An Infinitesimal Quantum Group Underlies Classical Fluid Mechanics

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Arnold showed that the Euler equations of an ideal fluid describe geodesics in the Lie algebra of incompressible vector fields. We will show that helicity induces a splitting of the Lie algebra into two isotropic subspaces, forming a Manin triple. Viewed another way, this shows that there is an infinitesimal quantum group (a.k.a. Lie bi-algebra) underlying classical fluid mechanics.
Infinite Dimensional Lie algebras are important in physics as symmetries of field theories, such as the standard model of particle physics. The best understood cases are 1 + 1 dimensional theories, such as the Kac–Moody\cite{Kac-Moody} and Virasoro algebras. The latter also arises in fluid mechanics in one spatial dimension\cite{Takens}: being essentially the Lie algebra of vector fields on a circle it is the phase space of a fluid with periodic boundary conditions. By now, its geometry, representation theory\cite{Kac-Moody} and physical meaning are all quite understood. The correlation functions of many conformal field theories (and related critical statistical models) can be found exactly using the representation theory\cite{Kac-Moody,Mu} of the Virasoro and Kac-Moody algebras.

A similar theory of Lie algebras that arise from 3 + 1 dimensional field theories would be of great interest, as they would apply to more realistic physical systems. A good candidate is the Lie algebra \( S \) of incompressible (i.e., satisfying \( \nabla \cdot u = 0 \)) vector fields. Arnold\cite{Arnold} showed that the Euler equations describe geodesics on it; its representations (more generally “quantum” deformations) are likely to be useful in determining the correlation functions of velocity in turbulent fluid. There are many analogies with the Virasoro and Kac-Moody algebras: \( S \) is a graded Lie algebra, with a non-trivial central extension \( \hat{S} \). It is known that \( S \) admits an invariant inner product (related to helicity in fluid mechanics). We will show in this paper that it can be extended to \( \hat{S} \), as for Kac-Moody algebras. Moreover, we will show that \( \hat{S} \) is a Lie bi-algebra. This opens up an as yet unsolved problem: exponentiate \( \hat{S} \) to a “quantum group” (more precisely, a Hopf algebra). This could generalize to three dimensions of the symplectic integration methods available in two dimensions\cite{Symplectic}.

The commutator \([u, w] = u \cdot \nabla w - w \cdot \nabla u\) is anti-symmetric and satisfies the Jacobi identity. The identity \([u, w] = \nabla \times (w \times u) + w \nabla u - u \nabla w\) shows that the commutator of incompressible vector fields is again incompressible; i.e., they form a Lie algebra \( S \). It shows a bit more: that the commutator is exact; i.e., the curl of some vector field. The exact vector fields form a sub-algebra (indeed an ideal) \( S' \subset S \).

We will impose periodic boundary conditions; i.e., space will be a torus \( \mathbb{T}^3 \) of side \( 2\pi \). A simple argument using Fourier analysis shows that, on \( \mathbb{T}^3 \), any vector field with zero average is exact (i.e., the curl operator is invertible in \( S' \)). So, any vector field can be written as the sum of a constant vector field (its average \( \bar{u} = \int u \frac{d^3x}{(2\pi)^3} \)) and an exact vector field \( u' = u - \bar{u} \). Since constant vector fields commute, we get a semi-direct product \( S = \mathbb{R}^3 \rtimes S' \).

\( S' \) has an invariant inner product \( \langle u', w' \rangle = \int u'.\text{curl}^{-1} w' \frac{d^3x}{(2\pi)^3} \). Symmetry follows by integration by parts. To see the invariance, use the commutator identity above to write \( \langle u', [v', w'] \rangle = \int u'.(v' \times w') \frac{d^3x}{(2\pi)^3} \) and use the anti-symmetry of the triple scalar product to get \( \langle [v', u'], w' \rangle + \langle u', [v', w'] \rangle = 0 \). It is annoying that this inner product does not extend to the full algebra \( S \), because curl is not invertible there.

There is a central extension \( \mathbb{R}^3 \to \hat{S} \to S \) (discovered in a different context\cite{Kac-Moody}): \[
\langle [u, \eta], (w, \mu) \rangle = \langle [u, w], \Omega(u, w) \rangle
\] with co-cycle \( \Omega(u, w) = \int w \times u \frac{d^3x}{(2\pi)^3} \). By pairing the constant vectors with the translations (by analogy with the Kac-Moody algebra\cite{Kac-Moody}), we can get an invariant inner product on the extended Lie algebra \( \hat{S} \):

\[
\langle [u, \eta], (w, \mu) \rangle = \int u'.\text{curl}^{-1} w' \frac{d^3x}{(2\pi)^3} + \eta.\bar{w} + \mu.\bar{u}.
\]

This central extension serves a useful physical purpose: it allows us to extend the geodesic interpretation of Euler’s equation to include translations. Recall\cite{Euler} that the Euler equations of an incompressible ideal fluid

\[
\frac{\partial v}{\partial t} + v.\nabla v = -\nabla p, \quad \nabla.v = 0
\]

can be written also as \( \frac{\partial v}{\partial t} + \omega \times v = -\nabla \left( p + \frac{1}{2}v^2 \right) \), where \( \omega = \text{curl} v \) is the vorticity. By taking the curl we get the vorticity form of these equations

\[
\frac{\partial \omega}{\partial t} + [v, \omega] = 0, \quad \omega = \nabla \times v
\]

But this is incomplete: we also need to know how the constant part of \( v \) (which is lost in \( \nabla \times v \)) evolves in time. By averaging Euler’s equation we also get

\[
\frac{\partial \bar{v}}{\partial t} + \int \omega \times v \frac{d^3x}{(2\pi)^3} = 0.
\]

This suggests that we should regard the curl operator as extended to the central extension by \( \text{curl} : S \to \hat{S}' \),

\[
\text{curl}v = \left( \nabla \times v, \int v \frac{d^3x}{(2\pi)^3} \right) \equiv \hat{\omega},
\]
So velocity belongs to $S$ (i.e., includes constant vectors). But vorticity belongs to $\hat{S}'$: total vorticity $\int_\omega \frac{dx}{(2\pi)^3}$ is zero, but extended vorticity $\hat{\omega}$ has a central component equal to average velocity. The extended notion of curl above is an invertible operator.

The Euler equation now becomes

$$\frac{\partial \hat{\omega}}{\partial t} + [v, \hat{\omega}] = 0$$

We can verify that

$$\langle \hat{\omega}, \text{curl}^{-1}\hat{\omega} \rangle = \int v^2 \frac{dx}{(2\pi)^3}$$

allowing us to interpret the total kinetic energy as $H = \frac{(2\pi)^3}{2} \langle \hat{\omega}, \text{curl}^{-1}\hat{\omega} \rangle$.

There is a close analogy to the rigid body: $\hat{\omega}$ is like the angular momentum $L$ and curl is analogous to moment of inertia. Moreover, $\langle \hat{\omega}, \hat{\omega} \rangle$ is analogous to $L \cdot L$; being invariant under the Lie algebra, it is a conserved quantity. In fluid mechanics, this is called helicity $[7, 8]$.

Thus, fluid mechanics should be thought of as geodesic motion in the extended Lie algebra $S'$. The central elements are trivially time-independent. It is amusing that total momentum of the fluid $\int m \hat{v} dx$ is not conserved: it has an “anomaly” proportional to the cocycle $\Omega(v, \omega)$.

Since parity changes its sign, the invariant inner product $\langle, \rangle$ is not positive. This raises the possibility that there is a splitting $\hat{S} = A \oplus B$ into isotropic subalgebras; i.e., such that $\langle A, A \rangle = 0 = \langle B, B \rangle$. The meaning of such a “Manin triple” $[12, 13]$ $(\hat{S}, A, B)$ is clearer in a basis $X_a \in A$ and $X^a \in B$ satisfying $\langle X_a, X_b \rangle = 0 = \langle X^a, X^b \rangle$, $\langle X_a, X^b \rangle = \delta_a^b$ and

$$[X_a, X_b] = \Gamma_{ab}^c X_c, \quad [X^a, X^b] = \mu_{ab}^c X_c$$

$$[X_a, X^b] = -\Gamma_{ad}^c X^d + \mu_{ad}^b X^c$$

These relations define a Lie bi-algebra $[12, 13]$ i.e., infinitesimal versions of quantum groups (Hopf algebras) which are also called Poisson-Hopf algebras. It is expected that every Lie bi-algebra (even infinite dimensional ones) are infinitesimal versions of Hopf algebras. There is a general procedure for exponentiating them in finite dimensions; but not yet in the general case.

To construct such a basis for $\hat{S}$ we will use Fourier analysis and an elementary trick using irrational numbers. Let $\alpha, \beta$ and $\gamma \equiv \alpha \times \beta$ be constant vectors satisfying the conditions that for $m \in \mathbb{Z}^3$,

$$\alpha \cdot m \neq 0, \quad \beta \cdot m \neq 0, \quad (\alpha \times \beta) \cdot m \neq 0.$$  

(Here $\mathbb{Z}^3$ is the set of non-zero 3-vectors with integer components). This “no resonance” condition implies that

$$\alpha \times m \neq 0, \quad \beta \times m \neq 0, \quad \gamma \times m \neq 0$$

as well. For example,

$$\alpha = \{1, \sqrt{2}, \sqrt{3}\}, \quad \beta = \{\sqrt{3}, 1, \sqrt{2}\}$$

leading to $\gamma = \{2 - \sqrt{3}, 3 - \sqrt{2}, 1 - \sqrt{6}\}$. (There are many other choices also, leading to equivalent bases.) The orbits of $\alpha, \beta, \gamma$ are dense in $\mathbb{T}^3$, so that the only continuous functions that vanish on them are zero everywhere. This allows us to invert operators such as curl and $\gamma \cdot \nabla$ subject to periodic boundary conditions. Such ideas occur also in the proof of the KAM theorem of classical mechanics $[11]$.

It is not difficult to verify that for $m \neq 0$, we can expand any vector satisfying $m.w = 0$ as $w = \frac{m.w \beta.m - m.m \beta.w}{m.m \gamma.m} \alpha \times m + \frac{m.w \alpha.m + m.m \alpha.w}{m.m \gamma.m} \beta \times w$. Thus

$$a_m = -\alpha \times \nabla e_m, \quad b_m = -\beta \times \nabla e_m,$$

(where $e_m = e^{im \cdot x}$) form a basis for $S'$. To extend the basis to $\hat{S}$ we add three central elements $c_j$ and the three translations along the co-ordinate axes $d_j$. We can calculate
\[ [a_m, a_n] = \alpha \cdot (m \times n)a_{m+n}, \quad [b_m, b_n] = \beta \cdot (m \times n)b_{m+n} \]

\[ [a_m, b_n] = \frac{\gamma \cdot m}{\gamma \cdot (m+n)} \beta \cdot (m \times n)a_{m+n} + \frac{\gamma \cdot n}{\gamma \cdot (m+n)} \alpha \cdot (m \times n)b_{m+n} + \gamma \cdot n \, c.n\delta(m+n) \]

In addition, \([*, c_i] = 0\) and \([d_j, d_k] = 0\) and

\[ [d_j, a_m] = i m_j a_m, \quad [d_j, b_m] = i m_j b_m. \]

The invariant inner product becomes,

\[ \langle a_m, a_n \rangle = \langle a_m, c_j \rangle = 0 = \langle b_m, b_n \rangle = \langle b_m, d_j \rangle \]

\[ \langle a_m, b_n \rangle = i \gamma \cdot n \delta_{m+n}, \quad \langle d_j, c_k \rangle = \delta_{j_k} \] (2)

We can change the basis slightly to make the Lie bi-algebra structure more obvious:

\[ X_m = a_m, \quad X^n = -\frac{1}{i \gamma.n}b_n \]

so that \( \langle X_m, X^n \rangle = \delta^n_m \). The commutation relations in this basis are of the required type with \( \Gamma^k_{mn} \equiv \delta^k_{m+n}\alpha.(m \times n), \mu^{mn}_k \equiv \delta^{m+n}_{k+1}(-i\gamma.k) (\gamma.m)(\gamma.n) \beta.(m \times n) \). Then \( \mathcal{A} \) is spanned by \( X_a = (X_m, c_j) \) and \( \mathcal{B} \) by \( X^a = (X^m, d_j) \).

The \( L^2 \) inner product is

\[ \langle a_m, a_n \rangle = (\alpha \times m)^2 \delta(m+n), \]

\[ \langle b_m, b_n \rangle = (\beta \times m)^2 \delta(m+n), \]

\[ \langle a_m, b_n \rangle = (\alpha \times m). (\beta \times n) \delta(m+n), \]

\[ (c_j, c_k) = \delta_{j_k} = (d_j, d_k). \]

The fluid flows along geodesics determined by this metric along with the commutation relations(1).

\( \hat{\mathcal{S}} \) is not a co-boundary Lie bi-algebra; i.e., there is no classical \( r \)-matrix such that \( \mu = \partial r \) in the Lie algebra co-homology of \( \mathcal{A} \). This makes it harder to construct the quantum group: it is not determined by an \( R \)-matrix.

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I. DETAILS OF PROOFS

A. The Central Extension

**Proposition 1.** Each component of \( \Omega(u, w) = \int \vec{w} \times \vec{u} \, dx \) is a 2-cocycle of \( S \). i.e., \( \partial \Omega(u, v, w) = \Omega(u, [v, w]) + \Omega(v, [w, u]) + \Omega(w, [u, v]) = 0 \)

*Proof.*

\[
- \partial \Omega(u, v, w) = \int u \times \left( v^i \partial_i w - w^i \partial_i v \right) \frac{dx}{(2\pi)^3} + \text{cyclic}
\]

Because of incompressibility,

\[
= \int u \times \partial_i \left( v^i w - w^i v \right) \frac{dx}{(2\pi)^3} + \text{cyclic}
\]

Integrating by parts

\[
= - \int \partial_i u \times \left( v^i w - w^i v \right) \frac{dx}{(2\pi)^3} + \text{cyclic}
\]

\[
= \int \left[ w^i \partial_i u \times v - v^i \partial_i u \times w \right] \frac{dx}{(2\pi)^3} + \text{cyclic}
\]

Cyclically permuting the first term

\[
= \int \left[ u^i \partial_i v \times w - v^i \partial_i u \times w \right] \frac{dx}{(2\pi)^3} + \text{cyclic}
\]

\[
= \int [u, v] \times w \frac{dx}{(2\pi)^3} + \text{cyclic}
\]

Anti-symmetry of cross product

\[
= - \int w \times [u, v] \frac{dx}{(2\pi)^3} + \text{cyclic}
\]

Cyclic permutation

\[
= - \int u \times [v, w] \frac{dx}{(2\pi)^3} + \text{cyclic}
\]

\[
= \partial \Omega(u, v, w)
\]

Thus \( \partial \Omega = 0 \).

This allows us to define a central extension. The vector space

\[
\hat{S} = \mathbb{R}^3 \oplus S
\]

is turned into a Lie algebra by

\[
[(u, \eta), (w, \mu)] = ([u, w], \Omega(u, w))
\]

The Jacobi identity follows from the co-cycle condition proved above.
Proposition 2. The Lie algebra $\hat{S}$ admits the invariant inner product $\langle (u, \eta), (w, \mu) \rangle = \int (u - \bar{u}) \cdot \text{curl}^{-1}(w - \bar{w}) \frac{dx}{(2\pi)^3} + \eta \cdot \bar{w} + \mu \cdot \bar{u}$, where $\bar{u} = \int u \frac{dx}{(2\pi)^3}$.

Proof. We have

$$\langle (u, \eta), [(w, \mu), (v, \sigma)] \rangle = \langle (u, \eta), ([w, v], \Omega(w, v)) \rangle$$

(14)

Since $[w, v] = 0$ and $[w, v] = \text{curl}(v \times w)$

$$= \int (u - \bar{u}) \cdot (v \times w) dx + \bar{u} \cdot \int v \times w \frac{dx}{(2\pi)^3}$$

(15)

$$= \int u \cdot (v \times w) \frac{dx}{(2\pi)^3}.$$  

(16)

The anti-symmetry of the triple scalar product now proves the invariance of the inner product, as before.  

Thus the central basis elements are dual to translations under the invariant inner product.

**B. Fourier Basis**

Lemma 3. For $m \neq 0$, we can expand any vector as $w = \frac{m \cdot w}{m \cdot m} m + \frac{m \cdot w \beta}{m \cdot m \gamma} m \cdot m \beta \cdot m + \frac{-m \cdot w \alpha}{m \cdot m \gamma} m \cdot m \alpha \cdot m$. 

Proof. Note that $m, \alpha \times m, \beta \times m$ form a basis because

$$\begin{vmatrix}
  m_1 & m_3 & \alpha_2 - \alpha_3 \\
  m_2 & m_1 & \alpha_3 - \alpha_1 \\
  m_3 & m_2 & \alpha_1 - \alpha_2
\end{vmatrix} = m \cdot m (\alpha \times \beta) \cdot m \neq 0.$$ 

Expand $w = \xi_1 \alpha + \xi_2 \beta + \xi_3 \times m$. Taking scalar products with $m, \beta, \alpha$ we get

$$m \cdot w = m \cdot \xi_1, \quad \alpha \cdot w = \alpha \cdot m \xi_1 + \gamma \cdot m \xi_3, \quad \beta \cdot w = \beta \cdot m \xi_1 - \gamma \cdot m \xi_2$$

(17)

Solving, we get

$$\xi_1 = \frac{m \cdot w}{m \cdot m}, \quad \xi_2 = \frac{m \cdot w \beta}{m \cdot m \gamma \cdot m}, \quad \xi_3 = \frac{-m \cdot w \alpha}{m \cdot m \gamma \cdot m}.$$ 

(18)

In particular, if $m \cdot w = 0$

$$w = -\frac{\beta \cdot w}{\gamma \cdot m} \alpha \times m + \frac{\alpha \cdot w}{\gamma \cdot m} \beta \times m.$$ 

(19)

Proposition 4. Any incompressible trigonometric polynomial vector field can be written as $u = \bar{u} + \alpha \times \nabla A + \beta \times \nabla B$ where $\bar{u}$ is a constant vector field and $A, B \in F$. Moreover, $A$ and $B$ are unique up to additive constants.

Proof. Expand in a Fourier series

$$u(x) = \bar{u} + \sum_{m \in \mathbb{Z}^3} u_m e^{i m \cdot x}, \quad m \cdot u_m = 0$$

(20)

where $\bar{u}$ is a constant vector. Using

$$u_m = -\alpha \times m \frac{\beta \cdot u_m}{\gamma \cdot m} + \beta \times m \frac{\alpha \cdot u_m}{\gamma \cdot m}.$$ 

(21)
\[a_m = -i\alpha \times me_m, \quad b_m = -i\beta \times me_m \quad (22)\]

we get the expansion

\[u = u_0 + \sum_{m \in \mathbb{Z}^3} \left[ -i\beta \cdot u_m a_m + i\alpha \cdot u_m b_m \right] \quad (23)\]

Finally, since \(\gamma \equiv \alpha \times \beta \neq 0\), it is obvious that \(\alpha, \beta, \gamma\) form a basis for the constant vectors.

We can also write this as

\[u = \bar{u} + \alpha \times \nabla A + \beta \times \nabla B \quad (24)\]

where the scalar fields \(A, B\) are given by

\[A = (\gamma \cdot \nabla)^{-1} \beta \cdot (u - \bar{u}), \quad B = -(\gamma \cdot \nabla)^{-1} \alpha \cdot (u - \bar{u}) \quad (25)\]

Also, \(\bar{u}\) is the average over the torus of \(u\).

The above decomposition shows that

**Corollary 5.** A constant vector field is incompressible, but is not the curl of any vector field on the torus (i.e., is not exact). An incompressible vector field whose average is zero can be written as the curl of another such vector field.

**Proof.** Fourier analysis shows that the average of \(\nabla \times U\) for any \(U\) with periodic components is zero: the constant terms in \(U\) have zero curl.

**Proposition.** In \(S\), we have the relations

\[[a_m, a_n] = \alpha \cdot (m \times n)a_{m+n}, \quad [b_m, b_n] = \beta \cdot (m \times n)b_{m+n} \quad (26)\]

\[[a_m, b_n] = \gamma \cdot (m + n)\beta \cdot (m \times n)a_{m+n} + \gamma \cdot n\alpha \cdot (m \times n)b_{m+n} \quad (27)\]

It is straightforward to verify (26). To prove (27) we need the lemma

**Lemma.** When \(\gamma \equiv \alpha \times \beta\), we have the identity

\[\alpha \cdot (m \times n) \{\gamma.n \beta \times m - \gamma.m \beta \times n\} - \beta \cdot (m \times n) \{\gamma.n \alpha \times m - \gamma.m \alpha \times n\} = 0 \quad (28)\]

**Proof.** Since \(\alpha \times \beta = \gamma\) and \(\alpha.(\beta \times m) = (\alpha \times \beta).m = \gamma.m\) we have

\[\alpha \cdot \{\gamma.n \beta \times m - \gamma.m \beta \times n\} = 0, \quad \beta \cdot \{\gamma.n \alpha \times m - \gamma.m \alpha \times n\} = 0 \quad (29)\]

Of course also \(\beta.(\beta \times m) = 0\) so that

\[\beta \cdot \{\gamma.n \beta \times m - \gamma.m \beta \times n\} = 0, \quad \alpha \cdot \{\gamma.n \alpha \times m - \gamma.m \alpha \times n\} = 0 \quad (30)\]

Since \(\{\alpha, \beta, \gamma\}\) is a basis, and we have proved that the \(\alpha\) and \(\beta\) components are zero, it is enough to prove that the \(\gamma\) component is zero as well. Now recall that

\[u.mw.n - w.m u.n = (u \times w) \cdot (m \times n) \quad (31)\]

so that
\[
\gamma.\{\gamma.n\beta \times m - \gamma.m\beta \times n\} = \gamma.n(\gamma \times \beta).m - \gamma.m(\gamma \times \beta).n
\]
(32)
\[
= -(\gamma \times (\gamma \times \beta))(m \times n)
\]
(33)
\[
= \gamma.\gamma \beta.(m \times n)
\]
(34)
Similarly
\[
\gamma.\{\gamma.n\alpha \times m - \gamma.m\alpha \times n\} = \gamma.\gamma\alpha.(m \times n)
\]
(35)
Thus
\[
\gamma.[\alpha.(m \times n)\{\gamma.n\beta \times m - \gamma.m\beta \times n\} - \beta.(m \times n)\{\gamma.n\alpha \times m - \gamma.m\alpha \times n\}] = \\
= \gamma.\gamma [\alpha.(m \times n)\beta.(m \times n) - \beta.(m \times n)\alpha.(m \times n)] = 0
\]
(36)
as needed. \hfill \Box

Now we can prove (27)

Proof. Start with
\[
a_m = -i\alpha \times me_m, \quad b_n = -i\beta \times ne_n
\]
(38)
Then
\[
a_m.\nabla b_n = -i(\alpha \times m).n \beta \times n \epsilon_{m+n}
\]
(39)
\[
= \frac{\gamma \cdot n}{\gamma \cdot (m + n)} \alpha \cdot (m \times n) b_{m+n} + \frac{\alpha.(m \times n)}{\gamma \cdot (m + n)} \{\gamma.n\beta \times (m + n) - \beta \times n\gamma.(m + n)\} ie_{m+n}
\]
(40)
\[
a_m.\nabla b_n = \frac{\gamma \cdot n}{\gamma \cdot (m + n)} \alpha \cdot (m \times n) b_{m+n} + \frac{\alpha.(m \times n)}{\gamma \cdot (m + n)} \{\gamma.m\beta \times m - \gamma.m\beta \times n\} ie_{m+n}
\]
(41)
Similarly
\[
b_n.\nabla a_n = \frac{\gamma \cdot m}{\gamma \cdot (m + n)} \beta \cdot (n \times m)a_{m+n} + \frac{\beta.(n \times m)}{\gamma \cdot (m + n)} \{\gamma.m\alpha \times n - \gamma.n\alpha \times m\} ie_{m+n}
\]
(42)
so that
\[
[a_m, b_n] = \frac{\gamma \cdot n}{\gamma \cdot (m + n)} \alpha \cdot (m \times n) b_{m+n} + \frac{\gamma \cdot m}{\gamma \cdot (m + n)} \beta \cdot (m \times n)a_{m+n}
\]
(43)
\[
+ [\alpha.(m \times n)\{\gamma.n\beta \times m - \gamma.m\beta \times n\} - \beta.(m \times n)\{\gamma.n\alpha \times m - \gamma.m\alpha \times n\}] \frac{ie_{m+n}}{\gamma \cdot (m + n)}
\]
(44)
The last term is zero by the Lemma above. \hfill \Box
C. Symmetric Version of the Commutation relations

\[ [X^m, X^n] = \frac{1}{(i\gamma.m)(i\gamma.n)}[b_m, b_n] = \frac{\beta.(m \times n)}{(i\gamma.m)(i\gamma.n)} b_{m-n} \]  

\[ = \frac{\beta.(m \times n)(-i\gamma.(m + n))}{(i\gamma.m)(i\gamma.n)} X^{m+n} \]  

Thus

\[ \mu_{mn}^k \equiv \delta_{k}^{m+n} \frac{(-i\gamma.k)}{(i\gamma.m)(i\gamma.n)} \beta.(m \times n) \]  

Finally,

\[ [X_m, X^n] = -\frac{1}{i\gamma.n}[a_m, b_n] \]  

\[ = -\frac{\gamma.m}{i\gamma.n \gamma.(m - n)} \beta \cdot (m \times n)a_{m-n} + \frac{\gamma \cdot n}{i\gamma.n \gamma.(m - n)} \alpha \cdot (m \times n)b_{m-n} \]  

\[ = -\frac{\gamma.m}{i\gamma.n \gamma.(m - n)} \beta \cdot (m \times n)a_{m-n} + \alpha \cdot (m \times n)X^{n-m} \]  

Now

\[ \Gamma_{mk}^n X^k = \delta_{k+m}^n \alpha.(m \times k)X^k = \alpha.(m \times n)X^{n-m} \]  

\[ \mu_{mk}^n X_k = \delta_{m+k}^n \frac{(-i\gamma.m)}{i\gamma.n \gamma.(m - n)} \beta.(n \times k)X_k = \frac{(-i\gamma.m)}{i\gamma.n \gamma.(m - n)} \beta.(n \times m)X_{m-n} \]  

\[ = -\frac{(-i\gamma.m)}{i\gamma.n \gamma.(m - n)} \beta.(m \times n)X_{m-n} \]  

\[ = \frac{\gamma.m}{i\gamma.n \gamma.(m - n)} \beta.(m \times n)X_{m-n} \]  

Thus

\[ [X_m, X^n] = \Gamma_{mk}^n X^k - \mu_{mk}^n X_k \]  

D. Proof that \( \mu \) is not a co-boundary

The structure constants \( \Gamma, \mu \) of a Lie bi-algebra satisfy the identities

\[ \Gamma_{ab}^c = -\Gamma_{ba}^c, \quad \Gamma_{ab}^c \Gamma_{de}^a + \Gamma_{be}^a \Gamma_{da}^e + \Gamma_{ea}^d \Gamma_{db}^e = 0 \]  

which are the just Jacobi identities for the Lie sub-algebras \( \mathcal{A}, \mathcal{B} \) spanned by \( X_a \) and \( X^a \) respectively. The mixed Jacobi identities \([X_a, X_b, X_c]\) and \([X^a, X^b, X_c]\) both lead to the condition

\[ \mu_{bd}^e \Gamma_{ac}^d = \left[ \Gamma_{bd}^e \mu_{ac}^d + \mu_{ad}^e \Gamma_{bc}^d \right] - b \leftrightarrow c \]
This has another meaning: it says that $\mu$ is a co-cycle in the Lie algebra cohomogy $H^1(\mathcal{A}, \mathcal{A} \otimes \mathcal{A})$ (or conversely, $\Gamma$ co-cycle in $H^1(\mathcal{B}, \mathcal{B} \otimes \mathcal{B})$).

If $\mathcal{A}$ is a finite dimensional simple algebra (such as $\mathfrak{sl}_2$) this is a co-boundary; i.e., there is an $r \in \mathcal{A} \otimes \mathcal{A}$ such that $\mu = \partial r$. The Jacobi identity of $\mu$ then becomes a quadratic condition on $r$ called the classical Yang-Baxter equation. This is the infinitesimal version of the famous Yang-Baxter equation of a quasi-triangular quantum group, which is the exponential of such a co-boundary Lie bi-algebra.

It would have been a simplification if such an $r-$ matrix existed for our Lie bi-algebra; being infinite dimensional, the usual arguments for its existence do not apply. A direct study is needed. We will now that

**Proposition 6.** For the Lie bi-algebra $\hat{\mathcal{S}}$ the structure constants $\mu$ are not a co-boundary; there is no classical $r-$matrix arising from it.

**Proof.** We have

$$\mu(X_k) = \sum_{m+n=k} \frac{(-i\gamma.k)}{(i\gamma.m)(i\gamma.n)} \beta.(m \times n) X_m \otimes X_n$$

Let us calculate the co-boundary of an element $\rho = \rho^{pq} X_p \otimes X_q \in \mathcal{A} \otimes \mathcal{A}$.

$$\partial \rho(X_k) = [X_k \otimes 1 + 1 \otimes X_k, \rho]$$

$$= [X_k \otimes 1 + 1 \otimes X_k, \rho^{pq} X_p \otimes X_q]$$

$$= \sum_{pq} \rho^{pq} \{[X_k, X_p] \otimes X_q + X_p \otimes [X_k, X_q]\}$$

$$= \sum_{pq} \rho^{pq} \{\alpha.(k \times p) X_{k+p} \otimes X_q + \alpha.(k \times q) X_p \otimes X_{k+q}\}$$

$$= \sum_{pq} \rho^{pq} \{\alpha.(k \times p) X_{k+p} \otimes X_q + \alpha.(k \times q) X_p \otimes X_{k+q}\}$$

Replace $p \mapsto p-k, q \mapsto q+k$ in the first term

$$= \sum_{pq} \{\rho^{p-k,q+k} \alpha.(k \times p) X_p \otimes X_{q+k} + \rho^{pq} \alpha.(k \times q) X_p \otimes X_{k+q}\}$$

$$= \sum_{pq} \{\rho^{p-k,q+k} \alpha.(k \times p) + \rho^{pq} \alpha.(k \times q)\} X_p \otimes X_{q+k}$$

Put $m = p, n = q + k$

$$\partial \rho(X_k) = \sum_{mn} \{\rho^{m-k,n} \alpha.(k \times m) + \rho^{m,n-k} \alpha.(k \times n)\} X_m \otimes X_n$$

Compare with

$$\mu(X_k) = \sum_{m,n} \delta^{m+n}_{k} \frac{(-i\gamma.k)}{(i\gamma.m)(i\gamma.n)} \beta.(m \times n) X_m \otimes X_n$$

$$\delta^{m+n}_{k} \frac{(-i\gamma.k)}{(i\gamma.m)(i\gamma.n)} \beta.(m \times n) = \rho^{m-k,n} \alpha.(k \times m) + \rho^{m,n-k} \alpha.(k \times n)$$
So rhs must vanish if $k \neq m + n$. This suggests the ansatz

$$\rho^{mn} = \delta(m + n)\rho^m$$

$$\delta_k^{m+n} \frac{(-i\gamma.k)}{(i\gamma.m)(i\gamma.n)} \beta.(m \times n) = \{-\rho^{-n}\alpha.(n \times m) - \rho^{m}\alpha.(m \times n)\} \delta(k + m + n)$$

$$\frac{(-i\gamma.(m + n))}{(i\gamma.m)(i\gamma.n)} \beta.(m \times n) = \{\rho^m + \rho^n\} \alpha.(m \times n)$$

Let

$$M_{mn} = \frac{(-i\gamma.(m + n))}{(i\gamma.m)(i\gamma.n)} \alpha.(m \times n)$$

If $\mu$ is a co-boundary, this should be equal to $\sigma^{mn} \equiv \rho^m + \rho^n$. But this is impossible.

For, $\sigma^{mn}$ is a rank two matrix, being of the form $\rho \otimes \xi + \xi \otimes \rho$ where $\xi$ is the vector all of whose components are equal to one. It is easy to check that $M$ has sub-matrices of rank higher than two. For example, we can very directly (with the choice $\alpha = (1, \sqrt{2}, \sqrt{3}), \beta = (\sqrt{3}, 1, \sqrt{2})$ given in the text) the sub-matrix $M_1 \subset M$ labeled by

$$m = \{(3, 2, 3), (4, 3, 4), (4, 1, 1)\}, \quad n = \{(2, 3, 2), (3, 4, 2), (2, 4, 3)\}$$

$$M_1 = i\begin{pmatrix}
\begin{array}{cccc}
-79\sqrt{7} + 221\sqrt{12} - 399 & 312\sqrt{7} + 258\sqrt{3} + 177 & 398\sqrt{7} - 285\sqrt{3} + 108 \\
-219\sqrt{2} + 16\sqrt{3} + 225\sqrt{6} + 2466 & 281\sqrt{2} + 242\sqrt{3} + 27\sqrt{6} + 4676 & 575\sqrt{2} - 396\sqrt{3} + 17\sqrt{6} - 174 \\
-219\sqrt{2} + 380\sqrt{3} + 214\sqrt{6} + 1923 & -915\sqrt{2} - 3910\sqrt{3} + 14\sqrt{6} + 8031 & -5019\sqrt{2} + 6604\sqrt{3} + 281\sqrt{6} - 5025 \\
-273\sqrt{2} + 47\sqrt{3} + 48\sqrt{6} + 164 & -459\sqrt{2} + 107\sqrt{3} + 66\sqrt{6} + 248 & -496\sqrt{2} + 83\sqrt{3} + 39\sqrt{6} + 332 \\
1805\sqrt{2} - 2682\sqrt{3} - 331\sqrt{6} + 2878 & 2385\sqrt{2} - 4306\sqrt{3} - 115\sqrt{6} + 4298 & 4399\sqrt{2} - 4718\sqrt{3} - 547\sqrt{6} + 3390
\end{array}
\end{pmatrix}$$

is of rank 3. In fact, we expect that $M$ is of infinite rank.

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