Integral-Input-Output to State Stability

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Abstract
A notion of detectability for nonlinear systems is discussed. Within the framework of “input to state stability” (ISS), a dual notion of “output to state stability” (OSS), and a more complete detectability notion, “input-output to state stability” (IOSS) have appeared in the literature. This note addresses a variant of the IOSS property, using an integral norm to measure signals, as opposed to the standard supremum norm that appears in ISS theory.

Keywords: detectability, zero-detectability, input to state stability, Lyapunov function, input-output to state stability, norm observer.

1 Introduction
We consider stability features for the system with output:

\[
\dot{x}(t) = f(x(t), u(t)), \quad y(t) = h(x(t)),
\]

where \(x \in \mathbb{R}^n\). The function \(f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n\) is assumed jointly continuous in \(x\) and \(u\), and locally Lipschitz in \(x\), uniformly in \(u\). The output map \(h : \mathbb{R}^n \to \mathbb{R}^p\) is assumed locally Lipschitz, and we suppose \(f(0, 0) = 0\) and \(h(0) = 0\). Inputs \(u(\cdot)\) take values in some set \(U \subseteq \mathbb{R}^m\) (where \(U = \mathbb{R}^m\) unless otherwise stated).

The notion of input to state stability (ISS), introduced in [22], provides a theoretical framework in which to formulate questions of robustness with respect to inputs (seen as disturbances) acting on a system. An ISS system is, roughly, one which has a “finite nonlinear gain” with respect to inputs and whose transient behavior can be bounded in terms of the size of the initial state; the precise definition is in terms of \(K\) function gains. The theory of ISS systems now forms an integral part of several texts ([1, 3, 6, 12, 13, 21]) as well as expository and research articles (see e.g. [6, 14, 17] as well as the recent [27]).

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Within the framework of ISS, a natural notion of detectability, and a dual to the ISS property, is the notion of output to state stability (OSS) addressed in [23, 24]. These references include a characterization of OSS in terms of a Lyapunov (or “storage”) function, as well as a discussion of the roles of OSS and the more general property of input-output to state stability (IOSS) in nonlinear observer theory. The IOSS property was further addressed in [11].

In each of the notions mentioned thus far, signals (i.e. inputs and outputs) are measured by a supremum (or $L^\infty$) norm. In many cases, it may be more natural to use an integral (or $L^1$-type) norm, which corresponds to a measure of the “total energy” of the signal. A variant of ISS using this norm, called integral-ISS (iISS) was introduced in [26] and further studied in [1].

This paper addresses a combination of the ideas described above: namely a notion of detectability making use of integral norms. This property is formulated as a natural combination of the IOSS and iISS properties. It was introduced as integral-input-output to state stability (iIOSS) in [11]. This notion has been called “integral-detectability” by Morse and Hespanha [19] and is closely related to the notion of a “convergent observer” used by Krener in [10]. In addition, all systems which are passive in the sense of [13] automatically satisfy the iIOSS property (cf. remark 12 in [24]).

The main result in this paper is a characterization of the iIOSS property in terms of the existence of an appropriate Lyapunov function. In general, the result provides for the existence of a continuous Lyapunov function, though we indicate an important case where the construction can be extended to show the existence of a smooth function. Such Lyapunov characterizations for detectability notions are especially insightful, since in some cases the notion of detectability has been defined in terms of the existence of an appropriate Lyapunov (or “storage”) function (e.g. [13, 18]).

While we refer to IOSS and iIOSS as notions of detectability, they should be called more precisely notions of zero-detectability, as they characterize the property that the information from the output is sufficient to deduce stability of the state to the origin. For linear systems, such a property is equivalent to “full-state detectability” – the property which allows construction of an observer which tracks arbitrary trajectories. For nonlinear systems, a zero-detectability condition cannot guarantee the existence of a “complete” observer. Given a nonlinear system which satisfies a zero-detectability property, the most one may expect is to be able to construct a norm-observer which is able to provide a bound on how far the state is from the origin. The existence of norm observers for IOSS systems was addressed in [11]. We shall see that a similar construction for iIOSS systems follows immediately from the definitions.

1.1 Basic Definitions and Notation

The Euclidean norm in a space $\mathbb{R}^k$ is denoted simply by $|\cdot|$. For each interval $\mathcal{I} \subseteq \mathbb{R}$ and any measurable function $u : \mathcal{I} \to \mathbb{R}^k$, we will use $\|u\|_\mathcal{I}$ to denote the (essential) supremum norm of $u(\cdot)$ over $\mathcal{I}$. That is, $\|u\|_\mathcal{I} = \text{ess sup} \{ |u(t)| : t \in \mathcal{I} \}$. An input (or control) will be a measurable, locally essentially bounded function $u : \mathcal{I} \to \mathbb{R}^m$, where $\mathcal{I}$ is a subinterval of $\mathbb{R}$ which contains the origin, such that $u(t) \in U \subseteq \mathbb{R}^m$ for almost all $t \in \mathcal{I}$. Unless otherwise specified, we assume $\mathcal{I} = \mathbb{R}_{\geq 0}$.

For each initial state $\xi$ and input $u$ we let $x(t, \xi, u)$ denote the unique maximal solution of (1), and we write the output signal as $y(t, \xi, u) := h(x(t, \xi, u))$. A system is forward complete if each $\xi \in \mathbb{R}^n$ and each input $u$ defined on $\mathbb{R}_{\geq 0}$ produce a solution $x(t, \xi, u)$ which is defined for all $t \geq 0$.

A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ (or a “$\mathcal{K}$ function”) if it is continuous, positive
definite, and strictly increasing; and is of class $K\infty$ if in addition it is unbounded. A function $\rho: \mathbb{R}_\geq \to \mathbb{R}_\geq$ is of class $K$ if it is continuous, decreasing, and tends to zero as its argument tends to $+\infty$. A function $\beta: \mathbb{R}_\geq \times \mathbb{R}_\geq \to \mathbb{R}_\geq$ is of class $KL$ if for each fixed $t \geq 0$, $\beta(\cdot, t)$ is of class $K$ and for each fixed $s \geq 0$, $\beta(s, \cdot)$ is of class $L$.

To formulate the statement that a nonsmooth function decreases in an appropriate manner, we will make use of the notion of the viscosity subgradient (cf. [3]).

**Definition 1.1** A vector $\zeta \in \mathbb{R}^n$ is a viscosity subgradient of the function $V: \mathbb{R}^n \to \mathbb{R}$ at $\xi \in \mathbb{R}^n$ if there exists a function $g: \mathbb{R}^n \to \mathbb{R}$ satisfying $\lim_{h \to 0} \frac{g(h)}{|h|} = 0$ and a neighbourhood $O \subset \mathbb{R}^n$ of the origin so that $V(\xi + h) - V(\xi) - \zeta \cdot h \geq g(h)$ for all $h \in O$. $\square$

The (possibly empty) set of viscosity subgradients of $V$ at $\xi$ is called the viscosity subdifferential and is denoted $\partial_D V(\xi)$. We remark that if $V$ is differentiable at $\xi$, then $\partial_D V(\xi) = \{\nabla V(\xi)\}$.

### 2 The integral-Input-Output to State Stability Property

The main property of interest in this paper is the following.

**Definition 2.1** We say that a forward complete system (1) satisfies the integral-input-output to state stability property (iIOSS) if there exist $\alpha \in K\infty$, $\beta \in KL$, and $\gamma_1, \gamma_2 \in K$ so that for every initial point $\xi \in \mathbb{R}^n$, and every input $u$,

$$
\alpha(|x(t, \xi, u)|) \leq \beta(|\xi|, t) + \int_0^t \gamma_1(|y(s, \xi, u)|) + \gamma_2(|u(s)|) \, ds
$$

for all $t \geq 0$. $\square$

**Remark 2.2** We note that, by causality, the iIOSS bound (2) can be expressed equivalently as

$$
\alpha(|x(t, \xi, u)|) \leq \beta(|\xi|, t) + \int_0^t \gamma_1(|y(s, \xi, u)|) \, ds + \int_0^\infty \gamma_2(|u(s)|) \, ds
$$

for all $t \geq 0$. We will make use of this alternate description. $\square$

**Remark 2.3** Recall that a forward complete system (1) satisfies the input-output to state stability property (IOSS) if there exist $\beta \in K\mathcal{L}$, and $\gamma_1, \gamma_2 \in K$ so that for every initial point $\xi \in \mathbb{R}^n$, and every input $u$,

$$
|x(t, \xi, u)| \leq \beta(|\xi|, t) + \gamma_1(||y(\cdot, \xi, u)||_{[0, t]}) + \gamma_2(||u||_{[0, t]})
$$

for all $t \geq 0$.

It is natural to compare the notion of iIOSS to this analogous property. We will show as a consequence of our main result that an IOSS system is in particular an iIOSS system. It has been shown (in [13] and [1] respectively), that the IOSS property is strictly weaker than OSS, and that the iISS property is strictly weaker than ISS. Either of these results show that iIOSS is a strictly weaker property than IOSS. $\square$
Remark 2.4 It is an easy exercise to show that for linear systems the iIOSS property is equivalent to detectability. However, as mentioned above, for general systems as in (1), iIOSS is a notion of zero-detectability. Given that a system satisfies the iIOSS property, one cannot hope to build a complete observer for the system, but rather only a norm observer which measures how far the state is from the origin. The construction of norm observers for IOSS systems was addressed in [11], where it was shown that a system satisfies the IOSS property if and only if it admits an appropriate norm observer.

For iIOSS systems, the situation is more transparent. It is immediate that if a system satisfies the iIOSS bound (2), then for the function 
\[ p(t) = \gamma_1(|y(t, \xi, u)|) + \gamma_2(|u(t)|), \quad p(0) = 0, \]
the system will satisfy
\[ \alpha(|x(t, \xi, u)|) \leq V(\xi) \leq \alpha(|\xi|) \quad \forall \xi \in \mathbb{R}^n \]
Thus the function \( p(\cdot) \) provides an asymptotic upper bound for the size of the state, i.e. it is a norm observer for the system.

To extend these ideas to the case where construction of a full-state observer may be possible, one must consider a notion of “complete” detectability for nonlinear systems. Such a notion was introduced in [24] under the name of incremental-IOSS.

2.1 iIOSS Lyapunov Functions

Definition 2.5 We call a continuous function \( V : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \) an iIOSS Lyapunov function if there exist \( \alpha, \varpi \in \mathcal{K}_\infty, \sigma_1, \sigma_2 \in \mathcal{K}, \) and \( \kappa : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) continuous positive definite so that
\[ \alpha(|\xi|) \leq V(\xi) \leq \varpi(|\xi|) \quad \forall \xi \in \mathbb{R}^n \]
and
\[ \zeta \cdot f(\xi, \mu) \leq -\kappa(|\xi|) + \sigma_1(|h(\xi)|) + \sigma_2(|\mu|) \quad \forall \xi \in \mathbb{R}^n, \forall \mu \in \mathbb{R}^m \]
for each \( \zeta \in \partial_D V(\xi) \).

We remark that the decrease statement (7) can be written equivalently in an integral formulation, using the following standard result.

Proposition 2.6 (e.g. [20] Proposition 14) Given a forward complete system as in (1), a continuous function \( V : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \), and a continuous function \( w : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \), the following are equivalent:

1. For all \( \xi \in \mathbb{R}^n \) and all \( \mu \in \mathbb{R}^m \)
\[ \zeta \cdot f(\xi, \mu) \leq w(\xi, \mu) \]
for each \( \zeta \in \partial_D V(\xi) \).

2. For each \( \xi \in \mathbb{R}^n \) and each input \( u \), the solution \( x(\cdot, \xi, u) \) satisfies
\[ V(x(t, \xi, u)) - V(\xi) \leq \int_0^t w(x(s, \xi, u), u(s)) \, ds \]
for any \( t \geq 0 \).
Remark 2.7 Applying Proposition 2.6 with
\[ w(\xi, \mu) = -\kappa(|\xi|) + \sigma_1(|h(\xi)|) + \sigma_2(|\mu|), \]
we conclude that the decrease statement (5) in the definition of an iIOSS Lyapunov function could be equivalently written as
\[ V(x(t, \xi, u)) - V(\xi) \leq \int_0^t -\kappa(|x(s, \xi, u)|) + \sigma_1(|h(x(s, \xi, u))|) + \sigma_2(|u(s)|) \, ds \quad (6) \]
for all \( \xi \in \mathbb{R}^n \), all inputs \( u \), and all \( t \geq 0 \). This alternative formulation will be used below. □

3 Lyapunov Characterization

Our main result is the following

Theorem 1 Suppose system (1) is forward complete. The following are equivalent.
1. The system is iIOSS.
2. The system admits an iIOSS Lyapunov function.

Remark 3.1 The main result in [11] is a Lyapunov characterization of the IOSS property. It is shown in that reference that a system is IOSS if and only if it admits an IOSS Lyapunov function, which can be defined as an iIOSS Lyapunov function for which the function \( \kappa \) is of class \( \mathcal{K}_\infty \). Thus it is an immediate consequence of Theorem [1] that the IOSS property implies the iIOSS property. □

Remark 3.2 We will prove a slightly stronger statement than \( (2 \Rightarrow 1) \) of Theorem [1]. The proof below shows that the existence of a lower semicontinuous iIOSS Lyapunov function implies that a system is iIOSS. □

It is still an open question whether every iIOSS system admits a smooth iIOSS Lyapunov function. However, as a minor extension of the proof of Theorem [1] we will also show the following.

Lemma 3.3 Suppose system (1) is forward complete and has compact input value set \( U \). Then, if the system is iIOSS, it admits a smooth iIOSS Lyapunov function, i.e. there exists a smooth \( (C^\infty) \) function \( V : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \), and \( \alpha, \beta, \kappa, \sigma_1, \sigma_2 \in \mathcal{K}_\infty \), and \( \kappa : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) continuous positive definite so that (4) holds and
\[ \nabla V(\xi) \cdot f(\xi, \mu) \leq -\kappa(|\xi|) + \sigma_1(|h(\xi)|) + \sigma_2(|\mu|) \quad \forall \xi \in \mathbb{R}^n, \forall \mu \in U. \]

Remark 3.4 Lemma [3.3] provides, in particular, a Lyapunov characterization in terms of a smooth function for the property of integral-output to state stability (iOSS) which is defined as iOSS for systems with no inputs (or equivalently, with \( U = \{0\} \)). □
3.1 Sufficiency

We begin with the proof of (2 $\Rightarrow$ 1) (sufficiency) in Theorem 1. Here we follow the sufficiency argument given in [1]. A few preliminary lemmas are needed.

Lemma 3.5 (Lemma 4.1) Let $\kappa : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a continuous positive definite function. Then there exists $\rho_1 \in K_\infty$ and $\rho_2 \in L$ such that

$$\kappa(s) \geq \rho_1(s)\rho_2(s) \quad \forall s \geq 0.$$ 

The following comparison result will be needed. This is a generalization of Corollary 4.3 in [1].

Proposition 3.6 Given any continuous positive definite $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, there exists a $KL$ function $\beta$ with the following property. For any $0 < \tilde{t} \leq \infty$, any lower semicontinuous function $y : [0, \tilde{t}) \to \mathbb{R}_{\geq 0}$, and any measurable, locally essentially bounded function $v : [0, \tilde{t}) \to \mathbb{R}_{\geq 0}$, if

$$y(t_2) \leq y(t_1) + \int_{t_1}^{t_2} -\alpha(y(s)) + v(s) \, ds \quad \forall 0 \leq t_1 \leq t_2 < \tilde{t},$$ 

then

$$y(t) \leq \beta(y(0), t) + \int_0^t 2v(s) \, ds \quad \forall t \in [0, \tilde{t}).$$

The following lemma will be needed to prove Proposition 3.6.

Lemma 3.7 Suppose given a locally Lipschitz positive definite function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, a time $0 < \tilde{t} \leq \infty$, and a measurable, locally essentially bounded function $v : [0, \tilde{t}) \to \mathbb{R}_{\geq 0}$. Let $y : [0, \tilde{t}) \to \mathbb{R}_{\geq 0}$ be any lower semicontinuous function which satisfies (7). Define $w(\cdot)$ to be the solution of the initial value problem

$$\dot{w}(t) = -\alpha(w(t)) + v(t), \quad w(0) = y(0).$$

Then $w(t)$ is defined for all $t \in [0, \tilde{t})$ and

$$y(t) \leq w(t) \quad \forall t \in [0, \tilde{t}).$$

Proof. (We follow the proof of Theorem III.4.1 in [1].) Let $y(\cdot)$ and $w(\cdot)$ be as above for given $\alpha$, $\tilde{t}$, and $v(\cdot)$. We first note that $w(\cdot)$ exists for all $t \in [0, \tilde{t})$, since $\alpha$ is nonnegative and $v(\cdot)$ is essentially bounded on each finite interval. For each integer $n \geq 1$, let $w_n(\cdot)$ be the solution of

$$\dot{w}_n(t) = -\alpha(w_n(t)) + v(t) + \frac{1}{n}, \quad w_n(0) = y(0),$$

which is also defined on $[0, \tilde{t})$. We will show that

$$y(t) \leq w_n(t) \quad \forall t \in [0, \tilde{t})$$
for all $n \geq 1$. Indeed, suppose not. Then there exists $n \geq 1$ and $\tau \in [0, \tilde{t})$ so that

$$y(\tau) > w_n(\tau).$$

Let

$$t_0 := \sup \{ t \in [0, \tau] : y(t) \leq w_n(t) \}.$$ 

Then, as $y(\cdot)$ is lower semicontinuous and $w_n(\cdot)$ is continuous,

$$y(t_0) \leq w_n(t_0).$$

We claim that in fact $y(t_0) = w_n(t_0)$. If this were not the case, there would be numbers $\delta_1, \delta_2$ so that

$$y(t_0) < \delta_1 < \delta_2 < w_n(t_0).$$

From (7), we have that

$$w_n(t_0 + t) = w_n(t_0) + \int_{t_0}^{t_0 + t} -\alpha(w_n(s)) + \varphi(s) ds$$

for each $t \in [0, \tilde{t} - t_0)$. Since

$$\lim_{t \to 0} \int_{t_0}^{t_0 + t} -\alpha(w_n(s)) + \varphi(s) ds = 0,$$

it follows from (11) that there is some $\varepsilon_1 > 0$ so that $y(t_0 + t) < \delta_1$ for all $t \in [0, \varepsilon_1]$. Since $w_n(\cdot)$ is continuous, (11) also gives an $\varepsilon_2 > 0$ so that $w_n(t_0 + t) > \delta_2$ for all $t \in [0, \varepsilon_2]$. Thus $y(t_0 + t) < w_n(t_0 + t)$ for all $t$ sufficiently small, which contradicts the definition of $t_0$. We conclude that $y(t_0) = w_n(t_0)$.

From (9) and Taylor’s Theorem, we have that for $\varepsilon \in [0, \tilde{t} - t_0)$,

$$y(t_0 + \varepsilon) \leq y(t_0) + \int_{t_0}^{t_0 + \varepsilon} -\alpha(y(s)) + v(s) ds$$

$$= y(t_0) - \varepsilon \alpha(y(t_0)) + \varepsilon v(t_0) + o(\varepsilon)$$

and from (9)

$$w_n(t_0 + \varepsilon) = w_n(t_0) + \int_{t_0}^{t_0 + \varepsilon} -\alpha(w_n(s)) + v(s) + \frac{1}{n} ds$$

$$= w_n(t_0) - \varepsilon \alpha(w_n(t_0)) + \varepsilon v(t_0) + \frac{\varepsilon}{n} + o(\varepsilon),$$

where $o(\cdot)$ signifies a function satisfying $\lim_{\varepsilon \to 0} \frac{o(t)}{\varepsilon} = 0$. Since $w_n(t_0) = y(t_0)$, it follows that $y(t_0 + \varepsilon) \leq w_n(t_0 + \varepsilon)$ for $\varepsilon$ sufficiently small, a contradiction. Thus (10) holds for all $t_0$. We conclude that $y(t_0) = w_n(t_0)$.

We note that $w_n(t) \to w(t)$ uniformly on each finite time interval (cf. e.g. Theorem 1 in [25]). Thus for any $T \in [0, t)$, as (10) holds for all $n$,

$$y(t) \leq \lim_{n \to \infty} w_n(t) = w(t) \quad \forall t \in [0, T].$$

As $T > 0$ is arbitrary, we conclude that $y(t) \leq w(t)$ for all $t \in [0, \tilde{t})$.

To complete the proof of Proposition 3.3 we will also need the following statement.
Lemma 3.8 (Corollary 4.3) Given any continuous positive definite $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{> 0}$, there exists a $KL$ function $\beta$ with the following property. For any $0 < \bar{t} \leq \infty$, any absolutely continuous function $w : [0, \bar{t}) \rightarrow \mathbb{R}_{\geq 0}$, and any measurable, locally essentially bounded function $v : [0, \bar{t}) \rightarrow \mathbb{R}_{> 0}$, if

$$\dot{w}(t) \leq -\alpha(w(t)) + v(t)$$

(12) for almost all $t \in [0, \bar{t})$, then

$$w(t) \leq \beta(w(0), t) + \int_0^t 2v(s) \, ds \quad \forall t \in [0, \bar{t}).$$

The proof of Proposition 3.6 is a straightforward combination of Lemma 3.7 and Lemma 3.8.

Proof. (Proposition 3.6) Let a continuous positive definite $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{> 0}$ be given. Without loss of generality, we assume $\alpha$ is locally Lipschitz (otherwise we replace $\alpha$ by a locally Lipschitz function majorized by $\alpha$). Let $\beta$ be the $KL$ function given by Lemma 3.8. Suppose $\bar{t}$, $y(\cdot)$ and $v(\cdot)$ are as in the statement of the Proposition so that (7) holds. Let $w(\cdot)$ be the solution of the initial value problem (8). Then Lemma 3.7 gives

$$y(t) \leq w(t) \quad \forall t \in [0, \bar{t}).$$

Also, since $w(\cdot)$ satisfies (12) (as an equality), Lemma 3.8 gives

$$w(t) \leq \beta(w(0), t) + \int_0^t 2v(s) \, ds \quad \forall t \in [0, \bar{t}).$$

Since $w(0) = y(0)$, the result follows.

We can now give the argument for sufficiency of the Lyapunov characterization. As mentioned earlier, this proof holds for lower semicontinuous Lyapunov functions.

Proof. Theorem 1 (2 $\Rightarrow$ 1)

Suppose the function $V$ satisfies the definition of an iIOSS Lyapunov function for the forward complete system (1) with functions $\alpha$, $\overline{\alpha}$, $\kappa$, $\sigma_1$ and $\sigma_2$ satisfying (4) and (5). Let $\rho_1 \in \mathcal{K}_\infty$ and $\rho_2 \in \mathcal{L}$ be functions as in Lemma 3.3 for $\kappa$. Let

$$\overline{\rho}(s) := \rho_1(\overline{\alpha}^{-1}(s))\rho_2(\overline{\alpha}^{-1}(s)).$$

By (4) and (5), we have, for each $\xi \in \mathbb{R}^n$ and each input $u$,

$$V(x(t_2, \xi, u)) \leq V(x(t_1, \xi, u)) + \int_{t_1}^{t_2} -\kappa(|x(s, \xi, u)|) + \sigma_1(|h(x(s, \xi, u))|) + \sigma_2(|u(s)|) \, ds$$

$$\leq V(x(t_1, \xi, u)) + \int_{t_1}^{t_2} -\rho_1(|x(s, \xi, u)|)\rho_2(|x(s, \xi, u)|)$$

$$+ \sigma_1(|h(x(s, \xi, u))|) + \sigma_2(|u(s)|) \, ds$$

$$\leq V(x(t_1, \xi, u)) + \int_{t_1}^{t_2} -\overline{\rho}(V(x(s, \xi, u))) + \sigma_1(|h(x(s, \xi, u))|) + \sigma_2(|u(s)|) \, ds$$

for all $0 \leq t_1 \leq t_2$. 

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Then, as $\tilde{\rho}$ is continuous positive definite, Proposition 3.6 gives the existence of a $\mathcal{KL}$ function $\beta$ so that for each $\xi \in \mathbb{R}^n$ and each input $u$

\[
\alpha(|x(t,\xi,u)|) \leq V(x(t,\xi,u)) \leq \beta(V(\xi),t) + \int_0^t 2\sigma_1(|h(x(s,\xi,u))|) + 2\sigma_2(|u(s)|) \, ds \\
\leq \beta(\pi(|\xi|),t) + \int_0^t 2\sigma_1(|h(x(s,\xi,u))|) + 2\sigma_2(|u(s)|) \, ds
\]

for all $t \geq 0$, which is the required bound. 

3.2 Necessity

We next prove ($1 \Rightarrow 2$) (necessity) for Theorem 1. We will construct an iIOSS Lyapunov function for a given iIOSS system. The proof combines ideas from the constructions in [28] and [1]. The following result will be needed.

This statement follows directly from Proposition 7 in [26].

**Proposition 3.9** For any given $\mathcal{KL}$ function $\beta$, there exist a family of mappings $\{T_r\}_{r \geq 0}$ with:

- for each fixed $r > 0$, $T_r : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ is strictly decreasing;
- for each fixed $\varepsilon > 0$, $T_r(\varepsilon)$ is strictly increasing as $r$ increases and $\lim_{r \to \infty} T_r(\varepsilon) = \infty$;
- the map $(r, \varepsilon) \mapsto T_r(\varepsilon)$ is jointly continuous in $r$ and $\varepsilon$;

such that

\[
\beta(s,t) \leq \varepsilon
\]

for all $s \leq r$, all $t \geq T_r(\varepsilon)$. 

Before giving the construction, we will cite a lemma on boundedness of reachable sets for forward complete systems which says that the reachable set from a given point over a finite time interval $[0, T]$ is bounded if the inputs are required to satisfy a bound of the type

\[
\int_0^T \gamma(|u(s)|) \, ds \leq M < \infty
\]

(13)

for an appropriate choice of $\gamma \in \mathcal{K}_\infty$.

**Remark 3.10** Note that for arbitrary $\mathcal{K}_\infty$ functions $\gamma$, this need not hold. Take, for example the one-dimensional system $\dot{x} = u^2$. With $\gamma(s) = s$, the inputs

\[
u_k(t) = \begin{cases} 
  k & 0 \leq t \leq \frac{1}{k} \\
  0 & \frac{1}{k} < t \leq 1 
\end{cases}
\]

defined on $[0, 1]$ satisfy $\int_0^1 \gamma(|u_k(s)|) \, ds = 1$ for each $k \geq 1$. However, the solution starting at the origin corresponding to the input $u_k(\cdot)$ satisfies $x(1) = k$, and so clearly one can reach an unbounded set in one time unit using controls satisfying (13).

The following lemma shows that one can always choose $\gamma$ so that the bound (13) on inputs implies a bounded reachable set. (In the example above, clearly $\gamma(s) = s^2$ will do.)
Lemma 3.11 (Corollary 2.13) Suppose system (1) is forward complete. Then there exist functions $\chi_1, \chi_2, \chi_3, \sigma$ of class $\mathcal{K}_\infty$ and a constant $c \geq 0$ such that

$$|x(t, \xi, u)| \leq \chi_1(t) + \chi_2(|\xi|) + \chi_3 \left(\int_0^t \sigma(|u(s)|) \, ds\right) + c$$

holds for all $\xi \in \mathbb{R}^n$, all inputs $u$, and all $t \geq 0$.

We now provide the Lyapunov construction.

Proof. Theorem (1 $\Rightarrow$ 2)

Suppose the system (1) is forward complete and satisfies the iIOSS property with gains $\alpha, \beta, \gamma_1$ and $\gamma_2$.

Pick any smooth, strictly increasing and bounded function $k : \mathbb{R} \to \mathbb{R}_{>0}$ whose derivative is strictly decreasing. Then there are two positive numbers $c_1 < c_2$ so that $k(t) \in [c_1, c_2]$ for all $t \geq 0$. Define $\lambda(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ by

$$\lambda(t) := \frac{d}{dt} k(t).$$

Since the system is forward complete, we may find a function $\sigma \in \mathcal{K}_\infty$ as in Lemma 3.11. Define $\bar{\gamma}_2(s) := \max\{\gamma_2(s), \sigma(s)\}$ for all $s \geq 0$. Note that the iOSS bound (3) holds with $\bar{\gamma}_2$ in the place of $\gamma_2$.

We define a Lyapunov function as

$$V_0(\xi) := \sup_{u} \sup_{t \geq 0} \left\{ \alpha(|x(t, \xi, u)|) - \int_0^t \gamma_1(|y(s, \xi, u)|) \, ds - \int_0^\infty 2\bar{\gamma}_2(|u(s)|) \, ds \right\} \lambda(t)$$

for each $\xi \in \mathbb{R}^n$. It is immediate that this function satisfies (4), as

$$c_1 \alpha(|\xi|) \leq V_0(\xi) \leq c_2 \beta(|\xi|, 0) \quad \forall \xi \in \mathbb{R}^n. \quad (14)$$

The first of these inequalities follows from considering the trajectory with input $u \equiv 0$ at time $t = 0$, and the second from the iOSS bound (3): for any $\xi \in \mathbb{R}^n$ and any input $u$,

$$\alpha(|x(t, \xi, u)|) - \int_0^t \gamma_1(|y(s, \xi, u)|) \, ds - \int_0^\infty 2\bar{\gamma}_2(|u(s)|) \, ds$$

$$\leq \alpha(|x(t, \xi, u)|) - \int_0^t \gamma_1(|y(s, \xi, u)|) \, ds - \int_0^\infty \gamma_2(|u(s)|) \, ds$$

$$\leq \beta(|\xi|, t)$$

$$\leq \beta(|\xi|, 0) \quad (15)$$

for all $t \geq 0$.

Next, we observe that for each $\xi$, the supremum over inputs in $V_0(\xi)$ can be taken to be a supremum over a restricted set, as follows. From the iOSS bound (3), we have, for any $\xi \in \mathbb{R}^n$ and any input $u$,

$$\alpha(|x(t, \xi, u)|) \leq \beta(|\xi|, 0) + \int_0^t \gamma_1(|y(s, \xi, u)|) \, ds + \int_0^\infty \bar{\gamma}_2(|u(s)|) \, ds$$

for all $t \geq 0$. Suppose now that $\xi$ and $u$ are such that

$$\int_0^\infty \bar{\gamma}_2(|u(s)|) \, ds > \beta(|\xi|, 0).$$
In this case it follows that
\[ \alpha(|x(t, \xi, u)|) \leq \int_0^t 2\gamma_2(|y(s, \xi, u)|) \, ds + \int_0^\infty \tilde{2}\gamma_2(|u(s)|) \, ds \]
for all \( t \geq 0 \). Then for all \( t \geq 0 \)
\[ \alpha(|x(t, \xi, u)|) - \int_0^t \gamma_1(|y(s, \xi, u)|) \, ds - \int_0^\infty 2\gamma_2(|u(s)|) \, ds \leq 0. \]

Since \( V_0(\xi) \geq 0 \) for each \( \xi \in \mathbb{R}^n \), it follows that for each \( \xi \in \mathbb{R}^n \)
\[ V_0(\xi) = \sup_{u \in U(\xi)} \sup_{t \geq 0} \left\{ \left( \alpha(|x(t, \xi, u)|) - \int_0^t \gamma_1(|y(s, \xi, u)|) \, ds - \int_0^\infty 2\gamma_2(|u(s)|) \, ds \right) k(t) \right\}, \]
where, for each \( r \geq 0 \), we define \( U(r) := \{ u(\cdot) : \int_0^\infty \tilde{2}\gamma_2(|u(s)|) \, ds \leq \beta(r, 0) \} \).

We next make the observation that the supremum in time can be taken over a restricted set as well. Let \( T_r(\varepsilon) \) be defined as in Proposition 3.9 for the function \( \beta \). From (14) and (15) we have
\[ V_0(\xi) = \sup_{u \in U(\xi)} \sup_{0 \leq t \leq T_\varepsilon} \left\{ \left( \alpha(|x(t, \xi, u)|) - \int_0^t \gamma_1(|y(s, \xi, u)|) \, ds - \int_0^t 2\gamma_2(|u(s)|) \, ds \right) k(t) \right\}, \]
where for each \( \xi \in \mathbb{R}^n \) we set \( T_\varepsilon := T_{2|\xi|}(\frac{c_2 \alpha(|\xi|)}{2}) \).

We will show that the function \( V_0 \) is continuous on \( \mathbb{R}^n \) by showing lower and upper semi-continuity in the next two lemmas.

**Proposition 3.12** The function \( V_0 \) is lower semicontinuous on \( \mathbb{R}^n \).

**Proof.** We will show
\[ \liminf_{\xi \to \xi_0} V_0(\xi) \geq V_0(\xi_0) \]
for all \( \xi_0 \in \mathbb{R}^n \).

Fix \( \xi_0 \in \mathbb{R}^n \) and let \( \varepsilon > 0 \) be given. There exists an input \( u_0 \) and a time \( t_0 \geq 0 \) so that
\[ \left( \alpha(|x(t_0, \xi_0, u_0)|) - \int_0^{t_0} \gamma_1(|y(s, \xi_0, u_0)|) \, ds - \int_0^\infty 2\gamma_2(|u_0(s)|) \, ds \right) k(t_0) \geq V_0(\xi_0) - \frac{\varepsilon}{2}. \]

By continuity of \( x(t_0, \cdot, u_0) \) and \( \alpha \), there exists a neighbourhood \( U_1 \) of \( \xi_0 \) so that
\[ |\alpha(|x(t_0, \xi_0, u_0)|) - \alpha(|x(t_0, \xi, u_0)|)| \leq \frac{\varepsilon}{4k(t_0)} \]
for all \( \xi \in U_1 \). Furthermore, as \( \xi \to \xi_0 \) implies \( y(t, \xi, u_0) \) converges uniformly to \( y(t, \xi_0, u_0) \) on the finite interval \([0, t_0]\), and since \( \gamma_1 \) is uniformly continuous on a compact containing an open neighbourhood of \( \{y(t, \xi_0, u_0) : t \in [0, t_0]\} \), we can find a neighbourhood \( U_2 \subseteq U_1 \) of \( \xi_0 \) so that each \( \xi \in U_2 \) satisfies
\[ |\gamma_1(|y(s, \xi_0, u_0)|) - \gamma_1(|y(s, \xi, u_0)|)| \leq \frac{\varepsilon}{4t_0 k(t_0)} \]
for all \( s \in [0, t_0] \). Thus, for all \( \xi \in U_2 \), we have
\[ \left( \alpha(|x(t_0, \xi_0, u_0)|) - \int_0^{t_0} \gamma_1(|y(s, \xi_0, u_0)|) \, ds - \int_0^\infty 2\gamma_2(|u_0(s)|) \, ds \right) k(t_0) \geq V_0(\xi_0) - \frac{\varepsilon}{2}. \]
for all \( s \in [0, t_0] \). Then for each \( \xi \in U_2 \),

\[
\left| \left( \alpha(|x(t_0, \xi, u_0)|) - \int_0^{t_0} \gamma_1(|y(s, \xi, u_0)|) \, ds \right) - \left( \alpha(|x(t_0, \xi, u_0)|) - \int_0^{t_0} \gamma_1(|y(s, \xi, u_0)|) \, ds \right) \right| \\
\leq \frac{\varepsilon}{4k(t_0)} + \int_0^{t_0} |\gamma_1(|y(s, \xi, u_0)|) - \gamma_1(|y(s, \xi, u_0)|) | \, ds \\
\leq \frac{\varepsilon}{2k(t_0)} .
\]

This gives, for each \( \xi \in U_2 \),

\[
V_0(\xi) \geq \left( \alpha(|x(t_0, \xi, u_0)|) - \int_0^{t_0} \gamma_1(|y(s, \xi, u_0)|) \, ds - \int_0^{\infty} 2\gamma_2(|u_0(s)|) \, ds \right) k(t_0) \\
\geq \left( \alpha(|x(t_0, \xi, u_0)|) - \int_0^{t_0} \gamma_1(|y(s, \xi, u_0)|) \, ds - \int_0^{\infty} 2\gamma_2(|u_0(s)|) \, ds \right) k(t_0) - \frac{\varepsilon}{2} \\
\geq V_0(\xi_0) - \varepsilon ,
\]

Hence \( V_0 \) is lower semicontinuous.

The next result will be needed to show upper semicontinuity.

**Proposition 3.13** For each \( T > 0 \) and each compact \( C \subset \mathbb{R}^n \), there exists \( L_{C,T} > 0 \) so that for any input \( u \in U(C) := \bigcup_{\xi \in C} U(||\xi||) \), each pair \( \eta, \xi \in C \) has the property that

\[
|x(t, \eta, u) - x(t, \xi, u)| \leq L_{C,T} |\eta - \xi| \quad \forall t \in [0, T] .
\]

That is, the trajectories are Lipschitz in the initial conditions, uniformly over inputs \( u \in U(C) \).

**Proof.** From Lemma 3.11, we have that the trajectories stay in a bounded set on the interval \([0, T]\). A standard Gronwall’s Lemma argument gives this Lipschitz condition from the local Lipschitz assumption on \( f \).

**Proposition 3.14** The function \( V_0 \) is upper semicontinuous on \( \mathbb{R}^n \).

**Proof.** We will show

\[
\limsup_{\xi \to \xi_0} V_0(\xi) \leq V_0(\xi_0) \tag{16}
\]

for all \( \xi_0 \in \mathbb{R}^n \).

Suppose \( \text{[16]} \) fails at some \( \xi_0 \in \mathbb{R}^n \). Then there exists \( \varepsilon > 0 \) and a sequence \( \{\xi_j\}_{j=1}^\infty \) so that \( \xi_j \to \xi_0 \) and

\[
V_0(\xi_j) > V_0(\xi_0) + \varepsilon \tag{17}
\]

for all \( j \geq 1 \). Choose \( r > 0 \) so that \( |\xi_0| \leq r \) and \( |\xi_j| \leq r \) for all \( j \geq 1 \). Then for each \( j \geq 1 \),

\[
V(\xi_j) = \sup_{u \in U(r)} \max_{t \in [0, T]} \left\{ \left( \alpha(|x(t, \xi_j, u)|) - \int_0^t \gamma_1(|y(s, \xi_j, u)|) \, ds - \int_0^{\infty} 2\gamma_2(|u(s)|) \, ds \right) k(t) \right\}
\]
which contradicts (17). We conclude that

\[ \text{Let } v \text{ and an input } u \in \eta \text{ be given, and consider the resulting trajectory. For } \tau > 0 \text{ small enough, we have } \frac{|x(\tau, \xi, v)|}{2} < |x(\tau, \xi, v)| < 2 |\xi|, \text{ so for such } \tau \text{ the supremum over time in the expression for } V_0(x(\tau, \xi, v)) \text{ may be taken over } [0, T_\xi]. \]

Finally, we show that the function \( V_0 \) satisfies the decrease statement (3). Let \( \xi \in \mathbb{R}^n \setminus \{0\} \) and an input \( v \) be given, and consider the resulting trajectory. For \( \tau > 0 \) small enough, we have \( \frac{|x(\tau, \xi, v)|}{2} < |x(\tau, \xi, v)| < 2 |\xi|, \) so for such \( \tau \) the supremum over time in the expression for \( V_0(x(\tau, \xi, v)) \) may be taken over \([0, T_\xi]\). We find, for such \( \tau \) sufficiently small,

\[
V_0(x(\tau, \xi, v)) = \sup_u \sup_{0 \leq s \leq T_\xi} \left\{ \alpha(|x(s, x(\tau, \xi, v), u)|) - \int_0^s \gamma_1(|y(r, x(\tau, \xi, v), u)|) \, dr \right\}
\]
where $v^\tau u$ is the concatenation of $u$ with $v$ at time $\tau$, that is

$$v^\tau u = \begin{cases} v(t) & \text{if } 0 \leq t \leq \tau \\ u(t-\tau) & \text{if } \tau < t \end{cases}$$

Rewriting, we arrive at

$$V_0(x(\tau, \xi, v)) - V_0(\xi) \leq V_0(\xi) \cdot \max_{0 \leq t \leq \tau + T_\xi} \left[ -1 + \frac{k(t-\tau)}{k(t)} \right] + c_2 \int_0^\tau \gamma_1(|y(r, \xi, v)|) + 2\gamma_2(|v(r)|) \, dr$$

for $\tau > 0$ sufficiently small. Recall that $\lambda(t) = \frac{4}{c_2}k(t)$ is decreasing, so for $\tau > 0$ small enough,
So for $\tau > 0$ sufficiently small,

$$V_0(x(\tau, \xi, v)) - V_0(\xi) \leq -V_0(\xi) \frac{k(T_\xi + \tau) - k(T_\xi)}{c_2} + c_2 \int_0^\tau \gamma_1(|y(r, \xi, v)|) + 2\gamma_2(|v(r)|) \, dr + c_2 \int_0^{\tau} \gamma_1(|y(r, \xi, v)|) + 2\gamma_2(|v(r)|) \, dr.$$

Recall that (18) has been verified for all $\xi \neq 0$. We next note that it also holds for $\xi = 0$. Let an input $v$ be given. The calculation above (with $T_\xi = \infty$) gives, for $\tau > 0$,

$$V_0(x(\tau, 0, v)) \leq V_0(0) \cdot \sup_{0 \leq t \leq \tau} \left\{ \frac{k(t - \tau)}{k(t)} \right\} + c_2 \int_0^\tau \gamma_1(|y(r, \xi, v)|) + 2\gamma_2(|v(r)|) \, dr,$$

as $V_0(0) = 0$. Clearly this gives (18) for $\xi = 0$.

Finally, we will make use of the following lemma to formulate the decrease statement (18) in the viscosity sense.

**Lemma 3.15** Suppose given a system as in (1), a function $V : \mathbb{R}^n \to \mathbb{R}$, a point $\xi \in \mathbb{R}^n$, and an element $\mu \in \mathbb{R}^m$. Then, if there exists a continuous $\alpha_{\xi, u} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and $\varepsilon > 0$ so that for all $0 \leq \tau < \varepsilon$

$$V(x(\tau, \xi, u)) - V(\xi) \leq \int_0^\tau \alpha_{\xi, u}(r) \, dr \leq \int_0^\tau \alpha_{\xi, u}(r) \, dr$$

where $u$ is the input constantly equal to $\mu$, then for any $\zeta \in \partial_D V(\xi)$, the instantaneous form of (19) holds in the viscosity sense:

$$\zeta \cdot f(\xi, \mu) \leq \alpha_{\xi, u}(0).$$

**Proof.** Suppose $V$, $\xi$, $\mu$, $\alpha_{\xi, u}$ and $\varepsilon$ are as above, and suppose $\zeta \in \partial_D V(\xi)$. Then, from the definition of the viscosity subgradient, we know that for $\tau$ small enough

$$\int_0^\tau \alpha_{\xi, u}(r) \, dr \geq V(x(\tau, \xi, u)) - V(\xi) \geq \zeta \cdot (x(\tau, \xi, u) - \xi) + g(x(\tau, \xi, u) - \xi)$$

where $g$ is some function satisfying $\lim_{s \to 0} \frac{g(s)}{|s|} = 0$. We note that since $u$ is constant valued, the trajectory $x(\cdot, \xi, u)$ is differentiable (not merely absolutely continuous). In particular, $

\frac{d}{dt} x(t, \xi, u)|_{t=0} = f(\xi, \mu).$ 

Now, dividing by $\tau$ in (19) gives

$$\frac{\int_0^\tau \alpha_{\xi, u}(r) \, dr}{\tau} \geq \zeta \cdot \frac{x(\tau, \xi, u) - \xi}{\tau} + \frac{g(x(\tau, \xi, u) - \xi)}{\tau}.$$

Taking the limit as $\tau$ tends to 0, we find

$$\alpha_{\xi, u}(0) \geq \zeta \cdot f(\xi, \mu) + \lim_{\tau \to 0} \frac{g(x(\tau, \xi, u) - \xi)}{\tau} \tau$$

$$= \zeta \cdot f(\xi, \mu) + \lim_{\tau \to 0} \frac{|x(\tau, \xi, u) - \xi|}{\tau} \frac{g(x(\tau, \xi, u) - \xi)}{|x(\tau, \xi, u) - \xi|} \tau$$

$$= \zeta \cdot f(\xi, \mu) + |f(\xi, \mu)| \lim_{\tau \to 0} \frac{g(x(\tau, \xi, u) - \xi)}{|x(\tau, \xi, u) - \xi|}$$

$$= \zeta \cdot f(\xi, \mu).$$

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Applying Lemma 3.14 to (18) with
\[\alpha_{x,u}(r) = -\frac{V_0(\xi)}{c_2} \lambda(T_\xi + r) + c_2 [\gamma_1(|y(r, \xi, u)|) + 2\tilde{\gamma}_2(|u(r)|)]\]

we conclude that
\[\zeta \cdot f(\xi, \mu) \leq -\frac{V_0(\xi)\lambda(T_\xi)}{c_2} + c_2 \gamma_1(|h(\xi)|) + 2c_2 \tilde{\gamma}_2(|\mu|) \quad \forall \xi \in \mathbb{R}^n, \forall \mu \in \mathbb{R}^m\]

for each \(\zeta \in \partial_D V_0(\xi)\). This implies (19), since \(V_0(\cdot)\lambda(T_\cdot) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}\) is continuous positive definite, so we can choose a continuous positive definite function \(\kappa : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}\) so that
\[\kappa(|\xi|) \leq \frac{1}{c_2} V_0(\xi) \lambda(T_\xi) \quad \forall \xi \in \mathbb{R}^n.\]

Then
\[\zeta \cdot f(\xi, \mu) \leq -\kappa(|\xi|) + c_2 \gamma_1(|h(\xi)|) + 2c_2 \tilde{\gamma}_2(|\mu|) \quad \forall \xi \in \mathbb{R}^n, \mu \in \mathbb{R}^m\]

for each \(\zeta \in \partial_D V_0(\xi)\).

This completes the construction of the iIOSS Lyapunov function.

Finally, we prove Lemma 3.3 by extending the proof in the case where the input value set \(U\) is compact.

**Proof.** (Lemma 3.3)

Suppose the forward complete system (11) satisfies the iIOSS property and has compact input value set \(U\). Given the construction above, it follows from Corollary 4.22 in [13] that there exists a smooth function \(\tilde{V} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}_{\geq 0}\), for which
\[\frac{V_0(\xi)}{2} \leq \tilde{V}(\xi) \leq 2V_0(\xi) \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}\]

and
\[\nabla \tilde{V}(\xi) \cdot f(\xi, \mu) \leq -\frac{1}{2} \kappa(|\xi|) + c_2 \gamma_1(|h(\xi)|) + 2c_2 \tilde{\gamma}_2(|\mu|) \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}, \forall \mu \in U.\]

We extend \(\tilde{V}\) to \(\mathbb{R}^n\) by setting \(\tilde{V}(0) = 0\). It is immediate that the resulting function is continuous on \(\mathbb{R}^n\) and that (21) holds on all of \(\mathbb{R}^n\).

By Proposition 4.2 in [13] there is a smooth \(\rho \in \mathcal{K}_\infty\) with \(\rho'(s) > 0\) for all \(s > 0\) such that \(\rho \circ \tilde{V}\) is smooth everywhere. Without loss of generality, we may assume that \(\rho'(s) \leq 1\) for all \(s > 0\). (If it is not, we may replace \(\rho\) by a smooth \(\mathcal{K}_\infty\) function \(\rho_0\) with the property that \(\rho_0(s) = \rho(s)\) in a neighbourhood of the origin where \(\rho'(s) \leq 1\) and \(\rho_0'(s) \leq 1\) everywhere else.) Let \(V = \rho \circ \tilde{V}\). It follows from (21) and (14) that
\[\alpha(|\xi|) \leq V(\xi) \leq \overline{\alpha}(|\xi|) \quad \forall \xi \in \mathbb{R}^n,\]

where \(\alpha(s) = \rho(\frac{1}{2} \alpha(s))\) and \(\overline{\alpha}(s) = \rho(2c_2 \beta(s, 0))\). Let \(\kappa_0 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}\) be a continuous positive definite function which satisfies \(\kappa_0(|\xi|) \leq \rho'(\tilde{V}(\xi)) \frac{1}{2} \kappa(|\xi|)\) for all \(\xi \in \mathbb{R}^n\). From (22), we have
\[\nabla V(\xi) \cdot f(\xi, \mu) \leq -\rho'\tilde{V}(\xi) \frac{1}{2} \kappa(|\xi|) + \rho'(\tilde{V}(\xi))(c_2 \gamma_1(|h(\xi)|) + 2c_2 \tilde{\gamma}_2(|\mu|)) \leq -\kappa_0(|\xi|) + c_2 \gamma_1(|h(\xi)|) + 2c_2 \tilde{\gamma}_2(|\mu|) \quad \forall \xi \in \mathbb{R}^n, \forall \mu \in U.\]

This holds at \(\xi = 0\) since \(V\) is smooth and has a minimum at the origin, so \(\nabla V(0) = 0\). Thus \(V\) is a smooth iIOSS Lyapunov function for the system.
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