Quantum cosmology and late-time singularities

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Abstract

The development of dark energy models has stimulated interest to cosmological singularities, which differ from the traditional Big Bang and Big Crunch singularities. We review a broad class of phenomena connected with soft cosmological singularities in classical and quantum cosmology. We discuss the classification of singularities from the geometrical point of view and from the point of view of the behavior of finite size objects, crossing such singularities. We discuss in some detail quantum and classical cosmology of models based on perfect fluids (anti-Chaplygin gas and anti-Chaplygin gas plus dust), of models based on the Born–Infeld-type fields and of the model of a scalar field with a potential inversely proportional to the field itself. We dwell also on the phenomenon of the phantom divide line crossing in the scalar field models with cusped potentials. Then we discuss the Friedmann equations modified by quantum corrections to the effective action of the models under considerations and the influence of such modification on the nature and the existence of soft singularities. We review also quantum cosmology of models, where the initial quantum state of the universe is presented by the density matrix (mixed state). Finally, we discuss the exotic singularities arising in the braneworld cosmological models.

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References

1. Introduction

The problem of cosmological singularities has been attracting the attention of theoreticians working in gravity and cosmology since the early 1950s [1–3]. In the 1960s, general theorems about the conditions for the appearance of singularities were proven [4, 5] and the oscillatory regime of approaching the singularity [6], called also ‘Mixmaster universe’ [7], was discovered. Basically, until the end of 1990s almost all discussions about singularities were devoted to the Big Bang and Big Crunch singularities, which are characterized by a vanishing cosmological radius.

However, kinematical investigations of Friedmann cosmologies have raised the possibility of sudden future singularity occurrence [8], characterized by a diverging \( \dot{a} \), whereas both the scale factor \( a \) and \( \dot{a} \) are finite. Then, the Hubble parameter \( H = \dot{a}/a \) and the energy density \( \rho \) are also finite, while the first derivative of the Hubble parameter and the pressure \( p \) diverge. Until recent years, however, the sudden future singularities attracted rather a limited interest of researchers. The situation has changed drastically in the new millennium, when a plenty of publications devoted to such singularities have appeared [9–25]. The arising interest to their studies is connected basically with two reasons. The recent discovery of the cosmic acceleration [26] has stimulated the elaboration of dark energy models, responsible for such a phenomenon (see e.g. for review [27]). Remarkably, in some of these models the sudden singularities arise quite naturally. Another source of the interest to sudden singularities is the development of brane models [10, 11, 18], where also singularity of this kind arise naturally (sometimes the singularities, arising in the brane models, are called ‘quiescent’ [10]).

In the investigations devoted to sudden singularities, one can distinguish three main topics. First of them deals with the question of the compatibility of the models, possessing soft singularities with observational data [15, 25, 28, 29]. The second direction is connected with the study of quantum effects [11, 17, 30–36]. Here one can see two subdirections: the study of quantum corrections to the effective Friedmann equation, which can eliminate classical singularities or, at least, change their form [10, 17, 30] and the study of solutions of the Wheeler–DeWitt equation for the quantum state of the universe in the presence of sudden
singularities [31–35]. The third direction is connected with the opportunity of the crossing of sudden singularities in classical cosmology [34, 37–40].

A particular feature of the sudden future singularities is their softness [37]. As the Christoffel symbols depend only on the first derivative of the scale factor, they are regular at these singularities. Hence, the geodesics are well behaved and they can cross the singularity [37]. One can argue that the particles crossing the singularity will generate the geometry of the spacetime, providing in such a way a soft rebirth of the universe after the singularity crossing [40]. Note that the opportunity of crossing of some kind of cosmological singularities were noticed already in the early paper by Tipler [41]. Rather a close idea of integrable singularities in black holes, which can give origin to a cosmogenesis, was recently put forward in [42, 43]. Besides, the results of [37, 38] were generalized for the case of general (non-Friedmann) universes in [44, 45]. For this purpose was used the formalism of the quasi-isotropic expansion of the solutions of the Einstein equations near cosmological singularities, which has been first proposed in [46]. (For some further developments of this formalism, see [47].)

The peculiarity of the sudden future singularities makes them to be a good tool for studying some general features of the general relativity, in particular the relations between classical and quantum gravity and cosmology. These relations are the main topic of this review. We shall also dwell on another aspect of general relativity, which from our point of view is a little bit underestimated. It is the fact that the requirement of the self-consistency of the system of laws of general relativity and particle physics, or, in other words, of the system of Einstein equations and of the equations describing the state or the motion of non-gravitational matter can induce some interesting transformations in the state of matter. Such transformations sometimes occur when the universe passes through the soft singularities, though there are some examples of such transformations which can be observed in the absence of singularities too. We shall consider some of such examples.

Generally, this review is devoted to three interrelated topics, which are connected in some way with the soft future singularities in cosmology—these are the problem of crossing of such singularities in classical cosmology, the relations between classical and quantum treatments of cosmological singularities and the changes of state of matter, induced by cosmological singularities or other geometrical irregularities in the framework of general relativity. The structure of the review is as follows. In the second section we shall give a brief and convenient classification of the future singularities, following [48]. In section 3, we present the classification of the types of singularities from the point of view of the finite objects, which approach these singularities. In section 4, we introduce the toy tachyon model [49] and shall discuss its basic properties. Section 5 is devoted to the cosmological model based on the mixture of the anti-Chaplygin gas and to the paradox of soft singularity crossing [50]. In section 6, we consider again the paradox of the soft singularity crossing in the presence of dust and shall discuss its possible resolution by introducing some transformation of matter [51]. In section 7, we shall give another example of the transformation of the Lagrangian of a scalar field due to its interaction with geometry, while its potential is not smooth [52]. Section 8 is devoted to the study of classical dynamics of the cosmological model with a scalar field whose potential is inversely proportional to the field, while in section 9 we study its quantum dynamics. Section 10 is devoted to attempts to apply the formalism of the Wheeler–DeWitt equation to the study of tachyon and pseudo-tachyon cosmological models. In section 11, we study Friedmann equations modified by quantum corrections and possible influence of these corrections on soft cosmological singularities [17, 30]. Section 12 is devoted to developing of such notions as the density matrix of the universe, quantum consistency and interplay between geometry and matter in quantum cosmology. In section 13, we consider singularities arising in some braneworld models, while section 14 contains some concluding remarks.
2. Classification of future cosmological singularities

In this section, we shall present rather a convenient classification of the future cosmological singularities, following [48]. We shall consider a flat Friedmann universe with the metric
\[ ds^2 = dt^2 - a^2(t) \, dl^2, \]
where \( a(t) \) is the cosmological radius (scale factor) and \( dl^2 \) is the spatial interval. We shall choose such a normalization of the gravitational constant which provides the following form of the first Friedmann equation:
\[ H^2 = \rho, \]
where
\[ H \equiv \frac{\dot{a}}{a} \]
is the Hubble parameter and \( \rho \) is the energy density of the universe. The second Friedmann or Raychaudhuri equation is
\[ \ddot{a} = -\frac{1}{2} (\rho + 3p), \]
where \( p \) is the pressure. The energy conservation equation looks as
\[ \dot{\rho} + 3H(\rho + p) = 0. \]

We shall write down also the expressions for the nonvanishing components of the Riemann–Christoffel curvature tensor, defined as [1]
\[ R^i_{\, klm} = \frac{\partial \Gamma^i_{\, km}}{\partial x^l} - \frac{\partial \Gamma^i_{\, km}}{\partial x^l} + \Gamma^i_{\, di} \Gamma^d_{\, km} - \Gamma^i_{\, mn} \Gamma^m_{\, kl}. \]
These nonvanishing components are
\[ R^a_{\, \beta \mu} = -\frac{\ddot{a}}{a} \delta^{a}_{\beta} = (-H + H^2) \delta^{a}_{\beta}, \]
where \( \alpha \) and \( \beta \) are spatial indices;
\[ R_{122}^{1} = R_{313}^{1} = R_{323}^{2} = a^2 \]
and the corresponding components arising from symmetry.

The singularities of the type I are the so-called Big Rip singularities [54, 55]. A type I singularity arises at some finite moment of the cosmic time \( t \rightarrow t_{\text{BR}} \), when \( a \rightarrow \infty, \dot{a} \rightarrow \infty, H \rightarrow \infty, \rho \rightarrow \infty, |p| \rightarrow \infty \). These singularities are present in the model, where the cosmological evolution is driven by the so-called phantom matter [56], when \( p < 0, |p| > \rho \) or, in other words the equation of state parameter \( w \equiv \frac{p}{\rho} < -1 \).

The conditions of arising and avoiding of such singularities were studied in detail in [57–59].

The singularities of the type II are characterized by the following behavior of the cosmological parameters: at a finite interval of time \( t = t_{\text{II}} \), a universe arrives with the finite values of the cosmological radius of the time derivative of the cosmological radius, of the Hubble parameter and of the energy density \( t \rightarrow t_{\text{II}}, a \rightarrow a_{\text{II}}, \dot{a} \rightarrow \dot{a}_{\text{II}}, H \rightarrow H_{\text{II}}, \rho \rightarrow \rho_{\text{II}}, \) while the acceleration of the universe and the first time derivative of the Hubble parameter tends to minus infinity \( \ddot{a} \rightarrow -\infty, H \rightarrow -\infty \) and the pressure tends to plus infinity \( p \rightarrow \infty \).

A particular case of the type II singularity is the Big Brake singularity, first found in [49]. At this singularity, the time derivative of the cosmological radius, the Hubble variable and the energy density are equal exactly to zero.
The type III singularities are the singularities occurring when the cosmological radius is finite, while its time derivative, the Hubble variable, the energy density and the pressure are divergent. The examples of such singularities were considered, for example in [9, 60].

The more soft singularities are the singularities of the type IV: at finite value of the cosmological factor both the energy density and the pressure tend to zero and only the higher derivatives of the Hubble parameter $H$ diverge. These singularities sometimes are called Big Separation singularities.

In [61] the type V singularities were added to the scheme proposed in [48]. These are the singularities which like the singularities of the type IV have the pressure and energy density tending to zero, but the higher time derivatives of the Hubble parameter are regular and only the barotropic index (equation of state parameter) $w$ is singular.

Sometimes the traditional Big Bang and Big Crunch singularities are called type 0 singularities, (see [62]).

In this review we shall mainly speak about type II singularities and their comparison with type 0 singularities.

3. The type of the singularity from the point of view of finite size objects, which approach these singularities

In this section, we shall present the classification of singularities, based on the point of view of finite size objects, which approach these singularities. In principle, finite size objects could be destroyed while passing through the singularity due to the occurring infinite tidal forces. A strong curvature singularity is defined by the requirement that an extended finite object is crushed to zero volume by tidal forces. We give below Tipler’s [41] and Królak’s [63] definitions of strong curvature singularities together with the relative necessary and sufficient conditions.

First of all, we shall write down the geodesics deviation equation. If $u^i$ are 4-velocities of test particles and $\eta^i$ is a 4-vector separating two spatially close geodesics, then the dynamics of this vector is given by the equation [1]

$$\frac{D^2 \eta^i}{ds^2} = R_{klm}^{i} u^k u^l \eta^m,$$  \hspace{1cm} (9)

where $D$ is the covariant derivative along a geodesics. In the case of a flat Friedmann universe (1) for the geodesics of particles, having zero spatial velocities (i.e. $u^0 = 0, u^t = 1$) equation (9) acquires, taking into account equation (7), a simple form

$$\ddot{\eta}^a = R_{np}^{a} \eta^p = \frac{\ddot{a}}{a} \eta^a.$$  \hspace{1cm} (10)

Looking at the above equation one can see that approaching a singularity, characterized by an infinite value of the deceleration, we experience an infinite force, stopping the farther increase of the separation of geodesics, while geodesics themselves can be quite regular if the velocity of expansion $\dot{a}$ is regular.

According to Tipler’s definition, if every volume element, defined by three linearly independent, vorticity-free, geodesic deviation vectors along every causal geodesic through a point $P$, vanishes, a strong curvature singularity is encountered at the respective point $P$ [37, 41]. The necessary and sufficient condition for a causal geodesic to run into a strong singularity at $\lambda_s$ ($\lambda$ is an affine parameter of the curve) [64] is that the double integral

$$\int_{0}^{\lambda_s} d\lambda \int_{0}^{\lambda_s} d\lambda' \left| R_{ab}^{c} \eta^a \eta^b \right|$$  \hspace{1cm} (11)
diverges as $\lambda \to \lambda_s$. A similar condition is valid for lightlike geodesics, with $R^i_{\alpha \beta \mu \nu} u^\alpha u^\mu u^\nu$ replacing $R^i_{\alpha \beta \mu \nu} u^\alpha u^\mu$ in the double integral.

Krółak’s definition is less restrictive. A future-endless, future-incomplete null (timelike) geodesic $\gamma$ is said to terminate in the future at a strong curvature singularity if, for each point $P \in \gamma$, the expansion of every future-directed congruence of null (timelike) geodesics emanating from $P$ and containing $\gamma$ becomes negative somewhere on $\gamma$ [63, 65]. The necessary and sufficient condition for a causal geodesic to run into a strong singularity at $\lambda_s$ [64] is that the integral

$$\int_0^{\lambda_s} d\lambda' |R^i_{\alpha \beta \mu \nu} u^\alpha u^\mu|$$

(12)

diverges as $\lambda \to \lambda_s$. Again, a similar condition is valid for lightlike geodesics, with $R^i_{\alpha \beta \mu \nu} u^\alpha u^\mu$ replacing $R_{\alpha \beta \mu \nu} u^\alpha u^\mu$ in the integral.

We conclude this section by mentioning that the singularities of the types 0 and I are strong and the singularities of the types II, IV and V are weak according to both the definitions (those by Tipler and Krółak), while the type III singularities are strong with respect to Krółak’s definition and weak with respect to Tipler’s definition [62].

The weakness of the type II singularities, which we shall study in some details in the following sections of this review, according to both the definitions, means that although the tidal forces become infinite, the finite objects are not necessarily crushed when reaching the singularity.

4. The tachyon cosmological model with the trigonometric potential

The tachyon field, born in the context of the string theory [66], provides an example of matter having a large enough negative pressure to produce an acceleration of the expansion rate of the universe. Such a field is today considered as one of the possible candidates for the role of dark energy and, also for this reason, in the recent years it has been intensively studied. The tachyon models represent a subclass of the models with non-standard kinetic terms [67], which descend from the Born–Infeld model, invented already in 1930s [68]. Before considering the model with the trigonometric potential [49], possessing the Big Brake singularity, we write down the general formulae of the tachyon cosmology.

The Lagrangian of the tachyon field $T$ is

$$L = -V(T) \sqrt{1 - g_{\mu \nu} T^\mu T^\nu}$$

(13)

or, for the spatially homogeneous tachyon field,

$$L = -V(T) \sqrt{1 - T^2}.$$  

(14)

The energy density and the pressure of this field are respectively

$$\rho = \frac{V(T)}{\sqrt{1 - T^2}}$$

(15)

and

$$p = -V(T) \sqrt{1 - T^2},$$

(16)

while the field equation is

$$\frac{\dddot{T}}{1 - T^2} + 3H \ddot{T} + \frac{V_T}{V(T)} = 0.$$  

(17)
We shall also introduce the pseudo-tachyon field with the Lagrangian \[49\]
\[ L = W(T)\sqrt{T^2 - 1} \quad (18) \]
and with the energy density
\[ \rho = \frac{W(T)}{\sqrt{T^2 - 1}} \quad (19) \]
and the pressure
\[ p = W(T)\sqrt{T^2 - 1}. \quad (20) \]
The Klein–Gordon equation for the pseudo-tachyon field is
\[ \dddot{T} + 3H\dot{T} + \frac{W_T}{W(T)} = 0. \quad (21) \]

We shall also write down the equations for the time derivative of the Hubble parameter in the tachyon and pseudo-tachyon models:
\[ \dot{H} = -\frac{3}{2} \frac{V(T)T^2}{\sqrt{1 - T^2}}, \quad (22) \]
\[ \dot{H} = -\frac{3}{2} \frac{W(T)T^2}{\sqrt{T^2 - 1}}. \quad (23) \]
We see that the Hubble parameter in both these models is decreasing.

Note that for the case when the potential of the tachyon field \(V(T)\) is a constant, the cosmological model with this tachyon coincides with the cosmological model with the Chaplygin gas [69]. The Chaplygin gas is the perfect fluid, satisfying the equation of state
\[ p = -\frac{A}{\rho}, \quad A > 0. \quad (24) \]

The cosmological model based on the Chaplygin gas was introduced in [70] and has acquired some popularity as a unified model of dark matter and dark energy [71]. Analogously, the pseudo-tachyon model with the constant potential coincides with the model with a perfect fluid, whose equation of state is
\[ p = +\frac{A}{\rho}, \quad A > 0. \quad (25) \]

This fluid can be called ‘anti-Chaplygin gas’. The corresponding model was introduced in [49] and we shall come back to it later. Curiously, a similar equation of motion arises in the theory of wiggly strings [72].

Now we shall study a very particular tachyon potential depending on the trigonometrical functions, which was suggested in [49]. Its form is
\[ V(T) = \frac{\Lambda}{\sin^3 \frac{3}{2}\sqrt{\frac{1}{2}(1+k)^2}} \times \sqrt{1 - (1 + k) \cos^2 \frac{3}{2}\sqrt{\frac{1}{2}(1+k)T}}, \quad (26) \]
where \(\Lambda\) is a positive constant and \(k\) is a parameter, which is chosen in the interval \(-1 < k < 1\). The case of the positive values of the parameter \(k\) is especially interesting. The set of possible cosmological evolutions, is graphically presented in figure 1, which is the phase portrait of our dynamical system, where the ordinate \(s\) is the time derivative of the tachyon field \(T\): \(s \equiv \dot{T}\).

The origin of the potential (26) is the following one: let us consider a flat Friedmann universe filled with two fluids, one of which is a cosmological constant with the equation of
state \( p = -\rho = -\Lambda \) and the second one is a barotropic fluid with the equation of state \( p = k\rho \). The Friedmann equation for such a model is exactly solvable and gives

\[
H(t) = \sqrt{\Lambda} \coth \frac{3\sqrt{\Lambda}(k + 1)t}{2}.
\]  

(27)

Then using the standard technique of the reconstruction of potentials, which was mainly used for the minimally coupled scalar field [73], but was easily generalized for the cases of non-minimally coupled fields [74, 85] and for tachyons [49, 75–77], we obtain expression (26). It is necessary to emphasize that the dynamics of the tachyon model with the potential (26) is much richer than the dynamics of the two fluid model with the unique cosmological evolution given by expression (27). In [49], both the cases \( k \leq 0 \) and \( k > 0 \) were considered. The case \( k > 0 \) is of a particular interest, because it reveals two unusual phenomena: a self-transformation of the tachyon into a pseudo-tachyon field and the appearance of the Big Brake cosmological singularity.

Let us discuss briefly the classical dynamics of the model with the trigonometric potential for the case \( k > 0 \). It is easy to see that the potential (26) is well defined at \( T_3 \leq T \leq T_4 \), where

\[
T_3 = \frac{2}{3\sqrt{(1+k)}\Lambda} \arccos \frac{1}{\sqrt{1+k}},
\]  

(28)

\[
T_4 = \frac{2}{3\sqrt{(1+k)}\Lambda} \left( \pi - \arccos \frac{1}{\sqrt{1+k}} \right).
\]  

(29)

In turn, the kinetic term \( \sqrt{1-T^2} \) is well defined at \(-1 \leq s \leq 1\). In other words, the Lagrangian (14) with the potential (26) is well defined inside the rectangle (see figure 1). The analysis of the dynamics of the equation of motion of the tachyon (17) and of the Friedmann
equations shows that a part of the trajectories end their evolution in the attractive node with the coordinates $T_0 = \frac{\pi}{\sqrt{\Lambda_1 + 1}}, s_0 = 0$, which describes an infinite de Sitter expansion. The upper and lower borders of the rectangle $s = 1, s = -1$, excluding the corner points, are the standard Big Bang cosmological singularities, while left and right borders $T = T_3$ and $T = T_4$ repel the trajectories. However, another part of the trajectories goes toward the corner points $(T = T_3, s = -1)$ and $(T = T_4, s = 1)$. These points are regular points from the point of view of the equations of motion of the corresponding dynamical system and besides, the direct calculation shows that there are no cosmological singularities there. Thus, there is no reason which prevents further evolution of the universe through these points. Indeed, one can see also that the equations of motion and their solutions can be continued into the vertical stripes (see figure 1). However, to reproduce these equations of motion in the stripes as Euler–Lagrange equations, we should substitute the tachyon Lagrangian (14) by the pseudo-tachyon Lagrangian (19) with the potential

$$W(T) = \frac{\Lambda}{\sin^2 \frac{3}{2} \sqrt{\Lambda(1 + k)} T} \times \sqrt{(1 + k) \cos^2 \frac{3}{2} \sqrt{\Lambda(1 + k)} T - 1}.$$  (30)

Thus, we have seen already the first unusual phenomenon—the self-transformation of the tachyon into the pseudo-tachyon field. Now, the question arises: what happens with the universe after the ‘crossing the corner’ and the transformation of the tachyon into the pseudo-tachyon? The analysis of equations of motion carried out in [49] shows that the universe in a finite moment of time $t = t_{BB}$ encounters the singularity, which is characterized by the following values of cosmological parameters:

$$a(t_{BB}) = a_{BB} < \infty,$$
$$\dot{a}(t_{BB}) = 0,$$
$$\ddot{a}(t) \to -\infty, \text{ at } t \to t_{BB},$$
$$T(t_{BB}) = T_{BB} > 0 \text{ (in lower left strip)},$$
$$s(t) \to -\infty, \text{ at } t \to t_{BB}, \text{ (in lower left strip)}$$
$$\rho(t_{BB}) = 0,$$
$$p(t) \to +\infty, \text{ at } t \to t_{BB}.$$  (31)

This singularity was called Big Brake singularity [49]. Obviously, it enters into the class II of singularities, according to the classification suggested in [48] and recapitulated in section 2 of this review.

Now, it is interesting to confront the prediction of this, a little bit artificial, but rather rich model with the observational data coming from the luminosity–redshift relation from Supernovae of type Ia. Such an attempt was undertaken in [29], where the set of supernovae studied in [78] was used. The strategy was as follows: the pairs of the present values of the tachyon field and of its time derivative were scanned (points in phase space) and then they were propagated backward in time, comparing the corresponding luminosity distance–redshift curves with the observational data from SNIa. Then, those pairs of values which appeared to be compatible with the data were chosen as initial conditions for the future cosmological evolution. Though the constraints imposed by the data were rather severe, both evolutions took place: one very similar to $\Lambda$CDM and ending in an exponential (de Sitter) expansion; another with the transformation of the tachyon into the pseudo-tachyon and the successive running toward the Big Brake singularity. It was found that a larger value of the model parameter $k$ enhances the probability to evolve into a Big Brake. The time intervals until the future encounter with the Big Brake were calculated and were found to be compatible with the present age of the universe [29].


The next question, which arises, is the fate of the universe after the encounter with the Big Brake singularity. As was already told above, this singularity is very soft and the geodesics can be continued across it. Then the matter, passing through the Big Brake singularity, reconstructs the spacetime. This process was studied in some detail in [40]. The analysis of the equation of motion for the universe approaching the Big Brake singularity gives the following expressions for the basic quantities:

\[ T = T_{BB} + \left( \frac{4}{3W(T_{BB})} \right)^{1/3} (t_{BB} - t)^{1/3}, \]  

\[ s = -\left( \frac{4}{81W(T_{BB})} \right)^{1/3} (t_{BB} - t)^{-2/3}, \]  

\[ a = a_{BB} - \frac{3}{4} a_{BB} \left( \frac{9W^2(T_{BB})}{2} \right)^{1/3} (t_{BB} - t)^{1/3}, \]  

\[ \dot{a} = a_{BB} \left( \frac{9W^2(T_{BB})}{2} \right)^{1/3} (t_{BB} - t)^{1/3}, \]  

\[ H = \left( \frac{9W^2(T_{BB})}{2} \right)^{1/3} (t_{BB} - t)^{1/3}. \] 

Expressions (32)–(36) can be continued into the region where \( t > t_{BB} \), which amounts to crossing the Big Bang singularity. Only the expression for \( s \) is singular at \( t = t_{BB} \), but this singularity is integrable and not dangerous.

Upon reaching the Big Brake, it is impossible for the system to stop there because the infinite deceleration leads to the decrease of the scale factor. This is because after the Big Brake crossing the time derivative of the cosmological radius (35) and of the Hubble variable (36) change their signs. The expansion is then followed by a contraction. Corresponding to given initial conditions, the values of \( T_{BB}, t_{BB} \) and \( a_{BB} \) were found numerically. Then the numerical integration of the equations of motion describes the contraction of the universe, culminating in the encounter with the Big Crunch singularity. Curiously, the time intervals between the Big Brake and Big Crunch singularities practically do not depend on the initial conditions and are equal approximately to \( 0.3 \times 10^9 \) yr [40].

Now, the next question arises: What happens if we consider a little bit more complicated model, adding to the tachyon matter some quantity of dust-like matter? Obviously, in this case instead of the Big Brake singularity the universe will encounter a soft type II singularity of a more general kind. Namely, due to the presence of dust, the energy density of the expanding universe cannot vanish and, hence, at the moment when the universe experiences an infinite deceleration its expansion should continue. This implies the appearance of some kind of contradictions, which can be resolved by the transformation of the pseudo-tachyon field into another kind of Born–Infeld-like field. The corresponding problem was considered in detail in [50, 51]. The first of these papers was devoted to a more simple model, based on the mixture of the anti-Chaplygin gas with dust. The next section will be devoted to this model.
5. The cosmological model based on the mixture of the anti-Chaplygin gas and the paradox of soft singularity crossing

The anti-Chaplygin gas with the equation of state (25) is one of the simplest cosmological models revealing the Big Brake singularity [49]. Indeed, combining the equation of state (25) with the energy conservation equation (5), one obtains immediately

$$\rho = \sqrt{\frac{B}{a^6}} - A,$$

where $B$ is a positive constant, characterizing the initial condition. Then, when in the process of the cosmological expansion the cosmological radius $a$ arrives to the critical value

$$a_S = \left(\frac{B}{A}\right)^{1/6},$$

the energy density of the universe vanishes while the pressure tends to infinity. Thus, the universe encounters the Big Brake singularity. Then, it begins contraction culminating in the encounter with the Big Crunch singularity.

Now, let us see what happens if we add some amount of dust with the energy density

$$\rho_m = \frac{\rho_0}{a^3},$$

where $\rho_0$ is a positive constant. In this case, the traversability of the singularity seems to be obstructed. The main reason for this is that while the energy density of the anti-Chaplygin gas vanishes at the singularity, the energy density of the matter component does not, leaving the Hubble parameter at the singularity with a finite value. Then some kind of the paradox arises: if the universe continues its expansion, and if the equation of state of the component of matter, responsible for the appearance of the soft singularity (in the simplest case, the anti-Chaplygin gas), is unchanged, then the expression for the energy density of this component becomes imaginary, which is unacceptable. The situation looks rather strange: indeed, the model, including dust should be in some sense more regular, than that containing only such an exotic fluid as the anti-Chaplygin gas. Thus, if the model, based on the pure anti-Chaplygin gas has a traversable Big Brake singularity, then the more general singularity arising in the model, based on the mixture of the anti-Chaplygin gas and dust, should also be traversable.

A possible way of resolution of this paradox, based on the use of the distributional cosmological quantities was suggested in [50]. Let us suppose that at the moment of the crossing of the soft cosmological singularity the expansion of the universe with the Hubble parameter $H$ is abruptly substituted by the cosmological contraction with the Hubble parameter $-H$. In this case, the value of the cosmological radius $a$ begins decreasing and expression (37) for the energy density just like the corresponding expression for the pressure remain well defined. The first Friedmann equation (2) and the energy conservation equation (5) also remain intact. A problem, however, arises with the second Friedmann equation (4). Let us rewrite this equation in the form

$$\dot{H} = -\frac{3}{2}(\rho + p).$$

If the Hubble parameter abruptly changes sign at the moment $t = t_S$ that means that it contains the term

$$H(t) = H_S(\theta(t_S - t) - \theta(t - t_S)),$$

where $\theta(x)$ is the Heaviside theta function. The derivative of the theta function is equal in the distributional sense to the Dirac delta function (see e.g. [79]). Hence, the left-hand side of equation (40) contains the Dirac delta function. Now, let us discuss in more detail the
expressions for the Hubble parameter and its time derivative in the vicinity of the singularity. The leading terms of the expression for \( H(t) \) are

\[
H(t) = H_S \text{sgn}(t_S - t) + \sqrt{\frac{3A}{2H_S a_S^4}} \text{sgn}(t_S - t) \sqrt{|t_S - t|},
\]

where \( \text{sgn}(x) \equiv \theta(x) - \theta(-x) \). Then

\[
\dot{H} = -2H_S \delta(t_S - t) - \sqrt{\frac{3A}{8H_S a_S^4}} \text{sgn}(t_S - t) \sqrt{|t_S - t|},
\]

Naturally, the \( \delta \)-term in \( \dot{H} \) arises because of the jump in \( H \), as the expansion of the universe is followed by a contraction. To restore the validity of the second Friedmann equation (40) we shall add a singular \( \delta \)-term to the pressure of the anti-Chaplygin gas, which will acquire the form

\[
p_{ACh} = \sqrt{\frac{A}{6H_S |t_S - t|}} + \frac{4}{3} H_S \delta(t_S - t).
\]

The equation of state of the anti-Chaplygin gas is preserved, if we also modify the expression for its energy density:

\[
\rho_{ACh} = \sqrt{\frac{A}{6H_S |t_S - t|}} + \frac{4}{3} H_S \delta(t_S - t).
\]

The last expression should be understood in the sense of the composition of distributions (see appendix A of [50] and references therein).

In order to prove that \( p_{ACh} \) and \( \rho_{ACh} \) represent a self-consistent solution of the system of cosmological equations, we shall use the following distributional identities:

\[
[\text{sgn}(\tau) g(|\tau|)] \delta(\tau) = 0,
\]

\[
[f(\tau) + C \delta(\tau)]^{-1} = f^{-1}(\tau),
\]

\[
\frac{d}{d\tau} [f(\tau) + C \delta(\tau)]^{-1} = \frac{d}{d\tau} f^{-1}(\tau).
\]

Here \( g(|\tau|) \) is bounded on every finite interval, \( f(\tau) > 0 \), and \( C > 0 \) is a constant. These identities were proven in [50], where was used the approach to the product and the composition of distributions developed in [80].

Due to equations (47)–(48), \( \rho_{ACh} \) vanishes at the singularity while still being continuous. The first term in the expression for the pressure (44) diverges at the singularity. Therefore, the addition of a Dirac delta term, which is not changing the value of \( p_{ACh} \) at any \( \tau \neq 0 \) (i.e. \( t \neq t_S \)), does not look too drastic and might be considered as some kind of renormalization.

To prove that the first and the second Friedmann equations and the continuity equation are satisfied we must only investigate those terms, appearing in the field equations, which contain Dirac \( \delta \)-functions. First, we check the continuity equation for the anti-Chaplygin gas. Due to the identities (47) and (48), the \( \delta(\tau) \)-terms occurring in \( \rho_{ACh} \) and \( \dot{\rho}_{ACh} \) could be dropped. We keep them, however, in order to have the equation of state explicitly satisfied. Then the \( \delta(\tau) \)-term appearing in \( 3H \rho_{ACh} \) vanishes, because the Hubble parameter changes sign at the singularity (see equation (46)).

The \( \delta(\tau) \)-term appearing in \( \rho_{ACh} \) does not affect the Friedmann equation due to the identity (47). Finally, the \( \delta \)-term arising in the time derivative of the Hubble parameter on the left-hand
side of the Raychaudhuri equation is compensated by the conveniently chosen \( \delta \)-term on the right-hand side of equation (44).

However, the mathematically self-consistent scenario, based on the use of generalized functions and on the abrupt change of the expansion into a contraction, looks rather counter-intuitive from the physical point of view. Such a behavior can be compared with the absolutely elastic bounce of a ball from a rigid wall, as studied in classical mechanics. In the latter case, the velocity and the momentum of the ball change their direction abruptly. Hence, an infinite force acts from the wall onto the ball during an infinitely small interval of time.

In reality, the absolutely elastic bounce is an idealization of a process of finite time-span during which inelastic deformations of the ball and of the wall occur. Thus, the continuity of the kinematics of the act of bounce implies a more complex and realistic description of the dynamical process of interaction between the ball and the wall. It is reasonable to think that something similar occurs also in the models, including dust and an anti-Chaplygin gas or a tachyon. The smoothing of the process of a transition from an expanding to a contracting phase should include some (temporary) geometrically induced change of the equation of state of matter or of the form of the Lagrangian. We know that such changes do exist in cosmology. In the tachyon model [49], there was the tachyon–pseudo-tachyon transformation driven by the continuity of the cosmological evolution. In a cosmological model with the phantom field with a cusped potential [52, 53], the transformations between phantom and standard scalar field were considered. Thus, it is quite natural that the process of crossing of the soft singularity should imply similar transformations.

However, now the situation is more complicated. It is not enough to require the continuity of evolution of the cosmological radius and Hubble parameter. It is necessary also to accept some hypothesis concerning the fate of the change of the equation of state of matter or of the form of the Lagrangian. This problem will be considered in the next section.

6. Paradox of soft singularity crossing and its resolution due to transformations of matter

The strategy of the analysis of the problem of soft singularity crossing in this section is the following [51]. First, we shall consider the model with the anti-Chaplygin gas and dust. We shall require a minimality of the change of the form of the dependence of the energy density and of the pressure, compatible with the continuation of the expansion while crossing the soft singularity. Such a requirement will bring us to the substitution of the anti-Chaplygin gas with the Chaplygin gas with a negative energy density. (Note, that in another context the Chaplygin gas with a negative energy density was considered in [81].) Then we shall consider the cosmological model based on the pseudo-tachyon field with a constant potential and dust. It is known that the energy–momentum tensor for such a pseudo-tachyon field coincides with that of the anti-Chaplygin gas (this fact relating the Chaplygin gas and the tachyon field with a constant potential was found in [69]). Thus, we would like to derive the form of the transformation of the pseudo-tachyon Lagrangian using its kinship with the anti-Chaplygin gas. As a result, we shall come to a new type of the Lagrangian, belonging to the ‘Born–Infeld family’. Finally, we shall extend the found form of transformation of the pseudo-tachyon field for the case of the field with the trigonometric potential.

As follows from equations (25) and (37), the pressure of the anti-Chaplygin gas

\[
p = \frac{A}{\sqrt{\frac{B}{\pi^2} - A}} \tag{49}
\]
and it tends to $+\infty$ when the universe approaches the soft singularity, when the cosmological radius $a \to a_S$ (see equation (38)). If we would like to continue the expansion into the region $a > a_S$, while changing minimally the equation of state we can require

$$\rho = \frac{A}{\sqrt{\frac{B}{a^6} - A}},$$

(50)

or, in other words,

$$p = \frac{A}{\sqrt{A - \frac{B}{a^6}}}, \text{ for } a > a_S.$$

(51)

We see that in some ‘generalized sense’ we conserve the continuity of the pressure crossing the soft singularity. It passes $+\infty$ conserving its sign. Combining expression (51) with the energy conservation law (5) we obtain

$$\rho = -\sqrt{A - \frac{B}{a^6}} \text{ for } a > a_S.$$

(52)

Thus, the energy density is also continuous passing through its vanishing value and changing its sign. It is easy to see that the energy density (52) and the pressure (51) satisfy the Chaplygin gas equation of state

$$p = -\frac{A}{\rho}.$$  

(53)

Thus, we have seen the transformation of the anti-Chaplygin gas into the Chaplygin gas with a negative energy density. The Friedmann equation after crossing the singularity is

$$H^2 = \frac{\rho_m}{a^2} - \sqrt{A} \sqrt{1 - \left(\frac{\dot{a}}{a}\right)^6}.$$  

(54)

It follows immediately from equation (54) that after achieving the point of maximal expansion $a = a_{\text{max}}$, where

$$a_{\text{max}} = \left(\frac{\rho_m}{A} + a_S^6\right)^{1/6},$$

(55)

the universe begins contracting. When the contracting universe arrives to $a = a_S$, it again stumbles upon a soft singularity and the Chaplygin gas transforms itself into the anti-Chaplygin gas with a positive energy density and the contraction continues until hitting the Big Crunch singularity.

Recall that in the preceding section and in [50], the process was described when the universe passed from the expanding to the collapsing phase instantaneously at the singularity causing a jump in the Hubble parameter. Here we showed that the continuous transition to the collapsing phase is possible if the equation of state of anti-Chaplygin gas has a some kind of a ‘phase transition’ at the singularity.

When the potential of the pseudo-tachyon field is constant, $W(T) = W_0$, then the energy density (19) and the pressure (20) satisfy the anti-Chaplygin gas equation of state (25) with

$$A = W_0^2.$$  

(56)

Solving the equation of motion for the pseudo-tachyon field (21) with $W(T) = W_0$ one finds

$$\dot{T}^2 = \frac{1}{1 - \left(\frac{\dot{a}}{a}\right)^6}$$  

(57)

and we see that the soft singularity arises at $a = a_S$, when $\dot{T}^2 \to +\infty$. 

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Now, we would like to change the Lagrangian (18) in such a way that the new Lagrangian gives us the energy density and the pressure satisfying the Chaplygin gas equation with a negative energy density. It is easy to check that the Lagrangian
\[ L = W_0 \sqrt{T^2 + 1} \] (58)
giving
\[ p = W_0 \sqrt{T^2 + 1} \] (59)
and
\[ \rho = -\frac{W_0}{\sqrt{T^2 + 1}} \] (60)
is what we are looking for.

Note, that the energy density and the pressure, passing through the singularity are continuous in the same sense in which they were continuous in the case of the anti-Chaplygin gas. Thus, we have introduced a new type of the Born–Infeld field, which can be called ‘anti-tachyon’. Generally, its Lagrangian is
\[ L = W(T) \sqrt{T^2 + 1} \] (61)
and the equation of motion is
\[ \frac{\ddot{T}}{T^2 + 1} + 3H\dot{T} - \frac{W_T}{W} = 0. \] (62)
For the case \( W(T) = W_0 \), the solution of equation (62) is
\[ \ddot{T}^2 = \frac{1}{\left( \frac{a}{a_0} \right)^6 - 1}, \] (63)
and the energy density evolves as
\[ \rho_T = -W_0 \sqrt{1 - \left( \frac{a}{a_0} \right)^6}, \] (64)
and the evolution of the universe repeats that for the model with the anti-Chaplygin gas and dust.

Let us emphasize once again that the transformation from the anti-Chaplygin gas to the Chaplygin gas corresponds to the transition from the pseudo-tachyon field with the Lagrangian (18) to the new type of the Born–Infeld field, which we can call ‘quasi-tachyon field’ with the Lagrangian (61).

Now, we shall consider the case of the toy model with the trigonometric potential in the presence of dust. We have seen that the Born–Infeld type pseudo-tachyon field runs into a soft Big Brake singularity with the expansion of the universe in this model. However, what happens in the presence of dust component? Does the universe still run into soft singularity?

To answer this question rewrite equation (21) as
\[ \ddot{T} = (\dot{T}^2 - 1) \left( 3H\dot{T} + \frac{W_T}{W} \right). \] (65)
It is easy to see that in the left lower and in the right upper stripes (see figure 1), where the trajectories describe the expansion of the universe after the transformation of the tachyon into the pseudo-tachyon field, the signs of \( T, \dot{T} \) and the term \( \frac{W_T}{W} \) coincide. The detailed analysis based on this fact was carried out in [49] and led to the conclusion that the universe encounters the singularity as \( T \to T_S \) (\( T_S > 0 \) or \( T_S > T_{\max} \), \(|T| \to \infty \). The presence of dust cannot
alter this effect because it increases the influence of the term $3H\dot{T}$, and hence, accelerates the encounter with the singularity.

However, the presence of dust changes the time dependence of the tachyon field close to the singularity in an essential way. As it was shown in [40] (see also section 4 of this review),

$$T = T_{BB} + \left( \frac{4}{3W(T_{BB})} \right)^{1/3} (t_{BB} - t)^{1/3},$$

(66)

while in the presence of dust one has

$$T = T_S + \frac{2}{3H_S} \sqrt{t_S - t},$$

(67)

where $H_S$ is the nonvanishing value of the Hubble parameter given by

$$H_S = \sqrt{\frac{\rho_{m,0}}{a_S^2}}.$$  

(68)

It is easy to see that the smooth continuation of expression (67) is impossible in contrast to the situation without dust (66) considered in [40].

Thus, the presence of dust is responsible for the appearance of similar paradoxes in both the anti-Chaplygin gas and tachyon models.

In the vicinity of the soft singularity, it is the ‘friction’ term $3H\dot{T}$ in the equation of motion (21) which dominates over the potential term $\frac{W_T}{T}$; hence, the dependence of $W(T)$ is not essential and a pseudo-tachyon field approaching this singularity behaves like one with a constant potential. Thus, it is quite reasonable to suppose that crossing the soft singularity, the pseudo-tachyon transforms itself into the quasi-tachyon with the Lagrangian (61).

Now, we can analyze the dynamics of the anti-tachyon field, driven by the equation of motion (62) and the Friedmann equation, where the right-hand side includes the dust contribution and the anti-tachyon energy density

$$\rho = -\frac{W(T)}{\sqrt{T^2 + 1}}.$$  

(69)

It is convenient to consider the processes developing in the left lower strip of the phase diagram of the model to facilitate the comparison with the earlier studies of the dynamics of the tachyon model without dust, undertaken in [40, 49].

One can see that the relative sign of the term with the second derivative $\ddot{T}$ with respect to the friction term $3H\dot{T}$ are opposite for the pseudo-tachyons and anti-tachyons. That means that after the crossing of the soft singularity the time derivative $\dot{T}$ is growing and its absolute value is diminishing. At the same time the value of the field $T$ is diminishing and the value of the potential $W(T)$ is growing. That means that the absolute value of the negative contribution to the energy density of the universe coming from the quasi-tachyon is growing while the energy density of the dust is diminishing due to the expansion of the universe. At some moment this process brings us to the vanishing value of the general energy density and we arrive to the point of maximal expansion of the universe. After that the expansion is replaced by the contraction and the Hubble variable changes sign. The change of sign of the friction term $3H\dot{T}$ implies the diminishing of the value of $\dot{T}$ and at some finite moment of time the universe again encounters the soft singularity when $T \to -\infty$. Passing this singularity the quasi-tachyon transforms itself back to the pseudo-tachyon and the relative sign of the terms with the second and first time derivatives in the equation of motion for this field changes once again. After that the time derivative of the pseudo-tachyon field begins growing and the universe continues its contraction until it encounters with the Big Crunch singularity.
It was shown in [40] that for the case of the purely tachyon model with the trigonometric potential the encounter of the universe with the Big Crunch singularity occurs at \( T = 0 \) and \( \dot{T} = -\sqrt{\frac{1+k}{k}} \). One can show that the presence of dust does not change these values. Indeed, let us consider the behavior of the pseudo-tachyon field when \( T \to 0 \), \( \dot{T} \to -\sqrt{\frac{1+k}{k}} \). It follows from expressions (19) and (20) that the ratio between the pressure and the energy density behaves as

\[ \frac{p}{\rho} = \dot{T}^2 - 1 \to \frac{1}{k}, \]  

i.e. in the vicinity of the Big Crunch singularity the pseudo-tachyon field behaves as a barotropic fluid with the equation of state parameter \( \frac{1}{k} > 1 \). That means that the energy density of the pseudo-tachyon field is growing as

\[ \rho \sim \frac{1}{a^{3(1+\frac{1}{k})}} \]  

as \( a \to 0 \), i.e. much more rapidly than the dust energy density. Thus, one can neglect the contribution of the dust in this regime of approaching the Big Crunch singularity and the description of the evolution of the universe to this point coincides with that of the pure tachyon model [40].

7. The transformations of the Lagrangian of a scalar field with a cusped potential

It is well known that the cosmological observations give as a best fit for the equation of state parameter \( w = \frac{p}{\rho} \) a value which is slightly inferior with respect to \(-1\) (see, e.g. [82]). The corresponding type of dark energy was called ‘phantom’ matter [56]. Wanting to realize such a dark matter using a minimally coupled scalar field, one has to introduce for the latter a negative kinetic term. Thus, its Lagrangian has the form

\[ L = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi). \]  

Some observations also indicate that the value of the equation of state parameter at some moment in the past has crossed the value \( w = -1 \), corresponding to the cosmological constant. Such a phenomenon has received the name of ‘phantom divide line crossing’ [83]. A minimally coupled scalar field, describing non-phantom dark energy has a kinetic term with the positive sign. So, it looks natural, to use two scalar fields, a phantom field with the negative kinetic term and a standard one, to describe the phantom divide line crossing [84]. Another possible way of the phantom divide line crossing, using a scalar field nonminimally coupled to gravity, was considered in [85].

However, in [52, 53] it was shown that considering potentials with cusps and choosing some particular initial conditions, one can describe the phenomenon of the phantom divide line crossing in the model with one minimally coupled scalar field. Curiously, a passage through the maximum point of the evolution of the Hubble parameter implies the change of sign of the kinetic term. Though a cosmological singularity is absent in these cases, this phenomenon is a close relative of those considered in the preceding sections, because here also we stumble upon some transformation of matter properties, induced by a change of geometry. One can add that in this aspect the phenomenon of the phantom divide line crossing is the close analogue of the transformation between the tachyon and pseudo-tachyon model, in the tachyon model with the trigonometric potential, described in section 4. Here, we shall present a brief sketch of the ideas, described in [52, 53], emphasizing the analogy and the differences between different geometrically induced matter transformations.
We begin with a simple mechanical analogue: a particle moving in a potential with a cusp [52]. Let us consider a one-dimensional problem of a classical point particle moving in the potential

\[ V(x) = \frac{V_0}{(1 + x^{2/3})^2}, \tag{73} \]

where \(V_0 > 0\). The equation of motion is

\[ \ddot{x} - \frac{4V_0}{3(1 + x^{2/3})^3x^{1/3}} = 0. \tag{74} \]

We consider three classes of possible motions characterized by the value of the energy \(E\). The first class consists of the motions when \(E < V_0\). Apparently, the particle with \(x < 0, \dot{x} > 0\) or with \(x > 0, \dot{x} < 0\) cannot reach the point \(x = 0\) and stops at the points \(x = \pm (\sqrt{V_0/E} - 1)^{3/2}\), respectively.

The second class includes the trajectories when \(E > V_0\). In this case, the particle crosses the point \(x = 0\) with a nonvanishing velocity.

If we have a fine tuning such that \(E = V_0\), we encounter an exceptional case. Now the trajectory satisfying equation (74) in the vicinity of the point \(x = 0\) can behave as

\[ x = C(t_0 - t)^{3/2}, \tag{75} \]

where

\[ C = \pm \left( \frac{16V_0}{9} \right)^{3/4} \tag{76} \]

and \(t \leq t_0\). It is easy to see that independently of the sign of \(C\) in equation (76) the signs of the particle coordinate \(x\) and of its velocity \(\dot{x}\) are opposite and hence, the particle can arrive in finite time to the point of the cusp of the potential \(V(x)\).

Another solution reads as

\[ x = C(t - t_0)^{3/2}, \tag{77} \]

where \(t \geq t_0\). This solution describes the particle going away from the point \(x = 0\). Thus, we can combine the branches of the solutions (75) and (77) in four different manners and there is no way to choose if the particle arriving to the point \(x = 0\) should go back or should pass the cusp of the potential (73). It can stop at the top as well. Such a ‘degenerate’ behavior of the particle in this third case is connected with the fact that this trajectory is the separatrix between two one-parameter families of trajectories described above. At the moment there is not yet any strict analogy between this separatrix and the cosmological evolution describing the phantom divide line. In order to establish a closer analogy and to understand what is the crucial difference between mechanical consideration and general relativistic one, we can try to introduce a friction term into the Newton equation (74)

\[ \ddot{x} + \gamma \dot{x} - \frac{4V_0}{3(1 + x^{2/3})^3x^{1/3}} = 0. \tag{78} \]

It is easy to check that if the friction coefficient \(\gamma\) is a constant, one does not have a qualitative change in respect to the discussion above. Let us assume for \(\gamma\) the dependence

\[ \gamma = 3\sqrt{\frac{\dot{x}^2}{2} + V(x)}. \tag{79} \]

then

\[ \ddot{x} = -\frac{3}{2} \dot{x}^2 \tag{80} \]
and
\[ \dot{\gamma} = -3\ddot{x}\dot{x} \] (81)
just like in the cosmological case, where the role of the friction coefficient is played by the Hubble parameter. The trajectory arriving to the cusp with vanishing velocity is still described by solution (75). Consider the particle coming to the cusp from the left \((C < 0)\). It is easy to see that the value of \(\dot{\gamma}\) at the moment \(t_0\) tends to zero, while its second derivative \(\ddot{\gamma}\) given by equation (81) is
\[ \ddot{\gamma}(t_0) = \frac{9}{8}C^2 > 0. \] (82)
Thus, it looks like the friction coefficient \(\gamma\) reaches its minimum value at \(t = t_0\). Let us suppose now that the particle is coming back to the left from the cusp and its motion is described by equation (77) with negative \(C\). A simple check shows that in this case
\[ \ddot{\gamma}(t_0) = -\frac{9}{8}C^2 < 0. \] (83)
Thus, from the point of view of the subsequent evolution this point looks as a maximum for the function \(\gamma(t)\). In fact, it means simply that the second derivative of the friction coefficient has a jump at the point \(t = t_0\). It is easy to check that if instead of choosing the motion to the left, we shall move forward our particle to the right from the cusp \((C > 0)\), the sign of \(\ddot{\gamma}(t_0)\) remains negative as in equation (83) and hence we have the jump of this second derivative again. If one would like to avoid this jump, one should try to change the sign in equation (81). To implement it in a self-consistent way one can substitute equation (79) by
\[ \dot{\gamma} = 3\sqrt{\frac{x^2}{2} + V(x)} \] (84)
and equation (78) by
\[ \ddot{x} + \gamma\dot{x} + \frac{4V_0}{3(1 + x^2/3)^{3/2}x^{3/2}} = 0. \] (85)
In fact, it is exactly that what happens automatically in cosmology, when we change the sign of the kinetic energy term for the scalar field, crossing the phantom divide line. Naturally, in cosmology the role of \(\gamma\) is played by the Hubble variable \(H\). The jump of the second derivative of the friction coefficient \(\gamma\) corresponds to the divergence of the third time derivative of the Hubble variable, which represents some kind of very soft cosmological singularity.

Thus, one seems to confront the problem of choosing between two alternatives: (1) to encounter a weak singularity in the spacetime geometry, (2) to change the sign of the kinetic term for matter field. We have pursued the second alternative insofar as we privilege the smoothness of the spacetime geometry and consider equations of motion for matter as less fundamental than the Einstein equations.

Now, we would like to say that the potential, considered in [52, 53], had the general structure
\[ V(\phi) = \frac{1}{A + B\phi^{2/3}}. \] (86)
The origin of this structure is the following: one considers the power-law expansion of the universe, it is well known that such an expansion could be provided by an exponential potential [86]. Then one can represent the Friedmann equation for the evolution of the scale factor of the universe as a second-order linear differential equation, where the potential is represented as a function of the time parameter [87]. This equation has two independent solutions: one of them is the power-law expansion and other corresponds to an evolution driven by a phantom matter. The linear combination of these two solutions with both nonvanishing coefficients gives an
evolution, where a universe crosses the phantom divide line. It is impossible to reconstruct the form of the potential as a function of the scalar field, which provides such an evolution explicitly; however, one can study its form around the point where the phantom divide crossing occurs and this form is exactly that of equation (86) [52].

At the end of this section, we would like to say that in the Newtonian mechanics there is rather a realistic example of motion when the dependence of the distance of time is given by some fractional power [88, 89]. Indeed, if one considers the motion of a car with a constant power (which is more realistic than the motion with a constant force, usually presented in textbooks), then the velocity behaves as \( t^{1/2} \), and if the initial values of the coordinate and velocity are equal to zero, then the acceleration behaves as \( t^{-1/2} \) and at the moment of start is singular. The motion at constant power is an excellent model of drag-car racing [88, 89]. Its analogy with the cosmology at the presence of sudden singularities was noticed in [45].

8. Classical dynamics of the cosmological model with a scalar field whose potential is inversely proportional to the field

We have considered earlier the simplest model, possessing a soft cosmological singularity (Big Brake)—the model based on the anti-Chaplygin gas. It was noticed that this model is equivalent to the model with the pseudo-tachyon field with constant potential. Here we would like to study a model, based on a minimally coupled scalar field, which possesses the same evolution as the model based on the anti-Chaplygin gas. Using the standard technique of the reconstruction of potential, the potential of the corresponding scalar field was found in [32] and it looks like

\[
V(\phi) = \pm \frac{\sqrt{A}}{2} \left( \sinh 3\phi - \frac{1}{\sinh 3\phi} \right).
\]

As a matter of fact we have two possible potentials which differ by the general sign. We choose the sign ‘plus’. Then, let us remember that the Big Brake occurs when the energy density is equal to zero (the disappearance of the Hubble parameter) and the pressure is positive and infinite (an infinite deceleration). To achieve this condition, in the scalar field model it is necessary to require that the potential is negative and infinite. It is easy to see from equation (87) that this occurs when \( \phi \to 0 \) being positive. Thus, to have the model with the Big Brake singularity we can consider the scalar field with a potential which is a little bit simpler than that from equation (87), but still possesses rather a rich dynamics. Namely, we shall study the scalar field with the potential

\[
V = -\frac{V_0}{\phi},
\]

(88)

where \( V_0 \) is a positive constant. The Klein–Gordon equation for the scalar field with the potential (88) is

\[
\ddot{\phi} + 3H \dot{\phi} + \frac{V_0}{\phi^2} = 0
\]

(89)

while the first Friedmann equation is

\[
H^2 = \frac{\dot{\phi}^2}{2} - \frac{V_0}{\phi}.
\]

(90)

We shall also need the expression for the time derivative of the Hubble parameter, which can be easily obtained from equations (89) and (90):

\[
\dot{H} = -\frac{3}{2} \dot{\phi}^2.
\]

(91)
Now we shall construct the complete classification of the cosmological evolutions (trajectories) of our model, using equations (89)–(91) [34].

First of all, let us announce briefly the main results of our analysis.

1. The transitions between the positive and negative values of the scalar field are impossible.
2. All the trajectories (cosmological evolutions) with positive values of the scalar field begin in the Big Bang singularity, then achieve a point of maximal expansion, then contract and end their evolution in the Big Crunch singularity.
3. All the trajectories with positive values of the scalar field pass through the point where the value of the scalar field is equal to zero. After that the value of the scalar field begins growing. The point \( \phi = 0 \) corresponds to a crossing of the soft singularity.
4. If the moment when the universe achieves the point of the maximal expansion coincides with the moment of the crossing of the soft singularity, then the singularity is the Big Brake.
5. The evolutions with the negative values of the scalar field belong to two classes—first, an infinite expansion beginning from the Big Bang and second, the evolutions obtained by the time reversion of those of the first class, which are contracting and end in the Big Crunch singularity.

To prove these results, we begin with the consideration of the universe in the vicinity of the point \( \phi = 0 \). We shall look for the leading term of the field \( \phi \) approaching this point in the form

\[
\phi(t) = \phi_1 (t_S - t)^\alpha, \tag{92}
\]

where \( \phi_1 \) and \( \alpha \) are positive constants and \( t_S \) is the moment of the soft singularity crossing. The time derivative of the scalar field is now

\[
\dot{\phi}(t) = \alpha \phi_1 (t_S - t)^{\alpha - 1}. \tag{93}
\]

Because of the negativity of the potential (88) at positive values of \( \phi \), the kinetic term should be stronger than the potential one to satisfy the Friedmann equation (4). This implies that \( \alpha \leq \frac{2}{3} \). However, if \( \alpha < \frac{2}{3} \), we can neglect the potential term and remain with the massless scalar field. It is easy to show considering the Friedmann (4) and Klein–Gordon (89) equations that in this case the scalar field behaves like \( \phi \sim \ln(t_S - t) \), which is incompatible with the hypothesis of its smallness (92). Thus, one remains with the only choice

\[
\alpha = \frac{2}{3}. \tag{94}
\]

Then, if the coefficient at the leading term in the kinetic energy is greater than that in the potential, it follows from the Friedmann equation (4) that the Hubble parameter behaves as \((t_S - t)^{-\frac{2}{3}}\) which is incompatible with equation (91). Thus, the leading terms of the potential and kinetic energy should cancel each other:

\[
\frac{1}{2} \alpha^2 \phi_1^2 (t_S - t)^{2\alpha - 2} = \frac{V_0}{\phi_1} (t_S - t)^{-\alpha}, \tag{95}
\]

that for \( \alpha = \frac{2}{3} \) gives

\[
\phi_1 = \left( \frac{9V_0}{2} \right)^{\frac{1}{3}}. \tag{96}
\]

Hence, the leading term for the scalar field in the presence of the soft singularity is

\[
\phi(t) = \left( \frac{9V_0}{2} \right)^{\frac{1}{3}} (t_S - t)^{\frac{2}{3}}. \tag{97}
\]
Now, integrating equation (91) we obtain

\[ H(t) = 2 \left( \frac{9V_0}{2} \right)^{\frac{1}{3}} (t_S - t)^{\frac{1}{3}} + H_S, \]  

(98)

where \( H_S \) is an integration constant giving the value of the Hubble parameter at the moment of the soft singularity crossing. If this constant is equal to zero, \( H_S = 0 \), the moment of the maximal expansion of the universe coincides with that of the soft singularity crossing and the universe encounters the Big Brake singularity. If \( H_S \neq 0 \), we have a more general type of the soft cosmological singularity where the energy density of the matter in the universe is different from zero. The sign of \( H_S \) can be both, positive or negative; hence, universe can pass through this singularity in the phase of its expansion or of its contraction.

The form of the leading term for the scalar field in the vicinity of the moment when \( \varphi = 0 \) (97) shows that, after passing the zero value, the scalar field begin growing being positive. Thus, it proves the first result from the list presented above about impossibility of the change of the sign of the scalar field in our model.

We have already noted that the time derivative of the scalar field had changed the sign crossing the soft singularity. It cannot change the sign in a non-singular way because the conditions \( \dot{\varphi}(t_0) = 0 \) and \( \varphi(t_0) \neq 0 \) are incompatible with the Friedmann equation (4). It is seen from equation (97) that before the crossing of the soft singularity the time derivative of the scalar field is negative and after its crossing it is positive. The impossibility of changing the sign of the time derivative of the scalar field without the soft singularity crossing implies the inevitability of the approaching of the universe to this soft singularity. Thus, the third result from the list above is proven.

It is easy to see from equation (91) that the value of the Hubble parameter is decreasing during all the evolution. At the same time, the absolute value of its time derivative (proportional to the time derivative squared of the scalar field) is growing after the soft singularity crossing. That means that at some moment the Hubble parameter should change its sign becoming negative. The change of the sign of the Hubble parameter is nothing but the passing through the point of the maximal expansion of the universe, after which it begin contraction culminating in the encounter with the Big Crunch singularity. Thus, the second result from the list presented above is proven.

Summing up, we can say that all the cosmological evolutions where the scalar field has positive values have the following structure: they begin in the Big Bang singularity with an infinite positive value of the scalar field and an infinite negative value of its time derivative, then they pass through the soft singularity where the value of the scalar field is equal to zero and where the derivative of the scalar field changes its sign. All the trajectories also pass through the point of the maximal expansion, and this passage through the point of the maximal expansion can precede or follow the passage through the soft singularity: in the case when these two moments coincide (\( H_S = 0 \)) we have the Big Brake singularity (see result (iv) from the list above). Thus, all the evolutions pass through the soft singularity, but only for one of them this singularity has a character of the Big Brake singularity. The family of the trajectories can be parameterized by the value of the Hubble parameter \( H_S \) at the moment of the crossing of the soft singularity. There is also another natural parameterization of this family—we can characterize a trajectory by the value of the scalar field \( \varphi \) at the moment of the maximal expansion of the universe and by the sign of its time derivative at this moment (if the time derivative of the scalar field is negative, it means that the passing through the point of maximal expansion precedes the passing through the soft singularity and if the sign of this time derivative is positive, then the passage through the point of maximal expansion follows the passage through the soft singularity). If at the moment when the universe achieves the point
of maximal expansion the value of the scalar field is equal to zero, then it is the exceptional trajectory crossing the Big Brake singularity.

For completeness, we shall say some words about result (v), concerning the trajectories with the negative values of the scalar field. Now, both the terms on the right-hand side of the Friedmann equation (4), potential and kinetic, are positive and, hence, the Hubble parameter cannot disappear or change its sign. It can only tend to zero asymptotically while both these terms tend asymptotically to zero. Thus, in this case there are two possible regimes: an infinite expansion which begins with the Big Bang singularity and an infinite contraction which culminates in the encounter with the Big Crunch singularity. The second regime can be obtained by the time reversal of the first one and vice versa. Let us consider the expansion regime. It is easy to check that the scalar field being negative cannot achieve the zero value, because the suggestion 

\[ \phi(t) = -\varphi_1(t_0 - t)^\alpha, \]

where \( \varphi_1 < 0, \alpha > 0 \), is incompatible with equations (4) and (91). Hence, the potential term is always non-singular and at the birth of the universe from the Big Bang singularity the kinetic term dominates and the dynamics is that of the theory with the massless scalar field. Namely,

\[ \phi(t) = \phi_0 + \frac{\sqrt{2}}{9} \ln t, \quad H(t) = \frac{1}{3t}, \]  

(99)

where \( \phi_0 \) is a constant. At the end of the evolution the Hubble parameter tends to zero, while the time grows indefinitely. Which means that both the kinetic and potential terms on the right-hand side of equation (4) should tend to zero. It is possible if the scalar field tends to infinity while its time derivative tends to zero. The joint analysis of equations (4) and (91) gives the following results for the asymptotic behavior of the scalar field and the Hubble parameter:

\[ \phi(t) = \tilde{\phi}_0 - \left(\frac{5}{6}\right)^\frac{1}{5} V_0^\frac{1}{5} t^\frac{4}{5}, \quad H(t) = \left(\frac{6}{5}\right)^\frac{1}{5} V_0^\frac{1}{5} t^{-\frac{1}{5}}, \]  

(100)

where \( \tilde{\phi}_0 \) is a constant.

9. The quantum dynamics of the cosmological model with a scalar field whose potential is inversely proportional to the field

The introduction of the notion of the quantum state of the universe, satisfying the Wheeler–DeWitt equation [90], has stimulated the diffusion of the hypothesis that in the framework of quantum cosmology, the singularities can disappear in some sense. Namely, the probability of finding of the universe with the parameters, which correspond to a classical cosmological singularity, can be equal to zero (for recent treatments, see [91–93]).

In this section, we shall study the quantum dynamics of the model, whose classical dynamics was described in the preceding section. Our presentation follows that of [32, 34].

As usual, we shall use the canonical formalism and the Wheeler–DeWitt equation [90]. For this purpose, instead of the Friedmann metric (1), we shall consider a more general metric

\[ ds^2 = N^2(t) dt^2 - a^2(t) dl^2, \]

(101)

where \( N \) is the so-called lapse function. The action of the Friedmann flat model with the minimally coupled scalar field looks now as

\[ S = \int dt \left( \frac{a^3 \dot{\phi}^2}{2N} - a^2 V(\phi) - \frac{aa^3}{N} \right). \]  

(102)

Variating the action (102) with respect to \( N \) and putting then \( N = 1 \) we come to the standard Friedmann equation. Now, introducing the canonical formalism, we define the canonically conjugated momenta as

\[ p_\phi = \frac{a^3 \phi}{N} \]  

(103)
\[ p_a = -\frac{a \dot{a}}{N}, \]  

(104)

The Hamiltonian is
\[ \mathcal{H} = N \left( -\frac{\dot{p}_a^2}{4a} + \frac{\dot{p}_\phi^2}{2a^3} + V(a) \right), \]  

(105)

and is proportional to the lapse function. The variation of the action with respect to \( N \) gives the constraint
\[ -\frac{\dot{p}_a^2}{4a} + \frac{\dot{p}_\phi^2}{2a^3} + V(a) = 0, \]  

(106)

and the implementation of the Dirac quantization procedure, i.e. the requirement that constraint eliminates the quantum state [94], gives the Wheeler–DeWitt equation
\[ \left( -\frac{\dot{p}_a^2}{4a} + \frac{\dot{p}_\phi^2}{2a^3} + V(a) \right) \psi(a, \phi) = 0. \]  

(107)

Here \( \psi(a, \phi) \) is the wavefunction of the universe and the hats over the momenta mean that the functions are substituted by the operators. Introducing the differential operators representing the momenta as
\[ \hat{p}_a = \frac{\partial}{i \partial a}, \quad \hat{p}_\phi = \frac{\partial}{i \partial \phi}, \]  

(108)

and multiplying equation (107) by \( a^3 \) we obtain the following partial differential equation:
\[ \left( a^2 \frac{\partial^2}{\partial a^2} + \frac{1}{2} \frac{\partial^2}{\partial \phi^2} + a^6 V_0 \right) \psi(a, \phi) = 0. \]  

(109)

Finally, for our potential inversely proportional to the scalar field we have
\[ \left( a^2 \frac{\partial^2}{\partial a^2} + \frac{1}{2} \frac{\partial^2}{\partial \phi^2} - \frac{a^6 V_0}{\phi} \right) \psi(a, \phi) = 0. \]  

(110)

Note that in equation (107) and in the subsequent equations we have ignored rather a complicated problem of the choice of the ordering of noncommuting operators, because the specification of such a choice is not essential for our analysis. Moreover, the interpretation of the wavefunction of the universe is rather an involved question [95–97]. The point is that to choose the measure in the space of the corresponding Hilbert space we should fix a particular gauge condition, eliminating in such a way the redundant gauge degrees of freedom and introducing a temporal dynamics into the model [96]. We shall not dwell here on this procedure, assuming generally that the cosmological radius \( a \) is in some way connected with the chosen time parameter and that the unique physical variable is the scalar field \( \phi \). Then, it is convenient to represent the solution of equation (110) in the form
\[ \psi(a, \phi) = \sum_{n=0}^{\infty} C_n(a) \chi_n(a, \phi), \]  

(111)

where the functions \( \chi_n \) satisfy the equation
\[ \left( -\frac{1}{2} \frac{\partial^2}{\partial \phi^2} - \frac{a^6 V_0}{\phi} \right) \chi(a, \phi) = -E_n(a) \chi_n(a, \phi), \]  

(112)
while the functions $C_n(a)$ satisfy the equation
\[
\frac{a^2}{4} \frac{\partial^2 C_n(a)}{\partial a^2} = E_n(a) C_n(a),
\]
(113)
where $n = 0, 1, \ldots$. Requiring the normalizability of the functions $\chi_n$ on the interval $0 \leq \phi < \infty$, which, in turn, implies their non-singular behavior at $\phi = 0$ and $\phi \to \infty$, and using the considerations similar to those used in the analysis of the Schrödinger equation for the hydrogen-like atoms, one can show that the acceptable values of the functions $E_n$ are
\[
E_n = \frac{V_0 a^{12}}{2(n + 1)^2},
\]
(114)
while the corresponding eigenfunctions are
\[
\chi_n(a, \phi) = \phi \exp \left( -\frac{V_0 a^6 \phi}{n + 1} \right) L_n^1 \left( \frac{2V_0 a^6 \phi}{n + 1} \right),
\]
(115)
where $L_n^1$ are the associated Laguerre polynomials.

Rather often the fact that the wavefunction of the universe disappears at the values of the cosmological parameters corresponding to some classical singularity is interpreted as an avoidance of such singularity. However, in the case of the soft singularity considered in the model at hand, such an interpretation does not look too convincing. Indeed, one can have a temptation to think that the probability of finding of the universe in the soft singularity state characterized by the vanishing value of the scalar field is vanishing because the expression for functions (115) entering into the expression for the wavefunction of the universe (111) is proportional to $\phi$. However, the wavefunction (111) can hardly have a direct probabilistic interpretation. Instead, one should choose some reasonable time-dependent gauge, identifying some combination of variables with an effective time parameter, and interpreting other variables as physical degrees of freedom [96]. The definition of the wavefunction of the universe in terms of these physical degrees of freedom is rather an involved question; however, we are in a position to make some semi-qualitative considerations. The reduction of the initial set of variables to the smaller set of physical degrees of freedom implies the appearance of the Faddeev–Popov determinant which as usual is equal to the Poisson bracket of the gauge-fixing condition and the constraint [96]. Let us, for example, choose as a gauge-fixing condition the identification of the new 'physical' time parameter with the Hubble parameter $H$ taken with the negative sign. Such an identification is reasonable, because as it follows from equation (91), the variable $H(t)$ is monotonously decreasing. The volume $a^3$ is the variable canonically conjugated to the Hubble variable. Thus, the Poisson bracket between the gauge-fixing condition $\chi = H - T_{\text{phys}}$ and the constraint (106) includes the term proportional to the potential of the scalar field, which is inversely proportional to this field itself. Thus, the singularity in $\phi$ arising in the Faddeev–Popov determinant can cancel zero, arising in (115).

Let us confront this situation with that of the Big Bang and Big Crunch singularities. As it was seen in section 3, such singularities classically arise at infinite values of the scalar field. To provide the normalizability of the wavefunction one should have the integral on the values of the scalar field $\phi$ convergent, when $|\phi| \to \infty$. This means that, independently of details connected with the gauge choice, not only the wavefunction of the universe but also the probability density of scalar field values should decrease rather rapidly when the absolute value of the scalar field is increasing. Thus, in this case, the effect of the quantum avoidance of the classical singularity is present.
10. The quantum cosmology of the tachyon and the pseudo-tachyon field

In this section, we would like to construct the Hamiltonian formalism for the tachyon and pseudo-tachyon fields. Using the metric (101), one can see that the contribution of the tachyon field into the action is

\[ S = -\int \text{d}t N a^3 V(T) \sqrt{1 - \frac{T^2}{N^2}}. \] (116)

The conjugate momentum for \( T \) is

\[ p_T = -\frac{a^3 V T}{N \sqrt{1 - \frac{T^2}{N^2}}} \] (117)

and so the velocity can be expressed as

\[ \dot{T} = \frac{N p_T}{\sqrt{p_T^2 + a^6 V^2}}. \] (118)

The Hamiltonian of the tachyon field is now

\[ H = N \sqrt{p_T^2 + a^6 V^2}. \] (119)

Analogously, for the pseudo-tachyon field, we have

\[ p_T = -\frac{a^3 W T}{N \sqrt{\frac{p_T^2}{N^2} - 1}} \] (120)

\[ \dot{T} = \frac{N p_T}{\sqrt{p_T^2 - a^6 W^2}} \] (121)

and

\[ H = N \sqrt{p_T^2 - a^6 W^2}. \] (122)

In what follows, it will be convenient for us to fix the lapse function as \( N = 1 \).

Now, adding the gravitational part of the Hamiltonians and quantizing the corresponding observables, we obtain the following Wheeler–DeWitt equations for the tachyons:

\[ \left( \sqrt{p_T^2 + a^6 V^2} - \frac{a^2 p_T^2}{4} \right) \psi(a, T) = 0 \] (123)

and for the pseudo-tachyons

\[ \left( \sqrt{p_T^2 - a^6 W^2} - \frac{a^2 p_T^2}{4} \right) \psi(a, T) = 0. \] (124)

The study of the Wheeler–DeWitt equation for the universe filled with a tachyon or a pseudo-tachyon field is rather a difficult task because the Hamiltonian depends non-polynomially on the conjugate momentum of such fields. However, one can come to interesting conclusions, considering some particular models.

First of all, let us consider a model with the pseudo-tachyon field having a constant potential. In this case the Hamiltonian in equation (124) does not depend on the field \( T \). Thus, it is more convenient to use the representation of the quantum state of the universe where it depends on the coordinate \( a \) and the momentum \( p_T \). Then the Wheeler–DeWitt equation will have the following form:

\[ \left( \sqrt{p_T^2 - a^6 W^2} + \frac{a^2}{4} \frac{\partial^2}{\partial a^2} \right) \psi(a, p_T) = 0. \] (125)
It becomes algebraic in the variable \( p_T \). Now, we see that the Hamiltonian is well defined at \( p_T^2 \geq d^6W^2 \). Looking at the limiting value \( p_T^2 = d^6W^2 \) and comparing it with the relation (121), we see that it corresponds to \( \tilde{T}^2 \to \infty \), which, in turn, corresponds to the encounter with the Big Brake singularity as was explained in section 5. The only way to ‘neutralize’ the values of \( p_T \), which imply the negativity of the expression under the square root on the left-hand side of equation (125), is to require that the wavefunction of the universe is such that

\[
\psi(a, p_T) = 0 \quad \text{at} \quad p_T^2 \leq d^6W^2. \tag{126}
\]

The last condition could be considered as a hint on the quantum avoidance of the Big Brake singularity. However, as it was explained in section 6 on the example of the scalar field model, to speak about the probabilities in the neighborhood of the point where the wavefunction of the universe vanishes, it is necessary to realize the procedure of the reduction of the set of variables to a smaller set of physical degrees of freedom. Now, let us suppose that the gauge-fixing condition is chosen in such a way that the role of time is played by a Hubble parameter. In this case, the Faddeev–Popov determinant, equal to the Poisson bracket between \( \frac{\partial^2}{\partial T^2} - d^6W^2(T) \), which is positively defined, and the function \( -d^6W^2(T) \), which is negatively defined. The complete expression should not be negative, but what does it mean in our case? It means that we should choose such wavefunctions for which the quantum average of the operator \( \hat{p}_T^2 - d^6W^2(T) \) is non-negative:

\[
\langle \psi | \hat{p}_T^2 - d^6W^2(T) | \psi \rangle = \int DT \psi^*(a, T) \left( -\frac{\partial^2}{\partial T^2} - d^6W(T)^2 \right) \psi(a, T) \geq 0. \tag{127}
\]

Here the symbol \( DT \) signifies the integration on the tachyon field \( T \) with some measure. It is easy to guess that the requirement (127) does not imply the disappearance of the wavefunction \( \psi(a, T) \) at some range or some particular values of the tachyon field, and one can always construct a wavefunction which is different from zero everywhere and, thus, does not show the phenomenon of the quantum avoidance of singularity. However, the forms of the potential \( V(T) \) given by equation (26) and of the corresponding potential \( W(T) \) for the pseudo-tachyon field arising in the same model [49] are too cumbersome to construct such functions explicitly. Thus, to illustrate our statement, we shall consider a more simple toy model.

Let us consider the Hamiltonian

\[
\mathcal{H} = \sqrt{\hat{p}^2} - V_0a^2. \tag{128}
\]
where $\hat{p}$ is the conjugate momentum of the coordinate $x$ and $V_0$ is some positive constant. Let us choose as a wavefunction a Gaussian function

$$\psi(x) = \exp(-\alpha x^2),$$

where $\alpha$ is a positive number and we have omitted the normalization factor, which is not essential in the present context. Then condition (127) will look like

$$\int dx \exp(-\alpha x^2) \left( - \frac{d^2}{dx^2} - V_0 x^2 \right) \exp(-\alpha x^2) = \sqrt{\frac{\pi}{2}} \left( \frac{3}{4} \sqrt{\alpha} - \frac{V_0}{2\alpha^2} \right) \geq 0,$$

which can be easily satisfied if

$$\alpha \geq \sqrt{\frac{3}{2} V_0}.$$

Thus, we have seen that for this very simple model one can always choose such a quantum state, which does not disappear at any value of the coordinate $x$ and which guarantees the positivity of the quantum average of the operator, which is not generally positively defined. Coming back to our cosmological model we can say that the requirement of the well-definiteness of the pseudo-tachyon part of the Hamiltonian operator in the Wheeler–DeWitt equation does not imply the disappearance of the wavefunction of the universe at some values of the variables and thus, does not reveal the effect of the quantum avoidance of the cosmological singularity.

At the end of this section we would like also to analyze the Big Bang and Big Crunch singularities in the tachyon model with the trigonometrical potential. As was shown in [49], the Big Bang singularity can occur at two occasions (the same is true also for the Big Crunch singularity [40])—either $W(T) \to \infty$ (for example for $T \to 0$) or at $\dot{T}^2 = 1$, $W(T) \neq 0$. One can see from equations (15) and (120) that when the universe approaches these singularities the momentum $p_T$ tends to infinity. As was explained before, the wavefunction of the universe in the momentum representation should vanish at $|p_T| \to \infty$ and hence, we have the effect of the quantum avoidance.

Finally, summing up the content of the last three sections, devoted to the comparative study of the classical and quantum dynamics in some models with scalar fields and tachyons, revealing soft future singularities, we can make the following remarks.

It was shown that in the tachyon model with the trigonometrical potential [49] the wavefunction of the universe is not obliged to vanish in the range of the variables corresponding to the appearance of the classical Big Brake singularity. In a more simple pseudo-tachyon cosmological model the wavefunction, satisfying the Wheeler–DeWitt equation and depending on the cosmological radius and the pseudo-tachyon field, disappears at the Big Brake singularity. However, the transition to the wavefunction depending only on the reduced set of physical degrees of freedom implies the appearance of the Faddeev–Popov factor, which is singular and which singularity compensates the terms, responsible for the vanishing of the wavefunction of the universe. Thus, in both these cases, the effect of the quantum avoidance of the Big Brake singularity is absent.

In the case of the scalar field model with the potential inversely proportional to this field, all the classical trajectories pass through a soft singularity (which for one particular trajectory is exactly the Big Brake). The wavefunction of the universe disappears at the vanishing value of the scalar field which classically corresponds to the soft singularity. However, also in this case the Faddeev–Popov factor arising at the reduction to the physical degrees of freedom provides nonzero value of the probability of finding of the universe at the soft singularity.

In spite of the fact that we have considered some particular scalar field and tachyon–pseudo-tachyon models, our main conclusions were based on rather general properties of these models. Indeed, in the case of the scalar field we have used the fact that its potential
at the soft singularity should be negative and divergent, to provide an infinite positive value of the pressure. In the case of the pseudo-tachyon field, both the possible vanishing of the wavefunction of the universe and its ‘re-emergence’ in the process of reduction were connected with the general structure of the contribution of such a field into the super-Hamiltonian constraint (122). Note that in the case of the tachyon model with the trigonometric potential, the wavefunction does not disappear at all.

On the other hand, we have seen that for the Big Bang and Big Crunch singularities not only the wavefunctions of the universe but also the corresponding probabilities disappear when the universe is approaching the corresponding values of the fields under consideration, and this fact is also connected with rather general properties of the structure of the Lagrangians of the theories. Thus, in these cases the effect of quantum avoidance of singularities takes place.

One can say that there is some kind of a classical–quantum correspondence here. The soft singularities are traversable at the classical level (at least for simple homogeneous and isotropic Friedmann models) and the effect of quantum avoidance of singularities is absent. The strong Big Bang and Big Crunch singularities cannot be passed by the universe at the classical level, and the study of the Wheeler–DeWitt equation indicates the presence of the quantum singularity avoidance effect.

It would be interesting also to find examples of the absence of the effect of the quantum avoidance of singularities, for the singularities of the Big Bang–Big Crunch type. Note that the interest to the study of the possibility of crossing of such singularities is growing and some models treating this phenomenon have been elaborated during last few years [98].

11. Friedmann equations modified by quantum corrections and soft cosmological singularities

As we have already mentioned in the introduction there are two main directions in the study of quantum cosmology of soft future singularities. One is connected with the analysis of the structure of the Wheeler–DeWitt equation and another concentrates on the study of quantum corrections to the Friedmann equations. While in two preceding sections we were studying the Wheeler–DeWitt equation, here we shall dwell on the quantum corrections to the Friedmann equations and on the possible influence of these corrections on the structure of soft singularities. Our presentation will be mainly based on [17, 30].

In [17], was considered a cosmological evolution described by

\[
a(t) = \left( \frac{t}{t_s} \right)^{1/2} (a_s - 1) + 1 - \left( 1 - \frac{t}{t_s} \right)^n,
\]

where \( t_s \) is the time, where the sudden singularity occurs, \( a_s \) is the value of the scale factor in this moment and \( 1 < n < 2 \). The matter responsible for this evolution was not specified. It is easy to see that at the beginning of the evolution (132) the universe passes through the radiation-dominated phase of the expansion, while when \( t \to t_s \) it enters into the singular regime. Then it was supposed that a massive scalar field conformally coupled to gravity is present. The general solutions describing the behavior of this scalar field in these two regimes were written down and the requirement of the matching of these conditions at \( a = a_s \) was imposed. Then, the solution in the first regime is chosen as

\[
\phi_k(\eta) = \frac{a_k \eta}{\sqrt{2k}},
\]

where \( \eta \) is the conformal time parameter. The solution in the regime of approaching the soft singularity will be

\[
\phi_k(\eta) = \xi_{01} e^{i\omega \eta} + \xi_{02} e^{-i\omega \eta},
\]
where \( \tilde{\omega} = \sqrt{k^2 + m^2 H^2 a_t^4} \) (135)

and the constants \( \xi_{01} \) and \( \xi_{02} \) are connected with the Bogoliubov coefficients:

\[
\alpha = \sqrt{2\tilde{\omega}\xi_{01}}, \quad \beta = \sqrt{2\tilde{\omega}\xi_{02}}.
\] (136)

The matching conditions permit to find the Bogoliubov coefficients and the number of created particles for each mode

\[
N_k = \beta_k \beta_k^* = \frac{1}{4} \left(1 - \frac{k}{\tilde{\omega}}\right)^2.
\] (137)

The total energy of the created particles

\[
\rho = \int \rho_k \, d^3k = \pi \int k^2 \tilde{\omega} \left(1 - \frac{k}{\tilde{\omega}}\right)^2 \, dk
\] (138)

is divergent in the ultraviolet limit. The authors of [17] renormalize expression (138) using the \( n \)-wave method [99] and show that the renormalized energy is equal to zero. Thus, they conclude that the quantum phenomena associated with the cosmological dynamics do not change the character of the sudden singularity or prevent its occurrence. Some arguments in favor of the hypothesis that birth of particles of a field which is not conformally invariant cannot change the Friedmann equation are also developed in [17].

More detailed analysis of the quantum contributions into energy–momentum tensor and, hence, into the Friedmann equations was undertaken in [30]. Here it was noticed that the analysis, presented in [17], is applicable only to situations when the frequency of the field under consideration is varying smoothly. Obviously, it is not the case here, because two different phases of evolution are considered and a naive matching of the value of the field and of its time derivative at the moment of arrival to the singularity is required. Moreover, the effect of polarization of the vacuum was not taken into account. Instead, the authors of [30] use the known expressions for the renormalized energy-density and pressure for a massless conformally coupled scalar field [100, 101]:

\[
\rho_{\text{ren}} = \frac{1}{480\pi^2} \left(3H^2 + \frac{1}{2}H^2 - \frac{1}{2}H^2\right) + \frac{1}{960\pi^2}H^4,
\] (139)

\[
p_{\text{ren}} = -\frac{1}{1440\pi^2} \left(\frac{\dot{H}}{2} + 6H\dot{H} + \frac{9}{2}H^2\right) - \frac{1}{960\pi^2}H^4.
\] (140)

Then proceeding as in [102], the authors of [30] consider the Friedmann semiclassical equation

\[
H^2 = \rho + \rho_{\text{ren}},
\] (141)

looking for its solution with the form

\[
H(t) = H_s - C \left(1 - \frac{t}{t_s}\right)^{n'},
\] (142)

where \( H_s \), \( C \) and \( n' \) are unknown parameters. They find, in particular, that

\[
n' = n + 1.
\] (143)

Then, since \( 3 < n' < 4 \), it turns out that \( \dot{H} \) and \( \ddot{H} \) do not diverge at \( t = t_s \), which means that, for these kinds of singular solutions, the singularity becomes much milder due to the quantum corrections. In fact, in the absence of the quantum corrections, one can see from equation (132) that \( \dot{H} \) diverges.
12. Density matrix of the universe, quantum consistency and interplay between geometry and matter in quantum cosmology

In this section, we shall speak about the quantum density matrix of the universe [103–107]—an approach to quantum cosmology, which permits the consideration of mixed quantum states of the universe instead of pure ones. Such an approach is based on rather a delicate interplay between geometry and matter and implies the existence of essential restrictions on the basic parameters of the theory. In the framework of this approach as a by-product arise also some new kinds of soft sudden quantum singularities [106].

As is well known, quantum cosmology predicts the initial conditions for the cosmological evolution of the universe, defining its quantum state—the wavefunction of the universe. The connection between the Euclidean quantum theory and the quantum tunneling is used in both the main approaches to the construction of such a function—the no-boundary prescription [108] and the tunneling one [109, 110]. In [103, 104], this traditional scheme of quantum cosmology was generalized for the case of fundamental mixed initial quantum states of the universe, in other words instead of wavefunction of the universe one can consider the density matrix of the universe, possessing some thermodynamical characteristics. Such a mixed state of the universe arises naturally if an instanton with two turning points (surfaces of vanishing external curvature) does exist. (The idea that instead of pure quantum state of the universe one can consider a density matrix of the universe, was suggested already in [111].)

In turn, an instanton with two turning points arises naturally, if we consider a closed Friedmann universe where two essential ingredients are present: an effective cosmological constant and radiation which corresponds to the presence of the conformally invariant fields. The Euclidean Friedmann equation in this case is written as

$$\frac{\dot{a}^2}{a^2} = \frac{1}{a^2} - H^2 - \frac{C}{a^4},$$

where $H^2$ is an effective cosmological constant and the constant $C$ characterizes the quantity of the radiation in the universe. The turning points are

$$a_{\pm} = \frac{1}{\sqrt{2}H} \sqrt{1 \pm (1 - 4CH^2)^{1/2}}, \quad 4CH^2 \leq 1.\quad (145)$$

(The same instanton was also considered in [112], where the conception of the universe, which gave birth to itself, was suggested.) Figure 2 gives the picture of the instanton representing the density matrix of the universe. For the pure quantum state, [108] the instanton bridge between $\Sigma$ and $\Sigma'$ breaks down (see figure 3). However, the radiation stress tensor prevents these half-instantons from closure. The relevant density matrix is the path integral

$$\rho[\varphi, \varphi'] = e^\gamma \int_{\Sigma_0} \Delta[g, \varphi] \exp(-S_E[g, \varphi]).\quad (146)$$
with the partition function $e^{-\Gamma}$ which follows from integrating out the field $\psi$ in the coincidence $\psi' = \psi$ corresponding to the identification of $\Sigma'$ and $\Sigma$, the underlying instanton acquiring the toroidal topology.

The metric of the instanton introduced above is conformally equivalent to the metric of the Einstein static universe:

$$d\bar{s}^2 = d\eta^2 + d^2\Omega^{(3)},$$

where $\eta$ is the conformal time parameter. We shall consider conformally invariant fields. As is well known, the quantum effective action for such fields has a conformal anomaly first studied in cosmology in [113, 114]. It has the form

$$g_{\mu\nu} \frac{\delta\Gamma_{1-loop}}{\delta g_{\mu\nu}} = \frac{1}{4(4\pi)^2} \bar{s}^{1/2} (\alpha E + \beta \bar{E})$$

(148)

where $E = R^2_{\text{average}} - 4R^2_{\mu\nu} + R^2$ and $\Delta$ is the four-dimensional Laplacian. This anomaly, when integrated functionally along the orbit of the conformal group, gives the relation between the actions on conformally related backgrounds [115]:

$$\Gamma_{1-loop}[g] = \Gamma_{1-loop}[\bar{g}] + \delta\Gamma[g, \bar{g}]$$

(149)

$$g_{\mu\nu}(x) = e^{\sigma(x)} \bar{g}_{\mu\nu}(x),$$

(150)

where

$$\delta\Gamma[g, \bar{g}] = \frac{1}{2(4\pi)^2} \int d^4x \bar{s}^{1/2} \left[ \frac{1}{2} \left[ \gamma C^2_{\mu\nu\rho} + \beta \left( \bar{E} - \frac{2}{3} \bar{\Delta} \bar{R} \right) \right] \sigma 
+ \frac{\beta}{2} \left[ (\Delta \sigma)^2 + \frac{2}{3} \bar{R} (\bar{\nabla}_\mu \sigma)^2 \right] \right] - \frac{1}{2(4\pi)^2} \left( \frac{\alpha}{12} + \frac{\beta}{18} \right) \times \int d^4x (\bar{g}^{1/2} R^2(g) - \bar{g}^{1/2} R^2(\bar{g})).$$

(151)

One can show that the higher derivative in $\sigma$-terms are all proportional to the coefficient $\alpha$. The $\alpha$-term can be arbitrarily changed by adding a local counterterm $\sim g^{1/2} R^2$. We fix this local renormalization ambiguity by an additional criterion of the absence of ghosts. The conformal contribution to the renormalized action on the minisuperspace background equals

$$\delta\Gamma[g, \bar{g}] = \Gamma_{\alpha}[g] - \Gamma_{\alpha}[\bar{g}] = m^2 B \int d\tau \left( \frac{a^2}{a} - \frac{1}{6} \frac{a^4}{a} \right),$$

(152)

$$m^2 B = \frac{1}{3} \beta,$$

(153)

with the constant $m^2 B$ which for scalars, two-component spinors and vectors equals respectively 1/240, 11/480 and 31/120. For a conformal scalar field,

$$S[\bar{g}, \phi] = \frac{1}{2} \sum_\omega \int_0^\infty d\eta' \left( \left( \frac{d\phi_\omega}{d\eta'} \right)^2 + \omega^2 \phi_\omega^2 \right).$$

(154)
where $\omega = n$, $n = 0, 1, 2, \ldots$ labels a set of eigenmodes and eigenvalues of the Laplacian on a unit 3-sphere. Thus,

$$e^{-\Gamma_{1-\text{loop}}[\bar{g}]} = \int d\phi_0 \int_{\phi_0 (\eta) = \phi_0 (0) = \phi_0} D[\phi] \exp (-S[\bar{g}, \phi]) = \text{const} \prod_{\omega} \left( \sinh \frac{\omega \eta}{2} \right)^{-1},$$

and the effective action equals the sum of contributions of the vacuum energy $E_0$ and free energy $F(\eta)$ with the inverse temperature played by $\eta$—the circumference of the toroidal instanton in units of a conformal time,

$$\Gamma_{1-\text{loop}}[\bar{g}] = \sum_{\omega} \left[ \frac{\omega}{2} + \ln (1 - e^{-\omega \eta}) \right] = m_p^2 E_0 \eta + F(\eta),$$

$$m_p^2 E_0 = \sum_{\omega} \frac{\omega}{2} = \sum_{n=1}^{\infty} \frac{n^3}{2},$$

$$F(\eta) = \sum_{\omega} \ln (1 - e^{-\omega \eta})$$

$$= \sum_{n=1}^{\infty} n^2 \ln (1 - e^{-n \eta}).$$

Similar expressions hold for other conformally invariant fields of higher spins. In particular, the vacuum energy (an analogue of the Casimir energy) on Einstein static spacetime is

$$m_p^2 E_0 = \frac{1}{960} \times \begin{pmatrix} 4 \\ 17 \\ 88 \end{pmatrix}$$

respectively for scalar, spinor and vector fields.

We should take into account the effect of the finite ghost-avoidance renormalization denoted below by a subscript $R$, which results in the replacement of $E_0$ above by a new parameter $C_0$:

$$\Gamma_R[\bar{g}] = m_p^2 C_0 \eta_0 + F(\eta),$$

$$m_p^2 C_0 = m_p^2 E_0 + \frac{3}{16} \alpha.$$  

A direct observation indicates the following universality relation for all conformal fields of low spins

$$m_p^2 C_0 = \frac{1}{2} m_p^2 B.$$  

Now we can write down the effective Friedmann equation governing the Euclidean evolution of the universe. First of all, the full conformal time on the instanton is

$$\eta = 2 \int_{\tau_-}^{\tau_+} d\tau \frac{N(\tau)}{a(\tau)},$$

where $\tau_{\pm}$ label the turning points for $a(\tau)$ – its minimal and maximal values.

The effective action is ($m_p^2 \equiv 3/4\pi G$)

$$\Gamma[\alpha(\tau), N(\tau)] = 2 m_p^2 \int_{\tau_-}^{\tau_+} d\tau \left( -\frac{\dot{a}^2}{N} - Na + NH^2 \dot{a}^2 \right) + 2B m_p^2 \int_{\tau_-}^{\tau_+} d\tau \left( \frac{\dot{a}^2}{N a} - \frac{1}{6} \frac{\dot{a}^4}{N^3 a} \right)$$

$$+ F \left( 2 \int_{\tau_-}^{\tau_+} d\tau \frac{N}{a} \right) + B m_p^2 \int_{\tau_-}^{\tau_+} d\tau \frac{N}{a}.$$  

(165)
and the effective Friedmann equation reads
\[
\frac{\delta \Gamma}{\delta N} = 2m_p^2 \left( \frac{a\dot{a}^2}{N^2} - a + H^2 a^3 \right) + 2Bm_p^2 \left( -\frac{\dot{a}^2}{N^2 a} + \frac{1}{2} \frac{\dot{a}^4}{N^4 a} \right) + \frac{2}{a} \left( \frac{dF(\eta)}{d\eta} + \frac{B}{2} m_p^2 \right) = 0.
\]
(166)

In the gauge \( N = 1 \) this equation takes the form
\[
\frac{\dot{a}^2}{a^2} + B \left( \frac{1}{2} \frac{\dot{a}^4}{a^4} - \frac{a^2}{a^4} \right) = \frac{1}{a^2} - H^2 - \frac{C}{a^4},
\]
(167)

where the amount of radiation constant \( C \) is given by the bootstrap equation
\[
m_p^2 C = m_p^2 B \frac{B}{2} + \frac{dF(\eta)}{d\eta} = B \frac{B}{2} m_p^2 + \sum_\omega \omega e^{\omega m} - 1.
\]
(168)

The Friedmann equation can be rewritten as
\[
\dot{a}^2 = \sqrt{\frac{(a^2 - B)^2}{B^2} + \frac{2H^2}{B} (a_v^2 - a^2)(a^2 - a_v^2) - \frac{(a^2 - B)}{B}}
\]
(169)

and has the same two turning points \( a_{\pm} \) as in the classical case provided
\[
a_{\pm}^2 \geq B.
\]
(170)

This requirement is equivalent to
\[
C \geq B - B^2 H^2, \quad BH^2 \leq \frac{1}{2}.
\]
(171)

Together with
\[
CH^2 \leq \frac{1}{2},
\]

the admissible domain for instantons reduces to the curvilinear wedge below the hyperbola and above the straight line to the left of the critical point (see figure 4).
The suggested approach allows us to resolve the problem of the so-called infrared catastrophe for the no-boundary state of the Universe based on the Hartle–Hawking instanton. This problem is related to the fact that the Euclidean action on this instanton is negative and inverse proportional to the value of the effective cosmological constant. This means that the probability of the universe creation with an infinitely big size is infinitely high. We shall show now that the conformal anomaly effect allows one to avoid this counter-intuitive conclusion.

Indeed, outside of the admissible domain for the instantons with two turning points, obtained above, one can also construct instantons with one turning point which smoothly close at $a_− = 0$ with $\dot{a}(\tau_−) = 1$. Such instantons correspond to the Hartle–Hawking pure quantum state. However, in this case the on-shell effective action, which reads for the set of solutions obtained above as

$$
\Gamma_0 = F(\eta) - \eta \frac{dF(\eta)}{d\eta} + 4m_p^2 \int_{a_−}^{a_+} \frac{da\dot{a}}{a} \left( B - a^2 - \frac{B\dot{a}^2}{3} \right),
$$

(172)

diverges to plus infinity. Indeed, for $a_− = 0$ and $\dot{a}_− = 1$

$$
\eta = \int_0^{a_+} \frac{da}{\dot{a}} = \infty, \quad F(\infty) = F'(\infty) = 0,
$$

(173)

and hence the effective Euclidean action diverges at the lower limit to $+\infty$. Thus,

$$
\Gamma_0 = +\infty, \quad \exp(-\Gamma_0) = 0,
$$

and this fact completely rules out all pure-state instantons, and only mixed quantum states of the universe, described by the cosmological density matrix appear to be admissible.

In connection with all said above a natural question arises: Where Euclidean quantum gravity comes from? The answer can be formulated briefly as follows: from the Lorentzian quantum gravity (LQG) [105]. Namely, the density matrix of the Universe for the microcanonical ensemble in Lorentzian quantum cosmology of spatially closed universes describes an equipartition in the physical phase space of the theory, but in terms of the observable spacetime geometry this ensemble is peaked about a set of cosmological instantons (solutions of the Euclidean quantum cosmology) limited to a bounded range of the cosmological constant. These instantons obtained above as fundamental in Euclidean quantum gravity framework, in fact, turn out to be the saddle points of the LQG path integral, belonging to the imaginary axis in the complex plane of the Lorentzian signature lapse function [105].

Now let us consider the cosmological evolution of the universe starting from the initial conditions described above. Making the transition from the Euclidean time to the Lorentzian one, $\tau = it$, we can write the modified Lorentzian Friedmann equation as [106]

$$
\frac{\dot{a}^2}{a^2} + \frac{1}{a^2} = \frac{1}{B} \left[ 1 - \sqrt{1 - \frac{16\pi G}{3} B \varepsilon} \right],
$$

(174)

$$
\varepsilon = \frac{3}{8\pi G} \left( H^2 + \frac{C}{a^2} \right),
$$

(175)

$$
C \equiv C - \frac{B}{2},
$$

(176)

where $\varepsilon$ is a total gravitating matter density in the model (including at later stages also the contribution of particles created during inflationary expansion and thermalized at the inflation exit). A remarkable feature of this equation is that the Casimir energy is totally screened here and only the thermal radiation characterized by $C$ weighs.
If one wants to compare the evolution described by equation (176) with the real evolution of the universe, first of all it is necessary to have a realistic value for an effective cosmological constant $\Lambda = 3H^2$. The only way to achieve this goal is to increase the number of conformal fields and the corresponding parameter $B$, (153), of the conformal anomaly (148). The mechanisms for growing number of the conformal fields exist in some string inspired cosmological models with extra dimensions [105]. If some of these mechanisms work we can encounter an interesting phenomenon: if the $B$ grows with $a$ faster than the rate of decrease of the energy density $\varepsilon$, one encounters a new type of the cosmological singularity—Big Boost. This singularity is characterized by finite values of the cosmological radius $a_{BB}$ and of its time derivative $\dot{a}_{BB}$, while the second time variable $\ddot{a}$ has an infinite positive value. The universe reaches this singularity at some finite moment of cosmic time $t_{BB}$:

$$a(t_{BB}) = a_{BB} < \infty,$$

$$\dot{a}(t_{BB}) = \dot{a}_{BB} < \infty,$$

$$\lim_{t \to t_{BB}} \ddot{a}(t) = \infty.$$

In [107], it was found that there exist some correspondences between quantum four-dimensional equations of motion and some classical five-dimensional equations of motion [106, 107]. Two five-dimensional models were considered: the Randall–Sundrum model[116] and the generalized Dvali–Gabadadze–Porrati (DGP) model [117].

The Randall–Sundrum braneworld model is a four-dimensional spacetime braneworld embedded into the five-dimensional anti-de Sitter bulk with the radius $L$. In the limit of small energy densities, the modified quantum Friedmann equations coincide with the modified four-dimensional Friedmann equations of the Randall–Sundrum model, provided

$$\beta G = \frac{\pi L^2}{2}.$$  

The five-dimensional action of the generalized DGP model includes the five-dimensional curvature term, the five-dimensional cosmological constant and the four-dimensional curvature term on the brane.

If we require the spherical symmetry, then we have the Schwarzschild–de Sitter solution, which depends also on the Schwarzschild radius $R_S$. The effective four-dimensional Friedmann equations on the four-brane coincide with the modified Friedmann equations in the quantum model, provided the quantity of the radiation is expressed through the Schwarzschild radius as

$$C = R_S^2.$$  

If we add the condition of the regularity of the Schwarzschild–de Sitter instanton (i.e. the condition of the absence of conical singularities), we obtain an additional relation for the parameters of the quantum cosmological model and the set of admissible values for the effective cosmological constant becomes discrete.

Concluding this section, we would like to say that relaxing the usual tacit requirement of the purity of the quantum state of the universe and imposing the conditions of quantum consistency of the system of equations governing the dynamics of the universe, one comes to non-trivial restrictions on the basic cosmological parameters. Besides, as a by-product one obtains a particular kind of future soft singularity—Big Boost. Finally, we can note that in the papers reviewed in this section both the main approaches to the study of quantum effects in cosmology were combined—the study of the modified Friedmann equations and the investigation of the structure of the quantum state of the universe. Usually, these two approaches are separated (see, section 11 and sections 9 and 10 of this review).
13. Quiescent singularities in braneworld models

One of the first examples of the soft future singularities in cosmology was presented in [10], where some braneworld cosmological models were considered. The higher dimensional models considered there were described by an action, where both the bulk and brane contained the corresponding curvature terms:

\[
S = M^3 \sum_i \int_{\text{bulk}} (R - 2\Lambda_i) - 2 \int_{\text{brane}} K + \int_{\text{brane}} (m^2 R - 2\sigma) + \int_{\text{brane}} L(h_{\alpha\beta}, \phi),
\]

(182)

where the sum is taken over the bulk components bounded by branes, and \(\Lambda_i\) is the cosmological constant on the \(i\)th bulk component. The Lagrangian \(L(h_{\alpha\beta}, \phi)\) corresponds to the presence of matter fields on the brane interacting with the induced metric \(h_{\alpha\beta}\); \(K\) is the trace of the extrinsic curvature. The Friedmann-type equation has the form

\[
H^2 + \frac{\kappa}{a^2} = \rho + \sigma + \frac{2}{l^2} \left[ 1 \pm \sqrt{1 + \frac{l^2}{\rho^3} \left( \frac{\rho + \sigma}{3m^2} - \frac{\Lambda}{6} - \frac{C}{a^4} \right)} \right],
\]

(183)

where \(\rho\) is the energy density of the matter on the brane, the integration constant \(C\) corresponds to the presence of a black hole in the five-dimensional bulk solution and the term \(C/a^4\), sometimes called ‘dark radiation’, arises due to the projection of the bulk gravitational degrees of freedom onto the brane. The length scale \(l\) is defined as

\[
l = \frac{m^2}{M^3}.
\]

(184)

The appearance of the quiescent singularities is connected with the fact that the expression under the square root in (183) turns to zero at some point during the evolution. There are essentially two types of singularities displaying this behavior.

A type 1 singularity (S1) is induced by the presence of the dark radiation term and arises in either of the following two cases: \(C > 0\) and the density of matter increases slower than \(a^{-4}\) as \(a \to 0\). An example is provided by dust.

The energy density of the universe is radiation dominated so that \(\rho = \rho_0/a^4\) and \(C > \rho_0\).

These singularities can take place either in the past of an expanding universe or in the future of a collapsing one.

A type 2 singularity (S2) arises if

\[
l^2 \left( \frac{\sigma}{3m^2} - \frac{\Lambda}{6} \right) < -1.
\]

(185)

In this case, the combination \(\rho/3m^2 - C/a^4\) decreases monotonically as the universe expands. The expression under the square root of (183) can therefore become zero at suitably late times.

For both S1 and S2, the scale factor \(a(t)\) and its first time derivative remain finite, while all the higher time derivatives of \(a\) tend to infinity as the singularity is approached. It is important that the energy density and the pressure of the matter in the bulk remain finite. This feature distinguishes these singularities from the singularities considered in the preceding sections, and justifies the special name ‘quiescent’ [10]. The point is that the existence of these singularities is connected not with special features of the matter on the brane, but with the particularity of the embedding of the brane into the bulk.

In [11], the question of influence of the quantum effects on a braneworld encountering a quiescent singularity during expansion was studied. The matter considered in [11] was constituted from conformally invariant fields. Hence, the particle production was absent and the only quantum effect was connected with the vacuum polarization. It was shown that this
effect boils down to the modification of the effective energy density of the matter on the brane. Namely, the quantum correction to this energy density is given by

\[ \rho_{\text{quantum}} = k_2 H^2 + k_3 (2H \dot{H} + 6H \ddot{H}^2 - \dot{H}^2). \]  

(186)

The insertion of this correction to the energy density changes drastically the form of the brane Friedmann-type equation (183)—the original algebraic equation becomes a differential equation. It implies essential changes in the possible behavior of the universe around singularities. First, the quiescent singularity changes its form and becomes much weaker, in fact, \( H \) and \( \dot{H} \) remain finite and only \( \cdots H \to \infty \). Second, vacuum polarization effects can also cause a spatially flat universe to turn around and collapse.

At the conclusion of this section we would like to mention another type of cosmological singularities, arising in the braneworld context. These are the so-called pressure singularities [118, 119]. These singularities arise in the generalized Friedmann branes, which can be asymmetrically embedded into the bulk and can include pull-backs on the brane, some non-standard field and geometric configurations, existing in the five-dimensional bulk [120]. It appears that it is possible to reproduce in this framework a Swiss cheese Einstein–Strauss model [121]. In this model, pieces of the Schwarzschild regions are inserted into Friedmann universes. At some conditions in the Friedmann regions of such branes the pressure of matter becomes infinite, while the cosmological radius and all its time derivatives remain finite. It was also shown [119] that at some critical value of the asymmetry in the embedding of the brane into the bulk, these singularities appear necessarily. It is interesting that these pressure singularities are in a way complementary to the quiescent singularities, discussed above, where the energy density and the pressure are always finite, while the time derivatives of the scale factor become divergent, beginning since the second or some higher order derivative.

### 14. Concluding remarks

In this review, we have considered a broad class of phenomena arising in cosmological models, possessing some exotic cosmological singularities, which differ from the traditional Big Bang and Big Crunch singularities. We have discussed the models, based on standard scalar fields, Born–Infeld-type fields and on perfect fluids, where soft future cosmological singularities exist and are traversable. The crossing of such singularities (or other geometrically peculiar surfaces in the spacetime) can imply such an interesting phenomenon as a transformation of matter properties, which is discussed in some detail here. Another interesting aspect of the study of both soft and ‘hard’ (Big Bang or Big Crunch) cosmological singularities is the existence of the correspondence between the phenomenon of quantum avoidance (or non-avoidance) of such singularities and the possibility of their crossing (or the absence of such a possibility) in classical cosmology.

Besides, the quantum cosmological approach, based on the study of the properties of solutions of the Wheeler–DeWitt equation, we have reviewed also some works based on the investigation of the modification of the Friedmann equation due to the quantum corrections and the influence of these corrections on the structure and the very existence of soft cosmological singularities.

While the main part of this review deals with the standard Einstein general relativity in the presence of non-standard matter, the last section is devoted to the exotic singularities arising in the braneworld cosmological models, which are very close in their nature to the soft sudden singularities arising in the general relativity.

Generally, we are convinced that the study of exotic singularities in classical and quantum cosmology is a promising branch of the theoretical physics, and nobody can exclude that it
can acquire some phenomenological value as well. Here, it is necessary to recognize that almost all studies in this field deal only with isotropic and homogeneous Friedmann universes. Thus, the extension of these studies to the anisotropic and inhomogeneous models represents a main challenge for people working in this field. Such an extension can bring some interesting surprises as it was with the study of the Big Bang–Big Crunch singularities, where the consideraton of the anisotropic Bianchi models, instead of Friedmann models, has given birth to the discovery of the oscillating approach to the singularity (Mixmaster Universe) [6, 7].

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