Dark energy in scalar-vector-tensor theories

Ryotaro Kase and Shinji Tsujikawa

Department of Physics, Faculty of Science, Tokyo University of Science,
1-3, Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan
E-mail: r.kase@rs.tus.ac.jp, shinji@rs.kagu.tus.ac.jp

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Abstract. The scalar-vector-tensor theories with second-order equations of motion can accommodate both Horndeski and generalized Proca theories as specific cases. In the presence of a perfect fluid, we study the cosmology in such a most general scheme of scalar-vector-tensor theories with parity invariance by paying particular attention to the application to dark energy. We obtain a closed-form expression of the background equations of motion by using coefficients appearing in the second-order action of scalar perturbations. We also derive conditions for the absence of ghost and Laplacian instabilities of tensor and vector perturbations and show that the existence of matter does not substantially modify the stabilities of dynamical degrees of freedom in the small-scale limit. On the other hand, the sound speed of scalar perturbations is affected by the presence of matter. Employing the quasi-static approximation for scalar perturbations deep inside the sound horizon, we derive analytic expressions of Newtonian and weak lensing gravitational potentials as well as two scalar perturbations arising from the scalar and vector fields. We apply our general framework to dark energy theories with the tensor propagation speed equivalent to the speed of light and show that the observables associated with the growth of matter perturbations and weak lensing potentials are generally affected by intrinsic vector modes and by interactions between scalar and vector fields.

Keywords: dark energy theory, modified gravity

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1 Introduction

Since the first discovery of late-time cosmic acceleration in 1998 [1, 2], the origin of dark energy has not been identified yet. The cosmological constant is overall consistent with the current observational data, but there is still a discrepancy between the values of today’s Hubble constant $H_0$ constrained from the Cosmic Microwave Background (CMB) data [3, 4] and from local measurements at low redshifts [5]. In dynamical models of dark energy whose equation of state $w_{\text{DE}}$ varies in time, there are some possibilities for relaxing this tension of $H_0$ [6–8].

The minimally coupled scalar fields like quintessence [9–14] and k-essence [15–17] predict the dynamical dark energy equation of state $w_{\text{DE}}$ larger than $-1$. The observational data of CMB combined with the data of baryon acoustic oscillations and type Ia supernovae have allowed not only $w_{\text{DE}}$ larger than $-1$ but also the region $w_{\text{DE}} < -1$ [18–20]. The latter region can be realized by the existence of a scalar or vector field coupled to gravity [21–25]. Indeed, it was shown that dark energy models based on extended scalar Galileons [26]
or generalized Proca theories \[27\] can reduce the tension of the Λ-cold-dark-matter model mentioned above \[28, 29\].

For a scalar field \(\phi\) coupled to gravity, Horndeski theories \[30\] are the most general scalar-tensor theories with second-order equations of motion \[31–33\], which accommodate (extended) Galileons \[26, 34–36\] as specific cases. The application of Horndeski theories to dark energy has been extensively carried out in the literature \[37–47\]. The modification of gravity from General Relativity (GR) gives rise to the speed of gravitational waves \(c_t\) which is not necessarily equivalent to that of light \(c\). On the other hand, the detection of the gravitational-wave event GW170817 \[48\] from a neutron star merger together with the gamma-ray burst GRB 170817A \[49\] constrained \(c_t\) to be very close to \(c\) \[50\]. If we impose \(c_t = c\) exactly and do not allow the tuning between functions, the Lagrangian of Horndeski theories needs to be of the form \[L = G_2(\phi, X_1) + G_3(\phi, X_1)\Box \phi + G_4(\phi)R\] \[51–59\], where \(G_2, G_3\) are functions of \(\phi\) and \(X_1 = -\partial_\mu \phi \partial^\mu \phi/2\), \(G_4\) is a function of \(\phi\), and \(R\) is the Ricci scalar. The Brans-Dicke theory \[60\], \(f(R)\) gravity \[61–64\], and cubic Galileons \[34–36\] belong to the theories with \(c_t = c\).

For a vector field \(A_\mu\) coupled to gravity with broken U(1) gauge symmetry, generalized Proca theories \[65–69\] are the most general vector-tensor theories with second-order equations of motion (see refs. \[70, 71\] for earlier works). If we apply generalized Proca theories to cosmology, the existence of vector derivative and nonminimal couplings to gravity can lead to the late-time cosmic acceleration with a temporal vector component \(A_0[27]\). Similar mechanisms of the Universe acceleration were also advocated in refs. \[67, 72–74\]. Dark energy models in generalized Proca theories can leave several interesting observational signatures such as the constant equation of state \(w_{DE} < -1\) in the matter era \[27\] and the possibility for realizing the cosmic growth rate slower than that in GR \[75\]. Moreover, the propagation of fifth forces in local regions of the universe can be suppressed under the operation of the Vainshtein mechanism \[76, 77\]. Imposing the constraint \(c_t = c\), the dependence of \(X_3 = -A_\mu A^\mu/2\) in quartic and quintic couplings \(G_4(X_3)\) and \(G_5(X_3)\) are not allowed, while all the other interactions including intrinsic vector modes are possible. This restricts the cosmic growth rate in the range larger than that in GR \[57\], but the evolution of \(w_{DE}\) mentioned above is still possible.

Horndeski and generalized Proca theories can be united in the form of scalar-vector-tensor (SVT) theories with second-order equations of motion \[78\]. In SVT theories with U(1) gauge symmetry, the longitudinal component of the vector field \(A_\mu\) does not propagate. Then, the dynamical degrees of freedom (DOFs) are the scalar field \(\phi\), two transverse vector modes associated with \(A_\mu\), and two tensor polarizations arising from the gravity sector. The Lagrangian of U(1)-invariant SVT theories was constructed in ref. \[78\], which was recently applied to the study of hairy black hole solutions and their stabilities \[79–82\]. The SVT theories with broken U(1) gauge symmetry give rise to the additional longitudinal propagation of \(A_\mu\), so there are six propagating DOFs in total. Moreover, the temporal vector component \(A_0\) contributes to the background cosmological dynamics besides the scalar field \(\phi\). If we apply such theories to inflation, for example, the standard single-field dynamics driven by \(\phi\) is modified by the auxiliary field \(A_0\) \[83\].

In the presence of new interactions arising in SVT theories with broken U(1) gauge symmetry, the authors of ref. \[83\] derived the second-order actions of tensor, vector, and scalar perturbations on the flat Friedmann-Lemaître-Robertson-Walker (FLRW) background to elucidate conditions for the absence of ghost and Laplacian instabilities in the small-scale limit. While these results can be directly applied to the inflationary epoch in which the
contribution of additional matter to the cosmological dynamics is neglected, this is not the case for the dynamics of late-time cosmic acceleration during which the energy densities of dark matter and baryons cannot be ignored relative to those of dark energy. In this paper, we consider SVT theories with broken U(1) gauge symmetry by implementing a perfect fluid matter described by a Schutz-Sorkin action \[84, 85\]. This gives rise to the additional scalar propagation, so there are three dynamical scalar DOFs in addition to two vector and two tensor propagating DOFs.

Besides the SVT interactions \( S_{\text{SVT}} \) studied in ref. \[83\], we take into account the Horndeski action \( S_{\text{HT}} \) to accommodate full parity-invariant SVT theories with second-order equations of motion. We derive the background equations of motion in a closed form and then expand the full SVT action in the presence of matter up to quadratic order in tensor, vector, and scalar perturbations. To test for SVT theories with the observations of large-scale structures and weak lensing, we analytically compute two gauge-invariant gravitational potentials as well as scalar perturbations arising from \( \phi \) and \( A_\mu \) by employing the so-called quasi-static approximation \[21, 42, 86\] for the modes deep inside the sound horizon.

Our general analysis encompasses both Horndeski and generalized Proca theories as specific cases. The extension of Horndeski and generalized Proca theories to the domain of SVT theories opens up a new window for the dynamics of dark energy and the cosmic growth history. At the background level, the interaction between scalar and vector fields affects the evolution of the dark energy equation of state. The growth of cosmological perturbations can be generally modified not only by scalar-vector interactions but also by intrinsic vector modes. We also study the case in which the condition \( c_t = c \) is imposed and discuss new features arising in Newtonian and weak lensing gravitational potentials.

Our paper is organized as follows. In section 2, we present the action of most general SVT theories with second-order equations of motion in the presence of matter. In section 3, we express the background equations of motion in a compact form by using coefficients arising in the second-order action of scalar perturbations. In section 4, we derive no-ghost conditions as well as the propagation speeds of tensor and vector perturbations in the small-scale limit. In section 5, we obtain the full scalar perturbation equations and clarify conditions for the absence of scalar ghost and Laplacian instabilities. In section 6, the analytic expressions of Newtonian and weak lensing gravitational potentials are derived under the quasi-static approximation to confront SVT theories with observations associated with the cosmic growth history. In section 7, we apply our general formulas of gravitational potentials for the SVT theories in which the speed of gravity is equivalent to \( c \). Section 8 is devoted to conclusions.

In what follows, we use the natural unit \( c = 1 \).

## 2 SVT theories with broken U(1) gauge invariance

The SVT theories with broken U(1) gauge symmetry \[78\] contain a scalar field \( \phi \) and a vector field \( A_\mu \) coupled to gravity. To describe kinetic terms of \( \phi \) and \( A_\mu \) and their interactions, we define

\[
X_1 = -\frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi, \quad X_2 = -\frac{1}{2} A^\mu \nabla_\mu \phi, \quad X_3 = -\frac{1}{2} A_\mu A^\mu, \tag{2.1}
\]

where \( \nabla_\mu \) represents a covariant derivative operator. We introduce a symmetric tensor \( S_{\mu\nu} \) constructed from \( A_\mu \), as

\[
S_{\mu\nu} = \nabla_\mu A_\nu + \nabla_\nu A_\mu, \tag{2.2}
\]
together with the antisymmetric field strength tensor \( F_{\mu\nu} \) and its dual \( \tilde{F}_{\mu\nu} \), as

\[
F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu, \quad \tilde{F}^\alpha_{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta},
\]

where \( \varepsilon^{\mu\nu\alpha\beta} \) is the anti-symmetric Levi-Civita tensor obeying the normalization \( \varepsilon^{\mu\nu\alpha\beta} \varepsilon_{\mu\nu\alpha\beta} = -4! \). We define several Lorentz-invariant quantities associated with intrinsic vector modes, as

\[
F^i = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad Y_1 = \nabla_\mu \phi \nabla_\nu \phi F^{\mu\nu} F^{\nu}_\alpha, \quad Y_2 = \nabla_\mu \phi A_\nu F^{\mu\nu} F^{\nu}_\alpha,
\]

\[
Y_3 = A_\mu A_\nu F^{\mu\nu} F^{\nu}_\alpha,
\]

which vanish by taking the scalar limit \( A_\mu \to \nabla_\mu \pi \).

We consider the SVT interactions described by the action \([78]\):

\[
S_{\text{SVT}} = \int d^4 x \sqrt{-g} \sum_{n=2}^{6} \mathcal{L}^{(n)}_{\text{SVT}},
\]

with the Lagrangians

\[
\mathcal{L}^{(2)}_{\text{SVT}} = f_2(\phi, X_1, X_2, X_3, F, Y_1, Y_2, Y_3),
\]

\[
\mathcal{L}^{(3)}_{\text{SVT}} = f_3(\phi, X_3) \rho^{\mu\nu} S_{\mu\nu} + \tilde{f}_3(\phi, X_3) A_\mu A_\nu S_{\mu\nu},
\]

\[
\mathcal{L}^{(4)}_{\text{SVT}} = f_4(\phi, X_3) R + f_4(\phi, X_3) \left[ (\nabla \mu A^\mu)^2 - \nabla_\mu A_\nu \nabla^\mu A^\nu \right],
\]

\[
\mathcal{L}^{(5)}_{\text{SVT}} = f_5(\phi, X_3) G^{\mu\nu} \nabla_\mu A_\nu - \frac{1}{6} f_5(\phi, X_3) \left[ (\nabla \mu A^\mu)^3 - 3 \nabla_\mu A^\mu \nabla_\rho A_\nu \nabla^\sigma A^\rho + 2 \nabla_\rho A_\sigma \nabla^\mu A^\nu \nabla^\sigma A^\mu \right] + M^{\mu\rho\nu\sigma} \nabla_\mu \phi \nabla_\nu \phi + N^{\mu\nu}_5 S_{\mu\nu},
\]

\[
\mathcal{L}^{(6)}_{\text{SVT}} = f_6(\phi, X_1) L^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} + M_6^{\mu\nu\alpha\beta} \nabla_\mu \phi \nabla_\nu \phi \nabla_\alpha \phi + \tilde{f}_6(\phi, X_3) L^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} + N_6^{\mu\nu\alpha\beta} S_{\mu\alpha} S_{\nu\beta},
\]

where \( g \) is a determinant of the metric tensor \( g_{\mu\nu} \), \( R \) and \( G^{\mu\nu} \) are the Ricci scalar and the Einstein tensor, respectively, and \( L^{\mu\nu\alpha\beta} \) is the double dual Riemann tensor defined by

\[
L^{\mu\nu\alpha\beta} = \frac{1}{4} \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma\delta} R_{\rho\sigma\gamma\delta},
\]

where \( R_{\rho\sigma\gamma\delta} \) is the Riemann tensor. The function \( f_2 \) depends on \( \phi, X_1, F, Y_1 \), where the subscript represents \( i = 1, 2, 3 \). The functions \( f_3, f_4, f_5, \tilde{f}_6 \) are dependent on \( \phi \) and \( X_3 \), while \( f_6 \) is a function of \( \phi \) and \( X_1 \). For partial derivatives with respect to \( \phi, X_1, F, Y_1 \), we use the notations like \( f_4, X_3 \equiv \partial f_4 / \partial X_3 \).

The 2-rank tensors \( M_5^{\mu\nu} \) and \( N_5^{\mu\nu} \) in \( \mathcal{L}^{(5)}_{\text{SVT}} \), which are associated with intrinsic vector modes, are given by

\[
M_5^{\mu\nu} = G_5^{\mu\nu} F^{\mu\nu} + N_5^{\mu\nu},
\]

with

\[
G_5^{\mu\nu} = g_{51}(\phi, X_1) \rho^{\mu\nu} + g_{52}(\phi, X_1) \nabla_\rho \phi \nabla_\sigma \phi + h_{53}(\phi, X_1) A_\mu A_\nu + h_{54}(\phi, X_1) A_\rho \nabla_\sigma \phi, \quad N_5^{\mu\nu} = G_5^{\mu\nu} F^{\mu\nu} + F_5^{\nu\sigma},
\]

\[
G_5^{\mu\nu} = h_{51}(\phi, X_1) \rho^{\mu\nu} + h_{52}(\phi, X_1) \nabla_\rho \phi \nabla_\sigma \phi + h_{53}(\phi, X_1) A_\mu A_\nu + h_{54}(\phi, X_1) A_\rho \nabla_\sigma \phi, \quad N_5^{\mu\nu} = G_5^{\mu\nu} F^{\mu\nu} + F_5^{\nu\sigma},
\]

\[
G_5^{\mu\nu} = h_{51}(\phi, X_1) \rho^{\mu\nu} + h_{52}(\phi, X_1) \nabla_\rho \phi \nabla_\sigma \phi + h_{53}(\phi, X_1) A_\mu A_\nu + h_{54}(\phi, X_1) A_\rho \nabla_\sigma \phi, \quad N_5^{\mu\nu} = G_5^{\mu\nu} F^{\mu\nu} + F_5^{\nu\sigma},
\]
where the effective metrics $G^{h_5}_{\mu \nu}$ and $G^b_{\mu \nu}$ contain possible combinations of $g_{\rho \sigma}, A_\mu,$ and $\nabla_\rho \phi$. The functions $h_{j5}$ and $h_{j5}$ (where $j = 1, 2, 3, 4$) depend on $\phi$ and $X_1, X_2, X_3$. For arbitrary curved backgrounds, the dependence of either $X_1$ or $X_3$ in $h_{5j}$ and $h_{5j}$ should appear dominantly to ensure that the temporal component of $A_\mu$ remains non-dynamical [78]. On the isotropic and homogenous cosmological background this dynamical property of $A_0$ does not manifest itself, so we do not restrict the $X_i$ dependence in the functions $h_{5j}$ and $h_{5j}$.

The Lagrangian $L^{(6)}_{\text{SVT}}$ corresponds to intrinsic vector modes that vanish in the scalar limit $A_\mu \to \nabla_\mu \pi$. In summary, the functional dependence of $F, Y_1, Y_2, Y_3$ in $f_2$ and the functions $h_{55}, h_{55}, f_6, f_5$ accommodate interactions of intrinsic vector modes.

In the action (2.5), we focused on the interactions invariant under the parity transformation $\mathcal{P}: \vec{x} \to -\vec{x}$. In other words, we did not take into account the dependence of parity-violating terms like $\tilde{F} = -F_{\mu \nu} \tilde{F}^{\mu \nu}/4$ in $f_2$. These parity-violating terms generate left-handed and right-handed helicity contributions to the vector perturbation equation, which makes the analysis more involved. The analysis containing parity-violating terms is left for a future separate work.

The action of scalar-tensor interactions, which corresponds to Horndeski theories [30, 32], is given by

$$S_{\text{ST}} = \int d^4 x \sqrt{-g} \sum_{n=3}^5 L_{\text{ST}}^{(n)},$$

with the Lagrangians

$$L_{\text{ST}}^{(3)} = G_3(\phi, X_1) \Box \phi,$$

$$L_{\text{ST}}^{(4)} = G_4(\phi, X_1) R + G_{4, X_1}(\phi, X_1) \left[ (\Box \phi)^2 - (\nabla_\mu \nabla_\nu \phi)(\nabla^\mu \nabla^\nu \phi) \right],$$

$$L_{\text{ST}}^{(5)} = G_5(\phi, X_1) G_{\mu \nu} (\nabla^\mu \nabla^\nu \phi)$$

$$- \frac{1}{6} G_{5, X_1}(\phi, X_1) \left[ (\Box \phi)^3 - 3(\Box \phi)(\nabla_\mu \nabla_\nu \phi)(\nabla^\mu \nabla^\nu \phi) + 2(\nabla_\mu \nabla_\alpha \phi)(\nabla^\mu \nabla_\beta \phi)(\nabla^\beta \nabla_\alpha \phi) \right],$$

where $\Box \phi = g^{\mu \nu} \nabla_\mu \nabla_\nu \phi$. The quadratic Lagrangian $L_{\text{ST}}^{(2)}$ of the form $G_2(\phi, X_1)$ is already included in the function $f_2$ of the SVT Lagrangian $L_{\text{ST}}^{(2)}$.

For the matter sector, we take into account a perfect fluid minimally coupled to gravity. This can be described by the Schutz-Sorkin action [27, 84, 85]:

$$S_m = -\int d^4 x \left[ \sqrt{-g} \rho_m(n) + J^\mu (\partial_\mu \ell + A_1 \partial_\mu B_1 + A_2 \partial_\mu B_2) \right],$$

where $J^\mu$ and $\ell$ describe scalar modes, while the contributions of vector modes are encoded in $A_{1,2}$ and $B_{1,2}$. The Schutz-Sorkin action has an advantage of appropriately dealing with vector perturbations on the FLRW background. The fluid density $\rho_m$ is a function of its number density $n$ defined by

$$n = \sqrt{\frac{J^\mu J^\nu g_{\mu \nu}}{g}}.$$
Varying the action (2.16) with respect to $J^\mu$, it follows that

$$u_\mu \equiv \frac{J_\mu}{n \sqrt{-g}} = \frac{1}{\rho_{m,n}} (\partial_\mu \ell + A_1 \partial_\mu B_1 + A_2 \partial_\mu B_2),$$

(2.18)

where $u_\mu$ is the normalized four-velocity, and $\rho_{m,n}$ is defined by $\rho_{m,n} = \partial \rho_m / \partial n$.

In this paper, we study the cosmology of full SVT theories with parity invariance given by the action

$$S = S_{SVT} + S_{ST} + S_m.$$  

(2.19)

Our analysis can accommodate both Horndeski and generalized Proca theories as specific cases. The Horndeski theories are characterized by the functions

$$f_2 = f_2(\phi, X_1), \quad f_3 = f_3(X_3), \quad f_5 = f_5(X_3),$$

$$f_5 = f_5(X_3), \quad f_6 = f_6(X_3), \quad h_{5j} = h_{5j} = 0.$$  

(2.20)

The generalized Proca theories, which are given by the Lagrangians (2.2)-(2.6) of ref. [75], correspond to

$$f_2 = f_2(X_3, F, Y_3), \quad f_3 = f_3(X_3), \quad f_5 = f_5(X_3), \quad f_6 = f_6(X_3),$$

$$h_{5j} = 0, \quad h_{51} = -\frac{1}{2} g_5(X_3), \quad h_{52} = h_{53} = h_{54} = 0, \quad G_3 = G_4 = G_5 = 0.$$  

(2.21)

We note that our SVT theories consist of one scalar field $\phi$ and one vector field $A_\mu$ coupled to gravity, so they do not accommodate scalar bi-galileons and their generalizations studied in refs. [87–90].

We will discuss how the background and scalar perturbation equations of motion in these two particular theories can be recovered in our general framework.

### 3 Background equations of motion

To derive the background equations of motion on the flat FLRW background, we take the line element

$$ds^2 = -N^2(t)dt^2 + a^2(t)\delta_{ij}dx^i dx^j,$$

(3.1)

where the lapse $N$ and scale factor $a$ depend on the cosmic time $t$. We consider a time-dependent scalar field $\phi(t)$ and a vector field $A_\mu(t)$ with a nonvanishing temporal component $A_0$ of the form $A_\mu(t) = (A_0(t) N(t), 0, 0, 0)$. Then, the quantities defined in eq. (2.1) reduce to

$$X_1 = \frac{\dot{\phi}^2}{2N^2}, \quad X_2 = \frac{\dot{\phi} A_0}{2N}, \quad X_3 = \frac{A_0^2}{2},$$

(3.2)

where a dot represents a derivative with respect to $t$. All the quantities associated with intrinsic vector modes, like those defined in eq. (2.4), vanish on the flat FLRW background.

From eq. (2.17), the temporal component $J^0$ corresponds to the total fluid number $N_0$, i.e.,

$$J^0 = N_0 = n_0 a^3,$$

(3.3)

where $n_0$ is the background value of $n$. On the background (3.1) the vector modes do not contribute to the matter action (2.16), so that

$$\bar{S}_m = - \int d^4x \left( N a^3 \tilde{\rho}_m + n_0 a^3 \tilde{\ell} \right),$$

(3.4)

where a bar is used to represent background values.
3.1 Full SVT theories

On the flat FLRW spacetime (3.1), we compute the action (2.5) and vary it with respect to \( N, a, \phi, \) and \( A_0, \) and finally set \( N = 1. \) Then, the resulting background equations of motion are

\[
6(f_4 + G_4) H^2 + f_2 - \ddot{\phi}^2 f_{2,X_1} - \frac{1}{2} \dddot{\phi} A_0 f_2 X_2 + A_0^2 \left( 3H \dddot{\phi} G_{3,X_1} - G_{3,\phi} \right) + 6H \left( \dddot{\phi} f_{4}\phi - HA_0^2 f_{4,X_3} \right) + 6H \dddot{\phi} \left( G_{4,\phi} + \dddot{\phi}^2 G_{4,X_1} - 2H \dddot{\phi} G_{4,X_1} - H \dddot{\phi}^3 G_{4,X_1,X_1} \right) + 2A_0 H^2 \left( 3\dddot{\phi} f_{5}\phi - HA_0^2 f_{5,X_3} \right) + H^2 \dddot{\phi}^2 \left( 9G_{5,\phi} + 3\dddot{\phi}^2 G_{5,X_1,\phi} - 5H \dddot{\phi} G_{5,X_1} - H \dddot{\phi}^3 G_{5,X_1,X_1} \right) = \rho_m, \quad (3.5)
\]

\[
2q_0 \dddot{H} - D_0 \dddot{\phi} + \frac{w_2}{A_0} \dddot{A}_0 + D_7 \dddot{\phi} = -\rho_m - P_m, \quad (3.6)
\]

\[
3D_6 \dddot{H} + 2D_1 \dddot{\phi} - D_8 \dddot{A}_0 + 3D_7 H - D_9 A_0 - D_5 = 0, \quad (3.7)
\]

\[
2 \left( f_{2,X_3} + 6H^2 f_{4,X_3} - 6H \dddot{\phi} f_{4,X_3,\phi} \right) A_0 \\
- 2 \left( 6H f_{5,X_3} + 6H f_{5,X_3} - 3H \dddot{\phi} f_{5,X_3} + 3H^2 \dddot{\phi} f_{5,X_3,\phi} \right) A_0 \\
+ 12H^2 f_{4,X_3} A_0 + 2H^2 f_{5,X_3} A_0 + \left( f_{2,X_3} + 4f_{3,\phi} - 6H^2 f_{3,\phi} \right) \dddot{\phi} = 0, \quad (3.8)
\]

where \( H = \dot{a}/a \) is the Hubble expansion rate. The matter pressure is given by

\[
P_m = n_0 \rho_{m,n} - \rho_m, \quad (3.9)
\]

where we used the relation \( \ddot{\ell} = -\rho_{m,n} \) and omitted the bar from the background quantities. The quantity \( q_t \) in eq. (3.6), which is associated with the no-ghost condition of tensor perturbations discussed later, is given by

\[
q_t = 2f_4 + 2G_4 - 2A_0^2 f_{4,X_3} - 2\dddot{\phi}^2 G_{4,X_1} + A_0 \dddot{\phi} f_{5,\phi} - HA_0^2 f_{5,X_3} + \dddot{\phi}^2 G_{5,\phi} - H \dddot{\phi}^3 G_{5,X_1}. \quad (3.10)
\]

The coefficients \( D_1, D_5, D_6, D_7, D_8, D_9 \) and \( w_2 \) in eqs. (3.6) and (3.7) are presented in appendix A. As we see later, they also appear as coefficients in the second-order action of scalar perturbations. The quantity \( w_2 \) is proportional to \( A_0, \) so the term \( w_2/A_0 \) in eq. (3.6) is not divergent in the limit that \( A_0 \to 0. \) We note that eq. (3.6) has been derived by varying the action with respect to \( a \) and then subtracting the corresponding equation of motion from eq. (3.5). The matter sector satisfies the continuity equation

\[
\dot{\rho}_m + 3H (\rho_m + P_m) = 0, \quad (3.11)
\]

which are consistent with eqs. (3.5)–(3.8).

Taking the time derivative of eq. (3.8), we obtain

\[
- \frac{3w_2}{A_0} \dddot{H} - D_8 \dddot{\phi} + \frac{2w_5}{A_0^2} \dddot{A}_0 - D_9 \dddot{\phi} = 0, \quad (3.12)
\]

where \( w_5 \) is given in appendix A. As long as the condition

\[
\mathcal{D} \equiv 2 \left( 4D_1 q_t w_5 + 3D_1 w_2^2 + 3D_6^2 w_5 - A_0^2 D_8^2 q_t - 3A_0 D_6 D_8 w_2 \right) \neq 0 \quad (3.13)
\]
is satisfied, the dynamical system is closed. In other words, we can solve eqs. (3.6), (3.7), and (3.12) for $\dot{H}$, $\dot{\phi}$, and $\dot{A}_0$ in the forms:

$$
\dot{H} = \frac{1}{D}[A_0^2 D_8(D_6 D_9 \dot{\phi} + D_7 D_9 \dot{\phi} - D_9 w_5) - A_0(2D_1 D_9 \phi w_2 - 3D_7 D_9 H w_2 + D_3 D_8 w_2 - 2D_6 D_9 w_5) - 2w_5(2D_1 D_7 \phi \dot{\phi} + 3D_6 D_7 H - D_5 D_6) + (A_0^2 D_6^2 - 4A_1 w_5)(\rho_m + P_m)], 
$$

(3.14)

$$
\dot{\phi} = \frac{1}{D}[2A_0^2 D_8 D_9 \dot{q}_t \dot{\phi} + A_0(3D_6 D_9 \phi w_2 - 3D_7 D_8 \dot{\phi} w_2 + 4D_9 \dot{q}_t w_5 + 3D_9 w_5^2) + 2w_5(3D_6 D_7 \phi - 6D_7 H \dot{q}_t + 2D_5 q_t) + 3w_5^2(D_5 - 3D_7 H) - 3(A_0 D_8 w_2 - 2D_6 w_5)(\rho_m + P_m)],
$$

(3.15)

$$
\dot{A}_0 = \frac{A_0}{D}[A_0 D_9(2A_0 D_8 q_t + 4D_1 q_t \dot{\phi} + 3D_6^2 \dot{\phi}) + A_0(3D_6 D_7 D_8 \phi - 6D_7 D_8 H q_t + 2D_5 D_8 q_t + 3D_6 D_9 w_2) - 3w_2(2D_1 D_7 \phi + 3D_6 D_7 H - D_5 D_6) + 3(A_0 D_6 D_8 - 2D_1 w_2)(\rho_m + P_m)].
$$

(3.16)

The initial conditions of $H$, $\dot{\phi}$, $A_0$ should be chosen to be consistent with eqs. (3.5) and (3.8). In section 5, we show that the determinant (3.13) is related to a quantity $q_s$ associated with the no-ghost condition of scalar perturbations.

We introduce the dark energy density $\rho_{DE}$ and pressure $P_{DE}$ in the forms:

$$
\rho_{DE} = \frac{3H^2}{8\pi G} - \rho_m, \tag{3.17}
$$

$$
P_{DE} = -\frac{2\dot{H} + 3H^2}{8\pi G} - P_m, \tag{3.18}
$$

where $G$ is the Newton gravitational constant. Then, the dark sector obeys the continuity equation

$$
\dot{\rho}_{DE} + 3H(\rho_{DE} + P_{DE}) = 0, \tag{3.19}
$$

where we used eq. (3.11). We can explicitly compute $\rho_{DE}$ and $P_{DE}$ by solving eqs. (3.5)–(3.6) for $\rho_m, P_m$ and substitute them into eqs. (3.17)–(3.18). To calculate $\dot{H}, \dot{\phi}, \dot{A}_0$ appearing in the expression of $P_{DE}$ explicitly, we need to employ eqs. (3.14)–(3.16).

### 3.2 Horndeski theories

In Horndeski theories the temporal vector component $A_0$ is absent, so eqs. (3.8) and (3.12) are redundant. In this case, as long as the condition

$$
D_{Ho} \equiv 4D_1 q_t + 3D_6^2 \neq 0 \tag{3.20}
$$

is satisfied, we can solve eqs. (3.6) and (3.7) for $\dot{H}$ and $\dot{\phi}$, as

$$
\dot{H} = -\frac{1}{D_{Ho}} \left[2D_1 D_7 \dot{\phi} - D_6(D_5 - 3D_7 H) + 2D_1(\rho_m + P_m)\right], \tag{3.21}
$$

$$
\dot{\phi} = \frac{1}{D_{Ho}} \left[3D_6 D_7 \dot{\phi} + 2(D_5 - 3D_7 H)q_t + 3D_6(\rho_m + P_m)\right]. \tag{3.22}
$$

The determinant $D_{Ho}$ is proportional to a quantity $q_{s, Ho}$ associated with the no-ghost condition of scalar perturbations discussed later in section 5.
3.3 Generalized Proca theories

In generalized Proca theories the scalar field $\phi$ is absent, in which case eq. (3.7) is redundant. Provided that the condition
\[ D_{GP} \equiv 4w_5 q_t + 3w_2^2 \neq 0 \] (3.23)
is satisfied, we can solve eqs. (3.6) and (3.12) for $\dot{H}$ and $\dot{A}_0$, as
\[ \dot{H} = -\frac{2w_5}{D_{GP}} (\rho_m + P_m) , \] (3.24)
\[ \dot{A}_0 = -\frac{3w_2}{D_{GP}} A_0 (\rho_m + P_m) . \] (3.25)

The determinant $D_{GP}$ is related to a quantity $q_{s,GP}$ associated with the no-ghost condition of scalar perturbations.

4 Tensor and vector perturbations

We proceed to the study of linear cosmological perturbations in SVT theories given by the action (2.19). In doing so, we decompose the perturbations into tensor, vector, and scalar modes on the flat FLRW background [91, 92]. We consider the perturbed line element in the flat gauge:
\[ ds^2 = -(1 + 2\alpha) dt^2 + 2(\partial_i \chi + V_i) dt dx^i + a^2(t) (\delta_{ij} + h_{ij}) dx^i dx^j , \] (4.1)
where $\alpha$ and $\chi$ are scalar perturbations, with the notation $\partial_i \chi \equiv \partial \chi / \partial x^i$. The vector perturbation $V_i$ obeys the transverse condition
\[ \partial^i V_i = 0 , \] (4.2)
whereas the tensor perturbation $h_{ij}$ satisfies the transverse and traceless conditions
\[ \partial^i h_{ij} = 0 , \quad h_i^i = 0 . \] (4.3)

For the scalar field $\phi$ and the vector field $A^\mu$, we decompose them into the background and perturbed parts, as
\[ \phi = \tilde{\phi}(t) + \delta \phi , \] (4.4)
\[ A^0 = -\tilde{A}_0(t) + \delta A , \quad A_i = \partial_i \psi + Z_i , \] (4.5)
where $\delta \phi, \delta A, \psi$ are scalar perturbations, and $Z_i$ is the vector perturbation obeying
\[ \partial^i Z_i = 0 . \] (4.6)

In the following, we omit the bar from the background quantities.

For the matter sector, the temporal and spatial components of $J^\mu$ in eq. (2.16) are decomposed as
\[ J^0 = N_0 + \delta J , \quad J^k = \frac{1}{a^2(t)} \delta^{ki} (\partial_i \delta j + W_i) , \] (4.7)
where $\delta J$ and $\delta j$ correspond to scalar perturbations, and $W_i$ is the vector perturbation satisfying
\[ \partial^i W_i = 0 . \] (4.8)
Without loss of generality, we can choose the vector fields \( V_i, Z_i, W_i \) obeying the transverse conditions (4.2), (4.6), and (4.8) in the forms

\[
V_i = (V_1(t, z), V_2(t, z), 0) , \quad Z_i = (Z_1(t, z), Z_2(t, z), 0) , \quad W_i = (W_1(t, z), W_2(t, z), 0) ,
\]

whose nonvanishing components depend on \( t \) and the third spatial coordinate \( z \).

The scalar quantity \( \ell \) is expressed as

\[
\ell = - \int^t \rho_{m,n}(\tilde{t}) d\tilde{t} - \rho_{m,n} v ,
\]

where \( v \) is the velocity potential. The quantities \( \mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2 \), which are related to intrinsic vector modes, can be chosen as \([27, 85]\)

\[
\mathcal{A}_1 = \delta \mathcal{A}_1(t, z) , \quad \mathcal{A}_2 = \delta \mathcal{A}_2(t, z) , \quad \mathcal{B}_1 = x + \delta \mathcal{B}_1(t, z) , \quad \mathcal{B}_2 = y + \delta \mathcal{B}_2(t, z) ,
\]

where \( \delta \mathcal{A}_{1,2} \) and \( \delta \mathcal{B}_{1,2} \) are perturbed quantities that depend on \( t \) and \( z \). Substituting eq. (4.10) into (2.18), the spatial component of \( u_\mu \) is expressed in the form

\[
u_i = - \partial_i v + v_i ,
\]

where the vector components \( v_i \) (with \( i = 1, 2 \)) are related to \( \delta \mathcal{A}_i \), as

\[
\delta \mathcal{A}_i = \rho_{m,n} v_i .
\]

The transverse condition \( \partial^i v_i = 0 \) is satisfied for \( \mathcal{A}_i \) given in eq. (4.11).

### 4.1 Tensor perturbations

We first compute the second-order action of tensor perturbations \( h_{ij} \). To satisfy the transverse and traceless conditions (4.3), we choose nonvanishing components of \( h_{ij} \) in the forms

\[
h_{11} = h_1(t, z) , \quad h_{22} = - h_1(t, z) , \quad h_{12} = h_{21} = h_2(t, z) ,
\]

where the functions \( h_1 \) and \( h_2 \) characterize two polarization states of the tensor sector.

Expanding the action \( S_{SVT} + S_{ST} \) up to second order in perturbations and integrating it by parts, the quadratic action contains the terms \( \hat{h}_i^2, (\partial h_i)^2, h_i^2 \) (where \( i = 1, 2 \)). The second-order action of \( S_m \) associated with the tensor sector can be written in the form

\[
(S_m^{(2)})_t = - \int \int d^3 x [(\sqrt{-g})^2 \rho_m + \sqrt{-g} \rho_{m,n} \partial n] \]

where \( (\sqrt{-g})^2 = -a^3 (h_1^2 + h_2^2)/2 \) and \( \partial n = n_0 (h_1^2 + h_2^2)/2 \) with \( \sqrt{-g} = a^3 \). Then, it follows that

\[
(S_m^{(2)})_t = - \int \int d^3 x \sum_{i=1}^{2} \frac{1}{2} a^3 P_m h_i^2 ,
\]

where \( P_m \) is the matter pressure given by eq. (3.9). Taking into account the contribution \( S_m^{(2)} \) to the second-order action of \( S_{SVT} + S_{ST} \) and eliminating \( P_m \) by using the background eqs. (3.5) and (3.6), the terms proportional to \( h_i^2 \) identically vanish. Then, the second-order action of tensor perturbations yields

\[
S_t^{(2)} = \int \int d^3 x \sum_{i=1}^{2} \frac{a^3}{4} \epsilon_{ij} \left[ \hat{h}_i^2 + \frac{a^2}{2} (\partial h_i)^2 \right] ,
\]

---
where \( q_t \) is given by eq. (3.10), and

\[
c_t^2 = \frac{2f_4 + 2G_4 - A_0 \phi f_{5,\phi} - \dot{\phi} A_0^2 f_{5,X_3} - \phi^2 G_{5,\phi} - \phi^2 \dot{\phi} G_{5,X_1}}{2f_4 + 2G_4 - 2A_0^2 f_{4,X_3} - 2\phi^2 G_{4,X_1} + A_0 \phi f_{5,\phi} - H A_0^2 f_{5,X_3} + \phi^2 G_{5,\phi} - H \phi^2 G_{5,X_1}}.
\]

(4.17)

The quantity \( q_t \) is associated with the no-ghost condition of tensor perturbations, while \( c_t^2 \) is the propagation speed squared of gravitational waves relevant to the Laplacian instability. To avoid the ghost and Laplacian instabilities, we require that

\[
q_t > 0, \quad c_t^2 > 0.
\]

(4.18)

Taking the limits \( f_4, f_{4,X_3}, f_{5,\phi}, f_{5,X_3} \to 0 \) in eqs. (3.10) and (4.17), we recover the values of \( q_t \) and \( c_t^2 \) in Horndeski theories [26, 32]. The values of \( q_t \) and \( c_t^2 \) in generalized Proca theories [27] also follow by taking the limits \( G_4, G_{4,X_1}, G_{5,\phi}, G_{5,X_1}, f_{5,\phi} \to 0 \).

Applying the SVT theories to today’s universe, there is a tight bound on \( c_t \) constrained from the GW170817 event [48] together with the electromagnetic counterpart [49]:

\[
-3 \times 10^{-15} \leq c_t - 1 \leq 7 \times 10^{-16}.
\]

(4.19)

If we strictly demand \( c_t^2 = 1 \) in eq. (4.17), the SVT theories need to satisfy the condition

\[
2A_0^2 f_{4,X_3} - 2A_0 \phi f_{5,\phi} + A_0^2 \left( H A_0 - \dot{\phi} \right) f_{5,X_3} + 2\phi^2 G_{4,X_1} - 2\phi^2 G_{5,\phi} + \phi^2 \left( H \phi^2 - \dot{\phi} \right) G_{5,X_1} = 0.
\]

(4.20)

If we consider the case in which each term on the left hand side of eq. (4.20) exactly vanishes without the cancellation between different terms, the couplings are constrained to be

\[
f_4 = f_4(\phi), \quad f_5 = \text{constant}, \quad G_4 = G_4(\phi), \quad G_5 = \text{constant}.
\]

(4.21)

The Lagrangians up to cubic order as well as intrinsic vector modes like \( L_{SVT}^{(6)} \) do not affect the tensor propagation speed.

We can also consider cases in which some of nonvanishing terms on the left hand side of eq. (4.20) cancel each other. In scalar-tensor theories beyond Horndeski gravity, it is also possible to construct similar tuned cosmological models with \( c_t^2 = 1 \) [39,41,43,44]. In our case, one of the simplest examples consistent with eq. (4.20) is the functions \( f_{4,X_3} = 1/X_3 \) and \( G_{4,X_1} = -1/X_1 \) with constant couplings \( f_5 \) and \( G_5 \). Of course, the fact that \( c_t^2 = 1 \) alone does not guarantee the stability of theories against vector and scalar perturbations, so we need to confirm whether such theories satisfy all the stability criteria required for the cosmological viability. In section 7, we will consider SVT theories given by the couplings (4.21), leaving the analysis of more general cases with \( c_t^2 = 1 \) as a future work.

4.2 Vector perturbations

Let us proceed to the derivation of second-order action of vector perturbations. Expanding the matter action \( S_m \) in terms of intrinsic vector perturbations \( W_i, \delta A_i, \delta B_i \), and \( V_i \), the resulting second-order action is given by [27, 75]

\[
(S_m^{(2)})_v = \int dt d^3 x \sum_{i=1}^{2} \left[ \frac{1}{2a^2 N_0} \left\{ \rho_{m,n} \left( W_i^2 + N_0^2 V_i^2 \right) + N_0 \left( 2\rho_{m,n} V_i W_i - a^3 \rho_{m,n} V_i^2 \right) \right\} V_i \delta A_i \right] .
\]

(4.22)
\[
\delta A = C W, \quad v = V_i - a^2 \delta B_i, \quad \delta A_i = \rho_{m,n} v_i = C_i, \tag{4.25}
\]

where \( C_i \) (\( i = 1, 2 \)) are constants in time. After integrating out the perturbations \( W_i \) and \( \delta A_i \), the matter action (4.22) reduces to

\[
(S_m^{(2)})_{\nu} = \int dt d^3 x \sum_{i=1}^{2} \frac{a}{2} \left[ (\rho_m + P_m) v_i^2 - \rho_m V_i^2 \right], \tag{4.26}
\]

where \( v_i \) contains \( V_i \) through eq. (4.24).

Taking into account the contribution (4.26) to the second-order action of \( S_{SVT} + S_{ST} \) and using the background equations of motion, the quadratic action of vector perturbations reads

\[
S_v^{(2)} = \int dt d^3 x \sum_{i=1}^{2} \left[ \frac{a q_e}{4a^2} \dot{Z}_i^2 - \frac{1}{2a} \alpha_1 (\partial Z_i)^2 - \frac{a}{4} \alpha_2 Z_i^4 + \frac{1}{2a} \alpha_3 (\partial V_i) (\partial Z_i) \right.
\]

\[+ \frac{a}{4a^2} (\partial V_i)^2 + \frac{a}{2} (\rho_m + P_m) v_i^2 \], \tag{4.27}
\]

where

\[
q_e = f_{2,F} + 2 \dot{\phi}^2 f_{2,Y1} + 2 \dot{\phi} A_0 f_{2,Y2} + 2 A_0^2 f_{2,Y3} - 4H \left( \dot{\phi} h_{51} + 2A_0 \ddot{h}_{51} \right)
+ 8H^2 \left( f_6 + \ddot{f}_6 + \dot{\phi}^2 f_{6,X1} + A_0^2 \ddot{f}_{6,X3} \right), \tag{4.28}
\]

\[
\alpha_1 = f_{2,F} - 4A_0 h_{51} + 8 \left( H^2 + \dot{H} \right) \left( f_6 + \ddot{f}_6 \right) - 2 \dot{\phi} h_{51} + H \left[ 2 \dot{\phi} \left( \dot{\phi}^2 h_{52} - h_{51} + 4 \ddot{\phi} f_{6,X1} \right) \right.
\]

\[+ A_0 \left( 4 \ddot{h}_{51} - 2 \dot{\phi} \left( h_{51} - 2 \ddot{h}_{51} + 2 \ddot{h}_{52} - 4A_0 \ddot{f}_{6,X3} \right) \right) + 2 \ddot{\phi} A_0^2 (h_{53} + 2 \ddot{h}_{54}) + 4 A_0^3 \ddot{h}_{53} \right], \tag{4.29}
\]

\[
\alpha_2 = f_{2,X3} + 4 \dot{H} f_{4,X3} - 2 \left( A_0 + 3HA_0 \right) \left( \dot{f}_{3,X3} + \ddot{f}_3 \right) - 2 \dot{\phi} A_0 \ddot{f}_3 \phi
+ 2H (3H f_{4,X3} + 3HA_0^2 f_{4,X3,X3} + 2A_0 \dot{A}_0 f_{4,X3,X3} - \dot{\phi} f_{4,X3} \phi) \tag{4.30}
\]

\[+ H \left( H A_0 + 2 \dot{H} A_0 + 3H^2 A_0 \right) f_{5,X3} + H^2 A_0 \left( H A_0^2 f_{5,X3} + A_0 \dot{A}_0 f_{5,X3} - 2 \dot{\phi} f_{5,X3} \phi \right),
\]

\[
\alpha_3 = -2A_0 f_{4,X3} - HA_0^2 f_{5,X3} + \dot{\phi} f_{5,X3}. \tag{4.31}
\]

Apart from the last term of eq. (4.27), all the other terms in \( S_v^{(2)} \) are exactly the same as those derived for the theories with the action \( S_{SVT} \) alone [83]. This reflects the fact that the action \( S_{ST} \) of scalar-tensor theories does not give rise to any modification to the vector sector.

Varying the action (4.27) with respect to \( V_i \), we obtain

\[
\ddot{\phi}^2 (\alpha_3 Z_i + q_i V_i) = 2a^2 (\rho_m + P_m) v_i. \tag{4.32}
\]
On using eq. (4.25), the term on the right hand side of eq. (4.32) can be expressed as $2a^2n_0C_i$. Taking the small-scale limit\(^1\) in eq. (4.32) under the condition that $C_i$ do not depend on scales, it follows that $V_i \simeq -\alpha_3Z_i/q_v$. Since the last term of eq. (4.27) is irrelevant to the dynamics of vector perturbations in the small-scale limit, the action (4.27) reduces to

$$S^{(2)}_v \simeq \int dt d^3x \sum_{i=1}^{2} a^2 \left[ \frac{a^2}{2} \left( \frac{Z_i^2}{\alpha_3^2} - \frac{C_v^2}{a^2} (\partial Z_i)^2 - \frac{\alpha_2}{q_v} Z_i^2 \right) \right],$$

where

$$C_v^2 = \frac{2\alpha_1 q_t + \alpha_3^2}{2q_v q_v}.\quad (4.34)$$

Hence there are two dynamically propagating fields $Z_1$ and $Z_2$ in the vector sector. The ghost and Laplacian instabilities are absent under the conditions

$$q_v > 0, \quad c_v^2 > 0,\quad (4.35)$$

which are exactly the same as those derived for the action $S_{SVT}$ alone [83]. This means that neither the action $S_{SYT}$ of scalar-tensor theories nor the matter action $S_m$ changes the stability conditions of vector perturbations in the small-scale limit.

### 5 Scalar perturbations

For the scalar sector, there are metric perturbations $\alpha, \chi$, scalar-field perturbation $\delta \phi$, and perturbations $\delta A, \psi$ arising from the vector field. The matter perfect fluid also contains the scalar perturbations $\delta J, \delta j, v$. We introduce the matter density perturbation $\delta \rho_m$, as

$$\delta \rho_m = \frac{\rho_{m,n}}{a^3} \delta J.\quad (5.1)$$

Defining $\delta \rho_m$ in this way, the perturbation of the fluid number density $n$, expanded up to second order, yields

$$\delta n = \frac{\delta \rho_m}{\rho_{m,n}} - \frac{N_0^2 (\partial \chi)^2}{2a^5} + 2N_0 \partial \chi \partial \delta j + (\partial \delta j)^2,\quad (5.2)$$

so that $\delta n$ is equivalent to $\delta \rho_m/\rho_{m,n}$ at first order. Expanding the Schutz-Sorkin action (2.16) up to quadratic order in perturbations, we obtain the second-order action:

$$\left( S^{(2)}_m \right)_s = \int dt d^3x \left\{ \frac{1}{2a^5 n_0 \rho_{m,n}} \right\} \times \left[ \rho_{m,n} \left( \rho_{m,n} \right)^2 + 2a^3 n_0 \rho_{m,n} \partial \delta j \partial v + 2a^8 n_0 \rho_{m,n} \dot{v} \delta \rho_m - 6a^8 n_0 \rho_{m,n} H \dot{v} \delta \rho_m \right. \left. - a^8 n_0 \rho_{m,n} (\partial \delta m)^2 \right] - a^3 \alpha \delta \rho_m + \frac{\rho_{m,n}}{a^3} \partial \chi \partial \delta j + \frac{1}{2} a^3 \rho_m \alpha^2 + \frac{1}{2} a (n_0 \rho_{m,n} - \rho_m) \left( \partial \chi \right)^2 \right\}.\quad (5.3)$$

Varying this action with respect to $\delta j$, it follows that

$$\partial \delta j = -a^3 n_0 (\dot{v} + \partial \chi).\quad (5.4)$$

\(^1\)Here and in the following, we use the word “small-scale limit” for the meaning of taking the large comoving wavelength limit ($k \to \infty$) in the perturbation equations of motion. In a strict sense, this limit can be applied to small-scale perturbations in the linear regime of gravity.
On using this relation, we can eliminate the perturbation $\delta j$ from eq. (5.3) to give

\[
(S^{(2)}_m)_s = \int dt d^3 x a^3 \left\{ \left( \dot{\psi} - 3H c_s^2 v - \alpha \right) \delta \rho_m - \frac{c_s^2 (\delta \rho_m)^2}{2n_0 c_s \rho_m} \right. \\
\left. - \frac{n_0 \rho_{m,n}}{2a^2} \left[ (\partial \nu)^2 + 2\partial \nu \partial \chi \right] + \frac{1}{2} \rho_m \alpha^2 - \frac{\rho_m}{2a^2} (\partial \chi)^2 \right\}, \tag{5.5}
\]

where $c_s^2$ is the matter sound speed squared defined by

\[
c_s^2 = \frac{P_{m,n}}{\rho_{m,n}} \frac{m_{n,n}}{\rho_{m,n}}. \tag{5.6}
\]

We also expand the action $S_{\text{SVT}} + S_{\text{ST}}$ up to quadratic order in scalar perturbations and take the sum with $(S^{(2)}_m)_s$. Using the background eq. (3.5), the last term of eq. (5.5) cancels out and the term $\rho_m \alpha^2/2$ can be absorbed into one of contributions arising from $S_{\text{SVT}} + S_{\text{ST}}$. After the integration by parts, the resulting second-order action is given by

\[
S^{(2)}_s = \int dt d^3 x (L_{s1} + L_{s2} + L_{s3}), \tag{5.7}
\]

where

\[
L_{s1} = a^3 \left[ D_1 \delta \phi^2 + D_2 \left( \frac{\partial \delta \phi}{a^2} \right)^2 + D_3 \delta \phi^2 + \left( D_4 \delta \phi + D_5 \delta \phi + D_6 \left( \frac{\partial \phi}{a^2} \right) \right) \delta \rho_m - \left( D_6 \delta \phi - D_7 \delta \phi \right) \frac{\partial^2 \chi}{a^2} \right], \tag{5.8}
\]

\[
L_{s2} = a^3 \left[ \left( w_1 \alpha - w_2 \frac{\delta A}{A_0} \right) \frac{\partial \delta \chi}{a^2} - w_3 \left( \frac{\partial \phi}{a^2} \right)^2 + w_4 \alpha^2 - \left( w_3 \frac{\partial^2 \delta A}{A_0} - w_8 \frac{\delta A}{A_0} + w_3 \frac{\partial \delta \phi}{A_0} + w_6 \frac{\partial \phi}{a^2} \right) \alpha \right. \\
- \left. w_3 \frac{\partial \delta A}{A_0} + w_5 \frac{\delta A}{A_0} + \left( w_3 \frac{\partial \phi}{a^2} \right) \frac{\partial^2 \delta A}{2a^2 A_0} \right], \tag{5.9}
\]

\[
L_{s3} = a^3 \left[ \left( \rho_m + P_m \right) v \frac{\partial \phi}{a^2} - \nu \delta \rho_m - 3H (1 + c_s^2) v \delta \rho_m \right. \\
- \left. \frac{1}{2} \left( \rho_m + P_m \right) \left( \frac{\partial \phi}{a^2} \right)^2 - \frac{c_s^2}{2(n_0 + P_m)} \left( \delta \rho_m \right)^2 - \alpha^2 \frac{\partial \phi}{a^2} \right]. \tag{5.10}
\]

In appendix A, we show explicit expressions of the coefficients $D_{1,\ldots,10}$ and $w_{1,\ldots,8}$. The Lagrangian $L_{s1}$ arises from the field perturbation $\delta \phi$, so it vanishes in generalized Proca theories. The Lagrangians $L_{s1}$ and $L_{s2}$ have the same structures as those of SVT theories with the action $S_{\text{SVT}}$ alone [83]. The difference arises only through the coefficients $D_{1,\ldots,10}$ and $w_{1,\ldots,8}$. The Lagrangian $L_{s3}$ newly arises from the matter sector. The intrinsic vector modes affect the scalar sector only through the quantity $w_3 = -2A_0^2 q$. The second-order action (5.7) contains scalar perturbations $\alpha, \chi, \delta A, v$ and $\psi, \delta \phi, \delta \rho_m$, among which the last three quantities correspond to dynamical propagating DOFs. Varying the action (5.7) with respect to $\alpha, \chi, \delta A, v$, we obtain their equations of motion in Fourier
space, as

\[ D_4 \dot{\delta \phi} + D_5 \delta \phi + 2w_4 \alpha + w_8 \frac{\delta A}{A_0} + \frac{k^2}{a^2} (w_6 \psi - w_1 \chi - D_6 \delta \phi - Y) - \delta \rho_m = 0, \quad (5.11) \]

\[ D_6 \dot{\delta \phi} - D_7 \delta \phi - w_1 \alpha - (\rho_m + P_m) v + w_2 \frac{\delta A}{A_0} = 0, \quad (5.12) \]

\[ D_8 \dot{\delta \phi} + D_9 \delta \phi + w_8 \frac{\alpha}{A_0} + 2w_5 \frac{\delta A}{A_0} + \frac{k^2}{a^2} \left( \frac{w_2 \chi - A_0 w_6 - w_2}{2A_0} \psi + \frac{1}{2} Y \right) = 0, \quad (5.13) \]

\[ \dot{\delta \rho}_m + 3 \left( 1 + \frac{c_m^2}{a^2} (\rho_m + P_m) (v + \chi) \right) = 0, \quad (5.14) \]

where \( k \) is a comoving wavenumber, and

\[ \mathcal{Y} \equiv - \frac{w_3}{A_0} \left( \dot{\psi} + \delta A - 2\alpha A_0 \right). \quad (5.15) \]

Variations of the action (5.7) with respect to \( \psi, \delta \phi, \delta \rho_m \) lead to the following equations of motion:

\[ \dot{\mathcal{Y}} + \left( \mathcal{H} - \frac{\dot{A}_0}{A_0} \right) \mathcal{Y} - \frac{1}{A_0} \left[ (2w_6 \alpha + 2w_7 \psi - 2D_{10} \delta \phi)A_0^2 \right] \right) = 0, \quad (5.16) \]

\[ \dot{Z} + 3H \dot{Z} - 2D_3 \delta \phi - D_5 \alpha - D_9 \delta A - \frac{k^2}{a^2} \left( 2D_2 \delta \phi - D_6 \alpha - D_7 \chi - D_{10} \psi \right) = 0, \quad (5.17) \]

\[ \dot{v} - 3H c_m^2 v - \frac{c_m^2}{\rho_m + P_m} \delta \rho_m - \alpha = 0, \quad (5.18) \]

where

\[ Z \equiv 2D_1 \delta \phi + D_4 \alpha + D_6 \frac{k^2}{a^2} \chi + D_8 \delta A. \quad (5.19) \]

### 5.1 Full SVT theories

In SVT theories, there are three dynamical DOFs characterized by the matrix

\[ \vec{X} = \left( \psi, \delta \phi, \frac{\delta \rho_m}{k} \right). \quad (5.20) \]

To eliminate the non-dynamical DOFs from the action (5.7), we solve eqs. (5.11)–(5.14) for \( \alpha, \chi, \delta A, v \) and substitute them into eq. (5.7). Then, the second-order scalar action can be expressed in the form

\[ S^{(2)}_s = \int dt d^3 x \left( \vec{X}_i \mathbf{K} \dot{\vec{X}} - \frac{k^2}{a^2} \vec{X}_i \mathbf{G} \vec{X} - \vec{X}_i \mathbf{M} \vec{X} - \vec{X}_i \mathbf{B} \dot{\vec{X}} \right), \quad (5.21) \]

where \( \mathbf{K}, \mathbf{G}, \mathbf{M}, \mathbf{B} \) are 3 × 3 matrices. We perform the integrations by parts such that the matrix components of neither \( \mathbf{B} \) nor \( \mathbf{M} \) contain the \( k^2 \) terms for \( k \to \infty \). Then, in the
there are the following particular relations: 

\[ K_{11} = \frac{w_2^2 w_5 + w_2^3 w_4 + w_1 w_2 w_8}{A_0^2 (w_1 - 2w_2)^2}, \]

\[ K_{22} = D_1 + \frac{D_6}{w_1 - 2w_2} \left( D_4 + \frac{w_4 + 4w_5 + 2w_8}{w_1 - 2w_2} D_6 + 2A_0 D_8 \right), \]

\[ K_{12} = K_{21} = -\frac{1}{2A_0 (w_1 - 2w_2)} \left[ w_2 D_4 + \frac{w_1(4w_5 + w_8) + 2w_2(w_4 + w_8)}{w_1 - 2w_2} D_6 + A_0 w_1 D_8 \right], \]

\[ K_{33} = \frac{\alpha^2}{2(\rho_m + P_m)}, \]  

(5.22) and

\[ G_{11} = \dot{E}_1 + HE_1 - \frac{4A_0^2}{w_3} E_1^2 - \frac{w_7}{2} - \frac{w_3^2 (\rho_m + P_m)}{2A_0^2 (w_1 - 2w_2)^2}, \]

\[ G_{22} = \dot{E}_2 + HE_2 - \frac{D_6 D_7 (w_1 - 2w_2)}{w_3} - \frac{4A_0^2}{w_3} E_3 - D_2 - \frac{D_6^2 (\rho_m + P_m)}{2(w_1 - 2w_2)^2}, \]

\[ G_{12} = G_{21} = \dot{E}_3 + HE_3 - \frac{4A_0^2}{w_3} E_1 E_3 + \frac{w_2}{2A_0 (w_1 - 2w_2)} D_7 + \frac{D_1 D_6}{2} + \frac{w_2 D_6 (\rho_m + P_m)}{2A_0 (w_1 - 2w_2)^2}, \]

\[ G_{33} = \frac{\alpha^2 m a^2}{2(\rho_m + P_m)}, \]

(5.23) where

\[ E_1 = \frac{w_6}{4A_0} - \frac{w_1 w_2}{4A_0^2 (w_1 - 2w_2)}, \quad E_2 = -\frac{D_6^2}{2(w_1 - 2w_2)}, \quad E_3 = \frac{w_2 D_6}{2A_0 (w_1 - 2w_2)}. \]  

(5.24)

These expressions are valid for the SVT theories with \( A_0 \neq 0 \), \( w_1 - 2w_2 \neq 0 \), and \( w_3 \neq 0 \). Since the off-diagonal components \( K_{13}, K_{23}, G_{13}, G_{23} \) vanish, the matter perturbation \( \delta \rho_m \) is decoupled from other fields \( \psi \) and \( \delta \phi \). Provided that the two conditions

\[ \rho_m + P_m > 0, \quad \alpha^2 m > 0 \]  

(5.25)

are satisfied, there are neither ghost nor Laplacian instabilities in the matter sector.

For the perturbations \( \psi \) and \( \delta \phi \), the conditions for the absence of ghosts are similar to those derived in ref. [83], i.e.,

\[ K_{11} > 0 \quad \text{or} \quad K_{22} > 0, \quad q_s = K_{11} K_{22} - K_{12}^2 > 0. \]  

(5.26)

(5.27)

Compared to the SVT theories with the action \( S_{SVT} \) alone, the action \( S_{ST} \) gives rise to modifications to no-ghost conditions through the change of coefficients \( D_i \) and \( w_i \), but the presence of matter does not affect no-ghost conditions. Among the coefficients \( D_i \) and \( w_i \), there are the following particular relations:

\[ w_2 - w_1 + \dot{\phi} D_6 = 2H q_6, \]  

(5.28)

\[ \dot{\phi}^2 D_1 + \dot{\phi} (D_4 + 3H D_6) - 3H w_1 + w_4 - w_5 = 3H^2 q_6, \]  

(5.29)

\[ w_8 = 3H w_1 - 2w_4 - \dot{\phi} D_4, \]  

(5.30)

\[ A_0 D_8 = -(2\dot{\phi} D_1 + D_4 + 3H D_6). \]  

(5.31)
By using these relations, one can show that the quantity $q_s$ is related to the determinant $D$ given by eq. (3.13), as

$$q_s = \frac{H^2 q_t D}{2A_0^2(w_1 - 2w_2)^2}.$$  \hspace{1cm} (5.32)

This means that, under the absence of scalar and tensor ghosts, the denominators in the background eqs. (3.14)–(3.16) do not cross 0.

The dispersion relation in the small-scale limit is given by $\det(c_s^2 K - G) = 0$, where $c_s$ is the propagation speed of scalar perturbations. One of the solutions is the matter propagation speed squared $c_m^2 = G_{33}/K_{33}$, while the other two solutions are

$$c_{s1}^2 = \frac{K_{11} G_{22} + K_{22} G_{11} - 2K_{12} G_{12} + \sqrt{(K_{11} G_{22} + K_{22} G_{11} - 2K_{12} G_{12})^2 - 4q_s (G_{11} G_{22} - G_{12}^2)}}{2q_s},$$  \hspace{1cm} (5.33)

$$c_{s2}^2 = \frac{K_{11} G_{22} + K_{22} G_{11} - 2K_{12} G_{12} - \sqrt{(K_{11} G_{22} + K_{22} G_{11} - 2K_{12} G_{12})^2 - 4q_s (G_{11} G_{22} - G_{12}^2)}}{2q_s}.$$  \hspace{1cm} (5.34)

The Laplacian instabilities are absent under the conditions

$$c_{s1}^2 > 0, \quad c_{s2}^2 > 0.$$  \hspace{1cm} (5.35)

Since the matrix components $G_{11}, G_{22}, G_{12}$ contain the term $\rho_m + P_m$, the existence of matter affects the sound speed squares $c_{s1}^2$ and $c_{s2}^2$. We note that there is the particular combination

$$q_s c_{s1} c_{s2} = G_{11} G_{22} - G_{12}^2,$$  \hspace{1cm} (5.36)

which is positive under the conditions (5.27) and (5.35).

The conditions (5.26), (5.27), and (5.35) were derived under the small-scale limit. One may wonder what happens for the large-scale limit ($k \to 0$). Since the Laplacian terms vanish in this limit, what we need to worry is the absence of scalar ghosts. We recall that the background eqs. (3.14)–(3.16) are nonsingular under the no-ghost condition $q_s > 0$ derived for $k \to \infty$. Since the homogenous background can be regarded as the $k \to 0$ limit of scalar perturbations, it is anticipated that the scalar perturbations in the large-scale limit may be stable under the condition $q_s > 0$. However we need a separate detailed analysis for concrete models to support this claim, which we do not address in this paper.

### 5.2 Horndeski theories

In Horndeski theories, there are two dynamical DOFs given by the matrix

$$\mathcal{X}^2 = \left( \begin{array}{c} \delta \phi \\ \delta \rho_m \\ k \end{array} \right),$$  \hspace{1cm} (5.37)

without the perturbations $\delta A$ and $\psi$ associated with the vector field $A_\mu$. From the coefficients given in appendix A, there are also the following relations

$$D_8 = D_9 = D_{10} = 0, \quad D_4 = -2\dot{\phi} D_1 - 3H D_6, \quad D_1 = -2\dot{\phi} D_6 - 2H q_t,$$

$$w_2 = w_3 = w_5 = w_6 = w_7 = w_8 = 0, \quad w_4 = \dot{\phi}^2 D_1 + 3H \dot{\phi} D_6 - 3H^2 q_t.$$  \hspace{1cm} (5.38)
The nondynamical perturbations $\alpha, \chi, v$ obey eqs. (5.11), (5.12), and (5.14), respectively, with $w_2 = w_6 = w_8 = 0$ and $\mathcal{Y} = 0$. After eliminating these fields from eq. (5.7), the second-order scalar action can be written in the form (5.21) with the $2 \times 2$ matrices $K, G, M, B$. In the small-scale limit, the nonvanishing components of $K$ and $G$ are

$$K_{11}^{\text{Ho}} = \frac{H^2 q_t (4 D_1 q_t + 3 D_0^2)}{w_1^2}, \quad K_{22}^{\text{Ho}} = \frac{a^2}{2(\rho_m + P_m)}, \quad (5.39)$$
$$G_{11}^{\text{Ho}} = \dot{E}_2 + H E_2 - \frac{D_6 D_7}{w_1} - D_2 - \frac{D_0^2 (\rho_m + P_m)}{2 w_1^2}, \quad G_{22}^{\text{Ho}} = \frac{c_m^2 a^2}{2(\rho_m + P_m)}, \quad (5.40)$$

where $E_2 = -D_0^2/(2w_1)$. We can obtain the value $K_{11}^{\text{Ho}}$ by using $K_{22}$ in eq. (5.22) with the correspondence (5.38). There is also the correspondence between $G_{11}^{\text{Ho}}$ and $G_{22}$ in eq. (5.23), but we need to caution that the limit $w_3 \to 0$ cannot be naively taken in $G_{22}$. Under the conditions (5.25), there are neither ghost nor Laplacian instabilities for the perfect fluid.

The scalar ghost associated with the perturbation $\delta \phi$ is absent under the condition

$$q_{s, \text{Ho}} \equiv K_{11}^{\text{Ho}} = \frac{H^2 q_t D_{\text{Ho}}}{w_1^2} > 0, \quad (5.41)$$

where $D_{\text{Ho}}$ is defined by eq. (3.20). Provided that the scalar and tensor ghosts are absent, the denominators in the background eqs. (3.21) and (3.22) do not cross 0. The propagation speed squared of $\delta \phi$ is given by

$$c_{s, \text{Ho}}^2 = \frac{G_{11}^{\text{Ho}}}{q_{s, \text{Ho}}} = \frac{w_1^2}{H^2 q_t (4 D_1 q_t + 3 D_0^2)} \left[ \dot{E}_2 + H E_2 - \frac{D_6 D_7}{w_1} - D_2 - \frac{D_0^2 (\rho_m + P_m)}{2 w_1^2} \right], \quad (5.42)$$

which is required to be positive to avoid the Laplacian instability. The values of $q_{s, \text{Ho}}$ and $c_{s, \text{Ho}}^2$ match with those derived in ref. [26].

### 5.3 Generalized Proca theories

In generalized Proca theories, the two dynamical DOFs are given by

$$\mathcal{X}^t = \left( \psi, \frac{\delta \rho_m}{k} \right), \quad (5.43)$$

without the scalar-field perturbation $\delta \phi$. In this case, we have the following relations

$$D_{1,2,\ldots,10} = 0, \quad w_1 = w_2 - 2H q_t, \quad w_4 = w_5 + \frac{3H (w_1 + w_2)}{2}, \quad w_8 = 3H w_1 - 2w_4. \quad (5.44)$$

Using these relations in eqs. (5.11)–(5.14), eliminating $\alpha, \chi, \delta A, v$ from (5.7), and taking the small-scale limit, the second-order scalar action reduces to the form (5.21) with non-vanishing components of the $2 \times 2$ matrices $K$ and $G$:

$$K_{11}^{\text{GP}} = \frac{H^2 q_t (4 w_3 q_t + 3 w_5^2)}{A_0^2 (w_1 - 2w_2)^2}, \quad K_{22}^{\text{GP}} = \frac{a^2}{2(\rho_m + P_m)}, \quad (5.45)$$
$$G_{11}^{\text{GP}} = \dot{E}_1 + H E_1 - \frac{4 A_0^2 E_2}{w_3} - \frac{w_7}{2} - \frac{w_5^2 (\rho_m + P_m)}{2 A_0^2 (w_1 - 2w_2)^2}, \quad G_{22}^{\text{GP}} = \frac{c_m^2 a^2}{2(\rho_m + P_m)}, \quad (5.46)$$

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where $K_{11}^{GP}$ and $G_{11}^{GP}$ are the same as $K_{11}$ and $G_{11}$ in eqs. (5.22) and (5.23), respectively, with the particular relations (5.44). The no-ghost condition for the perturbation $\psi$ is given by

$$q_{s,GP}^{GP} \equiv K_{11}^{GP} = \frac{H^2 q_{s} D_{GP}}{A_{0}^2 (w_1 - 2w_2)^2} > 0,$$

(5.47)

where $D_{GP}$ is given by eq. (3.23). Hence the denominators of eqs. (3.24) and (3.25) remain positive under the absence of scalar and tensor ghosts. The propagation speed squared of $\psi$ yields

$$c_{s,GP}^2 = G_{11}^{GP} = \frac{A_{0}^2 (w_1 - 2w_2)^2}{H^2 q_{s} (4w_5 q_{s} + 3w_2^2)} \left[ \ddot{E}_1 + H E_1 - \frac{4A_{0}^2}{w_3} E_1^2 - \frac{w_2^2 (\rho_m + P_m)}{2A_{0}^2 (w_1 - 2w_2)^2} \right],$$

(5.48)

which needs to be positive for the absence of Laplacian instabilities. The values of $q_{s,GP}$ and $c_{s,GP}^2$ coincide with those obtain in ref. [27].

6 Matter perturbations and gravitational potentials

In order to confront SVT theories with the observations associated with the evolution of matter perturbations and gravitational potentials, we consider non-relativistic matter characterized by

$$P_m = 0, \quad c_{m}^2 = 0.$$ 

(6.1)

We introduce the gauge-invariant matter density contrast $\delta_m$, as

$$\delta_m \equiv \frac{\delta \rho_m}{\rho_m} + 3H\nu.$$ 

(6.2)

Then, eqs. (5.14) and (5.18) can be expressed as

$$\dot{\delta}_m - 3\mathcal{B} + \frac{k^2}{a^2} (v + \chi) = 0,$$

$$\dot{v} = \alpha,$$

(6.3)

(6.4)

where $\mathcal{B} \equiv H\nu$. Taking the time derivative of eq. (6.3) and using eq. (6.4), we obtain

$$\ddot{\delta}_m + 2H \dot{\delta}_m + \frac{k^2}{a^2} \Psi = 3 \left( \mathcal{B} + 2H \dot{\mathcal{B}} \right),$$

(6.5)

where $\Psi$ is the gauge-invariant gravitational potential defined by

$$\Psi \equiv \alpha + \dot{\chi}.$$ 

(6.6)

We also introduce another gauge-invariant gravitational potential:

$$\Phi \equiv H \chi,$$

(6.7)

together with the gravitational slip parameter

$$\eta \equiv \frac{\Phi}{\Psi}.$$ 

(6.8)
We define the effective gravitational coupling $G_{\text{eff}}$ in the form
\[ \frac{k^2}{a^2}\Psi = -4\pi\mu G\delta \rho_m, \quad \text{with} \quad \mu = \frac{G_{\text{eff}}}{G}. \] (6.9)

Introducing the effective potential $\psi_{\text{eff}} = \Phi - \Psi$ associated with the light bending in weak lensing observations \[93, 94\], it follows that
\[ \frac{k^2}{a^2}\psi_{\text{eff}} = 8\pi G\Sigma \delta \rho_m, \quad \text{with} \quad \Sigma = \frac{1 + \eta}{2} - \mu. \] (6.10)

In what follows, we derive analytic solutions to $\Psi$, $\Phi$, $\psi$, and $\delta \phi$ under the quasi-static approximation for the modes deep inside the sound horizon ($c_s^2 k^2 \gg a^2 H^2$). This amounts to picking up terms containing $\delta \rho_m$ and $k^2/a^2$ in the perturbation equations of motion \[21, 42, 86\]. In some dark energy models like those in $f(R)$ gravity \[61–64\], the mass of field perturbation $\delta \phi$ can be much larger than $H$ in the early cosmological epoch. In such cases, we need to take into account the mass term $-2D_3 \delta \phi$ in eq. (5.17). If the field mass is heavy, however, the scalar field hardly propagates, so the evolution of perturbations is similar to that in GR \[42\]. Since we are interested in the growth of perturbations at the late cosmological epoch during which the field mass becomes as light as the Hubble expansion rate, it is a good approximation to neglect the masses of scalar and vector fields.

From eq. (6.3), the term $(k^2/a^2)v$ is at most of order $H\delta_m$. Then, for sub-horizon perturbations, we have $|Hv| \lesssim (aH/k)^2\delta_m \ll |\delta_m|$ and hence $\delta_m \simeq \delta \rho_m/\rho_m$ in eq. (6.2). Moreover, the time derivatives $\dot{\mathcal{B}}$ and $H\dot{\mathcal{B}}$ in eq. (6.5), which are at most of order $H^2 \mathcal{B} = H^2 v$ under the quasi-static approximation, can be neglected to the terms on its left hand side (which are of order $H^2 \delta_m$). Then, from eqs. (6.5) and (6.9), the density contrast $\delta_m$ obeys
\[ \ddot{\delta}_m + 2H \dot{\delta}_m - 4\pi \mu G \rho_m \delta_m \simeq 0. \] (6.11)

After deriving analytic expressions of $\mu$ and $\Sigma$, we can solve eqs. (6.11), (6.9), and (6.10) for $\delta_m$, $\Psi$, and $\psi_{\text{eff}}$, respectively.

### 6.1 Full SVT theories

We derive analytic solutions to $\mu$ and $\Sigma$ in full SVT theories with the three dynamical scalar perturbations $\psi$, $\delta \phi$, and $\delta \rho_m$. Applying the quasi-static approximation to eqs. (5.11) and (5.13), we have
\[ \delta \rho_m \simeq \frac{k^2}{a^2} \left( w_6 \psi - w_1 \chi - D_6 \delta \phi - \mathcal{Y} \right), \] (6.12)
\[ \mathcal{Y} \simeq \left( w_6 - \frac{w_2}{A_0} \right) \psi - 2w_2 \chi, \] (6.13)
and hence
\[ \delta \rho_m \simeq -\frac{k^2}{a^2} \left( \frac{w_1 - 2w_2}{H} \Phi - \frac{w_2}{A_0} \psi + D_6 \delta \phi \right). \] (6.14)

Eliminating the perturbation $v$ from eqs. (5.12) and (5.14), it follows that
\[ \delta \rho_m + 3H \delta \rho_m + \frac{k^2}{a^2} \left( \rho_m \chi - w_1 \alpha + w_2 \frac{\delta A}{A_0} + D_6 \dot{\delta \phi} - D_7 \delta \phi \right) = 0. \] (6.15)
Substituting eq. (6.14) and its time derivative into eq. (6.15), the derivative term \( \dot{\delta \phi} \) cancels out. After this substitution the time derivative \( \dot{\psi} \) appears, but it can be eliminated by using the relation

\[
\dot{\psi} \simeq 2A_0 \alpha - \delta A + \frac{1}{w_3} \left[(w_2 - A_0 w_6) \psi + 2w_2 A_0 \chi\right],
\]

which follows from eqs. (5.15) and (6.13). Then, we obtain

\[
w_3 A_0^2 \left( \kappa_1 \Psi + \kappa_2 \Phi + \kappa_4 \delta \phi \right) + \kappa_3 \psi \simeq 0,
\]

where

\[
\begin{align*}
\kappa_1 &= w_1 - 2w_2, \\
\kappa_2 &= \frac{1}{H} \left( \kappa_1 + H \kappa_1 - \rho_m - \frac{2w_2^2}{w_3} \right), \\
\kappa_3 &= w_2 w_6 A_0^2 - (\dot{w}_2 w_3 + H w_2 w_3 + w_2^2) A_0 + w_2 w_3 A_0, \\
\kappa_4 &= D_0 + H D_0 + D_7.
\end{align*}
\]

We also substitute eq. (6.13) and its time derivative into eq. (5.16). This leads to

\[
2w_2 w_3 A_0^2 \Psi - \frac{2A_0}{H} \kappa_3 \Phi - 2D_{10} w_3 A_0^3 \delta \phi = 0,
\]

where

\[
\kappa_5 = (2w_5 w_7 + w_6^2) A_0^3 - [(\dot{w}_6 + H w_6) w_3 + 2w_2 w_6] A_0^2 + \left[(\dot{w}_2 + H w_2 + w_6 A_0) w_3 + w_2^2\right] A_0 - 2w_2 w_3 A_0.
\]

From eqs. (5.17) and (5.19), it follows that

\[
H D_6 \Psi + \kappa_4 \Phi + H D_{10} \psi - 2H D_2 \delta \phi \simeq 0.
\]

Now, we can solve eqs. (6.14), (6.17), (6.22)(6.24) for \( \Psi, \Phi, \psi, \) and \( \delta \phi \), as

\[
\begin{align*}
\Psi &\simeq \frac{-4A_0^6 D_0^2 H \kappa_2 q^2 + 4A_0^5 D_0^3 \kappa_4 q_0 + 2A_0^4 D_0 H \kappa_2 q_0 + A_0^3 \kappa_2^2 \kappa_4 q_0 - 2D_{10}^2 q_0^2}{\Delta} \frac{a^2}{k^2} \delta \rho_m, \\
\Phi &\simeq \frac{A_0^5 \kappa_4 q_0 (4A_0^4 D_0^2 \kappa_1 q_0 + 4A_0^3 D_0^3 \kappa_4 q_0 + 2A_0 D_0 D_0^{10} \kappa_3 + 2D_{10}^2 \kappa_4 - 4D_{10}^2 \kappa_4 w_2 q_0 + D_0 \kappa_4 \kappa_1) a^2}{\Delta} \frac{a^2}{k^2} \delta \rho_m, \\
\psi &\simeq \frac{2A_0^3 \kappa_4 q_0 (2A_0^2 D_0 D_0^{10} H \kappa_2 q_0 - 2A_0 D_0^{10} \kappa_1 q_0 - 4A_0^3 D_0 H \kappa_2 q_0 - 2A_0^3 \kappa_2^2 q_0 w_2 q_0 + 2D_2 \kappa_1 \kappa_3 + D_0 \kappa_4 q_0) a^2}{\Delta} \frac{a^2}{k^2} \delta \rho_m, \\
\delta \phi &\simeq \frac{-4A_0^4 D_0^2 H \kappa_2 q_0^2 - 2A_0^3 D_0^{10} \kappa_1 q_0 + A_0^4 D_0 H \kappa_2 q_0 + A_0^3 \kappa_2^2 \kappa_4 q_0 + 2A_0^3 \kappa_4 q_0 w_2 q_0 - D_0 \kappa_2^2 q_0 a^2}{\Delta} \frac{a^2}{k^2} \delta \rho_m,
\end{align*}
\]

where we used the relation \( w_3 = -2A_0^2 q_0 \), and \( \Delta \) is defined by

\[
\Delta = -4A_0^8 D_0^2 \kappa_1^2 q_0^2 + 8A_0^5 D_0^3 H \kappa_2 q_0 - 2A_0^4 D_0^2 H \kappa_1 q_0 - 2A_0^3 D_0^3 \kappa_4 q_0 - 4A_0^2 w_3^2 q_0^2 (H D_0 \kappa_2 - \kappa_4 q_0) \\
-4A_0^2 w_3^2 q_0^2 (2H D_0 \kappa_2 + \kappa_4 q_0) - 4A_0^3 D_0 H \kappa_2 q_0 w_2 q_0 \\
+ A_0^3 q_0 \left[ D_0 (D_0 H \kappa_2 + 4\kappa_4 q_0 w_2) + 2\kappa_1 (4D_2 \kappa_3 w_2 - D_6 \kappa_4 q_0) - 2D_2 \kappa_1 \kappa_3 - D_0 \kappa_3^2. \right.
\]

(6.29)
We compute the right hand side of eq. (5.36) by using the definitions (5.23) with eq. (5.24). Then, the determinant $\Delta$ is simply related to the quantity $q_s c_s^2 c_{s,\text{H}}^2$, as
\[
\Delta = 16\kappa_1^2 A_0^2 q_s^2 c_{s,\text{H}}^2 c_{s,\text{H}}^2, \quad (6.30)
\]
which is positive under the absence of ghost and Laplacian instabilities of scalar perturbations.

From eqs. (6.8)–(6.10), the quantities $\mu$ and $\Sigma$ can be estimated as
\[
\mu = \frac{4A_0^2 D_{10}^2 H\kappa_2 q_s^2 + 4A_0^2 D_{10}\kappa_3 \kappa_6 q_s (2D_2 H\kappa_2 + \kappa_4^2) - 2D_2 \kappa_5^2}{64\pi G \kappa_1^2 A_0^2 q_s c_{s,\text{H}}^2 c_{s,\text{H}}^2}, \quad (6.31)
\]
\[
\Sigma = \frac{1}{128\pi G \kappa_1^2 A_0^2 q_s c_{s,\text{H}}^2 c_{s,\text{H}}^2} \left[ 4A_0^2 D_{10}^2 Hq_s^2 (\kappa_1 + \kappa_2) + 4A_0^2 D_{10} H\kappa_4 w_2 q_s^2 + 2A_0^2 D_{10} \kappa_3 q_s (D_6 H + 2\kappa_4) + A_0^2 q_s (2D_2 H\kappa_1 \kappa_5 + 2D_2 H\kappa_2 \kappa_5 - 4D_2 H\kappa_3 w_2 + D_6 H\kappa_4 \kappa_5 + \kappa_4^2 \kappa_5) - 2D_2 \kappa_3^2 \right]. \quad (6.32)
\]
The two quantities $\mu$ and $\Sigma$ evolve differently depending on the models of dark energy. Since the evolution of gravitational potentials as well as the growth of matter perturbations is affected by the changes of $\mu$ and $\Sigma$, one can distinguish between dark energy models in SVT theories from the observation data of large-scale structures, weak lensing, and CMB [95–97].

### 6.2 Horndeski theories

In Horndeski theories, the scalar perturbation $\psi$ does not exist as a propagating DOF. Under the quasi-static approximation, we have three perturbation equations of motion: (6.14) with $w_2 = 0$, (6.24) with $D_{10} = 0$, and
\[
w_1 \Psi + \kappa_2 \Phi + \kappa_4 \delta \phi \simeq 0, \quad (6.33)
\]
where the last equation is the analogue of eq. (6.17). Solving these equations for $\Psi, \Phi, \delta \phi$, it follows that
\[
\Psi \simeq -\frac{2D_2 H\kappa_2 + \kappa_4^2 a^2}{\Delta_{\text{Ho}}} \Delta_{\text{Ho}} \delta \rho_m, \quad \Phi \simeq \frac{H(2D_2 w_1 + D_6 \kappa_4)}{\Delta_{\text{Ho}}} \frac{a^2}{k^2} \delta \rho_m, \quad \delta \phi \simeq -\frac{D_6 H\kappa_2 - w_1 \kappa_4 a^2}{\Delta_{\text{Ho}}} \Delta_{\text{Ho}} \delta \rho_m, \quad (6.34)
\]
where
\[
\Delta_{\text{Ho}} = D_6^2 H\kappa_2 - 2D_2 w_1^2 - 2D_6 w_1 \kappa_4. \quad (6.35)
\]
The determinant $\Delta_{\text{Ho}}$ is related to the quantity $G_{11}^{\text{Ho}} = q_s \kappa_4 c_{s,\text{H}}^2$ in eq. (5.40), as
\[
\Delta_{\text{Ho}} = 2w_1^2 \frac{q_s \kappa_4 c_{s,\text{H}}^2}{\kappa_4}. \quad (6.36)
\]
Then, the quantities $\mu$ and $\Sigma$ reduce, respectively, to
\[
\mu = \frac{2D_2 H\kappa_2 + \kappa_4^2}{8\pi G w_1^2 \kappa_4 c_{s,\text{H}}^2}, \quad \Sigma = \frac{2D_2 H(\kappa_1 + \kappa_2) + D_6 H\kappa_4 + \kappa_4^2}{16\pi G w_1^2 q_s \kappa_4 c_{s,\text{H}}^2}. \quad (6.37)
\]
We confirmed that these results agree with those derived in ref. [42] in the limit that the scalar-field mass vanishes.
In generalized Proca theories the scalar-field perturbation \( \delta \phi \) is absent, so there are three independent perturbation equations: (6.14) with \( D_6 = 0 \), (6.17) with \( \kappa_4 = 0 \), and (6.22) with \( D_{10} = 0 \). Solving these equations for \( \Psi, \Phi, \psi \), we obtain

\[
\Psi \simeq -\frac{A_0 H \kappa_2 \kappa_5 w_3 + 2 \kappa_3^2 a^2}{\Delta_{GP}} k^2 \delta \rho_m, \quad \Phi \simeq \frac{A_0 w_3 H (\kappa_1 \kappa_5 - 2 \kappa_3 w_2)}{\Delta_{GP}} a^2 k^2 \delta \rho_m, \quad (6.38)
\]

where

\[
\Delta_{GP} = A_0 w_3 \left( 2 A_0 H \kappa_2 w_2 w_3 - \kappa_1^2 \kappa_5 + 4 \kappa_1 \kappa_3 w_2 \right). \quad (6.39)
\]

The determinant \( \Delta_{GP} \) can be expressed in terms of the quantity \( G_{11}^{GP} = q_{s,GP} c_{s,GP}^2 \) in eq. (5.46), as

\[
\Delta_{GP} = 16 A_0^8 \kappa_1^2 q_v^2 q_{s,GP} c_{s,GP}^2. \quad (6.40)
\]

On using this relation, it follows that

\[
\mu = \frac{\kappa_3^2 - A_0^2 H \kappa_2 \kappa_5 q_v}{32 \pi G A_0^2 \kappa_1^2 q_v^2 q_{s,GP} c_{s,GP}^2}, \quad \Sigma = \frac{\kappa_3^2 - A_0^2 H q_v [\kappa_5 (\kappa_1 + \kappa_2) - 2 \kappa_3 w_2]}{64 \pi G A_0^2 \kappa_1^2 q_v^2 q_{s,GP} c_{s,GP}^2}, \quad (6.41)
\]

which coincide with those derived in ref. [75].

### 7 SVT theories with \( c_t^2 = 1 \)

Finally, we estimate the quantities \( \mu \) and \( \Sigma \) associated with Newtonian and weak lensing potentials for SVT theories in which \( c_t^2 \) is exactly equivalent to 1. We consider the couplings of the forms

\[
G_4 = G_4(\phi), \quad G_5 = 0, \quad f_4 = 0, \quad f_5 = 0, \quad (7.1)
\]

where the \( \phi \) dependence of \( f_4 \) has been absorbed into \( G_4(\phi) \). We take into account all the Lagrangians associated with intrinsic vector modes, which affect scalar perturbations only through the quantity

\[
w_3 = -2 A_0^3 q_v. \quad (7.2)
\]

The variables \( w_1, w_2, w_6, w_7 \), which appear in \( \kappa_1, \kappa_2, \kappa_3, \kappa_5 \), are given by

\[
w_1 = -G_{3,X_1} \dot{\phi}^3 - 2 G_{4,\phi} \dot{\phi} + 2 A_0^3 \left( f_{3,X_3} + \ddot{f}_3 \right) - 4 G_4 H, \quad (7.3)
\]

\[
w_2 = -A_0 w_6 = 2 A_0^3 \left( f_{3,X_3} + \ddot{f}_3 \right), \quad (7.4)
\]

\[
w_7 = 2 A_0 \left( f_{3,X_3} + \ddot{f}_3 \right) + \frac{\dot{\phi}(f_{2,X_2} + 4 f_{3,\phi})}{2 A_0}. \quad (7.5)
\]

In eqs. (6.31) and (6.32) there are also other variables \( \kappa_4, D_2, D_6, D_{10} \), whose explicit forms are

\[
\kappa_4 = f_{2,X_1} \dot{\phi} + \frac{1}{2} f_{2,X_2} A_0 + \dot{\phi} \left( 2 G_{3,\phi} - 2 \ddot{\phi} G_{3,X_1} - 4 G_{3,X_1} H \dot{\phi} - G_{3,X_1} \dot{\phi}^2 - G_{3,X_1} \dot{\phi}^2 \dot{\phi} \right) + 2 f_{3,\phi} A_0 - 4 H G_{4,\phi} + 2 f_{4,\phi} \dot{\phi}, \quad (7.6)
\]

\[
D_2 = -\frac{1}{2} f_{2,X_1} - G_{3,\phi} + G_{3,X_1} \dot{\phi} + \frac{1}{2} \dot{\phi} \left( 4 G_{3,X_1} H + G_{3,X_1} \dot{\phi} + G_{3,X_1} \dot{\phi} \dot{\phi} \right), \quad (7.7)
\]

\[
D_6 = -G_{3,X_1} \dot{\phi}^2 - 2 G_{4,\phi}, \quad (7.8)
\]

\[
D_{10} = \frac{1}{2} f_{2,X_2} + 2 f_{3,\phi}. \quad (7.9)
\]
Note that $\mu$ and $\Sigma$ also depend on the combination $q_\nu c_1^2 c_2^2$, which can be expressed in terms of the variables mentioned above as well as $q_\nu$. Unless we impose further conditions, none of the variables given above vanish. As we see in eqs. (6.31) and (6.32), the effect of intrinsic vector modes on $\mu$ and $\Sigma$ is generally present through the quantity $q_\nu$ even for the theories with $c_1^2 = 1$.

There are classes of SVT theories with $c_1^2 = 1$ in which the dependence of $q_\nu$ in $\mu$ and $\Sigma$ disappears. In the following, we focus on the theories satisfying

$$f_3 = f_3(\phi), \quad \tilde{f}_3 = 0,$$

besides the conditions (7.1). In this case, the variables (7.3)–(7.5) reduce, respectively, to

$$w_1 = -G_{3,X_1}(\dot{\phi})^3 - 2G_{4,\phi}\dot{\phi} - 4G_4 H, \quad w_2 = w_6 = 0, \quad w_7 = -f_{2,X_3}.$$

For the derivation of $w_7$, we exploited the fact that the background eq. (3.8) gives

$$\langle f_{2,X_2} + 4f_{3,\phi} \rangle \phi = -2A_0 f_{2,X_3}.$$

On using the relations (7.11) in eqs. (6.20) and (6.23), it follows that

$$\kappa_3 = 0, \quad \kappa_5 = 2w_3 w_7 A_0^3 = 4A_0^3 q_\nu f_{2,X_3}.$$

Then, the terms containing $q_\nu^2$ in the denominators and numerators of eqs. (6.31) and (6.32) are factored out, such that

$$\mu = \frac{f_{2,X_3}(2D_2 H\kappa_2 + \kappa_4^2) + D_4^2 H\kappa_2}{16\pi G\kappa_4^2 q_\nu c_1^2 c_2^2},$$

$$\Sigma = \frac{f_{2,X_3}[H(2D_2\kappa_1 + 2D_2\kappa_2 + D_6\kappa_4) + \kappa_4^2] + D_4^2 H(\kappa_1 + \kappa_2)}{32\pi G\kappa_4^2 q_\nu c_1^2 c_2^2}.$$

Now, we substitute the values $\kappa_1 = w_1, \kappa_2 = (\dot{w}_1 + H w_1 - \rho_m)/H,$ and $\kappa_4$ into eqs. (7.14) and (7.15). In doing so, we take the time derivative of $w_1 = -G_{3,X_1}(\dot{\phi})^3 - 2G_{4,\phi}\dot{\phi} - 4G_4 H$ and then use the background eq. (3.6), i.e.,

$$4G_4 \dot{H} + \left(3G_{3,X_1}(\dot{\phi})^2 + 2G_{4,\phi}\right)\ddot{\phi} + \left(f_{2,X_1}(\dot{\phi})^2 + 2G_{3,\phi}\dot{\phi} - 3G_{3,X_1}H(\dot{\phi})^2 - 2G_{4,\phi}H + 2G_{4,\phi}\dot{\phi}\right)\dot{\phi}$$

$$-A_0^2 f_{2,X_3} = -\rho_m - P_m,$$

to eliminate $\rho_m$ in $\kappa_2$ (with $P_m = 0$). Then, we obtain

$$\mu = \frac{1}{16\pi G\kappa_4^2} \left[ 1 + \frac{(G_{3,X_1}(\dot{\phi})^2 - 2G_{4,\phi})^2}{\xi_s} \right],$$

$$\Sigma = \frac{1}{16\pi G\kappa_4^2} \left[ 1 + \frac{G_{3,X_1}(\dot{\phi})^2 (G_{3,X_1}(\dot{\phi})^2 - 2G_{4,\phi})}{\xi_s} \right],$$

where

$$\xi_s = \frac{w_1^2 q_\nu c_1^2 c_2^2}{f_{2,X_3} G_4 H^2} = \frac{2A_0}{\phi} \left( f_{2,X_2} + 4f_{3,\phi} \right) G_4 + \xi_{Ho},$$

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with

$$\xi_{\text{Ho}} \equiv 4G_4 \left[ f_{2,X_1} + 2G_{3,\phi} - 2G_{3,X_1} \left( \ddot{\phi} + 2H \dot{\phi} \right) - \dot{\phi}^2 \left( G_{3,X_1} + G_{3,X_1,\phi} \right) \right] - G_{3,X_1} \dot{\phi}^2 + 4G_{4,\phi} \left( G_{3,X_1} \dot{\phi}^2 + 3G_{4,\phi} \right).$$  \hspace{1cm} (7.20)$$

From eqs. (7.17) and (7.18), we observe that the $X_1$ dependence in $G_3$ and the $\phi$ dependence in $G_4$ lead to modifications to the values of $\mu$ and $\Sigma$ in GR. Since the two conditions $q_1 = 2G_4 > 0$ and $q_\phi c_2^2 c_{22}^2 > 0$ are required for the absence of ghost and Laplacian instabilities, the second term in the square bracket of eq. (7.17) is either positive or negative depending on the sign of $f_{2,X_1}$. If $f_{2,X_1} > 0$, then the gravitational attraction is enhanced by the couplings $G_3(X_1)$ and $G_4(\dot{\phi})$. From eq. (6.11), the growth of $\delta_m$ is also modified from that in GR.

A subclass of Horndeski theories allows the existence of no slip gravity scenario with $c_2^2 = 1$ in which the effective gravitational couplings to matter and light are equivalent to each other [99], such that $\mu = \Sigma$. In SVT theories given by the functions (7.10), it follows from eqs. (7.17) and (7.18) that the condition for no slip gravity corresponds to $G_{4,\phi} = 0$ or $G_{3,X_1} \dot{\phi}^2 = 2G_{4,\phi}$. In the former case the gravitational interactions are enhanced by the coupling $G_{3,X_1}$, while the enhancement is absent for the latter case.

For the theories in which the conditions

$$f_{2,X_2} = 0, \quad \text{and} \quad f_{3,\phi} = 0$$  \hspace{1cm} (7.21)

are satisfied, the term $\xi_\delta$ in eq. (7.19) is equivalent to $\xi_{\text{Ho}}$. Then, eqs. (7.17) and (7.18) reduce to

$$\mu = \frac{1}{16\pi G G_4} \left[ 1 + \frac{(G_{3,X_1} \ddot{\phi}^2 - 2G_{4,\phi})^2}{\xi_{\text{Ho}}} \right],$$  \hspace{1cm} (7.22)

$$\Sigma = \frac{1}{16\pi G G_4} \left[ 1 + \frac{G_{3,X_1} \ddot{\phi}^2 (G_{3,X_1} \ddot{\phi}^2 - 2G_{4,\phi})}{\xi_{\text{Ho}}} \right].$$  \hspace{1cm} (7.23)

These values are equivalent to those in the subclass of Horndeski theories given by the Lagrangian $\mathcal{L} = f_2(\phi, X_1) + G_3(\phi, X_1) \Box \phi + G_4(\phi) R$ [42]. This means that the effect of the vector field on $\mu$ and $\Sigma$ arises from the kinetic mixing between $A_0$ and $\dot{\phi}$ (i.e., $f_{2,X_2} \neq 0$) as well as from the cubic scalar-vector coupling $f_3(\phi) g^{\mu \nu} S_{\mu \nu}$. As long as the condition $f_{2,X_2} + 4f_{3,\phi} \neq 0$ is satisfied, eq. (7.12) shows that the temporal vector component $A_0$ does not vanish for $f_{2,X_1} \neq 0$. In this case, $\mu$ and $\Sigma$ in eqs. (7.17) and (7.18) depend on both the scalar-vector couplings $f_{2,X_2} + 4f_{3,\phi}$ and the contribution $\xi_{\text{Ho}}$ arising in Horndeski theories.

Finally, let us consider the theories satisfying

$$G_{3} = 0,$$  \hspace{1cm} (7.24)

without imposing the condition (7.21). Then, eqs. (7.17) and (7.18) yield

$$\mu = \frac{1}{16\pi G G_4} \left[ 1 + \frac{2G_{4,\phi}^2}{(A_0/\phi) (f_{2,X_2} + 4f_{3,\phi}) G_4 + 2(G_{4,f_{2,X_1}} + 3G_{4,\phi})} \right],$$  \hspace{1cm} (7.25)

$$\Sigma = \frac{1}{16\pi G G_4}.$$  \hspace{1cm} (7.26)

Provided that $f_{2,X_2} + 4f_{3,\phi} \neq 0$, the scalar-vector interactions give rise to the modification to $\mu$, while $\Sigma$ is not affected. Taking the limit $f_{2,X_2} + 4f_{3,\phi} \to 0$ in eq. (7.25), we recover
the value of \( \mu \) derived for a nonminimally coupled scalar field with the Lagrangian \( \mathcal{L} = f_2(\phi, X_1) + G_4(\phi)R \) [86, 98].

The fact that \( \mu \) and \( \Sigma \) are independent of \( q_v \) is attributed to the choice of functions \( f_3 \) and \( \tilde{f}_3 \) in eq. (7.10). For the theories in which \( f_3 \) depends on \( X_3 \) and \( \tilde{f}_3 \) is a non-vanishing function, the quantity \( \kappa_3 \) does not vanish in eqs. (6.31) and (6.32) and hence \( \mu \) and \( \Sigma \) depend on \( q_v \). In such cases, the effect of the vector field on \( \mu \) and \( \Sigma \) arises not only through scalar-vector interactions but also from intrinsic vector modes.

8 Conclusions

In full SVT theories with second-order equations of motion and parity invariance, we studied the behavior of linear cosmological perturbations on the flat FLRW background. The difference from previous study [83] is that we have taken into account the perfect fluid and the Horndeski action (2.12) besides the SVT action (2.5). As a result, the perturbation equations of motion can be directly applied to the growth of matter perturbations and the evolution of gravitational potentials for dark energy models in the framework of SVT theories. Moreover, our general analysis can accommodate both Horndeski and generalized Proca theories as subclasses of the action (2.19).

In section 3, we derived the background equations of motion with the matter density \( \rho_m \) and pressure \( P_m \). In particular, eqs. (3.6), (3.7) and the time derivative of (3.8) are expressed in compact forms by using the coefficients present in the second-order action of scalar perturbations. We showed that, for a nonvanishing determinant \( D \), the dynamical system in SVT theories can be solved for \( \dot{H}, \dot{\phi}, \dot{A}_0 \) in the forms (3.14)–(3.16). In Horndeski and generalized Proca theories, the dynamical systems are described by the combinations \((\dot{H}, \dot{\phi})\) and \((\dot{H}, \dot{A}_0)\), respectively. In all cases, the determinant \( D \) is directly related to a quantity \( q_s \) associated with the no-ghost condition of scalar perturbations.

In section 4, we decomposed the perturbations of metric, scalar and vector fields, and perfect fluid into tensor, vector, and scalar modes and obtained the second-order actions of tensor and vector perturbations. The conditions for the absence of ghost and Laplacian instabilities in the tensor sector correspond to eq. (4.18), with the tensor propagation speed squared given by eq. (4.17). If we apply SVT theories to dark energy and strictly demand that \( c_t^2 = 1 \) without allowing any tuning among functions, the theories are restricted to be of the forms (4.21). For vector perturbations, we showed that neither the scalar-tensor action \( S_{ST} \) nor the perfect-fluid action \( S_m \) give rise to modifications to stability conditions in the small-scale limit derived in ref. [83].

In section 5, we derived the second-order action of scalar perturbations and the resulting full perturbation equations of motion in the scalar sector. In SVT theories, there are three scalar dynamical DOFs \( \psi, \delta \phi, \delta \rho_m \), among which the matter perturbation \( \delta \rho_m \) is decoupled from others in the small-scale limit. The no-ghost conditions for the perturbations \( \psi \) and \( \delta \phi \) correspond to eqs. (5.26)–(5.27), while their propagation speed squares are given by eqs. (5.33)–(5.34). The perfect fluid affects the values of \( c_{s1}^2 \) and \( c_{s2}^2 \) through the term \( \rho_m + P_m \) appearing in \( G_{11}, G_{22}, G_{12} \) of eq. (5.23). We also showed that our general framework recovers the stability conditions of scalar perturbations derived in Horndeski and generalized Proca theories.

In section 6, we studied the behavior of matter perturbations \( \delta \rho_m \) and gauge-invariant gravitational potentials \( \Psi, \Phi \) by employing the quasi-static approximation for scalar perturbations deep inside the sound horizon. Under this approximation scheme, we derived
the closed-form expressions of $\Psi, \Phi, \psi, \delta \phi$ in SVT theories in the forms (6.25)–(6.28), where the determinant $\Delta$ in denominators can be expressed in terms of the quantity $q_e c_{x_1}^2 c_{x_2}^2$, as eq. (6.30). The dimensionless quantities $\mu$ and $\Sigma$ associated with Newtonian and weak lensing gravitational potentials $\Psi$ and $\psi_{\text{eff}}$ are given by eqs. (6.31) and (6.32), respectively. They contain quantities like $q_e$ and $q_e c_{x_1}^2 c_{x_2}^2$, whose positivities are required for the stability of vector and scalar perturbations. We also reproduced the values of $\mu$ and $\Sigma$ in Horndeski and generalized Proca theories as specific cases.

In section 7, we applied our general formulas of $\mu$ and $\Sigma$ to the SVT theories satisfying $c_t^2 = 1$. If the cubic couplings $f_3(X_3)$ and $\tilde{f}_3$ are present, the intrinsic vector modes generally affect $\mu$ and $\Sigma$ through the quantity $q_c$. For the theories with $f_3 = f_3(\phi)$ and $\tilde{f}_3 = 0$, we found that the terms containing $q_c^2$ in eqs. (6.31) and (6.32) are factored out. In latter theories, the evolution of scalar perturbations for the modes relevant to the observations of large-scale structures and weak lensing is not affected by intrinsic vector modes. However, as long as the scalar-vector couplings $f_2(X_2)$ and $\tilde{f}_3, \phi$ are present, their effects appear as the combination $f_2(X_2) + 4f_3, \phi$ in the expressions of $\mu$ and $\Sigma$ given by eqs. (7.17) and (7.18). In such cases, the behavior of matter perturbations and gravitational potentials is modified by the scalar-vector interactions.

Our general results about the stabilities of perturbations can be directly applied to the construction of viable dark energy models in the framework of SVT theories. Moreover, for the models with $c_t^2 = 1$, it will be of interest to study their observational signatures in more detail to extract some new features in SVT theories. These issues are left for future works.

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A Coefficients in the second-order action of scalar perturbations

The coefficients $D_{1, \ldots, 10}$ and $w_{1, \ldots, 8}$ appearing in the background eqs. (3.6), (3.7), (3.12) and the second-order action of scalar perturbations (5.7) are given by

$$D_1 = H^2 \dot{\phi} \left[ 3 G_5, X_1 + \frac{7}{2} \phi^2 G_5, X_1 X_1 + \frac{1}{2} \phi^4 G_5, X_1 X_1 X_1 \right] + 3 H^2 \left[ G_4, X_1 G_5, X_1 - G_5, X_1 \phi + \frac{\phi^2}{2} \left( 4 G_4, X_1 X_1 - \frac{5}{2} G_5, X_1 \phi \right) \right]$$

$$+ \frac{\phi^4}{2} \left[ G_4, X_1 G_5, X_1 X_1 - \frac{1}{2} G_5, X_1 \phi \right] - 3 H \dot{\phi} \left[ G_3, X_1 + 3 G_4, X_1 \phi + \phi^2 \left( \frac{1}{2} G_3, X_1 + G_4, X_1 \phi \right) \right]$$

$$+ \frac{1}{2} \left[ f_2, X_1 + 2 G_3, \phi + \dot{\phi} \left( f_2, X_1 + G_3, \phi \right) + \dot{\phi} A_0 f_2, X_2 \phi + \frac{A_0^2}{4} f_2, X_2 G_5, X_1 \phi \right] ,$$

$$D_2 = - \left[ 2 G_4, X_1 - G_5, \phi \right] + \phi^2 \left( 2 G_4, X_1 - G_5, \phi \right) + H \dot{\phi} \left( 2 G_5, X_1 + \phi^2 G_5, X_1 \right) \right] \dot{H}$$

$$+ \left[ G_4, X_1 + 3 G_4, X_1 \phi + \phi^2 \left( \frac{G_4, X_1 X_1}{2} + G_4, X_1 \phi \right) - 2 H \dot{\phi} \left( G_4, X_1 X_1 - 2 G_5, X_1 \phi \right) \right]$$

$$- H \phi^3 \left( G_4, X_1 X_1, G_5, X_1 \phi \right) - H^2 \left[ G_4, X_1 + \frac{5}{2} \phi G_5, X_1 + \frac{1}{2} \phi^3 G_5, X_1 \phi \right] \right] \dot{\phi}$$

$$- H^3 \dot{\phi} \left( 2 G_5, X_1 + \phi^2 G_5, X_1 \right) - H^2 \left[ 3 G_4, X_1 - G_5, \phi \right] + 5 \phi^2 \left( G_4, X_1 X_1 - \frac{1}{2} G_5, X_1 \phi \right) + \frac{1}{2} \phi^3 G_5, X_1 \phi \right]$$

$$+ 2 H \dot{\phi} \left( G_3, X_1 + 3 G_4, X_1 \phi \right) - H^3 \dot{\phi} \left( 2 G_4, X_1 X_1, G_5, X_1 \phi \right) + \phi^2 \left( \frac{1}{2} G_3, X_1 \phi + G_4, X_1 \phi \right) + \phi^2 \left( \frac{1}{2} G_3, X_1 \phi + G_4, X_1 \phi \right) - G_3, \phi - \frac{1}{2} f_2, X_1 ,$$

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\[ D_3 = 3 \left[ G_{4,\phi\phi} + f_{4,\phi\phi} + \phi^2 \left( \frac{1}{2} G_{3,\phi\phi} + G_{4,\phi\phi} \right) + H A_0 f_{5,\phi\phi} - 2H \phi (G_{4,\phi\phi} - G_{5,\phi\phi}) \right] \]
\[ - H \phi (2G_{4,\phi\phi} + G_{5,\phi\phi}) - H^2 \frac{\phi^2}{2} \left( 3G_{5,\phi\phi} + \phi^2 G_{5,\phi\phi} \right) \]
\[ - \left[ \frac{1}{2} f_{2,\phi\phi} + G_{3,\phi\phi} + \frac{1}{2} \phi A_0 f_{2,\phi\phi} + \frac{A_0^2}{8} f_{2,\phi\phi} \right] \]
\[ + \frac{1}{2} \phi^2 (f_{2,\phi\phi} + G_{3,\phi\phi}) - 3H \phi (G_{3,\phi\phi} + 3G_{4,\phi\phi}) \]
\[ - 3H^2 \phi^2 \left( \frac{1}{2} G_{3,\phi\phi} + G_{4,\phi\phi} \right) + 3H^2 (G_{4,\phi\phi} - G_{5,\phi\phi}) + 3H^2 \phi^2 (4G_{4,\phi\phi} - \frac{5}{2} G_{5,\phi\phi}) \]
\[ + 3H^2 \phi^3 \left( G_{4,\phi\phi} - \frac{5}{2} G_{5,\phi\phi} \right) + \frac{1}{2} A_0 (9f_{5,\phi\phi} + A_0^2 f_{5,\phi\phi}) - 9\phi (G_{4,\phi\phi} - G_{5,\phi\phi}) \]
\[ - \phi^3 \left( 9G_{4,\phi\phi} - \frac{7}{2} G_{5,\phi\phi} \right) + \frac{1}{2} \phi^3 G_{5,\phi\phi} + \frac{1}{2} G_{5,\phi\phi} - \phi^4 \left( G_{4,\phi\phi} - \frac{1}{2} G_{5,\phi\phi} \right) \]
\[ - 3H^2 \left[ A_0 (f_{5,\phi\phi} + f_{5,\phi\phi}) + \frac{1}{2} A_0 (f_{2,\phi\phi} + f_{5,\phi\phi}) - \phi (f_{2,\phi\phi} + f_{5,\phi\phi}) \right] \]
\[ - \frac{1}{4} \phi^2 (f_{2,\phi\phi} + f_{5,\phi\phi}) + \frac{1}{2} A_0 (f_{2,\phi\phi} + f_{5,\phi\phi}) + \frac{1}{2} A_0 (f_{2,\phi\phi} + f_{5,\phi\phi}) + \frac{1}{2} A_0 (f_{2,\phi\phi} + f_{5,\phi\phi}) \]
\[ D_4 = - H^3 \phi^2 \left( 15G_{5,\phi\phi} + 10\phi^2 G_{5,\phi\phi} + 10G_{5,\phi\phi} \right) + 3H^2 \left[ A_0 (f_{5,\phi\phi} - f_{5,\phi\phi}) - 6\phi (G_{4,\phi\phi} - G_{5,\phi\phi}) \right] \]
\[ - \phi^3 (12G_{4,\phi\phi} - 7G_{5,\phi\phi}) - \phi (2G_{4,\phi\phi} - 15G_{5,\phi\phi}) + 3H \left[ 2 (f_{4,\phi\phi} + G_{4,\phi\phi}) - 2A_0^2 f_{4,\phi\phi} \right] \]
\[ + \phi^2 (3G_{3,\phi\phi} + 8G_{4,\phi\phi}) + \phi (G_{3,\phi\phi} + G_{4,\phi\phi}) - \phi^3 (f_{2,\phi\phi} + f_{3,\phi\phi}) + \frac{1}{2} \phi^2 A_0 (f_{2,\phi\phi} + f_{5,\phi\phi}) \]
\[ - \phi (f_{2,\phi\phi} + f_{5,\phi\phi} + f_{2,\phi\phi} + A_0^2 f_{5,\phi\phi}) + \frac{1}{2} A_0 (f_{2,\phi\phi} + f_{5,\phi\phi}) \]
\[ D_5 = H^3 \left[ A_0^3 (f_{5,\phi\phi}) + A_0^4 (f_{5,\phi\phi}) \right] - \phi^3 (5G_{5,\phi\phi} + 10G_{5,\phi\phi}) + 3H^2 \left[ 2 (f_{4,\phi\phi} + G_{4,\phi\phi}) + A_0^2 f_{4,\phi\phi} \right] \]
\[ + \phi A_0 (f_{5,\phi\phi}) - A_0^2 f_{5,\phi\phi}) - \phi^2 (4G_{4,\phi\phi} + 3G_{5,\phi\phi}) - \phi (2G_{4,\phi\phi} - 15G_{5,\phi\phi}) \]
\[ - 3H \left[ 2 A_0^3 (f_{5,\phi\phi}) - 3f_{5,\phi\phi} + 2 (f_{4,\phi\phi}) - A_0^2 f_{5,\phi\phi} - A_0^2 f_{5,\phi\phi} \right] - \phi^3 (G_{3,\phi\phi} + G_{4,\phi\phi}) \]
\[ - \phi^2 (f_{2,\phi\phi} + f_{3,\phi\phi}) + 2 \phi A_0 (f_{4,\phi\phi} - A_0^2 f_{5,\phi\phi}) + f_{2,\phi\phi} + A_0^2 f_{2,\phi\phi} \]
\[ D_6 = H^2 \phi^2 (3G_{5,\phi\phi} + 10G_{5,\phi\phi}) - 2H \left[ A_0 (f_{5,\phi\phi}) - 2 (f_{4,\phi\phi}) + G_{4,\phi\phi} - \phi (2G_{4,\phi\phi} - 15G_{5,\phi\phi}) \right] \]
\[ - \phi^2 (G_{3,\phi\phi} + 2G_{4,\phi\phi}) + 2 (f_{4,\phi\phi}) + G_{4,\phi\phi} \]
\[ D_7 = H \phi^3 (3G_{5,\phi\phi} + 10G_{5,\phi\phi}) \]
\[ - H^2 \left[ A_0 (3f_{5,\phi\phi} + A_0^2 f_{5,\phi\phi}) - 6 \phi (G_{4,\phi\phi} - G_{5,\phi\phi}) - 2 \phi^3 (3G_{4,\phi\phi} - 2G_{5,\phi\phi}) \right] \]
\[ - H \left[ 2 f_{4,\phi\phi} + A_0^2 f_{4,\phi\phi} + G_{4,\phi\phi} \right] - 2A_0 \phi f_{5,\phi\phi} + \phi^2 (3G_{3,\phi\phi} + 10G_{4,\phi\phi} - 2G_{5,\phi\phi}) \]
\[ + \phi (f_{2,\phi\phi} + 2 f_{4,\phi\phi} + 2G_{3,\phi\phi} + G_{4,\phi\phi}) + \frac{1}{2} A_0 (f_{2,\phi\phi} + 4 f_{3,\phi\phi}) \]
\[ D_8 = - \frac{2D_A + D_4 + 3HD_6}{A_0} \]
\[ D_9 = - H^3 A_0^3 (3 f_{5,\phi\phi} + A_0^2 f_{5,\phi\phi}) - 3H^2 \left[ 2 A_0 (f_{4,\phi\phi} + A_0^2 f_{4,\phi\phi}) - \phi (f_{5,\phi\phi} + A_0^2 f_{5,\phi\phi}) \right] \]
\[ + 6H A_0 \left[ A_0 (f_{3,\phi\phi} + f_3) + \phi f_{4,\phi\phi} \right] - \phi \left( \frac{1}{2} f_{2,\phi\phi} + 2 f_{3,\phi\phi} - 2 A_0^2 \phi f_{5,\phi\phi} - A_0 f_{2,\phi\phi} \right) \]
\begin{align}
D_{10} &= -2\dot{H}f_{5,\phi} - H^2 (3f_{5,\phi} + A_0^2 f_{5,\phi,\phi}) - 2HA_0 \left( 2f_{4,\phi,X_3} + \dot{A}_0 f_{5,\phi} \right) - 2\dot{A}_0 f_{4,\phi,X_3} + 2f_{3,\phi} + \frac{1}{2} \dot{f}_{2,X_3}, \\
\text{and} \quad w_1 &= -H^2 \left[ A_0^3 (f_{5,\phi} + f_{5,\phi,X_3}) - \dot{\phi} (5G_{5,\phi,X_1} + \dot{\phi}^2 G_{5,\phi,X_1}) \right] - 2H \left[ 2(f_4 + A_0^2 f_{4,\phi,X_3}) + A_0 \right] \\
&+ A_0 \dot{\phi} (f_4 - A_0^2 f_{5,\phi,X_3}) - \dot{\phi} (4G_{4,X_1} - 3G_{5,\phi}) - \dot{\phi} (2G_{4,X_1} - G_{5,\phi}) \\
&- \dot{\phi}^3 (G_{3,\phi,X_1} + 2G_{4,X_1}) - 2\dot{\phi} (f_4 - A_0^2 f_{4,\phi,X_3} + G_{4,\phi}) + 2A_0^3 (f_{5,\phi} + f_3), \\
w_2 &= w_1 + 2H\dot{\phi}_0 - \dot{\phi} D_0, \\
&= A_0 \left[ -H^2 A_0^2 (f_{5,\phi} + f_{5,\phi,X_3}) - 2H \left[ 2A_0 (f_{4,\phi,X_3} + A_0^2 f_{4,\phi,X_3}) - \dot{\phi} (f_{5,\phi} + A_0^2 f_{5,\phi,X_3}) \right] \\
&+ 2A_0 \dot{\phi} f_{4,\phi,X_3} + 2A_0^3 (f_{3,\phi} + f_3) \right], \\
w_3 &= -2A_0^2 H, \\
w_4 &= w_5 - H^3 \left[ 3A_0^3 (f_{5,\phi} + f_{5,\phi,X_3}) - \dot{\phi} \left( 2G_{5,\phi,X_1} + \frac{13}{2} \dot{\phi}^2 G_{4,\phi,X_1} + \frac{1}{2} \dot{\phi}^4 G_{5,\phi,X_1,X_1} \right) \right] \\
&- 3H^2 \left[ 2(f_4 + A_0) + A_0 \left( 2f_{4,\phi,X_3} + 4A_0^2 f_{4,\phi,X_3} - 3A_0 \dot{\phi} f_{5,\phi,X_3} \right) - \dot{\phi}^2 (7G_{4,\phi,X_1} - 6G_{5,\phi}) \right] \\
&- \dot{\phi} \left( 8G_{4,\phi,X_1} - \frac{9}{2} G_{5,\phi,X_1} \right) \\
&- \dot{\phi} \left( 15G_{4,\phi,X_1} + \frac{13}{2} \dot{\phi}^2 G_{5,\phi,X_1} + \frac{1}{2} \dot{\phi}^4 G_{5,\phi,X_1} \right) + 3H \left[ 2A_0^3 (f_{3,\phi} + f_3) - 2\dot{\phi} (f_4 - 2A_0^2 f_{4,\phi,X_3} + G_{4,\phi}) \right] \\
&- \dot{\phi} \left( 2G_{3,\phi,X_1} + 5G_{4,\phi,X_1} \right) - \dot{\phi} \left( \frac{1}{2} G_{4,\phi,X_1} + G_{4,\phi,X_1} \right) + \frac{1}{2} \dot{\phi} \left( f_2 + \frac{1}{2} f_{2,X_3} + G_{3,\phi} \right) \\
&+ \dot{\phi} \left( \frac{1}{2} f_{2,X_3} - A_0^2 f_{2,X_3} + \frac{1}{8} A_0^2 f_{2,X_3} + G_{3,\phi} \right) - \frac{1}{2} A_0 \dot{\phi} \left( f_2 + A_0^2 f_{2,X_3} + 4f_{3,\phi} - 4A_0^3 f_{3,\phi} \right), \\
w_5 &= \frac{1}{2} H^3 A_0^2 (3f_{5,\phi} + 6A_0 f_{5,\phi,X_3} + A_0^4 f_{5,\phi,X_3}) + 3H^2 A_0 \left[ A_0^3 (f_{4,\phi,X_3} + A_0^2 f_{4,\phi,X_3} + A_0^3 f_{4,\phi,X_3}) \right] \\
&+ \frac{1}{2} \dot{\phi} (f_5,\phi - 2A_0^2 f_{5,\phi,X_3} + A_0^3 f_{5,\phi,X_3}) + 3H A_0^3 \left[ f_{3,\phi} + f_3 + A_0^2 (f_{3,\phi} + f_3) + A_0 \dot{\phi} f_{4,\phi,X_3} \right] \\
&+ \frac{1}{8} A_0^3 \dot{\phi} \left( f_{2,X_3} + 4f_{5,\phi} - 2A_0^2 f_{2,X_3} - 2f_{3,\phi} + 2f_{3,\phi} + 4A_0^3 f_{3,\phi} \right) + \frac{1}{2} A_0^3 f_{2,X_3} + \frac{1}{2} A_0^2 f_{2,X_3}, \\
w_6 &= -\frac{w_1 - \dot{\phi} D_0 + 2HA_0}{A_0} - 4H \left( 2A_0 f_{4,\phi,X_3} - \dot{\phi} f_{5,\phi} + H A_0^2 f_{5,\phi,X_3} \right), \\
&= -H^2 A_0^2 (f_{5,\phi} + A_0^2 f_{5,\phi,X_3}) - 2H \left[ 2A_0 (f_{4,\phi,X_3} - A_0^2 f_{4,\phi,X_3}) - \dot{\phi} (f_{5,\phi} - A_0^2 f_{5,\phi,X_3}) \right] - 2A_0 \dot{\phi} f_{4,\phi,X_3} \\
&- 2A_0^3 (f_{3,\phi} + f_3), \\
w_7 &= -2H \left( 2f_{4,\phi,X_3} + HA_0 f_{5,\phi,X_3} \right) - H^2 \left[ \frac{\dot{\phi} (3f_{5,\phi} + A_0^2 f_{5,\phi,X_3}) + A_0 (f_{5,\phi} + A_0^2 f_{5,\phi,X_3})}{A_0} \right] \\
&- 4H \left( \dot{\phi} f_{4,\phi,X_3} + A_0 f_{4,\phi,X_3} \right) + 2A_0 \left( f_{3,\phi} + f_3 \right) + \frac{\dot{\phi} (f_{2,X_3} + 4f_{3,\phi})}{2A_0}, \\
w_8 &= 3H w_1 - 2w_4 - \dot{\phi} D_1. 
\end{align}

We note that we used other background eqs. (3.5) and (3.8) for the derivation of these coefficients.

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