CARATHÉODORY COMPLETENESS ON THE PLANE

by Armen Edigarian

Dedicated to the memory of Professor Józef Siciak

Abstract. M. A. Selby [6] and, independently, N. Sibony [11] proved that on the complex plane \(c\)-completeness is equivalent to \(c\)-finitely compactness. Their proofs are quite similar and are based on [4]. We give more refined equivalent conditions and, along the way, simplify the proofs.

1. Introduction. Let \(D \subset \mathbb{C}^n\) be a domain and let \(\zeta \in \partial D\) be its boundary point. We denote by \(A(D \cup \{\zeta\})\) the set of all bounded holomorphic functions on \(D\) which extend continuously to \(D \cup \{\zeta\}\). Following [7], we say that \(\zeta\) is a weak peak point for \(D\) if there exists a function \(f \in A(D \cup \{\zeta\})\) such that \(|f| < 1\) on \(D\) and \(f(\zeta) = 1\).

Theorem 1. Let \(D \subset \mathbb{C}\) be a domain and let \(\zeta \in \partial D\) be its boundary point. Then the following conditions are equivalent:

1. \(\zeta\) is a weak peak point for \(D\);
2. there exists no finite Borel measure \(\mu\) on \(D\) such that
   \[|f(\zeta)| \leq \int |f|d\mu\quad \text{for any } f \in A(D \cup \{\zeta\});\]
3. we have
   \[
   \sum_{n=1}^{\infty} 2^n \gamma\left(A_n(\zeta, a) \setminus D\right) = +\infty,
   \]

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where $A_n(\zeta) = \{ z \in \mathbb{C} : \frac{1}{2^n+1} \leq |z - \zeta| \leq \frac{1}{2^n} \}$ and $\gamma$ is the analytic capacity (see the definition below).

The equivalency of (1) and (3) in Theorem 1 was proved by M. A. Selby (see [9]). Note that the implication (1) $\implies$ (2) is straightforward (also in a higher dimension). The implication (2) $\implies$ (1) in any dimension is claimed in [2]. However, the proof is based on a false version of Hahn–Banach theorem, claimed in [5]. So, we give a new proof on the complex plane. In a higher dimension, it is still an open problem whether (2) $\implies$ (1).

Let $D(\zeta, r) = \{ z \in \mathbb{C} : |z - \zeta| < r \}$ denote the disk on the complex plane and let $D = D(0, 1)$ denote the unit disk. We define the Poincaré function $p$ on $D$ as

$$p(\lambda_1, \lambda_2) = \frac{1}{2} \log \frac{1 + m(\lambda_1, \lambda_2)}{1 - m(\lambda_1, \lambda_2)}, \quad \lambda_1, \lambda_2 \in D,$$

where $m(\lambda_1, \lambda_2) = \frac{|\lambda_1 - \lambda_2|}{1 - \lambda_1 \overline{\lambda_2}}$ is the Möbius function.

Let $D \subset \mathbb{C}^n, n \geq 1$, be a domain. For $z_1, z_2 \in D$ put

$$\begin{align*}
(1) & \quad c_D(z_1, z_2) = \sup \{ p(f(z_1), f(z_2)) : f \in \mathcal{O}(D; \mathbb{D}) \}, \\
(2) & \quad c_D^*(z_1, z_2) = \sup \{ m(f(z_1), f(z_2)) : f \in \mathcal{O}(D; \mathbb{D}) \},
\end{align*}$$

where $\mathcal{O}(D; \mathbb{D})$ denotes the set of all holomorphic mappings $D \to \mathbb{D}$. $c_D$ is called the Carathéodory pseudodistance for $D$ (see e.g. [6]). In case when $c_D$ is indeed a distance we say that $D$ is $c$-hyperbolic. A $c$-hyperbolic domain $D$ is called $c$-complete if any $c_D$-Cauchy sequence $\{z_\nu\}_{\nu \geq 1} \subset D$ converges to a point $z_0 \in D$ (w.r.t. Euclidean topology).

The aim of this paper is to study more carefully the completeness on the complex plane. Along the way we simplify the proofs by M. A. Selby [8–10] and by N. Sibony [11].

We say that a measurable set $F \subset \mathbb{C}$ is of positive density at a point $\zeta \in \mathbb{C}$ if

$$\limsup_{r \to 0^+} \frac{\mathcal{L}(D(\zeta; r) \cap F)}{\pi r^2} > 0.$$ 

First we show the following result.

**Theorem 2.** Let $D \subset \mathbb{C}$ be a domain and let $\zeta \in \partial D$ be its boundary point. If $\zeta$ is not a weak peak point for $D$ then

$$\lim_{r \to 0^+} \frac{\mathcal{L}(D(\zeta; r) \cap D)}{\pi r^2} = 1.$$ 

We have the following inverse of Theorem 2.
Theorem 3. Let $D \subset \mathbb{C}$ be a domain and let $\zeta \in \partial D$ be its boundary point. Assume that
\[
\lim_{r \to 0^+} \frac{L(D; \zeta; r) \cap D}{\pi r^2} = 1.
\]
Then the following conditions are equivalent to conditions (1), (2), (3) in Theorem 1:

(4) there exists a set $A \subset D$ of positive density at $\zeta$ such that for any sequence $\{z_\nu\}_{\nu \geq 1} \subset A$ with $z_\nu \to \zeta$ we have $c_D(z_0, z_\nu) \to \infty$;

(5) there exists a set $A \subset D$ of positive density at $\zeta$ such that for any sequence $\{z_\nu\}_{\nu \geq 1} \subset A$ such that $z_\nu \to \zeta$ there follows that $\{z_\nu\}$ is not a $c_D$-Cauchy sequence.

Note that the implications (1) $\implies$ (4) $\implies$ (5) are straightforward. Essentially, the main result of the paper is showing that (5) $\implies$ (2). In case $A = \Omega$ in Theorem 3, the result is proved in [8] and [11].

2. Proof of Theorem 1. Recall the definition of the analytic capacity (see e.g. Chapter VIII in [3]). Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denote the Riemann sphere. The analytic capacity of a compact set $K$ is defined by
\[
\gamma(K) = \sup \{|f'(\infty)| : f \in \mathcal{O}(\Omega), \|f\| \leq 1, f(\infty) = 0\},
\]
where $\Omega$ is the unbounded component of $\hat{\mathbb{C}} \setminus K$ and
\[
f'(\infty) = \lim_{z \to \infty} z(f(z) - f(\infty)).
\]
For any set $F \subset \mathbb{C}$ we put
\[
\gamma(F) = \sup \{\gamma(K) : K \subset F \text{ compact}\}.
\]
Recall also the following characterization (see Theorem VIII.4.5 in [3]).

Theorem 4 (Melnikov’s criterion). Let $K \subset \mathbb{C}$ be a compact set and let $\zeta \in K$. Then $\zeta$ is a peak point for $R(K)$ if and only if
\[
\sum_{n=1}^{\infty} 2^n \gamma(A_n(\zeta) \setminus K) = +\infty.
\]

Note that the implication (1) $\implies$ (2) in Theorem 1 is immediate. Let us show the implication (2) $\implies$ (3).

Proof of (2) $\implies$ (3) in Theorem 1. Assume that
\[
\sum_{n=1}^{\infty} 2^n \gamma(A_n(\zeta) \setminus D) < +\infty.
\]
Using the continuity property of the analytic capacity (see the proof of Theorem 3.1 in [4]) one can show that there exists a compact set $K \subset D \cup \{\zeta\}$ such that

$$\sum_{n=1}^{\infty} 2^n \gamma(A_n(\zeta) \setminus K) < +\infty.$$  

By Melnikov’s criterion $\zeta$ is not a peak point for $R(K)$. Hence, by Bishop’s characterization of peak points (see e.g. [3]) there exists a Borel probability measure $\mu$ on $K$ such that $\mu(\{\zeta\}) = 0$ and

$$f(\zeta) = \int f d\mu \text{ for any } f \in R(K).$$

Note that $A(D \cup \{\zeta\}) \subset R(K)$ (see Corollary 8 below). Hence,

$$f(\zeta) = \int f d\mu \text{ for any } f \in A(D \cup \{\zeta\}).$$

A contradiction. \hfill \Box

3. Proof of Theorem 2. Let $\mathcal{L}$ denote the Lebesgue measure in $\mathbb{C}$. Recall the following well-known result (see e.g. [1], Lemma 1.5).

**Proposition 5.** Let $K \subset \mathbb{C}$ be a compact set. Then the function

$$f(z) = \int_K \frac{d\mathcal{L}(\eta)}{z-\eta}$$

is holomorphic on $\hat{\mathbb{C}} \setminus K$, continuous on $\hat{\mathbb{C}}$ and $f(\infty) = 0$. Moreover,

$$|f(z)| \leq \int_K \frac{1}{|z-\eta|} d\mathcal{L}(\eta) \leq 2\sqrt{\pi \mathcal{L}(K)}.$$  

As a corollary of Proposition 5 we get Theorem 2 (cf. Corollary VIII.4.2 in [3]).

**Proof of Theorem 2.** Assume that

$$\limsup_{r \to 0+} \frac{\mathcal{L}(\overline{B}\{\zeta; r\} \setminus D)}{r^2} > 0.$$  

Choose $r_n \to 0+$ and $b > 0$ such that $\mathcal{L}(K_n) > br_n^2$, where $K_n = \overline{B}\{\zeta; r_n\} \setminus D$. Put

$$g_n(z) = \frac{1}{\mathcal{L}(K_n)} \cdot (z-\zeta) \int_{K_n} \frac{d\mathcal{L}(\eta)}{z-\eta}.$$  

From Proposition 3 there follows that $g_n$ is a continuous function on $\hat{\mathbb{C}}$, holomorphic on $\hat{\mathbb{C}} \setminus K_n$, $g_n(\infty) = 1$. 

Note that for any $z \in \mathbb{C}$ such that $|z - \zeta| \leq r_n$ we have

$$|g_n(z)| \leq \frac{2r_n\sqrt{\pi\mathcal{L}(K_n)}}{\mathcal{L}(K_n)} \leq 2\sqrt{\frac{\pi}{5}}.$$

From the maximum principle we see that the above inequality holds on the whole $\mathcal{C}$. Now we proceed as in the proof of Theorem VIII.4.1 in [3] and get a weak peak function for $D$.

**4. Proof of Theorem 3.** We denote by $\mathcal{M}$ the set of all positive finite Borel measures in $\mathbb{C}$. For $\mu \in \mathcal{M}$ we define its Newton potential as

$$M(z) = M_\mu(z) = \int \frac{1}{|z - \eta|} d\mu(\eta).$$

From the inequality (3) we have

$$\frac{1}{\pi r^2} \int_{D(\eta, r)} |z - \eta| \cdot M(z) d\mathcal{L}(z) \leq 2\mu(\mathcal{C}),$$

and, therefore, $M < \infty$ a.e. on $\mathcal{C}$. The following result, which essentially is a corollary of Fubini’s theorem, shows the behaviour of the left side of (4) when $r \to 0$ (see e.g. [12], Lemma 26.16).

**Proposition 6.** Let $\mu \in \mathcal{M}$. For any $\eta \in \mathbb{C}$ we have

$$\lim_{r \to 0} \frac{1}{\pi r^2} \int_{D(\eta, r)} |z - \eta| \cdot M(z) d\mathcal{L}(z) = \mu(\{\eta\}).$$

In particular, if $\mu(\{\eta\}) = 0$, then for any $\epsilon > 0$ the set

$$\Pi(\epsilon) = \{z \in \mathbb{C} : |z - \eta| \cdot M(z) > \epsilon\}$$

is of zero density at $\eta$, i.e.,

$$\lim_{r \to 0} \frac{\mathcal{L}(\Pi(\epsilon) \cap D(\eta, r))}{r^2} = 0.$$

Recall the following approximation result (see e.g., Theorem 10.8 in Chapter VIII in [3]).

**Theorem 7.** Let $D \subset \mathbb{C}$ be a domain and let $\zeta \in \partial D$ be its boundary point. For any $f \in H^\infty(D)$ there exists a sequence $\{f_n\}_{n \geq 1} \subset H^\infty(D)$ with $\|f_n\|_D \leq 17\|f\|_D$ such that $f_n \to f$ locally uniformly on $D$ and each $f_n$ extends holomorphically to a neighborhood of $\zeta$. Moreover, if $f$ extends continuously to $\zeta$, then $f_n$ tends to $f$ uniformly on $D$.

From this we get.

**Corollary 8.** Let $D \subset \mathbb{C}$ be a domain and let $\zeta \in \partial D$ be its boundary point. Then for any compact set $K \subset D \cup \{\zeta\}$ we have $A(D \cup \{\zeta\}) \subset R(K)$. 
The following simple observation holds true.

**Proposition 9.** Let $D \subset \mathbb{C}$ be a domain and let $\zeta \in \partial D$. Assume that $\mu$ is a finite Borel measure in $D$ such that

$$|f(\zeta)| \leq \int |f| \, d\mu$$

for any $f \in A(D \cup \{\zeta\})$. Then for any $\eta \in D$ we have

$$|f(\eta) - f(\zeta)| \leq 2\|f\|_{\infty} M(\eta)|\eta - \zeta|.$$

In particular, for any $\eta_1, \eta_2 \in D$ we have

$$c_D^*(\eta_1, \eta_2) \leq 34 \left( |\zeta - \eta_1| M(\eta_1) + |\zeta - \eta_2| M(\eta_2) \right).$$

**Proof.** Fix $\eta \in D$. Then for any $f \in A(D \cup \{\zeta\})$ we have $\tilde{f}(z) = \frac{f(z) - f(\eta)}{z - \eta} \in A(D \cup \{\zeta\})$. Then

$$|\tilde{f}(\zeta)| \leq \int |\tilde{f}| \, d\mu.$$

Hence,

$$|f(\zeta) - f(\eta)| \leq |\zeta - \eta| \int \left| \frac{f(z) - f(\eta)}{z - \eta} \right| \, d\mu(z) \leq 2\|f\|_{\infty} M(\eta)|\eta - \zeta|.$$

Inequality (5) follows from Theorem 7. \[\square\]

We have the following corollary, which proofs the implication (5) $\Rightarrow$ (2).

**Corollary 10.** Let $D \subset \mathbb{C}$ be a domain and let $\zeta \in \partial D$. Assume that $\mu$ is a finite Borel measure in $D$ such that

$$|f(\zeta)| \leq \int |f| \, d\mu$$

for any $f \in A(D \cup \{\zeta\})$. Then for any measurable set $A \subset D$ of positive density at $\zeta$ there exists a $c$-Cauchy sequence $\{\eta_n\}_{n \geq 1} \subset A$ such that $\eta_n \to \zeta$.

**Proof.** If

$$\liminf_{r \to 0} \frac{\mathcal{L}(\mathbb{D}(\zeta; r) \cap D)}{\pi r^2} < 1$$

then by Theorem 2 $\zeta$ is a weak peak point, which contradicts the existence of the measure $\mu$. So,

$$\lim_{r \to 0} \frac{\mathcal{L}(\mathbb{D}(\zeta; r) \cap D)}{\pi r^2} = 1.$$

Hence,

$$\limsup_{r \to 0} \frac{\mathcal{L}(\mathbb{D}(\zeta; r) \cap A)}{\pi^2} > 0.$$
Then by Proposition 6 there exists a sequence \( \{\eta_n\}_{n \geq 1} \subset D \) with \( \eta_n \to \zeta \) such that \( |\zeta - \eta_n| M(\eta_n) \leq \frac{1}{2^n} \). From Theorem 9 we get the result.

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