A LIMITING ABSORPTION PRINCIPLE FOR HIGH-ORDER
SCHRÖDINGER OPERATORS IN CRITICAL SPACES

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Abstract. In this paper, we prove a limiting absorption principle for high-
order Schrödinger operators with a large class of potentials which generalize
some results by A. Ionescu and W. Schlag. Two key tools we use in this paper
are the Stein–Tomas theorem in Lorentz spaces and a sharp trace lemma given
by S. Agmon and L. Hörmander.

Key Words: Limiting absorption principle; High-order Schrödinger operator,
Stein–Tomas theorem, Sharp trace theorem.

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1. INTRODUCTION

In this paper, we prove a limiting absorption principle for the high-order Schrödinger
operators

\[ H = (-\Delta)^m + V, \quad m \geq 1, \]

with a large class of potentials \( V \) belonging to certain Banach spaces, which gen-
eralize some results by A. Ionescu and W. Schlag in [10]. Limiting absorption
principles play an important role in the spectral theory as they directly imply the
absence of singular spectrum. They are closely related to the spectral measures of
the Schrödinger operators, and the distorted Fourier transforms, and the long-time
behavior of the unitary group \( e^{itH} \).

Due to their wide applications, many different limiting absorption principles have
been proven over the decades. S. Agmon [1] established a limiting absorption prin-
ciple for elliptic operators with short-range potentials in weighted \( L^2 \) spaces using
the trace lemma. T. Ikebe and Y. Saito in [9] proved similar limiting absorption
principles for second-order differential operators with magnetic term using a pri-
or estimates. By introducing a critical space for the trace lemma, L. Hörmander
established a limiting absorption principle for simply characteristic differential op-
erators with generalized short-range potentials in [6]. Some quantitative limiting
absorption principles were investigated for Schrödinger operators on asymptotically
conic manifolds with short-range potentials by I. Rodnianski and T. Tao in [15].

By the connection between resolvent operators and restriction oper-
ators, M. Goldberg and W. Schlag established a limiting absorption principle in \( L^p \) spaces in-
stead of weighted \( L^2 \) spaces in [5] by the Stein–Tomas restriction theorem. This new
type of limiting absorption principle allowed them to handle the Schrödinger oper-
ators with singular potentials. However, the class of potentials investigated is quite
restricted in [5] since the results rely on the unique continuation results established
by A. Ionescu and D. Jerison in [8]. In order to avoid using the unique continuation
results, A. Ionescu and W. Schlag in [10] proved the asymptotic completeness of
Schrödinger operators for a class of potentials that include some singular potentials,
a “global” Kato class, as well as some first-order differential operators. Their main innovation was a new limiting absorption principle in some critical spaces which contains both the cases in [2] and [5]. This general limiting absorption principle extends the Agmon–Kato–Kuroda theorem to a large class of perturbations. See also [7] for more related results.

The goal of this paper is to establish a limiting absorption principle for high-order Schrödinger operators in some critical spaces inspired by the work of A. Ionescu and W. Schlag in [10]. In particular, we also extend some results in [10] by using the Stein–Tomas restriction theorem in Lorentz spaces instead of Lebesgue spaces. In order to establish our limiting absorption principle, we only need to consider the boundary operators for the resolvents using the maximum principle for analytic functions. The boundary operator is closely related to the Fourier restriction operators: the imaginary part of the boundary operator is the restriction operator. This relationship allows us to prove the required estimates in the critical spaces by using the Stein–Tomas restriction theorem and the sharp trace lemma.

The critical spaces we use in our paper allow us to include more general potentials, however we are unable to exclude the embedded eigenvalues of the corresponding operators. It therefore seems likely that other new techniques are required to attain this goal. We remark that this has been achieved for some special kinds of Schrödinger operators; for instance, T. Kato proved the non-existence of positive eigenvalues in [11], which implies the absence of embedded eigenvalues for the Schrödinger operators with short-range potentials. In [5], A. Ionescu and D. Jerison used unique continuation properties to exclude embedded eigenvalues for the Schrödinger operators with certain singular potentials. H. Koch and D. Tataru excluded the positive eigenvalues for Schrödinger operators with more general potentials in [13]. We leave this topic for further discussion.

1.1. Notations. In this subsection, we explain some notations used in this paper. Let $d \geq 2$ be the dimension and $m$ be any integer larger than zero.

We set $P_m(\xi) = |\xi|^{2m}$ so that $P_m(D) = (-\Delta)^m$ is the $m$th-order Laplacian (which is a differential operator of order $2m$). Let $R_m^0$ and $R_m$ denote the resolvents for $P_m(D)$ and $P_m(D) + V$ respectively.

For $r > 0$, we write $S_{d-1}^r = \{ \xi : P_m(\xi) = r^{2m} \}$, and write $d\sigma_r$ for the surface measure on $S_{d-1}^r$. In order to simplify notation, we will often reason with $\lambda > 0$ instead of $r$, where the two are related by

$$ r(\lambda) = \lambda^{\frac{1}{2m}}. $$

Let $B(X, Y)$ be the set of bounded linear operators $T : X \to Y$ between Banach spaces $X$ and $Y$.

For any $\alpha \in \mathbb{C}$, let $S_\alpha : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ denote the Fourier multiplier operators given by

$$ S_\alpha u(x) = \{(1 + |\xi|^2)^{\alpha/2}\hat{u}(\xi)\}^\vee(x). $$

For $1 < p < \infty$ and $\alpha \in \mathbb{R}$, we define the following spaces

$$ W_\alpha^p(\mathbb{R}^d) = \{ u \in \mathcal{S}'(\mathbb{R}^d) : S_\alpha u \in L^p(\mathbb{R}^d) \}, $$

$$ W_{\alpha,\text{loc}}^p(\mathbb{R}^d) = \{ u \in \mathcal{S}'(\mathbb{R}^d) : S_\alpha u \in L^p_{\text{loc}}(\mathbb{R}^d) \}, $$

and

$$ W_{\alpha}^{p,q}(\mathbb{R}^d) = \{ u \in \mathcal{S}'(\mathbb{R}^d) : S_\alpha u \in L^{p,q}(\mathbb{R}^d) \}, $$

where $L^{p,q}(\mathbb{R}^d)$ are the Lorentz spaces (see [17]).
Let $p_d = (2d + 2)/(d + 3)$ and $p'_d = (2d + 2)/(d - 1)$ denote the Stein–Tomas restriction exponents. The Stein–Tomas restriction theorem in Lorentz spaces, which states that $\|F\|_{L^2(S^{d-1})} \leq C_r\|f\|_{L^{p_d, \infty} (\mathbb{R}^d)}$, is due to Bak and Seeger [3]. Here, $Ff(x) = \int_{\mathbb{R}^n} f(x)e^{ix\xi} dx$ is the Fourier transform.

Another important result related to the restriction operator is given by the trace lemma (see [1]). The trace lemma asserts that if a function belongs to the weighted space $L^2_\alpha(\mathbb{R}^d)$ with $s > 1/2$, then its Fourier transform restricted to a sphere belongs to $L^2(d\sigma)$ (see [1]). It is not true for $s = 1/2$. However a critical space $B$ that was introduced by S. Agmon and L. Hörmander in [2] is a good replacement for $L^2_{1/2}(\mathbb{R}^d)$ in the sense that

$$F : B \rightarrow L^2(S^{d-1}) \quad \text{and} \quad F^{-1} : L^2(S^{d-1}) \rightarrow B^*,$$

as bounded operators.

The space $B$ and its dual $B^*$ is defined as follows. Let $D_j = \{x \in \mathbb{R}^d : 2^{j-1} \leq |x| \leq 2^j\}$ for $j \geq 1$, and $D_0 = \{x \in \mathbb{R}^d : |x| \leq 1\}.$ Then

$$B = \left\{ f : \mathbb{R}^d \rightarrow \mathbb{C} \mid \|f\|_B := \sum_{j=0}^{\infty} 2^{j/2} \|f\|_{L^2(D_j)} < \infty \right\},$$

and

$$B^* = \left\{ f : \mathbb{R}^d \rightarrow \mathbb{C} \mid \|f\|_{B^*} := \sup_{j \geq 0} 2^{-j/2} \|f\|_{L^2(D_j)} < \infty \right\}.$$

From Theorem 14.1.2 in [3], we have $B \hookrightarrow L^1(\mathbb{R}; L^2(\mathbb{R}^{d-1}))$,

$$\int_\infty^{\infty} \|f(\cdot, x_d)\|_{L^2(\mathbb{R}^d)} dx_d \leq \sqrt{2} \|f\|_B. \tag{1.2}$$

It follows by duality that $L^\infty(\mathbb{R}; L^2(\mathbb{R}^{d-1})) \hookrightarrow B^*$, i.e.

$$\|f\|_{B^*} \leq \sqrt{2} \sup_{x_d \in \mathbb{R}} \|f(x', x_d)\|_{L^2_{j'}}. \tag{1.3}$$

Let $S_\alpha(B)$ and $S_\alpha(B^*)$ denote the image of $B$ and $B^*$ under $S_\alpha$ respectively. As we mentioned before, we will prove a limiting absorption principle for $P_m(D)$ in some critical Banach spaces in $\mathbb{R}^d$ with $d \geq 2$. These spaces are

$$X := W^{p_d, 2}_{\theta_m, d}(\mathbb{R}^d) + S_m(B) \quad \text{and} \quad X^* := W^{p'_d, 2}_{\theta_m, d}(\mathbb{R}^d) \cap S_{-m}(B^*),$$

where $\theta_{m, d} = m - \frac{d}{d+1}$. Using the Sobolev embedding theorem and the fact that $B \hookrightarrow L^2(\mathbb{R}^d) \hookrightarrow B^* \hookrightarrow L^2_{loc}(\mathbb{R}^d)$, it is easy to get the following embeddings,

$$X \hookrightarrow W_{-m}^2, \quad W_m^2 \hookrightarrow X^* \hookrightarrow W_{m, loc}^2. \tag{1.4}$$

By sharp trace theorem and Stein-Tomas theorem, we have

$$F : X \rightarrow L^2(S^{d-1}) \quad \text{and} \quad F^{-1} : L^2(S^{d-1}) \rightarrow X^*$$

as bounded operators.
1.2. Main results. The first main result in this paper is the following limiting absorption principle for the high-order Laplacians:

**Theorem 1.1.** Assume that $\delta \in (0, 1]$. Then

$$
\sup_{\lambda \in [\delta, \delta - 1], \varepsilon \in [0, 1]} \| R_0^m(\lambda \pm i\varepsilon) \|_{X \to X^*} \leq C_{\delta, d} < \infty.
$$

(1.5)

In order to prove Theorem 1.1, we write the resolvent as an integral operator with a Schwartz kernel. By studying this Schwartz kernel, one is able to derive all the required estimates. Essentially, modulo a good operator, the resolvent operator is a restriction operator on a small spherical shell around a sphere, which leads to estimates similar to those for the restriction operator. Using the exact form of the kernel, we are able to reduce our operator to the restriction operator with a weight which is almost constant in the shell.

We emphasize that our method does not use the fundamental solution of $P_m(D)$. Therefore, we do not impose any upper bounds on $m$.

The second main result is the following properties for the perturbed operators:

**Theorem 1.2.** Assume that $V$ is an admissible perturbation (see Definition 3.1 below).

1. Then the operator $H_m = P_m(D) + V$ defines a self-adjoint operator on

$$
D(H_m) := \{ u \in W_2^m(\mathbb{R}^d) : H_m u \in L^2(\mathbb{R}^d) \}.
$$

In addition, $D(H_m)$ is dense in $L^2(\mathbb{R}^d)$, and $H_m$ is bounded from below on $D(H_m)$.

2. Let $I \subset (\mathbb{R} \setminus \{ 0 \}) \setminus \mathcal{E}$ be compact. Then

$$
\sup_{\lambda \in I, \varepsilon \in [0, 1]} \| R^m(\lambda \pm i\varepsilon) \|_{X \to X^*} \leq C(V, I) < \infty,
$$

(1.6)

where $\mathcal{E}$ is the set of nonzero discrete eigenvalues of $H_m$. This implies that the spectrum of the operator $H_m$ is purely absolutely continuous on $I$. In particular, $\Omega^\pm(H_m, P_m(D))$ exist and are complete, and $\sigma_{ac}(H_m) = \emptyset$, $\sigma_{ac}(H_m) = [0, \infty)$.

To get this result, we use the important Corollary 2.3 (see Section 2), which is deduced from Theorem 1.1. This Corollary and a Rellich theorem established in [2] gives the regularity for the eigenfunctions.

The rest of this paper is organized as follows. In Section 2, we prove a limiting absorption principle for high-order Laplacians. As a corollary, we deduce a weighted resolvent estimate. In Section 3, we define admissible potentials and prove that the same limiting absorption principle is also valid for admissible perturbations.

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For any \( z = \lambda + i\varepsilon \in \mathbb{C} \setminus \mathbb{R} \) and \( f, g \in S(\mathbb{R}^d) \), we have

\[
\langle R_0^m (\lambda + i\varepsilon) f, g \rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\hat{f}(\xi) \overline{\hat{g}}(\xi)}{P_m(\xi) - (\lambda + i\varepsilon)} \, d\xi
\]

\[
= \frac{1}{(2\pi)^d} \frac{1}{2m} \int_0^\infty \frac{t^{(d-2m)/2m}}{t - (\lambda + i\varepsilon)} \left( \int_{S^{d-1}} \hat{f}(t^{1/2m} w) \overline{\hat{g}}(t^{1/2m} w) \, dw \right) \, dt.
\]

If we denote \( \gamma(t) := t^{(d-2m)/2m} \int_{S^{d-1}} \hat{f}(t^{1/2m} w) \overline{\hat{g}}(t^{1/2m} w) \, dw. \)

It is easy to check that \( \gamma(t) \) is locally Hölder continuous in \( t \in (0, \infty) \). Hence, by Privalov’s theorem, the boundary operator \( R_0^m(\lambda \pm i0) = \lim_{\varepsilon \to 0} R_0^m(\lambda \pm i\varepsilon) \) exists as an \( \mathcal{B}(S(\mathbb{R}^d), S'(\mathbb{R}^d)) \)-valued function for every \( \lambda > 0 \) and are locally Hölder continuous. Furthermore, for any \( f, g \in S(\mathbb{R}^d) \), we can write the boundary operator as

\[
\langle R_0^m (\lambda \pm i0) f, g \rangle = \langle (P_m(\xi) - (\lambda \pm i0))^{-1} \hat{f}, \hat{g} \rangle
\]

\[
(2.1) = \pm \frac{1}{(2\pi)^d} \frac{i\pi}{2m} \int_{P_m(\xi) = \lambda} \hat{f}(\xi) \overline{\hat{g}}(\xi) \, d\sigma + \frac{1}{(2\pi)^d} \text{P.v.} \int_{R^d} \frac{\hat{f}(\xi) \overline{\hat{g}}(\xi)}{P_m(\xi) - \lambda} \, d\xi.
\]

Here, p.v. means that the integral is to be understood in the principal value sense.

Using the Fourier multiplier theorem and the first embedding in (1.4), it is easy to check for any \( z \in \mathbb{C} \setminus [0, \infty) \) that we have the estimate

\[
|\langle R_0^m(z) f, g \rangle| \leq C_z \|f\|_W^2 \|g\|_W^2 \leq C_z \|f\|_X \|g\|_X.
\]

That is, \( R_0^m(z) \) is a \( B(X, X^*) \)-valued analytic operator in \( \mathbb{C} \setminus [0, \infty) \). Using the maximum principle for analytic functions, we will have for all \( \lambda \in [\delta, \delta^{-1}] \) (\( \delta \in (0, 1] \)),

\[
\sup_{\lambda \in [\delta, \delta^{-1}], \xi \in [0, 1]} |\langle R_0^m (\lambda \pm i\varepsilon) f, g \rangle| \leq C_{\delta, d} \|f\|_X \|g\|_X,
\]

provided we can prove that

\[
\sup_{\lambda \in [\delta, \delta^{-1}]} |\langle R_0^m (\lambda \pm i0) f, g \rangle| \leq C_{\delta, d} \|f\|_X \|g\|_X.
\]

Thus, in the following, we give the uniform bound for the boundary operators which implies Theorem [1.4] using the maximum principle for analytic functions.

**Proposition 2.1.** Assume that \( \delta \in (0, 1] \). Then

\[
(2.2) \sup_{\lambda \in [\delta, \delta^{-1}]} \|R_0^m (\lambda \pm i0)\|_{X \to X^*} \leq C_{\delta, d} < \infty.
\]

**Proof.** Let \( \chi_\lambda : \mathbb{R}^d \to [0, 1] \) denote a smooth function supported in \( \{ \xi : P_m(\xi) \in [\lambda/2, 3/2\lambda] \} \) and equal to 1 in \( \{ \xi : P_m(\xi) \in [4/3\lambda, 5/3\lambda] \} \). We write \( \chi_\lambda(D) \) and \((1 - \chi_\lambda)(D)\) for the corresponding Fourier multiplier operators. This allows us to split

\[
R_0^m (\lambda \pm i0) = \chi_\lambda(D) R_0^m (\lambda \pm i0) + (1 - \chi_\lambda)(D) R_0^m (\lambda \pm i0).
\]

For the operator \((1 - \chi_\lambda)(D) R_0^m (\lambda \pm i0)\), notice that \((1 - \chi_\lambda)(\xi) \frac{(1 + |\xi|^2)^{m}}{1 + |\xi|^2} \in C^\infty (\mathbb{R}^d) \) with \( L^\infty \) norm bounded uniformly in \( \lambda \). Therefore, this part of the operator satisfies the stronger estimate

\[
(2.3) \quad \|(1 - \chi_\lambda)(D) R_0^m (\lambda \pm i0)\|_{W^{-m}_2 \to W^{-m}_2} \leq C_{\delta, d}.
\]
Thus, by Corollary 2.6 in [2], Fourier multiplier operator which by the definitions of $B$ following inequality is valid for all $f$ and by a density argument, in order to prove (2.4), we only need to prove the support of $\chi$ (2.7)

So we only need to prove that

\[
\|\chi(D) R^m_0 (\lambda \pm i0)\|_{X \to X^*} \leq C_{\delta,d},
\]

which by the definitions of $X$ and $X^*$ is equivalent to the following four inequalities:

\[
\begin{align*}
\|S_m \chi(D) R^m_0 (\lambda \pm i0) S_m\|_{B \to B^*} & \leq C_{\delta,d}, \\
\|S_{m,d} \chi(D) R^m_0 (\lambda \pm i0) S_{m,d}\|_{L^{p,d-2}(\mathbb{R}^d) \to L^{p,d-2}(\mathbb{R}^d)} & \leq C_{\delta,d}, \\
\|S_m \chi(D) R^m_0 (\lambda \pm i0) S_{m,d}\|_{L^{p,d-2}(\mathbb{R}^d) \to B^*} & \leq C_{\delta,d}, \\
\|S_{m,d} \chi(D) R^m_0 (\lambda \pm i0) S_{m,d}\|_{B \to L^{p,d-2}(\mathbb{R}^d)} & \leq C_{\delta,d}.
\end{align*}
\]

By duality, the $(B \to L^{p,d-2})$ bound follows from $(L^{p,d-2} \to B^*)$ bound.

Notice that, for any fixed $\lambda > 0$ and $\alpha > 0$, $S_\alpha(\xi) \chi(\lambda) \in C^{\infty}_c(\mathbb{R}^d)$, and for any $\beta \geq 0$,

\[
\|D^\beta [S_\alpha(\xi) \chi(\lambda)]\|_{\infty} \leq C_\beta (1 + \lambda^2)^{\alpha/2}.
\]

Thus, by Corollary 2.6 in [2], Fourier multiplier operator $S_\alpha(D) \chi(D)$ is bounded on $B$. Also, a standard multiplier theorem asserts that it is bounded on $L^p$ for $1 \leq p \leq \infty$. Thus using the interpolation theorem on Lorentz spaces, it is also bounded on $L^{p,d-2}(\mathbb{R}^d)$. Furthermore, for any $\lambda \in [\delta, \delta^{-1}]$, we have the following uniform bounds,

\[
\begin{align*}
\|S_\alpha \chi(D)f\|_B & \leq C_{\delta,d} \|f\|_B, \\
\|S_\alpha \chi(D)f\|_{L^{p,d-2}(\mathbb{R}^d)} & \leq C_{\delta,d} \|f\|_{L^{p,d-2}(\mathbb{R}^d)}.
\end{align*}
\]

Now, we define $T^+_\chi f := [(P_m(\xi) - (\lambda \pm i0))^{-1} \chi(\lambda) \hat{f}(\xi)]'(x)$. Using (2.5) and (2.6), and by a density argument, in order to prove (2.4), we only need to prove the following inequality is valid for all $f \in \mathcal{S}(\mathbb{R}^d)$,

\[
\|T^+_\chi f\|_{L^{p,d-2}(\mathbb{R}^d) \cap B} \leq C \|f\|_{L^{p,d-2}(\mathbb{R}^d) + B}
\]

In the following, we only give the details for $T^+_\chi$; $T^-_\chi$ can be treated similarly.

Let $A = \{\xi_1, \ldots, \xi_N\}$ denote a $\varepsilon/100$-net on $S_r(\lambda)$, where $\varepsilon$ is a small positive constant which depends on $\delta$. Then using a partition of unity, we can write $\chi = \chi_1^0 + \chi_1^1 + \cdots + \chi_1^N$, where $\chi_1^0$ is supported in the set $\{\xi : \|\xi - r(\lambda)\| \geq \frac{10}{10}\}$ and $\chi_1^j$ is supported in the set $\{\xi : \|\xi_j - \xi\| \leq \frac{4}{\varepsilon}\}$. We use $\chi_j(D)$ to denote the corresponding Fourier multiplier operators. Using the same arguments as (2.3), we have

\[
\|\chi_j^0(D) R^m_0 (\lambda \pm i0)\|_{W^2_{-m} \to W^2_{-m} \leq C_{\delta,d}}.
\]

For each $j \in \{1, \ldots, N\}$, using the rotation invariance, we may assume that the support of $\chi_j^j$ is near the north pole. From now on, we use $\chi$ to denote $\chi_j^j$ for simplicity. Thus, by the implicit function theorem, we can write $\xi_d = \varphi(\xi')$ near $S^{d-1}_{r(\lambda)}$ with $P_m(\xi', \varphi(\xi')) - \lambda = 0$. Note that

$$\frac{\chi(\xi')}{P_m(\xi') - \lambda}$$

has a removable singularity at $\xi_d = \varphi(\xi')$, with $\lim_{\xi_d \to \varphi(\xi')} \frac{\chi(\xi')}{P_m(\xi') - \lambda} = \frac{\chi(\xi)}{P_m(\xi) - \lambda}$. Hence,
by Taylor expansion around $\xi_d = \varphi_\lambda(\xi')$,

$$\chi_\lambda(\xi)(P_m(\xi) - \lambda - i0)^{-1} = \frac{1}{\xi_d - \varphi_\lambda(\xi')} \frac{\chi_\lambda(\xi_0)}{\xi_d - \varphi_\lambda(\xi)}$$

$$= \frac{1}{\xi_d - \varphi_\lambda(\xi')} \begin{pmatrix} \frac{\partial P_m(\xi)}{\partial \xi_d} (\xi', \varphi_\lambda(\xi')) + G_\lambda(\xi), \\
\end{pmatrix}$$

where $Q_\lambda(\xi') \in C^0_c(\mathbb{R}^d)$, $G_\lambda(\xi) \in C^0_c(\mathbb{R}^d)$ and both are uniformly bounded in $\lambda \in [\delta, \delta^{-1}]$. Recall that the inverse Fourier transform of $\frac{1}{\xi_d - \varphi_\lambda(\xi)}$ is given by

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{ix_d\xi_d}}{\xi_d - \varphi_\lambda(\xi')} \frac{d\xi_d}{\xi_d - \varphi_\lambda(\xi')} = ie^{ix_d\varphi_\lambda(\xi')} H(x_d),$$

where $H(t)$ is the Heaviside function. We define the main kernel for $T_\lambda^+$ as the inverse Fourier transform of $\frac{1}{\xi_d - \varphi_\lambda(\xi)}$, i.e.,

$$K_\lambda^+(x', x_d) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{d-1}} e^{ix_d\varphi_\lambda(\xi')} Q_\lambda(\xi') d\xi'.$$

We now write $T_\lambda^+$ as follows:

$$T_\lambda^+ f(x', x_d) = \int_{\mathbb{R}^d} K_\lambda^+(x' - y', x_d - y_d) f(y', y_d) dy' dy_d + \int_{\mathbb{R}^d} F^{-1} G_\lambda(x - y) f(y) dy$$

$$:= T_\lambda^{+\text{bad}} f(x) + T_\lambda^{+\text{good}} f(x),$$

where $F^{-1} G_\lambda$ is the inverse Fourier transform of $G_\lambda(\xi)$. Since $G_\lambda(\xi) \in C^0_c(\mathbb{R}^d)$, $F^{-1} G_\lambda \in S(\mathbb{R}^d)$. Thus, $T_\lambda^{+\text{good}} f(x)$ satisfies the following estimate

$$\left\| \int_{\mathbb{R}^d} F^{-1} G_\lambda(x - y) f(y) dy \right\|_{L^{p, \theta}_d(\mathbb{R}^d) \cap B^*} \leq C \|f\|_{L^{p, \theta}_d(\mathbb{R}^d) + B^*}.$$

It remains to prove the (2.11) is valid for the main term $T_\lambda^{+\text{bad}} f(x)$.

Firstly, we prove the $(B \to B^*)$ bound. Notice that for the kernel $K_\lambda^+(x')$, by (2.10) we have, writing $\bar{\mathcal{F}} f(\xi', x_d) = \int_{\mathbb{R}^d} f(x', x_d) e^{-ix\cdot\xi} dx'$ for the partial Fourier transform in $x'$,

$$\sup_{x_d} \left\| \bar{\mathcal{F}} K_\lambda^+(\xi', x_d) \right\|_{L^{\infty}(\mathbb{R}^d)} = \sup_{x_d} \sup_{\xi' \in \mathbb{R}^d} \left| ie^{ix_d\varphi_\lambda(\xi')} H(x_d) Q_\lambda(\xi') \right| < \infty,$$

By (1.2), (1.3), Minkowski’s inequality, Plancheral, and (2.11), we have

$$\|T_\lambda^{+\text{bad}} f(x)\|_{B^*} \leq C \sup_{x_d} \|T_\lambda^{+\text{bad}} f(\cdot, x_d)\|_{L^{2}_{\xi'}}$$

$$\leq C \sup_{x_d} \int_{\mathbb{R}} \left\| K_\lambda^+(x' - y', x_d - y_d) f(y', y_d) dy' \right\|_{L^{2}_{\xi'}} dy_d$$

$$\leq C \sup_{x_d} \int_{\mathbb{R}} \left\| e^{i(x_d - y_d)\varphi_\lambda(\xi')} Q_\lambda(\xi') \bar{\mathcal{F}} f(\xi', y_d) \right\|_{L^{2}_{\xi'}} dy_d$$

$$\leq C \int_{\mathbb{R}} \| \bar{\mathcal{F}} f(\xi', y_d) \|_{L^{2}_{\xi'}} dy_d \leq C \int_{\mathbb{R}} \| f(x', y_d) \|_{L^{2}_{\xi'}} dy_d \leq \|f\|_{B^*}.$$
Secondly, to get the \((L^{p_d,2} \to B^*)\) bound, let us define \(h_{x_d}(y', y_d) := i\mathcal{H}(x_d - y_d)f(y', y_d)\) for each fixed \(x_d\). It is easy to check that
\[
\sup_{x_d} \|h_{x_d}\|_{L^{p_d,2}(\mathbb{R}^d)} \leq \|f\|_{L^{p_d,2}(\mathbb{R}^d)}.
\]
Then,
\[
T^{+}_{\lambda} f(x', x_d) = \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} K^+_{\lambda}(x' - y', x_d - y_d)f(y', y_d)dy'dy_d
\]
\[
= \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} e^{ix'\xi'} \tilde{F}K^+_{\lambda}(\xi', x_d - y_d)\tilde{F}f(\xi', y_d)\xi'dy_d
\]
\[
= \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} e^{ix'\xi'}iH(x_d - y_d)e^{ix(x_d-y_d)}\varphi_{\lambda}(\xi')\tilde{F}f(\xi', y_d)Q_{\lambda}(\xi')\xi'dy_d
\]
(2.13) \[
= \int_{\mathbb{R}^{d-1}} e^{ix'\xi'} \int_{\mathbb{R}} iH(x_d - y_d)e^{-iyd\varphi_{\lambda}(\xi')}\tilde{F}f(\xi', y_d)dydQ_{\lambda}(\xi')\xi'd\xi'.
\]
That is,
\[
T^{+}_{\lambda} f(x', x_d) = \int_{\mathbb{R}^{d-1}} e^{ix'\xi'} e^{ixd\varphi_{\lambda}(\xi')}\mathcal{F}h_{x_d}(\xi', \varphi_{\lambda}(\xi'))Q_{\lambda}(\xi')\xi'd\xi'.
\]
By (1.13), the Plancherel theorem and the Stein–Tomas restriction theorem, we have
\[
\|T^+_{\lambda}f(x', x_d)\|_{B^*} \leq C\sup_{x_d} \left\| \int_{\mathbb{R}^{d-1}} e^{ix'\xi'} e^{ixd\varphi_{\lambda}(\xi')}Q_{\lambda}(\xi')\mathcal{F}h_{x_d}(\xi', \varphi_{\lambda}(\xi'))d\xi' \right\|_{L^2_{\xi'}(\mathbb{R}^{d-1})}
\]
\[
\leq C\sup_{x_d} \|Q_{\lambda}(\xi')\mathcal{F}h_{x_d}(\xi', \varphi_{\lambda}(\xi'))\|_{L^2_{\xi'}(\mathbb{R}^{d-1})}
\]
\[
\leq C_{\delta,d}\sup_{x_d} \|\mathcal{F}h_{x_d}\|_{L^2(B^{1/2}(x_d, \lambda))} \leq C_{\delta,d}\sup_{x_d} \|h_{x_d}\|_{L^{p_d,2}(\mathbb{R}^d)}
\]
\[
\leq C_{\delta,d}\|f\|_{L^{p_d,2}(\mathbb{R}^d)}.
\]
Finally, to prove the \((L^{p_d,2} \to L^{p_d,2})\) bound, we notice that \(|K^+_{\lambda}(x)| \leq C_{\delta,d}(1 + |x|)^{-d/2} + \frac{1}{2}\). Using a dyadic decomposition and interpolation, following the same argument to prove Theorem 6 in [4] gives
\[
(2.14) \quad \|T^{+}_{\lambda} f(x)\|_{L^{p_d,2}(\mathbb{R}^d)} \leq C\|f\|_{L^{p_d,2}(\mathbb{R}^d)}.
\]
This concludes the proof.

**Remark 2.2.** The proof shows that this limiting absorption principle (1.13) is also valid for all homogeneous elliptic operators \(P(D)\) satisfying the non-degeneracy condition, that is, \(P(\xi) = \lambda\) for \(\lambda > 0\) defines a compact hypersurface with nonvanishing Gaussian curvature. For example, one could use the finite type operators of (even) order \(m \geq 2\), e.g. \(P(D) = \xi^m + \cdots + \xi^m\).

For \(N \geq 0, \gamma \in (0, 1]\), and \(x \in \mathbb{R}^d\), for \(t \in [0, \infty)\) we define the weight function
\[
\mu_{N, \gamma}(t) = \frac{(1 + t^2)^N}{(1 + \gamma t^2)^N},
\]
and for \(x \in \mathbb{R}^d\), and \(\mu_{N, \gamma}(x) := \mu_{N, \gamma}(|x|)\). This weight function was introduced in [6] to prove the regularity of the eigenfunctions.

As an important corollary, we give the following weighted resolvent estimate.
Corollary 2.3. Assume that \( \delta \in (0, 1] \). Then
\[
(2.15) \quad \sup_{\lambda \in [\delta, \delta^{-1}]} \| \mu_{N, \gamma} R_0^m(\lambda \pm i0) \mu_{N, \gamma}^{-1} \|_{X \to X^*} \leq C_{\delta, d} < \infty.
\]

We start with following multiplier lemmas concerning the weight \( \mu_{N, \gamma} \), which follows from Lemma 3.2 in [10] and restricted weak type interpolation theorem.

Lemma 2.4. Let \( m(\xi) \in C^\infty(\mathbb{R}^d) \). If \( m \) satisfies the differential bounds
\[
| \partial^\alpha m(\xi) | \leq C_\alpha (1 + |\xi|^2)^{-|\alpha|/2}, \quad \text{for any } |\alpha| \geq 0,
\]
then if \( p \in \{p_d, 2, p_d' \} \) and \( \mu \in \{\mu_{N, \gamma}, \mu_{N, \gamma}^{-1} \} \), we have
\[
\| \mu \varphi(D) \mu^{-1} \|_{L^{p_d}(\mathbb{R}^d) \to L^{p_d'}(\mathbb{R}^d)} + \| \mu \varphi(D) \mu^{-1} \|_{B \to B} + \| \mu \varphi(D) \mu^{-1} \|_{B^* \to B^*} \leq C.
\]

Lemma 2.5. For \( \alpha \in [-2m, 2m] \), we have the following estimates
\[
\| \mu S_\alpha \mu^{-1} \|_{L^{p_d}(\mathbb{R}^d) \to L^{p_d'}(\mathbb{R}^d)} + \| \mu S_\alpha \mu^{-1} \|_{B \to B} + \| \mu S_\alpha \mu^{-1} \|_{B \to B} \leq C,
\]
\[
\| S_\alpha \mu S_\alpha \mu^{-1} \|_{L^{p_d}(\mathbb{R}^d) \to L^{p_d'}(\mathbb{R}^d)} + \| S_\alpha \mu S_\alpha \mu^{-1} \|_{B \to B} + \| S_\alpha \mu S_\alpha \mu^{-1} \|_{B \to B} \leq C,
\]
where \( p \in \{p_d, 2, p_d' \} \), \( \mu \in \{\mu_{N, \gamma}, \mu_{N, \gamma}^{-1} \} \).

Proof of Corollary 2.3. As before, we split the operator into a singular part on the shell and a regular part away from the shell. For the regular part, Lemma 2.4 and Lemma 2.5 gives the following stronger inequality,
\[
(2.16) \quad \| \mu_{N, \gamma}(1 - \chi_\lambda(D)) R_0^m(\lambda \pm i0) \mu_{N, \gamma}^{-1} \|_{X \to X^*} \leq C_{\delta, d}.
\]
Using the embedding result (1.24), we have
\[
\| \mu_{N, \gamma}(1 - \chi_\lambda(D)) R_0^m(\lambda \pm i0) \mu_{N, \gamma}^{-1} \|_{X \to X^*} < C_{\delta, d}.
\]
Thus, it remains to prove that
\[
(2.17) \quad \| \mu_{N, \gamma} \chi_\lambda(D) R_0^m(\lambda \pm i0) \mu_{N, \gamma}^{-1} \|_{X \to X^*} < C_{\delta, d}.
\]

By the proof of Theorem 1.2 in [10] and Lemma 2.5, this inequality is equivalent to the following boundness
\[
(2.18) \quad \| \mu_{N, \gamma} \chi_\lambda(D) R_0^m(\lambda \pm i0) \|_{L^{p_d}(\mathbb{R}^d) \cap B} \leq C_N \| f \|_{L^{p_d}(\mathbb{R}^d) + B},
\]
for all \( f \in \mathcal{S} \). Using the partition of unity and rotation invariance, we can assume that the support of \( \chi_\lambda \) is contained in a neighborhood of the north pole. In order to prove inequality (2.18), it suffices to prove
\[
(2.19) \quad \| \mu_{N, \gamma}(x_d) R_0^m(\lambda \pm i0) f \|_{L^{p_d}(\mathbb{R}^d) \cap B} \leq C_{N, \delta} \| \mu_{N, \gamma}(x_d) f \|_{L^{p_d}(\mathbb{R}^d) + B},
\]
for \( f \in \mathcal{S}(\mathbb{R}^d) \) and \( f f \) is supported in \( \{ \xi : |\xi - \xi_+| \leq \varepsilon_0 \} \), where \( \xi_+ = (0, \ldots, 0, r(\lambda)) \). In fact, let \( \xi_1^+, \ldots, \xi_d^+ \) be a basis of \( \mathbb{R}^d \) consisting of unit vectors in the ball \( \{ \xi : |\xi - \xi_+| \leq \varepsilon_0/2 \} \). Clearly, for any \( x \in \mathbb{R}^d \),
\[
|x| \leq C(|x \cdot \xi_1^+| + \cdots + |x \cdot \xi_d^+|).
\]
Since
\[
\mu_{N, \gamma}(x) \sim \mu_{N, \gamma}(x \cdot \xi_1^+) + \cdots + \mu_{N, \gamma}(x \cdot \xi_d^+),
\]
we can replace the weight \( \mu_{N, \gamma}(x) \) with \( \mu_{N, \gamma}(x \cdot \xi_j^+) \) in (2.18). In other words, (2.18) would follow from the following inequalities, \( j = 1, \ldots, d \):
\[
(2.20) \quad \| \mu_{N, \gamma}(x \cdot \xi_j^+) R_0^m(\lambda \pm i0) f \|_{L^{p_d}(\mathbb{R}^d) \cap B} \leq C_N \| \mu_{N, \gamma}(x \cdot \xi_j^+) f \|_{L^{p_d}(\mathbb{R}^d) + B}.
\]
To summarize, we only need to consider
\[ T_{\lambda}^{+, \mu} f(x', x_d) := \int_{\mathbb{R}^d} e^{ix \cdot \xi} \frac{\mu_{N, \gamma}(x_d)}{\mu_{N, \gamma}(y_d)} \frac{\chi_{\Lambda}(\xi) \hat{f}(\xi)}{P_m(\xi) - \lambda - i0} d\xi. \]

What we need to prove now is that $T_{\lambda}^{+, \mu} : L^{p, 2}(\mathbb{R}^d) + B \to L^{p, 2}(\mathbb{R}^d) \cap B^*$ boundedly.

Following the proof of Proposition 2.1, we can rewrite it as the following:

\[
T_{\lambda}^{+, \mu} f(x', x_d) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \frac{\mu_{N, \gamma}(x_d)}{\mu_{N, \gamma}(y_d)} K_{\lambda}^+(x' - y', x_d - y_d) f(y', y_d) dy' dy_d \\
+ \int \frac{\mu_{N, \gamma}(x_d)}{\mu_{N, \gamma}(y_d)} F^{-1} G_{\lambda}(x - y) f(y) dy \\
=: T_{\lambda}^{+, \mu, \text{bad}} f(x', x_d) + T_{\lambda}^{+, \mu, \text{good}} f(x', x_d)
\]

(2.21)

where $K_{\lambda}^+(x', x_d)$ is given by (2.10) and $F^{-1} G_{\lambda} \in S$. For $T_{\lambda}^{+, \mu, \text{good}} f(x', x_d)$, it is bounded from $L^{p, 2}(\mathbb{R}^d) + B$ to $L^{p, 2}(\mathbb{R}^d) \cap B^*$.

To deal with the $T_{\lambda}^{+, \mu, \text{bad}} f(x', x_d)$ term, we denote

\[
K_{\lambda}^{\mu, +}(x', x_d, y_d) = \frac{\mu_{N, \gamma}(x_d)}{\mu_{N, \gamma}(y_d)} K_{\lambda}^+(x', x_d - y_d).
\]

Then, its partial Fourier transform with respect to $x'$ is

\[
\tilde{F} K_{\lambda}^{\mu, +}(\xi', x_d, y_d) = \frac{\mu_{N, \gamma}(x_d)}{\mu_{N, \gamma}(y_d)} e^{i\xi' \cdot \phi_{\lambda}(\xi')} H(x_d - y_d) Q_{\lambda}(\xi').
\]

Since $\mu_{N, \gamma}(x_d) \leq \mu_{N, \gamma}(y_d)$ when $H(x_d - y_d)$ does not vanish. In this case,

\[
\sup_{x_d, y_d} \left\| \tilde{F} K_{\lambda}^{\mu, +}(\xi', x_d, y_d) \right\|_{L^{\infty}(\mathbb{R}^{d-1})} < \infty.
\]

(2.23)

By repeating the proof in (2.12), we see that $T_{\lambda}^{+, \mu}$ is $B \to B^*$ uniformly bounded in $\lambda$.

Similarly, defining $h_{\lambda}^{\mu, +}(y', y_d) := \frac{\mu_{N, \gamma}(x_d)}{\mu_{N, \gamma}(y_d)} i H(x_d - y_d) f(y', y_d)$ following the proof of (2.13), we can prove $T_{\lambda}^{+, \mu}$ is uniformly bounded $L^{p, 2} \to B^*$.

Also, $|K_{\lambda}^{\mu, +}(x' - y', x_d, y_d)| \leq C_{\delta, d}(1 + |x' - y'| + |x_d - y_d|)^{-\frac{\mu}{2}}$, thus we have that $T_{\lambda}^{+, \mu}$ is bounded $L^{p, 2} \to L^{p, 2}$. This finishes the proof of the Corollary. \qed

3. LIMITING ABSORPTION PRINCIPLE FOR HIGH-ORDER SCHröDINGER OPERATOR WITH ADMISSIBLE PERTURBATIONS

In this section, we prove that the same limiting absorption principle of Section 3 is also valid for high-order Schrödinger operators with admissible perturbations. Most of the results and methods are standard in this section, thus we omit some details and refer the reader to [10]. The main innovation is Lemma 3.4 which is a consequence of the Rellich theorem in [2].

We first give the definition of an admissible perturbation.

**Definition 3.1.** We say that $V$ is an admissible perturbation of $P_m(D)$ if:

1. $V \in B(X^*, X)$ and
   \[
   \langle V \phi, \psi \rangle = \langle \phi, V \psi \rangle
   \]
   for any $\phi, \psi \in \mathcal{S}(\mathbb{R}^d)$. 

(2) For any $\varepsilon > 0$ and $N \geq 0$, there exist constants $A_{N,\varepsilon}, R_{N,\varepsilon} \in [1,\infty)$ such that
\[
\|\mu_{N,\gamma}Vu\|_X \leq \varepsilon \|\mu_{N,\gamma}u\|_X + A_{N,\varepsilon}\|u1\{|x| \leq R_{N,\varepsilon}\}\|L^2
\]
for any $u \in X^*$ and any $\gamma \in (0,1]$.

(3) there is an integer $J \geq 1$ and operators $A_j, B_j \in \mathcal{B}(X^*, L^2)$ for $1 \leq j \leq J$
such that for any $f, g \in X^*$,
\[
(Vf, g) = \sum_{j=1}^{J} \langle B_j f, A_j g \rangle.
\]

Remark 3.1. Notice that $S$ is not dense in $X^*$, thus property (1) of an admissible perturbation alone is not enough to guarantee that $V$ is a symmetric operator on $X^*$. However, by the property (2) and the argument in [10], one can prove that $V$ is symmetric on $X^*$. In addition, the property (2) also guarantee that $V$ is a compact operator from $X^*$ to $X$. In particular, $R_m^0(i)V$ is a compact operator on $W^2_m$.

Some trivial modifications of Proposition 1.4 in [10] gives some examples of admissible perturbations of $P_m(D)$; we will omit the details (one can simply replace $q_0 = d/2$ by $q_m = \frac{d}{2m}$ or $q_m > 1$, $d \leq 2m$).

In this paper, we focus on the potentials in the completion of $C^\infty_c(\mathbb{R}^d)$ with respect to the norm $\| \cdot \|_{L^{q,m}}$, which we denote by $L^{q,m}(\mathbb{R}^d)$.

Proposition 3.2. Let $V$ be a real valued potential and
\[
V \in L^{q,m}_0(\mathbb{R}^d) \text{ for some } q \in \left[q_m, \frac{d+1}{2}\right],
\]
then $V$ is an admissible perturbation.

Proof. The symmetry follows since $V$ is real-valued.

Following Lemma 6.1 in [10], it is easy to prove that for any $f \in L^{q,m,2}(\mathbb{R}^d)$,
\[
\|\|V\|^{1/2}S_{-\theta_m,d}f\|_{L^{2}(\mathbb{R}^d)} \leq C\|\|V\|^{1/2}\|_{L^{q,m,\infty}(\mathbb{R}^d)}\|f\|_{L^{q,m,2}(\mathbb{R}^d)}.
\]
This implies that $V$ maps from $X^*$ to $X$.

Finally, for any $V$ that satisfies (3), we can find $V_\nu \in C^\infty_c(\mathbb{R}^d)$ such that $V_\nu$ converge to $V$ in the $L^{q,m,\infty}(\mathbb{R}^d)$ norm. So condition (3) can be easily deduced. \(\square\)

Next, we construct a function which shows that $L^{q,m}_0$ is not contained in $L^q$.

Example 3.1. Let $q \in [1,\infty)$ and let $\{E_j\}_{j=1}^\infty$ be a disjoint family of subsets of $\mathbb{R}^d$, each with measure $|E_j(x)| = \frac{1}{\ln(1+j)}$ and define $V(x) = \sum_{j=1}^{\infty} \frac{1}{j+j^q} 1_{E_j}(x)$. A direct computation shows that $V(x) \in L^{q,\infty}(\mathbb{R}^d)$, but $V(x) \notin L^q(\mathbb{R}^d)$.

Now we prove that it can be approximated by a bump function in $L^{q,\infty}$ norm. Let us define $V_N(x) = \sum_{j=1}^{N-1} \frac{1}{j+j^q} 1_{E_j}(x)$. In fact, for any given $\varepsilon > 0$, we can choose $N$ large enough so that (writing $\lfloor t \rfloor$ for the integer part of $t \geq 0$),
\[
\sup_{\lambda \in (0,\infty)} \lambda \left\{x \in \mathbb{R}^d \mid |V(x) - V_N(x)| > \lambda\right\}^{1/q} \leq \sup_{\lambda \in (0,N^{-1/q}]} \lambda \left(\sum_{j=N}^{\lfloor 1/\lambda \rfloor} \frac{1}{j+j^q}\right)^{1/q} \leq \sup_{\lambda \in (0,N^{-1/q}]} \frac{\lambda \left(\frac{1}{\lambda^{1/q}}\right)^{1/q}}{\ln(2+N)^{1/q}} < \frac{\varepsilon}{2}.
\]
Since $V_N \in L^2(\mathbb{R}^d)$ is compactly supported, there exists $\tilde{V}_N \in C_0^\infty$ such that $\|V_N - \tilde{V}_N\|_{L^\infty(\mathbb{R}^d)} < \frac{\varepsilon}{2}$. It follows

$$\|V - \tilde{V}_N\|_{L^{\infty}(\mathbb{R}^d)} \leq \|V - V_N\|_{L^{\infty}(\mathbb{R}^d)} + \|V_N - \tilde{V}_N\|_{L^{\infty}(\mathbb{R}^d)}$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

For the rest of this paper, we focus on proving Theorem 1.2.

**Proof of Theorem 1.2.** For the first part of the theorem, it is standard that $H_m = P_m(D) + V$ is self-adjoint on $D(H_m) := \{u \in W_0^2(\mathbb{R}^d) : H_m u \in L^2(\mathbb{R}^d)\}$; this follows from the proof of Theorem 1.3 in [10].

For the rest of this paper, we focus on proving Theorem 1.2.

Now, let us consider the limiting absorption principle for $H_m$. Since we already know that $H_m$ is self-adjoint on $D(H_m)$, let us temporarily define three apparently distinct sets

$$\mathcal{E} = \{\text{non-zero eigenvalues of } H_m\},$$

$$\tilde{\mathcal{E}}^\pm = \{\lambda \in \mathbb{R} \setminus \{0\} \mid \text{there exists } 0 \neq u \in X^* \text{ such that } u + R_0^m(\lambda \pm i0)V u = 0\}.$$ Notice that $R_0(\lambda + i0)f = R_0(\lambda - i0)\overline{f}$ for any $f \in X$. Thus, we have $\tilde{\mathcal{E}}^- = \tilde{\mathcal{E}}^+ =: \tilde{\mathcal{E}}$.

For any $\lambda \neq 0$, we denote the generalized eigenspaces by

$$E_\lambda^\pm := \{f \in X^* : (I + R_0(\lambda \pm i0)V)f = 0\}.$$ 

**Lemma 3.3.** Let $V$ be an admissible potential. Then:

(i) For any $\lambda \in \tilde{\mathcal{E}}$, $E_\lambda^\pm$ are the eigenspaces of self-adjoint operator $H_m$.

(ii) The set $\tilde{\mathcal{E}}$ is discrete in $\mathbb{R} \setminus \{0\}$, and for any $\lambda \in \tilde{\mathcal{E}}$, the vector spaces $E_\lambda^\pm$ are finite dimensional.

**Proof.** (i) For any $u \in E_\lambda^\pm$, by the definition of $\tilde{\mathcal{E}}$, we have

$$(3.5) \quad u + R_0^m(\lambda \pm i0)V u = 0.$$

Now, we only need to show that $u \in D(H_m)$ and $H_m u = \lambda u$. Note that $(P_m(D) - \lambda)R_0^m(\lambda \pm i0)g = g$ for any $g \in X$. Using this fact and (3.5), we have

$$(P_m(D) - \lambda)u + (P_m(D) - \lambda)R_0^m(\lambda \pm i0)V u = 0,$$

which is equivalent to

$$(3.6) \quad (P_m(D) + V)u = \lambda u.$$

By Corollary 2.3, we have that

$$\|\mu_{N,\gamma} u\|_{X^*} = \|\mu_{N,\gamma} R_0^m(\lambda \pm i0)V u\|_{X^*} \leq C_N \|\mu_{N,\gamma} V u\|_X \leq C_N \xi \|\mu_{N,\gamma} u\|_{X^*} + C_N \Lambda_N, \varepsilon \|u1_{|x| \leq R}\|_L^2.$$ We can take $\xi \ll 1$ so that $C_N \xi \ll 1/2$. Then

$$\|\mu_{N,\gamma} u\|_{X^*} \leq C_{N,\gamma} \|u\|_{B^*}.$$ Taking $\gamma \to 0$ yields

$$\|(1 + |x|^2)^N u\|_{X^*} \leq C_{N,\gamma, \varepsilon} \|u\|_{X^*}.$$ This shows that $u$ has sufficiently fast decay. In particular, for any $N \geq 0$, we have $(1 + |x|^2)^N u \in W_0^2$, and $u \in W_0^2$. Using (3.6) and the regularity of $u$, we have
shown that $u \in D(H_m)$, which implies that $u$ is the eigenfunction corresponding to eigenvalue $\lambda$ of $H$.

(ii) This follows from the argument of Lemma 4.5 in [10], using the compactness of the operators $R^m_0(-1)(1+|x|^2)^{-1}$ and $R^m_0(-1)V$ on $W^2_m$.

The above lemma implies that $\mathcal{E} \subset \mathcal{E}$. Next, we will prove that the reverse inclusion also holds. We first consider the equation

\begin{equation}
(P_m(D) - \lambda)u = f.
\end{equation}

The Rellich theorem established by S. Agmon and L. Hörmander in [2] asserts that if $u \in B^*$, $f \in B$ and

\begin{equation}
\lim_{R \to \infty} \frac{1}{R} \int_{|x| \leq R} |u|^2 dx = 0,
\end{equation}

then there is a unique solution given by $u = R^m_0(\lambda + i0)f = R^m_0(\lambda - i0)f$. The following lemma extends this result from $B^*$ to $X^*$.

**Lemma 3.4.** Let $u \in X^*$ and $\lambda \in \mathbb{R} \setminus \{0\}$ satisfy (3.7) where $f \in X$. If we further assume that $u$ satisfies (3.8), then we have

\begin{equation}
u = R^m_0(\lambda + i0)f = R^m_0(\lambda - i0)f\end{equation}

In particular, by Corollary (2.3), we have

\begin{equation}
\|\mu_{N, \gamma}u\|_{X^*} \leq C_N\|\mu_{N, \gamma}(P_m(D) - \lambda)u\|_X
\end{equation}

**Proof.** For any $f \in X$, a direct computation shows that

\begin{equation}
u_0 = [R^m_0(\lambda + i0) + R^m_0(\lambda - i0)]f/2
\end{equation}

is a special solution for (3.7) and $\nu_0 \in X^* \subset B^*$. Thus, for any other solution of (3.7) we have $(P_m(D) - \lambda)(u - \nu_0) = 0$. By Theorem 4.1 in [2], there exists an $L^2$-density $v_0$ on surface $S^{d-1}_{r(\lambda)}$, such that $u - \nu_0 = \mathcal{F}^{-1}(v_0d\sigma)$, where $d\sigma$ is the surface measure for $S^{d-1}_{r(\lambda)}$. That is,

\begin{equation}
u = [R^m_0(\lambda + i0) + R^m_0(\lambda - i0)]f/2 + \mathcal{F}^{-1}(v_0d\sigma).
\end{equation}

For $f \in \mathcal{S}$, by Theorem 6.1 in [2],

\begin{equation}
\int_{S^{d-1}_{r(\lambda)}} |v_0|^2 + |\pi f|^2|P^m_0|^{-2}d\sigma \leq \pi \liminf_{R \to \infty} \frac{1}{R} \int_{|x| < R} |u|^2 dx.
\end{equation}

In view of Proposition (2.1) and the sharp trace lemma, both sides of (3.13) are continuous with respect to the $X$ norm of $f$ and the $L^2(d\sigma)$ norm of $v_0$. As $\mathcal{S}$ is dense in $X$, it follows that the inequality (3.13) is valid for all $f \in X$. By assumption, the right hand side is 0, thus $v_0 = 0$ and $\hat{f} = 0$ on $S^{d-1}_{r(\lambda)}$. This shows that $u = R^m_0(\lambda + i0)f = R^m_0(\lambda - i0)f$.

**Proposition 3.5.** $\mathcal{E} = \tilde{\mathcal{E}}$.

**Proof.** By Lemma 3.3, we only need to prove that $\mathcal{E} \subset \tilde{\mathcal{E}}$. Take any $\lambda \in \mathcal{E}$, and $0 \neq u \in D(H_m) \subset X^*$ such that $(P_m(D) - \lambda)u = -Vu := f \in X$. In particular, $u \in L^2(\mathbb{R}^d)$, which implies that

\begin{equation}
\lim_{R \to \infty} \frac{1}{R} \int_{|x| \leq R} |u|^2 dx = 0.
\end{equation}
By Lemma 3.1, we have \( u = R_{m}^{0}(\lambda \pm i\epsilon)f = -R_{m}^{0}(\lambda \pm i\epsilon)Vu \), thus \( \lambda \in \tilde{\mathcal{E}} \). Hence, \( \tilde{\mathcal{E}} \subset \mathcal{E} \), which proves the proposition. 

Above results imply that the non-zero eigenvalues \( \mathcal{E} \) is discrete in \( \mathbb{R} \setminus \{0\} \) and each eigenvalue has finite multiplicity. 

Next, we prove the second part of Theorem 1.2. The following lemma gives locally uniform bounds for the inverse operators, which we use to establish the limiting absorption principle for the perturbed operator.

**Lemma 3.6.** For any compact set \( I \) contained in \( \mathbb{R} \setminus (\{0\} \cup \mathcal{E}) \),

\[
\sup_{\lambda \in I, 0 \leq \epsilon \leq 1} \| (\text{Id}_{\mathcal{X}^{*}} + R_{m}^{0}(\lambda \pm i\epsilon)V)^{-1} \|_{\mathcal{X}^{*} \rightarrow \mathcal{X}^{*}} \leq C(V, I).
\]

The proof for Lemma 3.6 is standard, the interested reader can check [10] for more details.

For \( \lambda \in I \), using fact the resolvent identity

\[
R_{m}^{0}(\lambda \pm i\epsilon) = (\text{Id}_{\mathcal{X}^{*}} + R_{m}^{0}(\lambda \pm i\epsilon)V)^{-1}R_{m}^{0}(\lambda \pm i\epsilon),
\]

together with Theorem 1.1 and Lemma 3.6, we can get the estimate (1.6). From (1.6) and Theorem XIII.20 in [14], we see that the singular continuous spectrum is null, i.e. \( \sigma_{sc}(H_{m}) = \emptyset \).

Next, we prove that \( \sigma_{ac}(H_{m}) \subset [0, \infty) \). Assume that \( \lambda \in (-\infty, 0) \setminus \mathcal{E} \), we can show that \( \lambda \) belongs to the resolvent set \( \rho(H_{m}) \) of \( H_{m} \) as follows. Since \( \lambda \) is not an eigenvalue of \( H_{m} \), the equation \( f + R_{m}^{0}(\lambda \pm i\epsilon)Vf = 0 \) has no solution in \( W_{m}^{2} \). Hence, by the Fredholm alternative, \( \text{Id}_{W_{m}^{2}} + R_{m}^{0}(\lambda)V \) is invertible on \( W_{m}^{2} \). But now, just notice that \( [\text{Id}_{W_{m}^{2}} + R_{m}^{0}(\lambda)V]^{-1}R_{m}^{0}(\lambda)(H_{m} - \lambda) = \text{identity map} \), and so is \( (H_{m} - \lambda)[\text{Id}_{W_{m}^{2}} + R_{m}^{0}(\lambda)V]^{-1} \), which is to say that \( \lambda \) belongs to the resolvent set of \( H_{m} \) as claimed.

In addition, using condition (3) of an admissible perturbation and the limiting absorption principle for both \( P_{m}(D) \) and \( H_{m} \), it is routine to check that the operators \( A_{j}, B_{j} \) are both \( H_{m} \)-smooth and \( P_{m}(D) \)-smooth on compact subsets \( I \subset \mathbb{R} \setminus (\{0\} \cup \mathcal{E}) \). Hence, the local wave operators \( \Omega^{\pm} \) exist and are complete. By Lemma 3.3, \( (0, \infty) \setminus \mathcal{E} = \bigcup_{i=1}^{\infty}(a_{i}, b_{i}) \). That the local wave operators exist and are complete implies in turn that \( (a_{i}, b_{i}) \subset \sigma_{ac}(H_{m}) \). Therefore, we in fact have \( [0, \infty) \subset \bigcup_{i=1}^{\infty}(a_{i}, b_{i}) \subset \sigma_{ac}(H_{m}) \). In conclusion, we have proven that \( \sigma_{ac}(H_{m}) = [0, \infty) \), which finishes the proof of Theorem 1.2. 

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