Determinants of Laplacians, the Ray-Singer Torsion on Lens Spaces and the Riemann zeta function

by

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Abstract: We obtain explicit expressions for the determinants of the Laplacians on zero and one forms for an infinite class of three dimensional lens spaces $L(p,q)$. These expressions can be combined to obtain the Ray-Singer torsion of these lens spaces. As a consequence we obtain an infinite class of formulae for the Riemann zeta function $\zeta(3)$. The value of these determinants (and the torsion) grows as the size of the fundamental group of the lens space increases and this is also computed. The triviality of the torsion for just the three lens spaces $L(6,1)$, $L(10,3)$ and $L(12,5)$ is also noted.

§ 1. Introduction

Topological phenomena are now known to play an important part in many quantum field theories. This is especially true of gauge theories. There are also topological quantum field theories in which the excitations are purely topological and the classical phase spaces of these theories usually reduce to a finite dimensional space: these spaces can be zero dimensional discrete sets, or whole moduli spaces. The semi-classical, or stationary phase, approximation to the functional integral of such a theory is then a weighted sum, or integral, over the finite dimensional phase space. In addition, for some of these theories, this approximation is exact providing thereby a reduction of the functional integral to a finite dimensional integral.

If a topological quantum field theory contains a gauge field the reduced functional integral mentioned above often consists of sums or integrals over flat connections; the non-triviality of such connections is determined purely by their holonomy, and, if $A$ is a flat connection over a manifold $M$, its holonomy is an element of the fundamental group $\pi_1(M)$. This means that an ideal laboratory within which to study such theories is provided by taking the manifold $M$ to have a non-trivial fundamental group but to be otherwise topologically rather simple. An ideal way to do this is to take $M$ to be the quotient of a sphere $S^n$ by a finite cyclic group $G$. This quotient $S^n/G$, described in more detail below, is what is called a lens space, written as $L(p,q)$. In this paper we take $M$ to be a lens space on which is placed a topological field theory whose classical phase space consists of flat connections.
Our approach is to take the model given by the field theory and analyse it in detail on a whole infinite class of lens spaces. We work in three dimensions and realise $M$ as the quotient of the manifold $S^3$ by the action of a discrete group $Z_p$. The resulting partition function on this manifold is a combinatorial invariant of the manifold known as the Ray-Singer torsion of the manifold. However the field theory gives this partition function as the ratio of a set of determinants. A standard technique in field theory has been to define these functional determinants through the analytic continuation of the zeta functions of the associated operators.

In this work we investigate the individual determinants that arise and obtain highly explicit expressions for them. Our expressions have an intriguing structure of their own. For example, on the lens space $L(2,1)$, we find that

$$\ln \text{Det } d^*d_0 = -\frac{3\zeta(3)}{2\pi^2} + \ln 2$$

$$\ln \text{Det } d^*d_1 = -\frac{3\zeta(3)}{\pi^2} - 2\ln 2$$

(1.1)

Similar, though more complicated, expressions occur for each of the lens spaces $L(p,1)$ for $p = 3, 4, \ldots$. This in turn leads to non-trivial formulae for $\zeta(3)$: to give two examples we find that

$$\zeta(3) = \frac{2\pi^2}{7} \ln(2) - \frac{8}{7} \int_0^{\pi/2} dz z^2 \cot(z)$$

$$\zeta(3) = \frac{2\pi^2}{13} \ln 3 - \frac{9}{13} \int_0^{\pi/3} dz (z + \frac{\pi}{3}) \cot(z) - \frac{9}{13} \int_0^{2\pi/3} dz (z - \frac{\pi}{3}) \cot(z)$$

(1.2)

these being the formulae that come from $L(2,1)$ and $L(3,1)$ respectively.

The structure of the paper is as follows. In Section 2 we describe the topological field theory under consideration. In section 3 we define the Ray-Singer torsion and describe the lens spaces with which we work; we also carry out the non-trivial task of obtaining the eigenvalues and degeneracies for the Laplacians on $p$-forms acting on these spaces. Section 4 deals with the lens space $L(2,1) (SO(3))$ and is a construction of the analytic continuation of the appropriate $p$-form zeta functions followed by a calculation of their associated determinants. Sections 5 and 6 describe the analogous calculation and expressions for the infinite classes of manifolds corresponding to $L(p,1)$ for $p$ odd and even respectively. Finally in section 7 we present our conclusions, some comments on the torsion of $L(p, q)$ for general $q$, and some graphical data for the resulting determinants and torsion.

§ 2. Topological Field Theory

The torsion studied in this paper has its origins in the 1930’s, cf. Franz [1], where it was combinatorially defined and used to distinguish various lens spaces from one another. Given a manifold $M$ and a representation of its fundamental group $\pi_1(M)$ in a flat bundle $E$, this Reidemeister-Franz torsion is a real number which is defined as a particular product of ratio’s of volume elements $V^i$ constructed from the cohomology groups $H^i(M; E)$. 
Since volume elements are essentially determinants then, for any alternative definition of a determinant, an alternative definition of the torsion can be given. Now if one uses de Rham cohomology to compute \( H^i(M; E) \) then these determinants become determinants of Laplacians \( \Delta_p^E \) on \( p \)-forms with coefficients in \( E \). But zeta functions for elliptic operators can be used to give finite values to such infinite dimensional determinants and so an analytic definition of the torsion results and this is the analytic torsion of Ray and Singer \([2,3,4]\) given in the 1970’s; furthermore this torsion was proved by them to be independent of the Riemannian metric used to define the Laplacian’s \( \Delta_p^E \).

This analytic torsion coincided, for the case of lens spaces, with the combinatorially defined Reidemeister-Franz torsion. Finally Cheeger and Müller \([5,6]\) independently proved that the analytic Ray-Singer torsion coincides with the combinatorial Reidemeister-Franz torsion in all cases.

Infinite dimensional determinants also occur naturally in quantum field theories when computing correlation functions and partition functions. In 1978 Schwarz \([7]\) showed how to construct a quantum field theory on a manifold \( M \) whose partition function is a power of the Ray-Singer torsion on \( M \).

Schwarz’s construction uses an Abelian gauge theory but in three dimensions a non-Abelian gauge theory—the \( SU(2) \) Chern-Simons theory—can be constructed and has deep and important properties established by Witten in 1988: Its partition function is the Witten invariant for the three manifold \( M \) and the correlation functions of Wilson loops give the Jones polynomial invariant for the link determined by the Wilson loops—cf. \([8,9]\). Finally the weak coupling limit of the partition function is a power of the Ray-Singer torsion.

To define the Ray-Singer torsion, or simply torsion, we take a closed compact Riemannian manifold \( M \) over which we have a flat bundle \( E \). Let \( M \) have a non-trivial fundamental group \( \pi_1(M) \) which is represented on \( E \)—this latter property arises very naturally in the physical gauge theory context where it corresponds simply to the space of flat connections all of whose content resides in their holonomy—In any case the torsion is then the real number \( T(M, E) \) where

\[
\ln T(M, E) = \sum_0^n (-1)^q \ln \text{Det} \Delta_q^E, \quad n = \text{dim } M
\]  

The metric independence of the torsion requires that we assume, in the above definition, that the cohomology ring \( H^*(M; E) \) is trivial; this means that the Laplacians \( \Delta_q^E \) have empty kernels and so are strictly positive definite. Given this fact one may use zeta functions to define \( \text{Det} \Delta_q^E \) in the standard way. Recall that if \( P \) is a positive elliptic differential or pseudo-differential operator with spectrum \( \{\mu_n\} \) and degeneracies \( \{\Gamma_n\} \) then its associated zeta function \( \zeta_P(s) \) is a meromorphic function of \( s \), regular at \( s = 0 \), which is given by

\[
\zeta_P(s) = \sum_{\mu_n} \frac{\Gamma_n}{\mu_n^s}
\]  

and its determinant \( \text{Det } P \) is defined by

\[
\ln \text{Det } P = - \frac{d\zeta_P(s)}{ds} \bigg|_{s=0}
\]
Using this we have

\[
\ln T(M, E) = - \sum_{q=0}^{n} (-1)^q q \frac{d\zeta_{\Delta E}(s)}{ds} \bigg|_{s=0} \tag{2.4}
\]

Quantum field theories of the type alluded to above are usually referred to as topological quantum field theories or simply topological field theories.

It turns out that more than one topological field theory can be used to give the torsion, for an excellent review of this question cf. Birmingham et al. [10]. For example one can take the action

\[
S[\omega] = i \int_M \omega_n d\omega_n, \quad \text{dim } M = 2n + 1 \tag{2.5}
\]

where \(\omega_n\) is an \(n\)-form. The partition function is then

\[
Z[M] = \int \mathcal{D}\omega \exp[S[\omega]] \tag{2.6}
\]

\(S[\omega]\) has a gauge invariance whereby \(S[\omega] = S[\omega + d\lambda]\) and therefore to define the partition function it is necessary to integrate over only inequivalent field configurations. The measure \(\mathcal{D}\omega \mu[\omega]\) thus contains functional delta functions which constrain the integration and play the role of gauge fixing, together with their associated determinants. This measure can be constructed using, for example, the Batalin-Vilkovisky BRST construction [11,12].

We shall be concerned here with the special situation of three dimensions and with the case where the three manifold \(M\) is a lens space. The topological field theory of interest to us in this paper is given by the action

\[
S[\omega] = i \int_M \omega_1 d\omega_1 \tag{2.7}
\]

where \(\omega_1\) is now a 1-form. To construct the integration measure we will follow the Batalin-Vilkovisky BRST construction [11,12]. The essential element of this construction is what is termed a “gauge Fermion” whose BRST variation gives the gauge fixing and ghost portion of the BRST invariant action. Integrating out these fields yields the contribution \(\mu[\omega]\) to the measure.

The gauge Fermion is constructed by choosing a gauge fixing for the field \(\omega_1\) (which we take to be \(d^*\omega = 0\)), and multiplying the condition by an anti-ghost \(c_{\bar{0}}\), which is a 3-form denoted by its conjugated Poincaré dual label, this indicates its anti-ghost nature also. Thus the gauge Fermion is given by

\[
\Psi = c_{\bar{0}} d^*\omega_1 \tag{2.8}
\]

The associated BRST variations of these fields are

\[
\begin{align*}
\delta \omega_1 &= -d c_{\bar{0}}, \\
\delta c_{\bar{0}} &= 0 \\
\delta c_{\bar{0}} &= i \omega_{\bar{0}}, \\
\delta \omega_{\bar{0}} &= 0
\end{align*} \tag{2.9}
\]
With these definitions it is easy to check that $\delta^2 = 0$. The BRST gauge fixed action is then
\[ L = i\omega_1 d\omega_1 + \delta \Psi \]
which expands to
\[ L = i\omega_1 d\omega_1 - c_0 d^* d c_0 + i\omega_0 d^* \omega_1 \tag{2.10} \]
If we integrate out all fields as they appear the resulting partition function is
\[ Z = (\text{Det } L_-)^{-\frac{1}{2}} \text{Det } d^* d_0 \tag{2.11} \]
where the operator $L_-$ is obtained by integrating out the $\omega_1$ fields and is a linear operator acting on odd forms. The partition function is therefore
\[ Z = \frac{\text{Det } \Delta_0}{\text{Det } \Delta_1^{\frac{1}{4}} \text{Det } \Delta_3^{\frac{1}{2}}} \]
Using Poincaré duality the logarithm of this partition function is then given by
\[ \ln Z = \frac{1}{4} (3 \ln \text{Det } \Delta_0 - \ln \text{Det } \Delta_1) \]
and we see it is proportional to the logarithm of the Ray-Singer torsion.

Our task in what follows is to evaluate the individual components of this expression both for their usefulness in their own right and to verify that the combined result agrees with the Ray-Singer torsion. We do this in the restricted setting where $M$ belongs to a class of three dimensional lens spaces. In the next section we specify the lens spaces that we work with and obtain the eigenvalues and their degeneracies of the Laplacians on these spaces.

§ 3. Lens Spaces

We now want to turn to field theories defined on lens spaces—for general background on lens spaces cf. [3,4] and references therein—briefly, a lens space can be constructed as follows: Take an odd dimensional sphere $S^{2n+1}$, considered as a subset of $\mathbb{C}^n$, on which a finite cyclic group of rotations $G$, say, acts. The quotient $S^{2n+1}/G$ of the sphere under this action is a lens space. More precisely, suppose that $G$ is of order $p$, $(z_1, \ldots, z_n) \in \mathbb{C}^n$ and the group action takes the form
\[ (z_1, \ldots, z_n) \mapsto (\exp(2\pi i q_1/p)z_1, \ldots, \exp(2\pi i q_n/p)z_n) \tag{3.1} \]
with $q_1, \ldots, q_n$ integers relatively prime to $p$ then the quotient $S^{2n+1}/G$ is a lens space often denoted by $L(p; q_1, \ldots, q_n)$. A formula for the torsion of these spaces was first worked out by Ray [2]. To our knowledge however there is no computation of the individual determinants of Laplacians on these spaces in the literature. Since these are of independent field theoretic significance and from these the torsion is constructed it is instructive to examine these separately and construct the
torsion from them. This we will proceed to do in the next sections focusing on the situation that obtains when $n = 2$ and $G$ is the group $\mathbb{Z}_p \equiv \mathbb{Z}/p\mathbb{Z}$. For simplicity we shall denote the resulting lens space $S^3/\mathbb{Z}_p = L(p; 1, 1)$ by $L(p)$, we shall also denote the lens space $L(p, 1, q)$ by $L(p, q)$; in passing we note that when $p = 2$ we have $L(2) = \mathbb{R}P^3 \simeq SO(3)$.

The group action above defines a representation $V$, say, of $\pi_1(L(p))$ and also determines a flat bundle $F = (V \times S^3)/\mathbb{Z}_p$, over $L(p)$. It is the determinants of Laplacians and the resulting torsion of this $F$ over $L(p)$ with which we are concerned here. Using zeta functions the torsion of these lens spaces is therefore given by

$$\ln T(L(p), F) = -\sum_{0}^{3} (-1)^q \left. \frac{d\zeta_{\Delta_F}(s)}{ds} \right|_{s=0}$$

(3.2)

As an aid to the calculation of $\ln T(L(p), F)$ it is useful to introduce the notation

$$\tau(p, s) = -\sum_{0}^{3} (-1)^q q \zeta_{\Delta_F}(s)$$

$$T(p) = T(L(p), F)$$

The relationship between the two functions being clearly

$$\ln T(p) = \left. \frac{d\tau(p, s)}{ds} \right|_{s=0}$$

(3.3)

For $\tau(p, s)$ itself we now have

$$\tau(p, s) = \zeta_{\Delta_1}(s) - 2\zeta_{\Delta_2}(s) + 3\zeta_{\Delta_3}(s)$$

$$= 3\zeta_{\Delta_3}(s) - \zeta_{\Delta_1}(s), \quad \text{using Poincaré duality}$$

(3.4)

Combining the standard decomposition $\Delta_p = (d^*d + dd^*)_p$, with the fact that $\ker d^* \cap \ker d = \emptyset$, we further obtain the formula

$$\tau(p, s) = 2\zeta_{d^*d_0}(s) - \zeta_{d^*d_1}(s)$$

(3.5)

We now simplify our notation by labelling

$$\tau_+(p, s) = 2\zeta_{d^*d_0}(s), \quad \tau_-(p, s) = \zeta_{d^*d_1}(s)$$

For the individual zeta functions we denote the eigenvalues and their degeneracies by $\lambda_n(q, p)$ and $\Gamma_n(q, p)$ respectively giving the expressions

$$\tau_+(p, s) = 2 \sum_{n} \frac{\Gamma_n(0, p)}{\lambda_n^2(0, p)}, \quad \tau_-(p, s) = \sum_{n} \frac{\Gamma_n(1, p)}{\lambda_n^2(1, p)}$$

(3.6)

It remains to compute these eigenvalues and degeneracies. The former are actually independent of $p$ and are fairly easily calculated by the technique of starting with harmonic
forms in $\mathbb{R}^{2n}$ and then restricting successively to $S^{2n+1}$ and $L(p)$. In any case they are given by

$$\begin{align*}
\lambda_n(0, p) &= n(n + 2), \ n = 1, 2, \ldots \\
\lambda_n(1, p) &= (n + 1)^2, \ n = 1, 2, \ldots
\end{align*} \tag{3.7}$$

To calculate the degeneracies is more difficult; we make use of the fact that $S^3$ is a group manifold and proceed as follows: Consider the Laplacians $d^*d_q$ on $S^3$, and $d^*d_q^F$ on $L(p)$ also, if $\lambda$ is an eigenvalue, denote the corresponding eigenspaces by $\Lambda_q(\lambda)$ and $\Lambda_q^F(\lambda)$ respectively. Let

$$v(z) \in \Lambda_q(\lambda), \text{ with } z \in S^3 \subset \mathbb{C}^2, \quad \text{and } g \in \mathbb{Z}_p, \text{ where } g \equiv \exp[2\pi ij/p], \ 0 \leq j \leq (p - 1) \tag{3.8}$$

The element $g$ acts on $v(z)$ to give $g \cdot v(z)$ where

$$g \cdot v(z) = v(gz)$$

$$gz = (\exp[2\pi ij/p]z_1, \exp[2\pi ij/p]z_2) \tag{3.9}$$

The above definitions allow us to define the projection $P(\lambda)$ on $\Lambda_q(\lambda)$ by

$$P(\lambda)v = \frac{1}{p} \sum_{g \in \mathbb{Z}_p} \exp[-2\pi ij/p]g \cdot v \tag{3.10}$$

Evidently

$$[P(\lambda), d^*d_q] = 0 \tag{3.11}$$

and so $P(\lambda)$ projects the space $\Lambda_q(\lambda)$ onto the space $\Lambda_q^F(\lambda)$. Finally this means that we obtain a formula for the degeneracy $\Gamma_n(q, p)$, namely

$$\begin{align*}
\Gamma_n(q, p) &= \text{tr} \left( P|_{\Lambda_q^F(\lambda)} \right) \\
&= \frac{1}{p} \sum_{j=0}^{(p-1)} \exp[-2\pi ij/p] \text{tr} \left( g|_{\Lambda_q^F(\lambda)} \right) \tag{3.12}
\end{align*}$$

To actually apply this formula we now add in the fact that $S^3$ is the group manifold for $SU(2)$. The Peter–Weyl theorem tells us, in this case where all representations are self-conjugate, that

$$L^2(S^3) = L^2(SU(2)) = \bigoplus_{\mu} c_{\mu} D_{\mu} = \bigoplus_{\mu} D_{\mu} \otimes D_{\mu} \tag{3.13}$$

where $c_{\mu}$ measures the multiplicity of the representation $\mu$ which must therefore be $\dim D_{\mu}$. But Hodge theory gives us the alternative decomposition

$$L^2(S^3) = \bigoplus_{\lambda} \Lambda_0(\lambda) \tag{3.14}$$
In addition, the Casimir operator for $SU(2)$ is a multiple of the Laplacian and, if the representation label $\mu$ is taken to be the usual half-integer $j$, then we know that this Casimir has eigenvalues $j(j+1)$, and also that $\dim D_j = 2j+1$. These facts identify the Laplacian $\Delta_0 = d^*d_0$ as four times the Casimir and identify $\Lambda_0(\lambda)$ as $\dim D_j$ copies of $D_j$. Thus if we set $n = 2j$, so that $n$ is always integral, then we have the degeneracy formula

$$\Gamma_n(0,p) = \frac{(n+1)}{p} \sum_{j=0}^{(p-1)} \exp[-2\pi ij/p] \chi^{n/2}(2\pi j/p)$$

where $\chi^j(\theta)$ denotes the $SU(2)$ character, on $D_j$, for rotation through the angle $\theta$; i.e.

$$\chi^j(\theta) = \frac{\sin((2j+1)\theta)}{\sin(\theta)}$$

Hence our explicit degeneracy formula for 0-forms on $L(p)$ is

$$\Gamma_n(0,p) = \frac{(n+1)}{p} \sum_{j=0}^{(p-1)} \exp[-2\pi ij/p] \sin(2\pi j/p)$$

We now have to find the analogous formula for the 1-forms. The formula that results is

$$\Gamma_n(1,p) = \frac{1}{p} \sum_{j=0}^{(p-1)} \exp[-2\pi ij/p] \left\{ n\chi^{(n+1)/2}(2\pi j/p) + (n+2)\chi^{(n-1)/2}(2\pi j/p) \right\}$$

or, more explicitly,

$$\Gamma_n(1,p) = \frac{1}{p} \sum_{j=0}^{(p-1)} \exp[-2\pi ij/p] \left\{ n\frac{\sin(2\pi j/p)}{\sin(2\pi j/p)} + (n+2)\frac{\sin(2\pi nj/p)}{\sin(2\pi j/p)} \right\}$$

To simplify the notation we introduce the ‘$p$-averaged character’ $\langle \chi^j \rangle_p$ which we define by

$$\langle \chi^j \rangle_p = \frac{1}{p} \sum_{j=0}^{(p-1)} \exp[-2\pi ij/p] \chi^j(2\pi j/p)$$

Finally this gives us a concrete expression for $\tau(p,s)$, i.e.

$$\tau(p,s) = \sum_n \left\{ \frac{2(n+1)}{\{n(n+2)\}^{2s}} - \frac{n\langle \chi^{(n+1)/2} \rangle_p + (n+2)\langle \chi^{(n-1)/2} \rangle_p}{(n+1)^{2s}} \right\}$$

where

$$\tau_+(p,s) = \sum_n \frac{2(n+1)}{\{n(n+2)\}^s} \langle \chi^{n/2} \rangle_p$$

$$\tau_-(p,s) = \sum_n \frac{n\langle \chi^{(n+1)/2} \rangle_p + (n+2)\langle \chi^{(n-1)/2} \rangle_p}{(n+1)^{2s}}$$

$$\tau(p,s) = \tau_+(p,s) - \tau_-(p,s)$$

(3.21)
To make further progress towards a computation of the determinants and torsion we need to be able to evaluate these \( p \)-averaged characters. This is a somewhat non-trivial combinatorial task but this task is eased if we use for \( \chi^j(\theta) \), the alternative expression

\[
\chi^j(\theta) = \sum_{m=-j}^j \exp[2im\theta] \tag{3.22}
\]

It is also necessary to divide \( n \) up into its conjugacy classes mod \( p \) by writing

\[
n = pk - j, \quad k \in \mathbb{Z}, \quad j = 0, 1, \ldots, (p - 1) \tag{3.23}
\]

We eventually discover that

\[
\left\langle \chi^{(pk-j)/2} \right\rangle_p = \begin{cases} 
   k & \text{for } j = 0, 2, \ldots, (p-1) \\
   (k-1) & \text{for } j = 3, 5, \ldots, (p-2) \\
   0 & \text{if } p \text{ is odd} \\
   2k & \text{for } j = 1 \\
   (2k-1) & \text{for } j = 3, 5, \ldots, (p-1) \\
   & \text{if } p \text{ is even}
\end{cases} \tag{3.24}
\]

We now lack only one ingredient among those necessary for a calculation of the determinants and the resulting torsion: this is the construction of the analytic continuation of the series for \( \tau(p, s) \). We shall construct this in the next section. The technique we shall use will be more easily followed if we first use it in a more simple case. Thus, to begin with, we set \( p = 2 \) and then construct the continuation.

\[\S\ 4. \ \textbf{The Analytic Continuation for } p = 2\]

The series to be continued are

\[
\tau_+(p, s) = \sum_{n=1}^{\infty} \frac{2(n+1) \left\langle \chi^{n/2} \right\rangle_p}{\{n(n+2)\}^s} \\
\tau_-(p, s) = \sum_{n=1}^{\infty} \frac{n \left\langle \chi^{(n+1)/2} \right\rangle_p + (n+2) \left\langle \chi^{(n-1)/2} \right\rangle_p}{(n+1)^{2s}}
\]

and their difference which leads to the torsion

\[
\tau(p, s) = \sum_n \left\{ \frac{2(n+1) \left\langle \chi^{n/2} \right\rangle_p}{\{n(n+2)\}^s} - \frac{n \left\langle \chi^{(n+1)/2} \right\rangle_p + (n+2) \left\langle \chi^{(n-1)/2} \right\rangle_p}{(n+1)^{2s}} \right\} \tag{4.1}
\]

These already converges for \( \text{Re } s > 3/2 \); however a calculation of the determinants and the torsion requires us to work at \( s = 0 \), hence we see the need for, and the extent of, the analytic continuation.
Our interest in this section for illustrative purposes is in the case \( p = 2 \) where we have

\[
\tau(2, s) = \tau_+ (2, s) - \tau_-(2, s)
\]

\[
= \sum_n \left\{ \frac{2(n + 1) \langle \chi^{n/2} \rangle}{n(n+2)} - \frac{n \langle \chi^{(n+1)/2} \rangle + (n + 2) \langle \chi^{(n-1)/2} \rangle}{(n + 1)^{2s}} \right\} \tag{4.2}
\]

But using 3.24 we find that

\[
\langle \chi^{(n+1)/2} \rangle = \langle \chi^{(2k-j+1)/2} \rangle, \quad (n = 2k - j)
\]

\[
= \left\{ \begin{array}{ll}
0, & j = 1 \\
2k + 2, & j = 0
\end{array} \right. \equiv \left\{ \begin{array}{ll}
0, & n \text{ odd} \\
2k + 2, & n \text{ even}
\end{array} \right.
\]

Similarly

\[
\langle \chi^{(n-1)/2} \rangle = \langle \chi^{(2k-j)/2} \rangle, \quad (n = 2k - j)
\]

\[
= \left\{ \begin{array}{ll}
2k, & j = 1 \\
0, & j = 0
\end{array} \right. \equiv \left\{ \begin{array}{ll}
(n + 1), & n \text{ odd} \\
0, & n \text{ even}
\end{array} \right.
\]

Thus \( \tau(2, s) \) becomes

\[
\tau(2, s) = \tau_+ (2, s) - \tau_-(2, s)
\]

\[
= \sum_{n \text{ odd}} \frac{2(n + 1)^2}{n(n+2)} - \sum_{n \text{ even}} \frac{2n(n+2)}{(n+1)^{2s}} \tag{4.5}
\]

Setting \( n = (2m - 1) \) in \( \tau_+ (2, s) \) and \( n = 2m \) in \( \tau_-(2, s) \) we have

\[
\tau_+ (2, s) = \sum_{m=1}^{\infty} \frac{8m^2}{(4m^2 - 1)^s}, \quad \tau_-(2, s) = \sum_{m=0}^{\infty} \frac{4m(2m + 2)}{(2m + 1)^{2s}}
\]

and

\[
\tau(2, s) = \sum_{m=1}^{\infty} \frac{8m^2}{(4m^2 - 1)^s} - \sum_{m=0}^{\infty} \frac{2}{(2m + 1)^{(2s-2)}} + \sum_{m=0}^{\infty} \frac{2}{(2m + 1)^{2s}} \tag{4.6}
\]

Now if we use the fact that

\[
\sum_{n=1,3,5, \ldots} \frac{1}{n^s} = (1 - 2^{-s})\zeta(s) \quad (4.7)
\]

where \( \zeta(s) \) is the usual Riemann zeta function then we get

\[
\tau_-(2, s) = 2(1 - 2^{-(2s-2)})\zeta(2s - 2) - 2(1 - 2^{-2s})\zeta(2s) \tag{4.8}
\]
and denoting

\[ A_2(m, 0, s) = \frac{(2m)^2}{(2m^2 - 1)s}, \quad \text{and} \quad A_2(0, s) = \sum_{m=1}^{\infty} A_2(m, 0, s) \]

(This notation is used to agree with the general case to be discussed in the next section. See also Appendix A.) Thus we have

\[ \tau_+(2, s) = 2A_2(0, s) \]

and these combine to give

\[ \tau(2, s) = 2A_2(0, s) - 2(1 - 2^{-(2s-2)})\zeta(2s - 2) + 2(1 - 2^{-2s})\zeta(2s) \quad (4.9) \]

Since the terms involving the Riemann zeta function already have a well defined continuation it remains to continue \( A_2(0, s) \). Now

\[ A_2(m, 0, s) = \frac{4m^2}{(4m^2 - 1)s} = \frac{4m^2}{(4m^2)^s} \left(1 - \frac{1}{4m^2}\right)^{-s} \]

\[ = \frac{1}{(4m^2)^{(s-1)}} \left\{1 + \frac{s}{4m^2} + \cdots \right\} \]

\[ = \frac{1}{(4m^2)^{(s-1)}} + \frac{s}{(4m^2)^s} + R(m, s), \quad \text{(def. of } R(m, s) \text{)} \]

So that the remainder term \( R(m, s) \) is given by

\[ R(m, s) = A_2(0, m, s) - \frac{1}{(4m^2)^{(s-1)}} - \frac{s}{(4m^2)^s} \]

\[ = \frac{4m^2}{(4m^2 - 1)^s} - \frac{1}{(4m^2)^{(s-1)}} - \frac{s}{(4m^2)^s} \quad (4.11) \]

The definition of the remainder term is chosen to ensure that

\[ |R(m, s)| \leq \frac{(\ln m)^{\alpha}}{m^2} \quad (4.12) \]

and this has the vital consequence that the operations \( d/ds \) (at \( s = 0 \)) and \( \sum_m \) commute when applied to \( R(m, s) \).

Defining

\[ R(s) = \sum_{m=0}^{\infty} R(m, s) \]

allows us to tidy our expressions up somewhat. Collecting our regulated expressions we therefore have

\[ \tau_+(2, s) = \frac{8}{4^s}\zeta(2s - 2) + \frac{2s}{4^s}\zeta(2s) + 2R(s) \quad \text{and} \]

\[ \tau_-(2, s) = 2(1 - 2^{-(2s-2)})\zeta(2s - 2) - 2(1 - 2^{-2s})\zeta(2s) \quad (4.14) \]
In fact the expression for $\tau(2, s)$ can be further tidied up to give

$$\tau(2, s) = 2 \left\{ \frac{8}{4(s)} - 1 \right\} \zeta(2s - 2) + 2 \left\{ 1 + \frac{(s - 1)}{4s} \right\} \zeta(2s) + 2R(s) \quad (4.15)$$

The series for $R(s)$ is *guaranteed* to be convergent and the analytic continuation is now complete.

Evaluating our expressions at $s = 0$ we find

$$\tau_{+}(2, 0) = 8\zeta(-2) + 2R(0) \quad \tau_{-}(2, 0) = -6\zeta(-2)$$

and

$$\tau(2, 0) = 14\zeta(-2) \quad (4.16)$$

Observe that with our continuation $R(0)$ is automatically zero. Thus noting also that $\zeta(-2) = 0$, we conclude

$$\tau_{+}(2, 0) = 0, \quad \tau_{-}(2, 0) = 0, \quad \text{and} \quad \tau(2, 0) = 0 \quad (4.17)$$

That $\tau_{\pm}(p, 0) = 0$ is quite generally true for arbitrary $p$; we shall see this in the next section and this agrees with general considerations for generalised zeta functions of second order operators on compact odd dimensional manifolds.

We can now take the final step which is to differentiate 4.14 and obtain $\tau_{+}'(2, 0)$ and $\tau_{-}'(2, 0)$, which we denote by $\tau_{+}'(2)$, $\tau_{-}'(2)$ respectively, and hence the torsion $T(2)$.

The resulting expressions are

$$\tau_{+}'(2) = 16\zeta'(-2) + 2\zeta(0) + 2R'(0)$$

$$\tau_{-}'(2) = -12\zeta'(-2) - 2\ln 4\zeta(0)$$

and for the torsion

$$\ln T(2) = \frac{d\tau(2, 0)}{ds} = 28\zeta'(-2) + 2(1 + \ln 4)\zeta(0) + 2R'(0) \quad (4.18)$$

But

$$\zeta(0) = -1/2, \quad \text{and} \quad \zeta'(-2) = -\frac{\zeta(3)}{4\pi^2} \quad \text{from the functional relation} \quad (4.19)$$

and by our remark above concerning the motive for our choice of definition for $R(m, s)$ we have

$$R'(0) = \frac{d}{ds} \sum_{m} R(m, s) \big|_{s=0}$$

$$\Rightarrow R'(0) = \sum_{m} \frac{dR(m, s)}{ds} \bigg|_{s=0}$$

$$= \sum_{m} \left[ 4m^2 \left\{ \ln(4m^2) - \ln(4m^2 - 1) \right\} - 1 \right] = -\sum_{m} \left[ 4m^2 \ln(1 - 1/4m^2) + 1 \right] \quad (4.20)$$
Hence
\[
\tau'_+(2) = -\frac{4}{\pi^2} \zeta(3) - 1 - 2 \sum_m \left[ 4m^2 \ln(1 - 1/4m^2) + 1 \right]
\]
\[
\tau'_-(2) = \frac{3}{\pi^2} \zeta(3) + 2 \ln 2 \quad \text{and}
\]
\[
\ln T(2) = -\frac{7}{\pi^2} \zeta(3) - 1 - 2 \ln(2) - 2 \sum_m \left[ 4m^2 \ln(1 - 1/4m^2) + 1 \right]
\]

However the series for \( R'(0) \) can be expressed as a trigonometric integral; in fact, as a special case of more general results which will be derived below, we have
\[
\sum_{m=1}^{\infty} \left[ 4m^2 \ln(1 - 1/4m^2) + 1 \right] = -\frac{1}{2} + \frac{4}{\pi^2} \int_0^{\pi/2} dz \frac{z^2 \cot(z)}{2}
\]

which means that
\[
\tau'_+(2) = -\frac{4}{\pi^2} \zeta(3) - \frac{8}{\pi^2} \int_0^{\pi/2} dz \frac{z^2 \cot(z)}{2}
\]
\[
\tau'_-(2) = \frac{3}{\pi^2} \zeta(3) + 2 \ln 2 \quad \text{and}
\]
\[
\ln T(2) = -\frac{7}{\pi^2} \zeta(3) - 2 \ln(2) - \frac{8}{\pi^2} \int_0^{\pi/2} dz \frac{z^2 \cot(z)}{2}
\]

The formula 4.23 above for \( T(2) \) can be pushed even further: By using it with Ray’s expression [2] for the torsion we can deduce that
\[
\ln T(p) = -4 \sum_{j=1}^{p-1} \sum_{k=1}^p \cos\left(\frac{2jk\pi}{p}\right) \ln(2 \sin\left(\frac{2k\pi}{p}\right)) \exp\left[\frac{2k\pi i}{p}\right]
\]
\[
= -4 \ln \left[ 2 \sin\left(\frac{\pi}{p}\right) \right]
\]

which, for \( p = 2 \), becomes simply
\[
\ln T(2) = -4 \ln(2)
\]

Hence we straightaway have the identity
\[
-4 \ln(2) = -\frac{7}{\pi^2} \zeta(3) - 2 \ln(2) - \frac{8}{\pi^2} \int_0^{\pi/2} dz \frac{z^2 \cot(z)}{2}
\]

Or
\[
\zeta(3) = \frac{2\pi^2}{7} \ln(2) - \frac{8}{7} \int_0^{\pi/2} dz \frac{z^2 \cot(z)}{2}
\]
In other words our computation of the torsion has given us a formula for the zeta function at its first odd argument. Equivalently we can use this relation to eliminate the integral and obtain quite simple expressions for the logarithms of the determinants of the Laplacians on zero and one forms respectively. We conclude this section by quoting these results.

Noting first of all that the expressions for \( \tau'_+ (2) \) and \( \tau'_- (2) \) in their simplest form now become

\[
\tau'_+ (2) = \frac{3}{\pi^2} \zeta (3) - 2 \ln 2 \quad \text{and} \quad \tau'_- (2) = \frac{3}{\pi^2} \zeta (3) + 2 \ln 2
\]

and from the definitions of these objects we have at once that

\[
\ln \text{Det } d^* d_0 = - \frac{3}{2\pi^2} \zeta (3) + \ln 2 \quad \quad (4.29)
\]

It is interesting to note the role that the Riemann zeta function \( \zeta (3) \) plays in these expressions. Since these are expressions for volume elements on the discrete moduli spaces associated with the Laplacians, we expect that there are deeper things to be learned from a further study of such expressions.

In the next section we tackle the continuation for arbitrary \( p \).

§ 5. The Determinants and the Torsion for \( p \) Odd.

The analytic continuation for a general value of \( p \) naturally divides into two cases: \( p \) odd and \( p \) even; in fact we shall see below that the case for \( p \) even further divides into two subcases which correspond to \( p = 0, 2 \mod 4 \). Due to the size of the expressions it is now much more convenient to continue \( \tau_+ (p, s) \) and \( \tau_- (p, s) \) separately and then combine them into expressions for the torsion. We deal first with \( \tau_+ (p, s) \) and begin with \( p \) odd.

§§ 5.1 The Function \( \tau_+ (p, s) \) and \( \ln \text{Det } d^* d_0 \)

Let us recall that \( \tau_+ (p, s) \) is given by

\[
\tau_+ (p, s) = \sum_{n=1}^{\infty} \frac{2(n+1) \langle \chi^{n/2} \rangle_{p}^{s}}{\{n(n+2)\}^s} \quad (5.1)
\]

Reference to the general character formula 3.24 shows that we must resolve \( n \) into its conjugacy classes mod \( p \) by writing

\[
n = pk - j, \quad j = 0, \ldots, (p - 1) \quad (5.2)
\]

and that we must distinguish the two parities of \( j \). To implement these requirements we set

\[
p = 2r + 1, \quad \text{and parametrise } \begin{cases} j & \text{odd by } j = 2l + 1, l = 0, 1, \ldots, (r - 1) \\ j & \text{even by } j = 2l, l = 0, 1, \ldots, r \end{cases} \quad (5.3)
\]
This gives
\[
\tau_+(p, s) = 2 \sum_{j=0}^{p-1} \sum_{k=1}^{\infty} \frac{(pk - j + 1) \langle \chi^{(pk-j)/2} \rangle_p}{((pk - j)(pk - j + 2))^s}
\]
\[
= 2 \sum_{k=1}^{\infty} \left[ \sum_{l=0}^{r-1} \frac{(pk - 2l) \langle \chi^{(pk-2l-1)/2} \rangle_p}{((pk - 2l - 1)(pk - 2l + 1))^s} + \sum_{l=0}^{r} \frac{(pk - 2l + 1) \langle \chi^{(pk-2l)/2} \rangle_p}{((pk - 2l)(pk - 2l + 2))^s} \right]
\] (5.4)

Then when we use 3.24 for \( p \) odd we get
\[
\tau_+(p, s) = \sum_{k=1}^{\infty} \left[ \frac{2pk^2}{((pk - 1)(pk + 1))^s} + \sum_{l=0}^{r-1} \frac{2(k - 1)(pk - 2l)}{((pk - 2l - 1)(pk - 2l + 1))^s} + \sum_{l=0}^{r} \frac{2k(pk - 2l + 1)}{((pk - 2l)(pk - 2l + 2))^s} \right]
\] (5.5)

To aid in marshalling the combinatorics of \( \tau_+(p, s) \) we define \( H_p(k, s, x) \) by
\[
H_p(k, s, x) = \frac{pk(pk + x)}{(pk + x - 1)(pk + x + 1)}
\] (5.6)

The point being that each of the three summands in 5.5 is of the form \( H_p(k, s, \lambda) \) for appropriate \( \lambda \). To see this we introduce precisely \( p \) constants of the type \( \lambda \) defined by
\[
\begin{cases}
\lambda_0 = 0 \\
\lambda_l = -2l + 1, \quad l = 0, \ldots, r \\
\bar{\lambda}_l = -2l + p, \quad l = 1, \ldots, (r - 1)
\end{cases}
\] (5.7)

With this notation it can be checked that \( \tau_+(p, s) \) is given by
\[
\tau_+(p, s) = 2 \frac{1}{p} \sum_{k=1}^{\infty} \left[ H_p(k, s, \lambda_0) + \sum_{l=1}^{r-1} H_p(k, s, \bar{\lambda}_l) + \sum_{l=0}^{r} H_p(k, s, \lambda_l) \right]
\] (5.8)

Also if we denote the entire set of \( \lambda \)'s by \( \{\lambda\} \) i.e.
\[
\{\lambda\} \equiv \{\lambda_0, \lambda_l, \bar{\lambda}_l\} = \{-(p - 2), \ldots, -5, -3, -1, 0, 1, 3, 5 \ldots (p - 2)\}
\] (5.9)

then we have the even more concise expression
\[
\tau_+(p, s) = 2 \frac{1}{p} \sum_{\{\lambda\}} \sum_{k=1}^{\infty} H_p(k, \lambda, s)
\] (5.10)

The functions \( H_p(\lambda, s) \) obtained by summing over \( k \) then form a set of \( p \) functions whose derivative at \( s = 0 \) can be viewed as living on the appropriate space of sections for the
Laplacian acting on 0 forms on the lens space $L(p)$; taking the trace over these functions viewed as forming a matrix then gives the analytic continuation of the determinant of the Laplacian.

Next observing

$H_p(k, x, s) = \frac{(pk + x)^2}{\{(pk + x)^2 - 1\}^s} - x \frac{(pk + x)}{\{(pk + x)^2 - 1\}^s}$

we see that there are therefore two additional functions of interest here i.e.

$A_p(k, x, s) = \frac{(pk + x)^2}{\{(pk + x)^2 - 1\}^s}$

and

$B_p(k, x, s) = \frac{(pk + x)}{\{(pk + x)^2 - 1\}^s}$

and we have

$H_p(k, x, s) = A_p(k, x, s) - xB_p(k, x, s)$

and we can equally write

$\tau_+(p, s) = \frac{2}{p} \sum_{\{\lambda\}} [A_p(\lambda, s) - \lambda B_p(\lambda, s)]$

Let us further note that in the set $\{\lambda\}$ the non zero elements come in pairs of the form $\{\lambda, -\lambda\}$. Thus we can further write

$\tau_+(p, s) = \frac{2}{p} A_p(0, s) + \frac{2}{p} \sum_{l=1}^{r-1} [A_p^+(2l + 1, s) - (2l + 1)B_p^-(2l + 1, s)]$

where the $\mp$ superscripts refer to the symmetric or anti-symmetric combination with respect to the first argument: i.e. $A_p^+(x, s) = A_p(x, s) + A_p(-x, s)$ and $B_p^-(x, s) = B_p(x, s) - B_p(-x, s)$. We relegate the details of the computation of these functions and their analytic continuation to appendices A and B—the calculations are generalisations of those performed for $p = 2$. Quoting here from appendices A and B we have that the relevant functions and their derivatives at $s = 0$ are given by

$A_p(x, 0) = p^2 \zeta(-2, 1 + \frac{x}{p}) = -\frac{x(x + p)(2x + p)}{6p}$

$B_p(x, 0) = p\zeta(-1, 1 + \frac{x}{p}) + \frac{1}{2} = -\frac{p}{12} - \frac{x}{p} + \frac{x^2 + 1}{2p}$

$H_p(x, 0) = -\frac{px}{12} - \frac{x}{2p} + \frac{x^3}{6p}$

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Which immediately implies that $H^+_p(x,0) = 0$ and we have our first result that for $p$ odd

$$
\tau_+(p,0) = 0
$$

The significance of this is that the analytic continuation of the scaling dimension of the determinant is zero.

Next we note again from appendices A and B that

$$
A^+_p(x) = - \frac{p^2}{\pi^2} \zeta(3) - \frac{p^2}{\pi^2} \int_0^{(x+1)\pi/p} \frac{dz}{z} (z - \frac{2\pi x}{p}) \cot(z) - \frac{p^2}{\pi^2} \int_0^{(x-1)\pi/p} dz (z - \frac{2\pi x}{p}) \cot(z)
$$

$$
- x^2 \ln \frac{2 \sin (((x+1)\pi/p)/x + 1)}{x + 1} - x^2 \ln \frac{2 \sin (((x-1)\pi/p)/x)}{x - 1}
$$

(5.17)

and

$$
B^-_p(x) = \frac{p}{\pi} \int_0^{(x+1)\pi/p} dz (z \cot(z)) + \frac{p}{\pi} \int_0^{(x-1)\pi/p} dz (z \cot(z))
$$

$$
- x \ln \frac{2 \sin (((x+1)\pi/p)/x + 1)}{x + 1} - x \ln \frac{2 \sin (((x-1)\pi/p)/x)}{x - 1}
$$

(5.18)

Now combining our expressions 5.17, 5.18 and summing over $\{\lambda\}$ we obtain for $\tau'_+(p)$ the expression

$$
\tau'_+(p) = -2 \frac{p^3}{2\pi^2} \zeta(3) + \frac{p^2}{\pi^2} \int_0^{\frac{2\pi}{p}} dz z^2 \cot(z) + \frac{p^2}{\pi^2} \int_0^{\frac{(p-1)\pi}{p}} dz (z - \frac{(p-2)\pi}{p}) \cot(z)
$$

$$
+ 2 \sum_{l=1}^{(p-3)/2} \frac{p^2}{\pi^2} \int_0^{\frac{2\pi}{p}} dz (z - \frac{2l\pi}{p}) \cot(z)
$$

(5.19)

for $p$ odd. But the expression $\tau'_+(p)$ above is $2 \ln \text{Det } d^*d_0$: i.e. twice the logarithm of the Laplacian on 0-forms for the lens space $L(p)$. More precisely our analytic continuation has shown us that

$$
\ln \text{Det } d^*d_0 = \frac{1}{p} \left[ \frac{p^3}{2\pi^2} \zeta(3) + \frac{p^2}{\pi^2} \int_0^{\frac{2\pi}{p}} dz z^2 \cot(z) + \frac{p^2}{\pi^2} \int_0^{\frac{(p-1)\pi}{p}} dz (z - \frac{(p-2)\pi}{p}) \cot(z)
$$

$$
+ 2 \sum_{l=1}^{(p-3)/2} \frac{p^2}{\pi^2} \int_0^{\frac{2\pi}{p}} dz (z - \frac{2l\pi}{p}) \cot(z) \right]
$$

(5.20)

§§ 5.2 The Function $\tau_-(p, s)$ and $\ln \text{Det } d^*d_1$

Let us now turn to $\tau_-(p, s)$ for $p$ odd.

$$
\tau_-(p, s) = \sum_{n=1}^{\infty} n \left< \chi^{(n+1)/p} \right> + (n + 2) \left< \chi^{(n-1)/p} \right>(n + 1)^2s
$$

(5.21)
which on decomposing \( n \) over the conjugacy classes mod \( p \) \( n = pk - j, \quad j = 0, \ldots, (p-1) \) as for \( \tau_+(p, s) \), distinguishing the two parities as in 5.3 yields

\[
\tau_-(p, s) = \sum_{k=1}^{\infty} \frac{kp\left< \frac{k+1}{p} \right> + (kp + 2)\left< \frac{k+1}{p} \right>}{(kp + 1)^{2s}} + \sum_{k=1}^{\infty} \frac{(kp - 2)\left< \frac{k-1}{p} \right> + kp\left< \frac{k-1}{p} \right>}{(kp - 1)^{2s}} \\
+ \sum_{k=1}^{\infty} \sum_{l=2}^{\infty} \frac{(kp - 2l)\left< \frac{k-2l+1}{p} \right> + (kp - 2l + 2)\left< \frac{k-2l-1}{p} \right>}{(kp - 2l + 1)^{2s}} \\
+ \sum_{k=1}^{\infty} \frac{(kp - p + 1)\left< \frac{k-p}{p} \right> + (kp - p + 3)\left< \frac{k-p-2}{p} \right>}{(kp - p + 2)^{2s}} \\
+ \sum_{k=1}^{\infty} \sum_{l=0}^{r-1} \frac{(kp - 2l - 1)\left< \frac{k-2l+1}{p} \right> + (kp - 2l + 1)\left< \frac{k-2l-2}{p} \right>}{(kp - 2l)^{2s}}
\]

(5.22)

Using our degeneracy formula 3.24 we have with a little re-arrangement

\[
\tau_-(p, s) = \sum_{k=1}^{\infty} \frac{2(kp + 1)k + kp}{(kp + 1)^{2s}} + \sum_{k=1}^{\infty} \frac{2(kp - 1)k - kp}{(kp - 1)^{2s}} \\
+ \sum_{l=2}^{\infty} \sum_{k=1}^{\infty} \frac{2(kp + 2l)k}{(kp + 2l)^{2s}}
\]

(5.23)

We now observe that this is of the form

\[
\tau_-(p, s) = \frac{2}{p} \sum_{\nu} \sum_{k=1}^{\infty} \frac{kp(kp + \nu)}{(kp + \nu)^{2s}} \\
+ \sum_{k=1}^{\infty} \frac{kp}{(kp + 1)^{2s}} - \sum_{k=1}^{\infty} \frac{kp}{(kp - 1)^{2s}}
\]

(5.24)

where we have denoted by \( \{\nu\} \) the set

\[
\{\nu\} \equiv \{\nu_0, \nu_1, \bar{\nu}_1\} = \{-(p-3), \ldots, -6, -4, -2, -1, 0, 1, 2, 4, 6 \ldots (p-3)\}
\]

(5.25)

The sums occurring in 5.24 are naturally expressible in terms of Hurwitz zeta functions and this provides the natural analytic continuation of this expression. We also note that

\[
\sum_{\nu} \sum_{k=1}^{\infty} \frac{1}{(kp + \nu)^{2s-2}} = \zeta(2s - 2) - 1 - \sum_{l=1}^{(p-3)} (2l)^{(2-2s)}
\]

(5.26)
On utilising these relations we find that

\[
\tau_-(p, s) = \frac{2}{p} \left[ \zeta(2s - 2) - 1 - \sum_{l=1}^{(p-3)/2} (2l)^{2-2s} \right] + (p - 2)p^{-2s} \left[ \zeta(2s - 1, 1 + \frac{1}{p}) - \zeta(2s - 1, 1 - \frac{1}{p}) \right] - 2p^{-2s} \sum_{l=-(p-3)/2}^{(p-3)/2} (2l)\zeta(2s - 1, 1 + \frac{2l}{p}) - p^{-2s} \left[ \zeta(2s, 1 + \frac{1}{p}) + \zeta(2s, 1 - \frac{1}{p}) \right]
\]

(5.27)

If we analytically continue the RHS of this expression to \( s = 0 \), and observe that \( \zeta(-1, 1 + x) - \zeta(-1, 1 - x) = -x \), then we obtain

\[
\tau_-(p, 0) = \frac{2}{p} \left[ 1 + \sum_{l=1}^{(p-3)/2} (2l)^2 \right] + (p - 2)(-\frac{1}{p}) - 2 \sum_{l=1}^{(p-3)/2} (2l)(-\frac{2l}{p})
\]

(5.28)

which immediately gives

\[
\tau_-(p, 0) = 0
\]

(5.29)

Thus again the scaling dimension of the associated determinant is zero.

Passing to \( \tau'_-(p) \) by taking the derivative at \( s = 0 \) gives

\[
\tau'_-(p) = \frac{4}{p} \left[ \zeta'(-2) - \ln p + \sum_{l=1}^{(p-3)/2} (2l)^2 \ln \left( \frac{2l}{p} \right) \right] + 2(p - 2) \left[ \zeta'(-1, 1 + \frac{1}{p}) - \zeta'(-1, 1 - \frac{1}{p}) \right] - 4 \sum_{l=1}^{(p-3)/2} (2l) \left[ \zeta'(-1, 1 + \frac{2l}{p}) - \zeta'(-1, 1 - \frac{2l}{p}) \right] - 2 \left[ \zeta'(0, 1 + \frac{1}{p}) + \zeta'(0, 1 - \frac{1}{p}) \right]
\]

(5.30)

A useful identity for Hurwitz zeta functions derived in appendix C is

\[
\zeta'(-1, 1 + x) - \zeta'(-1, 1 - x) = -x \ln \left[ \frac{2 \sin(\pi x)}{x} \right] + \frac{1}{\pi} \int_0^{\pi x} dz \cot(z)
\]
Using this and the expression $\zeta'(-2) = -\zeta(3)/4\pi^2$ we conclude that

$$
\tau'_-(p) = -\frac{1}{p\pi^2} \zeta(3) + \frac{4}{p} \ln \left( 2 \sin \left( \frac{\pi}{p} \right) \right) + \frac{4}{p} \sum_{l=1}^{(p-3)/2} (2l)^2 \ln \left( 2 \sin \left( \frac{2\pi l}{p} \right) \right) + \frac{2(p-2)}{\pi} \int_0^{\frac{\pi}{p}} dz \cot(z) - \frac{4}{\pi} \sum_{l=1}^{(p-3)/2} (2l) \int_0^{\frac{2\pi l}{p}} dz \cot(z) \tag{5.31}
$$

But $-\tau'_-(p)$ is the analytic continuation which gives $\ln \det d^*d_1$. Hence we find that

$$
\ln \det d^*d_1 = \frac{1}{p\pi^2} \zeta(3) + \frac{4}{p} \ln \left( 2 \sin \left( \frac{\pi}{p} \right) \right) + \frac{4}{p} \sum_{l=1}^{(p-3)/2} (2l)^2 \ln \left( 2 \sin \left( \frac{2\pi l}{p} \right) \right) + \frac{2(p-2)}{\pi} \int_0^{\frac{\pi}{p}} dz \cot(z) - \frac{4}{\pi} \sum_{l=1}^{(p-3)/2} (2l) \int_0^{\frac{2\pi l}{p}} dz \cot(z) \tag{5.32}
$$

§§ 5.3 The Torsion $T(p)$ and $\tau(p,s)$.

We are therefore now in a position to put these together and obtain an expression for the torsion. The first observation is that since $\tau_+(p,0)$ and $\tau_-(p,0)$ are both zero we have

$$
\tau(p,0) = 0 \tag{5.33}
$$

This vanishing of $\tau(p,0)$ is related to the metric independence of the torsion something which has been established quite generally by Ray and Singer [3].

The torsion $T(p)$ itself is given by the difference of 5.19 and 5.31. Combining these two expressions we find, upon a little simplification, that $T(p)$ is determined by the equation

$$
\ln T(p) = -\frac{2}{p} \left[ \frac{(p^3 - 1)}{2\pi^2} \zeta(3) + 2 \ln \left( 2 \sin \left( \frac{\pi}{p} \right) \right) + 2 \sum_{l=1}^{(p-3)/2} 4l^2 \ln \left( 2 \sin \left( \frac{2\pi l}{p} \right) \right) + \frac{p^2}{\pi^2} \int_0^{\frac{\pi}{p}} dz \left( \frac{p-2}{\pi} \right) \cot(z) + \frac{p^2}{\pi^2} \int_0^{\frac{(p-1)\pi}{p}} dz \left( \frac{p-2}{\pi} \right) \cot(z) \right] + 2 \sum_{l=1}^{(p-3)/2} \frac{p^2}{\pi^2} \int_0^{\frac{2\pi l}{p}} dz \left( \frac{4l\pi}{p} \right) \cot(z) \tag{5.34}
$$

Now the expression for the torsion from Ray’s calculation gave the alternative expression 4.24 i.e.

$$
\ln T(p) = -4 \ln \left( 2 \sin \left( \frac{\pi}{p} \right) \right) \tag{5.35}
$$
We have verified that these two expressions 5.34 and 4.24 agree numerically, yet it is not transparent by inspection that this should be so; also using C.41 of appendix C we can reduce 5.34 to Ray’s expression. One may conclude that, by following two alternate derivations, we have arrived at what is a sequence of non-trivial identities. As we saw in the case of \( p = 2 \) utilising these identities one can obtain non-trivial formulae for \( \zeta(3) \) and also can be used to further simplify the expressions for the individual determinants. The resulting expressions for \( \zeta(3) \) are

\[
\zeta(3) = \frac{2\pi^2}{(p^3 - 1)} \left[ 2(p - 1) \ln \left( 2 \sin \left( \frac{\pi}{p} \right) \right) - 2 \sum_{l=1}^{(p-3)/2} 4l^2 \ln \left( 2 \sin \left( \frac{2l\pi}{p} \right) \right) \right. \\
- \frac{p^2}{\pi^2} \int_0^{\frac{\pi}{p}} dz \left( z + \frac{(p - 2)\pi}{p} \right) \cot(z) - \frac{p^2}{\pi^2} \int_0^{\frac{(p-1)\pi}{p}} dz \left( z - \frac{(p - 2)\pi}{p} \right) \cot(z) \\
\left. - 2 \sum_{l=1}^{(p-3)/2} \frac{p^2}{\pi^2} \int_0^{\frac{2l\pi}{p}} dz \left( z - \frac{4l\pi}{p} \right) \cot(z) \right]
\] (5.36)

For the sake of illustration let us quote the implications of these formulae for the simplest odd case: \( p = 3 \). On utilising all of the information at our disposal we find that

\[
\tau'_+(3) = -\frac{\zeta(3)}{3\pi^2} - \frac{4}{3} \ln 3 + \frac{2}{\pi} \int_0^{\frac{\pi}{3}} dz \cot(z) \\
\tau'_-(3) = -\frac{\zeta(3)}{3\pi^2} + \frac{2}{3} \ln 3 + \frac{2}{\pi} \int_0^{\frac{2\pi}{3}} dz \cot(z) \quad \text{and} \\
\ln T(3) = -2 \ln 3
\]

The relation between the expressions we have obtained which we expressed in terms of a formula for \( \zeta(3) \) (the analog of 4.27 for \( p = 2 \)) becomes

\[
\zeta(3) = \frac{2\pi^2}{13} \ln 3 - \frac{9}{13} \int_0^{\frac{\pi}{3}} dz \left( z + \frac{\pi}{3} \right) \cot(z) - \frac{9}{13} \int_0^{\frac{2\pi}{3}} dz \left( z - \frac{\pi}{3} \right) \cot(z)
\]

We will now turn to the case of even \( p \).

§ 6. Determinants and the Torsion for \( p \) Even.

When \( p \) is even we follow a slightly different route to that used in the previous section but we arrive at expressions of a similar general form for the respective determinants and their corresponding torsion.

§§ 6.1 The Function \( \tau_+(p, s) \) and \( \ln \text{Det} \ d^*d_0 \)

Let us first obtain series expressions for \( \tau_+(p, s) \) for \( p \) even, and observe that the same functions as those encountered for \( p \) odd enter these also. Recalling our expression for \( \tau_+(p, s) \)

\[
\tau_+(p, s) = \sum_{n=1}^{\infty} \frac{2(n + 1) \langle \chi \hat{\phi} \rangle_p}{\left( (n + 1)^2 - 1 \right)^{s}} \] (6.1)
We again resolve \( n \) into its conjugacy classes mod \( p \) by writing
\[
n = pk - j, \quad j = 0, \ldots, (p - 1)
\] (6.2)
and distinguish the two parities of \( j \) by setting
\[
p = 2r, \quad \text{and parametrise } \begin{cases} j \text{ odd by } j = 2l + 1, & l = 0, 1, \ldots, (r - 1) \\ j \text{ even by } j = 2l, & l = 0, 1, \ldots, (r - 1) \end{cases}
\] (6.3)
Reference to the general character formula 3.24 shows that only \( j \) odd contributes and we obtain
\[
\tau_+(p, s) = \sum_{k=1}^{\infty} \frac{4pk}{(pk)^2 - 1} s + \sum_{l=0}^{(p-2)} \sum_{k=1}^{\infty} \frac{2(pk - 2l)(2k - 1)}{(pk - 2l)^2 - 1} s
\] (6.4)
which gives
\[
\tau_+(p, s) = \frac{4}{p} \sum_{k=1}^{(p-2)} \left[ \sum_{l=0}^{r-1} \frac{(pk - 2l)^2}{(pk - 2l)^2 - 1} s + \sum_{l=1}^{(p-2)} \frac{(pk - 2l)}{2l} \frac{(pk - 2l)}{(pk - 2l)^2 - 1} s \right]
\] (6.5)
We recognise the expressions arising as the functions from the preceding analysis in the case of \( p \) odd and which are analysed in appendices A and B. We can therefore write 6.5 as
\[
\tau_+(p, s) = \frac{4}{p} \sum_{l=0}^{r-1} [A_p(-2l, s) + (2l - r)B_p(-2l, s)]
\] (6.6)
Some further rearrangement will allow us to write these again in terms of the symmetric and anti-symmetric parts of \( A_p(x, s) \) and \( B_p(x, s) \) respectively. Note first of all, however that the term involving \( A_p \) is a sum over all even conjugacy classes i.e.
\[
\sum_{l=1}^{r-1} A(-2l, s) = \sum_{l=1}^{r-1} \sum_{k=1}^{\infty} \frac{(2rk - 2l)^2}{(2rk - 2l)^2 - 1} s
\] (6.7)
\[
= \sum_{m=1}^{\infty} \frac{(2m)^2}{(2m)^2 - 1} s
\]
and is therefore \( \tau_+(2, s)/2 \); our expression 6.6 for \( \tau_+(p, s) \) can hence be written as
\[
\tau_+(p, s) = \frac{2}{p} \left[ \tau_+(2, s) + 2 \sum_{l=1}^{r-1} (2l - r)B_p(-2l, s) \right]
\] (6.8)
Now when \( l \) ranges from 1 to \( r - 1 \), \( (2l - r) \) ranges over the set
\[
-(r - 2), (r - 4), \ldots, (r - 4), (r - 2)
\] (6.9)
This allows us to divide the range up into a sum from 1 up to the integer part of \((r − 1)/2\) which we denote by \([r − 1]/2\). Thus

\[
\tau_+(p, s) = \frac{2}{p} \left[ \tau_+(2, s) + 2 \sum_{l=1}^{[\frac{r-1}{2}]} (r − 2l) [B_p(−p + 2l, s) − B_p(−2l, s)] \right]
\]  \hspace{1cm} (6.10)

Observing that

\[
B_p(−p + x, s) = \frac{x}{\{x^2 − 1\}^s} + B_p(x, s)
\]

This gives

\[
\tau_+(p, s) = \frac{2}{p} \left[ \tau_+(2, s) + 2 \sum_{l=1}^{[\frac{r-2}{4}] (\frac{p}{2} − 2l) \left[ B_p^−(2l, s) + \frac{2l}{\{x^2 − 1\}^s} \right] \right]
\]  \hspace{1cm} (6.11)

Noting that \(\tau_+(2, 0) = 0\) and that \(B_p^−(x) = −x\) we see again immediately that

\[
\tau_+(p, 0) = 0
\]  \hspace{1cm} (6.12)

Again as we expect the scaling dimension of the associated determinant is zero.

Proceeding now to the expression for the determinant itself, we find the resulting expression from 6.11 for the derivative at \(s = 0\) is

\[
\tau'_p(p, 0) = \frac{2}{p} \left[ \tau'_+(2) + \sum_{l=1}^{[\frac{p-2}{4}]} (p − 4l) \left[ B_p^−(2l) − 2l \ln[2l^2 − 1] \right] \right]
\]  \hspace{1cm} (6.13)

Substituting for \(B_p^−(2l)\) from 5.18 gives

\[
\tau'_+(p) = \frac{2}{p} \left[ \tau'_+(2) + \frac{p}{\pi} \sum_{l=1}^{[\frac{p-2}{4}]} (p − 4l) \left[ \int_0^{(2l+1)\pi/p} z \cot(z) + \int_0^{(2l-1)\pi/p} dzz \cot(z) \right]
\]  \hspace{1cm} (6.14)

\[
−\frac{2l\pi}{p} \ln \left[ 2 \sin \left( \frac{(2l+1)\pi}{p} \right) \right] − \frac{2l\pi}{p} \ln \left[ 2 \sin \left( \frac{(2l-1)\pi}{p} \right) \right] \]
\]

where from 4.28 \(\tau'_+(2) = \frac{3}{\pi^2} \zeta(3) − 2 \ln 2\). As we see the case of even \(p\) divides naturally into two classes \(p = 2 \mod 4\) and \(p = 0 \mod 4\). Making this division we can further simplify things to obtain

\[
\tau'_+(p) = \frac{2}{r} \left[ \frac{3}{2\pi^2} \zeta(3) − 2 \sum_{l=1}^{[\frac{(r-3)}{2}]} [(2l + 1)(r − 2l − 1)] \ln \left[ 2 \sin \left( \frac{(2l+1)\pi}{2r} \right) \right] \right]
\]

\[
\hspace{1cm} + \frac{2r(r − 2)}{\pi} \int_0^{\pi/r} dzz \cot(z) + \frac{4r}{\pi} \sum_{l=1}^{[\frac{(r-2)}{2}]} (r − 2l − 1) \int_0^{(2l+1)\pi/2r} dzz \cot(z) \right] \hspace{1cm} p = 2 \mod 4
\]
\[\tau_\pm'(p) = \frac{2}{r} \left[ \frac{3}{2\pi^2} \zeta(3) - \ln 2 - 2 \sum_{l=1}^{(r-2)/2} [(2l+1)(r-2l-1) - 1] \ln \left(2 \sin \left(\frac{(2l+1)\pi}{2r}\right)\right)\right] + \frac{2r(r-2)}{\pi} \int_0^\pi dz \cot(z) + \frac{4r}{\pi} \sum_{l=1}^{(r-2)/2} (r-2l) \int_0^{(2l+1)\pi/2^s} dz \cot(z) \quad p = 0 \text{ mod } 4\]

We note that these expressions agree with the result for \(p = 2\) and, for \(p = 4\), we note in passing that inspection of the series shows that \(\tau_\pm'(4,0) = \tau_\pm'(2,0)/2\); this turns out to be also a property of (the logarithm of) the torsion itself, i.e. \(T(p)\) satisfies \(\ln T(4) = (\ln T(2))/2\).

We therefore have from 6.14 an expression for the appropriate logarithmic determinant on the lens space \(L(p)\) namely

\[
\ln \text{Det } d^*d_0 = -\frac{3}{p\pi^2} \zeta(3) + \frac{2}{p} \ln 2 - 2 \sum_{l=1}^{(p-2)/2} \left(\frac{p}{2} - 2l\right) \left[\frac{p}{\pi} \int_0^{(2l+1)p}\ z \cot(z) + \frac{p}{\pi} \int_0^{(2l-1)p} dz \cot(z)\right] p = 2r
\]

\[-2l \ln \left(2 \sin \left(\frac{(2l+1)p}{p}\right)\right) - 2l \ln \left(2 \sin \left(\frac{(2l-1)p}{p}\right)\right)\]

(6.15)

\[\tau_+(p, s) = \frac{\sum_{n=1}^\infty \langle \chi_n^{s+1}\rangle_p + (n+2) \langle \chi_n^{s-1}\rangle_p}{(n+1)^{2s}}\]

(6.16)

Decomposing the sum over \(n\) into the different conjugacy classes and using our general character formula we have

\[\tau_-(p, s) = \sum_{k=1}^\infty \frac{pk(pk+1) + (pk+2)2k}{(pk+1)^{2s}} + \sum_{k=1}^\infty \frac{(pk-2)2k + pk(2k-1)}{(pk-1)^{2s}} + \sum_{l=2}^{r-1} \frac{2(pk-2l+1)(2k-1)}{(pk-2l+1)^{2s}}\]

(6.17)
After some rearrangement we arrive at

$$
\tau_-(p, s) = \sum_{k=1}^{\infty} \left\{ \frac{4}{p} \sum_{l=0}^{r-1} \frac{1}{(pk - 2l + 1)^{2s-2}} \right. \\
+ (1 - \frac{4}{p}) \left[ \frac{1}{(pk + 1)^{2s-1}} - \frac{1}{(pk - 1)^{2s-1}} \right] \\
- \left[ \frac{1}{(pk + 1)^{2s}} + \frac{1}{(pk - 1)^{2s}} \right] \\
+ \frac{4}{p} \sum_{l=2}^{r-1} (2l - 1 - r) \frac{1}{(pk - 2l + 1)^{2s-1}} \right\} 
$$

(6.18)

Since the first term involves a sum over all odd conjugacy classes we have

$$
\sum_{k=1}^{\infty} \sum_{l=0}^{r-1} \frac{1}{(pk - 2l + 1)^{2s-2}} = \sum_{m=1}^{\infty} \frac{1}{(2m + 1)^{2s-2}} = (1 - \frac{1}{2^{2s-2}})\zeta(2s - 2) - 1
$$

(6.19)

Thus

$$
\tau_-(p, s) = \frac{4}{p} \left( (1 - \frac{1}{2^{2s-2}})\zeta(2s - 2) - 1 \right) \\
+ (p - 4) \frac{1}{p^{2s}} [\zeta(2s - 1, 1 + \frac{1}{p}) - \zeta(2s - 1, 1 - \frac{1}{p})] \\
- \frac{1}{p^{2s}} [\zeta(2s, 1 + \frac{1}{p}) + \zeta(2s, 1 - \frac{1}{p})] \\
- \frac{2}{p^{2s}} \sum_{l=2}^{r-1} (p - (2l - 1)) \zeta(2s - 1, 1 - \frac{2l - 1}{p})
$$

(6.20)

Noting that \( p - (2l - 1) \) ranges from \( -(p - 6) \) to \( (p - 6) \) in steps of 2 when \( l \) ranges from 2 to \( r - 1 \) the final sum in 6.20 is therefore of the form

$$
\sum_{l=2}^{r-1} (p - (2l - 1)) \zeta(2s - 1, 1 - \frac{2l - 1}{p}) = \sum_{l=2}^{[r-1]} (p - 2(2l - 1)) \left[ \zeta(2s - 1, 1 - \frac{2l - 1}{p}) \\
- \zeta(2s - 1, \frac{2l - 1}{p}) \right]
$$

(6.21)

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Substituting back we obtain
\[
\tau_-(p, s) = \frac{4}{p} \left(1 - \frac{1}{2^{2s-2}}\right) \zeta(2s - 2) - 1 \\
+ \left(p - 4\right) \frac{1}{p^{2s}} \left[\zeta(2s - 1, 1 + \frac{1}{p}) - \zeta(2s - 1, 1 - \frac{1}{p})\right] \\
- \frac{1}{p^{2s}} \left[\zeta(2s, 1 + \frac{1}{p}) + \zeta(2s, 1 - \frac{1}{p})\right] \\
+ 2 \frac{\lfloor \frac{r-1}{2} \rfloor}{p^{2s}} \sum_{l=2}^{r-1} \left(p - (2l - 1)\right) \left[\zeta(2s - 1, \frac{2l-1}{p}) - \zeta(2s - 1, 1 - \frac{2l-1}{p})\right]
\] (6.22)

Our first observation is that using \(\zeta(-2) = 0\), \(\zeta(-1, 1+a) - \zeta(-1, 1-a) = -a\), and \(\zeta(0, 1+a) + \zeta(0, 1-a) = -1\).
\[
\tau_-(p, 0) = 0 
\] (6.23)

Now differentiation of 6.22 with respect to \(s\) and evaluating the expression at \(s = 0\), and using some of our relations from Appendix C gives
\[
\tau'-(p) = -\frac{24}{p} \zeta'(-2) + 2(p-4)\left[\zeta'(-1, \frac{1}{p}) - \zeta'(-1, 1 - \frac{1}{p})\right] - 2\left[\zeta'(0, \frac{1}{p}) + \zeta'(0, 1 - \frac{1}{p})\right] \\
+ 4 \sum_{l=1}^{\lfloor \frac{r-3}{2} \rfloor} \left(p - (2l - 1)\right) \left[\zeta'(-1, \frac{2l-1}{p}) - \zeta'(-1, 1 - \frac{2l-1}{p})\right]
\] (6.24)

Further use or our the relations derived in Appendix C allows us to express the result in terms of integrals over trigonometric functions as in the preceding sections to yield
\[
\tau'-(p) = 4 \ln(2 \sin \frac{\pi}{p}) + \frac{6}{p \pi^2} \zeta(3) + 2(p-4)\left[\frac{1}{\pi} \int_0^{\pi} dzz \cot(z) - \frac{1}{p} \ln(2 \sin \frac{\pi}{p})\right] \\
+ 4 \sum_{l=1}^{\lfloor \frac{r-3}{2} \rfloor} \left(p - (2l + 1)\right) \left[\frac{1}{\pi} \int_0^{\pi(2l+1) / p} dzz \cot(z) - \frac{(2l + 1)}{p} \ln(2 \sin \frac{\pi(2l+1)}{p})\right]
\] (6.25)

This again decomposes into the two cases \(p = 0, 2 \mod 4\) and these yield the expressions
\[
\tau'-(p) = 4 \ln(2 \sin \frac{\pi}{p}) + \frac{6}{p \pi^2} \zeta(3) + 2(p-4)\left[\frac{1}{\pi} \int_0^{\pi} dzz \cot(z) - \frac{1}{p} \ln(2 \sin \frac{\pi}{p})\right] \\
+ 4 \sum_{l=1}^{t-1} \left(p - (2l + 1)\right) \left[\frac{1}{\pi} \int_0^{\pi(2l+1) / p} dzz \cot(z) - \frac{(2l + 1)}{p} \ln(2 \sin \frac{\pi(2l+1)}{p})\right]
\] (6.26)

\[p = 2 \mod 4\]
and
\[
\tau'_- (p) = 4 \ln(2 \sin \frac{\pi}{p}) + \frac{6}{p \pi^2} \zeta(3) + 2(p - 4) \left[ \frac{1}{p} \int_0^{\frac{\pi}{p}} dz \cot(z) - \frac{1}{p} \ln(2 \sin \frac{\pi}{p}) \right] \\
+ 4 \sum_{l=1}^{t-1} (p - (2l + 1)) \left[ \frac{1}{p} \int_0^{\frac{\pi(2l+1)}{p}} dz \cot(z) - \frac{(2l + 1)}{p} \ln(2 \sin \frac{\pi(2l+1)}{p}) \right] \\
p = 0 \mod 4
\] (6.27)

\[\Box \Box \Box 6.3 \text{ The Torsion } T(p) \text{ and } \tau(p, s)\]

We can now combine our results for \(\tau_+ (p, 0)\) and \(\tau_- (p, 0)\) to obtain expressions for the torsion in the present case where \(p\) is even. Note again that since \(\tau_\pm (p, 0) = 0\) we have

\[\tau(p, 0) = 0\] (6.28)

for the case of \(p\) even, and again this ensures that the torsion is metric independent.

Combining the expressions 6.14 and 6.25 for \(\tau'_+ (p)\) and \(\tau'_- (p)\) we obtain two expressions for \(\ln T(p)\): one for each conjugacy class; these are

\[
\ln T(p) = -4 \ln(2 \sin \frac{\pi}{p}) + \left[ \frac{2(p - 4)}{p} \ln(2 \sin \frac{\pi}{p}) \right] \\
+ \frac{4}{p} \sum_{l=1}^{(p-6)/4} [(2l + 1)^2 - 2] \ln \left[ 2 \sin \left( \frac{(2l + 1)\pi}{p} \right) \right] \quad p = 2 \mod 4 
\] (6.29)

and

\[
\ln T(p) = -4 \ln(2 \sin \frac{\pi}{p}) - \frac{4}{p} \ln 2 + 2(p - 4) \left[ \frac{1}{p} \ln(2 \sin \frac{\pi}{p}) \right] \\
+ \frac{4}{p} \sum_{l=1}^{(p-4)/4} [(2l + 1)(2l + 1) - 2] \ln \left[ 2 \sin \left( \frac{(2l + 1)\pi}{p} \right) \right] \\
- \frac{4}{p} \sum_{l=1}^{(p-4)/4} (2l - 1) \int_0^{\frac{(2l+1)\pi}{p}} dz \cot(z) \quad p = 0 \mod 4 
\] (6.30)

We note that for the above two formulae the torsion is already given by the first term on their RHS's and so we obtain somewhat non-trivial integration formulae for the integrals therein. On using the relations derived at the end of appendix B we can also reduce the above expressions to Ray's expression for the torsion.

We have therefore now obtained a complete list of the determinants of Laplacians for 0 and 1 forms on the lens spaces \(L(p)\) for all integer \(p \geq 2\), as well as the torsion \(T(p)\) for all \(p \geq 2\).
7. Conclusion

In the preceding sections we analysed by direct computation the determinants of Laplacians on 0 and 1-forms on the lens spaces $L(p)$, defined via the derivatives of their associated zeta functions.

In this concluding section we collect our results and present in graphical form the behaviour of the sequence of determinants we have analysed and their related torsion.

For 0-forms we found that

$$\ln \det d^* d_0 = \frac{1}{2p\pi^2} \zeta(3) + \ln \left[ 2 \sin \left( \frac{\pi}{p} \right) \right] + \frac{(p-2)}{\pi} \int_0^{\pi/p} \ln [2 \sin(z)]$$

$$- \frac{2}{\pi} \sum_{l=1}^{(p-3)/2} (2l) \int_0^{2l\pi/p} \ln [2 \sin(z)]$$

in the case of $p$ odd and

$$\ln \det d^* d_0 = -\frac{3}{p\pi^2} \zeta(3) + \frac{(p-4)}{\pi} \int_0^{\frac{\pi}{p}} dz \ln [2 \sin(z)]$$

$$+ \frac{t-1}{\pi} \sum_{l=1}^{p-1} (p + (2l + 1)) \frac{1}{\pi} \int_0^{\frac{\pi}{p}} dz \ln [2 \sin(z)]$$

$$\quad p = 2 \mod 4$$

and

$$\ln \det d^* d_0 = -\frac{6}{p\pi^2} \zeta(3)$$

$$+ \frac{(p-4)}{\pi} \int_0^{\frac{\pi}{p}} dz \ln [2 \sin(z)]$$

$$+ \frac{t-1}{\pi} \sum_{l=1}^{p-1} (p + (2l + 1)) \frac{1}{\pi} \int_0^{\frac{\pi}{p}} dz \ln [2 \sin(z)]$$

$$\quad p = 0 \mod 4$$

These are plotted in figure 1.

While for 1-forms we found

$$\ln \det d^* d_1 = \frac{1}{p\pi^2} \zeta(3) - 2 \ln \left[ 2 \sin \left( \frac{\pi}{p} \right) \right] + \frac{2(p-2)}{\pi} \int_0^{\pi/p} \ln [2 \sin(z)]$$

$$- \frac{4}{\pi} \sum_{l=1}^{(p-3)/2} (2l) \int_0^{2l\pi/p} \ln [2 \sin(z)]$$

$$\quad p = 3, 5, \ldots$$

while when $p$ is even we found that

$$\ln \det d^* d_1 = -4 \ln (2 \sin \frac{\pi}{p}) - \frac{6}{p\pi^2} \zeta(3) + 2(p-4) \frac{1}{\pi} \int_0^{\frac{\pi}{p}} dz \ln [2 \sin(z)]$$

$$+ \frac{4}{\pi} \sum_{l=1}^{t-1} (p + (2l + 1)) \frac{1}{\pi} \int_0^{\frac{\pi}{p}} dz \ln [2 \sin(z)]$$

$$\quad p = 2 \mod 4$$

(7.1.4)
and

\[ \ln \text{Det } d^*d_1 = -4 \ln(2 \sin \frac{\pi}{p}) - \frac{6}{p\pi^2} \zeta(3) \]

\[ + 2(p - 4) \frac{1}{\pi} \int_0^{\pi/p} dz \ln[2 \sin(z)] \]

\[ + 4 \sum_{l=1}^{p-1} (p - (2l + 1)) \frac{1}{\pi} \int_0^{\pi/2} dz \ln[2 \sin(z)] \]

\[ p = 0 \mod 4 \quad (7.6) \]

and these results are displayed in figure 2. The difference

\[ \ln T(p) = \ln \text{Det } d^*d_1 - 2 \ln \text{Det } d^*d_0 \]

\[ = -4 \ln \left[ 2 \sin \left( \frac{\pi}{p} \right) \right] \quad (7.7) \]

which gives the torsion itself, is plotted in figure 3.

A perusal of figure 3 shows that \( \ln T(p) \) is negative for small \( p \) and large and positive for large \( p \). This raises the question as to whether \( \ln T(p) \) crosses the \( p \) axis at an integer value or not. If so this corresponds to a trivial value for the torsion. In fact this clearly does happen for the value \( p = 6 \) i.e. we have

\[ \ln T(6) = 0 \quad (7.8) \]

We show the more detailed behaviour of the torsion for small \( p \) in figure 4.

Further interesting results are that that if we work with \( L(p, q) \) rather than \( L(p, 1) \) then the torsion, now denoted by \( T(p, q) \), is trivial for only two other three dimensional lens spaces, namely \( L(10, 3) \) and \( L(12, 5) \): we find that

\[ \ln T(p, q) = -2 \ln \left[ 4 \sin \left( \frac{\pi}{p} \right) \sin \left( \frac{\pi q^*}{p} \right) \right] \]

where \( q^* \) satisfies \( qq^* = 1 \mod p \)

\[ \quad (7.9) \]

It is then possible to prove that, for \( p > 12 \), \( \ln T(p, q) \) is strictly positive; while for \( p \leq 12 \) a check of the finite number of cases yields triviality in just the three cases given above. We conjecture that this may be understandable using some form of supersymmetry. These formulae have yet to be elucidated further.

The precise meaning of our formulae such as

\[ \zeta(3) = \frac{2\pi^2}{7} \ln(2) - \frac{8}{7} \int_0^{\pi/2} dz \cot(z) \]

\[ \zeta(3) = \frac{2\pi^2}{13} \ln 3 - \frac{9}{13} \int_0^{\pi/3} dz \cot(z) - \frac{9}{13} \int_{\pi/3}^{2\pi/3} dz \cot(z) \quad (7.10) \]
and the more general

\[
\zeta(3) = \frac{2\pi^2}{(p^3 - 1)} \left[ 2(p - 1) \ln \left( 2 \sin\left( \frac{\pi}{p} \right) \right) - 2 \sum_{l=1}^{(p-3)/2} 4l^2 \ln \left( 2 \sin\left( \frac{2l\pi}{p} \right) \right) - \frac{p^2}{\pi^2} \int_0^{\pi/p} dz \left( z + \frac{(p-2)\pi}{p} \right) \cot(z) - \frac{p^2}{\pi^2} \int_0^{(p-1)\pi/p} dz \left( z - \frac{(p-2)\pi}{p} \right) \cot(z) - 2 \sum_{l=1}^{(p-3)/2} \frac{p^2}{\pi^2} \int_0^{2l\pi/p} dz \left( z - \frac{4l\pi}{p} \right) \cot(z) \right] , \quad p = 3, 5, \ldots
\]

(7.11)

is, as yet, unclear. There may be some number theoretic matters underlying them as seems to be the case in other work on lens spaces, cf. [15]. A thought provoking fact is that \( \zeta(3) \) occurs in a recent paper of Witten [16] where, after multiplication by a known constant, it gives the volume of the symplectic space of flat connections over a non-orientable Riemann surface. The corresponding calculation for orientable surfaces (where the volume element is a rational cohomology class) allows a cohomological rederivation of the irrationality of \( \zeta(2), \zeta(4), \ldots \). This paper also involves the torsion but in two dimensions rather than three. The proof that \( \zeta(3) \) is irrational was only obtained in 1978 cf. [17] and the rationality of \( \zeta(5), \zeta(7), \ldots \) is at present open. However there are now other proofs [18], one of which uses the characters of conformal quantum field theory.

Finally, our technique, applied in five dimensions instead of three, would yield formulae for \( \zeta(5) \) but their nature has not yet been explored.
FIGURES: The Determinants

**Figure 1:** $2 \ln d^*d_0$ versus $p$.

**Figure 2:** $\ln d^*d_1$ versus $p$. 
FIGURES: The Torsion

Figure 3: $\ln T(p)$ versus $p$.

Figure 4: $\ln T(p)$ versus $p$ for small $p$. 
Appendix A. The function $A_p(x, s)$

In this appendix we analyze the function

$$A_p(x, s) = \sum_{k=1}^{\infty} \frac{(pk + x)^2}{(pk + x)^2 - 1}$$  \hfill (A.1)

We are interested in particular in the value of this function and its derivative with respect to $s$ at $s = 0$. For this purpose we denote

$$A_p(k, x, s) = \frac{(pk + x)^2}{(pk + x)^2 - 1}$$  \hfill (A.2)

which has the expansion

$$A_p(k, x, s) = \frac{1}{(pk)^2s-2} \left[ 1 + 2(1-s) \frac{x}{pk} + (s + (s - 1)(2s - 1)x^2) \frac{1}{(pk)^2} \right. \right.$$

$$\left. - 2sx \left( s + \frac{(s - 1)(2s - 1)}{3} x^2 \right) \frac{1}{(pk)^3} + \ldots \right]$$  \hfill (A.3)

Summing over $k$ leads to

$$A_p(x, s) = \frac{1}{p^{2s-2}} \zeta(2s - 2) + \frac{2x}{p^{2s-1}} (1-s) \zeta(2s - 1) + \frac{1}{p^{2s}} (s + (1 - s)(1 - 2s)x^2) \zeta(2s)$$

$$- \frac{1}{p^{2s+1}} \left( s + \frac{(1 - s)(1 - 2s)}{3} x^2 \right) x2s \zeta(2s + 1) + \hat{A}_p(x, s)$$  \hfill (A.4)

where

$$\hat{A}_p(x, s) = \sum_{k=1}^{\infty} \frac{(pk + x)^2}{(pk + x)^2 - 1} s - \frac{1}{(pk)^2s-2} \left[ 1 + 2(1-s) \frac{x}{pk} \right.$$

$$\left. + (s + (s - 1)(2s - 1)x^2) \frac{1}{(pk)^2} - 2sx \left( s + \frac{(s - 1)(2s - 1)}{3} x^2 \right) \frac{1}{(pk)^3} \right]$$  \hfill (A.5)

Now the function $\hat{A}_p(x, s)$ is such that the processes of summation and differentiation with respect to $s$ at $s = 0$ commute. Also it is such that $\hat{A}_p(x, 0) = 0$, which yields

$$A_p(x, 0) = -\frac{x(x + p)(2x + p)}{6p}$$  \hfill (A.6)
Next evaluating the derivative at $s = 0$ we have
\[
A'_p(x, 0) = -A_p(x, 0) \ln p^2 + 2p^2 \zeta'(-2) + \zeta(0) + 2px [2\zeta'(-1) - \zeta(-1)]
+ [2\zeta'(0) - 3\zeta(0)] x^2 - \frac{x}{p} + \frac{1}{3p} [3 - 2\gamma] x^3 + A'_p(x, 0) \tag{A.7}
\]
which on using $\zeta(0) = -\frac{1}{2}$, $\zeta(-1) = -\frac{1}{12}$, $2s\zeta(2s + 1) = 1 + 2\gamma s + \ldots$ and $\zeta'(0) = -\frac{1}{2} \ln 2\pi$ we have
\[
A'_p(x, 0) = -A_p(x, 0) \ln p^2 + 2p^2 \zeta'(-2) - \frac{1}{2} + \left[4p\zeta'(-1) + \frac{p}{6} - \frac{1}{p}\right] x
+ \left[\frac{3}{2} - \ln [2\pi]\right] x^2 + \frac{1}{3p} [3 - 2\gamma] x^3 + A'_p(x, 0) \tag{A.8}
\]
Since the processes of differentiation with respect to $s$ and performing the sum over $k$ commute for $A_p(x, s)$ we analyze this function by first taking the derivative and then performing the sum.
\[
\hat{A}'_p(k, x, 0) = -(pk + x)^2 \left[\ln \left[1 + \frac{(x + 1)}{pk}\right] + \ln \left[1 + \frac{(x - 1)}{pk}\right]\right]
+ (pk)^2 \left[\frac{2x}{pk} - \frac{(1 - 3x^2)}{(pk)^2} + \frac{2x^3}{3(pk)^3}\right] \tag{A.9}
\]
We note that
\[
\ln \left(1 + \frac{(x + 1)}{pk}\right) + \ln \left(1 + \frac{(x - 1)}{pk}\right) = \sum_{m=1}^{\infty} \frac{(-1)^m ((x + 1)^m + (x - 1)^m)}{m (pk)^m} \tag{A.10}
\]
Hence combining this with $\hat{A}'_p(k, x, 0)$ gives
\[
\hat{A}'_p(k, x, 0) = \sum_{m=4}^{\infty} \frac{(-1)^m [(1 + x)^m + (x - 1)^m]}{m (pk)^{m-2}}
+ 2x \sum_{m=3}^{\infty} \frac{(-1)^m [(1 + x)^m + (x - 1)^m]}{m (pk)^{m-1}}
+ x^2 \sum_{m=2}^{\infty} \frac{(-1)^m [(1 + x)^m + (x - 1)^m]}{m (pk)^m} \tag{A.11}
\]
which gives
\[
\hat{A}'_p(x, 0) = \sum_{m=2}^{\infty} \frac{(-1)^m [(1 + x)^{(m+2)} + (x - 1)^{(m+2)}]}{(m + 2) p^m} \zeta(m)
- 2x \sum_{m=2}^{\infty} \frac{(-1)^m [(1 + x)^{(m+1)} + (x - 1)^{(m+1)}]}{(m + 1) p^m} \zeta(m)
+ x^2 \sum_{m=2}^{\infty} \frac{(-1)^m [(1 + x)^m + (x - 1)^m]}{m p^m} \zeta(m) \tag{A.12}
\]
We now observe the identity
\[
\sum_{m=2}^{\infty} (-1)^m a^{m-1} \zeta(m, \alpha) = \psi(\alpha + a) - \psi(\alpha)
\] (A.13)
which yields
\[
\sum_{m=2}^{\infty} \frac{(-1)^m a^{m+\nu}}{m + \nu} \zeta(a, \alpha) = \int_0^a dy \ y^n \left[ \psi(y + \alpha) - \psi(\alpha) \right]
\] (A.14)
for $\nu \geq 0$. Now using this identity we have
\[
\hat{A}_p'(x, 0) = p^2 \int_0^{(x+1)/p} dy \ (y - \frac{x}{p})^2 \left[ \psi(1 + y) - \psi(1) \right]
+ p^2 \int_0^{(x-1)/p} dy \ (y - \frac{x}{p})^2 \left[ \psi(1 + y) - \psi(1) \right]
\] (A.15)
Finally using $\psi(1) = -\gamma$ we arrive at at our expression for the desired analytic continuation of the derivative at zero,
\[
A_p'(x, 0) = -A_p(x, 0) \ln p^2 - \frac{p^2}{2\pi^2} \zeta(3) - \frac{1}{2} + \frac{px}{6} \left[ 24\zeta'(-1) + 1 - \frac{6}{p^2} \right] + x^2 \left( \frac{3}{2} - \ln[2\pi] \right)
+ \frac{x^3}{p} + p^2 \int_0^{(x+1)/p} \left( y - \frac{x}{p} \right)^2 \psi(1 + y) + p^2 \int_0^{(x-1)/p} \left( y - \frac{x}{p} \right)^2 \psi(1 + y)
\] (A.16)
We are interested in particular in the symmetric sum
\[
A_p^+(x) = A_p'(x, 0) + A_p'(-x, 0)
\] (A.17)
which from A.16 on using
\[
\Gamma(1 + x)\Gamma(1 - x) = \frac{\pi x}{\sin(\pi x)}
\] (A.18)
we find to be
\[
A_p^+(x) = \frac{-p^2}{\pi^2} \zeta(3) - \frac{p^2}{\pi^2} \int_0^{(x+1)/p} \frac{d z}{p} \cot(z - \frac{2\pi x}{p}) - \frac{p^2}{\pi^2} \int_0^{(x-1)/p} \frac{d z}{p} \cot(z - \frac{2\pi x}{p})
- x^2 \ln \left[ \frac{2 \sin \left( \frac{(x+1)\pi}{p} \right)}{x + 1} \right]
- x^2 \ln \left[ \frac{2 \sin \left( \frac{(x-1)\pi}{p} \right)}{x - 1} \right]
\] (A.19)
Alternatively one can see this directly from the expansion of $\hat{A}_p^+$ in the series representation and the identities
\[
\sum_{l=1}^{\infty} \frac{1}{l+1} \frac{1}{a^{2l}} \zeta(2l) = \frac{1}{2} - \frac{a^2}{\pi^2} \int_0^{\pi} \cot(z) \frac{dz}{2}
\] (A.20)
\[ \sum_{l=1}^{\infty} \frac{1}{2l+1} \frac{1}{a^{2l}} \zeta(2l) = \frac{1}{2} - \frac{a}{2\pi} \int_{0}^{\pi} dz \cot(z) \] (A.21)

\[ \sum_{l=1}^{\infty} \frac{1}{l} \frac{1}{a^{2l}} \zeta(2l) = -\ln \left[ \frac{\sin \left( \frac{\pi}{a} \right)}{\frac{\pi}{a}} \right] \] (A.22)

A useful expression is obtained by summing over all conjugacy classes to obtain an expression independent of \( p \). Hence note that

\[ \sum_{j=0}^{p-1} \sum_{k=1}^{\infty} A_p(k, -j, s) = \sum_{\{r_p\}} \sum_{k=1}^{\infty} \frac{(pk - j)^2}{\{(pk - j)^2 - 1\}^s} \]

\[ = \sum_{n=1}^{\infty} \frac{(n+1)^2}{\{(n+1)^2 - 1\}^s} \] (A.23)

is independent of \( p \) since we have merely decomposed it as a sum over conjugacy classes mod \( p \). We label this sum \( A(s) \) and its derivative at \( s = 0 \) simply \( A \).

Now denoting

\[ A_p = A_p'(0, 0) + \sum_{l=0}^{(p-3)} A_p^+(2l + 1) \]

As a consequence of the above we have that

\[ A = A_p - \sum_{l=1}^{(p-3)} (2l + 1)^2 \ln[(2l + 1)^2 - 1] \] (A.24)

is independent of \( p \). This invariant can be written as

\[ A = -\frac{p^3}{2\pi^2} \zeta(3) - \sum_{l=1}^{(p-3)} (2l + 1)^2 \ln[(2l + 1)^2 - 1] - \frac{p^2}{\pi^2} \int_{0}^{\pi} dzz^2 \cot(z) \]

\[ -\frac{p^2}{\pi^2} \sum_{l=0}^{(p-3)} \left[ \int_{0}^{(2l+2)p} dzz(z - \frac{2\pi(2l+1)}{p}) \cot(z) + \int_{0}^{2\pi p} dzz(z - \frac{2\pi(2l+1)}{p}) \cot(z) \right] \]

\[ -\sum_{l=0}^{(p-3)} (2l + 1)^2 \left[ \ln \left( \frac{2 \sin \left( \frac{(2l+2)p}{p} \right)}{2l + 2} \right) + \ln \left( \frac{2 \sin \left( \frac{2l\pi}{p} \right)}{2l} \right) \right] \] (A.25)
Which on simplifying gives

\[
A = -\frac{p^3}{2\pi^2} \zeta(3) - (p - 2)^2 \ln[2\sin(\frac{\pi}{p})] - \ln(\frac{\pi}{p}) - \frac{p^2}{\pi^2} \int_0^{\frac{\pi}{p}} dz z^2 \cot(z)
\]

\[-\frac{p^2}{\pi^2} \int_0^{\frac{(p-1)\pi}{p}} dz z \left( -\frac{2(p-2)}{p} \right) \cot(z)
\]

\[-2 \sum_{l=1}^{(p-3)/2} \left[ \frac{p^2}{\pi^2} \int_0^{\frac{2l\pi}{p}} dz (z - \frac{4l\pi}{p}) \cot(z) + (4l^2 + 1) \ln[2\sin(\frac{2l\pi}{p})] \right]
\]

This constant has the value \( A = -1.20563 \)

One can use this together with the expression for \( B_p^{-}(x) \) obtained in the next appendix to give an alternative form of the expression for \( \tau'(p) \). Explicitly

\[
\tau'(p) = \frac{2}{p} \left[ A_p'(0,0) + \frac{2}{p} \sum_{l=0}^{(p-3)/2} \left[ A_p^+(2l+1) - (2l+1)B_p^-(2l+1) \right] \right]
\]

\[= \frac{2}{p} \left[ A - \sum_{l=1}^{(p-3)/p} (2l+1) \left[ B_p(2l+1) - (2l+1) \ln \left[ (2l+1)^2 - 1 \right] \right] \right]
\]

which works out to be

\[
\tau'(p) = \frac{2}{p} \left[ A + \ln(\frac{\pi}{p}) + (p - 2)^2 \ln[2\sin(\frac{\pi}{p})] - \frac{(p-2)p}{\pi} \int_0^{\frac{(p-1)\pi}{p}} dz z^2 \cot(z)
\]

\[-2 \sum_{l=1}^{(p-3)/2} 2l \int_0^{\frac{2l\pi}{p}} dz z \cot(z) + \sum_{l=1}^{(p-3)/2} (4l^2 + 1) \ln \left[ 2\sin(\frac{2l\pi}{p}) \right] \right]
\]

\[
(A.27)
\]

Appendix B. The function \( B_p(x, s) \)

In this appendix we analyze the function

\[
B_p(x, s) = \sum_{k=1}^{\infty} \frac{(pk + x)}{\left\{ (pk + x)^2 - 1 \right\}^s}
\]

We are again interested in the value of this function and its derivative with respect to \( s \) at \( s = 0 \). For this we denote

\[
B_p(k, x, s) = \frac{(pk + x)}{\left\{ (pk + x)^2 - 1 \right\}^s}
\]

(B.2)
The continuation now requires us to extract the appropriate large \( k \) behaviour from \( B_p(x, s) \). Proceeding to do this we find
\[
B_p(k, x, s) = (pk)^{-2s} - (2s - 1)x(pk)^{-2s} + s(1 + (2s - 1)x^2)(pk)^{-2s-1} + \hat{B}_p(k, x, s)
\] (B.3)

Hence summing over \( k \) yields
\[
B_p(x, s) = \frac{1}{p^{2s}} \left[ p\zeta(2s - 1) - x(2s - 1)\zeta(2s) + \frac{s(1 + (2s - 1)x^2)}{p} \zeta(2s + 1) \right] + \hat{B}_p(s, x)
\] (B.4)

where
\[
\hat{B}_p(x, s) = \sum_{k=1}^{\infty} \left[ \frac{(pk + x)}{(pk + x)^2 - 1} \right]^s - \frac{1}{(pk)^{2s-1}} \left\{ 1 - x\frac{(2s - 1)}{pk} \right\} + \frac{s(1 + (2s - 1)x^2)}{(pk)^2} - \frac{x}{3} \frac{s(2s + 1)(3 + (2s - 1)x^2)}{(pk)^{3}} \right]
\] (B.5)

\( \hat{B}_p(x, s) \) has the property that the process of taking the limit differentiating with respect to \( s \) and taking \( s \to 0 \) commutes with the sum over \( k \). The second property is that \( \hat{B}_p(x, 0) = 0 \). Hence this expression has the analytic continuation to \( s = 0 \)
\[
B_p(x, 0) = p\zeta(-1, 1 + \frac{x}{p}) + \frac{1}{2p}
\] (B.6)

Differentiating with respect to \( s \) and evaluating at \( s = 0 \) yields
\[
B'_p(x, 0) = -B_p(x, 0) \ln p^2 + 2p\zeta'(-1) + 2x [\zeta'(0) - \zeta(0)] + \frac{x^2}{p} + \frac{\gamma(1 - x^2)}{p} + \hat{B}'_p(x, 0)
\] (B.7)

We choose to do the evaluate \( \hat{B}'_p(x, 0) \) by doing the derivative first and the sum last. Therefore
\[
\hat{B}'_p(k, x, 0) = \sum_{m=2}^{\infty} \frac{(-1)^{(m+1)} [(1 + x)^{(m+1)} + (x - 1)^{(m+1)}]}{(m+1) pk^m} + x \sum_{m=2}^{\infty} \frac{(-1)^m [(1 + x)^{m} + (x - 1)^{m}]}{m pk^m}
\] (B.8)

and
\[
\hat{B}'_p(x, 0) = \sum_{m=2}^{\infty} \frac{(-1)^{(m+1)} [(1 + x)^{(m+1)} + (x - 1)^{(m+1)}]}{(m+1) p^m} \zeta(m) + x \sum_{m=2}^{\infty} \frac{(-1)^m [(1 + x)^{m} + (x - 1)^{m}]}{m p^m} \zeta(m)
\] (B.9)
Again using the summation A.14 as in appendix A we have

\[ \hat{B}'_p(x, 0) = -p \int_0^{(x+1)/p} dy \left( y - \frac{x}{p} \right) (\psi(1 + y) - \psi(1)) \]

\[ - p \int_0^{(x-1)/p} dy \left( y - \frac{x}{p} \right) (\psi(1 + y) - \psi(1)) \]  

(B.10)

Thus we arrive at

\[ B'_p(x, 0) = -B_p(x, 0) \ln p^2 + 2p\zeta'(-1) + [1 - \ln[2\pi]] x + \frac{x^2}{p} \]

\[ - p \int_0^{(x+1)/p} dy \left( y - \frac{x}{p} \right) \psi(1 + y) - p \int_0^{(x-1)/p} dy \left( y - \frac{x}{p} \right) \psi(1 + y) \]  

(B.11)

For the function \( B_p \) it is the anti-symmetric part that contributes to the quantities of interest thus defining

\[ B_p^-(x) = B'_p(x, 0) - B'_p(-x, 0) \]

\[ = x \ln p^2 + [1 - \ln[2\pi]] 2x + \hat{B}_p^-(x) \]  

(B.12)

we have

\[ B_p^-(x) = \frac{p}{\pi} \int_0^{(x+1)\pi/p} dz \cot(z) + \frac{p}{\pi} \int_0^{(x-1)\pi/p} dz \cot(z) \]

\[ - x \ln \left[ \frac{2 \sin \left( \frac{(x+1)\pi}{p} \right)}{x + 1} \right] - x \ln \left[ \frac{2 \sin \left( \frac{(x-1)\pi}{p} \right)}{x - 1} \right] \]  

(B.13)

B.1 The function \( H_p(x, s) \)

One can combine the results above with those in Appendix A to obtain an expression for \( H'_p(x, 0) \), where we have defined

\[ H_p(x, s) = A_p(x, s) - xB_p(x, s) \]  

(B.14)

Explicitly we have

\[ H'_p(x, 0) = -H_p(x, 0) \ln p^2 - \frac{p^2}{2\pi^2} \zeta(3) - \frac{1}{2} + \frac{px}{6} \left( 12\zeta'(-1) + 1 - \frac{6}{p^2} \right) \]

\[ + \frac{x^2}{2} + p^2 \int_0^{(x+1)/p} y(y - \frac{x}{p})\psi(1 + y) + p^2 \int_0^{(x-1)/p} dy y(y - \frac{x}{p}) \psi(1 + y) \]  

(B.15)

where

\[ H_p(x, 0) = -\frac{px}{12} - \frac{x}{2p} + \frac{x^3}{6p} \]  

(B.16)
By analogy with $A_p^+(x)$ we can define

$$H_p^+(x) = A_p^+(x) - xB_p^-(x)$$ (B.17)

which is given by

$$H_p^+(x) = -\frac{p^2}{\pi^2}\zeta(3) - \frac{p^2}{\pi^2} \int_0^{(x+1)\pi/p} dz \frac{z}{\pi x} \cot(z) - \frac{p^2}{\pi^2} \int_0^{(x-1)\pi/p} dz \frac{z}{\pi x} \cot(z)$$ (B.18)

Appendix C. Expressions involving the Hurwitz zeta function.

The task of this appendix is to obtain an expression for $\zeta'(0, 1 + a)$, $\zeta'(-1, 1 + a)$ and $\zeta'(-2, 1 + a)$. We begin with $\zeta'(0, 1 + a)$. Note first of all that

$$\zeta(s, 1 + a) = \sum_{n=1}^{\infty} \frac{1}{(n + a)^s}$$ (C.1)

has the series expansion

$$\zeta(s, 1 + a) = \zeta(s) + s\zeta(s + 1)a + \hat{\zeta}(s, 1 + a)$$ (C.2)

where

$$\hat{\zeta}(s, 1 + a) = \sum_{n=1}^{\infty} \left[ \frac{1}{(n + a)^s} - \frac{1}{n^s} + \frac{sa}{n^{s+1}} \right]$$

$$= \sum_{k=2}^{\infty} \frac{(-1)^k \Gamma(s + k)}{\Gamma(s) k!} \zeta(s + k)a^k$$ (C.3)

Which gives the analytic continuation for $\zeta(s, 1 + a)$ to $s = 0$. We therefore deduce

$$\zeta(0, 1 + a) = \zeta(0) - a$$ (C.4)

and

$$\zeta'(0, 1 + a) = -\frac{1}{2} \ln[2\pi] - \gamma a + \hat{\zeta}'(0, 1 + a)$$

where we used $\zeta'(0) = -\frac{1}{2} \ln[2\pi]$ and $s\zeta(s + 1) = 1 + \gamma s + \ldots$. Differentiation with respect to $s$ at $s = 0$ and summation over $n$ commute for $\hat{\zeta}(s, 1 + a)$ we therefore have

$$\hat{\zeta}'(0, 1 + a) = -\sum_{n=1}^{\infty} \left[ \ln\left[1 + \frac{a}{n}\right] - \frac{a}{n} \right]$$

$$= \sum_{k=2}^{\infty} \frac{(-1)^k a^k}{k} \zeta(k)$$ (C.5)
Thus using A.14 we have

\[ \hat{\zeta}'(0, 1 + a) = \int_0^a dy \left[ \psi(1 + y) - \psi(1) \right] \] (C.6)

We therefore obtain that

\[ \zeta'(0, 1 + a) = -\frac{1}{2} \ln[2\pi] + \ln \Gamma(1 + a) \] (C.7)

The symmetric part of this function on using A.18 can be expressed as is expressed as

\[ \zeta'(0, 1 + a) + \zeta'(0, 1 - a) = -\ln \left[ \frac{2 \sin(\pi a)}{a} \right] \] (C.8)

Finally by noting that

\[ \zeta(s) = p^{-s} \sum_{j=0}^{p-1} \zeta(s, 1 - \frac{j}{p}) \]

is a decomposition over conjugacy classes of \( \zeta(s) \) Differentiating and evaluating at \( s = 0 \) we obtain

\[ \zeta'(0) = -\sum_{j=0}^{p-1} \zeta(0, 1 - \frac{j}{p}) \ln p + \sum_{j=0}^{p-1} \zeta'(0, 1 - \frac{j}{p}) \]

Now the first term is just a decomposition of \( \zeta(0) \) over conjugacy classes therefore C.9 gives the identity

\[ \sum_{j=0}^{p-1} \zeta'(0, 1 - \frac{j}{p}) = \zeta'(0) + \zeta(0) \ln p \]

which for \( p \) odd gives

\[ \sum_{l=1}^{(p-1)/2} \ln \left[ 2 \sin \left( \frac{2l\pi}{p} \right) \right] = \frac{1}{2} \ln p \]

or

\[ \sum_{l=0}^{(p-1)/2} \ln \left[ 2 \sin \left( \frac{2l + 1}{p} \right) \pi \right] = \frac{1}{2} \ln p \]

Now for \( p \) even C.9 takes the form

\[ \sum_{l=0}^{r-1} \left[ \zeta'(0, 1 - \frac{l}{r}) + \zeta'(0, 1 - \frac{2l + 1}{2r}) \right] = \zeta'(0) + \zeta(0) \ln[2r] \]

The first term in C.9 is of the form of the original expression C.9 and therefore the resulting expression is

\[ \sum_{l=0}^{r-1} \zeta'(0, 1 - \frac{2l + 1}{2r}) = -\frac{1}{2} \ln 2 \]
For \( p = 4t + 2 \) we divide the \( l \)-summation into the two ranges, 0 to \( t - 1 \), and, \( t \) to \( 2t - 1 \); rearranging the latter sum and using C.8 then makes then gives us

\[
- \sum_{l=0}^{t-1} \ln \left[ 2 \sin \left( \frac{2l + 1}{p} \pi \right) \right] + \zeta'(0, \frac{1}{2}) = -\frac{1}{2} \ln 2
\]

Now noting that \( \zeta'(0, \frac{1}{2}) = -\frac{1}{2} \ln 2 \) we have the identity.

\[
\sum_{l=1}^{(p-6)/4} \ln \left[ 2 \sin \left( \frac{2l + 1}{p} \pi \right) \right] = 0
\]

We find for \( p = 4t \), by a similar procedure, that

\[
\sum_{l=0}^{(p-4)/2} \ln \left[ 2 \sin \left( \frac{2l + 1}{p} \pi \right) \right] = \frac{1}{2} \ln 2
\]

These are the first set of identities we obtain by decomposing over conjugacy classes.

**C.1 Relations involving \( \zeta'(-1, 1 + a) \)**

Next we turn to \( \zeta'(-1) \), thus note that

\[
\zeta(s - 1, 1 + a) = \sum_{n=1}^{\infty} \frac{1}{(n + a)(s-1)}
\]

has the series expansion in terms of \( a \)

\[
\zeta(s - 1, 1 + a) = \sum_{k=0}^{\infty} \frac{(-k)^k \Gamma(s + k - 1) a^k}{\Gamma(s-1)k!} \zeta(k + s - 1)
\]

which on extracting the divergent part gives

\[
\zeta(s - 1, 1 + a) = \zeta(s - 1) - (s - 1)a\zeta(s) + \frac{s(s - 1)a^2}{2} \zeta(s + 1) + \hat{\zeta}(s - 1, 1 + a)
\]

where

\[
\hat{\zeta}(s - 1, 1 + a) = \sum_{n=1}^{\infty} \left[ \frac{1}{(n + a)^{s-1}} - \frac{1}{n^{s-1}} + \frac{(s - 1)a}{n^s} - \frac{s(s - 1)a^2}{n^{s+1}} \right]
\]

This is now expressed in terms of our prescription for analytic continuation. Hence we have

\[
\zeta(-1, 1 + a) = -\frac{1}{12} - \frac{a(1-a)}{2}
\]
where \( \zeta(-1) = -1/2 \), \( \zeta(0) = -1/2 \) and \( s\zeta(s+1) = 1 + \gamma s + \ldots \) have been used. Differentiation with respect to \( s \) and setting it to zero yields

\[
\zeta'(-1, 1 + a) = \zeta'(-1) + [1 - \ln(2\pi)] \frac{a}{2} + \frac{(\gamma - 1)a^2}{2} + \hat{\zeta}'(-1, 1 + a) \tag{C.14}
\]

Since differentiation with respect to \( s \) at \( s = 0 \) and summation over \( n \) commute we have

\[
\hat{\zeta}'(-1, 1 + a) = -(n + a) \ln(1 + \frac{a}{n}) + a + \frac{a^2}{2n} \tag{C.15}
\]

\[
= -\sum_{k=2}^{\infty} \frac{(-1)^k a^{k+1}}{k+1} \zeta(k) + a \sum_{k=2}^{\infty} \frac{(-)^k a^k}{k} \zeta(k)
\]

Now using the identity

\[
\sum_{k=2}^{\infty} \frac{(-)^k a^{k+\nu}}{k+\nu} \zeta(k) = \int_{0}^{a} dy y^{\nu} (\psi(1+y) - \psi(1)) \tag{C.16}
\]

valid for \( a < 1 \) to remain away from the poles of \( \psi(y) \) which occur at \( y = -n \) for \( n = 0, 1, 2, \ldots \) we have

\[
\hat{\zeta}'(-1, 1 + a) = -\int_{0}^{a} dy (y-a) [\psi(1+y) - \psi(1)] \tag{C.17}
\]

and therefore

\[
\zeta'(-1, 1 + a) = \zeta'(-1) + [1 - \ln(2\pi)] \frac{a}{2} - \frac{a^2}{2} - \int_{0}^{a} dy (y-a) \psi(1+y) \tag{C.18}
\]

Finally we conclude that the anti-symmetrised sum is given by

\[
\zeta'(-1, 1 + a) - \zeta'(-1, 1 - a) = [1 - \ln(2\pi)] a - \int_{0}^{a} dy (y-a) [\psi(1+y) - \psi(1-y)] \tag{C.19}
\]

Now using

\[
\psi(1+y) - \psi(1-y) = -\frac{d}{dy} \ln \left[ \frac{\sin(\pi y)}{\pi y} \right] \tag{C.20}
\]

we obtain

\[
\zeta'(-1, 1 + a) - \zeta'(-1, 1 - a) = -a \ln \left[ \frac{2 \sin(\pi a)}{a} \right] + \frac{1}{\pi} \int_{0}^{\pi a} dz z \cot(z) \tag{C.21}
\]

An alternative way of expressing this which is useful in the main text and simplifies some of the expressions is

\[
\zeta'(-1, a) - \zeta'(-1, 1 - a) = -\frac{1}{\pi} \int_{0}^{\pi a} \ln [2 \sin(z)] \tag{C.22}
\]
Let us now examine the consequences of decomposing over conjugacy classes. Proceeding as for $\zeta'(0)$, on decomposing over conjugacy classes we obtain the identity

$$
\zeta'(-1) = \frac{1}{12} \ln p + p \sum_{j=0}^{p-1} \zeta'(-1, 1 - \frac{j}{p})
$$

For $p = 2r + 1$ this yields the identity

$$
\zeta'(-1) = \frac{1}{p^2 - 1} \left[ p \sum_{l=1}^{(p-1)/2} \left[ \frac{2l}{p} \ln \left( \frac{2l}{p} \right) + \left( \frac{2l}{p} \right)^2 \right]
+ \int_0^{2l/p} dy \left[ \psi(1 + y) + \psi(1 - y) \right] \right] - \frac{1}{12} \ln p
$$

i.e.

$$
\zeta'(-1) = \frac{1}{6} - \frac{1}{4} \ln p
+ \frac{1}{p^2 - 1} \left[ \sum_{l=1}^{(p-1)/2} \left[ 2l \ln[2l] + p \int_0^{2l/p} dy \left[ \psi(1 + y) + \psi(1 - y) \right] \right] - \frac{1}{12} \ln p \right]
$$

This can be used numerically to verify that $\zeta'(-1) = 0.16791$. Now noting that

$$
\int_0^{\pi(1-a)} dz \ln [2 \sin(z)] = - \int_0^{\pi a} dz \ln [2 \sin(z)]
$$

We find

$$
\sum_{j=1}^{p-1} \int_0^{j\pi/p} dz \ln [2 \sin(z)] = 0
$$

For $p$ even we find that by observing the two decompositions

$$
\zeta'(-1, 1 + \alpha) = -\zeta'(-1, 1 + \alpha) \ln p + \sum_{l=0}^{p-1} \zeta'(-1, 1 - \frac{l - \alpha}{p})
$$

for $\alpha$ not necessarily integer and

$$
\zeta'(-1, 1 + \alpha) = -\zeta(-1, 1 + \alpha) \ln p + \sum_{l=1}^{p} \zeta'(-1, \frac{l + \alpha}{p})
$$

valid for arbitrary $p$ can be used to simplify the expression for

$$
\frac{1}{p^2 - 1} \sum_{l=1}^{(p-2)/2} \int_0^{(2l+1)\pi/p} dz \ln [2 \sin(z)] = \frac{2 \ln 2}{p} - \frac{1}{2} \ln 2 - \frac{4}{p} \ln[p/2]
$$
These are used in simplifying the expression for the torsion in the case of \( p \) even.

### C.2 Relations involving \( \zeta'(-2, 1 + a) \)

We now turn to \( \zeta'(-2, 1 + a) \) which we obtain from

\[
\zeta(s - 2, 1 + a) = \sum_{n=1}^{\infty} \frac{1}{(n + a)^{s-2}} \quad (C.23)
\]

We again extract the divergent parts from the sum to obtain

\[
\zeta(s - 2, 1 + a) = \zeta(s - 2) - (s - 2)a\zeta(s - 1) + \frac{(s - 1)(s - 2)a^2}{2}\zeta(s) - \frac{s(s - 1)(s - 2)\zeta(s + 1)a^3}{6} + \hat{\zeta}(s - 2, 1 + a) \quad (C.24)
\]

where

\[
\hat{\zeta}(s - 2, 1 + a) = \sum_{n=1}^{\infty} \left[ \frac{1}{(n + a)^{s-2}} - \frac{1}{n^{s-2}} + \frac{(s - 2)a}{n^{s-1}} - \frac{(s - 1)(s - 2)a^2}{2n^s} + \frac{s(s - 1)(s - 2)a^3}{6n^{s+1}} \right] \quad (C.25)
\]

Thus

\[
\zeta(-2, 1 + a) = -\frac{a}{6} - \frac{a^2}{2} - \frac{a^3}{3} \quad (C.26)
\]

and differentiating with respect to \( s \) and evaluating at \( s = 0 \) we find

\[
\zeta'(-2, 1 + a) = \zeta'(-2) + 2\zeta'(-1) + \frac{1}{12} a + \left[ \frac{3}{4} - \frac{1}{2} \ln[2\pi] \right] a^2
\]

\[
- \left[ \frac{\gamma}{3} - \frac{1}{2} \right] a^3 + \hat{\zeta}'(-1, 1 + a) \quad (C.27)
\]

We evaluate \( \hat{\zeta}'(-2, 1 + a) \) by differentiating with respect to \( s \) first and then performing the sum over \( n \) obtaining

\[
\hat{\zeta}'(-2, 1 + a) = -\sum_{n=1}^{\infty} \left[ (n + a)^2 \ln[1 + \frac{a}{n}] - n + a - \frac{3a^2}{2} - \frac{a^3}{3n} \right]
\]

\[
= \sum_{k=2}^{\infty} \frac{(-1)^k a^{k+2}}{k + 2} \zeta(k) - 2a \sum_{k=2}^{\infty} \frac{(-1)^k a^{k+1}}{k + 1} \zeta(k) + a^2 \sum_{k=2}^{\infty} \frac{(-1)^k a^k}{k} \zeta(k) \quad (C.28)
\]

Thus using the summation formula C.16 we have

\[
\hat{\zeta}'(-2, 1 + a) = \int_0^a (y - a)^2 [\psi(1 + y) - \psi(1)] \quad (C.29)
\]
We therefore obtain
\[
\zeta'(-2, 1 + a) = \zeta'(-2) + \frac{1}{12} [24 \zeta'(-1) + 1] a + \frac{1}{4} [3 - 2 \ln(2\pi)] a^2 + \frac{1}{2} a^3 + \int_0^a dy (y - a)^2 \psi(1 + y) \tag{C.30}
\]

We note that our expression for \(\zeta'(-2, 1 + a)\) implies
\[
\zeta'(-2, 1 + a) + \zeta'(-2, 1 - a) = 2 \zeta'(-2) + \frac{1}{2} [3 - 2 \ln(2\pi)] a^2 + \int_0^a dy (y - a)^2 [\psi(1 + y) - \psi(1 - y)] \tag{C.31}
\]

Differentiating the functional relation
\[
\xi(s) = \xi(1 - s), \quad \text{where } \xi(s) = \Gamma(s/2) \pi^{-s/2} \zeta(s) \tag{C.32}
\]

with respect to \(s\) we find for \(s = -2\)
\[
\zeta'(-2) = -\frac{1}{4\pi^2} \zeta(3) \tag{C.33}
\]

Finally using C.20 and C.33 we have
\[
\zeta'(-2, 1 + a) + \zeta'(-2, 1 - a) = -\frac{1}{2\pi^2} \zeta(3) - a^2 \ln \left[ \frac{2 \sin(\pi a)}{a} \right] - \frac{1}{\pi^2} \int_0^{\pi a} dz z (z - 2\pi a) \cot(z) \tag{C.34}
\]

We can obtain an identity by decomposing over conjugacy classes mod \(p\), by noting that
\[
\zeta(s - 2) = p^{2-s} \sum_{j=0}^{p-1} \zeta(s - 2, 1 - \frac{j}{p}) \tag{C.35}
\]
implies
\[
\zeta'(-2) = p^2 \sum_{j=0}^{p-1} \zeta(-2, 1 - \frac{j}{p}) \ln p + p^2 \sum_{j=0}^{p-1} \zeta'(-2, 1 - \frac{j}{p}) \tag{C.36}
\]

The first sum is a decomposition of \(\zeta(-2)\) and therefore zero. If we decompose \(j\) into odd and even elements, by setting \(j = 2l + 1\) and \(j = 2l\) respectively, we find for \(p\) odd i.e. \(p = 2r + 1\)
\[
\zeta'(-2) = p^2 \zeta'(-2) + p^2 \sum_{l=1}^{r} \left[ \zeta'(-2, 1 - \frac{2l}{p}) + \zeta'(-2, 1 - \frac{2l + 1}{p}) \right] \tag{C.37}
\]
which can be rewritten as either

\[ \zeta'(-2) = p^2 \zeta'(-2) + p^2 \sum_{l=1}^{r} \left[ \zeta'(-2, 1 - \frac{2l}{p}) + \zeta'(-2, \frac{2l}{p}) \right] \tag{C.38} \]

or

\[ \zeta'(-2) = p^2 \zeta'(-2) + p^2 \sum_{l=0}^{r-1} \left[ \zeta'(-2, \frac{2l+1}{p}) + \zeta'(-2, 1 - \frac{2l+1}{p}) \right] \tag{C.39} \]

Therefore we have that

\[ \zeta'(-2) = p^3 \zeta'(-2) - \sum_{l=0}^{r} \left[ 4l^2 \ln \left( \frac{2 \sin \left( \frac{2l\pi}{p} \right)}{p} \right) + \frac{p^2}{\pi^2} \int_{0}^{2l\pi/p} dz \left( z - \frac{4l\pi}{p} \right) \cot(z) \right] \tag{C.40} \]

which on using C.33 gives

\[ \frac{(1-p^3)}{4\pi^2} \zeta(3) = \sum_{l=0}^{r} \left[ 4l^2 \ln \left( \frac{2 \sin \left( \frac{2l\pi}{p} \right)}{p} \right) + \frac{p^2}{\pi^2} \int_{0}^{2l\pi/p} dz \left( z - \frac{4l\pi}{p} \right) \cot(z) \right] \tag{C.41} \]

or

\[ \zeta'(-2) = p^3 \zeta'(-2) - \sum_{l=0}^{r-1} \left[ (2l+1)^2 \ln \left( 2 \sin \left( \frac{(2l+1)\pi}{p} \right) \right) \right. \]

\[ + \frac{p^2}{\pi^2} \int_{0}^{(2l+1)\pi/p} dz \left( z - \frac{2(2l+1)\pi}{p} \right) \cot(z) \] \tag{C.42}

which gives

\[ \frac{(1-p^3)}{4\pi^2} \zeta(3) = \sum_{l=0}^{r-1} \left[ (2l+1)^2 \ln \left( \frac{2 \sin \left( \frac{(2l+1)\pi}{p} \right)}{p} \right) \right. \]

\[ + \frac{p^2}{\pi^2} \int_{0}^{(2l+1)\pi/p} dz \left( z - \frac{2(2l+1)\pi}{p} \right) \cot(z) \] \tag{C.43}

We can equally establish that

\[ \frac{(1-p^3)}{4\pi^2} \zeta(3) = \sum_{j=1}^{r} \left[ j^2 \ln \left( \frac{2 \sin \left( \frac{j\pi}{p} \right)}{p} \right) + \frac{p^2}{\pi^2} \int_{0}^{j\pi/p} dz \left( z - \frac{2j\pi}{p} \right) \cot(z) \right] \tag{C.44} \]

Finally we tabulate here for future reference some useful identities regarding the Hurwitz and Riemann zeta functions

\[ \zeta(s, a) = a^{-s} + \zeta(s, 1 + a) \tag{C.45} \]
\[ \zeta(-2) = 0 \quad \zeta(-1, 1 + a) - \zeta(-1, 1 - a) = -a \]
\[ \zeta(0, a) + \zeta(0, 1 - a) = 0 \quad \zeta(0, 1 + a) + \zeta(0, 1 - a) = -1 \]
\[ \zeta'(0, 1 + a) = \zeta'(0, a) + \ln a \quad \zeta'(-1, 1 + a) = \zeta'(-1, a) + a \ln a \]
\[ \zeta'(0, 1 + a) + \zeta'(0, 1 - a) = -\ln \left[ \frac{2 \sin(\pi a)}{a} \right] \]  \hspace{1cm} (C.46)

\[ \zeta'(-1, 1 + a) - \zeta'(-1, 1 - a) = -a \ln \left[ \frac{2 \sin(\pi a)}{a} \right] + \frac{1}{\pi} \int_0^{\pi a} dz \cot(z) \]  \hspace{1cm} (C.47)

\[ \zeta'(-1, a) - \zeta'(-1, 1 - a) = -a \ln(2 \sin \pi a) + \frac{1}{\pi} \int_0^{\pi a} dz \cot(z) \]

and
\[ \zeta'(-2, 1 + a) + \zeta'(-2, 1 - a) = -\frac{1}{2\pi^2} \zeta(3) - a^2 \ln \left[ \frac{2 \sin(\pi a)}{a} \right] \]
\[ - \frac{1}{\pi^2} \int_0^{\pi a} dz \left( z - 2\pi a \right) \cot(z) \]  \hspace{1cm} (C.48)

\[ \zeta'(-2, a) + \zeta'(-2, 1 - a) = -\frac{1}{2\pi^2} \zeta(3) - a^2 \ln \left[ 2 \sin(\pi a) \right] - \frac{1}{\pi^2} \int_0^{\pi a} dz \left( z - 2\pi a \right) \cot(z) \]  \hspace{1cm} (C.49)

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