DECAY RATE OF GLOBAL SOLUTIONS TO THREE DIMENSIONAL GENERALIZED MHD SYSTEM

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Abstract. We investigate the initial value problem for the three dimensional generalized incompressible MHD system. Analyticity of global solutions was proved by energy method in the Fourier space and continuous argument. Then decay rate of global small solutions in the function space $\mathcal{X}^{1-2\alpha} \cap \mathcal{X}^{1-2\beta}$ follows by constructing time weighted energy inequality.

1. Introduction. In this paper, we investigate decay rate of global solutions to the three dimensional generalized incompressible magnetohydrodynamics (GMHD) system

$$
\begin{align*}
\partial_t u + u \cdot \nabla u - H \cdot \nabla H + \mu \Lambda^{2\alpha} u + \nabla P &= 0, \\
\partial_t H + u \cdot \nabla H - H \cdot \nabla u + \nu \Lambda^{2\beta} H &= 0, \\
\nabla \cdot u &= 0, \quad \nabla \cdot B = 0
\end{align*}
$$

associated with

$$
t = 0 : u = u_0(x), \quad H = H_0(x), \quad x \in \mathbb{R}^3.
$$

Here $u \in \mathbb{R}^3$ is the velocity, $H \in \mathbb{R}^3$ is the magnetic field, $p$ is the pressure of the flow and $P = p + \frac{1}{2}|H|^2$. $\mu > 0$ and $\nu > 0$ are viscosity, magnetic diffusivity, respectively. The fractional Laplacian operator $\Lambda^{2\alpha} = (-\Delta)^\alpha$ is defined by $\hat{\Lambda}^{2\alpha} \hat{f}(\xi) = |\xi|^{2\alpha} \hat{f}(\xi)$, where $\hat{f}(\xi)$ denotes Fourier transform of $f$.

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When $\alpha = \beta = 1$, (1) is reduced to the classical MHD system. Global well-posedness of MHD system is an important problem. We recall the global well-posedness and asymptotic decay of solutions in Lei-Lin function spaces for our purpose. Motivated by the argument of Lei and Lin [9] for the incompressible Navier-Stokes system, global existence of mild solutions in the critical function space was established in [17], proved that the norms of the initial data are bounded exactly by the minimal value of the viscosity coefficients. Later, the first author of this paper [19] proved asymptotic behavior and stability of global mild solution established in [17]. For a class of special large initial data with the critical norm of the initial data, proved that the norms of the initial data are bounded exactly by the minimal value of the viscosity coefficients.

Theorem 1.1. ([22], Theorem 1.1) Assume that $\frac{1}{2} \leq \alpha, \beta \leq 1$ and $u_0, H_0 \in \chi^{1-2\alpha} \cap \chi^{1-2\beta}$ with $\nabla \cdot u_0 = \nabla \cdot H_0 = 0$. Put

$$\delta = \|u_0\|_{\chi^{1-2\alpha} \cap \chi^{1-2\beta}} + \|H_0\|_{\chi^{1-2\alpha} \cap \chi^{1-2\beta}}.$$  

Then there exists $\delta_0 > 0$ such that if $\delta \leq \delta_0$, then the problem (1), (2) has a global solution $u \in C([0, \infty); \chi^{1-2\alpha} \cap \chi^{1-2\beta}) \cap L^1([0, \infty); \chi^{1} \cap \chi^{1-2\beta+2\alpha}), H \in C([0, \infty); \chi^{1-2\alpha} \cap \chi^{1-2\beta}) \cap L^1([0, \infty); \chi^{1} \cap \chi^{1-2\beta+2\beta})$ satisfying

$$\|u(t)\|_{\chi^{1-2\alpha} \cap \chi^{1-2\beta}} + \|H(t)\|_{\chi^{1-2\alpha} \cap \chi^{1-2\beta}} = \int_0^t \|u(\tau)\|_{\chi^{1} \cap \chi^{1-3\alpha+2\beta}} d\tau + \int_0^t \|H(\tau)\|_{\chi^{1} \cap \chi^{1-3\alpha+2\beta}} d\tau \leq C_0 \delta.$$  

Theorem 1.2. ([22], Theorem 1.2) Let $u \in C([0, \infty); \chi^{1-2\alpha} \cap \chi^{1-2\beta}) \cap L^1([0, \infty); \chi^{1} \cap \chi^{1-2\beta+2\alpha}), H \in C([0, \infty); \chi^{1-2\alpha} \cap \chi^{1-2\beta}) \cap L^1([0, \infty); \chi^{1} \cap \chi^{1-2\beta+2\beta})$ with $1/2 \leq \alpha, \beta \leq 1$ be global solutions to the problem (1), (2) given by Theorem 1.1, then

$$\limsup_{t \to \infty} (\|u(t)\|_{\chi^{1-2\alpha} \cap \chi^{1-2\beta}} + \|H(t)\|_{\chi^{1-2\alpha} \cap \chi^{1-2\beta}}) = 0.$$  

Our first result about analyticity of global small solutions, which will play a very important role in our decay rate.
Theorem 1.3. Assume that $\frac{1}{2} \leq \alpha, \beta \leq 1$ and $u_0, H_0 \in \mathcal{X}^{1-2\alpha} \cap \mathcal{X}^{1-2\beta}$ with $\nabla \cdot u_0 = \nabla \cdot H_0 = 0$. Put
\[
\delta = \|u_0\|_{\mathcal{X}^{1-2\alpha}} \cap \mathcal{X}^{1-2\beta} + \|H_0\|_{\mathcal{X}^{1-2\alpha}} \cap \mathcal{X}^{1-2\beta}.
\]
Let $(u, H)$ be the global solutions established in Theorem 1.1. Then there exists $\delta_1 > 0$ such that if $\delta \leq \delta_1$, then $(u, H)$ is analytic in the sense that
\[
\begin{align*}
&\|e^{\sqrt{\nu t} \Lambda^\alpha} u(t)\|_{\mathcal{X}^{1-2\alpha}} \cap \mathcal{X}^{1-2\beta} + \|e^{\sqrt{\nu t} \Lambda^\beta} H(t)\|_{\mathcal{X}^{1-2\alpha}} \cap \mathcal{X}^{1-2\beta} + \\
&\int_0^t \|e^{\sqrt{\nu \tau} \Lambda^\alpha} u(\tau)\|_{\mathcal{X}^{1-2\alpha+2\beta}} d\tau + \int_0^t \|e^{\sqrt{\nu \tau} \Lambda^\beta} H(\tau)\|_{\mathcal{X}^{1-2\alpha+2\beta}} d\tau \leq C_3 \delta.
\end{align*}
\] (7)

Remark 1. When $\alpha = \beta$, the result in Theorem 1.3 has been proved in [16]. Therefore, Theorem 1.3 generalizes the corresponding analyticity of global solutions result obtained in [16].

Theorem 1.4. Assume that the conditions of Theorem 1.1 and 1.2 hold. Furthermore, we assume $u_0, H_0 \in L^2$. Then
\[
\begin{align*}
&\|u(t)\|_{\mathcal{X}^{1-2\beta}} \leq C(1 + t)^{-(\frac{1}{\alpha} - \frac{2}{\beta})}, \\
&\|H(t)\|_{\mathcal{X}^{1-2\alpha}} \leq C(1 + t)^{-(\frac{1}{\beta} - \frac{2}{\alpha})}, \\
&\|u(t)\|_{\mathcal{X}^{1-2\alpha}} \leq C(1 + t)^{-(\frac{3}{2} - 1)}
\end{align*}
\] (8)-(10)
and
\[
\|H(t)\|_{\mathcal{X}^{1-2\beta}} \leq C(1 + t)^{-(\frac{3}{2} - 1)}.
\] (11)

Remark 2. On one hand, (8)-(11) implies that (5) holds. On the other hand, when $\alpha = \beta$, the result in Theorem 1.4 is reduced to the corresponding one in [20]. Hence this paper can be viewed as complementary results of the works [20] and [22].

Notations For $s \in \mathbb{R}$, the function space $\mathcal{X}^s$ is defined by
\[
\mathcal{X}^s(\mathbb{R}^3) \triangleq \left\{ f \in \mathcal{S}'(\mathbb{R}^3) \middle| \int_{\mathbb{R}^3} |\xi|^s \hat{f}(\xi) d\xi < \infty \right\},
\]
which is equipped with the norm
\[
\|f\|_{\mathcal{X}^s} \triangleq \int_{\mathbb{R}^3} |\xi|^s |\hat{f}(\xi)| d\xi.
\]

2. Proof of Theorem 1.3. In this section, our main aim is to prove the analyticity of global solutions established in Theorem 1.1. The following basic inequality will play a very important role in the proof of Theorem 1.3.

Lemma 2.1. ([21]) Let $\frac{1}{2} \leq \alpha \leq 1$. Then for any $\xi, \eta \in \mathbb{R}^3$, the following inequality
\[
|\xi|^{2-2\alpha} \leq 2^{1-2\alpha} (|\eta| |\xi - \eta|^{1-2\alpha} + |\eta|^{1-2\alpha} |\xi - \eta|)
\] (12)
holds.

In what follows, we give the proof of Theorem 1.3.
Proof. Let
\[ \hat{U}(\xi, t) = e^{\sqrt{\alpha^{2}}\xi^{n}} \hat{u}(\xi, t), \quad \hat{V}(\xi, t) = e^{\sqrt{\alpha^{2}}\xi^{n}} \hat{H}(\xi, t), \]
then \( \hat{U}(\xi, t) \) and \( \hat{H}(\xi, t) \) satisfy
\[
\hat{U}(\xi, t) = e^{\sqrt{\alpha^{2}}\xi^{n}-\mu(s^{2})\xi^{2n}} \hat{u}_{0}(\xi)-
\int_{0}^{t} e^{\sqrt{\alpha^{2}}\xi^{n}-\mu(s^{2})\xi^{2n}} \mathbb{P}(\xi) \cdot \left( \int_{\mathbb{R}^{2n}} \hat{u}(\xi - \eta, \tau) \hat{u}(\eta, \tau) d\eta d\tau \right) \] 
\[
+ \int_{0}^{t} e^{\sqrt{\alpha^{2}}\xi^{n}-\mu(s^{2})\xi^{2n}} \mathbb{P}(\xi) \cdot \left( \int_{\mathbb{R}^{2n}} \hat{H}(\xi - \eta, \tau) \hat{H}(\eta, \tau) d\eta d\tau \right) \]
\[\triangleq e^{\sqrt{\alpha^{2}}\xi^{n}-\mu(s^{2})\xi^{2n}} \hat{u}_{0}(\xi) - J_{1} + J_{2}.\]

\[ \hat{V}(\xi, t) = e^{\sqrt{\alpha^{2}}\xi^{n}-\nu(s^{2})\xi^{2n}} \hat{H}_{0}(\xi)-
\int_{0}^{t} e^{\sqrt{\alpha^{2}}\xi^{n}-\nu(s^{2})\xi^{2n}} (i^{2}) \mathbb{P}(\xi) \cdot \left( \int_{\mathbb{R}^{2n}} \hat{u}(\xi - \eta, \tau) \hat{H}(\eta, \tau) d\eta d\tau \right) \] 
\[
+ \int_{0}^{t} e^{\sqrt{\alpha^{2}}\xi^{n}-\nu(s^{2})\xi^{2n}} (i^{2}) \mathbb{P}(\xi) \cdot \left( \int_{\mathbb{R}^{2n}} \hat{H}(\xi - \eta, \tau) \hat{u}(\eta, \tau) d\eta d\tau \right) \]
\[\triangleq e^{\sqrt{\alpha^{2}}\xi^{n}-\nu(s^{2})\xi^{2n}} \hat{H}_{0}(\xi) - J_{3} + J_{4}.\]

Noting that (13), then \( J_{1} \) can be written as
\[
J_{1} = \int_{0}^{t} e^{\sqrt{\alpha^{2}}\xi^{n}-\mu(s^{2})\xi^{2n}} e^{-\frac{1}{2} (t-\tau) \xi^{2n}} \mathbb{P}(\xi) \cdot 
\left( \int_{\mathbb{R}^{2n}} \hat{u}(\xi - \eta, \tau) \hat{u}(\eta, \tau) d\eta d\tau \right) \]
\[
+ \int_{0}^{t} e^{\sqrt{\alpha^{2}}\xi^{n}-\mu(s^{2})\xi^{2n}} e^{-\frac{1}{2} (t-\tau) \xi^{2n}} \mathbb{P}(\xi) \cdot 
\left( \int_{\mathbb{R}^{2n}} \hat{H}(\xi - \eta, \tau) \hat{H}(\eta, \tau) d\eta d\tau \right) \]
\[\geq e^{\sqrt{\alpha^{2}}\xi^{n}-\mu(s^{2})\xi^{2n}} \hat{u}_{0}(\xi) - J_{1} + J_{2}.\]

Similarly, we have
\[
J_{2} = \int_{0}^{t} e^{\sqrt{\alpha^{2}}\xi^{n}-\nu(s^{2})\xi^{2n}} e^{-\frac{1}{2} (t-\tau) \xi^{2n}} \mathbb{P}(\xi) \cdot 
\left( \int_{\mathbb{R}^{2n}} \hat{u}(\xi - \eta, \tau) \hat{V}(\eta, \tau) d\eta d\tau \right) \]
\[
+ \int_{0}^{t} e^{\sqrt{\alpha^{2}}\xi^{n}-\nu(s^{2})\xi^{2n}} e^{-\frac{1}{2} (t-\tau) \xi^{2n}} \mathbb{P}(\xi) \cdot 
\left( \int_{\mathbb{R}^{2n}} \hat{H}(\xi - \eta, \tau) \hat{u}(\eta, \tau) d\eta d\tau \right) \]
\[\geq e^{\sqrt{\alpha^{2}}\xi^{n}-\nu(s^{2})\xi^{2n}} \hat{H}_{0}(\xi) - J_{3} + J_{4}.\]

\[
J_{3} = \int_{0}^{t} e^{\sqrt{\alpha^{2}}\xi^{n}-\mu(s^{2})\xi^{2n}} e^{-\frac{1}{2} (t-\tau) \xi^{2n}} \mathbb{P}(\xi) \cdot 
\left( \int_{\mathbb{R}^{2n}} \hat{u}(\xi - \eta, \tau) \hat{V}(\eta, \tau) d\eta d\tau \right) \]
\[
+ \int_{0}^{t} e^{\sqrt{\alpha^{2}}\xi^{n}-\mu(s^{2})\xi^{2n}} e^{-\frac{1}{2} (t-\tau) \xi^{2n}} \mathbb{P}(\xi) \cdot 
\left( \int_{\mathbb{R}^{2n}} \hat{H}(\xi - \eta, \tau) \hat{H}(\eta, \tau) d\eta d\tau \right) \]
\[\geq e^{\sqrt{\alpha^{2}}\xi^{n}-\mu(s^{2})\xi^{2n}} \hat{u}_{0}(\xi) - J_{1} + J_{2}.\]

and
\[
J_{4} = \int_{0}^{t} e^{\sqrt{\alpha^{2}}\xi^{n}-\nu(s^{2})\xi^{2n}} e^{-\frac{1}{2} (t-\tau) \xi^{2n}} \mathbb{P}(\xi) \cdot 
\left( \int_{\mathbb{R}^{2n}} \hat{u}(\xi - \eta, \tau) \hat{V}(\eta, \tau) d\eta d\tau \right) \]
\[
+ \int_{0}^{t} e^{\sqrt{\alpha^{2}}\xi^{n}-\nu(s^{2})\xi^{2n}} e^{-\frac{1}{2} (t-\tau) \xi^{2n}} \mathbb{P}(\xi) \cdot 
\left( \int_{\mathbb{R}^{2n}} \hat{H}(\xi - \eta, \tau) \hat{H}(\eta, \tau) d\eta d\tau \right) \]
\[\geq e^{\sqrt{\alpha^{2}}\xi^{n}-\nu(s^{2})\xi^{2n}} \hat{H}_{0}(\xi) - J_{3} + J_{4}.\]
Noting that the facts
\[
e^{\sqrt{\mu t}|\xi|^\alpha} - e^{\sqrt{\mu t}|\xi|^\alpha} = e^{-\sqrt{\mu t}(|\xi|^\alpha - |\eta|^\alpha)} + \frac{1}{2} e^{-\frac{1}{2} \mu t|\xi|^\alpha}
\]
and
\[
e^{\sqrt{\mu t}|\xi|^\alpha} - e^{\sqrt{\mu t}(t-\tau)|\xi|^\alpha} = e^{-\sqrt{\mu t}(t-\tau)(|\xi|^\alpha - |\eta|^\alpha)} + \frac{1}{2} e^{-\frac{1}{2} \mu t|\xi|^\alpha}
\]
are uniformly bounded independently \( t \) and \( \tau \). Therefore, combining (14)-(16) yields
\[
\|\hat{U}(\xi, t)\| \leq C e^{-\frac{1}{2}|\xi|^2\alpha t} |\hat{u}_0(\xi)| + C \int_0^t \int_{\mathbb{R}^3} e^{-\frac{1}{2}|\xi|^2(1-\tau)|\xi|} |\hat{U}(\xi - \eta, \tau)| |\hat{U}(\eta, \tau)| d\eta d\tau + \int_0^t \int_{\mathbb{R}^3} e^{-\frac{1}{2}|\xi|^2(1-\tau)|\xi|} |\hat{U}(\xi, \tau)| |\hat{U}(\eta, \tau)| d\eta d\tau.
\]
Similarly, from (14), (17) and (18), it holds that
\[
\|\hat{V}(\xi, t)\| \leq C e^{-\frac{1}{2}|\xi|^2\beta t} |\hat{H}_0(\xi)| + C \int_0^t \int_{\mathbb{R}^3} e^{-\frac{1}{2}|\xi|^2(t-\tau)|\xi|} |\hat{V}(\xi - \eta, \tau)| |\hat{V}(\eta, \tau)| d\eta d\tau.
\]
Let
\[
E_0(t) = \|U(t)\|_{H^{1-2\alpha}} + \|V(t)\|_{H^{1-2\beta}}
\]
and
\[
E_1(t) = \int_0^t \|U(\tau)\|_{H^{1-2\alpha}} + \|V(\tau)\|_{H^{1-2\beta}} d\tau.
\]
Multiplying (19) by \(|\xi|^{1-2\alpha}\) and then integrating the result over \( \mathbb{R}^3 \), using Lemma 2.1, we have
\[
\|U(t)\|_{H^{1-2\alpha}} \leq C \|u_0\|_{H^{1-2\alpha}} + C \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\eta||\xi - \eta|^{1-2\alpha} + C \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (|\eta||\xi - \eta|^{1-2\alpha} + |\xi|^{1-2\alpha} |\xi - \eta|) d\xi d\eta d\tau + C \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (|\eta||\xi - \eta|^{1-2\alpha} + |\xi|^{1-2\alpha} |\xi - \eta|) d\xi d\eta d\tau.
\]
Similarly, we can obtain from (19), (20) and Lemma 2.1
\[
\|U(t)\|_{H^{1-2\beta}} \leq C \|u_0\|_{H^{1-2\beta}} + C \sup_t E_0(t) E_1(t),
\]
\[ \|V(t)\|_{\mathcal{X}^{1-2\alpha}} \leq C\|H_0\|_{\mathcal{X}^{1-2\alpha}} + C(\sup_{t} \mathcal{E}_0(t))\mathcal{E}_1(t) \]  

(25)

and

\[ \|V(t)\|_{\mathcal{X}^{1-2\beta}} \leq C\|H_0\|_{\mathcal{X}^{1-2\beta}} + C(\sup_{t} \mathcal{E}_0(t))\mathcal{E}_1(t). \]  

(26)

Multiplying (19) by \(|\xi|^{1-2\alpha}\) and then integrating the result over \(\mathbb{R}^3 \times (0,t)\), using Young inequality, Lemma 2.1, we obtain

\[ \int_0^t \|U(\tau)\|_{\mathcal{X}^1}d\tau \leq C \int_0^t \int_{\mathbb{R}^3} e^{-\frac{\beta}{2}|\xi|^2 \tau} |\xi| |u_0(\xi)|d\xi d\tau + \]

\[ C \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-\frac{\beta}{2}|\xi|^2 \tau} |\xi|^2 \|\mathcal{U}(\xi - \eta, \tau)\|. \]

\[ |\mathcal{U}(\eta, \tau)|d\xi d\eta \left[ \int_0^\tau e^{-\frac{\beta}{2}|\xi|^2 \tau} d\tau \right] \]

\[ C \int_0^t \left[ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-\frac{\beta}{2}|\xi|^2 \tau} |\xi|^2 |\mathcal{V}(\xi - \eta, \tau)| |\mathcal{V}(\eta, \tau)|d\xi d\eta \right] \]

\[ \left[ \int_0^\tau e^{-\frac{\beta}{2}|\xi|^2 \tau} d\tau \right] \]

\[ \leq C \|u_0\|_{\mathcal{X}^{1-2\alpha}} + C \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\xi|^{2-2\alpha} |\mathcal{U}(\xi - \eta, \tau)|. \]

\[ |\mathcal{U}(\eta, \tau)|d\xi d\eta d\tau + \]

\[ C \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\xi|^{2-2\alpha} |\mathcal{V}(\xi - \eta, \tau)| |\mathcal{V}(\eta, \tau)|d\xi d\eta d\tau \]

\[ \leq C \|u_0\|_{\mathcal{X}^{1-2\alpha}} + C \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (|\eta||\xi - \eta|^{1-2\alpha} + |\xi|^{1-2\alpha} |\xi - \eta|) |\mathcal{U}(\xi - \eta, \tau)| |\mathcal{U}(\eta, \tau)|d\xi d\eta d\tau + \]

\[ C \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (|\eta||\xi - \eta|^{1-2\alpha} + |\xi|^{1-2\alpha} |\xi - \eta|) |\mathcal{V}(\xi - \eta, \tau)| |\mathcal{V}(\eta, \tau)|d\xi d\eta d\tau \]

\[ \leq C \|u_0\|_{\mathcal{X}^{1-2\alpha}} + C(\sup_{t} \mathcal{E}_0(t))\mathcal{E}_1(t). \]

(27)

The same procedure leads to

\[ \int_0^t \|U(\tau)\|_{\mathcal{X}^{1-2\beta+2\alpha}}d\tau \leq C \|u_0\|_{\mathcal{X}^{1-2\beta}} + C(\sup_{t} \mathcal{E}_0(t))\mathcal{E}_1(t), \]

(28)
Lemma 3.1. \([3]\) Let \(T > 0\) and \(f : [0, T] \to \mathbb{R}^+\) be a continuous function such that

\[
f(t) \leq K_0 + \theta_1 f(\theta_2 t), \quad \forall t \in [0, T],
\]

where \(K_0 \geq 0\) and \(0 < \theta_1, \theta_2 < 1\). Then

\[
f(t) \leq \frac{K_0}{1 - \theta_1}, \quad \forall t \in [0, T].
\]

Remark 3. It follows from a positive continuous function \(f : \mathbb{R}^+ \to \mathbb{R}^+\) satisfying

\[
f(t) \leq K_0 + \theta_1 f(\theta_2 t), \quad \forall t \geq 0
\]

where \(K_0 \geq 0\) and \(0 < \theta_1, \theta_2 < 1\). Then

\[
\lim_{t \to +\infty} f(t) \leq \frac{K_0}{1 - \theta_1}.
\]

Next we give the proof of Theorem 1.4.

Proof. Firstly, the energy equality

\[
\|u(t)\|_{L^2}^2 + \|H(t)\|_{L^2}^2 + \mu \int_0^t \|\Lambda^\alpha u(\tau)\|_{L^2}^2 d\tau + \nu \int_0^t \|\Lambda^\beta H(\tau)\|_{L^2}^2 d\tau = \|u_0\|_{L^2}^2 + \|H_0\|_{L^2}^2.
\]

holds. Owing to the definition of \(\mathcal{X}^{1-2\alpha}\), we obtain

\[
\|u(t)\|_{\mathcal{X}^{1-2\alpha}} = \int_{|\xi| \leq \varepsilon} |\xi|^{1-2\beta} |\widehat{u}(\xi, t)| d\xi + \int_{|\xi| \geq \varepsilon} |\xi|^{1-2\beta} |\widehat{u}(\xi, t)| d\xi
\]

\[
= I_1 + I_2.
\]

It follows from Hölder inequality and energy equality (32) that

\[
I_1 \leq \left( \int_{|\xi| \leq \varepsilon} |\xi|^{2(1-2\beta)} d\xi \right)^{\frac{1}{2}} \left( \int_{|\xi| \leq \varepsilon} |\widehat{u}(\xi, t)|^2 d\xi \right)^{\frac{1}{2}} \leq C \varepsilon^{\frac{1-4\beta}{2}} \|u(t)\|_{L^2}^2
\]

\[
\leq C \varepsilon^{\frac{1-4\beta}{2}} \left( \|u_0\|_{L^2}^2 + \|H_0\|_{L^2}^2 \right).
\]
By direct computation, we have

\[
I_2 \leq \int_{|\xi| \geq \varepsilon} e^{-\sqrt{\mu t/2} \xi \cdot \xi} e^{\sqrt{\mu t/2} \xi \cdot \xi} |\xi|^{1-2\beta} |\hat{u}(\xi, t)| d\xi \\
\leq C e^{-\sqrt{\mu t/2^{\alpha}} \xi \cdot \xi} \int_{|\xi| \geq \varepsilon} e^{\sqrt{\mu t/2} \xi \cdot \xi} |\xi|^{1-2\beta} |\hat{u}(\xi, t)| d\xi
\]

(35)

For fixed time \( t > 0 \), let \( v(x, s) = u(x, s + \frac{t}{2}) \), \( \hat{H}(x, s) = H(x, s + \frac{t}{2}) \). Then

\[
\|v(x, 0)\|_{X^{1-2\alpha} \cap X^{1-2\beta}} + \|\hat{H}(x, 0)\|_{X^{1-2\alpha} \cap X^{1-2\beta}} \leq C_1
\]

Let \((v, \hat{H})\) be the unique global solution to the initial value problem

\[
\begin{aligned}
\partial_s v + v \cdot \nabla v - \hat{H} \cdot \nabla \hat{H} + \mu \Lambda^{2\alpha} v + \nabla \hat{\rho} &= 0, \\
\partial_s \hat{H} + v \cdot \nabla \hat{H} - \hat{H} \cdot \nabla u + \nu \Lambda^{2\beta} \hat{H} &= 0, \\
\nabla \cdot v &= 0, \quad \nabla \cdot \hat{H} = 0, \\
\end{aligned}
\]

(36)

By (7), we obtain

\[
\int_{\mathbb{R}^3} e^{\sqrt{\mu t/2} \xi \cdot \xi} |\xi|^{1-2\alpha} |\hat{u}(\xi, s)| d\xi + \int_{\mathbb{R}^3} e^{\sqrt{\mu t/2} \xi \cdot \xi} |\xi|^{1-2\alpha} |\hat{H}(\xi, s)| d\xi \\
+ \int_{\mathbb{R}^3} e^{\sqrt{\mu t/2} \xi \cdot \xi} |\xi|^{1-2\beta} |\hat{v}(\xi, s)| d\xi + \int_{\mathbb{R}^3} e^{\sqrt{\mu t/2} \xi \cdot \xi} |\xi|^{1-2\beta} |\hat{\hat{H}}(\xi, s)| d\xi
\]

(37)

\[
\leq C(\|u(x, 0)\|_{X^{1-2\alpha} \cap X^{1-2\beta}} + \|H(x, 0)\|_{X^{1-2\alpha} \cap X^{1-2\beta}})
\]

That is

\[
\int_{\mathbb{R}^3} e^{\sqrt{\mu t/2} \xi \cdot \xi} |\xi|^{1-2\alpha} |\hat{u}(\xi, s + \frac{t}{2})| d\xi + \int_{\mathbb{R}^3} e^{\sqrt{\mu t/2} \xi \cdot \xi} |\xi|^{1-2\alpha} |\hat{H}(\xi, s + \frac{t}{2})| d\xi \\
+ \int_{\mathbb{R}^3} e^{\sqrt{\mu t/2} \xi \cdot \xi} |\xi|^{1-2\beta} |\hat{v}(\xi, s + \frac{t}{2})| d\xi + \int_{\mathbb{R}^3} e^{\sqrt{\mu t/2} \xi \cdot \xi} |\xi|^{1-2\beta} |\hat{\hat{H}}(\xi, s + \frac{t}{2})| d\xi
\]

\[
\leq C(\|u(x, 0)\|_{X^{1-2\alpha} \cap X^{1-2\beta}} + \|H(x, 0)\|_{X^{1-2\alpha} \cap X^{1-2\beta}})
\]

(38)

With \( s = \frac{t}{2} \) implies that

\[
\int_{\mathbb{R}^3} e^{\sqrt{\mu t/2} \xi \cdot \xi} |\xi|^{1-2\alpha} |\hat{u}(\xi, t)| d\xi + \int_{\mathbb{R}^3} e^{\sqrt{\mu t/2} \xi \cdot \xi} |\xi|^{1-2\alpha} |\hat{H}(\xi, t)| d\xi \\
+ \int_{\mathbb{R}^3} e^{\sqrt{\mu t/2} \xi \cdot \xi} |\xi|^{1-2\beta} |\hat{v}(\xi, t)| d\xi + \int_{\mathbb{R}^3} e^{\sqrt{\mu t/2} \xi \cdot \xi} |\xi|^{1-2\beta} |\hat{\hat{H}}(\xi, t)| d\xi
\]

(39)

\[
\leq C(\|u(x, 0)\|_{X^{1-2\alpha} \cap X^{1-2\beta}} + \|H(x, 0)\|_{X^{1-2\alpha} \cap X^{1-2\beta}})
\]

Then combining (35) and (39) yields

\[
I_2 \leq C e^{-\sqrt{\mu t/2^{\alpha}}}(\|u(x, 0)\|_{X^{1-2\alpha} \cap X^{1-2\beta}} + \|H(x, 0)\|_{X^{1-2\alpha} \cap X^{1-2\beta}}). 
\]

(40)
Inserting (34), (40) into (33) yields
\[ \|u(t)\|_{X^{1-2\beta}} \leq C\varepsilon^{\frac{5-4\beta}{5}} (\|u_0\|_{L^2} + \|H_0\|_{L^2}) + C\varepsilon^{-\sqrt{\mu/2\varepsilon}} (\|u(\frac{t}{2})\|_{X^{1-2\alpha}} \cap X^{1-2\beta} + \|H(\frac{t}{2})\|_{X^{1-2\alpha}} \cap X^{1-2\beta}). \]

Multiplying (41) by \( t^{\frac{\alpha}{2}} \), it holds that
\[ t^{\frac{\alpha}{2}} \|u(t)\|_{X^{1-2\beta}} \leq C\varepsilon^{\frac{5-4\beta}{5}} t^{\frac{\alpha}{2} - \frac{\beta}{\alpha}} (\|u_0\|_{L^2} + \|H_0\|_{L^2}) + C\varepsilon^{-\sqrt{\mu/2\varepsilon}} t^{\frac{\alpha}{2}} - \frac{\beta}{\alpha} (\|u(\frac{t}{2})\|_{X^{1-2\alpha}} \cap X^{1-2\beta} + \|H(\frac{t}{2})\|_{X^{1-2\alpha}} \cap X^{1-2\beta} + \|H^{(t)}(\frac{t}{2})\|_{X^{1-2\alpha}} \cap X^{1-2\beta}). \]

Taking \( \delta = \left( \frac{\sqrt{2}(\log C_2 + \frac{5-4\beta+4\alpha}{4\alpha} \log 2)}{\sqrt{\mu}} \right)^{\frac{1}{\mu}} \), then
\[ t^{\frac{\alpha}{2}} - \frac{\beta}{\alpha} \|u(t)\|_{X^{1-2\beta}} \leq C\left( \frac{\sqrt{2}(\log C_2 + \frac{5-4\beta+4\alpha}{4\alpha} \log 2)}{\sqrt{\mu}} \right)^{\frac{5-4\beta}{5}} (\|u_0\|_{L^2} + \|H_0\|_{L^2}) + \frac{1}{2}(t^{\frac{\alpha}{2}} - \frac{\beta}{\alpha} (\|u(\frac{t}{2})\|_{X^{1-2\alpha}} \cap X^{1-2\beta} + \|H(\frac{t}{2})\|_{X^{1-2\alpha}} \cap X^{1-2\beta} + \|H^{(t)}(\frac{t}{2})\|_{X^{1-2\alpha}} \cap X^{1-2\beta}). \]

Let
\[ K_0 = C\left( \frac{\sqrt{2}(\log C_2 + \frac{5-4\beta+4\alpha}{4\alpha} \log 2)}{\sqrt{\mu}} \right)^{\frac{5-4\beta}{5}} (\|u_0\|_{L^2} + \|H_0\|_{L^2}) \]
and
\[ f(t) = t^{\frac{\alpha}{2}} - \frac{\beta}{\alpha} \|u(t)\|_{X^{1-2\beta}}, \]
then
\[ f(t) \leq K_0 + \frac{1}{2} f\left( \frac{t}{2} \right), \quad \theta_1 = \theta_2 = \frac{1}{2} \]
It follows from Remark 3 that
\[ f(t) \leq \frac{K_0}{1-\theta_1} = 2K_0. \]

Then we complete the proof of (8). Similarly, we can prove (9), (10) and (11) can be obtained from (8) with \( \beta = \alpha \) and (9) with \( \alpha = \beta \), respectively. Theorem 1.4 is proved.

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