Bounds on the Dimension of Ext for Finite Groups of Lie Type

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Abstract

Let \( G \) be a finite group of Lie type defined in characteristic \( p \), and let \( k \) be an algebraically closed field of characteristic \( r > 0 \) (so, we are in the non-defining characteristic case). Let \( V \) be a finite-dimensional irreducible left \( kG \)-module. In 2011, Guralnick and Tiep found bounds on the dimension of \( H^1(G, V) \) in non-defining characteristic, which are independent of \( V \). The aim of this paper is to generalize the work of Guralnick and Tiep. We assume that \( G \) is split and use methods of modular Harish-Chandra theory to find bounds on the dimension of \( \text{Ext}^1_{kG}(Y, V) \), where \( Y \) and \( V \) are irreducible \( kG \)-modules. We then use Dipper and Du’s algorithms to illustrate our bounds in a series of examples.

1 Introduction

Let \( q \) be a power of a prime \( p \), let \( G \) be a finite group of Lie type over the finite field of \( q \) elements \( \mathbb{F}_q \), and let \( k \) be an algebraically closed field of characteristic \( r > 0, r \neq p \). We will work with left \( kG \)-modules, and all \( kG \)-modules will be assumed to be finite-dimensional over \( k \). When \( G \) is a finite group of Lie type, the \( BN \)-pair of \( G \) is split in the sense of [2, 2.5], i.e. this \( BN \)-pair satisfies the additional axioms (a) \( B = UT \), where \( T = B \cap N \) and \( U \) is the largest normal \( p \)-subgroup of \( B \), and (b) \( \cap_{n \in N} nBn^{-1} = T \). Thus, we can take advantage of the results of modular Harish-Chandra theory outlined in [11].

In this paper, we combine techniques of modular Harish-Chandra theory with Guralnick and Tiep’s methods for studying 1-cohomology in non-defining characteristic in order to find bounds on the dimension of \( \text{Ext}^1_{kG}(Y, V) \) when \( Y \) and \( V \) are certain irreducible \( kG \)-modules. In Section 4, we show that \( \text{Ext}^1_{kG}(Y, V) = 0 \) when \( Y \) and \( V \) are principal series representations belonging to distinct principal series. In Section 5, we show that \( \dim \text{Ext}^1_{kG}(Y, V) \leq [W : \]
\[ W(T, X) |W| + \min(\dim Y, \dim V)e \] (where \( e \) is the \( r \)-rank of \( T \)) when \( Y \) and \( V \) both belong to a principal series \( \text{Irr}_k(G|(T, X)) \) for some one-dimensional \( kT \)-module \( X \). In Section 7, we assume that \( Y \) is a unipotent principal series representation and that \( V \) is an irreducible \( kG \)-module which lies outside the unipotent principal series. Under certain additional assumptions on the group \( G \), we show that there exists a parabolic subgroup \( W_J \) of \( W \) (which depends only on \( V \)) such that \( \dim \text{Ext}^1_{kG}(Y, V) \leq [W : W_J] \). In Section 7, we give examples of this bound in the case that \( G \) is a finite general linear group. These results were originally proved in the author’s thesis \[19\].

2 Motivation

There are three distinct cases to consider in the representation theory of \( kG \) (where \( G \) is a finite group of Lie type over \( \mathbb{F}_q \), with \( q \) a power of a prime \( p \)): \( r = 0 \) (the characteristic 0 case), \( r = p \) (the defining characteristic case), and \( r > 0, r \neq p \) (the non-defining, or cross-characteristic case). One goal of research in the defining and non-defining characteristic cases is to compute the dimensions of Ext groups \( \text{Ext}^i(G, V) \), where \( V \) is an irreducible \( kG \)-module and \( i \geq 1 \). In the defining characteristic, it is known that such bounds exist when the rank \( i \) is fixed (this is due to Cline, Parshall, and Scott \[3\] and Parshall and Scott \[18\] in the \( i = 1 \) case and to Bendel, Nakano, Parshall, Pillen, Scott, and Stewart \[1\] in the \( i > 1 \) case).

In 2011, Guralnick and Tiep \[13\] published the following bounds on 1-cohomology in non-defining characteristic.

**Theorem 2.1.** (\[13, Cor. 3.3, 6.5\]) Let \( G \) be a finite group of Lie type defined in characteristic \( p \), and let \( \text{char}(k) = r > 0 \), with \( r \neq p \). Let \( W \) be the Weyl group of \( G \), and let \( e \) be the Lie rank of \( G \). If \( V \) is an irreducible \( kG \)-module, then
\[
\dim H^1(G, V) \leq \begin{cases} 
1 & \text{if } V^B = 0 \\
|W| + e & \text{if } V^B \neq 0 
\end{cases}
\]
(where \( B \) is a Borel subgroup of \( G \)).

An irreducible \( kG \)-module \( V \) satisfies the condition \( V^B \neq 0 \) if and only if \( V \) belongs to the unipotent principal Harish-Chandra series \( \text{Irr}_k(G|B) \) (see Section 3.4). Hence, \[13, Cor. 3.3, 6.5\] may be restated as follows: given an irreducible \( kG \)-module \( V \),
\[
\dim \text{Ext}^1_{kG}(k, V) \leq \begin{cases} 
1 & \text{if } V \not\in \text{Irr}_k(G|B) \\
|W| + e & \text{if } V \in \text{Irr}_k(G|B) 
\end{cases}
\]

The work presented in this paper stems from the observation that Guralnick and Tiep’s bounds can be interpreted in terms of Harish-Chandra theory. In Sections 4-7, we use techniques of modular Harish-Chandra theory to generalize \[13, Cor. 3.3, 6.5\] and find bounds on the dimension of \( \text{Ext}^1 \) between irreducible \( kG \)-modules.

\[3\] \( W(T, X) \) is the inertia group of \( X \) (see Definition \[3.2\]).
3 Modular Harish-Chandra Theory

Our summary of Harish-Chandra theory is based on [11, Sec. 4.2]. As above, let $G$ be a finite group of Lie type defined in characteristic $p$, and let $k$ be an algebraically closed field of characteristic $r > 0$, $r \neq p$ (for the remainder of this paper, we will restrict our attention to the non-defining characteristic case). Let $(W, S)$ be the Coxeter system corresponding to the $BN$-pair structure of $G$.

3.1 Harish-Chandra Induction and Restriction

Given a subset $J \subseteq S$, let $P_J$ be the standard parabolic subgroup of $G$ corresponding to $J$. Let $U_P$, be the largest normal $p$-subgroup of $P_J$, and let $L_J$ be a Levi subgroup of $P_J$. In the notation of [11, 4.2.1], let $\mathcal{P}_G = \{^n P_J \mid J \subseteq S, \ n \in N\}$ and $\mathcal{L}_G = \{^n L_J \mid J \subseteq S, \ n \in N\}$.

Let $n \in N$ and let $X$ be a (left) $kL$-module. We can define a $k(^n L)$-module structure on $X$ by setting $nln^{-1}.x = l.x$ for any $l \in L$ and $x \in X$. The resulting $k(^n L)$-module will be denoted by $^n X$. Let $w \in W = N/T$ and let $n \in N$ be a representative of $w$. In this case, we define $^w L := ^n L$ and $^w X := ^n X$ ($^n L$ and $^n X$ are well-defined since $^n t L = ^n L$ and $^n X \cong ^n t X$ for any $n \in N \leq N_G(T)$ and $t \in T$).

Let $P \in \mathcal{P}_G$ and let $L \in \mathcal{L}_G$ be a Levi complement in $P$; in this case, $P = U_P \rtimes L$. Let $kL \mod$ denote the category of (finite dimensional) left $kL$-modules, and let $kG \mod$ denote the category of (finite dimensional) left $kG$-modules. There is a Harish-Chandra induction functor

$$R^G_{L \subseteq P} : kL \mod \rightarrow kG \mod,$$

defined by $R^G_{L \subseteq P}(X) = \text{Ind}_P^G(\tilde{X})$ for all $X \in kL \mod$, where $\tilde{X}$ denotes the inflation of $X$ from $L$ to $P$ via the surjective homomorphism $P \rightarrow L$ with kernel $U_P$.

There is also a Harish-Chandra restriction functor

$$\ast R^G_{L \subseteq P} : kG \mod \rightarrow kL \mod.$$

Given a $kG$-module $Y$, $\ast R^G_{L \subseteq P}(Y) = Y^{U_P}$ (which has the structure of a $kL$-module since $U_P$ is a normal subgroup of $P$).

A key feature of Harish-Chandra induction is the following independence property, which was proved by Howlett and Lehrer [14] and Dipper and Du [4]. Let $L, M \in \mathcal{L}_G$, and suppose that $L$ is a Levi complement of $P \in \mathcal{P}_G$ and $M$ is a Levi complement of $Q \in \mathcal{P}_G$. Let $X \in kL \mod$ and $X' \in kM \mod$. If $M = ^n L$ and $X' \cong ^n X$ for some $n \in N$, then $R^G_{L \subseteq P}(X) \cong R^G_{M \subseteq G}(X')$. As a particular application of this independence property, we have that the Harish-Chandra induction functor $R^G_{L \subseteq P}$ is independent of the choice of parabolic subgroup $P \in \mathcal{P}_G$ containing $L$. Similarly, the Harish-Chandra restriction functor $\ast R^G_{L \subseteq P}$ is independent of the parabolic subgroup $P$ containing $L$. Therefore, we will omit the parabolic subgroup $P$ and write $R^G_L$ and $\ast R^G_L$ for the Harish-Chandra induction and restriction functors.

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4For any any subgroup $H \leq G$, $^n H = nHn^{-1}$.
3.2 Some Properties of Harish-Chandra Induction and Restriction

Adjointness. For any Levi subgroup $L \in \mathcal{L}_G$, $R^G_L$ and $^*R^G_L$ are exact. The functors $R^G_L$ and $^*R^G_L$ are each other’s two-sided adjoints.

Transitivity. Harish-Chandra induction and restriction are transitive. Suppose that $L, M \in \mathcal{L}_G$ are such that $L \subseteq M$. If $X \in kL - \text{mod}$, then $R^G_L(X) \cong R^G_M(R^L_M(X))$. If $Y \in kG - \text{mod}$, then $^*R^G_L(Y) \cong ^*R^M_L(^*R^G_M(Y))$.

Mackey decomposition. As in the case of ordinary induction and restriction, we have a Mackey decomposition formula for Harish-Chandra induction and restriction. Suppose $L, M \in \mathcal{L}_G$ are Levi complements of the parabolic subgroups $P, Q \in \mathcal{P}_G$, respectively. Let $X$ be a $kL$-module, and let $D(Q, P)$ denote a full set of $(Q, P)$-double coset representatives in $G$. The Mackey formula provides the following direct sum decomposition of the $kM$-module $^*R^G_M(R^G_L(X))$:

$$^*R^G_M(R^G_L(X)) \cong \bigoplus_{n \in D(Q, P)} R^M_{nL \cap M}(^*R^n_{L \cap M}(nX)).$$

Harish-Chandra induction and the linear dual. Let $L \in \mathcal{L}_G$ be the Levi complement of a parabolic subgroup $P \in \mathcal{P}_G$, and suppose that $X$ is a left $kL$-module. Let $X^*$ be the $k$-linear dual of $X$, viewed as a right $kL$-module. Since ordinary induction commutes with taking duals in the case of finite groups, we have $(R^G_L(X))^* \cong R^G_L(X^*)$.

3.3 Cuspidal Modules and Harish-Chandra Series.

A $kG$-module $Y$ is called cuspidal if $^*R^G_L(Y) = 0$ for all $L \in \mathcal{L}_G$ such that $L \subsetneq G$. (This definition extends to $kL$-modules for any $L \in \mathcal{L}_G$; a $kL$-module $X$ is cuspidal if $^*R^G_L(X) = 0$ for all $L' \in \mathcal{L}_G$ such that $L' \subsetneq L$.)

Let $\text{Irr}_k(G)$ denote a full set of non-isomorphic irreducible $kG$-modules. Given a pair $(L, X)$ with $L \in \mathcal{L}_G$ and $X$ an irreducible cuspidal $kL$-module, let $\text{Irr}_k(G|(L, X))$ be the subset of $\text{Irr}_k(G)$ consisting of all $Y \in \text{Irr}_k(G)$ such that $L \in \mathcal{L}_G$ is minimal with $^*R^G_L(Y) \neq 0$ and $X$ is a composition factor of $^*R^G_L(Y)$. The set $\text{Irr}_k(G|(L, X))$ is the Harish-Chandra series corresponding to the pair $(L, X)$. If $Y \in \text{Irr}_k(G|(L, X))$, we will say that $L$ is a Harish-Chandra vertex of $Y$ and that $X$ is a Harish-Chandra source of $Y$ (this terminology is used in [4] and [5]).

We summarize several properties of Harish-Chandra series, which are stated in [11] Sec. 4.2] and were originally proved by Hiss [14].

Proposition 3.1. (Properties of Harish-Chandra series)

(a) Let $L, L' \in \mathcal{L}_G$, let $X$ be a cuspidal irreducible $kL$-module, and let $X'$ be a cuspidal irreducible $kL'$-module. Then, $\text{Irr}_k(G|(L, X)) = \text{Irr}_k(G|(L', X'))$ if and only if there exists some $n \in \mathbb{N}$ with $L' = ^nL$ and $X' \cong ^nX$. 

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(b) The Harish-Chandra series \( \text{Irr}_k(G|(L, X)) \) consists of the irreducible \( kG \)-modules which occur in the head (or, equivalently, the socle) of \( R^G_L(X) \).

(c) The set \( \text{Irr}_k(G) \) is partitioned by the distinct Harish-Chandra series.

### 3.4 The Principal Series Representations

Since every \( kT \)-module is cuspidal, there is a Harish-Chandra series of the form \( \text{Irr}_k(G|(T, X)) \) for every irreducible \( kT \)-module \( X \). The irreducible representations of \( G \) belonging to a Harish-Chandra series of the form \( \text{Irr}_k(G|(T, X)) \) are called \textit{principal series representations}. (Since \( T \) is abelian, an irreducible \( kT \)-module \( X \) must be one-dimensional.)

The principal Harish-Chandra series corresponding to the pair \( (T, k) \) (where \( k \) is viewed as the trivial irreducible \( kT \)-module) is called the unipotent principal series and is denoted by \( \text{Irr}_k(G|B) \). Since \( R^G_T(k) \cong k|_B^G \), \( \text{Irr}_k(G|B) \) consists of the irreducible \( kG \)-modules which can be found in both the head and socle of the permutation module \( k|_B^G \). If \( V \in \text{Irr}_k(G) \), then the multiplicity of \( V \) in the head of \( k|_B^G \) is \( [k|_B^G : V] = \dim \text{Hom}_{kG}(k|_B^G, V) = \dim \text{Hom}_{kB}(k, V) = \dim V^B \). Similarly, \( [\text{soc}(k|_B^G) : V] = \dim V^B \). Thus, an irreducible \( kG \)-module \( V \) belongs to the unipotent principal series \( \text{Irr}_k(G|B) \) if and only if \( V^B \neq 0 \).

### 3.5 Hecke Algebra Associated with a Harish Chandra Series

Let \( L \in \mathcal{L}_G \) and let \( X \) be a cuspidal irreducible (left) \( kL \)-module. Let \( R^G_L \) be the Harish-Chandra induction functor from the category of left \( kL \)-modules to the category of left \( kG \)-modules. We define

\[
\mathscr{H}(L, X) := \text{End}_{kG}(R^G_L(X))^\text{op},
\]

where \( \text{End}_{kG}(R^G_L(X))^\text{op} \) denotes the opposite of the endomorphism algebra \( \text{End}_{kG}(R^G_L(X)) \). The algebra \( \mathscr{H}(L, X) \) is called the Hecke algebra associated with the pair \( (L, X) \).

The category of left \( kG \)-modules is related to the category of left \( \mathscr{H}(L, X) \)-modules via the Hom functor

\[
\mathfrak{F}_{R^G_L(X)} : kG - \text{mod} \to \mathscr{H}(L, X) - \text{mod}, \ Y \mapsto \text{Hom}_{kG}(R^G_L(X), Y).
\]

**Definition 3.2.** Given a pair \( (L, X) \), where \( L \in \mathcal{L}_G \) is a Levi subgroup of \( G \) and \( X \) is a cuspidal irreducible \( kL \)-module, the inertia group of \( X \) is \( \mathcal{W}(L, X) := \{n \in (N_G(L) \cap N)L \mid ^nX \cong X\}/L \) [11, 4.2].

The Hecke algebra \( \mathscr{H}(L, X) \) has a basis parameterized by \( \mathcal{W}(L, X) \) [11, Thm. 4.2.12] (the case of char \( k = 0 \) was originally handled in [15]) and the case of char \( k > 0 \) was originally handled in [10]).

### 4 A Bound on the Dimension of Ext\(^1\) Between Irreducible Modules in Distinct Principal Series

Let \( G \) be a finite group of Lie type defined in characteristic \( p \), and let \( k \) be an algebraically closed field of characteristic \( r > 0, r \neq p \). In this section, we will assume that the ir-
reducible $kG$-module $Y$ belongs to the principal series $\text{Irr}_k(G|(T, X))$ and that the irreducible $kG$-module $V$ belongs to the principal series $\text{Irr}_k(G|(T, X'))$, with $\text{Irr}_k(G|(T, X)) \neq \text{Irr}_k(G|(T, X'))$. Our goal is to prove that $\text{Ext}^1_{kG}(Y, V) = 0$. Since $Y \in \text{Irr}_k(G|(T, X))$, $Y$ is in the head of $R_T^G(X)$. Using a notation analogous to [13], let $\mathcal{L}^0$ denote a maximal submodule of $R_T^G(X)$ with $R_T^G(X)/\mathcal{L}^0 \cong Y$. In the next result, we generalize the proof of [13, Theorem 2.2] to show that for any irreducible $kG$-module $Z$ with $Z \notin \text{Irr}_k(G|(T, X))$, $\dim \text{Ext}^1_{kG}(Y, Z)$ is determined by the composition multiplicity of $Z$ in $\text{head}(\mathcal{L}^0)$.

**Theorem 4.1.** Let $Y$ be an irreducible $kG$-module in the principal series $\text{Irr}_k(G|(T, X))$ (where $X$ is a one-dimensional $kT$-module). If $Z$ is an irreducible $kG$-module such that $Z \notin \text{Irr}_k(G|(T, X))$, then $\dim \text{Ext}^1_{kG}(Y, Z) = [\text{head}(\mathcal{L}^0) : Z]$ (where $[\text{head}(\mathcal{L}^0) : Z]$ denotes the multiplicity of $Z$ as a composition factor of $\text{head}(\mathcal{L}^0)$).

**Proof.** By definition, $\mathcal{L}^0$ fits into a short exact sequence of $kG$-modules $0 \to \mathcal{L}^0 \to R_T^G(X) \to Y \to 0$. Since $Z \notin \text{Irr}_k(G|(T, X))$, $\text{Hom}_{kG}(Y, Z) = 0$ and $\text{Hom}_{kG}(R_T^G(X), Z) = 0$. Thus, the long exact sequence in $\text{Ext}$ yields

$$0 \to \text{Hom}_{kG}(\mathcal{L}^0, Z) \to \text{Ext}^1_{kG}(Y, Z) \to \text{Ext}^1_{kG}(R_T^G(X), Z) \to \cdots.$$ 

To prove the statement of the theorem, it is enough to show that $\text{Ext}^1_{kG}(R_T^G(X), Z) = 0$. For, if $\text{Ext}^1_{kG}(R_T^G(X), Z) = 0$, then $\text{Ext}^1_{kG}(Y, Z) \cong \text{Hom}_{kG}(\mathcal{L}^0, Z)$ by the exactness of the sequence above, which means that $\dim \text{Ext}^1_{kG}(Y, Z) = \dim \text{Hom}_{kG}(\mathcal{L}^0, Z) = [\text{head}(\mathcal{L}^0) : Z]$.

We will now prove that $\text{Ext}^1_{kG}(R_T^G(X), Z) = 0$. By definition of $R_T^G(X)$ and by the Eckmann-Shapiro Lemma, we have $\text{Ext}^1_{kG}(R_T^G(X), Z) = \text{Ext}^1_{kG}(\tilde{X}|_B, Z) \cong \text{Ext}^1_{kB}(\tilde{X}, Z)$ (where $\tilde{X}$ is the inflation of $X$ from $T$ to $B$ via the surjective homomorphism $B \to T$ with kernel $U$). Thus, it suffices to prove that $\text{Ext}^1_{kB}(\tilde{X}, Z) = 0$.

Using the notation of [13], let $A$ be the biggest normal subgroup of $B$ of order prime to $r = \text{char}(k)$. The quotient group $B/A$ is an $r$-group. We claim that $\text{Hom}_{kA}(\tilde{X}, Z) = 0$. Since $Z \notin \text{Irr}_k(G|(T, X))$, $0 = \text{Hom}_{kG}(R_T^G(X), Z) = \text{Hom}_{kG}(\tilde{X}|_B, Z) \cong \text{Hom}_{kB}(\tilde{X}, Z)$ (where the isomorphism $\text{Hom}_{kG}(\tilde{X}|_B, Z) \cong \text{Hom}_{kB}(\tilde{X}, Z)$ follows by Frobenius reciprocity). Now, the group $B$ acts on the $k$-vector space $\text{Hom}_k(\tilde{X}, Z)$ (given an element $b \in B$ and a $k$-vector space homomorphism $\phi \in \text{Hom}_k(\tilde{X}, Z)$, $b.\phi \in \text{Hom}_k(\tilde{X}, Z)$ is defined by $(b.\phi)(x) = b\phi(b^{-1}x)$ for any $x \in \tilde{X}$). We have an isomorphism $(\text{Hom}_k(\tilde{X}, Z))^B \cong \text{Hom}_{kB}(\tilde{X}, Z)$, from which it follows that $(\text{Hom}_k(\tilde{X}, Z))^B = 0$. Since $A$ is a normal subgroup of $B$, we also have $0 = (\text{Hom}_k(\tilde{X}, Z))^B = [(\text{Hom}_k(\tilde{X}, Z))^A]^B/A$. But, $B/A$ is an $r$-group and $\text{char}(k) = r$, so this is possible only if $(\text{Hom}_k(\tilde{X}, Z))^A = 0$ (otherwise, the $r$-group $B/A$ would have a non-zero fixed point on the $k$-vector space $(\text{Hom}_k(\tilde{X}, Z))^A$). Therefore, $\text{Hom}_{kA}(\tilde{X}, Z) \cong (\text{Hom}_k(\tilde{X}, Z))^A = 0$.

Applying the five-term inflation-restriction exact sequence on cohomology to the $kB$-module $\tilde{X}^* \otimes_k Z$ (where $\tilde{X}^*$ is the $k$-linear dual of $\tilde{X}$), we have:
0 \rightarrow H^1(B/A, (\tilde{X}^* \otimes_k Z)^A) \rightarrow H^1(B, \tilde{X}^* \otimes_k Z) \rightarrow H^1(A, \tilde{X}^* \otimes_k Z)^{B/A} \rightarrow H^2(B/A, (\tilde{X}^* \otimes_k Z)^A) \rightarrow H^2(B, \tilde{X}^* \otimes_k Z).

By assumption, \( r \nmid |A| \); therefore, \( kA \) is semisimple by Maschke’s Theorem, which means that \( H^1(A, \tilde{X}^* \otimes_k Z)^{B/A} = 0 \). Since \( (\tilde{X}^* \otimes_k Z)^A \cong \text{Hom}_{kA}(\tilde{X}, Z) = 0 \), we have \( H^1(B/A, (\tilde{X}^* \otimes_k Z)^A) = 0 \). We conclude that \( H^1(B, \tilde{X}^* \otimes_k Z) = 0 \) by exactness of the sequence above. But, \( H^1(B, \tilde{X}^* \otimes_k Z) \cong \text{Ext}^1_{kB}(k, \tilde{X}^* \otimes_k Z) \cong \text{Ext}^1_{kB}(\tilde{X}, Z) \), so it follows that \( \text{Ext}^1_{kB}(\tilde{X}, Z) = 0 \), as needed.

**Proposition 4.2.** Let \( Y \) be an irreducible \( kG \)-module in the principal series \( \text{Irr}_k(G|(T, X)) \) and let \( V \) be an irreducible \( kG \)-module in the principal series \( \text{Irr}_k(G|(T, X')) \), with \( \text{Irr}_k(G|(T, X)) \neq \text{Irr}_k(G|(T, X')) \). Then, \( [R^G_T(X) : V] = 0 \).

**Proof.** Since \( V \in \text{Irr}_k(G|(T, X')) \), \( V \) is a composition factor of the head and socle of \( R^G_T(X') \) and \( *R^G_T(V) \neq 0 \). Now, \( W \) gives a full set of \( (B, B) \)-double coset representatives in \( G \), so the Mackey decomposition yields

\[ *R^G_T(R^G_T(X')) \cong \bigoplus_{w \in W} R^T_{T \cap T} *R^w_{T \cap T}(w' X'). \]

But, since \( W \leq N_G(T)/T, wT = wT^{-1} = T \) for any \( w \in W \). Therefore, the functors \( R^w_{T \cap T} \) and \( *R^w_{T \cap T} \) are equal to the identity functor on \( kT - \mod \) for all \( w \in W \), and it follows that

\[ *R^G_T(R^G_T(X')) \cong \bigoplus_{w \in W} w' X'. \]

Since \( *R^G_T \) is exact, \( *R^G_T(V) \) is a non-zero \( kT \)-submodule of the completely reducible \( kT \)-module \( *R^G_T(R^G_T(X')) \). Thus, there must be some subset \( \Omega \subseteq W \) such that

\[ *R^G_T(V) \cong \bigoplus_{w \in \Omega} w' X'. \quad (3) \]

Suppose, for contradiction, that \( [R^G_T(X) : V] \neq 0 \). By the Mackey decomposition,

\[ *R^G_T(R^G_T(X)) \cong \bigoplus_{w \in W} wX. \]

Thus, if \( V \) is a composition factor of \( R^G_T(X) \), then \( *R^G_T(V) \) is a non-zero submodule of \( *R^G_T(R^G_T(X)) \), which means that \( w' X \subseteq *R^G_T(V) \) for some \( w \in W \). Then, (3) yields \( wX \cong w' X' \) for some \( w' \in \Omega \), so that \( X' \cong (w')^{-1}wX \). But, if \( X' \) is a twist of \( X \) by an element of \( W \), then \( \text{Irr}_k(G|(T, X)) = \text{Irr}_k(G|(T, X')) \), contradicting the assumption in the statement of the proposition.

**Corollary 4.3.** If \( Y \in \text{Irr}_k(G|(T, X)) \) and \( V \in \text{Irr}_k(G|(T, X')) \), with \( \text{Irr}_k(G|(T, X)) \neq \text{Irr}_k(G|(T, X')) \), then, \( \text{Ext}^1_{kG}(Y, V) = 0 \).

**Proof.** As above, let \( \mathcal{L}^0 \) be a submodule of \( R^G_T(X) \) with \( R^G_T(X)/\mathcal{L}^0 \cong Y \). Then, by Theorem 4.1 and Proposition 4.2, \( \dim \text{Ext}^1_{kG}(Y, V) = [\text{head}(\mathcal{L}^0) : V] \leq [\mathcal{L}^0 : V] \leq [R^G_T(X) : V] = 0 \).
5 A Bound on the Dimension of $\text{Ext}^1_{kG}(Y, V)$ when $Y$ and $V$ belong to the same Principal Series

Let $G$ be a finite group of Lie type defined in characteristic $p$, and let $k$ be an algebraically closed field of characteristic $r > 0$, $r \neq p$. In this section, we will assume that $Y$ and $V$ belong to the same principal series $\text{Irr}_k(G|(T, X))$ (where $X$ is any one-dimensional $kT$-module). Let $e$ be the $r$-rank of the maximal torus $T$ (that is, $e$ is the maximal dimension of an elementary abelian $r$-subgroup of $T$ as an $\mathbb{F}_p$-vector space). Our goal is to prove that $\dim \text{Ext}^1_{kG}(Y, V) \leq |W : W(T, X)| |W| + \min(\dim Y, \dim V)e$, where $W(T, X)$ is the inertia group of $X$ (see Definition 3.2). In Lemma 5.1, we provide an upper bound on the number of times an irreducible representation in the principal series $\text{Irr}_k(G|(T, X))$ can appear as a composition factor of the induced module $R^G_T(X)$. (The proof of Lemma 5.1 uses strategies of [11 Prop. 3.1].)

Lemma 5.1. If $Z$ is an irreducible $kG$-module in the principal series $\text{Irr}_k(G|(T, X))$, then $[R^G_T(X) : Z] \leq |W|$.

Proof. By the Mackey decomposition,

$$\ast R^G_T(R^G_T(X)) \cong \bigoplus_{w \in W} R^T_{wT} \ast R^T_{wT,T} (wX) \cong \bigoplus_{w \in W} wX.$$ 

Since $wX$ is an irreducible one-dimensional $kT$-module for every $w \in W$, the $kT$-module $\ast R^G_T(R^G_T(X))$ is completely reducible and $\dim \ast R^G_T(R^G_T(X)) = |W|$. On the other hand, since $\ast R^G_T(R^G_T(X))$ has the following direct sum decomposition as a $kT$-module:

$$\ast R^G_T(R^G_T(X)) \cong \bigoplus_{Z' \in \text{Irr}_k(G)} \ast R^G_T(Z') \oplus [R^G_T(X) : Z'].$$

Therefore,

$$\dim \ast R^G_T(R^G_T(X)) = \sum_{Z' \in \text{Irr}_k(G)} [R^G_T(X) : Z'] \dim \ast R^G_T(Z'). \tag{4}$$

Since $Z \in \text{Irr}_k(G|(T, X))$, we have $\ast R^G_T(Z) \neq 0$. Thus, $[R^G_T(X) : Z]$ appears with non-zero coefficient in (4), and it follows that $[R^G_T(X) : Z] \leq \dim \ast R^G_T(R^G_T(X)) = |W|$. \hfill \Box

To establish the desired bound on $\text{Ext}^1_{kG}(Y, V)$, we will work with the unique Sylow $r$-subgroup $T_r$ of the abelian group $T$. Hence, we will have to break our proof into two cases, depending on whether or not $|B|$ is divisible by the characteristic $r$ of $k$.

5.1 Case I: $r \nmid |B|

Assume that the characteristic $r$ of $k$ does not divide $|B|$.

Theorem 5.2. If $Y$ and $V$ are irreducible $kG$-modules in the principal series $\text{Irr}_k(G|(T, X))$, then, $\dim \text{Ext}^1_{kG}(Y, V) \leq |W|$.
Proof. Since $V \in \text{Irr}_k(G|(T, X))$, $V$ is contained in the socle of $R^G_T(X)$. Therefore, we have a short exact sequence of $kG$-modules

$$0 \to V \to R^G_T(X) \to M \to 0$$

(where $M \cong R^G_T(X)/V$). By assumption, $r \nmid |B|$, which means that $kB$ is semisimple and $\tilde{X}$ is an injective $kB$-module. Since induction from $B$ to $G$ is exact, $R^G_T(X) = \tilde{X}|^G_B$ is an injective $kG$-module and it follows that $\text{Ext}^1_{kG}(Y, R^G_T(X)) = 0$. So, the short exact sequence above induces the exact sequence

$$0 \to \text{Hom}_{kG}(Y, V) \to \text{Hom}_{kG}(Y, R^G_T(X)) \to \text{Hom}_{kG}(Y, M) \to \text{Ext}^1_{kG}(Y, V) \to 0.$$  

Therefore, $\dim \text{Ext}^1_{kG}(Y, V) \leq \dim \text{Hom}_{kG}(Y, M) = [\text{soc}(M) : Y] \leq [M : Y] \leq [R^G_T(X) : Y] \leq |W|$ (where the last inequality follows by Lemma 5.1).

5.2 Case II: $r \mid |B|$

Assume that $|B|$ is divisible by the characteristic $r$ of $k$. Let $T_r$ be the (unique) normal Sylow $r$-subgroup of the abelian group $T$. Since $r \mid |B| = |U||T|$ and $U$ is a $p$-group, $r \mid |T|$, so $T_r$ is a non-trivial $r$-subgroup of $T$. We will reduce the problem of bounding $\text{Ext}^1_{kG}(Y, V)$ to a problem of bounding a cohomology group of $T_r$. To achieve this reduction, we will work with the permutation module $k|T_r|^T$. Since $T_r$ is a normal subgroup of $T$, $T/T_r$ is a group and $k|T_r|^T \cong k[T/T_r]$. The group algebra $k[T/T_r]$ is semisimple by Maschke’s Theorem, which means that $k|T_r|^T$ is a completely reducible $kT$-module.

Let $Z$ be any irreducible (and, necessarily one-dimensional) $kT$-module. Since the only irreducible module for an $r$-group in characteristic $r$ is the trivial module $k$, we must have $Z \downarrow_{T_r} = k$. So, by Frobenius reciprocity, $\text{Hom}_{kT}(k|T_r|^T, Z) \cong \text{Hom}_{kT_r}(k, Z \downarrow_{T_r}) = \text{Hom}_{kT_r}(k, k) \cong k$, which means that $[k|T_r|^T : Z] = 1$. Thus, the completely reducible $kT$-module $k|T_r|^T$ contains every irreducible $kT$-module as a direct summand exactly once.

Lemma 5.3. Let $A$ be an abelian $r$-group or rank $e$. Then, $\dim H^n(A, k) = \binom{n + e - 1}{n}$ for all $n \geq 0$.

Proof. We proceed by induction on $e$. If $e = 1$, then $A = \mathbb{Z}/m\mathbb{Z}$ for some $m = r^d$ ($d \in \mathbb{Z}^+$), and $H^n(A, k) \cong k$ for all $n \geq 0$ since $A$ is a cyclic $r$-group. In particular, $\dim H^n(A, k) = 1 = \binom{n + 1 - 1}{n}$, and the statement of the lemma holds. Suppose now that $e > 1$ and $\dim H^n(A', k) = \binom{n + e - 2}{n}$ for all abelian $r$-groups $A'$ of rank $e - 1$ and all $n \geq 0$.

Let $m_1, \ldots, m_e$ be positive integers such that $m_i = r^{d_i}$ ($d_i \in \mathbb{Z}^+$) for $1 \leq i \leq e$ and $A = \mathbb{Z}/m_1\mathbb{Z} \times \mathbb{Z}/m_2\mathbb{Z} \times \cdots \times \mathbb{Z}/m_e\mathbb{Z}$. By the Künneth formula,
Let $\text{dim Ext}^1_{kT_r}(k, V) = ne$ (where $e$ is the $r$-rank of $T$).

**Proof.** We proceed by induction on the dimension $n$ of $V$. Throughout this proof, we will use the fact that the only irreducible module for an $r$-group in characteristic $r$ is the trivial module $k$. When $n = 1$, we have $V = k$, so we must show that $\text{dim Ext}^1_{kT_r}(k, k) \leq e$. But, $T_r$ is an abelian $r$-group of rank $e$. So, by Lemma 5.3, $\text{dim Ext}^1_{kT_r}(k, k) = \text{dim } H^1(T_r, k) = (1 + e - 1) = e$.

Suppose now that $\text{dim } V = n > 1$ and that the statement of the lemma holds for all $kT_r$-modules $V'$ of dimension $n - 1$. Let $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V$ be a composition series for $V$ as a $kT_r$-module. Note that every composition factor $V_i / V_{i-1}$ $(1 \leq i \leq n)$ is isomorphic to $k$. Therefore, we have a short exact sequence $0 \to V_{n-1} \to V \to k \to 0$, which induces the following long exact Ext sequence:

$$\cdots \to \text{Ext}^1_{kT_r}(V, k) \to \text{Ext}^1_{kT_r}(V_{n-1}, k) \to \cdots .$$

Thus, dim $\text{Ext}^1_{kT_r}(V, k) \leq \text{dim Ext}^1_{kT_r}(k, k) + \text{dim Ext}^1_{kT_r}(V_{n-1}, k)$. Now, dim $\text{Ext}^1_{kT_r}(k, k) = e$ and dim $\text{Ext}^1_{kT_r}(V_{n-1}, k) \leq (n - 1)e$ by the base case and the inductive hypothesis, respectively. Therefore, dim $\text{Ext}^1_{kT_r}(V, k) \leq e + (n - 1)e = ne$, as needed. 

**Theorem 5.5.** Let $Y$ and $V$ be irreducible $kG$-modules in the principal series $\text{Irr}_k(G)(T, X)$, let $W(T, X)$ be the inertia group of $X$, and let $e$ be the rank of a Sylow $r$-subgroup of $T$. In this case,

$$\text{dim Ext}^1_{kG}(Y, V) \leq [W : W(T, X)] |W| + (\text{dim } V)e.$$

**Proof.** Let $\tilde{\mathcal{L}} = \{wX | \text{$w \in W$}\}$. For any $w \in W$, $wT = T$, which means that the $k(wT)$-module $wX$ has the structure of a $kT$-module. Therefore, $\tilde{\mathcal{L}}$ is a subset of $\text{Irr}_k(T)$. Let $\mathcal{L}$ be a subset of $\tilde{\mathcal{L}}$ consisting of one representative of each isomorphism class of irreducible
\( kT \)-modules appearing in \( \widetilde{I} \). Since \( W(T, X) = \{ w \in W \mid wX \cong X \} \), \(|I| = |W : W(T, X)| \).

Since every irreducible \( kT \)-module occurs exactly once as a composition factor of the completely reducible \( kT \)-module \( k|T_r \), we have a direct sum decomposition

\[
k|T_r \cong M \oplus ( \oplus_{X' \in I} X') ,
\]

where \( M \) is a completely reducible \( kT \)-module which does not contain \( X' \) as a composition factor for any \( X' \in I \). Applying the (exact) Harish-Chandra induction functor \( R^G_T \), we can write

\[
R^G_T(k|T_r) \cong R^G_T(M) \oplus ( \oplus_{X' \in I} R^G_T(X')). \tag{5}
\]

Now, since \( Y \in \text{Irr}_k(G|(T, X)) \), \( Y \) is in the head of \( R^G_T(X); \) by (5), \( Y \) is also in the head of \( R^G_T(k|T_r) \). Therefore, we have a short exact sequence of \( kG \)-modules

\[
0 \to M' \to R^G_T(k|T_r) \to Y \to 0
\]

(where \( M' \) is a \( kG \)-submodule of \( R^G_T(k|T_r) \) with \( R^G_T(k|T_r)/M' \cong Y \)), which gives rise to the long exact sequence

\[
0 \to \text{Hom}_{kG}(Y, V) \to \text{Hom}_{kG}(R^G_T(k|T_r), V) \to \text{Hom}_{kG}(M', V) \to \text{Ext}^1_{kG}(Y, V) \to \text{Ext}^1_{kG}(R^G_T(k|T_r), V) \to \cdots.
\]

By exactness, \( \dim \text{Ext}^1_{kG}(Y, V) \leq \dim \text{Hom}_{kG}(M', V) + \dim \text{Ext}^1_{kG}(R^G_T(k|T_r), V) \). So, to prove the theorem, it is enough to show that \( \dim \text{Hom}_{kG}(M', V) \leq |W : W(T, X)| \cdot |W| \) and that \( \dim \text{Ext}^1_{kG}(R^G_T(k|T_r), V) \leq (\dim V) e \).

First, we show that \( \dim \text{Hom}_{kG}(M', V) \leq |W : W(T, X)| \cdot |W| \). We have

\[
\dim \text{Hom}_{kG}(M', V) = [\text{head}(M') : V] \leq [R^G_T(k|T_r) : V] = [R^G_T(M) : V] + \sum_{X' \in I} [R^G_T(X') : V].
\]

By the independence property of Harish-Chandra induction, \( R^G_T(X) \cong R^G_T(X') \) for all \( X' \in I \). So, using Lemma 5.1, we have that

\[
\sum_{X' \in I} [R^G_T(X') : V] = \sum_{X' \in I} [R^G_T(X) : V] \leq \sum_{X' \in I} |W| \leq |I| \cdot |W| = |W : W(T, X)| \cdot |W|.
\]

To show that \( \dim \text{Hom}_{kG}(M', V) \leq |W : W(T, X)| \cdot |W| \), it remains to check that \( [R^G_T(M) : V] = 0 \). By assumption, \( M \) has a direct decomposition \( M \cong \bigoplus_{i=0}^m Z_i \) such that each \( Z_i \) (\( 1 \leq i \leq m \)) is an irreducible \( kT \)-module with \( Z_i \not\cong X' \) for all \( X' \in I \). In particular, for any \( i \) (\( 1 \leq i \leq m \)) and for all \( w \in W \), \( Z_i \not\cong wX \), which means that the Harish-Chandra series \( \text{Irr}_k(G|(T, Z_i)) \) and \( \text{Irr}_k(G|(T, X)) \) are distinct. By Proposition 4.2, \( [R^G_T(Z_i) : V] = 0 \) for all \( i \), and it follows that \( [R^G_T(M) : V] = [\bigoplus_{i=0}^m R^G_T(Z_i) : V] = \sum_{i=0}^m [R^G_T(Z_i) : V] = 0. \)

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We now check that \( \dim \text{Ext}^1_{kG}(R_T^G(k|_{T_r}), V) \leq (\dim V)e \). Using the definition of the functor \( R_T^G \) and the Eckmann-Shapiro Lemma, we have

\[
\text{Ext}^1_{kG}(R_T^G(k|_{T_r}), V) = \text{Ext}^1_{kG}(\widetilde{(k|_{T_r})}^G, V) \cong \text{Ext}^1_{kB}(\widetilde{k|_{T_r}}, V).
\]

Since \( B = U \times T \) and \( r \not| U = [B : T] \), the restriction map \( \text{Ext}^1_{kB}(\widetilde{k|_{T_r}}, V) \to \text{Ext}^1_{kT}(\widetilde{k|_{T_r}}, V) = \text{Ext}^1_{kT}(k|_{T_r}, V) \) is injective. It follows that

\[
\dim \text{Ext}^1_{kG}(R_T^G(k|_{T_r}), V) = \dim \text{Ext}^1_{kB}(\widetilde{k|_{T_r}}, V) \leq \dim \text{Ext}^1_{kT}(k|_{T_r}, V).
\]

Applying the Eckmann-Shapiro Lemma once more, \( \text{Ext}^1_{kT}(k|_{T_r}, V) \cong \text{Ext}^1_{kT_r}(k, V) \). Therefore, \( \dim \text{Ext}^1_{kG}(R_T^G(k|_{T_r}), V) \leq \dim \text{Ext}^1_{kT_r}(k, V) \leq (\dim V)e \) by Lemma 5.4.

**Corollary 5.6.** If \( Y \) and \( V \) are irreducible \( kG \)-modules in the principal series \( \text{Irr}_k(G|(T, X)) \), then \( \dim \text{Ext}^1_{kG}(Y, V) \leq [W : W(T, X)] |W| + \min(\dim Y, \dim V)e \) (where \( e \) is the rank of a Sylow \( r \)-subgroup of \( T \)).

**Proof.** By Theorem 5.5, \( \dim \text{Ext}^1_{kG}(Y, V) \leq [W : W(T, X)] |W| + (\dim V)e \). Since \( Y \) and \( V \) belong to the principal series \( \text{Irr}_k(G|(T, X)) \), \( Y \) and \( V \) occur in both the head and the socle of \( R_T^G(X) \). Hence, the \( k \)-linear duals \( Y^* \) and \( V^* \) occur in both the head and the socle of \( (R_T^G(X))^* \cong R_T^G(X^*) \). Thus, \( Y^* \) and \( V^* \) both belong to the principal series \( \text{Irr}_k(G|(T, X^*)) \), and \( \dim \text{Ext}^1_{kG}(Y, V) = \dim \text{Ext}^1_{kG}(V^*, V^*) \leq [W : W(T, X)] |W| + (\dim Y^*)e = [W : W(T, X)] |W| + (\dim Y)e \) (where the inequality \( \dim \text{Ext}^1_{kG}(V^*, V^*) \leq [W : W(T, X)] |W| + (\dim Y^*)e \) follows by another application of Theorem 5.5).

**Corollary 5.7.** Let \( Y \) and \( V \) be irreducible \( kG \)-modules in the unipotent principal series \( \text{Irr}_k(G|B) \). Then, \( \dim \text{Ext}^1_{kG}(Y, V) \leq |W| + \min(\dim Y, \dim V)e \).

**Proof.** Since \( \text{Irr}_k(G|B) = \text{Irr}_k(G|(T, k)) \) and \( W(T, k) = W \), Corollary 5.6 yields

\[
\dim \text{Ext}^1_{kG}(Y, V) \leq [W : W(T, k)] |W| + \min(\dim Y, \dim V)e = |W| + \min(\dim Y, \dim V)e.
\]

Taking \( Y = k \) in Corollary 5.7, we recover the bound of [13, Cor. 3.3].

### 6 Some Preliminaries from [9, Section 4] (The Steinberg Module and Harish-Chandra Series)

In the next section, we will assume certain additional conditions on the group \( G \) in order to find a bound on \( \dim \text{Ext}^1_{kG}(Y, V) \) when \( Y \in \text{Irr}_k(G|B) \) and \( V \not\in \text{Irr}_k(G|B) \). Several key ideas behind the proof of the main theorem (Theorem 7.1) can be found in [9], so we begin with a summary of the relevant results of [9].
We define an element $e \in kG$ by $e = \sum_{w \in W} (-1)^{l(w)} n_w b$, where $b = \sum_{b \in B} b$ and $n_w \in N_G(T)$ is a representative of the coset $w \in W = N/T \leq N_G(T)/T$. If $k$ is a field, then $St_k = kG e$ is the Steinberg module of $G$ [20]. In [9, Sec. 4], Geck studies the relationship between $St_k$ and Harish-Chandra series of irreducible $kG$-modules, which allows him to determine the composition length of $St_k$ in certain cases. While the composition length of $St_k$ is not needed to prove any of the results presented in this paper, the Harish-Chandra series which Geck constructs with the aid of $St_k$ are crucial to the proof of Theorem 7.1.

6.1 An $r$-modular system.

Let $k$ be a field of characteristic $r > 0$, $r \neq p$. We will work with an $r$-modular system $(\mathcal{O}, K, k)$, where $\mathcal{O}$ is a complete discrete valuation ring with residue field $k$ and field of fractions $K$ (with $\text{char}(K) = 0$). We will assume that the fields $k$ and $K$ are both large enough, meaning that they are splitting fields for $G$ and all of its subgroups.

An $\mathcal{O}G$-module $M$ will be called a lattice if $M$ is finitely generated and free over $\mathcal{O}$. Given a lattice $M$, we let $KM := K \otimes_\mathcal{O} M$ and $\bar{M} := k \otimes_\mathcal{O} M$. It is a standard fact in modular representation theory that given a projective $\mathcal{O}G$-module $M$ and an $\mathcal{O}$-lattice $M'$ (with $M'$ not necessarily projective), $\dim \text{Hom}_{kG}(KM, KM') = \dim \text{Hom}_{kG}(\bar{M}, \bar{M}')$.

As recorded in [9] (pg. 14), Harish-Chandra induction and restriction are compatible with the $r$-modular system in the following sense. Let $J \subseteq S$ and let $L_J$ denote the Levi complement of the standard parabolic subgroup $P_J$ of $G$. Then, if $X$ is an $\mathcal{O}L_J$-lattice, $KR_{L_J}^G(X) \cong R_{L_J}^G(KX)$ and $\bar{R}_{L_J}^G(X) \cong \bar{R}_{L_J}^G(\bar{X})$.

If $Y$ is an $\mathcal{O}G$-lattice, $K^*R_{L_J}^G(Y) \cong *R_{L_J}^G(KY)$ and $*R_{L_J}^G(Y) \cong *R_{L_J}^G(\bar{Y})$.

6.2 An $\mathcal{O}G$-lattice which yields the Steinberg module

Let $St_\mathcal{O} = \mathcal{O}G e$ be the Steinberg module over $\mathcal{O}G$ (we will call $St_\mathcal{O}$ the Steinberg lattice). Then, $KSt_\mathcal{O} \cong St_K$ and $\bar{St}_\mathcal{O} = St_k$. Since $\text{char}(K) = 0$, the Steinberg module $St_K$ is irreducible. In [9] Section 4, Geck constructs another $\mathcal{O}G$-lattice which yields $St_K$ upon base change to $K$. In the remainder of this subsection, we will describe this alternate lattice.

Let $\sigma : U \rightarrow K^\times$ be a group homomorphism, and define $u_\sigma := \sum_{u \in U} \sigma(u) u \in KG$. Since $U$ is a $p$-subgroup of $G$, $r \nmid |U|$ and $\sigma(u) \in O^\times$ for all $u \in U$, which means $u_\sigma \in OG$. We have $u_\sigma^2 = \sum_{u, u' \in U} (\sigma(u)\sigma(u') u')(\sigma(u')u') = \sum_{u \in U} \sum_{u' \in U} \sigma(uu') uu' = \sum_{u \in U} u_\sigma = |U| u_\sigma$.

Since $|U|$ is a unit in $\mathcal{O}$, $\frac{1}{|U|} u_\sigma$ is an idempotent in $\mathcal{O}G$ and the $\mathcal{O}G$-lattice $\Gamma_\sigma := \mathcal{O}Gu_\sigma$ is projective.
Proposition 6.1. ([9, Proposition 4.2]) For any group homomorphism \( \sigma : U \to K^\times \), there exists a unique \( OG \)-sublattice \( \Gamma'_\sigma \subseteq \Gamma_\sigma \) such that \( K(\Gamma_\sigma/\Gamma'_\sigma) \cong St_K \). If \( S_\sigma = \Gamma_\sigma/\Gamma'_\sigma \), the \( kG \)-module \( D_\sigma := \overline{S_\sigma}/rad(\overline{S_\sigma}) \) is irreducible.

6.3 The Gelfand-Graev module

Since \( G \) is a finite group of Lie type defined in characteristic \( p \), there exists a connected reductive algebraic group \( G \) over the algebraic closure \( \overline{\mathbb{F}}_p \) of the finite field \( \mathbb{F}_p \) and a Steinberg endomorphism \( F \) of \( G \) such that \( G = G^F \). We will now assume that the center of \( G \) is connected. Let \( \sigma : U \to K^\times \) be a fixed regular character (see [9, (4.3)]). In this case, the projective \( OG \)-module \( \Gamma_\sigma \) is called a Gelfand-Graev module for \( G \) (the Gelfand-Graev module is unique up to isomorphism when the center of \( G \) is connected). Since \( \Gamma_\sigma \) is a projective \( OG \)-module, the \( r \)-modular reduction \( \Gamma_\sigma \) of \( \Gamma_\sigma \) is a projective \( kG \)-module.

For any \( J \subseteq S \), let \( L_J \) be the corresponding Levi subgroup of \( G \). By [9, 4.3], there is a Gelfand-Graev module \( \Gamma'_\sigma \) for \( OL_J \). Therefore, [9, Proposition 4.2] yields an \( OL_J \)-lattice \( \mathcal{S}_\sigma^J = \Gamma'_\sigma/\Gamma'_\sigma \) such that \( D_\sigma^J := \overline{\mathcal{S}_\sigma^J}/rad(\overline{\mathcal{S}_\sigma^J}) \) is an irreducible \( kL_J \)-module.

The \( OG \)-modules \( \Gamma_\sigma \) and \( \mathcal{S}_\sigma \) behave particularly well with respect to Harish-Chandra restriction. For any \( J \subseteq S \), we have the following isomorphisms of \( OL_J \)-modules [9, Lemma 4.4]:

\[ *R_{L_J}^G(\Gamma_\sigma) \cong \Gamma'_\sigma \quad \text{and} \quad *R_{L_J}^G(\mathcal{S}_\sigma) \cong \mathcal{S}_\sigma^J. \]

(The isomorphism \( *R_{L_J}^G(\Gamma_\sigma) \cong \Gamma'_\sigma \) is due to Rodier.)

Since the Harish-Chandra restriction functor is compatible with the \( r \)-modular system (\( O, K, k \)), we also have \( *R_{L_J}^G(K\Gamma_\sigma) \cong K\Gamma'_\sigma \), \( *R_{L_J}^G(K\mathcal{S}_\sigma) \cong K\mathcal{S}_\sigma^J \), \( *R_{L_J}^G(\mathcal{T}_\sigma) \cong \mathcal{T}_\sigma^J \), and \( *R_{L_J}^G(\overline{\mathcal{T}_\sigma}) \cong \overline{\mathcal{T}_\sigma} \).

6.4 Harish-Chandra series arising from the regular character \( \sigma \)

Let \( P^*_\sigma = \{ J \subseteq S \mid D_\sigma^J \text{ is a cuspidal } kL_J \text{-module} \} \). For every \( J \in P^*_\sigma \), \( D_\sigma^J \) is an irreducible cuspidal \( kL_J \)-module, which means that there is a Harish-Chandra series of the form \( \text{Irr}_k(G|(L_J, D_\sigma^J)) \).

Definition 6.2. We will say that the pair \( (G, k) \) satisfies property (P) if every composition factor of \( k|_B^G \) belongs to a Harish-Chandra series of the form \( \text{Irr}_k(G|(L_J, D_\sigma^J)) \) for some \( J \in P^*_\sigma \).

When the field \( k \) is clear from context we will simply say that \( G \) has property (P). There are many examples of finite groups of Lie type \( G \) such that \( (G, k) \) has property (P). If \( q \) is a power of \( p \), then the pair \( (G, k) \) has property (P) [9, Example 4.9]. If \( r \) is a linear prime, then the following pairs have property (P) [9, 4.14]:

\[ r \text{ is a linear prime for } SO_n(q) \text{ and } Sp_n(q) \text{ if } q^{-1} \not\equiv -1 \text{ mod } r \text{ for all } i \geq 1. \]
1. $(SO_n(q), k)$, $n$ odd and $q$ odd, and
2. $(Sp_n(q), k)$, $n$ even and $q$ a power of 2.

7 A Bound on the Dimension of $\text{Ext}^1$ between a Unipotent Principal Series Representation and an Irreducible Outside the Unipotent Principal Series

Let $G$ be a finite group of Lie type defined in characteristic $p$, and let $k$ be an algebraically closed field of characteristic $r > 0$, $r \neq p$. In this section, we will find a bound on $\dim \text{Ext}^1_{kG}(Y, V)$ when the pair $(G, k)$ has property (P) and $Y$ and $V$ are irreducible $kG$-modules such that $Y \in \text{Irr}_k(G|B)$ and $V \notin \text{Irr}_k(G|B)$. By Theorem 4.1 $\text{Ext}^1_{kG}(Y, V) = 0$ when $V$ is not a composition factor of $k|B^G$. So, it suffices to find a bound on $\dim \text{Ext}^1_{kG}(Y, V)$ in the case that $V$ is a composition factor of $k|B^G$. Since $(G, k)$ has property (P), we can assume that the irreducible $kG$-module $V$ (which is a composition factor of $k|B^G$) belongs to a Harish-Chandra series of the form $\text{Irr}_k(G|(L_J, D_J^j))$ for some $J \in P^*_\sigma$. The next theorem gives our bound on $\dim \text{Ext}^1_{kG}(Y, V)$; we note that some of the proof strategies of Theorem 7.1 were inspired by [9, Prop. 4.6].

**Theorem 7.1.** Suppose that the pair $(G, k)$ has property (P), and let $Y$ and $V$ be irreducible $kG$-modules such that $Y \in \text{Irr}_k(G|B)$ and $V \notin \text{Irr}_k(G|B)$. Assume that $V$ is a composition factor of $k|B^G$ and that $V$ belongs to the Harish-Chandra series $\text{Irr}_k(G|(L_J, D_J^j))$ ($J \in P^*_\sigma$). Then, $\dim \text{Ext}^1_{kG}(Y, V) \leq [W : W_J]$, where $W_J$ is the parabolic subgroup of $W$ generated by $J$.

**Proof.** By Theorem 4.1 $\dim \text{Ext}^1_{kG}(Y, V) \leq [k|B^G : V]$, so it suffices to prove that $[k|B^G : V] \leq [W : W_J]$. Since $V \in \text{Irr}_k(G|(L_J, D_J^j))$, $V$ is in the head of the $kG$-module $R^G_{L_J}(D_J^j)$. By definition, $D_J^j$ is in the head of the $r$-modular reduction $\Gamma_J^j$ of the Gelfand-Graev module $\Gamma_J^j$ for $L_J$, which means that there is a surjective $kL_J$-module homomorphism $\Gamma_J^j \twoheadrightarrow D_J^j$. Since the Harish-Chandra induction functor $R^G_{L_J}$ is exact, there is a surjective $kG$-module homomorphism $R^G_{L_J}(\Gamma_J^j) \twoheadrightarrow R^G_{L_J}(D_J^j)$, and it follows that $V \subseteq \text{head}(R^G_{L_J}(\Gamma_J^j))$.

Let $P_V$ denote the projective indecomposable $kG$-module with head $(P_V) = V$. Since $\Gamma_J^j$ is a projective $kL_J$-module and $R^G_{L_J}$ is exact, $R^G_{L_J}(\Gamma_J^j)$ is a projective $kG$-module. Thus, $P_V$ is a direct summand of $R^G_{L_J}(\Gamma_J^j)$ and we have

$$[k|B^G : V] = \dim \text{Hom}_{kG}(P_V, k|B^G) \leq \dim \text{Hom}_{kG}(R^G_{L_J}(\Gamma_J^j), k|B^G).$$

Now, $R^G_{L_J}(\Gamma_J^j) \cong R^G_{L_J}(\Gamma_J)$ is the reduction of the $OG$-lattice $R^G_{L_J}(\Gamma_J^j)$, and $k|B^G \cong \overline{O|B^G}$ is the reduction of the $\overline{OG}$-lattice $\overline{O|B}$. So, since $KR^G_{L_J}(\Gamma_J) \cong R^G_{L_J}(K\Gamma_J)$ and $K\overline{O|B} \cong K|B^G = R^G_T(K)$, we have

$$\dim \text{Hom}_{kG}(R^G_{L_J}(\Gamma_J^j), k|B^G) = \dim \text{Hom}_{kG}(R^G_{L_J}(K\Gamma_J), K|B^G) = \dim \text{Hom}_{kG}(R^G_{L_J}(K\Gamma_J), R^G_T(K)).$$

(The first equality above holds because $R^G_{L_J}(\Gamma_J)$ is a projective $OG$-lattice.)
We will now compute $\dim \text{Hom}_{kG}(R_{L_j}^G(K\Gamma^J_{\sigma}), R_{T}^G(K))$. First, since the functor $*R_{L_j}^G$ is right adjoint to $R_{L_j}^G$, we have

$$\text{Hom}_{kG}(R_{L_j}^G(K\Gamma^J_{\sigma}), R_{T}^G(K)) \cong \text{Hom}_{kL_j}(K\Gamma^J_{\sigma}, *R_{L_j}^G R_{T}^G(K)).$$

Let $P_J$ be the standard parabolic subgroup of $G$ containing $L_J$, and let $J^W$ denote the set of shortest right coset representatives of $W_J$ in $W$. Then, $J^W$ gives a full set of $(P_J, B)$-double coset representatives in $G$ and it follows by the Mackey decomposition that

$$\text{Hom}_{kL_j}(K\Gamma^J_{\sigma}, *R_{L_j}^G R_{T}^G(K)) \cong \bigoplus_{w \in J^W} \text{Hom}_{kL_j}(K\Gamma^J_{\sigma}, R_{L_j}^G R_{T}^G(wK)).$$

Since $K$ is the trivial one-dimensional $KT$-module, $wK$ is the trivial one-dimensional $K(wT)$-module. For all $w \in W$, $wT = T$, $R_{wT \cap L_j}^G = R_{T \cap L_j}^G = R_T$ is the identity functor on $KT$-mod, $wK = K$, and $R_{wT \cap L_j}^L = R_{L_j}^L$. Therefore, continuing the calculation above (and using the adjointness of $R_{T}^G$ and $*R_{L_j}^G$), we have

$$\bigoplus_{w \in J^W} \text{Hom}_{kL_j}(K\Gamma^J_{\sigma}, R_{L_j}^G R_{T}^G(wK)) \cong \bigoplus_{w \in J^W} \text{Hom}_{kL_j}(K\Gamma^J_{\sigma}, R_{T}^G(K))$$

$$\cong \bigoplus_{w \in J^W} \text{Hom}_{KT}(R_{T}^G(K\Gamma^J_{\sigma}), K).$$

Now, since $T = L_{\emptyset}$, $R_{T}^G(K\Gamma^J_{\sigma}) \cong K\Gamma^\emptyset_{\sigma}$ by Rodier’s result on the restriction of Gelfand-Graev modules in characteristic 0. But, $T$ has a trivial unipotent radical, which means that the Gelfand-Graev module $K\Gamma^\emptyset_{\sigma}$ for $T$ is equal to the group algebra $KT$. Therefore,

$$\bigoplus_{w \in J^W} \text{Hom}_{KT}(R_{T}^G(K\Gamma^J_{\sigma}), K) \cong \bigoplus_{w \in J^W} \text{Hom}_{KT}(KT, K).$$

$K$ appears once as a direct summand of the completely reducible $KT$-module $KT$, so $\dim \text{Hom}_{KT}(KT, K) = 1$ and, following our chain of calculations, we have

$$\dim \text{Hom}_{kG}(R_{L_j}^G(K\Gamma^J_{\sigma}), R_{T}^G(K)) = |J^W|.$$ 

Thus, $[k|_B^G : V] \leq \dim \text{Hom}_{kG}(R_{L_j}^G(K\Gamma^J_{\sigma}), k|_B^G) = \dim \text{Hom}_{kG}(R_{L_j}^G(K\Gamma^J_{\sigma}), R_{T}^G(K)) = |J^W| = [W : W_J]$, as needed. \hfill \square

The bound of Theorem 7.1 is particularly strong in the case that $V$ is a cuspidal irreducible $kG$-module.

**Corollary 7.2.** Suppose that the pair $(G, k)$ has property (P), and let $V$ be a cuspidal irreducible $kG$-module. Then, $\dim \text{Ext}_{kG}^1(Y, V) \leq 1$ for any irreducible $kG$-module $Y \in \text{Irr}_k(G|B)$.

**Proof.** If $V$ is not a composition factor of $k|_B^G$, then $\text{Ext}_{kG}^1(Y, V) = 0$ by Theorem 4.1. If $V$ is a composition factor of $k|_B^G$, the Harish-Chandra series containing $V$ is of the form $\{V\} = \text{Irr}_k(G|(G, D_\sigma^S))$. So, by Theorem 7.1, $\dim \text{Ext}_{kG}^1(Y, V) \leq [W : W_J] = 1$. \hfill \square
 Explicit Computations of Bounds on the Dimension of $\text{Ext}^1$ Between Irreducible Modules for $\text{GL}_n(q)$ in Cross Characteristic

We will explicitly demonstrate the bounds on $\dim \text{Ext}^1_{kG}(Y, V)$ given by Theorem 7.1 in a series of examples. In these examples, we will work with the general linear group $G = \text{GL}_n(q)$ over the finite field $\mathbb{F}_q$, where $q$ is a power of a prime $p$. Let $k$ be an algebraically closed field of characteristic $r > 0$, $r \neq p$. By [9, 4.9], the pair $(\text{GL}_n(q), k)$ satisfies property (P), and consequently the bounds of Theorem 7.1 apply.

We will use the parameterization of $k\text{GL}_n(q)$-modules given by Dipper and James in [7, (3.1)]. In this labeling, the irreducible constituents of $k|G_B$ are given by $D(1, \lambda)$, where $\lambda$ ranges over the partitions of $n$. (The trivial irreducible $kG$-module is parameterized as $k = D(1, (n))$.) We will present an algorithm of Dipper and Du [5, Sec. 4.3] which determines the Harish-Chandra vertex of any irreducible $D(1, \lambda)$, $\lambda \vdash n$. Then, we will apply Dipper and Du’s algorithm to certain irreducible $k\text{GL}_n(q)$-modules in order to obtain explicit bounds on the dimension of $\text{Ext}^1$ between these irreducibles.

8.1 The Harish-Chandra Vertex of the Module $D(1, \lambda)$

Let $|q \pmod{r}|$ denote the multiplicative order of $q$ modulo $r$, and define an integer $l \in \mathbb{Z}^+$ by

$$l = \begin{cases} r & \text{if } |q \pmod{r}| = 1 \\ |q \pmod{r}| & \text{if } |q \pmod{r}| > 1. \end{cases}$$

By [6, Thm. 7.6], $\text{Irr}_k(G|B) = \{D(1, \lambda) | \lambda \text{ is } l\text{-regular}\}$. Thus, when $\lambda$ is $l$-regular, the Harish-Chandra vertex of $\lambda$ is the maximal torus $T$. In general, the Harish-Chandra vertex of an irreducible $kG$-module $D(1, \lambda)$ (where $\lambda$ is any partition of $n$) may be determined as follows.

Let $\lambda'$ denote the dual partition of $\lambda$. Suppose that $\lambda' = \lambda'_{-1} + lr\lambda'_1 + lr^2\lambda'_2 + \cdots$ is the $l - r$-adic decomposition of $\lambda'$, meaning that $\lambda'_{-1} \vdash n_{-1}$ is $l$-restricted and $\lambda'_a \vdash n_a$ is $r$-restricted for $a \geq 0$. Since $\lambda' \vdash n$, the integers $n_a \in \mathbb{Z}_{\geq 0}$ must satisfy the relation

$$n = n_{-1} + \sum_{a \geq 0} lr^n n_a.$$ 

If $\lambda'$ has the $l - r$-adic decomposition given above, a Harish-Chandra vertex of the irreducible $k\text{GL}_n(q)$-module $D(1, \lambda)$ is the Levi subgroup

$$L = \text{GL}_1(q)^{\times n_{-1}} \times \text{GL}_d(q)^{\times n_0} \times \text{GL}_{lr}(q)^{\times n_1} \times \text{GL}_{lr^2}(q)^{\times n_2} \times \cdots$$

of $\text{GL}_n(q)$.

---

6A partition $\mu \vdash n$ is called $l$-restricted if its dual $\mu'$ is $l$-regular, meaning that every part of $\mu'$ occurs less than $l$ times.
Example 8.1. Let \(\text{char}(k) = r = 2\), let \(G = \text{GL}_2(q)\), and assume that \(2 \nmid q\). The only partitions of 2 are (2) and (1, 1) = (1^2), which means that the irreducible \(kG\)-modules which appear as composition factors of \(k|G|_B\) are \(D(1, (2))\) and \(D(1, (1^2))\). Here, \(D(1, (2)) = k \in \text{Irr}_k(G|B) = \text{Irr}_k(G|(T, k))\). Therefore, the Harish-Chandra vertex of \(D(1, (2))\) is the maximal torus \(T\).

We will apply Dipper and Du’s algorithm to find a Harish-Chandra vertex of \(D(1, (1^2))\). Since \(r = 2\) and \(2 \nmid q\), \(|q \text{ (mod } r)| = |q \text{ (mod } 2)| = 1\). Therefore, we set \(l = r = 2\). The 2-adic decomposition of \((1^2)^\prime = (2)\) is \((2) = (0) + 2(1) = 2(1)\). Since \(\lambda_0 = (1) \vdash 1\), \(n_0 = 1\); since \(\lambda_a = (0)\) for all \(a \neq 0\), \(n_a = 0\) for \(a \neq 0\). Thus, the Harish-Chandra vertex of \(D(1, (1^2))\) is \(\text{GL}_2(q) = G\), which means that \(D(1, (1^2))\) is cuspidal.

8.2 Some Examples of Bounds on the Dimension of \(\text{Ext}^1\)

As above, let \(G = \text{GL}_n(q)\) and let \(k\) be an algebraically closed field of characteristic \(r > 0\), \(r \nmid q\). In this context, Theorem 7.1 may be restated as follows.

Theorem 8.2. Let \(\lambda \vdash n\) and assume that the irreducible \(kG\)-module \(D(1, \lambda)\) belongs to the unipotent principal series \(\text{Irr}_k(G|B)\). Suppose that \(\mu \vdash n\) is such that \(D(1, \mu)\) is not a composition factor of \(k|G|_B\). If \(D(1, \mu)\) has Harish-Chandra vertex \(L_J \neq T\) (\(\emptyset \neq J \subseteq S\)), then

\[
\dim \text{Ext}^1_{kG}(D(1, \lambda), D(1, \mu)) \leq \frac{|W|}{|W_J|}.\]

In the remainder of this section, we will provide several explicit examples of the bound given in Theorem 8.2.

Example 8.3. \(G = \text{GL}_3(q)\), \(\text{char}(k) = r > 0\), \(r \nmid q\)

In this case, \(W = S_3\), the symmetric group on the set \(\{1, 2, 3\}\), and \(S\) is the set \(\{(1, 2), (2, 3)\}\) of fundamental reflections in \(W\). There are three partitions of 3: \((3)\), \((2, 1)\), and \((1^3)\). So, the irreducible \(kG\)-modules which occur as composition factors of \(k|G|_B\) are \(D(1, (3))\), \(D(1, (2, 1))\), and \(D(1, (1^3))\). Since \(l > 1\) by definition, the partitions \((3)\) and \((2, 1)\) are \(l\)-regular for any \(l\). Therefore, \(D(1, (3))\) and \(D(1, (2, 1))\) belong to \(\text{Irr}_k(G|B)\) for any \(r\) with \(r \nmid q\).

Thus, \(D(1, (1^3))\) is the only composition factor of \(k|G|_B\) whose Harish-Chandra vertex varies with \(r\). We will compute the Harish-Chandra vertex of \(D(1, (1^3))\) and find the corresponding bound on \(\text{Ext}^1\) in the cases of \(r = 2\) and \(r = 3\).

1. \(\text{GL}_3(q)\), \(r = 2 \nmid q\)

\footnote{In the case that \(\lambda = (n)\) and \(D(1, \lambda) = k\), this bound can be improved. As was shown by Guralnick and Tiep, \(\dim \text{Ext}^1_{kG}(k, D(1, \mu)) = \dim H^1(G, D(1, \mu)) \leq 1\) when \(D(1, \mu) \notin \text{Irr}_k(G|B)\).}
Since $r = 2 / q$, $|q \pmod{2}| = 1$. Thus, we set $l = r = 2$. The $2 - 2$-adic decomposition of $(3)$ is $(3) = (1) + 2(1)$. So, a Harish-Chandra vertex of $D(1, (1^3))$ is $L = \text{GL}_1(q) \times \text{GL}_2(q)$, which has a Weyl group of order 2. So, Theorem 8.2 yields the following bounds:

$$\dim \text{Ext}^1_{kG}(k, D(1, (1^3))) \leq \frac{|W|}{2} = \frac{3!}{2} = 3,$$

$$\dim \text{Ext}^1_{kG}(D(1, (2, 1)), D(1, (1^3))) \leq 3.$$

2. $\text{GL}_3(q), r = 3 \nmid q$

In this case $|q \pmod{3}|$ is equal to 1 or 2. If $|q \pmod{3}| = 1$, then $l = 3$ and the approach used above shows that $D(1, (1^3))$ is cuspidal. Thus, Theorem 8.2 yields the following bounds:

$$\dim \text{Ext}^1_{kG}(k, D(1, (1^3))) \leq \frac{|W|}{|W|} = 1,$$

$$\dim \text{Ext}^1_{kG}(D(1, (2, 1)), D(1, (1^3))) \leq 1.$$

If $|q \pmod{3}| = 2$, then $l = 2$ and our algorithm shows that a Harish-Chandra vertex of $D(1, (1^3))$ is $\text{GL}_2(q) \times \text{GL}_1(q)$. In this case, Theorem 8.2 yields the bounds:

$$\dim \text{Ext}^1_{kG}(k, D(1, (1^3))) \leq 3,$$

$$\dim \text{Ext}^1_{kG}(D(1, (2, 1)), D(1, (1^3))) \leq 3.$$

**Remark.** Guralnick and Tiep’s results yield sharper bounds on $\dim \text{Ext}^1_{kG}(k, D(1, (1^3)))$ in the cases of $G = \text{GL}_3(q), r = 2$ and $G = \text{GL}_3(q), r = 3, |q \pmod{3}| = 2$; by [13, Cor. 6.5], $\dim \text{Ext}^1_{kG}(k, D(1, (1^3))) \leq 1$ whenever $0 \neq r \neq p$.

**Example 8.4.** $G = \text{GL}_4(q), r = 2 \nmid q$

Here, $W = \mathfrak{S}_4$, and $l = 2$. The partitions of $(4)$ are: $(4)$, $(3, 1)$, $(2^2)$, $(2, 1^2)$, and $(1^4)$. Thus, the composition factors of $k|B$ are $D(1, (4)), D(1, (3, 1)), D(1, (2^2)), D(1, (2, 1^2))$, and $D(1, (1^4))$. The approach of the previous example yields the following results.

$$\text{Irr}_k(G|B) = \{D(1, (4)) = k, D(1, (3, 1))\}$$

| $\lambda \vdash n$ | $2 - 2$-adic decomposition of $\lambda$ | Harish-Chandra vertex $L_J$ of $D(1, \lambda')$ | $|W_J|$ | $\frac{|W|}{|W_J|}$ |
|-------------------|-----------------------------------|-----------------------------------------|----------|------------------|
| $(4)$             | $(4) = 4(1)$                      | $G$                                    | 4!       | 4! / 4! = 1     |
| $(3, 1)$          | $(3, 1) = (1, 1) + 2(1)$          | $\text{GL}_2(q) \times \text{GL}_1(q) \times 2$ | 2        | 4! / 2 = 12     |
| $(2^2)$           | $(2^2) = 2(1^2)$                  | $\text{GL}_2(q)^2$                     | 4        | 4! / 4 = 6      |
Thus, Theorem 8.2 (2) yields the bounds:
\[
\begin{align*}
\dim \text{Ext}^1_{kG}(k, D(1, (1^4))) &\leq 1, \\
\dim \text{Ext}^1_{kG}(k, D(1, (2^2))) &\leq 6, \\
\dim \text{Ext}^1_{kG}(D(1, (3, 1)), D(1, (1^4))) &\leq 1, \\
\dim \text{Ext}^1_{kG}(D(1, (3, 1)), D(1, (2^2))) &\leq 12, \\
\dim \text{Ext}^1_{kG}(D(1, (3, 1)), D(1, (2^2))) &\leq 6.
\end{align*}
\]

Remark. By [13, Cor. 6.5], \(\dim \text{Ext}^1_{kG}(k, D(1, (1^4))) \leq 1\) and \(\dim \text{Ext}^1_{kG}(k, D(1, (2^2))) \leq 1\). However, the other four bounds cannot be improved using Guralnick and Tiep’s results.

### 8.3 Improved bounds for \(\text{GL}_n(q), n \leq 10\)

In the case that \(G = \text{GL}_n(q)\) (with \(n \leq 10\)), we can use James’s decomposition matrices [17, Appendix 1] to obtain sharper bounds on \(\dim \text{Ext}^1_{kG}(Y, V)\), where \(Y\) and \(V\) are irreducible \(kG\)-modules with \(Y \in \text{Irr}_k(G|B)\) and \(V \not\in \text{Irr}_k(G|B)\).

Dipper and James ([12], etc.) associate two indecomposable \(kG\)-modules to every partition of \(n\). Given \(\lambda \vdash n\), there is an indecomposable \(kG\)-module \(S(1, \lambda)\) with \(\text{head}(S(1, \lambda)) = D(1, \lambda)\), which maps to a Specht module for a Hecke algebra under an appropriate Hecke functor [12, (3.1)]. There is also an indecomposable \(kG\)-module \(X(1, \lambda)\) (called a Young module) which contains \(S(1, \lambda)\) [17, pg. 43]. Each indecomposable direct summand of the permutation module \(k|_B^G\) is isomorphic to a Young module \(X(1, \lambda)\) for some \(\lambda \vdash n\). By [12, Thm. 1], each Young module \(X(1, \lambda)\) \((\lambda \vdash n)\) has an irreducible head and socle.

As above, let
\[
l = \begin{cases} 
q \pmod{r} & \text{if } |q \pmod{r}| = 1 \\
|q \pmod{r}| & \text{if } |q \pmod{r}| > 1.
\end{cases}
\]

Proposition 8.5. If \(\sigma \vdash n\) is \(l\)-regular and \(\mu \vdash n\) is \(l\)-restricted, then
\[
\dim \text{Ext}^1_{kG}(D(1, \sigma), D(1, \mu)) \leq \max\{\max\{|X(1, \lambda) : D(1, \mu)| : \lambda \vdash n\} \}
\]

Proof. Since \(\sigma\) is \(l\)-regular, \(D(1, \sigma) \in \text{Irr}_k(G|B)\), which means that \(D(1, \sigma)\) is in the head of \(k|_B^G\). Thus, there is a partition \(\tilde{\sigma} \vdash n\) such that the Young module \(X(1, \tilde{\sigma})\) is a direct summand of \(k|_B^G\) and \(\text{head}(X(1, \tilde{\sigma})) = D(1, \sigma)\). Let \(M\) be a maximal submodule of \(X(1, \tilde{\sigma})\) with \((X(1, \tilde{\sigma})/M) \cong D(1, \sigma)\) and write \(k|_B^G = X(1, \tilde{\sigma}) \oplus X\), where the \(kG\)-module \(X\) is a direct sum of various Young modules. Then, \(L^0 := M \oplus X\) is a maximal submodule of \(k|_B^G\) with \(k|_B^G/L^0 \cong D(1, \sigma)\). By Theorem 8.1, \(\dim \text{Ext}^1_{kG}(D(1, \sigma), D(1, \mu)) = \max\{\text{head}(L^0) : D(1, \mu)\}\). But, since \(D(1, \mu) \not\in \text{Irr}_k(G|B)\), \(D(1, \mu)\) is not in the head of \(k|_B^G\), and it follows that \(\text{head}(L^0) : D(1, \mu)\) = \(\text{head}(M) : D(1, \mu)\) ≤ \(X(1, \tilde{\sigma}) : D(1, \mu)\) ≤ \(\max\{|X(1, \lambda) : D(1, \mu)| : \lambda \vdash n\}\). \(\square\)

Following [17], let \(d_{\lambda\mu} = [S(1, \lambda) : D(1, \mu)]\) for any partitions \(\lambda, \mu\) of \(n\). James [17] shows how to find \(d_{\lambda\mu}\) for \(\text{GL}_n(q)\) when \(n \leq 10\) and records the integers \(d_{\lambda\mu}\) in the matrices \(\Delta_n\).
in [17, App. 1]. Since the composition factors of \(X(1, \lambda)\) are the same as the composition factors of \(\bigoplus_{\mu \vdash \lambda} S(1, \mu) \oplus d_{\mu', \nu'}\), with multiplicity (see [17, (3.4)]), we can use James’s matrices to compute \([X(1, \lambda) : D(1, \mu)]\) for any \(\lambda, \mu \vdash n\) when \(n \leq 10\). As illustrated in the next example, such computations (combined with the result of Proposition 8.5) can yield bounds on the dimensions of \(\text{Ext}^1\) groups between irreducible \(kG\)-modules.

**Example 8.6.** As in the first part of Example 8.3, consider \(G = \text{GL}_3(q)\) with \(r = 2 \nmid q\). In this case, \(l = 2\) (\(l\) is denoted by \(e\) in [17]). The partitions of 3 are \(3, (2, 1),\) and \(1^3\); in this case, \(\text{Irr}_k(G|B) = \{D(1, (3)), D(1, (2, 1))\}\) and \(D(1, (1^3))\) is the only irreducible \(kG\)-module outside of \(\text{Irr}_k(G|B)\).

We will use the information contained in the matrix \(\Delta_3\) on pg. 253 of [17] to find \([X(1,(1^3)) : D(1, (1^3))], [X(1,(2,1)) : D(1, (1^3))],\) and \([X(1,(3)) : D(1, (1^3))].\) We start with \([X(1,(1^3)) : D(1, (1^3))].\) Since \(d_{(3)(3)} = 1, d_{(2,1)(3)} = 0\) and \(d_{(1^3)(3)} = 1, X(1, (1^3))\) has the same composition factors as \(S(1, (1^3)) \oplus S(1, (3)).\) Therefore, \([X(1,(1^3)) : D(1, (1^3))] = [S(1, (1^3)) : D(1, (1^3))] + [S(1, (3)) : D(1, (1^3))] = d_{(1^3)(1^3)} + d_{(3)(3)} = 1 + 0 = 1.\) The same approach yields \([X(1,(2,1)) : D(1, (1^3))] = 0\) and \([X(1,(3)) : D(1, (1^3))] = 0\) (the fact that \([X(1,(3)) : D(1, (1^3))] = 0\) is obvious since \(X(1,(3)) = S(1, (3)) = D(1, (3)) = k).\) Thus, \(\max(\{[X(1,\lambda) : D(1, (1^3))] | \lambda \vdash 3\}) = 1\) and Proposition 8.5 yields the improved bounds \(\dim \text{Ext}^1_{kG}(k, D(1, (1^3))) \leq 1\) and \(\dim \text{Ext}^1_{kG}(D(1, (2, 1)), D(1, (1^3))) \leq 1.\)

**Example 8.7.** We use the approach of Example 8.6 to obtain new bounds in the other cases considered in Examples 8.3 and 8.4.

\(\text{GL}_3(q), r = 3 \nmid q, |q \pmod{3}| = 1\)

In this case, \(l = 3\), so \(\text{Irr}_k(G|B) = \{D(1, (3)), D(1, (2, 1))\}\) and \(D(1, (1^3))\) is the only irreducible \(kG\)-module outside of \(\text{Irr}_k(G|B).\) Using the matrix \(\Delta_3\) on pg. 258, we find that \([X(1,(1^3)) : D(1, (1^3))] = 1, [X(1,(2,1)) : D(1, (1^3))] = 0,\) and \([X(1,(3)) : D(1, (1^3))] = 0.\) Therefore, Proposition 8.5 yields the bounds:
\(\dim \text{Ext}^1_{kG}(D(1, (3)), D(1, (1^3))) \leq 1\) and \(\dim \text{Ext}^1_{kG}(D(1, (2, 1)), D(1, (1^3))) \leq 1.\)

\(\text{GL}_3(q), r = 3 \nmid q, |q \pmod{3}| = 2\)

In this case, \(l = 2\), so we use the matrix \(\Delta_3\) on pg. 253 to obtain the bounds:
\(\dim \text{Ext}^1_{kG}(D(1, (3)), D(1, (1^3))) \leq 1\) and \(\dim \text{Ext}^1_{kG}(D(1, (2, 1)), D(1, (1^3))) \leq 1.\)

\(\text{GL}_4(q), r = 2 \nmid q\)

The partitions of \(4\) are \(4, (3, 1), (2^2), (2, 1^2),\) and \(1^4.\) Since \(l = 2, (4)\) and \((3, 1)\) are the only \(l\)-regular partitions of \(4\). We have \(\text{Irr}_k(G|B) = \{D(1, (4)), k, D(1, (3, 1))\}\); the composition factors of \(k|_B^G\) outside the unipotent principal series are \(D(1, (2^2)), D(1, (2, 1^2)),\)
and $D(1, (1^4))$. The matrix $\Delta_4$ [17, pg. 253] (along with the appropriate adjustments for $r = 2$ described at the bottom of pg. 253) yields the following information about composition multiplicities of $D(1, (2^2))$, $D(1, (2, 1^2))$, and $D(1, (1^4))$ in the Young modules $X(1, \lambda)$, $\lambda \vdash 4$.

| $X(1, \lambda)$ | $[X(1, \lambda) : D(1, (2^2))]$ | $[X(1, \lambda) : D(1, (2, 1^2))]$ | $[X(1, \lambda) : D(1, (1^4))]$ |
|-----------------|-------------------------------|-------------------------------|-------------------------------|
| $X(1, (1^4))$   | 0                             | 0                             | 1                             |
| $X(1, (2, 1^2))$ | 1                             | 1                             | 0                             |
| $X(1, (2^2))$   | 1                             | 0                             | 0                             |
| $X(1, (3, 1))$  | 0                             | 0                             | 0                             |
| $X(1, (4))$     | 0                             | 0                             | 0                             |

Thus, Proposition 8.5 gives the following improved bounds:
\[
\dim \operatorname{Ext}^1_{kG}(k, D(1, (1^4))) \leq 1, \\
\dim \operatorname{Ext}^1_{kG}(k, D(1, (2, 1^2))) \leq 1, \\
\dim \operatorname{Ext}^1_{kG}(k, D(1, (2^2))) \leq 1, \\
\dim \operatorname{Ext}^1_{kG}(D(1, (3, 1)), D(1, (1^4))) \leq 1, \\
\dim \operatorname{Ext}^1_{kG}(D(1, (3, 1)), D(1, (2, 1^2))) \leq 1, \\
\dim \operatorname{Ext}^1_{kG}(D(1, (3, 1)), D(1, (2^2))) \leq 1.
\]

Remark. Though the results of Example 8.7 suggest that the dimensions of $\operatorname{Ext}^1$ groups are bounded by 1 when $G = \text{GL}_n(q)$ and $\text{char}(k) = r \nmid q$, this is not necessarily always the case. For instance, when $G = \text{GL}_4(q)$, $r > 2$, and $e = 3$, Proposition 8.5 yields the bound $\dim \operatorname{Ext}^1_{kG}(D(1, (3, 1)), D(1, (2, 1^2))) \leq 2$. We will explore whether the bounds obtained via Proposition 8.5 are optimal in future work. However, even if the bounds of Proposition 8.5 are not optimal, they do support the widely held belief that $\operatorname{Ext}$ groups are “small” in non-defining characteristic.

9 Conclusion and Outlook

We have made some progress toward a more complete understanding of $\operatorname{Ext}$ groups between irreducible $kG$-modules in non-defining characteristic. But, perhaps even more importantly, we have demonstrated that modular Harish-Chandra theory is a useful tool in the study of $\operatorname{Ext}$ groups between irreducible $kG$-modules. There are many open problems which may be viewed through the lens of modular Harish-Chandra theory, several of which are briefly outlined below.

First, there are more cases in which to compute bounds on $\dim \operatorname{Ext}^1_{kG}(Y, V)$ (where $Y$, $V$ are irreducible $kG$-modules.) In this paper, we have assumed that $Y \in \text{Irr}_k(G|(T, X))$ is a principal series representation. But, what if $Y$ belongs to a Harish-Chandra series of the form $\text{Irr}_k(G|(L, X))$ where $T \subset L$? A natural question to ask is whether we can find bounds on $\dim \operatorname{Ext}^1_{kG}(Y, V)$ analogous to those of Sections 4, 5 and 7 when $Y$ is not a principal series representation. Another question to consider is whether we can drop some of the additional assumptions on $G$ in Section 7. Specifically, is there a bound analogous to that found in
Section 7 when the pair $(G, k)$ does not satisfy property $(P)$? Additionally, it may be productive to look beyond the $\text{Ext}^1$ case and explore whether modular Harish-Chandra theory yields useful information on the dimension of higher $\text{Ext}$ groups in non-defining characteristic.

In another direction, we plan to continue to develop the computations of Section 8.3 involving the decomposition matrices for $\text{GL}_n(q)$. These computations yield significantly improved bounds on the dimension of $\text{Ext}^1$ in the cases considered. It may soon be possible to extend these computations to other finite groups of Lie type using recent work of Du, Parshall, and Scott [8], in which they construct an analog of the $q$-Schur algebra outside of Type A. Additionally, we may be able to use certain ideas involved in the computations of Section 8.3 to obtain new bounds on the dimension of $\text{Ext}^1$ (and perhaps higher $\text{Ext}$ groups) between irreducible modules for finite groups of Lie type.

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