A way finding $r(k,l)$ and $r(3,10)=41$

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Abstract: On basis of two definitions that 1. an induced subgraph by a vertex $v_i \in G$ and its neighbors in $G$ is defined a vertex adjacent closed subgraph denoted by $Q_i (=G[V(Nv_i)])$, with the vertex $v_i$ called the hub; 2. A $(r(k,l)-1)$ vertices connected graph is called a $(k,l)$-Ramsey graph denoted by $RG(k,l)$,if and only if 1. $RG(k,l)$ contains only cliques of degree $k-1$, and its complement contains only cliques of degree $l-1$; 2. the intersect $Q_i \cap Q_j$ of any two nonadjacent vertices $v_i$ and $v_j$ of $RG(k,l)$ contains $K_{k-2}$, and the intersect $Q_i \cap Q_j$ of any two nonadjacent vertices $v_i$ and $v_j$ of its complement $RG(l,k)$ contains $K_{l-2}$. Two theorems that theorem1: the biggest clique in $G$ is contained in some $Q_i$ of $G$, and theorem2: $r(k,l)=|V(RG(k,l))|+1$ are put forward and proved in this paper. With those definitions and theorems as well as analysis of property of chords a method for quick inspection and building $RG(k,l)$ is proposed. Accordingly, $RG(10,3)$ and its complement are built, which are respectively the strongly 29-regular graph and the strongly 10-regular graph on orders 40. We have tested $RG(10,3)$ and its complement $RG(3,10)$, and gotten $r(3,10)=41$.

Keyword: The vertex adjacent closed subgraph; the hub; the Ramsey number $r(10,3)$; Ramsey graph $RG(10,3)$

1. Introduction

Let $G$ be a simple graph. $P$ vertices of $G$ are numbered clockwise as $i=0,1,...,p-1$. A vertex $v_i$ could adjacent to vertices labeled as $i\pm1,i\pm2,...,\{i\pm(p-1)/2\pmod p\}$ (when $p$ is an odd number); $i+p/2$ (when $p$ is an even number) are called the chord length. An edge between two adjacent vertices is called a chord, the at least number of edges between two adjacent vertices (the numeral behind the $\pm$) is called the chord length. According to their values chords are classified into two categories, the odd-length and the even-length chords, called an odd and an even chord for short respectively. The undefined terms and symbols used in this paper can be found in [1].

A clique in $G$ is a subset $K$ of $V(G)$ such that $G[K]$ is complete. The vertex number of $K$ is called the clique degree, and denoted by $k$. A subset $L$ of $V(G)$ is called an independent set if no two vertices of $L$ are adjacent in $G$. The vertex number of $L$ is called the independent degree, denoted by $l$. Clearly, $K$ is a clique of $G$ if and only if $K$ is an independent set of $G$'s complement $G'$. Therefore, the two concepts are complementary.

If $G$ has no large cliques, then one might expect $G$ to have a large independent set. That this indeed the case was first proved by Ramsey (1930). He showed that given any positive integers $k$ and $l$, there exists a smallest integer $r(k,l)$ such that every graph on $r(k,l)$ vertices contains either a clique of $k$ vertices or an independent set of $l$ vertex. The number $r(k,l)$ is called the Ramsey number.

At present, the known nine Ramsey numbers are $r(3,3)=6$, $r(3,4)=9$, $r(3,5)=14$, $r(3,6)=18$, $r(3,7)=23$, $r(3,8)=28$, $r(3,9)=36$, $r(4,4)=18$ and $r(4,5)=25$.

The last one, $r(4,5)=25$, was obtained by S. P. Radziszowski and B. D. Mackay in 1995[2]. It nearly cost equivalent 11 years workload of a standard computer before $RG(4,5)$ was constructed and proved. So much computation workload was spent because resolving Ramsey number problems involved finding cliques and independent sets in $RG(4,5)$. They are determining NP-C problems[3]. Although the current computers are quite powerful, they still could do
nothing for NP-C problems of G with large orders while lacking of effective algorithms. But we have gotten a good method. By which we have manually constructed and validated RG(3,10) in a short time, and have gotten that r(3,10)=41.

2. Definitions and Theorems

Definition 1: an induced subgraph by \( v_i \in G \) and its neighbors in G is defined the vertex adjacent closed subgraph denoted by \( Q_i (=G[V(Nv_i)]) \), with the vertex \( v_i \) called the hub.

Theorem 1: the biggest clique in G must be contained in some \( Q_i \) of \( G \).

Proof end symbol: it can be proved directly by definitions of clique and \( Q_i \).

There is no clear and strict definition of \((k,l)\)-Ramsey graph. Its definition is extended out generally in accordance with the definition of Ramsey number. According to the fuzzy definition, there are multiple \((k,l)\)-Ramsey graphs, one of them is \( r \)-regular graph. And even if got all \((k,l)\)-Ramsey graphs, also just get a lower bound of the \( r(k,l) \). So that to find Ramsey number is leaded into error way, such that to waste a lot of manpower, material resources to find little significance of lower and upper bounds of \( r(k,l) \). And to make it very complex and hard!

We give below \((k,l)\)-Ramsey graph a clear strict definition.

Definition 2 A \( r(k,l) \)-1 vertices connected graph is called a \((k,l)\)-Ramsey graph, denoted by \( RG(k,l) \), if and only if 1. \( RG(k,l) \) contains only cliques of \( k-1 \) vertices, and its complement contains only cliques of \( l-1 \) vertices; 2. the intersect \( Q_i \cap Q_j \) of any two nonadjacent vertices \( v_i \) and \( v_j \) of \( RG(k,l) \) contains \( K_{k-2} \), and the intersect \( Q_i \cap Q_j \) of any two nonadjacent vertices \( v_i \) and \( v_j \) of its complement \( RG(l,k) \) contains a clique \( K_l \). The essence of definition 2 is that a complete graph on \( r(k,l) \)-1 vertices is decomposed into two graphs \( RG(k,l) \) and its complement \( RG(l,k) \) such that \( RG(k,l) \) contains only \( K_{k-1} \), its complement \( RG(l,k) \) contains only \( K_{l-1} \), and they cannot contain other cliques besides \( K_{k-1} \) and \( K_{l-1} \).

According to the definition and theorem 1 as well as analysis of chord property we ask required \( RG(k,l) \) must be a \( r \)-regular graph, it has a fixed point, and its each \( Q_i \) is symmetrical on the hub \( v_i \); all \( Q_i \) is homogeneous.

When \( k=l \), maybe, some a \( RG(k,k) \) may be not a regular graph. So we study only \( RG(k,l) \) when \( k \neq l \) in this paper.

Theorem 2 \( r(k,l) = |V(RG(k,l))| + 1 \).

Proof let required \( RG(k,l) \) be a \( r \)-regular graph, then its complement \( RG(l,k) \) must be \( (p-r-1) \)-regular graph. Next, if it can be proved that adding a vertex such that either \( RG(k,l) \) contains a clique \( K_k \) or its complement \( RG(l,k) \) contains a clique \( K_l \), then the theorem is established.

Adding a vertex \( u \) to \( RG(k,l) \) or its complement \( RG(l,k) \), the cases that \( u \) is adjacent to \( p \) vertices in \( RG(k,l) \) or its complement \( RG(l,k) \) must be the following one of two cases:

Case 1, \( u \) is adjacent to at least \( r+1 \) vertices in \( RG(k,l) \), or

Case 2, \( u \) is adjacent to at least \( p-r \) vertices in its complement \( RG(l,k) \).

3. Analysis and Construction of \( RG(k,l) \)

According to theorem 2 and definition 2, we should
construct directly a pair of regular graphs, and need only to construct and to test any Q_i of this pair of regular graphs, can determine the exact value of r(k,l). Don't cost time and effort to test large amounts of lower and up bounds of r(k,l).

Let G be a complete graph on p vertices. When p is an odd number it has (p-1)/2 kinds of chords, and the length of the longest chord = (p-1)/2, the quantity of chords of each kind is p.

When p is an even number, it has p/2 kinds of chords, and the length of the longest chord = p/2, and the quantity of other chords is p. Endpoint-labels of an even chord both are either two odd, or two even numbers. But for an odd chord’s endpoint-labels, one is an odd number the other is an even number.

Regardless of an odd or even chord, using a unique kind of chords can constitute cliques K_2. But even if using all odd chords, could not constitute any clique greater than K_2 when p>3. But using multiple kinds of even chords can constitute cliques with degree from 3 up to p/2.

From definition 2 known that RG(k,l) contains only K_{k-1}, its complement contains only K_{l-1}. According to the theorem 1, every Q_i of RG(k,l) contains too only K_{k-1}, and every Q_i of its complement contains too only K_{l-1}.

Therefore, we need only consider to construct Q_0 of RG(k,l) in addition to the hub, by which kinds of chords can compose of K_{k-2} to construct RG(k,l), and by which kinds of chords can compose of K_{l-2} to construct RG(l,k) step by step.

3.1 Constitute RG(10,3) and its complement

From above we known that RG(10,3) contains only K_9, its complement contains only K_2. According to the theorem 1, every Q_i of RG(10,3) contains too only K_9, and every Q_i of its complement contains too only K_2.

So let us consider only Q_0 of RG(10,3) besides the hub v_0, by which kinds of chord can compose of K_8 to construct RG(10,3), and by which kinds of chord will disjoin at any vertex to construct RG(3,10) step by step.

First, let ±2, ±4, ±6, ±7, ±8, ±10, ±12, ±13, ±14, ±15, ±17, ±18, ±19, ±20 (mod 40) ; i = 0, 1, 2, … 39. See Fig. 1.

The complementary graph of RG(10,3) is represented as below:

i: ±1, ±3, ±9, ±11, ±16, i = 0, 1, 2, … 39 (mod 40); i = 0, 1, 2, … 39. See Fig. 3.

4. The testing method that RG(k,l) contains only K_{k-1}

By constructing RG(k,l) and theorem 1 we know that all of Q_i of RG(k,l) are isomorphic and each of Q_i contains only K_{k-1} while not contain other cliques. Therefore, let’s select arbitrarily a vertex from RG(k,l), delete it and its nonadjacent vertices. As a result the remaining graph will contain only K_{k-2}. Repeating this operation k-2 times, the final remains should be
isolated vertices. It proves that RG(k,l) contains only \( K_{k-1} \). Because the vertex deleted each time and its nonadjacent vertices could not compose of any clique, but they can only with the rest of vertices every time to compose of \( K_{k-1} \).

### 4.1 Testing that RG(10,3) contains only \( K_9 \)

From RG(10,3), we select arbitrarily a vertex, such as \( v_0 \), delete it and its nonadjacent vertices. As a result, the remaining graph contains only \( K_9 \) (see Fig. 1). Next, repeat this operation, from the remained graph select and delete step by step a vertex and its non-adjacent vertices until doing 7 operations, the remains must be isolated vertices. This is the proof that RG(10,3) contains only \( K_9 \).

By above operations, we have tested every vertex in Fig. 2, the results are same. That RG(10,3) contains only \( K_9 \) is proved.

### 4.2 Testing that RG(3,10) contains only \( K_2 \)

By above operation, we delete a vertex and its nonadjacent vertices from RG(3,10) showed in Fig. 3. For example, delete \( v_0 \) and its nonadjacent vertices. The remaining graph is a null graph, i.e. some isolated vertices. This prove that RG(3,10) contains only \( K_2 \).

### 4.3 Testing that the Intersect \( Q_i \cap Q_j \) of any two Nonadjacent Vertices \( v_i \) and \( v_j \) of RG(3,10) is Isolated Vertices

Since RG(3,10) contains only \( K_2 \). And by definition2 the intersect \( Q_i \cap Q_j \) of any two nonadjacent vertices \( v_i \) and \( v_j \) of RG(3,10) should be isolated vertices. Due to \( Q_0 \)'s symmetry we need only to test that the \( Q_0 \cap Q_j \) of both \( v_0 \) and each of its nonadjacent vertices \( v_j \) to be respectively 2,4,5,6,7,8,10,12,13,14,
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15,17,18,19,20 is isolated vertices.

The testing results are: \( Q_0 \cap Q_2 \) is \( v_1,v_3,v_{11},v_{31},v_{39} \); \( Q_0 \cap Q_4 \) is \( v_1,v_3 \); \( Q_0 \cap Q_5 \) is \( v_{16},v_{29} \); \( Q_0 \cap Q_6 \) is \( v_3,v_9,v_{37} \); \( Q_0 \cap Q_7 \) is \( v_{16},v_{31} \); \( Q_0 \cap Q_8 \) is \( v_9,v_{11},v_{37},v_{39} \); \( Q_0 \cap Q_{10} \) is \( v_1,v_9,v_{11},v_{39} \); \( Q_0 \cap Q_{12} \) is \( v_1,v_3,v_9,v_{11} \); \( Q_0 \cap Q_{13} \) is \( v_{16},v_{24},v_{29},v_{37} \); \( Q_0 \cap Q_{14} \) is \( v_3,v_{11} \); \( Q_0 \cap Q_{15} \) is \( v_{16},v_{24},v_{31},v_{39} \); \( Q_0 \cap Q_{17} \) is \( v_1,v_{16} \); \( Q_0 \cap Q_{18} \) is \( v_9,v_{17},v_{29} \); \( Q_0 \cap Q_{19} \) is \( v_3,v_{16} \) and \( Q_0 \cap Q_{20} \) is \( v_9,v_{11},v_{29},v_{31} \).

4.4 Testing that the intersect \( Q_i \cap Q_j \) of any two nonadjacent vertices \( v_i \) and \( v_j \) of \( RG(10,3) \) contains a clique \( K_8 \)

Due to symmetry we need to test that the \( Q_0 \cap Q_j \) of vertex 0 and its nonadjacent vertices \( v_j \) to be respectively 1,3,9,11,16 contains a clique \( K_8 \)

The testing results are: \( Q_0 \cap Q_1 \) contains \( K_8(v_5,v_7,v_{13},v_{15},v_{19},v_{27},v_{33},v_{35}) \); \( Q_0 \cap Q_3 \) contains \( K_8(v_5,v_7,v_{13},v_{15},v_{17},v_{25},v_{30},v_{35}) \); \( Q_0 \cap Q_9 \) contains \( K_8(v_2,v_4,v_{17},v_{19},v_{21},v_{23},v_{27},v_{35}) \); \( Q_0 \cap Q_{11} \) contains \( K_8(v_{13},v_{15},v_{17},v_{19},v_{21},v_{23},v_{25},v_{38}) \) and \( Q_0 \cap Q_{16} \) contains \( K_8(v_2,v_4,v_6,v_8,v_{10},v_{12},v_{14},v_{34}) \).

They all meet the definition 2.

5. \( r(3,10)=41 \)

Since \(|V(RG(3,10))|=40\), and from theorem 2
\( r(k,l)=|V(RG(k,l))|+1 \), so we gotten:
\( r(3,10)=40+1=41 \).

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