Global regularity and stability of a hydrodynamic system modeling vesicle and fluid interactions

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Abstract

In this paper, we study a hydrodynamical system modeling the deformation of vesicle membranes in incompressible viscous fluids. In the three dimensional case, we prove the existence/uniqueness of local strong solutions for arbitrary initial data as well as global strong solutions under the large viscosity assumption. We also establish some regularity criteria in terms of the velocity for local smooth solutions. Finally, we study the stability of the system near local minimizers of the elastic bending energy.

1 Introduction

Biological vesicle membranes are interesting subjects widely studied in biology, biophysics and bioengineering. They are not only essential to the function of cells but exhibit rich physical and rheological properties as well [24]. The single component vesicles are possibly the simplest models for the biological cells and molecules, which are formed by certain amphiphilic molecules assembled in water to build bi-layers [12]. The equilibrium configurations of vesicle membranes can be characterized by the Helfrich bending elasticity energy of the surface [3, 7, 18] such that they are minimizers of the bending energy under possible constraints like prescribed surface area and bulk volume that account for the effects of density change and osmotic pressure [12, 32]. Let \( \Gamma \) be a smooth, compact surface without boundary representing the membrane of the vesicle. In the isotropic case, if the evolution of the vesicle membrane does not change its topology, the interfacial energy takes the simplified form [7]:

\[
E_{\text{elastic}} = \int_{\Gamma} \frac{k}{2} (H - H_0)^2 \, ds,
\]

where \( H \) is the mean curvature of the membrane surface; \( k \) is called the bending rigidity, which can depend on the local heterogeneous concentration of the species; \( H_0 \) is the spontaneous curvature that describes certain physical/chemical difference between the inside and the outside of the membrane.

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Recently, phase-field models have been derived within a general energetic variational framework to study vesicle deformations and numerical simulations of the membrane deformations were carried out (see e.g., [10–12] and references cited therein). As in [9], we denote by $\phi$ the phase function defined on the physical domain $\Omega$, which is used to label the inside and the outside of the vesicle $\Gamma$ such that $\phi$ takes the value 1 inside of the vesicle membrane and $-1$ outside. The sharp transition layer of the phase function gives a diffusive interface description of the vesicle membrane $\Gamma$, which is recovered by the level set $\{ x : \phi(x) = 0 \}$. The phase field approach describes geometric deformations in Eulerian coordinates and it provides a convenient way to capture topological transitions such as vessel fission or fusion via changes in the level set topology. This simplifies numerical approximations because it suffices to consider a fixed computational grid rather than tracking the position of the interface [11].

For the sake of simplicity, we assume that $k$ is a positive constant and $H_0 = 0$. The phase-field approximation of the Helfrich bending elasticity energy is given by a modified Willmore energy [12] (see, e.g., [10] the approximation energy for the elastic bending energy with non-zero spontaneous curvature)

$$ E_\epsilon(\phi) = \frac{k}{2\epsilon} \int_\Omega |f(\phi)|^2 dx, \text{ with } f(\phi) = -\epsilon \Delta \phi + \frac{1}{\epsilon}(\phi^2 - 1)\phi, \quad (1.1) $$

where $\epsilon$ is a small (compared to the vesicle size) positive parameter that characterizes the transition layer of the phase function. The convergence of the phase-field model to the original sharp interface model as the transition width of the diffuse interface $\epsilon \to 0$ was carried out in [9, 32]. Two constraints are widely used in the biophysical studies of vesicles [26] such that the total surface area and the volume of the vesicle are conserved (in time). The former is a consequence of the incompressibility of the membrane, while the latter is based on the consideration that, for a fluctuating vesicle with the inside pressure and outside pressure balanced by the osmotic pressure, the change in volume is normally a much slower process in comparison with the shape change [11]. The constraint functionals for the vesicle volume and surface area are given by (cf. [12])

$$ A(\phi) = \int_\Omega \phi dx, \quad B(\phi) = \int_\Omega \frac{\epsilon}{2} |\nabla \phi|^2 + \frac{1}{4\epsilon}(\phi^2 - 1)^2 dx. \quad (1.2) $$

Two penalty terms are introduced in order to enforce these constraints, and the approximate elastic bending energy is formulated in the following form [3, 10, 13, 14]:

$$ E(\phi) = E_\epsilon(\phi) + \frac{1}{2} M_1 (A(\phi) - \alpha)^2 + \frac{1}{2} M_2 (B(\phi) - \beta)^2, \quad (1.3) $$

where $M_1$ and $M_2$ are two penalty constants, $\alpha = A(\phi_0)$ and $\beta = B(\phi_0)$ are determined by the initial value of the phase function $\phi_0$. Alternatively, Lagrange multipliers could be used to conserve the vesicle volume and total vesicle surface area [11, 14].

In this paper, we consider a hydrodynamic system for the interaction of a vesicle with the fluid field, which describes the evolution of vesicles immersed in an incompressible, Newtonian fluid [8]. More precisely, we study the following phase-field Navier–Stokes equations for the velocity field $u$ of the fluid and the phase function $\phi$:

$$ u_t + u \cdot \nabla u + \nabla P = \mu \Delta u + \frac{\delta E(\phi)}{\delta \phi} \nabla \phi, \quad (1.4) $$
\[ \nabla \cdot u = 0, \quad (1.5) \]
\[ \phi_t + u \cdot \nabla \phi = -\gamma \frac{\delta E(\phi)}{\delta \phi}. \quad (1.6) \]

System \((1.4)-(1.6)\) can be obtained via an energetic variation approach \([33]\) (see \([11]\) for the derivation of a corresponding evolution system that adopts the Lagrange multiplier approach for the volume and surface area constants). The resulting membrane configuration and the flow field reflect the competition and the coupling of the kinetic energy and membrane elastic energies. Equations \((1.4)\) and \((1.5)\) are the Navier-Stokes equations of the viscous incompressible fluid with unit density and a force, which is derived from the variation of the elastic bending energy and it involves a nonlinear combination of higher-order spatial derivatives of the phase function. \(P\) denotes the pressure and \(\mu\) is the fluid viscosity, which is assumed to be a positive constant throughout both fluid phases and the interface. \((1.6)\) is a relaxed transport equation of \(\phi\) under the velocity field \(u\). Its right-hand side contains a regularization term, where \(\gamma\) is the mobility coefficient that is assumed to be a small positive constant.

Well-posedness of the system \((1.4)-(1.6)\) subject to no-slip boundary condition for the velocity field and Dirichlet boundary conditions for the phase function has been studied in \([8, 22]\). In \([8]\), the authors obtained the existence of weak solutions by using the Galerkin method. They also obtain the uniqueness of solutions in a more regular class than the one used for existence. Quite recently, existence and uniqueness of local strong solutions have been proved in \([22]\) via a fixed point argument. Existence of almost global strong solutions is obtained under the assumption that the initial data and the quantity \((|\Omega| + \alpha)^2\) are small enough. However, they were not able to prove global existence result because uniform-in-time a priori estimates were not available in their argument. On the other hand, since some compatibility conditions (at the boundary) are required in the fixed point strategy described in \([22]\) to obtain enough regular solutions, the authors have to confine them to function spaces with proper limited regularity.

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We note that, although Dirichlet type boundary conditions are more natural and physical conditions, the periodic boundary conditions can also be reasonably justified physically when the vesicle interface \(\Gamma\) is sufficient small compared with the overall physical domain \(\Omega\) (cf. \([14]\)). In our present paper, we study the system \((1.4)-(1.6)\) subject to the periodic boundary conditions (i.e., in torus \(T^3\)):

\[ u(x + e_i) = u(x), \quad \phi(x + e_i) = \phi(x), \quad \text{for } x \in \partial Q, \quad (1.7) \]

and to the initial conditions

\[ u|_{t=0} = u_0(x), \quad \text{with } \nabla \cdot u_0 = 0, \quad \int_Q u_0 dx = 0 \quad \text{and } \phi|_{t=0} = \phi_0(x), \quad \text{for } x \in Q, \quad (1.8) \]

where \(Q\) is a unit square in \(\mathbb{R}^3\).

The main purpose of this paper is to study the existence, regularity and stability of global strong solutions to problem \((1.4)-(1.8)\). In the subsequent proof, we shall see that problem \((1.4)-(1.8)\) has an energy dissipation mechanism (cf. \((2.16)\) below) that plays a crucial role in controlling the contribution to the momentum equation of the extra stress tensor due to the membrane deformation and the contribution of the convection term to the phase-field
evolution. The advantage to work in the periodic setting is that one can get rid of certain boundary terms when performing integration by parts. Due to the weak coupling in the phase-field equation (1.6) that is a gradient flow of the elastic bending energy under the fluid transport, we can derive uniform-in-time estimate for $H^3$-norm of $\phi$ (cf. Proposition 3.1) that enables us to derive some specific higher-order energy inequalities (cf. Lemma 3.3 and Lemma 3.4) in the spirit of [21] for a simplified nematic crystal system. Based on these higher-order inequalities, we can show existence and uniqueness of local strong solutions to problem (1.4)–(1.8) (cf. Theorem 3.1), existence of global strong solutions under properly large viscosity $\mu$ (cf. Theorem 3.2) and also the eventual regularity of the global weak solution (cf. Corollary 5.1). After a careful exploration of the nonlinear coupling between velocity field and membrane deformation, we establish some regularity criteria for solutions to problem (1.4)–(1.8) in 3D that only involve the velocity field (cf. Theorems 4.1, 4.2), which coincide with the results for conventional Navier–Stokes equations. This indicates that the velocity field indeed plays a dominant role in studying regularity for solutions $(u, \phi)$.

Finally, we prove the well-posedness and stability of global strong solutions (cf. Theorem 5.1) when the initial datum is close to a certain local minimizer of the elastic energy by using a suitable Lojasiewicz–Simon type inequality (cf. Lemma 5.2). The results obtained in this paper hold for any given (but fixed) penalty constants. Since now we are working with the penalty formulation to incorporate the volume and surface area conservation of the vesicle membrane, the constraints are satisfied only approximately. It would be interesting to investigate the corresponding results for the evolution system in the Lagrange multiplier formulation (cf. [11]) where the constraints are satisfied exactly.

The rest of the paper is organized as follows. In Section 2, we present the functional settings and some preliminary results. In Section 3, we prove the existence of local strong solutions and global ones under the large viscosity assumption. In Section 4, we establish some logarithmic-type regularity criteria for the smooth solutions only in terms of the velocity field. In Section 5, we study the well-posedness and stability of global strong solutions near local minimizers of the elastic energy. In the final Section 6, we sketch the proof of the Lojasiewicz–Simon type inequality that plays a key role in the analysis of Section 5.

2 Preliminaries

We recall the well established functional settings for periodic problems (cf. [29]):

\[ H^m_p(Q) = \{ u \in H^m(\mathbb{R}^n, \mathbb{R}) \ | \ u(x + e_i) = u(x) \}, \]

\[ H = \{ v \in L^2_p(Q), \ \nabla \cdot v = 0 \}, \text{ where } L^2_p(Q) = H^0_p(Q), \]

\[ V = \{ v \in H^1_p(Q), \ \nabla \cdot v = 0 \}, \]

\[ V' = \text{the dual space of } V. \]

For any Banach space $B$, we denote by $B$ the vector space $(B)^r$, $r \in \mathbb{N}$, endowed with the product norms. For any norm space $X$, its subspace that consists of functions in $X$ with zero-mean will be denoted by $\bar{X}$ such that $\bar{X} = \{ w \in X : \int_Q w \, dx = 0 \}$. We denote the inner product on $L^2_p(Q)$ (or $L^2_p(Q)$) as well as $H$ by $(\cdot, \cdot)$ and the associated norm by $\| \cdot \|$. The space $H^m_p(Q)$ will be short-handed by $H^m_p$. We denote by $C$ and $C_i, i = 0, 1, \cdots$ genetic constants.
which may depend only on \( \mu, \gamma, Q, \alpha, \beta \) and the initial data \((u_0, \phi_0)\). Special dependence will be pointed out explicitly in the text if necessary. Throughout the paper, the Einstein summation convention will be used. Following [29], one can define mapping
\[
Su = -\Delta u, \quad \forall u \in D(S) := \{ u \in \mathbf{H}, \Delta u \in \mathbf{H} \} = \bar{H}_0^2 \cap \mathbf{H}.
\] (2.9)

The Stokes operator \( S \) can be viewed as an unbounded positive linear self-adjoint operator on \( H \). If \( D(S) \) is endowed with the norm induced by \( \bar{L}_0^2 \), then \( S \) becomes an isomorphism from \( D(S) \) onto \( H \). More detailed properties of operator \( S \) can be found in [29]. We also recall the interior elliptic estimate, which states that for bounded domains \( U_1 \subset U_2 \) there is a constant \( C > 0 \) depending only on \( U_1 \) and \( U_2 \) such that \( \| \phi \|_{H^2(U_1)} \leq C(\| \Delta \phi \|_{L^2(U_2)} + \| \phi \|_{L^2(U_2)}) \). In our current case under periodic boundary conditions, we can choose \( Q' \) to be the union of \( Q \) and its neighborhood copies. Then we have
\[
\| \phi \|_{H^2(Q)} \leq C(\| \Delta \phi \|_{L^2(Q')} + \| \phi \|_{L^2(Q')}),
\] (2.10)

It follows from the periodic boundary condition that \( \int_Q \nabla \phi \, dx = 0 \) and \( \int_Q \Delta \phi \, dx = 0 \), then we infer from the Poincaré–Wirtinger inequality that
\[
\| \nabla \phi \| + \left| \int_Q \phi \, dx \right| \approx \| \phi \|_{H^1}, \quad \| \Delta \phi \| + \left| \int_Q \phi \, dx \right| \approx \| \phi \|_{H^2}, \quad \| \nabla \Delta \phi \| + \left| \int_Q \phi \, dx \right| \approx \| \phi \|_{H^3}. \tag{2.11}
\]

A direction calculation yields that the variation of the approximate elastic energy is given by
\[
\frac{\delta E(\phi)}{\delta \phi} = kg(\phi) + M_1(A(\phi) - \alpha) + M_2(B(\phi) - \beta)f(\phi), \tag{2.12}
\]
where
\[
g(\phi) = -\Delta f(\phi) + \frac{1}{\varepsilon^2}(3\phi^2 - 1)f(\phi).
\]

Since we are now dealing with the periodic boundary conditions, the average of velocity \( u \) is conserved.

**Lemma 2.1.** Let \((u, \phi)\) be a solution to the problem \((1.4) - (1.8)\) on \([0, T]\). It holds
\[
\int_Q u(t) \, dx = \int_Q u_0 \, dx, \quad \forall t \in [0, T]. \tag{2.13}
\]

**Proof.** It follows from (2.12) that
\[
\frac{\delta E}{\delta \phi} \nabla \phi = kg(\phi) \nabla \phi + M_1(A(\phi) - \alpha) \nabla \phi + M_2(B(\phi) - \beta)f(\phi) \nabla \phi := I_1 + I_2 + I_3. \tag{2.14}
\]

Since \( A(\phi) \) and \( B(\phi) \) are functions only depending on time, using integration by parts and the periodic boundary conditions, we deduce that
\[
I_2 = M_1(A(\phi) - \alpha) \int_Q \nabla \phi \, dx = 0,
\]

\[
I_3 = M_2(B(\phi) - \beta) \int_Q f(\phi) \nabla \phi \, dx.
\]
\[
M_2(B(\phi) - \beta) \int_Q \left( -\varepsilon \Delta \phi + \frac{1}{\varepsilon} (\phi^2 - 1) \phi \right) \nabla \phi \, dx = 0,
\]

where we have used the fact \(\Delta \phi \nabla \phi = \nabla \cdot (\nabla \phi \otimes \nabla \phi) - \frac{1}{2} \nabla (|\nabla \phi|^2)\). Finally,

\[
\frac{1}{k} I_1 = \varepsilon \int_Q \Delta^2 \phi \nabla \phi \, dx - \frac{1}{\varepsilon} \int_Q \Delta (\phi^3 - \phi) \nabla \phi \, dx
\]

\[
+ \frac{1}{\varepsilon^2} \int_Q \left[ -\varepsilon \Delta \phi + \frac{1}{\varepsilon} (\phi^2 - 1) \phi \right] \nabla (\phi^3 - \phi) \, dx
\]

\[
= \varepsilon \int_Q |\Delta \phi|^2 \, dx - \frac{1}{\varepsilon} \int_Q \Delta (\phi^3 - \phi) \nabla \phi \, dx
\]

\[
- \frac{1}{\varepsilon} \int_Q \Delta \phi \nabla (\phi^3 - \phi) \, dx + \frac{1}{2\varepsilon^3} \int_Q \nabla [(\phi^3 - \phi)^2] \, dx
\]

\[
= 0.
\]

Thus, we conclude that

\[
\int_Q \frac{\delta E}{\delta \phi} \nabla \phi \, dx = 0. \quad (2.15)
\]

After integrating (1.4) over \(Q\), we infer from (1.5), the periodic boundary condition (1.7) and (2.15) that (2.13) holds.

\textbf{Remark 2.1.} By Lemma 2.1, if one assumes that the average of the initial velocity vanishes, i.e., \(\frac{1}{|Q|} \int_Q u_0 \, dx = 0\), then we can apply the Poincaré–Wirtinger inequality to the solution \(u\) such that the \(H^1\)-norm of \(u\) can be controlled by \(|\nabla u|\). When a flow with non-vanishing average velocity \(u\) is considered, as for the single Navier–Stokes equation (cf. [29]), we can introduce the variable \(\tilde{u} = u - \frac{1}{|Q|} \int_Q u \, dx\) and transform the problem (1.4)–(1.8) into a new one in terms of \(\tilde{u}\) and \(\phi\). Since \(\frac{1}{|Q|} \int_Q u \, dx\) is a known constant determined by (2.13), it is not difficult to verify that our results on existence and uniqueness of weak/strong solutions for the initial velocity with zero mean can be extended to this case with minor modifications. However, results on long-time dynamics in Section 5 are no longer valid, because the velocity will not decay to zero (we also refer to [33] for a similar situation for the liquid crystal system).

For the sake of simplicity, in the remaining part of this paper, we will assume that the average flow vanishes. An important property of the coupling system (1.4)–(1.8) is that it has a basic energy law, which indicates the dissipative nature of the system. It states that the total sum of the kinetic and elastic energy is dissipated due to viscosity and other possible regularization/relaxations rates. A formal derivation can be carried out by multiplying (1.4) by \(u\), (1.6) by \(\frac{\delta \mathcal{E}(\phi)}{\delta \phi}\), respectively, and integrating over \(Q\). As a consequence, we have
Lemma 2.2 (Basic energy law). Let \((u, \phi)\) be a smooth solution to the problem (1.4)–(1.8). The following dissipative energy inequality holds:

\[
\frac{d}{dt} \left( \frac{1}{2} \|u(t)\|^2 + E(\phi(t)) \right) + \mu \|\nabla u\|^2 + \gamma \left\| \frac{\delta E}{\delta \phi} \right\|^2 = 0, \quad \forall t > 0. \tag{2.16}
\]

Based on Lemma 2.2, we can apply the Galerkin method similar to that in [8] to prove the following result on existence and uniqueness of weak solutions to the problem (1.4)–(1.8).

Theorem 2.1 (Existence of weak solutions). Let \(n = 3\). For any initial datum \((u_0, \phi_0) \in \dot{H} \times H_p^2, T > 0\), there exists at least one global weak solution \((u, \phi)\) to the problem (1.4)–(1.8) that satisfies

\[
u \in L^\infty(0, T; \dot{H}) \cap L^2(0, T; \dot{V}); \tag{2.17}
\]

\[
\phi \in L^\infty(0, T; H_p^2) \cap L^2(0, T; H_p^4) \cap H^1(0, T; L_p^2). \tag{2.18}
\]

In addition, the weak solution is unique provided that \(u \in L^8(0, T; L_p^4)\).

Besides, we can obtain the following uniform-in-time estimates on weak solutions from the basic energy law:

Proposition 2.1. Suppose \(n = 3\). For any initial data \(u_0 \in \dot{H}, \phi_0 \in H_p^2\), the corresponding weak solutions of the problem (1.4)–(1.8) have the following uniform estimates

\[
\|u(t)\| + \|\phi(t)\|_{H^2} \leq C, \quad \forall t \geq 0, \tag{2.19}
\]

\[
\int_0^{+\infty} \left( \mu \|\nabla u(t)\|^2 + \gamma \left\| \frac{\delta E}{\delta \phi} (t) \right\|^2 \right) dt \leq C, \tag{2.20}
\]

where \(C > 0\) is a constant depending on \(\|u_0\|, \|\phi_0\|_{H^2}\) and coefficients of the system except the viscosity \(\mu\).

Proof. We can derive a weaker version of the basic energy law rigorously via the Galerkin procedure such that the weak solution \((u, \phi)\) to problem (1.4)–(1.8) satisfies

\[
\frac{1}{2} \|u(t)\|^2 + E(\phi(t)) + \int_0^t \left( \mu \|\nabla u(s)\|^2 + \gamma \left\| \frac{\delta E}{\delta \phi} (s) \right\|^2 \right) ds \leq \frac{1}{2} \|u_0\|^2 + E(\phi_0), \quad \forall t \geq 0.
\]

Recalling the definition of \(E\), we know \(\frac{1}{2} \|u_0\|^2 + E(0)\) can be estimated by a constant depending on \(\|u_0\|, \|\phi_0\|_{H^2}\) and coefficients of the system, but not on \(\mu\). Thus (2.20) holds and \(\|u(t)\|, E(t)\) are bounded. On the other hand, we infer from the boundedness of \(E(t)\) that

\[
|A(\phi)| \leq C, \quad |B(\phi)| \leq C, \quad \|f(\phi)\| \leq C, \quad \forall t \geq 0.
\]

Hence, \(\int_Q \phi dx\), \(\|\nabla \phi\|\) are bounded. Then by the definition of \(f(\phi)\) and Sobolev embedding theorems, we can deduce that \(\|\phi\|_{H^2}\) is bounded. The proof is complete.

7
3 Existence of strong solutions

In this section, we study the existence of strong solutions. For this purpose, it suffices to derive proper higher-order uniform estimates for the Galerkin approximation of weak solutions and then pass to limit. We observe that the entire calculation is identical to that as we work with classical (smooth) solutions to problem (1.4)–(1.8). Thus, for the sake of simplicity, all the calculations below will be carried out formally for smooth solutions.

By the Sobolev embedding theorem in 3D, we can derive the following estimates that will be frequently used later.

**Lemma 3.1.** Suppose \( n = 3 \). We have

\[
\|\nabla \Delta \phi\| \leq C \left\| \frac{\delta E}{\delta \phi} \right\|^\frac{3}{2} + C, \quad \|\Delta^2 \phi\| \leq \frac{1}{k\varepsilon} \left\| \frac{\delta E}{\delta \phi} \right\| + C, \quad \forall \phi \in H^4_p,
\]

where \( C \) is a constant depending on \( \|\phi\|_{H^2} \) and coefficients of the system. Besides,

\[
\|\nabla \Delta^2 \phi\| \leq \frac{1}{k\varepsilon} \left\| \nabla \frac{\delta E}{\delta \phi} \right\| + C,
\]

where \( C \) is a constant depending on \( \|\phi\|_{H^3} \) and coefficients of the system.

**Proof.** Recalling (2.12), we can rewrite \( \frac{\delta E(\phi)}{\delta \phi} \) as

\[
\frac{\delta E(\phi)}{\delta \phi} = k\varepsilon \Delta^2 \phi + H(\phi),
\]

where

\[
H(\phi) = \frac{k}{\varepsilon} \Delta (\phi^3 - \phi) + \frac{k}{\varepsilon^2} (3\phi^2 - 1)f(\phi) + M_1(A(\phi) - \alpha) + M_2(B(\phi) - \beta)f(\phi)
\]

\[
= -\frac{6k}{\varepsilon} \phi|\nabla \phi|^2 - \frac{2k}{\varepsilon} (3\phi^2 - 1)\Delta \phi + \frac{k}{\varepsilon^2} (3\phi^2 - 1)(\phi^3 - \phi)
\]

\[
+ M_1(A(\phi) - \alpha) + M_2(B(\phi) - \beta)f(\phi).
\]

By the Hölder inequality and Sobolev embedding theorems, we infer that

\[
\|\Delta^2 \phi\| \leq \frac{1}{k\varepsilon} \left\| \frac{\delta E(\phi)}{\delta \phi} \right\| + C\|\phi\|_{L^\infty} \|\nabla \phi\|_{L^4}^2 + C\|\phi\|_{L^\infty}^2 \|\Delta \phi\|
\]

\[
+ C\|\Delta \phi\| + C(\|\phi\|_{L^\infty}^2 + 1) + M_1\|\phi\|_{L^1} + M_1\alpha
\]

\[
+ CM_2(\beta + \|\nabla \phi\|^2 + C\|\phi\|_{L^4}^4 + C)(\|\Delta \phi\| + C\|\phi\|_{L^6}^3 + C\|\phi\|)
\]

\[
\leq \frac{1}{k\varepsilon} \left\| \frac{\delta E}{\delta \phi} \right\| + C, \quad \forall \phi \in H^4_p,
\]

where \( C \) is a constant depending on \( \|\phi\|_{H^2} \) and coefficients of the system. The estimate for \( \|\nabla \Delta \phi\| \) easily follows from (3.3) and the fact \( \|\nabla \Delta \phi\|^2 = \int_\Omega \Delta \phi \Delta^2 \phi dx \leq C \|\Delta^2 \phi\| \).

Concerning the estimate for \( \|\nabla \Delta^2 \phi\| \), we just apply \( \nabla \) to \( \frac{\delta E(\phi)}{\delta \phi} \) and we can obtain our result by estimating \( \nabla H(\phi) \) via proper Sobolev embeddings. \( \square \)
Since our system (1.4)–(1.6) contains the Navier–Stokes equations as a subsystem, in the three dimensional case, one cannot expect that the weak solutions will become regular for strictly positive time. But it is worth noting that, due to the weak coupling in the phase-field equation (1.6) that \( u \) enters in the evolution equation only as a lower order term \( u \cdot \nabla \phi \), we can first derive certain regularity results for the phase function \( \phi \) and show that it turns out to be regular for \( t > 0 \).

**Lemma 3.2.** Let \( n = 3 \). For any smooth solution to the problem (1.4)–(1.8), it holds

\[
\frac{d}{dt} \|\nabla \Delta \phi\|^2 + k\gamma \varepsilon \|\nabla^2 \phi\|^2 \leq C(\|\nabla u\|^2 + 1)\|\nabla \Delta \phi\|^2 + C(1 + \|\nabla u\|^2),
\]

where \( C > 0 \) is a constant depending on \( \|u_0\|, \|\phi_0\|_{H^2} \) and coefficients of the system.

**Proof.** Multiplying (1.6) by \(-\Delta^3 \phi\), integrating over \( Q \), we have

\[
\frac{1}{2} \frac{d}{dt} \|\nabla \Delta \phi\|^2 = -\langle (u \cdot \nabla \phi), \nabla \Delta^2 \phi \rangle + \gamma \langle \frac{\delta E}{\delta \phi}, \nabla \Delta^2 \phi \rangle. \tag{3.5}
\]

Using the lower-order uniform estimates in Proposition 2.1, we estimate the first term on the right-hand side of (3.5) as follows

\[
\langle (u \cdot \nabla \phi), \nabla \Delta^2 \phi \rangle \\
\leq \frac{k\gamma \varepsilon}{8} \|\nabla\Delta^2 \phi\|^2 + C\|\nabla u \cdot \nabla \phi\|^2 + C\|u \cdot \nabla^2 \phi\|^2 \]

\[
\leq \frac{k\gamma \varepsilon}{8} \|\nabla\Delta^2 \phi\|^2 + C\|\nabla u\|^2 \|\nabla \phi\|^2_{L^\infty} + C\|u\|^2_{L^3} \|\nabla^2 \phi\|^2_{L^3} \]

\[
\leq \frac{k\gamma \varepsilon}{8} \|\nabla\Delta^2 \phi\|^2 + C\|\nabla u\|^2 \|\nabla \phi\|_{H^1} \|\nabla \phi\|_{H^2} \]

\[
\leq \frac{k\gamma \varepsilon}{8} \|\nabla\Delta^2 \phi\|^2 + C\|\nabla u\|^2 (\|\nabla \phi\|^2 + 1). \]

For the second term, we infer from (2.12) that

\[
\gamma \langle \frac{\delta E}{\delta \phi}, \nabla \Delta^2 \phi \rangle \\
= \gamma \langle \nabla (kg(\phi) + M_1(A(\phi) - \alpha) + M_2(B(\phi) - \beta)f(\phi)), \nabla \Delta^2 \phi \rangle \\
= k\gamma (\nabla \Delta f(\phi), \nabla \Delta^2 \phi) - \frac{k\gamma}{\varepsilon^2} \langle \nabla [(3\phi^2 - 1)f(\phi)], \nabla \Delta^2 \phi \rangle \\
+ M_2 \gamma (B(\phi) - \beta)(\nabla f(\phi), \nabla \Delta^2 \phi) \\
:= I_1 + I_2 + I_3,
\]

where

\[
I_1 \leq -k\gamma \varepsilon \|\nabla \Delta^2 \phi\|^2 + \frac{k\gamma}{\varepsilon} \|\nabla \Delta(\phi^3 - \phi)\| \|\nabla \Delta^2 \phi\| \\
\leq -\frac{7k\gamma \varepsilon}{8} \|\nabla \Delta^2 \phi\|^2 + C\|\nabla \Delta \phi\|^2 + C\|\phi\|^2_{L^\infty} \|\nabla \phi\|^2_{L^3} \|\Delta \phi\|^2_{L^3} + C\|\nabla \phi\|^6_{L^6} \]

\[
\leq -\frac{7k\gamma \varepsilon}{8} \|\nabla \Delta^2 \phi\|^2 + C(\|\nabla \Delta \phi\|^2 + 1),
\]

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Collecting the above estimates together, we arrive at our conclusion (3.4).

Then it follows from (3.4) and the uniform Gronwall lemma [30, Lemma III.1.1] that

where

\[ C \]

Proposition 3.1.

\[ (1.4) \]

The proof is complete. \( \square \)

Based on the higher-order differential inequality (3.4) for \( \phi \), we get

Proposition 3.1. Suppose \( n = 3 \). For any \( u_0 \in \dot{H} \), \( \phi_0 \in H^2_p \), the weak solution to problem (1.4)–(1.8) satisfies

\[ \| \phi(t) \|_{H^3} \leq C \left( 1 + \frac{1}{t} \right) \quad \text{and} \quad \| \nabla \phi(t) \|_{L^\infty} \leq C \left( 1 + \frac{1}{t} \right), \quad \forall t > 0, \quad (3.6) \]

where \( C \) is a constant depending on \( \| u_0 \| , \| \phi_0 \|_{H^2} \) and coefficients of the system. Moreover, if we assume in addition that \( \phi_0 \in H^3_p \), then

\[ \| \phi(t) \|_{H^3} \leq C \quad \text{and} \quad \| \nabla \phi(t) \|_{L^\infty} \leq C, \quad \forall t \geq 0, \quad (3.7) \]

where \( C \) is a constant depending on \( \| u_0 \| , \| \phi_0 \|_{H^3} \) and coefficients of the system.

Proof. We infer from Proposition 2.1 and Lemma 3.1 that for any \( r > 0 \) and \( t \geq 0 \),

\[ \sup_{t \geq 0} \int_{t}^{t+r} \| \nabla \Delta \phi(\tau) \|^2 \, d\tau \leq \sup_{t \geq 0} C \int_{t}^{t+r} \left\| \frac{\delta E}{\delta \phi}(\tau) \right\|^2 \, d\tau + Cr \]

\[ \leq C \int_{0}^{+\infty} \left\| \frac{\delta E}{\delta \phi}(\tau) \right\|^2 \, d\tau + Cr \leq C(1 + r), \quad (3.8) \]

\[ \sup_{t \geq 0} \int_{t}^{t+r} \| \nabla u(\tau) \|^2 \, d\tau \leq \int_{0}^{+\infty} \| \nabla u(\tau) \|^2 \, d\tau \leq C. \quad (3.9) \]

Then it follows from (3.4) and the uniform Gronwall lemma [30, Lemma III.1.1] that

\[ \| \nabla \Delta \phi(t + r) \|^2 \leq C \left( 1 + \frac{1}{r} \right), \quad \forall t \geq 0, \quad (3.10) \]

which yields (3.6). The estimate for \( \| \nabla \phi(t) \|_{L^\infty} \) follows from the continuous embedding \( H^2 \rightarrow L^\infty \) \( (n = 3) \).

If we further assume that \( \phi_0 \in H^3_p \), then by the standard Gronwall inequality, we see that \( \| \nabla \Delta \phi(t) \| \) is also bounded for \( t \in [0, 1] \). This combined with (3.10) yields our conclusion. The proof is complete. \( \square \)
Remark 3.1. We remark that the generic constant $C$ throughout the proof of Lemma 3.2 does not depend on the viscosity $\mu$, thus the uniform bounds for $\|\phi\|_{H^3}$ obtained in Proposition 3.1 is independent of $\mu$.

Define

$$A(t) = \|\nabla u\|^2(t) + \eta \|\delta E\|_{\partial \phi}^2(t),$$

where $\eta > 0$ is a proper constant to be determined later, which might depend on $\|u_0\|$, $\|\phi_0\|_{H^3}$ and coefficients of the system.

Lemma 3.3. Let $n = 3$. For any smooth solution to the problem (1.4) - (1.8), if

$$\|\phi(t)\|_{H^3} + \|\nabla \phi(t)\|_{L^\infty} \leq K, \quad \forall t \geq 0,$$

then for

$$\eta = \frac{\mu \gamma}{16k\varepsilon K^2},$$

the following higher-order energy inequality holds:

$$\frac{d}{dt} A(t) + \mu \|\Delta u\|^2 + k\gamma \eta \|\Delta \delta E\|_{\partial \phi}^2 \leq C_s (A^3(t) + A(t)),$$

where $C_s$ is a constant depending on $\|u_0\|$, $\|\phi_0\|_{H^3}$, $K$ and coefficients of the system.

Proof. By equation (1.4) and the periodic boundary conditions (1.7), we can see that

$$\frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 = -(u_t, \Delta u) = -\mu \|\Delta u\|^2 + (u \cdot \nabla u, \Delta u) - \left( \frac{\delta E}{\partial \phi} \nabla \phi, \Delta u \right).$$

Using the uniform estimates (2.19) and (3.12), the right-hand side of (3.15) can be estimated as follows

$$\left( u \cdot \nabla u, \Delta u \right) \leq \frac{\mu}{16} \|\Delta u\|^2 + C \|u \cdot \nabla u\|^2 \leq \frac{\mu}{16} \|\Delta u\|^2 + C \|u\|_{L^\infty} \|\nabla u\|^2$$

$$\leq \frac{\mu}{16} \|\Delta u\|^2 + C (\|\nabla u\| \|\Delta u\| + \|\nabla u\|^2) \|\nabla u\|^2$$

$$\leq \frac{\mu}{8} \|\Delta u\|^2 + C (\|\nabla u\|^6 + \|\nabla u\|^2),$$

and

$$\left( \frac{\delta E}{\partial \phi} \nabla \phi, \Delta u \right) \leq \frac{\mu}{8} \|\Delta u\|^2 + \frac{2}{\mu} \|\delta E\|_{\partial \phi}^2 \|\nabla \phi\|_{L^\infty} \leq \frac{\mu}{8} \|\Delta u\|^2 + C \|\delta E\|_{\partial \phi}^2.$$
\[ = \sum_{i=1}^{7} J_i. \quad (3.18) \]

The first term \( J_1 \) can be estimated as follows

\[
J_1 = -k\varepsilon \| \frac{\Delta E}{\delta \phi} \|^2 - k\varepsilon \left( \Delta (u \cdot \nabla \phi), \Delta \frac{\delta E}{\delta \phi} \right)
\]

\[
\leq -\frac{7k\varepsilon \gamma}{8} \| \frac{\Delta E}{\delta \phi} \|^2 + \frac{2k\varepsilon}{\gamma} \| \Delta (u \cdot \nabla \phi) \|^2
\]

\[
\leq -\frac{7k\varepsilon \gamma}{8} \| \frac{\Delta E}{\delta \phi} \|^2 + \frac{2k\varepsilon}{\gamma} (\| \Delta u \|^2 + 2 \| \Delta u \cdot \nabla \phi \|^2 + \| \Delta \phi \|^2)
\]

\[
\leq -\frac{7k\varepsilon \gamma}{8} \| \frac{\Delta E}{\delta \phi} \|^2 + \frac{7k\varepsilon R^2}{\gamma} \| \Delta u \|^2 + Ck^2 \| \Delta u \| \| \nabla u \|
\]

\[
\leq -\frac{7k\varepsilon \gamma}{8} \| \frac{\Delta E}{\delta \phi} \|^2 + \frac{4k\varepsilon R^2}{\gamma} \| \Delta u \|^2 + C \| \nabla u \|^2. \quad (3.19)
\]

Then for \( J_2, J_3, J_4 \), a direct computation yields that

\[
J_2 + J_3 + J_4
\]

\[
= -\frac{6k}{\varepsilon} \left( \phi \Delta (\nabla \phi^2 \phi), \frac{\delta E}{\delta \phi} \right) - \frac{3k}{\varepsilon} \left( \phi \delta (\phi^2 \Delta \phi), \frac{\delta E}{\delta \phi} \right) + \frac{k}{\varepsilon} \left( \delta \phi_t, \frac{\delta E}{\delta \phi} \right)
\]

\[
\leq -\frac{6k}{\varepsilon} \| (3\phi^2 - 1) \Delta \phi_t, \frac{\delta E}{\delta \phi} \| + \frac{k}{\varepsilon} \| (3\phi^2 - 2) \delta \phi_t, \frac{\delta E}{\delta \phi} \|
\]

\[
= -\frac{6k}{\varepsilon} \left( \| \nabla \phi \|^2 \phi_t, \frac{\delta E}{\delta \phi} \right) - \frac{12k}{\varepsilon} \left( \phi \delta \phi_t \Delta \phi, \frac{\delta E}{\delta \phi} \right)
\]

\[
+ \frac{k}{\varepsilon} \left( (2(3\phi^2) \delta \phi_t - 6 \nabla \phi^2 \cdot \nabla \phi_t), \frac{\delta E}{\delta \phi} \right)
\]

\[
+ \frac{k}{\varepsilon^3} \left( (15\phi^4 - 12\phi^2 + 1) \delta \phi, \frac{\delta E}{\delta \phi} \right)
\]

\[\quad := J_{2a} + J_{2b} + J_{2c} + J_{2d}.\]

Then we have

\[
J_{2a} \leq C \| \nabla \phi \|_{L\infty} \| \phi_t \| \| \frac{\delta E}{\delta \phi} \| \leq C \| u \cdot \nabla \phi \| \| \frac{\delta E}{\delta \phi} \| + C \| \frac{\delta E}{\delta \phi} \|^2
\]

\[
\leq C \| \frac{\delta E}{\delta \phi} \|^2 + C \| u \|_{L^2} \| \nabla \phi \|^2_{L^3}
\]

\[
\leq C \| \frac{\delta E}{\delta \phi} \|^2 + C \| \nabla u \|^2
\]

\[
J_{2b} \leq C \| \phi \|_{L\infty} \| \phi_t \| \| \Delta \phi \|_{L^6} \| \frac{\delta E}{\delta \phi} \|_{L^3}
\]

\[
\leq C \left( \| u \|_{L^6} \| \nabla \phi \|_{L^3} + \| \frac{\delta E}{\delta \phi} \| \right) \left( \| \Delta \frac{\delta E}{\delta \phi} \| \| \frac{\delta E}{\delta \phi} \| \|^2 + \| \frac{\delta E}{\delta \phi} \|^2 \right)
\]

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The proof is complete.

\[ \eta \]

Now multiplying (3.20) by \( \frac{\mu^2}{16k\varepsilon K^2} \) and adding the result to (3.13), we obtain (3.14).

The proof is complete.
**Theorem 3.1** (Local strong solution). Let \( n = 3 \). For any initial datum \((u_0, \phi_0) \in \tilde{V} \times H^4_p(Q)\), there exists \( T_0 \in (0, +\infty) \) such that problem (1.4)–(1.8) admits a unique strong solution \((u, \phi)\) satisfying

\[
\begin{align*}
  u &\in L^\infty(0, T_0; \tilde{V}) \cap L^2(0, T_0; H^2) ; \\
  \phi &\in L^\infty(0, T_0; H^4_p) \cap L^2(0, T_0; H^6_p) \cap H^1(0, T_0; H^2_p).
\end{align*}
\] (3.21) (3.22)

**Proof.** It follows from Proposition 3.1 that the assumption (3.12) in Lemma 3.3 is satisfied and \( K \) is a constant depending on \( \|u_0\|, \|\phi_0\|_{H^3} \) and coefficients of the system. As a consequence, (3.14) holds with \( \eta \) and \( C_* \) depending on \( \|u_0\|, \|\phi_0\|_{H^3} \) and coefficients of the system. A standard argument in ODE theory yields that there exists a \( T_0 = T_0(\mathcal{A}(0), C_*) \in (0, +\infty) \) such that \( \mathcal{A}(t) \) is bounded on \([0, T_0]\). The bound only depends on \( T_0, \mathcal{A}(0) \) and \( C_* \). This fact together with the lower-order estimates in Proposition 2.1 and the Galerkin scheme similar to that in [8] implies the existence of a local strong solution to problem (1.4)–(1.8) in the time interval \([0, t_0]\). Since \( u \in L^\infty(0, T_0; V) \subset L^8(0, T_0; L^4_p) \), uniqueness of the local strong solution follows from Theorem 2.1. The proof is complete. \( \square \)

In general, we cannot expect existence of global strong solutions to problem (1.4)–(1.8) for arbitrary initial data in \( \tilde{V} \times H^4_p \), due to the difficulty from the convection term in the 3D Navier–Stokes equations. However, if we assume that the viscosity \( \mu \) is properly large, then problem (1.4)–(1.8) will admit a unique global strong solution that is uniformly bounded in \( H^1 \times H^4 \) on \([0, +\infty)\). To verify this point, we first derive an alternative higher-order differential inequality.

**Lemma 3.4.** Let \( n = 3 \). For arbitrary \( \mu_0 > 0 \), if \( \mu \geq \mu_0 > 0 \), and (3.12) is satisfied, then choosing the parameter \( \eta \) in \( \mathcal{A}(t) \) to be

\[
\eta' = \frac{\mu_0 \gamma}{16k \varepsilon K^2},
\] (3.23)

the following inequality holds for the smooth solution \((u, \phi)\) to problem (1.4)–(1.8)

\[
\frac{d}{dt} \mathcal{A}(t) + \left( \mu - \mu^2 \mathcal{A}(t) \right) \| \Delta u \|^2 + k \varepsilon \gamma \eta' \| \Delta E \|_{\phi, \phi}^2 \leq C' \mathcal{A}(t),
\] (3.24)

where \( C' \) is a constant depending on \( \|u_0\|, \|\phi_0\|_{H^2} \), \( K, \mu_0 \) and coefficients of the system but except \( \mu \).

**Proof.** We only need to refine the estimate (3.16) in the proof of Lemma 3.3

\[
(u \cdot \nabla u, \Delta u) \leq \frac{\mu}{8} \| \Delta u \|^2 + \frac{2}{\mu} \| u \|_{L^\infty} \| \nabla u \|^2
\]
\[
\leq \frac{\mu}{8} \| \Delta u \|^2 + \frac{C}{\mu} \left( \| u \|^2 + \| \nabla u \|^2 \right) \| \nabla u \|^2
\]
\[
\leq \frac{\mu}{8} \| \Delta u \|^2 + \frac{\mu}{2} \| \nabla u \| \| \Delta u \| + C \left( \mu^{\frac{1}{2}} + \mu^{-1} \right) \| \nabla u \|^2.
\] (3.25)

Since \( \mu \geq \mu_0 \), we can choose \( \eta \) in \( \mathcal{A}(t) \) to be \( \eta' = \frac{\mu_0 \gamma}{16k \varepsilon K^2} \). Combining (3.25) with estimates for the other terms in the proof of Lemma 3.3, we can easily conclude (3.24) with our choice \( \eta' \). \( \square \)
Theorem 3.2 (Global strong solution under large viscosity). Let $n = 3$. For any initial data $(u_0, d_0) \in \dot{V} \times H^1_p$, if $\mu$ is sufficiently large such that (3.26) is satisfied, then the problem (1.4)–(1.8) admits a unique global solution.

Proof. We infer from (2.20) and the choice of $\eta'$ in Lemma 3.4 that
\[
\sup_{t \geq 0} \int_t^{t+1} A(\tau) d\tau \leq \int_0^{+\infty} A(t) dt \leq M,
\]
where $M$ is a constant depending on $\|u_0\|$, $\|\phi_0\|_{H^3}$, $\mu_0$ and coefficients of the system but except $\mu$. If the viscosity $\mu$ satisfies the following relation
\[
\mu^{\frac{3}{2}} \geq A(0) + C'M + 4M + \mu_0^{\frac{1}{2}},
\]
by applying the same idea as in [21, Section 4], we can deduce from (3.24) that $A(t)$ is uniformly bounded such that
\[
A(t) \leq \mu^{\frac{1}{2}}, \quad \forall \ t \geq 0.
\]
Based on the uniform-in-time estimates and the Galerkin scheme, we are able to prove the existence and uniqueness of a global strong solution to problem (1.4)–(1.8). We leave the details to interested readers.

4 Regularity criteria

In this section, we are going to establish some regularity criteria for solutions to problem (1.4)–(1.8) in the three dimensional case. These criteria only involve the velocity field, which indicate that in spite of the nonlinear coupling between the equations for velocity field and the phase function, the velocity field indeed plays a dominant role in regularity for solutions to system (1.4)–(1.6), just like the decoupled incompressible Navier–Stokes equations.

First, we provide a result on regularity criteria in terms of the velocity $u$ [27] or its gradient $\nabla u$ [2].

Theorem 4.1. Suppose $n = 3$. For $(u_0, \phi_0) \in \dot{V} \times H^1_p$, let $(u(t), \phi(t))$ be a local smooth solution to the problem (1.4)–(1.8) on $[0,T)$ for some $0 < T < +\infty$. Suppose that one of the following conditions holds,

(i) $\int_0^T \|\nabla u(t)\|_{L^p}^s dt < +\infty$, for $\frac{3}{p} + \frac{2}{s} \leq 2, \quad \frac{3}{2} < p \leq +\infty$,

(ii) $\int_0^T \|u(t)\|_{L^p}^s dt < +\infty$, for $\frac{3}{p} + \frac{2}{s} \leq 1, \quad 3 < p \leq +\infty$.

Then $(u(t), \phi(t))$ can be extended beyond $T$.

Proof. We keep in mind that uniform estimates (2.19) and (3.7) still hold for $t \geq 0$.

Suppose that (i) is satisfied. For $p > \frac{3}{2}$, we estimate (3.16) as follows
\[
(u \cdot \nabla u, \Delta u) = -\int_Q \nabla_k u_j \nabla_j u_i \nabla_k u_i dx - \int_Q u_j \nabla_j (\nabla_k u_i) \nabla_k u_i dx
\]
\[
\begin{align*}
\text{Proof. We recall that uniform estimates (2.19) and (3.7) still hold for}\,
& (\text{then multiplying the resultant with } \Delta)
\int_0^T \nabla u(t) \nabla u(t)\,dx
\leq C \|\nabla u\|_{L^p} \|\nabla u\|_{L^{p-1}}^2
\leq C \|\nabla u\|_{L^p} \left(\|\nabla u\|_{L^{p-1}}^2 \|\Delta u\|_{L^p} + \|\nabla u\|_{L^p}^2 \right)
\leq \frac{\mu}{8} \|\Delta u\|_{L^p}^2 + C \left(\|\nabla u\|_{L^p} + \|\nabla u\|_{L^{p-1}}^2\right) \|\nabla u\|_{L^p}^2. \tag{4.1}
\end{align*}
\]

Combining (4.1) with the other estimates in the proof of Lemma 3.3, we obtain that
\[
\frac{d}{dt} A(t) + \mu \|\Delta u\|_{L^p}^2 + k \gamma \eta \left(\Delta \frac{\delta E}{\delta \phi}\right)^2 \leq C \left(1 + \|\nabla u\|_{L^p}^\frac{2p}{p-1}\right) A(t). \tag{4.2}
\]

Then by the Gronwall inequality, we see that \(A(t) \leq C_T\) for \(t \in [0, T]\), which implies that the \(H^1 \times H^4\) norm of the strong solution \((u, \phi)\) is bounded on interval \([0, T]\). This yields that \([0, T]\) cannot be the maximal interval of existence, and the solution \((u, \phi)\) can be extended beyond \(T\).

Next, we suppose that (ii) is satisfied. For \(p > 3\), we estimate (3.16) in an alternative way such that
\[
\begin{align*}
(u \cdot \nabla u, \Delta u) &\leq C \|u\|_{L^p} \|\nabla u\|_{L^{p-1}} |\Delta u| \leq C \|u\|_{L^p} \|\Delta u\| \left(\|\nabla u\|_{L^p}^\frac{2p}{p-1} + \|\nabla u\|_{L^p}^\frac{2p}{p-3} + \|\nabla u\|_{L^{p-1}}^2\right)
\leq \frac{\mu}{8} \|\Delta u\|_{L^p}^2 + C \left(\|\nabla u\|_{L^p}^{\frac{2p}{p-1}} + 1\right) \|\nabla u\|_{L^p}^2. \tag{4.3}
\end{align*}
\]

then by the Gronwall inequality, \(A(t) \leq C_T\) for \(t \in [0, T]\), which again yields our conclusion. The proof is complete. \(\square\)

As for the conventional Navier–Stokes equations (see, for instance, \[35, 36\]), we can improve the results in Theorem 4.1 and obtain some logarithmical-type regularity criteria for our phase-field Navier–Stokes system (1.4)–(1.8).

**Theorem 4.2.** Suppose \(n = 3\). For \((u_0, \phi_0) \in (\nabla \cap H^3_{p}) \times H^5_p\), let \((u, \phi)\) be a local smooth solution to problem (1.4)–(1.8) on \([0, T]\) for some \(0 < T < +\infty\). If one of the following conditions holds,

\[
\begin{align*}
(i) &\quad \int_0^T \frac{\|\nabla u(t)\|_{L^p}^s}{1 + \ln(e + \|\nabla u(t)\|_{L^p})} dt < +\infty, \quad \text{for } \frac{3}{p} + \frac{2}{s} \leq 2, \quad \frac{3}{2} \leq p \leq 6, \quad \tag{4.4}
(ii) &\quad \int_0^T \frac{\|u(t)\|_{L^p}^{s}}{1 + \ln(e + \|u(t)\|_{L^\infty})} dt < +\infty, \quad \text{for } \frac{3}{p} + \frac{2}{s} \leq 1, \quad 3 < p \leq +\infty, \quad \tag{4.5}
\end{align*}
\]

then \((u, \phi)\) can be extended beyond \(T\).

**Proof.** We recall that uniform estimates (2.11) and (3.7) still hold for \(t \geq 0\).

**Case (i).** Suppose that (4.4) is satisfied. Applying \(\Delta\) to both sides of equation (1.4), multiplying the resultant with \(\Delta u\) and integrating over \(Q\), we get

\[
\frac{1}{2} \frac{d}{dt} \|\Delta u\|_{L^p}^2 + \mu \|\nabla u\|_{L^p}^2 = (\nabla (u \cdot \nabla u), \nabla \Delta u) - \left(\nabla \left(\frac{\delta E}{\delta \phi} \nabla \phi\right), \nabla \Delta u\right) = I_1 + I_2, \quad \tag{4.6}
\]
where

\[ I_1 \leq C \|
\n\frac{\delta E}{\delta \phi} \| ^2 + \| \nabla \Delta \| ^2 + \|
\n\left( \frac{\partial}{\partial t} \int \frac{\delta E}{\delta \phi} \right) \cdot \left( \frac{\delta E}{\delta \phi} \right) \]
For the sake of simplicity, we denote
\[
\delta E = \frac{\delta E_1}{\delta \phi}, \quad \delta \phi = \frac{\delta \phi_1}{\delta \phi},
\]
Collecting the above estimates together, we deduce that
\[
\text{Set } \sum \leq C(\|u\| \|\nabla \Delta u\| + \|u\|^2) \left( \|\delta E\|_{L^2} \right)^2 + C \|\nabla u\|^2 \|\nabla \Delta u\|^2
\]
\[
\leq \frac{1}{3} \|\nabla \Delta u\|^2 + C \left( \|\delta E\|^4 + \|\delta \phi\|^2 + \|\nabla u\|^2 \right).
\]
Summing up, we get
\[
J_1' \leq -\frac{7k \epsilon \gamma}{8} \|\nabla \Delta \delta E\|^2 + (1 + C_1) \|\nabla \Delta u\|^2 + C \left( \|\delta E\|^4 + \|\delta \phi\|^2 + \|\nabla u\|^2 \right),
\]
where \(C_1\) is a constant depending on \(\|u_0\|, \| \phi_0 \|_{H^3}\) and coefficients of the system due to estimates (2.19) and (3.7).

The remaining terms \(J_2', \ldots, J_6'\) can be estimated as for \(J_2, \ldots, J_7\) in the proof of Lemma 3.3 by using a similar argument with minor modifications (replacing \(\delta E\) in \(J_2, \ldots, J_7\) by \(\Delta \frac{\delta E}{\delta \phi}\)). Consequently,
\[
\sum_{i=2}^{6} J_i' \leq \frac{k \epsilon \gamma}{4} \|\nabla \Delta \delta E\|^2 + C \|\nabla u\|^2 + C \left( \|\delta E\|^2 + \|\delta \phi\| \right)^2.
\]
Set
\[
\eta_1 = \frac{\mu}{4(1 + C_1)}.
\]
Now we turn to estimate \(I_2\):
\[
I_2 \leq \frac{\mu}{8} \|\nabla \Delta u\|^2 + \|\nabla \phi\|_{L^\infty} \left( \|\nabla \delta E\|_{L^p} + \|\Delta \phi\|_{L^p} \right) \left( \|\delta E\|_{L^p} \right)^2
\]
\[
\leq \frac{\mu}{8} \|\nabla \Delta u\|^2 + C \left( \|\delta E\|^4 + \|\delta \phi\|^2 \right) + \|\nabla \Delta \delta E\|^2 + \|\delta \phi\| \left( \|\delta E\| \right)^2.
\]
Collecting the above estimates together, we deduce that
\[
\frac{d}{dt} \left( \|\Delta u\|^2 + \|\nabla \phi\|_{L^\infty} \right) \left( \|\nabla \delta E\|_{L^p} \right)^2 + \frac{k \epsilon \gamma}{2} \|\nabla \Delta \delta E\|^2 + \|\nabla \Delta u\|^2
\]
\[
\leq C \left( \|\delta E\|^4 + \|\delta \phi\|^2 + \|\nabla u\|^2 \right) \left( \|\nabla \Delta u\|^2 \right)\text{.}
\]
For the sake of simplicity, we denote
\[
Q(t) = \frac{1 + \|\nabla u(t)\|_{L^p}^{2p}}{1 + \ln(e + \|\nabla u(t)\|_{L^p})}.
\]
For \(\frac{t}{2} < p \leq 6\) we infer from (4.2) that
\[
\frac{d}{dt} A(t) \leq C \left( 1 + \|\nabla u(t)\|_{L^p}^{2p} \right) A(t)
\]
\[
\leq C \cdot Q(t) \left[ 1 + \ln(e + \|\Delta u(t)\|) \right] A(t),
\]
(4.9)
where \( C_* \) is a constant depending on \( \|u_0\|, \|\phi_0\|_{H^3} \) and coefficients of the system.

Because of (4.3), we denote \( \int_0^T Q(t) dt = M < +\infty \). Fix \( \epsilon \in \left( 0, \frac{1}{3M} \right) \). Then there exist
\( 0 = \hat{t}_0 < \hat{t}_1 < \ldots < \hat{t}_{N-1} < \hat{t}_N = T \) such that
\[
\int_{\hat{t}_{i-1}}^{\hat{t}_i} Q(t) dt \leq \frac{\epsilon}{2}, \quad \forall i = 1, 2, \ldots, N, \quad \text{with } N = \left\lceil \frac{2M}{\epsilon} \right\rceil + 1.
\]

Set \( t_0 = \hat{t}_0 = 0 \) and \( t_{N+1} = \hat{t}_N = T \). It follows from our assumption on the initial data that \( \mathcal{A}(0) < +\infty \). Due to (2.20), for each \( i = 1, 2, \ldots, N \), there exists \( t_i \in (\hat{t}_{i-1}, \hat{t}_i) \) such that \( \mathcal{A}(t_i) < +\infty \). Moreover,
\[
\int_{t_i}^{t_{i+1}} Q(t) dt \leq \int_{t_i}^{\hat{t}_{i+1}} Q(t) dt \leq \frac{\epsilon}{2} \leq \epsilon, \quad i = 0, N, \quad \text{(4.10)}
\]
\[
\int_{t_i}^{t_{i+1}} Q(t) dt \leq \int_{t_{i-1}}^{t_i} Q(t) dt + \int_{t_i}^{t_{i+1}} Q(t) dt \leq \epsilon, \quad \text{for } i = 1, 2, \ldots, N - 1. \quad \text{(4.11)}
\]

We can prove the required result by an iteration argument from \( i = 0 \) to \( i = N \). For \( i = 0 \), it follows from the Gronwall inequality and (4.10) that
\[
\mathcal{A}(t) \leq \mathcal{A}(0) \exp \left( C_* \left[ 1 + \ln \left( e + \sup_{[0,t]} \|\Delta u(\cdot)\| \right) \right] \int_0^t Q(s) ds \right) \leq \mathcal{A}(0) e^{C_* \epsilon \left( e + \sup_{[0,t]} \|\Delta u(\cdot)\| \right)} C_* \epsilon, \quad \forall t \in [0, t_1]. \quad \text{(4.12)}
\]

We infer from (1.8) and (4.12) that for \( t \in [0, t_1] \), it holds
\[
\frac{d}{dt} \left( \|\Delta u\|^2(t) + \eta_2 \left\| \frac{\nabla \Delta E}{\Delta \phi} \right\|^2(t) \right) \leq C \left( e + \sup_{[0,t]} \|\Delta u(\cdot)\| \right). \quad \text{(4.13)}
\]

Since \( \|\Delta u(0)\| \) and \( \left\| \frac{\nabla \Delta E}{\Delta \phi} \right\| (0) \) are bounded due to our assumption on the initial data, integrating (4.13) from \( 0 \) to \( t \), we get
\[
\|\Delta u(t)\|^2 + \eta_1 \left\| \frac{\nabla \Delta E}{\Delta \phi} \right\|^2(t) \leq \|\Delta u(0)\|^2 + \eta_1 \left\| \frac{\nabla \Delta E}{\Delta \phi} \right\|^2(0) + CT \left( e + \sup_{[0,t]} \|\Delta u(\cdot)\| \right), \quad \forall t \in [0, t_1].
\]

Then taking the supremum of both sides for \( t \in [0, t_1] \), we can see that
\[
\sup_{[0,t_1]} \left( \|\Delta u(\cdot)\|^2 + \eta_1 \left\| \frac{\nabla \Delta E}{\Delta \phi} \right\|^2 \right) \leq \|\Delta u(0)\|^2 + \eta_1 \left\| \frac{\nabla \Delta E}{\Delta \phi} \right\|^2(0) + CT \left( e + \sup_{[0,t_1]} \|\Delta u(\cdot)\| \right) \leq \|\Delta u(0)\|^2 + \frac{1}{2} \sup_{[0,t_1]} \|\Delta u(\cdot)\|^2 + C_T, \quad \text{(4.14)}
\]

which indicates that \( \|\Delta u\| \) and \( \left\| \frac{\nabla \Delta E}{\Delta \phi} \right\| \) are uniformly bounded on \( [0, t_1] \).

Then we can repeat the above argument for \( i = 1, \ldots, N \) such that on each interval \([t_i, t_{i+1}]\), it holds
\[
\sup_{[t_i, t_{i+1}]} \left( \|\Delta u(\cdot)\|^2 + \eta_1 \left\| \frac{\nabla \Delta E}{\Delta \phi} \right\|^2 \right) \leq \|\Delta u(t_i)\|^2 + \eta_1 \left\| \frac{\nabla \Delta E}{\Delta \phi} \right\|^2(t_i) + C_T, \quad \text{(4.15)}
\]
where the bound of \( \|\Delta u(t_i)\|, \left\| \nabla \frac{\delta E}{\delta \phi} \right\|(t_i) \) are given by the estimates in the previous step on \([t_{i-1}, t_i]\). As a consequence, we get
\[
\|u\|_{L^\infty(0,T;H^2)} \leq C; \quad \|\phi\|_{L^\infty(0,T;H^2)} \leq C;
\]
which indicate that \([0,T]\) cannot be the maximal interval of existence, and the solution \((u,\phi)\) can be extended beyond \(T\).

**Case (ii).** We re-estimate the terms \(I_1\) and \(J'_1\) in a different way by using the uniform estimates \((2.19)\) and \((3.7)\). The estimate for \(I_1\) can be done as follows:
\[
I_1 \leq C\|u\|_{L^\infty(0,T;H^4)} \leq C\|u\|_{L^\infty(0,T;H^5)} \|\phi\|_{H^5} \|\phi\|_{H^3} 
\]
\[
\leq C(\|\phi\|_{L^\infty(0,T;H^5)} + \|u\|^2) \left( \left\| \nabla \frac{\delta E}{\delta \phi} \right\| + C \right) 
\]
\[
\leq \frac{\mu}{8} \|\nabla \Delta u\|^2 + C \left( \left\| \nabla \frac{\delta E}{\delta \phi} \right\|^2 + \|\nabla u\|^2 \right). 
\]

The other terms in \(I_2, J'_1, ..., J'_6\) are estimated in the same way as in Case (i). Then we deduce that
\[
\frac{d}{dt} \left( \|\Delta u\|^2 + \eta_1 \left\| \nabla \frac{\delta E}{\delta \phi} \right\|^2 \right) + \frac{k\xi \eta_1}{2} \|\nabla \Delta u\|^2 + \mu \|\nabla \Delta u\|^2 
\]
\[
\leq C\|u\|_{L^\infty(0,T;H^4)} \|\Delta u\|^2 + C \left( \left\| \nabla \frac{\delta E}{\delta \phi} \right\|^2 + \left\| \frac{\delta E}{\delta \phi} \right\|^2 + \|\nabla u\|^2 \right) 
\]
\[
\leq C \left( 1 + \|u\|_{L^\infty(0,T;H^4)} \right) \left( e + \|\Delta u\|^2 + \eta_1 \left\| \nabla \frac{\delta E}{\delta \phi} \right\|^2 \right) 
\]
\[
\leq C \frac{1 + \|u\|_{L^\infty(0,T;H^4)}^2}{1 + \ln(e + \|u\|_{L^\infty(0,T;H^4)})} \left[ 1 + \ln(e + \|\Delta u\|^2) \right] \left( e + \|\Delta u\|^2 + \eta_1 \left\| \nabla \frac{\delta E}{\delta \phi} \right\|^2 \right), \quad (4.16)
\]
where we have used the Poincaré inequality
\[
\left\| \frac{\delta E}{\delta \phi} \right\|^2 \leq C \left\| \nabla \frac{\delta E}{\delta \phi} \right\|^2 + C \int_Q \frac{\delta E}{\delta \phi} dx \leq C \left\| \nabla \frac{\delta E}{\delta \phi} \right\|^2 + C(\|\phi\|_{H^2}).
\]
We infer from \((4.16)\) and the Gronwall inequality that for all \(t \in [0,T]\),
\[
\ln \left( 1 + \ln \left( e + \|\Delta u(t)\|^2 + \eta_1 \left\| \nabla \frac{\delta E}{\delta \phi} \right\|^2 (t) \right) \right) 
\]
\[
\leq C \ln \left( 1 + \ln \left( e + \|\Delta u_0\|^2 + \eta_1 \left\| \nabla \frac{\delta E}{\delta \phi} \right\|^2 (0) \right) \right) \int_0^T \frac{1 + \|u(t)\|^2_{L^\infty}}{1 + \ln(e + \|u(t)\|_{L^\infty})} dt,
\]
which together with \((4.15)\) implies that
\[
\|u\|_{L^\infty(0,T;H^2)} \leq C; \quad \|\phi\|_{L^\infty(0,T;H^2)} \leq C.
\]
The proof is complete.
5 Stability

Denote the total energy of the system (1.4)–(1.8) by

\[ \mathcal{E}(t) = \frac{1}{2} \|u(t)\|^2 + E(\phi(t)). \]

We recall that \( \mathcal{E}(t) \) satisfies the basic energy law (2.16), which characterizes the dissipative nature of the problem (1.4)–(1.8). Inspired by [21] for the liquid crystal system, we can show that if the initial datum is regular and the total energy \( \mathcal{E}(t) \) cannot drop too much for all time, then our problem (1.4)–(1.8) admits a unique bounded global strong solution.

**Proposition 5.1.** Let \( n = 3 \). For any initial data \((u_0, \phi_0) \in \mathring{V} \times H^4_p \), there exists a constant \( \varepsilon_0 \in (0, 1) \), depending on \( \|u_0\|_{H^1}, \|\phi_0\|_{H^4} \) and coefficients of the system such that either

1. The problem (1.4)–(1.8) has a unique global strong solution \((u, \phi)\) with uniform-in-time estimate

\[ \|u(t)\|_{V} + \|\phi(t)\|_{H^4} \leq C, \quad \forall t \geq 0, \]

or

2. there is a \( T_* \in (0, +\infty) \) such that \( \mathcal{E}(T_*) \leq \mathcal{E}(0) - \varepsilon_0 \).

**Proof.** The proof is based on the higher-order differential inequality (3.14) (cf. Lemma 3.3) and an argument similar to that in [21, 23]. We only sketch it here for convenience of the readers. For any initial data \((u_0, \phi_0) \in \mathring{V} \times H^4_p \), let \( L \) be a constant such that \( \|\nabla u_0\|^2 + \left\| \frac{\delta \mathcal{E}}{\delta \phi_0}(0) \right\|^2 \leq L \). It follows from Lemma 3.1 that \( \|\phi_0\|_{H^4} \) can be bounded in terms of \( L \) and \( \|\phi_0\|_{H^2} \). Then by Propositions 2.1 and (3.1), \( \|u(t)\| \) and \( \|\phi(t)\|_{H^4} \) can be bounded by a constant depending \( L \), \( \|\phi_0\|_{H^2} \) and coefficients of the system. Then we can fix the constant \( \eta \) in the definition of \( \mathcal{A}(t) \) (cf. (3.13)) and \( C_\star \) in (3.14). Consider the ODE problem

\[ \frac{d}{dt} Y(t) = C_\star \{Y(t)^3 + Y(t)\}, \quad Y(0) = \max\{1, \eta\} L \geq \mathcal{A}(0) \]

The maximal existence time \( T_{\text{max}} \) of the unique local solution \( Y(t) \) is determined by \( Y(0) \) and \( C_\star \). Now we take

\[ t_0 = \frac{1}{2} T_{\text{max}}(Y(0), C_\star), \quad \varepsilon_0 = \frac{LT_0}{2} \min \{\mu, \gamma\}. \]

If (ii) is not true, we have \( \mathcal{E}(t) \geq \mathcal{E}(0) - \varepsilon_0 \) for all \( t \geq 0 \). From the basic energy law (2.16), we infer that

\[ \int_{0}^{t_0} \mathcal{A}(t) dt \leq \int_{0}^{\infty} \mathcal{A}(t) dt \leq \kappa \varepsilon_0, \quad \text{with} \quad \kappa = \max\{1, \eta\} \max\{\mu^{-1}, \gamma^{-1}\}. \]

Hence, there exists a \( t_* \in [\frac{t_0}{2}, t_0] \) such that \( \mathcal{A}(t^*) \leq \frac{2\varepsilon_0}{\kappa} \leq Y(0) \). Take \( t_* \) as the initial time for (5.2) with \( Y(t^*) = Y(0) \), we infer from the above argument that \( Y(t) \) and thus \( \mathcal{A}(t) \) remain bounded at least on \([0, \frac{3t_0}{2}] \subset [0, t_* + t_0] \) with the same bound as that on \([0, t_0] \) (since the bound for \( Y(t) \) is the same). An iteration argument shows that \( \mathcal{A}(t) \) is bounded for all \( t \geq 0 \). The proof is complete.

**Corollary 5.1** (Eventual regularity of weak solutions in 3D). When \( n = 3 \), let \((u, \phi)\) be a global weak solution of the problem (1.4)–(1.8). Then there exists a time \( T_0 \in (0, +\infty) \) such that \((u, \phi)\) becomes a strong solution on \([T_0, +\infty) \).
Proof. It follows from (2.19), (2.20) and Lemma 3.1 that there exist a time $T_1 > 0$ such that $\|u(T_1)\|, \| \phi(T_1) \|_{H^4}$ are bounded. Taking $T_1$ as the initial time, we can fix $L$ in Proposition 5.1 and thus $\varepsilon_0$. (2.20) yields that there exists a $T_0 > T_1$ such that

$$\| \nabla u(T_0) \|^2 + \| \frac{\delta E}{\delta \phi}(T_0) \|^2 \leq L, \quad \int_{T_0}^{T_1} \left( \mu \| \nabla u(t) \|^2 + \gamma \| \frac{\delta E}{\delta \phi} \|^2 \right) dt \leq \varepsilon_0.$$ 

Taking $T_0$ as the initial time, we can apply the argument for Proposition 5.1 that $(u, \phi)$ will be bounded in $H^1 \times H^4$ after $T_0$.

Definition 5.1. We say $\phi^* \in H^2_p$ is a local minimizer of the elastic energy $E(\phi)$, if there exists a $\delta > 0$, $E(\phi^*) \leq E(\phi)$ for all $\phi \in H^2_p$ satisfying $\| \phi - \phi^* \|_{H^2} < \delta$. If for all $\phi \in H^2_p$, $E(\phi^*) \leq E(\phi)$, then $\phi^*$ is an absolute minimizer.

Lemma 5.1. Let $B$ be a bounded closed convex subset of $H^2_p$. The approximate elastic energy $E(\phi)$ admits at least one minimizer $\phi^* \in B$ such that $E(\phi^*) = \inf_{\phi \in B} E(\phi)$.

Proof. Since $E(\phi) \geq 0$ for all $\phi \in B$ and $\lim \phi \in B, \| \phi \|_{H^2} \to +\infty E(\phi) = +\infty$, $E(\phi)$ has a bounded minimizing sequence $\phi_n \in B$ such that

$$E(\phi_n) \to \inf_{\phi \in B} E(\phi).$$  

(5.3)

Recalling the definition of $E(\phi)$ (1.3), we can rewrite $E$ in the following form:

$$E(\phi) = \frac{k \varepsilon}{2} \| \Delta \phi \|^2 + F(\phi)$$

with

$$F(\phi) = \frac{k}{\varepsilon} \int_Q \phi \cdot \nabla (\phi^3 - \phi) dx + \frac{k}{2 \varepsilon^3} \int_Q (\phi^3 - \phi)^2 dx + \frac{1}{2} M_1 (A(\phi) - \alpha)^2 + \frac{1}{2} M_2 (B(\phi) - \beta)^2.$$ 

Since $\phi_n$ is bounded in $H^2$, there is a sequence, still denoted by $\phi_n$, such that $\phi_n$ weakly converges to a certain function $\phi^*$ in $H^2$. We infer from the compact Sobolev embedding theorem $(n = 3)$ that $\phi_n$ strongly converges to $\phi^*$ in $L^\infty$ and $H^1$. It turns out that $F(\phi_n) \to F(\phi^*)$. Since $\| \Delta \phi \|^2$ is weakly lower semi-continuous, it follows from (5.3) that $E(\phi^*) = \inf_{\phi \in B} E(\phi)$. Using the elliptic estimate and a bootstrap argument, we see that the minimizer $\phi^*$ is in fact smooth.

Remark 5.1. If $\phi$ is a minimizer of $E(\phi)$, then it is a critical point of $E(\phi)$. It is easy to verify that any critical point of $E(\phi)$ in $H^2_p$ is equivalent to a weak solution to the forth-order nonlocal elliptic problem

$$\frac{\delta E}{\delta \phi} = 0, \quad \text{with } \phi(x + \varepsilon_i) = \phi(x).$$  

(5.4)

In order to prove our stability result, we recall the following Lojasiewicz–Simon type inequality whose proof is postponed to the next section.

Lemma 5.2. Suppose $n = 3$. Let $\psi$ be a critical point of the elastic energy $E$. There exist constants $\beta > 0, \theta \in (0, \frac{1}{2})$ depending on $\psi$ such that for any $\phi \in H^4_p(Q)$ with $\| \phi - \psi \|_{H^2} < \beta$, it holds

$$\| \frac{\delta E}{\delta \phi} \| \geq |E(\phi) - E(\psi)|^{1-\theta}.$$  

(5.5)
Now we state the main result of this section.

**Theorem 5.1.** Let $\phi^* \in H^4_p(Q)$ be a local minimizer of $E(\phi)$. For any $R > 0$, consider the initial data

$$(u_0, \phi_0) \in B = \{(u, \phi) \in \hat{V} \times H^4_p(Q) : \|u\|_{H^1} \leq R, \|\phi - \phi^*\|_{H^4} \leq R\}.$$  

For any $\epsilon > 0$, there exists a constant $\sigma \in (0, \delta)$ that may depend on $\phi^*$, $R$, $\epsilon$ and coefficients of the system such that if the initial data $(u_0, \phi_0) \in B$ satisfies the condition

$$\|u_0\| + \|\phi_0 - \phi^*\|_{H^2} \leq \sigma,$$  

then problem (1.1)-(1.3) admits a unique global strong solution satisfying

$$\|\phi(t) - \phi^*\|_{H^2} \leq \epsilon, \quad \forall t \geq 0.$$  

**Proof.** If $\|u_0\|_{H^1} \leq R$ and $\|\phi_0 - \phi^*\|_{H^4} \leq R$, then the constant $\epsilon_0$ in Proposition 5.1 depends on $\phi^*$, $R$ and coefficients of the system. It follows from Proposition 4.1 and Proposition 3.1 that $\|u(t)\|$ and $\|\phi(t)\|_{H^2}$ are uniformly bounded (by a constant depending on $\phi^*$, $R$ and coefficients of the system). In what follows we denote by $C$, $C_i$ genetic constants that only depend on $R$, $\phi^*$ and coefficients of the system.

By a direction computation, we get

$$E(\phi_0) - E(\phi(t)) = [E_\epsilon(\phi_0) - E_\epsilon(\phi(t))] + \frac{1}{2} M_1 [(A(\phi_0) - \alpha)^2 - (A(\phi) - \alpha)^2]$$

$$+ \frac{1}{2} M_2 [(B(\phi_0) - \beta)^2 - (B(\phi) - \beta)^2]$$

$$:= F_1 + F_2 + F_3,$$

where

$$F_1 \leq C(\|\Delta \phi_0 - \Delta \phi\| + \|(\phi_0 - \phi)(\phi_0^2 + \phi^2 + \phi_0 \phi + 1)\|)$$

$$\leq C(\|\Delta \phi_0 - \Delta \phi\| + \|\phi_0 - \phi\|_2^2 + \phi_0^2 + \phi_0 \phi + 1\|_{L^\infty})$$

$$\leq C\|\phi_0 - \phi\|_{H^2}, \quad (5.8)$$

$$F_2 \leq C\|\phi_0 - \phi\|_{L^1}(\|\phi_0 + \phi\|_{L^1} + 2|\alpha|) \leq C\|\phi_0 - \phi\|, \quad (5.9)$$

$$F_3 \leq C\|\nabla \phi_0 + \nabla \phi\|_2 \|\nabla \phi_0 - \nabla \phi\| + C\|\phi_0 + \phi\|_2^2 + \phi_0^2 + \phi^2 - 2\|L^\infty\|_2 \|\phi_0 - \phi\|$$

$$\leq C\|\phi_0 - \phi\|_{H^1}. \quad (5.10)$$

Since the total energy $E$ is decreasing in time, we infer from the above estimate that

$$0 \leq E(0) - E(t) = \frac{1}{2}\|u_0\|^2 - \frac{1}{2}\|u(t)\|^2 + E(\phi_0) - E(\phi(t))$$

$$\leq \frac{1}{2}\|u_0\|^2 + E(\phi_0) - E(\phi(t))$$

$$\leq \frac{1}{2}\|u_0\|^2 + C_1\|\phi(t) - \phi_0\|_{H^2}. \quad (5.11)$$

Let $\beta$ denote the constant in Lemma 5.2 that depends only on $\psi = \phi^*$. Denote

$$w = \min \left\{1, \epsilon_0^{\frac{1}{2}}, \frac{\delta}{2}, \frac{\beta}{2}, \frac{3\epsilon_0}{4C_1}\right\}.$$
We assume that \( \sigma \leq \frac{1}{4} \varpi \). Let \( \tilde{T} \) be the smallest finite time for which \( \| \phi(\tilde{T}) - \phi^* \|_{H^2} \geq \varpi \). Then by the proof of Proposition 5.1, the problem admits a bounded strong solution on \([0, \tilde{T})\). If there exists \( t_* \in (0, \tilde{T}) \) such that \( \mathcal{E}(t_*) = E(\phi^*) \), since \( \phi^* \) is the local minimizer and \( \| \phi(t_*) - \phi^* \|_{H^2} < \varpi < \delta \), we deduce from \((5.14)\) that \( \| \nabla (u(t)) \| = \left\| \frac{\delta E}{\delta \phi}(t) \right\| = 0 \) for \( t \geq t_* \). It follows from

\[
\| \phi_t \| \leq \| u \cdot \nabla \phi \| + \gamma \left( \frac{\delta E}{\delta \phi} \right) \leq C \| \nabla u \| \| \nabla \phi \|_{L^2} + \gamma \left( \frac{\delta E}{\delta \phi} \right)
\]

that for \( t \geq t_* \), \( \| \phi_t \| = 0 \). Namely, \( \phi \) is independent of time for \( t \geq t_* \). As a result, \( u(t_*) = 0 \) and \( \phi(t_*) = \phi^* \), where \( \phi^* \) is also a local minimizer (but possibly different from \( \phi^* \)). Due to the uniqueness of strong solution, the evolution starting from \( t_* \) will be stationary. The proof is complete in this case.

We proceed to work with the case that \( \mathcal{E}(t) > E(\phi^*) \) for all \( t \in [0, \tilde{T}) \). From the definition of \( \tilde{T} \), we see that the conditions in Lemma 5.2 are satisfied with \( \psi = \phi^* \), on the interval \([0, \tilde{T})\). Consequently,

\[
- \frac{d}{dt} (\mathcal{E}(t) - E(\phi^*))^\theta \geq \frac{\mu \| \nabla u \|^2 + \gamma \left( \frac{\delta E}{\delta \phi} \right)^2}{\frac{1}{2} \| u \|^{2(1 - \theta)} + \left( \frac{\delta E}{\delta \phi} \right)} \geq C \left( \| \nabla u \| + \left\| \frac{\delta E}{\delta \phi} \right\| \right), \quad \forall \ t \in [0, \tilde{T}). \quad (5.13)
\]

We infer from \((5.12)\) that

\[
\int_0^{\tilde{T}} \| \phi_t(t) \| dt \leq C (E(0) - E(\phi^*))^\theta \leq C (\| u_0 \|^{2\theta} + |E(\phi_0) - E(\phi^*)|)^\theta \leq C \left( \| u_0 \|^{2\theta} + \| \phi_0 - \phi^* \|_{H^2}^\theta \right),
\]

which implies that

\[
\| \phi(\tilde{T}) - \phi^* \|_{H^2} \leq \| \phi(\tilde{T}) - \phi_0 \|_{H^2} + \| \phi_0 - \phi^* \|_{H^2}
\]

\[
\leq C \| \phi(\tilde{T}) - \phi_0 \|_{H^2} \| \phi(\tilde{T}) - \phi_0 \|_{H^2}^{\frac{1}{4}} + \| \phi_0 - \phi^* \|_{H^2}
\]

\[
\leq C \left( \int_0^{\tilde{T}} \| \phi_t(t) \| dt \right)^{\frac{1}{4}} + \| \phi_0 - \phi^* \|_{H^2}
\]

\[
\leq C_2 \left( \| u_0 \|^{2\theta} + \| \phi_0 - \phi^* \|_{H^2}^\theta \right) + \| \phi_0 - \phi^* \|_{H^2}. \quad (5.14)
\]

Taking

\[
\sigma \leq \min \left\{ \frac{\varpi}{4}, \left( \frac{\varpi}{4C_2} \right)^{\frac{1}{3}} \right\}, \quad (5.15)
\]

we easily deduce from \((5.14)\) that \( \| \phi(\tilde{T}) - \phi^* \|_{H^2} \leq \frac{\varpi}{4} \varpi < \varpi \), which leads to a contradiction with the definition of \( \tilde{T} \). Hence, \( \tilde{T} = +\infty \) and there holds

\[
\| \phi(t) - \phi^* \|_{H^2} \leq \varpi \leq \epsilon, \quad \forall \ t \geq 0. \quad (5.16)
\]
As a consequence,

\[ \| \phi(t) - \phi_0 \|_{H^2} \leq \| \phi(t) - \phi^* \|_{H^2} + \| \phi^* - \phi_0 \|_{H^2} \leq \varpi + \sigma \leq \frac{5}{4} \varpi. \quad (5.17) \]

then it follows from (5.11) that

\[ \mathcal{E}(t) \geq \mathcal{E}(0) - \varepsilon_0, \quad \forall \, t \geq 0. \quad (5.18) \]

By Proposition 5.1, we see that (1.4)–(1.8) admits a unique global strong solution that satisfies (5.7). The proof is complete.

**Corollary 5.2.** Assume that the assumptions of Theorem 5.1 are satisfied. The global strong solution \((u, \phi)\) has the following property:

\[ \lim_{t \to +\infty} (\| u(t) \|_{H^1} + \| \phi(t) - \phi_\infty \|_{H^4}) = 0, \quad (5.19) \]

where \( \phi_\infty \in H^4_p \) is a solution to (5.4) such that \( E(\phi^*) = E(\phi_\infty) \). Moreover, there exists a positive constant \( C \) depending on \( u_0, \phi_0 \) and coefficients of the system such that

\[ \| u(t) \|_{H^1} + \| \phi(t) - \phi_\infty \|_{H^4} \leq C(1 + t)^{-\theta'(1 - 2\theta')}, \quad \forall \, t \geq 0. \quad (5.20) \]

\( \theta' \in (0, \frac{1}{2}) \) is the Lojasiewicz exponent in Lemma 5.2 depending on \( \phi_\infty \).

**Proof.** We infer from the higher-order energy inequality (3.14) and uniform estimates (5.1) that \( \frac{d}{dt} A(t) \) is bounded for \( t > 0 \). On the other hand, the basic energy law implies that \( A(t) \in L^1(0, +\infty) \), then we have \( \lim_{t \to +\infty} A(t) = 0 \). Thus, we obtain the decay property of the velocity field \( u \) in \( V \) and

\[ \lim_{t \to +\infty} \left\| \frac{\delta E(t)}{\delta \phi} \right\| = 0. \quad (5.21) \]

Recalling the proof of Theorem 5.1, we have shown that \( \| \phi_t(t) \| \in L^1(0, +\infty) \). As a consequence, \( \phi(t) \) will converge in \( L^2 \) to a function \( \phi_\infty \in H^4_p(Q) \) that satisfies (5.4) due to (5.21).

It follows from (5.16) that for sufficiently large \( t \), we have

\[ \| \phi_\infty - \phi^* \|_{H^2} \leq \| \phi_\infty - \phi(t) \|_{H^2} + \| \phi(t) - \phi^* \|_{H^2} \leq C \| \phi_\infty - \phi(t) \|_{H^1}^{\frac{1}{2}} \| \phi_\infty - \phi(t) \|_{H^1}^{\frac{1}{2}} + \| \phi(t) - \phi^* \|_{H^2} \leq \min\{\delta, \beta\}. \]

Thus, applying Lemma 5.2 with \( \phi = \phi_\infty \) and \( \psi = \phi^* \), we have

\[ |E(\phi_\infty) - E(\phi^*)|^{1-\theta} \leq \left\| \frac{\delta E}{\delta \phi}(\phi_\infty) \right\| = 0. \]

The limit function \( \phi_\infty \) is also a local minimizer of the energy \( E \) and it will coincide with \( \phi^* \) if the latter is isolated. Finally, the proof for convergence rate (5.20) is based on Lemma 5.2 and higher-order differential inequalities as for the liquid crystal systems \([25, 33]\). Since the proof is lengthy but standard, we omit the details here. \( \square \)
Remark 5.2. We can also prove the long-time behavior for global weak solutions with arbitrary initial data in contrast with smallness assumption like (5.3). Indeed, Corollary 5.1 implies that any global weak solution \((u, \phi)\) to the problem (1.4)–(1.8) will become a bounded strong one after a sufficiently large time. Then we can just make a shift of time and consider the long-time behavior of bounded strong solutions. Applying Lemma 5.2, we can use the Lojasiewicz–Simon technique (cf. e.g., [1, 6, 17, 20, 28, 33] for various applications) to show that each weak solution does converge to a single pair \((0, \phi_\infty)\) with \(\phi_\infty\) satisfying the stationary problem (5.4). Besides, one can obtain the estimate on convergence rate as (5.20).

6 Appendix: The Łojasiewicz–Simon inequality

In this section, we prove that a Łojasiewicz–Simon type inequality holds in a proper neighborhood of every critical point of the functional \(E(\phi)\).

First, We recall the definition of analyticity on Banach spaces [37, Definition 8.8]: Suppose \(X, Y\) are two Banach spaces. The operator \(T : D(T) \subseteq X \to Y\) is analytic if and only if for any \(x_0 \in X\), there exists a small neighborhood of \(x_0\) such that

\[
T(x_0 + h) - T(x_0) = \sum_{n \geq 1} T_n(x_0)(h, \ldots, h), \quad \forall h \in X, \quad \|h\|_X < r << 1.
\]

Here \(T_n(x_0)\) is a continuous symmetrical \(n\)-linear operator on \(X^n \to Y\) and satisfies

\[
\sum_{n \geq 1} \|T_n(x_0)\|_{L(X^n, Y)} \|h\|^n_X < +\infty.
\]

Proposition 6.1. Suppose \(n = 3\), we have

(1) \(E(\phi) : H^4_p(Q) \to \mathbb{R}\) is analytic;

(2) \(\frac{dE}{d\phi} : H^4_p(Q) \to L^2_p(Q)\) is analytic;

(3) for any \(\psi \in H^4_p(Q)\), \(E''(\psi)\) is a Fredholm operator of index zero from \(H^4_p(Q)\) to \(L^2_p(Q)\).

Proof. (1) It follows from the definition of \(E\) that it is the sum of integrations of polynomials in terms of \(\Delta \phi, \nabla \phi\) and \(\phi\). Since \(\phi \in H^4_p(Q)\), then those functions belongs to \(H^2_p(Q)\) which is a Banach algebra for the pointwise multiplication when \(n = 3\). Thus, \(E(\phi) : H^4_p(Q) \to \mathbb{R}\) is analytic.

(2) Recalling (5.2), we see that \(\frac{dE}{d\phi} = k\varepsilon \Delta^2 \phi + H(\phi)\), where \(H(\phi)\) is the sum of polynomials in terms of \(\Delta \phi, \nabla \phi\) and \(\phi\) that belong to \(H^2_p(Q)\) when \(\phi \in H^4_p(Q)\). Thus, our conclusion follows.

(3) For any \(\psi, w_1, w_2 \in H^4_p(Q)\), we calculate that

\[
E''(\psi)(w_1, w_2)
= \frac{d}{ds}(E'(\psi + sw_1), w_2)|_{s=0}
= \frac{d}{ds} \int_Q (kg(\psi + sw_1) + M_1(A(\psi + sw_1) - \alpha) + M_2(B(\psi + sw_1) - \beta)f(\psi + sw_1)) w_2 dx \big|_{s=0}
= \int_Q k\varepsilon (\Delta^2 w_1)w_2 - \frac{6k}{\varepsilon} |\nabla \psi|^2 w_1 w_2 - \frac{12k}{\varepsilon} (\psi \nabla \psi \cdot \nabla w_1)w_2 - \frac{12k}{\varepsilon} (\psi \Delta \psi) w_1 w_2 dx
\]
8]). The remaining term \( R_k \) is some operators \( \Delta, \nabla \) operator of index zero from \( H \) is some \( \beta \) constants that belong to \( \text{rms of } \Delta \) with the symmetric bilinear form \( A : H^2_p(Q) \times H^2_p(Q) \rightarrow \mathbb{R} \) given by

\[
A(f, g) = k \varepsilon \int Q \Delta f \Delta g dx, \quad \forall f, g \in H^2_p(Q)
\]

such that by integration by parts \( \int Q k \varepsilon \Delta^2 f g dx = A(f, g) \) for any \( f, g \in H^2_p(Q) \). Obviously, \( A(\cdot, \cdot) \) is bounded on \( H^2_p(Q) \). By the elliptic estimate (2.10), for any \( \lambda > 0, \phi \in H^2_p(Q) \), there is some \( \eta' \) such that \( A(\phi, \phi) + \eta' \|\phi\|^2 \geq \eta \|\phi\|_{H^2}^2 \). Thus, it follows from the Lax–Milgram theorem that the self-adjoint operator \( k \varepsilon \Delta + \eta I : H^4_p(Q) \rightarrow L^2_p(Q) \) is an isomorphism. Then we see that \( k \varepsilon \Delta^2 : H^4_p(Q) \rightarrow L^2_p(Q) \) is a Fredholm operator of index zero (cf. e.g., [37, Section 8]). The remaining term \( R(\psi) \) consists of \( \psi, \nabla \psi, \Delta \psi \) and their integrals as well as differential operators \( \Delta, \nabla \). Therefore, \( R(\psi) \) is a compact operator from \( H^4_p(Q) \rightarrow L^2_p(Q) \) when \( \phi \in H^3_p(Q) \).

As a consequence, for any \( \phi \in H^3_p(Q) \), \( E''(\phi) \) is a compact perturbation of a Fredholm operator of index zero from \( H^4_p(Q) \) to \( L^2_p(Q) \), then itself is also a Fredholm operator of index zero from \( H^4_p(Q) \) to \( L^2_p(Q) \) (cf. e.g., [37, Section 8]). The proof is complete.

Using Proposition 6.1, we can infer from the abstract result [6, Corollary 3.11] that the following result holds

**Theorem 6.1.** Suppose \( n = 3 \). Let \( \psi \) be a critical point of energy \( E \). Then, there exist constants \( \beta_1 > 0, \theta \in (0, \frac{1}{2}) \) depending on \( \psi \) such that for any \( \phi \in H^3_p(Q) \) with \( \|\phi - \psi\|_{H^4} < \beta_1 \), there holds

\[
\left\| \frac{\delta E}{\delta \phi} \right\| \geq |E(\phi) - E(\psi)|^{1-\theta}.
\]  

**(Proof of Lemma 6.2).** Based on Theorem 6.1, we now relax the smallness condition and show that (5.5) holds if one only requires that \( \phi \) falls into a certain \( H^2 \)-neighbourhood of \( \psi \). For any \( \phi \in H^3_p(Q) \), using the regularity theory for elliptic problem, we can see that

\[
\|\phi - \psi\|_{H^4} \leq M(\|\Delta^2(\phi - \psi)\| + \|\phi - \psi\|),
\]

where \( M \) is a constant independent of \( \phi \) and \( \psi \). If \( \|\phi - \psi\|_{H^2} \leq 1 \), we take this assumption just to ensure that the fact \( \|\phi\|_{H^2} \leq \|\psi\|_{H^2} + 1 \) depends only on \( \phi \). Similar to (6.8)–(6.10), we see that

\[
|E(\phi) - E(\psi)|^{1-\theta} \leq C_1 \|\phi - \psi\|_{H^2}^{1-\theta}.
\]  

By Hölder inequality and Sobolev embedding theorem \( H^2_p(Q) \hookrightarrow L^\infty(Q) \), we get

\[
|B(\phi) - B(\psi)| \leq C \|\nabla(\phi + \psi)\| \|\nabla(\phi - \psi)\| + C(\|\phi\|_{L^\infty}^2 + \|\psi\|_{L^\infty}^2 + 1)\|\phi - \psi\|_{L^1}.
\]
\begin{align*}
\|f(\phi - f(\psi))\| &\leq C\|\phi - \psi\|_{H^1}, \\
&\leq C\|\Delta \phi - \Delta \psi\| + C(\|\phi\|_{L^\infty}^2 + \|\psi\|_{L^\infty}^2 + 1)\|\phi - \psi\| \leq C\|\phi - \psi\|_{H^2},
\end{align*}
where $C$ only depend on $\|\psi\|_{H^2}$. Recalling the expression of $H(\phi)$ given in \eqref{32}, we obtain
\begin{align*}
\|H(\phi) - H(\psi)\| &\leq C\|\phi - \psi\|_{L^\infty}\|\nabla \phi\|_{L^4}^2 + C\|\psi\|_{L^\infty}\|\nabla (\phi + \psi)\|_{L^4}\|\nabla (\phi - \psi)\|_{L^4} \\
&\quad + C\|\phi\|_{L^\infty}\|\Delta (\phi - \psi)\| + \|\Delta \psi\|\|\phi + \psi\|_{L^\infty}\|\phi - \psi\|_{L^\infty} \\
&\quad + C(\|\phi\|_{L^\infty}, \|\psi\|_{L^\infty})\|\phi - \psi\| + M_1\|\phi - \psi\|_{L^1} + M_2|B(\phi) - B(\psi)||f(\phi)|| \\
&\quad + M_2(|B(\psi)| + \beta)||f(\phi) - f(\psi)|| \\
&\leq C_2\|\phi - \psi\|_{H^2}.
\end{align*}

Since $C_1, C_2$-the constants above-only depend on $\|\psi\|_{H^2}$, there exists a (sufficiently small) constant $\beta$ independent of $\phi$ which satisfies
\begin{align*}
0 < \beta < \min \left\{ 1, \beta_1, \frac{\beta_1 k\varepsilon}{2M}, \left( \frac{\beta_1 k\varepsilon}{4M(C_1 + C_2)} \right)^{\frac{1}{1-\theta}} \right\}
\end{align*}
such that if $\|\phi - \psi\|_{H^2} < \beta$, then
\begin{align*}
\|H(\phi) - H(\psi)\| + |E(\phi) - E(\psi)|^{1-\theta} < \frac{\beta_1 k\varepsilon}{4M}. \tag{6.4}
\end{align*}

For any $\phi \in H^4_p(Q)$ satisfying $\|\phi - \psi\|_{H^2} < \beta$, there are only two possibilities: (i) if $\phi$ also satisfies $\|\phi - \psi\|_{H^4} < \beta_1$, then \eqref{6.1} holds; (ii) otherwise, if $\|\phi - \psi\|_{H^4} \geq \beta_1$, noticing that $\psi$ satisfies \eqref{5.3}, we deduce from \eqref{6.2} and \eqref{6.4} that
\begin{align*}
\left| \frac{\delta E}{\delta \phi} \right| &\geq k\varepsilon \Delta^2(\phi - \psi) + H(\phi) - H(\psi) \geq k\varepsilon \Delta^2(\phi - \psi) - \|H(\phi) - H(\psi)\| \\
&\geq \frac{3\beta_1 k\varepsilon}{4M} \|\phi - \psi\|_{H^4} - k\varepsilon \|\phi - \psi\| \\
&> \frac{\beta_1 k\varepsilon}{4M} > |E(\phi) - E(\psi)|^{1-\theta}.
\end{align*}

The proof is complete.

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**References**

[1] H. Abels, On a diffuse interface model for two-phase flows of viscous incompressible fluids with matched densities, Arch. Ration. Mech. Anal., 194 (2009), 463–506.

[2] H. Beirao da Veiga, A new regularity class for the Navier–Stokes equation in $\mathbb{R}^n$, Chinese Ann. Math., 16B (1995), 407–412.
[3] D. Boal, *Mechanics of the Cell*, Cambridge University Press, Cambridge, 2002.

[4] F. Boyer, Mathematical study of multi-phase flow under shear through order parameter formulation, Asymptotic Anal., 20 (1999), 175–212.

[5] C. Chan and A. Vasseur, Log improvement of the Prodi–Serrin criteria for Navier–Stokes equations, Methods Appl. Anal., 14 (2007), 197–212.

[6] R. Chill, On the Lojasiewicz–Simon gradient inequality, J. Funct. Anal., 201(2) (2003), 572–601.

[7] P.G. Ciarlet, Mathematical elasticity, vol. III, in: Studies in Mathematics and its Applications, 29, North-Holland, Amsterdam, 2000.

[8] Q. Du, M. Li and C. Liu, Analysis of a phase field Navier–Stokes vesicle-fluid interaction model, Discrete Contin. Dyn. Syst. B, 8(3) (2007), 539–556.

[9] Q. Du, C. Liu, R. Ryham and X. Wang, A phase field formulation of the Willmore problem, Nonlinearity, 18 (2005), 1249–1267.

[10] Q. Du, C. Liu, R. Ryham and X. Wang, Phase field modeling of the spontaneous curvature effect in cell membranes, Commun. Pure Appl. Anal., 4 (2005), 537–548.

[11] Q. Du, C. Liu, R. Ryham and X. Wang, Energetic variational approaches in modeling vesicle and fluid interactions, Physica D, 238 (2009), 923–930.

[12] Q. Du, C. Liu and X. Wang, A phase field approach in the numerical study of the elastic bending energy for vesicle membranes, J. Computational Physics, 198 (2004), 450–468.

[13] Q. Du, C. Liu and X. Wang, Retrieving topological information for phase field models, SIAM J. Appl. Math., 65 (2005), 1913–1932.

[14] Q. Du, C. Liu and X. Wang, Simulating the deformation of vesicle membranes under elastic bending energy in three dimensions, J. Computational Physics, 212 (2006), 757–777.

[15] Q. Du and Wang, Convergence of numerical approximations to a phase field bending elasticity model of membrane deformations, Inter. J. Numer. Anal and Modeling, 4(3&4) (2007), 441–459.

[16] X. Feng, Y. He and C. Liu, Analysis of finite element approximations of a phase-field model for two-phase fluids, Math. Comp., 76 (2007), 539–571.

[17] C.G. Gal, M. Grasselli, Asymptotic behavior of a Cahn–Hilliard–Navier–Stokes system in 2D, Ann. Inst. H. Poincaré Anal. Non Linéaire, 27 (2010), 401–436.

[18] W. Helfrich, Elastic properties of lipid bilayers: theory and possible experiments, Z. Naturforsch. C, 28 (1973), 693–703.

[19] O.A. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow*, Gordon and Breach, 1969.

[20] F.-H. Lin and Q. Du, Ginzburg–Landau vortices: dynamics, pinning, and hysteresis, SIAM J. Math. Anal., 28 (1997), 1265–1293.

[21] F.-H. Lin and C. Liu, Nonparabolic dissipative system modeling the flow of liquid crystals, Comm. Pure Appl. Math., XLVIII (1995), 501–537.
[22] Y. Liu, T. Takahashi and M. Tucsnak, Strong solution for a phase field Navier–Stokes vesicle fluid interaction model, J. Math. Fluid Mech., (2011), DOI 10.1007/s00021-011-0059-9.

[23] S. Montgomery-Smith, Conditions implying regularity of the three dimensional Navier–Stokes equation, Appl. Math., 50 (2005), 451–464.

[24] O. Mouritsen, Life–As a Matter of Fat: The Emerging Science of Lipidomics, Springer, Berlin, 2005.

[25] A. Segatti and H. Wu, Finite dimensional reduction and convergence to equilibrium for incompressible Smectic-A liquid crystal flows, SIAM J. Math. Anal., 43(6) (2011), 2445–2481.

[26] U. Seifert and R. Lipowsky, Morphology of Vesicles, in: Handbook of Biological Physics, vol. 1, 1995.

[27] J. Serrin, The initial value problem for the Navier–Stokes equations, In: Nonlinear Problems. Proc. Sympos., Madison, WI, University of Wisconsin Press, Madison, WI (1963), 69–98.

[28] L. Simon, Asymptotics for a class of nonlinear evolution equation with applicationa to geometric problems, Ann. of Math., 118 (1983), 525–571.

[29] R. Temam, Navier–Stokes equations and nonlinear functional analysis, SIAM, 1983.

[30] R. Temam, Infinite dimensional dynamical systems in mechanics and physics, Springer, New York, 1997.

[31] X. Wang and Q. Du, Modelling and simulations of multi-component liquid membranes and open membranes via diffuse interface approaches, J. Math. Biol., 56(3) (2008), 347–371.

[32] X. Wang, Asymptotic analysis of phase field formulation of bending elasticity models, SIAM J. Math Anal., 39(5) (2008), 1367–1401.

[33] H. Wu, X. Xu and C. Liu, Asymptotic behavior for a nematic liquid crystal model with different kinematic transport properties, Calc. Var. Partial Differential Equations, (2011), online first, DOI: 10.1007/s00526-011-0460-5.

[34] P. Yue, J.J. Feng, C. Liu and J. Shen, A diffuse-interface method for simulating two-phase flows of complex fluids, J. Fluid Mech., 515 (2004), 293–317.

[35] Y. Zhou and Z. Lei, Logarithmically improved criteria for Navier–Stokes equations, preprint, (2008), arXiv:0805.2784v2.

[36] Y. Zhou and S. Gala, Logarithmically improved regularity criteria for the Navier–Stokes equations in multiplier spaces, J. Math. Anal. Appl., 356 (2009), 498–501.

[37] E. Zeidler, Nonlinear Functional Analysis and Its Applications, Vol. I, Springer-Verlag, New York, 1988.