A NEW EXAMPLE OF A GENERIC 2-DISTRIBUTION ON A 5-MANIFOLD WITH LARGE SYMMETRY ALGEBRA

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Abstract. We discover a new example of a generic rank 2-distribution on a 5-manifold with a 6-dimensional transitive symmetry algebra, which is not present in Cartan’s classical five variables paper. It corresponds to the Monge equation \( z' = y + (y'')^{1/3} \) with invariant quartic having root type [4], and a 6-dimensional non-solvable symmetry algebra isomorphic to the semidirect product of \( \mathfrak{sl}(2) \) and the 3-dimensional Heisenberg algebra.

1. A NEW EXAMPLE

The classical work of Élie Cartan [1] on the geometry of rank 2 vector distributions on 5-dimensional manifolds is one of the most captivating papers of É. Cartan’s heritage and the source of many other constructions in modern differential geometry [2, 7, 9, 10, 12, 13, 14, 17].

The major part of this paper is devoted to the geometry of generic 2-distributions on 5-dimensional smooth manifolds. A 2-distribution \( D \) on 5-manifold \( M \) is called generic, if for any (local) basis \( X_1, X_2 \) of \( D \) the vector fields \( X_1, X_2, X_3 = [X_1, X_2], X_4 = [X_1, X_3], X_5 = [X_2, X_3] \) form a (local) frame on \( M \).

It is convenient to encode generic 2-distributions by so-called Monge equations, which are the underdetermined ODE’s of the form \( z' = F(x, y, y', y'', z) \). The corresponding vector distribution is defined on the mixed order jet space \( J^{2,0}(\mathbb{R}, \mathbb{R}^2) = J^2(\mathbb{R}, \mathbb{R}) \times \mathbb{R} \) with the standard coordinate system \( (x, y, y_1, y_2, z) \) by two vector fields:

\[
X_1 = \frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial x} + y_1 \frac{\partial}{\partial y} + y_2 \frac{\partial}{\partial y_1} + F \frac{\partial}{\partial z}.
\]

It is generic if and only if \( \frac{\partial^2 F}{(\partial y_2)^2} \neq 0 \). In fact, any generic 2-distribution can be encoded by a certain Monge equation [6] (see also [9, 10]).

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Given a generic 2-distribution $D$ on a 5-dimensional manifold $M$, Élie Cartan constructs a natural 14-dimensional principal bundle $\pi: G \to M$ and a $\mathfrak{g}_2$-valued absolute parallelism on $G$, where $\mathfrak{g}_2$ is the split real form of the exceptional simple Lie algebra of type $G_2$. In particular, this proves that the symmetry algebra of any generic 2-distribution is finite-dimensional and of dimension $\leq 14$.

Cartan also constructs the first non-trivial local invariant, which happens to be a quartic $F \in S^4(D^*)$. He proves that a given generic 2-distribution has exactly 14-dimensional symmetry algebra if and only if $F$ vanishes identically. In this case, it is locally equivalent to the distribution defined by the Monge equation $z' = (y'')^2$.

He then proceeds to analyze the cases when $F$ does not vanish identically. His main motivation is to find all cases when there is no frame on $M$ itself naturally associated to the distribution $D$ [11, p.162]:

Nous ne voulons pas entrer dans la discussion complète de tous les cas qui peuvent se présenter comme intermédiaires entre le cas général et le cas particulier où la forme $F$ est identiquement nulle. Nous allons simplement rechercher tous les cas dans lesquels il est impossible de trouver cinq expressions de Pfaff covariantes $\omega_1$, $\omega_1$, ..., $\omega_5$, ou, autrement dit, tous les cas dans lesquels il est impossible de former cinq paramètres différentiels linéaires indépendants. Il est nécessaire pour cela que la forme $F$ contienne ou, un facteur linéaire triple ou deux facteurs linéaires doubles, ou un facteur linéaire quadruple. C’est par ce dernier cas que nous commencerons.

Clearly, such a frame on the manifold $M$ does not exist, if the symmetry algebra of the distribution $D$ has non-trivial stabilizer on the manifold $M$. Cartan shows that such cases may happen only if the quartic $F$ has either one root of multiplicity 4 (root type $[4]$) or two different roots with multiplicity 2 (root type $[2,2]$).

In case of root type $[4]$ he states that there is exactly one family of such distributions parametrized by an additional local invariant $I$ (see also [15, Section 17.4] for a translation of Élie Cartan’s computations to English). When $I$ is constant\(^1\), the symmetry algebra is 7-dimensional, and the distribution itself is encoded by the following Monge equation

\[^1\text{It is a general belief (see, for example, [10, 17]) that these systems with constant } I \text{ can also be encoded by the simpler Monge equation } z' = (y'')^m \text{ with } m \neq -1, 1/3, 2/3, 2. \text{ However, this family misses a single case of } I = \pm 3/4, \text{ which is equivalent, for example, to } z' = \ln(y'').\]
A NEW EXAMPLE OF A (2,5)-DISTRIBUTION WITH LARGE SYMMETRY

(see [1] p.171, eqn.6)

(1) \[ z' = -\frac{1}{2} \left( (y'')^2 + \frac{10}{3} (y')^2 + (1 + I^2) y^2 \right). \]

If \( I \) is non-constant then the symmetry algebra is 6-dimensional non-transitive.

Note that the above description of all generic 2-distributions with exactly 7-dimensional symmetry algebra was also proven by other authors. For example, papers [9, 11] reprove that 7 is the maximal possible dimension for the symmetry algebra for non-flat case (i.e., for the case when \( \mathcal{F} \) does not vanish identically). The paper [11, Th.2] reproves that all such 2-distributions are described by the Monge equation:

\[ z' = (y'')^2 + r_1 (y')^2 + r_2 y^2, \]

where the pair \((r_1, r_2)\) is viewed up to the transformations \((r_1, r_2) \mapsto (c^2 r_1, c^4 r_2), \ c \neq 0\), and the roots of the equation \(\lambda^4 - r_1 \lambda^2 + r_2 = 0\) do not constitute an arithmetic progression. If the latter condition is not satisfied, then the symmetry algebra becomes 14-dimensional.

In case of root type \([2,2]\), Cartan finds a number of cases with exactly 6-dimensional symmetry algebra, which is isomorphic (after complexification) to one of the following: \(\text{sl}(2, \mathbb{C}) \times \text{sl}(2, \mathbb{C}), \ \text{so}(3, \mathbb{C}) \ltimes \mathbb{C}^3, \ \text{sl}(2, \mathbb{C}) \times (\text{so}(2, \mathbb{C}) \ltimes \mathbb{C}^2)\).

We show that the analysis of root type \([4]\) case is incomplete and misses the generic 2-distribution encoded by the Monge system:

(2) \[ z' = y + (y'')^{1/3} \]

This equation is a part of larger family \(z' = y + (y'')^m\), which has only 5-dimensional symmetry algebra for generic \(m\) (see the forthcoming paper [3]).

Lemma 1. The 2-distribution corresponding to (2) has invariant quartic with root type [4].

Proof. We use the formulas from [7, 8, 13] to compute the invariant quartic. The exact formulas are tedious and are omitted here. \(\square\)

\(^2\)Cartan’s paper has here the coefficient \(\frac{7}{6}\) in place of \(\frac{10}{3}\). But this is an arithmetic error corrected by Strazzullo [16].
Lemma 2. The symmetry algebra of \((2)\) is given by the following 6 vector fields:

\[
\begin{align*}
S_1 &= x \frac{\partial}{\partial y} + \frac{\partial}{\partial y_1} + \frac{1}{2} x^2 \frac{\partial}{\partial z}, \\
S_2 &= x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} - 2y_1 \frac{\partial}{\partial y_1} - 3y_2 \frac{\partial}{\partial y_2}, \\
S_3 &= y \frac{\partial}{\partial x} - y^2 \frac{\partial}{\partial y_1} - 3y_1 y_2 \frac{\partial}{\partial y_2} + \frac{1}{2} y^2 \frac{\partial}{\partial z}, \\
S_4 &= \frac{\partial}{\partial x}, \\
S_5 &= \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}, \\
S_6 &= \frac{\partial}{\partial z}.
\end{align*}
\]

It is isomorphic to \(\mathfrak{sl}(2) \triangleleft \mathfrak{n}_3\) and preserves the fibration \(\pi: J^{2,0}(\mathbb{R}, \mathbb{R}) \to J^2(\mathbb{R}, \mathbb{R})\). The restriction of \(\pi_\ast\) to this symmetry algebra has a 1-dimensional kernel spanned by the center \(\langle S_6 \rangle\) and a 5-dimensional image that coincides with the prolongation of the standard action of the equiaffine Lie algebra on the plane.

Proof. Simple computation shows that indeed all vector fields \(S_1, \ldots, S_6\) are symmetries of the distribution. As all these symmetries do not depend on \(z\), we see that the fibration \(\{z = \text{const}\}\) is indeed preserved by the symmetry algebra and that \(S_6\) lies in its center. Projecting these vector fields to \(J^2(\mathbb{R}, \mathbb{R})\) we get 5 vector fields (\(S_6\) projects to 0), which preserve the contact distribution on \(J^2(\mathbb{R}, \mathbb{R})\). It is also easy to see that they are prolongations of 5 generators for the equiaffine action on the plane:

\[
\begin{align*}
x \frac{\partial}{\partial y}, & \quad x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}, \quad y \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}.
\end{align*}
\]

We note that \(\langle S_1, S_2, S_3 \rangle\) is a subalgebra isomorphic to \(\mathfrak{sl}(2)\), while \(\langle S_4, S_5, S_6 \rangle\) is a radical isomorphic to the 3-dimensional Heisenberg algebra.

Let us now show that this is indeed the full symmetry algebra of the given distribution. Otherwise it would be a subalgebra of a certain bigger symmetry algebra. As this distribution has root type \([4]\) this bigger symmetry algebra could only be one of the 7-dimensional symmetry algebras of \([11]\). But these are all solvable and cannot contain a semisimple subalgebra isomorphic to \(\mathfrak{sl}(2)\). \(\Box\)
Remark 1. Apparently, the distribution corresponding to (2) was first discovered by Francesco Strazzullo in his PhD thesis [16] as Example 6.7.2. It is given there as a Monge equation

\[ z' = 1 + \exp(-4y/3) (y'' - 1/2(y')^2)^{2/3}. \]

However, as he used Maple for computing symmetry algebras, it was not clear whether the 6-dimensional symmetry algebra he computed, is the full symmetry algebra. Although the same argument as above could be used in his case as well.

Remark 2. To be independent of Lemma 1 and related calculations, let us show that equation (2) has non-vanishing invariant quartic. Indeed, otherwise its symmetry algebra would be embedded into \( g_2 \). But this is not possible, as up to conjugation \( g_2 \) has only one non-zero element that has a centralizer of dimension \( \geq 6 \) (see [5]). It is a nilpotent element that corresponds to the longest root of \( G_2 \). Its centralizer is 8-dimensional and is isomorphic to the semidirect product of \( \mathfrak{sl}(2) \) and a 5-dimensional Heisenberg algebra. But there are no Lie algebra injective homomorphisms \( \mathfrak{sl}(2) \ltimes \mathfrak{n}_3 \to \mathfrak{sl}(2) \ltimes \mathfrak{n}_5 \) which map \( \mathfrak{sl}(2) \) to \( \mathfrak{sl}(2) \) (we can assume this as all Levi subalgebras are conjugate to each other) and the center to the center.

Remark 3. We shall show in the forthcoming paper [3] that this is the only missing case of a generic 2-distribution with transitive symmetry algebra of dimension \( \geq 6 \). The proof is based on the observation that over complex numbers any such distribution admits a simply transitive 5-dimensional subalgebra of the full symmetry algebra.

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