Commutators of the four-current and sum rules in relativistic nuclear models

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There is a long-standing problem with the linearly energy-weighted sum of the excitation strengths in relativistic field theory and nuclear models: The sum value should be positive definite, while its naive calculation using the current commutator or the double commutator of the excitation operator with the Dirac Hamiltonian results in the value vanishing. This paradoxical contradiction is solved in an analytic way.

Subject Index D10

1. Introduction

The linearly energy-weighted sum \(S\) of the excitation strengths is expressed, using the double commutator of the excitation operator \(F\) with the Hamiltonian \(H\), as

\[
S = \sum_n (E_n - E_0) |\langle n | F | 0 \rangle|^2 = \frac{1}{2} \langle 0 | [F^\dagger, [H, F]] | 0 \rangle,
\]

where the closure property, \(\sum_n |n\rangle \langle n| = 1\), is employed, \(|n\rangle\) denoting the eigenstate of \(H\), \(H |n\rangle = E_n |n\rangle\). If there exist excited states with \((E_n - E_0) > 0\) and \(|\langle n | F | 0 \rangle| \neq 0\), the value of the sum should obviously be positive.

In non-relativistic models, for example, the well-known f-sum rule value \(S_f\) is obtained for an \(A\)-particle system as \([1]\):

\[
S_f = \frac{A}{2m} q^2,
\]

for

\[
F = \sum_{i=1}^A f(x_i), \quad f(x_i) = \exp(\text{i} q \cdot x_i),
\]

since the double commutator becomes a constant:

\[
[F^\dagger, [H, F]] = \sum_{i=1}^A \left[ f^*(x_i), \left[ \frac{p_i^2}{2m}, f(x_i) \right] \right] = \frac{A}{m} q^2.
\]

Here, the Hamiltonian is assumed to be

\[
H = \sum_{i=1}^A h_i, \quad h_i = \frac{p_i^2}{2m} + V(x_i),
\]

with potential \(V(x_i)\) which commutes with \(F\).
In relativistic models, however, the Dirac Hamiltonian contains the first derivative only,

\[ h = \alpha \cdot p + \beta m + V(x), \quad (6) \]

so that the double commutator vanishes,

\[ [f^*(x), [h, f(x)]] = 0, \quad (7) \]

in contradiction with \( S > 0 \).

Let us briefly review more generally the above result of the relativistic case according to the field theory. The nuclear four-current is given in terms of the nucleon field \( \psi(x) \) by,

\[ J^\mu(x) = \bar{\psi}(x) \gamma^\mu \psi(x). \quad (8) \]

Since the excitation operator \( F \) with a function \( f(x) \) is defined as

\[ F = \int d^3x f(x) J^0(x), \quad (9) \]

current conservation

\[ [H, J^0(x)] = i\nabla \cdot J(x) \quad (10) \]

provides us with

\[ [F^+, [H, F]] = \int d^3xd^3y f^*(x) f(y) i\nabla_y \cdot [J^0(x), J(y)]. \quad (11) \]

In using the anti-commutation relation as usual,

\[ \{\psi_m(x), \psi_n^\dagger(y)\} = \delta_{mn}\delta(x-y), \quad (12) \]

\( m \) and \( n \) being the Dirac matrix indices, the nuclear four-current satisfies

\[ [J^\mu(x), J^\nu(y)] = \bar{\psi}(x) [\gamma^0\gamma^\mu, \gamma^0\gamma^\nu] \psi(x) \delta(x-y). \quad (13) \]

Thus, the time component \( J^0(x) \) and the space component \( J(y) \) of the current commute with each other:

\[ [J^0(x), J(y)] = 0. \quad (14) \]

This fact makes Eq. (11) vanish, in contradiction with \( S > 0 \).

In the nonrelativistic framework, the commutation relation corresponding to Eq. (14) is written as [1]:

\[ [J^0(x), J(y)] = -\frac{i}{m} \sum_{k=1}^A \delta(y-x_k) \nabla_x \delta(x-y), \quad (15) \]

with the nonrelativistic four-current:

\[ J^0(x) = \sum_{k=1}^A \delta(x-x_k), \quad J(x) = \frac{1}{2m} \sum_{k=1}^A \{p_k, \delta(x-x_k)\}. \quad (16) \]

By inserting Eq. (15) into Eq. (11) and using

\[ \langle 0 | i [J^0(x), J(y)] | 0 \rangle = \frac{1}{m} \rho(y) \nabla_x \delta(x-y), \quad \rho(x) = \left\langle 0 \left| \sum_{k=1}^A \delta(x-x_k) \right| 0 \right\rangle, \quad (17) \]
we obtain, from Eq. (1), [1]:

\[ S = \frac{1}{2m} \int d^3x \rho(x) |\nabla f(x)|^2, \]  

(18)

where \( \rho(x) \) stands for the ground state density of the many-body system. If we set \( f(x) = \exp(ig \cdot x) \) in Eq. (18), we have the \( f \)-sum rule Eq. (2).

For the last more than 50 years, much has been written, from different points of view, on the above problem in relativistic field theory [2–11] and nuclear models [12,13].

In relativistic field theory, Schwinger [3] pointed out that Eq. (14) should have a gradient of a \( \delta \)-function on the right-hand side from Lorentz covariance considerations [5]. That additional term, called the Schwinger term, plays an important role, especially in current algebra, and has been widely explored [4–11], but its form is not well defined yet. For example, on the one hand, Schwinger reproduced the term by introducing the point-split current [3]. On the other hand, Gasiorowicz and Geffen [6] derived it by using the vector-meson dominance model, while Weinberg and Gross et al. [7–9] discussed it using SU(3) × SU(3) algebra.

In nuclear studies, Price et al. [12,13] tried to interpret Eq. (7) by invoking, in addition to usual particle-hole excitations, transitions of particles in the Fermi sea to negative energy states in the Walecka–Serot model [14,15]. The reason for the contradiction, however, is not made clear, and the role of the Schwinger term in this nuclear model has not been investigated so far.

The purpose of this paper is to show that within the framework of local field theory, the correct calculation of the right-hand side of Eq. (1) yields the Schwinger term which is responsible for a positive value of \( S \).

In the following section, we will define the relativistic four-current in the finite momentum space, where the time and space components do not commute with each other. If we make the space infinite, the commutator will disappear, as in Eq. (14), yielding the contradiction. In Sect. 3, however, it will be shown that the expectation value of the commutator should be calculated first, keeping the momentum space finite. That expectation value does not vanish, even in letting the momentum space be infinite later. The relationship of the present result with Schwinger's non-local current [3] will also be discussed. In Sect. 4, sum rule values of relativistic nuclear models [12,13,16,17] will be examined, according to the new insight of the present paper. Moreover, non-relativistic sum values will be derived from relativistic ones in the same framework. The final section will be devoted to a brief summary of the present work.

2. The four-current

It is clear that the contradiction in relativistic sum values stems from Eq. (12), which is normally used for calculations in field theory. In our formalism, therefore, we begin with the definition of the nucleon field,

\[ \psi(x) = \sum_\alpha \Theta_\alpha w_\alpha(x)a_\alpha. \]  

(19)

Here, we have used following abbreviations:

\[ \Theta_p = \theta(P_{\infty} - |p|), \quad w_\alpha(x) = \frac{1}{\sqrt{\mathcal{V}}} w_s(p\sigma)e^{ip\cdot x}, \quad a_\alpha = a_s(p\sigma), \]  

(20)
where \( \alpha \) denotes \( \{ s = \pm, p, \sigma \} \), \( V \) is the volume of the system, and \( w \) is the spinor

\[
\begin{align*}
    w_+(p\sigma) &= \sqrt{\frac{E_p + m}{2E_p}} \left( \frac{\chi_{\sigma}}{\sigma \cdot p} \right), \\
    w_-(p\sigma) &= \sqrt{\frac{E_p + m}{2E_p}} \left( -\frac{\sigma \cdot p}{E_p + m} \chi_{\sigma} \right),
\end{align*}
\]

with \( E_p = \sqrt{p^2 + m^2} \) and the 2-component spinor, \( \chi_{\sigma} \). The notations \( a_+(p\sigma) \) and \( a_-(p\sigma) \) stand for the annihilation operator of a particle and the creation operator of an antiparticle, respectively, satisfying

\[
\{ a_s(p\sigma), a^\dagger_{s'}(p'\sigma') \} = \delta_{pp'}\delta_{\sigma\sigma'}\delta_{\chi\chi'}, \quad \text{others} = 0.
\]

In the above field, the range of \( [p] \) is restricted by \( P_\infty \), which is finite for a while. In the limit \( P_\infty \rightarrow \infty \), Eq. (19) is reduced to the usual field, but we will take the limit later.

The anti-commutation relation of the new field becomes

\[
\{ \psi_m(x), \psi^\dagger_n(y) \} = D_{mn}(x, y),
\]

where we have defined

\[
D_{mn}(x, y) = \sum_\alpha \Theta_\alpha \left( w_\alpha(x) w^\dagger_\alpha(y) \right)_{mn} = \delta_{mn} d(x - y).
\]

\[
d(x) = \frac{1}{V} \sum_p \Theta_\alpha e^{ip \cdot x} = \frac{d^3p}{(2\pi)^3} \Theta_\alpha e^{ip \cdot x}.
\]

It can be seen that Eq. (24) is reduced to Eq. (12) in the limit \( P_\infty \rightarrow \infty \), since

\[
d(x) \rightarrow \delta(x), \quad (P_\infty \rightarrow \infty).
\]

The commutation relation between currents is calculated by using the following equation for arbitrary \( 4 \times 4 \) matrices \( \Gamma_1(x) \) and \( \Gamma_2(x) \):

\[
[\psi^\dagger(x) \Gamma_1(x) \psi(x), \psi^\dagger(y) \Gamma_2(y) \psi(y)] = \psi^\dagger(x) \Gamma_1(x) D(x, y) \Gamma_2(y) \psi(y)
\]

\[
- \psi^\dagger(y) \Gamma_2(y) D(y, x) \Gamma_1(x) \psi(x).
\]

For \( \Gamma_1 = 1 \) and \( \Gamma_2 = \gamma^0 \gamma \), we have

\[
[J^0(x), J(y)] = (\bar{\psi}(x) y \psi(y) - \bar{\psi}(y) y \psi(x)) d(x - y).
\]

which does not vanish for the finite value of \( P_\infty \), differently from Eq. (14).

Since Eq. (28) holds, even if \( \Gamma \) contains differential operators, we obtain

\[
[H, \psi^\dagger(x) \Gamma(x) \psi(x)] = (h_0(x) \psi(x)) \Gamma(x) \psi(x) - \psi^\dagger(x) \Gamma(x) h_0(x) \psi(x),
\]

where we have used the fact that

\[
\int d^3 y \ D(x, y) h_0(y) \psi(y) = h_0(x) \psi(x)
\]

for the one-body Hamiltonian,

\[
h_0(x) = -i \alpha \cdot \nabla + \beta m, \quad H = \int d^3 x \ \psi^\dagger(x) h_0(x) \psi(x).
\]
From Eq. (30), the equation of current conservation is obtained, like Eq. (10):

\[ [H, J^0(x)] = i \left( \nabla \psi^\dagger(x) \cdot \alpha \psi(x) + i \psi^\dagger(x) \alpha \cdot \nabla \psi(x) \right) = i \nabla \cdot J(x), \]

which gives the same expression for the double commutator as in Eq. (11).

For the one-body operator,

\[ F_i = \int d^3x \psi^\dagger(x) \Gamma_1(x) \psi(x), \]

we have, from Eq. (28),

\[ [F_1, F_2] = \int d^3x d^3y \psi^\dagger(x)(\Gamma_1(x)D(x, y)\Gamma_2(y) - \Gamma_2(x)D(x, y)\Gamma_1(y))\psi(y), \]

and, from Eq. (30),

\[ [H, F_1] = \int d^3x \psi^\dagger(x)[h_0(x), \Gamma_1(x)]\psi(x). \]

The above two equations give another expression of the double commutator of Eq. (1):

\[ [F^\dagger, [H, F]] = \int d^3x d^3y \psi^\dagger(x)(f^*(x)D(x, y)[h_0(y), f(y)] \]

\[ - [h_0(x), f(x)]D(x, y)f^*(y))\psi(y). \]

If we take the limit \( P_\infty \to \infty \) in the above equation, \( D(x, y) \) becomes \( \delta(x - y) \), so that we have the undesired result

\[ [F^\dagger, [H, F]] = \int d^3x \psi^\dagger(x)[f^*(x), [h_0(x), f(x)]]\psi(x) = 0, \]

as mentioned in Sect. 1. It will be shown that the limit \( P_\infty \to \infty \) should be taken after calculating the ground-state expectation value of Eq. (35) or Eq. (29) used in Eq. (11).

3. Expectation value of the current

Let us calculate the ground-state expectation value of Eq. (29). Assuming isospin symmetric nuclear matter with Fermi momentum \( k_F \), we have

\[ \langle 0|\bar{\psi}(x)\gamma_i\psi(y)|0 \rangle = \frac{4}{V} \sum_p \left( \theta_p - \Theta_p \right) \frac{P}{E_p} e^{-ip(x-y)} = j(x-y), \]

where the notation \( \theta_p = \theta(k_F - |p|) \) is used. The term with \( \Theta_p \) comes from the contributions of the Dirac sea. Using the above equation, the expectation value of Eq. (29) is expressed as

\[ \langle 0|[iJ^0(x), J(y)]|0 \rangle = 2ij(x-y)\delta(x-y) = K(x-y), \]

with

\[ K(x) = \frac{8i}{V^2} \sum_{p,q} \left( \theta_p - \Theta_p \right)\Theta_{p+q} \frac{P}{E_p} e^{iq \cdot x} \]

\[ = \frac{8i}{V^2} \sum_{p,q} \left( \theta_p - \Theta_p \right)\Theta_{p+q} \frac{q \cdot p / q^2}{E_p} e^{iq \cdot x} \]

\[ = \frac{2i}{V^2} \sum_q e^{iq \cdot x} \frac{q}{q^2} (S_N(q) + S_N(q)), \]

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where $S_N(q)$ and $S_N(q)$ stand for

$$S_N(q) = 4 \sum E_p \theta(p + q) \frac{p \cdot q}{E_p}, \quad S_N(q) = -4 \sum E_p \theta(p + q) \frac{p \cdot q}{E_p}. \tag{40}$$

The calculation of $S_N(q)$ is performed by replacing the sum with the integral. Defining $t = p \cdot q / pq$, $S_N(q)$ is expressed as

$$S_N(q) = \frac{V q}{\pi^2} \int_0^{k_F} \frac{dp}{E_p} \frac{p^3}{E_p} \int_{-1}^1 d\theta \Theta_{p+q} t = \frac{V q}{\pi^2} \int_0^{k_F} \frac{dp}{E_p} \frac{p^3}{E_p} \int_{-1}^1 d\theta (\xi - t), \tag{41}$$

with

$$\xi = \frac{p^2 \infty - p^2 - q^2}{2pq},$$

where $\xi$ should be $\xi > -1$. Finally, we have

$$S_N(q) = \begin{cases} 0, & |P_{\infty} - q| > k_F; \\ \frac{V}{8\pi^2 q} G(|P_{\infty} - q|, k_F), & |P_{\infty} - q| < k_F, \end{cases} \tag{42}$$

where $G$ is defined as

$$G(a, b) = \int_a^b dp \frac{E_p}{P_0} ((p + q)^2 - P_{\infty}^2)((p - q)^2 - P_{\infty}^2). \tag{43}$$

The above result can be understood as follows. The value of $S_N(q)$ vanishes in the region $q < P_{\infty} - k_F$ where $\Theta_{p+q} = 1$, since the sum of $p$ cancels the terms with $p$ and $-p$ in Eq. (40), while in the region $|P_{\infty} - q| < k_F$, there is no such perfect cancellation in the sum of $p$. In the region $q > P_{\infty} + k_F$, $\Theta_{p+q} = 0$ yields $S_N(q) = 0$.

The expression of $S_N(q)$ from the Dirac sea is obtained by replacing $k_F$ in Eq. (42) with $P_{\infty}$:

$$S_N(q) = \begin{cases} 0, & q > 2P_{\infty}; \\ -\frac{V}{8\pi^2 q} G(|P_{\infty} - q|, P_{\infty}), & q < 2P_{\infty}. \end{cases} \tag{44}$$

In the region $q > 2P_{\infty}$, the value of $S_N(q)$ disappears, because of $\Theta_p \Theta_{p+q} = 0$ in Eq. (40).

If there is no factor with $P_{\infty}$ in Eq. (40), both $S_N$ and $S_N$ vanish. The existence of $P_{\infty}$ yields a constraint on the states which contribute to Eq. (40). Thus, the operation of $P_{\infty} \to \infty$ and the calculation of Eq. (40) do not commute with each other.

The region which we may be physically interested in is in a range $q \ll P_{\infty}$ for $m \ll P_{\infty}$. In this region, the value of $S_N(q)$ vanishes, but that of $S_N(q)$ does not. When we expand the function $G$ in Eq. (44) in terms of $q / P_{\infty}$ for $m \ll P_{\infty}$, we have

$$G(|P_{\infty} - q|, P_{\infty}) = P_5 \int_0^{q/P_{\infty}} dx \frac{1 - x}{\sqrt{(1 - x)^2 + (m/P_{\infty})^2}} \times \left( (x^2 - 2x)^2 - 2(x^2 - 2x + 2) \frac{q^2}{P_{\infty}^2} + \frac{q^4}{P_{\infty}^4} \right) \approx -\frac{8}{3} P_5^3 \left( 1 - \frac{m^2}{P_{\infty}^2} \right) \frac{q^2}{P_{\infty}^3} \left( 1 - \frac{3q}{8P_{\infty}} - \frac{q^2}{5P_{\infty}^2} \right). \tag{45}$$
where \( x \) is defined by \( p = P_\infty (1 - x) \). The above equation shows that when \( P_\infty \to \infty \), the value of \( S_N(q) \) is divergent:

\[
S_N(q) = \frac{V}{3\pi^2} q^2 P_\infty^2.
\]  

(46)

More intuitive derivation of Eq. (46) may be performed by expanding the step function \( \Theta_{p+q} \) near \( |p| \approx P_\infty \) in Eq. (40):

\[
\Theta_{p+q} = \theta(P_\infty - |p| - \Delta p) = \Theta_p - \Delta p \delta(P_\infty - |p|) + \cdots, \quad \Delta p = |p + q| - |p|,
\]

(47)

which yields

\[
\Theta_p \Theta_{p+q} = \Theta_p - \frac{\Delta p}{2} \delta(P_\infty - |p|) + \cdots.
\]

(48)

This result, together with Eq. (40), provides us with

\[
S_N(q) \approx 2 \sum_p \Delta p \delta(P_\infty - |p|) \frac{p \cdot q}{E_p} \approx \frac{V}{3\pi^2} q^2 P_\infty^2,
\]

(49)

for \( m \ll P_\infty \), as in Eq. (46).

In the limit \( P_\infty \to \infty \) for \( q \ll P_\infty \), where \( S_N(q) \) vanishes and \( S_N(q) \) is given by Eq. (49), Eq. (39) is described as

\[
K(x) = \frac{2P_\infty^2}{3\pi^2} \nabla \sum_q e^{iq \cdot x} = \frac{2P_\infty^2}{3\pi^2} \nabla \delta(x).
\]

(50)

Consequently, we obtain the expression for the commutator of the currents,

\[
\langle 0 | [iJ^0(x), J(y)] | 0 \rangle = \frac{2P_\infty^2}{3\pi^2} \nabla_x \delta(x - y).
\]

(51)

This is nothing but the gradient of the \( \delta \)-function required by Schwinger.

Instead of Eq. (8), Schwinger [3] assumed the point-split current for the space part,

\[
J_\mu(x) = \overline{\psi}(x - \epsilon/2) \gamma^\mu \psi(x + \epsilon/2) \quad (\epsilon \to 0),
\]

(52)

which gives the commutation relation with the time-component,

\[
[J^0(x), J_\epsilon(y)] = (\delta(x - y + \epsilon/2) - \delta(x - y - \epsilon/2)) J_\epsilon(y)
\]

(53)

\[
= J_\epsilon(y) \epsilon \cdot \nabla_x \delta(x - y).
\]

The calculation of the ground-state expectation value provides us with [3]:

\[
\langle 0 | [iJ^0(x), J(y)] | 0 \rangle = \frac{4}{3\pi^2 \epsilon^2} \nabla_x \delta(x - y).
\]

(54)

This has a divergent limit of \( 1/\epsilon^2 \), and is the same as the present result Eq. (51) when setting \( P_\infty^2 = 2/\epsilon^2 \).

The relationship between the present current and Schwinger’s may also be seen qualitatively as follows. On the one hand, Eq. (53) is written as

\[
[J^0(x), J_\epsilon(y)] = \overline{\psi}(x) \gamma^\epsilon \psi(x + \epsilon/2) \delta(x - y + \epsilon/2) - \overline{\psi}(x - \epsilon) \gamma^\epsilon \psi(x) \delta(x - y - \epsilon/2).
\]

(55)

On the other hand, the present function \( d(x) \) in Eq. (26) is calculated as

\[
d(x) = -\frac{1}{2\pi^2} \frac{\sin P_\infty x}{x},
\]

(56)
which has the spreading width about $|x| \lesssim 2\pi/P_\infty$ around $x = 0$. Therefore, if we replace $\psi(y)d(x - y)$ by $\psi(x + \epsilon)\delta(x - y + \epsilon/2)$ with $\epsilon \sim 1/P_\infty$ in Eq. (29), we have the same form as Eq. (55) of Schwinger’s model.

Thus, the commutator with non-local current Eq. (52) assumed by Schwinger [3] is well understood, and the limit $P_\infty \to \infty$ should be taken after calculating the expectation value of the commutator in the local field theory.

4. Sum values in relativistic nuclear models

The sum value $S$ of the excitation strengths for the operator $f(x)$ is obtained by Eqs. (11) and (51) as

$$S = \frac{1}{2} \langle 0 | [F^+, [H, F]] | 0 \rangle = \frac{P_\infty^2}{3\pi^2} \int d^3x |\nabla f(x)|^2. \quad (57)$$

In writing the excitation operator in a momentum space,

$$f(x) = \sum_q f(q)e^{iq \cdot x}, \quad f(q) = \frac{1}{V} \int d^3x f(x)e^{-iq \cdot x}, \quad (58)$$

Eqs. (38) and (39) give the sum value of the form:

$$S = \frac{1}{2} \int d^3x d^3y f^*(x)f(y)\nabla_y \cdot K(x - y) = \sum_q |f(q)|^2 (S_N(q) + S_{\bar{N}}(q)). \quad (59)$$

Eq. (57) is for the case of $m, q \ll P_\infty (P_\infty \to \infty)$, where $S_N(q) = 0$, in the above equation.

When the function $f(x)$ is given by Eq. (3), Eq. (59) becomes

$$S = S_N(q) + S_{\bar{N}}(q). \quad (60)$$

It will be shown later how the above equation is reduced to the nonrelativistic f-sum rule value Eq. (2).

According to the result Eq. (59), let us discuss the sum values of relativistic nuclear models which have been extensively used for nuclear study, and shown to work well phenomenologically [14,15].

The full energy-weighted transition strengths of an $A$-nucleon system in the mean field are given by

$$S = S_{\text{ph}} + S_{\text{p}\bar{N}}, \quad (61)$$

with

$$S_{\text{ph}} = \sum_{ph} (E_p - E_h)|\langle p|f|h\rangle|^2, \quad S_{\text{p}\bar{N}} = \sum_{b<0,p} (E_p - E_b)|\langle p|f|b\rangle|^2,$$

where $\{\text{ph}\}$ and $\{\text{p}\bar{N}\}$ represent particle-hole and particle-antinucleon excitations, respectively, and $|b\rangle (b < 0)$ stands for the negative energy states. The second term $S_{\text{p}\bar{N}}$ is described as

$$S_{\text{p}\bar{N}} = S_{\text{vac}} - S_{\text{Pauli}}. \quad (62)$$

where $S_{\text{vac}}$ denotes the transitions of antinucleons in the vacuum, that is, the Dirac sea, to the positive energy states, and $S_{\text{Pauli}}$ the Pauli blocking terms due to the existence of $A$ nucleons:

$$S_{\text{vac}} = \sum_{b<0,a>0} (E_a - E_b)|\langle a|f|b\rangle|^2, \quad S_{\text{Pauli}} = \sum_{b<0,h} (E_h - E_b)|\langle h|f|b\rangle|^2. \quad (63)$$
Then, the total sum $S$ in Eq. (61) is written as

$$S = S_{\text{NoSA}} + S_{\text{vac}},$$

(64)

where $S_{\text{NoSA}}$ is defined by

$$S_{\text{NoSA}} = S_{\text{ph}} - S_{\text{Pauli}} = S_{\text{ph}} + S_{\text{Nh}},$$

(65)

Thus, $S_{\text{NoSA}}$ is nothing but the sum value in the no-sea approximation, which is extensively used in relativistic nuclear models [16,17]. In this approximation, the negative energy states are assumed to be empty, $S_{\text{vac}} = 0$. As a result, there is the second term $S_{\text{Nh}}$ which represents transitions of the particles in the Fermi sea to the negative energy states with negative excitation energies, yielding unphysical negative energy-weighted strengths in the $\bar{N}$ excitation energy region as a cost of neglecting $S_{\text{vac}}$ [18].

Since we have the identities

$$\sum_{h,h'} (E_{h'} - E_h)|h'|f|h|2 = 0, \quad \sum_{b<0,b'<0} (E_{b'} - E_b)|b'|f|b|2 = 0,$$

(66)

we can write $S_{\text{NoSA}}$ and $S_{\text{vac}}$ as

$$S_{\text{NoSA}} = \sum_{h,\alpha} (E_{\alpha} - E_h)|\alpha|f|h|2 = \sum_{h,\alpha} |h|f^*|\alpha|\langle h_0, f|h\rangle,$$

$$S_{\text{vac}} = \sum_{b<0,\alpha} (E_{\alpha} - E_b)|\alpha|f|b|2 = \sum_{b<0,\alpha} |b|f^*|\alpha|\langle h_0, f|b\rangle,$$

(67)

where $\alpha$ denotes both positive $|\alpha\rangle (\alpha > 0)$ and negative $|b\rangle (b < 0)$ energy states. When we express the time-reversal state of $|\alpha\rangle$ by $|\tilde{\alpha}\rangle$, they are also written as

$$S_{\text{NoSA}} = \sum_{h,\alpha} \langle \tilde{h}|f|\tilde{\alpha}\rangle\langle \tilde{\alpha}|h_0, f^*|\tilde{h}\rangle = -\sum_{h,\alpha} \langle h||h_0, f|\alpha\rangle\langle \alpha|f^*|h\rangle,$$

$$S_{\text{vac}} = \sum_{b<0,\alpha} \langle \tilde{b}|f|\tilde{\alpha}\rangle\langle \tilde{\alpha}|h_0, f^*|\tilde{b}\rangle = -\sum_{b<0,\alpha} \langle b||h_0, f|\alpha\rangle\langle \alpha|f^*|b\rangle,$$

which give the expressions

$$S_{\text{NoSA}} = \frac{1}{2} \sum_{h,\alpha} (\langle h|f^*|\alpha\rangle\langle h_0, f|\alpha\rangle - \langle h||h_0, f|\alpha\rangle\langle \alpha|f^*|h\rangle),$$

$$S_{\text{vac}} = \frac{1}{2} \sum_{b<0,\alpha} (\langle b|f^*|\alpha\rangle\langle b_0, f|\alpha\rangle - \langle b||h_0, f|\alpha\rangle\langle \alpha|f^*|b\rangle).$$

Now, if we were able to use the closure property in the intermediate states, $\sum_{\alpha} |\alpha\rangle\langle \alpha| = 1$, we would have

$$S_{\text{NoSA}} = \sum_h \langle h|f^*|h\rangle = 0, \quad S_{\text{vac}} = \sum_{b<0} \langle b|f^*|b\rangle = 0,$$

(68)

which led to misunderstanding the relativistic sum values [12,13]. We cannot use the closure property, since there are $|\alpha\rangle$ states which should be excluded by the step function with $P_\infty$. In fact, we have to
calculate Eq. (67) as follows:

\[ S_{\text{NoSA}} = \sum_{h, \alpha} \theta_h \Theta_{\alpha} \int d^3 x d^3 y f^*(x) w_h^\dagger(x) w_{\alpha}^\dagger(y) (-i\alpha \cdot \nabla f(y)) w_h(y) \]

\[ = -\frac{2i}{\sqrt{V}} \sum_{\rho} \theta_{\rho} \int d^3 x d^3 y e^{-ip \cdot (x-y)} d(x-y) f^*(x) \nabla f(y) \cdot \text{Tr}(\gamma^0 \Lambda^+_p), \]

\[ S_{\text{vac}} = \sum_{b<0, \alpha} \Theta_b \Theta_{\alpha} \int d^3 x d^3 y f^*(x) w_h^\dagger(x) w_{\alpha}^\dagger(y) (-i\alpha \cdot \nabla f(y)) w_h(y) \]

\[ = -\frac{2i}{\sqrt{V}} \sum_{\rho} \Theta_{\rho} \int d^3 x d^3 y e^{-ip \cdot (x-y)} d(x-y) f^*(x) \nabla f(y) \cdot \text{Tr}(\gamma^0 \Lambda^-_p), \]

where the projection operator \( \Lambda^\pm_p \) is defined as

\[ \Lambda^\pm_p = \sum_\sigma w_{\pm}(p\sigma) \bar{w}_{\pm}(p\sigma) = \frac{E_p \gamma^0 \mp \mathbf{p} \cdot \gamma \pm m}{2E_p}. \] (69)

By using the expressions of \( d(x) \) and \( f(x) \) in momentum space, and the fact that \( \text{Tr}(\gamma^0 \Lambda^\pm_p) = \pm 2p/E_p \), finally we obtain

\[ S_{\text{NoSA}} = 4 \sum_{p, q} \Theta_{p+q} |f(q)|^2 \frac{p \cdot q}{E_p} = \sum_q |f(q)|^2 S_N(q), \quad S_{\text{vac}} = \sum_q |f(q)|^2 S_N(q). \] (70)

When \( q \ll P_\infty \), we can replace \( \theta_{p+q} \) by \( \theta_p \) in the above \( S_N(q) \). Therefore, as far as discussions on \( S_{\text{NoSA}} \) are concerned, we can set \( P_\infty \to \infty \) at the beginning of calculations, which gives \( [f^*, [h_0, f]] = 0 \), and \( S_{\text{NoSA}} = 0 \).

The sum value of \( S_{ph} \) is given by,

\[ S_{ph} = \sum_{a>0, h} \langle h | f^* | a \rangle \langle a | [h_0, f] | h \rangle \]

\[ = \sum_{a>0, h} \theta_h \Theta_{\alpha} \int d^3 x d^3 y f^*(x) w_h^\dagger(x) w_{\alpha}^\dagger(y) (-i\alpha \cdot \nabla f(y)) w_h(y) \]

\[ = -\frac{2i}{\sqrt{V}} \sum_{\rho} \Theta_{\rho} \int d^3 x d^3 y e^{-ip \cdot (x-y)} f^*(x) \nabla f(y) \cdot \text{Tr}(D^+(x, y) \alpha \Lambda^+_p \gamma_0), \]

where \( D^+(x, y) \) is defined, as in Eq. (25), with

\[ D^+(x, y) = \sum_{\rho\sigma} \Theta_{\rho} w_{+\rho\sigma}(x) w_{+\rho\sigma}^\dagger(y) = \frac{1}{\sqrt{V}} \sum_{\rho} \Theta_{\rho} e^{ip \cdot (x-y)} \Lambda^+_p \gamma_0. \] (71)

It is calculated to be

\[ S_{ph} = \sum_q |f(q)|^2 S_{ph}(q), \quad S_{ph}(q) = 2 \sum_{p} \Theta_{p} \Theta_{p+q} \left( \frac{(p+q) \cdot q}{E_{p+q}} + \frac{p \cdot q}{E_p} \right). \] (72)

The transitions of particles in the Fermi sea to negative energy states are calculated in the same way:

\[ S_{\tilde{Nh}} = \sum_q |f(q)|^2 S_{\tilde{Nh}}(q), \quad S_{\tilde{Nh}}(q) = -2 \sum_{p} \Theta_{p} \Theta_{p+q} \left( \frac{(p+q) \cdot q}{E_{p+q}} - \frac{p \cdot q}{E_p} \right). \] (73)
The sum of $S_{\text{ph}}$ in Eq. (72) and $S_{\text{Nh}}(q)$ in Eq. (73) gives $S_N(q)$ as

$$S_N(q) = 4 \sum_p \theta_p \Theta_{p+q} \frac{P \cdot q}{E_p} = 0$$  \hspace{1cm} (74)

for $\theta_p \Theta_{p+q} = \theta_p (P_\infty - q > k_F)$, as seen in Eq. (42).

In nonrelativistic approximation, we may replace $E_{p+q}$ and $E_p$ by $m$ in Eq. (72), so that we obtain

$$S_{\text{ph}}^{\text{NR}} = \sum_q |f(q)|^2 S_0(q) = \frac{\rho}{2m} \int d^3 x |\nabla f(x)|^2, \quad \rho = \frac{A}{V} = \frac{2k_F^2}{3\pi^2},$$  \hspace{1cm} (75)

where $S_0(q)$ is defined in the limit $q \to 0$ as

$$S_{\text{ph}}(q) \to S_0(q) = \frac{A}{2E_F} q^2, \quad E_F = \sqrt{k_F^2 + m^2}.$$  \hspace{1cm} (76)

The above $S_{\text{ph}}^{\text{NR}}$ is the form of Eq. (18) for nuclear matter, and gives the f-sum rule Eq. (2) in the nonrelativistic framework.

As $q \to 0$, $S_{\text{ph}}(q)$ in Eq. (72) is proportional to $q^2$ like the f-sum rule, while as $q \to \infty$, it is proportional to $q$:

$$S_{\text{ph}}(q) \to \frac{k_F^3}{3\pi^2} V q \left(1 - \frac{m^2 + 2k_F^2}{2q^2}\right).$$  \hspace{1cm} (77)

It should be noted that, in relativistic models, $S_{\text{ph}}$, which is reduced to the nonrelativistic sum rule value, is exactly canceled by the subtraction of the Pauli blocking terms when $P_\infty - q > k_F$. Then, the relativistic sum value $S = S_{\text{vac}}$ stems from the transitions of antiparticles in the Dirac sea to positive energy states, which is infinite and is independent of the Fermi momentum or the $A$ of the nuclear system.

Before closing the present section, it may be useful to describe nonrelativistic sum rules in terms of the field theory developed in this paper. The nonrelativistic field is written as

$$\psi(x) = \frac{1}{\sqrt{V}} \sum_{p\sigma} \Theta_{p \sigma} e^{ip \cdot x} \psi_p(x),$$  \hspace{1cm} (78)

which satisfies the commutation relation $\{\psi_m(x), \psi_n^\dagger(y)\} = \delta_{mn} d(x - y)$. The nuclear four-current is given by

$$J_0(x) = \psi^\dagger(x) \psi(x), \quad J(x) = -\frac{i}{2m} (\psi^\dagger(x) \nabla \psi(x) - (\nabla \psi^\dagger(x)) \psi(x)).$$  \hspace{1cm} (79)

Their commutator is calculated as

$$[iJ_0(x), J(x)] = \frac{1}{m} (d(x - y) \nabla_x \rho(x, y) - \rho(x, y) \nabla_y d(x - y)),$$  \hspace{1cm} (80)

where we have defined

$$\rho(x, y) = \frac{1}{2} (\psi^\dagger(x) \psi(y) + \psi^\dagger(y) \psi(x)).$$  \hspace{1cm} (81)

Since we have

$$n(x - y) = \langle 0 | \rho(x, y) | 0 \rangle = \frac{4}{V} \sum_p \theta_p e^{-ip \cdot (x - y)},$$  \hspace{1cm} (82)

the expectation value of Eq. (80) is described as

$$\langle 0 | [iJ_0(x), J(x)] | 0 \rangle = \frac{1}{m} K(x - y),$$  \hspace{1cm} (83)
with
\[ K(x) = n(x) \nabla d(x) - d(x) \nabla n(x) = \frac{4i}{V^2} \sum_{p,q} \Omega_{p+q} (2p + q) e^{iq \cdot x}. \] (84)

In the case of \( P_\infty - q > k_F \), we can put \( \Omega_{p+q} = \Omega_p \) and \( P_\infty = \infty \) in the above equation. Therefore, we obtain
\[ K(x) = \rho \nabla \delta(x), \] (85)
which provides us with Eq. (17) for nuclear matter:
\[ \langle 0 | i [ J^0(x), J(y) ] | 0 \rangle = \frac{\rho}{m} \nabla_x \delta(x - y). \] (86)

5. Conclusions

It is known in relativistic field theory that a naive equal-time commutator between the time and space components of the local four-current vanishes, and that this fact leads to the paradoxical contradiction on the linearly energy-weighted sum of the excitation strengths [2–5]. The sum should be positive definite, but its expression in terms of the current disappears. In order to avoid this problem, Schwinger [3] introduced non-locality in the space part of the current from Lorentz covariance considerations [5]. The commutator of the non-local current with the time component yields the so-called Schwinger term, which has been shown to play an important role in relativistic field algebras [4–11], although its form is not fixed model-independently [3,5,6].

On the other hand, Walecka et al. [14] proposed a relativistic nuclear model, where nucleons are assumed to be Dirac particles. It has been shown that the nuclear response functions are well reproduced phenomenologically by the relativistic model [17], but that its energy-weighted sum value is not well defined, since the double commutator of the excitation operator with the Dirac Hamiltonian vanishes [12,13]. The role of the Schwinger term in this model has not been discussed so far.

In this paper, it has been shown that the ground state expectation value of the commutator with Schwinger’s nonlocal current is derived in an analytic way using the local current which is defined in the finite momentum space. By making the momentum space infinite after calculating the expectation value, the contradiction on the energy-weighted sum and a naive current commutator is solved. It has also been discussed why calculations of the expectation values cannot be exchanged with taking the infinite momentum space.

According to the same framework as the one for Schwinger term, the sum values of the relativistic nuclear models [14,15] have been examined. It has been shown that the vanishing double commutator of the excitation operator with the Dirac Hamiltonian can be used only in the no-sea approximation [16,17] where the Dirac sea is assumed to be empty, but should not be used for discussions of the total energy-weighted sum of relativistic nuclear models.

The Random Phase Approximation sum rules in relativistic nuclear models will be discussed elsewhere.

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