OPTIMAL DESTABILIZING CENTERS AND EQUIVARIANT K-STABILITY

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ABSTRACT. We give an algebraic proof of the equivalence of equivariant K-semistability (resp. equivariant K-polystability) with geometric K-semistability (resp. geometric K-polystability). Along the way we also prove the existence and uniqueness of minimal optimal destabilizing centers on K-unstable log Fano pairs.

1. INTRODUCTION

K-stability (see [Tia97, Don02]) is an algebraic condition that detects the existence of Kähler-Einstein metric on complex Fano varieties. By a well-known result of Matsushima [Mat57], on a Kähler-Einstein Fano manifold, the canonical metric is invariant under some maximal compact subgroup of the automorphism group. This motivates the folklore conjecture that a complex Fano variety (or more generally, a log Fano pair) with a group action is K-semistable (resp. K-polystable) if and only if it is equivariantly K-semistable (resp. equivariantly K-polystable). This conjecture has been confirmed on smooth Fano manifolds with a reductive group action [DS16] and on log Fano pairs with a finite group action [LZ20]. Despite the algebraic nature of the statement, the proofs in these two cases rely on deep analytic results and an algebraic proof of the conjecture is only known when the group action is given by a torus [LX20, LWX18]. On the other hand, if the log Fano pair is defined over a smaller field $\kappa$ (not necessarily algebraically closed), it comes with an induced Galois action. A variant of the above conjecture then predicts that it is enough to test the K-semistability (resp. K-polystability) of the log Fano pair using test configurations that are defined over $\kappa$. To our knowledge, there has been very little progress in this direction.

In this paper, we give a complete answer to these conjectures using purely algebraic argument. Our main result is as follows.

Theorem 1.1 (=Corollary 4.11). Let $k$ be a field of characteristic zero and let $\kappa$ be its algebraic closure. Let $G$ be an algebraic group and let $(X, \Delta)$ be a log Fano pair with an action of $G$ over $k$. Let $(X_\kappa, \Delta_\kappa) := (X, \Delta) \times_k \text{Spec}(\kappa)$.

1. If $(X, \Delta)$ is $G$-equivariantly K-semistable, then $(X_\kappa, \Delta_\kappa)$ is K-semistable.
2. If $G$ is reductive and $(X, \Delta)$ is $G$-equivariantly K-polystable, then $(X_\kappa, \Delta_\kappa)$ is K-polystable.

Note that in the K-semistable part we allow the group $G$ to be non-reductive; this can happen for automorphism groups of K-semistable Fano varieties, see [CS18, Example 1.4]. On the other hand, there exist Fano varieties that are equivariantly K-polystable with respect to their automorphism group but not K-polystable (see Example 4.12), thus the reductivity assumption in the K-polystable part is necessary. Since every K-polystable
A log Fano pair has a reductive automorphism group $[ABHLX20]$, this seems to be a natural assumption to add to the statement.

In fact, we prove a more precise statement than Theorem 1.1. Recall that the K-semistability of a log Fano pair is characterized by its stability threshold (or $\delta$-invariant), see $[FO18,BJ20]$ or Section 2.3. A priori, to define the stability threshold of a log Fano pair $(X, \Delta)$, one needs to take into account all divisors on various birational models of $X$. We show that in the K-unstable case, it is enough to consider divisors that not only have the same field of definition but also are invariant under the automorphism group. This is the key to our proof of Theorem 1.1.

Theorem 1.2 (=Theorem 4.1). Let $(X, \Delta)$ be a log Fano pair defined over a field $k$ of characteristic zero and let $G = \text{Aut}(X, \Delta)$. Assume that $(\bar{X}_k, \Delta_{\bar{k}})$ is not K-semistable. Then we have

$$\delta(X_{\bar{k}}, \Delta_{\bar{k}}) = \inf_E \frac{A_{X, \Delta}(E)}{S(E)}$$

where the infimum runs over all $G$-invariant geometrically irreducible divisors $E$ over $X$ that are lc places of complements.

Here $A_{X, \Delta}(E)$ is the log discrepancy of $E$ with respect to the pair $(X, \Delta)$ and $S(E)$ is the expected vanishing order of $-(K_X + \Delta)$ along the divisor $E$, see Definition 2.3. As a corollary, we obtain an algebraic proof of a generalization of Tian’s criterion $[Tia87,OS12]$ (see Section 2.3 for the definition of the $G$-alpha invariant $\alpha_G(X, \Delta)$ of a log Fano pair).

Corollary 1.3 (=Corollary 4.15). Let $G$ be an algebraic group and let $(X, \Delta)$ be a log Fano pair of dimension $n$ with a $G$-action over a field $k$ of characteristic zero.

1. If $\alpha_G(X, \Delta) \geq \frac{n}{n+1}$, then $(X_{\bar{k}}, \Delta_{\bar{k}})$ is K-semistable.
2. If $G$ is reductive and $\alpha_G(X, \Delta) > \frac{n}{n+1}$, then $(X_{\bar{k}}, \Delta_{\bar{k}})$ is K-polystable.

As another application, we recover the K-stability of Fermat hypersurfaces, as well as smooth complete intersections of two quadrics $[Tia00,AGP06]$. We remark that the previous proof of this fact involves analytic argument but our proof here is completely algebraic. Combined with $[Xu20,BLX19]$, this also gives an algebraic proof that a general Fano hypersurface is K-semistable (in fact K-stable if the degree is at least 3).

Corollary 1.4 (=Corollary 4.17). The following Fano manifolds are K-stable:

1. Fermat hypersurfaces $(x_0^d + \cdots + x_n^d = 0) \subseteq \mathbb{P}^{n+1}$ ($3 \leq d \leq n + 1$).
2. Complete intersection of two quadrics $Q_1 \cap Q_2 \subseteq \mathbb{P}^{n+2}$.

Along the proof of Theorem 1.2, we also discover the following result concerning optimal destabilizing centers (i.e. centers of valuations that compute the stability thresholds, see Definition 2.9) of K-unstable log Fano pairs, which may be of independent interest.

Theorem 1.5 (=Theorem 3.1). Let $(X, \Delta)$ be a log Fano pair. Assume that $(X, \Delta)$ is not K-semistable. Then there exists a (necessarily unique) subvariety $Z \subseteq X$ such that

1. $Z$ is an optimal destabilizing center of $(X, \Delta)$ and
2. $Z$ is contained in any other optimal destabilizing center of $(X, \Delta)$.

Let us briefly explain some key ideas behind the proofs of Theorems 1.2 and 1.5. By definition, an optimal destabilizing center of a log Fano pair $(X, \Delta)$ is (roughly speaking)
an lc center of \((X, \Delta + \delta D)\) for some basis type \(\mathbb{Q}\)-divisor \(D\) of the pair \((X, \Delta)\), where \(\delta = \delta(X, \Delta)\) is the stability threshold. Our first observation is that in fact every pair of optimal destabilizing centers can be realized as lc centers of a common pair \((X, \Delta + \delta D)\).

This allows us to show that optimal destabilizing centers exhibit properties that are similar to lc centers of a fixed lc pair. In particular, any two optimal destabilizing centers intersect (Lemma 3.3) due to Kollár-Shokurov’s connectedness theorem and the intersection is a union of optimal destabilizing centers (Lemma 3.5). As a result, the minimal optimal destabilizing center is unique and contained in any other optimal destabilizing centers. If the log Fano pair is defined over \(k\) and admits an action of an algebraic group \(G\), then this unique center is also defined over \(k\) and invariant under the \(G\)-action. While the valuations that compute the stability threshold may not be \(G\)-invariant a priori, we can at least identify a \(G\)-invariant one \(v\) that computes the “equivariant version” of the stability threshold at the minimal optimal destabilizing center. When \(v\) is divisorial and realized by some prime divisor \(E\) over \(X\), we may relate the stability threshold (and its geometric or equivariant version) on \(X\) to a similar invariant on \(E\) through the recent work [AZ20]. This suggests us to use induction on the dimension. However, as the divisor \(E\) is not necessarily log Fano and the valuation \(v\) may not even be divisorial, we need to carefully set up the inductive framework and work in a setting that’s slightly more general than those of [AZ20], see Theorem 4.4.

This paper is organized as follows. In Section 2 we recall some definitions and preliminary results. Theorem 1.5 is proved in Section 3 where we study the behaviour of optimal destabilizing centers. In Section 4, we prove Theorem 1.2 and deduce Theorem 1.1, Corollaries 1.3 and 1.4 as a consequence.

Postscript note. Since this paper appeared on the arXiv, there has been many progress on the algebraic study of K-stability that leads to different proofs of some results in this article. Firstly, it is shown in [XZ20b] that the minimizer of the normalized volume function is unique (up to rescaling) for any klt singularity. Combined with the cone construction argument (see e.g. [Li17, Theorem 3.1] and [Zhu21, Theorem B]), this gives a different proof of our Theorem 1.1). More recently, [LXZ21] proves the optimal destabilization conjecture. Combined with [BHLLX20], it implies that to any K-unstable Fano variety one can associate a unique destabilizing test configuration that minimizes the functional defined in [BHLLX20]. Such a test configuration is necessarily induced by an invariant divisor. This gives another proof of Theorem 1.2 and further shows that the infimum in Theorem 1.2 is in fact a minimum.

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2. Preliminaries

2.1. Notation and conventions. Throughout the paper let \(k\) be a field of characteristic zero and let \(\bar{k}\) be its algebraic closure. Unless otherwise specified, all varieties, morphisms and linear series are assumed to be defined over \(k\). Given an object \(X\) (e.g. a variety/divisor/linear series) over \(k\) and a field extension \(k \subseteq K\), we denote by \(X_K\) the
corresponding base change. A pair \((X, \Delta)\) consists of a normal geometrically irreducible variety \(X\) and an effective \(\mathbb{Q}\)-divisor \(\Delta\) such that \(K_X + \Delta\) is \(\mathbb{Q}\)-Cartier. The notions of klt and lc singularities are defined as in [Kol13, Definition 2.8]. A pair \((X, \Delta)\) is log Fano if \(X\) is projective, \((X, \Delta)\) is klt and \(-(K_X + \Delta)\) is ample. The non-klt center \(N_{\text{klt}}(X, \Delta)\) of a pair \((X, \Delta)\) is the union of points \(x \in X\) such that \((X, \Delta)\) is not klt at \(x\). If \(\pi : Y \to X\) is a projective birational morphism and \(E\) is a prime divisor on \(Y\), then we say \(E\) is a divisor over \(X\). We denote by \(C_X(E)\) the center of \(E\) on \(X\). Let \((X, \Delta)\) be a klt pair, \(Z \subseteq X\) a subvariety and \(D\) an effective divisor on \(X\), we denote by \(\text{lct}_Z(X, \Delta; D)\) the largest number \(\lambda \geq 0\) such that \((X, \Delta + \lambda D)\) is lc at the generic point of \(Z\) (by convention, we set \(\text{lct}_Z(X, \Delta; D) = +\infty\) if \(Z\) is not contained in the support of \(D\)). A \(\mathbb{Q}\)-ideal \(a\) on \(X\) is a formal linear combination \(a = \prod_{i=1}^n a_i^{\lambda_i}\) where \(a_i \subseteq O_X\) are ideal sheaves on \(X\) and \(\lambda_i \in \mathbb{Q}_+\). Its co-support \(\text{Cosupp}(a)\) is defined to be the union of \(\text{Supp}(O_X/\mathfrak{a}_i)\). We can similarly define the log canonical threshold \(\text{lct}_Z(X, \Delta; a)\) of a \(\mathbb{Q}\)-ideal \(a\) with respect to the pair \((X, \Delta)\). We denote by \(\text{Val}_X\) the set of \(\mathbb{R}\)-valued valuations on the function field \(k(X)^*\) that is trivial on \(k^*\) and has a center on \(X\). The log discrepancy \(A_{X, \Delta}(v)\) of a valuation \(v \in \text{Val}_X\) with respect to a pair \((X, \Delta)\) is defined as in [JM12, (5.2)] and we denote by \(\text{Val}^+_X\) the set of \(v \in \text{Val}_X\) with \(A_{X, \Delta}(v) < \infty\) for some \(\Delta\) such that \(K_X + \Delta\) is \(\mathbb{Q}\)-Cartier (it does not depend on the choice of \(\Delta\)). When \(X\) has a group \(G\) action, a valuation \(v\) is said to \(G\)-invariant if \(v(g \cdot s) = v(s)\) for any \(g \in G\) and any \(s \in k(X)^*\); a divisor \(E\) over \(X\) is \(G\)-invariant if the associated valuation \(\text{ord}_E\) is \(G\)-invariant. Given a \(\mathbb{Q}\)-divisor \(D\) on \(X\), we set
\[
H^0(X, D) := \{0 \neq s \in k(X) \mid \text{div}(s) + D \geq 0\} \cup \{0\}
\]
whose members can be viewed as effective \(\mathbb{Q}\)-divisors that are \(\mathbb{Z}\)-linearly equivalent to \(D\). If \(D\) is \(\mathbb{Q}\)-Cartier, then \(v(s) := v(\text{div}(s) + D)\) is well-defined for any \(0 \neq s \in H^0(X, D)\) and any valuation \(v \in \text{Val}^+_X\); we denote by \(\mathcal{F}_v^\lambda H^0(X, D) := \{s \in H^0(X, D) \mid v(s) \geq \lambda\}\).

2.2. Test configurations and K-stability. In this section, we recall the definition of K-stability and equivariant K-stability.

**Definition 2.1** ([Tia97, Don02, OS12, LX14]). Let \((X, \Delta)\) be an \(n\)-dimensional log Fano pair with an action of an algebraic group \(G\). Let \(L\) be an ample line bundle on \(X\) such that \(L \sim -r(K_X + \Delta)\) for some \(r \in \mathbb{N}^*\).

1. A (normal) \(G\)-equivariant test configuration \((\mathcal{X}, \Delta_{\text{tC}}; \mathcal{L})/\mathbb{A}^1\) of \((X, \Delta; L)\) consists of the following data:
   - a normal variety \(\mathcal{X}\), an effective \(\mathbb{Q}\)-divisor \(\Delta_{\text{tC}}\) on \(\mathcal{X}\), together with a flat projective morphism \(\pi : \mathcal{X} \to \mathbb{A}^1\);
   - a \(\pi\)-ample line bundle \(\mathcal{L}\) on \(\mathcal{X}\);
   - a \(G \times \mathbb{G}_m\)-action on \((\mathcal{X}, \Delta_{\text{tC}}; \mathcal{L})\) such that \(\pi\) is \(G \times \mathbb{G}_m\)-equivariant with respect to the trivial action of \(G\) on \(\mathbb{A}^1\) and the standard action of \(\mathbb{G}_m\) on \(\mathbb{A}^1\) via multiplication;
   - \((\mathcal{X} \smallsetminus \mathcal{X}_0, \Delta_{\text{tC}}|_{\mathcal{X} \smallsetminus \mathcal{X}_0}; \mathcal{L}|_{\mathcal{X} \smallsetminus \mathcal{X}_0})\) is \(G \times \mathbb{G}_m\)-equivariantly isomorphic to \((X, \Delta; L) \times (\mathbb{A}^1 \setminus \{0\})\).

2. A \(G\)-equivariant test configuration is called a product test configuration if
\[
(\mathcal{X}, \Delta_{\text{tC}}; \mathcal{L}) \cong (X \times \mathbb{A}^1, \Delta \times \mathbb{A}^1; \text{pr}_1^* L).
\]
A product test configuration is called a trivial test configuration if the above isomorphism is \( G \times \mathbb{G}_m \)-equivariant with respect to the given \( G \)-action on \( X \), trivial \( \mathbb{G}_m \)-action on \( X \), trivial \( G \)-action on \( \mathbb{A}^1 \), and the standard \( \mathbb{G}_m \)-action on \( \mathbb{A}^1 \) via multiplication.

(3) A \( G \)-equivariant test configuration \((\mathcal{X}, \Delta_{tc}; \mathcal{L})\) is said to be special if \((\mathcal{X}, \mathcal{X}_0 + \Delta_{tc})\) is plt and \( \mathcal{L} \sim_{\mathbb{Q}} -r(K_{\mathcal{X}} + \Delta_{tc}) \). In this case, we say that \((\mathcal{X}_0, (\Delta_{tc})_0)\) (which is necessarily log Fano) is a \( G \)-equivariant special degeneration of \((\mathcal{X}, \Delta)\).

(4) (cf. [Wan12, Oda13]) Assume \( \pi : (\mathcal{X}, \Delta_{tc}; \mathcal{L}) \rightarrow \mathbb{A}^1 \) is a \( G \)-equivariant test configuration of \((X, \Delta; L)\). Let \( \bar{\pi} : (\overline{\mathcal{X}}, \overline{\Delta_{tc}}; \overline{\mathcal{L}}) \rightarrow \mathbb{P}^1 \) be the natural \( G \times \mathbb{G}_m \)-equivariant compactification of \( \pi \). The generalized Futaki invariant of \((\mathcal{X}, \Delta_{tc}; \mathcal{L})\) is defined by the intersection formula

\[
\text{Fut}(\mathcal{X}, \Delta_{tc}; \mathcal{L}) := \frac{1}{(-K_{\mathcal{X}} - \Delta)^n} \left( \frac{n}{n+1} \cdot \frac{(\mathcal{L}^{n+1})^r}{r^{n+1}} + \frac{(\mathcal{L}^r \cdot (K_{\overline{\mathcal{X}}/\mathbb{P}^1} + \overline{\Delta_{tc}}))}{r^n} \right).
\]

(5) The log Fano pair \((X, \Delta)\) is said to be

- \textit{G-equivariantly K-semistable} if Fut\((\mathcal{X}, \Delta_{tc}; \mathcal{L})\) \(\geq 0\) for any \(G\)-equivariant test configuration \((\mathcal{X}, \Delta_{tc}; \mathcal{L})/\mathbb{A}^1\) and any \(r \in \mathbb{N}^*\) such that \(L\) is Cartier.

- \textit{G-equivariantly K-polystable} if it is \(G\)-equivariantly K-semistable and for any \(G\)-equivariant test configuration \((\mathcal{X}, \Delta_{tc}; \mathcal{L})/\mathbb{A}^1\) we have Fut\((\mathcal{X}, \Delta_{tc}; \mathcal{L}) = 0\) if and only if it is a product test configuration.

- \textit{Geometrically K-semistable} (resp. \textit{geometrically K-polystable}) if \((X_k, \Delta_k)\) is equivariantly K-semistable (resp. K-polystable) over \(k\) with respect to the trivial group action.

Taking \(G\) to be the trivial group and \(k = \mathbb{C}\), we recover the usual definition of the \(K\)-semistability (resp. \(K\)-polystability) of a complex log Fano pair. We say that a log Fano pair \((X, \Delta)\) is \(K\)-unstable if it is not \(K\)-semistable. The following fact will be frequently used in this paper.

\textbf{Theorem 2.2 (LWX18).} Let \((X, \Delta)\) be a geometrically \(K\)-semistable log Fano pair with an action of a group \(G\). Then it is \(G\)-equivariantly \(K\)-polystable if and only if every geometrically \(K\)-semistable \(G\)-equivariant special degeneration of \((X, \Delta)\) is isomorphic to \((X, \Delta)\).

\textit{Proof.} Let \((\mathcal{X}, \Delta_{tc}; \mathcal{L})\) be a \(G\)-equivariant test configuration such that Fut\((\mathcal{X}, \Delta_{tc}; \mathcal{L}) = 0\). Since \((X, \Delta)\) is geometrically \(K\)-semistable, the test configuration is special by [LX14, Theorem 7] and the central fiber \((X_0, \Delta_0)\) is geometrically \(K\)-semistable by [LWX18, Lemma 3.1]. The result then follows as \((\mathcal{X}, \Delta_{tc}; \mathcal{L})\) is a product test configuration if and only if \((X_0, \Delta_0) \cong (X, \Delta)\) (an isotrivial family over \(\mathbb{A}^1\) is automatically trivial).

\(\square\)

2.3. Valuative criterion and stability thresholds. We recall the valuative criterion of \(K\)-semistability developed by Fujita and Li as well as the definition of stability threshold.

\textbf{Definition 2.3 (POT18, BJ20).} Let \((X, \Delta)\) be a log Fano pair and let \(L = -(K_{\mathcal{X}} + \Delta)\). Let \(m > 0\) be an integer such that \(H^0(X, mL) \neq 0\).
(1) An \( m \)-basis type \( \mathbb{Q} \)-divisor of \((X, \Delta)\) is a divisor of the form

\[
D = \frac{1}{mN_m} \sum_{i=1}^{N_m} \{s_i = 0\}
\]

where \( N_m = h^0(X, mL) \) and \( s_1, \ldots, s_{N_m} \) is a basis of \( H^0(X, mL) \). We define \( \delta_m(X, \Delta) \) to be the largest number \( \lambda \geq 0 \) such that \((X, \Delta + \lambda D)\) is lc for every \( m \)-basis type \( \mathbb{Q} \)-divisor \( D \) of \((X, \Delta)\).

(2) Let \( v \in \text{Val}_X^* \) be a valuation. We define the following invariants:

\[
T_m(v) = \max\{v(D) \mid D \in |mL|\}, \quad S_m(v) = \max\{v(D) \mid D \text{ is an } m \text{-basis type } \mathbb{Q} \text{-divisor of } (X, \Delta)\}.
\]

We set \( T(v) = \lim_{m \to \infty} T_m(v), S(v) = \lim_{m \to \infty} S_m(v) \). If \( E \) is a divisor over \( X \), we define \( S(E) := S(\text{ord}_E), T(E) := T(\text{ord}_E) \), etc.

(3) The stability threshold (or \( \delta \)-invariant) of \((X, \Delta)\) is defined to be

\[
\delta(X, \Delta) := \inf_{v \in \text{Val}_X^*} \frac{A_{X, \Delta}(v)}{S(v)}.
\]

By [BJ20, Theorem A], we have \( \lim_{m \to \infty} \delta_m(X, \Delta) = \delta(X, \Delta) \).

**Definition 2.4** ([Tia87] and [CS08, Appendix]). Let \((X, \Delta)\) be a log Fano pair with an action of a group \( G \). The \( G \)-alpha invariant \( \alpha_G(X, \Delta) \) of \((X, \Delta)\) is defined to be the largest \( \lambda \geq 0 \) such that \((X, \Delta + \lambda \Delta_\mathcal{M}) \) is lc for any \( m \in \mathbb{N}^* \) and any \( G \)-invariant linear system \( \mathcal{M} \subseteq |−m(K_X + \Delta)| \). By definition, it is clear that

\[
\alpha_G(X, \Delta) \leq \frac{A_{X, \Delta}(v)}{T(v)}
\]

for any \( G \)-invariant valuation \( v \in \text{Val}_X^* \).

**Theorem 2.5** ([Fuj19b, Lj17, FO18, BJ20]). Let \((X, \Delta)\) be a log Fano pair. Assume that \( k = k \) is algebraically closed. The following are equivalent:

1. \((X, \Delta)\) is \( K \)-semistable.
2. \( \beta_{X, \Delta}(E) := A_{X, \Delta}(E) - S(E) \geq 0 \) for any divisor \( E \) over \( X \).
3. \( \delta(X, \Delta) \geq 1 \).

We also need the following equivariant version of the above valuative criterion.

**Definition 2.6** ([Fuj19b]). Let \((X, \Delta)\) be a log Fano pair and let \( E \) be a divisor over \( X \). Let \( r > 0 \) be an integer such that \( L := −r(K_X + \Delta) \) is ample. We say that \( E \) is dreamy if the graded algebra \( \bigoplus_{m \in \mathbb{N}} \mathcal{F}^1_E H^0(X, \mathcal{O}_X(mL)) \) is finitely generated.

**Remark 2.7.** If there exists some effective \( \mathbb{Q} \)-divisor \( D \sim \mathbb{Q} − (K_X + \Delta) \) such that \((X, \Delta + D)\) is lc and \( A_{X, \Delta + D}(E) = 0 \) (in this case we say that \( E \) is an lc place of complement), then \( E \) is dreamy. In fact, \((X, \Delta + (1 − \epsilon)D)\) is log Fano for some \( 0 < \epsilon \ll 1 \) and \( 0 < A_{X, \Delta + (1−\epsilon)D}(E) \), thus by [BCHM10, Corollary 1.4.3] there exists a proper birational morphism \( \pi: Y \to X \) with unique exceptional divisor \( E \). Using the crepant pullback of \((X, \Delta + (1−\epsilon)D)\), it is not hard to see that \( Y \) is of Fano type and hence the graded algebra \( \bigoplus_{m \in \mathbb{N}} \mathcal{F}^1_E H^0(Y, \mathcal{O}_Y(m\pi^*L−iE)) \) is finitely generated by [BCHM10, Corollary 1.3.2].
Proposition 2.8. Let \((X, \Delta)\) be a log Fano pair with an action of an algebraic group \(G\). Assume that \((X, \Delta)\) is \(G\)-equivariantly K-semistable. Then \(A_{X,\Delta}(E) \geq S(E)\) for any \(G\)-invariant geometrically irreducible dreamy divisor \(E\) over \(X\).

Proof. By [Fuj19] Theorem 5.2 and Section 6], the divisor \(E\) induces a test configuration \((\mathcal{X}, \Delta_{tc}; \mathcal{L})\) of \((X, \Delta)\) with geometrically integral central fiber such that
\[
\text{Fut}(\mathcal{X}, \Delta_{tc}; \mathcal{L}) = A_{X,\Delta}(E) - S(E).
\]
Since \(E\) is \(G\)-invariant, the test configuration is \(G\)-equivariant, hence \(\text{Fut}(\mathcal{X}, \Delta_{tc}; \mathcal{L}) \geq 0\) as \((X, \Delta)\) is \(G\)-equivariantly K-semistable and the result follows. \(\square\)

Finally we define the notion of optimal destabilizing centers.

Definition 2.9. Let \((X, \Delta)\) be a log Fano pair. A valuation \(v \in \text{Val}_X\) is called a \(\delta\)-minimizing (resp. destabilizing) valuation of \((X, \Delta)\) if
\[
\delta(X, \Delta) = \frac{A_{X,\Delta}(v)}{S(v)} \quad \text{resp.} \quad \frac{A_{X,\Delta}(v)}{S(v)} < 1.
\]
A subvariety \(Z \subseteq X\) is called a \(\delta\)-minimizing (resp. destabilizing) center of \((X, \Delta)\) if there exists a \(\delta\)-minimizing (resp. destabilizing) valuation \(v\) of \((X, \Delta)\) with center \(Z\). When \(\delta(X, \Delta) < 1\), a \(\delta\)-minimizing valuation (resp. center) of \((X, \Delta)\) is also called an optimal destabilizing valuation (resp. center).

Remark 2.10. By the proof of [BLX19 Theorem 4.5] (which doesn’t require the base field to be algebraically closed), optimal destabilizing valuations (resp. centers) exist on every log Fano pair \((X, \Delta)\) with \(\delta(X, \Delta) < 1\). In general, \(\delta\)-minimizing valuations (resp. centers) are only known to exist over an uncountable base field by the generic limit argument of [Bj20 Theorem E].

2.4. Graded sequence of ideals. A graded sequence of ideals (see [JM12] for a general discussion) on a variety is a sequence of ideals \(a_\bullet = (a_m)_{m \in \mathbb{N}}\) such that \(a_m \cdot a_n \subseteq a_{m+n}\) for all \(m, n \in \mathbb{N}\). As a typical example, every valuation \(v \in \text{Val}_X\) gives rise to a graded sequence of ideals \(a_\bullet(v)\) as follows: let \(U \subseteq X\) be an affine open set; we set \(a_m(v)(U) := \{ f \in \mathcal{O}_X(U) \mid v(f) \geq m \}\) if \(v\) has a center in \(U\) and otherwise \(a_m(v)(U) := \mathcal{O}_X(U)\).

Given \(v \in \text{Val}_X\) and a graded sequence of ideals \(a_\bullet\), we can evaluate \(a_\bullet\) along \(v\) by setting
\[
v(a_\bullet) = \inf_{m \in \mathbb{N}^*} \frac{v(a_m)}{m}.
\]
Let \((X, \Delta)\) be a sub-pair (i.e. \(K_X + \Delta\) is \(\mathbb{Q}\)-Cartier but \(\Delta\) is not necessarily effective), let \(a_\bullet\) be a graded sequence of ideals and let \(\lambda \in \mathbb{R}\). The pair \((X, \Delta + a_\lambda\bullet)\) is said to be klt (resp. lc) if \(A_{X,\Delta}(v) > \lambda \cdot v(a_\bullet)\) (resp. \(A_{X,\Delta}(v) \geq \lambda \cdot v(a_\bullet)\)) for all \(v \in \text{Val}_X^\lambda\). Note that if \((X, \Delta + a_\lambda^\bullet)\) is klt (resp. lc) for some \(m \in \mathbb{N}^*\) then the same is true for \((X, \Delta + a_\lambda\bullet)\). A subvariety \(Z \subseteq X\) is called a non-klt center of \((X, \Delta + a_\lambda\bullet)\) if there exists a valuation \(v \in \text{Val}_X^\lambda\) with center \(Z\) such that \(A_{X,\Delta}(v) \leq \lambda \cdot v(a_\lambda\bullet)\) (equivalently, \(A_{X,\Delta}(v) \leq \lambda \cdot \frac{v(a_m)}{m}\) for all \(m \in \mathbb{N}^*\)).

Lemma 2.11. Let \((X, \Delta)\) be a sub-pair, let \(a_\bullet\) be a graded sequence of ideals on \(X\) and let \(\lambda \in \mathbb{R}\). Assume that \(\bigcap_{m=1}^{\infty} \text{Nklt}(X, \Delta + a_\lambda^{\lambda/m}) \neq \emptyset\) and let \(S\) be an irreducible component
of the intersection. Then there exists a valuation \( v \in \text{Val}_X^* \) with center \( S \) such that
\[
\lambda \cdot v(a_\bullet) \geq A_{X,\Delta}(v).
\]
In other words, \( S \) is a non-klt center of \((X, \Delta + a_\bullet^\lambda)\).

Proof. This essentially follows from [JM12]. The hypothesis and statement are unaffected by taking log resolution and crepant pullbacks etc. so we may assume that \( S \) is a point and \((X, \Delta + a_\lambda^\lambda/m)\) is klt away from \( S \). In particular, \( D = -|\Delta| \geq 0 \). For simplicity, we further assume that \( \{\Delta\} = \emptyset \) (since the result and argument of [JM12] easily extends to the log smooth case). Recall that the multiplier ideal (see e.g. [Laz04]) \( \mathcal{J}(a_\lambda^\lambda/m) \) consists of those \( f \in \mathcal{O}_X \) such that
\[
\text{ord}_E(f) > \frac{\lambda}{m} \text{ord}_E(a_m) - A_X(E)
\]
for all divisors \( E \) over \( X \). Let \( \mathcal{J}(a_\bullet) = \cup_{m \in \mathbb{N}^*} \mathcal{J}(a_\lambda^\lambda/m) \) which equals \( \mathcal{J}(a_\lambda^\lambda/m) \) for sufficiently large \( m \). By assumption, we also have \( A_X(E) + \text{ord}_E(D) \leq \frac{\lambda}{m} \text{ord}_E(a_m) \) for some divisor \( E \) as \((X, -D + a_\lambda^\lambda/m)\) is not klt. Thus \( \mathcal{O}_X(-D) \not\subseteq \mathcal{J}(a_\lambda^\lambda/m) \) for all \( m \in \mathbb{N}^* \) and therefore
\[
\mathcal{O}_X(-D) \not\subseteq \mathcal{J}(a_\bullet).
\]
In the notation of [JM12] Section 1.4] this means \( \text{mult}_{\mathcal{O}_X(-D)}(a_\bullet) \leq \lambda \) and by [JM12] Theorem 7.3], there exists a valuation \( v \in \text{Val}_X^* \) such that
\[
\lambda \cdot v(a_\bullet) \geq A_X(v) + v(D) = A_{X,\Delta}(v).
\]
Since \((X, \Delta + a_\bullet^\lambda)\) is klt outside \( S \), the valuation \( v \) is necessarily centered at \( S \). This completes the proof. \( \square \)

3. Optimal destabilizing centers

In this section, we prove the existence and uniqueness of minimal optimal destabilizing centers (Definition 2.1) of a K-unstable log Fano pair.

Theorem 3.1. Let \((X, \Delta)\) be a log Fano pair with \( \delta(X, \Delta) < 1 \). Then there exists a (necessarily unique) subvariety \( Z \subseteq X \) such that
\begin{enumerate}
\item \( Z \) is an optimal destabilizing center of \((X, \Delta)\) and
\item \( Z \) is contained in any other optimal destabilizing center of \((X, \Delta)\).
\end{enumerate}

Note that the result is false for \( \delta \)-minimizing centers in general if we allow \((X, \Delta)\) to be K-semistable: for example, every closed point on \( \mathbb{P}^n \) is a \( \delta \)-minimizing center.

As we will see in the proof, optimal destabilizing centers behave like non-klt centers of a graded sequence of ideal. For this reason we need the following property of such non-klt centers.

Lemma 3.2. Let \((X, \Delta)\) be a pair, let \( (a_i)_{i \in \mathbb{N}^*} \) be a graded sequence of ideals and let \( \lambda \in \mathbb{R}_{\geq 0} \). Let \( Z_1, Z_2 \subseteq X \) be non-klt centers of \((X, \Delta + a_\lambda^\lambda)\). Assume that \( Z_1 \cap Z_2 \neq \emptyset \) and \((X, \Delta + a_\lambda^\lambda)\) is klt outside \( Z_1 \cup Z_2 \). Then \( Z_1 \cap Z_2 \) is a union of non-klt centers of \((X, \Delta + a_\lambda^\lambda)\).
Proof. We may assume that $Z_1 \not\subseteq Z_2$ and $Z_2 \not\subseteq Z_1$, otherwise there is nothing to prove. Passing to an open neighbourhood of a generic point of $Z_1 \cap Z_2$, we may assume that $X$ is affine and $Z := Z_1 \cap Z_2$ is irreducible. Let $\pi : Y \to X$ be a log resolution of $(X, \Delta)$, let $W$ (resp. $W_i$, $i = 1, 2$) be the union of $\pi$-exceptional divisors whose centers are contained in $Z_1 \cap Z_2$ (resp. contained in $Z_i$ but not in $Z_1 \cap Z_2$). Passing to a higher resolution, we may assume that $W_1$ is disjoint from $W_2$ (this can be achieved by blowing up strata in $W_1 \cap W_2$ until they don’t intersect). Let $(Y, \Delta_Y)$ be the crepant pullback of $(X, \Delta)$, i.e. $K_Y + \Delta_Y = \pi^*(K_X + \Delta)$. Note that $\Delta_Y$ is not necessarily effective. For every integer $m > 0$, we also let $V_m = \mathrm{Nklt}(Y, \Delta_Y + \pi^* a^{\lambda/m}_0)$. Since $a_\bullet$ is a graded sequence of ideals, we have $V_{m\ell} \subseteq V_m$ for all $m, \ell \in \mathbb{N}^*$. In particular $V_1 \supseteq V_2 \supseteq \cdots \supseteq V_{2^m} \supseteq \cdots$ and thus we have $V_{2^m} = V_{2^{m+1}} = \cdots = V$ for all sufficiently large $m$. By Kollár-Shokurov connectedness theorem [K+92, Theorem 17.4] (applied to the pair $(Y, \Delta_Y + \lambda/m \pi^* D)$ over $X$, where $D = \{ f = 0 \}$ and $f \in a_m$ is a general member), we know that $V_m$ is connected over the generic point of $Z$ and thus $V$ is connected over the generic point of $Z$ as well. By assumption and [JM12] Theorem A, $(X, \Delta + a^{\lambda/m}_0)$ is klt outside $Z$ for sufficiently large $m$ but fail to be klt along either $Z_i$ for any $m \in \mathbb{N}^*$. It follows that $V \subseteq W \cup W_1 \cup W_2$ and $V \cap W_i \neq \emptyset$ for both $i = 1, 2$. Since $W_1$ is disjoint from $W_2$, we deduce that $V \cap W \neq \emptyset$ (over the generic point of $Z$) and hence any irreducible component $S$ of $V \cap W$ that dominates $Z$ is a non-klt center of $(Y, \Delta_Y + f^* a_{\bullet})$ by Lemma 2.11. Since $\pi(S) = Z$ by construction and $(Y, \Delta_Y + f^* a_{\bullet})$ is the crepant pullback of $(X, \Delta + a_{\bullet})$, we conclude that $Z$ is a non-klt center of $(X, \Delta + a_{\bullet})$. □

Our proof of Theorem 3.1 now divides into two steps. We first show that the intersection of two optimal destabilizing centers is a union of optimal destabilizing centers (see [Kol13] for similar properties of lc centers).

Lemma 3.3. Let $(X, \Delta)$ be a log Fano pair and let $Z_1, Z_2$ be $\delta$-minimizing (resp. destabilizing) centers of $(X, \Delta)$. Then $Z_1 \cap Z_2$ is a union of $\delta$-minimizing (resp. destabilizing) centers.

Before giving the proof, let us recall a definition from [AZ20].

Definition 3.4 (c.f. [AZ20, Definition 1.5]). Let $(X, \Delta)$ be a log Fano pair and let $v \in \mathrm{Val}_X$. Let $m \in \mathbb{N}$ and let $D$ be an $m$-basis type $\mathbb{Q}$-divisor of $L = -(K_X + \Delta)$, i.e. there exists a basis $s_1, \cdots, s_{N_m}$ of $H^0(X, mL)$, where $N_m = h^0(X, mL)$, such that

$$D = \frac{1}{mN_m} \sum_{i=1}^{N_m} \{ s_i = 0 \}.$$  

We say that $D$ is compatible with $v$ if $F^\lambda_v H^0(X, mL)$ is spanned by some $s_i$ for every $\lambda \in \mathbb{R}$. It is not hard to see from the definition that $S_m(v) = v(D)$ for any $m$-basis type $\mathbb{Q}$-divisor $D$ that’s compatible with $v$.

Proof of Lemma 3.3. We only prove the lemma for $\delta$-minimizing centers since the argument is similar for destabilizing centers. By assumption, there exist valuations $v_1, v_2 \in \mathrm{Val}_X$ with centers $Z_1, Z_2$ such that

$$\delta(X, \Delta) = \frac{A_{X, \Delta}(v_1)}{S(v_1)} = \frac{A_{X, \Delta}(v_2)}{S(v_2)}.$$

(3.1)
Up to rescaling, we may assume that $A_{X,\Delta}(v_1) = A_{X,\Delta}(v_2) = 1$. For each integer $m > 0$, let
\[ a_m := a_m(v_1) \cap a_m(v_2) \subseteq \mathcal{O}_X \]
where $a_m(v_i)$ are the valuation ideals of $v_i$. Then $a_\bullet$ is a graded sequence of ideals with co-support $Z_1 \cup Z_2$; in particular, $(X, \Delta + a_m)$ is klt outside $Z_1 \cup Z_2$ for any $\lambda > 0$. Since $v_i(a_\bullet) \geq 1 = A_{X,\Delta}(v_i)$ ($i = 1, 2$), we see that both $Z_i$ are non-klt centers of $(X, \Delta + a_m)$, thus by Lemma [3.2] for any irreducible component $Z$ of $Z_1 \cap Z_2$, there exists a valuation $v \in \text{Val}_X$ with center $Z$ such that $v(a_\bullet) \geq A_{X,\Delta}(v)$. As before we may assume that $A_{X,\Delta}(v) = 1$. Then for any $f \in \mathcal{O}_{X,Z}$ we have $v(f) \geq \lambda$ if $v_i(f) \geq \lambda$ for both $i = 1, 2$; in other words, $v(f) \geq \min\{v_1(f), v_2(f)\}$. We claim that $v$ is a $\delta$-minimizing valuation of $(X, \Delta)$. To see this, let $m$ be a sufficiently divisible integer and let $D$ be an $m$-basis type $\mathbb{Q}$-divisor of $(X, \Delta)$ that’s compatible with both $v_i$ (which exists by [AZ20, Lemma 3.1]). Then we have $v_i(D) = S_m(v_i)$ ($i = 1, 2$) and thus $S_m(v) \geq v(D) \geq \min\{S_m(v_1), S_m(v_2)\}$. Letting $m \to \infty$ we obtain $S(v) \geq S(v_1) = S(v_2) = \frac{A_{X,\Delta}(v)}{S(v)}$ where the equalities follow from (3.1). It follows that $\delta(X, \Delta) \geq \frac{1}{S(v)} = \frac{A_{X,\Delta}(v)}{S(v)}$ but we always have $\delta(X, \Delta) \leq \frac{A_{X,\Delta}(v)}{S(v)}$ by definition; therefore $v$ is a $\delta$-minimizing valuation as desired.

The next step is to show that optimal destabilizing centers intersect with each other using Kollár-Shokurov’s connectedness theorem.

**Lemma 3.5.** Let $Z_1, Z_2$ be optimal destabilizing centers of a $K$-unstable log Fano pair $(X, \Delta)$. Then $Z_1$ has nonempty intersection with $Z_2$.

**Proof.** Suppose that $Z_1 \cap Z_2 = \emptyset$, we will derive a contradiction. Let $r$ be a sufficiently large and divisible integer such that $\mathcal{I}_{Z_1 \cup Z_2} \otimes \mathcal{O}_X(-r(K_X + \Delta))$ is globally generated (where $\mathcal{I}_{Z_1 \cup Z_2}$ denotes the ideal sheaf of $Z_1 \cup Z_2$) and fix some $\epsilon > 0$. Then we have $v_i(D) = S_m(v_i)$, hence by (3.2) we get $A_{X,\Delta + \delta_m D + \epsilon H}(v_i) < 0$ (where $\delta_m := \delta_m(X, \Delta)$). By the definition of $\delta_m$, we also know that $(X, \Delta + \delta_m D)$ is lc, therefore as $H$ is general, we see that $(X, \Delta + (1 - \gamma)\delta_m D + \epsilon H)$ is klt away from $Z_1 \cup Z_2$ while
\[ A_{X,\Delta + (1-\gamma)\delta_m D + \epsilon H}(v_i) < 0 \]
for all $0 < \gamma \ll 1$. In other words, $\text{Nkl}(X, \Delta + (1 - \gamma)\delta_m D + \epsilon H) = Z_1 \cup Z_2$. But as $-(K_X + \Delta + (1 - \gamma)\delta_m D + \epsilon H) \sim_{\mathbb{Q}} -(1 - (1 - \gamma)\delta_m - r\epsilon)(K_X + \Delta)$ is ample (by our choice of $m$ and $\epsilon$) and $Z_1$ is disjoint from $Z_2$, this contradicts Kollár-Shokurov’s connectedness theorem [K-S2, Theorem 17.4].

We are ready to prove Theorem 3.1.

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Proof of Theorem 3.1. Let \( Z \subseteq X \) be a minimal (with respect to inclusion) optimal destabilizing center of \((X, \Delta)\). We claim that it satisfies the statement of the theorem. Let \( Z' \) be another optimal destabilizing center of \((X, \Delta)\). By Lemma 3.5, \( Z' \) has nonempty intersection with \( Z \); on the other hand, by Lemma 3.3, \( Z \cap Z' \) is a union of optimal destabilizing centers of \((X, \Delta)\). Since \( Z \) is minimal with respect to inclusion, this implies \( Z \subseteq Z' \), which completes the proof. \( \Box \)

Example 3.6. From the uniqueness it is clear that the minimal optimal destabilizing center constructed in Theorem 3.1 is invariant under the automorphism group of \((X, \Delta)\). This helps us to identify the center in many cases. For example, if \( X \) is the blowup of one or two points on \( \mathbb{P}^2 \), then there is a unique \( \text{Aut}(X) \)-invariant \((-1)\)-curve on \( X \) which is necessarily the minimal optimal destabilizing center.

4. Equivariant K-stability

In this section, we show that to compute the stability threshold of a geometrically K-unstable log Fano pair, it is enough to use divisors that are defined over the base field and invariant under the automorphism group. It will imply all remaining results mentioned in the introduction.

Theorem 4.1. Let \((X, \Delta)\) be a log Fano pair and let \( G = \text{Aut}(X, \Delta) \). Assume that \((X_\bar{k}, \Delta_\bar{k})\) is not K-semistable. Then we have

\[
\delta(X_\bar{k}, \Delta_\bar{k}) = \inf_E \frac{A_X(\Delta)(E)}{S(E)}
\]

where the infimum runs over all \( G \)-invariant geometrically irreducible divisors \( E \) over \( X \) that are lc places of complements.

We remark that as in Theorem 3.1, the statement fails in general if \((X_\bar{k}, \Delta_\bar{k})\) is K-semistable; for example, \( \mathbb{P}^n \) does not even have \( \text{PGL}_{n+1} \)-invariant divisors over it.

We will prove Theorem 4.1 by induction on the dimension. However, for the induction to work, we need to slightly generalize the context and we are naturally led to consider boundaries of the following form. Similar objects have already appeared in the work [AZ20] as we refine a linear series by a divisor.

Definition 4.2. Let \((X, \Delta)\) be a pair. A boundary on \( X \) is a linear combination \( V = a_1V_1 + \cdots + a_rV_r \) where \( a_i \in \mathbb{Q}_+ \) and \( V_i \) are finite dimensional linear series associated to \( \mathbb{Q} \)-Cartier divisors (i.e. there exist some effective \( \mathbb{Q} \)-Cartier divisors \( L_i \) on \( X \) such that \( V_i \subseteq H^0(X, L_i) \) is a finite dimensional subspace). It’s divisor class \( c_1(V) \in \text{NS}(X)_{\mathbb{Q}} \) is defined in the obvious way. We say \( V \) is \( G \)-invariant (where \( G \subseteq \text{Aut}(X) \) is a subgroup) if the linear series \( V_i \) are all \( G \)-invariant. A filtration \( \mathcal{F} \) of \( V \) is given by a collection of subspaces \( \mathcal{F}^\lambda V_i \subseteq V_i \) for each \( \lambda \in \mathbb{R} \) and \( i = 1, \cdots, r \) such that

1. \( \mathcal{F}^\lambda V_i \subseteq \mathcal{F}^{\lambda'} V_i \) whenever \( \lambda \geq \lambda' \),
2. \( \mathcal{F}^\lambda V_i = V_i \) for \( \lambda \ll 0 \) and \( \mathcal{F}^\lambda V_i = 0 \) for \( \lambda \gg 0 \).

As a typical example, every valuation \( v \in \text{Val}_X \) induces a filtration \( \mathcal{F}_v \) on \( V \). A basis type \( \mathbb{Q} \)-divisor of \( V \) is a divisor of the form \( D = a_1D_1 + \cdots + a_rD_r \) where each \( D_i \sim_{\mathbb{Q}} L_i \) is a
basis type $\mathbb{Q}$-divisor of $V_i$, i.e.

$$D_i = \frac{1}{\dim V_i} \sum_{j=1}^{\dim V_i} \{s_j = 0\}$$

where the $s_j$ form a basis of $V_i$. We say that $D$ is compatible with a filtration $\mathcal{F}$ on $V$ if every $\mathcal{F}^s V_i$ is spanned by some $s_j$ in the above expression. In particular, we say that $D$ is compatible with a divisor $E$ over $X$ if it is compatible with the filtration $\mathcal{F}_{\text{ord}_E}$.

**Definition 4.3.** Let $U$ be a quasi-projective variety and let $f: (X, \Delta) \to U$ be a klt pair that’s projective over $U$. Let $V$ be a boundary on $X$, let $E$ be a divisor over $X$ and let $Z \subseteq X$ be a subvariety. We set

$$S(V; E) := \sup_D \text{ord}_E(D) = \max_D \text{ord}_E(D),$$

$$\delta_Z(V) = \delta_Z(X, \Delta; V) := \inf_D \text{lct}_Z(X, \Delta; D) = \min_D \text{lct}_Z(X, \Delta; D) \in \mathbb{R} \cup \{+\infty\}$$

where the supremum or infimum runs over all basis type $\mathbb{Q}$-divisors $D$ of $V$. Clearly $\delta_Z(V) < +\infty$ if and only if there exists some $1 \leq i \leq r$ and some $s \in V_i$ such that $Z \subseteq \{s = 0\}$. Note also that as the linear series $V_i$ are finite dimensional, the set of basis type $\mathbb{Q}$-divisors is bounded, hence the above supremum (resp. infimum) is a maximum (resp. minimum) by the constructibility of $\text{ord}_E$ (resp. log canonical thresholds) in family.

It is easy to see that

$$\delta_Z(V) = \delta_Z(X, \Delta; V) := \inf_E \frac{A_{X, \Delta}(E)}{S(V; E)} = \min_E \frac{A_{X, \Delta}(E)}{S(V; E)}$$

where the infimum runs over all divisors $E$ over $X$ whose center contains $Z$. If $Y \subseteq U$ is a subvariety, we also define the $\delta$-invariant of $V$ (with respect to $(X, \Delta)$) over $Y$ to be

$$\delta(V/Y) = \delta(X, \Delta; V/Y) := \inf_Z \delta_Z(V)$$

where the infimum runs over all subvarieties $Z \subseteq X$ whose image in $U$ contains $Y$. In other words, we only take the log canonical thresholds of basis type divisors along the fiber of $f$ over the generic point of $Y$. Let $G$ be an algebraic group. A $G$-action on $(X, \Delta)$ over $U$ is given by a $G$-action on both $(X, \Delta)$ and $U$ making the projection $f$ equivariant.

We now state a technical result which will imply Theorem 4.1. It is specifically designed for inductive purpose and will eventually be applied to the various complete linear series $|\mp m(K_X + \Delta)|$ of a K-unstable log Fano pair $(X, \Delta)$ to show that all $\delta_m(X, \Delta; \kappa) (m \gg 0)$ are computed by an $\text{Aut}(X, \Delta)$-invariant geometrically irreducible divisor over $X$.

**Theorem 4.4.** Let $G$ be an algebraic group, let $f: (X, \Delta) \to U$ be a klt pair that’s projective over $U$ with a $G$-action, let $Y \subseteq U$ be a $G$-invariant geometrically irreducible subvariety and let $V$ be a $G$-invariant boundary on $X$. Assume that $\delta(V_k/Y_k) < +\infty$ and that $-(K_X + \Delta + \delta(V_k/Y_k)c_1(V))$ is $f$-ample. Then there exists some $G$-invariant geometrically irreducible divisor $E$ over $X$ whose center dominates $Y$ such that

$$\delta(V_k/Y_k) = \frac{A_{X, \Delta}(E)}{S(V; E)}.$$
We divide the proof of Theorem 4.4 into several steps. Using ideas from the previous section, we first show that the it is enough to calculate the geometric $\delta$-invariant at some $G$-invariant geometrically irreducible center.

**Lemma 4.5.** Under the assumptions of Theorem 4.4, there exists some $G$-invariant geometrically irreducible subvariety $Z \subseteq X$ such that $Y \subseteq f(Z)$ and $\delta(V_k/Y_k) = \delta_{Z_k}(V_k)$.

**Proof.** Let $\delta = \delta(V_k/Y_k)$ and let $\eta$ be the generic point of $Y_k$. By definition, we have $\operatorname{lct}(X_k, \Delta_k; D) = \delta$ in a neighbourhood of $X_\eta$ for some $m$-basis type $\mathbb{Q}$-divisors $D$ of $V_k$. Among such $D$ we choose one (call it $D_0$) such that the minimal lc center (denoted by $Z$) of $(X_k, \Delta_k + D_0)$ that intersects $X_\eta$ has the smallest dimension. Clearly $\delta_{Z_k}(V_k) = \delta$. It remains to show that $Z$ can be defined over $k$ and is $G$-invariant. To this end, let $E_0$ be an lc place of $(X_k, \Delta_k + D_0)$ with $C_{X_k}(E_0) = Z$, let $g \in \operatorname{Gal}(\bar{k}/k) \times G(\bar{k})$ and let $E_1 = g(E_0)$ (viewed as a divisor over $X_k$). Since $V$ is $G$-invariant, we get $A_{X_k, \Delta_k}(E_0) = A_{X_k, \Delta_k}(E_1)$ and $S(V_k; E_0) = S(V_k; E_1)$. By [AZ20, Lemma 3.1], we can choose a basis type $\mathbb{Q}$-divisor $D$ of $V_k$ that’s compatible with both $E_0$ and $E_1$. In particular, we have $\operatorname{ord}_{E_i}(D) = S(V_k; E_i)$ for both $i = 0, 1$. By our choice of $E_i$ and the definition of $\delta(V_k/Y_k)$, we see that (in a neighbourhood of $X_\eta$)

$$\delta = \frac{A_{X_k, \Delta_k}(E_i)}{S(V_k; E_i)} = \frac{A_{X_k, \Delta_k}(E_i)}{\operatorname{ord}_{E_i}(D)} \geq \operatorname{lct}(X_k, \Delta_k; D) \geq \delta$$

for $i = 0, 1$. It follows that $\operatorname{lct}(X_k, \Delta_k; D) = \delta$ along $X_\eta$ and is computed by both divisors $E_i$. Clearly $g(Z) = C_{X_k}(E_1)$. Suppose that $C_{X_k}(E_0) \neq C_{X_k}(E_1)$. After possibly shrinking to an open neighbourhood of $\eta \in U$, we may assume that $U$ is affine (we don’t need the $G$-action any more), $(X_k, \Delta_k + D)$ is lc and $C_{X_k}(E_i)$ are minimal lc centers of $(X_k, \Delta_k + D)$ (otherwise the dimension of minimal lc center can be made smaller), hence $C_{X_k}(E_0)$ is disjoint from $C_{X_k}(E_1)$ by [Ko13, Corollary 4.41]. Let $H \sim_{\mathbb{Q}} -(K_{X_k} + \Delta_k + \delta c_1(V_k))$ be general among effective divisors that contains $C_{X_k}(E_0) \cup C_{X_k}(E_1)$ in its support (it is enough to ensure that $\operatorname{Supp}(H)$ doesn’t contain other lc centers of $(X_k, \Delta_k + D)$). Then for $0 < \epsilon \ll \gamma \ll 1$, we have

$$\operatorname{Nklt}(X_k, \Delta_k + (1 - \epsilon)\delta D + \gamma H) = C_{X_k}(E_0) \cup C_{X_k}(E_1)$$

and

$$-(K_{X_k} + \Delta_k + (1 - \epsilon)\delta D + \gamma H) \sim_{\mathbb{Q}} -(1 - \gamma)(K_{X_k} + \Delta_k + \delta D) + \epsilon \delta D$$

is ample. This contradicts Kollár-Shokurov’s connectedness theorem [K+92, Theorem 17.4]. Hence we must have $Z = C_{X_k}(E_0) = C_{X_k}(E_1) = g(Z)$. As $g \in \operatorname{Gal}(\bar{k}/k) \times G(\bar{k})$ is arbitrary, we see that $Z$ is defined over $k$ and $G$-invariant.

Given this $G$-invariant center, we would like to find a $G$-invariant geometrically irreducible divisor $E$ that computes the ‘$G$-equivariant’ $\delta$-invariant. A priori, there are two ways to define such equivariant $\delta$-invariants: one using $G$-invariant geometrically irreducible divisors over $X$ and the other using $G$-invariant filtrations of the linear series. The second version is more suitable for induction but it does not see the $G$-invariant divisor over $X$ directly. Therefore, the second step in our proof is to compare these two definitions and show that they are actually equivalent. To state the result, we need some more definitions.
Let $V = \sum a_i V_i$ be a boundary on $X$. For any filtration $\mathcal{F}$ of $V$, we define $a(\mathcal{F})$ to be the $\mathbb{Q}$-ideal given by
\[
a(\mathcal{F}) := \prod_{i, \lambda} b(\mathcal{F}^i V_i) \frac{a_i^{\dim \text{Gr}^i V_i}}{\dim \text{Gr}^i V_i},
\]
where $b(\mathcal{F}^i V_i)$ denotes the base ideal of the linear series $\mathcal{F}^i V_i$.

**Definition 4.6.** Let $G$ be an algebraic group, let $(X, \Delta)$ be a quasi-projective pair with a $G$-action and let $V$ be a $G$-invariant boundary on $X$. Let $Z \subseteq X$ be a $G$-invariant geometrically irreducible subvariety such that $(X, \Delta)$ is klt along the generic point of $Z$. We set
\[
(4.2) \quad \tilde{\delta}_{Z,G}(V) = \tilde{\delta}_{Z,G}(X, \Delta; V) := \inf_{\mathcal{F}} \text{lct}(X, \Delta; a(\mathcal{F}))
\]
where the infimum runs over all $G$-invariant filtrations $\mathcal{F}$ on $V$. We also define
\[
(4.3) \quad \delta_{Z,G}(V) = \delta_{Z,G}(X, \Delta; V) := \inf_{E} \frac{A_{X,\Delta}(E)}{S(V; E)}
\]
where the infimum runs over all $G$-invariant geometrically irreducible divisors $E$ over $X$ whose center contains $Z$.

**Lemma 4.7.** In the above notation, we have $\delta_{Z,G}(V) = \tilde{\delta}_{Z,G}(V)$. Moreover, the infimum in (4.3) is achieved by some $G$-invariant divisor over $X$ that’s of plt type at the generic point of $Z$.

Recall that given a klt pair $(X, \Delta)$, a divisor $E$ over $X$ is said to be of plt type at some $x \in C_X(E)$ if there exists an open neighbourhood $U \subseteq X$ of $x$ and a proper birational morphism $\phi: Y \to U$ such that $E$ is a geometrically irreducible divisor on $Y$, $(Y, \Delta_Y + E)$ is plt (where $\Delta_Y$ is the strict transform of $\Delta$) and $-E$ is $\phi$-ample. The map $\phi: Y \to U$ is called the associated plt blowup.

**Proof.** Let $E$ be a $G$-invariant geometrically irreducible divisor over $X$ whose center contains $Z$. Such divisor always exists by the following Lemma 4.8 (e.g. consider divisors that computes $\text{lct}(X, \Delta; \mathcal{I}_Z)$ where $\mathcal{I}_Z$ denotes the ideal sheaf of $Z$). Then it induces a $G$-invariant filtration $\mathcal{F} = \mathcal{F}_{\text{ord}_E}$ on $V$. It is not hard to see from the definition that $S(V; E) = \text{ord}_E(a(\mathcal{F}))$, thus
\[
\tilde{\delta}_{Z,G}(V) \leq \text{lct}(X, \Delta; a(\mathcal{F})) \leq \frac{A_{X,\Delta}(E)}{\text{ord}_E(a(\mathcal{F}))} = \frac{A_{X,\Delta}(E)}{S(V; E)}.
\]
Taking the infimum over all such $E$ we obtain $\delta_{Z,G}(V) \geq \tilde{\delta}_{Z,G}(V)$. It remains to prove the reverse inequality. To this end, let $\mathcal{F}$ be a $G$-invariant filtrations on $V$ that achieves the infimum in (4.2); this is possible since such filtrations are parametrized by a closed subset of a flag variety and let is constructible in family. If $Z$ is not contained in the co-support of $a(\mathcal{F})$ then $\tilde{\delta}_{Z,G}(V) = \text{lct}(X, \Delta; a(\mathcal{F})) = +\infty$ and there is nothing to prove as clearly $\delta_{Z,G}(V) \geq \tilde{\delta}_{Z,G}(V)$. Thus we may assume that $Z \subseteq \text{Cosupp}(a(\mathcal{F}))$. By the following Lemma 4.8, $\text{lct}(X, \Delta; a(\mathcal{F}))$ is computed by some $G$-invariant divisor $E$ over $X$ that’s of plt type at the generic point of $Z$. Let $m \gg 0$, let $D_1, \ldots, D_m$ be general basis type $\mathbb{Q}$-divisors of $V$ that are compatible with $\mathcal{F}$, and let $D = \frac{1}{m}(D_1 + \cdots + D_m)$. 

Then \( \text{lct}(X, \Delta; a(F)) = \text{lct}_Z(X, \Delta; D) \) and we have \( S(V; E) \geq \text{ord}_E(D) \) since \( S(V; E) \geq \text{ord}_E(D_i) \) for each \( i = 1, \cdots, m \). It follows that

\[
\delta_{Z,G}(V) = \text{lct}(X, \Delta; a(F)) = \text{lct}_Z(X, \Delta; D) = \frac{A_{X,\Delta}(E)}{\text{ord}_E(D)} \geq \frac{A_{X,\Delta}(E)}{S(V; E)} \geq \delta_{Z,G}(V).
\]

Combining with the inequality in the opposite direction we finish the proof. \( \square \)

The following result is used in the above proof.

**Lemma 4.8.** Let \( G \) be an algebraic group and let \( (X, \Delta) \) be a klt pair endowed with a \( G \)-action. Let \( a \subseteq O_X \) be a \( G \)-invariant \( \mathbb{Q} \)-ideal and let \( Z \subseteq \text{Cosupp}(a) \) be a \( G \)-invariant geometrically irreducible subvariety. Then \( \text{lct}_Z(X, \Delta; a) \) is computed by some \( G \)-invariant divisor \( E \) over \( X \) that’s of plt type at the generic point of \( Z \).

**Proof.** This is quite standard and should be well known to experts (c.f. [HX09; Proof of Theorem 1.3]). We provide a proof here for reader’s convenience. For ease of notation we assume that \( \Delta = 0 \); the proof of the general case is the same. Up to shrinking \( G \)-equivariant log resolution of \( (X, a) \) containing \( Z \) (see [Kol03 Corollary 4.41]), which is necessarily defined over the base field \( k \) and \( G \)-invariant. Let \( \mathcal{I} \) be the ideal sheaf of \( W \) and let \( \pi: Y \to X \) be a common \( G \)-equivariant log resolution of \( (X, a) \) and \( (X, \mathcal{I}) \) such that

1. the exceptional locus \( \text{Ex}(\pi) \subseteq Y \) supports a \( G \)-invariant \( \pi \)-ample divisor \( A \) (note that \( -A \) is effective by the negativity lemma [Kol03 Lemma 3.39]),

2. every irreducible component of \( \text{Ex}(\pi) \cup \text{Supp}(\pi^*a) \) is smooth (i.e. it is a log resolution over \( k \)) and disjoint from its \( G \)-translates (these can be achieved by further blowing up strata of \( \text{Ex}(\pi) \cup \text{Supp}(\pi^*a) \)).

Let \( 0 < \varepsilon \ll 1 \) and write

\[
K_Y + \sum_{i=1}^p a_i(\varepsilon)E_i + \sum_{j=1}^q b_j(\varepsilon)F_j + \sum_{k=1}^r c_k(\varepsilon)G_k = \pi^*(K_X + a^{1-\varepsilon})
\]

where \( E_1, \cdots, E_p \) are prime divisors on \( Y \) with center \( W \) such that \( A_X(E_i) = \text{ord}_{E_i}(a) \), \( F_1, \cdots, F_q \) are divisors with center \( W \) such that \( A_X(E_i) > \text{ord}_{E_i}(a) \) and \( G_1, \cdots, G_r \) are divisors on \( Y \) whose centers are different from \( W \) (but may contain \( W \)). We then have \( \pi^*\mathcal{I} = O_Y(-\sum_i a'_iE_i - \sum_j b'_jF_j) \) for some \( a'_i, b'_j > 0 \). Note that \( \lim_{\varepsilon \to 0} a_i(\varepsilon) = 1 > \lim_{\varepsilon \to 0} b_j(\varepsilon) \) (\( \forall i, j \)) by construction, thus we have

\[
\text{lct}(X, a^{1-\varepsilon}; \mathcal{I}) = \min_{1 \leq i \leq p} \frac{1 - a_i(\varepsilon)}{a_i^\varepsilon} < \frac{1 - b_j(\varepsilon)}{b_j^\varepsilon} \quad (\forall j)
\]

when \( \varepsilon \) is sufficiently small. It follows that for such \( \varepsilon \), if we set \( \lambda = \text{lct}(X, a^{1-\varepsilon}; \mathcal{I}) \), then \( (X, a^{1-\varepsilon}; \mathcal{I}^\lambda) \) is lc with a unique lc center \( W \) and every lc place of \( (X, a^{1-\varepsilon}; \mathcal{I}^\lambda) \) is also a lc place of \( (X, a) \). Replacing \( a \) by \( a^{1-\varepsilon} \), we may assume that \( (X, a) \) is lc with a unique lc center \( W \). In particular, we have \( \lim_{\varepsilon \to 0} c_k(\varepsilon) < 1 \) for all \( k \) in \( \{1, \cdots, r\} \). Let \( m \) be a sufficiently large and divisible integer and let \( b = \pi_*O_Y(mA) \). Then \( \text{ord}_E(b) = \text{ord}_E(-mA) \) for any divisor \( E \) over \( X \) and as before we see that \( \text{lct}(X, a^{1-\varepsilon}; b) \) is computed by some divisor \( E_i \) when \( 0 < \varepsilon \ll 1 \). By perturbing the coefficients of \( A \) in a \( G \)-equivariant manner, we can arrange that \( \text{lct}(X, a^{1-\varepsilon}; b) \) is computed by a unique \( G \)-orbit \( G \cdot E_i \). Replacing \( a \) by
for some \( \bar{k} \). By construction, the irreducible components of \( G \cdot E_i \) over \( \bar{k} \) are disjoint from each other, thus by the Kollár-Shokurov’s connectedness theorem [K+92, Theorem 17.4], \( G \cdot E_i \) is geometrically irreducible. In other words, \((X, a)\) has a unique lc place \( E = G \cdot E_i \). By [BCHM10, Corollary 1.4.3], there exists a proper birational morphism \( \phi : \tilde{X} \to X \) that extracts \( E \) as the unique exceptional prime divisor and \( -E \) is \( \phi \)-ample (such \( \tilde{X} \) is uniquely determined by \( E \), hence is defined over the same base field \( k \)). Since \( E \) is the unique lc place of \((X, a)\), we have \( \phi^*(K_X + a) \geq K_{\tilde{X}} + E \) and \( E \) is also the unique lc place of \((\tilde{X}, E)\). Hence \((\tilde{X}, E)\) is plt. In other words, \( E \) is of plt type over \( X \). Since \( E \) is \( G \)-invariant by construction, we are done. \( \square \)

We now come to the key inductive step in our proof of Theorem 4.4. Using Lemma 4.7, we may choose a \( G \)-invariant divisor \( E \) of plt type that computes the \( G \)-equivariant \( \delta \)-invariant. Using inversion of adjunction and techniques from [AZ20], we next compare the \( G \)-equivariant \( \delta \)-invariant of \( V \) with its ‘filtered restriction’ to \( E \) (to be defined in the proof below). Roughly speaking, the consequence is that the filtered restriction of \( V \) to \( E \) is \( G \)-equivariantly K-semistable; since \( E \) has smaller dimension, we can use our inductive hypothesis to conclude that the filtered restriction is indeed geometrically K-semistable. Using inversion of adjunction and [AZ20] again but without the equivariant information this time, we conclude that the geometric \( \delta \)-invariant of the origin linear series \( V \) is also computed by \( E \). These observations lead to the following statement.

**Lemma 4.9.** Assume Theorem 4.4 for pairs of dimension \( n - 1 \). Let \( G \) be an algebraic group, let \((X, \Delta)\) be a klt pair of dimension \( n \) with a \( G \)-action, let \( V \) be a \( G \)-invariant boundary on \( X \) and let \( Z \subseteq X \) be a \( G \)-invariant geometrically irreducible subvariety. Assume that \( \delta_{Z_k}(V_k) < +\infty \). Then it is computed by some \( G \)-invariant divisor \( E \) over \( X \) that’s of plt type at the generic point of \( Z \), i.e.

\[
\delta_{Z_k}(V_k) = \frac{A_{X,\Delta}(E)}{\mathcal{S}(V; E)}.
\]

**Proof.** Since \( \delta_{Z_k}(V_k) < +\infty \), there exists some basis type \( \mathbb{Q} \)-divisor of \( V_k \) whose support contains \( Z \). Thus if we let \( \mathcal{F}_Z \) be the \( G \)-invariant filtration of \( V \) induced by \( \text{mult}_Z \), then \( Z \subseteq \text{Cosupp}(a(\mathcal{F}_Z)) \) and hence \( \delta_{Z,G}(V) < +\infty \). We may assume that \( Z \) is not a divisor in \( X \), otherwise we can clearly take \( E = Z \). By Lemma 4.7, we have

\[
\delta := \delta_{Z,G}(V) = \delta_{Z,G}(V) = \frac{A_{X,\Delta}(E)}{\mathcal{S}(V; E)}
\]

for some \( G \)-invariant divisor \( E \) over \( X \) with \( Z \subseteq C_X(E) \) that’s of plt type at the generic point of \( Z \). We will show that \( \delta_{Z_k}(V_k) \) is also computed by this divisor \( E \). Since clearly \( \delta_{Z_k}(V_k) \leq \delta \), it suffices to prove the reverse inequality.

To this end, let \( \mathcal{F}_E \) be the \((G\text{-invariant})\) filtration on \( V \) induced by \( E \). Replacing \( X \) by a \( G \)-invariant open subset that intersects \( Z \), we may assume that \( E \) is of plt type over \( X \). Let \( \pi : Y \to X \) be the associated plt blowup. Note that \( E \) is exceptional over \( X \). We define a boundary \( W \) on \( E \subseteq Y \) (the ‘filtered restriction’ of \( V \) to \( E \)) as follows: if
$V_i \subseteq H^0(X, D_i)$ is a linear series on $X$, its filtered restriction is set to be

$$W_i := \sum_{\lambda \in \mathbb{Q}} \frac{\dim \text{Gr}^\lambda_{F_E} V_i}{\dim V_i} \cdot V_i(-\lambda E)|_E$$

where $V_i(-\lambda E)$ is the linear series in $H^0(Y, \pi^* D - \lambda E)$ given by $\text{Gr}^\lambda_{F_E} V_i$; we then extend the definition to boundaries by taking linear combination. The coefficients in the above definition is chosen such that if $W$ is the filtered restriction of $V$ to $E$ and $D$ is a basis type $\mathbb{Q}$-divisor of $V$ that’s compatible with $E$, then we have $\pi^* D = S(V; E) \cdot E + \Gamma$ for some divisor $\Gamma$ whose support doesn’t contain $E$ and $\Gamma|_E$ is a basis type $\mathbb{Q}$-divisor of $W$; conversely, every basis type $\mathbb{Q}$-divisor of $W$ can be obtained in this way. In particular, $c_1(W) \sim_{\mathbb{Q}} (\pi^* c_1(V) - S(V; E) \cdot E)|_E$ is $\pi$-ample.

Next, let $F$ be a $G$-invariant geometrically irreducible divisor over $E$ whose center dominates $Z$, let $m \gg 0$, and let $D_0^{(i)} (i = 1, \cdots, m)$ be general basis type $\mathbb{Q}$-divisors of $W$ that are compatible with $F$. Let $D_0 = \frac{1}{m}(D_0^{(1)} + \cdots + D_0^{(m)})$. By construction, the filtration on $W$ induced by $F$ lifts to a ($G$-invariant) refinement $\mathcal{F}$ of the filtration $F_E$ on $V$, and $D_0$ lifts to a divisor $D$ on $X$ that’s a convex linear combination of general basis type $\mathbb{Q}$-divisors of $V$ that are compatible with $F$ (hence are also compatible with $F_E$).

We have

$$\delta = \bar{\delta}_{Z,G}(V) = \frac{A_{X,\Delta}(E)}{S(V; E)} = \frac{A_{X,\Delta}(E)}{\text{ord}_E(D)} \geq \text{lct}_Z(X, \Delta; D) = \text{lct}_Z(X, \Delta; a(F)) \geq \bar{\delta}_{Z,G}(V),$$

thus equality holds everywhere and $E$ computes $\delta = \text{lct}_Z(X, \Delta; D)$. Let $\Delta_E = \text{Diff}_E(\Delta_Y)$ be the different; as $E$ is of plt type over $X$, $(E, \Delta_E)$ is klt and $-(K_E + \Delta_E)$ is $\pi$-ample. Recall that $\pi^* D = S(V; E) \cdot E + \Gamma$ where $\Gamma|_E = D_0$. We thus have $\pi^* (K_X + \Delta + \delta D) = K_Y + \Delta_Y + E + \delta \Gamma$; hence $K_E + \Delta_E + \delta c_1(W) \sim_{\mathbb{Q}} (K_Y + \Delta_Y + E + \delta \Gamma)|_E \sim_{\mathbb{Q}} 0$ and by inversion of adjunction we deduce that $(E, \Delta_E + \delta D_0)$ is lc over the generic point of $Z$. It follows that

$$(4.5) \quad \frac{A_{E,\Delta_E}(F)}{S(W; F)} = \frac{A_{E,\Delta_E}(F)}{\text{ord}_F(D_0)} \geq \delta$$

for all $G$-invariant geometrically irreducible divisors over $E$ whose centers dominate $Z$. Suppose that

$$\delta_0 := \delta(E_k, (\Delta_E)_k; W_k/Z_k) < \delta.$$

Then $-(K_E + \Delta_E + \delta_0 c_1(W)) \sim_{\mathbb{Q}} (\delta - \delta_0)c_1(W)$ is $\pi$-ample and by Theorem 4.3 (noting that $E$ has dimension $n - 1$), there exists some $G$-invariant geometrically irreducible divisor $F$ over $E$ whose center dominates $Z$ such that

$$\frac{A_{E,\Delta_E}(F)}{S(W; F)} = \delta_0 < \delta,$$

which contradicts (4.5). Therefore, we must have $\delta(E_k, (\Delta_E)_k; W_k/Z_k) \geq \delta$. But then by [AZ20] Proof of (3.4)], we get

$$\delta_{Z_k}(X_k, \Delta_E; V_k) \geq \min \left\{ \frac{A_{X,\Delta}(E)}{S(V; E)}, \delta(E_k, (\Delta_E)_k; W_k/Z_k) \right\} \geq \delta$$

as desired.

We are now ready to prove
Proof of Theorem 4.4. We prove by induction on \( n = \dim X \). The base case \( n = 0 \) is empty. Suppose the statement has been proved in dimension \( n - 1 \). By Lemma 4.3 there exists some \( G \)-invariant geometrically irreducible subvariety \( Z \subseteq X \) dominating \( Y \) such that \( \delta(V_k/Y_k) = \delta_Z(V_k) < \infty \). By induction hypothesis and Lemma 4.9 there exists some \( G \)-invariant geometrically irreducible divisor \( E \) over \( X \) whose center contains \( Z \) such that \( \delta_Z(V_k) = \frac{A_{X,\Delta}(E)}{S(V; E)} \). Thus
\[
\delta(V_k/Y_k) = \frac{A_{X,\Delta}(E)}{S(V; E)},
\]
proving the statement in dimension \( n \).

Proof of Theorem 4.4. Let \( \epsilon > 0 \). By [B120, Corollary 3.6], we have \( S_m(v) \leq (1 + \epsilon)S(v) \) for all \( m \gg 0 \) and all \( v \in \text{Val}_X \). Let \( m \gg 0 \) be also sufficiently divisible such that \( \delta_m := \delta_m(X,\Delta) < 1 \) and let \( V_m = \frac{1}{m} - m(K_X + \Delta) \) be the complete \( \mathbb{Q} \)-linear series. Then we have \( \delta((V_m)_{\overline{k}}) = \delta_m \) by definition and hence \( -(K_X + \Delta + \delta((V_m)_{\overline{k}})c_1(V_m)) \sim_{\mathbb{Q}} - (1 - \delta_m)(K_X + \Delta) \) is ample. By Theorem 4.4 (applied to the pair \( (X, \Delta) \) over \( U = \text{point} \) with boundary \( V_m \)), we see that there exists some \( G \)-invariant geometrically irreducible divisor \( E \) over \( X \) such that
\[
\frac{A_{X,\Delta}(E)}{S_m(E)} = \frac{A_{X,\Delta}(E)}{S(V; E)} = \delta_m.
\]
Let \( D \) be an \( m \)-basis type \( \mathbb{Q} \)-divisor of \( (X, \Delta) \) that’s compatible with \( E \). Then \( E \) computes \( \delta_m = \delta_m(X,\Delta) < 1 \) by the definition of \( \delta_m \) and the above equality. It follows that \( E \) is an lc place of the complement \( \delta_mD + (1 - \delta_m)H \) where \( H \sim_{\mathbb{Q}} -(K_X + \Delta) \) is effective and general. We also have
\[
\delta(X,\Delta) \leq \frac{A_{X,\Delta}(E)}{S(E)} \leq (1 + \epsilon)\frac{A_{X,\Delta}(E)}{S_m(E)} = (1 + \epsilon)\delta_m
\]
for \( m \gg 0 \). As \( \epsilon \) is arbitrary and \( \lim_{m \to \infty} \delta_m = \delta(X,\Delta) \), the equality (4.1) follows.

Remark 4.10. Using the argument of [BLX19], one can further show that \( \delta(X,\Delta) \) is computed by some \( \text{Aut}(X,\Delta) \)-invariant quasi-monomial valuation that’s an lc place of a bounded complement defined over \( k \). We leave the details to the reader.

As in [LZ20], Theorem 4.4 implies the equivalence of equivariant K-semistability (resp. K-polystability) with geometric K-semistability (resp. K-polystability), as well as a generalization of Tian’s criterion.

Corollary 4.11. Let \( G \) be an algebraic group and let \( (X, \Delta) \) be a log Fano pair with an action of \( G \).

1. If \( (X, \Delta) \) is \( G \)-equivariantly K-semistable, then \( (X,\Delta) \) is K-semistable.
2. If \( G \) is reductive and \( (X, \Delta) \) is \( G \)-equivariantly K-polystable, then \( (X,\Delta) \) is K-polystable.

The following example shows that the reductivity of \( G \) is necessary.

Example 4.12. Let \( k = \mathbb{C} \) and let \( X^a \) be the unique Fano threefold of degree 22 whose identity component \( \text{Aut}(X^a)_0 \) of the automorphism group is isomorphic to the additive group \( \mathbb{C}^+ \). As discussed in [CS18, Example 1.4], \( X^a \) is K-semistable but not K-polystable and its K-polystable degeneration is the Mukai-Umemura threefold \( X^{MU} \).
Moreover, \( X^{\text{MU}} \) is the only nontrivial K-semistable special degeneration of \( X^a \). To see this, let \( Y \) be a nontrivial K-semistable special degeneration of \( X^a \). Then it has a faithful \( \mathbb{C}^* \)-action. If \( Y \) is not isomorphic to \( X^{\text{MU}} \), then by [LWX18 Theorems 1.4 and 3.2], \( Y \) admits a nontrivial \( \mathbb{C}^* \)-equivariant special degeneration to \( X^{\text{MU}} \), giving rise to a faithful \( (\mathbb{C}^*)^2 \)-action on \( X^{\text{MU}} \), which is impossible as \( \text{Aut}(X^{\text{MU}}) \cong \text{PGL}(2, \mathbb{C}) \). Thus \( Y \cong X^{\text{MU}} \). However, none of the special degenerations of \( X^a \) to \( X^{\text{MU}} \) is \( \mathbb{C}^* \)-equivariant: otherwise we get a faithful \( \mathbb{C}^+ \times \mathbb{C}^* \)-action on \( X^{\text{MU}} \), which is again impossible. It follows that \( X^a \) is \( \mathbb{C}^* \)-equivariantly K-polystable but not K-polystable by Theorem 2.2.

**Proof of Corollary 4.11.** We first prove the K-semistable part. Suppose that \((X_k, \Delta_k)\) is not K-semistable, i.e. \(\delta(X_k, \Delta_k) < 1\). Then by Theorem 4.11 and Remark 2.7, there exists some \(G\)-invariant geometrically irreducible dreamy divisor \(E\) over \(X\) such that \(A_{X, \Delta}(E) < S(E)\), which implies that \((X, \Delta)\) is not \(G\)-equivariantly K-semistable by Proposition 2.8, a contradiction. Thus \((X_k, \Delta_k)\) is K-semistable.

Assume next that \(G\) is reductive and \((X, \Delta)\) is \(G\)-equivariantly K-polystable. Then from the previous part we know that \((X_k, \Delta_k)\) is K-semistable. Let \((X_0, \Delta_0)\) be its unique K-polystable degeneration [LWX18 Theorem 1.3] (a priori it is only defined over \(\bar{k}\)). Let

\[
W := \text{PGL}_{N+1}(k) \cdot [(X, \Delta)] \subseteq (\text{Hilb}(\mathbb{P}^N) \times \text{Chow}(\mathbb{P}^N)) \cap W^{\text{Kss}}
\]

and \(W_0 := \text{PGL}_{N+1}(\bar{k}) \cdot [(X_0, \Delta_0)] \subseteq W\) be the corresponding locus (with reduced scheme structure) in the moduli of K-semistable log Fano pairs (c.f. [XZ20a Proof of Theorem 2.21]). Then \(W_0\) is closed in \(W\): otherwise there exists some K-semistable log Fano pair \((X_1, \Delta_1)\) such that \(W_1 := \text{PGL}_{N+1}(\bar{k}) \cdot [(X_1, \Delta_1)]\); in particular, \(\dim W_1 < \dim W_0\) and \((X_0, \Delta_0)\) has an isotrivial degeneration to \((X_1, \Delta_1)\); since \((X_0, \Delta_0)\) is K-polystable, we deduce that \((X_1, \Delta_1)\) also specially degenerates to \((X_0, \Delta_0)\) by [BX19 Theorem 1.1(1) and Remark 1.2(2)], thus \(W_0 \not\subseteq W_1\) and \(\dim W_0 \not\leq \dim W_1\), a contradiction. We also have \(W_0\) is defined over \(k\) since all Galois conjugates of \((X_0, \Delta_0)\) are also K-polystable degenerations of \((X_k, \Delta_k)\), hence are isomorphic to \((X_0, \Delta_0)\) by the uniqueness of K-polystable degeneration. Since \(x := [(X, \Delta)] \in W\) is a \(k\)-rational point that’s fixed by the reductive group \(G\), by [Kem78 Corollary 4.5] we see that there exists a 1-parameter subgroup \(\rho: \mathbb{G}_m \to Z_G(\text{PGL}_{N+1})\) (defined over \(k\)) such that \(\lim_{t \to 0} \rho(t) \cdot x \in W_0\). In other words, there exists a \(G\)-equivariant special test configuration of \((X, \Delta)\) defined over \(k\) whose central fiber is isomorphic to \((X_0, \Delta_0)\) over the algebraic closure \(\bar{k}\); in particular, the central fiber is geometrically K-semistable. Since \((X, \Delta)\) is \(G\)-equivariantly K-polystable, we have \((X_k, \Delta_k) \cong (X_0, \Delta_0)\) by Theorem 2.2 and hence \((X_k, \Delta_k)\) is K-polystable. \(\square\)

**Corollary 4.13 (LZ20 Theorem 1.2).** Let \(\pi: (X, D) \to (Y, B)\) be a finite surjective Galois morphism between log Fano pairs such that \(K_X + D = \pi^*(K_Y + B)\). Then

1. \((X, D)\) is K-semistable (resp. K-polystable) if and only if \((Y, B)\) is K-semistable (resp. K-polystable).
2. If one of \((X, D)\) or \((Y, B)\) is K-unstable, then \(\delta(X, D) = \delta(Y, B)\).

**Proof.** Let \(G = \text{Aut}(f)\) be the Galois group. By Theorem 1.1 and Corollary 4.11 we may assume that the base field is algebraically closed. By [Fuj19a Corollary 1.7] and the computations in [Fuj19a Section 4.1], we know that \(\delta(X, D) \leq \delta(Y, B)\) and \((Y, B)\) is K-semistable if \((X, D)\) is. For every \(G\)-invariant prime divisor \(E\) over \(X\), there exists a
Let \( G \) be an algebraic group and let \((X, \Delta)\) be a log Fano pair with a \( G \)-action. Assume that \( A_{X, \Delta}(E) \geq S(E) \) (resp. \( G \) is reductive and \( A_{X, \Delta}(E) > S(E) \)) for all \( G \)-invariant geometrically irreducible divisors \( E \) over \( X \). Then \((X_k, \Delta_k)\) is \( K \)-semistable (resp. \( K \)-polystable).

Proof. We only prove the \( K \)-polystable part since the \( K \)-semistable part follows directly from Theorem 4.1. Assume that \( G \) is reductive and \( A_{X, \Delta}(E) > S(E) \) for all \( G \)-invariant geometrically irreducible divisors \( E \) over \( X \). Then \((X_k, \Delta_k)\) is \( K \)-semistable. If \((X_k, \Delta_k)\) is not \( K \)-polystable, then as in the proof of Corollary 4.14 we see that \((X, \Delta)\) has a non-trivial \( G \)-equivariant special test configuration with \( K \)-semistable central fiber. This is induced by a \( G \)-invariant geometrically irreducible divisor \( E \) over \( X \) with \( A_{X, \Delta}(E) = S(E) \) (see [BJ17] or [Fuj19b, Theorem 5.1]), a contradiction to our assumption. Thus \((X_k, \Delta_k)\) is \( K \)-polystable as desired.

Corollary 4.15. Let \( G \) be an algebraic group and let \((X, \Delta)\) be a log Fano pair of dimension \( n \) with a \( G \)-action.

1. If \( \alpha_G(X, \Delta) \geq \frac{n}{n+1} \), then \((X_k, \Delta_k)\) is \( K \)-semistable.
2. If \( G \) is reductive and \( \alpha_G(X, \Delta) > \frac{n}{n+1} \), then \((X_k, \Delta_k)\) is \( K \)-polystable.

Proof. Suppose that \( \alpha_G(X, \Delta) \geq \frac{n}{n+1} \). Then we have \( A_{X, \Delta}(E) \geq \frac{n}{n+1} T(E) \) for any \( G \)-invariant geometrically irreducible divisor \( E \) over \( X \). By [BJ17] Proposition 3.11, we also have \( S(E) \leq \frac{n}{n+1} T(E) \) for any such divisor \( E \), thus \( A_{X, \Delta}(E) \geq S(E) \) for any \( G \)-invariant geometrically irreducible divisor \( E \) over \( X \). By Corollary 4.14 this implies that \((X_k, \Delta_k)\) is \( K \)-semistable. The proof of the \( K \)-polystable part is similar.

Using equivariant K-stability and Corollary 4.11 we can also give algebraic proofs of the \( K \)-stability of some explicit Fano varieties. First we provide a short proof of the \( K \)-(poly)stability of del Pezzo surfaces (see e.g. [Tia90, Che08, PW18] for other proofs).

Corollary 4.16. Let \( X \) be a smooth complex del Pezzo surface of degree \( d \). Then \( X \) is \( K \)-polystable if and only if \( X \) is not the blowup of one or two points on \( \mathbb{P}^2 \) and it is \( K \)-stable if and only if \( d \leq 5 \).

Proof. It is well known that the blow up of one or two points on \( \mathbb{P}^2 \) is not \( K \)-polystable since the automorphism group is not reductive. Let \( X = \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1 \) or the blow up of \( \mathbb{P}^2 \) at three points and let \( G = \text{Aut}(X) \). Then it is easy to see that there are no \( G \)-invariant curves or \( G \)-fixed point on \( X \), thus \( X \) is \( K \)-polystable by Corollary 4.14. If \( X \) has degree 5 (resp. 4), then \( G = S_5 \) (resp. \( G = (\mathbb{Z}/2\mathbb{Z})^4 \)) acts on \( X \), \( \operatorname{Pic}(X)^G = \mathbb{Z} \cdot [-K_X] \) and
there are no $G$-fixed points on $X$ (see e.g. \cite{D109}). It follows that every $G$-invariant prime divisor over $X$ is a $G$-invariant curve $C \sim -rK_X$ on $X$ for some $r \geq 1$; an easy calculation shows that $\beta_X(C) = 1 - \frac{1}{3r} > 0$, thus $X$ is K-polystable by Corollary 4.14. Since $X$ has finite automorphism group in this case, it is in fact K-stable. Finally if $d \leq 3$ then $X$ is K-stable by \cite{AZ20}, Corollary 4.9).

As another example, we show that:

**Corollary 4.17.** The following Fano manifolds are K-stable:

- (1) \cite{Tia00} Fermat hypersurfaces $(x_0^d + \cdots + x_{n+1}^d = 0) \subseteq \mathbb{P}^{n+1}$ $(3 \leq d \leq n+1)$.
- (2) \cite{AGP06} Complete intersection of two quadrics $Q_1 \cap Q_2 \subseteq \mathbb{P}^{n+2}$.

Combined with \cite{Xu20, BLX19}, this gives an algebraic proof that a general Fano hypersurface of degree at least 3 is K-stable.

**Proof.** First let $X$ be the Fermat hypersurfaces $(x_0^d + \cdots + x_{n+1}^d = 0) \subseteq \mathbb{P}^{n+1}$ $(3 \leq d \leq n+1)$ and consider the morphism $f : X \to \mathbb{P}^n$ given by $[x_0 : \cdots : x_{n+1}] \mapsto [x_0^d : \cdots : x_n^d]$. Let $y_0, \cdots, y_n$ be the homogeneous coordinates on the target $\mathbb{P}^n$, let $H_i = (y_i = 0) \subseteq \mathbb{P}^n$ $(i = 0, 1, \cdots, n)$ and let $H_{n+1} = (y_0 + \cdots + y_n = 0)$.

Then it is clear that $f$ is Galois and $K_X = f^*(K_{\mathbb{P}^n} + (1 - \frac{1}{d})(H_0 + \cdots + H_{n+1}))$. By \cite{Pu17} Corollary 1.6, the pair $(\mathbb{P}^n, (1 - \frac{1}{d})(H_0 + \cdots + H_{n+1}))$ is K-polystable, thus by Corollary 4.13 the Fermat hypersurface $X$ is also K-polystable. It is indeed K-stable since $\text{Aut}(X)$ is finite when $d \geq 3$.

The argument is similar for the complete intersection of two quadrics $X = Q_1 \cap Q_2 \subseteq \mathbb{P}^{n+2}$. Up to change of coordinates, we may assume that $Q_1 = (x_0^d + \cdots + x_{n+2}^d = 0)$ and $Q_2 = (a_0x_0^d + \cdots + a_{n+2}x_{n+2}^d = 0)$ for some mutually distinct coefficients $a_0, \cdots, a_{n+2}$. Consider the Galois morphism $g : X \to \mathbb{P}^n$ given by $[x_0 : \cdots : x_{n+2}] \mapsto [x_0^d : \cdots : x_n^d]$. We may identify $\mathbb{P}^n$ with the linear subspace $(x_0 + \cdots + x_{n+2} = a_0x_0 + \cdots + a_{n+2}x_{n+2} = 0) \subseteq \mathbb{P}^{n+2}$ and let $H_i \subseteq \mathbb{P}^n$ $(i = 0, 1, \cdots, n+2)$ be the hyperplane ($x_i = 0$). Then we have $K_X = g^*(K_{\mathbb{P}^n} + \frac{1}{d}(H_0 + \cdots + H_{n+2}))$. By \cite{Pu17} Corollary 1.6, the pair $(\mathbb{P}^n, \frac{1}{d}(H_0 + \cdots + H_{n+2}))$ is K-polystable, thus by Corollary 4.13 the complete intersection $X$ is also K-polystable. Since $\text{Aut}(X)$ is again finite, we conclude that it is K-stable.

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