DYNAMICAL SYSTEM MODELING FERMIONIC LIMIT

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Abstract. The existence of multiple radial solutions to the elliptic equation modeling fermionic cloud of interacting particles is proved for the limiting Planck constant and intermediate value of mass parameters. It is achieved by considering the related nonautonomous dynamical system for which the passage to the limit can be established due to the continuity of the solutions with respect to the parameter going to zero.

1. Introduction and motivation. Consider the following elliptic boundary value problem

\[ \Delta \phi(u) = H_\eta^{-1}(c - \phi(u)) \]  \hspace{1cm} (1)

provided with the zero Dirichlet boundary condition, where \( \phi \) plays the role of the gravitational potential generated by the cloud of diffusive particles with the self-agreed density \( H_\eta^{-1}(c - \phi(u)) \) distributed over \( u \in B(0,1) \subset \mathbb{R}^d \) and the constant \( c \) satisfying, for given \( M > 0 \), the mass constraint

\[ \int_{B(0,1)} H_\eta^{-1}(c - \phi(u)) \, du = M. \]  \hspace{1cm} (2)

The mass \( M > 0 \) is a given parameter, while the parameter \( c \in \mathbb{R} \) related to the temperature and the potential \( \phi \) are unknown satisfying the equation (1) in the ball \( B(0,1) \) with the zero Dirichlet boundary condition

\[ \phi|_{\partial B(0,1)} = 0 \]  \hspace{1cm} (3)

and the mass constraint (2). The origins of the function \( H_\eta \) stems from the statistical mechanics approach, cf. [20]. The function \( H_\eta \) is given and depends on the parameter \( \eta \geq 0 \). The form of \( H_\eta \) encompasses the models arising from the Maxwell–Boltzmann and the Fermi–Dirac statistics.

We shall prove the multiplicity results for the above nonlocal BVP for the intermediate values of the mass parameter \( M > 0 \) while the parameter \( \eta > 0 \) is taken sufficiently close to zero. Thus it can be seen as a singular perturbation of the Maxwell–Boltzmann statistics with \( \eta = 0 \).

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The problem can be reduced, by appropriate substitution, to some dynamical system stated in (4) for given \( H_\eta \) by defining the new nonlinearity \( R_\eta \) as

\[
H'_\eta(z)R_\eta(z) = 1.
\]

We consider the following functions originating from the statistical mechanics:

- \( R_0(z) = z \) in the Maxwell–Boltzmann model with \( H_0(z) = \log(z) \).
- \( R_\eta(z) = (1/z + \eta/z^{1+2/d})^{-1} \) in the simplified Fermi–Dirac model with
  \[
  H_\eta(z) = \log(z) + \frac{\eta d}{2} z^{2/d}
  \]

- \( R_\eta(z) = \mu \frac{(d-2)}{4} f_{d/2-2}(f_{d/2-1}^{-1}(2z/\mu)) \) in the Fermi–Dirac model with
  \[
  \eta \mu^{2/d} = 2d^{2/d-1}
  \]

and the Fermi functions \( f_\alpha \) defined as

\[
f_\alpha(z) = \int_0^\infty \frac{x^\alpha}{1 + \exp(x-z)} \, dx.
\]

The Fermi–Dirac model was introduced to describe in a better way the existence of the galaxies or the gaseous stars than the Maxwell–Boltzmann model. In the Maxwell–Boltzmann model the existence of blowing–up solutions for (7) – the so called the gravo-thermal catastrophe was proved. It was accompanied by the lack of steady states for massive clouds but was not supported by observations of evolving galaxies or stars towards stable steady states, cf. [24]. The motivation for considering such form of equations comes from the models of self–gravitating diffusive particles introduced by Chavanis et al. in [7, 10] and developed further in [8].

Since the problem (1-3) is considered on the ball from the results of [18] all solutions are radially symmetric and therefore one can reduce the considerations to such solutions only. Relating the potential \( \phi \) to the new variables: \( x \) the reduced mass (related to the integral of \( \rho \)) and \( y \) the reduced energy (related to \( \rho H_\eta \)) leads to the possibly nonautonomous system

\[
\begin{align*}
x'(s) &= (2 - d) x(s) + y(s), \\
y'(s) &= 2 y(s) - x(s) e^{2s} R_\eta(e^{-2s} y(s)),
\end{align*}
\]

with parameters \( d \in \mathbb{N} \cap [3, 9], \eta \geq 0 \) that leads for \( R_0 = 1 \) to the autonomous one

\[
\begin{align*}
x'(s) &= (2 - d) x(s) + y(s), \\
y'(s) &= (2 - x(s)) y(s).
\end{align*}
\]

Indeed the system (4) can be derived from the elliptic equation (1), up to a constant stated in [26, 28], by considering

\[
- \, Q'' + (d-1) r^{-1} Q' = Q R_\eta(r^{1-d} Q')
\]

with \( Q(0) = 0, Q(1) = \sigma_d^{-1} M \) using the substitution relating \( s, x, y \) to \( r, Q, Q' \) given by

\[
Q(e^s) = x(s) e^{(d-2)s}, Q'(e^s) = y(s) e^{(d-3)s}
\]

The latter equation (6) describes the evolution of

\[
Q(r) = \sigma_d^{-1} \int_{B(0,r)} \rho(u) \, du
\]
the averaged (differing thus by a constant $\sigma_d$ measure of the unit sphere from notation adopted in [26] and [28]), i.e. integrated over the ball $B(0, r)$, radial density 

$$\rho(u) = H_\eta^{-1}(c - \phi(u))$$

of the particles preserving mass $\int_{B(0, 1)} \rho(u) du = M$.

The $x$ variable is related to the rescaled mass parameter, while $y$ can be vaguely referred to the energy of the system. The precise reference is stated in the sequel. One should note that while the system (5) referred to as Maxwell–Boltzmann case is well understood as been thoroughly examined in many papers, cf. [1, 22] and references therein, whilst the so called Fermi–Dirac like system (4) is less studied and not many results are available, cf. [26, 28]. This difficulty is generated by the nonlinear nature of the function $R_\eta$ causing some additional problems and posing some extra difficulties. The problem (4) can be also studied in a slightly more general framework allowing $R_\eta$ satisfying some condition cf. Theorem 3.1 encompassing also the Fermi–Dirac case. It should be noted that the results obtained for both models differ significantly for $d = 3$ and large values of mass parameter, while for small and intermediate values of mass parameter they share common features provided the parameter $\eta$ related to the Planck constant is small enough. The main result of this paper is the convergence of properly chosen solutions of the system (4) towards the solutions to (5) as $\eta \to 0$ and provided that the mass parameter $M$ attains some intermediate values. This results in the existence of multiple solutions for the Fermi–Dirac model for $\eta$ small enough and properly chosen mass parameter $M > 0$ with intermediate values as in the Maxwell–Boltzmann case, cf. [1, 21]. This can be depicted in the phase diagram on Figure 1 illustrating the main Theorem 3.2 and Corollary 1 of the manuscript. The results presented in this paper can be seen as continuous dependence of the solutions to the dynamical system on the parameter $\eta \geq 0$ but only for sufficiently small values of the parameter. It should be underlined that solutions for the dynamical systems are defined on the non-compact interval. Moreover, we choose some special family of the solutions characterized by the limit at minus infinity, not the whole set of possible solutions. The continuous dependence on parameters of the whole set of solutions for elliptic equations was established among others in [4, 5, 6]. One should point out that the passage to the singular limit was rigorously verified both for the related Navier–Stokes–Fourier–Poisson system by Laurencot and Feireisl in [15] while Golse and Saint–Raymond in [19] dealt with celebrated Navier–Stokes and Boltzmann equations.

The solutions of the BVP for the elliptic equation considered above (1) can be seen as steady states for the evolutions of the potential of particles with the density $\rho$ and with no flux boundary condition evolving by

$$\rho_t = \nabla \cdot N \left( \theta P'_\eta \nabla \rho + \rho \nabla \Delta^{-1} \rho \right),$$

with some positive coefficient $N$ possibly depending on other variables, where

$$P'_\eta(z) = H'_\eta(z)z.$$

2. **Derivation of the dynamical systems.** The derived results extend the results obtained for the case $d = 3$ in [11] to higher dimension $3 \leq d \leq 9$ and for the more general pressure formulae $P_\eta$ generating via

$$P'_\eta(z) = zH'_\eta(z), \quad H'_\eta(z)R_\eta(z) = 1$$
with the function $R_\eta$ appearing in the system (4) while $z = \rho \theta^{-d/2}$ where $\theta$ is the temperature of the system and $\eta$ is the parameter related to the Planck constant.

Let us analyze the limit system (5) for which the point $(0, 0)$ is a saddle, while the other stationary point $(2, 2(d-2))$ can change character depending on the dimension $d$. If any $d$ is considered with $3 \leq d \leq 9$ then $(2, 2(d-2))$ is a sink and a Lyapunov function

$$L(x, y) = \frac{1}{2}(x - 2)^2 + y - 2(d - 2) - 2(d - 2) \log(y/(2d - 4))$$

governs convergence towards this point as was established in [1] and started in [22].

Indeed multiplying the equations (5) for $x'$ by $x - 2$ and $y'/y$ by $2(2 - d)$ and summing them with added $y'$ one obtains

$$\frac{d}{dt} L(x(t), y(t)) = x'(t)(x(t) - 2) + y'(t) - 2(d - 2) y'(t)/y(t) = -(x(t) - 2)^2 \leq 0.$$

Moreover, using Taylor expansion in the neighborhood of $(x, y) \sim (2, 2(d-2))$ we can see that

$$L(x, y) \sim \frac{1}{2}(x - 2)^2 + \frac{1}{4(d - 2)}(y - 2(d - 2))^2.$$

Furthermore, note that the condition $Q(0) = 0$ can be translated to

$$\lim_{s \to -\infty} x(s)e^{(d-2)s} = 0,$$

while assuming $\rho \in L^\infty$ guarantees $Q(r)$ be of order $r^d$ at zero thus assuring $x(s)e^{-2s}$ to be bounded. Moreover, if $\rho$ is continuous then the following limit exists and is finite

$$\lim_{s \to -\infty} x(s)e^{-2s} < \infty.$$

Additionally,

$$\rho(0) = |\rho|_\infty = \lim_{s \to -\infty} y(s)e^{-2s} < \infty.$$

Indeed, one can see that using appropriate substitutions one can obtain

$$\lim_{s \to -\infty} x(s)e^{-2s} = \lim_{s \to -\infty} Q'(s)e^{(1-d)s} = \lim_{r \to 0^+} r^{d-1}\rho(r)r^{1-d} = \rho(0).$$

One assumes $R_\eta$ to be continuous on $[0, \infty)$ to claim the following lemma in the first positive quadrant.

**Lemma 2.1.** For any solution $(x, y)$ to (4), a finite $\rho_0 = \lim_{s \to -\infty} y(s)e^{-2s}$ implies

$$\lim_{s \to -\infty} \frac{x(s)}{y(s)} = \frac{1}{d}.$$

**Proof.** Using de l’Hospital rule together with the system (4) one gets the claim by

$$N = \lim_{s \to -\infty} \frac{x(s)}{y(s)} = \lim_{s \to -\infty} \frac{x'(s)}{y'(s)} = \lim_{s \to -\infty} \frac{1 + (2 - d) \frac{x(s)}{y(s)}}{2 - e^{2s}R_\eta(e^{-2s}y(s))\frac{x(s)}{y(s)}} = \frac{1 + (2 - d)N}{2}.$$
3. Convergence and multiplicity results. Consider the system describing the evolution of the difference

\[
\begin{cases}
  w_\eta = x_\eta - x_0 \\
  v_\eta = y_\eta - y_0
\end{cases}
\]  

of solutions \((x_\eta, y_\eta)\) to (4) and \((x_0, y_0)\) to (5) of the form

\[
\begin{cases}
  w'_\eta = (2 - d) w_\eta + v_\eta \\
  v'_\eta = (2 - x_0) v_\eta - y_\eta v_\eta + x_\eta e^{2s} S_\eta(e^{-2s} y_\eta)
\end{cases}
\]  

where

\[
S_\eta(z) = z - R_\eta(z).
\]

Now we shall prove a crucial a priori bound for the term \(x_\eta e^{2s} S_\eta(e^{-2s} y_\eta)\) appearing in (9). Set \(\rho_0 > 0\) and take for any \(\rho \leq \rho_0\) the solution \(y_\eta\) of (4) such that

\[
\rho = \lim_{s \to -\infty} y_\eta(s)e^{-2s}.
\]

Then by Lemma 2.1 we have that \(y_\eta e^{-2s} \not\to \rho\) and \(x_\eta e^{-2s} \not\to \frac{1}{2} \rho\) as \(s \to -\infty\) hence \(y_\eta \leq \rho_0 e^{2s}\) and \(x_\eta \leq \frac{1}{2} \rho_0 e^{2s}\) whence for any \(s \leq 0\) we get

\[
dx_\eta e^{2s} S_\eta(e^{-2s} y_\eta) \leq \rho_0 e^{4s} \max_{[0, \rho_0]} S_\eta = \rho_0 e^{4s} \overline{S}_{\eta, \rho_0} \leq \rho_0 \overline{S}_{\eta, \rho_0},
\]

where \(\overline{S}_{\eta, \rho_0} = \max_{[0, \rho_0]} S_\eta\) is increasing in \(\rho_0\) and converging to zero as \(\eta \to 0^+\) by (15).

Multiplying \(w'_\eta\) by \(w_\eta\) and \(v'_\eta\) by \(v_\eta\) respectively one obtains

\[
w'_\eta w_\eta = (2 - d) w_\eta^2 + v_\eta w_\eta
\]

and

\[
v'_\eta v_\eta = (2 - x_0) v_\eta^2 - y_\eta v_\eta v_\eta + x_\eta e^{2s} S_\eta(e^{-2s} y_\eta) v_\eta.
\]

Next setting \(\chi = w_\eta^2 + v_\eta^2\) one obtains

\[
\chi' \leq \alpha \chi + \beta
\]

where

\[
\alpha = 2 \max\{2 - d, |2 - x_0|\} + \max\{1 - y_\eta\} + \frac{\rho_0}{d} + \overline{S}_{\eta, \rho_0}
\]

and \(\beta = \overline{S}_{\eta, \rho_0}\) are the terms bounded with respect to \(\eta\). Indeed one can estimate

\[
|(1 - y_\eta) v_\eta w_\eta| \leq \frac{1}{2} (w_\eta^2 + v_\eta^2) \max\{1, \max y_\eta - 1\}
\]

and

\[
|2 - x_0| \leq \max\{2, \max x_0 - 2\}
\]

while

\[
y_\eta \leq \rho_0, x_\eta \leq \frac{1}{d} \rho_0, |v_\eta| \leq \frac{1}{2} (1 + v_\eta^2).
\]

Thus coming back to the estimate on \(\chi\) and by the Gronwall lemma

\[
\chi \leq \beta e^\alpha.
\]

But it should be noted that

\[
\beta \leq \frac{1}{2} x_\eta e^{2s} S_\eta \leq \frac{1}{2d} \rho_0 S_\eta \leq \frac{1}{2d} \rho_0 \overline{S}_{\eta, \rho_0}
\]

while \(S_\eta(z) \leq C(\eta) D(z)\) by the assumption (15) where \(C(\eta) \to 0\) as \(\eta \to 0\), cf. also Lemma 4.1 for the simplified and Lemma 4.2 for the full Fermi–Dirac models.
Recall that from the first equation of (9) one obtains
\[
\left( e^{(d-2)s} w_\eta(s) \right)' = e^{(d-2)s} v_\eta(s) ,
\]
while from the second equation of (9) one gets
\[
\left( e^{-2s} v_\eta(s) \right)' = -x_0(s) v_\eta(s) e^{-2s} - y_\eta(s) w_\eta(s) e^{-2s} + x_\eta(s) S_\eta(e^{-2s} y_\eta(s)) .
\]
Moreover, for any \( \eta \geq 0 \), one has
\[
y_\eta(s) \leq \rho_0 e^{2s} ,
\]
and
\[
y(s) \leq \rho_0 e^{2s} ,
\]
whence by (12) after integration over \((-\infty, t)\) and taking supremum one gets
\[
d A_\eta(t) \leq B_\eta(t) .
\]
Next by (13) and (14) one gets
\[
B_\eta(t) \leq \int^{-t}_{-\infty} \frac{2d}{\rho_0} e^{2s} B_\eta(s) + \frac{\rho_0}{d} e^{2s} \max_{[0, \rho_0]} S_\eta ds .
\]
Finally, using a Gronwall estimate, one gets by (10) and (15) for any \( t \leq 0 \) that
\[
0 \leq d A_\eta(t) \leq B_\eta(t) \leq \frac{\rho_0}{2d} e^{\rho_0/d} \max_{[0, \rho_0]} S_\eta \leq \frac{\rho_0}{2d} e^{\rho_0/d} C(\eta) \max_{[0, \rho_0]} D .
\]
Thus we have proved the following convergence theorem

**Theorem 3.1.** Fix any natural number \( 3 \leq d \leq 9 \) and \( \kappa_0 > 0 \) and take any \( \rho \in (0, \kappa_0) \) such that \( \lim_{\rho \to 0} \eta_\rho(s) e^{-2s} = \rho \) for some solution to the system (4). Assume that the continuous function \( R_\eta \) satisfies
\[
0 \leq z - R_\eta(z) \leq C(\eta) D(z)
\]
where \( C(\eta) \to 0 \) as \( \eta \to 0 \) and \( D : [0, \infty) \to [0, \infty) \) is a continuous function such that \( D(0) = 0 \). Then the solution \( (x_\eta, y_\eta) \) converges uniformly to \((x_0, y_0)\) on \((-\infty, 0]\) and in particular \( x_\eta(0) \) converges to \( x_0(0) \).

One can verify that the condition (15) is fulfilled for the simplified Fermi–Dirac model with \( C(\eta) = \eta \) and \( D(z) = z^{1+2/d} \) (cf. Lemma 4.1) and for the Fermi–Dirac model with \( C(\eta) = \eta \) and \( D(z) = D_1 z^{1+2/d} \) with suitably chosen constant \( D_1 \) defined in Lemma 4.2.

The role of the parameter \( \rho_0 > 0 \), which is arbitrary, is an a priori constraint imposed on the family of solutions with different values of the limit parameter \( \lim_{s \to \infty} g(s) e^{-2s} = \rho \). Looking at the right hand side the mass–density diagram we limit our considerations putting the constraint on the sup norm of the density \( \rho \) axis.

Recall from [11] with \( d = 3 \) the phase portrait for the Maxwell–Boltzmann case with \( R_0 = I \) identity function and \((2, 2(d - 2)) = (2, 2)\), cf. Figure 1. The structure of the phase portrait and the presence of the heteroclinic orbit is similar for any \( 3 \leq d \leq 9 \).
An easy corollary of the Theorem 3.1, due to the fact that the mass of the system is related to \( x_\eta(0) \), can be formulated as follows.

**Theorem 3.2.** For any mass parameter \( M \) in the corresponding, intermediate range for Fermi–Dirac like models modeled by \( R_\eta \), with \( \eta > 0 \) small enough, satisfying the condition from the Theorem 3.1 there exists as many solutions to (1)-(3) as for the Maxwell–Boltzmann case with \( R_0 = 1 \) depicted in the right part of Figure 1 and depending on the intersection of the vertical line (setting thus the mass \( M > 0 \)) with the bifurcation curves.

The details of the proof are the same as in Theorem 2 from [11] and are omitted herein. They actually focus on the continuous dependence of the mass \( M \) on the density \( \rho \) or in other words \( x_\eta(0) \) on \( \lim_{s \to -\infty} e^{-2s}y(s) \) expressed in the language of the dynamical system variables.

**Corollary 1.** For the intermediate values of the mass parameter \( M \) there exists multiple solutions to (1)-(3) for the Fermi–Dirac \( R_\eta \) and generalizations obeying the condition (15) from Theorem 3.1 provided the \( \eta \) parameter is sufficiently small.

We show that the phenomena appearing in the Fermi–Dirac model for large values of the mass (cf. [25]) parameter differentiating between dimensions \( d = 3 \) (solution for any mass) and \( d \geq 5 \) (solution only up to some mass parameter) are not present for intermediate value of the mass parameter, where the existence of solutions is generic for any dimension \( 3 \leq d \leq 9 \). This is accompanied by the existence of multiple solutions for any dimension if we are close enough to the Maxwell–Boltzmann case with \( \eta = 0 \). One should note also that for small values of mass parameter the uniqueness holds as was noted in [12].

4. Appendix.

**Lemma 4.1.** For simplified Fermi–Dirac model we have straightforward estimate

\[
z^{-1-2/d}(z - R_\eta(z)) \leq \eta.
\]

**Proof.** Indeed, recall the formula \( R_\eta(z) = (1/z + \eta/z^{1-2/d})^{-1} \). Then

\[
z^{-1-2/d}(z - R_\eta(z)) = \frac{\eta z^{4/d}}{z^{4/d} + \eta z^{2/d}} \leq \eta.
\]

\( \square \)
Lemma 4.2. For the Fermi–Dirac model we have
\[ z^{-1 - 2/d} (z - R_\eta(z)) \leq \left( \frac{2}{\mu} \right)^{2/d} C, \]
where
\[ C = \max_{w \in [0, \infty)} w^{-1 - 2/d} (w - \frac{d - 2}{2} \zeta(w)) \]
and
\[ \zeta(w) = f_{d/2 - 2}^{-1} (f_{d/2 - 2}^{-1}(w)). \]

Proof. Notice that due to the asymptotics of the Fermi functions [3] using
\[ w = \frac{2z}{\mu} \]
\[ z - R_\eta(z) \cdot \left( \frac{\mu}{2} \right)^{1+2/d} = \frac{\mu}{2} \frac{2z^2}{\mu} - \frac{\mu(d-2)}{4} \zeta(2z/\mu) \]
\[ = \frac{\mu w - d/2 \zeta(w)}{2w^{1+2/d}} \leq \frac{\mu}{2} C. \]
Indeed using the estimates from [3] or [27] we have that
\[ f_\alpha(w) \sim \frac{1}{\alpha + 1} w^{\alpha + 1}, w \sim \infty \]
while
\[ f_\alpha(w) \sim \Gamma(\alpha + 1) \exp(w), w \sim -\infty, \]
hence up to an unimportant positive constant we get
\[ \zeta(w) \sim w^{1 - 2/d}, w \sim \infty \]
and with an exact coefficient we have the asymptotics
\[ \zeta(w) \sim \Gamma(d/2 - 1) w, w \sim -\infty. \]
Recall the relation between constants that appear above to agree behavior of the functions at \( \infty \)
\[ \eta \mu^{2/d} = 2d^{2/d - 1}. \]
Hence \( \mu \to \infty \) when \( \eta \to 0^+ \).

The motivation for considering the pressure \( p \) in the model equation
\[ \rho_t = \nabla \cdot N \left( \nabla p + \rho \nabla \Delta^{-1} \rho \right), \]
in the form (7) with the specific dependence on the temperature \( \theta \), the density \( \rho \) and the dimension of the ambient space \( d \) reading
\[ p(\theta, \rho) = \theta^{d/2 + 1} P(\rho \theta^{-d/2}) \]
with some given \( P \) function (we drop dependence on \( \eta \)) is threefold. First of all one can for \( zH'(z) = P'(z) \) with \( z = \rho \theta^{-d/2} \) establish the entropy formula
\[ \mathcal{W} = \int_{B(0,1)} \left( \rho H(\rho \theta^{-d/2}) - \left( \frac{d}{2} + 1 \right) \theta^{d/2} P(\rho \theta^{-d/2}) \right) \]
due to this assumption on the pressure form, cf. [2]. Then the number of astrophysically motivated examples can be found as: Maxwell–Boltzmann, Fermi–Dirac, Bose–Einstein or polytropic statistics modeling clouds of particles, galaxies or stars. Finally, some monoatomic gases require this assumption which can be found in [13, 14, 16, 17, 23]. To this end we recall for \( d = 3 \) Maxwell’s equation with kinetic internal energy per molecule \( e \)
\[ \rho^2 e_\rho = p - \theta p_\theta. \]
While for monoatomic gas the relation holds
\[ 3p = 2\rho e. \]

Hence
\[ 3p\rho = 2\rho + 2\rho e. \]

plugged into Maxwell’s equation derived from the Gibb’s relation (cf. [16]) yields
\[ 2\rho - 2\theta p\theta = 2\rho^2\epsilon - 3\rho p\rho - 2\epsilon\rho = 3\rho p\rho - 3\rho. \]

This gives the linear first order partial differential equation
\[ 5p = 3\rho p + 2\theta p\theta \]

that can be solved with characteristics i.e. the system of equations
\[ \rho' = 3\rho, \theta' = 2\theta, p' = 5p \]

with two first integrals of the form
\[ p\theta^{-5/2}, \rho\theta^{-3/2}. \]

This yields the solution in the implicit form
\[ \Phi(p\theta^{-5/2}, \rho\theta^{-3/2}) = 0 \]

or explicit form
\[ p\theta^{-5/2} = P(\rho\theta^{-3/2}) \]

giving
\[ p = \theta^{5/2}P(\rho\theta^{-3/2}). \]

In higher dimension replacing 3 with \(d\) would yield the corresponding formula
\[ p = \theta^{d/2+1}P(\rho\theta^{-d/2}). \]

5. **Open problems and possible extensions.** One can consider the Dirichlet boundary value problem with elliptic equation
\[ \Delta\phi(u) = \rho(u) = H^{-1}_\eta(c - \phi(u)) \]

where the constant \(c\) is chosen so that the mass constraint holds
\[ \int_{B(0,1)} \rho(u)du = \int_{B(0,1)} H^{-1}_\eta(c - \phi(u))du = M. \]

The entropy can be used in dimensions \(d = 3\) with any mass \(M > 0\) or \(d = 4\) and the mass \(M\) sufficiently small to obtain the minimizer solving the related Euler-Lagrange equation. To be more specific the dual approach, cf. [25], uses the neg-entropy functional
\[ V = \int_{B(0,1)} \left( \rho H(\rho\theta^{-d/2}) - \theta^{d/2}P(\rho\theta^{-d/2}) + \frac{1}{2\theta} \rho\Delta^{-1} \rho \right) \]

over the space of integrable functions \(\rho \in L^{1+2/d}\). The functional is coercive and can be decomposed into compact and continuous part and lower–semicontinuous and convex part thus making the direct approach feasible to yield the existence of minimizer. It seems that the results of [4, 5, 6] can be used to get the continuity of the set of minimizers at least for sufficiently small mass \(M > 0\). The only obstacle is that the limiting functional is defined over the space of \(\rho \log \rho\) integrable functions as \(\eta \to 0^+\) while for \(\eta > 0\) one can consider the functional in a smaller \(L^{1+2/d}\) space.

Moreover, one can consider with necessary modifications the following nonlinearities
• $R_\eta(z) = \frac{\mu(d-2)}{4} g_{d/2-2}(g_{d/2-1}(2z/\mu))$ in the Bose–Einstein model with Bose functions $g_\alpha$ defined by

$$g_\alpha(z) = \int_0^\infty \frac{x^\alpha}{1 - \exp(x - z)} dx$$

requiring some limits for the density, or rather the ratio $\rho/\theta^{d/2}$,

• $R$ as in the classical King’s model, cf. [9], which has the intermediate form between Maxwell–Boltzmann and Fermi–Dirac cases.

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