Auslander correspondence

Osamu Iyama

ABSTRACT. We study Auslander correspondence from the viewpoint of higher dimensional Auslander-Reiten theory on maximal orthogonal subcategories. We give homological characterizations of Auslander algebras, especially an answer to a question of M. Artin. They are also closely related to Auslander’s representation dimension of artin algebras and Van den Bergh’s non-commutative crepant resolutions of Gorenstein singularities.

Let us recall M. Auslander’s classical theorem [A1] below, which introduced a completely new insight to the representation theory of algebras.

0.1 Theorem (Auslander correspondence) There exists a bijection between the set of Morita-equivalence classes of representation-finite finite-dimensional algebras $\Lambda$ and that of finite-dimensional algebras $\Gamma$ with $\text{gl.dim} \Gamma \leq 2$ and $\text{dom.dim} \Gamma \geq 2$ (3.3). It is given by $\Lambda \mapsto \Gamma := \text{End}_\Lambda(M)$ for an additive generator $M$ of $\text{mod} \Lambda$.

In this really surprising theorem, the representation theory of $\Lambda$ is encoded in the structure of the homologically nice algebra $\Gamma$ called an Auslander algebra. Since the category $\text{mod} \Gamma$ is equivalent to the functor category on $\text{mod} \Lambda$, Auslander correspondence gave us a prototype of the use of functor categories in representation theory. In this sense, Auslander correspondence was a starting point of later Auslander-Reiten theory [ARS] historically. Theoretically, Auslander correspondence gives a direct connection between two completely different concepts, i.e. a representation theoretic property ‘representation-finiteness’ and a homological property ‘$\text{gl.dim} \Gamma \leq 2$ and $\text{dom.dim} \Gamma \geq 2$’. It is a quite interesting project to find correspondence between representation theoretic properties and homological properties. Algebras $\Gamma$ with $\text{gl.dim} \Gamma \leq 2$ was studied in [I4,5] from this viewpoint.

The aim of this paper is to give a higher dimensional version of Auslander correspondence. Recently, it was pointed out in [I7] that Auslander-Reiten theory is ‘2-dimensional-like’ theory, and the concept of maximal $(n-1)$-orthogonal subcategories was introduced as a domain of ‘$(n+1)$-dimensional’ Auslander-Reiten theory. Thus it would be natural to study ‘$(n+1)$-dimensional’ Auslander correspondence from the viewpoint of maximal $(n-1)$-orthogonal subcategories. One of our main results is the theorem below, which is a special case of 4.2.2. We call an additive category finite if it has an additive generator.

0.2 Theorem ($(n+1)$-dimensional Auslander correspondence) For any $n \geq 1$, there exists a bijection between the set of equivalence classes of finite maximal $(n-1)$-orthogonal subcategories $\mathcal{C}$ of $\text{mod} \Lambda$ for finite-dimensional algebras $\Lambda$, and the set of Morita-equivalence classes of finite-dimensional algebras $\Gamma$ with $\text{gl.dim} \Gamma \leq n+1$ and $\text{dom.dim} \Gamma \geq n+1$. It is given by $\mathcal{C} \mapsto \Gamma := \text{End}_\Lambda(M)$ for an additive generator $M$ of $\mathcal{C}$.

Putting $n = 1$ in this theorem, we obtain the theorem 0.1 because $\text{mod} \Lambda$ has a unique maximal 0-orthogonal subcategory $\text{mod} \Lambda$.

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We study not only a higher global-dimensional version of 0.1 but also its higher Krull-
dimensional version. Auslander-Reiten theory plays a crucial role also in the study of
the category CM_\Lambda of Cohen-Macaulay modules over Cohen-Macaulay rings and orders
\Lambda (3.1) of Krull-dimension \( d \) \([A2,3,4][AR2][Y]\). A version of 0.1 for the case \( d = 1 \) and 2
was given by Auslander-Roggenkamp \([ARo]\) and Auslander \([Ar][RV]\) respectively. But it
seems that any version of 0.1 for the case \( d > 2 \) is unknown. Especially, M. Artin raised
a question in \([Ar]\) to characterize homologically the endomorphism rings \( \text{End}_\Lambda(M) \) of an
additive generator \( M \) of CM_\Lambda for representation-finite orders \( \Lambda \) with \( d > 2 \). In 4.2.3, we
give an answer to this question.

Since the category CM_\Lambda over an \( R \)-order \( \Lambda \) is the orthogonal category \( \perp T \) for \( T := \text{Hom}_R(\Lambda, R) \), we study the orthogonal category \( \mathcal{B} := \perp T \) for cotilting \( \Lambda \)-modules \( T \) with
\( \text{id}_\Lambda T = m \) (3.2) in general. The category \( \mathcal{B} \) seems to be still ‘2-dimensional-like’ even if \( m > 2 \)
from the viewpoint of \([I7]\), and we study ‘\( (n+1) \)-dimensional’ Auslander-Reiten theory
on maximal \( (n-1) \)-orthogonal subcategories \( \mathcal{C} \) of \( \mathcal{B} \) in §2. We call the endomorphism ring
\( \text{End}_\Lambda(M) \) of an additive generator \( M \) of \( \mathcal{C} \) an Auslander algebra of type \((d, m, n)\), and give
homological characterizations in §4. We see in 4.2.2 that Auslander correspondence of type
\((d, m, n)\) can be stated in terms of the \((m+1, n+1)\)-condition (3.3) which was introduced
in [I2,4] as a bridge between Auslander’s \( n \)-Gorenstein condition \([FGR][B][AR4][C]\) and
the dominant dimension \([T][H]\). Since ‘higher dimensional’ Auslander-Reiten theory for
the case \( d = m = n + 1 \) is quite peculiar \([I7]\), Auslander algebras of type \((d, d, d - 1)\) have
a very nice homological characterizations in 4.7, especially (3) is closely related to Artin-
Schelter regular ring of dimension \( d \). We observe in 4.7.1 that the \((n+1, n+1)\)-condition
means the existence of \( n \)-almost split sequence homologically.

Recently, in representation theory and non-commutative algebraic geometry, it seems
that the study of ‘nice’ subcategories becomes more and more important besides our max-
imal \( (n-1) \)-orthogonal subcategories. Especially, Van den Bergh introduced the concept
of non-commutative crepant resolutions \([V1,2]\) to study the Bondal-Orlov conjecture \([BO]\)
on derived categories of resolutions of a Gorenstein singularity. We see in 5.2.1 that
non-commutative crepant resolutions are almost one same concept as our maximal \((d-1)\-
orthogonal subcategories, and in 5.3.3 that all maximal 1-orthogonal subcategories are
derived equivalent, which supports Van den Bergh’s generalization \([V2]\) of the Bondal-
Orlov conjecture. As was pinted out by Leuschke \([L]\), we see in 5.4 that the concept
of non-commutative crepant resolutions is also closely related to the concept of Auslan-
der’s representation dimension \([A1]\), which measures how far an algebra is from being
representation-finite. A lot of recent results on the representation dimension (see refer-
ences in 5.4.4(1)) show that it is a really interesting and useful concept. Although the
representation dimension is always finite for \( d \leq 1 \) \([I1,3,6]\), we see in 5.4.3 that this is not
the case for \( d \geq 2 \). We give in 5.5 a boundedness conjecture for 1-orthogonal subcategories,
and prove it for algebras with the representation dimension at most three.

In §6, we give three remarkable examples. In 6.1, we observe a higher dimensional
version of Auslander’s theorem on McKay correspondence \([A4]\). In 6.2, we see that the
work of Geiss-Leclerc-Schröer \([GLS1,2]\) on rigid modules on preprojective algebras is closely
related to our study. In 6.3, we see that the work of Buan-Marsh-Reineke-Reiten-Todorov
\([BMRRT]\) on cluster categories is also closely related to our study.
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1 Preliminaries on functor categories

1.1 Let $\mathcal{A}$ be an additive category and $\mathcal{C}$ a subcategory of $\mathcal{A}$.

(1) We denote by $\mathcal{A}(X, Y)$ the set of morphisms from $X$ to $Y$, and by $fg \in \mathcal{A}(X, Z)$ the composition of $f \in \mathcal{A}(X, Y)$ and $g \in \mathcal{A}(Y, Z)$. We denote by $J_\mathcal{A}$ the Jacobson radical of $\mathcal{A}$, and by ind $\mathcal{A}$ the set of isoclasses of indecomposable objects in $\mathcal{A}$. An $\mathcal{A}$-module is a contravariant additive functor from $\mathcal{A}$ to the category of abelian groups. We denote by $\text{Mod} \mathcal{A}$ the abelian category of $\mathcal{A}$-modules. We call an $\mathcal{A}$-module $F$ finitely presented if there exists an exact sequence $\mathcal{A}(\cdot, X) \xrightarrow{f} \mathcal{A}(\cdot, Y) \rightarrow F \rightarrow 0$. We denote by $\text{mod} \mathcal{A}$ the category of finitely presented $\mathcal{A}$-modules [AR1]. We call $f \in J_\mathcal{A}(X, Y)$ a sink map if $\mathcal{A}(\cdot, X) \xrightarrow{f} J_\mathcal{A}(\cdot, Y) \rightarrow 0$ is exact, and a source map if $\mathcal{A}(Y, \cdot) \xrightarrow{g} J_\mathcal{A}(X, \cdot) \rightarrow 0$ is exact.

We call $\mathcal{C}$ contravariantly (resp. covariantly) finite in $\mathcal{A}$ if $\mathcal{A}(\cdot, X)|_\mathcal{C}$ (resp. $\mathcal{A}(X, \cdot)|_\mathcal{C}$) is a contravariantly generated $\mathcal{C}$-module for any $X \in \mathcal{A}$ [AS]. We call $\mathcal{C}$ functorially finite if it is contravariantly and covariantly finite. We call a complex $\cdots \xrightarrow{f_3} C_1 \xrightarrow{f_2} C_0 \xrightarrow{f_1} X$ a right $\mathcal{C}$-resolution of $X \in \mathcal{A}$ if $C_i \in \mathcal{C}$ and $\cdots \xrightarrow{f_3} \mathcal{A}(\cdot, C_1) \xrightarrow{f_2} \mathcal{A}(\cdot, C_0) \xrightarrow{f_1} \mathcal{A}(\cdot, X) \rightarrow 0$ is exact on $\mathcal{C}$. We write $\mathcal{C}\text{-dim} X \leq n$ if $X$ has a right $\mathcal{C}$-resolution with $C_{n+1} = 0$. Put $\mathcal{C}\text{-dim} \mathcal{A} := \sup_{X \in \mathcal{A}} \mathcal{C}\text{-dim} X$. Define a left $\mathcal{C}$-resolution, $\mathcal{C}\text{-op}\text{-dim} X$ and $\mathcal{C}\text{-op}\text{-dim} \mathcal{A}\text{op}$ dually. We denote by $[\mathcal{C}]$ the ideal of $\mathcal{A}$ consisting of morphisms which factor through $\mathcal{C}$.

(2) Let $R$ be a commutative local ring and $D : \text{Mod} R \rightarrow \text{Mod} R$ the Matlis dual. Assume that $\mathcal{A}$ is an $R$-category such that $\mathcal{A}(X, Y)$ is an $R$-module of finite length for any $X, Y \in \mathcal{A}$. For any $F \in \text{Mod} \mathcal{A}$ and $X \in \mathcal{A}$, $F(X)$ has an $R$-module structure naturally. Thus we have a functor $D : \text{Mod} \mathcal{A} \leftrightarrow \text{Mod} \mathcal{A}\text{op}$ by composing with $D$. We call $\mathcal{A}$ a dualizing $R$-variety if $D$ induces a duality $\text{mod} \mathcal{A} \leftrightarrow \text{mod} \mathcal{A}\text{op}$ [AR1]. If $\mathcal{A}$ is a dualizing $R$-variety, it is easily checked that $\text{mod} \mathcal{A}$ (resp. $\text{mod} \mathcal{A}\text{op}$) is an abelian subcategory of $\text{Mod} \mathcal{A}$ (resp. $\text{Mod} \mathcal{A}\text{op}$) which is closed under kernels, cokernels and extensions [A1;2.1]. In particular, $\mathcal{A}$ has pseudo-kernels and pseudo-cokernels.

1.2 The following version of a theorem of Auslander-Smalø [AS;2.3] gives a relationship between dualizing $\mathcal{A}$-varieties and functorially finite subcategories.

Proposition Let $\mathcal{A}$ be a dualizing $R$-variety. Then any functorially finite subcategory $\mathcal{C}$ of $\mathcal{A}$ is a dualizing $\mathcal{C}$-variety.

Proof (i) We will show that $F|_\mathcal{C} \in \text{mod} \mathcal{C}$ holds for any $F \in \text{mod} \mathcal{A}$.

Since $\text{mod} \mathcal{C}$ is closed under cokernels in general, we only have to show that $\mathcal{A}(\cdot, X)|_\mathcal{C} \in \text{mod} \mathcal{C}$ holds for any $X \in \mathcal{A}$. Let $f \in \mathcal{A}(C_0, X)$ be a right $\mathcal{C}$-resolution, $g \in \mathcal{A}(X_1, C_0)$ a pseudo-kernel of $f$, and $h \in \mathcal{A}(C_1, X_1)$ a right $\mathcal{C}$-resolution. Then $\mathcal{A}(\cdot, C_1) \xrightarrow{h} \mathcal{A}(\cdot, C_0) \xrightarrow{f} \mathcal{A}(\cdot, X) \rightarrow 0$ is exact on $\mathcal{C}$.

(ii) For any $F \in \text{mod} \mathcal{C}$, take an exact sequence $\mathcal{C}(\cdot, Y) \xrightarrow{f} \mathcal{C}(\cdot, X) \rightarrow F \rightarrow 0$. Define $F' \in \text{mod} \mathcal{A}$ by an exact sequence $\mathcal{A}(\cdot, Y) \xrightarrow{f} \mathcal{A}(\cdot, X) \rightarrow F' \rightarrow 0$. Since $\mathcal{A}$ is a dualizing
$R$-variety, $DF' \in \text{mod } \mathcal{A}^{\text{op}}$ holds. Thus $DF = (DF')|_C \in \text{mod } \mathcal{C}^{\text{op}}$ holds by (i). A dual argument shows that $DG \in \text{mod } \mathcal{C}$ holds for any $G \in \text{mod } \mathcal{C}^{\text{op}}$.

2 Higher dimensional Auslander-Reiten theory for dualizing $R$-varieties

In Auslander-Reiten theory, there are two approaches to showing the existence theorem of almost split sequences. One is based on an explicit calculation of extension groups [ARS], and higher dimensional Auslander-Reiten theory in [I7] was developed in this direction. Another is more general and suggestive but less concrete, and based on the concept of dualizing $R$-varieties [AR1][AS]. In this section, we will study higher dimensional Auslander-Reiten theory in the latter direction. This will enable us to treat the orthogonal category $^\perp T$ for a cotilting $\Lambda$-module $T$ in §3.

2.1 Throughout this section, assume that $R$ is a commutative local ring and $\mathcal{A}$ is an abelian $R$-category with enough projectives. For $X, Y \in \mathcal{A}$, we write $X \perp_n Y$ if $\text{Ext}_A^i(X, Y) = 0$ holds for any $i$ ($0 < i \leq n$). Put $\mathcal{C}^\perp_n := \{X \in \mathcal{A} \mid \mathcal{C} \perp_n X\}$ and $^\perp_n \mathcal{C} := \{X \in \mathcal{A} \mid X \perp_n \mathcal{C}\}$. Put $\perp := \perp_\infty$. Let $\mathcal{P} = \mathcal{P}(\mathcal{A}) := ^\perp \mathcal{A}$ be the category of projective objects in $\mathcal{A}$. Let $\mathcal{A} := \mathcal{A}/[\mathcal{P}]$ be the stable category (1.1), and $\Omega : \mathcal{A} \to \mathcal{A}$ the syzygy functor. One can easily check the facts below for any $X \in ^\perp \mathcal{P}$ and $Y \in \mathcal{P}$.

1. $\Omega^n : \mathcal{A}(X, Y) \to \mathcal{A}(\Omega^n X, \Omega^n Y)$ is bijective.
2. We have a functorial isomorphism $\mathcal{A}(\Omega^n X, Y) = \text{Ext}_A^n(X, Y)$.

2.2 In the rest of this section, we assume that $\mathcal{B}$ is a resolving subcategory of $\mathcal{A}$, i.e. $\mathcal{P} \subseteq \mathcal{B}$ and $\mathcal{B}$ is closed under extensions and kernels of surjections [AR3]. Thus $\Omega$ induces the syzygy functor $\Omega : \mathcal{B} \to \mathcal{B}$ for $\mathcal{B} := \mathcal{B}/[\mathcal{P}]$. Let $\mathcal{I} = \mathcal{I}(\mathcal{B}) := \mathcal{B}^\perp \cap \mathcal{B}$ be the category of injective objects in $\mathcal{B}$. Moreover, we assume that $\mathcal{B}$ is enough injectives, i.e. for any $X \in \mathcal{B}$, there exists an exact sequence $0 \to X \to I \to Y \to 0$ with $Y \in \mathcal{B}$ and $I \in \mathcal{I}$. Let $\mathcal{B} := \mathcal{B}/[\mathcal{I}]$ be the costable category, and $\Omega^\perp : \mathcal{B} \to \mathcal{B}$ the cosyzygy functor. For a subcategory $\mathcal{C}$ of $\mathcal{B}$, we denote by $\mathcal{C}$ (resp. $\mathcal{C}$) the corresponding subcategory of $\mathcal{B}$ (resp. $\mathcal{B}$). It is not difficult to check the proposition below (cf. [AR1]).

2.2.1 Proposition Let $0 \to X_2 \twoheadrightarrow X_1 \xrightarrow{f} X_0 \to 0$ be an exact sequence in $\mathcal{A}$ with $X_i \in \mathcal{B}$. Then we have the two long exact sequences below.

\[
\cdots \to \mathcal{B}(\Omega X_2, X_1) \xrightarrow{\Omega f} \mathcal{B}(\Omega X_1, X_0) \to \mathcal{B}(X_2, X_1) \xrightarrow{f} \mathcal{B}(X_0, X_1) \to \mathcal{B}(X_1, 0) \\
\to \text{Ext}^1_A(\Omega X_2, X_1) \xrightarrow{\Omega f} \text{Ext}^1_A(\Omega X_1, X_0) \to \text{Ext}^1_A(X_2, X_1) \xrightarrow{f} \text{Ext}^1_A(X_0, X_1) \to \cdots
\]

\[
\cdots \to \mathcal{B}(\Omega^\perp X_2, X_1) \xrightarrow{\Omega^\perp f} \mathcal{B}(\Omega^\perp X_1, X_0) \to \mathcal{B}(X_2, X_1) \xrightarrow{f} \mathcal{B}(X_0, X_1) \to \mathcal{B}(X_1, 0) \\
\to \text{Ext}^1_A(\Omega^\perp X_2, X_1) \xrightarrow{\Omega^\perp f} \text{Ext}^1_A(\Omega^\perp X_1, X_0) \to \text{Ext}^1_A(X_2, X_1) \xrightarrow{f} \text{Ext}^1_A(X_0, X_1) \to \cdots
\]

2.2.2 The following fundamental theorem is a version of [AR1].

Theorem (1) $\text{mod } \mathcal{B}$ is enough injectives, and $X \mapsto \text{Ext}_A^1(\Omega X, X)$ gives an equivalence from $\mathcal{B}$ to the category $\mathcal{I}(\text{mod } \mathcal{B})$ of finitely presented injective $\mathcal{B}$-modules.

(2) $\text{mod } \mathcal{B}^{\text{op}}$ is enough injectives, and $X \mapsto \text{Ext}_A^1(X, X)$ gives an equivalence from $\mathcal{B}^{\text{op}}$ to the category $\mathcal{I}(\text{mod } \mathcal{B}^{\text{op}})$ of finitely presented injective $\mathcal{B}^{\text{op}}$-modules.
Proof We only prove (1) since (2) is proved dually. Fix \(X,Y \in \mathcal{B}\). Let \(0 \to X \xrightarrow{g} I \xrightarrow{f} \Omega^- X \to 0\) be an injective resolution. Then \(\mathcal{A} : \mathcal{B}(, I) \xleftarrow{\tau} \mathcal{B}(, \Omega^- X) \to \text{Ext}^1_{\mathcal{A}}(, X) \to 0\) is exact by 2.2.1. Thus \(\text{Ext}^1_{\mathcal{A}}(, X) \in \text{mod} \mathcal{B}\). We have an exact sequence \(\text{Hom}(\mathcal{A}, \text{Ext}^1_{\mathcal{A}}(, Y)) : 0 \to \text{Hom}(\text{Ext}^1_{\mathcal{A}}(, X), \text{Ext}^1_{\mathcal{A}}(, Y)) \to \text{Ext}^1_{\mathcal{A}}(\Omega^- X, Y) \xrightarrow{f} \text{Ext}^1_{\mathcal{A}}(I, Y)\) by Yoneda’s lemma. Since \(0 \to \mathcal{B}(X, Y) \to \text{Ext}^1_{\mathcal{A}}(\Omega^- X, Y) \xrightarrow{f} \text{Ext}^1_{\mathcal{A}}(I, Y)\) is exact by 2.2.1, we have a bijection \(\mathcal{B}(X, Y) \to \text{Hom}(\text{Ext}^1_{\mathcal{A}}(, X), \text{Ext}^1_{\mathcal{A}}(, Y))\). Thus the functor \(\mathcal{B} \to \text{mod} \mathcal{B}\) given by \(X \mapsto \text{Ext}^1_{\mathcal{A}}(, X)\) is full and faithful.

For any \(F \in \text{mod} \mathcal{B}\), take an exact sequence \(\mathcal{B}(, Y_1) \xrightarrow{f} \mathcal{B}(, Y_0) \to F \to 0\). Then \(f\) is an isomorphism since \(F(\mathcal{P}) = 0\). Let \(0 \to Y_2 \xrightarrow{g} Y_1 \xrightarrow{f} Y_0 \to 0\) be an exact sequence in \(\mathcal{A}\). Then \(Y_2 \in \mathcal{B}\) (Be careful in the proof of (2)). By 2.2.1, \(\mathcal{P} : \mathcal{B}(, Y_2) \xrightarrow{\tau} \mathcal{B}(, Y_1) \xrightarrow{f} \mathcal{B}(, Y_0) \to F \to 0\) gives a projective resolution of \(F\). We have an exact sequence \(\text{Hom}(\mathcal{P}, \text{Ext}^1_{\mathcal{A}}(, X)) : \text{Ext}^1_{\mathcal{A}}(Y_0, X) \xrightarrow{g} \text{Ext}^1_{\mathcal{A}}(Y_1, X) \xrightarrow{f} \text{Ext}^1_{\mathcal{A}}(Y_2, X)\) by Yoneda’s lemma and 2.2.1. Thus \(\text{Ext}^1(F, \text{Ext}^1_{\mathcal{A}}(, X)) = 0\) holds, and \(\text{Ext}^1_{\mathcal{A}}(, X)\) is injective. Since we have an exact sequence \(0 \to F \to \text{Ext}^1_{\mathcal{A}}(, Y_2)\) by 2.2.1, \(\text{mod} \mathcal{B}\) is enough injectives.\[\Box\]

2.2.3 In the rest of this section, we assume that the conditions in the following version of a theorem of Auslander-Reiten [AR5;2.2] are satisfied.

**Proposition** Assume that \(\text{Ext}^1_{\mathcal{A}}(X, Y)\) is an \(R\)-module of finite length for any \(X,Y \in \mathcal{B}\). Then the following conditions are equivalent.

1. \(\mathcal{B}\) is a dualizing \(R\)-variety.
2. \(\overline{\mathcal{B}}\) is a dualizing \(R\)-variety.
3. There exists an equivalence \(\tau : \mathcal{B} \to \overline{\mathcal{B}}\) with a quasi-inverse \(\tau^-\) and a functorial isomorphism \(\overline{\mathcal{B}}(Y, \tau X) \simeq \text{DExt}^1_{\mathcal{A}}(X, Y) \simeq \overline{\mathcal{B}}(\tau^- Y, X)\) for any \(X,Y \in \mathcal{B}\).

**Proof**

(2)\(\Rightarrow\)(3) By Yoneda’s lemma, \(X \mapsto \mathcal{B}(, X)\) gives an equivalence \(\mathcal{F} : \mathcal{B} \to \mathcal{P}(\text{mod} \mathcal{B})\). By (2) and 2.2.2, \(X \mapsto \text{DExt}^1_{\mathcal{A}}(X, )\) gives an equivalence \(\mathcal{G} : \mathcal{B} \to \mathcal{P}(\text{mod} \mathcal{B})\). Let \(\mathcal{F}^-\) be a quasi-inverse of \(\mathcal{F}\), \(\tau := \mathcal{F}^- \circ \mathcal{G}\) and \(\tau^-\) a quasi-inverse of \(\tau\). Then the assertion follows. A dual argument shows (1)\(\Rightarrow\)(3).

(3)\(\Rightarrow\)(1)\&(2) Fix \(F \in \text{mod} \mathcal{B}\). By 2.2.1, we can take an exact sequence \(0 \to F \to \text{Ext}^1_{\mathcal{A}}(, X_2) \xrightarrow{\tau} \text{Ext}^1_{\mathcal{A}}(, X_1)\), which is induced by an exact sequence \(0 \to X_2 \xrightarrow{g} X_1 \xrightarrow{f} X_0 \to 0\) in \(\mathcal{B}\). Applying \(\text{D}\), we have an exact sequence \(\mathcal{B}(\tau^- X_1, ) \xrightarrow{\tau^- g} \mathcal{B}(\tau^- X_2, ) \to DF \to 0\) by (3). Thus \(DF \in \text{mod} \mathcal{B}^{\text{op}}\) holds. Dually, \(DG \in \text{mod} \mathcal{B}\) holds for any \(G \in \text{mod} \mathcal{B}^{\text{op}}\). Since we have equivalences \(\tau : \text{mod} \mathcal{B} \to \text{mod} \mathcal{B}\) and \(\tau : \text{mod} \mathcal{B}^{\text{op}} \to \text{mod} \mathcal{B}^{\text{op}}\) which commute with \(\text{D}\), we obtain (1) and (2).\[\Box\]

2.3 For \(n \geq 1\), we define functors \(\tau_n\) and \(\tau_n^-\) by

\[\tau_n := \tau \circ \Omega^{n-1} : \mathcal{B} \to \mathcal{B}\] and \(\tau_n^- := \tau^- \circ \Omega^{-(n-1)} : \mathcal{B} \to \mathcal{B}\),

where \(\Omega : \mathcal{B} \to \mathcal{B}\) is the syzygy functor and \(\Omega^- : \mathcal{B} \to \mathcal{B}\) is the cosyzygy functor. Put

\[X_n := \downarrow^n \mathcal{P} \cap \mathcal{B}\] and \(Y_n := \mathcal{I} \downarrow^n \cap \mathcal{B}\).

Let us give a version of [I7;1.4,1.5] for our situation.
2.3.1 Theorem (1) There exist functorial isomorphisms \( \mathcal{B}(Y, \tau_n X) \simeq D \text{Ext}^n_\mathcal{A}(X, Y) \simeq \mathcal{B}(\tau_n Y, X) \) for any \( X, Y \in \mathcal{B} \). Thus \( \tau_n : \mathcal{B} \to \mathcal{B} \) is a right adjoint of \( \tau_n^- : \mathcal{B} \to \mathcal{B} \).

(2) For any \( i \) (\( 0 < i < n \)), there exist functorial isomorphisms below for any \( X \in \mathcal{X}_{n-1} \), \( Y \in \mathcal{Y}_{n-1} \) and \( Z \in \mathcal{B} \).

\[
\begin{align*}
D \text{Ext}^n_\mathcal{A}(X, Z) &\simeq \mathcal{B}(Z, \tau_n X), & D \text{Ext}^{n-i}_\mathcal{A}(X, Z) &\simeq \text{Ext}^i_\mathcal{A}(Z, \tau_n X), & D \mathcal{B}(X, Z) &\simeq \text{Ext}^n_\mathcal{A}(Z, \tau_n X) \\
D \text{Ext}^n_\mathcal{A}(Z, Y) &\simeq \mathcal{B}(\tau_n Y, Z), & D \text{Ext}^{n-i}_\mathcal{A}(Z, Y) &\simeq \text{Ext}^i_\mathcal{A}(\tau_n Y, Z), & D \mathcal{B}(Z, Y) &\simeq \text{Ext}^n_\mathcal{A}(\tau_n Y, Z)
\end{align*}
\]

Proof (1) We have functorial isomorphisms \( \mathcal{B}(Y, \tau_n X) \stackrel{2.2.3(3)}{\simeq} D \text{Ext}^1_\mathcal{A}(\Omega^{n-1}X, Y) \simeq D \text{Ext}^n_\mathcal{A}(X, Y) \). The isomorphism for \( \tau_n^- \) is given dually.

(2) The left isomorphisms are given in (1). For \( i > 0 \), we have functorial isomorphisms \( \text{Ext}^i_\mathcal{A}(Z, \tau_n X) \simeq \text{Ext}^1_\mathcal{A}(\Omega^{i-1}Z, \tau_n X) \) \( \simeq \) \( D \mathcal{B}(\Omega^{n-1}X, \Omega^{i-1}Z) \simeq D \mathcal{B}(\Omega^{n-i}X, Z) \), which is \( \simeq \) \( D \text{Ext}^{n-i}_\mathcal{A}(X, Z) \) if \( n > i \). The isomorphisms for \( \tau_n^- \) are given dually.

2.3.2 Corollary \( \tau_n \) and \( \tau_n^- \) give mutually quasi-inverse equivalences \( \tau_n : \mathcal{X}_{n-1} \to \mathcal{Y}_{n-1} \) and \( \tau_n^- : \mathcal{Y}_{n-1} \to \mathcal{X}_{n-1} \).

Proof For any \( X \in \mathcal{X}_{n-1} \), \( \text{Ext}^i_\mathcal{A}(\mathcal{I}, \tau_n X) \) \( \simeq \) \( D \text{Ext}^{n-i}_\mathcal{A}(X, \mathcal{I}) = 0 \) holds for any \( i \) (\( 0 < i < n \)). Thus \( \tau_n \) gives a functor \( \mathcal{X}_{n-1} \to \mathcal{Y}_{n-1} \), which is full and faithful by 2.1(1) and 2.2.3(3). Dually, \( \tau_n^- \) gives a full and faithful functor \( \mathcal{Y}_{n-1} \to \mathcal{X}_{n-1} \). Since \( \mathcal{B}(X, \mathcal{I}) \) \( \simeq \) \( D \text{Ext}^n_\mathcal{A}(\mathcal{I}, \tau_n X) \) \( \simeq \) \( D \mathcal{B}(\tau_n \circ \tau_n, X) \) holds, \( \tau_n^- \circ \tau_n \) is isomorphic to the identity functor. Dually, \( \tau_n \circ \tau_n^- \) is isomorphic to the identity functor.

2.4 Let \( \mathcal{C} \) be a functorially finite subcategory of \( \mathcal{B} \), and \( l \geq 0 \). We call \( \mathcal{C} \) an \( l \)-orthogonal subcategory of \( \mathcal{B} \) if \( \mathcal{C} \perp \mathcal{I} \) and a maximal \( l \)-orthogonal subcategory of \( \mathcal{B} \) if \( \mathcal{C} = \mathcal{C}^{\perp 1} \cap \mathcal{B} \) holds. We call \( M = \mathcal{B} \) maximal \( l \)-orthogonal (resp. \( l \)-orthogonal) if so is \( M \). Any maximal \( l \)-orthogonal subcategory \( \mathcal{C} \) of \( \mathcal{B} \) satisfies \( \mathcal{P} \cup \mathcal{I} \subseteq \mathcal{C} \subseteq \mathcal{X}_l \cap \mathcal{Y}_l \). Since \( \mathcal{C} \) and \( \mathcal{C} \) are functorially finite subcategories of \( \mathcal{B} \) and \( \mathcal{B} \) respectively, \( \mathcal{C} \) and \( \mathcal{C} \) are dualizing \( R \)-varieties by 1.2. Thus \( \text{mod} \mathcal{C} \), \( \text{mod} \mathcal{C}^{op} \), \( \text{mod} \mathcal{C} \) and \( \text{mod} \mathcal{C}^{op} \) are closed under kernels, cokernels and extensions by 1.1. We have the following characterizations of maximal \( l \)-orthogonal subcategories [17;2.2.2].

2.4.1 Proposition Let \( \mathcal{C} \) be a functorially finite subcategory of \( \mathcal{B} \). Then the conditions (1), (2-i) and (3-i) are equivalent for any \( i \) (\( 0 \leq i \leq l \)).

(1) \( \mathcal{C} \) is a maximal \( l \)-orthogonal subcategory of \( \mathcal{B} \).

(2-i) \( \text{C-dim} \mathcal{B} \) \( \leq l \), \( \mathcal{C} \perp \mathcal{I} \), \( \mathcal{P} \subseteq \mathcal{C} \).

(3-i) \( \text{C-dim} \mathcal{B}^{op} \) \( \leq l \), \( \mathcal{C} \perp \mathcal{I} \), \( \mathcal{P} \subseteq \mathcal{C} \).

(2) \( \mathcal{C} \subseteq \mathcal{C}^{\perp 1} \cap \mathcal{B} \).

(3) \( \mathcal{C}^{op} \subseteq \mathcal{C}^{\perp 1} \cap \mathcal{B} \).

(2.4-l) \( \mathcal{C} \subseteq \mathcal{C}^{\perp 1} \cap \mathcal{B} \).

(3.4-l) \( \mathcal{C}^{op} \subseteq \mathcal{C}^{\perp 1} \cap \mathcal{B} \).

2.5 In the rest of this section, let \( \mathcal{C} \) be a maximal \( (n-1) \)-orthogonal subcategory of \( \mathcal{B} \) (\( n \geq 1 \)). Assume that \( \mathcal{C} \) is Krull-Schmidt, i.e. any object of \( \mathcal{C} \) is isomorphic to a finite direct sum of objects whose endomorphism rings are local.

2.5.1 The following fundamental theorem follows from previous results in 2.3 and 2.4 (cf. [17;2.3.2.3.1,2.2.3.3.5.2]).
Theorem (1) (n-Auslander-Reiten translation) For any $X \in \mathcal{C}$, $\tau_n X \in \mathcal{C}$ and $\tau_n^- X \in \mathcal{C}$ hold. Thus $\tau_n : \mathcal{C} \rightarrow \mathcal{C}$ and $\tau_n^- : \mathcal{C} \rightarrow \mathcal{C}$ are mutually quasi-inverse equivalences.
(2) (n-Auslander-Reiten duality) There exist functorial isomorphisms $\mathcal{C}(Y, \tau_n X) \simeq D \Ext^n_{\mathcal{A}}(X, Y) \simeq \mathcal{C}(\tau_n^- Y, X)$ for any $X, Y \in \mathcal{C}$.
(3) $\mathcal{C}$-dim $\mathcal{B} \leq n - 1$ and $\mathcal{C}$-$\op$-dim $\mathcal{B}$-$\op \leq n - 1$ hold.
(4) $X \mapsto \Ext^n_{\mathcal{A}}(X, )$ gives an equivalence $\mathcal{C} \rightarrow \mathcal{I}(\text{mod} \mathcal{C})$, and $X \mapsto \Ext^n_{\mathcal{A}}(X, )$ gives an equivalence $\mathcal{C}$-$\op \rightarrow \mathcal{I}(\text{mod} \mathcal{C}$-$\op$).

2.5.2 For any $F \in \text{mod} \mathcal{C}$, take a projective resolution $\mathcal{C}(, Y) \xrightarrow{f_2} \mathcal{C}(, X) \rightarrow F \rightarrow 0$. Define $\alpha F \in \text{mod} \mathcal{C}$-$\op$ by the exact sequence $0 \rightarrow \alpha F \rightarrow \mathcal{C}(X, ) \xrightarrow{f_2} \mathcal{C}(Y, )$. Then $\alpha$ gives a left exact functor $\alpha : \text{mod} \mathcal{C} \rightarrow \text{mod} \mathcal{C}$-$\op$. Define $\alpha : \text{mod} \mathcal{C}$-$\op \rightarrow \text{mod} \mathcal{C}$ dually. We denote by $R^n \alpha : \text{mod} \mathcal{C} \leftrightarrow \text{mod} \mathcal{C}$-$\op$ the $n$-th derived functor of $\alpha$ [FGR]. Then we have the following theorem (see [I7;3.6.1]).

Theorem Let $\mathcal{C}$ be a maximal $(n - 1)$-orthogonal subcategory of $\mathcal{B}$ $(n \geq 1)$.
(1) Any $0 \neq F \in \text{mod} \mathcal{C}$ satisfies $\text{pd}_c F = n + 1$ and $R^i \alpha F = 0$ $(i \neq n + 1)$. Any $0 \neq G \in \text{mod} \mathcal{C}$-$\op$ satisfies $\text{pd}_c \op G = n + 1$ and $R^i \alpha G = 0$ $(i \neq n + 1)$.
(2) $R^{n+1} \alpha$ gives a duality $\text{mod} \mathcal{C} \leftrightarrow \text{mod} \mathcal{C}$-$\op$, and the equivalence $D R^{n+1} \alpha : \text{mod} \mathcal{C} \leftrightarrow \text{mod} \mathcal{C}$ coincides with the equivalence induced by $\tau_n : \mathcal{C} \rightarrow \mathcal{C}$.

2.5.3 We can show the following theorem (see [I7;3.3.3.3.1]).

Theorem (n-almost split sequence) Let $\mathcal{C}$ be a maximal $(n - 1)$-orthogonal subcategory of $\mathcal{B}$ $(n \geq 1)$. Fix any non-projective $X \in \text{ind} \mathcal{C}$ (resp. non-injective $X \in \text{ind} \mathcal{C}$).
(1) There exists an exact sequence $A : 0 \rightarrow Y \xrightarrow{f_2} C_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} C_0 \xrightarrow{f_0} X \rightarrow 0$ with terms in $\mathcal{C}$ such that $f_i \in \mathcal{J}_\mathcal{C}$ and the following sequences are exact.

$$0 \rightarrow \mathcal{C}(, Y) \xrightarrow{f_2} \mathcal{C}(, C_{n-1}) \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} \mathcal{C}(, C_0) \xrightarrow{f_0} \mathcal{C}(, X) \rightarrow 0$$

$$0 \rightarrow \mathcal{C}(X, ) \xrightarrow{f_2} \mathcal{C}(C_0, ) \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} \mathcal{C}(C_{n-1}, ) \xrightarrow{f_0} \mathcal{C}(Y, ) \rightarrow 0$$

Such $A$ is unique up to isomorphisms of complexes, and satisfies $Y \simeq \tau_n X$ and $X \simeq \tau_n^- Y$.
(2) The simple modules $F := \mathcal{C} / \mathcal{J}_\mathcal{C}(, X)$ and $G := \mathcal{C} / \mathcal{J}_\mathcal{C}(Y, )$ satisfy $\text{pd}_c F = n + 1 = \text{pd}_c \op G$, $R^i \alpha F = 0 = R^i \alpha G$ $(i \neq n + 1)$, $F = R^{n+1} \alpha G$ and $G = R^{n+1} \alpha F$.

2.6 In the rest of this section, we fix $m \geq 0$ and impose the conditions below on $\mathcal{B}$.

Proposition For $m \geq 0$, the following conditions for $\mathcal{B}$ are equivalent.
(1) $\Omega^m X \in \mathcal{B}$ holds for any $X \in \mathcal{A}$.
(2) $\mathcal{B}$ is a contravariantly finite subcategory of $\mathcal{A}$ with $\mathcal{B}$-$\text{dim} \mathcal{A} \leq m$.
(3) If $0 \rightarrow Y \rightarrow B_{m-1} \rightarrow \cdots \rightarrow B_0 \rightarrow X \rightarrow 0$ is an exact sequence in $\mathcal{A}$ with $B_i \in \mathcal{B}$, then $Y \in \mathcal{B}$ holds.

Proof (3) $\Rightarrow$ (1) is obvious. We will show (2) $\Rightarrow$ (3). Take the following commutative diagram of exact sequences, where the upper sequence is a right $\mathcal{B}$-resolution of $X$.

$$0 \rightarrow C_m \rightarrow C_{m-1} \rightarrow C_{m-2} \rightarrow \cdots \rightarrow C_0 \rightarrow X \rightarrow 0$$

$$0 \rightarrow Y \rightarrow B_{m-1} \rightarrow B_{m-2} \rightarrow \cdots \rightarrow B_0 \rightarrow X \rightarrow 0$$
Taking mapping cone, we have an exact sequence $0 \to Y \to C_m \oplus B_{m-1} \to \cdots \to C_1 \oplus B_0 \to C_0 \to 0$. Since $\mathcal{B}$ is a resolving subcategory of $\mathcal{A}$, we obtain $Y \in \mathcal{B}$.

(1)$\Rightarrow$(2) Fix any $X \in \mathcal{A}$. By Auslander-Buchweitz approximation theory [AB;1.1], there exists an exact sequence $0 \to I_m \to \cdots \to I_1 \to B_0 \to X \to 0$ with $I_i \in \mathcal{I}$ and $B_0 \in \mathcal{B}$. It is easily checked that this is a right $\mathcal{B}$-resolution of $X$.

2.6.1 We can show the following theorem by a similar argument to [I7;3.6.2].

**Theorem** Any maximal $(n-1)$-orthogonal subcategory $\mathcal{C}$ $(n \geq 1)$ of $\mathcal{B}$ satisfies $\text{gl.dim}(\text{mod} \, \mathcal{C}) \leq \max\{n + 1, m\}$.

2.6.2 We end this section by pointing out the interesting result below, which realizes the category $\mathcal{C}^{{\perp_1}} \cap \mathcal{B}$ as the category of syzygies.

**Theorem** Let $\mathcal{C} = \text{add} \, M$ be an $(n-1)$-orthogonal subcategory of $\mathcal{B}$. Assume $\mathcal{P} \cup \mathcal{I} \subseteq \mathcal{C}$ and that $\Gamma := \text{End}_R(M)$ is a noetherian ring. For any $i \, (0 \leq i \leq n-1)$, we have full and faithful functors $\mathcal{F} := \mathcal{B}(\mathcal{M}, \cdot) : \mathcal{C}^{{\perp_1}} \cap \mathcal{B} \to \Omega^{i+2}(\text{mod} \, \Gamma)$ and $\mathcal{G} := \mathcal{B}(\mathcal{M}, M) : \mathcal{C}^{{\perp_1}} \cap \mathcal{B} \to \Omega^{i+2}(\text{mod} \, \Gamma^{op})$ such that $\mathcal{F} = (\cdot)^* \circ \mathcal{G}$ and $\mathcal{G} = (\cdot)^+ \circ \mathcal{G}$ for $(\cdot)^* = \text{Hom}_R(\mathcal{M}, \Gamma)$. If $m - 2 \leq i$, then $\mathcal{F}$ and $\mathcal{G}$ are equivalences.

**Proof** We only show the assertion for $\mathcal{F}$. For any $X \in \mathcal{B}$, take a right $\mathcal{C}$-resolution $C_1 \to C_0 \to X \to 0$, which is exact by $\mathcal{P} \subseteq \mathcal{C}$. We have exact sequences $0 \to \mathcal{B}(X, \cdot) \to \mathcal{B}(C_0, \cdot) \to \mathcal{B}(C_1, \cdot)$ and $0 \to \Gamma(\mathcal{F}X, \mathcal{F}(\cdot)) \to \Gamma(\mathcal{F}C_0, \mathcal{F}(\cdot)) \to \Gamma(\mathcal{F}C_1, \mathcal{F}(\cdot))$ on $\mathcal{B}$. Since $\mathcal{B}(C_i, \cdot) = \Gamma(\mathcal{F}C_i, \mathcal{F}(\cdot))$ holds on $\mathcal{B}$, $\mathcal{F}$ is full and faithful and $\mathcal{G} = (\cdot)^* \circ \mathcal{F}$ holds. For any $X \in \mathcal{C}^{{\perp_1}} \cap \mathcal{B}$, take an injective resolution $I : 0 \to X \to I_0 \to \cdots \to I_{i+1}$ in $\mathcal{B}$. Since $\mathcal{C} \perp_i X$ holds, $\mathcal{F}I : 0 \to \mathcal{F}X \to \mathcal{F}I_0 \to \cdots \to \mathcal{F}I_{i+1}$ is an exact sequence with $\mathcal{F}I_j \in \text{add} \, \Gamma$. Thus $\mathcal{F}X \in \Omega^{i+2}(\text{mod} \, \Gamma)$ holds.

Assume $m - 2 \leq i$. For any $Y \in \Omega^{i+2}(\text{mod} \, \Gamma)$, take an exact sequence $P : 0 \to Y \to P_{i+1} \to \cdots \to P_0$ with $P_j \in \text{add} \, \Gamma$. Since $\mathcal{F}$ gives an equivalence $\mathcal{C} \to \text{add} \, \Gamma$, we can take a complex $C : C_{i+1} \xrightarrow{f_{i+1}} \cdots \xrightarrow{f_1} C_0$ with $C_j \in \mathcal{C}$ such that $\mathcal{F}C$ is isomorphic to $P$. Since $\mathcal{P} \subseteq \mathcal{C}$ and $\mathcal{P}$ is exact, $\mathcal{C}$ is also exact. Put $X_j := \text{Ker} \, f_j$. Inductively, we can easily show that $\mathcal{C} \perp_j X_{j+2}$ holds for any $j$ by using exactness of $\mathcal{F}C$. In particular, $X_{i+2} \in \mathcal{C}^{{\perp_1}} \cap \mathcal{B}$ holds by $m - 2 \leq i$, and $\mathcal{F}X_{i+2} = Y$ holds.

3 Orders, cotilting modules and Auslander-type conditions

The aim of this section is to define a pair $(\mathcal{A}, \mathcal{B})$ to which we will apply our theory in §2, and to give preliminary facts which we will use in preceding sections.

3.1 Throughout this section, let $R$ be a complete regular local ring of dimension $d$ and $\Lambda$ a module-finite $R$-algebra. We call $\Lambda$ an isolated singularity [A3] if $\dim \Lambda \otimes_R R_p = \text{ht} \, p$ holds for any non-maximal prime ideal $p$ of $R$. We call a left $\Lambda$-module $M$ Cohen-Macaulay if it is a projective $R$-module. We denote by $\text{CM} \, \Lambda$ the category of Cohen-Macaulay $\Lambda$-modules. Then $D_d := \text{Hom}_R(\, R, R)$ gives a duality $\text{CM} \, \Lambda \leftrightarrow \text{CM} \, \Lambda^{op}$. We call $\Lambda$ an $R$-order (or Cohen-Macaulay $R$-algebra) if $\Lambda \in \text{CM} \, \Lambda$ [A2,3]. A typical example of an order is a commutative complete local Cohen-Macaulay ring $\Lambda$ containing a field since such $\Lambda$ contains a complete regular local subring $R$ [Ma;29.4]. Let $E : 0 \to R \to E_0 \to \cdots \to E_d \to 0$ be a minimal injective resolution of the $R$-module $R$. We denote by $D := \text{Hom}_R(\, E_d, R)$ the Matlis dual. Put $(\cdot)^* := \text{Hom}_\Lambda(\, \Lambda, \Lambda)$ and denote by $\nu_\Lambda := D \circ (\cdot)^*$ the Nakayama functor.
and by $\nu^-_\Lambda := (\ )^* \circ D_d$ the inverse Nakayama functor. If $\Lambda$ is an $R$-order, then $\nu_\Lambda$ and $\nu^-_\Lambda$ give mutually inverse equivalences $\text{add}_\Lambda \Lambda \leftrightarrow \text{add}(D_d\Lambda)$. The following observation in [I7;2.5.1] is useful.

3.1.1 Proposition Let $\Lambda$ be an $R$-order which is an isolated singularity, $X, Y \in \text{CM} \Lambda$ and $2 \leq n \leq d$. Then $\text{depth}_R \text{Hom}_\Lambda(X, Y) \geq n$ if and only if $X \perp_{n-2} Y$.

3.1.2 Let $\Lambda$ be a module-finite $R$-algebra which is an isolated singularity and $M \in \text{CM} \Lambda$. Let us recall the method of Goto and Nishida [GN1] to construct a minimal injective resolution of $M$ in $\text{mod} \Lambda$ from a minimal projective resolution $P : \cdots \to P_1 \to P_0 \to D_d M \to 0$ of $D_d M$. We have exact sequences $M \otimes_R E : 0 \to M \to M \otimes_R E_0 \to \cdots \to M \otimes_R E_{d-1} \to M \otimes_R E_d \to 0$ and $DP : 0 \to M \otimes_R E_d \to DP_0 \to DP_1 \to \cdots$. Connecting them, we obtain a minimal injective resolution

$$0 \to M \to M \otimes_R E_0 \to \cdots \to M \otimes_R E_{d-1} \to DP_0 \to DP_1 \to \cdots$$

of $M$ in $\text{mod} \Lambda$. Thus $\text{id}_\Lambda M = \text{pd}(D_d M)_{\Lambda} + d$ holds. In particular, if $\text{gl.dim} \Lambda = d$, then $\text{CM} \Lambda \subseteq \text{add}_\Lambda \Lambda$ holds.

3.2 Let $\Lambda$ be an $R$-order. For $m \geq d$, we call $T \in \text{CM} \Lambda$ an $m$-cotilting module [AR3][N] if $T \perp T$ (2.1) and there exist exact sequences $0 \to T \to I_0 \to \cdots \to I_{m-d} \to 0$ and $0 \to T_{m-d} \to \cdots \to T_0 \to D_d \Lambda \to 0$ with $I_i \in \text{add}_\Lambda(D_d \Lambda)$ and $T_i \in \text{add}_\Lambda T$. It is easy to check the facts below by 3.1.2 and 3.1.1.

1. $\text{id}_\Lambda T \leq m$ and $\perp T \subseteq \text{CM} \Lambda$ hold, and $\text{End}_\Lambda(T)$ is an $R$-order.
2. $T \in \text{CM} \Lambda$ is a $d$-cotilting module if and only if $\text{add}_\Lambda T = \text{add}_\Lambda(D_d \Lambda)$.

3.2.1 Let us recall the following classical cotilting theorem [M][Ha].

Proposition Let $T$ be an $m$-cotilting $\Lambda$-module and $\Lambda':=\text{End}_\Lambda(T)^{\text{op}}$. Then $T$ is an $m$-cotilting $\Lambda'$-module. We have mutually quasi-inverse equivalences $\text{Hom}_\Lambda(\ , T) : \perp(\Lambda T) \to \perp_{\Lambda'}(T)$ and $\text{Hom}_{\Lambda'}(\ , T) : \perp(\Lambda T) \to \perp_{\Lambda'}(T)$ which preserve $\text{Ext}^i$ for any $i \geq 0$.

3.2.2 We can apply our theory in §2 to $(\mathcal{A}, \mathcal{B}) := (\text{mod} \Lambda, \perp T)$ by the following version of a theorem of Auslander-Reiten [AR3].

Proposition Let $\Lambda$ be an $R$-order which is an isolated singularity, $T$ an $m$-cotilting $\Lambda$-module, $\mathcal{A} := \text{mod} \Lambda$ and $\mathcal{B} := \perp T$. Then the following assertions hold.

1. $\mathcal{B}$ is a enough injective resolving subcategory of $\mathcal{A}$ with $\mathcal{I}(\mathcal{B}) = \text{add} T$.
2. $\mathcal{B}$ is a functorially finite subcategory of $\mathcal{A}$ with $\mathcal{B}$-dim $\mathcal{A} \leq m$.
3. $\overline{\mathcal{B}}$ and $\mathcal{B}$ are dualizing $R$-varieties.

Proof
(1) It is easily checked that $\mathcal{B}$ is resolving with $T \in \mathcal{I}$. For any $X \in \mathcal{B}$, take an injection $X \hookrightarrow (D_d \Lambda)^l$ in $\text{CM} \Lambda$ by 3.2(1). Take an exact sequence $0 \to T_{m-d}^l \to \cdots \to T_0^l \to (D_d \Lambda)^l \to 0$ in 3.2. Then $a$ factors through $b$ by $X \perp T$. Thus $X$ is a submodule of $T_0^l$. We can take an exact sequence $0 \to X \to T' \to Y \to 0$ such that $c$ is a left (add $T'$)-resolution of $X$. Applying $\Lambda(\ , T)$, we obtain $Y \in \perp T = \mathcal{B}$. Thus $\mathcal{B}$ is enough injectives with $\mathcal{I} = \text{add} T$.

(2) Since $\Omega^n \mathcal{A} \subseteq \mathcal{B}$ holds by $\text{id}_\Lambda T \leq m$, $\mathcal{B}$ is a contravariantly finite subcategory of $\mathcal{A}$ with $\mathcal{B}$-dim $\mathcal{A} \leq m$ by 2.6. We will show that $\mathcal{B}$ is a covariantly finite subcategory
of \( \mathcal{A} \). Put \( \Lambda' := \text{End}_A(T)^{op} \) and \( \mathcal{B}' := \mathcal{A}' := \text{mod } \Lambda' \). Since \( T \) is an \( m \)-cotilting \( \Lambda' \)-module by 3.2.1, \( \mathcal{B}' \) is a contravariantly finite subcategory of \( \mathcal{A}' \). Fix \( X \in \mathcal{A} \).

Let \( \mathcal{B}' \xrightarrow{\sim} \Lambda'(X, T) \) be a right \( \mathcal{B}' \)-resolution. It is easily checked that the composition \( X \to \Lambda'(X, T), T \mapsto \Lambda'(\mathcal{B}', T) \) is a left \( \mathcal{B} \)-resolution of \( X \).

(3) Put \( \mathcal{B}_0 := \text{CMA} \). Since \( \Lambda \) is an isolated singularity, it is well-known that \( \mathcal{B}_0 \) satisfies the conditions in 2.2.3 (e.g. [A2:8.7][AR5:2.4]). Since \( \mathcal{B} \) is a functorially finite subcategory of \( \mathcal{B}_0 \) by (2), it is easily checked that \( \overline{\mathcal{B}} \) is that of \( \overline{\mathcal{B}_0} \). Thus \( \overline{\mathcal{B}} \) is a dualizing \( R \)-variety by 1.2, and so is \( \overline{\mathcal{B}} \) by 2.2.3.\( \square \)

\[ \text{3.2.3} \] Let us recall the theorem [I7:3.4.4] below, which tells us that higher dimensional Auslander-Reiten theory for the case \( d = m = n + 1 \) is quite peculiar. It means that \( \mathcal{C} \) has sequences which have properties like \( n \)-almost split sequences and connect projective modules and injective modules. We notice that \( f_0 \) below is not surjective in general.

**Theorem** \((n\text{-fundamental sequence})\) Let \( \mathcal{B} \) be in 3.2.2 and \( \mathcal{C} \) a maximal \((n - 1)\)-orthogonal subcategory of \( \mathcal{B} \). Assume \( d = m = n + 1 \). Fix any \( X \in \mathcal{C} \) (resp. \( Y \in \mathcal{C} \)).

(1) There exists an exact sequence \( \mathcal{A} : 0 \to Y \xrightarrow{f_0} C_n \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} C_0 \xrightarrow{f_0} X \) with terms in \( \mathcal{C} \) such that \( f_i \in J \mathcal{C} \) and the following sequences are exact.

\[
0 \to \mathcal{C}(, \mathcal{C}(, C_{n-1}) \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} \mathcal{C}(, C_0) \xrightarrow{f_0} \mathcal{C}(, X) \to 0
\]

\[
0 \to \mathcal{C}(X, ) \xrightarrow{f_0} \mathcal{C}(C_0, ) \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} \mathcal{C}(C_{n-1}, ) \xrightarrow{f_n} \mathcal{C}(Y, ) \to 0
\]

Such \( \mathcal{A} \) is unique up to isomorphisms of complexes, and satisfies \( Y \simeq \nu_{\Lambda} X \) and \( X \simeq \nu_{\Lambda} Y \).

(2) The simple modules \( F := \mathcal{C} / J \mathcal{C}(, X) \) and \( G := \mathcal{C} / J \mathcal{C}(Y, ) \) satisfy \( \text{pd}_\mathcal{C} F = n + 1 = \text{pd}_\mathcal{C} G, \text{R}^1 \alpha F = 0 = \text{R}^1 \alpha G \) \((i \neq n + 1)\), \( F = \text{R}^{n+1} \alpha G \) and \( G = \text{R}^{n+1} \alpha F \).

\[ \text{3.3 Definition} \] Let us introduce certain Auslander-type conditions on selfinjective resolutions, which will play a crucial role in this paper (see 4.2). Let \( \Gamma \) be a noetherian ring and \( 0 \to \Gamma \to I_0 \to I_1 \to \cdots \) a minimal injective resolution of the \( \Gamma \)-module \( \Gamma \). We say that \( \Gamma \) satisfies the \((m, n)\)-condition if \( \text{fd}_\Gamma I_i < m \) holds for any \( i \) \((i < n)\) [I2,4]. We can state many well-known homological conditions in terms of our \((m, n)\)-conditions. For example, the dominant dimension \( \text{dom.dim } \Gamma := \inf \{ i \geq 0 \mid \text{fd}_\Gamma I_i \neq 0 \} \) \([T][H]\) of \( \Gamma \) is the maximal number \( n \) such that \( \Gamma \) satisfies the \((1, n)\)-condition. Moreover, recall that \( \Gamma \) is called \( n \)-Gorenstein if \( \text{fd}_\Gamma I_i \leq i \) holds for any \( i \) \((0 \leq i \leq n)\) \([FGR][B][AR4][C]\). This is equivalent to that \( \Gamma \) satisfies the \((i, i)\)-condition for any \( i \) \((0 \leq i \leq n)\). We notice that our \((m, n)\)-condition itself is not left-right symmetric. We say that \( \Gamma \) satisfies the two-sided \((m, n)\)-condition if \( \Gamma \) and \( \Gamma^{op} \) satisfies the \((m, n)\)-condition.

\[ \text{3.3.1 Proposition} \] Let \( \Gamma \) be an \( R \)-order which is an isolated singularity, and \( 0 \to \Gamma \to I_0 \to I_1 \to \cdots \) a minimal injective resolution in \( \text{CMT} \).

(1) \( \Gamma \) is \( d \)-Gorenstein.

(2) \( \Gamma \) satisfies the \((m, n)\)-condition if and only if \( \text{pd}_\Gamma I_i < m - d \) for any \( i \) \((i < n - d)\).

(3) If \( I \in \text{add}(D_{d}\Gamma) \) satisfies \( \text{pd}_\Gamma I \leq n \), then \( I \in \text{add}(\oplus_{i=0}^{d} I_i) \) holds.

**Proof** (1) and (2) follow by 3.1.2 since \( \text{fd}_\Gamma (M \otimes_R E_i) = i \) \((i < d)\) and \( \text{fd}_\Gamma (I_i \otimes_R E_d) = \text{pd}_\Gamma I_i + d \) \((i \geq 0)\) hold by [GN1]. Miyachi’s theorem [Mi] implies (3).\( \square \)

\[ \text{3.4} \] Let us introduce \textit{m-extension pairs}, which will be used in 4.4. As we will see in 3.4.3, they are closely related to \( m \)-cotilting modules.
3.4.1 Proposition Let $\Gamma$ be a module-finite $R$-algebra, and $e$ and $f$ idempotents of $\Gamma$ such that $\Gamma f \in \text{CM} \Gamma$ and $e \Gamma \in \text{CM} \Gamma^{op}$. Put $P := \Gamma f$ and $I := D_d(e \Gamma)$. The conditions (1) and (2) below are equivalent.

(1) Put $\underline{\Gamma} := \Gamma / \Gamma e \Gamma$ and $\overline{\Gamma} := \Gamma / \Gamma f \Gamma$. For any $i \geq 0$, $\text{Ext}^i_{\Gamma}(\cdot, \Gamma)$ gives functors $\text{mod}_{\underline{\Gamma}} \to \text{mod}_{\Gamma^{op}}$ and $\text{mod}_{\overline{\Gamma}} \to \text{mod}_{\Gamma}$.

(2) There exist exact sequences $0 \to P \to I_0 \to I_1 \to \cdots$ and $\cdots \to P_1 \to P_0 \to I \to 0$ with $I_i \in \text{add} I$ and $P_i \in \text{add} P$.

If the conditions above and $\text{id}_\Gamma P \leq m$ and $\text{id}_\Gamma(D_d I) \leq m$ are satisfied, we call $(P, I)$ an $m$-extension pair. Then we can assume $I_{m-d+1} = 0 = P_{m-d+1}$ in (2).

PROOF Notice that $Y \in \text{mod} \Gamma^{op}$ is contained in $\text{mod} \Gamma^{op}$ if and only if $Y \otimes \Gamma P = 0$.

(1)$\Rightarrow$(2) Let $\cdots \to P'_1 \to P'_0 \to D_d P \to 0$ be a minimal projective resolution. By 3.1.2, we have a minimal injective resolution $0 \to P \to P \otimes_R E_0 \to \cdots \to P \otimes_R E_{d-1} \to D P'_0 \to D P'_1 \to \cdots$. Fix any simple $S \in \text{mod}_{\underline{\Gamma}}$. Since $\text{Ext}^i_{\Gamma}(S, \Gamma) \in \text{mod} \Gamma^{op}$ holds for any $i$, we obtain $D(P'_i \otimes_{\Gamma} S) = r(S, D P'_i) = \text{Ext}^{i+d}_{\Gamma}(S, P) = \text{Ext}^{i+d}_{\Gamma}(S, \Gamma) \otimes_{\Gamma} P = 0$ by $\text{Ext}^{i+d}_{\Gamma}(S, \Gamma) \in \text{mod} \Gamma^{op}$. Thus $P'_i \otimes_{\Gamma} S = 0$ holds, and $P'_i \in \text{add}(e \Gamma)_{\Gamma}$ for any $i$.

(2)$\Rightarrow$(1) By 3.4.2 below, $\text{Ext}^2_{\Gamma}(X, I_i) = 0$ holds for any $X \in \text{mod} \Gamma$ and $i, j \geq 0$. Since we have an exact sequence $0 \to P \to I_0 \to I_1 \to \cdots$, we have $\text{Ext}^2_{\Gamma}(X, \Gamma) \otimes_{\Gamma} P = \text{Ext}^2_{\Gamma}(X, P) = 0$ for any $j$. Thus $\text{Ext}^2_{\Gamma}(X, \Gamma) \in \text{mod} \Gamma^{op}$.

3.4.2 Lemma Let $\Gamma$ be a module-finite $R$-algebra, and $e$ an idempotent of $\Gamma$ such that $e \Gamma \in \text{CM} \Gamma^{op}$. Put $\underline{\Gamma} := \Gamma / \Gamma e \Gamma$ and $I := D_d(e \Gamma)$. Then $\text{Ext}^i_{\Gamma}(X, I \otimes_R Y) = 0$ holds for any $i \geq 0$, $X \in \text{mod} \underline{\Gamma}$ and $Y \in \text{Mod} R$.

PROOF Put $Q := \Gamma e$ and $\mathbb{Q} := \text{Hom}_{\Gamma}(Q, X)$. We have a functorial isomorphism $r(\cdot, I \otimes_R Y) = r(Q \otimes_R Y, Y) = r(Q, (\cdot), Y)$. Let $A : \cdots \to P_1 \to P_0 \to X \to 0$ be a projective resolution of $X \in \text{mod} \underline{\Gamma}$. We have an exact sequence $\mathbb{Q} A : \cdots \to \mathbb{Q} P_1 \to \mathbb{Q} P_0 \to 0$. Since $\mathbb{Q} P_i \in \text{CM} R$ holds for any $i$, the complex $\mathbb{Q} A$ splits as a complex of $R$-modules. Thus we obtain an exact sequence $r(\mathbb{Q} A, Y) : 0 \to r(\mathbb{Q} P_0, Y) \to r(\mathbb{Q} P_1, Y) \to \cdots$. By the remark above, $r(A, I \otimes_R Y) : 0 \to r(P_0, I \otimes_R Y) \to r(P_1, I \otimes_R Y) \to \cdots$ is exact. Thus $\text{Ext}^i_{\Gamma}(X, I \otimes_R Y) = 0$.

3.4.3 Proposition (1) Let $\Lambda$ be an $R$-order, $T$ an $m$-cotilting $\Lambda$-module and $\Lambda \oplus T \in \text{add}_\Lambda M \subseteq \Lambda T$. Put $\Gamma := \text{End}_{\Lambda}(M)$, $P := \text{Hom}_{\Lambda}(M, T)$ and $I := D_d M$. Then $(P, I)$ is an $m$-extension pair of $\Gamma$-modules.

(2) Let $\Gamma$ be an module-finite $R$-algebra and $(P, I)$ an $m$-extension pair. Put $Q := \nu_{\Gamma} I$, $\Lambda := \text{End}_{\Gamma}(Q)$, $M := \text{Hom}_{\Gamma}(Q, \Gamma) = D_d I$ and $T := \text{Hom}_{\Gamma}(Q, P)$. Then $\Lambda$ is an $R$-order, $T$ is an $m$-cotilting $\Lambda$-module and $\Lambda \oplus T \in \text{add}_\Lambda M \subseteq \Lambda T$.

PROOF (1) Take exact sequences $I : 0 \to T \to I_0 \to \cdots \to I_{m-d} \to 0$ and $T : 0 \to T_{m-d} \to \cdots \to T_0 \to D_d \Lambda \to 0$ in 3.2. Since $M \in \Lambda T$, we obtain exact sequences $\Lambda(M, I) : 0 \to P \to \Lambda(M, I_0) \to \cdots \to \Lambda(M, I_{m-d}) \to 0$ with $\Lambda(M, I_i) \in \text{add}_I I$ and $\Lambda(M, T) : 0 \to \Lambda(M, T_{m-d}) \to \cdots \to \Lambda(M, T_0) \to I \to 0$ with $\Lambda(M, T_i) \in \text{add}_P P$. Thus $(P, I)$ is an $m$-extension pair.

(2) By our assumption, $M = D_d I \in \text{CM} R$ holds and $\Lambda$ is an $R$-order. Put $Q := \text{Hom}_{\Gamma}(Q, )$. Then $Q I = D_d \Lambda$ and $\Lambda(M, Q I) = D_d M = I$ hold. Take exact sequences $I : 0 \to P \to I_0 \to \cdots \to I_{m-d} \to 0$ and $P : 0 \to P_{m-d} \to \cdots \to P_0 \to I \to 0$ with $I_i \in \text{add}_I I$.
and $P_i \in \text{add}_\Gamma P$. We have an exact sequence $\mathbb{Q}I : 0 \to T \to \mathbb{Q}I_0 \to \cdots \to \mathbb{Q}I_{m-d} \to 0$ with $\mathbb{Q}I_i \in \text{add}_\Lambda(D_d \mathbb{A})$, which gives an injective resolution of $T$. Thus $\text{id}_\Lambda T \leq m$ holds. Since $\Lambda(M, \mathbb{Q}I)$ is isomorphic to the exact sequence $I$ by the remark above, $M \perp T$ holds. On the other hand, we have an exact sequence $\mathbb{Q}P : 0 \to \mathbb{Q}P_{m-d} \to \cdots \to \mathbb{Q}P_0 \to D_d \mathbb{A} \to 0$ with $\mathbb{Q}P_i \in \text{add}_\Lambda T$. Thus $T$ is an $m$-cotilting $\Lambda$-module. Since $Q \oplus P \in \text{add}_\Gamma \Gamma$, we obtain $\Lambda \oplus T \in \text{add}_\Lambda M$.

3.5 Let us introduce $n$-superprojective modules, which will be used in 4.4.

3.5.1 Proposition Let $\Gamma$ be a module-finite $R$-algebra, and $e$ be an idempotent of $\Gamma$ such that $e \Gamma \subseteq \text{CM} \Gamma^{\text{op}}$. Put $Q := \Gamma e$, $I := D_d(e \Gamma)$ and $\Gamma := \Gamma / e \Gamma$. For $n \geq 1$, the conditions (1)-(3) below are equivalent.

1. grade $\Gamma, X \geq n + 1$ holds for any $X \in \text{mod} \Gamma$.
2. There exists an exact sequence $0 \to \Gamma \to I_0 \to \cdots \to I_n$ with $I_i \in \text{add}_\Gamma I$.
3. Put $\Lambda := \text{End}_\Gamma(Q)$. Then the functor $\mathbb{Q} := \text{Hom}_\Gamma(Q, ) : \text{add}_\Gamma \Gamma \to \text{mod} \Lambda$ is full and faithful and $\mathbb{Q} \Gamma \in \text{mod} \Lambda$ is $(n-1)$-orthogonal.

If the conditions above are satisfied, we call $Q$ $n$-superprojective. Moreover, if $n \geq d$ and $\Gamma$ is an isolated singularity, then the condition (4) below is also equivalent.

4. $\Gamma$ is an $R$-order with an injective resolution $0 \to \Gamma \to I_0 \to \cdots \to I_{n-d}$ in $\text{CM} \Gamma$ with $I_i \in \text{add}_\Gamma I$.

3.5.2 For the proof, we need the following easy lemma.

Lemma Let $P : P_{n+1} \to P_n \to \cdots \to P_0 \to 0$ be a complex with $P_i \in \text{add}_\Gamma \Gamma$, and $H_i$ the homology of $P$ at $P_i$. If grade $H_i > n - i$ holds for any $i$ $(0 \leq i \leq n)$, then $P^* : 0 \to P_0^* \to \cdots \to P_i^* \to P_{n+1}^*$ is exact for $(\ast)^* = \text{Hom}_\Gamma(\ast, \Gamma)$.

3.5.3 Proof of 3.5.1 By our assumption, $\Lambda$ is an $R$-order and $Q$ gives a functor $\mathbb{Q} = \text{Hom}_\Gamma(Q, ) : \text{add}_\Gamma \Gamma \to \text{CM} \Lambda$.

2) $\Rightarrow$ (1) By 3.4.2, $\text{Ext}^j_\Gamma(X, I_i) = 0$ holds for any $i, j \geq 0$. Since we have an exact sequence $0 \to \Gamma \to I_0 \to \cdots \to I_n$, $\text{Ext}^j_\Gamma(X, \Gamma) = 0$ holds for any $j$ $(j \leq n)$.

1) $\Rightarrow$ (3) Let $Q_n \to Q_0 \to \cdots$ be a right $(\text{add}_\Gamma \Gamma)$-resolution of $\Gamma$ and $H_i$ the homology of $Q$ at $Q_i$. Since $H_i = 0$ holds, we obtain grade $H_i > n$ for any $i$. By 3.5.2, $Q^* : 0 \to \Gamma \to Q_0^* \to \cdots \to Q_n^*$ is exact. On the other hand, we have a projective resolution $\mathbb{Q}Q : QQ_n \to QQ_0 \to \cdots \to QQ_0 \to QG \to 0$ of $QG \in \text{mod} \Lambda$. Thus we have an exact sequence $\Lambda(QQ, QG) : 0 \to \Lambda(QG, QG) \to \cdots \to \Lambda(QQ_n, QG)$ with a homology $\text{Ext}^i_\Lambda(QG, QG)$ at $\Lambda(QQ_i, QG)$ for any $i > 0$. Since we have the following commutative diagram of complexes, $Q$ is full and faithful and $QG$ is $(n-1)$-orthogonal.

\[
\begin{array}{cccccc}
Q^* & 0 & \Gamma & \cdots & Q_0^* & Q_1^* & \cdots & Q_n^* \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Lambda(QQ, QG) & 0 & \Lambda(QG, QG) & \cdots & \Lambda(QQ_0, QG) & \Lambda(QQ_1, QG) & \cdots & \Lambda(QQ_n, QG)
\end{array}
\]

(3) $\Rightarrow$ (2) Since $\Lambda$ is an $R$-order, we can take an injective resolution $A : 0 \to QG \to I_0' \to \cdots \to I_n'$ in $\text{CM} \Lambda$. Since $QG$ is $(n-1)$-orthogonal, $\Lambda(QG, A)$ is exact with $\Lambda(QG, I_i') \in \text{add}_\Gamma (D_d QG) = \text{add}_\Gamma I$. Thus $\Lambda(QG, A)$ gives the desired sequence.

We will show the assertion for (4). Obviously (2) $\Rightarrow$ (4) holds. We will show (4) $\Rightarrow$ (1). Let $E : 0 \to R \to E_0 \to E_1 \to \cdots \to E_d$ be a minimal injective resolution of the $R$-module
$R$ and $F_i := \Cok f_{i-1}$ for $i \leq d$. By 3.4.2, $\Ext^I_1(X, I \otimes_R F_i) = 0$ holds for any $j \geq 0$. Since we have an exact sequence $0 \to \Gamma \otimes_R F_i \to I_0 \otimes_R F_i \to \cdots \to I_{n-d} \otimes_R F_i$ with $I_i \in \add R I$, Ext$^I_1(X, \Gamma \otimes_R F_i) = 0$ holds for any $j$ ($j \leq n - d$). Since the $i$-th cosyzygy of $\Gamma$ in mod $\Gamma$ is $\Gamma \otimes_R F_i$ by 3.1.2, we obtain $\Ext^{i+j}_1(X, \Gamma) = \Ext^I_1(X, \Gamma \otimes_R F_i) = 0$ for any $i$ ($i \leq d$) and $j$ ($j \leq n - d$). Thus grade $\Gamma X > n$ holds.

4 Higher dimensional Auslander algebras
Throughout this section, fix a complete regular local ring $R$ of dimension $d \geq 0$.

4.1 Definition Let $m \geq d$ and $n \geq 1$. An Auslander (resp. quasi-Auslander) triple of type $(d, m, n)$ is a triple $(\Lambda, M, T)$ which satisfies (1)–(3) (resp. (1)(2) and (3)′) below.

(1) $\Lambda$ is an $R$-order which is an isolated singularity, and $T, M \in \mod \Lambda$.
(2) $T$ is an $m$-cotilting $\Lambda$-module.
(3) $\add \Lambda M$ is a maximal $(n - 1)$-orthogonal subcategory of $\perp T$.
(3)′ $\add \Lambda M$ is an $(n - 1)$-orthogonal subcategory of $\perp T$ and contains $\Lambda$ and $T$.

We call an $R$-algebra $\Gamma$ an Auslander (resp. quasi-Auslander) algebra of type $(d, m, n)$ if there exists an Auslander (resp. quasi-Auslander) triple $(\Lambda, M, T)$ of type $(d, m, n)$ such that $\Gamma = \End_\Lambda(M)$.

4.1.1 (1) We will consider triples $(\Lambda, M_1, M_2)$ of a noetherian ring $\Lambda$ and $M_i \in \mod \Lambda$. We say that two triples $(\Lambda^i, M^i_1, M^i_2)$ $(i = 1, 2)$ are equivalent if there exists an equivalence $\mod \Lambda^1 \to \mod \Lambda^2$ which induces equivalences $\add \Lambda^1 M^1_j \to \add \Lambda^2 M^2_j$ for $j = 1, 2$.

(2) For $m \geq d$ and $n \geq 1$, we denote by $\mathfrak{A}_{m,n}$ (resp. $\mathfrak{A}^q_{m,n}$) the set of equivalence classes of Auslander (resp. quasi-Auslander) triples of type $(d, m, n)$. Then $\mathfrak{A}^q_{m,n} \supseteq \mathfrak{A}_{m,n} \supseteq \mathfrak{A}_{m',n}$ and $\mathfrak{A}^q_{m,n} \supseteq \mathfrak{A}^q_{m',n'}$ hold for any $m \geq m'$ and $n \leq n'$. For any element of $\mathfrak{A}_{m,n}$ (resp. $\mathfrak{A}^q_{m,n}$), its associated Auslander (resp. quasi-Auslander) algebra is uniquely determined up to Morita-equivalence.

4.2 Main Results We collect our main results which will be proved in 4.6.

4.2.1 For the case $m \leq n$, we can give the homological characterization of (quasi-) Auslander algebras of type $(d, m, n)$ below by using Auslander-type condition in 3.3. The case $(d, m, n) = (0, 0, 1)$ is given by Auslander [A1] and Auslander-Solberg [ASo], the case $(d, m, n) = (1, 1, 1)$ is given by Auslander-Roggenkamp [ARo], and the case $(d, m, n) = (0, 1, 1)$ is given by the author [I5].

Theorem Let $\Gamma$ be an $R$-algebra. If $m \leq n$, then $\Gamma$ is an Auslander (resp. quasi-Auslander) algebra of type $(d, m, n)$ if and only if $\Gamma$ is an $R$-order which is an isolated singularity and satisfies the two-sided $(m + 1, n + 1)$-condition and $\text{gl.dim } \Gamma \leq n + 1$ (resp. the two-sided $(m + 1, n + 1)$-condition).

4.2.2 A more explicit result is given by Auslander correspondence below for the case $m \leq n$. A more general result for arbitrary case will be given in 4.4.1.

Theorem (Auslander correspondence of type $(d, m, n)$) Assume $m \leq n$. Then the map $(\Lambda, M, T) \mapsto \End_\Lambda(M)$ gives a bijection from $\mathfrak{A}_{m,n}$ (resp. $\mathfrak{A}^q_{m,n}$) to the set of Morita-equivalence classes of $R$-orders $\Gamma$ which are isolated singularities and satisfy the two-sided $(m + 1, n + 1)$-condition and $\text{gl.dim } \Gamma \leq n + 1$ (resp. the two-sided $(m + 1, n + 1)$-condition). In particular, two triples $(\Lambda_i, M_i, T_i) \in \mathfrak{A}^q_{m,n}$ ($i = 1, 2$) are equivalent if and only if two
categories add$_\Lambda M_i$ ($i = 1, 2$) are equivalent.

4.2.3 Let us study the case $d = m > n$. Then add$_\Lambda T = add_\Lambda (D_d\Lambda)$ and $^d T = CM \Lambda$ hold by 3.2(2). In this case, we can give the homological characterization of Auslander algebras below. In particular, putting $n := 1$, we obtain an answer to M. Artin’s question [Ar] to give a homological characterization of endomorphism rings $\text{End}_\Lambda (M)$ of additive generators $M$ of $CM \Lambda$ for representation-finite orders $\Lambda$.

**Theorem** Let $\Gamma$ be an $R$-algebra. If $d > n$, then $\Gamma$ is an Auslander algebra of type $(d, d, n)$ if and only if the conditions (1) and (2) below hold.

1. $\Gamma$ is a module-finite $R$-algebra which is an isolated singularity with $\text{gl.dim} \Gamma = d$ and depth$_R \Gamma \geq n + 1$.
2. There exists an idempotent $e$ of $\Gamma$ such that $e\Gamma \in \text{CM} \Gamma$, $\Gamma : = \Gamma / e\Gamma$ is artinian, and $\text{pd}_\Gamma X \leq n + 1$ holds for any $X \in \text{mod} \Gamma$.

4.2.4 For the arbitrary case, we can give a homological characterization of Auslander algebras in the theorem below. These conditions strongly reflect properties of maximal $(n - 1)$-orthogonal subcategories studied in §2.

**Theorem** Let $\Gamma$ be an $R$-algebra, $m \geq d$ and $n \geq 1$. Then $\Gamma$ is an Auslander algebra of type $(d, m, n)$ if and only if the conditions (1)–(4) below hold.

1. $\Gamma$ is a module-finite $R$-algebra which is an isolated singularity with $\text{gl.dim} \Gamma \leq \max\{n + 1, m\}$.
2. There exist idempotents $e$ and $f$ of $\Gamma$ such that $e\Gamma \in \text{CM} \Gamma$, $\text{id}(e\Gamma) \leq m$, $\Gamma f \in \text{CM} \Gamma$ and $\text{id}_\Gamma (\Gamma f) \leq m$.
3. Put $\Gamma := \Gamma / e\Gamma$ and $\Gamma := \Gamma / \Gamma f \Gamma$. Any $0 \neq X \in \text{mod} \Gamma$ satisfies $\text{pd}_\Gamma X = \text{grade}_\Gamma X = n + 1$, and any $0 \neq Y \in \text{mod} \Gamma^{op}$ satisfies $\text{pd}_\Gamma Y = \text{grade}_\Gamma Y = n + 1$.
4. $\text{Ext}^{2n}_{\Gamma + 1} (\cdot, \Gamma)$ gives a duality $\text{mod} \Gamma \leftrightarrow \text{mod} \Gamma^{op}$.

4.2.5 Let us start with collecting properties of (quasi-)Auslander algebras.

1. If $(\Lambda, M, T) \in \mathfrak{a}_{m, n}$ (resp. $\mathfrak{a}_{m, n}^q$), then $(\text{End}_\Lambda (T)^{op}, \text{Hom}_\Lambda (M, T), T) \in \mathfrak{a}_{m, n}$ (resp. $\mathfrak{a}_{m, n}^q$) and End$_{\text{End}_\Lambda (T)^{op}} (\text{Hom}_\Lambda (M, T)) = \text{End}_\Lambda (M)^{op}$ hold by 3.2.1. Consequently, $\Gamma$ is an Auslander (resp. quasi-Auslander) algebra of type $(d, m, n)$ if and only if so is $\Gamma^{op}$.

2. Assume $m := \text{gl.dim} \Lambda < \infty$. Then $\Lambda$ is an $m$-cotilting $\Lambda$-module with $^d \Lambda = \text{add} \Lambda$. Since $(\Lambda, \Lambda, \Lambda) \in \mathfrak{a}_{m, n}$ holds for any $n \geq 1$, $\Lambda$ is an Auslander algebra of type $(d, m, n)$. We call the equivalence class of such a triple *trivial*.

4.3.1 **Proposition** Let $(\Lambda, M, T) \in \mathfrak{a}_{m, n}^q$ and $\Gamma := \text{End}_\Lambda (M)$. Then (1)–(6) below hold. If $(\Lambda, M, T) \in \mathfrak{a}_{m, n}$, then (7)–(9) below hold.

1. $\Gamma$ is a module-finite $R$-algebra which is an isolated singularity.
2. We have mutually inverse equivalences $M := \text{Hom}_\Lambda (M, ) : \text{add} \Lambda M \rightarrow \text{add} \Gamma$ and $Q := \text{Hom}_\Gamma (Q, ) : \text{add} \Gamma \rightarrow \text{add} \Lambda M$ for $Q := \text{add} \Lambda$.
3. $M \in \text{CM} \Gamma^{op} \cap \text{add} \Gamma$ and $P := \text{Hom}_\Lambda (M, T) \in \text{CM} \Gamma \cap \text{add} \Gamma$. For $I := D_d \Lambda$, $(P, I)$ is an $m$-extension pair (3.4.1) and $Q := \nu_I (\text{add} \Lambda)$ is $n$-superprojective (3.5.1).
4. A left (add$_\Gamma P$)-resolution $0 \rightarrow \Gamma \rightarrow P_0 \rightarrow \cdots \rightarrow P_n$ of $\Gamma$ and a left (add$(D_d I)_\Gamma$)-resolution $0 \rightarrow \Gamma \rightarrow P'_0 \rightarrow \cdots \rightarrow P'_n$ of $\Gamma$ are exact.
5. $\Gamma$ satisfies depth$_R \Gamma \geq \min\{n + 1, d\}$ and the two-sided $(m + 1, n + 1)$-condition.
6. Take idempotents $e$ and $f$ of $\Gamma$ such that add$_\Gamma I = \text{add}_\Gamma D_d (e\Gamma)$ and add$_\Gamma P = \text{add}_\Gamma D_d (f\Gamma)$.
add \(f(\Gamma T)\). Then \(\Gamma := \Gamma/\Gamma e\Gamma\) and \(\Gamma := \Gamma/\Gamma f\Gamma\) are artin algebras with the following commutative diagram for \(C := \text{add} M, \ Q := C/[\text{add} \Lambda] \) and \(\overline{C} := C/[\text{add} T]\).

\[
\begin{array}{cccc}
\text{mod} \Gamma & \subset & \text{mod} \Gamma & \overset{\text{Ext}^1_i(\Gamma)}{\leftrightarrow} \text{mod} \Gamma^\text{op} & \supset & \text{mod} \Gamma^\text{op} \\
\uparrow i & & \uparrow i & & \uparrow i & & \uparrow i \\
\text{mod} \overline{C} & \subset & \text{mod} \overline{C} & \overset{\text{R}^i\text{op}}{\leftrightarrow} \text{mod} \overline{C}^\text{op} & \supset & \text{mod} \overline{C}^\text{op}
\end{array}
\]

(7) \(d \leq \text{gl.dim} \Gamma \leq \max\{n + 1, \text{id}_A T\}\) holds, and the right equality holds if \((\Lambda, M, T)\) is non-trivial \((4.3(2))\).

(8) Any \(0 \neq X \in \text{mod} \Gamma\) satisfies \(\text{pd} \Gamma X = \text{grade} \Gamma X = n + 1\), and any \(0 \neq Y \in \text{mod} \Gamma^\text{op}\) satisfies \(\text{pd} \Gamma Y = \text{grade} \Gamma Y = n + 1\).

(9) \(\text{Ext}^i_{\Gamma + 1}(\cdot, \Gamma)\) gives a duality \(\text{mod} \Gamma \leftrightarrow \text{mod} \Gamma^\text{op}\).

**Proof** (1) Obviously \(\Gamma\) is a finitely generated \(R\)-module. For any non-maximal prime ideal \(p\) of \(R, \ M_p\) is a progenerator of \(\Lambda_p\) (e.g. [R;3.5]). Thus \(\Gamma_p = \text{End}_{\Lambda_p}(M_p)\) is Morita-equivalent to \(\Lambda_p\). This implies that \(\Gamma\) is an isolated singularity.

(2) Obviously \(\mathbb{M}\) is an equivalence. Moreover, \(\mathcal{Q} \circ \mathbb{M} = \Gamma(Q, \Lambda(M, )) = \Lambda(M \otimes \Gamma Q, ) = \Lambda(M \otimes \Gamma \text{Hom}_\Lambda(M, \Lambda), ) = \Lambda(\Lambda, ) = 1\) holds.

(3) \((P, I)\) is an \(m\)-extension pair by 3.4.3(1), and \(Q\) is \(n\)-superprojective by 3.5.1(3).

(4) Take exact sequences \(T : 0 \to M \to T_0 \to \cdots \to T_n \) and \(P : P_n \to \cdots \to P_0 \to M \to 0\) with \(T_i \in \text{add} T\) and \(P_i \in \text{add} \Lambda\). Then \(\Lambda(M, P)\) and \(\Lambda(P, M)\) are exact by \(M \perp_{n-1} M\). They are left \((\text{add} \Gamma P)\) and \((\text{add} (D_{\phi} I) T)\)-resolutions by (2).

(5) The former assertion follows by (4). Take an exact sequence \(0 \to \Gamma \to I_0 \to \cdots \to I_n\) with \(I_i \in \text{add} I\) in 3.5.1(2). By 3.1.2, \(0 \to I \to I \otimes_R E_0 \to \cdots \to I \otimes_R E_d \to 0\) gives a minimal injective resolution of \(I\). Since \(\text{pd} \Gamma I \leq m - d\) holds, \(\text{fd} \Gamma I \otimes_R E_i \leq m\) holds. The mapping cone gives an injective resolution of \(\Gamma\), which shows that \(\Gamma\) satisfies the \((m + 1, n + 1)\)-condition. By 4.3(1), \(\Gamma^\text{op}\) satisfies the \((m + 1, n + 1)\)-condition.

(6) We have an equivalence \(\text{mod} \mathcal{C} \to \text{mod} \Gamma\) given by \(F \mapsto F(M)\), which makes our diagram commutative. Since \(\Lambda\) is an isolated singularity, \(\overline{\Gamma}\) and \(\overline{\Gamma}\) are artin algebras.

(7) Since \(\text{gl.dim} \Lambda \geq d\) holds by [R;3.2], the former assertion follows by 2.6.1. We will show the latter one. Since \((\Lambda, M, T)\) is non-trivial, there exists non-projective \(X \in \text{ind} \mathcal{C}\). Then the \(\Gamma\)-module \(F := C/J_C(M, X)\) satisfies \(\text{pd} \Gamma F = n + 1\) by 2.5.2. Let \(0 \to C_i \to \cdots \to C_1 \to J_\Lambda \to 0\) be a minimal right \(\mathcal{C}\)-resolution of \(J_\Lambda\) with \(C_i \neq 0\). Then the \(\Gamma\)-module \(G := C/J_C(M, \Lambda)\) satisfies \(\text{pd} \Gamma G = l\). Since we have an exact sequence \(0 \to C_i \to \cdots \to C_1 \to \Lambda \to \Lambda/J_\Lambda \to 0\) with \(C_i \oplus \Lambda \perp T\), we have \(l \geq d\) and \(\text{Ext}^l_{\Gamma + 1}(\Lambda/J_\Lambda, T) = 0\). By 3.1.2, \(\text{id}_\Lambda T \leq l\) holds. Thus \(\text{gl.dim} \Gamma \geq \max\{n + 1, \text{id}_\Lambda T\}\).

(8) follows by 2.5.2(1), and (9) follows by 2.5.2(2).

**4.4 Definition** To prove the theorems in 4.2, it is convenient to introduce \(\mathfrak{B}_{m,n}\) as follows. We denote by \(\mathfrak{B}_{m,n}\) (resp. \(\mathfrak{B}_{m,n}^\#\)) the set of equivalence classes \((A, P, I)\) of triples \((\Gamma, P, I)\) which satisfies the conditions (1)–(3) (resp. (1) and (2)) below.

(1) \(\Gamma\) is a module-finite \(R\)-algebra which is an isolated singularity.

(2) \((P, I)\) is an \(m\)-extension pair \((3.4.1)\) and \(Q := \nu_I\) is \(n\)-superprojective \((3.5.1)\).

(3) \(\text{gl.dim} \Gamma \leq \max\{n + 1, m\}\) and any \(X \in \text{mod} \Gamma \) satisfies \(\text{pd} \Gamma X \leq n + 1\).

We will show in 4.4.4 that the sets \(\mathfrak{B}_{m,n}\) and \(\mathfrak{B}_{m,n}^\#\) are 'left-right symmetric'.
4.4.1 Theorem (Auslander correspondence of type \((d, m, n)\))

(1) There exists a bijection \(\mathcal{A}_{m,n}^d \to \mathfrak{B}_{m,n}^d\) for any \(m \geq d\) and \(n \geq 1\). It is given by \(\alpha(\Lambda, M, T) := (\text{End}_\Lambda(M), \text{Hom}_\Lambda(M, T), D_dM)\), and the converse is given by \(\alpha^{-1}(\Gamma, P, I) := (\text{End}_\Gamma(Q), \text{Hom}_\Gamma(Q, \Gamma), \text{Hom}_\Gamma(Q, P))\) for \(Q := \nu_\Gamma I\).

(2) \(\alpha\) gives a bijection \(\mathcal{A}_{m,n}^d \to \mathfrak{B}_{m,n}^d\) for any \(m \geq d\) and \(n \geq 1\).

(3) \(\Gamma\) is an Auslander (resp. quasi-Auslander) algebra of type \((d, m, n)\) if and only if \((\Gamma, P, I) \in \mathfrak{B}_{m,n}^d\) (resp. \(\mathfrak{B}_{m,n}^q\)) holds for some \((P, I)\).

\[
\begin{array}{cccc}
\Lambda, & M, & T, & D_dA \\
\downarrow & \downarrow & \uparrow & \uparrow \\
Q, & \Gamma, & P, & I
\end{array}
\]

\[
\text{Hom}_\Lambda(M, ) \quad \text{Hom}_\Gamma(Q, )
\]

4.4.2 Lemma (1) For any \((\Lambda, M, T) \in \mathfrak{B}_{m,n}^d\), put \(\Gamma := \text{End}_\Lambda(M), P := \text{Hom}_\Lambda(M, T), I := D_dM\) and \(Q := \text{Hom}_\Lambda(M, \Lambda)\). Then \((\Gamma, P, I) \in \mathfrak{B}_{m,n}^d\). Moreover, \(\Lambda = \text{End}_\Gamma(Q), M = \text{Hom}_\Lambda(M, \Gamma)\) and \(T = \text{Hom}_\Gamma(Q, P)\) hold.

(2) For any \((\Gamma, P, I) \in \mathfrak{B}_{m,n}^d\), put \(Q := \nu_\Gamma I \in \text{add}_\Gamma\Lambda, \Lambda := \text{End}_\Gamma(Q), M := \text{Hom}_\Gamma(Q, \Gamma) = D_dI\) and \(T := \text{Hom}_\Gamma(Q, P)\). Then \((\Lambda, M, T) \in \mathfrak{B}_{m,n}^d\). Moreover, \(\Gamma = \text{End}_\Lambda(M), P = \text{Hom}_\Lambda(M, T)\) and \(I = \text{Hom}_\Lambda(M, D_d\Lambda)\) hold.

Proof (1) \((P, I)\) is an \(m\)-extension pair by 3.4.3(1). The latter assertion follows by 4.3.1(2). Thus \(Q : \text{add}_\Gamma\Lambda \to \text{mod}\Lambda\) is full and faithful and \(M = Q\Gamma\) is \((n - 1)\)-orthogonal. Hence \(Q\) is \(n\)-superprojective, and \((\Gamma, P, I) \in \mathfrak{B}_{m,n}^d\) holds.

(2) By 3.5.1(3), \(Q : \text{add}_\Gamma\Lambda \to \text{mod}\Lambda\) is full and faithful and \(M = Q\Gamma\) is \((n - 1)\)-orthogonal. Thus the former assertion holds by 3.4.3(2). Since \(\Lambda(M, Q( )) = \Lambda(Q\Gamma, Q( )) = r(\Gamma, \ ) = 1\) holds on add\(r\Gamma\), the latter assertion follows.

4.4.3 Proof of 4.4.1 (1) follows by 4.4.2. We will show (2). Fix \((\Lambda, M, T) \in \mathfrak{B}_{m,n}^d\) and the corresponding \((\Gamma, P, I) \in \mathfrak{B}_{m,n}^d\). If \((\Lambda, M, T) \in \mathfrak{B}_{m,n}^d\), then 4.4(3) holds by 4.3.1(7)(8), so \((\Gamma, P, I) \in \mathfrak{B}_{m,n}^d\) holds. We will show that \((\Gamma, P, I) \in \mathfrak{B}_{m,n}^d\) implies \((\Lambda, M, T) \in \mathfrak{B}_{m,n}^d\), i.e. \(\mathcal{C} := \text{add}_\Lambda M\) is a maximal \((n - 1)\)-orthogonal subcategory of \(\Gamma\). By 2.4.1, we only have to show that any \(X \in \mathcal{C}^\perp \cap \Gamma^\perp\) satisfies \(X \in \mathcal{C}\). Put \(g := \max\{n + 1, m\}\) and \(M := \Lambda(M, )\). Take an exact sequence \(T : 0 \to X \xrightarrow{f_0} T_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} T_{n-1}\) with \(T_i \in \text{add}_A T\). Applying \(\mathcal{M}\), we obtain a complex \(\mathcal{M}T : 0 \to \mathcal{M}X \to \mathcal{M}T_0 \to \cdots \to \mathcal{M}T_{n-1}\) with \(\mathcal{M}T_i \in \text{add}_A P\). Let \(H_i\) be the homology of \(\mathcal{M}T\) at \(\mathcal{M}T_i\). Since \(r(Q, \mathcal{M}( ))\) is the identity functor, \(r(Q, H_i) = 0\) holds for any \(i\). By 4.4(3), \(\text{pd}_r H_i \leq n + 1\) holds for any \(i\). Put \(X_0 := X\) and \(X_i := \text{Cok}\ f_{i-1}\) for \(i > 0\). Inductively, we will show that \(\text{pd}_r \mathcal{M}X_i \leq i\) holds for any \(i\) \((n - 1 \leq i \leq g)\). This is true for \(i = g\) by \(\text{gl.dim}\ \Gamma \leq g\). Assume that \(\text{pd}_r \mathcal{M}X_i \leq i\) holds for some \(i\) \((n - i \leq g)\). We have an exact sequence \(0 \to \mathcal{M}X_{i-1} \to \mathcal{M}T_{i-1} \to \mathcal{M}X_i \xrightarrow{g_i} H_i \to 0\). Since \(\text{pd}_r \mathcal{M}X_i \leq i\) and \(\text{pd}_r H_i \leq n + 1\) hold, we obtain \(\text{pd}_r \ker g_i \leq i\). Thus \(\mathcal{M}T_{i-1} \in \text{add}_A \Gamma\) implies \(\text{pd}_r \mathcal{M}X_{i-1} \leq i - 1\). In particular, \(\text{pd}_r \mathcal{M}X_{n-1} \leq n - 1\) holds. Since \(M\) is \((n - 1)\)-orthogonal, \(H_i = 0\) for any \(i\) \((0 \leq i < n)\). Thus \(\mathcal{M}X\) is an \((n - 1)\)-st syzygy of \(\mathcal{M}X_{n-1}\), and \(\mathcal{M}X \in \text{add}_A \Gamma\) holds. Thus we obtain \(X = r(Q, \mathcal{M}X) \in \mathcal{C}\).

4.4.4 Corollary Let \(m \geq d\) and \(n \geq 1\). If \((\Gamma, P, I) \in \mathfrak{B}_{m,n}^d\) (resp. \(\mathfrak{B}_{m,n}^q\)), then \((\Gamma^{op}, D_dI, D_dP) \in \mathfrak{B}_{m,n}^d\) (resp. \(\mathfrak{B}_{m,n}^q\)).

Proof Put \((\Lambda, M, T) := \alpha^{-1}(\Gamma, P, I) \in \mathfrak{B}_{m,n}^d\) (resp. \(\mathfrak{B}_{m,n}^q\)) and \(\Lambda' := \text{End}_\Lambda(T)^{op}\). Then \((\Lambda', P, T) = (\text{End}_\Lambda(T)^{op}, \Lambda(M, T), T) \in \mathfrak{A}_{m,n}^d\) (resp. \(\mathfrak{A}_{m,n}^q\)) and \(\text{End}_{\Lambda'}(P) = \Gamma^{op}\) hold by
4.3(1). Since \( \chi'(P, T) = M = D_dI \) holds by 3.2.1, we obtain \( (\Gamma, D_dI, D_dP) = \alpha(\Lambda', P, T) \in \mathfrak{B}_{m,n} \) (resp. \( \mathfrak{B}'_{m,n} \)).

4.5 The following proposition connects 4.2.2 and 4.4.1.

**Proposition** If \( m \leq n \), then the map \( (\Gamma, P, I) \mapsto \Gamma \) gives a bijection from \( \mathfrak{B}_{m,n} \) (resp. \( \mathfrak{B}'_{m,n} \)) to the set of Morita-equivalence classes of \( R \)-orders \( \Gamma \) which are isolated singularities and satisfy the two-sided \( (m+1, n+1) \)-condition and \( \text{gl.dim} \, \Gamma \leq n+1 \) (resp. the two-sided \( (m+1, n+1) \)-condition).

**Proof** We only have to show the assertion for \( \mathfrak{B}'_{m,n} \).

(i) Fix \( (\Gamma, P, I) \in \mathfrak{B}'_{m,n} \). Applying 4.3.1(5) to \( (\Lambda, M, T) := \alpha^{-1}(\Gamma, P, I) \in \mathfrak{B}_{m,n} \), \( \Gamma \) is an \( R \)-order and satisfies the two-sided \( (m+1, n+1) \)-condition.

We will show 4.2.3. The ‘only if’ part follows by 4.3.1. We will show the ‘if’ part. Put \( I := \bigoplus_{i=m}^d I_i \), \( Q := \nu_I^{\Gamma}, J := \bigoplus_{i=m}^d J_i \) and \( P := D_dJ \). By 3.5.1(4), \( Q \) is \( n \)-superprojective. Since \( \Gamma \) satisfies the two-sided \( (m+1, n+1) \)-condition, 3.3.1(2) implies \( \text{pd}_R I \leq m - d \) and \( \text{pd}_R J \leq m - d \). Thus \( (P, I) \) is an \( m \)-extension pair by \( m \leq n \). Thus \( (\Gamma, P, I) \in \mathfrak{B}'_{m,n} \).

We will show that any \( (\Gamma, P', I') \in \mathfrak{B}'_{m,n} \) is equivalent to \( (\Gamma, P, I) \). Since there exists an injective resolution \( 0 \rightarrow \Gamma \rightarrow I_0 \rightarrow \cdots \rightarrow I_{n-d} \) in \( \text{CM} \, \Gamma \) with \( I_i \in \text{add}_R I' \) by 3.5.1(4), \( I \in \text{add}_R I' \) holds. Since \( \text{pd}_R I' \leq m - d \leq n - d \) holds by 3.4.1, \( I' \in \text{add}_R I \) holds by 3.3.1(3). Thus \( \text{add}_R I' = \text{add}_R I \). A dual argument shows \( \text{add}_R P' = \text{add}_R P \).

4.6 Proof of Main results 4.2.2 follows immediately by 4.4.1 and 4.5, and 4.2.1 follows by 4.2.2. We will show 4.2.4. The ‘only if’ part follows by 4.3.1, and ‘if’ part follows by 4.4.1(3) since \( (\Gamma, \Gamma f_d, D_d(e\Gamma)) \in \mathfrak{B}_{m,n} \) holds by 3.4.1(1) and 3.5.1(1).

We will show 4.2.3. The ‘only if’ part follows by 4.3.1. We will show the ‘if’ part. Put \( I := D_d(e\Gamma) \). Since depth \( R I = d \) and \( \text{gl.dim} \, \Gamma = \text{d} \) hold by (2), \( I \in \text{add}_R \Gamma \) holds by 3.1.2. Take an idempotent \( f \) of \( \Gamma \) such that \( \text{add}_R I = \text{add}_R P \) holds for \( P := \Gamma f \). Then \( (P, I) \) is a \( d \)-extension pair by \( \text{id}_R P = d \) and \( \text{id}(D_dI) = d \). Since any \( X \in \text{mod} \, \Gamma \) has finite length by (2), we obtain \( \text{grade}_R X \geq n + 1 \) by \( \text{depth}_R \Gamma \geq n + 1 \) in (1). Thus \( Q \) is \( n \)-superprojective, and \( (\Gamma, P, I) \in \mathfrak{B}_{m,n} \) holds. The assertion follows by 4.4.1(3).

4.7 Recall that higher dimensional Auslander-Reiten theory for the case \( d = m + 1 \) is quite peculiar (3.2.3). Correspondingly, Auslander algebras of type \( (d, d, d - 1) \) have a very nice homological characterizations below. In particular, the condition (3) below means that \( \Gamma \) is a (non-local and non-graded version of) Artin-Schelter regular ring of dimension \( d \) [ArS]. The symmetry of projective resolutions of simple modules over Artin-Schelter regular rings corresponds to the selfduality of \( (d-1) \)-almost split sequences and \( (d-1) \)-fundamental sequences. See 5.2.2 and 6.1 for examples. The equivalence of (1) and (2) for \( d = 2 \) is a theorem of Auslander [Ar][RV].

**Theorem** For a module-finite \( R \)-algebra \( \Gamma \), the conditions below are equivalent.

(1) \( \Gamma \) is an Auslander algebra of type \( (d, d, d - 1) \).

(2) \( \Gamma \) is an \( R \)-order with \( \text{gl.dim} \, \Gamma = d \).

(3) \( \text{gl.dim} \, \Gamma = d \) holds, and any simple \( \Gamma \)-module \( S \) satisfies \( \text{Ext}_R^i(S, \Gamma) = 0 \) \((i \neq d)\) and \( \text{Ext}_R^d(S, \Gamma) \) is a simple \( \Gamma^{op} \)-module.
(4) Opposite side version of (3).

**Proof** (1)⇒(3) Immediate from 4.3.1(7) and 3.2.3.
(3)⇒(2) Immediate from depth$_R \Gamma = \inf\{i \geq 0 \mid \text{Ext}^i_\Gamma(\Gamma/J, \Gamma) \neq 0\}$ [GN2;3.2].
(2)⇒(1) Since CM$\Gamma = \text{add } \Gamma$ holds by 3.1.2, $\Gamma \in \text{CM} \Gamma$ is maximal $(d-1)$-orthogonal.

4.7.1 Let us recall the proposition below [I4;6.3]. An important example of such $\Gamma$ is an Auslander algebra of type $(d, n, n)$, which satisfies (1) and (2) below by 4.2.1 and 2.5.3 respectively. In this sense, the two-sided $(n+1, n+1)$-condition means the existence of $n$-almost split sequences homologically.

**Proposition** For a noetherian ring $\Gamma$ with gl.dim $\Gamma = n+1$, the conditions below are equivalent.

1. $\Gamma$ satisfies the two-sided $(n+1, n+1)$-condition.

2. Any simple $\Gamma$-module (resp. $\Gamma^{op}$-module) $S$ with pd $S = n+1$ satisfies Ext$_\Gamma^i(S, \Gamma) = 0$ $(i \neq n+1)$ and Ext$_\Gamma^{n+1}(S, \Gamma)$ is a simple $\Gamma^{op}$-module (resp. $\Gamma$-module).

4.8 Let us study how to get all quasi-Auslander triples with a fixed quasi-Auslander algebra. For any automorphism $\phi \in \text{Aut}(\Lambda)$, we have the induced auto-equivalence $\phi : \text{mod } \Lambda \to \text{mod } \Lambda$. Then any quasi-Auslander triple $(\Lambda, M, T)$ gives another quasi-Auslander triple $(\Lambda, \phi(M), \phi(T))$ with the same quasi-Auslander algebra. Since $\phi(X)$ is isomorphic to $X$ for any $\phi \in \text{Im}(\Lambda)$ and $X \in \text{mod } \Lambda$, this action of $\text{Aut}(\Lambda)$ factors through $\text{Out}(\Lambda)$. By the theorem below, $\text{Out}(\Lambda)$ is sufficient for our purpose. We call a triple $(\Lambda, M, T)$ basic if all algebras $\Lambda$, $\text{End}_\Lambda(M)$ and $\text{End}_\Lambda(T)$ are basic.

**Theorem** Let $(\Lambda, M_i, T_i)$ be a basic quasi-Auslander triple of type $(d, m, n)$ $(i = 1, 2)$. Assume $m \leq n$. Then $\text{End}_\Lambda(M_1)$ is isomorphic to $\text{End}_\Lambda(M_2)$ if and only if there exists $\phi \in \text{Out}(\Lambda)$ such that $\phi(M_1)$ and $\phi(T_1)$ are isomorphic to $M_2$ and $T_2$ respectively.

**Proof** By 4.2.2 and our definition of $\mathfrak{A}_{m,n}^d$ in 4.1.1, there exists an auto-equivalence $\mathcal{F} : \text{mod } \Lambda \to \text{mod } \Lambda$ which induces equivalences $\text{add } M_1 \to \text{add } M_2$ and $\text{add } T_1 \to \text{add } T_2$.

By Morita theory, there exists a progenerator $P \in \text{mod } \Lambda$ such that $\mathcal{F}$ is isomorphic to $\text{Hom}_\Lambda(P, )$ and $\text{End}_\Lambda(P) \simeq \Lambda$. Since $\Lambda$ is basic, $P$ is isomorphic to $\Lambda$ as a $\Lambda$-module. Thus we have an automorphism $\phi : \Lambda = \text{End}_\Lambda(\Lambda) \to \text{End}_\Lambda(P) \simeq \Lambda$. It is easily checked that $\mathcal{F}$ is isomorphic to $\phi$. Since $M_i$ and $T_i$ are basic, we obtain the assertion.

5 Non-commutative crepant resolution and representation dimension

Throughout this section, fix a complete regular local ring $R$ of dimension $d \geq 0$, an $R$-order $\Lambda$ which is an isolated singularity, and an $m$-cotilting $\Lambda$-module $T$. Put $\mathcal{A} := \text{mod } \Lambda$ and $\mathcal{B} := \perp T$ as in 3.2.2.

5.1 Let us start with studying properties of a $\Lambda$-module $M$ in terms of $\text{End}_\Lambda(M)$.

**Theorem** Let $M \in \mathcal{B}$, $\Gamma := \text{End}_\Lambda(M)$ and $n \geq 1$. Assume $\Lambda \oplus T \in \text{add } M$.

1. Assume $n < d$. Then $M$ is $(n-1)$-orthogonal if and only if depth$_R \Gamma \geq n+1$.

2. Assume that $M$ is $(m-1)$-orthogonal. Then $M$ is $(n-1)$-orthogonal if and only if $\Gamma$ satisfies the two-sided $(m+1, n+1)$-condition.

3. Assume that $M$ is $(n-1)$-orthogonal. If $M \in \mathcal{B}$ is maximal $(n-1)$-orthogonal, then gl.dim $\Gamma \leq \max\{m, n+1\}$ holds, and the converse holds if $m \leq n+1$. 18
Proof (1) follows by 3.1.1. By 4.4, \( \mathfrak{g}_{m,n} \subseteq \{(\Gamma, P, I) \in \mathfrak{g}_{m,n} \mid \text{gl.dim } \Gamma \leq \max\{n+1, m\}\} \) holds, and the equality holds if \( m \leq n+1 \). Thus (3) follows by 4.4.1. We will show (2). The ‘only if’ part follows by 4.3.1(5). To show the ‘if’ part, we can assume \( m \leq n \). For \((\Lambda, M, T) \in \mathfrak{g}_{m,m}\), put \((\Gamma, P, I) := \alpha(\Lambda, M, T) \in \mathfrak{g}_{m,m}\). Since \( \Gamma \) is an \( R \)-order and satisfies the two-sided \((m+1, n+1)\)-condition, 4.5 implies \((\Gamma, P, I) \in \mathfrak{g}_{m,n}\). Thus \((\Lambda, M, T) \in \mathfrak{g}_{m,n}\) holds, and we obtain \( M \perp_{n-1} M \).

5.2 Definition Let us generalize the concept of Van den Bergh’s non-commutative crepant resolution [V1,2] of commutative normal Gorenstein domains to our situation.

Again let \( \Lambda \) be an \( R \)-order which is an isolated singularity. We say that \( M \in \text{CM} \Lambda \) gives a Cohen-Macaulay non-commutative crepant resolution (CM NCCR for short) \( \Gamma := \text{End}_\Lambda(M) \) of \( \Lambda \) if \( \Lambda \oplus D_d \Lambda \in \text{add } M \) and \( \Gamma \) is an \( R \)-order with \( \text{gl.dim } \Gamma = d \).

Our definition is slightly stronger than original non-commutative crepant resolutions in [V2] where \( M \) is assumed to be reflexive (not Cohen-Macaulay) and \( \Lambda \oplus D_d \Lambda \in \text{add } M \) is not assumed. But all examples of non-commutative crepant resolutions in [V1,2] satisfy our condition. For the case \( d \geq 2 \), we have the remarkable relationship below between CM NCCR and maximal \((d-2)\)-orthogonal subcategories. In this case, \( \Gamma \) is an Auslander algebra of type \((d, d, d-1)\) and has remarkable properties (see 3.2.3 and 4.7).

5.2.1 Theorem Let \( d \geq 2 \). Then \( M \in \text{CM} \Lambda \) gives a CM NCCR of \( \Lambda \) if and only if \( M \in \text{CM} \Lambda \) is maximal \((d-2)\)-orthogonal.

Proof \( \Lambda \oplus D_d \Lambda \in \text{add } M \) holds. Put \( m := d \) and \( n := d-1 \) in 5.1(1) and (3).

5.2.2 Example Let \( k \) be a field of characteristic zero, \( G \) a finite subgroup of \( \text{GL}_d(k) \) with \( d \geq 2 \), \( \Omega := k[[x_1, \ldots, x_d]] \) and \( \Lambda := \Omega^G \) the invariant subring. Assume that \( G \) does not contain any pseudo-reflection except the identity, and that \( \Lambda \) is an isolated singularity. In [I7;2.5], it is shown that \( \mathcal{C} := \text{add } \Lambda \Omega \) is a maximal \((d-2)\)-orthogonal subcategory of \( \text{CM} \Lambda \), and \( \text{End}_G(\Lambda \Omega) \) is the skew group ring \( \Lambda \Omega \ast G \) [A4] (see also [Y;10.8]). Hence \( \Omega \) gives a CM NCCR \( \Lambda \Omega \ast G \) of \( \Lambda \) (see [V2;1.1]), and \( \Omega \ast G \) is an Auslander algebra of type \((d, d, d-1)\).

We will study this example in 6.1.

5.3 Conjecture Fix a pair \((A, B)\) in 3.2.2 again, and \( l \geq 1 \). It is interesting to study the relationship among all maximal \( l \)-orthogonal objects in \( B \). Especially, we conjecture that their endomorphism rings are derived equivalent. It is suggestive to relate this conjecture to Van den Bergh’s generalization [V2] of the Bondal-Orlov conjecture [BO], which asserts that all (commutative or non-commutative) crepant resolutions of a normal Gorenstein domain have the same derived category. Since maximal \( l \)-orthogonal subcategories are analogs of modules giving non-commutative crepant resolutions from the viewpoint of 5.2.1, our conjecture is analogous to the Bondal-Orlov-Van den Bergh conjecture. We will show in 5.3.3 that it is true for the case \( l = 1 \).

5.3.1 Lemma Let \( M_i \in B \) and \( t \geq 1 \). Assume \((\text{add } M_1)\)-dim \( M_2 \leq t \), \( M_2 \perp_1 M_2 \), \( M_1 \perp_{t-1} M_2 \) and that \( M_1 \) is a generator. Put \( \Gamma_i := \text{End}_A(M_i) \) and \( U := \text{Hom}_A(M_1, M_2) \). Then \( U \) satisfies \( \text{pd } \Gamma_i U \leq t \), \( (r_i U, \perp (r_i U) \text{ and } \text{End}_A U = \Gamma_i \).

Proof Take a right \((\text{add } M_1)\)-resolution \( X : 0 \to X_t \to \cdots \to X_0 \to M_2 \to 0 \) of \( M_2 \), which is exact since \( M_1 \) is a generator. Then \( \Lambda(M_1, X) : 0 \to \Lambda(M_1, X_t) \to \cdots \to \Lambda(M_1, X_0) \to U \to 0 \) gives a projective resolution of the \( \Gamma_1 \)-module \( U \). Thus \( \text{pd } \Gamma_1 U \leq t \)
holds. By $M_2 \perp_t M_2$ and $M_1 \perp_{l-1} M_2$, we have an exact sequence $\Lambda(X, M_2) : 0 \to \text{End}_\Lambda(M_2) \to \Lambda(X_0, M_2) \to \cdots \to \Lambda(X_t, M_2) \to 0$. Since $\Gamma_1(\Lambda(M_1, )\Lambda(M_1, M_2)) = \Lambda(\Lambda, M_2)$ holds on $\text{add } M_1$, the complex $\Gamma(\Lambda(M_1, X), U)$ is isomorphic to the exact sequence $\Lambda(X, M_2)$. Thus $(\Gamma_1 U) \perp (\Gamma_1 U)$ and $\text{End}_{\Gamma_1}(U) = \Gamma_2$ hold.

5.3.2 Theorem Let $M_i \in B$ be maximal $l$-orthogonal $(i = 1, 2)$ and $k \leq l \leq 2k + 1$. Assume $M_1 \perp_k M_2$. Put $\Gamma_i := \text{End}_\Lambda(M_i)$ and $U := \text{Hom}_\Lambda(M_1, M_2)$. Then $U$ is a tilting $(\Gamma_1, \Gamma_2)$-module with \text{pd} $\Gamma_1 U \leq l - k |M|$. Thus $\Gamma_1$ and $\Gamma_2$ are derived equivalent.

Proof Put $t := l - k$. By $M_1 \perp_k M_2$ and 2.4.1, $(\text{add } M_1)$-$\text{dim } M_2 \leq t$ and $(\text{add } M_2)^{\text{op}}$-$\text{dim } M_1 \leq t$ hold. We have $M_1 \perp_t M_i$ and $M_1 \perp_{l-1} M_2$. By 5.3.1, $\text{pd} \Gamma_1 U \leq t$, $(\Gamma_1 U) \perp (\Gamma_1 U)$ and $\text{End}_{\Gamma_1}(U) = \Gamma_2$ hold. Take a left (add $M_2$)-resolution $Y_0 \to M_1 \to \cdots \to Y_t \to 0$ of $M_1$. By $M_1 \perp_t M_i$ and $M_1 \perp_{l-1} M_2$, $\Lambda(M_1, Y_i) : 0 \to \Gamma_1 \to \Lambda(M_1, Y_i) \to \cdots \to \Lambda(M_1, Y_t) \to 0$ is exact with $\Lambda(M_1, Y_i) \in \text{add } \Gamma_1 U$. Thus $U$ is a tilting $(\Gamma_1, \Gamma_2)$-module, and $\Gamma_1$ and $\Gamma_2$ are derived equivalent by a result of Happel [Ha]\

5.3.3 Corollary (1) Let $C_i = \text{add } M_i$ be a maximal 1-orthogonal subcategory of $B$ and $\Gamma_i := \text{End}_\Lambda(M_i)$ $(i = 1, 2)$. Then $\Gamma_1$ and $\Gamma_2$ are derived equivalent. In particular, $\text{# ind } C_1 = \text{# ind } C_2$ holds.

(2) If $d \leq 3$, then all CM NCCR of $\Lambda$ have the same derived category.

Proof (1) Put $l := 1$ and $k := 0$ in 5.3.2. Since a derived equivalence preserves Grothendieck groups [Ha], the latter assertion follows.

(2) If $d = 3$, then the assertion follows by (1) and 5.2.1. If $d = 2$, then any $M$ giving a CM NCCR satisfies CM $\Lambda = \text{add } M$ by 5.2.1. Thus End$_\Lambda(M)$ is unique up to Morita-equivalences. Assume $d \leq 1$. It is well-known that, if an $R$-order $\Gamma$ satisfies $\text{gl}$.dim $\Gamma = d$, then $\text{gl}$.dim End$_\Gamma(P) = d$ holds for any $P \in \text{add } \Gamma$ [CR]. Thus $\Lambda$ has a CM NCCR if and only if $\text{gl}$.dim $\Lambda = d$. In this case, any CM NCCR is Morita-equivalent to $\Lambda$.

5.3.4 We obtain the corollary below by 2.6.2, 5.3.1 and 5.3.2. For the case $m = 0$, $\Gamma$ satisfies $\text{gl}$.dim $\Gamma \leq 3$ and $\text{dom}$.dim $\Gamma \geq 3$ by 4.2.1. Then Miyachi’s theorem [Mi] implies that $\Omega^2(\text{mod } \Gamma)$ coincides with the category of $\Gamma$-modules $X$ with $\text{pd} \Gamma X \leq 1$. Thus our corollary gives the (independent) result of Geiss-Leclerc-Schröer [GLS2].

Corollary Let $M \in B$ be maximal 1-orthogonal and $\Gamma := \text{End}_\Lambda(M)$. Assume $m \leq 2$. Then we have equivalences $\mathcal{F} := B(M, ) : B \to \Omega^2(\text{mod } \Gamma)$ and $\mathcal{G} := B(, M) : B \to \Omega^2(\text{mod } \Gamma^{\text{op}})$, which send maximal 1-orthogonal (resp. 1-orthogonal) objects in $B$ to tilting (resp. partial tilting) $\Gamma$ and $\Gamma^{\text{op}}$-modules respectively and satisfy $\mathcal{F} = (\mathcal{F})^* \circ \mathcal{G}$ and $\mathcal{G} = (\mathcal{G})^* \circ \mathcal{F}$ for $(\mathcal{F})^* = \text{Hom}_{\Gamma}(\, , \Gamma)$.

5.4 Definition Let us generalize the concept of Auslander’s representation dimension [A1] to relate it to non-commutative crepant resolutions. For $n \geq 1$, define the $n$-th representation dimension $\text{rep}$.dim$_n \Lambda$ of an $R$-order $\Lambda$ which is an isolated singularity by $\text{rep}$.dim$_n \Lambda := \inf \{ \text{gl}$.dim $\text{End}_\Lambda(M) \mid M \in \text{CM} \Lambda, \Lambda \oplus D_4 \Lambda \in \text{add } M, M \perp_{n-1} M \}$. In other words, we consider all $(\Lambda, M, D_4 \Lambda) \in \mathcal{X}_{d,n}^q$, and $\text{rep}$.dim$_n \Lambda$ is the infimum of global dimension of corresponding quasi-Auslander algebras $\text{End}_\Lambda(M)$ of type $(d, d, n)$. Obviously $d \leq \text{rep}$.dim$_1 \Lambda \leq \text{rep}$.dim$_n \Lambda$ holds for any $n \leq n'$ ([R;3.2]). Notice that $\text{rep}$.dim$_1 \Lambda$ coincides with the representation dimension defined in [A1][16].
5.4.1 We call \( \Lambda \) representation-finite if \( \# \text{ind}(\text{CM} \Lambda) < \infty \). In the sense of (1) below, \( \text{rep.dim}_1 \Lambda \) measures how far \( \Lambda \) is from being representation-finite (cf. [A1][16]).

**Theorem** (1) If \( \Lambda \) is representation-finite, then \( \text{rep.dim}_1 \Lambda \leq \max\{2, d\} \) holds, and the converse holds if \( d \leq 2 \). If \( d > 2 \), then the converse does not necessarily hold.

(2) \( \Lambda \) has a CM NCCR if and only if \( \text{rep.dim}_1 \Lambda = d \) holds for \( l := \max\{1, d - 1\} \).

(3) Let \( n \geq 1 \). If \( \text{CM} \Lambda \) has a maximal \((n - 1)\)-orthogonal subcategory \( \mathcal{C} \) with \( \# \text{ind} \mathcal{C} < \infty \), then \( \text{rep.dim}_n \Lambda \leq \max\{n + 1, d\} \) holds, and the converse holds if \( d \leq n + 1 \).

**Proof** (3) The assertion follows immediately by 5.1(3).

(2) If \( d \geq 2 \), then the assertion follows by (3) and 5.2.1. For the case \( d < 2 \), \( \Lambda \) has a CM NCCR if and only if \( \text{gl.dim} \Lambda = d \) if and only if \( \text{rep.dim}_1 \Lambda = d \) by the argument in the proof of 5.3.3(2).

(1) We obtain the assertion by putting \( n := 1 \) in (3). Let us give a counter-example for \( d > 2 \). Take \( \Lambda \) in 5.2.2, which has a CM NCCR. Thus \( \text{rep.dim}_1 \Lambda = \text{rep.dim}_{d-1} \Lambda = d \) by (2). But \( \Lambda \) is representation-infinite except the case \( d = 3 \) and \( G = \langle \text{diag}(-1, -1, -1) \rangle \) [AR2].

5.4.2 It is an interesting problem raised by Auslander [A1] to calculate the value of \( \text{rep.dim}_n \Lambda \). In particular, when \( \text{rep.dim}_n \Lambda \) is finite? For the case \( n = 1 \) and \( d \leq 1 \), we have the finiteness result below obtained by the author [I1,3,6] recently (see also [L]).

**Theorem** If \( d \leq 1 \), then \( \text{rep.dim}_1 \Lambda < \infty \).

5.4.3 If \( d \geq 2 \), then \( \text{rep.dim}_1 \Lambda < \infty \) does not necessarily hold. For example, if \( d = 2 \) and \( \Lambda \) is representation-infinite commutative Gorenstein, then \( \text{rep.dim}_1 \Lambda = \infty \) holds by (3) below, which we will prove by the argument of Van den Bergh in [V2;4.2]. We call a module-finite \( R \)-algebra \( \Lambda \) symmetric if \( D_\Lambda \Lambda \) is isomorphic to \( \Lambda \) as a \((\Lambda, \Lambda)\)-module. Any symmetric order is Gorenstein, and the converse holds if it is commutative.

**Proposition** Assume \( d \geq 2 \) and that \( \Lambda \) is a symmetric \( R \)-order.

(1) If \( M \in \text{mod} \Lambda \) is a reflexive \( R \)-module, then \( \text{End}_\Lambda(M) \) is a symmetric \( R \)-algebra.

(2) \( \text{rep.dim}_{d-1} \Lambda \) is either \( d \) or \( \infty \).

(3) If \( d = 2 \), then \( \text{rep.dim}_1 \Lambda < \infty \) if and only if \( \Lambda \) is representation-finite.

**Proof** (1) Put \( \Gamma := \text{End}_\Lambda(M) \). Since \( \Lambda \) is symmetric, \( M^* := \Lambda(M, \Lambda) \) is isomorphic to \( D_\Lambda M \) as a \((\Gamma, \Lambda)\)-module. We have a natural map \( f : M^* \otimes_\Lambda M \rightarrow \Gamma \). Since \( \Lambda \) is assumed to be an isolated singularity, \( f_p \) is an isomorphism for any \( p \in \text{Spec} R \) with \( \text{ht} p = 1 \). Now consider a \((\Gamma, \Gamma)\)-homomorphism \( D_\Gamma \Gamma \xrightarrow{D_\Gamma f} D_\Gamma(M^* \otimes_\Lambda M) \simeq D_\Gamma(D_\Gamma M \otimes_\Lambda M) = \Lambda(M, D_\Gamma D_\Gamma M) = \Gamma \). Since \( D_\Gamma f \) is a map between reflexive \( R \)-modules such that \((D_\Gamma f)_p \) is an isomorphism for any \( p \in \text{Spec} R \) with \( \text{ht} p = 1 \), it is an isomorphism.

(2) Take \( M \in \text{CM} \Lambda \) with \( M \perp_{d-2} M \). Then \( \Gamma := \text{End}_\Lambda(M) \) is an \( R \)-order by 5.1(1). Since \( \Gamma \) is Gorenstein by (1), \( \text{id}_\Gamma \Gamma = d \) holds by 3.1.2. Thus \( \text{gl.dim} \Gamma = d \) or \( \infty \).

(3) The assertion follows by 5.4.1(1).

5.4.4 We end this subsection by giving a few remarks on the value of \( \text{rep.dim}_n \Lambda \).

(1) Assume \( d = 0 \). Thus \( \text{rep.dim}_1 \Lambda < \infty \) holds by 5.4.2, and \( \text{rep.dim}_1 \Lambda \leq 2 \) if and only if \( \Lambda \) is representation-finite by 5.4.1. Dugas [D] and Guo [G] independently proved that \( \text{rep.dim}_1 \Lambda \) is preserved by stable equivalences. Recently, many results are
obtained on algebras with rep.dim_1 Λ ≤ 3, for example, they satisfy the famous finitistic dimension conjecture [IT]. Many classes of algebras are known to satisfy rep.dim_1 Λ ≤ 3, e.g. hereditary algebras [A1], tilted algebras [APT], algebras with radical square zero [A1], special biserial algebras [EHIS] and so on. See also [BHS][CP][Ho][X].

On the other hand, Rouquier showed that rep.dim_1 Λ = l + 1 holds for the exterior algebra Λ = ∧(k^l) of the l-dimensional vector space by applying his concept of the dimension of triangulated categories [Ro1,2] (see also [KK]). In general, it seems to be difficult to know the precise value of rep.dim Λ when this is larger than 3.

(2) Assume d = 2. Then rep.dim_1 Λ = 2 if and only if Λ has a CM NCCR if and only if Λ is representation-finite by 5.4.1. If Λ is commutative and contains its residue field C, then it is equivalent to be a quotient singularity [A4].

(3) A trivial example of an order Λ with rep.dim n Λ = ∞ is an order which does not satisfy D_d Λ ⊥_{n-1} Λ. Let us give a non-trivial Gorenstein example. Van den Bergh proved that Λ := k[[x, y, z, t]]/(x^2 + y^2 + z^2 + t^{2b+1}) does not have a non-commutative crepant resolution [V1:A.1]. More strongly, he proved that there is no non-free reflexive Λ-module M such that End_Λ(M) is an order, or equivalently, M ⊥_1 M holds (5.1(1)). Since Λ is not regular, rep.dim_2 Λ = ∞ holds.

5.5 Conjecture For l ≥ 1 and B in 3.2.2, it seems that no example of a maximal l-orthogonal subcategory C of B with # ind C = ∞ is known. This suggests us to study

\[ o(B) := \sup_{C \subseteq B, C \perp_1 C} # \text{ind} C. \]

We conjecture that o(B) is always finite. If Λ is a preprojective algebra of Dynkin type Δ, then Geiss-Schröer [GS] proved that o(mod Λ) equals the number of positive roots of Δ (see 6.2.1). It would be interesting to find an interpretation of o(B) for more general B.

5.5.1 For some classes of B, one can calculate o(B) by using the theorem below. Especially, (1) seems to be interesting in connection with known results in 5.4.4(1).

**Theorem** (1) rep.dim_1 Λ ≤ 3 implies o(CM Λ) < ∞.
(2) If B has a maximal 1-orthogonal subcategory C, then o(B) = # ind C.
(3) If B has a subcategory C such that Λ ∈ C and C-dim B ≤ 1, then o(B) ≤ # ind C.

**Proof** We will show (3). We can assume C = add_Λ M_1. For any 1-orthogonal M_2 ∈ B and t := 1, we apply 5.3.1. Consequently, U is a partial tilting module. Since any partial tilting module is a direct summand of a tilting module [Ha], we obtain # ind(add_Λ M_2) = # ind(add_Γ U) ≤ # ind(add_Γ Γ_1) = # ind C. Thus o(B) ≤ # ind C holds. In particular, (2) follows. We will show (1). Take M ∈ CM Λ such that Λ ⊕ D_d Λ ∈ add M and gl.dim End_Λ(M) ≤ 3. It is easily shown that (add M)-dim (CM Λ) ≤ 1 holds (e.g. [EHIS;2.1]). By (3), o(CM Λ) ≤ # ind(add M) holds.

5.5.2 Concerning our conjecture, let us recall the well-known proposition below which follows by a geometric argument due to Voigt ([P;4.2]). It is interesting to ask whether it is true without the restriction on R. If it is true, then any 1-orthogonal subcategory of B is ‘discrete’, and our conjecture asserts that it is finite. It is interesting to study the discrete structure of 1-orthogonal objects in B and the relationship to the whole structure of B.
Proposition Assume $d = 0$ and that $R$ is an algebraically closed field. For any $n > 0$, there are only finitely many isoclasses of $1$-orthogonal $\Lambda$-modules $X$ with $\dim_R X = n$.

6 Applications and examples

6.1 Let us recall Auslander’s contribution to McKay correspondence [Mc]. He proved in [A4] (see also [Y]) that the McKay quiver of a finite subgroup $G$ of $\text{GL}_2(k)$ coincides with the Auslander-Reiten quiver of the invariant subring $k[[x, y]]^G$. The aim of this section is to give a higher dimensional generalization 6.1.4 of this result.

6.1.1 Definition Let $(\mathcal{A}, \mathcal{B})$ be a pair in 3.2.2 and $\mathcal{C}$ a maximal $(n - 1)$-orthogonal subcategory of $\mathcal{B}$. We will define the Auslander-Reiten quiver $\mathfrak{A}(\mathcal{C})$ of $\mathcal{C}$. For simplicity, we assume that the residue field $k$ of $R$ is algebraically closed. The set of vertices of $\mathfrak{A}(\mathcal{C})$ is $\text{ind}\mathcal{C}$. For $X, Y \in \text{ind}\mathcal{C}$, we denote by $d_{XY}$ be the multiplicity of $X$ in $C$ for the sink map $C \to Y$ (1.1), which equals to the multiplicity of $Y$ in $C'$ for the source map $X \to C'$. Draw $d_{XY}$ arrows from $X$ to $Y$. Draw a dotted arrow from non-projective $X \in \text{ind}\mathcal{C}$ to non-injective $Y \in \text{ind}\mathcal{C}$. If $\mathcal{C} = \text{add} M$, then $\mathfrak{A}(\mathcal{C})$ coincides with the Gabriel quiver of $\text{End}_\Lambda(M)$ since $d_{XY} = \dim_k J_C/J_C^2(X, Y)$ holds.

6.1.2 Definition Let $k$ be a field of characteristic zero and $G$ a finite subgroup of $\text{GL}_d(k)$ with $d \geq 2$. Recall that the McKay quiver $\mathfrak{M}(G)$ of $G$ [Mc] is defined as follows: The set of vertices is the set $\text{irr} G$ of isoclasses of irreducible representations of $G$. Let $V$ be the representation of $G$ acting on $k^d$ through $\text{GL}_d(k)$. For $X, Y \in \text{irr} G$, we denote by $d_{XY}$ the multiplicity of $X$ in $\otimes k Y$, and draw $d_{XY}$ arrows from $X$ to $Y$. Let $S = \wedge^d V$ be the 1-dimensional representation of $G$ given by the determinant. Draw a dotted arrow from $X \in \text{irr} G$ to $\tau_{d-1}X := S \otimes_k X \in \text{irr} G$.

6.1.3 Let $G$ be in 6.1.2, $\Omega := k[[x_1, \cdots, x_d]]$, $\Lambda := \Omega^G$ the invariant subring and $\Gamma := \Omega * G$ the skew group ring. Assume that $G$ does not contain any pseudo-reflection except the identity, and that $\Lambda$ is an isolated singularity. Then $\mathcal{C} := \text{add}_\Lambda \Omega$ forms a maximal $(d - 2)$-orthogonal subcategory of $\text{CM} \Lambda$ and $\text{End}_\Lambda(\Omega) = \Gamma$ holds by 5.2.2. Let us compare $\mathfrak{A}(\mathcal{C})$ and $\mathfrak{M}(G)$ by applying the argument in [A4] (see also [Y]) to arbitrary $d \geq 2$.

The functor $F := \Omega \otimes_k : \text{mod} kG \to \text{mod} \Gamma$ induces a bijection $\text{irr} G \to \text{ind}(\text{add}_\Gamma \Gamma)$ [A4] (see also [Y;10.1]). Define a functor $G : \text{mod} \Gamma \to \text{mod} \Lambda$ by $G(X) := X^G$ and $G(f) := f|_{X^G}$ for $X, Y \in \text{mod} \Gamma$ and $f \in \text{Hom}_\Gamma(X, Y)$. Since $\text{End}_\Lambda(\Omega) = \Gamma$ holds, $G$ restricts to the equivalence $G : \text{add}_\Gamma \Gamma \to \text{add}_\Lambda \Omega = \mathcal{C}$. Composing $F$ and $G$, we obtain a functor $H := G \circ F : \text{mod} kG \to \mathcal{C}$, which gives a bijection $H : \text{irr} G \to \text{ind}\mathcal{C}$. Let

$$K : 0 \to \Omega \otimes_k \wedge^d V \to \cdots \to \Omega \otimes_k \wedge^2 V \to \Omega \otimes_k V \to \Omega \to k \to 0$$

be the Koszul complex of $\Omega$. Then $K$ forms an exact sequence of $\Gamma$-modules. For any $X \in \text{irr} G$,

$$K \otimes_k X : 0 \to F(\wedge^d V \otimes_k X) \to \cdots \to F(\wedge^2 V \otimes_k X) \to F(V \otimes_k X) \to F(X) \to X \to 0$$

gives a minimal projective resolution of the $\Gamma$-modules $X$. Taking $G$, we obtain an exact sequence

$$G(K \otimes_k X) : 0 \to H(\wedge^d V \otimes_k X) \to H(\wedge^2 V \otimes_k X) \to H(V \otimes_k X) \to H(X) \to G(X) \to 0$$

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with \( f_i \in J_C \) for any \( i \). Now \( \mathbb{G}(X) = X \) holds if \( X \) is a trivial \( G \)-module, and \( \mathbb{G}(X) = 0 \) otherwise. Since \( \mathbb{G} : \text{add}_\Gamma \Gamma \to \mathcal{C} \) was an equivalence, \( 0 \to \mathcal{C}(\cdot, \mathbb{H}(\wedge^d V \otimes_k X)) \xrightarrow{f_{d-1}} \cdots \xrightarrow{f_1} \mathcal{C}(\cdot, \mathbb{H}(X)) \to 0 \) is exact. Consequently, \( \mathbb{G}(K \otimes_k X) \) is a \( (d-1) \)-fundamental sequence (3.2.3) if \( X \) is trivial, and a \( (d-1) \)-almost split sequence (2.5.3) otherwise. Thus we obtain the theorem below, where we put \( \tau_{d-1} \Lambda := \nu \Lambda = D_d \Lambda \) from the viewpoint of 3.2.3.

6.1.4 Theorem The Auslander-Reiten quiver \( \Xi(\mathcal{C}) \) of \( \mathcal{C} := \text{add}_\Lambda \Omega \) coincides with the McKay quiver \( \mathfrak{m}(G) \) of \( G \). Precisely speaking, there exists a bijection \( \mathbb{H} : \text{irr} \mathcal{C} \to \text{ind} \mathcal{C} \) such that \( \mathbb{H} \circ \tau_{d-1} = \tau_{d-1} \circ \mathbb{H} \) and \( d_{XY} = d_{\mathbb{H}(X), \mathbb{H}(Y)} \) for any \( X, Y \in \text{irr} \mathcal{C} \).

6.2 Geiss-Leclerc-Schröer [GLS1,2] applied 1-orthogonal (=rigid in their papers) modules to study semicanonical basis of the quantum enveloping algebra. Their work is closely related to our study in this paper. Let \( \Lambda \) be a preprojective algebra of type \( A_n \) over an algebraically closed field \( k \). Thus \( \Lambda \) is defined by the following quiver with relations \( a_i b_i = 0 \), \( a_i b_{i+1} b_{i+1} = b_t a_i \) (\( 1 \leq i \leq n - 2 \)) and \( b_n - a_{n-1} = 0 \).

\[
\begin{array}{ccccccc}
e1 \leftrightarrow a1 & e2 \leftrightarrow a2 & e3 \leftrightarrow a3 & \cdots & e_{n-1} \leftrightarrow a_{n-1} & e_n
\end{array}
\]

We denote by \( \iota \) the automorphism of \( \Lambda \) defined by \( \iota(e_i) := e_{n+1-i} \), \( \iota(a_i) = b_{n-i} \) and \( \iota(b_i) = a_{n-i} \) for any \( i \). Let us collect results on 1-orthogonal subcategories of \( \text{mod} \Lambda \). Especially, (4) below answers a question raised by Schröer.

6.2.1 Theorem Let \( \Lambda \) be a preprojective algebra of type \( A_n \) over an algebraically closed field \( k \).

1.(Geiss-Leclerc-Schröer) Any 1-orthogonal subcategory of \( \text{mod} \Lambda \) is contained in a maximal 1-orthogonal subcategory of \( \text{mod} \Lambda \).

2. Any maximal 1-orthogonal subcategory \( \mathcal{C} \) of \( \text{mod} \Lambda \) satisfies \# ind \( \mathcal{C} = \frac{n(n+1)}{2} \).

3. Let \( M \in \text{mod} \Lambda \) be a generator and \( \Gamma := \text{End}_\Lambda(M) \). Then \( M \) is (maximal) 1-orthogonal if and only if \( \text{dom.dim} \Gamma \geq 3 \) (and \( \text{gl.dim} \Gamma \leq 3 \)).

4. Let \( M \) and \( M' \in \text{mod} \Lambda \) be basic 1-orthogonal generators. Then \( \text{End}_\Lambda(M) \) is isomorphic to \( \text{End}_\Lambda(M') \) if and only if \( M \) is isomorphic to \( M' \) or \( \iota(M') \).

Proof Geiss-Leclerc-Schröer proved (1) in [GLS2]. They constructed maximal 1-orthogonal \( M \in \text{mod} \Lambda \) with \# ind(\text{add} \( M \)) = \( \frac{n(n+1)}{2} \). Thus (2) follows by 5.3.3(1). (3) follows by 5.1(2)(3). We will show (4) in the rest of this section.

6.2.2 Let us start with calculating the group \( \text{Aut}(\Lambda) \) of \( k \)-algebra automorphisms.

Proposition Put \( H := \{ g \in \text{Aut}(\Lambda) \mid g \text{ fixes any } e_i \text{ and } a_i \} \). Then \( \text{Aut}(\Lambda) = (\text{Inn}(\Lambda) \times \{ \iota \}) \rtimes H \) and \( H \simeq \text{Aut}(k[x]/(x^{l+1})) \) hold, where \( l \) is the maximal integer which does not exceed \( n/2 \).

Proof (i) Let \( \{ f_1, \ldots, f_n \} \) be a complete set of orthogonal primitive idempotents of \( \Lambda \). We will show that there exist \( \lambda \in \Lambda^x \) and \( \sigma \in \mathfrak{S}_n \) such that \( e_i \lambda = \lambda f_{\sigma(i)} \) for any \( i \).

Since \( \bigoplus_{i=1}^{n} \Lambda e_i = \Lambda = \bigoplus_{i=1}^{n} \Lambda f_i \) holds, Krull-Schmidt theorem implies that there exists \( \sigma \in \mathfrak{S}_n \) such that \( \Lambda e_i = \Lambda f_{\sigma(i)} \) holds for any \( i \). Since \( \text{Hom}_\Lambda(\Lambda e_i, \Lambda e_i) \) (resp. \( \text{Hom}_\Lambda(\Lambda f_{\sigma(i)}, \Lambda e_i) \)) can be identified with \( e_i \Lambda f_{\sigma(i)} \) (resp. \( e_i \Lambda f_{\sigma(i)} \Lambda e_i \)), there exist \( \lambda_i \in e_i \Lambda f_{\sigma(i)} \) and \( \epsilon_{\lambda_i} \in e_i \Lambda f_{\sigma(i)} \Lambda e_i \) for any \( i \).
and \( \gamma_i \in f_{\sigma(i)} \Lambda e_i \) such that \( \lambda_i \gamma_i = e_i \) and \( \gamma_i \lambda_i = f_{\sigma(i)} \). Put \( \lambda := \sum_{i=1}^n \lambda_i \in \Lambda \) and 
\( \gamma := \sum_{i=1}^n \gamma_i \in \Lambda \). Then \( \lambda \gamma = 1 = \gamma \lambda \) holds by \( \lambda \in \Lambda^\times \), and \( e_i \lambda = \lambda_i = \lambda f_{\sigma(i)} \) holds.

(ii) We will show that \( \text{Aut}(\Lambda) \) is generated by \( \text{Inn}(\Lambda) \), \( \iota \) and \( H \).

Put \( G := \{ g \in \text{Aut}(\Lambda) \mid g \) fixes any \( e_i \} \) and \( g \in \text{Aut}(\Lambda) \). By (i), there exist 
\( h \in \text{Inn}(\Lambda) \) and \( \sigma \in \mathfrak{S}_n \) such that \( hg(e_i) = e_{\sigma(i)} \) for any \( i \). It is easily shown that \( \sigma \) is either identity or \( \sigma(i) = n + 1 - i \) for any \( i \). Thus \( hg \in G \) holds for the former case, and \( thg \in G \) holds for the latter case. Now we only have to show that \( G \) is generated by \( \text{Inn}(\Lambda) \) and \( H \). Again fix \( g \in G \). Inductively, we can take \( \lambda_i \in (e_i \Lambda e_i)^x \) such that \( \lambda_1 := e_1 \) and \( \lambda_i g(a_i) = a_i \lambda_{i+1} \) for any \( i \), since \( e_i \Lambda e_{i+1} \) is generated by \( a_i \) as an \((e_{i+1} \Lambda e_{i+1})^g\)-module.

Put \( \lambda := \sum_{i=1}^n \lambda_i \in \Lambda^x \). Then \( \lambda e_i \lambda^{-1} = e_i \) and \( \lambda g(a_i) \lambda^{-1} = a_i \) hold for any \( i \). Thus \((\lambda \cdot \lambda^{-1})g \in H \) holds.

(iii) We will show the latter equality. Fix \( g \in \text{Aut}(k[x]/(x^{l+1})) \). Then there exist 
\( s_1 \in k^x \) and \( s_j \in k \) \((1 < j \leq l)\) such that \( g(x) = \sum_{j=1}^l s_j x^j \). Define \( \phi(g) \in H \) by 
\( \phi(g)(b_i) := \sum_{j=1}^l s_j(b_i a_i)^j b_i \) for any \( i \). Thus we have a map \( \phi : \text{Aut}(k[x]/(x^{l+1})) \to H \), which is easily checked to be an injective homomorphism. We will show that \( \phi \) is surjective. Fix \( h \in H \). By the relation \( a_{i+1} h(b_{i+1}) = h(b_i) a_i \), it is easily checked that there exist 
\( s_1 \in k^x \) and \( s_j \in k \) \((1 < j \leq l)\) such that \( h(b_i) = \sum_{j=1}^l s_j(b_i a_i)^j b_i \) for any \( i \). Define 
\( g \in \text{Aut}(k[x]/(x^{l+1})) \) by \( g(x) = \sum_{j=1}^l s_j x^j \). Then \( \phi(g) = h \) holds.

(iv) We will show the former equality. We can assume \( n > 1 \). Take \( g \in \text{Inn}(\Lambda) \cap H \) and put 
\( g = (\lambda \cdot \lambda^{-1}) \) for \( \lambda \in \Lambda^x \). Since \( \lambda e_i = e_i \lambda \) holds for any \( i \), we can put \( \lambda = \sum_{i=1}^n \lambda_i \) with \( \lambda_i \in (e_i \Lambda e_i)^x \). Since \( \lambda_i a_i = a_i \lambda_i + 1 \) holds for any \( i \), we can easily check that \( \lambda \in Cen \Lambda \) and 
\( g = 1 \) hold. Take \( g \in \text{Inn}(\Lambda) \cap H \) and put 
\( g = (\lambda \cdot \lambda^{-1}) \) for \( \lambda \in \Lambda^x \). Then \( \lambda e_{n+1-i} = e_i \lambda \) holds for any \( i \). Thus \( e_1 \lambda e_1 = 0 \) holds by \( n > 1 \), a contradiction to \( \lambda \in (e_i \Lambda e_i)^x \). Consequently, \( \text{Inn}(\Lambda) \cap \{ \iota \} = 1 \) and \( \text{Inn}(\Lambda) \cap (\{ \iota \}) H = 1 \) hold. We only have to check that \( \text{Inn}(\Lambda) \cap (\{ \iota \}) H = 1 \) is normalized by \( H \). We will show that \( h^{-1} \iota h \in \text{Inn}(\Lambda) \) holds for any \( h \in H \). The isomorphism \( \phi \) in (iii) shows that there exists \( c \in (\text{Cen}(\Lambda))^x \) such that \( h(b_1) = b_1 c \) for any \( i \). Put \( d := h^{-1}(c) \in (\text{Cen}(\Lambda))^x \). Since \( \iota \) acts trivially on \( \text{Cen}(\Lambda) \) and \( h^{-1}(b_1) = b_i d^{-1} \) holds for any \( i \), we have \( h^{-1} \iota h(b_1) = a_i d \) and \( h^{-1} \iota h(b_i) = b_i d^{-1} \) for any \( i \). It is easily checked that \( \lambda := \sum_{i=1}^n e_i d^i \) satisfies \( h^{-1} \iota h = (\lambda \cdot \lambda^{-1}) \in \text{Inn}(\Lambda) \).

6.2.3 Lemma Let \( \Lambda \) be a finite-dimensional \( k \)-algebra and \( d > 0 \). Put \( F := \{ g \in \text{Aut}(\Lambda) \mid g(X) \) is isomorphic to \( X \) for any \( 1 \)-orthogonal \( \Lambda \)-module \( X \) with \( \text{dim}_k X = d \} \). Then \( F \) forms a Zariski-closed subgroup of \( \text{Aut}(\Lambda) \) of finite index.

Proof Fix a \( k \)-basis \( \{ x_1, \cdots, x_n \} \) of \( \Lambda \). We denote by \( \text{mod}_d \Lambda \) the set of \( \Lambda \)-module structure on the \( k \)-vector space \( k^d \). Denoting the action of \( x_i \) on \( k^d \) by \( X_i \in \text{M}_d(k) \simeq A_k^{d^2} \), we regard \( \text{mod}_d \Lambda \) as a closed subset of \( A_k^{d^2} \). Thus \( \text{GL}_d(k) \) acts on \( \text{mod}_d \Lambda \) by 
\( g(X_1, \cdots, X_n) := (gX_1 g^{-1}, \cdots, gX_n g^{-1}) \), and each \( \text{GL}_d(k) \)-orbit corresponds to an isoclass of \( \Lambda \)-modules. On the other hand, the closed subgroup \( \text{Aut}(\Lambda) \) of \( \text{GL}_d(k) \) acts on \( \text{mod}_d \Lambda \) by \( a(X_1, \cdots, X_n) := (\sum_{j=1}^n a_{ij} X_j, \cdots, \sum_{j=1}^n a_{nj} X_j) \) for \( a = (a_{ij})_{1 \leq i, j \leq n} \).

Put \( E := \{ X \in \text{mod}_d \Lambda \mid \text{Ext}_\Lambda^1(X, X) = 0 \} \). By Voight’s lemma, there are only finitely many \( \text{GL}_d(k) \)-orbits \( C_1, \cdots, C_l \) in \( E \), and each \( C_i \) forms a connected component of \( E \). Since the action of \( \text{GL}_d(k) \) and \( \text{Aut}(\Lambda) \) on \( \text{mod}_d \Lambda \) commute, \( \text{Aut}(\Lambda) \) acts on the set of orbits \( S := \{ C_1, \cdots, C_l \} \). Since \( F \) is the kernel of this action, our assertions follow.

6.2.4 Proof of 6.2.1(4) Fix a basic \( 1 \)-orthogonal generator \( X \). By 4.8, we only have
to show that $g(X)$ is isomorphic to $X$ or $ι(X)$ for any $g ∈ Aut(Λ)$. Put $F' := \{g ∈ Aut(Λ) \mid g(X)$ is isomorphic to $X$ or $ι(X)\}$. Then $F'$ contains $\text{Inn}(Λ) × \langle ι \rangle$. By 6.2.3, $F'$ forms a closed subgroup of $Aut(Λ)$ of finite index. By 6.2.2, $Aut(Λ)/(\text{Inn}(Λ) × \langle ι \rangle) = H ≃ Aut(k[x]/(x^{i+1})) = k^* × A^1_k$ is connected. Hence $F' = Aut(Λ)$ holds.] 6.3 Let $Λ$ be a representation-finite selfinjective algebra. It is well-known that the stable Auslander-Reiten quiver of $Λ$ has the form $\Lambda$. We have an isomorphism $\text{Ext}^g \Lambda$ to show that $H ≃ P$. Thus $\text{Ext}^g \Lambda$ or $\text{mod} Λ$ are characterized combinatorially in terms of $\Lambda$. Let us study them from a little bit different viewpoint.

Let $H$ be a hereditary $k$-algebra of Dynkin type $∆$, $D := D^b(\text{mod} H)$ the bounded derived category and $F := τ^{-1} \circ [1]$ the auto-equivalence of $D$. We put $\text{Ext}^1_D(X, Y) := \text{Hom}_D(X, Y[1])$ for $X, Y ∈ D$. Buan-Marsh-Reineke-Reiten-Todorov [BMRRT] introduced the cluster category $C_H := D/F$ and showed that $C_H$ is closely related to cluster algebras of Fomin-Zelevinsky [FZ1,2]. A key concept was an Ext-configuration, which is a subset $T$ of $\text{ind} D$ satisfying $T = \{X ∈ \text{ind} D \mid \text{Ext}^1_D(T, X) = 0\}$. Since this definition is essentially similar to our maximal 1-orthogonal subcategories in 2.4, we can regard Ext-configurations as ‘maximal 1-orthogonal subcategories of triangulated categories’.

6.3.1 It is well known [Ha] that the bounded derived category $D^b(\text{mod} H)$ is equivalent to the mesh category $k(\mathbb{Z} ∆)$ [Ri]. We identify $D^b(\text{mod} H)$ with $k(\mathbb{Z} ∆)$.

**Theorem** Let $∆$ be a Dynkin diagram and $Λ$ a standard selfinjective algebra with a covering functor $P : k(\mathbb{Z} ∆) → \text{mod} Λ$.

(1) The following diagrams are commutative.

\[
\begin{align*}
&D^b(\text{mod} H) \xrightarrow{[1]} D^b(\text{mod} H) \downarrow P \quad &D^b(\text{mod} H) \xrightarrow{F} D^b(\text{mod} H) \downarrow P \\
&\text{mod} Λ \xrightarrow{Ω^{-}} \text{mod} Λ &\text{mod} Λ \xrightarrow{τ^{-}} \text{mod} Λ
\end{align*}
\]

(2) For a subcategory $C$ of $\text{mod} Λ$ containing $Λ$, the conditions below are equivalent.

(i) $C$ is a maximal 1-orthogonal subcategory of $\text{mod} Λ$.

(ii) $P^{-1}(\text{ind} C)$ is an Ext-configuration.

**Proof** (1) Since $P$ is a triangle functor and $Ω^{-} : \text{mod} Λ → \text{mod} Λ$ and $[1] : D^b(\text{mod} H) → D^b(\text{mod} H)$ are shift functors, the left diagram is commutative. Since $P$ commutes with $τ$, we obtain $τ^{-} ◦ P = τ^{-} ◦ Ω^{-} ◦ P = P ◦ τ^{-} ◦ [1] = P ◦ F$.

(2) We can take an automorphism group $G$ of $\mathbb{Z} ∆$ such that $D/G = k(\mathbb{Z} ∆)/G ≃ \text{mod} Λ$. We have an isomorphism $\text{Ext}^1_A(PX, PY) ≃ \bigoplus_{g ∈ G} \text{Ext}^1_D(gX, Y)$ for any $X, Y ∈ D$ [BMRRT]. Thus $PX ⊥ PY$ if and only if $GX ⊥ Y$ if and only if $GY$. This implies $P^{-1}(C^{⊥1}) = (P^{-1}C)^{⊥1}$. Thus the assertion follows.

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Department of Mathematics, University of Hyogo, Himeji, 671-2201, Japan
iyama@sci.u-hyogo.ac.jp

Current address:
Graduate School of Mathematics, Nagoya University,
Chikusa-ku, Nagoya, 464-8602, Japan
iyama@math.nagoya-u.ac.jp