Universally Fisher-Symmetric Informationally Complete Measurements

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A quantum measurement is Fisher symmetric if it provides uniform and maximal information on all parameters that characterize the quantum state of interest. Using (complex projective) 2-designs, we construct measurements on a pair of identically prepared quantum states that are Fisher symmetric for all pure states. Such measurements are optimal in achieving the minimal statistical error without adaptive measurements. We then determine all collective measurements on a pair that are Fisher symmetric for the completely mixed state and for all pure states simultaneously. For a qubit, these measurements are Fisher symmetric for all states. The minimal optimal measurements are tied to the elusive symmetric informationally complete measurements, which reflects a deep connection between local symmetry and global symmetry. In the study, we derive a fundamental constraint on the Fisher information matrix of any collective measurement on a pair, which offers a useful tool for characterizing the tomographic efficiency of collective measurements.

Introduction.—Quantum state tomography is a primitive of various quantum information processing tasks, such as quantum communication and metrology. Crucial to achieving high tomographic efficiency is a judicious choice of the quantum measurement, which is usually represented by a positive-operator-valued measure (POVM). A POVM is informationally complete (IC) if all states can be determined uniquely by the measurement statistics. A symmetric informationally complete POVM (SIC for short) is a special IC POVM that is distinguished by global symmetry between POVM elements. SICs are optimal on average in minimal or linear quantum state tomography and are also interesting for many other reasons, including foundational studies.

In the paradigm of local tomography, the quantum state is known to be in the neighborhood of a fiducial state. In this case, the Fisher information matrix is a useful tool for analyzing the performance of a quantum measurement as its inverse sets a lower bound for the mean-square-error (MSE) matrix of any unbiased estimator. A quantum measurement is Fisher symmetric if it provides uniform and maximal information on all parameters that characterize the quantum state of interest. Here uniformity means that the Fisher information matrix is proportional to the quantum Fisher information matrix, and maximality means that no other measurement can provide more information. Such measurements, if they exist, are as efficient as possible for estimating all parameters of interest.

For pure states, Li et al. offered a method for constructing measurements that are Fisher symmetric for any arbitrary, but fixed state. The covariant measurement composed of all pure states weighted by the Haar measure is simultaneously Fisher symmetric for all pure states, but is not realistic to implement. For mixed states, Fisher-symmetric measurements in general cannot exist except for the completely mixed state. Even the covariant measurement is far from being Fisher symmetric for a generic state, which means mixed states cannot be estimated efficiently by fixed measurements if infidelity or Bures distance is the figure of merit.

In this paper, we show that many limitations mentioned above can be overcome if we can measure a pair of identically prepared quantum states together. Such collective measurements are known to provide more information than separable measurements and are tied to a number of nonclassical phenomena. Using complex projective 2-designs, we construct quantum measurements that are universally Fisher symmetric for all pure states and that have no more than $4d^2$ outcomes for a $d$-level system. These measurements are optimal in achieving the minimal statistical error without adaptive measurements. We then determine all measurements that are Fisher symmetric for the completely mixed state and all pure states. Interestingly, the minimal optimal measurements are tied to SICs, which reveals a deep connection between local symmetry and global symmetry. For a qubit, we determine all measurements on a pair that are universally Fisher symmetric for all states and show that they are significantly more efficient than all local measurements.

In the study, we derive a fundamental constraint on the Fisher information matrix of any collective measurement on a pair, which offers a useful tool for characterizing the efficiency of collective measurements.
Preliminaries.—Suppose the quantum state $\rho(\theta)$ acting on the $d$-dimensional Hilbert space $\mathcal{H}$ is characterized by parameters $\theta_1, \theta_2, \ldots, \theta_d$, where $g = 2d - 2$ for pure states and $g = d^2 - 1$ for mixed states. A POVM is a set of positive operators that sum up to the identity. Given a POVM $\{\Pi_\xi\}$ on $\mathcal{H}^\otimes N$ with positive integer $N$, the probability of obtaining the outcome $\xi$ is $p_\xi(\theta) = tr[\rho(\theta)^{\otimes N} \Pi_\xi]$. The Fisher information matrix $I(N)(\theta)$ has matrix elements

$$I_{ab}(N)(\theta) = \sum_{\xi,\eta > 0} \frac{1}{p_\xi} \frac{\partial p_\xi}{\partial \theta_a} \frac{\partial p_\xi}{\partial \theta_b}.$$  

(1)

The inverse Fisher information matrix sets a lower bound for the MSE matrix $C(N)(\theta)$ (also known as the covariance matrix) of any unbiased estimator, which is known as the Cramér-Rao bound [17]; see the supplement, which includes Refs. [32–44].

The quantum Fisher information matrix $J(\theta)$ [14] has matrix elements,

$$J_{ab}(\theta) = \frac{1}{2} \left[ tr(\rho L_a L_b + L_b L_a) \right],$$

(2)

where the Hermitian operator $L_a$ satisfies the equation $\frac{1}{2}(L_a + L_a^\dagger) = \rho_a := \frac{\partial \rho(\theta)}{\partial \theta_a}$ and is known as the symmetric logarithmic derivative associated with $\theta_a$. The matrix $J(\theta)$ is an upper bound for the scaled Fisher information matrix $I(N)(\theta)/N$, so its inverse is a lower bound for the scaled MSE matrix $N C(N)(\theta)$, which is known as the quantum Cramér-Rao bound [14]. The bound generally cannot be saturated except when different $L_a$ can be measured simultaneously.

Another fundamental constraint on the Fisher information matrix $I(N)(\theta)$ is the following inequality derived by Gill and Massar (GM) [19],

$$tr[J^{-1}(\theta) I(N)(\theta)] \leq N(d-1),$$

(3)

where $tr[J^{-1}(\theta) I(\theta)]$ is independent of the parametrization. For pure states, the GM inequality applies to arbitrary measurements [19, 43]. For mixed states, it applies to arbitrary separable measurements, but may fail for certain collective measurements [19, 21]. When $N = 1$, the GM inequality is saturated iff the POVM $\{\Pi_\xi\}$ is rank one and $tr[\rho(\theta) \Pi_\xi] \neq 0$; see the supplement. In addition, the GM inequality is useful to studying uncertainty relations [16] and quantum steering [17].

A POVM on $\mathcal{H}^\otimes t$ with positive integer $t$ is weakly Fisher symmetric for $\rho(\theta)$ if $I(t)(\theta)$ is proportional to $J(\theta)$ and Fisher symmetric if $tr[J^{-1}(\theta) I(t)(\theta)]$ further attains the maximum over all POVMs on $\mathcal{H}^\otimes t$ [18]. Such POVMs are as efficient as possible for estimating all parameters of interest. Pure states are specified by $2(d-1)$ parameters, and the inequality in Eq. (3) with $N = t$ holds for any POVM on $\mathcal{H}^\otimes t$, so Fisher symmetry means $I(t)(\theta) = \frac{1}{2} J(\theta)$. In that case, each parameter is determined with half of the maximum resolution for determining this parameter separately. When $t = 1$, Fisher symmetry for a mixed state means $I(\theta) = \frac{1}{2} J(\theta)$. Such measurements, if they exist, are optimal in minimizing the mean square Bures distance (MSB) and mean infidelity [21, 24]; see the supplement.

When $t = 1$, Fisher-symmetric measurements have been constructed by Li et al. for any given pure state [18]. In addition, the invariant measurement composed of all pure states weighted by the Haar measure is simultaneously Fisher symmetric for all pure states [20, 22]. However, no measurement with a finite number of outcomes has this property as shown in Theorem 1 below, which is proved in the supplement. For a mixed state, usually there is no Fisher-symmetric measurement except for the completely mixed state and qubit states [18, 21, 24]. Even in the case of a qubit, no measurement is Fisher symmetric for all states; even the covariant measurement is far from being Fisher symmetric for a generic state [22].

**Theorem 1.** No measurement on $\mathcal{H}$ with a finite number of outcomes is Fisher symmetric for all pure states.

Before studying collective measurements, we need to introduce several additional concepts. A weighted set of pure quantum states $\{|\psi_\xi\rangle, w_\xi\}$ in $\mathcal{H}$ with $w_\xi \geq 0$ is a (weighted) t-design [9–11] if $\sum_\xi w_\xi |\psi_\xi\rangle \langle \psi_\xi|$ is proportional to the projector $P_t$ onto the symmetric subspace $\mathcal{H}_+^t$ of $\mathcal{H}^\otimes t$. We are mostly interested in 2-designs and will assume $t = 2$ in the following discussion except when stated otherwise. Any 2-design $\{|\psi_\xi\rangle, w_\xi\}$ has at least $d^2$ elements, and the lower bound is saturated iff all weights $w_\xi$ are equal, and $\{|\psi_\xi\rangle\}$ forms a SIC [9, 11, 43] $|\langle \psi_\xi| \psi_\eta\rangle|^2 = \frac{d \delta_{\xi \eta} + 1}{d + 1}$, $\forall \xi, \eta$. (4)

**Fisher-symmetric measurements for pure states.—**Now we are ready to construct measurements on $\mathcal{H}^\otimes 2$ that are Fisher symmetric for all pure states. Since $\rho^{\otimes 2}$ is supported on the symmetric subspace $\mathcal{H}_+^2$ whenever $\rho$ is pure, it suffices to construct a POVM on this subspace. Let $\{|\psi_\xi\rangle, w_\xi\}$ be a 2-design with $\sum_\xi w_\xi = d(d+1)/2$. Then the operators $\Pi_\xi = w_\xi |\psi_\xi\rangle \langle \psi_\xi|$ form a POVM on $\mathcal{H}_+$. Such POVMs have been studied before and are known to be optimal on average for certain estimation problems [26, 49]. However, little is known about their performance with regard to local tomography. Here we shall show that these POVMs are optimal for every parameter point simultaneously. Note that the existence of such an efficient POVM itself is highly nontrivial.

**Theorem 2.** Let $\{|\psi_\xi\rangle, w_\xi\}$ be a 2-design with $\sum_\xi w_\xi = d(d+1)/2$ and $\Pi_\xi = w_\xi |\psi_\xi\rangle \langle \psi_\xi|^{\otimes 2}$. Then the POVM $\{\Pi_\xi\}$ on $\mathcal{H}_+$ is Fisher symmetric for all pure states.

**Proof.** The probability of obtaining outcome $\xi$ of $\{\Pi_\xi\}$ is $p_\xi = tr(\rho^{\otimes 2} \Pi_\xi) = w_\xi |\langle \psi_\xi| \psi_\xi\rangle|^2$, which is factorized. These probabilities determine probabilities associated with the companion POVM $\{2 w_\xi |\psi_\xi\rangle \langle \psi_\xi|\}$ on $\mathcal{H}$, which is IC. So $\{|\psi_\xi\rangle\}$ is also IC. The Fisher information
matrix \( I^+ \) provided by \( \{ \Pi_\xi \} \) has matrix elements
\[
I^+_{ab} = 4 \sum_\xi w_\xi \langle \psi_\xi | \rho_a | \psi_\xi \rangle \langle \psi_\xi | \rho_b | \psi_\xi \rangle = 4 \text{tr}[(\rho_a \otimes \rho_b)P_+] = 2 \text{tr}(\rho_a \rho_b),
\]
\( (5) \)
\( \)

note that \( \rho_a \) are traceless and that \( \langle \psi_\xi | \rho_a | \psi_\xi \rangle = 0 \) whenever \( \langle \psi_\xi | \rho_a | \psi_\xi \rangle = 0 \). Interestingly, the Fisher information matrix is independent of the specific measurement, as long as \( \{ | \psi_\xi \rangle, w_\xi \} \) is a 2-design. In particular, it is invariant under the unitary transformation \( \Pi \rightarrow U \otimes \Pi (U \otimes \Pi)^\dagger \) for any unitary \( U \) on \( \mathcal{H} \). Therefore, to show that \( \{ \Pi_\xi \} \) is Fisher symmetric for all pure states, it suffices to consider any given pure state, say \( \rho = |0\rangle \langle 0 | \), assuming \( |j \rangle \) for \( j = 0, 1, \ldots, d - 1 \) form an orthonormal basis. For pure states, we can choose a suitable parametrization such that \( \rho_a \) take on the form \( (19) \)
\[
\rho_a = \begin{cases} |a\rangle \langle 0 | + | 0 \rangle \langle a | & 1 \leq a \leq d - 1, \\ i(|a\rangle \langle 0 | - | 0 \rangle \langle a |) & d \leq a \leq 2(d - 1), \end{cases}
\]
\( (6) \)
\( \)

where \( a' = a - d + 1 \). Then \( L_a = 2\rho_a \) and
\[
I^+_{ab} = J_{ab} = 4\delta_{ab}.
\]
\( (7) \)
\( \)

So \( \{ \Pi_\xi \} \) is Fisher symmetric for all pure states. \( \square \)

If a SIC exists in dimension \( d \) \([8, 10]\), then we can construct a Fisher-symmetric measurement for all pure states with only \( d^2 \) outcomes. In every prime power dimension, such a measurement can be constructed using a complete set of mutually unbiased bases (MUB) \([50–52]\), which forms a 2-design with \( d^2 + d \) elements. In general, let \( d' \) be the smallest prime power that is not smaller than \( d \) (which satisfies \( d' \leq 2d - 2 \)); then a 2-design in dimension \( d \) can be constructed by projecting a complete set of mutually unbiased bases in dimension \( d' \) to a subspace of dimension \( d \). So we can always construct a Fisher-symmetric measurement for all pure states in dimension \( d \) with no more than \( 4d^2 \) outcomes. It is worth pointing out that tensor products of POVMs constructed above are also Fisher symmetric for all pure states. So are POVMs on \( \mathcal{H}^\otimes t \) constructed from \( t \)-designs with \( t \geq 3 \). However, such POVMs offer little advantage over those constructed from 2-designs, but are much more difficult to implement.

\begin{itemize}
  \item \textbf{Fisher-symmetric measurements for mixed states.}—
  \textbf{Here we need to generalize the concepts of 2-designs and SICs.} A set of positive operators \( \{ \Pi_\xi \} \) is called a \textit{generalized 2-design} if
\end{itemize}
\[
\sum_\xi \Pi_\xi \otimes \Pi_\xi \leq \frac{2}{\text{tr}(\Pi_\xi)} \sum_\xi w_\xi \left( \frac{1 + \varphi}{d + 1} P_+ + \frac{1 - \varphi}{d - 1} P_- \right),
\]
\( (8) \)
\( \)

where \( w_\xi = \text{tr}(\Pi_\xi), \varphi = \sum_\xi w_\xi \varphi_\xi / (\sum_\xi w_\xi), \varphi_\xi = \text{tr}(\Pi_\xi)/(\text{tr}(\Pi_\xi)^2), \text{ and } P_- \) is the projector onto the anti-symmetric subspace \( \mathcal{H}_- \) of \( \mathcal{H}^\otimes 2; \) cf. Ref. \([53]\). Here \( \varphi_\xi \)
may be interpreted as the purity of \( \Pi_\xi \), and \( \varphi \) as the purity of the set \( \{ \Pi_\xi \} \). A set of \( d^2 \) positive operators \( \{ \Pi_\xi \} \) is a \textit{generalized SIC} if \( \sum_\xi \Pi_\xi \) is proportional to the identity and \( \text{tr}(\Pi_\Pi) = \alpha \delta_{\eta} + \beta \) for some positive constants \( \alpha, \beta \) \([13, 54, 55]\). Any generalized SIC is a generalized 2-design; see the supplement for a partial converse.

No measurement on \( \mathcal{H} \) is Fisher symmetric for a mixed state \( \rho \) except when \( \rho \) is the completely mixed state or a qubit state \([15]\). In preparation for later applications, Proposition \([11]\) and Corollary \([11]\) below clarify the structure of (weakly) Fisher-symmetric measurements at the completely mixed state. The proofs are relegated to the supplement, which also explains the connection with tight IC measurements introduced by Scott \([11]\).

\begin{itemize}
  \item \textbf{Proposition 1.} A POVM \( \{ \Pi_\xi \} \) on \( \mathcal{H} \) is (weakly) Fisher symmetric at the completely mixed state iff \( \{ \Pi_\xi \} \) is a (generalized) 2-design.
  \textbf{Corollary 1.} Any POVM \( \{ \Pi_\xi \} \) on \( \mathcal{H} \) that is Fisher symmetric at the completely mixed state has at least \( d^2 \) elements; the lower bound is saturated iff \( \{ \Pi_\xi \} \) is a SIC.
\end{itemize}

It is much more difficult to study Fisher-symmetric measurements for mixed states when \( t \geq 2 \), because the GM inequality does not apply to collective measurements, and an extension of the GM inequality has been a long-standing open problem. Nevertheless, we have a simple solution in the case \( t = 2 \). For simplicity, \( \rho(\theta) \) has full rank in the rest of the paper.

\begin{itemize}
  \item \textbf{Theorem 3.} The Fisher information matrix \( I^{(2)}(\theta) \) at \( \rho(\theta) \) of any POVM \( \{ \Pi_\xi \} \) on \( \mathcal{H}^\otimes 2 \) satisfies
\end{itemize}
\[
\text{tr}[J^{-1}(\theta)I^{(2)}(\theta)] \leq 3d - 3.
\]
\( (9) \)
\( \)

The inequality is saturated iff each \( \Pi_\xi \) is proportional to either the tensor power of a pure state or a Slater-determinant state.

A variant of Theorem \([3]\) was proved in the thesis of the first author \([21]\); see the supplement for a simplified proof. Here a Slater–determinant state has the form \( U \otimes |\Psi_\rightarrow \rangle \langle \Psi_\rightarrow | U \otimes |\Psi_\rightarrow \rangle \langle \Psi_\rightarrow | \), where \( |\Psi_\rightarrow \rangle = (|01\rangle - |10\rangle) / \sqrt{2} \), and \( U \) is a unitary. Both Slater-determinant states and tensor powers of pure states are generalized coherent states \([56]\), which are least entangled and most classical for the given symmetry. Measurements (POVMs) composed of these states are referred to as \textit{coherent measurements} (POVMs) henceforth. The inequality in Eq. \([4]\) is saturated iff the POVM is coherent. Some POVMs known in the literature \([19, 57]\) are coherent by our definition, although they were introduced for different purposes.

\begin{itemize}
  \item \textbf{Theorem 3} implies \( \text{tr}[J^{-1}(\theta)I^{(N)}(\theta)] \leq 3N(d - 1)/2 \) if each time we can measure at most two copies of \( \rho(\theta) \) together, that is, if the POVM on \( \mathcal{H}^\otimes N \) is a tensor product of POVMs on \( \mathcal{H} \) or \( \mathcal{H}^\otimes 2 \). Compared with the GM inequality in Eq. \([3]\), here the upper bound is 50% larger, which reflects the advantage of collective measurements.
over separable measurements. The following corollary is a tomographic implication of Theorem 3 and the analog of the GM bound in Eq. (30) of Ref. 19 (note that a factor of \(1/(d-1)\) is missing there); see the supplement on the GM bound.

**Corollary 2.** In quantum state tomography with any collective measurement on \(\mathcal{H}^{\otimes 2}\), the scaled weighted mean square error (WMSE) \(N \text{tr}(WC(N))\) of any unbiased estimator is bounded from below by

\[
E_W = \frac{2(\text{tr}J^{-1/2}WJ^{-1/2}))^2}{3(d-1)}.
\]

The bound can be saturated iff there exists a measurement on \(\mathcal{H}^{\otimes 2}\) that yields the Fisher information matrix

\[
I_W^{(2)} = 3(d-1)J^{1/2}\frac{\sqrt{J^{-1/2}WJ^{-1/2}}}{\text{tr}J^{-1/2}WJ^{-1/2}}J^{1/2}.
\]

Here \(W\) is a positive semidefinite matrix that may depend on the parameter point. For MSB, \(W = J/4\) [3], so the bound in Eq. 10 is saturated iff the measurement yields \(I^{(2)} = 3J/(d+1)\) and is thus Fisher symmetric.

Any coherent POVM \(\{\Pi_\xi\}\) is the union of two POVMs \(\{\Pi_\xi^\top\}\) and \(\{\Pi_\xi\}\) on the symmetric and antisymmetric subspaces, respectively. The POVM \(\{\Pi_\xi^\top\}\) is tied to a 2-design and is thus Fisher symmetric for all pure states. Its contribution to the Fisher information matrix is independent of the specific POVM \(\{\Pi_\xi\}\) by Eq. 5. Any coherent POVMs yield the same Fisher information matrix. Since any 2-design has at least \(d^2\) elements, and minimal 2-designs are in one-to-one correspondence with SICs [10, 11, 13, 18], we deduce the following corollary.

**Corollary 3.** Any coherent POVM on \(\mathcal{H}^{\otimes 2}\) has at least \(\frac{1}{2}(3d^2 - d)\) elements. The bound is saturated iff \(\frac{d}{2}(d-1)\) elements are Slater-determinant states and form a projective measurement on \(\mathcal{H}_+\), and the other \(d^2\) elements have the form \(\frac{1}{2d}\{(\psi_\xi)\otimes(\psi_\xi)\}^{\otimes 2}\), where \(\{\psi_\xi\}\) is a SIC.

In the case of a qubit, a minimal coherent POVM has five elements, which take on the form [21, 57]

\[
\Pi_\xi = \frac{3}{4}(|\psi_\xi\rangle\langle\psi_\xi|)^{\otimes 2}, \quad \Pi_5 = |\psi_-\rangle\langle\psi_-|.
\]

where \(|\psi_\xi\rangle\) for \(\xi = 1, 2, 3, 4\) form a SIC, and \(|\psi_-\rangle\) is the singlet. This POVM is referred to as the collective SIC henceforth. Remarkably, it is universally Fisher symmetric, that is, Fisher symmetric for all states. To see this, parametrize the qubit state \(\rho\) by the Bloch vector \(s = (s_1, s_2, s_3)\) as \(\rho = \frac{1}{2}(1 + s \cdot \sigma)\). Then

\[
I_{ab}^{(2)} = J_{ab} = \delta_{ab} + \frac{s_\alpha s_\beta}{1 - s^2}, \quad a, b = 1, 2, 3.
\]

In conjunction with Theorem 3, we deduce the following.

**Theorem 4.** When \(d = 2\), a POVM \(\{\Pi_\xi\}\) on \(\mathcal{H}^{\otimes 2}\) is universally Fisher symmetric iff it is coherent.

![FIG. 1. (color online) Scaled MSE (left plot) and scaled MSB (right plot) achieved by universally Fisher-symmetric measurements (UFS), including the collective SIC, in qubit state tomography. Here \(s\) is the length of the Bloch vector. The performances of SIC, MUB, and covariant measurements (averaged over states with the same purity) are shown for comparison (reproduced from Ref. [22]).](image)

The scaled MSE (with respect to the Hilbert-Schmidt distance) and scaled MSB achieved by the collective SIC are respectively given by

\[
E(\rho) = 3 - s^2, \quad E_{SB}(\rho) = \frac{3}{2},
\]

as illustrated in Fig. 1. The tomographic efficiency is much higher than all POVMs on individual copies. In particular, the scaled MSB achieved by any fixed measurement on individual copies diverges in the pure-state limit [21, 22, 24]; accordingly, the mean infidelity scales as \((1/\sqrt{N})\) [23]. By contrast, the scaled MSB achieved by the collective SIC saturates the bound in Eq. 10 with \(W = J/4\), so that the mean infidelity scales as \((1/N)\). Recently, the collective SIC was successfully realized in experiments, which achieved the highest tomographic efficiency in qubit state tomography to date [31].

In general, to be Fisher symmetric, the Fisher information matrix should equal \(I^{(2)} = 3J/(d+1)\) according to Theorem 3. In the limit to pure states, this requirement is not compatible with Eq. 7 when \(d \geq 3\). Therefore, we believe that Fisher-symmetric measurements in general cannot exist for mixed states when \(d \geq 3\). Nevertheless, it is still desirable to construct POVMs that are Fisher symmetric for the completely mixed state and for all pure states simultaneously. Such POVMs are called tight: they are optimal in the tomography of pure states and highly mixed states. According to Theorem 3 all tight POVMs on \(\mathcal{H}^{\otimes 2}\) are coherent POVMs, which are automatically Fisher symmetric for all pure states. Theorems 5 and 6 below clarify the structure of such POVMs; the proofs are relegated to the supplement.

**Theorem 5.** A POVM \(\{\Pi_\xi\}\) on \(\mathcal{H}^{\otimes 2}\) is tight coherent (Fisher symmetric for the completely mixed state and all pure states) iff \(\{Q_\xi\}\) is a generalized 2-design of purity \(\frac{3d-1}{4}\), where \(Q_\xi = \text{tr}_1(\Pi_\xi) + \text{tr}_2(\Pi_\xi)\).

Theorem 5 offers a recipe for creating tight coherent POVMs. Let \(\{A_\xi\}\) be a 2-design with \(\sum_\xi A_\xi = (d+1)/2\) and \(\{B_\eta\}\) a generalized 2-design with \(\sum_\eta B_\eta = 2(d-1)\)
and with $B_\eta$ proportional to rank-2 projectors. Then the union of $\{\Pi_\xi^+\}$ and $\{\Pi_\xi^-\}$ is tight coherent, where

$$\Pi_\xi^\pm = \frac{A_\xi \otimes A_\xi}{\text{tr}(A_\xi^2)}, \quad \Pi_\eta^\pm = \frac{P_{\pm}(B_\eta \otimes B_\eta)P_{\pm}}{\text{tr}(B_\eta)}.$$  \hspace{1cm} (15)

**Theorem 6.** Any tight coherent POVM $\{\Pi_\xi\}$ on $\mathcal{H}^\otimes 2$ has at least $2d^2$ elements when $d \geq 3$. The lower bound is saturated iff $\{\Pi_\xi\}$ is a union of two POVMs $\{\Pi_\xi^+\}$ and $\{\Pi_\xi^-\}$ on $\mathcal{H}_+$ and $\mathcal{H}_-$, respectively, $\{Q_\xi^+\}$ forms a SIC, and $\{Q_\xi^-\}$ forms a generalized SIC of purity $\frac{1}{2}$.

**Summary.**—We introduced a general method for constructing two-copy collective measurements that are universally Fisher symmetric for all pure states. These measurements are optimal in achieving the minimal statistical error without adaptive measurements. We also determined all collective measurements on a pair that are Fisher symmetric for the completely mixed state and for all pure states. For a qubit, they are Fisher symmetric for all states and are substantially more efficient than all local measurements. In the study, we derived a fundamental constraint on the Fisher information matrix of any collective measurement on a pair, which provides a useful tool for characterizing the power of collective measurements. Our work is of interest not only to studying quantum measurements and estimation theory, but also to improving efficiency and precision in practical quantum state tomography and multiparameter quantum metrology.

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[1] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976).
[2] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982).
[3] M. Hayashi, ed., *Asymptotic Theory of Quantum Statistical Inference* (World Scientific, Singapore, 2005).
[4] S. L. Braunstein and C. M. Caves, “Statistical distance and the geometry of quantum states,” Phys. Rev. Lett. 72, 3439–3443 (1994).
[5] M. G. A. Paris and J. Řeháček, eds., *Quantum State Estimation*, Lecture Notes in Physics, Vol. 649 (Springer, Berlin, 2004).
[6] A. I. Lvovsky and M. G. Raymer, “Continuous-variable optical quantum-state tomography,” Rev. Mod. Phys. 81, 299 (2009).
[7] V. Giovannetti, S. Lloyd, and L. Maccone, “Advances in quantum metrology,” Nat. Photonics 5, 222 (2011).
[8] C. A. Fuchs, M. C. Hoang, and B. C. Stacey, “The SIC question: History and state of play,” Axioms 6, 21 (2017).
[9] G. Zauner, “Quantum designs: Foundations of a noncommutative design theory,” Int. J. Quantum Inf. 9, 445–507 (2011).
[10] J. M. Renes, R. Blume-Kohout, A. J. Scott, and C. M. Caves, “Symmetric informationally complete quantum measurements,” J. Math. Phys. 45, 2171 (2004).
[11] A. J. Scott, “Tight informationally complete quantum measurements,” J. Phys. A: Math. Gen. 39, 13507 (2006).
[12] H. Zhu and B.-G. Englert, “Quantum state tomography with fully symmetric measurements and product measurements,” Phys. Rev. A 84, 022327 (2011).
[13] H. Zhu, “Tomographic and Lie algebraic significance of generalized symmetric informationally complete measurements,” Phys. Rev. A 90, 032309 (2014).
[14] C. A. Fuchs and R. Schack, “Quantum-Bayesian coherence,” Rev. Mod. Phys. 85, 1603–1715 (2013).
[15] M. Appleby, C. A. Fuchs, B. C. Stacey, and H. Zhu, “Introducing the Qplex: a novel arena for quantum theory,” Eur. Phys. J. D 71, 197 (2017).
[16] H. Zhu, “Quasiprobability representations of quantum mechanics with minimal negativity,” Phys. Rev. Lett. 117, 120404 (2016).
[17] C. R. Rao, *Linear Statistical Inference and its Applications*, Wiley Series in Probability and Statistics (Wiley-Interscience, 2002).
[18] N. Li, C. Ferrie, J. A. Gross, A. Kalev, and C. M. Caves, “Fisher-symmetric informationally complete measurements for pure states,” Phys. Rev. Lett. 116, 180402 (2016).
[19] R. D. Gill and S. Massar, “State estimation for large ensembles,” Phys. Rev. A 61, 042312 (2000).
[20] M. Hayashi, “Asymptotic estimation theory for a finite-dimensional pure state model,” J. Phys. A: Math. Gen. 31, 4633 (1998).
[21] H. Zhu, *Quantum State Estimation and Symmetric Informationally Complete POMs*, Ph.D. thesis, National University of Singapore (2012).
[22] H. Zhu, “Quantum state estimation with informationally overcomplete measurements,” Phys. Rev. A 90, 012115 (2014).
[23] D. H. Mahler, L. A. Rozema, A. Darabi, C. Ferrie, R. Blume-Kohout, and A. M. Steinberg, “Adaptive quantum state tomography improves accuracy quadratically,” Phys. Rev. Lett. 111, 183601 (2013).
[24] Z. Hou, H. Zhu, G.-Y. Xiang, C.-F. Li, and G.-C. Guo, “Achieving quantum precision limit in adaptive qubit state tomography,” npj Quantum Information 2, 16001 (2016).
[25] A. Peres and W. K. Wootters, “Optimal detection of quantum information,” Phys. Rev. Lett. 66, 1119–1122 (1991).
[26] S. Massar and S. Popescu, “Optimal extraction of information from finite quantum ensembles,” Phys. Rev. Lett. 74, 1259–1263 (1995).
[27] E. Bagan, M. A. Ballester, R. D. Gill, R. Muñoz-Tapia, and O. Romero-Isart, “Separable measurement estimation of density matrices and its fidelity gap with collective protocols,” Phys. Rev. Lett. 97, 130501 (2006).
[28] M. D. Vidrighin, G. Donati, M. G. Genoni, X.-M. Jin, W. S. Koltzhammer, M. S. Kim, A. Datta, M. Barbieri, and I. A. Walmsley, “Joint estimation of phase and phase diffusion for quantum metrology,” Nat. Commun. 5, 3532 (2014).
[29] C. H. Bennett, D. P. DiVincenzo, C. A. Fuchs, T. Mor, E. Rains, P. W. Shor, J. A. Smolin, and W. K. Wootters, “Quantum nonlocality without entanglement,” Phys. Rev. A 59, 1070–1091 (1999).
[30] N. Gisin and S. Popescu, “Spin flips and quantum information for antiparallel spins,” Phys. Rev. Lett. 83, 432–435 (1999).
[31] Z. Hou, J.-F. Tang, J. Shang, H. Zhu, J. Li, Y. Yuan, K.-D. Wu, G.-Y. Xiang, C.-F. Li, and G.-C. Guo, “Deterministic realization of superefficient collective measurements via photonic quantum walks,” (2017), arXiv:1710.10045.
[32] M. Hayashi and K. Matsumoto, “Asymptotic performance of optimal state estimation in qubit system,” J. Math. Phys. 49, 102101 (2008).
[33] J. Kahn and M. Gutä, “Local asymptotic normality for finite dimensional quantum systems,” Commun. Math. Phys. 280, 597–652 (2009).
[34] H. Nagaoka, “A new approach to Cramér-Rao bounds for quantum state estimation,” IEICE Technical Report IT 89-42, 9–14 (1989), reprinted in Ref. [3].
[35] M. Hayashi, “A linear programming approach to attainable Cramér-Rao type bounds,” in Quantum Communication, Computing, and Measurement, edited by O. Hirota, A. S. Holevo, and C. M. Caves (Plenum, New York, 1997) pp. 99–108, reprinted in Ref. [3].
[36] I. Bengtsson and K. Życzkowski, Geometry of Quantum States: An Introduction to Quantum Entanglement (Cambridge University Press, Cambridge, UK, 2006).
[37] O. E. Barndorff-Nielsen and R. D. Gill, “Fisher information in quantum statistics,” J. Phys. A: Math. Gen. 33, 4481 (2000).
[38] C. Dankert, R. Cleve, J. Emerson, and E. Livine, “Exact and approximate unitary 2-designs and their application to fidelity estimation,” Phys. Rev. A 80, 012304 (2009).
[39] D. Gross, K. Audenaert, and J. Eisert, “Evenly distributed unitaries: On the structure of unitary designs,” J. Math. Phys. 48, 052104 (2007).
[40] A. Roy and A. J. Scott, “Unitary designs and codes,” Des. Codes Cryptogr. 53, 13–31 (2009).
[41] D. M. Appleby, “Symmetric informationally complete-positive operator valued measures and the extended Clifford group,” J. Math. Phys. 46, 052107 (2005).
[42] H. Zhu, “SIC POVMs and Clifford groups in prime dimensions,” J. Phys. A: Math. Theor. 43, 305305 (2010).
[43] L. P. Hughston and S. M. Salamon, “Surveying points in the complex projective plane,” Adv. Math. 286, 1017–1052 (2016).
[44] F. Szöllösí, “All complex equiangular tight frames in dimension 3,” (2014), arXiv:1402.6429.
[45] K. Matsumoto, “A new approach to the Cramér–Rao-type bound of the pure-state model,” J. Math. A: Math. Gen. 35, 3111 (2002).
[46] H. Zhu, “Information complementarity: A new paradigm for decoding quantum incompatibility,” Sci. Rep. 5, 14317 (2015).
[47] H. Zhu, M. Hayashi, and L. Chen, “Universal steering criteria,” Phys. Rev. Lett. 116, 070403 (2016).
[48] D. M. Appleby, C. A. Fuchs, and H. Zhu, “Group theoretic, Lie algebraic and Jordan algebraic formulations of the SIC existence problem,” Quantum Inf. Comput. 15, 61–94 (2015).
[49] A. Hayashi, T. Hashimoto, and M. Horibe, “Reexamination of optimal quantum state estimation of pure states,” Phys. Rev. A 72, 032325 (2005).
[50] W. K. Wootters and B. D. Fields, “Optimal state determination by mutually unbiased measurements,” Ann. Phys. 191, 363 (1989).
[51] T. Durt, B.-G. Englert, I. Bengtsson, and K. Życzkowski, “On mutually unbiased bases,” Int. J. Quantum Inf. 8, 535 (2010).
[52] H. Zhu, “Mutually unbiased bases as minimal Clifford covariant 2-designs,” Phys. Rev. A 91, 060301(R) (2015).
[53] M. A. Graydon and D. M. Appleby, “Quantum conical designs,” J. Phys. A: Math. Theor. 49, 085301 (2016).
[54] D. M. Appleby, “Symmetric informationally complete measurements of arbitrary rank,” Opt. Spectrosc. 103, 416–428 (2007).
[55] G. Gour and A. Kalev, “Construction of all general symmetric informationally complete measurements,” J. Phys. A: Math. Theor. 47, 335302 (2014).
[56] W.-M. Zhang, D. H. Feng, and R. Gilmore, “Coherent states: Theory and some applications,” Rev. Mod. Phys. 62, 867–927 (1990).
[57] G. Vidal, J. I. Latorre, P. Pascual, and R. Tarrach, “Optimal minimal measurements of mixed states,” Phys. Rev. A 60, 126 (1999).
Universally Fisher-Symmetric Informationally Complete Measurements: Supplement

In this supplement, we prove Theorems 1, 3, 5, 6, Proposition 1, and Corollary 1 presented in the main text. We also provide more details on quantum state tomography with collective measurements, the Gill-Massar inequality, Gill-Massar bound [19, 21, 24], generalized 2-designs, generalized symmetric informationally complete measurements (SICs for short) [9, 10, 13, 54, 55], and tight coherent measurements in dimension 3.

I. QUANTUM STATE TOMOGRAPHY WITH COLLECTIVE MEASUREMENTS

Quantum state tomography is a procedure for inferring the state of a quantum system from statistics of quantum measurements [1–3, 5, 6]. To achieve sufficient precision, usually many identically prepared quantum systems need to be measured. The simplest measurement strategy is to repeat a given measurement strategy is to repeat a given quantum measurement [1–3, 5, 6]. To achieve sufficient precision, usually many identically prepared systems are available for tomography, as illustrated in the left plot of Fig. S1. In this case, the efficiency of the quantum measurement is mainly determined by the Fisher information matrix since its inverse sets a lower bound for the mean-square-error (MSE) matrix of any unbiased estimator. In addition, the lower bound can be saturated asymptotically by the maximum-likelihood estimator [5, 17]. The Fisher-information matrices for independent measurements are additive, which means the MSE achievable by repeated measurements is inversely proportional to the sample size $N$ when the measurement is informationally complete (IC) with regard to the parameters of interest.

Repetition of a fixed measurement on individual quantum systems is not so efficient when the mean infidelity or mean square Bures distance (MSB) is the figure of merit. Adaptive measurements on individual quantum systems can improve the tomographic efficiency to some degree. To achieve the optimal performance, however, usually one needs to perform collective measurements on all $N$ quantum systems together [21, 32, 33], which is usually not realistic in practice. In the large-$N$ limit, the optimal performance is determined by the quantum Fisher information matrix [21, 32, 33]. To determine the optimal performance for a given sample size $N$, most researchers have adopted Bayesian approaches and employed the mean fidelity as the figure of merit [20, 26, 49, 57].

In this paper we are interested in the scenario in which we can perform limited collective measurements [21]. For example, suppose $N = tN'$; we measure $t$ identically prepared quantum systems together, and repeat this procedure $N'$ times, as illustrated in the right plot of Fig. S1. Such scenarios are more accessible to experiments and are sufficient to demonstrate the key distinction between collective measurements and individual measurements.

Suppose the density matrix $\rho(\theta)$ is characterized by a set of parameters denoted collectively by $\theta$, and we are interested in estimating these parameters. If we perform a collective measurement described by the POVM $M = \{\Pi_\xi\}$ on $\mathcal{H}^\otimes t$, then the probability of obtaining outcome $\xi$ is given by $p_\xi(\theta) := \text{tr}[\rho(\theta)^\otimes\Pi_\xi]$. After the measurement is repeated $N' = N/t$ times (see the right plot of...
The estimator \( p \) is given by

\[
I_{ab}^{(i)}(\theta) = \sum_{|\xi,p_i>0} \frac{1}{\xi_p} \frac{\partial p_{\xi}}{\partial \theta_a} \frac{\partial p_{\xi}}{\partial \theta_b}.
\]  (S1)

The MSE matrix \( C^{(N)}(\theta) \) of any unbiased estimator is bounded from below by the inverse Fisher information matrix \( [I^{(N)}(\theta)]^{-1} = [N I^{(i)}(\theta)]^{-1} \). When \( N \) is sufficiently large, the maximum-likelihood estimator can approximately saturate this lower bound, and the corresponding scaled MSE matrix reads \( NC^{(N)}(\theta) \approx t I^{(i)}(\theta)^{-1} \). To achieve high tomographic efficiency, the main task is to construct collective measurements that yield the most Fisher information.

Here, we should remark why we do not count the event \( \xi' \) with \( p_{\xi'}(\theta) = 0 \) in the definition of the Fisher information matrix \( I^{(i)}(\theta) \) at \( \tilde{\theta} \), as manifested in Eq. (S1), assuming that \( \rho(\theta) \) is pure. Note that the parametrization \( \rho(\theta) \) is differentiable, so \( p_{\xi'}(\theta) \) is also differentiable. In addition, \( \rho(\theta) \) is a pure state so it can be written as \( \rho(\theta) = |\psi(\theta)\rangle\langle\psi(\theta)| \), which implies that \( \frac{\partial p_{\xi'}(\theta)}{\partial \theta} \big|_{\theta=\tilde{\theta}} = 0 \) whenever \( p_{\xi'}(\tilde{\theta}) = 0 \).

We shall illustrate our argument by considering the estimation problem of one parameter denoted by \( \theta \). The significance of the Fisher information \( I(\theta) \) is tied to the mean square error (MSE) of a locally unbiased estimator. More precisely, its inverse sets a lower bound for the MSE of any locally unbiased estimator \( \hat{\theta} \) of \( \theta \), which is known as the Cramér-Rao bound, and the bound can be saturated asymptotically by the maximum-likelihood estimator \([5, 17]\). This fact underpins the definition of the Fisher information.

Recall that an estimator \( \hat{\theta} \) of \( \theta \) is a mapping from the data to the parameter \( \xi \rightarrow \hat{\theta}(\xi) \). Its MSE reads

\[
C(\hat{\theta}, \theta) := \sum_{\xi} p_{\xi}(\theta) \left| \hat{\theta}(\xi) - \theta \right|^2.
\]  (S2)

The estimator \( \hat{\theta} \) is locally unbiased at \( \tilde{\theta} \) if

\[
\sum_{\xi} \hat{\theta}(\xi) p_{\xi}(\tilde{\theta}) = \tilde{\theta}, \quad \sum_{\xi} \hat{\theta}(\xi) \frac{\partial p_{\xi}(\tilde{\theta})}{\partial \theta} \big|_{\theta=\tilde{\theta}} = 1.
\]  (S3)

When \( p_{\xi'}(\tilde{\theta}) = 0 \) and \( \frac{\partial p_{\xi}(\theta)}{\partial \theta} \big|_{\theta=\tilde{\theta}} = 0 \), the terms corresponding to \( \xi' \) can be eliminated from the above summation, so we get

\[
\sum_{\xi, p_{\xi}(\hat{\theta})>0} \left[ \hat{\theta}(\xi) - \tilde{\theta} \right] p_{\xi}(\tilde{\theta})^{1/2} p_{\xi}(\tilde{\theta})^{-1/2} \frac{\partial p_{\xi}(\theta)}{\partial \theta} \big|_{\theta=\tilde{\theta}} = \sum_{\xi} \left[ \hat{\theta}(\xi) - \tilde{\theta} \right] \frac{\partial p_{\xi}(\theta)}{\partial \theta} \big|_{\theta=\tilde{\theta}} = \sum_{\xi} \hat{\theta}(\xi) \frac{\partial p_{\xi}(\theta)}{\partial \theta} \big|_{\theta=\tilde{\theta}} = 1.
\]  (S4)

Applying the Cauchy-Schwarz inequality yields

\[
\left\{ \sum_{\xi, p_{\xi}(\hat{\theta})>0} p_{\xi}(\tilde{\theta}) \left[ \hat{\theta}(\xi) - \tilde{\theta} \right]^2 \right\} \left\{ \sum_{\xi, p_{\xi}(\hat{\theta})>0} \frac{1}{p_{\xi}(\tilde{\theta})} \left( \frac{\partial p_{\xi}(\theta)}{\partial \theta} \big|_{\theta=\tilde{\theta}} \right)^2 \right\} \geq 1,
\]  (S5)

which implies that

\[
C(\hat{\theta}, \tilde{\theta}) = \sum_{\xi} p_{\xi}(\tilde{\theta}) \left[ \hat{\theta}(\xi) - \tilde{\theta} \right]^2 = \sum_{\xi, p_{\xi}(\tilde{\theta})>0} p_{\xi}(\tilde{\theta}) \left[ \hat{\theta}(\xi) - \tilde{\theta} \right]^2 \geq \left\{ \sum_{\xi, p_{\xi}(\hat{\theta})>0} \frac{1}{p_{\xi}(\tilde{\theta})} \left( \frac{\partial p_{\xi}(\theta)}{\partial \theta} \big|_{\theta=\tilde{\theta}} \right)^2 \right\}^{-1}.
\]  (S6)

Further, the equality holds when the locally unbiased estimator at \( \tilde{\theta} \) is given as \([3, 34]\)

\[
\hat{\theta}_{\rho}(\xi) := \frac{1}{I(\theta)p_{\xi}(\theta)} \frac{\partial p_{\xi}(\theta)}{\partial \theta} \big|_{\theta=\tilde{\theta}} + \tilde{\theta}.
\]  (S7)

One might think that the estimator (S7) is meaningless because it depends on the true parameter \( \tilde{\theta} \). However, this estimator is asymptotically close to the maximum-likelihood estimator in the following
sense. The maximum-likelihood estimator $\hat{\theta}_{\text{ML}}$ is asymptotically locally unbiased at all points and attains the Cramér-Rao bound. When the true parameter is $\theta$ and we observe $N$ outcomes, the variable $\sqrt{N}(\hat{\theta}_{\text{ML}} - \theta)$ asymptotically approaches $\sqrt{N}$ times of the sample mean of $\hat{\theta} - \theta$, which converges to the Gaussian distribution with variance $1/I(\theta)$ according to the central limit theorem [17]. In this way, the asymptotic optimality of the maximum-likelihood estimator can be shown. The above discussion on the MSE of a locally unbiased estimator explains why the event $\xi'$ with $p_{\xi'}(\hat{\theta}) = 0$ and $\frac{\partial p_{\xi'}(\theta)}{\partial \theta} |_{\theta = \hat{\theta}} = 0$ does not contribute to the Fisher information $I(\theta)$ at $\hat{\theta}$.

II. GILL-MASSAR INEQUALITY AND GILL-MASSAR BOUND

In this section we provide more details on the Gill-Massar (GM) inequality and GM bound for the scaled weighted mean square error (WMSE) [13, 21, 24]. In particular we provide a self-contained proof of the GM inequality in the case of one-copy measurement and clarify the equality condition, which is useful to proving Theorem 1. In addition, our discussion on the GM bound is instructive to deriving Corollary 2 from Theorem 3.

A. Gill-Massar inequality

Recall that $\rho(\theta)$ is a quantum state on the Hilbert space $\mathcal{H}$ of dimension $d$, $J(\theta)$ is the quantum Fisher information matrix, and $J^{(N)}(\theta)$ is the Fisher information matrix of a (collective) measurement on $\mathcal{H}^{\otimes N}$. The GM inequality [19] states that

$$\text{tr}[J^{-1}(\theta) J^{(N)}(\theta)] \leq N(d - 1)$$

whenever the measurement is separable. Note that $\text{tr}[J^{-1}(\theta) J^{(N)}(\theta)]$ is invariant under reparametrization. For pure states, which are characterized by $2d - 2$ parameters, the GM inequality applies to arbitrary measurements, not necessarily separable. Here we provide a self-contained proof of the GM inequality in the case $N = 1$, assuming that $\rho(\theta)$ is either pure or of full rank. The main purpose of presenting this proof is to clarify the equality condition of the GM inequality, which was mentioned in Ref. [18] without proof.

Proposition S1. Suppose $\rho(\theta)$ is a pure state parametrized by $\theta_1, \theta_2, \ldots, \theta_{2d-2}$. Then the Fisher information matrix $I(\theta)$ at $\theta$ of any POVM $\{\Pi_\xi\}$ on $\mathcal{H}$ satisfies

$$\text{tr}[J^{-1}(\theta) I(\theta)] \leq d - 1.$$  

(S9)

The inequality is saturated iff all $\Pi_\xi$ are rank one and $\text{tr}[\rho(\theta) \Pi_\xi] > 0$ for all $\xi$.

This proposition is crucial to proving Theorem 1 in the main text, which states that no measurement with a finite number of outcomes on $\mathcal{H}$ is Fisher symmetric for all pure states.

Proof. The Fisher information matrix $I(\theta)$ has matrix elements

$$I_{ab} = \sum_{\xi, p_\xi > 0} \frac{1}{p_\xi} \frac{\partial p_\xi}{\partial \theta_a} \frac{\partial p_\xi}{\partial \theta_b} = \sum_{\xi, p_\xi > 0} \frac{\text{tr}(\rho_\xi \Pi_\xi) \text{tr}(\rho_\xi \Pi_\xi)}{\text{tr}(\rho_\xi)} = \sum_{\xi, p_\xi > 0} \frac{\text{tr}[(\rho_\xi \otimes \rho_\xi) \Pi_\xi^{\otimes 2}]}{\text{tr}(\rho_\xi)},$$

(S10)

where $p_\xi = p_\xi(\theta) = \text{tr}[\rho(\theta) \Pi_\xi]$ and $\rho_a(\rho, \theta) = \partial \rho(\theta)/\partial \theta_a$. According to Lemma S1 below,

$$\text{tr}(J^{-1}I) = \sum_{a,b=1}^{2d-2} (J^{-1})_{ab} I_{ab} = \sum_{\xi, p_\xi > 0} \frac{\text{tr}[(1 \otimes \rho_\xi) - 2(\text{tr}(\rho_\xi))^2]}{2 \text{tr}(\rho_\xi)}$$

$$= \sum_{\xi, p_\xi > 0} \left[ \frac{\text{tr}(\rho_\xi^2)}{\text{tr}(\rho_\xi)} - \text{tr}(\rho_\xi) \right] \leq \sum_{\xi, p_\xi > 0} \text{tr}(\Pi_\xi) - \sum_{\xi} \text{tr}(\rho_\xi) \leq \sum_{\xi} \text{tr}(\Pi_\xi) - 1 = \frac{d}{2} - 1.$$  

(S11)

Here the first inequality is saturated iff all $\Pi_\xi$ with $p_\xi > 0$ have rank one, the second inequality is saturated iff $p_\xi > 0$ for all $\xi$. This observation completes the proof of Proposition S1. 

□
Proposition S2. Suppose $\rho(\theta)$ is a state of full rank parametrized by $\theta_1, \theta_2, \ldots, \theta_{d^2-1}$. Then the Fisher information matrix $I(\theta)$ at $\theta$ of any POVM $\{\Pi_\xi\}$ on $\mathcal{H}$ satisfies

$$\text{tr}(J^{-1}(\theta)I(\theta)) \leq d - 1. \quad (S12)$$

The inequality is saturated iff all $\Pi_\xi$ are rank one.

Proof. The proof is similar to that of Proposition S1 except that Lemma S1 employed there should be replaced by Lemma S2. In addition, the requirement $p_\xi > 0$ is satisfied automatically because $\rho(\theta)$ has full rank.

Lemma S1. Suppose $\rho(\theta)$ is a pure state parametrized by $\theta_1, \theta_2, \ldots, \theta_{2d-2}$. Then

$$\sum_{a,b=1}^{2d-2} (J^{-1})_{ab}(\rho_a \otimes \rho_b) = \frac{1}{2} V(\rho \otimes 1 + 1 \otimes \rho) - \rho \otimes 2, \quad (S13)$$

where $V = \sum_{j,k} |jk\rangle \langle kj|$ is the swap operator.

Proof. Note that the left hand side of Eq. (S13) is invariant under changes of parametrization. Choose an orthonormal basis $\{|j\rangle\}_{j=0}^{d-1}$ such that $\rho = |0\rangle\langle 0|$ at the parameter point of interest. Choose a suitable parametrization such that $\rho_a$ take on the form [19]

$$\rho_a = \begin{cases} 
\rho_{j+}, & 1 \leq a \leq d - 1, \\
\rho_{k-}, & d \leq a \leq 2(d - 1),
\end{cases} \quad (S14)$$

where $j = a$, $k = a - d + 1$, and

$$\rho_{j+} = |j\rangle\langle 0| + |0\rangle\langle j|, \quad \rho_{j-} = i(|j\rangle\langle 0| - |0\rangle\langle j|), \quad 1 \leq j \leq d - 1. \quad (S15)$$

Then the symmetric logarithmic derivatives can be chosen to be $L_a = 2\rho_a$, so that

$$J_{ab} = 4\delta_{ab}. \quad (S16)$$

In addition,

$$\sum_{a=1}^{2d-2} \rho_a \otimes \rho_a = \sum_{j=1}^{d-1} (\rho_{j+} \otimes \rho_{j+} + \rho_{j-} \otimes \rho_{j-}) = 2 \sum_{j=1}^{d-1} (|0\rangle\langle j| + |j\rangle\langle 0|)

= 2 \sum_{j=0}^{d-1} (|0\rangle\langle j| + |j\rangle\langle 0|) - 4(|0\rangle\langle 0|) \otimes 2 = 2V(1 \otimes \rho + \rho \otimes 1) - 4\rho \otimes 2. \quad (S17)$$

Therefore,

$$\sum_{a,b=1}^{2d-2} (J^{-1})_{ab}(\rho_a \otimes \rho_b) = \frac{1}{4} \sum_{a=1}^{2d-2} \rho_a \otimes \rho_a = \frac{1}{2} V(\rho \otimes 1 + 1 \otimes \rho) - \rho \otimes 2. \quad (S18)$$

Lemma S2. Suppose $\rho(\theta)$ is a state of full rank parametrized by $\theta_1, \theta_2, \ldots, \theta_{d^2-1}$. Then

$$\sum_{a,b=1}^{d^2-1} (J^{-1})_{ab}(\rho_a \otimes \rho_b) = \frac{1}{2} V(\rho \otimes 1 + 1 \otimes \rho) - \rho \otimes 2. \quad (S19)$$

This lemma was reproduced from Lemma 6.2 in Ref. [21]. Note that the left hand side of Eq. (S19) is invariant under changes of parametrization. In addition, the right hand side of Eq. (S19) has the same form as that of Eq. (S13).
Proof. We first diagonalize $\rho$ at the given parameter point, so that it has the form $\rho = \sum \lambda_j \langle j | j \rangle$. Then we introduce a basis for the space of traceless Hermitian operators following Ref. [19],

$$
\rho_{jk^+} = |j\rangle \langle k| + |k\rangle \langle j|, \quad \rho_{jk^-} = -i(|j\rangle \langle k| - |k\rangle \langle j|), \quad j < k,
$$

$$
\rho, m = \sum_{j} c_{mj} |j\rangle \langle j|, \quad m = 1, 2, \ldots, d - 1,
$$

where the real coefficients $c_{mj}$ satisfy

$$
\sum_{j} c_{mj} = 0, \quad \sum_{j} \frac{1}{\lambda_j} c_{mj} c_{m'j} = \delta_{m'm},
$$

which implies that

$$
\sum_{m} c_{mj} c_{mk} = \lambda_j \delta_{jk} - \lambda_j \lambda_k.
$$

The operator basis in Eq. (S20) determines an affine parametrization in the state space,

$$
\rho(\theta) = \sum_{j} \lambda_j \langle j | j \rangle + \sum_{m=1}^{d-1} \theta_m \rho, m + \sum_{j<k} \left( \theta_{jk^+} \rho, jk^+ + \theta_{jk^-} \rho, jk^- \right).
$$

The associated symmetric logarithmic derivatives read

$$
L_{jk^+} = \frac{2\rho_{jk^+}}{\lambda_j + \lambda_k}, \quad L_{jk^-} = \frac{2\rho_{jk^-}}{\lambda_j + \lambda_k}, \quad L_m = \sum_{j} \frac{c_{mj}}{\lambda_j} |j\rangle \langle j|.
$$

With this parametrization, the quantum Fisher information matrix is diagonal at the given parameter point, with diagonal entries given by

$$
J_{jk^\pm, jk^\pm} = \frac{4}{\lambda_j + \lambda_k}, \quad J_{m,m} = 1.
$$

Now we are ready to prove Lemma S2

$$
\sum_{a,b=1}^{d^2-1} (J^{-1})_{ab} (\rho_a \otimes \rho_b) = \sum_{j<k} \frac{\lambda_j + \lambda_k}{4} \left[ (|j\rangle \langle k| + |k\rangle \langle j|)^{\otimes 2} - (|j\rangle \langle k| - |k\rangle \langle j|)^{\otimes 2} \right] + \sum_{m} \left( \sum_{j} c_{mj} |j\rangle \langle j| \right)^{\otimes 2}
$$

$$
= \sum_{j<k} \frac{\lambda_j + \lambda_k}{2} (|j\rangle \langle k| + |k\rangle \langle j|) + \sum_{j,k} \left[ \left( \sum_{m} c_{mj} c_{mk} \right) (|j\rangle \langle k|) \right]
$$

$$
= \sum_{j\neq k} \frac{\lambda_j + \lambda_k}{2} (|j\rangle \langle k|) + \sum_{j,k} (\lambda_j \delta_{jk} - \lambda_j \lambda_k) (|j\rangle \langle k|)
$$

$$
= \sum_{j,k} \frac{\lambda_j + \lambda_k}{2} (|j\rangle \langle k|) - \sum_{j,k} \lambda_j \lambda_k (|j\rangle \langle k|) = \frac{1}{2} V(\rho \otimes 1 + 1 \otimes \rho) - \rho^{\otimes 2}.
$$

Here the third equality follows from Eq. (S22).

B. Gill-Massar bound

In this section we provide more details on the Gill-Massar bound for the scaled WMSE [19, 21, 24]. Let $C^{(N)}(\theta)$ be the MSE matrix of any unbiased estimator, then $C^{(N)}(\theta)^{-1} \leq I^{(N)}(\theta)$ according to the Cramér-Rao bound [17]. In conjunction with the GM inequality in Eq. (S8), we deduce

$$
\text{tr} \left( J^{-1}(\theta)[C^{(N)}(\theta)^{-1}] \right) \leq N(d - 1).
$$

(S27)
This inequality imposes a lower bound for the scaled WMSE $N \tr(WC^{(N)})$ for any positive weighting matrix $W$ \cite{21, 21} (to simplify the notation the dependence on the parameter point $\theta$ is suppressed),

$$N \tr(WC^{(N)}) \geq \epsilon_W^{GM} := \frac{\left(\tr \sqrt{J^{-1/2}WJ^{-1/2}}\right)^2}{d-1}.$$(S28)

This bound applies to arbitrary measurements on $\mathcal{H}^\otimes N$ for pure states and to separable measurements for mixed states. In the case $N = 1$ and $d = 2$, this bound was first derived by Hayashi \cite{35}.

When independent and identical measurements are performed on individual copies of $\rho$, we have $I^{(N)}(\theta) = NI(\theta)$, where $I(\theta)$ is the Fisher information matrix associated with the measurement on $\mathcal{H}$. In addition, the bound $[C^{(N)}(\theta)]^{-1} \leq I^{(N)}(\theta)$ can be saturated by the maximum-likelihood estimator in the large-$N$ limit \cite{5, 17} (saturated approximately when $N$ is reasonably large). Then the lower bound in Eq. (S28) is saturated iff the measurement on $\mathcal{H}$ yields the Fisher information matrix \cite{21}.

$$I_W = (d-1)J^{1/2} \frac{\sqrt{J^{-1/2}WJ^{-1/2}}}{\tr \sqrt{J^{-1/2}WJ^{-1/2}}} J^{1/2}.$$ (S29)

The Bures distance between two quantum states $\rho$ and $\sigma$ is defined as $D_B(\rho, \sigma) = \sqrt{2 - 2\sqrt{F(\rho, \sigma)}}$, where $F(\rho, \sigma) = \left(\tr \sqrt{\rho^{1/2}\sigma\rho^{1/2}}\right)^2$ is the fidelity \cite{36}. The quantum Fisher information matrix $J$ allows defining a statistical distance in the state space that is equal to four times of the infinitesimal Bures distance $\delta$. So the weighting matrix for the mean square Bures distance (MSB) is equal to $J/4$, and the corresponding GM bound is \cite{21}.

$$\epsilon_{SB}^{GM} = \begin{cases} \frac{(d+1)^2(d-1)}{4} & \text{mixed states,} \\ d-1 & \text{pure states.} \end{cases}$$ (S30)

A measurement saturates this bound iff it yields the Fisher information matrix $I = J/(d+1)$ for mixed states or $I = J/2$ for pure states, in which case the measurement is Fisher symmetric. Note that the infinitesimal infidelity $1 - F$ is equal to the infinitesimal square Bures distance, so similar conclusion holds if the MSB is replaced by the mean infidelity when $N$ is sufficiently large.

For pure states, the infinitesimal square Hilbert-Schmidt distance is equal to two times of the infinitesimal Bures distance $\delta$. So the GM bound for the scaled MSE (with respect to the Hilbert-Schmidt distance) is $2(d-1)$, which is also saturated by Fisher-symmetric measurements.

Next, we discuss briefly the derivation of Corollary 2 from Theorem 3 in the main text. To this end, it is instructive to further clarify the problem setting; cf. Sec. I in this supplement. Here we assume that $N$ is even and $N$ identically prepared quantum systems available for tomography are divided into $N/2$ pairs. Each time we can measure a pair of quantum systems together; in other words, we can perform collective measurements on $\mathcal{H}^\otimes 2$. The simplest strategy is to repeat a given collective measurement $N/2$ times. In general, adaptive measurements are also allowed; that is, the choices of later measurements may depend on the outcomes of previous measurements. Nevertheless, each measurement setting is usually repeated many times to acquire reliable statistics.

Given the above explanation, it is clear that the derivation of Corollary 2 from Theorem 3 in the main text follows from a similar reasoning employed for deriving the GM bound as outlined above. The conclusion is also similar except that the bound for the scaled WMSE is reduced by a factor of 2/3 because the bound for $\tr(J^{-1}I^{(2)})$ is increased by 50%.

**III. PROOF OF THEOREM 1**

In this section we prove Theorem 1 which states that no single-copy measurement with a finite number of outcomes is Fisher symmetric for all pure states.

*Proof.* Suppose the POVM $\{\Pi_\xi\}$ on $\mathcal{H}$ is Fisher symmetric for all pure states. Then all $\Pi_\xi$ have rank one according to Proposition S1. Let $\rho$ be a pure state that has orthogonal support with $\Pi_\xi$, that is, $\tr(\rho \Pi_\xi) = 0$. Then the Fisher information matrix at $\rho$ provided by $\{\Pi_\xi\}$ cannot saturate the GM inequality again according to Proposition S1. Therefore, $\{\Pi_\xi\}$ is not Fisher symmetric at $\rho$. This contradiction confirms Theorem 1. \qed
Note that this proof also applies to POVMs with countable outcomes, but not to continuous POVMs. For a continuous POVM \( \{ \Pi_j \} \), the requirement \( \text{tr}(\rho \Pi_j) > 0 \) stated in Proposition S1 may be violated by a subset of outcomes that has measure zero. That is why Theorem 1 does not contradict the fact that the covariant measurement is Fisher symmetric for all pure states.

In view of Theorem 1, it is natural to ask whether there exists a finite POVM on \( \mathcal{H} \) that is Fisher symmetric for all pure states, except for a set of measure zero. We believe that such POVMs cannot exist, but have not found a rigorous proof so far. As an evidence in support of our belief, in the case of a qubit, calculation shows that POVMs constructed from plactic solids inscribed in the Bloch sphere are generally not Fisher symmetric except for certain special points on the Bloch sphere (depending on the plactic solids chosen). Nevertheless, it is possible to construct POVMs that are Fisher symmetric for almost all pure states on any given great circle on the Bloch sphere. For example, suppose pure qubit states are parameterized as

\[
\rho(\theta, \phi) = |\psi(\theta, \phi)\rangle \langle \psi(\theta, \phi)| = \cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|1\rangle, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi.
\]

Then the quantum Fisher information matrix is given by

\[
J(\theta, \phi) = \text{diag}(1, \sin^2(\theta)).
\]

Consider the POVM \( \{ \Pi_j \} \) composed of the four elements

\[
\Pi_1 = \frac{1}{2}(|0\rangle\langle 0|), \quad \Pi_2 = \frac{1}{2}(|1\rangle\langle 1|), \quad \Pi_3 = \frac{1}{2}(|+\rangle\langle +|), \quad \Pi_4 = \frac{1}{2}(|-\rangle\langle -|),
\]

where \(|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)\). The probability of obtaining the four outcomes are respectively given by

\[
p_1 = \frac{1 + \cos(\theta)}{2}, \quad p_2 = \frac{1 - \cos(\theta)}{2}, \quad p_3 = \frac{1 + \sin(\theta)\cos(\phi)}{2}, \quad p_4 = \frac{1 - \sin(\theta)\cos(\phi)}{2}.
\]

The Fisher information matrix provided by the POVM reads

\[
I(\theta, \phi) = \frac{1}{2}\left(\begin{array}{cc}
\frac{1 + \cos^2(\theta)\cos^2(\phi)}{\sin(\theta)\cos(\theta)\cos(\phi)} & \frac{\sin(\theta)\cos(\theta)\sin(\phi)\cos(\phi)}{\sin(\theta)\cos(\theta)\cos(\phi)} \\
\frac{\sin(\theta)\cos(\theta)\sin(\phi)\cos(\phi)}{\sin(\theta)\cos(\theta)\cos(\phi)} & \frac{1 - \sin^2(\theta)\cos^2(\phi)}{\sin(\theta)\cos(\theta)\cos(\phi)}
\end{array}\right).
\]

The POVM \( \{ \Pi_j \} \) is Fisher symmetric for the the parameter point \((\theta, \phi)\) if \(p_j \neq 0\) for \(j = 1, 2, 3, 4\) and \(I(\theta, \phi) = \frac{1}{2}J(\theta, \phi) = \frac{1}{2}\text{diag}(1, \sin^2(\theta))\). These conditions hold iff \(\theta = \pi/2\) and \(\phi \neq 0, \pi, \) or \(\phi = \pi/2, 3\pi/2\) and \(\theta \neq 0, \pi\). These parameter points form the union of two great circles with four points corresponding to \(\Pi_1, \Pi_2, \Pi_3, \Pi_4\) deleted. Incidentally, the POVM \( \{ |0\rangle\langle 0|, |1\rangle\langle 1| \} \) saturates the quantum Cramér-Rao bound for \(\theta\) whenever \(\theta \neq 0, \pi\); the POVM \( \{ |+\rangle\langle +|, |-\rangle\langle -| \} \) saturates the quantum Cramér-Rao bound for \(\phi\) on the great circle with \(\theta = \pi/2\) as long as \(\phi \neq 0, \pi\); cf. Ref. [37].

IV. GENERALIZED 2-DESIGNS AND GENERALIZED SICS

Here we provide additional details on generalized 2-designs and generalized SICs [9, 10, 13, 53–55], which are relevant to the current study. Most results presented here are more or less known before, but some of them have not been stated explicitly or clearly. Note that generalized 2-designs considered in this paper are slightly different from conical designs studied in Ref. [53], although they share a common spirit.

A. Generalized 2-designs

A weighted set of quantum states \( \{ |\psi_\xi\rangle, w_\xi \} \) in dimension \(d\) is a (weighted complex projective) 2-design [9, 11] if

\[
\sum_\xi w_\xi (|\psi_\xi\rangle\langle \psi_\xi|)^{\otimes 2} = \frac{2\sum_\xi w_\xi}{d(d+1)} P_+, \quad (S36)
\]
where $P_+$ ($P_-$) is the projector onto the symmetric subspace $\mathcal{H}_+$ (antisymmetric subspace $\mathcal{H}_-$) of $\mathcal{H}^{\otimes 2}$. It is straightforward to verify that
\[
\sum_{\xi, \eta} w_\xi w_\eta |\langle \psi_\xi | \psi_\eta \rangle|^4 \geq \frac{2}{d(d + 1)} \left( \sum_{\xi} w_\xi \right)^2,
\]  
(S37)
and the lower bound is saturated iff $\{|\psi_\xi\rangle, w_\xi\}$ is a 2-design. Any 2-design $\{|\psi_\xi\rangle, w_\xi\}$ has at least $d^2$ elements, and the lower bound is saturated iff all weights $w_\xi$ are equal, and $|\psi_\xi\rangle$ form a symmetric informationally complete measurement \cite{SIC} (SIC for short), that is,
\[
|\langle \psi_\xi | \psi_\eta \rangle|^2 = \frac{d \delta_{\xi \eta} + 1}{d + 1}, \quad \forall \xi, \eta.
\]  
(S38)
Another prominent example of 2-designs are complete sets of mutually unbiased bases (MUB) \cite{MUB}. In the case of a qubit, every SIC defines a regular octahedron on the Bloch sphere, and vice versa. By contrast, every complete set of MUB defines a regular tetrahedron on the Bloch sphere, and vice versa. For example, one SIC is composed of the four states with Bloch vectors $\frac{1}{\sqrt{3}} (1, 1, 1)$, $\frac{1}{\sqrt{3}} (1, -1, -1)$, $\frac{1}{\sqrt{3}} (-1, 1, -1)$, $\frac{1}{\sqrt{3}} (-1, -1, 1)$, respectively. One complete set of MUB is composed of the six states with Bloch vectors $(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)$.

A set of positive operators $\{\Pi_\xi\}$ is a generalized 2-design (cf. conical designs in Ref. \cite{Conical}) if
\[
\sum_{\xi} \frac{\Pi_\xi \otimes \Pi_\xi}{\text{tr}(\Pi_\xi)} = \frac{\sum_\xi w_\xi}{d} \left( \frac{1 + \varphi}{d + 1} P_+ + \frac{1 - \varphi}{d - 1} P_- \right),
\]  
(S40)
where
\[
w_\xi := \text{tr}(\Pi_\xi), \quad \varphi := \frac{\sum_\xi w_\xi \psi_\xi}{\sum_\xi w_\xi}, \quad \varphi_\xi := \frac{\text{tr}(\Pi_\xi^2)}{[\text{tr}(\Pi_\xi)]^2}.
\]  
(S41)
Here $\varphi$ may be interpreted as the purity of $\Pi_\xi$, recall that the purity of a positive operator $A$ is defined as $\text{tr}(A^2)/[\text{tr}(A)]^2$. Therefore, $\varphi$ is the (weighted) average purity of $\{\Pi_\xi\}$. Note that $1/d \leq \varphi \leq 1$. The lower bound is saturated iff all $\Pi_\xi$ are proportional to the identity, in which case $\{\Pi_\xi\}$ forms a trivial generalized 2-design. We will assume $\varphi > 1/d$ except when stated otherwise. The upper bound $\varphi \leq 1$ is saturated iff all $\Pi_\xi$ are rank one and thus can be expressed in the form $\Pi_\xi = w_\xi |\psi_\xi\rangle \langle \psi_\xi|$, in which case Eq. (S40) reduces to Eq. (S30). So a generalized 2-design of purity 1 is a 2-design, and vice versa. Many generalized 2-designs of lower purities can also be constructed from 2-designs. For example, let $\{\rho_\xi\}$ be the SIC defined by the Bloch vectors in Eq. (S39), then $\{\rho_\xi + a\}$ is a generalized 2-design for any positive constant $a$.

Taking the partial trace in Eq. (S40) yields
\[
\sum_\xi \Pi_\xi = \frac{\sum_\xi w_\xi}{d},
\]  
(S42)
which implies the following proposition.

**Proposition** S3. Any generalized 2-design forms a POVM after proper rescaling.

In the rest of this section, we assume that $\{\Pi_\xi\}$ is a set of nonzero positive operators in dimension $d$ with normalization $\sum_\xi \text{tr}(\Pi_\xi) = d$. Under this convention, $\{\Pi_\xi\}$ forms a POVM whenever it is a generalized 2-design. The general case can be analyzed by proper rescaling.

The following proposition provides a simple characterization of generalized 2-designs.

**Proposition** S4. Suppose $\{\Pi_\xi\}$ has purity $\varphi$. Then
\[
\sum_{\xi, \eta} \frac{[\text{tr}(\Pi_\xi \Pi_\eta)]^2}{\text{tr}(\Pi_\xi) \text{tr}(\Pi_\eta)} \geq \frac{d^2 (1 + \varphi^2) - 2d \varphi}{d^2 - 1},
\]  
(S43)
and the lower bound is saturated iff $\{\Pi_\xi\}$ is a generalized 2-design.
Proof. Let

\[ \mathcal{M} := \sum_{\xi} \Pi_{\xi} \otimes \Pi_{\xi}, \quad \mathcal{M}_{\pm} := P_{\pm} \mathcal{M} P_{\pm}. \]  

(\text{S44})

Then

\[ \text{tr}(\mathcal{M}_+) = \frac{1}{2} \sum_{\xi} \frac{[\text{tr}(\Pi_{\xi})]^2 + \text{tr}(\Pi_{\xi}^2)}{\text{tr}(\Pi_{\xi})} = \frac{1}{2} d(1 + \varphi), \quad \text{tr}(\mathcal{M}_-) = \frac{1}{2} \sum_{\xi} \frac{[\text{tr}(\Pi_{\xi})]^2 - \text{tr}(\Pi_{\xi}^2)}{\text{tr}(\Pi_{\xi})} = \frac{1}{2} d(1 - \varphi). \]  

(\text{S45})

Therefore,

\[ \text{tr}(\mathcal{M}^2) \geq \text{tr}(\mathcal{M}_+^2) + \text{tr}(\mathcal{M}_-^2) \geq \left(\frac{1 + \varphi}{d + 1}\right)^2 \text{tr}(P_+^2) + \left(\frac{1 - \varphi}{d - 1}\right)^2 \text{tr}(P_-^2) = \frac{d(1 + \varphi)^2}{2(d + 1)} + \frac{d(1 - \varphi)^2}{2(d - 1)} = \frac{d^2(1 + \varphi^2) - 2d\varphi}{d^2 - 1}. \]  

(\text{S46})

Here the first inequality is saturated iff \( \mathcal{M} = \mathcal{M}_+ + \mathcal{M}_- \), and the second one is saturated iff \( \mathcal{M}_+ \) and \( \mathcal{M}_- \) are proportional to \( P_+ \) and \( P_- \), respectively. The two inequalities are saturated simultaneously iff \( \text{Eq. (S40)} \) with \( \sum_{\xi} w_{\xi} = d \) is satisfied, in which case \( \{\Pi_{\xi}\} \) is a generalized 2-design. \( \square \)

Next, we clarify the connection between generalized 2-designs and tight IC measurements introduced by Scott [11], which is helpful to studying (weakly) Fisher-symmetric measurements. To this end, we need to introduce the formalism of superoperators following Refs. [12, 22, 48]. With respect to the Hilbert-Schmidt inner product, the set of operators on \( \mathcal{H} \) forms a Hilbert space. The vectors in this space are denoted using the double-ket notation; for example, the operator \( H \) is denoted by \( \langle \langle H \rangle \rangle \). The inner product between \( A \) and \( B \) is written as \( \langle \langle A \rangle \rangle \langle \langle B \rangle \rangle := \text{tr}(A^\dagger B) \). Operators acting on this space, so called superoperators, can be constructed from outer products, such as \( \langle \langle A \rangle \rangle \langle \langle B \rangle \rangle \), which induces the linear transformation \( \langle \langle C \rangle \rangle \rightarrow \langle \langle A \rangle \rangle \langle \langle B \rangle \rangle \langle \langle C \rangle \rangle \). The identity superoperator is denoted by \( \mathbf{I} \).

A POVM \( \{\Pi_{\xi}\} \) on \( \mathcal{H} \) is tight IC [11, 13] if it satisfies the following equation

\[ \sum_{\xi} \frac{\langle \langle \Pi_{\xi} \rangle \rangle \langle \langle \Pi_{\xi} \rangle \rangle}{\text{tr}(\Pi_{\xi})} = \frac{d\varphi - 1}{d^2 - 1} \mathbf{I} + \frac{d - \varphi}{d^2 - 1} |1\rangle \langle 1|, \]  

(\text{S47})

where \( \varphi \) is the purity of \( \{\Pi_{\xi}\} \). Note that the requirement \( \sum_{\xi} \Pi_{\xi} = 1 \) for a POVM is satisfied automatically if \( \text{Eq. (S47)} \) holds. According to Lemma 1 in Ref. [48], Eq. (S47) is equivalent to Eq. (S40) with \( \sum_{\xi} w_{\xi} = d \). This observation leads to the following proposition.

**Proposition S5.** A POVM \( \{\Pi_{\xi}\} \) is tight IC iff it is a generalized 2-design.

When \( \varphi > 1/d \), the superoperator in Eq. (S47) has full rank. It follows that any nontrivial generalized 2-design \( \{\Pi_{\xi}\} \) spans the whole operator space and thus has at least \( d^2 \) elements. Generalized 2-designs saturating the lower bound are called minimal. They are characterized by the following proposition, which follows from Theorem 1 in Ref. [48] and Lemma 1 in Ref. [13].

**Proposition S6.** Any nontrivial generalized 2-design has at least \( d^2 \) elements. The set \( \{\Pi_{\xi}\} \) is a minimal generalized 2-design of purity \( \varphi > 1/d \) iff

\[ \text{tr}(\Pi_{\xi} \Pi_{\eta}) = \frac{d\varphi - 1}{d^2 - 1} \sqrt{\text{tr}(\Pi_{\xi}) \text{tr}(\Pi_{\eta})} \delta_{\xi\eta} + \frac{d - \varphi}{d^2 - 1} \text{tr}(\Pi_{\xi}) \text{tr}(\Pi_{\eta}) \quad \forall \xi, \eta. \]  

(\text{S48})

**Remark S1.** If Eq. (S48) holds, then \( \{\Pi_{\xi}\} \) spans the operator space and \( \sum_{\xi} \Pi_{\xi} = 1 \). To see this, let \( L_{\xi} := \sqrt{\Pi_{\xi} / \text{tr}(\Pi_{\xi})} \), then \( \text{tr}(L_{\xi} L_{\eta}) = \sqrt{\text{tr}(\Pi_{\xi})} \) and

\[ \text{tr}(L_{\xi} L_{\eta}) = \frac{d\varphi - 1}{d^2 - 1} \delta_{\xi\eta} + \frac{d - \varphi}{d^2 - 1} \text{tr}(L_{\xi}) \text{tr}(L_{\eta}). \]  

(\text{S49})

Observing that the Gram matrix of \( \{L_{\xi}\} \) is positive definite whenever \( \varphi > 1/d \), we conclude that \( L_{\xi} \) are linearly independent and span the whole operator space and that the same holds for \( \{\Pi_{\xi}\} \). Summing over \( \xi \) in Eq. (S48), we can deduce that \( \sum_{\xi} \Pi_{\xi} \) is proportional to the identity. Then it is straightforward to show that \( \sum_{\xi} \text{tr}(\Pi_{\xi}) = d \), which implies that \( \sum_{\xi} \Pi_{\xi} = 1 \).
B. Generalized SIC

A POVM \{\Pi_\xi\} is a generalized SIC\cite{[13],[54],[55]} if it has \(d^2\) elements and satisfies
\[
\text{tr}(\Pi_\xi \Pi_\eta) = \alpha \delta_{\xi,\eta} + \beta \quad \forall \xi, \eta
\]  
(S50)
for some positive constants \(\alpha, \beta\). Together with the requirement \(\sum_\xi \Pi_\xi = 1\), Eq. (S50) implies that
\[
\text{tr}(\Pi_\xi) = \alpha + d^2 \beta = \frac{1}{d}, \quad \text{tr}(\Pi_\xi^2) = \alpha + \beta \quad \forall \xi.
\]  
(S51)
Therefore, all \(\Pi_\xi\) have the same trace of \(1/d\) and the same purity of \(\varphi = d^2(\alpha + \beta)\). The constants \(\alpha\) and \(\beta\) are determined by the purity as
\[
\alpha = \frac{d\varphi - 1}{d(d^2 - 1)}, \quad \beta = \frac{d - \varphi}{d^2(d^2 - 1)}.
\]  
(S52)
In this paper, \{c\Pi_\xi\} for any positive constant \(c\) is also called a generalized SIC whenever \{\Pi_\xi\} is. By virtue of Proposition [S4] or [S6] it is straightforward to verify that any generalized SIC is a generalized 2-design. Conversely, if \{\Pi_\xi\} is a minimal generalized 2-design with all \(\Pi_\xi\) having the same purity, then Eq. (S43) implies that all \(\Pi_\xi\) have the same trace of \(1/d\), so that \{\Pi_\xi\} is a generalized SIC (cf. Ref. [13]), which satisfies
\[
\text{tr}(\Pi_\xi \Pi_\eta) = \frac{d\varphi - 1}{d(d^2 - 1)} \delta_{\xi,\eta} + \frac{d - \varphi}{d^2(d^2 - 1)} \quad \forall \xi, \eta.
\]  
(S53)

Proposition S7. Any minimal generalized 2-design whose elements have the same purity is a generalized SIC, and vice versa.

Proposition S7 implies the following corollary proved in Refs. [10, 11]; see also Refs. [13, 48].

Corollary S1. Any minimal 2-design is a SIC, and vice versa.

C. Construction of generalized 2-designs from unitary 2-designs

In this section we introduce a method for constructing generalized 2-designs from unitary 2-designs. This method allows us to construct generalized 2-designs all of whose elements are proportional to projectors of a given rank, which are useful in constructing tight coherent measurements introduced in the main text.

A set of weighted unitary operators \{\(U_\xi, w_\xi\)\} on \(\mathcal{H}\) is a (weighted) unitary t-design\cite{[38],[39]} if it satisfies
\[
\sum_\xi w_\xi U_\xi^{\otimes t} M(U_\xi^{\otimes t})^\dagger = \sum_\xi w_\xi \int dU U^{\otimes t} M(U^{\otimes t})^\dagger
\]  
(S54)
for any operator \(M\) acting on \(\mathcal{H}^{\otimes t}\). Here the symbol \(^\dagger\) denotes the Hermitian conjugate, and the integral is taken with respect to the normalized Haar measure over the whole unitary group. It is known that a unitary t-design exists for any positive integer \(t\), assuming \(\mathcal{H}\) has a finite dimension [40].

Suppose \{\(U_\xi, w_\xi\)\} is a unitary 2-design, and \(\Pi\) is a positive operator. Let \(\Pi_\xi := w_\xi U_\xi \Pi U_\xi^{\dagger}\), then \{\(\Pi_\xi\)\} is a generalized 2-design. To see this, note that
\[
\sum_\xi \Pi_\xi \otimes \Pi_\xi \frac{\text{tr}(\Pi_\xi)}{\text{tr}(\Pi)} = \frac{\sum_\xi w_\xi (U_\xi \Pi U_\xi^{\dagger} \otimes (U_\xi \Pi U_\xi^{\dagger})^\dagger)}{\text{tr}(\Pi)} = \frac{1}{\text{tr}(\Pi)} \sum_\xi w_\xi U_\xi^{\otimes 2} \Pi^{\otimes 2} (U_\xi^{\otimes 2})^\dagger
\]  
(S55)
\[
= \sum_\xi \frac{w_\xi}{\text{tr}(\Pi)} \int dU U^{\otimes 2} \Pi^{\otimes 2} (U^{\otimes 2})^\dagger = \alpha P_+ + \beta P_-,
\]
where \(\alpha\) and \(\beta\) are nonnegative constants. Here the last equality follows from the fact that the second tensor power of the unitary group on \(\mathcal{H}\) has two inequivalent irreducible components on \(\mathcal{H}^{\otimes 2}\), which correspond to the symmetric and antisymmetric subspaces, respectively. This observation shows that \{\(\Pi_\xi\)\} is indeed a generalized 2-design. When \(\Pi\) is a pure state, we get a 2-design; when \(\Pi\) is a rank-\(k\) projector, we get a generalized 2-design all of whose elements are proportional to rank-\(k\) projectors.
V. FISHER-SYMMETRIC MEASUREMENTS AT THE COMPLETELY MIXED STATE

In this section we prove Proposition 1 in the main text, which states that a POVM \(\{\Pi_\xi\}\) is (weakly) Fisher symmetric at the completely mixed state iff \(\{\Pi_\xi\}\) is a (generalized) 2-design. In view of Proposition S5, it suffices to show that a POVM is weakly Fisher symmetric at the completely mixed state iff it is tight IC. Corollary 1 is an immediate consequence of Proposition S1.

Proof of Proposition 1. Let \(\{E_a\}_{a=1}^{d^2-1}\) be an orthonormal basis (with respect to the Hilbert-Schmidt inner product) for the space of traceless Hermitian operators. Choose the following affine parametrization on the state space

\[
\rho(\theta) = \frac{1}{d} + \sum_a \theta_a E_a,
\]

(S56)

then \(\rho_a = \partial \rho / \partial \theta_a = E_a\). At the completely mixed state, the symmetric logarithmic derivatives read

\[
L_a = d\rho_a = dE_a,
\]

(S57)

so the quantum Fisher information matrix has entries

\[
J_{ab} = \frac{d}{d+1} \text{tr}(E_a E_b) = \delta_{ab},
\]

(S57)

In addition, the Fisher information matrix has entries

\[
I_{ab} = d \sum_\xi \frac{\text{tr}(E_a \Pi_\xi) \text{tr}(E_b \Pi_\xi)}{\text{tr}(\Pi_\xi)} = d \langle \langle E_a | F | E_b \rangle \rangle,
\]

(S58)

where the superoperator \(F\) is defined as

\[
F := \sum_\xi \frac{||\Pi_\xi|| \langle \langle \Pi_\xi \rangle \rangle}{\text{tr}(\Pi_\xi)}. \tag{S59}
\]

The POVM is weakly Fisher symmetric at the completely mixed state iff \(I\) is proportional to the identity, which holds iff the superoperator \(F\) has the form of Eq. (S47), in which case \(\{\Pi_\xi\}\) is a tight IC POVM and a generalized 2-design. The POVM is Fisher symmetric iff

\[
I = \frac{J}{d^2 - d},
\]

(S62)

The POVM \(\{\Pi_\xi\}\) is weakly Fisher symmetric iff \(\text{tr}(I^2) = [\text{tr}(J)]^2 / (d^2 - 1)\), which holds iff the inequality in Eq. (S43) is saturated, in which case \(\{\Pi_\xi\}\) is a generalized 2-design. The POVM is Fisher symmetric if in addition \(\text{tr}(I) = \frac{d}{d+1} = d^2 - d\), which demands that \(\varphi = 1\), so that \(\{\Pi_\xi\}\) is a 2-design; the converse is also immediate.
VI. PROOF OF THEOREM 3

In this section we prove the inequality \( \text{tr}[J^{-1}(\theta)J^{(2)}(\theta)] \leq 3d - 3 \) in Theorem 3 and determine the condition for saturating the inequality. The simple idea can be explained as follows: the optimal POVM is always the union of two POVMs on the symmetric and antisymmetric subspaces, respectively; the value of \( \text{tr}(J^{-1}I^{(2)}) \) is maximized iff the marginal of each POVM element has the highest purity under the given symmetry. To simplify the notation, the parameter point \( \theta \) is omitted.

Proof of Theorem 3. The Fisher information matrix \( I^{(2)} \) provided by \( \{\Pi_\xi\} \) has matrix elements
\[
I^{(2)}_{ab} = \frac{\text{tr}((\rho \otimes \rho_a + \rho_a \otimes \rho)\Pi_\xi)}{\text{tr}(\rho \otimes \Pi_\xi)} = 2 \sum_\xi \frac{\text{tr}(\tilde{E}_a \hat{Q}_\xi) \text{tr}(\tilde{E}_b \hat{Q}_\xi)}{\text{tr}(\hat{Q}_\xi)},
\]
where
\[
\tilde{E}_a := \rho^{-1/2} \rho_a \rho^{-1/2}, \quad \hat{Q}_\xi := \text{tr}_1(\Pi_\xi) + \text{tr}_2(\Pi_\xi), \quad \Pi_\xi := (\rho^{1/2})^\otimes 2 \Pi_\xi (\rho^{1/2})^\otimes 2.
\]
According to Lemma S2
\[
\sum_{a,b=1}^{d^2-1} (J^{-1})_{ab}(\tilde{E}_a \otimes \tilde{E}_b) = \sum_{a,b=1}^{d^2-1} (J^{-1})_{ab}(\rho^{-1/2})^\otimes 2 (\rho_a \otimes \rho_b)(\rho^{-1/2})^\otimes 2
\]
= \( (\rho^{-1/2})^\otimes 2 [\frac{1}{2} V(\rho \otimes 1 + 1 \otimes \rho) - \rho^\otimes 2] (\rho^{-1/2})^\otimes 2 = \frac{1}{2} V(1 \otimes \rho^{-1} + \rho^{-1} \otimes 1) - 1 \otimes 1. \)

Therefore,
\[
\text{tr}(J^{-1}I^{(2)}) = \sum_{a,b=1}^{d^2-1} (J^{-1})_{ab} I^{(2)}_{ab} = 2 \sum_\xi \frac{\text{tr}(\rho^{-1} \hat{Q}_\xi^2) - [	ext{tr}(\hat{Q}_\xi)]^2}{\text{tr}(\hat{Q}_\xi)} = 2 \sum_\xi \frac{\text{tr}(\rho^{-1} \hat{Q}_\xi^2)}{\text{tr}(\hat{Q}_\xi)} - 4 \leq 3d - 3, \quad (S66)
\]
note that \( \sum_\xi \hat{Q}_\xi = 2\rho. \) Here the inequality follows from Lemma S3 below, which also shows that the inequality is saturated iff each \( \Pi_\xi \) is proportional to either the tensor power of a pure state or a Slater-determinant state.

Remark S2. The proof of Theorem 3 builds on the observation that the value of \( \text{tr}(J^{-1}I^{(2)}) \) associated with a POVM is connected to the purities of symmetrized marginals of POVM elements. The proof of Lemma S3 below that underpins Theorem 3 relies on the fact that the unitary group on the Hilbert space \( \mathcal{H} \) has only two irreducible components on \( \mathcal{H}^\otimes 2 \), which correspond to the completely symmetric subspace and completely antisymmetric subspace, respectively. The approach adopted here may also serve as a stepping stone for studying multi-copy collective measurements. However, new ideas are necessary to generalize our proof since the situation becomes more complicated in the multi-copy setting; see Ref. 21 for some partial progress along this direction.

Lemma S3. Suppose \( \rho \) is a density matrix that has full rank and \( \{\Pi_\xi\} \) is a POVM on \( \mathcal{H}^\otimes 2 \). Let \( \hat{\Pi}_\xi := (\rho^{1/2})^\otimes 2 \Pi_\xi (\rho^{1/2})^\otimes 2 \) and \( \hat{Q}_\xi := \text{tr}_1(\Pi_\xi) + \text{tr}_2(\Pi_\xi) \). Then
\[
\sum_\xi \frac{\text{tr}(\rho^{-1} \hat{Q}_\xi^2)}{\text{tr}(\hat{Q}_\xi)} \leq \frac{3d + 1}{2}, \quad (S67)
\]
and the inequality is saturated iff each \( \Pi_\xi \) is proportional to either the tensor power of a pure state or a Slater-determinant state.

Proof. Let \( \Pi_\xi^\pm := P_\pm \Pi_\xi P_\pm, \hat{\Pi}_\xi^\pm := (\rho^{1/2})^\otimes 2 \Pi_\xi^\pm (\rho^{1/2})^\otimes 2 = P_\pm \hat{\Pi}_\xi P_\pm, \) and \( \hat{Q}_\xi^\pm := \text{tr}_1(\hat{\Pi}_\xi^\pm) + \text{tr}_2(\hat{\Pi}_\xi^\pm) \); then \( \hat{Q}_\xi = \hat{Q}_\xi^+ + \hat{Q}_\xi^- \). Note that \( \hat{\Pi}_\xi^+ \) and \( \hat{\Pi}_\xi^- \) are supported on the symmetric and antisymmetric subspaces, respectively. Therefore,
\[
(\hat{Q}_\xi^+)^2 \leq \text{tr}(\hat{Q}_\xi^+) \hat{Q}_\xi^+, \quad (\hat{Q}_\xi^-)^2 \leq \frac{1}{2} \text{tr}(\hat{Q}_\xi^-) \hat{Q}_\xi^-, \quad (S68)
\]
where the first inequality is saturated iff $\Pi_\xi^+$ (or equivalently $\tilde{\Pi}_\xi^+$) is proportional to the tensor power of a pure state, and the second one is saturated iff $\Pi_\xi^-$ (or equivalently $\tilde{\Pi}_\xi^-$) is proportional to a Slater-determinant state. In addition, we have

$$\sum_\xi \tr(\rho^{-1}\tilde{Q}_\xi^\pm) = \sum_\xi \tr[(\rho^{-1} \otimes 1 + 1 \otimes \rho^{-1})(\rho^{1/2})^\otimes 2 \Pi_\xi^\pm (\rho^{1/2})^\otimes 2] = \tr[(1 \otimes \rho + \rho \otimes 1)P_\pm] = d \pm 1.$$  
(S69)

Consequently,

$$\sum_\xi \frac{\tr(\rho^{-1}\tilde{Q}_\xi^\pm)}{\tr(Q_\xi)} \leq \sum_\xi \left( \frac{\tr(\rho^{-1}(\tilde{Q}_\xi^+)^2)}{\tr(Q_\xi)} + \frac{\tr(\rho^{-1}(\tilde{Q}_\xi^-)^2)}{\tr(Q_\xi)} \right) \leq \sum_\xi \tr(\rho^{-1}\tilde{Q}_\xi^+) + \frac{1}{2} \sum_\xi \tr(\rho^{-1}\tilde{Q}_\xi^-) \leq (d + 1) + \frac{d - 1}{2} \leq \frac{3d + 1}{2},$$
(S70)

where $\tr[\rho^{-1}(\tilde{Q}_\xi^\pm)^2]/\tr(Q_\xi)$ is set to 0 when $\tilde{Q}_\xi^\pm = 0$. Here the first inequality follows from Lemma S4 below, and the second one from Eq. (S68). If $\Pi_\xi$ is proportional to either the tensor power of a pure state or a Slater-determinant state, then the two inequalities are saturated as well as the inequalities in Eq. (S68). Conversely, if the two inequalities are saturated, then $\Pi_\xi^+$ is proportional to the tensor power of a pure state, and $\Pi_\xi^-$ is proportional to a Slater-determinant state. In addition, either $\tilde{Q}_\xi^+$ or $\tilde{Q}_\xi^-$ must vanish in order to saturate the first inequality; that is, either $\Pi_\xi^+$ or $\Pi_\xi^-$ must vanish. Therefore, $\Pi_\xi$ is proportional to either the tensor power of a pure state or a Slater-determinant state.

**Lemma S4.** Suppose $\rho, A, B$ are nonzero positive operators on $\mathcal{H}$ with $\rho$ having full rank. Then

$$\frac{\tr[\rho(A + B)^2]}{\tr(A + B)} \leq \frac{\tr(A^2)}{\tr(A)} + \frac{\tr(B^2)}{\tr(B)},$$
(S71)

and the inequality is saturated iff $A$ is proportional to $B$.

**Proof.** According to the Cauchy-Schwarz inequality,

$$|\tr(\rho AB)| = |\tr(\rho^{1/2} AB \rho^{1/2})| \leq \sqrt{\tr(A^2) \tr(B^2)}.$$  
(S72)

The inequality is saturated iff $A^{1/2}$ is proportional to $B^{1/2}$, that is, $A$ is proportional to $B$. Therefore,

$$\frac{\tr[\rho(A + B)^2]}{\tr(A + B)} = \frac{\tr(A^2) + \tr(B^2) + \tr(\rho AB) + \tr(BA)}{\tr(A + B)} \leq \left[\frac{\tr(A^2) + \tr(B^2)}{\tr(A + B)}\right]^2 \leq \frac{\tr(A^2)}{\tr(A)} + \frac{\tr(B^2)}{\tr(B)}.$$  
(S73)

If the first inequality is saturated, then $A$ is proportional to $B$, in which case the second inequality is saturated automatically.

**VII. TIGHT COHERENT MEASUREMENTS**

In this section we prove Theorems 5 and 6 in the main text, thereby clarifying the structure of tight coherent measurements. We also provide more details on constructing minimal tight coherent measurements in dimension 3.

**A. Proofs of Theorems 5 and 6**

Theorem 5 is an immediate consequence of the following theorem, which refines Theorem 5.

**Theorem S1.** Let $\{\Pi_\xi\}$ be a POVM on $\mathcal{H}^\otimes 2$ and $Q_\xi := \tr_1(\Pi_\xi) + \tr_2(\Pi_\xi)$. Then the following statements are equivalent.

...
1. \{\Pi_\xi\} is tight coherent.
2. \{Q_\xi\} is a generalized 2-design of purity \(\frac{3d+1}{4d}\).
3. Each \(\Pi_\xi\) is proportional to either the tensor power of a pure state or a Slater-determinant state, and \(\{Q_\xi\}\) forms a generalized 2-design.
4. \(\{\Pi_\xi\}\) is a union of two POVMs \(\{\Pi_\xi^+\}\) and \(\{\Pi_\xi^-\}\) on \(\mathcal{H}_+\) and \(\mathcal{H}_-\), respectively; \(\{Q_\xi^+\}\) forms a 2-design, and \(\{Q_\xi^-\}\) forms a generalized 2-design of purity \(\frac{1}{2}\).

Remark S3. Here \(Q_\xi^+\) and \(Q_\xi^-\) are defined in analogy to \(Q_\xi\), that is, \(Q_\xi^+ = \text{tr}_1(\Pi_\xi^+) + \text{tr}_2(\Pi_\xi^+\) and \(Q_\xi^- = \text{tr}_1(\Pi_\xi^-) + \text{tr}_2(\Pi_\xi^-\). If any of the four statements in Theorem S1 holds, then \(Q_\xi^+\) are proportional to rank-1 projectors, and \(Q_\xi^-\) are proportional to rank-2 projectors. When \(d = 2\), \(Q_\xi^-\) are necessarily proportional to the identity, so the generalized 2-design \(\{Q_\xi^-\}\) is trivial.

Proof. Choose the affine parametrization specified in Eq. (S56), then the Fisher information matrix \(I^{(2)}\) at the completely mixed state associated with \(\{\Pi_\xi\}\) has matrix elements

\[
J_{ab}^{(2)} = 2 \sum_\xi \frac{\text{tr}(E_a Q_\xi) \text{tr}(E_b Q_\xi)}{\text{tr}(Q_\xi)},
\]

(S74)

Note that \(\sum_\xi Q_\xi = 2d\) and \(\sum_\xi \text{tr}(Q_\xi) = 2d^2\), so \(\{Q_\xi\}\) forms a POVM on \(\mathcal{H}\) up to scaling. Also, note the similarity between Eq. (S74) and Eq. (S55). According to Proposition B in the main text, \(\{\Pi_\xi\}\) is weakly Fisher symmetric at the completely mixed state iff \(\{Q_\xi\}\) is a generalized 2-design. In addition,

\[
\text{tr}(I^{(2)}) = 2 \sum_\xi \frac{d \text{tr}(Q_\xi^2) - [\text{tr}(Q_\xi)]^2}{d \text{tr}(Q_\xi)} = 4d^2 \varphi - 4d, \quad \text{tr}(J^{-1} I^{(2)}) = \frac{1}{d} \text{tr}(I^{(2)}) = 4d \varphi - 4.
\]

(S75)

where \(\varphi := \sum_\xi \text{tr}(Q_\xi^2)/[2d^2 \text{tr}(Q_\xi)]\) is the purity of \(\{Q_\xi\}\).

By Theorem B if \(\{\Pi_\xi\}\) is coherent, then \(\text{tr}(J^{-1} I^{(2)}) = 3d - 3\), which implies that \(\varphi = (3d + 1)/(4d)\). If \(\{\Pi_\xi\}\) is tight coherent, then it is also Fisher symmetric at the completely mixed state, so \(\{Q_\xi\}\) is a generalized 2-design of purity \((3d + 1)/(4d)\). Conversely, if \(\{Q_\xi\}\) is a generalized 2-design of purity \((3d + 1)/(4d)\), then \(\text{tr}(J^{-1} I^{(2)}) = 3d - 3\), and \(I\) is proportional to \(J\), so \(\{\Pi_\xi\}\) is Fisher symmetric. It follows that statements 1 and 2 are equivalent.

If statements 1 and 2 hold, then each \(\Pi_\xi\) is proportional to either the tensor power of a pure state or a Slater-determinant state according to Theorem B. In addition, \(\{Q_\xi\}\) is a generalized 2-design. So statements 1 and 2 imply statement 3. On the other hand, \(\{Q_\xi\}\) necessarily has purity \((3d + 1)/(4d)\) if each \(\Pi_\xi\) is proportional to either the tensor power of a pure state or a Slater-determinant state. So statement 3 implies statement 2. Consequently, statements 1, 2, and 3 are equivalent.

It is easy to verify that statement 4 implies statement 2, which in turn implies statements 1 and 3 according to the above discussion. Now suppose statement 3 holds. Then each \(\Pi_\xi\) is proportional to either the tensor power of a pure state or a Slater-determinant state, so \(\{\Pi_\xi\}\) is a union of two POVMs \(\{\Pi_\xi^+\}\) and \(\{\Pi_\xi^-\}\) on \(\mathcal{H}_+\) and \(\mathcal{H}_-\), respectively, \(\{Q_\xi^+\}\) is a 2-design, and \(\{Q_\xi^-\}\) has purity \(1/2\). Given that both \(\{Q_\xi\}\) and \(\{Q_\xi^+\}\) are generalized 2-designs, it follows that \(\{Q_\xi^-\}\) is also a generalized 2-design. Therefore, statement 3 implies statement 4. This observation completes the proof of Theorem S1.

Proof of Theorem 6. Suppose \(\{\Pi_\xi\}\) is tight coherent. According to Theorem S1, \(\{\Pi_\xi\}\) is a union of two POVMs \(\{\Pi_\xi^+\}\) and \(\{\Pi_\xi^-\}\) on \(\mathcal{H}_+\) and \(\mathcal{H}_-\), respectively, \(\{Q_\xi^+\}\) is a 2-design, \(\{Q_\xi^-\}\) is a generalized 2-design of purity \(1/2\), and \(Q_\xi^-\) are proportional to rank-2 projectors. According to Proposition S7, both \(\{Q_\xi^+\}\) and \(\{Q_\xi^-\}\) have at least \(d^2\) elements, so \(\{\Pi_\xi\}\) has at least \(2d^2\) elements. If the lower bound is saturated, then both \(\{Q_\xi^+\}\) and \(\{Q_\xi^-\}\) have \(d^2\) elements, so \(\{Q_\xi^+\}\) forms a SIC by Corollary S1 and \(\{Q_\xi^-\}\) forms a generalized SIC of purity \(1/2\) by Proposition S7. The converse is an easy consequence of Proposition S7 and Theorem S1.
B. Minimal tight coherent measurements in dimension 3

According to Theorem 6 in the main text, any tight coherent POVM on $\mathcal{H}$ with $2d^2$ elements when $d \geq 3$ is determined by a SIC and a generalized SIC composed of rank-2 projectors, and vice versa. When $d = 3$, if $\{B_\eta\}$ for $\eta = 1, 2, \ldots, 9$ is such a generalized SIC, then $\{1 - B_\eta\}$ is a SIC \[10\]. Therefore, minimal tight coherent POVMs in dimension 3 are in one-to-one correspondence with pairs of SICs.

Since all SICs in dimension 3 are known \[9, 10, 41–44\], all minimal tight coherent POVMs in dimension 3 can be constructed explicitly. More precisely, all SICs in dimension 3 are covariant with respect to the Heisenberg-Weyl group, which is generated by the cyclic shift operator $X$ and the phase operator $Z := \text{diag}(1, e^{2\pi i/3}, e^{4\pi i/3})$. In addition, every SIC is equivalent to a SIC of the form
\[
\{X^j Z^k |\psi(\phi)\rangle\}_{j,k=0,1,2}[16, 42],
\]
where
\[
|\psi(\phi)\rangle := \frac{1}{\sqrt{2}}(0, 1, -e^{i\phi})^T, \quad 0 \leq \phi \leq \frac{\pi}{9}. \tag{S76}
\]

Suppose $\{|\psi_\zeta\rangle\}$ and $\{|\varphi_\zeta\rangle\}$ are two SICs in dimension 3. Let
\[
\Pi^+_{\zeta} := \frac{2}{3} (|\psi_\zeta\rangle\langle\psi_\zeta|)^{\otimes 2}, \quad \Pi^-_{\zeta} := \frac{1}{3} P_{-}(1 - |\varphi_\zeta\rangle\langle\varphi_\zeta|)^{\otimes 2} P_{-}, \quad \zeta = 1, 2, \ldots, 9. \tag{S77}
\]

Then the union of $\{\Pi^+_{\zeta}\}$ and $\{\Pi^-_{\zeta}\}$, that is, the POVM composed of all $\Pi^+_{\zeta}$ and $\Pi^-_{\zeta}$, is tight coherent. Conversely, every minimal tight coherent POVM in dimension 3 has this form and thus can be constructed explicitly.