Influence of anisotropy and compressibility on anomalous scaling of a passive scalar field

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Abstract

Influence of uniaxial small-scale anisotropy and compressibility on the stability of scaling regime and on the anomalous scaling of structure functions of a scalar field is investigated in the model of a passive scalar field advected by the compressible Gaussian strongly anisotropic velocity field with the covariance \( \propto \delta(t - t') |x - x'|^2 \) ε by using the field theoretic renormalization group and the operator product expansion. The inertial-range stability of the corresponding scaling regime is established. The anomalous scaling of the single-time structure functions is studied and the corresponding anomalous exponents are calculated. Their dependence on the compressibility parameter and anisotropy parameters is analyzed. It is shown that the presence of compressibility leads to the decreasing of the critical dimensions of the important composite operators, i.e., the anomalous scaling is more pronounced in the compressible systems. This result is demonstrated for the structure function of the third order. All calculations are done to the first order in \( \varepsilon \).

Introduction

The so-called "rapid change model" of a scalar field passively advected by a self-similar Gaussian \( \delta \)--correlated in time velocity field introduced by Kraichnan¹ and number of its extensions play the central role in the theoretical investigation of intermittency and anomalous scaling²³. The main reason for this is the experimental fact that the deviations from the statements of the famous classical Kolmogorov-Obukhov phenomenological theory (see, e.g.,²³) are more noticeable for simpler models of passively advected scalar quantity (scalar field) than for the velocity field itself (see¹ and references cited therein) and, at the same time, the problem of the passive advection of a scalar field is much easier from theoretical point of view than the original problem of anomalous scaling in the framework of the Navier-Stokes velocity field. Within the rapid change model of a passive scalar advection the systematic analysis of the corresponding anomalous exponents was done for the first time on the microscopic level. For example, in the so-called "zero-mode approach" to the rapid change model (see survey paper¹ and references cited therein) the anomalous exponents are found from the homogenous solutions (zero modes) of the closed equations for the single-time correlations.

One of the most effective approach for studying self-similar scaling behavior is the method of the field theoretic renormalization group (RG)⁵⁶. This method can be also used in the theory of fully developed turbulence and related problems, e.g., in the problem of a passive scalar advection by the turbulent environment (see⁶⁷⁸ and references cited therein).

In⁹ the field theoretic RG and operator-product expansion (OPE) were used in the systematic investigation of the rapid-change model of a passive scalar. It was shown that within the field theoretic approach the anomalous scaling is related to the very existence of the so-called "dangerous" composite operators with negative critical dimensions in the OPE (see, e.g., Refs.⁶⁸ for details).
In subsequent papers a few generalizations of the rapid-change model towards more realistic ones were done. For example, in [10] the field theoretic RG and OPE were applied to the rapid change model of a passive scalar advected by the \( \delta \)-correlated, Gaussian strongly anisotropic velocity field. It was shown that in the small-scale anisotropic case the anomalous exponents of the structure functions and correlation functions are nonuniversal and they are functions of the parameters of anisotropy. Besides, they form the hierarchy with the leading exponent related to the most "isotropic" operator. On the other hand, in [11] the influence of compressibility and large-scale anisotropy on the anomalous scaling behavior was studied in the aforementioned model and the anomalous exponents of higher-order correlation functions were calculated as functions of the parameter of compressibility. It was shown that the presence of small-scale anisotropy is more pronounced for larger values of the compressibility parameter \( \alpha \). From this point of view, compressible systems are very interesting to be studied.

In the end, in [12] the combined effects of the small-scale anisotropy and compressibility on the anomalous scaling of the structure functions of a passive scalar within the rapid-change model were studied by using the simplest possible generalization of the incompressible and anisotropic tensor structure of the velocity field correlator to the compressible one where compressibility was introduced only to the isotropic part of the anisotropic tensor structure. In present paper, we shall investigate more general case of the model where compressibility will be introduced through all anisotropic tensor structures of the velocity field statistics. First of all, we shall establish stability of the scaling regime of the model and coordinates of the corresponding infrared (IR) stable fixed point will be found as functions of the compressibility and anisotropy parameters. These results will be then used in the analysis of the asymptotic behavior of the single-time structure functions of a passively advected scalar field.

Formulation of the model

We shall investigate the model of the advection of a passive "tracer" \( \theta(x) \equiv \theta(t, x) \) which is described by the following stochastic equation

\[
\frac{\partial_t \theta}{\nu_0 \Delta} = (v_\ell \partial_\ell) \theta + f,
\]

(1)

where \( \partial_t \equiv \partial/\partial t \), \( \partial_\ell \equiv \partial/\partial x_\ell \), \( \Delta \equiv \partial^2 \) is the Laplace operator, \( \nu_0 \) is the coefficient of molecular diffusivity (a subscript \( 0 \) will denote bare parameters of unrenormalized theory), \( v_\ell \equiv v_\ell(x) \) is the \( \ell \)-th component of the compressible velocity field \( v(x) \), and \( f \equiv f(x) \) is a Gaussian random noise with zero mean and correlation function

\[
D^f \equiv \langle f(x)f(x') \rangle = \delta(t-t')C(r/L), \quad r = x - x',
\]

(2)

where parentheses \( (...) \) hereafter denote average over corresponding statistical ensemble. The concrete form of the noise defined in (2) will not be essential in what follows. The only condition which must be satisfied by the function \( C(r/L) \) is that it must decrease rapidly for \( r \equiv |r| \gg L \), where \( L \) denotes an integral scale related to the stirring.

In real problems the velocity field \( v(x) \) satisfies Navier-Stokes equation but, in what follows, we shall work with a simplified model where we suppose that the velocity field obeys a Gaussian statistics with zero mean and pair correlation function

\[
\langle v_i(x)v_j(x') \rangle \equiv D^t_{ij}(x;x') = D_0 \delta(t-t') \int \frac{d^d k}{(2\pi)^d} \frac{R_{ij}(k)}{(k^2 + m^2)^{d/2+\varepsilon}} \exp[i k(x-x')],
\]

(3)

where \( d \) is the dimension of the space, \( k \) is the wave vector, and \( D_0 \) is an amplitude factor related to the coupling constant \( g_0 \) of the model (expansion parameter in the perturbation theory, see, e.g., [12]) by the relation \( D_0/\nu_0 \equiv R_0 \equiv \Lambda^{2\varepsilon} \), where \( \Lambda \) is the characteristic UV momentum scale. The parameter
of the energy spectrum of the velocity field \(0 < \varepsilon < 1\) is taken in such a way that its "Kolmogorov" value (the value which corresponds to the Kolmogorov scaling of the velocity correlation function in developed turbulence) is \(\varepsilon = 2/3\), and \(1/m\) is another integral scale. In our uniaxial anisotropic and compressible case the tensor \(R_{ij}(\mathbf{k})\) is taken in the following way \([10, 12]\)

\[
R_{ij}(\mathbf{k}) = \left(1 + \alpha_1 \xi_k^2 \right) \left(P_{ij} + \alpha Q_{ij}\right) + \alpha_2 n_s n_l \left(P_{is} + \alpha Q_{is}\right) (P_{jl} + \alpha Q_{jl}),
\]

where we denote \(\xi_k = \mathbf{n} \cdot \mathbf{k}/k\), \(P_{ij}(\mathbf{k}) \equiv \delta_{ij} - k_i k_j/k^2\) is the longitudinal projector, \(Q_{ij} = k_i k_j/k^2\) is the common isotropic transverse projector, \(\mathbf{n}\) determines the distinguished direction of uniaxial anisotropy, \(\alpha \geq 0\) is a free parameter of compressibility, and \(\alpha_1, \alpha_2\) are parameters characterizing the anisotropy. From the positiveness of the correlation tensor \(D_{ij}^v\) one finds restrictions on the values of the above parameters, namely, \(\alpha_{1,2} > -1\).

The stochastic problem \([11]-[13]\) can be rewritten in a field theoretic form with the following action functional \([5, 6]\)

\[
S(\Phi) = \int dt \, d^d \mathbf{x} \, \theta' \left[-\partial_t - v_i \partial_i + \nu_0 \Delta + \chi_0 \nu_0 (\mathbf{n} \cdot \partial)^2\right] \theta - \frac{1}{2} \int dt \, d^d \mathbf{x}_1 \, dt_2 \, d^d \mathbf{x}_2
\]

\[
\left(v_i(t_1, \mathbf{x}_1) [D^v_{ij}(t_1, \mathbf{x}_1; t_2, \mathbf{x}_2)] \right)^{-1} v_j(t_2, \mathbf{x}_2) - \theta'(t_1, \mathbf{x}_1) D^f(t_1, \mathbf{x}_1; t_2, \mathbf{x}_2) \theta'(t_2, \mathbf{x}_2),
\]

where \(D^v_{ij}\) and \(D^f_{ij}\) are given in \([4]\) and \([2]\) respectively, \(\theta'\) is an auxiliary scalar field (see, e.g., \([6]\)), and the required summations over the vector indices are implied. In action \([5]\) the term with new parameter \(\chi_0\) is related to the presence of small-scale anisotropy and the introduction of this term is necessary to make the model multiplicatively renormalizable. Model \([5]\) corresponds to a standard Feynman diagrammatic technique (see, e.g., \([10, 12]\) for details) and the standard analysis of canonical dimensions shows which one-irreducible Green functions can possess UV superficial divergences. The functional formulation \([5]\) gives possibility to extract large-scale asymptotic behavior of the correlation functions after an appropriate renormalization procedure which is needed to remove the UV-divergences.

**Influence of anisotropy and compressibility on the scaling regime of the model and on the anomalous scaling**

The IR scaling regimes of the model are given by the IR stable fixed points of the corresponding RG equations \([6, 7, 8]\). The fixed points of the RG equations can be determined from the requirement that all the so-called beta functions of the model vanish and the IR stability of the fixed point is given by the requirement that all the eigenvalues of the matrix of the first derivatives \(\Omega_{ij} = \partial \beta_i/\partial C_k\), where \(\beta_i\) denotes the full set of beta functions and \(C_k\) is the full set of charges of the model, must have positive real parts. In our case the coordinates of the fixed points are given by the following system of equations

\[
\beta_g(g_s, \chi_s, \alpha_j, \alpha, d, \varepsilon) = g_s (-2 \varepsilon + \gamma_1^*), \quad \beta_\chi(g_s, \chi_s, \alpha_j, \alpha, d, \varepsilon) = \chi_s (\gamma_1^* - \gamma_2^*) = 0,
\]

for \(j = 1, 2\) and variables with star denote the fixed point values of the corresponding variables. The explicit form of the so-called gamma functions \(\gamma_i, i = 1, 2\) is as follows

\[
\gamma_1 = \frac{g}{2} A_1, \quad \gamma_2 = \frac{g}{2\chi} A_2,
\]

where we denote \(\bar{g} = g S_d/(2 \pi)^d\), \(S_d = 2 \pi^{d/2}/\Gamma(d/2)\) is the \(d\)-dimensional sphere, and \(A_{1,2}\) have the following explicit form

\[
A_1 = \frac{(d + 2)(d - 1) + \alpha_1(d + 1) + \alpha_2 + \alpha(d + 2 + \alpha_1 - 2\alpha_2) + \alpha^2 \alpha_2}{d(d + 2)}.
\]


Figure 1: The restriction on the values of the anisotropy parameters $\alpha_1$ and $\alpha_2$ for different values of the compressibility parameter $\alpha$ for space dimensions $d = 2$ and $d = 3$ given by the physical condition $\chi_* > -1$. The figures show that the allowed regions of the parameters $\alpha_1$ and $\alpha_2$ decrease when the compressibility parameter $\alpha$ increases.

$$A_2 = \frac{-2(\alpha_1 + \alpha_2) + d^2\alpha_2 + 2\alpha(\alpha_1 + d\alpha_2) + 2\alpha^2\alpha_2}{d(d + 2)}.$$  \hspace{1cm} (9)

Thus, possible fixed points are found as solutions of the system of algebraic equations (6) and their IR stability is determined by the positive real parts of the eigenvalues of the matrix $\Omega = \{\Omega_{ik}\}$. Using the explicit expressions for gamma functions (7) we have found the coordinates of the nontrivial fixed point (trivial fixed point with $g_\ast = 0$ is not interesting for us)

$$g_\ast = \frac{4d(d + 2)\varepsilon}{(d + 2)(d - 1) + \alpha_1(d + 1) + \alpha_2 + \alpha(d + 2 + \alpha_1 - 2\alpha_2) + \alpha^2\alpha_2},$$  \hspace{1cm} (10)

$$\chi_\ast = \frac{-2(\alpha_1 + \alpha_2) + d^2\alpha_2 + 2\alpha(\alpha_1 + d\alpha_2) + 2\alpha^2\alpha_2}{(d + 2)(d - 1) + \alpha_1(d + 1) + \alpha_2 + \alpha(d + 2 + \alpha_1 - 2\alpha_2) + \alpha^2\alpha_2}.$$  \hspace{1cm} (11)

At fixed point the matrix of the first derivatives has the eigenvalues $\lambda_{1,2} = 2\varepsilon$. It means that the point is IR stable if $\varepsilon > 0$. It is also the only condition to have $g_\ast > 0$ (of course, together with the physical assumptions: $\alpha_{1,2} > -1$ and $\alpha > 0$). Besides, the fixed point value of $\chi$ must satisfy physical condition $\chi_* > -1$. This fact leads to restrictions on parametrical space of the model, namely, on parameters of anisotropy $\alpha_{1,2}$ and compressibility parameter $\alpha$. For space dimensions $d = 2$ and $d = 3$ these restrictions are shown in Fig. 1. The existence of these restrictions is not present in the model with simpler definition of the anisotropy and compressibility which was studied in [12].

Further we shall use the above obtained results to determine the anomalous scaling of the single-time structure functions of the scalar field inside the so-called inertial interval which is defined by inequalities $1/\Lambda \ll r \ll L$ [3, 6, 10, 12]. They are defined as follows

$$S_N(r) \equiv \langle [\theta(t, \mathbf{x}) - \theta(t, \mathbf{x}')]^N \rangle,$$  \hspace{1cm} (12)

where $N$ denotes the order of the structure function and $r = |\mathbf{r}| = |\mathbf{x} - \mathbf{x}'|$. We shall not discuss details of the corresponding analysis. It will be given elsewhere but we only stress that the anomalous scaling
An anomalous scaling of a passive scalar field is given by the existence in the model of the so-called dangerous composite operator with negative critical dimensions within the OPE [6, 7, 8]. In our case the main contribution to the anomalous scaling is given by the operators $F[N, p] = \partial_1 \theta \cdots \partial_p \theta (\partial_i \theta \partial_i \theta)^n$ with $N = 2n + p$ (see, e.g., [10, 12]).

After rather long additional renormalization procedure [10, 12] one comes to the following final expression for the inertial range behavior of the single-time structure functions

$$S_N(r) = D_0^{-N/2} r^{N(1-\epsilon)} \sum_{N'\leq N} \sum_p \{C_{N',p} (r/L)^{\Delta[N',p]} + \ldots \},$$

(13)

where $\Delta[N, p]$ denote the critical dimensions of the operators $F[N, p]$ (more precisely, they are eigenvalues of the corresponding matrix of the critical dimensions, see, e.g., [12]), $p$ obtains all possible values for given $N'$, $C_{N',p}$ are numerical coefficients which are functions of the parameters of the model, and dots means contributions by the operators others than $F[N, p]$ (see, e.g., [6, 10] for details).

Our aim is to analyze the influence of compressibility on critical dimensions of anisotropic operators and to find the answer on the question whether the compressibility makes the anomalous behavior more pronounced or not, i.e., whether the presence of compressibility decreases the critical dimensions of corresponding operators or not.

In Fig. 2 the dependence of the critical dimension of the principal eigenvalue (related to the isotropic case, see [10]) of the corresponding matrix of critical dimensions of the composite operators $F[3, p]$ is shown which play the central role in the asymptotic behavior of the structure function $S_3$ (it is well known that the structure function $S_2$ has not anomalous behavior [10, 12], therefore we start the analysis from $S_3$). It can be seen that in the case of the structure function $S_3$ the anomalous scaling is present (the existence of negative values for $\Delta[3, 1]$) and the effect is more pronounced for more compressible system. Detail analysis of the model for the structure functions of higher order will be given elsewhere.

Figure 2: Dependence of the critical dimension $\Delta[3, 1]/\epsilon$ on the anisotropy parameters $\alpha_1$ and $\alpha_2$ for different values of the compressibility parameter $\alpha$ and for $d = 3$. 
Conclusions

In present paper we have studied the influence of uniaxial small-scale anisotropy and compressibility on the anomalous scaling of the structure functions of a scalar field in the framework of the Kraichnan model by using the field theoretic RG and OPE. Using the RG technique we have shown the existence of the scaling regime within the inertial range which is defined by the corresponding IR stable fixed point of the RG equations. We have found restrictions on the anisotropy parameters which are related to the compressibility of the system. The restrictions become larger when compressibility parameter increases. Besides, using the OPE we have found the asymptotic form of the structure functions and, using the concrete example of the third order structure function, we have shown that the compressibility of the system makes the effects of anomalous scaling more pronounced, i.e., the critical dimensions of the corresponding composite operators are smaller in compressible case than in the incompressible case. Thus, we can conclude that the compressible environment is more suitable for experimental study of the anomalous scaling of the structure functions of a passive scalar than incompressible systems.

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