Regularity Criterion for a Two Dimensional Carreau Fluid Flow

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Abstract
Carreau fluids are a source of research from both theoretical and applied approaches. They have been considered to model different non-newtonian phenomena such as blood flow, plasma and viscoelastic materials. The purpose of this study is to develop the global regularity criteria for a Carreau fluid in two dimensions flowing in a strip. Firstly, a regularity criteria is shown for the initial set \((u_{10}, u_{20}) \in H^1(\Omega)\) where \(\Omega = [0, L] \times [0, \infty)\). Secondly, the analysis focuses on a regularity criteria when \((u_{10}, u_{20}) \in L^4(\Omega)\) and, lastly, similar results are obtained for \((u_{10}, u_{20}) \in H^2(\Omega)\) while the fluid velocity vertical component, \(u_2(x, y)\), is such that \(\frac{\partial u_2}{\partial x} \in L^4(\Omega)\) and \(\left(\frac{\partial^2 u_2}{\partial y}, \Delta u_2\right) \in L^2(\Omega)\).

Keywords Carreau fluid · Two Dimensional fluid · Unsteady flow · Regularity criterion

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1 Introduction

Most of the fluid conceptions used to model flows in typical media like air, water or oil, are based on a Newtonian description. In an important number of cases, the supposition of Newtonian behavior does not seem to be accurate enough, leading to expand the theory to introduce additional rheological properties that end in non-Newtonian descriptions. As a set of examples that preclude its ubiquity, the non-Newtonian flow is experienced in industries of mines, where sludges and muds are frequently dealt, in lubrication and biomedical flows.

The non-Newtonian fluids are characterized in different types depending on their rheological characteristics. One of such types is known as Carreau fluid (see [1–17] for few relevant studies). More particularly, an interesting research on Carreau fluids is given in [4] where the fluid is analyzed over a shrinking surface in presence of an infinite shear rate viscosity. In addition, the study of Carreau flows in spheres are widely analyzed in [5,6].

Currently, there is not a wide literature focused on developing the regularity criteria for equations obtained upon a Carreau fluid. Nonetheless, there exists an extensive literature developing the regularity criteria for Navier–Stokes equations (see [18–23]). Motivated by such facts, our intention in this study is to develop global regularity criteria for a magnetohydrodynamic (MHD) flow of Carreau fluid equations. The fluid is considered to flow between two stationary plates. Flow between the plates is induced by an initial velocity profile and a stationary linear law at $y = 0$ to the $x$-axis velocity component. In addition, a uniform magnetic field is applied in the transverse direction to the flow.

The paper layout is as follows: Firstly, the Carreau fluid model is discussed based on the general theory of fluid dynamics of non-Newtonian flows. In addition, some preliminary required results are introduced. Afterwards the theorems on existence of regular solutions are presented. The paper introduces progressively, in different sections, each of the required proofs for each of the regularity Theorems together with the supporting information required.

2 Model Proposal

We consider the two dimensional incompressible fluid flow equations of a Carreau fluid. The fluid is electrically conducting in the presence of an applied magnetic field $B_0$. The MHD flow is governed by the following set of equations:

$$\mathbf{V} = (u_1, u_2, 0), \quad \nabla \cdot \mathbf{V} = 0,$$

$$\rho \frac{d\mathbf{V}}{dt} = \nabla \cdot \tau + \mathbf{J} \times \mathbf{B},$$
where $\mathbf{V}$ is the velocity field, $t$ the time, $\rho$ is the fluid density, $\tau$ is the Cauchy stress tensor, $\mathbf{J}$ is the current density and $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$ is the magnetic field for a Carreau fluid, which is given by

$$\tau = -p\mathbf{I} + \eta \mathbf{A}_1, \quad (2.3)$$

with

$$\eta = \eta_\infty + (\eta_0 - \eta_\infty) \left[ 1 + \left( \Gamma \dot{\gamma} \right)^2 \right]^{\frac{n-1}{2}}, \quad (2.4)$$

where $p$ is the pressure field, $\mathbf{I}$ the identity tensor, $\eta_0$ the zero-shear-rate viscosity, $\eta_\infty$ the infinite-shear-rate viscosity, $\Gamma$ a material time constant, and $n$ express the power law index (since it describes the slope of $\frac{\eta_0 - \eta_\infty}{\eta_0 + \eta_\infty}$ in the power law region). The shear rate $\dot{\gamma}$ is defined by

$$\dot{\gamma} = \sqrt{\frac{1}{2} \sum_i \sum_j \dot{\gamma}_{ij} \dot{\gamma}_{ji}} = \sqrt{\frac{1}{2} \mathbf{II} = \sqrt{\frac{1}{2} \text{tr}(\mathbf{A}_1^2)}}. \quad (2.5)$$

Here $\mathbf{II}$ is the second invariant strain rate tensor and $\mathbf{A}_1$ is given by

$$\mathbf{A}_1 = (\nabla \mathbf{V}) + (\nabla \mathbf{V})^T. \quad (2.6)$$

Note that $(\cdot)^T$ denotes the transpose of a matrix. From (2.1) to (2.6), the governing equations in the absence of pressure gradient are given by

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} = \nu \frac{\partial^2 u_1}{\partial y^2} \left[ 1 + \Gamma^2 \left( \frac{\partial u_1}{\partial y} \right)^2 \right]^{\frac{n-1}{2}}$$

$$+ \nu (n-1) \Gamma^2 \frac{\partial^2 u_1}{\partial y^2} \left( \frac{\partial u_1}{\partial y} \right)^2 \left[ 1 + \Gamma^2 \left( \frac{\partial u_1}{\partial y} \right)^2 \right]^{\frac{n-1}{2}} - \frac{\sigma B_0^2}{\rho} u_1, \quad (2.7)$$

where $\nu = \frac{\eta_0}{\rho}$ is the kinematic viscosity and $\sigma$ is related with the electrical charges distribution.

Note that the subject boundary conditions are

$$u_1(x, y, t) = Ux, \quad u_2(x, y, t) = 0 \quad \text{at} \quad y = 0, \quad (2.8)$$

$$u_1(x, y, t) = 0, \quad u_2(x, y, t) = 0 \quad \text{at} \quad y \to \infty, \quad (2.9)$$

together with

$$u_1(x, y, t) = 0, \quad u_2(x, y, t) = 0 \quad \text{at} \quad x = 0, \quad x = L. \quad (2.10)$$

Where $U$ is a constant. In addition, the following initial conditions hold
such that \((u_{10}(x,y), u_{20}(x,y) )\) corresponds to the vector of initial velocities. Note that the domain is given as \(\Omega = \{(x,y) \in [0,L] \times [0,\infty)\}\).

In addition, consider that the shear stress is zero at \(y = 0\). As a consequence, the following holds

\[
\tau_{xy} = \nu \frac{\partial u_1}{\partial y} \left[ 1 + \Gamma^2 \left( \frac{\partial u_1}{\partial y} \right)^2 \right]^{\frac{n-1}{2}} = 0,
\]

leading to \(\frac{\partial u_1}{\partial y} = 0\) at \(y = 0\).

### 3 Preliminaries and Statement of Results

#### 3.1 Previous Results

Consider the well-known norm \(\|\cdot\|_{L^p}\) in the Lebesgue functional space \(L^p(\Omega)\) together with the usual Sobolev order \(m\) functional space defined by

\[
H^m(\Omega) = \left\{ u \in L^2(\Omega) : \nabla^m(u) \in L^2(\Omega) \right\}
\]

with the norm

\[
\|u\|_{H^m} = \left( \|u\|^2_{L^2} + \|\nabla^m u\|^2_{L^2} \right)^{\frac{1}{2}}.
\]

In addition, the following lemma is also needed (refer to Lemma 1 in [23])

**Lemma 1** The following anisotropic Sobolev inequality holds

\[
\iint_{\Omega} |fgh|dx\,dy \leq C_0 \|f\|_{L^2} \|g\|_{L^2}^\frac{1}{2} \left\| \frac{\partial g}{\partial x} \right\|_{L^2}^\frac{1}{2} \left\| \frac{\partial h}{\partial y} \right\|_{L^2}^\frac{1}{2} \left\| \frac{\partial h}{\partial y} \right\|_{L^2}^\frac{1}{2}.
\]

#### 3.2 Statement of Results

The main results are stated as follows:

**Theorem 3.1** Assuming \((u_{10}, u_{20}) \in H^1(\Omega)\), where \(\left|\nabla u_0\right| = \left|\frac{\partial u_0}{\partial y}\right|\), the equation (2.7) has solutions on \((0, T)\) in the defined strip \(\Omega = [0, L] \times [0, \infty)\).

**Theorem 3.2** Assuming \((u_{10}, u_{20}) \in L^4(\Omega)\), then the system (2.7) has solutions on \((0, T)\) in the defined strip \(\Omega = [0, L] \times [0, \infty)\).
Theorem 3.3 Assuming \((u_{10}, u_{20}) \in H^2(\Omega)\), then the system (2.7) has solutions on \((0, T]\) when \(\frac{\partial u}{\partial x} \in L^4(\Omega)\) and \(\left(\frac{\partial u}{\partial y}, \Delta u_2\right) \in L^2(\Omega)\), in the defined strip \(\Omega = [0, L] \times [0, \infty)\).

4 Proof of Theorem 3.1

Firstly, the following proposition is required to be shown

Proposition 4.1 Assume \(u_1\) is a solution departing from \(u_{10}\) to the set of the equation and conditions (2.7) to (2.11). Neglecting the higher powers of \(\Gamma^4\), then \(u_1(x, y, t)\) satisfies

\[
\begin{align*}
\sup_{0 \leq t \leq T} \left\| u_1 \right\|_{L^2}^2 + 2v \int_0^T \left\| \frac{\partial u_1}{\partial y} \right\|_{L^2}^2 dt + (n-1)v\Gamma^2 \int_0^T \left\| \frac{\partial u_1}{\partial y} \right\|_{L^4}^4 dt + \frac{1}{10}(n-3)(2n-5)v\Gamma^4 \int_0^T \left\| \frac{\partial u_1}{\partial y} \right\|_{L^6}^6 dt \leq \tilde{C}_1 \left\| u_{10} \right\|_{L^2}^2,
\end{align*}
\]

where \(\tilde{C}_1\) depends on a suitable constant \(M\) (to be defined in the proof) and \(T\).

Proof Multiplying the Eq. (2.7) by \(u_1\), operating and neglecting higher power of \(\Gamma^4\), the following holds

\[
\iint_{\Omega} u_1 \frac{\partial u_1}{\partial t} + I_1 = v \iint_{\Omega} \frac{\partial^2 u_1}{\partial y^2} u_1 dxdy + \frac{3}{2}(n-1)v\Gamma^2 \iint_{\Omega} \frac{\partial^2 u_1}{\partial y^2} \left( \frac{\partial u_1}{\partial y} \right)^2 u_1 dxdy + \frac{1}{4}(n-3)(2n-5)v\Gamma^4 \iint_{\Omega} \frac{\partial^2 u_1}{\partial y^2} \left( \frac{\partial u_1}{\partial y} \right)^4 u_1 dxdy
\]

\[- M^2 \iint_{\Omega} u_1^2 dxdy,
\]

After using integration by parts

\[
\frac{d}{dt} \left\| u_1 \right\|_{L^2}^2 = 2I_2 - 2v \left\| \frac{\partial u_1}{\partial y} \right\|_{L^2}^2 - (n-1)v\Gamma^2 \left\| \frac{\partial u_1}{\partial y} \right\|_{L^4}^4 \]

\[- \frac{1}{10}(n-3)(2n-5)v\Gamma^4 \left\| \frac{\partial u_1}{\partial y} \right\|_{L^6}^6 - 2M^2 \left\| u_1 \right\|_{L^2}^2,
\]

where

\[
I_1 = - \iint_{\Omega} u_1 \left( u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} \right) dxdy.
\]
Integrating by parts
\[ I_1 = \frac{1}{2} \int_{\Omega} u_1^2 \frac{\partial u_2}{\partial y} \, dx dy. \]

After using Eq. (2.1), the following reads
\[ I_1 = -\frac{1}{2} \int_{\Omega} u_1^2 \frac{\partial u_1}{\partial x} \, dx dy. \]

Integrating again we get \((I_1 = 0)\). Introducing the value of \(I_1\) into Eq. (4.1) and after Young’s inequality, the following applies
\[
\frac{d}{dt} \left\| u_1 \right\|_{L^2}^2 + 2\nu \left\| \frac{\partial u_1}{\partial y} \right\|_{L^2}^2 + (n - 1)\nu \Gamma^2 \left\| \frac{\partial u_1}{\partial y} \right\|_{L^4}^4 \\
+ \frac{1}{10} (n-3)(2n-5)\nu \Gamma^4 \left\| \frac{\partial u_1}{\partial y} \right\|_{L^6}^6 = -2M^2 \left\| u_1 \right\|_{L^2}^2
\]
\[
\leq \left| 2M^2 \right| \left\| u_1 \right\|_{L^2}^2.
\]

The Gronwall’s inequality yields
\[
\sup_{0 \leq t \leq T} \left\| u_1 \right\|_{L^2}^2 + 2\nu \int_0^T \left\| \frac{\partial u_1}{\partial y} \right\|_{L^2}^2 \, dt + (n - 1)\nu \Gamma^2 \int_0^T \left\| \frac{\partial u_1}{\partial y} \right\|_{L^4}^4 \, dt \\
+ \frac{1}{10} (n-3)(2n-5)\nu \Gamma^4 \int_0^T \left\| \frac{\partial u_1}{\partial y} \right\|_{L^6}^6 \, dt \leq \tilde{C}_1 \left\| u_{10} \right\|_{L^2}^2,
\]
where \(\tilde{C}_1\) depends on \(M\) and \(T\). \(\square\)

In addition, the following Proposition is also required.

**Proposition 4.2** Assume a solution \(u_1\) to set of the equation and conditions (2.7)–(2.11) departing from \(u_{10}\). Then, neglecting higher powers of \(\Gamma^4\), \(\frac{\partial u_1}{\partial y}\) satisfies
\[
\sup_{0 \leq t \leq T} \left\| \frac{\partial u_1}{\partial y} \right\|_{L^2}^2 + 2\nu \int_0^T \left\| \frac{\partial^2 u_1}{\partial y^2} \right\|_{L^2}^2 \, dt + 3(n - 1)\nu \Gamma^2 \int_0^T \left\| \frac{\partial^2 u_1}{\partial y^2} \frac{\partial u_1}{\partial y} \right\|_{L^2}^2 \, dt \\
+ \frac{1}{2} (n-3)(2n-5)\nu \Gamma^4 \int_0^T \left\| \frac{\partial^2 u_1}{\partial y^2} \left( \frac{\partial u_1}{\partial y} \right) \right\|_{L^2}^2 \, dt \leq \tilde{C}_2 \left\| \frac{\partial u_{10}}{\partial y} \right\|_{L^2}^2,
\]
where \(\tilde{C}_2\) depends on \(M\) and \(T\).

**Proof** Multiplying (2.7) by \(-\frac{\partial^2 u_1}{\partial y^2}\) and integrating by parts:
\[ \frac{1}{2} \frac{d}{dt} \left\| \frac{\partial u_1}{\partial y} \right\|_{L^2}^2 = \iint_{\Omega} \frac{\partial^2 u_1}{\partial y^2} (V \cdot \nabla u_1) \, dx \, dy - \nu \iint_{\Omega} \left( \frac{\partial^2 u_1}{\partial y^2} \right)^2 \left[ 1 + \Gamma^2 \left( \frac{\partial u_1}{\partial y} \right)^2 \right] \frac{\partial u_1}{\partial y} \, dx \, dy \]

\[ - \nu (n-1) \Gamma^2 \iint_{\Omega} \left( \frac{\partial^2 u_1}{\partial y^2} \right)^2 \left( \frac{\partial u_1}{\partial y} \right)^2 \left[ 1 + \Gamma^2 \left( \frac{\partial u_1}{\partial y} \right)^2 \right] \frac{\partial u_1}{\partial y} \, dx \, dy \]

\[ - M^2 \iint_{\Omega} \frac{\partial^2 u_1}{\partial y^2} \, dx \, dy. \]

Expanding the term on the right side and neglecting the higher powers of \( \Gamma^4 \)

\[ \frac{1}{2} \frac{d}{dt} \left\| \frac{\partial u_1}{\partial y} \right\|_{L^2}^2 = I_2 - \nu \iint_{\Omega} \left( \frac{\partial^2 u_1}{\partial y^2} \right)^2 \, dx \, dy - \frac{3}{2} (n-1) \nu \Gamma^2 \iint_{\Omega} \left( \frac{\partial^2 u_1}{\partial y^2} \right)^2 \left( \frac{\partial u_1}{\partial y} \right)^2 \, dx \, dy \]

\[ - \frac{1}{4} (n-3)(2n-5) \nu \Gamma^4 \iint_{\Omega} \left( \frac{\partial^2 u_1}{\partial y^2} \right)^2 \left( \frac{\partial u_1}{\partial y} \right)^4 \, dx \, dy \]

\[ - M^2 \iint_{\Omega} \left( \frac{\partial u_1}{\partial y} \right)^2 \, dx \, dy. \]  

\text{(4.2)}

where

\[ I_2 = \iint_{\Omega} \frac{\partial^2 u_1}{\partial y^2} (V \cdot \nabla u_1) \, dx \, dy \]

\[ = \iint_{\Omega} \frac{\partial^2 u_1}{\partial y^2} u_1 \frac{\partial u_1}{\partial x} \, dx \, dy + \iint_{\Omega} \frac{\partial^2 u_1}{\partial y^2} u_2 \frac{\partial u_1}{\partial y} \, dx \, dy. \]

Integrating \( I_2 \),

\[ I_2 = - \iint_{\Omega} u_1 \frac{\partial u_1}{\partial y} \frac{\partial^2 u_1}{\partial x \partial y} \, dx \, dy - \iint_{\Omega} \left( \frac{\partial u_1}{\partial y} \right)^2 \frac{\partial u_1}{\partial x} \, dx \, dy - \frac{1}{2} \iint_{\Omega} \left( \frac{\partial u_1}{\partial y} \right)^2 \frac{\partial u_2}{\partial y} \, dx \, dy \]

\[ = - \iint_{\Omega} u_1 \frac{\partial u_1}{\partial y} \frac{\partial^2 u_1}{\partial x \partial y} \, dx \, dy - \iint_{\Omega} \left( \frac{\partial u_1}{\partial y} \right)^2 \frac{\partial u_1}{\partial x} \, dx \, dy + \frac{1}{2} \iint_{\Omega} \left( \frac{\partial u_1}{\partial y} \right)^2 \frac{\partial u_1}{\partial x} \, dx \, dy \]

\[ = - \iint_{\Omega} u_1 \frac{\partial u_1}{\partial y} \frac{\partial^2 u_1}{\partial x \partial y} \, dx \, dy - \frac{1}{2} \iint_{\Omega} \left( \frac{\partial u_1}{\partial y} \right)^2 \frac{\partial u_1}{\partial x} \, dx \, dy, \]

where (2.1) has been employed. Now, integrating again we get \( I_2 = 0 \). Introducing \( I_2 \) in Eq. (4.2) and applying Young’s inequality, the following holds
\[
\frac{1}{2} \frac{d}{dt} \left\| \frac{\partial u_1}{\partial y} \right\|_{L^2}^2 \leq -\nu \left\| \frac{\partial^2 u_1}{\partial y^2} \right\|_{L^2}^2 - \frac{3}{2} (n-1) \nu \Gamma^2 \left\| \frac{\partial^2 u_1}{\partial y^2} \frac{\partial u_1}{\partial y} \right\|_{L^2}^2 \\
- \frac{1}{4} (n-3)(2n-5) \nu \Gamma^4 \int_{\Omega} \left\| \frac{\partial^2 u_1}{\partial y^2} \left( \frac{\partial u_1}{\partial y} \right) \right\|_{L^2}^2 dt - M^2 \left\| \frac{\partial u_1}{\partial y} \right\|_{L^2}^2,
\]
which implies that
\[
\frac{d}{dt} \left\| \frac{\partial u_1}{\partial y} \right\|_{L^2}^2 + 2\nu \left\| \frac{\partial^2 u_1}{\partial y^2} \right\|_{L^2}^2 + 3(n-1) \nu \Gamma^2 \left\| \frac{\partial^2 u_1}{\partial y^2} \frac{\partial u_1}{\partial y} \right\|_{L^2}^2 \\
+ \frac{1}{2} (n-3)(2n-5) \nu \Gamma^4 \int_{\Omega} \left\| \frac{\partial^2 u_1}{\partial y^2} \left( \frac{\partial u_1}{\partial y} \right) \right\|_{L^2}^2 dt \leq -2M^2 \left\| \frac{\partial u_1}{\partial y} \right\|_{L^2}^2.
\]

Now, applying Gronwall’s inequality
\[
\sup_{0 \leq t \leq T} \left\| \frac{\partial u_1}{\partial y} \right\|_{L^2}^2 + 2\nu \int_0^T \left\| \frac{\partial^2 u_1}{\partial y^2} \right\|_{L^2}^2 dt + 3(n-1) \nu \Gamma^2 \int_0^T \left\| \frac{\partial^2 u_1}{\partial y^2} \frac{\partial u_1}{\partial y} \right\|_{L^2}^2 dt \\
+ \frac{1}{2} (n-3)(2n-5) \nu \Gamma^4 \int_0^T \left\| \frac{\partial^2 u_1}{\partial y^2} \left( \frac{\partial u_1}{\partial y} \right) \right\|_{L^2}^2 dt \leq \tilde{C}_2 \left\| \frac{\partial u_{10}}{\partial y} \right\|_{L^2}^2,
\]
where \( \tilde{C}_2 \) depends on \( M \) and \( T \).

Finally, note that the Theorem 3.1 is proved by using Propositions 4.1 and 4.2.

## 5 Proof of Theorem 3.2

To this end, the following Proposition is required

**Proposition 5.1** Assume \( u_1 \) is a solution departing from \( u_{10} \) to the set of the equation and conditions (2.7) to (2.11). Neglecting the higher powers of \( \Gamma^4 \), then \( u_1(x,y,t) \) satisfies
\[
\sup_{0 \leq t \leq T} \left\| u_1 \right\|_{L^2}^4 + \nu \int_0^T \left\| \frac{\partial u_1}{\partial y} \right\|_{L^2}^2 dt + (n-1) \nu \Gamma^2 \int_0^T \left\| \frac{\partial^2 u_1}{\partial y^2} \right\|_{L^2}^4 dt \\
+ \frac{1}{10} (n-3)(2n-5) \nu \Gamma^4 \int_0^T \left\| \frac{\partial^2 u_1}{\partial y^2} \left( \frac{\partial u_1}{\partial y} \right) \right\|_{L^2}^6 dt \leq \tilde{C}_3 \left\| u_{10} \right\|_{L^4}^4,
\]
where \( \tilde{C}_3 \) depends on \( M \) and \( T \).
Proof Multiplying the Eq. (2.7) by $u_1^3$, operating, neglecting higher power of $\Gamma^4$, and after using integration by parts

$$
\iint_{\Omega} u_1^3 \frac{du_1}{dt} + I_3 = \nu \iint_{\Omega} \frac{\partial^2 u_1}{\partial y^2} u_1^3 dx dy + \frac{3}{2} (n-1) \nu \Gamma^2 \iint_{\Omega} \left( \frac{\partial u_1}{\partial y} \right)^2 u_1^3 dx dy
$$

$$
+ \frac{1}{4} (n-3)(2n-5) \nu \Gamma^4 \iint_{\Omega} \frac{\partial^2 u_1}{\partial y^2} \left( \frac{\partial u_1}{\partial y} \right)^4 u_1^3 dx dy
$$

$$
- M^2 \iint_{\Omega} u_1^4 dx dy.
$$

After using integration by parts

$$
\frac{d}{dt} \left\| u_1 \right\|_{L^4}^4 = 4I_3 - 12 \nu \left\| u_1 \frac{\partial u_1}{\partial y} \right\|_{L^2}^2 - 6(n-1) \nu \Gamma^2 \left\| \frac{1}{2} \frac{\partial u_1}{\partial y} \right\|_{L^4}^4
$$

$$
- \frac{3}{5} (n-3)(2n-5) \nu \Gamma^4 \left\| u_1 \frac{1}{2} \frac{\partial u_1}{\partial y} \right\|_{L^6}^6 - 4M^2 \left\| u_1 \right\|_{L^2}^4,
$$

where

$$
I_3 = \iint_{\Omega} u_1^4 \frac{\partial u_1}{\partial x} dx dy + \iint_{\Omega} u_1^3 u_2 \frac{\partial u_1}{\partial y} dx dy.
$$

Applying integration by parts on the second term in right side and making use of Eq. (2.1)

$$
I_3 = \frac{1}{2} \iint_{\Omega} u_1^4 \frac{\partial u_1}{\partial x} dx dy.
$$

Integrating again we get ($I_3 = 0$). Introducing the value of $I_3$ in Eq. (5.1) and after Young’s inequality

$$
\frac{d}{dt} \left\| u_1 \right\|_{L^4}^4 + \nu \left\| u_1 \frac{\partial u_1}{\partial y} \right\|_{L^2}^2 + (n-1) \nu \Gamma^2 \left\| \frac{1}{2} \frac{\partial u_1}{\partial y} \right\|_{L^4}^4
$$

$$
+ \frac{1}{10} (n-3)(2n-5) \nu \Gamma^4 \left\| u_1 \frac{1}{2} \frac{\partial u_1}{\partial y} \right\|_{L^6}^6 = -2M^2 \left\| u_1 \right\|_{L^4}^4
$$

$$
\leq \left| 2M^2 \left\| u_1 \right\|_{L^4}^4.
$$

The Grownwall’s inequality yields
\[
\sup_{0 \leq t \leq T} \left\| u_1 \right\|_{L^4}^4 + \nu \int_0^T \left\| \frac{\partial u_1}{\partial t} \right\|_{L^2}^2 dt + (n - 1)\nu \Gamma^2 \int_0^T \left\| \frac{\partial u_1}{\partial y} \right\|_{L^2}^4 dt \\
+ \frac{1}{10} (n - 3)(2n - 5)\nu \Gamma^4 \int_0^T \left\| \frac{\partial^2 u_1}{\partial y^2} \right\|_{L^6}^6 dt \leq \tilde{C}_3 \| u_{10} \|_{L^4}^4,
\]
where \( \tilde{C}_3 \) depends on \( M \) and \( T \).

Note that the Theorem 3.2 is shown as per the Proposition 5.1 introduced and proved.

6 Proof of Theorem 3.3

The Theorem is shown based on the coming Propositions.

Proposition 6.1 Assume \( u_1 \) is a solution departing from \( u_{10} \) to the set of the equation and conditions (2.7) to (2.11). In addition, assume the existence of \( \nabla u_{10} \) in \( L^2(\Omega) \) and that \( \frac{\partial u_1}{\partial x} \in L^4(\Omega) \). Neglecting the higher power of \( \Gamma^4 \) then

\[
\sup_{0 \leq t \leq T} \left\| \nabla u_1 \right\|_{L^2}^2 + (2\nu - \epsilon) \int_0^T \left\| \frac{\partial \nabla u_1}{\partial y} \right\|_{L^2}^2 dt + 3(n - 1)\nu \Gamma^2 \int_0^T \left\| \frac{\partial u_1}{\partial y} \frac{\partial \nabla u_1}{\partial y} \right\|_{L^2}^2 dt \\
+ \frac{1}{2} (n - 3)(2n - 5)\nu \Gamma^4 \int_0^T \left( \frac{\partial u_1}{\partial y} \right)^2 \left\| \nabla u_1 \right\|_{L^2}^2 dt \leq \tilde{C}_4 \| \nabla u_{10} \|_{L^2}^2,
\]
where \( \tilde{C}_4 \) depends on \( M \) and \( T \).

Proof Considering the inner product in Eq. (2.7) with \( \Delta u_1 \) and integrating

\[
\iint_{\Omega} \frac{\partial u_1}{\partial t} \Delta u_1 dx dy + \iint_{\Omega} \Delta u_1 \left( u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_2}{\partial y} \right) dx dy \\
= \nu \iint_{\Omega} \frac{\partial^2 u_1}{\partial y^2} \left[ 1 + \Gamma^2 \left( \frac{\partial u_1}{\partial y} \right)^2 \right]^2 \Delta u_1 dx dy \\
+ \nu(n - 1)\Gamma^2 \iint_{\Omega} \frac{\partial^2 u_1}{\partial y^2} \left( \frac{\partial u_1}{\partial y} \right)^2 \left[ 1 + \Gamma^2 \left( \frac{\partial u_1}{\partial y} \right)^2 \right]^2 \Delta u_1 dx dy \\
- M^2 \iint_{\Omega} u_1 \Delta u_1 dx dy.
\]
Expanding the term on the right side in the last equation and neglecting the higher powers of $\Gamma^4$,

$$\iint_\Omega \frac{\partial u_1}{\partial t} \Delta u_1 \, dx \, dy + I_4 = \nu \iint_\Omega \frac{\partial^2 u_1}{\partial y^2} \Delta u_1 \, dx \, dy$$

$$+ \frac{3}{2} (n - 1) \nu \Gamma^2 \iint_\Omega \frac{\partial^2 u_1}{\partial y^2} \left( \frac{\partial u_1}{\partial y} \right)^2 \Delta u_1 \, dx \, dy$$

$$+ \frac{1}{4} (n - 3)(2n - 5) \nu \Gamma^4 \iint_\Omega \frac{\partial^2 u_1}{\partial y^2} \left( \frac{\partial u_1}{\partial y} \right)^4 \Delta u_1 \, dx \, dy$$

$$- M^2 \iint_\Omega u_1 \Delta u_1 \, dx \, dy.$$ 

Aplying integration by parts

$$\frac{1}{2} \frac{d}{dt} \| \nabla u_1 \|_{L^2}^2 - I_4$$

$$= -\nu \iint_\Omega \left( \frac{\partial \nabla u_1}{\partial y} \right)^2 \, dx \, dy - \frac{3}{2} (n - 1) \nu \Gamma^2 \iint_\Omega \left( \frac{\partial u_1}{\partial y} \right)^2 \left( \frac{\partial \nabla u_1}{\partial y} \right)^2 \, dx \, dy$$

$$- \frac{1}{4} (n - 3)(2n - 5) \nu \Gamma^4 \iint_\Omega \left( \frac{\partial u_1}{\partial y} \right)^4 \left( \frac{\partial \nabla u_1}{\partial y} \right)^2 \, dx \, dy - M^2 \| \nabla u_1 \|_{L^2}^2,$$

which implies that

$$\frac{1}{2} \frac{d}{dt} \| \nabla u_1 \|_{L^2}^2 + \nu \left\| \nabla u_1 \right\|_{L^2}^2 + \frac{3}{2} (n - 1) \nu \Gamma^2 \left\| \frac{\partial u_1}{\partial y} \frac{\partial \nabla u_1}{\partial y} \right\|_{L^2}^2$$

$$+ \frac{1}{4} (n - 3)(2n - 5) \nu \Gamma^4 \left\| \frac{\partial u_1}{\partial y} \right\|_{L^2}^2 \left\| \frac{\partial \nabla u_1}{\partial y} \right\|_{L^2}^2 = I_4 - M^2 \| \nabla u_1 \|_{L^2}^2,$$ 

(6.1)

where

$$I_4 = \iint_\Omega \Delta u_1 \left( u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} \right) \, dx \, dy.$$ 

Now, the intention is to further develop the integral $I_4$, to this end.
From Eq. (2.1), the following holds

\[ I_4 = - \int_{\Omega} \nabla u_1 \nabla \left( u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} \right) dx dy \]

\[ = - \int_{\Omega} \left( \frac{\partial u_1}{\partial x} \right)^3 dx dy - \int_{\Omega} u_1 \frac{\partial u_1}{\partial x} \frac{\partial^2 u_1}{\partial x^2} dx dy - \int_{\Omega} \frac{\partial u_1}{\partial x} \frac{\partial u_1}{\partial y} \frac{\partial u_2}{\partial x} dx dy \]

\[ - \int_{\Omega} u_2 \frac{\partial^2 u_1}{\partial x \partial y} \frac{\partial u_1}{\partial x} dx dy - \int_{\Omega} \left( \frac{\partial u_1}{\partial y} \right)^2 \frac{\partial u_1}{\partial y} dx dy - \int_{\Omega} \frac{\partial u_1}{\partial y} \frac{\partial^2 u_1}{\partial y^2} dx dy, \]

\[ = - \frac{1}{2} \int_{\Omega} \left( \frac{\partial u_1}{\partial x} \right)^3 dx dy - \int_{\Omega} u_1 \frac{\partial u_1}{\partial x} \frac{\partial u_2}{\partial y} dx dy \]

\[ - \frac{1}{2} \int_{\Omega} \left( \frac{\partial u_1}{\partial y} \right)^2 \frac{\partial u_1}{\partial x} dx dy - \frac{1}{2} \int_{\Omega} \left( \frac{\partial u_2}{\partial y} \right)^2 \frac{\partial u_2}{\partial x} dx dy. \]

From Eq. (2.1), the following holds

\[ I_4 = - \frac{1}{2} \int_{\Omega} \left( \frac{\partial u_1}{\partial x} \right)^3 dx dy + \int_{\Omega} \frac{\partial u_2}{\partial y} \frac{\partial u_1}{\partial x} \frac{\partial u_2}{\partial x} dx dy. \]

Integrating again

\[ I_4 = - \frac{1}{2} \int_{\Omega} \left( \frac{\partial u_1}{\partial x} \right)^3 dx dy - \int_{\Omega} u_1 \frac{\partial^2 u_2}{\partial y^2} \frac{\partial u_2}{\partial x} dx dy - \int_{\Omega} \frac{\partial u_1}{\partial y} \frac{\partial u_2}{\partial x} dx dy \]

\[ = - \frac{1}{2} \int_{\Omega} \left( \frac{\partial u_1}{\partial x} \right)^3 dx dy - \int_{\Omega} u_1 \frac{\partial^2 u_2}{\partial y^2} \frac{\partial u_2}{\partial x} dx dy + \frac{1}{2} \int_{\Omega} \left( \frac{\partial u_2}{\partial y} \right)^2 \frac{\partial u_1}{\partial x} dx dy \]

\[ = - \int_{\Omega} u_1 \frac{\partial^2 u_1}{\partial x \partial y} \frac{\partial u_2}{\partial x} dx dy, \]

where Eq. (2.1) has been used, therefore \( I_4 \) becomes
After integration by parts, the following reads

\[ (2.7) \]

Introducing the assessed integral \( I_4 \) into Eq. (6.1) and after using Proposition 3 and \( \frac{\partial u_2}{\partial x} \in L^4(\Omega) \)

\[
\frac{d}{dt} \left\| \nabla u_1 \right\|_{L^2}^2 + (2\nu - \epsilon) \left\| \frac{\partial \nabla u_1}{\partial y} \right\|_{L^2}^2 + 3(n - 1)\nu \Gamma^2 \left\| \frac{\partial u_1}{\partial y} \frac{\partial \nabla u_1}{\partial y} \right\|_{L^2}^2 \\
+ \frac{1}{2} (n - 3)(2n - 5)\nu \Gamma^4 \int_0^T \left\| \frac{\partial u_1}{\partial y} \frac{\partial \nabla u_1}{\partial y} \right\|_{L^2}^2 \, dt \leq C_4 \left\| \nabla u_{10} \right\|_{L^2}^2,
\]

From Grownwall’s inequality

\[
\sup_{0 \leq t \leq T} \left\| \nabla u_1 \right\|_{L^2}^2 + (2\nu - \epsilon) \int_0^T \left\| \frac{\partial \nabla u_1}{\partial y} \right\|_{L^2}^2 \, dt + 3(n - 1)\nu \Gamma^2 \int_0^T \left\| \frac{\partial u_1}{\partial y} \frac{\partial \nabla u_1}{\partial y} \right\|_{L^2}^2 \, dt \\
+ \frac{1}{2} (n - 3)(2n - 5)\nu \Gamma^4 \int_0^T \left\| \frac{\partial u_1}{\partial y} \frac{\partial \nabla u_1}{\partial y} \right\|_{L^2}^2 \, dt \leq \sim C_4 \left\| \nabla u_{10} \right\|_{L^2}^2,
\]

where \( \sim C_4 \) depends on \( M \) and \( T \). \( \square \)

**Proposition 6.2** Assume \( u_1 \) is a solution departing from \( u_{10} \) to the set of the equation and conditions (2.7) to (2.11). Assume that in the strip \( \Omega \), there exists \( (\Delta u_{10}, \frac{\partial \nabla u_{10}}{\partial y}, \Delta u_2) \in L^2(\Omega) \). Neglecting the higher power of \( \Gamma^4 \), then

\[
\sup_{0 \leq t \leq T} \left\| \Delta u_1 \right\|_{L^2}^2 + 3(n - 1)\nu \Gamma^2 \int_0^T \left\| \frac{\partial u_1}{\partial y} \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^2 \, dt \\
+ \frac{1}{2} (n - 3)(2n - 5)\nu \Gamma^4 \int_0^T \left\| \frac{\partial u_1}{\partial y} \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^2 \, dt \\
+ 2(\nu - 8\epsilon) \int_0^T \left\| \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^2 \, dt \leq \sim C_5 \left\| \Delta u_{10} \right\|_{L^2}^2,
\]

where \( \sim C_5 \) depends on constants and \( T \).

**Proof** Apply the monotone operator \( \Delta \) to Eq. (2.7) and take inner product with \( \Delta u_1 \). After integration by parts, the following reads
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \iint_\Omega \Delta u_1 \frac{\partial u_1}{\partial t} dx dy &+ \iint_\Omega u_1 \Delta \left( u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} \right) dx dy \\
&= v \iint_\Omega \Delta \left( \frac{\partial^2 u_1}{\partial x^2} \right) \left[ 1 + \Gamma^2 \left( \frac{\partial u_1}{\partial y} \right)^2 \right]^{\frac{n-1}{2}} \Delta u_1 dx dy \\
&+ v(n-1)\Gamma^2 \iint_\Omega \Delta \left( \frac{\partial^2 u_1}{\partial y^2} \right) \left( \frac{\partial u_1}{\partial y} \right)^2 \left[ 1 + \Gamma^2 \left( \frac{\partial u_1}{\partial y} \right)^2 \right]^{\frac{n-3}{2}} \Delta u_1 dx dy \\
&- \frac{v\phi}{k} \iint_\Omega \Delta \left[ 1 + \Gamma^2 \left( \frac{\partial u_1}{\partial y} \right)^2 \right]^{\frac{n-1}{2}} u_1 \Delta u_1 dx dy - M^2 \iint_\Omega \Delta u_1 \Delta u_1 dx dy.
\end{align*}
\]

Expanding the terms on the right side and neglecting the higher powers of \(\Gamma^4\),

\[
\frac{1}{2} \frac{d}{dt} \iint_\Omega (\Delta u_1)^2 dx dy - I_5 = v \iint_\Omega \frac{\partial^2 u_1}{\partial y^2} \Delta u_1 dx dy \\
+ \frac{3}{2} (n-1)\Gamma^2 \iint_\Omega \Delta \left( \frac{\partial^2 u_1}{\partial y^2} \right) \left( \frac{\partial u_1}{\partial y} \right)^2 \Delta u_1 dx dy \\
+ \frac{1}{4} (n-3)(2n-5)\Gamma^4 \iint_\Omega \Delta \left( \frac{\partial^2 u_1}{\partial y^2} \right)^4 \Delta u_1 dx dy - M^2 \iint_\Omega \Delta u_1 dx dy.
\]

After Integration

\[
\frac{1}{2} \frac{d}{dt} \left\| \Delta u_1 \right\|_{L^2}^2 = I_5 - v \left\| \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^2 + \frac{3}{2} (n-1)\Gamma^2 I_6 \\
+ \frac{1}{4} (n-3)(2n-5)\Gamma^4 I_7 - M^2 \left\| \Delta u_1 \right\|_{L^2}^2,
\]

where

\[
I_5 = - \iint_\Omega \Delta \left( u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} \right) \Delta u_1 dx dy, \\
I_6 = \iint_\Omega \Delta \left( \frac{\partial^2 u_1}{\partial y^2} \left( \frac{\partial u_1}{\partial y} \right)^2 \right) \Delta u_1 dx dy, \\
I_7 = \iint_\Omega \Delta \left( \frac{\partial^2 u_1}{\partial y^2} \left( \frac{\partial u_1}{\partial y} \right)^4 \right) \Delta u_1 dx dy.
\]

Now for \(I_5\)
\[ I_5 = - \int \limits_\Omega \Delta \left( u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} \right) \Delta u_1 \, dx \, dy \]

\[ = - \int \limits_\Omega \left( \frac{\partial u_1}{\partial x} \left( \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} \right) + \frac{\partial u_1}{\partial y} \left( \frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} \right) \right) \Delta u_1 \, dx \, dy \]

\[ - \int \int \left( 2 \frac{\partial u_1}{\partial x} \frac{\partial^2 u_1}{\partial x^2} + 2 \frac{\partial u_1}{\partial y} \frac{\partial^2 u_1}{\partial y^2} + 2 \frac{\partial u_2}{\partial x} \frac{\partial^2 u_1}{\partial x \partial y} + 2 \frac{\partial u_2}{\partial y} \frac{\partial^2 u_1}{\partial y^2} \right) \Delta u_1 \, dx \, dy \]

\[ - \int \int \left( u_1 \frac{\partial^3 u_1}{\partial x^3} + u_1 \frac{\partial^3 u_1}{\partial y^2 \partial x} + u_2 \frac{\partial^3 u_1}{\partial x^2 \partial y} + u_2 \frac{\partial^3 u_1}{\partial y^3} \right) \Delta u_1 \, dx \, dy. \]

Applying integration by parts and continuity

\[ \int \int \left( u_1 \frac{\partial^3 u_1}{\partial x^3} + u_1 \frac{\partial^3 u_1}{\partial y^2 \partial x} + u_2 \frac{\partial^3 u_1}{\partial x^2 \partial y} + u_2 \frac{\partial^3 u_1}{\partial y^3} \right) \Delta u_1 \, dx \, dy = 0. \]

Then

\[ I_5 = - \int \limits_\Omega \Delta u_1 \frac{\partial u_1}{\partial x} \, dx \, dy - \int \limits_\Omega \Delta u_1 \frac{\partial u_1}{\partial y} \, dx \, dy - 2 \int \limits_\Omega \Delta u_1 \frac{\partial^2 u_1}{\partial x \partial y} \, dx \, dy \]

\[ - 2 \int \int \left( \frac{\partial u_1}{\partial y} \frac{\partial^2 u_1}{\partial x \partial y} \right) \, dx \, dy - 2 \int \int \frac{\partial u_2}{\partial x} \frac{\partial^2 u_1}{\partial x \partial y} \, dx \, dy - 2 \int \int \frac{\partial u_2}{\partial y} \frac{\partial^2 u_1}{\partial y^2} \, dx \, dy \]

\[ = k_1 + k_2 + k_3 + k_4 + k_5 + k_6. \]

(6.2)

In order to solve the above six terms, the Lemma 1 is employed so that

\[ k_1 = - \int \limits_\Omega \Delta u_1 \frac{\partial u_1}{\partial x} \, dx \, dy \]

\[ \leq C_0 \left\| \Delta u_1 \right\|_{L^2} \left\| \Delta u_1 \right\|_{L^2} \left\| \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2} \left\| \frac{\partial u_1}{\partial x} \right\|_{L^2} \left\| \frac{\partial^2 u_1}{\partial x^2} \right\|_{L^2} \]

\[ = C_0 \left\| \Delta u_1 \right\|_{L^2} \left\| \Delta u_1 \right\|_{L^2} \left\| \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2} \left\| \frac{\partial u_1}{\partial x} \right\|_{L^2} \left\| \frac{\partial^2 u_2}{\partial x \partial y} \right\|_{L^2} \]

\[ \leq C_0 \left\| \Delta u_1 \right\|_{L^2} \left\| \Delta u_1 \right\|_{L^2} \left\| \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2} \left\| \frac{\partial u_1}{\partial x} \right\|_{L^2} \left\| \frac{\partial^2 u_2}{\partial y} \right\|_{L^2} \]

\[ \leq \epsilon \left\| \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^2 + C_\epsilon \left\| \frac{\partial u_1}{\partial x} \right\|_{L^2}^2 \left\| \Delta u_1 \right\|_{L^2} \left\| \frac{\partial \Delta u_2}{\partial y} \right\|_{L^2} \]

(6.3)
\[ k_2 = -\int_\Omega \Delta u_1 \frac{\partial u_1}{\partial y} \Delta u_2 \, dx \, dy \]

\[ \leq C_0 \| \Delta u_2 \|_{L^2} \| \Delta u_1 \|_{L^2} \| \Delta u_1 \|_{L^2} \| \Delta u_2 \|_{L^2} \| \Delta u_1 \|_{L^2} \| \Delta u_2 \|_{L^2} \| \Delta u_1 \|_{L^2} \| \partial u_1 \|_{L^2} \| \partial u_1 \|_{L^2} \| \partial^2 u_1 \|_{L^2} \| \partial u_1 \|_{L^2} \| \partial u_1 \|_{L^2} \| \partial^2 u_1 \|_{L^2} \]

\[ \leq \epsilon \left( \frac{\partial \Delta u_1}{\partial y} \right)^2 + C_\epsilon \| \partial u_1 \|_{L^2} \| \Delta u_1 \|_{L^2} \| \partial \Delta u_1 \|_{L^2} + C_\epsilon \| \Delta u_2 \|_{L^2}. \tag{6.4} \]

\[ k_3 = 2 \int_\Omega \Delta u_1 \frac{\partial u_2}{\partial y} \frac{\partial^2 u_1}{\partial x^2} \, dx \, dy \]

\[ \leq C_0 \| \frac{\partial^2 u_2}{\partial x^2} \|_{L^2} \| \partial \Delta u_1 \|_{L^2} \| \partial u_1 \|_{L^2} \| \partial u_2 \|_{L^2} \| \partial^2 u_2 \|_{L^2} \| \partial u_1 \|_{L^2} \| \partial u_1 \|_{L^2} \| \partial^2 u_1 \|_{L^2} \]

\[ \leq C_0 \| \partial u_1 \|_{L^2} \| \Delta u_1 \|_{L^2} \| \partial u_1 \|_{L^2} \| \partial u_1 \|_{L^2} \| \partial \Delta u_1 \|_{L^2} \| \partial \Delta u_1 \|_{L^2} \]

\[ \leq \epsilon \left( \frac{\partial \Delta u_1}{\partial y} \right)^2 + C_\epsilon \| \partial u_1 \|_{L^2} \| \Delta u_1 \|_{L^2} \| \partial \Delta u_1 \|_{L^2} + C_\epsilon \| \Delta u_2 \|_{L^2}. \tag{6.5} \]

\[ k_4 = -2 \int_\Omega \frac{\partial u_1}{\partial y} \frac{\partial^2 u_1}{\partial x \partial y} \Delta u_1 \, dx \, dy \]

\[ \leq C_0 \| \frac{\partial u_1}{\partial y} \|_{L^2} \| \Delta u_1 \|_{L^2} \| \partial u_1 \|_{L^2} \| \partial u_1 \|_{L^2} \| \partial^2 u_1 \|_{L^2} \| \partial^2 u_1 \|_{L^2} \| \partial^3 u_1 \|_{L^2} \]

\[ \leq C_0 \| \frac{\partial u_1}{\partial y} \|_{L^2} \| \Delta u_1 \|_{L^2} \| \partial u_1 \|_{L^2} \| \partial u_1 \|_{L^2} \| \partial \Delta u_1 \|_{L^2} \| \partial \Delta u_1 \|_{L^2} \]

\[ \leq \epsilon \left( \frac{\partial \Delta u_1}{\partial y} \right)^2 + C_\epsilon \| \partial u_1 \|_{L^2} \| \Delta u_1 \|_{L^2} \| \partial \Delta u_1 \|_{L^2} + C_\epsilon \| \partial \Delta u_1 \|_{L^2}. \tag{6.6} \]

\[ k_5 = -2 \int_\Omega \frac{\partial u_1}{\partial y} \frac{\partial^2 u_1}{\partial x \partial y} \Delta u_1 \, dx \, dy \]

\[ \leq C_0 \| \frac{\partial u_1}{\partial x} \|_{L^2} \| \Delta u_1 \|_{L^2} \| \partial u_1 \|_{L^2} \| \partial u_1 \|_{L^2} \| \partial^2 u_1 \|_{L^2} \| \partial^2 u_1 \|_{L^2} \| \partial^3 u_1 \|_{L^2} \]

\[ \leq C_0 \| \frac{\partial u_1}{\partial x} \|_{L^2} \| \Delta u_1 \|_{L^2} \| \partial u_1 \|_{L^2} \| \partial u_1 \|_{L^2} \| \partial \Delta u_1 \|_{L^2} \| \partial \Delta u_1 \|_{L^2} \]

\[ \leq \epsilon \left( \frac{\partial \Delta u_1}{\partial y} \right)^2 + C_\epsilon \| \partial u_1 \|_{L^2} \| \Delta u_1 \|_{L^2} \| \partial \Delta u_1 \|_{L^2} + C_\epsilon \| \partial \Delta u_1 \|_{L^2}. \tag{6.7} \]
\[ k_6 = -2 \int_{\Omega} \frac{\partial u_2}{\partial y} \frac{\partial^2 u_1}{\partial y^2} \Delta u_1 \, dx \, dy \]
\[ \leq C_0 \| \Delta u_1 \|_{L^2} \| \frac{\partial u_2}{\partial y} \|_{L^2}^{\frac{1}{2}} \| \frac{\partial^2 u_1}{\partial y^2} \|_{L^2}^{\frac{1}{2}} \| \frac{\partial^2 u_1}{\partial x^2} \|_{L^2}^{\frac{1}{2}} \| \frac{\partial^3 u_1}{\partial x^3} \|_{L^2}^{\frac{1}{2}} \]
\[ \leq C_0 \| \Delta u_1 \|_{L^2} \| \frac{\partial u_1}{\partial x} \|_{L^2} \| \frac{\partial \nabla u_2}{\partial y} \|_{L^2}^{\frac{1}{2}} \| \nabla \Delta u_1 \|_{L^2}^{\frac{1}{2}} \]
\[ \leq \epsilon \left\| \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^2 + C_\epsilon \left\| \frac{\partial u_1}{\partial x} \right\|_{L^2} \left( \frac{\partial \nabla u_1}{\partial y} \right) \left( \frac{\partial \Delta u_1}{\partial y} \right) \]
\[ \leq \epsilon \left\| \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^2 + C_\epsilon \left( \frac{\partial u_1}{\partial x} \right) \left( \frac{\partial \nabla u_1}{\partial y} \right) \left( \frac{\partial \Delta u_1}{\partial y} \right) \|_{L^2}^2 - \left\| \frac{\partial u_1}{\partial y} \right\|_{L^2}^2 - \left\| \frac{\partial u_1}{\partial y} \right\|_{L^2}^2 \]
\[ \text{Integrating } I_6, \text{ we have} \]
\[ I_6 = -2 \int_{\Omega} \frac{\partial u_2}{\partial y} \left( \frac{\partial \nabla u_1}{\partial y} \right)^2 \frac{\partial \Delta u_1}{\partial y} \, dx \, dy - \int_{\Omega} \left( \frac{\partial u_1}{\partial y} \right)^2 \left( \frac{\partial \Delta u_1}{\partial y} \right) \, dx \, dy \]
\[ \leq \epsilon \left\| \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^2 + C_\epsilon \left\| \frac{\partial u_1}{\partial x} \right\|_{L^2} \left( \frac{\partial \nabla u_1}{\partial y} \right) \left( \frac{\partial \Delta u_1}{\partial y} \right) \|_{L^2}^2 - \left\| \frac{\partial u_1}{\partial y} \right\|_{L^2}^2 \]
\[ \text{where we made use of the Young’s inequality. Similarly to solve } I_7 \]
\[ I_7 \leq \epsilon \left\| \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^2 + C_\epsilon \left( \frac{\partial u_1}{\partial x} \right) \left( \frac{\partial \nabla u_1}{\partial y} \right) \left( \frac{\partial \Delta u_1}{\partial y} \right) \|_{L^2}^2 \]
\[ \text{Combining Eq. (2.13) to Eq. (2.22), After using Proposition 1, Proposition 2, Proposition 3 and } \Delta u_2, \frac{\partial \Delta u_2}{\partial y} \in L^2(\Omega), \text{ we get} \]
\[ \frac{d}{dt} \left\| \Delta u_1 \right\|_{L^2}^2 + 3(n-1)\nu^{\frac{1}{2}} \left\| \frac{\partial u_1}{\partial y} \right\|_{L^2}^2 \]
\[ + \frac{1}{2} (n-3)(2n-5)\nu^{\frac{3}{2}} \left( \frac{\partial u_1}{\partial y} \right) \left( \frac{\partial \Delta u_1}{\partial y} \right) \|_{L^2}^2 \]
\[ + 2(\nu - 8e) \left\| \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^2 \leq C_1 \left\| \Delta u_1 \right\|_{L^2}^2, \]
\[ \text{where } C_1 \text{ is suitable constant. Applying Gronwall’s inequality again} \]
\[
\sup_{0 \leq t \leq T} \| \Delta u_1 \|_{L^2}^2 + \frac{3}{2} (n - 1) \nu T^2 \int_0^T \left\| \frac{\partial u_1}{\partial y} \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^2 dt \\
+ \frac{1}{4} (n - 3)(2n - 5) \nu T^4 \int_0^T \left\| \left( \frac{\partial u_1}{\partial y} \right)^2 \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^2 dt \\
+ (\nu - 8 \epsilon) \int_0^T \left\| \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^2 dt \leq \sim C_5 \| \Delta u_{10} \|_{L^2},
\]

where \( C_5 \) depends on suitable constants and \( T \).

Finally, the Theorem 3.3 is shown by using Proposition 6.1 and Proposition 6.2.

7 Conclusion

The proposed Theorems 3.1, 3.2 and 3.3 have been shown in the different supporting propositions. Such Theorems lead to confirm on the existence of regular solutions departing from an initial data generalized under the defined functional spaces. The shown solutions are applicable to the two dimensional Carreau fluid on the defined strip \( \Omega = [0, L] \times [0, \infty) \).

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Declarations

Conflict of Interest The authors have no competing interests to declare that are relevant to the content of this article.

Consent for Publication Authors declare the consent for manuscript publication.

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References

1. Grabski, J.K., Kołodziej, J.A.: Analysis of Carreau fluid flow between corrugated plates. Comput. Math. Appl. 72(6), 1501–1514 (2016)
2. Carreau, P.J.: Rheological equations from molecular network theories. Trans. Soc. Rheol. 116, 99–127 (1972)
3. Siska, B., Bendova, H., MacHac, I.: Terminal velocity of non-spherical particles falling through a Carreau model Fluid. Chem. Eng. Process. 44(12), 1312–1319 (2005)
4. Khan, M., Sardar, H., Gulzar, M., Alshomrani, A.: On multiple solutions of non-Newtonian Carreau fluid flow over an inclined shrinking sheet. Results Phys. 8, 926–932 (2018)
5. Chhabra, R.P., Uhlherr, P.H.T.: Creeping motion of spheres through shear-thinning elastic fluids described by the Carreau viscosity equation. Rheo Acta 19, 187–195 (1980)
6. Bush, M.B., Phan-Thein, N.: Drag force on a sphere in creeping motion through a Carreau model fluid. J. Non-Newton. Fluid Mech. 16, 303–313 (1984)
7. Hyun, Y.H., Lim, S.T., Choi, H.B., John, M.S.: Rheology of poly (ethylene oxide) organoclay nanocomposites. Macromolecules 34(23), 8084–8093 (2001)
8. Corradini, D.: Buffer additives other than the surfactant sodium dodecyl sulfate for protein separations by capillary electrophoresis. J. Chromatogr. B Biomed. Sci. Appl. 699, 221–256 (1997)
9. Heller, C.: Principles of DNA separation with capillary electrophoresis. Electrophoresis 22(4), 629–643 (2001)
10. Chhabra, R.P., Uhlherr, P.H.T.: Creeping motion of Spheres through shear-thinning elastic fluids described by the Carreau viscosity equation. Rheo. Aeta. 19, 187–95 (1980)
11. Bush, M.B., Phan-Thein, N.: Drag force on a sphere in creeping motion through a Carreau model fluid. J. Non-Newton. Fluid Mech. 16, 303–313 (1984)
12. Hsu, J.P., Yeh, S.J.: Drag on two coaxial rigid spheres moving in the axis of a cylinder filled with Carreau fluid. Powder Technol. 182, 56–71 (2008)
13. Uddin, J., Marston, J.O., Thoroddsen, S.T.: Squeeze flow of a Carreau fluid during sphere impact. Phys. Fluid. 24, 073104 (2012)
14. Shadid, J.N., Eckert, E.R.G.: Viscous heating of a cylinder with finite length by a heigh viscosity fluid in steady longitudinal flow. II. Non-Newtonian Carreau model fluids. Int. J. Heat Mass Transf. 35(27), 39–49 (1992)
15. Tshehla, M.S.: The flow of Carreau fluid down an incline with a free surface. Int. J. Phys. Sci. 6, 3896–3910 (2011)
16. Khan, M., Sardar, H.: On steady two-dimensional Carreau nanofluid flow in the presence of infinite shear rate viscosity. Can. J. Phys. 97(4), 400–407 (2019)
17. Khan, M.: Hashim, Boundary layer flow and heat transfer to Carreau fluid over a nonlinear stretching sheet. AIP Adv. 5(10), 107203 (2015)
18. Zhong, X.: A continuation principle to the cauchy problem of two-dimensional compressible Navier–Stokes equations with variable viscosity. Math. Phys. Anal. Geom. 23(4), (2020)
19. Fan, J., Jia, X., Nakamura, G., Zhou, Y.: on well-posedness and blow-up criteria for the magnetohydrodynamics with the Hall and ion-slip effect. Z. Angew. Math. Phys. 66(4), 1695–1706 (2015)
20. Fan, J., Zhou, Y.: Local well-posedness for the isentropic compressible MHD system with vacuum. J. Math. Phys. 62(5), 051505 (2021)
21. Kang, L., Xuewei, D., Quanyi, B.: Energy conservation for the nonhomogeneous incompressible ideal Hall-MHD equations. J. Math. Phys. 62(3), 031506 (2021)
22. Fan, J., Quansen, J., Yanqing, W., Yuelong, X.: Blow up criterion for the 2D full compressible Navier–Stokes equations involving temperature in critical spaces. J. Math. Phys. 62(5), 051503 (2021)
23. Cao, C., Wu, J.: Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion. Adv. Math. 226, 1803–1822 (2011)