Dynamical multiple regression in function spaces, under kernel regressors, with ARH(1) errors

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Abstract A linear multiple regression model in function spaces is formulated, under temporal correlated errors. This formulation involves kernel regressors. A generalized least-squared regression parameter estimator is derived. Its asymptotic normality and strong consistency is obtained, under suitable conditions. The correlation analysis is based on a componentwise estimator of the residual autocorrelation operator. When the dependence structure of the functional error term is unknown, a plug-in generalized least-squared regression parameter estimator is formulated. Its strong-consistency is proved as well. A simulation study is undertaken to illustrate the performance of the presented approach, under different regularity conditions. An application to financial panel data is also considered.

Keywords ARH(1) errors · dynamical functional multiple regression · firm leverage maps · generalized least squared estimator · kernel regressors

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1 Introduction

Several authors highlight the advantages of the functional regression framework over discrete multivariate approaches (see, for example, Marx and Eilers, 1999; Ramsay and Silverman, 2005; Cuevas, Febrero and Fraiman, 2002). Indeed, only in the functional setting, we can incorporate smoothness assumptions on the predictors, and the regression parameter space. In particular, Crambes, Kneip and Sarda (2009) derive a smoothing splines estimator for the functional slope parameter. They prove that the rate of convergence of the prediction error depends on the smoothness of the slope function, and on the structure of the predictors. An overview on functional principal component regression and functional partial least-squared regression, in the parameter estimation of the functional linear model with scalar response, is presented in Febrero-Bande, Galeano and Gonzalez-Manteiga (2015). There exists an extensive literature on the asymptotic properties of functional regression estimators, in the case of scalar response and functional regressors (see, for example, Cai and Hall, 2006, and the references therein). A semi-functional partial linear approach for regression, based on nonparametric time series, is considered in Aneiros-Pérez and Vieu (2006; 2008). Applying the Projection Pursuit Regression principle, the approximation of the regression function in the case of a functional predictor and a scalar response is addressed in Ferraty et al. (2013) (see also Ferraty and Vieu, 2006; Ferraty and Vieu, 2011). In the nonparametric setting, the case of functional response and predictor is studied, for example, in Ferraty, Keilegom and Vieu (2012), where a kernel type estimator of the regression operator is derived, and its pointwise asymptotic normality is obtained. Goia and Vieu (2015) adopt a semiparametric approach, in a two-terms Partitioned Functional Single Index Model. Cuevas (2014) discusses central topics in Functional Data Analysis (FDA), related to probabilistic tools, definition and estimation of centrality parameters, and the main trends in regression, classification, dimension reduction, and bootstrap methods for FDA. Recent advances in the statistical analysis of high-dimensional data, including regression, from the parametric, semiparametric and nonparametric FDA frameworks, are collected in the Special Issue by Goia and Vieu (2016).

The kernel formulation of the regression parameters is usually adopted in the literature of parametric linear regression with functional response and regressors (see, for example, Chiou, Müller and Wang, 2004; Ruiz-Medina, 2011; Ruiz-Medina, 2012a; Ruiz-Medina, 2012b, and the references therein). An extensive review, and further references for functional regression approaches, including the case of functional response and regressors, can be found in Morris (2015). See also the monograph by Hsing and Eubank (2015), where several functional analytical tools are introduced, for the estimation of random elements in function spaces. The concept of $L^r - m$-approximable processes also allows to modeling the temporal dependence in the regression functional errors (see, for example, Horváth and Kokoszka, 2012). A central topic in this book is the analysis of functional data, displaying dependent structures in time and
space. A fixed effect approach in Hilbert spaces is adopted in Ruiz-Medina (2016), for FANOVA analysis under dependent errors. For simple regression, with explanatory variable taking values in some abstract space of functions, the rate of convergence of the mean squared error of the functional version of the Nadaraya–Watson kernel estimator is derived, in Benhenni, Hedli-Griche and Rachdi (2017), when the errors are represented by a stationary short or long memory process.

The present paper considers functional response and kernel regressors, and adopts the ARH(1) process framework (see Bosq, 2000), to represent the temporal correlation of the functional errors. The efficiency, consistency and asymptotic normality of a componentwise estimator of the residual autocorrelation operator can then be obtained, from the results derived, in the ARH(1) process framework (see, for example, Bosq, 2000; Bosq and Ruiz-Medina, 2014; Guillais, 2001; Mas, 2004; and Mas, 2007). The nonparametric time series model introduced in Ferraty, Goia and Vieu (2002) could also be adopted in the representation of the temporal dependence displayed by the regression error term. However, this paper focuses in the linear parametric time series framework. As proved in this paper, good asymptotic properties are displayed by the regression estimators in this framework, avoiding, in particular, some computational drawbacks, arising in the nonparametric functional statistical context. It is well-known that the functional nonparametric statistical modelling offers a more flexible framework, but suffers of the so-called curse of dimensionality, caused by the sparsity of data in high-dimensional spaces, affecting the asymptotic properties, in particular, of the nonparametric regression estimators. Geenens (2011) proposes slightly modified estimators, considering a semi-metric to measure the proximity between two random elements in an infinite-dimensional space. Furthermore, the implementation of nonparametric estimators requires the resolution of several selection problems. For example, in the implementation of the local-weighting-based approach, a smoothing parameter, and a suitable kernel must be previously selected. Recently, Kara et al. (2017a) investigate various nonparametric models, including regression, conditional distribution, conditional density and conditional hazard function, when the covariates are infinite dimensional. They prove uniform in bandwidth asymptotic results for kernel estimators of these functional operators. Data-driven bandwidth selection is also discussed for applications.

Inverse problems can be described as functional equations, where the value of the function is known or easily estimable, but the argument is unknown. In the finite-dimensional case, parameter estimation of the general linear model constitutes an example of inverse problem, where the unknown argument of the design matrix, the regression parameter, should be approximated. The usual two-dimensional definition of the design matrix involves the sample, and the covariate population dimensions. In the analysis of functional data, more complex dependence models arise, involving conditional distributions in abstract spaces. We refer to the reader to the recent contribution by Chaouch, Laib and Louani (2017), on kernel conditional mode estimation, from functional station-
ary ergodic data, in the context of random elements in semi-metric abstract spaces (see also Ling, Liu and Vieu, 2017).

This paper considers the problem of linear functional multiple regression estimation, when the response takes values in an abstract separable Hilbert space $H$, and the regressors are operators on $H$. The temporal dependence of the errors is represented, in terms of an ARH(1) time series model. Indeed, the presented approach provides a functional formulation of the parametric part, appearing in the above-referred semiparametric model adopted in Aneiros-Pérez and Vieu (2006; 2008) (but under a parametric framework, in the linear time series analysis of the temporal correlated random part).

The practical motivation of the kernel formulation of the regressors relies on the incorporation of possible correlations between the response and the regressors at different scales and domains in time, space or depth, among others. For example, the designed experiments could be run over time, with the control of the regressors over space and depth, in a period of time. This type of models arise, for instance, in the estimation of ocean surface temperature maps over time, from the evolution of related functional covariates observed at different ocean depth intervals (see Espejo, Fernández-Pascual and Ruiz-Medina, 2017).

In this paper, a financial panel data set is analyzed. Firm leverage mapping, during a given period of time, in the Spanish communities of the Iberian Peninsula, is addressed from a functional perspective. The kernel regressors are the firm factor determinants, involved in the analysis of the financing decisions of the company, depending on the industrial area sampled, and the Spanish community studied (see Section 6 and Supplementary Material II). The proposed functional estimation approach involves two steps: Generalized least-squared regression parameter estimation, and ARH(1) residual correlation analysis, for functional estimation of the response. The strong consistency of the generalized least-squared functional regression parameter estimator is derived. In the case where the auto-covariance matrix operator of the error term is unknown, the strong-consistency of the corresponding generalized least-squared plug-in estimator is obtained as well. Asymptotic normality of the generalized-least-squared functional parameter estimator is proved, in the case where the functional errors follow a known infinite-dimensional Gaussian distribution.

The outline of the paper is the following. Section 2 introduces the studied dynamical multiple regression model in Hilbert spaces, with ARH(1) error term. The generalized least-squared regression parameter estimator is derived in Section 3. Its asymptotic normality and strong consistency is obtained as well. When the functional correlation structure of the error process is unknown, sufficient conditions are considered, for the strong consistency of the generalized least-squared plug-in parameter estimator, in Section 4. A simulation study is undertaken in Section 5, to illustrate the performance of the presented approach, under different scenarios, assuming different regularity conditions on the regression functional parameters, kernel regressors, and error correlation structure. A real-data application is developed in Section 6, in the financial panel data context. Final comments are provided in Section 7.
Dynamical multivariate functional multiple regression also illustrated, in the simulation study (see also Supplementary Material I). In the Supplementary Material II, details on in the real-data application, and the practical implementation are provided.

2 The model

Let \((\Omega, \mathcal{A}, P)\) be the basic probability space, and \(H\) be a real separable Hilbert space. The following inverse problem formulation of a dynamical functional regression model is studied:

\[
Y_n = X^1_n(\beta_1) + \cdots + X^p_n(\beta_p) + \varepsilon_n, \quad n \in \mathbb{Z},
\]

where \(\beta = (\beta_1(\cdot), \ldots, \beta_p(\cdot))^T \in \mathbb{R}^p\); \(X^j_n \in \mathcal{S}(H), \ j = 1, \ldots, p, \ n \in \mathbb{Z}\), with \(\mathcal{S}(H)\) being the Hilbert space of Hilbert–Schmidt operators on \(H\), and \(Y_n, \ \varepsilon_n \in H\), for each \(n \in \mathbb{Z}\). For a given orthonormal basis \(\{\varphi_k\}_{k \geq 1}\) of \(H\), denote

\[
\langle X^j_n(\varphi_k), \varphi_l \rangle_H = x^j_{k,l}(n), \quad k, l \geq 1, \ \forall n \in \mathbb{Z}, \ j = 1, \ldots, p.
\]

Since \(X^j_n \in \mathcal{S}(H)\), then,

\[
\sum_{k,l} |x^j_{k,l}(n)|^2 < \infty,
\]

for every \(n \in \mathbb{Z}, \ j = 1, \ldots, p\), where \(=\) means the equality in the norm of \(H\).

The error term \(\varepsilon\) satisfies

\[
E[\varepsilon_n | X^1_n, \ldots, X^p_n] = 0, \quad \forall n \in \mathbb{Z}.
\]

Furthermore, \(\varepsilon\) is assumed to be a zero-mean ARH(1) process, i.e.,

\[
\varepsilon_n = \rho(\varepsilon_{n-1}) + \delta_n, \quad n \in \mathbb{Z},
\]

where \(\rho\) denotes the autocorrelation operator, which belongs to the space of bounded linear operators \(\mathcal{L}(H)\) on \(H\), satisfying \(\|\rho\|_{\mathcal{L}(H)} < 1\), for \(k \geq k_0\), for certain \(k_0 \in \mathbb{N}\). Here, \(\{\delta_n, \ n \in \mathbb{Z}\}\) is a sequence of independent and identically distributed \(H\)-valued zero-mean random variables, with trace covariance operator, i.e., defining strong-white noise in \(H\). They are uncorrelated with the random initial condition \(\varepsilon_0\) (see Bosq, 2000).

Remark 1 Let \(\{\varepsilon_n, \ n \in \mathbb{Z}\}\) be Gaussian with known auto-covariance and cross-covariance operators, the generalized least-squared estimator \(\hat{\beta}_N\), derived in Section 6 below, displays an asymptotic infinite-dimensional Normal distribution, as the functional sample size \(N \to \infty\).

A generalization of the classical linear statistical test, for checking the significance of the functional parameters \(\beta_1, \ldots, \beta_p\) can be obtained (see, for example, Theorem 3 in Section 6, in Ruiz-Medina, 2016). Indeed, under this Gaussian scenario, the adaptive selection, in time, of the regressors could be derived, from a temporal adaptive significance statistical test, keeping...
in mind the ARH(1) structure of the error term (see, for example, Kara et al., 2017b, where the same ideas motivate the use of kernel Nearest-Neighbor (kNN) estimators, in the nonparametric regression, conditional density, conditional distribution, and hazard operator based estimation).

Denote by

\[
R_0 = E[\varepsilon_0 \otimes \varepsilon_0] = E[\varepsilon_n \otimes \varepsilon_n], \quad \forall n \in \mathbb{Z},
\]

the trace autocovariance operator, and by

\[
R_1 = E[\varepsilon_0 \otimes \varepsilon_1] = E[\varepsilon_n \otimes \varepsilon_{n+1}], \quad \forall n \in \mathbb{Z},
\]

the nuclear cross-covariance operator.

The experiment is run, and a functional sample \(Y_1, \ldots, Y_N\) of size \(N\) of the response (1) is collected, under the control of the kernel regressors, \(X^1_1, \ldots, X^p_i\), for the times \(i = 1, \ldots, N\). From equations (1), and (4)–(5),

\[
\mu_{n, \mathcal{X}} = E[Y_n | X^1_n, \ldots, X^p_n] = X^1_n (\beta_1) + \cdots + X^p_n (\beta_p), \quad n = 1, \ldots, N
\]

\[
E[(Y_i - \mu_{i, \mathcal{X}}) \otimes (Y_j - \mu_{j, \mathcal{X}})] = E[\varepsilon_i \otimes \varepsilon_j] = \rho^{j-i} R_0,
\]

for \(i, j \in \{1, \ldots, N\}\), where, \(\mathcal{X}\) denotes the vector value of the covariates, to which we are conditioning. Here, in the last equation, we have applied that

\[
\varepsilon_n = \sum_{j=0}^{k} \rho^j \delta_{n-j} + \rho^{k+1} (\varepsilon_{n-k-1}), \quad k \geq 1
\]

(see equation (3.11) in Bosq, 2000). Thus, the covariance structure of the functional errors \(Y_1 - \mu_1, \ldots, Y_N - \mu_N, \mathcal{X}\) can be expressed, in matrix operator form, as follows:

\[
\mathbf{C} := E \left[ \left( (Y_1 - \mu_{1, \mathcal{X}}), \ldots, (Y_N - \mu_{N, \mathcal{X}}) \right)^T \otimes \left( (Y_1 - \mu_{1, \mathcal{X}}), \ldots, (Y_N - \mu_{N, \mathcal{X}}) \right) \right]
\]

\[
= \begin{bmatrix}
R_0 & \rho R_0 & \rho^2 R_0 & \ldots & \rho^{N-1} R_0 \\
\rho R_0 & R_0 & \rho R_0 & \ldots & \rho^{N-2} R_0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\rho^{N-1} R_0 & \rho^{N-2} R_0 & \ldots & R_0 & \rho R_0 \\
R_0 & 0 & 0 & \ldots & 0 \\
0 & R_0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & \ldots & R_0 \\
0 & 0 & \ldots & \ldots & \ldots
\end{bmatrix}
= \rho R_0,
\]

where \(I\) denotes the identity operator on \(H\).
Remark 2 The present approach can be easily extended to the case of an ARH(p), \( p \geq 2 \), error term, replacing operator \( \rho \) by

\[
\rho' = \begin{bmatrix}
\rho_1 & \rho_2 & \cdots & \rho_p \\
I & 0 & \cdots & 0 \\
0 & I & 0 & \cdots \\
0 & \cdots & I & 0
\end{bmatrix},
\]

where, as before, \( I \) denotes the identity operator on \( H \) (see Bosq, 2000, p.128).

If \( \mathbf{C}^{-1} \) exists, then

\[
\mathbf{C}^{-1} = \mathbf{R}_0^{-1} \rho^{-1}.
\]

(8)

It is clear that \( \mathbf{R}_0^{-1} \) exists if and only if \( R_0^{-1} \) exists, where

\[
\mathbf{R}_0^{-1} := \text{diag} \left( R_0^{-1}, \ldots, R_0^{-1} \right)_{N \times N}, \quad \text{with diag} \left( R_0^{-1}, \ldots, R_0^{-1} \right)_{N \times N} \text{ denoting an } N \times N \text{ diagonal matrix operator with functional diagonal entries equal to } R_0^{-1}.
\]

Assumption A1. The systems of eigenvalues of \( R_0 \) satisfy

\[
\lambda_1(R_0) > \lambda_2(R_0) > \cdots > \lambda_m(R_0) > \cdots > 0.
\]

Assumption A2. The autocorrelation operator \( \rho \) of the error term \( \varepsilon \) is a self-adjoint compact operator on \( H \).

Under Assumption A1, we can formally define the kernel \( k_{R_0} \) of the inverse \( R_0^{-1} \) of \( R_0 \) as \( k_{R_0} = \sum_{m=1}^{\infty} \frac{1}{\lambda_m(R_0)} \phi_m \otimes \phi_m \) (see Dautray and Lions, 1985, pp. 112-126). Since \( R_0 \) is a trace operator, \( \lambda_m(R_0) \to \infty \), as \( k \to \infty \).

Hence, a suitable orthonormal basis of \( H \) in \( R_1^2 / 2 \) of \( \varepsilon_n, n \in \mathbb{Z} \) (see Bosq, 2000; Da Prato and Zabczyk, 2002, Chapter 1, pp. 12–16).

Under Assumption A2, consider the system of eigenvectors \( \{ \psi_k \}_{k \geq 1} \) of the autocorrelation operator \( \rho \) satisfying

\[
\rho(\psi_k) = \lambda_k(\rho) \psi_k, \quad k \geq 1; \quad \rho(g) = \sum_{k=1}^{\infty} \lambda_k(\rho) \langle g, \psi_k \rangle_H \psi_k, \quad \forall g \in H.
\]

(9)

Lemma 1 Let \( \rho \) be the matrix operator introduced in [5]. Under Assumption A2, \( \rho \) admits the following series representation in \( H^N : \) For every \( f = (f_1, \ldots, f_N)^T \),

\[
\rho(f) = \sum_{k \geq 1} \Psi_k \begin{bmatrix}
1 & \lambda_k(\rho) & \cdots & \lambda_k(\rho)^{N-1} \\
\lambda_k(\rho) & 1 & \cdots & \lambda_k(\rho)^{N-2} \\
\vdots & \ddots & \ddots & \vdots \\
[\lambda_k(\rho)]^{N-1} & \cdots & 1
\end{bmatrix} \Psi_k^*(f),
\]

(10)
where for \( g = (g_1, \ldots, g_N)^T \in H^N \), and \( k \geq 1 \),

\[
\Psi_k^*(g) := \text{diag} (\psi_k, \ldots, \psi_k)_{N \times N} (g) = g_k
\]

\[
\Psi_k \Psi_k^*(g) = \Psi_k (g_k) = \begin{bmatrix}
\langle g_1, \psi_k \rangle_H & \psi_k \\
\langle g_2, \psi_k \rangle_H & \psi_k \\
\vdots & \vdots \\
\langle g_N, \psi_k \rangle_H & \psi_k
\end{bmatrix}
\]

\[
\Psi_k^* \Psi_k = \text{diag} ((\psi_k, \psi_k)_H, \ldots, (\psi_k, \psi_k)_H)_{N \times N} = I_{N \times N}.
\] (11)

Here, \( g_k = ((g_1, \psi_k)_H, \ldots, (g_N, \psi_k)_H)^T \), \( k \geq 1 \), and, as before \( \text{diag} (\cdots)_{N \times N} \) denotes an \( N \times N \) functional diagonal matrix. Also, \([\cdot]^{*}\) stands for the adjoint of the matrix operator \([\cdot]\), and \(I_{N \times N}\) denotes the \( N \times N \) identity matrix.

**Proof** Under **Assumption A2**, from equation (9), considering the identity

\[
\rho^j = \sum_{k=1}^{\infty} [\lambda_k (\rho)]^j \psi_k \otimes \psi_k, \quad j = 1, \ldots, N - 1,
\]

in equation (7), for \( f = (f_1, \ldots, f_N)^T \), \( \rho \) can then be expressed as

\[
\rho(f) = \sum_{k \geq 1} \begin{bmatrix}
\psi_k & 0 & \ldots & 0 \\
0 & \psi_k & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ldots & \ldots & \psi_k
\end{bmatrix} \begin{bmatrix}
1 & \lambda_k (\rho) & \ldots & [\lambda_k (\rho)]^{N-1} \\
\lambda_k (\rho) & 1 & \ldots & [\lambda_k (\rho)]^{N-2} \\
\vdots & \ddots & \ddots & \vdots \\
[\lambda_k (\rho)]^{N-1} & \ldots & \ldots & 1
\end{bmatrix} \begin{bmatrix}
f_1 \\
f_2 \\
\vdots \\
f_N
\end{bmatrix}
\]

\[
= \sum_{k \geq 1} \Psi_k A_k \Psi_k^*(f),
\] (12)

where

\[
A_k := \begin{bmatrix}
1 & \lambda_k (\rho) & \ldots & [\lambda_k (\rho)]^{N-1} \\
\lambda_k (\rho) & 1 & \ldots & [\lambda_k (\rho)]^{N-2} \\
\vdots & \ddots & \ddots & \vdots \\
[\lambda_k (\rho)]^{N-1} & \ldots & \ldots & 1
\end{bmatrix}, \quad k \geq 1.
\]

Then,

\[
\Psi_k^* \rho \Psi_k = A_k = A_k^T A_k, \quad k \geq 1.
\]
Lemma 2 below, involved in the computation of the generalized least-squared estimator \( \beta \) in model (1), defining the quadratic loss function in equation (24) are formally given by:

\[
\rho_i = \begin{bmatrix}
\lambda_1(ho) & \lambda_2(ho) & \ldots & \lambda_{N-1}(ho) \\
0 & \sqrt{1 - \lambda_1^2(ho)} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \sqrt{1 - \lambda_{N-1}^2(ho)}
\end{bmatrix}
\]

(13)

Remark 3 From Lemma 2 \( \rho \) admits an infinite-dimensional block diagonal representation, with respect to the orthonormal matrix functional system \{\Psi_k\}_{k \geq 1}, with matrix diagonal entries \( A_k \), \( k \geq 1 \). Equivalently, for \( k \geq 1 \),

\[
A_k = E \left[ (\langle Y_1 - \mu_1, x \rangle, \psi_k)_H \ldots (\langle Y_N - \mu_N, x \rangle, \psi_k)_H \right]^T
\times (\langle Y_1 - \mu_1, x \rangle, \psi_k)_H \ldots (\langle Y_N - \mu_N, x \rangle, \psi_k)_H \right] [\Psi_k^T \Psi]^{-1}
\]

\[
= E \left[ (\langle \varepsilon_1, \psi_k \rangle, \ldots, (\langle \varepsilon_N, \psi_k \rangle)_H \right]^T (\langle \varepsilon_1, \psi_k \rangle, \ldots, (\langle \varepsilon_N, \psi_k \rangle)_H \right] [\Psi_k^T \Psi]^{-1}
\]

The following lemma will be applied in the formal definition of the norm of the RKHS of \( \varepsilon \), in model (1), defining the quadratic loss function in equation (24) below, involved in the computation of the generalized least-squared estimator \( \hat{\beta} \) in the next section.

Lemma 2 For \( i, j = 1 \ldots N \), the functional entries \( \tilde{\rho}_{i,j} \) of \( \rho^{-1} = (\tilde{\rho}_{i,j})_{i,j=1\ldots N} \) are formally given by:

\[
\tilde{\rho}_{1,1} = \tilde{\rho}_{N,N} = (I - \rho^2)^{-1}
\]

\[
\tilde{\rho}_{i,i+1} = \tilde{\rho}_{j,j-1} = -(I - \rho^2)^{-1} \rho, \quad i = 1, \ldots, N - 1, \quad j = 2, \ldots, N
\]

\[
\tilde{\rho}_{i,i} = (I - \rho^2)^{-1} (I + \rho^2), \quad i = 2, \ldots, N - 1.
\]

(15)

Proof Operator \( \rho \) is invertible if and only if \( [A_k]_{N \times N} \), is invertible, for \( k \geq 1 \). The inverse \( \rho^{-1} \) then admits an infinite-dimensional block diagonal representation with respect to \{\Psi_k\}_{k \geq 1}, with matrix diagonal entries

\[
A_k^{-1} = \begin{bmatrix}
1 & \lambda_1(ho) & \ldots & \lambda_{k-1}(ho) \\
\lambda_1(ho) & 1 & \ldots & \lambda_{k-1}(ho) \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{k-2}(ho) & \lambda_{k-1}(ho) & \ldots & 1
\end{bmatrix}^{-1}
\]

\[
= [A_k^T A_k]^{-1} = A_k^{-1} [A_k^T]^{-1},
\]

where

\[
A_k^{-1} = \frac{1}{\sqrt{1 - \lambda_k^2(ho)}} \begin{bmatrix}
\sqrt{1 - \lambda_k^2(ho)} & -\lambda_k(ho) & 0 & \ldots & 0 \\
0 & 1 & -\lambda_k(ho) & \ldots & 0 \\
0 & 0 & \sqrt{1 - \lambda_k^2(ho)} & \ldots & 0 \\
0 & 0 & 0 & \ldots & 1 -\lambda_k(ho) \\
0 & 0 & 0 & \ldots & 1
\end{bmatrix}_{N \times N}, \quad k \geq 1
\]

(16)
(see, for example, Fitzmaurice et al., 2004). Thus, $\rho^{-1}$ in (8) admits the following series representation: For every $f = (f_1, \ldots, f_N)^T \in H^N$,

$$\rho^{-1}(f) = \sum_{k \geq 1} \Psi_k A_k^{-1} \Psi_k^*(f),$$  

where, for each $k \geq 1$, the $N \times N$ matrix $A_k^{-1}$ is given by

$$A_k^{-1} = \frac{1}{1 - \lambda_k^2(\rho)} \begin{bmatrix}
1 & -\lambda_k(\rho) & 0 & \cdots & \cdots & 0 \\
-\lambda_k(\rho) & 1 + \lambda_k^2(\rho) & -\lambda_k(\rho) & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & -\lambda_k(\rho) & 1 + \lambda_k^2(\rho) & -\lambda_k(\rho) \\
0 & 0 & \cdots & \cdots & -\lambda_k(\rho) & 1 
\end{bmatrix}.$$  

From (17)–(18), using Spectral Theorems on Spectral Calculus for continuous functions of self-adjoint operators on a Hilbert space (see Dautray and Lions 1985, pp. 112-126, for continuous functions, and p. 140, for the unbounded case), we obtain that, for $i, j = 1 \ldots N$, the functional entries $\tilde{\rho}_{i,j}$ of $\rho^{-1} = (\tilde{\rho}_{i,j})_{i,j=1 \ldots N}$ are defined as in equation (15).

From (8)–(15), the functional entries $\tilde{C}_{ij}$, $i, j = 1, \ldots, N$, of $C^{-1} = \left(\tilde{C}_{ij}\right)_{i,j=1,\ldots,N}$ are formally defined as

$$\tilde{C}_{i,1} = \tilde{C}_{N,N} = R_0^{-1}(I - \rho^2)^{-1}$$
$$\tilde{C}_{i,i+1} = \tilde{C}_{j,j-1} = -R_0^{-1}(I - \rho^2)^{-1} \rho, \quad i = 1, \ldots, N - 1, \quad j = 2, \ldots, N$$
$$\tilde{C}_{i,i} = R_0^{-1}(I - \rho^2)^{-1}(I + \rho^2), \quad i = 2, \ldots, N - 1.$$  

The following additional assumption is now considered:

**Assumption A3.** The eigenvectors $\{\psi_k\}_{k \geq 1}$ of $\rho$ satisfy $\{\psi_k\}_{k \geq 1} \subset R_0^{1/2}(H)$.

Under **Assumption A3**, the next lemma provides the series expansion of the functional entries of $C^{-1}$, leading to the derivation below of the generalized least-squared estimator $\hat{\beta}_N$ of $\beta$, under **Assumption A4**.

**Lemma 3** Under **Assumption A3**, since $\psi_k \in \rho(H)$, for every $k \geq 1$, the functional entries of matrix operator in (19) admit the following series expansion in the norm of $H$:
\[ \tilde{C}_{1,1}(f) = \tilde{C}_{N,N}(f) = R_0^{-1}(I - \rho^2)^{-1}(f) \]
\[ = \sum_{k,l} \frac{1}{1 - \lambda_k^2(\rho)} R_0^{-1}(\psi_k)(\psi_l) \langle \psi_k, f \rangle_H \psi_l \]
\[ = \sum_{k,l} a_{l,k} \langle \psi_k, f \rangle_H \psi_l, \quad \forall f \in H \]
\[ \tilde{C}_{i+1}(f) = \tilde{C}_{j-1}(f) = -R_0^{-1}(I - \rho^2)^{-1}(f) \]
\[ = -\sum_{k,l} \frac{\lambda_k(\rho)}{1 - \lambda_k^2(\rho)} R_0^{-1}(\psi_k)(\psi_l) \langle \psi_k, f \rangle_H \psi_l \]
\[ = \sum_{k,l} b_{l,k} \langle \psi_k, f \rangle_H \psi_l, \quad \forall f \in H, \quad i = 1, \ldots, N - 1, \quad j = 2, \ldots, N \]
\[ \tilde{C}_{i,j}(f) = R_0^{-1}(I - \rho^2)^{-1}(I + \rho^2)(f) = \sum_{k,l} \frac{1 + \lambda_k^2(\rho)}{1 - \lambda_k^2(\rho)} R_0^{-1}(\psi_k)(\psi_l) \langle \psi_k, f \rangle_H \psi_l \]
\[ = \sum_{k,l} c_{l,k} \langle \psi_k, f \rangle_H \psi_l, \quad \forall f \in H, \quad i = 2, \ldots, N - 1. \quad (20) \]

The proof follows from Assumption A3, and Spectral Theorems for compact self-adjoint operators (see Dautray and Lions 1985, pp. 112-126).

From (8)–(20), \( C^{-1} \) then admits the following series representation: For every \( f = (f_1, \ldots, f_N)^T, \ g = (g_1, \ldots, g_N)^T \in H^N, \)
\[ C^{-1}(f)(g) = \sum_{k,l} [\Psi^*_l(g)]^T H_{l,k}[\Psi^*_k(f)] \quad (21) \]
\[ H_{l,k} := \begin{bmatrix} a_{l,k} & b_{l,k} & 0 & \ldots & 0 \\ b_{l,k} & c_{l,k} & b_{l,k} & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & b_{l,k} & c_{l,k} & b_{l,k} \\ 0 & 0 & \ldots & b_{l,k} & a_{l,k} \end{bmatrix}, \quad (22) \]
where \( a_{l,k}, b_{l,k}, c_{l,k}, k, l \geq 1 \), have been introduced in (20). Equations (21)–(22) define the norm in the RKHS \( H(\epsilon) \) of \( \epsilon = (\epsilon_1, \ldots, \epsilon_N)^T \), given by
\[ \|f\|_{H(\epsilon)}^2 = C^{-1}(f)(f) = \sum_{k,l} [\Psi^*_l(f)]^T H_{l,k}[\Psi^*_k(f)], \quad \forall f \in H(\epsilon). \quad (23) \]

3 Functional regression parameter estimation

From a functional sample \( Y_1, \ldots, Y_N \), the functional parameter vector \( \beta \) in (1) is estimated by applying generalized least-squared. Thus, considering (23).
this estimator is computed as the solution to the minimization problem

\[
\hat{\beta}_N := \min_{\beta \in H_p} L_2(\beta) = \min_{\beta \in H_p} \|Y - X(\beta)\|_{H(\varepsilon)}^2
\]

\[
= \min_{\beta \in H_p} (Y - X(\beta))^T C^{-1}(Y - X(\beta))
\]

\[
= \min_{\beta \in H_p} \sum_{k,j} \langle \psi_j^* (Y - X(\beta)) \rangle^T H_{l,k} \psi_k^* (Y - X(\beta)),
\]

where

\[
X := \begin{bmatrix} X_1^T \\
\vdots \\
X_N^T \end{bmatrix} = [X_1^T, \ldots, X_p^T]
\]

\[
X_i^T := (X_1^i, \ldots, X_p^i), \quad i = 1, \ldots, N,
\]

\[
X_j = (X_1^j, \ldots, X_N^j)^T, \quad j = 1, \ldots, p
\]

\[
X_{i,n}^j(f)(g) = \sum_{k,l} x_{i,k,l}^j(n) \langle f, \psi_i \rangle_H \langle g, \psi_k \rangle_H,
\]

\[
\forall f, g \in H, \quad i = 1, \ldots, p, \quad n = 1, \ldots, N
\]

\[
Y := (Y_1, \ldots, Y_N)^T \quad \beta = (\beta_1, \ldots, \beta_p)^T.
\]

Consider

\[
\beta = \left( \sum_{k \geq 1} \langle \beta_1, \psi_k \rangle_H \psi_k, \ldots, \sum_{k \geq 1} \langle \beta_p, \psi_k \rangle_H \psi_k \right)^T
\]

\[
= \left( \sum_{k \geq 1} \beta_{1,k} \psi_k, \ldots, \sum_{k \geq 1} \beta_{p,k} \psi_k \right)^T,
\]

and the following assumption:

**Assumption A4.** Assume the regularity conditions ensuring the following identities hold:

\[
\frac{\partial \psi_j^* X(\beta)}{\partial \beta_{j_0 h_0}} = \left( \sum_{j=1}^P \sum_{h=1}^\infty \frac{\partial x_{j,k,h}^j(1)\beta_{j_0 h_0}}{\partial \beta_{j_0 h_0}} \ldots, \sum_{j=1}^P \sum_{h=1}^\infty \frac{\partial x_{j,k,h}^j(N)\beta_{j_0 h_0}}{\partial \beta_{j_0 h_0}} \right)^T
\]

\[
= \left( x_{j_0 k,h_0}^j(1), \ldots, x_{j_0 k,h_0}^j(N) \right)^T,
\]

with uniform convergence with respect to \( k \geq 1 \), for \( j_0 = 1, \ldots, p \), and \( h_0 \geq 1 \).
Under Assumption A4, denote, for $j_0 = 1, \ldots, p$,

$$
\frac{\partial \Psi^*_k X(j_0)}{\partial \beta_{j_0}} = \left( \left( \sum_{j=1}^{p} \sum_{h_0=1}^{\infty} \frac{\partial x_{j,h}(1) \beta_{j_0}}{\partial \beta_{j_0}} \right)_{h_0 \geq 1}, \ldots, \left( \sum_{j=1}^{p} \sum_{h_0=1}^{\infty} \frac{\partial x_{j,h}(N) \beta_{j_0}}{\partial \beta_{j_0}} \right)_{h_0 \geq 1} \right)^T
$$

$$
= \left( \left( x_{k,h_0}(1) \right)_{h_0 \geq 1}, \ldots, \left( x_{k,h_0}(N) \right)_{h_0 \geq 1} \right)^T \equiv \Psi^*_k X(j_0), \quad (29)
$$

where $X^{j_0}$ has been introduced in equations (25)–(26), and $\equiv$ denotes the identification $[\ell_2^N] \equiv H^N$ established by the isometry defined in terms of the orthonormal basis $\{\psi_k\}_{k \geq 1}$. Then, under Assumption A4, from equations (24)–(29), for each $j_0 = 1, \ldots, p$,

$$
\frac{\partial \|Y - X(\beta)\|^2_{H(\epsilon)}}{\partial \beta_{j_0}} = - \sum_{k,l} \frac{\partial \Psi^*_k (Y - X(\beta))^T H_{\ell,k} \Psi^*_k (Y - X(\beta))}{\partial \beta_{j_0}}
$$

$$
= - \sum_{k,l} \left[ X^{j_0} \right]^T \Psi^*_k H_{\ell,k} \Psi^*_k (Y - X(\beta)) + \left[ \Psi^*_k (Y - X(\beta))^T H_{\ell,k} \Psi^*_k X^{j_0} \right]. \quad (30)
$$

From (30), the minimizer of (24) with respect to $\beta$, i.e., the generalized least-squared estimator $\widehat{\beta}_N$ of $\beta$ is given by the solution to the following matrix functional equation:

$$
-X^T C^{-1} (Y - X(\beta)) = X^T C^{-1} X(\beta) - X^T C^{-1} Y = 0
$$

$$
-(Y - X(\beta))^T C^{-1} X = \beta^T X^T C^{-1} X - Y^T C^{-1} X = 0. \quad (31)
$$

Furthermore, under the condition that the inverse $(X^T C^{-1} X)^{-1}$ exists, the solution to (31) is defined as

$$
\widehat{\beta}_N = (X^T C^{-1} X)^{-1} X^T C^{-1} (Y_N)
$$

$$
= \beta + (X^T C^{-1} X)^{-1} X^T C^{-1} (\epsilon_N). \quad (32)
$$

Then, from (32),

$$
E[\widehat{\beta}_N] = \beta, \quad E[(\widehat{\beta}_N - \beta)(\widehat{\beta}_N - \beta)^T] = (X^T C^{-1} X)^{-1}
$$

$$
\widehat{\beta}_N \in H^p \Leftrightarrow \epsilon^T C^{-1} X (X^T C^{-1} X)^{-1} (X^T C^{-1} X)^{-1} X^T C^{-1} \epsilon < \infty, \quad \text{a.s.}, \quad (33)
$$

where a.s. denotes the almost surely equality, and the last condition in (33) should be assumed for the suitable definition of the parameter estimator $\widehat{\beta}_N$.\]
3.1 Asymptotic normality

From (33), applying Theorem 2.7 in Bosq (2000), the following central limit result provides the asymptotic normal distribution of the generalized least-squared estimator $\hat{\beta}_N$, as $N \to \infty$.

**Theorem 1** Under Assumptions A1–A4, let $\hat{\beta}_N$ be the generalized least-squared estimator defined in (32) satisfying (33). Assume that $\{\delta_n, n \in \mathbb{Z}\}$ is Gaussian strong-white noise in $H$. Then, as $N \to \infty$,

$$
\frac{(X^T C^{-1} X)^{1/2} (\hat{\beta}_N - \beta)}{\sqrt{N}} \to_d N(0, I_{N \times N}),
$$

where $I_{N \times N}$ denotes the identity operator on $H^N$.

**Proof** The proof directly follows from Theorem 2.7 in Bosq (2000), since, from equation (33), the $H$-valued components of the functional vector

$$Z = \begin{pmatrix} Z_1 \\ \vdots \\ Z_N \end{pmatrix} = (X^T C^{-1} X)^{1/2} (\hat{\beta}_N - \beta)$$

are independent and identically distributed $H$-valued random variables, with, for $i = 1, \ldots, p$,

$$Z_i = \sum_{j=1}^{N} B_{i,j}(\varepsilon_j) \sim N(0, I), \quad (34)$$

and, for $j = 1, \ldots, N$, $B_{i,j}$ denotes the $(i, j)$ functional entry of $(X^T C^{-1} X)^{1/2} (X^T C^{-1} X)^{-1} X^T C^{-1}$. As before, $I$ denotes the identity operator on $H$. The Central Limit Result provided in Theorem 2.7 in Bosq (2000) for i.i.d. $H$-valued random variables then lead to the desired result.

3.2 Strong consistency

The following conditions are required:

**Assumption A5.** There exists $Q \in \mathcal{L}(H^p)$ such that

$$\left\| \left( \frac{X^T C^{-1} X}{N} \right)^{-1} - Q \right\|_{\mathcal{L}(H^p)} \to 0, \quad N \to \infty, \quad (35)$$

where $\mathcal{L}(H^p)$ denotes the space of bounded linear operators on $H^p$.

**Assumption A6.** For every $N \geq 2$, $X$ is such that $C^{-1} XX^T C^{-1} \in \mathcal{L}(H^N)$, with $\mathcal{L}(H^N)$ denoting the space of bounded linear operators on $H^N$. 

**Theorem 2** Under Assumptions A1–A6, the generalized least-squared estimator $\hat{\beta}_N$ satisfying (32)–(33) is strong consistent in $H^p$, i.e.,

$$\|\hat{\beta}_N - \beta\|_{H^p} \to_{a.s.} 0, \quad N \to \infty. \quad (36)$$

From (32) and (35), as $N \to \infty$:

$$\left\| \hat{\beta}_N - \beta \right\|_{H^p}^2 \leq \left\langle \left( \frac{X^T C^{-1} X}{N} \right) \right\rangle_{L(H^p)} \left\| \frac{X^T C^{-1}(\varepsilon)}{N} \right\|_{H^p}^2, \quad \text{a.s.} \quad (37)$$

Furthermore, applying Cauchy-Schwarz inequality, the following a.s. identities hold:

$$\left\| \frac{X^T C^{-1}(\varepsilon)}{N} \right\|_{H^p}^2 = \left\langle \left( \frac{X^T C^{-1}(\varepsilon)}{N} \right) \cdot \left( \frac{X^T C^{-1}(\varepsilon)}{N} \right) \right\rangle_{H^p}$$

$$\left\| \frac{X^T C^{-1}(\varepsilon)}{N} \right\|_{H^p}^2 \leq \frac{1}{N^2} \left\langle \left( C^{-1} XX^T C^{-1}(\varepsilon) \right) \cdot \varepsilon \right\rangle_{H^N}$$

$$\left\| \frac{X^T C^{-1}(\varepsilon)}{N} \right\|_{H^p}^2 \leq \frac{1}{N^2} \left\| C^{-1} XX^T C^{-1}(\varepsilon) \right\|_{H^N} \left\| \varepsilon \right\|_{H^N}$$

$$\left\| \frac{X^T C^{-1}(\varepsilon)}{N} \right\|_{H^p}^2 \leq \frac{1}{N^2} \left\| C^{-1} XX^T C^{-1}(\varepsilon) \right\|_{L(H^N)} \left\| \varepsilon \right\|_{H^N}^2. \quad (38)$$

Now, consider

$$E \left\| \varepsilon \right\|_{H^N}^2 = \sum_{j=1}^{N} E \left\| \varepsilon_j \right\|_{H}^2 = N \| R_0 \|_{H^N}, \quad (39)$$

where $\| \cdot \|_{H^N}$ denotes the nuclear or trace operator norm. From (39),

$$\frac{\| \varepsilon \|_{H^N}^2}{N^2} \to_{a.s.} 0, \quad N \to \infty. \quad (40)$$

From equations (38) and (40), under Assumption A6,

$$\left\| \frac{X^T C^{-1}(\varepsilon)}{N} \right\|_{H^p}^2 \to_{a.s.} 0, \quad N \to \infty. \quad (41)$$

Under Assumption A5, from equations (37) and (41), the strong-consistency in $H^p$ of $\hat{\beta}_N$ holds.

**4 Practical implementation**

In practice, when $R_0$ and $R_1$ are unknown, ordinary least-squared is first applied, that is, $\tilde{\beta}_N = \left( X^T X \right)^{-1} X^T (Y)$, is computed, and the resulting residuals are used to approximate $R_0$ and $R_1$ as follows:

$$\tilde{R}_0 := \frac{1}{N} \sum_{m=1}^{N} [Y_m - X_m^T (\tilde{\beta}_N)] \otimes [Y_m - X_m^T (\tilde{\beta}_N)]$$

$$\tilde{R}_1 := \frac{1}{N-1} \sum_{m=1}^{N-1} [Y_m - X_m^T (\tilde{\beta}_N)] \otimes [Y_{m+1} - X_{m+1}^T (\tilde{\beta}_N)]. \quad (42)$$
In a second step, these empirical covariance operators are considered in the computation of equation (32), in terms of a suitable orthonormal empirical basis of $H$. Consider, in particular, $\{\phi_j(N)\}_{j \geq 1}$ the system of eigenvectors of the empirical autocovariance operator $\tilde{R}_0^N$, satisfying (see Bosq, 2000, pp. 102–103)

$$\tilde{R}_0^N \phi_{jN} = \lambda_{jN} \phi_{jN}, \quad j \geq 1,$$

$$\lambda_{1N} \geq \cdots \geq \lambda_{NN} \geq 0 = \lambda_{N,N+1} = \lambda_{N,N+2}, \ldots,$$  \hspace{1cm} (43)

where $\{\lambda_{jN}\}_{j \geq 1}$ is the system of eigenvalues of $\tilde{R}_0^N$. The operators $[\tilde{R}_0^N]^{-1}$ and $\tilde{\rho}_N = [\tilde{R}_1^N]^{-1}[\tilde{R}_0^N]^{-1}$ can then be computed in terms of such empirical eigenvectors. Thus, the $H$-valued residuals

$$\tilde{\varepsilon}_n := Y_n - \tilde{Y} = Y_n - X_0^1(\tilde{\beta}_1^N) - \cdots - X_p^p(\tilde{\beta}_p^N), \quad n = 1, \ldots, N,$$ \hspace{1cm} (44)

of the ordinary least-squared estimator $\tilde{\beta}_N = (\tilde{\beta}_1^N, \ldots, \tilde{\beta}_p^N)^T$ are considered, in the computation of the following estimator of the autocorrelation operator of the error process:

$$\tilde{\rho}_{kn} := \sum_{i=1}^{k_N} \sum_{j=1}^{k_N} \tilde{\rho}_{i,j,N} \phi_{iN} \otimes \phi_{jN}; \quad \tilde{\rho}_{i,j,N} = \frac{1}{N - 1} \sum_{n=1}^{N-1} \langle \tilde{\varepsilon}_n, \phi_{iN} \rangle_H \lambda_{jN} \int \langle \varepsilon_{n+1}, \phi_{jN} \rangle_H.$$ \hspace{1cm} (45)

Here, $k_N$ denotes the truncation parameter, with $k_N \leq N$, $k_N \to \infty$, and $k_N = o(N)$ (see Bosq, 2000). The estimator (45) has the same asymptotic properties as the estimator of $\rho$, computed from $\{\varepsilon_n, n = 1, \ldots, N\}$, in the case where the ordinary least-squared estimator $\tilde{\beta}_N$ of $\beta$ is strong consistent in $H^p$. In particular, $\tilde{\rho}_{kn}$ is also strong consistent, in the norm of $L(H)$ (see Chapter 8 in Bosq, 2000). Note that

$$\tilde{\varepsilon}_n := Y_n - \tilde{Y} = Y_n - X_0^1(\tilde{\beta}_1^N) - \cdots - X_p^p(\tilde{\beta}_p^N)$$

$$= \varepsilon_n + X_0^1(\beta_1 - \tilde{\beta}_1^N) + \cdots + X_p^p(\beta_p - \tilde{\beta}_p^N)$$

$$= \varepsilon_n + o_{a.s.}(1), \quad N \to \infty,$$

in view of the strong consistency of $\tilde{\beta}_N$, leading to

$$\tilde{R}_0^N = \frac{1}{N} \sum_{n=1}^{N} \tilde{\varepsilon}_n \otimes \tilde{\varepsilon}_n = \frac{1}{N} \sum_{n=1}^{N} \varepsilon_n \otimes \varepsilon_n + o_{a.s.}(1) = R_0^N + o_{a.s.}(1)$$

$$\tilde{R}_1^{N-1} = \frac{1}{N-1} \sum_{n=1}^{N-1} \tilde{\varepsilon}_n \otimes \tilde{\varepsilon}_{n+1} = \frac{1}{N-1} \sum_{n=1}^{N-1} \varepsilon_n \otimes \varepsilon_{n+1} + o_{a.s.}(1)$$

$$= R_1^{N-1} + o_{a.s.}(1),$$ \hspace{1cm} (46)

which also implies the strong consistency of $\tilde{R}_0^N$, and $\tilde{R}_1^{N-1}$, involved in the computation of (32), when $R_0$ and $R_1$ are unknown. For the strong consistency
of the ordinary least-squared parameter estimator $\tilde{\beta}_N$, under dependent errors, the following sufficient conditions are assumed:

**Assumption $\tilde{A}5$.** There exists $\tilde{Q} \in \mathcal{L}(H^p)$ such that

$$\left\| \left( \frac{X^TX}{N} \right)^{-1} - \tilde{Q} \right\|_{\mathcal{L}(H^p)} \to 0, \quad N \to \infty. \quad (47)$$

**Assumption $\tilde{A}6$.** $X$ is such that $XX^T \in \mathcal{L}(H^N)$, for every $N \geq 2$.

**Proposition 1.** Under **Assumptions $\tilde{A}5$–$\tilde{A}6$**, the ordinary least-squared parameter estimator $\tilde{\beta}_N$ is strong consistent.

Under **Assumptions $\tilde{A}5$–$\tilde{A}6$**, the proof of Proposition 1 is derived, in a similar way to Theorem 4, from the following a.s. inequality:

$$\|\tilde{\beta}_N - \beta\|^2_{H^p} \leq \left\| \left( \frac{X^TX}{N} \right)^{-1} - \tilde{Q} \right\|^2_{\mathcal{L}(H^p)} \left\| \left( \frac{X^T\varepsilon}{N} \right) \right\|^2_{H^p} \quad (48)$$

**Remark 4.** When $R_0$ and $R_1$ are unknown, the functional entries $\tilde{C}_{ij}$, $i, j = 1, \ldots, N$, of $C^{-1} = \left( \tilde{C}_{ij} \right)_{i,j=1,\ldots,N}$ in (29) can be replaced by $\tilde{R}_0$ and $\tilde{\rho}_{kN} = F(\tilde{R}_0^{-1}, \tilde{R}_0^{-1}) = \pi_{kN} \tilde{R}_0^{-1}\tilde{R}_0^{-1} \pi_{kN}$ (see equations (12)–(15)). Here, $\pi_{kN}$ denotes the orthogonal projector into the subspace of $H$ generated by the eigenvectors $\{\phi_{jN}, j = 1, \ldots, k_N\}$ of $\tilde{R}_0$, with, as before, $k_N \leq N$, $k_N \to \infty$, and $\frac{k_N}{N} \to 0$, $N \to \infty$. **Assumptions $\tilde{A}5$–$\tilde{A}6$** ensure the strong consistency of the ordinary least squared estimator $\tilde{\beta}_N$ of $\beta$. From equation (20), the eigenvectors $\{\phi_{jN}, j = 1, \ldots, k_N\}$ of $\tilde{R}_0$ a.s. converge to the eigenvectors of $R_0 = \frac{1}{N} \sum_{n=1}^{N} \varepsilon_n \otimes \varepsilon_n$, as $N \to \infty$, since $\tilde{R}_0 \to a.s. R_0$, $N \to \infty$. Under the conditions of Theorem 8.8 in Bosq (2000) (see Section 8.3 in Bosq, 2000), the strong consistency of $\tilde{\rho}_{kN}$ then holds, when $\rho$ is a Hilbert–Schmidt operator, considering $k_N$ such that

$$\frac{N \lambda_{kN}^2(R_0)}{\left( \sum_{j=1}^{k_N} a_j \right)^2 \log(N)} \to \infty, \quad N \to \infty, \quad (49)$$

where

$$a_1 = 2\sqrt{2}(\lambda_1(R_0) - \lambda_2(R_0))^{-1} \quad a_j = 2\sqrt{2} \max \left\{ (\lambda_{j-1}(R_0) - \lambda_j(R_0))^{-1}, (\lambda_j(R_0) - \lambda_{j+1}(R_0))^{-1} \right\}, \quad j \geq 2.$$

Thus, the strong consistency of the corresponding plug-in generalized least-squared estimator, $\tilde{\beta}_N$, holds from the strong consistency of $\tilde{\beta}_N$, under the conditions of Theorem 8.8 in Bosq (2000).
ARH(1)-based estimation of the functional response

The following estimator of the \( H \)-valued dynamical response is considered:

\[
\hat{Y}_N := X_1^N(\hat{\beta}_1^N) + \cdots + X_p^N(\hat{\beta}_p^N) + \hat{\rho}_{k_N}(\hat{\varepsilon}_{N-1}),
\]

(50)

where \( \hat{\rho}_{k_N}(\hat{\varepsilon}_{N-1}) \) is computed in a similar way to (45), from the residuals \( \hat{\varepsilon}_n = Y_n - X_1^N(\hat{\beta}_1^N) - \cdots - X_p^N(\hat{\beta}_p^N) \), \( n = 1, \ldots, N \), with \( \hat{\beta}_i^N \), \( i = 1, \ldots, p \), being the generalized least-squared estimators of the components of \( \beta \), based on the observation of \( Y_1, \ldots, Y_N \), computed in terms of \( C^{-1} \), or its empirical version, as given before, in the case where \( R_0 \) and \( R_1 \) are unknown.

5 Simulation Study

The performance of the presented approach is studied in the case where the eigenvectors of the autocovariance operator of the error process are unknown, as usually it occurs in practice. Model 2 below (see also Models 3 and 4 in Supplementary Material I), also illustrates the fact that the Hilbert–Schmidt assumption on the regressors can be relaxed to the compactness condition, under diagonal spectral design. Let us restrict our attention to the Gaussian case, and to the real separable Hilbert space \( H = L^2((a, b)) \), the space of square integrable functions on \((a, b)\), with \((a, b) = (0, 60)\). The following systems of eigenvectors and eigenvalues are considered:

\[
\phi_j(x) = \frac{2}{b-a} \sin \left( \frac{\pi j x}{b-a} \right), \quad j \geq 1
\]

(51)

\[
R_0(f)(x) = \sum_{k=1}^{\infty} \lambda_k(R_0) \int_a^b \phi_k(x) \phi_k(y) f(y) dy
\]

(52)

\[
R_\delta(f)(x) = \sum_{k=1}^{\infty} \lambda_k(R_\delta) \int_a^b \phi_k(x) \phi_k(y) f(y) dy
\]

(53)

\[
\rho(f)(x) = \sum_{k=1}^{\infty} \lambda_k(\rho) \int_a^b \phi_k(x) \phi_k(y) f(y) dy,
\]

(54)

\[
X_n^i(\beta_i)(x) = \sum_{k=1}^{\infty} \beta_i^1(n) \int_a^b \phi_k(x) \phi_k(y) \beta_i(y) dy, \quad i = 1, \ldots, p
\]

\[
\beta_i(x) = \sum_{k=1}^{\infty} \langle \beta_i, \phi_k \rangle_{L^2((a, b))} \phi_k(x) = \sum_{k=1}^{\infty} \beta_i(k) \phi_k(x), \quad i = 1, \ldots, p.
\]

(55)

Equation (51) defines \( \{\phi_j\}_{j \geq 1} \) as the eigenvectors of the Dirichlet negative Laplacian operator on \((a, b)\). The sequences \( \{\lambda_k(R_0)\}_{k \geq 1} \), \( \{\lambda_k(R_\delta)\}_{k \geq 1} \) and \( \{\lambda_k(\rho)\}_{k \geq 1} \) are the respective systems of eigenvalues of \( R_0 \), \( R_\delta \) and \( \rho \). Note that, in the examples below, \( \{\psi_k\}_{k \geq 1} \) coincide with the eigenvectors \( \{\phi_k\}_{k \geq 1} \) of \( R_0 \). Six models have been analyzed, displaying different regularity orders. The
observations \(Y_1, \ldots, Y_N\) of the response are generated from equations (1)–(5), in terms of (51)–(55) (one realization of a functional sample of size \(N = 200\) of the response and its estimation is represented in Supplementary Material I, for the six Models analyzed). The results for the most regular and singular scenarios are displayed here, corresponding to Model 1 and 2, respectively (results on Models 3–6 are displayed, for \(k_N = 2, 3, 4, \) and \(N = 200, 600, 1000,\) in Supplementary Material I). Tables 1 and 2 show the functional empirical mean quadratic errors

\[
EFMQE(n) = \frac{1}{r} \sum_{i=1}^{r} \frac{1}{60} \sum_{x \in (0,60)} [Y^i_n(x) - \hat{Y}^i_n(x)]^2, \tag{56}
\]

for the most unfavorable case, i.e., for the largest truncation parameter value \(k_N = 4\), and the smallest sample size \(N = 200\). Here, \(r\) denotes the number of repetitions generated. See also Supplementary Material I, where additional truncation parameter values and sample sizes are showed. The CEMQEs,

\[
CEMQE(x,n) = \frac{1}{r} \sum_{i=1}^{r} [Y^i_n(x) - \hat{Y}^i_n(x)]^2, \quad x \in (0,60), \quad n = 1, \ldots, N,
\]

are also represented, in that supplementary material. Here, \(Y^i_n(x)\) denotes the value of the response at point \(x \in (0,60)\), and \(\hat{Y}^i_n(x)\) is its estimated value, for times \(n = 1, \ldots, N = 200\), computed from the \(i\)-th generation of a functional sample of size \(N\), for \(i = 1, \ldots, r\). As given in Remark \(\#\) the optimal \(k_N\) is determined from the sample size, the convergence rate to zero of the empirical eigenvalues of \(R_0\), and the distance between the empirical eigenvalues of \(R_0\). Indeed, the optimal \(k_N\) value lies in the interval \([2, 4]\), for \(N = 200, 600, 1000\) (see Supplementary Material I).

Models 1 and 2 are defined from the following parameter values: For each \(k \geq 1,\) and \(n \geq 1,\)

\[
\text{Model 1} \quad \lambda_k(R_0) = \frac{1}{(k+1)^3}, \quad \lambda_k(R_0) = \frac{1}{(k+1)^2}, \quad \lambda_k(\rho) = \frac{1}{(k+1)},
\]

\[
x_1^k(n) = \exp(-nk^{1/10}), \quad x_2^k(n) = \exp(-nk^{15/100}),
\]

\[
x_3^k(n) = \exp(-nk^{2/10}), \quad \langle \beta_1, \phi_k \rangle_{L^2(a,b)} = \frac{1}{(k+1)^{7/5}}, \quad \langle \beta_2, \phi_k \rangle_{L^2(a,b)} = \frac{1}{(k+1)^{7/10}}, \quad \langle \beta_3, \phi_k \rangle_{L^2(a,b)} = \frac{1}{(k+1)^{9/10}}. \tag{57}
\]
Model 2  \[ \lambda_k(R_0) = \frac{1}{(k+1)^{11/10}}, \quad \lambda_k(R_\delta) = \frac{1}{(k+1)^{12/10}}, \quad \lambda_k(\rho) = \frac{1}{(k+1)^{51/100}} \]

\[ x_k^1(n) = \frac{1}{n(k+1)^{1/10}}, \quad x_k^2(n) = \frac{1}{n(k+1)^{2/100}}, \quad (\beta_1, \phi_k)_{L^2(a,b)} = \frac{1}{(k+1)^3/5} \]

\[ (\beta_2, \phi_k)_{L^2(a,b)} = \frac{1}{(k+1)^7/15}, \quad (\beta_3, \phi_k)_{L^2(a,b)} = \frac{1}{(k+1)^4/5}. \] (58)

In Model 1, the fastest velocity decay of the eigenvalues of the autocovariance and autocorrelation kernels is displayed. The regressor kernels define respective Hilbert–Schmidt integral operators. Model 2 corresponds to the most singular scenario, with the regressors being defined by compact but not Hilbert–Schmidt operators. The empirical functional mean quadratic errors, obtained from \( r = 100 \) realizations of a functional sample of size \( N = 200 \), are shown, in Table 1 for Model 1, and in Table 2 for Model 2, considering the times \( n = 10t, \ t = 1, \ldots, 20, \) from the 200 times computed. The regularity properties, i.e., continuity and differentiability properties of the regression parameter functions, and of the autocovariance kernels of the response and innovations, as well as of the autocorrelation and regressor kernels, directly affect the performance of the presented approach. For the sample sizes \( N = 200, 600, 1000, \) and truncation parameter values \( k_N = 2, 3, 4, \) tested, the best performance corresponds to Model 1, providing the most regular parametric scenario. The worst performance is observed in Model 2, corresponding to the most singular scenario, leading to the largest values of \( A_{k_N} = \sup_{j=1, \ldots, k_N} \frac{1}{\lambda_j(R_0) - \lambda_{j+1}(R_0)}. \) See Theorem 2 of Guillas (2001), which provides the convergence to zero of the functional mean-square error, in the norm of \( L(H) \). Note that according to this result, the optimal choice of \( k_N \) is such that

\[ \lambda_{k_N}^{4+2\gamma}(R_0) = \frac{c A_{k_N}^2}{N^{1-2\epsilon}}, \quad c > 0, \ \epsilon < 1/2, \ \gamma \geq 1. \]

The rate of convergence in quadratic mean is then of order

\[ \lambda_{k_N}^2(R_0) \simeq \left( \frac{A_{k_N}^2}{N^{1-2\epsilon}} \right)^{1/(\gamma+2)} \]

(see Supplementary Material I, to compare with Models 3–6).
Table 1 Model 1. Empirical Functional Mean Quadratic Errors (EFMQEs), based on $r = 100$ repetitions of a functional response sample of size $N = 200$, considering the truncation order $k_N = 4$.

| Time | EFMQE | Time | EFMQE |
|------|-------|------|-------|
| 10   | 0.0075| 110  | 0.0030|
| 20   | 0.0072| 120  | 0.0038|
| 30   | 0.0058| 130  | 0.0023|
| 40   | 0.0039| 140  | 0.0036|
| 50   | 0.0048| 150  | 0.0018|
| 60   | 0.0042| 160  | 0.0033|
| 70   | 0.0020| 170  | 0.0052|
| 80   | 0.0062| 180  | 0.0056|
| 90   | 0.0036| 190  | 0.0023|
| 100  | 0.0031| 200  | 0.0045|

Table 2 Model 2. Empirical Functional Mean Quadratic Errors (EFMQEs), based on $r = 100$ repetitions of a functional response sample of size $N = 200$, considering the truncation order $k_N = 4$.

| Time | EFMQE | Time | EFMQE |
|------|-------|------|-------|
| 10   | 0.2960| 110  | 0.0652|
| 20   | 0.3068| 120  | 0.0629|
| 30   | 0.2970| 130  | 0.0625|
| 40   | 0.3145| 140  | 0.0588|
| 50   | 0.2289| 150  | 0.0372|
| 60   | 0.2491| 160  | 0.0655|
| 70   | 0.2339| 170  | 0.0709|
| 80   | 0.1496| 180  | 0.1048|
| 90   | 0.1200| 190  | 0.1011|
| 100  | 0.0922| 200  | 0.1237|

6 Application

In this section, a panel of small and medium size Spanish companies, in different industrial areas of the 15 autonomous Spanish communities, in the Iberian Peninsula, is analyzed during the period 1999 – 2007, considering 4 industry sectors (Factories, Building, Commerce and Several). Data were collected from the SABI (Sistema de Análisis de Balances Ibéricos) database. The firm factor determinants of the leverage, considered in the analysis of the financing decisions, are: Firm size, Asset structure, Profitability, Growth, Firm risk, Age. Specifically, the leverage is measured as the ratio of the total debt to the total assets; the firm size is measured as the log of the total assets; the asset structure consists of the net fixed assets divided by the total assets of the firm; the profitability is computed as the ratio between earnings before interest, taxes amortization and depreciation, and the total assets; growth is measured in terms of the growth of the assets, calculated as the annual change of the total assets of the firm; the firm risk is given by the business risk, and it is defined as the standard deviation of the earnings before the
interest, and the taxes over book value of the total assets, during the sample period; and, finally, the age is measured as the logarithm of the number of years that the firm has been operating. These firm factor determinants depend on the Spanish community studied (spatial location in the Iberian Peninsula), and on the industrial area sampled (located by the radial argument, in the corresponding autonomous community). They are inspected during the period 1999-2007 (see Supplementary Material II, where the response, and these kernel regressors are represented for the Factory sector). Beals smoothing has been traditionally consider in Ecology to predict the probability of appearance of different species in the sample units (see, for example, Cáceres and Legendre, 2008). The overall firm structure of the Spanish communities studied, during the temporal period analyzed, has been taken into account, in the selection procedure of suitable target ‘industry sub-sectors’, in our implementation of Beals smoothing. Specifically, the following target ‘industry subsectors’ (i.e., target ‘species’) are considered: 11 target industry subsectors in Factory sector (food; beverages and tobacco; paper, cardboard, desktop and graphic arts; articles and automotive; textile manufacture and footwear; manufacturer for construction and equipment; industry wood, cork and furniture; metal-mechanical industry; chemical and paraquimica industry; diverse industries; information technology and the knowledge economy), 3 target industry subsectors in Building sector (specialized construction activities; edification; civil work), 9 target industry subsectors in Commerce sector (household items, furniture and appliances; electronic, computer and telecommunication equipment and components; hardware, glass and construction materials; machinery, furniture and equipment for agricultural and industrial activities; raw materials, agricultural, for industry and waste materials; pharmaceuticals, perfumery, clothing accessories; books and others; textile products and footwear; vehicles, motor, spare parts, fuels and lubricants), and 6 target industry subsectors in Several sector (hostelry; service to the company; distribution service; social service; consumer services; transport). The estimated probability values (by Beals

|      | 1999  | 2000  | 2001  | 2002  | 2003  | 2004  | 2005  | 2006  | 2007  |
|------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 1     | 0.0053 | 0.0066 | 0.0146 | 0.0141 | 0.0050 | 0.0061 | 0.0055 | 0.0040 | 0.0182 |
| 2     | 0.0459 | 0.0492 | 0.0285 | 0.0366 | 0.0299 | 0.0273 | 0.0198 | 0.0252 | 0.0280 |
| 3     | 0.0487 | 0.0213 | 0.0288 | 0.0384 | 0.0197 | 0.0175 | 0.0169 | 0.0146 | 0.0256 |
| 4     | 0.0038 | 0.0051 | 0.0102 | 0.0070 | 0.0065 | 0.0035 | 0.0037 | 0.0052 | 0.0092 |
| 5     | 0.0110 | 0.0127 | 0.0097 | 0.0106 | 0.0065 | 0.0088 | 0.0173 | 0.0106 | 0.0141 |
| 6     | 0.0162 | 0.0069 | 0.0161 | 0.0208 | 0.0105 | 0.0107 | 0.0115 | 0.0078 | 0.0180 |
| 7     | 0.0058 | 0.0039 | 0.0186 | 0.0212 | 0.0043 | 0.0037 | 0.0046 | 0.0037 | 0.0204 |
| 8     | 0.0070 | 0.0052 | 0.0267 | 0.0309 | 0.0057 | 0.0061 | 0.0124 | 0.0058 | 0.0376 |
| 9     | 0.0662 | 0.0515 | 0.0237 | 0.0372 | 0.0221 | 0.0265 | 0.0585 | 0.0352 | 0.0237 |
| 10    | 0.0326 | 0.0273 | 0.0467 | 0.0501 | 0.0453 | 0.0452 | 0.0445 | 0.0417 | 0.0537 |
| 11    | 0.0087 | 0.0921 | 0.0086 | 0.0017 | 0.0076 | 0.0096 | 0.0086 | 0.0059 | 0.0082 |
| 12    | 0.0062 | 0.0087 | 0.0102 | 0.0220 | 0.0054 | 0.0053 | 0.0060 | 0.0046 | 0.0107 |
| 13    | 0.0129 | 0.0073 | 0.0104 | 0.0103 | 0.0094 | 0.0109 | 0.0179 | 0.0099 | 0.0240 |
| 14    | 0.0170 | 0.0097 | 0.0249 | 0.0235 | 0.0048 | 0.0053 | 0.0085 | 0.0063 | 0.0440 |
| 15    | 0.0123 | 0.0086 | 0.0130 | 0.0137 | 0.0112 | 0.0102 | 0.0127 | 0.0057 | 0.0170 |

Table 3 Factory Sector. Mean LOOCV errors at each one of the Spanish Autonomous Communities analyzed, for the years studied in the period 1999 – 2007.
smoothing), that a given target industry subsector occurs in a specific sampling unit, play the role of weights, in the computation of a smoothed spatial version of the observed firm leverage (see mean firm leverage per community, and the Beals smoothed leverage mapping in Supplementary Material II). Spatial interpolation on a regular grid is then performed. The proposed functional regression model is fitted from such spatially interpolated and smoothed data set, in terms of the empirical eigenvectors and eigenvalues (see Supplementary Material II for more details). Given the small functional sample size \( N = 9 \), and the distance between the empirical eigenvalues of the autocovariance operator of the regression residuals, associated with the ordinary least-squared estimator (see Section 4), only one empirical eigenvector \( (k_N = 1) \) is considered in equation (45) (see also Bosq, 2000). Leave One Out Cross Validation (LOOCV) is applied to check model fitting. The mean Leave One Out Cross Validation errors at the 15 Spanish communities, for the years in the period 1999 – 2007, are displayed, in Tables 3–6, for the four industry sectors studied, respectively. (The Spanish Community Codes (SCC) are given in Table 3). Note that, a worse fitting of the model is observed for \( k_N = 2 \) and \( k_N = 3 \) (see Supplementary Material II). The best results correspond to the Factory sector followed by the Building and Commerce sectors, where the target firm subsectors seem to be selected, according to the enterprise structure of most of the Spanish communities. While in the Several industry sector the worst performance is observed, since this sector includes a greater diversity of industrial areas with little spatial dependence. Despite these observed Beals smoothing effects, the magnitude of the mean LOOCV errors are quite stable through time and space (see also mean LOOCV error maps in the Supplementary Material II, for \( k_N = 1 \)). Given the absence of records in the used database, in the Building sector in Cantabria, and in the Commerce sector in La Rioja, we omit these lines, in the corresponding mean LOOCV error tables. The effect of these missing data can be observed in the mean LOOCV error maps in Supplementary Material II. The development of the presented approach, under missing data, constitutes the subject of future work.

7 Final comments

This paper extends the generalized least-squared estimation results obtained in Ruiz-Medina (2016), on FANOVA analysis of fixed effects models in Hilber spaces, under dependent errors. Specifically, the approach presented allows the analysis of functional responses over a period of time, under the control of kernel regressor in that period. While, in Ruiz-Medina (2016), a scalar fixed effect design is considered, and the experiment is not running over time. In Benhenni, Hedli-Griche and Rachdi (2017), a functional random design is assumed in simple regression under dependent errors. Here, a kernel random design is considered in multiple regression under dependent errors. Furthermore, sufficient conditions are obtained for the explicit derivation of the generalized least-squared regression parameter estimator, beyond the restriction,
considered in Ruiz-Medina (2016), of the spectral diagonalization of the functional parameters, in terms of a common eigenvector system. In the practical implementation of the proposed methodology, a suitable orthonormal basis \( \{ \varphi_k = \psi_k, \ k \geq 1 \} \) of \( H \) must be considered. When \( H \) is an element of the scale of fractional Sobolev spaces, including \( L^2 \) spaces, wavelet bases provide unconditional bases for these spaces. In particular, \( \{ \psi_k, \ k \geq 1 \} \) can be an orthonormal wavelet basis providing an \([s]+1\)-regular multiresolution analysis of an \( L^2 \) space, for a suitable \( s > 0 \), allowing the continuous inversion of the autocovariance operator \( R_0 \). Here, \([\cdot]\) denotes the integer part. The simulation study highlights the interaction between the regularity properties of the functional data, and the performance of the presented approach, depending on the truncation order and the sample size. On the other hand, the real-data example illustrates its performance, from very small functional sample sizes, requiring small truncation orders, after applying a suitable smoothing technique. The role of the kernel regressors is illustrated as well. In our example, they soft the effect of industrial areas, in the representation of the annual Beals smoothed firm leverage maps (response), as the output of a linear filter, with input the regression parameters, incorporating the information from firm factor determinants (kernel regressors), depending on the industrial area sampled, and on the Spanish community studied.

| SCC | 1999 | 2000 | 2001 | 2002 | 2003 | 2004 | 2005 | 2006 | 2007 |
|-----|------|------|------|------|------|------|------|------|------|
| 1   | 0.0238 | 0.0163 | 0.0332 | 0.0359 | 0.0154 | 0.0169 | 0.0261 | 0.0378 | 0.0157 |
| 2   | 0.0628 | 0.0680 | 0.0703 | 0.0494 | 0.0715 | 0.0937 | 0.0648 | 0.0445 | 0.0557 |
| 4   | 0.0416 | 0.0301 | 0.0382 | 0.0474 | 0.0336 | 0.0165 | 0.0376 | 0.0477 | 0.0365 |
| 5   | 0.0290 | 0.0301 | 0.0261 | 0.0808 | 0.0191 | 0.0399 | 0.0898 | 0.0756 | 0.0389 |
| 6   | 0.0245 | 0.0163 | 0.0375 | 0.0370 | 0.0122 | 0.0507 | 0.0407 | 0.0489 | 0.0158 |
| 7   | 0.0148 | 0.0136 | 0.0230 | 0.0276 | 0.0195 | 0.0149 | 0.0177 | 0.0471 | 0.0216 |
| 8   | 0.0540 | 0.0538 | 0.0664 | 0.0465 | 0.0684 | 0.0314 | 0.0610 | 0.1226 | 0.0795 |
| 9   | 0.0639 | 0.0457 | 0.0636 | 0.1043 | 0.0554 | 0.0937 | 0.0599 | 0.1636 | 0.0498 |
| 10  | 0.0294 | 0.0306 | 0.0337 | 0.0311 | 0.0260 | 0.0330 | 0.0487 | 0.0689 | 0.0461 |
| 11  | 0.0199 | 0.0333 | 0.0190 | 0.0255 | 0.0413 | 0.0092 | 0.0144 | 0.0418 | 0.0147 |
| 12  | 0.0251 | 0.0248 | 0.0316 | 0.0262 | 0.0246 | 0.0315 | 0.0432 | 0.0600 | 0.0222 |
| 13  | 0.0226 | 0.0224 | 0.0300 | 0.0310 | 0.0190 | 0.0190 | 0.0190 | 0.0179 | 0.0177 |
| 14  | 0.0335 | 0.0504 | 0.0546 | 0.0620 | 0.0298 | 0.0289 | 0.0245 | 0.1275 | 0.0336 |
| 15  | 0.0316 | 0.0321 | 0.0413 | 0.0432 | 0.0092 | 0.0397 | 0.0225 | 0.0332 | 0.0560 |

Table 4 Building Sector. Mean LOOCV errors at each one of the Spanish Autonomous Communities analyzed, for the years studied in the period 1999 – 2007.

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References

1. Aneiros-Pérez G, Vieu P (2006) Semi-functional partial linear regression. Stat. Probab. Letters 76:1102–1110
Table 5  Commerce Sector. Mean LOOCV errors at each one of the Spanish Autonomous Communities analyzed, for the years studied in the period 1999 – 2007.

| SCC | 1999 | 2000 | 2001 | 2002 | 2003 | 2004 | 2005 | 2006 | 2007 |
|-----|------|------|------|------|------|------|------|------|------|
| 1   | 0.0094 | 0.0100 | 0.0071 | 0.0092 | 0.0089 | 0.0090 | 0.0115 | 0.0120 | 0.0078 |
| 2   | 0.0258 | 0.0258 | 0.0233 | 0.0247 | 0.0208 | 0.0250 | 0.0260 | 0.0274 | 0.0251 | 0.0154 |
| 3   | 0.0211 | 0.0216 | 0.0153 | 0.0153 | 0.0180 | 0.0236 | 0.0274 | 0.0251 | 0.0154 |
| 4   | 0.0049 | 0.0052 | 0.0052 | 0.0047 | 0.0054 | 0.0064 | 0.0057 | 0.0064 | 0.0051 |
| 5   | 0.0879 | 0.0850 | 0.0821 | 0.0789 | 0.0833 | 0.0877 | 0.0810 | 0.0826 | 0.0794 |
| 6   | 0.0129 | 0.0172 | 0.0126 | 0.0128 | 0.0149 | 0.0166 | 0.0188 | 0.0171 | 0.0109 |
| 7   | 0.0042 | 0.0057 | 0.0045 | 0.0067 | 0.0060 | 0.0061 | 0.0048 | 0.0064 | 0.0058 |
| 8   | 0.0176 | 0.0165 | 0.0178 | 0.0175 | 0.0169 | 0.0157 | 0.0169 | 0.0148 | 0.0187 |
| 9   | 0.0084 | 0.0085 | 0.0106 | 0.0093 | 0.0082 | 0.0094 | 0.0090 | 0.0105 | 0.0097 |
| 10  | 0.0099 | 0.0101 | 0.0105 | 0.0114 | 0.0130 | 0.0190 | 0.0145 | 0.0132 |
| 11  | 0.0099 | 0.0138 | 0.0068 | 0.0052 | 0.0072 | 0.0122 | 0.0183 | 0.0205 | 0.0074 |
| 12  | 0.0079 | 0.0075 | 0.0082 | 0.0079 | 0.0093 | 0.0110 | 0.0092 | 0.0082 | 0.0088 |
| 13  | 0.0216 | 0.0271 | 0.0206 | 0.0209 | 0.0235 | 0.0251 | 0.0228 | 0.0241 | 0.0197 |
| 14  | 0.0088 | 0.0090 | 0.0072 | 0.0069 | 0.0086 | 0.0110 | 0.0106 | 0.0106 | 0.0074 |

Table 6  Several Sector. Mean LOOCV errors at each one of the Spanish Autonomous Communities analyzed, for the years studied in the period 1999 – 2007.

| SCC | 1999 | 2000 | 2001 | 2002 | 2003 | 2004 | 2005 | 2006 | 2007 |
|-----|------|------|------|------|------|------|------|------|------|
| 1   | 0.0578 | 0.0577 | 0.0547 | 0.0595 | 0.0226 | 0.0342 | 0.0364 | 0.0137 | 0.0528 |
| 2   | 0.0351 | 0.0085 | 0.0157 | 0.0253 | 0.1956 | 0.0228 | 0.0157 | 0.0341 | 0.0440 |
| 3   | 0.0360 | 0.0385 | 0.0354 | 0.0334 | 0.3637 | 0.0357 | 0.0480 | 0.0406 | 0.0449 |
| 4   | 0.0190 | 0.0257 | 0.0214 | 0.0341 | 0.2277 | 0.0197 | 0.0197 | 0.0253 | 0.0307 |
| 5   | 0.0674 | 0.0379 | 0.0397 | 0.0711 | 0.2124 | 0.0416 | 0.0407 | 0.0389 | 0.0472 |
| 6   | 0.0205 | 0.0311 | 0.0376 | 0.0578 | 0.7336 | 0.0279 | 0.0360 | 0.0279 | 0.0305 |
| 7   | 0.0440 | 0.0401 | 0.0109 | 0.0373 | 0.0876 | 0.0192 | 0.0232 | 0.0351 | 0.0371 |
| 8   | 0.0215 | 0.0264 | 0.0137 | 0.0714 | 0.5700 | 0.0308 | 0.0136 | 0.0204 | 0.0202 |
| 9   | 0.0406 | 0.0592 | 0.0689 | 0.0707 | 0.2756 | 0.0732 | 0.0560 | 0.0533 | 0.0631 |
| 10  | 0.0464 | 0.0479 | 0.0315 | 0.1038 | 0.1239 | 0.0416 | 0.0364 | 0.0450 | 0.0514 |
| 11  | 0.0674 | 0.0579 | 0.0353 | 0.0292 | 0.0718 | 0.0250 | 0.0183 | 0.0418 | 0.0433 |
| 12  | 0.0273 | 0.0288 | 0.0206 | 0.0465 | 0.1548 | 0.0556 | 0.0243 | 0.0569 | 0.0532 |
| 13  | 0.0190 | 0.0330 | 0.0315 | 0.0554 | 0.4012 | 0.0475 | 0.0399 | 0.0398 | 0.0392 |
| 14  | 0.0624 | 0.0092 | 0.0223 | 0.0237 | 0.2590 | 0.0245 | 0.0351 | 0.0307 | 0.0483 |
| 15  | 0.0247 | 0.0346 | 0.0116 | 0.0240 | 0.3455 | 0.0468 | 0.0277 | 0.0848 | 0.0948 |

2. Aneiros-Pérez G, Vieu P (2008) Nonparametric time series prediction: A semi-functional partial linear modeling. J. Multivariate Anal. 99:834–857
3. Benhenni K, Hedli-Griche S, Rachdi M (2017) Regression models with correlated errors based on functional random design. Test 26:1–21
4. Bosq D (2000) Linear Processes in Function Spaces. Springer-Verlag, New York
5. Bosq D, Ruiz-Medina MD (2014) Bayesian estimation in a high dimensional parameter framework. Electronic Journal of Statistics 8:1604–1640
6. Cáceres MD, Legendre P (2008) Beals smoothing revisited. Oecologia 156:657–669
7. Cai T, Hall P (2006) Prediction in functional linear regression. Annals of Statistics 34:2159–2179
8. Chaoch M, Laib N, Louani, D (2017) Rate of uniform consistency for a class of mode regression on functional stationary ergodic data. Statistical Methods & Applications 26:19–47
9. Chiu J, Müller HG, Wang JL (2004) Functional response models. Statistica Sinica 14: 659–677
10. Crambes C, Kneip A, Sarda P (2009) Smoothing splines estimators for functional linear regression. Annals of Statistics 37:35–72
11. Cuevas A, Febrero M, Fraiman R (2002) Linear functional regression: The case of a fixed design and functional response. Canadian J. Statistics 30:285–300
12. Cuevas A (2014) A partial overview of the theory of statistics with functional data. Journal of Statistical Planning and Inference 147:1–23
13. Dautray R, Lions JL (1985) Mathematical Analysis and Numerical Methods for Science and Technology, Vol. 3, Spectral Theory and Applications. Springer, New York
14. Da Prato G., Zabczyk J. (2002) Second Order Partial Differential Equations in Hilbert Spaces. University Press, Cambridge
15. Espejo RM, Fernández-Pasqual R, Ruiz-Medina MD (2017) Spatial-depth functional estimation of ocean temperature from non-separable covariance models. Stoch. Environ. Res. Risk Assess. 31:39–51
16. Febrero-Bande M, Galeano P, González-Manteiga W (2015) Functional principal component regression and functional partial least-squares regression: an overview and a comparative study. International Statistical Review 10.1111/insr.12116
17. Ferraty F, Goia A, Salinelli E, Vieu P (2013) Functional projection pursuit regression. Test 22:293–320
18. Ferraty F, Goia A, Vieu, P (2002) Functional nonparametric model for time series: a fractal approach for dimension reduction. Test 11:317–344
19. Ferraty F, Keilegom IV, Vieu P (2012) Regression when both response and predictor are functions. J. Multivariate Anal. 109:10–28
20. Ferraty F, Vieu P (2006) Nonparametric Functional Data Analysis: Theory and Practice. Springer, New York
21. Ferraty F, Vieu P (2011) Kernel regression estimation for functional data. In: Ferraty F, Romain Y (eds) The Oxford Handbook of Functional Data Analysis. Oxford University Press, Oxford, pp. 72–129
22. Fitzmaurice GM, Laird NM, Ware JH (2004) Applied Longitudinal Analysis. John Wiley and Sons, New York
23. Geenens G (2011) Curse of dimensionality and related issues in nonparametric functional regression. Statistics Surveys 5:30–43
24. Goia A, Vieu P (2015) A partitioned single functional index model. Computational Statistics 30:673–692.
25. Goia A, Vieu P (2016) An introduction to recent advances in high/infinite dimensional statistics. Journal of Multivariate Analysis 146:1–6.
26. Guillou S (2001) Rates of convergence of autocorrelation estimates for autoregressive Hilbertian processes. Statistics & Probability Letters 55:281–291
27. Horváth L, Kokoszka P (2012) Inference for Functional Data with Applications. Springer, New York
28. Hsing T, Eubank R (2015) Theoretical Foundations of Functional Data Analysis, with an Introduction to Linear Operators. In: Wiley Series in Probability and Statistics, John Wiley & Sons, Chichester
29. Kara L Z, Laksaci A, Rachdi M, Vieu P (2017a) Uniform in bandwidth consistency for various kernel estimators involving functional data. Journal of Nonparametric Statistics 29:85–107
30. Kara L Z, Laksaci A, Rachdi M, Vieu P (2017b) Data-driven kNN estimation in nonparametric functional data analysis. Journal of Multivariate Analysis 153:176–188
31. Ling N, Liu Y, Vieu P (2017) On asymptotic properties of functional conditional mode estimation with both stationary ergodic and responses MAR. In: Functional Statistics and Related Fields, pp 173–178, Springer, Switzerland
32. Marx BD, Eilers PH (1999) Generalized linear regression on sampled signals and curves: A P-spline approach. Technometrics 41:1–13
33. Mas A (2004) Consistance du prédicteur dans le modèle ARH(1): le cas compact. Ann. I.S.U.P. 48:39–48
34. Mas A (2007) Weak-convergence in the functional autoregressive model. J. Multivariate Anal. 98:1251–1261
35. Morris JS (2015) Functional regression. Annual Review of Statistics and Its Application 2:321–359
36. Ramsay JO and Silverman BW (2005) Functional data analysis, Second Ed. Springer Series in Statistics. Springer, New York
37. Ruiz-Medina MD (2011) Spatial autoregressive and moving average Hilbertian processes. Journal of Multivariate Analysis 102:292–305
38. Ruiz-Medina MD (2012a) New challenges in spatial and spatiotemporal functional statistics for high-dimensional data. Spatial Statistics 1:82–91
39. Ruiz-Medina MD (2012b) Spatial functional prediction from spatial autoregressive Hilbertian processes. Environmetrics 23:119–128
40. Ruiz-Medina MD (2016) Functional analysis of variance for Hilbert-valued multivariate fixed effect models. Statistics 50:689–715