CRYSTALLINE TEMPERATE DISTRIBUTIONS WITH UNIFORMLY DISCRETE SUPPORT AND SPECTRUM

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Abstract. We prove that a temperate distribution on \( \mathbb{R} \) whose support and spectrum are uniformly discrete sets, can be obtained from Poisson’s summation formula by a finite number of basic operations (shifts, modulations, differentiations, multiplication by polynomials, and taking linear combinations).

1. Introduction

By a crystalline measure on \( \mathbb{R} \) (or \( \mathbb{R}^d \)) we mean a pure point measure \( \mu \) which is a temperate distribution and whose distributional Fourier transform \( \hat{\mu} \) is also a pure point measure,

\[
\mu = \sum_{\lambda \in \Lambda} a(\lambda) \delta_\lambda, \quad \hat{\mu} = \sum_{s \in S} b(s) \delta_s,
\]

(1.1)

where the support \( \Lambda \) and the spectrum \( S \) are locally finite sets \( \text{[Mey16]} \). This notion may be considered as a mathematical model for quasicrystals, i.e. atomic arrangements having a discrete diffraction pattern (see \( \text{[Lag00]} \)).

A classical example of a crystalline measure is

\[
\mu = \sum_{\lambda \in L} \delta_\lambda,
\]

(1.2)

where \( L \) is a lattice. Indeed, by the Poisson summation formula, the Fourier transform \( \hat{\mu} \) is the sum of equal masses on the dual lattice \( L^* \). By applying a finite number of shifts, modulations, and taking linear combinations, one can construct more general examples of crystalline measures \( \mu \) whose supports are contained in finite unions of translates of the lattice \( L \), while the spectra contained in finite unions of translates of \( L^* \).

However there exist also examples of crystalline measures \( \mu \), such that the support \( \Lambda \) is not contained in any finite union of translates of a lattice. Constructions of such examples, using different approaches, were given in \( \text{[LO16, Kol16, Mey16, Mey17, RV19, KS20, Mey20, OU20]} \).

On the other hand, it was proved in \( \text{[LO13, LO15]} \) that if the support \( \Lambda \) and the spectrum \( S \) of a crystalline measure \( \mu \) on \( \mathbb{R} \) are uniformly discrete sets, then the measure can be obtained from Poisson’s formula by a finite number of shifts, modulations, and taking linear combinations.
One can introduce a more general notion of a crystalline temperate distribution, which is by definition a temperate distribution $\alpha$ whose support $\Lambda$ is a locally finite set and such that the spectrum $S$ (the support of the Fourier transform $\hat{\alpha}$) is also a locally finite set. See [Pal17], [Fav18], [LR20] where distributions of this type were studied.

Examples of crystalline temperate distributions which are not crystalline measures may be constructed by starting from a crystalline measure $\mu$ and applying, in addition to the operations mentioned above, also a finite number of differentiations and multiplication by polynomials. The main result of the present paper is that any temperate distribution on $\mathbb{R}$ whose support and spectrum are uniformly discrete sets, can be obtained from Poisson’s summation formula by these basic operations:

**Theorem 1.1.** Let $\alpha$ be a temperate distribution on $\mathbb{R}$ such that $\Lambda = \text{supp}(\alpha)$ and $S = \text{supp}(\hat{\alpha})$ are uniformly discrete sets. Then $\alpha$ can be represented in the form

$$\alpha = \sum_{(\tau,\omega,l,p)} c(\tau,\omega,l,p) \sum_{\lambda \in L} \lambda \cdot e^{2\pi i \lambda \omega} \delta_{\lambda+\tau}$$

where $L$ is a lattice, $(\tau,\omega,l,p)$ goes through a finite set of quadruples such that $\tau,\omega$ are real numbers and $l,p$ are nonnegative integers, and $c(\tau,\omega,l,p)$ are complex numbers.

Conversely, if $\alpha$ is a distribution of the form (1.3) then its support $\Lambda$ and spectrum $S$ are uniformly discrete sets. In fact, $\Lambda$ is contained in a finite union of translates of the lattice $L$, while $S$ is contained in a finite union of translates of the dual lattice $L^*$. The proof of Theorem 1.1 is based on the approach developed in [LO13], [LO15] which is generalized from the context of measures to temperate distributions.

2. Preliminaries

In this section we briefly recall some preliminary background in the theory of Schwartz distributions (see [Rud91] for more details).

The Schwartz space $\mathcal{S}(\mathbb{R})$ consists of all infinitely smooth functions $\varphi$ on $\mathbb{R}$ such that for each $n,k \geq 0$, the norm

$$\|\varphi\|_{n,k} := \sup_{x \in \mathbb{R}} (1 + |x|)^n \sum_{j=0}^k |\varphi^{(j)}(x)|$$

is finite. A temperate distribution on $\mathbb{R}$ is a linear functional on the Schwartz space which is continuous with respect to the topology generated by this family of norms.

We use $\alpha(\varphi)$ to denote the action of a temperate distribution $\alpha$ on a Schwartz function $\varphi$. For each temperate distribution $\alpha$ there exist $n$ and $k$ such that

$$|\alpha(\varphi)| \leq C \|\varphi\|_{n,k}, \quad \varphi \in \mathcal{S}(\mathbb{R}),$$

where $C = C(\alpha, n, k)$ is a constant which does not depend on $\varphi$.

If $\varphi$ is a Schwartz function then its Fourier transform is defined by

$$\hat{\varphi}(t) = \int_{\mathbb{R}} \varphi(x) e^{-2\pi i tx} \, dx.$$

If $\alpha$ is a temperate distribution then its Fourier transform is defined by $\hat{\alpha}(\varphi) = \alpha(\hat{\varphi})$.

We denote by $\text{supp}(\alpha)$ the closed support of a temperate distribution $\alpha$. 
If $\alpha$ is a temperate distribution and if $\varphi$ is a Schwartz function, then the product $\alpha \cdot \varphi$ is a temperate distribution defined by $(\alpha \cdot \varphi)(\psi) = \alpha(\varphi \cdot \psi)$, $\psi \in \mathcal{S}(\mathbb{R})$. If $\varphi$ does not vanish at any point of supp($\alpha$) then we have supp($\alpha \cdot \varphi$) = supp($\alpha$).

The convolution $\alpha \ast \varphi$ of a temperate distribution $\alpha$ and a Schwartz function $\varphi$ is an infinitely smooth function which is also a temperate distribution, and whose Fourier transform is $\hat{\alpha} \cdot \hat{\varphi}$. If $\varphi$ has compact support then supp($\alpha \ast \varphi$) is contained in the Minkowski sum supp($\alpha$) + supp($\varphi$).

**Lemma 2.1.** Let $\alpha$ be a temperate distribution whose support $\Lambda$ is a uniformly discrete set. Then $\alpha$ has the form

$$\alpha = \sum_{p=0}^{k} \sum_{\lambda \in \Lambda} a_p(\lambda) \delta^{(p)}(\lambda)$$

where the coefficients $a_p(\lambda)$ satisfy the condition

$$\sum_{p=0}^{k} |a_p(\lambda)| \leq C(1 + |\lambda|)^n, \quad \lambda \in \Lambda,$$

for certain constants $n$ and $C$.

For a proof see e.g. [Pal17, Proposition 2] or [Fav18, Proposition 3.1].

### 3. Spectral Gap and Density

We say that a temperate distribution $\gamma$ has a spectral gap of size $a > 0$, if its Fourier transform $\hat{\gamma}$ vanishes on an interval of length $a$. There is a well-known principle stating that if a set $\Gamma \subset \mathbb{R}$ supports a nonzero measure, or a distribution, with a spectral gap, then $\Gamma$ cannot be “too sparse”. Several concrete versions of this general principle can be found in [KM58, Proposition 7], [MP10], [Pol12], [LO15, Section 4].

Let $\Gamma \subset \mathbb{R}$ be a locally finite set (that is, a set with no finite accumulation points) and consider a distribution $\gamma$ of the form

$$\gamma = \sum_{p=0}^{k} \sum_{\lambda \in \Gamma} c_p(\lambda) \delta^{(p)}(\lambda),$$

where the coefficients $c_p(\lambda)$ are assumed to satisfy the condition

$$\sum_{p=0}^{k} \sum_{\lambda \in \Gamma} |c_p(\lambda)| < +\infty.$$  

The condition (3.2) implies in particular that the sum in (3.1) converges in the space of temperate distributions.

We define the density $D(\Gamma)$ of the set $\Gamma$ to be

$$D(\Gamma) := \liminf_{R \to +\infty} \frac{1}{2R} \int_{1}^{R} \frac{n_{\Gamma}(r)}{r} dr,$$

where we denote $n_{\Gamma}(r) := \#(\Gamma \cap [-r, r])$. We have the following result:
Theorem 3.1. Let $\Gamma \subset \mathbb{R}$ be a locally finite set, and let $\gamma$ be a nonzero distribution of the form (3.1) and such that condition (3.2) is satisfied. If the Fourier transform $\hat{\gamma}$ vanishes on an interval of length $a$, then we must have

$$D(\Gamma) \geq \frac{a}{k+1}. \quad (3.4)$$

The result actually holds under more general assumptions, where instead of conditions (3.1) and (3.2) one merely assumes that $\gamma$ is a nonzero distribution with $\text{supp}(\gamma) \subset \Gamma$ and satisfying the condition $|\gamma(\varphi)| \leq C\|\varphi\|_{n,k}, \varphi \in \mathcal{S}(\mathbb{R})$, where $n, k$ and $C$ do not depend on $\varphi$. However we will not use this more general version in the paper and we do not give its proof. The proof of Theorem 3.1 given below is based on a classical approach which involves an application of Jensen’s formula to the Cauchy transform of the distribution $\gamma$, see [KM58, p. 73].

We note that the estimate (3.4) is sharp, see Example 3.2 below.

Proof of Theorem 3.1. Consider the Fourier transform $\hat{\gamma}$ of the distribution $\gamma$,

$$\hat{\gamma}(t) = \sum_{p=0}^{k} (2\pi i t)^p \sum_{\lambda \in \Gamma} c_p(\lambda) e^{-2\pi i \lambda t}. \quad (3.5)$$

Then $\hat{\gamma}$ is a continuous function satisfying the estimate

$$|\hat{\gamma}(t)| \leq K(1 + |t|)^k, \quad (3.6)$$

where $K$ is a certain constant which does not depend on $t$. Define the function

$$f(z) := -\int_{0}^{\infty} \hat{\gamma}(t) e^{2\pi izt} dt, \quad \text{Im}(z) > 0, \quad (3.7)$$

and

$$f(z) := \int_{-\infty}^{0} \hat{\gamma}(t) e^{2\pi izt} dt, \quad \text{Im}(z) < 0. \quad (3.8)$$

The integrals in (3.7) and (3.8) converge absolutely, due to the estimate (3.6).

The function $f$ is called the Fourier-Carleman transform of the distribution $\gamma$. It follows from the uniqueness property of the Fourier transform that the function $\hat{\gamma}$, and hence also the distribution $\gamma$, are uniquely determined by $f$. In particular, since $\gamma$ is assumed to be nonzero, it follows that $f$ does not vanish identically.

If we substitute (3.5) into (3.7) and (3.8) and exchange the order of summation and integration (which is justified using (3.2) and the dominated convergence theorem) then we obtain

$$f(z) = \frac{1}{2\pi i} \sum_{p=0}^{k} \sum_{\lambda \in \Gamma} c_p(\lambda) \frac{p!(-1)^p}{(z - \lambda)^{p+1}} \quad (3.9)$$

for every $z \in \mathbb{C} \setminus \mathbb{R}$. Using again the assumption (3.2) this implies that $f$ can be extended to a meromorphic function in $\mathbb{C}$ whose poles are contained in $\Gamma$, and such that the multiplicity of each pole is at most $k + 1$.

The right hand side of (3.9) is called the Cauchy transform of the distribution $\gamma$. The equality in (3.9) states a well-known relation between the Cauchy transform and the Fourier-Carleman transform, see e.g. [Ben84, Section 3].
We now use the assumption that \( \hat{\gamma} \) vanishes on an interval of length \( a \). By applying a translation we may assume, with no loss of generality, that \( \hat{\gamma} \) vanishes on the interval \((-a/2, a/2)\). Using this together with (3.12), (3.13) and (3.14) we obtain

\[
|f(x + iy)| \leq K \int_{a/2}^{\infty} (1 + t^k)e^{-2\pi|y|t} dt = K|y|^{-k-1}p_k(|y|)e^{-\pi a|y|},
\]

where \( p_k \) is a polynomial of degree \( k \) whose coefficients depend on \( k \) and \( a \) but do not depend on \( x \) and \( y \). This implies the estimate

\[
|f(x + iy)| \leq C(|y|^{-k-1} + |y|^{-1})e^{-\pi a|y|}, \quad x \in \mathbb{R}, \quad y \neq 0,
\]

where \( C \) is a constant not depending on \( x \) and \( y \).

Finally we apply Jensen’s formula (see e.g. [Lan99, Chapter XII, Section 1] or [Lev96, Sections 2.4 and 2.5]) to the function \( f \). It yields that for \( R \geq 1 \) we have

\[
\int_1^R \frac{n_f(r, 0) - n_f(r, \infty)}{r} dr = \int_0^{2\pi} \log |f(Re^{i\theta})| \frac{d\theta}{2\pi} + c_f,
\]

where \( n_f(r, 0) \) is the number of zeros (counted with multiplicities) of \( f \) in the disk \( \{z : |z| \leq r\} \), \( n_f(r, \infty) \) is the number of poles in the same disk (again counted with multiplicities), and \( c_f \) is a constant which depends on \( f \) but does not depend on \( R \).

The estimate (3.11) implies that

\[
\int_0^{2\pi} \log |f(Re^{i\theta})| \frac{d\theta}{2\pi} \leq -2aR + o(R), \quad R \to +\infty.
\]

We also have

\[
n_f(r, \infty) \leq (k + 1)n_\Gamma(r), \quad (3.14)
\]

since the poles of \( f \) are contained in \( \Gamma \) and the multiplicity of each pole is at most \( k+1 \). It then follows from (3.12), (3.13) and (3.14) that

\[
(k + 1) \int_1^R \frac{n_\Gamma(r)}{r} dr \geq 2aR - o(R), \quad R \to +\infty.
\]

If we now divide both sides of (3.15) by \( 2(k + 1)R \) and take the lim inf as \( R \to +\infty \), we arrive at (3.4). This concludes the proof of Theorem 3.1. \( \square \)

Example 3.2. The following example shows that the estimate (3.4) is sharp. Let \( \Gamma := \mathbb{Z} \), then \( D(\Gamma) = 1 \). Given \( \varepsilon > 0 \) we construct a nonzero distribution \( \gamma \) satisfying (3.1) and (3.2), and such that the Fourier transform \( \hat{\gamma} \) vanishes on an interval of length \( a = k + 1 - \varepsilon \). Let \( \alpha := \sum_{p=0}^k a_p \sum_{\lambda \in \mathbb{Z}} \delta_\lambda^{(p)} \), where we choose the coefficients \( \{a_p\} \) so that \( P(t) := \sum_{p=0}^k a_p(2\pi it)^p \) is a nonzero polynomial vanishing on the set \( \{1, 2, \ldots, k\} \). Then \( \alpha \) is a nonzero temperate distribution, \( \text{supp}(\alpha) \subset \Gamma \), and \( \hat{\alpha} \) vanishes on the open interval \((0, k + 1)\). We then let \( \gamma := \alpha \cdot \varphi \), where \( \varphi \) is a Schwartz function such that \( |\varphi| > 0 \) and \( \text{supp}(\hat{\varphi}) \subset (-\varepsilon, 0) \). Then the distribution \( \gamma \) is nonzero, it satisfies the conditions (3.1) and (3.2), and \( \hat{\gamma} \) vanishes on the interval \([0, k + 1 - \varepsilon]\).
4. Arithmetic structure of the support

In this section our goal is to prove the following result:

**Theorem 4.1.** Let $\alpha$ be a temperate distribution on $\mathbb{R}$ such that $\Lambda = \text{supp}(\alpha)$ and $S = \text{supp}(\hat{\alpha})$ are both uniformly discrete sets. Then $\Lambda$ is contained in a finite union of translates of some lattice.

The result was proved in [LO13], [LO15] in the case where $\alpha$ and $\hat{\alpha}$ are measures, and it is generalized here to the context of temperate distributions. Theorem 4.1 constitutes the first step in the proof of Theorem 1.1, where the next step consists of showing that the distribution $\alpha$ must be of the form (1.3).

4.1. Recall that a set $\Lambda \subset \mathbb{R}$ is said to be uniformly discrete if there is $\delta > 0$ such that $|\lambda' - \lambda| \geq \delta$ for any two distinct points $\lambda, \lambda'$ in $\Lambda$. The maximal constant $\delta$ with this property is called the separation constant of $\Lambda$, and will be denoted by $\delta(\Lambda)$.

One can check that if $\Lambda$ is a uniformly discrete set then $D(\Lambda) \leq 1/\delta(\Lambda)$. In particular, the density $D(\Lambda)$ is finite.

We say that a set $\Lambda \subset \mathbb{R}$ is relatively dense if there is $\alpha > 0$ such that any interval $[x, x + a]$ contains at least one point from $\Lambda$.

If $\Lambda$ is uniformly discrete and also relatively dense, then $\Lambda$ is called a Delone set.

4.2. Let $\Lambda \subset \mathbb{R}$, and for each $h \in \Lambda - \Lambda$ denote

$$\Lambda_h := \Lambda \cap (\Lambda - h) = \{\lambda \in \Lambda : \lambda + h \in \Lambda\}.$$  

(4.1)

Then $\Lambda_h$ is a nonempty subset of $\Lambda$. We will use the following key result:

**Theorem 4.2 ([LO13], [LO15]).** Let $\Lambda \subset \mathbb{R}$ be a Delone set, and suppose that there exists a constant $c = c(\Lambda) > 0$ such that $D(\Lambda_h) \geq c$ for every $h \in \Lambda - \Lambda$. Then $\Lambda$ is contained in a finite union of translates of a certain lattice.

This was actually proved in [LO13], [LO15] with the density

$$D_\#(\Lambda_h) := \liminf_{r \to +\infty} \frac{\#(\Lambda_h \cap [-r, r])}{r}$$  

(4.2)

instead of $D(\Lambda_h)$ in the statement, but both the result and its proof are valid for either one of these densities. The proof involves the concept of Meyer sets and is based on a characterization of these sets that is due to Meyer [Mey72].

4.3. The following result can be found in [KM58, Proposition 7].

**Lemma 4.3.** Let $\alpha$ be a nonzero temperate distribution whose spectrum $S = \text{supp}(\hat{\alpha})$ is uniformly discrete. Then the support $\Lambda = \text{supp}(\alpha)$ is a relatively dense set.

**Proof.** Suppose that $[x, x + a]$ is an interval of length $a$ disjoint from the support $\Lambda$. Let $\varphi$ be a Schwartz function supported on a sufficiently small neighborhood of the origin so that $\alpha * \varphi$ vanishes on $[x, x + a]$. It follows from Lemma 2.1 (applied to the Fourier transform $\hat{\alpha}$ of $\alpha$) that the distribution $\gamma := \hat{\alpha} \cdot \hat{\varphi}$ satisfies the conditions (3.1) and (3.2) with $\Gamma = S$ and with $k$ that does not depend on $\varphi$. If we choose $\varphi$ such that also $\hat{\varphi} > 0$, then $\gamma$ is a nonzero distribution whose Fourier transform vanishes on an interval of length $a$. Theorem 3.1 then yields that $a$ cannot be greater than $(k + 1)D(S)$.  \[\square\]
4.4. We now use the previous results in order to prove Theorem 4.1.

Proof of Theorem 4.1. By Lemma 4.3 the support \( \Lambda \) is relatively dense, so \( \Lambda \) is a Delone set. Due to Theorem 4.2 it will therefore suffice to show that there is \( c > 0 \) such that
\[
D(\Lambda_h) \geq c \quad \text{for every } h \in \Lambda - \Lambda.
\]
We know from Lemma 2.1 that there exist \( n \) and \( k \) such that the distribution \( \alpha \) has the form (2.4) and that (2.5) is satisfied. We will prove that the condition
\[
D(\Lambda_h) \geq \frac{\delta(S)}{(k+1)}
\]
holds for every \( h \in \Lambda - \Lambda \).

To prove (4.3) we will show that given any \( h \in \Lambda - \Lambda \) and any \( r > 0 \) one can find a nonzero distribution
\[
\gamma_h = \sum_{p=0}^{k} \sum_{\lambda \in \Lambda_h} c_{p,h}(\lambda) \delta^{(p)}_\lambda
\]
such that the coefficients \( c_{p,h}(\lambda) \) satisfy
\[
\sum_{p=0}^{k} \sum_{\lambda \in \Lambda_h} |c_{p,h}(\lambda)| < +\infty,
\]
and such that the Fourier transform \( \hat{\gamma}_h \) vanishes on the set
\[
U := \mathbb{R} \setminus [(S - S) + (-r, r)].
\]
Observe that the last property implies that the distribution \( \gamma_h \) has a spectral gap of size \( \delta(S) - 2r \), and so it follows from Theorem 3.1 that \( D(\Lambda_h) \geq (\delta(S) - 2r)/(k + 1) \). Since \( r \) may be chosen arbitrarily small, this yields (4.3).

In order to construct the distribution \( \gamma_h \) we use the approach in [LO13, Section 2.1]. We fix \( r > 0 \) and choose a Schwartz function \( \varphi > 0 \) such that \( \text{supp}(\hat{\varphi}) \subset (-r/2, r/2) \). It follows from (2.5) that the product \( \alpha \cdot \varphi \) has the form
\[
\alpha \cdot \varphi = \sum_{p=0}^{k} \sum_{\lambda \in \Lambda} b_p(\lambda) \delta^{(p)}_\lambda
\]
where the coefficients \( b_p(\lambda) \) satisfy
\[
\sum_{p=0}^{k} \sum_{\lambda \in \Lambda} |b_p(\lambda)| < +\infty.
\]
(The coefficients \( b_p(\lambda) \) depend on the function \( \varphi \), but \( k \) does not.)

Let \( f \) be the Fourier transform of the distribution \( \alpha \cdot \varphi \). Then we have \( f = \hat{\alpha} \ast \hat{\varphi} \) and hence \( f \) is an infinitely smooth function vanishing outside the \((r/2)\)-neighborhood of the set \( S \). On the other hand, by (4.7) we have
\[
f(x) = \sum_{p=0}^{k} (2\pi ix)^p \sum_{\lambda \in \Lambda} b_p(\lambda) e^{-2\pi i \lambda x}.
\]
Let \( g(x, u) := f(x)f(x-u) \), then using (4.9) and opening the brackets we obtain
\[
g(x, u) = \sum_{l=0}^{2k} (2\pi ix)^l \sum_{h \in \Lambda - \Lambda} A_{h,l}(u) e^{-2\pi ih x}.
\]
where the coefficients $A_{h,l}(u)$ are given by

$$A_{h,l}(u) = \sum_{(p,q,j) \in J(k,l)} (-1)^{p+j} \binom{p}{j} (2\pi i u)^j \sum_{\lambda \in \Lambda_h} b_q(\lambda + h) \overline{b_p(\lambda)} e^{-2\pi i \lambda u}$$  \hfill (4.11)

and where $J(k,l)$ denotes the finite set of all triples $(p, q, j) \in \mathbb{Z}^3$ satisfying

$$0 \leq p, q \leq k, \quad 0 \leq j \leq p, \quad p + q - j = l. \hfill (4.12)$$

We observe that $g(x, u)$ is, as a function of $x$, the Fourier transform of the distribution

$$\beta_u = \sum_{l=0}^{2k} \sum_{h \in \Lambda - \Lambda} A_{h,l}(u) \delta_h^j,$$  \hfill (4.13)

and the coefficients $A_{h,l}(u)$ satisfy

$$\sum_{l=0}^{2k} \sum_{h \in \Lambda - \Lambda} |A_{h,l}(u)| < +\infty. \hfill (4.14)$$

On the other hand, for every $u \in U$ the function $g(x, u)$ vanishes identically with respect to $x$. It follows from the uniqueness of the representation (4.13) that $A_{h,l}(u) = 0$, $u \in U$.

For each $h \in \Lambda - \Lambda$ and $0 \leq l \leq 2k$ we now define

$$\gamma_{h,l} := \sum_{(p,q,j) \in J(k,l)} (-1)^{p+j} \binom{p}{j} \sum_{\lambda \in \Lambda_h} b_q(\lambda + h) \overline{b_p(\lambda)} \delta_h^j. \hfill (4.15)$$

The fact that

$$\sum_{\lambda \in \Lambda_h} |b_q(\lambda + h) \overline{b_p(\lambda)}| < +\infty, \quad (p, q, j) \in J(k,l), \hfill (4.16)$$

ensures that the sum (4.15) converges in the space of temperate distributions. Moreover, the distribution $\gamma_{h,l}$ has the form (4.4), (4.5), since the condition $(p, q, j) \in J(k,l)$ implies that $j \leq k$. Moreover, due to (4.11) we have $\widehat{\gamma}_{h,l}(u) = h_{h,l}(u)$, $u \in \mathbb{R}$, which yields

$$\widehat{\gamma}_{h,l}(u) = 0, \quad u \in U. \hfill (4.17)$$

It thus remains to show that for each $h \in \Lambda - \Lambda$ there is at least one $l_0 = l_0(h)$ such that the distribution $\gamma_{h,l_0}$ is nonzero. Indeed, given such $h$ we choose $\lambda_0 \in \Lambda$ such that $\lambda_0 + h$ is also in $\Lambda$. Since $\varphi > 0$ we have $\text{supp}(\alpha \cdot \varphi) = \Lambda$, hence by (4.7) there is a largest integer $j_0$ such that the coefficient $b_{j_0}(\lambda_0)$ is nonzero, and there is also at least one integer $l_0$ such that $b_{l_0}(\lambda_0 + h)$ is nonzero (in particular we have $0 \leq l_0 \leq k$). Let us show that for this choice of $l_0$ the distribution $\gamma_{h,l_0}$ is nonzero. It would suffice to verify that there is precisely one triple $(p, q, j) \in J(k,l_0)$ with $j = j_0$ and such that $b_q(\lambda_0 + h) \overline{b_p(\lambda_0)}$ is nonzero, since this would imply that in the sum (4.15) the coefficient of $\delta_{j_0}^1$ is nonzero. Indeed, if $(p, q, j)$ is such a triple then we have $p \leq j_0$ due to the maximality of $j_0$. But at the same time $p \geq j = j_0$, so $p = j_0$. Since $p + q - j = l_0$, this implies in turn that $q = l_0$. It follows that $(j_0, l_0, j_0)$ is the unique triple in $J(k,l_0)$ with the properties above, and the coefficient $b_{l_0}(\lambda_0 + h) \overline{b_{j_0}(\lambda_0)}$ is indeed nonzero. We thus obtain that the distribution $\gamma_{h,l_0}$ is nonzero which concludes the proof. \hfill \Box
5. Poisson type structure of the distribution

In this section we complete the proof of Theorem 1.1 by establishing the following:

**Theorem 5.1.** Let \( \alpha \) be a temperate distribution on \( \mathbb{R} \). Suppose that \( \Lambda = \text{supp}(\alpha) \) is contained in a finite union of translates of a lattice \( L \), and that \( S = \text{supp}(\hat{\alpha}) \) is a locally finite set. Then \( \alpha \) can be represented in the form \( (1.3) \).

Theorem 1.1 follows as a consequence of Theorem 4.1 and Theorem 5.1.

The representation \( (1.3) \) implies that the distribution \( \alpha \) can be obtained from the Poisson summation formula by a finite number of basic operations: shifts, modulations, differentiations, multiplication by polynomials, and taking linear combinations.

5.1. We will use the following two lemmas.

**Lemma 5.2.** Let \( z_1, \ldots, z_N \) be distinct nonzero complex numbers, let \( k_1, \ldots, k_N \) be positive integers, and let \( K = k_1 + \cdots + k_N \). For each integer \( m \), \( 0 \leq m \leq K - 1 \), and for each pair of integers \((j, l)\) satisfying \( 1 \leq j \leq N \), \( 0 \leq l \leq k_j - 1 \), we denote
\[
M_{m,j,l} := (2\pi im)^l z_j^m.
\]
(5.1)

Let \( M \) be a \( K \times K \) matrix whose rows are indexed by \( m \) and columns indexed by pairs \((j, l)\), such that the entry at row \( m \) and column \((j, l)\) is \( M_{m,j,l} \). Then \( M \) is invertible.

This is due to the fact that the matrix \( M \) can be obtained by a finite number of elementary column operations from a confluent Vandermonde matrix (sometimes also referred to as a generalized Vandermonde matrix) which is known to be invertible, see, for instance, [Kal84]. This type of matrix arises e.g. in the linear interpolation problem asking to find a polynomial \( p(z) \) of degree not greater than \( K - 1 \) such that at each point \( z_j \) the values \( p(z_j), p'(z_j), \ldots, p^{(k_j-1)}(z_j) \) are prescribed.

**Lemma 5.3.** Let \( M(t) \) be a \((k+1) \times (k+1)\) matrix with entries \( M_{p,l}(t) \), \( 0 \leq p, l \leq k \), defined by
\[
M_{p,l}(t) = \binom{p}{l}(2\pi it)^{p-l}, \quad l \leq p,
\]
(5.2)
and \( M_{p,l}(t) = 0 \), \( p < l \). Then the matrix \( M(t) \) is invertible, and the entries of the inverse matrix \( M^{-1}(t) \) are polynomials in \( t \).

This lemma is obvious since \( M(t) \) is a triangular matrix whose nonzero entries are polynomials in \( t \), and whose entries on the main diagonal are all equal to 1.

5.2. Next we prove Theorem 5.1. The approach is inspired by [LO15, Section 7].

**Proof of Theorem 5.1.** By assumption we have \( \Lambda \subset L + \{\tau_1, \ldots, \tau_N\} \), where \( L \subset \mathbb{R} \) is a lattice and the \( \tau_j \) are real numbers. We may suppose that the \( \tau_j \) are distinct modulo the lattice \( L \). Moreover, by rescaling it would suffice to consider the case \( L = \mathbb{Z} \).

By Lemma 2.1 the distribution \( \alpha \) can be represented in the form
\[
\alpha(x) = \sum_{p=0}^{k} \sum_{j=1}^{N} \mu_j^{(p)}(x - \tau_j)
\]
(5.3)
where each \( \mu_{j,p} \) is a measure supported on \( \mathbb{Z} \),

\[
\mu_{j,p} = \sum_{\lambda \in \mathbb{Z}} a_{j,p}(\lambda) \delta_{\lambda},
\]

(5.4)

and where the masses \( a_{j,p}(\lambda) \) satisfy

\[
\sum_{p=0}^k \sum_{j=1}^N |a_{j,p}(\lambda)| \leq C(1 + |\lambda|)^n, \quad \lambda \in \mathbb{Z},
\]

(5.5)

for certain constants \( n \) and \( C \).

It follows from (5.5) that each measure \( \mu_{j,p} \) is a temperate distribution. We have

\[
\hat{\alpha}(t) = \sum_{p=0}^k (2\pi it)^p \sum_{j=1}^N \hat{\mu}_{j,p}(t) e^{-2\pi it\tau_j},
\]

(5.6)

according to (5.3). Since \( \mu_{j,p} \) is a measure supported on \( \mathbb{Z} \), its Fourier transform \( \hat{\mu}_{j,p} \) is a \( \mathbb{Z} \)-periodic distribution. This implies that for every \( m \in \mathbb{Z} \) we have

\[
\hat{\alpha}(t + m) = \sum_{l=0}^k (2\pi im)^l \sum_{j=1}^N \beta_{j,l}(t) e^{-2\pi im\tau_j},
\]

(5.7)

where \( \beta_{j,l} \) denotes the temperate distribution defined by

\[
\beta_{j,l}(t) := e^{-2\pi it\tau_j} \sum_{p=0}^k \binom{p}{l} (2\pi it)^{p-l} \hat{\mu}_{j,p}(t)
\]

(5.8)

for \( 1 \leq j \leq N \) and \( 0 \leq l \leq k \).

If we now apply (5.8) with \( m = 0, 1, 2, \ldots, (k+1)N - 1 \) then we obtain a system of \((k+1)N\) equations. We consider this as a linear system with unknowns \( \beta_{j,l} \). We then invoke Lemma 5.2 with \( z_j := e^{-2\pi it\tau_j} \) (which are distinct nonzero complex numbers) and with \( k_j := k + 1 \) (\( 1 \leq j \leq N \)). It follows from the lemma that the system is invertible, and hence there exist coefficients \( b_{j,l,m} \) such that

\[
\beta_{j,l}(t) = \sum_{m=0}^{(k+1)N-1} b_{j,l,m} \hat{\alpha}(t + m).
\]

(5.9)

In other words, each \( \beta_{j,l} \) is a finite linear combination of integer translates of \( \hat{\alpha} \).

Next we apply (5.8) with \( l = 0, 1, \ldots, k \) and consider the obtained \( k+1 \) equations as a linear system with unknowns \( \hat{\mu}_{j,p}(t) e^{-2\pi it\tau_j} \) \((0 \leq p \leq k)\). By Lemma 5.3 this system is invertible and the coefficients of the inverse system are polynomials in \( t \). We conclude that there exist polynomials \( \chi_{p,l}(t) \) such that

\[
\hat{\mu}_{j,p}(t) = e^{2\pi it\tau_j} \sum_{l=0}^k \chi_{p,l}(t) \beta_{j,l}(t).
\]

(5.10)

Now we use the assumption that \( S = \text{supp}(\hat{\alpha}) \) is a locally finite set. This together with (5.9) and (5.10) implies that the distribution \( \hat{\mu}_{j,p} \) is supported on the locally finite
set $S = \{0, 1, 2, \ldots, (k + 1)N - 1\}$. At the same time, we know that $\hat{\mu}_{j,p}$ is a $\mathbb{Z}$-periodic distribution. Hence $\hat{\mu}_{j,p}$ must have the form
\[
\hat{\mu}_{j,p} = \nu_{j,p} \ast \sum_{\lambda \in \mathbb{Z}} \delta_{\lambda},
\]
(5.11)
where $\nu_{j,p}$ is a distribution with finite support. In turn this implies that
\[
\mu_{j,p} = \sum_{\lambda \in \mathbb{Z}} \hat{\nu}_{j,p}(-\lambda) \delta_{\lambda},
\]
(5.12)
that is, the masses $a_{j,p}(\lambda)$ in (5.4) are given by
\[
a_{j,p}(\lambda) = \hat{\nu}_{j,p}(-\lambda), \quad \lambda \in \mathbb{Z}.
\]
(5.13)
Finally we combine (5.3) and (5.12) to conclude that
\[
\alpha = \sum_{p=0}^{k} \sum_{j=1}^{N} \sum_{\lambda \in \mathbb{Z}} \hat{\nu}_{j,p}(-\lambda) \delta_{\lambda+\tau_j}.
\]
(5.14)
We also observe that each one of the functions $\hat{\nu}_{j,p}(x)$ is a finite linear combination of products of polynomials and complex exponentials. Hence (1.3) follows from (5.14).

6. Remarks

6.1. We say that a set $\Lambda \subset \mathbb{R}$ has bounded density if it satisfies the condition
\[
\sup_{x \in \mathbb{R}} \#(\Lambda \cap [x, x+1)) < +\infty.
\]
(6.1)
This holds if and only if $\Lambda$ is the union of a finite number of uniformly discrete sets.

One can prove the following version of Theorem 1.1 where the support $\Lambda$ is not assumed to be uniformly discrete but only to have bounded density:

**Theorem 6.1.** Let $\alpha$ be a temperate distribution on $\mathbb{R}$ satisfying (2.4) and (2.5), such that the support $\Lambda = \text{supp}(\alpha)$ has bounded density, while the spectrum $S = \text{supp}(\hat{\alpha})$ is uniformly discrete. Then the conclusion of Theorem 1.1 holds.

This is an extension of [LO17, Theorem 2.2] where the result was proved in the case where $\alpha$ and its Fourier transform $\hat{\alpha}$ are measures. The proof is based on the fact that Theorem 4.2 remains valid under the weaker assumption that $\Lambda$ is a relatively dense set of bounded density (not assumed to be uniformly discrete), see [LO17, Lemma 6.3].

6.2. There is an interesting question as to whether the result in Theorem 1.1 holds in several dimensions. The problem is open even for measures:

Let $\Lambda, S$ be two uniformly discrete sets in $\mathbb{R}^d$, $d > 1$. Suppose that there is a measure $\mu$, $\text{supp}(\mu) = \Lambda$, whose distributional Fourier transform $\hat{\mu}$ is also a measure, $\text{supp}(\hat{\mu}) = S$. Does it follow that $\Lambda$ can be covered by a finite union of translates of several lattices?

It was proved in [LO15] that the answer is affirmative if $\mu$ is a positive measure, and in this case the support $\Lambda$ can in fact be covered by a finite union of translates of a single lattice. However an example in [Fav16, Section 2] shows that for signed measures $\mu$ the support need not be contained in a finite union of translates of a single lattice.
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