Geometry of the Welch Bounds

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Abstract — A geometric perspective is used to derive the entire family of Welch bounds. This perspective unifies a number of observations that have been made regarding tightness of the bounds and their connections to symmetric $k$-tensors, tight frames, homogeneous polynomials, and $t$-designs.

Index Terms — Frames, Grammian, Homogeneous polynomials, Symmetric tensors, $t$-designs, Welch bounds

I. INTRODUCTION

In a brief but important 1976 paper [23], L. R. Welch considered the situation of unit vectors $\{x_1, \ldots, x_m\}$ in $\mathbb{C}^n$ with $m > n$. He developed a family of lower bounds on the maximal cross correlation $c_{\text{max}} = \max_{i \neq j} |\langle x_i, x_j \rangle|$ among the vectors and described the implications of these bounds in the design of sequences having desirable correlation properties for multichannel communications applications. In the years following their original derivation, the Welch bounds became a standard tool in waveform design for both communications and radar.

Although Welch’s original derivation was analytical, several subsequent authors have noted that the Welch bounds have a geometric character. Geometric derivations of the first Welch bound were published in 2003 by Strohmer and Heath [21] and Waldron [22]. Shapiro gave a similar argument in unpublished notes a few years earlier [19]. In this paper, the geometric perspective is extended to derive the entire family of Welch bounds. The derivation can be formulated using either grammian or frame operator points of view, both of which hinge upon an inequality relating the Hilbert-Schmidt norm of a positive semidefinite operator to the ratio of its trace and rank.

Additionally, conditions under which the first Welch bound is satisfied with equality have been studied [12], [22], [21] and design methods for “Welch bound equality” (WBE) sequences that meet the bound with equality, typically with emphasis on the real unimodular case, have been proposed [12], [16], [21], [24], [6]. The motivation for identifying such sequences that is cited in most of the literature has generally involved communications (e.g., CDMA), though they have application in waveform design for radar and sonar as well. Section III of this paper gives frame conditions for equality in the complete family of Welch bounds and also comments on conditions under which the “higher-order” bounds are relevant.

Peng and Waldron [13] point out the existence of isometries between certain spaces of homogeneous polynomials and symmetric tensors in the context of their discussion of Hadamard products of Gram matrices, which play a role in the Welch bound derivations given here. Strohmer and Heath [21] developed Welch-like bounds in infinite-dimensional settings, whereas this paper gives new results for infinite collections of vectors that frame a finite-dimensional space. The foundation of the relationship between the Welch bounds and symmetric $k$-tensors is elucidated in the derivation given in Section III. Section IV addresses the connection to homogeneous polynomials. An extension to generalized frames, which subsumes both the finite and countably infinite frame cases, is presented in VI. The paper concludes with some remarks relating tight generalized frames Haar measures and linking homogeneous polynomials and $t$-designs.

Before beginning the mathematical sections of the paper, a few comments on notation and terminology are needed. For $x = [x^{(1)} \ldots x^{(n)}]^T$ and $y = [y^{(1)} \ldots y^{(n)}]^T$ in $\mathbb{C}^n$, their inner product will be denoted by

$$\langle x, y \rangle = \sum_{j=1}^n \overline{x^{(j)}} y^{(j)}$$

where the bar denotes complex conjugate; i.e., the inner product is conjugate linear in its first argument and linear in its second argument. The corresponding convention will be used for inner products in other complex Hilbert spaces. Given a finite frame $\Phi = \{x_1, \ldots, x_m\}$ for an $n$-dimensional complex vector space $V$, the function $F : V \to \ell_2(\{1, \ldots, m\}) = \mathbb{C}^m$ by $F(w) = [\langle x_1, w \rangle \ldots \langle x_m, w \rangle]^T$ will be called the frame operator associated with $\Phi$, while $F^* F : V \to V$ (i.e., the composition of the adjoint of $F$ with $F$) will be called the metric operator associated with $\Phi$. This follows the terminology used in [9] and will be carried over to the setting of generalized frames in Section VI.

II. THE WELCH BOUNDS

With the notation introduced above, Welch showed that

$$c_{\text{max}}^{2k} \geq \frac{1}{m-1} \left[ \frac{m}{(n+k-1)} \right] - 1 \quad (1)$$

for all integers $k \geq 1$. In fact, he obtained this inequality as a corollary to a more fundamental one:

$$\sum_{i=1}^m \sum_{j=1}^m |\langle x_i, x_j \rangle|^{2k} \geq \frac{m^2}{(n+k-1)} \quad (2)$$

Since the $x_i$ have unit norm, (2) is equivalent to

$$\sum_{i \neq j} |\langle x_i, x_j \rangle|^{2k} \geq \frac{m^2}{(n+k-1)} - m$$

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Because the \(m(m-1)\) terms in the sum are all non-negative, their maximum must be at least as large as their average and thus \((1)\) follows directly from \((2)\). Indeed, most recent work on this topic recognizes \((2)\) as Welch’s main result and refers to these inequalities as the Welch bounds.

Some variations on this basic result have been noted. For example, relaxing the unit-norm assumption to allow the \(x_i\) to be any non-zero vectors yields

\[
\sum_{i=1}^{m} \sum_{j=1}^{m} |\langle x_i, x_j \rangle|^{2k} \geq \frac{1}{(n-k-1)} \left( \sum_{i=1}^{m} \|x_i\|^{2k} \right)^{\frac{k}{n}}
\]

Despite the availability of such slightly more general forms as well as corollary results, the bounds given in \((2)\) are at the heart of the subject and will be the focus of attention in what follows.

A. The first Welch bound

The bound in \((2)\) with \(k = 1\) has received, by far, the most attention in the literature. As noted above, geometric proofs of this particular bound have appeared in published work and were known at least as early as 1998. The “first Welch bound” (i.e., for \(k = 1\)) is derived in this section. This derivation introduces the essential geometric foundations for obtaining the general case, which is carried out in the following section.

Consider a finite-dimensional subspace \(W\) of a complex Hilbert space \(\mathbb{H}\) and let \(T : W \to \mathbb{H}\) be a positive semidefinite linear operator\(^1\). Denote \(n = \dim W\) and let \(\lambda_1, \ldots, \lambda_n\) be the eigenvalues of \(T\). Then the Hilbert-Schmidt (Frobenius) norm of \(T\) satisfies

\[
\|T\|^2 = \sum_{i=1}^{n} |\lambda_i|^2
\]

So, by the Cauchy-Schwarz inequality,

\[
\|T\|^2 \geq \frac{1}{n} \left( \sum_{i=1}^{n} |\lambda_i| \right)^2 = \frac{\|\text{tr } T\|^2}{\dim W}
\]

(3)

Now, suppose the unit vectors \(\{x_1, \ldots, x_m\}\) in \(\mathbb{H}\) span a subspace \(V\) of dimension \(n\). Denote by \(F\) the associated frame operator on \(V\); i.e., \(F : V \to \mathbb{C}^m\) by

\[
F(y) = [(\langle x_1, y \rangle, \ldots, \langle x_m, y \rangle)^T
\]

and denote its adjoint by \(F^*\). Note that some authors (e.g., \([2]\)) refer to \(F^*F\) as the frame operator. Then the gramian \(G = FF^*\) is an operator of rank \(n\) on \(\mathbb{C}^m\) whose norm is

\[
\|G\| = \left( \sum_{i=1}^{m} \sum_{j=1}^{m} |\langle x_i, x_j \rangle|^2 \right)^{\frac{1}{2}}
\]

and whose trace is \(m\). Further, the rank of \(G\) is exactly \(n\), so it operates non-trivially on a subspace \(W \subset \mathbb{C}^m\) of that dimension. Thus applying \((3)\) to \(G\) yields the Welch bound \((2)\) with \(k = 1\).

A “dual” argument is obtained by considering the metric operator \(F = F^*F : \mathbb{H} \to \mathbb{H}\). The non-zero eigenvalues of \(F\) are identical to those of \(G\), so its trace and rank are also equal to those of \(G\). So \((3)\) applied to \(F\) also yields \((2)\) with \(k = 1\).

B. Higher-order Welch bounds

Alternatives to Welch’s original analytical derivation of the bounds \((2)\) for \(k > 1\) do not seem to appear in published literature. In fact, these cases also follow from \((3)\) by considering \(k\)-fold Hadamard or tensor products.

With \(k \geq 1\), the left-hand side of \((2)\) is the Hilbert-Schmidt norm of the \(k\)-fold Hadamard product \(G^{\otimes k}\) of the gramian \(G\). In \([13]\), it is shown that the rank of \(G^{\otimes k}\) is \((\binom{n+k-1}{k})\) provided \(m > \binom{n+k-1}{k}\). The Schur product theorem \([8], [13]\) ensures that \(G^{\otimes k}\) is positive semidefinite. Since \(tr\ G^{\otimes k} = \sum_{i=1}^{m} \|x_i\|^{2k} = m\), \((3)\) gives

\[
\|G^{\otimes k}\|^2 = \sum_{i=1}^{m} \sum_{j=1}^{m} |\langle x_i, x_j \rangle|^{2k} \geq \frac{m^2}{\binom{n+k-1}{k}}
\]

In this argument, the binomial coefficient in the denominator of the Welch bounds has an explicit geometric interpretation as the dimension of the subspace on which \(G^{\otimes k}\) operates non-trivially. This will be discussed further in later sections of this paper.

As in the \(k = 1\) case, there is a similar proof of \((2)\) using the metric operator associated with the frame \(\{x_1, \ldots, x_m\}\) for \(V\). In this argument, the tensor product plays a role similar to that of the Hadamard product in the preceding proof. Recall that the \(k\)-fold tensor product \(V^{\otimes k}\) of an \(n\)-dimensional vector space \(V\) is a vector space spanned by elements of the form \(v_1 \otimes \cdots \otimes v_k\) where each \(v_i \in V\) \([7], [20]\). The vector \(v_1 \otimes \cdots \otimes v_k\) has \(n^k\) coordinates \(\{v_i^{(\ell)}\}_{\ell = 1}^{k}\) where \(v_i^{(\ell)}\) denotes the \(\ell\)th coordinate of the vector \(v_i\). A choice of basis \(\{e_1, \ldots, e_n\}\) for \(V\) gives rise to a basis for \(V^{\otimes k}\) consisting of the \(n^k\) product elements \(e_{i_1} \cdots e_{i_k} \equiv e_{i_1} \otimes \cdots \otimes e_{i_k}\) \(1 \leq i_1, \ldots, i_k \leq n\). In particular, \(V^{\otimes k}\) has dimension \(n^k\).

The space of symmetric \(k\)-tensors associated with \(V\), denoted \(\text{Sym}^k(V)\), is the subspace of \(V^{\otimes k}\) consisting of those tensors which remain fixed under permutation. Specifically, denote by \(S_k\) the symmetric group on \(k\) symbols and define an action of \(S_k\) on \(V^{\otimes k}\) by

\[
A_{\sigma}(v_1 \otimes \cdots \otimes v_k) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}
\]

Then \(\text{Sym}^k(V)\) consists of all elements of \(V^{\otimes k}\) such that \(A_{\sigma}(v_1 \otimes \cdots \otimes v_k) = v_1 \otimes \cdots \otimes v_k\) for all \(\sigma \in S_k\) (see Chapter 10 of \([20]\)). \(\text{Sym}^k(V)\) is spanned by the tensor powers \(v^{\otimes k}\) where \(v \in V\). If \(V\) has dimension \(n\) then

\[
\dim \text{Sym}^k(V) = \binom{n+k-1}{k}
\]

\(\text{Sym}^k(V)\) has a natural inner product with the property

\[
\langle v^{\otimes k}, w^{\otimes k} \rangle_{\text{Sym}^k(V)} = \langle v, w \rangle^k
\]

(4)

Now consider the space \(\text{Sym}^k(V)\) where \(V\) is the \(n\)-dimensional span of \(\{x_1, \ldots, x_m\}\). This space has dimension

\[
\dim \text{Sym}^k(V) = \binom{n+k-1}{k}
\]
\( (n+k-1) \) and is framed by \( X^{(k)} = \{ x_1^{\otimes k}, \ldots, x_m^{\otimes k} \} \). Denoting the metric operator associated with this frame by \( F^{(k)} \), note
\[
\text{tr } F^{(k)} = \sum_{i=1}^m \langle x_i^{\otimes k}, x_i^{\otimes k} \rangle = \sum_{i=1}^m \langle x_i, x_i \rangle^k = m
\]
Thus applying inequality (3) gives
\[
\| F^{(k)} \|^2 = \sum_{i=1}^m \sum_{j=1}^m | \langle x_i, x_j \rangle |^{2k} \geq \frac{m^2}{(n+k-1)}
\]
as desired.

III. TIGHTNESS OF THE WELCH BOUNDS

As noted in the introduction to this paper, several authors have investigated conditions under which the Welch bound with \( k = 1 \) is satisfied with equality. A condition for all \( k \geq 1 \) is given below, followed by some discussion about when these higher-order Welch bounds are important. Methods for design of sequences meeting the higher-order bounds with equality, building on earlier results in \([12],[16],[21],[24],[6]\) for example, may follow from the criterion presented here.

A. Conditions for Equality

Well-known conditions for equality to hold in the Cauchy-Schwarz inequality imply that equality holds in (3) if and only if all the eigenvalues of \( T \) are equal. In the \( k = 1 \) case, this means all the non-zero eigenvalues of \( G \) (and of \( F \)) must be equal to \( m/n \). This holds if and only if \( \{ x_1, \ldots, x_m \} \) is a tight frame for the \( n \)-dimensional subspace \( V \) of \( H \), in which case \( F = \frac{m}{n} I_V \) (i.e., a constant times the identity on \( V \)).

Extending to \( k \geq 1 \), the \( m \times m \) Gram matrix associated with the gramian \( G^{(k)} \) is
\[
\begin{bmatrix}
\langle x_1, x_1 \rangle^k & \cdots & \langle x_1, x_m \rangle^k \\
\vdots & \ddots & \vdots \\
\langle x_m, x_1 \rangle^k & \cdots & \langle x_m, x_m \rangle^k
\end{bmatrix}
\]
which is the Gram matrix corresponding to the gramian of the set \( \{ x_1^{\otimes k}, \ldots, x_m^{\otimes k} \} \) in \( \text{Sym}^k(V) \). Thus equality holds in the Welch bound if and only if the eigenvalues of \( G^{(k)} \) are all equal to \( \frac{m}{(n+k-1)} \), in which case the set \( \{ x_1^{\otimes k}, \ldots, x_m^{\otimes k} \} \) is a tight frame for \( \text{Sym}^k(V) \) with metric operator
\[
F^{(k)} = \frac{m}{(n+k-1)} I_{\text{Sym}^k(V)}
\]

The preceding paragraphs set forth conditions for equality to hold in (2). In order for equality to hold in (1), an additional condition must hold. Given non-negative numbers \( \{ a_1, \ldots, a_L \} \) it is clear that
\[
\frac{1}{L} \sum_{j=1}^L a_j \leq \max_{j=1, \ldots, L} a_j
\]
As noted earlier, this is the inequality used to obtain (1) from (2). The unit vectors \( \{ x_1, \ldots, x_m \} \) are equiangular in \( H \) if \( | \langle x_i, x_j \rangle | \) is constant for all \( i \neq j \). In this situation,
\[
\max_{i \neq j} | \langle x_i, x_j \rangle | = \frac{1}{m(m-1)} \sum_{i \neq j} | \langle x_i, x_j \rangle |
\]
Thus (1) holds with equality if and only if \( \{ x_1^{\otimes k}, \ldots, x_m^{\otimes k} \} \) is an equiangular tight frame for \( \text{Sym}^k(V) \). Connection of equiangular lines to tight frames and their existence in dimensions of up to \( d = 45 \) has been discussed in [15]. See also [4].

B. Non-triviality of the bounds

As mentioned earlier, the bound with \( k = 1 \) has received most attention in literature subsequent to Welch’s original paper. Taking the \( k \)-th root of both sides of (1), it is easy to find values of \( m \) and \( n \) where \( k = 1 \) does not give the smallest bound on \( c_{\max}^2 \). For given \( n \) and \( m \), bounds in (2) corresponding to different values of \( k \) are not directly comparable. But the tight frame conditions given above give some clues about how to investigate the existence of collections of vectors that simultaneously satisfy multiple Welch bounds with equality.

It is clear that (2) is vacuous for some combinations of values for \( n, m, \) and \( k \). To avoid this, it is necessary that
\[
m > \binom{n+k-1}{k}
\]
This suggests that \( m > O(n^k) \) as \( k \to \infty \), thereby implying that for higher values of \( k \) one can hope for meaningful bounds only when \( m \gg n \). Similarly, if \( k \) is fixed, useful bounds require \( m > \max(n, \binom{n+k-1}{k}) \). This implies that \( m > O(k^{n-1}) \) as \( n \to \infty \). In any case, it is evident that the bounds for large \( k \) are only significant when \( m \gg n \).

IV. A REMARK ON SAMPLING OF HOMOGENEOUS POLYNOMIALS

It is well known (see, e.g., [20]) that \( H_{(0,k)} \), the linear space of homogeneous polynomials of total degree \( k \) in variables \( \bar{z}^{(1)}, \ldots, \bar{z}^{(n)} \) is isomorphic to \( \text{Sym}^k(V) \). This section points out a connection between the condition that \( X^{(k)} = \{ x_1^{\otimes k}, \ldots, x_m^{\otimes k} \} \) is a frame for \( \text{Sym}^k(V) \) and the reconstructability of polynomials in \( H_{(0,k)} \) from the values they take at sets of \( m \) points in \( \mathbb{C}^n \).

Beginning with \( k = 1 \), let \( w \in V = \text{Sym}^1(V) \) and denote by \( [w^{(1)} \ldots w^{(n)}]^T \) the coordinates of \( w \) in some orthonormal basis for \( V \). There is an obvious isomorphism that takes \( w \) to \( V \) to the polynomial \( p_w \in H_{(0,1)} \) defined by \( p_w(z^{(1)}, \ldots, z^{(n)}) = w^{(1)}z^{(1)} + \cdots + w^{(n)}z^{(n)} \). If \( X = \{ x_1, \ldots, x_m \} \) is a frame for \( V \), the associated frame operator \( F : V \to \mathbb{C}^m \) is given by
\[
F(w) = \begin{bmatrix}
\langle x_1, w \rangle \\
\vdots \\
\langle x_m, w \rangle
\end{bmatrix} = \begin{bmatrix}
p_w(x_1^{(1)}, \ldots, x_1^{(n)}) \\
\vdots \\
p_w(x_m^{(1)}, \ldots, x_m^{(n)})
\end{bmatrix}
\]
In other words, \( F(w) \) is a vector of values obtained by evaluating (i.e., “sampling”) \( p_w \) at the points \( x_1, \ldots, x_m \).
may ask whether this set of \( m \) sample values is sufficient to uniquely determine \( p_w \).

To address this question, define a sampling function \( P_X : H_{(0,1)} \to \mathbb{C}^m \) by

\[
P_X(p) = \begin{bmatrix}
p(x_1^{(1)}, \ldots, x_1^{(n)}) \\
\vdots \\
p(x_m^{(1)}, \ldots, x_m^{(n)})
\end{bmatrix}
\]

and note that (5) shows the frame operator is given by \( F(w) = P_X(p_w) \). Because the frame operator is invertible, \( w \) is uniquely determined by \( F(w) \). Hence any \( p_w \in H_{(0,1)} \) is uniquely determined by its samples \( P_X(p_w) \).

Conversely, if \( X \) fails to frame \( V \), the mapping \( F \) defined by (5) is still well-defined, but has non-trivial kernel \( K \). In this case, \( P_X(p_w) = P_X(p_{w+u}) \) for all \( u \in K \). So, in particular, \( p_w \) is not uniquely determined from its samples at \( x_1, \ldots, x_m \).

A similar situation occurs for \( k > 1 \), where the space of interest is Sym\(^k(V) \) and the frame is \( X(k) = \{x_1^\otimes k, \ldots, x_m^\otimes k\} \). As in the \( k = 1 \) case, mapping a polynomial to its coefficient sequence defines an isomorphism between \( H_{(0,k)} \) and Sym\(^k(V) \) for \( k > 1 \). If \( v = w^\otimes k \in \text{Sym}^k(V) \) is a pure tensor power of \( w \in V \), then

\[
F(k)(v) = \begin{bmatrix}
\langle x_1^\otimes k, w^\otimes k \rangle \\
\vdots \\
\langle x_m^\otimes k, w^\otimes k \rangle
\end{bmatrix} = \begin{bmatrix}
p(x_1(x_1)) \\
\vdots \\
p(x_m(x_m))
\end{bmatrix}
\]

where \( p_v \in H_{(0,k)} \) defined by \( p_v(z) = \langle z, w^\otimes k \rangle \). Sym\(^k(V) \) is spanned by pure tensor powers of elements in \( V \) [20]. Thus, for arbitrary \( v \in \text{Sym}^k(V) \), \( F(k)(v) \) is a vector of \( m \) samples of a polynomial in \( H_{(0,k)} \) taken at points \( x_1, \ldots, x_m \). Thus, as in the \( k = 1 \) case, polynomials in \( H_{(0,k)} \) are uniquely determined by the samples

\[
P_X^{(k)}(p) = \begin{bmatrix}
p(x_1^\otimes k) \\
\vdots \\
p(x_m^\otimes k)
\end{bmatrix}
\]

if and only if \( X^{(k)} \) frames \( \text{Sym}^k(V) \).

**V. Relationship of the Metric Operator and the Grammian**

Suppose a rank-\( n \) grammmian \( \mathcal{G} \) has associated \( m \times m \) Gram matrix \( G = [g_{ij}] \). From this starting point, it is possible to deduce vectors \( \{x_1, \ldots, x_m\} \) in \( V \) having grammmian \( \mathcal{G} \). This set of vectors is a frame for \( V \) and denoting its frame operator by \( F \), it is apparent that \( \mathcal{G} = FF^\ast \) and the corresponding metric operator is \( F = F^\ast F \). This follows directly from \( G \) being Hermitian and positive semidefinite. Writing \( G = UD^\ast U^\ast \) with \( U \) unitary and \( D = \text{diag}(\lambda_1 \ldots \lambda_n \ 0 \ldots 0) \) with the eigenvalues \( \lambda_i \) positive for \( i = 1, \ldots, n \). \( X = U\sqrt{D} \) is an \( m \times m \) matrix whose last \( m-n \) columns are zero. Defining \( x_i \) to be the \( n \)-vector formed from the first \( n \) components of the \( i \)-th row of \( X \) for \( i = 1, \ldots, m \) yields \( \langle x_i, x_j \rangle = g_{ij} \). Thus \( \mathcal{G} \) is the grammmian of \( \{x_1, \ldots, x_m\} \). If the original \( G \) happens to have ones on the main diagonal, then this construction produces unit vectors.

Similarly, starting with \( \mathcal{F} \), one can find a corresponding frame \( \{x_1, \ldots, x_m\} \) for \( V \) and thus a frame operator \( F \) and a grammmian \( \mathcal{G} = FF^\ast \) having identical non-zero eigenvalues to those of \( \mathcal{F} \).

In view of this connection between \( \mathcal{G} \) and \( \mathcal{F} \), it is no surprise that inner product geometry (i.e., via \( \mathcal{G} \)) and outer product geometry (i.e., via \( F \)) yield similar derivations of the Welch bounds.

**VI. Generalized Frames**

A good overview of generalized frames is given in [9]. The following paragraphs set forth the essentials needed for the Welch bound results that follow.

Let \( \mathbb{H} \) be a complex Hilbert space and \((M, \mu)\) a measure space. A *generalized frame* in \( \mathbb{H} \) indexed by \( M \) is a family of vectors \( X_M = \{x_\alpha \in \mathbb{H} : \alpha \in M\} \) such that:

(a) For every \( y \in \mathbb{H} \), the function \( \hat{y} : M \to \mathbb{C} \) defined by

\[
\hat{y}(\alpha) = \langle x_\alpha, y \rangle
\]

is \( \mu \)-measurable.

(b) There exist constants \( 0 < A \leq B < \infty \) such that, for every \( y \in \mathbb{H} \),

\[
A||y||^2_\mathbb{H} \leq \int_M |\langle h_\alpha, y \rangle|^2 d\mu(\alpha) \leq B||y||^2_\mathbb{H}
\]

or

\[
A||y||^2_\mathbb{H} \leq ||\hat{y}||^2_{L^2(M, \mu)} \leq B||y||^2_\mathbb{H}
\]

If \( \mu \) is a counting measure on \( M \), condition (a) is trivial and (b) is the familiar frame condition for a discrete frame with bounds \( A \) and \( B \).

The frame operator \( F : \mathbb{H} \to L^2(M, \mu) \) is given by \( F(y) = \langle x_\alpha, y \rangle \sum_{\alpha \in M} g(\alpha) x_\alpha d\mu(\alpha) \). The metric operator \( \mathcal{F} : \mathbb{H} \to \mathbb{H} \) is \( \mathcal{F} = F^{\ast} F \); i.e., for \( y \in \mathbb{H} \)

\[
\mathcal{F}(y) = \int_M \langle x_\beta, y \rangle x_\alpha d\mu(\alpha)
\]

The grammmian \( \mathcal{G} : L^2(M, \mu) \to L^2(M, \mu) \) is defined by \( \mathcal{G} = \mathcal{F} F^{\ast} \); i.e.,

\[
(\mathcal{G} f)(\beta) = \int_M \langle x_\beta, x_\alpha \rangle f(\alpha) d\mu(\alpha)
\]

**A. Welch bounds for generalized frames**

With \( V \) an \( n \)-dimensional subspace of \( \mathbb{H} \), denote by \( S^{n-1} \) the set of unit vectors in \( V \). For each \( x \in S^{n-1} \), there is an associated projector \( \Pi_x : V \to \text{span}(x) \) (i.e., onto the one-dimensional subspace spanned by \( x \)) given by

\[
\Pi_x(v) = \langle x, v \rangle x
\]

Since \( \Pi_x = \Pi_{x^\theta \alpha_x} \) for any \( \theta \in [0, 2\pi) \), the collection of projectors \( \Pi_x \) is parameterized by the complex projective space \( \mathbb{CP}^{n-1} \). Given a normalized measure \( \mu \) on \( \mathbb{CP}^{n-1} \) i.e.,
This bound is achieved if and only if the generalized frame is
frame setting are obtained by considering

\[ \mu \]

Taking \( \{e_1, \ldots, e_n\} \) to be an orthonormal basis of \( V \), the trace of \( \mathcal{F}_\mu \) is given by

\[ \text{tr} \mathcal{F}_\mu = \sum_{k=1}^{n} \langle \mathcal{F}_\mu(e_k), e_k \rangle = 1 \]

Also, the Hilbert-Schmidt norm of \( \mathcal{F}_\mu \) is

\[ ||\mathcal{F}_\mu||^2 = \text{tr} \mathcal{F}_\mu^* \mathcal{F}_\mu = \int \int_{C^2} |\langle x, y \rangle|^2 d\mu(x) d\mu(y) \]

so that the Welch bound obtained from (3) in this setting for

\[ k = 1 \]

is

\[ \int \int_{C^2} |\langle x, y \rangle|^2 d\mu(x) d\mu(y) \geq \frac{1}{n} \]

This bound is achieved if and only if the generalized frame is tight; i.e., if and only if

\[ \mathcal{F}_\mu = \frac{1}{n} \mathcal{I}_V \]

For \( k \geq 1 \), higher-order Welch bounds for the generalized frame setting are obtained by considering \( \text{Sym}^k(V) \). In this setting, the projector \( \Pi_x \otimes k \) maps \( \text{Sym}^k(V) \) onto the one-dimensional subspace spanned by the tensor power \( x \otimes k \) with \( x \in S^{n-1} \). Direct calculation using (4) yields

\[ \Pi_x \otimes k = \Pi_x \otimes k \]

and, for \( v \in V \),

\[ \Pi_x \otimes k v = \langle x, v \rangle^k x \otimes k \]

This collection of projectors is parameterized by \( C^{n-1} \). Corresponding to each \( x \in C^{n-1} \), choosing a representative unit vector in \( V \) yields a collection of unit vectors

\[ X_{C^{n-1}}^{(k)} = \{ u_x \otimes k \mid u_x \in V, x \in C^{n-1} \} \]

Given a normalized measure \( \mu \) on \( C^{n-1} \), \( X_{C^{n-1}}^{(k)} \) becomes a generalized frame for \( \text{Sym}^k(V) \) with metric operator \( \mathcal{F}_\mu : \text{Sym}^k(V) \to \text{Sym}^k(V) \) by

\[ \mathcal{F}_\mu \left( \sum_{j=1}^{n} w_j \right) \]

Noting that \( \text{tr} \mathcal{F}_\mu = 1 \), (4) implies

\[ ||\mathcal{F}_\mu||^2_{\text{Sym}^k(V)} = \int \int_{C^{n-1}} |\langle x, y \rangle|^2 d\mu(x) d\mu(y) \geq \frac{1}{(n+k-1)} \]

with equality if and only if \( (X_{C^{n-1}}^{(k)}, \mu) \) is a generalized tight frame for \( \text{Sym}^k(V) \); i.e.,

\[ \mathcal{F}_\mu = \frac{1}{(n+k-1)} \mathcal{I}_\text{Sym}^k(V) \]

This generalized frame perspective immediately yields the Welch bounds for finite frames \( X = \{x_1, \ldots, x_m\} \) of unit vectors by considering the (normalized) discrete measure

\[ \mu = \frac{1}{m} \sum_{x \in X} \delta_x \]

Using this measure in (5) yields

\[ \frac{1}{m^2} \sum_{x,y \in X} |\langle x, y \rangle|^2 \geq \frac{1}{(n+k-1)} \]

which is equivalent to (3). Equality is obtained if and only if \( X^{(k)} = \{x \otimes k \mid x \in X\} \) is a tight frame for \( \text{Sym}^k(V) \); i.e., if and only if

\[ \frac{1}{m} \sum_{x \in X} \Pi_x \otimes k = \frac{1}{k} \mathcal{I}_{\text{Sym}^k(V)} \]

This perspective also produces Welch bounds for countably infinite frames. If \( X = \{x_1\}_{i=1}^{\infty} \) in \( C^{n-1} \) and \( \{w_i\}_{i=1}^{\infty} \) is a summable set of non-negative numbers, defining a discrete measure by

\[ \mu = \sum_{i=1}^{\infty} w_i \delta_{x_i} \sum_{j=1}^{n} w_j \]

yields a generalized frame \( (X, \mu) \). With this measure (6) becomes

\[ \left( \sum_{j=1}^{n} w_j \right)^2 \sum_{i,j} |\langle w_i x_j, w_k x_l \rangle|^2 \geq \frac{1}{(n+k-1)} \]

B. The gramian associated with \( X_{C^{n-1}}^{(k)} \)

The gramian for the generalized frame \( (X_{C^{n-1}}^{(k)}, \mu) \) is

\[ G^{(k)}_\mu : L^2(C^{n-1}, \mu) \to L^2(C^{n-1}, \mu) \]

by

\[ (G^{(k)}_\mu f)(\beta) = \int_{C^{n-1}} \langle x_{\beta}, x_{\alpha} \rangle^k f(\alpha) \ d\mu(\alpha) \]

Denote the coordinates of \( x_{\beta} \) in some orthonormal basis by \( [x^{(1)}} \ldots x^{(n)}] \). Then, for each \( \alpha \), \( \langle x_{\beta}, x_{\alpha} \rangle f(\alpha) \) is a homogeneous polynomial of degree one in the \( n \) variables \( x^{(1)} \), \ldots, \( x^{(n)} \). Thus \( \langle x_{\beta}, x_{\alpha} \rangle^k f(\alpha) \) is a homogeneous polynomial of degree \( k \) in \( x^{(1)} \), \ldots, \( x^{(n)} \) and \( G^{(k)}_\mu \) defines a projection into \( H_{0,k} \). This means that the rank of \( G^{(k)}_\mu \) is at most \( \text{dim} H_{0,k} = \binom{n+k-1}{k} \). So applying (5) to \( G^{(k)}_\mu \) implies

\[ ||G^{(k)}_\mu||^2_{H_{0,k}} = \int_{C^{n-1}} |\langle x, y \rangle|^2 d\mu(x) d\mu(y) \geq \frac{1}{(n+k-1)} \]

Further, equality holds if and only if

\[ G^{(k)}_\mu = \frac{1}{(n+k-1)} \mathcal{I}_{H_{0,k}} \]

The argument just outlined is the generalized frame analogy to the gramian-based derivation of the Welch bounds presented in Section II for the finite frame case.
C. Tight generalized frames and Haar measure

Equality in (5) is attained if and only if \((X_{\mathbb{C}^{n-1}}^{(k)}, \mu)\) is a generalized tight frame for \(\text{Sym}^k(V)\). The following paragraph sketches an argument showing that this occurs when \(\mu\) is Haar measure.

Denote by \(\mathcal{U}_n\) the \(n\)-dimensional unitary group and suppose \(\mu\) is the normalized \(\mathcal{U}_n\)-invariant Haar measure on \(\mathbb{C}^{n-1}\). \(\mathcal{U}_n\) acts transitively on \(V^\otimes k\) by

\[
\Phi_U^{(k)}(y) = U^\otimes k y \quad y \in V^\otimes k
\]

where \(U \in \mathcal{U}_n\) [14]. \(\text{Sym}^k(V)\) is an invariant subspace of \(V^\otimes k\) under this action. Further, for any \(U \in \mathcal{U}_n\),

\[
U^\otimes k \mathcal{F}_\mu^{(k)}(U^\otimes k)^* = \int_{\mathbb{C}^{n-1}} U^\otimes k \Pi_{\mu}^{(k)}(U^\otimes k)^* d\mu(x)
\]

\[
= \int_{\mathbb{C}^{n-1}} \Pi_{Ux}^\otimes k d\mu(x)
\]

\[
= \int_{\mathbb{C}^{n-1}} \Pi_x^\otimes k d\mu(x)
\]

where the last equality holds because \(\mu\) is \(\mathcal{U}_n\)-invariant. This shows that \(\mathcal{F}_\mu^{(k)}\) commutes with all \(U^\otimes k\). Since \(\Phi_U^{(k)}\) is irreducible, Schur’s lemma implies \(\mathcal{F}_\mu^{(k)} = \lambda \mathcal{I}_{\text{Sym}^k(V)}\). Because \(\text{tr} \mathcal{F}_\mu^{(k)} = 1\), dimensionality considerations imply that

\[
\lambda = \frac{1}{\dim \text{Sym}^k(V)} = \left(\frac{n+k-1}{k}\right)
\]

D. Homogeneous polynomials and t-designs

As in the finite frame case, the generalized frame perspective yields connections to homogeneous polynomials and, further, to spherical \(t\)-designs. Suppose that \((X_{\mathbb{C}^{n-1}}^{(k)}, \mu)\) is a tight frame for \(\text{Sym}^k(V)\). Then

\[
\mathcal{F}_\mu^{(k)} = \int_{\mathbb{C}^{n-1}} \Pi_{\mu}^\otimes k d\mu(x) = \frac{1}{\left(\frac{n+k-1}{k}\right)} \mathcal{I}_{\text{Sym}^k(V)}
\]

The frame operator \(F^{(k)}: \text{Sym}^k(V) \to L^2(\mathbb{C}^{n-1}, \mu)\) is given by \(F^{(k)}(w) = \{x^\otimes k, w\}_{x \in \mathbb{C}^{n-1}}\) for \(w \in \text{Sym}^k(V)\). Since the tensor powers \(\{v^\otimes k : v \in V\}\) span \(\text{Sym}^k(V)\), \(\langle x^\otimes k, w \rangle\) can be written as a linear combination of terms of the form \(\langle x, w \rangle^k\) and hence is in \(H_{0(k)}\). Denoting this polynomial associated with \(w\) by \(p_w\) and using (7) gives, for any \(v, w \in \text{Sym}^k(V)\),

\[
\langle v, w \rangle = \left(\frac{n+k-1}{k}\right) \int_{\mathbb{C}^{n-1}} \langle v, x^\otimes k \rangle \langle x^\otimes k, w \rangle d\mu(x)
\]

\[
= \left(\frac{n+k-1}{k}\right) \int_{\mathbb{C}^{n-1}} p_v(x)p_w(x) d\mu(x)
\]

If \(\mu\) is the normalized discrete measure discussed in Section VI-A, the metric operator \(\mathcal{F}_\mu^{(k)}\) can be written as

\[
\frac{1}{m} \sum_{x \in X} \Pi_{\mu}^\otimes k = \frac{1}{\left(\frac{n+k-1}{k}\right)} \mathcal{I}_{\text{Sym}^k(V)}
\]

Using this representation of \(\mathcal{I}_{\text{Sym}^k(V)}\) gives

\[
\langle v, w \rangle = \frac{\left(\frac{n+k-1}{k}\right)}{m} \sum_{x \in X} p_v(x)p_w(x)
\]

so that

\[
\int_{\mathbb{C}^{n-1}} p_v(x)p_w(x) d\mu(x) = \frac{1}{m} \sum_{x \in X} p_v(x)p_w(x)
\]

This implies that, for any \(g \in H_{0(k)}\), the space of homogeneous polynomials of total degree \(k\) in \(x_1, \ldots, x_m\) and total degree \(k\) in \(\bar{x}_1, \ldots, \bar{x}_m\),

\[
\int_{\mathbb{C}^{n-1}} g(x) d\mu(x) = \frac{1}{m} \sum_{x \in X} g(x)
\]

If \(X^{(k)} = \{x_1^\otimes k, \ldots, x_m^\otimes k\}\) is a tight frame for \(\text{Sym}^k(V)\) for all \(k \leq t\), then

\[
\int_{\mathbb{C}^{n-1}} g(x) d\mu(x) = \frac{1}{m} \sum_{x \in X} g(x)
\]

for all \(g \in \bigoplus_{k=1}^t H_{0(k)}\). Equation (8) defines \(X^{(k)} = \{x_1^\otimes k, \ldots, x_m^\otimes k\}\) as a complex projective \(t\)-design [10].

VII. CONCLUSION

The classical Welch bounds have been shown to arise from dimensionality considerations in connection with frame and grammian operators. Geometric derivations of the first Welch bound have been given in previous work. This paper has extended the geometric perspective to obtain the higher-order Welch bounds, with the \(k\)th bound for \(k \geq 1\) arising naturally from observing either the \(k\)-fold Hadamard product of the grammian or the metric operator associated with a frame on a space of symmetric \(k\)-tensors.

Welch bounds for generalized frames have been derived and the classical case shown to follow from this more general result. The role of frame tightness in achieving the Welch bounds with equality, which has been previously recognized in the \(k = 1\) case, has been established in this general setting. Further, specific connections have been clarified between the circle of ideas entailed in the geometric understanding of the Welch bounds and related topics involving symmetric tensors, homogeneous polynomials, and \(t\)-designs.

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