Comment on the Bound State Problem in $N = 4$ Super Yang-Mills Theory

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Abstract

We re-examine the threshold bound state problem on the wrong sign Taub-Nut space; the metric on which describes the relative moduli space of well separated BPS monopoles. The quantum mechanics gives rise to a continuous family of threshold bound states, in distinction to the unique one found on the Atiyah-Hitchin metric.
1 Introduction

The quantum aspects of monopole dynamics has been under much study in the past few years to a large extent because of the evidence found for S-duality in $N = 4$ supersymmetric Yang-Mill theory. In particular, Sen first produced evidence [1] for the existence of a threshold bound state of magnetic charge two in a spontaneously broken $SU(2) N = 4$ supersymmetric Yang-Mills theory. Such states are found by looking for smooth normalizable ground states of the quantum mechanics model describing the slowly moving magnetic monopoles.

In this note we re-examine this problem in the limit where we decouple the massive vector boson while maintaining the monopole degree of freedom (i.e. $m_W = v g \to \infty$ and $m_m = v/g \to \text{constant}$). We point out that the quantum mechanics of point-like dyons is smooth, even in the case where the equations of motion governing the particles in this approximation are singular. We show that the Hamiltonian which arises within the low-energy approximation to the dynamics of point-like dyons possesses a one-parameter family of smooth ground states. This enlargement in the number of threshold bound states is surprising because the moduli space metric is singular.

2 Quantum Mechanics of Point-like dyons

The slow-motion dynamics of monopoles is governed by geodesic motion derived from the metrics on particular hyper-Kähler manifolds [2]; although in principle the metrics on these spaces are difficult to find explicitly, approximate forms describing the asymptotic regime may be found through a Lagrangian description [5]. Namely, in the limit of long-range interactions, where the effects of the $W$-bosons are neglected we may use the well known world-line formulation of point-like dyons to derive the forces (and corresponding metric) [3, 4]. In the following we reanalyze the classic work of Sen [1] in this approximation.

The two-monopole moduli space metric is uniquely determined to be the Atiyah-Hitchin metric [2]. Its asymptotic form limits to the double cover of the Taub-NUT metric, which possesses a tri-holomorphic isometry reflecting the fact that there is an additional charge conservation. The equations of
motion describing the time evolution of the zero modes is derived from the bosonic quantum mechanics problem,

\[ \mathcal{L} = \int dt \ g^{ab} \dot{x}_a \dot{x}_b . \]  

(1)

The Taub-NUT metric is explicitly:

\[ ds^2 = (1 - \frac{1}{r})d\vec{r} \cdot d\vec{r} + \frac{r}{r-1}(d\psi + \vec{\omega} \cdot d\vec{r})^2 , \]

(2)

which has a real singularity at \( r = 1/m_w \), the Compton wavelength of the \( w \)-boson (in the remainder of this note we absorb the mass parameter into the radial coordinate). Including the \( w \)-bosons, in which non-abelian monopoles are described, smooths out the singularity appearing in the charge-rotator degree of freedom; in this case one obtains the Atiyah-Hitchin metric \[1\].

The quantum mechanics describing the system \[1\] possesses states in one-to-one correspondence with the Hamiltonian found from the Lagrangian \[1\] \[7\]. We first introduce the vielbein basis

\[ v^0 = (1 - \frac{1}{r})^{1/2}dr \quad v^1 = r(1 - \frac{1}{r})^{1/2}\sigma_1 \]

(3)

\[ v^2 = r(1 - \frac{1}{r})^{1/2}\sigma_2 \quad v^3 = -(1 - \frac{1}{r})^{-1/2}\sigma_3 , \]

(4)

in which the metric may be expressed as \( g = \sum_{j=0}^{3} v^j \otimes v^j \). The Hamiltonian written in terms of these is simply

\[ H = d^i d^i + dd^i , \]

(5)

and the quantum states are in correspondence with the eigenfunctions of the Hamiltonian. In the following we examine only the 2-form eigenfunctions of the Hamiltonian.

In terms of \( v_i \), we make an ansatz for the self-dual 2-forms which are annihilated by \( H \) with

\[ \omega_i^\pm = F_i(r) \left( v^0 \wedge v^i \pm \frac{1}{2} \varepsilon_{ijk} v^j \wedge v^k \right) . \]

(6)
We first introduce the left-invariant 1-forms of $SO(3)$, $\sigma_j$. They satisfy $d\sigma_i = \frac{1}{2}\epsilon_{ijk}\sigma^j \wedge \sigma^k$ and have an explicit form:

$$\sigma_1 = -\sin \psi d\theta + \cos \psi \sin \theta d\phi$$  \hspace{1cm} (7)

$$\sigma_2 = \cos \psi d\theta + \sin \psi \sin \theta d\phi$$  \hspace{1cm} (8)

$$\sigma_3 = d\psi + \cos \theta d\phi$$  \hspace{1cm} (9)

On wrong-sign Taub-NUT we find the 2-form solutions to the Hamiltonian in (5),

$$\omega_1^+ = e^{-r}\{(r-1)dr \wedge \sigma_1 - r d\sigma_1\}$$  \hspace{1cm} (10)

$$\omega_1^- = \frac{1}{r^2}e^r\{(r-1)dr \wedge \sigma_1 + r d\sigma_1\}$$  \hspace{1cm} (11)

$$\omega_2^- = \frac{r^2}{(r-1)^2}\{dr \wedge \sigma_2 + r^2(1 - \frac{1}{r})d\sigma_2\}$$  \hspace{1cm} (12)

and

$$\omega_2^+ = \frac{1}{(r-1)^2}\{-dr \wedge \sigma_2 + r^2(1 - \frac{1}{r})d\sigma_2\}.$$  \hspace{1cm} (13)

The solutions to $\omega_3^\pm$ follow by replacing $\sigma_2$ with $\sigma_3$.

The solutions for $\omega_2^\pm$ are found by replacing $\sigma_1$ with $\sigma_2$ in $\omega_1^\pm$. All of the two-forms obey

$$\omega_i^{(\alpha)} \wedge \omega_j^{(\beta)} = 0,$$  \hspace{1cm} (14)

for $(i, \alpha) \neq (j, \beta)$. Only the solutions $\omega_1^+$ and $\omega_2^+$ are smooth and normalizable. They both may serve as zero-energy ground states of the Hamiltonian.

Both $\omega_1^+$ and $\omega_2^+$ are odd under $\psi \to \psi + \pi$. They would correspond to odd electrically charged states if interpreted as threshold bound states of the abelian monopole system. Further, under $\psi \to \psi + \theta$ they change as:
\[
\begin{pmatrix} \omega_1^+ \\ \omega_2^+ \end{pmatrix} \rightarrow \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \omega_1^+ \\ \omega_2^+ \end{pmatrix}
\]

(15)

The Hamiltonian (3) possesses a family of normalizable ground states of the form

\[
\omega_\alpha = \cos \theta \omega_1^+ + \sin \theta \omega_2^+ ,
\]

(16)

where both \( \omega_1^\pm \) are normalized to one. In a complex basis

\[
\omega = \omega_1^+ + i\omega_2^+ \quad \bar{\omega} = \omega_1^+ - i\omega_2^+ ,
\]

(17)

these states are written as

\[
\omega_\alpha = e^{i\alpha} \omega + e^{-i\alpha} \bar{\omega} .
\]

(18)

The harmonic 2-forms \( \omega^\pm \) both are odd under shifts of the coordinate \( \theta \) by \( \pi \). The total moduli space for the two monopoles has the form \( M_2 = R^3 \otimes S^2 \otimes \mathbb{Z}_2 \), where we have approximated to only the asymptotic region (i.e. the Taub-NUT metric). The complete wave function is found by tensoring the relative bound states \( \omega^\pm \) with the wave function on center of mass component \( R^3 \) of the total moduli space. We see, according to the analysis of Sen [1], that the states in (18) correspond to dyonic bound states with electric charge \( \pm 1 \): the negative (positive) electrically charged bound states are \( \omega^+ (\omega^-) \) as may be seen under a continuous rotation of the \( \theta \) coordinate.

The existence of these bound states is surprising for two reasons. First, the zero energy eigenstates are smooth and normalizable which means that despite the singularity on the wrong sign Taub-NUT the quantum mechanics is sensible. Second, the appearance of a \( U(1) \) set of bound states appears to be in contradiction with the \( SL(2,\mathbb{Z}) \) invariance of the \( N = 4 \) supersymmetric Yang-Mills spectrum.

The spectrum of \( N = 4 \) super Yang-Mills theory is invariant under \( SL(2,\mathbb{Z}) \) transformations, which takes a BPS soliton with magnetic and electric quantum numbers \( (m, n) \) to

\[
\begin{pmatrix} m \\ n \end{pmatrix} \rightarrow \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix}
\]

(19)
with \( pr - sq = 1 \). In the limit we are taking all of the BPS states satisfying the mass formula, \( m^2 = 2g|n_m v_D + n_e v|^2 (v_D = v/g) \), with non-vanishing electric quanta decouple. Under the \( S \)-type duality transformation the coupling constant gets mapped from \( g \rightarrow 1/g \) and the decoupled electric states change to purely magnetic ones with zero mass. Our \( U(1) \) set of bound states reflects a condensation of an infinite tower of massless magnetic states.

However, in \( N = 4 \) super Yang-Mills theory there are two distinct phases, namely \( v = 0 \) and \( v \neq 0 \). In the analysis of the threshold bound state problem originally done in [1], one works in the latter case: In the Atiyah-Hitchin metric there is only one mass scale and the (non-zero) vacuum value of the scalar field may be eliminated via a coordinate transformation. The threshold bound state calculated in [1] and its implications towards the \( SL(2, Z) \) duality symmetry of \( N = 4 \) super Yang-Mills theory is valid only in this case, as opposed to the pure abelian system considered here.

3 Discussion

We have re-examined the threshold bound state problem first analyzed by Sen [1], but for the abelian monopole system. In the two-monopole sector there arises a continuous family of bound states with magnetic charge two and electric charge \( \pm 1 \); such states do not violate the \( SL(2, Z) \) duality symmetry which exists in a completely broken \( N = 4 \) super Yang-Mills theory at finite coupling. Similar bound states exist on the higher dimensional analogs of the Taub-Nut metric. In future work we plan to investigate the relation of the enlargement of the number of threshold bound states to the problem of enhanced gauge symmetry of the Yang-Mills system describing coincident \( p \)-branes [8].

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References

[1] Ashoke Sen, Phys. Lett. B329:217-221 (1994), hep-th/9402032
[2] M.F. Atiyah, N.S. Hitchin, The Geometry and Dynamics of Magnetic Monopoles, M.B. Porter Lectures, Princeton, USA: Univ. Pr. (1988).

[3] T. T. Wu, C. N. Yang, Nucl. Phys. B107:365 (1976).

[4] T. T. Wu, C. N. Yang, Phys. Rev. D14:437-445 (1976)

[5] N.S. Manton, Phys. Lett. 154B:397 (1985); erratum ibid. 157B:475 (1985).

[6] G. W. Gibbons, N. Manton, Phys. Lett. B356:32-38 (1995).

[7] E. Witten, Nucl. Phys. B202:253 (1982).

[8] E. Witten, Nucl. Phys. B460:335-350 (1996), [hep-th/9510135](https://arxiv.org/abs/hep-th/9510135).