The Asymptotic Expansion of Kummer Functions for Large Values of the \(a\)-Parameter, and Remarks on a Paper by Olver⋆

Hans VOLKMER

Department of Mathematical Sciences, University of Wisconsin-Milwaukee, P.O. Box 413, Milwaukee, WI, 53201, USA
E-mail: volkmer@uwm.edu

Received January 10, 2016, in final form May 01, 2016; Published online May 06, 2016
http://dx.doi.org/10.3842/SIGMA.2016.046

Abstract. It is shown that a known asymptotic expansion of the Kummer function \(U(a, b, z)\) as \(\alpha\) tends to infinity is valid for \(z\) on the full Riemann surface of the logarithm. A corresponding result is also proved in a more general setting considered by Olver (1956).

Key words: Kummer functions; asymptotic expansions

2010 Mathematics Subject Classification: 33B20; 33C15; 41A60

1 Introduction

Recently, the author collaborated on a project [1] investigating the maximal domain in which an integral addition theorem for the Kummer function \(U(a, b, z)\) due to Magnus [2, 3] is valid. In this work it is important to know the asymptotic expansion of \(U(a, b, z)\) as \(\alpha\) tends to infinity. Such an expansion is well-known, and, for instance, can be found in Slater’s book [8]. Slater’s expansion is in terms of modified Bessel functions \(K_{\nu}(z)\), and it is derived from a paper by Olver [5]. However, there are two problems when we try to use the known result. As Temme [9] pointed out, there is an error in Slater’s expansion. Moreover, in all known results the range of validity for the variable \(z\) is restricted to certain sectors in the \(z\)-plane.

The purpose of this paper is two-fold. Firstly, we correct the error in [8], and we show that the corrected expansion based on [5] agrees with the result in [9] which was obtained in an entirely different way. Secondly, we show that the asymptotic expansion of \(U(a, b, z)\) as \(\alpha\) tends to infinity is valid for \(z\) on the full Riemann surface of the logarithm. This is somewhat surprising because often the range of validity of asymptotic expansions is restricted by Stokes’ lines. Olver’s results in [5] are valid for a more general class of functions (containing confluent hypergeometric functions as a special case.) He introduces a restriction on \(\arg z\), and on [5, p. 76] he writes “In the case of the series with the basis function \(K_{\mu}\) we establish the asymptotic property in the range \(|\arg z| \leq \frac{3}{2}\pi\). It is, in fact, unlikely that the valid range exceeds this . . . .” However, we show in this paper that the restriction \(|\arg z| \leq \frac{3}{2}\pi\) can be removed at least under an additional assumption (2.4).

In Section 2 of this paper we review the results that we need from Olver [5]. We discuss these results in Section 3. In Section 4 we prove that Olver’s asymptotic expansion holds on the full Riemann surface of the logarithm. Sections 5, 6 and 7 deal with extensions to more general values of parameters. In Section 8 we specialize to asymptotic expansions of Kummer functions. In Section 9 we make the connection to Temme [9].

⋆This paper is a contribution to the Special Issue on Orthogonal Polynomials, Special Functions and Applications. The full collection is available at http://www.emis.de/journals/SIGMA/OPSFA2015.html
2 Olver’s work

Olver [5, (7.3)] considers the differential equation

\[ w''(z) = \frac{1}{z} w'(z) + \left( u^2 + \frac{\mu^2 - 1}{z^2} + f(z) \right) w(z). \]  

(2.1)

The function \( f(z) \) is even and analytic in a simply-connected domain \( D \) containing 0. It is assumed that \( \Re \mu \geq 0 \). The goal is to find the asymptotic behavior of solutions of (2.1) as \( 0 < u \to \infty \).

Olver [5, (7.4)] starts with a formal solution to (2.1) of the form

\[ w(z) = z \mathcal{Z}_\mu(uz) \sum_{s=0}^{\infty} \frac{A_s(z)}{u^{2s}} + \frac{z}{u} \mathcal{Z}_{\mu+1}(uz) \sum_{s=0}^{\infty} \frac{B_s(z)}{u^{2s}}, \]

where either \( \mathcal{Z}_\mu = I_\mu, \mathcal{Z}_{\mu+1} = I_{\mu+1} \) or \( \mathcal{Z}_\mu = K_\mu, \mathcal{Z}_{\mu+1} = -K_{\mu+1} \) are modified Bessel functions. The functions \( A_s(z) = A_s(\mu, z), B_s(z) = B_s(\mu, z) \) are defined by \( A_0(z) = 1 \), and then recursively, for \( s \geq 0 \),

\[
2B_s(z) = -A'_s(z) + \int_0^z \left( f(t)A_s(t) - \frac{2\mu + 1}{t} A'_s(t) \right) dt,
\]

(2.2)

\[
2A_{s+1}(z) = \frac{2\mu + 1}{z} B_s(z) - B'_s(z) + \int f(z)B_s(z)dz.
\]

(2.3)

The integral in (2.3) denotes an arbitrary antiderivative of \( f(z)B_s(z) \). The functions \( A_s(z), B_s(z) \) are analytic in \( D \), and they are even and odd, respectively.

If the domain \( D \) is unbounded, Olver [5, p. 77] requires that \( f(z) = O(|z|^{-1-\alpha}) \) as \( |z| \to \infty \), where \( \alpha > 0 \). In our application to the confluent hypergeometric equation in Section 8 the function \( f(z) = z^2 \) does not satisfy this condition. Therefore, throughout this paper, we will take

\[ D = \{ z : |z| < R_0 \}, \]

(2.4)

where \( R_0 \) is a positive constant. Olver [5, p. 77] introduces various subdomains \( D', D_1, D_2 \) of \( D \). We may choose \( D' = \{ z : |z| \leq R \} \), where \( 0 < R < R_0 \). The domain \( D_1 \) comprises those points \( z \) in \( D' \) which can be joined to the origin by a contour which lies in \( D' \) and does not cross either the imaginary axis, or the line through \( z \) parallel to the imaginary axis. For our special \( D' \) the contour can be taken as the line segment connecting \( z \) and \( 0 \), so \( D_1 = D' \). The domain \( D_1 \) appears in Olver [5, Theorem D(i)]. According to this theorem, (2.1) has a solution \( W_1(u, z) \) of the form

\[
W_1(u, z) = zI_\mu(uz) \left( \sum_{s=0}^{N-1} \frac{A_s(z)}{u^{2s}} + g_1(u, z) \right) + \frac{z}{u} I_{\mu+1}(uz) \left( \sum_{s=0}^{N-1} \frac{B_s(z)}{u^{2s}} + zh_1(u, z) \right),
\]

(2.5)

where

\[
|g_1(u, z)| + |h_1(u, z)| \leq \frac{K_1}{u^{2N}} \quad \text{for} \quad 0 < |z| \leq R, \quad u \geq u_1.
\]

(2.6)

Remarks 2.1.

1. The parameter \( \mu \) is considered fixed. We may write \( W_1(u, \mu, z) \) to indicate the dependence of \( W_1 \) on \( \mu \).
2. Every solution $w(z)$ of (2.1) is defined on the Riemann surface of the logarithm over $D$. Note that there is no restriction on $\arg z$ in (2.6), see [5, p. 76].

3. The precise statement is this: for every positive integer $N$ there are functions $g_1, h_1$ and positive constants $K_1, u_1$ (independent of $u, z$) such that (2.5), (2.6) hold.

4. The functions $A_s(z), B_s(z)$ are not uniquely determined because of the free choice of integration constants in (2.3). Even if we make a definite choice of these integration constants, the solution $W_1(u, z)$ is not uniquely determined by (2.5), (2.6). For example, one can replace $W_1(u, z)$ by $(1 + e^{-u})W_1(u, z)$.

5. Olver’s construction of $W_1(u, z)$ is independent of $N$ but may depend on $R$. In our application to the confluent hypergeometric differential equation we have $f(z) = z^2$. Then $R$ can be any positive number but $W_1(u, z)$ may depend on the choice of $R$. 

6. Olver has the term $\frac{s}{1 + |s|}$ in place of $z$ in front of $h_1$ in (2.5) but since we assume $|z| \leq R$ this makes no difference.

For the definition of $D_2$ we suppose that $\alpha$ is an arbitrary point of the sector $|\arg a| < \frac{1}{2} \pi$ and $\epsilon > 0$. Then $D_2$ comprises those points $z \in D'$ for which $|\arg z| \leq \frac{3}{4} \pi, \Re z \leq \Re a$, and a contour can be found joining $z$ and $\alpha$ which satisfies the following conditions:

(i) it lies in $D'$,

(ii) it lies wholly to the right of the line through $z$ parallel to the imaginary axis,

(iii) it does not cross the negative imaginary axis if $\frac{1}{2} \pi \leq \arg z \leq \frac{3}{2} \pi$, and does not cross the positive imaginary axis if $-\frac{3}{2} \pi \leq \arg z \leq -\frac{\pi}{2}$,

(iv) it lies outside the circle $|t| = \epsilon |z|$.

In our special case $D' = \{ z : |z| \leq R \}$ we choose $a = R$. If $0 \leq \arg z \leq \frac{3}{2} \pi$ and $0 < |z| \leq R$, we choose the contour starting at $z$ moving in positive direction parallel to the imaginary axis until we hit the circle $|t| = R$. Then we move clockwise along the circle $|t| = R$ towards $\alpha$. Taking into account condition (iv), we see that $D_2$ is the set of points $z$ with $-\frac{3}{2} \pi + \delta \leq z \leq \frac{3}{2} \pi - \delta, 0 < |z| \leq R$, where $\delta > 0$. The domain $D_2$ appears in Olver [5, Theorem D(ii)]. According to this theorem, (2.1) has a solution $W_2(u, z)$ of the form

$$W_2(u, z) = zK_\mu(uz) \left( \sum_{s=0}^{N-1} \frac{A_s(z)}{u^{2s}} + g_2(u, z) \right)$$

$$- \frac{z}{u} K_{\mu+1}(uz) \left( \sum_{s=0}^{N-1} \frac{B_s(z)}{u^{2s}} + zh_2(u, z) \right).$$

(2.7)

where

$$|g_2(u, z)| + |h_2(u, z)| \leq \frac{K_2}{u^{2N}} \quad \text{for} \quad 0 < |z| \leq R, \quad |\arg z| \leq \frac{3}{2} \pi - \delta, \quad u \geq u_2. \quad (2.8)$$

Note that in (2.8) there is a restriction on $\arg z$.

In the rest of this paper we choose the functions $A_s(z)$ such that

$$A_s(0) = 0 \quad \text{if} \quad s \geq 1. \quad (2.9)$$

Then the functions $A_s(z), B_s(z)$ are uniquely determined.
3 Properties of solutions $W_1$ and $W_2$

The differential equation (2.1) has a regular singularity at $z = 0$ with exponents $1 \pm \mu$. Substituting $x = z^2$ we obtain an equation which has a regular singularity at $x = 0$ with exponents $\frac{1}{2}(1 \pm \mu)$. Therefore, for every $\mu$ which is not a negative integer, (2.1) has a unique solution $W_+(z) = W_+(u, \mu, z)$ of the form

$$W_+(z) = z^{1+\mu} \sum_{n=0}^{\infty} c_n z^{2n},$$

where the $c_n$ are determined by $c_0 = 1$, and

$$4n(\mu + n)c_n = u^2 c_{n-1} + \sum_{j=0}^{n-1} f_j c_{n-1-j} \quad \text{for } n \geq 1$$

when

$$f(z) = \sum_{n=0}^{\infty} f_n z^{2n}.$$ 

If $\mu$ is not an integer, then $W_+(u, \mu, z)$ and $W_+(u, -\mu, z)$ form a fundamental system of solutions of (2.1). If $\Re \mu \geq 0$, there is a solution $W_-(z)$ linearly independent of $W_+(z)$ such that

$$W_-(z) = z^{1-\mu} p(z^2) + d \ln z W_+(z),$$

where $p$ is a power series and $d$ is a suitable constant. If $\mu \neq 0$ we choose $p(0) = 1$. If $\mu$ is not an integer then $d = 0$.

**Lemma 3.1.** Suppose $\Re \mu \geq 0$. There is a function $\alpha(u)$ such that

$$W_1(u, z) = \alpha(u) W_+(u, z),$$

and, for every $N = 1, 2, 3, \ldots$,

$$\alpha(u) = \frac{2^{-\mu} u^\mu}{\Gamma(\mu + 1)} \left(1 + O\left(\frac{1}{u^{2N}}\right)\right) \quad \text{as } 0 < u \to \infty.$$

**Proof.** There are functions $\alpha_+(u), \alpha_-(u)$ such that

$$W_1(u, z) = \alpha_+(u) W_+(u, z) + \alpha_-(u) W_-(u, z).$$

Suppose $\Re \mu > 0$. Then (3.2) implies

$$\lim_{z \to 0^+} z^{\mu-1} W_1(u, z) = \alpha_-(u).$$

We use [7, (10.30.1)]

$$\lim_{z \to 0^+} I_\nu(z) z^{-\nu} = \frac{2^{-\nu}}{\Gamma(\nu + 1)}.$$ 

Then (2.5), (2.6) give

$$\lim_{z \to 0^+} z^{\mu-1} W_1(u, z) = 0.$$

It follows from (3.3), (3.4) that $\alpha_-(u) = 0$. 

Now suppose that $\Re \mu = 0$, $\mu \neq 0$. Then we argue as before but instead of $z \to 0^+$ we approach 0 along a spiral $z = re^{\pm i\nu}, 0 < r \to 0$, when $\pm 3\mu > 0$. Then along this spiral $z^{2\mu} \to 0$. We obtain again that $\alpha_-(u) = 0$. In a similar way, we also show that $\alpha_-(u) = 0$ when $\mu = 0$.

Therefore, (3.2) gives

$$
\lim_{z \to 0^+} z^{-\mu-1} W_1(z, u) = \alpha_+(u)
$$

and, from (2.5), (2.6), (2.9)

$$
\lim_{z \to 0^+} z^{-\mu-1} W_1(u, z) = \frac{2^{-\mu} u^\mu}{\Gamma(\mu + 1)} \left( 1 + O\left( \frac{1}{u^{2N}} \right) \right)
$$

which implies (3.1) with $\alpha(u) = \alpha_+(u)$. $\blacksquare$

Let us define

$$
W_3(u, \mu, z) = \frac{2^{-\mu} u^\mu}{\Gamma(\mu + 1)} W_+(u, \mu, z).
$$

Then Lemma 3.1 gives

$$
W_3(u, z) = \tilde{\alpha}(u) W_1(u, z), \quad \text{where} \quad \tilde{\alpha}(u) = 1 + O\left( \frac{1}{u^{2N}} \right).
$$

Therefore, $W_3$ admits the asymptotic expansion (2.5), (2.6), so we can replace $W_1$ by $W_3$. Note that in contrast to $W_1$, $W_3$ is a uniquely defined function which is identified as a (Floquet) solution of (2.1) and not by its asymptotic behavior as $u \to \infty$.

Unfortunately, it seems impossible to replace $W_2$ by an easily identifiable solution of (2.1). However, we will now prove several useful properties of $W_2$.

**Lemma 3.2.** Suppose that $\Re \mu \geq 0$. There is a function $\beta(u)$ such that

$$
W_2(u, ze^{\pi i}) - e^{\pi i (1-\mu)} W_2(u, z) = \beta(u) W_3(u, z),
$$

and, for every $N = 1, 2, 3, \ldots$,

$$
\beta(u) = \pi i \left( 1 + O\left( \frac{1}{u^{2N}} \right) \right) \quad \text{as} \quad 0 < u \to \infty.
$$

**Proof.** We set $\lambda_\pm = e^{\pi i (1 \pm \mu)}$. Equation (2.1) has a fundamental system of solutions $W_+, W_-$ such that

$$
W_+(ze^{\pi i}) = \lambda_+ W_+(z), \quad W_-(ze^{\pi i}) = \lambda_- W_-(z) + \rho W_+(z).
$$

Let $w(z) = c_+ W_+(z) + c_- W_-(z)$ be any solution of (2.1). Then

$$
w(ze^{\pi i}) - \lambda_- w(z) = ((\lambda_+ - \lambda_-) c_+ + \rho c_-) W_+(z).
$$

If we apply this result to $w = W_2$ we see that there is a function $\beta(u)$ such that (3.5) holds.

Let $z > 0$ and set $z_1 = ze^{\pi i}$. We use (2.7) for $z_1$ in place of $z$, and [7, (10.34.2)]

$$
K_\nu(ze^{\pi im}) = e^{-\pi i m} K_\nu(z) - \frac{\sin(\pi \nu m)}{\sin(\pi \nu)} I_\nu(z)
$$

(3.7)
with $m = 1$. Then

$$W_2(u, z_1) = z (\lambda K_\mu(uz) + \pi i I_\mu(uz)) \left( \sum_{s=0}^{N-1} A_s(z) + g_2(u, z_1) \right)$$

$$+ \frac{z}{u} (-\lambda K_{\mu+1}(uz) + \pi i I_{\mu+1}(uz)) \left( \sum_{s=0}^{N-1} B_s(z) + z h_2(u, z_1) \right).$$

Using (2.7) a second time, we find that

$$W_2(u, z_1) - \lambda W_2(u, z) = \pi iz I_\mu(uz) \left( \sum_{s=0}^{N-1} A_s(z) + g_2(u, z_1) \right)$$

$$+ \frac{z}{u} I_{\mu+1}(uz) \left( \sum_{s=0}^{N-1} B_s(z) + z h_2(u, z_1) \right)$$

$$+ \lambda z K_\mu(uz)(g_2(u, z_1) - g_2(u, z)) - \lambda \frac{z^2}{u} K_{\mu+1}(uz)(h_2(u, z_1) - h_2(u, z)).$$

We now expand the right-hand side of (3.5) using (2.5), and compare the expansions. Setting $z = R$ and dividing by $RI_\mu(uR)$, we obtain

$$(\beta(u) - \pi i) \left( 1 + O \left( \frac{1}{u} \right) \right) = O \left( \frac{1}{u^{2N}} \right) \text{ as } 0 < u \to \infty,$$

where we used [7, (10.40.1)]

$$I_\nu(x) = \frac{e^x}{\sqrt{2\pi x}} \left( 1 + O \left( \frac{1}{x} \right) \right) \text{ as } 0 < x \to \infty,$$  \hspace{1cm} (3.8)

and [7, (10.40.2)]

$$K_\nu(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \left( 1 + O \left( \frac{1}{x} \right) \right) \text{ as } 0 < x \to \infty.$$ \hspace{1cm} (3.9)

This proves (3.6).

\textbf{Lemma 3.3.}

(a) If $\Re \mu > 0$ then, for every $N = 1, 2, 3, \ldots$, we have

$$\limsup_{z \to 0^+} \left| z^{\mu-1} W_2(u, z) - \Gamma(\mu) 2^{\mu-1} u^{-\mu} \left( 1 - 2\mu \sum_{s=0}^{N-1} \frac{B_s'(0)}{u^{2s+2}} \right) \right| = O \left( \frac{u^{-\mu}}{u^{2N+2}} \right) \hspace{1cm} (3.10)$$

as $0 < u \to \infty$.

(b) If $\Re \mu = 0$, $\mu \neq 0$, (3.10) holds when we replace $z^{\mu-1} W_2(u, z)$ by

$$z^{\mu-1} W_2(u, z) - \Gamma(-\mu) 2^{-\mu-1} u^\mu z^{2\mu}.$$

(c) If $\mu = 0$ then

$$\limsup_{z \to 0^+} \left| \frac{W_2(u, z)}{z \ln z} + 1 \right| = O \left( \frac{1}{u^{2N}} \right).$$
Applying the reflection formula for the Gamma function, we obtain (3.15). If \(\Re \mu > 0\). Then we use [7, (10.30.2)]

\[
\lim_{x \to 0^+} x^\nu K_\nu(x) = \Gamma(\nu)2^{\nu-1} \quad \text{for} \quad \Re \nu > 0.
\]

It follows that

\[
\lim_{z \to 0^+} z^\mu K_\mu(uz) = \Gamma(\mu)2^{\mu-1}u^{-\mu},
\]

(3.11)

\[
\lim_{z \to 0^+} \frac{1}{u} z^{\mu+1} K_{\mu+1}(uz) = \Gamma(\mu + 1)2^\mu u^{-\mu-2}.
\]

(3.12)

Using (2.7), (2.9), (3.11), (3.12), we obtain

\[
\limsup_{z \to 0^+} \left| z^{\mu-1} W_2(u, z) - \Gamma(\mu)2^{\mu-1}u^{-\mu} \left( 1 - 2\mu \sum_{s=0}^{N-1} \frac{B'_s(0)}{u^{2s+2}} \right) \right|
\]

\[
\leq \limsup_{z \to 0^+} \left| \Gamma(\mu)2^{\mu-1}u^{-\mu}g_2(u, z) - \Gamma(\mu + 1)2^\mu u^{-\mu-2}h_2(u, z) \right|.
\]

Now (2.8) gives (3.10) with \(N - 1\) in place of \(N\). If \(\Re \mu = 0\), \(\mu \neq 0\), then we use [5, (9.7)]

\[
K_\mu(x) = \Gamma(\mu)2^{\mu-1}x^{-\mu} + \Gamma(-\mu)2^{-\mu-1}x^\mu + o(1) \quad \text{as} \quad 0 < x \to 0
\]

and argue similarly. If \(\mu = 0\) we use [7, (10.30.3)]

\[
\lim_{x \to 0^+} \frac{K_0(x)}{\ln x} = -1.
\]

Theorem 3.4. Suppose that \(\Re \mu \geq 0\) and \(\mu\) is not an integer. There are functions \(\gamma(u), \delta(u)\) such that

\[
W_2(u, z) = \gamma(u)W_3(u, \mu, z) + \delta(u)W_3(u, -\mu, z),
\]

(3.13)

and, for every \(N = 1, 2, 3, \ldots\),

\[
\gamma(u) = -\frac{\pi}{2\sin(\pi \mu)} \left( 1 + O \left( \frac{1}{u^{2N}} \right) \right),
\]

(3.14)

\[
\delta(u) = \frac{\pi}{2\sin(\pi \mu)} \left( 1 - 2\mu \sum_{s=0}^{N-1} \frac{B'_s(0)}{u^{2s+2}} + O \left( \frac{1}{u^{2N+2}} \right) \right).
\]

(3.15)

Proof. Since \(\mu\) is not an integer, \(W_3(u, \mu, z)\) and \(W_3(u, -\mu, z)\) are linearly independent so (3.13) holds for some suitable functions \(\gamma, \delta\). From (3.13) we get

\[
W_2(u, ze^{\pi i}) - e^{\pi i(1-\mu)}W_2(u, z) = \gamma(u)(e^{\pi i(1+\mu)} - e^{\pi i(1-\mu)})W_3(u, z).
\]

Comparing with Lemma 3.2, we find \(-2i\gamma(u)\sin(\pi \mu) = \beta(u)\). Now (3.6) gives (3.14).

Suppose that \(\Re \mu > 0\). Then (3.13) yields

\[
\lim_{z \to 0^+} z^{\mu-1} W_2(u, z) = \delta(u) \frac{2^\mu u^{-\mu}}{\Gamma(1 - \mu)}.
\]

Using Lemma 3.3(a) we obtain

\[
\Gamma(\mu)2^{\mu-1}u^{-\mu} \left( 1 - 2\mu \sum_{s=0}^{N-1} \frac{B'_s(0)}{u^{2s+2}} + O \left( \frac{1}{u^{2N+2}} \right) \right) = \delta(u) \frac{2^\mu u^{-\mu}}{\Gamma(1 - \mu)}.
\]

Applying the reflection formula for the Gamma function, we obtain (3.15). If \(\Re \mu = 0\), \(\mu \neq 0\), the proof of (3.15) is similar. ■
4 Removal of restriction on arg $z$

Using $\beta(u)$ from Lemma 3.2 we define

$$W_4(u, z) = \frac{\pi i}{\beta(u)} W_2(u, z).$$

Then we have

$$W_4(u, ze^{\pi i}) = e^{\pi i(1-\mu)} W_4(u, z) + \pi i W_3(u, z). \quad (4.1)$$

Moreover, (3.6) shows that $W_4$ shares the asymptotic expansion (2.7), (2.8) with $W_2$. From (4.1) we obtain

$$W_4(u, ze^{\pi im}) = e^{\pi i(1-\mu)m} W_4(u, z) + \pi i \frac{\sin(\pi(\mu + 1)m)}{\sin(\pi(\mu + 1))} W_3(u, z) \quad (4.2)$$

for every integer $m$. We will use (4.2) and the asymptotic expansions (2.5), (2.7) for $|\arg z| \leq \frac{1}{2}\pi$ to prove that in (2.8) we can remove the restriction on arg $z$ completely.

**Theorem 4.1.** Suppose that $\Re \mu \geq 0$. For every $N = 1, 2, 3, \ldots$, $W_2(u, z)$ can be written as the right-hand side of (2.7), and (2.8) holds without a restriction on arg $z$:

$$|g_2(u, z)| + |h_2(u, z)| \leq \frac{K_2}{u^{2N}} \quad \text{for} \quad 0 < |z| \leq R, \ u \geq u_2.$$

**Proof.** Without loss of generality we replace $W_2$ by $W_4$. We assume that $|\arg z| \leq \frac{1}{2}\pi$, $0 < |z| \leq R$, $u > 0$, $m$ is an integer and $z_1 := ze^{\pi im}$. We insert (2.5), (2.7) on the right-hand side of (4.2). Using (3.7) we obtain

$$W_4(u, z_1) = z_1 K_\mu(u z_1) \sum_{s=0}^{N-1} \frac{A_s(z_1)}{u^{2s}} - \frac{z_1}{u} K_{\mu+1}(u z_1) \sum_{s=0}^{N-1} \frac{B_s(z_1)}{u^{2s}} + f(u, z), \quad (4.3)$$

where

$$f = E_1 g_2 + E_2 h_2 + E_3 g_1 + E_4 h_1,$$

with

$$E_1(u, z) = e^{-\pi i(\mu+1)m} z K_\mu(u z), \quad E_2(u, z) = -e^{-\pi i(\mu+1)m} \frac{z^2}{u} K_{\mu+1}(uz),$$

$$E_3(u, z) = \pi i \frac{\sin(\pi(\mu + 1)m)}{\sin(\pi(\mu + 1))} z I_\mu(u z), \quad E_4(u, z) = \pi i \frac{\sin(\pi(\mu + 1)m)}{\sin(\pi(\mu + 1))} \frac{z^2}{u} I_{\mu+1}(uz).$$

We will construct functions $G_j(u, z)$ and $H_j(u, z)$ such that

$$E_j(u, z) = z_1 K_\mu(u z_1) G_j(u, z) - \frac{z_1^2}{u} K_{\mu+1}(u z_1) H_j(u, z)$$

for $j = 1, 2, 3, 4$. Then (4.3) becomes

$$W_4(u, z_1) = z_1 K_\mu(u z_1) \left( \sum_{s=0}^{N-1} \frac{A_s(z_1)}{u^{2s}} + g_3(u, z) \right) - \frac{z_1}{u} K_{\mu+1}(u z_1) \left( \sum_{s=0}^{N-1} \frac{B_s(z_1)}{u^{2s}} + z_1 h_3(u, z) \right), \quad (4.4)$$
where

\[ g_3 = G_1 g_2 + G_2 h_2 + G_3 g_1 + G_4 h_1, \quad h_3 = H_1 g_2 + H_2 h_2 + H_3 g_1 + H_4 h_1. \]

We now use [7, (10.28.2)]

\[ K_\mu(x) I_{\mu+1}(x) + K_{\mu+1}(x) I_\mu(x) = \frac{1}{x}. \] (4.5)

From (4.5) and the relation

\[ I_\mu(ze^{\pi i m}) = e^{\pi i \mu m} I_\mu(z) \] (4.6)

we obtain

\[ uz_1 K_\mu(uz_1)e^{\pi i (\mu+1)m} I_{\mu+1}(uz) + uz_1 K_{\mu+1}(uz_1)e^{\pi i \mu m} I_\mu(uz) = 1. \]

Therefore, we can choose

\[ G_1(u, z) = uz K_\mu(uz) I_{\mu+1}(uz), \quad H_1(u, z) = -u^2 K_\mu(uz) I_\mu(uz). \]

We set

\[ l_0(x) = \ln \frac{1 + 2|x|}{|x|}, \quad l_\mu(x) = 1 \quad \text{if} \quad \mu \neq 0, \]

and note the estimates [5, (9.12)]

\[ |I_\mu(x)K_\mu(x)| \leq \frac{C|l_\mu(x)|}{1 + |x|}, \quad |I_{\mu+1}(x)K_\mu(x)| \leq \frac{C|x|l_\mu(x)}{1 + |x|^2}, \] (4.7)

\[ |I_{\mu+1}(x)K_{\mu+1}(x)| \leq \frac{C}{1 + |x|}, \quad |I_\mu(x)K_{\mu+1}(x)| \leq \frac{C}{|x|} \] (4.8)

valid when \(|\arg x| \leq \frac{1}{2} \pi\) with \(C\) independent of \(x\). At this point we assume that \(\mu \neq 0\) (the case \(\mu = 0\) is mentioned at the end of the proof). The estimates (4.7) give

\[ |G_1(u, z)| \leq C, \quad |H_1(u, z)| \leq Cu^2. \] (4.9)

Similarly, we choose

\[ G_2(u, z) = -z^2 K_{\mu+1}(uz) I_{\mu+1}(uz), \quad H_2(u, z) = uz K_{\mu+1}(uz) I_\mu(uz). \]

The estimates (4.8) give

\[ |G_2(u, z)| \leq C|z|^2, \quad |H_2(u, z)| \leq C. \] (4.10)

It follows from (3.7) that

\[ E_3(u, z) = -E_1(u, z) + z K_\mu(uz_1), \quad E_4(u, z) = -E_2(u, z) - \frac{z^2}{u} K_{\mu+1}(uz_1). \]

Therefore, we can choose

\[ G_3(u, z) = 1 - G_1(u, z), \quad H_3(u, z) = -H_1(u, z), \]
\[ G_4(u, z) = -G_2(u, z), \quad H_4(u, z) = 1 - H_2(u, z). \]
From (4.9), (4.10), we get

\[ |G_3(u, z)| \leq C + 1, \quad |H_3(u, z)| \leq Cu^2, \quad (4.11) \]
\[ |G_4(u, z)| \leq C|z|^2, \quad |H_1(u, z)| \leq C + 1. \quad (4.12) \]

The estimates (4.9), (4.10), (4.11), (4.12) give

\[ |g_3(u, z)| \leq C|g_2(u, z)| + C|z|^2|h_2(u, z)| + (C + 1)|g_1(u, z)| + C|z|^2|h_1(u, z)|, \]
\[ |h_3(u, z)| \leq Cu^2|g_2(u, z)| + C|h_2(u, z)| + Cu^2|g_1(u, z)| + (C + 1)|h_1(u, z)|. \]

Since we assumed that

\[ |g_1(u, z)| + |h_1(u, z)| + |g_2(u, z)| + |h_2(u, z)| \leq \frac{K}{u^{2N}} \]

for \(|\arg z| \leq \frac{1}{2} \pi, 0 < |z| \leq R, u \geq u_0\), the expansion (4.4) has the desired form with \(N\) replaced by \(N - 1\).

Suppose \(\mu = 0\). We use [7, (10.31.2)]

\[ K_0(x) = -\left( \ln \left( \frac{1}{2}x \right) + \gamma \right) I_0(x) + \frac{1}{4}x^2 \frac{\Gamma(1)}{\Gamma(3)} + \left( 1 + \frac{1}{2} \right) \frac{(\frac{1}{4}x^2)^2}{(2!)^2} + \cdots. \quad (4.13) \]

It follows from (4.13) that there exist positive constants \(r > 0, D > 0\) such that

\[ \frac{|K_0(x)|}{|K_0(xe^{im})|} \leq D \quad \text{for} \quad 0 < |x| \leq r, \quad |\arg x| < \frac{1}{2} \pi, \quad m \in \mathbb{Z}. \]

Then we set

\[ G_1(u, z) = \frac{K_0(uz)}{K_0(uz_1)}, \quad H_1(u, z) = 0 \quad \text{if} \quad 0 < |uz| \leq r \]

with \(G_1\) and \(H_1\) the same as before when \(|uz| > r\). The estimates (4.9) are valid with a suitable constant \(C\). The rest of the proof is unchanged. This completes the proof of the theorem. \(\blacksquare\)

5 Extension to complex \(u\)

So far we considered only \(0 < u \to \infty\). Now we set \(u = te^{i\theta}\), where \(t > 0\) and \(\theta \in \mathbb{R}\). In (2.1) we substitute \(z = e^{-i\theta}x, \tilde{w}(x) = w(z)\). Then we obtain the differential equation

\[ \frac{d^2}{dx^2} \tilde{w}(x) = \frac{1}{x} \frac{d}{dx} \tilde{w}(x) + \left( t^2 + \frac{u^2 - 1}{x^2} + e^{-2i\theta}f(e^{-i\theta}x) \right) \tilde{w}(x). \quad (5.1) \]

Assuming \(R_0 \mu \geq 0\), we can apply Olver’s theory to this equation, and obtain functions \(\tilde{W}_1(t, x)\) and \(\tilde{W}_2(t, x)\). Since we assumed that \(f(z)\) is analytic in the disk \(\{z : |z| < R_0\}\), the new function \(\tilde{f}(x) = e^{-2i\theta}f(e^{-i\theta}x)\) is analytic in the same disk. Therefore, the domains \(D_1, D_2\) are the same as before. The functions \(\tilde{A}_s(x), \tilde{B}_s(x)\) that appear in place of \(A_s(z), B_s(z)\) satisfy

\[ \tilde{A}_s(x) = e^{-2si\theta}A_s(z), \quad \tilde{B}_s(x) = e^{-(2s+1)i\theta}B_s(z), \]

so

\[ \frac{\tilde{A}_s(x)}{t^{2s}} = \frac{A_s(z)}{u^{2s}}, \quad \frac{\tilde{B}_s(x)}{t^{2s+1}} = \frac{B_s(z)}{u^{2s+1}}. \]
Therefore, the functions $e^{-iθ}\tilde{W}_1(t,x)$ and $e^{-iθ}\tilde{W}_2(t,x)$ have the asymptotic expansions (2.5), (2.6) and (2.7), (2.8) with $(t, x)$ replacing $(u, z)$.

Let $\tilde{W}_3(t, \mu, x)$ be the function $W_3$ for the differential equation (5.1). Then

$$W_3(te^{iθ}, \mu, e^{-iθ}x) = e^{-iθ}\tilde{W}_3(t, \mu, x).$$

It follows that $W_3(u, \mu, z)$ can be expanded in the form of the right-hand side of (2.5), and (2.6) holds for $0 < |z| \leq R$ and $u = te^{iθ}$ for any fixed real $θ$.

We would like to connect $\tilde{W}_2$ to $W_2$ in a similar manner but this is not possible at this point because $W_2(u, z)$ is only defined for $u > 0$, and so we cannot substitute $u = te^{iθ}$.

6 Properties of $A_s, B_s$

For any $μ ∈ \mathbb{C}$ we consider the solution $A_s(z) = A_s(μ, z)$, $B_s(z) = B_s(μ, z)$ of the recursion (2.2), (2.3) which is uniquely determined by $A_0(z) = 1$ and (2.9). The following lemma is mentioned by Olver [4, p. 327], [5, p. 81, line 6].

**Lemma 6.1.** Let $\hat{A}_s(z)$, $\hat{B}_s(z)$ be any solution of (2.2), (2.3) with $\hat{A}_0(z) = 1$. Then, for all $s ≥ 0$,

$$\hat{A}_s(z) = \sum_{r=0}^{s} A_r(z)\hat{A}_{s-r}(0), \quad \hat{B}_s(z) = \sum_{r=0}^{s} B_r(z)\hat{A}_{s-r}(0). \tag{6.1}$$

**Proof.** Let us denote the right-hand sides of equations (6.1) by $A^*_s(z)$, $B^*_s(z)$, respectively. It is easy to show that $A^*_s(z)$, $B^*_s(z)$ is a solution of (2.2), (2.3). Since $A^*_0(z) = 1$ and $A^*_s(0) = \hat{A}_s(0)$, this solution must agree with $A_s(z)$, $\hat{B}_s(z)$.

We now define $a_0(z) = 1$ and, for $s ≥ 0$,

$$a_{s+1}(z) := A_{s+1}(-μ, z) + \frac{2μ}{z}B_s(-μ, z), \tag{6.2}$$
$$b_s(z) := B_s(-μ, z). \tag{6.3}$$

**Theorem 6.2.** The functions $a_s(z)$, $b_s(z)$ satisfy (2.2), (2.3) with $A_s$, $B_s$ replaced by $a_s$, $b_s$, respectively, and, for all $s ≥ 0$,

$$a_s(z) = A_s(μ, z) + 2μ \sum_{r=0}^{s-1} A_r(μ, z)B^*_r(-μ, 0), \tag{6.4}$$
$$b_s(z) = B_s(μ, z) + 2μ \sum_{r=0}^{s-1} B_r(μ, z)B^*_r(-μ, 0). \tag{6.5}$$

**Proof.** We have

$$2A_{s+1}(-μ, z) = \frac{-2μ+1}{z}B_s(-μ, z) - B^*_s(-μ, z) + \int f(z)B_s(-μ, z)dz.$$

We add $\frac{4μ}{z}B_s(-μ, z)$ on both sides and get

$$2a_{s+1}(z) = \frac{2μ+1}{z}b_s(z) - b^*_s(z) + \int f(z)b_s(z)dz. \tag{6.6}$$

This is (2.3) for $a_s(z)$, $b_s(z)$.
Equation (2.2) is true for \( a_s(z), b_s(z) \) when \( s = 0 \). Suppose \( s \geq 1 \). We have

\[
2B_s'(-\mu, z) = -A''_s(-\mu, z) + f(z)A_s(-\mu, z) + \frac{2\mu - 1}{z} A'_s(-\mu, z).
\]

Using the definitions of \( a_s(z), b_s(z) \) we get

\[
2b'_s(z) = -a''_s(z) + f(z)a_s(z) - \frac{2\mu + 1}{z} a'_s(z) + \frac{4\mu}{z} a'_s(z) + G,
\]

where

\[
G := \frac{d^2}{dz^2} \left( \frac{2\mu}{z} b_{s-1}(z) \right) - f(z) \frac{2\mu}{z} b_{s-1}(z) - \frac{2\mu - 1}{z} \frac{d}{dz} \left( \frac{2\mu}{z} b_{s-1}(z) \right).
\]

In (6.7) we replace \( \frac{4\mu}{z} a'_s(z) \) through (6.6). Then we obtain

\[
2b'_s(z) = -a''_s(z) + f(z)a_s(z) - \frac{2\mu + 1}{z} a'_s(z) + H + G,
\]

where

\[
H := \frac{2\mu}{z} \left[ \frac{d}{dz} \left( \frac{2\mu + 1}{z} b_{s-1}(z) \right) - b''_{s-1}(z) + f(z)b_{s-1}(z) \right].
\]

By direct computation, we show \( H + G = 0 \) for any function \( b_{s-1}(z) \). Therefore, by integrating (6.8) noting that \( a_s(z) \) is even and \( b_s(z) \) is odd, we obtain (2.2) for \( a_s(z), b_s(z) \).

We now get (6.4), (6.5) from Lemma 6.1.

Using multiplication of formal series, we can write (6.4), (6.5) as

\[
F(u, -\mu) \sum_{s=0}^{\infty} \frac{A_s(z)}{u^{2s}} = \sum_{s=0}^{\infty} \frac{a_s(z)}{u^{2s}},
\]

\[
F(u, -\mu) \sum_{s=0}^{\infty} \frac{B_s(z)}{u^{2s}} = \sum_{s=0}^{\infty} \frac{b_s(z)}{u^{2s}},
\]

where

\[
F(u, \mu) = 1 - 2\mu \sum_{s=0}^{\infty} \frac{B'_s(\mu, 0)}{u^{2s+2}}.
\]

We differentiate (6.5) with respect to \( z \) and set \( z = 0 \). Then we find

\[
B'_s(-\mu, 0) = B'_s(\mu, 0) + 2\mu \sum_{r=0}^{s-1} B'_r(\mu, 0) B'_{s-1-r}(-\mu, 0),
\]

or, equivalently,

\[
F(u, \mu) F(u, -\mu) = 1.
\]

In particular, it follows that

\[
F(u, \mu) \sum_{s=0}^{\infty} \frac{a_s(z)}{u^{2s}} = \sum_{s=0}^{\infty} \frac{A_s(z)}{u^{2s}},
\]

\[
F(u, \mu) \sum_{s=0}^{\infty} \frac{b_s(z)}{u^{2s}} = \sum_{s=0}^{\infty} \frac{B_s(z)}{u^{2s}}.
\]
7 Asymptotic expansion of $W_3$ when $\Re \mu < 0$

In Section 3 we saw that $W_3(u, \mu, z)$ can be written as the right-hand side of (2.5), and (2.6) holds. However, this was proved only when $\Re \mu \geq 0$. Now we remove this restriction.

**Theorem 7.1.** Suppose that $\mu \in \mathbb{C}$ is not a negative integer, and $u = t e^{i \theta}$ with $t > 0$, $\theta \in \mathbb{R}$. Then $W_3(u, \mu, z)$ can be written as the right-hand side of (2.5) and, for each $R > 0$ and $N \geq 1$, there are constants $L_1$ and $t_1$ such that

$$|g_1(u, z)| + |h_1(u, z)| \leq \frac{L_1}{t_1 N} \quad \text{for} \quad 0 < |z| \leq R, \quad t \geq t_1.$$  

**Proof.** In Sections 3 and 5 we proved this statement for $\Re \mu \geq 0$. Therefore, it will be sufficient to treat $W_3(u, -\mu, z)$ with $\Re \mu > 0$. By the considerations in Section 5, it is sufficient to consider $\theta = 0$, so $u > 0$. Suppose $|\arg z| \leq \frac{1}{2} \pi$, $0 < |z| \leq R$. By (3.13), we have

$$c \delta(u) W_3(u, -\mu, z) = c W_2(u, \mu, z) - c \gamma(u) W_3(u, \mu, z), \quad (7.1)$$

where $c = \frac{2}{\pi} \sin(\pi \mu)$. On the right-hand side of (7.1) we insert the expansions (2.5) for $W_3$ and (2.7) for $W_2$. Taking into account (3.14), we can expand $-c \gamma(u) W_3(u, \mu, z)$ the same way as $W_3$. Then using [7, (10.27.4)]

$$K_{\nu}(x) = \frac{\pi}{2 \sin(\pi \nu)} (I_{-\nu}(x) - I_{\nu}(x)), \quad (7.2)$$

we obtain

$$c \delta(u) W_3(u, -\mu, z) = z I_{-\mu}(uz) \sum_{s=0}^{N-1} \frac{A_s(z)}{u^{2s}} + \frac{z}{u} I_{-\mu-1}(uz) \sum_{s=0}^{N-1} \frac{B_s(z)}{u^{2s}} + f(u, z), \quad (7.3)$$

where

$$f = E_1 g_2 + E_2 h_2 + E_3 g_1 + E_4 h_1$$

with

$$E_1(u, z) = cz K_{\mu}(uz), \quad E_2(u, z) = -c z^2 K_{\mu+1}(uz),$$

$$E_3(u, z) = z I_{\mu}(uz), \quad E_4(u, z) = z^2 I_{\mu+1}(uz).$$

We will construct functions $G_j(u, z)$ and $H_j(u, z)$ such that

$$E_j(u, z) = z I_{-\mu}(uz) G_j(u, z) + \frac{z^2}{u} I_{1-\mu}(uz) H_j(u, z)$$

for $j = 1, 2, 3, 4$. Also using [7, (10.29.1)]

$$I_{\nu-1}(x) - I_{\nu+1}(x) = \frac{2\nu}{x} I_{\nu}(x), \quad (7.4)$$

(7.3) becomes

$$c \delta(u) W_3(u, -\mu, z) = z I_{-\mu}(uz) \left( \sum_{s=0}^{N-1} \frac{\tilde{A}_s(z)}{u^{2s}} + g_3(u, z) \right)$$

$$+ \frac{z}{u} z I_{1-\mu}(uz) \left( \sum_{s=0}^{N-1} \frac{B_s(z)}{u^{2s}} + zh_3(u, z) \right), \quad (7.5)$$
where
\[
\tilde{A}_0(z) = 1, \quad \tilde{A}_s(z) = A_s(z) - \frac{2\mu}{z} B_{s-1}(z) \quad \text{for} \quad s = 1, \ldots, N - 1,
\]
and
\[
g_3 = -\frac{2\mu}{z} B_{N-1}(z) u^{-2N} + G_1 g_2 + G_2 h_2 + G_3 g_1 + G_4 h_1,
\]
\[
h_3 = H_1 g_2 + H_2 h_2 + H_3 g_1 + H_4 h_1.
\]
The identities (4.5) and [7, (10.29.1)]
\[
K_{\nu-1}(x) - K_{\nu+1}(x) = -\frac{2\nu}{x} K_\nu(x)
\]
give
\[
uz I_{-\mu}(uz) \left( K_{\mu+1}(uz) - \frac{2\mu}{uz} K_\mu(uz) \right) + uz I_{1-\mu}(uz) K_\mu(uz) = 1.
\]
Therefore, we can choose
\[
G_3(u, z) = uz \left( K_{\mu+1}(uz) - \frac{2\mu}{uz} K_\mu(uz) \right) I_\mu(uz),
\]
\[
H_3(u, z) = u^2 K_\mu(uz) I_\mu(uz).
\]
The estimates (4.7), (4.8) give
\[
|G_3(u, z)| \leq C_3, \quad |H_3(u, z)| \leq D_3 u^2. \quad (7.7)
\]
Similarly, we choose
\[
G_4(u, z) = z^2 \left( K_{\mu+1}(uz) - \frac{2\mu}{uz} K_\mu(z) \right) I_{\mu+1}(uz),
\]
\[
H_4(u, z) = uz K_\mu(uz) I_{\mu+1}(uz),
\]
and estimate
\[
|G_4(u, z)| \leq C_4 |z|^2, \quad |H_4(u, z)| \leq D_4. \quad (7.8)
\]
It follows from (7.2), (7.4) that
\[
E_1(u, z) = z I_{-\mu}(uz) - E_3(u, z),
\]
\[
E_2(u, z) = \frac{z^2}{u} \left( -\frac{2\mu}{uz} I_{-\mu}(uz) + I_{1-\mu}(uz) \right) - E_4(u, z).
\]
Therefore, we can choose
\[
G_1(u, z) = 1 - G_3(u, z), \quad H_1(u, z) = -H_3(u, z),
\]
\[
G_2(u, z) = -\frac{2\mu}{u^2} - G_4(u, z), \quad H_2(u, z) = 1 - H_4(u, z).
\]
From (7.7), (7.8), we get
\[
|G_1(u, z)| \leq C_1, \quad |H_1(u, z)| \leq D_1 u^2, \quad (7.9)
\]
\[
|G_2(u, z)| \leq C_2 (1 + |z|^2), \quad |H_2(u, z)| \leq D_2. \quad (7.10)
\]
Since we know that 
\[ |g_1(u, z)| + |h_1(u, z)| + |g_2(u, z)| + |h_2(u, z)| \leq \frac{K}{u^{2N}} \]
for \(|\arg z| \leq \frac{1}{2} \pi, 0 < |z| \leq R, u \geq u_0\), the estimates (7.7), (7.8), (7.9), (7.10) give
\[ |g_3(u, z)| + |h_3(u, z)| \leq \frac{L}{u^{2N-2}} \quad \text{if} \quad |\arg z| \leq \frac{1}{2} \pi, \ 0 < |z| \leq R, \ u \geq u_3. \]

Now we divide both sides of (7.5) by \(c \delta(u)\) and use (3.15), (6.12), (6.13) (with \(\mu\) replaced by \(-\mu\)). Then we obtain the desired expansion of \(W_3(u, -\mu, z)\) for \(\Re \mu < 0\) and \(|\arg z| \leq \frac{1}{2} \pi, 0 < |z| \leq R\). The restriction on \(\arg z\) is easily removed using (4.6) and \(W_3(e^{\pi i \mu} z) = e^{\pi i (\mu + 1) m} W_3(z)\). \(\blacksquare\)

8 Application to the confluent hypergeometric equation

The confluent hypergeometric differential equation
\[ xe''(x) + (b - x)v'(x) - av(x) = 0 \]
has solutions \(M(a, b, x)\) and \(U(a, b, x)\). Substituting \(x = z^2, w = e^{-\frac{1}{2}z^2} z^b v\) we obtain the differential equation
\[ w''(z) = \frac{1}{z} w'(z) + \left( a^2 + \frac{\mu^2 - 1}{z^2} + z^2 \right) w(z), \]
where
\[ a = \frac{1}{4} u^2 + \frac{1}{2} b, \quad \mu = b - 1. \]

Equation (8.1) agrees with (2.1) when \(f(z) = z^2\). Let \(A_s, B_s\) be defined as in Section 2 for \(f(z) = z^2\). In this case, \(A_s(z), B_s(z)\) are polynomials. Throughout this section, we assume that \(a, b, u, \mu\) satisfy (8.2).

The function \(M(a, b, x)\) is given by a power series in \(x\) and \(M(a, b, 0) = 1\). Therefore, the function \(W_3\) associated with (8.1) is given by
\[ W_3(u, \mu, z) = \frac{2^{1-b} u^{b-1}}{\Gamma(b)} e^{-\frac{1}{2} z^2} z^b M(a, b, z^2). \]

Theorem 7.1 implies the following theorem.

**Theorem 8.1.** Suppose that \(b \in \mathbb{C}\) is not 0 or a negative integer, \(u = te^{i\theta}\) with \(t > 0, \ \theta \in \mathbb{R}, \ and \ N \geq 1, \ R > 0\). Then we can write
\[ \frac{2^{1-b} u^{b-1}}{\Gamma(b)} e^{-\frac{1}{2} z^2} z^b M\left(\frac{1}{4} u^2 + \frac{1}{2} b, b, z^2\right) = z I_{b-1}(uz) \left( \sum_{s=0}^{N-1} A_s(z) \frac{u^{2s}}{u^{2s}} + g_1(u, z) \right) + \frac{z}{u} I_{b}(uz) \left( \sum_{s=0}^{N-1} B_s(z) \frac{u^{2s}}{u^{2s}} + z h_1(u, z) \right), \]
where
\[ |g_1(u, z)| + |h_1(u, z)| \leq \frac{L_1}{t^{2N}} \quad \text{for} \quad 0 < |z| \leq R, \ t \geq t_1. \]
and \(L_1, t_1\) are positive constants independent of \(z\) and \(u\) (but possibly depending on \(b, \theta, N, R\)). There is no restriction on \(\arg z\). The polynomials \(A_s(z), B_s(z)\) appearing in (8.4) are determined by the recursion (2.2), (2.3) with \(f(z) = z^2\) and the conditions \(A_0(z) = 1, A_0(0) = 0\) for \(s \geq 1\).
Suppose that $\Re b \geq 1$. Let $W_2(u, z)$ be the function associated with equation (8.1) which satisfies (2.7), (2.8). There are functions $\beta_1(u), \beta_2(u)$ such that

$$W_2(u, z) = \beta_1(u)e^{-\frac{1}{4}z^2}z^b M(a, b, z^2) + \beta_2(u)e^{-\frac{1}{4}z^2}z^b U(a, b, z^2). \quad (8.5)$$

The determination of $\beta_1(u), \beta_2(u)$ is not obvious. It is in this part of the analysis where there is an error in [8]. Slater [8, p. 79] derives $\beta_2(u) \sim \Gamma(a)2b^{-2}u^{1-b}$, and claims “we can take $\beta_1(u) = 0$” without proof. When comparing with [8], note that our $\beta_2(u)$ is denoted by $1/\beta_2(u)$ in [8]. Actually, the stated formula for $\beta_2(u)$ is correct but it is only the leading term of the required full asymptotic expansion given in the following lemma.

**Lemma 8.2.** Suppose $\Re b \geq 1$. For every $N = 1, 2, 3, \ldots$, as $0 < u \to \infty$,

$$\beta_2(u) = \Gamma(a)2b^{-2}u^{1-b} \left(1 + 2(1-b)\sum_{s=0}^{N-1} \frac{B_s'(0)}{u^{2s+2}} + O\left(\frac{1}{u^{2N+2}}\right)\right). \quad (8.6)$$

**Proof.** Suppose $\Re b > 1$. Then [7, (13.2.18)]

$$\lim_{z \to 0^+} z^{2b-2}U(a, b, z^2) = \frac{\Gamma(b-1)}{\Gamma(a)}$$

and (8.5) give

$$\lim_{z \to 0^+} z^{b-2}W_2(u, z) = \beta_2(u) \frac{\Gamma(b-1)}{\Gamma(a)}.$$

Comparing with (3.10), we obtain (8.6).

If $\Re b = 1$, $b \neq 1$, the proof is similar using Lemma 3.3(b) and [7, (13.2.18)]

$$U(a, b, x) = \frac{\Gamma(b-1)}{\Gamma(a)}x^{1-b} + \frac{\Gamma(1-b)}{\Gamma(a-b+1)} + O(x) \quad \text{as} \quad x \to 0^+.$$

If $b = 1$ we use Lemma 3.3(c) and [7, (13.2.19)]

$$\lim_{x \to 0^+} \frac{U(a, 1, x)}{\ln x} = -\frac{1}{\Gamma(a)}.$$  \hfill \blacksquare$$

We cannot show that $\beta_1(u) = 0$ but we can prove that $|\beta_1(u)|$ is very small as $u \to \infty$. To this end we need the following lemma.

**Lemma 8.3.** Let $b \in \mathbb{C}, \Re x > 0$, and $\epsilon > 0$. There is a constant $Q$ independent of $a$ such that

$$|\Gamma(a)U(a, b, x)| \leq Q \quad \text{if} \quad \Re a \geq \epsilon.$$  

**Proof.** We use the integral representation [7, (13.4.4)]

$$\Gamma(a)U(a, b, x) = \int_0^\infty e^{-xt}t^{a-1}(1+t)^{b-a-1}dt.$$

Therefore, if $\Re a \geq \epsilon$,

$$|\Gamma(a)U(a, b, x)| \leq \int_0^\infty e^{-\Re xt} \left(\frac{t}{1+t}\right)^{\Re a-\epsilon} \left(\frac{t}{1+t}\right)^{-1} (1+t)^{\Re b-2}dt$$

$$\leq \int_0^\infty e^{-\Re xt} \left(\frac{t}{1+t}\right)^{-1} (1+t)^{\Re b-2}dt =: Q.$$  \hfill \blacksquare
Lemma 8.4. Suppose $\Re b \geq 1$. For every $q < R$ we have $\beta_1(u) = O(e^{-qu})$ as $0 < u \to \infty$.

Proof. In the following let $0 < z \leq R$ (and $b$) be fixed. By Lemmas 8.2, 8.3, there is a constant $C_1 > 0$ such that, for sufficiently large $u > 0$,

\[ |\beta_2(u) e^{-\frac{1}{2} z^2} z^b U(a, b, z^2)| \leq C_1 |u^{1-b}| Q. \]  

(8.7)

Using (3.8) we get from Theorem 8.1 with $N = 1$, for some constant $C_2 > 0$,

\[ |e^{-\frac{1}{2} z^2} z^b M(a, b, z^2)| \geq C_2 |u^{\frac{1}{2}-b}| e^{zu}. \]  

(8.8)

Similarly, (3.9), (2.7), (2.8) yield

\[ |W_2(u, z)| \leq C_3 u^{-\frac{3}{2}} e^{-zu}. \]  

(8.9)

Substituting (8.7), (8.8) and (8.9) in (8.5), we find

\[ |\beta_1(u)| \leq \frac{C_3}{C_2} |u^{b-1}| e^{-2zu} + \frac{C_1 Q}{C_2} u^{\frac{1}{2}} e^{-zu}. \]

If we choose $z = R$, we obtain the desired estimate. \[\blacksquare\]

Lemma 8.5. Suppose $\Re b \geq 1$. For every $N = 1, 2, 3, \ldots$, the function

\[ \beta_2(u) e^{-\frac{1}{2} z^2} z^b U(a, b, z^2) \]

can be written in the form of the right-hand side of (2.7), and (2.8) holds with $R$ replaced by $\frac{1}{3} R$.

Proof. Let

\[ L(u, z) := \beta_1(u) e^{-\frac{1}{2} z^2} z^b M(a, b, z^2). \]

Applying Theorem 8.1 and Lemma 8.4, we estimate, for $0 < |z| \leq R$,

\[ |L(u, z)| \leq C_1 e^{-qu} |z| (|I_\mu(uz)| + u^{-1} |I_{\mu+1}(uz)|), \]  

(8.10)

where $q < R$ will be chosen later. We use the estimate

\[ |I_\nu(x)| \leq C_2 e^{\epsilon |x|} \quad \text{for} \quad |\arg x| \leq \frac{3}{2} \pi \]  

(8.11)

provided that $\Re \nu \geq 0$. This inequality follows from \([5, (9.2), (9.3)]\). Therefore, (8.10) yields

\[ |L(u, z)| \leq C_3 |z| e^{-qu+\frac{1}{3} Ru} \quad \text{for} \quad 0 < |z| \leq \frac{1}{3} R, \quad |\arg z| \leq \frac{3}{2} \pi, \quad u \geq u_0. \]  

(8.12)

Using (4.5), we have

\[ L(u, z) = z K_{\mu}(uz) g(u, z) - \frac{z}{u} K_{\mu+1}(uz) z h(u, z), \]

where

\[ g(u, z) = u I_{\mu+1}(uz) L(u, z), \quad h(u, z) = -\frac{u}{z} I_\mu(uz) L(u, z). \]

From (8.11), (8.12), we get, for $0 < |z| \leq \frac{1}{3} R, \quad |\arg z| \leq \frac{3}{2} \pi$,

\[ |g(u, z)| \leq C_2 C_3 R u e^{u \left(\frac{1}{3} R - q\right)}, \quad |h(u, z)| \leq C_2 C_3 u^2 e^{u \left(\frac{1}{3} R - q\right)}. \]

By (8.5), we can write $\beta_2(u) e^{-\frac{1}{2} z^2} z^b U(a, b, z^2)$ as the right-hand side of (2.7) with $g_2$ replaced by $g_2 - g$ and $h_2$ replaced by $h_2 - h$. If we choose $q = \frac{5}{6} R$, $g$ and $h$ become exponentially small as $u \to \infty$, and the theorem is proved. \[\blacksquare\]
Lemma 8.6. Suppose \( \Re b \geq 1 \). For all \( N = 1, 2, 3, \ldots \), we have, as \( 0 < u \to \infty \),

\[
\frac{\beta_2(u) 2^b u^{1-b}}{\Gamma(1 + a - b)} = 1 + O\left( \frac{1}{u^{2N}} \right).
\]

(8.13)

Moreover, for all \( b \in \mathbb{C} \) and all \( N = 1, 2, 3, \ldots \), we have, as \( 0 < u \to \infty \),

\[
\frac{\Gamma(1 + a - b)}{\Gamma(a)} 2^{2-2b} u^{2b-2} = 1 + 2(1 - b) \sum_{s=0}^{N-1} \frac{B_s^b(0)}{u^{2s+2}} + O\left( \frac{1}{u^{2N+2}} \right).
\]

(8.14)

Proof. We set

\[
T(u, z) := \beta_2(u) e^{-\frac{1}{2} z^2} z^b U(a, b, z^2).
\]

Using [7, (13.2.12)]

\[
U(a, b, xe^{2i\pi}) = e^{-2\pi ib} U(a, b, x) + \frac{2\pi ie^{-\pi ib}}{\Gamma(b) \Gamma(1 + a - b)} M(a, b, x)
\]

and (8.3) we obtain

\[
T(u, ze^{i\pi}) - e^{-\pi ib} T(u, z) = \beta_2(u) \frac{\pi i 2^b u^{1-b}}{\Gamma(1 + a - b)} W_3(u, z).
\]

Now we argue as in the proof of Lemma 3.2 (applying Lemma 8.5 twice) and arrive at (8.13).

If \( \Re b \geq 1 \) the asymptotic formula (8.14) follows from (8.13) and Lemma 8.2. If \( \Re b < 1 \) we use (6.11).

Theorem 8.7. Suppose that \( b \in \mathbb{C}, N \geq 1 \) and \( R > 0 \). Then we can write

\[
\Gamma\left(1 + \frac{1}{4} u^2 - \frac{1}{2} b\right) 2^{-b} u^{b-1} e^{-\frac{1}{2} z^2} z^b U\left(\frac{1}{4} u^2 + \frac{1}{2} b, b, z^2\right)
\]

\[
= zK_{b-1}(uz) \left( \sum_{s=0}^{N-1} \frac{A_s(z)}{u^{2s}} + g_2(u, z) \right) - \frac{z}{u} K_b(uz) \left( \sum_{s=0}^{N-1} \frac{B_s(z)}{u^{2s}} + zh_2(u, z) \right),
\]

(8.15)

where

\[
|g_2(u, z)| + |h_2(u, z)| \leq \frac{K_2}{u^{2N}} \quad \text{for} \quad 0 < |z| \leq R, \quad u \geq u_2,
\]

(8.16)

and \( K_2, u_2 \) are constants independent of \( z \) and \( u \). There is no restriction on arg \( z \). The polynomials \( A_s(z), B_s(z) \) appearing in (8.15) are determined by the recursion (2.2), (2.3) with \( f(z) = z^2 \) and the conditions \( A_0(z) = 1, A_s(0) = 0 \) for \( s \geq 1 \).

Alternatively, we have

\[
\Gamma\left(\frac{1}{4} u^2 + \frac{1}{2} b\right) 2^{b-2} u^{1-b} e^{-\frac{1}{2} z^2} z^b U\left(\frac{1}{4} u^2 + \frac{1}{2} b, b, z^2\right)
\]

\[
= zK_{b-1}(uz) \left( \sum_{s=0}^{N-1} \frac{a_s(z)}{u^{2s}} + g_2(u, z) \right) - \frac{z}{u} K_b(uz) \left( \sum_{s=0}^{N-1} \frac{b_s(z)}{u^{2s}} + zh_2(u, z) \right),
\]

(8.17)

where again (8.16) holds. The polynomials \( a_s(z), b_s(z) \) are defined by (6.2), (6.3).
It is easy to see that Lemma 8.2 remains valid. Since we allow \( -t \) otic expansions for \( u \to \infty \),

we use the integral representation \([7, (13.4.14)]\)

\[
V(u, \mu, z) := \Gamma(1 + a - b)2^{-b}a^{b-1}e^{-\frac{1}{2}z^2}z^b U(a, b, z^2).
\]

Then we have

\[
V(u, -\mu, z) = \Gamma(a)2^{b-2}a^{1-b}e^{-\frac{1}{2}z^2}z^b U(a, b, z^2)
\]

which follows from \([7, (13.2.40)]\)

\[
U(a, b, x) = x^{1-b}U(1 + a - b, 2 - b, x).
\]

For any \( b \in \mathbb{C} \), \( (6.9) \), \( (6.10) \), \( (6.12) \), \( (6.13) \), \( (8.14) \) show that the expansions \( (8.15) \) and \( (8.17) \) are equivalent. We will prove \( (8.15) \) and \( (8.17) \) for \( \Re\mu \geq 0 \) and \( \Re\mu < 0 \), respectively.

Suppose \( \Re\mu \geq 0 \). Then \( (8.15), (8.16) \) follow from Lemmas 8.5 and 8.6 when \( |\arg z| \leq \frac{3}{2}\pi - \delta \). Since the function \( V(u, \mu, z) \) is independent of \( R \) we can replace \( \frac{1}{R} \) by \( R \). By Theorem 4.1, we can remove the restriction on \( \arg z \). Note that in the proof of Theorem 4.1 we only used that \( W_2(u, z) \) solves \( (2.1) \) and admits the asymptotic expansions \( (2.7), (2.8) \). Therefore, we can apply the theorem to the function \( V(u, \mu, z) \) in place of \( W_2(u, z) \).

Now suppose that \( \Re\mu < 0 \). Then, using the expansion we just proved,

\[
V(u, -\mu, z) = zK_{-\mu}(uz) \left( \sum_{s=0}^{N-1} \frac{A_s(-\mu, z)}{u^{2s}} + g_2(u, z) \right) - \frac{z}{u}K_{-\mu+1}(uz) \left( \sum_{s=0}^{N-1} \frac{B_s(-\mu, z)}{u^{2s}} + zh_2(u, z) \right).
\]

Using \( (6.2) \), \( (6.3) \), \( (7.6) \) and \( K_\nu(x) = K_{-\nu}(x) \), we obtain \( (8.17), (8.16) \).

So far we considered only asymptotic expansions of \( U(a, b, z^2) \) as \( 0 < u \to \infty \). Now we set \( u = te^{i\theta} \), where \( t > 0 \) and \( -\frac{1}{2}\pi < \theta < \frac{1}{2}\pi \). Using the notation of Section 5, we have

\[
e^{-it}W_2(t, x) = \beta_1(u)e^{-\frac{1}{2}t^2}t^bM(a, b, z^2) + \beta_2(u)e^{-\frac{1}{2}t^2}z^bU(a, b, z^2).
\]

It is easy to see that Lemma 8.2 remains valid. Since we allow \( -\frac{1}{2}\pi < \theta < \frac{1}{2}\pi \), \( a = \frac{1}{4}u^2 + \frac{1}{2}b \) may have negative real part. We need a modification of Lemma 8.3.

**Lemma 8.8.** Let \( b \in \mathbb{C}, -\pi < \arg x < 0, |\arg(a - 1)| \leq \pi - \delta \) for some \( \delta > 0 \). Then there is a constant \( Q \) independent of \( a \) such that

\[
|\Gamma(a)U(a, b, x)| \leq Q.
\]

**Proof.** We use the integral representation \([7, (13.4.14)]\)

\[
(e^{2\pi(a-1)} - 1)\Gamma(a)U(a, b, x) = \int_C e^{-xt}t^{a-1}(1 + t)^{b-a-1}dt,
\]

where the contour \( C \) starts at \( +\infty i \) and follows the positive imaginary axis, then describes a loop around \( 0 \) in positive direction and returns to \( +\infty i \). The argument of \( t \) starts at \( \frac{1}{2}\pi \) and increases to \( \frac{3}{2}\pi \). It will be sufficient to estimate \( \Gamma(a)U(a, b, x) \) in the sector \( \frac{1}{2}\pi \leq \arg(a - 1) \leq \alpha_0 \), where \( \frac{1}{2}\pi < \alpha_0 < \pi \). The loop is chosen so that \( w = \frac{1}{1+t} \) describes the circle \( |w| = \cos\theta_0 \), where \( \theta_0 \in (0, \frac{1}{2}\pi) \) is the unique solution of the equation

\[
\cos\theta_0 = e^{\theta_0 \tan \alpha_0}.
\]

Then one obtains \( |w^{a-1}| \leq 1 \) on the contour \( C \) which implies the desired estimate.

The proofs of Lemma 8.6 and Theorem 8.7 can be easily modified to give the desired asymptotic expansions for \( u = te^{i\theta} \) as \( 0 < t \to \infty \) for fixed \( \theta \in (-\frac{1}{2}\pi, \frac{1}{2}\pi) \). In \( (8.16) \) we now have \( u = te^{i\theta}, t \geq t_2 \) and \( 0 < |z| \leq R \).
9 Comparison with Temme [9]

It is known [7, (5.11.13)] that, as $z \to \infty$, $|\arg z| \leq \pi - \delta$,

$$\frac{\Gamma(z + r)}{\Gamma(z + s)} \sim z^{r-s} \sum_{n=0}^{\infty} \binom{r-s}{n} B_n^{(r-s+1)}(r) \frac{1}{z^n}, \quad (9.1)$$

where the generalized Bernoulli polynomials $B_n^{(\ell)}(x)$ are defined by the Maclaurin expansion

$$\left( \frac{t}{e^t - 1} \right)^\ell e^{xt} = \sum_{n=0}^{\infty} B_n^{(\ell)}(x) \frac{t^n}{n!}.$$

We apply (9.1) with $z = \frac{1}{4} u^2$, $0 < u \to \infty$, and $r = 1 - \frac{1}{2} b$, $s = \frac{1}{2} b$. Then we obtain with $a = \frac{1}{4} u^2 + \frac{1}{2} b$,

$$\frac{\Gamma(1 + a - b)}{\Gamma(a)} 2^{2-2b} u^{2b-2} \sim \sum_{n=0}^{\infty} \frac{d_n}{u^{2n}},$$

where

$$d_n = 4^n \binom{1-b}{n} B_n^{(2-b)} \left( 1 - \frac{1}{2} b \right).$$

We notice that

$$\left( \frac{t}{e^t - 1} \right)^{2-b} e^{(1-\frac{1}{2} b)t} = \left( \frac{1}{2} \frac{t}{\sinh \frac{1}{2} t} \right)^{2-b}$$

is an even function of $t$. Therefore, $d_n = 0$ for odd $n$.

It follows from (8.14) that

$$B_n'(0) = \frac{1}{2} \frac{1}{1 - b} d_{n+1},$$

and then from (6.2), (6.3)

$$a_n(0) = \tilde{d}_n, \quad b_n'(0) = -\frac{1}{2} \frac{1}{1 - b} \tilde{d}_{n+1}, \quad (9.2)$$

where $\tilde{d}_n$ is obtained from $d_n$ by replacing $b$ by $2 - b$, that is,

$$\tilde{d}_n = 4^n \binom{b-1}{n} B_n^{(b)} \left( \frac{1}{2} b \right).$$

Temme [9, (3.22)] obtained the asymptotic expansion of (8.17) involving polynomials $a_n^\dagger(z)$, $b_n^\dagger(z)$ in place of $a_n(z)$, $b_n(z)$. The polynomials $a_n^\dagger(z)$, $b_n^\dagger(z)$ as follows. Introduce the function

$$f(s, z) = e^{2z} \mu(s) \left( \frac{1}{2} \frac{s}{\sinh \frac{1}{2} s} \right)^b, \quad \mu(s) = \frac{1}{s} - \frac{1}{e^s - 1} - \frac{1}{2},$$

and its Maclaurin expansion

$$f(s, z) = \sum_{k=0}^{\infty} c_k(z) s^k.$$
Then recursively, set $c_k^{(0)} = c_k$ and
\[ c_k^{(n+1)} = 4(z^2 c_{k+2}^{(n)} + (1 - b + k)c_{k+1}^{(n)}), \]
where $k \geq 0$ and $n \geq 0$. Then set
\[ a_n^\dagger = c_0^{(n)}, \quad b_n^\dagger = -2zc_1^{(n)}. \]

**Theorem 9.1.** For every $n = 0, 1, 2, \ldots$, we have $a_n = a_n^\dagger$ and $b_n = b_n^\dagger$.

**Proof.** The function $f$ satisfies the partial differential equation
\[ 4\frac{\partial f}{\partial s} = \frac{\partial^2 f}{\partial z^2} + \frac{1}{z} \left( 2b - 1 - 4 \frac{z^2}{s} \right) \frac{\partial f}{\partial z} - z^2 f. \]
This implies
\[ 4(k+1)c_{k+1} + 4zc_{k+1}' = c_k'' + \frac{2b-1}{z} c_k' - z^2 c_k, \quad (\cdot)' = \frac{d}{dz}. \tag{9.3} \]
By induction on $n$ one can show that (9.3) is also true with $c_k$ replaced by $c_k^{(n)}$ for any $n = 0, 1, 2, \ldots$. If we use this extended equation with $k = 0$ and $k = 1$, then we obtain (2.2), (2.3) with $a_s^\dagger, b_s^\dagger$ in place of $A_s, B_s$, respectively.

When $z = 0$, we have
\[ a_n^\dagger(0) = c_0^{(n)}(0) = 4^n(1-b)_n c_n(0) = 4^n \frac{(1-b)_n}{n!} B_n^{(b)} \left( \frac{1}{2} b \right). \]
Comparing with (9.2) and using that $c_n(0) = 0$ for odd $n$, we find, for all $n$,
\[ a_n^\dagger(0) = a_n(0). \tag{9.4} \]
Since both $a_n, b_n$ and $a_n^\dagger, b_n^\dagger$ solve (2.2), (2.3), (9.4) implies that $a_n^\dagger = a_n, b_n^\dagger = b_n$ for all $n$. ■

**10 Concluding remark**

In this paper we started from Olver’s paper [5], added some results, and then applied them to the confluent hypergeometric functions. A referee pointed out that Chapter 12 of Olver’s book [6] contains a reworked version of [5] also involving error bounds. It would be interesting to start from this book chapter and derive results analogous to the ones obtained in the present paper. However, in contrast to [5] the book chapter assumes that $\mu$ is positive while in our original problem [1] $\mu$ is complex. Therefore, an extension of the results in [6, Chapter 12] to complex $\mu$ would be required to obtain results for the confluent hypergeometric functions in full generality.

**References**

[1] Cohl H.S., Hirtenstein J., Volkmer H., Convergence of Magnus integral addition theorems for confluent hypergeometric functions, arXiv:1601.02566.

[2] Magnus W., Zur Theorie des zylindrisch-parabolischen Spiegels, Z. Physik 118 (1941), 343–356.

[3] Magnus W., Über eine Beziehung zwischen Whittakerschen Funktionen, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. 1946 (1946), 4–5.
[4] Olver F.W.J., The asymptotic solution of linear differential equations of the second order for large values of a parameter, *Philos. Trans. Roy. Soc. London, Ser. A.* **247** (1954), 307–327.

[5] Olver F.W.J., The asymptotic solution of linear differential equations of the second order in a domain containing one transition point, *Philos. Trans. Roy. Soc. London, Ser. A.* **249** (1956), 65–97.

[6] Olver F.W.J., Asymptotics and special functions, *Computer Science and Applied Mathematics*, Academic Press, New York – London, 1974.

[7] Olver F.W.J., Lozier D.W., Boisvert R.F., Clark C.W. (Editors), NIST handbook of mathematical functions, U.S. Department of Commerce National Institute of Standards and Technology, Washington, DC, Cambridge University Press, Cambridge, 2010, available at [http://dlmf.nist.gov](http://dlmf.nist.gov).

[8] Slater L.J., Confluent hypergeometric functions, Cambridge University Press, New York, 1960.

[9] Temme N.M., Remarks on Slater’s asymptotic expansions of Kummer functions for large values of the $\alpha$-parameter, *Adv. Dyn. Syst. Appl.* **8** (2013), 365–377, [arXiv:1306.5328](http://arxiv.org/abs/1306.5328).