Magnetization Plateaus in a Solvable 3-Leg Spin Ladder

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We present a solvable ladder model which displays magnetization plateaus at fractional values of the total magnetization. Plateau signatures are also shown to exist along special lines. The model has isotropic Heisenberg interactions with additional many-body terms. The phase diagram can be calculated exactly for all values of the rung coupling and the magnetic field. We also derive the anomalous behaviour of the susceptibility near the plateau boundaries. There is good agreement with the phase diagram obtained recently for the pure Heisenberg ladders by numerical and perturbative techniques.

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One of the surprising aspects of low-dimensional quantum systems with long range finite interactions is the occurrence of fractional magnetization plateaus. Theoretical considerations have revealed that such plateaus originate from interactions beyond those of nearest neighbours. Spin ladders provide natural examples of systems with non trivial magnetization plateaus since they can be reformulated as spin chains with longer range interactions. They have the advantage that they are accessible experimentally and spin ladders in a field have attracted quite some attention over the past few years \[1\]. Fractional values of the total magnetization were measured only very recently in 1D compounds \[2\] and in a 2D system \[3\].

Groundstate phase diagrams and magnetization plateaus have been calculated for a special family of ladder models for which partially exact results could be obtained \[4\]. We consider an integrable 3-leg spin ladder, or spin tube for periodic boundary conditions \[5\]. Its phase diagram can be calculated for all values of the rung coupling and the magnetic field from the Bethe Ansatz solution. The spins along each leg and each rung have an isotropic Heisenberg interaction, with the introduction of many-body terms to retain integrability. The model is a generalisation of Wang’s 2-leg ladder model \[6\]. The overall phase diagram of the 2- and 3-leg ladders compare well with those obtained recently for the pure Heisenberg ladders, by DMRG, bosonization, series expansions and mappings to effective Hamiltonians (see, e.g., Refs. \[7\]).

The Hamiltonian of our model is

\[ H = \sum_{i=1}^{L} H_{i,i+1}^{\text{leg}} + \sum_{i=1}^{L} H_{i}^{\text{rung}} + \sum_{i=1}^{L} H_{i}^{\text{field}}, \]

where,

\[ H_{i,j}^{\text{leg}} = \frac{1}{2} \prod_{l=1}^{3} \left( 1 + \sigma_{i}^{(l)} \cdot \sigma_{j}^{(l)} \right), \]

\[ H_{i}^{\text{rung}} = \frac{3}{2} \sum_{l=1}^{3} J_{l} \left( \sigma_{i}^{(l)} \cdot \sigma_{i}^{(l+1)} - 1 \right), \]

\[ H_{i}^{\text{field}} = -h \sum_{l=1}^{3} (\sigma_{i}^{z})^{(l)}. \]

The operators \((\sigma_{i}^{x})^{(l)}, (\sigma_{i}^{y})^{(l)}\) and \((\sigma_{i}^{z})^{(l)}\) act as the corresponding Pauli matrices on the \((i,l)\)th factor in the Hilbert space.

As shown in \[8\], the Hamiltonian \((\text{1})\) is integrable for \(h = 0\). The addition of the magnetic field term however, does not destroy integrability since \([H_{i}^{\text{leg}}, H_{i}^{\text{field}}] = 0\). In the following we set \(J_{1} = J_{2} = J\) and \(J_{3} = J’\), so that we can go from the isotropic tube, \(J’ = J\), to the ladder, \(J’ = 0\).

For \(h = 0\) it is convenient to change to the basis where the square and the \(z\)-component of the total spin of a given triangle, \(S = \sigma^{(1)} + \sigma^{(2)} + \sigma^{(3)}\) are diagonal \[9\]. It follows that the eight states on a given triangle fall into a spin-\(\frac{1}{2}\) quadruplet and two spin-\(\frac{1}{2}\) doublets. We will denote these states by \(|s_{z})_{q}\) for the quadruplet and \(|s_{z})_{d}, (i = 1, 2)\) for each of the doublets. Switching on the magnetic field breaks this symmetry further due to the Zeeman splitting. The energies of the rung and field Hamiltonians are given by

\[ H_{\text{rung}}^{\text{field}} = \text{diag}\{-3J - h, -3J + h, -2J’ - J - h, -2J’ - J + h, -3h, -h, h, 3h\}, \]

on the states

\(|\pm)_{d_{1}, \pm}d_{2}, |\pm)_{d_{1}, \mp}d_{2}, |\pm)_{d_{1}, \pm}d_{2}, |\pm)_{q}|\pm)_{q}, |\pm)_{q}, |\pm)_{q}, |\pm)_{q}\rangle, \]

Since \(S^{2}\) and \(S^{z}\) commute with \(H\), the total spin and its \(z\)-component are good quantum numbers, as in Ref. \[10\].

\(H\) can be diagonalized using the Bethe Ansatz. It is important to note that the Hamiltonian \((\text{1})\) does not change under the change of basis given above. Furthermore, \(\text{1}\) is invariant under any choice of reference state (or pseudo-vacuum) \(|\Omega\rangle\) and any assignment of Bethe
Ansatz pseudo particles. For each choice however, one has to re-interpret this assignment. The rung and field Hamiltonians do alter with the choice of ordering, but the change is just a rearrangement of their eigenvalues along the diagonal. We use this property to our advantage by doing calculations with that choice of ordering for which the Bethe Ansatz reference state is closest to the true groundstate of the system. The eigenenergies of \( \sum_{i=1}^{L} H^{\text{leg}}_{i,i+1} \) are given by

\[
E^{\mathrm{deg}} = -\sum_{j=1}^{M_{\lambda}} \frac{1}{(\lambda^{(1)}_{j})^2 + \frac{1}{4}},
\]

where the number \( \lambda^{(1)}_{j} \) satisfy the well known Bethe Ansatz equations (17),

\[
\left( \frac{\lambda^{(1)}_{j} - \frac{i}{2}}{\lambda^{(1)}_{j} + \frac{i}{2}} \right)^{L} = \prod_{k \neq j}^{M_{\lambda}} \frac{\lambda^{(1)}_{j} - \lambda^{(1)}_{k}}{\lambda^{(1)}_{j} + \lambda^{(1)}_{k}} + i \prod_{k=1}^{M_{\lambda}} \frac{\lambda^{(1)}_{j} - \lambda^{(2)}_{k}}{\lambda^{(1)}_{j} + \lambda^{(2)}_{k}} - \frac{1}{2},
\]

and for \( r = 2, \ldots, 7 \) with \( M_{\lambda} = 0, \)

\[
\prod_{k \neq j}^{M_{\lambda}} \frac{\lambda^{(r)}_{j} - \lambda^{(r)}_{k}}{\lambda^{(r)}_{j} + \lambda^{(r)}_{k}} + i \prod_{k=1}^{M_{\lambda}} \frac{\lambda^{(r)}_{j} - \lambda^{(r+1)}_{k}}{\lambda^{(r)}_{j} + \lambda^{(r+1)}_{k}} - \frac{1}{2},
\]

where \( j = 1, \ldots, M_{r} \).

In the following we will restrict ourselves to the quadrant \( J, h \geq 0 \) and consider the magnetization, which is defined by

\[
M = \frac{1}{nL} \sum_{i=1}^{L} \sum_{\lambda=1}^{n} \left( \sigma^{z} \right)_{i}^{(r)} \left( \lambda^{(r)}_{j} \right),
\]

where \( n \) is the number of legs.

THE 2-LEG LADDER

In this section we briefly review and expand the results of Wang before treating the 3-leg case. For the 2-leg case the rung states fall into a singlet and a triplet. The phase diagram in the \( J, h > 0 \) quadrant is determined by the competition between the singlet state and the spin up state of the triplet. The difference between their respective energies changes sign at \( h = J \). This line therefore divides phase space into two regions. In each of these regions a convenient choice of Bethe Ansatz reference state and pseudo particles may be made. The value of the gaps can then easily be calculated and it follows that there is a massive phase for \( h - J > 2 \) where the groundstate is the simple product of the spin up triplet state. In this phase \( \langle M \rangle = 1 \). For \( J - h > 2 \) there is another massive phase where the groundstate consists of singlets on each rung. Here evidently \( \langle M \rangle = 0 \). In between these two phases lies a massless phase where the magnetization varies continuously. On approaching the lines \( h = J \pm 2 \) from within the massless phase, the susceptibility shows the familiar square root singularity (18).

On the line \( h = J \) the spin up triplet and the singlet state are degenerate. Therefore, on this line the magnetization \( \langle M \rangle = \frac{1}{2} \). At the point \( J = h = \log 2 \) the other excitations become massless and a completely massless phase is entered. This phase actually covers a finite region around the origin which seems to be common to this type of solvable ladder model. The other point that can be calculated exactly marking its phase boundary is \( J = 2, h = 0 \). The line \( h = J \) can be regarded as the onset of a plateau boundary. Although there is no singularity in the magnetic susceptibility in the present case, it will appear as soon as the plateau opens. It is expected that the opening of this plateau is governed by anisotropy (19).

FIG. 1. The solvable 2-leg ladder phase diagram. The dashed line \( h = J \) divides phase space into two regions. The bold lines given by \( h = J \pm 2 \) are phase boundaries.

THE TUBE: \( J' = J \)

Due to the extra symmetry in this case, the two doublets are degenerate for the rung Hamiltonian and only three eigenvalues of the Hamiltonian \( H^{\text{rung}} + H^{\text{field}} \) are competing, \(-3J + h\), \(-3J - h\) and \(-3h\). Their differences change sign at the lines, \( 2h = 3J \) and \( 4h = 3J \), dividing the phase space in three regions. In each of these regions we can make a convenient choice of ordering the states to facilitate our calculations.

\[ i ) \ h \geq \frac{3}{2} J \] We choose our states to be ordered in increasing energy with respect to \( H^{\text{rung}} + H^{\text{field}} \), i.e. \( \{ | + \rangle_{d_{1}}, | + \rangle_{d_{2}}, \ldots \} \). The energy, up to an irrelevant constant, is given by
\[
E = - \sum_{j=1}^{M_1} \left( \frac{1}{(\lambda_j^{(1)})^2 + \frac{1}{4}} - (2h - 3J) \right) + 3JM_3 + (2h - 3J)M_4 + 3JM_6 + 2hM_7. \tag{10}
\]

From this expression it follows straightforwardly that for \(2h - 3J > 4\) we have \(M_3 = 0\) for the groundstate energy. In this phase, the pseudo vacuum is the true groundstate and the total magnetization \(\langle M \rangle = 1\) and all excitations are gapped. Below the line \(h = \frac{3}{4}J + 2\), a finite density of \(|+\rangle_{d_1}\) and \(|+\rangle_{d_2}\) states appear in the groundstate. This phase is massless and the mean magnetization varies continuously. As the magnetization reaches its maximum value, the susceptibility diverges as,

\[
\chi \sim (\frac{3}{2}J + 2 - h)^{-\frac{1}{2}}. \tag{11}
\]

Upon reaching the line \(h = \frac{3}{4}J\), the three states have become degenerate in energy while all other excitations remain massive for large enough \(J\). On this line the magnetization is \(\langle M \rangle = \frac{5}{9}\) [18]. This line can be regarded as a plateau boundary in a similar way as the line \(h = J\) for the 2-leg ladder. The plateau will open upon introduction of anisotropy.

The first excitations to become massless on this line are \(|+\rangle_q, |-\rangle_{d_1}\) and \(|-\rangle_{d_2}\). This happens at the point \(3J = \pi/\sqrt{3} - \log 3\).

\(\text{ii)} \ \frac{3}{4}J \leq h \leq \frac{3}{2}J\). Here we choose our state to be ordered as \(|+\rangle_{d_1}, |+\rangle_{d_2}, |+\rangle_q, \ldots\rangle\). The energy in this case takes the form,

\[
E = - \sum_{j=1}^{M_1} \frac{1}{(\lambda_j^{(1)})^2 + \frac{1}{4}} + (3J - 2h)M_2 + (4h - 3J)M_3 + (3J - 2h)M_5 + 2h(M_6 + M_7). \tag{12}
\]

Following the same reasoning as above, we find that for \(3J - 2h\) large enough \(M_2 = 0\) for the groundstate. In this region the magnetization is \(\langle M \rangle = \frac{1}{4}\) and every magnetic excitation is massive. Because of the degeneracy of the spin up states of both doublets, there are massless excitations because the groundstate is essentially that of a spin-\(\frac{1}{2}\) chain. The \(|+3\rangle_q\) excitation becomes massless at \(3J - 2h = 2\log 2\) where the massless phase is entered. Here \(\langle M \rangle\) may vary continuously. The susceptibility shows the square root singularity upon approaching the plateau from within this phase.

\(\text{iii)} \ h \leq \frac{3}{4}J\). Here, the ordering for the lowest energy rung states is given by \(|+\rangle_{d_1}, |+\rangle_{d_2}, |+\rangle_q, \ldots\rangle\). The energy in this case takes the form

\[
E = - \sum_{j=1}^{M_1} \frac{1}{(\lambda_j^{(1)})^2 + \frac{1}{4}} + 2hM_2 + (3J - 4h)M_4 + 2h(M_5 + M_6 + M_7). \tag{13}
\]

Above the line \(h = \log 2\) there is the plateau phase with \(\langle M \rangle = \frac{1}{3}\). Below this line, the excitations \(|-\rangle_{d_1}\) and \(|-\rangle_{d_2}\) become massless and the magnetization is allowed to vary continuously. On the line \(h = 0\) the two doublets are degenerate and the magnetization \(\langle M \rangle = 0\). As before, we may regard the line \(h = 0\) as a plateau boundary.

Our results for the tube are summarised in Fig. 2.

**FIG. 2.** The solvable 3-leg tube phase diagram. The dashed lines \(2h = 3J\) and \(4h = 3J\) divide phase space into three regions. The bold lines \(h = \frac{3}{2}J + 2, h = \frac{3}{2}J - \log 2\) and \(h = \log 2\) are phase boundaries.

**THE LADDER:** \(J' = 0\)

For the ladder the degeneracies are lifted and the massless regions display more structure. Phase space is now divided into six regions due to level crossings, of which the boundaries are given by \(h = \frac{k}{2}J\) for \(k = 1, 2, 3, 4, 6\). The only boundaries where level crossings concerning the groundstate occur are those with \(k = 3\) and \(k = 6\), i.e. the same as those for the tube. A similar analysis as for the tube gives the following results.

\(\text{i)} \ h \geq \frac{3}{4}J\). As in the case for the tube, for strong magnetic fields the ladder is completely magnetized. Below the line \(h = \frac{3}{2}J + 2\), some of the \(|+\rangle_{d_1}\) states appear and the magnetization may vary continuously. Again, the susceptibility diverges with a square root singularity as this plateau is approached. On the line \(2h = 3J\), the states \(|+3\rangle_q\) and \(|+\rangle_{d_1}\) are degenerate and the magnetization thus is given by \(\langle M \rangle = \frac{3}{4}\). As before this line is interpreted as a plateau boundary. The line does not extend to the origin, but at some finite value of \(J = \log 2\) other magnetic excitations become massless.

\(\text{ii)} \ \frac{3}{4}J \leq h \leq \frac{3}{2}J\). For \(3J - 2h\) large enough only the state \(|+\rangle_{d_1}\) appears in the groundstate. In this region the magnetization is \(\langle M \rangle = \frac{1}{3}\) and every excitation is massive. As opposed to the tube, the two doublets are not degenerate and the groundstate is completely polarized, as in Ref. [11]. The \(|+3\rangle_q\) excitation becomes massless at \(3J - 2h = 2\) where the massless phase of the previous paragraph is entered. Here \(\langle M \rangle\) may vary continuously.
but the phase is still polarized. Also here the square root singularity shows up when approaching this plateau.

On the line $4h = 3J$, the $\langle M \rangle = \frac{3}{7}$ plateau extends into the massless phase. On this line the states $| + 3 \rangle_d$ and $| - \rangle_d$ are degenerate so that they may combine to a net nonmagnetic excitation. It ends at some finite value of $J$ which cannot be calculated analytically.

$$iii) h \leq \frac{3}{4}J.$$ Here, the ordering for the lowest energy rung states is given by $\{ |d_1 \rangle, |d_1 \rangle, |d_2 \rangle, |d_2 \rangle, |d_3 \rangle \}$. Above the line $h = 2$ there is the plateau phase with $\langle M \rangle = \frac{3}{7}$. Below this line, the excitation $| - \rangle_d$ becomes massless and the magnetization is allowed to vary continuously. For $J$ not too small, the phase remains polarized. The line $h = 0$, where the magnetization $\langle M \rangle = 0$, may again be regarded as a plateau boundary. The second doublet enters the groundstate at $J = \log 2$ and the polarized excitations become massless. As before, the susceptibility shows the square root singularity upon approaching this line from above and we may regard the line $h = 0$ as a plateau boundary.

Our results for the tube are summarised in Fig. 3.

FIG. 3. The solvable 3-leg ladder phase diagram. The dashed lines $2h = 3J$ and $4h = 3J$ divide phase space into three regions. The bold lines $h = \frac{3}{4}J \pm 2$ and $h = 2$ are phase boundaries.

CONCLUSION

In summary we have considered a simple solvable model that displays fractional magnetization plateaus. Such plateaus have been studied theoretically only by perturbative or numerical techniques. The phase diagram and in particular the location of the plateaus and the expected square root singularity have been calculated exactly. The phase diagrams in Figs. 3 are to be compared with those of the pure Heisenberg ladders given in Ref. 3. The overall agreement is seen to be excellent and our results should provide a useful benchmark for further studies. Consideration of the complete eigen-

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