The satisfiability threshold for constraint satisfaction problems is that value of the ratio of constraints (or clauses) to variables, above which the probability that a random instance of the problem has a solution is zero in the large system limit. Two different approaches to obtaining this threshold have been discussed in the literature - using first or second-moment methods which give rigorous bounds or using the non-rigorous but powerful replica-symmetry breaking (RSB) approach, which gives very accurate predictions on random graphs. In this paper, we lay out a different route to obtaining this threshold on a Bethe lattice. We need make no assumptions about the solution-space structure, a key assumption in the RSB approach. Despite this, our expressions and threshold values exactly match the best predictions of the cavity method under the 1-RSB (one-step RSB) hypothesis. Our method hence provides alternate interpretations as well as motivations for the key equations in the RSB approach.

The random $k$-satisfiability problem may be stated as follows. Fix $k$ and consider $N$ boolean variables $x_1, x_2, \cdots, x_N$. An expression of the sort $\phi_1 = x_1 \lor x_5 \lor \overline{x}_9$ is called a clause ($k = 3$ in the above example) and evaluates to TRUE, if at least one of the $k$ variables evaluates to TRUE, i.e. $x_1 = 1$ or $x_5 = 1$ or $\overline{x}_9 = 0$. For each clause $i$, the boolean information set $\eta_1^i, \eta_2^i, \cdots, \eta_k^i$, called a literal assignment, carries information on which of the variables in $i$ are negated. In the most standard version of the $k$-SAT problem, the random $k$-SAT, each literal takes the value 0 or 1 with equal probability and hence all literal assignments are equiprobable. A formula is an expression $\phi_1 \land \phi_2 \cdots \phi_M$ with $M$ clauses generated randomly from the $N$ variables. This formula has a solution if there is any assignment of variables for which all $M$ clauses are satisfied or evaluate to TRUE. A formula may be represented as a factor graph, with $N$ vertices (or variables), $M$ clauses (denoted by a filled square in Fig. 1) and (two types of) edges connecting the vertices to clauses. If the further constraint is imposed of each vertex participating only in $d+1$ clauses, then the factor graph represents a random regular graph.

Random $k$-satisfiability is only one of many so-called constraint satisfaction problems. Several variations have been considered in which the constraints (or clauses) have different forms. Literal assignments may also be chosen differently. For example, in the $k$-NAE-SAT problem, a clause is unsatisfied both if all variables that participate in it evaluate to FALSE, or if they evaluate to TRUE, with all literal assignments being equiprobable.

One can vary a formula by varying only the literal assignment (or instance) and ask what is the probability that a randomly chosen literal assignment has a solution. If an instance has a solution then it is a satisfiable instance. As the constraint density ($\alpha = M/N$) increases, this probability decreases. In the limit of $M, N \rightarrow \infty$, the system is known to have a sharp threshold $\alpha_s$ below which the probability of finding satisfiable instances approaches 1 and above which it vanishes.

The location of $\alpha_s$ has long been an area of active research in the computer science field, and is known rigorously for $k = 2$ [3]. For higher $k$, several rigorous results on upper and lower bounds to the solvability threshold exist [4], usually obtained using first moment arguments or the second moment method [5, 6] respectively.

Physicists meanwhile, have tackled this problem using other methods, namely the replica and cavity approaches from spin-glass theory [7]. In these approaches, it is assumed that, for any typical literal assignment, the solutions start to cluster beyond a certain clustering threshold $\alpha_d$, which lies strictly below the satisfiability transition $\alpha_s$. $\alpha_d$ arises naturally as the point where the cavity recursions develop a new non-trivial solution. $\alpha_s$ on the other hand, is the location at which the log of the number of solution clusters goes to zero. This latter quantity evaluated per variable, called complexity, results in the 1-RSB prediction for $\alpha_s$ [8, 9]. The cavity approach also predicts a number of other thresholds between the clustering and satisfiability thresholds, such as the condensation threshold [10] and the freezing threshold [11].

These two different theoretical routes for obtaining $\alpha_s$, have come together in recent work, with both the existence of clusters proven rigorously [12] as well as the 1-RSB prediction for $\alpha_s$ established explicitly for the $k$-NAE-SAT problem [13] and the $k$-SAT problem [14, 12] for large $k$. Both approaches obtain $\alpha_s$ by looking at properties of solution clusters. In this paper, we obtain instead the same expression for $\alpha_s$ by calculating directly the fraction of literal assignments that have solutions. We demonstrate how to do this for both $k$-SAT as well as $k$-NAE-SAT, building on previous work [10, 15], where we computed the fraction of satisfiable literal assignments exactly on trees. We outline a procedure, equivalent to performing this calculation now on a Bethe lattice, which
gives us an easy way to derive an analog of the complexity for this whole class of problems.

Our method can easily be applied even to obtaining an expression for the moments of the number of solutions on the Bethe lattice. Applying the procedure to the first moment results in the replica-symmetric expression for the complexity. Applying this method to the second moment results, for the first time to our knowledge, in an analog of the complexity (or a rate function) for the second moment. This expression, derived for a Bethe lattice, again matches its counterpart on a random graph \[5\] in so far as all quantitative predictions go, and in addition, brings to light different transitions connected with the change in the nature of the overlap of solutions.

The Analog of Complexity: We consider first a factor graph which is a rooted tree with all nodes having a degree \(d + 1\). The boundaries, or leaf-nodes, are assigned a fixed value 0 or 1 randomly, and have a degree = 1. Every node on this tree (except the boundary nodes) is the root of its subtree and considered as such, we can vary the literal assignment of the edges on this subtree and calculate the fraction of instances \(P_0\) in which this node can take no value (since either value would violate a clause that it participates in), only one value (0 or 1) \(P_1\) or both values \(P_2\) \[16\] \[17\]. Clearly \(P_0 + P_1 + P_2 = 1\). The quantity \(\sum \log(1 - P_0)\) then, where the sum is over every node in the tree, is the log of the probability that an arbitrary assignment of literals over the whole tree, is satisfiable \[16\]. In general, for a tree, \(P_0\) will depend on the level of a node (its distance from the leaves, with the leaves at level 0). For an infinite tree, the quantity \(P_0\) eventually becomes independent of the level of the node (except for the central node which has \(d + 1\) subtrees) and the value it takes is given by the fixed point of the tree recursions \[16\] \[17\]. These tree-recursions are written for the quantity \(Q\): \(Q = \frac{P_0}{1 - P_0}\) is the fraction of instances (taken only over all satisfiable instances on the sub-tree) in which the root can only take the one value not satisfying the clause connecting it to the node at the higher level, for any literal assignment. For \(d < d_c\), the fixed-point equation (FPE) only has a trivial solution \(Q = 0\). Above \(d_c\), a second non-trivial stable solution exists at a non-zero value of \(Q\).

For an infinite tree, let us now consider only 'interior' nodes with high enough levels such that the tree recursions have reached a fixed point. For this system, we can use the relation \(\alpha = \frac{d}{d + 1}\), \(\alpha_d = \frac{d + 1}{k}\) then indicates the branching beyond which the fraction of literal assignments that have solutions goes to 0. The value \(\alpha_d\) is exactly the same as obtained earlier \[1\] for the clustering transition (for reasons we explain a little later). However for our model, \(\alpha_d\) signifies a satisfiability threshold on the tree.

If however, instead of a tree, we consider a graph which is only locally tree-like, with all nodes having a degree \(d + 1\) and with neither a central node nor a surface, then as detailed below, this system displays a non-trivial satisfiability threshold exactly as predicted by the cavity method. This graph is what we mean by a Bethe lattice.

The problem now is to calculate the analog of \(\sum \log(1 - P_0)\) on our Bethe lattice. We do not expect it to be possible to calculate \(P_0\) node-by-node as we did for the tree. However, if it is possible to calculate the fraction of satisfiable instances for two systems (obeying all the same constraints) differing only by a known number of nodes, then a logarithm of the ratio of the two quantities can provide an estimate per node. A general and simple prescription for calculating partition functions in this manner on the Bethe lattice has been given by Gujral \[19\]. The idea is to consider two separate trees which differ only by a certain number of internal nodes but are so constructed that they have exactly the same number of leaves. The fraction of satisfiable instances is exactly calculable for both of these systems. A logarithm of the ratio then provides an estimate of the logarithm of the fraction of satisfiable instances per node.

FIG. 1. The three systems considered in the text are shown. For the example above, \(k = 3\) and \(d + 1 = 3\). In this case, system B is a set of \(4\) independent trees of the type shown. Systems A and B differ by exactly \(k = 3\) nodes and systems A and C differ by 1 node.

The idea outlined above applied to \(k\)-SAT leads to the consideration of the following systems (see Fig. 1). System A has a central node (or root node) \(s_0\), with \((k - 1)(d + 1)\) neighbours \(s_1\). Each of these \(s_1\) neighbours have \((k - 1)d\) other neighbours besides the root \((s_2\) nodes). System B is a collection of \((k - 1)d\) \(s_1\) nodes, each however with \((k - 1)(d + 1)\) \(s_2\) neighbours. It is easy to see that the two situations have the same number of leaf nodes, and differ from each other only by \(k\) internal nodes \((k - 1)s_1\) nodes \(+ s_0\) node), or \(d + 1\) clauses. Alternatively, we could also consider a system C (see Fig. 1), which differs from system A by exactly one node. Note that all three systems satisfy the same constraints, namely all nodes have a degree \(d + 1\) and all clauses have a degree \(k\).

A logarithm of the ratio of the fraction of satisfiable instances of system A to system B (or system C) should then result in a value of this quantity for \(k\) nodes (respectively one node). As we will see, this is the quantity
that plays the role of complexity.

In terms of the quantity \( Q \), the logarithm of the ratio of the two probabilities (we call this \( \Sigma \) in analogy with all the earlier work) is

\[
k\Sigma = \log \left\{ \frac{f_{d+1}(Q)f_d(Q)^{(k-1)(d+1)}}{f_{d+1}(Q)^{(k-1)d}} \right\} \tag{1}
\]

where \( f_{d+1} \) is the probability \( 1 - P_0 \) calculated for a node with degree \( d + 1 \) and \( f_d \) is the equivalent quantity calculated for a node with degree \( d \). In what follows, we present the details of our calculations for both \( k \)-NAE-SAT and \( k \)-SAT.

\( k \)-NAE-SAT on the Bethe lattice: The recursions for \( P_0, P_1 \) (with \( P_2 = 1 - P_0 - P_1 \)) on a rooted tree with branching number \( d \) are:

\[
P_0 = 1 + (1 - 2Q^{k-1})^d - 2(1 - Q^{k-1})^d \equiv 1 - f_d(Q) \tag{2}
\]

\[
P_1 = 2(1 - Q^{k-1})^d - 2(1 - 2Q^{k-1})^d
\]

The two equations above may be written as a recursion for one single quantity \( Q \). The FPE is then

\[
Q = \frac{(1 - Q^{k-1})^d - (1 - 2Q^{k-1})^d}{2(1 - Q^{k-1})^d - (1 - 2Q^{k-1})^d} \tag{3}
\]

Eq. 3 with a change of variables, is the same equation derived in [18] by the cavity method. Above a critical value \( d_c(k) \), Eq. 3 has a second non-trivial solution for any \( k > 2 \) in which \( Q \) is non-zero.

For both systems A and B, it is easy to calculate the fraction of satisfiable instances exactly and so also the logarithm of their ratio (Eq. 1). For \( k \)-NAE-SAT, substituting the expressions from Eq. 2 into Eq. 1 we get

\[
k\Sigma = (d + 1) \log(1 - Q^{k-1}) + (d + 1)(k - 1) \log(2 - g(Q)^d) + (d + 1 - dk) \log(2 - g(Q)^{d+1}) \tag{4}
\]

where \( g(Q) = \frac{1 - 2Q^{k-1}}{1 - Q^{k-1}} \). \( \Sigma \) is evaluated at the fixed point of the recursion for \( Q \) (Eq. 3) (see Fig. 2). It is easy to show that Eq. 3 is exactly the same as the \( m = 0 \), 1-RSB expression for complexity, obtained for \( k \)-NAE-SAT on random regular graphs by Dall’asta et al [18], as well as for the exact expression for the rate function obtained by Ding et al [13] for \( k \)-NAE-SAT (for large \( k \)) on random regular graphs. (Fig. 2).

It is interesting at this point to consider the interpretation of the complexity function in our case. When \( Q = 0 \), \( \Sigma = 0 \) and the fraction of satisfiable instances is \( 1 \) for both the numerator and the denominator. As \( d \) increases, beyond a critical value \( d_c \), as explained above, Eq. 3 has two stable solutions, in one of which \( Q \) is non-zero. The value of \( d_c \) is the same for systems A and B, but the value of \( \Sigma \) is non-zero and positive up to a value \( d_c \), for non-zero \( Q \) (see Fig. 2). A positive value of \( \Sigma \) is however not consistent with its interpretation as the logarithm of a probability, since in this case, it should either be \( 0 \) or negative. For \( d < d_c \), \( Q = 0 \) and \( \Sigma = 0 \), which is consistent. For \( d > d_c \), \( \Sigma \) can take a physically acceptable value only if the chosen solution continues to be \( Q = 0 \), till the value of \( d \) when \( \Sigma \) crosses over to negative values. This value of \( d \) is indeed the satisfiability threshold \( d_0 \), since as soon as \( Q \) becomes non-zero, the contribution of each node to \( \Sigma \) is negative and the fraction of satisfiable instances goes to \( 0 \) in the large \( N \) limit.

\( k \)-SAT on the Bethe lattice: For completeness we show the results for \( k \)-SAT as well. The FPE for the quantity \( Q \) for \( k \)-SAT on a tree with branching number \( d \) is [16]

\[
Q = \frac{(1 - 0.5Q^{k-1})^d - (1 - Q^{k-1})^d}{2(1 - 0.5Q^{k-1})^d - (1 - Q^{k-1})^d} \tag{5}
\]

The function \( f_d(Q) = 2(1 - 0.5Q^{k-1})^d - (1 - Q^{k-1})^d \), and substituting this in Eq. 1 we get,

\[
k\Sigma = (d + 1) \left[ \log(1 - 0.5Q^{k-1}) + (k - 1) \log(2 - g(Q)^d) \right] + (d + 1 - dk) \log(2 - g(Q)^{d+1}) \tag{6}
\]

where \( Q \) satisfies the FPE Eq. 3 and \( g(Q) = \frac{1 - 0.5Q^{k-1}}{1 - Q^{k-1}} \). The cavity approach predicts for this problem the expression [21]

\[
\Sigma' = \log(2 - g(Q)^{d+1}) + (d + 1) \log(1 - 0.5Q^{k-1}) - (d + 1)(1 - 1/k) \log(1 - Q^k) \tag{7}
\]

Again, \( \Sigma = \Sigma' \) as long as the FPE Eq. 3 is satisfied.

The second moment method applied to clusters of \( k \)-SAT solutions has been used recently to obtain the exact threshold in a regular symmetrized \( k \)-SAT problem [14] as well as for \( k \)-SAT on random graphs [15]. Both these works confirm the 1-RSB prediction for the complexity for these problems, implying in turn that our procedure
First and Second Moment of the total number of solutions: The procedure detailed above may be utilized to obtain a rate function for any quantity that varies exponentially with the number of variables \( N \). We now define our quantity of interest to be the first moment \( \langle Z \rangle \) and second moment \( \langle Z^2 \rangle \) of the number of solutions, where \( Z \) is the total number of solutions for a given literal assignment. For the \( k \)-SAT problem on a tree with branching number \( d \), fixed boundary nodes and a given literal assignment, it is easy to write a recursion relation for \( Z \) as a function of the level \( \alpha \). The corresponding recursion relation for \( k \)-NAE-SAT is only slightly different [22]. The first moment \( \langle Z \rangle \) is readily obtained from the recursion and the corresponding rate function results in the replica-symmetric expression for the complexity (as expected).

If we follow the same procedure to now obtain a rate function for the second moment \( \langle Z^2 \rangle \), we obtain the expression

\[
\frac{\log\langle Z^2 \rangle}{N} = \log(2) + \frac{k-1}{k} (d+1) \log \left( \frac{h^d}{2} \right) + \left( \frac{d+1}{k} - 1 \right) \log \left[ h^{d+1} + \hat{h}^{d+1} \right]
\]

where the functions \( h = \left[ (2k-1) (1+r)^{k-1} + 0.5 \right] \), \( \hat{h} = h - 0.5 \) and \( r \) is the solution of the fixed point equation (16)

\[
r = \frac{\hat{h}^d}{\hat{h}^{d+1}}
\]

![FIG. 3. The expression for the logarithm of the second moment (per variable) from Eq. 8 is plotted for \( k = 5 \). The various transitions are also indicated in the figure with \( \alpha = \frac{44}{11} \). The clustering transition, at \( \alpha_d = 16.16 \) is not shown. At \( \alpha_f \sim 20.3 \), the fixed point equation for \( r \) (Eq. 9) develops a second solution, which is however chosen only at \( \alpha_o = 20.72 \) where the expression for the second moment (Eq. 8) develops a discontinuity in the derivative. The satisfiability threshold occurs at \( \alpha_s = 21.27 \). At \( \alpha_M = 22.8 \), the logarithm goes to 0. Beyond this value, the second moment vanishes in the large \( N \) limit.](https://example.com/figure3.png)

| \( k \) | \( \alpha_d \) | \( \alpha_f \) | \( \alpha_s \) | \( \alpha_o \) | \( \alpha_M \) |
|---|---|---|---|---|---|
| 2 | 1.5 | 1.5 | - | 2.8 |
| 3 | 4.16 | 4.55 | - | 5.83 |
| 4 | 8.4 | 10.15 | - | 11.56 |
| 5 | 16.2 | 20.3 | 21.26 | 20.7 | 22.8 |
| 6 | 30.5 | 38.5 | 43.41 | 42.95 | 45 |
| 7 | 57.28 | 72.1 | 87.84 | 87.35 | 89.42 |
| 8 | 107.13 | 134.2 | 176.57 | 176.06 | 178.1 |

For \( k < 5 \), Eq. 9 has only one solution for any \( d \). For \( d > d_f \) and for \( k > 5 \), the fixed point equation develops two solutions (interestingly, as \( d \) continues to increase, the fixed point equation goes back to having only one solution, but this value of \( d \) lies well beyond the point where Eq. 8 vanishes, for large \( k \)). The value of \( r \) which leads to the larger value of the second moment (Eq. 8) is the chosen one. At \( d = d_o(k) \), this value changes, leading to a discontinuity in the derivative of the second moment with respect to \( d \) (or \( \alpha \)); see Fig. [3](https://example.com/fig3.png)

\( d_o \) occurs before the satisfiability threshold for \( k > 5 \), though well after the clustering threshold. In terms of the overlap between two solutions, the discontinuity of the derivative of the second moment translates to a discontinuous transition from a low to a high value, at \( d_o \). This was first remarked in [23], and is also exhibited by the exact expression of the second moment in terms of the overlap, for a random graph [3]. The precise value \( \alpha_M \) at which Eq. 8 equals zero, also matches the random-graph values for all \( k \). It would be very interesting to understand if \( d_f \) and \( d_o \) are connected to the condensation [10] and freezing [11] transitions. We include a table of values of \( \alpha \) for the different transitions for varying \( k \) (see table [I](https://example.com/table1.png)).

In analogy with the work on tree reconstruction [24], which related \( \alpha_d \) to a process on trees, we are able to give an interpretation for both \( \alpha_d \) and \( \alpha_s \) entirely in terms of the fraction of satisfiable instances. The calculations though explicitly performed on a Bethe lattice, match in every case, the relevant expressions on random (regular) graphs. Our methods are easily applicable even to \( k \)-SAT problems on random graphs with other degree distributions, or for models with variable clause sizes, making this perhaps a useful tool for studying real-world SAT applications. Acknowledgements: We would like to thank Deepak Dhar and Guilhem Semerjian for a critical reading of the manuscript and Cris Moore for several helpful suggestions.
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