Existence and uniqueness of the Boussinesq equations for MHD convection

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Abstract. This paper is concerned with the Boussinesq-MHD system with constant viscosity, thermal diffusivity, and electrical conductivity. The existence of this Boussinesq-MHD system was estimated by Littlewood-Paley decomposition, Bony’s para product and commutator estimates. Meanwhile, the uniqueness was estimated by Grönwall inequality. In this paper, the space is the optimal Sobolev spaces for the Boussinesq-MHD system.

1. Introduction

The Boussinesq equation for magnetohydrodynamic convection (Boussinesq-MHD) is a dynamo model of turbulent plasma flows as stated in [6, 9, 14] and the references therein. In this paper, we consider the Cauchy problem of the following Boussinesq-MHD model in $\mathbb{R}^n$,

$$\begin{cases}
  u_t + u \cdot \nabla u - b \cdot \nabla b + \nabla p - \nu \Delta u = \theta e_n, \\
  b_t + u \cdot \nabla b - b \cdot \nabla u - \mu \Delta b = 0, \\
  \theta_t + u \cdot \nabla \theta - \kappa \Delta \theta = 0, \\
  \nabla \cdot u = 0, \quad \nabla \cdot b = 0,
\end{cases} \tag{1}$$

with the initial conditions

$$u(x, 0) = u_0(x), \quad b(x, 0) = b_0(x), \quad \theta(x, 0) = \theta_0(x), \quad \nabla \cdot u_0 = \nabla \cdot b_0 = 0. \tag{2}$$

Here, $u$ denotes fluid velocity, $b$ is the magnetic field, $p$ is the scalar pressure, $\theta$ is the scalar temperature in the content of thermal convection (or density in the modeling of geophysical fluids), the constant $\nu > 0$ is the viscosity, $\kappa > 0$ is the thermal diffusivity, $\mu$ is the electrical resistivity, and $e_n = (0, 0, \cdots, 0, 1)$ is the unit vector in the $x_n$-direction, representing the unit vector in the direction of gravity.

In mathematics, the system (1) is a combination of the incompressible Boussinesq equations of fluid dynamics and Maxwell’s equations of electromagnetism, where the displacement current can be neglected. If the fluid is not affected by the temperature (i.e., $\theta \equiv 0$), then the equations (1) reduces to the MHD equations, which govern the dynamics of the velocity and the magnetic field in electrically conducting fluids such as plasmas and reflect the basic physics conservation laws. On the other hand, if the fluid is not affected by the Lorentz force (i.e., $B \equiv 0$), then the system (1) becomes the classical Boussinesq system.
For the full Boussinesq-MHD system (1), much work has been concentrated on the 2D case. For the 2D Boussinesq-MHD system, Bian et al. established the global well-posedness of weak or strong solutions of the initial boundary value problems under various boundary conditions and rigorously justified the stability and instability in a fully nonlinear, dynamical setting from a mathematical point of view as stated in the references [2, 3, 1, 15]. For the 3D case, Larios-Pei [8] proved the local well-posedness in $H^5$ space, which is not optimal. Recently, Bian-Pu [4] and Liu-Bian-Pu [10] obtained the global well-posedness for the axisymmetric, the Boussinesq-MHD system and the Boussinesq-MHD equations with a nonlinear damping term. Based on the previous works, our interest is to prove that the solutions of the Boussinesq-MHD system are unique and exists in optimal Sobolev spaces and Liu-Bian-Pu [10] obtained the global well-posedness for the axisymmetric, the Boussinesq-MHD system and the Boussinesq-MHD equations with a nonlinear damping term. Based on the previous works, our interest is to prove that the solutions of the Boussinesq-MHD system are unique and exists in optimal Sobolev spaces $H^s(\mathbb{R}^n) \times H^{s+1-\epsilon}(\mathbb{R}^n) \times H^{s+1-\epsilon}(\mathbb{R}^n)$ with $s > \frac{n}{2} - 1$ and any small enough $\epsilon > 0$ such that $s + 1 - \epsilon > \frac{n}{2}$. On this topic, it was first shown in [5, 12, 13] for Hall-MHD system very recently.

In the following, we simplify the presentation by denoting $\mathcal{X}_{s,r,l} = H^s(\mathbb{R}^n) \times H^r(\mathbb{R}^n) \times H^l(\mathbb{R}^n)$ for any real numbers $r, s$ and $l$. Our main result is the following

**Theorem 1.1.** Let $(u_0, b_0, \theta_0) \in \mathcal{X}_{s,s+1-\epsilon,s+1-\epsilon}$ with $s > \frac{n}{2} - 1$ and any small enough $\epsilon > 0$ such that $s + 1 - \epsilon > \frac{n}{2}$ and assume $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. There exists a time $T = T(\nu, \mu, \kappa, \|u_0\|_{H^s}, \|b_0\|_{H^{s+1-\epsilon}}, \|\theta_0\|_{H^{s+1-\epsilon}}) > 0$ and a unique solution $(u, b, \theta)$ of (1) on $[0, T)$ such that $(u, b, \theta) \in C([0, T); \mathcal{X}_{s,s+1-\epsilon,s+1-\epsilon})$.

The following estimates use much of the Littlewood-Paley decomposition, Bony’s paraproduct calculus and commutator estimates, for which the interested readers may refer to [7, 11]. In the following, we always use $\| \cdot \|_p$ to denote $\| \cdot \|_{L^p}$.

2. Existence

In this section, we establish a priori estimates for smooth solutions in $\mathcal{X}_{s,r,l} := H^s(\mathbb{R}^n) \times H^r(\mathbb{R}^n) \times H^l(\mathbb{R}^n)$ with appropriate indices $s, r$ and $l$. We only present the following theorem and its proof.

**Theorem 2.1.** Let $(u_0, b_0, \theta_0) \in \mathcal{X}_{s,r,l}$ with $\frac{n}{2} - 1 < s \leq l$ and $\frac{n}{2} < r \leq l \leq s + 1 - \epsilon$ for small enough $\epsilon > 0$. There exists a time $T = T(\nu, \mu, \kappa, \|u_0\|_{H^s}, \|b_0\|_{H^r}, \|\theta_0\|_{H^l}) > 0$ such that the Boussinesq-MHD system (1) has a solution $(u, b, \theta)$ satisfying

$$(u, b, \theta) \in L^\infty(0, T; \mathcal{X}_{s,r,l}) \cap L^2(0, T; \mathcal{X}_{s+1,r+1,l+1}).$$

The proof involves certain amount of computations and estimates which will be divided into several lemmas. To start, applying the operator $\Delta_q$ to the equations (1), then multiplying the first equation of (1) by $\lambda_q^{2s} u_q$ and the second one by $\lambda_q^{2r} b_q$, and the third one by $\lambda_q^{2l} \theta_q$, and adding up for all $q \geq -1$, we obtain

$$\frac{1}{2} \frac{d}{dt} \sum_{q \geq -1} \lambda_q^{2s} \|u_q\|_2^2 + \nu \sum_{q \geq -1} \lambda_q^{2s+2} \|u_q\|_2^2 \leq -I_1 + I_2 + I_3,$$

$$\frac{1}{2} \frac{d}{dt} \sum_{q \geq -1} \lambda_q^{2r} \|b_q\|_2^2 + \mu \sum_{q \geq -1} \lambda_q^{2r+2} \|b_q\|_2^2 \leq I_4 - I_5,$$

$$\frac{1}{2} \frac{d}{dt} \sum_{q \geq -1} \lambda_q^{2l} \|\theta_q\|_2^2 + \kappa \sum_{q \geq -1} \lambda_q^{2l+2} \|\theta_q\|_2^2 \leq -I_6,$$
The estimates for $I_1, I_2, I_4$ and $I_5$ can be similarly conducted as in [12] in the following lemmas.

**Lemma 2.1.** Let $s > \frac{n}{2} - 1$ and $(u, b, \theta)$ be a smooth solution, then for some absolute constants $\gamma_1, \gamma_2 > 0$, it holds that

$$|I_1| \leq \frac{\nu}{8} \sum_{q \geq -1} \lambda^2 q^s + 2 \nu \| u_q^2 + C_r \| u \|_{H^s}^{2 + \gamma_1} + C_r \| u \|_{H^s}^{2 + \gamma_2}.$$

**Lemma 2.2.** Let $\frac{n}{2} + s - 2r \leq 0$ and $s < r$. The following estimate holds

$$|I_2| \leq \frac{\nu}{8} \sum_{q \geq -1} \lambda^2 q^s + 2 \nu \| u_q^2 + C_r \| b \|_{H^r}^4.$$

**Lemma 2.3.** Let the index $r$ and $s$ satisfy conditions in Lemma 2.5. In addition, assume $r \leq s + 1 - \epsilon$ for a small enough constant $\epsilon > 0$. We have

$$|I_4| \leq \frac{\nu}{32} \sum_{q \geq -1} \lambda^2 q^s + 2 \nu \sum_{q \geq -1} \lambda^2 q^{s+2} + C_r \| u \|_{H^s}^{2 + \gamma_3} + C_r \| u \|_{H^s}^{2 + \gamma_5} + C_r \| b \|_{H^r}^{2 + \gamma_5},$$

for various constants $C_r \| u \|_{H^s}$ depending on $\nu, \mu$, and some constants $\gamma_3, \gamma_4, \gamma_5 > 0$.

**Lemma 2.4.** Let $s > \frac{n}{2} - 1$ and $\frac{n}{2} + s < r < s + 2 - \epsilon$ with small enough $\epsilon > 0$. We have the estimate

$$|I_5| \leq \frac{\nu}{8} \sum_{q \geq -1} \lambda^2 q^s + 2 \nu \sum_{q \geq -1} \lambda^2 q^{s+2} \| b_q^2 + C_r \| u \|_{H^s}^{2 + \gamma_6} + C_r \| u \|_{H^s}^{2 + \gamma_7}$$

for some constants $\gamma_6, \gamma_7 > 0$.

Now we give the estimate about $I_3$.

**Lemma 2.5.** Let $\frac{n}{2} - 1 < s \leq l$. The following estimate holds

$$|I_3| \leq C \| u \|_{H^s}^2 + C \| \theta \|_{H^r}^2.$$

**Proof.** By Hölder’s inequality and Young’s Inequality, we obtain

$$|I_3| = \sum_{q \geq -1} \lambda^2 q^s \int_{\mathbb{R}^3} \Delta_q \theta e_n u_q \, dx \leq \sum_{q \geq -1} \lambda^2 q^s \int \theta u_q \, dx \leq \sum_{q \geq -1} \lambda^2 q^s \| \theta_q \|_{2} \| u_q \|_{2}^2$$

$$\leq \sum_{q \geq -1} \lambda^2 q^s \left( \frac{\| \theta_q \|_{2}^2}{2} + \| u_q \|_{2}^2 \right) \leq \frac{1}{2} \sum_{q \geq -1} \lambda^2 q^s \| u_q \|_{2}^2 + \frac{1}{2} \sum_{q \geq -1} \lambda^2 q^s \| \theta_q \|_{2}^2$$

$$\leq \frac{1}{2} \| u \|_{H^s}^2 + C \| \theta \|_{H^r}^2.$$
Next, we establish the estimate about $I_6$.

**Lemma 2.6.** Let $s > \frac{n}{2} - 1$ and $\frac{n}{4} + \frac{s}{2} < r < s + 2 - \epsilon$ with small enough $\epsilon > 0$. We have the estimate

$$|I_6| \leq \frac{\nu}{8} \sum_{q \geq -1} \lambda_q^{2s+2} \|u_q\|_2 + \frac{K}{8} \sum_{q \geq -1} \lambda_q^{2s+2} \|\theta_q\|_2^2 + C_{\nu,\mu} \|u\|_{H^s}^{2+\gamma_8} + C_{\nu,\mu} \|\theta\|_{H^s}^{2+\gamma_9}$$

for some constants $\gamma_8, \gamma_9 > 0$.

**Proof.** We first decompose $I_6$ by Bony’s paraproduct

$$I_6 = \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2l} \int_{\mathbb{R}^3} \Delta_q(u \cdot \nabla \theta) \theta_q dx$$

$$= \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2l} \int_{\mathbb{R}^3} \Delta_q(u_{p-2} \cdot \nabla \theta_{q-p}) \theta_q dx + \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2l} \int_{\mathbb{R}^3} \Delta_q(u_{p} \cdot \nabla \theta_{q-p-2}) \theta_q dx$$

$$+ \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2l} \int_{\mathbb{R}^3} \Delta_q(u_{p} \cdot \nabla \theta_{q-p}) \theta_q dx = I_{61} + I_{62} + I_{63}$$

with

$$I_{61} = \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2l} \int_{\mathbb{R}^3} [\Delta_q, u_{p-2} \cdot \nabla] \theta_{q-p} \theta_q dx + \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2l} \int_{\mathbb{R}^3} u_{p-2} \cdot \nabla \Delta_{q-p} \theta_{q-p} \theta_q dx = I_{611} + I_{612}.$$ 

By the commutator estimate, we infer

$$|I_{611}| \lesssim \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2l} \|\nabla u_{p-2}\|_\infty \|\theta_{q-p}\|_2 \|\theta_q\|_2 \lesssim \sum_{q \geq -1} \lambda_q^{2l} \|\theta_q\|_2^2 \sum_{p \leq q} \lambda_p^{\frac{n}{2}+1} \|u_p\|_2$$

$$\lesssim \sum_{q \geq -1} \lambda_q^{2l} \|\theta_q\|_2^2 \lambda_p^{2(2-s)} \|\theta_q\|_2^{-2s} \lambda_p^{s+1-\delta} \|u_p\|_2 \lambda_p^{1-\delta} \left(\lambda_q^{\nu} \lambda_p^{\frac{n}{2}+1-s-\delta}\right)$$

$$\lesssim \sum_{q \geq -1} \lambda_q^{2l} \|\theta_q\|_2^2 \lambda_p^{2(2-s)} \|\theta_q\|_2^{-2s} \lambda_p^{s+1-\delta} \|u_p\|_2 \lambda_p^{1-\delta} \lambda_p^{q-2\vartheta}$$

for parameters $\vartheta$ and $\delta$ satisfying $0 < \vartheta < 2, 0 < \delta < 1$ and $s > \frac{n}{4} + 1 - \vartheta - \delta$.

From Young’s inequality, for some constants $\gamma_{10}, \gamma_{11} > 0$, it follows that

$$|I_{611}| \lesssim \frac{\nu}{32} \sum_{q \geq -1} \lambda_q^{2s+2} \|u_q\|_2^2 + \frac{K}{32} \sum_{q \geq -1} \lambda_q^{2s+2} \|\theta_q\|_2^2 + C_{\nu,\mu} \left(\sum_{q \geq -1} \lambda_q^{2s} \|u_q\|_2^2\right)^{1+\gamma_{10}} + C_{\nu,\mu} \left(\sum_{q \geq -1} \lambda_q^{2s} \|\theta_q\|_2^2\right)^{1+\gamma_{11}}.$$ 

For the term $I_{612}$, noting that

$$I_{612} = \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2l} \int_{\mathbb{R}^3} u_{q-2} \cdot \nabla \Delta_q \theta_{p} \theta_q dx + C_{\nu,\mu} \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2l} \int_{\mathbb{R}^3} (u_{p-2} - u_{q-2}) \cdot \nabla \Delta_q \theta_{p} \theta_q dx,$$

by the facts that $\sum_{q-2 \leq p \leq q+2} \Delta_q \theta_p = \theta_p$ and $\nabla \cdot u_{q-2} = 0$, one can prove that $|I_{612}| \lesssim |I_{611}|$. 


Following similar strategy as \( I_{611} \), for \( 0 < \theta < 1 \) and \( s \geq \frac{n}{2} - \theta \), we can estimate \( I_{62} \) as follows,

\[
|I_{62}| \leq \sum_{q \geq 1} \sum_{|q-p| \leq 2} \lambda_q^q \| u_{q,p} \|_2 \| \nabla \theta_{p-2} \|_\infty \| \theta_q \|_2 \lesssim \sum_{q \geq -1} \lambda_q^{q} \| u_q \|_2 \| \theta_q \|_2 \sum_{p \leq q} \lambda_p^{q+1} \| \theta_p \|_2
\]

\[
\lesssim \sum_{q \geq -1} \lambda_q^{q+1} \| u_q \|_2 \lambda_q^{(r+1)q} \| b_q \|_2 \lambda_q^{r(1-\theta)} \| b_q \|_2 \| \theta_q \|_2 \| \theta_p \|_2 \lambda_{q-p}^{q-1-\theta} \| \lambda_{q-p}^{q-1-\theta} \|
\]

Similarly, from Young’s inequality and Jensen’s inequality, with triplet \( \frac{q}{2}, \frac{q}{2}, \frac{2}{1-\theta} \) satisfying \( l - s - 1 - \theta < 0 \) such that for some constant \( \gamma_{12} > 0 \),

\[
|I_{62}| \lesssim \frac{\nu}{32} \sum_{q \geq -1} \lambda_{q}^{2q+2} \| u_q \|_2^2 + \frac{\mu}{32} \sum_{q \geq -1} \lambda_{q}^{2q+2} \| \theta_q \|_2^2 + C_{\nu,\mu,\kappa} \| \theta \|_{H^l}^{2+\gamma_{12}}.
\]

Finally, we obtain for some constant \( \gamma_{13} > 0 \) that

\[
|I_{63}| \lesssim \frac{\nu}{32} \sum_{q \geq -1} \lambda_{q}^{2q+2} \| u_q \|_2^2 + \frac{\kappa}{32} \sum_{q \geq -1} \lambda_{q}^{2q+2} \| \theta_q \|_2^2 + C_{\nu,\mu,\kappa} \left( \sum_{q \geq -1} \lambda_{q}^{2q} \| \theta_q \|_2^2 \right)^{1+\frac{\gamma_{13}}{2}}.
\]

Combining the above estimates, we can get the following Lemma which concludes the proof of Theorem 2.1.

**Lemma 2.7.** Let \( s > \frac{n}{2} - 1 \), \( r > \frac{n}{2} \), \( l > \frac{n}{2} \) and \( \frac{n}{2} + \frac{2}{\theta} < l \leq r \leq s + 1 - \epsilon \) for a small enough constant \( \epsilon > 0 \). There exist \( T = T(\nu, \mu, \kappa, \| u_0 \|_{H^r}, \| b_0 \|_{H^r}, \| \theta_0 \|_{H^l}) \) and \( C_{\nu,\mu,\kappa} \) depending on \( \nu, \mu \) and \( \kappa \) such that

\[
\| u(t) \|_{H^r}^2 + \| b(t) \|_{H^r}^2 + \| \theta(t) \|_{H^l}^2 \leq C_{\nu,\mu,\kappa} \left( \| u_0 \|_{H^r}^2 + \| b_0 \|_{H^r}^2 + \| \theta_0 \|_{H^l}^2 \right), \quad \forall t \in [0, T].
\]

**Proof.** Combining the above estimates, there exists constant \( C_{\nu,\mu,\kappa} \) depending on \( \nu, \mu \) and \( \kappa \) such that

\[
\frac{d}{dt} \left( \| u \|_{H^r}^2 + \| b \|_{H^r}^2 + \| \theta \|_{H^l}^2 \right) + \nu \| \nabla u \|_{H^r}^2 + \mu \| \nabla b \|_{H^r}^2 + \kappa \| \nabla \theta \|_{H^l}^2
\]

\[
\leq C_{\nu,\mu,\kappa} \left( \| u \|_{H^r}^2 + \| b \|_{H^r}^2 + \| \theta \|_{H^l}^2 \right)^{1+\gamma} + C_{\nu,\mu,\kappa} \left( \| u \|_{H^r}^2 + \| b \|_{H^r}^2 + \| \theta \|_{H^l}^2 \right)^{1+\gamma},
\]

with \( \gamma = \min\{\gamma_1, \ldots, \gamma_{13}\} \) and \( \gamma = \max\{\gamma_1, \ldots, \gamma_{13}\} \).

Denote \( \psi(t) = \| u(t) \|_{H^r}^2 + \| b(t) \|_{H^r}^2 + \| \theta(t) \|_{H^l}^2 \). Let

\[
T = \frac{1}{2} \min \left\{ \frac{1}{C_{\nu,\mu,\kappa} \| \psi \|_2(0)}, \frac{1}{C_{\nu,\mu,\kappa} \| \psi \|_\infty^\gamma(0)} \right\}.
\]

It follows from the above energy inequality that for \( t \in [0, T] \),

\[
\psi'(t) \leq C_{\nu,\mu,\kappa} \psi(t)^{1+\frac{\gamma}{2}} + C_{\nu,\mu,\kappa} \psi(t)^{1+\gamma} = C_{\nu,\mu,\kappa} \left( \psi(t) + (\psi(t)^\gamma + \psi(t)^\gamma) \right)
\]

which follows by Grönewall inequality that

\[
\| u(t) \|_{H^r}^2 + \| b(t) \|_{H^r}^2 + \| \theta(t) \|_{H^l}^2 \leq C_{\nu,\mu,\kappa} \left( \| u_0 \|_{H^r}^2 + \| b_0 \|_{H^r}^2 + \| \theta_0 \|_{H^l}^2 \right).
\]

This completes the proof of the lemma and therefore implies the result in Theorem 2.1. \( \Box \)
3. Uniqueness

In this section, we establish the uniqueness of solutions stated in Theorem 1.1. We use Grönwall inequality to estimate the uniqueness of this Boussinesq-MHD system.

**Theorem 3.1.** Let $\epsilon > 0$ be small enough. Assume $(u_1, b_1, \theta_1, p_1)$ and $(u_2, b_2, \theta_2, p_2)$ are solutions of (1)-(2) in $C([0, T]; X_{s+1-\epsilon,s+1-\epsilon})$ satisfying the estimates in Theorem 2.1. Then $(u_1, b_1, \theta_1) = (u_2, b_2, \theta_2)$ for all $t \in [0, T]$.

**Proof.** The difference $(U, B, \theta, \pi) = (u_1 - u_2, b_1 - b_2, \theta_1 - \theta_2, p_1 - p_2)$ satisfies

$$
U_t + u_2 \cdot \nabla U - b_2 \cdot \nabla B + U \cdot \nabla u_1 - B \cdot \nabla b_1 + \nabla \pi = \nu \Delta U + \theta e_n,\\
B_t + u_2 \cdot \nabla B - b_2 \cdot \nabla U + U \cdot \nabla b_1 - B \cdot \nabla u_1 = \mu \Delta B,\\
\theta_t + u_2 \cdot \nabla \theta + U \cdot \nabla \theta_1 = \kappa \Delta \theta.
$$

Taking inner product of the system (7) with $(U, B, \theta)$ gives that

$$
\frac{d}{dt} \left( \|U\|^2_t + \|B\|^2_t + \|\theta\|^2_t \right) + \nu \|\nabla U\|^2_t + \mu \|\nabla B\|^2_t + \kappa \|\nabla \theta\|^2_t = \int \nabla U \cdot U dx + \int \nabla B \cdot B dx - \int \nabla U \cdot U dx - \int \nabla B \cdot B dx
$$

Using integration by parts, the terms $K_3, K_7, K_9$ and $K_1 + K_5$ vanish. For the other terms, we have

$$
|K_2| = \left| \int (B \cdot \nabla) U \cdot b_1 dx \right| \leq \frac{\nu}{8} \|\nabla U\|^2_t + C_\nu \|B\|^2_t |b_1|^2_{H^{s+1-\epsilon}},
$$

where we used the embedding $H^{s+1-\epsilon} \subset L^\infty$ for $s+1-\epsilon > \frac{n}{2}$, since we can choose $\epsilon = \frac{1}{2} [s - (\frac{n}{2} - 1)]$ and $s > \frac{n}{2} - 1$. Analogous computation shows that

$$
|K_{11}| \leq \|\theta\|^2_t + \|U\|^2_t,\\
|K_4| + |K_6| \leq \frac{\nu}{8} \|\nabla U\|^2_t + \frac{\mu}{8} \|\nabla B\|^2_t + C_{\mu,\nu} \|U, B\|^2_t |u_1|^2_{H^{s+1-\epsilon}},\\
|K_8| + |K_{10}| \leq \frac{\mu}{8} \|\nabla B\|^2_t + \frac{\kappa}{8} \|\nabla \theta\|^2_t + C_{\mu,\kappa} \|U\|^2_t |b_1, \theta_1|^2_{H^{s+1-\epsilon}}.
$$

These estimates along with (8) give that

$$
\frac{d}{dt} \left( \|U\|^2_t + \|B\|^2_t + \|\theta\|^2_t \right) + \nu \|\nabla U\|^2_t + \mu \|\nabla B\|^2_t + \kappa \|\nabla \theta\|^2_t \leq C_{\nu,\mu,\kappa} \left( \|u_1|^2_{H^{s+1-\epsilon}} + |b_1|^2_{H^{s+1-\epsilon}} + |\theta_1|^2_{H^{s+1-\epsilon}} \right) \left( \|U\|^2_t + \|B\|^2_t + \|\theta\|^2_t \right),
$$

from which it follows that $\|U(t)\|^2_3 + \|B(t)\|^2_3 + \|\theta(t)\|^2_3 = 0$ for all $t \in [0, T]$. Here we have used Grönwall inequality and the fact that $U(0) = B(0) = \theta(0) = 0, u_1 \in L^2(0, T; H^{s+1})$ and $(b_1, \theta_1) \in (L^2(0, T; H^{s+1-\epsilon}))^2$.  

\qed
4. Conclusion
Based on the results of main Theorem 1.1, we can conclude that the solution of Boussinesq-MHD system with constant viscosity, thermal diffusivity, and electrical conductivity in the optimal Sobolev spaces $H^s(\mathbb{R}^n) \times H^{s+1-\epsilon}(\mathbb{R}^n) \times H^{s+1-\epsilon}(\mathbb{R}^n)$ exists through Littlewood-Paley decomposition, Bony’s para product, and commutator estimates. Moreover, the uniqueness of solution is also obtained through Grönwall inequality.

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