HIERARCHICAL PINNING MODELS, QUADRATIC MAPS
AND QUENCHED DISORDER

GIAMBATTISTA GIACOMIN, HUBERT LACOIN, AND FABIO LUCIO TONINELLI

Abstract. We consider a hierarchical model of polymer pinning in presence of quenched disorder, introduced by B. Derrida, V. Hakim and J. Vannimenus [11], which can be reinterpreted as an infinite dimensional dynamical system with random initial condition (the disorder). It is defined through a recurrence relation for the law of a random variable \{R_n\}_{n=1,2,...}, which in absence of disorder (i.e., when the initial condition is degenerate) reduces to a particular case of the well-known Logistic Map. The large-\(n\) limit of the sequence of random variables \(2^{-n}\log R_n\), a non-random quantity which is naturally interpreted as a free energy, plays a central role in our analysis. The model depends on a parameter \(\alpha \in (0,1)\), related to the geometry of the hierarchical lattice, and has a phase transition in the sense that the free energy is positive if the expectation of \(R_0\) is larger than a certain threshold value, and it is zero otherwise. It was conjectured in [11] that disorder is relevant (respectively, irrelevant or marginally relevant) if \(1/2 < \alpha < 1\) (respectively, \(\alpha < 1/2\) or \(\alpha = 1/2\)), in the sense that an arbitrarily small amount of randomness in the initial condition modifies the critical point with respect to that of the pure (i.e., non-disordered) model if \(\alpha \geq 1/2\), but not if \(\alpha < 1/2\). Our main result is a proof of these conjectures for the case \(\alpha \neq 1/2\). We emphasize that for \(\alpha > 1/2\) we find the correct scaling form (for weak disorder) of the critical point shift.

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1. Introduction

1.1. The model. Consider the dynamical system defined by the initial condition \(R_0^{(i)} > 0\), \(i \in \mathbb{N} := \{1,2,...\}\) and the array of recurrence equations

\[
R_{n+1}^{(i)} = \frac{R_n^{(2i-1)} R_n^{(2i)}}{B} + (B-1), \quad i \in \mathbb{N},
\]

for \(n = 0,1,...\) and a given \(B > 2\). Of course if \(R_0^{(i)} = r_0\) for every \(i\), then the problem reduces to studying the quadratic recurrence equation

\[
r_{n+1} = \frac{r_n^2 + (B-1)}{B},
\]

a particular case of a very classical problem, the logistic map, as it is clear from the fact that \(z_n := 1/2 - r_n/(2(B-1))\) satisfies the recursion

\[
z_{n+1} = \frac{2(B-1)}{B} z_n (1 - z_n).
\]

We are instead interested in non-constant initial data and, more precisely, in initial data that are typical realizations of a sequence of independent identically distributed (IID)
random variables. In its random version, the model was first considered in [11] (see § 1.2 and § 1.6 below for motivations in terms of pinning/wetting models and for an informal discussion of what the interesting questions are and what is expected to be true). We will consider rather general distributions, but we will assume that all the moments of $R_0^{(i)}$ are finite. As it will be clear later, for our purposes it is actually useful to write

$$R_0^{(i)} = \exp(\beta \omega_i - \log M(\beta) + h),$$

(1.4)

with $\beta \geq 0$, $h \in \mathbb{R}$, $\{\omega_i\}_{i \in \mathbb{N}}$ a sequence of exponentially integrable IID centered random variables normalized to $E \omega_i^2 = 1$ and for every $\beta$

$$M(\beta) := E \exp(\beta \omega_1) < \infty.$$  

(1.5)

The law of $\{\omega_i\}_{i \in \mathbb{N}}$ is denoted by $P$ and we will often alternatively denote the average $E(\cdot)$ by brackets $\langle \cdot \rangle$.

Note that, for every $n$, $\{R_n^{(i)}\}_{i \in \mathbb{N}}$ are IID random variables and therefore this dynamical system is naturally re-interpreted as the evolution of the probability law $L_n$ (the law of $R_n^{(1)}$): given $L_n$, the law $L_{n+1}$ is obtained by constructing two IID variables distributed according to $L_n$ and applying

$$R_{n+1} = \frac{R_n^{(1)} R_n^{(2)} + (B - 1)}{B}.$$  

(1.6)

Of course, the iteration (1.6) is well defined for every $B \neq 0$. In particular, as detailed in Appendix A.3 the case $B \in (1, 2)$ can be mapped exactly into the case $B > 2$ we explicitly consider here, while for $B < 1$ one loses the direct statistical mechanics interpretation of the model discussed in Section 1.6.

1.2. Quadratic maps and pinning models. The model we are considering may be viewed as a hierarchical version of a class of statistical mechanics models that go under the name of (disordered) pinning or wetting models [13, 15], that are going to be described in some detail in § 1.6. It has been introduced in [11, Section 4.2], where the partition function $R_n = R_n^{(1)}$ is defined for $B = 2, 3, \ldots$ as

$$R_n = E_B^B \left[ \exp \left( \sum_{i=1}^{2^n} (\beta \omega_i - \log M(\beta) + h) 1_{\{S_i = (d_{i-1}, d_i)\}} \right) \right],$$  

(1.7)

with $\{S_i\}_{i=0,\ldots,2^n}$ a simple random walk (of law $P_B^B$) on a hierarchical diamond lattice with growth parameter $B$ and $d_0, \ldots, d_{2^n}$ are the labels for the vertices of a particular path that has been singled out and dubbed defect line. The construction of diamond lattices and a graphical description of the model are detailed in Figure 1 and its caption.

The phenomenon that one is trying to capture is the localization at (or delocalization away from) the defect line, that is one would like to understand whether the rewards (that could be negative, hence penalizations) force the trajectories to stick close to the defect line, or the trajectories avoid the defect line. A priori it is not clear that there is necessarily a sharp distinction between these two qualitative behaviors, but it turns out that it is the case and which of the two scenarios prevails may be read from the asymptotic behavior of $R_n$. The Laplace asymptotics carries already a substantial amount of information, so we define the quenched free energy

$$F(\beta, h) := \lim_{n \to \infty} \frac{1}{2^n} \log R_n^{(1)},$$

(1.8)
Figure 1. Given $B = 2, 3, \ldots$ ($B = 3$ in the drawing) we build a diamond lattice by iterative steps (left to right): at each step one replaces every bond by $B$ branches consisting of two bonds each. A trajectory of our process in a diamond lattice at level $n$ is a path connecting the two poles $d_0$ and $d_2^n$: two trajectories, $a$ and $b$, are singled out by thick lines. Note that at level $n$, each trajectory is made of $2^n$ bonds and there are $N_n$ trajectories, $N_0 := 1$ and $N_{n+1} = BN_n^2$. A simple random walk at level $n$ is the uniform measure over the $N_n$ trajectories. A special trajectory, with vertices labeled $d_0, d_1, \ldots, d_2^n$, is chosen (and marked by a triple line: the right-most trajectory in the drawing, but any other trajectory would lead to an equivalent model), we may call it defect line or wall boundary, and rewards $u_j := \beta \omega_j - \log M(\beta) + h$ (negative or positive) are assigned to the bonds of this trajectory. The energy of a trajectory depends on how many and which bonds it shares with the defect line: trajectory $a$ carries no energy, while trajectory $b$ carries energy $u_1 + u_2$. The pinning model is then built by rewarding or penalizing the trajectories according to their energy in the standard statistical mechanics fashion and the partition function of such a model is therefore given by $R_n$ in (1.7). It is rather elementary, and fully detailed in [11], how to extract from (1.7) the recursion (1.6). But the recursion itself is well defined for arbitrary real value $B \neq 0$ and one may forget the definition of the hierarchical lattice, as we do here. The definition of $P_n^B$ can also be easily generalized to $B > 1$, see (A.11) of Appendix A.

where the limit is in the almost sure sense: the existence of such a limit and the fact that it is non-random may be found in Theorem [1.1]. Note in fact that $\partial_h F(\beta, h)$ coincides with the $n \to \infty$ limit of $E_{n, \omega}^B[2^{-n} \sum_1 \mathbf{1}_{\{(S_{i+1}, S_i) = (d_{i+1}, d_i)\}}]$, where $P_{n, \omega}^B$ is the probability measure associated to the partition function $R_n$, when $\partial_h F(\beta, h)$ exists (that is for all $h$ except at most a countable number of points, by convexity of $F(\beta, \cdot)$, see below). Therefore
\( \partial_h \mathcal{V}(\beta, h) \) measures the density of contacts between the walk and the defect line and below we will see that \( \partial_h \mathcal{V}(\beta, h) \) is zero up to a critical value \( h_c(\beta) \), and positive for \( h > h_c(\beta) \): this is a clear signature of a localization transition.

1.3. A first look at the role of disorder. Of course if \( \beta = 0 \) the disorder \( \omega \) plays no role and the model reduces to the one-dimensional map (1.2) (in our language \( \beta > 0 \) corresponds to the model in which disorder is present). This map has two fixed points: 1, which is stable, and \( B - 1 \), which is unstable. More precisely, if \( r_0 < B - 1 \) then \( r_n \) converges monotonically (and exponentially fast) to 1. If \( r_0 > B - 1 \), \( r_n \) increases to infinity in a super-exponential fashion, namely \( 2^{-n} \log r_n \) converges to a positive number which is of course function of \( r_0 \). The question is whether, and how, introducing disorder in the initial condition (\( \beta > 0 \)) modifies this behavior.

There is also an alternative way to link (1.1) and (1.2). In fact, by taking the average of (1.1) we obtain
\[
\langle R_{n+1} \rangle = \frac{\langle R_n \rangle^2 + (B-1)}{B},
\]
where we have dropped the superscript in \( \langle R^{(i)}_n \rangle \). Therefore the behavior of the sequence \( \{\langle R_n \rangle\} \) is (rather) explicit, in particular such a sequence tends (monotonically) to 1 if \( \langle R_0 \rangle < B - 1 \), while \( \langle R_n \rangle = B - 1 \) for \( n = 1, 2, \ldots \) if \( \langle R_0 \rangle = B - 1 \). This is already a strong piece of information on \( R^{(1)}_n \) (the sequence \{\mathcal{L}_n\} is tight). Less informative is instead the fact that \( \langle R_n \rangle \) diverges if \( \langle R_0 \rangle > B - 1 \), even if we know precisely the speed of divergence: in fact the sequence of random variables can still be tight! In principle such an issue may be tackled by looking at higher moments, but while \( \langle R_n \rangle \) satisfies a closed recursion, the same is not true for higher momenta in the sense that the recursions they satisfy depend on the behavior of the lower-order moments. For instance, if we set \( \Delta_n := \text{var}(R_n) \), we have
\[
\Delta_{n+1} = \frac{\Delta_n (2\langle R_n \rangle^2 + \Delta_n)}{B^2}.
\]
In principle such an approach can be pushed further, but most important for understanding the behavior of the system is capturing the asymptotic behavior of \( \log R_n^{(i)} \), i.e. (1.8).

1.4. Quenched and annealed free energies. Our first result says, in particular, that the quenched free energy (1.8) is well defined:

**Theorem 1.1.** The limit in (1.8) exists \( \mathbb{P}(d\omega) \)-almost surely and in \( L^1(\mathbb{P}) \), it is almost-surely constant and it is non-negative. The function \( (\beta, h) \mapsto \mathcal{V}(\beta, h + \log M(\beta)) \) is convex and \( \mathcal{V}(\beta, \cdot) \) is non-decreasing (and convex). These properties are inherited from \( \mathcal{F}_N(\cdot, \cdot) \), defined by
\[
\mathcal{F}_N(\beta, h) = \frac{1}{2N} \langle \log R_N \rangle.
\]
Moreover \( \mathcal{F}_N(\beta, h) \) converges to \( \mathcal{V}(\beta, h) \) with exponential speed, more precisely for all \( N \geq 1 \)
\[
\mathcal{F}_N(\beta, h) - 2^{-N} \log B \leq \mathcal{V}(\beta, h) \leq \mathcal{F}_N(\beta, h) + 2^{-N} \log \left( \frac{B^2 + B - 1}{B(B - 1)} \right).
\]

Let us also point out that \( \mathcal{V}(\beta, h) \geq 0 \) is immediate in view of the fact that \( R_n^{(i)} \geq (B - 1)/B \) for \( n \geq 1 \), cf. (1.1). The lower bound \( \mathcal{V}(\beta, h) \geq 0 \) implies that we can split the parameter space (or phase diagram) of the system according to \( \mathcal{V}(\beta, h) = 0 \) and \( \mathcal{V}(\beta, h) > 0 \) and this clearly corresponds to sharply different asymptotic behaviors of \( R_n \). In conformity with related literature, see §1.6, we define localized and delocalized phases.
as $L := \{ (\beta, h) : F(\beta, h) > 0 \}$ and $D := \{ (\beta, h) : F(\beta, h) = 0 \}$ respectively. It is therefore natural to define, for given $\beta \geq 0$, the critical value $h_c(\beta)$ as

$$h_c(\beta) = \sup \{ h \in \mathbb{R} : F(\beta, h) = 0 \}. \quad (1.13)$$

Theorem 1.2 says in particular that

$$h_c(\beta) = \inf \{ h \in \mathbb{R} : F(\beta, h) > 0 \}, \quad (1.14)$$

and that $F(\beta, \cdot)$ is (strictly) increasing on $(h_c(\beta), \infty)$. Note that, thanks to the properties we just mentioned, the contact fraction, defined in the end of §1.2, is zero $h < h_c(\beta)$ and is instead positive if $h > h_c(\beta)$ (define the contact fraction by taking the inferior limit for the values of $h$ at which $F(\beta, \cdot)$ is not differentiable).

Another important observation on Theorem 1.1 is that it yields also the existence of $\lim_{n \to \infty} 2^{-n} \log(R_n)$ and this limit is simply $F(0, h)$, in fact $r_n(h, 0) = 2^{-n} \log(R_n)$ for every $n$. In statistical mechanics language $(R_n)$ is an annealed quantity and $\lim_{n \to \infty} 2^{-n} \log(R_n)$ is the annealed free energy: by Jensen inequality it follows that $F(\beta, h) \leq F(0, h)$ and $h_c(\beta) \geq h_c(0)$. It is also a consequence of Jensen inequality (see Remark A.1) the fact that $F(\beta, h + \log M(\beta)) \geq F(0, h)$, so that $h_c(\beta) \leq h_c(0) + \log M(\beta)$. Summing up:

$$h_c(0) \leq h_c(\beta) \leq h_c(0) + \log M(\beta). \quad (1.15)$$

Therefore, by the convexity properties of $F(\cdot, \cdot)$ (Theorem 1.1) and by (1.15), we see that $h_c(\cdot) - \log M(\cdot)$ is concave and may diverge only at infinity, so that $h_c(\cdot)$ is a continuous function.

The following result on the annealed system, i.e. just the non-disordered system, is going to play an important role:

**Theorem 1.2.** (Annealed system estimates). The function $h \mapsto F(0, h)$ is real analytic except at $h = h_c := h_c(0)$. Moreover $h_c = \log(B - 1)$ and there exists $c = c(B) > 0$ such that for all $h \in (h_c, h_c + 1)$

$$c(B)^{-1}(h - h_c)^{1/\alpha} \leq F(0, h) \leq c(B)(h - h_c)^{1/\alpha}, \quad (1.16)$$

where

$$\alpha := \frac{\log(2(B - 1)/B)}{\log 2}. \quad (1.17)$$

Bounds on the annealed free energy can be extracted directly from (1.12), namely that for every $n \geq 1$

$$\frac{B(B - 1)}{B^2 + B - 1} \exp(2^n F(0, h)) \leq < R_n > \leq B \exp(2^n F(0, h)). \quad (1.18)$$

Moreover let us note from now that $\alpha \in (0, 1)$ and that $1/\alpha > 2$ if and only if $B < B_c := 2 + \sqrt{2}$, and $1/\alpha = 2$ for $B = B_c$. It follows that $F(0, h) = o((h - h_c)^2)$ for $B < B_c (\alpha < 1/2)$, while this is not true for $B > B_c (\alpha > 1/2)$.

**Remark 1.3.** For models defined on hierarchical lattices, in general one does not expect the (singular part of the) free energy to have a pure power-law behavior close to the critical point $h_c$, but rather to behave like $H(\log(h - h_c))(h - h_c)^\nu$, with $\nu$ the critical exponent and $H(\cdot)$ a periodic function, see in particular [12]. Note that, unless $H(\cdot)$ is trivial (i.e. constant), the oscillations it produces become more and more rapid for $h \searrow h_c$.

We have observed numerically such oscillations in our case and therefore we expect that estimate (1.16) cannot be improved at a qualitative level as $h$ approaches $h_c$ (the problem
of estimating sharply the size of the oscillations appears to be a non-trivial one, but this is not particularly important for our analysis).

1.5. **Results for the disordered system.** The first result we present gives information on the phase diagram: we use the definition

\[ \Delta = \Delta(\beta) := (B - 1)^2 \left( \frac{M(2\beta)}{M(\beta)^2} - 1 \right) \geq 0, \tag{1.19} \]

so that \( \text{Var}(R_0)^{h = h_c} \Delta. \) The quantity \( \Delta \) should be thought of as the size of the disorder at a given \( \beta. \)

**Theorem 1.4.** Recall that the critical value for the annealed system is \( h_c = \log(B - 1). \) We have the following estimates on the quenched critical line:

1. Choose \( B \in (2, B_c). \) If \( \Delta(\beta) \leq B^2 - 2(B - 1)^2 \) then \( h_c(\beta) = h_c. \)
2. Choose \( B > B_c. \) Then \( h_c(\beta) > h_c \) for every \( \beta > 0. \) Moreover for \( \beta \) small (say, \( \beta \leq 1 \)) one can find \( c \in (0, 1) \) such that

\[ c^{\beta^{2\alpha/(2\alpha-1)}} \leq h_c(\beta) - h_c \leq c^{-1}\beta^{2\alpha/(2\alpha-1)}. \tag{1.20} \]

3. If \( B = B_c \) then one can find \( C > 0 \) such that, for \( \beta \leq 1, \)

\[ 0 \leq h_c(\beta) - h_c \leq \exp(-C/\beta^2). \tag{1.21} \]

Moreover if \( \omega_1 \) is such that \( \mathbb{P}(\omega_1 > t) > 0 \) for every \( t > 0, \) then for every \( B > 2 \) we have \( h_c(\beta) - h_c > 0 \) for \( \beta \) sufficiently large, in fact \( \lim_{\beta \to \infty} h_c(\beta) = \infty. \)

Of course (1.21) leaves open an evident question for \( B = B_c, \) that will be discussed in §1.6. We point out that the constant \( C \) is explicit (see Proposition 3.4) but it does not have any particular meaning. It is possible to show that \( C \) can be chosen arbitrarily close to the constant given in [11], but here, for the sake of simplicity, we have decided to prove a weaker result (i.e., with a smaller constant). This is not a crucial issue, since the upper bound on \( h_c(\beta) \) is not comforted by a suitable lower bound.

The next result is about the free energy.

**Theorem 1.5.** We have the following:

1. Choose \( B \in (2, B_c) \) and \( \beta \) such that \( \Delta(\beta) < B^2 - 2(B - 1)^2. \) Then for every \( \eta \in (0, 1) \) one can find \( \epsilon > 0 \) such that

\[ F(\beta, h) \geq (1 - \eta)F(0, h), \tag{1.22} \]

for \( h \in (h_c, h_c + \epsilon). \)

2. Choose \( B > B_c. \) Then for every \( \eta \in (0, 1) \) one can find \( c > 0 \) and \( \beta_0 > 0 \) such that (1.22) holds for \( \beta < \beta_0 \) and \( h - h_c \in (c^{\beta^{2\alpha/(2\alpha-1)}}, 1). \)

While the relevance of the analysis of the free energy will be discussed in depth in the next subsection, it is natural to address the following issue: in a *sharp* sense, how does the random array \( R_0^{(1)} \) behave as \( n \) tends to infinity? We recall that the non-disordered system displays only three possible asymptotic behaviors: \( r_n \to 1, \) \( r_n = B - 1 \) for all \( n \) and \( r_n \not\to \infty \) in a super-exponentially fast fashion.

What can be extracted directly from the free energy is quite satisfactory if the free energy is positive: \( R_0^{(1)} \) diverges at a super-exponential speed that is determined to leading order. However, the information readily available from the fact that the free energy is zero
is rather poor; this can be considerably improved, starting with the fact that, by the lower bound in (1.12), if the free energy is zero then \( \sup_n \langle \log R_n \rangle \leq \log B \), which implies the tightness of the sequence.

**Theorem 1.6.** If \( f(\beta, h) = 0 \) then the sequence \( \{R_n\}_n \) is tight. Moreover if \( h < h_c(\beta) \) then

\[
\lim_{n \to \infty} R_n^{(1)} = 1 \quad \text{in probability.} \tag{1.23}
\]

Let us mention that we also establish almost sure convergence of \( R_n \) toward 1 when we are able to find \( \gamma \in (0, 1) \) and \( n \in \mathbb{N} \) such that \( \mathbb{E}[(R_n - 1)^\gamma] \) is smaller than an explicit constant (see Section 4, in particular Remark 4.4). It is interesting to compare such results with the estimates on the size of the partition function \( Z_{N,\omega} \) of non-hierarchical pinning/wetting models, which are proven in [25, end of Sec. 3.1] in the delocalized phase, again via estimation of fractional moments of \( Z_{N,\omega} \) (which plays the role of our \( R_n \)).

What one should expect at criticality is rather unclear to us (see however [23] for a number of predictions and numerical results on hierarchical pinning and also [6, 7] for some theoretical considerations on a different class of hierarchical models).

### 1.6. Pinning models: the role of disorder.

Hierarchical models on diamond lattices, homogeneous or disordered [3, 4, 5, 6, 7, 9], are a powerful tool in the study of the critical behavior of statistical mechanics models, especially because real-space renormalization group transformations à la Migdal-Kadanoff are exact in this case. In most of the cases, hierarchical models are introduced in association with a more realistic non-hierarchical one. It should however be pointed out that hierarchical models on diamond lattices are not rough simplifications of non-hierarchical ones. They are in fact meant to retain the essential features of the associated non-hierarchical models (notably: the critical properties!). In particular, it would be definitely misleading to think of the hierarchical model as a mean field approximation of the real one.
Non-hierarchical pinning models have an extended literature (e.g. [13, 15]). They may be defined like in (1.7), with $S$ a symmetric random walk with increment steps in $\{-1,0,+1\}$, energetically rewarded or penalized when the bond $(S_{n-1}, S_n)$ lies on the horizontal axis (that is $d_j = 0$ for every $j$ in (1.7)), but they can be restated in much greater generality by considering arbitrary homogeneous Markov chains that visit a given site (say, the origin) with positive probability and that are then rewarded or penalized when passing by this site. In their non-disordered version [13], this general class of models has the remarkable property of being exactly solvable, while displaying a phase transition – a localization-delocalization transition – and the order of such a transition depends on a parameter of the model (the tail decay exponent of the distribution of the first return of the Markov chain to the origin: we call $\alpha$ such an exponent and it is the analog of the quantity $\alpha$ in our hierarchical context, cf. (1.17); one should however note that for non-hierarchical models values $\alpha \geq 1$ can also be considered, in contrast with the model we are studying here). As a matter of fact, transitions of all order, from first order to infinite order, can be observed in such models. They therefore constitute an ideal set-up in which to address the natural question: how does the disorder affect the transition?

Such an issue has often been considered in the physical literature and a criterion, proposed by A. B. Harris in a somewhat different context, adapted to pinning models [14, 11], yields that the disorder is irrelevant if $\beta$ is small and $\alpha < 1/2$, meaning by this that quenched and annealed critical points coincide and the critical behavior of the free energy is the same for annealed and quenched system (note that the annealed system is a homogeneous pinning system, and therefore exactly solvable). The disorder instead becomes relevant when $\alpha > 1/2$, with a shift in the critical point (quenched is different from annealed) and different critical behaviors (possibly expecting a smoother transition, but the Harris criterion does not really address such an issue). In the marginal case, $\alpha = 1/2$, disorder could be marginally relevant or marginally irrelevant, but this is an open issue in the physical literature, see [14, 11] and [15] for further literature.

Much progress has been made very recently in the mathematical literature on non-hierarchical pinning models, in particular:

(1) The irrelevant disorder regime is under control [11, 24] and even more detailed results on the closeness between quenched and annealed models can be established [20].

(2) Concerning the relevant disorder regime, in [19] it has been shown that the quenched free energy is smoother than the annealed free energy if $\alpha > 1/2$. The non-coincidence of quenched and annealed critical points for large disorder (and for every $\alpha$) has been proven in [25] via an estimation of non-integer moments of the partition function. The idea of considering non-integer moments (this time, of $R_n - 1$) plays an important role also in the present paper.

(3) A number of results on the behavior of the paths of the model have been proven addressing the question of what can be said about the trajectories of the system once we know that the free energy is zero (or positive) [17, 18]. One can in fact prove that if $F(\beta, h) > 0$ then the process sticks close to the origin (in a strong sense) and it is therefore in a localized ($L$) regime. When $F(\beta, h) = 0$, and leaving aside the critical case, one expects that the process essentially never visits the origin, and we say that we are in a delocalized regime ($D$). We refer to [15] for further discussion and literature on this point.
In this work we rigorously establish the full Harris criterion picture for the hierarchical version of the model. In particular we wish to emphasize that we do show that there is a shift in the critical point of the system for arbitrarily small disorder if \( \alpha > 1/2 \) and we locate such a point in a window that has a precise scaling behavior, cf. (1.20) (a behavior which coincides with that predicted in [11]).

As a side remark, one can also generalize the smoothing inequality proven in [19] to the hierarchical context and show that for every \( B > 2 \) there exists \( c(B) < \infty \) such that, if \( \omega_1 \sim \mathcal{N}(0, 1) \), for every \( \beta > 0 \) and \( \delta > 0 \) one has
\[
\mathcal{V}(\beta, h_c(\beta) + \delta) \leq \delta^2 c(B)/\beta^2,
\]
which implies that annealed and quenched free energy critical behaviors are different for \( \alpha > 1/2 \), cf. (1.16) (as in [19], such inequality can be generalized well beyond Gaussian \( \omega_1 \), but we are not able to establish it only assuming the finiteness of the exponential moments of \( \omega_1 \)). The proof of (1.24) is detailed in [22].

Various intriguing issues remain open:

1. Is there a shift in the critical point at small disorder if \( B = B_c \) (that is \( \alpha = 1/2 \))?
   We stress that in [11] is predicted that \( h_c(\beta) - h_c(0) \approx \exp(-\log 2/\beta^2) \) for \( \beta \) small.

2. Can one go beyond (1.24)? That is, can one find sharp estimates on the critical behavior when the disorder is relevant?

3. With reference to the caption of Figure 2, can one prove \( \beta_c > \hat{\beta} \) (for \( B < B_c \))?

4. Does the law of \( R_n \) converge to a non-trivial limit for \( n \to \infty \), when \( h = h_c(\beta) \)?

Of course, all these issues are open also in the non-hierarchical context and, even if not every question becomes easier for the hierarchical model, it may be the right context in which to attack them first.

### 1.7. Some recurrent notation and organization of the subsequent sections.

Aside for standard notation like \( \lceil x \rceil := \min\{n \in \mathbb{Z} : n \geq x\} \) and \( |x| := |x| - 1 \), or \( |.|^+ := \max(0, .) \), we will repeatedly use \( \Delta_n \) for the variance of \( R_n^{(1)} \), see (1.10), and \( Q_n := \Delta_n/(\langle R_n \rangle^2) \) so that from (1.9) and (1.10), one sees that
\[
Q_{n+1} = 2 \left( \frac{B - 1}{B} \right)^2 \left( \frac{\langle R_n \rangle^4}{\langle R_{n+1} \rangle^2(B-1)^2} \right) \left( Q_n + \frac{1}{2} Q_n^2 \right),
\]
and we observe that
\[
Q_0 = \left( \frac{M(2\beta)}{M(\beta)^2} - 1 \right)^{\beta/2} \beta^2.
\]

Note that \( 2(B-1)^2/B^2 \) is smaller than 1 if and only if \( B < B_c \) and
\[
\left( \frac{\langle R_n \rangle^4}{\langle R_{n+1} \rangle^2(B-1)^2} \right) \leq \left( \frac{B}{B-1} \right)^2.
\]

We will also frequently use \( P_n := \langle R_n \rangle - (B - 1) \), which satisfies
\[
P_{n+1} = 2 \left( \frac{B - 1}{B} \right) P_n + \frac{1}{B} P_n^2,
\]
and \( P_0 = \varepsilon \) in our notations (see (1.30) below). With some effort, one can explicitly verify that for every \( n \)
\[
\left( \frac{\langle R_n \rangle^4}{\langle R_{n+1} \rangle^2(B-1)^2} \right) \leq 1 + \frac{4P_n}{B(B-1)}.
\]
Finally, there is some notational convenience at times in making the change of variables
\[ \varepsilon := \langle R_0 \rangle - (B - 1) = e^h - (B - 1), \quad (1.30) \]
and
\[ \hat{f}(\beta, \varepsilon) := f(\beta, h(\varepsilon)), \quad (1.31) \]
and when we write \( h(\varepsilon) \) we refer to the invertible map defined by (1.30).

The work is organized as follows. Part (1) of Theorem 1.4 and of Theorem 1.5 are proven in Section 2. In Section 3 we prove part (2) of Theorem 1.5 and, as a consequence, part (2) of Theorem 1.4 except the lower bound in (1.20). Part (3) of Theorem 1.4 is proven in Section 3.1 and the lower bound of (1.20) in Section 4 (after a brief sketch of our method). The proof of Theorem 1.6 is given in Section 5. Finally, the proofs of Theorems 1.1 and 1.2 are based on more standard techniques and can be found in Appendix A.

2. Free energy lower bounds: \( B < B_c = 2 + \sqrt{2} \)

We want to give a proof of part (1) of Theorem 1.5 which in particular implies part (1) of Theorem 1.4.

The strategy goes roughly as follows: since \( h > h_c \) is close to \( h_c \), that is \( \varepsilon(= R_0) > 0 \) is close to 0, \( P_n \) keeps close to zero for many values of \( n \) and \( P_{n+1} \approx (2(B - 1)/B)P_n \) (recall (1.28) and the fact that \( 2(B - 1)/B > 1 \) for \( B > 2 \). This is going to be true up to \( n \) much smaller than \( \log(1/\varepsilon)/\log(2(B - 1)/B) \). At the same time for the normalized variance \( Q_n \) we have the approximated recursion \( Q_{n+1} \approx 2((B - 1)/B)^2(Q_n + (1/2)Q_n^2) \), which one derives from (1.25) by using \( P_n \approx 0 \). Since \( 2((B - 1)/B)^2 < 1 \) is equivalent to \( B < B_c \), we easily see that (if \( Q_0 \) is not too large) \( Q_n \) shrinks at an exponential rate. This scenario actually breaks down when \( P_n \) is no longer small, but at that stage \( Q_n \) is already extremely small (such a value of \( n \) is precisely defined and called \( n_0 \) below). From that point onward \( Q_n \) starts growing exponentially and eventually it diverges, but after \( (1 + \gamma)n_0 \) steps, for some \( \gamma > 0 \), \( Q_n \) is still small while \( P_n \) is large, so that a second moment argument, combined with (1.12) which yields a control on \( f(\beta, h) \) via \( F_n(\beta, h) \), allows to conclude.

Before starting the proof we give an upper bound on the size of \( Q_n(= \Delta_n/\langle R_n \rangle^2) \) in the regime in which the recursion for \( \langle R_n \rangle \) can be linearized (for what follows, recall (1.25), (1.26) and (1.27)).

**Lemma 2.1.** Let \( B \in (2, B_c) \) and \( \beta \) such that \( \Delta = \Delta(\beta) < B^2 - 2(B - 1)^2 \). There exist \( c := c(B, \Delta) > 0 \), \( c_1 := c_1(B, \Delta) > 0 \) and \( \delta_0 := \delta_0(B, \Delta) > 0 \) with
\[ 2(1 + \delta_0) \left( \frac{B - 1}{B} \right)^2 < 1, \quad (2.1) \]
such that for every \( \varepsilon \) satisfying \( 0 < \varepsilon/(B - 1) < ((B^2 - 2(B - 1)^2)/\Delta)^{1/2} - 1 \) (recall the definition (1.30) of \( \varepsilon \)) and
\[ n \leq n_0 := \left\lfloor \log \left( c \delta_0/\varepsilon \right)/\log \left( \frac{2(B - 1)}{B} \right) \right\rfloor, \quad (2.2) \]
one has
\[ Q_n \leq c_1 \left( 2(1 + \delta_0) \left( \frac{B - 1}{B} \right)^2 \right)^n Q_0. \quad (2.3) \]
Note that the condition on $\varepsilon$ simply guarantees $\Delta_0 = (1 + \varepsilon/(B-1))^2\Delta$ is smaller than $B^2 - 2(B-1)^2$.

**Proof of Lemma 2.1.** Recall that $P_n = \langle R_n \rangle - (B-1)$ and that it satisfies the recursion (1.28) (and that $P_0 = \varepsilon$).

For $G_n := (P_n/P_0)(2(B-1)/B)^n$, we have from (1.28) and (1.30)

$$G_{n+1} = G_n + \frac{\varepsilon}{B} \left( \frac{2(B-1)}{B} \right)^{n-1} G_n^2,$$

(2.4)

and $G_0 = 1$. If $G_m \leq 2$ for $m \leq n$, then

$$\frac{G_{n+1}}{G_n} \leq 1 + 2\frac{\varepsilon}{B} \left( \frac{2(B-1)}{B} \right)^n,$$

(2.5)

which entails

$$G_{n+1} \leq \exp \left( \frac{2\varepsilon}{B} \sum_{j=0}^n \left( \frac{2(B-1)}{B} \right)^{j-1} \right) \leq 1 + \varepsilon C(B) \left( \frac{2(B-1)}{B} \right)^{n+1},$$

(2.6)

for a suitable constant $C(B) < \infty$.

As we have already remarked, our assumption on $\varepsilon$ yields $\Delta_0 < B^2 - 2(B-1)^2$, so

$$Q_0 < \left( \frac{B}{B-1} \right)^2 - 2.$$  

(2.7)

Choose $\delta_0 > 0$ sufficiently small so that (2.1) is satisfied and moreover

$$2 \left( \frac{B-1}{B} \right)^2 \left( 1 + \delta_0 \right) (Q_0 + \frac{1}{2}Q_0^2) < Q_0,$$

(2.8)

(the latter can be satisfied in view of (2.7)). It is immediate to deduce from (2.6) that if $c$ in (2.2) is chosen sufficiently small (in particular, $c \leq B(B-1)/8$), then $G_n \leq 2$ for $n \leq n_0$ and, as an immediate consequence,

$$0 < P_n \leq 2\varepsilon \left( \frac{2(B-1)}{B} \right)^n \leq 2c\delta_0 \leq \delta_0 \frac{B(B-1)}{4},$$

(2.9)

where the first inequality is immediate from (1.28) and $P_0 = \varepsilon > 0$. Now we apply (1.29)

$$Q_{n+1} \leq 2 \left( \frac{B-1}{B} \right)^2 \left( 1 + \delta_0 \right) (Q_n + \frac{1}{2}Q_n^2).$$

(2.10)

Notice also that $Q_1 < Q_0$ thanks to (2.3). From this it is easy to deduce that, as long as $n \leq n_0$, $Q_n$ is decreasing and satisfies (2.3) for a suitable $c_1$. In particular, $c_1(B, \Delta_0)$ can be chosen such that $\lim_{\Delta_0 \searrow 0} c_1(B, \Delta_0) = 1$. \hfill $\square$

**Proof of Theorem 1.5, part (1).** We use the bound (1.27) to get

$$Q_{n+1} \leq 2 \left( Q_n + \frac{1}{2}Q_n^2 \right) \leq 3Q_n,$$

(2.11)

where the last inequality holds as long as $Q_n \leq 1$. Then we apply Lemma 2.1 (recall in particular $\delta_0$ and $n_0$ in there). Combining (2.3) and (2.11) we get

$$Q_n \leq Q_{n_0} 3^{n-n_0} \leq c_1 Q_0 \left( 2(1 + \delta_0) \left( \frac{B-1}{B} \right)^2 \right)^{n_0} 3^{n-n_0},$$

(2.12)
for every $n \geq n_0$ satisfying $Q_n \leq 1$ (which implies $Q'_n \leq 1$ for all $n' \leq n$ as $Q_n$ is increasing). Of course this boils down to requiring that the right-most term in (2.12) does not get larger than 1. Since $n_0$ diverges as $\varepsilon \searrow 0$, if we choose $\gamma > 0$ such that $3^*2(1 + \delta_0)(B - 1)^2/B^2 < 1$, then the right-most term in (2.12) is bounded above for every $n \leq (1 + \gamma)n_0$ by a quantity $o_{\varepsilon}(1)$ which vanishes for $\varepsilon \to 0$. Summing all up:

$$Q_{[(1+\gamma)n_0]} = o_{\varepsilon}(1).$$  \hspace{1cm} (2.13)

Next, note that

$$(\log R_{[(1+\gamma)n_0]} \geq \log \left( \frac{1}{2} \langle R_{[(1+\gamma)n_0]} \rangle \right) \cdot \mathbb{P}\left( R_{[(1+\gamma)n_0]} \geq \frac{1}{2} \langle R_{[(1+\gamma)n_0]} \rangle \right) + \log \left( \frac{B - 1}{B} \right),$$

where we have used the fact that $R_n \geq (B - 1)/B$ for $n \geq 1$. Applying the Chebyshev inequality one has

$$\mathbb{P}\left( R_{[(1+\gamma)n_0]} \geq (1/2) \langle R_{[(1+\gamma)n_0]} \rangle \right) \geq 1 - 4Q_{[(1+\gamma)n_0]} = 1 + o_{\varepsilon}(1).$$

Therefore, from (1.15), (2.14) and (2.15) one has

$$F_{[(1+\gamma)n_0]}(\beta, h) \geq (1 + o_{\varepsilon}(1))\overline{F}(0, \varepsilon) - 2^{-[(1+\gamma)n_0]}c(B),$$

for some $c(B) < \infty$ and, from (1.12) (or, equivalently, (1.12)),

$$\overline{F}(\beta, \varepsilon) \geq (1 + o_{\varepsilon}(1))\overline{F}(0, \varepsilon) - 2^{-[(1+\gamma)n_0]}c_1(B).$$

Since $\overline{F}(0, \varepsilon)2^{-[(1+\gamma)n_0]}$ diverges for $\varepsilon \to 0$ if $\gamma > 0$, as one may immediately check from (2.2) and (1.16), one directly extracts that for every $\eta > 0$ there exists $\varepsilon_0 > 0$ such that

$$F(\beta, h) = F(\beta, \varepsilon) \geq (1 - \eta)\overline{F}(0, \varepsilon) = (1 - \eta)\overline{F}(0, h),$$

for $\varepsilon \leq \varepsilon_0$, i.e. $h \leq h_c(0) + \log(1 + \varepsilon/(B - 1))$, and we are done. \hspace{1cm} $\Box$

3. **Free energy lower bounds:** $B \geq B_c = 2 + \sqrt{2}$

The arguments in this section are close in spirit to the ones of the previous section. However, since $B > B_c$, the constant $2(B - 1)/B^2$ in the linear term of the recursion equation (1.23) is larger than one, so the normalized variance $Q_n$ grows from the very beginning. Nonetheless, if $Q_0$ is small, it will keep small for a while. The point is to show that, if $P_0$ is not too small (this concept is of course related to the size of $Q_0$), when $Q_n$ becomes of order one $P_n$ is sufficiently large. Therefore, once again, a second moment argument and (1.12) yield the result we are after, that is:

**Proposition 3.1.** Let $B > B_c$. For every $\eta \in (0, 1)$ there exist $c > 0$ and $\beta_0 > 0$ such that

$$F(\beta, h) \geq (1 - \eta)\overline{F}(0, h),$$

for $\beta \leq \beta_0$ and $c\beta^{2\alpha/(2\alpha - 1)} \leq h - h_c(0) \leq 1$. This implies in particular that $h_c(\beta) < h_c(0) + c\beta^{2\alpha/(2\alpha - 1)}$, for every $\beta \leq \beta_0$.

Of course this proves part (2) of Theorem 1.5 and the upper bound in (1.20).

In this section $q := 2(B - 1)^2/B^2$ and $\bar{q} := 2(B - 1)/B$: note that in full generality $q < \bar{q} < 2$ and $\bar{q} > 1$, while $q > 1$ because we assume $B > B_c$. One can easily check that

$$\frac{\alpha}{2\alpha - 1} = \frac{\log \bar{q}}{\log q},$$

(3.2)
Moreover in what follows some expressions are in the form \( \max A, A \subset \mathbb{N} \cup \{0\} \); also when we do not state it explicitly, we do assume that \( A \) is not empty (in all cases this boils down to choosing \( \beta \) sufficiently small).

We start with an upper bound on the growth of \( \langle R_n \rangle = (B - 1) + P_n \) (recall (1.28)) for \( n \) not too large.

**Lemma 3.2.** If \( P_0 = c_1 \beta^{2\alpha/(2\alpha - 1)} \), \( c_1 > 0 \), then
\[
P_n \leq 2c_1 \beta^{2\alpha/(2\alpha - 1)} q^n \leq 1,
\]
for \( n \leq N_1 := \max \{ n : C_1(B) c_1 \beta^{2\alpha/(2\alpha - 1)} q^n \leq 1 \} \), where
\[
C_1(B) := 2 \max \left( \frac{1}{(q - 1)B \log 2}, 1 \right).
\]

The next result controls the growth of the variance of \( R_n \) in the regime when \( \langle R_n \rangle \) is close to \( (B - 1) \), i.e. \( P_n \) is small. Let us set
\[
N_2 := \max \{ n : (2c_1/(q - 1)) \beta^{2\alpha/(2\alpha - 1)} q^n \leq (\log 2)/2 \}.
\]

Observe that \( N_2 \leq N_1 \) and recall that \( Q_0 \beta \sim 0 \beta^2 \), cf. (1.29).

**Lemma 3.3.** Under the same assumptions as in Lemma 3.2, for \( Q_0 \leq 2\beta^2 \) and assuming \( c_1 \geq 20 \log q / \log q \) we have
\[
Q_n \leq 2Q_0 q^n,
\]
for \( n \leq N_2 \).

**Proof of Proposition 3.1** Let us choose \( c_1 \) as in Lemma 3.3. Let us observe also that, thanks to (3.2), \( N_2 = [ \log(1/\beta^2)/\log q - \log(C_1)/\log q ] \) for a suitable choice of the constant \( C = C(B) \). Therefore Lemma 3.3 ensures that
\[
Q_{N_2} \leq 4(C_1)^{-\log q / \log q}.
\]

From the definition of \( Q_n \) we directly see that \( Q_{n+1} \leq 3Q_n \) if \( Q_n \leq 1 \), as in (2.11). Therefore for any fixed \( \delta \in (0, 1/16) \)
\[
Q_{N_2+n} \leq 3^n 4(C_1)^{-\log q / \log q} \leq 4\delta,
\]
if
\[
n \leq N_3 := \left\lfloor \frac{\log q \log(C_1)}{\log q \log 3} - \frac{\log(1/\delta)}{\log 3} \right\rfloor.
\]
Since \( Q_{N_2+N_3} \leq 4\delta \) (by definition of \( N_3 \)), we have then
\[
P \left( R_{N_2+N_3} \leq \frac{1}{2} \langle R_{N_2+N_3} \rangle \right) \leq 16\delta.
\]

As a consequence, applying (1.12) and (1.18) with \( N = N_2 + N_3 \) one finds
\[
f(\beta, h) \geq (1 - 16\delta)f(0, h) - 2^{-(N_2+N_3)} c_3(B),
\]
of course with \( h \) such that \( P_0 = c_1 \beta^{2\alpha/(2\alpha - 1)} \), i.e.,
\[
h = \log \left( (B - 1) + c_1 \beta^{2\alpha/(2\alpha - 1)} \right).
\]

The last step consists in showing that if the last term in the right-hand side of (3.10) is negligible with respect to the first one. A look at (3.8) shows that \( N_3 \) can be made arbitrarily large by choosing \( c_1 \) large; moreover, by definition of \( N_2 \) we have
\[
2^{N_2 c_1^{1/\alpha}} \beta^{2/(2\alpha - 1)} \geq \frac{1}{2} C^{-1/\alpha},
\]
for \( \beta \) sufficiently small. From these two facts and from the critical behavior of \( F(0, \cdot) \) (cf. (1.16)) one deduces that for any given \( \delta \) one may take \( c_1 \) sufficiently large so that

\[
2^{-(N_2+N_3)}/F(0, h) \leq \delta,
\]

(3.13)

provided that \( h \leq h_c(0) + 1 \). For a given \( \eta \in (0, 1) \) this proves (3.1) whenever \( \beta \) is sufficiently small and \( c_1\beta^{2\alpha/(2\alpha-1)} \leq h - h_c(0) \leq 1 \), with \( c \) sufficiently large (when \( \eta \) is small) but independent of \( \beta \). \( \square \)

Proof of Lemma 3.2 Call \( N_0 \) the largest value of \( n \) for which \( P_n \leq 2c_1\beta^{2\alpha/(2\alpha-1)}q^n \) (for \( c_1 \) and \( \beta \) such that \( P_0 \leq 1 \)). Recalling (1.28), for \( n \leq N_0 \) we have

\[
\frac{P_{n+1}}{P_n} \leq \bar{q} \left( 1 + \frac{2c_1}{B\bar{q}}\beta^{2\alpha/(2\alpha-1)}q^n \right),
\]

(3.14)

so that for \( N \leq N_0 \), using the properties of \( \exp(\cdot) \) and the elementary bound \( \sum_{n=0}^{N-1} a^n \leq a^N/(a-1) \) (\( a > 1 \)), we obtain

\[
P_N \leq P_0 q^N \exp \left( \frac{2c_1}{(\bar{q}-1)B}\beta^{2\alpha/(2\alpha-1)}q^N \right).
\]

(3.15)

The latter estimate yields a lower bound on \( N_0 \):

\[
N_0 \geq \max \left\{ n : \frac{2c_1}{(\bar{q}-1)B}\beta^{2\alpha/(2\alpha-1)}q^n \leq \log 2 \right\}.
\]

(3.16)

\( N_1 \) is found by choosing it as the minimum between the right-hand side in (3.16) and the maximal value of \( n \) for which the second inequality in (3.3) holds. \( \square \)

Proof of Lemma 3.3 Let us call \( N_1' \) the largest \( n \) such that \( Q_n \leq 2Q_0q^n \) (\( N_1' \) is introduced to control the nonlinearity in (1.25)) and let us work with \( n \leq \min(N_1', N_2) \). Since \( N_2 \leq N_1 \), \( (N_1 \) given in Lemma 3.2), the bound (3.3) holds and \( P_n \leq 1 \). Therefore, by using first (1.25) and (1.29), and then (3.3), we have

\[
\frac{Q_{n+1}}{Q_n} \leq q(1 + P_n) \left( 1 + 2\beta^2 q^n \right) \leq q \left( 1 + 2c_1\beta^{2\alpha/(2\alpha-1)}q^n + 4\beta^2 q^n \right),
\]

(3.17)

which implies

\[
Q_n \leq Q_0q^n \exp \left( \frac{2c_1}{q-1}\beta^{2\alpha/(2\alpha-1)}q^n + \frac{4}{q-1}\beta^2 q^n \right).
\]

(3.18)

By definition of \( N_2 \) the first term in the exponent is at most \( (\log 2)/2 \). Moreover \( n \leq N_2 \) implies, via (3.2),

\[
n \leq \frac{\log(1/\beta^2)}{\log q} - \frac{\log ((4/\log 2)c_1/(\bar{q}-1))}{\log \bar{q}},
\]

(3.19)

and one directly sees that for such values of \( n \) we have \( \beta^2 q^n \leq (4c_1/(\bar{q}-1) \log 2)^{-\log q/\log \bar{q}} \). Therefore also the second term in the exponent (cf. (3.18)) can be made smaller than \( (\log 2)/2 \) by choosing \( c_1 \) larger than a number that depends only on \( B \), see the statement for an explicit expression.

Summing all up, for \( c_1 \) chosen suitably large, \( Q_n \leq 2Q_0q^n \) for \( n \leq \min(N_0', N_2) \). But, by definition of \( N_0' \), this just means \( n \leq N_2 \) and the proof is complete. \( \square \)
3.1. The $B = B_c$ case.

**Proposition 3.4.** Set $B = B_c$. There exists $\beta_0$ such that for all $\beta \leq \beta_0$

$$h_\epsilon(\beta) - h_\epsilon(0) < \exp\left(\frac{-(\log 2)^2}{2\beta^2}\right).$$

(3.20)

**Remark 3.5.** The constant $(\log 2)^2/2$ that appears in the exponential is certainly not the best possible. In fact, one can get arbitrarily close to the optimal constant $\log 2$ given in [11], but we made the choice to keep the proof as simple as possible.

**Proof of Proposition 3.4.** Choose

$$h = e^{-(\log 2)^2/(2\beta^2)} + \log(B_c - 1),$$

so that

$$P_0 = \exp(h) - (B_c - 1)^{\beta \sim 0} (B_c - 1) \exp(-(\log 2)^2/(2\beta^2)).$$

(3.22)

Given $\delta > 0$ small (for example, $\delta = 1/70$), we let $n_\delta$ be the integer uniquely identified (because of the strict monotonicity of $\{P_n\}$) by

$$P_{n_\delta} < \delta \leq P_{n_\delta + 1},$$

(3.23)

(we assume that $P_0 < \delta$, which just means that we take $\beta$ small enough). We observe that (1.28) implies $P_{n+1}/P_n \geq \sqrt{2}$ for every $n$, from which follows immediately that (say, for $\beta$ sufficiently small)

$$n_\delta \leq \left\lfloor \frac{\log 2}{\beta^2} \right\rfloor.$$

(3.24)

We want to show first of all that $Q_{n_\delta}$ is of the same order of magnitude as $Q_0$, and therefore much smaller than $P_{n_\delta}$ (for $\beta$ small) in view of $Q_0 \beta \sim 0 \beta^2$.

From (1.25), recalling the definition of $P_n$ (cf. (1.28)) and the bound (1.29), we derive

$$Q_{n+1} = \left(\frac{(R_n)^2}{(R_{n+1})^2(B - 1)^2}\right) \left(Q_n + \frac{1}{2}Q_n^2\right) \leq Q_n (1 + P_n) \left(1 + \frac{Q_n}{2}\right).$$

(3.25)

If we define $c(\delta)$ through

$$c(\delta) = \prod_{k=0}^{\infty} \left(1 + \delta 2^{-k/2}\right) \leq \exp(\delta (2 + \sqrt{2})) \leq \frac{21}{20},$$

(3.26)

from (3.25) we directly obtain that, as long as $Q_n \leq 3Q_0$ and $n \leq n_\delta$,

$$Q_n \leq Q_0 (1 + (3/2)Q_0)^{n-1} \prod_{k=0}^{n-1} (1 + P_n) \leq c(\delta) Q_0 e^{(3/2)Q_0 n}.$$

(3.27)

It is then immediate to check, using (1.26), that $Q_{n_\delta} \leq 3Q_0$ for $\beta$ small.

But, as already exploited in (2.11), $Q_{n+1}/Q_n \leq 3$ for every $n$ such that $Q_n \leq 1$, so that $Q_{n_\delta + n} \leq 4\beta^2 3^n \leq 1$ for $n \leq n_1 := \log_3(1/(4\beta^2)) - 1$. But for such values of $n$

$$P_{n_\delta + n} \geq \delta 2^{(n-1)/2},$$

(3.28)

so that we directly see that $P_{n_\delta + n_1}$ diverges as $\beta$ tends to zero, and therefore $\langle R_{n_\delta + n_1}\rangle$, can be made large for $\beta$ small, while $Q_{n_\delta + n_1}$, that is the ratio between the variance of $R_{n_\delta + n_1}$ and $\langle R_{n_\delta + n_1}\rangle^2$ is bounded by 1. By exploiting $R_n \geq (B - 1)/B$ for $n \geq 1$ and
using Chebyshev inequality it is now straightforward to see that \( \langle \log(R_{n+1}/B) \rangle > 0 \) and by (1.12) (or, equivalently, (A.5)) we have \( F(\beta, h) > 0 \).

\[ \square \]

4. Free energy upper bounds beyond annealing

In this section we introduce our main new idea, which we briefly sketch here. In order to show that the free energy vanishes for \( h \) larger than, but close to, \( h_c(0) \), we take the system at the \( n \)-th step of the iteration, for some \( n = n(\beta) \) that scales suitably with \( \beta \) (in particular, \( n(\beta) \) diverges for \( \beta \to 0 \)) and we modify (via a tilting) the distribution \( P \) of the disorder. If \( \alpha > 1/2 \), it turns out that one can perform such tilting so to guarantee on one hand that, under the new law, \( R_{n(\beta)} \) is concentrated around 1, and, on the other hand, that the two laws are very close (they have a mutual density close to 1). This in turn implies that \( R_{n(\beta)} \) is concentrated around 1 also under the original law \( P \), and the conclusion that \( F(\beta, h) = 0 \) follows then via the fact that if some non-integer moment (of order smaller than 1) of \( R_{n(\beta)} - 1 \) is sufficiently small for some integer \( n_0 \), then it remains so for every \( n \geq n_0 \) (cf. Proposition 4.1).

4.1. Fractional moment bounds. The following result says that if \( R_{n_0} \) is sufficiently concentrated around 1 for some \( n_0 \geq 0 \), then it remains concentrated for every \( n > n_0 \) and the free energy vanishes. In other words, we establish a finite-volume condition for delocalization.

Proposition 4.1. Let \( B > 2 \) and \( (\beta, h) \) be given. Assume that there exists \( n_0 \geq 0 \) and \( (\log 2/\log B) < \gamma < 1 \) such that \( \langle (|R_{n_0} - 1|^\gamma) \rangle < B^\gamma - 2 \). Then, \( F(\beta, h) = 0 \).

Proof of Proposition 4.1. We rewrite (1.6) as

\[ R_{n+1} - 1 = \frac{1}{B} \left[ \left( R_{n}^{(1)} - 1 \right) \left( R_{n}^{(2)} - 1 \right) + \left( R_{n}^{(1)} - 1 \right) + \left( R_{n}^{(2)} - 1 \right) \right], \]

and we use the inequalities \( |rs + r + s|^\gamma \leq |r|^\gamma |s|^\gamma + |r|^\gamma + |s|^\gamma \), that holds for \( r, s \geq -1 \), and \( (a+b)^\gamma \leq a^\gamma + b^\gamma \), that holds for \( \gamma \in (0, 1] \) and \( a, b \geq 0 \). If we set \( A_n := \langle (|R_{n} - 1|^\gamma) \rangle \)

we have

\[ A_{n+1} \leq \frac{1}{B^\gamma} \left[ A_n^2 + 2A_n \right] \]

and therefore \( A_n \to 0 \) for \( n \to \infty \) under the assumptions of the Proposition. Deducing \( F(\beta, h) = 0 \) (and actually more than that) is then immediate:

\[ \langle \log R_n \rangle = \frac{1}{\gamma} \langle \log(R_n)^\gamma \rangle \leq \frac{1}{\gamma} \langle \log \left[ \langle (|R_{n} - 1|^\gamma) + 1 \rangle \right] \rangle \leq \frac{1}{\gamma} \log(A_n + 1) \to 0. \]

Proposition 4.1 will be essential in Section 4 to prove that, for \( B > B_c \), an arbitrarily small amount of disorder shifts the critical point. Let us also point out that it implies that, if \( \omega_1 \) is an unbounded random variable, then for any \( B > 2 \) and \( \beta \) sufficiently large quenched and annealed critical points differ (the analogous result for non-hierarchical pinning models was proven in [25, Corollary 3.2]):

Corollary 4.2. Assume that \( P(\omega_1 > t) > 0 \) for every \( t > 0 \). Then, for every \( h \in \mathbb{R} \) and \( B > 2 \) there exists \( \beta_0 < \infty \) such that \( F(\beta, h) = 0 \) for \( \beta \geq \beta_0 \).
Proof of Corollary 4.2 Choose some $\gamma \in (\log 2/\log B, 1)$. One has $\lim_{\beta \to \infty} R_0 = 0$ $\mathbb{P}(d\omega)$-a.s. (see [1.4]) and note that log $M(\beta)/\beta \to \infty$ for $\beta \to +\infty$ under our assumption on $\omega_1$, while $\langle ((R_0 - 1)^+)\rangle^{1/\gamma} \leq 1 + (R_0) = 1 + \exp(h)$, so $\lim_{\beta \to \infty} A_0 = 0$. □

Remark 4.3. Note moreover that if we set $X = \exp(\beta \omega_1 - \log M(\beta))$ we have (without requiring $\omega_1$ unbounded) that $\langle ((B - 1)X - 1)^+)\rangle^{B \sim} B^\gamma(X^{\gamma})$. The right-hand side is smaller than $B^\gamma - 2$ for $X$ non-degenerate and $B$ large, so that if we choose $\delta > 0$ such that $\exp(\delta X^{\gamma}) < 1$ we have

$$\langle ((B - 1)\exp(\delta)X - 1)^+)\rangle^{B \sim} < B^\gamma - 2,$$

(4.4)

for $B$ sufficiently large. Therefore, by applying Proposition 4.1 we see that for every $\beta > 0$ there exists $\delta > 0$ such that $\mathbb{P}(\beta, h(0) + \delta) = 0$ for $B$ sufficiently large. This observation actually follows also from the much more refined Proposition 4.5 below, which by the way says precisely how large $B$ has to be taken: $B > B_c$.

Remark 4.4. It follows from inequality (1.2) that, if the assumptions of Proposition 4.1 are verified, then $A_n$ actually vanishes exponentially fast for $n \to \infty$. Therefore, for $\epsilon > 0$ one has

$$\mathbb{P}(R_n \geq 1 + \epsilon) = \mathbb{P}((R_n - 1)^+ \geq \epsilon) \leq \frac{A_n}{\epsilon^\gamma},$$

(4.5)

and from the Borel-Cantelli lemma follows the almost sure convergence of $R_n$ to 1 when we recall that $R_n(i) \geq r_n$ with $r_0 = 0$ ($r_n$ is the solution of the iteration scheme (1.2) and converges to 1).

4.2. Upper bounds on the free energy for $B > B_c$. Here we want to prove the lower bound in (1.20), plus the fact that $h_c(\beta) > h_c$ whenever $\beta > 0$ and $B > B_c$. This follows from

Proposition 4.5. Let $B > B_c$. For every $\beta > 0$ one has $h_c(\beta) > h_c(0)(= \log(B - 1))$. Moreover, there exists a positive constant $c$ (possibly depending on $B$) such that for every $0 \leq \beta \leq 1$

$$h_c(\beta) - h_c(0) \geq c\beta^{2\alpha/(2\alpha - 1)}.$$  

(4.6)

Proposition 4.5 is proven in section 4.4 but first we need to state a couple of technical facts.

4.3. Auxiliary definitions and lemmas. For $\lambda \in \mathbb{R}$ and $N \in \mathbb{N}$ let $\mathbb{P}_{N,\lambda}$ be defined by

$$\frac{d\mathbb{P}_{N,\lambda}}{d\mathbb{P}}(x_1, x_2, \ldots) = \frac{1}{M(-\lambda)^N} \exp\left(-\lambda \sum_{i=1}^N x_i\right).$$

(4.7)

Lemma 4.6. There exists $1 < C < \infty$ such that for $a \in (0, 1)$, $\delta \in (0, a/C)$ and $N \in \mathbb{N}$ we have

$$\mathbb{P}_{N, \frac{\delta}{\sqrt{N}}}(\frac{d\mathbb{P}}{d\mathbb{P}_{N, \frac{\delta}{\sqrt{N}}}}(\omega) < \exp(-a)) \leq C \left(\frac{\delta}{a}\right)^2.$$  

(4.8)

Proof of Lemma 4.6 We write

$$\mathbb{P}_{N, \frac{\delta}{\sqrt{N}}}(\frac{d\mathbb{P}}{d\mathbb{P}_{N, \frac{\delta}{\sqrt{N}}}}(\omega) < \exp(-a)) = \mathbb{P}_{N, \frac{\delta}{\sqrt{N}}}(\frac{\delta}{\sqrt{N}} \sum_{i=1}^N \omega_i + N \log M \left(\frac{\delta}{\sqrt{N}}\right) < -a).$$

(4.9)
Since all exponential moments of \( \omega_1 \) are assumed to be finite, one has

\[
0 \geq \log M(-\lambda) - \lambda \frac{d}{d\lambda} \log M(-\lambda) \geq -\frac{C}{2} \lambda^2, \tag{4.10}
\]

for some \( 1 < C < \infty \) and \( 0 \leq \lambda \leq 1 \) (the first inequality is due to convexity of \( \lambda \mapsto \log M(-\lambda) \)). Note also that

\[
\mathbb{E}_{N,\lambda}(\omega_1) = -\frac{d}{d\lambda} \log M(-\lambda). \tag{4.11}
\]

Therefore, the right-hand side of (4.9) is bounded above by

\[
\mathbb{P}_{N,\delta/\sqrt{N}}\left( \frac{\sum_{i=1}^{N} \omega_i}{\sqrt{N}} - \mathbb{E}_{N,\delta/\sqrt{N}} \left[ \frac{\sum_{i=1}^{N} \omega_i}{\sqrt{N}} \right] \right) \leq \frac{4\delta^2}{a^2} \mathbb{E}_{N,\delta/\sqrt{N}} \left( \omega_1 - \mathbb{E}_{N,\delta/\sqrt{N}}(\omega_1) \right)^2, \tag{4.12}
\]

where we have used Chebyshev inequality and the fact that, under the assumptions we made, \((a/\delta) - (C/2)\delta > a/(2\delta)\). The proof of (4.8) is then concluded by observing that the variance of \( \omega_1 \) under \( \mathbb{P}_{N,\lambda} \) is \( d^2/d\lambda^2 \log M(-\lambda) \), which is bounded uniformly for \( 0 \leq \lambda \leq 1 \). \qed

We define the sequence \( \{a_n\}_{n=0,1,\ldots} \) by setting \( a_0 = a > 0 \) and \( a_{n+1} = f(a_n) \) with

\[
f(x) := \sqrt{Bx + (B - 1)^2} - (B - 1). \tag{4.13}
\]

We define also the sequence \( \{b_n\}_{n=0,1,\ldots} \) by setting \( b_0 = b \in (-B, 0) \) and \( b_{n+1} = f(b_n) \). Note that \( a_n = g(a_{n+1}) \) and \( b_n = g(b_{n+1}) \) for \( g(x) = (2(B - 1)x + x^2)/B \).

**Lemma 4.7.** There exist two constants \( G_a > 0 \) et \( H_b > 0 \) such that for \( n \to \infty \)

\[
a_n \sim G_a \left( \frac{B}{2(B - 1)} \right)^n = G_a 2^{-an} \quad \text{and} \quad b_n \sim -H_b \left( \frac{B}{2(B - 1)} \right)^n = -H_b 2^{-bn}. \tag{4.14}
\]

Moreover, \( G_a \overset{a \to 0}{\sim} a \) and \( H_b \overset{b \to 0}{\sim} |b| \).

**Proof of Lemma 4.7.** In order to lighten the proof we put \( s := B/(2(B - 1)) \) and we observe that \( 0 < s < 1 \) since \( B > 2 \). The function \( f(\cdot) \) is concave and \( f'(0) = s \), so \( a_n \) vanishes exponentially fast:

\[
a_n \leq a s^n. \tag{4.15}
\]

Moreover,

\[
a_n \leq a_{n-1} \frac{1}{s^{n-1} 1 + a_n/(2(B - 1))} \geq a_{n-1} \frac{1}{s^{n-1} 1 + as^n/(2(B - 1))}, \tag{4.16}
\]

so that for every \( n > 0 \)

\[
a_n s^n \geq a \prod_{\ell=1}^{n-1} \frac{1}{1 + as^\ell/(2(B - 1))} > 0. \tag{4.17}
\]

From (4.16) we see that \( a_n s^{-n} \) is monotone increasing in \( n \), so that the first statement in (4.11) holds with \( G_a \in (0, a) \) by (4.15) and (4.17). The fact that \( G_a \sim a \) for \( a \to 0 \) follows from the fact that the product in (4.17) converges to 1 in this limit.

The second relation is proven in a similar way. Since \( b_n < 0 \) for every \( n \), one has first of all

\[
b_n s^n = b_{n-1} \frac{1}{s^{n-1} 1 + b_n/(2(B - 1))} < b_{n-1} s^{n-1}. \tag{4.18}
\]
Moreover, since $|b_n|$ decreases to zero and $f(x) \geq c_1(b)x$ for $b \leq x \leq 0$ for some $c_1(b) < 1$ if $b < -(B - 2)$, one sees that $|b_n|$ actually vanishes exponentially fast. Therefore, from (4.18)

$$\frac{b_n}{s^n} \geq \frac{b_{n-1}}{s^{n-1}} \frac{1}{1 - c_2(b) c_1(b)^n} \geq \frac{b}{s^\infty} \frac{1}{1 - c_2(b) c_1(b)^\ell}.$$  (4.19)

One has then the second statement of (4.14) with $H_0 \in (|b|, \infty)$.  

4.4. Proof of Proposition 4.5. In this proof $C_i, i = 1, 2, \ldots$ denote constants depending only on $\beta_0$ and (possibly) on $B$. Recall that the exponent $\alpha$ defined in (1.17) satisfies $1/2 < \alpha < 1$ for $B > B_c$. Fix $\beta_0 > 0$, let $0 < \beta < \beta_0$ and choose $h = h(\beta)$ such that

$$\langle R_0 \rangle = (B - 1) + \eta\beta^{2\alpha - 1},$$  (4.20)

where $\eta > 0$ will be chosen sufficiently small and independent of $\beta$. Call $n_0 := n_0(\eta, \beta)$ the integer such that

$$\langle R_{n_0} \rangle \leq B \leq \langle R_{n_0+1} \rangle,$$  (4.21)

i.e., $P_{n_0} < 1 < P_{n_0+1}$. Note that $n_0(\eta, \beta)$ becomes larger and larger as $\beta \searrow 0$: this can be quantified since from (1.28) one sees that $a_n := P_{n_0-n}$ satisfies for $0 \leq n < n_0$ the iteration $a_{n+1} = f(a_n)$ introduced in §4.3 and therefore it follows from Lemma 4.7 that

$$|n_0(\eta, \beta) - \log (\eta^{-1}\beta^{-\frac{2\alpha}{2\alpha - 1}})/(\alpha \log 2)| \leq C_1,$$  (4.22)

for every $0 < \eta < 1/C_1$ and $\beta \in [0, \beta_0]$. With the notations of Section 4.3 let $\overline{P} := \mathbb{P}_{\delta_0, \delta_0/2}$, where $\delta := \delta(\eta)$ will be chosen suitably small later. Note that, with $\lambda := \delta_0^{-n_0/2}$, one has from (4.22)

$$\frac{1}{C_2} \delta \eta^{1/(2\alpha)} \beta^{1/(2\alpha - 1)} \leq \lambda \leq C_2 \delta \eta^{1/(2\alpha)} \beta^{1/(2\alpha - 1)}.$$  (4.23)

In particular, since $\alpha < 1$, if $\eta$ is small enough then $\lambda \leq \beta$ uniformly for $\beta \leq \beta_0$. Observe also that

$$\overline{E}(R_0) = \langle R_0 \rangle \frac{M(\beta - \lambda)}{M(\beta)M(-\lambda)},$$  (4.24)

and call $\phi(\cdot) := \log M(\cdot)$. Since $\phi(\cdot)$ is strictly convex, one has

$$\phi(\beta - \lambda) - \phi(\beta) - \phi(-\lambda) = -\int_0^\lambda dx \int_0^\beta dy \phi''(x + y) \in \left(-\frac{\lambda \beta}{C_3}, -C_3 \lambda \beta\right),$$  (4.25)

for some $C_3 > 0$, uniformly in $\beta \leq \beta_0$ and $0 \leq \lambda \leq \beta$ and, thanks to (4.23), if $\eta$ is chosen sufficiently small,

$$1 - \frac{\beta \lambda}{C_4} \leq \frac{M(\beta - \lambda)}{M(\beta)M(-\lambda)} \leq 1 - C_4 \beta \lambda.$$  (4.26)

Therefore, from (4.24) and (4.23) and choosing

$$\eta^{1-1/(2\alpha)} \ll \delta(\eta) \ll 1,$$  (4.27)

(which is possible with $\eta$ small since $\alpha > 1/2$) one has

$$- C_5^{-1} \delta(\eta) \eta^{1/(2\alpha)} \beta^{2\alpha - 1} < \overline{E}(R_0) - (B - 1) \leq - C_5 \delta(\eta) \eta^{1/(2\alpha)} \beta^{2\alpha - 1},$$  (4.28)

always uniformly in $\beta \leq \beta_0$.  

□
Since $b_n := \mathbb{E}(R_{n_0-n}) - (B - 1)$ satisfies the recursion $b_{n+1} = f(b_n)$, from the second statement of (4.14) if follows that

$$\mathbb{E} R_{n_1} \leq \frac{B}{2},$$

(4.29)

for some integer $n_1 := n_1(\eta, \beta)$ satisfying

$$n_1 \leq \left( \log \left( \delta(\eta)^{-1} \eta^{-1/(2\alpha)} \beta^{-2\alpha/(2\alpha-1)} \right) \right) + C_6.$$

(4.30)

It is immediate to see that $n_0(\eta, \beta) - n_1(\eta, \beta)$ gets large (uniformly in $\beta$) for $\eta$ small, if condition (4.27) is satisfied. Therefore, since the fixed point 1 of the iteration for $\mathbb{E} R_n$ is attractive, one has that

$$\mathbb{E} R_{n_0} \leq 1 + r_1(\eta),$$

(4.31)

(here and in the following, $r_i(\eta)$ with $i \in \mathbb{N}$ denotes a positive quantity which vanishes for $\eta \searrow 0$, uniformly in $\beta \leq \beta_0$). On the other hand, one has deterministically

$$\lim_{n \to \infty} \left[ 1 - R_n \right]^+ = 0,$$

(4.32)

as one sees immediately comparing the evolution of $R_n$ with that obtained setting $R_0 = 0$ for every $i$. In particular, $R_{n_0} \geq 1 - r_2(\eta)$. An application of Markov’s inequality gives

$$\mathbb{P}(R_{n_0} \geq 1 + r_3(\eta)) \leq r_3(\eta).$$

(4.33)

It is immediate to prove that, given a random variable $X$ and two mutually absolutely continuous laws $\mathbb{P}$ and $\mathbb{P}'$, one has for every $x, y > 0$

$$\mathbb{P}(X \leq 1 + x) \geq e^{-y} \left[ \mathbb{P}(X \leq 1 + x) - \mathbb{P}\left( \frac{\text{d}\mathbb{P}}{\text{d}\mathbb{P}'} \leq e^{-y} \right) \right].$$

(4.34)

Applying this to the case $X = R_{n_0}$ and using Lemma 4.6 with $r_4(\eta) > C\delta(\eta)$ gives

$$\mathbb{P}(R_{n_0} \leq 1 + r_3(\eta)) \geq e^{-r_4(\eta)} \left[ 1 - r_3(\eta) \right] C\delta(\eta)^{r_4(\eta) - C\delta(\eta)}.$$ 

(4.35)

In particular, choosing

$$\delta(\eta) \ll r_4(\eta) \ll 1,$$

(4.36)

one has

$$\mathbb{P}(R_{n_0} \leq 1 + r_3(\eta)) \geq 1 - r_5(\eta),$$

(4.37)

and we emphasize that this inequality holds uniformly in $\beta \leq \beta_0$.

At this point (4.6) is essentially proven: choose some $\gamma \in (\log 2/\log B, 1)$ and observe that

$$\langle \langle [R_{n_0} - 1]^+ \rangle \rangle^\gamma \leq r_3(\eta)^\gamma + \langle \mathbb{E}[R_{n_0} - 1]^+ \rangle^\gamma \langle \mathbb{P}(R_{n_0} \geq 1 + r_3(\eta)) \rangle^{1-\gamma}$$

$$\leq r_3(\eta)^\gamma + B^\gamma r_5(\eta)^{1-\gamma},$$

(4.38)

where in the first inequality we have used Hölder inequality and in the second one we have used (4.21) and (4.37). Finally, we remark that the quantity in (4.38) can be made smaller than $B^\gamma - 2$ choosing $\eta$ small enough. At this point, we can apply Proposition 4.11 to deduce that $F(\beta, h) = 0$ for $h = \log(B - 1) + \eta^\beta 2\alpha/(2\alpha - 1)$ with $\eta$ small but finite, which proves (4.16).

We complete the proof by observing that $h_c(\beta) > \log(B - 1)$ for every $\beta > 0$ follows from the arbitrariness of $\beta_0$. \qed
5. The delocalized phase

Here we prove Theorem 1.6 using the representation (A.11), given in Appendix A, for $R_n$. With reference to (A.11), let us observe that

$$\lim_{n \to \infty} p(n, \emptyset) = 1,$$

which is just a way to interpret

$$\lim_{n \to \infty} r_n = 1.$$  \hspace{1cm} (5.1)

when $r_0 = 0$, that follows directly from (1.2).

Fix $\varepsilon > 0$ arbitrarily small and consider $h < h_c(\beta)$. Let $\bar{R}_n$ be the partition function which corresponds to $h_c(\beta)$ and $R_n$ the one that corresponds to $h$. We can find $K$ large enough such that

$$\mathbb{P}(\bar{R}_n \geq K) \leq \varepsilon/2 \quad \text{for all } n \geq 1.\hspace{1cm} (5.3)$$

This follows from the fact that $\bar{R}_n \geq (B - 1)/B$, and from (A.4). We define $C := (\log(2K/\varepsilon))/(h_c(\beta) - h)$ and we write, using (A.11),

$$R_n = p(n, \emptyset) + \sum_{I \subset \{1, \ldots, 2^n\}} p(n, I) \exp \left( \sum_{i \in I} (\beta \omega_i - \log M(\beta) + h) \right)$$

$$+ \sum_{I \subset \{1, \ldots, 2^n\}} p(n, I) \exp \left( \sum_{i \in I} (\beta \omega_i - \log M(\beta) + h) \right) =: T_1 + T_2 + T_3.\hspace{1cm} (5.4)$$

$T_1$ is smaller than 1 and

$$T_3 \leq \exp(-C(h_c(\beta) - h)) \bar{R}_n,\hspace{1cm} (5.5)$$

so that $T_3 \leq \varepsilon/2$ with probability greater than $(1 - \varepsilon/2)$ (cf. (5.3)) for all $n$. As for $T_2$, its easy to compute and bound its expectation:

$$\left\langle \sum_{I \subset \{1, \ldots, 2^n\}} \sum_{1 \leq |I| \leq C} p(n, I) \exp \left( \sum_{i \in I} (\beta \omega_i - \log M(\beta) + h) \right) \right\rangle \leq \exp(CH)[1 - p(n, \emptyset)],\hspace{1cm} (5.6)$$

and (5.1) tells us that the right-hand side tends to zero when $n$ goes to infinity. In particular we can find $N$ (depending on $C$) such that for all $n \geq N$ we have

$$\left\langle \sum_{I \subset \{1, \ldots, 2^n\}} \sum_{1 \leq |I| \leq C} p(n, I) \exp \left( \sum_{i \in I} (\beta \omega_i - \log M(\beta) + h) \right) \right\rangle \leq \varepsilon^2/4.\hspace{1cm} (5.7)$$

Then for $n \geq N$ we have $\mathbb{P}(T_2 \geq \varepsilon/2) \leq \varepsilon/2$. Altogether we have

$$\mathbb{P}(R_n \geq 1 + \varepsilon) \leq \varepsilon,\hspace{1cm} (5.8)$$

and since $R_n$ is bounded from below by $p(n, \emptyset)$ which tends to 1, the proof is complete. □
Appendix A. Existence of the free energy and annealed system estimates

A.1. Proof of Theorem 1.1. Since the basic induction (1.6) gives \( R_n \geq (B - 1)/B \) for every \( n \geq 1 \), one has

\[
\frac{R_{n+1}}{B} \geq \frac{R_n^{(1)} R_n^{(2)}}{B},
\]

(A.1)

and

\[
R_{n+1} \leq \frac{R_n^{(1)} R_n^{(2)}}{B} + \frac{B}{B-1} R_n^{(1)} R_n^{(2)},
\]

(A.2)

so that

\[
(K_B R_{n+1}) \leq (K_B R_n^{(1)})(K_B R_n^{(2)}) \quad \text{with} \quad K_B = \frac{B^2 + B - 1}{B(B - 1)}.
\]

(A.3)

Taking the logarithm of (A.1) and (A.3), we get that

\[
\{2^{-n} \mathbb{E}[\log(R_n/B)]\}_{n=1,2,...} \quad \text{is non-decreasing},
\]

(A.4)

while

\[
\{2^{-n} \mathbb{E}[\log(K_B R_n)]\}_{n=1,2,...} \quad \text{is non-increasing},
\]

(A.5)

so that both sequences are converging to the same limit

\[
f(\beta, h) = \lim_{n \to \infty} 2^{-n} \langle \log R_n \rangle
\]

(A.6)

and (1.12) immediately follows. It remains to be proven that the limit of \( 2^{-n} \log R_n \) exists \( \mathbb{P}(d\omega) \)-almost surely and in \( L^1(d\mathbb{P}) \). Fixing some \( k \geq 1 \) and iterating (A.1) one obtains for \( n > k \)

\[
2^{-n} \log(R_n/B) \geq 2^{-k}(2^{k-n} \sum_{i=1}^{2^{n-k}} \log(R_k^{(i)}/B)).
\]

(A.7)

Using the strong law of large numbers in the right-hand side, we get

\[
\liminf_{n \to \infty} 2^{-n} \log(R_n/B) \geq 2^{-k} \langle \log(R_k/B) \rangle \quad \mathbb{P}(d\omega) - \text{a.s.}
\]

(A.8)

Hence taking the limit for \( k \to \infty \) in the right-hand side again we obtain

\[
\liminf_{n \to \infty} 2^{-n} \log R_n = \liminf_{n \to \infty} 2^{-n} \log(R_n/B) \geq f(\beta, h) \quad \mathbb{P}(d\omega) - \text{a.s.}
\]

(A.9)

Doing the same computations with (A.3) we obtain

\[
\limsup_{n \to \infty} 2^{-n} \log R_n = \limsup_{n \to \infty} 2^{-n} \log(K_B R_n) \leq f(\beta, h) \quad \mathbb{P}(d\omega) - \text{a.s.}
\]

(A.10)

This ends the proof for the almost sure convergence. The proof of the \( L^1(d\mathbb{P}) \) convergence is also fairly standard, and we leave it to the reader.

The fact that \( f(\beta, \cdot) \) is non-decreasing follows from the fact that the same holds for \( R_n(\beta, \cdot) \), and this is easily proved by induction on \( n \). Convexity of \( (\beta, h) \mapsto f(\beta, h + \log M(\beta)) \) is immediate from (1.7) (hence for \( B = 2, 3, \ldots \)). But (1.7) can be easily generalized to every \( B > 1 \): this follows by observing that from (1.6) and (1.4) one has that

\[
R_n = \sum_{\mathcal{I} \subset \{1, \ldots, 2^n\}} p(n, \mathcal{I}) \exp \left( \sum_{i \in \mathcal{I}} (\beta \omega_i - \log M(\beta) + h) \right),
\]

(A.11)

for suitable positive values \( p(n, \mathcal{I}) \), which depend on \( B \): by setting \( \beta = h = 0 \) we see that \( \sum_{\mathcal{I}} p(n, \mathcal{I}) = 1 \) and hence \( R_n \) can be cast in the form of the expectation of a Boltzmann factor, like (1.7). This yields the desired convexity.

\( \square \)
**Remark A.1.** Another consequence of (A.11) is that \( f(\beta, h + \log M(\beta)) \geq f(0, h) \) [15, Ch. 5, Prop. 5.1].

**A.2. Proof of Theorem 1.2** When \( \beta = 0 \) the iteration (1.6) reads

\[
R_{n+1} = \frac{R_n^2 + (B - 1)}{B}.
\]

(A.12)

A quick study of the function \( x \mapsto [x^2 + (B - 1)]/B \), gives that \( R_n \xrightarrow{n \to \infty} \infty \) if and only if \( R_0 > (B - 1) \). Initial conditions \( R_0 < B - 1 \) are attracted by the stable fixed point, 1, while the fixed point \( (B - 1) \) is unstable. The inequality (A.1) guaranties that \( f(0, h) > 0 \) when \( R_N > B \) for some \( N \). This immediately shows that that \( h_c(0) = \log(B - 1) \).

Next we prove (1.16), i.e., that (with the notations in (1.30) and (1.31)) there exists a constant \( C \) such that

\[
\frac{1}{C} \varepsilon^{1/\alpha} \leq \hat{f}(0, \varepsilon) \leq C \varepsilon^{1/\alpha}
\]

for all \( \varepsilon \in (0, 1) \). To that purpose take \( a := a_0 \) such that \( \hat{f}(0, a) = 1 \) (this is possible because of the convexity of \( f(\beta, \cdot + \log M(\beta)) \) we obtain both continuity and \( \lim_{\varepsilon \to 0} \hat{f}(0, \varepsilon) = \infty \) and note that the sequence \( \{a_n\}_{n \geq 0} \) defined just before Lemma 4.7 is such that \( 2\hat{f}(0, a_{n+1}) = \hat{f}(0, a_n) \), so that \( \hat{f}(0, a_{n+1}) = 2^{-n} \). Thanks to Lemma 4.7 we have that along this sequence

\[
\hat{f}(0, a_n) \sim 2G_a^{-1/\alpha} a_n^{1/\alpha}.
\]

(A.14)

Let \( K_a \) be such that \( a_n \leq K_a a_{n+1} \) for all \( n \), and \( c_a \) such that \( c_a^{-1} a_n^{1/\alpha} \leq \hat{f}(0, a_n) \leq c_a a_n^{1/\alpha} \). Then, for all \( n \) and all \( \varepsilon \in [a_{n+1}, a_n] \), since \( \hat{f}(0, \cdot) \) is increasing we have

\[
\hat{f}(0, \varepsilon) \geq \hat{f}(0, a_{n+1}) \geq c_a^{-1} a_{n+1}^{1/\alpha} \geq c_a^{-1} K_a^{-1/\alpha} \varepsilon^{1/\alpha},
\]

\[
\hat{f}(0, \varepsilon) \leq \hat{f}(0, a_n) \leq c_a a_n^{1/\alpha} \leq c_a K_a^{1/\alpha} \varepsilon^{1/\alpha}.
\]

(A.15)

Finally, the analyticity of \( f(0, \cdot) \) on \( (h_c, \infty) \) follows for example from [8, Lemma 4.1].

\( \square \)

**A.3. About models with \( B \leq 2 \).** We have chosen to work with the model (1.1), with positive initial data and \( B > 2 \), because this is the case that is directly related to pinning models and because in this framework we had the precise aim of proving the physical conjectures formulated in (1.1). But of course the model is well defined for all \( B \neq 0 \) and in view of the direct link with the logistic map \( z \mapsto Az(1 - z) \), cf. (1.3), also the case \( B \leq 2 \) appears to be intriguing. Recall that \( A = 2(B - 1)/B \) and note that \( A \in (1, 2) \) if \( B \in (2, \infty) \). What we want to point out here is mainly that the case of (1.1) with positive initial data and \( B \in (1, 2) \), i.e. \( A \in (0, 1) \), is already contained in our analysis. This is simply the fact that there is a duality transformation relating this new framework to the one we have considered. Namely, if we let \( B \in (1, 2) \) and we set \( \tilde{R}_n := R_n/(B - 1) \), then \( \tilde{R}_n \) satisfies (1.1) with \( B \) replaced by \( \tilde{B} := B/(B - 1) > 2 \). Of course the fixed points of \( x \mapsto (x^2 + \tilde{B} - 1)/\tilde{B} \) are again 1 (stable) and \( \tilde{B} - 1 \) (unstable). This transformation allows us to generalize immediately all the theorems we have proven in the obvious way, in particular the marginal case corresponds to \( \tilde{B} = \tilde{B}_c := 2 + \sqrt{2} \), i.e., \( B = \sqrt{2} \) and in the irrelevant case \( (B \in (\sqrt{2}, 2)) \) the condition on \( \Delta(\beta) \) in Theorem 1.4 now reads \( M(2\beta)/M(\beta)^2 < B^2 - 1 \).

This discussion leaves open the cases \( B = 1 \) and \( B = 2 \) to which we cannot apply directly our theorems, but:
(1) If $B = 1$ the model is exactly solvable and $R_n$ is equal to the product of $2^n$ positive IID random variables distributed like $R_0$, so $v(\beta, h) = h - \log M(\beta)$. The model in this case is a bit anomalous, since the stable fixed point is 0 and therefore the free energy can be negative and no phase transition is present (this appears to be the analogue of the non-hierarchical case with inter-arrival probabilities that decay exponentially fast \cite{[15], Ch. 1, Sec. 9}).

(2) If $B = 2$ then, with reference to (1.2), $r_n \rightarrow \infty$ if $r_0 > 1$ and $r_n \rightarrow 1$ if $r_0 < 1$. The basic results like Theorem 1.1 are quickly generalized to cover this case. Only slightly more involved is the generalization of the other results, notably Theorem 1.4(1). In fact we cannot apply directly our results because the iteration for $P_n$, that is $(R_n) - 1$, reads $P_{n+1} = P_n + (P_n^2 / 2)$ (cf. (1.28)) so that the growth of $P_n$, for $P_0 > 0$, is just due to the nonlinear term and it is therefore slow as long as $P_n$ is small. However the technique still applies (note in particular that, by (1.25) and (1.29), the variance of $R_n$ decreases exponentially if $\Delta_0 < 2$ as long as $P_n$ is sufficiently small) and along this line one shows that the disorder is irrelevant, at least as long as $\Delta_0 < 2$.

If we now let $B$ run from 1 to infinity, we simply conclude that the disorder is irrelevant if $B \in (\sqrt{2}, 2 + \sqrt{2})$, and it is instead relevant in $B \in (1, \sqrt{2}) \cup (2 + \sqrt{2}, \infty)$. In the case $B = 1$ (and, by duality, $B = \infty$) there is no phase transition.

Finally, a word about the models with $B < 1$. Various cases should be distinguished: going back to the logistic map, we easily see that playing on the values of $B$ one can obtain values of $|A|$ larger than 2 and the very rich behavior of the logistic map sets in \cite{[3]}. non-monotone convergence to the fixed point, oscillations in a finite set of points, chaotic behavior, unbounded trajectories for any initial value. It appears that it is still possible to generalize our approach to deal with some of these cases, but this would lead us far from our original aim. Moreover, for $B < 1$ the property of positivity of $R_n$, and therefore its statistical mechanics interpretation as a partition function, is lost.

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Note added in proof. After this work was completed, a number of results have been proven by developing further the ideas set forth here, solving some of questions raised at the end of Section 1.6. First of all we were able (in collaboration with B. Derrida \cite{[10]}) to extend the main idea of this work to the non-hierarchical set-up and we have shown that the quenched critical point (of the non-hierarchical model) is shifted with respect to the annealed value for arbitrarily small disorder, if $\alpha > 1/2$ (this result has been sharpened in \cite{[2]}, taking a different approach). Then one of us \cite{[21]} has been able to show the shift of the critical point for arbitrarily small disorder for $\alpha = 1/2$ in a hierarchical with site disorder (the case considered here is bond disorder, cf. Figure 1) by using a location-dependent shift of the disorder variables in the change-of-measure argument (in the present paper, the shift is the same for each variable). Finally, very recently \cite{[16]} we have also been able to treat the case $\alpha = 1/2$ ($B = B_c$), both for the hierarchical and non-hierarchical model, by introducing long range correlations in the auxiliary measure $\tilde{P}$.
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Université Paris Diderot (Paris 7) and Laboratoire de Probabilités et Modèles Aléatoires (CNRS), U.F.R. Mathématiques, Case 7012 (site Chevaleret) 75205 Paris cedex 13, France
E-mail address: giacomin@math.jussieu.fr

Université Paris Diderot (Paris 7) and Laboratoire de Probabilités et Modèles Aléatoires (CNRS), U.F.R. Mathématiques, Case 7012 (site Chevaleret) 75205 Paris cedex 13, France
E-mail address: lacoin@math.jussieu.fr

CNRS and Laboratoire de Physique, ENS Lyon, 46 Allée d’Italie, 69364 Lyon, France
E-mail address: fabio-lucio.toninelli@ens-lyon.fr