The operator-sum-difference representation of a quantum noise channel

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Abstract When a model for quantum noise is exactly solvable, a Kraus (or operator-sum) representation can be derived from the spectral decomposition of the Choi matrix for the channel. More generally, a Kraus representation can be obtained from any positive-sum (or ensemble) decomposition of the matrix. Here we extend this idea to any Hermitian-sum decomposition. This yields what we call the “operator-sum-difference” (OSD) representation, in which the channel can be represented as the sum and difference of “subchannels.” As one application, the subchannels can be chosen to be analytically diagonalizable, even if the parent channel is not (on account of the Abel-Galois irreducibility theorem), though in this case the number of the OSD representation operators may exceed the channel rank. Our procedure is applicable to general Hermitian (completely positive or non-completely positive) maps and can be extended to the more general, linear maps. As an illustration of the application, we derive an OSD representation for a two-qubit amplitude-damping channel.

Keywords Kraus representation · Amplitude-damping channel · Correlated quantum noise
1 Introduction

Any practical use of a quantum operation involves taking into account the effect of the ambient environment, and the systematic study of such an effect constitutes the theory of open quantum systems. This now pervades a vast arena of studies, see, for example, the Refs. [1,2] for two distinct flavors of the subject. The effect of the environment, interchangeably called the reservoir or the bath, affects the system dynamics, in general, in two ways depending upon the commutability of the system and interaction Hamiltonian. If the two commute, then the process is a quantum non-demolition one, that is, there is dephasing without any energy exchange [3], while if they do not commute, then there is dephasing along with dissipation [4,5]. These effects have been brought within the ambit of practical implementation by a number of very impressive experiments, for example, [6] involving ion traps and [7] using high-Q cavity quantum electrodynamics.

As a result, the use of ideas from open quantum systems has become widespread in quantum information processing [8]. A very useful theoretical tool in this regard is the Kraus or operator-sum representation of the noise channel [9], which encodes the effect of the environment on the system of interest. The Kraus operators are canonically derived by spectrally decomposing the Choi matrix $B$, which is proportional to the density matrix obtained by acting the channel on one half of a maximally entangled bipartite system. Map $E$ is defined to be CP if it is positive, and furthermore, its extension $I \otimes E$ to a larger Hilbert space is also positive [8]. The map $E$ on density operators corresponding to a noise channel is completely positive (CP) if and only if the associated Choi matrix is positive [10–12]. If at least one of the eigenvalues in the Choi matrix is negative for positive map $E$, then $E$ is non-completely positive (NCP). Partial trace is a familiar NCP map.

A quantum noise model posits a system $S$ interacting with the environment $E$ via a Hamiltonian $H_S + H_E + H_{SE}$, where $H_S$ ($H_E$) is the free Hamiltonian for the system (environment) and $H_{SE}$ is the interaction Hamiltonian [3,4]. The Schrödinger equation for the joint state $\rho_{SE}$ of the system and environment is solved to obtain the unitary evolution $U(t)$ of the initial state $\rho_{SE}(0)$ of the joint system $S - - E$. Tracing out the environment then yields the map $E$ on $\rho_S$, the reduced density operator of the system. If the initial state has the product form $\rho_{SE} \equiv \rho_S \otimes \rho_E$, the result is a CP map: $E^{CP}(\rho_S) = \text{Tr}_E[U(t)(\rho_S(0) \otimes \rho_E(0))U^\dagger(t)]$. If $E$ is NCP, it would mean that $\rho_{SE}(0)$ lacks the product form. An NCP map can be represented as the difference of two CP maps [12].

Choi’s procedure to construct Kraus operators for a CP map can be extended to any ensemble (or positive-sum) decomposition of the Choi matrix [13]. Here we show that Choi’s procedure can be adapted to the more general decomposition in terms of a sum over Hermitian operators. Moreover, this can be applied to more general channels than CP, namely Hermitian and even linear maps. This will allow us to express any given channel as a combination (possibly including negative summands) of component channels. Accordingly, we call this the “operator-sum-difference representation” (OSDR). As an application of our method, we construct the OSD
representation of decoherence based on a two-qubit model much discussed in the literature [14].

For sufficiently complicated noisy situations, the noise model is only numerically solvable. When the model is exactly solvable, we obtain an analytical expression for $\mathcal{E}$. Even so, when we attempt to derive the canonical Kraus operators by diagonalizing the Choi matrix, we may encounter a problem which has its origin in the Abel-Galois irreducibility theorem, according to which there is no general method to solve quintic and higher-order polynomials by radicals [15].

This difficulty can be circumvented by employing a general ensemble decomposition. Not requiring diagonalization, it is not thwarted by the Abel-Galois no-go theorem. An alternative is to use the OSD representation, whereby we can choose the component channels to be analytically diagonalizable, even when the parent channel is not (though in this case the total number of Kraus-like terms may exceed $d^2$). The noise model to which we apply this, discussed in Sect. 4, is exactly solvable, but lacks analytic diagonalization.

The two-qubit model considered is that of two qubits interacting with a bath, e.g., an electromagnetic field in a squeezed thermal state, via the dipole interaction, which has been considered in detail for both pure dephasing [16] and dissipative [17] system–reservoir interactions. The system–reservoir coupling constant is dependent upon the position of the qubit, leading to interesting dynamical consequences. In particular, we consider here the special case of a vacuum bath, because this is exactly solvable [17]. One can continuously shift the noise from the independent to the collective regime, elaborated in Sect. 4. The former noise admits a Kraus representation based on a spectral decomposition, while the latter is not, and requires a Kraus representation based on a more general positive-sum decomposition or, as we show, an OSD representation.

The plan of the paper is as follows. In the next section, we show how any Hermitian map (CP or NCP) can always be expressed as a sum of Hermitian component maps or “submaps,” and thus yield an OSD representation. Like the conventional Kraus representation, the OSD representation is not unique. In Sect. 3, we point out that the representation of NCP maps as the difference of two CP maps may be considered as implementing a special case of the present approach: The OSD representation even of a CP map may include negative Kraus-like terms, because the submaps are only Hermitian and not necessarily positive. In Sect. 4, we apply this method to derive an OSD representation of the two-qubit amplitude-damping (2AD) channel. This is followed by a discussion on how our method can be extended to general linear maps in Sect. 5. Finally, we make our conclusions in Sect. 6.

### 2 Hermitian decomposition

A linear map, transforming bounded operators in an input Hilbert space $\mathcal{H}_N$ of dimension $N$ to bounded operators in an output Hilbert space $\mathcal{H}_M$ of dimension $M$, is given by [18]

$$\rho \longrightarrow \mathcal{E}^L(\rho) = \sum_j A_j \rho A_j^\dagger,$$

(1)
where \( \{A_j\} (\{A'_j\}) \) are \( M \times N (N \times M) \) matrices satisfying the completeness condition \( \sum_j A_j^{\dagger} A_j = I \). Two linear maps \( \{A_j, A'_j\} \) and \( \{B_j, B'_j\} \) give rise to the same quantum operation if they are unitarily connected: \( B_j = \sum_k U_{jk} A_k \) and \( B'_j = \sum_k U_{jk} A'_k \).

A linear map is Hermitian if it maps Hermitian operators \( \rho \) in its domain to Hermitian operators. It is described according to the prescription [18]:

\[
\rho \rightarrow \mathcal{E}^H(\rho) = \sum_j c_j A_j \rho A_j^{\dagger}, \quad c_j \in \mathbb{R},
\]

where \( \rho \) is the density operator and \( A_j^{\dagger} A_j \) are positive operators that satisfy the completeness condition \( \sum_j c_j A_j^{\dagger} A_j = I \). In Eq. (2), if \( c_j \geq 0 \), then we obtain a completely positive (CP) map \( \mathcal{E}^{CP} \) [11]. If this positivity requirement on \( c_j \) is not met, the result is a non-completely positive (NCP) map.

Here we show that any CP or NCP map can be decomposed into simpler Hermitian submaps. Given a Hermitian map \( \mathcal{E}^H \), we can form a decomposition of the kind

\[
\mathcal{E}^H(\rho) = \sum_{a=1}^p \mathcal{E}^H_a(\rho) = \sum_a \left( \sum_{j_a=1}^\mu c_a^{(j_a)} A_a^{(j_a)} \rho A_a^{(j_a)^\dagger} \right), \quad c_a^{(j_a)} \in \mathbb{R},
\]

where \( \mathcal{E}_a^H \) may be selected according to required criteria, e.g., that they should be analytically diagonalizable (though that is not necessary). In (3), the operators \( A_a^{(j)} \) satisfy the completeness condition

\[
\sum_a \sum_{j_a} c_a^{(j_a)} A_a^{(j_a)^\dagger} A_a^{(j_a)} = I,
\]

where \( I \) is the identity operator. If the noise \( \mathcal{E}^H \) is exactly solvable, then the operators \( A_{j_a} \) have a closed, analytic form.

**Theorem 1** Any Hermitian map \( \mathcal{E}^H \) can always be decomposed as the sum of Hermitian maps \( \mathcal{E}_a^H \) in the form Eq. (3).

**Proof** We essentially extend the method of Choi to an arbitrary Hermitian decomposition of \( \mathcal{B} \) [11]. Consider the bipartite unnormalized state \( \tilde{\Phi} = \sum_{j,k} |j \rangle \langle k| \otimes |j \rangle \langle k| = n |\phi \rangle \langle \phi| \), where \( |\phi\rangle = \frac{1}{\sqrt{n}} \sum_j |j \rangle, j \rangle \) is the maximally entangled state in the Hilbert space \( \mathcal{H}_n \otimes \mathcal{H}_n \), represented in the basis \( |j \rangle \) for \( \mathcal{H}_n \). Now \( \tilde{\Phi} \) can be considered as an \( n \times n \) array of \( n \times n \) blocks, where the \( (j,k) \)th block is \( |j \rangle \langle k| \) (cf. Ref. [13]). We will now construct two equivalent representations of the Choi matrix \( \mathcal{B} \equiv (I \otimes \mathcal{E})[\tilde{\Phi}] \).
(i) In the first construction, \((\mathcal{I} \otimes \mathcal{E})[\hat{\Phi}] = \sum_{j,k} |j\rangle \langle k| \otimes \mathcal{E}(|j\rangle \langle k|)\) is an array of \(n \times n\) array of \(n \times n\) blocks, such that the \((j,k)\)th block is \(\mathcal{E}(|j\rangle \langle k|)\):

\[
(\mathcal{I} \otimes \mathcal{E})[\hat{\Phi}] = \begin{pmatrix}
\mathcal{E} \begin{pmatrix} 1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \\
\mathcal{E} \begin{pmatrix} 0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \\
\vdots \\
\mathcal{E} \begin{pmatrix} 0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \\
\mathcal{E} \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\end{pmatrix}.
\tag{5}
\]

(ii) In the second construction, we spectrally decompose the Hermitian matrix corresponding to each submap \(\mathcal{E}^H_a\). We then have

\[
\mathcal{B} = \sum_a \mathcal{B}_a = \sum_{a=1} \mathcal{U}_a \mathcal{D}_a \mathcal{U}_a^\dagger = \sum_a \left( \sum_{j_a} c_{a}^{(j_a)} |j_a\rangle \langle j_a| \right),
\tag{6}
\]

where \(\mathcal{D}_a\) is a diagonal matrix of real numbers, which are the eigenvalues \(c_{j_a}\) of \(\mathcal{B}_a\). More generally, the spectral decompositions in Eq. (6) can be replaced by any pure-state decompositions (with possibly negative weight factors).

Now divide the column (row) vector \(|j_a\rangle (\langle j_a|)\) into \(n\) segments of length \(n\). In the columnar (row) case, define a \(n \times n\) matrix \(A_a^{(j_a)} (A_a^{(j_a)\dagger})\) whose \(l\)th column (row) is the \(l\)th segment. Then, in \(|j_a\rangle \langle j_a|\), the \((j,k)\)th block will be \(A_a^{(j_a)} |j\rangle \langle k| A_a^{(j_a)\dagger}\), where \(|j\rangle\) and \(|k\rangle\) are vectors in the \(n\)-dimensional space. To see this, we note that

\[
(\mathcal{I} \otimes \mathcal{E})[\hat{\Phi}] = \sum_{a,j_a} c_{a}^{(j_a)} \begin{bmatrix} A_a^{(j_a)} |1\rangle \\
A_a^{(j_a)} |2\rangle \\
\vdots \\
A_a^{(j_a)} |n\rangle
\end{bmatrix} \times \begin{bmatrix} |1\rangle A_a^{(j_a)\dagger} \\
|2\rangle A_a^{(j_a)\dagger} \\
\vdots \\
n |A_a^{(j_a)\dagger}\rangle
\end{bmatrix}.
\tag{7}
\]
Comparing the two constructions for their description of the action on $|j⟩⟨k|$, we find that
\[ \mathcal{E}(|j⟩⟨k|) = \sum_{a} \sum_{ja} c_{a}^{(ja)} A_{a}^{(ja)} |j⟩⟨k| A_{a}^{(ja)\dagger}, \] (8)
from which Eq. (3) follows.

The basic idea is that, because of linearity, Choi’s method works even when
the spectral or ensemble decomposition of $B$ is replaced by the a pure-state (not
necessarily positive-sum) decomposition of elements $B_{a}$ in its Hermitian paritition.
Each such element induces a “submap.” Suppose $B_{a}$ is non-positive for some $a$. Let
$r_{a} \equiv \text{rank} (B_{a}) \leq 4$, and further, let the $r_{a}$ OSDR operators derived from it by the
method described in Theorem 1 be denoted $k_{a}^{(ja)} (j_{a} = 1, 2, \ldots, r_{a})$. Then the submap
induced by $B_{a}$, namely $\mathcal{E}_{B_{a}} : \rho \mapsto \sum_{ja} c_{a}^{(ja)} k_{a}^{(ja)} \rho (k_{a}^{(ja)})^{\dagger}$, has at least one $c_{a}^{(ja)} < 0$.
The result is a Kraus-like representation, except that some terms will appear with a
negative sign. The parent map will be the sum of such submaps. By this construc-
tion, it will have a representation in terms of the sum or difference of OSDR terms
$k_{a}^{(ja)} \rho (k_{a}^{(ja)})^{\dagger}$.

3 Non-positive submaps and NCP maps: an example

We note that the representation of an NCP map as the difference of two CP maps is a
special case of representation (3) in which we set $B = B_{(+)} + B_{(-)}$, where $B_{(+)} \geq 0$
and $B_{(-)} \leq 0$. Now $B_{(\pm)}$ may itself be broken up into possibly lower-rank pieces using
the method of Theorem 1.

Further, an OSD representation of a CP map may contain negative Kraus terms. As
an example of a such non-positive decomposition of a CP map, consider an OSR repre-
sentation of the generalized amplitude-damping channel (GAD) [4]. The Choi matrix
corresponding to GAD channel with elements $\sqrt{p} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\sqrt{p} \begin{pmatrix} 0 & 0 \\ \sqrt{1} & 0 \end{pmatrix}$,
$\sqrt{1-p} \begin{pmatrix} \sqrt{1} & 0 \\ 0 & 1 \end{pmatrix}$, and $\sqrt{1-p} \begin{pmatrix} 0 & \sqrt{1} \\ 0 & 0 \end{pmatrix}$ is
\[ B = (I \otimes \mathcal{E}_{\text{GAD}}) \Phi = \begin{pmatrix} 1 - \lambda + p\lambda & 0 & 0 & \sqrt{1 - \lambda} \\ 0 & p\lambda & 0 & 0 \\ 0 & 0 & (1 - p)\lambda & 0 \\ \sqrt{1 - \lambda} & 0 & 0 & 1 - p\lambda \end{pmatrix}, \] (9)
where $\Phi \equiv (|00⟩⟨00| + |11⟩⟨11|)$ and $0 \leq p, \lambda \leq 1$. A Hermitian decomposition of
$B$ in Eq. (9) is $B = B_{(+)} + B_{(-)}$, such that $B_{(+)} \geq 0$ while $B_{(-)} \leq 0$, is
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\[ B_{(\pm)} = \begin{pmatrix}
1 - \lambda + p\lambda + \frac{\sqrt{1-\lambda}}{2} & 0 & 0 & \frac{3\sqrt{1-\lambda}}{4} \\
0 & p\lambda & 0 & 0 \\
0 & 0 & (1 - p)\lambda & 0 \\
\frac{3\sqrt{1-\lambda}}{4} & 0 & 0 & 1 - p\lambda + \frac{\sqrt{1-\lambda}}{2}
\end{pmatrix}, \]

\[ B_{(-)} = \begin{pmatrix}
-\frac{\sqrt{1-\lambda}}{2} & 0 & 0 & \frac{\sqrt{1-\lambda}}{4} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{\sqrt{1-\lambda}}{4} & 0 & 0 & -\frac{\sqrt{1-\lambda}}{2}
\end{pmatrix}. \tag{10}\]

In the manner of Eq. (6), the matrix \( B_{(\pm)} \) can be diagonalized, from which, per prescription (8), we can form the OSDR elements by “folding” the eigenvectors, and thereby construct the submap induced by \( B_{(\pm)} \). This is given by \( E_{(\pm)} : \rho \mapsto \sum_{j=1}^{4} c_{(j)} A_{(j)} \rho A_{(j)}^\dagger \), where

\[ c_{(1)}^{(1)} = 4 + 2\sqrt{1-\lambda} - 2\lambda - \sqrt{a}; \quad A_{(1)}^{(1)} = \begin{pmatrix}
-2\lambda(1-2p)+\sqrt{a} \\
3\sqrt{1-\lambda} \\
0 \\
1
\end{pmatrix}; \tag{11}\]

\[ c_{(2)}^{(2)} = 4 + 2\sqrt{1-\lambda} - 2\lambda + \sqrt{a}; \quad A_{(2)}^{(2)} = \begin{pmatrix}
-2\lambda(1-2p)-\sqrt{a} \\
3\sqrt{1-\lambda} \\
0 \\
1
\end{pmatrix}; \tag{12}\]

where \( a = 9(1-\lambda) + 4\lambda^2(1 - 2p)^2 \). As with \( B_{(\pm)} \), in the manner of Eq. (6), the matrix \( B_{(-)} \) can be diagonalized, from which, per prescription (8), we can construct the submap induced by \( B_{(-)} \). This is given by \( E_{(-)} : \rho \mapsto \sum_{j=1}^{2} c_{(-j)} A_{(-j)} \rho A_{(-j)}^\dagger \), where

\[ c_{(-1)}^{(1)} = -\frac{(1 - \lambda)^{1/2}}{4}; \quad A_{(-1)}^{(1)} = \mathbb{I}, \]

\[ c_{(-2)}^{(2)} = -\frac{3(1 - \lambda)^{1/2}}{4}; \quad A_{(-2)}^{(2)} = \sigma_z, \tag{13}\]

which correspond to the purely dephasing channel (in which the diagonal terms are unaffected and only off-diagonal terms are killed off). The OSD representation for the full GAD noise map is obtained by combining the above two submaps, yielding:

\[ E_{\text{GAD}} : \rho \mapsto \sum_{j=1}^{4} c_{(j)}^{(j)} A_{(j)} \rho A_{(j)}^\dagger + \sum_{j=1}^{2} c_{(-j)}^{(-j)} A_{(-j)} \rho A_{(-j)}^\dagger, \tag{14}\]
where the elements $c^{(j±)}_\pm$ and $A^{(j±)}_\pm$ are taken from Eqs. (12) and (13). The form (14) allows us to interpret the GAD channel as a dissipative part minus a purely dephasing part. This example serves to underscore that the negative OSDR terms do not necessarily indicate that the map is not CP. That the map is indeed CP is verified by observing that the matrix $B$ in Eq. (9) has no negative eigenvalues.

In the above example, we note that the positive and negative OSDR elements are, up to a constant factor, trace preserving. The basic idea behind this separation is that the Choi matrix can always be written as the difference of two positive operators: $B = B_1 - B_2$. One way to do this would be to form the difference $B'' = B - B'$, where $B'$ is also a density operator. In general, $B''$ will be Hermitian. Denote the spectral decomposition of $B''$ by $\sum_\lambda + \lambda_+ |\lambda_+ \rangle \langle \lambda_+ | - \sum_\lambda - \lambda_- |\lambda_- \rangle \langle \lambda_- | = B''_+ - B''_-$, where $\lambda_\pm > 0$. Then we can set $B_1 = B' + B''_+$ while $B_2 = B''_-$.

Now let $B_1 \equiv c_1 B'_1$ and $B_2 \equiv c_2 B'_2$, where $B'_j > 0$ and $\text{Tr} (B'_j) = 1$, i.e., each $B'_j$ is a density operator. Each $B'_j$, interpreted as a Choi matrix, gives a quantum channel via the Choi-Jamiolkowski isomorphism, and we obtain the required construction.

4 Application to two-qubit noise

It is a frequent assumption that the noise $E$ acting on a $N$-qubit system affects each qubit independently, in which case the Choi matrix is diagonalizable, because then $E = E^{(1)} \otimes E^{(2)} \otimes \cdots \otimes E^{(N)}$ (where $E^{(j)}$ is the independent noise acting on the $j$th qubit) and each $E^{(j)}$ yields a diagonalizable Choi matrix, of rank at most 4. This independent noise model, while reasonable, is not always accurate in practice, because the noise affecting two or more qubits can be correlated when they are sufficiently close to experience the same bath. When the noise is correlated, as it happens in the collective regime of the 2AD noise discussed here, the rank of $B$ will exceed 4, and the Kraus operators may not be obtainable by diagonalizing $B$. One can then use an ensemble decomposition of $B$ instead or resort to an OSD representation.

In Ref. [17], the problem of two qubits interacting collectively with a bath via a dissipative interaction was studied. In the case of a vacuum bath, the model is exactly solvable. The degree of correlation in the noise is governed by the inter-qubit distance $r$ such that the noise is in the collective regime when $r/L \to 0$ and in the independent regime when $r/L \gg 0$. Here $L$ is the typical length scale defined by the bath mode.

The Choi matrix in this case, $(I \otimes E^{2\text{AD}})(|\Phi \rangle \langle \Phi |)$, where $|\Phi \rangle \equiv |00\rangle|00\rangle + |01\rangle|01\rangle + |10\rangle|10\rangle + |11\rangle|11\rangle$, is given by:
\[ B = \begin{pmatrix} A & 0 & 0 & 0 & 0 & J & 0 & 0 & 0 & 0 & M & 0 & 0 & 0 & 0 & L \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ J^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ M^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ L^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \]

where

\[ A = e^{-2\Gamma t}; \quad B = e^{-(\Gamma + \Gamma_{12}^t)t}; \quad C = \frac{\Gamma + \Gamma_{12}^t}{T - \Gamma_{12}^t} (1 - e^{-(\Gamma - \Gamma_{12}^t)t}) e^{-(\Gamma + \Gamma_{12}^t)t}; \]

\[ D = e^{-(\Gamma - \Gamma_{12}^t)t}; \quad E = \frac{\Gamma - \Gamma_{12}^t}{T + \Gamma_{12}^t} (1 - e^{-(\Gamma + \Gamma_{12}^t)t}) e^{-(\Gamma - \Gamma_{12}^t)t}; \quad F = 1 - e^{-(\Gamma + \Gamma_{12}^t)t}; \]

\[ G = 1 - e^{-(\Gamma - \Gamma_{12}^t)t}; \quad J = e^{-i(\omega_0 - \Omega_{12})t} e^{(3\Gamma + \Gamma_{12}^t)t/2}; \quad L = e^{-i\omega_0^t} e^{-\Gamma t}; \quad M = e^{-i(\omega_0 + \Omega_{12})t} e^{-(3\Gamma - \Gamma_{12}^t)t/2}; \quad P = e^{-i2\Omega_{12}^t} e^{-\Gamma t}; \quad Q = e^{-i(\omega_0 - \Omega_{12})t} e^{-(\Gamma - \Gamma_{12}^t)t/2}; \quad T = e^{-i(\omega_0 + \Omega_{12})t} e^{-(\Gamma + \Gamma_{12}^t)t/2}, \]

\[ H = \frac{\Gamma + \Gamma_{12}^t}{2\Gamma} \left[ 1 - \frac{2}{\Gamma - \Gamma_{12}^t} \left( \frac{\Gamma + \Gamma_{12}^t}{2} (1 - e^{-(\Gamma - \Gamma_{12}^t)t}) + \frac{\Gamma - \Gamma_{12}^t}{2} \right) e^{-(\Gamma + \Gamma_{12}^t)t} \right] \]

\[ + \frac{\Gamma - \Gamma_{12}^t}{\Gamma + \Gamma_{12}^t} \left[ (1 - e^{-(\Gamma - \Gamma_{12}^t)t}) - \frac{\Gamma - \Gamma_{12}^t}{2\Gamma} (1 - e^{-2\Gamma t}) \right], \]

\[ Y = i \left[ \frac{\Gamma + \Gamma_{12}^t}{\Gamma^2 + 4\Omega_{12}^2} e^{-i(\omega_0 - \Omega_{12})t} e^{-(\Gamma - \Gamma_{12}^t)t/2} \left[ 2\Omega_{12}^t (1 - e^{-\Gamma t} \cos(2\Omega_{12}^t)) \right] \right. \]

\[ - \frac{\Gamma - \Gamma_{12}^t}{\Gamma^2 + 4\Omega_{12}^2} e^{-i(\omega_0 - \Omega_{12})t} e^{-(\Gamma - \Gamma_{12}^t)t/2} \left[ 2\Omega_{12}^t e^{-\Gamma t} \sin(2\Omega_{12}^t) \right] \]

\[ + i \left. \Gamma (1 - e^{-\Gamma t} \cos(2\Omega_{12}^t)) \right], \]

\[ X = \frac{\Gamma + \Gamma_{12}^t}{\Gamma^2 + 4\Omega_{12}^2} e^{-i(\omega_0 + \Omega_{12})t} e^{-(\Gamma + \Gamma_{12}^t)t/2} \left[ 2\Omega_{12}^t e^{-\Gamma t} \sin(2\Omega_{12}^t) \right] \]

\[ + \frac{\Gamma - \Gamma_{12}^t}{\Gamma^2 + 4\Omega_{12}^2} e^{-i(\omega_0 + \Omega_{12})t} e^{-(\Gamma + \Gamma_{12}^t)t/2} \left[ 2\Omega_{12}^t (1 - e^{-\Gamma t} \cos(2\Omega_{12}^t)) \right] \]

\[ + i \left. \Gamma (1 - e^{-\Gamma t} \cos(2\Omega_{12}^t)) \right], \]

\[ - \frac{\Gamma + \Gamma_{12}^t}{\Gamma^2 + 4\Omega_{12}^2} e^{-i(\omega_0 + \Omega_{12})t} e^{-(\Gamma + \Gamma_{12}^t)t/2} \left[ 2\Omega_{12}^t (1 - e^{-\Gamma t} \cos(2\Omega_{12}^t)) \right] \]

\[ - \frac{\Gamma - \Gamma_{12}^t}{\Gamma^2 + 4\Omega_{12}^2} e^{-i(\omega_0 + \Omega_{12})t} e^{-(\Gamma + \Gamma_{12}^t)t/2} \left[ 2\Omega_{12}^t (1 - e^{-\Gamma t} \cos(2\Omega_{12}^t)) \right], \]
where $F = |F| e^{i\phi_F}$, $J = |J| e^{i\phi_J}$, $M = |M| e^{i\phi_M}$, $P = |P| e^{i\phi_P}$, $Q = |Q| e^{i\phi_Q}$, $X = |X| e^{i\phi_X}$, $Y = |Y| e^{i\phi_Y}$. The terms $\omega_0$ and $\Omega_{12}$ are bath mode frequency parameters, while $\Gamma$ and $\Gamma_{12}$ are bath coupling parameters, details of which are given in Ref. [17].

In Eq. (15), rank($B$) = 9, which exceeds 4, and an analytic spectral decomposition does not exist, as far as we know. A Kraus representation can be obtained by an ensemble decomposition. In this Section, we will instead derive an OSD representation by breaking up $E_{2AD}$ into submaps that yield diagonalizable Choi matrices. This partitioning of $E_{2AD}$ into submaps is not unique. We give a particular decomposition as a worked out illustration of a practical application of OSDR. To simplify the presentation, we note that the Choi matrix only has support in the span of the basis states $\mathcal{B} \equiv \{|0000\}, |0100\}, |0101\}, |1000\}, |1010\}, |1100\}, |1101\}, |1110\}, |1111\}$. The following decomposition is given restricted to span($\mathcal{B}$).

$$B = B_1 + B_2 + B_3$$

$$\equiv \begin{pmatrix} A & 0 & 0 & J & 0 & 0 & 0 & 0 & 0 \\ 0 & C & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ J^* & 0 & B & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & E & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & M & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & X & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & P & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ M^* & 0 & P^* & 0 & D & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & X^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\equiv \sum_{j_1=1}^{5} |K^{(j_1)}_1\rangle\langle K^{(j_1)}_1| + \sum_{j_2=1}^{4} |K^{(j_2)}_2\rangle\langle K^{(j_2)}_2| + \sum_{j_3=1}^{4} |K^{(j_3)}_3\rangle\langle K^{(j_3)}_3|; \quad (18)$$

where we have absorbed the eigenvalues $c_j$ into the eigenvectors, since all $B_a$ above are positive, as are the eigenvalues.

Thus, in Eq. (18), we first decompose $B$ into three Hermitian parts $B_a$, and then diagonalize each, according to step (6). From each of the submatrices, per prescription (8), we can form Kraus operators and construct the submap induced by each of these pieces. This is done below. In particular, from the spectral decomposition of the matrix $B_1$, we obtain the following OSDR operators (using the above renormalized eigenvectors):
The operator-sum-difference representation of a quantum noise channel

\[ A_1^{(1)} = \sqrt{\frac{1}{2} \left( A + B - \sqrt{(A - B)^2 + 4J^2} \right)^2 + 1} \begin{pmatrix} e^{i\phi J} \left( -A + B + \sqrt{(A - B)^2 + 4J^2} \right) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ; \]

\[ A_1^{(2)} = \sqrt{\frac{1}{2} \left( A + B + \sqrt{(A - B)^2 + 4J^2} \right)^2 + 1} \begin{pmatrix} e^{i\phi J} \left( A - B + \sqrt{(A - B)^2 + 4J^2} \right) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ; \]

\[ A_1^{(3)} = \sqrt{C} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ; \quad A_1^{(4)} = \sqrt{E} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ; \quad A_1^{(5)} = \sqrt{H} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} . \quad (19) \]

One can verify that the initial state

\[ \rho \equiv \begin{pmatrix} \rho_{ee} & \rho_{es} & \rho_{ea} & \rho_{eg} \\ \rho_{se} & \rho_{ss} & \rho_{sa} & \rho_{sg} \\ \rho_{ae} & \rho_{as} & \rho_{aa} & \rho_{ag} \\ \rho_{ge} & \rho_{gs} & \rho_{ga} & \rho_{gg} \end{pmatrix} , \quad (20) \]

under the submap \( E(1) \), defined by the OSDR operators in Eq. (19), transforms to

\[ \rho_1' = \sum_{j_1=1}^{5} A_1^{(j_1)} \rho A_1^{(j_1)^\dagger} = \begin{pmatrix} A\rho_{ee} & J e^{-i\phi J} \rho_{es} & 0 & 0 \\ J e^{i\phi J} \rho_{se} & B \rho_{ss} + C \rho_{ee} & 0 & 0 \\ 0 & 0 & E \rho_{ee} & 0 \\ 0 & 0 & 0 & H \rho_{ee} \end{pmatrix} . \quad (21) \]

The following notation will be useful for compactness: \( D_1 \equiv \sqrt{D^2 + 4(M^2 + P^2)} \) and \( D_\pm \equiv D \pm D_1 \). Further \( F_1 \equiv \sqrt{F^2 + 4X^2} \) and \( F_\pm = F \pm F_1 \). From \( B_2 \), we obtain the OSDR operators:

\[ A_2^{(1)} = \sqrt{D_-} \left[ \frac{2M}{D_-} \right]^2 \left( \frac{2P}{D_-} \right)^2 + 1 \right]^{-1} \times \text{Diag} \left( -2e^{-i\phi M} \frac{M}{D_-}, -2e^{-i\phi P} \frac{P}{D_-}, 1, 0 \right) , \]

\[ A_2^{(2)} = \sqrt{D_+} \left[ \frac{2M}{D_+} \right]^2 \left( \frac{2P}{D_+} \right)^2 + 1 \right]^{-1} \times \text{Diag} \left( 2e^{-i\phi M} \frac{M}{D_+}, 2e^{-i\phi P} \frac{P}{D_+}, 1, 0 \right) , \]
\[
A_2^{(\frac{7}{2} \pm \frac{1}{2})} = \sqrt{\frac{F_{\pm}}{2} \left[ \left( \frac{F_{\mp}}{2X} \right)^2 + 1 \right]^{-1} \left( \begin{array}{cc}
0 & 0 \\
-\frac{e^{-i\phi_x} F_{\mp}}{2X} & 0
\end{array} \right) \oplus \left( \begin{array}{cc}
0 & 0 \\
1 & 0
\end{array} \right) .
\]

The following notation will be useful for compactness: \( W = \sqrt{4(L^2 + Q^2 + T^2)} \) and \( L_{\pm} = 1 \pm W \). Further \( G_{\pm} = \sqrt{G^2 + 4^2 \pm G} \). Then, from \( B_3 \) we obtain

\[
A_3^{(1)} = \sqrt{\frac{L_-}{2} \left[ \left( \frac{W}{L_-} \right)^2 + 1 \right]^{-1}} \times \text{Diag} \left( \begin{array}{ccc}
-2e^{-i\phi_L} L & -2e^{-i\phi_T} T & -2e^{-i\phi_Q} Q \\
L_- & L_- & L_-
\end{array} \right) .
\]

\[
A_3^{(2)} = \sqrt{\frac{L_+}{2} \left[ \left( \frac{W}{L_+} \right)^2 + 1 \right]^{-1}} \times \text{Diag} \left( \begin{array}{ccc}
2e^{-i\phi_L} L & 2e^{-i\phi_T} T & 2e^{-i\phi_Q} Q \\
L_+ & L_+ & L_+
\end{array} \right) .
\]

\[
A_3^{(\frac{7}{2} \pm \frac{1}{2})} = \sqrt{\frac{G_{\pm}}{2} \left[ \left( \frac{G_{\mp}}{2Y} \right)^2 + 1 \right]^{-1}} \left( \begin{array}{ccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\pm \frac{e^{-i\phi_Y} G_{\mp}}{2Y} & 0 & 0 & 1
\end{array} \right) .
\]

In the above expressions, \( \phi_L = -2\omega_0 t \), \( \phi_M = -(\omega_0 + \Omega_{12}) t \), \( \phi_P = -2\Omega_{12} t \), \( \phi_T = -(\omega_0 + \Omega_{12}) t \), \( \phi_X = \text{Arg}(X) \), \( \phi_Q = -(\omega_0 - \Omega_{12}) t \), \( \phi_Y = \text{Arg}(Y) \).

We then obtain the full evolution in the OSD representation combining the operators in Eqs. (19), (22), and (23), to be given by:

\[
\rho \rightarrow E^{2\Delta D}(\rho) = \sum_{a=1}^{3} \sum_{j_a=1}^{r_a} A_{a}^{(j_a)} \rho A_{a}^{(j_a)\dagger} ,
\]

where \( r_1 = 5 \), while \( r_2 = r_3 = 4 \).

5 Extension to general linear maps

The OSD representation scheme for Hermitian maps can be readily extended to linear maps (1). Here |\( \Phi \rangle \rangle = |1, 1 \rangle + |2, 2 \rangle + \cdots + |M, M \rangle \), and the Choi matrix is given by

\[
\sum_{j,k} |j \rangle \langle k| \otimes E^L(|j \rangle \langle k|) ,
\]

a square matrix of size \( MN \times MN \) (i.e., \( M \times M \) blocks of size \( N \times N \)). We do this essentially by representing a linear \( M \times N \) matrix as a sum of simpler matrices corresponding to the linear submaps.

The proof of Theorem 1 goes through here again, except that the spectral decomposition of \( (I \otimes E)[\hat{\Phi}] \) of individual Hermitian submaps is replaced by singular value decomposition (SVD) of individual linear submaps. Given an \( M \times N \) matrix \( G \), SVD is the factorization

\[
G = V \Delta U^\dagger ,
\]

where \( \Delta \) is a diagonal positive semi-definite \( M \times N \) matrix and \( V \) (\( U \)) is an \( M \times M \) (\( N \times N \) matrix).
In Eq. (6), we have instead for the r.h.s $\sum_{a=1}^{\infty} U_a D_a V_a^\dagger \equiv \sum_{a} \left( \sum_{ja} c_a^{(ja)} |j_a\rangle \langle j_a'| \right)$ where $U_a$ and $V_a$ are unitaries and $D_a$ is a diagonal matrix with singular values $c_a^{(ja)} \geq 0$. Now divide the column (row) vector $|j_a\rangle$ into $n$ segments of length $m$. In the columnar (row) case, define an $M \times N$ ($N \times M$) matrix $A_a^{(ja)} ((A_a^{(ja)})^\dagger)$ whose $l$th column (row) is the $l$th segment. Then, in $|j_a\rangle \langle j_a'|$, the $(j, k)$th block will be $A_a^{(ja)} |j\rangle \langle (A_a^{(ja)})_{k}^\dagger$, and the rest of the proof of Theorem 1 follows.

6 Conclusions and discussion

The operator-sum representation of a noise channel is derived by spectral decomposition of the Choi matrix $B$ of a channel, or more generally, by decomposing $B$ into a positive sum. We introduced the OSD representation of a channel, which is based on the more general Hermitian-sum decomposition. The method yields a set of OSDR operators, which are Kraus operators, but may sometimes appear with a negative sign. Our method is applicable to CP and NCP maps, and more generally, to an arbitrary noise represented by a linear map. We applied our method to derive an OSD representation for a two-qubit amplitude-damping channel.

Two sets of Kraus operators, $\{G_j\}$ and $\{H_k\}$, give the same quantum operation iff they are related by a transformation [5,8]:

$$G_j = \sum_k U_{jk} H_k,$$

(26)

where the numbers $U_{jk}$ constitute a unitary matrix. This unitary freedom in the operator-sum representation is a consequence of two equivalences: (i) that of the two “Choi ensembles”: $|g_j\rangle = \sum_i |i\rangle G_j |i\rangle$ and $|h_k\rangle = \sum_i |i\rangle H_k |i\rangle$, by virtue of the Choi-Jamiolkowski isomorphism, i.e.,

$$B_G \equiv \sum_j |g_j\rangle \langle g_j| = \sum_k |h_k\rangle \langle h_k| \equiv B_H;$$

(27)

further, (ii) the unitary freedom for ensembles that correspond to the same density matrix [8]:

$$\sum_j |g_j\rangle \langle g_j| = \sum_k |h_k\rangle \langle h_k| \iff |g_j\rangle = U_{jk} |h_k\rangle.$$

(28)

The equivalence (28) holds precisely when the density operator is a “sum-ensemble” and not a “sum-difference-ensemble.” Therefore in OSDR, a unitary freedom of the kind (26) does not exist across the positive and negative terms.

However, the equivalence (27) still holds, meaning that two sets of OSDR operators $\{G'_j\}$ and $\{H'_k\}$ give the same quantum operation iff they generate the same Choi matrix. We can thus regard the Choi matrix as defining an equivalence class, with equivalent sets of OSDR operators being members of the same class. Now, since a map is CP if and only if the corresponding Choi matrix is positive [10,11], the complete positivity

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of a set of OSDR operators can be determined by verifying that all eigenvalues of the Choi matrix that they generate are positive. Departure from complete positivity occurs precisely if the Choi matrix is non-positive. In our first example, we know that the operator-sum-difference representation gives a completely positive operator, despite the occurrence of negative OSDR elements (13), because the Choi matrix (9) is positive.

Finally, we note that the OSDR of the GAD channel in Sect. 3 shows that the channel can be regarded as the difference (apart from a constant factor) between a non-Pauli channel and a Pauli channel (in this case, a phase flip channel). Here by non-Pauli channel, we mean one whose process matrix is not diagonal in the Pauli representation. An interesting application of OSDR would be to generalize this idea and study how one could decompose a quantum operation into sums and differences of well-studied quantum operations.

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References

1. Breuer, H.-P., Petruccione, F.: The Theory of Open Quantum Systems. Oxford University Press, Oxford (2002)
2. Weiss, U.: Quantum Dissipative Systems. World Scientific, Singapore (2008)
3. Banerjee, S., Ghosh, R.: Dynamics of decoherence without dissipation in a squeezed thermal bath. J. Phys. A: Math. Theor. 40, 13735 (2007)
4. Srikanth, R., Banerjee, S.: Squeezed generalized amplitude damping channel. Phys. Rev. A 77, 012318 (2008)
5. Omkar, S., Srikanth, R., Banerjee, S.: Dissipative and non-dissipative single-qubit channels: dynamics and geometry. Quant. Info. Proc. 12, 3725 (2013)
6. Turchette, Q.A., Myatt, C.J., King, B.E., Sackett, C.A., et al.: Decoherence and decay of motional quantum states of a trapped atom coupled to engineered reservoirs. Phys. Rev. A 62, 053807 (2000)
7. Brune, M., Hagley, E., Dreyer, J., Maitre, X., et al.: Observing the progressive decoherence of the meter in a quantum measurement. Phys. Rev. Lett. 77, 4887 (1996)
8. Nielsen, M., Chuang, I.: Quantum Computation and Quantum Information. Cambridge University Press, Cambridge (2000)
9. Kraus, K.: States, Effects and Operations. Springer, Berlin (1983)
10. Choi, M.D.: Positive linear maps on C*-algebras. Can. J. Math. 24, 520 (1972)
11. Choi, M.D.: Completely positive linear maps on complex matrices. Linear Algebra Appl. 10, 285–290 (1975)
12. Jordan, T.F., Shaji, A., Sudarshan, E.C.G.: Dynamics of initially entangled open quantum systems. Phys. Rev. A 70, 052110 (2004)
13. Leung, D.W.: Choi’s proof and quantum process tomography. J. Math. Phys. 44, 528–33 (2003)
14. Ficek, Z., Tanaś, R.: Entangled states and collective nonclassical effects in two-atom systems. Phys. Rep. 372, 369 (2002)
15. Artin, E.: Galois Theory. Dover, (2004)
16. Banerjee, S., Ravishankar, V., Srikanth, R.: Entanglement dynamics in two-qubit open system interacting with a squeezed thermal bath via quantum nondemolition interaction. Euro. Phys. J. D. 56, 277 (2010)
17. Banerjee, S., Ravishankar, V., Srikanth, R.: Dynamics of entanglement in two-qubit open system interacting with a squeezed thermal bath via dissipative interaction. Ann. Phys. 325, 816 (2010)
18. Shabani, A., Lidar, D.A.: Maps for general open quantum systems and a theory of linear quantum error correction. Phys. Rev. A 80, 012309 (2009)