Experimental and Numerical Studies on Flow behind a Circular Cylinder Based on POD and DMD

Masayuki SAKAI, Yasuto SUNADA, Taro IMAMURA, and Kenichi RINOE

Department of Aeronautics and Astronautics, University of Tokyo, Tokyo 113–8656, Japan

Proper orthogonal decomposition (POD) or dynamic mode decomposition (DMD) are useful analytical techniques for describing the behavior of a complicated flow by decomposing the flow into several components called modes. In POD analysis, the flow is most efficiently decomposed into orthonormal modes in terms of energy, so most of the flow energy can be extracted using a small number of modes. On the other hand, in DMD analysis, flow structures that have specific frequencies can be observed by managing the sequence of the given time series data. In this study, POD and DMD analyses were applied to the experimental and numerical results for velocity fields around a circular cylinder, and the analytical results were compared. As a result, flow structures of POD and DMD modes from the experimental results were mostly in agreement with those from the numerical results, but differences were seen in energy distribution. In the POD analysis, the rates of energy in the numerical results were clearly divided in each pair of POD modes, but in the experimental results, they were not clear except for the first and second POD modes.

Key Words: Circular Cylinder, Proper Orthogonal Decomposition, Dynamic Mode Decomposition

Nomenclature

- $a_{ni}$: coefficients of POD modes
- $E_n$: time-averaged energies of POD modes
- $f$: vortex shedding frequency
- $F_n$: energies which DMD modes have
- $m$: the number of the acquired data
- $n$: dimension of the acquired data
- $q$: non-dimensional vorticity
- $Re$: Reynolds number
- $St$: Strouhal number
- $T$: non-dimensional time
- $U$: velocity component in the $x$-direction
- $U_{inf}$: freestream velocity
- $V$: velocity component in the $y$-direction
- $v_i$: instantaneous velocity vectors
- $v_f$: fluctuation velocity vectors
- $x$: flow direction
- $y$: direction perpendicular to freestream
- $\xi$: DMD modes
- $\psi$: POD modes

Subscripts

- exp: in the experimental results
- num: in the numerical results

1. Introduction

Today wind-tunnel experiments and numerical simulations are two main techniques for reproducing actual flow fields. Wind-tunnel experiments provide reliable measuring results, but measurement technology is limited and the operating cost is high. On the other hand, in numerical simulations, we can reproduce complicated flow fields at a low cost, but it is necessary to verify the code and assess the accuracy of the results. It is important to understand these factors and to combine experiments with numerical simulations.

As the quantity of data obtained by experiments and simulations grows due to the development of measurement technologies and computers, it becomes more difficult to analyze the data directly. Therefore, proper orthogonal decomposition (POD, otherwise known as principal component analysis) or Karhunen-Loève decomposition) and dynamic mode decomposition (DMD) have become popular. POD is an analytical method to efficiently abstract low-dimensional components from given multi-dimensional data, and is used widely in various fields such as statistics and photo processing. Fluid-dynamically, it is possible to extract flow structures that occupy a large amount of energy. Since Lumley applied POD analysis to turbulent flow fields in 1967, there have been many applications to various flow fields such as flow around a circular cylinder or separated and reattachment flows. On the other hand, in DMD analysis, it is possible to take in dynamic information by managing the sequence of the given time series data and observing flow structures that have specific frequencies. Though DMD is a comparatively new analytical method, Rowley et al. (2009) applied the DMD analysis to a jet flow and decomposed the flow into several frequency components. Other application examples can also be found in Schmid (2010, 2011), Schmid et al. (2011) and Chen et al. (2012). In our previous research, we measured flow around circular cylinders arranged independently, in a series or parallel by time-resolved PIV, and applied POD and DMD analyses to the data. In this paper, to verify the analysis results ob-
tained from those experiments, flow around a circular cylinder at a Reynolds number of $5.0 \times 10^3$ is numerically simulated using a two-dimensional CFD solver. POD and DMD analyses are applied to the data, and the experimental and numerical results are compared. Features of the POD and DMD analysis results and the influence of experimental and numerical errors are also considered.

2. Experimental Setup

In this section, the experimental technique is explained on the basis of our previous research.\textsuperscript{10}

Experiments were conducted in a small-sized, blow-down wind tunnel at the Department of Aeronautics and Astronautics, The University of Tokyo. The size of the air outlet was $300 \text{mm} \times 300 \text{mm}$. The freestream velocity was set to $U_\infty = 8.1 \times 10^2 \text{mm/s}$ in all experiments. From past measurements, the turbulence intensity of the mainstream direction in the test section measured using constant temperature anemometry was below 0.3% of the freestream velocity.

A circular cylinder model with a diameter of 10 mm was used. The Reynolds number based on the diameter was $Re = 5.0 \times 10^3$. The aspect ratio was set to $AR = 30$ so that observed mean flow fields at the mid-span could be treated as two-dimensional flow fields.

The time-resolved measurements of the flow velocity were performed using high-speed particle image velocimetry (PIV). The PIV system used in this work consisted of a Nd:Yag laser (Ray Power 2000, power 2 W, wavelength 532 nm) as a light source and a high-speed camera with a resolution of 1,024 $\times$ 1,024 pixels. Due to the restrictions of the irradiation width of the laser sheet and the resolution of the camera, the measurement domain was $50 \text{mm} \times 50 \text{mm}$ and the wake region of the model was mainly measured. The flow was seeded with tracer particles generated from Onion Oil by an atomizer. A series of images was acquired at a rate of 1,000 Hz for 2 s. Initial and final interrogation window sizes were $128 \times 128$ pixels and $32 \times 32$ pixels, respectively. The window overlap was 50% and $63 \times 63$ two-dimensional velocity vectors were obtained each time. After that, unnecessary or incorrect vectors were removed by masking and range validation. The series of operations was conducted using the commercial software “Dantec Dynamic Studio v3.14.”

In the PIV measurements, there were various error factors, such as a calibration error and an error associated with the measurement principle. When they were taken into consideration and the uncertainty was estimated according to the Handbook of Particle Image Velocimetry,\textsuperscript{11} the overall uncertainty of the PIV measurements to the freestream velocity $U_\infty$ was estimated at about $\pm 1.9\%$.

3. Numerical Simulation

3.1. Computational methods

Two-dimensional flow simulations are performed for comparison with the experimental results. The governing equations are the two-dimensional compressible Navier-Stokes equations in integral form and they are discretized using the cell-centered finite volume method.

$$\frac{\partial}{\partial t} \int \mathbf{Q} \, dV + \oint \mathbf{F} \cdot d\mathbf{A} = \oint \mathbf{F}^\tau \cdot d\mathbf{A},$$  \hspace{1cm} (1)

where $\mathbf{Q}$, $\mathbf{F}$ and $\mathbf{F}^\tau$ are conservative variables, inviscid flux and viscous flux on a cell boundary, respectively. The convection term is evaluated using the simple low-dissipation AUSM (SLAU) scheme.\textsuperscript{12} The viscous term is treated by the conventional second-order central difference method. For time integration, the lower-upper symmetric-Gauss-Seidel (LU-SGS) method\textsuperscript{13,14} is used with the dual-time stepping method. Cartesian grids with adaptive mesh refinement (AMR) are used for rapid grid generation around an arbitrary shape.

3.2. Computational conditions

Table 1 shows the computational conditions. The Reynolds number is the same value as that in the experiments. The freestream Mach number is set to $M = 0.2$, which is higher than the experiments. When a compressible flow solver is used for very low-speed flow simulation ($M < 0.1$), it is well known that the solution will become unphysical. Therefore, the freestream velocity is increased to a speed of which compressibility does not influence the flow field. The minimum grid size $\Delta x_{\text{min}}$ is 0.01. Grid refinement is applied in the wake region of the circular cylinder by addition. The total number of cells is 176,732. Figure 1 shows the grid generated around a circular cylinder. Global time step is adopted and non-dimensional time step $\Delta t$ is 0.05.

After flow simulations were performed, the numerical data in the wake region were extracted and vectors were thinned out, so that the data area and the density of vectors were set to the almost same level as those in the experimental data.

4. Analysis Methods

4.1. Proper orthogonal decomposition (POD)

Proper orthogonal decomposition (POD) is an analysis
method for finding orthonormal bases that express the given multi-dimensional data following Gauss distribution most efficiently. The bases found using POD analysis are often called POD modes and are used to understand the flow field. The mathematical principle of POD analysis is as follows.

First, the time-averaged velocity vector \( \bar{V} \) is removed from each instantaneous velocity vector \( v_i \), then only the fluctuation velocity vectors \( v'_i \) are left.

\[
v_i - \bar{V} = v'_i \quad (i = 1, 2, \ldots, m),
\]

(2)

When POD modes from the first to the \( r-1 \)th have been found and the goal is to find the \( r \)th POD mode, the \( r \)th POD mode \( \phi_r \) is decided so as to minimize the value of the following formula

\[
g(\phi_r) = \sum_{i=1}^{m} \left( v'_i - \sum_{k=1}^{r} (v'_i, \phi_k) \phi_k \right)^2 - \lambda_r \left( |\phi_r|^2 - 1 \right),
\]

(3)

where \((a, b)\) is an inner product of vectors \( a \) and \( b \). The first term deals with the magnitude of differential vectors between actual velocity vectors and those restored from POD modes. The second term deals with the normality of POD modes using a Lagrange multiplier \( \lambda_r \).

By differentiating Eq. (3) with respect to each component of \( \phi_r \), the necessary conditions for taking the extreme value are as follows.

\[
\sum_{i=1}^{m} \left( v'_i, \phi_{r,i} \right) = \lambda_r \phi_{r,i} \quad (i = 1, 2, \ldots, n)
\]

(4)

Using the matrix \( X = [v'_1 \ v'_2 \ \cdots \ v'_m] \) and \( \phi_r \), Eq. (4) is rewritten as Eq. (5).

\[
XX^T \phi_r = \lambda_r \phi_r.
\]

(5)

The above process does not depend on the value of \( r \). This fact and Eq. (5) mean all POD modes are the eigenvectors of the matrix \( XX^T \). \( XX^T \) is a symmetric matrix, and it is given that eigenvectors of a symmetric matrix are normalized, so the orthogonality of the POD modes is completed automatically.

The fluctuation velocity \( v'_i \) can be expressed as follows using POD modes \( \{\phi_i\}_{i=1}^{m+1} \).

\[
v'_i = a_{r_1} \phi_1 + a_{r_2} \phi_2 + \cdots + a_{r_m} \phi_m
\]

(6)

where \( a_{r_i} \) denotes the coefficient of the \( r \)th POD mode as a function of time. \( a_{r_i} \) is easily calculated as in Eq. (7) by multiplying both sides of Eq. (6) by \( \phi_r \) and using the orthonormality of the POD modes.

\[
a_{r_i} = (\phi_r, v'_i)
\]

(7)

\( a_{r_i} \) has a dimension of velocity. \( a_{r_i}^2 \) is also defined as the energy that the \( r \)th POD mode has at time \( t \). Then, the time-averaged energy of the \( r \)th POD mode is calculated as

\[
E_r = \frac{1}{m} \sum_{i=1}^{m} a_{r_i}^2 = \frac{1}{m} \sum_{i=1}^{m} (\phi_r, v'_i)^2.
\]

(8)

It is known that there are pairs of POD modes that have almost the same time-averaged energies when the POD analysis is applied to a periodic flow without amplification or attenuation. This is because as long as the real-value functions are used, at least two functions are required for expressing a periodic flow, like a pair of sine and cosine functions.

In this research, first Gappy POD is used and the experimental results are revised as preprocessing. The presumed vectors are interpolated at the points where measured velocity vectors are removed by masking and range validation. Next, Snapshot POD is used and the main POD analysis is conducted using the revised experimental results and numerical results.

### 4.2 Dynamic mode decomposition (DMD)

In POD analysis, it is difficult to take in dynamic information on the flow field because time series data are treated one-by-one. On the other hand, DMD is an analysis method for managing the sequence of the given time series data and extracting modes that have dynamic information.

The mathematical principle of DMD analysis is as follows.

In DMD, it is assumed that, by making linear mapping of velocity data at a certain time \( t \), velocity data at the next time \( t+1 \) is obtained. That is, the following formula is fulfilled.

\[
v'_{t+1} = A v'_t.
\]

(9)

where matrix \( A \) is not a function of time \( t \). DMD modes are eigenvectors of \( A \). \( A \) is generally unknown, but by using the mathematical approach called singular value decomposition, eigenvalues and eigenvectors of \( A \) can be approximated.

Equation (9) is fulfilled for any \( t \), so it is rewritten as Eq. (10).

\[
A X^m_t = [v'_1 \ v'_2 \ \cdots \ v'_m] = X^m_{t+1},
\]

(10)

where \( X^m_t \) is defined as \( X^m_{t+1} = [v'_1 \ v'_2 \ \cdots \ v'_m] \). Additionally, by singular value decomposition using POD modes, \( X^m_t \) is decomposed into three matrices \( U, \Sigma \) and \( V \), as

\[
X^m_t = U \Sigma V^T,
\]

(11)

where \( U \) and \( V \) are matrices in which columns have orthogonality, and \( \Sigma \) is a diagonal matrix. Then, based on Eqs. (10) and (11), \( S \) is defined as follows.

\[
S = U^T A U = U^T (A U \Sigma V^T) V \Sigma^{-1} = U^T X^m_{t+1} V \Sigma^{-1}
\]

(12)

\( U, X^m_{t+1}, V \) and \( \Sigma \) on the right side of Eq. (12) are known or calculated, so \( S \) can be calculated. \( U^T A U \) in Eq. (12) means \( A \) is projected on a certain orthonormal basis, and eigenvalues of \( A \) can be approximated as those of \( S \). Eigenvectors of \( A \), that are DMD modes \( \{\xi_i\}_{i=1}^{m+1} \), are calculated from eigenvectors of \( S \) as

\[
\xi_i = U \phi_i,
\]

(13)

where \( \{\phi_i\}_{i=1}^{m} \) are eigenvectors of \( S \). If \( \{\phi_i\}_{i=1}^{m} \) are normalized, DMD modes \( \{\xi_i\}_{i=1}^{m} \) are also normalized because \( U \) is a matrix in which columns have orthogonality.

The fluctuation velocity \( v'_i \) can be expressed as follows using eigenvalues \( \{\kappa_i\}_{i=1}^{m} \) and DMD modes \( \{\xi_i\}_{i=1}^{m} \).

\[
v'_i = A^{-1} v'_i = \sum_{i=1}^{m} c_i \kappa_i^{-1} \xi_i
\]

(14)
where $c_i$ is the coefficient decided for each mode and has a dimension of velocity. However, unlike $a_{ci}$ in POD analysis, $c_i$ is independent from time $t$. When $|\kappa_i|$ is not equal to 1, the flow expressed in $\xi_i$ is amplified or decreased. That is to say, when it is assumed that the observed flow is a periodic flow without amplification or attenuation, it is necessary to fulfill $|\kappa_i| \approx 1$. On this assumption, $c_i$ denotes the degree of contribution of $\xi_i$, and $\kappa_i^{-1}$ denotes the phase of $\xi_i$ at time $t$. Then, in this paper, the energy of $\xi_i$ is defined as

$$ F_i = |c_i|^2, $$

and is used for comparing DMD modes. The frequency of $\xi_i$ is also calculated from $\kappa_i$ as

$$ f_i = \text{Im} (\ln \kappa_i) / 2\pi \Delta t, $$

where $\Delta t$ is the time interval between the acquired discrete data.

The POD and DMD analyses in this paper were conducted using self-written programs.

5. Results and Discussion

5.1. Situations of flow fields

Figures 2 and 3 show examples of instantaneous flow fields obtained as the experimental and numerical results, respectively. Color contour denotes strength of vorticity defined as

$$ q = (\partial V / \partial x - \partial U / \partial y) \times D / U_\infty, $$

and a range of values of the contour is optimized for each figure. This applies to the subsequent figures. In both figures, vortex shedding is generated and the periodic flow is formed in the wake region of the circular cylinder. However, a small difference is found at the vortex formation point. While this point is immediately behind the circular cylinder in Fig. 3, it is located slightly on the downstream side in Fig. 2. Then, the freestream Mach number was set to $M = 0.1$ and the numerical simulation was repeated, but the position of the vortex formation point did not change. Therefore, it is thought that the Mach number does not affect numerical results.

The vortex shedding frequency in the experimental results is $f_{\exp} = 17.5$ Hz. When the frequency is converted into a Strouhal number, it is $St_{\exp} = 0.216$. In this paper, the Strouhal number is defined as

$$ St = f D / U_\infty, $$

where $f$ is the vortex shedding frequency and $D$ is the diameter of the circular cylinder. On the other hand, the Strouhal number in the numerical results is $St_{\text{num}} = 0.220$ and these values are mostly in agreement.

5.2. POD analysis results

In our previous research, a pair of POD modes was observed from the experimental results, and it was found that the coefficients of these modes had almost the same frequency as the vortex shedding frequency. In this section, these results are compared with the numerical results. Note, however, that our previous POD analysis was conducted without removing the time-averaged velocity from the instantaneous velocity, so the conditions are not completely the same as those explained in this paper. Therefore, the POD analysis according to the procedure in section 4.1 was applied to the experimental results again in this paper. The POD analysis was conducted on 1,998 snapshots acquired in the experiments and 601 snapshots from numerical simulations.

First, the rate of the time-averaged energy of each POD mode is shown in Fig. 4. The vertical axis is scaled logarithmically. In the experimental results, the first and second POD modes have almost the same time-averaged energies and they are much larger than the others. From this finding, these two modes are a pair of POD modes. In the numerical results, there are many pairs of POD modes, not only the first and second modes, but also the third and fourth modes, and sub-

![Fig. 2. Instantaneous flow field (experiment).](image)

![Fig. 3. Instantaneous flow field (simulation).](image)

![Fig. 4. The rates of time-averaged energies of POD modes.](image)
sequent modes. When the experimental and numerical results are compared, the energy rates of the first to sixth modes are in good agreement, but the flow structures of these modes are different. Details of each flow structure are mentioned later. Also, in regard to the energy rates after the seventh mode, the experimental results are higher than the numerical results. This is probably because, in the experiments, periodicity is affected by the disturbance or turbulence in the freestream and more modes are required to express the flow fields with accuracy. Measuring errors in the experimental data may also affect the analysis results.

Figures 5 and 6 show flow structures of the first and second POD modes in the experimental and numerical results, respectively. When the first and second POD modes are compared, it is common in both modes that the vortices are located on a line in the $x$-direction. However, the positions of the vortices differ and the spatial phase shifts by $\pi/2$ from the other mode. This feature was described in our previous research\cite{Taira10} and is in agreement with the feature of POD modes described in Taira.\cite{Taira11} This feature is shared by both the experimental and numerical results, but the positions of the vortices in the POD modes differ. This is because the vortex formation points in Figs. 2 and 3 are different.

Next, Fig. 7 shows flow structures of the third and fifth POD modes in the numerical results. The structures of vortices in Fig. 7 are finer in the $x$- or $y$-direction than those in Fig. 6. When the experimental results are examined in detail, it is found that similar flow structures are extracted in the sixth and seventh POD modes and the ninth and twelfth POD modes (see Fig. 8). These pairs of POD modes cannot clearly be found in Fig. 4. Judging by the flow structures in

Figs. 7 and 8, the ninth, twelfth, sixth and seventh POD modes in the experimental results correspond to the third, fourth, fifth and sixth POD modes in the numerical results, respectively. The pairs of POD modes and their correspondent relations between the experimental and numerical results are summarized in Table 2. The upper and lower rows in each cell mean a mode number and dominant frequency of the mode coefficient, respectively. Meanwhile, the third,
fourth, fifth, eighth, tenth and eleventh POD modes in the experimental results are not described in Table 2. These modes have no partner and have physically meaningless flow structures. These modes are obtained due to the disorder of the periodicity in the experimental results; so these modes are not provided in the numerical results.

Lastly, Figs. 9 and 10 show the time histories of the coefficients of the first, second, sixth, seventh, ninth and twelfth POD modes in the experimental results, and the first six POD modes in the numerical results, respectively. The horizontal axis is the non-dimensional time $T$, which is defined as

$$T = tU_\infty/D,$$  (19)

where $t$ is dimensional actual time. In the experimental results, the disorder of the periodicity is a little striking, so the part where the periodicity is clearly seen is extracted. The time histories in Fig. 9(b) and (c) are slightly disordered, but every coefficient undergoes a periodic variation. The frequencies in Fig. 9(a), (b) and (c) and Fig. 10(a), (b) and (c) are given by $St = 0.216, 0.437, 0.647, 0.220, 0.660$ and $0.440$, respectively. These are the vortex shedding frequency and its higher harmonics. In the experimental results, the higher order mode has a higher frequency, but it is not necessarily applied in the numerical results. That is, the fifth and sixth modes have a higher frequency than the third and fourth modes. The order is reversed.

### 5.3. DMD analysis results

In our previous research, DMD modes that have the vortex shedding frequency and its harmonics indicated high energy rates. Additionally, as frequencies of DMD modes increased, the size of the vortices in the modes decreased. Similar to the POD analysis, the analysis conditions in our previous research are different from those in this paper. Therefore, the DMD analysis according to the procedure in section 4.2 is applied to the experimental results again.

First, Fig. 11 shows the eigenvalues acquired in the DMD analysis plotted on a complex plain. The eigenvalues are mainly located on a unit circle, and $|\xi| \approx 1$ is fulfilled. This supports the notion that the observed flow is a periodic flow without amplification or attenuation. In Fig. 11(b), the distribution of eigenvalues is slightly disordered in the second and third quadrants because the calculation accuracy of the eigenvalues $\xi$ worsens when one of the eigenvalues $\lambda$ calculated in the POD analysis is nearly zero. Additionally, the inverse of a nearly-singular matrix is calculated in the DMD analysis rather than the periodicity of the observed flow being disordered.
Next, energy distributions vs. frequency are shown in Figs. 12 and 13. Both horizontal axes are the Strouhal number and the vertical axes are scaled logarithmically. In both figures, the highest peak is seen near $St = 0.220$, corresponding to the vortex shedding frequency, and other peaks are seen at their higher harmonics. In the numerical results, the second highest peak is at $St = 0.657$, corresponding to the third harmonics, and the third is at $St = 0.437$, corresponding to the second. These findings denote the same phenomenon as the reversal of the order of POD modes in the numerical results. Such a phenomenon is not seen in the experimental results or the results reported in Rowley et al.,\textsuperscript{5}) where the DMD analysis was applied to the flow around a circular cylinder at a Reynolds number of 60. These differences may be caused by the three-dimensionality of the flow fields or Reynolds number effects, so further research is required.

Finally, Fig. 14 shows flow structures of DMD modes with $St = 0.216$, 0.432 and 0.649 from the experimental results. These modes correspond to the peak points in Figs. 12 and 13. Since DMD modes are generally expressed in the form of complex numbers, Figs. 14 and 15 show flow structures in real parts of the DMD modes. These flow structures are very similar to the flow structures of POD modes, which are shown in Figs. 5–8. This is because the coefficient of each POD mode has almost the same frequency as each DMD mode. The correspondent relations between POD and DMD modes are shown in Table 3. In the numerical results, Fig. 15(b) and (c) correspond to Fig. 7(b) and (a), respectively, and these correspondent relations are reversed compared to the experimental results. This is because, as stated above, the third harmonic modes have a higher energy than the second harmonic modes in the numerical results.

6. Conclusion

In this study, POD and DMD analyses were applied to the experimental and numerical results for velocity fields around...
agreement with each other. In addition, the numerical results showed fair agreement with each other. As a result, the following conclusion was reached.

In terms of flow structures extracted in POD or DMD modes, the experimental and numerical results showed fair agreement with each other. In addition, flow structures of POD modes were often similar to those of DMD modes with vortex shedding frequency or higher harmonics. On the other hand, differences were found in the energy rates of POD and DMD modes. In POD analysis, the energy rates in the numerical results were clearly divided in every pair of POD and DMD modes. In POD analysis, the third harmonic mode had a higher energy than the second harmonic mode in the numerical results. A similar phenomenon was observed as a reversal of the order of POD modes in the numerical results. Since such a phenomenon was not found in the experimental results or the results reported in other references, further research is required.

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