Necessary Conditions for Infinite Horizon Optimal Control Problems Revisited

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Abstract

Necessary optimality conditions in the form of the maximum principle for control problems with infinite time horizon are considered. Both finite and infinite values of objective functional are allowed since the concept of overtaking and weakly overtaking optimality is used. New form of optimality condition is obtained and compared with the transversality conditions usually used in the literature. The examples, where these transversality conditions may fail while the new condition holds are presented. For Ramsey problem of capital accumulation a simple form of necessary optimality conditions is derived, which is also valid in the case of zero discounting.

1 Introduction

Optimal control problems with infinite horizon play an important role in economic theory. For instance, in the theory of economic growth, Pontragin’s maximum principle is the workhorse for many researchers. The proof of the maximum principle for infinite time horizon one can find, e.g., in [7]. The proved theorem does not include transversality conditions. Moreover it is known, [7, 8, 14], that usually used forms of transversality conditions:

$$\lim_{t \to \infty} \psi(t) = 0,$$

$$\lim_{t \to \infty} \langle \dot{x}(t), \psi(t) \rangle = 0,$$

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where
can be not necessary, where \( \hat{x} \) is the optimal state variable, \( \psi \) is the corresponding adjoint variable, and brackets \( \langle \cdot, \cdot \rangle \) denote scalar product of two vectors.

Transversality condition obtained in [11], under assumptions including that the objective functional takes only finite values, has the form of Hamiltonian \( \mathcal{H} \) converging to zero

\[
\lim_{t \to \infty} \mathcal{H}(\hat{x}(t), \hat{u}(t), t, \psi(t)) = 0,
\]

where \( \hat{x} \) and \( \hat{u} \) are the optimal state trajectory and control. [14] Lecture III proposes, without prove, transversality condition as

\[
\limsup_{t \to \infty} \langle \psi(t), x(t) \rangle \geq \liminf_{t \to \infty} \langle \psi(t), \hat{x}(t) \rangle,
\]

where \( x \) is any admissible path of the state variable. [14] footnote 4 for Lecture III says that “this conjecture is related to a conjecture made by Kenneth J. Arrow in private correspondence.” Notice that Arrow’s sufficiency theorem contains condition that follows from \( (3) \), which one can expect for problems, where the maximum principle provides both necessary and sufficient conditions of optimality.

In [1, 2, 3, 4] the authors determine the adjoint variable uniquely by a Cauchy-type formula, that solves adjoint equation with transversality conditions in the following form

\[
\lim_{t \to \infty} Y(t) \psi(t) = 0,
\]

where \( Y(t) \) is the fundamental matrix of the state equation linearized about the optimal solution, see eg. [9, 10].

Due to their limitations all aforementioned transversality conditions fail to select the optimal solution of Ramsey problem without discounting, [13], where we consider diverging objective functional. Condition \( (3) \) is proved only for converging functionals, but it can hold if we modify the objective improper integral subtracting from its integrand the constant, such that for optimal solution the integral converges, see e.g., [6, Section 7]. Condition \( (4) \) fails because \( x \) is any admissible path of the state variable, it could hold if \( x \) have been requited to satisfy the maximum principle. Condition \( (5) \) fails because it is proved for problems, where optimal trajectory is in the interior of the domain of admissible trajectories, which is not the case for Ramsey problem.
Ramsey problem without discounting can be solved with the necessary conditions obtained in this paper. In contrast to (1)–(5) new conditions do not contain explicitly the adjoint variable. The paper in hand considers another example, where some of the conditions (1)–(5) do not hold, while the new conditions are valid. The proved conditions include condition (5) as a special case and extend its domain of applicability.

2 Statement of the problem

Let $X$ be a nonempty open convex subset of $R^n$, $U$ be an arbitrary nonempty set in $R^m$. Let us consider the following optimal control problem:

$$\int_{t_0}^{\infty} g(x(t), u(t), t) \, dt \to \max_u$$

$$\dot{x}(t) = f(x(t), u(t), t), \quad x(t_0) = x_0,$$

where $u(t) \in U$ and exists state variable $x(t) \in X$ for all $t \in [t_0, +\infty)$. We call such control $u(\cdot)$ and state variable $x(\cdot)$ trajectories admissible. Functions $f$ and $g$ are differentiable w.r.t. their first argument, $x$, and together with these partial derivatives are defined and locally bounded, measurable in $t$ for every $(x, u) \in X \times U$, and continuous in $(x, u)$ for almost every $t \in [0, \infty)$.

Improper integral in (6) might not converge for any candidate for optimal control $\hat{u}(\cdot)$, i.e. the limit

$$\lim_{T \to \infty} J(\hat{u}(\cdot), x_0, t_0, T),$$

might fail to exist, or might be infinite, where we introduce the finite time horizon functional:

$$J(u(\cdot), x_0, t_0, T) = \int_{t_0}^{T} g(x(t), u(t), t) \, dt,$$

subject to state equation (7). Thus functional $J$ may be unbounded or oscillating as $T \to \infty$. So we consider more general definitions of optimality.

**Definition 2.1.** An admissible control $\hat{u}(\cdot)$ is *overtaking optimal* (OO) if for every admissible control $u(\cdot)$ and every scalar $\varepsilon > 0$ there exists time $T = T(\varepsilon, u(\cdot)) > t_0$ such that for all $T' \geq T$

$$J(u(\cdot), x_0, t_0, T') - J(\hat{u}(\cdot), x_0, t_0, T') \leq \varepsilon.$$  

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Definition 2.2. An admissible control $\hat{u}(\cdot)$ is weakly overtaking optimal (WOO) if for every admissible control $u(\cdot)$, scalar $\varepsilon > 0$, and time $T > t_0$ one can find $T' = T'(\varepsilon, T, u(\cdot)) \geq T$ such that

$$J(u(\cdot), x_0, t_0, T') - J(\hat{u}(\cdot), x_0, t_0, T') \leq \varepsilon.$$  (11)

These two definitions imply that for all admissible controls $u(\cdot)$

$$\limsup_{T \to \infty} (J(u(\cdot), x_0, t_0, T) - J(\hat{u}(\cdot), x_0, t_0, T)) \leq 0,$$

for OO $\hat{u}(\cdot)$ in Definition 2.1 and

$$\liminf_{T \to \infty} (J(u(\cdot), x_0, t_0, T) - J(\hat{u}(\cdot), x_0, t_0, T)) \leq 0,$$

for WOO $\hat{u}(\cdot)$ in Definition 2.2. It is clear that if $\hat{u}(\cdot)$ is OO, then it is also WOO. When usual optimality holds, i.e. finite limit exists in (8) and for all admissible controls $u(\cdot)$

$$\limsup_{T \to \infty} J(u(\cdot), x_0, t_0, T) \leq \lim_{T \to \infty} J(\hat{u}(\cdot), x_0, t_0, T),$$

then $\hat{u}(\cdot)$ is also both OO and WOO.

There are many definitions of optimality in the literature, for example, corresponding uniform optimalities, when $T$ and $T'$ do not depend on $u(\cdot)$ in Definitions 2.1 and 2.2, see e.g., [9]. These definitions are stronger and may lead to absence of corresponding optimal solutions. WOO seems to be one of the weakest concepts for which optimality conditions can be proved in the form of Pontryagin’s maximum principle.

3 Optimality conditions

With the use of the adjoint variable $\psi$ we introduce Hamiltonian

$$\mathcal{H}(x, u, t, \psi, \lambda) = \lambda g(x, u, t) + \langle \psi, f(x, u, t) \rangle$$  (12)

where brackets $\langle \cdot, \cdot \rangle$ denote scalar product of two vectors. It is known, see [7, 12, 14], that there exist scalar $\lambda \geq 0$ and vector $\psi_0$, such that $(\lambda, \psi_0) \neq 0$ and the maximum principle holds:

$$\mathcal{H}(\hat{x}(t), \hat{u}(t), t, \psi(t), \lambda) = \max_{u \in U} \mathcal{H}(\hat{x}(t), u, t, \psi(t), \lambda),$$  (13)
along with the adjoint equation:

\[- \dot{\psi}(t) = \frac{\partial H}{\partial x}(\dot{x}(t), \dot{u}(t), t, \psi(t), \lambda), \quad \psi(t_0) = \psi_0. \quad (14)\]

In order to select such \( \lambda \) and \( \psi_0 \) with the use of transversality condition we consider the linearization of system (7)

\[ \dot{y}(t) = \left( \frac{\partial f}{\partial x}(\dot{x}(t), \dot{u}(t), t) \right) y(t). \quad (15) \]

Its solution, for the given initial condition \( y(\tau) \), can be written with the use of the state-transition matrix \( K(t, \tau) \):

\[ y(t) = K(t, \tau) y(\tau). \quad (16) \]

The following assumption determines how well linearized system (15) should approximate the original system, so that the proposed optimality condition holds along with the maximum principle.

**Assumption 1.** For almost all time instances \( \tau \geq t_0 \) directional derivative \( \langle J_x, \zeta \rangle \) of the functional at the optimal trajectory \( \dot{x} \) uniformly bounds from below the finite difference approximation of the directional derivative:

\[ \lim_{\alpha \to 0} \liminf_{T \to \infty} \left( \frac{J(\dot{u}(\cdot), \dot{x}(\tau), \tau, T) - J(\dot{u}(\cdot), \dot{x}(\tau), \tau, T)}{\alpha} - \langle \dot{J}_x(\tau, T), \zeta \rangle \right) \geq 0, \]

only with such perturbations of the initial conditions, \( x(\tau) = \dot{x}(\tau) + \alpha \zeta \), that result in admissible trajectories, i.e. \( x(t) \in X \) in \( [\tau, \infty) \), where we denote

\[ \dot{J}_x(\tau, T) := \int_{\tau}^{T} K^*(t, \tau) \frac{\partial g}{\partial x}(\dot{x}(t), \dot{u}(t), t) \, dt. \quad (17) \]

**Remark 1.** If function \( g \) does not depend on \( x \), then Assumption 1 trivially holds. If functions \( f \) and \( g \) are linear w.r.t. \( x \), then Assumption 1 holds as equality for any scalar \( \alpha > 0 \) and direction \( \zeta \).

The next assumption is needed for admissibility of optimal trajectory variations in all directions.

\(^1\)Due to inequality in Assumption 1 it is also satisfied when \( g \) is convex and \( f \) is linear w.r.t. \( x \).
Assumption 2. There exists a number $\beta(\tau) > 0$ such that for all $x(\tau) \in X$ satisfying the inequality $|x(\tau) - \hat{x}(\tau)| < \beta(\tau)$, the initial value problem with $u = \hat{u}$ and the initial condition $x(t_0) = x(\tau)$ at $t_0 = \tau$ has a solution $x(t) \in X$ for all $t \geq \tau$.

Proposition 3.1 (Necessary optimality condition). Let Assumptions 1 and 2 be fulfilled for an admissible pair $(\hat{u}(\cdot), \hat{x}(\cdot))$ and the following limit finite valued

$$\limsup_{T \to \infty} |\hat{J}_x(\tau, T)| < +\infty.$$  \hfill (18)

1) If control $\hat{u}$ is WOO, then for all $\tau \in [t_0, \infty)$ and $u \in U$

$$\liminf_{T \to \infty} \left( \mathcal{H}(\hat{x}(\tau), u, \tau, \hat{J}_x(\tau, T), 1) - \mathcal{H}(\hat{x}(\tau), \hat{u}(\tau), \tau, \hat{J}_x(\tau, T), 1) \right) \leq 0, \hfill (19)$$

2) If control $\hat{u}$ is OO, then for all $\tau \in [t_0, \infty)$ and $u \in U$

$$\limsup_{T \to \infty} \left( \mathcal{H}(\hat{x}(\tau), u, \tau, \hat{J}_x(\tau, T), 1) - \mathcal{H}(\hat{x}(\tau), \hat{u}(\tau), \tau, \hat{J}_x(\tau, T), 1) \right) \leq 0. \hfill (20)$$

Proof. See Appendix A.

Example 4.3 demonstrate the application of this new optimality condition, when $\hat{J}_x(\tau, T)$ oscillating in $T$.

Under additional assumption, that $\hat{J}_x(\tau, T)$ converges as $T \to \infty$, referred to as condition of dominating discount in general form, see [4], the following corollary proves that maximum principle holds in normal case ($\lambda = 1$) and provides an explicit expression for the adjoint variable, which is equivalent to transversality condition (5).

Corollary 1 (Dominating discounting). If control $\hat{u}$ is WOO, Assumptions 1 and 2 hold, and the following limit exists

$$\hat{\psi}(\tau) := \lim_{T \to \infty} \hat{J}_x(\tau, T) = \int_{\tau}^{\infty} K^*(t, \tau) \frac{\partial g}{\partial x}(\hat{x}(t), \hat{u}(t), t) \, dt, \hfill (21)$$

see (17), then $\hat{\psi}$ solves adjoint system (14) in the normal case ($\lambda = 1$) and the maximum principle holds:

$$\mathcal{H}(\hat{x}(t), \hat{u}(t), t, \hat{\psi}(t), 1) \leq \mathcal{H}(\hat{x}(t), u, t, \hat{\psi}(t), 1), \hfill \text{for all } u \in U. \hfill (22)$$
Proof. Conditions (19) and (20) take the form of maximum principle (22). Differentiation w.r.t. $\tau$ of the integral expression for vector-function $\hat{\psi}$ in (21) shows that this is a solution of the adjoint system (14) in the normal case. Alternatively, see Proposition 3.2 for $a_0 = 0$ and $\lambda = 1$.

The following proposition, without Assumption 2, allows not only for normal case ($\lambda = 1$), but also for abnormal one, when $\lambda = 0$, see Example 4.2.

**Proposition 3.2 (Special transversality conditions).** If control $\hat{u}$ is WOO, Assumption 7 holds, and the following limit exists

$$\lim_{T \to \infty} K^*(T, t_0) \psi(T) = a_0, \quad (23)$$

where $\psi(\cdot)$ is the solution of adjoint equation (14) such that maximum condition (13) is fulfilled, and the limit in (21) exists, then

$$\psi(\tau) = K^*(t_0, \tau) a_0 + \lambda \hat{\psi}(\tau).$$

**Proof.** See Appendix B.

The similar expression was obtained in [1, Section 6] under certain assumptions ensuring the existence of the limit in (21). It was proved that vector $a_0$ has nonnegative components when the problem is autonomous and monotonic in the state variable, i.e. $\frac{\partial y}{\partial x}(x, u) > 0$, $\frac{\partial f}{\partial x}(x, u) > 0$ for all $(x, u) \in X \times U$ and for all optimal trajectories $\hat{x}$ there exist $\tau \geq t_0$ and vector $u_{\tau}$ such that $f(\hat{x}(t), u_{\tau}) > 0$. In [10] was noted the statement on the following corollary.

**Corollary 2.** The limit in (21) exists if, and only if, (5) holds.

**Proof.** Taking $a_0 = 0$ in Proposition 3.2 we have

$$\lim_{T \to \infty} K^*(T, t_0) \hat{\psi}(T) = \lim_{T \to \infty} Y(T) \hat{\psi}(T) = 0, \quad (24)$$

where the state transition matrix $K(T, \tau) = Y(T) Y(\tau)^{-1}$ is expressed via the non-degenerate fundamental matrix $Y$, such that $Y(t_0) = I$.

Notice that, in contrast to $\psi(\cdot)$, the solution $\hat{\psi}(\cdot)$ of the adjoint equation is not necessarily a correct adjoint variable, such that maximum condition is fulfilled for an optimal control, since we do not require Assumption 2 in
Proposition 3.2. Assumption 2 is indeed violated in some classical models such as Ramsey model, see Example 4.1.

In order to find optimal control without Assumption 2, we can formulate similar necessary optimality conditions to (19)–(20) when state variable $x$ is one-dimensional ($n = 1$) and state equation (17) does not depend explicitly on time. Consider set $G \subset U \times X$ of all admissible pairs $(u(\cdot), x(\cdot))$ satisfying maximum condition (13) with adjoint equation (14). Then, in the following Proposition we take the set $\hat{U}(x) = \{u : (u, x) \in G\} \subset U$ instead of $U$. This means that we find synthesis of control for each $\psi_0$ in (14) which results in an admissible control.

Proposition 3.3 (Special necessary optimality condition for autonomous one-dimensional state equation). Let one-dimensional state variable $x$ be governed by an autonomous equation, Assumption 7 fulfilled for an admissible pair $(\hat{u}(\cdot), \hat{x}(\cdot))$, and limit (18) finite valued.

1) If control $\hat{u}$ is WOO, then for all $\tau \in [t_0, \infty)$ and $u \in \hat{U}(\hat{x}(\tau))$
\[ \liminf_{T \to \infty} \left( H(\hat{x}(\tau), u, \tau, \hat{J}_x(\tau, T), 1) - H(\hat{x}(\tau), \hat{u}(\tau), \tau, \hat{J}_x(\tau, T), 1) \right) \leq 0, \] (25)

2) If control $\hat{u}$ is OO, then for all $\tau \in [t_0, \infty)$ and $u \in \hat{U}(\hat{x}(\tau))$
\[ \limsup_{T \to \infty} \left( H(\hat{x}(\tau), u, \tau, \hat{J}_x(\tau, T), 1) - H(\hat{x}(\tau), \hat{u}(\tau), \tau, \hat{J}_x(\tau, T), 1) \right) \leq 0, \] (26)

where $\hat{U}(\hat{x}(\tau))$ is the set of all admissible controls satisfying maximum principle (13)–(14), taken at $x = \hat{x}(\tau)$.

Proof. See Appendix A.

Remark 2. This case is equivalent to imposing mixed constraints which borders coincide with some admissible pairs $(u(\cdot), x(\cdot))$ satisfying maximum principle (13)–(14).

When integrand $g$ does not depend on state variable $x$, like in Ramsey model in Example 4.1, then we can formulate the following corollary.

Corollary 3. Let state variable $x$ be one-dimensional, the state equation autonomous, and integrand $g$ independent of state variable. If control $\hat{u}$ is WOO, then for almost all $\tau \in [t_0, \infty)$ the integrand of the objective functional is maximal on admissible control values:
\[ g(\cdot, \hat{u}(\tau), \tau) = \max_u \{ g(\cdot, u, \tau) \mid (u, \hat{x}(\tau)) \in G \}. \] (27)
Proof. Assumption \( \hat{J}_x(\tau, T) \equiv 0 \) trivially holds and limit (18) is finite valued, due to \( \hat{J}_x(\tau, T) \equiv 0 \). Necessary condition (27) follows from Proposition 3.3. \( \square \)

4 Examples

Example 4.1 (centralized Ramsey model without discounting). We maximize aggregated constant relative risk aversion utility

\[
\int_0^\infty \frac{c(t)^{1-\theta}}{1-\theta} \, dt \to \max_{c \geq 0} \quad \text{s.t.} \quad \dot{k}(t) = k(t)^\alpha - \delta k(t) - c(t), \quad k(t) \geq 0,
\]

where \( k(0) = k_0 > 0 \), \( \theta \neq 1 \), \( \theta > 0 \), and \( \alpha \in (0, 1) \). We write Hamiltonian in normal case (\( \lambda = 1 \)):

\[
\mathcal{H}(k, c, t, \psi, 1) = \frac{c^{1-\theta}}{1-\theta} + \psi (k^\alpha - \delta k - c),
\]

since the abnormal case (\( \lambda = 0 \)) would lead to \( \psi = 0 \) and thus impossible due to \( (\lambda, \psi) \neq 0 \). Maximum condition \( c(t)^{-\theta} = \psi(t) \) and the adjoint equation \( -\dot{\psi}(t) = (\alpha k(t)^{\alpha-1} - \delta) \psi(t) \) result in the Euler equation

\[
\frac{\dot{c}(t)}{c(t)} = \frac{\alpha k(t)^{\alpha-1} - \delta}{\theta}.
\]

It follows from the Euler equation and the state equation, that any admissible pair \( (k, c) \), that does not violate constraint \( k(t) \geq 0 \), converges either to steady state \( (k_s, c_s) \), where \( k_s = (\delta/\alpha)^{\frac{1}{\alpha-1}} \) and \( c_s = (1-\alpha)(\delta/\alpha)^{\frac{\alpha}{\alpha-1}} > 0 \), or to \( (\delta^{\frac{1}{\alpha-1}}, 0) \), where \( k_s < \delta^{\frac{1}{\alpha-1}} \). Necessary conditions (27) formulated in Corollary 3 single out the optimal pair converging to \( (k_s, c_s) \).

\[
\frac{\dot{c}(t)^{1-\theta}}{1-\theta} \geq \frac{c^{1-\theta}}{1-\theta}, \quad \text{for all } c \text{ such that } (c, k(t)) \in G, \quad (28)
\]

where \( G \) is the set of the trajectories governed by the state and Euler equations, not violating conditions \( c(t) \geq 0 \) and \( k(t) > 0 \), see solid lines in Figure 1.

Let us check transversality conditions (1)–(5):

1. Adjoint variable \( \psi(t) = c(t)^{-\theta} \to c_s^{-\theta} > 0 \) does not converge to zero as \( t \to \infty \).
Figure 1: Phase diagram for $\alpha = 0.4$, $\delta = 0.05$, and $\theta = 0.5$. Bold lines denote stationary points, where $\dot{k} = 0$ and $\dot{c} = 0$. Solid lines are the trajectories satisfying optimality conditions in the form of Euler equation. Dashed lines are those trajectories which eventually violate nonnegativity of capital $k$.

2. The product $\langle \psi(t), \hat{k}(t) \rangle \rightarrow k_* c_*^{\theta} > 0$ does not converge to zero.

3. Hamiltonian $\mathcal{H}(\hat{x}(t), \hat{u}(t), t, \psi(t), 1) \rightarrow \frac{1-\theta}{1-\theta} > 0$ does not converge to zero, because condition $[3]$ requires convergence of the objective functional.

4. Each admissible pair of constants $(k, c)$, such that $k < k_*$ and $c = k^\alpha - \delta k$ would violate condition $[4]$. Notice that condition $[4]$ would be satisfied if we considered only such admissible trajectories that satisfy the maximum principle, i.e. the Euler equation.

5. Vector-function $\hat{J}_x$, obtained in $[37]$, does not converge as $T \rightarrow \infty$, because condition $[5]$ requires Assumption $[2]$.
The following example of an autonomous monotonic problem illustrates Proposition 3.2.

**Example 4.2.**

\[
\int_0^\infty e^{-\rho t} x(t) \, dt \rightarrow \max_u, \quad \text{s.t.} \quad \dot{x}(t) = u(t), \quad x(0) = 0, \quad u(t) \in [0, 1]
\]

where obvious optimal control is \( u \equiv 1 \).

For \( \rho > 0 \) we have from Proposition 3.2 the adjoint variable \( \psi(t) = a_0 + \lambda e^{-\rho t}/\rho \) in normal case, where \( \lambda > 0 \). Overtaking optimal control \( u \equiv 1 \) provides maximum to the Hamiltonian \( \mathcal{H}(x, u, t, \psi, \lambda) = \lambda e^{-\rho t} x + \psi u \) for all \( t \geq 0 \) if, and only if, \( a_0 \geq 0 \). Thus, \( \hat{\psi}(t) = \lambda e^{-\rho t}/\rho \) resulting from (21) is also valid. Notice that in the abnormal case of \( \lambda = 0 \) the adjoint is strictly positive, \( \psi \equiv a_0 > 0 \), since \( (\lambda, \psi_0) \neq 0 \).

For \( \rho = 0 \) maximum principle holds only in the abnormal form, \( \psi \equiv a_0 > 0 \), otherwise \((\lambda > 0)\) maximum condition (13) would be violated for any solutions \( \psi(t) = a_0 - \lambda t \). In this case all conditions (1)–(5) are not satisfied.

All aforementioned correct adjoint solutions in this example, including the abnormal case, satisfy our new optimality condition (20) that reads as \( \psi_0 \geq 0 \), since \( K \equiv 1 \) and \( u \leq 1 \), even though requirement (18) of Proposition 3.1 is not fulfilled.

Let us check transversality conditions (1)–(5) the case of \( \rho = 0 \), when maximum principle holds only with \( \lambda = 0 \):

1. Adjoint vector \( \psi(t) \equiv a_0 > 0 \) does not converge to zero as \( t \rightarrow \infty \).

2. Due to (36) the scalar product \( \langle \psi(t), \dot{x}(t) \rangle = a_0 t \) does not converge to zero.

3. Hamiltonian does not converge to zero.

4. Condition (11) is not satisfied since \( \langle \psi(t), \dot{x}(t) - x(t) \rangle = -a_0 t < 0 \) for \( x(t) \equiv 0 \). Notice that condition (11) would be trivially satisfied, \( \langle \psi(t), \dot{x}(t) - x(t) \rangle \equiv 0 \), if we considered only such admissible trajectories that satisfy the maximum principle, i.e. \( x(t) = a_0 t \).

5. Vector-function \( \hat{J}_x(\tau, T) = T - \tau \) does not converge as \( T \rightarrow \infty \).
Example 4.3 ([5], Example 1.2], if $b = 0$). Let us maximize the following integral

$$
\max_u \int_0^\infty (x_2(t) + bu(t)) \, dt,
$$

where $b \geq 0$, subject to the system describing a linear oscillator

\begin{align*}
\dot{x}_1(t) &= x_2(t), \quad x_1(0) = 0, \\
\dot{x}_2(t) &= u(t) - x_1(t), \quad x_2(0) = 0,
\end{align*}

(29) \tag{29}

(30) \tag{30}

with bounded control $u(t) \in [-1, 1]$. We have the state-transition matrix

$$
K(t, \tau) = \left(\exp \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \right)^{t-\tau} = \left( \begin{array}{cc} \cos(t-\tau) & \sin(t-\tau) \\ -\sin(t-\tau) & \cos(t-\tau) \end{array} \right)
$$

and the state trajectory

$$
x(t) = \int_0^t K(t, s) \begin{pmatrix} 0 \\ u(s) \end{pmatrix} \, ds = \int_0^t u(s) \begin{pmatrix} \sin(t-s) \\ \cos(t-s) \end{pmatrix} \, ds.
$$

Functional (31) reads as follows

$$
J(u(\cdot), 0, 0, T) = x_1(T) + \int_0^T bu(t) \, dt = \int_0^T (\sin(T-t) + b) u(t) \, dt. \tag{31}
$$

For $b \geq 1$ control $\hat{u} \equiv 1$ is OO. For $b \in [0, 1)$ control $\hat{u} \equiv 1$ is WOO rather than OO, see proof in Appendix C. This control maximizes the Hamiltonian

$$
\mathcal{H}(x, u, t, \psi, \lambda) = \lambda (x_2 + bu) + \psi_1 x_2 + \psi_2 (u - x_1), \tag{32}
$$

only in normal case ($\lambda = 1$) and only for solutions

$$
\psi_1(t) = -r \cos(t + \phi) - 1, \quad \psi_2(t) = r \sin(t + \phi), \tag{33}
$$

of the adjoint system

\begin{align*}
\dot{\psi}_1(t) &= \psi_2(t), \tag{34} \\
\dot{\psi}_2(t) &= -\psi_1(t) - \lambda. \tag{35}
\end{align*}
where \(|r| \leq b\) and \(\phi\) is any phase shift. Indeed, in the abnormal case \((\lambda = 0)\) control \(\hat{u} \equiv 1\) would maximize Hamiltonian \((32)\) only for \(\psi_1 \equiv \psi_2 \equiv 0\), that contradicted \((\lambda, \psi_0) \neq 0\). Optimal trajectory reads as

\[
\dot{x}_1(t) = 1 - \cos(t), \quad \dot{x}_2(t) = \sin(t).
\]  

(36)

Vector-function \(\hat{J}_x\), defined in \((17)\), is oscillating in \(T\):

\[
\hat{J}_x(\tau, T) = \int_\tau^T \begin{pmatrix} \cos(t - \tau) & -\sin(t - \tau) \\ \sin(t - \tau) & \cos(t - \tau) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt
= \begin{pmatrix} \cos(T - \tau) - 1 \\ \sin(T - \tau) \end{pmatrix}.
\]  

(37)

We denote \(\Delta \hat{\mathcal{H}}(u, \tau, T) := \mathcal{H}(\dot{x}(\tau), u, \tau, \hat{J}_x(\tau, T), 1) - \mathcal{H}(\dot{x}(\tau), \hat{u}(\tau), \tau, \hat{J}_x(\tau, T), 1)\)

Since \(\Delta \hat{\mathcal{H}}(u, \tau, T) = (u - 1) (\sin(T - \tau) + b)\), condition \((19)\) for \(\hat{u} \equiv 1\) being WOO takes the form \((u - 1) (1 + b) \leq 0\) for all \(u \in [-1, 1]\). Condition \((20)\) for \(\hat{u} \equiv 1\) being OO reads as \((u - 1) (-1 + b) \leq 0\) for all \(u \in [-1, 1]\). Let us check transversality conditions \((1)\)–\((5)\):

1. Adjoint vector \(\psi(t)\), obtained in \((33)\), does not converge to zero as \(t \to \infty\).

2. Due to \((36)\) the scalar product

\[
\langle \psi(t), \dot{x}(t) \rangle = \cos(t) - r \cos(t + \phi) + r \cos(\phi) - 1 \to 0
\]

as \(t \to \infty\) only for \(\psi_1(t) = -\cos(t) - 1\) and \(\psi_2(t) = \sin(t)\) which belong to the correct adjoints in \((33)\) only if \(b \geq 1\), when \(\hat{u} \equiv 1\) is OO.

3. Due to \((36)\) and \((33)\) Hamiltonian \(\mathcal{H}(\dot{x}(t), \hat{u}(t), t, \psi(t), 1) = r \sin(\phi) + b\) can converge to zero only if \(r \sin(\phi) = -b\). Since \(|r| \leq b\) we have the only such adjoint solution: \(\psi_1(t) = -b \sin(t) - 1\) and \(\psi_2(t) = -b \cos(t)\).

4. Condition \((1)\) seems to be satisfied for all correct adjoint variables \((33)\), such that \(|r| \leq b\).

5. Vector-function \(\hat{J}_x\), obtained in \((37)\), does not converge as \(T \to \infty\).
A Proofs of Propositions 3.1 and 3.3

Proof of Proposition 3.1. Needle variation at time $\tau$ can be defined as

$$u_\alpha(t) := \begin{cases} \hat{u}(t), & t \notin (\tau - \alpha, \tau] \\ u, & t \in (\tau - \alpha, \tau] \end{cases},$$

where $u \in U$ is some constant. It is implied that $u = u(\tau)$. Assumption 2 guarantees that for sufficiently small $\alpha > 0$ control $u_\alpha$ is admissible, i.e. corresponding trajectory $x_\alpha(t) \in X$ for all $t > t_0$.

The corresponding increment in the value of the functional can be written as follows:

$$\Delta J_\alpha(T) := J(u_\alpha(\cdot), x_0, t_0, T) - J(\hat{u}(\cdot), x_0, t_0, T)$$

$$= \int_{\tau - \alpha}^{T} (g(x_\alpha(t), \hat{u}(t), t) - g(x(t), \hat{u}(t), t)) \, dt$$

$$= J(\hat{u}(\cdot), x_\alpha(\tau), \tau, T) - J(\hat{u}(\cdot), \hat{x}(\tau), \tau, T)$$

$$+ \int_{\tau - \alpha}^{\tau} (g(x_\alpha(t), u, t) - g(x(t), \hat{u}(t), t)) \, dt,$$

where $x_\alpha$ is the trajectory corresponding to control $u_\alpha$. Then, due to Assumption 1 we have that for all $\varepsilon > 0$ there exists $\alpha(\varepsilon) > 0$ such that for all $T \geq \tau$ the following inequality holds

$$\frac{\Delta J_\alpha(T)}{\alpha} \geq - \varepsilon + \langle \hat{J}_x(\tau, T), \zeta \rangle$$

$$+ \frac{1}{\alpha} \int_{\tau - \alpha}^{\tau} (g(x_\alpha(t), u, t) - g(x(t), \hat{u}(t), t)) \, dt.$$ (39)

We take the vector $\zeta$ in (39) as

$$\zeta_\alpha(\tau) = \frac{x_\alpha(\tau) - \hat{x}(\tau)}{\alpha}.$$ (40)

Due to the differentiability of function $f$ with respect to $x$ we have from differential equation (7) the following limit

$$\lim_{\alpha \to 0} \zeta_\alpha(\tau) = y(\tau),$$ (40)

where $y$ is the solution (16) of the linearized system (15), $y(t) = K(t, \tau) y(\tau)$, with the following initial condition at $t = \tau $:

$$y(\tau) = f(\hat{x}(\tau), u, \tau) - f(\hat{x}(\tau), \hat{u}(\tau), \tau).$$ (41)
Taking into account expression (40), we have the following approximation of the last term in (39), due to continuity of \( u \) and \( \hat{u} \) at \( \tau \) and continuity of \( g \) w.r.t. \((x,u)\):

\[
\frac{1}{\alpha} \int_{\tau-\alpha}^{\tau} (g(x_\alpha(t), u, t) - g(\hat{x}(t), \hat{u}(t), t)) \, dt = g(\hat{x}(\tau), u, \tau) - g(\hat{x}(\tau), \hat{u}(\tau), \tau) + O(\alpha),
\]

where \( \lim_{\alpha \to 0} O(\alpha) = 0 \). Hence, inequality (39) takes the form

\[
\frac{\Delta J_\alpha(T)}{\alpha} \geq -\varepsilon + \langle \dot{J}_x(\tau, T), \zeta_\alpha(\tau) \rangle + g(\hat{x}(\tau), u, \tau) - g(\hat{x}(\tau), \hat{u}(\tau), \tau) + O(\alpha).
\]

Limits (18) and (40) imply that for all \( \varepsilon > 0 \) there exist \( T_1(\varepsilon) \) and \( \alpha_1(\varepsilon) > 0 \) such that for all \( T \geq T_1(\varepsilon) \) and \( \alpha \in (0, \alpha_1(\varepsilon)) \) holds

\[
\langle \dot{J}_x(\tau, T), y(\tau) - \zeta_\alpha(\tau) \rangle \leq |\dot{J}_x(\tau, T)||y(\tau) - \zeta_\alpha(\tau)| < \varepsilon,
\]

that can be written as

\[
\langle \dot{J}_x(\tau, T), y(\tau) \rangle < \langle \dot{J}_x(\tau, T), \zeta_\alpha(\tau) \rangle + \varepsilon.
\]

Hence, we have inequality (39) in the form

\[
\frac{\Delta J_\alpha(T)}{\alpha} \geq -2\varepsilon + \langle \dot{J}_x(\tau, T), y(\tau) \rangle + g(\hat{x}(\tau), u(\tau), \tau) - g(\hat{x}(\tau), \hat{u}(\tau), \tau) + O(\alpha).
\]

**Definition 2.2** of WOO means, that for all \( \varepsilon > 0 \) and \( T_2 > t_0 \) there exists \( T_1(T_2) \geq T_2 \) such that holds \( \Delta J_\alpha(T_1(T_2)) \leq \varepsilon_2 \). Let us take \( T_2 \geq T_1 \) and \( \varepsilon_2 = \alpha \varepsilon \).

Then inequality \( \Delta J_\alpha(T'') \leq \alpha \varepsilon \) results in

\[
3\varepsilon \geq \langle \dot{J}_x(\tau, T''), y(\tau) \rangle + g(\hat{x}(\tau), u(\tau), \tau) - g(\hat{x}(\tau), \hat{u}(\tau), \tau) + O(\alpha). \quad (42)
\]

Suppose that (19) is violated, i.e. there exist \( \varepsilon > 0 \) and \( T \geq t_0 \) such that for all \( T' \geq T \)

\[
\langle \dot{J}_x(\tau, T'), y(\tau) \rangle + g(\hat{x}(\tau), u, \tau) - g(\hat{x}(\tau), \hat{u}(\tau), \tau) \geq 4\varepsilon,
\]

Then we have contradiction with (42) taking \( \alpha \) small enough, i.e (19) should hold.

Similar calculations can be done for OO condition (20).
Proof of Proposition 3.3. If set $\hat{U}(\hat{x}(\tau))$ is a singleton, then $\hat{u}(\tau) = \hat{U}(\hat{x}(\tau))$. If set $\hat{U}(\hat{x}(\tau))$ contains many values we can construct an admissible needle variation at time $\tau$ as

$$u_\alpha(t) := \begin{cases} \hat{u}(t), & t \notin (\tau - \alpha, \tau] \\ u(t), & t \in (\tau - \alpha, \tau] \end{cases},$$

where $u(t) \in \hat{U}(\hat{x}(t))$ for all $t \in (\tau - \alpha, \tau]$. Admissibility of control $u_\alpha$ follows from convexity of $X$. Indeed, one-dimensional autonomous state equation will have an admissible solution for any subsequent combination of admissible controls. Thus for all $\alpha > 0$ corresponding trajectory $x_\alpha(t) \in X$ for all $t > t_0$.

The rest of the proof is the same as that of Proposition 3.1.  

B Proof of Proposition 3.2

First, we prove the following Lemma.

Lemma B.1. Adjoint equation (14) can be written with the use of (17) as

$$\psi(\tau) = K^*(T, \tau) \psi(T) + \lambda \hat{J}_x(\tau, T), \quad \psi(t_0) = \psi_0. \quad (44)$$

Proof. Since vector $y(\tau)$ in (16) is arbitrary, from (15) one can find the matrix derivative

$$\frac{\partial K}{\partial t}(t, \tau) = \left( \frac{\partial f}{\partial x}(\hat{x}(t), \hat{u}(t), t) \right) K(t, \tau).$$

Taking the Hermitian transpose we have

$$\frac{\partial K^*}{\partial t}(t, \tau) = K^*(t, \tau) \left( \frac{\partial f}{\partial x}(\hat{x}(t), \hat{u}(t), t) \right) \ast.$$

Hence, if we multiply the adjoint equation (14) by matrix $K^*(t, \tau)$ as

$$-K^*(t, \tau) \dot{\psi}(t) = K^*(t, \tau) \left( \frac{\partial f}{\partial x}(\hat{x}(t), \hat{u}(t), t) \right) \ast \psi(t) + \lambda K^*(t, \tau) \frac{\partial g}{\partial x}(\hat{x}(t), \hat{u}(t), t),$$

then we have

$$-\frac{\partial}{\partial t}(K^*(t, \tau) \psi(t)) = \lambda K^*(t, \tau) \frac{\partial g}{\partial x}(\hat{x}(t), \hat{u}(t), t).$$

Integration of the letter equation from $\tau$ till $T$ yields (44).  

\[ \Box \]
B.1 Main proof

Proof. 1) It follows from (23) and (44) with \( \lambda = 1 \) that

\[
\psi(\tau) - \lim_{T \to +\infty} \hat{J}_x(\tau, T) = \lim_{T \to +\infty} K^*(T, \tau) \psi(T) \\
= K^*(t_0, \tau) \lim_{T \to +\infty} (K^*(T, t_0) \psi(T)) \\
= K^*(t_0, \tau) a_0,
\]

where we use the expression \( K^*(T, \tau) = (K(T, t_0) K(t_0, \tau))^* = K^*(t_0, \tau) K^*(T, t_0) \).

Taking into account (21) we have

\[
\psi(\tau) - \hat{\psi}(\tau) = K^*(t_0, \tau) a_0.
\]

2) It follows from (44) with \( \lambda = 0 \), that \( \psi(\tau) = K^*(t_0, \tau) a_0 \).

\( \square \)

C Proof of optimality in example 4.2

We show that for \( b \in [0, 1) \) control \( u(t) \equiv 1 \) is weak overtaking optimal among all controls in \([-1, 1]\), see Definition 2.2 where

\[
J(u(\cdot), 0, 0, T) - J(1, 0, 0, T) = \int_0^T (\sin(T - t) + b) (u(t) - 1) \, dt. \tag{45}
\]

It suffice to prove that for the function

\[
\Delta x_1(T) := \int_0^T \sin(T - t) (u(t) - 1) \, dt \tag{46}
\]

for all \( T' \geq 0 \) there exists \( T \geq T' \) such that \( \Delta x_1(T) \leq 0 \). Assume the opposite, i.e. there exists \( T' \geq 0 \) such that for all \( T \geq T' \) holds \( \Delta x_1(T) > 0 \).

Take the integer number \( n \) such that \( 2\pi (n - 1) > T' \), so that

\[
\Delta x_1(2\pi (n - 1)) = \int_0^{2\pi (n-1)} \sin(t) (u(t) - 1) \, dt > 0.
\]
The function $\Delta x_1$ of $T = 2n\pi$ can be written as
\[
\Delta x_1(2n\pi) = -\int_0^{2n\pi} \sin(t) (u(t) - 1) \, dt
\]
\[
= -\Delta x_1(2\pi (n - 1)) - \int_{(2n-1)\pi}^{2n\pi} \sin(t) (u(t) - 1) \, dt. \tag{47}
\]

The last integral is not negative
\[
\int_{(2n-1)\pi}^{2n\pi} \sin(t) (u(t) - 1) \, dt \geq 0,
\]
since $u(t) \leq 1$ and $\sin(t) \leq 0$ for all $t \in [(2n - 1) \pi, 2n\pi]$. Thus, recalling that $\Delta x_1(2\pi (n - 1)) > 0$, we have inequality $\Delta x_1(2n\pi) < 0$, which contradicts the assumption. \qed

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