Gamow factors and current densities in cold field emission theory: a comparative study

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The factors that contribute to the accuracy of the cold field emission current within the contemporary frameworks are investigated. It is found that so long as the net current is evaluated using an expression for the local current density obtained by linearizing the Gamow factor, the primary source of error is the choice of the energy at which the Taylor expansion is done, but not as much on the choice of the method used to arrive at the approximate Gamow factor. A suitable choice of linearization energy and the implementation of the Kemble correction, allows the restriction of errors to below 3% across a wide range of local fields.

I. INTRODUCTION

The evaluation of the cold field emission current from a metallic surface has been the subject of interest for almost a century starting with the works of Fowler-Nordheim (FN), Murphy-Good (MG) and several others in the past decades. The analytical expression for the current density is based on the free-electron model of metals, the fermionic nature of electrons, the WKB approximation for the tunneling transmission coefficient and finally an integration over the electron states using a linearization of the Gamow factor. Having a ready-to-use expression is helpful since it speeds up the numerical calculation of the net emission current by integrating over the surface. It also allows for an approximate analytical expression for the net emission current using (in several cases) a knowledge of the local field variation around the emitter apex. Finally, an analytical expression for the distribution of emitted electrons also leads to a better and faster modelling of the emission process in Particle-In-Cell codes.

The continued theoretical interest in the subject stems from the fact that even for cold field emission, there remains sufficient scope for improving the modelling of emitters based on its validation with experimental results. For instance, the Fowler-Nordheim expression for current density (see for instance Eq. (6) of Ref. [21]) that uses the exact triangular (ET) barrier potential, is almost certainly invalid for its neglect of image charge contributions included in the Murphy and Good form, a point emphasized by Forbes. A manifestation of its shortcoming is that emitter characteristics, such as the field enhancement factor or the emission area, inferred from the experimental data are highly inconsistent with the physical dimensions of the emitter. The Murphy-Good current density on the other hand, seems to capture the essential physics but is inadequate for nano-tipped emitters having apex radius of curvature less than 100nm. Thus, incorporation of curvature effects is an area of concern and future efforts in this direction may be aimed at further improvement of accuracy.

The focus in this work is on emitters where curvature effects are negligible and hence they can be adequately described within the standard MG approach. The errors in the Murphy-Good current density (MGCD) in the high field regime is a motivation for the present study. There are three basic ingredients that go into the derivation of MGCD that need to be looked at afresh. The first of these centres around the Gamow factor,

\[ G(\mathcal{E}) = g \int_{s_1}^{s_2} \sqrt{V_T(s) - \mathcal{E}^2} \, ds \]  (1)

where \( V_T \) is the tunneling potential energy, \( s_1 \) and \( s_2 \) are the turning points determined using \( V_T(s) - \mathcal{E} = 0 \), \( g = \sqrt{8m/\hbar} \), \( m \) is the mass of the electron and \( \mathcal{E} \) is the normal-energy of the electron incident on the barrier. In the standard Murphy-Good approach, the Gamow factor \( G \) for the image charge modified potential energy (the Schottky-Nordheim or SN barrier; see Eq. (3)), is expressed as a product of the Gamow factor for the exact triangular barrier \( G_{ET} = (2/3)\sqrt{qE}^{3/2}/(qE) \), and a barrier form correction factor (BFCF). For the SN barrier, the BFCF is the WKB integral

\[ \nu(y) = \frac{3}{2} \int_{\xi_1}^{\xi_2} d\xi \left( 1 - \xi + \frac{y^2}{4\xi} \right)^{1/2} \]  (2)

where \( \xi_1 \) and \( \xi_2 \) are the roots of \( 1 - \xi + \frac{y^2}{4}\xi \) with \( y = 2\sqrt{BqE}/\varphi \). Here \( B = q^2/(16\pi\epsilon_0) \), \( E \) is the local field, \( \varphi = \mathcal{E}_F + \phi - \mathcal{E} \), \( q \) is the magnitude of the electronic charge, \( \mathcal{E}_F \) is the Fermi energy, and \( \phi \) is the workfunction. The BFCF \( \nu(y) \) due to the image charge can also be expressed in terms of complete elliptic integrals (see Eq. (16) of Murphy and Good). Importantly, for this communication, \( \nu(y) \) has a convenient algebraic approximation \( \nu(y) \approx 1 - y^2 + \left( y^4/3 \right) \ln(y) \) due to Forbes, which allows the MGCD to be readily used to evaluate the field emission current density. In the following, we shall refer to the Gamow-factor evaluated using the Forbes approximation as the MG method of evaluating the Gamow factor and refer to this as \( G_{MG} \). Obviously, \( G_{MG} \) provides a useful approximate value for the ‘exact’ Gamow factor, \( G \), which can be obtained by numerically integrating.
Eq. [2] and multiplying this by $G_{ET}$. The ‘exact-WKB’ method that we shall use in the following sections as a benchmark, uses the ‘exact’ Gamow factor.

An alternate, though related, approach to the evaluation of the Gamow factor, is the so-called shape-factor (SF) method due to Jensen[24,25] which recasts the Gamow factor as $G = 2L(y)\kappa(y)\sigma(y)$, a product form (see Eqns. [11]) applicable to all barriers, with individual terms dependent on the shape of the barrier. The so-called shape-factor term in the product, $\sigma(y)$, is an integral (see Eq. [11]) that still needs to be evaluated. Note that that $G = G_{ET} \nu(y)$ is exactly equivalent to $G = 2L(y)\kappa(y)\sigma(y)$. Both forms should lead to the exact Gamow factor if $\nu(y)$ in Eq. [2] and $\sigma(y)$ in Eq. [11] are computed accurately.

For the SF method to be readily used, an algebraic approximation of the shape-factor integral is required just like the Forbes approximation for $\nu(y)$. Useful approximations for the shape-factor $\sigma(y)$ for the SN barrier exist, expressed as a second or fourth degree polynomial in $(1-y)/(1+y)$, with the coefficients determined from a fit. We shall refer to Gamow factor evaluated thus as $G_{SF}$. Both $G_{MG}$ and $G_{SF}$ represent basic approximations that can be used to arrive at expressions for the field emission current density and the present comparative study involves these approximate forms of the Gamow factor. As we shall see, $G_{SF}$ leads to a more accurate evaluation of the transmission coefficient compared to $G_{MG}$, for both the second and fourth degree approximate polynomial forms of the shape factor.

The second of the three ingredients necessary for an analytical expression for the current density, involves the point on the energy scale at which a linearization of the Gamow factor should be made. In the standard MG approach, the Fermi energy has been the point of linearization. While this may be adequate for cold field emission at low to moderate fields, it is clear that this would lead to large errors at higher fields or in the thermal-field region. The emerging point of view is that the location of the peak in the normal energy distribution provides a suitable energy for linearization. The shifted point of linearization, together with the use of the shape factor approximation, results in a more accurate determination of the current density in case of thermal-field emission[9].

In case of cold field emission, the shape factor method (even for second degree approximation) yields a somewhat more involved expression for the current density compared to the MGCD as we shall see in section [III]. Whether this translates to a more accurate expression for the current density will be a subject of investigation.

The third ingredient concerns the use of $e^{-G}$ for the transmission coefficient. While this is adequate at energies where the barrier is strong, it contributes to larger errors at energies close to the top of the barrier. At higher field strengths, the SN-barrier peak comes closer to the Fermi energy while the peak of the normal energy distribution shifts away from the Fermi energy. Thus, the second approximation (linearization at Fermi energy) as well as the third approximation (use of $e^{-G}$) contribute to the errors at higher fields. An alternate and better approximation near the barrier-top is the so-called Kemble form which uses $(1 + e^{-G})^{-1}$ to approximate the transmission coefficient within the WKB method.

While there has been a need to correct the errors in the prediction of the MGCD in the high field regime, it is of interest to know whether the shape-factor method has an equally important role to play in improving the accuracy, as the point of linearization along the energy axis and the Kemble form of transmission coefficient. The interest in such a question is manifold apart from the purely academic one. To begin with, if the simplicity of the MGCD can be retained without compromising with the accuracy by merely shifting the point of linearization and introducing a correction term to account for the Kemble form, such an approach might be worthwhile. Besides, the entire paraphernalia of curvature-corrections to the existing expressions for the net field emission current density from locally parabolic tips and the electron distributions[10] we shall thus compare the field emission current density and net emission current using $G_{MG}$ and $G_{SF}$, and study the relative importance of the point of linearization, the two approximate methods of evaluating the Gamow factor, and the use of Kemble form in the context of cold field emission. To keep matters simple, we shall assume that the emitters have tip-radius large enough to ignore curvature corrections.

Before embarking on this comparative study, it is important to decide on the benchmark to be used. Since the WKB method is central to the field emission formalism, it is essential to use an exact numerical evaluation of the Gamow factor, and the Kemble form of the transmission coefficient to compute the current-density by integrating over the electron states. We shall refer to this as the exact-WKB method, the word ‘exact’ referring to the use of the exact Gamow factor and numerical integration over the energy states but not to the transmission coefficient. This approach provides a natural benchmark for comparing other approximate results which invoke an approximation to the Gamow factor and its subsequent linearization in order to carry out the energy integration. It is important to note that the exact current density, which can be computed for instance by using the transfer matrix approach[26], may differ substantially from the benchmark itself[27,28] depending on the ratio of the Fermi energy and the workfunction[29].

In section [II] we shall first compare the transmission coefficient for different ranges of energy using $G_{MG}$ and $G_{SF}$, and also look at the net emission current without resorting to linearization. Section [III] deals with analytical expressions for current density using an approximate Kemble form and a linearization of the Gamow factors $G_{MG}$ and $G_{SF}$ at an arbitrary energy $E_m$ at or below the
Fermi energy. These are then used to compare the net emission current in the linearized framework with the benchmark. Our conclusions and discussions form the final section.

II. COMPARISON OF THE TRANSMISSION COEFFICIENT

The central object in field emission is the tunneling transmission coefficient, $T$. The WKB-method provides a handy method for determining $T(E)$ for a particle with incident normal-energy $E$. It may be expressed as:

$$T(E) \approx \frac{1}{1 + e^{G(E)}}$$  

(3)

$$G(E) = g \int_{s_1}^{s_2} \sqrt{V_T(s) - E} \, ds$$  

(4)

where $g = \sqrt{8m/\hbar} \approx 10.246$ (eV)$^{-1/2}$ (nm)$^{-1}$ while $s_1, s_2$ are the zeroes of the integrand. The tunneling potential energy

$$V_T(s) = E_F + \phi - qE_F s - \frac{B}{s}$$  

(5)

where $q$ is magnitude of the electronic charge, $E_i$ is the local field a point on the emitter-surface, $s$ is the normal distance from the point, $B = q^2/(16\pi\epsilon_0)$ while $E_F$ and $\phi$ are the Fermi energy and workfunction respectively. To simplify matters, we have not included any curvature correction to the tunneling potential energy. Note that the exact-triangular potential energy can be obtained by neglecting the image charge contribution $B/s$ in Eq. (5).

In the Murphy-Good (MG) approach, the Gamow factor $G(E)$ is expressed as:

$$G(E) = \frac{2}{3} \frac{\varphi^{3/2}}{qE_i} \nu(y) = G_{E_{F \nu}(y)}$$  

(6)

where $\varphi = E_F + \phi - E$, $y = 2\sqrt{BqE_i}/\varphi$ and $E_i$ is the local electric field. The BFCF or the image-charge correction factor, $\nu(y)$ is well approximated by:

$$\nu(y) \approx 1 - y^2 + \frac{y^4}{3}\ln(y).$$  

(7)

Eq. (6) together with Eq. (7) gives an approximate expression for the Gamow factor and is referred to as $G_{MG}$. In the more recent shape-factor (SF) approach, the Gamow factor is expressed as:

$$G(E) = 2\sigma(E)\kappa(E)L(E).$$  

(8)

For the Schottky-Nordheim barrier,

$$L(E) = \frac{1}{E_i} \sqrt{\varphi^2 - 4BE_i} = \frac{\varphi}{E_i} (1 - y^2)^{1/2}$$  

(9)

$$\kappa(E) = \frac{9}{2} (\varphi - \sqrt{4BE_i})^{1/2} = \frac{9\sqrt{\varphi}}{2} (1 - y)^{1/2}$$  

(10)

$$\sigma(y(E)) = \frac{\sqrt{2}}{4} (1 + y)^{1/2} \int_{-1}^{1} ds \left[ \frac{1 - s^2}{1 + s\sqrt{1 - y^2}} \right]^{1/2}$$  

(11)

Useful approximate forms for the shape factor $\sigma(y)$ exist, expressed as:

$$\sigma(y) = \sum_{j=0}^{n} C_j \left( \frac{1 - y}{1 + y} \right)^j.$$  

(12)

We shall, for the most part, use the one with $n = 2$ with $C_0 = 0.785398$, $C_1 = -0.092385$ and $C_2 = -0.026346$ in order to obtain a manageable expression for the linearized current density in section III. Eqn. (8) together with Eqs. (9), (10) and (12) gives an approximate expression for the Gamow factor in the SF approach and is referred to as $G_{SF}$.

These approximate expressions for the Gamow factor can be used to arrive at expressions for the current density

$$J = \frac{2mq}{(2\pi\hbar)^3} \int_{0}^{E_F} T(E) (E_F - E) \, dE$$  

(13)

which can finally be integrated over the surface to arrive at the net emission current.

It is instructive to compare the transmission coefficient and net emission current obtained using the two approaches before proceeding with the linearization of the Gamow factor to obtain an analytical expression for the current density.

Figure (1) shows the error in transmission coefficient as a function of energy for 3 different applied fields, $E_i = 4.7$ and 9V/nm, for a material with $\phi = 4.5$eV and $E_F = 8.5$eV. The label MG refers to the transmission coefficient evaluated using $T_{MG}(E) \approx 1/(1 + e^{G_{MG}(E)})$ with $G_{MG}$ evaluated using Eq. (6) and Eq. (7). Similarly, SF refers to the transmission coefficient evaluated as $T_{SF} \approx 1/(1 + e^{G_{SF}})$ with $G_{SF}$ evaluated using Eqs. (8), (9), (10) and (12). Clearly, the shape factor approximation of the Gamow factor provides much better results for both $n = 2$ and $n = 4$ at all energies while the MG transmission coefficient scores well at higher energies. Note that the maximum value of energy for each applied field corresponds to the top of the barrier. It is thus higher for lower field strengths.

Note that insofar as the net emission current is concerned, the electrons that contribute at a lower apex field correspond to the top of the barrier. It is thus higher for lower field strengths.
FIG. 1. The relative error in transmission coefficient \( T(\varepsilon) \) with respect to the exact-WKB result. The maximum value of energy \( \varepsilon \) for each applied field corresponds to the top of the barrier. MG refers to Murphy-Good while SF refers to the Shape-Factor method. In (a) the second degree polynomial form of the shape factor is used with \( C_0 = 0.785398, C_1 = -0.092385 \) and \( C_2 = -0.026346 \). In (b) the fourth degree polynomial form is used with \( C_0 = 0.785398, C_1 = -0.0961, C_2 = -0.029092, C_3 = 0.034482 \) and \( C_4 = -0.027987 \). Also marked are the local fields in V/nm. At higher fields, the barrier comes closer to the Fermi energy.

FIG. 2. The relative error in net-emission current with respect to the exact-WKB result. Both SF and MG methods use the Kemble form and numerical integration over energy. Both under-predict the current. In (a) the second degree polynomial form approximation of the shape factor is used while in (b) the fourth degree polynomial form is used. The coefficients \( C_i \) are as in Fig. 1.

on errors in the the emission current for both MG and SF approximations. A plot of the net-emission current using these transmission coefficient confirms this observation. In Fig. 2, we consider a hemiellipsoidal emitter in a parallel-plate configuration with \( h/R_a = 300 \) and \( R_a = 10\mu m \). The net emission current is evaluated by integrating the current density obtained using Eq. [13] over the surface using the local cosine law of field variation \( E_l = E_a \cos \theta \) where \( \cos \theta = (z/h)/\sqrt{(z/h)^2 + (\rho/R_a)^2} \). The label ‘Murphy-Good’ refers to the current obtained using \( T_{MG} \) in Eq. [13] while ‘Shape-Factor’ refers to the use of \( T_{SF} \) in Eq. [13]. In both cases, the energy integration is performed numerically. The errors computed are relative to the exact-WKB method where the Gamow factor is obtained numerically. Clearly \( G_{SF} \) gives better results for both \( n = 2 \) and \( n = 4 \) compared to \( G_{MG} \) when the current density is obtained by numerically integrating over the electron energy states.

III. LINEARIZATION AND THE ANALYTICAL CURRENT DENSITY

The linearization of the Gamow factor allows us to perform the energy integration in Eq. [13] analytically. If the Taylor expansion is done at \( \varepsilon = \varepsilon_F \), the errors are larger at higher field strengths since the peak of the normal energy distribution lies below the Fermi energy for cold field emission and moves further away as the local electric field is increased. A suitable alternate energy value may be chosen in one of several ways. The one suggested for thermal-field emission corresponds to the peak (or maxima) of the normal energy distribution. We shall thus compare the two expressions for the linearized current densities and the net emission current obtained using these.

Eq. [13] for the current density can be written approximately as,

\[
J = \frac{2me}{(2\pi)^2\hbar^3} \int_{0}^{\varepsilon_F} (\varepsilon_F - \varepsilon) \frac{1}{1 + e^{G(\varepsilon)}} d\varepsilon \tag{14}
\]

\[
\approx \frac{2me}{(2\pi)^2\hbar^3} \int_{0}^{\varepsilon_F} (\varepsilon_F - \varepsilon) e^{-G(\varepsilon)} \left[ 1 - e^{-G(\varepsilon)} + \ldots \right] d\varepsilon
\]

The integration can now be carried out easily. Since the algebra is quite straightforward, we shall merely state the final result. In the MG case, a linearization at \( \varepsilon = \varepsilon_m \) leads to

\[
J_{MG}^{m} \approx A_{FN} \frac{l_{m}^{2}}{t_{m}} e^{-B_{MG}} \left( 1 - \frac{e^{-B_{MG}}}{4} \right) \tag{15}
\]

\[
B_{MG} = B_{FN} \frac{\nu_{m}}{E_l} + \frac{t_{m}}{d_{m}} (\varepsilon_F - \varepsilon_m) \tag{16}
\]

where \( A_{FN} \approx 1.541434 \mu A \text{eV}^{-2} \), \( B_{FN} \approx 6.830890 \text{eV}^{-3/2} \text{V nm}^{-1} \), \( g = \sqrt{8m_{e}/\hbar} \approx 10.246 \text{(eV)}^{-1/2} \text{(nm)}^{-1} \), \( E_l \) is the local field, while \( \varphi_{m} = \varepsilon_F + \phi - \varepsilon_m \), \( d_{m} = \varphi_{m}^{1/2}/E_l \), \( y_m = c_S \sqrt{E_l/\varphi_{m}} \) with \( c_S = 1.199958 \text{eV/V nm}^{-1/2} \) and...
\[ \nu_m = 1 - y_m^2 + \frac{y_m^2}{3} \ln y_m \]  
\[ t_m = 1 + \frac{y_m^2}{9} - \frac{y_m^2}{9} \ln y_m. \]  

Note that the value of \( \mathcal{E}_m \) has not been specified so far and hence Eq. (15) applies to an arbitrary point of linearization between 0 and \( \mathcal{E}_F \). Recall that the standard MG approach uses \( \mathcal{E}_m = \mathcal{E}_F \). Also, the factor \( 1 - e^{-B_{MG}/4} \) in Eq. (15) is a first correction arising from the use of the Kemble form of transmission coefficient. Neglecting the correction factor, \( 1 - e^{-B_{MG}/4} \), would amount to using \( T(\mathcal{E}) = e^{-G} \) thereby reducing Eq. (15) at \( \mathcal{E}_m = \mathcal{E}_F \) to the standard MGCD. \( J_{MG} \). \( J^m_{MG} \) may thus be referred to as MG-like current density to emphasize (a) that \( \mathcal{E}_m \) may be different from \( \mathcal{E}_F \) and (b) the inclusion of the correction term in \( J^m_{MG} \).

A similar expression can be obtained using the shape-factor method. We shall restrict ourselves to the second degree polynomial approximation in order to obtain a manageable expression for the current density. The use of \( G_{SF} \) with \( n = 2 \) and its linearization at \( \mathcal{E} = \mathcal{E}_m \) leads to a form for the current density that can be expressed as

\[ J^m_{SF} \approx A_{FN} \frac{1}{\varphi_m} \frac{E_1^2}{T_m^2} e^{-B_{SF}} \left( 1 - \frac{e^{-B_{SF}}}{4} \right) \]  
\[ B_{SF} = B_{FN} \sqrt[n]{e^{-t_m}} + \frac{T_m}{d_m} (\mathcal{E}_F - \mathcal{E}_m) \]

where

\[ N_m = \frac{3}{2} \left[ C_0 y_1 y_2^{1/2} + C_1 \frac{y_3^2}{y_2^{1/2}} + C_2 \frac{y_4^2}{y_2^{3/2}} \right] \]  
\[ P_m = \frac{C_0}{2} \frac{y_3}{y_2^{1/2}} + \frac{C_1}{2} \frac{y_3 y_4}{y_2^{1/2}} + \frac{3}{2} C_2 \frac{y_5 y_1}{y_2^{5/2}} \]  
\[ T_m = N_m + y_m P_m \]

with \( y_1 = 1 - y_m, y_2 = 1 + y_m, y_3 = 1 + 3y_m, y_4 = 5 + 3y_m \) and \( y_5 = 3 + y_m \). The coefficients \( C_0 = 0.785398, C_1 = -0.092385 \) and \( C_2 = -0.026346 \). As in the MG case, the correction factor \( (1 - e^{-B_{SF}/4}) \) in Eq. (19) accounts for, to a first approximation, the use of the Kemble form of the transmission coefficient.

We are now in a position to compare the relative importance of the approximate forms of the Gamow factor, the energy at which they are linearized and the use of Kemble correction to the transmission coefficient. For both current densities, the superscript \( m \) refers to the point of linearization.

We shall first compare the net emission current using a linearization at \( \mathcal{E}_F \) by setting \( \mathcal{E}_m = \mathcal{E}_F \) for a hemispherical emitter with \( h/R_a = 300 \) and \( R_a = 10\mu m \). The errors as before are computed with respect to the exact-WKB result which acts as a natural benchmark. The plot labels refer to the Gamow factor approximation used (MG vs SF), the value of \( \mathcal{E}_m \) and the approximation used for the transmission coefficient. Figure 3 shows the errors in \( J_{MG}^m \) and \( J_{SF}^m \), without the respective correction factors (i.e. using \( e^{-G} \) for the transmission coefficient; thus \( J_{MG}^m \) is \( J_{MG} \) in this case), plotted against the apex field \( E_a \). Clearly linearization at \( \mathcal{E}_m = \mathcal{E}_F \) produces large errors at high fields for both MG and SF. Surprisingly, it affects the shape factor method more. Note that both MG and SF over-predict the net current. Fig. 4 shows a similar comparison with the correction factors in place.
(i.e. Kemble form). The errors reduce at higher fields but MG continues to perform better.

The errors in both MG and SF reduce compared to the results for $E = E_F$. The MG case undergoes a transition from under-prediction to over-prediction (compared to the benchmark) as the field increases while the SF method consistently over-predicts the net current for all values of $E_a$. As in case of the expansion at $E_F$, the MG approximation gives better results which improves further on use of the correction factor due to the Kemble form of transmission coefficient (see Fig. 6). In summary, the use of $J_{MG}^m$ with $E_m = E_F - d_F/t_F$ is found to have errors within 3% of the exact WKB result over a wide range of fields.

IV. DISCUSSIONS AND CONCLUSIONS

The shape-factor $\sigma$, even with the quadratic approximation, is found to give very good results for all local fields, so long as the integration over the electron energies is carried out numerically. The error is found to be below 1% compared to the WKB result using the exact Gamow factor, and decreases at higher fields. When the transmission coefficient is determined using the Forbes approximation for the barrier form correction factor, the errors are much larger but improves at higher fields. In contrast, a linearization of the Gamow factor using the quadratic approximation for the shape-factor, leads to errors that are somewhat larger compared to the Murphy-Good-Forbes approach. Thus, if a reasonably accurate analytic form for the cold field emission current density is required, the current density $J_{MG}^m$ as in Eq. (15) with $E_m = E_F - d_F/t_F$, corresponding approximately to the peak of the normal energy distribution, is suitable for its accuracy and relative ease of use. The ease of use can be further improved by ignoring the correction factor provided the fields involved are not too high. Note that the errors on using $J_{SF}^m$ with $E_m = E_F - d_F/t_F$, is only marginally higher and hence there is very little to choose between the two except for a simpler expression for $J_{MG}^m$.

It is possible that the average error may reduce further on choosing another point of linearization or improving upon the algebraic approximations for $\nu(y)$ and $\sigma(y)$ considered here. Such improvements would obviously lead to more involved expressions for the current density making them more cumbersome to use for analyzing experimental data. It must also be noted that the benchmark chosen here is the exact-WKB method and errors may be quite different, and even large compared to the errors presented here, if the exact current is instead chosen for comparison. Such errors are found to follow an approximate trend and can be minimized (though not eliminated) by using a correction factor dependent on $E_F/\phi$ (as shown in Ref [13]) along with $J_{MG}^m$.

Finally, since a shifted point of linearization and the use of the Kemble correction are found to be important in reducing errors, the notation $J_{MG}^m$ may be reserved for the current density given by Eqns. [15] and [16] for
\( \mathcal{E}_m \neq \mathcal{E}_F \), in order to distinguish it from the standard MGCD denoted by \( J_{\text{MG}} \), which corresponds to \( \mathcal{E}_m = \mathcal{E}_F \) and does not have any Kemble-correction factor.

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VI. AUTHOR DECLARATIONS

A. Conflict of interest

There is no conflict of interest to disclose.

B. Data Availability

The data that supports the findings of this study are available within the article.

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