Revisiting Linformer with a modified self-attention with linear complexity

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Abstract

Although Transformer models such as Google’s BERT and OpenAI’s GPT-3 are successful in many natural language processing tasks, training and deploying these models are costly and inefficient. Even if pre-trained models are used, deploying these models still remained a challenge due to their large size. Apart from deployment, these models take higher time during inference restricting user-friendliness. The main bottleneck is self-attention which uses quadratic time and space with respect to the sequence length. In order to reduce the quadratic time complexity of the self-attention mechanism, Linformer by Facebook’s AI research team was introduced where they showed that the self-attention mechanism can be approximated by a low-rank matrix and exploiting this finding, a new method for self-attention with linear time and space complexity was proposed by them. In the Linformer, the time complexity depends on the projection mapping dimension which acts as a hyperparameter and affects the performance of the model, tuning this hyperparameter can be time-consuming. In this paper, I proposed an alternative method for self-attention with linear complexity in time and space and is independent of the projection mapping dimension. Since this method works for long sequences this can be used for images as well as audios.

1. Introduction and related work

Transformer(Vaswani et al 2017) has become the popular model for natural language processing including text classification, translation(Ott et al., 2018), or question answering system. Models that uses transformer have a huge number of parameters starting from 340 million in BERT-large to 175 billion in GPT-3. Due to this training and deploying such models are slow and require extensive distillation or compression to use for real-life applications.

The main bottleneck is self-attention which requires $O(n^2)$ There were prior works done to reduce this complexity One method was to introduce sparsity into the attention layers by making each token to attend only a subset of tokens of an entire sequence. But this method suffers from a large performance drop with limited efficiency gain. Then Reformer was introduced which uses locally sensitive hashing was used to avoid costly computation they also proposed to use reversible layers to allow storing the only once instead of for each layer but its efficiency gain appears only after sequence length >;2048.
| Model                  | Complexity per layer | Sequential Operation |
|------------------------|----------------------|----------------------|
| Recurrent Neural Network | $O(n)$               | $O(n)$               |
| Transformer            | $O(n^2)$             | $O(1)$               |
| Sparse Transformer     | $O(n\sqrt{n})$       | $O(1)$               |
| Reformer               | $O(n \log(n))$       | $O(\log(n))$         |
| Linformer              | $O(nk)$              | $O(1)$               |
| This model             | $O(nd^2)$            | $O(1)$               |

**Table 1:** Per-layer time complexity and minimum number of sequential operations as a function of sequence length (n) for various architectures, k is the projection dimension and d is the embedding dimension.

2. **Background**

Let $Q, K, V$ be the key, query and value the attention is defined as

$$\text{soft max} \left( \frac{QW_i^O(KW_i^K)^T}{\sqrt{d_k}} \right) V W_i^V$$

Where $W_i^O$ and $W_i^K$ are the matrices learned during training.

If we use dot product and use a general function then the attention becomes

$$f\left( QW_i^O(KW_i^K)^T \right) V W_i^V$$

Now if $f$ is scaling function $f(x) = \frac{x}{n}$

Then above equation becomes

$$\left( \frac{QW_i^O(KW_i^K)^T}{n} \right) V W_i^V$$

If we take $f(\left( QW_i^O(KW_i^K)^T \right) V W_i^V)$ as attention and $f(x) = \frac{x}{\sqrt{n}}$ as scaling function then both are equivalent.
\[
\frac{QW_i^0}{\sqrt{n}} \left( \frac{KW_i^x}{\sqrt{n}} V W_i^y \right)
\]

\[\text{Step a utilizes the fact that scalar multiplication is commutative and matrix multiplication is associative.}\]

**JL lemma:**
For a given \(0 < \varepsilon < 1\) and a set \(Y\) of \(q\) points in \(\mathbb{R}^n\) such that \(n > 8 \ln(q)/\varepsilon^2\) there exists a linear map \(f: \mathbb{R}^n \rightarrow \mathbb{R}^n\) such that
\[
(1 - \varepsilon)\|x - y\|^2 \leq \|f(x) - f(y)\|^2 \leq (1 + \varepsilon)\|x - y\|^2
\]
for all \(x, y \in Y\).

**Compact region:**
Let’s first look at the definition of compact set for real numbers. A set \(S\) of real numbers is called compact if every sequence in \(S\) has a subsequence that converges to an element again contained in \(S\). Formally, a topological space \(X\) is called compact if each of its open covers has a finite subcover.

**Lipschitz continuous functions:**
Given two metric spaces \((X, d_X)\) and \((Y, d_Y)\), where \(d_X\) denotes the metric on the set \(X\) and \(d_Y\) is the metric on set \(Y\), a function \(f: X \rightarrow Y\) is called Lipschitz continuous if there exists a real constant \(K \geq 0\) such that, for all \(x_1\) and \(x_2\) in \(X\),
\[
d_Y(f(x_1), f(x_2)) \leq Kd_X(x_1, x_2)
\]

**3. Method**

**3.1 Proof that the new context mapping matrix is low rank**
The following proof is based on JL Lemma (Lindenstrauss, 1984) the following version is from (Arriaga & Vempala, 2006).

Let \(R\) be an \(k \times n\) matrix with i.i.d entries from \(N\left(0, \frac{1}{k}\right)\). For any \(x, y \in \mathbb{R}^n\) we have

**Lemma 1**
\[
\Pr(\|Rx\| \leq (1 + \varepsilon)\|x\|) \geq 1 - 2e^{-(\varepsilon^2 - \varepsilon)}k/4
\]

**Lemma 2**
\[
\Pr(\|x^T R y\| \leq \varepsilon \|xy\|) \geq 1 - 2e^{-(\varepsilon^2 - \varepsilon)}k/4
\]
Let $A = \frac{Q W_i^Q}{\sqrt{d_k}}, B = \frac{(K W_i^*)^T}{\sqrt{d_k}}$

Then,

$$P = \text{soft max}(A) \text{soft max}(B) = \exp(A) D_A^{-1} \exp(B) D_B^{-1}$$

Where $D_A$ and $D_B$ are diagonal matrices

$$(D_A)_j = \sum_{i=1}^n \exp(A_{ji}) \quad \text{and} \quad (D_B)_j = \sum_{i=1}^n \exp(B_{ji})$$

Let’s define

$$\tilde{P} = \exp(A) D_A^{-1} \exp(B) D_B^{-1} R^T R$$

We know that for any two matrices compatible for multiplication

$$\text{rank}(LM) \leq \min\{\text{rank}(L), \text{rank}(M)\}$$

$$\text{rank}\left(\exp(A) D_A^{-1} \exp(B) D_B^{-1} R^T R\right) \leq \text{rank}\left(\exp(A) D_A^{-1} \exp(B) D_B^{-1} R^T\right) \text{rank}(R) \leq \text{rank}(R) = k$$

Now we will prove that for any column vector $c \in R^n$

$$\Pr\left(\|P c - P c\| < \varepsilon\|P c\| \right) > 1 - o(1)$$

for some $k$

Applying lemma 2 for any row vector $u \in R^n$ of $P$ matrix and for any column vector $w \in R^n$ of $VW_i^T$ we obtain

$$\Pr\left(\|R^T R W - u w^T\| \leq \|u w^T\| \right) > 1 - 2e^{-(\varepsilon^2 - \varepsilon^4)k^4/4}$$

Therefore we have

$$\Pr\left(\|\hat{P} W - P w\| \leq \|P w\|\right) = \Pr\left(\|\hat{P} R^T R W - P w\| \leq \|P w\|\right)$$

$$\geq 1 - \sum_{x \in F} \left(\|x R^T R W - P w\| \leq \|x w\|\right)$$

$$> 1 - 2ne^{-(\varepsilon^2 - \varepsilon^4)k^4/4}$$
Where step (a) is based on union bound and (b) is using equation

setting $k = 5\log(n)/(\varepsilon^2 - \varepsilon^3)$ the result follows

Here is the demonstration for the same in figure 1

![Graph showing normalized cumulative eigenvalue vs eigen value index](image)

**Figure 1:** The long tail indicates that the matrix can be approximated by a low rank matrix

### 3.2 Method to construct low-rank matrix and it is close to new context mapping matrix

Now we define a new method to find attention,

\[
\text{head}_i = \text{Attention}(QW_i^Q, E_iKW_i^K, F_iW_i^V)
\]

\[
= \text{soft max} \left( \frac{QW_i^Q}{\sqrt{d_k}} \right) \text{soft max} \left( \frac{(E_iKW_i^K)^T}{\sqrt{d_k}} \right) F_iW_i^V
\]

Let \( H = \text{Soft max} \left( \frac{QW_i^Q}{\sqrt{d_k}} \right) \), \( L = \text{Soft max} \left( \frac{(E_iKW_i^K)^T}{\sqrt{d_k}} \right) \) and \( X = F_iW_i^V \)
Size of $H$ is $n \times d, L$ is $d \times k$ and $X$ is $k \times d$ according to the paper for Linformer
Since Matrix multiplication is associative
Time complexity of $(HL)X = ndk + ndk = 2ndk = O(ndk)$
Time complexity of $H(LX) = ndd + ndk = O(nd^2)$
If we multiply in the second way we can see it is independent of $k.$

Now we need to prove that for $x_1 \in H$ and $x_2 \in L$

$$\Pr(\|\exp(x_1) \exp(x_2 E_i) F y - \exp(x_1) \exp(x_2) V W_i^T\| \leq \varepsilon \|\exp(x_1) \exp(x_2) V W_i^T\|) \geq 1 - o(1)$$

This can be proved by proving for any $y \in V W_i^T$

$$\Pr(\|\exp(x_1) \exp(x_2 E_i) F y - \exp(x_1) \exp(x_2) y^T\| \leq \varepsilon \|\exp(x_1) \exp(x_2) y^T\|) \geq 1 - 2e^{-(\varepsilon^2 - \varepsilon^4)k/4}$$

(2)

Since for two vectors $a$ and $b$ $\|ab\| = \|a\|\|b\|$

Equation (1) can be rewritten as

$$\Pr(\|\exp(x_2 E_i) F y - \exp(x_2) y^T\| \leq \varepsilon \|\exp(x_2) y^T\|) \geq 1 - 2e^{-(\varepsilon^2 - \varepsilon^4)k/4}$$

(3)

Let $E_i = \delta R$ and $F_i = e^{-\delta} R$ where $R \in \mathbb{R}^{k \times n}$ with i.i.d entries from $N\left(0, \frac{1}{k}\right)$ and $\delta = 1/2^n$

By Triangle inequality of norms, we have the following

$$\|\exp(x_2 E_i) F y - \exp(x_2) y^T\| \leq \varepsilon \|\exp(x_2)\| \leq \|\exp(x_2 E_i) F y - \exp(x_2 R^T Ry)\| + \|\exp(x_2 R^T Ry) - \exp(x_2) y^T\|$$

(a) $\leq (1 + \varepsilon)\|\exp(x_2 E_i) - \exp(x_2)\| + \|\exp(x_2 R^T Ry) - \exp(x_2) y^T\|$

(b) $\leq \|\exp(x_2 R^T Ry) - \exp(x_2) y^T\| + o(\|\exp(x_2)\|\|y\|)$

(c) $\leq o(\|\exp(x_2)\|\|y\|)$

(a) is based on Cauchy inequality and (b) is based on the fact that exponential function is Lipschitz continuous in compact region then we can choose small enough $\delta = o\left(\frac{1}{n}\right)$ such that $\|\exp(\delta x R) - \exp(\delta x) R\| = o(\|\exp(x)\|)$

And $c$ is by Lemma 2
Setting \( k = 5 \log(n)(\varepsilon^2 - \varepsilon^3) \) the result follows.

Note that \( k \) is dependent on sequence length \( n \), but it can be proved for \( k \) independent of \( n \) using the fact that \( P \) is low rank.

Here is the proof for the same.

Let \( \text{rank}(P) = d \), then we can find a submatrix \( P' \in R^{2d \times d} \) of \( \exp(A) \exp(BE^T)FV \) such that rank of this submatrix is also \( d \).

Applying the result in equation - for every row of matrix \( P' \) and every column of \( V \) \( k = 9 \log(d)(\varepsilon^2 - \varepsilon^3) \) and we obtain for any row \( P_i' \) of \( P' \)

\[
\Pr\left\| \exp(P_i'E)V - \exp(P_i)V \right\| \leq 6\left\| \exp(P_i)V \right\| \geq 1 - o(1) \quad (4)
\]

\( P = AB \) in a similar way we can write \( P' = A'B' \)

Let’s define a matrix

\[
\Gamma = \begin{bmatrix}
\exp(A)\exp(BE^T)FV \\
\exp(A)\exp(B)V
\end{bmatrix}^{-1}
\begin{bmatrix}
\exp(A')\exp(B'E^T)FV \\
\exp(A')\exp(B')V
\end{bmatrix}
\]

Size of the above matrix is \( n \times 2d \)

Then, for every row \( B_i \) of \( B \) \( P_i' \) of \( P' \) and \( \Gamma_i \) of \( \Gamma \)

\[
\left\| \exp(B_iE^T)FV - \exp(B)V \right\| = \left\| \Gamma_i \exp(P_i'E^T)FV - \Gamma_i \exp(P')V \right\|
\]

\[
\overset{(a)}{\leq} \left\| \exp(P_i'E^T)FV - \exp(P')V \right\| \left\| \Gamma_i \right\|
\]

\[
\overset{(b)}{\leq} \Theta(d) \sum_{i=1}^{2d} \left\| \exp(P_i'E^T)FV - \exp(P')V \right\|
\]

\[
= \Theta(d) \sum_{i=1}^{2d} \left\| \exp(P_i'E^T)FV - \exp(P_i)V \right\|
\]

\[
\overset{(c)}{\leq} \varepsilon \Theta(d) \sum_{i=1}^{2d} \left\| \exp(P_i') \right\| \left\| V \right\|
\]

\[
\leq \varepsilon \Theta(d) \left\| \exp(P_i') \right\| \left\| V \right\|
\]

Step (a) is from the fact that \( \left\| Ax \right\| \leq \left\| A \right\| \left\| x \right\| \) where \( \left\| A \right\| = \sqrt{\lambda_{\max}(A^T A)} \) Step (b) uses matrix norm inequality \( \left\| A \right\| \leq \left\| A \right\|_F \). (c) is obtained by using equation (4).
4. Conclusion

In this paper, I introduced a new method for calculating self-attention. I showed theoretically and empirically that there exists a low-rank matrix for this matrix also. I then proposed a method to calculate this low-rank matrix and proved theoretically that the project matrix is not far from the original matrix.

5. References

Zhuoran Shen, Mingyuan Zhang, Haiyu Zhao, Shuai Yi, Hongsheng Li, Efficient Attention: Attention with Linear Complexities arXiv:1812.01243

Sinong Wang, Belinda Z. Li, Madian Khabsa, Han Fang, Hao Ma, Linformer: Self-Attention with Linear Complexity arXiv:2006.04768, 2020.

Rosa I Arriaga and Santosh Vempala. An algorithmic theory of learning: Robust concepts and random projection. Machine Learning, 63(2):161–182, 2006.

Iz Beltagy, Matthew E Peters, and Arman Cohan. Longformer: The long-document transformer. arXiv preprint arXiv:2004.05150, 2020.

Tom B Brown, Benjamin Mann, Nick Ryder, Melanie Subbiah, Jared Kaplan, Prafulla Dhariwal, Arvind Neelakantan, Pranav Shyam, Girish Sastry, Amanda Askell, et al. Language models are few-shot learners. arXiv preprint arXiv:2005.14165, 2020.

Tianqi Chen, Bing Xu, Chiyuan Zhang, and Carlos Guestrin. Training deep nets with sublinear memory cost. arXiv preprint arXiv:1604.06174, 2016.

Zihan Chen, Hongbo Zhang, Xiaoji Zhang, and Leqi Zhao. Quora question pairs, 2018. Rewon Child, Scott Gray, Alec Radford, and Ilya Sutskever. Generating long sequences with sparse transformers. arXiv preprint arXiv:1904.10509, 2019.

Jacob Devlin, Ming-Wei Chang, Kenton Lee, and Kristina Toutanova. Bert: Pre-training of deep bidirectional transformers for language understanding. In Proceedings of the 2019 Conference of the North American Chapter of the Association for Computational Linguistics: Human Language Technologies, Volume 1 (Long and Short Papers), pp. 4171–4186, 2019.

Carl Eckart and Gale Young. The approximation of one matrix by another of lower rank. Psychometrika, 1(3):211–218, 1936. Angela Fan, Pierre Stock, Benjamin Graham, Edouard Grave, Remi Gribonval, Herve Jegou, and Armand Joulin. Training with quantization noise for extreme fixed-point compression. arXiv preprint arXiv:2004.07320, 2020. Geoffrey Hinton, Oriol Vinyals, and Jeff Dean. Distilling the knowledge in a neural network. arXiv preprint arXiv:1503.02531, 2015.

Yanping Huang, Youlong Cheng, Ankur Bapna, Orhan Firat, Dehao Chen, Mia Chen, HyoukJoong Lee, Jiquan Ngiam, Quoc V Le, Yonghui Wu, et al. Gpipe: Efficient
training of giant neural networks using pipeline parallelism. In Advances in Neural Information Processing Systems, pp. 103–112, 2019.

Benoit Jacob, Skirmantas Kligys, Bo Chen, Menglong Zhu, Matthew Tang, Andrew Howard, Hartwig Adam, and Dmitry Kalenichenko. Quantization and training of neural networks for efficient integer-arithmetic-only inference. In Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition, pp. 2704–2713, 2018.

Nikita Kitaev, Lukasz Kaiser, and Anselm Levskaya. Reformer: The efficient transformer. In International Conference on Learning Representations, 2020.

Mike Lewis, Yinhan Liu, Naman Goyal, Marjan Ghazvininejad, Abdelrahman Mohamed, Omer Levy, Ves Stoyanov, and Luke Zettlemoyer. Bart: Denoising sequence-to-sequence pre-training for natural language generation, translation, and comprehension. ACL, 2019.

W Johnson J Lindenstrauss. Extensions of lipschitz maps into a hilbert space. Contemp. Math, 26: 189–206, 1984.

Myle Ott, Sergey Edunov, Alexei Baevski, Angela Fan, Sam Gross, Nathan Ng, David Grangier, and Michael Auli. fairseq: A fast, extensible toolkit for sequence modeling. In Proceedings of the 2019 Conference of the North American Chapter of the Association for Computational Linguistics (Demonstrations), pp. 48–53, 2019.
Alec Radford, Jeffrey Wu, Rewon Child, David Luan, Dario Amodei, and Ilya Sutskever. Language models are unsupervised multitask learners. OpenAI Blog, 1(8):9, 2019.

Colin Raffel, Noam Shazeer, Adam Roberts, Katherine Lee, Sharan Narang, Michael Matena, Yanqi Zhou, Wei Li, and Peter J Liu. Exploring the limits of transfer learning with a unified text-to-text transformer. arXiv preprint arXiv:1910.10683, 2019.

Pranav Rajpurkar, Jian Zhang, Konstantin Lopyrev, and Percy Liang. Squad: 100,000+ questions for machine comprehension of text. In Proceedings of the 2016 Conference on Empirical Methods in Natural Language Processing, pp. 2383–2392, 2016.

Victor Sanh, Lysandre Debut, Julien Chaumond, and Thomas Wolf. Distilbert, a distilled version of bert: smaller, faster, cheaper and lighter. arXiv preprint arXiv:1910.01108, 2019.

Richard Socher, Alex Perelygin, Jean Wu, Jason Chuang, Christopher D Manning, Andrew Y Ng, and Christopher Potts. Recursive deep models for semantic compositionality over a sentiment treebank. In Proceedings of the 2013 conference on empirical methods in natural language processing, pp. 1631–1642, 2013.

Ashish Vaswani, Noam Shazeer, Niki Parmar, Jakob Uszkoreit, Llion Jones, Aidan N Gomez, Łukasz Kaiser, and Illia Polosukhin. Attention is all you need. In Advances in neural information processing systems, pp. 5998–6008, 2017.

Alex Wang, Amanpreet Singh, Julian Michael, Felix Hill, Omer Levy, and Samuel R. Bowman. GLUE: A multi-task benchmark and analysis platform for natural language understanding. CoRR, abs/1804.07461, 2018. URL http://arxiv.org/abs/1804.07461.

Yukun Zhu, Ryan Kiros, Rich Zemel, Ruslan Salakhutdinov, Raquel Urtasun, Antonio Torralba, and Sanja Fidler. Aligning books and movies: Towards story-like visual explanations by watching movies and reading books. In Proceedings of the IEEE international conference on computer vision, pp. 19–27, 2015.