Disturbance in weak measurements and the difference between quantum and classical weak values

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Abstract

The role of measurement induced disturbance in weak measurements is of central importance for the interpretation of the weak value. Uncontrolled disturbance can interfere with the postselection process and make the weak value dependent on the details of the measurement process. Here we develop the concept of a generalized weak measurement for classical and quantum mechanics. The two cases appear remarkably similar, but we point out some important differences. A priori it is not clear what the correct notion of disturbance should be in the context of weak measurements. We consider three different notions and get three different results: (1) For a ‘strong’ definition of disturbance, we find that weak measurements are disturbing. (2) For a weaker definition we find that a general class of weak measurements are non-disturbing, but that one gets weak values which depend on the measurement process. (3) Finally, with respect to an operational definition of the ‘degree of disturbance’, we find that the AAV weak measurements are the least disturbing, but that the disturbance is always non-zero.

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I. INTRODUCTION

It has been proposed that weak values\(^1\) could serve as an operational definition of the expectation values of observables in the intermediate time between the preparation of a system and a postselection on the final state of the system. If one were to employ standard (strong) measurements between the preparation and postselection the inevitable disturbance caused by the measurement (see e.g. the “No Information Gain Without Disturbance” theorem of Ref. \(^4\)) would interfere with the postselection. The central idea of weak measurements is to avoid the issue of disturbance by making the interaction between the measurement apparatus and the system during the intermediate measurement arbitrarily small. This was expressed explicitly in Ref. \(^3\) as:

...the [weak] measurements hardly disturb the ensemble, and therefore they characterize the ensemble during the whole intermediate time [between preparation and postselection].

Motivated by recent work\(^5, 6\) we explore role of disturbance and the relation between classical and quantum mechanical weak measurements.

For a classical systems the expectation value of some observable between a preparation and a postselection has a perfectly unambiguous meaning, and if one applies the weak measurement procedure with a non-disturbing measurement one recovers the ‘correct’ expectation value. However, if one allows the intermediate measurement to disturb the system, even if this disturbance goes to zero along with the interaction strength, one can get results that deviates from this value (see Section III). This demonstrates the importance of understanding how the disturbance vanishes in the weak limit.

To gain a better understanding of this problem, we find it useful to develop the theory of both classical and quantum weak measurements. The parallels between quantum and non-ideal classical measurements have recently been highlighted\(^5, 7\), see also Refs. \(^8–10\). We start by introducing a notion of generalized weak measurements within an operational framework\(^39\) which is general enough to encompass both classical and quantum mechanics. When applied to a classical system, the (generalized) weak expectation value takes the form

\[
\frac{\sum_{j,k} q_k A_{kj} p_j}{\sum_j q_j p_j}. \quad \text{(classical)}
\]  

(1)
Where $\tilde{A}$ is a real matrix, $p_j$ is the probability for the system to be prepared in state $j$, and $q_j$ is the probability for the postselection to succeed given the system is in state $j$. In the quantum case the generalized weak expectation value takes the form

$$\text{Re} \frac{\langle \phi | \hat{A} | \psi \rangle}{\langle \phi | \psi \rangle}, \quad \text{(quantum)} \quad (2)$$

which is the standard AAV form, except that $\hat{A}$ is not Hermitian in general. Weak values of non-Hermitian operators have been considered previously. We show that for any real matrix $\tilde{A}$ there is a weak measurements procedure that yields (1) as its expectation value by an explicit example. Similarly we show that any operator $\hat{A}$ can in principle appear in (2).

While (1) and (2) look similar, the possibility of (quantum) interference in the denominator of (2) makes an important difference. In particular the classical weak measurement only exhibits anomalous weak values when $\tilde{A}$ is not diagonal (which implies the measurement process disturbs the system), while anomalous weak values occur in the quantum case for any non-trivial (i.e. not proportional to the identity) $\hat{A}$.

For the generalized weak measurements, the only constraint on the disturbance induced by the measurement is that it should vanish at vanishing interaction. To control the disturbance we introduce two different constraints in the general framework. Both constraints lead to the usual notion of a non-disturbing measurement when applied to classical mechanics. For quantum mechanics the situation is more intriguing; One of the constraints is impossible to satisfy, while the other one can be satisfied by all measurements (strictly speaking you have to change the measurement procedure slightly, but this change has no effect on the actual measurement outcome).

Another way to control the disturbance is to introduce some quantitative measure of the amount of disturbance. We introduce such a measure following Ref. [15], and we show under quite general assumptions that the measurements minimizing this quantity lead to the usual AAV weak value. This results should be compared to the uniqueness theorem of Ref. [16].

Disturbance in weak measurements has previously been analyzed through Leggett-Garg inequalities. In particular it has been shown that anomalous weak values imply either that the measurement is disturbing or that macrorealism fails. In this paper we will take a purely operational point of view, and as a consequence assumptions such as macrorealism will play no role.
The outline of the article is as follows: In Section II we introduce generalized weak measurements in a general operational formalism. We then consider weak measurements in classical mechanics in Section III. Section IV forms the main part of these notes and deals with weak measurements in quantum mechanics. We end in Section V with a discussion and outlook on some questions that would be interesting to address in further work. Appendix A addresses some ways to generalize the formalism, while appendix C deals with the special case of von Neumann measurements. Finally appendices D and E contains some technical details.

II. GENERAL FORMALISM

Here we will described generalized weak measurements in an operational framework which is independent of the details of the physical system under consideration. Our framework can be seen as a variation of the Generalized Probabilistic Theories, see Refs. [20, 21] for recent expositions.

Let $S$ be the set of preparation procedures of the system. In classical mechanics an element of $S$ would be a probability distribution on the systems phase space, while for quantum systems the elements are density matrices. For brevity we will often refer to the elements of $S$ as states. Our measurement apparatus will have a finite number of outcomes, and we will use the index $m$ to denote a specific outcome. Given a state $s \in S$, the probability of getting outcome $m$ is denoted $P^\lambda(m|s)$. The non-negative number $\lambda$ quantifies the interaction strength between the system and the apparatus. The important point is that both the disturbance caused by the measurement apparatus and the information extracted about the system should go to zero as $\lambda \to 0$.

In order to define expectation values, we need to assign numerical values to the measurement outcomes. We thus introduce a real number $A_m$ to each $m$, and, considering $A$ as a random variable, we define the expectation value

$$E^\lambda_s[A] := \sum_m A_m P^\lambda(m|s).$$  \hfill (3)

The $A_m$ can be understood as contextual values as introduced in Ref. [22] (see also Ref. [2]). In these notes we will only be interested in the weak limit $\lambda \to 0$. With no interaction, $\lambda = 0$, the probability $P^\lambda(m|s)$ is assumed to be independent of the state $s$, and will be
denote \( P^0(m) \). We further assume that we have an asymptotic expansion around \( \lambda = 0 \),

\[
P^\lambda(m|s) = P^0(m) + \lambda \delta P(m|s) + O(\lambda^2).
\] (4)

For simplicity we assume that

\[
\mathbb{E}^{\lambda=0}[A] = \sum_m A_m P^0(m) = 0.
\] (5)

We can then define the following (non-postselected) weak limit of the expectation value:

\[
\mathcal{E}_w[A] := \lim_{\lambda \to 0} \lambda^{-1} \mathbb{E}_d^{\lambda}[A] = \sum_m A_m \delta P(m|s).
\] (6)

Note that we have to amplify the signal by a factor \( \lambda^{-1} \) to get something non-trivial. As discussed in Appendix A there is no loss of generality in assuming (5), and we will continue doing so in the following.

In order to discuss postselection we need to know the state of the system once it leaves the measurement apparatus. The state after the measurement conditioned on a given outcome is specified by the map \( s \mapsto s' = M^\lambda_m(s) \).

\[
s \mapsto s' = M^\lambda_m(s).
\] (7)

Note that the map is non-trivial even for non-disturbing measurements, since the outcome \( m \) in general increases our knowledge about the system. We denote the the of postselection procedures by \( \mathcal{S} \). An element of \( \mathcal{S} \) is a map \( \tilde{s} : S \to [0, 1] \) giving the probability that the postselection will succeed on a given state,

\[
\tilde{s}(s) := P(\tilde{s} \text{ will accept } s).
\] (8)

By only considering the experimental runs where a given postselection procedure succeeds, we get the following expectation value (the product \( P^\lambda(m|s)\tilde{s}(M^\lambda_m(s)) \) is a joint probability, in more standard notation it might be written \( P(m, \tilde{s}|s) \))

\[
\tilde{s} \mathbb{E}_d^{\lambda}[A] := \frac{\sum_m A_m P^\lambda(m|s)\tilde{s}(M^\lambda_m(s))}{\sum_m P^\lambda(m|s)\tilde{s}(M^\lambda_m(s))}.
\] (9)

In words \( \tilde{s} \mathbb{E}_d^{\lambda}[A] \) is the conditional expectation value of \( A \) given a initial preparation \( s \), and conditioned on the success of a final postselection \( \tilde{s} \). To take the weak limit of this we need to demand that

\[
M^{\lambda=0}_m(s) = s, \quad \text{for all } m, s,
\] (10)
in accordance with our interpretation of $\lambda$ as interaction strength. We can then define the 
*generalized weak value* by

$$
\hat{s}E_w^s[A] := \lim_{\lambda \to 0} \lambda^{-1} s E^\lambda_s[A] = E^w_s[A] + \sum_m A_m \frac{P_0(m) \delta \hat{s}(M_m(s))}{\hat{s}(s)},
$$

(11)

where $\delta \hat{s}(M_m(s))$ is defined by the following small $\lambda$ expansion:

$$
\hat{s}(M^\lambda_m(s)) = \hat{s}(s) + \lambda \delta \hat{s}(M_m(s)) + O(\lambda^2).
$$

(12)

The RHS of (11) is only defined when $\hat{s}(s)$ is non-zero, and this will be tacitly assumed in
the following.

We will use that $S$ and $\hat{S}$ are convex set. I.e. if $s$ and $s'$ are preparation procedures,
then one can construct a combined procedure by selecting procedure $s$ with probability $\alpha$
and $s'$ with probability $1 - \alpha$. This combined state is denoted $\alpha s + (1 - \alpha)s'$. By a similar
construction $\hat{S}$ is also convex. From basic probability theory we get the following relations:

$$
P^\lambda(m|\alpha s + (1 - \alpha)s') = \alpha P^\lambda(m|s) + (1 - \alpha) P^\lambda(m|s'),
$$

(13)

$$
P^\lambda(m|\alpha s + (1 - \alpha)s') M^\lambda_m(\alpha s + (1 - \alpha)s') = \alpha P^\lambda(m|s) M^\lambda_m(s) + (1 - \alpha) P^\lambda(m|s') M^\lambda_m(s'),
$$

(14)

and

$$
\hat{s}(\alpha s + (1 - \alpha)s') = \alpha \hat{s}(s) + (1 - \alpha) \hat{s}(s'),
$$

(15)

$$
(\alpha \hat{s} + (1 - \alpha) \hat{s}')(s) = \alpha \hat{s}(s) + (1 - \alpha) \hat{s}'(s).
$$

From these relations it follows that the function

$$
G(s, \hat{s}) := \hat{s}(s) \hat{s}E_w^s[A]
$$

(16)

is bilinear with respect to convex combinations,

$$
G(\alpha s + (1 - \alpha)s', \hat{s}) = \alpha G(s, \hat{s}) + (1 - \alpha) G(s', \hat{s}),
$$

(17)

and

$$
G(s, \alpha \hat{s} + (1 - \alpha) \hat{s}') = \alpha G(s, \hat{s}) + (1 - \alpha) G(s, \hat{s}').
$$

(18)

### A. Disturbance

Without postselection we do not need to worry about how the measurement apparatus
affects the system, but, as we have seen, the generalized weak expectation value (11) will
depend on this disturbance. In order to associate an unique postselected expectation value with a given ordinary observable we thus have to constrain the disturbance. Here we formulate two simple condition within the general operational framework. Later we will see that both of these have the desired effect on classical measurements, but that the situation is not so simple for quantum mechanics.

Morally, we want to say that the measurement apparatus does not change the ontic state of the system, but since our operational framework lack the notion of an ontic state, we cannot express this directly. Instead we can assume that there exists a subset of the states $S' \subset S$ such that every state $s$ can be written as a convex combination of states in $S'$,

$$s = \sum_j p_j s_j, \quad s_j \in S', \quad \sum_j p_j = 1. \tag{19}$$

We will then say that a measurement procedure is non-disturbing in the strong sense if

$$M^\lambda_m(s) = s + O(\lambda^2) \quad \text{for all } s \in S', m. \tag{20}$$

Assume that we have an expansion of $s$ as in (19). Given a $\tilde{s}$ we can then define a new state by

$$s \cdot \tilde{s} := (\tilde{s}(s))^{-1} \sum_j p_j \tilde{s}(s_j)s_j. \tag{21}$$

If now (20) holds, we find

$$s E^w_s[A] = E^w_{s \cdot \tilde{s}}[A] := \frac{1}{\sum_j p_j \tilde{s}(s_j)} \sum_j p_j \tilde{s}(s_j)E^w_{s_j}[A]. \tag{22}$$

Thus, if a measurement is non-disturbing in the strong sense, then the postselected weak value is equal to the non-postselected weak value in the combined ensemble $s \cdot \tilde{s}$.

Another possibility is to say that the state we obtain if we ignore the measurement outcome $m$, i.e.

$$M^\lambda_?(s) := \sum_m P^\lambda(m|s)M^\lambda_m(s), \tag{23}$$

is just $s$ to first order in $\lambda$. We will thus call a measurement procedure such that

$$M^\lambda_?(s) = s + O(\lambda^2) \tag{24}$$

non-disturbing in the weak sense. This definition is adopted in Ref. [14, 23]. We note that (20) indeed implies (24) in accordance with the naming.
III.  CLASSICAL MECHANICS

In order to clarify the ideas of the previous section, and to provide a background to understand quantum weak measurements, let us consider the situation in classical mechanics. A model of weak measurements with disturbance on a classical system was recently given in Ref. [6]. That model does, however, not strictly fall within our framework, since the dependence of the disturbance on the interaction strength is different. Models of weak measurements on classical fields have also been considered [8–10]. A conceptual difference between the models we will consider and the field models is that for the field models the measurement disturbance is deterministic, while we will only consider stochastic disturbance.

For simplicity we will consider systems with a finite number of ontic states (i.e. the ‘phase space’ of the system consists of a finite number of points), and we will denote these \( s_j \). The preparation procedures are then specified by probability distributions on the ontic states, that is

\[
S = \left\{ \sum_j p_j s_j \left| \sum_j p_j = 1, p_j \geq 0 \right. \right\},
\]

where \( p_j \) is the probability of preparing the system in state \( j \). Defining dual states by

\[
\tilde{s}_j(s_k) := \delta_{jk},
\]

we can also expand \( \tilde{s} \) as

\[
\tilde{s} = \sum_j q_j \tilde{s}_j, \quad q_j := \tilde{s}(s_j).
\]

Using the bilinearity of the \( G(s, \tilde{s}) \) function (Eq. (16)), we find

\[
G(s, \tilde{s}) = \sum_{j,k} q_k \tilde{A}_{kj} p_j,
\]

with the real matrix \( \tilde{A} \) defined by

\[
\tilde{A}_{kj} := \lim_{\lambda \to 0} \lambda^{-1} \sum_m A_m P^\lambda(m|s_j) \tilde{s}_k(M^\lambda_m(s_j)).
\]

It follows immediately that the generalized weak value is

\[
\tilde{s} \mathbb{E}_s^w[A] = \frac{1}{\tilde{s}(s)} \sum_{j,k} q_k \tilde{A}_{kj} p_j.
\]

A natural question is whether all real matrices \( \tilde{A} \) can appear in (30)? The answer is positive, as can be seen by the following simple construction. Let the real matrix \( \tilde{A} \) be
given. We consider a measurement with two outcomes, denoted by \( m = \pm \). Take the probability to get a given outcome to be (note that \( \lambda \) has to be sufficiently small for the model to make sense)

\[
P^\lambda(m = \pm | s_j) = \frac{1}{2} + \frac{\lambda}{2} \sum_k \tilde{A}_{kj},
\]

(31)

and the post-measurement state to be

\[
M^\lambda_{\pm}(s_j) = (1 - 2\lambda \sum_{k \neq j} (\pm \tilde{A}_{kj})_+ s_j + 2\lambda \sum_{k \neq j} [\pm \tilde{A}_{kj}]_+ s_k.
\]

(32)

In the last equation \([\cdot]_+\) denotes the positive part, as defined by

\[
[x]_+ := \max\{x, 0\}.
\]

(33)

A calculation now shows that (30) is indeed satisfied. We conclude that the space of generalized weak measurements on a classical system with \( d \) states is in one-to-one correspondence with the space of real \( d \times d \) matrices [41].

Before we turn to quantum mechanics let us note the following result: if a classical weak measurement is non-disturbing in the weak sense if and only if it is non-disturbing in the strong sense. One direction has already been shown to hold in general. To see the other direction we assume that the measurement is non-disturbing in the weak sense. We take \( S' \) to be the set of ontological states. By assumption we have

\[
M^\lambda_m(s) := \sum_{m} P^\lambda(m | s) M^\lambda_m(s) = s + O(\lambda^2)
\]

(34)

for all states \( s \in S \). Using that every state can uniquely [42] be written as

\[
s = \sum_j p_j s_j, \quad s_j \in S'
\]

(35)

it is now easy to check that (34) can only hold for ontic states \( s \in S' \) if we have

\[
M^\lambda_m(s) = s + O(\lambda^2) \quad \text{for all } s \in S', m.
\]

(36)

Going back to (30) we see that for non-disturbing classical weak measurements \( \tilde{A} \) will be diagonal (the converse is however not true in general).

For classical mechanics we thus have the following simple picture: If a generalized weak measurement is non-disturbing in the usual sense that it does not change the ontic state of
the system, then it will be described by a diagonal matrix \( \tilde{A}_{kj} \) (furthermore is easy to see that all diagonal matrices appear this way). By the above result it is actually sufficient to assume that the measurement is non-disturbing in the weak sense. If one does not put any constraints on the disturbance, then the measurement is described by a general real matrix \( \tilde{A}_{kj} \).

IV. QUANTUM MECHANICS

Having discussed the simpler classical case, we go on the main topic of the paper, namely weak measurements in quantum mechanics. We take it as an axiom of quantum mechanics that the space of preparation procedures, \( \mathcal{S} \), is identified with the set of density matrices (positive operators of trace one) on some Hilbert space \( \mathcal{H} \),

\[
\mathcal{S} = \{ s \in \text{End}(\mathcal{H}) \mid s = s^\dagger, \ s \geq 0, \ \text{tr}[s] = 1 \}.
\]  

(37)

In the remainder of the article we will keep the finite dimensional system space \( \mathcal{H} \) fixed. For the set of postselection conditions the most general choice is the effects on \( \mathcal{H} \). We will thus take \( \tilde{s} \) to be a positive operator with eigenvalues \( \leq 1 \),

\[
\tilde{\mathcal{S}} = \{ \tilde{s} \in \text{End}(\mathcal{H}) \mid \tilde{s} = \tilde{s}^\dagger, \ 0 \leq \tilde{s} \leq 1 \}.
\]  

(38)

The probability for a system in state \( s \) to be postselected is then

\[
\tilde{s}(s) := \text{tr}[\tilde{s}s].
\]  

(39)

In particular, having no postselection (i.e. accepting all runs of the experiment) is represented by setting \( \tilde{s} = 1 \).

Before we perform an explicit calculation of \( s \mathbb{E}_s^w[A] \), let us anticipate the result using a more heuristic argument. We recall that the function \( G \) satisfies

\[
G(\alpha s + (1 - \alpha)s', \tilde{s}) = \alpha G(s, \tilde{s}) + (1 - \alpha)G(s', \tilde{s}),
\]  

(40)

and

\[
G(s, \alpha \tilde{s} + (1 - \alpha)\tilde{s}') = \alpha G(s, \tilde{s}) + (1 - \alpha)G(s, \tilde{s}')
\]  

(41)

The simplest non-trivial family of real functions with this property is \( \text{Re} \text{tr}[\tilde{s}\hat{A}s] \), where \( \hat{A} \) is a (not necessarily Hermitian) operator on \( \mathcal{H} \). One could also imagine having terms of the
form $\text{Re} \, \text{tr}[\hat{s} \hat{A} \hat{B}]$, but because we only expand to first order in $\lambda$ we will not see this more general type of term, however see Appendix A. We thus claim that the post-selected weak value must take the form

$$s E_w^w[A] = \frac{\text{Re} \, \text{tr}[\hat{s} \hat{A}s]}{\text{tr}[\hat{s}s]}. \quad (42)$$

Note that this expression has both the real and imaginary part of the usual weak value as special cases. Indeed, if we set $s = |\psi\rangle\langle\psi|$, $\hat{s} = |\phi\rangle\langle\phi|$ and $\hat{A} = \hat{O}$, where $\hat{O}$ is Hermitian, we recover the real part of the usual AAV expression

$$s E_w^w[A] = \text{Re} \frac{\langle \phi | \hat{O} | \psi \rangle}{\langle \phi | \psi \rangle}. \quad (\hat{A} = \hat{O}) \quad (43)$$

On the other hand, setting $\hat{A} = -i \hat{O}$, we obtain the imaginary part

$$s E_w^w[A] = \text{Im} \frac{\langle \phi | \hat{O} | \psi \rangle}{\langle \phi | \psi \rangle}. \quad (\hat{A} = -i \hat{O}) \quad (44)$$

We will call $\hat{A}$ a \textit{generalized observable}.

Some operators give the same expectation values when plugged in to (42). To be precise one should thus define a generalized observable to be an element of

$$\text{End}(\mathcal{H})/\sim, \quad (45)$$

where $\hat{A} \sim \hat{A}'$ iff $\hat{A} - \hat{A}'$ is a purely imaginary multiple of the identity. See Appendix B for further details.

Let us now verify (42) by a more careful calculation. The most general measurement on a quantum system can be described by a \textit{quantum instrument} [24]. For our purposes it will be convenient to express the instrument in terms of Kraus operators. For each measurement outcome $m$ we thus have a family of operators $\hat{K}^\lambda_{m,n}$ on $\mathcal{H}$ such that

$$\sum_{m,n} (\hat{K}^\lambda_{m,n})^\dagger \hat{K}^\lambda_{m,n} = 1. \quad (46)$$

The probability of obtaining outcome $m$ is

$$P^\lambda(m|s) := \sum_n \text{tr}[s (\hat{K}^\lambda_{m,n})^\dagger \hat{K}^\lambda_{m,n}], \quad (47)$$

and the post-measurement state is

$$M^\lambda_m(s) := \frac{\sum_n \hat{K}^\lambda_{m,n}s(\hat{K}^\lambda_{m,n})^\dagger}{P^\lambda(m|s)}. \quad (48)$$
We assume that the Kraus operators have an expansion in \( \lambda \),

\[
\hat{K}_{\lambda}^{m,n} = K_{0}^{m,n} + \lambda \delta K_{m,n} + \frac{1}{2} \lambda^2 \delta^2 \hat{K}_{m,n} + O(\lambda^3). \tag{49}
\]

The basic assumption that \( M_{m=0}^{\lambda=0}(s) = s \) is then equivalent to

\[
K_{0}^{m,n} \propto 1, \quad \text{for all } m,n. \tag{50}
\]

It is clear from (47) and (48) that the physics is invariant under a change of phase of the \( \hat{K}_{\lambda}^{m,n} \) operators. We will thus assume that \( K_{0}^{m,n} \) is real and positive (for all \( m,n \)). Plugging (47) and (48) into (11) we obtain (42) with \( \hat{A} \) explicitly given by

\[
\hat{A} := 2 \sum_{m} A_{m} \delta \hat{K}_{m}, \tag{51}
\]

and where we define the averaged \( \delta \hat{K} \) by

\[
\delta \hat{K}_{m} := \sum_{n} K_{0}^{m,n} \delta \hat{K}_{m,n}. \tag{52}
\]

Similarly to the classical case, we can show that any generalized observable \( \hat{A} \) is realized by a measurement scheme. To show this we consider the following explicit model, which has been previously discussed in Ref. [13]: Let \( \hat{A} \in \text{End}(\mathcal{H}) \) be given, and let \( \mathcal{H}_{\text{aux}} \) be a two dimensional Hilbert space with orthonormal basis \( |\pm\rangle \). On \( \mathcal{H} \otimes \mathcal{H}_{\text{aux}} \) we define the operator

\[
2\hat{H} := i \hat{A}^{R} \otimes |-\rangle \langle +| - i \hat{A}^{I} \otimes |+\rangle \langle -| + \hat{A}^{I} \otimes |+\rangle \langle +| - \hat{A}^{I} \otimes |-\rangle \langle -|. \tag{53}
\]

The model is then defined by setting (we omit the \( n \) index on \( \hat{K} \), since it is trivial)

\[
\hat{K}_{\pm}^{\lambda} = \frac{1}{\sqrt{2}} \text{tr}_{\text{aux}}[e^{i\lambda H}(|+\rangle \langle -|)\langle \pm|] = \frac{1}{\sqrt{2}} \pm \frac{\lambda}{2\sqrt{2}} \hat{A} + O(\lambda^2), \tag{54}
\]

\[
A_{\pm} = \pm 1, \tag{55}
\]

and (42) is verified. Physically the model can be understood as letting the system \( \mathcal{H} \) interact weakly with an auxiliary qubit, and then performing a projective measurement on the qubit. It is easy to show that this model is non-disturbing in the weak sense, for all operators \( \hat{A} \).

Let us rewrite the expression for the (generalized) quantum weak value in a way that makes comparison with the classical case easier. We will focus on pure states, so we set \( s = |\psi\rangle \langle \psi| \) and \( \bar{s} = |\phi\rangle \langle \phi| \). Choose an orthonormal basis \( |j\rangle \) for \( \mathcal{H} \), and define

\[
u_{j} = \langle j | \psi \rangle, \quad v_{j} = \langle j | \phi \rangle, \quad A_{kj} := \langle k | \hat{A} | j \rangle. \tag{56}
\]
The weak value is then given by
\[ \hat{s} E^w_s[A] = \text{Re} \sum_{jk} v_k^* \hat{A}_{kj} u_j \sum_j v_j^* u_j. \] (quantum) (57)

On the other hand, the classical weak value is given by (Eq. (30))
\[ \hat{s} E^w_s[A] = \frac{\sum_{jk} q_k \hat{A}_{kj} p_j}{\sum_j q_j p_j}. \] (classical) (58)

The two expressions look very similar, but it is important to keep in mind that \( p_j \) and \( q_j \) are (positive) probabilities, while \( u_j \) and \( v_j \) are (complex) amplitudes. This makes an important difference. Let us say that a measurement allows for anomalous weak values if one can make \( \hat{s} E^w_s[A] \) arbitrarily large by choosing \( s \) and \( \hat{s} \) appropriately. In the classical case we see that this is possible iff \( \hat{A}_{jk} \) is not diagonal (anomalous weak values in classical systems are also discussed in Ref. [6]). In the quantum case, however, we can get anomalous weak values for any non-trivial (i.e. not proportional to the identity) \( \hat{A}_{jk} \) due to the possibility of destructive interference in the denominator of (57).

In this section we avoid discussing the details of the measurement apparatus. Since the concept of weak measurement is often presented in the context of von Neumann measurements, we consider this case in detail in Appendix C.

A. (Non-)Disturbance in the weak and strong sense

Let us first show that a non-trivial weak measurement cannot be non-disturbing in the strong sense. In order that every state can be written as
\[ s = \sum_j p_j s_j, \quad s_j \in \mathcal{S}', \] (59)

it is well known that \( \mathcal{S}' \) must contain all pure states [43]. To first order in \( \lambda \), \( M^\lambda_m \) sends pure states to pure state:
\[ M^\lambda_m(|\psi\rangle\langle\psi|) = |\psi'\rangle\langle\psi'| + O(\lambda^2) \] (60)

with
\[ |\psi'\rangle = \left(1 + \frac{\lambda}{\text{P}_0(m)}[\delta \hat{K}_m - \langle \psi | \delta \hat{K}_m | \psi \rangle 1] \right) |\psi\rangle. \] (61)

The only way that \( |\psi'\rangle \) can be in the same ray as \( |\psi\rangle \) for all \( m \) and \( \psi \) is if all \( \delta \hat{K}_m \) are proportional to the identity. But then we also have \( \hat{A} \propto 1 \) and \( \hat{s} E^w_s[A] \) becomes a trivial constant independent of \( s \) and \( \hat{s} \).
The situation for the weak condition of Section II A is quite different. We first note that
\[
M^\lambda_s(s) := \sum_m P^\lambda(m|s)M^\lambda_m(s) = s + \lambda i\hat{D}, s + O(\lambda^2),
\]
(62)
where
\[
\hat{D} := -i \sum_m \delta \hat{K}_m.
\]
(63)
Here we have used that from (66) it follows that \( \hat{D} \) is Hermitian. Since (62) is a unitary
transformation to order \( \lambda \), we can eliminate the disturbance by performing the inverse
unitary after \( M \). Moreover, this compensating transformation does change the generalized
observable \( \hat{A} \). In more detail, the replacement
\[
\hat{K}^\lambda_{m,n} \to e^{-i\lambda \hat{D}} \hat{K}^\lambda_{m,n}
\]
(64)
ensures that \( M^\lambda_s(s) = s + O(\lambda^2) \) and using (63) one can check that it leaves \( \hat{A} \) invariant.

From (62) it follows that the overall probability for successful postselection is
\[
\tilde{s}(M^\lambda_s(s)) = \text{tr}[\tilde{s}] \left( 1 - 2\lambda \text{Im} \frac{\text{tr}[\tilde{s} \hat{D}s]}{\text{tr}[\tilde{s}s]} \right) + O(\lambda^2).
\]
(65)
Thus, the relative change (due to the intermediate measurement) of the postselection proba-
bility is proportional to the imaginary part of the AAV weak value of \( \hat{D} \) (see also [25]). Note
that in the general setting we are considering there need not to be any connection between
\( \hat{D} \) and \( \hat{A} \), but for von Neumann measurements one has \( \hat{D} \propto \hat{A} \), see Ref. [25] and Appendix
C.

We see that disturbance in quantum mechanics behaves quite different from classical
mechanics. One the one hand a quantum mechanical measurement cannot be non-disturbing
in the strong sense (except in the trivial case), whereas this is usually implicitly assumed
for classical measurements. On the other hand being non-disturbing in the weak sense is
rather restrictive in the classical setting (since it implies being non-disturbing in the strong
sense), while it does not restrict the class quantum mechanical measurements at all (in the
sense that the generalized observable \( \hat{A} \) is unconstrained).

**B. Minimal disturbance and uniqueness of the weak value**

By setting \( \tilde{s} = 1 \) in (62) we obtain the expectation value without postselection,
\[
\mathbb{E}_s^w[A] = \text{tr}[s \hat{A}^R] =: \langle \hat{A}^R \rangle_s.
\]
(66)
Here $\hat{A}^R$ denotes the Hermitian part of $\hat{A}$, i.e.

$$\hat{A} = \hat{A}^R + i\hat{A}^I, \quad (\hat{A}^R)^\dagger = \hat{A}^R, \quad (\hat{A}^I)^\dagger = \hat{A}^I.$$  \hspace{1cm} (67)

We have just seen that the conditions of non-disturbance discussed in Section II A are not useful in restricting the allowed generalized observable. This means that, given an ordinary observable $\hat{O}$, it is not given which generalized observable $\hat{A}$ (satisfying $\hat{A}^R = \hat{O}$) we should associate with it. This is in contrast to the classical case, where either of the conditions of non-disturbance selects a unique $\hat{A}$ (namely the diagonal one) for a given observable. For an extended discussion of the uniqueness of the weak value see Ref. [16] and references therein.

Instead of requiring the measurement to be non-disturbing, one can look for a way to quantify the amount of disturbance, and then demand this quantity to be minimal. In Refs. [16, 22] it is shown that one recovers the AAV weak value if one requires that the Kraus operators are positive and Hermitian (this is taken as the definition of a minimally disturbing measurement in Ref. [26]). Note that the assumptions of Refs. [16, 22] are somewhat different from ours.

Here we want to highlight a numerical quantity measuring disturbance [15] which is minimized, and show how it appears from an operational point of view. Note that, in contrast to the various error-disturbance relations discussed recently (see e.g. [27–32]), here we are interested in the disturbance of the system as such, rather than one of its observables. In fact, there is no good candidate for the observable in the case we are considering (in particular $\hat{A}^R$ would be a bad choice, since then the disturbance would simply be zero for a large class of measurement procedures).

For a system prepared in a pure state, a natural way to measure the disturbance is by the survival probability (alternatively, the quantum fidelity between the initial and final state)

$$F^\lambda(\psi) := \text{tr} \left[ M^\lambda_{\psi} |\psi\rangle\langle\psi|\right].$$ \hspace{1cm} (68)

This is simply the probability that the system was not kicked into an orthogonal state by the measurement process.

Expanding in $\lambda$ we find that

$$F^\lambda(\psi) = 1 - \lambda^2 \sum_{m,n} \left( |\langle\psi|\delta\hat{K}_{m,n}^\dagger\delta\hat{K}_{m,n}|\psi\rangle - |\langle\psi|\delta\hat{K}_{m,n}|\psi\rangle|^2 \right) + O(\lambda^3).$$ \hspace{1cm} (69)
Here we have simplified the expression using the relation

\[ \sum_{m,n} (\delta \hat{K}_{m,n})^\dagger \delta \hat{K}_{m,n} + \frac{1}{2} \sum_{m,n} K_{m,n}^0 [\delta^2 \hat{K}_{m,n}^\dagger + \delta^2 \hat{K}_{m,n}] = 0, \]  

(70)

which follows from (46). Note that the leading order term of \( F^\lambda \) only depends on the first order terms of \( \hat{K}_{m,n}^\lambda \). To get a state independent number we now average over \( \psi \) with respect to the Haar measure[15, 33]. We use the integral

\[ \int d\psi \langle \psi | \hat{B} | \psi \rangle \langle \psi | \hat{C} | \psi \rangle = \frac{1}{d(d+1)} (\text{tr}[\hat{B} \hat{C}] + \text{tr}[\hat{B}] \text{tr}[\hat{C}]), \]

(71)

and find

\[ \bar{F}^\lambda := \int d\psi F^\lambda(\psi) = 1 - \frac{\lambda^2}{d(d+1)} \mathcal{F} + O(\lambda^3), \]

(72)

with

\[ \mathcal{F} := \sum_{m,n} \left( d \text{tr}[\delta \hat{K}_{m,n}^\dagger \delta \hat{K}_{m,n}] - | \text{tr}[\delta \hat{K}_{m,n}]|^2 \right). \]

(73)

We will take \( \mathcal{F} \) as our measure of disturbance. Note that \( \mathcal{F} \) can be understood as a weak limit of \( \bar{F}^\lambda \),

\[ \mathcal{F} = d(d+1) \lim_{\lambda \to 0} \lambda^{-2}(1 - \bar{F}^\lambda). \]

(74)

It is convenient to write

\[ \mathcal{F} = \sum_{m,n} f(\delta \hat{K}_{m,n}^R) + f(\delta \hat{K}_{m,n}^I), \]

(75)

where

\[ f(\hat{B}) := d \text{tr}[\hat{B}^2] - (\text{tr}[\hat{B}])^2, \quad \text{for Hermitian } \hat{B}. \]

(76)

The function \( f(\hat{B}) \) is non-negative, and vanishes iff \( \hat{B} \) is proportional to the identity. It follows immediately that \( \mathcal{F} \) is strictly positive for all non-trivial measurements. We can now show the following (the proof and exact statement is in Appendix D): Fix the number of measurement outcomes and an observable \( \hat{O} \). Bound (or fix) the values \( A_m \). Among the generalized weak measurements with \( \hat{A}^R \sim \hat{O} \) those which minimize \( \mathcal{F} \) have \( \hat{A} \sim \hat{O} \). More loosely, the minimally disturbing generalized weak measurements yield the AAV weak value.

As an explicit example, let us mention that for the model define by (53) and (54), we find

\[ \mathcal{F} = \frac{1}{4} \left( f(\hat{A}^R) + f(\hat{A}^I) \right). \]

(77)

Here we see explicitly that the disturbance is minimal exactly when \( \hat{A}^I \propto 1 \). In Appendix C we calculate \( \mathcal{F} \) for von Neumann like models.
V. DISCUSSION

Let us outline some different attitudes one can take towards weak values in light of the above remarks: (a) Generalized weak measurements that are non-disturbing in the weak sense should be considered non-disturbing. By non-disturbing (without the weak or strong qualifier) we mean that the disturbance is sufficiently weak that it does not affect the weak value which is the result of the measurement. (b) All (non-trivial) generalized weak measurements should be considered disturbing. The measurements of a given observable that are least disturbing yield the AAV weak value. (c) There are some generalized weak measurements that are non-disturbing, and these always yield the AAV weak value.

Consider a weak measurement procedure which is non-disturbing in the weak sense. Without postselection it will measure some ordinary observable \( \hat{O} \). If we consider the measurement to be non-disturbing, as postulated in option (a), the intermediate measurement should not interfere with postselection. Thus the experiment with postselection should still be a measurement of \( \hat{O} \), just in a different ensemble (namely the one defined both by the preparation and postselection). But \( \mathcal{E}_w^{\hat{A}}[A] \) also depends on \( \hat{A}' \), which is not determined by \( \hat{O} \). In other words, the weak value depends on how we measure \( \hat{O} \), even though the measurement is non-disturbing. It seems that to understand option (a), one is faced with the task of making sense of this additional dependence in the weak value. The relation between contextuality and weak values was recently discussed in Ref. [34].

If weak measurements disturb the system, then it is difficult to understand why the weak value should be considered the expectation value of an observable in the postselected ensemble. We have seen that if one allows for (weak) disturbance in a classical setting, one does not get the ‘right’ answer when turning on postselection. The main question arising from position (b) then seems to be: What is the fundamental interpretation of the weak value, other than the result of a specific measurement procedure? Of course, it is possible that there is no such interpretation.

Option (c) is attractive because it allows for a straightforward interpretation of the weak value as the expectation value of some observable between preparation and postselection. The measure of disturbance \( \mathcal{F} \) lends some support to this position in that, when it is minimal, the measurement yields the AAV weak value. On the other hand, the minimum of \( \mathcal{F} \) cannot be zero (unless the measured observable is a trivial constant), even in the original AAV
setup (see also Eq. \((\text{C15})\)). It is possible that there exists ways of quantifying the disturbance such that option \((c)\) is realized, but the author is not aware of any.

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**Appendix A: Some further generalizations**

Here we discuss two generalizations of the framework considered in the main part of the article. We will focus on the quantum case. First, let us consider the constraint Eq. \((5)\),

\[
\mathbb{E}^{\lambda=0}[A] = \sum_{m} A_{m} P^{0}(m) = 0.
\]  

(A1)

If we drop this constraint the conditional expectation value \((9)\) becomes

\[
\hat{s}\mathbb{E}^{\lambda} s[A] = \mathbb{E}^{\lambda=0}[A] + \lambda \frac{\text{Re tr}[\hat{s}\hat{A}'s]}{\text{tr}[\hat{s}s]} + O(\lambda^{2}),
\]  

(A2)

where \(\hat{A}'\) contains an additional contribution proportional to \(\mathbb{E}^{\lambda=0}[A]\),

\[
\hat{A}' := \hat{A} - i 2 \mathbb{E}^{\lambda=0}[A] \hat{D}.
\]  

(A3)

Here \(\hat{A}\) is defined by Eq. \((51)\), while \(\hat{D}\) is defined by Eq. \((63)\). It is now natural to define the generalized weak value to be the coefficient of \(\lambda\) in Eq. \((A2)\),

\[
\hat{s}\mathbb{E}^{w'} s[A] := \lim_{\lambda \to 0} \lambda^{-1} (\hat{s}\mathbb{E}^{\lambda} s[A] - \mathbb{E}^{\lambda=0}[A]) = \frac{\text{Re tr}[\hat{s}\hat{A}'s]}{\text{tr}[\hat{s}s]}.
\]  

(A4)

Let us now note that the shift

\[
A_{m} \to A_{m} - \mathbb{E}^{\lambda=0}[A]
\]  

(A5)
leaves (A4) invariant while ensuring that (A1) is satisfied. We thus conclude that there is no loss of generality in restricting to the case where (A1) holds.

A more substantial generalization comes about by reconsidering the asymptotic expansion of the Kraus operators. In Section IV we assumed the $G(s, \hat{s})$ function to take the form $G(s, \hat{s}) = \text{tr}[\hat{s}\hat{A}s]$, however, the most general bilinear real function takes the form

$$G(s, \hat{s}) = \sum_j \varepsilon_j \text{tr}[\hat{s}\hat{A}_j s\hat{A}_j^\dagger],$$

(A6)

where $\hat{A}_j$ is some set of (non-Hermitian) operators on $\mathcal{H}$ and $\varepsilon_j = \pm 1$. This follows from the polarization identity

$$\text{Re} \text{tr}[\hat{s}\hat{A}\hat{B}] = \frac{1}{4} \left( \text{tr}[\hat{s}(\hat{A} + \hat{B}^\dagger)s(\hat{A} + \hat{B}^\dagger)^\dagger] - \text{tr}[\hat{s}(\hat{A} - \hat{B}^\dagger)s(\hat{A} - \hat{B}^\dagger)^\dagger] \right).$$

(A7)

Terms of this more general form are obtained if some of the Kraus operators behave as

$$\hat{K}_{\lambda}^{m,n} = \lambda^{1/2} \hat{L}_{m,n} + O(\lambda^{3/2})$$

(A8)

in the weak limit. Note that (A8) is compatible with $M^\lambda_m(s)$ and $P^\lambda(m|s)$ having expansions in integer powers of $\lambda$. However, for indirect measurements where the Hamiltonian is an analytical function of $\lambda$ (i.e. von Neumann measurements or the qubit scheme discussed in Section IV) the Kraus operators will also be analytical in $\lambda$.

From (A6) it follows that

$$\hat{s}\mathcal{E}_s^w[A] = \frac{\sum_j \varepsilon_j \text{tr}[\hat{s}\hat{A}_j s\hat{A}_j^\dagger]}{\text{tr}[\hat{s}s]}.$$ 

(A9)

A particular example of this is the so-called null weak values [35] where

$$\hat{s}\mathcal{E}_s^w[A] = \frac{\text{tr}[\hat{O}s]}{\text{tr}[\hat{s}s]},$$

(A10)

for some Hermitian $\hat{O}$. The most general form (A9) can be obtained by considering a measurement with two outcomes $m = \pm$. Indeed, setting $A_\pm = \pm 1$ and (here $[\cdot]_+$ is defined by (33))

$$P^\lambda(m = \pm|s)M^\lambda_\pm(s) = \frac{1}{2} s + \lambda \sum_j \left( [\pm \varepsilon_j)_+ \hat{A}_j s \hat{A}_j^\dagger - \frac{1}{4} \hat{A}_j^\dagger \hat{A}_j s - \frac{1}{4} s \hat{A}_j^\dagger \hat{A}_j \right) + O(\lambda^2)$$

(A11)

one recovers (A9). We leave the extension of the model (A11) to finite $\lambda$ to further work.
Allowing Kraus operators of the form (A8) we can embed the classical model of weak measurements in the quantum model. To see this, let us choose some basis $|j\rangle$ for the system Hilbert space, and take $s$ and $\tilde{s}$ to be diagonal,

$$s = \sum_j p_j |j\rangle \langle j|, \quad \tilde{s} = \sum_j q_j |j\rangle \langle j|.$$  \hspace{1cm} (A12)

With

$$\hat{A}_{kj} := \sqrt{|\hat{A}_{kj}|} |k\rangle \langle j|$$  \hspace{1cm} (A13)

we then find

$$\sum_{j,k} \text{sgn}(\tilde{A}_{kj}) \left( \frac{\text{tr}[\tilde{s} \hat{A}_{kj} s \hat{A}_{kj}^\dagger]}{\text{tr}[\tilde{s}s]} \right) = \sum_{j,k} q_k \tilde{A}_{kj} p_j \frac{\sum_j q_j p_j}{\sum_j q_j p_j},$$  \hspace{1cm} (A14)

which is just the classical weak value (30).

Let us finally note that having Kraus operators with expansions of the form (A8) (with $\hat{L}_{m,n}$ not proportional to the identity) implies that the measurement cannot be non-disturbing in the weak sense.

**Appendix B: ‘Gauge invariance’ of generalized observables**

Given two generalized observables $\hat{A}, \hat{A}'$ we want to know whether they give rise to the same expectation values, i.e. whether it holds that

$$\frac{\text{Re tr}[\tilde{s} \hat{A}s]}{\text{tr}[\tilde{s}s]} = \frac{\text{Re tr}[\tilde{s} \hat{A}'s]}{\text{tr}[\tilde{s}s]},$$  \hspace{1cm} (B1)

for all $s \in S, \tilde{s} \in \tilde{S}$ such that $\text{tr}[\tilde{s}s] \neq 0$.

This is clearly equivalent to finding the operators $\hat{B}$ that satisfy

$$\text{Re tr}[\tilde{s} \tilde{s} \hat{B}] = 0,$$  \hspace{1cm} (B2)

for all $s \in S, \tilde{s} \in \tilde{S}$.

Note that if $\hat{B}$ satisfy this equation then the same is true of $\hat{B}^\dagger$. It is thus sufficient to consider Hermitian and anti-Hermitian solutions of (B2).

Let us first consider $\hat{B}$ Hermitian (and non-zero). Then, by letting $s \tilde{s}$ be the projection on the eigenspace of a non-zero eigenvalue, we see that (B2) does not hold. Next we consider anti-Hermitian $\hat{B}$. Clearly $B \propto i\mathbb{1}$ solves (B2). We claim that these are the only solutions. To see this, consider a $\hat{B}$ with two different eigenvalues,

$$\hat{B}|1\rangle = i\lambda_1 |1\rangle, \quad \hat{B}|2\rangle = i\lambda_2 |2\rangle, \quad \lambda_1 \neq \lambda_2.$$  \hspace{1cm} (B3)
If we now set
\[
s = \frac{1}{2}(|1\rangle + e^{i\pi/4}|2\rangle)(\langle 1\rangle + e^{-i\pi/4}\langle 2\rangle), \quad \tilde{s} = \frac{1}{2}(|1\rangle + e^{-i\pi/4}|2\rangle)(\langle 1\rangle + e^{i\pi/4} \langle 2\rangle)
\] (B4)
we find
\[
\text{Re} \text{tr}[s\tilde{s}\hat{B}] = \frac{1}{4}(\lambda_1 - \lambda_2) \neq 0
\] (B5)
and the claim follows. This justifies the equivalence \(\sim\) in (45).

**Appendix C: The von Neumann measurement scheme**

Originally, weak measurements were discussed in the context of a specific physical implementation of the measurement process due to von Neumann. Here we review this formulation of weak measurements and relate it to the results of the present paper.

One imagines performing the measurement by coupling the system of interest \(\mathcal{H}\) to an auxiliary meter system \(\mathcal{H}_{\text{aux}}\). More specifically, let \(\mathcal{H}_{\text{aux}} = L^2(\mathbb{R})\) with the usual operators \([\hat{X}, \hat{P}] = i\). Given an observable \(\hat{O}\) on \(\mathcal{H}\), we take the interaction between the system and the meter to be given by the unitary
\[
\hat{U} := e^{-i\hat{O}\hat{P}}. \tag{C1}
\]
The physical intuition is that the position of the meter (\(\hat{X}\)) is shifted by the eigenvalue of \(\hat{O}\), but we will see that the situation is more complicated if we postselect on the system. The initial state of the meter, \(s_{\text{aux}}\), is taken to be peaked around \(x = 0\), with width \(\sigma\),
\[
\langle \hat{X} \rangle_{s_{\text{aux}}} = 0, \quad \langle \hat{X}^2 \rangle_{s_{\text{aux}}} = \sigma^2. \tag{C2}
\]
The expectation value of \(\hat{X}\), after the interaction between the meter and the system, is simply the expectation value of \(\hat{O}\),
\[
\text{tr}[(1 \otimes \hat{X})\hat{U}(s \otimes s_{\text{aux}})\hat{U}^\dagger] = \langle \hat{O} \rangle_{s}. \tag{C3}
\]
When the initial width of meter state is much larger than the eigenvalues of \(\hat{O}\) the measurement becomes weak, with \(\sigma^{-1}\) playing the role of the interaction strength. From the discussion in Section IV we then expect the expectation value of \(\hat{X}\) conditioned on successful postselection (on the original system) to take the form
\[
\frac{\text{tr}[(\tilde{s} \otimes \hat{X})\hat{U}(s \otimes s_{\text{aux}})\hat{U}^\dagger]}{\text{tr}[(\tilde{s} \otimes 1)\hat{U}(s \otimes s_{\text{aux}})\hat{U}^\dagger]} = \text{Re} \frac{\text{tr}[\tilde{s}\hat{A}s]}{\text{tr}[\tilde{s}s]} + O(\sigma^{-1}) \tag{C4}
\]
in the weak limit. On one hand it is clear from (C3) that we must have $\hat{A}^R = \hat{O}$, on the other hand $\hat{O}$ is the only operator on $\mathcal{H}$ in the game, so we should also have $A^I \propto \hat{O}$. Indeed, an explicit calculation shows that

$$\hat{A} = \hat{O} - i\langle\{\hat{X}, \hat{P}\}\rangle_{s_{\text{aux}}} \hat{O}. \tag{C5}$$

The AAV weak value is thus recovered when

$$\langle\{\hat{X}, \hat{P}\}\rangle_{s_{\text{aux}}} = 0. \tag{C6}$$

There are many possible ways to generalize this model of measurement such that $\hat{A}^I$ does not have to be proportional to $\hat{O}$. One possibility is to replace $\hat{U} \rightarrow \hat{U}^{\sigma}$,

$$\hat{U}^{\sigma} := e^{i\frac{\sigma-2}{2} \hat{B} \hat{X}} e^{-i\hat{O}\hat{P}}. \tag{C7}$$

Here $\hat{B}$ is an arbitrary Hermitian operator on $\mathcal{H}$. In this generalized model (C3) still holds (for any finite $\sigma$), but now the conditional expectation value is

$$\frac{\text{tr}[\hat{s} \otimes \hat{X} \hat{U}^{\sigma}(s \otimes s_{\text{aux}}^{\sigma})(\hat{U}^{\sigma})^\dagger]}{\text{tr}[\hat{s} \otimes 1 \hat{U}^{\sigma}(s \otimes s_{\text{aux}}^{\sigma})(\hat{U}^{\sigma})^\dagger]} = \text{Re} \frac{\text{tr}[\hat{s} \hat{A}' s]}{\text{tr}[\hat{s} \hat{s}]} + O(\sigma^{-1}), \tag{C8}$$

with

$$\hat{A}' = \hat{O} + i(\hat{B} - \langle\{\hat{X}, \hat{P}\}\rangle_{s_{\text{aux}}} \hat{O}). \tag{C9}$$

Before we turn to disturbance, let us briefly examine how the meter system is affected by the interaction. The probability distribution of the meter position $\hat{X}$ is initially

$$P_i(x) := \langle\hat{\Pi}_x\rangle_{s_{\text{aux}}}, \quad \Pi_x := |x\rangle\langle x| \tag{C10}$$

After the interaction and postselection of the system it becomes

$$P_f(x) := \frac{\text{tr}[\hat{s} \otimes \hat{\Pi}_x \hat{U}^{\sigma}(s \otimes s_{\text{aux}}^{\sigma})(\hat{U}^{\sigma})^\dagger]}{\text{tr}[\hat{s} \otimes 1 \hat{U}^{\sigma}(s \otimes s_{\text{aux}}^{\sigma})(\hat{U}^{\sigma})^\dagger]} = P_i(x) - \left(\text{Re} \frac{\text{tr}[\hat{s} \hat{O} s]}{\text{tr}[\hat{s} \hat{s}]}\right) \partial_x P_i(x)$$

$$+ \left(\text{Im} \frac{\text{tr}[\hat{s} \hat{O} s]}{\text{tr}[\hat{s} \hat{s}]}\right) \langle\{\hat{\Pi}_x, \hat{P}\}\rangle_{s_{\text{aux}}} - \left(\text{Im} \frac{\text{tr}[\hat{s} \hat{B} s]}{\text{tr}[\hat{s} \hat{s}]}\right) \sigma^{-2} x P_i(x) + O(\sigma^{-3}) \tag{C11}$$

to lowest non-trivial order. With no postselection only the two first terms contribute, and we see that the meter (distribution) is simply translated, in accordance with the physical intuition. However, once we postselect this picture is in general ruined by the additional terms, even if $\hat{B}$ is zero (i.e. in the original von Neumann model). This shows that one
should be careful about applying intuition to the quantum measurement process, even for simple models like von Neumann’s.

The state of the system after the weak measurement is

$$M_\sigma^s(s) = \text{tr}_{\text{aux}}[\hat{U}\sigma(s \otimes s^\sigma_{\text{aux}})(\hat{U}_{\sigma})^\dagger] = s - i\langle \hat{P}\rangle_{s^\sigma_{\text{aux}}} + O(\sigma^{-2}).$$

We thus conclude that the measurement is non-disturbing in the weak sense iff $$\langle \hat{P}\rangle_{s^\sigma_{\text{aux}}} = 0.$$ Note that this condition does not put any constraints on $$\langle{\hat{X}, \hat{P}}\rangle$$ or $$\hat{B}$$. The average survival probability is

$$\int d\psi \text{ tr}[M_\sigma^s(|\psi\rangle\langle\psi|)] = 1 - \frac{\sigma^{-2}}{d(d+1)}F + O(\sigma^{-3}).$$

This expression is, for fixed $$\hat{O}$$, bounded from below. In fact,

$$F = \sigma^2\langle{\hat{P}}^2\rangle_{s^\sigma_{\text{aux}}} f(\hat{O}) + \frac{1}{4} f(\hat{B}) - \frac{1}{2} \langle{\hat{X}, \hat{P}}\rangle_{s^\sigma_{\text{aux}}} (d \text{ tr}[\hat{O}\hat{B}] - \text{ tr}[\hat{O}] \text{ tr}[\hat{B}]).$$

This expression is, for fixed $$\hat{O}$$, bounded from below. In fact,

$$F = \left(\sigma^2\langle{\hat{P}}^2\rangle_{s^\sigma_{\text{aux}}} - \frac{1}{4} \langle{\hat{X}, \hat{P}}\rangle_{s^\sigma_{\text{aux}}}ight) f(\hat{O}) + \frac{1}{4} f \left(\langle{\hat{X}, \hat{P}}\rangle_{s^\sigma_{\text{aux}}} \hat{O} - \hat{B}\right)$$

$$\geq \left(\sigma^2\sigma_p^2 - \frac{1}{4} \langle{\hat{X}, \hat{P}}\rangle_{s^\sigma_{\text{aux}}}ight) f(\hat{O})$$

$$\geq \frac{1}{4} f(\hat{O}),$$

where the last inequality is the Schrödinger uncertainty relation. Note that for non-trivial observables $$\hat{O}$$, the inequality implies that $$F$$ is strictly larger than zero. The situation considered by AAV[1] corresponds to $$\langle{\hat{X}, \hat{P}}\rangle = 0$$, $$\hat{B} = 0$$ and $$\langle\hat{P}^2\rangle = \sigma^{-2}/4$$ which implies that $$F = f(\hat{O})/4$$. We see that the AAV measurement procedure minimizes the value of $$F$$, in accordance with the general result of Section IV B.

Appendix D: Minimally disturbing measurements

We want to characterize the minimally disturbing (in the sense of having the smallest $$F$$ as defined in Sec. IV B) generalized weak measurements. More concretely, consider the collection $$\mathcal{M}_\hat{O}$$ of weak measurements measuring a fixed observable without postselection, i.e. such that $$\hat{A}^R = \hat{O}$$. As a first guess one might try to minimize $$F$$ on $$\mathcal{M}_\hat{O}$$, but this fails because the simple rescaling

$$A_m \rightarrow \epsilon^{-1} A_m, \quad K_{m,n}^0 \rightarrow K_{m,n}^0, \quad \delta K_{m,n} \rightarrow \epsilon \delta K_{m,n}$$

(D1)
shows that there are elements of $\mathcal{M}_\hat{O}$ with arbitrarily small $\mathcal{F}$.

Let $N$ be the number of measurement outcomes, which we consider fixed. A natural choice is to consider the subset $\mathcal{M}_{\hat{O},A^*} \subset \mathcal{M}_\hat{O}$ where $(A_{m=1}, \ldots, A_{m=N})$ is constrained to belong to some compact set $A^* \subset \mathbb{R}^N$. In this way we avoid the problem of the rescaling (D1), since compact sets are bounded. The exact nature of the set $A^*$ is not important, except that $\mathcal{M}_{\hat{O},A^*}$ should be non-empty. Unfortunately $\mathcal{M}_{\hat{O},A^*}$ is not compact, so the existence of a minimal elements is still not obvious. To remedy this problem we define a better behaved subset $\mathcal{M}'_{\hat{O},A^*} \subset \mathcal{M}_{\hat{O},A^*}$ such that to each element $x$ of $\mathcal{M}_{\hat{O},A^*}$ there corresponds an element $y$ of $\mathcal{M}'_{\hat{O},A^*}$ with $\mathcal{F}(y) \leq \mathcal{F}(x)$. It is then clear that a minimal element of $\mathcal{M}'_{\hat{O},A^*}$ is also a minimal element of $\mathcal{M}_{\hat{O},A^*}$. What we will show is:

If the set $\mathcal{M}_{\hat{O},A^*}$ is non-empty then it contains elements minimal with respect to $\mathcal{F}$. Furthermore, these minimal elements satisfy $\hat{A} \sim \hat{O}$ which implies that the weak values are given by the AAV formula.

Let us first note that setting all $\delta \hat{K}_{m,n} = 0$ decreases $\mathcal{F}$ (see Eq. (75)) and does not change $\hat{A}^R$. We can thus restrict $\mathcal{M}'_{\hat{O},A^*}$ to having Hermitian $\delta \hat{K}_{m,n}$. We can also restrict to having only one Kraus operator per measurement outcome. To see this fix an $m$ and consider the contribution $\mathcal{F}_m$ to $\mathcal{F}$ from this outcome. We then have the inequalities

$$P_m^0 \mathcal{F}_m = \left( \sum_n (K^0_{m,n})^2 \right) \left( \sum_n \text{tr}[\delta \hat{K}^2_{m,n}] \right)$$

$$\geq \left| \sum_n K^0_{m,n} \sqrt{\text{tr}[\delta \hat{K}^2_{m,n}]} \right|^2$$

$$\geq \text{tr} \left[ \left( \sum_n K^0_{m,n} \delta \hat{K}_{m,n} \right)^2 \right]$$

$$= \text{tr}[\delta \hat{K}^2_m],$$

that is

$$\mathcal{F}_m \geq \frac{\text{tr}[\delta \hat{K}^2_m]}{P_m^0}.$$ 

But this shows that replacing $\delta \hat{K}_{m,n}$ by a single operator $\delta \hat{K}_m$ given by

$$\delta \hat{K}_m = \frac{\sum_n K^0_{m,n} \delta \hat{K}_{m,n}}{\sqrt{P_m^0}}$$

(along with $K^0_{m,n} \rightarrow K^0_m = \sqrt{P_m^0}$) decreases $\mathcal{F}_m$ and thus $\mathcal{F}$.  

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To simplify matters slightly let us assume $\text{tr} \hat{O} = 0$ for now. We note that $f(\delta \hat{K} + c \mathbb{1}) = f(\delta \hat{K})$ for any $c \in \mathbb{R}$. It follows that the replacement

$$
\delta \hat{K}_m \rightarrow \delta \hat{K}_m - (d^{-1} \text{tr}[\delta \hat{K}_m]) \mathbb{1}
$$

(D8)

leaves $\mathcal{F}$ invariant. This allows us to restrict $\mathcal{M}_O^\prime \hat{A}^\ast$ to measurements with $\text{tr}[\delta \hat{K}_m] = 0$.

On elements of $\mathcal{M}_O^\prime \hat{A}^\ast$, $\mathcal{F}$ is given by

$$
\mathcal{F} = d \sum_m \text{tr}[\delta \hat{K}_m^2].
$$

(D9)

For sufficiently big $C$ the set

$$
\mathcal{M}_O^\prime \hat{A}^\ast := \{ x \in \mathcal{M}_O^\prime \hat{A}^\ast \mid \mathcal{F}(x) \leq C \}
$$

(D10)

is seen to be non-empty and compact (here the compactness of $\hat{A}^\ast$ is needed), and a minimal element of $\mathcal{M}_O^\prime \hat{A}^\ast$, is also minimal in $\mathcal{M}_O^\prime \hat{A}^\ast$ and hence in $\mathcal{M}_O \hat{A}^\ast$. We have thus shown that there are minimally disturbing measurements. The general case of $\text{tr} \hat{O} \neq 0$ is easily reduced to the case we have covered by shifting by the identity (similar to Eq. (D8)).

Using that $f(\delta \hat{K}_{m,n}^I) = 0$ iff $\delta \hat{K}_{m,n}$ is proportional to the identity and Eq. (75) it is clear that for minimally disturbing measurements in $\mathcal{M}_O \hat{A}^\ast$, we must have $\hat{A} \sim \hat{O}$, which is what we wanted to show. We leave a more thorough characterization of the minimally disturbing measurements to further work.

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[39] By *operational* we mean that the framework is expressed in terms of notions directly related to the experimental situation, e.g. preparation procedures and outcome probabilities.

[40] Note that \( \mathcal{S} \) does not contain all maps \( \mathcal{S} \to [0, 1] \).

[41] Here we identity measurement procedures where the weak values are identical, i.e. where the maps \( (s, \hat{s}) \mapsto \hat{s} \mathbb{E}_s^w[A] \) agree.

[42] This uniqueness fails in the quantum mechanical case.

[43] Note that \( \mathcal{S}' \) must then be (uncountably) infinite, but that we will still only need to consider finite sums of states from \( \mathcal{S}' \).

[44] For instance it is assumed in Refs. [16, 22] that the effect operators (i.e. \( \sum_n (\hat{K}_{m,n})^\dagger \hat{K}_{m,n} \) in our notation) all commute with the observable.

[45] When expanding we take \( \hat{X} \) to be of order \( \sigma \) and \( \hat{P} \) to be of order \( \sigma^{-1} \). To make the calculations rigorous, it is necessary to add regularity conditions on \( s_{aux}^\sigma \). We omit the details.

[46] As in Sec. [IV] we consider a weak measurement procedure to be specified by constants \( A_m \) and \( K^0_{m,n} \), and operators \( \delta \hat{K}_{m,n} \).