The disc expansion model to determine the MIC of a planar bounded connected domain

Min Zhou
College of Engineer, China Agricultural University, 100083 Beijing, China

*E-mail: zhoumin2016@cau.edu.cn

Abstract. To determine the maximum inscribe circle (MIC) of a planar bounded connected domain which can be represented as a generalized polygon (with arcs edges), a disco expansion model is proposed. The model simulates a disc continues expanding process in a polygon without friction. When it cannot expand, the three contact points which are both on the circle and the sides of polygons are just the points to determine the MIC. The implementation algorithm of the model for convex polygon is also given. For generalized polygons with arcs or concave points, bottleneck lines as soft edge are adopted to partitioned the polygons first. Then, the proposed model can be applicable for generalized polygons with little modification.

1. Introduction

The solution of the maximum inscribed circle (MIC) in a planar bounded connected domain (PBCD) has wide application value. For example, in the measurement technique, the evaluation of roundness error is required to calculate the maximum inscribed circle of the contour [1]; and the MIC of the machining field is used as the constraint reference for the optional maximum diameter tool when the tool is selected for pocket milling [2], and so on. For the problem of the MIC determination, there are mainly the exploration method for the single facility maximin location problem (MAX G(Ω)) [3], the map algebra method [4], Voronoi diagram method [5], the method of central axis transformation [6], least squares method [7] and other methods [8]. The mainly problems of the above methods are described as follows: (1) MAX G (Ω) only requires the center of the MIC in the bounded domain; (2) The method of the central axis transformation is less efficient in the case of complex curves or too many points being on the contour; Voronoi diagrams, least squares, and other methods cannot be used directly in the presence of holes in the domain. Therefore, this paper proposes the simulation model of the disk expansion to determine the MIC of a planar bounded connected domain.

The MIC calculation of a PBCD can be transferred to the MIC determination of a generalized polygon which represents its boundary. It is easy to know that three non-collinear points can uniquely determine a circle. Thus, the MIC of a polygon can be determined by three points. These three points may be points on the edge of the polygon or concave vertices, and the edge or concave vertices determining the MIC are defined as touch elements. Obviously, the shortest distance from the center of the MIC to the three touch elements is equal, which is the MIC radius. Specially, when there are mutually parallel line segments in the polygon, if the MIC exists between the two parallel line segments, there are an infinite number of MIC, and the center line is a line segment parallel to the two parallel line segments. At this time, any MIC is determined by two touch elements, that is, the two...
parallel line segments. Based on the above principle, a simulation model of disc expansion is proposed and its implementation algorithm is given.

2. The simulation model of expansion disk

Let \( \Omega \) be a PBCD, \( \Gamma \) be the boundary of \( \Omega \), \( P \) be a polygon representing \( \Gamma \), and \( P \) is a convex polygon. \( C \) is a disc in \( \Omega \), and the disc is continuously free to expand outward under the unobstructed condition by the action of internal force \( F \), which is called a free expansion disc, referred to as an expansion disc. The simulation model of expansion disk is as following:

1. When \( C \) does not touch the boundary of \( \Omega \), the resultant external force on the disk is 0 (assuming all friction coefficients are 0).

2. When the disc expands to touch a certain edge of \( P \) at point \( p \) (point \( p \) is called a contact point), the disc is subjected to a reaction force \(-F\) at point \( p \) in direction of \( \overrightarrow{po} \) (as shown in figure 1), and the condition is not limited by the force state of the disc.

![Figure 1. Force diagram of the expansion disk touching the point p1.](image)

3. Under the action of external force, the center of the disc moves in the same direction as the direction of reaction force \(-F\) at contact point \( p \), while the disc continues to expand in the direction of unobstructed.

4. The speed at which the disc expands causes the disc to be always tangential to the edge of the obstacle edge until it hits the next obstacle edge.

5. When the disc expands to the state that the radius cannot continue to increase when the center moves towards the direction of the resultant force of the reaction forces of contact points, and the disc stop expanding, then, the disc at this time is the MIC of \( P \).

Based on the simulation model of the disk expansion, let the moving direction of center be \( \mathbf{n}_c \). The figure 2 illustrates the expansion process of disc in convex polygon \( P \). A expansion disk \( C \) (figure 2a) is expanding in a polygon \( P \). When \( C \) expands to touch a certain edge of \( P \) at point \( p_1 \) (figure 2b), center of \( C \) moves towards to the direction of \( \overrightarrow{p_1o} \) (\( \mathbf{n}_3 \)), while the disc continues expanding. When \( C \) expand to touch another edge of \( P \) at point \( p_2 \) (figure 2c), the disc is affected by two reaction forces at directions \( \overrightarrow{p_1o} \) and \( \overrightarrow{p_2o} \), thus, the center \( o \) moves along \( \mathbf{n}_o \), the direction of the resultant force, while the disc continues expanding. When \( C \) expand to touch the edge of \( P \) at \( p_3 \) as shown in figure 2d, the disc is affected by three reaction forces at directions \( \overrightarrow{p_1o} \), \( \overrightarrow{p_2o} \) and \( \overrightarrow{p_3o} \), \( p_1' \) and \( p_2' \) are new contact points for disc and \( P \). Since only three non-collinear points can uniquely determine a circle, it is necessary to judge whether the disc can continue to expand. If the disc continues to expand, and any contact point will becomes a line while distance between the center and the contact point/line no longer increases, then, the disc should stop expanding and the disc at this time is the MIC of \( P \); otherwise, the circle continues to expand and the center moves towards the direction of the new resultant force till the disk cannot to continue expanding to obtain the MIC of \( P \) (figure 2e).

If \( P \) is not a convex polygon, when the contact point \( p \) is a concave vertex, the direction of the reaction is \( \overrightarrow{po} \). Or, the polygon is a generalized polygon which has arcs, the simulation model of disc expansion is still adoptable. However, in above both cases, the simulation model of disc expansion may result in the final solution being a local optimal solution rather than a global optimal solution, as shown in figure 3. For this situation, the polygon can be first partitioned according to the bottleneck...
line, and the bottleneck line of the partition is regarded as a soft boundary which can intersect with the MIC. Then, the disk expansion model with little modification can be used to determine the MICs of each sub-region. And compare them. The MIC of the entire $P$ can be obtained by comparing MICs of regional areas.

![Diagram](image1)

**Figure 2.** Schematic diagram of the solution for MIC of $P$ by expansion disk method. (a. Initial disk; b. $o$ moving in the direction of $\overrightarrow{p_o o}$; c. $o$ moving in the $n_o$ direction; d. $o$ moving in the $n_o$ direction; e. MIC of $P$).

![Diagram](image2)

**Figure 3.** Schematic diagram of local optimal solution. (a. The resultant force of the reaction forces is 0; b. Having reached the condition of stopping expansion).

3. Algorithm for the disc expansion model

![Algorithm](image3)

**Figure 4.** Algorithm for MIC calculation based on simulated expansion disc method.
Since the geometric center of non-porous polygons is very close to the center of its MIC in most cases, and the solution for geometric center determination is simple, this paper first finds the geometric center of the polygon (convex-hull \( CH(P) \) for generalized polygon) as a starting point for searching the center of the MIC to speed up the search. For polygon or \( CH(P) \) whose geometric center is not inside of \( P \), find a point in \( P \) as a starting point. The specific algorithm is shown in Figure 4.

where:

**Step1.** Enter \( P \) and find the geometric center \( o \) of \( P \) or \( CH(P) \). If \( o \) is not inside \( P \), find a point in \( P \) as the initial center \( o \) of the expansion disk;

**Step2.** Calculate the distance from \( o \) to each edge of \( P \), and let \( m_{op_i} = \min \{ m_{op} \} \), where, \( i = 1, 2, \ldots, N \), \( N \) is the number of edges of \( P \), \( p_1 \) is the point on \( P \) and closest to \( o \);

**Step3.** Let the vector of expansion disc’s center \( o \)’s moving direction \( n_o = \frac{p_1 - o}{||p_1 - o||} \), and the radius of the expansion disc \( r = m_{op_1} \);

**Step4.** Find a point \( o' \) to make it on the extension line of \( o' \), that is \( \vec{oo'} \cdot \vec{n_o} = 0 \), and make \( ||oo'|| = s \), where, \( s \) is the moving step of the center.

**Step5.** Calculate the shortest distance from \( o' \) to each edge of \( P \) except the edge with \( p_1 \), and let \( m_{op_{i-1}} = \min \{ m_{op} \} \), \( i = 1, 2, \ldots, N \);

**Step6.** If the distance from \( o' \) to \( p_1 \) is equal to the distance from \( o' \) to \( p_2 \), that is \( m_{op_1} - (r + s) < \varepsilon \), where \( \varepsilon \) is any small positive number, then go to Step10; otherwise, go to Step7.

**Step7.** If the distance from \( o' \) to \( p_2 \) is smaller than the distance from \( o' \) to \( p_1 \), that is, \( m_{op_{i-1}} - (r + s) < 0 \), then, reduce the step size to make \( s = s/2 \), and go back to Step4.

**Step8.** If the distance from \( o' \) to \( p_2 \) is bigger than the distance from \( o' \) to \( p_1 \), that is, \( m_{op_{i-1}} - (r + s) > 0 \), then, increase the step size to make \( s = s + s/2 \), and go back to Step4.

**Step9.** Update the values of step, center and radius to make \( s = t, o = o', r = r + s \);

where, \( t \) is the given value of step size and equal to the previously given step \( s \) in Step4.

**Step10.** Recalculate the moving direction of the center, which is the sum of the directions of the reaction forces of \( p_1 \) and \( p_2 \), that is, \( n_o = \frac{op_1 + op_2}{||op_1 + op_2||} \);

**Step11.** Determine whether vector \( n_o \) is equal to \( 0 \), if it is equal to \( 0 \), it means that the edge with \( p_1 \) are parallel to the edge with \( p_2 \), and the MIC of \( P \) has been obtained, then, go to Step21; otherwise, go to Step12.

**Step12.** Find a point \( o' \) and make \( oo' \) and \( n_o \) in the same direction, that is \( \vec{oo'} \cdot \vec{n_o} = 0 \), and let \( ||oo'|| = s \).

**Step13.** Calculate the distance from \( o' \) to each side of \( P \) and arrange them in ascending order;

**Step14.** If \( m_{op_1} \) is bigger than the radius obtained by Step10, namely, \( m_{op_1} - r > 0 \), then go to Step16; otherwise, go to Step15.

**Step15.** Reduce the value of step size to make \( s = s/2 \), and determine whether inequality \( s < \varepsilon \) is satisfied. If it is satisfied, go to step21; Otherwise, go to step12.

**Step16.** Determine whether the value of \( m_{op_1} \) is equal to the value of \( m_{op_2} \), that is, whether \( m_{op_1} - m_{op_2} < \varepsilon \) is satisfied, if yes, go to Step17; otherwise, go to Step15;

**Step17.** Update the values of step, center and radius to make \( r = m_{op_2}, s = t, o = o', r = r + s \).
Step 18. Determine whether $|\overline{op}_1|$ and $|\overline{op}_3|$ are equal, that is, whether $|\overline{op}_1| - |\overline{op}_3| < \varepsilon$ is satisfied, if yes, go to Step 19; otherwise, go to Step 10.

Step 19. Recalculate the moving direction of the center: find the two sides with the two smallest angles among the three edges with the three contact points, then the moving direction of the center is just the direction of the resultant force of the reaction forces generated by the two sides.

Step 20. Determine whether the vector $n_o$ is equal to $0$, if yes, go to Step 21; Otherwise, go to Step 12.

Step 21. Let $r_{MIC} = r$, $o_{MIC} = o'$, output the value of the radius and the position information of the center.

4. Conclusion
A disc expansion model is proposed to determine the MIC of generalized polygon with arc lines. The polygon with concave point is portioned into convex polygons by bottleneck lines which are dealt as soft edges first. Then, the model is applicable. The experiment results also show the correctness of the proposed model and the corresponding algorithm.

References
[1] X.Q. Lei, C. Zhang, Y.J. Xue, J.S. Li, Measurement., 44: 345-350(2011)
[2] M. Zhou, G.L. Zheng, Z.C. Chen, Int J Adv Manue Tech, 83(1-4): 407-420(2016)
[3] F.G. Zhang, Melachrinoudis E, Comput Optim Appl, 19(2): 209-234(2001)
[4] Hu Hai, Liu Jingnan, Geo Inf Sci of Wuhan University, 31(008): 700-703(2006)
[5] Toussaint G.T., Int J Com Info Sci, 12(5): 347-358(1983)
[6] J.T. Chen, Dalian: Dalian University of Technology(2009)
[7] K. Kim, S. Lee, H.B. Jung, Int J Adv Manue Tech, 16(8): 559-563(2000)
[8] Y. Sun, R. Che, Opt.Precision Eng., 11(2): 181-187(2003)