REAL TIME ANALYSIS OF THERMAL ACTIVATION VIA SPHALERON TRANSITIONS

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Abstract

We study the process of thermal activation mediated by sphaleron transitions by analyzing the real-time dynamics of the decay out of equilibrium in a 1 + 1 dimensional field theory with a metastable state. The situation considered is that of a rapid supercooling in which the system is trapped in a metastable state at a temperature larger than the mass of the quanta, but smaller than the energy to create a critical droplet. The initial density matrix is evolved in time and the nucleation rate (probability current at the saddle point) is computed. The nucleation rate is time dependent, vanishing at early times, reaching a maximum at a time $t \approx 1/m$ with $m$ the mass of quanta in the metastable state, and decreasing at long times as a consequence of unitarity. An estimate for the average number of particles of “true vacuum” produced as a function of time during the nucleation process is obtained.
1 Introduction and Motivation

Thermal activation plays a very important role in the dynamics of evolution out of equilibrium of thermodynamically metastable states. Such a situation arises for example in first order phase transitions which may have occurred in the early universe\textsuperscript{[1, 2, 3, 4]}, where false vacuum decay provides a mechanism for ending inflation. More recently thermal activation has been conjectured to be the mechanism responsible for unsuppressed baryon number violation at high temperatures\textsuperscript{[5, 6, 7]}. Within the context of baryon number violation, Kuzmin, Rubakov and Shaposnikov\textsuperscript{[5]}, argued that in non-Abelian gauge theories topologically different vacua (characterized by different Chern-Simons number) are separated by energy barriers and that there are configurations (in functional space) that are responsible for over the barrier transitions from one vacuum to another. Klinkhamer and Manton\textsuperscript{[7]} suggested that these over-the-barrier transitions correspond to static configurations that extremize the energy functional with at least one unstable mode. These authors named this configuration the “sphaleron” and provided a topological proof for the existence of such configurations in the Standard Model. In non-Abelian gauge theories this configuration carries half a unit of winding number, and corresponds to a saddle point in the space of functions. These transitions are necessarily accompanied by a change in baryon number\textsuperscript{[8, 9, 10, 11]}.

Within the context of first order phase transitions at finite temperature, Linde\textsuperscript{[12]} argued that the relevant configuration responsible for the decay of the metastable state is an $O(3)$ (static) “bounce”, which is in a sense the high temperature limit of the zero temperature bounce that describes a tunneling event\textsuperscript{[13]} between the metastable and stable phases.

An important quantity is the transition rate (per unit volume), as this quantity determines the relevant time scales for the completion of first order phase transitions in inflationary cosmologies and for the rate of baryon number violation in gauge theories at finite temperature.

The usual calculation of the rate follows Langer’s original theory of homogeneous nucleation in classical statistical mechanics\textsuperscript{[14]}, and Affleck’s\textsuperscript{[15]} generalization to the quantum case.

Langer’s treatment is based on the computation of the probability current at the saddle point corresponding to a steady state solution of an appropriate Fokker-Planck equation. It is assumed in this treatment, and clearly spelled out in Langer’s calculation, that the steady state condition corresponds to the situation in which the metastable phase is replenished at the same rate at which probability is flowing out across the saddle. There is a “source” of probability on the metastable well and a corresponding “sink” on the stable side. The
saddle point configuration corresponds to a “critical droplet”. The probability flows from the metastable to the stable phase along the unstable direction in functional space. Under the steady-state assumption the rate is calculated in equilibrium and identified to be proportional to the imaginary part of an analytically continued free energy. The same assumptions are implicit in Affleck’s approach in which the transition rate is computed as an equilibrium average. As a result of the steady-state assumption, the rate is independent of time and basically determined by an Arrhenius (thermal activation) equilibrium Boltzmann factor.

Langer’s treatment of nucleation has been recently generalized to relativistic field theory (under basically the same assumption of a steady state) by Csernai and Kapusta. The “critical droplets” in Langer’s theory of homogeneous nucleation are thus identified with the “sphalerons” in the broader sense studied by Manton and Samols, i.e., static field configurations with (at least) one unstable mode. These are saddle points of the energy functional corresponding to field configurations “sitting” at the top of the energy barrier (in functional space). Numerical simulations of sphaleron transitions have been carried out by Hellmund and Kripfganz and Ambjorn et al. These authors integrate numerically the classical equations of motion by starting from a sphaleron configuration. Recently a numerical study of the process of thermal activation assuming a simple Langevin description has been reported.

The motivation of this article is to provide a different approach to the study of the decay of a metastable state by considering the process of thermal activation through “sphaleron transitions” as a non equilibrium evolution in real time.

We consider the situation of a 1 + 1 dimensional scalar field theory in which a metastable state has been formed after a period of rapid supercooling and decays via the formation of “critical droplets” and their subsequent evolution. We propose to describe the supercooled metastable state in terms of a functional density matrix corresponding to a thermal distribution of free-field quanta (harmonic oscillator states) of the false vacuum. The temperature of the ensemble is assumed to be much larger than the mass of the quanta but smaller than the energy to create a sphaleron configuration. The “sphaleron” determines an unstable direction in functional space along which probability flows. We solve for the real time evolution of the initial density matrix along this unstable direction. The transition rate is identified with the probability current flowing across the saddle from the metastable to the stable state. This approach provides a description of the decay process out of equilibrium from an initial state. We find that the transition rate is time dependent and that it strongly depends on the initial state.

In section II we study the properties of the “sphaleron” and quantize the theory around
this static semiclassical configuration. Section III presents a detailed analysis of the time evolution of the field theoretical density matrix and the probability current along the unstable direction on the saddle point is evaluated. Following Langer’s original treatment, this probability current is identified with the transition rate. In this section we also mention some recent experiments on nucleation in classical fluids that report a time dependent rate, and dependence on initial conditions.

In section IV we present an approximate calculation for the production of particles of the true vacuum during the decay process. Section V summarizes our conclusions, establishes the range of validity of our results, and explores potential implications.

Two appendices contain relevant technical details.

## 2 The Sphaleron and Thermal Activation:

We consider a scalar field theory in 1 + 1 space-time dimensions with Hamiltonian

\[
H = \int dx \left\{ \frac{1}{2} \Pi^2(x) + \left( \frac{d\Phi(x)}{dx} \right)^2 + V(\Phi(x)) \right\}
\]

where

\[
V(\Phi) = \lambda (\Phi - \Phi_-)^2(\Phi - \Phi^*)
\]

The potential is depicted in figure 1(a). The point \(\Phi^*\) is parametrized by the mass \(m\) of harmonic oscillations around the minimum \(\Phi_-\) as

\[
\Phi^* = \Phi_- \left[ 1 - \frac{m^2}{2\lambda \Phi^2} \right]
\]

In two space time dimensions, the scalar field is dimensionless and the coupling constant has dimensions of \((mass)^2\).

The “sphaleron” is defined as a static field configuration that corresponds to an extremum of the energy functional with at least one unstable direction in functional space. In our case, a stationary solution of the equations of motion (extremum of the energy functional) satisfies

\[
-\frac{d^2\Phi(x)}{dx^2} + \frac{\partial V(\Phi)}{\partial \Phi} = 0
\]

This equation resembles that of a particle moving in “time” (labeled by \(x\)) down a potential \(-V\). The solution that starts at \(\Phi_-\) at \(x \to -\infty\) and returns to \(\Phi_-\) as \(x \to \infty\) is easily found to be

\[
\Phi_{sph} = \Phi_- + \frac{m}{2\sqrt{2\lambda}} \left[ \tanh\left[ \frac{m}{2}(x - x_o) + s_o \right] - \tanh\left[ \frac{m}{2}(x - x_o) - s_o \right] \right]
\]
\[ s_o = \frac{1}{2} \cosh^{-1} \left[ \frac{\epsilon + 1}{\epsilon - 1} \right] \]  
where \( x_o \) is an arbitrary integration constant and reflects the translational invariance of the equations of motion. This solution corresponds to a kink-antikink pair or “droplet” with a “radius” \( 2s_o/m \), and is similar to a “polaron solution” found in quasi-one dimensional polymers\[23\]. This solution and its stability was also studied by Baltar et. al.\[24\].

The dimensionless ratio \( \epsilon \) is a measure of the depth of the global minimum (\( \Phi_+ \)). As \( \epsilon \to 1 \) the minima of the potential become degenerate and \( s_o \to \infty \). The situation \( \epsilon \approx 1; \ s_o \gg 1 \) corresponds to a “thin-wall” droplet for which the radius is much larger than the wall thickness or “skin” of the one-dimensional droplet \( \xi = 2/m \), as depicted in figure 2. For a “thin-wall” droplet we find the energy of the sphaleron configuration to be

\[ E_{sph} = \frac{m^3}{6\lambda} + \mathcal{O}(\epsilon^{-2s_o}) \]  
where we have shown that the corrections are exponentially small in the “thin-wall” limit.

It is convenient to expand the field and its canonical momentum around the sphaleron configuration as

\[ \Phi(x) = \Phi_{sph}(x - x_o) + \sum_l \phi_l^{[x_o]}(x - x_o) \]  
\[ \Pi(x) = \sum_l \pi_l^{[x_o]}(x - x_o) \]  
\[ [\pi_l^{[x_o]}, \phi_{l'}^{[x_o]}] = -i\hbar \delta_{l,l'} \]  
with the \( f_l(x - x_o) \) being the eigenfunctions of the operator of quadratic fluctuations around the sphaleron configuration

\[ \left[ -\frac{d^2}{dx^2} + V''(\Phi_{sph}(x - x_o)) \right] f_l(x - x_o) = \omega_l^2 f_l(x - x_o) \]  
chosen to be real and orthonormal (in a volume L). We have explicitly written the dependence of the eigenfunctions on \( x_o \) as a consequence of translational invariance, and indicated that the operators \( \phi_l, \pi_l \) depend parametrically on \( x_o \). This will become important later when we introduce collective coordinates to treat translational invariance.

Translational invariance guarantees that there is a zero frequency mode whose normalized eigenfunction is given by

\[ f_0(x - x_o) = \frac{1}{\sqrt{E_{sph}}} \frac{d\Phi_{sph}(x - x_o)}{dx} \]
This eigenfunction is parity odd and has one node, and in the thin-wall limit it is identified as the antisymmetric linear combination of the zero modes for kink and anti-kink. There is an orthogonal symmetric combination that is parity even and nodeless corresponding to the orthogonal linear combination, which is the eigenfunction with the lowest eigenvalue. In the thin-wall limit we find it to be

\[ f_{-1} = \frac{m}{2\sqrt{E_{sp}} \ ds_o} d\Phi_{sp}(x - x_o, s_o) \]

(14)

This mode has a negative eigenvalue \( \omega_{-1}^2 = -\Omega^2 \) (\( \Omega > 0 \)), in the “thin-wall” limit this eigenvalue is exponentially small, determined by the overlap integral of the kink-antikink zero modes \( \Omega^2 \approx O(e^{-2 s_o}) \). In this approximation, and based on the known results on bound-states of the one kink case, we conclude that the rest of the spectrum is positive definite.

The Hamiltonian in this basis, to be referred to as the “sphaleron basis”, becomes

\[ H = E_{sp} + H_q + H_I \]

(15)

\[ H_q = \frac{\pi_0^2}{2} - \frac{\pi_{-1}^2}{2} - \frac{\Omega^2}{2} \phi_{-1} + \frac{1}{2} \sum_{l \geq 1} [\pi_l^2 + \omega_l^2 \phi_l] \]

(16)

where \( H_I \), has terms cubic and quartic in terms of the \( \phi_l \) and we have suppressed the upper index \([x_o]\) in both the \( \pi_l ; \phi_l \) to avoid clutttering of notation, but keeping in mind that these operators depend parametrically on \((x_o)\).

The nature of the instability represented by the mode \( l = -1 \) is physically clear. For small amplitudes of \( \phi_0 \); \( \phi_{-1} \) the field may be written as

\[ \Phi(x) \approx \Phi_{sp}[x - (x_o + \delta x_o); (s_o + \delta s_o)] + \sum_{l \geq 1} f_l (x - x_o) \phi_l \]

(17)

\[ \delta x_o = \frac{\phi_0}{\sqrt{E_{sp}}} \ ; \ \delta s_o = \frac{m}{2\sqrt{E_{sp}}} \phi_{-1} \]

(18)

Whereas \( \phi_0 \) represents a translation of the center of mass of the sphaleron configuration, \( \phi_{-1} \) represents an expansion or contraction of the radius of the droplet. Thus we identify the collective coordinate that describes the radius of the droplet (in this approximation) as

\[ s = s_o + \frac{m}{2\sqrt{E_{sp}}} \phi_{-1} \]

(19)

The coordinate \( \phi_{-1} \) measures the departure from the critical droplet (for small amplitudes of this coordinate).
When the droplet expands, the configuration gains volume energy because it is sampling a larger region in space with a lower energy density. The gain in volume energy grows linearly (asymptotically) in one space dimension.

The gradient terms, giving rise to a surface energy of the droplet, saturate at about twice the kink mass in one space dimension. To see this clearly, let us define a field configuration (droplet) parametrized by the radius \( s \), as

\[
\Phi_{\text{drop}}(x - x_\circ, s) = \Phi_{\text{sph}}(x - x_\circ, s_\circ = s)
\]

and use \( s \) as a parameter. The (classical) energy density as a function of this parameter \( E(x, s) \) is depicted in figure 3 (a, b, c) for \( s < s_\circ ; s = s_\circ ; s > s_\circ \) respectively, for the case of the potential with a metastable minimum (figure 1(a)).

The total (classical) energy \( E[s] \) is depicted in figure (4) for the potentials shown in figure (1). The maximum of \( E[s] \) for the case of a potential with a metastable minimum (figure 5) is given by \( s_\circ \) given by equation (6), as this is the value that corresponds to the solution of the equation of motion for the static configuration, that is, the “sphaleron”. The value \( s_\circ \) corresponds to a “critical droplet”, for \( s < s_\circ \) the droplet will shrink, as the cost in elastic surface term is greater than the gain in volume energy. A droplet with \( s = s_\circ \) is in unstable equilibrium, whereas for \( s > s_\circ \) the droplet will grow as the gain in volume energy offsets the cost of elastic surface (wall) energy.

The sphaleron configuration corresponds to a saddle point in functional space; there is one unstable direction corresponding to the dilation or contraction of the droplet, one flat direction corresponding to translational invariance, and the remaining (infinite) directions are all stable, with the energy increasing quadratically for small amplitudes away from the saddle.

This unstable direction in functional space plays a fundamental role: consider an initial quantum mechanical state localized in the metastable minimum, for example either a ground state wave functional or a thermal density matrix for the quadratic potential centered on the metastable minimum (see figure 1(a)). If the probability for finding a droplet with a size greater than the critical size in the initial state is non-zero, the time evolution of this state will inexorably tend to spread the state in such a way that field configurations which will sample the global minimum in larger regions in space will acquire larger probabilities as time evolves. This translates into the notion that the time evolved state will give rise to a probability current at the saddle along the unstable direction towards field configurations corresponding to growing droplets. The “decay rate” of the initial state (or ensemble) will be determined by the total probability current along this direction in functional space. Furthermore, by
translational invariance, these droplets appear with equal probability at all points in space, and the total current passing through the saddle point along the unstable direction will be proportional to the volume.

At finite temperature, there will be a non-zero probability of finding a critical (and larger) droplet in the initial ensemble and the instability will take the initial ensemble away from equilibrium.

The “decay” of the initial ensemble will thus correspond to a process of thermal activation, in which the initial state is driven “over the barrier” in functional space. This barrier should not be confused with the hump in the scalar potential, but it must be identified with the maximum of $E[s]$ in figure 5 for the metastable case, that is to say, the energy of a critical droplet.

We now study this process of thermal activation in real time by following the time evolution of an initially prepared density matrix.

3 Real Time evolution of Initial Ensembles:

We consider the situation in which there is a very rapid supercooling from a high temperature phase in thermal equilibrium to a situation in which the system gets trapped in a metastable state. This corresponds to the case in which the initial ensemble is described by a density matrix in thermal equilibrium, but “centered” at the metastable minimum of the potential. We approximate this initial density matrix as that of a free field theory centered at the metastable minimum and with particles with mass $m$ determined by the second derivative of the potential at $\Phi_-$. In the Schroedinger representation the density matrix elements are given by

$$\rho(\Phi, \Phi', t = 0) = N \exp \left\{ -\frac{1}{2} \int dx \int dy [(\Phi(x) - \Phi_-)K_1(x - y)(\Phi(y) - \Phi_-) + (\Phi'(x) - \Phi_-)K_1(x - y)(\Phi'(y) - \Phi_-) - 2(\Phi(x) - \Phi_-)K_2(x - y)(\Phi'(y) - \Phi_-)] \right\}$$

$$K_1(x - y) = \int \frac{dk}{2\pi} e^{-ik(x-y)}K_1(k) ; \quad K_1(k) = \left[ \frac{\omega_k \cosh[\beta \hbar \omega_k]}{\hbar \sinh[\beta \hbar \omega_k]} \right]$$

$$K_2(x - y) = \int \frac{dk}{2\pi} e^{-ik(x-y)}K_2(k) ; \quad K_2(k) = \left[ \frac{\omega_k}{\hbar \sinh[\beta \hbar \omega_k]} \right]$$

$$N = \left( \text{Det} \left[ \frac{K_1 - K_2}{\pi} \right] \right)^{1/2} ; \quad \omega_k = \sqrt{k^2 + m^2}$$

8
We have chosen the normalization factor in such a way that

\[ \text{Tr} \rho = \int \mathcal{D}\Phi \rho(\Phi, \Phi) = 1 \]  

(25)

With this normalization, the quantity \( \rho(\Phi, \Phi) \) has the interpretation of a probability density in functional space.

In terms of the Fourier components of the field \( \Phi \), the above density matrix is easily identified with a density matrix in the coordinate representation for a collection of independent harmonic oscillators with frequencies \( \omega_k \).

In particular, a meaningful question is: what is the probability (density) to find the sphaleron configuration in the initial ensemble? This quantity is easily calculated in the high temperature limit where the Fourier transform of \( (\Phi_{sph}(x) - \Phi) \) is non-negligible only for momenta \( k \approx m \). Thus in the convolution of the sphaleron configuration and the kernels, for temperatures \( T \gg m \), the kernels in the density matrix may be approximated by

\[ K_1(k) - K_2(k) = \frac{\omega_k}{\hbar} \tanh \left[ \frac{\hbar \omega_k}{2k_B T} \right] \approx \frac{\omega_k^2}{2k_B T} \]  

(26)

after some straightforward algebra we find in the “thin-wall” limit

\[ \rho(\Phi_{sph}, \Phi_{sph}) \approx N \exp \left[ -\frac{E_{sph}}{2k_B T}(1 + 12s_o) \right] \]  

(27)

with \( s_o \) the radius of the droplet.

Then we see that the probability of finding the “sphaleron” configuration in the initial ensemble is very different from \( \exp[-E_{sph}/k_B T] \). The latter corresponds to a density matrix of the form \( \rho_0 = \exp[-(E_{sph} + H_q)/k_B T] \), with \( H_q \) the Hamiltonian \( H_0 \). This density matrix represents an ensemble “centered” at the sphaleron configuration.

We can also compute the “average radius” of the droplets in the initial state. We find (see appendix B)

\[ \langle \phi \rangle \approx \frac{3}{m} \sqrt{E_{sph}} \]  

(28)

\[ \langle s \rangle \approx s_o - \frac{3}{2} \]  

(29)

in the thin wall limit \( s_o \gg 1 \) most of the “droplets” in the initial ensemble are slightly smaller than the critical droplet. This is clearly in agreement with the condition that our initial ensemble is describing the metastable phase and the field configurations that sample large regions of the stable phase are very rare.
The time evolution of the initial density matrix (21) is easily obtained in the quadratic approximation for (16), that is setting $H_I = 0$. The reason is that both the initial density matrix and the Hamiltonian (around the sphaleron configuration) are quadratic, thus at all times the density matrix will be of the gaussian form. By expanding the field in the sphaleron basis and defining

$$\Phi_\pm - \Phi_{sph}(x - x_o) = \sum_l f_l(x - x_o) \bar{\phi}_l \quad ; \quad \bar{\phi}_0 = 0$$  \hspace{1cm} (30)

the initial density matrix becomes

$$\rho(\phi, \phi', t = 0) = N \exp \left\{ -\frac{1}{2} \left[ (K_1)_{l,l'} (\eta_l \eta_{l'} + \eta'_l \eta'_{l'}) - 2(K_2)_{l,l'} \eta_l \eta'_{l'} \right] \right\}$$  \hspace{1cm} (31)

$$(K_1)_{l,l'} = \int dx \int dy \int \frac{dk}{2\pi} e^{-ikx} K_1(k) f_l(x - x_o) f_{l'}(y - x_o)$$  \hspace{1cm} (32)

$$(K_2)_{l,l'} = \int dx \int dy \int \frac{dk}{2\pi} e^{-ikx} K_2(k) f_l(x - x_o) f_{l'}(y - x_o)$$  \hspace{1cm} (33)

$$\eta_l = (\phi_l - \bar{\phi}_l) \quad ; \quad \eta'_l = (\phi'_l - \bar{\phi}_l)$$  \hspace{1cm} (34)

Thus we see that in the sphaleron basis, the different modes are mixed in the initial density matrix.

The time evolution of the density matrix is formally determined by the Liouville equation

$$i\hbar \frac{\partial \rho(\Phi, \Phi', t)}{\partial t} = [H, \rho(\Phi, \Phi', t)]$$ \hspace{1cm} (35)

whose formal solution is

$$\rho(\Phi, \Phi', t) = e^{-\frac{i}{\hbar} Ht} \rho(\Phi, \Phi', 0) e^{\frac{i}{\hbar} Ht}$$  \hspace{1cm} (36)

In the Schroedinger representation with $\Pi(x) = -i\hbar \frac{\delta}{\delta \Phi(x)}$ it is straightforward to see that the functional probability density obeys a continuity equation

$$\frac{\partial \rho(\Phi, \Phi', t)}{\partial t} = -\int dx \frac{\delta J[\Phi(x), t]}{\delta \Phi(x)}$$  \hspace{1cm} (37)

$$J[\Phi(x), t] = -\frac{i\hbar}{2} \left( \frac{\delta}{\delta \Phi(x)} - \frac{\delta}{\delta \Phi'(x)} \right) \rho(\Phi, \Phi', t)|_{\Phi = \Phi'}$$  \hspace{1cm} (38)

In the “sphaleron basis”, the above continuity equation becomes

$$\frac{\partial \rho(\Phi, \Phi', t)}{\partial t} = -\sum_l \frac{\delta J_l[\phi, t]}{\delta \phi_l}$$  \hspace{1cm} (39)

$$J_l[\phi, t] = -\frac{i\hbar}{2} \left( \frac{\delta}{\delta \phi_l} - \frac{\delta}{\delta \phi'_l} \right) \rho(\phi, \phi', t)|_{\phi = \phi'}$$  \hspace{1cm} (40)
we have suppressed the label \( (x_o) \), but we must keep in mind (and this will become relevant later) that the \( \phi_l \) depend on it (though the \( \bar{\phi}_l \) do not).

Following Langer\[14\], we now identify the transition (nucleation) rate with the total probability current flowing across the saddle point along the unstable direction (since the initial density matrix has been normalized to one), i.e, the rate is given by

\[
\Gamma = \int_{0}^{0} d\phi_{-1} \int_{-\infty}^{\infty} \prod_{l \neq -1} d\phi_{l} \frac{\partial \rho(\phi, \phi, t)}{\partial t} = -\int_{-\infty}^{\infty} J[\phi_{-1} = 0; \phi_{l}] \prod_{l \neq -1} d\phi_{l}
\]

(41)

The integral over the \( l \neq -1 \) modes (path integral) corresponds to the trace of the time dependent density matrix in the reduced functional space perpendicular to the unstable mode.

Although we can formally find the time evolution of the density matrix by solving the Liouville equation with the Hamiltonian to quadratic order in the sphaleron basis, the resulting equations of motion mix the different modes and become very difficult to solve.

It becomes clear from the above expression that, in order to compute (41), we need to compute the “reduced density matrix”

\[
\rho_r(\eta_{-1}, \eta'_{-1}, t) = Tr_{l \neq -1} \left( e^{-\hat{\kappa}H_{l}^t} \rho(0) e^{\hat{\kappa}H_{l}^t} \right)
\]

(42)

Because in the quadratic approximation around the sphaleron solution, the Hamiltonian is a sum of mutually commuting Hamiltonians for each \( l \), it is clear that the above trace and thus the reduced density matrix evolves in time only through \( H_{-1} \)

\[
\rho_r(t) = e^{-\frac{1}{\hbar}H_{-1}t} \rho_r(0) e^{\frac{1}{\hbar}H_{-1}t}
\]

(43)

In terms of the variable \( \eta_l = \phi_l - \bar{\phi}_l \), the reduced density matrix initially is

\[
\rho_r(\eta_{-1}, \eta'_{-1}, 0) = Tr' \rho(0) = \int_{-\infty}^{\infty} \prod_{l \neq -1} d\eta_l \rho(\eta_{-1}, \eta; \eta'_{-1}, \eta_l)
\]

(44)

The time evolution of the reduced density matrix will be solely determined by the Liouville equation with the Hamiltonian corresponding to the unstable mode. Clearly, this is a consequence of the quadratic approximation. Thus, by taking the trace (integrating) over all functional directions perpendicular to the unstable mode and obtaining a reduced density matrix for the unstable coordinate, we have cast the multidimensional problem in terms of only one quantum mechanical degree of freedom corresponding to the collective coordinate representing the “radius of the droplet”. Now the problem becomes very similar to the quantum mechanical example studied recently\[25\]. It is at this stage that we recognize that by introducing a basis which depends on the center of mass of the sphaleron \( (x_o) \), we are not
treating translational invariance properly. For each \((x_o)\), the chosen basis corresponds to an orthonormal coordinate system in functional space. Changing \((x_o)\) amounts to performing a linear transformation on the coordinates \(\phi_l^{[x_o]}\) (see appendix A), thus, by integrating over all the possible values of the coordinates \(\phi_l^{[x_o]}\) one recovers translational invariance. However, one of the coordinates is only integrated up to the saddle point and translational invariance is not explicit in the above expression for the rate. To remedy this situation, it is convenient at this point to perform a non-linear transformation and treat \((x_o)\) as a collective coordinate. This is achieved by going from the (cartesian) coordinate system \(\{\phi_0, \phi_l \neq 0\}\) to new (curvilinear) coordinates \(\{x_o, \phi_l \neq 0\}\). There is a Jacobian associated with this transformation. To leading semiclassical order it is given by (see appendix A)

\[
J = \sqrt{E_{sph} + \mathcal{O}(\phi)}
\]  

(45)

The collective coordinate is introduced into the trace (functional integral) as\[27, 28\] (see appendix A)

\[
\int d\phi_0^{[x_o]} \prod_{l \neq 0} d\phi_l = \int d(x_o) \delta(\phi_0^{[x_o]}_0) J d\phi_0 \prod_{l \neq 0} d\phi_l
\]  

(46)

Postponing the integration over the collective coordinate to the end of the calculation, the trace over the \(l \neq -1\) modes can be performed easily and we find the reduced density matrix for the unstable coordinate to leading semiclassical order

\[
\rho_r(\eta, \eta') = \mathcal{N} \exp\left\{-\frac{1}{2\hbar} [\alpha(\eta^2 + \eta'^2) + 2\gamma \eta \eta']\right\}
\]  

(47)

\[
\frac{\alpha}{\hbar} = (K_1)_{-1,-1} - \frac{1}{2} \tilde{Q}^T \tilde{K}^{-1} \tilde{Q}
\]  

(48)

\[
\frac{\gamma}{\hbar} = -(K_2)_{-1,-1} - \frac{1}{2} \tilde{Q}^T \tilde{K}^{-1} \tilde{Q}
\]  

(49)

\[
Q_l = (K_1 - K_2)_{l,-1} ; \ l \neq 0, -1
\]  

(50)

\[
\tilde{K}_{l,l'} = (K_1 - K_2)_{l,l'} ; \ l, l' \neq 0, -1
\]  

(51)

\[
\mathcal{N} = \sqrt{E_{sph} \left( \frac{\text{Det}[K_{\pi}]}{\text{Det}[\frac{K_{\pi}}{\pi}]} \right)^{\frac{1}{2}}}
\]  

(52)

with \(K = (K_1 - K_2)\) and \(\tilde{K}\) is the same operator but without the row and columns of matrix elements with the zero mode and the unstable mode in the “sphaleron” basis. Since the reduced density matrix evolves in time only through \(H_1\), the continuity equation leads to

(restoring the integral over the collective coordinate)

\[
\int dx_o J \int_{-\infty}^0 d\phi_{-1} \frac{\partial \rho_r(\phi_{-1}, \phi_{-1}, t)}{\partial t} = -\int dx_o J_{-1}[\phi_{-1} = 0, t]
\]  

(53)

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\[ J_{-1}[\phi_{-1}, t] = -\frac{i\hbar J}{2} \left( \frac{\delta}{\delta\phi_{-1}} - \frac{\delta}{\delta\phi'_{-1}} \right) \rho_r(\phi_{-1}, \phi'_{-1}, t) |_{\phi_{-1} = \phi'_{-1}} \quad (54) \]

The current at the saddle turns out to be translational invariant (independent of \( x_o \)) because it only depends on \( \tilde{\phi}_{-1} \) (which is independent of \( x_o \)) and the integration over the collective coordinate will yield to a volume factor.

The remaining task is to find the time evolution of the reduced density matrix \((\{1\})\) with the Hamiltonian (in the Schroedinger representation)

\[ H_{-1} = \frac{1}{2} \left[ -\hbar^2 \frac{\delta^2}{\delta\phi_{-1}^2} - \Omega^2 \phi_{-1}^2 \right] \quad (55) \]

Since the initial reduced density matrix is quadratic and so is the evolution Hamiltonian, we propose a gaussian ansatz for the time dependent reduced density matrix (the label \( \{ -1 \} \) is not explicitly written for \( \eta \) to avoid cluttering of notation)

\[ \rho_r(\eta, \eta', t) = \mathcal{N}(t) \exp \left\{ -\frac{1}{2\hbar} \left[ \alpha(t)\eta^2(t) + \alpha^*(t)\eta'^2(t) + 2\gamma(t)\eta(t)\eta'(t) \right] + \right. \]
\[ \left. \frac{i}{\hbar} \left[ P(t)\eta(t) - P^*(t)\eta'(t) \right] \right\} \quad (56) \]

\[ \eta(t) = \phi_{-1} - \tilde{\phi}_{-1}(t) ; \quad \eta'(t) = \phi'_{-1} - \tilde{\phi}_{-1}(t) \quad (57) \]

Clearly, \( \tilde{\phi}_{-1}(t) ; P(t) \) are the expectation value of the coordinate and canonical momentum along the unstable direction. The Liouville equation in the Schroedinger representation for the \( \{-1\} \) (unstable coordinate) now reads

\[ i\hbar \frac{\partial \rho_r(\eta, \eta', t)}{\partial t} = \left[ -\frac{\hbar^2}{2} \left( \frac{\delta^2}{\delta\phi_{-1}^2} - \frac{\delta^2}{\delta\phi'_{-1}^2} \right) - \frac{1}{2}\Omega^2 \left( \phi_{-1}^2 - \phi'^{2}_{-1} \right) \right] \rho_r(\eta, \eta', t) \quad (58) \]

The form of the above density matrix is dictated by hermiticity. The time dependence of the parameters \( \mathcal{N}(t), \phi_{-1}(t), P(t) \), is obtained by comparing the left and right hand side of (58) and matching the coefficients of the powers of \( \eta ; \eta' \). We find the following equations

\[ \dot{\mathcal{N}} = \frac{1}{2}(\alpha - \alpha^*) \quad (59) \]
\[ -i\dot{\alpha} = -\alpha^2 + \gamma^2 - \Omega^2 \quad (60) \]
\[ -i\dot{\alpha}^* = \alpha^2 - \gamma^2 + \Omega^2 \quad (61) \]
\[ i\dot{\gamma} = \gamma(\alpha - \alpha^*) \quad (62) \]
\[ \dot{P} = \Omega^2 \phi_{-1} ; \quad P = P^* \quad (63) \]
\[ \dot{\phi}_{-1} = P \quad (64) \]
The last two equations are the classical equations of motion for the unstable coordinate in an inverted harmonic oscillator. The boundary conditions are:

\[ P(0) = 0; \quad \bar{\phi}_{-1}(0) = \bar{\phi}_{-1}; \quad N(0) = N; \quad \alpha(0) = \alpha^*(0) = \alpha; \quad \gamma(0) = \gamma \]

In terms of the real and imaginary parts of \( \alpha \) we find the solution to the above equations to be

\[
\alpha_I(t) = -\frac{\Omega \sinh[2\Omega t]}{\cosh[2\Omega t] - \cos(2\delta)} \quad (65)
\]

\[
\alpha_R(t) = \alpha(0) \left[ \frac{1 - \cos(2\delta)}{\cosh[2\Omega t] - \cos(2\delta)} \right] \quad (66)
\]

\[
\gamma(t) = \gamma(0) \left[ \frac{1 - \cos(2\delta)}{\cosh[2\Omega t] - \cos(2\delta)} \right] \quad (67)
\]

\[
N(t) = N(0) \left[ \frac{1 - \cos(2\delta)}{\cosh[2\Omega t] - \cos(2\delta)} \right]^{\frac{1}{2}} \quad (68)
\]

\[
\bar{\phi}_{-1}(t) = \bar{\phi}_{-1}(0) \cosh[\Omega t] \quad (69)
\]

\[
\tan \delta = \frac{\Omega}{\sqrt{\alpha_R^2(0) - \gamma^2(0)}} = \frac{\Omega}{W} \quad (70)
\]

We obtain the expression for the rate per unit volume

\[
\frac{\Gamma(t)}{L} = J[\bar{\phi}_{-1}, t] \quad (71)
\]

\[
\frac{\Gamma(t)}{L} = -\Omega \bar{\phi}_{-1}(0) A(t) \sqrt{E_{esp}N(0)} \exp \left\{ -\frac{1}{\hbar} \bar{\phi}_{-1}(0)(\alpha_R(0) + \gamma(0)) B(t) \right\} \quad (72)
\]

\[
A(t) = \frac{W^2}{\Omega^2} \frac{\sinh[\Omega t]}{\cosh^2[\Omega t] + \frac{W^2}{\Omega^2} \sinh^2[\Omega t]} \quad (73)
\]

\[
B(t) = \frac{1}{\left[ 1 + \frac{W^2}{\Omega^2} \tanh^2[\Omega t] \right]} \quad (74)
\]

There are two competing effects that lead to the final expression for the prefactor \( A(t) \) in the rate. The first corresponds to the “rolling” of the expectation value of the unstable coordinate \( (\bar{\phi}_{-1}) \) down the inverted quadratic potential, the canonical momentum of this mode contributes to the prefactor a term proportional to \( \sinh[\Omega t] \). The second contribution has its origin in the spread of the probability distribution function (width of the gaussian) associated with the growth of the unstable fluctuations contributing typical factors \( \sinh[2\Omega t] \). The width of the reduced gaussian density matrix gives the two-point correlation function of the fluctuation of the unstable mode. This correlation function grows exponentially because
of the instability. Thus we see that the spread of the probability distribution is the dominant term in the time dependent rate. This observation is in agreement with the arguments of classical homogeneous nucleation in that the fluctuations are the main responsible for the growth of droplets and the decay of the metastable state.

In order to find a more compact expression for the rate we need to obtain the coefficients \( \alpha(0) ; \gamma(0) ; \tilde{\phi}(0) \). In appendix B we find a simple expression for these coefficients and compute them in the high temperature limit and in the “thin-wall” approximation. Finally our expression for the rate is

\[
\Gamma(t) = \frac{3\Omega}{L} \left( \frac{E_{sph} k_B T}{\pi m \hbar^2} \right) DA(t) \exp \left\{ -\frac{C E_{sph}}{k_B T} B(t) \right\}
\]

with \( C \approx 5.1759 \) (see appendix B) and where we have absorbed temperature and \( \hbar \) factors in the kernels (making them dimensionless) and defined

\[
D = \left[ \frac{\text{Det}(D)}{\text{Det}(D'')} \right]^{\frac{1}{2}}
\]

\[
D' = \prod_k [\beta \hbar \omega_k] \tanh \left( \frac{\beta \hbar \omega_k}{2} \right)
\]

and \( D'' \) the same operator in the sphaleron basis but without the rows and columns corresponding to the zero mode and the unstable mode. The rate as a function of time is depicted in figure (6) for an arbitrary choice of the parameters. We see that the rate vanishes at \( t = 0 \), reaches a maximum and eventually falls off exponentially to zero at very long times. In the thin wall limit, the maximum occurs at \( t \approx 1/W \approx 1/m \). The reason why the rate vanishes at \( t = 0 \) is because the initial state needs to spread out to reach the saddle and that at the initial time the expectation value of the canonical momentum conjugate to the unstable coordinate vanishes and that the density matrix has real kernels. Thus the current vanishes at \( t = 0 \).

The fact that at very large times the rate must fall off to zero is a consequence of unitary time evolution. The total probability is conserved. As current is passing over the saddle, the probability density in the metastable state is depleted and thus the current must diminish.

This is the main result of this work, a time dependent rate that incorporates the realistic condition of a supercooled non-equilibrium initial state. This result is certainly very different from the familiar approach to decay of metastable states described as steady state processes in thermal equilibrium. We want to emphasize the main differences with previous work on this problem: our expression for the rate obtains from a real time evolution of an initial supercooled state (ensemble). The vanishing of the rate at early times is a consequence of
the zero average canonical momentum and real kernels of the initial ensemble (compatible with local thermodynamic equilibrium), and the vanishing of the rate at late times is an unavoidable consequence of unitary time evolution and the depletion of probability in the initial state.

Certainly, our approximation of keeping the quadratic fluctuations around the sphaleron configuration is not valid at long times as the amplitudes of the fluctuations become very large and the form of the decay rate obtained will not be accurate at long times. However, unitary time evolution and conservation of probability will necessarily constrain the time dependent rate to fall-off to zero at long times.

This clearly illuminates the fact that the familiar results for the decay rate correspond to the very particular situation of a steady state and replenishing of probability in the metastable state.

In a remarkable experiment on nucleation of classical fluids under shear, Min and Goldburg have recently reported[26] a time dependent nucleation rate that is strongly dependent on the initial conditions (shear). The nucleation rate reported by these authors has a qualitative behavior very similar to the time dependent rate (75), vanishing at early times, reaching a maximum and falling off at late times. Although our calculation is clearly quantum mechanical, we conjecture that its classical limit will offer a description of classical statistical systems.

4 Particle Production:

We are now in condition to understand the production of particles of “true vacuum” as the metastable phase decays into the true phase via the process of thermal activation. Consider the operators that create and annihilate particles of momentum \( \mathbf{k} \) of the “true vacuum”, that is the vacuum centered at \( \Phi^+ \). Quantizing in a volume \( L \) these are given by

\[
a_k^\dagger = \frac{1}{\sqrt{2\hbar L}} \int dx e^{ikx} \left[ \sqrt{\omega_k^+} (\Phi(x) - \Phi^+) - \frac{i}{\sqrt{\omega_k^+}} \Pi(x) \right]
\]

(76)

\[
a_k = \frac{1}{\sqrt{2\hbar L}} \int dx e^{-ikx} \left[ \sqrt{\omega_k^+} (\Phi(x) - \Phi^+) + \frac{i}{\sqrt{\omega_k^+}} \Pi(x) \right]
\]

(77)

with \( \omega_k^+ \) the frequencies of harmonic oscillator quanta around the “true vacuum”. Now we can expand the field and its canonical momentum in the “sphaleron” basis by writing

\[
\Phi(x) - \Phi^+ = \Phi(x) - \Phi^- + \Delta \Phi
\]

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Since we expect that the maximum contribution to the rate of particle production will arise from the evolution of the unstable mode, we will only keep this contribution to the creation and annihilation operators. The average number of “true” particles as a function of time is then given by

\[ \langle N_k \rangle(t) = \frac{Tr a_k^\dagger a_k \rho(t)}{Tr \rho(t)} = \frac{Tr a_k^\dagger(t) a_k(t) \rho(0)}{Tr \rho(0)} \]  

(78)

where \( a_k^\dagger(t) ; a_k(t) \) are the operators in the Heisenberg picture with the Hamiltonian [16].

Using, in this approximation

\[ \phi_{-1}(t) = \phi_{-1}(0) \cosh[\Omega t] + \frac{\pi_{-1}(0)}{\Omega} \sinh[\Omega t] \]

\[ \pi_{-1}(t) = \dot{\phi}_{-1}(t) \]

and that in the initial density matrix

\[ \langle \eta_{-1} \rangle(0) = \langle \pi_{-1}(0) \rangle = 0 \]

\[ \langle \eta_{-1}^2 \rangle(0) = \frac{\hbar}{2(\alpha(0) + \gamma(0))} \]

\[ \langle \pi_{-1}^2(0) \rangle = \frac{\hbar}{2(\alpha(0) - \gamma(0))} \]

and using the results obtained in the appendices we find the leading contribution of the unstable evolution to the total number of particles of true vacuum produced as a function of time (we are neglecting time independent and subleading contributions)

\[ \sum_k \langle N_k(t) \rangle \approx \frac{3\pi^2}{4} \left( \frac{k_B T}{\hbar m} \right) \left\{ \cosh^2[\Omega t] \left[ S_1 \left( \frac{m^+}{m} \right) + S_2 \left( \frac{m^+}{m} \right) \left( \frac{9E_{sph}}{k_B T} \right) \right] + \sinh^2[\Omega t] S_3 \left( \alpha(0); m^+; \frac{\Omega}{m} \right) \right\} \]

(79)

The functions \( S_1 ; S_2 ; S_3 \) are determined by the structure factor of the unstable mode \( f_{-1}(x - x_0) \), the unstable frequency \( \Omega \) and the frequencies of the quanta in the true vacuum.

In the thin wall limit the momentum integrals that determine these structure factors are all dominated by a peak at \( k \approx 0 \) of width \( \approx m \). In this limit \( m^+/m^- \approx 1 \); \( \Omega/m \ll 1 \), in this limit these three functions become pure numbers of \( O(1) \).

**Validity of the Approximations:**

Our result for the transition rate for thermal activation relies on several approximations.

The first one corresponds to keeping the quadratic fluctuations around the droplet (“sphaleron”) configuration. This approximation provides the leading semiclassical expression for the rate
and is justified at early and intermediate times, since the fluctuations of the unstable coordinate will grow typically as \( \cosh[2\Omega t] \) at early times the amplitude for the fluctuation will grow and presumably higher order terms (cubic and quartic) may have to be kept to fully understand the long time behavior.

The thin wall approximation is justified in the case of a small supercooling (small energy density difference between the metastable and the stable state), this condition may be relaxed in the case of strong supercooling. In this latter case, the “droplet” configuration will be indistinguishable from a localized large amplitude fluctuation.

The initial state condition will be justified in the case of a very rapid supercooling. Since the maximum of the rate occurs at a time \( t \approx 1/m \) (in the thin wall limit) the initial supercooled state condition will be justified if the time during which the supercooling takes place is much smaller than \( 1/m \). Clearly this situation is not general and will have to be understood case by case. In particular in the case of inflationary cosmologies, this inequality results in that the Hubble constant must be much larger than the mass of the quanta in the metastable state. This seems to be the case in the most popular theories on inflation that require a supercooled first order phase transition.

Clearly at low temperatures (much smaller than \( E_{sph} \)) tunneling will be the most important mechanism for metastable decay. Eventually there will be a crossover between tunneling and thermal activation that must be understood better. We must say however that within the gaussian approximation for the fluctuations around the sphaleron, our solution for the time evolution of the density matrix \( \text{does incorporate quantum corrections} \) as may be seen from our result prior to taking the high temperature (classical) limit.

Although we presented the analysis in a 1+1 dimensional field theory, we see no problem in considering our approach in 3+1 dimensions. We are currently extending our results to the three dimensional case and expect to report on it in a forthcoming article.
5 Conclusions and Implications

In this article we have studied the process of decay of a metastable state via thermal activation mediated by “sphaleron” (droplet) configurations. We offered a real time analysis based on the time evolution of an initially prepared density matrix corresponding to a supercooled state. The initial density matrix is assumed to describe the harmonic oscillations of the metastable state at an initial temperature much larger than the mass of the quanta in this state but smaller than the energy of the sphaleron configuration.

We obtain a time dependent nucleation rate. This rate vanishes at early times as a consequence of the quantum mechanical spreading of the density matrix, reaches a maximum at a time $t \approx 1/m$ with $m$ the mass of the quanta in the metastable state and vanishes at long times as a consequence of unitary time evolution. The most important contribution to the rate arises from the fluctuations along the unstable direction in functional space.

This behavior is similar to nucleation rates obtained experimentally recently in classical fluids under shear, but strikingly different from the familiar expressions for the decay rate. The difference with the usual result is that it corresponds to a steady state process in thermal equilibrium in which the metastable state is replenished at the same rate at which probability is flowing across the saddle.

We also provided an estimate for the number of particles of the stable state produced as a function of time during the process of decay of the metastable state.

For applications in the early universe or to describe processes that involve baryon number violation one must understand better the physics of the initial conditions. We believe that a steady state assumption, leading to the familiar result for the rate is not warranted in either case and considering an initially supercooled state may be closer to the physical situation.

Within the context of the inflationary scenario, our results offer a rather pessimistic outlook. It is widely accepted that in order to complete the phase transition a fairly large nucleation rate is required at long times, precisely when unitarity forces the rate to vanish. The actual time at which the rate starts to fall off as as a consequence of unitarity will depend on the details of the potentials, and the amount of supercooling. A deeper understanding of this time regime will require to go beyond the quadratic approximation.

With respect to sphaleron mediated baryon number violating processes, a time dependent rate may have important consequences for obtaining a net baryon asymmetry. In particular, when the thermal activation rate falls below the expansion rate in an expanding cosmology, sphaleron transitions go out of equilibrium. If furthermore the CP violating effects may be accommodated in the theory, the necessary elements for baryon number violation are in
place, but the question must be addressed within the realm of a realistic gauge theory with fermions. The influence of fermions on the dynamics of the droplets has not received much attention but will perhaps be an important ingredient for a deeper understanding of these processes.

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6 Appendix A:

Here we collect some relevant results for the treatment of collective coordinates. From the expansion around the sphaleron solution centered at \( x_o \) one finds

\[
\phi_i^{[x_o]} = \int dx f_i(x - x_o) \left( \Phi(x) - \Phi_{sph}(x - x_o) \right) \tag{80}
\]

\[
\frac{d\phi_i^{[x_o]}}{dx_o} = \int dx f_i(x - x_o) \frac{d\Phi(x)}{dx} = \sqrt{E_{sph}} \delta_{l,0} + \sum_p \int dx f_i(x - x_o) \frac{df_p(x - x_o)}{dx} \phi_p^{[x_o]} \tag{81}
\]

where we have used (13). Treating \( x_o \) as a collective coordinate implies \( \Phi(x) = \Phi_{sph}(x - x_o) + \sum_{l \neq 0} \phi_l f_l(x - x_o) \tag{82} \)

and treating \( \{ x_o ; \phi_l \neq 0 \} \) as coordinates. In the functional integrals (traces) the change over to collective coordinates is done by introducing \( \int dx_o \delta\left(\phi_0^{[x_o]}\right) J \tag{83} \)

\[
J = \left\| \frac{d\phi_0^{[x_o]}}{dx_o} \right\| \approx \sqrt{E_{sph}} + \mathcal{O}(\phi) \tag{84}
\]

20
In this appendix we derive an expression for the coefficients $\alpha(0) ; \gamma(0)$ that enter in the reduced density matrix. From the relations (48,49) we find

$$\frac{1}{\hbar}(\alpha(0) - \gamma(0)) = \int \frac{dk}{2\pi \hbar} \int dx \int dy f_{-1}(x - x_o)f_{-1}(y - x_o)e^{-ik(x-y)} \times \omega_k \left( \cosh[\beta \hbar \omega_k] + 1 \over \sinh[\beta \hbar \omega_k] \right) \tag{85}$$

since the Fourier transform of $f_{-1}(x)$ is localized in k-space, we can use the high temperature expansion of the kernels to find in the high temperature limit $\beta m \ll 1$

$$\frac{1}{\hbar}(\alpha(0) - \gamma(0)) \approx \frac{2k_B T}{\hbar^2} \tag{86}$$

We also need the sum of the coefficients; it is obtained as follows: for a fixed $x_o$ consider the quantity (here $\eta = \eta_{-1}$)

$$\langle \left( \eta_{-1} \right)^2 \rangle = \frac{\int \prod_l d\phi_l(x_o) \rho_l(\phi_l, \phi_l)}{\int \prod_l d\phi_l(x_o) \rho_l(\phi_l, \phi_l)} \tag{87}$$

Where the expectation value is in the initial density matrix. Introducing the collective coordinate as in (46,45), and performing the integrals over all directions perpendicular to $l = -1$, this expectation value can be recast in leading semiclassical order as

$$\int dx_o \int d\phi_{-1} \left( \eta_{-1} \right)^2 \rho_r(\phi_{-1}, \phi_{-1}) = \frac{\hbar}{2(\alpha(0) + \gamma(0))} \tag{88}$$

Thus

$$\frac{\hbar}{2(\alpha(0) + \gamma(0))} = \int dx \int dy \int \frac{dk}{2\pi} f_{-1}(x - x_o)f_{-1}(y - x_o) \frac{\hbar e^{-ik(x-y)}}{2\omega_k \tanh[\beta \hbar \omega_k/2]} \tag{89}$$

Again, the Fourier transform may be calculated in the high temperature limit because the mode functions are localized in k-space; using $\tanh[\beta \hbar \omega_k/2] \approx \beta \hbar \omega_k/2$, we finally find

$$\frac{1}{\hbar}(\alpha(0) + \beta(0)) \approx \frac{m^2}{9k_B T} \left( \frac{3\pi}{2\hbar(s_o)} \right) \tag{90}$$

With $\hbar(s_o)$ a function of only the radius of the droplet. In the thin-wall approximation, we find numerically
\[ C = \left( \frac{3\pi}{2h(s_o \gg 1)} \right) \approx 5.1759 \]

We also find in the thin-wall approximation that

\[ \bar{\phi}_{-1}(0) \approx -\frac{3}{m} \sqrt{E_{sph}} \]  

(91)

\[ \tan \delta \approx \delta \approx \frac{3}{\sqrt{2C}} \frac{\Omega}{m} \]  

(92)
Figure Captions:

**Figure 1:** $V(\Phi)$ vs. $\Phi$ for when $\Phi_-$ corresponds to: a metastable state (a), a degenerate state (b), and the true vacuum state (c).

**Figure 2:** $\Phi_{sph}(x)$ vs $x$ for a “thin-wall” sphaleron (droplet).

**Figure 3:** Energy density as a function of the radius of the sphaleron $\mathcal{E}(x; s)$ in the metastable case for $s < s_o$ (a); $s = s_o$ (b) and $s > s_o$ (c).

**Figure 4:**
Total energy of a droplet $E(s)$ as a function of the “radius” $s$, for the cases in which $\Phi_-$ represents the metastable state (a), degenerate state (b) and true vacuum state (c). (See figure 1).

**Figure 5:**
Total energy of a droplet configuration $E(s)$ as a function of $s$ detailing the maximum at $s = s_o$.

**Figure 6:**
Decay rate per unit volume $\Gamma(t)/L$ as a function of $Wt$ for the arbitrary values $CE_{sph}/k_B T = 2.0$, $\Omega/W = 0.01$. (The constant coefficients in the expression for the rate were set to 1).
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