta}^{-2}) E_\beta \otimes F_\beta \right],$

where the product is over all the positive roots of $\mathfrak{g}$, and the order of the terms is such that $\beta_r$ appears to the left of $\beta_s$ if $r > s$.

**Proof.** Follows immediately from Theorems 3.3 and 7.7, since the action of $R$ on $V_\lambda \otimes V_\mu$ is invertible. $\square$

As discussed in [KT], the following is equivalent to Theorem 7.7:

**Corollary 7.9.** (see [KT, Comment 7.3]) Let $X = J T_{w_0}$. Then
\[R = (X^{-1} \otimes X^{-1}) \Delta(X).\]

\[\square\]

**Lemma 7.10.** Fix type $1$ finite dimensional $U_q(\mathfrak{g})$ representations with chosen global bases $(V, B)$ and $(W, C)$. The operator $(\Gamma_{(V, B)} \otimes \Gamma_{(W, C)}) \Gamma_{(V \otimes W, A)}^{-1}$ acts on $V \otimes W$ as the identity, where $A$ is the global basis of $V \otimes W$ constructed from $B$ and $C$ in Definition 6.6.

**Proof.** It suffices to consider the case when $V = V_\lambda$ and $W = V_\mu$ are irreducible. Set
\begin{equation}
m^ \Gamma := (\Gamma_{(V_\lambda, B_\lambda)} \otimes \Gamma_{(V_\mu, B_\mu)}) (\Gamma_{(V_\lambda \otimes V_\mu, A)})^{-1} : V_\lambda \otimes V_\mu \rightarrow V_\lambda \otimes V_\mu
\end{equation}
(19)

We must show that $m^ \Gamma$ is the identity. $C_1$ is a Hopf algebra automorphism of $U_q(\mathfrak{g})$, so, as in Section 4, it follows that $m^ \Gamma$ is an automorphism of $U_q(\mathfrak{g})$ representations.
In particular, \( m^\Gamma \) preserves isotypic components of \( V_\lambda \otimes V_\mu \) and acts on each subquotient \( Q^\nu_{\lambda,\mu} \) (see Definition 6.4). It is sufficient to show that the action on \( Q^\nu_{\lambda,\mu} \) is the identity for all \( \nu \). In fact it is sufficient to consider the action on the highest weight space of \( Q^\nu_{\lambda,\mu} \), since this generates \( Q^\nu_{\lambda,\mu} \). This highest weight space has a basis consisting of \( \{ b_\lambda \otimes b : b \in S^\nu_{\lambda,\mu} \} \), where \( S^\nu_{\lambda,\mu} \) is as in Definition 5.7 and we use the notation \( a \otimes b \) to denote the image of \( a \otimes b \) in \( Q^\nu_{\lambda,\mu} \).

By Lemma 7.6 part (i) and Corollary 7.8,
\[
\begin{align*}
    m^\Gamma &= (\text{bar}_{(V_\lambda,B_\lambda)} \otimes \text{bar}_{(V_\mu,B_\mu)})(T_{w_0}^{-1} \otimes T_{w_0}^{-1}) \Delta(T_{w_0}) \text{bar}_{(V_\lambda \otimes V_\mu,A)} \\
    &\quad = (\text{bar}_{(V_\lambda,B_\lambda)} \otimes \text{bar}_{(V_\mu,B_\mu)}) \prod_{\beta} \exp_{q_\beta} \left[ (1 - q_\beta^{-2})E_\beta \otimes F_\beta \right] \text{bar}_{(V_\lambda \otimes V_\mu,A)},
\end{align*}
\]
For convenience, set
\[
    \Psi := (\text{bar}_{(V_\lambda,B_\lambda)} \otimes \text{bar}_{(V_\mu,B_\mu)}) \prod_{\beta} \exp_{q_\beta} \left[ (1 - q_\beta^{-2})E_\beta \otimes F_\beta \right].
\]
Both \( m^\Gamma \) and \( \text{bar}_{(V_\lambda \otimes V_\mu,A)} \) act in a well defined way on each \( Q^\nu_{\lambda,\mu} \), which implies that \( \Psi \) does as well.

The global basis \( A \) was chosen so that \( \text{bar}_{(V_\lambda \otimes V_\mu,A)}(b_\lambda \otimes b) = b_\lambda \otimes b \) (see Definition 6.6). Since all \( E_\beta \) kill \( b_\lambda \) and \( \text{bar}_{(V_\lambda,B_\lambda)} \otimes \text{bar}_{(V_\mu,B_\mu)} \) preserves \( b_\lambda \otimes b \) by definition, we see that \( \Psi(b_\lambda \otimes b) = b_\lambda \otimes b \), and, taking the image in \( Q^\nu_{\lambda,\mu} \), \( \Psi(b_\lambda \otimes b) = b_\lambda \otimes b \). Thus, using (20), we see that \( m^\Gamma \) acts on \( b_\lambda \otimes b \) as the identity. The lemma follows.

**Theorem 7.11.** Fix type 1 finite dimensional \( U_q(\mathfrak{g}) \) representations with chosen global bases \( (V,B) \) and \( (W,C) \). Then \( (\Theta^{-1}_{(V,B)} \otimes \Theta^{-1}_{(W,C)})\Theta_{(V \otimes W,A)} \) acts on \( V \otimes W \) as the standard \( R \)-matrix, where \( A \) is the global basis of \( V \otimes W \) constructed from \( B \) and \( C \) in Definition 6.6. This holds independently of the choices of global bases \( B \) and \( C \).

**Proof.** By Corollary 7.9 and Lemma 7.6 part (v)
\[
    R = ((JT_{w_0})^{-1} \otimes (JT_{w_0})^{-1}) \Delta(JT_{w_0}) \\
    = (\Theta^{-1}_{(V,B)} \otimes \Theta^{-1}_{(W,C)})(\Gamma_{(V,B)} \otimes \Gamma_{(W,C)})(\Gamma_{(V \otimes W,A)}^{-1})^{-1}\Theta_{(V \otimes W,A)}).
\]
By Lemma 7.10, the \( (\Gamma_{(V,B)} \otimes \Gamma_{(W,C)})(\Gamma_{(V \otimes W,A)}^{-1})^{-1} \) that appears acts as the identity.

**Comment 7.12.** By Theorem 7.11, the composition
\[
    (\Theta^{-1}_{(V,B)} \otimes \Theta^{-1}_{(W,C)})\Theta_{(V \otimes W,A)}
\]
does not depend on the choices on global bases \( B \) and \( C \). Introducing the notation \( \Delta(\Theta) \) to mean \( \Theta_{(V \otimes W,A)} \) and dropping the subscripts, we can interpret \( (\Theta^{-1} \otimes \Theta^{-1})\Delta(\Theta) \) as (23) calculated using any global bases \( B \) and \( C \). Then Theorem 7.11 becomes (2) from the introduction. We also note that \( \Theta_{(V,B)} \) is easily seen to be an involution, so the inverses in (23) are perhaps unnecessary.
8. Future directions

Although we have only proven Theorem 7.11 when \( g \) is of finite type, much of the construction works in greater generality. We did not assume \( g \) was finite type in Section 6, so the expression \( (\Theta^{-1}_{(V,B)} \otimes \Theta^{-1}_{(W,C)}) \Theta_{(V \otimes W, A)} \) makes sense for any symmetrizable Kac-Moody algebra. Since \( C_\Theta \) is a coalgebra-antiautomorphism, the methods from Section 4 imply that

\[
\text{Flip} \circ (\Theta^{-1}_{(V,B)} \otimes \Theta^{-1}_{(W,C)}) \Theta_{(V \otimes W, A)}
\]

is an isomorphism of representations. Furthermore, it is true in general that (24) does not depend on the choice of \( B \) and \( C \). To see why, it is sufficient to consider the case when \( V = V_\lambda \) and \( W = V_\mu \) are irreducible. Then the global bases \( B_\lambda \) and \( B_\mu \) are unique up multiplication by an overall scalar. It is straightforward to see that if \( B_\lambda \) (or \( B_\mu \)) is scaled by a constant \( z \), then \( A \) is scaled by \( z \) as well, and from there that both \( \Theta_{(V_\lambda, B_\lambda)} \) and \( \Theta_{(V_\lambda \otimes V_\mu, A)} \) are scaled by \( z/\bar{z} \), where \( \bar{z} \) is obtained from \( z \) by inverting \( q \). Thus the composition is unchanged.

As in Comment 7.12, we can now make sense of the expression \( (\Theta^{-1} \otimes \Theta^{-1}) \Delta(\Theta) \) for all symmetrizable Kac-Moody algebras \( g \). The fact that (24) defines an isomorphism is one of the properties required of a universal \( R \)-matrix. However, we have not proven the crucial equalities (5). Thus we ask:

**Question 1.** Is \( (\Theta^{-1} \otimes \Theta^{-1}) \Delta(\Theta) \) a universal \( R \)-matrix for \( U_q(g) \) if \( g \) is a general symmetrizable Kac-Moody algebra? If yes, is it the standard \( R \)-matrix?

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