$n$-gr-Coherent rings and Gorenstein graded modules

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Abstract. Let $R$ be a graded ring and $n \geq 1$ an integer. In this paper, we introduce and study the notions of Gorenstein $n$-FP-gr-injective and Gorenstein $n$-gr-flat modules by using the notion of special finitely presented graded modules. On $n$-gr-coherent rings, we investigate the relationships between Gorenstein $n$-FP-gr-injective and Gorenstein $n$-gr-flat modules. Among other results, we prove that any graded module in $R$-gr (resp. gr-$R$) admits Gorenstein $n$-FP-gr-injective (resp. Gorenstein $n$-gr-flat) cover and preenvelope.

Keywords: $n$-gr-coherent ring; Gorenstein $n$-FP-gr-injective modules; Gorenstein $n$-gr-flat modules, covers, (pre)envelopes.

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1 Introduction

In 1990s, Enochs, Jenda and Torrecillas, introduced the concepts of Gorenstein injective and Gorenstein flat modules over arbitrary rings [14, 16]. In 2008, Mao and Ding introduced a special case of the Gorenstein injective modules and they called Gorenstein FP-injective modules, which renamed by Gillespie by Ding injective [21]. These Gorenstein FP-injective modules are stronger than the Gorenstein injective modules, and in general an FP-injective module is not necessarily Gorenstein FP-injective [25, Proposition 2.7]. For this reason, Gao and Wang introduced and studied in [19] another notion called Gorenstein FP-injective modules which is weaker than the usual Gorenstein injective modules. Furthermore, all FP-injective modules are in the class of Gorenstein FP-injective modules (see Section 2 for the definitions of these notions).

In this paper we deal with the graded aspect of some extensions of these notions. As it is known, graded rings and modules are a classical notions in algebra which build their values and strengths from their connection with algebraic geometry (see for instance [29, 30, 31]). Several authors have investigated the graded aspect of some notions in relative homological algebra. For example, Asensio, López Ramos and Torrecillas in [1, 2] introduced the notions of Gorenstein gr-projective, gr-injective and gr-flat modules. In the recent years, the Gorenstein homological theory for graded rings have become an important area of research (see for instance [4, 20]). The notions of FP-gr-injective modules was introduced in [3], and in [35] homological behavior of the FP-gr-injective modules on gr-coherent rings were investigated. Along the same lines, it is natural to generalize the notion of “FP-gr-injective modules and gr-flat modules” to “n-FP-gr-injective modules and n-gr-flat modules”. This done by Zhao, Gao and Huang in [36] basing on the notion of special finitely presented graded modules which they defined via projective resolutions of n-presented graded modules. Recently, in 2017, Mao via FP-gr-injective modules gave a definition of Ding gr-injective modules [26]. Under this definition these Ding gr-injective modules are stronger than the Gorenstein gr-injective modules, and an FP-gr-injective module is not necessarily Ding gr-injective in general [26, Corollary 3.7]. So, for any $n \geq 1$, we study the consequences of extending the notion of n-FP-gr-injective and n-gr-flat modules to that of Gorenstein n-FP-gr-injective and Gorenstein n-gr-flat modules, respectively. Then, in this paper, for any $n \geq 1$ by using n-FP-gr-injective modules and n-gr-flat modules, we introduce a concept of Gorenstein n-FP-gr-injective and Gorenstein n-gr-flat modules, and under this definition, Gorenstein n-FP-gr-injective and Gorenstein n-gr-flat modules are weaker than the usual Gorenstein gr-injective and Gorenstein gr-flat modules, respec-
tively. Also, for any $n \geq 1$, all gr-injective, $n$-FP-gr-injective modules and gr-flat, $n$-gr-flat modules are Gorenstein $n$-FP-gr-injective and Gorenstein $n$-gr-flat, respectively, and in general, Gorenstein $n$-FP-gr-injective and Gorenstein $n$-gr-flat $R$-modules need not be $n$-FP-gr-injective and $n$-gr-flat, unless in certain cases, see Proposition 3.18.

The paper is organized as follows:

In Sec. 2, some fundamental concepts and some preliminary results are stated.

In Sec. 3, we introduce Gorenstein $n$-FP-gr-injective and Gorenstein $n$-gr-flat modules for an integer $n \geq 1$ and then we give some characterizations of these modules. Among other results, we prove that, for an exact sequence $0 \to A \to B \to C \to 0$ of graded left $R$-modules, if $A$ and $B$ are Gorenstein $n$-FP-gr-injective, then $C$ is Gorenstein $n$-FP-gr-injective if and only if every $n$-presented module in $R$-gr with $\text{gr-pd}_R(U) < \infty$ is $(n + 1)$-presented, and it follows that $(\perp G_{\text{gr-FI}_n}, G_{\text{gr-FI}_n})$ is a hereditary cotorsion pair if and only if every $n$-presented module in $R$-gr with $\text{gr-pd}_R(U) < \infty$ is $(n + 1)$-presented and every $M \in (\perp G_{\text{gr-FI}_n})^\perp$ has an exact left $(\text{gr-FI}_n)$-resolution, where $G_{\text{gr-FI}_n}$ and $\text{gr-FI}_n$ denote the classes of Gorenstein $n$-FP-gr-injective and $n$-FP-gr-injective modules in $R$-gr, respectively. Also, for a graded left (resp. right) $R$-module $M$ over a left $n$-gr-coherent ring $R$: $M$ is Gorenstein $n$-FP-gr-injective (resp. Gorenstein $n$-gr-flat) if and only if $M^*$ is Gorenstein $n$-gr-flat (resp. Gorenstein $n$-FP-gr-injective). Furthermore, the class of Gorenstein $n$-FP-gr-injective (resp. Gorenstein $n$-gr-flat) modules are closed under direct limits (resp. direct products). In this section, examples are given in order to show that Gorenstein $m$-FP-gr-injectivity (resp. Gorenstein $m$-gr-flatness) does not imply Gorenstein $n$-FP-gr-injectivity (resp. Gorenstein $n$-gr-flatness) for any $m > n$. Also, examples are given showing that Gorenstein $n$-FP-gr-injectivity does not imply gr-injectivity. In this paper, $\text{gr-I}$ denote the classes of gr-injective modules in $R$-gr and $\text{gr-F}$, $\text{gr-F}_n$ and $G_{\text{gr-F}_n}$ denote the classes of gr-flat, $n$-gr-flat and Gorenstein $n$-gr-flat modules in gr-$R$, respectively.

In Sec. 4, it is shown that the class of Gorenstein $n$-FP-gr-injective and Gorenstein $n$-gr-flat modules are covering and preenveloping on $n$-gr-coherent rings. We also establish some equivalent characterizations of $n$-gr-coherent rings in terms of Gorenstein $n$-FP-gr-injective and Gorenstein $n$-gr-flat modules.
2 Preliminaries

Throughout this paper, all rings considered are associative with identity element and the $R$-modules are unital. By $R$-Mod and Mod-$R$ we will denote the category of all left $R$-modules and right $R$-modules, respectively.

In this section, some fundamental concepts and notations are stated.

Let $n$ be a non-negative integer and $M$ a left $R$-module. Then, $M$ is said to be Gorenstein injective (resp. Gorenstein flat) \cite{14,16} if there is an exact sequence

$$
\cdots \rightarrow I_1 \rightarrow I_0 \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots
$$

of injective (resp. flat) left $R$-modules with $M = \ker(I^0 \rightarrow I^1)$ such that $\text{Hom}_R(U, -)$ (resp. $U \otimes_R -$) leaves the sequence exact whenever $U$ is an injective left (resp. right) $R$-module.

$M$ is said to be $n$-presented \cite{10,13} if there is an exact sequence

$$
F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow U \rightarrow 0
$$

of left $R$-modules, where each $F_i$ is finitely generated free, and a ring $R$ is called left $n$-coherent, if every $n$-presented left $R$-module is $(n + 1)$-presented. $M$ is said to be $n$-FP-injective \cite{11} if $\text{Ext}_R^n(U, M) = 0$ for any $n$-presented left $R$-module $U$. In case $n = 1$, $n$-FP-injective modules are nothing but the well-known FP-injective modules. A right module $N$ is called $n$-flat if $\text{Tor}_n^R(N, U) = 0$ for any $n$-presented left $R$-module $U$.

$M$ is said to be Gorenstein FP-injective \cite{25} if there is an exact sequence

$$
E = \cdots \rightarrow E_1 \rightarrow E_0 \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots
$$

of injective left modules with $M = \ker(E^0 \rightarrow E^1)$ such that $\text{Hom}_R(U, E)$ is an exact sequence whenever $U$ is an FP-injective left $R$-module. Then, in \cite{19}, Gao and Wang introduced other concept of Gorenstein FP-injective modules as follows: $M$ is said to be Gorenstein FP-injective \cite{19} if there is an exact sequence

$$
E = \cdots \rightarrow E_1 \rightarrow E_0 \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots
$$

of FP-injective left modules with $M = \ker(E^0 \rightarrow E^1)$ such that $\text{Hom}_R(P, E)$ is an exact sequence whenever $P$ is a finitely presented module with $\text{pd}_R(P) < \infty$. 


Let $G$ be a multiplicative group with neutral element $e$. A graded ring $R$ is a ring with identity 1 together with a direct decomposition $R = \bigoplus_{\sigma \in G} R_{\sigma}$ (as additive subgroups) such that $R_{\sigma}R_{\tau} \subseteq R_{\sigma \tau}$ for all $\sigma, \tau \in G$. Thus, $R e$ is a subring of $R$, $1 \in Re$ and $R_{\sigma}$ is an $Re$-bimodule for every $\sigma \in G$. A graded left (resp. right) $R$-module is a left (resp. right) $R$-module $M$ endowed with an internal direct sum decomposition $M = \bigoplus_{\sigma \in G} M_{\sigma}$, where each $M_{\sigma}$ is a subgroup of the additive group of $M$ such that $R_{\sigma}M_{\tau} \subseteq M_{\sigma \tau}$ for all $\sigma, \tau \in G$. For any graded left $R$-modules $M$ and $N$, set $\text{Hom}_{R^{-}\text{gr}}(M, N) := \{ f : M \to N \mid f \text{ is } R\text{-linear and } f(M_{\sigma}) \subseteq N_{\sigma} \text{ for any } \sigma \in G \}$, which is the group of all morphisms from $M$ to $N$ in the class $R\text{-gr}$ of all graded left $R$-modules ($gr-R$ will denote the class of all graded right $R$-modules). It is well known that $R\text{-gr}$ is a Grothendieck category. An $R$-linear map $f : M \to N$ is said to be a graded morphism of degree $\tau$ with $\tau \in G$ if $f(M_{\sigma}) \subseteq N_{\sigma \tau}$ for all $\sigma \in G$. Graded morphisms of degree $\sigma$ build an additive subgroup $\text{HOM}_{R}(M, N)_{\sigma}$ of $\text{Hom}_{R}(M, N)$. Then $\text{HOM}_{R}(M, N) = \bigoplus_{\sigma \in G} \text{HOM}_{R}(M, N)_{\sigma}$ is a graded abelian group of type $G$. We will denote by $\text{Ext}_{R^{-}\text{gr}}^{i}$ and $\text{EXT}_{R}^{i}$ the right derived functors of $\text{Hom}_{R^{-}\text{gr}}$ and $\text{HOM}_{R}$, respectively. Given a graded left $R$-module $M$, the graded character module of $M$ is defined as $M^{*} := \text{HOM}_{Z}(M, Q/Z)$, where $Q$ is the rational numbers field and $Z$ is the integers ring. It is easy to see that $M^{*} = \bigoplus_{\sigma \in G} \text{HOM}_{Z}(M_{\sigma^{-1}}, Q/Z)$.

Let $M$ be a graded right $R$-module and $N$ a graded left $R$-module. The abelian group $M \otimes_{R} N$ may be graded by putting $(M \otimes_{R} N)_{\sigma}$ with $\sigma \in G$ to be the additive subgroup generated by elements $x \otimes y$ with $x \in M_{\alpha}$ and $y \in N_{\beta}$ such that $\alpha \beta = \sigma$. The object of $Z\text{-gr}$ thus defined will be called the graded tensor product of $M$ and $N$.

If $M$ is a graded left $R$-module and $\sigma \in G$, then $M(\sigma)$ is the graded left $R$-module obtained by putting $M(\sigma)_{\tau} = M_{\tau \sigma}$ for any $\tau \in G$. The graded module $M(\sigma)$ is called the $\sigma$-suspension of $M$. We may regard the $\sigma$-suspension as an isomorphism of categories $T_{\sigma} : R\text{-gr} \to R\text{-gr}$, given on objects as $T_{\sigma}(M) = M(\sigma)$ for any $M \in R\text{-gr}$. The forgetful functor $U : R\text{-gr} \to R\text{-Mod}$ associates to $M$ the underlying ungraded $R$-module. This functor has a right adjoint $F$ which associated to $M \in R\text{-Mod}$ the graded $R$-module $F(M) = \bigoplus_{\sigma \in G}(\sigma M)$, where each $\sigma M$ is a copy of $M$ written $\{ \sigma x : x \in M \}$ with $R$-module structure defined by $r^{*} \sigma x = \sigma (r^{\ast} x)$ for each $r \in R_{\sigma}$. If $f : M \to N$ is $R$-linear, then $F(f) : F(M) \to F(N)$ is a graded morphism given by $F(f)(\sigma x) = \sigma f(x)$.

The injective (resp. flat) objects of $R\text{-gr}$ (resp. $gr-R$) will be called gr-injective (resp. gr-flat) modules, because $M$ is gr-injective (resp. gr-flat) if and only if it is a injective (resp. flat) graded
module. By $\text{gr-pd}_R(M)$ and $\text{gr-fd}_R(M)$ we will denote the gr-projective and gr-flat dimension of a graded module $M$, respectively. A graded left (resp. right) module $M$ is said to be Gorenstein gr-injective (resp. Gorenstein gr-flat) \cite{1,2,4} if there is an exact sequence
\[ \cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots \]
of gr-injective (resp. gr-flat) left (resp. right) modules with $M = \ker(I^0 \to I^1)$ such that $\text{Hom}_{R-\text{gr}}(E, \_)$ (resp. $\_ \otimes_R E$) leaves the sequence exact whenever $E$ is a gr-injective $R$-module. The gr-injective envelope of $M$ is denoted by $E_{g}(M)$. A graded left module $M$ is said to be Ding gr-injective \cite{26} if there is an exact sequence
\[ \cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots \]
of gr-injective left modules, with $M = \ker(I^0 \to I^1)$ such that $\text{Hom}_{R-\text{gr}}(E, \_)$ leaves the sequence exact whenever $E$ is an FP-gr-injective left $R$-module.

**Definition 2.1** (\cite{36}, Definition 3.1). Let $n \geq 0$ be an integer. Then, a graded left module $U$ is called $n$-presented, if there exists an exact sequence $F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to U \to 0$ in $R$-gr with each $F_i$ is finitely generated free left $R$-module.

Set $K_{n-1} = \text{Im}(F_{n-1} \to F_{n-2})$ and $K_n = \text{Im}(F_n \to F_{n-1})$. Then we get a short exact sequence $0 \to K_n \to F_{n-1} \to K_{n-1} \to 0$ in $R$-gr with $F_{n-1}$ is a finitely generated free module. The modules $K_n$ and $K_{n-1}$ will be called special finitely gr-generated and special finitely gr-presented, respectively. The sequence $(\Delta) : 0 \to K_n \to F_{n-1} \to K_{n-1} \to 0$ in $R$-gr will be called a special short exact sequence.

Moreover, a short exact sequence $0 \to A \to B \to C \to 0$ in $R$-gr is called special gr-pure if the induced sequence
\[ 0 \to \text{HOM}_{R}(K_{n-1}, A) \to \text{HOM}_{R}(K_{n-1}, B) \to \text{HOM}_{R}(K_{n-1}, C) \to 0 \]
is exact for every special finitely gr-presented module $K_{n-1}$. In this case $A$ is said to be special gr-pure in $B$.

Analogously to the classical case, a graded ring $R$ is called left $n$-gr-coherent if each $n$-presented module in $R$-gr is $(n + 1)$-presented.

Ungraded $n$-presented modules have been used by many authors in order to extend some homological notions. For example, in \cite{9}, let $R$ be an associative ring and $M$ a left $R$-module, then module $M$ is called $FP_{n}$-injective if $\text{Ext}^1_R(L, M) = 0$ for all $n$-presented left $R$-modules $L$. In
2018, Zhao, Gao and Huang in [36] showed that if we similarly use the derived functor $\text{Ext}^1$ to define the $FP_n$-gr-injective and $FP_\infty$-gr-injective modules, then they are just the $FP_n$-injective and $FP_\infty$-injective objects in the class of graded modules, respectively. If $L$ is an $n$-presented graded left $R$-module with $n \geq 2$, then $\text{Ext}^1_R(L, M) = \text{Ext}^1_R(L, M)$ for any graded $R$-module $M$. For this reason, they introduced the concept of $n$-FP-gr-injective modules as follows: A graded left $R$-module $M$ is called $n$-FP-gr-injective [36] if $\text{Ext}^n_R(N, M) = 0$ for any finitely $n$-presented graded left $R$-module $N$. If $n = 1$, then $M$ is FP-gr-injective. A graded right $R$-module $M$ is called $n$-gr-flat [36] if $\text{Tor}_n^R(M, N) = 0$ for any finitely $n$-presented graded left $R$-module $N$.

If $U$ is an $n$-presented graded left $R$-module and $0 \to K_n \to F_{n-1} \to K_{n-1} \to 0$ is a special short exact sequence in $R$-gr with respect to $U$, then $\text{Ext}^n_R(U, M) \cong \text{Ext}^n_R(K_{n-1}, M)$ for every graded left $R$-module $M$, and $\text{Tor}_n^R(M, U) \cong \text{Tor}_n^R(M, K_{n-1})$ for every graded right $R$-module $M$. The $n$-FP-gr-injective dimension of a graded left $R$-module $M$, denoted by $n.\text{FP-gr-id}_R(M)$, is defined to be the least integer $k$ such that $\text{Ext}^{k+1}_R(K_{n-1}, M) = 0$ for any special gr-presented module $K_{n-1}$ in $R$-gr. The $n$-gr-flat dimension of a graded right $R$-module $M$, denoted by $n.\text{gr-fd}_R(M)$, is defined to be the least integer $k$ such that $\text{Tor}_{k+1}^R(M, K_{n-1}) = 0$ for any special gr-presented module $K_{n-1}$ in $R$-gr. Also, $r.\text{n.FP-gr-dim}(R) = \sup \{n.\text{FP-gr-id}_R(M) \mid M \text{ is a graded left module} \}$ and $r.\text{n-gr-dim}(R) = \sup \{n.\text{gr-fd}_R(M) \mid M \text{ is a graded right module} \}$.

3 Gorenstein $n$-FP-gr-Injective and Gorenstein $n$-gr-Flat Modules

In this section, we introduce and study Gorenstein $n$-FP-gr-injective and Gorenstein $n$-gr-flat modules which are defined as follows:

**Definition 3.1.** Let $R$ be a graded ring and $n \geq 1$ an integer. Then, a module $M$ in $R$-gr is called Gorenstein $n$-FP-gr-injective if there exists an exact sequence of $n$-FP-gr-injective modules in $R$-gr of this form:

$$A = \cdots \longrightarrow A_1 \longrightarrow A_0 \longrightarrow A^0 \longrightarrow A^1 \longrightarrow \cdots$$

with $M = \ker(A^0 \to A^1)$ such that $\text{HOM}_R(K_{n-1}, A)$ is an exact sequence whenever $K_{n-1}$ is a special gr-presented module in $R$-gr with $\text{gr-pd}_R(K_{n-1}) < \infty$. 


The class of Gorenstein $n$-FP-gr-injective will be denoted $\mathcal{G}_{gr-FI_n}$.

A module $N$ in $gr-R$ is called Gorenstein $n$-gr-flat if there exists the following exact sequence of $n$-gr-flat modules in $gr-R$ of this form:

$$F = \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F^0 \rightarrow F^1 \rightarrow \cdots$$

with $N = \ker(F^0 \rightarrow F^1)$ such that $F \otimes_R K_{n-1}$ is an exact sequence whenever $K_{n-1}$ is a special gr-presented module in $R$-gr with $gr\text{-}fd_R(K_{n-1}) < \infty$.

The class of Gorenstein $n$-FP-gr-flat will be denoted $\mathcal{G}_{gr-F_n}$.

In the ungraded case, the $R$-modules $A_i$ and $A^i$ (resp. $F_i$ and $F^i$) as in the definition above, are called $n$-FP-injective (resp. $n$-flat). Also, $R$-modules $M$ and $N$ are called Gorenstein $n$-FP-injective and Gorenstein $n$-flat, respectively, and $K_{n-1}$ is a special presented left module with respect to any $n$-presented left $R$-module $U$.

**Remark 3.2.** Let $R$ be a graded ring. Then:

1. $gr-I \subseteq gr-FI_1 \subseteq gr-FI_2 \subseteq \cdots \subseteq gr-FI_n \subseteq G_{gr-FI_n}$. But, Gorenstein $n$-FP-gr-injective $R$-modules need not be gr-injective, see Example 3.3(1). Also, $gr-F \subseteq gr-F_1 \subseteq gr-F_2 \subseteq \cdots \subseteq gr-F_n \subseteq G_{gr-F_n}$.

   In general, every Gorenstein $n$-FP-gr-injective (resp. Gorenstein $n$-gr-flat) $R$-module is not $n$-FP-gr-injective (resp. $n$-gr-flat), except in a certain state, see Proposition 3.18.

2. $G_{gr-FI_1} \subseteq G_{gr-FI_2} \subseteq \cdots \subseteq G_{gr-FI_n}$ and $G_{gr-F_1} \subseteq G_{gr-F_2} \subseteq \cdots \subseteq G_{gr-F_n}$. But for any integers $m > n$, Gorenstein $m$-FP-gr-injective (resp. Gorenstein $m$-gr-flat) $R$-modules need not be Gorenstein $n$-FP-gr-injective (resp. Gorenstein $n$-gr-flat), see Example 3.3(2, 3).

3. In Definition 3.1 it is clear that $\ker(A_i \rightarrow A_{i-1})$ and $\ker(A^i \rightarrow A^{i+1})$ are Gorenstein $n$-FP-gr-injective, and $\ker(F_i \rightarrow F_{i-1})$, $\ker(F^i \rightarrow F^{i+1})$ are Gorenstein $n$-gr-flat for any $i \geq 1$.

It is known that the trivial extension of a commutative ring $A$ by an $A$-module $M$, $R = A \ltimes M$, is a $\mathbb{Z}_2$-graded ring, see [7,8].
Example 3.3. (1) Let $K$ be a field with characteristic $p \neq 0$ and let $G = \bigcup_{k \geq 1} G_k$, where $G_k$ is the cyclic group with generator $a_k$, the order of $a_k$ is $p^k$ and $a_k = a_{k+1}^p$. Let $R = K[G]$. Then, by Remark 3.2, $R[H]$ is Gorenstein $n$-FP-gr-injective for every group $H$, since by [5] Example iii], $R[H]$ is $n$-FP-gr-injective but it is not gr-injective.

(2) Let $A$ be a field, $E$ a nonzero $A$-vector space and $R = A \times E$ be a trivial extension of $A$ by $E$. If $\dim_A E = 1$, then by Remark 3.2 every $R$-module in $R$-gr is Gorenstein $n$-FP-gr-injective, see [6, Corollary 2.2]. If $E$ is an $A$-vector space with infinite rank, then by [27, Theorem 3.4], every 2-presented module in $R$-gr is projective. So, every module in $R$-gr is 2-FP-gr-injective and hence, every module in $R$-gr is Gorenstein 2-2-FP-gr-injective. If every module in $R$-gr is Gorenstein 1-FP-gr-injective, then $R$ is gr-regular, contradiction.

(3) Let $R = k[X]$, where $k$ is a field. Then, by Theorem 3.16 every graded right $R$-module is Gorenstein 2-gr-flat, and there is a graded right $R$-module that is not Gorenstein 1-gr-flat, since $l.FP-\dim(R) \leq 1$, see Proposition 3.18 and [36, Example 3.6].

We start with the result which proves that the behaviour of Gorenstein $n$-FP-gr-injective (resp. Gorenstein $n$-gr-flat) modules in short exact sequences is the same as the one of the classical homological notions.

Proposition 3.4. Let $R$ be a graded ring. Then:

(1) For every short exact sequence $0 \to A \to B \to C \to 0$ in $R$-gr, $B$ is Gorenstein $n$-FP-gr-injective if $A$ and $C$ are Gorenstein $n$-FP-gr-injective.

(2) For every short exact sequence $0 \to A \to B \to C \to 0$ in gr-$R$, $B$ is Gorenstein $n$-gr-flat if $A$ and $C$ are Gorenstein $n$-gr-flat.

Proof. (1) By Definition 3.1, there is an exact sequence $\cdots \to A_1 \to A_0 \to A^0 \to A^1 \to \cdots$ of $n$-FP-gr-injective modules in $R$-gr, where $A = \text{Ker}(A^0 \to A^1)$, $K'_i = \text{Ker}(A_i \to A_{i-1})$ and $(K')^i = \text{Ker}(A^i \to A^{i+1})$. Also, there is an exact sequence $\cdots \to C_1 \to C_0 \to C^0 \to C^1 \to \cdots$ of $n$-FP-gr-injective modules in $R$-gr, where $C = \text{Ker}(C^0 \to C^1)$, $K''_i = \text{Ker}(C_i \to C_{i-1})$ and $(K'')^i = \text{Ker}(C^i \to C^{i+1})$. For any $n$-presented graded left module $P$, $\text{EXT}^n_R(P, A_i \oplus C_i) = \text{EXT}^n_R(P, A_i) \oplus \text{EXT}^n_R(P, C_i) = 0$, then $A_i \oplus C_i$ is $n$-FP-gr-injective for any $i \geq 0$. Similarly, $A^i \oplus C^i$ is $n$-FP-gr-injective for any $i \geq 0$. Therefore, there is an exact sequence

$$\mathcal{Y} = \cdots \to A_1 \oplus C_1 \to A_0 \oplus C_0 \to A^0 \oplus C^0 \to A^1 \oplus C^1 \to \cdots$$
of \( n \)-FP-gr-injective modules in \( R \)-gr, where \( B = \text{Ker}(A^0 \oplus C^0 \to A^1 \oplus C^1), K_i = K_i' \oplus K_i'' = \text{Ker}(A_i \oplus C_i \to A_{i-1} \oplus C_{i-1}) \) and \( K_i = (K_i')' \oplus (K_i'')' = \text{Ker}(A_i \oplus C_i \to A^{i+1} \oplus C^{i+1}) \). Let \( K_{n-1} \) be a special gr-presented module in \( R \)-gr with gr-\( \text{pd}_R(K_{n-1}) < \infty \). Then, \( \text{EXT}_R^1(K_{n-1}, B) = 0 \), and also we have: \( \text{EXT}_R^1(K_{n-1}, K_i) = \text{EXT}_R^1(K_{n-1}, K_i' \oplus K_i'') = 0 \). Similarly, \( \text{EXT}_R^1(K_{n-1}, K^i) = 0 \). Consequently, \( \text{HOM}_R(K_{n-1}, \mathcal{Y}) \) is exact and so, \( B \) is Gorenstein \( n \)-FP-gr-injective.

(2) By Definition 3.1 there is an exact sequence \( \cdots \to A_1 \to A_0 \to A^0 \to A^1 \to \cdots \) of \( n \)-gr-flat modules in \( \text{gr}-R \), where \( A = \text{Ker}(A^0 \to A^1), K_i' = \text{Ker}(A_i \to A_{i-1}) \) and \( (K_i)' = \text{Ker}(A^i \to A^{i+1}) \). Also, there is an exact sequence \( \cdots \to C_1 \to C_0 \to C^0 \to C^1 \to \cdots \) of \( n \)-gr-flat modules in \( \text{gr}-R \), where \( C = \text{Ker}(C^0 \to C^1), K_i'' = \text{Ker}(C_i \to C_{i-1}) \) and \( (K_i'')' = \text{Ker}(C_i \to C^{i+1}) \). Similarly to (1), there is an exact sequence

\[
\mathcal{Y} = \cdots \to A_1 \oplus C_1 \to A_0 \oplus C_0 \to A^0 \oplus C^0 \to A^1 \oplus C^1 \to \cdots
\]

of \( n \)-gr-flat modules in \( \text{gr}-R \), where \( B = \text{Ker}(A^0 \oplus C^0 \to A^1 \oplus C^1) \), and if \( K_{n-1} \) is a special gr-presented module in \( R \)-gr with gr-\( \text{fd}_R(K_{n-1}) < \infty \), then \( \mathcal{Y} \otimes_R K_{n-1} \) is exact and so, \( B \) is Gorenstein \( n \)-gr-flat.

Transfer results of \( n \)-FP-injective and Gorenstein \( n \)-FP-injective modules with respect to the functor \( F \) is given in the following result.

**Proposition 3.5.** Let \( R \) be a ring graded by a group \( G \).

(1) If \( M \) is an \( n \)-FP-injective left \( R \)-module, then \( F(M) \) is \( n \)-FP-gr-injective.

(2) If \( M \) is a Gorenstein \( n \)-FP-injective left \( R \)-module, then \( F(M) \) is Gorenstein \( n \)-FP-gr-injective.

**Proof.** (1) If \( 0 \to K_n \to F_{n-1} \to K_{n-1} \to 0 \) is special short exact sequence in \( R \)-gr with respect to an \( n \)-presented graded left \( R \)-module \( U \), then similar to the proof of [35, Lemma 2.3],

\[
0 = \text{Ext}_R^1(K_{n-1}, F(M)(\sigma)) = \text{Ext}_R^n(U, F(M)(\sigma)),
\]

and hence by [36, Proposition 3.10], \( F(M) \) is \( n \)-FP-gr-injective.

(2) Let \( M \) be a Gorenstein \( n \)-FP-injective left \( R \)-module. Then, there exists an exact sequence of \( n \)-FP-injective left modules:

\[
B = \cdots \to B_1 \to B_0 \to B^0 \to B^1 \to \cdots
\]
with \( M = \ker(B^0 \to B^1) \) such that \( \text{Hom}_R(K_{n-1}, B) = \ker(F(B^0) \to F(B^1)) \) is an exact sequence whenever \( K_{n-1} \) is a special finitely presented module in \( R\text{-gr} \) with \( \text{pd}_R(K_{n-1}) < \infty \). By (1), \( F(B_i) \) and \( F(B_i') \) are \( n\text{-FP-gr-injective} \) for any \( i \geq 0 \). Since the functor \( F \) is exact, we get the following exact sequence

\[
F(B) = \cdots \to F(B_1) \to F(B_0) \to F(B^0) \to F(B^1) \to \cdots
\]

of \( n\text{-FP-gr-injective} \) left \( R \)-modules with \( F(M) = \ker(F(B^0) \to F(B^1)) \). If \( K_{n-1} \) is special gr-presented left module with \( \text{gr-pd}_R(K_{n-1}) < \infty \), then \( U(K_{n-1}) \) is finitely presented with \( \text{pd}_R(U(K_{n-1})) < \infty \). By hypothesis, \( \text{Hom}_R(U(K_{n-1}), B) = \text{Hom}_{R\text{-gr}}(K_{n-1}, F(B)) \) is exact. Therefore, from \( \text{Hom}_R(U(K_{n-1}), B) = \text{Hom}_{R\text{-gr}}(K_{n-1}, F(B)) \), it follows that \( \text{Hom}_{R\text{-gr}}(K_{n-1}, F(B)) \) is exact and consequently, the isomorphism

\[
\text{HOM}_R(K_{n-1}, F(B)) = \bigoplus_{\sigma \in G} \text{HOM}_R(K_{n-1}, F(B))_\sigma \cong \bigoplus_{\sigma \in G} \text{Hom}_{R\text{-gr}}(K_{n-1}, F(B))(\sigma)
\]

implies that \( F(M) \) is Gorenstein \( n\text{-FP-gr-injective} \).

Now, we give a characterization of a graded ring \( R \) on which \( n \)-presented modules in \( R\text{-gr} \) with \( \text{gr-pd}_R(U) < \infty \) (resp. \( \text{gr-fd}_R(U) < \infty \)) are \( (n+1) \)-presented. For this, we need the following lemma.

**Lemma 3.6.** Assume that every \( n \)-presented module in \( R\text{-gr} \) with \( \text{gr-fd}_R(U) < \infty \) is \( (n+1) \)-presented. Then, for any \( t \geq 1 \):

1. \( \text{EXT}^t_R(K_{n-1}, M) = 0 \) for any Gorenstein \( n\text{-FP-gr-injective} \) left \( R \)-module \( M \) and any special gr-presented left \( R \)-module \( K_{n-1} \) with \( \text{gr-pd}_R(K_{n-1}) < \infty \).

2. \( \text{Tor}^t_R(M, K_{n-1}) = 0 \) for any Gorenstein \( n\text{-gr-flat} \) right \( R \)-module \( M \) and any special gr-presented left \( R \)-module \( K_{n-1} \) with \( \text{gr-fd}_R(K_{n-1}) < \infty \).

**Proof.** (1) Assume that \( K_{n-1} \) is a special gr-presented module in \( R\text{-gr} \) with \( \text{gr-pd}_R(K_{n-1}) \leq m \) respect to any \( n \)-presented module \( U \) in \( R\text{-gr} \). If \( M \) is a Gorenstein \( n\text{-FP-gr-injective} \) left \( R \)-module, then, there is a left \( n\text{-FP-gr-injective} \) resolution of \( M \) in \( R\text{-gr} \). So, we have:

\[
0 \to N \to E_{m-1} \to \cdots \to E_0 \to M \to 0,
\]

where every \( E_j \) is \( n\text{-FP-gr-injective} \) for every \( 0 \leq j \leq m-1 \). Since \( \text{gr-fd}_R(U) < \infty \), \( U \) is \( (n+1) \)-presented, and so \( \text{EXT}^{i+1}_R(K_{n-1}, E_j) = 0 \) for any \( i \geq 0 \). Hence, \( \text{EXT}^{i+1}_R(K_{n-1}, M) \cong \)
EXT\(^{m+i+1}(K_{n-1}, N)\), and since gr-pd\(_R(K_{n-1}) \leq m\), it follows that , EXT\(^{i+1}(K_{n-1}, M) = 0\) for any \(i \geq 0\).

(2) Assume that \(K_{n-1}\) is a special gr-presented module in \(R\)-gr with gr-fd\(_R(K_{n-1}) \leq m\) for any \(n\)-presented module \(U\) in \(R\)-gr. If \(M\) is a Gorenstein \(n\)-gr-flat right \(R\)-module, then there is a right \(n\)-gr-flat resolution of \(M\) in gr-\(R\) of the form:

\[
0 \rightarrow M \rightarrow F^0 \rightarrow \cdots \rightarrow F^{m-1} \rightarrow N \rightarrow 0,
\]

where every \(F^j\) is \(n\)-gr-flat for every \(0 \leq j \leq m-1\). Since \(U\) is \((n+1)\)-presented, we have \(\text{Tor}_{i+1}^R(F^j, K_{n-1}) = 0\) for any \(i \geq 0\). If gr-fd\(_R(K_{n-1}) \leq m\), then \(\text{Tor}_{i+1}^R(M, K_{n-1}) \cong \text{Tor}_{m+i+1}^R(N, K_{n-1}) = 0\), and so \(\text{Tor}_{i+1}^R(M, K_{n-1}) = 0\) for any \(i \geq 0\).

**Theorem 3.7.** Let \(R\) be a graded ring. Then, the following statements are equivalent:

1. Every \(n\)-presented module in \(R\)-gr with gr-pd\(_R(U) < \infty\) is \((n+1)\)-presented;

2. For every short exact sequence \(0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0\) in \(R\)-gr, \(C\) is Gorenstein \(n\)-FP-gr-injective if \(A\) and \(B\) are Gorenstein \(n\)-FP-gr-injective.

**Proof.** (1) \(\implies\) (2) If \(B\) is a Gorenstein \(n\)-FP-gr-injective module in \(R\)-gr, then by Definition 3.1 and Remark 3.2, there is an exact sequence \(0 \rightarrow K \rightarrow B_0 \rightarrow B \rightarrow 0\) in \(R\)-gr, where \(B_0\) is \(n\)-FP-gr-injective and \(K\) is Gorenstein \(n\)-FP-gr-injective. Consider the following commutative diagram with exact rows exists:

By Proposition 3.4(1), \(D\) is Gorenstein \(n\)-FP-gr-injective, and so we have a commutative diagram

\[
\begin{array}{c}
0 \\
0 \\
0 \\
0 \\
0
\end{array}
\]

\[
\begin{array}{c}
K \\
B_0 \\
B \\
C \\
N
\end{array}
\]

\[
\begin{array}{cc}
0 & 0 \\
0 & D \\
0 & A \\
0 & 0
\end{array}
\]

\[
\begin{array}{cc}
K & K \\
B & C \\
C & 0 \\
0 & 0
\end{array}
\]
in $R$-gr:

\[ \cdots \rightarrow D_1 \rightarrow D_0 \rightarrow B_0 \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots \]

where $D_i$ and $B_0$ are $n$-FP-gr-injective, $E^i$ is gr-injective, $C = \text{Ker}(E^0 \rightarrow E^1)$, $D = \text{Ker}(B_0 \rightarrow C)$, $L_i = \text{Ker}(D_i \rightarrow D_{i-1})$ and $L^i = \text{Ker}(E^i \rightarrow E^{i+1})$. By Remark 3.2, $E^i$ and $L_i$ are Gorenstein $n$-FP-gr-injective and hence by Lemma 3.6(1), $\text{EXT}^i_R(K_{n-1}, L_i) = \text{EXT}^i_R(K_{n-1}, D) = 0$ for any special gr-presented $K_{n-1}$ module in $R$-gr with gr-$\text{pd}_R(K_{n-1}) < \infty$. Therefore, we have the following exact commutative diagram:

\[ \text{Hom}(K_{n-1}, D_1) \quad \text{Hom}(K_{n-1}, D_0) \quad \cdots \]

\[ \text{Hom}(K_{n-1}, L_0) \quad \text{Hom}(K_{n-1}, D) \quad 0 \]

Hence, $C$ is Gorenstein $n$-FP-gr-injective.

(2) $\implies$ (1) Let $U$ be an $n$-presented graded left $R$-module with gr-$\text{pd}_R(U) < \infty$, and let $0 \rightarrow K_n \rightarrow F_{n-1} \rightarrow K_{n-1} \rightarrow 0$ be a special short exact sequence in $R$-gr with respect to $U$, where $K_n$ is a special gr-generated module. We show that $K_n$ is special gr-presented. Let $M$ be a Gorenstein $n$-FP-injective module and $0 \rightarrow M \rightarrow E \rightarrow L \rightarrow 0$ an exact sequence in $R$-$\text{Mod}$, where $E$ is injective. Then, $0 \rightarrow F(M) \rightarrow F(E) \rightarrow F(L) \rightarrow 0$ is exact, where $F(M)$ and $F(E)$ are Gorenstein $n$-FP-gr-injective in $R$-gr by Proposition 3.5. So by (2), we deduce that $F(L)$ is Gorenstein $n$-FP-gr-injective. We have:

\[ 0 = \text{Ext}^1_{R-\text{gr}}(F_{n-1}, F(M)) \rightarrow \text{Ext}^1_{R-\text{gr}}(K_n, F(M)) \rightarrow \text{Ext}^2_{R-\text{gr}}(K_{n-1}, F(M)) \rightarrow 0. \]

So, $\text{Ext}^1_{R-\text{gr}}(K_n, F(M)) \cong \text{Ext}^2_{R-\text{gr}}(K_{n-1}, F(M))$. On the other hand,

\[ 0 = \text{Ext}^1_{R-\text{gr}}(K_{n-1}, F(E)) \rightarrow \text{Ext}^1_{R-\text{gr}}(K_{n-1}, F(L)) \rightarrow \text{Ext}^2_{R-\text{gr}}(K_{n-1}, F(M)) \rightarrow 0. \]
Hence, \( \text{Ext}^{1}_{\text{R-gr}}(K_{n-1}, F(L)) \cong \text{Ext}^{2}_{\text{R-gr}}(K_{n-1}, F(M)) \). Since \( F(L) \) is Gorenstein \( n \)-FP-gr injective, we get \( 0 = \text{EXT}^{1}_{\text{R}}(K_{n-1}, F(L))_{\sigma} \cong \text{Ext}^{1}_{\text{R-gr}}(K_{n-1}, F(L)(\sigma)) \) for any \( \sigma \in G \). This implies that \( \text{Ext}^{1}_{\text{R-gr}}(K_{n-1}, F(L)) = 0 \) and consequently \( \text{Ext}^{1}_{\text{R-gr}}(K_{n}, F(M)) = 0 \). So, the following commutative diagram exists:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \text{Hom}_{\text{R-gr}}(K_{n}, F(M)) & \longrightarrow & \text{Hom}_{\text{R-gr}}(K_{n}, F(E)) & \longrightarrow & \text{Hom}_{\text{R-gr}}(K_{n}, F(L)) & \longrightarrow & 0 \\
& & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & \text{Hom}_{\text{R}}(K_{n}, M) & \longrightarrow & \text{Hom}_{\text{R}}(K_{n}, E) & \longrightarrow & \text{Hom}_{\text{R}}(K_{n}, L) & & \\
\end{array}
\]

So, \( \text{Ext}^{1}_{\text{R-gr}}(K_{n}, F(M)) \cong \text{Ext}^{1}_{\text{R}}(K_{n}, M) = 0 \) for any Gorenstein \( n \)-FP-injective left \( R \)-module \( M \). Since every FP-injective left module is Gorenstein \( n \)-FP-injective, \( \text{Ext}^{1}_{\text{R-gr}}(K_{n}, F(N)) \cong \text{Ext}^{1}_{\text{R}}(K_{n}, N) = 0 \) for any FP-injective left module \( N \) and so \( K_{n} \) is \( 1 \)-presented. Therefore, \( U \) is \( (n + 1) \)-presented in \( R \)-gr.

**Corollary 3.8.** Let every \( n \)-presented module in \( \text{R-gr} \) with \( \text{gr-pd}_R(U) < \infty \) be \( (n + 1) \)-presented. Then, a module \( M \) in \( \text{R-gr} \) is Gorenstein \( n \)-FP-gr-injective if and only if every gr-pure submodule and any gr-pure epimorphic image of \( M \) are Gorenstein \( n \)-FP-gr-injective.

**Proof.** (\( \Rightarrow \)) Let \( M \) be a Gorenstein \( n \)-FP-gr-injective module in \( \text{R-gr} \). If the exact sequence \( 0 \to K \to M \to M_K \to 0 \) is gr-pure, then by [3] Proposition 2.2, \( \text{EXT}^{1}_{\text{R}}(K_{n-1}, K) = 0 \) for every special gr-presented module \( K_{n-1} \) in \( \text{R-gr} \). So, we have \( 0 = \text{EXT}^{1}_{\text{R}}(K_{n-1}, K) \cong \text{EXT}^{\text{n}}_{\text{R}}(U, K) \) for any \( n \)-presented module \( U \) in \( \text{R-gr} \). Thus, \( K \) is \( n \)-FP-gr-injective, and hence \( K \) is Gorenstein \( n \)-FP-gr-injective by Remark [3,2]. Therefore, by Theorem [3,7], \( M_K \) is Gorenstein \( n \)-FP-gr-injective.

(\( \Leftarrow \)) Assume that the exact sequence \( 0 \to K \to M \to L \to 0 \) in \( \text{R-gr} \) is gr-pure, where \( L \) and \( K \) are Gorenstein \( n \)-FP-gr-injective. Then, by Proposition [3,4(1)], \( M \) is Gorenstein \( n \)-FP-gr-injective.

The following definition is the graded version of [15,23].

**Definition 3.9.** Let \( \bar{\emptyset} \) be a class of graded left \( R \)-module. Then:

1. \( \bar{\emptyset}^{\perp} = \ker \text{Ext}^{1}_{\text{R-gr}}(\bar{\emptyset}, -) = \{ C \mid \text{Ext}^{1}_{\text{R-gr}}(L, C) = 0 \text{ for any } L \in \bar{\emptyset} \} \).

2. \( \perp \bar{\emptyset} = \ker \text{Ext}^{1}_{\text{R-gr}}(-, \bar{\emptyset}) = \{ C \mid \text{Ext}^{1}_{\text{R-gr}}(C, L) = 0 \text{ for any } L \in \bar{\emptyset} \} \).
A pair \((\mathcal{F}, \mathcal{C})\) of classes of graded \(R\)-modules is called a cotorsion theory, if \(\mathcal{F}^\perp = \mathcal{C}\) and \(\mathcal{F} = \mathcal{C}^\perp\). A cotorsion theory \((\mathcal{F}, \mathcal{C})\) is called hereditary, if whenever \(0 \to F' \to F \to F'' \to 0\) is exact in \(R\)-gr with \(F, F'' \in \mathcal{F}\) then \(F'\) is also in \(\mathcal{F}\), or equivalently, if \(0 \to C' \to C \to C'' \to 0\) is an exact sequence in \(R\)-gr with \(C, C' \in \mathcal{C}\), then \(C''\) is also in \(\mathcal{C}\).

**Corollary 3.10.** Let \(R\) be a graded ring. Then, the following statements are equivalent:

1. \((^\perp \mathcal{G}_{gr-\mathcal{F}I_n}, \mathcal{G}_{gr-\mathcal{F}I_n})\) is a hereditary cotorsion pair;

2. Every \(n\)-presented module in \(R\)-gr with \(gr-pd(U) < \infty\) is \((n + 1)\)-presented and every \(M \in (^\perp \mathcal{G}_{gr-\mathcal{F}I_n})^\perp\) has an exact left \((gr-\mathcal{F}I_n)\)-resolution.

**Proof.** (1) \(\implies\) (2) Let \(M\) be a Gorenstein \(n\)-FP-injective left \(R\)-module and \(0 \to M \to E \to L \to 0\) an exact sequence in \(R\)-Mod, where \(E\) is injective. Then, \(0 \to F(M) \to F(E) \to F(L) \to 0\) is exact in \(R\)-gr, where \(F(M)\) and \(F(E)\) are Gorenstein \(n\)-FP-injective by Proposition 3.5. So by hypothesis, \(F(L)\) is Gorenstein \(n\)-FP-gr-injective. If \(U\) is an \(n\)-presented graded left \(R\)-module with \(gr-pd_R(U) < \infty\), then similar to the proof (2) \(\implies\) (1) of Theorem 3.7, it follows that \(U\) is \((n + 1)\)-presented. Since \((^\perp \mathcal{G}_{gr-\mathcal{F}I_n})^\perp = \mathcal{G}_{gr-\mathcal{F}I_n}\) and every \(N \in \mathcal{G}_{gr-\mathcal{F}I_n}\) has an left exact \((gr-\mathcal{F}I_n)\)-resolution, then \(M \in (^\perp \mathcal{G}_{gr-\mathcal{F}I_n})^\perp\) as well.

(2) \(\implies\) (1) Note that we have to show that \((^\perp \mathcal{G}_{gr-\mathcal{F}I_n})^\perp = \mathcal{G}_{gr-\mathcal{F}I_n}\). If \(M \in (^\perp \mathcal{G}_{gr-\mathcal{F}I_n})^\perp\), then \(n\)-FP-gr-injective resolution \(\cdots \to A_3 \to A_1 \to A_0 \to M \to 0\) of \(M\) in \(R\)-gr exists. Also, we have an exact sequence \(0 \to M \to E_0 \to E_1 \to \cdots\) in \(R\)-gr, where any \(E_i\) is gr-injective. So, there exists an exact sequence

\[V : \cdots \to A_3 \to A_1 \to A_0 \to E_0 \to E_1 \to \cdots\]

of \(n\)-FP-gr-injective modules in \(R\)-gr with \(M = \text{Ker}(E_0 \to E_1)\). Let \(0 \to K_n \to F_{n-1} \to K_{n-1} \to 0\) be a short exact sequence in \(R\)-gr with \(gr-pd_R(K_{n-1}) < \infty\). Then, by hypothesis, \(K_n\) is a gr-presented module with \(gr-pd_R(K_n) < \infty\). So, by [12] Theorem 6.10] and by using the inductive presumption on \(gr-pd_R(K_{n-1})\), we deduce that \(\text{HOM}_R(K_{n-1}, V)\) is exact. Thus, \(M\) is Gorenstein \(n\)-FP-gr-injective and hence \(M \in \mathcal{G}_{gr-\mathcal{F}I_n}\).

Now, if \(0 \to A \to B \to C \to 0\) is a short exact sequence in \(R\)-gr, where \(A, B \in \mathcal{G}_{gr-\mathcal{F}I_n}\), then by Theorem 3.7 \(C \in \mathcal{G}_{gr-\mathcal{F}I_n}\). Hence, the pair \((^\perp \mathcal{G}_{gr-\mathcal{F}I_n}, \mathcal{G}_{gr-\mathcal{F}I_n})\) is a hereditary cotorsion pair. 

\[\blacksquare\]
Proposition 3.11. Assume that every $n$-presented module in $R$-$gr$ with $gr$-$fd_{R}(U) < \infty$ is $(n+1)$-presented. Then, for every short exact sequence $0 \to A \to B \to C \to 0$ in $gr$-$R$, $A$ is Gorenstein $n$-$gr$-flat if $B$ and $C$ are Gorenstein $n$-$gr$-flat.

Proof. If $B$ is a Gorenstein $n$-$gr$-flat module in $gr$-$R$, then by Definition 3.1 and Remark 3.2, there is an exact sequence $0 \to B \to F^0 \to L \to 0$ in $gr$-$R$, where $F^0$ is $n$-$gr$-flat and $L$ is Gorenstein $n$-$gr$-flat. We have the following pushout diagram with exact rows:

\[
\begin{array}{ccc}
0 & 0 & 0 \\
0 & A & B & C & 0 \\
0 & A & F^0 & D & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
L & L & 0 & 0 \\
\end{array}
\]

By Proposition 3.4(2), $D$ is Gorenstein $n$-$gr$-flat, and so we have the following commutative diagram in $gr$-$R$:

\[
\cdots \to P_1 \to P_0 \to F^0 \to D^0 \to D^1 \to \cdots,
\]

\[
\begin{array}{cccc}
\downarrow & \downarrow & & \downarrow & \downarrow \\
L_0 & A & D & L^1 \\
\end{array}
\]

where $D^i$ and $F^0$ are $n$-$gr$-flat modules, $P_i$ is gr-flat, $A = \text{Ker}(F^0 \to D)$, $D = \text{Ker}(D^0 \to D^1)$, $L_i = \text{Ker}(P_i \to P_{i-1})$ and $L^i = \text{Ker}(D^i \to D^{i+1})$. By Remark 3.2, $P_i$ and $L^i$ are Gorenstein $n$-$gr$-flat and hence by Lemma 3.6(2), $\text{Tor}_t^R(L^i, K_{n-1}) = \text{Tor}_t^R(D, K_{n-1}) = 0$ for any special gr-presented module $K_{n-1}$ in $R$-$gr$ with $gr$-$fd_{R}(K_{n-1}) < \infty$ and any $t \geq 0$. So, similar to the proof $(1) \implies (2)$ of Theorem 3.7, it follows that $- \otimes_R K_{n-1}$ on the above horizontal sequence in diagram is exact and so $A$ is Gorenstein $n$-$gr$-flat.
Corollary 3.12. Let every \( n \)-presented module in \( R \)-gr with \( \text{gr-fd}_R(U) < \infty \) be \((n+1)\)-presented. Then, a module \( M \) in \( gr-R \) is Gorenstein \( n \)-gr-flat if and only if every gr-pure submodule and any gr-pure epimorphic image of \( M \) are Gorenstein \( n \)-gr-flat.

Proof. \( (\Rightarrow) \) Let \( M \) be a Gorenstein \( n \)-gr-flat module in \( gr-R \) and \( K \) a gr-pure submodule in \( M \). Then, the exact sequence \( 0 \to K \to M \to \frac{M}{K} \to 0 \) is gr-pure. So, if \( K_{n-1} \) is special gr-presented module in \( R \)-gr, then \( \text{Tor}^R_1(\frac{M}{K}, K_{n-1}) = 0 \) and consequently by \([18]\) Lemma 2.1, \( \text{Tor}^R_1(\frac{M}{K}, K_{n-1})^* \cong \text{Ext}^1_R(K_{n-1}, (\frac{M}{K})^*) = 0 \). Therefore, the exact sequence \( 0 \to (\frac{M}{K})^* \to M^* \to K^* \to 0 \) is special gr-pure in \( R \)-gr, and using \([36]\) Proposition 3.10, we deduce that \((\frac{M}{K})^*\) is \( n \)-FP-gr-injective. By \([36]\) Proposition 3.8, \( \frac{M}{K} \) is \( n \)-gr-flat, and then Proposition 3.11 shows that \( K \) is Gorenstein \( n \)-gr-flat.

(\(\Leftarrow\)) Let \( K \) be a gr-pure submodule in \( M \). Then, the exact sequence \( 0 \to K \to M \to \frac{M}{K} \to 0 \) is gr-pure. So, it follows, by Proposition 3.4(2), that \( M \) is Gorenstein \( n \)-gr-flat.

Proposition 3.13. Let \( R \) be a graded ring. Then:

1. The class \( G_{gr-FI} \) in \( R \)-gr is closed under direct products.

2. The class \( G_{gr-F} \) in \( gr-R \) is closed under direct sums.

Next definition contains some general remarks about resolving classes of graded modules which will be useful in Sections 3 and 4. We use \( gr-\mathcal{I}(R) \) to denote the class of finite injective graded left modules and the symbol \( gr-\mathcal{P}(R) \) denotes the class of finite projective graded right modules (the graded version of \([22]\) 1.1. Resolving classes).

Definition 3.14. Let \( R \) be a graded ring and \( \mathcal{X} \) a class of graded modules. Then:

1. We call \( \mathcal{X} \) \( gr \)-injectively resolving if \( gr-\mathcal{I}(R) \subseteq \mathcal{X} \), and for every short exact sequence \( 0 \to A \to B \to C \to 0 \) with \( A \in \mathcal{X} \) the conditions \( B \in \mathcal{X} \) and \( C \in \mathcal{X} \) are equivalent.

2. We call \( \mathcal{X} \) \( gr \)-projectively resolving if \( gr-\mathcal{P}(R) \subseteq \mathcal{X} \), and for every short exact sequence \( 0 \to A \to B \to C \to 0 \) with \( C \in \mathcal{X} \) the conditions \( A \in \mathcal{X} \) and \( B \in \mathcal{X} \) are equivalent.
By Definition 3.14 Propositions 3.4 3.11 3.13 Theorem 3.7 and the graded version of [22 Proposition 1.4], we have the following easy observations.

**Proposition 3.15.** Assume that every \( n \)-presented module in \( R\)-gr with \( \text{gr-fd}_R(U) < \infty \) is \((n+1)\)-presented. Then:

1. The class \( \mathcal{G}_{gr-FI_n} \) is gr-injectively resolving.
2. The class \( \mathcal{G}_{gr-FI_n} \) is closed under direct summands.
3. The class \( \mathcal{G}_{gr-F_n} \) is gr-projectively resolving.
4. The class \( \mathcal{G}_{gr-F_n} \) is closed under direct summands.

We know that, if \( R \) is a left \( n \)-gr-coherent ring, then every \( n \)-presented module in \( R\)-gr with \( \text{gr-fd}_R(U) < \infty \) is \((n+1)\)-presented. So in the following theorem according to previous results, we investigate the relationships between Gorenstein \( n \)-FP-gr-injective and Gorenstein \( n \)-gr-flat modules on \( n \)-gr-coherent rings.

**Theorem 3.16.** Let \( R \) be a left \( n \)-gr-coherent ring. Then,

1. Module \( M \) in \( R\)-gr is Gorenstein \( n \)-FP-gr-injective if and only if \( M^* \) is Gorenstein \( n \)-gr-flat in \( gr-R \).
2. Module \( M \) in \( gr-R \) is Gorenstein \( n \)-gr-flat if and only if \( M^* \) is Gorenstein \( n \)-FP-gr-injective in \( R\)-gr.

**Proof.** (1) \( \Rightarrow \) By Definition 3.1 there is an exact sequence \( \cdots \to A_1 \to A_0 \to M \to 0 \) in \( R\)-gr, where every \( A_i \) is \( n \)-FP-gr-injective, and by [36 Theorem 3.17], every \( (A_i)^* \) is \( n \)-gr-flat in \( gr-R \). So by [32 Lemma 3.53], there is an exact sequence \( 0 \to M^* \to (A_0)^* \to (A_1)^* \to \cdots \) in \( gr-R \). Hence, we have:

\[
\mathcal{E} : \cdots \to P_1 \to P_0 \to (A_0)^* \to (A_1)^* \to \cdots,
\]

where \( P_i \) is gr-projective and \( n \)-gr-flat in \( gr-R \) by Remark 3.2 and also \( M^* = \ker((A_0)^* \to (A_1)^*) \). Let \( 0 \to K_n \to F_{n-1} \to K_{n-1} \to 0 \) be a special short exact sequence in \( R\)-gr with
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Then, $K_n$ is a gr-presented module with $\text{gr-fd}_R(K_n) < \infty$, since $R$ is $n$-gr-coherent. By [32, Theorem 6.10] and by using the inductive presumption on $\text{gr-fd}_R(K_{n-1})$, we deduce that $\mathcal{Y} \otimes_R K_{n-1}$ is exact and then $M^*$ is Gorenstein $n$-gr-flat.

$(\Leftarrow)$ Let $M^*$ be a Gorenstein $n$-gr-flat module in $\text{gr}-R$. Then, by (2)$(\Longrightarrow)$, $M^{**}$ is Gorenstein $n$-FP-gr-injective in $R$-gr. By [34, Proposition 2.3.5], $M$ is gr-pure in $M^{**}$, and so by Corollary 3.8 $M$ is Gorenstein $n$-FP-gr-injective.

$(\Rightarrow)$ By Definition 3.1 there is an exact sequence $0 \to M \to F^0 \to F^1 \to \cdots$ of $n$-gr-flat modules in $\text{gr}-R$. By [36 Proposition 3.8], $(F^i)^*$ is $n$-FP-gr-injective for any $i \geq 0$. So by [32 Lemma 3.53], there is an exact sequence $\cdots \to (F^1)^* \to (F^0)^* \to M^*$ in $R$-gr. For a module $M^*$, there is an exact sequence $0 \to M^* \to E_0 \to E_1 \to \cdots$ in $R$-gr, where $E_i$ is gr-injective. Consider the following exact sequence:

\[ \cdots \to (F^1)^* \to (F^0)^* \to E_0 \to E_1 \to \cdots \]

with $M^* = \ker(E_0 \to E_1)$. Hence, by analogy with the proof (2)$(\Rightarrow)$ (1) of Corollary 3.10 we obtain that $M^*$ is Gorenstein $n$-FP-gr-injective.

$(\Leftarrow)$ Let $M^*$ be a Gorenstein $n$-FP-gr-injective module in $R$-gr. Then, by (1)$(\Rightarrow)$, $M^{**}$ is Gorenstein $n$-gr-flat in $R$-gr. By [34 Proposition 2.3.5], $M$ is gr-pure in $M^{**}$, and so by Corollary 3.12 $M$ is Gorenstein $n$-gr-flat.

Next, we are given other results of Gorenstein $n$-FP-gr-injective and $n$-gr-flat modules on $n$-gr-coherent rings.

**Proposition 3.17.** Let $R$ be a left $n$-gr-coherent ring. Then,

1. the class $G_{gr-FI_n}$ in $R$-gr is closed under direct limits.

2. the class $G_{gr-F_n}$ in $gr-R$ is closed under direct products.

**Proof.** (1) Let $U \in R$-gr be an $n$-presented module and let $\{A_i\}_{i \in I}$ be a family of $n$-FP-gr-injective modules in $R$-gr. Then by [36 Theorem 3.17], $\lim_{\longrightarrow} A_i$ is $n$-FP-gr-injective. So, if $\{M_i\}_{i \in I}$ is a family of Gorenstein $n$-FP-gr-injective modules in $R$-gr, then the following $n$-FP-gr-injective complex

\[ \mathcal{Y}_i = \cdots \to (A_i)_1 \to (A_i)_0 \to (A_i)^0 \to (A_i)^1 \to \cdots, \]


where $M_i = \ker((A_i)^0 \to (A_i)^1)$, induces the following exact sequence of $n$-FP-gr-injective modules in $R$-gr:

$$\lim M_i = \cdots \to \lim (A_i)_1 \to \lim (A_i)_0 \to \lim (A_i)^0 \to \lim (A_i)^1 \to \cdots,$$

where $\lim M_i = \ker(\lim (A_i)^0 \to \lim (A_i)^1)$. Assume that $K_{n-1}$ is special gr-presented module in $R$-gr with $\text{gr-pd}_R(K_{n-1}) < \infty$, then by [36 Proposition 3.13],

$$\text{HOM}_R(K_{n-1}, \lim \mathcal{M}) \cong \lim \text{HOM}_R(K_{n-1}, \mathcal{M}).$$

By hypothesis, $\text{HOM}_R(K_{n-1}, \mathcal{M})$ is exact, and consequently $\lim M_i$ is Gorenstein $n$-FP-gr-injective.

(2) Let $U \in R$-gr be an $n$-presented and let $\{F_i\}_{i \in I}$ be a family of $n$-gr-flat modules in gr-$R$. Then by [36 Theorem 3.17], $\text{gr-flat}(\text{gr-flat}(\text{gr-flat}(\text{gr-flat})))$ is $n$-gr-flat. So, if $\{M_i\}$ is a family of Gorenstein $n$-gr-flat modules in gr-$R$, then the following $n$-gr-flat complex

$$\mathcal{X}_i = \cdots \to (F_i)_1 \to (F_i)_0 \to (F_i)^0 \to (F_i)^1 \to \cdots,$$

where $M_i = \ker((F_i)^0 \to (F_i)^1)$, induces the following exact sequence of $n$-gr-flat modules in gr-$R$:

$$\prod_{i \in I} \mathcal{X}_i = \cdots \to \prod_{i \in I} (F_i)_1 \to \prod_{i \in I} (F_i)_0 \to \prod_{i \in I} (F_i)^0 \to \prod_{i \in I} (F_i)^1 \to \cdots,$$

where $\prod_{i \in I} M_i = \ker(\prod_{i \in I} (F_i)^0 \to \prod_{i \in I} (F_i)^1)$. If $K_{n-1}$ is special gr-presented, then

$$(\prod_{i \in I} \mathcal{X}_i \otimes_R K_{n-1}) \cong \prod_{i \in I} (\mathcal{X}_i \otimes_R K_{n-1}).$$

By hypothesis, $\mathcal{X}_i \otimes_R K_{n-1}$ is exact, and consequently $\prod_{i \in I} M_i$ is Gorenstein $n$-gr-flat.

In the following proposition, we show that if $R$ is $n$-gr-coherent, then every Gorenstein $n$-FP-gr-injective module in $R$-gr is $n$-FP-gr-injective if $\text{gr-FP-dim}(R) < \infty$, and every Gorenstein $n$-gr-flat module in gr-$R$ is $n$-gr-flat if $\text{gr-FP-dim}(R) < \infty$.

**Proposition 3.18.** Let $R$ be a left $n$-gr-coherent ring.

(1) If $\text{gr-FP-dim}(R) < \infty$, then every Gorenstein $n$-FP-gr-injective module in $R$-gr is $n$-FP-gr-injective.
(2) If $n\cdot \text{gr-dim}(R) < \infty$, then every Gorenstein $n$-gr-flat module in $gr-R$ is $n$-gr-flat.

Proof. (1) Let $l\cdot \text{FP-gr-dim}(R) \leq k$. If $M$ is a Gorenstein $n$-FP-gr-injective module in $R$-gr, then there exists an exact sequence

$$0 \to N \to A_{k-1} \to A_{k-2} \to \cdots \to A_0 \to M \to 0$$

in $R$-gr, where every $A_i$ is $n$-FP-gr-injective for any $0 \leq i \leq k - 1$. Since $R$ is $n$-gr-coherent for any $t \geq 1$, $\text{EXT}^t_R(K_{n-1}, A_i) = 0$ for all special gr-presented left modules $K_{n-1}$ with respect to every $n$-presented module $U$ in $R$-g. Let $L_i = \ker(A_i \to A_{i-1})$. Then, we have

$$\text{EXT}^{k+1}_R(K_{n-1}, N) \cong \text{EXT}^k_R(K_{n-1}, L_{k-2}) \cong \cdots \cong \text{EXT}^2_R(K_{n-1}, L_0) \cong \text{EXT}^1_R(K_{n-1}, M).$$

Since $n$-FP-gr-id$_R(N) \leq k$, then $0 = \text{EXT}^{k+1}_R(K_{n-1}, N) \cong \text{EXT}^1_R(K_{n-1}, M) \cong \text{EXT}^1_R(U, M)$ and consequently $M$ is $n$-FP-gr-injective.

(2) The proof is similar to that of (1).

4 Covers and Preenvelopes by Gorenstein graded Modules

For a graded ring $R$, let $\mathcal{F}$ be a class of graded left $R$-modules and $M$ a graded left $R$-module. Following [5, 35], we say that a graded morphism $f : F \to M$ is an $\mathcal{F}$-precover of $M$ if $F \in \mathcal{F}$ and $\text{Hom}_{R-\text{gr}}(F', F) \to \text{Hom}_{R-\text{gr}}(F', M) \to 0$ is exact for all $F' \in \mathcal{F}$. Moreover, if whenever a graded morphism $g : F \to F$ such that $fg = f$ is an automorphism of $F$, then $f : F \to M$ is called an $\mathcal{F}$-cover of $M$. The class $\mathcal{F}$ is called (pre)covering, if each object in $R$-gr has an $\mathcal{F}$-(pre)cover. Dually, the notions of $\mathcal{F}$-preenvelopes, $\mathcal{F}$-envelopes and (pre)enveloping are defined.

In this section, by using of duality pairs on $n$-gr-coherent rings, we show that the classes $\mathcal{G}_{gr-\mathcal{F}L_n}$ (resp. $\mathcal{G}_{gr-\mathcal{F}n}$) or other signs are covering and preenveloping.

Definition 4.1 (The graded version of Definition 2.1 of [23]). Let $R$ be a graded ring. Then, a duality pair over $R$ is a pair $(\mathcal{M}, \mathcal{C})$, where $\mathcal{M}$ is a class of graded left (respectively, right) $R$-modules and $\mathcal{C}$ is a class of graded right (respectively, left) $R$-modules, subject to the following conditions:
(1) For any graded module $M$, one has $M \in \mathcal{M}$ if and only if $M^* \in \mathcal{C}$.

(2) $\mathcal{C}$ is closed under direct summands and finite direct sums.

A duality pair $(\mathcal{M}, \mathcal{C})$ is called (co)product-closed, if the class of $\mathcal{M}$ is closed under graded direct (co)products, and a duality pair $(\mathcal{M}, \mathcal{C})$ is called perfect, if it is coproduct-closed, $\mathcal{M}$ is closed under extensions and $R$ belongs to $\mathcal{M}$.

**Proposition 4.2.** Let $R$ be a left $n$-gr-coherent ring. Then, the pair $(\mathcal{G}_{gr-FI_n}, \mathcal{G}_{gr-F_n})$ is a duality pair.

**Proof.** Let $M$ be an $R$-module in $R$-gr. Then by Theorem 3.16(1), $M \in \mathcal{G}_{gr-FI_n}$ if and only if $M^* \in \mathcal{G}_{gr-F_n}$. By Proposition 3.13(2), any finite direct sum of Gorenstein $n$-gr-flat modules is Gorenstein $n$-gr-flat. Also, by Proposition 3.15(4), $\mathcal{G}_{gr-F_n}$ is closed under direct summands. So, by Definition 4.1 the pair $(\mathcal{G}_{gr-FI_n}, \mathcal{G}_{gr-F_n})$ is a duality pair.

**Proposition 4.3.** Let $R$ be a left $n$-gr-coherent ring. Then, the pair $(\mathcal{G}_{gr-F_n}, \mathcal{G}_{gr-FI_n})$ is a duality pair.

**Proof.** Let $M$ be an $R$-module in $gr-R$. Then by Theorem 3.16(2), $M \in \mathcal{G}_{gr-F_n}$ if and only if $M^* \in \mathcal{G}_{gr-FI_n}$. By Proposition 3.13(1), any finite direct sum of Gorenstein $n$-gr-FP-injective modules is Gorenstein $n$-FP-gr-injective and by Proposition 3.15(2), $\mathcal{G}_{gr-FI_n}$ is closed under direct summands. So, by Definition 4.1 the pair $(\mathcal{G}_{gr-F_n}, \mathcal{G}_{gr-FI_n})$ is a duality pair.

**Theorem 4.4.** Let $R$ be a left $n$-gr-coherent ring. Then:

(1) The class $\mathcal{G}_{gr-FI_n}$ is covering and preenveloping.

(2) The class $\mathcal{G}_{gr-F_n}$ is covering and preenveloping.

**Proof.** (1) Every direct limit of Gorenstein $n$-FP-gr-injective modules and every direct product of Gorenstein $n$-FP-gr-injective modules in $R$-gr are Gorenstein $n$-FP-gr-injective by Propositions 3.17(1) and 3.13(1), respectively. Also, by Corollary 3.8 the class of Gorenstein $n$-FP-gr-injective modules in $R$-gr is closed under gr-pure submodules, gr-pure quotients and gr-pure extensions. So, by Proposition 4.2 and [36, Theorem 4.2], we deduce that every $R$-module in $R$-gr has a Gorenstein $n$-FP-gr-injective cover and a Gorenstein $n$-FP-gr-injective preenvelope.
Every direct sum of Gorenstein $n$-gr-flat modules and every direct product of Gorenstein $n$-gr-flat modules in $\text{gr}-R$ are Gorenstein $n$-gr-flat by Propositions 3.13(2) and 3.17(2), respectively. Also, by Corollary 3.12, the class of Gorenstein $n$-gr-flat modules in $\text{gr}-R$ is closed under gr-pure submodules, gr-pure quotients and gr-pure extensions. So, by Proposition 4.3 and [36, Theorem 4.2], we deduce that every $R$-module in $\text{gr}-R$ has a Gorenstein $n$-gr-flat cover and a Gorenstein $n$-gr-flat preenvelope.

Now we give some equivalent characterizations for $R$ being Gorenstein $n$-FP-gr-injective in terms of the properties of Gorenstein $n$-FP-gr-injective and Gorenstein $n$-gr-flat modules.

**Theorem 4.5.** Let $R$ be a left $n$-gr-coherent ring. Then, the following statements are equivalent:

1. $R$ is Gorenstein $n$-FP-gr-injective;
2. Every graded module in $\text{gr}-R$ has a monic Gorenstein $n$-gr-flat preenvelope;
3. Every gr-injective module in $\text{gr}-R$ is Gorenstein $n$-gr-flat;
4. Every $n$-FP-gr-injective module in $\text{gr}-R$ is Gorenstein $n$-gr-flat;
5. Every flat module in $\text{R-gr}$ is Gorenstein $n$-FP-gr-injective;
6. Every graded module in $\text{R-gr}$ has an epic Gorenstein $n$-FP-gr-injective cover.

Moreover, if $l.n$-FP-gr-$\text{dim}(R) < \infty$, then the above conditions are also equivalent to:

7. Every Gorenstein gr-flat module in $\text{gr}-R$ is Gorenstein $n$-FP-gr-injective;
8. Every graded module in $\text{R-gr}$ is Gorenstein $n$-FP-gr-injective;
9. Every Gorenstein gr-injective module in $\text{gr}-R$ is Gorenstein $n$-gr-flat.

**Proof.** (8) $\implies$ (7), (7) $\implies$ (5) and (9) $\implies$ (3) are obvious.

(1) $\implies$ (2) By Theorem 4.4(2), every module $M$ in $\text{gr}-R$ has a Gorenstein $n$-gr-flat preenvelope $f : M \rightarrow F$. By Theorem 3.16(1), $R^+\ast$ is Gorenstein $n$-gr-flat in $\text{gr}-R$, and so $\prod_{i \in I}^{\text{gr}-R} R^+\ast$ is Gorenstein $n$-gr-flat by Proposition 3.17. On the other hand, $(R^+)^\ast$ is a cogenerator in $\text{gr}-R$. Therefore, exact sequence of the form $0 \rightarrow M \xrightarrow{g} \prod_{i \in I}^{\text{gr}-R} R^+\ast$ exists, and hence homomorphism $0 \rightarrow F \xrightarrow{h} \prod_{i \in I}^{\text{gr}-R} R^+\ast$ such that $hf = g$ shows that $f$ is monic.
Let $E$ be a gr-injective module in gr-$R$. Then $E$ has a monic Gorenstein $n$-gr-flat preenvelope $f : E \to F$ by assumption. So, the split exact sequence $0 \to E \to F \to \frac{F}{E} \to 0$ exists, and so $E$ is direct summand of $F$. Hence, by Proposition 3.15, $E$ is Gorenstein $n$-gr-flat.

(3) $\implies$ (1) By (3), $R^*$ is Gorenstein $n$-gr-flat in gr-$R$, since $R^*$ is gr-injective. So, $R$ is Gorenstein $n$-FP-gr-injective in gr-$R$ by Theorem 3.16(1).

(3) $\implies$ (4) Let $M$ be an $n$-FP-gr-injective module in gr-$R$. Then by [36, Proposition 3.10], the exact sequence $0 \to M \to E^g(M) \to \frac{E^g(M)}{M} \to 0$ is special gr-pure. Since by (3), $E^g(M)$ is Gorenstein $n$-gr-flat, from Corollary 3.12, we deduce that $M$ is Gorenstein $n$-gr-flat.

(4) $\implies$ (5) Let $F$ be a flat module in gr-$R$. Then, $F^*$ is gr-injective in gr-$R$, so $F^*$ is Gorenstein $n$-gr-flat by (4), and hence $F$ is Gorenstein $n$-FP-gr-injective by Theorem 3.16(1).

(5) $\implies$ (6) By Theorem 4.4(1), every module $M$ in gr-$R$ has a Gorenstein $n$-FP-gr-injective cover $f : A \to M$. On the other hand, there exists an exact sequence $\bigoplus_{\gamma \in S} R(\gamma) \to M \to 0$ for some $S \subseteq G$. Since $R(\gamma)$ is Gorenstein $n$-FP-gr-injective by assumption, we have that $\bigoplus_{\gamma \in S} R(\gamma)$ is Gorenstein $n$-FP-gr-injective by Proposition 3.17. Thus $f$ is an epimorphism.

(6) $\implies$ (1) By hypothesis, $R$ has an epic Gorenstein $n$-FP-gr-injective cover $f : D \to R$, then we have a split exact sequence $0 \to \ker f \to D \to R \to 0$ with $D$ is a Gorenstein $n$-FP-gr-injective module in gr-$R$. So, by Proposition 3.15, $R$ is Gorenstein $n$-FP-gr-injective in gr-$R$.

(1) $\implies$ (8) Let $M$ be a graded left $R$-module. Then, there is an exact sequence $\cdots \to F_1 \to F_0 \to M \to 0$ in gr-$R$, where each $F_i$ is gr-flat. If $R$ is a Gorenstein $n$-FP-gr-injective module in gr-$R$, then by Proposition 3.18(1), $R$ is n-FP-gr-injective. Hence, by [36, Theorem 4.8], we deduce that every $F_i$ is n-FP-gr-injective. Also, for module $M$, there is an exact sequence $0 \to M \to E_0 \to E_1 \to \cdots 0$ in gr-$R$, where every $E_i$ is gr-injective. So, we have:

$$\cdots \to F_1 \to F_0 \to E_0 \to E_1 \to \cdots,$$

where $F_i$ and $E_i$ are n-FP-gr-injective and $M = \ker(E_0 \to E_1)$. Thus, similar to the proof (2) $\implies$ (1) of Corollary 3.10, we get that $M$ is Gorenstein n-FP-gr-injective.

(8) $\implies$ (9) If $M$ is a Gorenstein gr-injective module in gr-$R$, then $M^*$ is in gr-$R$. So by hypothesis, $M^*$ is Gorenstein n-FP-gr-injective, and hence by Theorem 3.16, it follows that $M$ is Gorenstein n-gr-flat.

Example 4.6. Let $R$ be a commutative, Gorenstein Noetherian, complete, local ring, $m$ its maximal ideal. Let $E = E(R/m)$ be the $R$-injective hull of the residue field $R/m$ of $R$. By [33, Theorem A],
\[ \lambda \dim(R \ltimes E) = \dim R, \text{ where } \dim R \text{ is the Krull dimension of } R. \] We suppose that \( \dim R = n \), then \((R \ltimes E)\) is \(n\)-gr-coherent. And if we take in [28 Theorem 4.2] \( n = 1 \) and \( B = \{0\} \), we get \( \text{Hom}_R(E, E) = R \). Then, by [17 Corollary 4.37], \((R \ltimes E)\) is self gr-injective which implies that \((R \ltimes E)\) is a left \(n\)-FP-gr-injective module over itself. Hence, \( R \ltimes E \) is \(n\)-FC graded ring \((n\text{-gr-coherent and } n\text{-FP-gr-injective})\), and then by Remark 3.2 \((R \ltimes E)\) is Gorenstein \(n\)-FP-gr-injective. For example, the ring \( R = K[[X_1, \ldots, X_n]] \) of formal power series in \( n \) variables over a field \( K \) which is commutative, Gorenstein Noetherian, complete, local ring, with \( \mathfrak{m} = (X_1, \ldots, X_n) \) its maximal ideal. We obtain \( \lambda \dim(R \ltimes E(R/\mathfrak{m})) = n \), that is, \( R \ltimes E(R/\mathfrak{m}) \) is \(n\)-gr-coherent ring. So according to the above \( R \ltimes E(R/\mathfrak{m}) \) is \(n\)-FC graded ring. So, every left \( R \ltimes E(R/\mathfrak{m})\)-module is Gorenstein \(n\)-FP-gr-injective.

**Proposition 4.7.** Let \( R \) be a left \( n\)-gr-coherent. Then, \((G_{gr-F_n}, (G_{gr-F_n})^\perp)\) is hereditary perfect cotorsion pair.

**Proof.** Let \( G_{gr-F_n} \) be a class of Gorenstein \(n\)-gr-flat modules in \( gr-R \). Then, by Corollary 3.12 \( G_{gr-F_n} \) is closed under gr-pure submodules, gr-pure quotients and gr-pure extensions. On the other hand, \( R \in G_{gr-F_n} \) by Remark 3.2 and \( G_{gr-F_n} \) is closed under graded direct sums by Proposition 3.13 So, it follows that duality pair \((G_{gr-F_n}, G_{gr-F_n})^\perp)\) is perfect. Consequently by [36 Theorem 4.2], \((G_{gr-F_n}, (G_{gr-F_n})^\perp)\) is perfect cotorsion pair. Consider the short exact sequence \( 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \) in \( gr-R \), where \( B \) and \( C \) are Gorenstein \(n\)-gr-flat. Then, by Proposition 3.11 \( A \) is Gorenstein \(n\)-gr-flat and hence perfect cotorsion pair \((G_{gr-F_n}, (G_{gr-F_n})^\perp)\) is hereditary. \( \blacksquare \)

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