HODGE THEORY OF CUBIC FOURFOLDS, THEIR FANO VARIETIES, AND ASSOCIATED K3 CATEGORIES

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Abstract. These are notes of lectures given at the school 'Birational Geometry of Hypersurfaces' in Gargnano in March 2018. The main goal was to discuss the Hodge structures that come naturally associated with a cubic fourfold. The emphasis is on the Hodge and lattice theoretic aspects with many technical details worked out explicitly. More geometric or derived results are only hinted at.

The primitive Hodge structure of a smooth cubic fourfold \( X \subset \mathbb{P}^5 \) is concentrated in degree four and it is of a very particular type. Once a Tate twist is applied and the sign of the intersection form is changed, it reveals its true nature. It very much looks like the Hodge structure of a K3 surface. In his thesis Hassett [Ha00] studied this curious relation and the intricate lattice theory behind it in greater detail. He established a transcendental correspondence between polarized K3 surfaces of certain degrees and special cubic fourfolds, some aspects of which are reminiscent of the Kuga–Satake construction. The geometric nature of the Hassett correspondence is still not completely understood but it seems that derived categories are central for its understanding. Work of Addington and Thomas [AT14] represents an important step in this direction, combining Hassett’s Hodge theory with Kuznetsov’s categorical approach to hypersurfaces.

The aim of the lectures was to discuss the Hodge structures \( H^4(X, \mathbb{Z}) \), \( H^4(X, \mathbb{Z})_{pr} \), \( \tilde{H}(X, \mathbb{Z}) \), and \( H^2(F(X), \mathbb{Z}) \), all naturally associated with a cubic fourfold \( X \), and their relation to the Hodge structures \( H^2(S, \mathbb{Z}) \), \( H^2(S, \mathbb{Z})_{pr} \), \( \tilde{H}(S, \mathbb{Z}) \), and \( \tilde{H}(S, \alpha, \mathbb{Z}) \) that come with a (polarized, twisted) K3 surface \( S \). For a discussion of more motivic aspects, partially covered by the original lectures, and of derived aspects, not touched upon at all, we have to refer to the existing literature. Most of the content of the lectures is also covered by [Hu19].

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1. LATTICE AND HODGE THEORY FOR CUBIC FOURFOLDS AND K3 SURFACES

In the first section, we collect all facts from Hodge and lattice theory relevant for the study of cubic fourfolds. The curious relation between the lattice theory of cubic fourfolds and K3 surfaces has been systematically studied first by Hassett [Ha00]. Earlier results in this direction are due to Beauville and Donagi [BD85].

1.1. As abstract lattices, the middle cohomology and the primitive cohomology of a smooth cubic fourfold \( X \subset \mathbb{P}^5 \) are described by

\[
\begin{align*}
H^4(X, \mathbb{Z}) &= I_{21,2} \cong E_8^{\oplus 2} \oplus U^{\oplus 2} \oplus I_{3,0}, \\
H^4(X, \mathbb{Z})_{\text{pr}} &= E_8^{\oplus 2} \oplus U^{\oplus 2} \oplus A_2,
\end{align*}
\]

where the square of the hyperplane class \( h \) is given as \( h^2 = (1, 1, 1) \in I_{3,0} \). Here, we use the common notation \( E_8 \) and \( U \) for the unique, unimodular, even lattice of signature \((8,0)\) and \((1,1)\), respectively, and \( I_{m,n} \) for the unique, unimodular, odd lattice of signature \((m,n)\), see [Hu19, Sec. 1.1.5] for details and references. It will be convenient to change the sign and introduce the cubic lattice \( \bar{\Gamma} := I_{21,2} \cong E_8^{\oplus 2} \oplus U^{\oplus 2} \oplus I_{0,3} \) and the primitive cubic lattice \( \Gamma := E_8^{-1} \oplus U^{\oplus 2} \oplus A_2 \). In particular, from now on \( h^2 = (1, 1, 1) \in I_{3,0} \). Here, we use the common notation \( E_8 \) and \( U \) for the unique, unimodular, even lattices of signature \((8,0)\) and \((1,1)\), respectively, and \( I_{m,n} \) for the unique, unimodular, odd lattice of signature \((m,n)\), see [Hu19, Sec. 1.1.5] for details and references. It will be convenient to change the sign and introduce the cubic lattice and the primitive cubic lattice as

\[
\begin{align*}
\bar{\Gamma} &:= I_{2,21} \cong E_8^{-1} \oplus U^{\oplus 2} \oplus I_{0,3} \cong H^4(X, \mathbb{Z})(-1), \\
\Gamma &:= E_8^{-1} \oplus U^{\oplus 2} \oplus A_2(-1) \cong H^4(X, \mathbb{Z})_{\text{pr}}(-1).
\end{align*}
\]

In particular, from now on \( (h^2)^2 = -3 \). The twist should not be confused with the Tate twist of the Hodge structure. It turns out that \( E_8(-1)^{\oplus 2} \), certainly the most interesting part of these lattices, will hardly play any role in our discussion. We shall henceforth abbreviate it by \( E := E_8(-1)^{\oplus 2} \) and consequently write

\[
\bar{\Gamma} \cong E \oplus U^{\oplus 2} \oplus I_{0,3} \text{ and } \Gamma \cong E \oplus U^{\oplus 2} \oplus A_2(-1).
\]

Although there is a priori no geometric reason why K3 surfaces should enter the picture at all, their intersection form will play a central role in our discussion. We will first address this first purely on the level of abstract lattice theory and later add Hodge structures.

Recall that for a complex K3 surface \( S \), its middle cohomology with the intersection form is the lattice

\[
H^2(S, \mathbb{Z}) \cong E \oplus U^{\oplus 3} \cong E \oplus U_1 \oplus U_2 \oplus U_3 =: \Lambda,
\]

see [Hu16, Ch. 14]. The summands \( U_i, i = 1, 2, 3 \), are copies of the hyperbolic plane \( U \). Indexing them will make the discussion more explicit and will help us to avoid ambiguities later on.

The full cohomology \( H^*(S, \mathbb{Z}) \) is also endowed with a unimodular intersection form. It is customary to introduce a sign in the pairing on \( (H^0 \oplus H^4)(S, \mathbb{Z}) \), which, however, does not
change the abstract isomorphism type, for $U \simeq U(-1)$. The resulting lattice is the Mukai lattice
\[
\tilde{H}(S, \mathbb{Z}) := H^2(S, \mathbb{Z}) \oplus (H^0 \oplus H^4)(S, \mathbb{Z}) \simeq E \oplus U^3 \oplus U_4
\]
\[
\simeq E \oplus U_1 \oplus U_2 \oplus U_3 \oplus U_4 =: \tilde{\Lambda}.
\]

The standard basis of $U$ consists of isotropic vectors $e, f$ with $(e, f) = 1$. We shall denote the standard bases in the first three copies of $U$ as $e_i, f_i \in U_i, i = 1, 2, 3$. However, in order to take into account the sign change in the Mukai pairing, we shall use the convention that $(e_4, f_4) = -1$ and that $e_4 = [S] \in H^0(S, \mathbb{Z})$ and $f_4 = [x] \in H^4(S, \mathbb{Z})$ with $x \in S$ a point.

Next, we introduce an explicit embedding $A_2 \subset \tilde{\Lambda}$. Here, $A_2 = \mathbb{Z} \lambda_1 \oplus \mathbb{Z} \lambda_2$ is the lattice of rank two given by the intersection form $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ and we define

(1.1) $A_2 \subset U_3 \oplus U_4 \subset \tilde{\Lambda}$

by $\lambda_1 \mapsto e_4 - f_4$ and $\lambda_2 \mapsto e_3 + f_3 + f_4$. The orthogonal complement $\langle \lambda_1, \lambda_2 \rangle = A_2^\perp \subset \tilde{\Lambda}$ is the lattice

$A_2^\perp = E \oplus U_1 \oplus U_2 \oplus A_2(-1),$

where $A_2(-1) \subset U_3 \oplus U_4$ is spanned by $\mu_1 := e_3 - f_3$ and $\mu_2 := -e_3 - e_4 - f_4$ satisfying $(\mu_i)^2 = -2$ and $(\mu_1, \mu_2) = 1$.

**Remark 1.1.** We observe that $\lambda_1^\perp = E \oplus U_1 \oplus U_2 \oplus U_3 \oplus \mathbb{Z}(-2)$, where the last direct summand is generated by $e_4 + f_4$. Hence, $\lambda_1^\perp \simeq \Lambda \oplus \mathbb{Z}(-2)$, which is a lattice of discriminant $\text{disc}(\lambda_1^\perp) = 2$ and which contains $A_2^\perp \oplus \mathbb{Z}(\lambda_1 + 2\lambda_2)$ as a sublattice of index three. As $H^2(S, \mathbb{Z}) \simeq \Lambda$ and $H^2(S^{[2]}, \mathbb{Z}) \simeq H^2(S, \mathbb{Z}) \oplus \mathbb{Z}(-2)$ for the Hilbert scheme $S^{[2]}$ of any K3 surface $S$, this can be read as a lattice isomorphism $\lambda_1^\perp \simeq H^2(S^{[2]}, \mathbb{Z})$.

The discussion so far leads to the fundamental observation that there exists an isomorphism

$\Gamma \supset \Gamma \simeq A_2^\perp \subset \tilde{\Lambda}$

between the primitive cubic lattice $\Gamma$ and the lattice $A_2^\perp$ inside the Mukai lattice $\tilde{\Lambda}$.

For later use, we record that (1.1) induces inclusions of index three:

$A_2 \oplus A_2(-1) \subset U_3 \oplus U_4$ and $A_2 \oplus A_2^\perp \subset \tilde{\Lambda}$,

where, for example, the quotient of the latter is generated by the image of the class $(1/3)(\mu_1 - \mu_2 - \lambda_1 + \lambda_2) = e_3 + f_4$.

Another technical result that will be crucial at some point later, is the following elementary statement which is surprisingly difficult to prove, cf. [AT14, Prop. 3.2].

---

1 The sign of the discriminant will be of no importance in our discussion, we tacitly work with its absolute value.
Lemma 1.2. Consider $A_2 \subset \tilde{\Lambda}$ as before, let $\overline{A_2 + U}$ be an isometric embedding of a copy of the hyperbolic plane, and denote by $\overline{A_2 + U}$ the saturation of $A_2 + U \subset \tilde{\Lambda}$. Then there exists an isometric embedding of a copy of the hyperbolic plane $\overline{A_2 + U}$ such that $\text{rk}(A_2 + U') = 3$.

Proof. See [AT14] for the proof. □

Remark 1.3. To motivate the notion of Noether–Lefschetz (or Heegner) divisors for cubic fourfolds, let us recall the corresponding concept for K3 surfaces: For a primitive class $\ell \in \Lambda$ with $(\ell)^2 = d$, we write

$$\Lambda_d := \ell^\perp \subset \Lambda.$$ 

As $\ell$ is in the same $O(\Lambda)$-orbit as the class $e_2 + (d/2) f_2$, cf. [Hu16] Cor. 14.1.10, it can abstractly be described as

$$\Lambda_d \simeq E \oplus U^{\oplus 2} \oplus \mathbb{Z}(-d).$$

It is important to note that the lattices $\Lambda_d$ are in general not contained in $A_2^\perp \subset \tilde{\Lambda}$.

We shall call any primitive vector $v \in \Gamma \simeq A_2^\perp$ with $(v)^2 < 0$ a Noether–Lefschetz vector. With such Noether–Lefschetz vector one naturally associates two lattices. On the cubic side, one defines

$$\mathbb{Z} h^2 \oplus \mathbb{Z} v \subset K_v \subset \tilde{\Gamma}$$

as the saturation of $\mathbb{Z} h^2 \oplus \mathbb{Z} v \subset \tilde{\Gamma}$. On the K3 side, we introduce the saturation

$$A_2 \oplus \mathbb{Z} v \subset L_v \subset \tilde{\Lambda}.$$ 

Note that $L_v$ is of rank three and signature $(2,1)$, while $K_v$ is of rank two and signature $(0,2)$. Clearly, their respective orthogonal complements are isomorphic:

$$\tilde{\Gamma} \supset K_v^\perp \simeq L_v^\perp \subset \tilde{\Lambda},$$

as they are both described as $v^\perp \subset \Gamma \simeq A_2^\perp$. In particular, for the discriminants we have

$$d := \text{disc}(L_v) = \text{disc}(K_v).$$

The situation has been studied in depth in [Ha00] Prop. 3.2.2:

Lemma 1.4 (Hassett). Only the following two cases can occur:

(i) Either $\mathbb{Z} h^2 \oplus \mathbb{Z} v = K_v$, $A_2 \oplus \mathbb{Z} v = L_v$, and

$$d = \text{disc}(K_v) = \text{disc}(L_v) = -3(v)^2 \equiv 0 \ (6)$$

(ii) or $\mathbb{Z} h^2 \oplus \mathbb{Z} v \subset K_v$, $A_2 \oplus \mathbb{Z} v \subset L_v$ are both of index three, and

$$d = \text{disc}(K_v) = \text{disc}(L_v) = -\frac{1}{3}(v)^2 \equiv 2 \ (6).$$
Proof. The main ingredient is the standard formula, see e.g. [Hu16 Sec. 14.0.2],

\[ \text{disc}(K_v) \cdot [K_v : \mathbb{Z} h^2 \oplus \mathbb{Z} v]^2 = \text{disc}(\mathbb{Z} h^2 \oplus \mathbb{Z} v) = -3(v)^2. \]

Any \( y \in K_v \) is of the form \( y = s h^2 + t v \), with \( s, t \in \mathbb{Q} \). From \((h,y) \in \mathbb{Z} \) one concludes \( s \in (1/3) \mathbb{Z} \) and hence also \( t \in (1/3) \mathbb{Z} \). This shows that \([K_v : \mathbb{Z} h^2 \oplus \mathbb{Z} v] = 1,3, \) or \( = 9 \), but the last possibility is excluded as \((1/3) h^2 \not\in \bar{\Gamma} \).

In the first case, i.e. \( K_v = \mathbb{Z} h^2 \oplus \mathbb{Z} v \), one finds \( d = \text{disc}(K_v) = -3(v)^2 \equiv 0 (6) \). In the second case, so when the index is three, then \( 3d = -(v)^2 \equiv 0, 2, 4 (6) \). On the other hand, \( K_v \) admits a basis consisting of \( h^2 \) and another class \( x \). Indeed, pick any class \( x \in K_v \) whose image generates the quotient \( K_v/(\mathbb{Z} h^2 \oplus \mathbb{Z} v) \simeq \mathbb{Z}/3\mathbb{Z} \). We may assume \( 3x = s h^2 + t v \) with \( s, t = \pm 1 \) and, therefore, \( K_v = \mathbb{Z} h^2 \oplus \mathbb{Z} x \). Hence, its discriminant satisfies \( d = -(3x)^2 - (x.h^2)^2 \equiv 0, 2, 3, 5 (6) \). Altogether this shows that \( d \equiv 0, 2 (6) \).

We claim that \( d \equiv 0 (6) \) holds if and only if \( K_v = \mathbb{Z} h^2 \oplus \mathbb{Z} v \). The ‘if’-direction was proven already. For the ‘only if’-direction, assume that \( d \equiv 0 (6) \) but \([K_v : \mathbb{Z} h^2 \oplus \mathbb{Z} v] = 3 \). Pick \( x \in K_v \) as above. Then, write \( v = s h^2 + t x, s, t \in \mathbb{Z} \), and use \((v,h^2) = 0 \) and the primitivity of \( v \) to show \( v = r ((x.h^2) h^2 + 3x) \) with \( r = \pm 1, \pm (1/3) \) as \( v \) is primitive. However, \((x.h^2) \equiv 0 (3) \) under the assumption that \( d \equiv 0 (6) \). Hence, \( \pm v = m h^2 + x, m \in \mathbb{Z} \), and, therefore, \( x \in \mathbb{Z} h^2 \oplus \mathbb{Z} v \). This yields a contradiction and thus proves the assertion.

The assertions for the lattice \( L_v \) follows directly from the ones for \( K_v \). \( \Box \)

Remark 1.5. Depending on the perspective, it may be useful to study the various cases from the point of view of \( d \) or, alternatively, of \((v)^2 \). To have the results handy for later use, we restate the above discussion as

\[
\begin{align*}
d \equiv 0 (6) & \quad \Rightarrow \quad (v)^2 = -d/3 \equiv 0 (6) \quad \text{or} \quad \equiv \pm 2 (6), \\
d \equiv 2 (6) & \quad \Rightarrow \quad (v)^2 = -3d \equiv 0 (6)
\end{align*}
\]

and

\[
\begin{align*}
(v)^2 \equiv \pm 2 (6) & \quad \Rightarrow \quad d = -3(v)^2 \equiv 0 (6), \\
(v)^2 \equiv 0 (6) & \quad \Rightarrow \quad d = -3(v)^2 \equiv 0 (6) \quad \text{or} \quad d = -(1/3)(v)^2 \equiv 2 (6).
\end{align*}
\]

In particular, \( d \) determines \((v)^2 \) uniquely, but not vice versa unless \((v)^2 \equiv \pm 2 (6) \).

Proposition 1.6 (Hassett). Let \( v, v' \in \bar{\Gamma} \) be two primitive vectors and assume that \( \text{disc}(L_v) = \text{disc}(L_{v'}) \) or, equivalently, \( \text{disc}(K_v) = \text{disc}(K_{v'}) \). Then there exist an orthogonal transformations \( g \in \bar{O}(\Gamma) \) such that \( g(v) = \pm v' \) and, in particular,

\[ L_{v'} \simeq L_{g(v)} \quad \text{and} \quad K_{v'} \simeq K_{g(v)}. \]

The definition of \( \bar{O}(\Gamma) \) will be recalled below.

Proof. We apply Eichler’s criterion, cf. [GHS09 Prop. 3.3]. If an even lattice \( N \) is of the form \( N \simeq N' \oplus U^\oplus 2 \), then a primitive vector \( v \in N \) with prescribed \((v)^2 \in \mathbb{Z} \) and \((1/n) \bar{v} \in \mathbb{A}_N \), with \( n \) determined by \((v.N) = n \mathbb{Z} \), is unique up to the action of \( \bar{O}(N) \). Apply this to \( v \in \bar{\Gamma} \simeq \bar{O}(1) \) and \( \bar{O}(\Gamma) \).
\(A_2^+ \simeq E \oplus U^\oplus_2 \oplus A_2(-1)\) and use that for any primitive \(v \in \Gamma\), either \((v, \Gamma) = Z\) or \(3Z\). This follows from \([\Gamma : \Gamma \oplus Z h^2] = 3\) and the unimodularity of \(\bar{\Gamma}\).

(i) If \((v)^2 \equiv 0 (6)\), there are two cases: Assume first that \(d \equiv 2 (6)\) or, equivalently, that \(Zv + Z h^2\) is not saturated. Then, one finds an element of the form \(\alpha := (1/3)v + t h^2 \in \bar{\Gamma}\). As \((\alpha, w) \in Z\) for all \(w \in \Gamma\), this shows \((v, \Gamma) \subset 3Z\). Hence, \(n = 3\) and \((1/3) \bar{v} = \pm 1 \in A_\Gamma \simeq Z/3Z\).

Assume now that \(d \equiv 0 (6)\) and write \(v = n_1v_1 + n_2v_2\) with \(v_1 \in E \oplus U_1 \oplus U_2\) and \(v_2 \in A_2(-1)\), both primitive, and \(n_1, n_2 \in Z\). If \(n_1 \not\equiv 0 (3)\), then there exists a class \(w\) in the unimodular lattice \(E \oplus U_1 \oplus U_2 \subset \Gamma\) with \((v, w) \not\equiv 3Z\) and hence \((v, \Gamma) = Z\). If \(n_1 \equiv 0 (3)\), then \(n_2 \not\equiv 0 (3)\), as \(v\) is primitive. However, in this case \((1/3)(v \pm h^2) = (n_1/3)v_1 + (1/3)(n_2v_2 \pm h^2) = \bar{\Gamma}\) and so \(Zv + Z h^2\) is not saturated, contradicting \(d \equiv 0 (6)\).

(ii) If \((v)^2 \equiv 2 (6)\) and hence \((v)^2 \not\equiv 0 (3)\), then \((v, \Gamma) = Z\), \(n = 1\), and \(\bar{v} \in A_\Gamma\) is trivial.

Hence, in case (i) and (ii), if indeed \(d\) and not only \((v)^2\) is fixed, then \((v)^2 = (v')^2\) and \((1/n) \bar{v} = (1/n) \bar{v}' \in A_\Gamma\) (up to sign). \(\square\)

**Remark 1.7.** Due to the uniqueness, no information is lost when explicit classes \(v \in \Gamma \simeq A_2^+\) are chosen for any given \(d\). In the sequel, we will work with the following ones.

(i) For \(d \equiv 0 (6)\), one may choose \(v_d := e_1 - (d/6)f_1 \in U_1 \subset \Gamma\). Observe that indeed, as explained in the general context above, \((v_d)^2 = -d/3\) and that the lattice \(A_2 \oplus Z v_d\) is saturated (use \(A_2 \subset U_2 \oplus U_3\) and \(v_d \in U_1\)), i.e.

\[L_d := L_{v_d} = A_2 \oplus Z v_d.\]

Similarly,

\[K_d := K_{v_d} = Z h^2 \oplus Z v_d,\]

which again shows \((v_d)^2 = -d/3\). Their orthogonal complement is

\[\Gamma_d := L_d^\perp \simeq K_d^\perp \simeq E \oplus U_2 \oplus A_2(-1) \oplus Z(e_1 + (d/6)f_1)\]

and their discriminant group

\[A_{K_d^\perp} \simeq A_{K_d} \simeq Z/3\mathbb{Z} \oplus Z/(d/3)\mathbb{Z}\]

is cyclic if and only if \(9 \nmid d\).

(ii) For \(d \equiv 2 (6)\), one sets \(v_d := 3(e_1 - ((d - 2)/6)f_1) + \mu_1 - \mu_2 \in U_1 \oplus A_2(-1)\). Then both inclusions

\[A_2 \oplus Z v_d \subset L_d := L_{v_d}\]

and \(Z h^2 \oplus Z v_d \subset K_d := K_{v_d}\)

are of index three, for example \(v_d - \lambda_1 + \lambda_2\) and \(v_d - h^2\) are divisible by 3. Use \(\lambda_1 = e_4 - f_4, \lambda_2 = e_3 + f_3 + f_4, \mu_1 = e_3 - f_3,\) and \(\mu_2 = -e_3 - e_4 - f_4\), the latter corresponding to \((1, -1, 0), (0, 1, -1) \in \mathbb{Z}^E_3\). In this case, see \[\text{[Ad16, Ha00, TV19]}\],

\[\Gamma_d := L_d^\perp \simeq K_d^\perp \simeq E \oplus U_2 \oplus (Z^\oplus_3, (\ , )_A)\]

with \(A := \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & (d-2)/3 \end{pmatrix}\).
and \( L_d \) and \( K_d \) are given by the matrices \(-A\) and
\[
\begin{pmatrix}
-3 & 1 \\
1 & -(d+1)/3
\end{pmatrix},
\]
respectively. The discriminant groups for \( d \equiv 2 \pmod{6} \) are cyclic, indeed \( A_{K_d} \cong A_{K_d} \cong \mathbb{Z}/d\mathbb{Z} \).

In addition to the orthogonal group
\[
\tilde{O}(\Gamma) := \{ \ g \in O(\Gamma) \ | \ g(h^2) = h^2 \ \},
\]
which we will also think of as \( \tilde{O}(\Gamma) = \{ \ g \in O(\Gamma) \ | \ \bar{g} \equiv \text{id} \text{ on } A_\Gamma \} \), we need to consider
\[
\tilde{O}(\Gamma, K_d) := \{ \ g \in \tilde{O}(\Gamma) \ | \ g(K_d) = K_d, \text{i.e. } g(v_d) = \pm v_d \ \}
\]
\[
\bigcup \tilde{O}(\Gamma, v_d) := \{ \ g \in \tilde{O}(\Gamma) \ | \ g|_{K_d} = \text{id}, \text{i.e. } g(v_d) = v_d \ \}.
\]
Observe that \( \tilde{O}(\Gamma, v_d) \) can be identified with the subgroup of all \( g \in O(\Gamma_d) \) with trivial action on the discriminant group \( A_{\Gamma_d} \cong A_{K_d} \). Also, by definition, \( \tilde{O}(\Gamma, v_d) \subset \tilde{O}(\Gamma, K_d) \) is a subgroup of index one or two. Note that the natural homomorphism \( \tilde{O}(\Gamma, K_d) \rightarrow O(K_d) \) is neither surjective (let alone injective) nor is its image contained in the subgroup of transformations acting trivially on the discriminant \( \tilde{O}(K_d) \).

**Lemma 1.8** (Hassett). The subgroup \( \tilde{O}(\Gamma, v_d) \subset \tilde{O}(\Gamma, K_d) \) is of index at most two. More precisely, one distinguishes the following cases:

(i) If \( d \equiv 0 \pmod{6} \), then
\[
\tilde{O}(\Gamma, v_d) \subset \tilde{O}(\Gamma, K_d)
\]
has index two.

(ii) If \( d \equiv 2 \pmod{6} \), then
\[
\tilde{O}(\Gamma, v_d) = \tilde{O}(\Gamma, K_d).
\]

**Proof.** (i) According to Lemma 1.4 \( d \equiv 0 \pmod{6} \) if and only if \( \mathbb{Z} h^2 \oplus \mathbb{Z} v_d = K_d \), which is contained in \( I_{0,3} \oplus U_1 \). Let \( g \in \tilde{O}(\Gamma) \) be the orthogonal transformation defined by \( g = \text{id} \text{ on } E \oplus U_2 \oplus I_{0,3} \) and by \( g = -\text{id} \text{ on } U_1 \). Then \( g \) is an element in \( \tilde{O}(\Gamma, K_d) \setminus \tilde{O}(\Gamma, v_d) \).

(ii) Now, \( d \equiv 2 \pmod{6} \) if and only if \( \mathbb{Z} h^2 \oplus \mathbb{Z} v_d \subset K_d \) has index three and then \( v_d = 3(e_1 - ((d-2)/6)f_1) + \mu_1 - \mu_2 \) with \( \mu_1 = (1, -1, 0), \mu_2 = (0, 1, -1) \in A_2(-1) \subset I_{0,3} \) and \( h^2 = (1, 1, 1) \). Now observe that \( (1/3)(v_d - h^2) \in K_d \), but \( (1/3)(-v_d - h^2) \notin K_d \). \( \square \)

1.2. It turns out that certain geometric properties of cubic fourfolds are encoded by lattice-theoretic properties of Noether–Lefschetz vectors \( v \in \Gamma \). The following ones are relevant for our purposes. It is a matter of choice, whether they are read as conditions on \( d \) or on the primitive
For \( d \in \mathbb{Z} \) one considers the conditions:

\[(*) \iff \text{There exists an } L_d.\]

\[(**') \iff \text{There exists an } L_d \text{ and an embedding } U(n) \hookrightarrow L_d \text{ for some } n \neq 0.\]

\[(**) \iff \text{There exists an } L_d \text{ and a primitive embedding } U \hookrightarrow L_d.\]

\[(***) \iff \text{There exists an } L_d \text{ and a primitive embedding } U \hookrightarrow L_d \text{ with } \lambda_1 \in U.\]

**Remark 1.9.** (i) The following implications trivially hold

\[(***) \Rightarrow (**) \Rightarrow (**') \Rightarrow (*).\]

(ii) Each of the conditions in fact splits in two, distinguishing between \( d \equiv 0 \mod 6 \) and \( d \equiv 2 \mod 6 \).

We shall write accordingly \((*)_0, (**)_0, (**')_0, (**')_2, \) etc.

**Lemma 1.10.** Condition \((**)\) holds if and only if there exists an isomorphism of lattices

\[\varepsilon : \Gamma_d \sim L_d.\]

In this case, one also has an isomorphism of groups

\[\tilde{O}(\Gamma, v_d) \simeq \tilde{O}(\Lambda_d).\]

**Proof.** Assume that there exists a (primitive) hyperbolic plane \( U \hookrightarrow L_d \). As the composition with the inclusion \( L_d \subset \Lambda \) can be identified with \( U \hookrightarrow \Lambda \) up to the action of \( O(\Lambda) \), see [Hu16, Thm. 14.1.12], one has \( U \perp \sim \Lambda \). Hence, \( \Gamma_d = L_d^+ \subset U^\perp \sim \Lambda \) is a primitive sublattice of corank one, signature \((2,19)\), discriminant \( d \), and is, therefore, isomorphic to \( \Lambda_d \). Conversely, if \( L_d^+ = \Gamma_d \simeq \Lambda_d \subset \Lambda \subset \tilde{\Lambda} \), then \( U_1 \subset L_d \). Here, one again uses that up to \( O(\tilde{\Lambda}) \), there exists only one primitive embedding \( \Lambda_d \hookrightarrow \tilde{\Lambda} \).

For the isomorphism between the two orthogonal groups, just recall that they are both described as the subgroup of all orthogonal transformations of \( \Gamma_d \simeq \Lambda_d \) acting trivially on the discriminant \( A_{\Gamma_d} \simeq A_{\Lambda_d} \simeq \mathbb{Z}/d\mathbb{Z} \).

**Remark 1.11.** As any isometric embedding \( U \hookrightarrow L_d \) splits, see [Hu16, Ex. 14.0.3], one concludes that for \( d \) satisfying \((**)_0\) and \((**)_2\), respectively, that

\[(**)_0: \quad A_2 \oplus \mathbb{Z} v_d \simeq L_d \simeq U \oplus \mathbb{Z}(d) \text{ and } (v_d)^2 = -(1/3) d \]

\[(**)_2: \quad A_2 \oplus \mathbb{Z} v_d \hookrightarrow L_d \simeq U \oplus \mathbb{Z}(d) \text{ index three and } (v_d)^2 = -3 d.\]

**Remark 1.12.** For a numerical description of these conditions one needs the following classical facts determining which numbers are represented by \( A_2 \), see [Co89, Kn02]. (i) For a given even, positive integer \( d \) there exists a vector \( w \in A_2 \) with \((w)^2 = d\) if and only if the prime factorization of \( d/2 \) satisfies

\[d/2 = \prod p^{n_p} \text{ with } n_p \equiv 0 \mod 2 \text{ for all } p \equiv 2 \mod 3.\]
(ii) For a given even, positive integer \(d\) there exists a primitive vector \(w \in A_2\) with \((w)^2 = d\) if and only if

\[
(1.4) \quad d = \prod p^{n_p} \text{ with } n_p = 0 \text{ for all } p \equiv 2 (3) \text{ and } n_3 \leq 1.
\]

**Proposition 1.13.** Numerically, \((\ast), \,**\,'\), \((\ast\ast)\), and \((\ast\ast\ast)\) are described by:

(i) \((\ast)\) \iff \(d \equiv 0, 2 (6)\).

(ii) \((\ast\ast\,')\) \iff \(\exists w \in A_2: (w)^2 = d \iff (1.3)\).

(iii) \((\ast\ast)\) \iff \(\exists w \in A_2 \text{ primitive: } (w)^2 = d \iff (1.4) \iff \exists a, n \in \mathbb{Z}: d = \frac{2n^2+2n+2}{a}\).

(iv) \((\ast\ast\ast)\) \iff \(\exists a, n \in \mathbb{Z}: d = \frac{2n^2+4n+2}{a}\).

**Proof.** The first assertion follows from Lemma 1.3.

To prove (ii), one has to distinguish between the two cases \(d \equiv 0 (6)\) and \(d \equiv 2 (6)\). Assume first that \((\ast\ast\,')\) holds. Then \(L_d = A_2 \oplus \mathbb{Z} v_d\), which contains the isotropic vector \(e \in U(n) \subset L_d\). Writing \(e = w_0 + a v_d\) for some \(w_0 \in A_2\) and \(a \in \mathbb{Z}\), one has \((w_0)^2 = a^2 d/3\). Hence, \(a^2 d/6\) satisfies \((1.3)\) and, therefore, \(d/2\) does. The latter then yields the existence of some \(w \in A_2\) with \((w)^2 = d\). Assume now we are in case \((\ast\ast\,')\), then the standard basis vector \(e \in U(1) \subset L_d\) itself might not be contained in \(A_2 \oplus \mathbb{Z} v_d\), but \(3e\) is and replacing \(e\) by \(3e\) and \((1/3)\) by \(3\), one can argue as before.

Conversely, if \(d/2\) satisfies \((1.3)\), then we can pick \(w \in A_2\) with \((w)^2 = d/3\) for \(d \equiv 0 (6)\) and with \((w)^2 = 3d\) for \(d \equiv 2 (6)\). Then \(e := w + v_d\) is isotropic. Furthermore, there exists \(w' \in A_2\) with \(m := (e.w') = (w,w') \neq 0\). Then \(f := m w' - ((w')^2/2) e\) satisfies \((f)^2 = 0\) and \((e,f) = (e.w')^2 := n\), which yields an embedding \(U(n) \hookrightarrow A_2 \oplus \mathbb{Z} v_d \subset L_d\) proving \((\ast\ast\,')\).

Turning to (iii) and using the notation in (ii), observe that in case \((\ast\ast\,')\), the class \(w_0\) has to be primitive. Indeed, if \(w_0 = p w_1\) for some prime \(p\), then \(p \mid a\) or \(p \mid d/3\). On the other hand, writing \(f = w_0 + a' v_d\) yields the contradiction \(1 = (e,f) = p(w_1 w_0') + aa' d/3 \equiv 0 (p)\). Hence, \(a^2 d/6\) satisfies \((1.4)\) and, therefore, \(d/2\) does, i.e. there exists a primitive \(w \in A_2\) with \((w)^2 = d/2\). The argument for \((\ast\ast)\) is similar: If \(e = w_0 + a v_d\), one argues as before. If not, then \(3e = w_0 + a v_d\) and if \(w_0 = p w_1\), then \(p \neq 3\). All other primes are excluded as before.

For the converse in this situation, we use the arguments above and pick a primitive \(w \in A_2\) with \((w)^2 = d/3\) or \(= 3d\), respectively. As \(A_2 \simeq \mathbb{Z}/3\mathbb{Z}\), either \((w,A_2) = \mathbb{Z}\) or \(= 3 \mathbb{Z}\). If \((w)^2 = d/3\), then the former holds (because \(3^2 \nmid d\)) and, therefore, \(w'\) above can be chosen such that \(m = 1\). Hence, there exists \(U \rightarrow L_d\). If \((w,A_2) = 3 \mathbb{Z}\), so in particular \((w)^2 = 3d\) and \(d \equiv 2 (6)\), then the class \(e := w \pm v_d\) is of the form \(e = 3e'\) with \(e' \in L_d\). Therefore, the two classes \(e'\) and \(f' := w' - ((w')^2/2) e'\), where \(w' \in A_2\) is chosen such that \((w,w') = 3\), define an embedding \(U \rightarrow L_d\).

As we will not use the presentation of \(d\) as \((2n^2+2n+2)/a\) and \((2n^2+2n+2)/a^2\), respectively, we leave the proof of the other equivalences to the reader, see [Ha00, Prop. 6.1.3] and [Ad16, Sec. 3].
The following table lists the first special discriminants, highlighting the difference between the four conditions.

| (***) | 14 | 26 | 38 | 42 |
| --- | --- | --- | --- | --- |
| (**) | 14 | 26 | 38 | 42 |
| (**') | 8 | 14 | 18 | 24 | 26 | 32 | 38 | 42 |
| (*) | 8 | 12 | 14 | 18 | 20 | 24 | 26 | 30 | 32 | 36 | 38 | 42 |

1.3. In the theory of K3 surfaces, there are good reasons to pass from the K3 lattice $\Lambda \simeq H^2(S, \mathbb{Z})$ to the Mukai lattice $\tilde{\Lambda} \simeq \tilde{H}(S, \mathbb{Z}) \simeq H^2(S, \mathbb{Z}) \oplus U_4$, see [Hu16, Ch. 16] for a survey and references. A similar extension of lattices, though slightly more technical due to the non-triviality of the canonical bundle, turns out to be useful for cubics and their comparison with K3 surfaces.

We have already constructed and fixed an isomorphism $\Gamma \simeq E \oplus U_1 \oplus U_2 \oplus A_2(-1) \simeq A_2^3$, where $A_2(-1) \oplus A_2^3 \cong U_3 \oplus U_4$. On the cubic side, one also finds a natural sublattice isomorphic to $U_3 \oplus U_4$, namely $H^{*\neq 4}(X, \mathbb{Z})$. However, the distinguished $A_2(-1) \subset \Gamma$ sits in $H^4(X, \mathbb{Z})$, so this has to be modified. Moreover, we will embed $A_2$ into rational cohomology $H^*(X, \mathbb{Q})$ and the intersection product on $H^*(X, \mathbb{Q})$ is modified by more than a mere sign.

**Definition 1.14.** The Mukai pairing on $H^*(X, \mathbb{Q})$ is defined as

$$(\alpha, \alpha') := -\int e^{\frac{\alpha^!(X)}{2}} \cdot \alpha^* \cdot \alpha'.$$

Here, $(\alpha_0 + \alpha_2 + \alpha_4 + \alpha_6 + \alpha_8)^* := \alpha_0 - \alpha_2 + \alpha_4 - \alpha_6 + \alpha_8$ and

$$e^{\frac{\alpha^!(X)}{2}} = e^{\frac{3h}{4}} = 1 + \frac{3}{2}h + \frac{9}{8}h^2 + \frac{27}{48}h^3 + \frac{81}{384}h^4.$$

**Warning:** Unlike the Mukai pairing for K3 surfaces, the pairing (1.5) is not symmetric.

**Definition 1.15.** The Mukai vector of a coherent sheaf $E \in \text{Coh}(X)$, or a complex $E \in \text{D}^b(X)$, or simply a class $E \in K_{\text{top}}(X)$ is defined as

$$v(E) := \text{ch}(E) \cdot \sqrt{\text{td}(X)}.$$

One easily computes

$$\sqrt{\text{td}(X)} = 1 + \frac{3}{4}h + \frac{11}{32}h^2 + \frac{15}{128}h^3 + \frac{121}{6144}h^4.$$
Using the general fact \( \sqrt{\text{td}^*} = e^{-\frac{\Delta}{2}} \cdot \sqrt{\text{td}} \) and the Grothendieck–Riemann–Roch formula, one expresses the Euler–Poincaré pairing of two coherent sheaves as
\[
(1.5) \quad \chi(E, E') = -v(E).v(E').
\]
Note that the left hand side is not symmetric, as \( \omega_X \) is not trivial. This confirms the observation that (1.5) is not symmetric.

**Example 1.16.** For our purposes the following classes are of importance:
\[
w_0 := v(\mathcal{O}_X) = \sqrt{\text{td}(X)}, \quad w_1 := v(\mathcal{O}_X(1)) = e^{i} \cdot \sqrt{\text{td}(X)},
\]
and
\[
w_2 := v(\mathcal{O}_X(2)) = e^{2i} \cdot \sqrt{\text{td}(X)}.
\]
In a sense to be made more precise, these classes are responsible for ( . ) not being symmetric. Explicitly, they are
\[
w_0 = 1 + \frac{3}{4}h + \frac{15}{32}h^2 + \frac{121}{6144}h^4, \quad w_1 = 1 + \frac{7}{4}h + \frac{128}{384}h^2 + \frac{385}{384}h^3 + \frac{2921}{6144}h^4,
\]
and
\[
w_2 = 1 + \frac{132}{32}h^2 + \frac{1397}{384}h^3 + \frac{16025}{6144}h^4.
\]
In addition to the classes \( w_0, w_1, w_2 \), one also needs the following ones
\[
v(\lambda_1) := 3 + \frac{5}{4}h - \frac{7}{32}h^2 - \frac{77}{384}h^3 + \frac{41}{2048}h^4.
\]
\[
v(\lambda_2) := -3 - \frac{1}{4}h + \frac{15}{32}h^2 + \frac{1}{384}h^3 - \frac{153}{2048}h^4.
\]

**Remark 1.17.** The notation suggests that the \( v(\lambda_i), i = 1, 2, \) are Mukai vectors of some natural (complexes of) sheaves. This is almost true, as we explain next. Consider an arbitrary line \( L \subset X \) and the two natural sheaves \( \mathcal{O}_L(i), i = 1, 2 \), on \( X \). Their Mukai vectors are
\[
u_i := v(\mathcal{O}_L(i)) = \begin{cases} \frac{1}{2}h^3 + \frac{5}{12}h^4 & \text{if } i = 1 \\ \frac{1}{2}h^3 + \frac{9}{12}h^4 & \text{if } i = 2. \end{cases}
\]
Under the right orthogonal projection \( H^*(X, \mathbb{Q}) \twoheadrightarrow \{w_0, w_1, w_2\}^\perp \) they are mapped to \( \lambda_i \). Explicitly,
\[
(1.7) \quad v(\lambda_1) = u_1 - w_1 + 4w_0 \quad \text{and} \quad v(\lambda_2) = u_2 - w_2 + 4w_1 - 6w_0.
\]
Here, one uses \((u_i, u_j) = 0 \) for all \( i, j \) and
\[
(w_i, w_j) = \chi(\mathcal{O}_X(i), \mathcal{O}_X(j)) = \chi(\mathcal{O}_X(j - i)),
\]
\[
(w_i, u_j) = \chi(\mathcal{O}_X(i), \mathcal{O}_L(j)) = \chi(\mathcal{O}_L(j - i)),
\]
\[
(u_i, u_j) = \chi(\mathcal{O}_L(i), \mathcal{O}_X(j)) = \chi(\mathcal{O}_L(i - j - 3)).
\]

**Lemma 1.18.** If \( H^*(X, \mathbb{Q}) \) is considered with the negative Mukai pairing, then
\[
A_2 \xrightarrow{\lambda_i} H^*(X, \mathbb{Q}), \quad \lambda_i \mapsto v(\lambda_i)
\]
defines an isometric embedding. Furthermore,
(i) $\mathbf{v}(\lambda_1), \mathbf{v}(\lambda_2) \in \{w_0, w_1, w_2\}^\perp$.
(ii) $w_0, w_1, w_2, v(\lambda_1), v(\lambda_2) \in \mathbb{Q}[h]$ are linearly independent.
(iii) $\{w_0, w_1, w_2, v(\lambda_1), v(\lambda_2)\}^\perp = H^4(X, \mathbb{Q})_{\text{pr}} = \{w_0, w_1, w_2, v(\lambda_1), v(\lambda_2)\}$, on which the Mukai pairing coincides with the intersection product (up to sign).
(iv) The Mukai pairing $(\;\cdot\;)$ is symmetric on the right orthogonal complement
\[ \{w_0, w_1, w_2\}^\perp \subset H^*(X, \mathbb{Q}). \]

**Proof.** The first assertion can be verified by a computation or using (1.7). Similarly, (i) follows from the observation that $\mathbf{v}(\lambda_i)$ is the orthogonal projection of $u_i$ and (ii) is again proven by a computation. Finally, (ii) implies (iii) and (iv) can be deduced from (iii). \hfill \Box

**Corollary 1.19.** The lattices $A^2_2 \simeq \Gamma \simeq H^4(X, \mathbb{Z})_{\text{pr}} \subset H^*(X, \mathbb{Q})$ and $A_2 \simeq \mathbb{Z} \mathbf{v}(\lambda_1) \oplus \mathbb{Z} \mathbf{v}(\lambda_2) \subset H^*(X, \mathbb{Q})$ are orthogonal with respect to the Mukai pairing (1.5). The induced embedding of their direct sum $A^2_2 \oplus A_2$ extends to
\[ (1.8) \qquad A^2_2 \oplus A_2 \subset \tilde{\Lambda} \hookrightarrow H^*(X, \mathbb{Q}). \]  

A more conceptual understanding of these calculations is provided by the discussion in [AT14]. In particular, cohomology with rational coefficients $H^*(X, \mathbb{Q})$ is replaced by integral topological $K$-theory. Denote by $K_{\text{top}}(X)$ the topological $K$-theory of all complex vector bundles. Traditionally, the Chern character is used to identify $K_{\text{top}}(X) \otimes \mathbb{Q}$ with $H^*(X, \mathbb{Q}) = H^{2*}(X, \mathbb{Q})$. For our purposes the Mukai vector is better suited
\[ \mathbf{v}: K_{\text{top}}(X) \hookrightarrow K_{\text{top}}(X) \otimes \mathbb{Q} \iso H^*(X, \mathbb{Q}). \]

Note that the torsion freeness of $K_{\text{top}}(X)$ follows from the torsion freeness of $H^*(X, \mathbb{Z})$ and the Atiyah–Hirzebruch spectral sequence. Then $K_{\text{top}}(X)$ is equipped with a non-degenerate but non-symmetric linear form with values in $\mathbb{Q}$. Due to (1.6), it takes values in $\mathbb{Z}$ on the image of the highly non-injective map $K(X) \longrightarrow K_{\text{top}}(X)$. Clearly, the classes $[\mathcal{O}_X(i)], i = 0, 1, 2,$ and $[\mathcal{O}_L(i)], i = 1, 2,$ are all contained in the image. We shall be interested in the right orthogonal complement of the former three classes and introduce the notation:
\[ K'_{\text{top}}(X) := \{ [\mathcal{O}_X], [\mathcal{O}_X(1)], [\mathcal{O}_X(2)] \}^\perp \subset K_{\text{top}}(X). \]

**Proposition 1.20** (Addington–Thomas). The restriction of the Mukai pairing $(\;\cdot\;)$ to $K'_{\text{top}}(X)$ is symmetric and integral. Moreover, as abstract lattices
\[ \tilde{\Lambda} \simeq K'_{\text{top}}(X). \]

**Proof.** Note that $v: K'_{\text{top}}(X) \otimes \mathbb{Q} \iso \{w_0, w_1, w_2\}^\perp$. Hence, Lemma 1.18 implies the first assertion. The original proof [AT14] of the second assertion uses derived categories. Here is a sketch of a more direct, purely topological argument. Consider the right orthogonal projection $p: K_{\text{top}}(X) \longrightarrow K'_{\text{top}}(X)$. It really is defined over $\mathbb{Z}$, as $(w_i)^2 = 1$. Analogously to (1.7), one has $p[\mathcal{O}_L(1)] = [\mathcal{O}_L(1)] - [\mathcal{O}_X(1)] + 4[\mathcal{O}_X]$ and $p[\mathcal{O}_L(2)] = [\mathcal{O}_L(2)] - [\mathcal{O}_X(2)] + 4[\mathcal{O}_X(1)] - 6[\mathcal{O}_X]$. Hence, $\lambda_i \mapsto p[\mathcal{O}_L(i)]$ defines an isometric embedding $A_2 \hookrightarrow K'_{\text{top}}(X)$.
First, $H^4(X, \mathbb{Z})_{\text{pr}} \subset H^4(X, \mathbb{Q})$ is contained in $\nu(K'_{\text{top}}(X))$. Indeed, $H^4(X, \mathbb{Z})_{\text{pr}}$ is spanned by classes of all vanishing spheres and those lift to $K_{\text{top}}(X)$. After fixing an isometry $E \oplus U^\oplus 2 \oplus A_2(-1) \simeq A_2^1 \simeq \Gamma \simeq H^4(X, \mathbb{Z})_{\text{pr}} \subset K'_{\text{top}}(X)$, this yields an isometric embedding $\Gamma \oplus A_2 \hookrightarrow K'_{\text{top}}(X)$ and allows one to view $\mu_1, \mu_2 \in A_2(-1)$ as classes in $K'_{\text{top}}(X)$.

Second, one needs to show that the class $(1/3)(\mu_1 - \mu_2 - \lambda_1 + \lambda_2) \in (A_2(-1) \oplus A_2) \otimes \mathbb{Q} \subset K_{\text{top}}(X) \otimes \mathbb{Q}$ is integral, i.e. contained in $K_{\text{top}}(X)$. This presumably can be achieved algebraically on some particular cubic fourfold. Hence, the embedding in step one extends to an isometric embedding $\tilde{\Lambda} \hookrightarrow K'_{\text{top}}(X)$ of finite index. The unimodularity of $\tilde{\Lambda}$ then implies the second assertion. □

1.4. We now endow the various lattices considered above with natural Hodge structures. Let us first briefly recall the well known theory for K3 surfaces, see [Hu16, Ch. 16] for further details and references.

For any complex K3 surface $S$ its second cohomology $H^2(S, \mathbb{Z})$, which as a lattice is isomorphic to $\Lambda$, comes with a natural Hodge structure of weight two given by the $(2,0)$-part $H^{2,0}(S)$. The full Hodge structure is then determined by additionally requiring $H^{1,1}(S) \perp H^{2,0}(S)$ with respect to the intersection pairing.

The global Torelli theorem for complex K3 surfaces asserts that two K3 surfaces $S$ and $S'$ are isomorphic if and only if there exists a Hodge isometry $H^2(S, \mathbb{Z}) \simeq H^2(S', \mathbb{Z})$, i.e. an isomorphism of integral Hodge structures that is compatible with the intersection pairing:

$$ S \simeq S' \iff \exists H^2(S, \mathbb{Z}) \simeq H^2(S', \mathbb{Z}) \text{ Hodge isometry.} $$

Let $(S, L)$ be a polarized K3 surface. Then the primitive cohomology $H^2(S, \mathbb{Z})_{L-\text{pr}} \subset H^2(S, \mathbb{Z})$ is endowed with the induced structure. Its $(2,0)$-part is again $H^{2,0}(S)$ and its $(1,1)$-part is the primitive part of $H^{1,1}(S)$, i.e. the kernel of $(L, \cdot) : H^{1,1}(S) \longrightarrow \mathbb{C}$. The polarized version of the global Torelli theorem is the statement that two polarized K3 surfaces $(S, L)$ and $(S', L')$ are isomorphic if and only if there exists a Hodge isometry $H^2(S, \mathbb{Z}) \simeq H^2(S', \mathbb{Z})$ inducing $H^2(S, \mathbb{Z})_{L-\text{pr}} \simeq H^2(S', \mathbb{Z})_{L'-\text{pr}}$:

$$ (S, L) \simeq (S', L) \iff \exists H^2(S, \mathbb{Z}) \simeq H^2(S', \mathbb{Z}), \quad L \longrightarrow L', \text{ Hodge isometry} $$

The result will be stated again in moduli theoretic terms in Theorem 2.1.

**Warning:** A Hodge isometry $H^2(S, \mathbb{Z})_{L-\text{pr}} \simeq H^2(S', \mathbb{Z})_{L'-\text{pr}}$ does not necessarily extend to a Hodge isometry between the full cohomology. Hence, in general, the existence of a Hodge isometry between the primitive Hodge structures of two polarized K3 surfaces does not imply that $(S, L)$ and $(S', L')$ are isomorphic. In fact, even the unpolarized K3 surfaces $S$ and $S'$ may be non-isomorphic.

Next comes the Mukai Hodge structure $\tilde{H}(S, \mathbb{Z})$. The underlying lattice is $H^4(S, \mathbb{Z})$ with the sign change in $U_4 = (H^0 \oplus H^4)(S, \mathbb{Z})$. The Hodge structure of weight two is again given by the $(2,0)$-part being $\tilde{H}^{2,0}(S) := H^{2,0}(S)$ and the condition that $\tilde{H}^{1,1}(S) \perp H^{2,0}(S)$ with respect

---

2 This mysterious class would need to satisfy the two equations $(\lambda_1, \alpha) = -1$ and $(\lambda_2, \alpha) = 1$. 
to the Mukai pairing. In particular, \( U \simeq U_4 = (H^0 \oplus H^4)(S, \mathbb{Z}) \) is contained in \( \bar{H}^{1,1}(S, \mathbb{Z}) \).

The derived global Torelli theorem is the statement that for two projective K3 surfaces \( S \) and \( S' \) there exists an exact, \( \mathbb{C} \)-linear equivalence \( D^b(S) \simeq D^b(S') \) between their bounded derived categories of coherent sheaves if and only if there exists a Hodge isometry \( \bar{H}(S, \mathbb{Z}) \simeq \bar{H}(S', \mathbb{Z}) \):

\[
D^b(S) \simeq D^b(S') \Leftrightarrow \exists \bar{H}(S, \mathbb{Z}) \simeq \bar{H}(S', \mathbb{Z}) \text{ Hodge isometry.}
\]

A twisted K3 surface \((S, \alpha)\) consists of a K3 surface \( S \) together with a Brauer class \( \alpha \in \text{Br}(S) \simeq H^2(S, \mathcal{O}_S^*) \) (we work in the analytic topology). Choosing a lift \( B \in H^2(S, \mathbb{Q}) \) of \( \alpha \) under the natural morphism \( H^2(S, \mathbb{Q}) \longrightarrow \text{Br}(S) \) induced by the exponential sequence allows one to introduce a natural Hodge structure \( \bar{H}(S, \alpha, \mathbb{Z}) \) of weight two associated with \((S, \alpha)\). As a lattice, this is just \( \bar{H}(S, \mathbb{Z}) \), but the \((2,0)\)-part is now given by \( \bar{H}^{2,0}(S, \alpha) := \mathbb{C}(\sigma + \sigma \wedge B) \), where \( 0 \neq \sigma \in H^{2,0}(S) \). This defines a Hodge structure by requiring, as before, that \( \bar{H}^{1,1}(S, \alpha) \perp \bar{H}^{2,0}(S, \alpha) \) with respect to the Mukai pairing. Although the definition depends on the choice of \( B \), the Hodge structures induced by two different lifts \( B \) and \( B' \) of the same Brauer class \( \alpha \) are Hodge isometric albeit not canonically, see [HS05].

The twisted version of the derived global Torelli theorem is the statement that the bounded derived categories of twisted coherent sheaves on \((S, \alpha)\) and \((S', \alpha')\) are equivalent if and only if there exists a Hodge isometry \( \bar{H}(S, \alpha, \mathbb{Z}) \simeq \bar{H}(S', \alpha', \mathbb{Z}) \) preserving the natural orientation of the four positive directions, cf. [Hu16] Ch. 16.4 and [Re17]:

\[
D^b(S, \alpha) \simeq D^b(S', \alpha') \Leftrightarrow \exists \bar{H}(S, \alpha, \mathbb{Z}) \simeq \bar{H}(S', \alpha', \mathbb{Z}) \text{ oriented Hodge isometry.}
\]

Next consider \( H^4(X, \mathbb{Z}) \) and \( H^4(X, \mathbb{Z})_{\text{pr}} \) of a smooth cubic fourfold \( X \). These are Hodge structures of weight four determined by the one-dimensional \( H^{3,1}(X) \) and the condition that \( H^{3,1}(X) \perp H^{2,2}(X) \) with respect to the intersection product.

The global Torelli theorem for smooth cubic fourfolds, which we will state again as Theorem 2.12 in moduli theoretic terms, is the statement that two smooth cubic fourfolds \( X \) and \( X' \) are isomorphic (as abstract complex varieties) if and only if there exists a Hodge isometry \( H^4(X, \mathbb{Z})_{\text{pr}} \simeq H^4(X', \mathbb{Z})_{\text{pr}} \):

\[
X \simeq X' \Leftrightarrow \exists H^4(X, \mathbb{Z})_{\text{pr}} \simeq H^4(X', \mathbb{Z})_{\text{pr}} \text{ Hodge isometry.}
\]

Note that any such Hodge isometry can be extended to a Hodge isometry \( H^4(X, \mathbb{Z}) \simeq H^4(X', \mathbb{Z}) \) that maps \( h^2_X \) to \( \pm h^2_{X'} \). The situation here is easier compared to the case of polarized K3 surfaces as the discriminant of \( H^4(X, \mathbb{Z})_{\text{pr}} \) is just \( \mathbb{Z}/3\mathbb{Z} \).

To relate \( H^4(X, \mathbb{Z}) \) of a cubic fourfolds to K3 surfaces one has to change the sign of the intersection product, so that as abstract lattices \( H^4(X, \mathbb{Z}) \simeq \tilde{\Gamma} \) and \( H^4(X, \mathbb{Z})_{\text{pr}} \simeq \tilde{\Gamma} \) (with an implicit sign change), and Tate shift the Hodge structure to obtain \( H^4(X, \mathbb{Z})(1) \) and \( H^4(X, \mathbb{Z})_{\text{pr}}(1) \), which are now Hodge structures of weight two.

\footnote{We will encounter yet another Torelli theorem in Section 3.3}
Definition 1.21. The integral Hodge structure $\tilde{H}(X, \mathbb{Z})$ of K3 type associated with a smooth cubic fourfold $X$ is the lattice

$$\tilde{H}(X, \mathbb{Z}) := K_{\text{top}}'(X)$$

with the Hodge structure of weight two given by $\tilde{H}^{2,0}(X) := v^{-1}(H^3,1(X))$ and the requirement that $\tilde{H}^{1,1}(X)$ and $\tilde{H}^{2,0}(X)$ are orthogonal with respect to the Mukai pairing on $K_{\text{top}}(X)$.

The Mukai vector $K_{\text{top}}(X) \otimes \mathbb{Q} \xrightarrow{\sim} H^*(X, \mathbb{Q})$ induces an isometry

$$\tilde{H}(X, \mathbb{Z}) = K_{\text{top}}(X) \simeq \Lambda \subset H^*(X, \mathbb{Q})$$

with $\tilde{\Lambda} \subset H^*(X, \mathbb{Q})$ provided by [18]. Observe that there is a natural isometric inclusion of Hodge structures

$$H^4(X, \mathbb{Z})_{\text{pr}}(1) \subset \tilde{H}(X, \mathbb{Z}).$$

Moreover, the sublattice $A_2$ is algebraic, i.e. $A_2 \subset \tilde{H}^{1,1}(X, \mathbb{Z})$, and its orthogonal Hodge structure is $A_2^\perp \simeq H^4(X, \mathbb{Z})_{\text{pr}}(1)$. Also note that according to Remark [11, 12] $\Lambda_1^\perp \subset \tilde{H}(X, \mathbb{Z})$ is a sub Hodge structure with underlying lattice isomorphic to $\Lambda \oplus \mathbb{Z}(-2)$.

Remark 1.22. Once the Kuznetsov category $A_X \subset D^b(X)$ is introduced, one also writes $\tilde{H}(A_X, \mathbb{Z}) = \tilde{H}(X, \mathbb{Z})$. The notation $\tilde{H}(X, \mathbb{Z})$ is analogous to the notation $\tilde{H}(S, \mathbb{Z})$ for K3 surfaces and the Hodge structure plays a similar role. In fact, as a consequence of the above discussion we know that as lattices $\tilde{H}(X, \mathbb{Z}) \simeq \tilde{H}(S, \mathbb{Z})$ and the analogy goes further: For a K3 surface, the algebraic part naturally contains a hyperbolic plane:

$$U \simeq (H^0 \oplus H^4)(S, \mathbb{Z}) \hookrightarrow \tilde{H}^{1,1}(S, \mathbb{Z}).$$

Similarly, for a smooth cubic fourfold the algebraic part naturally contains a copy of $A_2$:

$$v: A_2 \simeq \mathbb{Z} \cdot p[\mathcal{O}_L(1)] \oplus \mathbb{Z} \cdot p[\mathcal{O}_L(2)] \hookrightarrow \tilde{H}^{1,1}(X, \mathbb{Z}).$$

Here, $p: K_{\text{top}}(X) \rightarrow K_{\text{top}}'(X)$ is the projection as in the proof of Proposition [11, 20] so the composition maps $\lambda \mapsto p(\mathcal{O}_L(i)) \mapsto v(\lambda)$. Their respective orthogonal complements are

$$H^2(S, \mathbb{Z}) = U^\perp \hookrightarrow \tilde{H}(S, \mathbb{Z}) \quad \text{and} \quad H^4(X, \mathbb{Z})_{\text{pr}}(1) = A_2^\perp \hookrightarrow \tilde{H}(X, \mathbb{Z}),$$

in terms of which the global Torelli theorem is formulated in both instances. Also, $e_4 - f_4 = (1, 0, -1) \in \tilde{H}^{1,1}(S, \mathbb{Z})$ and $v(\lambda_1) \in \tilde{H}^{1,1}(X, \mathbb{Z})$ are both algebraic classes satisfying $(e_4 - f_4)^2 = 2 = (v(\lambda_1))^2$. Their orthogonal complements are isometric.

Definition 1.23. Let $(S, L)$ be a polarized K3 surface and $X$ a smooth cubic fourfold.

(i) We say $(S, L)$ and $X$ are associated, $(S, L) \sim X$, if there exists an isometric embedding of Hodge structures

$$H^2(S, \mathbb{Z})_{L,\text{pr}} \hookrightarrow H^4(X, \mathbb{Z})_{\text{pr}}(1).$$

(ii) We say $S$ and $X$ are associated, $S \sim X$, if there exists a Hodge isometry

$$\tilde{H}(S, \mathbb{Z}) \simeq \tilde{H}(X, \mathbb{Z}).$$
(iii) For $\alpha \in \text{Br}(S)$ we say that the twisted K3 surface $(S, \alpha)$ and $X$ are associated, $(S, \alpha) \sim X$, if there exists a Hodge isometry

$$\tilde{H}(S, \alpha, \mathbb{Z}) \simeq \tilde{H}(X, \mathbb{Z}).$$

First observe the immediate implication:

$$(S, L) \sim X \Rightarrow S \sim X.$$}

Indeed, any isometric embedding (1.9) can be extended to an isometry $\tilde{H}(S, \mathbb{Z}) \simeq \tilde{H}(X, \mathbb{Z})$. This follows from the existence of the hyperbolic plane $U \subset H^2(S, \mathbb{Z})_{\text{pr}}$, cf. [Hu16] Rem. 14.1.13].

As an aside, observe that a K3 surface $S$ that is associated with a cubic fourfold in any sense is necessarily projective. Indeed, if for example $S \sim X$, then $\tilde{H}^{1,1}(S, \mathbb{Z}) \simeq \tilde{H}^{1,1}(X, \mathbb{Z})$ contains the positive plane $A_2$ and, therefore, $H^{1,1}(S, \mathbb{Z})$ contains at least one class of positive square.

The key to link $S \sim X$, $(S, L)$, and $(S, \alpha) \sim X$ to the properties $(**)$ and $(**')$ is the following result in [AT14] generalized to the twisted case in [Hu17].

**Proposition 1.24** (Addington–Thomas, Huybrechts). Assume $X$ is a smooth cubic fourfold.

(i) There exists a K3 surface $S$ with $S \sim X$ if and only if there exists a (primitive) embedding $U \hookrightarrow \tilde{H}^{1,1}(X, \mathbb{Z})$.

(ii) There exists a twisted K3 surface $(S, \alpha)$ with $(S, \alpha) \sim X$ if and only if there exists an embedding $U(n) \hookrightarrow \tilde{H}^{1,1}(X, \mathbb{Z})$ for some $n \neq 0$.

**Proof.** Any Hodge isometry $\tilde{H}(S, \mathbb{Z}) \simeq \tilde{H}(X, \mathbb{Z})$ yields a hyperbolic plane $U \simeq (H^0 \oplus H^4)(S, \mathbb{Z}) \subset \tilde{H}^{1,1}(S, \mathbb{Z}) \simeq \tilde{H}^{1,1}(X, \mathbb{Z})$. Conversely, if $U \subset \tilde{H}^{1,1}(X, \mathbb{Z}) \subset \tilde{H}(X, \mathbb{Z})$, then as a lattice $U^\perp \simeq \Lambda$. Moreover, the Hodge structure of $\tilde{H}(X, \mathbb{Z})$ induces a Hodge structure on $U^\perp \simeq \Lambda$ which due to the surjectivity of the period map [Hu16] Thm. 7.4.1] is Hodge isometric to $H^2(S, \mathbb{Z})$ for some K3 surface $S$. However, as before, $U^\perp \simeq H^2(S, \mathbb{Z})$ extends to $\tilde{H}(X, \mathbb{Z}) \simeq \tilde{H}(S, \mathbb{Z})$. This proves (i).

For (ii), again one direction is easy, as $\tilde{H}^{1,1}(S, \mathbb{Z})$ contains the B-field shift of $(H^0 \oplus H^4)(S, \mathbb{Z})$, cf. [Hu16] Ch. 14. More precisely, $\tilde{H}^{1,1}(S, \alpha, \mathbb{Z}) = (\exp(B) \tilde{H}^{1,1}(S, \mathbb{Q})) \cap \tilde{H}(S, \mathbb{Z})$, which clearly contains the lattice $((1, B, B^2/2) \cap \tilde{H}(S, \mathbb{Z})) \oplus H^4(S, \mathbb{Z}) \simeq U(n)$, where $n$ is minimal with $n(1, B, B^2) \in \tilde{H}(S, \mathbb{Z})$. The other direction needs a surjectivity statement for twisted K3 surfaces which is an easy consequence of the surjectivity of the untwisted period map. □

**Proposition 1.25.** Assume a smooth cubic fourfold $X$ is associated with some K3 surface $S$, so $S \sim X$. Then there exists a polarized K3 surface $(S', L') \sim X$:

$$\tilde{H}(S, \mathbb{Z}) \simeq \tilde{H}(X, \mathbb{Z}) \Rightarrow H^2(S', \mathbb{Z})_{\text{pr}} \hookrightarrow H^4(X, \mathbb{Z})_{\text{pr}}.$$

**Proof.** Assume $S \sim X$. Then there exists a Hodge isometry $\tilde{H}(S, \mathbb{Z}) \simeq \tilde{H}(X, \mathbb{Z})$. On the left hand side, one finds $U \simeq (H^0 \oplus H^4)(S, \mathbb{Z}) \subset \tilde{H}^{1,1}(S, \mathbb{Z})$ and, on the right hand side, $A_2 \subset \tilde{H}^{1,1}(X, \mathbb{Z})$. Consider the saturation of the sum of both as a lattice $U + A_2 \subset \tilde{H}^{1,1}(S, \mathbb{Z})$. 

According to Lemma 1.2, there exists another hyperbolic plane $U' \subset U + A_2$ with $\text{rk}(U' + A_2) = 3$. Using the surjectivity of the period map, one finds another K3 surface $S'$ and a Hodge isometry

$$\tilde{H}(S', \mathbb{Z}) \simeq \tilde{H}(S, \mathbb{Z}) \simeq \tilde{H}(X, \mathbb{Z})$$

inducing $H^2(S', \mathbb{Z}) \simeq U'$. But then $H^2(S', \mathbb{Z}) \cap A_2^\perp \subset H^2(S', \mathbb{Z})$ is of corank one and we can assume it to be of the form $H^2(S', \mathbb{Z})_{L' - \text{pr}}$. However, being contained in $A_2^\perp$ implies that under $(1.10)$ $H^2(S', \mathbb{Z})_{L' - \text{pr}}$ embeds into $H^4(X, \mathbb{Z})_{\text{pr}(1)}$, which ensures $(S', L') \sim X$. □

**Corollary 1.26.** A smooth cubic fourfold $X$ is associated with some polarized K3 surface, $(S, L) \sim X$, if and only if there exists an isometric embedding $U \xrightarrow{\sim} H^{1,1}(X, \mathbb{Z})$. □

## 2. Period domains and moduli spaces

The comparison of the Hodge theory of K3 surfaces and cubic fourfolds is now considered in families. Via period maps, this leads to an algebraic correspondence between the moduli space of polarized K3 surfaces of certain degrees and the moduli space of cubic fourfolds. The approach has been initiated by Hassett [Ha00] and has turned out to be very valuable indeed.

### 2.1. Here is a very brief reminder on some results, mostly due to Borel and Baily–Borel, on arithmetic quotients of orthogonal type. Let $(N, \langle \cdot, \cdot \rangle)$ be a lattice of signature $(2, n_-)$ and set $V := N \otimes \mathbb{R}$. Then the period domain $D_N$ associated with $N$ is the Grassmannian of positive, oriented planes $W \subset V$, which alternatively can be described as

$$D_N \simeq \{ x | \langle x \rangle^2 = 0, \langle x, x \rangle > 0 \} \subset \mathbb{P}(N \otimes \mathbb{C})$$

$$\simeq \text{O}(2, n_-)/\text{O}(2) \times \text{O}(n_-).$$

By definition, the period domain $D_N$ associated with $N$ has the structure of a complex manifold. This is turned into an algebraic statement by the following fundamental result [BB66]. It uses the fact that under the assumption on the signature of $N$ the orthogonal group $\text{O}(N)$ acts properly discontinuously on $D_N$.

**Theorem 2.1** (Baily–Borel). Assume $G \subset \text{O}(N)$ is a torsion free subgroup of finite index. Then the quotient

$$G \setminus D_N$$

has the structure of a smooth, quasi-projective complex variety.

As $G$ acts properly discontinuously as well, the stabilizers are finite and hence trivial. This already proves the smoothness of the quotient $G \setminus D_N$. The difficult part of the theorem is to find a Zariski open embedding into a complex projective variety.

Finite index subgroups $G \subset \text{O}(N)$ with torsion are relevant, too. In this situation, one uses Minkowski’s theorem stating that the map $\pi_p: \text{Gl}(n, \mathbb{Z}) \longrightarrow \text{Gl}(n, \mathbb{F}_p), p \geq 3$, is injective on finite subgroups or, equivalently, that its kernel is torsion free. Hence, for every finite index subgroup
$G \subset O(N)$ there exists a normal and torsion free subgroup $G_0 := G \cap \text{Ker} (\pi_p) \subset G$ of finite index.

**Corollary 2.2.** Assume $G \subset O(N)$ is a subgroup of finite index. Then the quotient $G \backslash D_N$ has the structure of a normal, quasi-projective complex variety with finite quotient singularities. □

We remark that not only these arithmetic quotients, but also holomorphic maps into them are algebraic. This is the following remarkable GAGA style result, see [Bor72].

**Theorem 2.3 (Borel).** Assume $G \subset O(N)$ is a torsion free subgroup of finite index. Then any holomorphic map $\varphi: Z \rightarrow G \backslash D_N$ from a complex variety $Z$ is regular.

**Remark 2.4.** Often, the result is applied to holomorphic maps to singular quotients $G \backslash D_N$, i.e. in situations when $G$ is not necessarily torsion free. This is covered by the above only when $Z \rightarrow G \backslash D_N$ is induced by a holomorphic map $Z' \rightarrow G_0 \backslash D_N$, where $Z' \rightarrow Z$ is a finite quotient and $G_0 \subset G$ is a normal, torsion free subgroup of finite index.

2.2. We shall be interested in (at least) three different types of period domains: For polarized K3 surfaces and for (special) smooth cubic fourfolds. These are the period domains associated with the lattices $\Gamma, \Gamma_d, \text{and } \Lambda_d$:

$$D \subset \mathbb{P} (\Gamma \otimes \mathbb{C}), \quad D_d \subset \mathbb{P} (\Gamma_d \otimes \mathbb{C}), \quad \text{and } Q_d \subset \mathbb{P} (\Lambda_d \otimes \mathbb{C}).$$

These period domains are endowed with the natural action of the corresponding orthogonal groups $O(\Gamma), O(\Gamma_d), \text{and } O(\Lambda_d)$ and we will be interested in the following quotients by distinguished finite index subgroups of those:

$$\mathcal{C} := \tilde{O}(\Gamma) \backslash D = O(\Gamma) \backslash D, \quad \mathcal{C}_d := \tilde{O}(\Gamma, K_d) \backslash D_d, \quad \mathcal{C}_d := \tilde{O}(\Gamma, v_d) \backslash D_d, \quad \text{and } \mathcal{M}_d := \tilde{O}(\Lambda_d) \backslash Q_d.$$

For the first equality note that $\tilde{O}(\Gamma) \subset O(\Gamma)$ is of index two, but $-\text{id} \in O(\Gamma) \setminus \tilde{O}(\Gamma)$ acts trivially on $D$. The subgroup $\tilde{O}(\Lambda_d) \subset O(\Lambda_d)$ is defined analogously to (1.2).

Due to Theorem 2.1 and 2.3 see also Remark 2.4 the induced maps $\tilde{\mathcal{C}}_d \longrightarrow \tilde{\mathcal{C}}_d \longrightarrow \tilde{\mathcal{C}}$ are regular morphisms between normal quasi-projective varieties. The image in $\mathcal{C}$ shall be denoted by $\mathcal{C}_d$, so that

$$\tilde{\mathcal{C}}_d \longrightarrow \tilde{\mathcal{C}}_d \longrightarrow \mathcal{C}_d \subset \mathcal{C}.$$  

The condition $(\ast)$ will in the sequel be interpreted as the condition that $\mathcal{C}_d \neq \emptyset$.

**Corollary 2.5 (Hassett).** Assume $d$ satisfies $(\ast)$. The naturally induced maps

$$\tilde{\mathcal{C}}_d \longrightarrow \tilde{\mathcal{C}}_d \longrightarrow \mathcal{C}_d$$

are surjective, finite, and algebraic.

Furthermore, $\tilde{\mathcal{C}}_d \longrightarrow \mathcal{C}_d$ is the normalization of $\mathcal{C}_d$ and $\tilde{\mathcal{C}}_d \longrightarrow \tilde{\mathcal{C}}_d$ is a finite morphism between normal varieties, which is an isomorphism if $d \equiv 2 \ (6)$ and of degree two if $d \equiv 0 \ (6)$. 

Proof. Clearly, if \( d \) satisfies \((\ast)_2\), then \( \tilde{O}(\Gamma, K_d) = \tilde{O}(\Gamma, v_d) \) by Lemma 1.8 and, therefore, \( \tilde{C}_d \simeq \tilde{C}_d \). Otherwise, \( \tilde{C}_d \longrightarrow \tilde{C}_d \) is the quotient by the involution \( g \in \tilde{O}(\Gamma) \) defined by \( g = \text{id} \) on \( E \oplus U_2 \oplus I_{0,3} \) and \( g = -\text{id} \) on \( U_1 \), which indeed acts non-trivially on \( \tilde{C}_d \).

To prove that \( \tilde{C}_d \longrightarrow \tilde{C}_d \) is quasi-finite, use that \( \tilde{C}_d \longrightarrow \tilde{C}_d \) is algebraic with discrete and hence finite fibres. For a very general \( x \in D_d \) such that there does not exist any proper primitive sublattice \( N \subset \Gamma_d \) with \( x \in N \otimes \mathbb{C} \), any \( g \in \tilde{O}(\Gamma) \) with \( g(x) = x \) also satisfies \( g(\Gamma_d) = \Gamma_d \) and, therefore, \( g(K_d) = K_d \), i.e. \( g \in \tilde{O}(\Gamma, K_d) \). This proves that \( \tilde{C}_d \longrightarrow \tilde{C}_d \) is generically injective. Thus, once \( \tilde{C}_d \longrightarrow \tilde{C}_d \) is shown to be finite, and not only quasi-finite, it is the normalization of its image \( \tilde{C}_d \). We refer to [Br18, Ha00] for more details on this point. \( \square \)

**Remark 2.6.** Note that while the fibre of \( \tilde{C}_d \longrightarrow \tilde{C}_d \) consists of at most two points, the fibres of \( \tilde{C}_d \longrightarrow \tilde{C}_d \) may contain more points, depending on the singularity type of the points in \( \tilde{C}_d \). For fixed \( d \), the cardinality of the fibres is bounded. However, it is unbounded when \( d \) is allowed to grow.

Lemma 1.10 immediately yields the following result which eventually leads to the mysterious relation between K3 surfaces and cubic fourfolds.

**Corollary 2.7.** Assume \( d \) satisfies \((\ast)_\circ \). We choose an isomorphism \( \varepsilon : \Gamma_d \longrightarrow \Lambda_d \).

(i) If \( d \) satisfies \((\ast)_0 \), then \( \varepsilon \) naturally induces an isomorphism \( \mathcal{M}_d \simeq \tilde{C}_d \). Therefore, \( \mathcal{M}_d \) comes with a finite morphism onto \( \mathcal{C}_d \) generically of degree two:

\[
\Phi_\varepsilon : \mathcal{M}_d \simeq \tilde{C}_d \longrightarrow \tilde{C}_d \quad \text{norm} \quad \mathcal{C}_d \subset \mathcal{C}.
\]

(ii) If \( d \) satisfies \((\ast)_2 \), then \( \varepsilon \) naturally induces an isomorphism \( \mathcal{M}_d \simeq \tilde{C}_d \simeq \tilde{C}_d \). Therefore, \( \mathcal{M}_d \) can be seen as the normalization of \( \tilde{C}_d \subset \mathcal{C} \):

\[
\Phi_\varepsilon : \mathcal{M}_d \simeq \tilde{C}_d \simeq \tilde{C}_d \quad \text{norm} \quad \mathcal{C}_d \subset \mathcal{C}.
\] \( \square \)

**Remark 2.8.** As indicated by the notation, the morphism \( \Phi_\varepsilon : \mathcal{M}_d \longrightarrow \mathcal{C}_d \subset \mathcal{C} \), which will be seen to link polarized K3 surfaces \((S, L)\) of degree \( d \) with special cubic fourfolds \( X \), depends on the choice of \( \varepsilon : \Gamma_d \longrightarrow \Lambda_d \). There is no distinguished choice for \( \varepsilon \) and, therefore, one should not expect to find a distinguished morphism \( \mathcal{M}_d \longrightarrow \mathcal{C}_d \) that can be described by a geometric procedure associating a cubic fourfold \( X \) to a polarized K3 surface \((S, L)\).

To avoid any dependance on \( \varepsilon \), one could think of defining a morphism from the finite quotient

\[
\pi_d : \mathcal{M}_d = \tilde{O}(\Lambda_d) \setminus Q_d \longrightarrow \tilde{M}_d := O(\Lambda_d) \setminus Q_d
\]
to some meaningful quotient of \( \mathcal{C} \). But, as the degree of \( \pi_d \) grows with \( d \), there definitely is no reasonable quotient of \( \mathcal{C} \) that would receive all of them. However, it seems plausible that a quotient \( \mathcal{C}_d \longrightarrow \tilde{C}_d \) can be constructed that allows for a morphism \( \tilde{M}_d \longrightarrow \tilde{C}_d \). The derived point of view to be explained later will shed more light on this.

\[ \underline{4} \] I wish to thank E. Brakkee and P. Magni for discussions concerning this point.
2.3. We start by recalling the central theorem in the theory of K3 surfaces: the global Torelli theorem. In the situation at hand, it is due to Pjateckiĭ-Šapiro and Šafarevič, see [Hu16] for details, generalizations, and references.

Consider the coarse moduli space $M_d$ of polarized K3 surfaces $(S, L)$ with $(L)^2 = d$, which can be constructed as a quasi-projective variety either by (not quite) standard GIT methods, by using the theorem below, or as a Deligne–Mumford stack.

The period map associates with any $[(S, L)] \in M_d$ a point in $M_d$. For this, choose an isometry $H^2(S, \mathbb{Z}) \simeq \Lambda$, called a marking, that maps $c_1(L)$ to $\ell = e_2 + (d/2)f_2$ and, therefore, induces an isometry $H^2(S, \mathbb{Z})_{L-pr} \simeq \Lambda_d$. Then the $(2,0)$-part $H^2(S, \mathbb{C})_{L-pr} \simeq \Lambda_d$ defines a point in the period domain $Q_d$. The image point in the quotient $\hat{O}(\Lambda_d) \setminus Q_d$ is then independent of the choice of any marking. This defines the period map $P: M_d \to M_d$ which Hodge theory reveals to be holomorphic. Note that both spaces, $M_d$ and $\hat{M}_d$, are quasi-projective varieties with quotient singularities.

**Theorem 2.9** (Pjateckiĭ-Šapiro and Šafarevič). The period map is an algebraic, open embedding

$$P: M_d \to \hat{M}_d = \hat{O}(\Lambda_d) \setminus Q_d.$$  

**Remark 2.10.** Coming back to Remark 2.8, one might wonder how the image of $M_d$ under the finite quotient $\pi_d: \mathcal{M}_d \to \mathcal{M}_d$ can be interpreted geometrically in terms of the polarized K3 surfaces $(S, L)$ parametrized by $M_d$. There is no completely satisfactory answer to this, i.e. the image $\pi_d(M_d)$ is not known (and should probably not expected) to be the coarse moduli space of a nice geometric moduli functor. The best one can say is that for $(S, L) \in M_d$ with $\rho(S) = 1$, the fibre $\pi_d^{-1}(\pi_d(S, L))$ can be viewed as the set of all Fourier–Mukai partners of $S$, which come with a unique polarization, cf. [HP13, Hu18].

To understand the complement of the open embedding (2.1), note first that any $x \in Q_d$ is the period of some K3 surface $S$. This surface then comes with a natural line bundle $L$ (up to the action of the Weyl group) corresponding to $\ell = e_2 + (d/2)f_2 \in \Lambda$. Furthermore, $L$ is ample (again, possibly after applying the Weyl group action) if and only if there exists no $\delta \in \Lambda_d$ with $(\delta)^2 = -2$ orthogonal to $x$, i.e. $x \in Q_d \setminus \bigcup \delta$ with $\delta \in \Delta_d := \Delta(\Lambda_d)$, the set of all $(-2)$-classes in $\Lambda_d$. Hence, the complement of $M_d \subset \hat{M}_d$ can be described as the quotient

$$\hat{O}(\Lambda_d) \setminus \bigcup \delta \subset \hat{M}_d.$$  

Note that $\hat{O}(\Lambda_d)$ acts on $\Delta_d$ and that the quotient (2.2) really is a finite union. In fact, it consists of at most two components due to the following result.

**Proposition 2.11.** The complement $\mathcal{M}_d \setminus M_d$ consists of either one or two irreducible Noether–Lefschetz divisors depending on $d$:

(i) If $d/2 \not\equiv 1 (4)$, then the complement (2.2) of $M_d \subset \mathcal{M}_d$ is irreducible.

---

5Thanks to O. Debarre for pointing this out to me.
(ii) If \( d/2 \equiv 1 \pmod{4} \), then the complement \((2.2)\) of \( M_d \subset \mathcal{M}_d \) has of two irreducible components.

**Proof.** This is again an application of Eichler’s criterion, see the proof of Proposition 1.6. For \( \delta \in \Lambda_d \) with \((\delta)^2 = -2\), one has \( \delta, \Lambda_d = n \mathbb{Z} \) with \( n = 1 \) or \( n = 2 \). In the first case, the residue class \((1/n)\delta \in A_{\Lambda_d} \cong \mathbb{Z}/d\mathbb{Z}\) is trivial. In the second case, \((1/2)\delta \equiv 0 \) or \( \equiv d/2(\delta) \) in \( \mathbb{Z}/d\mathbb{Z} \). However, the second case is only possible if \( d/2 \equiv 1 \pmod{4} \). Indeed, write \( \delta = \delta' + \delta'' \in U_2^\perp \oplus U_2 \) with \( \delta'' \in \ell^\perp \cap U_2 = \mathbb{Z}(e_2 - (d/2)f_2) \). Then \((1/2)\delta' + (1/2)\delta'' + (m/2)\ell \in \Lambda\) for some \( m \in \mathbb{Z} \). Hence, \((1/2)\delta' \in \Lambda\) and, therefore, \(-2 = (\delta)^2 = (\delta'')^2 (8) \). Combine this with \((1/2)\delta'' + (m/2)\ell \in U_2\), which implies \((\delta'')^2 \equiv -m^2d(8)\). \( \Box \)

To be more explicit, one can write

\[
M_d = \begin{cases} 
\mathcal{M}_d \setminus \delta_0^\perp & \text{if } \frac{d}{2} \not\equiv 1 \pmod{4} \\
\mathcal{M}_d \setminus (\delta_0^\perp \cup \delta_1^\perp) & \text{if } \frac{d}{2} \equiv 1 \pmod{4},
\end{cases}
\]

where \( \delta_0, \delta_1 \) are chosen explicitly as \( \delta_0 = e_1 - f_1 \) and \( \delta_1 = 2e_1 + \frac{d^2-1}{2}f_1 + e_2 - (d/2)f_2 \).

2.4. We now switch to the cubic side. The moduli space \( M \) of smooth cubic fourfolds can be constructed by means of standard GIT methods as the quotient

\[
M = |\mathcal{O}_{\mathbb{P}^5}(3)|_{\text{sm}}//\text{PGl}(6).
\]

As in the case of K3 surfaces, mapping a smooth cubic fourfold \( X \) to its period \( H^{3,1}(X) \subset H^4(X, \mathbb{C})_{\text{pr}} \cong \Gamma \otimes \mathbb{C} \), which is a point in the period domain \( D \subset \mathbb{P}(\Gamma \otimes \mathbb{C}) \), defines a holomorphic map \( \mathcal{P} : M \rightarrow \mathcal{C} \). In analogy to the situation for K3 surfaces, the following global Torelli theorem has been proven \([\text{Vo86}, \text{Vo08}, \text{Lo09}, \text{Ch12}, \text{HR18}]\).

**Theorem 2.12** (Voisin, Looijenga,...,Charles, Huybrechts–Rennemo,...). The period map is an algebraic, open embedding

\[
\mathcal{P} : M \longrightarrow \mathcal{C} = O(\Gamma) \setminus D.
\]

This central result is complemented by a result of Laza and Looijenga, which can be seen as an analogue of Proposition 2.11 see \([\text{La10}, \text{Lo09}]\). First note that for \( d = 2 \) and \( d = 6 \) the lattice \( K_d \) is given by the matrices

\[
\begin{pmatrix} -3 & 1 \\ 1 & -1 \end{pmatrix}
\quad \text{and} \quad 
\begin{pmatrix} -3 & 0 \\ 0 & -2 \end{pmatrix},
\]

respectively, see Remark 1.7. Hence, if a smooth cubic fourfold \( X \) defined a point in \( \mathcal{C}_6 \), then \( H^{2,2}(X, \mathbb{Z})_{\text{pr}} \) would contain a class \( \delta \) with \((\delta)^2 = 2\) contradicting \([\text{Vo86}, \S4, \text{Prop. 1}]\). In \([\text{Ha00}]\) one finds an argument using limiting mixed Hodge structures to also exclude the case \([X] \in \mathcal{C}_2 \). So, \( M \subset \mathcal{C} \setminus (\mathcal{C}_2 \cup \mathcal{C}_6) \).

**Theorem 2.13** (Laza, Looijenga). The period map identifies the moduli space \( M \) of smooth cubic fourfolds with the complement of \( \mathcal{C}_2 \cup \mathcal{C}_6 \):

\[
\mathcal{P} : M \longrightarrow \mathcal{C} \setminus (\mathcal{C}_2 \cup \mathcal{C}_6).
\]
To complete the picture, we state the following result. We refrain from giving a proof, but refer to similar results in the theory of K3 surfaces [Hu16, Prop. 6.2.9].

**Proposition 2.14.** The union $\bigcup \mathcal{C}_d \subset \mathcal{C}$ of all $\mathcal{C}_d$ with $d$ satisfying (***) is analytically dense in $\mathcal{C}$. Consequently, the union of all $\mathcal{C}_d$ for satisfying (**') (or (**) or (**) is analytically dense.

**Remark 2.15.** On the level of moduli spaces, the theory of K3 surfaces is linked with the theory of cubic fourfolds in terms of the morphism $\Phi_\varepsilon: M_d \subset M_d \to \mathcal{C}_d \subset \mathcal{C}$, cf. Corollary 2.7. Note that the image of a point $[(S, L)] \in M_d$ corresponding to a polarized K3 surface $(S, L)$ can a priori be contained in the boundary $\mathcal{C} \setminus M = \mathcal{C}_2 \cup \mathcal{C}_6$. However, unless $d = 2$ or $d = 6$, generically this is not the case and the map defines a rational map $\Phi_\varepsilon: M_d \to \Lambda_d$, which is of degree one or two.

2.5. In Section 1.4 we have linked Hodge theory of K3 surfaces and Hodge theory of cubic fourfolds. We will now cast this in the framework of period maps and moduli spaces, i.e. in terms of the maps $\Phi_\varepsilon$.

**Proposition 2.16.** A smooth cubic fourfold $X$ and a polarized K3 surface $(S, L)$ are associated, $(S, L) \sim X$, in the sense of Definition 1.23 if and only if $\Phi_\varepsilon[(S, L)] = [X]$ for some choice of $\varepsilon: \Gamma_d \sim \to \Lambda_d$:

$$(S, L) \sim X \iff \exists \varepsilon: \Phi_\varepsilon[(S, L)] = [X].$$

**Proof.** Assume $\Phi_\varepsilon[(S, L)] = [X]$. Pick an arbitrary marking $H^2(S, \mathbb{Z}) \sim \to \Lambda_\varepsilon$ with $L \sim \to \ell$. Composing the induced isometry $H^2(S, \mathbb{Z})_{L-pr} \sim \to \Lambda_\varepsilon$ with $\varepsilon^{-1}: \Lambda_\varepsilon \sim \to \Gamma_d \subset \Gamma$ yields a point in $D_d \subset D$. Then there exists a marking $H^4(X, \mathbb{Z})_{pr} \simeq \Gamma$ such that $X$ yields the same period point in $D$, which thus yields a Hodge isometric embedding $H^2(S, \mathbb{Z})_{L-pr} \sim \to H^4(X, \mathbb{Z})_{pr}(1)$. Conversely, any such Hodge isometric embedding defines a sublattice of $\Gamma \simeq H^4(X, \mathbb{Z})_{pr}$ isomorphic to some $\nu^{1}$ which after applying some element in $O(\Gamma)$ becomes $\Gamma_d$, see Proposition 1.6. Composing with a marking of $(S, L)$ yields the appropriate $\varepsilon$. □

**Corollary 2.17.** Let $X$ be a smooth cubic fourfold.

(i) For fixed $d$, there exists a polarized K3 surface $(S, L)$ of degree $d$ with $X \sim (S, L)$ if and only if $X \in \mathcal{C}_d$ and $d$ satisfies (**).

(ii) There exists a twisted K3 surface $(S, \alpha)$ with $X \sim (S, \alpha)$ if and only if $X \in \mathcal{C}_d$ for some $d$ satisfying (**').

**Proof.** Consider $\mathcal{M}_d$ as the moduli space of quasi-polarized K3 surfaces $(S, L)$, i.e. with $L$ only big and nef but not necessarily ample. One then has to show that whenever there exists a Hodge isometric embedding $H^2(S, \mathbb{Z})_{L-pr} \sim \to H^4(X, \mathbb{Z})_{pr}(1)$, then $L$ is not orthogonal to any
There are finitely many choices of \( \varepsilon \) will be explained in the next section, see Proposition 3.4. The conditions \( d \) only generically injective for \( \Phi \) of Remark 2.18. It may occur there again as well.

A geometric interpretation, we consider the Fano correspondence \( (3.1) \)

\[
F(X) \xleftarrow{p} \mathbb{L} \xrightarrow{q} X.
\]

Here, \( F(X) \) is the Fano variety of lines contained in \( X \), \( p: \mathbb{L} \rightarrow F(X) \) is the universal line, and \( q \) is the natural projection, cf. [Hu19 Ch. 3] for details and references. Due to work of
Beauville and Donagi [BD85], it is known that \( F(X) \) is a four-dimensional hyperkähler manifold deformation equivalent to the Hilbert scheme \( S^{[2]} \) of a K3 surface \( S \).

3.1. The fact that \( F(X) \) is of K3\(^{[2]} \)-type implies that \( H^2(F(X), \mathbb{Z}) \) with the Beauville–Bogomolov pairing is isometric to the lattice \( H^2(S^{[2]}, \mathbb{Z}) \cong \Lambda \oplus \mathbb{Z}(-2) \). But the cohomology of the Fano variety can also be compared to \( \tilde{H}(X, \mathbb{Z}) \) by the following combination of [Ad16, BD85].

**Theorem 3.1** (Beauville–Donagi, Addington). The Fano correspondence \( \mathcal{B} \) induces two compatible Hodge isometries

\[
\begin{align*}
H^4(X, \mathbb{Z})_{pr}(1) & \xrightarrow{\sim} H^2(F(X), \mathbb{Z})_{pr} \\
\cap v(\lambda_1) & \xrightarrow{\sim} H^2(F(X), \mathbb{Z}) \\
\cap \tilde{H}(X, \mathbb{Z}).
\end{align*}
\]

On the left hand side, \( H^4(X, \mathbb{Z})_{pr}(1) \subset v(\lambda_1) \subset \tilde{H}(X, \mathbb{Z}) \) is the Hodge structure introduced earlier on the sublattice \( v(\lambda_1) \cong \Lambda^1 \cong \Lambda \oplus \mathbb{Z}(-2) \). As before, the sign of the intersection pairing on \( H^4(X, \mathbb{Z})_{pr} \) is changed. On the right hand side, \( H^2(F(X), \mathbb{Z})_{pr} \) is the primitive cohomology with respect to the Plücker polarization \( g \in H^2(F(X), \mathbb{Z}) \). It is endowed with a natural quadratic form, the Beauville–Bogomolov form on the hyperkähler fourfold \( F(X) \). We shall not attempt to prove the result but we will define the maps that are used and indicate the main steps of the argument.

First, it has been observed in [BD85] that

\[
\varphi := p_* \circ q^*: H^4(X, \mathbb{Z})(1) \longrightarrow H^2(F(X), \mathbb{Z})
\]

maps \( h^2 \) to the Plücker polarization \( g \in H^2(F(X), \mathbb{Z}) \) and that for four-dimensional cubics the map induces an isomorphism

\[
H^4(X, \mathbb{Z})_{pr}(1) \xrightarrow{\sim} H^2(F(X), \mathbb{Z})_{pr}
\]

of Hodge structures of weight two satisfying \( (\alpha)^2 = -\frac{1}{6} \int_{F(X)} \varphi(\alpha)^2 \cdot g^2 \), cf. [Hu19] Sec. 3.4] for statements and further references.

Now, as \( v(\lambda_1) \subset \tilde{H}(X, \mathbb{Z}) \subset H^*(X, \mathbb{Q}) \) is not concentrated in degree four, we need to extend the above to the full cohomology. As was observed by Mukai, the natural map \( p_* \circ q^* \) needs to be modified to enjoy certain functoriality properties. More precisely, it is known that the following diagram commutes

\[
\begin{array}{ccc}
K_{\text{top}}(X) & \xrightarrow{p_* \circ q^*} & K_{\text{top}}(F(X)) \\
\downarrow v & & \downarrow v \\
H^*(X, \mathbb{Q}) & \longrightarrow & H^*(F(X), \mathbb{Q}).
\end{array}
\]
Here, the top and bottom rows are given by $E \longrightarrow p_*(q^*E)$ and $\alpha \longrightarrow p_*(q^*\alpha \cdot v(i_*\mathcal{O}_L))$, respectively, where $i: \mathbb{L} \subset X \times F(X)$ is the inclusion, see [Hu06] Ch. 5. The Mukai vector $i_*\mathcal{O}_L$ can be computed by means of the Grothendieck–Riemann–Roch formula as

$$v(i_*\mathcal{O}_L) = i_*(\text{td}(p)) \cdot \left(\text{td}(X)^{-1} \boxtimes \text{td}(F(X))\right)^{1/2}.$$  

From here it is a straightforward computation to show that the commutativity of the diagram (3.3) implies the commutative diagram

$$
\begin{array}{ccc}
K_{\text{top}}(X) & \xrightarrow{p_*q^*} & K_{\text{top}}(F(X)) \\
\text{ch} & & \text{ch} \\
H^*(X, \mathbb{Q}) & \xrightarrow{\varphi} & H^*(F(X), \mathbb{Q}),
\end{array}
$$

where now the bottom row is defined as $\varphi: \alpha \longrightarrow p_*(q^*\alpha \cdot \text{td}(p))$. In particular, for any class $\gamma \in K_{\text{top}}(X)$ one finds $c_1(p_*(q^*(\gamma))) = \{p_*(q^*\text{ch}(\gamma) \cdot \text{td}(p))\}_2$.

The restriction of $c_1 \circ p_* \circ q^*: K_{\text{top}}(X) \longrightarrow H^2(F(X), \mathbb{Z})$ to the primitive part $A_2^\bot \subset K_{\text{top}}(X)$, i.e. the part mapping to $H^2(X, \mathbb{Q})_{\text{pr}}$ under ch (or, equivalently, under the Mukai vector $v$), factors over the original isometry $H^4(X, \mathbb{Z})_{\text{pr}}(1) \sim H^2(F(X), \mathbb{Z})_{\text{pr}}$. As observed in Remark 1.1, $\lambda_1^\bot \subset K'_{\text{top}}(X)$ contains $A_2^\bot \oplus \mathbb{Z}(\lambda_1 + 2\lambda_2)$ as a sublattice of index three. A computation reveals where the second summand is mapped to, cf. [Ad16].

**Lemma 3.2** (Addington). Under the map $c_1 \circ p_* \circ q^*: K_{\text{top}}(X) \longrightarrow H^2(F(X), \mathbb{Z})$ the class $\lambda_1 + 2\lambda_2$ is mapped to the Plücker polarization $g \in H^2(F(X), \mathbb{Z})$. Furthermore, $(\lambda_1 + 2\lambda_2)^2 = (g)^2 = 6$, where the second square is with respect to the Beauville–Bogomolov form.

Therefore, there exists an isometric embedding of the sublattice

(3.3) $A_2^\bot \oplus \mathbb{Z}(\lambda_1 + 2\lambda_2) \sim H^2(F(X), \mathbb{Z})$,

where $A_2^\bot \oplus \mathbb{Z}(\lambda_1 + 2\lambda_2)$ is a sublattice of $\lambda_1^\bot$ of index three and discriminant disc $= 18$. On the other hand, as abstract lattices $H^2(F(X), \mathbb{Z}) \simeq \lambda_1^\bot$. Using this, one then proves that (3.3) indeed extends to an isometry $\lambda_1^\bot \sim H^2(F(X), \mathbb{Z})$. Composition with $\lambda_1^\bot \simeq v(\lambda_1)^\bot$ yields the Hodge isometry $v(\lambda_1)^\bot \sim H^2(F(X), \mathbb{Z})$. Here, the orthogonal complements $\lambda_1^\bot$ and $v(\lambda_1)^\bot$ are taken in $K'_{\text{top}}(X)$ and $\tilde{H}(X, \mathbb{Z})$, respectively.

3.2. In the sequel, we will think of $H^2(F(X), \mathbb{Z})$ as a natural sub Hodge structure of $\tilde{H}(X, \mathbb{Z})$: $H^2(F(X), \mathbb{Z}) \subset \tilde{H}(X, \mathbb{Z})$, orthogonal to the distinguished class $v(\lambda_1) \in \tilde{H}^{1,1}(X, \mathbb{Z})$. This should be thought of as analogous to the inclusion

$$H^2(S^{[2]}, \mathbb{Z}) \subset \tilde{H}(S, \mathbb{Z}),$$

which is orthogonal to $v(\mathcal{I}_x) = (1, 0, -1) \in (\mathcal{H}^0 \oplus H^4)(S, \mathbb{Z}) \subset \tilde{H}^{1,1}(S, \mathbb{Z})$. Note that both vectors, $v(\lambda_1)$ and $v(\mathcal{I}_x)$, are of square two, which immediately leads to the following observation.
Lemma 3.3. Let $X$ be a smooth cubic fourfold and $S$ a K3 surface. Then every Hodge isometry $H^2(F(X), \mathbb{Z}) \cong H^2(S^{[2]}, \mathbb{Z})$ extends to a Hodge isometry $	ilde{H}(X, \mathbb{Z}) \cong H(S, \mathbb{Z})$ mapping $v(\lambda_1)$ to $v(\mathbb{Z}_S)$. \hfill \square

The result should be compared to the observation made earlier that every Hodge isometry $H^4(X, \mathbb{Z})_{\text{pr}} \cong H^4(X', \mathbb{Z})_{\text{pr}}$ extends to $H^4(X, \mathbb{Z}) \cong H^4(X', \mathbb{Z})$ with $h^2_X = \pm h^2_{X'}$.

This enables one to prove the Fano analogue of Proposition \[1.24\] see \[Ad16, Ha00, Hu19\].

Proposition 3.4 (Addington, Hassett, Huybrechts). Assume $X$ is a smooth cubic fourfold.

(i) There exist a K3 surface $S$ and a Hodge isometry

\begin{equation}
H^2(S^{[2]}, \mathbb{Z}) \cong H^2(F(X), \mathbb{Z})
\end{equation}

if and only if there exists an embedding $U \hookrightarrow \tilde{H}^{1,1}(X, \mathbb{Z})$ with $v(\lambda_1)$ contained in its image.

(ii) There exist a K3 surface $S$ and a Hodge isometry

\begin{equation}
H^2(M_S(v), \mathbb{Z}) \cong H^2(F(X), \mathbb{Z})
\end{equation}

for some smooth, projective, four-dimensional moduli space $M_S(v)$ of stable sheaves on $S$ if and only if there exists a K3 surface $S$ with $\sim X$ if and only if there exists an embedding $U \hookrightarrow \tilde{H}^{1,1}(X, \mathbb{Z})$.

(iii) There exist a twisted K3 surface $(S, \alpha)$ and a Hodge isometry

\begin{equation}
H^2(M_{S,\alpha}(v), \mathbb{Z}) \cong H^2(F(X), \mathbb{Z})
\end{equation}

for some smooth, projective, four-dimensional moduli space $M_{S,\alpha}(v)$ of twisted stable sheaves on $S$ if and only if there exists a twisted K3 surface $(S, \alpha)$ with $(S, \alpha) \sim X$ if and only if there exists an embedding $U(n) \hookrightarrow \tilde{H}^{1,1}(X, \mathbb{Z})$ for some $n \neq 0$.

Proof. Any Hodge isometry (3.4) extends to a Hodge isometry $\tilde{H}(S, \mathbb{Z}) \cong H^2(F(X), \mathbb{Z})$ with $(1, 0, -1) \mapsto v(\lambda_1)$. As $(1, 0, -1) \in U \simeq (H^0 \oplus H^4)(S, \mathbb{Z}) \subset \tilde{H}^{1,1}(S, \mathbb{Z})$, this proves one direction in (i). For the other direction use the arguments in the proof of Proposition \[1.24\] to show that there exists a K3 surface $S$ with $\sim X$ such that the given $U \hookrightarrow \tilde{H}^{1,1}(X, \mathbb{Z})$ corresponds to $(H^0 \oplus H^4)(S, \mathbb{Z})$.

For (ii) and (iii) recall that there exists a Hodge isometry $H^2(M_{S,\alpha}(v), \mathbb{Z}) \simeq v^\perp \subset \tilde{H}(S, \alpha, \mathbb{Z})$, cf. \[Hu16\, Ch. 10\] for references in the untwisted case and \[HS05\] for the twisted case. Then, if a Hodge isometry $\tilde{H}(S, \alpha, \mathbb{Z}) \simeq \tilde{H}(X, \mathbb{Z})$ is given, let $v \in \tilde{H}^{1,1}(S, \alpha, \mathbb{Z})$ be the vector that is mapped to $v(\lambda_1)$. Then (3.5) and (3.6) hold. The remaining assertions follow from Proposition \[1.24\]. \hfill \square

This leads to the following analogue of Corollary \[2.17\].

Corollary 3.5. For a smooth cubic fourfold $X$ the condition (i) (or (ii) or (iii)) is equivalent to $X \in \mathcal{C}_d$ for some $d$ satisfying $(\ast\ast\ast)$ (or $(\ast\ast)$ or $(\ast\ast\ast\ast)$, respectively). \hfill \square
3. The purely Hodge and lattice theoretic considerations above can now be combined with the global Torelli theorem for hyperkähler fourfolds due to Verbitsky [Ve13] and Markman [Ma11], see also [Hu12]: Two hyperkähler fourfolds $Y$ and $Y'$ of $\mathds{K}^{[2]}$-type are birational if and only if there exists a Hodge isometry $H^2(Y, \mathbb{Z}) \cong H^2(Y', \mathbb{Z})$:

$$Y \sim Y' \iff H^2(Y, \mathbb{Z}) \cong H^2(Y', \mathbb{Z}).$$

This then implies the following reformulation of the above results:

$$S^{[2]} \sim F(X) \iff (***), \quad M_S(v) \sim F(X) \iff (**),$$

and

$$M_{S,\alpha}(v) \sim F(X) \iff (**').$$

More precisely, one has:

**Corollary 3.6.** Let $X$ be a smooth cubic fourfold and $F(X)$ its Fano variety of lines.

(i) There exists a $K3$ surface $S$ such that $F(X)$ is birational to $S^{[2]}$ if and only if $X \in \mathcal{C}_d$ for some $d$ satisfying (**).

(ii) There exists a $K3$ surface $S$ such that $F(X)$ is birational to a certain smooth, projective moduli space $M_S(v)$ of stable sheaves on $S$ if and only if $X \in \mathcal{C}_d$ for some $d$ satisfying (**).

(iii) There exists a twisted $K3$ surface $(S, \alpha)$ such that $F(X)$ is birational to a certain smooth, projective moduli space $M_{S,\alpha}(v)$ of twisted stable sheaves on $S$ if and only if $X \in \mathcal{C}_d$ for some $d$ satisfying (**').

\[
\Box
\]

**Remark 3.7.** For $d \equiv 0 \mod 6$ and very general $(S, L) \in M_d$, i.e. $\text{Pic}(S) \cong \mathbb{Z}L$, there exists exactly one other polarized $K3$ surface $(S', L') \in M_d$ with $\Phi_\varepsilon[(S, L)] = \Phi_\varepsilon[(S', L')] = [X]$. In particular, the Fano variety $F(X)$ of lines in the corresponding cubic fourfold $X$ is a natural four-dimensional hyperkähler manifold associated with $(S, L)$ and $(S', L')$. Other hyperkähler manifolds that come naturally with $S$ and $S'$ would be $S^{[2]}$ and $S'^{[2]}$. From Corollary 3.6 we know that for $d$ not satisfying (***) the Hilbert scheme $S^{[2]}$ and the Fano variety $F(X)$ are not isomorphic. It was recently shown in [Br18] that also $S^{[2]}$ and $S'^{[2]}$ need not be isomorphic (nor birational). More precisely, they are isomorphic if and only if the Pell equation $3p^2 - (d/6)q^2 = -1$ has an integral solution.

4. The Hodge theory of Kuznetsov’s category

In this short last section we touch upon the Hodge theoretic aspects of Kuznetsov’s triangulated category $\mathcal{A}_X$ naturally associated with every smooth cubic fourfold $X \subset \mathbb{P}^5$. For the more categorical aspects we refer to the original [Ku04, Ku10] or the lecture notes in this volume.
The Hodge theoretic investigation of $\mathcal{A}_X$ was initiated by Addington and Thomas [AT14], the algebraic part of it played a crucial role already in [Ku10].

4.1. We consider the bounded derived category $D^b(X) = D^b(\text{Coh}(X))$ of the abelian category $\text{Coh}(X)$ of coherent sheaves on $X$. The three line bundles $\mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \in D^b(X)$ form an exceptional collection, i.e. $\text{Hom}(\mathcal{O}_X(i), \mathcal{O}_X(j)[*]) = 0$ for $i > j$ and $\mathbb{C}[0]$ for $i = j$. According to a result of Bondal and Orlov [BO02], the derived category $D^b(X)$ determines $X$ uniquely. More precisely, if there exists an exact, linear equivalence $D^b(X) \simeq D^b(X')$ for two smooth cubic fourfolds $X, X' \subset \mathbb{P}^5$, then $X \simeq X'$. This could be called a categorical global Torelli theorem, although the existence of such an equivalence is almost as hard as writing down an explicit isomorphism between them. However, it turns out that $D^b(X)$ contains a natural subcategory which is a much subtler invariant.

**Definition 4.1.** For a smooth cubic fourfold $X \subset \mathbb{P}^5$, we denote by

$$\mathcal{A}_X := (\mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2))^\perp \subset D^b(X)$$

the full triangulated subcategory of all objects $F \in D^b(X)$ right orthogonal to $\mathcal{O}_X, \mathcal{O}_X(1)$, and $\mathcal{O}_X(2)$, i.e. such that $\text{Hom}(\mathcal{O}_X(i), F[*]) = 0$ for $i = 0, 1, 2$.

**Theorem 4.2** (Kuznetsov). The triangulated category $\mathcal{A}_X$ is a Calabi–Yau category of dimension two, i.e. $F \overset{\sim}{\longrightarrow} F[2]$ defines a Serre functor.

In other words, for all $E, F \in \mathcal{A}_X$ there exist functorial isomorphisms

$$\text{Hom}(E, F) \simeq \text{Hom}(F, E[2])^*.$$

Other examples of such categories are provided by $D^b(S)$ and $D^b(S, \alpha)$ associated with K3 surfaces $S$ and twisted K3 surfaces $(S, \alpha)$. A natural question in this context is now to determine when the Kuznetsov category $\mathcal{A}_X$ associated with a cubic fourfold is equivalent to the derived category $D^b(S)$ or $D^b(S, \alpha)$ for some (twisted) K3 surface.

4.2. The goal of [AT14] was to compare Hassett’s condition (***) with the condition $\mathcal{A}_X \simeq D^b(S)$. Building upon [AT14], the twisted version was later dealt with in [Hu17].

**Theorem 4.3** (Addington–Thomas, Huybrechts). Let $X$ be a smooth cubic fourfold and $(S, \alpha)$ a twisted K3 surface.

(i) Any exact, linear equivalence $\mathcal{A}_X \simeq D^b(S)$ induces a Hodge isometry $\widetilde{H}(X, \mathbb{Z}) \simeq \widetilde{H}(S, \mathbb{Z})$. In particular, $X$ is contained in $\mathcal{C}_d$ with $d$ satisfying (**).

(ii) Any exact, linear equivalence $\mathcal{A}_X \simeq D^b(S, \alpha)$ induces a Hodge isometry $\widetilde{H}(X, \mathbb{Z}) \simeq \widetilde{H}(S, \alpha, \mathbb{Z})$. In particular, $X$ is contained in $\mathcal{C}_d$ with $d$ satisfying (**').

In fact, it is also known that for very general $X \in \mathcal{C}_d$ with $d$ satisfying (***) or (**') respectively, the converse in (i) and (ii) hold true. The proof, however, requires a fair amount of deformation theory for Fourier–Mukai kernels developed in [HIM09, To09, AT14, Hu17]. For non-special cubic fourfolds one has the following result.
Proposition 4.4 (Huybrechts). Let $X$ and $X'$ be smooth cubic fourfolds. Then any Fourier–Mukai equivalence $A_X \simeq A_{X'}$ induces a Hodge isometry $\tilde{H}(X, \mathbb{Z}) \simeq \tilde{H}(X', \mathbb{Z})$. The converse holds for all non-special $X$ and for general special ones.

The results of the forthcoming [B+18] complete this picture, so that eventually we will have

$$A_X \simeq D^b(S) \iff \tilde{H}(S, \mathbb{Z}) \simeq \tilde{H}(X, \mathbb{Z}) \text{ Hodge isometry,}$$
$$A_X \simeq D^b(S, \alpha) \iff \tilde{H}(S, \alpha, \mathbb{Z}) \simeq \tilde{H}(X, \mathbb{Z}) \text{ Hodge isometry,}$$
$$A_X \simeq A_{X'} \iff \tilde{H}(X, \mathbb{Z}) \simeq \tilde{H}(X', \mathbb{Z}) \text{ Hodge isometry.}$$

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