Euclidean 4-simplices and invariants of four-dimensional manifolds: I. Moves $3 \rightarrow 3$

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Abstract

We construct invariants of four-dimensional piecewise-linear manifolds, represented as simplicial complexes, with respect to rebuildings that transform a cluster of three 4-simplices having a common two-dimensional face in a different cluster of the same type and having the same boundary. Our construction is based on the usage of Euclidean geometric values.

1 Introduction

The present work is planned to be the first one in a series of papers whose aim is the construction of an invariant of four-dimensional piecewise-linear manifolds similar to the invariants of three-dimensional manifolds constructed in papers [1] and [2]. We recall that in those papers we were considering orientable manifolds represented as simplicial complexes (or “pre-complexes” where the boundary of a simplex may contain several identical components). We assigned a Euclidean length to every edge, and a sign $+$ or $-$ to every 3-cell (tetrahedron) in such way that the following “zero curvature” condition was satisfied: the algebraic sum of dihedral angles abutting on each edge was zero modulo $2\pi$. Here each dihedral angle is determined by the lengths of edges of the corresponding tetrahedron and is taken with the sign that we have assigned to that tetrahedron. Then we considered infinitesimal variations of edge lengths and infinitesimal “deficit angles” at edges depending on those length variations. The matrix that expressed these (linear) dependencies played the key rôle in the resulting explicit formula for manifold invariants.

The paper [2] shows that the invariants constructed in this way are sensitive enough in the three-dimensional case: they distinguish (in particular) the homotopy equivalent lens spaces $L(7,1)$ and $L(7,2)$. On the other hand, a formula derived in preprint [3] suggests the possibility of generalizing these constructions to the four-dimensional case and, most likely, higher dimensions. That formula deals with the re-building of one
cluster of three 4-simplices, i.e. is local in a natural sense. The present work is devoted to constructing invariants of any sequences of such $3 \rightarrow 3$ re-buildings (three 4-simplices are replaced with three different 4-simplices with the same boundary) in a “large” simplicial complex.

Now we describe the contents of the following sections. In Section 2 we derive the “local” formula, that is, the formula for one re-building. In comparison with paper [3], the formula is derived with some refinements due to the fact that we now take into account the orientations of 4-simplices and, consequently, the signs of dihedral angles. In Section 3 we introduce the basic values — lengths, areas and deficit angles of two types — from which our invariants will be built in the global case. In Section 4 we prove some useful properties of matrices that express the relations between the differentials of those values. Here, a key rôle is played by the symmetry property of one of those matrices, which is deduced with the help of the four-dimensional version of Schlӓfi differential identity (see [4]). In Section 5 we obtain our “global” invariant of moves $3 \rightarrow 3$. To do that, we need to use the local formula in combination with some differential forms. In the concluding Section 6 we discuss our results.

## 2 The local formula

Consider a Euclidean 4-simplex $ABCDE$ with all edge lengths fixed except the length of edge $AE$. Opposite $AE$, there is a two-dimensional face $BCD$ where two three-dimensional faces come together, forming an inner dihedral angle $\vartheta_{BCD}$. The starting point of our constructions is the formula

$$\frac{dL_{AE}}{V_{ABCDE}} = 24 \frac{d\vartheta_{BCD}}{S_{BCD}},$$

(1)

where we denoted by $L_{AE}$ the squared length $AE$, while $V_{ABCDE}$ and $S_{BCD}$ are the corresponding hypervolume and area (cf. [5, formula (15)]).

Now we consider six points $A, \ldots, F$ in $\mathbb{R}^4$ with all distances between them fixed except $AE$ and $AF$. Then (cf. [3, (16)])

$$\frac{dL_{AE}}{V_{ABCDE}} = \frac{dL_{AF}}{V_{ABCDF}}.$$

(2)

Proof: the angles at the face $BCD$ in 4-simplices $ABCDE$ and $ABCD$ differ by the value of the angle at the same face in a “rigid” 4-simplex $BCDEF$ where all lengths are fixed. Formula (2) follows from the equalness of length differentials, with (1) taken into account.

**Important remark:** in formula (2), one has to regard $V_{ABCDE}$ and $V_{ABCDF}$ as oriented volumes and to take the dihedral angles therein with the signs coinciding with the signs of volumes (to be exact, we assume that they lie in the interval $(0, \pi)$ for a positive volume and in $(-\pi, 0)$ for a negative one).

It is clear that we can obtain relations between $dL$ for any two edges sharing a common vertex by permutations of letters in (2). If, however, we want to relate, e.g.,
\( dL_{AB} \) with \( dL_{DE} \) assuming the rest of the lengths constant, we can act as follows: consider \( L_{AD} \) as a function of \( L_{AB} \) and \( L_{DE} \) (the other lengths are fixed) and find

\[
\frac{dL_{DE}}{dL_{AB}} = -\frac{\partial L_{AD}/\partial L_{AB}}{\partial L_{AD}/\partial L_{DE}}
\]

as the derivative of an implicit function \( L_{AD}(L_{AB}, L_{DE}) = \text{const.} \) In this way we find

\[
\frac{dL_{DE}}{V_A V_B} = -\frac{dL_{AB}}{V_D V_E},
\]

where we adopt the following notations: \( \hat{A}, \hat{B} \) etc. is the sequence \( ABCDEF \) with a removed letter \( A, B \) etc.; \( V_{\ldots} \) means oriented four-dimensional volume.

Consider now three 4-simplices \( ABCEF = \hat{D}, ABCFD = -\hat{E} \) and \( ABCDE = \hat{F} \) with the common two-dimensional face \( ABC \). We denote the angles in those respective simplices at that face as \( \alpha, \beta \) and \( \gamma \). They form together the full angle: \( \alpha + \beta + \gamma = 0 \pmod{2\pi} \). Assuming that everything is taking place in a Euclidean space and only \( AB \) and \( DE \) of all lengths can change, we find:

\[
0 = d(\alpha + \beta + \gamma) = \frac{\partial \gamma}{\partial L_{DE}} dL_{DE} + \frac{\partial(\alpha + \beta + \gamma)}{\partial L_{AB}} dL_{AB}.
\]

(4)

Here, of course, \( \partial \alpha/\partial L_{AB} \) is calculated from simplex \( \hat{D} \), while \( \partial \beta/\partial L_{AB} \) — from \( \hat{E} \) and \( \partial \gamma/\partial L_{AB} \) — from \( \hat{F} \).

We denote the sum \( \alpha + \beta + \gamma \), regarded as a function of ten formally independent squared lengths, as \( -\omega_{ABC} \) (cf. below formula (9)). It can be regarded this way because each of angles \( \alpha, \beta, \gamma \) is calculated separately from its own 4-simplex. Clearly, all simplices can be placed together in \( \mathbb{R}^4 \) only if \( \omega_{ABC} = 0 \pmod{2\pi} \).

Taking into account that

\[
\frac{\partial \gamma}{\partial L_{DE}} = \frac{1}{24} \frac{S_{ABC}}{V_{\hat{F}}}
\]

(cf. (1)) and comparing (4) with (3), we get:

\[
\frac{\partial \omega_{ABC}}{\partial L_{AB}} = -\frac{S_{ABC}}{24} \frac{V_A V_B}{V_D V_E V_{\hat{F}}}. \tag{5}
\]

Similarly, we define \( \omega_{DEF} \) as minus sum of the dihedral angles at face \( DEF \) in 4-simplices \( \hat{A}, -\hat{B} \) and \( \hat{C} \), and get the equality

\[
\frac{\partial \omega_{DEF}}{\partial L_{DE}} = -\frac{S_{DEF}}{24} \frac{V_{\hat{D}} V_{\hat{E}}}{V_A V_B V_C}. \tag{6}
\]

The value \( \omega_{DEF} \) can, too, be considered as a function of ten formally independent \( L \).

We will be dealing only with infinitesimal deviations of values \( L \) from the “flat” case \( \omega_{ABC} = \omega_{DEF} = 0 \). In such case, the differentials \( d\omega_{ABC} \) and \( d\omega_{DEF} \) are proportional, because they are two linear forms, both depending on ten \( dL \), whose null spaces
(kernels) coincide (because both conditions \( d\omega_{ABC} = 0 \) and \( d\omega_{DEF} = 0 \) mean that the corresponding infinitesimal deformation of edge lengths can be realized in \( \mathbb{R}^4 \)). In particular, if only \( L_{AB} \) and \( L_{DE} \) can change then
\[
d\omega_{ABC} = c_1 dL_{AB} + c_2 dL_{DE},
d\omega_{DEF} = c_3 dL_{AB} + c_4 dL_{DE},
\] (7)
where the equal ratios \( c_1/c_2 \) and \( c_3/c_4 \) are found by comparing (7) with the relation (8) that was obtained, of course, under the condition \( d\omega_{ABC} = d\omega_{DEF} = 0 \). Namely,
\[
c_1/c_2 = c_3/c_4 = V_A^\hat{A} V_B^\hat{B}/V_D^\hat{D} V_E^\hat{E}.
\]
Besides, \( c_1 \) and \( c_4 \) are right-hand sides of (7) and (8), respectively.

All this together allows us to calculate the proportionality factor between \( d\omega_{ABC} \) and \( d\omega_{DEF} \). We write the result in the following form:
\[
V_D^\hat{D} V_{-E}^\hat{E} V_{-F}^\hat{F} d\omega_{ABC}/S_{ABC} = V_{-B}^\hat{B} V_C^\hat{C} d\omega_{DEF}/S_{DEF}.
\] (8)

The most remarkable thing in relation (8) is that all the values entering in its l.h.s. belong to 4-simplices \( \hat{D}, \hat{E} \) and \( \hat{F} \), while those in its r.h.s. — to \( \hat{A}, \hat{B} \) and \( \hat{C} \). Thus, (8) is an algebraic relation corresponding to the move \( 3 \rightarrow 3 \).

We make also the following remarks. Of course, \( V_{-E} \) and \( V_{-B} \) are nothing but \( -V_E \) and \( -V_B \). The way we have written the formula (8) is due to the fact that 4-simplices \( \hat{D}, -\hat{E} \) and \( \hat{F} \) have consistent orientations, and the same applies to \( \hat{A}, -\hat{B} \) and \( \hat{C} \). On the other hand, the common boundary of 4-simplices \( \hat{A}, -\hat{B} \) and \( \hat{C} \) coincides with that of \( \hat{D}, -\hat{E} \) and \( \hat{F} \) with the orientation taken into account. The sign of each of dihedral angles entering in \( -\omega_{ABC} \) and \( -\omega_{DEF} \) coincides with the sign of the respective quantity \( V_A^\hat{A}, V_{-B}^\hat{B}, V_C^\hat{C}, V_D^\hat{D}, V_{-E}^\hat{E}, \) or \( V_F^\hat{F} \).

3 The global case: metric values and their differentials

In this Section, we introduce the principal quantities — lengths, areas, and two types of deficit angles — that will take part in the game in the global case. Relations between variations, i.e. differentials, of those values will play the main rôle.

Consider an orientable four-dimensional manifold \( M \). We represent it as a simplicial complex. Choose a consistent orientation of the 4-simplices in this complex (by orientation of a simplex we understand an ordering of its vertices taken up to even permutations).

We assign lengths to the edges of the complex and signs to its 4-simplices in full analogy with the three-dimensional case described in the Introduction to paper [2]. Namely, we consider the universal cover of the triangulation of manifold \( M \) which is
itself, of course, a simplicial complex. Its vertices are divided in classes which are inverse images of each given vertex w.r.t. the covering map. Let a homomorphism (representation) \( \varphi: \pi_1(M) \to E_4 \) be fixed of the fundamental group of manifold \( M \) in the group of motions of the four-dimensional Euclidean space. We construct a mapping \( f \) from the set of vertices of the universal cover to \( \mathbb{R}^4 \) in the following way. Map one arbitrarily chosen vertex in each class to an arbitrary point. Any of the remaining vertices, say, vertex \( B \), can be represented as \( B = g(A) \) for some element \( g \in \pi_1(M) \) and some vertex \( A \) whose image has already been constructed. Demand then that \( f(B) = \varphi(g) f(A) \).

Having constructed the images of the vertices of our complex universal cover in \( \mathbb{R}^4 \), we can also put in correspondence to every cell of nonzero dimension — an edge, a triangle, a tetrahedron, a 4-simplex — its image in \( \mathbb{R}^4 \). That will be, of course, the convex hull of its vertex images. We emphasize that we are not afraid of possible intersections of the resulting simplices in \( \mathbb{R}^4 \) but we demand that all the 4-simplices in \( \mathbb{R}^4 \) — the images of 4-simplices from the universal cover — be nondegenerate (have nonzero volume).

Thus we get lengths associated to the edges of the universal cover — they are, of course, the distances between the respective vertices in \( \mathbb{R}^4 \). It is clear that if two given edges are mapped into the same one by the covering map then their lengths obtained from our construction are equal (because their images in \( \mathbb{R}^4 \) are taken one into the other by some element \( \varphi(g) \in E_4 \)).

We assume that we have fixed an orientation of \( \mathbb{R}^4 \). Then a 4-simplex from the triangulation (recall that we have chosen a consistent orientation for them) either preserves its orientation under the mapping in \( \mathbb{R}^4 \), or changes it. In the first case, we assign to it the sign “+”, in the second — the sign “−”. Clearly, if two 4-simplices are sent to the same simplex by the covering map then they get the same sign.

This means that we have assigned lengths to the edges and signs to the 4-simplices of the manifold \( M \)’s triangulation itself and not only of its universal cover.

As soon as the edges of each 4-simplex have acquired Euclidean lengths, one can calculate (inner) dihedral angles formed by pairs of three-dimensional faces of the 4-simplex intersecting at a two-dimensional face. Below, we consider every such angle with the sign + or − coinciding with the one assigned to the 4-simplex. That is what we mean when we speak below of algebraic sums of such angles.

We call the deficit angle \( \omega_{ABC} \) at a two-dimensional face \( ABC \) of the complex the algebraic sum of all dihedral angles around that face taken with the minus sign and considered to within the multiples of \( 2\pi \):

\[
\omega_{ABC} \overset{\text{def}}{=} -\sum_k \vartheta_k \pmod{2\pi} \quad (9)
\]

(subscript \( k \) numbers the angles). Clearly, all deficit angles in our construction are so far zero: the cluster of all 4-simplices having a given common two-dimensional face can be mapped in the Euclidean space so that every separate 4-simplex is mapped isometrically and the angles are taken with the proper signs. We are going, however,
to consider two kinds of deformations of Euclidean values attributed to every 4-simplex that bring about nonzero deficit angles.

The first kind of deformations consists in small changes of edge lengths: we vary each of them arbitrarily in a neighbourhood of the value obtained from the above construction. Every separate 4-simplex remains Euclidean, but a mapping of their cluster in the Euclidean space of the above type may no longer exist. As in Section 2, we prefer to deal with squared edge lengths, denoting, for instance, the squared length of edge $AB$ as $L_{AB}$. We use letters $a, b, \ldots$ for numbers of edges of the complex, and letters $i, j, \ldots$ — for numbers of two-dimensional faces. An example of these notations is the matrix of partial derivatives $(\partial \omega_i / \partial L_a)$ which will play one of the main rôles in our work.

The second kind of deformations of Euclidean quantities looks, probably, more unusual. As is known, the variations of all metric values in a given 4-simplex can be calculated from variations of the areas of its ten two-dimensional faces (although a “global” statement of such kind does not hold: areas, in contrast with lengths, do not determine a 4-simplex uniquely). We take the values of areas obtained from the above construction as initial ones and then give them arbitrary independent variations. These variations determine the variations of dihedral angles in each 4-simplex and, consequently, the variations of deficit angles. In doing so, we ignore the fact that the variations of the length of one and the same edge may not coincide if we calculate them from two neighbouring 4-simplices.

Besides the angles at two-dimensional faces, we will need the angles at edges. Consider at first one single 4-simplex. We define the angle at its edge $a$ as

$$\Theta_a \overset{\text{def}}{=} \sum_i \frac{\partial S_i}{\partial L_a} \vartheta_i,$$

where $i$ numbers the two-dimensional faces of the simplex (there are four nonzero terms in the sum), while $S_i$ is, of course, the area of the $i$th face. The reasonableness of such definition can be seen, by the way, from the fact that thus defined angles obey an analogue of the Schlåfli differential identity (see [4]).

Recall that the usual Schlåfli identity for a 4-simplex states that

$$\sum_i S_i \, d\vartheta_i = 0$$

for any infinitesimal deformations. We are going to prove that

$$\sum_a L_a \, d\Theta_a = 0$$

as well. To do so, it is enough to consider instead of $d\Theta_a$ the partial derivatives w.r.t. an arbitrary $L_b$:

$$\sum_a L_a \frac{\partial \Theta_a}{\partial L_b} = \sum_a L_a \sum_i \frac{\partial^2 S_i}{\partial L_a \partial L_b} \vartheta_i + \sum_a L_a \sum_i \frac{\partial S_i}{\partial L_a} \frac{\partial \vartheta_i}{\partial L_b}.$$
Now we note that $S_i$ are homogeneous functions of degree 1 in squared lengths $L_a$. This yields $\sum_a L_a \partial S_i / \partial L_a = S_i$ and $\sum_a L_a \partial^2 S_i / (\partial L_a \partial L_b) = 0$. Now we see that both terms in (12) vanish (the second one — because of the usual Schlafli identity (11)).

To conclude this Section, we give a natural definition of the deficit angles around edges:

$$\Omega_a \overset{\text{def}}{=} -\sum_k (\Theta_a)_k \pmod{2\pi}, \quad (13)$$

where $k$ numbers 4-simplices that contain edge $a$.

### 4 Matrices relating various differentials: symmetry properties

We now consider matrices that express the linear dependencies of the differentials of deficit angles of two types introduced in Section 3 on the differentials of squared lengths or areas. These latter differentials, as was already explained, are regarded as independent variables. Everything takes place in an infinitesimal neighbourhood of the flat case (where all deficit angles are zero).

**Theorem 1** The matrix $(\partial \omega_i / \partial S_j)$ is symmetric:

$$\frac{\partial \omega_i}{\partial S_j} = \frac{\partial \omega_j}{\partial S_i}. \quad (14)$$

**Proof.** Consider first a similar statement for one 4-simplex:

$$\partial \vartheta_i / \partial S_j = \partial \vartheta_j / \partial S_i. \quad (15)$$

To prove it, we consider a function $f(S_1, \ldots, S_{10}) = \sum_i S_i \vartheta_i$, where the sum is taken over all two-dimensional faces of the 4-simplex. The relation (15) follows from the equalness of the mixed derivatives $\partial^2 f / (\partial S_i \partial S_j) = \partial^2 f / (\partial S_j \partial S_i)$ if we use the Schlafli formula (11) while calculating them. Now formula (14) is obtained by summing over the dihedral angles entering in the given deficit angle.

Consider now the rectangular matrix $(\partial \Omega_a / \partial S_i)$. Note that the elements of this matrix are determined correctly in spite of the fact that the definition of $\Omega_a$ involves some quantities (partial derivatives of the areas w.r.t. the lengths) whose variations are not determined uniquely (they depend on the given 4-simplex, like the variations of lengths do). The point is that all such quantities are multiplied by the vanishing deficit angles $\omega_j$ when we calculate $\partial \Omega_a / \partial S_i$.

**Theorem 2** Matrices $(\partial \Omega_a / \partial S_i)$ and $(\partial \omega_i / \partial L_a)$ are mutually conjugate:

$$\frac{\partial \Omega_a}{\partial S_i} = \frac{\partial \omega_i}{\partial L_a}. \quad (16)$$
Proof follows from the chain of equalities:

\[
\frac{\partial \Omega_a}{\partial S_i} = \frac{\partial}{\partial S_i} \left( \sum_j \frac{\partial S_j}{\partial L_a} \omega_j \right) = \sum_j \frac{\partial \omega_j}{\partial S_i} \frac{\partial S_j}{\partial L_a} = \sum_j \frac{\partial \omega_i}{\partial S_j} \frac{\partial S_j}{\partial L_a} = \frac{\partial \omega_i}{\partial L_a}.
\]

Here we have used, first, the expression of \( \Omega_a \) in terms of \( \omega_j \) (obtained from (9), (10) and (13)), secondly, the fact that the derivatives are taken with all \( \omega_j = 0 \) and, finally, the equality (14).

5 The invariant of moves 3 \( \rightarrow \) 3

The preparatory work made in Sections 3 and 4 allows us now to pass from the local formula (8) to its global analogue.

**Theorem 3** Choose in matrix \((\partial \omega_i/\partial L_a)\) a largest square submatrix \( \mathcal{B} \) with nonzero determinant. Let \( \mathcal{B} \) contain a row corresponding to such face \( i = ABC \) that belongs to exactly three 4-simplices. Then those latter can be replaced by three new 4-simplices, as in Section 3. After such replacement, take in the new matrix \((\partial \omega_i/\partial L_a)\) the new submatrix \( \mathcal{B} \) containing the same rows and columns, with the only following change: the row corresponding to face \( ABC \) is replaced by the row corresponding to the new face \( DEF \).

The expression

\[
\det \mathcal{B} \cdot \frac{\prod_{V \text{ over all 4-simplices}} V}{\prod_{S \text{ over all 2-dim. faces}} S}
\]  

(17)

...does not change under such re-building.

Proof. The only thing in formula (17) which is not completely evident when compared with (8) is that there is a determinant \( \det \mathcal{B} \) instead of just one differential \( d\omega \). It is convenient to write that determinant as \( \det \mathcal{B} = \bigwedge d\omega / \bigwedge dL \), where the exterior product in the numerator is taken over the two-dimensional faces corresponding to the rows of matrix \( \mathcal{B} \), while the one in the denominator is taken over the edges corresponding to the columns of \( \mathcal{B} \). Now we note that if \( \omega_{ABC} = 0 \) or, equivalently, \( \omega_{DEF} = 0 \), then the deficit angles at the remaining faces do not change under the re-building (because they are obtained from the same dihedral angles). Thus, the difference \( d\omega_j - d\omega'_j \) for any of the remaining faces (with those differentials considered as depending on \( dL \)'s), taken before and after the re-building, is proportional to \( d\omega_{ABC} \) or,
equivalently, to $d\omega_{DEF}$. Consequently, $\det B$ gets multiplied by $d\omega_{ABC}/d\omega_{DEF}$ under the re-building. The theorem is proven.

Expression (17) depends on a specific choice of the subsets of faces and edges corresponding to the rows and columns of matrix $B$. In getting rid of these dependencies, we will be helped, like in the three-dimensional case of papers [1, 2], by some differential forms.

We denote by $D$ the set of two-dimensional faces corresponding to the rows of matrix $B$. By $\overline{D}$ we denote the set of the rest of the two-dimensional faces in the complex. Similarly, the set of edges corresponding to the columns of $B$ will be denoted $C$, while $\overline{C}$ will be the set of the remaining edges in the complex. We write, somewhat loosely:

$$B = D \left( \frac{\partial \omega_i}{\partial L_a} \right) C.$$

The looseness consists in the fact that, for matrix $B$ to be determined completely, we need also the ordering of rows and columns, and this ordering influences the sign of determinant $\det B$.

At the moment, we assume that the set $D \cup \overline{D}$ of all two-dimensional faces is fixed. The set $C \cup \overline{C}$ is fixed, too, because the set of all edges does not change under re-buildings $3 \rightarrow 3$ which we are considering.

**Theorem 4** The expression

$$\left( \det B \right)^{-1} \cdot \bigwedge_{i \in D} dS_i \cdot \bigwedge_{a \in \overline{C}} dL_a,$$

taken to within its sign, does not depend on the choice of sets $C$ and $D$.

We explain the meaning of expression (18). Set $\overline{C}$ is, in full analogy with papers [1, 2], a largest subset in the set of all edges such that the lengths of edges from $\overline{C}$ can be given arbitrary infinitesimal increments $dL_a$ without violating the “zero curvature” conditions $d\omega_i = 0$ for all $i$. The rest of $dL_a$ (for $a \in C$) are determined uniquely from those conditions. This allows us to express $\bigwedge_{a \in \overline{C}} dL_a$ in terms of $\bigwedge_{a \in C} dL_a$, where $\overline{C}_1$ is another set similar to $\overline{C}$. To be exact, these exterior products are proportional, and we can calculate the proportionality factor from matrix $(\partial \omega_i / \partial L_a)$. This means that we consider exterior products of such kind as products of linear forms defined on the linear space of such infinitesimal deformations of lengths which leave the deficit angles zero.

It turns out that set $\overline{D}$ admits a similar description: $\overline{D}$ is a largest subset in the set of all two-dimensional faces in the complex such that the areas of faces in $\overline{D}$ can be given arbitrary infinitesimal increments $dS_i$ without violating the “zero curvature around edges” conditions $d\Omega_a = 0$ for all $a$. Here we regard $d\Omega_a$ as functions of all $dS_i$, 
as in Section 4. The values $dS_i$ for $i \in \mathcal{D}$ are determined from the conditions $d\Omega_a = 0$ unambiguously.

Indeed, Theorem 2 states that matrices $(\partial \Omega_a / \partial S_i)$ and $(\partial \omega_i / \partial L_a)$ are obtained one from the other by matrix transposing (which we denote by superscript T). Consequently, $\mathcal{B}^T = c(\partial \Omega_a / \partial S_i)_{\mathcal{D}}$ is a maximal nondegenerate square submatrix in $(\partial \Omega_a / \partial S_i)$, and this yields our statements.

Thus, we consider exterior products of area differentials on the linear space of such infinitesimal deformations of the areas that leave the deficit angles around edges zero. If $\mathcal{D}_1$ is a different set of two-dimensional faces similar to $\mathcal{D}$ then the forms $\bigwedge_{i \in \mathcal{D}_1} dS_i$ and $\bigwedge_{i \in \mathcal{D}} dS_i$ are proportional, with the proportionality factor calculated again from matrix $(\partial \omega_i / \partial L_a)$.

**Proof of Theorem 4.** We prove first that expression (18) does not change when $\mathcal{C}$ and $\bar{\mathcal{C}}$ exchange their elements. For simplicity, we limit ourselves to the case where only one edge $b$ is carried from $\bar{\mathcal{C}}$ to $\mathcal{C}$, while an edge $c$ takes its place. In doing so, $\bigwedge_{a \in \mathcal{C}} dL_a$ is of course multiplied by

$$\left. \frac{\partial L_c}{\partial L_b} \right|_{\text{with all } d\omega_i = 0 \text{ and } dL_a = 0, \text{ where } a \in \bar{\mathcal{C}}, \ a \neq b}.$$  

(19)

On the other hand, we can calculate the partial derivative (19) by considering the squared edge lengths in $\mathcal{C}$ as implicit functions of the squared edge lengths in $\bar{\mathcal{C}}$ given by the conditions $\omega_i = 0$, $i \in \mathcal{C}$. This yields

$$\frac{\partial L_c}{\partial L_b} = -\frac{\det \mathcal{B}_{\text{new}}}{\det \mathcal{B}}.$$  

Consequently, $\det \mathcal{B}$ gets multiplied, up to a sign, by the same expression (19) under such exchange $b \leftrightarrow c$.

We prove now that expression (18) does not change when $\mathcal{D}$ and $\bar{\mathcal{D}}$ exchange their elements. Again, we assume for simplicity that only one pair of elements is exchanged. To be exact, let there be $n$ two-dimensional faces in the complex, and let set $\mathcal{D}$ include the faces with numbers from 1 through $m$, while $\bar{\mathcal{D}}$ — those with numbers from $m+1$ through $n$. We are going to consider the interchange $m \leftrightarrow m+1$.

Certainly, $m = \text{rank}(\partial \omega_i / \partial L_a)$. Thus, in particular, $(m+1)$th row of matrix $(\partial \omega_i / \partial L_a)$ is a linear combination of the first $m$ rows with some coefficients $c_1, \ldots, c_m$. This can be written as the relation

$$d\omega_{m+1} = c_1 d\omega_1 + \cdots + c_m d\omega_m,$$  

(20)

and also as

$$(-c_1 \ldots -c_m 1 0 \ldots 0) \left( \frac{\partial \omega_i}{\partial L_a} \right) = 0.$$  

(21)
Matrix transposing relation (21), we find
\[
\left( \frac{\partial \Omega_a}{\partial S_i} \right) \begin{pmatrix} -c_1 \\ \vdots \\ -c_m \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0.
\]

This means that all \( d\Omega_a = 0 \) as soon as the following proportions are satisfied:
\[
dS_1 : dS_2 : \cdots : dS_m : dS_{m+1} : dS_{m+2} : \cdots : dS_n = \\
= -c_1 : -c_2 : \cdots : -c_m : 1 : 0 : \cdots : 0.
\]

In particular,
\[
\frac{\partial S_m}{\partial S_{m+1}} \bigg|_{m \leq m+1} \text{ with all } d\Omega_a = 0 \text{ and } dS_{m+2} = \cdots = dS_n = 0 = -c_m. \tag{22}
\]

Note that \( \bigwedge_{i \in D} dS_i \) gets multiplied exactly by (22) under the interchange of faces \( m \leftrightarrow m+1 \).

On the other hand, equality (20) shows that \( \det B \) gets multiplied by \( c_m \) under such interchange of faces. Thus, Theorem 4 is proven.

Combining Theorem 4 with Theorem 3, we get the desired invariant of moves 3 \( \rightarrow \) 3.

**Theorem 5** The value
\[
\prod_{\text{over all \ 2-dim. \ faces}} S \cdot \bigwedge_{\mathcal{S}} dS \cdot \bigwedge_{\mathcal{C}} dL \]
\[
\frac{\det B \cdot \prod_{\text{over all \ 4-simplices}} V}{\text{(23)}}
\]
does not depend on the choice of sets \( \mathcal{C} \) and \( \mathcal{D} \) and is an invariant of moves 3 \( \rightarrow \) 3.

**6 Discussion**

It is not hard to see in the expression (23) a direct generalization of the similar invariant of three-dimensional manifolds. To be exact, (23) looks very much like the squared expression for the three-dimensional manifold invariant from paper [1] (see [1], formula (30)). Note that the mentioned three-dimensional manifold invariant is an invariant of all kinds of re-buildings of three-dimensional simplicial complexes, or
Pachner moves, namely, $2 \leftrightarrow 3$ and $1 \leftrightarrow 4$. This makes us think that in the four-dimensional case, too, the applicability of expression (23) goes beyond the limits of just moves $3 \rightarrow 3$. Besides, this makes us hope for nontrivial consequences of Theorem 4, because the mentioned three-dimensional invariant passes the nontriviality test [2].

In the further works we plan to study the behaviour of expression (23) under rebuildings $2 \leftrightarrow 4$ and $1 \leftrightarrow 5$. Meanwhile, we propose the following argument in defence of investigating the moves $3 \rightarrow 3$ even by themselves, by referring to the analogy with the Yang–Baxter equation (YBE). As is well-known, YBE corresponds, from the knot theory viewpoint, to only one of the Reidemeister moves which are fundamental there. Nevertheless, YBE proves to be interesting enough by itself from an algebraic point of view, and finds nontrivial applications, in particular, in statistical physics and quantum field theory.

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