Compact Group Actions On Operator Algebras
and Their Spectra

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Abstract. We consider a class of dynamical systems with compact non abelian groups that include C*--, W*- and multiplier dynamical systems. We prove results that relate the algebraic properties such as simplicity or primeness of the fixed point algebras as defined in Section 3., to the spectral properties of the action, including the Connes and strong Connes spectra.

1 Introduction

In [2], Connes introduced the invariant \( \Gamma(U) \) known as the Connes spectrum of the action \( U \) of a locally compact abelian group on a von Neumann algebra and used it in his seminal classification of type III von Neumann factors. Soon after, Olesen [10] defined the Connes spectrum of an action of a locally compact abelian group on a C*-algebra. In [11], using the definition of the Connes spectrum in [10], it is proven an analog of a result in [3, Chapter III, Corollary 3.4.] regarding the significance of the Connes spectrum of a locally compact abelian group action on a C*-algebra for the ideal structure of the crossed product. In particular, in [11] is discussed a spectral characterization for the crossed product to be a prime C*-algebra. This definition of the Connes spectrum in [10] cannot be used to prove similar results for the simplicity of the crossed product, unless the group is discrete [11]. Kishimoto [8] defined the strong Connes spectrum for C*-dynamical systems with locally compact abelian groups that coincides with the Connes spectrum for the W*-dynamical system and with the Connes spectrum defined by Olesen for discrete abelian group actions on C*-algebras and he proved the Connes-Takesaki result for simple crossed products. In [2] Connes obtained results that relate the spectral properties of the von Neumann algebra with the algebraic properties of the fixed point algebra. These results were extended in [12] to C*-algebras and compact abelian groups. In [6, 14] we considered the problems of simplicity and primeness of the crossed product by compact, non abelian group actions. In particular, in [6] we have defined the Connes and strong Connes spectra for such actions that coincide with Connes spectra [2, 10], respectively with the strong Connes spectra [8] for compact abelian groups. Further, in [15] we have considered the case of one-parameter F-dynamical systems that include the C*- the W*- and the multiplier one-parameter dynamical systems. In particular, we have
obtained extensions of some results in [2, 12] for \( F \)-dynamical systems with compact abelian group actions [see 15, Theorems 3.2 and 3.4]. In this paper we will consider the extension of [2 Proposition 2.2.2. b) and Theorem 2.4.1], [12, Theorem 2], [13, Theorem 8.10.4] and [15, Theorems 3.2. and 3.4.] to the case of \( F \)-dynamical systems with compact non abelian group actions. In Section 2, we will set up the framework and state some results that will be used in the rest of the paper. In Section 3, we discuss the connection between the strong Connes spectrum, \( \Gamma_F(\alpha) \), of the action and the \( F \)-simplicity of the fixed point algebras \( (X \otimes B(H_\pi))^\alpha \otimes \text{ad}_\pi \). In Section 4, we will get similar results about the connection between the \( F \)-primeness of the fixed point algebras and the Connes spectrum, \( \Gamma_F(\alpha) \), of the action.

2 Notations and preliminary results

This section contains the definitions of the basic concepts used in the rest of the paper, the notations and some preliminary results.

2.1. Definition. ([1], [16]) A dual pair of Banach spaces is, by definition, a pair \( (X, F) \) of Banach spaces with the following properties:

a) \( F \) is a Banach subspace of the dual \( X^* \) of \( X \).

b) \( \|x\| = \sup \{|\varphi(x)| : \varphi \in F, \|\varphi\| \leq 1\}, x \in X \).

c) \( \|\varphi\| = \sup \{|\varphi(x)| : x \in X, \|x\| \leq 1\}, \varphi \in F \).

d) The convex hull of every relatively \( F \)-compact subset of \( X \) is relatively \( F \)-compact.

e) The convex hull of every relatively \( X \)-compact subset of \( F \) is relatively \( X \)-compact.

In the rest of the paper \( X \) will be assumed to be a \( C^* \)-algebra with the additional property

f) The involution of \( X \) is \( F \)-continuous and the multiplication in \( X \) is separately \( F \)-continuous.

The property d) implies the existence of the weak integrals of continuous functions defined on a locally compact measure space, \((S, \mu)\) with values in \( X \) endowed with the \( F \)-topology:

If \( f \) is such a function, we will denote by

\[ \int f(s) d\mu \]

the unique element \( y \) of \( X \) such that

\[ \varphi(y) = \int_S \varphi(f(s)) d\mu \]
for every $\varphi \in \mathcal{F}$. The property c) was used by Arveson [1, Proposition 1.4.] to prove the continuity in the $\mathcal{F}$-topology of some linear mappings on $X$ (in particular the mappings $P_\alpha(\pi)$ and $(P_\alpha)_{ij}(\pi)$ defined below).

### 2.2. Examples.

a) [1] If $X$ is a $C^*$-algebra and $\mathcal{F} = X^*$, conditions 1)-5) are satisfied.

b) [1] If $X$ is a $W^*$-algebra and $\mathcal{F} = X_*$ is its predual then conditions 1)-5) are satisfied.

c) [4] If $X = M(Y)$ is the multiplier algebra of $Y$ and $\mathcal{F} = Y^*$ then conditions 1)-5) are satisfied. In addition, in this case, the $\mathcal{F}$-topology on $X$ is compatible with the strict topology on $X = M(Y)$.

Let $(X, \mathcal{F})$ be a dual pair of Banach spaces $G$ a compact group and $\alpha : G \to Aut(X)$ a homeomorphism of $G$ into the group of $*$-automorphisms of $X$. We say that $(X, G, \alpha)$ is an $\mathcal{F}$-dynamical system if the mapping

$$g \to \varphi(\alpha_g(x))$$

is continuous for every $x \in X$ and $\varphi \in \mathcal{F}$.

### 2.3. Examples.

a) If $\mathcal{F} = X^*$, the dual of $X$ then, by [7 p. 306] the above condition is equivalent to the continuity of the mapping $g \to \alpha_g(x)$ from $G$ to $X$ endowed with the norm topology for every $x \in X$, so, in this case $(X, G, \alpha)$ is a $C^*$-dynamical system.

b) If $X$ is a von Neumann algebra and $\mathcal{F} = X_*$, the predual of $X$ then $(X, G, \alpha)$ is a $W^*$-dynamical system.

c) If $X = M(Y)$ is the multiplier algebra of $Y$ and $\mathcal{F} = Y^*$, then $(X, G, \alpha)$ is said to be a multiplier dynamical system.

Let $(X, G, \alpha)$ be an $\mathcal{F}$-dynamical system with $G$ compact. Denote by $\hat{G}$ the set of unitary equivalence classes of irreducible representations of $G$. For each $\pi \in \hat{G}$ denote also by $\pi$ a fixed representative of that class. If $\chi_\pi(g) = d_\pi \sum d_\pi \pi_{ii}(g^{-1}) = d_\pi \sum \pi_{ii}(g)$ is the character of $\pi$, denote by

$$P_\alpha(\pi)(x) = \int_G \chi_\pi(g) \alpha_g(x)dg.$$

Then $P_\alpha(\pi)$ is a projection of $X$ onto the spectral subspace

$$X_1(\pi) = \{x \in X : P_\alpha(\pi)(x)\}.$$

where the integral is taken in the weak sense defined in (1) above. As in [14] one can also define for every $1 \leq i, j \leq d_\pi$

$$(P_\alpha)_{ij}(\pi)(x) = \int_G \overline{\pi_{ji}(g)} \alpha_g(x)dg.$$
where \( d_\pi \) is the dimension of the Hilbert space \( H_\pi \) of \( \pi \) and show that

\[
(P_\alpha)_{ij}(\pi)(X) \subset X_1(\pi).
\]

Using [1, Proposition 1.4.] it follows that \( P_\alpha(\pi) \), \((P_\alpha)_{ij}(\pi)\) are \( \mathcal{F} \)-continuous. If \( \pi \) is the identity one dimensional representation \( \iota \) of \( X \), we will denote

\[
P_\alpha(\iota) = P_\alpha,
\]

and

\[
X_1(\iota) = X^\alpha.
\]
is the fixed point algebra of the action.

**2.4. Remark.** \( \sum_{\pi \in \hat{G}} X_1(\pi) = X \), where \( \sum_{\pi \in \hat{G}} X_1(\pi) \) denotes the closure of \( \sum_{\pi \in \hat{G}} X_1(\pi) \) in the \( \mathcal{F} \)-topology of \( X \).

**Proof.** Suppose that there exists \( \varphi \in \mathcal{F} \) such that \( \varphi(X_1(\pi)) = \{0\} \) for every \( \pi \in \hat{G} \). Since, as noticed above, \((P_\alpha)_{ij}(\pi)(X) \subset X_1(\pi)\), it follows that

\[
\int_G \pi_{ij}(g) \varphi(\alpha_g(x)) dg = 0.
\]

for every \( x \in X \) and every \( \pi \in \hat{G} \). Since \( \{\pi_{ij}(g) : \pi \in \hat{G}, 1 \leq i, j \leq d_\pi\} \) is an orthogonal basis of \( L^2(G) \), and \( \varphi(\alpha_g(x)) \) is a continuous function of \( g \), for every \( x \in X \), it follows that \( \varphi(x) = 0 \) for every \( x \in X \) so \( \varphi = 0 \) and we are done.

In ([9], [14], [6]) it is pointed out that the spectral subspaces

\[
X_2(\pi) = \{ a \in X \otimes B(H_\pi) : (\alpha_g \otimes \iota)(a) = a(1 \otimes \pi_g) \},
\]

where \( \iota \) is the identity automorphism of \( B(H_\pi) \) are, in some respects more useful. In [14] it is shown that \( X_2(\pi) \) consists of all matrices

\[
\{ a = [(P_\alpha)_{ij}(\pi)(x) = [a_{ij}]) \in X \otimes B(H_\pi) : x \in X, 1 \leq i, j \leq d_\pi \}.
\]

It is straightforward to prove that, if \( a \in X_2(\pi) \) and \( x = \sum_i a_{ii} \), then \( a_{ij} = (P_\alpha)_{ij}(\pi)(x) \). In what follows, if \( b \in X \otimes B(H_\pi) \) we will denote

\[
tr(b) = \sum b_{ii}
\]

which is an \( \mathcal{F} \)-continuous linear mapping from \( X \otimes B(H_\pi) \) to \( X \). The following lemma is proven for compact non abelian group actions on \( C^* \)-algebras in [6, Lemma 2.3.] and for compact abelian \( \mathcal{F} \)-dynamical systems in [15]. Since the proof is very similar with the proof of [6, Lemma 2.3.] we will state it without proof.
2.5. Lemma. Let \((X,G,\alpha)\) be an \(\mathcal{F}\)-dynamical system with \(G\) compact and \(J\) a two sided ideal of \(X^\alpha\). Then

\[
(XJ\mathcal{X})^\alpha = \mathcal{F}\text{-closed linear span of } \left\{ \text{tr}(X_2(\pi)JX_2(\pi)^*) : \pi \in \hat{G} \right\}.
\]

where, if \(a = [a_{kl}] \in X \otimes B(H_\pi)\) and \(j \in X\), by \(ja\) we mean the matrix \([ja_{kl}]\) and the multiplications \(XJ\mathcal{X}\), \(X_2(\pi)JX_2(\pi)^*\) are defined in 1.6. below.

We will use the following notations

2.6. Notation. Let \((X,\mathcal{F})\) be a dual pair of Banach spaces with \(X\) a \(C^*\)-algebra satisfying conditions 1)-6). If \(Y, Z\) are subsets of \(X\) denote:

a) \(\text{lin}\{Y\}\) is the linear span of \(Y\).

b) \(Y^* = \{y^* : y \in Y\}\).

c) \(YZ = \text{lin}\{yz : y \in Y, z \in Z\}\).

d) \(Y^\|\| = \mathcal{F}\text{-closure of } Y\) in \(X\).

e) \(Y^\omega = \text{w}^*\text{-closure of } Y\) in \(\mathcal{F}^*\).

If \((X,G,\alpha)\) is an \(\mathcal{F}\)-dynamical system denote

g) \(\mathcal{H}_\alpha^\omega(X)\) the set of all non-zero globally \(\alpha\)-invariant \(\mathcal{F}\)-closed hereditary \(C^*\)-subalgebras of \(X\).

Notice that if \((X,G,\alpha)\) is an \(\mathcal{F}\)-dynamical system and if \(X_2(\pi)\) is the spectral subspace defined above, then \(X_2(\pi)X_2(\pi)^*\) is a two sided ideal of \(X^\alpha \otimes B(H_\pi)\) and \(X_2(\pi)^*X_2(\pi)\) is a two sided ideal of \((X \otimes B(H_\pi))^\omega \otimes \text{ad} \pi\) where \(\alpha \otimes \text{ad} \pi\) is the action

\[
(\alpha_g \otimes \text{ad} \pi_g)(a) = (1 \otimes \pi_g)[\alpha_g(a_{ij})](1 \otimes \pi_{g^{-1}}).
\]
on \(X \otimes B(H_\pi)\).

2.7. Definition. a) \(\text{sp}(\alpha) = \left\{ \pi \in \hat{G} : X_1(\pi) \neq \{0\} \right\}\).

b) \(\text{sp}_\mathcal{F}(\alpha) = \left\{ \pi \in \hat{G} : X_2(\pi)^*X_2(\pi)^* \text{ is essential in } (X \otimes B(H_\pi))^\omega \otimes \text{ad} \pi \right\}\).

c) \(\tilde{\text{sp}}_\mathcal{F}(\alpha) = \left\{ \pi \in \hat{G} : X_2(\pi)^*X_2(\pi)^* = (X \otimes B(H_\pi))^\omega \otimes \text{ad} \pi \right\}\).

Corresponding to the above Arveson type spectra b) and c) we define two Connes type spectra
d) \(\Gamma_\mathcal{F}(\alpha) = \cap \{\text{sp}_\mathcal{F}(\alpha |_Y) : Y \in \mathcal{H}_\alpha^\omega(X)\}\).

e) \(\tilde{\Gamma}_\mathcal{F}(\alpha) = \cap \{\tilde{\text{sp}}_\mathcal{F}(\alpha |_Y) : Y \in \mathcal{H}_\alpha^\omega(X)\}\).

Clearly, \(\tilde{\text{sp}}_\mathcal{F}(\alpha) \subset \text{sp}_\mathcal{F}(\alpha) \subset \text{sp}(\alpha)\), so \(\tilde{\Gamma}_\mathcal{F}(\alpha) \subset \Gamma_\mathcal{F}(\alpha)\). The definition of \(\tilde{\Gamma}_\mathcal{F}(\alpha)\) is a direct generalization of the strong Connes spectrum of Kishimoto to compact non abelian groups. Our motivation for the definition of \(\Gamma_\mathcal{F}(\alpha)\) above (and \(\Gamma(\alpha)\) for \(C^*\)-dynamical systems in [6]) is the following observation.
2.8. Remark a) If \((X, G, \alpha)\) is an \(\mathcal{F}\)–dynamical system with \(G\) compact abelian, then
\[
\cap \{ \text{sp}(\alpha|_Y) : Y \in \mathcal{H}_\alpha(X) \} = \cap \{ \text{sp}_F(\alpha|_Y) : Y \in \mathcal{H}_\alpha(X) \}.
\]
and the left hand side of the above equality is the Connes spectrum for \(W^*\)–as well as for \(C^*\)-dynamical systems.

b) If \(G\) is not abelian, the equality in part a) is not true.

Proof. a) We have to prove only one inclusion, the opposite one being obvious. Let \(\gamma \in \cap \{ \text{sp}(\alpha|_Y) : Y \in \mathcal{H}_\alpha(X) \} \) and \(Y \in \mathcal{H}_\alpha(X)\). Suppose that \(aY^*_\gamma Y_\gamma = \{0\}\) for some \(a \in Y^\alpha, a \neq 0\). Then \(aY^*_\gamma = \{0\}\). Therefore, if we denote \(Z = aY^\alpha\), it follows that \(Z \in \mathcal{H}_\alpha(X)\) and \(Z^*_\gamma = \{0\}\) which is in contradiction with the hypothesis that \(\gamma \in \gamma \in \cap \{ \text{sp}(\alpha|_Y) : Y \in \mathcal{H}_\alpha(X) \} \subset \text{sp}(\alpha|_Z)\).

b) In [14, Example 3.9.] we provided an example of an action of an action of \(G = S_3\) the permutation group on three elements on the algebra \(X\) of \(2 \times 2\) matrices such that \(\text{sp}(\alpha) = \hat{G}, \mathcal{H}_\alpha = \{X\}\), so
\[
\cap \{ \text{sp}(\alpha|_Y) : Y \in \mathcal{H}_\alpha(X) \} = \text{sp}(\alpha)
\]
and we have shown that there exists \(\pi \in \hat{G}\) such that \((X \otimes B(\mathcal{H}_\pi))^\alpha \otimes \text{ad}\pi\) has nontrivial center and, therefore, it is not a prime \(C^*\)-algebra. By [6, Thm. 2.2.], it follows that \(\cap \{ \text{sp}(\alpha|_Y) : Y \in \mathcal{H}_\alpha(X) \} \neq \cap \{ \text{sp}_F(\alpha|_Y) : Y \in \mathcal{H}_\alpha(X) \} = \Gamma(\alpha)\).

3 \(\mathcal{F}\)–simple fixed point algebras

Let \((X, G, \alpha)\) be an \(\mathcal{F}\)-dynamical system with \(G\) compact. In the rest of this paper we will study how the \(\mathcal{F}\)-simplicity (respectively \(\mathcal{F}\)-primeness) as defined below, of the fixed point algebras \((X \otimes B(\mathcal{H}_\pi))^\alpha \otimes \text{ad}\pi\) is reflected in the spectral properties of the action.

3.1. Definition. Let \((B, \mathcal{F})\) be a dual pair of Banach spaces with \(B\) a \(C^*\)-algebra.

a) \(B\) is called \(\mathcal{F}\)-simple if every non zero two sided ideal of \(B\) is \(\mathcal{F}\)-dense in \(B\).

b) \(B\) is called \(\mathcal{F}\)-prime if the annihilator of every non zero two sided ideal of \(B\) is trivial, or, equivalently, every non zero two sided ideal of \(B\) is an essential ideal (using Definition 2.1. f) it is easy to see that \(X\) is \(\mathcal{F}\)-prime if and only if \(X\) is prime as a \(C^*\)-algebra).

Let \((X, G, \alpha)\) be an \(\mathcal{F}\)-dynamical system.

c) \(X\) is called \(\alpha\)-simple if every non zero \(\alpha\)-invariant two sided ideal of \(X\) is \(\mathcal{F}\)-dense in \(X\).
d) $X$ is called $\alpha$–prime if every non zero $\alpha$–invariant two sided ideal of $X$ is an essential ideal.

In the particular case when $B$ is a C*-algebra and $\mathcal{F} = B^*$ is its dual, then, clearly, the concepts of $\mathcal{F}$–simple, (respectively $\mathcal{F}$–prime) in the above Definition 3.1. a) (respectively b)) coincide with the usual concepts of simple (respectively prime) C*-algebras. Similarly, if $(X, G, \alpha)$ is a C*-dynamical system, that is if $X$ is a C*-algebra and $\mathcal{F} = X^*$ is its dual, then the notions of $\alpha$–simple and $\alpha$–prime coincide with the usual ones for C*-dynamical systems.

If $B$ is a von Neumann algebra and $\mathcal{F} = B_*$ is its predual, then, since the weak closure of every essential ideal equals $B$, it follows that $B$ is $\mathcal{F}$-simple if and only if $B$ is $\mathcal{F}$-prime, so, if and only if $B$ is a factor. It is also obvious that if $(X, G, \alpha)$ is a W*-dynamical system, that is if $X$ is a von Neumann algebra and $\mathcal{F} = X_*$ is its predual, then $X$ is $\alpha$–simple if and only if it is $\alpha$–prime, and this holds if and only if $\alpha$ acts ergodically on the center of $X$ (i.e. every fixed element in the center of $X$ is a scalar).

The above observations and the next Remark show that for W*-dynamical systems, $(X, G, \alpha)$ with $G$ compact, the results in the current Section 3 and Section 4 are equivalent.

3.2. Remark. Let $(X, G, \alpha)$ be a W*-dynamical system, that is, an $\mathcal{F}$-dynamical system with $X$ a von Neumann algebra and $\mathcal{F} = X_*$ its predual. Then $\tilde{\Gamma}_X(\alpha) = \Gamma_F(\alpha)$.

Proof. This follows from the fact that if $X$ is a von Neumann algebra, $p \in X^\alpha$ an $\alpha$-invariant projection and $pX_2(\pi)p^*X_2(\pi)p^*$ is essential in $(pXp \otimes B(H_\pi))^\alpha \otimes \text{ad} \pi$, then $pX_2(\pi)p^*X_2(\pi)p^* = (pXp \otimes B(H_\pi))^\alpha \otimes \text{ad} \pi$. ■

The next lemma will be used in the proofs of the main results of the current Section 3 and the next Section.

3.3. Lemma. Let $(B, G, \alpha)$ be an $\mathcal{F}$-dynamical system with $G$ compact. Then

a) If $\{e_\lambda\}$ is an approximate identity of $B^\alpha$ in the norm topology, then

$$(\text{norm}) \lim_\lambda e_\lambda x = (\text{norm}) \lim_\lambda xe_\lambda = (\text{norm}) \lim_\lambda e_\lambda xe_\lambda = x.$$

for every $x \in \sum_{\pi \in \hat{G}} B_1(\pi)$.\\
b) If $b \in B$ is such that $B^\alpha bB^\alpha = \{0\}$ then $b = 0$.

c) $B^\alpha BB^\alpha = B^\alpha BB^\alpha = B$.

d) $B_1^\alpha (\pi) = B_1(\pi)B^\alpha = B_1(\pi)$, $\pi \in \hat{G}$.

Proof. a) This follows from the proof of [5, Lemma 2.7] in the more general case of compact quantum group actions.
b) If \( \{e_\lambda\} \) is an approximate identity of \( B^\alpha \), then \( e_\lambda b e_\lambda = 0 \) implies

\[
 e_\lambda P_\alpha(\pi_{ij})(b)e_\lambda = P_\alpha(e_\lambda b e_\lambda) = 0.
\]

for every \( \pi \in \hat{G}, 1 \leq i, j \leq d_\pi \), so, by a), \( P_\alpha(\pi_{ij})(b) = 0 \). Therefore,

\[
 \varphi(P_\alpha(\pi_{ij})(b)) = \int_G \pi_{ij}(g)\varphi(\alpha_g(b)) dg = 0.
\]

for every \( \varphi \in \mathcal{F}, \pi \in \hat{G}, 1 \leq i, j \leq d_\pi \). Since \( \{\pi_{ij}(g) : \pi \in \hat{G}, 1 \leq i, j \leq d_\pi \} \) form an orthogonal basis of \( L^2(G) \), and \( \varphi(\alpha_g(b)) \) is continuous on \( G \), it follows that \( \varphi(\alpha_g(b)) = 0 \) for every \( g \in G, \varphi \in \mathcal{F} \), so \( b = 0 \).

c) We will prove only that \( B^\alpha BB^\sigma = B \), the proofs of the other equalities being similar. Let \( \{e_\lambda\} \) be an approximate identity of \( B^\alpha \). By a),

\[
 (\text{norm}) \lim_\lambda e_\lambda xe_\lambda = x \quad \text{for every } x \in \sum_{\pi \in \hat{G}} B_1(\pi).
\]

Therefore

\[
 \sum_{\pi \in \hat{G}} B_1(\pi) \subset B^\alpha BB^\sigma \subset B^\alpha BB^\sigma^*.
\]

Since, by Remark 2.4., the \( \mathcal{F} \)-closure of \( \sum_{\pi \in \hat{G}} B_1(\pi) \) equals \( B \) it follows that

\[
 B = \sum_{\pi \in \hat{G}} B_1(\pi) \subset B^\alpha BB^\sigma^*.
\]

so \( B^\alpha BB^\sigma^* = B \).

d) The proof is similar with the proof of part c). □

Theorem 3.4. below is an extension of [2, Proposition 2.2.2. b)] to the case of \( F \)-dynamical systems with compact groups, not necessarily abelian, for the strong Connes spectrum, \( \tilde{\Gamma}_F(\alpha) \).

3.4. Theorem. Let \((X, G, \alpha)\) be an \( F \)-dynamical system with \( G \) compact. Then

\[
 \tilde{\Gamma}_F(\alpha) = \cap \{ \tilde{s}_F(\alpha|_{JXJ^*}) : J \subset X^\alpha, \mathcal{F}\text{-closed two sided ideal} \}
\]

Proof. Clearly, since \( JXJ^* \in \mathcal{H}_\sigma^*(X) \),

\[
 \tilde{\Gamma}_F(\alpha) \subset \cap \{ \tilde{s}_F(\alpha|_{JXJ^*}) : J \subset X^\alpha, \mathcal{F}\text{-closed two sided ideal} \}.
\]

Let \( \pi \in \cap \{ \tilde{s}_F(\alpha|_{JXJ^*}) : J \subset X^\alpha, \mathcal{F}\text{-closed two sided ideal} \} \) and \( Y \in \mathcal{H}_\sigma^*(X) \), so \( Y^\alpha \in \mathcal{H}_\sigma(X^\alpha) \). We will prove that \( Y_2(\overline{\pi})Y_2(\overline{\pi})^* = (Y \otimes B(H_\pi))^{d_H} \).
thus \( \pi \in \tilde{s}_F(\alpha|_Y) \). Since \( Y \in \mathcal{H}_\sigma^0(X) \) is arbitrary, it will follow that \( \pi \in \tilde{\Gamma}_F(\alpha) \).

Denote by \( J \) the following ideal of \( X^\alpha \)

\[
J = X^\alpha Y^\alpha X^\alpha. 
\]

It is clear that \( J = \tilde{J}X^\alpha \tilde{J}' \) (actually it is quite easy to show that this equality holds without the closure, but we do not need this fact). Also

\[
\bar{Y}^\alpha J Y^\alpha = Y^\alpha X^\alpha Y^\alpha X^\alpha Y^\alpha = (Y^\alpha X^\alpha Y^\alpha)(Y^\alpha X^\alpha Y^\alpha)^\sigma = \bar{Y}^\alpha Y^\alpha = Y^\alpha. 
\]

Denote \( Z = \tilde{J}X^\alpha \tilde{J}' \). Notice that, since \( Y \in \mathcal{H}_\sigma^0(X) \), we have \( \bar{Y}^\alpha X^\alpha Y^\alpha = Y \), so

\[
\bar{Z}_2(\pi)^* \bar{Z}_2(\pi)^* = (Z \otimes B(H_\pi))^{\alpha \otimes ad \pi}. 
\]

Using the equalities (2) above, the fact that \( Y \) is a hereditary C*-subalgebra of \( X \), and the obvious equality

\[
P_{ij}(\pi)(xyz) = xP_{ij}(\pi)(yz) 
\]

for every \( x, z \in X^\alpha \), \( y \in X \), and \( 1 \leq i, j \leq \dim H_\pi \), the relation (4) becomes

\[
\bar{X}^\alpha Y_2(\pi)^* Y_2(\pi) \bar{X}^\alpha = \bar{X}^\alpha (Y \otimes B(H_\pi))^{\alpha \otimes ad \pi} \bar{X}^\alpha. 
\]

where, for \( x \in X^\alpha \) and \( a \in X \otimes B(H_\pi) \), \( a = [a_{kl}] \), by \( xa \) we mean the matrix whose \( kl \) entry is \( x_{0kl} \). Therefore, by applying Lemma 3.3. d) to \( B = Y \), we get

\[
\bar{X}^\alpha Y^\alpha Y_2(\pi)^* Y_2(\pi) \bar{Y}^\alpha X^\alpha = \bar{X}^\alpha Y^\alpha (Y \otimes B(H_\pi))^{\alpha \otimes ad \pi} \bar{X}^\alpha. 
\]

By multiplying (3) on the right and on the left by \( Y^\alpha \) and taking into account that, by Lemma 3.3. c) \( \bar{Y}^\alpha Y^\alpha Y^\alpha = Y \) and consequently, \( \bar{Y}^\alpha X^\alpha Y^\alpha = Y^\alpha \), it follows that

\[
\bar{Y}_2(\pi)^* Y_2(\pi)^* = (Y \otimes B(H_\pi))^{\alpha \otimes ad \pi}. 
\]

Therefore, \( \pi \in \tilde{s}_F(\alpha|_Y) \) and the proof is complete. \( \Box \)

In the next Lemma and the rest of the paper, a subalgebra of \( X \otimes B(H_\pi) \) will be called \( \mathcal{F} \)-simple (respectively \( \mathcal{F} \)-prime) if it is \( \mathcal{F} \otimes B(H_\pi)^* \)-simple (respectively \( \mathcal{F} \otimes B(H_\pi)^* \)-prime) where \( B(H_\pi)^* \) denotes the dual of \( B(H_\pi) \). Clearly, a subalgebra of \( X \otimes B(H_\pi) \) is \( \mathcal{F} \)-prime if and only if it is a prime C*-algebra. The similar statement for the \( \mathcal{F} \)-simple case is not true.
3.5. Lemma. Let \((X, G, \alpha)\) be an \(\mathcal{F}\)-dynamical system with \(G\) compact. Then, if \(X^\alpha\) is \(\mathcal{F}\)-simple, it follows that \((X \otimes B(H_\pi))^{\alpha \otimes ad_\pi}\) is \(\mathcal{F}\)-simple.

**Proof.** Let \(\pi \in \tilde{sp}_\mathcal{F}(\alpha) \subset sp(\alpha)\). Since \(X^\alpha\) is \(\mathcal{F}\)-simple, so \(X^\alpha \otimes B(H_\pi)\) is also \(\mathcal{F}\)-simple and \(X_2(\pi)X_2(\pi)^*\) is an ideal of \(X^\alpha \otimes B(H_\pi)\), it follows that \(X_2(\pi)^*X_2(\pi)^* = X^\alpha \otimes B(H_\pi)\). To prove that \((X \otimes B(H_\pi))^{\alpha \otimes ad_\pi}\) is simple, let \(I \subset (X \otimes B(H_\pi))^{\alpha \otimes ad_\pi}\) be a non-zero ideal. Then it can be easily verified that

\[
J = \overline{\text{lin}} \{yy^*: y \in X_2(\pi)I\} = X_2(\pi)I X_2(\pi)^*.
\]

is an ideal of \(X^\alpha \otimes B(H_\pi)\) and, since the latter algebra is \(\mathcal{F}\)-simple, it follows that \(J = X^\alpha \otimes B(H_\pi)\). Therefore, since \(\pi \in \tilde{sp}_\mathcal{F}(\alpha)\), we have \(X_2(\pi)^*X_2(\pi)^* = (X \otimes B(H_\pi))^{\alpha \otimes ad_\pi}\). Consequently, since, by Lemma 3.3. d) \(X^\alpha X_2(\pi) = X_2(\pi)\), we have

\[
(X \otimes B(H_\pi))^{\alpha \otimes ad_\pi} = X_2(\pi)^*JX_2(\pi) \subset X_2(\pi)^*X_2(\pi)I X_2(\pi)^*X_2(\pi)^* \subset I.
\]

Thus \(I = (X \otimes B(H_\pi))^{\alpha \otimes ad_\pi}\) and we are done. ■

The following result extends [2, Théorème 2.4.1], [12, Theorem 2. i)\(\Leftrightarrow\) ii)] and [15, Theorem 3.4.] to the more general case of \(\mathcal{F}\)-dynamical systems and non abelian compact groups \(G\).

3.6. Theorem. Let \((X, G, \alpha)\) be an \(\mathcal{F}\)-dynamical system with \(G\) compact. The following conditions are equivalent:

i) \((X \otimes B(H_\pi))^{\alpha \otimes ad_\pi}\) is \(\mathcal{F}\)-simple for all \(\pi \in sp(\alpha)\).

ii) \(X\) is \(\alpha\)-simple and \(sp(\alpha) = \tilde{\Gamma}_\mathcal{F}(\alpha)\).

**Proof.** i) \(\Rightarrow\) ii) Suppose that \((X \otimes B(H_\pi))^{\alpha \otimes ad_\pi}\) is \(\mathcal{F}\)-simple for all \(\pi \in sp(\alpha)\). Then, it follows immediately from the definitions that \(sp(\alpha) = \tilde{sp}_\mathcal{F}(\alpha)\). Let \(\pi \in sp(\alpha)\) be arbitrary. Since, in particular, \(X^\alpha\) is \(\mathcal{F}\)-simple, so it has no non-trivial \(\mathcal{F}\)-closed ideals, from Theorem 3.4. it follows that \(\pi \in \tilde{\Gamma}_\mathcal{F}(\alpha)\), so \(sp(\alpha) = \tilde{\Gamma}_\mathcal{F}(\alpha)\). Let us prove that \(X\) is \(\alpha\)-simple. If \(I\) is an \(\mathcal{F}\)-closed \(\alpha\)-invariant ideal of \(X\), then \(I^\alpha\) is an \(\mathcal{F}\)-closed ideal of \(X^\alpha\), so \(I^\alpha = X^\alpha\). By Lemma 3.3. c) applied to \(B = I\), and to \(B = X\) it follows that \(I^\alpha I^\alpha^\perp = I\) and \(X^\alpha X^\alpha^\perp = X\), so

\[
X = X^\alpha X^\alpha^\perp = I^\alpha X^\alpha^\perp \subset I.
\]

Therefore, \(I = X\), hence \(X\) is \(\alpha\)-simple.

ii) \(\Rightarrow\) i). Suppose that \(X\) is \(\alpha\)-simple and \(sp(\alpha) = \tilde{\Gamma}_\mathcal{F}(\alpha)\). We will prove first that \(X^\alpha\) is \(\mathcal{F}\)-simple. Let \(J \subset X^\alpha\) be a non zero ideal and \(\pi \in \tilde{\Gamma}_\mathcal{F}(\alpha)\). Since \(JXJ^\perp \in \mathcal{H}_\mathcal{F}(X)\), and \(\pi \in \tilde{\Gamma}_\mathcal{F}(\alpha)\), it follows that

\[
JX_2(\pi)^*JX_2(\pi)J^\perp = J(X \otimes B(H_\pi))^{\alpha \otimes ad_\pi}J^\perp.
\]

(6)
where, for $j \in J \subset X$ and $a \in X \otimes B(H_\pi)$, $a = [a_{kl}]$, by $ja$ we mean the matrix whose $kl$ entry is $ja_{kl}$. By multiplying the above relation on the left by $X_2(\pi)$ and on the right by $X_2(\pi)^*$, we get

$$X_2(\pi)JX_2(\pi)^*JX_2(\pi)JX_2(\pi)^* = X_2(\pi)J(X \otimes B(H_\pi))^{\alpha \otimes ad \pi} JX_2(\pi)^*.$$

From the above relations (6) and (7) it follows that

$$X_2(\pi)JX_2(\pi)^* = \overline{X_2(\pi)JX_2(\pi)JX_2(\pi)JX_2(\pi)^*} = X_2(\pi)JX_2(\pi)^*JX_2(\pi)JX_2(\pi)^* \subset (X \otimes B(H_\pi))^{\alpha \otimes ad \pi} JX_2(\pi)^* \subset J \otimes B(H_\pi)$$

It follows that $tr(X_2(\pi)JX_2(\pi)^*) \subset J$. From Lemma 2.5. it follows that $(XJX)^\alpha \subset J$. Since $X$ is $\alpha$-simple, we have $XJX = X$, so $J = X^\alpha$ and therefore, $X^\alpha$ is $F$-simple. Applying lemma 3.5. it follows that $(X \otimes B(H_\pi))^{\alpha \otimes ad \pi}$ is $F$-simple for all $\pi \in sp(\alpha) = \Gamma_F(\alpha)$. ■

### 4 $F$-prime fixed point algebras

This section is concerned with the relationship between the $F$-primeness of the fixed point algebras and the spectral properties, involving the Connes spectrum $\Gamma_F(\alpha)$ of the $F$-dynamical system $(X, G, \alpha)$.

Theorem 4.1. below is an extension of [2, Proposition 2.2.2. b)] to the case of $F$-dynamical systems with compact groups, not necceessarily abelian, for the Connes spectrum, $\Gamma_F(\alpha)$. By Remark 3.2. and the discussion preceding it, if $(X, G, \alpha)$ is a $W^\alpha$-dynamical system (that is $X$ is a von Neumann algebra and $F = X_\alpha$ its predual), then the next Theorem 4.1. is equivalent with Theorem 3.4.

#### 4.1. Theorem. Let $(X, G, \alpha)$ be an $F$-dynamical system. Then

$$\Gamma_F(\alpha) = \cap \{ sp_F(\alpha |_JX\alpha) : J \subset X^\alpha, \text{ a non-zero } F\text{-closed two sided ideal} \}.$$

**Proof.** Since $JX\alpha \subset \mathcal{H}_\alpha^\alpha(X)$ for every non-zero $F$-closed two sided ideal $J \subset X^\alpha$, we have,

$$\Gamma_F(\alpha) \subset \cap \{ sp_F(\alpha |_JX\alpha) : J \subset X^\alpha, \text{ } F\text{-closed two sided ideal} \}.$$  

Now let $\pi \in \cap \{ sp_F(\alpha |_JX\alpha) : J \subset X^\alpha, \text{ } F\text{-closed two sided ideal} \}$ and $Y \in \mathcal{H}_\alpha^\alpha(X)$, so $Y^\alpha \in \mathcal{H}_\alpha(X^\alpha)$. We will prove that $\overline{Y_2(\pi)^*Y_2(\pi)}^{\alpha \otimes ad \pi}$ is essential in $(Y \otimes B(H_\pi))^{\alpha \otimes ad \pi}$. As in the proof of Theorem 3.4., let $J = X^\alpha Y \alpha X^{\alpha^*}$ and $Z =$
\[ JX' \in H_{\alpha}^2(X) \]. Since \( \pi \in \cap \{ sp_{\mathcal{F}}(\alpha|JX') : J \subset X^{\alpha} \}, \) \( \mathcal{F} \)-closed two sided ideal, we have that \( \pi \in sp_{\mathcal{F}}(\alpha|Z) \). Therefore, \( Z_2(\pi)^*Z_2(\pi) \) is essential in \( (Z \otimes B(H_\pi))^{\alpha \otimes ad\pi} \). As noticed in the proof of Theorem 3.4,
\[
Z_2(\pi)^*Z_2(\pi) = X^{\alpha}Y_2(\pi)^*Y_2(\pi)X^{\alpha}\gamma.
\]
and
\[
(Z \otimes B(H_\pi))^{\alpha \otimes ad\pi} = X^{\alpha}(Y \otimes B(H_\pi))^{\alpha \otimes ad\pi}X^{\alpha}\gamma.
\]

Let \( a \in (Y \otimes B(H_\pi))^{\alpha \otimes ad\pi} \) be such that
\[
aY_2(\pi)^*Y_2(\pi) = \{0\}.
\]
Then, by Lemma 3.3. c) \( Y = Y^\alpha Y^\gamma = Y^\alpha X^\alpha Y^\gamma \) and, by Lemma 3.3. d), \( Y_2(\pi)^* = Y^\alpha Y_2(\pi)^* \). Then, it follows that
\[
aY_2(\pi)^*Y_2(\pi) = aY^\alpha Y_2(\pi)^*Y_2(\pi) = aY^\alpha X^\alpha Y_2(\pi)^*Y_2(\pi) \subset \{0\}.
\]
Therefore
\[
\overline{Y^\alpha aY^\alpha X^\alpha Y_2(\pi)^*Y_2(\pi)X^{\alpha}\gamma} = \{0\}.
\]
so, since \( Z_2(\pi)^*Z_2(\pi) = X^{\alpha}Y_2(\pi)^*Y_2(\pi)X^{\alpha}\gamma \) is essential in \( X^{\alpha}(Y \otimes B(H_\pi))^{\alpha \otimes ad\pi}X^{\alpha}\gamma \), we have \( Y^\alpha aY^\alpha = 0 \) and therefore, by Lemma 3.3 b) applied to \( B = Y \), it follows that \( a = 0 \). ■

4.2. **Lemma.** Let \( (X, G, \alpha) \) be an \( \mathcal{F} \)-dynamical system with \( G \) compact. Then, if \( X^{\alpha} \) is \( \mathcal{F} \)-prime, it follows that \( (X \otimes B(H_\pi))^{\alpha \otimes ad\pi} \) is \( \mathcal{F} \)-prime for every \( \pi \in sp_{\mathcal{F}}(\alpha) \).

**Proof.** Since \( X^{\alpha} \) is \( \mathcal{F} \)-prime, it follows that \( X^{\alpha} \otimes B(H_\pi) \) is \( \mathcal{F} \)-prime for every \( \pi \in \hat{G} \). Since \( X_2(\pi)X_2(\pi)^* \) is a non zero ideal of \( X^{\alpha} \otimes B(H_\pi) \), it follows that \( X_2(\pi)X_2(\pi)^* \) is an essential ideal. To prove that \( (X \otimes B(H_\pi))^{\alpha \otimes ad\pi} \) is \( \mathcal{F} \)-prime, let \( I \subset (X \otimes B(H_\pi))^{\alpha \otimes ad\pi} \) be a non-zero ideal. Then, as in the proof of Lemma 3.5., consider the following ideal of \( X^{\alpha} \otimes B(H_\pi) \)
\[
J = \overline{\text{lin} \{ yy^* : y \in X_2(\pi)I \}} = \overline{X_2(\pi)IX_2(\pi)^*}.
\]
Since \( X^{\alpha} \otimes B(H_\pi) \) is \( \mathcal{F} \)-prime, it follows that \( J \) is essential in \( X^{\alpha} \otimes B(H_\pi) \). Therefore, if \( a \in (X \otimes B(H_\pi))^{\alpha \otimes ad\pi} \) and \( aI = \{0\} \), we have
\[
aX_2(\pi)^*JX_2(\pi) \subset aX_2(\pi)^*X_2(\pi)IX_2(\pi)^*X_2(\pi) \subset aI = \{0\}.
\]
so
\[
(X_2(\pi)aX_2(\pi)^*JX_2(\pi))X_2(\pi)^* = \{0\}
\]
Thus, since \( X_2(\pi)X_2(\pi)^* \) is essential in \( X^{\alpha} \otimes B(H_\pi) \), it follows that \( X_2(\pi)aX_2(\pi)^*J = \{0\} \). Since \( X_2(\pi)aX_2(\pi)^* \subset X^{\alpha} \otimes B(H_\pi) \), \( J \) is essential in \( X^{\alpha} \otimes B(H_\pi) \) and \( \pi \in sp_{\mathcal{F}}(\alpha) \), it follows that \( a = 0 \). ■
The next result extends [2, Théorème 2.4.1.], and [13, Theorem 8.10.4.] to the case of $\mathcal{F}$-dynamical systems with compact non abelian groups.

### 4.3. Theorem

Let $(X, G, \alpha)$ be an $\mathcal{F}$-dynamical system with $G$ compact. The following conditions are equivalent:

i) $(X \otimes B(H_\pi))^{\alpha \otimes ad_\pi}$ is $\mathcal{F}$-prime for all $\pi \in sp(\alpha)$.

ii) $X$ is $\alpha$-prime and $sp(\alpha) = \Gamma_\mathcal{F}(\alpha)$.

**Proof.** i) $\Rightarrow$ ii) Suppose that $(X \otimes B(H_\pi))^{\alpha \otimes ad_\pi}$ is $\mathcal{F}$-prime for all $\pi \in sp(\alpha)$. Then, it follows immediately from i) and the definitions that $sp(\alpha) = sp_\mathcal{F}(\alpha)$.

Let $\pi$ be arbitrary. We will use Theorem 4.1. to show that $\pi \in \Gamma_\mathcal{F}(\alpha)$. Indeed, let $J$ be a non-trivial ideal of $X^\alpha$ and $Z = \bigcap XJ$ in $H_\pi^2(X)$. We will show that $\pi \in sp(\alpha|_Z)$, that is $Z_2(\pi)^* Z_2(\pi)$ is essential in $(Z \otimes B(H_\pi))^{\alpha \otimes ad_\pi}$. Notice that

$$Z_2(\pi) = JX_2(\pi).$$

So

$$Z_2(\pi)^* Z_2(\pi) = JX_2(\pi)^* JX_2(\pi).$$

Since, in particular, $X^\alpha$ is prime, and $J$ is an essential ideal of $X^\alpha$, we have $Z_2(\pi) \neq \{0\}$. Indeed as observed after 2.6., $X_2(\pi) X_2(\pi)^*$ is an ideal of $X^\alpha \otimes B(H_\pi)$, so, as $X^\alpha$ is $\mathcal{F}$-prime it follows that $X^\alpha \otimes B(H_\pi)$ is a prime C*-algebra and therefore $JX_2(\pi)^* X_2(\pi) \neq \{0\}$, hence $JX_2(\pi)^* JX_2(\pi) \neq \{0\}$, so $X_2(\pi)^* JX_2(\pi) \neq \{0\}$. Using the hypothesis that $(X \otimes B(H_\pi))^{\alpha \otimes ad_\pi}$ is $\mathcal{F}$-prime and the fact that $J$ is a non-trivial ideal of $X^\alpha$, it follows that $X_2(\pi)^* JX_2(\pi) J \neq \{0\}$, so $Z_2(\pi) \neq \{0\}$. As noticed above, $X_2(\pi)^* JX_2(\pi)$ is a non-trivial ideal of $(X \otimes B(H_\pi))^{\alpha \otimes ad_\pi}$. If $a \in (Z \otimes B(H_\pi))^{\alpha \otimes ad_\pi}$, $a \geq 0$ is such that $a Z_2(\pi)^* Z_2(\pi) = \{0\}$, then

$$a JX_2(\pi)^* JX_2(\pi) J = \{0\}.$$ 

Hence

$$Ja JX_2(\pi)^* JX_2(\pi) J = \{0\}.$$ 

Since, as noticed above, $X_2(\pi)^* JX_2(\pi)$ is non-trivial and $(X \otimes B(H_\pi))^{\alpha \otimes ad_\pi}$ is $\mathcal{F}$-prime it follows that

$$Ja J = \{0\}.$$ 

so $Ja = \{0\}$. Hence $Jtr(a) = \{0\}$. Since $X^\alpha$ is $\mathcal{F}$-prime, we deduce that $tr(a) = 0$, so $a = 0$ because $a$ was assumed to be non negative. Therefore, $\pi \in \Gamma_\mathcal{F}(\alpha)$, so $sp(\alpha) = \Gamma_\mathcal{F}(\alpha)$. It remains to prove that $X$ is $\mathcal{F}$-prime. Let $I \subset X$ be an $\alpha$-invariant non-trivial ideal and $x \in X$, $x \geq 0$ be such that $xI = \{0\}$. Then, in particular, $xI^\alpha = \{0\}$, so $P(x)I^\alpha = \{0\}$. Since $X^\alpha$ is $\mathcal{F}$-prime and $I^\alpha$ is a non trivial ideal of $X^\alpha$ we have $P(x) = 0$ so, since $P$ is faithful, $x = 0$.

ii) $\Rightarrow$ i) Suppose that $X$ is $\alpha$-prime and $sp(\alpha) = \Gamma_\mathcal{F}(\alpha)$. We will prove first that $X^\alpha$ is $\mathcal{F}$-prime. Let $J \subset X^\alpha$ be a non zero ideal and $a \in X^\alpha$, $a \geq 0$, $a \neq 0$ such
that \( Ja = \{0\} \). Since \( X \) is \( \alpha \)-prime, and \( XJX \) is a non zero \( \alpha \)-invariant ideal of \( X \), it follows that \( XJXa \neq \{0\} \) so \( JXa \neq \{0\} \). Therefore, since by Remark 2.4,

\[
X = \sum_{\pi \in sp(\alpha)} X_1(\pi) .
\]

there exists \( \pi \in sp(\alpha) \) such that \( JX_1(\pi)a \neq \{0\} \). Denote \( Z = aXa = H_\alpha^\sigma(X). \)

Then, since \( \pi \in sp(\alpha) = \Gamma_F(\alpha) \), \( Z_2(\pi)^*Z_2(\pi) \) is essential in \( (Z \otimes B(H_\pi))^{\alpha \otimes ad_\pi} \).

But

\[
\overline{Z_2(\pi)^*Z_2(\pi)} = aX(\pi)^*a^2X(\pi)a^- .
\]

and

\[
(Z \otimes B(H_\pi))^{\alpha \otimes ad_\pi} = a((X \otimes B(H_\pi))^{\alpha \otimes ad_\pi})a^- .
\]

Taking into account that \( X_2(\pi)a^2X_2(\pi)^* \subset X^\alpha \otimes B(H_\pi) \) and \( Ja = \{0\} \) we immediately get that \( JX_2(\pi)a^2X_2(\pi)^* \subset J \otimes B(H_\pi) \).

Hence

\[
\overline{(aX(\pi)^*JX_2(\pi)a^2X_2(\pi)^*\{a^2X_2(\pi)a\})} = \{0\} .
\]

Therefore

\[
\overline{(aX(\pi)^*JX_2(\pi)a\{aX(\pi)^*a^2X_2(\pi)a\})} = \{0\} .
\]

It follows that

\[
(aX(\pi)^*JX_2(\pi)a)(Z_2(\pi)^*Z_2(\pi)) = \{0\} .
\]

Since \( Z_2(\pi)^*Z_2(\pi) \) is essential in \( (Z \otimes B(H_\pi))^{\alpha \otimes ad_\pi} \) and obviously \( aX(\pi)^*JX_2(\pi)^*a \subset (Z \otimes B(H_\pi))^{\alpha \otimes ad_\pi} \) it follows that \( JX_2(\pi)a = \{0\} \) and hence \( JX_1(\pi)a = \{0\} \), but this is in contradiction with our choice of \( \pi \) in \( sp(\alpha) \) so \( X^\alpha \) is \( F \)-prime.

From Lemma 4.2. it follows that \( (X \otimes B(H_\pi))^{\alpha \otimes ad_\pi} \) is \( F \)-prime for all \( \pi \in \Gamma_F(\alpha) = sp_F(\alpha) = sp(\alpha) \) and we are done. 

\[\square\]

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