TOPOLOGICAL PRISMATOIDS AND SMALL NON-HIRSCH SPHERES

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ABSTRACT. We introduce topological prismatoids, a combinatorial abstraction of the (geometric) prismatoids used in the recent counter-examples to the Hirsch Conjecture. We show that the “strong $d$-step Theorem” that allows to construct non-Hirsch polytopes from prismatoids of large width still works at this combinatorial level. Then, using metaheuristic methods on the flip graph, we construct four combinatorially different “non-$d$-step” 4-dimensional topological prismatoids with 14 vertices. This implies the existence of 8-dimensional spheres with 18 vertices which do not satisfy the Hirsch bound, which is smaller that the previously known examples by Mani and Walkup [20] (24 vertices, dimension 11).

Our non-Hirsch spheres are shellable but we do not know whether they are polytopal.

1. Introduction

One of the most important open questions in polytope theory is how big can the graph-diameter of a polytope $P$ be in terms of its dimension $d$ and number $n$ of facets. The gap between the known lower and upper bounds for this function, that we denote $H_{\text{poly}}(n, d)$, is extremely big: no polynomial upper bound for it is known, and no polytope is known whose diameter exceeds $1.05(n - d)$.

Since more than 50 years ago [17] we know that $H_{\text{poly}}(n, d) \leq H_{\text{poly}}(2n - 2d, n - d)$. This, in practice, means that to answer the diameter question one can restrict to the case $n = 2d$. The famous Hirsch Conjecture from 1957 stated that $H_{\text{poly}}(n, d) \leq n - d$. The conjecture is now disproved [24] but known counter-examples to it are still rare. We call such counter-examples non-Hirsch polytopes.

The problem can be stated topologically, by looking at simplicial spheres. In this context we denote by $H_{\text{sph}}(n, d)$ the greatest diameter of the adjacency graph among simplicial $(d - 1)$-spheres with $n$ vertices. We call such a sphere non-Hirsch if this diameter exceeds $n - d$. Since $H_{\text{poly}}(n, d)$ is known to be attained at some simple polytope for every $n$ and $d$, we have that $H_{\text{sph}}(n, d) \geq H_{\text{poly}}(n, d)$: for a simple $P$ attaining $H_{\text{poly}}(n, d)$, the boundary complex of the polar of $P$ is a sphere showing the inequality. Even though there is no reason to believe these two functions to coincide for every value of $n$ and $d$, one does expect their asymptotics to be similar. (For example, all known upper bounds for diameters of polytopes hold also for
spheres, including the Klee-Walkup result that $H_{sph}(n, d) \leq H_{sph}(2n - 2d, n - d)$. See Proposition 2.2.

The Hirsch bound $H_{poly}(n, d) \leq n - d$ is only known to hold for $n - d \leq 6$ [7], $n \leq 12$ [3], and $d \leq 3$, but the smallest known non-Hirsch polytopes have $d = n - d = 20$ [21]. The smallest known non-Hirsch sphere previous to our work was constructed by Mani and Walkup [20]. It is a shellable 11-sphere with 24-vertices ($d = n - d = 12$) shown to be non-polytopal by Altshuler [2].

We here construct non-Hirsch $(d - 1)$-spheres with $d = n - d = 9$.

**Theorem 1.1.** There is a non-Hirsch 8-sphere with 18 vertices. That is, $H_{sph}(18, 9) > 9$.

This is close to minimal since the inequality $H_{sph}(n, d) \leq n - d$ is known to hold for $n - d \leq 5$. This was shown for polytopal spheres by Klee and Walkup [17], and we show in Section 2.2 (Theorem 2.3) how to modify their proof for non-polytopal ones.

One reason to concentrate on small examples is that from them it is very easy to construct bigger ones. In particular, from the example mentioned in Theorem 1.1 one easily derives the following:

**Corollary 1.2.** $H_{sph}(2d, d) > d$ for every $d \geq 9$.

**Proof.** The suspension (or “double-pyramid”) operation shows $H_{sph}(n + 2, d + 1) \geq H_{sph}(n, d) + 1$. □

With the slightly more sophisticated operations of suspension and connected sum we prove a more refined asymptotic bound (see Theorem 2.4). Applying this to the non-Hirsch sphere of Theorem 1.1 gives:

**Corollary 1.3.** For every $n$ and $d$,

$$H_{sph}(n, d) > \left\lfloor \frac{n - d}{d} \right\rfloor \cdot \left(\left\lfloor \frac{10d}{9} \right\rfloor - 1 \right) \simeq 1.11(n - d).$$

Our approach for the construction leading to Theorem 1.1 uses the same prismatoid technique developed by the second author and used in all non-Hirsch polytopes known so far [21, 24]. But we here abstract it to a combinatorial/topological context. Recall that a $d$-prismatoid $P$ is a $d$-dimensional polytope whose vertices lie in two parallel facets. Removing from $\partial P$ the relative interiors of these two facets produces a polyhedral complex homeomorphic to the Cartesian product of a $(d - 2)$-sphere and a segment and with all its vertices in the boundary. This complex, that we assume to be simplicial, is what we call a topological $(d - 1)$-prismatoid. See Section 3 for details. The width of a topological prismatoid $\mathcal{C}$ is defined to equal 2 plus the minimum distance, in the adjacency graph, between facets incident to one and the other component of $\partial \mathcal{C}$. In this setting we prove the following analogue of [24, Theorem 2.6]:

**Theorem (Theorem 3.5).** Let $\mathcal{C}$ be a topological prismatoid of dimension $(d - 1)$, of width $l$ and with $n > 2d$ vertices. Assume that its two bases are polytopal. Then, there exists a topological $(n - d - 1)$-prismatoid $\mathcal{C}'$ with $2n - 2d$ vertices and width at least $l + n - 2d$. 
In particular, if $l > d$ then $C'$ is a non-Hirsch simplicial sphere of dimension $D - 1 := n - d - 1$, with $N := 2D = 2n - 2d$ vertices, and of adjacency diameter larger than $N - D$.

Since this theorem is related to the $d$-step property of Klee and Walkup \cite{17}, we say that topological $(d - 1)$-prismatoids of width larger than $d$ are non-$d$-step. Our goal is to find non-$d$-step topological prismatoids with $n - d$ as small as possible.

We do this by a simulated annealing approach on the graph of bistellar flips among non-$d$-step prismatoids of a given dimension. That is, we start with a topological $(d - 1)$-prismatoid of width $l > d$ and do flips in it at random, but preserving the width and giving higher probability to the flips that go in the direction of decreasing $n$. See Section 3.4 for the definition and properties of flips in topological prismatoids, and Section 4 for details of our heuristics and implementation.

We choose as starting point the 28-vertex prismatoid constructed in \cite[Corollary 2.9]{21}. It has dimension $d = 5$ as a polytope, that is, $d - 1 = 4$ as a topological prismatoid. We take this one, instead of the smaller one with 25 vertices also constructed in \cite{21}, because this one has much more symmetry.

From it we find thousands of non-$d$-step topological 4-prismatoids with number of vertices bounded by 28. In particular, we find four combinatorially different ones with 14 vertices. Any of these four implies Theorem 1.1, via Theorem 3.5.

The constructed prismatoids are analyzed a bit in Section 5. For the four smallest ones we have checked that they are shellable, with respect to a natural notion of shelling of topological prismatoids that we introduce in Section 3.4, and which implies shellability for the resulting non-Hirsch spheres (see Remark 3.16). Shellability is necessary for polytopality, but we do not know whether our prismatoids (or spheres) are polytopal.

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2. Preliminaries

2.1. Pure simplicial complexes, simplicial spheres. Here we compile several notions from combinatorial topology.

A simplicial complex $C$ is a collection of subsets of a finite ground set $V$ (typically, $V = [n] := \{1, \ldots, n\}$), that is closed under taking subsets. It is pure of dimension $d - 1$ (in which case we call it a $(d - 1)$-complex) if all maximal elements of $C$ have the same cardinality, equal to $d$. The elements of $C$ are called faces and maximal faces are facets; more specifically, a face of size $i$ is called an $(i - 1)$-face. Some faces have special names according to their cardinality:

- Faces of size 1 and 2 are called vertices and edges; together they form a graph, the 1-skeleton of $C$.
- Faces of size $d$ and $d - 1$ are called facets and ridges; they also define a graph, called the adjacency graph (or dual graph) of $C$: its vertices are the facets and two facets are adjacent if they share a ridge.
- Every complex has a face of size 0, the empty face.

We call Hasse diagram of $C$ the Hasse diagram of the partial order of faces by inclusion. That is, it is a directed graph with an arc $f_1 \to f_2$ for every pair of faces $f_1$ and $f_2$ with $f_2 = f_1 \cup \{v\}$ for some $v \in V$. Observe that the 1-skeleton and the
adjacency graph of a pure complex contain the information about the lower two and the higher two levels of the Hasse diagram, respectively.

**Example 2.1.** The boundary complex of a simplicial $d$-polytope is a pure simplicial complex of dimension $d - 1$.

There are the following three subcomplexes of $C$ associated to every face $f \in C$.

- **The deletion** of $f$ in $C$ is the set of faces disjoint from $f$.
- **The star** of $f$ in $C$ is the set of faces $f'$ such that $f \cup f'$ is also a face. Equivalently, it is the simplicial complex whose facets are the facets of $C$ containing $f$.
- **The link** of $f$ in $C$ is the set of faces of $C \setminus f$ that, joined with $f$, belong to the complex. That is, it equals the deletion of $f$ in the star of $f$.

We call **neighborhood** of $f$ the set of vertices of its star.

Every simplicial complex $C$ has a realization as a topological space obtained as follows: consider a disjoint family of simplices in $\mathbb{R}^N$ (for sufficiently big $N$) consisting of a simplex of dimension $i$ for each $i$-face in $C$. Then take the topological quotient of this set of simplices, by identifying faces as indicated by containment in $C$. A simplicial $(d - 1)$-manifold or triangulated manifold (with or without boundary) is a pure $(d - 1)$-complex whose realization is a manifold. A simplicial $(d - 1)$-sphere or $(d - 1)$-ball is defined in the same way.

Every ridge of a simplicial manifold is contained in either two or one facets. Ridges of the first type are called **interior** and those of the second type are called **boundary**. The boundary of a $(d - 1)$-manifold $C$ is the $(d - 2)$-pure complex having as facets the boundary ridges. That is, all faces contained in boundary ridges are **boundary faces**.

### 2.2. The Hirsch conjecture and its relatives

The (combinatorial) diameter of a polytope is the diameter (in the sense of graph theory) of its 1-skeleton. Let $H_{\text{poly}}(n, d)$ denote the maximum diameter among all $d$-polytopes with $n$ facets. The Hirsch Conjecture stated that $H_{\text{poly}}(n, d) \leq n - d$. The first counter-examples\(^1\) have been obtained in \([24, 21]\) and exceed the conjecture by a 5% or less.

It is known for long that $H_{\text{poly}}(n, d)$ is attained at a simple polytope for every $n$ and $d$. In particular, $H_{\text{poly}}(n, d)$ equals the maximum diameter of the adjacency graphs of simplicial $d$-polytopes with $n$ vertices. Generalizing this a little bit we define $H_{\text{sph}}(n, d)$ to be the maximum diameter of the adjacency graphs of simplicial $(d - 1)$-spheres with $n$ vertices, and $H_{\text{simp}}(n, d)$ to be the maximum diameter of the adjacency graphs of pure simplicial $(d - 1)$-complexes with $n$ vertices.

The latter is known to be exponential, but the first two are conjectured to be polynomial and perhaps not far from the Hirsch bound. More precisely, Table 1 sums up the known bounds.

Additionally, the Hirsch bound is known for 2-spheres (all of which are polytopal) and for spheres with $n - d \leq 5$. The latter is stated in \([17]\) only for polytopes, but the proof works for spheres with minor changes as we sketch below.

In the proof we need the **one-point suspension** construction: the one-point suspension of a simplicial complex $C$ at a vertex $w$ is the complex $C'$ obtained by considering the usual suspension $C \ast \{\{w_1\}, \{w_2\}\}$ of $C$ and then merging the stars

\(^1\)...to the version we just stated; the original Conjecture was for perhaps-unbounded polyhedra and was disproved in 1967 by Klee and Walkup [17].
of edges $ww_1$ and $ww_2$ into the star of a single edge $w_1w_2$ so that the vertex $w$ disappears. For a more direct and explicit description:

$$C' := \{ F, F \cup \{ w_1 \}, F \cup \{ w_2 \}, F \cup \{ w_1, w_2 \} : F \in \text{link}_C(w) \}$$

$$\cup \{ F, F \cup \{ w_1 \}, F \cup \{ w_2 \} : w \notin F \in C \}.$$ 

The one-point suspension of a sphere is a sphere with one more dimension, one more vertex and (at least) the same diameter as the original one. (See details for example in [10] [24]).

**Proposition 2.2** (Sphere version of [17] Proposition 2.10)).

$$H_{sph}(n, d) \leq H_{sph}(2n - 2d, n - d) \quad \forall n, d.$$ 

**Proof.** Let $S$ be a $(d-1)$-sphere with $n$ vertices and let $F$ and $G$ be two facets in it. By induction on $|n - 2d|$ we only need to show that the distance between $F$ and $G$ is bounded by $H_{sph}(n + 1, d + 1)$ if $n < 2d$ and by $H_{sph}(n - 1, d - 1)$ if $n > 2d$.

In the first case $F$ and $G$ cannot be disjoint, so let $v \in F \cap G$. Denote $S' = \text{link}_S(v)$, and consider $F' = F \setminus \{ v \}$ and $G' = G \setminus \{ v \}$, which are facets in $S'$. The distance from $F$ to $G$ in $S$ is clearly bounded by the distance from $F'$ to $G'$ in $S'$, and $S'$ is a $(d-2)$-sphere with at most $n - 1$ vertices.

The second part follows from the one-point suspension of $S$ at any vertex. 

**Theorem 2.3** (Sphere version of [17] Theorem 4.2)). For any $n, d$ with $n - d \leq 5$, $H_{sph}(n, d) \leq n - d$.

**Proof.** By Proposition 2.2 we can assume $n = 2d$. So, let $S$ be a $(d-1)$-sphere with $2d$ vertices and let $F$ and $G$ be two facets in it achieving the diameter. We want to show the distance from $F$ to $G$ to be at most $d$.

The proof follows the one in [17] for the polytopal case and consists of the following three claims. Claims 1 and 2 do not need the assumption that $d \leq 5$, and correspond to Theorem 2.8 and Proposition 3.4.(b) in [17].

Claim 1: there is no loss of generality in assuming $F \cap G = \emptyset$. If this is not the case, let $S' = \text{link}_S(F \cap G)$. Then, $F' := F \setminus G$ and $G' := G \setminus F$ are facets in $S'$ at distance at least equal to the distance between $F$ and $G$ in $S$. Observe that $S'$ is a $(d-1-k)$-sphere, where $k = |F \cap G|$, and it has between $2d - 2k$ and $2d - 1$ vertices. If $S'$ has $2d - 2k$ vertices, the diameter of $S'$ is bounded by $H_{sph}(2d - 2k, d - k)$ which, by induction on $d$, is at most $d - k$. If $S'$ has more than $2d - 2k$ vertices, let $S''$ be the iterated one-point suspension of $S'$ at each vertex $w$ not in $F' \cup G'$. 

| $H_{simp}(n, d)$ | Lower bound | $\left(\frac{n}{d+1}\right)^{d-1} - 3$ (Criado-Santos 2016 [8]) | Upper bound | $\binom{n}{d-1}$ |
|-----------------|-------------|-----------------------------------------------|-------------|----------------|
| $H_{sph}(n, d)$ | $\simeq 1.08(n - d)$ (Mani-Walkup 1980 [20]) | $\simeq 1.11(n - d)$ (This paper) | $\min\{2^{d-3}n, n^{\log d - d - 1}\}$ (Larman 1970 [18], Matschke-Weibel-Santos 2017 [21]) | $\min\{2^{d-3}n, n^{\log d - d - 1}\}$ (Larman 1970 [18], Matschke-Weibel-Santos 2017 [21]) |
| $H_{poly}(n, d)$ | $\simeq 1.05(n - d)$ (Matschke-Weibel-Santos 2017 [21]) | $\min\{2^{d-3}n, n^{\log d - d - 1}\}$ (Larman 1970 [18], Matschke-Weibel-Santos 2017 [21]) | $\min\{2^{d-3}n, n^{\log d - d - 1}\}$ (Larman 1970 [18], Matschke-Weibel-Santos 2017 [21]) |

**Table 1.** Known bounds for maximum diameters of classes of simplicial complexes
\(S''\) is a sphere of dimension \(d - k - 1 + l\) and with \(2d - 2k + 2l\) vertices, where \(l\) is the number of times we did the one-point suspension. In \(S', F'\) and \(G'\), each together with one of the two vertices of each suspension, give two complementary facets which are at least at the same distance as \(F'\) and \(G'\) were in \(S'\).

**Claim 2:** assume \(F \cap G = \emptyset\). There are vertices \(v \in F\) and \(w \in G\) such that \(\{v, w\}\) is an edge in \(S\) and such that \(F\) and \(G\) are adjacent respectively to facets \(F'\) and \(G'\) with \(\{u, v\} \in F' \cap G'\). Let \(F'\) be any facet adjacent to \(F\). Let \(w\) be the unique vertex in \(F'\) \(\setminus\) \(F\) and \(u\) the unique vertex in \(F'\) \(\setminus\) \(F'\). It is impossible for the \(d - 1\) facets adjacent to \(G\) and containing \(w\) to all of them use \(w\); indeed, if that happened then these \(d - 1\) facets together with \(G\) form a ball with \(w\) in its interior, implying that there are no other facets in the star of \(w\); this is impossible because \(F'\) is in the star of \(w\) too. Hence, there is a facet \(G'\) adjacent to \(G\), using \(w\), and using a vertex \(v\) of \(F\) other than \(u\). This proves the claim for the edge \(vw\) thus obtained.

**Claim 3:** with \(v, w\) as above, there is a path in \(\text{star}_S(\{v, w\})\) of length at most \(d - 2\) between a facet \(F'\) adjacent to \(F\) and a facet \(G'\) adjacent to \(G\). The proof of this claim is the complicated part, occupying most of Section 4 (pages 69–71) of [17]. The good thing is that we do not even check that the proof extends to \(d\)-polytopes for \(d \geq 5\). The lower bounds in the last two rows of Table 1 are meant asymptotically: they hold for fixed but sufficiently large \(n\) and as \(n\) goes to infinity. They follow from the known smallest known non-Hirsch polytope \((n = 40, d = 20\) [21]) and sphere \((n = 24, d = 12\) [20]) via Theorem 2.4 below, whose proof follows from the following considerations.

We call a \(d\)-polytope or \((d - 1)\) sphere with \(n\) vertices non-Hirsch if it has diameter \(l\) greater than \(n - d\). Its excess is defined to be \(\frac{n - d}{d - 1} - 1\). From any non-Hirsch sphere or ball infinitely many additional ones can be obtained by the following procedures:

- The join of a \((d_1 - 1)\)-sphere \(S_1\) with \(n_1\) vertices and diameter \(l_1\) and a \((d_2 - 1)\)-sphere \(S_2\) with \(n_2\) vertices and diameter \(l_2\) is a \((d_1 + d_2 - 1)\)-sphere with \(n_1 + n_2\) vertices and diameter \(l_1 + l_2\). It is denoted \(S_1 \ast S_2\).
- The connected sum of two \((d - 1)\)-spheres \(S_1\) and \(S_2\) with \(n_1\) and \(n_2\) vertices and of diameters \(l_1\) and \(l_2\) is a \((d - 1)\)-sphere with \(n_1 + n_2 - d\) vertices and of diameter at least \(l_1 + l_2 - 1\). It is denoted \(S_1 \# S_2\). (Strictly speaking, the diameter of a connected sum of two spheres depends on the choice of facets to glue; we here assume that they are glued in the worst possible way).
- The suspension of \(S\) has one more dimension and two more vertices than \(S\), and its diameter is one more than that of \(S\).

These constructions, which all preserve polytopality, lead to the following:
**Theorem 2.4** (Variation of [24, Theorem 6.5]). If for a certain $d_0$ we know that $H_{\text{sph}}(2d_0, d_0) = l_0$, then

$$H_{\text{sph}}(n, d) > \left\lfloor \frac{n - d}{d} \right\rfloor \cdot \left\lfloor \frac{d}{d_0} \right\rfloor (l_0 - d_0) + d - 1 \geq \frac{l_0}{d_0} (n - d), \quad \forall n, d.$$  

In particular, we have $H_{\text{sph}}(n, d) \geq (n - d)\frac{l_0}{d_0}$ for $n \gg d \gg d_0$.

The same holds for $H_{\text{poly}}$.

**Proof.** Let $S_0$ be the initial sphere, of dimension $d_0 - 1$, with $2d_0$ vertices, and with diameter $l_0$. Then, for every $k$ the $k$-fold suspension $S^{*k}$ of $S$ has dimension $kd_0 - 1$, diameter $k$, and $2kd_0$ vertices. Letting $k = [d/d_0]$ and performing $d - d_0k$ suspensions on $S^{*k}$ we obtain that

$$H_{\text{sph}}(2d, d) \geq \left\lfloor \frac{d}{d_0} \right\rfloor (l_0 - d_0) + d, \quad \forall d, k.$$  

Let $T^d$ be the $(d - 1)$-sphere on $2d$ vertices obtained so far. By a connected sum of $[n/d] - 1 = [(n - d)/d]$ copies of $T^d$ we conclude that

$$H_{\text{sph}}(n, d) \geq H_{\text{sph}}(d[n/d], d) \geq \left\lfloor \frac{n - d}{d} \right\rfloor \cdot \left( \left\lfloor \frac{d}{d_0} \right\rfloor (l_0 - d_0) + d \right) - \left\lfloor \frac{n - d}{d} \right\rfloor + 1$$

$$= \left\lfloor \frac{n - d}{d} \right\rfloor \cdot \left( \left\lfloor \frac{d}{d_0} \right\rfloor (l_0 - d_0) + d - 1 \right) + 1.$$  

The smallest non-Hirsch spheres constructed in this paper have dimension 8, 18 vertices, and excess 1/9, which gives the bound in Corollary 1.3.

### 2.3. Prismatoids and the strong $d$-step theorem

The $d$-step theorem of Klee and Walkup is the statement that $H_{\text{poly}}(n, d) \leq H_{\text{poly}}(2(n - d), n - d)$ for every $n$ and $d$. In particular, it reduces the study of the Hirsch conjecture or the asymptotic behavior of $H_{\text{poly}}(n, d)$ to the case $n = 2d$. The proof works with no change for $H_{\text{sph}}(n, d)$, since it is purely combinatorial.

Santos’ construction of non-Hirsch polytopes is based on a version of this result for a particular class of polytopes, the so-called prismatoids.

**Definition 2.5.** A *prismatoid* is a polytope $Q$ with two parallel facets $Q^+$ and $Q^-$, that we call the bases, containing all the vertices. We call a prismatoid simplicial if all faces except perhaps $Q^+$ and $Q^-$ are simplices. Observe that the faces of a prismatoid of dimension $d$, excluding the two bases, form a simplicial complex of dimension $d - 1$ and homeomorphic to the product of $S^{d-2}$ with a segment. We call this complex the **prismatoid complex** of $Q$.

The width of a prismatoid is the distance from one base to the other, measured in the adjacency graph.

**Theorem 2.6** (Strong $d$-step theorem for prismatoids [24]). If $Q$ is a simplicial $d$-prismatoid of width $l$ and with $n > 2d$ vertices, there exists a simplicial $n - d$-prismatoid $Q'$ with $2n - 2d$ vertices and width at least $l + n - 2d$.

In particular, if $l > d$ then (the simple polytope dual to) $Q'$ violates the Hirsch Conjecture.
Santos’ original counterexample applies this result to a 5-prismatoid with 48 vertices and of width six, thus obtaining a non-Hirsch 23-polytope with 46 facets. This was improved in [21] to a 5-prismatoid of the same width but with only 25 vertices, which provides a non-Hirsch polytope in dimension 20.

3. Topological prismatoids and the topological strong $d$-step theorem

3.1. Prismatoids and bistellar flips in them. We now define the main object we work with:

**Definition 3.1.** A $((d - 1)$-dimensional) topological prismatoid $C$ is a $(d - 1)$-dimensional pure simplicial complex homeomorphic to $S_{d-2} \times [0, 1]$ (that is, it is homeomorphic to a cylinder), and such that every face with all its vertices in the same boundary component is a boundary face. Put differently, the two boundary components, each homeomorphic to $S_{d-2}$, are induced subcomplexes.

Bistellar flips were introduced for general manifolds in [22] and they are a standard tool in combinatorial topology by now, as local modifications that preserve the PL-type. The main result of Pachner [22] is the converse: every two PL-homeomorphic simplicial manifolds can be transformed into one another via a sequence of bistellar flips. We here adapt the general definition to the case of topological prismatoids. The main new feature of our definition is that we want to allow flips to change the boundary but we need to guarantee that the two bases are still induced subcomplexes.

**Definition 3.2.** A bistellar flip in a topological prismatoid $C$ is a triple $(f, l, v)$ of pairwise disjoint subsets of $V(C)$ such that $f$ is a face, $l$ is a minimal nonface, $\text{link}_C(f) = \partial(l) * v$, and one of the following two things happens:

- $|f| + |l| = d + 1$ and $v = \emptyset$, in which case $l$ is required to intersect both bases of $C$. (Observe that in this case $\text{link}_C(f) = \partial(l) * v = \partial(l)$).
- $|f| + |l| = d$ and $v$ is a vertex, in which case $f$ and $l$ are required to be contained in the base opposite to $v$.

In both cases, the result of the flip is the prismatoid $C' = C \setminus \text{star}_C(f) \cup (l * \partial(f) * v)$.

Flips with $v = \emptyset$ are called interior flips and flips where $v$ is a vertex are called boundary flips. The support of the flip is $f \cup l \cup v$.

Put differently, an $(f, l, v)$ flip removes all faces containing $f$ and inserts as new faces all subsets of $f \cup l \cup v$ that contain $l$ but not $f$.

Only a boundary flip can modify the boundary. In particular, only boundary flips can add or remove vertices.

**Remark 3.3.** In Definition 3.2, $V(C)$ is understood as the ground set of the prismatoid, which may contain points that are not used as vertices. In particular, a boundary flip may have $l = \{w\}$ for a $w$ that is not a vertex, and $f$ a facet in a base. The result of the flip is that this facet is stellarly subdivided with the new vertex $w$.

We need this type of flips because we want flips to be reversible for the Simulated Annealing framework. These flips are the inverse of vertex-removing flips.
An important feature used in our implementation is that knowing only the support $u$ of a flip we can recover the sets $f$, $l$ and $v$ and thus perform the flip:

- If $u$ has a single vertex from one of the bases then the flip is a boundary flip, and that vertex is $v$. Indeed, in an interior flip $l$ has at least one vertex from each component by definition, and $f$ has at least another from each base because the condition $\text{link}(f) = \partial(l)$, with $|f| + |l| = d + 1$, implies that $f$ is an interior face.
- In both cases, the set $f \cup v$ equals the intersection of all facets of $C$ contained in $u$. This allows us to recover $f$, and hence $l$, once we know $v$ by the previous point.

The support $u$ of a flip must have $d + 1$ vertices, since it is the vertex set of a $d$-ball of the form $l \ast \partial(f) \ast v$, that is, the join of an $i$-simplex and the boundary of a $j$-simplex, with $i + j = d - 1$.

Moreover, the following result allows us to detect flips:

**Proposition 3.4.** Given a set $u$ of $d + 1$ vertices (or $d$ vertices and an unused point in the case of insertion flips) not all in one base, let $f$, $l$ and $v$ be as above. The following conditions are necessary and sufficient for $u$ to support a flip in $(f, l, v)$:

1. $u$ is the neighborhood of a ridge.
2. $\text{neigh}(f)$ has $d + 1$ vertices (or $d$ vertices in case of insertion flips).
3. $l$ is not a face of $C$.

### 3.2. The strong $d$-step theorem for topological prismatoids

We here prove the main theoretical result that allows us to use topological prismatoids to search for non-Hirsch spheres. The **width** of $C$ is two plus the distance, in the adjacency graph, between the set of facets incident to one base and the set of facets incident to the other (the distance between two sets is, as customary, the minimum distance between respective elements).

**Theorem 3.5** (Strong $d$-step theorem for topological prismatoids). Let $C$ be a topological prismatoid of dimension $(d - 1)$, width $l$ and with $n > 2d$ vertices. Assume that its two bases are polytopal. Then, there exists a topological $(n - d - 1)$-prismatoid $C'$ with $2n - 2d$ vertices and width at least $l + n - 2d$.

In particular, if $l > d$ then $C'$ is a simplicial sphere of dimension $D - 1 := n - d - 1$, with $N := 2D = 2n - 2d$ vertices whose adjacency graph has diameter larger than $N - D$.

**Remark 3.6.** The excess of the non-Hirsch sphere produced via Theorem 3.5 from a topological $(d - 1)$-prismatoid $C$ of width $l$ and $n$ vertices equals

$$\frac{l - d}{n - d}.$$  

Thus, we call that quotient the **(prismatoid) excess** of $C$.

**Proof.** By induction on $n - 2d$ it suffices to construct a prismatoid of dimension $d$ with $n + 1$ vertices and width at least $l + 1$. Repeating this procedure $n - 2d$ times we must arrive at a $(D - 1)$-prismatoid with $2D$ vertices. In such a prismatoid the bases are simplices, so the prismatoid is a $(D - 1)$-sphere with $2D$-vertices and diameter at least $l + (n - 2d)$.

For the inductive step, let $B^+$ and $B^-$ be the two bases of $C$. Since $C$ has more than $2d$ vertices, at least one of them (say $B^+$) is not a simplex. Let $S^+$ be a
simplicial polytopal $d$-sphere containing $B^+$ and with no additional vertices. $S^+$ exists since $B^+$ is polytopal: Let $P \subset \mathbb{R}^d$ be a $d$-polytope realizing $B^+$ and choose a sufficiently generic lifting function $h: \text{vertices}(P) \to \mathbb{R}$. Then $S^+$ can be chosen to be the boundary complex of $\text{conv}\{(v, h(v)) : v \in \text{vertices}(P)\}$.

Let $S^+_1$ and $S^+_2$ be the two closed $d$-balls whose intersection is $B^+$ and whose union is $S^+$. Let $v_1'$ and $v_2'$ be two additional vertices. Consider the following simplicial complex:

$$C' := (C \cup B^+_1) \ast v_1 \cup (C \cup B^+_2) \ast v_2.$$ 

$C'$ is a topological $d$-prismatoid with bases $S^+$ and $S^- := B^- \ast \{v_1, v_2\}$ (see Figure 1). It is not yet the prismatoid we want since it has $n + 2$ vertices, but let’s not care about that for the time being. Instead, for reasons that will become apparent later, when computing the length of a path between facets adjacent to $S^+$ and to $S^-$ we will neglect the steps of the form $F \ast v_1$ to $F \ast v_2$, and vice versa, for facets $F$ of $C$. We claim that even with this reduced way of counting steps $C'$ has width strictly larger than $C$.

For this, let $F'_0, \ldots, F'_t$ be a path in $C'$ from a facet $F'_0$ adjacent to $S^-$ to a facet $F'_t$ adjacent to $S^+$. Since we want our path as short as possible, there is no loss of generality in assuming that $F'_i$ is the only facet adjacent to $S^+$. That is, each facet $F'_i, i \in \{0, \ldots, t - 1\}$, is of the form $F_i \ast v_j$ for a certain facet $F_i$ of $C$ and $j \in \{1, 2\}$. We claim that:

- Facets $F_i$ and $F_{i+1}$ are either adjacent or the same; the latter happens if and only if $F'_i$ and $F'_{i+1}$ are of the form $F \ast v_1$ and $F \ast v_2$ for the same facet $F$ of $C$.
- The first facet $F_0$ is adjacent to $B^-$, since $F'_0$ is adjacent to $S^- = B^- \ast \{v_1, v_2\}$.
- The last facet $F_{t-1}$ is adjacent to $B^+$, since $F'_{t-1}$ is obtained from the facet $F'_t = F_t \ast v_j$ by changing a single vertex, and $F_t$ is a facet of $S^+$, not of $C$.
Thus, as claimed, the width of $C$ is strictly larger than that of $C$, even neglecting the steps $F \ast v_1 \leftrightarrow F \ast v_2$.

We now get rid of one vertex without decreasing the width of $C'$. For this, let $v$ be any vertex of $B^-$ and observe that

$$\text{link}_{C'}(vv_1) = \text{link}_{C'}(vv_2) = \text{link}_C(v).$$

This implies that we can substitute in $C'$ the stars of edges $vv_1$ and $vv_2$ by the star of a single edge $v_1v_2$, to obtain a topological prismatoid with one less vertex, that is, with $n + 1$ vertices. More precisely, we let:

$$C'' := C' \setminus (vv_1 \ast \text{link}_C(v) \cup vv_2 \ast \text{link}_C(v)) \cup v_1v_2 \ast \text{link}_C(v).$$

This is indeed a prismatoid $C''$, with bases $S^+$ and the one-point suspension of $B^-$ at $v$. It has $n + 1$ vertices, dimension $d$, and it has the same width as $C'$ if we neglect the steps of the form $F \ast v_1 \leftrightarrow F \ast v_2$; in particular, it has width strictly larger than that of $C$.

\begin{remark}
Analyzing the proof of Theorem 3.5, the condition of the bases being polytopal can be changed to the following weaker one: we require each base $B$ to be embeddable in a chain of simplicial spheres $B = S_{d-2} \subset S_{d-1} \subset \cdots \subset S_{m-2}$ where $m$ is the number of vertices of $B$ and each $S_i$ is a simplicial sphere with the same vertex set as $B$. (In particular, $S_{m-2}$ is the boundary of an $(m-1)$-simplex).

Indeed, the only place where we use polytopality in the proof is to guarantee the existence of the $(d-1)$-sphere $S^+$ containing $B^+$ and without additional vertices. $S^+$ becomes a base of the prismatoid $C''$ obtained in the proof, which is why we need the condition recursively. It is important to notice that this property is closed under one-point suspension, because $S_i \subset S_{i+1}$ implies that the one-point suspension of $S_i$ is contained in the one-point suspension of $S_{i+1}$. In particular, if both bases of $C$ satisfy the property then both bases of $C''$ satisfy it too: $S^+$ by definition of the class and the other base because it equals the one-point suspension of $B^{-}$.

We do not know whether all spheres have to this property but we suspect not, in the light of Example 3.5 below. But the class of spheres with the property contains at least the spheres whose face lattice is isomorphic to the Las Vergnas lattice of a uniform acyclic oriented matroid [4, Definition 4.1.2]. Recall that the Las Vergnas lattice $B_M$ of an acyclic oriented matroid $M$ is the lattice of non-negative covectors ordered by reverse inclusion. When $M$ is uniform $B_M$ is a PL-sphere. That these spheres belong to our class follows from the existence of “reoriented Lawrence lifts” (which correspond to one-point suspensions) and “acyclic uniform lifts” (which produce the increasing sequence of spheres). See, e.g., [23].

\end{remark}

\begin{example}
In [4], Altshuler constructs a 10-vertex 3-sphere with $f$-vector $(10, 45, 70, 35)$ and with the following peculiar property: for each vertex $v$ in $S_{10}$ and for every 3-ball $B_v$ with no additional vertices and with $\partial B_v = \text{link}_{S_{10}}(v)$, there is an interior face in $B_v$ that is already used in $S_{10}$. That is, the hole created by removing the (open) star of $v$ from $S_{10}$ cannot be closed by gluing a simplicial ball without additional vertices. The sphere $S_{10}$ also has the property that the pulling triangulations from any two different vertices $v$ and $w$ of $S_{10}$ have common interior faces. (This is equivalent to saying that $S_{10}$ has some minimal non-face containing $v$ and $w$).

Put differently, neither a “pushing” nor a “pulling” strategy can give two 4-balls with boundary $S_{10}$ and with disjoint interiors. This made $S_{10}$ a good candidate for

\end{example}
not being embeddable in a 4-sphere without additional vertices. However, we have checked that two 10-vertex 4-balls with boundary $S_{10}$ and with disjoint interiors do exist. (We constructed them by flipping away the common interior faces from two pulling triangulations).

3.3. **Prismatoids of large width via reduced incidence patterns.** In all previous constructions of non-Hirsch polytopes, the proof that the prismatoids to which Theorem 3.5 is applied is non-d-step uses the following result.

**Proposition 3.9.** Let $C$ be a (geometric or topological) prismatoid with bases $B^+$ and $B^-$. A necessary condition for $C$ to be d-step is that there are vertices $v \in B^+$ and $w \in B^-$ such that $vw$ is an edge and the star of $vw$ contains facets incident to both bases.

This proposition is a rephrasing of [21, Proposition 2.1], and used in part (3) of [24, Lemma 5.9]. Observe that the necessary condition in the statement is exactly Claim 2 in the proof of Theorem 2.3. Following [21] we introduce the following graph-theoretical way to visualize this property:

**Definition 3.10.** Let $C$ be a topological prismatoid with bases $B^+$ and $B^-$. The incidence pattern of $C$ is the bipartite directed graph having a node for each vertex of $C$ with bipartition given by the bases and with the following arcs: for each $v \in B^+$ and $w \in B^-$ we have an arc $v \to w$ (resp., $w \to v$) if there is a facet $F$ in $C$ containing $vw$ and incident to $B^+$ (resp., incident to $B^-$). The reduced incidence pattern is the subgraph induced by vertices that are not sources.

In this language, Proposition 3.9 becomes:

**Proposition 3.11 (Topological version of [21, Proposition 2.3]).** Let $C$ be a topological prismatoid. If there is no directed cycle of length two (that is, a “bidirectional arc”) in its reduced incidence pattern then $C$ is non-d-step.

**Proof.** Suppose $C$ is d-step, so that there is a sequence of facets $F_1, \ldots, F_d$, each adjacent to the next, and with $F_0$ adjacent to $B^+$ and $F_1$ adjacent to $B^-$. In particular, $F_1$ consists of a vertex $v$ of $B^-$ and $d-1$ vertices of $B^+$, and $F_d$ consists of a vertex $w$ of $B^+$ and $d-1$ vertices of $B^-$. This can only happen if $v$ is already in $F_{d-1}$ and $w$ in $F_1$, so that $v$ and $w$ form a 2-cycle in the reduced incidence pattern.

Using the fact that in a reduced incident pattern without cycles of length two all vertices must have out-degree at least two, the minimum possible patterns were classified in [21]. The proof works without changes for topological prismatoids:

**Lemma 3.12 ([21, Proposition 2.4]).** Let $C$ be a topological prismatoid whose reduced incidence pattern has no cycles of length two. Then, the reduced incident pattern has at least eight vertices.

Moreover, the only two possible patterns with eight vertices are those of Figure 2 (vertices of one base are white, vertices of the other are grey).

It is interesting to note that the prismatoids constructed in [21, 24] have the reduced incidence pattern on the left of Figure 2 while the ones we obtain in this paper are related to the pattern on the right. See Section 5 for details.
Remark 3.13. The absence of cycles of length two is sufficient but not necessary for being non-$d$-step. For example, two of the four small non-Hirsch prismatoids described in Section 5 do have cycles of length two in their reduced incidence patterns. In fact, a cycle of length two in the reduced incidence pattern is the same as an edge in the prismatoid whose star contains facets adjacent to both bases. Hence, Claim 2 in the proof of Theorem 2.3 is equivalent to: “if $C$ is a topological $(d-1)$-prismatoid with $2d$ vertices then its reduced incidence pattern does not contain cycles of length two”.

3.4. Shellability of (topological) prismatoids.

Definition 3.14. Let $C$ be a topological prismatoid with bases $B^+$ and $B^-$. A prismatoid shelling of $C$ from $B^+$ to $B^-$ is an ordering $F_1, \ldots, F_K$ of the facets of $C$ with the following property: for each $i = 1, \ldots, K$, the intersection of $|F_i|$ with $|B^+| \cup |F_1| \cup \cdots \cup |F_{i-1}|$ is a $(d-2)$-ball in the boundary complex of $F_i$. Here, the notation $|\cdot|$ applied to a subset of vertices means the subcomplex induced by them.

This definition is a special case of the definition of shellability of relative simplicial complexes. Indeed, $C$ is shellable from $B^+$ to $B^-$ in the sense of Definition 3.14 if and only if the relative complex $(C, B^+)$ is shellable in the sense of [11, Section 4.2].

Our definition is also related to the classical notion of shellability of regular cell complexes as follows: Consider the bases $B^+$ and $B^-$ as triangulations of the boundary of respective $d$-balls $S^+$ and $S^-$ (the interiors of $S^+$ and $S^-$ are left unrefined. That is, $S^+$ and $S^-$ are not simplicial complexes but regular cell complexes with a single $d$-cell and with all other cells simplicial). If we glue $S^+$ and $S^-$ to $C$ we obtain a “regular cell complex with the intersection property” (see [3]) that we here denote $C'$. The standard notion of shellability for such a complex is the following. (This is a slight modification of the definition in [19], where the complex is assumed to be polyhedral. Our complex $C'$ is polyhedral if and only if both bases of $C$ are polytopal spheres):

$C$ is shellable if facets have shellable boundary and there is an ordering $F_0, \ldots, F_{K+1}$ of the facets such that for every $i \geq 1$ the intersection of $|F_i|$ with $|F_0| \cup |F_1| \cup \cdots \cup |F_{i-1}|$ is a $(d-2)$-ball or a $(d-2)$-sphere in the boundary complex of $F_i$, and this ball/sphere is the beginning of a shelling of the boundary of $F_i$.
Observe that the definition is recursive on the dimension. In our case, where all the facets of $C'$ other than $S^+$ and $S^-$ are simplices, the recursive condition is automatic except for the fact that $C'$ cannot be shellable unless $B^+$ and $B^-$ are shellable. In fact:

**Lemma 3.15.** Any shelling of $C'$ that starts with $F_0 = S^+$ and ends with $F_{K+1} = S^-$ induces (by forgetting the first and last cell) a prismatoid shelling of $C$ from $B^+$ to $B^-$. Conversely, if $F_1, \ldots, F_K$ is a prismatoid shelling of $C$ from $B^+$ to $B^-$ and $B^+$ and $B^-$ are shellable then $S^+, F_1, \ldots, F_K, S^-$ is a shelling of $C'$.

*Proof.* The first implication is obvious.

For the other direction, if $F_1, \ldots, F_K$ is a shelling of $C$ from $B^+$ to $B^-$, for every $F_i$ the intersection of $|F_i|$ with $|B^+| \cup |F_1| \cup \cdots \cup |F_{i-1}|$ is a ball by definition, so the corresponding $F_i$ also verifies the shellability condition in the shelling of $C'$. Furthermore, since $F_i$ is a simplex, the intersection is the beginning of a shelling of $F_i$, no matter what intersection this is.

Finally, we have to check that the second base also verifies the shelling condition. $B^-$ intersects the previous facets in the shelling in its complete boundary, which is a $(d-2)$-sphere. Since $B^-$ is also shellable, this is also the beginning (and the end) of a shelling of $B^-$. \qed

Because of this lemma there is no ambiguity in calling “prismatoid shellings” just “shellings” from now on. Another consequence is that every polytopal prismatoid is shellable.

**Remark 3.16.** Shellability is preserved under the “strong $d$-step construction”, as follows: Suppose that the sphere $S^+$ has a shelling that starts with $S_1^+$ and ends with $S_2^*$ (this happens for polytopes, since a line shelling does it). Then shell first $S_1^+ \ast v_1$, then the prismatoid part in order (if $F_1, F_2, \ldots$, is a shelling order of the original prismatoid, do $F_1 \ast v_1 F_1 \ast v_2, F_2 \ast v_1, F_2 \ast v_2, \ldots$, and finally $S^* \ast v_2$.

**Remark 3.17.** It is not clear to us whether the bases of a prismatoid that is shellable in the sense of Definition 3.14 have to be shellable themselves.

Related to shellability, we now define the $h$-vector of a topological prismatoid:

**Definition 3.18.** Let $C$ be a topological $(d-1)$-prismatoid with bases $B^+$ and $B^-$. Let $h' = (h'_0, \ldots, h'_d)$, $h^+ = (h_0^+, \ldots, h_{d-1}^+)$, and $h^- = (h_0^-, \ldots, h_{d-1}^-)$, be the $h$-vectors of the spheres $C'$, $B^+$ and $B^-$, respectively. We define the $h$-vector of $C$ from $B^+$ to $B^-$ to be $h = (h_0, \ldots, h_{d-1})$ with

$$h_i = h'_i - h_i^+ - h_i^-.$$

Put differently,

$$h = (h_0', \ldots, h_d') - (h_0^-, \ldots, h_{d-1}^-) - (0, h_0^-, \ldots, h_{d-1}^-).$$

Observe that $h_0 = h_d = 0$, since $h'_0 = h'_d = h_0^- = h_d^- = h_{d-1}^+ = h_{d-1}^-$.

This definition is designed to match the following statement.

**Lemma 3.19.** Let $F_1, \ldots, F_K$ be a shelling from $B^+$ to $B^-$ of a prismatoid $C$ and assume $B^+$ and $B^-$ are shellable. Then, the entry $h_j$ of the $h$-vector of $C$ from $B^+$ to $B^-$ equals the number of facets $F_i$ that share exactly $j$ ridges with $F_1, \ldots, F_{i-1}$.
Proof. Let $C''$ be the $(d - 1)$-sphere obtained from $C$ by gluing to $B^+$ and $B^-$ respective cones $B^+ * v^+$ and $B^- * v^-$. Lemma 3.15 implies that the shelling of $C$ can be extended to a shelling of $C''$ that has as initial segment a shelling of $B^+ * v^+$ and as final segment a shelling of $B^- * v^-$. Since shelling a complex and a cone over it is the same thing, the result follows from the usual relation of the $h$-vector of a shellable sphere to a shelling of it.

4. Metaheuristics and implementation

In this section, we show our approach to find non-$d$-step topological prismatoids with few vertices.

The general idea is to start with a 28 vertices prismatoid as defined in [21], and perform bistellar flips on it attempting to remove vertices while preserving its width. A general, well known framework to do this is simulated annealing.

Simulated annealing is a very common metaheuristic algorithm for optimization problems, used when we have a search space and an “adjacency relation” between pairs of feasible solutions. The idea is to perform a random walk through the state graph of a problem, but favoring moves that improve the desired objective function over moves that do not. It has been used successfully in combinatorial topology to simplify simplicial complexes while preserving a condition (typically their homeomorphism type) [5] or, in conjunction with other strategies, to tackle the problem of sphere recognition [14, 13].

There is a variable, the temperature, regulating the probability assigned to each possible step as a function of how much it improves or worsens the objective. At higher temperatures the choice is more random; when the system cools down it converges to accepting only improving moves. Formally, the probability of accepting a step that increases cost by $\Delta c$ at temperature $t$ is:

\[
\begin{cases}
1 & \text{if } \Delta c < 0 \\
\exp(-\Delta c/t) & \text{if } \Delta c \geq 0
\end{cases}
\]

Note that it is very important to choose the potential step with uniform probability, among all the neighbor of the current state. This “a priori” probability distribution, together with the cooling schedule, produces an “a posteriori” probability distribution of performing the step. This gives higher probability to improving steps, but also gives chance to worsening steps at high temperatures.

That is, areas of the graph with smaller values of the objective function are more likely to be explored. As the temperature cools down, the random walk will focus on these areas and make optimizations with more detail. Loosely speaking, the first few iterations of the algorithm are more exploration-focused, and the last iterations are exploitation-focused.

Formally, simulated annealing requires the following aspects to be decided:

- A state graph representing the feasible states of the problem and an adjacency relation, plus an initial state.
- A cooling schedule, that defines temperature as a function of time, thus modulating the probability of acceptance of a cost-increasing step.
- An appropriate objective function that we aim to minimize.

In our particular problem, our state graph consists of all non-$d$-step topological 4-prismatoids, with an edge between a pair of prismatoids if they differ by a flip. This graph is undirected, since every flip is reversible by another flip.
There is a lot of research on cooling schedules for different problems. It is known that SA converges to the global optimum for a certain cooling schedule \[10\], but it is too slow for any practical application. Since the best schedule depends on the problem, several adaptive schedules have been proposed too \[12\]. However, the most common approach is to define a geometric cooling schedule, of the form \(T_t = t_0 \cdot e^{st}\), where the parameters \(t_0\) (initial temperature), \(s\) (cooling speed) and the number of iterations are adjusted manually. Since the flipping operation is very fast, we have chosen a slow schedule with a high number of iterations.

The particular parameters have been obtained by trial and error. We used cooling schedule \(T(k) = 1000 \cdot 0.99997^k\) and 500000 iterations for each run.

4.1. The objective function. The objective function guides our algorithm towards non-d-step topological prismatoids with few vertices. This is, prismatoids with many vertices should have higher cost.

A naive approach would be to just take the number of vertices as an objective function. But this objective function has large plateaus, connected subgraphs of the state graph with constant number of vertices, and it does not push our state towards less vertices. So we have to find a way to break ties between prismatoids with the same number of vertices by giving less cost to prismatoids from which we expect it to be easier to remove vertices. That is, the objective function we want has the number of vertices as a main component, plus a smaller (heuristic) tie-breaker that pushes the algorithm in the right direction.

A flip that removes a vertex must be of type \((1, d)\); therefore, in order to perform it we must have a vertex with exactly \(d + 1\) neighbors, which is the smallest possible size for the neighborhood of a vertex in a prismatoid. So, a good approach is to try to reduce the size of the neighborhood of a vertex until it is precisely \(d + 1\).

However, just taking the size of the smallest neighborhood as a tie-breaker is not sensitive enough because the algorithm will then not try to reduce the neighborhoods of other vertices. For this reason, we use as a tie-breaker a generalized mean of the sizes of neighborhoods of vertices. More precisely, the objective function that achieved the best performance in practice among the ones we tried is

\[
\text{cost}(C) = |V(C)| + \varepsilon \left( \frac{\sum_{v \in V(C)} |\text{neigh}(v)|^{-3}}{|V(C)|} \right)^{-1/3}.
\]

4.2. Data structures. A proper “topological prismatoid” data structure for our problem needs to allow for the following operations:

- Construction from the list of facets.
- Check if a set of vertices is a face.
- Iterate through the maximal subfaces of a face.
- Iterate through the minimal superfaces of a face.
- Perform a flip.
- Compute the width of the prismatoid.
- Get a valid random flip (with uniform probability).

We implement a prismatoid \(C\) as a map of pairs (face, neighborhood), indexed by the faces in \(C\). Faces and neighborhoods themselves are of type “set of integers”. We also store the bases of \(C\) in the same manner.

Observe this implicitly gives us the Hasse diagram (maximal subfaces and minimal superfaces): Each face \(F\) is directly above those of the form \(F \setminus \{v\}\) for \(v \in F\),
and directly below those of the form \( F \cup \{w\} \) for \( w \in \text{neigh}(F) \). There is no need to store the adjacency graph, because it is implicit in the Hasse diagram. The facets adjacent to a facet \( F \) are computed from the neighborhood of the non-boundary ridges in \( F \). Boundary and interior ridges are distinguished by their neighborhoods having \( d + 1 \) and \( d \) elements, respectively.

To compute and update the width we store, for each facet, the distance to the first base (chosen arbitrarily but once and for all) and the number of paths achieving that distance. In this way we do not need to explore again all facets to compute the new width after performing a flip, we just need to update the values that change. For this, when we perform the flip the new facets are inserted into a queue, and the distances are updated by cascading through the prismatoid.

A flip is implemented simply by removing the old faces and inserting the new inserted faces. What is not so straightforward, and needs to be addressed, is how to implement an unbiased generation of random flips among all the possible ones.

For this, we imitate to some extent the technique used in polymake \[9\]. In polymake, there is a set of pairs \((f, l)\), called “options” satisfying some conditions for flipability, in particular conditions (1) and (2) of Section 3.1. The flips are categorized by dimension of \( f \). But the list of candidate pairs \((f, l)\) is very hard to update after a flip is performed. Among other things, some potential flips may change their \( f \) and \( l \) while preserving their support \( f \cup l \).

Since the support of every flip is the neighborhood of a ridge, one could simplify this by using the list of ridges instead of the pairs \((f, l)\) as input to generate a random flip. But choosing randomly from the list of ridges creates bias: some flips are more likely than others since several ridges (actually \(|l|\) of them) correspond to the same flip.

To avoid these drawbacks we store and update the list of ridge-neighborhoods. That is, for each ridge \( F \) we store the facet \( F_1 \) containing \( F \) or the union \( F_1 \cup F_2 \) of the two facets containing \( F \) depending on whether \( F_1 \) is in the boundary or the interior. This is very easy to update, and it also makes it very easy to spot vertex-adding flips (which correspond to boundary ridge-neighborhoods and are characterized by having \( d \) instead of \( d + 1 \) elements). Since there is a bijection between flip-defining ridge-neighborhoods and flips, via the \((f, l, v)\) formalism introduced in Section 3.1 it is easy to choose flips uniformly at random: choose a random ridge-neighborhood and discard non-valid ones.

We find this approach more stable and requiring less changes to the data structure than the ones based on \((f, l)\) pairs or in ridges alone.

5. Results. Small non-\(d\)-step prismatoids and spheres

As said in the previous section, we ran our algorithm with cooling schedule \( T(k) = 1000 \cdot 0.99997^k \) and 500000 iterations for each run. We let it run for three days on an openSuse 42.3 Linux machine with 16 GB of RAM and an AMD Phenom X6 1090T processor, after which we had concluded 4093 runs. We thus obtained 4093 non-\(d\)-step topological 4-prismatoids, with number of vertices ranging between 14 and 28. Figure 3 shows the distribution we obtained for the number of vertices alone (top) and for number of vertices versus number of facets (bottom).

For the rest of this section we focus on the 4 smallest examples, with 14 vertices. We call them #1039, #1963, #2669 and #3513 since these are their indices among the 4093 experiments that we did. They are listed in Tables 2–5. Vertices from one
base are labelled 0 to 6 and vertices from the other are a to g. In the tables, the facets of each prismatoid are grouped by layers, where a layer consists of all facets sharing the number of vertices they have from each base. It is remarkable that the four examples obtained have a lot of similarities:

- They have combinatorially isomorphic bases. Indeed, in all cases the list of facets of the bases are as follows. (To relate this to the tables, observe that the bases correspond to the first and last layer in the prismatoid, removing from each facet the unique vertex from the other base).

| Facets         | 0123          | a06d  |
|----------------|---------------|-------|
| 0134 0234 1234 | a0de          |
| 0145 0245 1245 | acde          |
| 0156 0256 1256 | bcde          |
| 0126           | abcd          |

Observe that both are the face complex of the stacked 4-polytope with seven vertices. That is, they are the boundaries of the stacked 4-balls \{01234, 01245, 01256\} and \{abcd, abce, abcfg\}, respectively.

- They have the same $f$-vector $(14, 85, 220, 241, 92)$. Together with the combinatorics of their bases this fixes their prismatoid $h$-vector (as defined in Section 3.4) to be $(0, 7, 39, 39, 7, 0)$.

- They are all shellable, with a shelling that is monotone on layers. That is, no facet of one layer is used until finishing the previous layer. In the tables, facets within each layer are given in a shelling order.

- The vector of number of facets in different layers is the same $(11, 35, 35, 11)$, and symmetric, in three of them. In #2669 we get the slightly asymmetric vector $(11, 34, 36, 11)$. 

![Figure 3. Top: distribution of the 4093 prismatoids by number of vertices. Bottom: distribution by number of vertices and facets. The initial prismatoid has 28 vertices and 272 facets.](image)
Table 2. Prismatoid #1039

| Prismatoid #1039 | 0126d | 014ae | 023bd | 126cg | 014be | 24ac | 24cd | 25bd | 5abe | 6abe | 5abe | 6abe | 5abe | 6abe |
|-----------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0123d           | 024ae | 123bd | 125cg | 24abe | 12ac | 25bd | 26bc | 3abe | 3abe | 6abe | 6abe | 6abe | 6abe | 6abe |
| 0134e           | 013ae | 023be | 125cf | 24abe | 25ac | 26bc | 13acg | 5acde | 3abe | 3abe | 6abe | 6abe | 6abe | 6abe |
| 0234e           | 014ae | 026be | 015cf | 25abe | 25ac | 26bc | 13acg | 5acde | 3abe | 3abe | 6abe | 6abe | 6abe | 6abe |
| 1234e           | 024af | 123be | 013cf | 25abe | 25ac | 26bc | 13acg | 5acde | 3abe | 3abe | 6abe | 6abe | 6abe | 6abe |
| 0145f           | 025af | 124be | 015cg | 25abe | 25ac | 13acg | 15abe | 3abfg | 5abe | 6abe | 6abe | 6abe | 6abe | 6abe |
| 0245f           | 025ae | 124bd | 025cg | 03abe | 06ace | 06bce | 5abe | 6abe | 6abe | 6abe | 6abe | 6abe | 6abe | 6abe |
| 1245f           | 014bf | 016bg | 026ce | 03baf | 13ace | 15abe | 6abe | 6abe | 6abe | 6abe | 6abe | 6abe | 6abe | 6abe |
| 0156g           | 013af | 026bg | 023cd | 014ae | 15abe | 13ace | 26bce | 4abe | 4abe | 4abe | 4abe | 4abe | 4abe | 4abe |
| 0256g           | 013ad | 015cg | 026af | 23abg | 26bce | 04abf | 5abde | 6abef | 5abde | 6abef | 5abde | 6abef | 5abde | 6abef |

Table 3. Prismatoid #1963

| Prismatoid #1963 | 0156g | 023ad | 126cg | 024af | 24abg | 06bce | 04abf | 5abde | 6abef | 5abde | 6abef | 5abde | 6abef |
|-----------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0256g           | 013ad | 015cg | 026af | 23abg | 26bce | 04abf | 5abde | 6abef | 5abde | 6abef | 5abde | 6abef | 5abde | 6abef |
| 1256g           | 026ad | 016cd | 026bf | 13acg | 23bce | 06abe | 5abe | 6abe | 5abe | 6abe | 5abe | 6abe | 5abe | 6abe |
| 0123d           | 013ae | 015cd | 025bf | 14ace | 05bce | 14abe | 4abe | 4abe | 4abe | 4abe | 4abe | 4abe | 4abe | 4abe |
| 0126d           | 034ae | 123cf | 125bf | 13acg | 15bce | 14abe | 4abe | 4abe | 4abe | 4abe | 4abe | 4abe | 4abe | 4abe |
| 0134e           | 234ae | 123cg | 124bf | 23acg | 15abe | 13bfg | 5abe | 6abe | 5abe | 6abe | 5abe | 6abe | 5abe | 6abe |
| 1234e           | 123ae | 123aq | 015bf | 26acg | 15abe | 13bfg | 5abe | 6abe | 5abe | 6abe | 5abe | 6abe | 5abe | 6abe |
| 0234a           | 124ae | 124ag | 015be | 06ace | 15bce | 06abf | 3acfg | 6acef | 3acfg | 6acef | 3acfg | 6acef | 3acfg | 6acef |
| 0145f           | 026af | 124bg | 014bf | 14ace | 05bce | 23ace | 6abe | 6abe | 6abe | 6abe | 6abe | 6abe | 6abe | 6abe |
| 0245f           | 025bg | 015cd | 014be | 05ace | 06bce | 26ace | 3bce | 3bce | 3bce | 3bce | 3bce | 3bce | 3bce | 3bce |
| 1245f           | 125bg | 015ad | 05ace | 23abf | 23bce | 6acef | 6acef | 6acef | 6acef | 6acef | 6acef | 6acef | 6acef |
| 016cg           | 015ae | 06ace | 23abf | 26bce | 3bce | 3bce | 3bce | 3bce | 3bce | 3bce | 3bce | 3bce | 3bce | 3bce |

Table 4. Prismatoid #2669

| Prismatoid #2669 | 0156g | 015ag | 014bg | 014ce | 04abg | 14bce | 04bce | 5abde | 6abef | 5abde | 6abef | 5abde | 6abef | 5abde |
|-----------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0256g           | 015ag | 014bg | 013ce | 05abg | 14acg | 16abe | 3abfg | 6acef | 3abfg | 6acef | 3abfg | 6acef | 3abfg | 6acef |
| 1256g           | 015ag | 014bg | 013ce | 15bce | 14ace | 26abe | 6abe | 6abe | 6abe | 6abe | 6abe | 6abe | 6abe | 6abe |
| 0134e           | 015ag | 014bg | 013ce | 05abg | 04ace | 25bce | 4abeg | 4abeg |
| 0234e           | 1256g | 015ag | 014bg | 013ce | 15bce | 14ace | 26abe | 6abe | 6abe | 6abe | 6abe | 6abe | 6abe | 6abe |
| 0123d           | 016bg | 0123c | 013cd | 05abg | 04ace | 25bce | 4abeg | 4abeg |
| 0126d           | 026bd | 0123c | 013cd | 25abg | 13ace | 26ace | 5abde | 5abde |
| 0134e           | 015ag | 014bg | 013ce | 13abf | 04ace | 13bce | 6acef | 6acef |
| 0145f           | 014ag | 014be | 013cd | 13abg | 01bcd | 6acef | 6acef |

Table 5. Prismatoid #3513
Their reduced incidence patterns, shown in Figure 4, are very similar. In #1963 and #3513 the reduced incidence patterns coincide with the one in the right part of Figure 2, minimal by Lemma 3.12. In #1039 and #2669 they are almost the same, except each of them has a single “outlier” facet incident to a base (obde in #1039 and 0234a in #2669) that introduces a new vertex in the pattern and creates directed cycles of length two. In particular, in these two Proposition 3.11 is not enough to prove the non-d-step property.

Note that the starting prismatoid of the algorithm had the other reduced incidence pattern of minimal size, the one in the left in Figure 2. We do not know whether any of the four is polytopal.

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