Semi-Dirichlet forms, Feynman-Kac functionals and the Cauchy problem for semilinear parabolic equations

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Abstract
In the first part of the paper we prove various results on regularity of Feynman-Kac functionals of Hunt processes associated with time dependent semi-Dirichlet forms. In the second part we study the Cauchy problem for semilinear parabolic equations with measure data involving operators associated with time-dependent forms. Model examples are non-symmetric divergence form operators and fractional laplacians with possibly variable exponents. We first introduce a definition of a solution resembling Stampacchia’s definition in the sense of duality and then, using the results of the first part, we prove the existence, uniqueness and regularity of solutions of the problem under mild assumptions on the data.

1 Introduction
Let\( E \) be a locally compact separable metric space, \( m \) be an everywhere dense Borel measure on \( E \) and let \( \{B(t); t \in \mathbb{R}\} \) be a family of regular semi-Dirichlet forms on \( L^2(E;m) \) with common domain \( F \). Let us consider a time-dependent semi-Dirichlet form
\[
\mathcal{E}(u,v) = \begin{cases} (-\frac{\partial u}{\partial t},v) + B(u,v), & (u,v) \in \mathcal{W} \times L^2(0,T;F), \\ (u,\frac{\partial v}{\partial t}) + B(u,v), & (u,v) \in L^2(0,T;F) \times \mathcal{W}, \\ \end{cases}
\]
where \( \mathcal{W} = \{u \in L^2(0,T;F); \frac{\partial u}{\partial t} \in L^2(0,T;F')\} \), \((\cdot,\cdot)\) stands for the duality pairing between \( L^2(0,T;F) \) and \( L^2(0,T;F') \), and
\[
B(u,v) = \int_{\mathbb{R}} B^{(t)}(u(t),v(t)) \, dt.
\]
Let \( \mathcal{M} = (\{X_t, t \geq 0\}, \{P_z, z \in E \times \mathbb{R}\}) \) be a Hunt process with life-time \( \zeta \) properly associated with \( \mathcal{E} \). The main object of the present paper is to study regularity of the Feynman-Kac functionals of the form
\[
u(z) = E_z 1_{\{\zeta > T - \tau(0)\}} \varphi(X_{T - \tau(0)}) + E_z \int_{0}^{\zeta \wedge T} dA^\mu_t, \quad z \in E_{0,T} \equiv (0,T] \times E. \quad (1.1)
\]

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Here $E_z$ denotes the expectation with respect to $P_z$, $\zeta_\tau = \zeta \land (T - \tau(0))$, where $\tau$ is the uniform motion to the right, $\varphi : E \to \mathbb{R}$ and $A^\mu$ is the additive functional of $M$ in Revuz correspondence with a smooth measure $\mu$ on $E_0,T$.

Our interest in functionals of the form (1.1) comes from the fact that regularity of $u$ implies regularity of solutions of the Cauchy problem

$$-rac{\partial u}{\partial t} - L_t u = \mu, \quad u(T) = \varphi,$$

where $L_t$ is the operator associated with the form $B^{(t)}$. The study of equations of the form (1.2) and more general semilinear equations of the form

$$-rac{\partial u}{\partial t} - L_t u = f(t, x, u) + \mu, \quad u(T) = \varphi$$

is the second main goal of the paper. With $\varphi \in L^1(E; m)$ and “true” measure data. Therefore in the paper we assume that $\mu$ belongs to the space $\mathcal{R}(E_0,T)$ of all smooth (with respect to the capacity determined by $\mathcal{E}$) measures on $E_0,T$ such that $E_z A_{[\varphi]}^{[\mu]} < \infty$ for $\mathcal{E}$-quasi-every (q.e.) $z \in E_0,T$, and that $\delta_{\{T\}} \otimes \varphi \cdot m \in \mathcal{R}(E_0,T)$. These are minimal assumptions on $\mu, \varphi$ under which $u$ is finite $m_1$-a.e., and hence finite $\mathcal{E}$-q.e. The class $\mathcal{R}(E_0,T)$ is quite wide. If $\mathcal{E}$ satisfies some duality condition (see condition $(\Delta)$ below) then it includes the space $\mathcal{M}_{0,b}(E_0,T)$ of all bounded smooth measures on $E_0,T$. Our general framework of time-dependent semi-Dirichlet forms associated with the family of semi-Dirichlet forms allows us to study (1.2), (1.3) for wide class of local and nonlocal operators $L_t$. Model examples are diffusion operators with drift terms and fractional laplacians with constant and variable exponents (for more examples see [11, 15, 16, 21, 24]). We think that applicability of our general results to parabolic equations with measure data involving nonlocal operators is of particular interest, because to our knowledge, with the exception of [17], no such result has appeared in the literature.

Large majority of known results on the regularity of $u$ given by (1.1) concerns the case where $\mu = g \cdot m_1$. One can roughly divide them into two groups. In the first group of results one shows that $u$ is continuous and then that it is a viscosity solution of (1.2). To show this one assumes that $\varphi, g$ are continuous with polynomial growth and $L_t$ is a non-divergent form diffusion operator or Lévy type operator with diffusion part in the non-divergent form with Lipschitz continuous coefficients (see, e.g., [3, 25]). The results of the second group say that $u$ is a Sobolev space weak (in the variational sense) solution of (1.2). In the known results $f, \varphi$ are assumed to be square integrable and $L_t$ is a diffusion operator with regular coefficients (see [1, 4]), uniformly elliptic diffusion operator with measurable coefficients (see [2, 19, 30]) or Lévy type operator whose diffusion part has regular coefficients (see [34]). In [38] diffusion operators with singular coefficients are considered. However, in the case considered in [38] the regularity of $u$ follows from that for diffusion with no singular part and from the stochastic representation of the divergence (see [13, 31, 37]).

In [17] regularity of $u$ given by (1.1) and connections of (1.1) with solutions of (1.2) are investigated in case $L_t$ is a uniformly elliptic divergence form operator, $\varphi \in L^1(E; m)$ and $\mu$ is a general bounded smooth measure. We generalize considerably these results. The remarkable feature of [17] and the present paper is that in both papers the regularity of Feynman-Kac functionals of the form (1.1) plays an important
role in the proof of their connections with PDEs. Secondly, as in [17], in the present paper the proof of regularity of Feynman-Kac functionals (and hence of solutions to related PDEs) relies purely on the theory of Dirichlet forms. In the existing literature the proofs of regularity of Feynman-Kac functionals are usually based on results on stochastic flows (in the case of regular coefficients) or on regularity results from the theory of PDEs combined with approximation methods based on regularization of the data involved in the functional.

In the first part of the paper we prove that in general, if \( \varphi \in L^1(E; m) \) and \( \mu \in \mathcal{R}(E_{0,T}) \), then \( u \) given by (1.1) is quasi-l.s.c and quasi-càdlàg, and if \( u \in L^2(E_{0,T}; m_1) \), where \( m_1 = dt \otimes m \), then \( (0,T) \ni t \mapsto u(t) \in L^2(E; m) \) is càdlàg. If \( \alpha \mu \) is continuous then \( u \) is quasi-continuous, and if moreover \( u \in L^2(E_{0,T}; m_1) \), then \( (0,T) \ni t \mapsto u(t) \in L^2(E; m) \) is continuous. We also show that if the following duality condition is satisfied:

\[(\Delta) \text{ for some } \alpha \geq 0 \text{ there exists a nest } \{F_n\} \text{ on } E_{0,T} \text{ such that for every } n \geq 1 \text{ there is a non-negative } \eta_n \in L^2(E_{0,T}; m_1) \text{ such that } \eta_n > 0 \text{ } m_1\text{-a.e. on } F_n \text{ and } \hat{G}_\alpha^{0,T} \eta_n \text{ is bounded,} \]

where \( \hat{G}_\alpha^{0,T} \) is the adjoint operator to the resolvent \( G_\alpha^{0,T} \) of the operator \( -\partial_t - L_t \), then

\[\mathcal{M}_{0,b}(E_{0,T}) \subset \mathcal{R}(E_{0,T}).\]  

(1.4)

Condition \((\Delta)\) is satisfied for instance if \( \alpha \hat{G}_\gamma^{0,T} \) is Markovian for some \( \gamma \geq 0 \). From (1.4) it follows in particular that if \( \varphi \in L^1(E; m) \) then \( \delta_{\{T\}} \otimes \varphi \cdot m \in \mathcal{R}(E_{0,T}) \). We next prove some energy estimates for \( u \). To this end, we first prove that if \( \varphi \in L^2(E; m) \) and \( \mu \in S_0(E_{0,T}) \), i.e. \( \mu \) is a finite energy measure on \( E_{0,T} \), then \( u \in L^2(0,T; F) \) and \( u \) is a weak solution of (1.2) in the variational sense. We then use this result to show that if \( \varphi \in L^1(E; m) \), \( \mu \in \mathcal{M}_{0,b}(E_{0,T}) \) and for some \( \gamma \geq 0 \) the form \( \mathcal{E}_\gamma = \mathcal{E} + \gamma \langle \cdot \rangle^2_L \) has the dual Markov property then \( u \in L^1(E_{0,T}; m_1) \), \( T_k(u) = ((-k) \vee u) \wedge k \in L^2(0,T; F) \) for every \( k \geq 0 \) and

\[\int_0^T B^\gamma(t)(T_k(u)(t), T_k(u)(t)) \, dt \leq k(\|\mu\|_{TV} + \|\varphi\|_{L^1} + \gamma\|u\|_{L^1}).\]

In the second part of the paper we study the Cauchy problems (1.2), (1.3). Before describing briefly our main results let us mention that one delicate issue one encounters when considering (1.2), (1.3) with measure data is to give proper definition of a solution. This is caused by the fact that even in the linear case the distributional solution may be not unique (see [33] for a suitable example of linear equation with uniformly elliptic divergence form operator). The problem of existence and uniqueness of solutions of equations with measure data was first addressed in Stampacchia’s paper [35] devoted to the Dirichlet problem for elliptic equations with uniformly elliptic divergence form operator. To overcome the difficulty with the uniqueness of solutions Stampacchia introduced the so-called solutions by the method of duality and showed that in his class of solutions the problem is well posed. A drawback to the original Stampacchia’s definition of solutions, and perhaps the main reason why the theory of solutions by duality have not been developed, is that it applies mainly to linear equations. In the early nineties of the last century the so-called entropy and renormalized solutions were introduced (see, e.g., [5, 8] and the references therein), and an extensive study of nonlinear equations with measure data and local operators began. For a selection of
important results on the subject we refer the reader to [5, 8] (elliptic equations) and [10, 27] (parabolic equations).

In the present paper by a solution to (1.2) we mean \( u \) satisfying (1.1). In case (\( \Delta \)) is satisfied we show that equivalently \( u \) can be defined as a measurable function on \( E_{0,T} \) satisfying the equation

\[
(u, \eta)_{L^2(E_{0,T}; m_1)} = (\varphi, (G^{0,T} \eta)(T))_{L^2(E;m)} + \int_{E_{0,T}} G^{0,T} \eta \, d\mu \tag{1.5}
\]

for every non-negative \( \eta \in L^2(E_{0,T}; m_1) \) such that \( G^{0,T} \eta \) is bounded. It follows in particular that under (\( \Delta \)) there is at most one \( u \) satisfying (1.5). The definition of a solution to (1.1) via (1.5) resembles Stampacchia's definition given in [35]. In case of local operators, it coincides with the original definition from [35]. Note also that our definition (1.5) extends to the parabolic case and semi-Dirichlet forms the definition introduced in [15] (see also [16]) in case of elliptic equations with measure data involving operators associated with Dirichlet forms.

In the semilinear case the definitions of solutions are similar to those in the linear case. We call a measurable \( u : E_{0,T} \to \mathbb{R} \) a solution to (1.3) if (1.1) is satisfied with \( \mu \) replaced by \( f_u \cdot m + \mu \), where \( f_u = f(\cdot, \cdot, u) \). In case (\( \Delta \)) is satisfied, \( u \) is a solution of (1.3) if \( f_u \in L^1(E_{0,T}; m_1) \) and (1.5) is satisfied with \( \mu \) replaced by \( f_u \cdot m + \mu \). We prove the existence and uniqueness of solutions to (1.3) for \( f \) satisfying the monotonicity condition, continuous with respect to \( u \) and such that \( f(\cdot, \cdot, 0) \in \mathcal{R}(E_{0,T}) \) and

\[
\forall y \in \mathbb{R} \quad f(t, x, y) \in qL^1(E_{0,T}; m_1), \tag{1.6}
\]

where \( qL^1(E_{0,T}; m_1) \) is the space of quasi-integrable functions on \( E_{0,T} \) (see Section 3).

Let us note that equations of the form (1.3) with local operators (nonlinear of Leray-Lions type) were considered in [5, 6]. In these papers it is assumed that \( f(\cdot, \cdot, 0) \) is bounded. It follows in particular that under (\( \Delta \)) there is at most one \( u \) satisfying (1.5). The definition of a solution to (1.1) via (1.5) resembles Stampacchia's definition given in [35]. In case of local operators, it coincides with the original definition from [35]. Note also that our definition (1.5) extends to the parabolic case and semi-Dirichlet forms the definition introduced in [15] (see also [16]) in case of elliptic equations with measure data involving operators associated with Dirichlet forms.

Elliptic problems with Laplace operator and right-hand side satisfying weak growth condition of the form (1.6) were considered in [23] for \( f \) independent of \( x \) and in [14] for diagonal systems. Let us also mention the papers [7, 9] in which \( L \) (independent of \( t \)) is assumed to be accretive on \( L^1(E_{0,T}; m_1) \), \( \mu \in L^1(E_{0,T}; m_1) \) and \( f \) satisfies some condition which implies (1.7). It is worth noting that except for [14] in all the mentioned papers \( f(\cdot, \cdot, 0), \mu \) are assumed to be in \( L^1(E_{0,T}, m_1) \) or in \( \mathcal{M}_{0,b}(E_{0,T}) \). In the present paper we consider the class \( \mathcal{R}(E_{0,T}) \), which for some classes of operators defined on bounded smooth domains \( D \subset \mathbb{R}^d \) includes weighted Lebesgue spaces \( L^1(D_{0,T}; \delta^\alpha \cdot m_1) \) for some \( \alpha \geq 0 \), where \( \delta(x) = \text{dist}(x, \partial D) \). These classes of spaces are important in applications to elliptic systems (see [29]).

Finally, let us note that in the paper we assume that \( \{B^{0}\} \) appearing in the definition of \( \mathcal{E} \) is a family of regular semi-Dirichlet forms. However, at the end of Section 4 we show that in the case where \( \{B^{0}\} \) is a family of non-negative quasi-regular Dirichlet forms one can apply the so-called transfer method to the form \( \mathcal{E} \). Therefore all the results of the paper on regularity of (1.1) and solutions of (1.2), (1.3) also hold true under the last assumption on \( \{B^{0}\} \).
2 Preliminaries

In the paper $E$ denotes a locally compact separable metric space and $m$ denotes an everywhere dense measure on the Borel $\sigma$-algebra $B(E)$.

Let $F$ be a dense subspace of $H \equiv L^2(E; m)$ and $B : F \times F \to \mathbb{R}$ be a bilinear form. We say that $B$ is closed on $F$ if

(B1) there exists $\alpha_0 \geq 0$ such that

$$B_{\alpha_0}(u, u) \geq 0, \quad u \in F,$$

where $B_{\alpha_0}(u, v) = B(u, v) + \alpha_0(u, v)_{L^2},$

(B2) there exists $K \geq 0$ such that

$$|B(u, v)| \leq KB_{\alpha_0}(u, u)^{1/2}B_{\alpha_0}(v, v)^{1/2}, \quad u, v \in F,$$

(B3) $F$ is a Hilbert space with the inner product

$$(u, v)_F \equiv \frac{1}{2}(B_{\alpha_0}(u, v) + B_{\alpha_0}(v, u)).$$

We say that $B$ has the Markov property if

(B4) for all $u \in F$ and $a \geq 0$, $u \land a \in F$ and $B(u \land a, u - u \land a) \geq 0$.

We say that $B$ has the dual Markov property if

(â€¢B4) for all $u \in F$ and $a \geq 0$, $u \land a \in F$ and $B(u - u \land a, u \land a) \geq 0$.

We say that a form $(B, F)$ is a Dirichlet form if it is closed and has the Markov property (B4). A Dirichlet form $(B, F)$ is called non-negative if $\alpha_0 = 0$.

It is known (see [24, Theorem 1.1.5]) that if $(B, F)$ is a Dirichlet form then (B4) is equivalent to the following condition: $\alpha G_{\alpha_0}^f$ is Markovian for every $\alpha > 0$, i.e. if $0 \leq f \leq 1$ then $0 \leq \alpha G_{\alpha_0}^f \leq 1$, where $\{\alpha G_{\alpha_0}, \alpha > \alpha_0\}$ is the resolvent associated with $(B, F)$.

We say that $(B, F)$ is regular if there exists a subset $C$ of the space $C_0(E)$ of continuous functions on $E$ with compact support such that $F \cap C$ is $B_{\alpha_0}$-dense in $F$ and dense in $C_0(E)$ with uniform norm.

For $k \geq 0$ put

$$T_k(u) = \max\{\min\{u, k\}, -k\}, \quad u \in \mathbb{R}.$$

Then for every $\alpha > \alpha_0$ and $u \in F$,

$$\alpha(T_k(u) - \alpha G_{\alpha}^0T_k(u), u - T_k(u))_{L^2} \geq 0,$$

because $-k \leq \alpha G_{\alpha}^0T_k(u) \leq k$ by the Markovovian property of $\alpha G_{\alpha}^0$. By Theorems 1.1.4 and 1.1.5 in [24] the above inequality implies that

(B4a) $T_k(u) \in F$ for every $u \in F$ and

$$B(T_k(u), T_k(u)) \leq B(T_k(u), u).$$
It follows that if \((B, F)\) is closed then condition (B4) is equivalent to (B4a). Similarly, if \((B, F)\) is closed then (B4) is equivalent to

\begin{equation}
\hat{(B4a)} \quad T_k(u) \in F \text{ for every } u \in F \text{ and } B(T_k(u), T_k(u)) \leq B(u, T_k(u)).
\end{equation}

In what follows \(E^1 = \mathbb{R} \times E, \ m_1 = \lambda \otimes m\) and \(\lambda^1\) is the Lebesgue measure on \(\mathbb{R}\). We set \(F = L^2(\mathbb{R}; F), \ F_{0,T} = L^2(0, T; F), \ F_T = L^2(-\infty, T; F)\) and \(H = L^2(E^1; m_1), \ H_{0,T} = L^2(0, T; H), \ H_T = L^2(-\infty, T; H)\). Let \(F' = L^2(\mathbb{R}; F')\) denotes the dual space to \(F\). We set

\[ \mathcal{W} = \{ u \in F; \frac{\partial u}{\partial t} \in F' \} \]

and define \(\mathcal{W}_T, \mathcal{W}_{0,T}\) analogously to \(\mathcal{W}\) but with \(F\) replaced by \(F_T, F_{0,T}\), respectively. For \(u \in \mathcal{W}\) we put

\[ \|u\|_{\mathcal{W}} = \|\frac{\partial u}{\partial t}\|_F + \|u\|_F. \]

The norms \(\|u\|_{\mathcal{W}_{0,T}}\) and \(\|u\|_{\mathcal{W}_T}\) are defined analogously: we replace \(F\) in the above definition by \(F_{0,T}\) and \(F_T\), respectively.

For \(a, b \in \mathbb{R} \cup \{\pm \infty\} \cup \{-\infty\}\) let \(C(a, b; H)\) denote the space of all functions \(u \in \mathcal{B}((a, b) \times E)\) such that the mapping \((a, b] \ni t \mapsto u(t) \in H\) is continuous and let \(C(\mathbb{R}; H) = C(-\infty, +\infty; H)\). It is well known (see [20]) that \(\mathcal{W} \subset C(\mathbb{R}; H)\).

By \(D(a, b; H)\) we denote the space of those functions \(u \in \mathcal{B}((a, b] \times E)\) for which the mapping \((a, b] \ni t \mapsto u(t) \in H\) is c\'adl\'ag, i.e. right continuous with left limits.

Let \(\{B^{(t)}, t \in \mathbb{R}\}\) be a family of regular Dirichlet forms on \(F\). In the paper we assume that for every \(u, v \in F\) the mapping

\[ \mathbb{R} \ni t \mapsto B^{(t)}(u, v) \]

is measurable and the constant \(\alpha_0\) of conditions (B2), (B3) does not depend on \(t\). We may and will assume that \(\alpha_0 < 1\). We also assume that there exists \(\lambda > 0\) such that

\begin{equation}
\frac{1}{\lambda} B^{(0)}_{\alpha_0}(u, u) \leq B^{(t)}_{\alpha_0}(u, u) \leq \lambda B^{(0)}_{\alpha_0}(u, u), \quad u \in F, \quad t \in \mathbb{R}. \tag{2.1}
\end{equation}

For \((u, v) \in (\mathcal{F} \times \mathcal{W}) \cup (\mathcal{W} \times \mathcal{F})\) we put

\[ \mathcal{E}(u, v) = \begin{cases} \left(-\frac{\partial u}{\partial t}, v\right) + B(u, v), & (u, v) \in \mathcal{W} \times \mathcal{F}, \\ (u, \frac{\partial v}{\partial t}) + B(u, v), & (u, v) \in \mathcal{F} \times \mathcal{W} \end{cases} \tag{2.2} \]

and \(\mathcal{E}_\alpha(u, v) = \mathcal{E}(u, v) + \alpha(u, v)_{L^2}\), where \((\cdot, \cdot)\) stands for the duality pairing between \(\mathcal{F}\) and \(\mathcal{F'}\) and

\[ B(u, v) = \int_{\mathbb{R}} B^{(t)}(u(t), v(t)) dt. \]

It is known (see, e.g., [24, 36]) that for every \(\alpha > \alpha_0\) and \(f \in H\) there exist unique \(G_\alpha f, \hat{G}_\alpha \in \mathcal{F}\) such that

\[ \mathcal{E}_\alpha(G_\alpha f, v) = (f, v), \quad v \in \mathcal{W}, \]

\[ \mathcal{E}_\alpha(v, \hat{G}_\alpha f) = (f, v), \quad v \in \mathcal{W}. \]
Moreover, there exist strongly continuous semigroups \( \{T_t, t \geq 0\} \) on \( \mathcal{H} \) such that \( \|T_t\|_{L^2} \leq e^{\alpha t}, \|\hat{T}_t\|_{L^2} \leq e^{\alpha t}, \ t \geq 0, \) and

\[
G_\alpha f = \int_0^\infty e^{-\alpha t}T_t f \, dt, \quad \hat{G}_\alpha f = \int_0^\infty e^{-\alpha t}\hat{T}_t f \, dt. \tag{2.3}
\]

It is also known that \( T_t \) (resp. \( \hat{T}_t \)) can be extended to \( L^\infty(E^1; m_1) \) (resp. \( L^1(E^1; m_1) \)) and that \( T_t \) (resp. \( \hat{T}_t \)) is a contraction on \( L^\infty(E^1; m_1) \) (resp. \( L^1(E^1; m_1) \)). Therefore \( \alpha G_\alpha \) (resp. \( \alpha \hat{G}_\alpha \)) given by (2.3) is well defined and is a contraction on \( L^\infty(E^1; m_1) \) (resp. \( L^1(E^1; m_1) \)).

We say that a form \( E \) has the dual Markov property if \((\tilde{B}4)\) holds with \( B \) replaced by \( E \) and \( F \) replaced by \( W \). This is equivalent to say that \( \alpha \hat{G}_\alpha \) is a contraction on \( L^1(E^1; m_1) \) for every \( \alpha > 0 \) (see the reasoning in the proof of [24, Theorem 3.10]). By standard approximation arguments (see the proof of Theorem 3.10), if \( E \) has the dual Markov property then

\[
\int_0^T B(t)(u(t) - \langle u \rangle(t), (u \wedge a)(t)) \geq 0
\]

for every \( u \in \mathcal{F}_{0,T} \) and \( a \in \mathbb{R} \).

A function \( u \in B^+(E^1) \) satisfying \( \beta G_{\beta+\alpha}u \leq u \) (resp. \( \beta \hat{G}_{\beta+\alpha}u \leq u \)) for every \( \beta \geq 0 \) is called an \( \alpha \)-excessive (resp. \( \alpha \)-co-excessive) function. By \( \mathcal{P}_\alpha \) (resp. \( \hat{\mathcal{P}}_\alpha \)) we denote the set of all \( \alpha \)-excessive (resp. \( \alpha \)-co-excessive) functions.

Let \( \psi \in L^2(E^1; m_1), \ 0 < \psi \leq 1, \ m_1 \)-a.e. For an open set \( U \subset E^1 \) we put

\[
\text{Cap}_\psi(U) \equiv (h_U, \psi)_{L^2},
\]

where \( h = G_1 \psi \) and \( h_U \) is the reduced function of \( h \) on \( U \) (see [36]). For an arbitrary set \( B \subset E^1 \) we put

\[
\text{Cap}_\psi(B) = \inf\{\text{Cap}_\psi(U); B \subset U, U \subset E^1, \ U \text{-open}\}.
\]

We say that set \( B \) is \( \mathcal{E} \)-exceptional if \( \text{Cap}_\psi(B) = 0 \). We say that some property is satisfied quasi everywhere (q.e.) if the set of those \( z \in E^1 \) for which it does not hold is \( \mathcal{E} \)-exceptional. The capacity \( \text{Cap}_\psi \) is equivalent to the capacity considered in [24, Section 6.2]. It is known (see the argument following (6.2.2) on page 237 in [24]) that for every \( f \in \mathcal{W} \),

\[
\|e_f\|_\mathcal{F} \leq c\|f\|_W, \tag{2.4}
\]

where

\[
e_f = \min\{u \in \mathcal{P}_1 \cap \mathcal{F} : u \geq f \ \text{a.e.}\}
\]

(see [24, Theorem 6.2.6]). By [36, Proposition 3.6] (see also the reasoning in the proof of [36, Proposition 3.7]), for every \( u \in \mathcal{H} \) and \( \lambda > 0 \),

\[
\text{Cap}_\psi(\{|u| > \lambda\}) \leq \frac{1}{\lambda}\|u\|_{L^2} \cdot \|\psi\|_{L^2}.
\]

Combining the above with (2.4) we conclude that for every \( f \in \mathcal{W} \),

\[
\text{Cap}_\psi(\{|f| > \lambda\}) \leq \frac{c}{\lambda}\|f\|_W \cdot \|\psi\|_{L^2}. \tag{2.5}
\]
Let \( \{F_k\} \) be an increasing sequence of closed subsets of \( E^1 \). It is called a nest if \( \text{Cap}_\psi(F_k^c) \to 0 \) as \( k \to \infty \). We say that a function \( u \in B(E^1) \) is quasi-continuous (resp. quasi-l.s.c.) if there exists a nest \( \{F_k\} \) such that \( u|_{F_k} \) is continuous (resp. l.s.c.) for every \( k \geq 1 \).

A Borel measure \( \mu \) on \( E^1 \) is called smooth if it does not charge \( \mathcal{E} \)-exceptional sets and there exists a nest \( \{F_k\} \) such that \( |\mu|(F_k) < \infty \) for \( k \geq 1 \), where \( |\mu| \) is the variation of \( \mu \). By \( S \) we denote the set of all smooth measures on \( E^1 \). \( S_0 \) is the set of all measures of finite energy integrals, i.e. the subset of \( S \) consisting of all measures \( \mu \) having the property that there is \( K \geq 0 \) such that

\[
\int_{E^1} |\eta| d|\mu| \leq K \|\eta\|_W, \quad \eta \in W.
\]

Let us remark that each \( \eta \in W \) possesses a quasi-continuous version, so that the integral on the left-hand side of the above inequality is well defined.

For a given Borel measure \( \mu \) on \( E^1 \) and a Borel measurable function \( f \) on \( E^1 \) let \( f \cdot \mu \) denote the Borel measure on \( E^1 \) given by the formula

\[
(f \cdot \mu)(B) = \int_B f \, d\mu, \quad B \in B(E^1).
\]

We will also use the notation

\[
\langle f, \mu \rangle = \int_{E^1} f \, d\mu.
\]

Since \( \text{Cap}_\psi \) is strongly subadditive (see [36]), using and (2.5) and repeating the proofs of Lemmas 2.2.8, 2.2.9 in [11] (it is enough to replace the capacity appearing there by \( \text{Cap}_\psi \) one can show that for every \( \mu \in S \) there exists a nest \( \{F_k\} \) such that \( 1_{F_k} \cdot \mu \in S_0 \).

Let us recall that for every \( \mu \in S_0 \) and \( \alpha > \alpha_0 \) there exists unique \( U_\alpha \mu, \hat{U}_\alpha \mu \in \mathcal{F} \) such that

\[
\mathcal{E}_\alpha(U_\alpha \mu, \eta) = \mathcal{E}_\alpha(\eta, \hat{U}_\alpha \mu) = \int_{E^1} \eta \, d\mu
\]

for every \( \eta \in \mathcal{W} \).

It is known (see [21, 24]) that with a regular Dirichlet form \((B, F)\) one can associate a Hunt process \( \mathcal{M}^0 = (\{X_t, t \geq 0\}, \{P_x, x \in E \cup \{\Delta\}\}, \mathcal{F}^0, \{\theta_t^0, t \geq 0\}, \zeta^0) \) such that for every \( f \in B_b(E) \) and \( \alpha > 0 \) the function

\[
(R^0_\alpha f)(x) = E_x \int_0^\infty e^{-\alpha t} f(X_t) \, dt, \quad x \in E
\]

is a \( B \)-quasi-continuous \( m \)-version of \( G^0_\alpha \). It is also known (see [24, 36]) that with the time-dependent form \( \mathcal{E} \) defined by (2.2) one can associated a Hunt process \( \mathcal{M} = (\{X_t, t \geq 0\}, \{P_z, z \in E^1 \cup \{\Delta\}\}, \mathcal{F}, \{\theta_t, t \geq 0\}, \zeta) \) such that for every \( f \in B_b(E^1) \) and \( \alpha > 0 \) the function

\[
(R_\alpha f)(z) = E_z \int_0^\infty e^{-\alpha t} f(X_t) \, dt, \quad z \in E^1
\]

is an \( \mathcal{E} \)-quasi-continuous \( m_1 \)-version of \( G_\alpha f \). Moreover,

\[
X_t = (\tau(t), X_{\tau(t)}), \quad t \geq 0,
\]

\[
\mathcal{X}_t = (\tau(t), X_{\mathcal{\tau}(t)}), \quad t \geq 0,
\]
where \( \tau(t) \) is the uniform motion to the right, i.e. \( \tau(t) = \tau(0) + t, \tau(0) = s, P_z\text{-a.s.} \) for \( z = (s,x) \), and for each \( s \in \mathbb{R} \) the process \( M^{(s)} = (\{X_t^{(s)}, t \geq 0\}, \{P_s,x, x \in E\}, F^{(s)} = \{\mathcal{F}_{s+t}, t \geq 0\}) \) is a Hunt process associated with the form \((B^{(s)}, F)\).

A real valued \( F \) adapted process \( A \) is called an additive functional (AF) of \( M \) if there exists a set \( \Lambda \subset \Omega \) (called defining set) and an \( E \)-exceptional set \( N \subset E^1 \) such that \( P_t(\Lambda) = 1, z \in E_1 \setminus N, \theta_t \Lambda \subset \Lambda, t \geq 0 \) and for every \( \omega \in \Lambda \),

(a) \( [0, \infty) \ni t \mapsto \Lambda_t(\omega) \) is càdlàg,

(b) \( \Lambda_0(\omega) = 0, |\Lambda_t(\omega)| < \infty, t \in [0, \zeta), \Lambda_t(\omega) = \Lambda_{\zeta}(\omega), t \geq \zeta(\omega), \)

(c) \( \Lambda_{s+t}(\omega) = \Lambda_t(\omega) + \Lambda_s(\theta_t \omega), s, t \geq 0. \)

An AF \( A \) of \( M \) is called natural (NAF) if \( A \) is a continuous process. Finally, an AF \( A \) of \( M \) is called positive (PAF) if \( A_t \) is non-negative for every \( t \geq 0 \).

Let \( \mu \) be a non-negative Borel measure on \( E^1 \) and \( A \) be a PAF of \( M \). We say that \( \mu \) and \( A \) are in the Revuz correspondence if for every \( m_1 \)-integrable \( \alpha \)-co-excessive function \( h \) and every \( f \in \mathcal{B}_0^\infty(E_1), \)

\[
\int_{E^1} f(z) h(z) \mu(dz) = \lim_{t \to 0} \frac{1}{t} E_{h,m_1}(f \cdot A)_t = \lim_{\beta \to \infty} \beta(h, U_{\beta}^A f)_{L^2},
\]

where

\[
(f \cdot A)_t = \int_0^t f(X_s) dA_s,
\]

\[
(U_{\beta}^A f)(z) = E_{\zeta} \int_0^\infty e^{-\beta t} f(X_t) dA_t.
\]

By [24, Theorem 6.4.7], for every \( \mu \in S_0 \) there exists a unique NAF \( A \) of \( M \) in the Revuz correspondence with \( \mu \). Since each measure \( \mu \in S \) may be approximated by measures in \( S_0 \), repeating step by step the proof of [24, Theorem 4.1.16] one can show that for every \( \mu \in S \) there exists a unique NAF \( A \) of \( M \) in the Revuz correspondence with \( \mu \). We will denote it by \( A^\mu \).

### 3 Feynman-Kac functionals

In this section we prove basic regularity results for \( u \) defined by (1.1). We begin with continuity properties. Then we prove that \( u \) is the usual weak solution to (1.2) if \( \psi \in L^2(E; m) \) and \( \mu \in S_0(E_{0,T}) \). In the last part we derive energy estimates for \( u \) in case \( \psi \in L^1(E; m), \mu \in M_{0,0}(E_{0,T}) \).

#### 3.1 General continuity properties

Let \( E_T = (-\infty, T] \times E, E_{0,T} = (0, T] \times E \). By \( \hat{C}(E^1) \) (resp. \( \hat{C}(E_T), \hat{C}(E_{0,T}) \)) we denote the set of Borel measurable functions \( u \) on \( E^1 \) (resp. \( E_T, E_{0,T} \)) such that for \( m_1 \)-a.e. \( z \in E^1 \) (resp. \( E_T, E_{0,T} \)) the process \( t \mapsto u(X_t) \) is right continuous on \( [0, \zeta) \) (resp. \( [0, \zeta_T) \), where \( \zeta_T = \zeta \wedge (T - \tau(0)) \)) and the process \( t \mapsto u(X_{t-}) \) is left continuous on \( (0, \zeta) \) (resp. \( (0, \zeta_T) \)) \( P_z\text{-a.s.} \).

In [21] it is proved that if \( B \) is a quasi-regular Dirichlet form and \( u \in \hat{C}(E) \) then \( u \) is quasi-continuous. From this it follows that for every finely open \( U \subset E \), if \( u \in \hat{C}(U) \) then \( u \) is quasi-continuous on \( U \) (to see this it is enough to consider the part of the form \( B \) on \( U \), which is also quasi-regular). If \( U \) is not finely open then in general the
last implication does not hold. In the following lemma we show that it is true, however, if \( U = E_T \).

**Lemma 3.1.** Assume that \( f \in \hat{C}(E_T) \). Then \( f \) is quasi-continuous on \( E_T \).

**Proof.** We may assume that \( f \geq 0 \). Let us extend \( f \) by zero to the whole \( E^1 \). As in the proof of [21, Lemma V.2.6], with [21, Proposition V.1.6] replaced by [36, Proposition IV.3.4], we show that for every open \( B \subset E^1 \),

\[
\text{Cap}_\psi(B) = E_\mu e^{-SB},
\]

where

\[
S_B = \inf\{t \geq 0, \bar{X}_t \cap B \neq \emptyset\}.
\]

We next repeat, with some obvious changes, arguments from the proof of [21, Lemma V.2.19] to show that (3.1) holds for every \( B \in \mathcal{B}(E^1) \). Since \( \mathcal{M} \) is special standard, \( S_B = \inf\{0 \leq t < \zeta; X_t \in A \lor X_{t-} \in A\} \land \zeta \). For \( f \in \mathcal{B}_0(E^1) \) set

\[
\|f\| = E_\mu \sup_{t \geq 0} e^{-t}(|f(X_t)| \lor |f(X_{t-})|),
\]

\[
\|f\|_T = E_{\mu,T} \sup_{t \geq 0} e^{-t}(|f(X_t)| \lor |f(X_{t-})|),
\]

where \( P_{\mu,T}(\cdot) = \int_{E_T} P_{\mu}(\cdot) \varphi(z)\,dz, \ P_{\mu}(\cdot) = \int_{E} P_{\mu}(\cdot) \varphi(z)\,dz \). Arguing as in the proof of [21, Lemma 5.23] (with the norm \( \| \cdot \|_T \) on \( \hat{C}(E_T) \)) we show that \( \overline{C_0(E_T)} = \hat{C}(E_T) \), where \( \overline{C_0(E_T)} \) is the closure of \( C_0(E_T) \) in \( \hat{C}(E_T) \) with respect to the norm \( \| \cdot \|_T \). Let \( \{f_n\} \subset C_0(E_T) \) be such that \( \|f - f_n\|_T \to 0 \) and \( \|f_{n+1} - f_n\|_T < 2^{-2n}, \ n \geq 1 \). For \( n, N \in \mathbb{N} \) set

\[
A_n = \{z \in E_T; |f_{n+1} - f_n| > 2^{-\nu}\}, \quad B_N = \bigcup_{n \geq N} A_n.
\]

By (3.1), for every \( N \in \mathbb{N} \),

\[
\text{Cap}_\psi(B_N) \leq \sum_{n \geq N} \text{Cap}_\psi(A_n) = \sum_{n \geq N} E_\mu e^{-SB} = \sum_{n \geq N} \|1_{A_n}\| = \sum_{n \geq N} \|1_{A_n}\|_T \\
\leq \sum_{n \geq N} 2^n \|f_{n+1} - f_n\|_T \leq 2^{-N+1}.
\]

By standard argument (see the proof of [21, Proposition 5.24]) we can now show that the function

\[
\tilde{f}(z) = \begin{cases}
\lim_{n \to \infty} f_n(z), & \text{if } z \in \bigcup_{N \in \mathbb{N}} (E_T \setminus B_N), \\
f(z), & \text{otherwise}
\end{cases}
\]

is quasi-continuous on \( E_T \) and \( \tilde{f} = f \) q.e. on \( E_T \). \( \square \)

Let \( S(E_T), S(E_{0,T}) \) denote the spaces of smooth measures with support in \( E_T, E_{0,T} \), respectively. By \( \mathcal{R} \) (resp. \( \mathcal{R}(E_T), \mathcal{R}(E_{0,T}) \)) we denote the space of all \( \mu \in S \) (resp. \( \mu \in S(E_T), \mu \in S(E_{0,T}) \)) such that \( E_\mu A^+_T < \infty \) for q.e. \( z \in E^1 \) (resp. \( E_T, E_{0,T} \)). Observe that if \( \mu \in S(E_T) \) then \( \mathcal{R} = \mathcal{R}(E_T) \).

We say that a Borel measurable function \( u \) is quasi-càdlàg on \( E^1 \) (resp. \( E_T, E_{0,T} \)) if for q.e. \( z \in E^1 \) (resp. \( E_T, E_{0,T} \)) the process \( t \mapsto u(X_t) \) is càdlàg on \([0, \zeta] \) (resp. \([0, \zeta_T] \)) \( P_z \)-a.s.
Proposition 3.2. Assume that $\delta_{\{T\}} \otimes \varphi \cdot m$, $\mu \in \mathcal{R}(E_{0,T})$ and $\varphi \geq 0$, $\mu \geq 0$. Let $u : E_T \to \mathbb{R}$ be defined as

$$u(z) = E_z 1_{\{\zeta > T - \tau(0)\}} \varphi(X_{T - \tau(0)}) + E_z \int_0^{\zeta_r} dA^\mu_r, \quad z \in E_T. \quad (3.2)$$

Then

(i) $u$ is quasi-l.s.c. and quasi-càdlàg, and if $u \in L^2(E_T; m_1)$ then $u \in D(-\infty, T; H)$. If moreover $A^\mu$ is continuous on $[0, \zeta_r]$ then $u$ is quasi-continuous on $E_T$, and if $u \in L^2(E_T; m_1)$ then $u \in C(-\infty, TH)$,

(ii) there exists a MAF $M$ such that

$$u(X_t) = 1_{\{\zeta > T - \tau(0)\}} \varphi(X_{T - \tau(0)}) + \int_t^{\zeta_r} dA^\mu_r - \int_t^{\zeta_r} dM_r, \quad t \in [0, \zeta_r] \quad (3.3)$$

$P_z$-a.s. for q.e. $z \in E^1$.

Proof. Let us consider the following additive functional

$$A_t = 1_{\{\tau(0) < T \leq \tau(0) + t\}} \varphi(X_{T - \tau(0)}), \quad t \geq 0. \quad (3.4)$$

Observe that $A = A^\nu$, where $\nu = \delta_{\{T\}} \otimes \varphi \cdot m$. Set $\delta = \nu + \mu$ and

$$w(z) = E_z A^\delta_\infty, \quad z \in E^1. \quad (3.5)$$

By the assumptions, $w(z) < \infty$ for a.e. $z \in E^1$. Using argument analogous to that in the proof of [15, Lemma 4.2] one can show that in fact $w(z) < \infty$ for q.e. $z \in E^1$. Observe that

$$w(z) = u(z), \quad z \in (-\infty, T) \times E. \quad (3.6)$$

Since for every $B \in \mathcal{B}(E)$, $\text{Cap}_\varphi\{\{T\} \times B\} = 0$ iff $m(B) = 0$, it follows from (3.6) that $u(z) < \infty$ for q.e. $z \in E_T$. Let $N = \{z \in E^1; u(z) = \infty\}$. We may assume that $N$ is properly exceptional. By the strong Markov property, for every $z \in E_T \setminus N$ and $\sigma \in T$ such that $0 \leq \sigma \leq \zeta_r$ we have

$$u(X_\sigma) = E_z (1_{\{\zeta > T - \tau(0)\}} \varphi(X_{T - \tau(0)})|\mathcal{F}_\sigma) + E_z (A^\mu_\infty|\mathcal{F}_\sigma) - A_\sigma.$$

By the section theorem it follows that $u(X)$ has the representation (3.3) with $M$ being a càdlàg version of the martingale

$$E_z (1_{\{\zeta > T - \tau(0)\}} \varphi(X_{T - \tau(0)}) + \int_0^{\zeta_r} dA^\mu_r|\mathcal{F}_t) - u(X_0).$$

This shows (ii) because by [11, Lemma A.3.5] one can choose such a version independently of $z$. From (3.3) we conclude that $u$ is quasi-càdlàg on $E_T$. Now let us assume additionally that $A^\mu$ is continuous on $[0, \zeta_r]$. Then from (3.3) we deduce that $u(X)_-$ is right-continuous on $[0, \zeta_r]$, $(u(X)_-)_-$ is left-continuous on $[0, \zeta_r]$ and $(u(X)_-)_- = u(X_{t-})$, $t \in (0, \zeta_r)$ (see the reasoning in the proof of Claim 2 in [21, Proposition IV.5.14]). Hence
Let $u \in \mathcal{C}(E_T)$, which when combined with Lemma 3.1 implies that $u$ is quasi-continuous on $E_T$. For $\alpha > 0$ put

$$
u_\alpha(z) = E_z 1_{\{z > -T(0)\}} \varphi(X_{T-\tau(0)}) + \alpha E_z \int_0^{\tau} e^{-\alpha t} u(X_t) dt.$$ 

By what has already been proved, $u_\alpha$ is quasi-continuous on $E_T$. By the strong Markov property,

$$u_\alpha(z) = E_z 1_{\{z > -T(0)\}} \varphi(X_{T-\tau(0)}) + E_z \int_0^{\tau} (1 - e^{-\alpha t}) dA_t^z,$$

which implies that $u_\alpha(z) \not\geq u(z)$, $z \in E_T$. Hence $u$ is quasi-l.s.c on $E_T$. If $\alpha > \alpha_0$ and $u \in L^2(E^1;m_1)$ then $w \in L^2(E^1,m_1)$ (since $w \leq u$) and $w_\alpha$ defined as

$$w_\alpha(z) = \alpha E_z \int_0^\infty e^{-\alpha t} w(X_t) dt, \quad z \in E^1$$

belongs to $W(E^1)$. Since $w_\alpha$ is quasi-continuous, $w_\alpha \in C(\mathbb{R};H)$. Moreover, since $w_\alpha = \alpha R_\alpha w$,

$$E_\alpha(w_\alpha, \eta) = \alpha(w, \eta), \quad \eta \in W(E^1)$$

or, equivalently,

$$E(w_\alpha, \eta) = \alpha(w - w_\alpha, \eta), \quad \eta \in W(E^1). \quad (3.7)$$

Let $t_1 < t_2$ and let $v$ be a measurable function on $E^1$. Put

$$v^\varepsilon(t, x) = \begin{cases} 
\frac{1}{\varepsilon} v(t_1 + \varepsilon, x)(t - t_1), & (t, x) \in [t_1, t_1 + \varepsilon] \times E, \\
v(t, x), & (t, x) \in [t_1, t_2] \times E, \\
-\frac{1}{\varepsilon} v(t_2, x)(t - t_2 - \varepsilon), & (t, x) \in [t_2, t_2 + \varepsilon] \times E, \\
0, & \text{otherwise}. 
\end{cases} \quad (3.8)$$

Then taking $w_\varepsilon^\varepsilon$ as a test function in (3.7) and letting $\varepsilon \to 0^+$ we get

$$\|w_\alpha(t_1)\|_{L^2}^2 - \|w_\alpha(t_2)\|_{L^2}^2 + 2 \int_{t_1}^{t_2} B(t)(w_\alpha(t), w_\alpha(t)) dt$$

$$= 2\alpha \int_{t_1}^{t_2} \|w(t) - w_\alpha(t)\|_{L^2}^2 dt, \quad (3.9)$$

whereas taking $\eta^\varepsilon$ with nonnegative $\eta \in F$ as a test function and letting $\varepsilon \to 0^+$ we get

$$(w_\alpha(t_1), \eta)_{L^2} - (w_\alpha(t_2), \eta)_{L^2} + \int_{t_1}^{t_2} B(t)(w_\alpha(t), \eta) dt$$

$$= \alpha \int_{t_1}^{t_2} (w(t) - w_\alpha(t), \eta)_{L^2} dt. \quad (3.10)$$

Write

$$x(t) = 2 \int_0^t B(s)(w(s), w(s)) ds, \quad x^n(t) = \int_0^t B(s)(w(s), \eta) ds,$$

$$x_\alpha(t) = 2 \int_0^t B(s)(w_\alpha(s), w_\alpha(s)) ds, \quad x^n_\alpha(t) = \int_0^t B(s)(w_\alpha(s), \eta) ds,$$
and
\[ y_\alpha(t) = \|w_\alpha(t)\|_{L^2}^2 - x_\alpha(t), \quad y(t) = \|w(t)\|_{L^2}^2 - x(t), \]
\[ y'^\alpha(t) = (w_\alpha(t), \eta)_{L^2} - x'^\alpha(t), \quad y'(t) = (w(t), \eta)_{L^2} - x'(t). \]

By what has already been proved, \( w_\alpha(z) \not\to w(z), z \in E^1 \). It is also known that \( w_\alpha \to w \in \mathcal{F} \) (see [24, Theorem 6.1.2]). Therefore from (3.9), (3.10) it may be concluded that
\[ y_\alpha(t) \to y(t), \quad y'^\alpha(t) \to y'(t), \quad t \in \mathbb{R}, \]
\[ x_\alpha(t) \to x(t), \quad x'^\alpha(t) \to x'(t), \quad t \in \mathbb{R}. \]

Moreover, \( y \) is nonincreasing and for every \( \eta \in F \) such that \( \eta \geq 0 \) the function \( y_\eta \) is nonincreasing. Since the sequences \( \{\|w_\alpha(t)\|_{L^2}^2\} \), \( \{(w_\alpha(t), \eta)_{L^2}\} \) are nondecreasing we get by [26] that the mappings \( t \mapsto \|w(t)\|_{L^2}^2, t \mapsto (w(t), \eta)_{L^2} \) are càdlàg on \( \mathbb{R} \). By the classical results they are also l.s.c. We now show that \( w \in D(\mathbb{R}, H) \). Let \( t_n \to t_0^+ \). Then
\[ \|w(t_n) - w(t_0)\|_{L^2}^2 = \|w(t_n)\|^2 + \|w(t_0)\|^2 - 2(w(t_n), w(t_0))_{L^2}. \]

Since \( t \to \|w(t)\|_{L^2}^2 \) is càdlàg,
\[ \limsup_{n \to \infty} \|w(t_n) - w(t_0)\|_{L^2}^2 = 2\|w(t_0)\|^2 - 2\liminf_{n \to \infty} (w(t_n), w(t_0))_{L^2}. \]

But the mapping \( t \to (w(t), w(s)) \) is l.s.c. Hence
\[ \limsup_{n \to \infty} \|w(t_1) - w(t_0)\|_{L^2}^2 \leq 0. \]

Let \( t_n \not\to t_0^- \). Since \( t \to \|w(t)\|_{L^2}^2 \) is locally bounded and \( t \to (w(t), \eta)_{L^2} \) is càdlàg, it follows that there exists \( v \in H \) not depending on the choice of the sequence \( \{t_n\} \) such that \( w(t_n) \to v \) weakly in \( H \). By [28] there exists an \( m_1 \)-version \( \tilde{w} \) of \( w \) such that the mapping \( \mathbb{R} \ni t \mapsto \tilde{w}(t) \in H \) is càglàd, i.e. left continuous with right limits. Without loss of generality we may assume that \( \tilde{w}(t_n) = w(t_n) \) m.a.e. for \( n \geq 1 \). Therefore \( \{w(t_n)\} \) is strongly convergent in \( H \) and of course \( w(t_n) \to v \) in \( H \). In particular, \( \|w(t_n)\|_{L^2} \to \|v\|_{L^2} \). Since \( t \mapsto \|w(t)\|_{L^2}^2 \) is càdlàg, there exists the limit \( \lim_{t \to t_0^-} \|w(t)\|_{L^2}^2 \) and obviously \( \lim_{t \to t_0^+} \|w(t)\|_{L^2}^2 \). Therefore \( \lim_{t \to t_0} \|w(t)\|_{L^2} \) is strongly in \( H \) Finally, since \( w(z) = u(z) \) for \( z \in (-\infty, T) \times E, u \in D(-\infty, T; H) \). \( \square \)

**Remark 3.3.** If in Lemma 3.1 and Proposition 3.2 we consider the form \( \mathcal{E} \) on \([0, \infty) \times E \) instead of the form \( \mathcal{E} \) on \( E^1 \), then their assertions remains valid if we replace \( E_T \) by \( E_{0,T} \), replace \( (-\infty, T] \) by \((0, T] \) and \( \mathcal{R} \) by \( \mathcal{R}(E_{0,T}) \).

### 3.2 Energy estimates: the case of finite energy integral measures

In the sequel \( f_a^b \) stands for \( \int_{(a,b]} \).

**Definition.** Let \( \varphi \in L^2(E; m) \) and \( \mu \in S_0(E_{0,T}) \). We say that a measurable function \( u : E_{0,T} \to \mathbb{R} \) is a weak solution of the Cauchy problem
\[ -\frac{\partial u}{\partial t} - L_t u = \mu, \quad u(T) = \varphi \quad (3.11) \]
if
(a) $u \in \mathcal{F}_{0,T}, u \in D(0, T; H)$,
(b) for every $t \in (0, T]$ and $\eta \in \mathcal{W}(E_{0,T})$,

\[ (u(t), \eta(t))_{L^2} + \int_t^T (u(s), \frac{\partial \eta}{\partial t}(s))_{L^2} \, ds + \int_t^T B(s)(u(s), \eta(s)) \, ds = (\varphi, \eta(T))_{L^2} + \int_t^T \int_E \eta(z) \, d\mu(z). \]

Proposition 3.4. There exists at most one weak solution of (3.11).

Proof. Without loss of generality we may assume that $\alpha_0 = 0$. Assume that $u_1, u_2$ are solutions of (3.11) and set $u = u_1 - u_2$. Then for every $\eta \in \mathcal{W}(E_{0,T})$ and $t \in [0, T]$,

\[ (u(t), \eta(t))_{L^2} + \int_t^T (u(s), \frac{\partial \eta}{\partial s}(s))_{L^2} \, ds + \int_t^T B(s)(u(s), \eta(s)) \, ds = 0. \]

From this we easily deduce that $u \in \mathcal{W}(E_{0,T})$. Replacing $\eta$ by $u$ in (3.12) we get

\[ \|u(t)\|_{L^2}^2 + 2 \int_t^T B(s)(u(s), u(s)) \, ds = 0, \]

which implies that $u = 0$ a.e. \qed

Theorem 3.5. Assume that $\varphi \in L^2(E; m)$ and $\mu \in S_0(E_{0,T})$. Then $u : E_{0,T} \rightarrow \mathbb{R}$ defined by (3.2) is a weak solution of the Cauchy problem (3.11).

Proof. Let $\nu = \delta_{\{T\}} \otimes \varphi \cdot m$ and $\eta \in \mathcal{W}$. Then

\[ \nu(\eta) = \int_{E^1} \eta(z) \nu(dz) = \int_E \eta(T, x)\varphi(x) \, dx \leq \|\eta(T)\|_{L^2} \cdot \|\varphi\|_{L^2} \leq \sup_{t \geq 0} \|\eta(t)\|_{L^2} \|\varphi\|_{L^2} \leq \|\eta\|_{W} \cdot \|\varphi\|_{L^2}. \]

Hence $\nu \in S_0$. Let $\alpha > \alpha_0$ and let $\mu^\alpha = e^{-\alpha(T-\cdot)} \cdot \mu$, $\delta^\alpha = \nu + \mu^\alpha$. Observe that

\[ A_t^{\mu^\alpha} = \int_0^t e^{-\alpha(T-r)} \, dA_r^{\mu}, \quad t \geq 0. \]

Put

\[ w_\alpha(z) = E_z \int_0^{\infty} e^{-\alpha t} \, dA_t^{\delta^\alpha}. \]

It is known that $w_\alpha = U_\alpha \delta^\alpha$, i.e. $E_\alpha(w_\alpha, \eta) = \langle \delta^\alpha, \eta \rangle, \eta \in \mathcal{W}$. Hence

\[ (w_\alpha, \frac{\partial \eta}{\partial t})_{L^2} + B(w_\alpha, \eta) = \langle \delta^\alpha, \eta \rangle - \alpha(w_\alpha, \eta)_{L^2}, \quad \eta \in \mathcal{W}. \]

Therefore for any $t \in [0, T]$ and $\varepsilon > 0$ we have

\[ (w_\alpha, \frac{\partial \eta^\varepsilon}{\partial t}) + B(w_\alpha, \eta^\varepsilon) = \langle \delta^\alpha, \eta^\varepsilon \rangle - \alpha(w_\alpha, \eta^\varepsilon), \quad \eta \in \mathcal{W}, \]

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where the approximation $\eta^\varepsilon$ is defined by (3.8) with $t_1 = t, t_2 = T$. Letting $\varepsilon \to 0^+$ and using Proposition 3.2 we get
\[
(w_\alpha(t), \eta(t))_{L^2} + \int_t^T (w_\alpha(s), \frac{\partial \eta}{\partial s}(s))_{L^2} + \int_t^T B^{(s)}(w_\alpha(s), \eta(s)) \, ds \\
= \int_t^T \eta(z) \delta^\alpha(dz) - \int_t^T \alpha(w_\alpha(s), \eta(s))_{L^2} \, ds
\]
for every $\eta \in \mathcal{W}(E_{0,T})$. The second term on the left-hand side of the above equation is equal to
\[
-\alpha \int_t^T (u_\alpha(s), e^{\alpha(T-s)} \eta(s)) \, ds + \int_t^T (u_\alpha(s), e^{\alpha(T-s)} \frac{\partial \eta}{\partial s}) \, ds.
\]
Putting $\tilde{u}(t) = e^{\alpha(T-t)} u_\alpha(t)$ we conclude that
\[
(w(\tilde{u}(s), \eta(t))_{L^2} + \int_t^T (\tilde{u}(s), \frac{\partial \eta}{\partial s}(s))_{L^2} \, ds + \int_t^T B^{(s)}(\tilde{u}(s), \eta(s)) \, ds \\
= (\varphi, \eta(T))_{L^2} + \int_t^T \eta(z) \, d\mu(z)
\]
for every $\eta \in \mathcal{W}(E_{0,T})$. Let us put $w(t) = e^{\alpha(T-t)} w_\alpha(t), t \in [0,T]$. Then for $z = (s, x), s \in [0,T]$ we have
\[
w(z) = e^{\alpha(T-s)} E_z \int_0^\infty e^{-\alpha t} \, dA_\delta t = e^{\alpha(T-s)} E_z \int_0^\infty e^{-\alpha t} e^{-\alpha(T-t)} \, dA_\delta t \\
= e^{\alpha(T-s)} E_z \int_0^\infty e^{-\alpha t} e^{-\alpha(T-t)} \, dA_\delta t = E_z A_\infty.
\]
Since $u(z) = w(z)$ for $z \in [0,T] \times E$ and $u(T) = \varphi, u(z) = \tilde{u}(z), z \in E_{0,T}$. \qed

### 3.3 Energy estimates: the case of bounded smooth measures

Let $\mathcal{W}_T = \{ u \in \mathcal{W}(E_{0,T}); u(T) = 0 \}, \mathcal{W}_0 = \{ u \in \mathcal{W}(E_{0,T}); u(0) = 0 \}$ and let
\[
\mathcal{E}^{0,T}(u, v) = \left\{ \begin{array}{ll}
\int_0^T \left( -\frac{\partial u}{\partial t}, v \right) \, dt + \int_0^T B^{(t)}(u(t),v(t)) \, dt, & (u, v) \in \mathcal{W}_T \times \mathcal{F}_{0,T}, \\
\int_0^T \left( u, \frac{\partial v}{\partial t} \right) \, dt + \int_0^T B^{(t)}(u(t),v(t)) \, dt, & (u, v) \in \mathcal{F}_{0,T} \times \mathcal{W}_0.
\end{array} \right.
\]

It is known (see [36, Example I.4.9(iii)]) that $\mathcal{E}^{0,T}$ is a generalized semi-Dirichlet form and
\[
\mathcal{L} = -\frac{\partial}{\partial t} - L_t, \quad D(\mathcal{L}) = \{ u \in \mathcal{W}_T; \mathcal{L} u \in \mathcal{H}_{0,T} \}
\]
is the operator associated with $\mathcal{E}^{0,T}$. Note that the adjoint operator $\hat{\mathcal{L}}$ to $\mathcal{L}$ is given by
\[
\hat{\mathcal{L}} = \frac{\partial}{\partial t} - \hat{L}_t, \quad D(\hat{\mathcal{L}}) = \{ u \in \mathcal{W}_0; \hat{\mathcal{L}} u \in \mathcal{H}_{0,T} \}.
\]

Let $\{ T^{0,T}_t, t > 0 \}$ (resp. $\{ T^{0,T}_t, t \geq 0 \}$) be a $C_0$-semigroup on $\mathcal{H}_{0,T}$ associated with the operator $\mathcal{L}$ (resp. $\hat{\mathcal{L}}$). By (2.1), $\| T^{0,T}_t \|_{L^2 \to L^2} \leq e^{\alpha t}, \| T^{0,T}_t \|_{L^2 \to L^2} \leq e^{\alpha t}, t \geq 0,$ and the corresponding resolvents are given by
\[
G^{0,T}_t f = \int_0^\infty e^{-\alpha t} T^{0,T}_t f \, dt, \quad \hat{G}^{0,T}_t f = \int_0^\infty e^{-\alpha t} \hat{T}^{0,T}_t f \, dt.
\] (3.13)
The Hunt process $M_{0,T}^0$ properly associated with the form $\mathcal{E}_{0,T}$ is the process $M$ with lifetime $\zeta$. Therefore $T_{t}^{0,T} = 0$, $\hat{T}^{0,T} = 0$, $t \geq T$. It follows that $G_{0,T}^{0,T}, G_{0,T}^{0}$ are well defined as operators on $L^2(E_0,T;m_1)$ for every $\alpha \geq 0$. It is standard that $T^{0,T}$ (resp. $\hat{T}^{0,T}$) can be extended to $L^{\infty}(E_0,T;m_1) \cup L^2(E_0,T;m_1)$ (resp. $L^1(E_0,T;m_1) \cup L^2(E_0,T;m_1)$) and that the extension of $T_{t}^{0,T}$ (resp. $\hat{T}_{t}^{0,T}$) is a contraction on $L^{\infty}(E_0,T;m_1)$ (resp. $L^1(E_0,T;m_1)$). Therefore for every $\alpha \geq 0$ we can extend $G_{0,T}^{0}$ (resp. $\hat{G}_{0,T}^{0}$) defined by (3.13) to an operator on $L^{\infty}(E_0,T;m_1)$ (resp. $L^1(E_0,T;m_1)$). For $\mu \in S$ and $\alpha \geq 0$ we define

$$R_{0,T}^{0,T} \mu(z) = E_z \int_{0}^{\zeta} e^{-\alpha t} dA_{t}, \quad \hat{R}_{0,T}^{0,T} \mu(z) = \hat{E}_z \int_{0}^{\zeta} e^{-\alpha t} d\hat{A}_{t}, \quad z \in E^1,$$

where $\zeta = \zeta \land \tau(0)$, $\hat{A}^{\mu}$ is the dual additive functional associated with $\mu$ (see [24]). It is clear that for every $f \in L^{\infty}(E_0,T;m_1) \cup L^2(E_0,T;m_1)$ and $g \in L^1(E_0,T;m_1) \cup L^2(E_0,T;m_1)$,

$$R_{0,T}^{0,T} f = G_{0,T}^{0,T} f, \quad \hat{R}_{0,T}^{0,T} g = \hat{G}_{0,T}^{0,T} g$$

(3.14)

$m_1$-a.e. for every $\alpha \geq 0$, and for every non-negative $\mu, \nu \in S$,

$$\langle R_{0,T}^{0,T} \mu, \nu \rangle = \langle \mu, \hat{R}_{0,T}^{0,T} \nu \rangle.$$ 

for $\alpha \geq 0$.

By (3.14) and Proposition 3.2, for every $f \in L^{\infty}(E_0,T;m_1) \cup L^2(E_0,T;m_1)$ and $g \in L^1(E_0,T;m_1) \cup L^2(E_0,T;m_1)$ the resolvents $G_{0,T}^{0,T} f, \hat{G}_{0,T}^{0,T} g$ have quasi-continuous $m_1$-versions. In what follows we adopt the convention that they are already quasi-continuous. If $\alpha = 0$ then we write $R_{0,T}^{0,T}, R_{0,T}^{0,T}, G_{0,T}^{0,T}, \hat{G}_{0,T}^{0,T}$ instead of $R_{0,T}^{0,T}, R_{0,T}^{0,T}, G_{0,T}^{0,T}, \hat{G}_{0,T}^{0,T}$.

### Proposition 3.6

**Assume that $\mathcal{E}$ satisfies the dual condition $(\Delta)$.** Then

$$\mathcal{M}_{0,b}(E_0,T) \subset \mathcal{R}(E_0,T).$$

(3.15)

**Proof.** Let $\{\eta_n\}$ be the sequence of the definition of condition $(\Delta)$. Since

$$(G_{0,T}^{0,T} \mu, \eta_n)_{L^2} = \langle \mu, \hat{G}_{0,T}^{0,T} \eta_n \rangle \leq \| \mu \| \cdot \| \hat{G}_{0,T}^{0,T} \eta_n \|_{\infty},$$

(3.16)

it follows that $R_{0,T}^{0,T} \mu < \infty m_1$-a.e. on $E_0,T$. \qed

**Remark 3.7.** If for some $\gamma \geq 0$ the form $\mathcal{E}_{\gamma}$ has the dual Markov property then the duality condition $(\Delta)$ is satisfied. Indeed, by [24, Theorem 1.1.5], $\alpha \hat{G}_{0,T}^{0}$ is Markovian for every $\alpha > 0$, so $(\Delta)$ is satisfied with $\eta \equiv 1$, $\alpha = 1$ and $F_n = \Phi_n$, $n \geq 1$.

The following example shows that the condition that $\mathcal{E}_{\gamma}$ has the dual Markov property for some $\gamma \geq 0$ is not necessary for $(\Delta)$ to hold.

### Example 3.8

Let $D \subset \mathbb{R}^d$, $d \geq 3$, be an open bounded set with smooth boundary, and let

$$\mathcal{L} = -\frac{\partial}{\partial t} - L_t = -\frac{\partial}{\partial t} - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}(a_{ij} \frac{\partial}{\partial x_i}) + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i},$$
where \(a_{ij}, b_i : [0, T] \times D \to \mathbb{R}\) are measurable functions such that \(b_i\) is bounded, \(a_{ij} = a_{ji}\) and
\[
\lambda^{-1} |\xi|^{2} \leq \sum_{i,j=1}^{d} a_{ij} \xi_i \xi_j \leq \lambda |\xi|^{2}, \quad \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^{d}
\]
for some \(\lambda \geq 1\). Then
\[
G^{0,T} f(z) = \mathbb{E}_{z} \int_{0}^{\infty} f(X_{t}) \, dt = \mathbb{E}_{s,x} \int_{0}^{(T - \tau(0)) \wedge \zeta} f(s + t, X_{s+t}) \, dt
\]
\[
= \mathbb{E}_{s,x} \int_{0}^{T-s} f(s + t, X^{D}_{s+t}) \, dt = \int_{0}^{T} \int_{D} p_{D}(s + t, x, t) f(s + t, y) \, dt \, dy
\]
\[
= \int_{0}^{T} \int_{D} p_{D}(s, x, t, y) f(t, y) \, dt \, dy,
\]
where \(X^{D}\) denotes the process \(X\) killed upon leaving \(D\) and \(p_{D}\) is the transition density of \(X^{D}\). We also have
\[
\tilde{G}^{0,T} f(s, x) = \int_{0}^{s} \int_{D} p_{D}(t, y, s, x) f(t, y) \, dt \, dy.
\]
Let \(f \equiv 1\). Then by Aronson’s estimates,
\[
(\tilde{G}^{0,T} 1)(s, x) \leq c_{1} \int_{0}^{s} \int_{D} (s - t)^{-\frac{d}{2}} \exp \left( \frac{-c_{2} |y - x|^{2}}{2(s - t)} \right) \, dt \, dy \leq c' T.
\]
Thus condition (\(\Delta\)) is satisfied. On the other hand it follows from the formula preceding [24, Corollary 1.5.4] (page 33) that if we take \(b(x) = \sqrt{|x|}\) and \(D = B(0, 1)\) then there is no \(\gamma \in \mathbb{R}\) such that \(\mathcal{E}_{\gamma}\) has the dual Markov property.

**Corollary 3.9.** If for some \(\gamma \geq 0\) the form \(\mathcal{E}_{\gamma}\) has the dual Markov property then (3.15) is satisfied.

**Proof.** Follows from Proposition 3.6 and Remark 3.7. \(\square\)

**Theorem 3.10.** Assume that \(\varphi \in L^{1}(E; m), \mu \in \mathcal{M}_{0,b}(E_{0,T})\) and there exists \(\gamma \geq \alpha_{0}\) such that \(\mathcal{E}_{\gamma}\) has the dual Markov property. Let \(u\) be defined by (3.2). Then \(u \in L^{1}(E_{0,T}; m_{1}), T_{k}(u) \in \mathcal{F}_{0,T}\) and
\[
\int_{0}^{T} B^{(t)}_{\gamma}(T_{k}(u)(t), T_{k}(u)(t)) \, dt \leq k(\|\mu\| + \|\varphi\|_{L^{1}} + \gamma \|u\|_{L^{1}}) \quad (3.17)
\]
for every \(k \geq 0\), and moreover, for every \(\alpha > 0\),
\[
\|u\|_{L^{1}} \leq \alpha^{-1} e^{T (\alpha + \gamma)} (\|\varphi\|_{L^{1}} + \|\mu\|). \quad (3.18)
\]

**Proof.** Since \(\mathcal{E}_{\gamma}\) has the dual Markov property, \(\alpha \tilde{T}^{0,T}_{\alpha + \gamma} 1 \leq 1\) for \(\alpha > 0\), which implies that \(\tilde{T}^{0,T}_{\alpha + \gamma} 1 \leq \alpha^{-1} e^{T (\alpha + \gamma)}\). Since \(u = R^{0,T}_{\nu} m_{1}\)-a.e. on \(E_{0,T}\), where \(\nu = \delta_{T} \otimes \varphi \cdot m + \mu\), we therefore have
\[
\|u\|_{L^{1}} = (R^{0,T}_{\nu}, 1)_{L^{2}} = (\nu, \tilde{R}^{0,T}_{\nu} 1) \leq \alpha^{-1} e^{T (\alpha + \gamma)} (\|\varphi\|_{L^{1}} + \|\mu\|),
\]

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which proves (3.18). To prove (3.17), let us consider a nest \( \{F_n\} \) such that \( \varphi_n = \mathbf{1}_{F_n}(T, \cdot) \varphi \in L^2(E; m) \), \( \mu_n = \mathbf{1}_{F_n} \cdot \mu \in S_0 \). Write

\[
u_n(z) = E_\zeta \mathbf{1}_{(\zeta > T - \tau(0))} \varphi_n(X_{T - \tau(0)}) + E_\zeta \int_0^{\zeta} dA_{\mu_n}^\nu.
\]

By Theorem 3.5, \( u_n \in \mathcal{F} \cap D(0, T; H) \) and for every \( \eta \in \mathcal{W}(E_{0,T}) \),

\[
(u_n(0), \eta(0))_{L^2} + \int_0^T (u_n(s), \frac{\partial \eta}{\partial s}(s))_{L^2} ds + \int_0^T \mathcal{B}^{(s)}(u_n(s), \eta(s)) ds = \langle \varphi_n, \eta(T) \rangle_{L^2} + \int_{E_{0,T}} \eta d\mu_n.
\] (3.19)

Given \( w \in \mathcal{F}_{0,T} \cap D(0, T; H) \) set

\[
[w]_m(t) = \int_{-T}^t m e^{-m(t-s)} w(s) ds.
\]

It is well known that \( [w]_m \to w \) in \( \mathcal{F}_{0,T} \) and \( [w]_m(t) \to w(t) \) in \( H \) for every \( t \in [0, T] \) (in the proof of the last property it is usually assumed that \( w \in C(0, T; H) \) but in fact it is enough to know that \( w \in D(0, T; H) \)). Moreover, \( [w]_m \in \mathcal{W}(E_{0,T}) \),

\[
([w]_m)_2 = m(w - [w]_m)
\] (3.20)

and \( [w]_m \leq w, [w]_m \leq [w]_{m+1}, m \geq 1 \). Let us fix \( n \in \mathbb{N} \) for a moment and put \( v = u_n \).

Taking \( \eta = [T_k(v)]_m \) as a test function in (3.19) we obtain

\[
\langle v(0), [T_k(v)]_m(0) \rangle_{L^2} + \int_0^T \langle v(s), ([T_k(v)]_m)_s(s) \rangle_{L^2} ds\]

\[
+ \int_0^T \mathcal{B}^{(s)}(v(s), ([T_k(v)]_m)_s(s)) ds = \langle \varphi_n, [T_k(v)]_m(T) \rangle_{L^2} + \int_{E_{0,T}} [T_k(v)]_m(z) \mu_n(dz)
\] (3.21)

and, by (3.20),

\[
\int_0^T \langle v(s), ([T_k(v)]_m)_s(s) \rangle_{L^2} ds = \int_0^T \langle [T_k(v)]_m(s), ([T_k(v)]_m)_s(s) \rangle_{L^2} ds
\]

\[
+ \int_0^T \langle v(s) - [T_k(v)]_m(s), ([T_k(v)]_m)_s(s) \rangle_{L^2} ds
\]

\[
= I_1(m) + \int_0^T \langle v(s) - [T_k(v)]_m(s), T_k(v)(s) - [T_k(v)]_m(s) \rangle_{L^2} ds
\]

\[
= I_1(m) + I_2(m).
\] (3.22)

Let us denote by \( F \) the integrand in the integral \( I_2(m) \). Observe that

\[
-k \leq [T_k(v)]_m(z) \leq k, \quad z \in E_{0,T}.
\] (3.23)
If \(-k \leq v(z) \leq k\) then \(F(z) \geq 0\). If \(v(z) > k\) then by (3.23),

\[
F(z) = (v(z) - [T_k(v)]_m(z)) - k - [T_k(v)]_m(z) \geq 0.
\]

Similarly, \(F(z) \geq 0\) if \(v(z) < -k\). Therefore \(I_2(m) \geq 0\) for \(m \in \mathbb{N}\). Moreover,

\[
I_1(m) = \frac{1}{2} \| [T_k(v)]_m(T) \|_{L^2}^2 - \frac{1}{2} \| [T_k(v)]_m(0) \|_{L^2}^2.
\]

By the above equality, (3.21)–(3.23) and the convergence properties of the sequence \([T_k(v)]_m\) we get

\[
\int_0^T B^{(s)}(v(s), T_k(v(s))) \, ds \leq k(\| \varphi \|_{L^1} + \| \mu \|).
\]

By the above inequality and the assumptions,

\[
\int_0^T B^{(s)}(T_k(u_n(s)), T_k(u_n(s))) \, ds \leq k(\| \varphi \|_{L^1} + \| \mu \|) + k_\gamma \| u \|_{L^1}.
\]

Letting \(n \to \infty\) we get (3.17). \(\Box\)

**Proposition 3.11.** Assume that \(\mu, \nu\) are non-negative smooth measures on \(E_{0,T}\) and

\[
E_z \int_{0}^{\xi^\tau} \, dA_t^\mu \leq E_z \int_{0}^{\xi^\tau} \, dA_t^\nu
\]

for q.e. \(z \in E_{0,T}\). If \(E\) has the dual Markov property then \(\| \mu \|_{TV} \leq \| \nu \|_{TV} \).

**Proof.** It is well known that for every measurable \(h\) on \(E_{0,T}\) such that \(\eta \geq h\) \(m_1\)-a.e. for some \(\eta \in \mathcal{W}(E_{0,T})\) there exists a (unique) minimal solution \(u \in \mathcal{F}_{0,T}\) of the obstacle problem

\[
\hat{L}u = 0 \text{ on } \{u > h\}, \quad \hat{L}u \geq 0, \quad u(0) = 0, \quad u \geq h
\]

(see [22, 28]). By Riesz’s theorem there exists a Radon measure \(\delta\) on \(E_{0,T}\) such that \(\hat{L}u = \delta\) (in \(C_0(E_{0,T})\)). From this one can easily deduce that \(\delta \in \mathcal{S}_0(E_{0,T})\). Therefore \(u = \hat{R}^{0,T}\delta\). Since \(E\) is regular, there exists a sequence \(\{E_n\}\) of compact subsets of \(E\) with the property that for each \(n \in \mathbb{N}\) there exists \(\eta \in \mathcal{W}(E_{0,T})\) such that \(\eta_n \geq h_n \equiv 1_{[0,T] \times E_n}\). Therefore for every \(n \in \mathbb{N}\) there exists a solution \(u_n\) of the obstacle problem (3.24) with the barrier \(h_n\). Let \(\delta_n \in \mathcal{S}_0(E_{0,T})\) be such that \(u_n = \hat{R}^{0,T}\delta_n\). Since \(u_n\) is the smallest potential majorizing \(h_n\) such that \(u_n(0) = 0, u_n \leq u_n \wedge 1\). Therefore \(u_n = 1\) q.e. on \([0,T] \times E_n\), which implies that \(u_n \nRightarrow 1\) q.e. on \((0,T] \times E\). By the assumptions, \(\hat{R}^{0,T}\mu \leq \hat{R}^{0,T}\nu\). Hence

\[
\| \mu \|_{TV} = \lim_{n \to \infty} \langle \mu, u_n \rangle = \lim_{n \to \infty} \langle \mu, \hat{R}^{0,T}\delta_n \rangle = \lim_{n \to \infty} \langle \hat{R}^{0,T}\mu, \delta_n \rangle
\]

\[
\leq \lim_{n \to \infty} \langle \hat{R}^{0,T}\nu, \delta_n \rangle = \lim_{n \to \infty} \langle \nu, \hat{R}^{0,T}\delta_n \rangle = \| \nu \|_{TV},
\]

which proves the proposition. \(\Box\)
4 Linear equations with measure data

In this section we consider linear problems of the form (1.2) under the assumption that \( E \) satisfies the duality condition. The case of general forms will be considered in a more general setting of semilinear equations in the next section.

**Definition.** Let \( \varphi \in L^1(E;m) \), \( \mu \in M_{0,b}(E_0,T) \) and assume that \( E \) satisfies the duality condition (\( \Delta \)). We say that a measurable function \( u : E_0,T \to \mathbb{R} \) is a solution of (3.11) in the sense of duality if \( u \in L^1(E_0,T;\eta \cdot m_1) \) and

\[
(u,\eta)_{L^2} = (\varphi, \tilde{G}^{0,T} \eta(T))_{L^2} + \int_{E_0,T} \tilde{G}^{0,T} \eta \, d\mu
\]

(4.1)

for every non-negative \( \eta \in L^2(E_0,T;m_1) \) such that \( \tilde{G}^{0,T} \eta \) is bounded.

**Proposition 4.1.** Let \( \mu, \varphi, E \) be as in the above definition and let \( u : E_0,T \to \mathbb{R} \) be defined by (3.2). Then \( u \) is a unique solution of (3.11) in the sense of duality.

**Proof.** Uniqueness easily follows from condition (\( \Delta \)). Let \( \{F_n\} \) be a generalized nest such that \( \mu_n = 1_{F_n} \cdot \mu \in S_0 \) and \( \varphi_n = 1_{F_n} \varphi \in L^2(E;m) \). Set

\[
u_n(z) = E_z 1_{\{z_t > \tau(0)\}} \varphi_n(X_{T-\tau(0)}) + E_z \int_0^{\zeta_r} dA^\mu_t.
\]

By Theorem 3.5, \( u_n \in \mathcal{F}_0,T \cap D(0,T;H) \) and for every \( \psi \in \mathcal{W}(E_0,T), \)

\[
(u_n(0), \psi(0))_{L^2} + \int_0^T (u_n(t), \frac{\partial \psi}{\partial t}(t))_{L^2} \, dt + \int_0^T B(t)(u_n(t), \psi(t)) \, dt = (\varphi_n, \psi(T)) + \int_{E_0,T} \psi(z) \, d\mu_n(z).
\]

Taking \( \psi = \tilde{G}^{0,T} \eta \) with non-negative \( \eta \in L^2(E_0,T;m_1) \) such that \( \tilde{G}^{0,T} \eta \) is bounded as a test function we obtain

\[
(u_n,\eta)_{L^2} = (\varphi_n, \tilde{G}^{0,T} \eta(T))_{L^2} + \int_{E_0,T} \tilde{G}^{0,T} \eta(z) \, d\mu_n(z).
\]

(4.2)

Observe that \( |u_n| \leq v, u_n \to u \text{ m}_1\text{a.e.}, \) where \( v(z) = G^{0,T} \nu \) and \( \nu = \delta_T \otimes |\varphi| \cdot m + |\mu| \).

Moreover, for every \( \eta \) as above,

\[
\int_{E_0,T} |u| \eta \, dm_1 = (G^{0,T} \nu, \eta)_{L^2} = (\nu, \tilde{G}^{0,T} \eta) \leq \|\nu\|_{TV} \|\tilde{G}^{0,T} \eta\|_{\infty}
\]

\[
\leq (\|\varphi\|_{L^1} + \|\mu\|_{TV}) \|\tilde{G}^{0,T} \eta\|_{\infty},
\]

so \( u \in L^1(E_0,T;\eta \cdot m_1) \). Letting \( n \to \infty \) in (4.2) we get (4.1). \( \Box \)

**Corollary 4.2.** Let assumptions of Proposition 4.1 hold and let \( u \) be a solution of (3.11) in the sense of duality. Then there exists an \( m_1 \)-version of \( u \) satisfying (3.2).
Remark 4.3. Let \( \{B(t); t \in \mathbb{R}\} \) be a family of non-negative quasi-regular Dirichlet forms satisfying (2.1). A careful inspection of the proof of [21, Theorem VI.1.2] reveals that there exist a \( B^{(t)} \) nest \( \{E_k\}_{k \geq 1} \) consisting of compact metrizable sets in \( E \) and a locally compact separable metric space \( Y^\# \) such that \( Y^\# \) is a local compactification of \( Y = \bigcup_{k \geq 1} E_k \). Moreover, the trace topologies of \( E_k \) induced by \( E \) and \( Y^\# \) coincide and \( (B^{(s)}, D(B^{(s)})) \), which is the image of \((B^{(s)}, D(B^{(s)}))\) under the inclusion map \( i : Y \rightarrow Y^\# \), is a regular Dirichlet form on \( L^2(Y^\#; m^\#) \), where \( m^\# = m \circ i^{-1} \). By [21, Theorem VI.1.6] the Hunt process \( M^{(s)} = (\{P_{s,x}, x \in E\}, \{X_{s+t}, t \geq 0\}, \zeta^\#) \) associated with the regular form \( B^{(s)}\# \) is the trivial extension of the special standard process \( M^{(s)} = (\{P_{s,x}, x \in E\}, \{X_{s+t}, t \geq 0\}, \zeta^\#) \) associated with the form \( B^{(s)} \). Let \( E^\# \) be the time-dependent Dirichlet form on \( L^2(E^1\#; m^\#) \), where \( E^1\# = \mathbb{R} \times E^\# \), constructed from the family \( \{B^{(s)}, \#; s \in \mathbb{R}\} \) as in Section 2 (see (2.2)). Then the process \( M^\# = (\{X_{t^\#}, t \geq 0\}, \{P_{z^\#}, z \in \mathbb{R} \times E^\#\}, \zeta^\#) \) on \( \Omega' = \Omega \cup (E^1\# \setminus E^1) \) associated with the form \( E^\# \) is given by

\[
X_{t^\#}(\omega) = X_t(\omega), \quad t \geq 0, \quad \omega \in \Omega, \quad X_{t^\#}(\omega) = \omega, \quad t \geq 0, \quad \omega \in E^1\# \setminus E^1
\]

and \( P_{z^\#} = P_z \) for \( z \in E^1 \), \( P_{z^\#} = \delta_{\{z\}} \) for \( z \in E^1\# \setminus E^1 \). It is clear that the trace topologies on \( \mathbb{R} \times E_k \) induced by \( \mathbb{R} \times E \) and by \( \mathbb{R} \times Y^\# \) coincide. It follows that [21, Corollary VI.1.4] holds true for the form \( E^\# \) and capacity \( \text{Cap}_{h,\#} \) considered in [21] replaced by \( \text{Cap}_{\psi} \).

Remark 4.4. The above remark shows that one can apply the so-called “transfer method” (see [21, Section VI], [16]) to the form \( E \) defined by (2.2). Therefore the results of the present paper hold true for \( E \) with \( B^{(t)} \) being quasi-regular Dirichlet forms.

5 Semilinear equations with measure data

In this section we assume that \( \mu \in \mathcal{R}(E_{0,T}), \delta_{(T)} \otimes \varphi \cdot m \in \mathcal{R}(E_{0,T}) \) and \( f \in \mathcal{B}(E_{0,T}) \). In what follows given \( u \in \mathcal{B}(E_{0,T}) \) we set

\[
f_u(t,x) = f(t,x,u(t,x)), \quad (t,x) \in E_{0,T}.
\]

5.1 General semi-Dirichlet forms

Definition. We say that \( u \) is a solution of the Cauchy problem

\[
- \frac{\partial u}{\partial t} - L_t u = f(t,x,u) + \mu, \quad u(T) = \varphi
\]

if \( f_u \in \mathcal{R}(E_{0,T}) \) and for q.e. \( z \in E_{0,T} \),

\[
u(z) = \text{E}_z \left( 1_{\{\zeta > T - \tau(T)\}} \varphi(X_{T - \tau(T)}) + \int_0^{\zeta_T} f(X_t, u(X_t)) \, dt + \int_0^{\zeta_T} dA_t^\mu \right).
\]

Definition. We say that \( f : E_{0,T} \rightarrow \mathbb{R} \) is quasi-integrable \( (f \in qL^1(E_{0,T}; m_1) \) in notation) if \( f \in \mathcal{B}(E_{0,T}) \) and \( P_z(\int_0^{\zeta_T} |f(X_t)| \, dt < \infty) = 1 \) for q.e. \( z \in E_{0,T} \).

Let us consider the following hypotheses.
(H1) \( u \mapsto f(t, x, u) \) is continuous for every \((t, x) \in E_{0,T}\).

(H2) There is \( \alpha \in \mathbb{R} \) such that
\[
(f(t, x, y) - f(t, x, y'))(y - y') \leq \alpha |y - y'|^2
\]
for every \((t, x) \in E_{0,T}\) and \(y, y' \in \mathbb{R}\).

(H3) \( f(\cdot, 0) \in \mathcal{R}(E_{0,T}) \).

(H4) \( f(\cdot, y) \in qL^1(E_{0,T}; m_1) \) for every \( y \in \mathbb{R} \).

**Remark 5.1.** It is clear that \( \mathcal{B}(E_{0,T}) \cap \mathcal{R}(E_{0,T}) \subset qL^1(E_{0,T}; m_1) \). By Proposition 3.6, under the dual condition \( (\Delta) \), \( L^1(E_{0,T}; m_1) \subset \mathcal{R}(E_{0,T}) \). It follows that
\[
L^1(E_{0,T}; m_1) \subset qL^1(E_{0,T}; m_1) \tag{5.3}
\]
under \( (\Delta) \). Let us consider the following condition
\[
\forall \varepsilon > 0 \ \exists F_\varepsilon \subset E_{0,T}, \ \text{\( F_\varepsilon \)-closed}, \ \text{\( \text{Cap}_\psi(E_{0,T} \setminus F_\varepsilon) < \varepsilon \)}, \ 1_{F_\varepsilon} f \in L^1(E_{0,T}; m_1). \tag{5.4}
\]
Assume that \( f \in \mathcal{B}(E_{0,T}) \) satisfies (5.4) and the dual condition \( (\Delta) \) holds. Let \( \{F_n\} \) be an increasing sequence of closed subsets of \( E_{0,T} \) such that \( \text{Cap}_\psi(G_n) \to 0 \), where \( G_n = E_{0,T} \setminus F_n \). By (5.3),
\[
P_{m_1}(\int_0^{\xi_T} |f|(X_r) \, dr = \infty) \leq P_{m_1}(\int_0^{\xi_T} |f|1_{F_n}(X_r) \, dr = \infty)
+ P_{m_1}(\int_0^{\xi_T} |f|1_{G_n}(X_r) \, dr = \infty).
\]
The first term on the right-hand side of the above inequality equals zero. From the above and [36, Remark IV.3.6] it follows that
\[
P_{m_1}(\int_0^{\xi_T} |f|(X_r) \, dr = \infty) = P_{m_1}(\int_{\sigma G_n}^{\xi_T} |f|1_{G_n}(X_r) \, dr = \infty) \leq P_{m_1}(\lim_{n \to \infty} \sigma G_n < \xi_T)
= 0.
\]
Consequently, \( P_{\varepsilon}(\int_0^{\xi_T} |f|(X_r) \, dr = \infty) = 0 \) for \( m_1 \)-a.e., and hence for q.e. \( z \in E_{0,T} \) by standard argument. Thus \( f \in qL^1(E_{0,T}; m_1) \).

From Proposition 3.6 we know that if \( \mathcal{E} \) satisfies \( (\Delta) \) then \( \mathcal{M}_{0,\delta}(E_{0,T}) \subset \mathcal{R}(E_{0,T}) \).

The following example shows that the inclusion may be strict.

**Example 5.2.** Let \( f \) be a non-negative measurable function on \( E_{0,T} \). Then
\[
(G_0^{0,T} f, 1)_{L^2} = (f, \hat{G}^{0,T} 1)_{L^2}.
\]

Let \( D \) and \( L_t \) be as in Example 3.8. Then for \( x \in D \),
\[
\hat{R}^{0,T}(s, x) \leq c_1 \int_0^s \int_D t^{-d/2} \exp\left(\frac{-c_2|y-x|^2}{2t}\right) \, dt \, dy \leq c_3 \int_D G^1_D(x, y) \, dy \leq c_4 \delta(x),
\]

\[22\]
where $\delta(x) = \text{dist}(x, \partial D)$ and $G^1_D(\cdot, \cdot)$ is the Green function for the operator $\Delta$ on $D$. If $L_t = \Delta^{\alpha/2}$, where $\alpha \in (0, 2)$, then for $x \in D$ we have

$$\bar{R}^{0,T}1(s, x) \leq \int_D G^2_D(x, y) \, dy \leq c\delta^{\alpha/2}(x),$$

where $G^2_D(\cdot, \cdot)$ is the Green function for the operator $\Delta^{\alpha/2}$ on $D$ (For the last inequality see [18, Proposition 4.9]). We see that $L^1(E_{0,T}; \delta \cdot m_1) \subset \mathcal{R}(E_{0,T})$ if $L_t$ is the operator of Example 3.8 and $L^1(E_{0,T}, \delta^{\alpha/2} \cdot m_1) \subset \mathcal{R}(E_{0,T})$ if $L_t = \Delta^{\alpha/2}$, so in both cases $\mathcal{M}_{0,b}(E_{0,T}) \subseteq \mathcal{R}(E_{0,T})$.

The next example shows that in general quasi-integrable functions need not be locally integrable.

**Example 5.3.** Let $L_t$ be as in Remark 3.8 and let $D = \{x \in \mathbb{R}^d; |x| < 1\}$, $d \geq 2$. Set $f(t, x) = |x|^{-d}$, $(t, x) \in E_{0,T}$. Direct calculation shows that $\int_0^T \int_{B(0, \varepsilon)} f(t, x) \, dt \, dx = \infty$ for every $\varepsilon \in (0, 1)$, i.e. $f$ is not locally integrable. It is, however, quasi-integrable, because if $F_n = \{(t, x) \in E_{0,T}; t \in [0, T], |x| \geq \frac{1}{n}\}$ then $1_{F_n}f \in L^1(E_{0,T}; m_1)$, $n \geq 1$. Moreover, if we set $G_n = E_{0,T} \setminus F_n$ then $G_{n+1} \subset G_n$ and $\bigcap_n G_n = \bigcap_n \bar{G}_n$. Since $\text{Cap}_\psi$ is a Choquet capacity (see [36, Proposition III.2.8]), it follows that

$$\lim_{n \to \infty} \text{Cap}_\psi(G_n) = \text{Cap}_\psi(\bigcap_n G_n) = \text{Cap}_\psi((0, T] \times \{0\}) = 0.$$

From the above example it follows in particular that in general $\mathcal{R}(E_{0,T}) \subseteq qL^1(E_{0,T})$. For instance, the inclusion is strict if $L$ is the Laplace operator on smooth bounded domain, because in this case each function from $\mathcal{R}(E_{0,T})$ is locally integrable thanks to the positivity and continuity of the corresponding Green function.

Let $(\Omega, \mathcal{F} = \{\mathcal{F}_t, t \in [0, T]\}, P)$ be a fixed stochastic basis. Suppose we are given an $\mathcal{F}_T$ measurable random variable $\xi$, an $\mathcal{F}$ progressively measurable function $F : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ and an $\mathcal{F}$ adapted càdlàg process $A$ of finite variation.

**Definition.** We say that a pair $(Y, M)$ of processes on $[0, T]$ is a solution of the backward stochastic differential equation

$$Y_t = \xi + \int_t^T F(r, Y_r) \, dr + \int_t^T dA_r - \int_t^T dM_r, \quad t \in [0, T] \tag{5.5}$$

(BSDE($\xi, F + dA$) in notation) if $Y$ is a progressively measurable process of class (D), $t \to F(t, Y_t) \in L^1(0, T)$, $M$ is a $\mathcal{F}$-martingale such that $M_0 = 0$ and (5.5) holds $P$-a.s.

We will need the following assumptions.

(A1) $y \to F(t, y)$ is continuous for a.e. $t \in [0, T]$,

(A2) there is $\alpha \in \mathbb{R}$ such that

$$(F(t, y) - F(t, y'))(y - y') \leq \alpha|y - y'|^2$$

for a.e. $t \in [0, T]$ and every $y, y' \in \mathbb{R}$,
(A3) \( E \int_0^T |F(t,0)| \, dt < \infty \), \( E|\xi| + E|A_T| < \infty \) (\(|A_T|\) denotes the variation of \( A \) on \([0,T]\)),

(A4) \([0,T] \ni t \to F(t,y) \in L^1(0,T)\) for every \( y \in \mathbb{R} \).

**Theorem 5.4.** Assume (A1)–(A4). Then there exists a unique solution \((Y, M)\) of BSDE\((\xi, f + dA)\). Moreover, \( Y, M \in S^q \), \( q \in (0,1) \) and

\[
E \int_0^T |F(t,Y_t)| \, dt \leq C(\alpha, T) \left( E \int_0^T |F(t,0)| \, dt + E \int_0^T d|A_t| \right).
\]

**Proof.** By using the standard change of variable one can reduce the proof to the case where \( \alpha = 0 \) in (H2). But then the desired result follows from [15, Theorem 2.7]. \( \square \)

**Definition.** Let \( z \in E \). We say that a pair \((Y, M)\) is a solution of BSDE\((\varphi, f + d\mu)\) if \((Y, M)\) is a solution of the BSDE

\[
Y_t = 1_{\{\zeta > T - \tau(0)\}} \varphi(X_{\zeta^r}) + \int_t^{\zeta^r} f(X_r, Y_r) \, dr + \int_t^{\zeta^r} dA_r + \int_t^{\zeta^r} dM_r, \quad t \in [0, \zeta^r]
\]
on the probability space \((\Omega, \mathcal{F}, P_z)\).

**Proposition 5.5.** Assume (H1)–(H4). Then for q.e. \( z \in E_{0,T} \) there exists a unique solution \((Y^z, M^z)\) of BSDE\((\varphi, f + d\mu)\). Moreover, there exists a pair of processes \((Y, M)\) such that for q.e. \( z \in E_{0,T}\),

\[
(Y_t, M_t) = (Y^z_t, M^z_t), \quad t \in [0, T - \tau(0)], \quad P_z\text{-a.s.}
\]

**Proof.** If \( \varphi, f, \mu \) satisfy (H1)–(H4) then \( \xi = 1_{\{\zeta > T - \tau(0)\}} \varphi(X_{\zeta^r}) \), \( F = f(\cdot, X, \cdot) \), \( A = A^\mu \) satisfy (A1)–(A4) under the measure \( P_z \) for q.e. \( z \in E_{0,T} \). Therefore the first part of the proposition follows from Theorem 5.4. The second part follows from [15, Remark 3.6]. \( \square \)

**Lemma 5.6.** Assume (H1)–(H4) and let \((Y, M)\) be the pair of Proposition 5.5. Then for q.e. \( z \in E_{0,T} \) and every \( h \in [0, T - \tau(0)] \),

\[
Y_t \circ \theta_h = Y_{t+h}, \quad t \in [0, T - \tau(0) - h], \quad P_z\text{-a.s.}
\]

**Proof.** Since for q.e. \( z \in E_{0,T} \) the solution of BSDE\((\varphi, f + d\mu)\) is unique, to prove the proposition it suffices to repeat the proof of [15, Proposition 3.5] (see also the proof of [17, Proposition 3.24]). \( \square \)

**Theorem 5.7.** Assume (H1)–(H4). Then there exists a unique solution \( u \) of (5.1). Moreover, for q.e. \( z \in E_{0,T} \) there exists a unique solution \((Y^z, M^z)\) of BSDE\((\varphi, f + d\mu)\). In fact,

\[
Y^z_t = u(X_t), \quad t \in [0, \zeta^r],
\]

\[
M^z_t = E_z\left( 1_{\{\zeta > T - \tau(0)\}} \varphi(X_{T-\tau(0)}) + \int_0^{\zeta^r} f_u(X_r) \, dr + \int_0^{\zeta^r} dA^\mu_r |\mathcal{F}_t \right) - u(X_0).
\]
Proof. By Proposition 5.5 for q.e. \( z \in E_{0,T} \) there exists a unique solution \((Y^z, M^z)\) of BSDE\(_z(\varphi, f + d\mu)\). Let \((Y, M)\) be the pair of Proposition 5.5 and let \( u(z) = E_z Y_0 \). Then by Lemma 5.6 and the strong Markov property,

\[
u(X_t) = E_{X_t} Y_0 = E_z (Y_0 \circ \theta_t | \mathcal{F}_t) = E_z (Y_t | \mathcal{F}_t) = Y_t
\]

every \( t \in [0, T - \tau(0)] \). From this, (H3), Theorem 5.7 and the definition of a solution of BSDE\(_z(\varphi, f + d\mu)\) we deduce that (5.2) is satisfied for q.e. \( z \in E_{0,T} \), i.e. \( u \) is a solution of (5.1). By Proposition 3.2, \( u \) is quasi-càdlàg. Therefore \( Y_t = u(X_t), t \in [0, \zeta_t] \). From this and the definition of a solution of BSDE\(_z(\varphi, f + d\mu)\) the representation formula for \( M^z \) immediately follows. Suppose now that \( v \) is another solution of (5.1). Then by the strong Markov property,

\[
v(X_t) = \mathbf{1}_{\{\zeta > T - \tau(0)\}} \varphi(X_{T - \tau(0)}) + \int_t^{\zeta} f_v(X_r) \, dr + \int_t^{\zeta} dA_r - \int_t^{\zeta} dM_r,
\]

where \( M \) is a càdlàg and independent of \( z \) version of the martingale \( N^z \) given by

\[
N_t^z = E_z \left( \mathbf{1}_{\{\zeta > T - \tau(0)\}} \varphi(X_{T - \tau(0)}) + \int_0^{\zeta} f_v(X_r) \, dr + \int_0^{\zeta} dA_r | \mathcal{F}_t \right) - v(X_0)
\]

(Existence of such version follows from [11, Lemma A.3.5]). We see that the pair \((v(X), M)\) is a solution of BSDE\(_z(\varphi, f + d\mu)\) for q.e. \( z \in E_{0,T} \). Consequently, \( u = v \) q.e. by Proposition 5.5.

Corollary 5.8. Let assumptions of Theorem 5.7 hold and let \( u_i \) be a solution of (5.1) with terminal condition \( \varphi_i \), and right-hand side \( f_i + d\mu_i \), \( i = 1, 2 \). If \( \varphi_1 \leq \varphi_2 \) \( m_1 \)-a.e., \( \mu_1 \leq \mu_2 \) and either \( f_1 \) satisfies (H2) and \( f_{1,u_2} \leq f_{2,u_2} \) \( m_1 \)-a.e. or \( f_2 \) satisfies (H2) and \( f_{1,u_1} \leq f_{2,u_1} \) \( m_1 \)-a.e., then then \( u_1(z) \leq u_2(z) \) for q.e. \( z \in E_{0,T} \).

Proof. Follows immediately from Theorem 5.7 and [15, Proposition 2.1].

Proposition 5.9. Let assumptions of Theorem 5.7 hold, \( \varphi \in L^1(E; m), \mu \in M_{0,b}(E_{0,T}) \) and for some \( \gamma \geq 0 \) the form \( \mathcal{E}_\gamma \) has the dual Markov property. Then if \( u \) is a solution of (5.1) then \( f_u \in L^1(E_{0,T}; m_1) \) and

\[
\|f_u\|_{L^1} \leq C(\alpha, T, \gamma)(\|\mu\| + \|\varphi\|_{L^1} + \|f(\cdot, 0)\|_{L^1}).
\]

Proof. Let \((Y, M)\) be as in Proposition 5.5 and let \((\tilde{Y}_t, \tilde{M}_t) = (e^{\gamma t} Y_t, e^{\gamma t} M_t), t \in [0, \zeta_T] \). Applying Itô’s formula shows that \((\tilde{Y}, \tilde{M})\) is a solution of BSDE\(_z(\tilde{\varphi}, \tilde{f} + d\tilde{\mu})\) with \( \tilde{\varphi}(x) = e^{\gamma \zeta_T} \varphi(x) = e^{\gamma T} f(t, x, y) - \gamma y \) and \( d\tilde{\mu}(t, x) = e^{\gamma t} d\mu(t, x) \). By Theorem 5.4 applied to the pair \((\tilde{Y}, \tilde{M})\),

\[
E_z \int_0^{\zeta_T} e^{\gamma t} |f(X_t, Y_t)| \, dt \leq C(\alpha, T) \left( E_z \mathbf{1}_{\{\zeta > T - \tau(0)\}} e^{\gamma \zeta_T} \varphi(X_{T - \tau(0)}) \right)
\]

for q.e. \( z \in E_{0,T} \). Since \( Y_t = u(X_t), t \in [0, \zeta_T] \), \( P_z \)-a.s. for q.e. \( z \in E_{0,T} \) by Theorem 5.7 and \( \mathcal{E}_\gamma \) has the dual Markov property, it follows from the above inequality and Proposition 3.11 that

\[
\|f_u\|_{L^1} \leq C(\alpha, T)(\|\varphi\|_{L^1} + \|f(\cdot, 0)\|_{L^1} + \gamma \|\mu\|_{L^1}).
\]

From this and Theorem 3.10 we get the desired inequality.

\[
= 0
\]
Corollary 5.10. Let assumptions of Proposition 5.9 hold. If $u$ is a solution of (5.1) then $u \in L^1(E_{0,T}, m_1)$ and $T_k(u) \in \mathcal{F}_{0,T}$ for $k \geq 0$. Moreover, (3.17) and (3.18) hold true with $\mu$ replaced by $\mu + f(\cdot, 0) \cdot m$.

Proof. Follows from Theorem 3.10 and Proposition 5.9. \qed

5.2 Semi-Dirichlet forms satisfying the duality condition

Let us recall that in Section 4 we have defined a solution in the sense of duality of linear equations. In the semilinear case we adopt the following natural definition.

Definition. Let $\varphi \in L^1(E; m)$, $\mu \in \mathcal{M}_{0,b}(E_{0,T})$ and assume that $\mathcal{E}$ satisfies the dual condition ($\Delta$). We say that a measurable function $u : E_{0,T} \to \mathbb{R}$ is a solution of (5.1) in the sense of duality if $f_u \in L^1(E_{0,T}; m_1)$ and (4.1) is satisfied with $\mu$ replaced by $f_u \cdot m_1 + \mu$.

Theorem 5.11. Assume (H1)–(H4) and that there is $\gamma \geq 0$ such that $\mathcal{E}_\gamma$ has the dual Markov property. Then there exists a unique solution of (5.1) in the sense of duality.

Proof. The existence part follows from Theorem 5.7 and Corollaries 4.2 and 5.10. The uniqueness follows from Corollaries 4.2 and 5.8. \qed

Example 5.12. Let $\alpha$ be a measurable function on $\mathbb{R}^d$ such that $\alpha_1 \leq \alpha(x) \leq \alpha_2$, $x \in \mathbb{R}^d$, for some constants $0 < \alpha_1 \leq \alpha_2 < 2$. Let $L_t = L = \Delta^{\alpha(x)}$, i.e. $L$ is a pseudodifferential operator such that

$$Lu(x) = \int_{\mathbb{R}^d} e^{i x \cdot \xi} |\xi|^{\alpha(x)} \hat{u}(\xi) \, d\xi, \quad u \in C^\infty_c(\mathbb{R}^d).$$

For $r > 0$ set $\beta(r) = \sup_{|x-y| \leq r} |\alpha(x) - \alpha(y)|$. By [32, Proposition 3.1], if

$$\int_0^1 \frac{(\beta(r) |\log r|)^2}{r^{1+\alpha_2}} \, dr < \infty$$

then the form $B^{(t)} = B$ associated with $L$ is a regular semi-Dirichlet form. It is known (see [12, 32]) that for $u, v \in C^\infty_c(\mathbb{R}^d)$ the form $B$ is given by

$$B(u, v) = - \int_{\mathbb{R}^d} \int_{\{z \neq 0\}} w(x)v(x)(u(x+z) - u(x) - \nabla u(x) \cdot z 1_{\{|z| \leq 1\}}(z))|z|^{-\alpha(x)} \, dx \, dz,$$

where

$$w(x) = \alpha(x) 2^{\alpha(x)-1} \frac{\Gamma(\frac{d}{2} \alpha(x) + \frac{1}{2} d)}{\pi^{d/2} \Gamma(1 - \frac{d}{2} \alpha(x))}.$$

By [32, Theorem 2.1],

$$L^* u(x) = \Lambda u(x) + \kappa(x) u(x), \quad u \in C^\infty_c(\mathbb{R}^d)$$

(5.6)

for some measurable function $\kappa$ and some operator $\Lambda$ associated with a semi-Dirichlet form. By [32, Remark 3.2], under the additional condition that $\alpha \in C^2_b(\mathbb{R}^d)$ the function $\kappa$ is bounded on $\mathbb{R}^d$. Therefore from (5.6) it follows that there exists $\gamma \geq 0$ such that $B_\gamma$ has the dual Markov property, which implies that $\mathcal{E}_\gamma$ has the dual Markov property.

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