Abstract

This paper follows very closely a famous paper by Csiszár and Körner about classical (non-quantum) wiretap coding. Our paper gives a self-contained and slightly novel review of some important results of the paper by Csiszár and Körner. Then we present a generalization of those results to the quantum realm, thus giving one of the first half-decent treatments of quantum wiretap coding. Like Csiszár and Körner, we too find a capacity region (i.e., the maximal achievable region of rates) characterized in terms of one-letter informations. We try to make our treatment of quantum wiretap coding as parallel a possible to our treatment of classical wiretap coding. This parallel treatment is facilitated by the use of CB nets (classical Bayesian networks) for the classical case and QB nets (quantum Bayesian networks) for the quantum one.
1 Introduction

For a good textbook on classical (non-quantum) Shannon Information Theory (SIT), see, for example, Ref.\[1\] by Cover and Thomas. For a good textbook on quantum SIT, see, for example, Ref.\[2\] by Wilde. Wiretap coding is an example of network information theory, which in turn is part of SIT. For a good textbook on classical network information theory, see, for example, Ref.\[3\] by El Gamal and Kim.

Wiretap coding is a famous topic in classical SIT. In classical discussions of wiretap coding, one considers a situation wherein Alice sends a private message to Bob and a shared message to both Bob and Eve. Many workers have pointed out that an analogous situation arises when doing quantum communication. In that case, the environment plays the role of Eve.

The subject of classical wiretap coding has an illustrious history. In Ref.\[4\], Wyner found the so called capacity region (i.e., the maximal achievable region of rates) for a special type of channel, a “degraded wiretap channel”. Wyner characterized his capacity region using one-letter informations. In Ref.\[5\], Csiszár and Körner generalized the results of Wyner to encompass any type of discrete memoryless wiretap channel. They removed the degraded channel assumption, but still characterized the capacity region in terms of one-letter informations.

This paper follows Ref.\[5\] by Csiszár and Körner very closely. Our paper gives a self-contained and slightly novel review of some important results of Ref.\[5\]. Then it presents a generalization of those results to the quantum realm, thus giving one of the first half-decent treatments of quantum wiretap coding. Like Csiszár and Körner, we too find a capacity region characterized in terms of one-letter informations.

We try to make our treatment of quantum wiretap coding as parallel a possible to our treatment of classical wiretap coding. This parallel treatment is facilitated by the use of CB nets (classical Bayesian networks) for the classical case and QB nets (quantum Bayesian networks) for the quantum one.

The capacity regions found by Wyner, then Csiszár/Körner, and now us, are 3 dimensional. They are a closed, convex set of points $(R_s, R_t, R_e)$. Here $R_s$ is the rate of the secret, private message, $R_t$ is the rate of the tapped, shared message, and $R_e$ is the so called equivocation rate which measures how ignorant Eve is about the secret message $s$.

Many authors have previously discussed the topic of sending classical and quantum messages through a quantum channel, in some cases with “entanglement assistance”. See Ref.\[2\] for a masterful review with ample references. However, few authors have specifically discussed quantum wiretap coding. No previous paper, as far as I know, gives a treatment of quantum wiretap coding that includes proofs of achievability and optimization for a rate region characterized by one-letter informations.

This paper is written assuming that the reader has first read 3 previous papers by the same author.
• Ref. [6] is an introduction to quantum Bayesian networks for mixed states.

• Ref. [7] redoes the classics (the basic theorems of classical SIT) using p-type integration techniques and CB nets.

• Refs. [8] discusses well-known inequalities of classical and quantum SIT from a Bayesian networks perspective.

2 Preliminaries and Notation

Reading Refs. [6], [7], and [8] is a prerequisite to reading this paper. This section will introduce only notation which hasn’t been defined already in those 3 papers.

We will use the symbol \((y \leftrightarrow z)\) after an equation to indicate that the same equation is also valid with \(y\) and \(z\) swapped.

Suppose \(n\) is any positive integer. Let \(x^n = (x_1, x_2, \ldots, x_n)\) be the random variable that takes on values \(x^n = (x_1, x_2, \ldots, x_n) \in S^n_x\). For any integer \(j\) such that \(1 < j < n\), let \(x^<j = (x_1, x_2, \ldots, x_{j-1})\), and \(x^>j = (x_{j+1}, x_2, \ldots, x_n)\).

Given \(\{P(x)\}_{\forall x \in pd(S_x)}\), we used in previous papers the expected value operator:

\[
E_x = \sum_x P(x) .
\] (1)

Given a probability amplitude \(A(x)\) such that \(\sum_x |A(x)|^2 = 1\), we will now use the following 2 operators which are sort of “square roots” of \(E_x\):

\[
E_x = \sum_x A(x) , \quad E_x^* = \sum_x A^*(x) .
\] (2)

For both CB nets and QB nets, define a **source (or root)** node as a node that has only outgoing arrows, no incoming ones. Define a **sink (or leaf)** node as one that has only incoming arrows, no outgoing ones. Sometimes, to simplify a graph, rather than drawing a root node \(r\) and its outgoing arrows explicitly, we will just put the line \(r \rightarrow x_1, x_2, \ldots, x_n\) beneath the graph, where \(x_1, x_2, \ldots, x_n\) are the children nodes of \(r\). Likewise, sometimes, rather than drawing a leaf node \(l\) and its incoming arrows explicitly, we will just put the line \(l \leftarrow x_1, x_2, \ldots, x_n\) beneath the graph, where \(x_1, x_2, \ldots, x_n\) are the parent nodes of \(l\).

For QB nets, a **reservoir** is defined as a sink or source node that is traced over. A reservoir that is a sink can always be traded for one that is a source or vice versa. Indeed, if

\[
\rho_\mathcal{Z} = \sum_r \left[ \sum_x A(x|\rho)|x\right] A(\rho) \right) \right] \text{ h.c. } = \text{ tr}_\mathcal{Z} \left[ \begin{array}{c} x \end{array} \right] \text{ h.c. } ,
\] (3)
and we define amplitudes $A(r|x)$ and $A(x)$ so that

$$A(x|r)A(r) = A(r|x)A(x)$$  \hspace{1cm} (4)

for all $r, x$, then we can also write

$$\rho_x = \sum_r \left[ \frac{\sum_x A(x|x)}{A(r|x)} \right] \left[ \text{h.c.} \right] = \text{tr}_r \left[ \quad \right] \left[ \text{h.c.} \right].$$ \hspace{1cm} (5)

Eq. (4), call it the “reservoir as past history” version, is more directly related to the eigenvalue decomposition of the density matrix $\rho_x$. Indeed, suppose $\rho_x$ has eigenvalues $\lambda_x$ and eigenvectors $|\lambda_x\rangle$ for all $x \in S_x$. Then one can set $S_x = S_x'$, $A(x|r) = \langle x|\lambda_r\rangle$ and $A(r) = \sqrt{\lambda_r}$. This makes $A(x|r)$ an isometry.

Note that reservoirs can be merged. For example,

$$\sum_{r_1, r_2} \left[ \frac{\sum_{a,b,c} A(a)A(a)}{A(r_1|a,c)A(b|b)c} \right] \left[ \text{h.c.} \right] = \sum_r \left[ \frac{\sum_{a,b,c} A(a)A(a)}{A(r|a,c)A(c|b)c} \right] \left[ \text{h.c.} \right],$$ \hspace{1cm} (6)

where $r = (r_1, r_2)$. In diagrammatic language,

$$\text{tr}_{x_1, x_2} \left[ \quad \right] \left[ \text{h.c.} \right] = \text{tr}_x \left[ \quad \right] \left[ \text{h.c.} \right].$$ \hspace{1cm} (7)

This example involved merging 2 sink reservoirs. Two source reservoirs can likewise be merged.

We will often denote the eigenvalues and eigenvectors of a matrix $\{M_{x,y}\}_{x,y \in S_x}$ by $\lambda_x(M)$ and $|\lambda_x(M)\rangle$ for all $x \in S_x$. Note that if $\rho_x \in \text{dm}(H_x)$, then $\{\lambda_x(\rho_x)\}_{x \in S_x} \in \text{pd}(S_x)$. Also if $\rho_{x,y} \in \text{dm}(H_{x,y})$, then $\{\lambda_{x,y}(\rho_{x,y})\}_{x,y \in S_x} \in \text{pd}(S_x)$. Let $\lambda_{x,y,e} = \lambda_{x,y,e}(\rho_{x,y})$ for some fixed density matrix $\rho_{x,y}$. Since $\lambda_{x,y,e}$ behaves like a classical probability distribution, it is convenient to define $\lambda_{x,y} = \sum_e \lambda_{x,y,e}$, $\lambda_{y|x} = \frac{\lambda_{y|x}}{\lambda_x}$, $\lambda_{yx} = \frac{\lambda_{x,y}}{\lambda_x}\lambda_x$, etc.

Consider a density matrix $\rho_{x,y}$. Just like we refer to $\rho_y = \text{tr}_x \rho_{x,y}$ as a marginal (or partial, or reduced) density matrix of $\rho_{x,y}$, we will refer to $\lambda_y(\rho_{x,y}) = \sum_x \lambda_{x,y}(\rho_{x,y})$ as a marginal (or partial) eigenvalue of $\rho_{x,y}$. It is important to realize that

$$\lambda_y(\rho_{x,y}) \neq \lambda_y(\rho_{x}) \; .$$ \hspace{1cm} (8)
That is, in general, a marginal eigenvalue of $\rho_{\bar{x}, \bar{y}}$ is not equal to the eigenvalue of a marginal density matrix of $\rho_{\bar{x}, \bar{y}}$. The provenance of the eigenvalue matters. As with famous paintings, one can’t trace away part of that provenance with impunity. For a diagonal density matrix $\rho_{\bar{x}, \bar{y}}$ (i.e., the classical case), Eq. (5) becomes an equality.

3 Conditioning on Two Parents

The following inequality will prove useful later on.

Claim 1 If $\rho_{\bar{z}, \bar{x}, \bar{y}} \in dm(H_{\bar{z}, \bar{x}, \bar{y}})$, then

$$0 \leq S(a | x) + S(a | y).$$

(9)

proof: Suppose $\rho_{\bar{z}, \bar{x}, \bar{y}, \bar{w}} \in dm(H_{\bar{z}, \bar{x}, \bar{y}, \bar{w}})$ is a pure state with partial trace $\rho_{\bar{z}, \bar{x}, \bar{y}}$. Because of CMI $\geq 0$, we must have

$$S(a | x, r) \leq S(a | x).$$

(10)

Hence

$$S(a, x, r) - S(x, r) \leq S(a | x).$$

(11)

But since $\rho_{\bar{z}, \bar{x}, \bar{y}, \bar{w}}$ is pure, Eq. (11) implies that

$$S(y) - S(y, a) \leq S(a | x).$$

(12)

Thus

$$- S(a | y) \leq S(a | x).$$

(13)

QED

I like to interpret inequality Eq. (9) as saying that conditioning on two parents yields a net positive entropy.

4 Chain Rule Extravaganza

The theories for classical and quantum wiretap coding both rely on some identities that are direct consequences (one might even call then souped-up chain rules) of the the chain rules of classical and quantum SIT. This section will derive those souped-up chain rules.

All identities in this section will be stated for classical entropies $H$, but they also hold in the quantum case if we simply replace all $H$’s by $S$’s.
When trying to find the counterpart in quantum SIT of a result in classical SIT, it is often helpful to keep in mind that classical and quantum SIT satisfy identical identities if those identities follow from the chain rule. On the other hand, they sometimes satisfy different inequalities. (For instance, \( H(a | b) \) is non-negative but \( S(a | b) \) can be negative.)

Classically, the chain rule for probabilities can be stated as

\[
P(x^n) = \prod_{j=1}^{n} P(x_j | x_{>j}) = \prod_{j=1}^{n} P(x_j | x_{<j})
\]  

(14)

for all \( x^n \in S_x^n \), for \( n \) random variables \( x^n \) that are not necessarily i.i.d.. We see that there are two versions of this chain rule: one puts conditions on the past and the other on the future.

An alternative notation that I find more symmetrical is as follows. Eschew the vertical bar that indicates that conditions will follow. Instead, put a superscript of 1 on a random variable that is being conditioned on, and put a superscript of 0 on a random variable that is absent (i.e., it’s just serving as a placeholder). In this “binary” notation, Eq. (14) becomes

\[
P(x^n) = \prod_{j=1}^{n} P(x_0^j, x_j, x_1^j) = \prod_{j=1}^{n} P(x_1^j, x_j, x_0^j).
\]

(15)

Note that in this binary notation, one can go from one version of the chain rule to the other simply by swapping 0 and 1 superscripts.

The chain rule for probabilities immediately implies a chain rule for plain entropies and one for conditional entropies. For plain entropies, one gets the following chain rule

\[
H(x^n) = \sum_{j=1}^{n} H(x_j | x_{>j}) = \sum_{j=1}^{n} H(x_j | x_{<j}),
\]

(16)

or, expressed in binary notation,

\[
H(x^n) = \sum_{j=1}^{n} H(x_0^j, x_j, x_1^j) = \sum_{j=1}^{n} H(x_1^j, x_j, x_0^j).
\]

(17)

For conditional entropies, one gets the following chain rule

\[
H(x : y^n) = \sum_{j=1}^{n} H(x_j : y_j | y_{>j}) = \sum_{j=1}^{n} H(x_j : y_j | y_{<j}),
\]

(18)

or, expressed in binary notation,
\[ H(x : y^n) = \sum_{j=1}^{n} H(x : y^0_j, y^1_j, y^{>}_{j}) = \sum_{j=1}^{n} H(x : y^0_{<j}, y^1_{<j}, y^{>}_{j}). \] (19)

Claim 2

\[ H(x : b) - H(x : a) = H(x : b|a) - H(x : a|b), \] (20)

or, equivalently,

\[ H(x : a^0, b) - H(x : a, b^0) = H(x : a^1, b) - H(x : a, b^1). \] (21)

\[ H(x : a, b) = H(x : b) + H(x : a|b) \] (22a)

\[ = H(x : b|a) + H(x : a). \] (22b)

QED

Define

\[ \Sigma_{< \Delta} = \sum_{j=1}^{n} H(y_j : z_{>j}|y_{<j}, \lambda) = \sum_{j=1}^{n} H(y_j : y^1_{<j}, z_{>j}, \lambda^1), \] (23)

and

\[ \Sigma_{> \Delta} = \sum_{j=1}^{n} H(z_{<j} : y_{<j}|z_{>j}, \lambda) = \sum_{j=1}^{n} H(z_{<j} : y^1_{<j}, z_{>j}, \lambda^1). \] (24)

Claim 3

\[ \Sigma_{< \Delta} = \Sigma_{> \Delta}. \] (25)

\[ \Sigma_{< \Delta} = \sum_{j=1}^{n} \sum_{k=j+1}^{n} H(y_j : y^1_{<j}, z_{>j}, \lambda^1). \] (26)

\[ \Sigma_{> \Delta} = \sum_{j=1}^{n-1} \sum_{k=1}^{n} H(y_k : y^1_{<k}, z_{>j}, \lambda^1) \] (27a)

\[ = \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} H(y_k : y^1_{<k}, z_{>j}, \lambda^1) \] (27b)

\[ = \sum_{k=1}^{n-1} \sum_{j=1}^{n-1} H(y_j : y^1_{<j}, z_{>k}, \lambda^1). \] (27c)
Finally, note that \( \sum_{j=1}^{n} \sum_{k=j+1}^{n} = \sum_{k=1}^{n} \sum_{j=1}^{k-1} \) when acting on any function of \((j, k)\).

QED

Henceforth, we will use the shorthand 
\[
\alpha_j = (y_{<j}, z_{>j}) .
\] (28)

Claim 4

\[
H(y^n : t) = \sum_j H(y_j : \alpha_j, t) - \sum_j H(y_j : y_{<j}) - \Sigma_{<t} \tag{29}
\]

\[
H(z^n : t) = \sum_j H(z_j : \alpha_j, t) - \sum_j H(z_j : z_{>j}) - \Sigma_{>t} \tag{30}
\]

proof: Just apply the appropriate chain rule to the the first \( H \) on left hand side and
the first \( H \) on the right hand side of both equations.

QED

Claim 5

\[
H(y^n : s \mid t) = \sum_j H(y_j : s \mid \alpha_j, t) + \Sigma_{<t} - \Sigma_{<s \mid t} \tag{31}
\]

\[
H(z^n : s \mid t) = \sum_j H(z_j : s \mid \alpha_j, t) + \Sigma_{>t} - \Sigma_{>s \mid t} \tag{32}
\]

proof:

\[
H(y^n : s \mid t) = \sum_j H(y_j : y_{<j}^1, z_{>j}^0, s, t^1) \tag{33a}
\]

\[
= \sum_j \left\{ \begin{array}{ll}
H(y_j : y_{<j}^1, z_{>j}^0, s, t^1) \\
+ H(y_j : y_{<j}^1, z_{>j}^1, s, t^1) \\
- H(y_j : y_{<j}^1, z_{>j}^0, s, t^1)
\end{array} \right\} \tag{33b}
\]

\[
= \Sigma_{<t} + \sum_j H(y_j : s \mid \alpha_j, t) - \Sigma_{<s \mid t} \tag{33c}
\]

QED

Claim 6 (Csiszár & Körner’s Extreme Chain Ruling Identity)

\[
H(y^n : s \mid t) - H(z^n : s \mid t) = \sum_j H(y_j : s \mid \alpha_j, t) - \sum_j H(z_j : s \mid \alpha_j, t) \tag{34}
\]
proof: Follows immediately from Claims 3 and 5
QED

Define

\[
\begin{align*}
n\delta_y &= H(s, t | y^n), \\
n\delta_{ty} &= H(t | y^n), \\
n\delta_{tz} &= H(t | z^n)
\end{align*}
\] (35)

Claim 7

\[
H(s | z^n) = H(y^n : s | t) - H(z^n : s | t) - H(t | z^n, s) + n(\delta_{tz} + \delta_y - \delta_{ty})
\] (36)

\[
H(s | t) = H(y^n : s | t) + n(\delta_y - \delta_{ty})
\] (37)

\[
H(t) = H(y^n : t) + n\delta_{ty}
\] (38)

\[
H(t) = H(z^n : t) + n\delta_{tz}
\] (39)

proof:

\[
\begin{align*}
H(s | z^n) &= H(s) - H(s : z^n) \\
&= H(s) + \left[ \frac{H(s : t, z^n : y^n) - H(s : t, y^n)}{-H(s : t, z^n, y^n)} \right] - H(s : t) \\
&= H(s) + \left[ \frac{H(s : t, z^n : y^n) - H(s : t, y^n)}{-H(s : t, y^n)} \right] \\
&= H(s) + H(s : t | z^n) - H(s : z^n | t) - H(s : t, y^n) + H(s : y^n | t) \\
&= H(s) + \left[ \frac{-n\delta_{tz}}{-H(s : t, z^n)} \right] - H(s : z^n | t) - \left[ \frac{-H(s | z^n)}{-n\delta_y - n\delta_{ty}} \right] + H(s : y^n | t)
\end{align*}
\] (40a)

\[
\begin{align*}
H(s | t) &= H(s : y^n | t) + H(s | y^n, t) \\
&= H(s : y^n | t) + n(\delta_y - \delta_{ty})
\end{align*}
\] (41a)

\[
\begin{align*}
H(t) &= H(t : y^n) + H(t | y^n) \\
&= H(t : y^n) + n\delta_{ty}
\end{align*}
\] (42a)
Claim 8

\[ H(s | z^n) = \sum_j H(y_j : s | \alpha_j, t) - \sum_j H(z_j : s | \alpha_j, t) - H(t | z^n, s) + n(\delta_{tz} + \delta_y - \delta_{ty}) \]  

(43)

\[ H(s | t) = \sum_j H(y_j : s | \alpha_j, t) + \Sigma_{<t} - \Sigma_{<z,t} + n(\delta_y - \delta_{ty}) \]  

(44)

\[ H(t) = \sum_j H(y_j : \alpha_j, t) - \sum_j H(y_j : y_{<j}) - \Sigma_{<t} + n\delta_{ty} \]  

(45)

\[ H(t) = \sum_j H(z_j : \alpha_j, t) - \sum_j H(z_j : z_{>j}) - \Sigma_{<t} - n\delta_{tz} \]  

(46)

proof: Combine Claims 3 to 7.

QED

5 Revisiting Classical Channel Coding

We will assume that the reader of this paper has first read Ref.[7]. Ref.[7] provides a derivation of the capacity for classical channel coding, a derivation based on p-type integration techniques. The same p-type integration techniques will be used in this paper to derive capacities for classical and quantum wiretap coding.

In this section, we will mention just a few highlights of the theory of classical channel coding as given in Ref.[7]. We will state those highlights in a form that is slightly more general than the form presented in Ref.[7]. The more general form clarifies the connections between channel coding and wiretap coding.

Channel coding is based on the following CB net:

\[ \hat{m} \rightarrow y^n \rightarrow \hat{x}^n \rightarrow m \]

\[ C \]

The probability distribution \( P_{y|x} \), called the channel, is given and fixed. Define \[ \mathcal{P}_{\text{gen}} = \{ P_{z|y} \in \text{pd}(S_z, y) : P_{y|x} \text{ fixed} \} . \]

For any \( P \in \text{pd}(S_z, y) \), we can define a convex hull of \( R \)’s by

\[ ^1_{\text{gen} = \text{“general”}} \]
\[
\mathcal{R}(P) = \{ R_m : R_m \leq H_P(y : x) \}.
\] (49)

It’s also useful to consider, for any \( \mathcal{P} \subset \mathcal{P}(S_x, y) \), the set
\[
\mathcal{R}(\mathcal{P}) = \bigcup_{P \in \mathcal{P}} \mathcal{R}(P).
\] (50)

The following Lemma will be used later on.

**Claim 9** Suppose \( a(j) \) and \( b(j) \) are two random variables that are functions of a random variable \( j \). Furthermore, suppose \( b(j) = (\beta, j) \) for some random variable \( \beta \). Then\(^2\)

\[
H(a(j) : b(j) | j) \leq H(a(j) : b(j)).
\] (51)

**proof:**

\[
H(a(j) : b(j) | j) \overset{(i)}{=} H(a(j) : b(j), j) \overset{(ii)}{\geq} H(a(j) : b(j) | j).
\] (52a)

(i) follows by cloning \( j \), which is a component of \( b(j) \). (ii) follows from the chain rule.

QED

Later on, we will use random variables \( a_j \) that are labeled by an index \( j = 1, 2, \ldots, n \). The index \( j \) can be promoted to a random variable that is uniformly distributed. \( j \) is often called a “time sharing variable”. We can then use \( E_j = \frac{1}{n} \sum_j \).

\( a_j \) can be thought of as a special case of \( a(j) \) when \( j \) is uniformly distributed.

**Claim 10** Achievability: \( \forall R_m, \) if \( R_m \in \mathcal{R}(\mathcal{P}_{gen}) \), then \( \exists \) an encoding and a decoding that satisfy \( \lim_{n \to \infty} P_{err} = 0 \) for the CB net of Eq.(47).

**proof:** A proof of this is given in Ref.[7]. We will not repeat that proof here. That proof is very similar to the achievability proofs that will be presented later in this paper.

QED

**Claim 11** Optimality: \( \forall R_m, \) if \( \exists \) an encoding and a decoding that satisfy \( \lim_{n \to \infty} P_{err} = 0 \) for the CB net of Eq.(47), then \( R_m \in \mathcal{R}(\mathcal{P}_{gen}) \).

\(^2\)Note that the inequality in Eq.(51) is different from \( H(a | j) \leq H(a) \), which is just MI\( \geq 0 \).
proof:

A proof of this is given in Ref. [7]. We will not repeat that proof here. That proof is of the type most commonly found in introductory SIT textbooks. Let us present here an alternative proof which is more akin to the optimality proofs which will be presented later in this paper.

\[ nR_m = \ln N_m = H(m) = H(m : y^n) + H(m | y^n) \]  
\[ \leq H(m : y^n) + n\delta \]  
\[ = \sum_{j=1}^{n} H(y_j : y_{<j}, m) + n\delta \]  
\[ = nH(y_j : y_{<j}, m | j) + n\delta \]  
\[ = n(H(y : u | j) + \delta) \]  
\[ \leq n(H(y : u) + \delta) \]  
\[ \leq n(H(y : x) + \delta) \]  

(53a) This follows from Fano's inequality, which says that if \( \lim_{n \to \infty} P_{err} = 0 \), then \( H(m | y^n) \leq n\delta \) with \( \lim_{n \to \infty} \delta = 0 \).

(53b) This follows from the chain rule for MI.

(53c) This follows from defining a new, uniformly distributed random variable \( z \) with states \( z = 1, 2, \ldots, n \).

(53d) This follows from a change of notation \( y_j \to y \) and \( (y_{<j}, m, j) \to u \).

(53e) This follows from Claim 9.

(53f) This follows from the classical data processing inequalities.

\[ (54a) \]

QED

In the above optimality proof, we start with the CB net of Eq. (47). Then we do some “chain ruling” reminiscent of peeling away all \( n \) layers except one. We end up with a different CB net with random variables \( y, x, u \). It’s instructive to present a chain of CB nets connecting the beginning and ending CB nets of this process.

One starts with the CB net (\( \mathcal{C} \) implicit)

\[ (54a) \]

Tracing over (i.e., adding over all states of) the node \( \hat{m} \) (highlighted with a double circle) gives
The previous CB net equals

\[ y^n \rightarrow x^n \rightarrow m \]  \hspace{2cm} (54b)

Tracing over all the nodes highlighted with a double circle gives

\[ y_{<j} \rightarrow x_{<j} \rightarrow m \]  \hspace{2cm} (54c)

Merging the \( m \) and \( y_j \) nodes and promoting the index \( j \) to a random variable, we get

\[ y_j \rightarrow x_j \rightarrow m \]  \hspace{2cm} (54d)

In the last graph, we leave implicit a source node \( j \) with outgoing arrows pointing into all nodes that mention \( j \).

From Claims 10 and 11, we see that \( \mathcal{R}(\mathcal{P}_{\text{gen}}) \) is the maximal achievable region (MAR) of \( R_m \)'s. Some authors refer to the MAR as the “capacity region”. One defines the channel capacity \( C \) by

\[ C = \max_{R_m \in \mathcal{R}(\mathcal{P}_{\text{gen}})} R_m = \max_{P \in \mathcal{P}_{\text{gen}}} H_P(y : x) \].  \hspace{2cm} (55)

Simply put, for channel coding, the MAR is just a closed interval \([0, C]\). The upper limit point \( C \) of this MAR is called the channel capacity.

\section{Classical Wiretap Coding}

In this section, we derive the channel capacity for classical wiretap coding. Everything we prove in this section has already been proven by Csiszár and Körner in Ref.[3].
However, some of our proofs (most notably our achievability proof) differ significantly from those presented in Ref.[5]. We find that our alternative proof methods are easier to generalize to quantum wiretap coding.

To help those readers who want to compare our proofs to those of Ref.[5], we try in this paper to use names of variables that are identical or at least very similar to those of Ref.[5]. As in Ref.[5], the channel considered in this paper takes $x^n$ to $(y^n, z^n)$. Like Ref.[5], we use the letter $s$ to denote the “secret” signal, one that is visible only to the observer that measures the $y^n$ output. And we use the letter $t$ to denote the “tapped” or shared signal, one that is visible to both the $y^n$ and $z^n$ observers.

We consider all wiretap coding protocols that can be described by the following CB net

\[
\begin{align*}
\hat{s} & \rightarrow y^n \\
\hat{t}_y & \rightarrow x^n \\
z^n & \rightarrow \hat{t}_z \\
& \rightarrow C
\end{align*}
\]

with

\[
P(s) = \frac{1}{N_s}, \quad P(t) = \frac{1}{N_t},
\]

\[
P(x^n|s, t, C) = \delta(x^n, x^n(s, t)),
\]

\[
P(y^n, z^n|x^n) = \prod_j P(y_j, z_j|x_j),
\]

\[
P(\hat{s}|y^n, C), P(\hat{t}_y|y^n, C), P(\hat{t}_z|z^n, C) = \text{to be specified},
\]

and

\[
P(C) = \text{to be specified}.
\]

Assume that we are given a channel \(\{P(y, z|x)\}_{y, z} \in pd(S_{y, z})\) for all \(x \in S_x\). The encoding \(P(C)\) and decoding \(P(\hat{s}|y^n, C), P(\hat{t}_y|y^n, C), P(\hat{t}_z|z^n, C)\) probability distributions are yet to be specified.
We will consider 3 different coding rates:

\[ R_e = \frac{H(s^k | z^n)}{nk}, \quad R_s = \frac{\ln N_s}{n}, \quad R_t = \frac{\ln N_t}{n}. \]  

(62)

By definition, all 3 R’s are non-negative. To define \( R_e \), called the equivocation rate, we’ve replaced \( s \) by a block of \( k \) letters \( s^k \). CMI \( \geq 0 \) implies that \( \frac{1}{nk}H(s^k | z^n) \leq \sum_{j=1}^{k} \frac{1}{nk}H(s_j | z^n) \). If we assume \( H(s_j | z^n) \) is the same for all \( j = 1, 2, \ldots k \), then \( \frac{1}{nk}H(s^k | z^n) \leq \frac{1}{n}H(s_1 | z^n) \). Furthermore, MI \( \geq 0 \) implies that \( \frac{1}{n}H(s_1 | z^n) \leq \frac{1}{n}H(s_1) = R_s \). We can now set \( s_1 \) to \( s \). We have established that

\[ 0 \leq R_e \leq \frac{1}{n}H(s | z^n) \leq R_s. \]  

(63)

Note that \( R_e = R_s \) implies \( H(s) = H(s | z^n) \) or, equivalently, \( H(s : z^n) = 0 \). Thus the case \( R_e = R_s \) can be described as perfect secrecy. The case \( R_t = 0 \) can be described as zero tapping rate.

Let

\[ \vec{R} = (R_e, R_s, R_t). \]  

(64)

We will always assume in this paper that \( s \) and \( t \) are independent. (Ref.\[5\] sometimes assumes they aren’t). Ref.\[5\] defines \( R_s \) as \( \frac{H(s | t)}{n} \). When \( s \) and \( t \) are independent, \( H(s | t) = H(s) = \ln N_s \).

For a given, fixed channel \( P_{y,z|x} \), define

\[ \mathcal{P}_{gen} = \left\{ P_{y,z,v,u} \in pd(S_{y,z,v,u}) : \begin{array}{l}
    P(y,z,v,u) = \sum_x P(y,z|x)P(x|v)P(v|u)P(u), \\
    P_{y,z|v,u} \text{ fixed}
  \end{array} \right\}. \]  

(65)

In terms of CB nets, the elements of \( \mathcal{P}_{gen} \) must have the following graph topology

\[ P_{y,z,v,u} = \sum_x P_{y,z,v,u|x}. \]  

(66)

We will say that \( P(y,z|v,u) \) factors in \( y \) and \( z \) if \( P(y,z|v,u) = P(y|v,u)P(z|v,u) \) for all \( y, z, v, u \). (i.e., conditional independence). It is convenient to define the following subset of \( \mathcal{P}_{gen} \)

\[ \mathcal{P}_{fac} = \{ P \in \mathcal{P}_{gen} : P_{y,z|v,u} \text{ factors in } y \text{ and } z \} . \]  

(67)
The elements of $\mathcal{P}_{\text{fac}}$ have the same graph as Eq. (66) except without the arrow connecting $\tilde{z}$ and $\tilde{y}$.

For any $P \in pd(S_{\tilde{\underline{y}}, \tilde{z}, \underline{v}, \underline{u}})$, we can define a convex hull of $\overrightarrow{R}$'s by

$$
\mathcal{R}(P) = \begin{cases} 
\overrightarrow{R} = (R_e, R_s, R_t) \in \mathbb{R}^3 : & 0 \leq R_e \leq R_s, 0 \leq R_s, 0 \leq R_t, \\
& R_e \leq H(y : v | u) - H(z : v | u), \\
& R_s + R_t \leq H(y : v | u) + \ell, \\
& R_t \leq \ell, \\
& \ell = \min\{H(y : u), H(z : u)\} \\
& \text{all } H \text{ evaluated at } P 
\end{cases}. \quad (68)
$$

It’s also useful to consider, for any $\mathcal{P} \subset pd(S_{\tilde{\underline{y}}, \tilde{z}, \underline{v}, \underline{u}})$, the set

$$
\mathcal{R}(\mathcal{P}) = \bigcup_{P \in \mathcal{P}} \mathcal{R}(P). \quad (69)
$$

![Diagram](image)

Figure 1: $R_t = 0$ crosssections of the sets $\mathcal{R}(\mathcal{P}_{\text{gen}})$ and $\mathcal{R}(P)$ for some particular point $P \in \mathcal{P}_{\text{gen}}$. $\mathcal{R}(P)$ is a convex hull, which is a set bounded by straight lines (or by planes in higher dimensions). $\mathcal{R}(\mathcal{P}_{\text{gen}})$ doesn’t necessarily have completely straight sides but it must still be convex.

$\mathcal{R}(\mathcal{P})$ for arbitrary $\mathcal{P}$ is not always a convex set. For instance, it isn’t if $\mathcal{P}$ consisting of just two $P$’s. But how about $\mathcal{R}(\mathcal{P})$ when $\mathcal{P} = \mathcal{P}_{\text{gen}}$?

Claim 12 $\mathcal{R}(\mathcal{P}_{\text{gen}})$ is a closed convex set.

Proof: $\mathcal{R}(\mathcal{P}_{\text{gen}})$ is a union of closed sets so it is closed. Next we prove convexity.

Suppose $\underline{b}$ is a random variable with $S_{\underline{b}} = \{0, 1\}$. Let $E_{\underline{b}} = \sum_{\underline{b}} P(\underline{b})$.

In this proof, we will use the shorthand

$$
\xi(\underline{b}) = (\underline{y}(\underline{b}), \tilde{z}(\underline{b}), \underline{v}(\underline{b}), \underline{u}(\underline{b})). \quad (70)
$$
for \( b = 0, 1 \). Furthermore, we will use \( \xi_j(b) \) with \( j \in \{1, 2, 3, 4\} \) to denote the four components of this vector. \( \xi(0) \) and \( \xi(1) \) represent two different CB nets. Note that in general \( S_{\xi_j(0)} \neq S_{\xi_j(1)} \) for all \( j \).

We want to prove that if \( \vec{R}(b) \in \mathcal{R}(P_{\text{gen}}) \) for \( b = 0, 1 \), then \( E_b \vec{R}(b) \in \mathcal{R}(P_{\xi}) \). This means that if we are given 2 CB nets \( \xi(0) \) and \( \xi(1) \) such that \( P_{\xi(b)} \in P_{\text{gen}} \) and \( \vec{R}(b) \in \mathcal{R}(P_{\xi(b)}) \) for \( b = 0, 1 \), the we can find an average CB net \( \xi \) such that \( P_{\xi} \in P_{\text{gen}} \) and \( E_b \vec{R}(b) \in \mathcal{R}(P_{\xi}) \). Define \( \bar{R} = E_b \vec{R}(b) \). We want to prove that \( \bar{R} \in \mathcal{R}(P_{\xi}) \).

We define the average CB net \( \xi \) to have the topology prescribed by \( P_{\text{gen}} \), with nodes

\[
\xi_j = [\xi_j(0), \xi_j(1), b]
\]

for all \( j \). If we assume that

\[
P(\xi = \xi|b = b) = P(\xi(b) = \xi(b)),
\]

then

\[
P(\xi = \xi) = \sum_b P(\xi = \xi|b = b)P(b) = \sum_b P(\xi(b) = \xi(b))P(b) = E_bP(\xi(b) = \xi(b)).
\]

One has

\[
0 \leq R_{\xi}(b) \leq R_{\bar{\xi}}(b)
\]

for \( b = 0, 1 \). Hence

\[
0 \leq E_bR_{\xi}(b) \leq E_bR_{\bar{\xi}}(b).
\]

Hence

\[
0 \leq R_{\xi} \leq R_{\bar{\xi}}.
\]

Next note that

\[
H_{P_{\xi}}(y : v|u) - H_{P_{\bar{\xi}}}(y : v|u) =
\]

\[
= H_{P_{\bar{\xi}}}(y : v|u, b) - H_{P_{\bar{\xi}}}(y : v|u, b)
\]

\[
= E_b\left\{H_{P_{\bar{\xi}}}(y(b) : v(b)|u(b)) - H_{P_{\bar{\xi}}}(y(b) : v(b)|u(b))\right\}
\]

\[
\geq E_bR_{\xi}(b) = R_{\bar{\xi}}.
\]
Next note that
\[
H_{P_{\hat{\xi}}}(y : v | u) + H_{P_{\hat{\xi}}}(y : u) = \\
= H_{P_{\hat{\xi}}}(y : v, \hat{b}) + H_{P_{\hat{\xi}}}(y : u, \hat{b}) \\
\geq H_{P_{\hat{\xi}}}(y : v, \hat{b}) + H_{P_{\hat{\xi}}}(y : u | \hat{b}) \\
= E_{\hat{b}}\left\{ H_{P_{\hat{\xi}}}(y(b) : v(b) | u(b)) + H_{P_{\hat{\xi}}}(y(b) : u(b)) \right\} \\
\geq E_{\hat{b}}\{ R_{\hat{\xi}}(b) + R_{\parallel}(b) \} \\
= R_{\hat{\xi}} + R_{\parallel}.
\]

Likewise,
\[
H_{P_{\hat{\xi}}}(y : v | u) + H_{P_{\hat{\xi}}}(z : u) \geq R_{\hat{\xi}} + R_{\parallel}.
\]

Next note that
\[
H_{P_{\hat{\xi}}}(y : u) = H_{P_{\hat{\xi}}}(y : u, \hat{b}) \\
\geq H_{P_{\hat{\xi}}}(y : u | \hat{b}) \\
= E_{\hat{b}} H_{P_{\hat{\xi}}}(y(b) : u(b)) \\
\geq E_{\hat{b}} R_{\parallel}(b) \\
= R_{\parallel}.
\]

Likewise,
\[
H_{P_{\hat{\xi}}}(z : u) \geq R_{\parallel}.
\]

QED

Claim 13
\[
\mathcal{R}(\mathcal{P}_{\text{gen}}) = \mathcal{R}(\mathcal{P}_{\text{fac}}).
\]

proof: \( \mathcal{P}_{\text{fac}} \subset \mathcal{P}_{\text{gen}} \) so \( \mathcal{R}(\mathcal{P}_{\text{fac}}) \subset \mathcal{R}(\mathcal{P}_{\text{gen}}) \). Next let’s prove the reverse inclusion. Suppose \( P_{y, z, v, u} \in \mathcal{P}_{\text{gen}} \). \( P_{y, z | v, u} \) only appears in the definition of \( \mathcal{R}(P) \) through its marginals \( P_{y | v, u} \) and \( P_{z | v, u} \). The set \( \mathcal{R}(P) \) doesn’t change if in its definition, one replaces everywhere the probability distribution \( P_{y, z | v, u} \) by the product probability distribution \( P_{y | v, u} P_{z | v, u} \). Both probability distributions belong to \( \text{pd}(S_{y, z}) \) for all \( v, u \). Given \( P \in \mathcal{P}_{\text{gen}} \), we have found \( \bar{P} \in \mathcal{P}_{\text{fac}} \) such that \( \mathcal{R}(P) = \mathcal{R}(\bar{P}) \). QED
6.1 Optimality

Claim 14 Optimality: ∀ \vec{R}, if ∃ an encoding and a decoding that satisfy \( \lim_{n \to \infty} P_{\text{err}} = 0 \) for the CB net of Eq. (56), then \( \vec{R} \in \mathcal{R}(\mathcal{P}_{\text{gen}}) \).

proof: The proof starts from the result Claim 8. As pointed out in the section that ends and culminates with Claim 8, Claim 8 is basically a souped-up version of the chain rule. To prove the current claim, we will need to add 3 new ingredients that are not consequences of only the chain rule. First, we will use Fano’s inequality (see Ref. [1]). Second, we will use the fact that classical clone random variables can be merged. By this we mean that

\[ H(a : b, c | c) = H(a : b | c) \]  

(83)

for any random variables \( a, b, c \). Third, we will use Claim 9. Note that, by assumption, \( \lim_{n \to \infty} P_{\text{err}} = 0 \). Hence, by Fano’s inequality, the \( \delta_y, \delta_{ty}, \delta_{tz} \) used in Claim 8 go to zero as \( n \to \infty \). Furthermore, classical conditional entropies are non-negative and CMI ≥ 0 so

\[ 0 \leq \frac{1}{n} H(t | z^n, s) \leq \frac{1}{n} H(t | z^n) . \]  

(84)

But Fano’s inequality implies that \( \lim_{n \to \infty} \frac{1}{n} H(t | z^n) = 0 \) so also

\[ \lim_{n \to \infty} \frac{1}{n} H(t | z^n, s) = 0 . \]  

(85)

Let \( j \) be a random variable that is uniformly distributed and has states \( j = 1, 2, \cdots, n \). Let \( E_j = \frac{1}{n} \sum_j \). Define

\[ \vec{u} = (y_{< j}, z_{> j}, t, j) , \]  

\[ \vec{v} = (u, \tilde{z}) , \]  

\[ \vec{x} = x_j , \]  

\[ \vec{y} = y_j , \]  

\[ \vec{z} = z_j . \]  

(86)

After dropping \( \frac{1}{n} H(t | z^n, s) \) and the deltas \( \delta_y, \delta_{ty}, \delta_{tz} \), the 4 identities (souped-up chain rules) of Claim 8 yield the following 4 inequalities.

- Eq. (83) yields

\[ R_{\vec{z}} \leq \frac{1}{n} H(s | z^n) \]  

(87a)

\[ = E_j H(y_{j} : s | \alpha_{j}, t) - E_j H(z_{j} : s | \alpha_{j}, t) \]  

(87b)

\[ = H(y_{j} : s | \alpha_{j}, t, j) - H(z_{j} : s | \alpha_{j}, t, j) \]  

(87c)

\[ = H(y : v | u) - H(z : v | u) . \]  

(87d)
• Eq. (44) yields

\[
R_s = E_jH(y_j : s | \alpha_j, t) + \frac{\Sigma_{<s,t}}{n} - \frac{\Sigma_{<s,t}}{n} \tag{88a}
\]
\[
= H(y_j : s | \alpha_j, t, j) + \frac{\Sigma_{<s,t}}{n} - \frac{\Sigma_{<s,t}}{n} \tag{88b}
\]
\[
= H(y_j : v | u) + \frac{\Sigma_{<s,t}}{n} - \frac{\Sigma_{<s,t}}{n} \tag{88c}
\]
\[
\leq H(y_j : v | u) + \frac{\Sigma_{<s,t}}{n}. \tag{88d}
\]

• Eq. (45) yields

\[
R_t = E_jH(y_j : \alpha_j, t) - E_jH(y_j : y_{<j}) - \frac{\Sigma_{<s,t}}{n} \tag{89a}
\]
\[
= H(y_j : \alpha_j, t, j) - H(y_j : y_{<j}, j) - \frac{\Sigma_{<s,t}}{n} \tag{89b}
\]
\[
= H(y_j : v | u) - H(y_j : y_{<j}, j) - \frac{\Sigma_{<s,t}}{n} \tag{89c}
\]
\[
\leq H(y_j : v | u) - \frac{\Sigma_{<s,t}}{n} \tag{89d}
\]
\[
\leq H(y_j : u) - \frac{\Sigma_{<s,t}}{n}. \tag{89e}
\]

Eq. (89e) follows from Claim 9.

• Eq. (46) yields

\[
R_k = E_jH(z_j : \alpha_j, t) - E_jH(z_j : z_{>j}) - \frac{\Sigma_{<s,t}}{n} \tag{90a}
\]
\[
= H(z_j : \alpha_j, t, j) - H(z_j : z_{>j}, j) - \frac{\Sigma_{<s,t}}{n} \tag{90b}
\]
\[
= H(z_j : v | u) - H(z_j : z_{>j}, j) - \frac{\Sigma_{<s,t}}{n} \tag{90c}
\]
\[
\leq H(z_j : v | u) - \frac{\Sigma_{<s,t}}{n} \tag{90d}
\]
\[
\leq H(z_j : u) - \frac{\Sigma_{<s,t}}{n}. \tag{90e}
\]

Eqs. (89e) and (90e) can be combined by writing

\[
R_k \leq \ell - \frac{\Sigma_{<s,t}}{n}. \tag{91}
\]

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From all we’ve said so far, it’s clear that $\bar{R} \in \mathcal{R}(\mathcal{P}_{gen})$.

**QED**

In the above optimality proof, we start with the CB net of Eq.$(56)$. Then we do some “chain ruling” reminiscent of peeling away all $n$ layers except one. We end up with a different CB net with random variables $y, z, x, v, u$. It’s instructive to present a chain of CB nets connecting the beginning and ending CB nets of this process.

One starts with the CB net (Ç implicit)

![Diagram](92a)

Tracing over all the nodes highlighted with a double circle gives

![Diagram](92b)

The previous CB net equals
Tracing over all the nodes highlighted with a double circle gives

Merging the \( y_{<j} \) and \( z_{>j} \) nodes gives

(92e)
The previous CB net can be “accommodated” or modeled by the following CB net.

\[
\begin{align*}
\hat{u} &= u = y < z > t \\
\hat{v} &= v = y < z > t \\
\end{align*}
\]  \quad (92f)

In the last graph, we leave implicit a source node \( j \) with outgoing arrows pointing into all nodes that mention \( j \).

### 6.2 Achievability

This section will give a proof of achievability for classical wiretap coding. *The proof is very similar to the one given in Ref.[7] of achievability for classical channel coding.*

**Claim 15** Achievability: \( \forall \tilde{R}, \text{ if } \tilde{R} \in \mathcal{R}(\mathcal{P}_{gen}), \text{ then } \exists \text{ an encoding and a decoding that satisfy } \lim_{n \to \infty} P_{err} = 0 \text{ for the CB net of Eq.}(56).\)

**proof:**

Suppose \( P \in \mathcal{P}_{gen} \) and \( \tilde{R} \in \mathcal{R}(P). \)

We consider all wiretap coding protocols that can be described by the following CB net

\[
\begin{align*}
\hat{s} &\rightarrow u^n \\
\hat{z} &\rightarrow z^n \\
u^n &\rightarrow \hat{s} \\
x^n &\rightarrow \hat{z} \\
 &\rightarrow \hat{t}
\end{align*}
\]

We will use a codebook \( \mathcal{C} = (\mathcal{C}_s, \mathcal{C}_u, \mathcal{C}_u) \) composed of 3 sub-codebooks. Codebook \( \mathcal{C}_s \) is as an \( N_s \times N_t \times n \) matrix given by \( \mathcal{C}_s = \{x^n(s, t)\}_{s, t} = x^n(\cdot) \) where
where $x^n(s, t) = x^n(v^n(s, t))$, \(\forall s, t\).

Similarly, \(C_u = \{v^n(s, t)\}_{s, t} = v^n(\cdot)\) where $v^n(s, t)$ for all $s, t$ and \(C_u = \{u^n(t)\}_{t} = u^n(\cdot)$ where $u^n(t) \in S^n_u$ for all $t$.

We will use the shorthand notations $\hat{M} = (\hat{s}, \hat{t}, \hat{t}_y, \hat{t}_z)$ and $m = (s, t)$. Let $f$ be the function that maps $f(m) = f(s, t) = (s, t, t)$ for all $s, t$.

We assign to the CB net of Eq.(93) the following node transition matrices:

\[
P(s) = \frac{1}{N_\Delta}, \quad P(t) = \frac{1}{N_\Delta},
\]

\[
P(u^n|t, C) = \delta(u^n, u^n(t)),
\]

\[
P(v^n|u^n, s, C) = \delta(v^n, v^n(s, t)),
\]

where $v^n(s, t) = v^n(s, u^n(t))$,

\[
P(x^n|u^n, C) = \delta(x^n, x^n(s, t)),
\]

where $x^n(s, t) = x^n(v^n(s, t))$,

\[
P(y^n, z^n|x^n) = \prod_j P(y_j, z_j|x_j),
\]

\[
P(C) = \prod_{s, t} P(x^n(s, t), v^n(s, t), u^n(t)) = \prod_{s, t, j} P_{\hat{s}, \hat{t}, \hat{t}_y}(x_j(s, t), v_j(s, t), u_j(t)),
\]

where $P_{\hat{s}, \hat{t}, \hat{t}_y}(x, v, u) = P_{\hat{s}|\hat{t}}(x|v)P_{\hat{t}|\hat{t}_y}(v|u)P_{\hat{t}_y}(u)$, and

\[
P(\hat{M}|y^n, z^n, C) = \prod_{\mu=1}^5 \prod_{m_\mu \neq \hat{m}_\mu} \theta(R_{\mu} < \Gamma_{\mu}),
\]

where the quantities $\hat{m}_\mu$, $R_{\mu}$ and $\Gamma_{\mu}$ are defined in Appendix A. Assume that we are given a channel $\{P(y, z|x)\}_{y, z} \in pd(S_{\hat{y}, \hat{z}})$ for all $x$.

The probability of success is defined by

\[
P_{\text{suc}} = P(\hat{M} = f(m)).
\]

One has

\[
P_{\text{suc}} = \sum_{\hat{M}, m} \theta(\hat{M} = f(m))P(\hat{M}, m)
\]

\[
= \frac{1}{N_\Delta} \sum_{\hat{s}, \hat{t}, \hat{t}_y, \hat{t}_z} \sum_{C} P(C) \sum_{y^n, z^n} P(\hat{M} = (\hat{s}, \hat{t}, \hat{t}_y, \hat{t}_z)|y^n, z^n)P(y^n, z^n|x^n(s, t))
\]

\[
= E_C \sum_{y^n, z^n} P(y^n, z^n|x^n(s, t)) \prod_{\mu=1}^5 \prod_{m_\mu \neq \hat{m}_\mu} \theta(R_{\mu} < \Gamma_{\mu}),
\]

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where \( \hat{t} \) means evaluated at \( \hat{t}_y = \hat{t}_z = \hat{t} \).

Let
\[
\oint \kappa = 5 \prod_{\mu=1}^{5} \prod_{m_\mu \neq \hat{m}_\mu} \left\{ \int_{-\infty}^{\infty} \frac{dk_\mu(m_\mu)}{2\pi i} \left( \frac{1}{k_\mu(m_\mu) - i\epsilon} \right) \right\},
\]
and
\[
K_\mu = \sum_{m_\mu \neq \hat{m}_\mu} k_\mu(m_\mu).
\]

We will also use the following shorthand notation
\[
\hat{x} = x(s, \hat{t}), \quad \hat{v} = v(s, \hat{t}), \quad \hat{u} = u(\hat{t}),
\]
and
\[
\xi = (y, z, x(\cdot), v(\cdot), u(\cdot)).
\]

Expressing the \( \theta \) functions in Eq.(102c) as integrals, we get
\[
P_{\text{suc}} = \oint \kappa e^{-i \sum_{\mu} K_\mu R_\mu} \sum_{y^n, z^n, x^n(\cdot), v^n(\cdot), u^n(\cdot)} e^{\sum_{\xi \nu} nP_1(\xi) \ln Z(\xi)},
\]
where
\[
Z(\xi) = P(y, z|\hat{x}) \prod_{s, t} \left\{ P(x(s, t), v(s, t), u(t)) \right\} \prod_{\mu=1}^{5} \prod_{m_\mu \neq \hat{m}_\mu} \left\{ (\gamma_{\mu (m_\mu)}, \hat{t}) \right\},
\]
and
\[
\xi = (y, z, x(\cdot), v(\cdot), u(\cdot)).
\]

Next we express the sum over \( y^n, z^n, x^n(\cdot), v^n(\cdot), u^n(\cdot) \) in Eq.(107) as a p-type integral and do the p-type integration by the steepest descent method. We get for the leading order term:
\[
P_{\text{suc}} = \oint \kappa e^{-i \sum_{\mu} K_\mu R_\mu} e^{\sum_{\xi} nP_1(\xi) \ln Z(\xi)},
\]
where
\[
Z = \sum_{\xi} Z(\xi).
\]

Note that in Eq.(110), the sum over the \( x(\cdot) \) component of \( \xi \) can be done easily because none of the \( \gamma_{\mu} \) depend on \( x(\cdot) \) so we may set
\[
\sum_x P(y, z|x) \prod_{s,t} \{P(x(s,t)|v(s,t), u(t))\} = P(y, z|\hat{v}, \hat{u}) .
\]

(111)

Hence, using the shorthand notations

\[
E_{y,z} = \sum_{y,z} P(y, z),
\]

(112)

and

\[
E_{v(s,t), u(t)} = \sum_{v(s,t), u(t)} P(v(s,t), u(t)) ,
\]

(113)

\[
Z \text{ can be expressed as }
\]

\[
Z = E_{y,z} E_{v(\cdot), u(\cdot)} \left[ P(y, z : \hat{v}, \hat{u}) \prod_{\mu} \prod_{m_\mu \neq \hat{m}_\mu} \left\{ \frac{\gamma_{\mu}^{k_\mu(m_\mu)}}{n} \right\} \right] .
\]

(114)

Define

\[
Z_0 = E_{y,z} E_{\hat{v}, \hat{u}} \left[ P(y, z : \hat{v}, \hat{u}) \prod_{\mu} \left\{ \Phi_{\mu}^{K_\mu} \right\} \right] ,
\]

(115)

where the quantities \( \Phi_{\mu} \) are defined in Appendix A.

Define the integration operators

\[
1_{h,K} = \prod_{\mu} \left\{ \int_{-\infty}^{+\infty} dh_{\mu} \int_{-\infty}^{+\infty} dK_{\mu} \frac{1}{2\pi} e^{-ih_{\mu}K_{\mu}} \right\} ,
\]

(116)

and

\[
1_{h>0,K} = 1_{h,K} \prod_{\mu} \theta(h_{\mu} > 0) .
\]

(117)

Note that 1 equals

\[
1 = \prod_{\mu} \left\{ \int_{-\infty}^{+\infty} dK_{\mu} \delta\left( \sum_{m_\mu \neq \hat{m}_\mu} \{k_\mu(m_\mu)\} - K_{\mu} \right) \right\} \quad (118a)
\]

\[
= 1_{h,K} e^{i\sum_{\mu} h_{\mu} \sum_{m_\mu \neq \hat{m}_\mu} k_{\mu}(m_\mu)} . \quad (118b)
\]

Multiplying \( P_{\text{suc}} \) by 1 certainly doesn’t change it. Thus the right hand sides of Eqs. \((109)\) and \((118b)\) can be multiplied to get

\[
P_{\text{suc}} = 1_{h,K} e^{-i\sum_{\mu} K_{\mu} R_{\mu}} \oint_k e^{i\sum_{\mu} h_{\mu} \sum_{m_\mu \neq \hat{m}_\mu} k_{\mu}(m_\mu)} e^{n \ln Z} .
\]

(119)
Next we will assume that, for all $m_{\mu}$, when doing the contour integration over $k_{\mu}(m_{\mu})$ in Eq. (119) with $Z$ given by Eq. (114), the $e^{n \ln Z}$ can be evaluated at the value $k_{\mu}(m_{\mu}) = i \epsilon \to 0$ of the pole. Symbolically, this means we will assume

$$
\oint e^{i \sum_{\mu} h_{\mu} \sum_{m_{\mu} \neq \hat{m}_{\mu}} k_{\mu}(m_{\mu})} e^{n \ln Z} = e^{n \ln Z_0} \prod_{\mu} \{\theta(h_{\mu} > 0)\} .
$$

(120a)

Applying Eq. (120b) to Eq. (119) gives

$$
P_{\text{suc}} = 1_{h > 0, K} e^{-i \sum_{\mu} K_{\mu} R_{\mu} e^{n \ln Z_0}} .
$$

(121)

Expanding $\ln Z_0$ to first order in $K_{\mu}$ and doing the $h$ and $K$ integrals yields

$$
P_{\text{suc}} \approx \prod_{\mu=1}^{5} \theta(R_{\mu} < H_{\mu})
$$

(122a)

$$
= \theta(\vec{R} \in \mathcal{R}(P)) ,
$$

(122b)

where the quantities $H_{\mu}$ are defined in Appendix A.

QED

6.3 Capacity

From Sections 6.1 and 6.2, we see that $\mathcal{R}(P_{\text{gen}})$ is the maximal achievable region (MAR) of $\vec{R}$'s. The MAR is a closed convex set. We define the channel capacity as the value of $R_{\hat{z}}$ at one of the corners of the MAR.

Define the line

$$
\mathcal{L} = \{\vec{R} \in \mathbb{R}^3 : R_{\hat{x}} = 0, R_{\hat{z}} = R_{\hat{z}}\} .
$$

(123)

One defines the channel capacity $C_{\hat{z}}$ by

$$
C_{\hat{z}} = \max_{R_{\hat{z}} \in \mathcal{R}(P_{\text{gen}}) \cap \mathcal{L}} R_{\hat{z}} = \max_{P \in P_{\text{gen}}} \{H_P(y : v | u) - H_P(z : v | u)\} .
$$

(124)

Some authors put extra restrictions on the wiretap channel $P(y, z | x)$ and use Claim 16 given below. They do this in order to reduce and simplify the MAR and to simplify the formula for the channel capacity. It is also possible to trim $P_{\text{fac}}$. That is, to find $P \subset P_{\text{fac}}$ with $\mathcal{R}(P) = \mathcal{R}(P_{\text{fac}})$. One can, for example, place bounds on $N_{\hat{x}}$ and $N_{\hat{z}}$ by polynomial functions of $N_{\hat{z}}$. This reduces the size of the space $\mathcal{P}$ one must search over in order to calculate the channel capacity. See Ref. [5] for more details.
Claim 16 For a Markov chain \( a \leftarrow e \leftarrow b \) (or any tri-node Markov-like chain with \( e \) in the middle),

\[
H(a : e | b) = H(a : e) - H(a : b) .
\] (125)

proof: Claim 2 implies

\[
H(a : e | b) = H(a : b | e) + H(a : e) - H(a : b) .
\] (126)

But for the type of graph that is assumed as a premise, one has \( H(a : b | e) = 0 \).

QED

7 Quantum Wiretap Coding

In this section, we consider quantum wiretap coding. We try to make our treatment of quantum wiretap coding as parallel as possible to our treatment in the previous sections of classical wiretap coding. This parallel treatment is facilitated by the use of CB nets for the classical case and QB nets for the quantum one.

We consider all wiretap coding protocols that can be described by the following QB net

![QB net diagram]

with

\[
A(s) = \frac{1}{\sqrt{N_s}} , \quad A(t) = \frac{1}{\sqrt{N_t}} ,
\] (128)

\[
A(x^n | s, t, C) = \delta(x^n, x^n(s, t)) ,
\] (129)

\[
A(y^n, z^n | x^n) = \prod_j A(y_j, z_j | x_j) ,
\] (130)
\[ A(\tilde{s}|y^n, C), A(\hat{t}_y|y^n, C), A(\hat{t}_z|z^n, C) = \text{to be specified}, \]  
and
\[ A(C) = \text{to be specified}. \]  

Assume that we are given the reservoir amplitudes \( A(r_\varrho |s) \) and \( A(r_\lambda |t) \). Assume that we are also given an isometry \( A(y, z|x) \) (called the wiretap channel amplitude for this problem). The encoding \( A(C) \) and decoding \( A(\tilde{s}|y^n, C), A(\hat{t}_y|y^n, C), A(\hat{t}_z|z^n, C) \) probability amplitudes are yet to be specified.

We will consider 3 different coding rates:

\[ R_\varrho = \frac{S(\hat{s}_k|\hat{z}^n)}{nk} , \quad R_\lambda = \frac{\ln N_\varrho}{n} , \quad R_\lambda = \frac{\ln N_\lambda}{n}. \]  

Clearly, \( R_\varrho \) and \( R_\lambda \) must be non-negative. \( R_\lambda \) must be non-negative too even though it is defined as a quantum conditional entropy and those can sometimes be negative. But not this time, at least not for very large \( n \). This is why, \( -\frac{1}{n}S(\hat{s}_k|y^n) \leq \frac{1}{n}S(\hat{s}_k|\hat{z}^n) \) by Claim 1. Furthermore, \( \lim_{n\to\infty} -\frac{1}{n}S(\hat{s}_k|y^n) = 0 \) by Alicki-Fannes's inequality, assuming an encoding and decoding that satisfy \( \lim_{n\to\infty} P_{\text{err}} = 0 \). Thus, \( R_\varrho \geq 0 \). To define \( R_\lambda \), called the equivocation rate, we’ve replaced \( \hat{z}_k \) by a block of \( k \) letters \( \hat{s}_k \). CMI \( \geq 0 \) implies that \( \frac{1}{nk}S(\hat{s}_k|\hat{z}^n) \leq \sum_{j=1}^{k} \frac{1}{nk}S(\hat{s}_j|\hat{z}^n) \). If we assume \( S(\hat{s}_j|\hat{z}^n) \) is the same for all \( j = 1, 2, \ldots k \), then \( \frac{1}{nk}S(\hat{s}_k|\hat{z}^n) \leq \frac{1}{n}S(\hat{s}_1|\hat{z}^n) \). Furthermore, \( \text{MI} \geq 0 \) implies that \( \frac{1}{n}S(\hat{s}_1|\hat{z}^n) \leq \frac{1}{n}S(\hat{s}_1) = R_\lambda \). We can now set \( \hat{s}_1 \) to \( \hat{z}_k \).

We have established that
\[ 0 \leq R_\varrho \leq \frac{S(\hat{s}_k|\hat{z}^n)}{n} \leq R_\lambda. \]  

Let
\[ \bar{R} = (R_\varrho, R_\lambda, R_\lambda). \]  

For a given, fixed channel \( A_{y, z|x} \), define

\[ \mathcal{P}_{\text{gen}} = \left\{ \rho_{y, z, x, u} \in dm(H_{y, z, x, u}) : \rho_{y, z, x, u} = \sum_r A(y, z|x) |y\rangle \langle y| \right\}. \]  

In terms of QB nets, the elements of \( \mathcal{P}_{\text{gen}} \) must have the following graph topology.
\[
\rho_{y, z, v, u} = \text{tr}_x \begin{bmatrix}
\delta \\
y \\
z \\
\delta \\
y \\
v \\
u \\
\end{bmatrix} \text{[ h.c. ]}.
\tag{137}
\]

Note that, unlike its classical counterpart Eq. (66), this graph has an arrow from \(u\) to \(x\). Define

\[
A(y, z | v, u) = \sum_x A(y, z | x)A(x | v, u).
\tag{138}
\]

We will say that \(A(y, z | v, u)\) factors in \(y\) and \(z\) if \(A(y, z | v, u) = A(y | v, u)A(z | v, u)\) for all \(y, z, v, u\) (i.e., conditional independence). If also \(A(y | v, u)\) and \(A(z | v, u)\) are both isometries, we will say that \(A(y, z | v, u)\) iso-factors in \(y\) and \(z\). It is convenient to define the following subset of \(P_{\text{gen}}\)

\[
\mathcal{P}_{\text{fac}} = \{ P \in P_{\text{gen}} : A(y, z | v, u) \text{ iso-factors in } y \text{ and } z \}.
\tag{139}
\]

The elements of \(\mathcal{P}_{\text{fac}}\) have the same graph as Eq. (137) because, by convention, the marginalizer nodes \(y\) and \(z\) are drawn without an arrow connecting them regardless of whether these two nodes depend on each other or not.

For any \(\rho \in dm(H_{y, z, v, u})\), we can define a convex hull of \(\mathcal{R}\)'s by

\[
\mathcal{R}(\rho) = \bigg\{ \tilde{R} = (R_{x}, R_{\bar{x}}, R_{\bar{u}}) \in \mathbb{R}^3 : \begin{align*}
0 &\leq R_{\bar{x}} \leq R_{x}, \quad 0 \leq R_{\bar{u}} \leq R_{u}, \\
R_{x} &\leq S(y : v | u) - S(z : v | u), \\
R_{\bar{x}} + R_{\bar{u}} &\leq S(y : v | u) + \ell, \\
R_{\bar{u}} &\leq \ell, \\
\ell &\leq \min\{S(y : u), S(z : u)\} \\
\text{all } S \text{ evaluated at } \rho
\end{align*} \bigg\}.
\tag{140}
\]

It's also useful to consider, for any \(\mathcal{P} \subset dm(H_{y, z, v, u})\), the set

\[
\mathcal{R}(\mathcal{P}) = \bigcup_{\rho \in \mathcal{P}} \mathcal{R}(\rho).
\tag{141}
\]

Fig. was given initially for classical wiretap coding, but it is still valid for quantum wiretap coding.

Claim 17 \(\mathcal{R}(P_{\text{gen}})\) is a closed convex set.
**proof:** The proof is almost identical to that of Claim 12.

The random variable $b$ is still classical, even in the quantum case.

Here are some small differences between the 2 proofs. Instead of 2 CB nets with probability distributions $P_{\xi(b)} \in \mathcal{P}_{\text{gen}}$, we consider now 2 QB nets with density matrices $\rho_{\xi(b)} \in \mathcal{P}_{\text{gen}}$. Instead of an average CB net with probability distribution $P_{\xi} = E_b P_{\xi(b)}$, we consider a QB net with density matrix $\rho_{\xi} = E_b \rho_{\xi(b)}$. Instead of $H(\cdot)$’s, we use $S(\cdot)$’s.

**QED**

Claim 18

$$\mathcal{R}(\mathcal{P}_{\text{gen}}) = \mathcal{R}(\mathcal{P}_{\text{fac}})$$

**proof:** We won’t give a rigorous proof of this, just a plausibility argument.

$\mathcal{P}_{\text{fac}} \subset \mathcal{P}_{\text{gen}}$ so $\mathcal{R}(\mathcal{P}_{\text{fac}}) \subset \mathcal{R}(\mathcal{P}_{\text{gen}})$. Next let’s prove the reverse inclusion. Suppose $\rho_{y,z,u} \in \mathcal{P}_{\text{gen}}$. $\rho_{y,z,u}$ only appears in the definition of $\mathcal{R}(\rho)$ through its marginals $\rho_{y,z} \rho_{z,u}$.

For the rest of this proof we will use the shorthand notation $\xi = (v, u)$.

$\rho_{z,y,\xi}$ has the form

$$\rho_{z,y,\xi} = \sum_r \left[ \begin{array}{c} \sum_{y,z,\xi} A(y,z|\xi) \\ A(R|y,z,\xi) \end{array} \right] \left[ \begin{array}{c} \sum_{y,z,\xi} A(y,z|\xi) \\ A(\xi|\xi) \end{array} \right] \text{ h.c.} .$$

We would like to find a new density matrix $\tilde{\rho}_{z,y,\xi}$ in terms of the original density matrix $\rho_{z,y,\xi}$. We would like $\tilde{\rho}_{z,y,\xi}$ to be of the form

$$\tilde{\rho}_{z,y,\xi} = \sum_R \left[ \begin{array}{c} \sum_{y,z,\xi} \tilde{A}(y|\xi) \tilde{A}(z|\xi) \\ \tilde{A}(R|y,z,\xi) \end{array} \right] \left[ \begin{array}{c} \sum_{y,z,\xi} \tilde{A}(y|\xi) \tilde{A}(z|\xi) \\ \tilde{A}(\xi|\xi) \end{array} \right] \text{ h.c.} ,$$

where $\tilde{A}(y|\xi)$ and $\tilde{A}(z|\xi)$ are both isometries, and such that $\tilde{\rho}_{z,y,\xi}$ and $\rho_{z,y,\xi}$ have the same marginal density matrices $\rho_{y,z,\xi}$ and $\rho_{y,\xi}$.

We will set $\tilde{A}(\xi) = \tilde{A}(\xi)$ for all $\xi$. Stated more explicitly, we want $\tilde{\rho}_{z,y,\xi}$ to satisfy the constraints:

$$\sum_{R,y} \left[ \sum_{z,\xi} \tilde{A}(y|\xi) \tilde{A}(z|\xi) \right] \left[ \begin{array}{c} \text{h.c.} \\ \xi \to \xi' \end{array} \right] = \sum_{R,y} \left[ \sum_{z,\xi} \tilde{A}(y|\xi) \tilde{A}(z|\xi) \right] \left[ \begin{array}{c} \text{h.c.} \\ \xi \to \xi' \end{array} \right] , (y \leftrightarrow z)$$

(145a)

which is equivalent to $\tilde{\rho}_{z,\xi} = \rho_{z,\xi}$ and $\tilde{\rho}_{y,\xi} = \rho_{y,\xi}$.

$$\sum_y \left[ \tilde{A}(y|\xi) \right] \left[ \begin{array}{c} \text{h.c.} \\ \xi \to \xi' \end{array} \right] = \delta_{\xi'}^\xi , (y \leftrightarrow z) ,$$

(145b)
and
\[
\sum_{R} |\tilde{A}(R|y, z, \xi)|^2 = 1 . \tag{145c}
\]

Let’s count the number of unknowns (i.e., real-valued degrees of freedom).

- \(\tilde{A}(y|\xi) \in \mathbb{C}\) so it contains \(2N_y N_\xi\) unknowns.
- \(\tilde{A}(z|\xi) \in \mathbb{C}\) so it contains \(2N_z N_\xi\) unknowns.
- \(\tilde{A}(R|y, z, \xi) \in \mathbb{C}\) so it contains \(2N_R N_y N_z N_\xi\) unknowns.

That’s a total of
\[
2N_y N_\xi + 2N_z N_\xi + 2N_R N_y N_z N_\xi
\]
unknowns.

Next, let’s count the number of independent equations (i.e., independent real-valued constraints). Recall that an \(N \times N\) Hermitian matrix contains \(N^2\) degrees of freedom.

- Eqs.\(^{145a}\) give \((N_z N_\xi)^2 + (N_y N_\xi)^2\) independent equations.
- Eqs.\(^{145b}\) give \(2N^2_\xi\) independent equations.
- Eqs.\(^{145c}\) give \(N_y N_z N_\xi\) independent equations.

That’s a total of
\[
(N_z N_\xi)^2 + (N_y N_\xi)^2 + 2N^2_\xi + N_y N_z N_\xi
\]
independent equations.

Now note that we can always make \(N_R\) large enough so that the number of unknowns is greater or equal to the number of equations. The moral is that when we replace \(\tilde{A}(y,z|\xi)\) by a product of two isometries \(\tilde{A}(y|\xi)\) and \(\tilde{A}(z|\xi)\), we are losing degrees of freedom. Those extra degrees of freedom can be transferred to the reservoir whenever the only part of \(\rho_{y,z,v,u}\) that is visible are the marginal density matrices \(\rho_{y,v,u}\) and \(\rho_{z,v,u}\).

Given \(\rho \in \mathcal{P}_{\text{gen}}\), we have found \(\tilde{\rho} \in \mathcal{P}_{\text{fac}}\) such that \(\mathcal{R}(\rho) = \mathcal{R}(\tilde{\rho})\).

\begin{equation*}
\text{QED}
\end{equation*}

For quantum wiretap coding, we will prove below an optimality theorem for \(\mathcal{R}(\mathcal{P}_{\text{gen}})\) and an achievability theorem for \(\mathcal{R}(\mathcal{P}_{\text{fac}})\). Since \(\mathcal{R}(\mathcal{P}_{\text{gen}}) = \mathcal{R}(\mathcal{P}_{\text{fac}})\), this will show achievability and optimality over the same set of \(\tilde{R}\)'s.

\textsuperscript{3}Let the Hermitian matrix have entries \(H_{i,j}\). If \(i < j\), store the real part of \(H_{i,j}\) at position \((i,j)\) and the imaginary part at position \((j,i)\). If \(i = j\), \(H_{i,i}\) is real. Store it at position \((i,i)\).
7.1 Measuring Success in Quantum Communication

When considering communication through any channel, whether it be a classical or a quantum one, it behooves us to define a measure of the success of that communication. In this section, we will review the measures of success in communication that we used previously for classical channel coding and classical wiretap coding. Then we will explain how we will measure success in communication for quantum wiretap coding.

In classical channel coding, an $N_m \times N_m$ transition matrix $\{P(\hat{m}|m)\}_{\forall \hat{m}, m}$ sends $P_m \in \text{pd}(S_m)$ to $P_{\hat{m}} \in \text{pd}(S_{\hat{m}})$ as follows

$$P_{\hat{m}}(\hat{m}) = \sum_{m} P(\hat{m}|m)P_m(m) ,$$

(146)

where

$$P(\hat{m}|m) = \sum_{y^n} P(\hat{m}|y^n)P(y^n|x^n(m)) .$$

(147)

$P_{err}$ was defined in Ref.[7] as the sum of the off-diagonal entries of the transition matrix divided by $N_m$. Thus, $P_{err} = 0$ if and only if $P(\hat{m}|m) = \delta_{\hat{m} m}$. We found that this occurs for small enough values of $R_m$:

$$P(\hat{m}|m) = \delta(\hat{m}, m)\theta(R_m \leq H(y : x)) .$$

(148)

Note that our definition of $P_{err}$, as the sum of the off-diagonal entries of $P(\hat{m}|m)/N_m$, is just one possible definition of a distance between $P(\hat{m}|m)$ and $\delta_{\hat{m} m}$. We could, for example, have chosen instead, as a measure of that distance, the maximum of the off diagonal entries of $P(\hat{m}|m)$.

In classical wiretap coding, an $N_{\hat{M}} \times N_m$ transition matrix $\{P(\hat{M}|m)\}_{\forall \hat{M}, m}$ sends $P_m \in \text{pd}(S_m)$ to $P_{\hat{M}} \in \text{pd}(S_{\hat{M}})$ as follows

$$P_{\hat{M}}(\hat{M}) = \sum_{m} P(\hat{M}|m)P_m(m) ,$$

(149)

where

$$P(\hat{M}|m) = \sum_{y^n, z^n} P(\hat{M}|y^n, z^n)P(y^n, z^n|x^n(m)) .$$

(150)

Let the non-diagonal entries of the transition matrix be defined as those entries whose indices $\hat{M}, m$ satisfy $\hat{M} \neq m$. $P_{err}$ was defined earlier in this paper as the sum of the non-diagonal entries of the transition matrix divided by $N_m$. Thus, $P_{err} = 0$ iff $P(\hat{M}|m) = \delta(\hat{M}, f(m))$. We found that this occurs for small enough values of $\tilde{R}$:

$$P(\hat{M}|m) = \delta(\hat{M}, f(m))\theta(\tilde{R} \in \mathcal{R}(P)) .$$

(151)
Note that our definition of $P_{\text{err}}$ is just one possible definition of a distance between $P(\hat{M}|m)$ and $\delta(\hat{M}, f(m))$.

In quantum wiretap coding, a channel superoperator $\mathcal{T}_{\hat{M}|m}$ sends a density matrix $\rho_{\text{in}}^m \in dm(\mathcal{H}_m)$ into a density matrix $\rho_{\text{out}}^m \in dm(\mathcal{H}_{\hat{M}})$ as follows

$$\rho_{\text{out}}^m = \mathcal{T}_{\hat{M}|m}(\rho_{\text{in}}^m).$$

(152)

In general, the density matrix $\rho_{\text{in}}^m$ has matrix elements of the form:

$$\langle m| \rho_{\text{in}}^m |m'\rangle = \sum_{r,m} \left[ A(r_m|m)A(m) \right] \begin{pmatrix} \text{h.c.} \\ m \rightarrow m' \end{pmatrix}$$

(153)

for all $m, m' \in S_m$. One can also find the matrix elements of the channel superoperator $\mathcal{T}_{\hat{M}|m}$. They have two indices $\hat{M}, \hat{M}' \in S_{\hat{M}}$ and two indices $m, m' \in S_m$. From Eq.(152) one gets

$$\langle \hat{M}| \rho_{\text{out}}^m |\hat{M}'\rangle = \sum_{m,m'} \langle \hat{M}| \mathcal{T}_{\hat{M}|m}(|m\rangle\langle m'|)|\hat{M}'\rangle$$

(154a)

$$= \sum_{m,m'} \langle m| \rho_{\text{in}}^m |m'\rangle \langle \hat{M}| \mathcal{T}_{\hat{M}|m}(|m\rangle\langle m'|)|\hat{M}'\rangle$$

(154b)

where the matrix elements $\langle \hat{M}| \mathcal{T}_{\hat{M}|m}(|m\rangle\langle m'|)|\hat{M}'\rangle$ of the channel superoperator $\mathcal{T}_{\hat{M}|m}$ have the general form

$$\langle \hat{M}| \mathcal{T}_{\hat{M}|m}(|m\rangle\langle m'|)|\hat{M}'\rangle = \sum_{y^n,z^n} \left[ A(\hat{M}, y^n, z^n|m) \right] \begin{pmatrix} \text{h.c.} \\ \hat{M} \rightarrow \hat{M}' \\ m \rightarrow m' \end{pmatrix}.$$  

(155)

For quantum wiretap coding, rather than give one of many possible definitions of $P_{\text{err}}$, we shall prove below that: the matrix elements of the channel superoperator equal $\delta(\hat{M}, f(m))\delta(\hat{M}', f(m'))$ for small enough $\vec{R}$’s. That is, we will prove that

$$\langle \hat{M}| \mathcal{T}_{\hat{M}|m}(|m\rangle\langle m'|)|\hat{M}'\rangle = \delta(\hat{M}, f(m))\delta(\hat{M}', f(m'))\theta(\vec{R} \in \mathcal{R}(\rho)),$$

(156)

Thus, the channel superoperator $\mathcal{T}_{\hat{M}|m}$ can transmit density matrices $\rho_{\text{in}}^m$ faithfully for small enough $\vec{R}$’s.

### 7.2 Optimality

**Claim 19** Optimality: $\forall \vec{R}$, if $\exists$ an encoding and a decoding that satisfy $\lim_{n \to \infty} P_{\text{err}} = 0$ for the QB net of Eq.(127), then $\vec{R} \in \mathcal{R}(\mathcal{P}_{\text{gen}})$. 

34
proof: The proof of optimality for quantum wiretap coding is very similar to the proof of optimality for classical wiretap coding. The main difference is that Eq.(83) cannot be used in the quantum case. In the quantum case, random variables cannot be cloned unless they are classical. (See Ref.[8] for a discussion of this). Because of this, we will end up defining $\underline{v}$ and $\underline{u}$ differently for quantum and classical wiretap coding.

The proof starts from the result Claim 8. As pointed out in the section that ends and culminates with Claim 8, Claim 8 is basically a souped-up version of the chain rule. To prove the current claim, we will need to add 2 new ingredients that are not consequences of only the chain rule. First, we will use Alicki Fannes’s inequality (see Ref.[2]), which is a generalization to the quantum realm of Fano’s inequality. Second, we will use Claim 9 with $H(\cdot) \to S(\cdot)$. This claim is still valid in the quantum realm as long as $j$ is a classical random variable.

Note that, by assumption, $\lim_{n \to \infty} P_{\text{err}} = 0$. Hence, by Alicki Fannes’s inequality, the $\delta_y, \delta_{ty}$ and $\delta_{tz}$ used in Claim 8 go to zero as $n \to \infty$. Furthermore, Claim 1 and CMI $\geq 0$ imply that

$$-\frac{1}{n}S(t|y^n) \leq \frac{1}{n}S(t|z^n, s) \leq \frac{1}{n}S(t|z^n).$$

(157)

But Alicki Fannes’s inequality implies that both sides of the last equation, $-\frac{1}{n}S(t|y^n)$ and $\frac{1}{n}S(t|z^n)$, tend to zero as $n \to \infty$. Hence, also

$$\lim_{n \to \infty} \frac{1}{n}S(t|z^n, s) = 0.$$ (158)

Let $j$ be a classical random variable that is uniformly distributed and has states $j = 1, 2, \cdots, n$. Let $E_j = \frac{1}{n} \sum_j$. In the quantum case, we define

$$\underline{u} = (y < j, z > j, t, j),$$

$$\underline{v} = \underline{s},$$

$$\underline{z} = \underline{z} j, \quad \underline{y} = \underline{y} j, \quad \underline{z} = \underline{z} j.$$ (159)

(Compare this with Eq.(86) for the classical case).

After dropping $\frac{1}{n}S(t|z^n, s)$ and the deltas $\delta_y, \delta_{ty}, \delta_{tz}$, the 4 identities (souped-up chain rules) of Claim 8 yield the following 4 inequalities.

* Eq.(43) yields

$$R_{\underline{z}} \leq \frac{1}{n}S(s|z^n)$$

$$= E_j S(y_j : s | \alpha_j, t_j) - E_j S(z_j : s | \alpha_j, t_j)$$

$$= S(y_j : s | \alpha_j, t_j) - S(z_j : s | \alpha_j, t_j)$$

$$= S(y : v | u) - S(\underline{z} : v | u).$$ (160a) (160b) (160c) (160d)
• Eq.(44) yields

\[ R_s = E_j S(y_j : s | a_j, t) + \frac{\sum_{<t}}{n} - \frac{\sum_{<s,t}}{n} \]  

(161a)

\[ = S(y_j : s | a_j, t, j) + \frac{\sum_{<t}}{n} - \frac{\sum_{<s,t}}{n} \]  

(161b)

\[ = S(y : u | u) + \frac{\sum_{<t}}{n} \]  

(161c)

\[ \leq S(y : u | u) + \frac{\sum_{<t}}{n} . \]  

(161d)

• Eq.(45) yields

\[ R_t = E_j S(y_j : a_j, t) - E_j S(y_j : y_{<j}) - \frac{\sum_{<t}}{n} \]  

(162a)

\[ = S(y_j : a_j, t | j) - S(y_j : y_{<j} | j) - \frac{\sum_{<t}}{n} \]  

(162b)

\[ = S(y : u | j) - S(y_{<j} : y_{<j} | j) - \frac{\sum_{<t}}{n} \]  

(162c)

\[ \leq S(y : u | j) - \frac{\sum_{<t}}{n} \]  

(162d)

\[ \leq S(y : u) - \frac{\sum_{<t}}{n} . \]  

(162e)

Note that \( j \) can be cloned because it’s classical. Eq.(162c) follows from Claim 9.

• Eq.(46) yields

\[ R_z = E_j S(\tilde{z}_j : a_j, t) - E_j S(\tilde{z}_j : \tilde{z}_{>j}) - \frac{\sum_{<t}}{n} \]  

(163a)

\[ = S(\tilde{z}_j : a_j, t | j) - S(\tilde{z}_j : \tilde{z}_{>j} | j) - \frac{\sum_{<t}}{n} \]  

(163b)

\[ = S(\tilde{z} : u | j) - S(\tilde{z}_{>j} : \tilde{z}_{>j} | j) - \frac{\sum_{<t}}{n} \]  

(163c)

\[ \leq S(\tilde{z} : u | j) - \frac{\sum_{<t}}{n} \]  

(163d)

\[ \leq S(\tilde{z} : u) - \frac{\sum_{<t}}{n} . \]  

(163e)

Eqs.(162c) and (163c) can be combined by writing
\[ R_R \leq \ell - \frac{\sum_{\ell} \xi}{n}. \] (164)

From all we’ve said so far, it’s clear that \( \bar{\mathcal{R}} \in \mathcal{R}(\mathcal{P}_{\text{gen}}). \)

**QED**

In the above optimality proof, we start with the QB net of Eq.\((127)\). Then we do some “chain ruling” reminiscent of peeling \( n - 1 \) layers away down to just one. We end up with a different QB net with random variables \( y, z, x, v, u \). It’s instructive to present a chain of QB nets connecting the beginning and ending QB nets of this process.

One starts with the QB net (\( \mathcal{C} \) implicit)

\[
\begin{bmatrix}
\hat{y} \\
\hat{z} \\
\hat{t}
\end{bmatrix}
\xrightarrow{\delta}
\begin{bmatrix}
y^n \\
y^n, z^n \\
z^n
\end{bmatrix}
\xrightarrow{\delta}
\begin{bmatrix}
s \\
x^n \\
t
\end{bmatrix}
\]

[ h.c. ]. \hspace{1cm} (165a)

Tracing over all the nodes highlighted with a double circle and lumping the traced nodes into a reservoir \( r \) gives

\[
\begin{bmatrix}
y^n \\
y^n, z^n \\
z^n
\end{bmatrix}
\xrightarrow{\delta}
\begin{bmatrix}
s \\
x^n \\
t
\end{bmatrix}
\]

[ h.c. ]. \hspace{1cm} (165b)

The previous QB net equals
Tracing over all the nodes highlighted with a double circle gives

\[ \Lambda \leftarrow y^n, z^n \]

Merging the \( y_{<j} \) and \( z_{>j} \) nodes gives

\[ \Lambda \leftarrow y_j, z_j, y_{<j}, z_{>j} \]
The previous QB net can be “accommodated” or modeled by the following QB net.

In the last graph, we leave implicit a (classical) source node \( j \) with outgoing arrows pointing into all nodes that mention \( j \).

Note that in Claim 19 (the optimality proof for quantum wiretap coding) all the quantum entropies \( S() \) were evaluated for the density matrix Eq. (165f).

7.3 Achievability

Claim 20 Achievability: \( \forall \bar{R}, \text{if } \bar{R} \in \mathcal{R}(\mathcal{P}_\text{gen}), \text{ then } \exists \text{ an encoding and a decoding that satisfy } \lim_{n \to \infty} P_{err} = 0 \text{ for the QB net of Eq. (127)}. \)

proof: Suppose \( \rho \in \mathcal{P}_\text{fac} \) and \( \bar{R} \in \mathcal{R}(\rho) \).

We consider all wiretap coding protocols that can be described by the following
We will use a codebook $C = (C_x, C_v, C_u)$ composed of 3 sub-codebooks. Codebook $C_x$ is an $N_x N_z \times n$ matrix given by $C_x = \{x^n(s,t)\}_{s,t} = x^n(\cdot)$ where $x^n(s,t) \in S^n_x$ for all $(s,t) \in S_x \times S_z$. Similarly, $C_v = \{v^n(s,t)\}_{s,t} = v^n(\cdot)$ where $v^n(s,t) \in S^n_v$ for all $(s,t)$ and $C_u = \{u^n(t)\}_{t} = u^n(\cdot)$ where $u^n(t) \in S^n_u$ for all $t$.

We will use the shorthand notations $\hat{M} = (\hat{s}, \hat{t}, \hat{z})$ and $m = (s, t)$. Let $f$ be the function that maps $f(m) = f(s, t) = (s, t, t)$ for all $s, t$.

We assign to the QB net of Eq.(166) the following node transition matrices:

$$A(s) = \frac{1}{\sqrt{N_x}}, \quad A(t) = \frac{1}{\sqrt{N_t}},$$

$$A(u^n|t, C) = \delta(u^n, u^n(t)),$$

$$A(v^n|s, u^n, C) = \delta(v^n, v^n(s, t)),$$

where $v^n(s,t) = v^n(s, u^n(t))$,

$$A(x^n|v^n, u^n, C) = \delta(x^n, x^n(s, t)),$$

where $x^n(s,t) = x^n(v^n(s, t), u^n(t))$,

$$A(y^n, z^n|x^n) = \prod_j A(y_j, z_j|x_j),$$

$$A(C) = \prod_{s,t} A(x^n(s,t), v^n(s,t), u^n(t)) = \prod_{s,t,j} A(x_j(s, t), v_j(s, t), u_j(t)).$$
where \( A(v, u) = A_{v|u}(v|u)A_u(u) \) and \( A(x, v, u) = A_{x|v, u}(x|v, u)A(v, u) \), and

\[
A(\hat{M}|y^n, z^n, C) = \prod_{\mu=1}^{5} \prod_{m_{\mu} \neq \hat{m}_{\mu}} \theta(R_{\mu} < \Gamma_{\mu}), \tag{173}
\]

where the quantities \( m_{\mu}, R_{\mu} \) and \( \Gamma_{\mu} \) are defined in Appendix A. Assume that we are given the reservoir amplitudes: \( A(r_{\text{out}}|y^n, z^n) \) (we choose this one to be an isometry), \( A(r^n|y^n, z^n, v^n, u^n) = \prod_j A(r_j|y_j, z_j, v_j, u_j) \), \( A(r_{\hat{x}}|s) \) and \( A(r_{\hat{t}}|t) \). Assume that we are also given an isometry \( A(y, z|x) \) (called the channel amplitude for this problem).

The reservoir amplitude \( A(r_{\text{out}}|y^n, z^n) \) is applied by the decoder person and is designed to be an isometry. In a moment, we shall explain the reason why we need it to be an isometry. The reservoir amplitudes \( A \) and \( \mu \) are independent of its value. Our results will also be independent of \( m \) where the quantities \( n \) as it factors into an isometry. It is out of the control of any human agent. As we shall see, as long as it factors into \( n \) amplitudes, our results (for the measure of success in quantum communication) are independent of its value. Our results will also be independent of the reservoir amplitudes \( A(r_{\hat{x}}|s) \) and \( A(r_{\hat{t}}|t) \) acting on the inputs.

Henceforth we will use \( \langle T \rangle \) as shorthand for

\[
\langle T \rangle = \langle \hat{M}|T_{\hat{M}|m} (|m_0\rangle\langle m'_0|) |\hat{M}' \rangle, \tag{174}
\]

and \( A_{vn} \) for

\[
A_{vn} = A(r^n|y^n, z^n, v^n, u^n), \tag{175}
\]

where \( m_0 = (s_0, t_0) \). One finds for the above QB net with the given node transition matrices that

\[
\langle T \rangle = \sum_{r^n, r_{\text{out}}} \left[ \sum_{y^n, z^n} \mathcal{E}_C \left\{ A(r_{\text{out}}|y^n, z^n) A_{vn} A(y^n, z^n|x^n(m_0)) \prod_{\mu=1}^{5} \prod_{m_{\mu} \neq \hat{m}_{\mu}} \theta(R_{\mu} < \Gamma_{\mu}) \right\} \right] \begin{bmatrix} \text{h.c.} \\ m_0 \rightarrow m'_0 \end{bmatrix} \begin{bmatrix} \text{h.c.} \\ \hat{M} \rightarrow \hat{M}' \end{bmatrix}, \tag{176}
\]

where \( \mathcal{E}_C = \sum_C A(C) \).

We immediately perform the sum over \( r_{\text{out}} \). Since \( A(r_{\text{out}}|y^n, z^n) \) is an isometry, upon summing over \( r_{\text{out}} \), the random variables \( y^n \) and \( z^n \) become classical. Making \( y^n \) and \( z^n \) classical random variables is the raison d’être for the isometry \( A(r_{\text{out}}|y^n, z^n) \). Thus, we get

\[
\langle T \rangle = \sum_{r^n, y^n, z^n} \left[ \mathcal{E}_C A_{vn} A(y^n, z^n|x^n(m_0)) \prod_{\mu=1}^{5} \prod_{m_{\mu} \neq \hat{m}_{\mu}} \theta(R_{\mu} < \Gamma_{\mu}) \right] \begin{bmatrix} \text{h.c.} \\ m_0 \rightarrow m'_0 \end{bmatrix} \begin{bmatrix} \text{h.c.} \\ \hat{M} \rightarrow \hat{M}' \end{bmatrix}. \tag{177}
\]

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Let
\[ \oint k = 5 \prod_{\mu=1, m_{\mu} \neq \hat{m}_{\mu}} 1 \int_{-\infty}^{+\infty} \frac{dk_{\mu}(m_{\mu})}{2\pi i} \left( \frac{1}{k_{\mu}(m_{\mu}) - i\epsilon} \right) \], \hspace{1cm} (178)
and
\[ K_{\mu} = \sum_{m_{\mu} \neq \hat{m}_{\mu}} k_{\mu}(m_{\mu}) . \hspace{1cm} (179) \]

We will also use the following shorthand notation
\[ \hat{x} = x(\hat{s}, \hat{t}) , \hspace{0.5cm} \hat{v} = v(\hat{s}, \hat{t}) , \hspace{0.5cm} \hat{u} = u(\hat{t}) , \hspace{1cm} (180) \]

\[ \xi = (r, y, z, x(\cdot), v(\cdot), u(\cdot)) , \hspace{1cm} (181) \]

\[ A_r = A(r|y, z, v(m_0), u(t_0)) , \hspace{1cm} (182) \]
and
\[ \hat{A}_r = A(r|y, z, \hat{v}, \hat{u}) . \hspace{1cm} (183) \]

Expressing the \( \theta \) functions in Eq.(177) as integrals, we get
\[ \langle T \rangle = \sum_{r^n, y^n, z^n} \left[ \oint k e^{-i \sum_{\mu} K_{\mu} R_{\mu} \sum_{x^n(\cdot), v^n(\cdot), u^n(\cdot)} e^{i \sum_t nP_t(\xi) \ln Z(\xi)}} \right] \left[ \begin{array}{c} \text{h.c.} \\ m_0 \rightarrow m'_0 \\ \hat{M} \rightarrow \hat{M}' \end{array} \right] , \hspace{1cm} (184) \]

where
\[ Z(\xi) = A_r A(y, z|x(m_0)) \prod_{s,t} \left\{ A(x(s, t), v(s, t), u(t)) \right\} 5 \prod_{\mu=1, m_{\mu} \neq \hat{m}_{\mu}} 1 \prod_{\mu=1, m_{\mu} \neq \hat{m}_{\mu}} \left\{ \gamma_{\mu} \right\} , \hspace{1cm} (185a) \]

\[ = \left\{ \begin{array}{c} A_r A(y, z|x(m_0)) \prod_{s,t} \left\{ A(x(s, t)|v(s, t), u(t)) \right\} \\ \prod_{s,t} \left\{ A(v(s, t), u(t)) \right\} 5 \prod_{\mu=1, m_{\mu} \neq \hat{m}_{\mu}} \left\{ \gamma_{\mu} \right\} \end{array} \right\} , \hspace{1cm} (185b) \]

where the quantities \( \gamma_{\mu} \) are defined in Appendix [A].

Next we express the sum over \( x^n(\cdot), v^n(\cdot), u^n(\cdot) \) in Eq.(184) as a p-type integral and do the p-type integration by the steepest descent method. We get for the leading order term:
\[ \langle T \rangle = \sum_{r^n, y^n, z^n} \left[ \oint k e^{-i \sum_{\mu} K_{\mu} R_{\mu} e^{i \sum_{r, y, z} nP_t(r, y, z) \ln Z(r, y, z)}} \right] \left[ \begin{array}{c} \text{h.c.} \\ m_0 \rightarrow m'_0 \\ \hat{M} \rightarrow \hat{M}' \end{array} \right] , \hspace{1cm} (186) \]
where

\[ Z(r, y, z) = \sum_{x(\cdot), v(\cdot), u(\cdot)} Z(\xi). \]  \hspace{1cm} (187)

If we define

\[ A(y, z | v(\cdot), u(\cdot), m_0) = \sum_{x(\cdot)} A(y, z | x(m_0)) \prod_{s,t} A(x(s, t) | v(s, t), u(t)), \]  \hspace{1cm} (188)

then \( Z(r, y, z) \) can be expressed as

\[ Z(r, y, z) = \mathcal{E}_{v(\cdot), u(\cdot)} A_r A(y, z | v(\cdot), u(\cdot), m_0) \prod_{\mu \neq \hat{\mu}} \left\{ \gamma_{\mu}^{-\frac{k_{\mu}(m_{\mu})}{\alpha}} \right\}. \]  \hspace{1cm} (189)

Define

\[ Z_{\text{mid}}(r, y, z) = \mathcal{E}_{\hat{v}, \hat{u}} A_r A(y, z | \hat{v}, \hat{u}) \prod_{\mu} \left\{ \Phi_{\mu}^{\frac{K_{\mu}}{\alpha}} \right\}, \]  \hspace{1cm} (190)

where

\[ A(y, z | v, u) = \sum_{x} A(y, z | x) A(x | v, u), \]  \hspace{1cm} (191)

and where the quantities \( \Phi_{\mu} \) are defined in Appendix A.

Define the integration operators

\[ 1_{h,K} = \prod_{\mu} \left\{ \int_{-\infty}^{+\infty} dK_{\mu} \int_{-\infty}^{+\infty} \frac{dh_{\mu}}{2\pi} e^{-iK_{\mu}h_{\mu}} \right\} \]  \hspace{1cm} (192)

and

\[ 1_{h>0,K} = 1_{h,K} \prod_{\mu} \theta(h_{\mu} > 0). \]  \hspace{1cm} (193)

Note that 1 equals

\[ 1 = \prod_{\mu} \left\{ \int_{-\infty}^{+\infty} dK_{\mu} \delta \left( \sum_{m_{\mu} \neq \hat{\mu}} \{k_{\mu}(m_{\mu})\} - K_{\mu} \right) \right\} \]  \hspace{1cm} (194a)

\[ = 1_{h,K} e^{i \sum_{\mu} h_{\mu} \sum_{m_{\mu} \neq \hat{\mu}} k_{\mu}(m_{\mu})}. \]  \hspace{1cm} (194b)

Multiplying \( \langle T \rangle \) by 1 certainly doesn’t change it. Thus the right hand sides of Eqs. (186) and (194b) can be multiplied to get
\[ \langle T \rangle = \sum_{r^n, y^n, z^n} \left[ 1_{h-K} e^{-i \sum_{\mu} K_{\mu} R_{\mu}} \oint_k e^{i \sum_{m \neq \hat{\mu}} k_m(m) e^{-i \sum_{r, y, z} n \mathcal{P}^1_{r, y, z} \ln Z(r, y, z)}} \right] \left[ \begin{array}{c} \text{h.c.} \\ m_0 \to m'_0 \\ \hat{M} \to \hat{M}' \end{array} \right]. \] (195)

Next we will assume that, for all \( m_\mu \), when doing the contour integration over \( k_\mu(m_\mu) \) in Eq. (195) with \( Z(r, y, z) \) given by Eq. (189), the \( e^{\ln Z(r, y, z)} \) can be evaluated at the value \( k_\mu(m_\mu) = i \epsilon \to 0 \) of the pole. Furthermore, we will assume that only the terms with \( \hat{M} = m_0 \) give a non-vanishing contribution to the integral. Symbolically, this means we will assume

\[
\oint_k e^{i \sum_{\mu} h_\mu \sum_{m \neq \hat{\mu}} k_m(m) e^{-i \sum_{r, y, z} n \mathcal{P}^1_{r, y, z} \ln Z(r, y, z)}} =
= \delta_{\hat{M} m_0} e^{\sum_{r, y, z} n \mathcal{P}^1_{r, y, z} \ln Z_{mid}(r, y, z)} \oint_k e^{i \sum_{\mu} h_\mu \sum_{m \neq \hat{\mu}} k_m(m)}
= \delta_{\hat{M} m_0} e^{\sum_{r, y, z} n \mathcal{P}^1_{r, y, z} \ln Z_{mid}(r, y, z)} \prod_{\mu} \{ \theta(h_\mu > 0) \}. \tag{196a}
\]

Applying Eq. (196b) to Eq. (195) gives

\[
\langle T \rangle = \delta_{\hat{M} m_0} \delta_{\hat{M}' m'_0} \sum_{r^n, y^n, z^n} \left[ 1_{h>0, K} e^{-i \sum_{\mu} K_{\mu} R_{\mu} e^{-i \sum_{r, y, z} n \mathcal{P}^1_{r, y, z} \ln Z_{mid}(r, y, z)}} \right] \left[ \begin{array}{c} \text{h.c.} \\ \end{array} \right]. \tag{197}
\]

Next we express the sum over \( r^n, y^n, z^n \) in Eq. (197) as a p-type integral and do the p-type integration by the steepest descent method. We get for the leading order term:

\[
\langle T \rangle = \delta_{\hat{M} m_0} \delta_{\hat{M}' m'_0} 1_{h>0, K} 1_{h'>0, K'} e^{-i \sum_{\mu} (K_{\mu} - K'_{\mu}) R_{\mu}} e^{\ln Z_{fin}}, \tag{198}
\]

where

\[
Z_{fin} = \sum_{r, y, z} \left[ Z_{mid}(r, y, z) \right] \left[ \begin{array}{c} \text{h.c.} \\ K_{\mu} \to K'_{\mu} \end{array} \right], \tag{199a}
\]

\[
= \sum_{r, y, z} \mathcal{E}_{\hat{v}, \hat{u}} \mathcal{E}_{\hat{v}', \hat{u}'} \left[ \hat{A}_r A(y, z | \hat{v}, \hat{u}) \right] \left( \begin{array}{c} + i n \sum_{\mu} K_{\mu} \ln \Phi_{\mu} \\ - i n \sum_{\mu} K'_{\mu} \ln \Phi'_{\mu} \end{array} \right) \left[ \begin{array}{c} \text{h.c.} \\ \hat{v} \to \hat{v}' \\ \hat{u} \to \hat{u}' \end{array} \right] \tag{199b}
\]

where we approximated

\[
\prod_{\mu} \Phi_{\mu}^{i K_{\mu} / n} \approx 1 + \frac{i}{n} \sum_{\mu} K_{\mu} \ln \Phi_{\mu}. \tag{200}
\]
Because \( \rho \in P_{\text{fac}} \), we may set in the right hand side of Eq.(199b) \[
A(y, z|\hat{v}, \hat{u}) = A(y|\hat{v}, \hat{u})A(z|\hat{v}, \hat{u}),
\]
where \( A(y|\hat{v}, \hat{u}) \) and \( A(z|\hat{v}, \hat{u}) \) are both isometries. It is important to notice that \( \sum_{\mu} K_{\mu} \ln \Phi_{\mu} \) is a sum of terms each of which only depends either on \( y \) or \( z \) but not both. Thus in Eq.(199b), either the \( y \) sum or the \( z \) sum (or both) kills all terms except those with \( \hat{u} = \hat{u}' \) and \( \hat{v} = \hat{v}' \). Hence

\[
Z_{\text{fin}} = \sum_{r,y,z,\hat{v},\hat{u}} |\tilde{A}_r A(y, z|\hat{v}, \hat{u})A(\hat{v}, \hat{u})|^2 \left( 1 + \frac{i}{n} \sum_{\mu} (K_{\mu} - K'_{\mu}) \ln \Phi_{\mu} \right) \tag{202a}
\]

\[
= 1 + \frac{i}{n} \sum_{\mu} (K_{\mu} - K'_{\mu}) E_{y,z,\hat{v},\hat{u}} \ln \Phi_{\mu}, \tag{202b}
\]

\[
n \ln Z_{\text{fin}} \approx i \sum_{\mu} (K_{\mu} - K'_{\mu}) E_{y,z,\hat{v},\hat{u}} \ln \Phi_{\mu}. \tag{203}
\]

Combining Eqs.(198) and (203) yields

\[
\langle T \rangle = \delta_{M}^{\prime} \delta_{M'}^{\prime} \left| 1_{h>0, K} e^{-i \sum_{\mu} K_{\mu} R_{\mu}} e^{i \sum_{\mu} K_{\mu} E_{y,z,\hat{v},\hat{u}} \ln \Phi_{\mu}} \right|^2 \tag{204a}
\]

\[
= \delta_{M}^{\prime} \delta_{M'}^{\prime} \prod_{\mu} \theta(R_{\mu} < S_{\mu}) \tag{204b}
\]

\[
= \delta_{M}^{\prime} \delta_{M'}^{\prime} \theta(\bar{R} \in \mathcal{R}(\rho)), \tag{204c}
\]

where the quantities \( S_{\mu} \) are defined in Appendix A.

QED

7.4 Capacity

All statements\(^4\) made in Section 6.3 for the channel capacity for classical wiretap coding are also valid for a quantum wiretap coding, as long as (1) classical informations \( H_{P} \) are replaced by quantum informations \( S_{\rho} \) throughout, and (2) the symbols \( P_{\text{gen}}, P_{\text{fac}} \) and \( \mathcal{R}(P) \) are defined as they were defined for quantum instead of classical wiretap coding.

\(^4\) Except for Claim 16. In the quantum case, Claim 16 is true with \( H() \to S() \) if one adds to the claim the additional assumption that \( e \) is a classical random variable. For if \( e \) is a classical random variable at the middle of a Markov chain, then we can assert that \( S(\hat{a} : \hat{h} | \hat{e}) = 0 \).
Appendix: Table of Quantities Used In Both Classical and Quantum Wiretap Coding

| µ  | m_µ | R_µ | Γ_µ | γ_µ | Φ_µ | H_µ |
|----|-----|-----|-----|-----|-----|-----|
| 1  | (s, t_y, t_z) | R_µ | Γ_2y - Γ_2z | γ_2y | Φ_2y | H(y : v | u) - H(z : v | u) |
| 2  | (s, t_y) | R_µ + R_µ | Γ_2y + Γ_1y | γ_2y γ_1y | Φ_2y Φ_1y | H(y : v | u) + H(y : u) |
| 3  | (s, t_z) | R_µ + R_µ | Γ_2y + Γ_1z | γ_2y γ_1z | Φ_2y Φ_1z | H(y : v | u) + H(z : u) |
| 4  | t_y     | R_µ | Γ_1y | γ_1y | Φ_1y | H(y : u) |
| 5  | t_z     | R_µ | Γ_1z | γ_1z | Φ_1z | H(z : u) |

(205)

The last column of the table of Eq.(205) is for the classical case. The H’s are evaluated at some P ∈ R(µ). In the quantum case, H_µ should be replaced by S_µ, and all H’s should be replaced by S’s. The S’s are evaluated at some ρ ∈ R(µ).

In the classical case, one defines

\[ Γ_1y = \frac{1}{n} \ln \frac{P(y^n : u^n(\hat{t}_y))}{P(y^n : u^n(t_y))}, \quad (y \leftrightarrow z), \]  

(206)

\[ γ_1y = \frac{P(y : u(\hat{t}_y))}{P(y : u(t_y))}, \quad (y \leftrightarrow z), \]  

(207)

\[ Φ_1y = P(y : u(\hat{t})), \quad (y \leftrightarrow z), \]  

(208)

\[ Γ_2y = \frac{1}{n} \ln \frac{P(y^n : v^n(\hat{s}, \hat{t}_y)|u^n(\hat{t}_y))}{P(y^n : v^n(s, t_y)|u^n(t_y))}, \quad (y \leftrightarrow z), \]  

(209)

\[ γ_2y = \frac{P(y : v(\hat{s}, \hat{t}_y)|u(\hat{t}_y))}{P(y : v(s, t_y)|u(t_y))}, \quad (y \leftrightarrow z) \]  

(210)

and

\[ Φ_2y = P(y : v(\hat{s}, t)|u(t)), \quad (y \leftrightarrow z). \]  

(211)

Note that Γ_µ and γ_µ are related by

\[ Γ_µ = \sum_{y \in S_µ} \sum_{z \in S_µ} \sum_{v(\cdot) \in S_{N_L}} \sum_{u(\cdot) \in S_{N_L}} P(\cdot)(y, z, v(\cdot), u(\cdot)) \ln(γ_µ). \]  

(212)

Note that Φ_µ is evaluated at \( \hat{t}_y = \hat{t}_z = \hat{t} \).

In the quantum case, Eqs.(206) to (212) must be changed as follows.
In the quantum case, $\rho_{x,y,z,u,v,w}$ is given. From it, one can define $P_{x,y,z,v,u} = |A_{x,y,z,v,u}|^2$. Suppose $x, y, e$ are three distinct random variables from the set $\{ y, z, v, u \}$. Define the function $\phi : \mathbb{R}^0 \to \mathbb{R}$ by

$$\phi(r) = r^r.$$  \hspace{1cm} (213)

In Eqs. (206) to (212), replace any $\frac{1}{n} \ln P(x^n : y^n | e^n)$ by $\frac{1}{n} \ln P^{br}(x^n : y^n | e^n) = \sum_{x,y,e} P_{[x^n,y^n,e^n]}(x,y,e) \left\{ \begin{array}{c} \frac{1}{P(x,y,e)} \ln \phi(\lambda_{x,y,e}(\rho_{x,y,e})) \\ + \frac{1}{P(e)} \ln \phi(\lambda_e(\rho_e)) \\ - \frac{1}{P(x,e)} \ln \phi(\lambda_{x,e}(\rho_{x,e})) \\ - \frac{1}{P(y,e)} \ln \phi(\lambda_{y,e}(\rho_{y,e})) \end{array} \right\}$, \hspace{1cm} (214)

and any $\ln P(x : y | e)$ by

$$\ln P^{br}(x : y | e) = \text{expression inside curly brackets in Eq. (214)}.$$  \hspace{1cm} (215)

Also replace any $\frac{1}{n} \ln P(x^n : y^n)$ by

$$\frac{1}{n} \ln P^{br}(x^n : y^n) = \sum_{x,y} P_{[x^n,y^n]}(x,y) \left\{ \begin{array}{c} \frac{1}{P(x,y)} \ln \phi(\lambda_{x,y}(\rho_{x,y})) \\ - \frac{1}{P(x)} \ln \phi(\lambda_x(\rho_x)) \\ - \frac{1}{P(y)} \ln \phi(\lambda_y(\rho_y)) \end{array} \right\}$, \hspace{1cm} (216)

and any $\ln P(x : y)$ by

$$\ln P^{br}(x : y) = \text{expression inside curly brackets in Eq. (216)}.$$  \hspace{1cm} (217)

Note that the bridge functions $P^{br}$ depend on both $\rho_{x,y,z,u}$ and $P_{x,y,z,u}$. Note also that the bridge functions equal the classical functions they are replacing when $\rho_{x,y,z,u}$ is diagonal (the classical case).

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$^{5}\text{br}=\text{bridge}$

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