A GENERALIZATION OF AN INTEGRABILITY THEOREM OF DARBOUX

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Abstract. In his monograph “Systèmes Orthogonaux” Darboux stated three theorems providing local existence and uniqueness of solutions to first order systems of the type

\[ \partial_i u_\alpha(x) = f^i_\alpha(x, u(x)), \quad i \in I_\alpha \subseteq \{1, \ldots, n\}. \]

For a given point \( \bar{x} \in \mathbb{R}^n \) it is assumed that the values of the unknown \( u_\alpha \) are given locally near \( \bar{x} \) along \( \{ x \mid x_i = \bar{x}_i \text{ for each } i \in I_\alpha \} \). The more general of the theorems, Théorème III, was proved by Darboux only for the cases \( n = 2 \) and \( 3 \).

In this work we formulate and prove a generalization of Darboux’s Théorème III which applies to systems of the form

\[ r_i (u_\alpha) \bigg|_\alpha = f^i_\alpha(x, u(x)), \quad i \in I_\alpha \subseteq \{1, \ldots, n\} \]

where \( R = \{ r_i \}_{i=1}^n \) is a fixed local frame of vector fields near \( \bar{x} \). The data for \( u_\alpha \) are prescribed along a manifold \( \Xi_\alpha \) containing \( \bar{x} \) and transverse to the vector fields \( \{ r_i \mid i \in I_\alpha \} \). We identify a certain Stable Configuration Condition (SCC). This is a geometric condition that depends on both the frame \( R \) and on the manifolds \( \Xi_\alpha \); it is automatically met in the case considered by Darboux. Assuming the SCC and the relevant integrability conditions are satisfied, we establish local existence and uniqueness of a \( C^1 \)-solution via Picard iteration for any number of independent variables \( n \).

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1. Introduction

Darboux, in Chapitre I, Livre III in his monograph “Systèmes Orthogonaux” [3], stated three integrability theorems (“Théorèmes I-III”) for certain types of first order systems of PDEs. The theorems apply to systems of the form

$$\partial_x u_\alpha(x) = f^\alpha_i(x, u(x))$$

where $u = (u_1, \ldots, u_m)$ denotes the vector of unknown functions, the independent variables $x = (x_1, \ldots, x_n)$ range over an open set about a fixed point $\bar{x} \in \mathbb{R}^n$, and the $f^\alpha_i$ are given $C^1$-functions from an appropriate open subset of $\mathbb{R}^{n+m}$ to $\mathbb{R}$. The data for these systems consist of $C^1$-functions $g_\alpha$ prescribing the unknowns $u_\alpha$ along certain affine subspaces through the point $\bar{x} \in \mathbb{R}^n$.

For each $\alpha = 1, \ldots, m$, we let $I_\alpha$ be the set of all indices $i \in \{1, \ldots, n\}$ for which the system contains the equation $\partial_x u_\alpha = f^\alpha_i(x, u(x))$. The cases covered by Darboux’s three theorems can then be described as follows:

**Théorème I** applies to determined systems, which are characterized by the requirement that $|I_\alpha| = 1$ for all $\alpha = 1, \ldots, m$. For each $\alpha$, letting $I_\alpha = \{i_\alpha\}$, the data for $u_\alpha$ are prescribed near $\bar{x}$ along the hyperplane $\{x | x_{i_\alpha} = \bar{x}_{i_\alpha}\}$. In the special case that $i_\alpha$ is the same index for all $\alpha$, Darboux’s Théorème I reduces to the standard local existence and uniqueness result for ODEs with parameters (see [4]).

**Théorème II** is the PDE version of Frobenius’ theorem for completely overdetermined systems. This situation is characterized by $|I_\alpha| = n$ for all $\alpha = 1, \ldots, m$, i.e. the derivatives of all unknowns are prescribed in all coordinate directions. In this case, the data prescribes the value of each $u_\alpha$ at the point $\bar{x}$. The integrability conditions require the partial derivatives given by the system to be consistent with equality of 2nd order mixed partial derivatives. Under these conditions Darboux’s Théorème II guarantees a unique local solution of the PDE system with the assigned data.

**Théorème III** applies to the general case where the elements, as well as the cardinality, of $I_\alpha$ may vary with $\alpha$. The data are assigned as follows: if $I_\alpha = \{i_1, \ldots, i_{p_\alpha}\}$, we prescribe a function $g_\alpha$ along the affine subspace $\Xi_\alpha := \{x | x_{ij} = \bar{x}_{ij}, 1 \leq j \leq p_\alpha\}$, and require that $u_\alpha|_{\Xi_\alpha} = g_\alpha$. Under the appropriate integrability conditions, detailed below in Section 2.1, Darboux’s Théorème III guarantees a unique local solution of the PDE system with the assigned data.

We note that Théorèmes I and II are particular cases of Théorème III. Another special case of systems, for which the index sets $I_\alpha$ are the same for all $\alpha$, was addressed separately in [1].

Darboux stated his Théorème III for any number of independent variables. He provided a proof only in the cases with two and three independent variables ($n = 2$ or 3), which sufficed for his investigation of triply orthogonal systems in [3]. For $n = 2$ his proof of Théorème III used Théorème I; for $n = 3$ he used both the result for $n = 2$ as well as his Théorème I. Given
Darboux's partial proof, it is natural to try and establish his Théorème III via induction on the number \( n \) of independent variables. While this is possible, we have been able to do so only through an involved, combinatorial argument (see our unpublished note [2]). Furthermore, this inductive approach does not apply to the more general situation we consider in the present paper. Instead, we shall provide a direct proof that applies to more general systems with any number of independent variables.

Our results generalize Darboux’s Théorème III in two ways:

(i) The unknowns may be differentiated along vector fields in a fixed frame \( \mathcal{R} = \{ r_i \}_{i=1}^n \) defined near \( \bar{x} \). That is, for each \( \alpha = 1, \ldots, m \), there is an index set \( I_\alpha \subseteq \{1, \ldots, n\} \) such that the system contains the equations

\[
    r_i(u_\alpha) \big|_{x} = f_\alpha^i(x, u(x)) \quad \text{for each } i \in I_\alpha.
\]

As in Darboux’s Théorème III, the elements and cardinality of the index sets \( I_\alpha \) may vary with \( \alpha \).

(ii) The prescribed data \( g_\alpha \) for the unknown \( u_\alpha \) may be given along a manifold \( \Xi_\alpha \) through the point \( \bar{x} \) which is transverse to the vector fields \( r_i \) with \( i \in I_\alpha \).

The claim is that, under the appropriate integrability conditions (generalizing those of Darboux’s Théorème III), the PDE system (2) has a unique local solution which takes on the assigned data. A precise formulation is provided in our Theorem 1 in Section 3.

However, our proof requires what we refer to as a Stable Configuration Condition (SCC) to be satisfied. The formulation of the SCC is somewhat technical (see Section 2.2 and Definitions 3.1 and 3.2 below). Roughly speaking, this condition is required to guarantee that the natural Picard iteration scheme is well defined. We note that the validity of the SCC depends on both the frame \( \mathcal{R} \) and on the relative location of the manifolds \( \Xi_\alpha \) that carry the data; see Section 2.2 below for a concrete example. Also, it is immediate to verify that the SCC is met in the setting of Darboux’s original treatment where \( r_i \equiv \partial_{x_i}, i = 1, \ldots, n \), and \( \Xi_\alpha = \{ x \mid x_i = \bar{x}_i, i \in I_\alpha \} \).

Concerning regularity, we assume that the frame \( \{ r_i \}_{i=1}^n \), the functions \( f_\alpha^i \), the manifolds \( \Xi_\alpha \), and the data \( g_\alpha \) are all \( C^1 \)-smooth. A solution refers to a \( C^1 \)-smooth function \( u = (u_1, \ldots, u_m) \) which satisfies the PDEs and the data in a classic, pointwise manner on a neighborhood of the given point \( \bar{x} \).

The rest of the present paper is organized as follows. In Section 2.1 we review Darboux’s original Théorème III and the partial proof provided by Darboux. We also indicate how our approach in this paper differs from that of Darboux. Section 2.2 considers a simple system of equations to highlight the role of the Stable Configuration Condition (SCC): for a determined system of two equations for two unknowns in the plane, we show how the relative location of the two data manifolds \( \Xi_1 \) and \( \Xi_2 \) can yield radically different behavior in terms of the domains of definition of the natural Picard iterates. Finally, in Section 3 we formulate and prove our Theorem. A
key part of the proof is a technical lemma about the “restricted” system obtained by considering the same set of equations as in the original system, but restricted to certain sub-manifolds defined in terms of the frame vector fields $r_i$; see Lemma 3.3 below.

2. Review of Darboux’s work and the Stable Configuration Condition (SCC)

2.1. Darboux’s setup and result. We first consider the situation addressed by Darboux in his Théorème III: for each unknown $u_\alpha$, the system consists of the equations

\[ \partial x_i u_\alpha(x) = f_i^\alpha(x, u(x)) \quad \text{for } i \in I_\alpha \subseteq \{1, \ldots, n\}, \]

where $f_i^\alpha$ are $C^1$-smooth functions on $\mathbb{R}^{n+m}$. Setting

\[ \Xi_\alpha := \{x \mid x_i = \bar{x}_i \text{ for } i \in I_\alpha\}, \]

we prescribe the data

\[ u_\alpha|_{\Xi_\alpha} = g_\alpha, \]

for a given $C^1$-smooth function $g_\alpha : \Xi_\alpha \to \mathbb{R}$.

Next consider the integrability conditions which need to be imposed. Let $u_\alpha$ be an unknown for which the system prescribes two distinct partial derivatives, say

\[ \partial x_i u_\alpha(x) = f_i^\alpha(x, u(x)) \quad \text{and} \quad \partial x_j u_\alpha(x) = f_j^\alpha(x, u(x)) \]

where $i \neq j$ and $i, j \in I_\alpha$. The derivatives prescribed by the system need to be consistent with equality of mixed partial derivatives. That is, the expressions

\[ \partial^2 x_{ix_i} u_\alpha(x) = \partial x_i f_i^\alpha(x, u(x)) + \sum_{\beta=1}^m \partial u_\beta f_i^\alpha(x, u(x)) \partial x_j u_\beta(x) \]

and

\[ \partial^2 x_{ix_j} u_\alpha(x) = \partial x_j f_j^\alpha(x, u(x)) + \sum_{\beta=1}^m \partial u_\beta f_j^\alpha(x, u(x)) \partial x_i u_\beta(x) \]

should agree. Since the system may not prescribe all the partials $\partial x_i u_\beta$ and $\partial x_j u_\beta$ appearing on the right-hand sides of (6) and (7), this puts constraints on which dependent variables $u_\beta$ the functions $f_i^\alpha$ and $f_j^\alpha$ may depend on. This is brought out in the following example which is the simplest case of an overdetermined system where Darboux’s Théorème III applies.

Example 2.1. Consider a system of 3 equations for 2 unknowns in 2 independent variables. Let the unknowns be $u$ and $v$, the independent variables be $x$ and $y$, and assume that the equations are

\[ u_x = f(x, y, u, v) \]
\[ v_x = \phi(x, y, u, v) \]
\[ v_y = \psi(x, y, u, v). \]
The data in this case take the form

\[ u(\bar{x}, y) = g_1(y) \]  
\[ v(\bar{x}, \bar{y}) = g_2, \]

where \( g_1 \) is a given function and \( g_2 \) is a given constant. The integrability condition is imposed to ensure that the prescription of the two partial derivatives of the unknown \( v \) is consistent with the equality of the partial derivatives \( (v_x)_y = (v_y)_x \). To derive these conditions we expand \( \partial_y[\phi(x, y, u, v)] = \partial_x[\psi(x, y, u, v)] \) applying the chain rule, to obtain

\[ \phi_y + \phi_u u_y + \phi_v v_y = \psi_x + \psi_u u_x + \psi_v v_x. \]

We next substitute the derivatives given by the system (8)-(10) into (13). However, the system does not provide an expression for \( u_y \), and we must therefore impose the condition

\[ \phi_u = 0. \]

All other partial derivatives of \( u \) and \( v \) appearing in (13) are prescribed by (8)-(10), and we obtain the condition:

\[ \phi_y + \phi_u u_y + \phi_v v_y = \psi_x + \psi_u u_x + \psi_v v_x. \]

Conditions (14) and (15) comprise the integrability conditions for the system (8)-(10). If these conditions hold as identities in an \( (x, y, u, v) \)-neighborhood of \((\bar{x}, \bar{y}, g_1(\bar{y}), g_2)\), then Darboux’s Théorème III guarantees the existence of a unique local \( C^1 \)-smooth solution \((u(x, y), v(x, y))\) to (8)-(10) near \((\bar{x}, \bar{y})\) taking on the data (11)-(12).

In the general setting of the system (3), the integrability conditions require that, whenever \( \alpha \in \{1, \ldots, m\} \) and \( i, j \in I_\alpha \) with \( i \neq j \), then the following should hold: for all \( \beta \in \{1, \ldots, m\} \) with \( i \notin I_\beta \) we have

\[ \partial_{u_\beta} f^\alpha_j = 0 \]

and

\[ \partial_i f^\alpha_j + \sum_{\beta: i \in I_\beta} (\partial_{u_\beta} f^\alpha_j) f^\beta_i = \partial_{x_i} f^\alpha_j + \sum_{\beta: j \in I_\beta} (\partial_{u_\beta} f^\alpha_i) f^\beta_j. \]

If these conditions hold as identities in a neighborhood of \((\bar{x}, g(\bar{x}))\), then Darboux’s Théorème III guarantees the existence of a unique local \( C^1 \)-smooth solution \( u(x) \) to (3) in a neighborhood of \( \bar{x} \) that takes on the data (5).

Due to the particular structure of the systems under consideration (viz., each equation contains a single derivative for which it is solved), it is natural to base a proof of existence on Picard iteration. Indeed, it is immediate to write down a functional map for which any solution of (3)-(5) must be a fixed point (see (27) below).

Now, in the particular situation addressed by Darboux’s Théorème I (for determined systems), one can verify that a fixed point exists and provides a solution of the original system. This is how Darboux established his
Théorème I. However, in the more general situation of overdetermined systems addressed by his Théorème III, such an approach appears to be more challenging. This circumstance might explain why Darboux [3] did not provide a general proof based directly on Picard iteration for his Théorème III. Instead, for $n = 2$ case he exploited Théorème I. For $n = 3$ case he identified sub-systems that can be treated by Théorème I or by $n = 2$ case. These sub-systems are solved in a “right” order so that the solution of one sub-system provides initial data to the next. Darboux states that the general proof will be too technical and, therefore, he restricts himself to the cases of $n = 2$ and $n = 3$ as they are sufficient for the applications he considers.\footnote{“Pour établir cette importante proposition, sans employer un trop grand luxe de notations, nous nous bornerons au cas de deux et de trois variables indépendantes, qui suffira d’ailleurs pour les applications que nous avons en vue.” [3] p. 336.}

Darboux’s treatment of $n = 3$ case indicates a possibility of an inductive proof for an arbitrary $n$, which we accomplish in [2]. The inductive proof turns out, indeed, to be quite technical. Moreover, the same approach cannot be applied to a more general problem considered here, because even if the initial system satisfies the hypothesis of our Theorem 1 the sub-systems appearing in the inductive proof may not satisfy these hypothesis.

In fact, as we shall show below, it is possible to provide a direct argument based on Picard iteration also for overdetermined systems of the more general type [2] with any number of independent variables. The key observation is that it suffices to consider a certain “restricted” system which consists of the same equations as the original system, but now required to hold only along certain submanifolds containing the given point $\bar{x}$. For this, seemingly weaker, restricted system, we establish existence of a solution $\tilde{u}$ via Picard iteration. This result (Lemma 3.3 below) is the first main new ingredient in our approach. (We note that this part of the argument does not involve any use of the integrability conditions.)

In the more general setting described in (i) and (ii) above, the argument of Lemma 3.3 becomes complicated by the need to work with different coordinate systems for different components $u_\alpha$ of the solution $u$. This is where we have found it necessary to introduce the Stable Configuration Condition (SCC) illustrated in the next section. We stress that, in the more general setting, the SCC is relevant already for generalizing Darboux’s Théorème I (i.e., the it is not about determinacy or over-determinacy of the system). However, in the original setting described by Darboux [3], the SCC condition is trivially satisfied. Therefore, our paper contains a direct proof of Darboux’s Théorème III, for an arbitrary number of variables.

Before considering an example explaining the relevance of the SCC, we outline the last step of the proof: showing that the fixed point $\tilde{u}$ is a solution of the original system on a full $\mathbb{R}^n$-neighborhood of $\bar{x}$. This is accomplished by showing that the quantities

$$A^{\alpha}_i(x) = r_i(\tilde{u}_\alpha)|_x - f^{\alpha}_i(x, \tilde{u}(x)),$$

$$1 \leq \alpha \leq m, \ i \in I_\alpha,$$
satisfy certain linear, homogeneous equations which form a restricted system of the type covered by our Lemma. Only at this point are the integrability conditions used. As this latter system admits the trivial solutions $A^\alpha_i \equiv 0$ on a full neighborhood of $\bar{x}$, it follows from the uniqueness part of the Lemma that $r_i(\tilde{u}_\alpha)|_{x} = f^\alpha_i(x, \tilde{u}(x))$ for all $x$ near $\bar{x}$, thereby completing the proof.

2.2. The Stable Configuration Condition (SCC). To illustrate the relevance of the SCC we consider the following simple example of a system for the two unknown scalar functions $u(x, y)$ and $v(x, y)$ of two independent variables:

\begin{align*}
  u_x &= v, \quad (16) \\
  v_y &= u. \quad (17)
\end{align*}

The system \[(16)-(17)\] is a determined system of the form \[(2)\] with $m = n = 2$, $u_1 = u$, $u_2 = v$, $r_1 = \partial_x$, and $r_2 = \partial_y$. We consider the system near the origin in the $(x, y)$-plane and let $M$ and $N$ be the two straight lines

\[ M := \{(x, y) \mid x = ay\} \quad N := \{(x, y) \mid y = bx\}, \]

where $0 < \frac{1}{a} < b$. Next, we consider separately the two cases:

(a) The data for $u$ are prescribed along $\Xi_1 = N$, and the data for $v$ are prescribed along $\Xi_2 = M$.
(b) Vice versa: $u$ is prescribed along $\Xi_1 = M$ and $v$ is prescribed along $\Xi_2 = N$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{stable_configuration}
\caption{Stable configuration.}
\end{figure}
Clearly, in both cases the transversality condition in (ii) above is met. Letting $g$ and $h$ be scalar functions of a single argument and defined near zero, we now have:

(a) With $u(x, bx) = g(x)$ and $v(ay, y) = h(y)$, say, the natural iteration scheme is to set:

$$u^{(0)}(x, y) := g(x), \quad v^{(0)}(x, y) := h(y),$$

and then define

$$u^{(n+1)}(x, y) := g\left(\frac{y}{b}\right) + \int_{\frac{x}{b}}^{x} v^{(n)}(s, y) \, ds,$$

and

$$v^{(n+1)}(x, y) := h\left(\frac{x}{a}\right) + \int_{\frac{y}{a}}^{y} u^{(n)}(x, s) \, ds,$$

for $n \geq 0$. Consider any rectangular neighborhood $R$ of the origin with the property that its upper-right corner and its lower-left corner both lie between the lines $M$ and $N$ in the first and third quadrants, respectively; see Figure 1. It is assumed that $R$ is sufficiently small so that $u^{(0)}$ and $v^{(0)}$ are both defined on $R$. It is immediate to verify that any such neighborhood provides a stable configuration in the following sense: whenever we start at a point $(x_0, y_0) \in R$ and move toward $N$ or $M$ along the integral curves of $r_1 = \partial_x$ or $r_2 = \partial_y$, respectively, we remain within $R$ until we meet $N$ and $M$, respectively.
In particular, this guarantees that all iterates \( u^{(n)} \) and \( v^{(n)} \), as defined above, are well-defined on \( R \). We express this by saying that the Stable Configuration Condition (SCC) is satisfied in this case, and this is the situation covered by our theorem below.

(b) The situation changes if we instead prescribe \( u \) along \( M \) and \( v \) along \( N \), say, \( u(ay, y) = g(y) \) and \( v(x, bx) = h(x) \). The natural iteration scheme is now to set:

\[
u^{(0)}(x, y) := g(y), \quad v^{(0)}(x, y) := h(x),
\]

and then define

\[
u^{(n+1)}(x, y) := g(ay) - \int_x^{ay} v^{(n)}(s, y) \, ds,
\]

and

\[
v^{(n+1)}(x, y) := h(bx) - \int_y^{bx} u^{(n)}(x, s) \, ds,
\]

for \( n \geq 0 \). The trouble with this is that there is no bounded neighborhood of the origin which provides a stable configuration in the above sense; see Figure 2. More precisely, one can verify that given any bounded neighborhood \( U \) of the origin, there is always a point \( (x_0, y_0) \in U \), such that, either in moving horizontally till intersecting \( M \), or in moving vertically till intersecting \( N \), one leaves \( U \). The upshot is that there is no fixed, bounded neighborhood on which all iterates are defined; the SCC is not met in this case.

Of course, given locally defined functions \( g \) and \( h \), we could extend them to all of \( \mathbb{R} \), and then run the iteration defined above on all of \( \mathbb{R}^2 \). However, even assuming convergence to a solution of the original system, it appears that the limiting solution would depend on the choice of extensions.

Indeed, a simple example illustrates that the uniqueness part of our Theorem might fail when the SCC is not met. To describe this accurately, we introduce the following terminology:

- we say that strong uniqueness holds if there is a fixed neighborhood \( U \) of the point \( \bar{x} \) with the property that any two solutions of the system, both of which are defined on \( U \), must agree on \( U \);
- we say that weak uniqueness holds if, given any two solutions of the system, each of which is defined on an open set about the point \( \bar{x} \), there is a neighborhood \( V \) (possibly depending on the given solutions) on which the two solutions agree.

The uniqueness claim in the Theorem below is that strong uniqueness holds under the SCC. In contrast, the following simple example shows that strong uniqueness might fail when the SCC is not met. Consider now the trivial system \( u_x = 0, v_y = 0 \), with constantly vanishing initial data prescribed along the lines \( M \) and \( N \) depicted in Figure 2. Consider the system as defined on any open ball about the origin. For any neighborhood \( U \) of
the origin on which we wish to define a solution, since the SCC is not met, there is either

- an interval \((y_0, y_1)\) such that every horizontal line \(y = \overline{y}\) with \(\overline{y} \in (y_0, y_1)\) intersects \(U\) but only meets \(M\) outside of \(U\), or
- an interval \((x_0, x_1)\) such that every vertical line \(x = \overline{x}\) with \(\overline{x} \in (x_0, x_1)\) intersects \(U\) but only meets \(N\) outside of \(U\).

Suppose the former. Then let \(u = F(y)\), where \(F\) is a bump function whose bump is located in \((y_0, y_1)\), and let \(v = 0\). These functions are a solution to the system on \(U\). Since \(u = 0, v = 0\) is clearly also a solution, we do not have uniqueness on \(U\). (If we instead have the latter situation, we instead let \(v = G(x)\) be a bump function.)

**Remark 2.2.** We note that weak uniqueness does hold for the example just considered, even though the SCC is not met. It would be of interest to know if this is the case for more general systems.

**Remark 2.3.** Returning to the system (16)-(17) above, we note that applying \(\partial_y\) to the first equation and using the second equation, yield the second order hyperbolic equation \(u_{xy} = 0\). Conversely, a solution of the latter equation yields, upon setting \(v := u_x\), a solution to the system (16)-(17). Thus, at least at the level of \(C^2\)-solutions, these are equivalent problems.

Equations of the form \(u_{xy} = F(x,y,u,u_x,u_y)\) have been studied extensively, starting with classical treatments by Riemann, Darboux, and Goursat. In particular, various types of boundary value problems have been considered; see [5, 6] and references therein. However, we are not aware of results that cover the particular situation above, i.e. with general data for \(u\) and \(u_x\) prescribed along two different non-characteristic curves through the origin, and a solution is sought on a full neighborhood of the origin.

### 3. Statement and Proof

We start by stating and proving a Lemma about certain restricted systems; and then state and prove our main theorem.

We are given open sets \(\Omega \subseteq \mathbb{R}^n\) and \(\Upsilon \subseteq \mathbb{R}^m\) together with a \(C^1\) frame \(\{r_i\}_{i=1}^n\) on \(\Omega\), and we fix a point \(\bar{x} \in \Omega\). For all positive integers \(p\), on all open subsets of \(\mathbb{R}^p\), we will use the max norm, denoted by \(\|\cdot\|_\infty\). The corresponding open ball of radius \(\epsilon > 0\) about a point \(y \in \mathbb{R}^p\) is denoted \(B_\epsilon(y)\).

Throughout this section we use the following conventions: the integers \(i, j, k\) satisfy \(1 \leq i, j, k \leq n\) and index the vector fields in the frame (derivations). The integers \(\alpha\) and \(\beta\) satisfy \(1 \leq \alpha, \beta \leq m\) and index the unknown functions. For each index \(\alpha\), \(I_\alpha\) will denote the set the indices of vector fields with respect to which the unknown function \(u_\alpha\) is differentiated in the original system (2). Both the elements and the cardinality of \(I_\alpha\) may vary with \(\alpha\). However, in order to avoid an extra index, whenever \(\alpha\)
is fixed, we simply write $I_\alpha = \{i_1, i_2, \ldots, i_{p_\alpha}\}$, where it is assumed that $1 \leq i_1 < i_2 < \cdots < i_{p_\alpha} \leq n$.

Finally, for each $\alpha$ we fix an $(n - p_\alpha)$-dimensional $C^1$ submanifold $\Xi_\alpha$ of $\Omega$ that is transverse to the span of vector fields $\{r_i \mid i \in I_\alpha\}$. We assume that the point $\bar{x}$ belongs to $\cap_{\alpha=1}^m \Xi_\alpha$ and that each $\Xi_\alpha$ is small enough to be covered by a single coordinate chart centered at $\bar{x}$. In other words, there exist $C^1$-diffeomorphisms $\xi_\alpha : \mathcal{U}_\alpha \to \Xi_\alpha$, where $\mathcal{U}_\alpha \subset \mathbb{R}^{n-p_\alpha}$ is an open neighborhood of the origin and $\xi_\alpha(0) = \bar{x}$. The submanifold $\Xi_\alpha$ will be where the the unknown function $u_\alpha$ is prescribed. We shall refer to the $\Xi_\alpha$ as data manifolds.

To formulate and prove our key lemma, we introduce a collection of new local coordinate charts near $\bar{x}$, one for each $\alpha$. For this let $W^t_i$ denote the flow of $r_i$:

$$\frac{d}{dt} W^t_i(x) = r_i|_{W^t_i(x)}, \quad W^0_i(x) = x \quad (i = 1, \ldots, n).$$

As the vector fields $r_i$ are $C^1$-smooth, their flows are defined and $C^1$-smooth with respect to $(t, x)$ on an $\mathbb{R} \times \mathbb{R}^n$-neighborhood of $(0, \bar{x})$ (see Theorem 2.6 in p. 81 in [8]).

For each $\alpha$, with $I_\alpha = \{i_1, i_2, \ldots, i_{p_\alpha}\}$, there exists an open neighborhood of the origin $\Theta_\alpha \subset \mathbb{R}^n$ small enough so that for each $t = (t_1, \ldots, t_n) \in \Theta_\alpha$ the map $\psi_\alpha : \Theta_\alpha \to \mathbb{R}^n$ given by

$$\psi_\alpha(t) := W^{t_{p_\alpha}}_{i_{p_\alpha}} \cdots W^{t_2}_{i_2} W^{t_1}_{i_1} \xi_\alpha(t_{p_\alpha+1}, \ldots, t_n).$$

is well-defined and $C^1$-smooth. Since, by assumption, the manifold $\Xi_\alpha$ is everywhere transverse to the span of vector fields $\{r_i \mid i \in I_\alpha\}$, it follows from the $C^1$-version of the inverse mapping theorem (Theorem 5.2 in p. 13 in [8]) that, after possibly shrinking $\Theta_\alpha$ near the origin, we may assume that $\psi_\alpha$ is a $C^1$-diffeomorphism between $\Theta_\alpha$ and its image $\Omega_\alpha := \psi_\alpha(\Theta_\alpha) \subset \mathbb{R}^n$. Thus we now have $m$ coordinate charts $(\Omega_\alpha, \psi_\alpha^{-1})$ near $\bar{x}$. From now on we shrink $\Xi_\alpha$ to $\Xi_\alpha \cap \Omega_\alpha$.

Lemma 3.3 below can be regarded as a “restricted” form of our main theorem, in which integrability conditions are not assumed and, consequently, we cannot conclude that the equations in the original system [2] have solutions defined on a full $\mathbb{R}^n$-neighborhood of $\bar{x}$. Instead, we will only conclude that each equation is satisfied on a certain submanifold of $\Omega$ (which varies from equation to equation). To define these submanifolds, for each $\alpha$, we introduce a sequence of manifolds $\Xi_\alpha^0, \ldots, \Xi_\alpha^{p_\alpha}$ by:

$$\Xi_\alpha^t := \{\psi_\alpha(t) \mid t = (t_1, \ldots, t_j, 0, \ldots, 0, t_{p_\alpha+1}, \ldots, t_n) \in \Theta_\alpha\}. \quad (19)$$

We observe that $\Xi_\alpha^0 \equiv \Xi_\alpha$, while $\Xi_\alpha^t$ is the set of points obtained by starting from a point in $\Xi_\alpha$ and then applying the flows $W^{t_1}_{i_1}, \ldots, W^{t_j}_{i_j}$, in that order. In particular, $\Xi_\alpha^{p_\alpha} \equiv \Omega_\alpha$.

The existence of the restricted solutions in Lemma 3.3 will be obtained via Picard iteration. As outlined in Section 2.2 on the Stable Configuration
Condition (SCC), we shall need to impose constraints on the coordinate charts \((\Omega, \psi^{-1}_\alpha)\) in order to guarantee that the various iterates are well-defined. To avoid a further level of technical details we choose to require that the SCC is satisfied in arbitrarily small neighborhoods. We first introduce the following terminology.

**Definition 3.1.** Let \(p \in \{1, \ldots, n\}\). An open neighborhood \(\Theta \subset \mathbb{R}^n\) of the origin is called \(p\)-accessible if for each \(t = (t_1, t_2, \ldots, t_n) \in \Theta\), the piecewise linear path from \((0, \ldots, 0, t_{p+1}, \ldots, t_n)\) to \(t\), with vertices

\[
(0, \ldots, 0, t_{p+1}, \ldots, t_n), \\
(t_1, 0, \ldots, 0, t_{p+1}, \ldots, t_n), \\
(t_1, t_2, 0, \ldots, 0, t_{p+1}, \ldots, t_n), \\
\vdots \\
(t_1, t_2, \ldots, t_n),
\]

belongs to \(\Theta\).

**Definition 3.2** (Stable Configuration Condition). In the context described above, let \(\mathcal{R} = \{r_1, \ldots, r_n\}\) be a local frame near \(\bar{x}\), and let a set of \(m\) submanifolds \(\Xi := \{\Xi_1, \ldots, \Xi_m\}\), of co-dimensions \(p_1, \ldots, p_m\) in \(\mathbb{R}^n\), respectively, and containing \(\bar{x}\), be given. We say that \((\mathcal{R}, \Xi)\) is a stable configuration near \(\bar{x} \in \Omega\) if for every \(\epsilon > 0\), there exists an open neighborhood \(\Omega_\epsilon \subset B_\epsilon(\bar{x})\) of \(\bar{x}\) and \(p_\alpha\)-accessible neighborhoods \(\Theta_{\alpha, \epsilon} \subset \mathbb{R}^n\), \(\alpha = 1, \ldots, m\), such that for all \(\alpha\) the map \(\psi_\alpha: \Theta_{\alpha, \epsilon} \to \Omega_\epsilon\), defined by \(\Xi_\alpha\), is a \(C^1\)-diffeomorphism. The neighborhoods \(\Omega_\epsilon\) are called stable neighborhoods.

The significant part of this definition is that the same neighborhood \(\Omega_\epsilon\) works for all \(\alpha = 1, \ldots, m\). Also, \(\Omega_\epsilon\) is stable (or invariant) in the sense that, given any point \(x \in \Omega_\epsilon\), we can flow back to each \(\Xi_\alpha\) along integral curves of the vector fields \(\{r_i\}_{i \in I_\alpha}\) (in decreasing order) without leaving \(\Omega_\epsilon\). The following lemma provides a key technical step and will be applied twice in the proof of the main theorem.

**Lemma 3.3.** Let \(\Omega \subset \mathbb{R}^n\) and \(\Upsilon \subset \mathbb{R}^m\) be open subsets. For \(\alpha = 1, \ldots, m\), let \(I_\alpha = \{i_1, i_2, \ldots, i_{p_\alpha}\}\) be an ordered set of indices: \(1 \leq i_1 < i_2 < \cdots < i_{p_\alpha} \leq n\). Assume the following:

1. \(\mathcal{R} = \{r_i\}_{i=1}^n\) is a \(C^1\) frame on \(\Omega\);
2. for each \(\alpha\), \(\Xi_\alpha\) is an embedded \(C^1\) manifold in \(\Omega\), and \(\bar{x} \in \cap_\alpha \Xi_\alpha\);
3. the manifold \(\Xi_\alpha\) is of codimension \(p_\alpha\) and is everywhere transverse to the span of \(\{r_i\}_{i \in I_\alpha}\);
4. the frame \(\mathcal{R}\) and the set of manifolds \(\Xi := \{\Xi_1, \ldots, \Xi_m\}\) is a stable configuration near \(\bar{x} \in \Omega\) according to Definition 3.2;
5. for each \(\alpha\), \(g_\alpha: \Xi_\alpha \to \mathbb{R}\) is \(C^1\)-smooth and bounded, and \(\bar{g} := (g_1(\bar{x}), g_2(\bar{x}), \ldots, g_m(\bar{x})) \in \Upsilon\);
Then there is a neighborhood \( \tilde{\Omega} \ni \tilde{x} \) on which there is a unique solution \( u : \tilde{\Omega} \to \mathbb{Y} \) to the system
\[
\mathbf{r}_{ij}(u_\alpha)\big|_x = f^\alpha_i(x,u(x)) \quad \text{for } 1 \leq \alpha \leq m, \ i_j \in I_\alpha, \text{ and } x \in \Xi^i_{\alpha} \cap \tilde{\Omega} \tag{20}
\]
satisfying the data
\[
u_\alpha(x) = g_\alpha(x) \quad \text{for } x \in \Xi_\alpha \cap \tilde{\Omega}. \tag{21}
\]

**Remark 3.4.** Note that, in (20), we need to employ double subscripts \( i_j \) for the elements of the ordered set \( I_\alpha \) since the position (the \( j \)th, say) of an index in \( I_\alpha \) relates to which submanifold \( \Xi^i_{\alpha} \) is considered.

**Proof.** Let \( L \) be a common Lipschitz constant for the functions \( f^\alpha_i \): \[
|f^\alpha_i(x,u) - f^\alpha_i(x,v)| \leq L \|u - v\|_\infty \quad \text{whenever } x \in \Omega, \ u, v \in \mathbb{Y}. \tag{22}
\]
We let \( M \) be a common bound for the functions \( f^\alpha_i \):
\[
|f^\alpha_i(x,u)| \leq M \quad \text{for } (x,u) \in \Omega \times \mathbb{Y}. \tag{23}
\]
Choose \( r > 0 \) such that \( \bar{B}_r(\tilde{g}) \) (the closed ball under the sup norm) is contained in \( \mathbb{Y} \) and put
\[
\bar{\mathbb{Y}} := \bar{B}_r(\tilde{g}). \tag{24}
\]
Shrink \( \Omega \) to \( \Omega' \ni \tilde{x} \) such that
\[
|g_\alpha(x) - g_\alpha(\tilde{x})| \leq \frac{1}{2} r \quad \text{for each } \alpha \text{ and for } x \in \Xi_\alpha \cap \Omega'. \tag{25}
\]
Finally, using assumption (4), we choose a bounded and stable neighborhood \( \bar{\Omega} \subset \Omega' \), with accessible neighborhoods \( \Theta_\alpha \) and diffeomorphisms \( \psi_\alpha : \Theta_\alpha \to \bar{\Omega} \), according to Definitions 3.1 and 3.2. If necessary, we may shrink \( \bar{\Omega} \), and hence each \( \Theta_\alpha \), so as to have
\[
nL\|t\|_\infty \leq \frac{1}{2} \quad \text{and} \quad nM\|t\|_\infty \leq \frac{1}{2} r \quad \text{for all } t \in \Theta_\alpha, \ \alpha = 1, \ldots, m. \tag{26}
\]
Next, let \( \mathcal{C} \) denote the set of continuous functions \( \bar{\Omega} \to \bar{\mathbb{Y}} \). Define a functional \( \Phi : \mathcal{C} \to \mathcal{C} \) by defining its components according to
\[
\Phi[u]_\alpha(x) := g_\alpha(\xi_\alpha) + \int_0^{t_1} f^\alpha_{i_1}(\psi_\alpha(s,0,\ldots,0,t_{p_\alpha}), u(\psi_\alpha(s,0,\ldots,0,t_{p_\alpha}))) \, ds
+ \int_0^{t_2} f^\alpha_{i_2}(\psi_\alpha(t_1,0,\ldots,0,t_{p_\alpha}), u(\psi_\alpha(t_1,0,\ldots,0,t_{p_\alpha}))) \, ds
+ \cdots
+ \int_0^{t_{p_\alpha}} f^\alpha_{i_{p_\alpha}}(\psi_\alpha(t_1,\ldots,t_{p_{\alpha-1}},0,t_{p_\alpha}), u(\psi_\alpha(t_1,\ldots,t_{p_{\alpha-1}},0,t_{p_\alpha}))) \, ds. \tag{27}
\]
In the above equation, the values \( (t_1,t_2,\ldots,t_{n}) \) are chosen so that \( x = \psi_\alpha(t_1,\ldots,t_n) \). Since \( \psi_\alpha \), given by (18) is a diffeomorphism, this choice
is unique. We have also used the abbreviations: $t_{p_\alpha} := (t_{p_\alpha+1}, \ldots, t_n)$, $\xi_\alpha := \xi_\alpha(t_{p_\alpha})$. Note that the function $\Phi[u]$ is well-defined since $u \in \mathcal{C}$ and since the neighborhoods $\Theta_\alpha$ are accessible; this is the technical reason for imposing the Stable Configuration Condition.

To verify that $\Phi$ in fact maps $\mathcal{C}$ into itself, we assume $u \in \mathcal{C}$ and $x \in \bar{\Omega}$ and show that the right hand side of (27) belongs to $\bar{\Upsilon} = \overline{B}_r(\bar{g})$. To this end, observe that

$$\|\Phi[u](x) - \bar{g}\|_\infty = \max_{\alpha} |\Phi[u]_\alpha(x) - g_\alpha(\bar{x})|$$

$$\leq \max_{\alpha} \left\{ |g_\alpha(\xi) - g_\alpha(\bar{x})| + \sum_{j=1}^{p_\alpha} \int_0^{t_j} f_{ij}^\alpha(\psi_\alpha, u(\psi_\alpha)) \, ds \right\}$$

$$\leq \max_{\alpha} \left(\frac{1}{2}r + nM \cdot \left( \max_{1 \leq j \leq p_\alpha} |t_j| \right) \right) \leq r,$$

where we have omitted the arguments of $\psi_\alpha$. Here, to obtain line (28) we used the definition (27) of the functional $\Phi$ and the triangle inequality. To obtain (29) we used statement (25) for the first term and the triangle inequality together with (23) and the fact that $p_\alpha \leq n$ for the second term. Finally, we have used (26).

Equip $\mathcal{C}$ with the metric $d(u, v) = \sup_{x \in \Omega} \|u(x) - v(x)\|_\infty$. With this metric, $\mathcal{C}$ is a complete metric space. We now show that $\Phi$ is a contraction mapping. Let $u, v \in \mathcal{C}$, and estimate

$$d(\Phi[u], \Phi[v]) = \sup_{x \in \Omega} \|\Phi[u](x) - \Phi[v](x)\|_\infty = \sup_{x \in \Omega} \max_{\alpha} |\Phi[u]_\alpha(x) - \Phi[v]_\alpha(x)|$$

$$\leq \sup_{x \in \Omega} \max_{\alpha} \sum_{j=1}^{23} \int_0^{t_0} \left| f_{ij}^\alpha(\psi_\alpha, u(\psi_\alpha)) - f_{ij}^\alpha(\psi_\alpha, v(\psi_\alpha)) \right| \, ds$$

$$\leq n \max_{1 \leq j \leq m} |t_j| L \cdot \sup_{y \in \Omega} \|u(y) - v(y)\|_\infty$$

$$\leq \frac{1}{2} d(u, v),$$

where, again, we have omitted the arguments of $\psi_\alpha$. To obtain line (30), we used the definition (27) of $\Phi$ and the triangle inequality. To obtain line (31), we used the fact that each $f_{ij}^\alpha$ has Lipschitz constant $L$ and that $p_\alpha \leq n$. To obtain line (32), we used the first inequality in (26) and the definition of $d(u, v)$. Thus $\Phi$ is a (uniformly) strict contraction. It follows that $\Phi$ has a unique fixed point in $\mathcal{C}$, which we denote $\bar{u}$. Here we used notation $t_0^{\alpha_j}$, to
emphasize that the pre-image of $x$ under $\psi_\alpha$ depends on $\alpha$. Thus

$$\tilde{u}_\alpha(x) = g_\alpha(\xi_\alpha) + \int_0^{t_1} f^\alpha_{i_1}(\psi_\alpha(s, 0, \ldots, 0, t_{p_\alpha+}), \tilde{u}(\psi_\alpha(s, 0, \ldots, 0, t_{p_\alpha+}))) \, ds$$

$$+ \int_0^{t_2} f^\alpha_{i_2}(\psi_\alpha(t_1, s, 0, \ldots, 0, t_{p_\alpha+}), \tilde{u}(\psi_\alpha(t_1, s, 0, \ldots, 0, t_{p_\alpha+}))) \, ds$$

$$+ \cdots$$

$$+ \int_0^{t_{p_\alpha}} f^\alpha_{i_{p_\alpha}}(\psi_\alpha(t_1, t_2, \ldots, t_{p_\alpha-1}, s, t_{p_\alpha+}), \tilde{u}(\psi_\alpha(t_1, t_2, \ldots, t_{p_\alpha-1}, s, t_{p_\alpha+}))) \, ds.$$  \hfill (33)

Since $t_1 = \cdots = t_{p_\alpha} = 0$ and $\xi_\alpha \equiv \xi_\alpha(t_{p_\alpha+}) = x$, whenever $x \in \Xi_\alpha$, the function $\tilde{u}$ satisfies the data (21). Note that on the manifold $\Xi_\alpha$, we have $t_k = 0$ for $j < k \leq p_\alpha$, and also $r_{ij} = \partial_{ij}$. The latter follows since, for any smooth function $h : \tilde{\Omega} \to \mathbb{R}$, whenever $x \in \Xi_\alpha$, we have

$$\partial_i h(x) = \partial_{ij} h(W_j \cdots W_{i_1} \xi_\alpha) = \nabla h \cdot r_{ij} \bigg|_{W_j \cdots W_{i_1} \xi_\alpha} = r_{ij} h(x).$$ \hfill (34)

Thus, with equation (33) restricted to $\Xi_\alpha$, we apply the fundamental theorem of calculus and obtain that, for $x \in \Xi_\alpha$,

$$r_{ij} \tilde{u}_\alpha(x) = \partial_{ij} \tilde{u}_\alpha(x)$$

$$= f^\alpha_{i_j}(\psi_\alpha(t_1, \ldots, t_j, 0, \ldots, 0, t_{p_\alpha+1}), \tilde{u}(\psi_\alpha(t_1, \ldots, t_j, 0, \ldots, 0, t_{p_\alpha+1})))$$

$$= f^\alpha_{i_j}(x, \tilde{u}(x)),$$

showing that $\tilde{u}$ is indeed a solution of (20)-(21).

It remains to show that $\tilde{u}$ is the unique solution of (20)-(21) on $\tilde{\Omega}$. Assuming $v : \tilde{\Omega} \to \mathcal{T}$ is also a solution to (20)-(21) in $\tilde{\Omega}$, we have

$$d(\tilde{u}, v) = \sup_{x \in \tilde{\Omega}} \| \tilde{u}(x) - v(x) \|_\infty = \sup_{x \in \tilde{\Omega}} \max_\alpha | \tilde{u}_\alpha(x) - v_\alpha(x) |$$

$$= \sup_{x \in \tilde{\Omega}} \max_\alpha \left| \sum_{j=1}^{p_\alpha} \int_0^{t_{i_j}} f^\alpha_{i_j}(\psi_\alpha, \tilde{u}(\psi_\alpha)) - f^\alpha_{i_j}(\psi_\alpha, v(\psi_\alpha)) \, dx \right|$$ \hfill (35)

$$\leq \sup_{y \in \tilde{\Omega}} \max_\alpha \sum_{j=1}^{p_\alpha} |t_{i_j}| L \| \tilde{u}(y) - v(y) \|_\infty$$ \hfill (36)

$$\leq \frac{1}{2} \sup_{y \in \tilde{\Omega}} \| \tilde{u}(y) - v(y) \|_\infty = \frac{1}{2} d(\tilde{u}, v),$$ \hfill (37)

where, as above, $x = \psi_\alpha(t_1, \ldots, t_{p_\alpha})$ and we have omitted the arguments of $\psi_\alpha$. Line (35) follows from the fact that both $\tilde{u}$ and $v$, being solutions to (20)-(21), satisfy (33). Line (36) follows from the triangle inequality, the Lipschitz property (22), and the fact that the neighborhoods $\Theta_\alpha$ are accessible (so that the $\psi_\alpha$ take values in $\tilde{\Omega}$). The inequality in (37) follows from the first inequality in (26). Now, $\tilde{\Omega}$ is, by choice, a bounded set in which each point can be reached by starting on any data manifold $\Xi_\alpha$ and then
moving along a finite number of integral curves of the vector fields $r_i$. Since we assume that the initial data $g_\alpha$ are bounded, and also that the functions $f_i^\alpha : \Omega \times \Upsilon \to \mathbb{R}$ are uniformly bounded, it follows that the solutions $\tilde{u}$ and $v$ are both bounded on $\tilde{\Omega}$. Therefore, $\sup_{x \in \tilde{\Omega}} \|\tilde{u}(x) - v(x)\|_\infty < \infty$ and it follows from the inequalities above that $d(\tilde{u}, v) = 0$, i.e. $\tilde{u} \equiv v$ on $\tilde{\Omega}$. 

**Theorem 1.** Suppose that, in addition to hypotheses (1)–(6) of Lemma 3.3, we also have that:

1. the functions $f_i^\alpha$ belong to $C^1(\Omega \times \Upsilon)$;
2. for each $\alpha$, for each $i, j \in I_\alpha$ with $i \neq j$, and for each $\beta$: if $i \notin I_\beta$, then $\partial_{x^\beta} f_j^\alpha = 0$;
3. for each $\alpha$, the vector fields $\{r_i\}_{i \in I_\alpha}$ are in involution, i.e. $[r_j, r_k] \in \text{span}\{r_i\}_{i \in I_\alpha}$ whenever $j, k \in I_\alpha$;
4. for each $\alpha$, for each $i, j \in I_\alpha$ with $i \neq j$, and for all $(x, u) \in \Omega \times \Upsilon$:
   \[
   (\nabla_x f_i^\alpha)|_{(x, u)} \cdot r_i|_x + \sum_{\beta \in I_\beta} \partial_{x^\beta} f_j^\alpha(x, u) f_j^\beta(x, u) - \nabla_x (f_i^\alpha)|_{(x, u)} \cdot r_j|_x - \sum_{\beta \in I_\beta} \partial_{x^\beta} f_j^\alpha(x, u) f_j^\beta(x, u) = \sum_{k \in I_\alpha} c_{ij}^k(x) f_k^\alpha(x, u),
   \]  
   where here and below $\nabla_x (f_i^\alpha)$ denotes the gradient with respect to the variables $x$ and $c_{ij}^k$ denote the structure coefficients of the frame:
   \[
   [r_i, r_j]|_x = \sum_{k=1}^n c_{ij}^k(x) r_k|_x \quad \text{for } x \in \Omega \text{ and } 1 \leq i, j \leq n.
   \]
5. for each $\alpha$ and each $i, j, k \in I_\alpha$, the structure coefficient $c_{ij}^k(x)$ is uniformly bounded on $\Omega$.

Then there is a neighborhood $\bar{\Omega}$ of $\bar{x}$ on which there is a unique solution to the system

$$r_i(u_\alpha)|_x = f_i^\alpha(x, u(x)) \quad \text{for } 1 \leq \alpha \leq m, i \in I_\alpha, \text{ and } x \in \bar{\Omega},$$

satisfying the data

$$u_\alpha(x) = g_\alpha(x) \quad \text{for } x \in \Xi_\alpha \cap \bar{\Omega}.$$

Before giving the proof we make a few remarks. First, as remarked above, the difference between the conclusion of Theorem 1 and the conclusion of Lemma 3.3 is that in the Theorem, the equations of system (39) are satisfied everywhere in $\tilde{\Omega}$, while in the Lemma, each equation of system (20) is only guaranteed to be satisfied only for $x \in \Xi_{\alpha} \cap \tilde{\Omega}$. The data (21) and (40) are identical.

The integrability conditions appearing in equations (38) are the generalization of the condition of mixed partial derivatives being equal, to the case of non-commutative derivations. They correspond to the integrability conditions in the PDE version of the classic Frobenius Theorem (see the
which should hold for any function $u_\alpha$, and then, once fully expanded, making substitutions of the form

$$u \equiv f_i^\alpha(x,u),$$

which should hold for any function $u_\alpha$, and then, once fully expanded, making substitutions of the form $r_i(u_\alpha) = f_i^\alpha(x,u)$, which should hold for any solution $u = (u_1, \ldots, u_m)$ of (39).

The restricted summations of the form $\{\beta : i \in I_\beta\}$, $\{\beta : j \in I_\beta\}$, and $\{k \in I_\alpha\}$ in (38) ensure that (38) only contains functions $f_i^\alpha$ which actually are defined by the system (39). For instance, examining the first summation in (38), if we included an index $\beta$ with $i \notin I_\beta$, then the factor $f_i^\alpha(x,u)$ would be a function that is not given by the system (39). Similar remarks apply to the other two summations.

Hypotheses (8) and (9) are necessary so that in making the restrictions on summation indices just described, we have not actually omitted any terms that should appear in the expansion of (41). For instance, for the omitted indices $\beta$ with $i \notin I_\beta$ from the first summation of (38), hypothesis (8) guarantees that $\partial u_\beta f_i^\alpha(x,u) \equiv 0$, and so we have not actually missed any terms. Similar remarks apply to hypothesis (9) and the last summation.

**Proof.** Apply Lemma 3.3 to obtain a neighborhood $\bar{\Omega}$ of $\bar{x}$ on which there is a unique solution $\bar{u}$ to the system (20) satisfying (40). It remains only to show that $\bar{u}$ is a solution to the full system (39); uniqueness is already established since any solution of (39) is also a solution of (20).

Fix (for now) an index $\alpha$. As in the formulation of Lemma 3.3 we will use double indices $i_j$ for the elements of the ordered set $I_\alpha = \{i_1, \ldots, i_{p_\alpha}\}$. For each $i_j \in I_\alpha$ (i.e., $1 \leq j \leq p_\alpha$) we define the function $A_{ij}^\alpha : \Omega \to \mathbb{R}$ by

$$A_{ij}^\alpha(x) := r_{ij}(\bar{u}_\alpha) - f_i^\alpha(x, \bar{u}(x)).$$

Then, for each $i_k \in I_\alpha$ with $j < k \leq p_\alpha$, we apply $r_{ik}$ to (42) to obtain

$$r_{ik} (A_{ij}^\alpha) \big|_x = r_{ik}(r_{ij}(\bar{u}_\alpha)) \big|_x - \langle \nabla_x f_i^\alpha \rangle_{(x,\bar{u}(x))} \cdot r_{ik} \big|_x$$

$$- \sum_{\beta : i_k \in I_\beta} \partial u_{\beta} f_i^\alpha(x, \bar{u}(x)) r_{ik}(\bar{u}_\beta) \big|_x$$

$$= r_{ij}(r_{ik}(\bar{u}_\alpha)) \big|_x + \sum_{l \in I_\alpha} c_{ij}^l(x) r_{ij}(\bar{u}_\alpha) \big|_x$$

$$- \langle \nabla_x f_i^\alpha \rangle_{(x,\bar{u}(x))} \cdot r_{ik} \big|_x - \sum_{\beta : i_k \in I_\beta} \partial u_{\beta} f_i^\alpha(x, \bar{u}(x)) r_{ik}(\bar{u}_\beta) \big|_x.$$

We note that the summation in line (43) is restricted to $\{\beta : i_k \in I_\beta\}$ by hypothesis (8), and that the summation in line (44) is restricted to $\{l \in I_\alpha\}$ by hypothesis (9). We now restrict the last equation above to $x \in \Xi_\alpha^\alpha$, etc.
where, according to the conclusion of Lemma 3.3, \( r_k(\tilde{u}_\alpha) = f_{ik}^\alpha(x, \tilde{u}) \). Thus, for \( x \in \Xi_k^\alpha \) we have

\[
(45)
\]

We now consider (48) as a system of differential equations for the unknowns \( A_{ij}^\alpha \), with vanishing data prescribed along appropriate submanifolds. The trivial functions \( A_{ij}^\alpha \equiv 0 \) clearly provide a solution. The goal is to apply the uniqueness part of Lemma 3.3 to the new system (48), and conclude that the trivial solution is the only one. That is, \( A_{ij}^\alpha \) defined by (42) must vanish identically near \( \bar{x} \). We proceed with verifying that the assumptions of Lemma 3.3 are satisfied for the system (48).

To do so we introduce the following notations. Let \( J = \{(j, \alpha) | 1 \leq \alpha \leq m; 1 \leq j \leq p_\alpha \} \), and for each double index \((j, \alpha) \in J \) define \( \Lambda_{j, \alpha} := \Xi_k^\alpha \), where the right-hand side is given by (19). Thus, the \( C^1 \) submanifolds \( \Lambda_{j, \alpha} \)
can be parametrized by the functions
\[
\lambda_{j,\alpha}(s_1, \ldots, s_{n-p_a+j}) := W_{i_j}^{s_j} \cdots W_{i_1}^{s_1} \xi_\alpha(s_{j+1}, \ldots, s_{n-p_a+j})
\]
\[= \psi_\alpha(s_1, s_j, 0, \ldots, 0, s_{j+1}, \ldots, s_{n-p_a+j}),\]
which are defined for all \((s_1, \ldots, s_{n-p_a+j})\) with the property that the point \((s_1, s_j, 0, \ldots, 0, s_{j+1}, \ldots, s_{n-p_a+j})\) belongs to \(\Theta_\alpha\). The domain of \(\lambda_{j,\alpha}\) is, therefore, the intersection of \(\Theta_\alpha\) with the coordinate subset of \(\mathbb{R}^n\) where \(t_{j+1} = \cdots = t_{p_a} = 0\).

We let the data \(h_{j,\alpha}\) for the unknown \(A^\alpha_i\) in the system (48) vanish identically on the manifolds \(\Lambda_{j,\alpha}: h_{j,\alpha}(x) \equiv 0\) for \(x \in \Lambda_{j,\alpha}\). Finally, setting \(J_{j,\alpha} := \{k \mid j < k \leq p_a\}\), we have that (48) yields an equation for \(r_k(A^\alpha_i)\) whenever \((j, \alpha) \in J\) and \(k \in J_{j,\alpha}\).

Now consider the hypotheses of Lemma 3.3 in the context of system (48):

(1)’ \(\mathcal{R} = \{r_i\}_{i=1}^p\) is a \(C^1\) frame on \(\tilde{\Omega}\);

(2)’ \(\Lambda_{j,\alpha}, (j, \alpha) \in J\), are embedded \(C^1\) submanifolds of \(\tilde{\Omega}\), and they share a common point \(\tilde{x} \in \cap_{(j,\alpha) \in J} \Lambda_{j,\alpha}\);

(3)’ each manifold \(\Lambda_{j,\alpha}\) is of codimension \(p_a - j\) and is everywhere transverse to the span of \(\{r_k\}_{k \in J_{j,\alpha}}\);

(4)’ the frame \(\mathcal{R}\) and the set of manifolds \(\Lambda = \{\Lambda_{j,\alpha} \mid (j, \alpha) \in J\}\) is a stable configuration near a point \(\tilde{x} \in \tilde{\Omega}\);

(5)’ for each double index \((j, \alpha) \in J\), \(h_{j,\alpha}: \Lambda_{j,\alpha} \to \mathbb{R}\) is bounded and \(C^1\)-smooth, with \(h(\tilde{x}) \in \mathbb{R}^{|J|}\);

(6)’ for each double index \((j, \alpha) \in J\) and each \(k \in J_{j,\alpha}\), the right-hand side of (48) is a function \(\tilde{\Omega} \times \mathbb{R}^{|J|} \to \mathbb{R}\) which is bounded, continuous, and also uniformly Lipschitz in its second argument.

The statements (1)’, (2)’, (3)’, and (5)’ are self-evident, while statement (6)’ follows from assumption (6) of Lemma 3.3 together with assumptions (7) and (11) of the present theorem.

Finally, to verify (4)’ we show that stability of the configuration \((\mathcal{R}, \Lambda)\) follows from that of the configuration \((\mathcal{R}, \Xi)\) (which holds according to assumption (6) of Lemma 3.3). By stability of the configuration \((\mathcal{R}, \Xi)\), let \(\tilde{\Omega}\) be a neighborhood of \(\tilde{x}\) for which there is, for each \(\alpha = 1, \ldots, m\), a \(p_a\)-accessible neighborhood \(\Theta_\alpha \subset \mathbb{R}^n\) such that the map \(\psi_\alpha: \Theta_\alpha \to \tilde{\Omega}\) given by (18) is a \(C^1\)-diffeomorphism (see Definitions 3.1-3.2). Then, for \((j, \alpha) \in J\), the map
\[
\tilde{\psi}_{j,\alpha}(t_1, \ldots, t_n) := W_{t_{p_a-j}}^{t_{p_a}} \cdots W_{t_1}^{t_1} \lambda_{j,\alpha}(t_{p_a-j+1}, \ldots, t_n)
\]
\[= \psi_\alpha(t_{p_a-j+1}, \ldots, t_{p_a}, t_1, \ldots, t_{p_a-j}, t_{p_a+1}, \ldots, t_n)\]
is defined for all \((t_1, \ldots, t_n) \in \mathbb{R}^n\) with the property that
\[\left(t_{p_a-j+1}, \ldots, t_{p_a}, t_1, \ldots, t_{p_a-j}, t_{p_a+1}, \ldots, t_n\right) \in \Theta_\alpha.\]
Thus, the domain $\hat{\Theta}_{j,\alpha}$ of $\hat{\psi}_{j,\alpha}$ is the image of $\Theta_{\alpha}$ under a permutation-of-coordinates map $\Pi$ that interchanges the block of 1st-through-$j$th coordinates with the block of $(j+1)$th-through-$(p_{\alpha}-1)$th coordinates. The map $\hat{\psi}_{j,\alpha} = \psi_{\alpha} \circ \Pi^{-1}$ is therefore a $C^1$-diffeomorphism $\hat{\Theta}_{j,\alpha} \to \hat{\Omega}$. It is straightforward to verify that $(p_{\alpha}-j)$-accessibility of $\hat{\Theta}_{j,\alpha}$ follows from $p_{\alpha}$-accessibility of $\Theta_{\alpha}$ (see Definition 3.1). This shows that assumption (4)' is satisfied. Thus, according to Lemma 3.3, there is a neighborhood $\tilde{\Omega}'$ of $\bar{x}$ in $\mathbb{R}^n$ on which there exists a unique set of functions $\{A^a_{ij}(x) \mid (j, \alpha) \in J\}$ solving (48) (with vanishing data on the $\Lambda_{k,j,\alpha}$), for all $x \in \Lambda_{k,j,\alpha} \cap \tilde{\Omega}'$, where $\Lambda_{k,j,\alpha}$ are defined similarly to (19) by

$$\Lambda_{k,j,\alpha} := \{\hat{\psi}_{j,\alpha}(t) \mid t = (t_1, \ldots, t_{k-j}, 0, \ldots, 0, t_{p_{\alpha}-j+1}, \ldots, t_n) \in \hat{\Theta}_{j,\alpha}\} \quad (49)$$

Unwinding the definition of $\hat{\psi}_{j,\alpha}(t)$ we see that $\Lambda_{k,j,\alpha}$ is obtained by starting at a point $\Lambda_{j,\alpha}$, and then flowing in turn along the vectors fields whose indices appear no later than the $(k-j)$th member of $J_{j,\alpha}$. Recalling that $\Lambda_{j,\alpha} = \Xi_{\alpha}^j$ and observing that the $(k-j)$th member of $J_{j,\alpha}$ is the $k$-th member of $I_{\alpha}$, we conclude that $\Lambda_{k,j,\alpha}$ equals to $\Xi_{\alpha}^k$, on which (48) is to hold.

Finally, since the identically vanishing functions $A^a_{ij}(x) \equiv 0$ satisfy (48) as well as the data, it follows from the uniqueness part of Lemma 3.3 that the functions defined by (42) are identically zero on on $\tilde{\Omega}'$, that is,

$$r_{ij}(\tilde{u}_{\alpha}) \mid_{x = f^a_{ij}(x, \tilde{u}(x))} = f^a_{ij}(x, \tilde{u}(x)) \quad \text{for all } x \in \tilde{\Omega}',$$

as was to be shown. \hfill \Box

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