Anisotropy screening in Horndeski cosmologies

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We consider anisotropic cosmologies in a particular shift-symmetric Horndeski theory containing the $G^{\mu \nu} \partial^\mu \phi \partial^\nu \phi$ coupling, where $G^{\mu \nu}$ is the Einstein tensor. This theory admits stable in the future self-accelerating cosmologies whose tensor perturbations propagate with the velocity very close to the speed of light such that the theory agrees with the gravity wave observations. Surprisingly, we find that the anisotropies within the Bianchi I homogeneous spacetime model are screened at early time by the scalar charge, whereas at late times they are damped in the usual way, hence they are damped at all times. Therefore, contrary to what one would normally expect, the initial state of the universe close to the singularity cannot be anisotropic, although it seems it should be inhomogeneous. At the same time, we find that in the Bianchi IX case the universe can be strongly anisotropic close to the initial singularity even in the presence of a scalar charge.
I. INTRODUCTION

It is usually assumed that the state of the universe close to the initial singularity should be strongly anisotropic [1], [2], [3]. This belief is based on the fact that spatial anisotropies produce in the Einstein equations terms proportional to the inverse square of the volume, $1/V^2$, which become dominant when one goes backwards in time. In other words, anisotropic perturbations grow to the past. When the universe expands, the anisotropy terms decrease faster than the contribution of other forms of energy subject to the dominant energy condition, and the universe rapidly approaches a locally isotropic state during inflation [4], [5] (without the inflationary stage this process may require a longtime or may not happen at all due to the possibility of recollapse). Therefore, thinking about the early history of the universe, one could expect the isotropic phase of inflation to be generically preceded by an anisotropic phase. Although this argument seems quite robust, we shall present in what follows a peculiar cosmology whose anisotropies are damped at early times, hence the existence of a primary anisotropic phase is not as universal as one might think.

The theory we wish to discuss is the particular subset of the Horndeski theory for a gravitating scalar field [6] defined by the action (2.1) below. Its homogeneous and isotropic cosmologies were studied in [7], [8], but later it was discovered that theories of this type should be disfavoured because they predict the speed of gravity waves (GW) different from the speed of light [9–11], whereas the recent GW170817 event shows that the GW speed is equal to the speed of light with very high precision [12]. However, this constraint applies rather to some solutions of the theory than to the theory itself. The theory admits stable in the future self-accelerating cosmologies whose tensor perturbations propagate with the velocity very close to the speed of light, the relative difference being proportional to $1/V$. Therefore, the theory can perfectly agree with the GW observation of [12] at late times, and we can extrapolate it to the early times as well since no observational data about the GW speed at redshifts $z > 0.3$ are currently available.

We shall therefore study anisotropic cosmologies of the simplest Bianchi I homogeneous spacetime type within this Horndeski model. Surprisingly, we find that the anisotropies are screened at early times by the scalar charge, therefore the universe remains isotropic. However, the universe turns out to be unstable in this limit with respect to inhomogeneous perturbations, which suggests that at early times it should be inhomogeneous. Therefore,
the standard argument in favour of strong anisotropies at early times does not always apply. However, our numerics suggest that the universe can be strongly anisotropic close to the initial singularity within the Bianchi IX class, therefore the anisotropy screening is not generic for all Bianchi models.

II. ISOTROPIC CASE

To begin with, we summarise the essential properties of the isotropic solutions, some of which have not been discussed before. We consider the theory

$$S = \frac{1}{2} \int (\mu R - (\alpha g_{\mu\nu} + \varepsilon g_{\mu\nu}) \nabla^\mu \phi \nabla^\nu \phi - 2\Lambda) \sqrt{-g} \, d^4x \equiv \frac{1}{2} \int L \, d^4x ,$$  \hspace{1cm} (2.1)

where \(\mu = M_{Pl}^2\) and \(\alpha, \varepsilon, \Lambda\) are parameters, while the metric is

$$ds^2 = -dt^2 + a^2(t) \left[ dx_1^2 + dx_2^2 + dx_3^2 \right].$$ \hspace{1cm} (2.2)

The case of more general homogeneous and isotropic metrics, including also an extra matter, was considered in [8]. Assuming \(\phi\) to depend only on time, the Friedmann equation for the Hubble parameter \(H = \dot{a}/a\) is

$$3\mu H^2 = \frac{1}{2} (\varepsilon - 9\alpha H^2) \dot{\phi}^2 + \Lambda.$$

(2.3)

The equation for the scalar can be integrated once to give

$$\left( 3\alpha H^2 - \varepsilon \right) \dot{\phi} = \frac{C}{\alpha^3},$$

(2.4)

where the integration constants \(C\) is the scalar charge associated with the invariance of the action under shifts \(\phi \to \phi + \phi_0\). If \(C = 0\) then one has either

$$H^2 = \alpha \frac{\Lambda}{3\mu}, \quad \dot{\phi} = 0 ,$$

(2.5)

or

$$H^2 = \alpha \frac{\varepsilon}{3\alpha}, \quad \dot{\phi}^2 = \alpha \frac{\Lambda}{\varepsilon} - \alpha \frac{\mu}{\varepsilon},$$

(2.6)

in both cases the metric is pure de Sitter. If the charge \(C\) does not vanish then its effect should become negligible for \(a \to \infty\), as seen from (2.4), hence the solutions should approach either (2.5) or (2.6) at late times. If \(C \neq 0\) then \(\dot{\phi}\) can be algebraically expressed in terms of \(a, H\). Using the values of the Hubble parameter and scale factor at present, \(H_0, a_0, \)
we introduce dimensionless variables \( y = (H/H_0)^2 \), \( a = a/a_0 \) and \( \psi = (3\alpha H_0^2 a_0^3/C) \dot{\phi} \) and also dimensionless parameters \( \Omega_0 = \Lambda/(3\mu H_0^2) \), \( \Omega_0 = C^2/(18\alpha a_0^6 H_0^4 \mu) \) and \( \zeta = \varepsilon/(3\alpha H_0^2) \). Equations (2.3), (2.4) then assume the form

\[
y = \Omega_0 + \frac{\Omega_0}{a^6} \left[ \frac{\zeta - 3y}{\zeta - y} \right]^2, \quad \psi = \frac{1}{a^3(y - \zeta)},
\]

and since they should hold if \( y = a = 1 \), it follows that

\[
\Omega_0 = (\zeta - 1)^2 \frac{(1 - \Omega_0)}{\zeta - 3}. \tag{2.8}
\]

These equations determine \( y(a) \) and \( \psi(a) \), which determine \( a(t) \) and \( \phi(t) \).

Before considering solutions of the equations, let us study conditions for their stability. Considering small fluctuations \( g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}, \ \phi \rightarrow \phi + \delta \phi \), the metric perturbation can be decomposed into the scalar, vector, and tensor parts in the standard way [13], while \( \delta \phi \) can be gauged away using the residual freedom of infinitesimal reparametrisations of the time coordinate, hence \( \delta \phi = 0 \). The second variation of the action then splits into three independent parts describing the two tensor polarisations and the scalar mode (the vector sector contains no dynamics). Each of these parts has the structure

\[
I = \frac{M_{Pl}^2}{2} \int K \left( \ddot{X}^2 - c_s^2 \frac{p^2}{a^2} X^2 \right) a^3 d^4x, \tag{2.9}
\]

where \( p \) is the spatial momentum. In the case of tensor perturbations, \( X \) is the tensor mode amplitude, while the kinetic term and the sound speed squared are

\[
K_T = 1 + \Omega_6 \psi^2, \quad c_T^2 = \frac{1 - \Omega_6 \psi^2}{1 + \Omega_6 \psi^2}. \tag{2.10}
\]

In the scalar sector one has \( X = \delta g_{00} \) while

\[
K_S = \frac{3\Omega_6 \psi (\Omega_6 \psi^2 + 1) \left[ \Omega_6 \psi^2 (4\zeta a^3 \psi - 3) + 1 \right]}{a^3 (3\Omega_6 \psi^2 + 1)^2 y}, \tag{2.11}
\]

\[
c_S^2 = \frac{(3\Omega_6 \psi^2 + 1) \left[ \Omega_6^2 \psi^4 (16\zeta^2 \psi^2 a^6 - 52\zeta \psi a^3 + 39) - 2\Omega_6 \psi^2 (10\zeta \psi a^3 - 13) + 3 \right]}{3(\Omega_6 \psi^2 + 1) \left[ \Omega_6 \psi^2 (4\zeta a^3 \psi - 3) + 1 \right]^2}.
\]

The functions \( K_S, K_T, c_S^2, c_T^2 \) should be positive for the background solutions to be stable.

We can now analyse solutions of (2.7), and the simplest way to get them is to transform the first equation in (2.7) to

\[
a^6 = \frac{\Omega_6 (\zeta - 3y)}{(y - \zeta)^2(y - \Omega_0)}. \tag{2.12}
\]
Therefore, if \( a \to 0 \) then \( y \to \zeta/3 \), while if \( a \to \infty \) then either \( y \to \Omega_0 \) or \( y \to \zeta \). It is natural to assume that \( 0 < \Omega_0 < 1 \), and then the positivity of \( \Omega_6 \) defined by (2.8) requires that \( \zeta > 3 \). In this case there exists only one solution \( y(a) \) of (2.12), which fulfils

\[
\frac{\zeta}{3} \leftarrow y \to \Omega_0 \quad \text{as} \quad 0 \leftarrow a \to \infty. \tag{2.13}
\]

A direct verification reveals that \( K_T > 0 \) and \( K_S > 0 \) everywhere for this solution, hence the ghost is absent, while at large \( a \) one has \( c_T^2 > 0 \) and \( c_S^2 > 0 \), hence the solution is free in this limit also from gradient instabilities. The profile of this solution is shown in Fig.1. The solution has two inflationary phases: an early inflation driven by the scalar \( \phi \), with the Hubble rate being determined by \( \zeta \sim \varepsilon/\alpha \), and a late inflation driven by the cosmological constant \( \Lambda \sim \Omega_0 \). It is therefore natural to assume the parameter \( \zeta \) to be large, which can be achieved by requiring the coefficient \( \alpha \) in (2.1) to be small.

![Graph showing the profile of \( \sqrt{y} = H/H_0 \) for \( \Omega_0 = 0.7 \) and \( \zeta = 50 \). The kinetic terms \( K_S \) and \( K_T \) are always positive hence the ghost is absent. One has \( K_T \to 1 \) while \( K_S \propto 1/(\zeta - \Omega_0)\Omega_0 a^6 \) at large \( a \). The sound speeds \( c_S^2 \) and \( c_T^2 \) approach unity at late times but become negative at small \( a \), showing gradient instabilities with respect to inhomogeneous perturbations. The amplitude \( \chi \) is the anisotropy defined by (3.27) (assuming that \( B = 1 \)).](image)

As \( a \to \infty \) one has \( K_T = 1 + \mathcal{O}(1/a^6) \) hence the GW speed approaches the speed of light. At the same time, the present time moment corresponds to finite values \( y = a = 1 \), when

\[
K_T - 1 = \frac{1 - \Omega_0}{\zeta - 3}, \tag{2.14}
\]
Since $\zeta$ and $\Omega_0$ determine the Hubble rates of the early and late inflations, the ratio $\zeta/\Omega_0$ should be of the order of the early inflation energy scale divided by the late inflation energy scale, in which case $K_T - 1 \ll 10^{-15}$. This agrees with the observed bound on the relative difference between the GW velocity and the speed of light [12].

The solution develops gradient instabilities for small $a$ when $c_T^2$ and $c_S^2$ become negative. Before discussing these instabilities, let us consider another interesting property of the solution – the screening of anisotropies.

### III. BIANCHI I – ANISOTROPY SCREENING

Let us consider the Bianchi I metric

$$ds^2 = -N^2 dt^2 + a_1^2 dx_1^2 + a_2^2 dx_2^2 + a_3^2 dx_3^2,$$

where $N, a_k$ and also $\phi$ are functions of $t$. Injecting to (2.1), the Lagrangian is

$$L = -3a^3 \left( \frac{2\mu}{N} + \frac{\alpha \dot{\phi}^2}{N^2} \right) \mathcal{H}^2 + \left( \frac{\varepsilon \dot{\phi}^2}{N} - 2N\Lambda \right) a_1 a_2 a_3,$$

with

$$3a^3 \mathcal{H}^2 \equiv a_1 \dot{a}_2 \dot{a}_3 + a_2 \dot{a}_1 \dot{a}_3 + a_3 \dot{a}_1 \dot{a}_2 = 3a^3 \left( \frac{\dot{a}_2^2}{a^2} - \dot{\beta}_+^2 + \dot{\beta}_-^2 \right),$$

where $a_1 = a e^{\beta_+ + \sqrt{3}\beta_-}, a_2 = a e^{\beta_+ - \sqrt{3}\beta_-}, a_3 = a e^{-2\beta_+}$. The field equations can be obtained by varying $L$ with respect to $N, \beta_\pm, \phi$ and then setting $N = 1$. This yields

$$3\mu \mathcal{H}^2 = \frac{1}{2} (\varepsilon - 9a \mathcal{H}^2) \dot{\phi}^2 + \Lambda,$$

$$\left( \sigma a^3 \dot{\beta}_\pm \right) = 0,$$

$$a^3 (3\alpha \mathcal{H}^2 - \varepsilon) \dot{\phi} = C,$$

with $\sigma = 2\mu + \alpha \dot{\phi}^2$, where $C$ is the scalar charge. If the anisotropies $\beta_\pm$ vanish, these equations reduce to (2.3),(2.4). If anisotropies do not vanish, then one has from (3.19)

$$\dot{\beta}_\pm = 2\mu \frac{B_\pm}{\sigma a^3},$$

where $B_\pm$ are integration constants. Let us see what this implies first in the case when the scalar charge is zero, $C = 0$. Then (3.20) can be solved by $\dot{\phi} = 0$ while (3.18) and (3.21) yield

$$\frac{\dot{a}^2}{a^2} = \dot{\beta}_+^2 + \dot{\beta}_-^2 + \frac{\Lambda}{3\mu}, \quad \dot{\beta}_\pm = \frac{B_\pm}{a^3},$$

(3.22)
The anisotropy terms on the right in the first equation decay with time, hence anisotropies contribution becomes irrelevant and the universe rapidly approaches the isotropic de Sitter phase (2.5). However, the anisotropy terms become dominant at small \( a \), when one can neglect the \( \Lambda \)-term and the universe is described by the Kasner metric for which \( a_k \propto t^{p_k} \) where the exponents \( p_k \) are expressed in terms of \( B_\pm \) and fulfil \( p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1 \). This supports the standard view according to which anisotropies should be important close to the initial singularity.

If \( C = 0 \) then (3.20) can be solved also by setting \( 3\alpha \mathcal{H}^2 - \varepsilon = 0 \), which gives

\[
\frac{\dot{a}^2}{a^2} = \dot{\beta}_+^2 + \dot{\beta}_-^2 + \frac{\varepsilon}{3\alpha}, \quad \dot{\beta}_\pm = \frac{2\mu\varepsilon B_\pm}{(\mu \varepsilon + \alpha \Lambda) a^3}, \quad \dot{\phi}^2 = \frac{\Lambda}{\varepsilon} - \frac{\mu}{\alpha}. \tag{3.23}
\]

The solution approaches the isotropic de Sitter phase (2.6) at late times, but at early times the anisotropies are again dominant.

Assume now that the scalar charge does not vanish, \( C \neq 0 \). Then one obtains from (3.20)

\[
\dot{\phi} = \frac{C}{a^3(3\alpha \mathcal{H}^2 - \varepsilon)}. \tag{3.24}
\]

Injecting this to (3.18), setting \( \mathcal{H}^2 = H_0^2 y \) and introducing the same \( a, \psi, \Omega_0, \Omega_6, \zeta \) as above, one obtains exactly the same equations as in (2.7). Their solution \( y(a) \) and \( \psi(a) \) is the same as the one described above and shown in Fig.1. This time, however, it describes a Bianchi I spacetime with the anisotropies expressed by (3.21),

\[
\dot{\beta}_\pm = \frac{B_\pm}{a^3(1 + \Omega_6 \psi^2)}, \tag{3.25}
\]

and with the Hubble rate

\[
\frac{\dot{a}^2}{a^2} = \dot{\beta}_+^2 + \dot{\beta}_-^2 + \mathcal{H}^2 = \left( \frac{\dot{\beta}_+^2 + \dot{\beta}_-^2}{\mathcal{H}^2} + 1 \right) \mathcal{H}^2 \equiv H_0^2 (\chi + 1) y. \tag{3.26}
\]

Here

\[
\chi = \frac{B}{a^6(1 + \Omega_6 \psi^2)^2 y} \tag{3.27}
\]

is the relative contribution of the anisotropies to the total energy balance, its amplitude is \( B = (B_+^2 + B_-^2)/(H_0 a_0^3)^2 \). Since the universe is highly isotropic at present, when \( y = a = 1 \), one should assume that \( B \ll 1 \), but this does not mean that isotropies have always been small.

Notice however that, according to (2.13), one has at early times

\[
y \approx \frac{\zeta}{3}, \quad \psi = \frac{1}{a^3(y - \zeta)} \propto a^{-3} \quad \Rightarrow \quad a^3(1 + \Omega_6 \psi^2) \propto a^{-3} \tag{3.28}
\]
and hence
\[ \dot{\beta}_\pm \propto a^3 \Rightarrow \dot{\beta}_+^2 + \dot{\beta}_-^2 \propto a^6. \] (3.29)

As a result, the anisotropies tend to zero for \( a \to 0 \) and their contribution to the total energy balance is \( \propto a^6 \) instead of \( \propto 1/a^6 \). Therefore, the anisotropy effect is totally negligible at early times. This is true if only the scalar charge \( C \) is non-zero, hence one can say that anisotropies are “screened by the scalar charge”. Of course, the anisotropy contribution is suppressed at late time as well by the factor \( 1/a^6 \) in (3.26) (since \( \psi \to 0 \) as \( a \to \infty \)). As seen in Fig.1, the anisotropy \( \chi \) defined by (3.27) approaches zero at early and late times.

As there is no reason to assume the scalar charge to be zero, it follows that the anisotropies are screened at all times in our theory. This means that, unlike what one would normally expect, the state of the universe close to the singularity cannot be anisotropic. Instead, the universe must become inhomogeneous, which follows form the fact that the homogeneous and isotropic solution develops a gradient instability at small \( a \) when the sound speed squared \( c^2 \) becomes negative. This instability is present both in the tensor and scalar sectors. Since the corresponding potential term in the effective action (2.9) contains the factor of \( p^2 \), the instability exists only for modes with a non-vanishing spatial momentum \( p \), that is for inhomogeneous modes.

The absence of homogeneous instabilities agrees with the fact that the homogeneous and anisotropic deformations are dumped at early times. Therefore, the initial state of the universe should be inhomogeneous, perhaps similar to the Gowdy metrics [14].

IV. BIANCHI IX CASE

One may wonder if the anisotropy screening is typical only for the Bianchi I class or it occurs also for other Bianchi types. We shall therefore analyse the Bianchi IX class, in which case the spacetime metric is
\[ ds^2 = -N^2 dt^2 + \frac{1}{4} \left( a_1^2 \omega_1 \otimes \omega_1 + a_2^2 \omega_2 \otimes \omega_2 + a_3^2 \omega_3 \otimes \omega_3 \right), \] (4.30)
where \( \omega_a \) are the invariant forms on \( S^3 \) subject to \( d\omega_a + \epsilon_{abc} \omega_b \wedge \omega_c = 0 \), while \( a_k \) and the scalar \( \phi \) depends only on time. The Lagrangian in (3.16) generalises to
\[ 8L = \frac{6\mu a^3}{N} \left( \frac{KN^2}{a^2} - \mathcal{H}^2 \right) - \frac{3\alpha a^3}{N^3} \phi^2 \left( \mathcal{H}^2 + \frac{KN^2}{a^2} \right) + \left( \frac{\varepsilon}{N} \phi^2 - 2N \Lambda \right) a^3, \] (4.31)
where $H$ is the same as in (3.17), while the anisotropy potential is

$$
K = -\frac{1}{3} e^{-8\beta_+} \left(4e^{6\beta_+} \cosh^2(\sqrt{3}\beta_-) - 1\right) \left(4e^{6\beta_+} \sinh^2(\sqrt{3}\beta_-) - 1\right). \tag{4.32}
$$

Varying the Lagrangian and setting $N = 1$ gives the equations (with $\sigma = 2\mu + \alpha\dot{\phi}^2$)

$$
3\mu \left(\mathcal{H}^2 + \frac{K}{a^2}\right) + \frac{3}{2} \alpha \dot{\phi}^2 \left(3\mathcal{H}^2 + \frac{K}{a^2}\right) = \frac{\varepsilon}{2} \dot{\phi}^2 + \Lambda, \tag{4.33}
$$

$$
\frac{1}{a^2} \left(\sigma a\dot{a}\right)' = \sigma \left(\frac{1}{2} \mathcal{H}^2 - \dot{\beta}_+^2 - \dot{\beta}_-^2\right) + \left(\frac{\alpha}{2} \dot{\phi}^2 - \mu\right) \frac{K}{a^2} - \frac{\varepsilon}{2} \dot{\phi}^2 + \Lambda, \tag{4.34}
$$

$$
\left(\sigma a^3 \dot{\beta}_\pm\right)' = a \left(\mu - \frac{\alpha}{2} \dot{\phi}^2\right) \frac{\partial K}{\partial \beta_\pm}, \tag{4.35}
$$

$$
a^3 \left(3\alpha \left(\mathcal{H}^2 + \frac{K}{a^2}\right) - \varepsilon\right) \dot{\phi} = C. \tag{4.36}
$$

The effect of anisotropies is encoded in Eqs. (4.33), (4.36) only through the term $K \geq 1$.

Therefore, applying the same transformations as before, one obtains instead of (2.7) the equations, with $\Omega_2 = -K/(H_0^2 a_0^2)$,

$$
y = \Omega_0 + \frac{\Omega_2}{a^2} + \frac{\Omega_0}{a^6} \left[\zeta - 3y + \Omega_2/a^2\right], \quad \psi = \frac{1}{a^2(y - \zeta) + \Omega_2}. \tag{4.37}
$$

Let us first consider the isotropic case. One can consistently set in the equations $\beta_\pm = 0$.

Then $K = 1$ and the solutions of (4.37) are such that $y(a)$ approaches a finite value as $a \to \infty$ but vanishes at $a = a_{\text{min}} > 0$ and becomes negative for $a < a_{\text{min}}$. Such solutions describe bouncing universes which shrink from infinity to the minimal size $a_{\text{min}}$ when the Hubble parameter $H = H_0\sqrt{y}$ vanishes, and then expand again [8]. Such a bouncing behaviour is due to the positive spatial curvature, and for $C = 0$ the solutions reduce to the de Sitter metric expressed in coordinates with compact spatial slicings.

These bounces can be generalised to include small anisotropies, because expanding the equations (4.35) for $\beta_\pm$ up to the fist order yields $\left(\sigma a^3 \dot{\beta}_\pm\right)' = 0$ and hence $\dot{\beta}_\pm = 2\mu B_\pm/(\sigma a^3)$. Since the value of $\sigma a^3$ is bounded below for a bounce, it follows that if the integration constants $B_\pm$ are small, the anisotropies always remain small and only produce a small correction to the Hubble rate. The zero of $H$ shifts slightly due to the anisotropies, since one has

$$
H^2 = \dot{\beta}_+^2 + \dot{\beta}_-^2 + H_0^2 y. \tag{4.38}
$$

Therefore, if $\beta_\pm = 0$ then $H$ vanishes at $a_{\text{min}}$ where $y$ vanishes, but if $\dot{\beta}_\pm \neq 0$ then the zero of $H$ shifts to the region $a < a_{\text{min}}$ where $y < 0$. 
Now, if \( C = \dot{\phi} = 0 \) then the equations also admit the slightly anisotropic bounces, but they admit as well strongly anisotropic solutions with initial singularity. In other words, increasing the amplitude of anisotropies shifts the bounce position more and more until it reaches \( a = 0 \), after which the solutions are no longer bounces and show the initial curvature singularity. Eq. (4.33) then reduce to

\[
\frac{\dot{a}^2}{a^2} = \dot{\beta}_-^2 + \dot{\beta}_+^2 - \frac{K}{a^2} + \frac{\Lambda}{3\mu}.
\]  

(4.39)

where the positive anisotropy terms on the right are large enough to overcome the negative term, thereby eliminating the bouncing behaviour. The solutions are then characterised by a sequence of “Kasner epochs” during which \( \partial K/\partial \beta_{\pm} \approx 0 \) and the universe is approximately described by the Kasner metric with \( a^3 \dot{\beta}_\pm \equiv B_\pm \approx \text{const.} \). The term \( \partial K/\partial \beta_{\pm} \) becomes important only during short moments when \( B_\pm \) change, after which the next Kasner epoch with new values of \( B_\pm \) starts [1].

One can wonder if a similar evolution near singularity is possible also when the scalar charge \( C \) does not vanish? It seems at first that the answer should be negative, since the Kasner epochs during which the solution is approximately Bianchi I seem to be forbidden by the anisotropy screening.

To clarify the situation, we solved numerically the system of second order equations (4.34),(4.35) together with the equation obtained by differentiating (4.36). The first order equation (4.33) was used to constraint the initial values, and we checked that the constraint propagates. We also checked that the scalar charge defined by (4.36) remains constant during the evolution, as it should. We chose the initial data to describe a slightly anisotropic universe of a finite size and then integrated the equations to the past. It turns out that if the initial anisotropy is small, then the scale factor \( a(t) \) first decreases to the past, then passes through a minimal non-zero value, and then starts increasing. The solution is of the bounce type and the anisotropies always remain small. However, if the initial anisotropy is large enough, then the scale factor always decreases to the past while the anisotropies grow. The singularity is strongly anisotropic.

A typical solution is shown in Fig. 2. Surprisingly, it demonstrates a sequence of Kasner epochs during which it must approach the Bianchi I regime. This seems to contradict the fact that the anisotropies should then be screened. However, the explanation is the following. In the Bianchi I case the anisotropies are screened because \( \psi \propto a^{-3} \), which makes large the
denominator in (3.25). In the Bianchi IX case Eq.(4.37) yields $y = \Omega_2/(3a^2) + O(1)$ and $\psi = 3/(2\Omega_2 a) + O(1)$ for small $a$. Since $\Omega_2 \sim K$ one has $\psi \propto 1/(Ka)$, and since $K \geq 1$, it follows that $\psi$ grows not faster than $a^{-1}$. This implies that the denominator in (3.25) behaves as $a/K^2$ and tends to zero, hence $\dot{\beta}_\pm$ expressed by (3.25) are large. Therefore, the anisotropies are not screened in the Bianchi IX case, hence their screening is not a generic feature for all Bianchi models. A more detailed analysis is needed to find out if the solutions can be chaotic [1].

![Graph](image)

**FIG. 2.** Solution of Eqs.(4.33)–(4.36) for $\mu = \Lambda = \varepsilon = 1$, $\alpha = 0.1$ and with the initial values $a = 5$, $\dot{\beta}_\pm = 0$, $\beta_+ = 0$, $\beta_- = 0.05$, $\dot{\phi} = 0.02$. The solution shows a sequence of Kasner epochs during which $\ln(a_k)/\ln(a) \approx \text{const}$. The scalar $\dot{\phi}$ oscillates but the charge $C$ is constant.

V. CONCLUSION

We studied anisotropic cosmologies in the shift-symmetric, non-minimally coupled Horndeski model (2.1). Even though this model is thought to be disfavoured by the GW observations, its homogeneous and isotropic solution propagates tensor perturbations with the velocity that can be insensitively close to the speed of light. Surprisingly, it turns out that the spatial anisotropies in this theory get damped near the singularity in the Bianchi I case, instead of being amplified. Therefore, the standard argument in favour of strong anisotropies at early times does not always apply. However, it seems that the anisotropy screening is not generic for all Bianchi types, since our numerics suggest that the universe can be strongly
anisotropic close to the initial singularity within the Bianchi IX class.

To the best of our knowledge, a similar systematic analysis of anisotropic cosmologies for generic Horndeski models has never been carried out before, although anisotropic cosmologies with scalars have been studied. For example, anisotropies in the theory with a conformally coupled scalar field [15] and also in the $R + R^2$ gravity [16], [17] have been discussed, both cases being conformally dual to the ordinary gravity with a scalar field. But in our case the theory cannot be conformally transformed to the Einstein frame. As a result, qualitatively new behaviour of anisotropy takes place.

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