Blow-up estimates for a system of semilinear SPDEs with fractional noise

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Abstract

In this paper, we obtain lower and upper bounds for the blow-up time to a system of semilinear stochastic partial differential equations driven by two-dimensional fractional Brownian motion. Under suitable assumptions, lower and upper bounds for the blow-up time of the solution are obtained by using explicit solutions of an associated system of random partial differential equations and formula due to Yor. We also provide an estimate for the finite-time blow-up solutions of a system by choosing suitable parameters. Further, a lower bound for the blow-up probability of solutions is provided by using Malliavin calculus.

Keyword: Semilinear SPDEs, fractional Brownian motion, blow-up times, stopping times, lower and upper bounds, Malliavin calculus.

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1 Introduction

Fujita \cite{9, 10}, in his pioneering work, proved that the semilinear heat equation
\[
\frac{\partial u(t, x)}{\partial t} = \Delta u(t, x) + u^{1+\alpha}(t, x), \quad x \in D,
\]
defined on a smooth bounded domain \(D \subset \mathbb{R}^d, \quad d \geq 1\) with the Dirichlet boundary condition, where \(\alpha > 0\) is a constant, explodes in finite time for all non-negative initial data \(f(x) \in L^2(D)\) satisfying
\[
\int_D f(x)\psi(x)dx > \lambda^\frac{1}{\alpha},
\]
where \(\lambda > 0\) being the first eigenvalue of the Laplacian on \(D\) and \(\psi > 0\), the corresponding normalized eigenfunction with \(\|\psi\|_{L^1} = 1\).

The problem of existence of global positive solutions of semilinear stochastic partial differential equations (SPDEs) has been extensively investigated \cite{1, 2}. Dozzi \cite{3} investigated the blow-up problem for the stochastic analog of the above deterministic problem, namely
\[
\begin{cases}
    du(t, x) = [\Delta u(t, x) + G(u(t, x))]dt + \kappa u(t, x)dB(t), \quad t > 0, \\
    u(0, x) = f(x) \geq 0, \quad x \in D, \\
    u(t, x) = 0, \quad t \geq 0, \quad x \in \partial D,
\end{cases}
\]

\begin{equation}
\text{(1.1)}
\end{equation}

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where $D \subset \mathbb{R}^d$ is a smooth bounded domain, $G : \mathbb{R} \to \mathbb{R}^+$ is locally Lipschitz and satisfies

$$G(z) \geq Cz^{1+\beta} \quad \text{for all} \quad z > 0,$$

(1.2)

$C, \kappa$ and $\beta$ are given positive numbers and $B(\cdot)$, a standard one-dimensional Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. By transforming the above equation to a random partial differential equation, the bounds for the explosion time are estimated in terms of exponential functionals of $B(\cdot)$. For the case, $G(u) = u^{1+\beta}$, $\beta > 0$, the upper and lower bounds for the explosion time are estimated in [12] by using the formula derived by Yor [19].

The impact of Gaussian noises on the time of blowup of a nonlinear SPDE

$$du(t, x) = [\Delta u(t, x) + G(u(t, x))] dt + k_1 u(t, x) dB_1(t) + k_2 u(t, x) dB_2(t),$$

with Dirichlet boundary condition, where $(B_1(t), B_2(t))_{t \geq 0}$ is a two-dimensional Brownian motion was investigated by Niu and Xie [16]. Dozzi [4] extended the results to a system of semilinear SPDEs of the form:

$$
\begin{align*}
\begin{cases}
    du_1(t, x) = [(\Delta + V_1)u_1(t, x) + u_2^p(t, x)] dt + k_1 u_1(t, x) dB(t), \\
    du_2(t, x) = [(\Delta + V_2)u_2(t, x) + u_1^p(t, x)] dt + k_2 u_2(t, x) dB(t),
\end{cases}
\end{align*}
$$

(1.4)

with the Dirichlet boundary data

$$u_i(0, x) = f_i(x) \geq 0, \quad x \in D \quad \text{and} \quad u_i(t, x) = 0, \quad t \geq 0, \quad x \in \partial D.$$

Here $p \geq q > 1$ are constants, $D \subset \mathbb{R}^d$ is a bounded smooth domain, $V_i > 0$ and $k_i \neq 0$ are constants, $i = 1, 2$. In a recent work [13], lower and upper bounds for the blow-up times to a more general system of semilinear SPDEs are obtained.

Dozzi [6] derived lower and upper bounds for the explosion time to the system (1.1) driven by a one-dimensional fractional Brownian motion $(B^H(t))_{t \geq 0}$ with Hurst index $H > \frac{1}{2}$. Further, sufficient conditions are derived for blow-up in finite time and upper bounds for the blow-up time of positive solutions. Recently, Dung [7] provided upper and lower bounds for the probability of the finite-time blow-up of solutions of the system (1.1) with $B(\cdot)$ replaced by $B^H(\cdot)$, in terms of Hurst parameter $H$, where $H \in (0, 1)$, by using Malliavin calculus techniques. Note that $B^H(\cdot)$ reduces to a standard Brownian motion $W(\cdot)$ for the case $H = \frac{1}{2}$. However, there are limited results available in the literature for the blow-up of solutions of SPDEs driven by fractional Brownian motions.

Motivated by the works [6, 7], our objective in this article is to obtain lower and upper bounds for the blow-up times for solutions to the following system of semilinear SPDEs:

$$
\begin{align*}
\begin{cases}
    du_1(t, x) = \left[ \Delta u_1(t, x) + \gamma_1 u_1(t, x) + u_2^{1+\beta_1}(t, x) \right] dt \\
    \quad \quad \quad + k_{11} u_1(t, x) dB_1^H(t) + k_{12} u_1(t, x) dB_2^H(t), \\
    du_2(t, x) = \left[ \Delta u_2(t, x) + \gamma_2 u_2(t, x) + u_1^{1+\beta_2}(t, x) \right] dt \\
    \quad \quad \quad + k_{21} u_2(t, x) dB_1^H(t) + k_{22} u_2(t, x) dB_2^H(t),
\end{cases}
\end{align*}
$$

(1.3)

for $x \in D$, $t > 0$, along with the Dirichlet boundary conditions

$$
\begin{align*}
\begin{cases}
    u_i(0, x) = f_i(x) \geq 0, \quad x \in D, \\
    u_i(t, x) = 0, \quad x \in \partial D, \quad t \geq 0, \quad i = 1, 2,
\end{cases}
\end{align*}
$$

(1.4)

where $D \subset \mathbb{R}^d$ is a bounded domain with smooth boundary $\partial D$. Here $\beta_1 \geq \beta_2 > 0$, $\gamma_i > 0$ and $k_{ij} \neq 0, i, j = 1, 2$ are constants. Here $f_i$ are of class $C^2$ and not identically zero for $i = 1, 2$ and
$(B^H_1(t), B^H_2(t))_{t \geq 0}$ is a standard two-dimensional fractional Brownian motion (fBm) with Hurst index $H \in (0, 1)$ with respect to a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. In the case of $H = \frac{1}{2}$ and $\gamma_1 = \gamma_2 = 0$, lower and upper bounds for the blow-up times of solutions are obtained in [13]. Conditions for finite-time blowup of positive weak solutions to a fractional semilinear equation perturbed by a fractional Brownian noise are established in [5].

The main aim of this article is to obtain estimates for lower and upper bounds for the blow-up probabilities of solutions $u = (u_1, u_2)^T$ to the system (1.3)-(1.4), and a lower bound for the blow-up probability of solutions using Malliavin calculus techniques. Using the Yor formula (cf. [11] [14]) and explicit solutions of an associated system of random PDEs, lower and upper bounds of explosion times are also obtained. Recall that $\lambda > 0$ is the first eigenvalue of the Laplacian operator $-\Delta$ on $D$ and $\psi > 0$ is the corresponding normalized eigenfunction so that $\|\psi\|_{L^1} = 1$. Initial data components are assumed of the form $f_i = C_i\psi$, where $C_i > 0$ are constants. Choices of these quantities are essential to derive the explicit solutions of an associated system of random PDEs by using a suitable transformation.

The article is organized as follows: The next section is devoted to obtain an associated system of random PDEs, by using a random transformation of the system (1.3)-(1.4), which is useful to establish lower and upper bounds for the explosion time $\tau$. The special case of $(1 + \beta_1)k_{21} - k_{11} = (1 + \beta_2)k_{11} - k_{21} =: \rho_1$, $(1 + \beta_1)k_{22} - k_{12} = (1 + \beta_2)k_{12} - k_{22} =: \rho_2$ is considered in Sections 3 and 4. Under the above setting, the relevant exponential functionals of the form $\int_0^t \exp\{\rho_1 B^H_i(r) + \rho_2 B^H_j(r) - ar\}dr$, and the lower and upper bounds of the finite-time blow-up $\tau$ ($\tau_*$ and $\tau^*$) are obtained explicitly from Yor’s formula (Theorems 3.1 and 4.1). By relaxing the above conditions, the lower and upper bounds ($\tau_{**}$, $\tau_{**}^*$ and $\tau^*_{**}$) for the blow-up time $\tau$ of the solution to (1.3)-(1.4) are established (Theorems 5.1 and 5.2) in Section 5. Moreover, in Section 6 for $\lambda > \gamma = \max\{\gamma_1, \gamma_2\}$, we explicitly provide the bounds for the probability of finite-time blow-up of the system (1.3)-(1.4) for an appropriate choice of parameters, by using the Malliavin calculus and the method adopted in [7] (Theorem 6.2 and Corollary 6.1). We further extend the probability of finite-time blow-up results to a more general set of parameters (Theorems 6.3 and Corollary 6.2).

2 A System of Random PDEs

In this section, we obtain a system of random PDEs by using the random transformations

$$v_i(t, x) = \exp\{-k_{i1}B_1^H(t) - k_{i2}B_2^H(t)\}u_i(t, x), \quad i = 1, 2,$$

for $t \geq 0$, $x \in D$ and $\frac{1}{2} < H < 1$, the system (1.3)-(1.4) can be transformed into a system of random PDEs:

$$\begin{cases}
\frac{\partial v_i(t, x)}{\partial t} = (\Delta + \gamma_i) v_i(t, x) + e^{-k_{i1}B_1^H(t) - k_{i2}B_2^H(t)} \left(e^{-k_{j1}B_1^H(t) - k_{j2}B_2^H(t)} v_j(t, x)\right)^{1+\beta_i},
\quad v_i(0, x) = f_i(x), \quad x \in D, \\
v_i(t, x) = 0, \quad t \geq 0 \quad x \in \partial D,
\end{cases}
(2.1)$$

for $i = 1, 2$, $j \in \{1, 2\}/\{i\}$. This system is understood in the pathwise sense and thus classical results for parabolic PDEs can be applied to show existence, uniqueness and positivity of solution $v = (v_1, v_2)^T$ up to an eventual blow-up (Friedman [8]).

**Theorem 2.1.** Let $u = (u_1, u_2)^T$ be the weak solution of (1.3)-(1.4). Then the function $v = (v_1, v_2)^T$ defined by

$$v_i(t, x) = \exp\{-k_{i1}B_1^H(t) - k_{i2}B_2^H(t)\}u_i(t, x), \quad t \geq 0, \quad x \in D, \quad i = 1, 2,$$
solves the system (2.1).

Proof. By using Ito’s formula (Lemma 2.7.1 in [15]), we have for $\frac{1}{2} < H < 1$

$$e^{-k_1 B^H_1(t) - k_2 B^H_2(t)} = 1 - \int_0^t e^{-k_1 B^H_1(s) - k_2 B^H_2(s)}(k_{11} dB^H_1(s) + k_{22} dB^H_2(s)).$$

For any smooth function $\varphi_i$ with compact support, we set

$$u_i(t, \varphi_i) = \int_D u_i(t, x)\varphi_i(x)dx, \quad i = 1, 2.$$

Then the weak solution of (1.3)-(1.4) is given by

$$u_i(t, \varphi_i) = u_i(0, \varphi_i) + \int_0^t u_i(s, \Delta \varphi_i)ds + \gamma_i \int_0^t u_i(s, \varphi_i)ds + \int_0^t u_{1+\beta_i}(s, \varphi_i)ds$$

$$+ k_{11} \int_0^t u_i(s, \varphi_i)dB^H_1(s) + k_{22} \int_0^t u_i(s, \varphi_i)dB^H_2(s),$$

where $i = 1, 2$, $\{j\} = \{1, 2\}/\{i\}$. By applying the integration by parts formula, we obtain

$$v_i(t, \varphi_i) := \int_D v_i(t, x)\varphi_i(x)dx$$

$$= v_i(0, \varphi_i) + \int_0^t e^{-k_1 W^H_1(t) - k_2 B^H_2(t)}du_i(s, \varphi_i)$$

$$+ \int_0^t u_i(s, \varphi_i)\left(e^{-k_1 B^H_1(s) - k_2 B^H_2(s)}(k_{11} dB^H_1(s) + k_{22} dB^H_2(s))ds\right).$$

Therefore,

$$v_i(t, \varphi_i) = v_i(0, \varphi_i) + \int_0^t v_i(s, \Delta \varphi_i)ds + \gamma_i \int_0^t v_i(s, \varphi_i)ds$$

$$+ \int_0^t e^{-k_1 B^H_1(t) - k_2 B^H_2(t)}\left(e^{k_{11} B^H_1(t) + k_{22} B^H_2(t)}v_j\right)_{1+\beta_i}(s, \varphi_i)ds$$

$$= v_i(0, \varphi_i) + \int_0^t \left[\Delta v_i(s, \varphi_i) + \gamma_i v_i(s, \varphi_i)\right]ds$$

$$+ \int_0^t e^{-k_1 B^H_1(t) - k_2 B^H_2(t)}\left(e^{k_{11} B^H_1(t) + k_{22} B^H_2(t)}v_j\right)_{1+\beta_i}(s, \varphi_i)ds. \quad (2.3)$$

The preceding equalities mean that $v = (v_1, v_2)^\top$ is a weak solution of (2.1), which is differentiable with respect to $t$. From the uniqueness of weak solutions, $v = (v_1, v_2)^\top$ solves (2.1). \[\square\]

The equation (2.3) can be rewritten as

$$v_i(t, x) = \exp\{t\gamma_i\} S_t f_i(x)$$

$$+ \int_0^t e^{(t-r)\gamma_i} S_{t-r}\left[e^{-k_1 B^H_1(r) - k_2 B^H_2(r)}\left(e^{k_{11} B^H_1(r) + k_{22} B^H_2(r)}v_j(r, \cdot)\right)_{1+\beta_i}\right](x)dr,$$

for $i = 1, 2$, $t \geq 0$ and $x \in D$. Here, the semigroup $\{S_t\}_{t \geq 0}$ of bounded linear operators is defined by

$$S_t f(x) = \mathbb{E}[f(X_t), \ t < \tau_D|X_0 = x], \quad x \in D,$$
for all bounded and measurable $f : D \to \mathbb{R}$, where $\{X_t\}_{t \geq 0}$ is the Brownian motion in $\mathbb{R}^d$ with variance parameter 2, killed at the time $\tau_D$ at which it strike the boundary $\partial D$. As discussed above, $\lambda > 0$ is the first eigenvalue of $-\Delta$ on $D$, which satisfies
\[
-\Delta \psi(x) = \lambda \psi(x), \quad x \in D,
\]
\[
(1 + \beta_1)k_{21} - k_{11} = (1 + \beta_2)k_{11} - k_{21} =: \rho_1,
\]
\[
(1 + \beta_1)k_{22} - k_{12} = (1 + \beta_2)k_{12} - k_{22} =: \rho_2,
\]
and
\[
\tau \quad \text{being the corresponding eigenfunction, which is strictly positive on } D \text{ and } \psi|_{\partial D} = 0. \text{ Remember that}
\]
\[
S_t \psi = \exp\{-\lambda t\} \psi, \quad t \geq 0,
\]
where we have assumed that $\psi$ is normalized so that $\int_D \psi(x) dx = 1$.

Let $\tau$ be the blow-up time of the system (2.1) with the initial values of the above form. Due to Theorem 2.1 and a.s continuity of $B_i^H$, the blow-up time for the system (1.3)-(1.4). Our aim is to find random times $\tau_*$ and $\tau^*$ such that $0 < \tau_* \leq \tau \leq \tau^*$.

### 3 A Lower Bound for $\tau$

This section aims to find a lower bound $\tau_*$ to the blow-up times such that $\tau_* \leq \tau$ when $\gamma_1 = \gamma_2 = \lambda$ and $\frac{1}{2} < H < 1$. First, we consider the equation (1.3) with parameters
\[
(1 + \beta_1)k_{21} - k_{11} = (1 + \beta_2)k_{11} - k_{21} =: \rho_1,
\]
\[
(1 + \beta_1)k_{22} - k_{12} = (1 + \beta_2)k_{12} - k_{22} =: \rho_2,
\]

**Theorem 3.1.** Assume that the conditions given in (3.1) hold, and let the initial values be of the form
\[
f_1 = C_1 \psi \quad \text{and} \quad f_2 = C_2 \psi,
\]
for some positive constants $C_1$ and $C_2$. Then $\tau_* > \tau$, where $\tau_*$ is given by
\[
\tau_* = \inf \left\{ t \geq 0 : \int_0^t \exp\{\rho_1 B_1^H(r) + \rho_2 B_2^H(r)\} dr \geq \min \left\{ \frac{1}{\beta_1 C_1^* \|\psi\|_\infty^{\beta_1}}, \frac{1}{\beta_2 C_2^* \|\psi\|_\infty^{\beta_2}} \right\} \right\}.
\]

**Proof.** Let $v_1$ and $v_2$ solve (2.4). Then, we have
\[
v_1(t,x) = \exp\{\lambda t\} S_t f_1(x) + \int_0^t \exp\{\lambda (t-r)\} S_{t-r} \left( \exp\{\rho_1 B_1^H(r) + \rho_2 B_2^H(r)\} v_2^{1+\beta_1}(r,x) \right) dr,
\]
\[
v_2(t,x) = \exp\{\lambda t\} S_t f_2(x) + \int_0^t \exp\{\lambda (t-r)\} S_{t-r} \left( \exp\{\rho_1 B_1^H(r) + \rho_2 B_2^H(r)\} v_1^{1+\beta_2}(r,x) \right) dr,
\]
for all $x \in D$, $t \geq 0$. Let us define the operators $T_1$, $T_2$ as
\[
T_1 v(t,x) = \exp\{\lambda t\} S_t f_1(x) + \int_0^t \exp\{\rho_1 B_1^H(r) + \rho_2 B_2^H(r) + \lambda(t-r)\} (S_{t-r} v)^{1+\beta_1} dr,
\]
\[
T_2 v(t,x) = \exp\{\lambda t\} S_t f_2(x) + \int_0^t \exp\{\rho_1 B_1^H(r) + \rho_2 B_2^H(r) + \lambda(t-r)\} (S_{t-r} v)^{1+\beta_2} dr,
\]
where $v$ is any non-negative, bounded and measurable function. Moreover, on the set $t \leq \tau_*$, we set
\[
G_1(t) = \left[ 1 - \beta_1 \int_0^t e^{\rho_1 B_1^H(r) + \rho_2 B_2^H(r)} \|\exp\{\lambda r\} S_r f_1\|_\infty^{\beta_1} dr \right]^{-\frac{1}{\beta_1}},
\]
Thus, we have
\[
\mathcal{G}_2(t) = \left[ 1 - \beta_2 \int_0^t \exp\{\lambda r\} S_r f_2(t) \|\exp\{\lambda r\} S_r f_2(t)\|_{\infty}^2 \right]^{\frac{1}{2}}.
\]
Then, it can be easily seen that
\[
\frac{d\mathcal{G}_1(t)}{dt} = e^{\rho_1 B^H_1(t) + \rho_2 B^H_2(t)} \|\exp\{\lambda t\} S_t f_1(t)\|_{\infty}^{\beta_1} \mathcal{G}_1^{1+\beta_1}(t), \quad \mathcal{G}_1(0) = 1,
\]
so that
\[
\mathcal{G}_1(t) = 1 + \int_0^t e^{\rho_1 B^H_1(s) + \rho_2 B^H_2(s)} \|\exp\{\lambda r\} S_r f_1(r)\|_{\infty}^{\beta_1} \mathcal{G}_1^{1+\beta_1}(r) dr.
\]
Similarly, we have
\[
\mathcal{G}_2(t) = 1 + \int_0^t e^{\rho_1 B^H_1(s) + \rho_2 B^H_2(s)} \|\exp\{\lambda r\} S_r f_2(r)\|_{\infty}^{\beta_2} \mathcal{G}_2^{1+\beta_2}(r) dr.
\]
Let us choose \( v \geq 0 \) such that
\[
v(t, x) = \exp\{\lambda t\} S_t f_1(x) \mathcal{G}_1(t),
\]
for \( x \in D \) and \( t < \tau_* \). Then \( \exp\{\lambda t\} S_t f_1(x) \leq \mathcal{T}_1 v(t, x) \) and
\[
\mathcal{T}_1 v(t, x) = \exp\{\lambda t\} S_t f_1(x) + \int_0^t \exp\{\rho_1 B^H_1(r) + \rho_2 B^H_2(r) + \lambda(t-r)\} (S_{t-r} v) \|S_t f_1\|_{\infty}^{1+\beta_1} dr
\]
\[
\leq \exp\{\lambda t\} S_t f_1(x) + \int_0^t \exp\{\rho_1 B^H_1(r) + \rho_2 B^H_2(r) + \lambda(t-r)\}
\times \left[ \exp\{\lambda r\} \mathcal{G}_1(r) \exp\{\beta_1 \lambda r\} \mathcal{G}_1^{\beta_1}(r) \|S_r f_1\|_{\infty}^{\beta_1} (S_{t-r} f_1(x)) \right] dr
\]
\[
= \exp\{\lambda t\} S_t f_1(x) + \int_0^t \exp\{\rho_1 B^H_1(r) + \rho_2 B^H_2(r) + \lambda t\}
\times \left[ \exp\{\beta_1 \lambda r\} \mathcal{G}_1^{1+\beta_1}(r) \|S_r f_1\|_{\infty}^{\beta_1} S_{t-r} f_1(x) \right] dr
\]
\[
= \exp\{\lambda t\} S_t f_1(x) \left\{ 1 + \int_0^t \exp\{\rho_1 B^H_1(r) + \rho_2 B^H_2(r)\} \|\exp\{\lambda r\} S_r f_1\|_{\infty}^{\beta_1} \mathcal{G}_1^{1+\beta_1}(r) dr \right\}
\]
\[
= \exp\{\lambda t\} S_t f_1(x) \mathcal{G}_1(t).
\]
Thus, we have
\[
\exp\{\lambda t\} S_t f_1(x) \leq \mathcal{T}_1 v(t, x) \leq \exp\{\lambda t\} S_t f_1(x) \mathcal{G}_1(t).
\]
Similarly, we get
\[
\exp\{\lambda t\} S_t f_2(x) \leq \mathcal{T}_2 w(t, x) \leq \exp\{\lambda t\} S_t f_2(x) \mathcal{G}_2(t),
\]
for all \( w \) such that
\[
0 \leq w(t, x) \leq \exp\{\lambda t\} S_t f_2(x) \mathcal{G}_2(t).
\]
Let us take
\[
u_1^{(0)}(t, x) = \exp\{\lambda t\} S_t f_1(x), \quad \nu_2^{(0)}(t, x) = \exp\{\lambda t\} S_t f_2(x),
\]
for all \( t < \tau_* \).
Moreover, we have
\[ u_1^{(n)}(t, x) = T_1 u_2^{(n-1)}(t, x), \quad u_2^{(n)}(t, x) = T_2 u_1^{(n-1)}(t, x), \quad n \geq 1, \]
for \( x \in D, \ 0 \leq t \leq \tau_* \). Our aim is to show that the sequences of functions \( \{u_1^{(n)}\} \) and \( \{u_2^{(n)}\} \) are increasing. We consider
\[
u_1(t, x) = \lim_{n \to \infty} u_1^{(n)}(t, x), \quad v_2(t, x) = \lim_{n \to \infty} u_2^{(n)}(t, x),
\]
each existing for \( x \in D \) and \( 0 \leq t < \tau_* \). Then by the monotone convergence theorem, we obtain
\[
u_1(t, x) = T_1 u_2(t, x), \quad v_2(t, x) = T_2 u_1(t, x), \quad x \in D, \quad 0 \leq t < \tau_*.
\]
Moreover, we have
\[
T_1 v(t, x) \leq \exp\{\lambda t\} S_t f_1(x) \mathcal{G}_1(t) \quad \text{and} \quad T_2 u(t, x) \leq \exp\{\lambda t\} S_t f_2(x) \mathcal{G}_2(t),
\]
so that
\[
v_1(t, x) \leq \frac{\exp\{\lambda t\} S_t f_1(x)}{\left[ 1 - \beta_1 \int_0^t \exp\{\rho_1 B_1^H(r) + \rho_2 B_2^H(r)\} \| \exp\{\lambda r\} S_r f_1 \|_{\infty}^\beta dr \right]^{\frac{1}{\beta_1}},
\]
\[
v_2(t, x) \leq \frac{\exp\{\lambda t\} S_t f_2(x)}{\left[ 1 - \beta_2 \int_0^t \exp\{\rho_1 B_1^H(r) + \rho_2 B_2^H(r)\} \| \exp\{\lambda r\} S_r f_2 \|_{\infty}^\beta dr \right]^{\frac{1}{\beta_2}}}
\]
By the choice of initial values as in (3.2), the proof of the theorem follows.

\[ \square \]

**Remark 3.1.** For general bounded, measurable and positive \( f_i, \ i = 1, 2 \), the blow-up time of (1.3) is bounded below by the random time
\[
\inf \left\{ t \geq 0 : \int_0^t \exp\{\rho_1 B_1^H(r) + \rho_2 B_2^H(r)\} \| \exp\{\lambda r\} S_r f_1 \|_{\infty}^\beta dr \geq \frac{1}{\beta_1} \right\},
\]
or
\[
\inf \left\{ t \geq 0 : \int_0^t \exp\{\rho_1 B_1^H(r) + \rho_2 B_2^H(r)\} \| \exp\{\lambda r\} S_r f_2 \|_{\infty}^\beta dr \geq \frac{1}{\beta_2} \right\},
\]
which coincides with the lower bound \( \tau_* \) in Theorem 3.1 when the initial values satisfy (3.2).
4 Upper Bound

In this section, we obtain the upper bound \( \tau^* \) for the explosion time \( \tau \), under the assumption (5.1) and \( \frac{1}{2} < H < 1 \). By setting \( \varphi_i = \psi, \ i = 1, 2 \), the system (2.3) reduces to

\[
v_i(t, \psi) = v_i(0, \psi) + \int_0^t \Delta v_i(s, \psi)ds + \gamma_i \int_0^t v_i(s, \psi)ds
+ \int_0^t \exp\{-k_1 B_1^H(s) - k_2 B_2^H(s)\}(\exp\{k_1 B_1^H(s) + k_2 B_2^H(s)\}v_j)^{1+\beta_i}(s, \psi)ds. \tag{4.1}
\]

Using the fact that \( \psi(x) = 0 \) for \( x \in \partial D \), we have

\[
v_i(s, \Delta \psi) = \int_D v_i(s, x)\Delta \psi(x)dx = \int_D \Delta v_i(s, x)\psi(x)dx = \Delta v_i(s, \psi) = -\lambda v_i(s, \psi).
\]

Therefore, (4.1) becomes

\[
\frac{dv_i(t, \psi)}{dt} = (-\lambda + \gamma_i)v_i(t, \psi) + \exp\{-k_1 B_1^H(s) - k_2 B_2^H(s)\}(\exp\{k_1 B_1^H(s) + k_2 B_2^H(s)\}v_j)^{1+\beta_i}(t, \psi),
\]

for \( i = 1, 2, \ j \in \{1, 2\}/\{i\} \). By using Jensen’s inequality, we obtain

\[
(\exp\{k_1 B_1^H(s) + k_2 B_2^H(s)\}v_j)^{1+\beta_i}(s, \psi) \\
\geq \left[ \int_D (\exp\{k_1 B_1^H(s) + k_2 B_2^H(s)\}v_j(s, x))\psi(x)dx \right]^{1+\beta_i} \\
= \exp\{(1 + \beta_i)k_1 B_1^H(s) + (1 + \beta_i)k_2 B_2^H(s)\}v_j(s, \psi)^{1+\beta_i}. \tag{4.2}
\]

Thus, it is immediate that

\[
\frac{dv_i(t, \psi)}{dt} \geq (-\lambda + \gamma_i)v_i(t, \psi) + \exp\{-k_1 B_1^H(s) - k_2 B_2^H(s)\} \\
\times \exp\{(1 + \beta_i)k_1 B_1^H(s) + (1 + \beta_i)k_2 B_2^H(s)\}v_j(s, \psi)^{1+\beta_i},
\]

\( i = 1, 2, \ j \in \{1, 2\}/\{i\} \). In this way, \( v_i(t, \psi) \geq h_i(t) \geq 0 \), for \( i = 1, 2 \), where

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{dh_1(t)}{dt} = (-\lambda + \gamma_1)h_1(t) + \exp\{\rho_1 B_1^H(t) + \rho_2 B_2^H(t)\}h_2^{1+\beta_1}(t) \\
\frac{dh_2(t)}{dt} = (-\lambda + \gamma_2)h_2(t) + \exp\{\rho_1 B_1^H(t) + \rho_2 B_2^H(t)\}h_1^{1+\beta_2}(t), \\
h_i(0) = v_i(0, \psi), \ i = 1, 2.
\end{array} \right.
\tag{4.3}
\]

Let us define \( E(t) := h_1(t) + h_2(t) \geq 0, \ t \geq 0 \), so that \( E(\cdot) \) satisfies

\[
\frac{dE(t)}{dt} \geq (-\lambda + \gamma)E(t) + \exp\{\rho_1 B_1^H(t) + \rho_2 B_2^H(t)\}\left[h_1^{1+\beta_2}(t) + h_2^{1+\beta_1}(t)\right], \tag{4.4}
\]

where \( \gamma = \max\{\gamma_1, \gamma_2\} \).

**Theorem 4.1.**

1. Assume that \( \beta_1 = \beta_2 = \beta \) (say) > 0. Then \( \tau \leq \tau^* \), where

\[
\tau^* = \inf\left\{ t \geq 0 : \int_0^t \exp\{\rho B_1^H(s) + \rho B_2^H(s) + \beta(-\lambda + \gamma)s\}ds \geq 2^\beta \beta^{-1}E^{-\beta}(0) \right\}
\]

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Remark 4.1. Note that

\[ \text{Case 1:} \]

Proof of Theorem 4.1. By substituting \( a = 1, b = \frac{h_1}{h_2}, n = 1 + \beta \) into the inequality

\[ a^n + b^n \geq 2^{-n}(a + b)^n, \]

which is valid for \( a, b \geq 0 \), we obtain

\[ h_1^{1+\beta}(t) + h_2^{1+\beta}(t) \geq 2^{-(1+\beta)}(h_1(t) + h_2(t))^{1+\beta} \geq 2^{-(1+\beta)}E^{1+\beta}(t). \]

From (4.7), we infer that

\[ \frac{dE(t)}{dt} \geq (-\lambda + \gamma)E(t) + 2^{-\beta} \exp\{\rho B_1^H(t) + \rho B_2^H(t)\} E^{1+\beta}(t). \]

Thus, \( E(t) \) blows up not later than the solution \( I(t) \) of the differential equation

\[ \begin{cases} 
\frac{dI(t)}{dt} = (-\lambda + \gamma)I(t) + 2^{-\beta} \exp\{\rho B_1^H(t) + \rho B_2^H(t)\} I(t)^{1+\beta}, \\
I(0) = E(0), 
\end{cases} \]

which immediately gives

\[ I(t) = \exp\{(\lambda - \gamma)t\} \left\{ E(0)^{-\beta} - 2^{-\beta} \int_0^t \exp\{\rho B_1^H(t) + \rho B_2^H(t) + \beta(-\lambda + \gamma)s\} ds \right\}^{-\frac{1}{\beta}}. \]
From the above equation, the explosion time is given by
\[
\tau^* = \inf \left\{ t \geq 0 : \int_0^t \exp\{\rho B_1^H(t) + \rho B_2^H(t) + \beta(-\lambda + \gamma)s\} ds \geq 2^\beta \beta^{-1} E^{-\beta}(0) \right\}.
\]

**Case 2:** Suppose \( \beta_1 > \beta_2 > 0 \) in (4.4), then we have
\[
\frac{dE(t)}{dt} \geq (-\lambda + \gamma) E(t) + \exp\{\rho_1 B_1^H(t) + \rho_2 B_2^H(t)\} \left[ h_1^{1+\beta_2}(t) + h_2^{1+\beta_1}(t) \right].
\] (4.9)

The Young inequality states that if \( 1 < b < \infty \), \( \beta > 0 \), and \( a = \frac{b}{b-1} \), then
\[
xy \leq \frac{\delta^a x^a}{a} + \frac{\delta^{-b} y^b}{b}, \quad x, y \geq 0.
\] (4.10)

By setting \( b = \frac{1+\beta_2}{1+\beta_1} \), \( y = h_2^{1+\beta_2}(t) \), \( x = \epsilon \), \( \delta = \left( \frac{1+\beta_1}{1+\beta_2} \right)^{\frac{1+\beta_1}{1+\beta_2}} \), and using the fact that \( \beta_2 < \beta_1 \) in (4.10), it follows that for any \( \epsilon > 0 \),
\[
h_2^{1+\beta_1}(t) \geq \epsilon h_2^{1+\beta_2}(t) - D_1 \epsilon^{\frac{1+\beta_1}{\beta_1-\beta_2}}.
\]

Since \( \epsilon_0 \) is the minimum of the quantities given in (4.5), in particular, we have \( \epsilon_0 \leq (h_1(0)/D_1^{1/1+\beta_2})^{\beta_1-\beta_2} \), so that
\[
\epsilon_0 h_2^{1+\beta_2}(0) - D_1 \epsilon_0^{\frac{1+\beta_1}{\beta_1-\beta_2}} \geq 0.
\]

From (4.9), we have
\[
\frac{dE(t)}{dt} \geq (-\lambda + \gamma) E(t) + \exp\{\rho_1 B_1^H(t) + \rho_2 B_2^H(t)\} \left[ h_1^{1+\beta_2}(t) + \epsilon_0 h_2^{1+\beta_2}(t) - D_1 \epsilon_0^{\frac{1+\beta_1}{\beta_1-\beta_2}} \right].
\] (4.11)

Utilizing the inequality (4.8) again with \( n = 1 + \beta_2 \), \( a = 1 \) and \( b = \epsilon_0 \frac{1+\beta_1}{1+\beta_2} h_1(t) \), and by using (4.5), we see that
\[
h_2^{1+\beta_2}(t) + \epsilon_0 h_1^{1+\beta_2}(t) \geq 2^{-(1+\beta_2)} \left( h_2(t) + \epsilon_0 \frac{1+\beta_1}{1+\beta_2} h_1(t) \right)^{1+\beta_2} \geq 2^{-(1+\beta_2)} \epsilon_0 E^{1+\beta_2}(t).
\]

From (4.11), we deduce that
\[
\frac{dE(t)}{dt} \geq (-\lambda + \gamma) E(t) + \exp\{\rho_1 B_1^H(t) + \rho_2 B_2^H(t)\} \left[ 2^{-(1+\beta_2)} \epsilon_0 E^{1+\beta_2}(t) - D_1 \epsilon_0^{\frac{1+\beta_1}{\beta_1-\beta_2}} \right].
\]

Note that the condition (4.6) gives \( E(t) \geq E(0) > 0 \), and so
\[
\frac{dE(t)}{dt} \geq (-\lambda + \gamma) E(t) + \exp\{\rho_1 B_1^H(t) + \rho_2 B_2^H(t)\} \left[ \epsilon_0 \frac{2^{1+\beta_2}}{2^{1+\beta_2}} - D_1 \epsilon_0^{\frac{1+\beta_1}{\beta_1-\beta_2}} \right] dt.
\]

Thus, \( E(t) \) blows up not later than the solution \( I(t) \) of the equation
\[
\begin{cases}
\frac{dI(t)}{I^{1+\beta_2}(t)} = (-\lambda + \gamma) dt + \exp\{\rho_1 B_1^H(t) + \rho_2 B_2^H(t)\} \left[ \epsilon_0 \frac{2^{1+\beta_2}}{2^{1+\beta_2}} - D_1 \epsilon_0^{\frac{1+\beta_1}{\beta_1-\beta_2}} \right] dt,
\end{cases}
\]
\[
I(0) = E(0),
\]
and
\[ I(t) = e^{(\lambda - \gamma)t} \left\{ E^{-\beta_2}(0) - \beta_2 \left[ \frac{\epsilon_0}{21 + \beta_2} - \frac{\epsilon_0}{E^{1 + \beta_2}(0)} \right] \int_0^t e^{\beta_1 B_i^H(s) + \rho_2 B_i^H(s) + \beta_2(-\lambda + \gamma)s} ds \right\}^{1/\beta_2}. \]

From the above equality, the explosion time is given by
\[ \tau^* = \inf \left\{ t \geq 0 : \int_0^t e^{\beta_1 B_i^H(s) + \rho_2 B_i^H(s) + \beta_2(-\lambda + \gamma)s} ds \geq \left[ \beta_2 E^{\beta_2}(0) \left( \frac{\epsilon_0}{21 + \beta_2} - \frac{\epsilon_0}{E^{1 + \beta_2}(0)} \right) \right]^{-1} \right\}, \]

which completes the proof. \( \square \)

5 A More General Case

In this section, we consider the system (1.3)-(1.4) with the assumption that \( \beta_1 \geq \beta_2 > 0 \) and \( \frac{1}{2} < H < 1 \). From [2.3], we have
\[ v_i(t, x) = \exp\{\gamma_i t\} S_i f_i(x) + \int_0^t e^{((1 + \beta_1)k_{21} - k_{11})B_i^H(r) + ((1 + \beta_2)k_{21} - k_{11})B_i^H(r) + \gamma_i(t-r)S_{t-r}(v_i(r, \cdot))^{1+\beta_i}(x) dr. \]

where \( i = 1, 2 \).

Theorem 5.1. Assume that \( \beta_1 \geq \beta_2 > 0 \) and \( \gamma_1 = \gamma_2 = \lambda \) and let
\[ f_1 = C_1 \psi \text{ and } f_2 = C_2 \psi, \]
for some positive constant \( C_1 \) and \( C_2 \). Then \( \tau_{**} \leq \tau \), where \( \tau_{**} \) be given by
\[ \tau_{**} = \inf \left\{ t \geq 0 : \int_0^t \exp\{((1 + \beta_1)k_{21} - k_{11})B_i^H(r) + ((1 + \beta_2)k_{21} - k_{11})B_i^H(r)\} dr \right\}
\[ \geq \frac{1}{\beta_1 C_1 \|\psi\|_{\infty}^{\beta_1}} \]
or
\[ \int_0^t \exp\{((1 + \beta_1)k_{22} - k_{12})B_i^H(r) + ((1 + \beta_2)k_{12} - k_{22})B_i^H(r)\} dr \right\}
\[ \geq \frac{1}{\beta_2 C_2 \|\psi\|_{\infty}^{\beta_2}} \]

Proof. For \( i = 1, 2 \), we define
\[ T_i v(t, x) = \exp\{\lambda t\} S_i f_i(x) + \int_0^t e^{((1 + \beta_1)k_{21} - k_{11})B_i^H(r) + ((1 + \beta_2)k_{21} - k_{11})B_i^H(r) + \lambda(t-r)S_{t-r}v(r, x)}^{1+\beta_i} dr, \]
and
\[ G_i(t) = \left[ 1 - \beta_i \int_0^t \exp\{((1 + \beta_1)k_{21} - k_{11})B_i^H(r) + ((1 + \beta_2)k_{11} - k_{21})B_i^H(r) + \lambda r\} \|S_r f_i\|_{\infty}^{\beta_i} dr \right]^{\frac{1}{\beta_i}}. \]
Proceeding in the same way as in the proof of Theorem 3.1, we get
\[ v_1(t, x) = \mathcal{T}_1 u_2(t, x), \quad v_2(t, x) = \mathcal{T}_2 u_1(t, x), \]
whenever \( t \leq \tau^*, \) and \( x \in D. \) Moreover
\[ v_i(t, x) \leq \exp\{\lambda t\}S_tf_i(x) \]
\[ = \frac{\exp\{(1 + \beta_i)k_{2i} - k_{1i}\}B_t^H(r) + ((1 + \beta_i)k_1 - k_{2i})B_t^H(r)}{C_i\psi(x)} \]
\[ \leq \frac{\exp\{(1 + \beta_i)k_{2i} - k_{1i}\}B_t^H(r) + ((1 + \beta_i)k_1 - k_{2i})B_t^H(r)}{\beta_1C_1\psi_t^\beta_1, \beta_2C_2\psi_t^\beta_2} \]
by the choice of \( f_1 \) and \( f_2. \)

**Corollary 5.1.** Let the random time \( \tau' \) be defined by
\[ \tau' = \inf \left\{ t \geq 0 : \int_0^t \max \left\{ \exp\{(1 + \beta_i)k_{2i} - k_{1i}\}B_t^H(r) + ((1 + \beta_i)k_1 - k_{2i})B_t^H(r) \right\} dr \right\} \]
\[ \geq \min \left\{ \frac{1}{\beta_1C_1\psi_t^\beta_1}, \frac{1}{\beta_2C_2\psi_t^\beta_2} \right\}. \]
Then \( \tau' \leq \tau^*. \)

To estimate upper bounds for \( \tau, \) when \( \beta_1 \geq \beta_2 > 0, \) we first notice that a sub-solution of equation 4.1 is given by
\[ \begin{cases} dh_1(t) = (-\lambda + \gamma_1)h_1(t) + \exp\{(1 + \beta_i)k_{2i} - k_{1i}\}B_t^H(r) + ((1 + \beta_i)k_1 - k_{2i})B_t^H(r)h_2^{1+\beta_i}(t), \\ dh_2(t) = (-\lambda + \gamma_2)h_2(t) + \exp\{(1 + \beta_i)k_{2i} - k_{1i}\}B_t^H(r) + ((1 + \beta_i)k_1 - k_{2i})B_t^H(r)h_1^{1+\beta_i}(t), \\ h_i(0) = v_i(0, \psi), \quad i = 1, 2. \end{cases} \]
(5.1)

Implementing the same idea as in the proof of Theorem 4.1 to the system (5.1) with \( \gamma = \max\{\gamma_1, \gamma_2\} \) and if \( \beta_1 = \beta_2 = \beta \) (say) > 0, we deduce that
\[ \frac{dE(t)}{dt} \geq (-\lambda + \gamma)E(t) + \min \left\{ \exp\{(1 + \beta)k_{2i} - k_{1i}\}B_t^H(t) + ((1 + \beta)k_1 - k_{2i})B_t^H(t) \right\} 2^{-\beta} E^{1+\beta}(t), \]
If \( \beta_1 > \beta_2 > 0, \) we obtain
\[ \frac{dE(t)}{E^{1+\beta_2}(t)} \geq \left( -\lambda + \gamma \right) \frac{dt}{E^{\beta_2}(t)} + \min \left\{ \exp\{(1 + \beta_1)k_{2i} - k_{1i}\}B_t^H(t) + ((1 + \beta_1)k_1 - k_{2i})B_t^H(t), \right. \]
\[ \exp\{(1 + \beta_2)k_{2i} - k_{1i}\}B_t^H(t) + ((1 + \beta_2)k_1 - k_{2i})B_t^H(t) \left\} \right. \]
\[ \times \left[ \frac{\epsilon_0}{2^{1+\beta_2} - \frac{D_1\epsilon_0}{E^{1+\beta_2}(0)} \right] dt, \]
which also follows from the proof of Theorem 4.1. In this manner we obtain the upper bounds for the explosion time to the solutions of the system (1.3)- (1.4) as follows:
Theorem 5.2. Let $\beta_1 \geq \beta_2 > 0$ and $\gamma = \max\{\gamma_1, \gamma_2\}$.

1. If $\beta_1 = \beta_2 = \beta$ (say) then $\tau \leq \tau_1^{**}$, where

$$
\tau_1^{**} = \inf \left\{ t \geq 0 : \int_0^t \min\{\exp\{((1 + \beta)k_{21} - k_{11})B_1^H(s) + ((1 + \beta)k_{11} - k_{21})B_2^H(s) + \beta(-\lambda + \gamma)s\}, \exp\{((1 + \beta)k_{22} - k_{12})B_1^H(s) + ((1 + \beta)k_{12} - k_{22})B_2^H(s) + \beta(-\lambda + \gamma)s\}\} ds \geq 2^2\beta^{-1}E^{-\beta}(0) \right\}.
$$

2. If $\beta_1 > \beta_2 > 0$, then $\tau \leq \tau_2^{**}$, where

$$
\tau_2^{**} = \inf \left\{ t \geq 0 : \int_0^t \min\{\exp\{((1 + \beta_1)k_{21} - k_{11})B_1^H(s) + ((1 + \beta_1)k_{11} - k_{21})B_2^H(s) + \beta_1(-\lambda + \gamma)s\}, \exp\{((1 + \beta_2)k_{11} - k_{21})B_1^H(s) + ((1 + \beta_2)k_{22} - k_{12})B_2^H(s) + \beta_2(-\lambda + \gamma)s\}\} ds \geq \beta_2E^{\beta_2}(0) \left( \frac{e_0}{(1 + \beta_1 - \gamma)D_1} - \frac{e_0}{E^{\beta_2}(0)} \right)^{-1} \right\}.
$$

Corollary 5.2. Assume that $f_i \geq 0$, $i = 1, 2$ such that

$$
\beta_i \int_0^t \exp\{\rho_1 B_1^H(r) + \rho_2 B_2^H(r)\}\|\exp\{\lambda r\}S_r f_i\|_\infty dr < 1, \ t \geq 0. \tag{5.2}
$$

Then the equation (1.3)-(1.4) admits a global solution $u = (u_1, u_2)^T$ that satisfies

$$
0 \leq v_i(t, x) \leq \frac{\exp\{\lambda t\}S_r f_i(x)}{\left[1 - \beta_i \int_0^t \exp\{\rho_1 B_1^H(r) + \rho_2 B_2^H(r)\}\|\exp\{\lambda r\}S_r f_i\|_\infty dr\right]^\beta_i}, \ t \geq 0.
$$

Proof. Proof of this immediately follows from Theorem 5.1 \qed

6 Probability of finite time blow-up

In this section, we estimate the blow-up probability of solutions to the system (1.3)-(1.4). Even though we have proved the upper and lower bounds for the blow-up times for the solutions to the system (1.3)-(1.4) for $\frac{1}{2} < H < 1$, the results obtained in this section are true for $0 < H < 1$. Let $B_i^H = (B_i^H(t), B_2^H(t))_{t \geq 0}$ be a two dimensional fBm defined on a complete probability space $(\Omega, F, (F_t)_{t \geq 0}, \mathbb{P})$, where $B_1^H(t)$ and $B_2^H(t)$ are independent. For $h = (h_1, h_2) \in \mathbb{H} := L^2(\mathbb{R}^+; \mathbb{R}^2)$, the Wiener integral $W(h)$ is denoted by

$$
W(h) = \int_0^\infty h^1(t_1)dW(t_1) + \int_0^\infty h^2(t_2)dW(t_2).
$$

Let $S$ denote the dense subset of $L^2(\Omega, F, \mathbb{P})$ consisting of smooth random variables of the form

$$
F = f(W(h_1), W(h_2), \ldots, W(h_n)), \text{ and for each } h_i = (h^1_i, h^2_i), \tag{6.1}
$$

where $n \in \mathbb{N}$, $i \in \{1, 2, \ldots, n\}$, $f \in C_0^\infty(\mathbb{R}^n)$ and $h^1_i, h^2_i \in L^2(\mathbb{R}^+)$. If $F$ is of the form give in (6.1), we define its Malliavin derivative as the process

$$
DF := (D_{t_1}F, D_{t_2}F)_{t_1, t_2 \geq 0}.
$$
given by
\[ D_t F = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(W(h_1), W(h_2), \ldots, W(h_n))h_k(t_i), \quad i = 1, 2. \]

For any \(1 \leq p < \infty\), we shall denote by \(\mathbb{D}^{1,p}\), the closure of \(\mathcal{S}\) with respect to the norm
\[ \| F \|_{1,p}^p := \mathbb{E}[|F|^p] + \mathbb{E}[\|DF\|_{\mathbb{H}}^p], \]
where \(\|DF\|_{\mathbb{H}}^p = \int_0^\infty |D_t F|^p dt_1 + \int_0^\infty |D_t^2 F|^p dt_2\). A random variable \(F\) is said to be Malliavin differentiable if it belongs to \(\mathbb{D}^{1,2}\).

**Lemma 6.1.** (Lemma 2.1 in \([7]\)). Let \(Z\) be a centered random variable in \(\mathbb{D}^{1,2}\). Assume that there exists a non-random constant \(M\) such that \(\int_0^\infty |D_\theta Z|^2 d\theta \leq M\), \(\mathbb{P}\)-a.s. Then the following tail probability estimate holds:
\[ \mathbb{P}(Z \geq x) \leq \exp \left\{ -\frac{x^2}{2M} \right\}, \quad x > 0. \]

An fBm of Hurst parameter \(H \in (0, 1)\) is a centered Gaussian process \(B^H = (B^H_1(t_1), B^H_2(t_2))_{t_1, t_2 \geq 0}\), with covariance function
\[ R_H(t, s) := \frac{1}{2} (s^{2H} + t^{2H} - |t - s|^{2H}) I_{2 \times 2}. \]

It is known that \(B^H_1(t)\) admits the so called Volterra representation (Nualart \([17]\), 277-279),
\[ B^H_1(t) := \int_0^t K_H(t, s) dW_i(s), \quad i = 1, 2, \]
where \((W_i(t))_{t \geq 0}, \quad i = 1, 2,\) are standard Brownian motions, the Volterra kernel \(K_H(t, s)\) is defined by
\[ K_H(t, s) = CH \left[ \left( \frac{t^{H-\frac{1}{2}}}{s^{H-\frac{1}{2}}} (t-s)^{H-\frac{1}{2}} - \left( H - \frac{1}{2} \right) \int_s^t \frac{u^{H-\frac{1}{2}}}{s^{H-\frac{1}{2}} (u-s)^{H-\frac{1}{2}}} du \right), \quad s \leq t, \right] \]
where \(C_H\) is a constant depending only on \(H\). We establish an upper bound for probability of the form
\[ \mathbb{P} \left[ \int_0^\infty \exp\{-as + \sigma_1 X_s + \sigma_2 Y_s\} ds < x \right], \quad (6.2) \]
where \(x > 0, \sigma_i > 0, \quad i = 1, 2\) and \(a\) are real numbers, \((X_t)_{t \geq 0}\) and \((Y_t)_{t \geq 0}\) are continuous stochastic processes, which contain fBms \((B^H_1(t))_{t \geq 0}\) and \((B^H_2(t))_{t \geq 0}\), respectively.

**Assumption 1.** The stochastic process \((X_t, Y_t)_{t \geq 0}\) is \(\mathcal{F}_t\)-adapted and satisfies the following properties:

1. \(\int_0^\infty \exp\{-as\} \mathbb{E}[\exp\{\sigma_1 X_s\}] \mathbb{E}[\exp\{\sigma_2 Y_s\}] ds < +\infty.\)
2. For each \(t \geq 0, \ X_t, Y_t \in \mathbb{D}^{1,2}.\)
3. There exists a function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\lim_{t \to \infty} f(t) = +\infty$ and for each $x > 0$

$$\sup_{t \geq 0} \frac{\int_{s \in [0,t]} |D_{\theta_1} X_s|^2 d\theta_1 + \sup_{s \in [0,t]} |D_{\theta_2} Y_s|^2 d\theta_2}{(\ln(x+1) + f(t))^2} \leq M_x < +\infty \ P\text{-a.s.}$$

**Theorem 6.1.** Suppose that Assumption 1 holds, then we have

$$\mathbb{P}\left[ \int_0^\infty \exp\{-as + \sigma_1 X_s + \sigma_2 Y_s\} ds < x \right] \leq \exp\left\{ -\frac{(N_x - 1)^2}{2\sigma^2 M_x} \right\} , \quad (6.3)$$

where

$$N_x = \mathbb{E}\left[ \sup_{t \geq 0} \frac{\ln\left( \int_0^t \exp\{-as + \sigma_1 X_s + \sigma_2 Y_s\} ds + 1 \right) + f(t)}{\ln(x+1) + f(t)} \right] \geq 1. \quad (6.4)$$

**Proof.** We have

$$\mathbb{P}\left[ \int_0^\infty \exp\{-as + \sigma_1 X_s + \sigma_2 Y_s\} ds < x \right] = \mathbb{P}\left[ \int_0^t \exp\{-as + \sigma_1 X_s + \sigma_2 Y_s\} ds < x \text{ for all } t > 0 \right]$$

$$= \mathbb{P}\left[ \ln\left( \int_0^t \exp\{-as + \sigma_1 X_s + \sigma_2 Y_s\} ds + 1 \right) < \ln(x+1) \text{ for all } t > 0 \right]$$

$$= \mathbb{P}\left[ \ln\left( \int_0^t \exp\{-as + \sigma_1 X_s + \sigma_2 Y_s\} ds + 1 \right) + f(t) \frac{\ln(x+1) + f(t)}{\ln(x+1) + f(t)} < 1 \text{ for all } t > 0 \right]$$

$$= \mathbb{P}\left[ \sup_{t \geq 0} Z_t < 1 \right], \quad (6.4)$$

where the stochastic process $\{Z_t\}_{t \geq 0}$ is defined by

$$Z_t := \frac{\ln\left( \int_0^t \exp\{-as + \sigma_1 X_s + \sigma_2 Y_s\} ds + 1 \right) + f(t)}{\ln(x+1) + f(t)} \leq \frac{\ln\left( \int_0^t \exp\{-as + \sigma_1 X_s + \sigma_2 Y_s\} ds + 1 \right)}{\ln(x+1)} + 1, \quad t \geq 0. \quad (6.5)$$

By using the fundamental inequality $\ln x \leq \sqrt{x}$, when $x > 0$, we obtain

$$Z_t \leq \sqrt{\frac{\int_0^t \exp\{-as + \sigma_1 X_s + \sigma_2 Y_s\} ds + 1}{\ln(x+1)}} + 1, \quad t \geq 0.$$

Making use of the inequality $(a + b)^p \leq 2^{p-1}(a^p + b^p)$, for any $p > 1$ and $a, b > 0$, we have

$$\mathbb{E}\left[ \sup_{t \geq 0} Z_t^2 \right] \leq 2 \int_0^\infty \mathbb{E}[\exp\{-as\}]\mathbb{E}[\exp\{\sigma_1 X_s\}]\mathbb{E}[\exp\{\sigma_2 Y_s\}] ds + 1 < +\infty. \quad (6.6)$$
Let us compute the Malliavin derivative of $Z_t$ for $\theta = (\theta_1, \theta_2)$ as

$$D_\theta Z_t = (D_{\theta_1} Z_t, D_{\theta_2} Z_t)_{\theta_1, \theta_2 \geq 0}.$$ 

Since $\{X_t\}_{t \geq 0}$ is $\mathcal{F}_t$-adapted, we have $D_{\theta_1} X_t = 0$ when $\theta_1 > t$, so that for $\theta_1 \leq t$, we estimate $D_{\theta_1} Z_t$ as

$$D_{\theta_1} Z_t = \frac{D_{\theta_1} \left[ \ln \left( \int_0^t \exp \{-as + \sigma_1 X_s + \sigma_2 Y_s\} ds + 1 \right) \right]}{(\ln(x + 1) + f(t)) \left( \int_0^t \exp \{-as + \sigma_1 X_s + \sigma_2 Y_s\} ds + 1 \right)}$$

$$\quad = \frac{\int_0^t D_{\theta_1} \left[ \exp \{-as + \sigma_1 X_s + \sigma_2 Y_s\} \right] ds}{(\ln(x + 1) + f(t)) \left( \int_0^t \exp \{-as + \sigma_1 X_s + \sigma_2 Y_s\} ds + 1 \right)}$$

$$\quad = \frac{\sigma_1 \int_0^t \exp \{-as + \sigma_1 X_s + \sigma_2 Y_s\} D_{\theta_1} X_s ds}{(\ln(x + 1) + f(t)) \left( \int_0^t \exp \{-as + \sigma_1 X_s + \sigma_2 Y_s\} ds + 1 \right)}.$$ 

Since $\{X_s\}_{s \geq 0}$ is $\mathcal{F}_s$-adapted, we have $D_{\theta_1} X_s = 0$ when $\theta_1 > s$. Thus, we obtain

$$D_{\theta_1} Z_t = \frac{\sigma_1 \int_0^t \exp \{-as + \sigma_1 X_s + \sigma_2 Y_s\} D_{\theta_1} X_s ds}{(\ln(x + 1) + f(t)) \left( \int_0^t \exp \{-as + \sigma_1 X_s + \sigma_2 Y_s\} ds + 1 \right)},$$

and

$$|D_{\theta_1} Z_t|^2 = \frac{\left| \sigma_1 \int_0^t \exp \{-as + \sigma_1 X_s + \sigma_2 Y_s\} D_{\theta_1} X_s ds \right|^2}{(\ln(x + 1) + f(t))^2 \left( \int_0^t \exp \{-as + \sigma_1 X_s + \sigma_2 Y_s\} ds + 1 \right)^2}.$$ 

By using Hölder’s inequality, we have

$$|D_{\theta_1} Z_t|^2 \leq \frac{\left( \sigma_1 \int_0^t \exp \{-as + \sigma_1 X_s + \sigma_2 Y_s\} |D_{\theta_1} X_s|^2 ds \right)^2}{(\ln(x + 1) + f(t))^2 \left( \int_0^t \exp \{-as + \sigma_1 X_s + \sigma_2 Y_s\} ds + 1 \right)^2}$$

$$\quad \leq \frac{\sigma^2_1 \int_0^t \exp \{-as + \sigma_1 X_s + \sigma_2 Y_s\} \left| D_{\theta_1} X_s \right|^2 ds}{(\ln(x + 1) + f(t))^2 \left( \int_0^t \exp \{-as + \sigma_1 X_s + \sigma_2 Y_s\} ds + 1 \right)^2}, \text{ for } \theta_1 \leq t,$$

and hence

$$\int_0^t |D_{\theta_1} Z_t|^2 d\theta_1 \leq \frac{\sigma^2_1 \int_0^t \int_{\theta_1}^t \exp \{-as + \sigma_1 X_s + \sigma_2 Y_s\} \left| D_{\theta_1} X_s \right|^2 ds d\theta_1}{(\ln(x + 1) + f(t))^2 \left( \int_0^t \exp \{-as + \sigma_1 X_s + \sigma_2 Y_s\} ds + 1 \right)^2}$$

$$\quad = \frac{\sigma^2_1 \int_0^s \exp \{-as + \sigma_1 X_s + \sigma_2 Y_s\} \left| D_{\theta_1} X_s \right|^2 d\theta_1 ds}{(\ln(x + 1) + f(t))^2 \left( \int_0^t \exp \{-as + \sigma_1 X_s + \sigma_2 Y_s\} ds + 1 \right)^2}$$

$$\quad \leq \frac{\sup_{s \in [0,t]} \int_0^s \left| D_{\theta_1} X_s \right|^2 d\theta_1}{(\ln(x + 1) + f(t))^2}, \text{ P-a.s.}$$
Similarly, we obtain

$$
\int_0^t |D_{\theta_2} Z_t|^2 d\theta_2 \leq \sigma_2^2 \sup_{s \in [0,t]} f_0^s |D_{\theta_2} Y_s|^2 d\theta_2, \quad \mathbb{P}\text{-a.s.}
$$

Now if \( \sigma^2 = \max\{\sigma_1^2, \sigma_2^2\} \), then we have

$$
\sup_{t \geq 0} \int_0^\infty |D_{\theta} Z_t|^2 d\theta = \sup_{t \geq 0} \int_0^t |D_{\theta_1} Z_t|^2 d\theta_1 + \sup_{t \geq 0} \int_0^t |D_{\theta_2} Z_t|^2 d\theta_2
$$

$$
\leq \sigma_1^2 \sup_{t \geq 0} \int_0^t |D_{\theta_1} X_t|^2 d\theta_1 + \sigma_2^2 \sup_{t \geq 0} \int_0^t |D_{\theta_2} Y_t|^2 d\theta_2
$$

$$
\leq \sigma^2 \int_0^\infty \left( \sup_{s \in [0,t]} f_0^s \right) \left( \sup_{s \in [0,t]} (\ln(x+1) + f(t))^2 \right) d\theta,
$$

$$
= \sigma^2 M_x < +\infty, \quad \mathbb{P}\text{-a.s.}
$$

(6.7)

The estimates (6.6) and (6.7) point out that the stochastic process \((Z_t)_{t \geq 0}\) satisfies the conditions of Proposition 2.1.10, [17]. Thus, one can conclude that the random variable \(\sup_{t \geq 0} Z_t\) belongs to \(D^{1,2}\).

We now claim that

$$
\int_0^\infty \left| D_{\theta} \left( \sup_{t \geq 0} Z_t \right) \right|^2 d\theta \leq \sup_{t \geq 0} \int_0^\infty |D_{\theta} Z_t|^2 d\theta, \quad \mathbb{P}\text{-a.s.}
$$

(6.8)

The above computations can be repeated so that \(\sup_{0 \leq t \leq T} Z_t\) belongs to \(D^{1,2}\) for each \(T > 0\). Moreover, it is easy to see that \(\sup_{0 \leq t \leq T} Z_t\) converges to \(\sup_{t \geq 0} Z_t\) in \(L^2(\Omega)\) as \(T \to \infty\). Therefore, by the closability of Malliavin derivatives, \(D_{\theta} \left( \sup_{0 \leq t \leq T} Z_t \right)\) converges to \(D_{\theta} \left( \sup_{t \geq 0} Z_t \right)\) in \(L^2(\mathbb{R}^+ \times \Omega; \mathbb{R}^2)\). In order to prove the claim (6.8), it is enough to show that

$$
\int_0^\infty \left| D_{\theta} \left( \sup_{0 \leq t \leq T} Z_t \right) \right|^2 d\theta \leq \sup_{t \geq 0} \int_0^\infty |D_{\theta} Z_t|^2 d\theta, \quad \mathbb{P}\text{-a.s.},
$$

(6.9)

for any \(T > 0\). For each \(\nu > 0\), we define

$$
Z_T^{(\nu)} := \frac{1}{\nu} \log \left( \int_0^T \exp\{\nu Z_s\} ds \right).
$$

By Laplace’s principle, we have

$$
\lim_{\nu \to \infty} Z_T^{(\nu)} = \sup_{0 \leq s \leq T} Z_s, \quad \mathbb{P}\text{-a.s.}
$$

(6.10)

By considering a subsequence if necessary, the convergence also holds in \(L^2(\Omega)\). Note that

$$
D_{\theta} Z_T^{(\nu)} = \left( D_{\theta_1} Z_T^{(\nu)}, D_{\theta_2} Z_T^{(\nu)} \right) \quad \text{and} \quad |D_{\theta} Z_T^{(\nu)}|^2 = |D_{\theta_1} Z_T^{(\nu)}|^2 + |D_{\theta_2} Z_T^{(\nu)}|^2.
$$
We have $D_{\theta_1} Z_T^{(\nu)} = 0$ when $\theta_1 > T$ and by chain rule

$$D_{\theta_1} Z_T^{(\nu)} = \frac{D_{\theta_1} \left[ \int_0^T \exp\{\nu Z_s\} ds \right]}{\nu \int_0^T \exp\{\nu Z_s\} ds} = \frac{\int_0^T \nu \exp\{\nu Z_s\} D_{\theta_1} Z_s ds}{\int_0^T \exp\{\nu Z_s\} ds}, \quad \theta_1 \leq T.$$  

We use Hölder’s inequality to get

$$|D_{\theta_1} Z_T^{(\nu)}|^2 \leq \left( \frac{\int_0^T \exp\{\nu Z_s\} |D_{\theta_1} Z_s|^2 ds}{\int_0^T \exp\{\nu Z_s\} ds} \right) \left( \frac{\int_0^T \exp\{\nu Z_s\} |D_{\theta_1} Z_s|^2 ds}{\int_0^T \exp\{\nu Z_s\} ds} \right) = \frac{\int_0^T \exp\{\nu Z_s\} |D_{\theta_1} Z_s|^2 ds}{\int_0^T \exp\{\nu Z_s\} ds}, \quad \theta_1 \leq T,$$

and

$$\int_0^T |D_{\theta_1} Z_T^{(\nu)}|^2 d\theta_1 \leq \frac{\sup_{0 \leq s \leq T} \int_0^s |D_{\theta_1} Z_s|^2 d\theta_1}{\int_0^T \exp\{\nu Z_s\} ds} \leq \sup_{t \geq 0} \int_0^T |D_{\theta_1} Z_t|^2 d\theta_1.$$  

Likewise, we have

$$\int_0^\infty |D_{\theta_2} Z_T^{(\nu)}|^2 d\theta_2 \leq \sup_{0 \leq s \leq T} \int_0^s |D_{\theta_2} Z_s|^2 d\theta_2 \leq \sup_{t \geq 0} \int_0^\infty |D_{\theta_2} Z_t|^2 d\theta_2,$$

so that

$$\int_0^\infty |D_{\theta} Z_T^{(\nu)}|^2 d\theta \leq \sup_{t \geq 0} \left[ \int_0^\infty |D_{\theta_1} Z_t^{(\nu)}|^2 d\theta_1 + \int_0^\infty |D_{\theta_2} Z_t^{(\nu)}|^2 d\theta_2 \right] = \sup_{t \geq 0} \int_0^\infty |D_{\theta} Z_t^{(\nu)}|^2 d\theta.$$  

The estimate (6.9) is obtained by allowing $\nu \to \infty$ and using (6.10). Thus the claim (6.8) holds.

Finally, we use Lemma 6.1 to complete the proof. Consider the centered random variable

$$Z = -\sup_{t \geq 0} Z_t + \mathbb{E} \left[ \sup_{t \geq 0} Z_t \right].$$

Recalling (6.4), we have

$$\mathbb{P} \left[ \int_0^\infty \exp\{-as + \sigma_1 X_s + \sigma_2 Y_s\} ds < x \right] = \mathbb{P} \left[ Z \leq \mathbb{E} \left[ \sup_{t \geq 0} Z_t \right] - 1 \right]. \quad (6.11)$$  

By the definition of $Z_t$ (cf. (6.5)) and the property of the function $f$,

$$Z_t \geq \frac{f(t)}{\ln(x+1)} \to 1 \text{ as } t \to \infty.$$  

This implies $\mathbb{E} \left[ \sup_{t \geq 0} Z_t \right] \geq \mathbb{E} \left[ \sup_{t \geq 0} Z_t \right] = 1$. From (6.7) and (6.8), it follows that

$$\int_0^\infty |D_{\theta} Z|^2 d\theta \leq \sigma^2 M_x, \quad \mathbb{P}\text{-a.s.}$$

By (6.11) and Lemma 6.1 we have

$$\mathbb{P} \left[ \int_0^\infty \exp\{-as + \sigma_1 X_s + \sigma_2 Y_s\} ds < x \right] \leq \exp \left( -\frac{1}{2\sigma^2 M_x} \left( \mathbb{E} \left[ \sup_{t \geq 0} Z_t \right] - 1 \right)^2 \right),$$

which completes the proof. \square
6.1 Blow-up probability

In this section, we apply the results obtained in Section 6 to find a lower bound for the blow-up probability of positive solutions to \((1.3)-(1.4)\). We first find an upper bound for the quantity

\[
\mathbb{P}\left[\int_0^\infty \exp\{-as + \sigma_1 B^H_1(s) + \sigma_2 B^H_2(s)\} ds < x\right],
\]

where \(a, \sigma_1, \sigma_2, x\) are positive real numbers. Here we fix \(\sigma_1 = \rho_1^2\) and \(\sigma_2 = \rho_2^2\), where \(\rho_1, \rho_2\) are defined in (3.1).

**Theorem 6.2.** Let \((B^H_1(t), B^H_2(t))_{t \geq 0}\) be a two dimensional fBm with Hurst parameter \(H \in (0, 1)\), for any \(\alpha > H\) it holds that

\[
\mathbb{P}\left[\int_0^\infty \exp\{-as + \sigma_1 B^H_1(s) + \sigma_2 B^H_2(s)\} ds < x\right] \leq \exp\left\{-\frac{1}{4\sigma^2} \left(\frac{\alpha}{\alpha - H}\right)^{2-\frac{2H}{\alpha}} (\ln(x + 1))^{2-\frac{2H}{\alpha}} (N_x(\alpha) - 1)^2\right\}, \quad (6.12)
\]

where

\[
\sigma^2 = \max\{\sigma_1^2, \sigma_2^2\}, \quad N_x(\alpha) := \mathbb{E}\left[\sup_{t \geq 0} \frac{\ln\left(\int_0^t \exp\{-as + \sigma_1 X_s + \sigma_2 Y_s\} ds + 1\right)}{\ln(x + 1) + t^\alpha}\right]. \quad (6.13)
\]

**Proof.** We have

\[
\mathbb{P}\left[\int_0^\infty \exp\{-as + \sigma_1 B^H_1(s) + \sigma_2 B^H_2(s)\} ds < x\right] \leq \mathbb{P}\left[\int_0^\infty \exp\left\{-as - \frac{\sigma_1^2}{2}s^{2H} - \frac{\sigma_2^2}{2}s^{2H} + \sigma_1 B^H_1(s) + \sigma_2 B^H_2(s)\right\} ds < x\right].
\]

We choose \(X_t = -\frac{\sigma_1^2}{2}t^{2H} + B^H_1(t)\) and \(Y_t = -\frac{\sigma_2^2}{2}t^{2H} + B^H_2(t)\) in Theorem 6.1. The stochastic processes \((X_t)_{t \geq 0}\) and \((Y_t)_{t \geq 0}\) satisfy the conditions (1) and (2) of Assumption 1.

Since

\[
\int_0^\infty \mathbb{E}\left[\exp\left\{-as - \frac{\sigma_1^2}{2}s^{2H} - \frac{\sigma_2^2}{2}s^{2H} + \sigma_1 B^H_1(s) + \sigma_2 B^H_2(s)\right\}\right] ds = \int_0^\infty \exp\{-as\} ds = \frac{1}{a} < +\infty,
\]

and

\[
D_{\theta_1} X_t = D_{\theta_1} B^H_1(t) = K_H(t, \theta_1), \quad \theta_1 \leq t,
\]

\[
D_{\theta_2} Y_t = D_{\theta_2} B^H_2(t) = K_H(t, \theta_2), \quad \theta_2 \leq t.
\]

Furthermore, we have

\[
\sup_{s \in [0, t]} \int_0^s |D_{\theta_1} X_s|^2 d\theta_1 = \sup_{s \in [0, t]} \int_0^s K_H^2(t, \theta_1) d\theta_1 = \mathbb{E}\left[|B^H_1(s)|^2\right] = t^{2H}.
\]
and also
\[ \sup_{s \in [0, t]} \int_0^s |D_{\theta_2} Y_s|^2 d\theta_2 = t^{2H}. \]

Moreover, \(X_t\) and \(Y_t\) satisfy the condition (3) of the Assumption 1 with \(f(t) = t^\alpha, \alpha > H\) and
\[ M_x := \sup_{t \geq 0} \frac{2t^{2H}}{t^\alpha} = 2(\ln(x + 1) + f(t))^2 = 2(\ln(x + 1) + \alpha H - 2 \frac{f(t)}{\alpha} - \frac{2H}{\alpha}). \]

The estimate (6.12) follows by inserting \(M_x\) into (6.3).

**Remark 6.1.** When \(H < \frac{1}{2}\), we have
\[ \int_0^\infty \mathbb{E}[\exp\{-as + \sigma_1 B_1^H(s) + \sigma_2 B_2^H(s)\} ds] = \int_0^\infty \exp\{-as + \frac{\sigma_1^2 + \sigma_2^2 s}{2} 2^{2H}\} ds < \infty. \]

Hence, one can choose \(X_t = B_1^H(t)\) and \(Y_t = B_2^H(t)\). The constant \(N_\alpha(\alpha)\) defined by (6.13) can be replaced by
\[ N_\alpha(\alpha) := \mathbb{E}\left[ \sup_{t \geq 0} \frac{\ln(\int_0^t \exp\{-as + \sigma_1 B_1^H(s) + \sigma_2 B_2^H(s)\} ds + 1) + t^\alpha}{\ln(x + 1) + t^\alpha} \right]. \]

**Remark 6.2.** We can choose \(f(t) = t^H\). In this case, we have \(M_x = 2\) for all \(x\) and so
\[ \mathbb{P}\left[ \int_0^\infty \exp\{-as + \sigma_1 X_s + \sigma_2 Y_s\} ds < x \right] \leq \exp\left(-\frac{(N_\alpha(H) - 1)^2}{4\sigma^2}\right), \]
which is the limit of (6.12) as \(\alpha \to H\). Thus (6.12) holds for any \(\alpha \geq H\) and by taking infimum over \(\alpha\), we obtain the following the optimal estimate
\[ \mathbb{P}\left[ \int_0^\infty \exp\{-as + \sigma_1 B_1^H(s) + \sigma_2 B_2^H(s)\} ds < x \right] \leq \inf_{\alpha \geq H} \exp\left(-\frac{1}{4\sigma^2} \frac{\alpha}{\alpha - H} \frac{2^{2H}}{\alpha} (\ln(x + 1))^2 - \frac{2H}{\alpha} (N_\alpha(\alpha) - 1)^2\right). \]

**Corollary 6.1.**

1. Suppose \(\lambda > \gamma\) and \(\beta_1 = \beta_2 = \beta > 0\). The probability that the solution of (1.3)-(1.4) blow-up in finite time is lower bounded by
\[ \mathbb{P}(\tau^*_1 < \infty) \geq 1 - \exp\left(-\frac{1}{4\rho^2} \frac{\alpha}{\alpha - H} \frac{2^{2H}}{\alpha} (\ln(2^\beta E^{-1}(0) + 1))^2 - \frac{2H}{\alpha} (N_\alpha(\alpha) - 1)^2\right), \]
where \(\alpha \geq H\) is an arbitrary real number, \(\sigma_1 = \sigma_2 = \rho\) and
\[ N_\alpha(\alpha) := \mathbb{E}\left[ \sup_{t \geq 0} \frac{\ln(\int_0^t \exp\{\beta(-\lambda + \gamma)s + \rho B_1^H(s) + \rho B_2^H(s)\} ds + 1) + t^\alpha}{\ln(2^\beta E^{-1}(0) + 1) + t^\alpha} \right]. \]
2. Suppose that \( \lambda > \gamma \) and \( \beta_1 > \beta_2 \). The probability that the solution of (1.34)- (1.41) blow-up in finite time is lower bounded by

\[
\mathbb{P}(\tau^*_2 < \infty) \geq 1 - \exp \left\{ - \frac{1}{4\rho^2} \left( \frac{\alpha}{\alpha - H} \right)^{2 - \frac{2H}{\alpha}} \left( \ln \left( \left[ \beta_2 E^{\beta_2}(0) \left( \frac{\epsilon_0}{2^{1+\beta_2}} - \frac{\epsilon_0^{\frac{1+\beta_1}{1-\beta_2}} D_1}{E^{1+\beta_2}(0)} \right) \right]^{-1} + 1 \right) \right)^{2 - \frac{2H}{\alpha}} \times (N(\alpha) - 1)^2 \right\},
\]

where \( \alpha \geq H \) is an arbitrary real number, \( \rho = \frac{\sigma^2 + \sigma_0^2}{2} \) and

\[
N(\alpha) := \mathbb{E} \sup_{t \geq 0} \left[ \ln \left( \int_0^t \exp\{\beta(-\lambda + \gamma)s + \rho B_1^H(s) + \rho B_2^H(s)\} ds + 1 \right) + t^\alpha \right],
\]

where \( \beta \) is an arbitrary real number.

### 6.2 Blow-up probability for a general case

In this section, we apply the results obtained in Section 6 to find a lower bound for the blow-up probability of solutions to (1.34)- (1.41). We first find an upper bound for probability of the form given in (6.2) with the set of parameters

\[
(1 + \beta_1)k_{21} - k_{11} = \sigma_{11}, \ (1 + \beta_2)k_{11} - k_{21} = \sigma_{21},
\]

\[
(1 + \beta_1)k_{22} - k_{12} = \sigma_{12}, \ (1 + \beta_2)k_{12} - k_{22} = \sigma_{22},
\]

and \( H \in (0,1) \).

**Remark 6.3.** Note that if \( \sigma_{12} \geq \sigma_{11} > 0 \) and \( \sigma_{22} \geq \sigma_{21} > 0 \), we have

\[
\int_0^t \exp\{\sigma_{11} B_1^H(s) + \sigma_{21} B_2^H(s) - as\} \exp\{\sigma_{12} B_1^H(s) + \sigma_{22} B_2^H(s) - as\} ds
\]

\[
= \int_0^t \exp\{\sigma_{12} B_1^H(s) + \sigma_{22} B_2^H(s) - as\} \chi_{\{B_1^H < 0, B_2^H < 0\}} ds
\]

\[
+ \int_0^t \exp\{\sigma_{11} B_1^H(s) + \sigma_{21} B_2^H(s) - as\} \exp\{\sigma_{12} B_1^H(s) + \sigma_{22} B_2^H(s) - as\} \chi_{\{B_1^H > 0, B_2^H \leq 0\}} ds
\]

\[
+ \int_0^t \exp\{\sigma_{11} B_1^H(s) + \sigma_{21} B_2^H(s) - as\} \exp\{\sigma_{12} B_1^H(s) + \sigma_{22} B_2^H(s) - as\} \chi_{\{B_1^H \leq 0, B_2^H > 0\}} ds
\]

\[
+ \int_0^t \exp\{\sigma_{11} B_1^H(s) + \sigma_{21} B_2^H(s) - as\} \chi_{\{B_1^H \geq 0, B_2^H \geq 0\}} ds
\]

\[
\leq \int_0^\infty \exp\{-as\} ds + \int_0^t \exp\{(\sigma_{11} + \sigma_{12}) B_1^H(s) - as\} ds
\]

\[
+ \int_0^t \exp\{(\sigma_{21} + \sigma_{22}) B_2^H(s) - as\} ds + \int_0^t \exp\{\sigma_{11} B_1^H(s) + \sigma_{21} B_2^H(s) - as\} ds
\]

\[
= \frac{1}{\alpha} + \int_0^t \exp\{(\sigma_{11} + \sigma_{12}) B_1^H(s) - as\} ds + \int_0^t \exp\{(\sigma_{21} + \sigma_{22}) B_2^H(s) - as\} ds
\]
By Remark 6.3, it follows that
\[ f_{B^H} \text{ with Hurst parameter } H \]

where

\[ 1. \text{ If } \beta \]

By Theorem 5.2, we have the blow-up times \( \tau^{**} \).

Proof. Suppose \( \sigma_{12} \geq \sigma_{11} > 0 \) and \( \sigma_{22} \geq \sigma_{21} > 0 \). Let \( (B_{1}^H, B_{1}^H)_{t \geq 0} \) be a two dimensional fBm with Hurst parameter \( H \in (0, 1) \), for any \( \alpha > H \), then the blow-up times \( \tau^{***} \) and \( \tau^{**} \) are obtained and the probability that the solution blow-up is up per bounded by

\[
\mathbb{P}\left\{ \int_0^\infty \left\{ \exp\{(\sigma_{11} + \sigma_{12})B_1^H(s) - as\} + \exp\{(\sigma_{21} + \sigma_{22})B_2^H(s) - as\} \right\} ds \text{ } < \text{ } x \right\} 
\]

\[
\leq \exp\left\{ \frac{-1}{4(\sigma_{11} + \sigma_{12})^2} \left( \frac{\alpha}{\alpha - H} \right)^{2-\frac{2H}{\alpha}} \left( \ln\left(\frac{x}{3} + 1\right)\right)^{2-\frac{2H}{\alpha}} (m_1^x(\alpha) - 1)^2 \right\} 
\]

\[
+ \exp\left\{ \frac{-1}{4(\sigma_{21} + \sigma_{22})^2} \left( \frac{\alpha}{\alpha - H} \right)^{2-\frac{2H}{\alpha}} \left( \ln\left(\frac{x}{3} + 1\right)\right)^{2-\frac{2H}{\alpha}} (m_2^x(\alpha) - 1)^2 \right\} 
\]

\[
+ \exp\left\{ \frac{-1}{4\sigma^2} \left( \frac{\alpha}{\alpha - H} \right)^{2-\frac{2H}{\alpha}} \left( \ln\left(\frac{x}{3} + 1\right)\right)^{2-\frac{2H}{\alpha}} (N_x(\alpha) - 1)^2 \right\} 
\]

where

\[
\hat{\sigma}^2 = \max\{\sigma_{11}^2, \sigma_{21}^2\}, \\
m_i^x(\alpha) := \mathbb{E}\left[ \sup_{t \geq 0} \frac{\ln(\int_0^t \exp\{-as - \frac{(\sigma_{11} + \sigma_{12})^2}{2}s^{2H} + (\sigma_{11} + \sigma_{12})B_1^H(s) ds\} + 1) + t^\alpha}{\ln(x + 1) + t^\alpha} \right] \\
N_x(\alpha) := \mathbb{E}\left[ \sup_{t \geq 0} \frac{\ln(\int_0^t \exp\{-as - \frac{\sigma_{11}^2}{2}s^{2H} - \frac{\sigma_{12}^2}{2}s^{2H} + 11B_1^H(s) + \sigma_{21}B_2^H(s) ds\} + 1) + t^\alpha}{\ln(x + 1) + t^\alpha} \right] 
\]

for \( i = 1, 2 \).

Proof. By Theorem 5.2 we have the blow-up times

1. If \( \beta_1 = \beta_2 = \beta \) (say), then the blow-up time \( \tau \leq \tau^{**} \), where

\[
\tau^{**} = \inf\left\{ t \geq 0 : \int_0^t \exp\{\sigma_{11}B_1^H(s) + \sigma_{21}B_2^H(s) - \beta(\lambda - \gamma)s\} \right. \\
\left. \wedge \exp\{\sigma_{12}B_1^H(s) + \sigma_{22}B_2^H(s) - \beta(\lambda - \gamma)s\} ds \geq 2^\beta \beta^{-1} E^{-\beta}(0) \right\} 
\]

By Remark 6.3, it follows that

\[
\tau^{***} = \inf\left\{ t \geq 0 : \int_0^t \left\{ \exp\{(\sigma_{11} + \sigma_{12})B_1^H(s) - \beta(\lambda - \gamma)s\} \\
+ \exp\{(\sigma_{21} + \sigma_{22})B_2^H(s) - \beta(\lambda - \gamma)s\} + \exp\{\sigma_{11}B_1^H(s) + \sigma_{21}B_2^H(s) - \beta(\lambda - \gamma)s\} \right\} ds \right\} 
\]
\[
\geq 2^2 \beta^{-1} E^{-\beta}(0) - \frac{1}{\beta(\lambda - \gamma)} \Bigg\}.
\]

It can be seen that \( \tau_{1}^{*} \leq \tau_{1}^{**} \).

2. If \( \beta_1 > \beta_2 > 0 \), then \( \tau \leq \tau_{2}^{**} \), where

\[
\tau_{2}^{**} = \inf \left\{ t \geq 0 : \int_{0}^{t} \exp\{\sigma_{11} B_{1}^{H}(s) + \sigma_{21} B_{2}^{H}(s) - \beta_2 (\lambda - \gamma) s\} \wedge \exp\{\sigma_{12} B_{1}^{H}(s) + \sigma_{22} B_{2}^{H}(s) - \beta_2 (\lambda - \gamma) s\} ds \geq \left[ \frac{\beta_2 E^{\beta_2}(0)}{2^{1+\beta_2}} - \frac{e_{0}^{1+\beta_2} D_{1}}{E^{1+\beta_2}(0)} \right]^{-1} - \frac{1}{\beta_2 (\lambda - \gamma)} \right\}.
\]

By Remark 6.3, it follows that

\[
\tau_{2}^{**} = \inf \left\{ t \geq 0 : \int_{0}^{t} \exp\{\sigma_{11} B_{1}^{H}(s) + \sigma_{21} B_{2}^{H}(s) - \beta_2 (\lambda - \gamma) s\} + \exp\{\sigma_{12} B_{1}^{H}(s) + \sigma_{22} B_{2}^{H}(s) - \beta_2 (\lambda - \gamma) s\} ds \geq \left[ \frac{\beta_2 E^{\beta_2}(0)}{2^{1+\beta_2}} - \frac{e_{0}^{1+\beta_2} D_{1}}{E^{1+\beta_2}(0)} \right]^{-1} - \frac{1}{\beta_2 (\lambda - \gamma)} \right\}.
\]

and \( \tau_{2}^{**} \leq \tau_{2}^{***} \).

By definition of \( \tau_{1}^{***} \) and \( \tau_{2}^{***} \), we find an upper bound for the blow-up probability of solutions to (1.3)-(1.4) as

\[
P \left\{ \int_{0}^{x} \exp\{\sigma_{11} B_{1}^{H}(s) - as\} + \exp\{\sigma_{21} + \sigma_{22} B_{2}^{H}(s) - as\} \right\} ds < \frac{x}{3} \leq P \left\{ \int_{0}^{x} \exp\{\sigma_{11} B_{1}^{H}(s) - as\} ds < \frac{x}{3}, \int_{0}^{x} \exp\{\sigma_{21} + \sigma_{22} B_{2}^{H}(s) - as\} ds < \frac{x}{3}, \right\}
\]

\[
\int_{0}^{x} \exp\{\sigma_{11} B_{1}^{H}(s) + \sigma_{21} B_{2}^{H}(s) - as\} ds < \frac{x}{3} \}
\]

\[
\leq P \left\{ \int_{0}^{x} \exp\{\sigma_{11} + \sigma_{12} B_{1}^{H}(s) - as\} ds < \frac{x}{3}\right\} + P \left\{ \int_{0}^{x} \exp\{\sigma_{21} + \sigma_{22} B_{2}^{H}(s) - as\} ds < \frac{x}{3}\right\}
\]

\[
+ P \left\{ \int_{0}^{x} \exp\{\sigma_{11} + \sigma_{12} B_{1}^{H}(s) - as\} ds < \frac{x}{3}\right\}.
\]

By using Theorem 4.1 [7] and Theorem 6.2 we have

\[
P \left\{ \int_{0}^{x} \exp\{\sigma_{11} + \sigma_{12} B_{1}^{H}(s) - as\} + \exp\{\sigma_{21} + \sigma_{22} B_{2}^{H}(s) - as\} \right\} ds < \frac{x}{3} \}.
\]
where

\[ \delta_2^2 = \max\{\sigma_{1i}, \sigma_{2i}\}, \]

\[ m_i^x(\alpha) := \mathbb{E} \sup_{t \geq 0} \ln \left( \int_0^t \exp\left( -as - \frac{(\sigma_{1i} + \sigma_{2i})^2}{2} s^{2H} + (\sigma_{1i} + \sigma_{2i}) B_1^{sH}(s) \right) ds + 1 \right) + t^\alpha, \]

\[ N_x(\alpha) := \mathbb{E} \sup_{t \geq 0} \ln \left( \int_0^t \exp\left( -as - \frac{\sigma_{1i}^2}{2} s^{2H} - \sigma_{2i}^2 s^{2H} + \sigma_{1i} B_1^{sH}(s) + \sigma_{2i} B_1^{sH}(s) \right) ds + 1 \right) + t^\alpha, \]

for \( i = 1, 2. \)

Next, we have to find lower bounds for the blow-up probability of positive solutions to (1.3)-(1.4).

**Corollary 6.2.** 1. Suppose that \( \lambda > \gamma \) and \( \beta_1 = \beta_2 = \beta > 0 \). The probability that the solution of (1.3)-(1.4) blows up in finite time is lower bounded by

\[ \mathbb{P}(\tau_1^{**} < \infty) \geq 1 - \left\{ \exp \left\{ \frac{-1}{4(\sigma_{1i} + \sigma_{2i})^2} \left( \frac{\alpha}{\alpha - H} \right)^{2 - 2H/\alpha} \left( \ln \left( \frac{x}{3} + 1 \right) \right)^{2 - 2H/\alpha} (m_i^x(\alpha) - 1)^2 \right\} \right. + \exp \left\{ \frac{-1}{4(\sigma_{2i} + \sigma_{2i})^2} \left( \frac{\alpha}{\alpha - H} \right)^{2 - 2H/\alpha} \left( \ln \left( \frac{x}{3} + 1 \right) \right)^{2 - 2H/\alpha} (m_i^x(\alpha) - 1)^2 \right\} \]

\[ + \exp \left\{ \frac{-1}{4\sigma_2^2} \left( \frac{\alpha}{\alpha - H} \right)^{2 - 2H/\alpha} \left( \ln \left( \frac{x}{3} + 1 \right) \right)^{2 - 2H/\alpha} (N_x(\alpha) - 1)^2 \right\}, \]

where

\[ x = \frac{2\beta - 1}{3} - \frac{1}{3\beta(\lambda - \gamma)}. \]

2. Suppose that \( \lambda > \gamma \) and \( \beta_1 > \beta_2 > 0 \). The probability that the solution of (1.3)-(1.4) blows up in finite time is lower bounded by

\[ \mathbb{P}(\tau_2^{**} < \infty) \geq 1 - \left\{ \exp \left\{ \frac{-1}{4(\sigma_{1i} + \sigma_{2i})^2} \left( \frac{\alpha}{\alpha - H} \right)^{2 - 2H/\alpha} \left( \ln \left( \frac{x}{3} + 1 \right) \right)^{2 - 2H/\alpha} (m_i^x(\alpha) - 1)^2 \right\} \right. + \exp \left\{ \frac{-1}{4(\sigma_{2i} + \sigma_{2i})^2} \left( \frac{\alpha}{\alpha - H} \right)^{2 - 2H/\alpha} \left( \ln \left( \frac{x}{3} + 1 \right) \right)^{2 - 2H/\alpha} (m_i^x(\alpha) - 1)^2 \right\} \]

\[ + \exp \left\{ \frac{-1}{4\sigma_2^2} \left( \frac{\alpha}{\alpha - H} \right)^{2 - 2H/\alpha} \left( \ln \left( \frac{x}{3} + 1 \right) \right)^{2 - 2H/\alpha} (N_x(\alpha) - 1)^2 \right\}, \]
\begin{equation*}
+ \exp \left\{ -\frac{1}{4\sigma_2^2} \left( \frac{\alpha}{\alpha - H} \right)^{2-\frac{2H}{\alpha}} \left( \ln \left( \frac{x}{3} + 1 \right) \right)^{2-\frac{2H}{\alpha}} \left( N_x(\alpha) - 1 \right)^2 \right\},
\end{equation*}

where $\alpha \geq H$ is an arbitrary real number and

\begin{equation*}
x = \left[ \beta_2 E^{\beta_2}(0) \left( \frac{\epsilon_0}{2^{1+\beta_2}} - \frac{\epsilon_0^{1-\beta_2}}{E^{1+\beta_2}(0)} \right)^{-1} - \frac{1}{\beta_2(\lambda - \gamma)} \right].
\end{equation*}

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