Global perinormality in a generalized $D + M$ construction

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ABSTRACT
A domain $R$ is perinormal if every going-down overring is flat and a perinormal domain $R$ is globally perinormal if every flat overring is a localization of $R$ [2]. I show that global perinormality is preserved in a pullback construction which encompasses a classical $D + M$ construction. In doing so, a result is given for the transfer of the property that every flat overring is a localization in the pullback construction considered.

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1. Introduction
It is well-known that many classes of domains are preserved in the classical $D + M$ construction. A domain $R$ is perinormal if every going-down overring is flat and a perinormal domain $R$ is globally perinormal if every flat overring is a localization of $R$ [2]. In [3, Theorem 2.7], Epstein and Shapiro showed that perinormality is preserved in a more general version of the classical $D + M$ pullback in which one removes the condition that $V$ is of the form $k + M$ where $k$ is the residue field of $V$. In a generalization of the classical $D + M$ construction considered in [4, Section 2], one starts with a local domain rather than a valuation domain. The purpose of this paper is to consider the preservation of global perinormality in this more general construction. Let $(T, M)$ be a local domain, $k$ the residue field of $T$, and $D$ be an integral domain with fraction field $k$. Building on the results in [4, Section 2] and making use of [7, Theorem 2], we will study the transfer of the property that every flat overring is a localization to obtain that if the pullback $D \times_k T$ is globally perinormal, then $D$ and $T$ are globally perinormal. A partial converse is obtained by adding the assumption that $T$ is a valuation domain. This is analogous to the result in [4, Theorem 2.7(h)] for going-down domains.

2. Preliminary material
The following theorem by Richman gives a criterion for an overring of an integral domain $R$ to be flat as an $R$-module and is utilized in Section 3.

Theorem 2.1. [7, Theorem 2] Let $R$ be an integral domain and $S$ an overring of $R$. Then $S$ is flat over $R$ if and only if $S_m = R_m \cap R$ for every $m \in \text{Max}(S)$.
Given a diagram

\[
\begin{array}{ccc}
S & \rightarrow & T \\
\downarrow & & \downarrow \\
R & \rightarrow & R
\end{array}
\]

in the category of commutative rings, let

\[ S \times_R T := \{ t \in T \mid \pi(t) \in \iota(S) \} \]

denote the pullback of \( S \) and \( T \) over \( R \). Note that \( S \times_R T \) is a subring of \( T \).

**Observation 2.2.** Let \( \pi : T \rightarrow R \) be a surjective ring homomorphism and \( D \) and \( S \) be subrings of \( R \). If \( S \times_R T = D \times_R T \), then \( S = D \).

**Proof.** This follows immediately because \( \pi \) is surjective and \( S \) and \( D \) are subrings of \( R \).

**Lemma 2.3.** Let \( D \) be an integral domain with fraction field \( k \) and let \( \pi : T \rightarrow k \) be a surjective homomorphism. Let \( W \) be a multiplicative subset of \( D \). Then \( D \times_k T \) is a subring of \( k \times_k T \).

**Proof.** Let \( W \) be a multiplicative subset of \( D \) and let \( V = \pi^{-1}(W) \). Let \( y \in (D \times_k T)_V \). Then there is some \( v \in V \) such that \( vy \in D \times_k T \) so \( \pi(vy) = \pi(v)\pi(y) \in D \). Because \( \pi(v) \in W \), \( \pi(y) = \frac{\pi(vy)}{\pi(v)} \in D_W \). Hence, \( y \in D_W \times_k T \).

Let \( x \in D_W \times_k T \). Then \( \pi(x) \in D_W \) and hence \( \pi(x) = \frac{d}{w} \) for some \( d \in D \), \( w \in W \). Then there exists \( t \in T \) and \( v \in V \) such that \( \pi(t) = d \) and \( \pi(v) = w \). So \( \pi(xv) = \pi(x)\pi(v) = \pi(t) \in D \). Thus, \( xv \in D \times_k T \) and hence \( x = \frac{1}{v}(xv) \in (D \times_k T)_V \).

The following propositions from [4] are essential for the work which follows and are stated here for the reader’s convenience.

**Proposition 2.4.** Let \( D \) be an integral domain with fraction field \( k \) and let \( \pi : T \rightarrow k \) be a surjective homomorphism. If \( W \) is a multiplicative subset of \( D \times_k T \), then \( (D \times_k T)_W = D_{WD} \times_{k_{W_D}} T_W \) where \( W_D \) is the image of \( W \) in \( D \) and \( W_k \) is the image of \( W \) in \( k \).

**Proof.** This is a special case of part of [4, Proposition 1.9]. Equality holds with the pullback defined as above.

**Proposition 2.5.** Let \( D \) be an integral domain with fraction field \( k \) and \( (T, M) \) a local domain with residue field \( k \). Let \( P \in \text{Spec}(D \times_k T) \).

1. The ideal \( M \) is a common ideal of \( T \) and \( D \times_k T \) and every \( P \in \text{Spec}(D \times_k T) \) is comparable with \( M \) with respect to inclusion.
2. There is an isomorphism between the lattice of all the ideals of \( D \) and all the ideals of \( D \times_k T \) containing \( M \).
3. If \( P \subseteq M \) and \( Q \) is the unique prime ideal of \( T \) corresponding to \( P \), \( (D \times_k T)_P = T_Q \).
4. If \( M \subseteq P \), \( (D \times_k T)_P = D_p \times_k T \) where \( p \) is the unique prime ideal of \( D \) corresponding to \( P \).

**Proof.** Because \( D \times_k T \) is a subring of \( T \), \( M = \ker(T \rightarrow k) \) is a common ideal of \( T \) and \( D \times_k T \) by [5, Lemma 1.1.4(1)]. The other claims are [4, Proposition 2.1(4), 2.2(2), 2.2(3) and 2.2(6)] respectively. Equality holds with the pullback defined as above.
3. Preservation of flat overrings and global perinormality

Proposition 3.3 gives a result for the transfer of the property that every flat overring is a localization. From this, one readily obtains a result for the preservation of global perinormality as a corollary. To prove Proposition 3.3, we will utilize the following analog of [1, Theorem 3.1] in which we observe that the valuation domain need not be of the form $k + M$ where $k$ is the residue field $V$.

**Proposition 3.1** (Analog of Theorem 3.1 in [1] with $V$ not necessarily equal to $k + M$). Let $(V, M)$ be a valuation domain with residue field $k$ and fraction field $K$. Let $D$ be an integral domain that is a subring of $k$. Then each $D \times_k V$-submodule of $K$ compares with $V$ with respect to inclusion. Furthermore, the set of overrings of $D \times_k V$ is $\{S_\alpha\}_{\alpha \in A} \cup \{T_\beta \times_k V\}_{\beta \in B}$ where $\{S_\alpha\}_{\alpha \in A}$ is the set of overrings of $V$ and $\{T_\beta\}_{\beta \in B}$ is the set of subrings of $k$ containing $D$.

**Proof.** Let $\pi : V \rightarrow k$ be the canonical mapping.

The proof in [1, proof of Theorem 3.1] that each $D \times_k V$-submodule of $K$ compares with $V$ with respect to inclusion follows through without modification.

The set of overrings of $D \times_k V$ contains the set $\{S_\alpha\}_{\alpha \in A} \cup \{T_\beta \times_k V\}_{\beta \in B}$ where $\{S_\alpha\}_{\alpha \in A}$ is the set of overrings of $V$ and $\{T_\beta\}_{\beta \in B}$ is the set of subrings of $k$ containing $D$. Any overring of $D \times_k T$ compares with $V$ with respect to inclusion. Clearly any overring of $V$ is in the described set. Let $T$ be an overring of $D \times_k V$ such that $T \subseteq V$. Then

\[
\pi^{-1}(D)/M \subseteq T/M \subseteq V/M = k
\]

so $T/M$ is a subring of $k$ containing $\pi^{-1}(D)/M = D$. Note that

\[
T = \{v \in V : v \in T\} = \{v \in V : v + M \in T/M\} = \{v \in V : \pi(v) \in T/M\} = \pi^{-1}(T/M).
\]

Before proving Proposition 3.3, we prove the following lemma using Richman’s Criterion for flatness [7, Theorem 2].

**Lemma 3.2.** Let $(T, M)$ be a local domain with residue field $k$ and $D$ be a domain with fraction field $k$. Let $S$ be an overring of $D$. Then $S$ is a flat overring of $D$ if and only if $S \times_k T$ is a flat overring of $D \times_k T$.

**Proof.** In what follows, let $\varphi : S \times_k T \rightarrow S$ be the canonical projection. Suppose that $S$ is a flat overring of $D$. Let $\tilde{N} \in \text{Max}(S \times_k T)$ be arbitrary. Then there exists a unique $N \in \text{Max}(S)$ corresponding to $\tilde{N}$ by Proposition 2.5 (1) and (2). Because $S$ is a flat overring of $D$, by [7, Theorem 2], $S_N = D_{N \cap D}$. Again note that $N \cap D \subseteq S$ and

\[
\varphi^{-1}(N \cap D) = \varphi^{-1}(N) \cap \varphi^{-1}(D) = \tilde{N} \cap (D \times_k T)
\]

where $\varphi : S \times_k T \rightarrow S$ is the canonical projection. Hence, $\tilde{N} \cap (D \times_k T)$ is the unique prime ideal of $D \times_k T$ corresponding to $N \cap D$. Then

\[
(S \times_k T)_{\tilde{N}} = S_N \times_k T \quad \text{by Proposition 2.5(4)}
\]

\[
= D_{N \cap D} \times_k T \quad \text{because } S_N = D_{N \cap D}
\]

\[
= (D \times_k T)_{\tilde{N} \cap (D \times_k T)} \quad \text{by Proposition 2.5(4)}.
\]
Because $N \in \text{Max}(S \times_k T)$ was arbitrary, $S \times_k T$ is flat over $D \times_k T$ by [7, Theorem 2].

Conversely, suppose that $S \times_k T$ is a flat overring of $D \times_k T$. Let $N \in \text{Max}(S)$ be arbitrary. Then there is a unique $\tilde{N} \in \text{Max}(S \times_k T)$ corresponding to $N$ by Proposition 2.5 (1) and (2). Because $S \times_k T$ is a flat overring of $D \times_k T$,

$$(S \times_k T)\tilde{N} = (D \times_k T)\tilde{N} \cap (D \times_k T)$$

by [7, Theorem 2]. Note that $N \cap D \subseteq S$ and

$$\varphi^{-1}(N \cap D) = \varphi^{-1}(N) \cap \varphi^{-1}(D) = \tilde{N} \cap (D \times_k T).$$

So $\tilde{N} \cap (D \times_k T)$ is the unique prime ideal of $D \times_k T$ corresponding to $N \cap D$. Then

$$S_N \times_k T = (S \times_k T)\tilde{N} \text{ by Proposition 2.5(4)}$$

$$= (D \times_k T)\tilde{N} \cap (D \times_k T) \text{ as noted above}$$

$$= D_{N \cap D} \times_k T \text{ by Proposition 2.5(4)}.$$ 

So $S_N = D_{N \cap D}$ by Observation 2.2. Because $N \in \text{Max}(S)$ was arbitrary, $S$ is a flat overring of $D$ by [7, Theorem 2]. \hfill \square

**Proposition 3.3.** Let $D$ be an integral domain with fraction field $k$. Let $(T, M)$ be a local domain with residue field $k$. If $D \times_k T$ has the property that every flat overring is a localization of $D \times_k T$, then so does $D$. The converse holds if $T$ is a valuation domain.

**Proof.** Let $D$ be an integral domain with fraction field $k$. Let $(T, M)$ be a local domain with residue field $k$ and let $\pi : T \rightarrow k$ be the canonical homomorphism. Suppose that every flat overring of $D \times_k T$ is a localization of $D \times_k T$. Let $S$ be a flat overring of $D$. By Lemma 3.2, $S \times_k T$ is a flat overring of $D \times_k T$. Because every flat overring of $D \times_k T$ is a localization of $D \times_k T$, $S \times_k T = (D \times_k T)_W$ where $W$ is a multiplicative subset of $D \times_k T$. By Proposition 2.4, $(D \times_k T)_W = D_{W_D} \times_{k W} T_W$ where $W_D$ is the image of $W$ in $D$ and $W$ is the image of $W$ in $k$. If $W \cap \ker(\pi) = 0$, then $W$ consists of units of $T$ and hence $S \times_k T = D_{W_D} \times_{k W} T_W = D_{W_D} \times_k T$. Hence, $S = D_{W_D}$ by Observation 2.2. If $W \cap \ker(\pi) \neq 0$, then

$$S \times_k T = D_{W_D} \times_{k W} T_W = 0 \times_0 T_W = T_W.$$

Hence, $S \times_k T = T$ because $S \times_k T \subseteq T$. It follows that $S = k = D_{D \setminus \{0\}}$. \hfill \square

Now let $(T, M)$ be a valuation domain. Suppose that every flat overring of $D$ is a localization of $D$. Let $S$ be an arbitrary flat overring of $D \times_k T$. Then $S$ is an overring of $T$ or $S = R \times_k T$ where $R$ is an overring of $D$ by Proposition 3.1. First, suppose $S$ is an overring of $T$. Then $S = T_P$ for some prime ideal $P$ of $T$ because every overring of a valuation domain is a localization at a prime ideal (see [6, Theorem 10.1]). By Proposition 2.5(3), $T$ is a localization of $D \times_k T$. Hence $S$ is a localization of $D \times_k T$. Now suppose $S = R \times_k T$ where $R$ is an overring of $D$. By Lemma 3.2, $R$ is flat over $D$. Then $R = D_W$ where $W$ is a multiplicative subset of $D$ because every flat overring of $D$ is a localization of $D$. Then $S = R \times_k T = D_W \times_k T$ is a localization of $D \times_k T$ by Lemma 2.3. \hfill \square

In [3, Theorem 2.7] which is given below for the reader’s convenience, Epstein and Shapiro showed that perinormality is preserved in a more general version of the classical $D + M$ pullback.

**Theorem 3.4.** [3, Theorem 2.7] Let $D$ be an integral domain with fraction field $k$. Let $V$ be a valuation domain with residue field $k$. Then $D \times_k V$ is perinormal if and only if $D$ is perinormal.

In order to give the corollary in the generality in which it will be stated, it is necessary to note that the valuation domain assumption was not necessary in one direction.
**Observation 3.5.** Let $D$ be an integral domain with fraction field $k$. Let $(T, M)$ be a local domain with residue field $k$. If the pullback $D \times_k T$ is perinormal, then $D$ and $T$ are perinormal.

**Proof.** Let $D$ be an integral domain with fraction field $k$. Let $(T, M)$ be a local domain with residue field $k$. By Proposition 2.5(3), $T$ is a localization of $D \times_k T$ which is perinormal. Thus, $T$ is perinormal by [2, Proposition 2.5].

The perinormality of $D$ follows from the proof of [3, Theorem 2.7].

As a corollary, we obtain the following analog of [2, Theorem 2.7] for globally perinormal domains.

**Corollary 3.6.** Let $D$ be an integral domain with fraction field $k$. Let $(T, M)$ be a local domain with residue field $k$. If the pullback $D \times_k T$ is globally perinormal, then $D$ and $T$ are globally perinormal. If $T$ is a valuation domain and $D$ is globally perinormal, then $D \times_k T$ is globally perinormal.

**Proof.** Let $D$ be an integral domain with fraction field $k$. Let $(T, M)$ be a local domain with residue field $k$. Suppose $D \times_k T$ is globally perinormal. So every flat overring of $D \times_k T$ is a localization of $D \times_k T$. By Observation 3.5, $D$ is perinormal. To show that $D$ is globally perinormal, it suffices to show that every flat overring of $D$ is a localization of $D$ but this follows directly by Proposition 3.3.

By Proposition 2.5(3), $T$ is a localization of $D \times_k T$ which is globally perinormal. Hence, $T$ is globally perinormal by [2, Proposition 6.1].

Now suppose $T$ is a valuation domain and $D$ is globally perinormal. Then $D \times_k T$ is perinormal by [3, Theorem 2.7]. Because every flat overring of $D$ is a localization of $D$, it follows by Proposition 3.3 that every flat overring of $D \times_k T$ is a localization of $D \times_k T$. Hence, $D \times_k V$ is globally perinormal.

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