A SOBOLEV SPACE THEORY FOR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS WITH TIME-FRACTIONAL DERIVATIVES

ILDOO KIM, KYEONG-HUN KIM, AND SUNGBIN LIM

ABSTRACT. In this article we present an $L^p$-theory ($p \geq 2$) for the time-fractional quasi-linear stochastic partial differential equations (SPDEs) of type

$$\partial^\alpha_t u = L(\omega, t, x)u + f(u) + \partial^\beta_t \sum_{k=1}^\infty \int_0^1 (\Lambda^k(\omega, t, x)u + g^k(u))dw^k_t,$$

where $\alpha \in (0, 2)$, $\beta < \alpha + 1/2$, and $\partial^\alpha_t$ and $\partial^\beta_t$ denote the Caputo derivative of order $\alpha$ and $\beta$ respectively. The processes $w^k_t$, $k \in \mathbb{N} = \{1, 2, \cdots\}$, are independent one-dimensional Wiener processes defined on a probability space $\Omega$, $L$ is a second order operator of either divergence or non-divergence type, and $\Lambda^k$ are linear operators of order up to two. The coefficients of the equations depend on $\omega(\in \Omega), t, x$ and are allowed to be discontinuous. This class of SPDEs can be used to describe random effects on transport of particles in medium with thermal memory or particles subject to sticking and trapping.

We prove uniqueness and existence results of strong solutions in appropriate Sobolev spaces, and obtain maximal $L^p$-regularity of the solutions. Our result certainly includes the $L^p$-theory for SPDEs driven by space-time white noise if the space dimension $d < 4 - 2(2\beta - 1)\alpha^{-1}$. In particular, if $\beta < 1/2 + \alpha/4$ then we can cover space-time white noise driven SPDEs with space dimension $d = 1, 2, 3$. This is a quite interesting result since in case of the classical SPDE, i.e. $\alpha = \beta = 1$, $d$ must be one.

1. Introduction

Fractional calculus and related equations have become important topics in many science and engineering fields. For instance, they appear in mathematical modeling [27, 40], control engineering [5, 34], biophysics [15, 25], electromagnetism [14, 44], polymer science [2, 32], hydrology [4, 41], and even finance [36, 40]. While the classical heat equation $\partial_t u = \Delta u$ describes the heat propagation in homogeneous mediums, the time-fractional diffusion equation $\partial^\alpha_t u = \Delta u$, $\alpha \in (0, 1)$, can be used to model the anomalous diffusion exhibiting subdiffusive behavior, due to particle sticking and trapping phenomena [28, 31]. The fractional wave equation $\partial^\alpha_t u = \Delta u$, $\alpha \in (1, 2)$ governs the propagation of mechanical diffusive waves in viscoelastic media [26]. The fractional differential equations have an another important issue in

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the probability theory related to non-Markovian diffusion processes with a memory  
However, so far, the study of time-fractional partial differential equations is mainly restricted to deterministic equations. For the results on deterministic equations, we refer the reader e.g. to 
(L2-theory), (Lp-theory), and (Lq(Lp)-theory). Also see [38] for BUC1−β([0,T];X)-type estimates, [7] for Schauder estimates, [51] for DeGirogi-Nash type estimate, and [54] for Harnack inequality.

In this article we prove existence and uniqueness results in Sobolev spaces for the time-fractional SPDEs of non-divergence type

\[
\partial_t^\alpha u = [a^{ij}u_{x^i x^j} + b^i u_x + cu + f(u)] \\
+ \partial_t^\beta \int_0^t [\sigma^{ijk} u_{x^i x^j x^k} + \mu^k u_{x^k} + \nu^k u + g^k(u)] dw_s^k
\]

(1.1)
as well as of divergence type

\[
\partial_t^\alpha u = \left[D_{x^i} (a^{ij}u_{x^j} + b^i u + f^i(u)) + cu + h(u)\right] \\
+ \partial_t^\beta \int_0^t [\sigma^{ijk} u_{x^i x^j x^k} + \mu^k u_{x^k} + \nu^k u + g^k(u)] dw_s^k.
\]

(1.2)

Here, \( \alpha \) and \( \beta \) are arbitrary constants satisfying

\[
\alpha \in (0, 2), \quad \beta \in (-\infty, \alpha + \frac{1}{2}),
\]

(1.3)

and the equations are interpreted by their integral forms (see Definition 2.5). The notation \( \partial_t^\gamma \) denotes the Caputo derivative and the Riemann-Liouville integral of order \( \gamma \) if \( \gamma \geq 0 \) and if \( \gamma \leq 0 \), respectively (see Section 2). The coefficients \( a^{ij}, b^i, c, \sigma^{ijk}, \mu^k, \) and \( \nu^k \) are functions depending on \( (\omega, t, x) \in \Omega \times [0, \infty) \times \mathbb{R}^d \), and the nonlinear terms \( f, f^i, h, \) and \( g^k \) depend on \( (\omega, t, x) \) and the unknown \( u \). The indices \( i \) and \( j \) go from 1 to \( d \) and \( k \) runs through \( \{1, 2, 3, \cdots \} \). Einstein’s summation convention on \( i, j, \) and \( k \) is assumed throughout the article. By having infinitely many Wiener processes in the equations, we cover SPDEs for measure valued processes, for instance, driven by space-time white noise (see Section 7.3).

Concerning the leading coefficients \( a^{ij} \), while in the divergence case we assume they are measurable in \( \omega \), piecewise continuous in \( t \), and uniformly continuous in \( x \), in the non-divergence case we further assume \( \sup_{\omega, t} |a^{ij}(t, \cdot)|_{C^{\gamma}(\mathbb{R}^d)} < \infty \) to get \( H^{p+2}_p \)-valued solutions. Here \( \gamma \in \mathbb{R} \) and \( p \geq 2 \).

Our motivation of studying time-fractional SPDEs of type (1.1) and (1.2) naturally arises from the consideration of the heat equation in a material with thermal memory and random external forces. For a detailed derivation of these equations, we refer the reader to [6]. If \( \alpha = \beta = 1 \) then (1.1) and (1.2) are classical second-order SPDEs of non-divergence and divergence types. Hence our equations are far-reaching generalizations. The condition \( \beta < \alpha + \frac{1}{2} \) in (1.3) is necessary to make sense of the equations. We refer again to [3] for the reason.

To the best of our knowledge, [11][10][12] made first attempt to study the mild solutions to time-fractional SPDEs. The authors in [11][10][12] applied \( H^{\infty} \)-functional calculus technique to obtain \( L_p(L_q) \)-estimates \( (p, q \geq 2) \) for the mild solution to the integral equation of the type

\[
U(t) + A \int_0^t k_1(t-s)U(s)ds = \int_0^t k_2(t-s)G(s)dW_s.
\]

(1.4)
Here, \(k_1(t) = \frac{1}{t^{(1/2)}}t^{\alpha-1},\) \(k_2 = \frac{1}{t^{\beta'}}t^{\beta'-1},\) and \(A\) is the generator of a bounded analytic semigroup on \(L_q\) and assumed to admit a bounded \(H^{\infty}\)-calculus on \(L_q.\)

Actually, due to Lemma 2.2(iii), \((1.4)\) is equivalent to

\[
\partial_t u = -Au + \partial_t^\beta \int_0^t G(s) dW_s,
\]

where \(\beta := 1 + \alpha - \beta'.\) Thus equation \((1.5)\) is similar to our equations, but it is quite simple compared to ours. For instance, the operator \(A\) in \((1.4)\) is independent of \((\omega, t),\) and equation \((1.4)\) contains only a additive noise. The extra condition \(\beta > \alpha - 1\) is also assumed there. We also refer to a recent article [6], where an \(L_2\)-theory for time-fractional SPDEs is presented under the extra condition \(\alpha, \beta \in (0, 1).\) The main tool used in [6] for the \(L_2\)-estimate is the Parseval’s identity.

In this article we exploit Krylov’s analytic approach to study the strong solutions of much general equations. To obtain \(L_p\)-estimates of solutions, we control the sharp functions of the solutions in terms of the maximal functions of free terms \(f, h,\) and \(g,\) and then apply Hardy-Littlewood theorem and Fefferman-Stein theorem. The lack of integrability of derivatives of kernels related to the representation of solutions causes main difficulties in carrying out this procedure. We prove that for any \(\gamma \in \mathbb{R}\) and \(p \geq 2,\) under a minimal regularity assumption (depending on \(\gamma\)) on the coefficients and the nonlinear terms, equation \((1.1)\) with zero initial condition has a unique \(H_p^{\gamma+2}\)-valued solution, and for this solution the following estimate holds:

\[
\|u\|_{H_p^{\gamma+2}(T)} \leq N \left( \|f(0)\|_{\mathbb{H}_p^{\gamma+2}(T)} + \|g(0)\|_{\mathbb{H}_p^{\gamma+2}(T,l_2)} \right),
\]

where \(\mathbb{H}_p^\gamma(T) = L_p(\Omega \times [0, T); H_p^\gamma),\) \(\mathbb{H}_p^\gamma(T, l_2) = L_p(\Omega \times [0, T); H_p^\gamma(l_2))\) and \(c_0 > (2\beta-1)/2 > 0\) if \(\beta = 1/2,\) and \(c_0 = 0\) if \(\beta \neq 1/2.\) The result for \(\gamma \leq 0\) is needed to handle SPDEs driven by space-time white noise with the space dimension \(d < 4 - 2(2\beta - 1)\alpha^{-1}.\) For divergence type equation \((1.2),\) we prove uniqueness, existence, and a version of \((1.6)\) for \(\gamma = -1.\)

Our main results, Theorem 2.17 and Theorem 2.15, substantially improve the results of \([11, 10, 12]\) in the sense that (i) we study the strong solutions, (ii) our coefficients are much general and allowed to be discontinuous and depend on \((\omega, t, x),\) (iii) the second and lower order derivatives of solutions appear in the stochastic part of our equations, (iv) non-linear terms are also considered, (v) we do not impose the lower bound of \(\beta\) and there is no restriction on \(\gamma,\) and (vi) we also cover SPDEs driven by space-time white noise with space dimension \(d < 4 - 2(2\beta - 1)\alpha^{-1}.\)

This article is organized as follows. In Section 2 we present some preliminaries on the fractional calculus and introduce our main results. We prove a parabolic Littlewood-Paley inequality for a model time-fractional SPDE in Section 3. The unique solvability and a priori estimate for the model equation are obtained in Section 4. We prove Theorems 2.17 and 2.15 in Section 5 and 6, respectively. In Section 7.3 we give an application to SPDE driven by space-time white noise.

Finally we introduce some notation used in this article. We use “:=” to denote a definition. As usual, \(\mathbb{R}^d\) stands for the \(d\)-dimensional Euclidean space of points \(x = (x_1, \ldots, x_d),\) \(B_r(x) := \{y \in \mathbb{R}^d : |x - y| < r\},\) and \(B_r := B_r(0)\). \(\mathbb{N}\) denotes the natural number system and \(\mathbb{C}\) indicates the complex number system. For \(i =\)
1, . . . , d, multi-indices \( a = (a_1, \ldots, a_d) \), \( a_i \in \{0, 1, 2, \ldots\} \), and functions \( u(x) \) we set
\[
  u_{x^i} = \frac{\partial u}{\partial x^i} = D_i u, \quad D_x^a u = D_1^{a_1} \cdots D_d^{a_d} u, \quad |a| = a_1 + \cdots + a_d.
\]
We also use the notation \( D_x^m \) for a partial derivative of order \( m \) with respect to \( x \). By \( C_c^\infty(\mathbb{R}^d; H) \), we denote the collection of \( H \)-valued smooth functions having compact support in \( \mathbb{R}^d \), where \( H \) is a Hilbert space. In particular, \( C_c^\infty := C_c^\infty(\mathbb{R}^d; \mathbb{R}) \). \( S(\mathbb{R}^d) \) denotes the Schwartz class on \( \mathbb{R}^d \). For \( p \geq 1 \) and a normed space \( F \) by \( L_p(\mathcal{O}; F) \) we denote the set of \( F \)-valued Lebesgue measurable function \( u \) on \( \mathcal{O} \) satisfying
\[
  \|u\|_{L_p(\mathcal{O}; F)} := \left( \int_{\mathcal{O}} \|u(x)\|_F^p dx \right)^{1/p} < \infty.
\]
We write \( L_p(\mathcal{O}) = L_p(\mathcal{O}; \mathbb{R}) \) and \( L_p = L_p(\mathbb{R}^d) \). Generally, for a given measure space \( (X, \mathcal{M}, \mu) \), \( L_p(X, \mathcal{M}, \mu; F) \) denotes the space of all \( F \)-valued \( \mathcal{M}^\mu \)-measurable functions \( u \) so that
\[
  \|u\|_{L_p(X, \mathcal{M}, \mu; F)} := \left( \int_X \|u(x)\|_F^p \mu(dx) \right)^{1/p} < \infty,
\]
where \( \mathcal{M}^\mu \) denotes the completion of \( \mathcal{M} \) with respect to the measure \( \mu \). If there is no confusion for the given measure and \( \sigma \)-algebra, we usually omit the measure and the \( \sigma \)-algebra. We denote by
\[
  \mathcal{F}(f)(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} f(x) dx, \quad \mathcal{F}^{-1}(g)(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\xi \cdot x} g(\xi) d\xi,
\]
the Fourier and the inverse Fourier transforms of \( f \) in \( \mathbb{R}^d \) respectively. \( |a| \) is the greatest integer which is less than or equal to \( a \), whereas \( \lfloor a \rfloor \) denotes the smallest integer which is greater than or equal to \( a \). \( a \wedge b := \min\{a, b\} \), \( a \vee b := \max\{a, b\} \), \( a_+ := a \vee 0 \), and \( a_- := -(a \wedge 0) \). If we write \( N = N(a, b, \cdots) \), this means that the constant \( N \) depends only on \( a, b, \cdots \). Throughout the article, for functions depending on \( (\omega, t, x) \), the argument \( \omega \in \Omega \) will be usually omitted.

2. Main Results

First we introduce some elementary facts related to the fractional calculus. We refer the reader to [33, 39, 3, 16] for more details. For \( \varphi \in L_1((0, T)) \) and \( n = 1, 2, \cdots \), define \( n \)-th order integral
\[
  I_n^\alpha \varphi(t) := \int_0^t (I_{n-1}^\alpha \varphi)(s) ds, \quad (I_0^0 \varphi := \varphi).
\]
In general, the Riemann-Liouville fractional integral of the order \( \alpha \geq 0 \) is defined as
\[
  I_1^\alpha \varphi := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s) ds, \quad 0 \leq t \leq T.
\]
By Jensen’s inequality, for \( p \in [1, \infty] \),
\[
  \|I_n^\alpha \varphi\|_{L_p(0,T)} \leq N(T, \alpha) \|\varphi\|_{L_p(0,T)}. \tag{2.1}
\]
Thus \( I_n^\alpha \varphi(t) \) is well-defined and finite for almost all \( t \leq T \). This inequality shows that if \( 1 \leq p < \infty \) and \( \varphi_n \to \varphi \) in \( L_p([0, T]) \), then \( I_n^\alpha \varphi_n \) also converges to \( I_n^\alpha \varphi \) in \( L_p([0, T]) \). The inequality for \( p = \infty \) implies that if \( f_n(\omega, t) \) converges in probability uniformly in \([0, T]\) then so does \( I_n^\alpha f_n \).
Using Fubini’s theorem one can easily show for any \( \alpha, \beta \geq 0 \)
\[
I_t^\alpha I_t^\beta \varphi = I_t^{\alpha + \beta} \varphi, \quad (a.e.) \ t \leq T. \tag{2.2}
\]
It is known that if \( p > \frac{1}{\alpha} \) and \( \alpha - \frac{1}{p} \not\in \mathbb{N} \) then (see [33] Theorem 3.6)
\[
\|I_t^\alpha \varphi\|_{C^{\alpha-\beta}([0,T])} \leq N(p,T,\alpha)\|\varphi\|_{L_p(0,T)}.
\tag{2.3}
\]
Let \( \alpha \geq 0, \varphi \in C^\alpha([0,T]) \), and \( m \) be the maximal integer such that \( m < \alpha \). It is also known that, for any \( \beta \geq 0 \) (see [33] Theorem 3.2)
\[
\left\| I_t^\alpha \left( \varphi - \sum_{k=0}^{m} \frac{\varphi^{(k)}(0)}{k!} t^k \right) \right\|_{C^{\alpha-\beta}([0,T])} \leq N(\beta) \left\| \varphi - \sum_{k=0}^{m} \frac{\varphi^{(k)}(0)}{k!} t^k \right\|_{C^{\alpha}([0,T])}
\tag{2.4}
\]
if either \( \alpha + \beta \notin \mathbb{N} \) or \( \alpha, \beta \in \mathbb{N} \cup \{0\} \).

Next we introduce the fractional derivative \( D_t^\alpha \), which is (at least formally) the inverse operator of \( I_t^\alpha \). Let \( \alpha \geq 0 \) and \( |\alpha| = n - 1 \) for some \( n \in \mathbb{N} \). Then obviously
\[
n - 1 \leq \alpha < n, \quad n - \alpha \in (0,1].
\]
For a function \( \varphi(t) \) which is \( (n - 1) \)-times differentiable and \( (\frac{d}{dt})^{n-1} I_t^{n-\alpha} \varphi \) is absolutely continuous on \([0,T]\), the Riemann-Liouville fractional derivative \( D_t^\alpha \) and the Caputo fractional derivative \( \partial_t^\alpha \) are defined as
\[
D_t^\alpha \varphi := \left( \frac{d}{dt} \right)^n (I_t^{n-\alpha} \varphi),
\tag{2.5}
\]
and
\[
\partial_t^\alpha \varphi := D_t^{\alpha-(n-1)} \left( \varphi^{(n-1)}(t) - \varphi^{(n-1)}(0) \right).
\tag{2.6}
\]
By (2.2) and (2.5), for any \( \alpha, \beta \geq 0 \),
\[
D_t^\alpha I_t^\beta \varphi = \begin{cases} 
D_t^{\alpha-\beta} \varphi & : \alpha > \beta \\
I_t^{\beta-\alpha} \varphi & : \alpha \leq \beta,
\end{cases}
\tag{2.6}
\]
and \( D_t^{\alpha} D_t^{\beta} = D_t^{\alpha+\beta} \). Using (2.7), one can check
\[
\partial_t^\alpha \varphi = D_t^\alpha \left( \varphi(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} \varphi^{(k)}(0) \right),
\tag{2.7}
\]
Thus if \( \varphi(0) = \varphi^{(1)}(0) = \cdots = \varphi^{(n-1)}(0) = 0 \) then \( D_t^\alpha \varphi = \partial_t^\alpha \varphi \) and by (2.7) and (2.8),
\[
\|\partial_t^\beta \varphi\|_{C^{\alpha-\beta}([0,T])} \leq \|I_t^\beta \varphi\|_{C^{\beta+1-\beta}([0,T])} \leq N\|\varphi\|_{C^{\alpha}([0,T])} \quad \forall \beta \leq \alpha,
\tag{2.8}
\]
where either \( \alpha - \beta \notin \mathbb{N} \) or \( \alpha, \beta \in \mathbb{N} \cup \{0\} \).

**Remark 2.1.** Banach space valued fractional calculus can be defined as above on the basis of Bochner’s integral and Pettis’s integral. See e.g. [1] and references therein.

Let \((\Omega, \mathcal{F}, P)\) be a complete probability space and \( \{ \mathcal{F}_t, t \geq 0 \} \) be an increasing filtration of \( \sigma \)-fields \( \mathcal{F}_t \subset \mathcal{F} \), each of which contains all \( (\mathcal{F}, P) \)-null sets. We assume that an independent family of one-dimensional Wiener processes \( \{ w^k_t \}_{k \in \mathbb{N}} \) relative to the filtration \( \{ \mathcal{F}_t, t \geq 0 \} \) is given on \( \Omega \). By \( \mathcal{P} \), we denote the predictable \( \sigma \)-field generated by \( \mathcal{F}_t \), i.e. \( \mathcal{P} \) is the smallest \( \sigma \)-field containing every set \( A \times (s,t] \), where \( s < t \) and \( A \in \mathcal{F}_s \).
For \( p \geq 2 \) and \( \gamma \in \mathbb{R} \), let \( H_\gamma^p = H_\gamma^p(\mathbb{R}^d) \) denote the class of all tempered distributions \( u \) on \( \mathbb{R}^d \) such that

\[
\|u\|_{H_\gamma^p} := \|(1 - \Delta)^{\gamma/2} u\|_{L_p} < \infty,
\]

where

\[
(1 - \Delta)^{\gamma/2} u = \mathcal{F}^{-1} \left( (1 + |\xi|^2)^{\gamma/2} \mathcal{F}(u) \right).
\]

It is well-known that if \( \gamma = 1, 2, \cdots \), then

\[
H_\gamma^p = W_\gamma^p := \{ u : D_\gamma^p u \in L_p(\mathbb{R}^d), \ |a| \leq \gamma \}, \quad H_\gamma^{-\gamma} = (H_\gamma^p)^{\gamma}. \]

For a tempered distribution \( u \in H_\gamma^p \) and \( \phi \in \mathcal{S}(\mathbb{R}^d) \), the action of \( u \) on \( \phi \) (or the image of \( \phi \) under \( u \)) is defined as

\[
(u, \phi) = \left( (1 - \Delta)^{\gamma/2} u, (1 - \Delta)^{-\gamma/2} \phi \right) = \int_{\mathbb{R}^d} (1 - \Delta)^{\gamma/2} u \cdot (1 - \Delta)^{-\gamma/2} \phi \, dx.
\]

Let \( l_2 \) denote the set of all sequences \( a = (a^1, a^2, \cdots) \) such that

\[
|a|_{l_2} := \left( \sum_{k=1}^{\infty} |a|^2 \right)^{1/2} < \infty.
\]

By \( H_\gamma^p(l_2) = H_\gamma^p(\mathbb{R}^d, l_2) \) we denote the class of all \( l_2 \)-valued tempered distributions \( v = (v^1, v^2, \cdots) \) on \( \mathbb{R}^d \) such that

\[
\|v\|_{H_\gamma^p(l_2)} := \|1 - \Delta\|^\gamma/2 v\|_{l_2} < \infty.
\]

We introduce stochastic Banach spaces:

\[
\mathbb{H}^\gamma_p(T) := L_p(\Omega \times [0, T], \mathcal{P}, H_\gamma^p), \quad \mathbb{L}^\gamma_p(T) = H^0_p(T)
\]

\[
\mathbb{H}^\gamma_p(T, l_2) := L_p(\Omega \times [0, T], \mathcal{P}, H_\gamma^p(l_2)), \quad \mathbb{L}^\gamma_p(T, l_2) = H^0_p(T, l_2).
\]

For instance, \( u \in \mathbb{H}^\gamma_p(T) \) if and only if \( u \) is a \( H_\gamma^p \)-valued \( \mathcal{P} \times dt \)-measurable process defined on \( \Omega \times [0, T] \) such that

\[
\|u\|_{\mathbb{H}^\gamma_p(T)} := \left( \mathbb{E} \int_0^T \|u\|_{H_\gamma^p}^p \, dt \right)^{1/p} < \infty.
\]

Here \( \mathcal{P} \times dt \) is the completion of \( \mathcal{P} \) w.r.t. \( dP \times dt \). We write \( g \in \mathbb{H}^{\infty}_{\gamma}(T, l_2) \) if \( g^k = 0 \) for all sufficiently large \( k \), and each \( g^k \) is of the type

\[
g^k(t, x) = \sum_{i=1}^{n} 1(\tau_{i-1, i}) (t) g^{i k}(x),
\]

where \( \tau_{i} \leq T \) are stopping times with respect to \( \mathcal{F}_t \) and \( g^{i \cdot} \in C_c^\infty(\mathbb{R}^d) \). It is known \cite{21} Theorem 3.10 that \( \mathbb{H}^{\infty}_{\gamma}(T, l_2) \) is dense in \( \mathbb{H}^\gamma_p(T, l_2) \) for any \( \gamma \). We use \( U_\gamma^{(\alpha)} \) to denote the family of \( H_\gamma^{(2-2/(\alpha p))} \)-valued \( \mathcal{F}_0 \)-measurable random variables \( u_0 \) such that

\[
\|u_0\|_{H_\gamma^{(2-2/(\alpha p))}} := \left( \mathbb{E} \|u_0\|_{H_\gamma^{(2-2/(\alpha p))}}^p \right)^{1/p} < \infty,
\]

where \( (2 - 2/(\alpha p))_+ = \frac{(2-2/(\alpha p)) + 2 - 2/(\alpha p)}{2} \).

(i) and (ii) of Lemma \cite{22} below are used e.g. when we apply \( I_\gamma^p \) and \( D_\gamma^p \) to the time-fractional SPDEs, and (ii) can be used in the approximation arguments.
Lemma 2.2. (i) Let $\alpha \geq 0$ and $h \in L_2(\Omega \times [0,T], \mathcal{P}; l_2)$. Then the equality
\[ I^\alpha \left( \sum_{k=1}^\infty \int_0^t h_k(s) dw_k^s \right) (t) = \sum_{k=1}^\infty \left( I^\alpha \int_0^t h_k(s) dw_k^s \right) (t) \] (2.10)
holds for all $t \leq T$ (a.s.) and also in $L_2(\Omega \times [0,T])$, where the convergence of the series in both sides is understood in probability sense.

(ii) Suppose $\alpha \geq 0$ and $h_n \to h$ in $L_2(\Omega \times [0,T], \mathcal{P}; l_2)$ as $n \to \infty$. Then
\[ \sum_{k=1}^\infty \left( I^\alpha \int_0^t h_n_k(s) dw_k^s \right) (t) \to \sum_{k=1}^\infty \left( I^\alpha \int_0^t h_k(s) dw_k^s \right) (t) \]
in probability uniformly on $[0,T]$. (iii) If $\alpha > 1/2$ and $L_2(\Omega \times [0,T], \mathcal{P}; l_2)$, then
\[ \frac{\partial}{\partial t} \left( I^\alpha \sum_{k=1}^\infty \int_0^t h_k(s) dw_k^s \right) (t) = \frac{1}{\Gamma(\alpha)} \sum_{k=1}^\infty \int_0^t (t-s)^{\alpha-1} h_k(s) dw_k^s \]
(a.e.) on $\Omega \times [0,T]$. □

Proof. See Lemmas 3.1 and 3.3 of [6].

Remark 2.3. By [21, Remark 3.2], for any $g \in H_p^\gamma(T, l_2)$ and $\phi \in C_c^\infty(\mathbb{R}^d)$
\[ \mathbb{E} \left[ \sum_k \left( \int_0^T (g_k^s, \phi)^2 \right) ds \right] \leq N(p, \phi) \| g \|_{H_p^\gamma(T, l_2)}^2. \] (2.11)
Thus if $g_n \to g$ in $H_p^\gamma(T, l_2)$, then $(g_n, \phi) \to (g, \phi)$ in $L_2(\Omega \times [0,T], \mathcal{P}; l_2)$. Therefore, one can apply Lemma 2.2 (ii) with $h_n(t) = (g_n(t, \cdot), \phi)$ and $h(t) = (g(t, \cdot), \phi)$.

Let $\alpha \in (0,2), \beta < \alpha + \frac{1}{2}$ and set
\[ \Lambda := \max([\alpha], [\beta]). \]

Definition 2.4. Define
\[ H_p^\alpha(T) := H_p^{\alpha+2}(T) \cap \{ u : I_t^{\Lambda-\alpha}u \in L_p(\Omega; C([0,T]; H_p^{\gamma})) \}, \]
that is, $u \in H_p^{\alpha+2}(T)$ iff $u \in H_p^{\gamma+2}(T)$ and $I_t^{\Lambda-\alpha}u$ has a $H_p^{\gamma}$-valued continuous version $I_t^{\Lambda-\alpha}u$. The norm in $H_p^{\alpha+2}(T)$ is defined as
\[ \| u \|_{H_p^{\alpha+2}(T)} := \| u \|_{H_p^{\gamma+2}(T)} + \left( \mathbb{E} \sup_{t \leq T} \| I_t^{\Lambda-\alpha}u(t, \cdot) \|_{H_p^{\gamma}}^p \right)^{1/p}. \]

Definition 2.5. Let $u \in H_p^{\alpha+2}(T), f \in H_p^{\gamma}(T), g \in H_p^{\gamma}(T, l_2), u_0 \in U_p^{\gamma_1}$, and $v_0 \in U_p^{\alpha-1,\gamma_1}$ for some $\gamma_i \in \mathbb{R}$ $(i = 1, 2, 3, 4)$. We say that $u$ satisfies
\[ \partial_t^\alpha u(t, x) = f(t, x) + \partial^2_s \int_0^t g_k(s, x) dw_k^s, \quad t \in (0,T], \] (2.12)
\[ u(0, \cdot) = u_0, \quad \partial_t u(0, \cdot) = v_0 \] (if $\alpha > 1$)
if for any $\phi \in \mathcal{S}(\mathbb{R}^d)$ the equality
\[ (I_t^{\Lambda-\alpha}u(t) - I_t^{\Lambda-\alpha}(u_0 + t v_0 1_{\alpha > 1}), \phi) \]
\[ = I_t^{\Lambda} (f(t, \cdot), \phi) + \sum_{k=1}^\infty I_t^{\Lambda-\beta} \int_0^t (g_k(s, \cdot), \phi) dw_k^s \] (2.13)
holds for all $t \in [0, T]$ (a.s.) (see Remark 2.8 for an equivalent version of (2.13)).
In this case we say (2.12) holds in the sense of distributions. We say $u$ (or (2.12))
has zero initial condition if (2.13) holds with $u_0 = v_0 = 0$.

Below we discuss how the space $U_{p}^{\alpha, \gamma}$ is chosen and show why (2.13)
is an appropriate interpretation of (2.12).

**Remark 2.6.** In this article we always assume $u(0) = 1_{\alpha>1}\partial_t u(0) = 0$.
The space $U_{p}^{\alpha, \gamma}$ is defined for later use. It turns out that for the solution to
the equation

$$\partial_t^\alpha u = \Delta u, \quad t > 0; \quad u(0, \cdot) = u_0, \quad 1_{\alpha>1}\partial_t u(0, \cdot) = 1_{\alpha>1}v_0,$$

we have, for any $\gamma \in \mathbb{R}$ and $\kappa > 0$,

$$\|u\|_{L_p((0, T), H_{p}^{\gamma+2})} \leq N \left( \|u_0\|_{U_{p}^{\alpha, \gamma'}} + 1_{\alpha>1}\|v_0\|_{U_{p}^{\alpha-1, \gamma'}} \right),$$

where $\gamma' = \gamma + \kappa_1\beta_1 = 1/2$.

**Remark 2.7.** If $\alpha = \beta = 1$ then $\Lambda = 1$ and (2.13) coincides with classical
definition of the weak solution [21, Definition 3.1].

**Remark 2.8.** (i) Let $u, f, g, u_0,$ and $v_0$ be given as in Definition 2.6. We claim that
(2.13) holds for all $t \leq T$ (a.s.) if and only if the equality

$$(u(t) - u_0 - t\partial_t v_01_{\alpha>1}, \phi) = I_t^\alpha(f(t), \phi) + \sum_{k=1}^{\infty} I_t^{\alpha-\beta} \int_0^t (g^k(s), \phi) dw_s^k$$

(2.14)

holds for almost all $t \leq T$ (a.s.). Indeed, applying $D_t^{1-\alpha}$ to (2.13) and using (2.6),
we get equality (2.14) for almost all $t \leq T$ (a.s.). Here $I_t^{1-\beta} := D_t^{\beta-\alpha}$ if $\alpha \leq \beta$.
Note that if $\alpha \leq \beta$, the last term of (2.14) makes sense due to Lemma 2.2(iii) and
the assumption $\beta - \alpha < 1/2$. For the other direction, we apply $I_t^{1-\alpha}$ to (2.14) and
get (2.13) for all $t \leq T$ (a.s.). This is because $(\Lambda_t^{1-\alpha} u, \phi)$ is continuous in $t$
by the assumption $u \in H_{p}^{\gamma+2}(T)$.

Also, taking $D_t^{\alpha}$ to (2.14), we formally get a distributional version of (2.12):

$$(\partial_t^\alpha u, \phi) = (f(t), \phi) + \partial_t^\alpha \int_0^t (g^k(s), \phi) dw_s^k, \quad (a.e.) t \leq T.$$  

(ii) Let $\beta < 1/2$ and $u(0) = 1_{\alpha>1}u'(0) = 0$. Denote

$$\tilde{f}(t) = \frac{1}{\Gamma(1-\beta)} \sum_{k=1}^{\infty} \int_0^t (t-s)^{-\beta} g^k(s) dw_s^k.$$ 

Then from (2.14) and Lemma 2.2(iii) it follows that the equality

$$(u(t), \phi) = I_t^\alpha(f(t) + \tilde{f}(t), \phi)$$

holds for almost all $t \leq T$ (a.s.). Therefore (2.13) holds for all $t \leq T$ (a.s.) with $f + \tilde{f}$ and $0$
in place of $f$ and $g$, respectively.

To use some deterministic results later in this article we show our interpretation of (2.12)
coincides with the one in [17, 48, 49]. In the following remark $u$ is not random and
$\gamma_1 = \gamma_2 = \gamma$.

**Remark 2.9.** Denote $H_{p}^{\gamma+2}(T) = L_p((0, T); H_{p}^{\gamma+2})$ and $L_p(T) = H_{p}^0(T)$. We denote
by $H_{p,\gamma+2}^{\alpha+2}(T)$ the completion of $C_c^\infty((0, \infty) \times \mathbb{R}^d)$ with the norm

$$\|\cdot\|_{H_{p,\gamma+2}^{\alpha+2}(T)} := \|\cdot\|_{H_{p}^{\gamma+2}(T)} + \|\partial_t^\alpha \cdot\|_{H_{p}^{\gamma+2}(T)}.$$
That is, $u \in H_{p,0}^{\alpha,\gamma+2}(T)$ if and only if there exists a sequence $u_n \in C^\infty((0, \infty) \times \mathbb{R}^d)$ such that $\|u_n - u\|_{H_{p,0}^{\alpha,\gamma+2}(T)} \to 0$ and $f_n := \partial_t^\alpha u_n$ is a Cauchy sequence in $H_{p,0}^\gamma(T)$, whose limit is defined as $\partial_t^\alpha u$.

The following two statements are equivalent:

- $u \in H_{p,0}^{\alpha,\gamma+2}(T)$ and $\partial_t^\alpha u = f$ in $H_{p,0}^\gamma(T)$.
- $u \in \mathcal{H}_{p}^{\gamma+2}(T)$, $f \in \mathcal{H}_{p}^{\gamma}(T)$, and $u$ satisfies $\partial_t^\alpha u = f$ with zero initial condition in the sense of Definition 2.4.

First, let $u \in H_{p,0}^{\alpha,\gamma+2}(T)$ and $\partial_t^\alpha u = f$ in $H_{p,0}^\gamma(T)$. Take $u_n$ and $f_n$ as above. Then since $u_n, f_n \in C([0, T]; H_{p}^\gamma)$, we have

$$u_n(t) = I_t^\alpha f_n(t), \quad \forall t \leq T,$$

and letting $n \to \infty$ we conclude

$$u(t) = I_t^\alpha f(t), \quad (a.e.) \; t \leq T. \quad (2.15)$$

Taking $I^{\lambda - \alpha}$ to both sides of (2.15) and recalling $\lambda \geq 1$, one easily finds that $I^{\lambda - \alpha} u$ has an $H_{p}^\gamma$-valued continuous version. Therefore, by Remark 2.4 $u \in \mathcal{H}_{p}^{2+2}(T)$ and it satisfies $\partial_t^\alpha u = f$ with the zero initial condition in the sense of Definition 2.4.

Next, let $u \in \mathcal{H}_{p}^{\gamma+2}(T)$ satisfy $\partial_t^\alpha u = f$ in the sense of Definition 2.4 with zero initial condition. Then by (2.13),

$$u(t) = I_t^\alpha f(t) \quad \text{in} \quad \mathcal{H}_{p}^\gamma(\mathbb{R}^d), \quad (a.e.) \; t \in [0, T].$$

Extend $u$ so that $u(t) = 0$ for $t < 0$. Take $\eta \in C_c^\infty((1, 2))$ with the unit integral, and denote $\eta_{\varepsilon}(t) = \varepsilon^{-1}\eta(t/\varepsilon)$,

$$u^\varepsilon(t) := u \ast \eta_{\varepsilon}(t) := \int_{\mathbb{R}} u(s)\eta_{\varepsilon}(t-s)\, ds = \int_{0}^{t} u(s)\eta_{\varepsilon}(t-s)\, ds,$$

and $f^\varepsilon := f \ast \eta_{\varepsilon}$. Note $u^\varepsilon(t) = 0$ for $t < \varepsilon$, and thus $u^\varepsilon \in C^\alpha([0, T]; H_{p}^\gamma)$ for any $n$. Multiplying by a smooth function which equals one for $t \leq T$ and vanishes for $t > T + 1$, we may assume $u^\varepsilon \in C^\infty((0, \infty); H_{p}^\gamma)$. Obviously $\partial_t^\alpha u^\varepsilon = f^\varepsilon$ in $H_{p,0}^\gamma(T)$, $\|u^\varepsilon - u\|_{H_{p,0}^{\alpha,\gamma+2}(T)} \to 0$ and $\|f^\varepsilon - f\|_{H_{p,0}^\gamma(T)} \to 0$ as $\varepsilon \downarrow 0$. Next choose a smooth function $\zeta(x) \in C_c^\infty(B_1(0))$ with unit integral, and denote $u^{\varepsilon,\delta}(t, x) = u^\varepsilon * \zeta(\cdot/\delta)$ and define $f^{\varepsilon,\delta} = f * \zeta(\cdot/\delta)$ similarly. Then we still have $\partial_t^\alpha u^{\varepsilon,\delta} = f^{\varepsilon,\delta}$. For any $\varepsilon' > 0$, choose $\varepsilon$ and $\delta$ so that $\|u^{\varepsilon,\delta} - u\|_{H_{p,0}^{\alpha,\gamma+2}(T)} + \|\partial_t^\alpha(u^{\varepsilon,\delta} - u^\varepsilon)\|_{H_{p,0}^\gamma(T)} \leq \varepsilon'$. After this, multiplying by appropriate smooth cut-off functions of $x$, we can approximate $u^{\varepsilon,\delta}$ and $f^{\varepsilon,\delta}$ with functions in $C_c^\infty((0, \infty) \times \mathbb{R}^d)$, and therefore we may assume $u^{\varepsilon,\delta}, f^{\varepsilon,\delta} \in C_c^\infty((0, \infty) \times \mathbb{R}^d)$. Thus it follows that $u \in H_{p,0}^{\alpha,\gamma+2}(T)$ and it satisfies $\partial_t^\alpha u = f$ as the limit in $H_{p,0}^\gamma(T)$.

**Theorem 2.10.** (i) For any $\gamma, \nu \in \mathbb{R}$, the map $(1 - \Delta)^{\nu/2} : \mathcal{H}_{p}^{\gamma+2} \to \mathcal{H}_{p}^{-\nu+2}(T)$ is an isometry.

(ii) Let $u \in \mathcal{H}_{p}^{\gamma+2}(T)$ satisfy (2.18). Then

$$\mathbb{E}\sup_{t \leq T} \|I^{\lambda - \alpha} u(t, \cdot)\|_{H_{p,0}^\gamma} \leq N(\|u(0)\|_{H_{p,0}^\gamma} + 1_{\alpha > 1}\mathbb{E}\|\partial_t u(0)\|_{H_{p,0}^\gamma} + \|f\|_{H_{p,0}^\gamma} + \|g\|_{L^2(H_{p,0}^\gamma(T, L^2)))}, \quad (2.16)$$

where $N = N(d, p, T)$. 
(iii) $\mathcal{H}^{\gamma+2}_p(T)$ is a Banach space,
(iv) Let $\theta := \min\{1, \alpha, 2(\alpha - \beta) + 1\}$. Then there exists a constant $N = N(d, \alpha, \beta, p, T)$ so that for any $t \leq T$ and $u \in \mathcal{H}^{\gamma+2}_p(T)$ satisfying (2.12) with the zero initial condition,

$$
\|u\|_{\mathcal{H}^{\gamma+2}_p(T)}^p \leq N \int_0^t (t-s)^{\theta-1} \left(\|f\|_{H^\theta_p(s)}^p + \|g\|_{H^\theta_p(s, t)}^p\right) ds.
$$

(2.17)

**Proof.** (i) For any $u \in \mathcal{H}^{\gamma+2}_p(T)$, $(1 - \Delta)^{\nu/2} I_t^{\Lambda - \alpha} u$ is an $H^{\gamma+\nu+2}_p$-valued continuous version of $(1 - \Delta)^{\nu/2} I_t^{\Lambda - \alpha} u$. Thus it is obvious.

(ii) Due to (i), we may assume that $\gamma = 0$. Take a nonnegative function $\zeta \in C_0^\infty(\mathbb{R}^d)$ with unit integral. For $\varepsilon > 0$ define $\zeta_{\varepsilon}(x) = \varepsilon^{-d} \zeta(x/\varepsilon)$, and for tempered distributions $\nu$ on $\mathbb{R}^d$ put $\nu(x) := \nu * \zeta_{\varepsilon}(x)$. Note that for each $t \in (0, T)$, $\nu(x)$ is an infinitely differentiable function of $x$. By plugging $\zeta_{\varepsilon}(\cdot - x)$ in (2.13) in place of $\phi$, for any $x$

$$(I^{\Lambda - \alpha}_t u)(x, t) = I^{\Lambda}_t f(x, t) + I^{\Lambda - \beta}_t \int_0^t g(x, s) ds + \int_0^t h(x, s) ds, \quad \forall t \leq T (a.s.).$$

(2.18)

Observe that

$$
\mathbb{E} \sup_{t \leq T} \left\| I^{\Lambda}_t f(x, \cdot) \right\|_p^p \leq N \mathbb{E} \int_0^T \| f(x, \cdot) \|_p^p ds.
$$

(2.19)

Also, by (2.1), the Burkholder-Davis-Gundy inequality, and the Hölder inequality,

$$
\mathbb{E} \sup_{t \leq T} \left\| I^{\Lambda - \beta}_t \int_0^t g(x, s, \cdot) ds \right\|_p^p \leq N \int_\mathbb{R}^d \mathbb{E} \sup_{t \leq T} \left\| \sum k \int_0^t g(x, s, \cdot) ds \right\|_p^p dx
$$

$$
\quad \leq N \mathbb{E} \int_0^T \| g(x, \cdot) \|_{L_p(t, 0)}^p ds.
$$

(2.20)

Thus from (2.18),

$$
\mathbb{E} \sup_{t \leq T} \left\| (I^{\Lambda - \alpha}_t u)(x, \cdot) \right\|_p^p \leq N(\| f(x) \|_{L_p(T)} + \| g(x) \|_{L_p(T, l_2)})
$$

$$
\quad \leq N(\| f \|_{L_p(T)} + \| g \|_{L_p(T, l_2)}).
$$

(2.21)

By considering $(I^{\Lambda - \alpha}_t u)(x) - (I^{\Lambda - \alpha}_t u)(x)$ instead of $(I^{\Lambda - \alpha}_t u)(x)$, we easily see that $(I^{\Lambda - \alpha}_t u)(x)$ is a Cauchy sequence in $L_p(\Omega; C([0, T]; L_p))$. Let $\bar{u}$ be the limit in this space. Then since $(I^{\Lambda - \alpha}_t u)(x)$ converges to $I^{\Lambda - \alpha}_t u$ in $L_p(T)$, we conclude $\bar{u} = I^{\Lambda - \alpha}_t u$, and get (2.10) by considering the limit of (2.21) as $\varepsilon \to 0$ in the space $L_p(\Omega; C([0, T]; L_p))$.

(iii) By (2.1), $I^{\Lambda - \alpha}_t u_n$ converges to $I^{\Lambda - \alpha}_t u$ in $\mathcal{H}_p^{\gamma+2}(T)$ if $u_n$ converges to $u$ in $\mathcal{H}_p^{\gamma+2}(T)$. Moreover, both $\mathcal{H}_p^{\gamma+2}(T)$ and $L_p(\Omega; C([0, T]; H^\gamma_p))$ are Banach spaces. Therefore, $\mathcal{H}_p^{\gamma+2}(T)$ is a Banach space.

(iv) As in the proof of (ii), we only consider the case $\gamma = 0$. By (2.14), for each $x \in \mathbb{R}^d$, (a.s.)

$$
\nu(x, t) = I^{\alpha}_t f(x, t) + I^{\alpha - \beta}_t \int_0^t g(x, s) ds (a.e.) \quad t \in [0, T].
$$

Note

$$
\| I^{\alpha}_t f(x) \|_{L_p(t)}^p \leq NI^{\alpha}_t \| f(x) \|_{L_p(t)}^p \leq NI^{\alpha}_t \| f \|_{L_p(t)}^p \quad \forall t \in [0, T].
$$
By Lemma 2.2 and the stochastic Fubini theorem (note if $\alpha < \beta$ then we define $I_t^{\alpha-\beta} = \frac{\partial}{\partial t} I_t^{\alpha+1-\beta}$), for each $x$ (a.s.)

$$v^\varepsilon(t, x) := I_t^{\alpha-\beta} \int_0^t g(e^{(x)}k(s, x)) \, dw^k_s = c(\alpha, \beta) \int_0^t (t-s)^{\alpha-\beta} g(e^{(x)}k(s,x)) \, dw^k_s$$

for almost all $t \in [0, T]$. Thus by the Burkholder-Davis-Gundy inequality and the Hölder inequality, for any $t \leq T$,

$$\|v^\varepsilon\|_{L^p(t)}^p \leq N \mathbb{E} \int_0^t \int_{\mathbb{R}^d} \left( I_s^{2(\alpha-\beta)+1} \left( |g(e^{(x)})|^2 \right)(s) \right)^{p/2} \, dx \, ds$$

$$\leq NI_t^{2(\alpha-\beta)+1} \left( \|g\|_{L^p(\cdot, t_2)}^p \right)(t).$$

Observe that for $s \leq t \leq T$,

$$(t-s)^{\alpha-1} + (t-s)^{2(\alpha-\beta)} \leq N(t-s)^{\beta-1}$$

where $N$ depends on $\alpha, \beta$ and $T$. Thus, for any $t \leq T$

$$\|u^{(\varepsilon)}\|_{L^p(t)}^p \leq NI_t^{\alpha} \left( \|f\|_{L^p(\cdot)}^p \right)(t) + NI_t^{2(\alpha-\beta)+1} \left( \|g\|_{L^p(\cdot, t_2)}^p \right)(t)$$

$$\leq NI_t^{\beta} \left( \|f\|_{L^p(\cdot)}^p + \|g\|_{L^p(\cdot, t_2)}^p \right)(t).$$

The claim of (iv) follows from Fatou’s lemma.

Assumption 2.11 below will be used for both divergence type and non-divergence type equations. As mentioned before, the argument $\omega$ is omitted for functions depending on $(\omega, t, x)$.

**Assumption 2.11.** (i) The coefficients $a^{ij}, b^i, c, \sigma^{ijk}, \mu^k, \nu^k$ are $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable.

(ii) The leading coefficients $a^{ij}$ are continuous in $x$ and piecewise continuous in $t$ in the following sense: there exist stopping times $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_M = T$ such that

$$a^{ij}(t, x) = \sum_{n=1}^{M_0} a^{ij}_n(t, x) 1_{(\tau_{n-1}, \tau_n]}(t),$$

where each $a^{ij}_n$ are uniformly continuous in $(t, x)$, that is for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|a^{ij}_n(t, x) - a^{ij}_n(s, y)| \leq \varepsilon, \quad \forall \omega \in \Omega$$

whenever $|(t, x) - (s, y)| \leq \delta$.

(iii) There exists a constant $\delta_0 \in (0, 1)$ so that for any $n, \omega, t, x$

$$\delta_0 |\xi|^2 \leq a^{ij}_n(t, x) \xi^i \xi^j \leq \delta_0^{-1} |\xi|^2, \quad \forall \xi \in \mathbb{R}^d,$$

$$|b^i(t, x)| + |c(t, x)| + |\sigma^{ij}(t, x)||_2 + |\mu^i(t, x)||_2 + |\nu(t, x)||_2 \leq \delta_0^{-1}.$$

(iv) $\sigma^{ijk} = 0$ if $\beta \geq 1/2$, and $\mu^k = 0$ if $\beta \geq 1/2 + \alpha/2$ for every $i, j, k, \omega, t, x$.

Recall for $a \in \mathbb{R}$, $a_+ := a \vee 0$. For $\kappa \in (0, 1)$, denote

$$c_0 = c_0(\alpha, \beta) = \frac{(2\beta - 1)\alpha}{\alpha}, \quad c_0' = c_0'(\kappa) = c_0 + \kappa 1_{\beta=1/2}.$$

Note that $c_0' \in [0, 2)$ because $\beta < \alpha + \frac{1}{2}$, and $c_0 = c_0' = 0$ if $\beta < 1/2$. 

□
Remark 2.12. (i) Assumption 2.11(iv) is made on the basis of the model equation
\[
\partial_t^2 u = (\Delta u + \tilde{f})dt + \partial_t^2 \int_0^t g^k dw^k_t, \quad u(0) = 1_{\alpha > 1}u'(0) = 0,
\]
for which the following sharp estimate holds (see Lemma 3.6 and Theorem 4.3): for any \( \gamma \in \mathbb{R} \) and \( \kappa > 0 \),
\[
\|u\|_{\mathcal{H}^{\gamma + 2} (T)} \leq c \left( \|\tilde{f}\|_{\mathcal{H}^{\gamma} (T)} + \|g\|_{\mathcal{H}^{\gamma + \kappa} (T, l_2)} \right).
\]
Thus to have \( \mathcal{H}^{\gamma + 2}_{1} \)-valued solutions we need \( \tilde{f} \in \mathcal{H}^{\gamma} (T) \) and \( g \in \mathcal{H}^{\gamma + \kappa} (T, l_2) \). In particular if \( \beta < 1/2 \) then the solution is twice more differentiable than \( g \). This enables us to have the second derivatives of solutions in the stochastic parts of equations (1.1) and (1.2).

(ii) For the solution of stochastic heat equation \( du = \Delta u dt + g(u)dw_t \) (this is the case when \( \alpha = \beta = 1 \), the solution is once more differentiable than \( g \) (i.e. \( \|\nabla u\|_{L_2} \approx \|g\|_{L_p} \)), and if \( g \) contains any second-order derivatives of \( u \) then one cannot control \( \nabla u \) and any other derivatives of \( u \).

Remark 2.13. Due to (2.25) we need \( \epsilon_0' > c_0 \) if \( \beta = 1/2 \). This is why in Assumption 2.10 below we impose extra smoothness on the coefficients and free terms of the stochastic parts when \( \beta = 1/2 \).

To describe the regularity of the coefficients we introduce the following space introduced e.g. in [21]. Fix \( \delta_1 > 0 \), and for each \( r \geq 0 \), let
\[
B^r := \begin{cases} L_{\infty} (\mathbb{R}^d) & : r = 0 \\
C^{r-1,1} (\mathbb{R}^d) & : r = 1, 2, 3, \cdots \\
C^{r+\delta_1} (\mathbb{R}^d) & : \text{otherwise},
\end{cases}
\]
where \( C^{r+\delta_1} (\mathbb{R}^d) \) and \( C^{r-1,1} (\mathbb{R}^d) \) are the Hölder space and the Zygmund space respectively. We also define the space \( B^r (l_2) \) for \( l_2 \)-valued functions using \( | \cdot |_{l_2} \) in place of \( | \cdot | \).

It is well-known (e.g. [21] Lemma 5.2) that for any \( \gamma \in \mathbb{R} \), \( u \in H^\gamma_1 \) and \( a \in B^{\gamma_1} \),
\[
\|au\|_{H^\gamma_1} \leq N(d, p, \delta_1, \gamma) |a|_{B^{\gamma_1}} \|u\|_{H^\gamma_1},
\]
and similarly for any \( b \in B^{\gamma_1} (l_2) \),
\[
\|bu\|_{H^\gamma_1 (l_2)} \leq N(d, p, \delta_1, \gamma) |b|_{B^{\gamma_1} (l_2)} \|u\|_{H^\gamma_1}.
\]

The following assumption is only for the divergence type equation. We use the notation \( f^i (u), h(u), \) and \( g(u) \) to denote \( f^i (t, x, u), h(t, x, u), \) and \( g(t, x, u) \), respectively. Take \( \epsilon_0' \) from (2.24) and note \( \epsilon_0' - 1 < 1 \).

Assumption 2.14. (i) There exists a \( \kappa \in (0, 1) \) so that for any \( u \in \mathcal{H}^1_1 (T), \)
\[
f^i (u) \in \mathbb{L}_p (T), \quad h (u) \in \mathcal{H}^{-1}_p (T), \quad g (u) \in \mathcal{H}^{\kappa - 1}_p (T, l_2).
\]
(ii) For any \( \epsilon > 0 \) there exists \( K_1 = K_1 (\epsilon) \) so that
\[
\|f^i (t, \cdot, u) - f^i (t, \cdot, v)\|_{L_p} + \|h (t, \cdot, u) - h (t, \cdot, v)\|_{H^{-1}_p (l_2)}
+ \|g (t, \cdot, u) - g (t, \cdot, v)\|_{\mathcal{H}^{\kappa - 1}_p (l_2)} \leq \varepsilon \|u - v\|_{L_p} + K_1 \|u - v\|_{L_p}
\]
for all \( u, v \in \mathcal{H}^1_1 \) and \( \omega, t \).
(iii) There exists a constant $K_2 > 0$ such that
\[ |\sigma^{ij}(t, \cdot)|_{B^1(t_2)} + |\mu^i(t, \cdot)|_{B^{1-\delta}(t_2)} + |\nu(t, \cdot)|_{B^{1-\delta}(t_2)} \leq K_2, \quad \forall i, j, t. \]

Note that \((2.28)\) is certainly satisfied if $f^i(v), h(v)$, and $g(v)$ are Lipschitz continuous with respect to $v$ in their corresponding spaces uniformly on $\omega$ and $t$. Indeed, if $g(v)$ is Lipschitz continuous then using $\epsilon_0' - 1 < 1$ and an interpolation inequality (see e.g. [15 Section 2.4.7]), we get for any $\epsilon > 0$,
\[ \|g(u) - g(v)\|_{H^{\epsilon_0'-1}(t_2)} \leq N\|u - v\|_{H^{\epsilon_0'}(t_2)} \leq \epsilon\|u - v\|_{H^1} + K(\epsilon)\|u - v\|_{L_p}. \]

Finally we give our main result for divergence equation \((1.2)\).

**Theorem 2.15.** Let $p \geq 2$. Suppose that Assumptions \([2.11]\) and \([2.14]\) hold. Then divergence type equation \((1.2)\) with the zero initial condition has a unique solution $u \in H_p^1(T)$ in the sense of Definition \([2.4]\) and for this solution we have
\[ \|u\|_{H_p^1(T)} \leq N \left( \|f^i(0)\|_{L_p(T)} + \|h(0)\|_{H^{\epsilon}_p(T)} + \|g(0)\|_{H^{\epsilon_0'-1}_p(T)} \right), \tag{2.29} \]
where the constant $N$ depends only on $d, p, \alpha, \beta, \kappa, \delta_0, \delta_1, K_1, K_2$, and $T$.

Next we introduce our result for non-divergence equation. To have $H_p^{\gamma+2}$-valued solution we assume the following conditions.

**Assumption 2.16.** (i) There exists a $\kappa \in (0,1)$ so that for any $u \in H_p^{\gamma+2}(T)$,
\[ f(u) \in H_p^{\gamma+2}(T), \quad g(u) \in H_p^{\gamma+\epsilon_0}(T, l_2). \]

(ii) There exists a constant $K_3$ so that for any $\omega$, $i, j, t$
\[ |a^{ij}(t, \cdot)|_{B^1(t_2)} + |b^j(t, \cdot)|_{B^{1-\delta}(t_2)} + |c(t, \cdot)|_{B^{1-\delta}(t_2)} \leq K_3, \tag{2.30} \]
and
\[ |\sigma^{ij}(t, \cdot)|_{B^1(t_2)} + |\mu^i(t, \cdot)|_{B^{1-\delta}(t_2)} + |\nu(t, \cdot)|_{B^{1-\delta}(t_2)} \leq K_3. \]

(iii) For any $\epsilon > 0$, there exists a constant $K_4 = K_4(\epsilon) > 0$ such that
\[ \|f(t, u) - f(t, v)\|_{H_p^\gamma} + \|g(t, u) - g(t, v)\|_{H_p^{\gamma+\epsilon_0}(t_2)} \leq \epsilon\|u - v\|_{H_p^{\gamma+2}} + K_4\|u - v\|_{H_p^\gamma}, \tag{2.31} \]
for any $u, v \in H_p^{\gamma+2}$ and $\omega, t$.

See [21] for some examples of \((2.31)\). Here we introduce only one nontrivial example. Let $\gamma + 2 - d/p > n$ for some $n \in \{0,1,2,\cdots\}$ and $f_0 = f_0(x) \in H_p^\gamma$. Take
\[ f(u) = f_0(x) \sup_x |D_p^n u|. \]

Take a $\delta > 0$ so that $\gamma + 2 - d/p - n > \delta$. Using a Sobolev embedding $H_p^{\gamma+2-\delta} \subset C^{\gamma+2-\delta-d/p} \subset C^\gamma$, we get for any $u, v \in H_p^{\gamma+2}$ and $\epsilon > 0$,
\[ \|f(u) - f(v)\|_{H_p^\gamma} \leq \|f_0\|_{H_p^\gamma} \sup_x |D_p^n(u - v)| \leq N\|u - v\|_{C^n}, \]
\[ \leq N\|u - v\|_{H_p^{\gamma+2-\delta}} \leq \epsilon\|u - v\|_{H_p^{\gamma+2}} + K(\epsilon)\|u - v\|_{H_p^\gamma}. \]

Here is our main result for non-divergence equation \((1.1)\).
Theorem 2.17. Let $\gamma \in \mathbb{R}$ and $p \geq 2$. Suppose that Assumptions 2.7 and 2.17 hold. Then non-divergence type equation \((\ref{eq:1.1})\) with zero initial condition has a unique solution $u \in H^2_p(T)$ in the sense of Definition 2.4 and for this solution
\[
\|u\|_{H^2_p(T)} \leq N \left( \|f(0)\|_{H^2_p(T)} + \|g(0)\|_{H^2_p(T)} \right),
\]
where the constant $N$ depends only on $d$, $p$, $\alpha$, $\beta$, $\kappa$, $\delta_0$, $\delta_1$, $K_3$, $K_4$, and $T$.

3. PARABOLIC LITTLEWOOD-Paley INEQUALITY

In this section we obtain a sharp $L_p$-estimate for solutions to the model equation
\[
\partial_t^\alpha u = \Delta u + \partial_t^\beta \int_0^t g^k dw_s.
\]
For this, we prove the parabolic Littlewood-Paley inequality related to the equation. For the classical case $\alpha = \beta = 1$ we refer to \([19, 20, 23]\).

Consider the fractional diffusion-wave equation
\[
\partial_t^\alpha u(t, x) = \Delta u(t, x), \quad u(0) = u_0, \quad 1_{\alpha > 1} u'(0) = 0.
\]
By taking the Fourier transform and the inverse Fourier transform with respect to $x$, we formally find that $u(t) = p(t) * u_0$ is a solution to this problem if $p(t, x)$ satisfies
\[
\partial_t^\alpha \mathcal{F}(p) = -|\xi|^{2 \gamma} \mathcal{F}(p), \quad \mathcal{F}(p)(0, \xi) = 1, \quad 1_{\alpha > 1} \mathcal{F} \left( \frac{\partial p}{\partial t} \right)(0, \xi) = 0.
\]
It turns out that (see \([13, 18]\) or Lemma 3.1 below) there exists a function $p(t, x)$, called the fundamental solution, such that it satisfies \((\ref{eq:3.3})\). It is also true that $p$ is infinitely differentiable in $(0, \infty) \times \mathbb{R}^d \setminus \{0\}$ and $\lim_{t \to 0} \frac{\partial^\alpha p(t, x)}{\partial t^\alpha} = 0$ if $x \neq 0$. Define
\[
q_{\alpha, \beta}(t, x) := \begin{cases} I_t^{\alpha-\beta} p(t, x) & : \alpha \geq \beta \\ D_t^{\beta-\alpha} p(t, x) & : \alpha < \beta, \end{cases}
\]
and
\[
q(t, x) := q_{\alpha, 1}(t, x).
\]
Note that $q_{\alpha, \beta}$ is well defined due to above mentioned properties of $p$. Moreover $D_t^{\beta-\alpha} p(t, x) = \partial_t^{\beta-\alpha} p(t, x)$ since $p(0, x) = 0$ if $x \neq 0$.

In the following lemma we collect some important properties of $p(t, x)$, $q(t, x)$, and $q_{\alpha, \beta}(t, x)$ taken from \([13, 18]\).

Lemma 3.1. Let $d \in \mathbb{N}$, $\alpha \in (0, 2)$, $\beta < \alpha + \frac{1}{2}$, and $\gamma \in [0, 2)$.

(i) There exists a fundamental solution $p(t, x)$ satisfying above mentioned properties. It also holds that for any $t \neq 0$ and $x \neq 0$,
\[
\partial_t^\alpha p(t, x) = \Delta p(t, x), \quad \frac{\partial p(t, x)}{\partial t} = \Delta q(t, x),
\]
and for each $x \neq 0$, $\frac{\partial p(t, x)}{\partial t} \to 0$ as $t \downarrow 0$. Moreover, $\frac{\partial p(t, \cdot)}{\partial t}$ is integrable in $\mathbb{R}^d$ uniformly on $t \in [\varepsilon, T]$ for any $\varepsilon > 0$.

(ii) If $n \leq 3$, $D_t^\alpha q(t, \cdot)$ is integrable in $\mathbb{R}^d$ uniformly on $t \in [\varepsilon, T]$ for any $\varepsilon > 0$.

(iii) There exist constants $c = c(d, \alpha)$ and $N = N(d, \alpha)$ such that if $|x|^2 \geq t^\alpha$,
\[
|p(t, x)| \leq N|x|^{-d} \exp \left\{ -c|x|^\frac{\alpha}{2} t^{-\frac{\alpha}{2}} \right\}.
\]

(iv) It holds that
\[ F\{D^\alpha_\sigma q_{\alpha,\beta}(t, \cdot)\}(\xi) = t^{\alpha-\sigma} E_{\alpha,1+\alpha-\sigma}(-|\xi|^2 t^\alpha), \quad (3.7) \]
where \( E_{a,b}(z) \), \( a > 0 \), is the Mittag-Leffler function defined as
\[ E_{a,b}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak+b)}, \quad z \in \mathbb{C}. \]

(v) There exists a constant \( N = N(d, \gamma, \alpha, \beta) \) such that
\[ \left| D^\gamma_\sigma q_{\alpha,\beta}(1, x) \right| + \left| D^\gamma_\sigma q_{\alpha,\beta}(1, x) \right| \leq N \left( |x|^{-d+2-\gamma} \wedge |x|^{-d-\gamma} \right) \]
if \( d \geq 2 \), and
\[ \left| D^\gamma_\sigma q_{\alpha,\beta}(1, x) \right| + \left| D^\gamma_\sigma q_{\alpha,\beta}(1, x) \right| \leq N \left( \{ |x|^{-1-\gamma} (1 + \ln |x|)_{1=1} \} \wedge |x|^{-1-\gamma} \right) \]
if \( d = 1 \). Furthermore, for each \( n \in \mathbb{N} \)
\[ \left| D^\gamma_\sigma D^\alpha_\sigma q_{\alpha,\beta}(1, x) \right| + \left| D^\gamma_\sigma D^\alpha_\sigma q_{\alpha,\beta}(1, x) \right| \leq N(d, \gamma, \alpha, \beta, n) \left( |x|^{-d+2-\gamma-n} \wedge |x|^{-d-\gamma-n} \right). \quad (3.8) \]

(vi) The scaling properties hold:
\[ q_{\alpha,\beta}(t, x) = t^{-\frac{d-\gamma}{2} + \alpha-\beta} q_{\alpha,\beta}(1, xt^{-\frac{d}{2}}), \quad (3.9) \]
\[ D^\gamma_\sigma q_{\alpha,\beta}(t, x) = t^{-\sigma-\frac{\alpha(d+1)}{2} + \alpha-\beta} D^\gamma_\sigma q_{\alpha,\beta}(1, xt^{-\frac{d}{2}}). \quad (3.10) \]

Proof. (i), (ii), (iii), and (v) are easily obtained from Theorem 2.1 and Theorem 2.3 of [18]. The proof of (iv) can be found in Section 6 of [18]. For the scaling property (vi), see [18, (5.2)].

\[ \square \]

The following result is well-known, for instance if \( \alpha \in (0, 1] \). For the completeness of the article, we give a proof.

**Corollary 3.2.** Let \( f \in C^2_0(\mathbb{R}^d) \). Then
\[ \int_{\mathbb{R}^d} p(t, x-y)f(y)dy \]
converges to \( f(x) \) uniformly as \( t \downarrow 0 \).

Proof. By (3.7), for any \( t > 0 \),
\[ \int_{\mathbb{R}^d} p(t, y)dy = \mathcal{F}p(0) = E_{\alpha,1}(0) = 1. \]
Also \((3.9)\) shows that \(\|p(t, \cdot)\|_{L_1(\mathbb{R}^d)}\) is a constant function of \(t\). For any \(\delta > 0\),
\[
\begin{align*}
&\left| \int_{\mathbb{R}^d} p(t, x - y)f(y)dy - f(x) \right| \\
&= \left| \int_{\mathbb{R}^d} p(t, y)(f(x - y) - f(x))dy \right| \\
&\leq \int_{|y| < \delta} |p(t, y)(f(x - y) - f(x))| dy + \int_{|y| > \delta} |p(t, y)(f(x - y) - f(x))| dy \\
&=: \mathcal{I}(\delta) + \mathcal{J}(\delta).
\end{align*}
\]
Since \(f \in C^2_0(\mathbb{R}^d)\), for any \(\varepsilon > 0\), one can take a small \(\delta\) so that \(\mathcal{I}(\delta) < \varepsilon\). Moreover due to \((3.6)\), for fixed \(\delta > 0\), \(\mathcal{J}(\delta) \to 0\) as \(t \downarrow 0\). The corollary is proved. \(\square\)

In the remainder of this section, we restrict the range of \(\beta\) so that
\[
\frac{1}{2} < \beta < \alpha + \frac{1}{2}. \tag{3.11}
\]
Thus by definition \((2.24)\), we have
\[
c_1 := 2 - c_0' = 2 - \frac{2\beta - 1}{\alpha} \in (0, 2).
\]

In the following section (i.e. Section 4) we prove that if \(g \in \mathbb{H}^\infty(\mathcal{T}, l_2)\) then the unique solution (in the sense of Definition \((2.4)\)) to equation \((3.1)\) with the zero initial condition is given by the formula
\[
u = \int_0^t \int q_{\alpha, \beta}(t - s, x - y)g^k(s, y)dydw^k_s. \tag{3.12}
\]
By Burkerholder-Davis-Gundy inequality
\[
\|(\Delta)^{c_1/2}u\|_{L_p(T)}^p \leq N\mathbb{E} \int_{\mathbb{R}^d} \int_0^T \left[ \int_0^t \left( \int_{\mathbb{R}^d} (\Delta)^{c_1/2}q_{\alpha, \beta}(t - s, x - y)g(s, y)dy \right)_2^2 ds \right]^{p/2} dt dx.
\tag{3.13}
\]
Our goal is to control the right hand side of \((3.13)\) in terms of \(\|g\|_{\mathbb{H}^\infty(\mathcal{T}, l_2)}\). For this, we introduce some definitions as follows. Let \(H\) be a Hilbert space. For \(g \in C_\infty(\mathbb{R}^{d+1}; H)\), define
\[
T_{t-s}^{\alpha, \beta}g(s, \cdot)(x) := \int_{\mathbb{R}^d} q_{\alpha, \beta}(t - s, x - y)g(s, y)dy.
\]
Note that, due to Lemma \((3.1)\), \((\Delta)^{c_1/2}q_{\alpha, \beta}(t, \cdot) \in L_1(\mathbb{R}^d)\) for all \(t > 0\). Therefore, for any \(t > s\)
\[
(\Delta)^{c_1/2}T_{t-s}^{\alpha, \beta}g(s, \cdot) \in L_1(\mathbb{R}^d)
\]
and
\[
(\Delta)^{c_1/2}T_{t-s}^{\alpha, \beta}g(s, \cdot)(x) = \int_{\mathbb{R}^d} (\Delta)^{c_1/2}q_{\alpha, \beta}(t - s, x - y)g(s, y)dy.
\]
We also define the sublinear operator \(\mathcal{T}\) as
\[
\mathcal{T}g(t, x) := \left[ \int_{-\infty}^t \left( \int_{\mathbb{R}^d} (\Delta)^{c_1/2}T_{t-s}^{\alpha, \beta}g(s, \cdot)(x)_2^2 _H ds \right)^{1/2} \right]^{1/2},
\]
where $| \cdot |_H$ denotes the given norm in the Hilbert space $H$. $T$ is sublinear due to the Minkowski inequality

$$
\|f + g\|_{L^2((-\infty,t);H)} \leq \|f\|_{L^2((-\infty,t);H)} + \|g\|_{L^2((-\infty,t);H)}.
$$

(3.14)

Now we introduce a parabolic version of Littlewood-Paley inequality. The proof is given at the end of this section.

**Theorem 3.3.** Let $H$ be a separable Hilbert space, $p \in [2, \infty)$, $T \in (-\infty, \infty]$, and $\alpha \in (0, 2)$. Assume that (3.11) holds. Then for any $g \in C^\infty_c(\mathbb{R}^{d+1};H)$,

$$
\int_{\mathbb{R}^d} \int_{-\infty}^{T} |Tg(t,x)|^p dt dx \leq N \int_{\mathbb{R}^d} \int_{-\infty}^{T} |g(t,x)|^p dt dx,
$$

(3.15)

where $N = N(d, p, \alpha, \beta)$.

**Remark 3.4.** By Theorem 3.3, the operator $T$ can be continuously extended onto $L^p(\mathbb{R}^{d+1};H)$. We denote this extension by the same notation $T$.

**Remark 3.5.** Take $u$ and $g$ from (3.12). Extend $g(t) = 0$ for $t \leq 0$. Note that the right hand side of (3.13) is $\mathbb{E} \int_{\mathbb{R}^d} \int_{-\infty}^{T} |Tg(t,x)|^p dt dx$. Thus, using (3.15) (actually Remark 3.4) for each $\omega$ and taking the expectation, we get

$$
\|(\Delta)^{\alpha/2} u\|_{L^p(T)}^p \leq N \|g\|_{L^p(T)}^p.
$$

First we prove Theorem 3.3 for $p = 2$. The following lemma is a slight extension of [6, Lemma 3.8], which is proved only for $\alpha \in (0, 1)$ with constant $N$ depending also on $T$. For the proof we use the following well-known property of the Mittag-Leffler function: if $\alpha \in (0, 2)$ and $b \in \mathbb{C}$, then there exist positive constants $\varepsilon = \varepsilon(\alpha)$ and $C = C(\alpha, b)$ such that

$$
|E_{\alpha,b}(z)| \leq C(1 \wedge |z|^{-1}), \quad \pi - \varepsilon \leq |\text{arg}(z)| \leq \pi.
$$

(3.16)

See [38, Lemma 3.1] for the proof of (3.16).

**Lemma 3.6.** Suppose that the assumptions in Theorem 3.3 hold. Then for any $T \in (-\infty, \infty]$ and $g \in C^\infty_c(\mathbb{R}^{d+1};H)$,

$$
\int_{\mathbb{R}^d} \int_{-\infty}^{T} |Tg(t,x)|^2 dt dx \leq N \int_{\mathbb{R}^d} \int_{-\infty}^{T} |g(t,x)|^2 dt dx,
$$

(3.17)

where $N = N(d, p, \alpha, \beta)$ is independent of $T$.

**Proof.** Step 1. First, assume $g(t,x) = 0$ for $t \leq 0$. In this case we may assume $T > 0$ because the left hand side of (3.17) is zero if $T \leq 0$. 


We prove (3.15) for $T = 1$. Since $g(t, x) = T g(t, x) = 0$ for $t \leq 0$, by Parseval’s identity and (3.7),
\[
\int_{\mathbb{R}^d} \int_{-\infty}^{1} |T g(t, x)|^2 dt dx
\]
\[
= \int_{0}^{1} \int_{0}^{t} \int_{\mathbb{R}^d} |\xi|^{2c_1} |\mathcal{F} \{q_{\alpha, \beta}(t - s, \cdot)\} (\xi)|^2 |\mathcal{F} \{g\} (s, \xi)|_H^2 d\xi ds dt
\]
\[
\leq \int_{|\xi| \leq 1} \int_{0}^{1} |\mathcal{F} \{g\} (s, \xi)|_H^2 \left( \int_{s}^{1} |\xi|^{2c_1} |t^{\alpha - \beta} E_{\alpha, 1 - \beta + \alpha} (\cdot, \xi^2) t^\alpha|^2 dt \right) ds d\xi
\]
\[
+ \int_{|\xi| \geq 1} \int_{0}^{1} |\mathcal{F} \{g\} (s, \xi)|_H^2 \left( \int_{s}^{1} |\xi|^{2c_1} |t^{\alpha - \beta} E_{\alpha, 1 - \beta + \alpha} (\cdot, \xi^2) t^\alpha|^2 dt \right) ds d\xi
\]
\[
\leq N \int_{0}^{1} \int_{\mathbb{R}^d} |g(t, x)|_H^2 dt dx
\]
\[
+ N \int_{|\xi| \geq 1} \int_{0}^{1} |\mathcal{F} \{g\} (s, \xi)|_H^2 \left( \int_{s}^{1} |\xi|^{2c_1} |t^{\alpha - \beta} E_{\alpha, 1 - \beta + \alpha} (\cdot, \xi^2) t^\alpha|^2 dt \right) ds d\xi,
\]
where the last inequality is due to (3.10) and the condition $\alpha - \beta > -1/2$. Thus to prove our assertion for $T = 1$ we only need to prove
\[
\sup_{\xi} \left( 1_{|\xi| \geq 1} |\xi|^{2c_1} \int_{0}^{1} |t^{\alpha - \beta} E_{\alpha, 1 - \beta + \alpha} (\cdot, \xi^2) t^\alpha|^2 dt \right) < \infty.
\]
By (3.16), if $|\xi| \geq 1$ (recall we assumed $\beta > 1/2$ in this section),
\[
|\xi|^{2c_1} \int_{0}^{1} |t^{\alpha - \beta} E_{\alpha, 1 - \beta + \alpha} (\cdot, \xi^2) t^\alpha|^2 dt
\]
\[
\leq N |\xi|^{2c_1} \int_{|\xi|^{-2/\alpha}}^{1} t^{2(\alpha - \beta)} dt + N |\xi|^{2c_1} \int_{|\xi|^{-2/\alpha}}^{1} \left| \frac{t^{\alpha - \beta}}{|\xi|^2 t^\alpha} \right|^2 dt
\]
\[
\leq N |\xi|^{2(c_1 - 2 + 2\beta - 1)} + N |\xi|^{2c_1 - 4} \left( |\xi|^{2(\frac{2\beta - 1}{2})} - 1 \right) \leq 3N.
\]
Therefore, the case $T = 1$ is proved.

For arbitrary $T > 0$, we use (3.10), which implies
\[
(-\Delta)^{c_1/2} q_{\alpha, \beta}(T(t - s), x) = T^{-\frac{\alpha(d + 1)}{2} + \alpha - \beta} (-\Delta)^{c_1/2} q_{\alpha, \beta}(t - s, T^{-\frac{d}{2}} x), \tag{3.18}
\]
and consequently
\[
T g(Tt, x) = T g(Tt, T^{-\frac{d}{2}} x), \tag{3.19}
\]
where $g(Tt, T^{-\frac{d}{2}} x)$. By using the result proved for $T = 1$,
\[
\int_{\mathbb{R}^d} \int_{-\infty}^{T} |T g(t, x)|^2 dt dx = T^{1 + \frac{d}{2}} \int_{\mathbb{R}^d} \int_{-\infty}^{1} |T g(t, x)|^2 dt dx
\]
\[
\leq NT^{1 + \frac{d}{2}} \int_{\mathbb{R}^d} \int_{-\infty}^{1} |g(t, x)|^2 dt dx
\]
\[
= N \int_{\mathbb{R}^d} \int_{-\infty}^{T} |g(t, x)|^2 dt dx.
\]
Thus (3.15) holds for all $T > 0$ with a constant independent of $T$. It follows that (3.15) also holds for $T = \infty$. 


Step 2. General case. Take \( a \in \mathbb{R} \) so that \( g(t, x) = 0 \) for \( t \leq a \). Then obviously, for \( \tilde{g}(t, x) := g(t + a, x) \) we have \( \tilde{g}(t) = 0 \) for \( t \leq 0 \). Thus it is enough to apply the result for Step 1 with \( \tilde{g} \) and \( T - a \) in place of \( g \) and \( T \) respectively.

For a real-valued measurable function \( h \) on \( \mathbb{R}^d \), define the maximal function

\[
\mathbb{M}_x h(x) := \sup_{r > 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |h(y)|dy = \sup_{r > 0} \int_{B_r(x)} |h(y)|dy.
\]

The Hardy-Littlewood maximal theorem says

\[
\| \mathbb{M}_x h \|_{L^p(\mathbb{R}^d)} \leq N(d, p)\| h \|_{L^p(\mathbb{R}^d)}, \quad \forall p > 1.
\]

(3.20)

For a function \( h(t, x) \), set

\[
\mathbb{M}_x h(t, x) = \mathbb{M}_x (h(t, \cdot))(x), \quad \mathbb{M}_t h(t, x) = \mathbb{M}_t (h(\cdot, x))(t),
\]

and

\[
\mathbb{M}_x \mathbb{M}_t h(t, x) = \mathbb{M}_t (\mathbb{M}_x h(\cdot, x))(t).
\]

To evaluate \( \mathbb{M}_x \mathbb{M}_t h(t, x) \), we first fix \( t \) and estimate \( (\mathbb{M}_x h(t, \cdot))(x) \). After this, we fix \( x \) and regard \( (\mathbb{M}_x h(t, \cdot))(x) \) as a function of \( t \) only to estimate the maximal function with respect to \( t \).

Denote

\[
Q_0 := [-2^{\frac{2}{d}}, 0] \times [-1, 1]^d.
\]

(3.21)

Lemma 3.7. Let \( g \in C_c(\mathbb{R}^{d+1}; H) \) and assume that \( g = 0 \) outside of \([-4^{\frac{2}{d}}, 4^{\frac{2}{d}}] \times B_{3d} \). Then for \((t, x) \in Q_0 \),

\[
\int_{Q_0} |\mathcal{T}g(s, y)|^2 dsdy \leq N\mathbb{M}_x |g|^2_H(t, x),
\]

where \( N = N(d, \alpha, \beta) \).

Proof. By Lemma 3.6

\[
\int_{Q_0} |\mathcal{T}g(s, y)|^2 dsdy \leq \int_{-4^{\frac{2}{d}}}^{0} \int_{B_{3d}} |g(s, y)|^2_H dyds.
\]

For any \((t, x) \in Q_0 \) and \( y \in B_{3d} \), since \(|x - y| \leq |x| + |y| \leq \sqrt{d} + 3d \leq 4d \), we obtain

\[
\int_{-4^{\frac{2}{d}}}^{0} \int_{B_{3d}} |g(s, y)|^2_H dyds \leq \int_{-4^{\frac{2}{d}}}^{0} \int_{|x - y| \leq 4d} |g(s, y)|^2_H dyds \leq N \int_{-4^{\frac{2}{d}}}^{0} \mathbb{M}_x |g(s, x)|^2_H ds \leq N\mathbb{M}_x |g|^2_H(t, x).
\]

The lemma is proved.

\[
\square
\]

Here is a generalization of Lemma 3.7

Lemma 3.8. Let \( g \in C_c(\mathbb{R}^{d+1}; H) \) and assume that \( g(t, x) = 0 \) for \(|t| \geq 4^{\frac{2}{d}} \). Then for any \((t, x) \in Q_0 \),

\[
\int_{Q_0} |\mathcal{T}g(s, y)|^2 dsdy \leq N(d, \alpha, \beta)\mathbb{M}_x |g|^2_H(t, x).
\]
Proof. Take \( \zeta \in C_\infty(\mathbb{R}^d) \) such that \( \zeta = 1 \) in \( B_{2d} \) and \( \zeta = 0 \) outside \( B_{3d} \). Recall that \( T \) is a sublinear operator, and therefore

\[
T g \leq T(\zeta g) + T((1 - \zeta)g).
\]

Since \( T(\zeta g) \) can be estimated by Lemma 3.7, we may assume that \( g(t, x) = 0 \) for \( x \in B_{2d} \). Let \( 0 > s > r > -\frac{d}{2} \). Then by (3.10),

\[
\left| (-\Delta)^{c_2/2} T^{\alpha, \beta}_{s-r} g(r, \cdot)(y) \right|_{H} \leq (s - r)^{-d \alpha - \frac{d}{2} + \frac{\alpha - 2}{4}} \int_{\mathbb{R}^d} \left| (-\Delta)^{c_2/2} q_{\alpha, \beta}(1, (s - r)^{-\frac{2}{d}} y) \right| |g(r, y - z)|_{H} \, dz
\]

\[
= (s - r)^{-d \alpha - \frac{d}{2} + \frac{\alpha - 2}{4}} \int_{\mathbb{R}^d} \left| (-\Delta)^{c_2/2} q_{\alpha, \beta}(1, (s - r)^{-\frac{2}{d}} y) \right| |g(r, y - z)|_{H} \, dz. \quad (3.22)
\]

To proceed further we use the following integration by parts formula: if \( F \) and \( G \) are smooth enough then for any \( 0 < \varepsilon < R < \infty \),

\[
\int_{|\eta| \leq R} F(\eta) G(|\eta|) d\eta = -\int_{\varepsilon}^{R} G'(\rho) \left( \int_{|\eta| \leq \rho} F(\eta) d\eta \right) d\rho + G(R) \int_{|\eta| \leq R} F(\eta) d\eta - G(\varepsilon) \int_{|\eta| \leq \varepsilon} F(\eta) d\eta. \quad (3.23)
\]

Indeed, (3.23) is obtained by applying integration by parts to

\[
\int_{\varepsilon}^{R} G(\rho) \frac{d}{d\rho} \left( \int_{B_{\rho}(0)} F(z) \, dz \right) d\rho = \int_{\varepsilon}^{R} G(\rho) \left( \int_{\partial B_{\rho}(0)} F(s) \, dS_{\rho} \right) d\rho
\]

\[
= \int_{R \geq |z| \geq \varepsilon} F(z) G(|z|) \, dz.
\]

Observe that if \( (s, y) \in Q_0 \) and \( \rho > 1 \), then

\[
|x - y| \leq 2d, \quad B_{\rho}(y) \subset B_{2d+\rho}(x) \subset B_{(2d+1)\rho}(x), \quad (3.24)
\]

whereas if \( \rho \leq 1 \) then for \( z \in B_{\rho}(0) \), \( |y - z| \leq \sqrt{d} + 1 \leq 2d \) and thus \( g(r, y - z) = 0 \). Therefore by (3.23) and (3.23),

\[
(s - r)^{-\frac{\alpha - 2}{4} - \frac{d}{2}} \int_{\mathbb{R}^d} \left| (-\Delta)^{c_2/2} q_{\alpha, \beta}(1, (s - r)^{-\frac{2}{d}} y) \right| |g(r, y - z)|_{H} \, dz
\]

\[
\leq N(s - r)^{-\frac{\alpha - 2}{4} - \frac{d}{2} - \frac{\alpha - 2}{4}} \int_{1}^{\infty} (s - r)^{-\frac{2}{d} - 1 - c_1} \left[ \int_{|z| \leq \rho} |g(r, y - z)|_{H} \, dz \right] d\rho
\]

\[
\leq N(s - r)^{\alpha - \beta} \int_{1}^{\infty} \rho^{-d - 1 - c_1} \left[ \int_{|z| \leq \rho} |g(r, y - z)|_{H} \, dz \right] d\rho
\]

\[
\leq N(s - r)^{\alpha - \beta} \int_{1}^{\infty} \rho^{-1 - c_1} \left[ \int_{B_{\rho}(x)} |g(r, z)|_{H} \, dz \right] d\rho
\]

\[
\leq N(s - r)^{\alpha - \beta} M_{x} |g|_{H} (r, x).
\]
Then due to the fact that $\{M_x|g|_H\}^2 \leq M_x|g|_H^2$,

$$
\int_{Q_0} |Tg(s,y)|^2\,dsdy = \int_{Q_0} \int_{-\infty}^{t} \left|\left(-\Delta\right)^{c_1/2}T_{s-r}^{\alpha,\beta}g(r,\cdot)(y)\right|^2_H \,drdsdy
$$

$$
\leq N \int_{Q_0} \int_{-\frac{3}{2}}^{t} \left[ M_x|g|_H^2 (r,x)(s-r)^{2(\alpha-\beta)} \right] \,drds
$$

$$
\leq N \int_{-\frac{3}{2}}^{t} \left( \int_{r}^{t} (s-r)^{2(\alpha-\beta)} \,ds \right) M_x|g|_H^2 (r,x)dr \leq N M_x^2 |g|_H^2 (t,x).
$$

The lemma is proved. \hfill \Box

**Lemma 3.9.** Let $g \in C_0^\infty(\mathbb{R}^{d+1};H)$ and assume $g(t,x) = 0$ outside of $(-\infty, -3\frac{3}{2}) \times B_3$. Then for any $(t,x) \in Q_0$,

$$
\int_{Q_0} |Tg(s,y)|^2\,dsdy \leq N M_x^2 |g|_H^2 (t,x),
$$

where $N = N(d,\alpha,\beta)$.

**Proof.** Note that $g(s,\cdot) = 0$ for $s \geq -3\frac{3}{2}$. Recalling (3.10), we have

$$
|Tg(s,y)|^2 \leq \int_{-\infty}^{t} \left|\left(-\Delta\right)^{c_1/2}T_{s-r}^{\alpha,\beta}g(r,\cdot)(y)\right|^2_H \,dr
$$

$$
= \int_{-\infty}^{-3\frac{3}{2}} \left| (s-r)^{-\frac{\alpha}{2} - \frac{3}{2}} \int_{\mathbb{R}^d} \left(-\Delta\right)^{c_1/2}q_{\alpha,\beta} (1, (s-r)^{-\frac{3}{2}}z) \,dz \right|^2_H \,dr
$$

$$
\leq \int_{-\infty}^{-3\frac{3}{2}} \left| (s-r)^{-\alpha d - 1} \int_{\mathbb{R}^d} \left(-\Delta\right)^{c_1/2}q_{\alpha,\beta} (1, (s-r)^{-\frac{3}{2}}z) \,dz \right|^2_H \,dr.
$$

If $|z| \geq 4d$, then $g(r,y-z) = 0$ since $y \in Q_0$ and $|y-z| \geq |z| - |y| \geq 3d$. Therefore, by Minkowski’s inequality and Lemma 3.4

$$
\int_{[-1,1]^d} \int_{\mathbb{R}^d} \left|\left(-\Delta\right)^{c_1/2}q_{\alpha,\beta} (1, (s-r)^{-\frac{3}{2}}z)\right| |g(r,y-z)|_H \,dz \,dy
$$

$$
\leq \int_{[-1,1]^d} \int_{|z| \leq 4d} \left|\left(-\Delta\right)^{c_1/2}q_{\alpha,\beta} (1, (s-r)^{-\frac{3}{2}}z)\right| |g(r,y-z)|_H \,dz \,dy
$$

$$
\leq \left( \int_{|z| \leq 4d} \left( \int_{[-1,1]^d} |g(r,y-z)|_H^2 \,dy \right)^{1/2} \left|\left(-\Delta\right)^{c_1/2}q_{\alpha,\beta} (1, (s-r)^{-\frac{3}{2}}z)\right| \,dz \right)^2
$$

$$
\leq \left( \int_{|z| \leq 4d} \left( \int_{B_{3d}(0)} |g(r,y)|_H^2 \,dy \right)^{1/2} \left|\left(-\Delta\right)^{c_1/2}q_{\alpha,\beta} (1, (s-r)^{-\frac{3}{2}}z)\right| \,dz \right)^2
$$

$$
\leq N M_x^2 |g|_H^2 (r,x) \left( \int_{|z| \leq 4d} \left|\left(-\Delta\right)^{c_1/2}q_{\alpha,\beta} (1, (s-r)^{-\frac{3}{2}}z)\right| \,dz \right)^2
$$

$$
\leq N (s-r)^{\alpha(d+\varepsilon-2)} M_x^2 |g|_H^2 (r,x),
$$
where \( \hat{c} \in (1, 2) \) if \( c_1 = 1 \) and \( d = 1 \), and otherwise \( \hat{c} = c_1 \). Since \( |s - r| \sim |r| \) for \( r < -3^{2\alpha} \) and \(-2^{\frac{1}{2}} < s < 0\), we have

\[
\int_{Q_0} |Tg(s, y)|^2 \, dsdy = \int_{-2^{\frac{1}{2}}}^{0} \int_{-1,1} |Tg(s, y)|^2 \, dsdy \\
\leq N \int_{-2^{\frac{1}{2}}}^{0} \int_{-\infty}^{-3^{\frac{1}{2}}} (s - r)^{\alpha(\hat{c} - 2) - 1} \mathcal{M}_x |g|_H^2 (r, x) \, drds \\
\leq N \int_{-\infty}^{-3^{\frac{1}{2}}} \mathcal{M}_x |g|_H^2 (r, x) \frac{dr}{|r|^\alpha(2-\hat{c})+1} \\
\leq N \int_{-\infty}^{-3^{\frac{1}{2}}} \left( \int_{-r}^{0} \mathcal{M}_x |g|^2_H (s, x) \, ds \right) \frac{dr}{|r|^\alpha(2-\hat{c})+2} \\
\leq N \mathcal{M}_t \mathcal{M}_x |g|^2_H (t, x) \int_{-\infty}^{-3^{\frac{1}{2}}} \frac{dr}{|r|^\alpha(2-\hat{c})+1} \leq N \mathcal{M}_t \mathcal{M}_x |g|^2_H (t, x).
\]

The lemma is proved. \( \square \)

**Lemma 3.10.** Let \( g \in C^\infty_c (\mathbb{R}^{d+1}; H) \) and assume that \( g(t, x) = 0 \) outside of \((-\infty, -3^{\frac{1}{2}}) \times B_{2d}^c\). Then for any \((t, x) \in Q_0\),

\[
\int_{Q_0} \int_{Q_0} |Tg(s, y) - Tg(r, z)|^2 \, dsdydrdz \leq N \mathcal{M}_t \mathcal{M}_x |g|^2_H (t, x),
\]

where \( N = N(d, \alpha, \beta) \).

**Proof.** Due to Poincaré’s inequality, it is enough to show

\[
\int_{Q_0} \left( \frac{\partial}{\partial s} Tg \right|^2 + |D_y Tg|^2 \right) \, dsdy \leq N \mathcal{M}_t \mathcal{M}_x |g|^2_H (t, x). \tag{3.25}
\]

Because of the similarity, we only prove

\[
\int_{Q_0} |D_y Tg|^2 \, dsdy \leq N \mathcal{M}_t \mathcal{M}_x |g|^2_H (t, x). \tag{3.26}
\]

Note that since \( g(s, \cdot) = 0 \) for \( s \geq -3^{\frac{1}{2}}\),

\[
D_x Tg(t, x) = D_x \left[ \int_{-\infty}^{-3^{\frac{1}{2}}} (-\Delta)^{c_1/2} T^{\alpha, \beta}_{t-s} g(s, \cdot) \right]_{H}^2 \, ds \right]^{1/2} \\
\leq \left[ \int_{-\infty}^{-3^{\frac{1}{2}}} \left| D_x (-\Delta)^{c_1/2} T^{\alpha, \beta}_{t-s} g(s, \cdot) \right|_H^2 \, ds \right]^{1/2},
\]
where the above inequality is from Minkowski’s inequality. Recall (3.10). Thus for any \((s, y) \in Q_0\),

\[
\|D_y T g(s, y)\|^2_H \\
\leq \int_{-\infty}^{-\frac{3}{2}} \left| D_y (-\Delta)^{c_1/2} T^{\alpha, \beta}_{s-r} g(r, \cdot)(y) \right|^2_H \, dr \\
= \int_{-\infty}^{-\frac{3}{2}} (s-r)^{-\frac{d+1}{2} - \frac{1}{2} \alpha} \int_{\mathbb{R}^d} D_x (-\Delta)^{c_1/2} q_{\alpha, \beta} \left( 1, (s-r)^{-\frac{d}{2}} z \right) g(r, y-z) dz \right|^2_H \, dr \\
\leq \int_{-\infty}^{-\frac{3}{2}} (s-r)^{-\alpha(d+1)-1} \\
\times \left[ \int_{|z| \geq d} \left| D_x (-\Delta)^{c_1/2} q_{\alpha, \beta} \left( 1, (s-r)^{-\frac{d}{2}} z \right) \right| |g(r, y-z)|_H \, dz \right]^2 \, dr ds dy.
\]

Let \((t, x) \in Q_0\). By using (3.23) and Lemma 3.1(v),

\[
\int_{|z| \geq d} \left| D_x (-\Delta)^{c_1/2} q_{\alpha, \beta} \left( 1, (s-r)^{-\frac{d}{2}} z \right) \right| |g(r, y-z)|_H \, dz \\
\leq N (s-r)^{-\frac{d}{2}} \int_{d}^{\infty} \left((s-r)^{-\frac{d}{2}} \rho \right)^{-\frac{d}{2} - c_1 - \frac{1}{2}} \left( \int_{B_{\rho}(y)} |g(r, z)|_H \, dz \right) d \rho \\
\leq N (s-r)^{\frac{d}{2} (d+c_1+\varepsilon-1) M_x} |g(r, x)|_H,
\]

where \(\varepsilon \in [0, 2]\) is taken so that \(c_1 + \varepsilon \in (1, 2)\). Therefore,

\[
\int_{Q_0} \|D_y T g(s, y)\|^2_H \, ds dy \leq N \int_{-\infty}^{0} \left[ \int_{-\infty}^{-\frac{3}{2}} (s-r)^{-\alpha(c_1+\varepsilon-2)-1} M_x |g(r, x)|^2_H \, dr \right] ds \\
\leq N \int_{-\infty}^{-\frac{3}{2}} \left( \int_{r}^{0} M_x |g(r, x)|^2_H \, ds \right) \frac{dr}{|r|^{\alpha(c_1+\varepsilon-2)-2}} \\
\leq N M_x M_x |g(t, x)|^2_H \int_{-\frac{3}{2}}^{\infty} \frac{dr}{r^{\alpha(c_1+\varepsilon-2)-1}} \\
\leq M_x M_x |g(t, x)|^2_H.
\]

Thus (3.26) and the lemma are proved. \(\square\)

For a measurable function \(h(t, x)\) on \(\mathbb{R}^{d+1}\), we define the sharp function

\[
h^\#(t, x) = \sup_{Q} \int_{Q} |h(r, z) - h_Q| \, dr dz,
\]
where
\[ h_Q = \int_Q h(s, y) \, dy \]
and the supremum is taken over all \( Q \subset \mathbb{R}^{d+1} \) containing \((t, x)\) of the form
\[ Q = Q_R(s, y), \quad R > 0 \]
\[ = (s - R^{\frac{d}{2}}/2, s + R^{\frac{d}{2}}/2) \times (y^1 - R/2, y^1 + R/2) \times \cdots \times (y^d - R/2, y^d + R/2). \]
By Fefferman-Stein theorem,
\[ \|h^R\|_{L_p(\mathbb{R}^{d+1})} \leq N\|h\|_{L_p(\mathbb{R}^{d+1})}, \quad p > 1. \]  
(3.27)
Also note that for any \( h \in L^2(\mathbb{R}^d) \),
\[ \int_Q |h(r, z) - h_Q|^2 \, dr dz = \int_Q \left( \int_Q (h(r, z) - h(s, y)) \, dy \right)^2 \, dr dz \leq 4 \int_Q |h(r, z) - c|^2 \, dr dz. \]  
(3.28)
Proof of Theorem 3.3: If \( p = 2 \), (3.15) follows from Lemma 3.6. Hence we assume \( p > 2 \).
First we prove for each \( Q = Q_R(s, y) \) and \((t, x) \in Q\),
\[ \int_Q |Tg - (Tg)_Q|^2 \, ds \, dy \leq N M_x |g|_{H_t}^2 (t, x). \]  
(3.29)
Note that for any \( h_0 \in \mathbb{R} \) and \( h \in \mathbb{R}^d \),
\[ Tg(t - h_0, x - h) = \left[ \int_{-\infty}^{t-h_0} \left| (-\Delta)^{c_1/2} T^{\alpha, \beta}_{t-h_0-s} g(s, \cdot)(x - h) \right|^2 \, ds \right]^{1/2} \]
\[ = \left[ \int_{-\infty}^{t-h_0} \left| (-\Delta)^{c_1/2} \int_{\mathbb{R}^d} q_{\alpha, \beta}(t - h_0 - s, x - h - y) g(s, y) \, dy \right|^2 \, ds \right]^{1/2} \]
\[ = \left[ \int_{-\infty}^{t} \left| (-\Delta)^{c_1/2} \int_{\mathbb{R}^d} q_{\alpha, \beta}(t - s, x - y) \bar{g}(s, y) \, dy \right|^2 \, ds \right]^{1/2} \]
\[ = Tg(t, x) \]
where \( \bar{g}(s, y) := g(s - h_0, y - h) \). This shows that to prove (3.29) we may assume \((s + R^{\frac{d}{2}}, y) = (0, 0)\).
Also, due to (3.18) (or (3.19)),
\[ Tg(c^{\frac{d}{2}} \cdot, c \cdot)(t, x) = Tg(c^{\frac{d}{2}} t, cx). \]
Since dilations do not affect averages, it suffices to prove (3.29) with \( R = 2 \), i.e.
\[ Q = Q_0 = [-2^{\frac{d}{2}}, 0] \times [-1, 1]^d. \]
Now we take a function \( \zeta \in C_c^\infty(\mathbb{R}) \) such that \( \zeta = 1 \) on \([ -3^{\frac{d}{2}}, 3^{\frac{d}{2}}] \), \( \zeta = 0 \) outside of \([ -4^{\frac{d}{2}}, 4^{\frac{d}{2}}] \), and \( 0 \leq \zeta \leq 1 \). We also choose a function \( \eta \in C_c^\infty(\mathbb{R}^d) \) such that \( \eta = 1 \) on \( B_2 \), \( \eta = 0 \) outside of \( B_3 \), and \( 0 \leq \eta \leq 1 \). Set
\[ g_1(t, x) = g\zeta, \quad g_2 = g(1 - \zeta)\eta, \quad g_3 = g(1 - \zeta)(1 - \eta). \]
Observe that \( g = g_1 + g_2 + g_3 \) and
\[
(-\Delta)^{c_1/2} T_{t,s}^{\alpha,\beta} g_1(s, y) = \zeta(s) (-\Delta)^{c_1/2} T_{t,s}^{\alpha,\beta} g(s, y),
\]
\[
T g \leq T g_1 + T (g_2 + g_3),
\]
(3.30)
\[
T g_3 \leq T (g_2 + g_3) \leq T g.
\]
(3.31) \( (3.30) \) is because \( T \) is sublinear (see (3.14)), and \( (3.31) \) comes from the facts \( g_3 = (1 - \eta)(g_2 + g_3) \), \( g_2 + g_3 = (1 - \zeta)g \), \( |1 - \eta(s)| \leq 1 \), and \( |1 - \zeta(s)| \leq 1 \). Hence for any constant \( c \),
\[
|T g - c| \leq |T g_1| + |T (g_2 + g_3) - c|
\]
and
\[
|T (g_2 + g_3) - c| \leq |T g_2| + |T g_3 - c|.
\]
(3.32)
Indeed, \( (3.32) \) is from \( (3.30) \) if \( c \leq T g \), and if \( c > T g \) then it follows from \( T (g_2 + g_3) \leq T g \). Similarly, \( (3.33) \) is obvious if \( c \leq T (g_2 + g_3) \), and \( c > T (g_2 + g_3) \) we use \( T g_3 \leq T (g_2 + g_3) \).

Therefore, for any \( c \in \mathbb{R} \),
\[
|T g(s, y) - c| \leq |T g_1(s, y)| + |T (g_2 + g_3)(s, y) - c|
\]
\[
\leq |T g_1(s, y)| + |T g_2(s, y)| + |T g_3(s, y) - c|,
\]
and by \( (3.28) \)
\[
\int_{Q_0} \left| T g(s, y) - (T g)_{Q_0} \right|^2 dy ds \leq 4 \int_{Q_0} |T g(s, y) - c|^2 dy ds
\]
\[
\leq 16 \int_{Q_0} |T g_1(s, y)|^2 dy ds + 16 \int_{Q_0} |T g_2(s, y)|^2 dy ds + 16 \int_{Q_0} |T g_3(s, y) - c|^2 dy ds.
\]
Note \( g_1 \) and \( g_2 \) satisfy the conditions of Lemma 3.8 and 3.9 respectively, and thus
\[
\int_{Q_0} |T g_1(s, y)|^2 dy ds + \int_{Q_0} |T g_2(s, y)|^2 dy ds
\]
\[
\leq N \left( M_{t} M_{x} |g_1|_{H}^2 (t, x) + M_{t} M_{x} |g_2|_{H}^2 (t, x) \right) \leq N M_{t} M_{x} |g|_{H}^2 (t, x).
\]
The second inequality above is due to \( |g_i| \leq |g| \) \( (i = 1, 2, 3) \). Take \( c = (T g_3)_{Q_0} \) and note that
\[
\int_{Q_0} \left| T g_3(s, y) - (T g_3)_{Q_0} \right|^2 dy ds \leq \int_{Q_0} \int_{Q_0} |T g_3(s, y) - T g_3(r, z)|^2 dr dz dy ds.
\]
(3.34)
Note also, on \( Q_0 \), \( T g_3 \) does not depend on the values of \( g_3(t, x) \) for \( t > 0 \). Hence the above two integrals do not change if we replace \( g_3 \) by \( g_3 \xi \), where \( \xi \in C^\infty (\mathbb{R}) \) so that \( 0 \leq \xi \leq 1 \), \( \xi = 1 \) for \( t \leq 1 \), and \( \xi = 0 \) for \( t \geq 2^{1/\alpha} \). Now it is easy to check that \( g_3 \xi \) satisfies the assumptions of Lemma 3.10 and therefore the right hand side of \( (3.34) \) is controlled by
\[
M_{t} M_{x} |g_3 \xi|_{H}^2 (t, x) \leq M_{t} M_{x} |g|_{H}^2 (t, x).
\]
Hence \( (3.29) \) is finally proved.

We continue the proof of the theorem. By \( (3.28) \) and Jensen’s inequality
\[
(T g)^\#(t, x) \leq N \left( M_{t} M_{x} |g|_{H}^2 (t, x) \right)^{1/2}.
\]
Therefore by Fefferman-Stein theorem ([13, Theorem 4.2.2]) and Hardy-Littlewood maximal theorem ([13, Theorem 1.3.1]),

\[
\|Tg\|_{L^p(\mathbb{R}^{d+1})} \leq N \|M_x|g|_H^{1/2}\|_{L^p(\mathbb{R}^{d+1})} \leq N \|g|_H\|_{L^p(\mathbb{R}^{d+1})}.
\]

This proves the theorem if \( T = \infty \). Note that if \( T < \infty \) the left hand side of (3.3) does not depend on the value of \( g \) for \( t \geq T \). Take \( \tilde{\xi} \in C^\infty(\mathbb{R}) \) such that \( 0 \leq \tilde{\xi} \leq 1 \), \( \tilde{\xi} = 1 \) for \( t \leq T \) and \( \tilde{\xi} = 0 \) for \( t \geq T + \varepsilon \), \( \varepsilon > 0 \). Then it is enough to apply the result for \( T = \infty \) with \( g\tilde{\xi} \). Since \( \varepsilon > 0 \) is arbitrary the theorem is proved. \( \square \)

4. Model Equation

Let \( \alpha \in (0, 2) \) and \( \beta \in (-\infty, \alpha + \frac{1}{2}) \). In this section we obtain the uniqueness, existence, and sharp estimate of strong solutions to the model equation

\[
\partial_t^\alpha u(t, x) = \Delta u(t, x) + \partial_t^\beta \int_0^t g^k(s, x) dw^k_s, \quad t > 0
\]

with the zero initial condition \( u(0, x) = 0 \) (additionally \( \partial_t u(0, x) = 0 \) if \( \alpha > 1 \)).

The following lemma is used to estimate solutions to the equation when \( \beta < 1/2 \).

Lemma 4.1. Let \( \gamma \in \mathbb{R} \), \( p > 2 \), \( \beta < \frac{1}{2} \), and \( g \in H_2^\gamma(T, l_2) \). Then for any \( t \in [0, T] \),

\[
\mathbb{E} \int_0^t \left\| \sum_{k=1}^{\infty} \partial_t^\beta \int_0^r g^k(s, \cdot) dw^k_s \right\|_{H_2^\gamma}^p \ dr \leq N(d, p, \beta, T) I_t^{1-2\beta} \|g\|_{H_2^\gamma}^p.
\]

In particular,

\[
\mathbb{E} \int_0^t \left\| \sum_{k=1}^{\infty} \partial_t^\beta \int_0^r g^k(s, \cdot) dw^k_s \right\|_{H_2^\gamma}^p \ dr \leq N\|g\|_{H_2^\gamma}^p.
\]

Proof. Due to the isometry \( (I - \Delta)^{\gamma/2} : H_2^\gamma \rightarrow L_p \), we only need to prove the case \( \gamma = 0 \). By Lemma 2.3 (iii),

\[
\partial_t^\beta \left( \sum_{k=1}^{\infty} \int_0^r g^k(s, x) dw^k_s \right) = \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{\infty} \int_0^t (t-s)^{-\beta} g^k(s, x) dw^k_s,
\]
for almost all \( t \leq T \) (a.s.). By the Burkholder-Davis-Gundy inequality and the Hölder inequality, for all \( t \leq T \),

\[
\mathbb{E} \int_0^t \left\| \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{\infty} \int_0^t (r-s)^{-\beta} g^k(s, \cdot) \, dw_s^k \right\|_{L_p}^p \, dr \\
\leq N \mathbb{E} \int_0^t \int_0^t \left( \int_0^r (r-s)^{-2\beta} |g(s, x)|^2_{l_2} ds \right)^{p/2} \, dr \, dx \\
\leq N \mathbb{E} \int_0^t \int_0^t \left( \int_0^r (r-s)^{-2\beta} |g(s, x)|^2_{l_2} ds \right)^{p/2} \, dr \, dx \\
\leq N \mathbb{E} \int_0^t \int_0^t (r-s)^{-2\beta} |g(s, x)|^p_{l_2} \, ds \, dx \\
= N \int_0^t (t-s)^{-2\beta} \|g\|_{L_p(s, l_2)}^p \, ds = NI_1^{-2\beta} \|g\|_{L_p(\cdot, l_2)}^p(t).
\]

The lemma is proved. □

A version of Lemma 4.2 can be found in [5] for \( p = 2 \) and \( \alpha, \beta \in (0,1) \). But solutions spaces are slightly different, and our proof is more rigorous.

**Lemma 4.2.** Let \( g \in \mathbb{H}_0^\infty(T, l_2) \) and define

\[
u(t, x) := \sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{R}^d} q_{\alpha, \beta}(t-s, x-y) g^k(s, y) \, dy \, dw_s^k.
\] (4.2)

Then \( \nu \in \mathcal{H}_p^\infty(T) \) and satisfies with the zero initial condition in the sense of distributions (see Definition 2.4).

**Proof.** Let \( (t, x) \in [0, T] \times \mathbb{R}^d \). Set

\[
v(t, x) := \sum_{k=1}^{\infty} \int_0^t g^k(s, x) \, dw_s^k, \quad w(t, x) := I_t^{\alpha-\beta} v(t, x),
\]

where \( I_t^{\alpha-\beta} v = D_t^{\beta-\alpha} v \) if \( \alpha < \beta \). Note that since \( g \in \mathbb{H}_0^\infty(T, l_2) \), by the Kolmogorov continuity theorem

\[v \in C^{1/2-\varepsilon}([0, T], H_p^m)\]

for any \( \varepsilon > 0 \) and \( m \). Thus \( w \in C^\delta([0, T], H_p^m) \) for some \( \delta > 0 \) (see [5, Lemma 2.3]).

By Fubini’s theorem if \( \alpha \geq \beta \) and fractional integration by parts (e.g. [5, Lemma 2.3]) if \( \alpha < \beta \),

\[
\int_0^t I_s^{\alpha-\beta} p(s, x-y) \left( \int_0^{t-s} g^k(r, y) \, dw_r^k \right) ds = \int_0^t p(t-s, x-y) I_s^{\alpha-\beta} \int_0^s g^k(r, y) \, dw_r^k ds.
\]
Here $I^a_tp(s, x − y)$ and $I^a_t\int_0^t g^k(r, y)dw^k_r$ are used to denote $(I^a_t\int_0^t g^k(r, y)dw^k_r)(s)$ and $(I^a_t\int_0^t g^k(r, y)dw^k_r)(s)$, respectively. Thus, using the stochastic Fubini theorem (see [24] Lemma 2.7) we get, for each $(t, x)$, (a.s.)

$$
\int_0^t u(s, x)ds = \sum_{k=1}^{\infty} \int_{\mathbb{R}^d} \int_0^t I^a_0 I^a_{t-s} p(s, x - y) \int_0^{t-s} g^k(r, y)dw^k_r ds dy
$$

$$
= \sum_{k=1}^{\infty} \int_{\mathbb{R}^d} \int_0^t I^a_0 I^a_{t-s} \int_0^{t-s} g^k(r, y)dw^k_r dy ds
$$

$$
= \int_0^t \int_{\mathbb{R}^d} p(t-s, x-y)w(s, y)dy ds.
$$

Due to the continuity with respect to $t$, for each $x$ we get

$$
\int_0^t u(s, x)ds = \int_0^t \int_{\mathbb{R}^d} p(t-s, x-y)w(s, y)dy ds, \quad \forall t \leq T \text{ (a.s.)}
$$

and therefore (a.s.)

$$
u(t, x) = \frac{\partial}{\partial t} \int_0^t \int_{\mathbb{R}^d} p(t-s, x-y)w(s, y)dy ds, \quad (a.e.) \ t \leq T. \quad (4.3)
$$

In other words, the above equality holds (a.e.) on $\Omega \times [0, T] \times \mathbb{R}^d$.

Next we claim that

$$
u(t, x) - w(t, x) = \int_0^t \int_{\mathbb{R}^d} q(t-s, x-y)\Delta w(s, y)dy ds \quad (4.4)
$$

(a.e.) on $\Omega \times [0, T] \times \mathbb{R}^d$. By the definition of the differentiation, for each $(\omega, t, x)$,

$$
\frac{\partial}{\partial t} \int_0^t \int_{\mathbb{R}^d} p(t-s, x-y)w(s, y)dy ds
$$

$$
= \lim_{h \downarrow 0} \frac{1}{h} \int_t^{t+h} \int_{\mathbb{R}^d} (p(t+h-s, x-y))w(s, y)dy ds
$$

$$
+ \lim_{h \downarrow 0} \int_0^t \int_{\mathbb{R}^d} \left[ \frac{p(t+h-s, x-y) - p(t-s, x-y)}{h} \right] w(s, y)dy ds.
$$

By the mean value theorem, the integration by parts, and Lemma 3.1(i) and (ii),

$$
\lim_{h \downarrow 0} \int_0^t \int_{\mathbb{R}^d} \left[ \frac{p(t+h-s, x-y) - p(t-s, x-y)}{h} \right] w(s, y)dy ds
$$

$$
= \lim_{h \downarrow 0} \int_0^t \int_{\mathbb{R}^d} \frac{\partial p}{\partial h}(t + \theta h-s, x-y)w(s, y)dy ds, \quad \theta \in (0, 1)
$$

$$
= \lim_{h \downarrow 0} \int_0^t \int_{\mathbb{R}^d} q(t + \theta h-s, x-y)\Delta w(s, y)dy ds
$$

$$
= \int_0^t \int_{\mathbb{R}^d} q(t-s)\Delta w(s, y)dy ds.
$$

For the last equality above we used the $L_1$-continuity of the integrable function [34] Theorem 9.5], which implies that for any $f \in L_1([0, t + \varepsilon])$, where $\varepsilon > 0$, it holds that $\lim_{h \to 0} \int_0^t |f(s+h) - f(s)|dt = 0$. 

On the other hand, due to Corollary \(3.2\)
\[
\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} \int_{\mathbb{R}^d} (p(t+h-s,x-y)) w(s,y) dy ds = w(t,x).
\]
Thus \((4.3)\) is proved due to \((3.9)\), and from \((4.4)\) it easily follows that \(u\) has a \(H^2_p\)-valued continuous version since \(g \in H^\infty(T,T_2)\). It only remains to show that \(u\) satisfies \((4.1)\). By representation formula \((4.4)\), it follows that \(u - w \in H^{3,2}_p(T)\) (a.s.), and
\[
\partial^\alpha_t (u - w) = \Delta (u - w) + \Delta w(t,x) = \Delta u
\]
in \(L_p(T)\). See Remark \(2.9\) for spaces \(H^\alpha_p(T)\) and \(L_p(T)\). Actually in \((17\text{ Lemma }3.5)\), it is proved that \((4.4)\) gives the unique solution to \((4.5)\) in the space \(H^\alpha_p(T)\) if \(\Delta w\) is sufficiently smooth. However one can easily check that this representation holds even if \(\Delta w \in L_p([0,T] \times \mathbb{R}^d)\) by using an approximation argument. It follows from \((2.14)\) and Remark \(2.9\) that for any \(\phi \in C^\infty_c(\mathbb{R}^d)\), (a.s.)
\[
(u(t) - w(t), \phi) = \int_{0}^{T} (\Delta u(t) + \Delta w(t,x), \phi) dt, \quad (a.e.) \ t \leq T.
\]
Taking \((w(t), \phi)\) to the right hand side of the equality and using the continuity of \(u\) with respect to \(t\), we get
\[
(u(t), \phi) = I^\alpha_t (\Delta u(t), \phi) + I^\alpha_t - \beta \int_{0}^{t} (g^k, \phi) dw^k_x, \quad \forall t \leq T \ (a.s.).
\]
Therefore \(u\) is a solution to \((4.1)\) in the sense of distributions because \(u\) itself is an \(H^2_p\)-valued continuous process. The lemma is proved.

Recall, for \(\kappa \in (0, 1)\),
\[
e_0 = e_0(\kappa) = \frac{(2\beta - 1) + }{\alpha} + \kappa 1_{\beta = 1/2} \in [0, 2).
\]

**Theorem 4.3.** Let \(\gamma \in \mathbb{R}\) and \(p \geq 2\). Suppose \(g \in H^{\gamma + \epsilon_0}_p(T,T_2)\) for some \(\kappa > 0\). Then, equation \((4.1)\) with zero initial condition has a unique solution \(u \in H^{\gamma,2}_p(T)\) in the sense of distributions, and for this solution we have
\[
\left\| u \right\|_{H^{\gamma,2}_p(T)} \leq N \left\| g \right\|_{H^{\gamma + \epsilon_0}_p(T,T_2)},
\]
where \(N = N(d,p,\alpha,\beta,\kappa,T)\). Furthermore, if \(\beta > 1/2\) then
\[
\left\| u_{xx} \right\|_{H^{\gamma}_p(T)} \leq N \left\| \Delta^{\epsilon_0/2} g \right\|_{H^{\gamma}_p(T,T_2)},
\]
where \(N = N(d,p,\alpha,\beta)\) is independent of \(T\).

**Proof.** Due to the isometry \((I - \Delta)^{\gamma/2} : H^\gamma_p \to L_p\), we only need to prove the case \(\gamma = 0\).

Recall that as discussed in Remark \(2.9\) for the deterministic case, our sense of solutions introduced in Definition \(2.3\) coincides with the one in \([17]\). Therefore the uniqueness result easily follows from the deterministic result \((17\text{ Theorem }2.9)\), cf. \([18]\). Therefore it is sufficient to prove the existence of the solution and estimates \((4.6)\) and \((4.7)\).
Step 1. First, assume \( g \in \mathcal{H}_0^\infty(T,l_2) \). Define
\[
u(t,x) = \sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{R}^d} \alpha, \beta(t-s, y) g^k(s, x-y) dy dw_k.
\]

Then by Lemma 4.2, \( u \in \mathcal{H}_0^2(T) \) is a solution to equation (4.1) with the zero initial condition. Thus we only need to prove the estimates. We divide the proof according to the range of \( \beta \).

Case 1: \( \beta > \frac{1}{2} \).

Due to the inequality (e.g. p.41 of [22]),
\[
\| \Delta u \|_{L_p(T)} \leq N \| \Delta c_0/2 g \|_{L_p(T,l_2)}, \tag{4.8}
\]
Denote
\[
\nu = (-\Delta) c_0/2 u, \quad \bar{g} = (-\Delta) c_0/2 g.
\]
By the Burkholder-Davis-Gundy inequality and Remark 3.4,
\[
\| \Delta u \|_{L_p(T)} \leq N \mathbb{E} \int_0^T \int_{\mathbb{R}^d} |\bar{g}(t,x)|^p \ dx \ dt \leq N \| \bar{g} \|_{L_p(T,l_2)}^p \| \Delta c_0/2 u \|_{L_p(T,l_2)}^p.
\]

Next we prove (4.6). By Lemma 2.10(iv) and (4.8),
\[
\| u \|_{L_p(T)} \leq N \int_0^T (T-s)^{\theta-1} \left( \| \Delta u \|_{L_p(s)}^p + \| g \|_{L_p(s,l_2)}^p \right) ds \leq N \| g \|_{L_p(T,l_2)}^p \int_0^T (T-s)^{\theta-1} ds \leq N \| g \|_{L_p(T,l_2)}^p.
\]

Case 2: \( \beta < \frac{1}{2} \).

In this case, \( c_0 = 0 \) and we apply the result of the deterministic equation from [17]. By Remarks 2.8(ii) and 2.9, \( u \) satisfies
\[
\partial_\tau^\nu u = \Delta u + \bar{f}
\]
in the sense of [17] Definition 2.4, where
\[
\bar{f}(t) = \frac{1}{\Gamma(1-\beta)} \sum_k \int_0^t (t-s)^{-\beta} g^k(s) dw_k^k.
\]

Due to [17] Theorem 2.9] and Lemma 1.1,
\[
\| u \|_{L_p(T)}^p \leq N \| \bar{f} \|_{L_p(T)}^p \leq N \| g \|_{L_p(T,l_2)}^p,
\]
which together with (2.16) yields (4.6).

Case 3: \( \beta = \frac{1}{2} \).
Indeed, to check $v(x, t)$, write $\hat{\beta} = \frac{1}{2} + \delta$ and define

$$v(t, x) = \sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{R}^d} q_{\alpha, \hat{\beta}}(t - s, x - y) g^k(s, y) dy dw_s^k.$$ 

Since $0 < \delta < \alpha$ and $\frac{1}{2} < \hat{\beta} < 2$, the result from Case 1 with $c_0' = (2\hat{\beta} - 1)/\alpha = \kappa$ implies that $v \in \mathcal{H}^2_p$ satisfies

$$\partial_t^\alpha v(t, x) = \Delta v(t, x) + \sum_{k=1}^{\infty} \partial_t^k \int_0^t g^k(s, x) dw_s^k,$$

with the zero initial condition and

$$\|v\|_{\mathcal{H}^2_p(T)} \leq N \|g\|_{\mathcal{H}^{c_0'}_p(T, I_2)}.$$ 

Since $I^t_1 v$ satisfies (4.11), by the uniqueness of solutions, we conclude that $I^t_1 v(t, x) = u(t, x)$. Therefore,

$$\|u\|_{\mathcal{H}^2_p(T)} = \|I^t_1 v\|_{\mathcal{H}^2_p(T)} \leq N \|v\|_{\mathcal{H}^2_p(T)} \leq N \|g\|_{\mathcal{H}^{c_0'}_p(T, I_2)}.$$

Thus the theorem is proved if $g \in \mathbb{H}^\infty_0(T, I_2)$.

**Step 2.** For general $g \in \mathbb{H}^\infty_0(T, I_2)$, take a sequence $g_n \in \mathbb{H}^\infty_0(T, I_2)$ so that $g_n \to g$ in $\mathbb{H}^\infty_0(T, I_2)$. Define $u_n$ as the solution of equation (4.1) with $g_n$ in place of $g$. Then

$$\|u_n\|_{\mathcal{H}^2_p(T)} \leq N \|g_n\|_{\mathcal{H}^{c_0'}_p(T, I_2)},$$

$$\|u_n - u_m\|_{\mathcal{H}^2_p(T)} \leq N \|g_n - g_m\|_{\mathcal{H}^{c_0'}_p(T, I_2)}.$$ 

Thus, $u_n$ converges to $u$ in $\mathcal{H}^2_p(T)$ and $u$ becomes a solution to equation (4.1). Indeed, to check $u$ is a solution, let $\phi \in \mathcal{S}$ and then we have

$$(I^\Lambda_0 u_n(t), \phi) = I^t_1 (\Delta u_n(t, \cdot), \phi) + \sum_{k=1}^{\infty} I^t_1 \left( (\delta_n(s, \cdot), \phi) dw_s^k \right), \quad \forall t \leq T.$$ 

Taking the limit and using (4.11) we conclude that $I^\Lambda_0 u$ has a continuous version and therefore the above equality holds for all $t \leq T$ (a.s.) with $u$ and $g$ in place of $u_n$ and $g_n$ respectively. The theorem is proved.

5. **Proof of Theorem 2.17**

First, we introduce a version of method of continuity used in this article. Later we will take $L_0 = \Delta$ and $L_0 = 0$.

**Lemma 5.1** (Method of continuity). Let $L_0, L_1$ be continuous operators from $\mathcal{H}_p^{\gamma + 2}(T)$ to $\mathbb{H}^\gamma_p(T)$ and $\Lambda_0, \Lambda_1$ be continuous operators from $\mathcal{H}_p^{\gamma + 2}(T)$ to $\mathbb{H}^{\gamma + c_0'}_p(T, I_2)$. For $\lambda \in [0, 1]$ and $u \in \mathcal{H}_p^{\gamma + 2}(T)$, denote $L\lambda u = \lambda L_1 u + (1 - \lambda) L_0 u$ and $\Lambda\lambda u = \lambda \Lambda_1 u + (1 - \lambda) \Lambda_0 u$. Suppose that for any $f \in \mathbb{H}^\gamma_p(T)$ and $g \in \mathbb{H}^{\gamma + c_0'}_p(T, I_2)$ the equation

$$\partial_t^\alpha u = L_0 u + f + \partial_t^\beta \int_0^t (\Lambda_0 u + g^k) dw_s^k$$

...
with zero initial condition has a solution \( u \) in \( H^{n+2}_p(T) \). Also assume that if \( u \in H^{n+2}_p(T) \) has zero initial condition and satisfies (in the sense of distributions) the equation

\[
\partial_t^\alpha u = L_\lambda u + f + \partial_t^\beta \int_0^t (\Lambda_\lambda^k u + g^k)dw^k,
\]

then the following “a priori estimate” holds:

\[
\|u\|_{H^{n+2}_p(T)} \leq N_0 \left( \|f\|_{H^p(T)} + \|g\|_{H^{n+2}_p(T)} \right),
\]

where \( N_0 \) is independent of \( \lambda, u, f, \) and \( g \). Then for any \( \lambda \in [0,1], f \in H^p_p(T), \) and \( g \in H^{\gamma+\phi}_p(T,t_2) \) the equation

\[
\partial_t^\alpha u = L_\lambda u + f + \partial_t^\beta \int_0^t (\Lambda_\lambda^k u + g^k)dw^k
\]

with zero initial condition has a unique solution \( u \) in \( H^{n+2}_p(T) \).

**Proof.** The uniqueness easily follows from (5.2). Let \( J \) be the set of all \( \lambda \in [0,1] \) for which equation (5.3) has a solution in \( H^{n+2}_p(T) \) for any \( f \in H^p_p(T) \) and \( g \in H^{\gamma+\phi}_p(T,t_2) \). By the assumption \( 0 \in J \). Thus to prove the lemma it suffices to show that there exists \( \varepsilon > 0 \) depending only on \( N_0 \) and the boundedness of the operators \( L_i \) and \( \Lambda_i \) \( (i = 0, 1) \) such that \( \lambda \in J \) whenever \( \lambda_0 \in J \) and \( |\lambda - \lambda_0| < \varepsilon \).

Let \( \lambda_0 \in [0,1] \) and \( \lambda \in [0,1] \). Fix \( u^0 \in H^{n+2}_p(T) \). By the assumption, we can inductively define \( u^{n+1} \in H^{n+2}_p(T) \) as the solution to

\[
\partial_t^\alpha u^{n+1} = L_{\lambda_0} u^{n+1} + (-L_{\lambda_0} u^n + L_\lambda u^n + f)
\]

\[
+ \partial_t^\beta \int_0^t (\Lambda_\lambda u^{n+1} + (-\Lambda_{\lambda_0} u^n + \Lambda_\lambda u^n + g^k))dw^k.
\]

Note that for \( u^{n+1} - u^n \in H^{\gamma+2}_p(T) \) satisfies

\[
\partial_t^\alpha (u^{n+1} - u^n) = L_{\lambda_0} (u^{n+1} - u^n) + (\lambda - \lambda_0)(L_1 - L_0)(u^n - u^{n-1})
\]

\[
+ \partial_t^\beta \int_0^t (\Lambda_\lambda u^{n+1} - u^n) + (\lambda - \lambda_0)(\Lambda_1 - \Lambda_0)(u^n - u^{n-1})dw^k.
\]

By a priori estimate (5.3), we have

\[
\|u^{n+1} - u^n\|_{H^{\gamma+2}_p(T)} \\
\leq N_0|\lambda - \lambda_0|\|((L_1 - L_0)(u^n - u^{n-1}))\|_{H^p(T)} + \|(\Lambda_1 - \Lambda_0)(u^n - u^{n-1})\|_{H^{\gamma+\phi}_p(T,t_2)}
\]

\[
\leq N|\lambda - \lambda_0|\|u^n - u^{n-1}\|_{H^{\gamma+2}_p(T)},
\]

where the second inequality is due to the continuity of operators \( L_0, L_1, \Lambda_0, \) and \( \Lambda_1 \). Note that the constant \( N \) above does not depend on \( \lambda \) and \( \lambda_0 \) as well. Thus if \( \varepsilon N < 1/2 \) and \( |\lambda - \lambda_0| \leq \varepsilon \) then \( u_n \) becomes a Cauchy sequence in \( H^{\gamma+2}_p(T) \) and therefore the limit \( u \) of \( u^n \) becomes a solution to equation (5.3), which is easily checked by taking the limit in (5.4). The lemma is proved.

Next we present an estimate for a deterministic equation of non-divergence type. We use the space \( H^{\gamma+2}_{p,0}(T) \) introduced in Remark 2.9.
Lemma 5.2. Let \( a^{ij} \) be given as in (2.23), that is
\[
a^{ij}(t, x) = \sum_{n=1}^{M_0} a_{ij}^n(t, x)1_{(\tau_n-1, \tau_n)}(t)
\]  
(5.5)
where \( \tau_n \) and \( a_{ij}^n \) are non-random, and \( a^{ij} \) satisfy (2.26) and (2.30) with the constants \( \delta_0 \) and \( K_3 \) given there. Then for any solution \( u \in H_{p, \gamma}^{6+2}(T) \) to the deterministic equation
\[
\partial_t^\alpha u = a^{ij} u_{x^i x^j} + f
\]  
(5.6)
in \( H_{p}^3(T) \), it holds that
\[
\|u\|_{H_{p}^3(T)} \leq N \|f\|_{H_{p}^3(T)},
\]  
(5.7)
where \( N \) depends only on \( \alpha, p, \gamma, \delta_0, K_3, T, M_0 \), and the modulus of continuity of \( a_{ij}^n \). In particular, \( N \) depends on \( M_0 \) but independent of the choice of \( \tau_1, \cdots, \tau_{M_0-1} \).

Proof. If \( \gamma = 0 \) then this lemma is proved in [17, Theorem 2.9] under the condition that \( a_{ij}^n \) are uniformly continuous in \((t, x)\), but without the condition \( |a^{ij}|_{B^{1/2}} \leq K_3 \). The proof for the case \( \gamma \neq 0 \) depends on the one for \( \gamma = 0 \).

We divide the proof into several steps.

(Step 1). Assume that \( a^{ij} \) are independent of \((t, x)\). In this case (5.7) holds due to [17, Theorem 2.9] (or see [48, 49]) if \( \gamma = 0 \). For the case \( \gamma \neq 0 \) it is enough to apply the operator \((1 - \Delta)^{\gamma/2}\) to the equation.

We show that (5.7) leads to
\[
\|u_{xx}\|_{H_{p}^3(T)} \leq N_0 \|f\|_{H_{p}^3(T)},
\]  
(5.8)
where \( N_0 = N_0(\alpha, p, \gamma, \delta_0) \), and thus independent of \( T \). Obviously, to prove the independency of \( T \) we only need to consider the case \( \gamma = 0 \), and for this case, it is enough to notice that \( v(t, x) := u(Tt, T^{\alpha/2}x) \) satisfies \( \partial_t^\alpha v = a^{ij} v_{x^i x^j} + T^\alpha f(Tt, T^{\alpha/2}x) \) in \( L_0(1) \) and use the result for \( T = 1 \).

(Step 2). (Perturbation in \( x \)). Assume that \( a^{ij} \) depends only on \( x \). Recall we are assuming
\[
\sup_{i,j, \omega} |a^{ij}(\cdot)|_{B^{1/2}} \leq K_3.
\]  
(5.9)
In this step we prove that there exists a positive constant \( \varepsilon_1 = \varepsilon_1(N_0) \), thus independent of \( T \) and \( K_3 \), so that (5.8) holds with new constant \( N = N(N_0, K_3) \) if
\[
\sup_{i,j, t, x, y} |a^{ij}(x) - a^{ij}(y)| \leq \varepsilon_1.
\]  
(5.10)
Set \( a_{ij}^0 := a^{ij}(0) \), and rewrite (5.6) as
\[
\partial_t^\alpha u = a_{ij}^0 u_{x^i x^j} + f + (a^{ij} - a_{ij}^0) u_{x^i x^j}.
\]
By the result of Step 1, for each \( t \leq T \)
\[
\|u_{xx}\|_{H_{p}^3(t)} \leq N_0 \left( \|f\|_{H_{p}^3(t)} + \|(a^{ij} - a_{ij}^0) u_{x^i x^j}\|_{H_{p}^3(t)} \right).
\]  
(5.11)
By (2.26),
\[
\|(a^{ij} - a_{ij}^0) u_{x^i x^j}\|_{H_{p}^3} \leq N(d, \gamma) \|a^{ij} - a_{ij}^0\|_{B^{1/2}} \|u_{xx}\|_{H_{p}^3}.
\]
It follows from (5.11) that
\[
\|u_{xx}\|_{H_{p}^3(t)} \leq N_0 \|f\|_{H_{p}^3(t)} + N_0 N(d, \gamma) \|a^{ij}(t, \cdot) - a_{ij}^0\|_{B^{1/2}} \|u_{xx}\|_{H_{p}^3(t)}.
\]  
(5.12)
Hence we get (5.8) with $2N_0$ in place of $N_0$ if

$$|a^{ij} - a^{ij}_{\rho}|_{B^{\gamma|}} \leq \frac{1}{2N(d, \gamma)N_0} =: \varepsilon_2.$$  (5.13)

Now we take $\varepsilon_1 = \varepsilon_2/2$ and assume (5.10) holds. Fix a small constant $\rho > 0$ so that $\rho^{(1|\gamma)|1}K_3 \leq \varepsilon_2/2$, and set

$$a^{ij}_\rho(t, x) := a^{ij}(\rho x), \quad u_\rho(t, x) := u(\rho^{2|}t, \rho x), \quad f_\rho(t, x) := \rho^2 f(\rho^{2|}t, \rho x).$$

Note that $\varepsilon_{\rho} = 2 \varepsilon_2/2$. Consequently, for each $t \leq \rho^{-2/\alpha}T$, we get for each $t \leq \rho^{-2/\alpha}T$.

By the definition of $B^{\gamma|}$, (5.9), and the choice of $\rho$,

$$|a^{ij}_{\rho}(-) - a^{ij}_{\rho}(0)|_{B^{\gamma|}} \leq \sup_x |a^{ij} - a^{ij}_0| + 1_{\gamma \neq 0}\rho^{(1|\gamma)|1}K_3 \leq \varepsilon_2.$$

Thus by the above arguments which lead to (5.12) and (5.13), we get for each $t \leq \rho^{-2/\alpha}T$,

$$\|u_\rho(t)\|_{H^t_{\varepsilon}(t)} \leq 2N_0\|f_\rho\|_{H^t_{\varepsilon}(t)}.$$

Consequently, for each $t \leq T$,

$$\|u_{xx}\|_{H^t_{\varepsilon}(t)} \leq N(K_4, N_0)\|f\|_{H^t_{\varepsilon}(t)}.$$

As before, this and (2.14) yield (5.7). Before moving to the next step we emphasize that we take $\varepsilon_1 = (4N(d, \gamma)N_0)^{-1}$ and therefore it does not depend on $T$ and $K_3$.

(Step 3). (Partition of unity). We still assume $a^{ij}$ is independent of $t$. Choose a $\delta_1$ so that

$$|a^{ij}(x) - a^{ij}(y)| \leq \varepsilon_1/2$$

whenever $|x - y| \leq 4\delta_1$. For this $\delta_1$, take a sequence of functions $\zeta_n \in C^\infty_c(\mathbb{R}^d)$, $n \in \mathbb{N}$, so that $0 \leq \zeta_n \leq 1$, the support of $\zeta_n$ lies in $B_{\delta_1}(x_n)$ for some $x_n \in \mathbb{R}^d$,

$$\sup_{x \in \mathbb{R}^d} \sum_{n \in \mathbb{N}} |D_x^n \zeta_n(x)| \leq M(\delta_1, \eta) < \infty$$

for any multi-index $n \in \mathbb{Z}^d$ and,

$$\inf_{x \in \mathbb{R}^d} \sum_{n \in \mathbb{N}} |\zeta_n(x)| \geq \theta > 0.$$

It is well-known ([21 Lemma 6.7]) that for any $\gamma \in \mathbb{R}$ and $n \in \mathbb{N},$

$$\|h\|_{H^\gamma_p} \leq N \sum_{n \in \mathbb{N}} \|h\|_{H^\gamma_p} \leq N \sum_{n \in \mathbb{N}} \|u D_x^n \zeta_n\|_{H^\gamma_p} \leq N \|u\|_{H^\gamma_p}$$

where $N$ depend only on $d, \gamma, M(\delta_1, \eta)$, and $\theta$. Take a nonnegative $\eta \in C^\infty_c(\mathbb{R}^d)$ so that $0 \leq \eta \leq 1$, $\eta = 1$ on $B_1$, and $\eta = 0$ outside $B_2$. Write

$$u_n = u\zeta_n, \quad \eta_n(x) = \eta \left(\frac{x - x_n}{\delta_1}\right)$$

and define

$$a^{ij}_n(x) := \eta_n(x)a^{ij}(x) + (1 - \eta_n(x))a^{ij}(x_n)$$

Then, because $\eta_n = 1$ on the support of $\zeta_n$, $u_n(t, x)$ satisfies

$$\partial_t^n u_n(t, x) = a^{ij}_n(u_n)_{x, ij} + f_n,$$
We take \( \varepsilon \) the constant \( N \) wall’s lemma (\cite[Corollary 2]{47}), we obtain (5.7). We remark that up to this step, where the last inequality is due to (5.17). Therefore by applying fractional Gronwall’s lemma (\cite[Corollary 2]{17}), for each \( t \leq T \),

\[
\|u\|^p_{H^{n+2}_p(t)} \leq N \sum_{n \in \mathbb{N}} \|u_n\|^p_{H^{n+2}_p(t)} \leq N \sum_{n \in \mathbb{N}} \|\tilde{f}_n\|^p_{H^{n}_p(t)} \\
\leq N \|u\|^p_{H^{n+1}_p(t)} + N \|f\|^p_{H^{n}_p(t)} \\
\leq \varepsilon \|u\|^p_{H^{n+2}_p(t)} + N(\varepsilon) \|u\|^p_{H^{n}_p(t)} + N \|f\|^p_{H^{n}_p(t)}. \tag{5.17}
\]

We take \( \varepsilon = 1/2 \), and to drop the term \( \|u\|^p_{H^{n}_p(t)} \) above we use (5.17), which implies

\[
\|u\|^p_{H^{n+2}_p(t)} \leq N \int_0^t (t-s)^{\sigma - 1} \|a^{ij} u_{x^j x^i} + f\|^p_{H^{n}_p(s)} ds \\
\leq N \int_0^t (t-s)^{\sigma - 1} (\|u\|^p_{H^{n+2}(s)} + \|f\|^p_{H^{n}_p(s)}) ds \\
\leq N \int_0^t (t-s)^{\sigma - 1} (\|u\|^p_{H^{n}_p(s)} + \|f\|^p_{H^{n}_p(s)}) ds
\]

where the last inequality is due to (5.17). Therefore by applying fractional Gronwall’s lemma (\cite[Corollary 2]{17}), we obtain (5.17). We remark that up to this step, the constant \( N \) of (5.17) depends only on \( \delta_0, p, K, \alpha, \gamma, T \), and the modulus of continuity of \( a^{ij} \).

(Step 4) (general case). Recall that in Step 3 we proved the lemma when \( a^{ij} \) are independent of \( t \). For the general case, it is enough to repeat Steps 5 and 6 of the proof of \cite[Theorem 2.9]{17}. Indeed, in \cite{17} the lemma is proved when \( \gamma = 0 \), and the proof is first given for time-independent \( a^{ij} \), and then this result is extended for the general case. This method of generalization works exactly same for any \( \gamma \in \mathbb{R} \). \( \square \)

We continue the proof of Theorem 2.17

**Case A. Linear case.** Suppose \( f \) and \( g \) are independent of \( u \), and \( b^i = c = \mu^k = \nu^k = 0 \). To apply the method of continuity, for all \( \lambda \in [0, 1] \) denote

\[
(a^{ij}_\lambda) = \lambda (a^{ij}) + (1 - \lambda) I_{d \times d}, \quad \sigma^{ij}_\lambda = \lambda \sigma^{ij},
\]

where \( I_{d \times d} \) is the \( d \times d \)-identity matrix. Then

\[
L^\lambda u := \lambda a^{ij} u_{x^j x^i} + (1 - \lambda) \Delta u = a^{ij}_\lambda u_{x^j x^i}
\]

and

\[
\Lambda^\lambda u := \lambda \sigma^{ijk} u_{x^k x^j} = \sigma^{ijk}_\lambda u_{x^k x^j}.
\]

Due to the method of continuity and Theorem 4.3 we only need to prove a priori estimate (5.2) holds given that a solution \( u \in H^{n+2}_p(0, T) \) to equation (5.1) already...
exists. Note that for any \( \lambda \in [0, 1] \) the coefficients \( a_{ij}^{\lambda} \) and \( \sigma_\lambda \) satisfy the same conditions assumed for \( a_{ij} \) and \( \sigma_{ijk} \), that is, conditions specified in Assumptions 2.11 and 2.14 with the same constants used there. This shows that by considering \( a_{ij}^{\lambda} \) and \( \sigma_\lambda \) in place of \( a_{ij} \) and \( \sigma \), we only need to prove (5.2) for \( \lambda = 1 \).

By Theorem 4.3, the equation
\[
\partial_t^\alpha v(t, x) = \Delta v(t, x) + \sum_{k=1}^\infty \partial_t^\beta_k \int_0^t (\sigma_{ijk} u_{x^i x^j} + g_k u) dw_k
\]
(5.18)
has a unique solution \( v \in \mathcal{H}_{p,0}^{\gamma+2}(T) \), and moreover
\[
\|v\|_{\mathcal{H}_{p,0}^{\gamma+2}(T)} \leq N \|g\|_{\mathcal{H}_{p,0}^{\gamma+2}(T)}.
\]
(5.19)
Indeed, (5.19) is obvious if \( \beta > 1/2 \) because \( \sigma_{ijk} \) is 0 in this case. If \( \beta < 1/2 \), then by Theorem 4.3 and Lemma 4.4 for each \( t \leq T \),
\[
\|v\|^p_{\mathcal{H}_{p,0}^{\gamma+2}(t)} \leq N \|\sigma_{ijk} u_{x^i x^j} + g\|^p_{\mathcal{H}_{p,0}^{\gamma+2}(t)} + N \|g\|^p_{\mathcal{H}_{p,0}^{\gamma+2}(T)}. \]
Therefore (5.19) follows from the fractional Gronwall’s lemma.

Note that \( \tilde{u} = u - v \) satisfies the equation
\[
\partial_t^\alpha \tilde{u}(t, x) = a_{ij} \tilde{u}_{x^i x^j}(t, x) + a_{ij}^{(T)} t v_{x^i x^j}(t, x) - \Delta v(t, x) + f(t, x).
\]
By Lemma 5.2
\[
\|\tilde{u}\|_{\mathcal{H}_{p,0}^{\gamma+2}(T)} \leq N \|a_{ij}^{(T)} t v_{x^i x^j}(t, x) - \Delta v(t, x) + f(t, x)\|_{\mathcal{H}_{p,0}^{\gamma+2}(T)} + N \|v\|_{\mathcal{H}_{p,0}^{\gamma+2}(T)}.
\]
(5.20)
Since \( u = \tilde{u} + v \), the desired estimate follows from (5.19) and (5.20).

**B: General case.** Write
\[
\hat{f} := b^i u_{x^i} + cu + f(u), \quad \hat{g}^k := \mu^k u_{x^i} + \nu^k u + g^k(u).
\]
Note that \( \mu^k = 0 \) if \( c^0 \geq 1 \). Then by (2.26), (2.27), and Assumption 2.16(iii),
\[
\|\hat{f}(u) - \hat{g}(v)\|_{\mathcal{H}_{p,0}^{\gamma+2}(T)} \leq N \left( \|u - v\|_{\mathcal{H}_{p,0}^{\gamma+2}(T)} + \|\mu^k (u - v)\|_{\mathcal{H}_{p,0}^{\gamma+2}(T)} \right)
\]
where \( N \) depends on \( d, p, m, \kappa, K_3, K_4, \) and \( \varepsilon \). Hence considering \( \hat{f} \) and \( \hat{g}^k \) in place of \( f \) and \( g^k \), we may assume \( b^i = c^0 = \mu^k = \nu^k = 0 \).

For each \( u \in \mathcal{H}_{p,0}^{\gamma+2}(T) \), consider the equation
\[
\partial_t^\alpha v = a_{ij} v_{x^i x^j} + f(u) + \sum_{k=1}^\infty \int_0^t (\sigma_{ijk} v_{x^i x^j} + g_k(u)) dw_k
\]
with zero initial condition. By the result of Case A, this equation admit a unique solution \( v \in \mathcal{H}_{p,0}^{\gamma+2}(T) \). By denoting \( v = Ru \), we can define an operator
\[
\mathcal{R} : \mathcal{H}_{p,0}^{\gamma+2}(T) \to \mathcal{H}_{p,0}^{\gamma+2}(T).
\]
By Lemma 2.10(ii), (2.31), and the result of Case A, for each $t \leq T$,
\[
\| \mathcal{R} u - \mathcal{R} v \|^p_{H^{n+2}(t)} \leq N_0 \left( \| f(u) - f(v) \|^p_{H^p(t)} + \| g(u) - g(v) \|^p_{H^{p+\delta_0}(t,t_2)} \right) 
\leq N_0 \varepsilon^p \| u - v \|^p_{H^{n+2}(t)} + N_1 \| u - v \|^p_{H^{p+\delta_0}(t)} 
\leq N_0 \varepsilon^p \| u - v \|^p_{H^{n+2}(t)} + N_1 \int_0^t (t-s)^{\theta-1} \| u - v \|^p_{H^{n+2}(s)} ds,
\]
where $N_1$ depends also on $\varepsilon$. Next, we fix $\varepsilon$ so that $\Theta := N_0 \varepsilon^p < 2^{-2}$. Then repeating the above inequality and using the identity
\[
\int_0^t (t-s)^{\theta-1} \int_0^{s_1} (s_1 - s_2)^{\theta-1} \cdots \int_0^{s_{n-1}} (s_{n-1} - s_n)^{\theta-1} ds_n \cdots ds_1 
= \frac{\{\Gamma(\theta)\}^n}{\Gamma(n \theta + 1)} t^n \theta,
\]
we get
\[
\| \mathcal{R}^n u - \mathcal{R}^n v \|^p_{H^{n+2}(t)} \leq \sum_{k=0}^n \binom{n}{k} \Theta^{n-k} \left( T^\theta N_1 \right)^k \frac{\{\Gamma(\theta)\}^k}{\Gamma(k \theta + 1)} \| u - v \|^p_{H^{n+2}(t)} 
\leq 2^n \Theta^n \max_k \left[ \frac{\{\Theta^{-1} T^\theta N_1 \Gamma(\theta)\}^k}{\Gamma(k \theta + 1)} \right] \| u - v \|^p_{H^{n+2}(t)} 
\leq \frac{1}{2^n} N_2 \| u - v \|^p_{H^{n+2}(t)}.
\]
For the second inequality above we use $\sum_{k=0}^n \binom{n}{k} = 2^n$. It follows that if $n$ is sufficiently large then $\mathcal{R}^n$ is a contraction in $H^{n+2}(T)$, and this yields all the claims. The theorem is proved.

6. Proof of Theorem 2.15

We first prove a result for a deterministic equation of divergence type.

**Lemma 6.1.** Let $a^{ij}$ be given as in (5.3) with non-random $\tau_n$ and $a^i_n$. Suppose $a^{ij}$ satisfy the uniform ellipticity in (2.29) and $a^i_n$ are uniformly continuous in $(t,x)$. Then for any solution $u \in H^{a,i}_p(T)$ to the deterministic equation
\[
\partial^i_t u = D_x (a^{ij} u_{x_i}) + f^i + h 
\tag{6.1}
\]
in $H^{-1}_p(T)$, it holds that
\[
\| u \|^p_{H^1_p(T)} \leq N(\| f^i \|^p_{L_p(T)} + \| h \|^p_{H^{-1}_p(T)}), \tag{6.2}
\]
where $N$ depends only on $\alpha$, $p$, $\gamma$, $\delta_0$, $T$, $M_0$, and the modulus of continuity of $a^i_n$.

**Proof.** We divide the proof into three steps.

(Step 1). Let $a^{ij}$ depend only on $t$. In this case, (6.2) is a consequence of (5.4) with $\gamma = -1$, which is because $\| D_x f^i \|^p_{H^{-1}_p} \leq N \| f^i \|^p_{L_p}$.

(Step 2). We prove there exists $\varepsilon_2 > 0$, which depends also on $T$, such that (5.2) holds if
\[
\sup_{t,x} |a^{ij}(t,x) - a^{ij}(t,y)| \leq \varepsilon_2. \tag{6.3}
\]
Denote $a_{ij}^j(t) := a^{ij}(t,0)$, and rewrite the equation as

$$\partial_t^a u = D_x (a_{ij}^j u_x^j + \bar{f}^i) + h$$

where

$$\bar{f}^i := f^i + \sum_{j=1}^d \left( a^{ij} - a_{ij}^j \right) u_x^j.$$ 

Note that $a_{ij}^j$ is independent of $x$. By the result of Step 1, for each $t \leq T$,

$$\|u\|_{H^p_p(t)} \leq N_3(\|f\|_{L^p_p(t)} + \|a^{ij} - a_{ij}^j\|_{L^p_p(t)} + N\|h\|_{H^{-1}_p(t)}).$$

Observe that

$$\|(a^{ij}(t,\cdot) - a_{ij}^j(t))u_x^j(t,\cdot)\|_{L^p_p} \leq N(d,p) \sup_{t,x} |a^{ij}(t,x) - a_{ij}^j(t)|\|u(t,\cdot)\|_{H^p_p}.$$ 

Therefore, our claim follows if $\varepsilon > 0$, with $\varepsilon = (2N(d,p)N_3)^{-1}$.

(Step 3). We introduce a partition of unity $\zeta_n$ as in the proof of Lemma 5.2, and define $\eta$ and $a_n^{ij}$ as in (5.16) so that each $(a_n^{ij})$ satisfies (6.3). Note $u^n(t,x) = u \zeta_n$ satisfies

$$\partial_t^a u^n = D_x (a_n^{ij} u_n^j + \bar{f}^{n,i}) + \bar{h}^{n}$$

where

$$\bar{f}^{n,i} = f^i \zeta_n - a^{ij} u_{x} \zeta_n^j, \quad \bar{h}^{n} = h \zeta_n - a^{ij} u_{x} \zeta_n^j.$$ 

Therefore, using Step 2 and $\| \cdot \|_{H^{-1}} \leq N \| \cdot \|_{L^p}$, we get

$$\|u\|_{H^p_p(t)} \leq N \sum_{n \in \mathbb{N}} \|u^n\|_{H^p_p(t)} \leq N \sum_{n \in \mathbb{N}} (\|\bar{f}^n\|_{L^p_p(t)} + \|\bar{h}^n\|_{H^{-1}_p(t)}) \leq N(\|f\|_{L^p_p(t)} + \|h\|_{H^{-1}_p(t)}) + N\|u\|_{L^p_p(t)} + N\|a^{ij} u_x^j\|_{H^{-1}_p(t)}.$$ 

Here we claim that for any $\varepsilon > 0$,

$$\|a^{ij} u_x^j\|_{H^{-1}} \leq \varepsilon \|u\|_{H^1_p} + N(\varepsilon)\|u\|_{L^p}.$$ 

(6.4)

Indeed, since $a^{ij}$ are uniformly continuous in $x$ uniformly in $t$, considering appropriate convolution we can take a sequence of $C^1$-functions $a_n^{ij}$ which uniformly converges to $a^{ij}$ with respect to $x$ uniformly in $t$. Thus

$$\|a^{ij} u_x^j\|_{H^{-1}} \leq \|a_n^{ij} - a^{ij}\|_{H^{-1}} + \|a_n^{ij} u_x^j\|_{H^{-1}} \leq \sup_{t,x} |a_n^{ij} - a^{ij}|\|u_x^j\|_{L^p_p} + |a_n^{ij}|_{L^1} \|u_x^j\|_{H^{-1}}.$$ 

This certainly proves (6.4). Taking small $\varepsilon$ and using the interpolation $\|u\|_{L^p} \leq \varepsilon' \|u\|_{H^1_p} + N(\varepsilon')\|u\|_{H^{-1}_p}$, we get for each $t \leq T$,

$$\|u\|_{H^p_p(t)} \leq N\|f\|_{L^p_p(t)} + N\|h\|_{H^{-1}_p(t)} + N\|u\|_{H^{-1}_p(t)}.$$ 

The last term $\|u\|_{H^{-1}_p(t)}$ can be easily dropped as before using (2.17) and Gronwall’s lemma. The lemma is proved.
Now we prove Theorem 2.15.

(Step 1). Suppose $f^i$, $h$ and $g$ are independent of $u$ and $b^i = c = \nu^k = 0$. In this case, by the method of continuity and Theorem 4.3 we only need to show an a priori estimate (2.29) holds given that a solution $u \in H^1_p(T)$ already exists. See the proof of Theorem 2.17 for details.

In this case estimate (2.29) follows from Lemma 6.1 and the arguments in Case 1 of the proof of Theorem 2.17. Indeed, take the function $v \in H^1_p(T)$ from (5.18), which is a solution to

$$\partial_t^\alpha v(t, x) = \Delta v(t, x) + \sum_{k=1}^{\infty} \partial_i^\beta \int_0^t (\sigma^{ijk} u_{x^i, x^j} + g^k) dw^k_t.$$ 

By (5.19),

$$\|v\|_{H^1_p(T)} \leq N\|g\|_{H^{-(\beta-1)}_p(T \times \Omega)}.$$ 

Note that $\bar{u} := u - v$ satisfies

$$\partial_t^\alpha \bar{u} = D_x^i \left(a^{ij} \bar{v}_{x^j} + \bar{f}^i\right) + h, \quad \bar{f}^i := (a^{ij} - \delta^{ij})v_{x^j}.$$ 

Thus one can estimate $\|\bar{u}\|_{H^1_p(T)}$ using Lemma 6.1 and this leads to (2.29) since $u = \bar{u} + v$.

(Step 2). General case. The proof is almost identical to that of Case B of the proof of Theorem 2.17. We put

$$\bar{f}^i = b^i u + f(u), \quad \bar{h}(u) = cu + h(u), \quad \bar{g}^k = \nu^k u + \nu^k u + g^k(u).$$

Then, as before, one can check these functions satisfy condition (2.28), and therefore we may assume $b^i = c = \nu^k = \nu^k = 0$. Then, using Step 1, we define the operator $\mathcal{R} : H^1_p(T) \rightarrow H^1_p(T)$ so that $v = \mathcal{R}u$ is the solution to the problem

$$\partial_t^\alpha v = D_x^i (a^{ij} v_{x^j} + f^i(u)) + h(u) + \partial_t^\beta \int_0^t (\sigma^{ijk} v_{x^i, x^j} + g^k(u)) dw^k_t$$

with zero initial condition. After this, using the arguments used in the proof of Theorem 2.17 one easily finds that $\mathcal{R}^n$ is a contraction in $H^1_p(T)$ if $n$ is large enough. This proves the theorem. □

7. SPDE Driven by Space-Time White Noise

In this section we assume

$$\beta < \frac{3}{4} \alpha + \frac{1}{2},$$

and the space dimension $d$ satisfies

$$d < 4 - \frac{2(2\beta - 1)_+}{\alpha} := d_0.$$ 

Note $d_0 \in (1, 4]$ due to (7.1). If $\beta < \frac{3}{4} + 1/2$ then one can take $d = 1, 2, 3$. Also, $\alpha = \beta = 1$ then $d$ must be 1.

In this section we study the SPDE

$$\partial_t^\alpha u = (a^{ij} u_{x^i, x^j} + b^i u_{x^i} + cu + f(u)) + \partial_t^\beta \int_0^t h(u) dB_t$$

(7.3)

where the coefficients $a^{ij}$, $b^i$, $c$ are functions depending on $(\omega, t, x)$, the functions $f$ and $h$ depend on $(\omega, t, x)$ and the unknown $u$, and $B_t$ is a cylindrical Wiener process on $L_2(\mathbb{R}^d)$. 
Let \( \{ \eta^k : k = 1, 2, \cdots \} \) be an orthogonal basis of \( L_2(\mathbb{R}^d) \). Then (see [21] Section 8.3)
\[
    dB_t = \sum_{k=1}^{\infty} \eta^k dw^k_t
\]
where \( w^k_t := (B_t, \eta^k)_{L_2} \) are independent one dimensional Wiener processes. Hence one can rewrite (7.3) as
\[
    \partial_t^\alpha u = (a^{ij} u_{x_i x_j} + b^i u_{x_i} + cu + f(u)) + \sum_{k=1}^{\infty} \partial_t^\beta \int_0^t g^k(u) \, dw^k_t,
\]
where
\[
    g^k(t, x, u) = h(t, x, u) \eta^k(x).
\]

**Lemma 7.1.** Assume
\[
    \kappa_0 \in \left( \frac{d}{2} - \frac{d}{2r} \right], \quad 2 \leq 2r \leq p, \quad 2r < \frac{d}{d - \kappa_0}, \quad (7.4)
\]
and \( h(x, u), \xi(x) \) are functions of \( (x, u) \) and \( x \) respectively such that \( |h(x, u) - h(x, v)| \leq \xi(x)|u - v| \). For \( u \in L_p(\mathbb{R}^d) \), set \( g^k(u) = h(x, u(x)) \eta^k(x) \). Then
\[
    \|g(u) - g(v)\|_{H^{-\kappa_0}_{p}(\mathbb{R}^d)} \leq N \|\xi\|_{L_2} \|u - v\|_{L_p},
\]
where \( s = r/(r - 1) \) is the conjugate of \( r \) and \( N = N(r) < \infty \). In particular, if \( r = 1 \) and \( \xi \) is bounded, then
\[
    \|g(u) - g(v)\|_{H^{-\kappa_0}_{p}(\mathbb{R}^d)} \leq N \|u - v\|_{L_p}.
\]

**Proof.** It is well-known (e.g. [12] p.132, [22] Exercise 12.9.19) that there exists a Green function \( G(x) \), which decays exponentially fast at infinity and behaves like \( |x|^\kappa_0 - d \) so that the equality holds:
\[
    \|g(u) - g(v)\|_{H^{-\kappa_0}_{p}(\mathbb{R}^d)} = \|\tilde{h}\|_{L_p},
\]
where
\[
    \tilde{h}(x) := \left( \int_{\mathbb{R}} |G(x - y)|^2 |h(y, u(y)) - h(y, v(y))|^2 \, dy \right)^{1/2}
\]
\[
    \leq \left( \int_{\mathbb{R}} |G(x - y)|^2 \xi^2(y) |u(y) - v(y)|^2 \, dy \right)^{1/2} =: \tilde{h}(x).
\]
By Hölder’s inequality,
\[
    |\tilde{h}(x)| \leq \|\xi\|_{L_2} \cdot \left( \int_{\mathbb{R}} |G(x - y)|^{2r} |u(y) - v(y)|^{2r} \, dy \right)^{1/(2r)}.
\]
Note that \( \|G\|_{L_{2r}} < \infty \) since \( 2r < \frac{d}{d - \kappa_0} \). Therefore applying Minkowski’s inequality, we have
\[
    \|\tilde{h}\|_{L_2} \leq N \|\xi\|_{L_2} \|G\|_{2r} \|u - v\|_{L_p} \leq N \|\xi\|_{L_2} \|u - v\|_{L_p}.
\]
The lemma is proved. \( \square \)

**Remark 7.2.** By following the proof of Lemma 7.1, one can easily check that
\[
    \|g(u)\|_{H^{-\kappa_0}_{p}(\mathbb{R}^d)} \leq N \|h(u)\|_{L_p}. 
\]
Assumption 7.3. (i) The coefficients $a^{ij}, b^i$, and $c$ are $\mathcal{P} \otimes \mathcal{B}(\mathbb{R})$-measurable.
(ii) The functions $f(t, x, u)$ and $g(t, x, u)$ are $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d \times \mathbb{R})$-measurable.
(iii) For each $\omega, t, x, u$ and $v$,
\[
|f(t, x, u) - f(t, x, v)| \leq K|u - v|, \quad |h(t, x, u) - h(t, x, v)| \leq \xi(t, x)|u - v|,
\]
where $\xi$ depends on $(\omega, t, x)$. Denote
\[
f_0 = f(t, x, 0), \quad h_0 = h(t, x, 0).
\]

Theorem 7.4. Suppose Assumption 7.3 holds and
\[
\|f_0\|_{\mathcal{H}^{s_0 - \epsilon \delta_0}_p(T)} + \|h_0\|_{L_p(T)} + \sup_{\omega, t} \|\xi\|_{2s} \leq K < \infty,
\]
where $\kappa_0$ and $s$ satisfy
\[
\frac{d}{2} < \kappa_0 < \left(2 - \frac{\beta - 1}{\alpha}\right) \wedge d, \quad \frac{d}{2\kappa_0 - d} < s
\]
and $\epsilon_0$ from (2.21). Also assume that the coefficients $a^{ij}, b^i$, and $c$ satisfy Assumption 2.11 and (2.30) with $\gamma := -\kappa_0 - c_0'$. Then equation (7.3) with zero initial condition has a unique solution $u \in \mathcal{H}^{2\kappa_0 - c_0}_p(T)$, and for this solution we have
\[
\|u\|_{\mathcal{H}^{2\kappa_0 - c_0}_p(T)} \leq N\|f_0\|_{\mathcal{H}^{s_0 - \epsilon \delta_0}_p(T)} + N\|h_0\|_{L_p(T)}.
\]

Proof. We only need to check if the conditions for Theorem 2.17 are satisfied with $\gamma := -\kappa_0 - c_0'$. Since $f(u)$ is Lipschitz continuous, we only check the conditions for $g^k(u) := h(u)\eta_k$. Let $r$ be the conjugate of $s$ and then $2r < \frac{d}{2\kappa_0}$ due to the assumption $\frac{d}{\kappa_0 - d} < s$. Recall $\gamma$ is chosen such that $\gamma + c_0' = -\kappa_0$. Thus, By Lemma 7.1 for any $\varepsilon > 0$,
\[
\|g(u) - g(v)\|_{\mathcal{H}^{2\kappa_0 - c_0}_p(L_2)} \leq N\|\xi\|_{L_2} \|u - v\|_{L_p} \leq \varepsilon \|u - v\|_{\mathcal{H}^{2\kappa_0 - c_0}_p} + N(\varepsilon)\|u - v\|_{\mathcal{H}^{2\kappa_0 - c_0}_p},
\]
where the second inequality is due to $\gamma + 2 > 0$, which is equivalent to $\kappa_0 + c_0' < 2$. Therefore all the conditions for Theorem 2.17 are checked. The theorem is proved.

Remark 7.5. (i) By (7.2), there always exists $\kappa_0$ satisfying (7.3).
(ii) The constant $2 - \kappa_0 - c_0'$ gives the regularity of the solution $u$. To see how smooth the solution is, recall $c_0' = (2\beta - 1)/\alpha + \kappa_1\beta = 1/2$. It follows
\[
0 < 2 - \kappa_0 - c_0' < \begin{cases} 2 - \frac{d}{2} - \frac{2\beta - 1}{\alpha} & \text{if } \beta > 1/2 \\
2 - \frac{d}{2} & \text{if } \beta \leq 1/2.
\end{cases}
\]
If $\xi$ is bounded one can take $r = 1$ and $\kappa_0 \approx \frac{d}{2}$, thus $2 - \kappa_0 - c_0'$ can be as close as one wishes to the above upper bounds.

Remark 7.6. Take $\alpha = 1$ and $\beta \leq 1$ so that the integral form of (7.3) becomes
\[
u(t, x) = \int_0^t (a^{ij}u_{x^j}x^i) + b^i u_{x^i} + cu + f(u))dt + \int_0^t h(u)dB_t.
\]
By the stochastic Fubini theorem, at least formally
\[
\int_0^t h(u)dB_t = \frac{1}{\Gamma(2 - \beta)} \int_0^t h(u(s))(t - s)^{1-\beta}d_\beta.
\]
If $\beta = 1$ then the classical theory (see e.g. [21, Section 8]) requires $d = 1$ to have meaningful solutions, that is locally integrable solutions. By Theorem 7.4 if $\beta < 3/4$ then it is possible to take $d = 1, 2, 3$. This might be because the operator $I_t^{1-\beta}$ gives certain smoothing effect to $B_t$ in the time direction.

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