REMARKS ON THE SCHRÖDINGER EQUATION

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Abstract. Various origins of linear and nonlinear Schrödinger equations are discussed in connection with diffusion, hydrodynamics, and fractal structure. The treatment is mainly expository, emphasizing the quantum potential, with a few new observations.

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1. INTRODUCTION

Perhaps no subject has been the focus of as much mystery as “classical” quantum mechanics (QM) even though the standard Hilbert space framework provides an eminently satisfactory vehicle for determining accurate conclusions in many situations. This and other classical viewpoints provide also seven decimal place accuracy in QED for example. So why all the fuss? The erection of the Hilbert space edifice and the subsequent development of operator algebras (extending now into noncommutative (NC) geometry) has an air of magic. It works but exactly why it works and what it really represents remain shrouded in ambiguity. Also geometrical connections of QM and classical mechanics (CM) are still a source of new work and a modern paradigm focuses on the emergence of CM from QM (or below). Below could mean here a micro structure of space time (quantum foam, Cantorian spacetime, etc.). In addition there are beautiful stochastic theories for diffusion and QM. In terms of background information in book form we mention here e.g. [4, 6, 11, 12, 31, 27, 28, 31, 65, 74, 77, 79, 82, 84, 86, 90, 92, 111, 114, 128, 134, 136] (the
2. BACKGROUND FOR THE SCHRÖDINGER EQUATION

First consider the SE in the form (A1) $- (\hbar^2 / 2m) \psi'' + V \psi = i \hbar \psi_t$ so that for $\psi = R \exp(iS/\hbar)$ one obtains

$$S_t + \frac{S^2_X}{2m} + V - \frac{\hbar^2 R''}{2mR} = 0; \quad \partial_t (R^2) + \frac{1}{m} (R^2 S')' = 0$$

where $S' \sim \partial S/\partial X$. Writing $P = R^2$ (probability density $\sim |\psi|^2$) and $Q = -(\hbar^2 / 2m)(R''/R)$ (quantum potential) this becomes

$$S_t + \frac{(S')^2}{2m} + Q + V = 0; \quad P_t + \frac{1}{m} (PS')' = 0$$

and this has some hydrodynamical interpretations in the spirit of Madelung. Indeed going to [41] for example we take $p = S'$ with $p = m\dot{q}$ for $\dot{q}$ a velocity (or “collective” velocity - unspecified). Then (2.2) can be written as ($\rho = mP$ is an unspecified mass density)

$$S_t + \frac{p^2}{2m} + Q + V = 0; \quad P_t + \frac{1}{m} (Pp)' = 0; \quad P = R^2; \quad Q = -\frac{\hbar^2}{2m} \frac{R''}{R} = -\frac{\hbar^2}{2m} \frac{\partial^2 \sqrt{\rho}}{\sqrt{\rho}}$$

Note here

$$\frac{\partial^2 \sqrt{\rho}}{\sqrt{\rho}} = \frac{1}{4} \left[ \frac{2 \rho''}{\rho} - \left( \frac{\rho'}{\rho} \right)^2 \right]$$

Now from $S' = p = m\dot{q} = mv$ one has

$$P_t + (P\dot{q})' = 0 \equiv \rho_t + (\rho \dot{q})' = 0; \quad S_t + \frac{p^2}{2m} + V - \frac{\hbar^2}{2m} \frac{\partial^2 \sqrt{\rho}}{\sqrt{\rho}} = 0$$

Differentiating the second equation in X yields ($\partial \sim \partial/\partial X, \; v = \dot{q}$)

$$mv_t + mvv' + \partial V - \frac{\hbar^2}{2m} \partial \left( \frac{\partial \sqrt{\rho}}{\sqrt{\rho}} \right) = 0$$

Consequently, multiplying by $p = mv$ and $\rho$ respectively in (2.5) and (2.6), we obtain

$$m \rho v_t + m \rho vv' + \rho \partial V - \frac{\hbar^2}{2m} \rho \partial \left( \frac{\partial \sqrt{\rho}}{\sqrt{\rho}} \right) = 0; \quad mv \rho_t + mv(\rho' v + vv') = 0$$
Then adding in (2.7) we get

\[(2.8) \quad \partial_t(\rho v) + \partial(\rho v^2) + \frac{F}{m}\partial V - \frac{\hbar^2}{2m^2}\rho \partial \left( \frac{\partial^2 \sqrt{\rho}}{\sqrt{\rho}} \right) = 0 \]

This is similar to an equation in [41] (called an “Euler” equation) and it definitely has a hydrodynamic flavor (cf. also [60]).

Now go to [124] and write (2.6) in the form \((mv = p = S')\)

\[(2.9) \quad \frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\frac{1}{m} \nabla(V + Q); \quad v_t + vv' = -(1/m)\partial(v + Q) \]

The higher dimensional form is not considered here but matters are similar there. This equation (and (2.8)) is incomplete as a hydrodynamical equation as a consequence of a missing term \(-\rho^{-1}\nabla p\) where \(p\) is the pressure (cf. [31]). Hence one “completes” the equation in the form

\[(2.10) \quad m \left( \frac{\partial v}{\partial t} + (v \cdot \nabla)v \right) = -\nabla(V + Q) - \nabla F; \quad mv_t + mvv' = -\partial(V + Q) - F' \]

where (A2) \(\nabla F = (1/R^2)\nabla p\) (or \(F' = (1/R^2)p'\)). By the derivations above this would then correspond to an extended SE of the form

\[(2.11) \quad i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta \psi + V \psi + F \psi \]

provided one can determine \(F\) in terms of the wave function \(\psi\). One notes that it a necessary condition here involves \(\text{curl}grad(F) = 0\) or (A3) \(\text{curl}(R^{-2}\nabla p) = 0\) which enables one to take e.g. (A4) \(p = -bR^2 = -b|\psi|^2\). For one dimension one writes (A5) \(F' = -b(1/R^2)|\psi|^2 = -(2bR'/R) \Rightarrow F = -2b\log(R) = -b\log(|\psi|^2)\). Consequently one has a corresponding SE

\[(2.12) \quad i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\psi'' + V \psi - b(\log|\psi|^2)\psi \]

This equation has a number of nice features discussed in [124] (but serious drawbacks as indicated in [23] - cf. also [37] [40] [42] [43] [56] [107] [108] [109]). For example (A6) \(\psi = \beta G(x - vt)\text{exp}(ikx - i\omega t)\) is a solution of (2.2) with \(V = 0\) and for \(v = \hbar k/m\) one gets (A7) \(\psi = c\text{exp}(-B/4)(x - vt + d)^2\text{exp}(ikx - i\omega t)\) where \(B = 4mb/\hbar^2\). Normalization \(\int_{-\infty}^{\infty} |\psi|^2 = 1\) is possible with (A8) \(|\psi|^2 = \delta_m(\xi) = \sqrt{\alpha m/\pi}\text{exp}(-am\xi^2)\) where \(\alpha = 2b/\hbar^2\), \(d = 0\), and \(\xi = x - vt\). For \(m \to \infty\) we see that \(\delta_m\) becomes a Dirac delta and this means that motion of a particle with big mass is strongly localized. This is impossible for ordinary QM since \(\text{exp}(ikx - i\omega t)\) cannot be localized as \(m \to \infty\). Such behavior helps to explain the so-called collapse of the wave function and since superposition does not hold Schrödinger’s cat is either dead or alive. Further \(v = k\hbar/m\) is equivalent to the deBroglie relation \(\lambda = \hbar/p\) since \(\lambda = (2\pi/k) = 2\pi(h/mv) = 2\pi(h/2\pi)(1/p)\).

**Remark 2.1.** We go now to [70] and the linear SE in the form (A9) \(i(\partial\psi/\partial t) = -(1/2m)\Delta \psi + U(\vec{r})\psi\); such a situation leads to the Ehrenfest equations which have the form (A10) \(<\vec{v}> = (d/dt) <\vec{r}>\) and \(<\vec{r}^2> = \int d^3x|\psi(\vec{r},t)|^2\vec{r}\) with \(\vec{F}(t) = -\int d^3x|\psi(\vec{r},t)|^2\nabla U(\vec{r})\). Thus the quantum expectation values of position and
velocity of a suitable quantum system obey the classical equations of motion and the amplitude squared is a natural probability weight. The result tells us that besides the statistical fluctuations quantum systems possess an extra source of indeterminacy, regulated in a very definite manner by the complex wave function. The Ehrenfest theorem can be extended to many point particle systems and in [70] one singles out the kind of nonlinearities that violate the Ehrenfest theorem. A theorem is proved that connects Galilean invariance, and the existence of a Lagrangian whose Euler-Lagrange equation is the SE, to the fulfillment of the Ehrenfest theorem.

**REMARK 2.2.** There are many problems with the quantum mechanical theory of derived nonlinear SE (NLSE) but many examples of realistic NLSE arise in the study of superconductivity, Bose-Einstein condensates, stochastic models of quantum fluids, etc. and the subject demands further study. We make no attempt to survey this here but will give an interesting example later from [23] related to fractal structures where a number of the difficulties are resolved. For further information on NLSE, in addition to the references above, we refer to [7, 38, 54, 56, 70, 71, 72, 79, 111, 140, 141, 144] for some typical situations (the list is not at all complete and we apologize for omissions). Let us mention a few cases.

- The program of [70] introduces a Schrödinger Lagrangian for a free particle including self-interactions of any nonlinear nature but no explicit dependence on the space of time coordinates. The corresponding action is then invariant under spatial coordinate transformations and by Noether’s theorem there arises a conserved current and the physical law of conservation of linear momentum. The Lagrangian is also required to be a real scalar depending on the phase of the wave function only through its derivatives. Phase transformations will then induce the law of conservation of probability identified as the modulus squared of the wave function. Galilean invariance of the Lagrangian then determines a connection between the probability current and the linear momentum which insures the validity of the Ehrenfest theorem.

- We turn next to [72] for a statistical origin for QM (cf. also [11, 38, 71, 73, 111, 118, 133]). The idea is to build a program in which the microscopic motion, underlying QM, is described by a rigorous dynamics different from Brownian motion (thus avoiding unnecessary assumptions about the Brownian nature of the underlying dynamics). The Madelung approach gives rise to fluid dynamical type equations with a quantum potential, the latter being capable of interpretation in terms of a stress tensor of a quantum fluid. Thus one shows in [72] that the quantum state corresponds to a subquantum statistical ensemble whose time evolution is governed by classical kinetics in the phase space. The equations take the form

\[
\begin{align*}
\rho_t + \partial_x (\rho u) &= 0; \quad \partial_t (\rho v) + \partial_j (\rho v_{ij}) + \rho \partial_i V = 0; \quad \partial_t (\rho E) + \partial_i (\rho S) - \rho \partial_t V = 0 \\
\frac{\partial S}{\partial t} + \frac{1}{2\mu} \left( \frac{\partial S}{\partial x} \right)^2 + W + V &= 0
\end{align*}
\]
for two scalar fields \( \rho, S \) determining a quantum fluid. These can be rewritten as

(2.15) \[
\frac{\partial \xi}{\partial t} + \frac{1}{\mu} \frac{\partial^2 S}{\partial x^2} + \frac{1}{\mu} \frac{\partial \xi}{\partial x} \frac{\partial S}{\partial x} = 0;
\]

\[
\frac{\partial S}{\partial t} - \frac{\eta^2}{4 \mu} \frac{\partial^2 \xi}{\partial x^2} - \frac{\eta^2}{8 \mu} \left( \frac{\partial \xi}{\partial x} \right)^2 + \frac{1}{2 \mu} \left( \frac{\partial S}{\partial x} \right)^2 + V = 0
\]

where \( \xi = \log(\rho) \) and for \( \Omega = (\xi/2) + (i/\eta)S = \log \Psi \) with \( m = N\mu, \ V = NV, \) and \( h = N\eta \) one arrives at a SE

(2.16) \[
\frac{\hbar}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V\Psi = 0
\]

Further one can write \( \Psi = \rho^{1/2} \exp(i\mathcal{G}/\hbar) \) with \( \mathcal{G} = NS \) and here \( N = \int |\Psi|^2 dm x. \)

The analysis is very interesting.

**REMARK 2.3.** Now in \[44\] one is obliged to use the form \( \psi = R \exp(iS/\hbar) \) to make sense out of the constructions (this is no problem with suitable provisos, e.g. that \( S \) is not constant - cf. \[8, 11, 47, 48\]). Thus note from (A12) \( \psi/\psi = (R'/R) + i(S'/h) \) with \( \Im(\psi'/\psi) = (1/m)S' \sim p/m \) (see also (2.19) below). Also note (A13) \( J = (h/m)3\psi^* \psi' \) and \( \rho = R^2 = |\psi|^2 \) represent a current and a density respectively. Then using \( p = mv = m\dot{q} \) one can write (A14) \( \psi = (h/m)3\psi'/\psi \) and \( J = (h/m)3|\psi|^2(\psi^* \psi'/|\psi|^2) = (h/m)3(\rho v) \). Then look at the SE in the form \( \hbar \dot{\psi}_t = -(\hbar^2/2m)|\psi''| + V\psi \) with \( \dot{\psi}_t = (R_t + iS_t R/h) \exp(iS/h) \) and \( \psi_{xx} = [(R' + iS'R/h) \exp(iS/h)]' = [R'' + (2iS'R/h + (iS'R/h) + (iS'/h)^2)R \exp(iS/h)] \) which means

(2.17) \[
\frac{h^2}{2m} \left[ R'' - \left( \frac{S'}{h} \right)^2 + \frac{2iS'R'}{h} + \frac{iS''R}{h} \right] + VR = i\hbar \left[ R_t + iS_t R/h \right] \Rightarrow \partial_t R^2 + \frac{1}{m} (R'^2 S')' = 0 ; \ S_t + (S')^2 2mR \right] - \frac{h^2 R''}{2mR} + V = 0
\]

This can also be written as

(2.18) \[
\partial_t \rho + \frac{1}{m} \partial(p\rho) = 0 ; \ S_t + \frac{p^2}{2m} + Q + V = 0
\]

where \( Q = -h^2 R''/2mR. \) Now we sketch the philosophy of \[11, 45\] in part. Most of such aspects are omitted and we try to isolate the essential mathematical features. First one emphasizes configurations based on coordinates whose motion is choreographed by the SE according to the rule (1-D only here)

(2.19) \[
\dot{q} = v = \frac{h}{m} \Im(\psi^* \psi')
\]

where (A15) \( i\hbar \dot{\psi}_t = -(\hbar^2/2m)|\psi''| + V\psi. \) The argument for (2.19) is based on obtaining the simplest Galilean and time reversal invariant form for velocity, transforming correctly under velocity boosts. This leads directly to (2.19) (~ (A14)) so that Bohmian mechanics (BM) is governed by (2.19) and (A15). It’s a fairly convincing argument and no recourse to Floydian time seems possible (cf. \[11, 48, 50, 51\]). Note however that if \( S = c \) then \( \dot{q} = v = (h/m)3(R'/R) = 0 \) while \( p = S' = 0 \) so perhaps this formulation avoids the \( S = 0 \) problems indicated in \[11, 48, 50, 51\]. One notes also
that BM depends only on the Riemannian structure $g = (g_{ij}) = (m_i \delta_{ij})$ in the form (A16) $\dot{q} = h \Im(\text{grad}\psi/\psi); \ i\psi = -(h^2/2)\Delta \psi + V \psi$. What makes the constant $h/m$ in (2.21) important here is that with this value the probability density $|\psi|^2$ on configuration space is equivariant. This means that via the evolution of probability densities $\rho_t = \text{div}(v \rho) = 0$ (as in (2.18) with $v \sim p/m$) the density $\rho = |\psi|^2$ is stationary relative to $\psi$, i.e. $\rho(t)$ retains the form $|\psi(q, t)|^2$. One calls $\rho = |\psi|^2$ the quantum equilibrium density (QED) and says that a system is in quantum equilibrium when its coordinates are randomly distributed according to the QED. The quantum equilibrium hypothesis (QHP) is the assertion that when a system has wave function $\psi$ the distribution $\rho$ of its coordinates satisfies $\rho = |\psi|^2$.

REMARK 2.4. We extract here from [61, 62, 63] (cf. also the references there for background and [52, 53, 68] for some information geometry). There are a number of interesting results connecting uncertainty, Fisher information, and QM and we make no attempt to survey the matter. Thus first recall that the classical Fisher information associated with translations of a 1-D observable $X$ with probability density $P(x)$ is

$$F_X = \int dx\ P(x)(| \text{log}(P(x))|^2)' > 0 \tag{2.20}$$

One has a well known Cramer-Rao inequality (A17) $\text{Var}(X) \geq F_X^{-1}$ where $\text{Var}(X) \sim$ variance of $X$. A Fisher length for $X$ is defined via (A18) $\delta X = F_X^{-1/2}$ and this quantifies the length scale over which $p(x)$ (or better $\text{log}(p(x))$) varies appreciably. Then the root mean square deviation $\Delta X$ satisfies (A19) $\Delta X \geq \delta X$. Let now $P$ be the momentum observable conjugate to $X$, and $P_{cl}$ a classical momentum observable corresponding to the state $\psi$ given via (A20) $p_{cl}(x) = (h/2i)[(\psi'/\psi) - (\psi'/\psi)]$ (cf. (2.19)). One has the identity (A21) $\langle p >_{\psi} = \langle p_{cl} >_{\psi}$ following from (A20) with integration by parts. Now define the nonclassical momentum by $p_{nc} = p - p_{cl}$ and one shows then (A21) $\Delta X \Delta p \geq \delta X \Delta p_{nc} = h/2$. Now go to [62] now where two proofs are given for the derivation of the SE from the exact uncertainty principle (as in (A21)). Thus consider a classical ensemble of n-dimensional particles of mass $m$ moving under a potential $V$. The motion can be described via the HJ and continuity equations

$$\frac{\partial s}{\partial t} + \frac{1}{2m}|\nabla s|^2 + V = 0; \ \frac{\partial P}{\partial t} + \nabla \cdot \left[ P \frac{\nabla s}{m} \right] = 0 \tag{2.21}$$

for the momentum potential $s$ and the position probability density $P$ (note that we have interchanged $p$ and $P$ from [62] - note also there is no quantum potential and this will be supplied by the information term). These equations follow from the variational principle $\delta \mathcal{L} = 0$ with Lagrangian

$$L = \int dt\ d^nx\ P \left[ \frac{\partial s}{\partial t} + \frac{1}{2m}|\nabla s|^2 + V \right] \tag{2.22}$$

It is now assumed that the classical Lagrangian must be modified due to the existence of random momentum fluctuations. The nature of such fluctuations is immaterial for (cf. [62] for discussion) and one can assume that the momentum associated with position $x$ is given by (A22) $p = \nabla s + N$ where the fluctuation term $N$ vanishes on average at each point $x$. 
Thus s changes to being an average momentum potential. It follows that the average kinetic energy \( < |\nabla s|^2 > /2m \) appearing in (2.22) should be replaced by \( < |\nabla s + N|^2 > /2m \) giving rise to

\[
L' = L + (2m)^{-1} \int dt < N \cdot N > = L + (2m)^{-1} \int dt (\Delta N)^2
\]

where \( \Delta N = < N \cdot N >^{1/2} \) is a measure of the strength of the fluctuations. The additional term is specified uniquely, up to a multiplicative constant, by the following three assumptions

1. **Action principle:** \( L' \) is a scalar Lagrangian with respect to the fields \( P \) and \( s \) where the principle \( \delta L' = 0 \) yields causal equations of motion. Thus (A23) \((\Delta N)^2 = \int d^n x \, pf(P, \nabla P, \partial P/\partial t, s, \nabla s, \partial s/\partial t, x, t) \) for some scalar function \( f \).

2. **Additivity:** If the system comprises two independent noninteracting subsystems with \( P = P_1 P_2 \) then the Lagrangian decomposes into additive subsystem contributions; thus (A24) \( f = f_1 + f_2 \) for \( P = P_1 P_2 \).

3. **Exact uncertainty:** The strength of the momentum fluctuation at any given time is determined by and scales inversely with the uncertainty in position at that time. Thus (A25) \( \Delta N \to k \Delta N \) for \( x \to x/k \). Moreover since position uncertainty is entirely characterized by the probability density \( P \) at any given time the function \( f \) cannot depend on \( s \), nor explicitly on \( t \), nor on \( \partial P/\partial t \).

The following theorem is then asserted (see [13, 62] for the proofs).

**THEOREM 2.1.** The above 3 assumptions imply (A26) \((\Delta N)^2 = c \int d^n x \, P |\nabla \log(P)|^2 \)

where \( c \) is a positive universal constant.

**COROLLARY 2.1.** It follows from (2.23) that the equations of motion for \( p \) and \( s \) corresponding to the principle \( \delta L' = 0 \) are

\[
(2.24) \quad i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V \psi
\]

where \( \hbar = 2\sqrt{c} \) and \( \psi = \sqrt{P} \exp(is/\hbar) \).

**REMARK 2.5.** We sketch here for simplicity and clarity another derivation of the SE along similar ideas following [13, 62]. Let \( P(y^i) \) be a probability density and \( P(y^i + \Delta y^i) \) be the density resulting from a small change in the \( y^i \). Calculate the cross entropy via

\[
J(P(y^i + \Delta y^i) : P(y^i)) = \int P(y^i + \Delta y^i) \log \frac{P(y^i + \Delta y^i)}{P(y^i)} \, d^n y \approx
\]

\[
\left[ \frac{1}{2} \int \frac{1}{P(y^i)} \frac{\partial P(y^i)}{\partial y^i} \frac{\partial P(y^i)}{\partial y^k} \, d^n y \right] \Delta y^i \Delta y^k = I_{jk} \Delta y^j \Delta y^k
\]

The \( I_{jk} \) are the elements of the Fisher information matrix. The most general expression has the form

\[
(2.26) \quad I_{jk}(\theta^i) = \frac{1}{2} \int \frac{1}{P(x^i | \theta^i)} \frac{\partial P(x^i | \theta^i)}{\partial \theta^j} \frac{\partial P(x^i | \theta^i)}{\partial \theta^k} \, d^n x
\]

where \( P(x^i | \theta^i) \) is a probability distribution depending on parameters \( \theta^i \) in addition to the \( x^i \). For (A27) \( P(x^i | \theta^i) = P(x^i + \theta^i) \) one recovers (2.25) (straightforward - cf. [13, 62]). If
P is defined over an n-dimensional manifold with positive inverse metric $g^{ik}$ one obtains a natural definition of the information associated with P via

$$I = g^{ik} I_{ik} = \frac{g^{ik}}{2} \int \frac{1}{P} \frac{\partial P}{\partial y^i} \frac{\partial P}{\partial y^k} d^m y$$

(2.27)

Now in the HJ formulation of classical mechanics the equation of motion takes the form

$$\frac{\partial S}{\partial t} + \frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} + V = 0$$

(2.28)

where $g^{\mu\nu} = \text{diag}(1/m, \ldots, 1/m)$. The velocity field $u^\mu$ is given by (A28) $u^\mu = g^{\mu\nu} (\partial S/\partial x^\nu)$. When the exact coordinates are unknown one can describe the system by means of a probability density $P(t, x^\mu)$ with (A29) $\int P d^n x = 1$ and (A30) $(\partial P/\partial t) + (\partial/\partial x^\mu)(P g^{\mu\nu} (\partial S/\partial x^\nu)) = 0$. These equations completely describe the motion and can be derived from the Lagrangian

$$L_{CL} = \int P \left\{ \frac{\partial S}{\partial t} + \frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} + V \right\} dtd^n x$$

(2.29)

using fixed endpoint variation in S and P. Quantization is obtained by adding a term proportional to the information entropy I defined in (2.27). This leads to

$$L_{QM} = L_{CL} + \lambda I = \int P \left\{ \frac{\partial S}{\partial t} + \frac{1}{2} g^{\mu\nu} \left[ \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} + \frac{\lambda}{P^2} \frac{\partial P}{\partial x^\mu} \frac{\partial P}{\partial x^\nu} \right] + V \right\} dtd^n x$$

(2.30)

Fixed endpoint variation in S leads again to (A30) while variation in P leads to

$$\frac{\partial S}{\partial t} + \frac{1}{2} g^{\mu\nu} \left[ \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} + \frac{\lambda}{P^2} \frac{\partial P}{\partial x^\mu} \frac{\partial P}{\partial x^\nu} \right] + V = 0$$

(2.31)

These equations are equivalent to the Schrödinger equation if (A31) $\psi = \sqrt{P} \exp(iS/\hbar)$ with $\lambda = (2\hbar)^2$ (cf. Section 6).

**REMARK 2.6.** The SE gives to a probability distribution $\rho = |\psi|^2$ (with suitable normalization) and to this one can associate an information entropy $S(t)$ (actually configuration information entropy) (A32) $S = - \int \rho \log(\rho) d^3 x$ which is typically not a conserved quantity (S is an unfortunate notation here but we retain it momentarily since no confusion should arise). The rate of change in time of S can be readily found by using the continuity equation (A33) $\partial_t \rho = - \nabla \cdot (\nu \rho)$ where $\nu$ is a current velocity field Note here (cf. also [126])

$$\frac{\partial S}{\partial t} = - \int \rho (1 + \log(\rho)) dx = \int (1 + \log(\rho)) \partial(\nu \rho)$$

(2.32)

Note that a formal substitution of $\nu = -u$ in (A33) implies the standard free Brownian motion outcome (A34) $dS/dt = D \cdot \int \frac{|(\nabla \rho)^2/\rho|}{d^3 x} = D \cdot Tr \mathcal{F} \geq 0$ - use (A35) $u = D \nabla \log(\rho)$ with $D = h/2m$ and (2.32) with $\int (1 + \log(\rho)) \partial(\nu \rho) = - \int \nu \rho \partial \log(\rho) = - \int \nu \rho' \sim \int ((\rho')^2/\rho)$ modulo constants involving D etc. Recall here $m F \sim -(2/D^2) \int \rho Q dx = \int dx [(\nabla \rho)^2/\rho]$ is a functional form of Fisher information. A high rate of information entropy production corresponds to a rapid spreading (flattening down) of the probability density. This delocalization feature is concomitant with the decay in time property quantifying the time rate at which the far from equilibrium system approaches its stationary state of
equilibrium \( \frac{d}{dt} \text{Tr} \mathfrak{F} \leq 0 \).

REMARK 2.7. Now going back to the quantum context one admits general forms of the current velocity \( v \). For example consider a gradient field \( v = b - u \) where the so-called forward drift \( b(x,t) \) of the stochastic process depends on a particular diffusion model. Then one can rewrite the continuity equation as a standard Fokker-Plank equation (A37) \( \partial_t \rho = D \Delta \rho - \nabla \cdot (b \rho) \). Boundary restrictions requiring \( \rho, v \rho, \) and \( b \rho \) to vanish at spatial infinities or at boundaries yield the general entropy balance equation

\[
\frac{dS}{dt} = \int \left[ \rho (\nabla \cdot b) + D \cdot (\nabla \rho)^2 \right] d^3x = \int D (\nabla \rho) \cdot (\nabla \rho) = \frac{1}{2} \langle v^2 \rangle
\]

The first term in the first equation is not positive definite and can be interpreted as an entropy flux while the second term refers to the entropy production proper. The flux term represents the mean value of the drift field divergence \( \nabla \cdot b \) which by itself is a local measure of the flux incoming to or outgoing from an infinitesimal surrounding of \( x \) at time \( t \). If locally \( (\nabla \cdot b)(x,t) > 0 \) on an infinitesimal time scale we would encounter a local entropy increase in the system (increasing disorder) while in case \( (\nabla \cdot b)(x,t) < 0 \) one thinks of local entropy loss or restoration or order. Only in the situation \( < \nabla \cdot b > = 0 \) is there no entropy production. Quantum dynamics permits more complicated behavior. One looks first for a general criterion under which the information entropy (A32) is a conserved quantity. Consider (2.8) and invoke the diffusion current to write (recall \( u = D(\nabla \rho) / \rho )\)

\[
D \frac{dS}{dt} = - \int \left[ \rho^{-1/2} (\rho v) \right] \cdot \left[ \rho^{-1/2} (D \nabla \rho) \right] d^3x
\]

Then by means of the Schwarz inequality one has (A38) \( D |dS/dt| \leq \langle v^2 \rangle^{1/2} < u^2 >^{1/2} \) so a necessary (but insufficient) condition for \( dS/dt \neq 0 \) is that both \( < v^2 > \) and \( < u^2 > \) are nonvanishing. On the other hand a sufficient condition for \( dS/dt = 0 \) is that either one of these terms vanishes. Indeed in view of (A39) \( < u^2 > = D^2 \int |(\nabla \rho)^2 / \rho | d^3x \) the vanishing information entropy production implies \( dS/dt = 0 \); the vanishing diffusion current does the same job.

REMARK 2.8. We develop a little more perspective now (following [55] - first paper). Recall Q written out as

\[
Q = 2D^2 \frac{\Delta \rho^{1/2}}{\rho^{1/2}} = D^2 \left[ \frac{\Delta \rho}{\rho} - \frac{1}{2\rho^2} (\nabla \rho)^2 \right] = \frac{1}{2} \langle u^2 \rangle + D \nabla \cdot u
\]

where \( u = D \nabla \log(\rho) \) is called an osmotic velocity field. The standard Brownian motion involves \( v = -u \), known as the diffusion current velocity and (up to a dimensional factor) is identified with the thermodynamic force of diffusion which drives the irreversible process of matter exchange at the macroscopic level. On the other hand, even while the thermodynamic force is a concept of purely statistical origin associated with a collection of particles, in contrast to microscopic forces which have a direct impact on individual particles themselves, it is well known that this force manifests itself as a Newtonian type entry in local conservation laws describing the momentum balance; in fact it pertains to the average (local average) momentum taken over by the particle cloud, a statistical ensemble property quantified in terms of the probability distribution at hand. It is precisely the (negative)
gradient of the above potential $Q$ in (2.35) which plays the Newtonian force role in the momentum balance equations. The second analytical expression of interest here involves
\begin{equation}
- \int Q \rho \, dx = (1/2) \int u^2 \rho \, dx = (1/2) D^2 \cdot F_X; \quad F_X = \int \frac{(\nabla \rho)^2}{\rho} \, dx
\end{equation}
where $F_X$ is the Fisher information, encoded in the probability density $\rho$ which quantifies its gradient content (sharpness plus localization/disorder) (note $- \int Q \rho = - \int [(1/2)u^2 \rho + Du \rho^\prime] = - \int (1/2)u^2 \rho + \int Du \rho^\prime = -(1/2) \int D^2(\rho/\rho)^2 \rho + D^2 \int \rho^\prime \rho^\prime/\rho = (D^2/2) \int (\rho^\prime)^2/\rho = (1/2) \int u^2 \rho$). On the other hand the local entropy production inside the system sustaining an irreversible process of diffusion is given via
\begin{equation}
\frac{dS}{dt} = D \cdot \int \frac{(\nabla \rho)^2}{\rho} \, dx = D \cdot F_X \geq 0
\end{equation}
This stands for an entropy production rate when the Fick law induced diffusion current (standard Brownian motion case) $j = -D \nabla \rho$, obeying $\partial_t \rho + \nabla j = 0$, enters the scene. Here $S = - \int \rho \log(\rho) \, dx$ plays the role of (time dependent) information entropy in the nonequilibrium statistical mechanics framework for the thermodynamics of irreversible processes. It is clear that a high rate of entropy increase corresponds to a rapid spreading (flattening) of the probability density. This explicitly depends on the sharpness of density gradients. The potential type $Q(x,t)$, the Fisher information $F_X$, the nonequilibrium measure of entropy production $dS/dt$, and the information entropy $S(t)$ are thus mutually entangled quantities, each being exclusively determined in terms of $\rho$ and its derivatives.

In the standard statistical mechanics setting the Euler equation gives a prototypical momentum balance equation in the (local) mean
\begin{equation}
(\partial_t + v \cdot \nabla) v = \frac{F}{m} - \frac{\nabla P}{\rho}
\end{equation}
where $F = -\nabla F$ represents normal Newtonian force and $P$ is a pressure term. $Q$ appears in the hydrodynamical formalism of QM via
\begin{equation}
(\partial_t + v \cdot \nabla) v = \frac{1}{m} F - \nabla Q = \frac{1}{m} F + \frac{\hbar^2}{2m^2} \nabla \frac{\Delta \rho^{1/2}}{\rho^{1/2}}
\end{equation}
Another spectacular example pertains to the standard free Brownian motion in the strong friction regime (Smoluchowski diffusion), namely
\begin{equation}
(\partial_t + v \cdot \nabla) v = -2D^2 \nabla \frac{\Delta \rho^{1/2}}{\rho^{1/2}} = -\nabla Q
\end{equation}
where $v = -D(\nabla \rho/\rho)$ (formally $D = \hbar/2m$).

**Remark 2.9.** The papers in [39] contain very interesting derivations of Schrödinger equations via diffusion ideas à la Nelson, Markov wave equations, and suitable “applied” forces (e.g. radiative reactive forces).
3. DIFFUSION AND FRACTALS

We go now to Nagasawa \[SS\] \[90\] to see how diffusion and the SE are really connected (cf. also \[3\] \[10\] \[23\] \[57\] \[83\] \[97\] \[111\] \[119\] \[120\] \[121\] for related material, some of which is discussed later in detail); for now we simply sketch some formulas for a simple Euclidean metric where (B1) $\Delta = \sum (\partial / \partial x^i)^2$. Then $\psi(t,x) = exp[R(t,x) + iS(t,x)]$ satisfies a SE (B2) $i\partial_t \psi + (1/2)\Delta \psi + ia(t,x) \cdot \nabla \psi - V(t,x)\psi = 0$ ($\hbar$ and $m$ omitted) if and only if

$$V = -\frac{\partial S}{\partial t} + \frac{1}{2} \Delta R + \frac{1}{2}(\nabla R)^2 - \frac{1}{2}(\nabla S)^2 - a \cdot \nabla S;$$

$$0 = \frac{\partial R}{\partial t} + \frac{1}{2} \Delta S + (\nabla S) \cdot (\nabla R) + a \cdot \nabla R$$

in the region (B3) $D = \{(s,x) : \psi(s,x) \neq 0\}$. Solutions are often referred to as weak or distributional but we do not belabor this point. From \[SS\] \[90\] there results

**THEOREM** 3.1. Let $\psi(t,x) = exp[R(t,x) + iS(t,x)]$ be a solution of the SE (B2); then (B4) $\phi(t,x) = exp[R(t,x) + S(t,x)]$ and $\hat{\phi} = exp[R(t,x) - S(t,x)]$ are solutions of

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} \Delta \phi + a(t,x) \cdot \nabla \phi + c(t,x,\phi)\phi = 0;$$

$$-\frac{\partial \hat{\phi}}{\partial t} + \frac{1}{2} \Delta \hat{\phi} - a(t,x) \cdot \nabla \hat{\phi} + c(t,x,\hat{\phi})\hat{\phi} = 0$$

where the creation and annihilation term $c(t,x,\phi)$ is given via

$$c(t,x,\phi) = -V(t,x) - 2\frac{\partial S}{\partial t}(t,x) - (\nabla S)^2(t,x) - 2a \cdot \nabla S(t,x)$$

Conversely given $(\phi,\hat{\phi})$ as in (B4) satisfying (B2) it follows that $\psi$ satisfies the SE (B2) with $V$ as in (3.3) (note $R = (1/2)\log(\hat{\phi}\phi)$ and $S = (1/2)\log(\hat{\phi}/\phi)$ with $exp(R) = (\hat{\phi}\phi)^{1/2}$).

We will discuss this later in more detail and give proofs along with probabilistic content (note that the equations (3.2) are not imaginary time SE). From this one can conclude that nonrelativistic QM is diffusion theory in terms of Schrödinger processes (described by $(\phi,\hat{\phi})$ - more details later). Further it is shown that key postulates in Nelson’s stochastic mechanics or Zambrini’s Euclidean QM (cf. \[144\]) can both be avoided in the connection to diffusion processes (since they are automatically valid). Look now at Theorem 3.1 for one dimension and write $T = \hbar t$ with $X = (h/\sqrt{m})x$; then the SE (B2) becomes (B5) $i\hbar \psi_T = -(h^2/2m)\psi_{XX} - iA\psi_X + V\psi$ where $A = ah/\sqrt{m}$. In addition (B6) $i\hbar R_T + (h^2/m^2)R_XS_X + (h^2/2m^2)S_{XX} + ARX = 0$ and (B7) $V = -i\hbar S_T + (h^2/2m)R_{XX} + (h^2/2m^2)R_X - (h^2/2m^2)S_X^2 - ASX$. Hence

**PROPOSITION** 3.1. Equation (B2), written in the variables (B8) $X = (h/\sqrt{m})x$, $T = \hbar t$, with $A = (\sqrt{m}/h)a$ and $V = V(X,T) \sim V(x,t)$ is equivalent to (B5).

Making a change of variables in (3.2) now, as in Proposition 3.1, yields
COROLLARY 3.1. Equation (3.5), written in the variables of Proposition 3.1, becomes

\[ h\phi_T + \frac{h^2}{2m} \phi_{XX} + A\phi_X + \tilde{c}\phi = 0; -h\hat{\phi}_T + \frac{h^2}{2m} \hat{\phi}_{XX} - A\hat{\phi}_X + \hat{c}\hat{\phi} = 0; \]

\[ \tilde{c} = -\tilde{V}(X, T) - 2hS_T - \frac{h^2}{m} S_X^2 - 2AS_X \]

Thus the diffusion processes pick up factors of \( \hbar \) and \( h/\sqrt{m} \).

REMARK 3.1. We extract here from the Appendix to \[90\] for some remarks on competing points of view regarding diffusion and the the SE. First some work of Fenyes \[49\] is cited where a Lagrangian is taken as

\[ \frac{\partial S}{\partial t} + 1/2 (\nabla S)^2 + 1/2 \left( \frac{1}{2} \frac{\nabla \mu}{\mu} \right)^2 \]

where \( \mu(x) = \exp(2R(t, x)) \) denotes the distribution density of a diffusion process and \( V \) is a potential function. The term (B9) \( \Pi(\mu) = (1/2)[(1/2)(\nabla \mu/\mu)]^2 \) is called a diffusion pressure and since \( (1/2)(\nabla \mu/\mu) \sim \nabla R \) the Lagrangian can be written as

\[ L = \int \left[ \frac{\partial S}{\partial t} + 1/2 (\nabla S)^2 + 1/2 (\nabla R)^2 + V \right] \mu dx \]

Applying the variational principle \( \delta \int_a^b L(t)dt = 0 \) one arrives at

\[ \frac{\partial S}{\partial t} + 1/2 [ (\nabla (R + S))^2 - (\nabla (R + S)) \cdot \left( \frac{1}{2} \frac{\nabla \mu}{\mu} \right) ] + \left( \frac{1}{2} \frac{\nabla \mu}{\mu} \right)^2 - 1/4 \frac{\Delta \mu}{\mu} + V = 0 \]

which is called a motion equation of probability densities. From this he shows that the function \( \psi = \exp(R + iS) \) satisfies the SE (B10) \( i\partial_t + (1/2) \Delta \psi - V(t, x) \psi = 0 \). Indeed putting (B9) and the formula (B11) \( (1/2)(\Delta \mu/\mu) + (1/2) \Delta R + (\nabla R)^2 \) into (3.6) one obtains

\[ \frac{\partial S}{\partial t} + 1/2 (\nabla S)^2 - 1/2 (\nabla R)^2 - 1/2 \Delta R + V = 0 \]

which goes along with the duality relation (B12) \( R_t + (1/2) \Delta S + \nabla S \cdot \nabla R + b \cdot \nabla R = 0 \)

(B13) \( u = (1/2)(a + \hat{a}) = \nabla R \) and \( v = (1/2)(a - \hat{a}) = \nabla S \) as derived in the Nagasawa theory.

Hence \( \psi = \exp(R + iS) \) satisfies the SE by previous calculations. One can see however that the equation (3.6) is not needed since the SE and diffusion equations are equivalent and in fact the equations of motion are the diffusion equations. Moreover it is shown in \[90\] that (3.6) is an automatic consequence in diffusion theory with \( V = -c - 2S_t - (\nabla S)^2 \) and therefore it need not be postulated or derived by other means. This is a simple calculation from the theory developed above.

REMARK 3.2. Nelson’s important work in stochastic mechanics \[111\] produced the SE from diffusion theory but involved a stochastic Newtonian equation which is shown in \[90\] to be automatically true. Thus Nelson worked in a general context which for our purposes here can be considered in the context of Brownian motions

\[ B(t) = \partial_t + (1/2) \Delta + b \cdot \nabla + a \cdot \nabla; \quad \hat{B}(t) = -\partial_t + (1/2) \Delta - b \cdot \nabla + \hat{a} \cdot \nabla \]
and used a mean acceleration (B14) \( \alpha(t, x) = -(1/2)[B(t)\dot{B}(t)x + \dot{B}(t)B(t)x] \). Assuming the duality relations (B12) - (B13) he obtains a formula

\[
(3.10) \quad \alpha(t, x) = -\frac{1}{2}[B(t)(-b + \dot{a}) + \dot{B}(b + a)] = b_t + (1/2)\nabla(b)^2 - (b + v) \times \text{curl}(b) -
\]

\[-v_t + (1/2)\Delta a + (1/2)(\dot{a} \cdot \nabla)a + (1/2)(a \cdot \nabla)\dot{a} - (b \cdot \nabla)v - (v \cdot \nabla)b - v \times \text{curl}(b)\]

Then it is shown that the SE can be deduced from the stochastic Newton’s equation

\[
(3.11) \quad \alpha(t, x) = -\nabla V + \frac{\partial b}{\partial t} + \frac{1}{2} \nabla(b^2) - (b + v) \times \text{curl}(b)
\]

Nagasawa shows that this serves only to reproduce a known formula for \( V \) yielding the SE; he also shows that (3.10) also is an automatic consequence of the duality formulation of diffusion equations above. This equation (3.10) is often called stochastic quantization since it leads to the SE and it is in fact correct with the \( V \) specified there. However the SE is more properly considered as following directly from the diffusion equations in duality and is not correctly an equation of motion. There is another discussion of Euclidean QM by Zambrini \[144\]. This involves (B15) \( \tilde{\alpha}(t, x) = (1/2)[B(t)B(t)x + \dot{B}(t)\dot{B}(t)x] \) (with \( (\sigma_\alpha^T)^{ij} = \delta^{ij} \)). It is postulated that this equals (B16) \( -\nabla c + b_t + (1/2)\nabla(b)^2 - b + v \times \text{curl}(b) \) which in fact leads to the same equation for \( V \) as above with \( V = -c - 2S_t - (\nabla S)^2 - 2b \cdot \nabla S \) so there is nothing new. Indeed it is shown in [20] that (B16) holds automatically as a simple consequence of time reversal of diffusion processes.

3.1. SCALE RELATIVITY. There are several excellent and exciting approaches here. The method of Nottale \[13\, 14\, 15\] is preeminent (cf. also \[19\, 20\, 21\, 22\] and there is also a nice derivation of a nonlinear SE via fractal considerations in \[23\] (indicated below). The most elaborate and rigorous approach is due to Cresson \[33\], with elaboration and updating in \[2\, 31\, 35\]. We refer here to \[14\, 13\, 26\, 33\, 34\, 31\, 14\). There are various derivations of the SE and we follow \[14\] here (cf. also \[15\, 15\]). The philosophy is discussed in \[13\, 14\, 33\, 34\, 14\] and we just write down equations here. First a bivelocity structure is defined (recall that one is dealing with fractal paths). One defines first

\[
(3.12) \quad \frac{d_+}{dt}y(t) = \lim_{\Delta t \to 0^+} \left\langle \frac{y(t + \Delta t) - y(t)}{\Delta t} \right\rangle;
\]

\[
\frac{d_-}{dt}y(t) = \lim_{\Delta t \to 0^+} \left\langle \frac{y(t) - y(t - \Delta t)}{\Delta t} \right\rangle
\]

Applied to the position vector \( x \) this yields forward and backward mean velocities, namely (B17) \( (d_+/dt)x(t) = b_+ \) and \( (d_-/dt)x(t) = b_- \). Here these velocities are defined as the average at a point \( q \) and time \( t \) of the respective velocities of the outgoing and incoming fractal trajectories; in stochastic QM this corresponds to an average on the quantum state. The position vector \( x(t) \) is thus “assimilated” to a stochastic process which satisfies respectively after \( (dt > 0) \) and before \( (dt < 0) \) the instant \( t \) a relation (B18) \( dx(t) = b_+ x(t)dt + d\xi_+(t) = b_- x(t)dt + d\xi_-(t) \) where \( \xi(t) \) is a Wiener process (cf. \[111\]). It is in the description of \( \xi \) that the \( D = 2 \) fractal character of trajectories is inserted; indeed
that $\xi$ is a Wiener process means that the $d\xi$’s are assumed to be Gaussian with mean 0, mutually independent, and such that

$$<d\xi_i(t)d\xi_j(t)> = 2D\delta_{ij} dt; <d\xi_i(t)d\xi_{-j}(t)> = -2D\delta_{ij} dt$$

where $<>$ denotes averaging and $D$ is the diffusion coefficient. Nelson’s postulate (cf. [114]) is known to be of fractal (Hausdorff) dimension 2. Note also that any value of $\nabla S$ (note this is $\nabla'$) is indeed a consequence of fractal (Hausdorff) dimension 2 of trajectories follows from

$$<d\xi_1(t) d\xi_2(t)>$$

which becomes

$$<d\delta(t)>$$

and $\xi$ is a Wiener process means that the $d\xi$’s are Gaussian with mean 0, mutually independent, and such that then define a complex operator $(B26)$ $V = (1/2)[b_+ + b_-]$ and $U = (1/2)[b_+ - b_-]$. Consequently adding and subtracting one obtains $(B22)$ $\rho_1 + \text{div}(\rho U) = 0$ (continuity equation) and $(B23)$ $\text{div}(\rho U) - D\partial U = 0$ which is equivalent to $(B24)$ $\text{div}[\rho(U - D\nabla\log(\rho))] = 0$. One can show, using $(\ref{eqn:feynman_result})$ that the term in square brackets in $(B24)$ is zero leading to $(B25)$ $U = D\nabla\log(\rho)$. Now place oneself in the $(U, V)$ plane and write $(B26)$ $V = V - iU$. Then write $(B27)$ $(dV/dt) = (1/2)(d_+ + d_-)/dt$ and $(dU/dt) = (1/2)(d_+ - d_-)/dt$. Combining the equations in $(B28)$ one defines $(B28)$ $(dV/dt) = \partial U/V$ and $(dU/dt) = D\partial U/V$; then define a complex operator $(B29)$ $(dV/dt) = (dV/dt) - i(dU/dt)$ which becomes

$$\frac{d^d}{dt} = \left(\frac{\partial}{\partial t} - iD \Delta\right) + V \cdot \nabla$$

One now postulates that the passage from classical mechanics to a new nondifferentiable process considered here can be implemented by the unique prescription of replacing the standard $d/dt$ by $d^d/dt$. Thus consider $(B30)$ $S = \int_{t_1}^{t_2} \mathcal{L}(x, V, t) dt$ yielding by least action $(B31)$ $(d^d/dt)(\partial\mathcal{L}/\partial V_i) = \partial\mathcal{L}/\partial x_i$. Define then $P_i = \partial\mathcal{L}/\partial V_i$ leading to $(B32)$ $P = \nabla S$ (note this is $S$ and not $S$). Now for Newtonian mechanics write $(B33)$ $L(x, v, t) = (1/2)mv^2 - U$ which becomes $\mathcal{L}(x, V, t) = (1/2)mV^2 - \mathcal{U}$ leading to $(B34)$ $-\nabla \mathcal{U} = m(dV/dt)V$. One separates real and imaginary parts of the complex acceleration $\gamma = (d^dV/dt)$ to get

$$d^dV = (dV - idU)(V - iU) = (dV - dU) - i(dU - dV)$$

The force $F = -\nabla \mathcal{U}$ is real so the imaginary part of the complex acceleration vanishes; hence

$$\frac{dU}{dt} V + \frac{dv}{dt} U = \frac{\partial U}{\partial t} + U \cdot \nabla V + V \cdot \nabla U + D \Delta V = 0$$
from which \(\partial U/\partial t\) may be obtained. Differentiating the expression \(U = D \nabla \log(\rho)\) and using the continuity equation yields another expression (B35) \((\partial U/\partial t) = -D \nabla (\text{div} V) - \nabla (V \cdot U)\). Comparison of these relations yields \(\nabla (\text{div} V) = \Delta V - U \wedge \text{curl} V\) where the \(\text{curl} V\) term vanishes since \(U\) is a gradient. However in the Newtonian case \(P = mV\) so (B32) implies that \(V\) is a gradient and hence a generalization of the classical action \(S\) can be defined via \((B36)\, V = 2D \nabla S\) (note then \(\nabla (\text{div} V) = \Delta V\) and \(\text{curl} V = 0\)). Combining this with the expression for \(U\) one obtains (B37) \(S = \log(\rho^{1/2}) + iS\). One notes that this is compatible with [111] for example. The way to the SE is now short; set (B38) \(V\) vanishes since \(U\) is a gradient. However in the Newtonian case \(P \sim -\nabla g\) where \(g\) is the gravitational acceleration (note this gives (3.21)).

Comparison of these relations yields (3.22) \(\Delta \psi = -D \nabla \log(\rho) + 2D \nabla S = V - iU\); thus for \(P = mV\) the relation (B40) \(P \sim -ih\nabla \text{or} P \psi = -ih\nabla \psi\) has a natural interpretation. Putting \(\psi\) in (B34), which generalizes Newton’s law of universal gravitation, the equation of motion takes the form (B41) \(\nabla U = 2iDm(d'/dt)(\nabla \log(\psi))\). Noting that \(d'\) and \(\nabla\) do not commute one replaces \(d'/dt\) by (3.15) to obtain (3.18) \(\nabla U = 2iDm [\partial_t \nabla \log(\psi) - iD \Delta (\nabla \log(\psi)) - 2iD(\nabla \log(\psi) \cdot \nabla)(\nabla \log(\psi))\)

This expression can be simplified via (3.19) \(\nabla \Delta = \Delta \nabla; (\nabla f \cdot \nabla)(\nabla f) = (1/2)(\nabla f)^2; \frac{\Delta f}{f} = \Delta \log(f) + (\nabla \log(f))^2\)

This implies (3.20) \(\frac{1}{2}(\nabla \log(\psi)) + (\nabla \log(\psi) \cdot \nabla)(\nabla \log(\psi)) = \frac{1}{2} \nabla \frac{\Delta \psi}{\psi}\)

Integrating this equation yields (B42) \(D^2 \Delta \psi + iD\partial_t \psi - (\Omega/2m)\psi = 0\) up to an arbitrary phase factor \(\alpha(t)\) which can be set equal to 0 by a suitable choice of phase \(S\). Replacing \(D\) by \(h/2m\) one arrives at the SE (B43) \(ih\psi_t = -(h^2/2m)\Delta \psi + \Omega \psi\). This suggests an interpretation of QM as mechanics in a nondifferentiable (fractal) space.

**REMARK 3.3.** Some of the relevant equations for dimension one are collected together in Section 6. We note that it is the presence of \(\pm\) derivatives that makes possible the introduction of a complex plane to describe velocities and hence QM; one can think of this as the motivation for a complex valued wave function and the nature of the SE.

We go now to [23] and will sketch some of the material. Here one extends ideas of Nottale and Ord in order to derive a nonlinear Schrödinger equation (NLSE). Using the hydrodynamic model in [24] one added a hydrostatic pressure term to the Euler-Lagrange equations and another possibility is to add instead a kinematic pressure term. The hydrostatic pressure is based on an Euler equation \(-\nabla p = \rho g\) where \(\rho\) is density and \(g\) the gravitational acceleration (note this gives \(p = \rho gx\) in 1-D). In [23] one took \(\rho = \psi^* \psi\), \(b\) a mass-energy parameter, and \(p = \rho;\) then the hydrostatic potential is (for \(\rho_0 = 1\)) (3.21) \(b \int g(x) \cdot dr = -b \int \nabla p / \rho \cdot dr = -b \log(\rho/\rho_0) = -b \log(\psi^* \psi)\)

Here \(-b \log(\psi^* \psi)\) has energy units and explains the nonlinear term of [3] which involved (3.22) \(ih \frac{\partial \psi}{\partial t} = \frac{h^2}{2m}(\nabla^2 \psi + U \psi - b[\log(\psi^* \psi)] \psi)\)
A derivation of this equation from the Nelson stochastic QM was given by Lemos (cf. [79]). There are however some problems since this equation does not obey the homogeneity condition saying that the state $\lambda |\psi >$ is equivalent to $|\psi >$; however $\lambda^2$ is not invariant under $\psi \rightarrow \lambda \psi$. Further, plane wave solutions to (3.22) do not seem to have a physical interpretation due to extraneous dispersion relations. Finally one would like to have a SE in terms of $\psi$ alone. Note that another NLSE could be obtained by adding kinetic pressure terms $(1/2)\rho v^2$ and taking $\rho = a \psi^* \psi$ where $v = p/m$. Now using the relations from HJ theory (B44) $(\psi/\psi^*) = exp[2i\bar{\Theta}(x)/\hbar]$ and $p = \nabla \bar{S}(x) = mv$ one can write (B45) $v = -i(\hbar/2m)\nabla \log(\psi/\psi^*)$ so that the energy density becomes (B46) $(1/2)\rho |v|^2 = (\hbar^2/8m^2)\psi^* \nabla \log(\psi/\psi^*) \cdot \nabla \log(\psi/\psi^*)$. This leads to a corresponding nonlinear potential associated with the kinematical pressure via (B47) $(\hbar^2/8m^2)\nabla \log(\psi/\psi^*) \cdot \nabla \log(\psi/\psi^*)$. Hence a candidate NLSE is

\begin{equation}
(3.23) \quad \hbar \partial_t = -\frac{\hbar^2}{2m} \nabla^2 \psi + U\psi - b[\log(\psi^* \psi)]\psi + \frac{\hbar^2}{8m^2} \left( \nabla \log \frac{\psi^*}{\psi} \cdot \nabla \log \frac{\psi}{\psi^*} \right)
\end{equation}

(apparently this equation has not yet been derived in the literature). Here the Hamiltonian is Hermitian and $a \neq b$ are both mass-energy parameters to be determined experimentally. The new term can also be written in the form (B48) $\nabla \log(\psi/\psi^*) \cdot \nabla \log(\psi^*/\psi) = -[\nabla \log(\psi/\psi^*)]^2$. The goal now is to derive a NLSE directly from fractal space time dynamics for a particle undergoing Brownian motion. This does not require a quantum potential, a hydrodynamic model, or any pressure terms as above.

**REMARK 3.4.** One should make some comments about the kinematic pressure terms (B49) $(1/2)\rho v^2 \iff (\hbar^2/2m)(a/m)[\nabla \log(\rho)]^2$ versus hydrostatic pressure terms of the form (B50) $f(\nabla \rho) \iff -b \log(\psi^* \psi)$. The hydrostatic term breaks homogeneity whereas the kinematic pressure term preserves homogeneity (scaling with a $\lambda$ factor). The hydrostatic pressure term is also not compatible with the motion kinematics of a particle executing a fractal Brownian motion. The fractal formulation will enable one to relate the parameters $a, b$ to $h$.

Following Nottale nondifferentiabilty implies a loss of causality and one is thinking of Feynmann paths with $< v^2 > \propto (dx/dt)^2 \propto d\tau^{(2(D-1)/D)}$ with $D = 2$. Now a fractal function $f(x, \epsilon)$ could have a derivative $\partial f/\partial \epsilon$ and renormalization group arguments lead to (B51) $(\partial f(x, \epsilon)/\partial \log \epsilon) = a(x) + b f(x, \epsilon)$ (cf. [123]). This can be integrated to give (B52) $f(x, \epsilon) = f_0(x)[1 - \zeta(x)/(\lambda/\epsilon)^b]$. Here $\lambda^{-b} \zeta(x)$ is an integration constant and $f_0(x) = -a(x)/b$. This says that any fractal function can be approximated by the sum of two terms, one independent of the resolution and the other resolution dependent; one expects $\zeta(x)$ to be a fluctuating function with zero mean. Provided $a \neq 0$ and $b < 0$ one has two interesting cases (i) $\epsilon \ll \lambda$ with $f(x, \epsilon) \sim f_0(x)(\lambda/\epsilon)^{-b}$ and (ii) $\epsilon >> \lambda$ with $f$ independent of scale. Here $\lambda$ is the deBroglie wavelength. Now one writes

\begin{equation}
(3.24) \quad r(t + dt, dt) - r(t, dt) = b_+(r(t, dt) + x_+(t, dt) \left( \frac{dt}{\tau_0} \right)^\beta; \quad b_-(r(t, dt) - b_-(r(t, dt) + x_-(t, dt) \left( \frac{dt}{\tau_0} \right)^\beta)
\end{equation}
where $\beta = 1/D$ and $b_{\pm}$ are average forward and backward velocities. This leads to (B53) $v_{\pm}(r, t, dt) = b_{\pm}(r, t) + \xi_{\pm}(t, dt)(dt/\tau_0)^{\beta - 1}$. In the quantum case $D = 2$ one has $\beta = 1/2$ so $dt^{\beta - 1}$ is a divergent quantity (so nondifferentiability ensues). Following [23] one defines

$$\frac{d_{\pm}r(t)}{dt} = \lim_{\Delta t \to 0} \left< \frac{r(t + \Delta t) - r(t)}{\Delta t} \right>$$

from which (B54) $d_{\pm}r(t)/dt = b_{\pm}$. Now following Nottale one writes

$$\frac{\delta}{\delta t} = \frac{1}{2} \left( \frac{d_{+}}{dt} + d_{-} \right) - i \frac{1}{2} \left( \frac{d_{+}}{dt} - d_{-} \right)$$

which leads to (B55) $(\delta/dt) = (\partial/\partial t) + v \cdot \nabla - iD\nabla^2$. Here in principle D is a real valued diffusion constant to be related to $h$. (A symbol D for the fractal dimension is no longer needed here - e.g. $D = 2$ with (B56) $< d\xi_{\pm +}d\xi_{\pm -} >= \pm 2D\delta_{ij}dt$.) Now for the complex time dependent wave function we take $\psi = \exp[iS/2mD]$ with $p = \nabla S$ so that (B57) $v = -2iD\nabla \log(\psi)$. The SE is obtained from the Newton equation ($F = ma$) via (B58) $-\nabla U = m(\delta/dt)v = -2imD(\delta/dt)\nabla \log(\psi)$. Inserting (B55) gives

$$-\nabla U = -2im[D\partial_t \nabla \log(\psi)] - 2D\nabla \left( \frac{\nabla^2 \psi}{\psi} \right)$$

(see [114] for identities involving $\nabla$). Integrating (B27) yields (B59) $D^2\psi + iD\partial_t \psi - (U/2m)\psi = 0$ up to an arbitrary phase factor which may be set equal to zero. Now replacing D by $h/2m$ one gets the SE (B60) $ih\partial_t \psi + (h^2/2m)\nabla^2 \psi = U\psi$. Here the Hamiltonian is Hermitian, the equation is linear, and the equation is homogeneous of degree 1 under the substitution $\psi \to \lambda \psi$.

Next one generalizes this by relaxing the assumption that the diffusion coefficient is real. Some comments on complex energies are needed - in particular constraints are often needed (cf. [129]). However complex energies are not alien in ordinary QM (cf. [23] for references). Now the imaginary part of the linear SE yields the continuity equation $\partial_t \rho + \nabla \cdot (\rho v) = 0$ and with a complex potential the imaginary part of the potential will act as a source term in the continuity equation. Instead of (B61) $< d\xi_{\pm +}d\xi_{\pm -} >= \pm 2Ddt$ with D and $2mD = h$ real one gets (B62) $< d\xi_{\pm +}d\xi_{\pm -} >= \pm (D + D^*)dt$ with D and $2mD = h = \alpha + i\beta$ complex. The complex time derivative operator becomes (B63) $(\delta/dt) = \partial_t + v \cdot \nabla - (i/2)(D + D^*)\nabla^2$. Writing again (B64) $\psi = \exp[iS/2mD] = \exp(iS/h)$ one obtains (B65) $v = -2iD\nabla \log(\psi)$. The NLSE is then obtained (via the Newton law) as (B66) $-\nabla U = m(\delta/dt)v = -2imD(\delta/dt)\nabla \log(\psi)$. Inserting (B63) one gets

$$\nabla U = 2im \left[ D\partial_t \nabla \log(\psi) - 2iD^2(\nabla \log(\psi) \cdot \nabla)(\nabla \log(\psi) - \frac{i}{2}(D + D^*)D\nabla^2(\nabla \log(\psi)) \right]$$

Now using the identities (i) $\nabla \nabla^2 = \nabla^2 \nabla$, (ii) $2(\nabla \log(\psi) \cdot \nabla)(\nabla \log(\psi) = \nabla(\nabla \log(\psi))^2$ and (iii) $\nabla^2 \log(\psi) = \nabla^2 \psi/\psi - (\nabla \log(\psi))^2$ leads to a NLSE with nonlinear (kinematic pressure) potential, namely

$$ih\partial_t \psi = -\frac{h^2}{2m} \frac{\alpha}{h} \nabla^2 \psi + U\psi - i\frac{h^2}{2m} \frac{\beta}{h} (\nabla \log(\psi))^2 \psi$$
Note the crucial minus sign in front of the kinematic pressure term and also that \( h = \alpha + \beta \) is complex. When \( \beta = 0 \) one recovers the linear SE. The nonlinear potential is complex and one defines (B67) \( W = -(h^2/2m)(\beta/h)(\nabla \log(\psi))^2 \) with \( U \) the ordinary potential; then the NLSE is (B68) \( i\hbar \partial_t \psi = -(h^2/2m)(\alpha/h)\nabla^2 + U + iW) \psi. \) This is the fundamental result of [23]; it has the form of an ordinary SE with complex potential \( U + iW \) and complex \( h \). The Hamiltonian is no longer Hermitian and the potential itself depends on \( \psi \). Nevertheless one can have meaningful physical solutions with real valued energies and momenta; the homogeneity breaking hydrostatic pressure term \( -b(\log(\psi^* \psi) \psi \) is not present (it would be meaningless) and the NLSE is invariant under \( \psi \to \lambda \psi \).

**REMARK 3.5.** One could ask why not simply propose as a valid NLSE an equation

\[
(3.30) \quad i\hbar \partial_t \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + U \psi + \frac{\hbar^2}{2m} \alpha \nabla \log(\psi)^2 \psi
\]

Here one has a real Hamiltonian satisfying the homogeneity condition and the equation admits soliton solutions of the form (B69) \( \psi = C A(x-vt) \exp[i(kx-\omega t)] \) where \( A(x-vt) \) is to be determined by solving the NLSE. The problem here is that the equation suffers from an extraneous dispersion relation. Thus putting in the plane wave solution \( \psi \sim \exp[-i(Et - px)] \) one gets an extraneous EM relation (after setting \( U = 0 \)), namely (B70) \( E = (p^2/2m)[1 + (a/m)] \) instead of the usual \( E = p^2/2m \) and hence \( E_{QM} \neq E_{FT} \) where FT means field theory.

**REMARK 3.6.** It has been known since e.g. [129] that the expression for the energy functional in nonlinear QM does not coincide with the QM energy functional, nor is it unique. To see this write down the NLSE of [3] in the form (B71) \( i\hbar \partial_t \psi = \partial H(\psi, \psi^*)/\partial \psi^* \) where the real Hamiltonian density is

\[
(3.31) \quad H(\psi, \psi^*) = -\frac{\hbar^2}{2m} \psi^* \nabla^2 \psi + U \psi^* \psi - b \psi^* \log(\psi^* \psi) \psi + b \psi^* \psi
\]

Then using \( E_{FT} = \int H d^3r \) we see it is different from \( < \hat{H}>_{QM} \) and in fact \( E_{FT} - E_{QM} = \int b \psi^* \psi d^3r = b \). This problem does not occur in the fractal based NLSE since it is written entirely in terms of \( \psi \).

**REMARK 3.7.** In the fractal based NLSE there is no discrepancy between the QM energy functional and the FT energy functional. Both are given by

\[
(3.32) \quad N_{fractal}^{NLSE} = -\frac{\hbar^2}{2m} \frac{\alpha}{h} \psi^* \nabla^2 \psi + U \psi^* \psi - i\frac{\hbar^2}{2m} \frac{\beta}{h} \psi^* (\nabla \log(\psi)^2 \psi
\]

The NLSE is unambiguously given by (B71) and \( H(\psi, \psi^*) \) is homogeneous of degree 1 in \( \lambda \). Such equations admit plane wave solutions with dispersion relation \( E = p^2/2m \); indeed, inserting the plane wave solution into the fractal based NLSE one gets (after setting \( U = 0 \))

\[
(3.33) \quad E = \frac{\hbar^2}{2m} \frac{\alpha}{h} \frac{p^2}{2m} + i\frac{\beta}{h} \frac{p^2}{2m} = \frac{p^2}{2m} \frac{\alpha + i\beta}{h} = \frac{p^2}{2m}
\]

since \( h = \alpha + i\beta \). The remarkable feature of the fractal approach versus all other NLSE considered sofar is that the QM energy functional is precisely the FT one. The complex
diffusion constant represents a truly new physical phenomenon insofar as a small imaginary correction to the Planck constant is the hallmark of nonlinearity in QM (see [23] for more on this).

4. REMARKS ON A FRACTAL SPACETIME

There have been a number of articles and books involving fractal methods in spacetime or fractal spacetime itself with impetus coming from quantum physics and relativity. We refer here especially to [1, 14, 13, 24, 58, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 105, 106] for background to this paper. Many related papers are omitted here and we refer in particular to the journal Chaos, Solitons, and Fractals (CSF) for further information. For information on fractals and stochastic processes we refer for example to [4, 5, 27, 28, 29, 46, 59, 74, 75, 80, 85, 110, 117, 125, 127, 134, 138, 139, 143]. We discuss here a few background ideas and constructions in order to indicate the ingredients for El Naschie’s Cantorian spacetime $\mathcal{E}\infty$, whose exact nature is elusive. Suitable references are given but there are many more papers in the journal CSF by El Naschie (and others) based on these fundamental ideas and these are either important in a revolutionary sense or a fascinating refined form of science fiction. In what appears at times to be pure numerology one manages to (rather hastily) produce amazingly close numerical approximations to virtually all the fundamental constants of physics (including string theory). The key concepts revolve around the famous golden ratio $\phi = (\sqrt{5} - 1)/2$ and a strange Cantorian space $\mathcal{E}\infty$ which we try to describe below. It is very tempting to want all of these (heuristic) results to be true and the approach seems close enough and universal enough to compel one to think something very important must be involved. Moreover such scope and accuracy cannot be ignored so we try to examine some of the constructions in a didactic manner in order to possibly generate some understanding.

4.1. COMMENTS ON CANTOR SETS.

**EXAMPLE 4.1.** In the paper [85] one discusses random recursive constructions leading to Cantor sets, etc. Associated with each such construction is a universal number $\alpha$ such that almost surely the random object has Hausdorff dimension $\alpha$ (we assume that ideas of Hausdorff and Minkowski-Bouligand (MB) or upper box dimension are known - cf. [5, 14, 46, 80]). One construction of a Cantor set goes as follows. Choose $x$ from $[0, 1]$ according to the uniform distribution and then choose $y$ from $[x, 1]$ according to the uniform distribution on $[x, 1]$. Set $J_0 = [0, x]$ and $J_1 = [y, 1]$ and recall the standard $1/3$ construction for Cantor sets. Continue this procedure by rescaling to each of the intervals already obtained. With probability one one then obtains a Cantor set $S_c^0$ with Hausdorff dimension (C1) $\alpha = \phi = (\sqrt{5} - 1)/2 \sim 0.618$. Note that this is just a particular random Cantor set; there are others with different Hausdorff dimensions (there seems to be some - possibly harmless - confusion on this point in the El Naschie papers). However the golden ratio $\phi$ is a very interesting number whose importance rivals that of $\pi$ or $e$. In particular (cf. [4]) $\phi$ is the hardest number to approximate by rational numbers and could be called the most irrational number. This is because its continued fraction representation involves all 1’s.

**EXAMPLE 4.2.** From [91] the Hausdorff (H) dimension of a traditional triadic Cantor set is $d_H^{(0)} = \log(2)/\log(3)$. To determine the equivalent to a triadic Cantor set in 2 dimensions one looks for a set which is triadic Cantorian in all directions. The analogue
of an area \( A = 1 \times 1 \) is a quasi-area \( A_c = d_c^{(0)} \times d_c^{(0)} \) and to normalize \( A_c \) one uses \( \rho_2 = (A/A_c)^2 = 1/(d_c^{(0)})^2 \) (for n-dimensions (C2) \( \rho_n = 1/(d_c^{(n)})^{n-1} \)). Then the \( n^{th} \) Cantor like H dimension \( d_c^{(n)} \) will have the form (C3) \( d_c^{(n)} = \rho_n d_c^{(0)} = 1/(d_c^{(0)})^{n-1} \). Note also that the H dimension of a Sierpinski gasket is (C4) \( d_c^{(n+1)} = 1/(d_c^{(n)}) = \log(3)/\log(2) \) and in any event the straightforward interpretation of \( d_c^{(2)} = \log(3)/\log(2) \) is a scaling of \( d_c^{(0)} = \log(2)/\log(3) \) proportional to the ratio of areas \( (A/A_c)^2 \). One notes that (C5) \( d_c^{(4)} = 1/(d_c^{(0)})^3 = (\log(3)/\log(2))^3 \approx 3.997 \sim 4 \) so the 4-dimensional Cantor set is essentially “space filling”.

Another derivation goes as follows. Define probability quotients \( \Omega = \text{dim(subset)}/\text{dim(set)} \).

For a triadic Cantor set in 1-D (C6) \( \Omega^{(1)} = d_c^{(0)}/d_c^{(1)} = d_c^{(0)} (d_c^{(1)} = 1) \). To lift the Cantor set

to n-dimensions look at the multiplicative probability law (C7) \( \Omega^{(n)} = (\Omega^{(1)})^n = (d_c^{(0)})^n \).

However since \( \Omega^{(1)} = d_c^{(0)}/d_c^{(n)} \) we get (C8) \( d_c^{(0)}/d_c^{(n)} = (d_c^{(0)})^n \Rightarrow d_c^{(n)} = 1/(d_c^{(0)})^{n-1} \). Since \( \Omega^{(n-1)} \) is the probability of finding a Cantor point (Cantorian) one can think of the H dimension \( d_c^{(n)} = 1/\Omega^{(n-1)} \) as a measure of ignorance. One notes here also that for \( d_c^{(0)} = \phi \) (the Cantor set \( S_c^{(0)} \) of Example 2.1) one has \( d_c^{(4)} = 1/\phi^3 = 4 + \phi^3 \approx 4.236 \) which is surely space filling.

Based on these ideas one proves in [95, 96, 98] a number of theorems and we sketch some of this here. One picks a “backbone” Cantor set with H dimension \( d_c^{(0)} \) (the choice of \( \phi = d_c^{(0)} \) will turn out to be optimal for many arguments). Then one imagines a Cantorian spacetime \( \mathcal{E}^\infty \) built up of an infinite number of spaces of dimension \( d_c^{(n)} \) (\( -\infty \leq n < \infty \)). The exact form of embedding etc. here is not specified so one imagines e.g. \( \mathcal{E}^\infty = \cup \mathcal{E}^{(n)} \) (with unions and intersections) in some amorphous sense. There are some connections of this to vonNeumann’s continuous geometries indicated in [100]. In this connection we remark that only \( \mathcal{E}^{(-\infty)} \) is the completely empty set (\( \mathcal{E}^{(-1)} \) is not empty). First we note that \( \phi^2 + \phi - 1 = 0 \) leading to (C9) \( 1 + \phi = 1/\phi, \phi^3 = (2 + \phi)/\phi, (1 + \phi)/(1 - \phi) = 1/\phi(1 - \phi) = 4 + \phi^3 = 1/\phi^3 \) (a very interesting number indeed). Then one asserts that

**THEOREM 4.1.** Let \( (\Omega^{(1)})^n \) be a geometrical measure in n-dimensional space of a multiplicative point set process and \( \Omega^{(1)} \) be the Hausdorff dimension of the backbone (generating) set \( d_c^{(0)} \). Then \( < d > = 1/d_c^{(0)}(1 - d_c^{(0)}) \) (called curiously an average Hausdorff dimension) will be exactly equal to the average space dimension \( < n > = (1 + d_c^{(0)})(1 - d_c^{(0)}) \) and equivalent to a 4-dimensional Cantor set with H-dimension \( d_c^{(4)} = 1/(d_c^{(0)})^3 \) if and only if \( d_c^{(0)} = \phi \).

To see this take \( \Omega^{(n)} = (\Omega^{(1)})^n \) again and consider the total probability of the additive set described by the \( \Omega^{(n)} \), namely (C10) \( Z_0 = \sum_0^\infty (\Omega^{(1)})^n = 1/(1 - \Omega^{(1)}) \). It is conceptually easier here to regard this as a sum of weighted dimensions (since \( d_c^{(n)} = 1/(d_c^{(0)})^{n-1} \)) and consider \( w_n = n(d_c^{(0)})^n \). Then the expectation of \( n \) becomes (note \( d_c^{(n)} \sim 1/(d_c^{(0)})^{n-1} \)

\[ \sum_0^\infty (\Omega^{(1)})^n = 1/(1 - \Omega^{(1)}) \]
\[
\frac{1}{\Omega^{(n-1)}} \text{ so } n(d_c^{(0)})^{n-1} \sim n/d_c^{(n)} \\
\]

(\text{4.1})

\[
E(n) = \sum_{1}^{\infty} \frac{n^2(d_c^{(0)})^{n-1}}{\sum_{1}^{\infty} n(d_c^{(0)})^{n-1}} = -< n > = \frac{1 + d_c^{(0)}}{1 - d_c^{(0)}}
\]

Another average here is defined via (blackbody gamma distribution)

(\text{4.2})

\[
<n> = \int_{0}^{\infty} \frac{n^2(\Omega^{(1)})^n dn}{\int_{0}^{\infty} n(\Omega^{(1)})^n dn} = \frac{-2}{\log(\Omega^{(1)})}
\]

which corresponds to \(-< n >\) after expanding the logarithm and omitting higher order terms. However \(-< n >\) seems to be the more valid calculation here. Similarly one defines (somewhat ambiguously) an expected value for \(d_c^{(n)}\) via

(\text{4.3})

\[
<d> = \sum_{1}^{\infty} \frac{n(d_c^{(0)})^{n-1}}{\sum_{1}^{\infty} (d_c^{(0)})^n} = \frac{1}{d_c^{(0)}(1 - d_c^{(0)})}
\]

This is contrived of course (and cannot represent \(E(d_c^{(n)})\) since one is computing reciprocals \(\sum(n/d_c^{(n)})\) but we could think of computing an expected ignorance and identifying this with the reciprocal of dimension. Thus the label \(< d >\) does not seem to represent an expected dimension but if we accept it as a symbol then for \(d_c^{(0)} = \phi\) one has from \((C9)\)

(\text{4.4})

\[
-< n > = \frac{1 + \phi}{1 - \phi} = < d > = \frac{1}{\phi(1 - \phi)} = d_c^{(4)} = 4 + \phi^3 = \frac{1}{\phi^3} \sim 4.236
\]

**Remark 4.1.** We note that the normalized probability \((C11)\) \(N = \Omega^{(1)}/Z_0 = \Omega^{(1)}(1 - \Omega^{(1)}) = 1/< d >\) for any \(d_c^{(0)}\). Further if \(< d >= 4 = 1/d_c^{(0)}(1 - d_c^{(0)})\) one has \(d_c^{(0)} = 1/2\) while \(-< n > = 3 < 4 = < d >\). One sees also that \(d_c^{(0)} = 1/2\) is the minimum (where \(d < d > /d(d_c^{(0)}) = 0)\). \(
\)

Remark 4.2. The results of Theorem 4.1 should really be phrased in terms of \(\mathcal{E}^\infty\) (cf. [101]), thus \((H \sim \text{Hausdorff dimension and } T \sim \text{topological dimension})\)

(\text{4.5})

\[
dim_H \mathcal{E}^{(n)} = d_c^{(n)} = \frac{1}{(d_c^{(0)})^{n-1}};
\]

\[
<d> = \frac{1}{d_c^{(0)}(1 - d_c^{(0)})}; -< dim_T \mathcal{E}^\infty >= \frac{1 + d_c^{(0)}}{1 - d_c^{(0)}} = -< n >
\]

In any event \(\mathcal{E}^\infty\) is formally infinite dimensional but effectively it is \(4 \pm\) dimensional with an infinite number of internal dimensions. We emphasize that \(\mathcal{E}^\infty\) appears to be constructed from a fixed backbone Cantor set with \(H\) dimension \(1/2 \leq d_c^{(0)} < 1\); thus each such \(d_c^{(0)}\) generates an \(\mathcal{E}^\infty\) space. Note that in \([101]\) \(\mathcal{E}^\infty\) is looked upon as a transfinite discretum (?) underpinning the continuum. \(\blacksquare\)

**Remark 4.3.** An interesting argument from \([100]\) goes as follows. Thinking of \(d_c^{(0)}\) as a geometrical probability one could say that the spatial (3-dimensional) probability of finding a Cantorian “point” in \(\mathcal{E}^\infty\) must be given by the intersection probability \((C12)\) \(P = (d_c^{(0)})^3\).
where $3 \sim 3$ topological spatial dimension. P could then be regarded as a Hurst exponent (cf. \[1] \[14] \[13]) and the Hausdorff dimension of the fractal path of a Cantorian would be (C13) $d_{\text{path}} = 1/H = 1/P = 1/(d_{c}^{(0)}/3)$. Given $d_{c}^{(0)} = \phi$ this means $d_{\text{path}} = 4 + \phi^{3} \sim 4^+$ so a Cantorian in 3-D would sweep out a 4-D world sheet; i.e. the time dimension is created by the Cantorian space $E^{\infty}$ (! - ?). Conjecturing further (wildly) one could say that perhaps space (and gravity) is created by the fractality of time. This is a typical form of conjecture to be found in the El Naschie papers - extremely thought provoking but ultimately heuristic. Regarding the Hurst exponent one recalls that for Feynmann trajectories in 1+1 dimensions (C14) $d_{\text{path}} = 1/H = 1/d_{c}^{(0)} = d_{c}^{(2)}$. Thus we are concerned with relating (C13) and (C14) (among other matters). Note that path dimension is often thought of as a fractal dimension (M-B or box dimension), which is not necessarily the same as the Hausdorff dimension. However in [H] one shows that quantum mechanical free motion produces fractal paths of Hausdorff dimension 2 (cf. also [76]).

REMARK 4.4. Following [25] let $S_{c}^{(0)}$ correspond to the set with dimension $d_{c}^{(0)} = \phi$. Then the complementary dimension is $d_{\bar{c}}^{(0)} = 1 - \phi = \phi^{2}$. The path dimension is given as in (C14) by (C15) $d_{\text{path}} = d_{c}^{(2)} = 1/\phi = 1 + \phi$ and $d_{\bar{c}}^{(2)} = 1/(1 - \phi) = 1/\phi^{2} = (1 + \phi)^{2}$. Following El Naschie for an equivalence between unions and intersections in a given space one requires (in the present situation) that

\[
(4.6) \quad d_{\text{crit}} = d_{c}^{(2)} + d_{\bar{c}}^{(2)} = 1 + \frac{1}{\phi^{2}} = \frac{\phi(1 + \phi)}{\phi^{2}} = \frac{1}{\phi^{2}} = \frac{1}{\phi} \cdot \frac{1}{\phi^{2}} = d_{c}^{(2)} \cdot d_{\bar{c}}^{(2)} = 4 + \phi^{3}
\]

where (C16) $d_{\text{crit}} = 4 + \phi^{3} = d_{c}^{(4)} \sim 4.236$. Thus the critical dimension coincides with the Hausdorff dimension of $S_{c}^{(4)}$ which is embedded densely into a smooth space of topological dimension 4. On the other hand the backbone set of dimension $d_{c}^{(0)} = \phi$ is embedded densely into a set of topological dimension zero (a point). Thus one thinks in general of $d_{c}^{(n)}$ as the H dimension of a Cantor set of dimension $\phi$ embedded into a smooth space of integer topological dimension $n$.

REMARK 4.5. In [25] it is also shown that realization of the spaces $E^{(n)}$ comprising $E^{\infty}$ can be expressed via the fractal sprays of Lapidus-van Frankenhuyzen (cf. [80]). Thus we refer to [80] for graphics and details and simply sketch some ideas here (with apologies to M. Lapidus). A fractal string is a bounded open subset of $\mathbb{R}$ which is a disjoint union of an infinite number of open intervals $I = \ell_{1}, \ell_{2}, \ldots$. The geometric zeta function of $I$ is (C17) $\zeta_{I}(s) = \sum_{j} \ell_{j}^{-s}$. One assumes a suitable meromorphic extension of $\zeta_{I}$ and the complex dimensions of $I$ are defined as the poles of this meromorphic extension. The spectrum of $I$ is the sequence of frequencies $f = k \cdot \ell_{j}^{-1} \ (k = 1, 2, \ldots)$ and the spectral zeta function of $I$ is defined as (C18) $\zeta_{I}(s) = \sum_{f} f^{-s}$ where in fact $\zeta_{I}(s) = \zeta_{I}(s)\zeta(s)$ (with $\zeta(s)$ the classical Riemann zeta function). Fractal sprays are higher dimensional generalizations of fractal strings. As an example consider the spray $\Omega$ obtained by scaling an open square $B$ of size 1 by the lengths of the standard triadic Cantor string $CS$. Thus $\Omega$ consists of one open square of size $1/3$, 3 open squares of size $1/9$, 4 open squares of size $1/27$, etc. (see [80] for pictures and explanations). Then the spectral zeta function for the Dirichlet Laplacian
on the square is \( \zeta_B (s) = \sum_{n_1, n_2 = 1}^{\infty} (n_1^2 + n_2^2)^{s/2} \) and the spectral zeta function of the spray is \( \zeta\nu (s) = \zeta_{CS} (s) \cdot \zeta_B (s) \). Now \( \mathcal{E}^\infty \) is composed of an infinite hierarchy of sets \( \mathcal{E}^{(j)} \) with dimension \((1 + \phi)^{-1} = 1/\phi^{-1} \) and these sets correspond to a special case of boundaries \( \partial \Omega \) for fractal sprays \( \Omega \) whose scaling ratios are suitable binary powers of \( 2^{-\phi^{-1}} \). Indeed for \( n = 2 \) the spectral zeta function of the fractal golden spray indicated above is \( \zeta\nu (s) = (1/(1 - 2 \cdot 2^{s\phi})) \zeta_B (s) \). The poles of \( \zeta_B (s) \) do not coincide with the zeros of the denominator \( 1 - 2 \cdot 2^{s\phi} \) so the (complex) dimensions of the spray correspond to those of the boundary \( \partial \Omega \) of \( \Omega \). One finds that the real part \( \Re s \) of the complex dimensions coincides with \( \dim \mathcal{E}^{(2)} = 1 + \phi = 1/\phi^2 \) and one identifies then \( \partial \Omega \) with \( \mathcal{E}^{(2)} \).

The procedure generalizes to higher dimensions (with some stipulations) and for dimension \( n \) there results \( \Re s = 1/\phi^{n-1} = \dim \mathcal{E}^{(n)} \). This produces a physical model of the Cantorian fractal space from the boundaries of fractal sprays (see \[25\] for further details and \[80\] for precision). Other (putative) geometric realizations of \( \mathcal{E}^\infty \) are indicated in \[104\] in terms of wild topologies, etc.

5. HYDRODYNAMICS AND THE FRACTAL SCHRÖDINGER EQUATION

We sketch first some material from \[3\] (see also \[14\], \[114\], \[115\], \[116\] and Sections 2-4 for background). Thus let \( \psi \) be the wave function of a test particle of mass \( m_0 \) in a force field \( U(r, t) \) determined via \( (D1) \) \( i \hbar \partial_t \psi = U \psi - (\hbar^2/2m) \nabla^2 \psi \) where \( \nabla^2 = \Delta \). One writes \( (D2) \) \( \psi(r, t) = R(r) \exp(iS(r, t)) \) with \( (D3) \) \( v = (\hbar/2m) \nabla S \) and \( \rho = R \cdot R \) (one assumes \( \rho \neq 0 \) for physical meaning). Thus the field equations of QM in the hydrodynamic picture are

\[
(5.1) \quad d_t (m_0 \rho v) = \partial_t (m_0 \rho v) + \nabla (m_0 \rho v) = -\rho \nabla (U + Q); \quad \partial_t \rho + \nabla \cdot (\rho v) = 0
\]

where \( (D4) \) \( Q = -(\hbar^2/2m_0)(\Delta \sqrt{\rho}/\sqrt{\rho}) \) is the quantum potential (or interior potential). Now because of the nondifferentiability of spacetime an infinity of geodesics will exist between any couple of points A and B. The ensemble will define the probability amplitude (this is a nice assumption but what is a geodesic here). At each intermediate point C one can consider the family of incoming (backward) and outgoing (forward) geodesics and define average velocities \( b_+ (C) \) and \( b_- (C) \) on these families. These will be different in general and following Nottale this doubling of the velocity vector is at the origin of the complex nature of QM. Even though Nottale reformulates Nelson’s stochastic QM the former’s interpretation is profoundly different. While Nelson (cf. \[111\]) assumes an underlying Brownian motion of unknown origin which acts on particles in a still Minkowskian spacetime, and then introduces nondifferentiability as a byproduct of this hypothesis, Nottale assumes as a fundamental and universal principle that spacetime itself is no longer Minkowskian nor differentiable. While with Nelson’s Browian motion hypothesis, nondifferentiability is but an approximation which expected to break down at the scale of the underlying collisions (\(?\)), where a new physics should be introduced, Nottale’s hypothesis of nondifferentiability is essential and should hold down to the smallest possible length scales. (This sentence is interesting but needs elaboration). Following Nelson one defines now the mean forward and backward derivatives

\[
(5.2) \quad \frac{d_+}{dt} y(t) = \lim_{\Delta t \to 0_+} \left( \frac{y(t + \Delta t) - y(t)}{\Delta t} \right)
\]

\[
\frac{d_-}{dt} y(t) = \lim_{\Delta t \to 0_-} \left( \frac{y(t) - y(t - \Delta t)}{\Delta t} \right)
\]
This gives forward and backward mean velocities \((D5)\) \((d_+/dt)x(t) = b_+\) and \((d_-/dt)x(t) = b_-\) for a position vector \(x\). Now in Nelson’s stochastic mechanics one writes two systems of equations for the forward and backward processes and combines them in the end in a complex equation, Nottale works from the beginning with a complex derivative operator

\[
\frac{\delta}{dt} = \frac{(d_+ + d_-) - i(d_+ - d_-)}{2dt}
\]

leading to \((D6)\) \(V = (\delta/dt)x(t) = v - iu = (1/2)(b_+ + b_-) - (i/2)(b_+ - b_-)\). One defines also \((D7)\) \((d_x/dt) = (1/2)(d_+ + d_-)/dt\) and \((d_u/dt) = (1/2)(d_+ - d_-)/dt\) so that \(d_x x/dt = v\) and \(d_u x/dt = u\). Here \(v\) generalizes the classical velocity while \(u\) is a new quantity arising from nondifferentiability. This leads to a stochastic process satisfying (respectively for the forward \((dt > 0)\) and backward \((dt < 0)\) processes) \((D8)\) \(dx(t) = b_+ [x(t)] + d\xi_+(t) = b_- [x(t)] + d\xi_-(t)\). The \(d\xi(t)\) terms can be seen as fractal functions and they amount to a Wiener process when \(D = 2\) (presumably the fractal dimension). Then the \(d\xi(t)\) are Gaussian with mean zero, mutually independent, and satisfy \((D9)\) \(<d\xi_\pm, d\xi_{\pm,j}> = \pm 2D\delta_{ij}\) where \(D\) is a diffusion coefficient. \(D\) can be found via \(D = \hbar/2m_0\) given \(\tau_0 = \hbar/(m_0c^2)\) (deBroglie time scale in the rest frame - cf [13] for more on this). Now \((D9)\) allows one to give a general expression for the complex time derivative, namely

\[
df = \frac{\partial f}{\partial t} + \nabla f \cdot dx + \frac{1}{2} \frac{\partial^2 f}{\partial x_j \partial x_j} dx_j dx_j
\]

Next compute the forward and backward derivatives of \(f\). Then \(<dx_i dx_j> \rightarrow <d\xi_\pm, d\xi_{\pm,j}>\) so that the last term in \((D4)\) amounts to a Laplacian via \((D9)\) and one obtains \((D10)\) \((d_\pm f/dt) = [\partial_t + b_\pm \cdot \nabla \pm D\Delta] f\). This is an important result. Thus assume the fractal dimension is not 2 in which case there is no longer a cancellation of the scale dependent terms in \((D3)\) and instead of \(D\Delta f\) one would obtain an explicitly scale dependent behavior \(D\delta t^{(2/D)}\Delta f\). In other words the value \(D = 2\) implies that the scale symmetry becomes hidden in the operator formalism. Using \((D10)\) one obtains the complex time derivative operator in the form \((D11)\) \((\delta/dt) = \partial_t + V \cdot \nabla - iD\Delta\) (cf. \((D6)\) for \(V\)). Nottale’s prescription is then to replace \(d/dt\) by \(\delta/dt\). In this spirit one can write now \((D12)\) \(\psi = exp(i\mathfrak{S}/2m_0D)\) so that \((D13)\) \(V = -2iD\nabla (\log(\psi))\) and then the generalized Newton equation \((D14)\) \(-\nabla U = m_0(\delta/dt)\) \(\psi\) reduces to the SE.

Now assume the velocity field from the hydrodynamic model agrees with the real part \(v\) of the complex velocity \(V\) and equate the wave functions from the two models \((D12)\) and \((D2)\); one obtains for \(\mathfrak{S} = s + i\sigma\) \((D15)\) \(s = 2m_0DS, D = (h/2m_0), and \sigma = -m_0Dlog(\rho)\). Using the definition \((D16)\) \(V = (1/m_0)\nabla \mathfrak{S} = (1/m_0)\nabla s + (i/m_0)\nabla \sigma = v - iu\) (which results via \((D6)\) by putting \((D12)\) into \((D13)\)) we get \((D17)\) \(v = (1/m_0)\nabla s = 2D\nabla S\) and \(u = -(1/m_0)\nabla \sigma = D\nabla log(\rho)\). Note that the imaginary part of the complex velocity given in \((D17)\) coincides with Nottale. Dividing the time dependent SE \((D1)\) by \(2m_0\) and taking the gradient gives \((D18)\) \(\nabla U/m_0 = 2D\nabla [i\hbar log(\psi) + D(\Delta \psi/\psi)]\) where \(h/2m_0\) has been replaced by \(D\). Then consider the identities

\[
\Delta \nabla = \nabla \Delta; (\nabla f \cdot \nabla) (\nabla f) = (1/2)(\nabla (\nabla f))^2; \frac{\Delta f}{f} = \Delta log(f) + (\nabla log(f))^2
\]
Now the second term in the right of (D18) becomes (D19) \( \nabla(\Delta \psi/\psi) = \Delta(\nabla \log(\psi)) + 2(\nabla \log(\psi) \cdot \nabla)/(\nabla \log(\psi)) \) so (D18) can be written as (D20) \( \nabla U = 2iDm_0[\partial_t \nabla \log(\psi) - iD \nabla(\nabla \log(\psi)) - 2iD(\nabla \log(\psi) \cdot \nabla)/(\nabla \log(\psi))]. \) One can show that (D20) is nothing but the generalized Newton equation (D14). Now if we replace the complex velocity (D13), taking into account (D6) and (D17) we get

\[
(5.6) \quad -\nabla U = m_0[\partial_t(v - iD\nabla \log(\rho) + [i(v - iD\nabla \log(\rho)) \cdot \nabla](v - iD\nabla \log(\rho)) - \\
- iD\Delta(v - iD\nabla \log(\rho))]
\]

Equation (5.6) is a complex differential equation and reduces to (using (5.5))

\[
(5.7) \quad m_0[\partial_t v + (v \cdot \nabla)v] = -\nabla \left( U - 2m_0D^2 \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) : \nabla \left\{ \frac{1}{\rho} [\partial_t \rho + \nabla \cdot (\rho v)] \right\}
\]

The last equation in (5.7) reduces to the continuity equation up to a phase factor \( \alpha(t) \) which can be set equal to zero (note again that \( \rho \neq 0 \) is posited). Thus (5.6) is nothing but the fundamental equations (5.1) of the hydrodynamic model. Further combining the imaginary part of the complex velocity in (D17) with the quantum potential (D4) and using (5.5) one gets (D21) \( Q = -m_0D\nabla \cdot u - (1/2)m_0u^2. \) Since \( u \) arises from nondifferentiability according to our nondifferentiable space model of QM it follows that the quantum potential comes from the nondifferentiability of the quantum spacetime (very nice but where is \( \mathcal{E}^{\infty} \) from the title of [3] - also the \( x \) derivatives should be clarified).

Putting \( U = 0 \) in the first equation of (5.7), multiplying by \( \rho \), and taking the second equation into account yields

\[
(5.8) \quad \partial_t(m_0\rho v_k) + \frac{\partial}{\partial x_i}(m_0\rho v_i v_k) = -\rho \frac{\partial}{\partial x_k} \left[ 2m_0D^2 \frac{1}{\sqrt{\rho}} \frac{\partial^2}{\partial x_i \partial x_k}(\sqrt{\rho}) \right]
\]

(here \( \nu_k \sim v_k \) seems indicated). Now set (D22) \( \Pi_{ik} = m_0\rho v_i v_k - \sigma_{ik} \) along with \( \sigma_{ik} = m_0\rho D^2(\partial/\partial x_i)(\partial/\partial x_k)(\log(\rho)). \) Then (5.8) takes the simple form (D23) \( \partial_t(m_0\rho v_k) = -\partial \Pi_{ik}/\partial x_i. \) The analogy with classical fluid mechanics works well if one introduces the kinematic (D24) \( \mu = D/2 \) and dynamic \( \eta = (1/2)m_0D \rho \) viscosities. Then \( \Pi_{ik} \) defines the momentum flux density tensor and \( \sigma_{ik} \) the internal stress tensor (D25) \( \sigma_{ik} = \eta(\partial u_i/\partial x_k + (\partial u_k/\partial x_i)). \) From (D22) one can see that the internal stress tensor is build up using the quantum potential while the equations (5.1) or (5.7) are nothing but systems of Navier-Stokes type for the motion where the quantum potential plays the role of an internal stress tensor. In other words the nondifferentiability of the quantum spacetime manifests itself like an internal stress tensor. For clarity in understanding (D23) we put this in one dimensional form so (5.8) becomes

\[
(5.9) \quad \partial_t(m_0\rho v) + \partial_x(m_0\rho v^2) = -\rho \partial \left( 2m_0D^2 \frac{1}{\sqrt{\rho}} \partial^2 \sqrt{\rho} \right) = \rho \partial Q
\]

and \( \Pi = m_0\rho v^2 - \sigma \) with \( \sigma = m_0\rho D^2 \partial^2 \log(\rho). \) This agrees in the standard formulas (cf. [14]). Now note \( \partial \sqrt{\rho} = (1/2)\rho^{-1/2}\rho' \) and \( \partial^2 \sqrt{\rho} = (1/2)\rho^{-3/2}(\rho')^2 + \rho^{-1/2}\rho'' \) with \( \partial^2 \log(\rho) = \partial(\rho'/\rho) = (\rho'/\rho) - (\rho'/\rho)^2 \) while

\[
(5.10) \quad -\rho \partial \left[ 2m_0D^2 \frac{1}{\sqrt{\rho}} \left( \partial^2 \sqrt{\rho} \right) \right] = -2m_0D^2 \rho \partial \left[ \frac{1}{2\sqrt{\rho}} \left( -\frac{1}{2} \rho^{-3/2}(\rho')^2 + \rho^{-1/2}\rho'' \right) \right]
\]
\[
= -2m_0 D^2 \rho \frac{\dot{\rho}}{\rho} \left[ \frac{\ddot{\rho}}{2 \rho} - \frac{1}{4} \left( \frac{\dot{\rho}}{\rho} \right)^2 \right] = -m_0 D^2 \rho \frac{\ddot{\rho}}{\rho} - \frac{1}{2} \left( \frac{\dot{\rho}}{\rho} \right)^2 \]
\]

One wants to show that (D23) holds or equivalently \(-\partial \sigma = \text{(5.10)}\). Here

\[(5.11) \quad -\partial \sigma = -\partial [m_0 \rho D^2 \partial^2 \log(\rho)] = -m_0 D^2 \left[ \rho \left( \frac{\dot{\rho}}{\rho} - \left( \frac{\dot{\rho}}{\rho} \right)^2 \right) + \rho \dot{\sigma} \left( \frac{\dot{\rho}}{\rho} - \left( \frac{\dot{\rho}}{\rho} \right)^2 \right) \right] \]

so we want \(\text{(5.11)} = \text{(5.10)}\) and this is easily verified.

### 6. Recapitulation

We write down now some of the main formulas here (with some unification of notation) in order to help provide perspective. The goal is not entirely clear but many questions will arise as we go along and at the end. Hopefully we will be able to answer some of the questions.

1. We write from Section 2 (E1) \(\psi = R \exp(i S/\hbar)\) with

\[(6.1) \quad S_t + \frac{(S')^2}{2m} + V - \frac{\hbar^2}{2m} R'' R = 0; \quad \partial_t (R^2) + \frac{1}{m} (R^2 S')' = 0 \]

For \(P = R^2\) and \(Q = -(\hbar^2/2m)(R''/R)\) this yields

\[(6.2) \quad S_t + \frac{(S')^2}{2m} + Q + V = 0; \quad P_t + \frac{1}{m} (P S')' = 0 \]

Writing \(\rho = mP\) and \(p = m\dot{x}\) leads to

\[(6.3) \quad \partial_t (\rho v) + \partial (\rho v^2) + \frac{\rho}{m} \partial V - \frac{\hbar^2}{2m^2} \rho \dot{\sigma} \left( \frac{\partial^2 \sqrt{\rho}}{\sqrt{\rho}} \right) = 0 \]

Along the way one arrived at (2.8) and “completed” this with a pressure term \(\nabla F = \rho^{-1} \nabla p\) or \(F' = (1/R^2)p'\) to arrive at (E2) \(m v_t + m v' = -\partial (V + Q) - F'\) corresponding to a SE (E3) \(i \hbar \psi_t = -(\hbar^2/2m) \psi'' + V \psi + F \psi\). One wants then \(F = F(\psi)\).

2. Consider a quantum state corresponding to a “subquantum” statistical ensemble governed by classical kinetics in a phase space. One arrives at \(\psi = \rho^{1/2} \exp(i \mathcal{S}/\hbar)\) with (E4) \(i \hbar \psi_t = -(\hbar^2/2m) \psi_{xx} + V \psi\) where \(\mathcal{S} = NS, \quad N = \int |\psi|^2 d^3 x, \quad h = N \eta, \quad m = N \mu, \quad V = NV, \quad \text{and} \log(\psi) = (1/2) \log(\rho) + (i/\eta)S\). The fields \(\rho, S\) or \(\xi, S\) determine a quantum fluid with (cf. (2.15))

\[(6.4) \quad \frac{\partial \xi}{\partial t} + \frac{1}{\mu} \frac{\partial S}{\partial x} + \frac{1}{\mu} \frac{\partial \xi}{\partial x} \frac{\partial S}{\partial x} = 0; \quad \frac{\partial S}{\partial t} - \frac{\eta^2}{4 \mu} \frac{\partial^2 \xi}{\partial x^2} - \frac{\eta^2}{8 \mu} \left( \frac{\partial \xi}{\partial x} \right)^2 + \frac{1}{2 \mu} \left( \frac{\partial S}{\partial x} \right)^2 + V = 0 \]

which for \(\psi = \rho^{1/2} \exp(i \mathcal{S}/\hbar)\) leads to

\[(6.5) \quad i \hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V \Psi \]
The Nagasawa theory (based in part on Nelson’s work) is very revealing and fascinating. Remarks 2.4-2.5 involves a classical ensemble and the manner in which $Q$ appears in the hydrodynamical formalism is exhibited in the Lagrangian

$$S_t + \frac{1}{2m}(S')^2 + V = 0; \quad P_t + \frac{1}{m}\partial(PS')' = 0$$

where $S$ is a momentum potential; note that no quantum potential is present but this will be added on in the form of a term $(1/2m)\int dt(\Delta N)^2$ in the Lagrangian which measures the strength of fluctuations. This can then be specified in terms of the probability density $P$ as indicated in Remark 2.4 leading to a SE $\Sigma$. A “neater” approach is given in Remark 2.5 leading in 1-D to

$$S_t + \frac{1}{2m}(S')^2 + V + \frac{\lambda}{m}\left(\frac{(P')^2}{P^2} - \frac{2P''}{P}\right) = 0$$

Note that $Q = -(\hbar^2/2m)(R'/R)$ becomes for $R = P^{1/2} (E5) Q = -(2\hbar^2/2m)[(2P''/P) - (P'/P)^2]$ (cf. $E3$). Thus the addition of the Fisher information serves to quantize the classical system.

One defines an information entropy (IE) in Remark 2.6 via

$$(E6) \mathcal{S} = -\int \rho \log(\rho) d^3x (\rho = \vert\psi\vert^2)$$

leading to

$$\frac{\partial \mathcal{S}}{\partial t} = \int (1 + \log(\rho))\partial(v\rho) \sim \int \frac{(\rho')^2}{\rho}$$

modulo constants involving $D \sim h/2m$. $\mathcal{S}$ is typically not conserved and $\partial_t \rho = -\nabla \cdot (v\rho)$ ($u = D\nabla \log(\rho)$ with $v = -u$ corresponds to standard Brownian motion with $d\mathcal{S}/dt \geq 0$. Then high IE production corresponds to rapid flattening of the probability density. Note here also that $\mathfrak{F} \sim -(2/D^2)\int \rho Q dx = \int dx[(\rho')^2/\rho]$ is a functional form of Fisher information. Entropy balance is discussed in Remark 2.8 and the manner in which $Q$ appears in the hydrodynamical formalism is exhibited in $E3$-

The Nagasawa theory (based in part on Nelson’s work) is very revealing and fascinating (see $[10],[11]$). The essence of Theorem 3.1 is that $\psi = \exp(R + iS)$ satisfies the SE $E7$: $i\psi_t + (1/2)\psi'' + i\psi' - V\psi = 0$ if and only if

$$V = -S_t + \frac{1}{2}R'' + \frac{1}{2}(R')^2 - \frac{1}{2}(S')^2 - aS; \quad 0 = R_t + \frac{1}{2}S'' + S'R' + aR'$$

Changing variables in $E8$: $X = (h/\sqrt{m})x$ and $T = \hbar t$ one arrives at $E9$ $i\hbar \psi_T = -(h^2/2m)\psi_{XX} - iA\psi_X + V\psi$ where $A = ah/\sqrt{m}$ and

$$i\hbar R_T + (h^2/m^2)R_XS_X + (h^2/2m^2)S_{XX} + AR_X = 0;$$

$$V = -i\hbar S_T + (h^2/2m)R_{XX} + (h^2/2m^2)R_X^2 - (h^2/2m^2)S_X^2 - AS_X$$

The diffusion equations then take the form

$$h\Phi_T + \frac{h^2}{2m}\Phi_{XX} + A\Phi_X + c\phi = 0; \quad -h\Phi_T + \frac{h^2}{2m}\Phi_{XX} - A\Phi_X + \tilde{c}\phi = 0;$$

$$c = -\tilde{V}(X,T) - 2hS_T - \frac{h^2}{m}S_X - 2AS_X$$
It is now possible to introduce a role for the quantum potential in this theory. Thus from \( \psi = \exp(R + iS) \) (with \( \hbar = m = 1 \) say) we have \( \psi = \rho^{1/2} \exp(iS) \) with \( \rho^{1/2} = \exp(R) \) or \( R = (1/2) \log(\rho) \). Hence \( (1/2)(\rho'/\rho) = R' \) and \( R'' = (1/2)[(\rho''/\rho) - (\rho'/\rho)^2] \) while the quantum potential is \( Q = (1/2)(\partial^2 \rho^{1/2}/\rho^{1/2}) = -(1/8)[(2\rho''/\rho) - (\rho'/\rho)^2] \) (cf. Theorem 3.1). Equation (6.9) becomes then

\[
V = -S_t + \frac{1}{8} \left( \frac{2\rho''}{\rho} - \frac{(\rho')^2}{\rho^2} \right) - \frac{1}{2}(S')^2 - aS \equiv S_t + \frac{1}{2}(S')^2 + V + Q + aS = 0;
\]

and

\[
\rho_t + \rho S'' + S' \rho' + a\rho' = 0 \equiv \rho_t + (\rho S')' + a\rho' = 0
\]

Thus \(-2S_t - (S')^2 = 2V + 2Q + 2aS \) and one has

**PROPOSITION 6.1.** The creation-annihilation term \( c \) in the diffusion equations (cf. Theorem 3.1) becomes

\[
\frac{d}{dt} y(t) = \lim_{\Delta t \to 0} \left\langle \frac{\pm y(t + \Delta t) \mp y(t)}{\Delta t} \right\rangle
\]

and we collect equations in \( \rho = |\psi|^2 \)

\[
dx = b_+ dt + d\xi_+ = b_- dt + d\xi_-; <d\xi_+^2> = 2Ddt = -<d\xi_-^2>
\]

\[
\frac{d_+}{dt} f = (\partial_t + b_+ \partial + D\partial^2) f; \quad \frac{d_-}{dt} f = (\partial_t + b_- \partial - D\partial^2) f
\]

\[
V = \frac{1}{2}(b_+ + b_-); \quad U = \frac{1}{2}(b_+ - b_-); \quad \rho_t + \partial(\rho V) = 0; \quad U = D\partial(\log(\rho));
\]

\[
V = V - iU; \quad d_V = \frac{1}{2}(d_+ + d_-); \quad d_U = \frac{1}{2}(d_+ - d_-)
\]

\[
\frac{dV}{dt} = \partial_t + V \partial; \quad \frac{dU}{dt} = D\partial^2 + U \partial; \quad \frac{d'}{dt} = (\partial_t - iD\partial^2) + V \partial
\]

\[
V = 2D\partial S; \quad S = \log(\rho^{1/2}) + iS; \quad \psi = \sqrt{\rho} e^{is} = e^{is}; \quad V = -2iD\partial \log(\psi)
\]

For Lagrangian \( L = (1/2)m\psi^2 - m\Omega \) one gets a SE

\[
\frac{ih\psi}{t} = -\frac{\hbar^2}{2m} \partial^2 \psi + \Omega \psi
\]

coming from Newton’s law \( \text{(E10)} \) - \( \partial \Omega = -2iDm(d'/dt)\partial \log(\psi) = m(d'/dt)V \).
(7) The development in Section 3 based on [23] involves thinking of nonlinear QM as a fractal Brownian motion with complex diffusion coefficient. We note (E10) corresponds to (B58) and (B55) arises in (6.18). These give rise to

\begin{equation}
- \nabla U = -2im[D \partial_t \nabla \log(\psi)] - 2D \nabla \left( \frac{\nabla^2 \psi}{\psi} \right)
\end{equation}

Thus putting in a complex diffusion coefficient leads to the NLSE

\begin{equation}
i\hbar \partial_t \psi = -\frac{\hbar^2}{2m} \alpha \nabla^2 \psi + U \psi - i \frac{\hbar^2}{2m} \beta (\nabla \log(\psi))^2 \psi
\end{equation}

with \( \hbar = \alpha + i \beta = 2mD \) complex.

(8) In [3] one writes again \( \psi = R \exp(iS/\hbar) \) with field equations in the hydrodynamical picture

\begin{equation}
d_t (m_0 \rho v) = \partial_t (m_0 \rho v) + \nabla (m_0 \rho v) = -\rho \nabla (u + Q); \quad \partial_t \rho + \nabla \cdot (\rho v) = 0
\end{equation}

where \( Q = -(\hbar^2/2m_0)(\Delta \sqrt{\rho}/\sqrt{\rho}) \). One works with the Nottale approach as above with \( d_u \sim d_V \) and \( d_u \sim d_U \) (cf. (6.18)). One assumes that the velocity field from the hydrodynamical model agrees with the real part \( v \) of the complex velocity \( V = v - i u \) so (cf. (6.17)) \( v = (1/m_0) \nabla s \sim 2D \partial s \) and \( u = -(1/m_0) \nabla \sigma \sim D \partial \log(\rho) \) where \( D = \hbar/2m_0 \). In this context the quantum potential \( Q = -(\hbar^2/2m_0) \Delta \sqrt{\rho} / \sqrt{\rho} \) becomes (E11) \( Q = -m_0 \nabla u - (1/2)m_0 u^2 \sim -(h/2) \partial u - (1/2)m_0 u^2 \). Consequently \( Q \) arises from the fractal derivative and the nondifferentiability of spacetime. Further one can relate \( u \) (and hence \( Q \)) to an internal stress tensor (D25) whereas the \( v \) equations correspond to systems of Navier-Stokes type. Note here that [59] involves a term relating the stress tensor \( \Pi \) and \( Q \) directly.

7. CONCLUSIONS

One feature either exhibited or suggested in the examples displayed involves the role of a quantum potential in either quantization or "classicalization" of certain systems of equations of hydrodynamic type. Now with numbers referring to Section 6 we have:

(1) One arrived at an equation of hydrodynamic type directly from the SE upon addition of a pressure term which served to augment the original potential \( V \) (however this could have simply been included in \( V \)). On the other hand \( Q \) does not appear in the SE but is generated by the decomposition \( \psi = R \exp(iS/\hbar) \)

(2) In a general statistical mechanical approach, with the dynamics determined by classical kinetics in a phase space, the quantum potential has an interpretation in terms of an internal stress tensor for a quantum fluid. The equations are again described in terms of a probability density \( \rho \) and a phase factor \( S \).

(3) In #3-#4 one takes a classical statistical ensemble with \( S \) a momentum potential and expresses momentum fluctuations in terms of Fisher information; this leads to a SE with quantization term \( Q \) expressed as Fisher information. In Remarks 2.6-2.8 we show how Fisher information, entropy, and the quantum potential are mutually entangled (cf. also [39]). In [23]-[39] (based on [55]) we see how the Euler equation \( (\partial_t + v \cdot \nabla) v = (F/m) - (\Delta P/\rho) \) (where \( P \) is a pressure term) is related to the quantized form \( (\partial_t + v \cdot \nabla) v = (F/m) - \nabla Q \) arising from a SE.
(4) The Nagasawa-Nelson approach in #5 views matters rather differently in showing the equivalence of the SE to a pair of diffusion equations. The full theory is very elegant and extends to singular situations, etc. (cf. [90, 92]). It would be of interest here to further examine the quantum potential in this context.

(5) In #6-#8 one arrives at a pair of equations by virtue of the “fractal” structure of space (where fractal here simply means that nondifferentiable paths are considered which generate a complex velocity). In [3] (as exhibited in #8) one relates the quantum potential to the velocity $u$, showing its origin in the “fractal” derivative idea.

We emphasize that in fact the quantum potential comes up in a serious manner in the Bohm theory, with refinements as in [11, 18, 19, 20, 21, 22, 43, 44, 45, 47, 48, 50, 51, 65, 66]. In fact, given that trajectories are at the base of this theory one can foresee a fractal Bohm theory in the future (cf. [67, 112]). On the other hand one can make convincing arguments for fields as the fundamental objects (except perhaps in the Bohmian type theories) with particles “emerging” (cf. [64, 142]) as in quantum field theory (or perhaps via ripples or fractal structure in spacetime itself).

It is not entirely clear how to handle derivatives in statistical or fractal theories. There are of course many powerful techniques available for Brownian motion and stochastic differential equations and there is a developing literature about differential calculus on fractals. Random walks and general discretization methods are also useful. Somehow one would like to imagine that the formal power of calculus (and duality via distribution like theories) might be strong enough to override the microscopic details about the domains of differential operators. Perhaps the coordinate derivative operators in situations such as #6-#8 could be defined so that their domains are various fractal sets densely embedded in $\mathbb{R}^n$ (in this connection see e.g. [30, 74, 75, 87, 110, 123, 125, 137]). In the end the most attractive formulation would seem to be some (more or less rigorous) version of a Feynmann path integral where precise definitions of the path space are not critical.
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