PERTURBATION THEORY IN ANGULAR QUANTIZATION APPROACH AND THE EXPECTATION VALUES OF EXPONENTIAL FIELDS IN SIN-GORDON MODEL

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Abstract

In angular quantization approach a perturbation theory for the Massive Thirring Model (MTM) is developed, which allows us to calculate Vacuum Expectation Values of exponential fields in sin-Gordon theory near the free fermion point in first order of MTM coupling constant $g$. The Hankel-transforms play an important role when carrying out this calculations. The expression we have found coincides with that of the direct expansion over $g$ of the exact formula conjectured by S. Lukyanov and A. Zamolodchikov.
1 Introduction

Sin-Gordon theory is one of the most studied examples of exactly integrable relativistic QFT in two dimensions. Its action

\[ S_{SG} = \int d^2x \left\{ \frac{1}{16\pi} \partial_\nu \varphi \partial^\nu \varphi + 2\mu \cos \beta \varphi \right\} \]  

(1.1)

may be viewed as a perturbation of the free massless Bose field by the relevant operator \( \cos \beta \varphi \) with scaling dimension \( d = 2\beta^2 < 2 \). It is convenient to normalize exponential fields by the condition, that in UV limit their two point functions approach to those of \( c = 1 \) conformal free Bose field theory

\[ \langle e^{ia\varphi(x)} e^{-ia\varphi(y)} \rangle_{SG} \to |x - y|^{-4a^2} \quad \text{as} \quad |x - y| \to 0. \]

(1.2)

The on-shell properties of this theory i.e. the mass spectrum and the S-matrix are well known \[1\]. The lightest particles of the theory are solitons and antisolitons in term of which sin-Gordon theory has an equivalent Lagrangian formulation with the action of Massive Thirring Model (MTM) \[2\]

\[ S_{MTM} = \int d^2x \left\{ i\overline{\Psi} \gamma^\nu \partial_\nu \Psi - M \overline{\Psi} \Psi - \frac{g}{2} \left( \overline{\Psi} \gamma^\nu \Psi \right) \left( \overline{\Psi} \gamma_\nu \Psi \right) \right\}, \]

(1.3)

where \( \overline{\Psi}, \Psi \) are two component Dirac spinors and the corresponding (anti-) particles are identified with the (anti-) solitons of (1.1). The famous Coleman relations

\[ \frac{g}{\pi} = \frac{1}{2\beta^2} - 1; \quad J^\nu = \overline{\Psi} \gamma^\nu \Psi = -\frac{\beta}{2\pi} \varepsilon^{\nu\eta} \partial_\eta \varphi \]

(1.4)

serve as a dictionary between bosonic and fermionic languages. More recently comparing the results of the thermodynamic Bethe-Ansatz analyses with those of the Conformal Perturbation Theory an exact relation between the soliton mass \( M \) and the perturbation parameter \( \mu \) is established \[3\]

\[ \mu = \frac{\Gamma(\beta^2)}{\pi \Gamma(1-\beta^2)} \left[ \frac{M \sqrt{\pi} \Gamma \left( \frac{1+\xi}{2} \right)}{2 \Gamma \left( \frac{\xi}{2} \right)} \right]^{2-2\beta^2}, \]

(1.5)

where

\[ \xi = \frac{\beta^2}{1-\beta^2}. \]

(1.6)
Starting from the expressions for the special cases $\beta \rightarrow 0$ (semiclassical limit) and $\beta^2 = 1/2$ (free fermion case) S.Lukyanov and A.Zamolodchikov in [4] the following general formula for the Vacuum Expectation Value (VEV) $G_a = \langle \exp ia\varphi(0) \rangle$ have conjectured

$$G_a = \left( \frac{m \Gamma \left( \frac{1+i\xi}{2} \right) \Gamma \left( \frac{1-i\xi}{2} \right)}{4\sqrt{\pi}} \right)^{2a^2} \times \exp \left\{ \int_0^\infty \frac{dt}{t} \left[ \frac{\sinh^2 (2a\beta t)}{2 \sinh^2 \beta^2 t \sinh t \cosh ((1-\beta^2)t)} - 2a^2 e^{-2t} \right] \right\}. \quad (1.7)$$

In the subsequent papers [5], [6] some convincing arguments have been presented, showing that the VEV’s $\tilde{G}_a = \langle \exp a\varphi(0) \rangle_{sG}$ in sinh-Gordon theory, the action of which formally can be obtained simply replacing $\beta \rightarrow ib$ in (1.1), obey the functional relations

$$\tilde{G}_a = R(a) \tilde{G}_{Q-a} = \tilde{G}_{-a}, \quad (1.8)$$

where $Q = b + 1/b$ and $R(a)$ is related to the Liouville reflection amplitude $S(p)$ [7]

$$R \left( \frac{Q}{2} + ip \right) = S(p) = - \left( \frac{\pi \mu \Gamma \left( b^2 \right)}{\Gamma \left( 1 - b^2 \right)} \right)^{-2ip/6} \frac{\Gamma \left( 1 + 2ip/b \right) \Gamma \left( 1 + 2ipb \right)}{\Gamma \left( 1 - 2ip/b \right) \Gamma \left( 1 - 2ipb \right)}. \quad (1.9)$$

It is natural to expect that $\tilde{G}_a$ can be obtained from $G_a$ making the analytic continuation

$$\beta \rightarrow ib; \quad a \rightarrow -ia. \quad (1.10)$$

And indeed it can be shown that after substitution (1.10), the expression (1.7) obeys the relation (1.8). Unfortunately the functional relations (1.8) alone are not sufficient to find $\tilde{G}_a$ uniquely: the multiplication by any even, periodic with period $Q$ function of $a$ gives a different solution to (1.8). However, $\tilde{G}_a$ obtained from (1.7) is the only meromorphic solution, obeying the extra requirement of ”minimality” (i.e. the condition that only the poles and zeros, imposed by the functional relations (1.8) are allowed). So, any independent test of the Lukyanov-Zamolodchikov formula (1.7) will support the minimality assumption as well. This is important also because there are other interesting models for which some functional relations like (1.8) are hold and the minimality condition makes it possible to find exact VEV’s.
In this article a perturbation theory based on the angular quantization \[8\] of the MTM (1.3) is developed, using which Lukyanov-Zamolodchikov formula (1.7) near the free fermion point in first order over the MTM coupling constant $g$ is tested.

The section 2 is devoted to the angular quantization of the MTM (1.3). A particular attention is payed to the local field product regularization procedure, which has some additional features in comparison with the case of the ordinary quantization in Cartesian coordinates.

In section 3 we calculate VEV $\langle \exp ia \varphi (0) \rangle$ near the free fermion point. It appears that the Hankel-transform is a very useful tool to carry out this calculation. Some of the related mathematical details are presented in Appendix A. The choices we have made to regularize the traces over the fermionic Fock space and the local field product, result in a finite multiplicative renormalization of the field $\exp ia \varphi (0)$. The corresponding renormalization factor is calculated using the methods of Boundary CFT \[9\] in Appendix B. The final expression we obtained for the expansion of the VEV $\langle \exp ia \varphi (0) \rangle$ up to the first order in MTM coupling constant $g$ is in complete agreement with the Lukyanov-Zamolodchikov conjecture (1.7).

\section{Angular Quantization of the Massive Thirring Model}

It is convenient to use the ciral representation of the Dirac matrices

$$\gamma^0 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^1 = -i \sigma_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \tag{2.1}$$

and denote the components of Dirac spinors as

$$\Psi \equiv \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \quad \overline{\Psi} \equiv \Psi^\dagger \gamma^0 \tag{2.2}$$

In this notations the action (1.3) in Euclidean space reads
\[ A_{MTM} = \int d^2 z \left[ \psi_R^\dagger \partial \psi_R + \psi_L^\dagger \overline{\partial} \psi_L - \frac{iM}{2} \left( \psi_L^\dagger \psi_R - \psi_R^\dagger \psi_L \right) \right. \\
\left. + g \psi_L^\dagger \psi_R \psi_R^\dagger \psi_L \right], \quad (2.3) \]

where \( z = \tau + ix^1, \ \overline{z} = \tau - ix^1 \) are complex coordinates on the plane (\( \tau = ix^0 \) is the Euclidean time), \( \partial \equiv \partial/\partial z, \ \overline{\partial} \equiv \partial/\partial \overline{z} \) and the measure \( d^2 z \equiv dx^1 d\tau \).

As the VEV’s of local fields \( \langle e^{i\alpha x(0)} \rangle \) have rotational symmetry, it is natural to use the conformal polar coordinates \( \eta, \theta \) defined by

\[ z \equiv e^{\eta+i\theta}; \ \ \ \overline{z} \equiv e^{\eta-i\theta} \quad (2.4) \]

and treat \( \eta, \theta \) as space and (Euclidean) time respectively. Under the coordinate transformation (2.4) the Fermi fields transform as follows:

\[ \psi_L(z, \overline{z}) = e^{-i\pi \eta/2} \psi_L(\eta,\theta); \ \psi_R(z, \overline{z}) = e^{i\pi \eta/2} \psi_R(\eta,\theta), \quad (2.5) \]

and similarly for the conjugate fields \( \psi_L^\dagger, \psi_R^\dagger \). In this coordinates the action (2.3) takes the form:

\[ A_{MTM} = \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} d\eta \left[ \psi_L^\dagger (\partial_\theta - i\partial_\eta) \psi_L + \psi_R^\dagger (\partial_\theta + i\partial_\eta) \psi_R - iMe^\eta \left( \psi_L^\dagger \psi_R^\dagger \psi_L \psi_R \right) + 2g \psi_L^\dagger \psi_L \psi_R^\dagger \psi_R \right]. \quad (2.6) \]

The usual canonical quantization yields the following "equal time" anti-commutation relations:

\[ \{ \psi_L(\eta), \psi_L^\dagger(\eta') \} = \delta (\eta - \eta'), \ \ \ \{ \psi_R(\eta), \psi_R^\dagger(\eta') \} = \delta (\eta - \eta'). \quad (2.7) \]

From (2.6) one deduces that the Hamiltonian defining the evolution along \( \theta \) is given by

\[ K = \int_{-\infty}^{\infty} d\eta \left[ -\psi_L^\dagger i\partial_\eta \psi_L + \psi_R^\dagger i\partial_\eta \psi_R - iMe^\eta \left( \psi_L^\dagger \psi_R^\dagger \psi_L \psi_R \right) + 2g \psi_L^\dagger \psi_L \psi_R^\dagger \psi_R \right]. \quad (2.8) \]
To regularize the theory, following [4] let us restrict the range of the "space" coordinate \( \eta \) to the semi-infinite box \( \eta \in [\log \varepsilon; \infty) \) with \( M\varepsilon \ll 1 \) and impose the boundary conditions

\[
\psi_L + \psi^\dagger_L |_{\eta=\log\varepsilon} = \psi_R + \psi^\dagger_R |_{\eta=\log\varepsilon} = 0.
\] (2.9)

As usual, to develop perturbation theory in interaction picture, one first has to diagonalize the quadratic part of the Hamiltonian (2.8). This can be achieved using the decomposition [4]

\[
\psi_L (\eta, \theta) = \sum_{\nu \in \mathcal{N}_\varepsilon} \frac{1}{\sqrt{2\pi \rho (\nu)}} c_{\nu} u_{\nu} (\eta) e^{-i\nu \theta},
\]

\[
\psi_R (\eta, \theta) = \sum_{\nu \in \mathcal{N}_\varepsilon} \frac{1}{\sqrt{2\pi \rho (\nu)}} c_{\nu} v_{\nu} (\eta) e^{-i\nu \theta},
\]

\[
\psi^\dagger_L (\eta, \theta) = \sum_{\nu \in \mathcal{N}_\varepsilon} \frac{1}{\sqrt{2\pi \rho (\nu)}} c^\dagger_{\nu} u^{*}_{\nu} (\eta) e^{i\nu \theta},
\]

\[
\psi^\dagger_R (\eta, \theta) = \sum_{\nu \in \mathcal{N}_\varepsilon} \frac{1}{\sqrt{2\pi \rho (\nu)}} c^\dagger_{\nu} v^{*}_{\nu} (\eta) e^{i\nu \theta},
\] (2.10)

where the set of admissable \( \nu \)'s \( \mathcal{N}_\varepsilon \) and the density of states \( \rho (\nu) \) are specified below (see (2.14), (2.15)) and the wave functions [4]

\[
\begin{pmatrix}
    u_{\nu} (\eta) \\
    v_{\nu} (\eta)
\end{pmatrix} = \frac{\sqrt{2Me^2}}{\Gamma \left( \frac{1}{2} - i\nu \right)} \left( \frac{M}{2} \right)^{-i\nu} \begin{pmatrix}
    K_{\frac{1}{2}-i\nu} (Me^\eta) \\
    K_{\frac{1}{2}+i\nu} (Me^\eta)
\end{pmatrix},
\] (2.11)

are solutions of the free Dirac equation (\( K_{\nu} (x) \) is the MacDonald function). The wave functions (2.11) have the asymptotic behavior

\[
\begin{pmatrix}
    u_{\nu} (\eta) \\
    v_{\nu} (\eta)
\end{pmatrix} \rightarrow \begin{pmatrix}
    1 \\
    0
\end{pmatrix} e^{i\nu \eta} + S_F (\nu) \begin{pmatrix}
    0 \\
    1
\end{pmatrix} e^{-i\nu \eta} \text{ as } \eta \rightarrow -\infty,
\] (2.12)

where

\[
S_F (\nu) = \left( \frac{M}{2} \right)^{-2i\nu} \frac{\Gamma \left( \frac{1}{2} + i\nu \right)}{\Gamma \left( \frac{1}{2} - i\nu \right)}
\] (2.13)

is the fermion scattering amplitude off the "mass barrier" [4]. It follows from (2.9), (2.12) that the set \( \mathcal{N}_\varepsilon \) of admissable \( \nu \)'s over which the sum in (2.10) is carried out consists of the solutions of the equation

\[
2\pi \left( n + \frac{1}{2} \right) = 2\nu \log \frac{1}{\varepsilon} + \frac{1}{i} \log S_F (\nu),
\] (2.14)
where \( n \) is arbitrary integer. Therefore the density of states is given by

\[
\rho(\nu) = \frac{dn}{d\nu} = \frac{1}{\pi} \log \frac{1}{\varepsilon} + \frac{1}{2\pi i} \log S_F(\nu),
\]

(2.15)

Below we’ll use the notation \( \mathcal{N}_\varepsilon^+ \subset \mathcal{N}_\varepsilon \) for the subset of positive \( \nu \)'s. The operators \( c_\nu, c^{\dagger}_\nu \) satisfy the anti-commutation relations

\[
\{c_\nu, c_{\nu'}\} = \{c^{\dagger}_\nu, c^{\dagger}_{\nu'}\} = 0,
\]

\[
\{c_\nu, c^{\dagger}_{\nu'}\} = \delta_{\nu,\nu'}.
\]

(2.16)

In terms of this operators the free part of the Hamiltonian (2.8) takes a very simple form

\[
K_0 = \sum_{\nu \in \mathcal{N}_\varepsilon^+} \nu \left( c^{\dagger}_\nu c_\nu + c_{-\nu} c^{\dagger}_{-\nu} \right),
\]

(2.17)

which shows that \( c^{\dagger}_\nu, c_{-\nu} \) (\( c_\nu, c^{\dagger}_{-\nu} \)) are fermion and anti-fermion creation (annihilation) operators. Therefore the vacuum state \( |0\rangle \) can be defined by

\[
c_\nu |0\rangle = c^{\dagger}_{-\nu} |0\rangle = 0, \quad \nu \in \mathcal{N}_\varepsilon^+,
\]

(2.18)

and the Hilbert space of states \( \mathcal{H} \) is spanned over the base vectors

\[
\prod_{\nu \in \mathcal{N}_\varepsilon^+} c^{\dagger}_{n_\nu} c_{n_{\nu}} |0\rangle,
\]

(2.19)

where \( n_\nu \in \{0, 1\} \) (\( \pi_\nu \in \{0, 1\} \)) are the occupation numbers of fermions (anti-fermions) with energy \( \nu \).

Standard arguments show, that the expectation value of any quantity \( \langle X \rangle \) in interacting theory (2.8) may be evaluated using the formula

\[
\langle X \rangle \equiv \frac{\int D\Psi D\overline{\Psi} e^{-A_{MTM}} X}{\int D\Psi D\overline{\Psi} e^{-A_{MTM}}} = \frac{Tr_{\mathcal{H}} \left[ e^{-2\pi K_0 T} \left( e^{-\int K_I(\eta, \theta) d\eta d\theta} X \right) \right]}{Tr_{\mathcal{H}} \left[ e^{-2\pi K_0 T} \left( e^{-\int K_I(\eta, \theta) d\eta d\theta} \right) \right]},
\]

(2.20)

where \( X \) in second line of (2.20) as well as the interaction term of the Hamiltonian

\[
K_I = 2g \int_{\log \varepsilon}^{\infty} N \left( \psi^L_\nu \psi^L_\nu \psi^R_\nu \psi^R_\nu \right) d\eta
\]

(2.21)
is taken in interaction picture and \( T \) indicates the time ordering operation. In (2.21) we introduced the notation \( N (\cdots) \) for the suitably regularized product of the local operators in coinciding points (see below). Appearance of trace instead of conventional vacuum matrix element in (2.20) is due to compactification of the ”time” \( \theta \).

Now let us turn to the regularization procedure of the products of local operators at the coinciding points. In the ordinary case of non-compactified time one simply implies normal ordering prescription which is well known to be equivalent to the suppression of all contractions among the fields inside normal ordering symbol. In contrary to the vacuum matrix element, the trace of normal ordered product of creation and annihilation operators doesn’t vanish, therefore the analogues regularization in the case of compactified time is slightly changed. It is easy to see that in this case the correctly regularized perturbing operator \( N (\psi_L^+ \psi_L \psi_R^+ \psi_R) \) is given by

\[
N (\psi_L^+ \psi_L \psi_R^+ \psi_R) = \psi_L^+ \psi_L^+ \psi_R^+ \psi_R^+ - \langle \psi_L^+ \psi_L \rangle_0 \psi_R^+ \psi_R^+ - \langle \psi_R^+ \psi_R \rangle_0 \psi_L^+ \psi_L^+ + \\
\langle \psi_L^+ \psi_R \rangle_0 \psi_R^+ \psi_L^+ + \langle \psi_R^+ \psi_L \rangle_0 \psi_L^+ \psi_R^+ + \\
\langle \psi_L^+ \psi_L \rangle_0 \langle \psi_R^+ \psi_R \rangle_0 - \langle \psi_L^+ \psi_R \rangle_0 \langle \psi_R^+ \psi_L \rangle_0 = \\
\psi_L^+ \psi_L \psi_R^+ \psi_R^+ + \langle : \psi_L^+ \psi_R \psi_L^+ : \rangle_0 : \psi_R^+ \psi_L^+ : + \langle : \psi_R^+ \psi_L \psi_R^+ : \rangle_0 : \psi_L^+ \psi_R^+ : - \\
\langle : \psi_L^+ \psi_R : \rangle_0 \langle : \psi_R^+ \psi_L : \rangle_0 ,
\]

where

\[
\langle X \rangle_0 = \frac{Tr e^{-2\pi K_0} X}{Tr e^{-2\pi K_0}} ,
\]

for any operator \( X \), \( : \) denotes the ordinary normal ordering with respect to the mode decomposition (2.10) and in the second equality of (2.22) we have taken into account that

\[
\langle : \psi_L \psi_R : \rangle_0 = \langle : \psi_L^+ \psi_R^+ : \rangle_0 = \langle : \psi_L^+ \psi_L^+ : \rangle_0 = \langle : \psi_R^+ \psi_R^+ : \rangle_0 = 0 .
\]

Note that, while to give a proper mining to the first expression for the perturbing operator \( N (\psi_L^+ \psi_L \psi_R^+ \psi_R) \) (see (2.22)) one has to apply point splitting, the separate terms of the second expression are already finite.
3 VEV’s of Exponential Fields

The Dirac fields $\Psi(z, \overline{z})$, $\overline{\Psi}(z, \overline{z})$ have non-trivial monodromy with respect to the exponential fields $\exp ia\phi(0)$

$$
\Psi(z, \overline{z}) \ast e^{ia\phi(0)} = e^{\frac{2\pi a}{\beta}} \Psi(z, \overline{z}) e^{ia\phi(0)}, \\
\overline{\Psi}(z, \overline{z}) \ast e^{ia\phi(0)} = e^{-\frac{2\pi a}{\beta}} \overline{\Psi}(z, \overline{z}) e^{ia\phi(0)},
$$

(3.1)

where $\ast$ denotes the analytic continuation around the point 0. The VEV $\langle \exp ia\phi(0) \rangle$ can be expressed in terms of Grassmanian functional integral \[4\]

$$
I(a) = \int_{\mathcal{F}_a} \left[ D\Psi D\overline{\Psi} \right] e^{-A_{MTM}} \\
\int_{\mathcal{F}_0} \left[ D\Psi D\overline{\Psi} \right] e^{-A_{MTM}},
$$

(3.2)

where the functional integration in the numerator is carried out over the space $\mathcal{F}_a$ of twisted field configurations with monodromy (3.1). In angular quantization picture the insertion of the operator $\exp ia\phi(0)$ changes the boundary conditions along ”time” direction so that the Hilbert space remains untouched but the Hamiltonian due to (3.1) acquires an additional term $-iaQ/\beta$, where

$$
Q = \sum_{\nu \in \mathcal{N}_f^+} \left( c_{\nu}^c c_{\nu} - c_{-\nu} c_{-\nu}^c \right)
$$

(3.3)

is the fermion charge operator. Thus, due to (2.20), for the regularized version of the functional integral (3.2) we have

$$
I_\varepsilon(a, g) = \frac{\text{Tr}_H \left[ e^{-2\pi K_0 + \frac{2\pi i a}{\beta} Q_T} \left( e^{-\int K_I(\eta, \theta)d\eta d\theta} \right) \right]}{\text{Tr}_H \left[ e^{-2\pi K_0 T} \left( e^{-\int K_I(\eta, \theta)d\eta d\theta} \right) \right]}. \\
$$

(3.4)

The VEV $\langle \exp ia\phi(0) \rangle$ can be expressed in terms of $I_\varepsilon(a, g)$ as

$$
\left\langle e^{ia\phi(0)} \right\rangle = \lim_{\varepsilon \to 0} Z^{-1} \varepsilon^{-2a^2} I_\varepsilon(a, g),
$$

(3.5)

where $Z$ is some renormalization factor. This point, as well as the appearance of the factor $\varepsilon^{-2a^2}$, which has purely CFT origin, will be discussed later on.
The main goal of this paper is the evaluation of (3.4) and (3.5) perturbatively up to the linear over $g$ terms

$$I_{\varepsilon} (a, g) = I_{\varepsilon} (a, 0) \left(1 + gI_{\varepsilon}^1 (a) + O (g^2) \right).$$

(3.6)

The calculation of

$$I_{\varepsilon} (a, 0) = \frac{\text{Tr}_H \left[ e^{-2\pi K_0 + 2\pi i a \sqrt{2}Q} \right]}{\text{Tr}_H \left[ e^{-2\pi K_0} \right]}$$

(3.7)

is carried out in [4] and the result is

$$I_{\varepsilon} (a, 0) = \varepsilon^{2a^2} \left< e^{i a \varphi (0)} \right> \mid_{g=0} =
\left( \frac{M \varepsilon}{2} \right)^{2a^2} \exp \left\{ \int_0^\infty dt \left[ \frac{\sinh^2 \left( \sqrt{2} at \right)}{\sinh^2 t} - 2a^2 e^{-2t} \right] \right\},$$

(3.8)

so that we’ll concentrate our attention on the second term

$$I_{\varepsilon}^1 (a) = \frac{a}{2\pi} \partial_a \log I_{\varepsilon} (a, 0) -
4\pi \frac{\text{Tr}_H \left[ e^{-2\pi K_0 + 2\pi i a \sqrt{2}Q} \int_0^\infty d\eta N \left( \psi^L \psi^L \psi^R \psi^R \right) \right]}{\text{Tr}_H \left[ e^{-2\pi K_0 + 2\pi i a \sqrt{2}Q} \right]}.$$

(3.9)

Using mode decomposition (2.10) and evaluating traces over $\mathcal{H}$ in the bases (2.19) for (3.9) we obtain

$$I_{\varepsilon}^1 (a) = \frac{a}{2\pi} \partial_a \log I_{\varepsilon} (a, 0) + \sum_{\nu_1, \nu_2 \in \mathbb{N}^*} \left\{ \frac{\cosh \pi \nu_1 \cosh \pi \nu_2}{\nu_1 \rho (\nu_1) \nu_2 \rho (\nu_2)} \times
\int_{M \varepsilon} A_{\nu_1, \nu_2} \left( \left| K_{\frac{1}{2} + i \nu_1} (x) \right|^2 \left| K_{\frac{1}{2} + i \nu_2} (x) \right|^2 - K_{\frac{1}{2} + i \nu_1} (x) K_{\frac{1}{2} + i \nu_2} (x) \right) +
A_{\nu_1, \nu_2}^* \left( K_{\frac{1}{2} + i \nu_1} (x) K_{\frac{1}{2} + i \nu_2} (x) - \left| K_{\frac{1}{2} + i \nu_1} (x) \right|^2 \left| K_{\frac{1}{2} + i \nu_2} (x) \right|^2 \right) -
A_{\nu_1} \left( A_{\nu_2} - \frac{1}{2} A_{\nu_2}^* \right) \left( K_{\frac{1}{2} + i \nu_1} (x) - K_{\frac{1}{2} - i \nu_1} (x) \right) \right\} x d x \right},$$

(3.10)

where

$$A_{\nu} = \frac{e^{-2\pi \nu + 2i \sqrt{2}a}}{1 + e^{-2\pi \nu + 2i \sqrt{2}a}}.$$

(3.11)
and \( A^0_\nu \equiv A_\nu \mid_{a=0} \).

The following formulae for the integrals over \( x \) included in (3.10) are proved in Appendix A (below and later on we’ll omit vanishing in the limit \( \varepsilon \to 0 \) terms)

\[
\begin{align*}
\pi^2 \cosh \pi \nu_1 \cosh \pi \nu_2 & \int_{M\varepsilon}^{\infty} K_{\frac{1}{2}+i\nu_1}^2 (x) K_{\frac{1}{2}+i\nu_2}^2 (x) \, dx = \\
\int_{0}^{\infty} \left[ \frac{\sin^2 \nu_1 t + \sin^2 \nu_2 t}{2 \sinh t} + \frac{\cos 2\nu_1 t \cos 2\nu_2 t - 1}{2 \sinh 2t} \right] \, dt - \\
n & \frac{\gamma + 2 \log 2 + \log M\varepsilon}{4},
\end{align*}
\]

(3.12)

\[
\begin{align*}
\pi^2 \cosh \pi \nu_1 \cosh \pi \nu_2 & \int_{M\varepsilon}^{\infty} K_{\frac{1}{2}+i\nu_1} K_{\frac{1}{2}+i\nu_2}^2 (x) \, dx = \\
\int_{0}^{\infty} \left[ \frac{\sinh (1+2i\nu_1) t \sinh (1+2i\nu_2) t}{2 \sinh 2t} - \frac{1}{4} e^{2i(\nu_1+\nu_2)t} \right] \, dt + \\
\frac{\gamma \left( \frac{1}{2} + i\nu_1 \right) \gamma \left( \frac{1}{2} + i\nu_2 \right) \left( \frac{2}{M\varepsilon} \right)^{2i(\nu_1+\nu_2)} - 1}{8i (\nu_1 + \nu_2)},
\end{align*}
\]

(3.13)

where \( \gamma = 0.577216 \cdots \) is the Euler constant and

\[
\gamma(x) \equiv \frac{\Gamma(x)}{\Gamma(1-x)}.
\]

(3.14)

Let us imagine that \( x \) integration in (3.10) with the help of (3.12) and (3.13) is already performed. Then the resulting expression can be represented as a sum of two parts

\[
I^1_\varepsilon(a) = \sum_{\nu_1,\nu_2 \in \mathbb{N}_\varepsilon^+} \frac{1}{\rho(\nu_1) \rho(\nu_2)} [i] + \sum_{\nu_1,\nu_2 \in \mathbb{N}_\varepsilon^+} \frac{1}{\rho(\nu_1) \rho(\nu_2)} [ii],
\]

(3.15)

where in the second part, symbolically denoted as \([ii]\), are collected all the terms induced by the term

\[
\frac{\gamma \left( \frac{1}{2} + i\nu_1 \right) \gamma \left( \frac{1}{2} + i\nu_2 \right) \left( \frac{2}{M\varepsilon} \right)^{2i(\nu_1+\nu_2)} - 1}{8i (\nu_1 + \nu_2)}
\]

(3.16)

of the equation (3.13). As the \( \nu_1, \nu_2 \) dependence of terms collected in the remaining part \([i]\) (in contrary to those of \([ii]\)) are free of rapid, comparable with \( \log 1/M\varepsilon \) frequency oscillations, it is safe to make replacement

\[
\sum_{\nu_1,\nu_2 \in \mathbb{N}_\varepsilon^+} \frac{1}{\rho(\nu_1) \rho(\nu_2)} [i] \to \int_{0}^{\infty} \int_{0}^{\infty} d\nu_1 d\nu_2 [i].
\]

(3.17)
Surprisingly enough it is possible to factorize these integrals in such a way, that they can be performed explicitly using the formula (for proof see Appendix A)

\[
Im \int_0^\infty d\nu \frac{e^{-2\pi\nu+i\pi\alpha+2\nu t}}{1 + e^{-2\pi\nu+i\pi\alpha}} = \frac{1}{4t} - \frac{e^{-\alpha t}}{4\sinh t}.
\] (3.18)

The sum over \( \nu_1, \nu_2 \) in second term of (3.15) also can be converted into the integrals provided one notices that (3.16) is exactly zero when \( \nu_1, \nu_2 \in \mathbb{N}_\varepsilon \), \( \nu_1 + \nu_2 \neq 0 \) and is equal to \( \pi \rho (\nu_1) / 4 \) (up to nonessential, finite in the \( \varepsilon \to 0 \) limit term) if \( \nu_1 + \nu_2 = 0 \), so that (3.16) can be effectively replaced by \( \pi \delta (\nu_1 + \nu_2) / 4 \). This leaves us with elementary one dimensional integrals. As a result of above described calculation for (3.10) one obtains

\[
2\pi I^1_\varepsilon (a) = \pi \alpha \cot \frac{\pi \alpha}{2} - 2 + \frac{\alpha^2}{2} \left( \psi \left( \frac{1}{2} \right) - \log 2 \right) + \\
\int_0^\infty dt \left[ \frac{\sinh^2 \alpha t}{\sinh 2t \sinh^2 t} + \frac{\alpha^2 (2 \cosh t - 1)}{\sinh 2t} - \frac{\alpha \sinh \alpha t}{2 \sinh^2 t} + \\
\frac{2 \cosh t \sinh^4 \frac{\alpha t}{2t}}{\sinh^3 t} + \frac{\cosh \alpha t - 1}{\sinh^2 t} - \frac{\sinh^2 \alpha t}{\sinh 2t} - \frac{\alpha^2 e^{-2t}}{2t} \right],
\] (3.19)

where \( \psi (x) \) is the logarithmic derivative of the \( \Gamma \) -function. In (3.19) and later on we set

\[
\alpha = 2 \sqrt{2a}.
\] (3.20)

Using the table of integrals presented at the end of Appendix A it is not difficult to perform \( t \) integration too

\[
2\pi I^1_\varepsilon (a) = -\frac{1}{2} \psi \left( \frac{1}{2} \right) - 1 + \frac{\pi \alpha}{2} \cot \frac{\pi \alpha}{2} + \alpha^2 \left( \frac{1}{4} - \log 2 \right) + \\
\frac{\alpha^2}{4} \left( \psi \left( \frac{\alpha}{2} \right) + \psi \left( -\frac{\alpha}{2} \right) \right) + \frac{1 - \alpha^2}{4} \left( \psi \left( \frac{1 + \alpha}{2} \right) + \psi \left( \frac{1 - \alpha}{2} \right) \right).
\] (3.21)

Now let us turn to the computation of the renormalization factor \( Z \) in (3.5). If \( \varepsilon \) is small enough we can split the region \( |z| \geq \varepsilon \) into two pieces by the circle \( |z| = \tilde{\varepsilon} \) with some \( \tilde{\varepsilon} \) satisfying the conditions

\[
\log \frac{\tilde{\varepsilon}}{\varepsilon} \gg 1; \quad \log M \tilde{\varepsilon} \ll 1
\] (3.22)
so that inside the first region \( U_1 = \{ z; \varepsilon \leq |z| \leq \bar{\varepsilon} \} \) the theory is nearly conformal invariant and at the same time in the region \( U_2 = \{ z; |z| > \bar{\varepsilon} \} \) the influence of the boundary at \(|z| = \varepsilon\) could be neglected. Note that in contrary to the region \( U_2 \) where the regularization prescription (2.22) is standard and in Cartesian coordinates transforms to the usual normal ordering, due to the influence of the boundary in region \( U_1 \) the interaction term of the Hamiltonian (2.21) results in an extra multiplicative renormalization of the field \( \exp ia\varphi \) (besides the usual charge renormalization \( a_r = (1 - g/2\pi) a \)). The actual computation of the renormalization constant \( Z \) is significantly simplified owing to the existence of conformal invariance inside the region \( U_1 \). As usual in CFT it is convenient to use radial quantization [10]. let us denote by \( |B, a\rangle \) the boundary state [9] corresponding to the boundary conditions (2.9) and belonging to the conformal family [10] of the state \( |a\rangle = \exp ia\varphi(0)|0\rangle \). Note that during the evaluation from \( \varepsilon \) to \( \bar{\varepsilon} \), the state \( \varepsilon^{-2a^2} |B, a\rangle \) approaches to \( \bar{\varepsilon}^{-2a^2} |a\rangle \) thus correctly imitating the insertion of the field \( \exp ia\varphi(0) \). This consideration makes transparent the appearance of the factor \( \varepsilon^{-2a^2} \) in (3.5). It is not difficult to see that the renormalization factor \( Z \) is given by

\[
Z - 1 = -4\pi g \int_{\varepsilon}^{\bar{\varepsilon}} \langle a| \, N \left( \psi_L^\dagger \psi_L \psi_R^\dagger \psi_R \right) \varepsilon^{L_0 + \bar{L}_0} |B, a\rangle |z| \, d|z| - \frac{g\alpha^2}{4\pi} \log \frac{\bar{\varepsilon}}{\varepsilon}. \tag{3.23}
\]

Here and in what follows we take the Fermi fields in initial coordinates \( z, \bar{z} \) (i.e. the transformation (2.3) is not applied). The operator \( \varepsilon^{L_0 + \bar{L}_0} (L_0, \bar{L}_0 \) are the Virasoro generators) is included in (3.23) to take into account that the boundary state \( |B, a\rangle \) is associated to the circle \(|z| = \varepsilon\). The second term in (3.23) subtracts the contribution of the charge renormalization. In Appendix B we have presented the details of the computation of the matrix element included in (3.23). Inserting (B.6) into (3.23) and performing integration with the help of (A.25) we obtain

\[
Z - 1 = -\frac{g}{2\pi} \left( 1 - \frac{\pi\alpha}{2} \cot \frac{\pi\alpha}{2} \right). \tag{3.24}
\]

As it should be expected the choice of \( \bar{\varepsilon} \) satisfying the conditions (3.22) has no effect on the value of \( Z \). Now taking into account (3.3), (3.8), (B.24) and (B.24) we can write down

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a final expression for the expansion of VEV up to linear over $g$ terms

$$\frac{\langle e^{ia\varphi(0)} \rangle}{\langle e^{ia\varphi(0)} \rangle |_{g=0}} = 1 + \frac{g}{8\pi} \left[ -2\psi\left(\frac{1}{2}\right) + \alpha^2 (1 - 4 \log 2) + \alpha^2 \left( \psi\left(\frac{\alpha}{2}\right) + \psi\left(-\frac{\alpha}{2}\right) \right) + (1 - \alpha^2) \left( \psi\left(\frac{1 + \alpha}{2}\right) + \psi\left(\frac{1 - \alpha}{2}\right) \right) \right] + O\left(g^2\right). \tag{3.25}$$

It is not difficult to check that (3.25) exactly coincides with the expression which one obtains directly expanding Lukyanov-Zamolodchikov formula \[4\].

**Acknowledgements**

The author is grateful to his colleagues H.Babujian and A.Sedrakyan for interesting discussions. This research is partially supported by INTAS grant 96-524.

**Appendix A**

It appears that the Hankel-transforms are appropriate tools allowing us to perform the integration over $x$ in (3.10). In polar coordinates the Hankel-transforms play the same role, as the ordinary Fourier-transforms in the Cartesian one.

Let me briefly recall the main formulae concerning to the Hankel-transforms (for details see [11] and references therein). The $\nu$-th order ($\nu > -1$) direct and inverse Hankel-transforms of the function $f(x)$ defined on $(0, \infty)$ are given by

$$f(x) = \int_{0}^{\infty} J_{\nu}(sx) \tilde{f}_{\nu}(s) sds, \tag{A.1}$$

$$\tilde{f}_{\nu}(s) = \int_{0}^{\infty} J_{\nu}(sx) f(x) xdx, \tag{A.2}$$

where $J_{\nu}$ is the Bessel function. In complete analogy with the case of Fourier-transform, it follows from (A.1), (A.2), that the ”scalar product” of any two functions $f(x), g(x)$ in

\[1\] Since the functions we are dealing with are regular in the interval $(0, \infty)$, the only thing one has to care of is the convergence of integrals at the extreme points 0 and $\infty$. 

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representation coincides with the "scalar product" of their images in s representation:
\[
\int_0^\infty f(x) g(x) \, dx = \int_0^\infty \tilde{f}_\nu(s) \tilde{g}_\nu(s) \, ds.
\]  
(A.3)

The starting point of our consideration is the formula [11]
\[
[K_\nu(x)]^2 \sin(\nu \pi) = \pi \int_0^\infty J_0(2x \sin t) \sinh(2\nu t) \, dt,
\]  
(A.4)

\[|\text{Re} \nu| < \frac{3}{4}; x > 0,\]

which obviously can be considered (after substitution \(s = \sinh t\)) as the 0-th order Hankel-transform of the function \([K_\nu(x)]^2\). Assuming \(0 < \text{Re} \nu_1 < 1/2, 0 < \text{Re} \nu_2 < 1/2\) and using (A.3) one obtains
\[
\frac{\sin(\nu_1 \pi) \sin(\nu_2 \pi)}{\pi^2} \int_0^\infty [K_{\nu_1}(x)]^2 [K_{\nu_2}(x)]^2 \, dx = \int_0^\infty \frac{\sin(2\nu_1 t) \sinh(2\nu_2 t)}{2 \sinh 2t} dt =
\]
\[
\frac{1}{16} \left[ \psi \left( \frac{1+\nu_1 - \nu_2}{2} \right) + \psi \left( \frac{1-\nu_1 + \nu_2}{2} \right) - 
\psi \left( \frac{1+\nu_1 + \nu_2}{2} \right) + \psi \left( \frac{1-\nu_1 - \nu_2}{2} \right) \right],
\]  
(A.5)

where the second equality follows from (A.25). For our purposes we have to restrict the range of integration in (A.3) to \([M\varepsilon, \infty)\) and substitute \(\nu_1 \to 1/2 + i\nu_1, \nu_1 \to 1/2 + i\nu_2\). This has to be done carefully, since for such values of parameters the integral (A.3) doesn’t converge at \(x \to 0\). To avoid this difficulty, one first has to subtract the integral
\[
\frac{\sin(\nu_1 \pi) \sin(\nu_2 \pi)}{\pi^2} \int_0^{M\varepsilon} [K_{\nu_1}(x)]^2 [K_{\nu_2}(x)]^2 \, dx =
\]
\[
\frac{1}{2(1-\nu_1 - \nu_2)} \left( \frac{2}{M\varepsilon} \right)^{2(\nu_1+\nu_2-1)} \left( 1 + O \left( \varepsilon^2 \right) \right)
\]  
(A.6)

from (A.5) and only after it to make above mentioned substitutions \(\nu_1 \to 1/2 + i\nu_1, \nu_2 \to 1/2 + i\nu_2\). Note that in (A.6) we have used the following expansion of \(K^2_{\nu}(x)\) near 0:
\[
K^2_{\nu}(x) = \frac{\pi \gamma(\nu)}{4 \sin(\nu \pi)} \left( \frac{2}{x} \right)^{2\nu} \left( 1 + O \left( x^2 \right) \right).
\]  
(A.7)

As a result one easily arrives at (3.13).
The proof of \((3.12)\) we begin with the formula
\[
\frac{\sin (\nu \pi)}{\pi} K_{\nu} (x) K_{1-\nu} (x) = \frac{1}{2x} - \int_0^\infty [J_1 (2x \sinh t) \cosh ((2\nu - 1) t)] \, dt,
\]
(A.8)
which can be verified differentiating (A.4) over \(x\) and using the identities
\[
K'_{\nu} (x) = -K_{1-\nu} (x) - \frac{\nu}{x} K_{\nu} (x),
\]
\[
J'_{0} (x) = -J_{1} (x).
\]
(A.9)

Since the image of the function \(1/x\) under the first order Hankel-transform is equal to 1, the equation (A.8) (after substitution \(\nu \to 1/2 + i\nu\)) can be rewritten as
\[
\frac{\cosh (\nu \pi)}{\pi} \left| K_{\nu} (x) \right|^2 = \int_0^\infty J_1 (2x \sinh t) (\cos 2\nu t - \cosh t) \, dt.
\]
(A.10)

Unfortunately (A.10) can not be used directly to calculate integral (3.12) because of the singularity at \(x = 0\):
\[
\frac{\cosh (\nu \pi)}{\pi} \left| K_{\nu} (x) \right|^2 = \frac{1}{2x} \left( 1 + O \left( x^2 \right) \right).
\]
(A.11)

To overcome this difficulty we replace the functions
\[
\frac{\cosh (\nu \pi)}{\pi} \left| K_{\nu} (x) \right|^2, \frac{\cosh (\nu \pi)}{\pi} \left| K'_{\nu} (x) \right|^2
\]
with their regularized versions
\[
\frac{\cosh (\nu \pi)}{\pi} \left| K_{\nu} (x) \right|^2 = e^{-\Lambda x} \frac{1}{2x}, \frac{\cosh (\nu \pi)}{\pi} \left| K'_{\nu} (x) \right|^2 = e^{-\Lambda x} \frac{1}{2x},
\]
(A.12)

where \(\Lambda\) is some large positive cut-off. The first order Hankel images of the functions (A.13) can be obtained using the formulae
\[
\frac{e^{-\Lambda x}}{2x} = \frac{1}{2} \int_0^\infty J_1 (sx) \left( 1 - \frac{\Lambda}{\sqrt{\Lambda^2 + s^2}} \right) \, ds
\]
(A.14)

and (A.10). Then one first applies (A.3) to calculate the regularized version of the integral (3.12) over the full interval \((0, \infty)\), next subtracts the integral over the interval \((0, M\varepsilon)\) (as \(M\varepsilon \ll 1\), one simply applies the asymptotic formula (A.11) ) and, afterwards gets
rid of the cut-off Λ tending it to ∞. The final result is the equation (3.12). Note that using (A.25), integration over \( t \) on the right hand side of (3.12) may be expressed via the ψ function. For the sake of completeness let us present here this expression too

\[
\frac{\cosh \pi \nu_1 \cosh \pi \nu_2}{\pi^2} \int_\varepsilon^\infty \left| K_{\frac{1}{2}+i\nu_1}(x) \right|^2 \left| K_{\frac{1}{2}+i\nu_2}(x) \right|^2 dx = \\
\sum_{\sigma=\pm 1} \left[ \frac{1}{8} \left( \psi \left( \frac{1}{2} + i\sigma \nu_1 \right) + \psi \left( \frac{1}{2} + i\sigma \nu_2 \right) \right) - \\
\frac{1}{16} \left( \psi \left( \frac{1}{2} + i\sigma \frac{\nu_1 + \nu_2}{2} \right) + \psi \left( \frac{1}{2} + i\sigma \frac{\nu_1 - \nu_2}{2} \right) \right) \right] - \\
\frac{1}{4} \left( 1 + O \left( \varepsilon^2 \right) \right) \log \varepsilon . \tag{A.15}
\]

Now let us sketch the proof of the equation (3.18). Consider the function \( L(a, b) \) (\( a, b \) are real variables and \( |a| < \pi \)) defined by the integral

\[
L(a, b) = \int_0^\infty e^{ibx} \log \left( 1 + e^{-x+ia} \right) dx \tag{A.16}
\]

and the contour integral

\[
J(C; b) = \int_C e^{bz} \log \left( 1 + e^{-z} \right) dz \tag{A.17}
\]

for any contour \( C \) on the complex \( z \)-plain. Let \( C_{1,a}, C_{2,a} \) be the contours

\[
C_{1,a} : \quad z = -i\pi + it; \quad 0 < t \leq \pi - a,
\]

\[
C_{2,a} : \quad z = -ia + t; \quad t > 0 . \tag{A.18}
\]

As the integration contour can be freely deformed inside analyticity domain it is clear that

\[
J(C_{1,a}; b) + J(C_{2,a}; b) = J(C_{2,\pi}; b) = J(C_{1,-\pi}; b) + J(C_{2,-\pi}; b) . \tag{A.19}
\]

Taking into account the evident relations

\[
J(C_{2,a}; b) = e^{ab} L(a, b), \quad L(\pi, b) = L(-\pi, b) \tag{A.20}
\]
we obtain
\[ J(C_1,a;b) + e^{ab}L(a,b) = \frac{e^{\pi b}}{e^{\pi b} - e^{-\pi b}} J(C_1,-\pi;b). \]  
(A.21)

Thus
\[ e^{ab}L(a,b) + L(-a,-b) = -J(C_1,a;b) - J(C_1,-a;-b) + \]
\[ \frac{e^{\pi b}}{2 \sinh \pi b} J(C_1,-\pi;b) - \frac{e^{-\pi b}}{2 \sinh \pi b} J(C_1,-\pi;-b). \]  
(A.22)

But an easy exercise shows that the r.h.s. of (A.22) is equal to
\[ \frac{e^{-\pi b}}{2 \sinh \pi b} \int_{-\pi}^{\pi} x e^{-bx} dx + \int_{-a}^{a} x e^{-bx} dx = e^{ab} \left( \frac{1 - ab}{b^2} - \frac{\pi e^{-ab}}{b \sinh \pi b} \right). \]  
(A.23)

So we have obtained
\[ \text{Re} \int_{0}^{\infty} e^{ibx} \log \left( 1 + e^{-x+ia} \right) dx = 1 - \frac{ab}{2b^2} - \frac{\pi e^{-ab}}{2b \sinh \pi b}. \]  
(A.24)

Differentiating the equation (A.24) over \( a \) and changing the notations in an obvious way we explicitly arrive at (3.18).

We’ll complete this section presenting a table of integrals, which has been used when performing \( t \) integration in (3.19):
\[ \int_{0}^{\infty} \frac{\cosh \alpha t - 1}{\sinh^2 t} dt = 1 - \frac{\pi \alpha}{2} \cot \frac{\pi \alpha}{2}, \]  
(A.25)

\[ \int_{0}^{\infty} \frac{\cosh \alpha t - 1}{\sinh t} dt = \psi \left( \frac{1}{2} \right) - \frac{1}{2} \left( \psi \left( \frac{1 + \alpha}{2} \right) + \psi \left( \frac{1 - \alpha}{2} \right) \right), \]  
(A.26)

\[ \int_{0}^{\infty} \left( \frac{\sinh^2 \alpha t}{\sinh 2t \sinh^2 t} - \frac{\alpha^2}{\sinh 2t} \right) dt = \frac{3\alpha^2 + (2\alpha^2 - 1) \psi \left( \frac{1}{2} \right)}{4} - \]
\[ \frac{\alpha^2}{8} \left( \psi \left( \frac{\alpha}{2} \right) + \psi \left( -\frac{\alpha}{2} \right) \right) + \frac{1 - \alpha^2}{8} \left( \psi \left( \frac{1 + \alpha}{2} \right) + \psi \left( \frac{1 - \alpha}{2} \right) \right), \]  
(A.27)

\[ \int_{0}^{\infty} \left( \frac{\alpha \sinh \alpha t}{\sinh^2 t} - \frac{\alpha^2}{\sinh t} \right) dt = \frac{\alpha^2}{2} \left( 2 + 2\psi \left( \frac{1}{2} \right) - \psi \left( \frac{\alpha}{2} \right) - \psi \left( -\frac{\alpha}{2} \right) \right), \]  
(A.28)

\[ \int_{0}^{\infty} \frac{\cosh t \sinh^4 \alpha t}{\sinh^3 t} dt = \frac{\alpha^2}{8} \left( \psi \left( \frac{\alpha}{2} \right) + \psi \left( -\frac{\alpha}{2} \right) - \psi (\alpha) - \psi (-\alpha) \right). \]  
(A.29)
Note that (A.26) is a direct consequence of the standard integral representation for the function \( \psi \) and that (A.27)-(A.29) after some algebraic manipulations and integrations by part may be reduced to (A.26).

Appendix B

The usual mode decomposition of massless Fermi fields obeying the monodromy relation (3.1) takes the form

\[
\psi_L(z) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}^-} \frac{c_n}{z^{n+1/2}}; \quad \psi^\dagger_L(z) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}^+} \frac{c_n^\dagger}{z^{n+1/2}}, \\
\psi_R(\bar{z}) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}^+} \frac{\tilde{c}_n}{\bar{z}^{n+1/2}}; \quad \psi^\dagger_R(\bar{z}) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}^-} \frac{\tilde{c}_n^\dagger}{\bar{z}^{n+1/2}},
\]

where the operators \( c_n, c_n^\dagger, \tilde{c}_n, \tilde{c}_n^\dagger \) satisfy the anti-commutation relations

\[
\{ c_n, c_m^\dagger \} = \delta_{n+m,0}, \quad \{ \tilde{c}_n, \tilde{c}_m^\dagger \} = \delta_{n+m,0},
\]

with other anti-commutators being 0. The state \( |a\rangle \) is defined by the condition that it is annihilated by the positive mode operators and the defining property of the boundary state \( |B, a\rangle \) is

\[
(c_n - \tilde{c}_{-n}) |B, a\rangle = \left( c_n^\dagger - \tilde{c}_{-n}^\dagger \right) |B, a\rangle = 0 \quad \text{for every } n \in \mathbb{Z} - \frac{a}{2}.
\]

Using (B.1)-(B.3) it is easy to find all nonzero two point functions

\[
\langle a | \psi^\dagger_L(z_1) \psi_L(z_2) | B, a \rangle = \frac{1}{2\pi (z_1 - z_2)} \left( \frac{z_2}{z_1} \right)^{\alpha/2}, \\
\langle a | \psi^\dagger_R(z_1) \psi_R(z_2) | B, a \rangle = \frac{1}{2\pi (\bar{z}_1 - \bar{z}_2)} \left( \frac{\bar{z}_2}{\bar{z}_1} \right)^{-\alpha/2}, \\
\langle a | \psi^\dagger_L(z_1) \psi_R(z_2) | B, a \rangle = \frac{1}{2\pi (z_1 \bar{z}_2 - 1)} (z_1 \bar{z}_2)^{-\alpha/2}, \\
\langle a | \psi^\dagger_R(z_1) \psi_L(z_2) | B, a \rangle = \frac{1}{2\pi (\bar{z}_1 z_2 - 1)} (\bar{z}_1 z_2)^{\alpha/2}
\]

Using the first definition of the perturbing operator \( N (\psi^\dagger_L \psi_L \psi^\dagger_R \psi_R) \) (see the first equality in (2.22)) and the "propagators" (B.4) it is straightforward to find the matrix element

\[
\langle a | N (\psi^\dagger_L \psi_L \psi^\dagger_R \psi_R) | B, a \rangle = \frac{1}{4\pi^2 |z|^2} \left( \frac{\sinh^2 \left( \frac{\alpha}{2} \log |z| \right)}{\sinh^2 (\log |z|)} - \frac{\alpha^2}{4} \right).
\]
During all this consideration we have assumed the boundary state $|B, a\rangle$ to be associated with the unite circle $|z| = 1$, but simple scaling arguments allow us immediately to write down the corresponding expression for the case, when $|B, a\rangle$ is attached to the circle $|z| = \varepsilon$ as well:

$$
\langle a | N \left( \psi_L \psi_L \psi_R \psi_R \right) \varepsilon^{L_0 + \tilde{L}_0} | B, a \rangle = \frac{1}{4\pi^2 |z|^2} \left( \frac{\sin \left( \frac{\alpha \log |z|}{2\varepsilon} \right)}{\sin \left( \frac{\log |z|}{\varepsilon} \right)} - \frac{\alpha^2}{4} \right). \quad (B.6)
$$

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