HILBERT–SAMUEL MULTIPlicITIES OF CERTAIN DEFORMATION RINGS

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Abstract. We compute presentations of crystalline framed deformation rings of a two-dimensional representation $\overline{\rho}$ of the absolute Galois group of $\mathbb{Q}_p$, when $\overline{\rho}$ has scalar semi-simplification, the Hodge–Tate weights are small and $p > 2$. In the non-trivial cases, we show that the special fibre is geometrically irreducible, generically reduced and the Hilbert–Samuel multiplicity is either 1, 2 or 4 depending on $\overline{\rho}$. We show that in the last two cases the deformation ring is not Cohen–Macaulay.

1. Introduction

Let $p > 2$ be a prime. Let $k$ be a finite field of characteristic $p$, $E$ be a finite totally ramified extension of $W(k)[\frac{1}{p}]$ with ring of integers $\mathcal{O}$ and uniformizer $\pi$. For a given continuous representation $\overline{\rho}: G_{\mathbb{Q}_p} \to \text{GL}_2(k)$ we consider the universal framed deformation ring $R_{\overline{\rho}}$ and the universal framed deformation $\rho^{\text{univ}}: G_{\mathbb{Q}_p} \to \text{GL}_2(R_{\overline{\rho}})$. For all $p \in \text{m-Spec}(R_{\overline{\rho}}[\frac{1}{p}])$, the set of maximal ideals of $R_{\overline{\rho}}[\frac{1}{p}]$, we can specialize the universal representation at $p$ to obtain the representation $\rho_p: G_{\mathbb{Q}_p} \to \text{GL}_2(R_{\overline{\rho}}[\frac{1}{p}]/\mathfrak{p})$, where $R_{\overline{\rho}}[\frac{1}{p}]/\mathfrak{p}$ is a finite extension of $\mathbb{Q}_p$.

Let $\tau: I_{G_{\mathbb{Q}_p}} \to \text{GL}_2(E)$ be a representation with an open kernel, where $I_{G_{\mathbb{Q}_p}}$ is the inertia subgroup of $G_{\mathbb{Q}_p}$. We also fix integers $a, b$ with $b \geq 0$ and a continuous character $\psi: G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ such that $\psi \epsilon = \det(\overline{\rho})$, where $\epsilon$ is the cyclotomic character. Kisin showed in [10] that there exist unique reduced $\mathcal{O}$-torsion free quotients $R_{\overline{\rho}, \psi}^{\text{cris}}(a, b, \tau)$ and $R_{\overline{\rho}, \psi}^{\text{cris}}(a, b, \tau)$ of $R_{\overline{\rho}}$ with the property that $\rho_p$ factors through $R_{\overline{\rho}, \psi}^{\text{cris}}(a, b, \tau)$ if and only if $\rho_p$ is potentially semi-stable resp. potentially crystalline with Hodge–Tate weights $(a, a+b+1)$ and has determinant $\psi \epsilon$ and inertial type $\tau$. If $\tau$ is trivial then $R_{\overline{\rho}, \psi}^{\text{cris}}(a, b) := R_{\overline{\rho}, \psi}^{\text{cris}}(a, b, 1 \oplus 1)$ parametrizes all the crystalline lifts of $\overline{\rho}$ with Hodge–Tate weights $(a, a+b+1)$ and determinant $\psi \epsilon$. The Breuil–Mézard conjecture, proved by Kisin for almost all $\overline{\rho}$, see also [2,3,7,8,14], says that the Hilbert–Samuel multiplicity of the ring $R_{\overline{\rho}, \psi}^{\text{cris}}(a, b, \tau)/\pi$ can be determined by computing certain automorphic multiplicities, which do not depend on $\overline{\rho}$, and the Hilbert–Samuel multiplicities of $R_{\overline{\rho}, \psi}^{\text{cris}}(a, b)$ in low weights for $0 \leq a \leq p-2$, $0 \leq b \leq p-1$. For most $\overline{\rho}$, the Hilbert–Samuel multiplicities of $R_{\overline{\rho}, \psi}^{\text{cris}}(a, b)$ have already been determined. Our goal in this paper is to compute the...
Hilbert–Samuel multiplicity of the ring $R_{\bar{\rho},\text{cris}}(a,b)$ with $0 \leq a \leq p-2$, $0 \leq b \leq p-1$ when

$$\bar{\rho}: G_{\mathbb{Q}_p} \to \text{GL}_2(k), \quad g \mapsto \begin{pmatrix} \chi(g) & \phi(g) \\ 0 & \chi(g) \end{pmatrix}.$$

One may show that $R_{\bar{\rho},\text{cris}}(a,b)$ is zero if either $b \neq p-2$ or the restriction of $\chi$ to $I_{\mathbb{Q}_p}$ is not equal to $\epsilon^a$ modulo $\pi$.

**Theorem 1.** Let $a$ be an integer with $0 \leq a \leq p-2$ such that $\chi|_{I_{\mathbb{Q}_p}} \equiv \epsilon^a \pmod{\pi}$. Then $R_{\bar{\rho},\text{cris}}(a,p-2)/\pi$ is geometrically irreducible, generically reduced and

$$e(R_{\bar{\rho},\text{cris}}(a,p-2)/\pi) = \begin{cases} 1 & \text{if } \bar{\rho} \otimes \chi^{-1} \text{ is ramified}, \\
2 & \text{if } \bar{\rho} \otimes \chi^{-1} \text{ is unramified, indecomposable}, \\
4 & \text{if } \bar{\rho} \otimes \chi^{-1} \text{ is split.} \end{cases}$$

In the last two cases, $R_{\bar{\rho},\text{cris}}(a,p-2)$ is not Cohen–Macaulay.

The multiplicity 4 does not seem to have been anticipated in the literature, see for example [11, 1.1.6]. Our method is elementary in the sense that we do not use any integral $p$-adic Hodge theory. The only $p$-adic Hodge theoretic input is that if $\rho$ is a crystalline lift of $\bar{\rho}$ with Hodge–Tate weights $(0,p-1)$, then we have an exact sequence

$$0 \to \epsilon^{p-1} \chi_1 \to \rho \to \chi_2 \to 0,$$

where $\chi_1, \chi_2: G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ are unramified characters. This allows us to convert the problem into a linear algebra problem, which we solve in Lemma 2. This gives us an explicit presentation of the ring $R_{\bar{\rho},\text{cris}}(a,p-2)$, using which we compute the multiplicities in Section 4. Our argument gives a proof of the existence of $R_{\bar{\rho},\text{cris}}(a,p-2)$ independent of [10]. After writing this note we discovered that the idea to convert the problem into linear algebra already appears in [15].

**2. The universal ring**

After twisting we may assume that $\chi = 1$ and $a = 0$ so that

$$\bar{\rho}(g) = \begin{pmatrix} 1 & \phi(g) \\ 0 & 1 \end{pmatrix}.$$

Since the image of $\bar{\rho}$ in $\text{GL}_2(k)$ is a $p$-group, the universal representation factors through the maximal pro-$p$ quotient of $G_{\mathbb{Q}_p}$, which we denote by $G$. We have the following commuting diagram:

$$\begin{array}{cccc}
G_{\mathbb{Q}_p} & \longrightarrow & G \\
\downarrow & & \downarrow \\
G_{\mathbb{Q}_p}^{ab} & \longrightarrow & G_{\mathbb{Q}_p}^{ab}(p) \cong G^{ab},
\end{array}$$
where $G_{Q_p}^{ab} := \text{Gal}(Q_p^{ab}/Q_p)$ is the maximal abelian quotient of $G_{Q_p}$ and can be described by the exact sequence

$$1 \longrightarrow \text{Gal}(Q_p^{ab}/Q_p^{ur}) \longrightarrow G_{Q_p}^{ab} \longrightarrow G_{F_p} \longrightarrow 1$$

where $Q_p^{ur}$ is the maximal unramified extension of $Q_p$ inside $\bar{Q}_p$. Local class field theory implies that the natural map

$$G_{Q_p}^{ab} \rightarrow \text{Gal}(Q_p^{ur}/Q_p) \times \text{Gal}(Q_p(\mu_{p^\infty})/Q_p)$$

is an isomorphism, where $\mu_{p^\infty}$ is the group of $p$-power order roots of unity in $\bar{Q}_p$. The cyclotomic character $\epsilon$ induces an isomorphism

$$\text{Gal}(Q_p(\mu_{p^\infty})/Q_p) \xrightarrow{\epsilon} \mathbb{Z}_p^\times$$

and $\text{Gal}(Q_p^{ur}/Q_p) \cong \hat{\mathbb{Z}}$, hence

$$G_{Q_p}^{ab} \cong (1 + p\mathbb{Z}_p) \times \mathbb{Z}_p,$$

where the map onto the first factor is given by $\epsilon^{p-1}$. We choose a pair of generators $\gamma, \delta$ of $G_{Q_p}^{ab}$ such that $\gamma \mapsto (1 + p, 0)$ and $\delta \mapsto (1, 1)$. With [1, Lemma 3.2] we obtain that $G$ is a free pro-$p$ group in two letters $\gamma, \delta$ which project to $\bar{\gamma}, \bar{\delta}$. The way we choose these generators will be of importance in the following.

**Lemma 1.** Let $\eta: G_{Q_p} \rightarrow \mathbb{Z}_p^\times$ be a continuous character such that $\eta \equiv 1(p)$. Then $\eta = \epsilon^k \chi$ for an unramified character $\chi$ if and only if $\eta(\gamma) = \epsilon(\gamma)^k$ and $p - 1|k$.

**Proof.** “$\Rightarrow$”: Since $\gamma$ maps to identity in $\text{Gal}(Q_p^{ur}/Q_p)$, we clearly have $\chi(\gamma) = 1$ for every unramified character $\chi$. Hence $\epsilon(\gamma)^k \equiv 1(p)$, which implies $p - 1|k$.

“$\Leftarrow$”: From $\eta \epsilon^{-k}(\gamma) = 1$ and the fact that $\delta$ maps to the image of identity in the maximal pro-$p$ quotient of $\text{Gal}(Q_p(\mu_{p^\infty})/Q_p)$, we see that $\eta \epsilon^{-k} = \chi$ for an unramified character $\chi$. \hfill $\square$

Since $G$ is a free pro-$p$ group generated by $\gamma$ and $\delta$, to give a framed deformation of $\tilde{\rho}$ to $(A, \mathfrak{m}_A)$ is equivalent to give two matrices in $\text{GL}_2(A)$ which reduce to $\bar{\rho}(\gamma)$ and $\bar{\rho}(\delta)$ modulo $\mathfrak{m}_A$. Thus

$$R_{\tilde{\rho}} = \mathcal{O}[[x_{11}, \tilde{x}_{12}, x_{21}, t_{\gamma}, y_{11}, \tilde{y}_{12}, y_{21}, t_{\delta}]]$$

and the universal framed deformation is given by

$$\rho^{\text{univ}}: G \rightarrow \text{GL}_2(R_{\tilde{\rho}}),
\gamma \mapsto \left(\begin{array}{cc} 1 + t_{\gamma} + x_{11} & x_{12} \\ x_{21} & 1 + t_{\gamma} - x_{11} \end{array}\right),
\delta \mapsto \left(\begin{array}{cc} 1 + t_{\delta} + y_{11} & y_{12} \\ y_{21} & 1 + t_{\delta} - y_{11} \end{array}\right),$$

where $x_{12} := \tilde{x}_{12} + [\phi(\gamma)]$, $y_{12} := \tilde{y}_{12} + [\phi(\delta)]$ where $[\phi(\gamma)], [\phi(\delta)]$ denote the Teichmüller lifts of $\phi(\gamma)$ and $\phi(\delta)$ to $\mathcal{O}$. 

\[\text{Hilbert–Samuel multiplicities of certain deformation rings} \ 607\]
Remark 1. We note that there are essentially three different cases:

1. $\rho$ is ramified $\Leftrightarrow \phi(\gamma) \neq 0 \Leftrightarrow x_{12} \in (R_{\rho}^\square)^\times$;
2. $\rho$ is unramified, non-split $\Leftrightarrow \phi(\gamma) = 0, \phi(\delta) \neq 0 \Leftrightarrow x_{12} \in m_{R_{\rho}^\square}, y_{12} \in (R_{\rho}^\square)^\times$;
3. $\rho$ is split $\Leftrightarrow \phi(\gamma) = 0, \phi(\delta) = 0 \Leftrightarrow x_{12}, y_{12} \in m_{R_{\rho}^\square}$.

Let $\psi: G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ be a continuous character, such that $\det(\bar{\rho}) = \overline{\psi \epsilon}$, and let $R_{\rho,\psi}^\square$ be the quotient of $R_{\rho}^\square$ which parametrizes lifts of $\bar{\rho}$ with determinant $\psi \epsilon$. Since $\gamma, \delta$ generate $G$ as a group, we obtain

$$R_{\rho,\psi}^\square \cong R_{\rho}^\square / (\det(\rho_{\text{univ}}^\psi(\gamma) - \psi \epsilon(\gamma)), \det(\rho_{\text{univ}}(\delta) - \psi \epsilon(\delta)))$$

$$\cong \mathcal{O}[[x_{11}, \hat{x}_{12}, x_{21}, y_{11}, \hat{y}_{12}, y_{21}]],$$

because we can eliminate the parameters $t_\gamma, t_\delta$ due to the relations $(1 + t_\gamma)^2 = \psi \epsilon(\gamma) + x_{11}^2 + x_{12}x_{21}, t_\gamma \equiv 0(\mathfrak{m}), (1 + t_\delta)^2 = \psi \epsilon(\delta) + y_{11}^2 + y_{12}y_{21}, t_\delta \equiv 0(\mathfrak{m})$. We let $v := \frac{1 - e_{\psi(\gamma)^{-1}}}{2}$ and define four polynomials

1. $I_1 := (v + x_{11})(v - x_{11}) - x_{12}x_{21},$
2. $I_2 := (v + x_{11})^2 y_{12} - 2(v + x_{11})x_{12}y_{11} - x_{12}^2y_{21},$
3. $I_3 := x_{21}y_{12} - 2x_{21}(v - x_{11})y_{11} - (v - x_{11})^2y_{21},$
4. $I_4 := (v + x_{11})x_{21}y_{12} - 2x_{12}x_{21}y_{11} - x_{12}(v - x_{11})y_{12}.$

Since for every representation with Hodge–Tate weights $(0, p - 1)$ the determinant is a character of Hodge–Tate weight $p - 1$ and $R_{\rho,\psi}^\square(0, p - 2)$ parametrizes all lifts $\rho_p$ with determinant $\psi \epsilon$, we let from now on $\psi$ have Hodge–Tate weight $p - 2$, as otherwise $R_{\rho,\psi}^\square(0, p - 2)$ would be trivial.

Definition 1. We set

$$R := R_{\rho,\psi}^\square / (I_1, I_2, I_3, I_4).$$

Our goal is to show that $R_{\rho,\psi}^\square(0, p - 2)$ is isomorphic to $R$.

Lemma 2. If $p \in \text{m-Spec}(R_{\rho,\psi}^\square(0, p - 2))$, then $p \in \text{m-Spec}(R_{\rho,\psi}^\square(1))$ if and only if $\rho_p(\gamma)$ is reducible and $\rho_p(\gamma)$ acts on the $G$-invariant subspace with eigenvalue $e^{p-1}(\gamma)$.

Proof. Let $p \in \text{m-Spec}(R_{\rho,\psi}^\square(0, p - 2))$, such that $\rho_p$ is reducible and $\rho_p(\gamma)$ acts on the $G$-invariant subspace with eigenvalue $e^{p-1}(\gamma)$. Since $\det(\rho_p(\gamma)) = \psi \epsilon(\gamma) = \epsilon(\gamma)^{p-1}$ and $e(\gamma)^{p-1}$ is an eigenvalue of $\rho_p(\gamma)$, the other eigenvalue must be 1. Therefore we can write $1 + t_\gamma = \frac{e(\gamma)^{p-1} + 1}{2}$ and obtain

$$0 = \det \begin{pmatrix} 1 + t_\gamma + x_{11} - \epsilon(\gamma)^{p-1} & x_{12} \\ x_{21} & 1 + t_\gamma - x_{11} - \epsilon(\gamma)^{p-1} \end{pmatrix} = (v + x_{11})(v - x_{11}) - x_{12}x_{21}.$$
If we now take $p$ as above but with $I_1 := (v+x_{11})(v-x_{11}) - x_{12}x_{21} \in p$, it is easy to see that the vectors $v_1 = \begin{pmatrix} -x_{12} \\ v+x_{11} \end{pmatrix}$ and $v_2 = \begin{pmatrix} v-x_{11} \\ -x_{21} \end{pmatrix}$ are eigenvectors for $\rho_p(\gamma)$ with eigenvalue $\epsilon(\gamma)^{p-1}$ if they are non-zero. But at least one of them is non-zero because otherwise we obtain $v = 0$ and thus $\epsilon(\gamma)^{p-1} = 1$, which is a contradiction to the definition of $\gamma$. So $\rho_p$ is reducible with an invariant subspace on which $\rho_p(\gamma)$ acts by $\epsilon(\gamma)^{p-1}$ if and only if the vectors $v_1, v_2, \rho^\text{univ}(\delta)v_1, \rho^\text{univ}(\delta)v_2$ are pairwise linear dependent. It is easy to check that this is equivalent to the satisfaction of the equations $I_1 = I_2 = I_3 = I_4 = 0$. □

**Lemma 3.**

$$m\text{-Spec} \left( R \left[ \frac{1}{p} \right] \right) = m\text{-Spec} \left( R^\square,\psi(0, p-2) \left[ \frac{1}{p} \right] \right).$$

**Proof.** From Khare and Wintenberger [9, Proposition 3.5(i)] we know that every crystalline lift $\rho_p$ of a reducible two-dimensional representation $\bar{\rho}$, such that $\rho_p$ has Hodge–Tate-weights $(0, p-1)$, is reducible itself. Moreover, Brinon and Conrad [4, Theorem 8.3.5] say that if $\rho$ is a reducible two-dimensional crystalline representation, then we have an exact sequence

$$0 \rightarrow \epsilon^{p-1} \chi_1 \rightarrow \rho \rightarrow \chi_2 \rightarrow 0.$$

Thus $\rho_p(\gamma)$ acts on the invariant subspace as $\epsilon(\gamma)^{p-1}$ and hence from Lemma 2 it is clear that

$$m\text{-Spec} \left( R \left[ \frac{1}{p} \right] \right) \supset m\text{-Spec} \left( R^\square,\psi(0, p-2) \left[ \frac{1}{p} \right] \right).$$

For the other inclusion we note that it is also clear from Lemma 2 that any maximal ideal $p \in m\text{-Spec}(R[\frac{1}{p}])$ gives rise to a reducible representation $\rho_p$ such that $\rho_p(\gamma)$ acts on the invariant subspace as $\epsilon(\gamma)^{p-1}$ and that the other eigenvalue of $\rho_p(\gamma)$ is 1. So we obtain with Lemma 1 that $\rho_p$ is an extension of two crystalline characters

$$0 \rightarrow \eta_1 \rightarrow \ast \rightarrow \eta_2 \rightarrow 0,$$

where the Hodge–Tate weight of $\eta_1$ is equal to $p-1$ and the weight of $\eta_2$ is equal to 0. Then we can conclude from [13, Proposition 128] that it is semi-stable and from [4, Theorem 8.3.5, Proposition 8.38] that it is crystalline and hence $p \in m\text{-Spec} \left( R^\square,\psi(0, p-2) \left[ \frac{1}{p} \right] \right)$. □

**Remark 2.** We have the following identities mod $I_1$:

(5) \[ x_{21}I_2 = (v+x_{11})I_4, \]

(6) \[ (v-x_{11})I_2 = x_{12}I_4, \]

(7) \[ x_{21}I_4 = (v+x_{11})I_3, \]

(8) \[ (v-x_{11})I_4 = x_{12}I_3. \]
3. Reducedness

In order to show that $R^{\square, \psi}(0, p - 2)$ is equal to $R$, it is enough to show that $R$ is reduced and $\mathcal{O}$-torsion free, since then the assertion follows from Lemma 3, as $R[\frac{1}{p}]$ is Jacobson because $R$ is a quotient of a formal power series ring over a complete discrete valuation ring.

Lemma 4. If $\mathcal{O} = W(k)$, then $R$ is an $W(k)$-torsion-free integral domain.

Proof. We distinguish two cases.

If $\bar{\rho}$ is ramified, i.e., $x_{12}$ is invertible, we consider the fact that for every complete local ring $A$ with $a \in \mathfrak{m}_A$, $u \in A^\times$, there is a canonical isomorphism $A[[z]]/(uz - a) \cong A$. Using this we see from (1),(2),(6) and (8) that

$$R = \mathcal{O}[[x_{11}, \hat{x}_{12}, x_{21}, y_{11}, \hat{y}_{12}, y_{21}]]/(I_1, I_2)$$

$$\cong \mathcal{O}[[x_{11}, \hat{x}_{12}, y_{11}, \hat{y}_{12}]],$$

which shows the claim.

In the second case, where $\bar{\rho}$ is unramified, i.e., $x_{12} \notin R^\times$, we consider the ideal $I := (\pi, x_{11}, x_{12}, x_{21})$ and have

$$\text{gr}_I R^{\square, \psi} \cong k[[y_{11}, \hat{y}_{12}, y_{21}]][(\bar{\pi}, \bar{x}_{11}, \bar{x}_{12}, \bar{x}_{21}]].$$

Since $\mathcal{O} = W(k)$ we have $v \in I \setminus I^2$ and hence the elements $I_1, I_2, I_3, I_4$ are homogeneous of degree 2, so that

$$\text{gr}_I R \cong k[[y_{11}, \hat{y}_{12}, y_{21}]][(\bar{\pi}, \bar{x}_{11}, \bar{x}_{12}, \bar{x}_{21})/(I_1, I_2, I_3, I_4)],$$

see [6, Example 5.3]. Because $R$ is noetherian it follows from [6, Corollary 5.5] that it is enough to show that $\text{gr}_I R$ is an integral domain.

We define

$$A := k[[y_{11}, \hat{y}_{12}, y_{21}]][(\bar{x}_{11}, \bar{x}_{12}, \bar{x}_{21}, \bar{\pi})/(\bar{I}_1)]$$

and look at the map

$$\phi: A \to A[\bar{x}_{12}^{-1}]/(\bar{I}_2).$$

The latter ring is isomorphic to $(k[[y_{11}, \hat{y}_{12}, y_{21}]][(\bar{x}_{11}, \bar{x}_{12}, \bar{x}_{12}^{-1}, \bar{\pi})/(I_2)])$ and since $I_2$ is irreducible it is an integral domain. So we would be done by showing that $\ker(\phi) = (\bar{I}_2, \bar{I}_3, \bar{I}_4)$. The inclusion $(I_2, I_3, I_4) \subset \ker(\phi)$ is clear from (6) and (8). For the other one we consider the fact that

$$\ker(\phi) = \{a \in A : \exists n \in \mathbb{N} \cup \{0\}, b, c, d \in A : \bar{x}_{12}^n a = b\bar{I}_2 + c\bar{I}_3 + d\bar{I}_4\}.$$

To show that $\ker(\phi) \subset (I_2, I_3, I_4)$, we let $a \in A$ and $n$ be minimal with the property that there exist $b,c,d \in A$ such that

$$\bar{x}_{12}^n a = b\bar{I}_2 + c\bar{I}_3 + d\bar{I}_4. \tag{9}$$
If \( n = 0 \) there is nothing to show. Now we assume that \( n > 0 \) and consider the prime ideal \( p := (\bar{x}_{12}, \bar{v} - \bar{x}_{11}) \subset A \) and see that

\[
A/p \cong k[[y_{11}, y_{12}, y_{21}]][[\bar{x}_{11}, \bar{x}_{12}]]
\]

is a unique factorization domain. We also observe that

\[
\begin{align*}
I_2 &\equiv y_{12}(\bar{v} + \bar{x}_{11})^2 \mod p, \\
I_3 &\equiv y_{12}\bar{x}_{21}^2 \mod p, \\
I_4 &\equiv y_{12}(\bar{v} + \bar{x}_{11})\bar{x}_{21} \mod p.
\end{align*}
\]

Modulo \( p \) (9) becomes

\[
0 \equiv y_{12}b(\bar{v} + \bar{x}_{11})^2 + y_{12}c\bar{x}_{21}^2 + y_{12}d(\bar{v} + \bar{x}_{11})\bar{x}_{21}.
\]

Since \( A/p \) is a unique factorization domain there are \( b_1, c_1 \in A \) such that

\[
\begin{align*}
y_{12}b &\equiv b_1\bar{x}_{21} \mod p, \\
y_{12}c &\equiv c_1(\bar{v} + \bar{x}_{11}) \mod p
\end{align*}
\]

and we see that

\[
d \equiv -\frac{b_1\bar{x}_{21} + c_1(\bar{v} + \bar{x}_{11})}{2} \mod p.
\]

Hence we can find \( b_2, b_3, c_2, c_3, d_1, d_2 \in A \) such that

\[
\begin{align*}
b &= b_1\bar{x}_{21} + b_2\bar{x}_{12} + b_3(\bar{v} - \bar{x}_{11}), \\
c &= c_1(\bar{v} + \bar{x}_{11}) + c_2\bar{x}_{12} + c_3(\bar{v} - \bar{x}_{11}), \\
d &= -\frac{b_1\bar{x}_{21} + c_1(\bar{v} + \bar{x}_{11})}{2} + d_1\bar{x}_{12} + d_2(\bar{v} - \bar{x}_{11}).
\end{align*}
\]

Substituting this in (9) we get

\[
\begin{align*}
\tilde{x}_{12}^{\alpha}a &= b\tilde{I}_2 + c\tilde{I}_3 + d\tilde{I}_4 \\
&= \bar{x}_{12}(b_2I_2 + b_3I_4 + c_2I_3 + d_1I_4 + d_2I_3) \\
&\quad + \frac{1}{2}(b_1(\bar{v} + \bar{x}_{11}) + c_1\bar{x}_{21})I_4 + (\bar{v} - \bar{x}_{11})c_3I_3.
\end{align*}
\]

Modulo \( p \) we have \( b_1(\bar{v} + \bar{x}_{11}) + c_1\bar{x}_{21} \equiv 0 \) and hence there are \( b_4, b_5, b_6, c_4, c_5, c_6 \) with

\[
\begin{align*}
b_1 &= \bar{x}_{21}b_4 + \bar{x}_{12}b_5 + (\bar{v} - \bar{x}_{11})b_6, \\
c_1 &= (\bar{v} + \bar{x}_{11})c_4 + \bar{x}_{12}c_5 + (\bar{v} - \bar{x}_{11})c_6.
\end{align*}
\]

Hence we can rewrite (18) to

\[
\begin{align*}
\bar{x}_{12}^{\alpha}a &= \bar{x}_{12}\tilde{z} + \frac{1}{2}(b_4 + c_4)(\bar{v} + \bar{x}_{11})^2I_3 + (\bar{v} - \bar{x}_{11})c_3I_3
\end{align*}
\]
for a certain $z \in (I_2, I_3, I_4)$. So with (21) we see that $b_4 + c_4 \equiv 0$ modulo $p$ and $c_3 \equiv 0$ modulo the prime ideal $p' := (\bar{x}_{12}, \bar{v} + \bar{x}_{11})$. Therefore we can find some $c_7, c_8, e_1, e_2 \in A$ with

\[ c_3 = c_7 \bar{x}_{12} + c_8 (\bar{v} + \bar{x}_{11}), \]
\[ b_4 + c_4 = e_1 \bar{x}_{12} + e_2 (\bar{v} - \bar{x}_{11}).\]

But since we have $(v + x_{11}) (v - x_{11}) = x_{12} x_{21}$ in $A$ we can finally transform (21) to

\[ \bar{x}_{12}^n a = \bar{x}_{12} z' \]

for some $z' \in (I_2, I_3, I_4)$ which shows that $\bar{x}_{12}^{n-1} a \in (I_2, I_3, I_4)$, since $A$ is an integral domain. But this is a contradiction to the minimality of $n$. □

**Proposition 1.** $R$ is reduced and $\mathcal{O}$-torsion free for any choice of $\mathcal{O}$.

**Proof.** Since $\mathcal{O}$ is flat over $W(k)$ and we have seen in Lemma 3 that

\[ S := W(k)[[x_{11}, \bar{x}_{12}, x_{21}, y_{11}, \bar{y}_{12}, y_{21}]]/(I_1, I_2, I_3, I_4) \]

is an integral domain, we get an injection

\[ \mathcal{O} \otimes_{W(k)} S \rightarrow \mathcal{O} \otimes_{W(k)} \text{Quot}(S). \]

As $S$ is $W(k)$-torsion free by Lemma 3, we obtain an isomorphism

\[ \mathcal{O} \otimes_{W(k)} \text{Quot}(S) \cong \mathcal{O} \left[ \frac{1}{p} \right] \otimes_{W(k)[\frac{1}{p}]} \text{Quot}(S). \]

Since $\mathcal{O}[\frac{1}{p}]$ is a separable field extension of $W(k)[\frac{1}{p}]$, we deduce that $\mathcal{O}[\frac{1}{p}] \otimes_{W(k)[\frac{1}{p}]} \text{Quot}(S)$ is reduced and $\mathcal{O}$-torsion free. □

### 4. The multiplicity

We want to compute the Hilbert–Samuel multiplicity of the ring $R/\pi$ for the given representation

\[ \bar{\rho} : G_{Q_p} \rightarrow \text{GL}_2(k), \quad g \mapsto \begin{pmatrix} 1 & \phi(g) \\ 0 & 1 \end{pmatrix}. \]

We denote the maximal ideal of $R/\pi$ by $m$.

**Theorem 2.**

\[ e(R/\pi) = \begin{cases} 1 & \text{if } \bar{\rho} \text{ is ramified,} \\ 2 & \text{if } \bar{\rho} \text{ is unramified, indecomposable,} \\ 4 & \text{if } \bar{\rho} \text{ is split.} \end{cases} \]

**Proof.** If we set $J := y_{12} x_{21} + 2 x_{11} y_{11} + x_{12} y_{21}$ we obtain modulo $\pi$ the relations

\[ I_2 \equiv -x_{12} J, \]
\[ I_3 \equiv x_{21} J, \]
\[ I_4 \equiv x_{11} J. \]
We split the proof into three cases as in Remark 1. If \( \tilde{\rho} \) is ramified, i.e., \( x_{12} \) is invertible, we see as in the proof of Lemma 4 that
\[
R/\pi \cong k[[x_{11}, \hat{x}_{12}, x_{21}, y_{11}, \hat{y}_{12}, y_{21}]]/(x_{11}^2 + x_{12}x_{21}, J)
\cong k[[x_{11}, \hat{x}_{12}, y_{11}, \hat{y}_{12}]].
\]
Hence it is a regular local ring and therefore \( e(R/\pi) = 1 \).

Let us assume in the following that \( \tilde{\rho} \) is unramified, i.e., \( x_{12} = \hat{x}_{12} \in m_R \), and we can consider the exact sequence
\[
(25) \quad 0 \to (R/\pi)/\text{Ann}_{R/\pi}(J) \to R/\pi \to R/(\pi, J) \to 0.
\]

Since \( x_{11}, x_{12}, x_{21} \in \text{Ann}_{R/\pi}(J) \), see (22)–(24), we have \( \dim((R/\pi)/\text{Ann}_{R/\pi}(J)) \leq 3 \). But \( \dim R/\pi = 4 \) so that (25) gives us \( e(R/\pi) = e(R/(\pi, J)) \), see [12, Theorem 14.6]. We obtain that
\[
R/(\pi, J) \cong k[[x_{11}, x_{12}, x_{21}, y_{11}, \hat{y}_{12}, y_{21}]]/(x_{11}^2 + x_{12}x_{21}, J)
\cong (k[[x_{11}, x_{12}, x_{21}]]/(x_{11}^2 + x_{12}x_{21}))(y_{111}, \hat{y}_{12}, y_{211})/(J)
\]
is a complete intersection of dimension 4. So if \( q \subset R/(\pi, J) \) is an ideal generated by four elements, such that \( R/(\pi, J, q) \) has finite length as a \( R/(\pi, J) \)-module, then these elements form a regular sequence in \( R/(\pi, J) \) and \( e_q(R/(\pi, J)) = l(R/(\pi, J, q)) \), see [12, Theorem 17.11]. Besides, if there exists an integer \( n \) such that \( qm^n = m^{n+1} \), then \( e(R/(\pi, J)) = e_q(R/(\pi, J)) \), see [12, Theorem 14.13]. So to finish the proof it would suffice to find such an ideal \( q \).

If \( \tilde{\rho} \) is indecomposable, i.e., \( \phi(\delta) \) is non-zero and therefore \( y_{12} \) is a unit in \( R \), we can write the equation \( J = 0 \) as
\[
x_{21} = -y_{12}^{-1}(2x_{11}y_{11} + y_{21}x_{12})
\]
and \( I_1 = 0 \) as
\[
x_{11}^2 = x_{12}y_{12}^{-1}(2x_{11}y_{11} + y_{21}x_{12})
\]
so that
\[
R/(\pi, J) \cong k[[x_{11}, x_{12}, y_{11}, \hat{y}_{12}, y_{21}]]/(x_{11}^2 - x_{12}y_{12}^{-1}(2x_{11}y_{11} + y_{21}x_{12})).
\]
Hence it is clear that \( x_{12}, x_{21}, y_{11}, \hat{y}_{12} \) is a system of parameters for \( R/(\pi, J) \) that generates an ideal \( q \) with \( qm = m^2 \). So we obtain
\[
e_q(R/(\pi, J)) = l(R/(\pi, J, q)) = l(k[[x_{11}]]/(x_{11}^2)) = 2
\]
and hence \( e(R/\pi) = 2 \).

If \( \tilde{\rho} \) is split, which is equivalent to \( x_{12}, y_{12} \notin R^x \), we take \( q := (x_{12} - x_{21}, x_{12} - y_{12}, x_{12} - y_{21}, y_{11}) \) and claim that \( qm^2 = m^3 \). If we write \( m = (x_{12} - x_{21}, x_{12} - y_{12}, x_{12} - y_{21}, y_{11}, x_{11}, x_{12}) \) we just have to check that \( x_{11}^3, x_{11}^2x_{12}, x_{11}x_{12}^2, x_{12}^3 \in qm^2 \). Therefore it is enough to see that
\[
x_{11}^2 = x_{11}y_{11} - \frac{1}{2}(x_{12} - y_{12})x_{21} - \frac{1}{2}(x_{21} - y_{21})x_{12} \in mq,
x_{12}^2 = -x_{11}^2 + x_{12}(x_{12} - x_{21}) \in mq.
\]
Hence
\[ e(R/\pi) = l(R/(\pi, J, q)) = l(k[[x_{11}, x_{12}]]/(x_{11}^2, x_{12}^2)) = 4. \]

**Corollary 1.** If \( \bar{\rho} \) is unramified, then the ring \( R \) is not Cohen–Macaulay.

**Proof.** Since \( R \) is \( \mathcal{O} \)-torsion free, \( \pi \) is \( R \)-regular and hence \( R \) is CM if and only if \( R/\pi \) is CM. In (25) we have constructed a non-zero submodule of \( R/\pi \) of dimension strictly less than the dimension of \( R/\pi \). It follows from [5, Theorem 2.1.2(a)] that \( R/\pi \) cannot be CM. \( \square \)

**Proposition 2.** \( \text{Spec}(R/\pi) \) is geometrically irreducible and generically reduced.

To prove the proposition we need the following lemma. As in the proof of Theorem 2 we define \( J := y_{12}x_{21} + 2x_{11}y_{11} + x_{12}y_{21} \).

**Lemma 5.** \( R/\pi, J \) is an integral domain.

**Proof.** We again distinguish between three cases as in Remark 1. If \( \bar{\rho} \) is ramified, i.e., \( x_{12} \) is invertible, we have already seen in the proof of Theorem 2 that
\[
R/\pi \cong k[[x_{11}, \hat{x}_{12}, x_{21}, y_{11}, \hat{y}_{12}, y_{21}]]/(x_{11}^2 + x_{12}x_{21}, J)
\]
\[ \cong k[[x_{11}, \hat{x}_{12}, y_{11}, \hat{y}_{12}]]. \]

If \( \bar{\rho} \) is unramified and indecomposable, i.e., \( x_{12} = \hat{x}_{12} \in m_R, y_{12} \in R^\times \) we saw that
\[
R/\pi \cong k[[x_{11}, x_{12}, y_{11}, \hat{y}_{12}, y_{21}]]/(x_{11}^2 - x_{12}y_{12}^{-1}(2x_{11}y_{11} + y_{21}x_{12})),
\]
which is easily checked to be an integral domain. If \( \bar{\rho} \) is unramified and split, i.e., \( x_{12}, y_{12} \in m_R \), let \( n \) denote the maximal ideal of \( R/\pi, J \). It is enough to show that the graded ring \( \text{gr}_n R/\pi, J \) is a domain. Since \( J \) is homogeneous we have
\[
\text{gr}_n R/\pi, J \cong k[[x_{11}, x_{12}, x_{21}, y_{11}, y_{12}, y_{21}]]/(x_{11}^2 + x_{12}x_{21}, J).
\]

We set \( A := k[[x_{11}, x_{12}, x_{21}, y_{11}, y_{12}, y_{21}]]/(x_{11}^2 + x_{12}x_{21}) \) and have to prove that \( (J) \subset A \) is a prime ideal. We look at the localization map \( A \to A[y_{21}^{-1}] \), which is an inclusion because \( y_{21} \) is regular in \( A \). This gives us a map \( A \to A[y_{21}^{-1}]/(J) \). Since
\[
A[y_{21}^{-1}]/(J) \cong k[[x_{11}, x_{12}, y_{11}, y_{12}, y_{21}]]/(x_{11}^2 - x_{21}y_{21}^{-1}(2x_{11}y_{11} + x_{21}y_{12})),
\]
is a domain, we would be done by showing that \( \text{ker}(\bar{\iota}) = (J) \). We have
\[ \text{ker}(\bar{\iota}) = \{ a \in A : y_{21}^{-1}a = bJ \text{ for some } i \in \mathbb{Z}_{\geq 0}, b \in A : y_{21} \not| b \}. \]

But since \( (y_{21}) \subset A \) is a prime ideal and \( y_{21} \) does not divide \( J \), we see that \( i = 0 \) in all these equations and hence \( \text{ker}(\bar{\iota}) = (J) \). \( \square \)

**Proof of Proposition 2.** Let \( \mathfrak{p} \) be a minimal prime ideal of \( S := R/\pi \). It follows from (22)–(24) that \( J^2 = 0 \) and thus \( J \in \text{rad}(S) = \bigcap \mathfrak{p} \) minimal \( \mathfrak{p} \). So Lemma 5 gives us that \( JS \) is the only minimal prime ideal of \( S \), hence \( \text{Spec}(S) \) is irreducible. If we replace the field \( k \) by an extension \( k' \), we obtain the irreducibility of \( \text{Spec}(S \otimes_k k') \) analogously, thus \( \text{Spec}(S) \) is geometrically irreducible.
Spec(S) is called generically reduced if $S_p$ is reduced for any minimal prime ideal $p$. We have already seen that there is only one minimal prime ideal $p = JS$. By localizing (25) we obtain $S_p \cong R/(\pi, J)$. Lemma 5 implies that $S_p$ is reduced. □

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