Quantum oscillations in the spin-boson model: Reduced visibility from non-Markovian effects and initial entanglement

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Abstract. The loss of coherence of quantum oscillations is of fundamental interest as well as of practical importance in quantum computing. In solid-state experiments the oscillations show, next to the familiar exponential decay on time scales $T_{1/2}$, an overall loss of amplitude. We solve the spin-Boson for a large class of initial conditions without the Markov approximation at the pure dephasing point. It is shown that a loss of visibility occurs in the form of a fast initial drop for factorized initial conditions and an overall reduction for entangled initial conditions. This loss of amplitude is distinct from $T_2$-decoherence with the difference being most drastic for environments with real or pseudo-gaps. This result is explained by bandwidth effects in quantum noise as well as in terms of higher-order phase-breaking processes. For several experiments, such gapped environments are identified. We confirm that this physics is valid beyond the pure dephasing point.

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1. Introduction

Quantum coherence driven by the goal of quantum computation is a central theme of present-day research in mesoscopic condensed-matter physics. In particular in the field of superconducting qubits [1 2 3 4], spectacular successes have been achieved, such as Rabi oscillations, charge and flux echo, and a controlled-not gate [5 6 7 8 9 10]. In order to achieve this, the decoherence due to the ubiquitous environmental degrees of freedom in solid-state systems had to be overcome. In fact, careful modelling of the environment allows to make predictions of the relaxation and dephasing times, $T_1$ and $T_2$, which are extracted either from spectroscopic line widths or from fitting exponential envelopes to Rabi- or Ramsey oscillation data [11 12]. Theoretical predictions of $T_1$ are in reasonable agreement with experiments. $T_2$ is mostly attributed to $1/f$-noise [6 12] for which self-consistent theories can be formulated [12 13] even though the detailed origins of that noise are not quantitatively predictable yet. Technically, the notion of exponential decay of coherences on scales $T_{1/2}$ is based on a Markov approximation of some, potentially implicit, type.

On the other hand, a variety [6 7 14] of controlled experiments show an additional loss of visibility: Next to the exponential decay given by $T_{1/2}$, the amplitude of coherent oscillations is reduced even further, i.e., it does not extrapolate back to the full expected amplitude at $t = 0$. This behavior is not explained by a simple $T_{1/2}$-picture. This reduced visibility is an obstacle for the demonstration of macroscopic quantum effects in these devices, e.g., of the violation of Bell’s inequality in a coupled qubit system [15], and for quantum computing applications. It has so far mostly been attributed to technological shortcomings of the detector. This paper introduces a different, additional mechanism for a reduced visibility based on non-Markovian effects in the spin boson model. This generic effect is induced i) by higher order processes involving virtual intermediate states, which rapidly entangle system and environment and ii) potential initial entanglement between system and environment. We show, that the reduction originates from the off-resonant high-frequency parts of the environmental spectrum which do not contribute to $T_{1/2}$. We demonstrate that reduced visibility is compatible with large $T_2$-values in superohmic and gapped environments and has to be considered as an independent quantifier of decoherence. We identify such environments in recent experiments and estimate the loss of visibility they induce.

2. The model

As a generic model, we consider the spin-boson Hamiltonian [16]

$$H_0 = \frac{E}{2} \left( \cos \theta \hat{\sigma}_z + \sin \theta \hat{\sigma}_x \right) + \frac{1}{2} \hat{\sigma}_z \sum_i \lambda_i (a_i + a_i^\dagger) + \sum_i \omega_i \left( a_i^\dagger a_i + \frac{1}{2} \right). \quad (1)$$

where the $\hat{\sigma}_i$ are Pauli matrices and $a$ and $a^\dagger$ are Boson annihilation and creation operators. The bath is characterized by the spectral density $J(\omega) = \sum_i |\lambda_i|^2 \delta(\omega - \omega_i)$ which is related to the equilibrium spectral noise power $S(\omega) = J(\omega) \coth(\omega/2T)$. This
model is realized in superconducting qubits, where the bath is the electromagnetic environment [17, 18] or by phonons [19], which also play an important role in quantum dots [20]. Here and henceforth, we chose $\hbar = 1$ and $k_B = 1$.

The spin-boson model is in general not exactly solvable. It has been treated with a number of approaches [16, 21, 22]. For quantum computing, the $\lambda_i$ are small at $\omega_i \approx E$ by design and the system-bath coupling is usually treated perturbatively. If the system-bath interaction also defines the longest time in the problem, a Markov approximation is justified (see, e.g., [18] for a recent review). This procedure leads to variants of the well-known Bloch-Redfield master equation which predicts strictly exponential decay of the spin projections contained in the system+bath density matrix $\rho$, $s_i = \text{Tr} [ (\sigma_i \otimes 1) \rho ]$ with time scales $T_{1/2}$. Thus, conceivably, such an approach cannot describe the loss of visibility. Moreover, many designed environments use the option to allow for larger $\lambda_i$ at high frequencies [6], $\omega_i \gg E$.

In order to go beyond Bloch-Redfield, we use two approaches: For $\theta = 0$, the spin-boson model reduces to the exactly solvable independent boson model [23, 24, 25, 26]. For $\theta \neq 0$ we will use perturbation theory in the qubit Hamiltonian to obtain approximate insights. We will recover similar physics in both cases.

3. Pure dephasing point, $\theta = 0$

We now proceed to the exact solution for a very general initial state in the independent boson limit. We can perform a Schmid decomposition of the initial density matrix in the qubit $\otimes$ bath Hilbert space as

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}. \tag{2}$$

As the bath is composed of noninteracting oscillators, the submatrices $\rho_{ij} = \prod_k \rho_{ij}^{(k)}$ remain factorized within the bath modes $k$. Note that the submatrices do not have to be valid density matrices, only the complete $\rho$ has to be. All matrices with bounded trace can be parameterized with the characteristic function $\chi$ of the Wigner function $\chi_{ij}^{(k)}(\alpha^{(k)}, \alpha^{(k)*}) = \text{Tr} [ \rho_{ij}^{(k)} \hat{D}^{(k)}(-\alpha^{(k)}) ]$ with $\hat{D}^{(k)}(\alpha^{(k)}) = \exp(\alpha^{(k)} a^{(k)}_i - \alpha^{*(k)} a^{(k)}_j)$ the displacement operator for mode $k$. Preparing such an initial state requires to use controls with $\theta(t) \neq 0$ at $t < 0$.

The fact that for $\theta = 0$ the Hamiltonians at different times $t > 0$ commute allows us to exactly compute the propagator in interaction representation. The result can be written as a product of $D$-operators, $\hat{U}(t) = \prod_k \hat{D}^{(k)}(\hat{\mu}_k(t))$ with $\hat{\mu}_k(t) = -\frac{\lambda_k \sigma^z}{2 \hbar \omega_k} (e^{i \omega_k t} - 1)$. The application of these to the density matrix is straightforward, the essentially induce a coordinate transformation on the $\alpha^{(k)}$: The matrix of characteristic functions will be

$$\chi_{ij}^{(k)}(\alpha_{ij}^{(k)}) = \text{Tr}_k \left( \rho_{ij}(t = 0) D^{(k)}(\mu_i^{(k)}) D^{(k)}(-\alpha_{ij}^{(k)}) D^{(k)*}(\mu_j^{(k)}) \right)$$

$$= e^{i \phi_{ij}^{(k)}} \text{Tr}_k \left( \rho_{ij}(t = 0) D^{(k)}(-\alpha_{ij}^{(k)} + \mu_i^{(k)} - \mu_j^{(k)}) \right). \tag{3}$$
where the phase factor comes from multiplying the displacement operators and reads \( \phi^{(k)}_{ij} = \text{Im} \left( \mu_i \alpha_{ij}^{(k)*} + \alpha_{ij}^{(k)*} \mu_j - \mu_i \mu_j \right) \). We can conclude that \( \chi_{ij}^{(k)}(\alpha, \alpha^*, t) = e^{i \phi_{ij}^{(k)}} \chi_{ij}^{(k)}(\alpha_{ij}^{(k)}(t), \alpha_{ij}^{(k)*}(t), 0) \) with \( \alpha_{ij}^{(k)}(t) = \alpha_{ij}^{(k)}(t) + \delta \alpha_{ij}^{(k)}(t) \) with \( \delta \alpha_{ij}^{(k)}(t) = \mu_i^{(k)}(t) - \mu_j^{(k)}(t) \).

This simple property that time evolution is a mere change of coordinate frame is the main rationale for resorting to the phase factor \( \chi \) as a phase-space parameterization of the density matrix. The other rationale is that \( \chi \) makes it in particular easy to compute expectation values of qubit operators \( q \) using the corrolary \( \text{Tr} [\rho_{ij}] = \text{Tr} \left[ \rho_{ij} \Pi_k D_{ij}^{(k)}(0) \right] = \prod_k \chi_{ij}^{(k)}(0, 0, t) \) as

\[
\langle q \otimes \hat{1} \rangle = \text{Tr} \left[ (q \otimes \hat{1}) \rho \right] = \sum_{i,j \in \{0,1\}} \langle i|q|j \rangle \prod_k \chi_{ij}^{(k)}(-\delta \alpha(t), -\delta \alpha^*(t), 0) \tag{4}
\]

without any further integration.

From knowing the full density matrix at any time \( t > 0 \) we can calculate any property of the system we like, including correlation functions. The main purpose of this paper is, however, to discuss the quantum system alone in order to establish the connection to master equation approaches.

Being at the pure dephasing point we can see that there is no dynamics on the diagonal elements: \( \alpha_{ii}(t) = \alpha_{ii}(0) \) and \( \phi_{ii} = 0 \).

On the off-diagonal, things get more involved. We focus on studying the coherence given in terms of the characteristic Wigner function as

\[
s_+ = \frac{1}{2} (s_x + is_y) = \prod_k \chi_{01}(-\delta \alpha_{01}(t), -\delta \alpha_{01}^*(t), 0). \tag{5}
\]

We can work out explicitly

\[
\alpha_{01}^{(k)}(t) = \frac{\lambda}{\hbar \omega_i} (e^{i \omega k t} - 1) \tag{6}
\]

and it is easy to show that the phase factor drops out.

We now apply this technique to a physical realistic realization by assuming a specific \( \chi_{01}(\alpha, \alpha^*, 0) \). The main restriction on this function is that \( \rho \) has to be a valid density matrix, i.e. Hermitian, normalized, and positive. We are restricting ourselves to symmetrically entangling a qubit that is initially in an eigenstate of \( \sigma_x \) with classical states of the bath oscillators, i.e. by choosing for the complete initial density matrix

\[
\rho_{ij}^{(k)} = \left( U_D^{(k)} \right)_{ii} \rho_{th} \left( U_D^{(k)*} \right)_{jj} \tag{7}
\]

with \( \rho_{th} \) the thermal density matrix for the bath and \( U_D = D^{(k)}(d^{(k)} \sigma_x) \) displaces each bath mode in opposite directions in phase space conditioned on the two states of the qubit, i.e. it performs a controlled unitary displacement by some mode-specific amound \( d^{(k)} \) of the bath oscillators. Reusing the multiplication property of displacement operators this means that

\[
\chi_{01}^{(k)}(\alpha, \alpha^*, 0) = \chi_{th}^{(k)}(\alpha - 2d^{(k)}, \alpha - 2d^{(k)}) \tag{8}
\]

with the phase space function of a thermal state being

\[
\chi_{th}^{(k)} = e^{-2 \alpha \alpha^* \coth(\omega_k/2T)} \tag{9}
\]
Visibility and non-Markovian effects

In this limit, we thus obtain
\[ s_+ = \prod_k e^{-2\coth(\omega_K/2T)|2d_k+\mu_k(t)|^2}. \] (10)
taking the continuum limit and substituting \( d_k = (\lambda_k/2\hbar\omega_k)(u_k + iv_k) \) with real dimensionless coefficients \( u_i \) and \( v_i \). Taking the continuum limit we find \( s_+ = \Re s_+ = e^{-K(t)}\cos et \) with
\[ K(t) = -\frac{1}{2} \int_0^\infty \frac{d\omega}{\omega^2} S(\omega) \left[ (u(\omega) + 1)^2 + v^2(\omega) + 1 - 2((1 + u(\omega))\cos\omega t + v(\omega)\sin\omega t) \right]. \] (11)

Different choices of \( u(\omega) \) and \( v(\omega) \) can lead to rich dynamics. We single out prominent quantities: The initial amplitude \( e^{-K(0)} = \exp \left[ -\int_0^\infty \frac{d\omega}{\omega^2} S(\omega)(u^2(\omega) + v^2(\omega)) \right] \), the time constant \( T_2 \) of exponential decay in the long-time limit, \( 1/T_2 = \lim_{t \to \infty} \partial_t K(t) = S(0)(u(0) + 1) \) and the visibility, the amplitude found when extrapolating the long-time behavior back to \( t \to 0 \)
\[ V[u, v] = \lim_{t \to \infty} e^{\frac{t}{T_2} - K(t)} = e^{-\mathcal{P} \int_0^\infty d\omega S(\omega)((1+u(\omega))^2+v^2(\omega))}. \] (12)
We analyze this result in important limiting cases.

3.1. Factorizing initial conditions

Factorizing initial conditions imply for the density matrix \( \rho(t = 0) = \rho_S \otimes \rho_B \), i.e. in our notation \( u(\omega) = v(\omega) = 0 \). This case corresponds to environments which are part of a detector or any piece of electromagnetic environment that is switched on during the experiment, e.g. the DC-SQUID detecting a flux qubit \[17\ 27\ 28\]. It also applies to the case of excitons \[29\ 25\] as the two-state system that are created by ultrafast laser control in the beginning of the experiment. From the above result, we can see that this is the only case in which the initial amplitude is unity, \( K(0) = 0 \), i.e. when the actual oscillations starts at full amplitude. In this case, \( K(t) \) can be written as \[24\ 30\]
\[ K(t) = \int_0^\infty d\omega \frac{1 - \cos \omega t}{\omega^2} S(\omega) = \frac{t^2}{2} \int_0^\infty \frac{\sin^2 \left( \frac{\omega t}{2} \right)}{(\omega t/2)^2} S(\omega). \] (13)
The long-term expansion of this result can be written as \( K(t) = t/T_2 + \log V[0, 0] + O(1/t) \) with \( 1/T_2 = S(0) \). Note, that the rate \( 1/T_2 \) is identical to the Bloch-Redfield result, but additionally the constant term \( V[0, 0] = \exp \left[ -\mathcal{P} \int_0^\infty \frac{S(\omega)}{\omega} \right] \) describes an overall loss of amplitude. Here \( \mathcal{P} \) denotes the Cauchy mean value.

The dynamics resulting from Eq. (13) can be interpreted in terms of environmental quantum noise. This is accomplished by looking at the second equality of eq. (13) which multiplies \( S(\omega) \) with a spectral weight of width \( 1/t \). When the system has been coupled to the environment for a time \( t \), it samples its spectrum over a bandwidth of \( \delta \omega = 1/t \) due to frequency-time uncertainty. This bandwidth goes to zero only in the long-time limit. Quantum-mechanically, the long-time limit only captures direct energy-conserving processes between system and environment, whereas at shorter times also higher-order processes involving energetically forbidden intermediate states play a role.
The leading process beyond $T_2$ is the excitation of an environmental mode followed by the relaxation of another, excited state in the environment with infinitesimally different energy. These modes have to be distinct in order to leave a trace in the environment which is a necessary condition to entangle system and environment and hence cause true dephasing. Thus, it is crucial that the environment is in an excited state in the beginning of this cycle. In our case, this is guaranteed by the factorized initial conditions: The ground state of the spin-Boson Hamiltonian [16] is an appropriate dressed state of the spin. When the interaction is switched on, energy is redistributed: Forming this dressed state forms lowers the extra energy compared to the initial state. The extra energy is accommodated in bath excitations necessary for true dephasing as just explained.

We will now study the consequences of these results for a number of important spectra, starting with the ones where $T_2 \rightarrow \infty$, i.e. a constant amplitude at long times. These structured environments are an important test-bed for our approach because all the decoherence is from non-Markovian effects. The gapped Ohmic model approximately describes the effect of the quasiparticle transport channel shunting a Josephson junction [31]. Its spectral density reads $J_g(\omega) = \alpha \omega e^{-\omega/\omega_c} \Theta(|\omega| - E_g)$ where $\Theta$ is the Heaviside unit step function. In the limit of $T \ll E_g \ll \omega_c$, we find $V[0,0] = (E_g/\omega_c)^{\alpha_1} e^{-\alpha_1 \gamma}$ where $\gamma \approx 0.577\ldots$ it the Euler-Mascheroni constant. The amplitude as shown in Fig. 1 drops to this level during a time $t \approx 1/E_g$, indicating the time permitted by the energy-time uncertainty relation over which virtual excitations above the gap edge may be maintained.

Figure 1. Dynamics of $S_x$ and its envelope in the gapped Ohmic model using $\alpha = 0.1$, $\omega_c = 10E$, $\omega_{IR} = 2E$, and $T = 0$. The exact result is compared to the envelope which would be obtained within Bloch-Redfield theory for the same model.
The same physics holds for soft gaps such as in the standard superohmic case, \( J_q(\omega) = \alpha_q \omega^q \omega_c^1 - q e^{-\omega/\omega_c} \). This model describes phonon baths as well as electromagnetic environments blocked off at low frequency by a serial capacitance \([6]\). Here, \( K(t) \) can be given in closed form \([32]\). For \( q \geq 3 \) we find a visibility of \( V[0, 0] = \exp \left[ 2\alpha_q \Gamma(q - 1) \right] \) for \( k_B T \ll \omega_c \). For \( 1 < q < 3 \), the amplitude decays non-exponentially, also at long times, given by 

\[
e^{-K(t)} \rightarrow \exp \left( 2\alpha_q T \Gamma(q - 2) \sin \left( \frac{\pi(q - 1)}{2} \right) \omega_c^1 - q t^2 \right)
\]

and for \( q = 2 \) we find 

\[
e^{-K(t)} \rightarrow e^{-2\alpha_2 (1 + \kappa) \left( \frac{tT}{(1 + \kappa)} \right)^{2\alpha_2 \kappa}}.
\]

These sub-exponential decay features cannot be characterized by a time scale \( T_2 \) and may resemble reduced visibility if the observation does not span enough time.

Loss of visibility is not restricted to gapped models. It is usually joined by genuine exponential \( T_2 \) decoherence. This is illustrated in the Ohmic Spin-Boson model 

\[
J_1(\omega) = \alpha_1 \omega e^{-\omega/\omega_c},
\]

which describes the quantum version of classical friction and e.g. describes standard resistive environments as well as gapless electron-hole excitations.

At finite \( T \), we identify a finite \( 1/T_2 = \alpha_1 T \). The full shape of the envelope in the long-time limit, \( tT \gg 1 \), is however 

\[
e^{-K(t)} \rightarrow (T/\omega_c)^\alpha e^{-t/T_2},
\]

i.e. there is a visibility prefactor \( v[0, 0] = (T/\omega_c)^\alpha \). At \( T = 0 \), the long-time limit is never reached as there is no low-energy scale in the problem, and we find power law decay, 

\[
e^{-K(t)} = (1 + \omega_c^2 t^2)^{-\alpha/2}.
\]

Thus, at \( T \rightarrow 0 \), the very small \( v[0, 0] \) reflects the fact that for most of the time the power-law decay dominates and the long-time expansion involving \( T_2 \) becomes valid at extremely long times only. The finite temperature behavior is illustrated in Fig. 2. Note that this initial decay followed by exponential decay has been discussed by an entirely different approach in Ref. \[33\].

### 3.2. Entangled initial condition

Factorized initial conditions are standard assumptions in the theory of open quantum systems, however, they are not always realistic. It is already seen in our above result Eq. (12), that our prediction of an initial drop qualitatively holds for essentially any nonequilibrium initial condition. As a complementary case, we chose the \( \alpha_i^\pm \) such that in eq. (12) \( u_i = -1, v_i = 0 \). Physically, this state corresponds to a qubit dressed by environmental oscillators at all frequencies. It is the variational (in \( u \) and \( v \)) ground state as long as \( E \ll \int d\omega J(\omega)/\omega \). It is thus a two-state analog to the initial condition discussed in Ref. \[34\]. Inspection of eq. (12) shows that these initial conditions are special in two ways: i) the envelope \( e^{-K(t)} \) is constant in time, \( 1/T_2 = 0 \) and ii) visibility \( V[-1, 0] \) has the maximum possible value. It is related to the visibility of the factorized system by \( V[-1, 0] = \sqrt{V[0, 0]} \). These observations can be physically motivated from the minimization of the total energy, which implies that no further rearrangement of the bath is necessary for forming the proper dressed states and the dressing is optimum by accommodating the smallest possible total energy. Consequently, any other initial condition is not optimal and contains irreversible parts of the interaction.
Figure 2. Coherent oscillations and their envelope in the Ohmic spin-boson model with $\omega_c = 10\epsilon$, $\alpha = 0.1$ and $k_B T = 0.5\epsilon$. The envelope predicted by Bloch-Redfield theory has the correct slope at long times, but misses overall amplitude.

with the environment, i.e., genuine $T_2$ decoherence (i.e. finite $T_2$), which we have already identified above as redistribution of surplus energy.

We appreciate from these results, that the initial conditions play a decisive role. In fact, the visibility prefactor is a long-time consequence of short-time physics governed by the initial conditions. The choice of initial condition depends on the type of physical environment and experimental procedure: A detector is typically switched on during the experiment and is well described by factorizing initial conditions. If the switch is non-adiabatic and it happens on a time-scale $\tau_c$, this can be taken into account by choosing $u(\omega) = 0$ for $\omega \tau_c \ll 1$ and $u(\omega) = 1$ for $\omega \tau_c \gg 1$ as an initial state in the expression for the visibility. The baths in the material and/or the control environment are permanently coupled to the qubit. In a typical experiment, where on top of the pure dephasing Hamiltonian eq. (1) a large $\Delta \sigma_x$ term is applied at times $t < 0$, the entangled initial condition gives a realistic approximation for the initial state, predicting a reduced visibility at all times. Realistic experiments, involving a non-trivial preparation sequence encoded in $\theta(t)$ at $t < 0$ may be described by variants of these initial conditions. However, the fact that the variational ground state shows the highest possible visibility outlines the strength of our result: In many experiments, the detector signal representing $s_x = 1$ is not known a priori and is obtained by ground state measurements. Following our results, the visibility obtained in a dynamical experiment is always smaller, $V[u, v] \leq V[-1, 0]$. Thus, our theory does even apply to
these data, the loss of visibility is not an artefact of the initial condition, it rather is a generic consequence of the fact that quantum computing takes place far from thermal equilibrium. In fact, the short time slip accommodates the extra energy of the system relative to the dressed ground state. Moreover, experiments with good short-time resolution such as the phase qubit setup show an explicit sign of an initial drop of the oscillation amplitude \[35\].

4. Beyond pure dephasing

So far, we have primarily discussed the exactly solvable pure dephasing point of Hamiltonian eq. (11), mainly for being able to easily circumvent the Born and Markov approximations. Our qualitative results do however not depend on that assumption. Although a more general solution may require more elaborate methods such as path integral expansions [21], flow equations [36, 37] or real-time RG [38], we can verify this conclusion by elementary means for the gapped environment, when \(\omega_{\text{IR}} > E\). As shown before, this environment does not lead to Markovian decoherence and thus all decoherence effects are necessarily purely nonmarkovian. In this case, it is legitimate to proceed by perturbation theory in \(E/\omega_i \ll 1\). The eigenstates of the unperturbed, \(E = 0\), Hamiltonian are two-fold degenerate and can be written as \(|\psi_{\pm},\{n_i\}\rangle = \pm \prod_k D^{(k)}(\pm \gamma_i)|n_i\rangle\) with eigenenergies \(E(\{n_i\}) = E_0 + \sum_i(n_i + 1/2)\omega_i\). The perturbation Hamiltonian lifts the degeneracy and splits the doublets into sublevels

\[
\begin{align*}
|g, \{n_i\}\rangle &= -\sin \theta_{\text{eff}}/2|+, \{n_i\}\rangle + \cos \theta_{\text{eff}}/2|-, \{n_i\}\rangle \\
|e, \{n_i\}\rangle &= \cos \theta_{\text{eff}}/2|+, \{n_i\}\rangle + \sin \theta_{\text{eff}}/2|-, \{n_i\}\rangle.
\end{align*}
\]

Here, the angle depends on the \(n_i\) and is given by \(\tan \theta_{\text{eff}}(\{n_i\}) = c(\{n_i\})\tan \theta\) with \(c_i = \langle \{n_i\}|D(2\gamma)|\{n_i\}\rangle\). This can be viewed as a down-scaling of the effective tunnel splitting connected to that specific energies. The shifted energies are \(E_{g/e}(\{n_i\}) = \pm E_{\text{eff}}(\{n_i\})/2 + E(\{n_i\})\) with \(E_{\text{eff}}(\{n_i\}) = E\sqrt{\cos^2 \theta + \sin^2 \theta c(\{n_i\})^2}\).

We now use these approximate eigenstates for the Gedanken experiment analogous to the pure dephasing case: We prepare the system in the equal superposition \(|\psi_0\rangle = \frac{1}{\sqrt{2}}(|g, \{0\}\rangle + |e, \{0\}\rangle)\) factorized to the bath and compute the probability of returning onto the initial state, \(S(t) = |\langle \psi_0|\psi(t)\rangle|^2\). \(|\psi(t)\rangle\) is obtained by expanding the \(|\psi_0\rangle\) in the basis of the approximate eigenstates, \(|\psi_0\rangle = \sum_i(d_+(\{n_i\})|e, \{n_i\}\rangle + d_-(\{n_i\})|g, \{n_i\}\rangle)\). The expansion coefficients read

\[
\begin{align*}
d_-(\{n_i\}) &= \sqrt{\frac{\gamma(\{n_i\})}{2}} \left( \sin \left( \frac{\theta - \theta_{\text{eff}}}{2} \right) + \cos \left( \frac{\theta - \theta_{\text{eff}}}{2} \right) \right) \\
d_+(\{n_i\}) &= \sqrt{\frac{\gamma(\{n_i\})}{2}} \left( -\sin \left( \frac{\theta - \theta_{\text{eff}}}{2} \right) + \cos \left( \frac{\theta - \theta_{\text{eff}}}{2} \right) \right) i
\end{align*}
\]

where \(\gamma = \langle \{n_i\}|D(\alpha)|0\rangle\). This expression nicely illustrates our previous point that the initial state is broken up into entangled states that contain significant bath excitations.
We now propagate these in time. We find without further approximations

\[ S(t) = \sum_{\{n_i\}} \gamma(\{n_i\}) e^{-i \sum_i n_i \omega_i t} \left[ \cos(E_{\text{eff}}(\{n_i\}) t) + \sin(\theta - \theta_{\text{eff}}) \sin(E_{\text{eff}}(\{n_i\}) t) \right] \]  

We recover the factorized pure dephasing result in the case of \( \theta = 0 \). For the other extreme, \( \theta = \pi/2 \), we recognize that the last term is dropping out but the sum cannot be performed analytically. We observe because the \( \{n_i\} \) dependence of the argument of the cosine provided additional decoherence. For general \( \theta \), the second, phase shifted term that originates from misalignment of the pure and effective magnetic field \( \theta \neq \theta_{\text{eff}} \) on the Bloch sphere provides yet another dephasing channel. A full numerical analysis of these cases goes far beyond the scope of this paper. Qualitatively, we see that the dephasing familiar from the pure dephasing case combines with further dephasing channels, rendering our previous discussion to remain valid beyond the pure dephasing point.

Note that \( E_{\text{eff}} \) at \( n_i \equiv 0 \) leads to the same expression that appears in the adiabatic renormalization treatment at \( \theta = \pi/2 \) \[16\].

Clearly, the results of this section recover the features of the above discussion and shows, that the assumption of pure dephasing is not crucial for the physics of the loss of visibility. Any more quantitative statement should build on more quantitative numerical methods. A wealth of these methods has been developed in the last years, including work on the numerical renormalization group \[39\], analytical RG \[40\], and non-Markovian master equations \[41\].

5. Possible relevance for superconducting qubits

We have seen that for experiments with long \( T_2 \), a pronounced loss of visibility is governed by gapped baths, unlike the temporal decay, which is dominated by \( 1/f \)-noise. Such baths can be identified in experiments. We list a few cases and give their factorized visibility. Note that higher values of the visibility indicate the inadequacy of factorized initial conditions.

In the flux qubit\[7\], the contribution of the junction quasiparticles can be approximated by a gapped Ohmic model with \( \omega_{\text{IR}} = 2 \Delta, \omega_c = (R_N C_J)^{-1} \) where \( R_N \) is the normal-state resistance of the Josephson junctions and \( C_J \) the junction capacitance. \( \alpha = R_Q/R_N (\Phi/\Phi_0 - 1/2)^2 \) where \( R_Q = h/4e^2 \) is the quantum resistance for Cooper pairs. Using the numbers of Ref. \[7\] we obtain \( V[0,0] = 0.8 \).

In the Quantronium and other samples, the quasiparticles are largely shunted through the capacitor, \( \omega_{\text{IR}} > \omega_c \). There, however, an RC-element is fabricated on chip \[6\]. The capacitor is used to decouple the resistor at low frequencies. Indeed, the environmental spectral density is super-Ohmic, \( J(\omega) \propto \omega^3 \) at \( \omega RC \ll 1 \). Using the expression in Ref. \[42\] with eq. \[12\] we obtain a reduction of visibility from this term alone of \( V[0,0] = 0.6 \).
Phase qubits exhibit a large number of spurious resonances. There is no exact mapping onto an oscillator bath, however, we can still estimate a visibility log $v = -\sum_i \frac{T_i^2}{|E_i - E|}$ where $E_i$ is the energy of the ith resonance and $T_i$ is the coupling matrix element. This expression can be checked from experimental data when the $T_j$ and $E_j$ have been mapped out.

Recent experiments in charge qubits report extremely high visibility [43]. This result agrees with our scenario: The off-resonant degrees of freedom which appear in $v$ but not in $T_{1/2}$ are filtered out by a cavity.

The impact of phonons has been studied in Ref. [19]. Due to the super-Ohmic spectrum they would be a clear candidate for our scenario, however, the resulting number even for factorized initial condition is extremely close to unity. The topic of visibility has been studied recently for phase qubits [44, 45] and for the Ohmic spin-boson model [22]. The connection between exact solutions and approximate master equations is discussed in depth in [46]. Both of these work assume weak coupling to the bath and factorizing initial conditions. If we expand our results to lowest order, they cover those papers as special cases. Similar calculations have been done in Refs. [24, 25] for $\theta = 0$ and factorized initial conditions.

6. Conclusions

We have studied the Spin-Boson model beyond the Born and Markov approximations. We have shown, that for nonequilibrium initial conditions, a long-time observer will measure a reduced visibility of quantum oscillations as a consequence of short-time physics involving higher-order processes. This is well compatible with long $T_2$ for gapped or superohmic environmental spectra and with a range of experimental data.

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