The initial evolution of millisecond magnetars: an analytical solution

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ABSTRACT
Millisecond magnetars are often invoked as the central engine of some gamma-ray bursts (GRBs), specifically the ones showing a plateau phase. We argue that an apparent plateau phase may not be realized if the magnetic field of the nascent magnetar is in a transient rapid decay stage. Some GRBs that lack a clear plateau phase may also be hosting millisecond magnetars. We present an approximate analytical solution of the coupled set of equations describing the evolution of the angular velocity and the inclination angle between rotation and magnetic axis of a neutron star in the presence of a co-rotating plasma. We also show how the solution can be generalized to the case of evolving magnetic fields. We determine the evolution of the spin period, inclination angle, magnetic dipole moment and braking index of six putative magnetars associated with GRB 091018, GRB 070318, GRB 080430, GRB 090618, GRB 110715A, GRB 140206A through fitting, via Bayesian analysis, the X-ray afterglow light curves by using our recent model [Shaşmaz Muş et al. 2019]. We find that within the first day following the formation of the millisecond magnetar, the inclination angle aligns rapidly, the magnetic dipole field may decay by a few times and the braking index varies by an order of magnitude.

Key words: gamma-ray burst; general – stars: magnetars

1 INTRODUCTION
Gamma-ray bursts (GRBs) are highly energetic explosions with durations of milliseconds to minutes (Lyutikov & Blandford 2003; Zhang & Mészáros 2004; Piran 2005; Kumar & Zhang 2015). The prompt emission is followed by an X-ray afterglow (Costa et al. 1997). It is considered that the central engine of some of the GRBs, specifically the ones showing a plateau stage in their afterglows, could be strongly magnetized rapidly spinning neutron stars i.e. millisecond magnetars (Usov 1992; Duncan & Thompson 1992; Dai & Lu 1998; Klužniak & Ruderman 1998; Bucciantini et al. 2008). The spin-down power of a millisecond magnetar, $L_{\text{sd}} = -\dot{I}\Omega$, where $I$ is the moment of inertia of the star, $\Omega$ is the angular velocity, and the dot denotes the derivative with respect to time, is employed for explaining the X-ray afterglows:

$$L_X = \eta L_{\text{sd}}. \quad (1)$$

where $\eta$ is an efficiency coefficient. In the case of spin-down under magnetic dipole torque $\dot{I}\Omega = -2\mu^2 \sin^2 \alpha \Omega^3 / 3c^3$ where $\mu$ is the magnetic dipole moment and $\alpha$ is the inclination angle between rotation and magnetic axis, an exact analytic solution, $\Omega = \Omega_0 (1 + t / t_0)^{-1/2}$, can be obtained where $\Omega_0$ is the initial angular velocity and $t_0 = 3I^3 / (2\mu^2 \sin \alpha)^2$ is the spin-down time-scale. This leads to

$$L_X = \eta \Omega_0 (1 + t / t_0)^{-2}$$

where $L_0 = 2\mu^2 \Omega_0^3 \sin^2 \alpha / 3c^3$. The spin-down time-scale $t_0$ determines the duration of the plateau phase which is followed by the rapid-decay stage $L_X \propto t^{-2}$. This model has been generalized by Lasky et al. (2017) to infer the braking indices of nascent magnetars (see also Lü et al. 2019).

The solution given above, employed by many previous work, neglects the alignment component of the dipole torque (Michel & Goldwire 1970; Davis & Goldstein 1970). It also assumes a constant magnetic dipole moment rotating in vacuum. Initially, the magnetar is far from an equilibrium stage and its just generated magnetic field may be in a rapid relaxation stage. Because of this rapid decay of the field, the spin-down power may decline so fast that a clear ‘plateau phase’ may not be realized.

In this work we employ the recent model proposed by Şaşmaz Muş et al. (2019) for modeling the X-ray afterglows of six putative magnetars associated with GRB afterglow light curves, GRBs 091018, 070318, 080430, 090618, 110715A and 140206A. This model assumes the magnetar has a corotating plasma (Goldreich & Julian 1969) and employs the appropriate spin-down (Spitkovsky 2006)
alignment (Philippov et al. 2014) torque components. It also assumes an exponential relaxation of the magnetic field (Saşmaz Muş et al. 2019).

We review the model equations in Section 2.1. We present an approximate analytical solution for the model equations in Section 2.2, GRB sample used in this work in Section 2.3 and the method for fitting the model to the GRB afterglow light curves in Section 2.4. We present our results in Section 3 and discuss the implications of our findings in Section 4.

2 METHOD

2.1 Model equations

We employ the model recently set-up by Saşmaz Muş et al. (2019) to fit the X-ray afterglow light curves of 7 GRBs. This model is a set of three ordinary differential equations (ODEs) which employs the spin-down (Spitkovsky 2006)

$$\frac{d \Omega}{dt} = -\frac{\mu^2 \Omega^3}{c^3}(1 + \sin^2 \alpha),$$

and alignment (Philippov et al. 2014)

$$\frac{d \alpha}{dt} = -\frac{\mu^2 \Omega^2}{c^3} \sin \alpha \cos \alpha, \tag{3}$$

components of the magnetic dipole torque in the presence of a corotating plasma (Hones & Bergeson 1965; Goldreich & Julian 1969), and a simple prescription for the evolution of the magnetic dipole moment

$$\dot{\mu} = -(\mu - \mu_0)/t_m, \tag{4}$$

(Saşmaz Muş et al. 2019) where $\mu_0$ is the settling value of the magnetic moment and $t_m$ is its evolution time-scale. This model predicts the braking index to be

$$n = \frac{\Omega \dot{\Omega}}{\dot{\Omega}^2} = 3 + 2 \left[ \frac{\sin \alpha \cos \alpha}{1 + \sin^2 \alpha} \right]^2 + 2 \frac{\Omega \dot{\mu}}{\dot{\Omega} \mu}, \tag{5}$$

The first two equations, Eqn.(2) and Eqn.(3), are coupled while Equation (4) can be solved independently to give

$$\mu = \mu_0 + (\mu_0 - \mu_0)e^{-1/t_m}, \tag{6}$$

where $\mu_0$ is the initial magnetic dipole moment of the magnetar.

2.2 Approximate analytical solutions of the angular velocity and inclination angle

In Saşmaz Muş et al. (2019) we have solved the above set of equations numerically to find the evolution of $\Omega$, $\alpha$ and thus $L_X$. Although a single numerical solution takes less than a second, the Bayesian fitting procedure coupled with the MCMC simulation, requires solving the ODE set several hundred thousand times which is computationally expensive. An exact solution for Equations (2) and (3) is given in Philippov et al. (2014), but as this solution is implicit, employing it would require solving the algebraic equation numerically at each time step, therefore, using this method does not improve the computational expense. We also note that an exact explicit solution for the vacuum case exists (Pandey & Prasad 1996), but we are unable to generalize it to the case of non-vacuum equations we employ in this work.

We, thus, present a very accurate approximate solution of the angular velocity and inclination angle, i.e. the ODE set in Section 2.1. This significantly (by ~5 times) reduces the computational time and gives insight into the dependencies of the spin and inclination angle.

Equations (2)-(3) implies an integration constant

$$\Omega \frac{1 - \sin^2 \alpha}{\sin \alpha} = \Omega_0 \frac{1 - \sin^2 \alpha_0}{\sin \alpha_0}, \tag{7}$$

where $\Omega_0$ and $\alpha_0$ are the initial values of the spin and the inclination angle, respectively (Philippov et al. 2014). By...
using the integration constant, the angular velocity can be eliminated from Equation (3),

\[ \frac{dy}{d\tau} = \frac{y^3}{1 - y^2} \left( 1 - \frac{y_0^2}{y^2} \right), \]  

(8)

where \( y = \sin \alpha \), \( y_0 = \sin \alpha_0 \) and the dimensionless time, \( \tau \), is defined as

\[ \tau = \frac{\Omega_0^2}{Ic^3} \int_0^\tau \mu^2 \, d\tau. \]  

(9)

Integrating Equation (8) gives

\[ -y_0^2 \ln(1 + \xi) + \xi = 2 \left( 1 - \frac{y_0^2}{y^2} \right)^2 \tau, \]  

(10)

where \( \xi = \frac{y_0^2}{y^2} - 1 \). By applying the approximate solution,

\[ \xi = \xi_0 + y_0^2 \xi_1 + y_0^4 \xi_2, \]  

(11)

into Equation (10) and then solving it to the order of \( y_0 \), an approximate solution for the inclination angle is obtained as

\[ y(\tau) = \frac{y_0}{\sqrt{1 + f(\tau)}}, \]  

(13)

where

\[ f(\tau) = 2(1 - \frac{y_0^2}{y^2})^2 \left[ y_0^2 \ln(1 + 2\tau) + y_0^4 \ln(1 + 2\tau) - 4\tau \right] \frac{y_0^3}{1 + 2\tau} - y_0^2 \ln(1 + 2\tau) + \frac{y_0^4}{1 + 2\tau} \ln(1 + 2\tau) - 4\tau. \]  

(14)

The spin evolution can be easily obtained from Equation (7)

\[ \Omega(\tau) = \Omega_0 \left[ 1 - \frac{y_0^2}{y^2} \frac{1 + f(\tau)}{1 - y_0^2 \frac{1 + f(\tau)}{1 - y_0^2}} \right]. \]  

(15)

These approximate solutions are well-consistent with the numerical solutions as shown in Figure 1. The relative difference between the approximate solution and numerical solution is less than 6% for \( \alpha_0 < 70^\circ \). Also, the form of the solutions is not altered in the case of changing magnetic field as the field evolution only modifies the relation between the time, \( t \), and the dimensionless time, \( \tau \) given in Equation (9).

The linear term of \( \tau \) increases faster than the logarithmic term \( \ln(1 + \tau) \). So, in the limit of \( \tau \gg 1 \), the approximate solution of the inclination angle reduces to

\[ y(\tau) \simeq \frac{y_0}{\sqrt{1 + 2 \left( 1 - \frac{y_0^2}{y^2} \right)^2 \tau}}, \]  

(16)

as well as the spin of the star reduces to

\[ \Omega(\tau) = \Omega_0 \left[ 1 - \frac{y_0^2}{y^2} \right] \frac{1 + f(\tau)}{1 - y_0^2 \frac{1 + f(\tau)}{1 - y_0^2}} \]. \]  

(17)

\[ \begin{array}{lll}
\text{GRB} & z & \Gamma \\
091018 & 0.971 & 2.0 \pm 0.115 \\
070318 & 0.84 & 2.01 \pm 0.12 \\
080430 & 0.767 & 1.98 \pm 0.09 \\
090618 & 0.54 & 1.83 \pm 0.04 \\
110715A & 0.82 & 1.76 \pm 0.105 \\
140206A & 2.73 & 1.80 \pm 0.05 \\
\end{array} \]

\[ ^a \text{Redshifts and photon indices are obtained from the Swift/XRT GRB light curve repository (Evans et al. 2007, 2009).} \]

For later times, \( \tau \gg 1/(1 - \frac{y_0^2}{y^2})^2 \), both the inclination angle and the spin of the star approximate to

\[ y(\tau) \simeq \frac{y_0}{(1 - \frac{y_0^2}{y^2}) \sqrt{2\tau}}, \quad \text{and} \quad \Omega(\tau) \simeq \frac{\Omega_0}{\sqrt{2\tau}} \]  

(18)

Accordingly, both the spin and the inclination angle decrease with \( \tau^{-1/2} \) for the late time. The magnetic field and the rotation axis almost aligned (\( \alpha < 11^\circ \)) in this limit. If the magnetic field is constant, this limit indicates a time-scale

\[ t_{\text{alignment}} \sim 10^{-1} \frac{y_0^2}{(1 - \frac{y_0^2}{y^2})^2} \left[ \frac{\mu_3}{1\text{ms}} \right] \left[ \frac{P_0}{1\text{ms}} \right]^2 \text{day}, \]  

(19)

where \( \mu_3 = \mu/10^{33}\text{Gcm}^3 \). Alignment takes longer if the magnetic field decreases with time.

\[ \frac{\mu_3}{1\text{ms}} \]  

2.3 GRB sample

Our sample in this work contains GRBs 070318, 080430, 090618, 110715A, 140206A and 091018. We included GRB 091018, a source which is also presented in Şaşmaz Muş et al. (2019), in order to compare the numerical and approximate analytical solutions.

In contrast to Şaşmaz Muş et al. (2019), we did not restrict our sample only to GRBs with plateau phases since we now have clue that the magnetic dipole moment might be changing in the first day of a nascent magnetar. Thus, it is possible to model GRB afterglow light curves with steeper evolution.

The unabsorbed flux values, redshifts and photon indices of the GRB sample are obtained from the Swift-XRT GRB light curve repository\(^1\) (Evans et al. 2007, 2009) and listed in Table 1. We converted the unabsorbed flux values, \( F_X \), to luminosity values using

\[ L = 4\pi d_L^2(z) F_X k(z). \]  

(20)

Luminosity distance, \( d_L(z) \), is calculated in a flat \( \Lambda \)CDM cosmological model using astropy.cosmology subpackage (Price-Whelan et al. 2018). The cosmological parameters are taken as \( H_0 = 71 \text{ km s}^{-1} \text{ Mpc}^{-1} \) and \( \Omega_M = 0.27 \). The cosmological k-correction (Bloom et al. 2001), \( k(z) \), is calculated with \( k(z) = (1+z)(l-2) \) using redshift and photon index (\( \Gamma \)) values listed in Table 1 for each GRB.

\[ ^1 \text{http://www.swift.ac.uk/xrt_curves/} \]
2.4 Parameter estimation

We estimated the period, inclination angle, magnetic dipole moment of nascent magnetars at the start of the plateau phase as well as the value of the magnetic dipole moment which the star settles down and the evolution time-scale of this relaxation by using a Bayesian framework. We have given the details of this analysis in Sa¸smaz Mu¸s et al. (2019). The light curves of the selected GRBs are modelled with

\[ L_X = \eta \mu^2 \Omega^4 (1 + \sin^2 \alpha). \]

Here, \( \alpha \) and \( \Omega \) are calculated using approximate analytical solutions presented in Section 2.2 by Equations (13) and (15).

We used Gaussian likelihood and uniform prior probability to construct the posterior probability distribution. We used the same prior probabilities given in Sa¸smaz Mu¸s et al. (2019) except for GRB 140206A. For this source we decreased the lower limit of the initial rotation period from 0.7 ms to 0.5 ms as initial analysis suggested a lower value for this source. Finally, we sampled the posterior probability distribution of the parameters with \textit{emcee} (Foreman-Mackey et al. 2013, 2018) as described in Sa¸smaz Mu¸s et al. (2019) in detail and obtained the parameter values from the posterior probability distributions.

3 RESULTS

We have modelled the X-ray afterglow light curves of GRB 091018, GRBs 070318, 080430, 090618, 110715A and 140206A with the model described above to determine the initial parameters of the putative magnetars with the Bayesian method introduced above. The estimated values of the putative nascent magnetar parameters of the selected GRBs are presented in Table 2. The evolution as well as the 1D and 2D posterior distributions of the parameters plotted with \textit{getdist} (Lewis et al. 2018) are presented in Figure 2, Figure 3, Figure 4, Figure 5,  Figure 6 and Figure 7. We included GRB 090108 in our data set to compare numeric solution presented in Sa¸smaz Mu¸s et al. (2019) and analytic solution presented in this paper.

We have found that, within the time frame of the X-ray afterglow—about a few days following the birth of the magnetar—the inclination angle of putative magnetars change from ~ 30°–40° to ~ 5°–10° and the magnetic dipole moments decrease by a factor of 2–5. As a result, the braking index varies significantly in the episodes considered, confirming the previous results of Sa¸smaz Mu¸s et al. (2019).

The initial periods and magnetic moments determined in this work depend on the choice of the efficiency factor \( \eta \) and moment of inertia \( I \). The efficiency factor, \( \eta \), varies in a wide range; it can be as low as \( 10^{-5} \) or high as 50 due to the beaming and other effects (Frail et al. 2001; Kargaltsev et al. 2012). The moment of inertia of a neutron star takes values around \( 10^{45} \) g cm\(^2\) depending on the equation of state, the central mass density and the spin of the star (Haensel et al. 2007). In this work we chose \( \eta = 1 \) and \( I_L = 1 \) as is usual to choose. We note that, \( \eta \) and \( I \) can be eliminated from the equations by defining new variables as \( \sqrt{\eta I_L} \) and \( \mu/(\sqrt{\eta I_L}) \). Therefore, for different values of \( \eta \) and \( I \), the initial parameters transform as \( P_0 \rightarrow P_0 \sqrt{\eta I_L} \) and \( \mu_0 \rightarrow \mu_0 \sqrt{\eta I_L} \) while the others do not change. In Figure 8 we present all possible values for each source on the \( \mu_0 - P_0 \) plane. We emphasize that the evolution of the inclination angle and the braking index are not affected by the choice of \( \eta \) or \( I \).

4 SUMMARY AND DISCUSSION

We have invoked the ‘millisecond magnetar model’ (Usov 1992; Duncan & Thompson 1992; Dai & Lu 1998; Kluzniak & Ruderman 1998; Bucciantini et al. 2008) to infer the initial parameters of nascent magnetars from the X-ray afterglow light curves of GRBs.

We have presented an explicit approximate analytical solution of the system of equations describing the evolution of spin and inclination angle of a magnetized neutron star. We have shown that this solution is very accurate except for highly orthogonal initial conditions (\( \alpha_0 > 70^\circ \)).

We have fitted, via a Bayesian procedure, the light curves of 6 GRB afterglows by using the analytical solution to determine the evolution of the period, inclination angle, magnetic dipole moment and the braking index. The spin and magnetic parameters we obtained are consistent with the initial parameters suggested for the ‘millisecond magnetar model’.

We have shown that the inclination angle, just like the spin period, varies rapidly within the time-frame of the X-ray afterglows. This is compatible with the recent result we have obtained that the inclination angles of magnetars align rapidly within the first ~ 10 days (Sa¸smaz Mu¸s et al. 2019).
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As a consequence of the alignment and magnetic field decay the braking index is greater than three \( (n > 3) \) and varies rapidly confirming the results of \( \text{(Saçmaç Muş et al. 2019)} \). According to this picture the constant braking indices inferred by Lasky et al. (2017) and Li et al. (2019) are effective average values.

The 'millisecond magnetar model' is often invoked as an explanation to the 'plateau phase' observed in some X-ray afterglows. We have shown that the magnetic field of a nascent magnetar may decline immediately after its birth. As a result the spin-down power of the magnetar decreases more rapidly than it would if magnetic field remained constant, and thus an extended 'plateau phase' may not be realized. According to this picture the systems with the extended 'plateau phase' host magnetars with relatively longer field decay time-scales. This suggests that the relevance of the 'millisecond magnetar model' may not be restricted to the GRB afterglows with a plateau phase.

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Figure 2. Left: Evolution of luminosity, period, inclination angle, magnetic dipole moment and braking index of putative nascent magnetar in GRB 091018. The red line in the upper panel represents the luminosity model obtained from the median value of all samples. Solid black lines represent randomly chosen 500 models from the posterior distribution. Right: 2D joint (with 1 and 2 sigma contours) and 1D marginalized posterior probability distributions of the parameters.

Figure 3. Same as Figure 2 but for GRB 070318.
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Figure 4. Same as Figure 2 but for GRB 080430.

Figure 5. Same as Figure 2 but for GRB 090618.
Figure 6. Same as Figure 2 but for GRB 110715A.

Figure 7. Same as Figure 2 but for GRB 140206A. We excluded the flare data that comes after the first data point of the presented light curve.
Figure 8. Possible values of the initial period and the initial magnetic dipole moment for different values of $\eta$ and $I_{45}$. On solid lines, $\eta$ varies and $I_{45}$ fixed at 1. On dashed lines, $\eta$ fixed at 1 and $I_{45}$ varies. The grey shaded area is the possible minimum period range of various equation of states (Cook et al. 1994).