The Harder–Narasimhan stratification of the moduli stack of $G$-bundles via Drinfeld’s compactifications

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Abstract We use Drinfeld’s relative compactifications $\overline{\text{Bun}}_P$ and the Tannakian viewpoint on principal bundles to construct the Harder–Narasimhan stratification of the moduli stack $\text{Bun}_G$ of $G$-bundles on an algebraic curve in arbitrary characteristic, generalizing the stratification for $G = \text{GL}_n$ due to Harder and Narasimhan to the case of an arbitrary reductive group $G$. To establish the stratification on the set-theoretic level, we exploit a Tannakian interpretation of the Bruhat decomposition and give a new and purely geometric proof of the existence and uniqueness of the canonical reduction in arbitrary characteristic. We furthermore provide a Tannakian interpretation of the canonical reduction in characteristic 0 which allows to study its behavior in families. The substack structures on the strata are defined directly in terms of Drinfeld’s compactifications $\overline{\text{Bun}}_P$, which we generalize to the case where the derived group of $G$ is not necessarily simply connected. Using $\overline{\text{Bun}}_P$, we establish various properties of the stratification, including finer information about the structure of the individual strata and a simple description of the strata closures.

Keywords Harder–Narasimhan stratification · Drinfeld’s relative compactifications · Canonical reduction in arbitrary characteristic · Tannakian formalism for bundles · Geometric Langlands program

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1 Introduction

1.1 Overview

Let $X$ be a smooth complete curve over an algebraically closed field $k$ of arbitrary characteristic, and let $G$ be a reductive linear algebraic group over $k$. The main goal of
this article is to generalize the Harder–Narasimhan stratification of the moduli stack of vector bundles to the moduli stack $\text{Bun}_G$ of principal $G$-bundles on $X$, in the form stated in our main theorem, Theorem 2.1 below.

The main technical tool in our construction of the stratification, as well as in establishing various properties, is Drinfeld’s relative compactification $\text{Bun}_P$. The latter object first appeared in the context of geometric Eisenstein series (see [6]) and has since been of great importance in many areas of geometric representation theory (see [5, Sec. 0.1] for an overview). Our interest in stratifying the moduli stack $\text{Bun}_G$ stems from the geometric Langlands program (see, for example, [8]), and more specifically the study of D-modules on $\text{Bun}_G$. For instance, the main theorem of the present paper, Theorem 2.1, is applied by Drinfeld and Gaitsgory in their forthcoming paper [7].

We now briefly describe the stratification on the set-theoretic level. It is known (and we will also prove) that every $G$-bundle on $X$ possesses a unique canonical reduction to a unique parabolic subgroup $P$ of $G$, i.e., a reduction to $P$ such that its corresponding Levi bundle is semistable and such that its coweight degree $\check{\lambda}_P$ enjoys a certain regularity property. To describe the strata, we associate to any given $G$-bundle the pair $(P, \check{\lambda}_P)$ obtained from its canonical reduction. Then, on the level of $k$-points, the strata are precisely those loci in $\text{Bun}_G$ on which the pair $(P, \check{\lambda}_P)$ remains constant. The open strata together comprise the locus of semistable $G$-bundles.

In the case $G = \text{GL}_n$, the idea of stratifying the moduli stack of vector bundles by loci of instability is due to G. Harder and M. S. Narasimhan. It was carried out set-theoretically by Harder and Narasimhan [9] and scheme-theoretically by Shatz [17]. The study in the context of the geometric Langlands program and in the language of stacks was initiated by Laumon [12]. The case of a general reductive group $G$ in arbitrary characteristic has been considered by Behrend [1] in his thesis, although the parts containing the stratification results obtained by his methods seem to remain unpublished. Our approach to stratifying $\text{Bun}_G$ is, however, quite different, as will be explained in the next sections in more detail. One the one hand, our use of Drinfeld’s compactifications $\overline{\text{Bun}}_P$ greatly simplifies the construction of the strata and the analysis of their geometry. On the other hand, our approach to the canonical reduction uses only basic algebraic geometry and representation theory and avoids any involved combinatorics.

1.2 Main results

1.2.1 Results about the canonical reduction

To establish the stratification on the set-theoretic level, we give a direct and—to our knowledge—new proof of the existence and uniqueness of the canonical reduction in arbitrary characteristic. Recall that in the case $G = \text{GL}_n$ the canonical reduction reduces to the Harder–Narasimhan filtration of a vector bundle, as defined by Harder and Narasimhan [9]. For a general reductive group $G$ in characteristic 0, the existence and uniqueness of the canonical reduction were proven by Ramanathan [13, 14].

In arbitrary characteristic, the existence and uniqueness of the canonical reduction are more involved and were established much later by Behrend [1, 2] by first passing
from principal bundles to reductive group schemes over $X$ and then using the structure theory for the latter as well as a detailed analysis of certain combinatorial devices called *complementary polyhedra on root systems*.

Our present approach to the existence and uniqueness of the canonical reduction is different and proceeds in a fairly pedestrian manner, featuring algebraic geometry and representation theory instead of combinatorics. Namely, we give a direct bundle-theoretic and in some sense purely geometric proof which exploits the Bruhat decomposition of the double quotient $P_1 \backslash G/P_2$ for parabolics $P_1$ and $P_2$ of $G$ and consistently uses the Tannakian perspective on principal bundles. Our use of the double quotient $P_1 \backslash G/P_2$ is to a certain extent reminiscent of the geometry underlying the “Geometric Lemma” of I. N. Bernstein and A. V. Zelevinsky in the representation theory of reductive groups over non-archimedean local fields (see [3]). The Tannakian viewpoint is employed to obtain a modular interpretation of the Bruhat decomposition, which yields finer information than the set-theoretic decomposition alone.

Furthermore, we exhibit a certain extremal property of the canonical reduction in arbitrary characteristic (the *comparison theorem*, Theorem 4.1), which strengthens the uniqueness assertion and which also immediately yields a strategy to prove existence.

In characteristic 0, and for reductions to the Borel $B$ of $G$ in arbitrary characteristic, we obtain stronger results: We provide a Tannakian interpretation of the canonical reduction, from which we for example deduce that the canonical reduction is also “unique in families.” The latter is in turn equivalent to Behrend’s conjecture (see Sect. 2.4.3).

Throughout the article, we employ a notion of *slope* for $G$-bundles and their reductions (see Sect. 2.2 for the definition). It appears to us that our definitions (such as semistability of $G$-bundles), results (such as the comparison theorem), and proofs are most naturally phrased in terms of this notion. Furthermore, its use makes the more technical constructions in Sects. 4 and 5 more transparent.

### 1.2.2 Construction of the strata via Drinfeld’s compactifications

To define our strata as locally closed substacks and not only on the level of $k$-points as above, we use the relative compactifications $\overline{\text{Bun}}_P$, which are due to V. Drinfeld and were introduced by Braverman and Gaitsgory [6].

To motivate the use of $\overline{\text{Bun}}_P$ in the present context, let $P$ be a parabolic subgroup of $G$, let $\text{Bun}_P$ denote the moduli stack of $P$-bundles, and consider the natural projection map $p_P : \text{Bun}_P \to \text{Bun}_G$. Then, even though we can identify our desired strata on the level of $k$-points with the set-theoretic images of certain open substacks of $\text{Bun}_P$, these images do a priori not carry a natural stack structure since the map $p_P$ is not proper.

This is remedied by Drinfeld’s compactification $\overline{\text{Bun}}_P$, which contains $\text{Bun}_P$ as an open dense substack and comes equipped with a proper map $\overline{p}_P : \overline{\text{Bun}}_P \to \text{Bun}_G$ extending the projection $p_P$. Thus, we define the strata as the stack-theoretic images under $\overline{p}_P$ of certain substacks of $\overline{\text{Bun}}_P$ and use some well-known properties of the latter to establish that these images indeed possess the desired $k$-points as described in Sect. 1.1 above.
The definition of Drinfeld’s compactification $\overline{\text{Bun}}_P$ in [6] requires the derived group $[G, G]$ of $G$ to be simply connected in order for $\overline{\text{Bun}}_P$ to have the desired properties (which are stated in Sect. 6.1). To avoid having a similar restriction in our main theorem, we show in the last section, Sect. 7, how the definition of $\overline{\text{Bun}}_P$ can be modified so that it satisfies the desired properties for an arbitrary reductive group $G$. The same strategy applies to Drinfeld’s compactifications $\tilde{\text{Bun}}_P$, which are, however, not used in the present article.

1.2.3 Properties of the stratification

Under the assumption that the characteristic of $k$ is 0 or that $P = B$, we use Drinfeld’s compactifications $\overline{\text{Bun}}_P$ to show that each individual stratum is isomorphic to its corresponding locus of reductions in the stack $\text{Bun}_P$. This implies for example that all strata are smooth in these cases. For a general parabolic $P$ in arbitrary characteristic, we show that the strata are still almost-isomorphic (in a precise sense, see Sect. 2.3.2) to their corresponding loci of reductions, which is sufficient for applications in the theory of D-modules or etale cohomology.

Apart from various elementary properties, we also provide a formula for the closure of a stratum. This formula follows immediately from the construction of the strata via $\overline{\text{Bun}}_P$ and some well-known properties of the latter. In general, the closure of a stratum need, however, not be a union of strata, as we illustrate with an example for $G = \text{GL}_3$.

1.3 Structure of the article

We now briefly discuss the content of the individual sections.

Section 2 can be viewed as an extension of the introduction; it contains no proofs. Its goal is to state the main theorem (Theorem 2.1) after introducing the necessary notation and definitions. We conclude the section with a series of remarks detailing those in the introduction and provide some very basic examples in the case $G = \text{GL}_n$.

Section 3 is preparatory. Here, we collect several combinatorial lemmas that are used throughout the article, and record some basic results from the Tannakian formalism for principal bundles.

In Sect. 4, we first analyze the notion of relative position of two reductions to parabolics $P_1$ and $P_2$ of the same $G$-bundle on a scheme, in terms of the Bruhat cells of the double quotient stack $P_1 \backslash G / P_2$ (see, for example, Corollary 4.1). After specializing to the case of a curve and providing a Tannakian interpretation of the Bruhat decomposition of $P_1 \backslash G / P_2$ (Proposition 4.4), we prove the aforementioned comparison theorem (Theorem 4.1). Since in characteristic 0 the comparison theorem will also be deduced from the results of Sect. 5, the reader who is only interested in Theorem 2.1 in characteristic 0 can skip Sect. 4 entirely.

In Sect. 5, we provide the Tannakian characterization of the canonical reduction in characteristic 0 (Proposition 5.1). We furthermore prove the above-mentioned result about the uniqueness of the canonical reduction in families (Proposition 5.3) and give another proof of the comparison theorem in characteristic 0.
In Sect. 6, we define the strata using Drinfeld’s compactifications \( \text{Bun}_P \) and combine the results of the previous sections to complete the proof of Theorem 2.1.

Finally, Sect. 7 contains the aforementioned generalization of Drinfeld’s compactifications to the case of an arbitrary reductive group, and the proof that it satisfies the desired properties. This section can be read independently from the rest of the article.

2 The main theorem

2.1 The setting

2.1.1 Notation related to the group

Let \( k \) be an algebraically closed field of arbitrary characteristic, and let \( G \) be a connected reductive linear algebraic group over \( k \). Fix a Borel subgroup \( B \) of \( G \) and let \( T = B/U(B) \) denote the abstract Cartan, where \( U(B) \) is the unipotent radical of \( B \). We also fix a splitting of the surjection \( B \to T \), i.e., a realization of \( T \) as a maximal torus in \( B \). By a parabolic subgroup of \( G \), we will always mean a standard parabolic subgroup, i.e., one containing the fixed Borel \( B \). The connected component of the center of \( G \) will be denoted by \( Z_0(G) \), the collection of roots and coroots of \( G \) by \( \mathcal{R} \) and \( \check{\mathcal{R}} \), their positive parts by \( \mathcal{R}^+ \) and \( \check{\mathcal{R}}^+ \), and the set of vertices of the Dynkin diagram by \( \mathcal{I} \). As usual the Weyl group of \( G \) will be denoted by \( \mathcal{W} \) and its longest element by \( w_0 \).

Next, let \( \check{\mathcal{R}}^+ \) denote the coweight lattice of \( G \), let \( \Lambda^\ast \) denote the weight lattice, and let \( \langle ., . \rangle : \check{\mathcal{R}}^+ \times \mathcal{R} \to \mathbb{Z} \) denote the natural pairing between the two. Let furthermore \( \check{\mathcal{R}}^\ast \) be the semigroup of dominant coweights, and similary for \( \Lambda \). Set \( \check{\mathcal{R}}^Q := \check{\mathcal{R}} \otimes_{\mathbb{Z}} \mathbb{Q} \), and let \( \check{\mathcal{R}}^Q_\ast \) and \( \check{\mathcal{R}}^Q_\ast \) denote the rational cones of \( \check{\mathcal{R}}^\ast \) and \( \check{\mathcal{R}}^\ast \) and analogously for the weight lattice. As usual, given two coweights \( \check{\lambda}, \check{\mu} \in \check{\mathcal{R}}^Q \), we write \( \check{\lambda} \geq \check{\mu} \) if the difference \( \check{\lambda} - \check{\mu} \) lies in \( \check{\mathcal{R}}^Q_\ast \).

2.1.2 Notation related to a parabolic

Let \( P \) be a parabolic subgroup of \( G \), let \( U(P) \) be its unipotent radical, and let \( M = P/U(P) \) be its Levi quotient. Having fixed a splitting of the surjection \( B \to T \), we also obtain an induced splitting of the surjection \( P \to M \). The subset of vertices of \( \mathcal{I} \) corresponding to \( P \) will be denoted by \( \mathcal{I}_M \), the collection of roots of the Levi \( M \) by \( \mathcal{R}_M \), and the root lattice of \( M \) by

\[ \mathbb{Z}\mathcal{R}_M := \text{span}_\mathbb{Z}(\mathcal{R}_M) \subset \Lambda.
\]
Furthermore, for any vertex \( i \in I \), we will denote by \( P_i \) the corresponding maximal parabolic subgroup of \( G \), i.e., the parabolic corresponding to the subset \( I \setminus \{i\} \subset I \).

Next, define the sublattice \( \Lambda_{G,P} \subset \Lambda_G \) as

\[
\Lambda_{G,P} := \{ \lambda \in \Lambda_G \mid \langle \check{\alpha}_i, \lambda \rangle = 0 \text{ for all } i \in I_M \}.
\]

Thus, for \( P = B \), we have \( \check{\Lambda}_{G,B} = \check{\Lambda}_G \). Consider furthermore the sublattice \( \check{\Lambda}_{[M,M]}_{sc} \subset \check{\Lambda}_G \) spanned by the simple coroots \( \check{\alpha}_i \) for \( i \in I_M \) and set

\[
\check{\Lambda}_{G,P} := \check{\Lambda}_G / \check{\Lambda}_{[M,M]}_{sc}.
\]

By definition, we have \( \Lambda_{G,P} = \Lambda_{M,M} \) and \( \check{\Lambda}_{G,P} = \check{\Lambda}_{M,M} \). Furthermore, the pairing \( \langle ., . \rangle \) above induces a pairing

\[
\langle ., . \rangle : \check{\Lambda}_{G,P} \times \Lambda_{G,P} \to \mathbb{Z},
\]

which becomes perfect after quotienting out by the torsion part of \( \check{\Lambda}_{G,P} \). As before we let \( \check{\Lambda}_{G,P}^\mathbb{Q} := \check{\Lambda}_{G,P} \otimes_{\mathbb{Z}} \mathbb{Q} \) and similarly for \( \check{\Lambda}_{G,P}^\mathbb{Q} \). Finally, we denote by \( \check{\Lambda}_{G,P}^{\text{pos}} \) the image of \( \check{\Lambda}_{G,P}^\mathbb{Q} \) in \( \check{\Lambda}_{G,P} \), and analogously for \( \check{\Lambda}_{G,P}^{\text{pos}} \subset \check{\Lambda}_{G,P}^\mathbb{Q} \).

### 2.1.3 The slope map \( \phi_P \)

We now define a map

\[
\phi_P : \check{\Lambda}_{G,P} \longrightarrow \check{\Lambda}_{G}^\mathbb{Q}
\]

which we will call the slope map. Its importance, as well as our choice of terminology, will become apparent in Sects. 2.2 and 2.3 below.

Let \( Z_0(M) \) denote the connected component of the center of \( M \). The splitting of the surjection \( P \to M \) gives rise to a natural inclusion

\[
\check{\Lambda}_{Z_0(M)}^\mathbb{Q} \hookrightarrow \check{\Lambda}_G^\mathbb{Q},
\]

and the composition of this inclusion with the projection \( \check{\Lambda}_G^\mathbb{Q} \to \check{\Lambda}_{G,P}^\mathbb{Q} \) is an isomorphism

\[
\check{\Lambda}_{Z_0(M)}^\mathbb{Q} \cong \check{\Lambda}_{G,P}^\mathbb{Q}.
\]

We then define the map \( \phi_P \) as the composition

\[
\check{\Lambda}_{G,P} \longrightarrow \check{\Lambda}_{G,P} \cong \check{\Lambda}_{Z_0(M)}^\mathbb{Q} \hookrightarrow \check{\Lambda}_G^\mathbb{Q}.
\]
It follows directly from the definition of $\phi_P$ that for any elements $\tilde{\lambda}_P \in \tilde{\Lambda}_{G,P}$ and $\lambda_P \in \Lambda_{G,P}$ we have
\[
(\tilde{\lambda}_P, \lambda_P) = (\phi_P(\tilde{\lambda}_P), \lambda_P),
\] (2.1)
where the pairing on the right-hand side is the natural one between $\tilde{\Lambda}^\mathbb{Q}_{G}$ and $\Lambda^\mathbb{Q}_{G}$.

Finally, observe that for any element $\tilde{\lambda}_P \in \tilde{\Lambda}_{G,P}$ we have $\langle \phi_P(\tilde{\lambda}_P), \alpha_i \rangle = 0$ for all $i \in I_{M}$. We call $\tilde{\lambda}_P$ dominant $P$-regular if $\langle \phi_P(\tilde{\lambda}_P), \alpha_i \rangle > 0$ for all $i \in I \setminus I_{M}$.

2.2 Slope and semistability of $G$-bundles

2.2.1 Moduli stacks of bundles

Let $X$ be a smooth and complete curve over $k$ and let $\text{Bun}_G$ denote the moduli stack of principal $G$-bundles on $X$ and similarly for other linear algebraic groups. Extension of structure group along the homomorphisms $P \hookrightarrow G$ and $P \twoheadrightarrow M$ defines maps of stacks
\[
\begin{array}{ccc}
\text{Bun}_P & \xrightarrow{q_P} & \text{Bun}_M \\
\downarrow p_P & & \downarrow \\
\text{Bun}_G & & 
\end{array}
\]

It is well known that the map $q_P$ induces a bijection on the sets of connected components of $\text{Bun}_P$ and $\text{Bun}_M$ and that
\[
\pi_0(\text{Bun}_P) \cong \pi_0(\text{Bun}_M) \cong \tilde{\Lambda}_{G,P}.
\]

2.2.2 Degree and slope of a bundle

We denote the connected component of $\text{Bun}_P$ and $\text{Bun}_M$ corresponding to $\tilde{\lambda}_P \in \tilde{\Lambda}_{G,P}$ by $\text{Bun}_{P,\tilde{\lambda}_P}$ and $\text{Bun}_{M,\tilde{\lambda}_P}$, respectively. If $F_P$ is a $P$-bundle on $X$ which lies in $\text{Bun}_{P,\tilde{\lambda}_P}$, we will refer to the element $\tilde{\lambda}_P$ as the degree of $F_P$.

Furthermore, we propose to call the element $\phi_P(\tilde{\lambda}_P)$ the slope of $F_P$. We have several reasons for this choice of terminology. First, for $G = \text{GL}_n$, this notion indeed reduces to the usual notion of slope of vector bundles (see Sect. 2.2.4). Second, for an arbitrary reductive group $G$, the element $\phi_P(\tilde{\lambda}_P)$ in fact determines the slopes of certain naturally associated vector bundles (see Proposition 3.2 and Remark 3.2.4); in other words, the element $\phi_P(\tilde{\lambda}_P)$ yields the correct Tannakian version of slope. Third, by analogy with the case $G = \text{GL}_n$ (see Sect. 2.2.4), the terminology is motivated by the re-definition of semistability of $G$-bundles that we introduce next.
2.2.3 Re-definition of semistability

Using the above notion of slope, we propose the following re-definition of semistability of $G$-bundles, which is equivalent, though not tautologically, to the usual definition.

Let $F_G \in \text{Bun}_{G,\tilde{\lambda}_G}$ be a $G$-bundle on $X$. Then we call $F_G$ semistable if one of the following equivalent conditions holds:

(a) For every parabolic $P$ and for every element $\tilde{\lambda}_P \in \tilde{\Lambda}_{G,P}$ such that $F_G$ admits a reduction $F_P \in \text{Bun}_{P,\tilde{\lambda}_P}$ of degree $\tilde{\lambda}_P$, we have

$$\phi_P(\tilde{\lambda}_P) \leq \phi_G(\tilde{\lambda}_G).$$

(b) For every element $\tilde{\lambda}_B \in \tilde{\Lambda}_{G,B} = \tilde{\Lambda}_G$ such that $F_G$ admits a reduction $F_B \in \text{Bun}_{B,\tilde{\lambda}_B}$ of degree $\tilde{\lambda}_B$, we have

$$\tilde{\lambda}_B \leq \phi_G(\tilde{\lambda}_G).$$

(c) For every maximal parabolic $P_i$ and every element $\tilde{\lambda}_{P_i} \in \tilde{\Lambda}_{G,P_i}$ such that $F_G$ admits a reduction $F_{P_i} \in \text{Bun}_{P_i,\tilde{\lambda}_{P_i}}$, we have

$$\phi_{P_i}(\tilde{\lambda}_{P_i}) \leq \phi_G(\tilde{\lambda}_G).$$

The claimed equivalence of the three conditions follows easily from Proposition 3.1 below; the argument is carried out in Lemma 3.3. In the same lemma, we also use Proposition 3.1 to show that our definition of semistability of $G$-bundles agrees with the usual definition from [13,15,16]; in the language of Sect. 2.1.2, the usual definition can be expressed as follows:

Note first that for every parabolic $P$ of $G$ the vector space $\tilde{\Lambda}_{G,P}^Q$ is the direct sum of the subspaces $\tilde{\Lambda}_{Z_0(G)}^Q$ and $\sum_{i \in \mathcal{I} \setminus \mathcal{I}_M} \mathbb{Q} \tilde{\alpha}_i$. We denote by $\text{proj}_P$ the projection onto the second summand, i.e., the composition

$$\text{proj}_P : \tilde{\Lambda}_{G,P}^Q \rightarrow \sum_{i \in \mathcal{I} \setminus \mathcal{I}_M} \mathbb{Q} \tilde{\alpha}_i \rightarrow \tilde{\Lambda}_{G,P}^Q.$$

Let now $F_G \in \text{Bun}_{G,\tilde{\lambda}_G}$ be a $G$-bundle on $X$. Then $F_G$ is semistable in the sense of the usual definition if it satisfies the following condition:

(d) For every parabolic $P$ and every element $\tilde{\lambda}_P \in \tilde{\Lambda}_{G,P}$ such that $F_G$ admits a reduction $F_P \in \text{Bun}_{P,\tilde{\lambda}_P}$, the image of $\tilde{\lambda}_P$ under the composition

$$\tilde{\Lambda}_{G,P} \rightarrow \tilde{\Lambda}_{G,P}^Q \xrightarrow{\text{proj}_P} \tilde{\Lambda}_{G,P}^Q$$

lies in the negative cone $-\tilde{\Lambda}_{G,P}^{Q,\text{pos}}$. 
Our new definition turns out to be more convenient for our purposes (see, for example, Theorem 4.1, Proposition 6.1, Lemmas 3.4, 4.7, Proposition 6.2) and might help streamline various arguments in the reduction theory of a general reductive group. Furthermore, it closely resembles the well-known definition of slope-semistability of vector bundles, to which it reduces for $G = \text{GL}_n$ and which we now recall.

2.2.4 The case $G = \text{GL}_n$

It is easy to see that a $\text{GL}_n$-bundle is semistable in the above sense if and only if the corresponding vector bundle is slope-semistable in the following sense. Recall first that the slope $\mu(E)$ of a vector bundle $E$ on the curve $X$ is defined as

$$\mu(E) := \frac{\text{deg} \ E}{\text{rk} \ E}$$

where $\text{deg} \ E$ and $\text{rk} \ E$ denote the degree and rank of $E$, respectively. Now, $E$ is called semistable if

$$\mu(F) \leq \mu(E)$$

for every locally free subsheaf $F \subseteq E$, or equivalently for every subbundle $F \subseteq E$.

For $G = \text{GL}_n$, the maps $\phi_G$ and $\phi_P$ have the following meaning. Let $E$ be a vector bundle on $X$ corresponding to a $G$-bundle of degree $\tilde{\lambda}_G$ and identify $\tilde{\lambda}_G^\mathbb{Q}$ with $\mathbb{Q}^n$ in the canonical way. Then

$$\phi_G(\tilde{\lambda}_G) = (\mu(E), \ldots, \mu(E)) \in \mathbb{Q}^n.$$

More generally, let

$$0 \neq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_m = E$$

be the flag of vector bundles corresponding to a $P$-bundle $F_P$ of degree $\tilde{\lambda}_P$ for some parabolic $P \subset \text{GL}_n$. Then

$$\phi_P(\tilde{\lambda}_P) = (\mu(E_1), \ldots, \mu(E_1), \ldots, \mu(E_m/E_{m-1}), \ldots, \mu(E_m/E_{m-1})) \in \mathbb{Q}^n$$

where each $\mu(E_i/E_{i+1})$ is repeated $\text{rk}(E_i)$ times.

2.3 Statement of the theorem

2.3.1 Loci of semistability

It is well known that for any reductive group $G$ the collection of semistable $G$-bundles constitutes an open substack $\text{Bun}^{ss}_G$ of $\text{Bun}_G$ (see also Proposition 6.1 below, where we give a quick proof of this fact using Drinfeld’s relative compactifications $\overline{\text{Bun}}_P$) and that its intersection with each connected component of $\text{Bun}_G$ is quasi-compact.
For a parabolic $P$ of $G$, we denote by $\text{Bun}^{ss}_P$ the inverse image of $\text{Bun}^{ss}_M$ under the projection $q_P$ and by $\text{Bun}^{ss}_{P,\tilde{\lambda}_P}$ the intersection of $\text{Bun}^{ss}_P$ with a given connected component $\text{Bun}_{P,\tilde{\lambda}_P}$. As the projection $q_P$ is quasi-compact, we see that the open substacks $\text{Bun}^{ss}_{P,\tilde{\lambda}_P}$ are quasi-compact as well.

A direct definition of $\text{Bun}^{ss}_P$ in the spirit of Definition 2.2.3 can be found in Lemma 3.4 below.

2.3.2 Almost-isomorphisms

We call a morphism of algebraic stacks $\mathcal{X} \to \mathcal{Y}$ an almost-isomorphism if it is schematic, finite, and if the fiber of every geometric point of $\mathcal{Y}$ consists, as a topological space, of precisely one point.

This terminology stems from the fact that in certain contexts, such as in the theory of D-modules or etale cohomology, an almost-isomorphism can play the same role as an actual isomorphism.

We can now formulate the main theorem:

**Theorem 2.1** (a) Let $P$ be a parabolic in $G$ and let $\tilde{\lambda}_P \in \tilde{\Lambda}_{G,P}$ be dominant $P$-regular. Then there exists a unique reduced quasi-compact locally closed substack $\text{Bun}^{P,\tilde{\lambda}_P}_G$ of $\text{Bun}_G$ such that the projection $p_P : \text{Bun}_P \to \text{Bun}_G$ induces an almost-isomorphism between $\text{Bun}^{ss}_{P,\tilde{\lambda}_P}$ and $\text{Bun}^{P,\tilde{\lambda}_P}_G$.

(b) If the field $k$ is of characteristic 0, or if $P = B$, then the map $p_P$ in fact induces an isomorphism

$$\text{Bun}^{ss}_{P,\tilde{\lambda}_P} \cong \text{Bun}^{P,\tilde{\lambda}_P}_G.$$  

(c) The substacks $\text{Bun}^{P,\tilde{\lambda}_P}_G$ for all pairs $(P, \tilde{\lambda}_P)$ as in (a) define a stratification of $\text{Bun}_G$ in the sense that every $k$-point of $\text{Bun}_G$ lies in a unique $\text{Bun}^{P,\tilde{\lambda}_P}_G$.

(d) Let $P'$ be another parabolic and let $\tilde{\lambda}'_{P'}$ be any element of $\tilde{\Lambda}_{G,P'}$. If the image of $\text{Bun}^{P',\tilde{\lambda}'_{P'}}$ in $\text{Bun}_G$ meets a stratum $\text{Bun}^{P,\tilde{\lambda}_P}_G$, then we have

$$\phi_P(\tilde{\lambda}_P) \geq \phi_{P'}(\tilde{\lambda}'_{P'}).$$

(e) The stratification of $\text{Bun}_G$ by the strata $\text{Bun}^{P,\tilde{\lambda}_P}_G$ is locally finite in the sense that there exists a covering of $\text{Bun}_G$ by open substacks, each of which intersects only finitely many strata.

(f) On the level of $k$-points, the closure of the stratum $\text{Bun}^{P,\tilde{\lambda}_P}_G$ in $\text{Bun}_G$ is equal to the (non-disjoint) union

$$\text{Bun}^{P,\tilde{\lambda}_P}_G = \bigcup_{\tilde{\theta} \in \tilde{\Lambda}_{G,P}^{\text{pass}}} p_P(\text{Bun}_{P,\tilde{\lambda}_P+\tilde{\theta}}).$$
(g) The closure of a stratum $\text{Bun}_G^{P, \lambda_p}$ need not be a union of strata, or equivalently, it need not contain every stratum it meets. However, if the closure of $\text{Bun}_G^{P, \lambda_p}$ meets a stratum $\text{Bun}_G^{P, \lambda'_p}$ corresponding to the same parabolic $P$, then it contains that entire stratum.

2.4 Remarks and complements

2.4.1 The canonical reduction

A reduction $F_P$ of a $G$-bundle on $X$ is called canonical if $F_P \in \text{Bun}_P^{ss, \lambda_p}$ for $\lambda_p$ dominant $P$-regular. Thus, parts (a) and (c) of Theorem 2.1 together imply that every $G$-bundle on $X$ has a unique canonical reduction $F_P$ to a unique parabolic $P$. The strata of the theorem are precisely the loci in $\text{Bun}_G$ obtained by fixing the numerical invariants of the canonical reduction.

2.4.2 The Harder–Narasimhan filtration

One sees easily that in the case $G = \text{GL}_n$ the canonical reduction of a vector bundle $E$ is precisely the Harder–Narasimhan filtration of $E$, i.e., the unique filtration

$$0 \neq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_m = E$$

of $E$ by subbundles with the property that all quotients $E_{i+1}/E_i$ are semistable and that

$$\mu(E_1) > \mu(E_2/E_1) > \cdots > \mu(E_m/E_{m-1}).$$

It is easy to see directly that every vector bundle on $X$ possesses a unique Harder–Narasimhan filtration. Its first term equals the maximal destabilizing subsheaf of $E$, i.e., the unique subbundle $F \subseteq E$ with the properties that $\mu(F) \geq \mu(F')$ for any other locally free subsheaf $F' \subseteq E$, and that $F' \subseteq F$ if $\mu(F') = \mu(F)$. It is the unique subbundle of $E$ of maximal rank among all subbundles of maximal slope.

For $G = \text{GL}_n$, Theorem 2.1 yields a stratification of the moduli stack of rank $n$ vector bundles on $X$ by loci on which the ranks and slopes of the subbundles occurring in the Harder–Narasimhan filtration remain constant. In this case, the stratification was already defined set-theoretically by Harder and Narasimhan [9]. By analogy with this special case, one might call the stratification of $\text{Bun}_G$ in Theorem 2.1 the Harder–Narasimhan stratification of $\text{Bun}_G$ and its strata the Harder–Narasimhan strata. As has already been remarked in the introduction, several of the assertions of Theorem 2.1 can also be extracted, in different language and with different proofs, from Behrend’s unpublished thesis [1].
2.4.3 Behrend’s Conjecture

Let $P$ be any parabolic of $G$ and let $F_P$ be the canonical reduction of a $G$-bundle on the curve $X$. Denote the Lie algebras of $G$ and $P$ by $\mathfrak{g}$ and $\mathfrak{p}$, respectively, and consider the vector bundle $(\mathfrak{g}/\mathfrak{p})_{F_P}$ obtained by twisting the $P$-representation $\mathfrak{g}/\mathfrak{p}$ by $F_P$. Then, Behrend conjectured in [1] that the space of global sections of this vector bundle vanishes:

$$H^0(X, (\mathfrak{g}/\mathfrak{p})_{F_P}) = 0.$$ 

By considering the differential of the projection $p_P : \text{Bun}_P \to \text{Bun}_G$ and using part (a) of Theorem 2.1, it is easy to show that Behrend’s conjecture holds for the group $G$ if and only if the conclusion of part (b) of Theorem 2.1 is valid, i.e., if and only if the map $\text{Bun}^{ss}_{P, \lambda_P} \to \text{Bun}^P_{G, \lambda_P}$ is an isomorphism for any parabolic $P$ and any degree $\lambda_P$.

However, in the present approach, part (b) of Theorem 2.1 is proven directly, and thus Behrend’s conjecture in characteristic 0 follows. Since Behrend’s conjecture is known to be false in positive characteristic (see, for example, [10]), the restriction to characteristic 0 in part (b) is necessary.

2.5 Examples for $G = GL_n$

We now collect some very basic examples illustrating certain aspects of Theorem 2.1 in the case $G = GL_n$.

2.5.1 Strata closure for vector bundles of rank 2

Consider the moduli stack $\text{Bun}_{GL_2}$ of vector bundles of rank 2 on any smooth projective curve $X$, and let $\text{Bun}^{ss}_{GL_2, d}$ denote the connected component consisting of all bundles of degree $d$. As the group $GL_2$ has semisimple rank 1, part (g) of Theorem 2.1 shows that the Harder–Narasimhan stratification of $\text{Bun}^{ss}_{GL_2, d}$ has the property that the closure of a stratum is a union of strata.

Using part (f) of Theorem 2.1, one easily determines the strata occurring in this union. To do so, recall first that the open stratum $\text{Bun}^{ss}_{GL_2, d}$ consists of all semistable bundles and that the remaining strata $\text{Bun}^{(\ell, d-\ell)}_{GL_2}$ are parametrized by integers $\ell$ satisfying $\ell > \frac{d}{2}$. Namely, the stratum $\text{Bun}^{(\ell, d-\ell)}_{GL_2}$ consists of precisely those bundles $E$ whose Harder–Narasimhan flag is of the form $0 \neq L \subsetneq E$ for a line bundle $L$ of degree $\ell$, which must then be larger than the slope $\mu(E) = \frac{d}{2}$ of $E$. Specializing part (f) of Theorem 2.1 to the case $G = GL_2$ then yields:

Example 2.1 The closure of the stratum $\text{Bun}^{(\ell, d-\ell)}_{GL_2}$ is equal to the union of all strata $\text{Bun}^{(\ell', d-\ell')}_{GL_2}$ for all $\ell' \geq \ell$. 
2.5.2 Specialization of vector bundles on the projective line

Up to isomorphism, every vector bundle $E$ on the projective line $\mathbb{P}^1$ is of the form

$$E = \mathcal{O}(d_1) \oplus \mathcal{O}(d_2) \oplus \cdots \oplus \mathcal{O}(d_n)$$

for integers $d_1 \geq d_2 \geq \cdots \geq d_n$. A natural deformation-theoretic question is to ask which vector bundles on $\mathbb{P}^1$ the bundle $E$ can specialize to. This question has the following easy and well-known answer, which for simplicity we state in the case that $d_1 > d_2 > \cdots > d_n$; the general case is analogous.

**Example 2.2** Let $E' = \mathcal{O}(d'_1) \oplus \cdots \oplus \mathcal{O}(d'_n)$ be another vector bundle on $\mathbb{P}^1$ of the same rank and degree as $E$. Then $E$ specializes to $E'$ if and only if some permutation of the tuple $(d'_1, \ldots, d'_n)$ lies in the subset

$$(d_1, \ldots, d_n) + \sum_{i=1}^{n-1} \mathbb{Z}_{\geq 0} (e_i - e_{i+1})$$

of the lattice $\mathbb{Z}^n$.

Though easy to prove directly, this fact is also a very special case of part (f) of Theorem 2.1. Namely, for $X = \mathbb{P}^1$, we let $G = \text{GL}_n$ and let $P$ be the standard Borel $B$ of $\text{GL}_n$. Then the claim follows immediately from the formula in part (f) and from the fact that for $G = \text{GL}_n$ the simple coroots are given by $\tilde{\alpha}_i = e_i - e_{i+1}$ in the coweight lattice $\tilde{\Lambda}_{\text{GL}_n} = \mathbb{Z}^n$. (The author would like to thank the referee for pointing out that Example 2.2 has already been discussed in great detail in Laumon’s article [12]).

2.5.3 Counterexample to strata closure

As asserted in part (g) of Theorem 2.1, the closure of a stratum need not be a union of strata. By the second assertion of part (g), this, however, cannot happen if the semisimple rank of $G$ is 1, as in Example 2.1 for $G = \text{GL}_2$ above.

In the general case, we provide the following counterexample in Sect. 6.5 below. Let $G = \text{GL}_3$, let $B$ denote the standard Borel subgroup of $\text{GL}_3$, and assume that the genus of the curve $X$ is at least 2. Then, using part (f) of Theorem 2.1, we show that the closure of the stratum $\text{Bun}_{\text{GL}_3}^{B,(2,1,0)}$, which consists of those vector bundles $E$ of rank 3 whose Harder–Narasimhan flag

$$0 \neq L \subset F \subset E$$

is complete and satisfies

$$\text{deg}(L) = 2, \quad \text{deg}(F/L) = 1, \quad \text{and} \quad \text{deg}(E/F) = 0,$$

is not a union of strata.
2.5.4 The almost-isomorphism for $G = \text{GL}_n$

The almost-isomorphism

$$Bun^s_{P, \lambda, P} \rightarrow Bun^p_{G, \hat{\lambda}, P}$$

from part (a) of Theorem 2.1 is always an isomorphism for $G = \text{GL}_n$, i.e., the assertion of part (b) of the theorem holds for any parabolic and in any characteristic. This follows immediately, and exactly as in the proof of part (b) of the theorem in Sect. 6.2 below, once one establishes that Proposition 5.3 always holds for $G = \text{GL}_n$; the latter states that the above map is a monomorphism. To prove this, one only needs to adapt the usual proof of the uniqueness of the Harder–Narasimhan flag to families using the theorem on cohomology and base change, in the same way it is used in the proof of Proposition 5.3 below.

3 Preparations

3.1 Lemmas about the slope map $\phi_P$

In this preparatory section, we collect several easy lemmas about the slope map $\phi_P : \hat{\Lambda}_{G, P} \rightarrow \hat{\Lambda}^\mathbb{Q}_{G}$ defined in Sect. 2.1.3 which will be used throughout the article, and expound on the definition of semistability in Sect. 2.2.3 above.

3.1.1 Preservation of the partial ordering

Part (a) of the following elementary proposition turns out to be of surprising importance and will be used frequently:

**Proposition 3.1** (a) For any $j \in \mathcal{I} \setminus \mathcal{I}_M$, the element $\phi_P(\hat{\alpha}_j)$ lies in $\hat{\Lambda}_{G, P}^{\mathbb{Q}, \text{pos}}$. In other words, the map $\phi_P$ preserves the partial orders “$\leq$” on $\hat{\Lambda}_{G, P}^{\mathbb{Q}}$ and $\hat{\Lambda}_{G}^{\mathbb{Q}}$ in the sense that it maps $\hat{\Lambda}_{G, P}^{\mathbb{Q}, \text{pos}}$ to $\hat{\Lambda}_{G}^{\mathbb{Q}, \text{pos}}$.

(b) For any $j \in \mathcal{I} \setminus \mathcal{I}_M$, we have $\langle \phi_P(\hat{\alpha}_j), \alpha_j \rangle > 0$.

**Proof of Proposition 3.1** Fix a $W$-invariant scalar product $(\ldots)$ on $\hat{\Lambda}_{G}^{\mathbb{Q}}$. By the definition of the map $\phi_P$, we can write $\phi_P(\hat{\alpha}_j)$ as

$$\phi_P(\hat{\alpha}_j) = \sum_{i' \in \mathcal{I}_M'} c_{i'} \hat{\alpha}_{i'} - \sum_{i'' \in \mathcal{I}_M''} c_{i''} \hat{\alpha}_{i''} + \hat{\alpha}_j$$

where $\mathcal{I}_M'$ and $\mathcal{I}_M''$ are disjoint subsets of $\mathcal{I}_M$, and all $c_{i'}$ and $c_{i''}$ are positive. Pairing both sides of this equation with the sum $\sum_{i''} c_{i''} \hat{\alpha}_{i''}$, the left-hand side becomes

$$\left( \phi_P(\hat{\alpha}_j), \sum_{i''} c_{i''} \hat{\alpha}_{i''} \right) = 0.$$
On the right-hand side, we find that
\[
\left( \sum_{i'} c'_{i'} \tilde{\alpha}_{i'}, \sum_{i''} c''_{i''} \tilde{\alpha}_{i''} \right) \leq 0
\]
and
\[
\left( \tilde{\alpha}_j, \sum_{i''} c''_{i''} \tilde{\alpha}_{i''} \right) \leq 0.
\]
Together this implies that
\[
\left( \sum_{i''} c''_{i''} \tilde{\alpha}_{i''}, \sum_{i''} c''_{i''} \tilde{\alpha}_{i''} \right) \leq 0,
\]
but then positive definiteness forces \( \sum_{i''} c''_{i''} \tilde{\alpha}_{i''} = 0 \), as desired.

To prove the second assertion, we pair both sides of the equality
\[
\phi_P(\tilde{\alpha}_j) = \sum_{i'} c'_{i'} \tilde{\alpha}_{i'} + \tilde{\alpha}_j
\]
with the sum \( \sum_{i'} c'_{i'} \tilde{\alpha}_{i'} + \tilde{\alpha}_j \). Then, the left-hand side equals \( (\phi_P(\tilde{\alpha}_j), \tilde{\alpha}_j) \) since
\[
\left( \phi_P(\tilde{\alpha}_j), \sum_{i'} c'_{i'} \tilde{\alpha}_{i'} \right) = 0.
\]
But, by positive definiteness, the right-hand side must be positive, and the second assertion of the lemma follows.

### 3.1.2 Comparison of slope maps

We now record two simple lemmas relating the slope maps \( \phi_P \) and \( \phi_{P'} \) for different parabolics \( P \) and \( P' \) in two specific circumstances which arise frequently. To state the first lemma, we denote by
\[
\phi^Q_P : \tilde{\Lambda}^Q_{G,P} \leftrightarrow \tilde{\Lambda}^Q_G
\]
the natural map induced by \( \phi_P \), i.e., the composition
\[
\tilde{\Lambda}^Q_{G,P} \cong \tilde{\Lambda}^Q_{Z_0(M)} \leftrightarrow \tilde{\Lambda}^Q_G.
\]
We then have:
Lemma 3.1 Let $P$ and $P'$ be parabolics in $G$ and assume that $P \subset P'$. Let furthermore $\tilde{\lambda}_{P'}$ be an element of $\tilde{\Lambda}_{G,P'}$, and let the image of $\phi_{P'}(\tilde{\lambda}_{P'})$ under the projection $\tilde{\Lambda}_G^Q \to \tilde{\Lambda}_{G,P}^Q$ be denoted by $\phi_{P'}(\tilde{\lambda}_{P'})$. Then we have

$$\phi_P(\phi_{P'}(\tilde{\lambda}_{P'})) = \phi_{P'}(\tilde{\lambda}_{P'}).$$

Proof Follows directly from the definitions and the fact that, denoting the Levi of $P'$ by $M'$, the inclusion $\tilde{\Lambda}_{Z_0(M')}^Q \subset \tilde{\Lambda}_{Z_0(M)}^Q$ holds in $\tilde{\Lambda}_G^Q$. \(\Box\)

We now record the second lemma:

Lemma 3.2 Let $P$ and $P'$ be parabolic subgroups of $G$, and let $\tilde{\lambda}_P \in \tilde{\Lambda}_{G,P}$ and $\tilde{\lambda}_{P'} \in \tilde{\Lambda}_{G,P'}$.

(a) Assume that $\tilde{\lambda}_P$ and $\tilde{\lambda}_{P'}$ map to the same element of $\tilde{\Lambda}_{G,G}$ under the natural projections. Then the difference

$$\phi_P(\tilde{\lambda}_P) - \phi_{P'}(\tilde{\lambda}_{P'})$$

lies in the subspace

$$\sum_{i \in \mathcal{I}} \mathbb{Q}\tilde{\alpha}_i \subset \tilde{\Lambda}_G^Q.$$

(b) Assume now that $P' \subset P$, and that $\tilde{\lambda}_{P'}$ maps to $\tilde{\lambda}_P$ under the natural projection. Then the difference

$$\phi_P(\tilde{\lambda}_P) - \phi_{P'}(\tilde{\lambda}_{P'})$$

lies in the subspace

$$\sum_{i \in \mathcal{I}_M} \mathbb{Q}\tilde{\alpha}_i \subset \tilde{\Lambda}_G^Q,$$

where $\mathcal{I}_M \subset \mathcal{I}$ denotes the subset corresponding to $P$.

Proof Since the kernel of the projection $pr : \tilde{\Lambda}_G^Q \to \tilde{\Lambda}_{G,G}^Q$ is precisely the subspace $\sum_{i \in \mathcal{I}} \mathbb{Q}\tilde{\alpha}_i$, we can prove part (a) by showing that

$$pr(\phi_P(\tilde{\lambda}_P)) = pr(\phi_{P'}(\tilde{\lambda}_{P'})).$$
Denoting the image of \( \tilde{\lambda}_G \) under the natural map \( \tilde{\Lambda}_{G,G} \to \tilde{\Lambda}_{G,G}^Q \) by \( \tilde{\lambda}_G \otimes 1 \), the commutativity of the diagram

\[
\begin{array}{ccc}
\tilde{\Lambda}_{G,P} & \xrightarrow{\phi_P} & \tilde{\Lambda}_G^Q \\
\downarrow & & \downarrow \text{pr} \\
\tilde{\Lambda}_{G,G} & \to & \tilde{\Lambda}_{G,G}^Q
\end{array}
\]

shows that \( \text{pr}(\phi_P(\tilde{\lambda}_P)) = \tilde{\lambda}_G \otimes 1 \). The exact same reasoning for \( P' \) instead of \( P \) yields that \( \text{pr}(\phi_{P'}(\tilde{\lambda}_{P'})) = \tilde{\lambda}_G \otimes 1 \) as well, finishing the proof of part (a).

Part (b) can be verified analogously. Alternatively, part (b) also follows from part (a). To see this, consider the parabolic subgroup \( P'/U(P) \) of the Levi quotient \( M \). We then have natural identifications

\[
\tilde{\Lambda}_{G,P} = \tilde{\Lambda}_{M,M} \quad \text{and} \quad \tilde{\Lambda}_{G,P'} = \tilde{\Lambda}_{M,P'/U(P)},
\]

and, furthermore, the slope maps \( \phi_P \) and \( \phi_{P'} \) for the group \( G \) agree with the slope maps \( \phi_M \) and \( \phi_{P'/U(P)} \) for the group \( M \) under these identifications. Thus, part (b) follows from part (a) applied to the reductive group \( M \) and its parabolic subgroups \( M \) and \( P'/U(P) \).

3.1.3 Lemmas about semistability

We now establish the claimed equivalence of conditions (a), (b), (c) in our re-definition of semistability in Sect. 2.2.3 above and, furthermore, show that our definition agrees with the usual definition of semistability, condition (d), from \([13, 15, 16]\).

**Lemma 3.3** The four conditions (a), (b), (c), (d) of Definition 2.2.3 are equivalent.

**Proof** The implications (a) \( \Rightarrow \) (b) and (a) \( \Rightarrow \) (c) are formal. We now prove the implication (b) \( \Rightarrow \) (a). Thus, assuming condition (b), we need to show that given a reduction \( F_P \in \text{Bun}_{P,\tilde{\lambda}_P} \) of \( F_G \), the difference \( \phi_G(\tilde{\lambda}_G) - \phi_P(\tilde{\lambda}_P) \) lies in the positive cone \( \tilde{\Lambda}_G^{Q,+} \). To do so, choose a reduction \( F_B \) of \( F_P \) to the Borel \( B \) and let \( \tilde{\lambda}_B \in \tilde{\Lambda}_G \) denote its degree. Then the difference \( \phi_G(\tilde{\lambda}_G) - \tilde{\lambda}_B \) lies in the positive cone \( \tilde{\Lambda}_G^{Q,+} \), and thus its image in \( \tilde{\Lambda}_{G,P}^Q \) lies in \( \tilde{\Lambda}_{G,P}^{Q,+} \). But since applying the slope map \( \phi_P^Q \) to this image yields precisely the difference \( \phi_G(\tilde{\lambda}_G) - \phi_P(\tilde{\lambda}_P) \) by Lemma 3.1, the claim follows from part (a) of Proposition 3.1.

Next, we prove the implication (c) \( \Rightarrow \) (b). Thus, assuming condition (c), we have to prove that given a reduction \( F_B \in \text{Bun}_{B,\tilde{\lambda}_B} \) of \( F_G \), the difference \( \phi_G(\tilde{\lambda}_G) - \tilde{\lambda}_B \) lies in \( \tilde{\Lambda}_G^{Q,+} \). By part (a) of Lemma 3.2, we have that

\[
\phi_G(\tilde{\lambda}_G) - \tilde{\lambda}_B = \sum_{i \in I} c_i \tilde{\alpha}_i
\]
for certain rational numbers $c_i \in \mathbb{Q}$. We hence need to show that $c_i \geq 0$ for any $i$. To do so, consider the $P_i$-bundle $F_{P_i}$ induced from $F_B$, and let $\lambda_{P_i} \in \tilde{\Lambda}_{G,P_i}$ denote its degree. We now first project both sides of the last equality to $\tilde{\Lambda}_{G,P_i}$ and then apply the slope map $\phi_{P_i}$ to both projections. Using Lemma 3.1, this yields the equality

$$\phi_{G}(\lambda_G) - \phi_{P_i}(\lambda_{P_i}) = c_i \cdot \phi_{P_i}(\tilde{\alpha}_i).$$

But then condition (c) applied to the reduction $F_{P_i}$ of $F_G$ and part (a) of Proposition 3.1 together show that $c_i \geq 0$, as desired.

We have now established the equivalence of the conditions (a), (b), (c). We finish the proof by showing that conditions (a) and (d) are equivalent. Let $P$ and $\lambda_P$ be as in the formulation of these conditions. Using part (a) of Lemma 3.2, one checks easily that

$$\phi_{P}(\text{proj}_{P}(\lambda_P)) = \phi_{P}(\lambda_P) - \phi_{G}(\lambda_G).$$

Hence, part (a) of Proposition 3.1 implies that the element $\text{proj}_{P}(\lambda_P)$ lies in the negative cone $-\tilde{\Lambda}_{G,P}^{\text{pos}}$ if and only if the element $\phi_{P}(\lambda_P) - \phi_{G}(\lambda_G)$ lies in the negative cone $-\tilde{\Lambda}_{G}^{\text{pos}}$, and the equivalence follows.

We also record the following easy characterization of those $P$-bundles on the curve $X$ whose induced $M$-bundles are semistable, i.e., of those lying in $\text{Bun}_{P}^{\text{ss}}$.

**Lemma 3.4** A $P$-bundle $F_P \in \text{Bun}_{P,\lambda_P}$ on $X$ lies in $\text{Bun}_{P}^{\text{ss}}$ if and only if for any smaller parabolic $P' \subset P$ and any element $\lambda_{P'} \in \tilde{\Lambda}_{G,P'}$ such that $F_P$ admits a reduction $F_{P'} \in \text{Bun}_{P',\lambda_{P'}}$ we have

$$\phi_{P'}(\lambda_{P'}) \leq \phi_{P}(\lambda_P).$$

In fact, it suffices to check this condition for reductions $F_{P'}$ to maximal proper sub-parabolics $P' \subset P$, i.e., to those parabolics corresponding to the subsets $\mathcal{I}_{M} \setminus \{i\}$ for any $i \in \mathcal{I}_{M}$.

**Proof** Recall that the assignment $P' \mapsto P'/U(P)$ defines a bijection between the collection of parabolic subgroups of $G$ contained in $P$ and the collection of parabolic subgroups of the Levi $M$ and, furthermore, that there is a natural one-to-one correspondence between reductions of $F_P$ to $P'$ and reductions of the induced Levi bundle $F_M$ to $P'/U(P)$. Finally, this correspondence induces the identity map

$$\tilde{\Lambda}_{G,P'} = \tilde{\Lambda}_{M,P'/U(P)}$$

on the level of degrees. Thus, using Lemma 3.2(b), we see that the first assertion of the lemma is nothing but a restatement of part (a) of the definition of semistability 2.2.3 for the Levi bundle $F_M$. Similarly, the second assertion is a restatement of part (c) of the same definition. □
3.2 Associated bundles and Tannakian formalism for reductions

In this section, we record some basic but frequently used lemmas about certain associated vector bundles of a given \( G \)-bundle \( F_G \), as well as about certain subbundles induced by a given reduction of \( F_G \) to a parabolic subgroup. For the majority of the article, we will use bundles associated to an arbitrary \( G \)-representation of highest weight \( \lambda \). However, in certain applications in Sect. 5, it will be essential to use specifically the Weyl modules of \( G \), due to Proposition 3.3 below.

3.2.1 Notation

For any linear algebraic group \( H \), we will denote its algebra of distributions at the element \( 1 \in H \) by \( \text{Dist}(H) \). Its use in the sequel stems from the fact that in arbitrary characteristic the algebra of distributions of \( H \) retains many of the features that the Lie algebra of \( H \) enjoys only in characteristic 0. For an exposition of its basic properties, see [11, Sec. I.7 and Sec. II.1.12].

Next, given a \( G \)-bundle \( F_G \) on a scheme \( S \) and a finite-dimensional representation \( V \) of \( G \), we denote by

\[
V_{F_G} := V \times F_G
\]

the associated vector bundle on \( S \), and similarly for any linear algebraic group over \( k \). For any weight \( \lambda \in \Lambda_G \), we let \( k_\lambda \) be the corresponding 1-dimensional representation of the maximal torus \( T \), and we let

\[
\mathcal{O}(\lambda) := k_\lambda \times G
\]

be the associated line bundle on the flag variety \( G/B \). Finally, for a dominant weight \( \lambda \in \Lambda_G^+ \), we denote by \( V^\lambda \) the corresponding Weyl \( G \)-module

\[
V^\lambda := H^0(G/B, \mathcal{O}(-w_0\lambda))^*.
\]

3.2.2 Representations of highest weight \( \lambda \)

Fix a dominant weight \( \lambda \in \Lambda_G^+ \) and let \( V \) be any finite-dimensional \( G \)-representation of highest weight \( \lambda \), i.e., if

\[
V = \bigoplus_{\nu} V[\nu]
\]

is the weight decomposition of \( V \), then \( \lambda \geq \nu \) for all \( \nu \). Given any parabolic \( P \) of \( G \), we define the subspace \( V[\lambda + \mathbb{Z}R_M] \subset V \) as the sum of weight spaces

\[
V[\lambda + \mathbb{Z}R_M] := \bigoplus_{\nu \in \lambda + \mathbb{Z}R_M} V[\nu].
\]
We record the following basic fact:

**Lemma 3.5** (a) The subspace \( V[\lambda + \mathbb{Z}R_M] \) is a \( P \)-subrepresentation of \( V \). Furthermore, the unipotent radical \( U(P) \) of \( P \) acts trivially on \( V[\lambda + \mathbb{Z}R_M] \), and hence the action of \( P \) descends to an action of the Levi \( M \).

(b) Let now \( V = V^\lambda \) be the Weyl module corresponding to the dominant weight \( \lambda \). Then the space \( V^\lambda[\lambda + \mathbb{Z}R_M] \) is 1-dimensional if and only if \( \lambda \) lies in \( \Lambda_{G,P} \). In this case, this space is then equal to the highest weight line \( V^\lambda[\lambda] \subset V^\lambda \).

**Proof** It suffices to check the two assertions of part (a) on the level of the algebras of distributions \( \text{Dist}(P) \) and \( \text{Dist}(U(P)) \), for which they are immediate.

To prove part (b), note first that every weight \( \nu \) of the \( M \)-representation \( V^\lambda[\lambda + \mathbb{Z}R_M] \) satisfies \( \lambda \geq \nu \geq w_0, M(\lambda) \) where \( w_0, M \) denotes the longest element of the Weyl group \( W_M \) of \( M \). As the weight \( \lambda \) occurs with multiplicity 1 in \( V^\lambda[\lambda + \mathbb{Z}R_M] \), it follows that \( V^\lambda[\lambda + \mathbb{Z}R_M] \) is 1-dimensional if and only if \( w_0, M(\lambda) = \lambda \).

If now \( \lambda \) lies in \( \Lambda_{G,P} \), then every simple reflection \( s_{\alpha_i} \) with \( i \in I_M \) leaves \( \lambda \) invariant, and thus \( w_0, M(\lambda) = \lambda \), proving one direction. Conversely, assume that \( V^\lambda[\lambda + \mathbb{Z}R_M] \) is 1-dimensional. Then, since \( s_{\alpha_i}(\lambda) \) is again a weight of \( V^\lambda[\lambda + \mathbb{Z}R_M] \) for any \( i \in I_M \), we conclude that \( s_{\alpha_i}(\lambda) = \lambda \) and thus \( \langle \tilde{\alpha}_i, \lambda \rangle = 0 \) as desired. \( \square \)

### 3.2.3 Subbundles induced by reductions

Let \( F_P \) be a reduction of a \( G \)-bundle \( F_G \) on a scheme \( S \) to a parabolic \( P \), and let \( F_M \) denote the corresponding Levi bundle. Then, for any dominant weight \( \lambda \in \Lambda^+_G \) and any \( G \)-representation \( V \) of highest weight \( \lambda \), the inclusion map \( V[\lambda + \mathbb{Z}R_M] \hookrightarrow V \) gives rise to a subbundle map

\[ \kappa^\lambda : V[\lambda + \mathbb{Z}R_M]_{F_M} \hookrightarrow V_{F_G} \]

between associated vector bundles.

The maps \( \kappa^\lambda \) will play a prominent role in the study of the collection of reductions of a given \( G \)-bundle. One instance of this is the use of Proposition 3.3 below in Sect. 5. Another instance is that we will use the subbundles \( V[\lambda + \mathbb{Z}R_M]_{F_M} \) to compare numerical data attached to two reductions of a \( G \)-bundle on the curve \( X \) in Sect. 4.4 below. In that section, as well as in other parts of the article, the following calculation will be used frequently:

**Proposition 3.2** Let \( V \) be a finite-dimensional representation of \( G \), let \( V = \bigoplus_v V[v] \) be its weight decomposition with weights \( v \in \Lambda_G \), and let \( m_v \) denote the multiplicity of \( v \). Furthermore, let \( F_G \in \text{Bun}_G, \tilde{\lambda}_G \) be a \( G \)-bundle on the curve \( X \) of degree \( \tilde{\lambda}_G \in \tilde{\Lambda}_{G,G} \). Then we have:
(a) The sum $\sum_v m_v v$ lies in $\Lambda_{G,G}$, and the degree of the associated vector bundle $V_{FG}$ is

$$\deg V_{FG} = \left\langle \tilde{\lambda}_G, \sum_v m_v v \right\rangle = \left\langle \phi_G(\tilde{\lambda}_G), \sum_v m_v v \right\rangle.$$ 

(b) If $V$ is of highest weight $\lambda \in \Lambda^+_G$, then the slope of the associated vector bundle $V_{FG}$ is

$$\mu(V_{FG}) = \langle \phi_G(\tilde{\lambda}_G), \lambda \rangle.$$ 

Next, let $P$ be a parabolic in $G$ and let $F_P \in \text{Bun}_P, \tilde{\lambda}_P$ be a $P$-bundle of degree $\tilde{\lambda}_P \in \tilde{\Lambda}_{G,P}$ with corresponding Levi bundle $F_M$. Consider the $M$-representation $V[\lambda + \mathbb{Z}R_M]$ obtained from a $G$-representation $V$ of highest weight $\lambda$. Then we have:

(c) The slope of the associated vector bundle $V[\lambda + \mathbb{Z}R_M]_{FM}$ is

$$\mu(V[\lambda + \mathbb{Z}R_M]_{FM}) = \langle \phi_P(\tilde{\lambda}_P), \lambda \rangle.$$ 

Proof The first assertion of part (a) holds since the weight $\sum_v m_v v$ corresponds to the action of the torus $T$ on the determinant representation $\text{det}(V)$ of $G$. For the second assertion of part (a), choose a reduction $F_B$ of $F_G$ to the Borel $B$, and let $\tilde{\lambda} \in \tilde{\Lambda}_G = \tilde{\Lambda}_{G,B}$ denote its degree. Recall that if $k_{\lambda}$ denotes the 1-dimensional representation of $B$ corresponding to a character $\lambda \in \Lambda_G$, then the degree of the associated line bundle $(k_{\lambda})_{F_B}$ is

$$\deg((k_{\lambda})_{F_B}) = \langle \tilde{\lambda}, \lambda \rangle.$$ 

Considering $V$ as a $B$-representation, it possesses a filtration by $B$-subrepresentations such that each successive quotient is 1-dimensional. Each such quotient must be isomorphic to some $k_v$ for $v$ a weight of $V$, and the number of times $k_v$ appears among these quotients is precisely $m_v$. Hence,

$$\deg(V_{FG}) = \deg(V_{FB})$$
$$= \sum_v m_v \deg(k_{FB}^v)$$
$$= \left\langle \tilde{\lambda}, \sum_v m_v v \right\rangle$$
$$= \left\langle \tilde{\lambda}_G, \sum_v m_v v \right\rangle,$$

proving the first half of the formula. The second half is just Eq. (2.1) above.
To prove part (c), note that $\lambda - \nu$ lies in $\Lambda^\text{pos}_M$ for every weight $\nu$ of $V[\lambda + \mathbb{Z} R_M]$, and hence

$$\langle \phi_P(\tilde{\lambda}_P), \lambda \rangle = \langle \phi_P(\tilde{\lambda}_P), \nu \rangle$$

as $\langle \phi_P(\tilde{\lambda}_P), \alpha_i \rangle = 0$ for all $i \in I_M$. Combined with part (a) applied to the reductive group $M$, we obtain

$$\text{deg}(V[\lambda + \mathbb{Z} R_M]_{F_M}) = \left( \sum_\nu m_\nu \right) \cdot \langle \phi_P(\tilde{\lambda}_P), \lambda \rangle,$$

giving rise to the desired formula for the slope $\mu(V[\lambda + \mathbb{Z} R_M]_{F_M})$.

Finally, part (b) is just the case $P = G$ in part (c). \hfill \Box

### 3.2.4 Remark

Proposition 3.2 provides another reason for calling the element $\phi_P(\tilde{\lambda}_P)$ the “slope” of a $P$-bundle $F_P$ of degree $\tilde{\lambda}_P$. Namely, by part (c), the function

$$\langle \phi_P(\tilde{\lambda}_P), - \rangle : \Lambda^G_G \longrightarrow \mathbb{Q}$$

sends any weight $\lambda \in \Lambda_G$ to the slope of the subbundle corresponding to the inclusion $\kappa^\lambda$.

### 3.2.5 The case of Weyl modules

We now consider the induced subbundles $\kappa^\lambda$ in the case where the representation $V$ of highest weight $\lambda$ is specifically the Weyl module $V^\lambda$ of $G$. First recall from Lemma 3.5(b) above that $V^\lambda[\lambda + \mathbb{Z} R_M] = V^\lambda[\lambda]$ if the weight $\lambda$ lies in $\Lambda_G$, and hence the inclusions $\kappa^\lambda$ correspond to line subbundles for such $\lambda$. These line subbundles are of great importance in the study of reductions of $G$-bundles to parabolic subgroups (see [6, Ch. 1], [5, Ch. 1]). We will only need the following fact:

**Proposition 3.3** Let $F_G$ be a $G$-bundle on a scheme $S$ and let $F_P$ and $\tilde{F}_P$ be two reductions of $F_G$ to the same parabolic $P$. Then, if the line subbundles

$$V^\lambda[\lambda]_{F_M} \xleftarrow{\kappa^\lambda} V^\lambda_{F_G} \xrightarrow{\tilde{\kappa}^\lambda} V^\lambda[\lambda]_{\tilde{F}_M}$$

are equal for all $\lambda \in \Lambda^+_G \cap \Lambda_{G,P}$, then the reductions $F_P$ and $\tilde{F}_P$ are already equal.

**Proof** Follows from the description of Plücker data in, for example, [6, Sec. 1.3.2] or [5, Sec. 1.1], taking into account that the action of $M$ on $V^\lambda[\lambda]$ factors through the torus $M/\{ M, M \}$ and that the induced $M/\{ M, M \}$-bundle $F_{M/[M,M]}$ is uniquely determined by the inclusions of line subbundles $\kappa^\lambda$ for $\lambda \in \Lambda^+_G \cap \Lambda_{G,P}$. \hfill \Box
According to the proposition, a reduction of $F_G$ to a parabolic $P$ is already uniquely determined by the subbundle maps $\kappa^\lambda$ for $\lambda \in \Lambda^+_G \cap \Lambda_{G,P}$. However, unlike in Proposition 3.3, we will be interested in comparing reductions to possibly different parabolics of $G$, and hence it is important to consider the subbundles $\kappa^\lambda$ for all $\lambda \in \Lambda^+_G$ above.

4 Comparing two reductions

Fix two parabolic subgroups $P_1$ and $P_2$ of $G$. The purpose of this section is to compare two given reductions $F_{P_1}$ and $F_{P_2}$ of the same $G$-bundle on the curve $X$. Its main goal, and its sole application toward the proof of Theorem 2.1, is to prove the comparison theorem, Theorem 4.1 below.

The comparison theorem is used in the proofs of almost all parts of Theorem 2.1 in Sect. 6 below. For example, it immediately implies the uniqueness of the canonical reduction and thus establishes half of part (c). It furthermore directly implies part (d) and is also used in the proofs of parts (a) and (g). As we include a simpler proof of the comparison theorem in characteristic 0 in Sect. 5 below, the reader only interested in that case can skip the present Sect. 4 entirely.

4.1 Relative position of two reductions

Consider the double quotient stack $P_1 \backslash G / P_2$. By definition, it parametrizes triples $(F_{P_1}, F_{P_2}, \gamma)$ consisting of a $P_1$-bundle $F_{P_1}$, a $P_2$-bundle $F_{P_2}$, and an isomorphism

$$\gamma : (F_{P_1})_G \cong (F_{P_2})_G$$

of their induced $G$-bundles. Thus, given two reductions $F_{P_1}$ and $F_{P_2}$ of the same $G$-bundle on a scheme $S$, we obtain a map

$$S \longrightarrow P_1 \backslash G / P_2.$$  

The main idea of the present Sect. 4 is to use the geometry of the stack $P_1 \backslash G / P_2$—namely, a Tannakian interpretation of its Bruhat stratification—to compare the two reductions. We begin by introducing the notion of relative position of two reductions.

4.1.1 Notation

The Levi quotients of $P_1$ and $P_2$ will be denoted by $M_1$ and $M_2$ and will also be considered as subgroups via the fixed splitting of $B \rightarrow T$, see Sect. 2.1.2 above. Furthermore, the subsets of the set of vertices of the Dynkin diagram $\mathcal{I}$ corresponding to $M_1$ and $M_2$ will be denoted by

$$\mathcal{I}_{M_1} \subset \mathcal{I} \supset \mathcal{I}_{M_2},$$

the roots of $M_1$ and $M_2$ by
and their corresponding subgroups of the Weyl group $W$ of $G$ by

$$W_{M_1} \subset W \supset W_{M_2}.$$ 

Now consider the set of double cosets $W_{M_1} \setminus W / W_{M_2}$ for the action of $W_{M_1}$ on $W$ from the left and of $W_{M_2}$ from the right. We obtain a canonical set of representatives by taking the unique shortest element in each coset. In other words, we consider the subset

$$W_{1,2} = \{ w \in W \mid \ell(w) \leq \ell(w_1 w w_2) \text{ for all } w_1 \in W_{M_1}, w_2 \in W_{M_2} \}$$

of $W$. Furthermore, we also choose representatives in $G$ for each $w \in W_{1,2}$ and again denote them by $w$ by abuse of notation.

### 4.1.2 Stratification of $P_1 \setminus G / P_2$

By the Bruhat decomposition, the group $G$ is the disjoint union of double cosets

$$G = \bigcup_{w \in W_{1,2}} P_1 w P_2,$$

and the cosets $P_1 w P_2$ are locally closed subvarieties of $G$. Quotienting out by the left action of $P_1$ and the right action of $P_2$ thus yields a stratification of the double quotient stack

$$P_1 \setminus G / P_2 = \bigcup_{w \in W_{1,2}} (P_1 \setminus G / P_2)_w$$

by locally closed substacks

$$(P_1 \setminus G / P_2)_w := P_1 \setminus (P_1 w P_2) / P_2.$$ 

### 4.1.3 Definition of relative position

Let $F_{P_1}$ and $F_{P_2}$ be two reductions of the same $G$-bundle on a scheme $S$ and let

$$\psi : S \longrightarrow P_1 \setminus G / P_2$$

be the induced map. Moreover, let $w \in W_{1,2}$. Then we say that $F_{P_1}$ is in relative position $w$ with respect to $F_{P_2}$ if the map $\psi$ factors through the substack $(P_1 \setminus G / P_2)_w$ of $P_1 \setminus G / P_2$.

We furthermore say that $F_{P_1}$ is generically in relative position $w$ with respect to $F_{P_2}$ if there exists an open dense subscheme $U$ of $S$ such that the restriction of $\psi$ to $U$ factors through $(P_1 \setminus G / P_2)_w$.

For example, we have the following immediate lemma:
**Lemma 4.1** Let $F_{P_1}$ and $F_{P_2}$ be two reductions of the same $G$-bundle on an integral scheme $S$. Then there exists a unique element $w \in W_{1,2}$ such that $F_{P_1}$ is generically in relative position $w$ with respect to $F_{P_2}$.

4.2 Deeper reductions

4.2.1 Overview

Let $F_{P_1}$ and $F_{P_2}$ be two reductions of the same $G$-bundle $F_G$ on a scheme $S$ and assume that $F_{P_1}$ is in relative position $w \in W_{1,2}$ with respect to $F_{P_2}$. In this section we construct certain “deeper” reductions, i.e., reductions $F_{Q_1}$ of $F_{P_1}$ and $F_{Q_2}$ of $F_{P_2}$ to smaller parabolics $Q_1 \subset P_1$ and $Q_2 \subset P_2$.

The parabolics $Q_1$ and $Q_2$ only depend on the element $w$ and have the property that their corresponding Levi subgroups $L_1$ and $L_2$ are $w$-conjugate inside the group $G$. Furthermore, the reductions $F_{Q_1}$ and $F_{Q_2}$ have the property that their induced Levi bundles $F_{L_1}$ and $F_{L_2}$ are naturally isomorphic when $L_1$ and $L_2$ are identified via the element $w$ (see Corollary 4.1 below):

![Diagram of deeper reductions](image)

The geometry we employ in this section to construct the deeper reductions $F_{Q_1}$ and $F_{Q_2}$ is to a certain extent reminiscent of the geometry underlying the “Geometric Lemma” of Bernstein and Zelevinsky in the representation theory of reductive groups over non-archimedean local fields (see [3]).

4.2.2 Lemmas about the Weyl group

We start with the following two easy lemmas about the subset $W_{1,2}$ of the Weyl group $W$:

**Lemma 4.2** Let $w \in W_{1,2}$. Then we have:

\[
\forall i \in \mathcal{I}_{M_1} : w^{-1}(\alpha_i) \in R_+ \\
\forall i \in \mathcal{I}_{M_2} : w(\alpha_i) \in R_+
\]

**Proof** To prove the second assertion, suppose there exists some $i \in \mathcal{I}_{M_2}$ such that $w(\alpha_i) \in -R_+$. Then, as the simple reflection $s_{\alpha_i} \in W_{M_2}$ permutes the set $R_+ \setminus \{\alpha_i\}$,
the element \( w_s \alpha_i \) is shorter than \( w \), in contradiction to the fact that \( w \in W_{1,2} \). The first assertion is checked analogously.

**Lemma 4.3** Let \( w \in W_{1,2} \). If \( i \in I_{M_1} \) and \( w^{-1}(\alpha_i) \in R_{I_{M_2}} \), then \( w^{-1}(\alpha_i) \) is again a simple root. Similarly, if \( i \in I_{M_2} \) and \( w(\alpha_i) \in R_{I_{M_1}} \), then \( w(\alpha_i) \) is again a simple root.

**Proof** For \( i \in I_{M_1} \), the assumption together with the first half of Lemma 4.2 implies that \( w^{-1}(\alpha_i) \) lies in the positive part \( R_{I_{M_2}}^+ \). Now, if \( w^{-1}(\alpha_i) \) was not simple, then by the second half of Lemma 4.2 its image under \( w \) could not be simple either, but of course \( w(w^{-1}(\alpha_i)) = \alpha_i \). The second statement is verified analogously.

**4.2.3 The parabolics \( Q_1 \) and \( Q_2 \)**

Fix an element \( w \in W_{1,2} \). We now define the parabolics \( Q_1 \subset P_1 \) and \( Q_2 \subset P_2 \). Namely, we define the following sets of vertices:

\[
\mathcal{I}_{L_1} := \{ i \in I_{M_1} \mid \exists j \in I_{M_2} : w(\alpha_j) = \alpha_i \}
\]

\[
\mathcal{I}_{L_2} := \{ i \in I_{M_2} \mid \exists j \in I_{M_1} : w^{-1}(\alpha_j) = \alpha_i \}
\]

Let \( Q_1 \) and \( Q_2 \) denote the parabolic subgroups of \( G \) corresponding to the sets of vertices \( \mathcal{I}_{L_1} \) and \( \mathcal{I}_{L_2} \), and let \( L_1 \) and \( L_2 \) denote their Levi quotients, which as before we simultaneously regard as subgroups. Since \( \mathcal{I}_{L_1} \subset I_{M_1} \) and \( \mathcal{I}_{L_2} \subset I_{M_2} \), we have \( Q_1 \subset P_1 \) and \( Q_2 \subset P_2 \). Furthermore, we denote by \( R_{L_1} \subset R_{M_1} \) and \( R_{L_2} \subset R_{M_2} \) the corresponding sets of roots. Finally, it follows directly from the definition of \( \mathcal{I}_{L_1} \) and \( \mathcal{I}_{L_2} \) that conjugation by \( w \) maps \( L_2 \) isomorphically onto \( L_1 \):

\[
wL_2w^{-1} = L_1
\]

**4.2.4 Lemmas about \( Q_1 \) and \( Q_2 \)**

Next, we record two lemmas about the parabolics \( Q_1 \) and \( Q_2 \) that will be needed for the construction of the deeper reductions \( F_{Q_1} \) and \( F_{Q_2} \) in Propositions 4.1 and 4.2 below.

First, the element \( w \) was chosen from the set of coset representatives \( W_{1,2} \subset W \) in order to make the following lemma hold true:

**Lemma 4.4**

\[
R_{L_1} = R_{M_1} \cap w(R_{M_2})
\]

\[
R_{L_2} = w^{-1}(R_{M_1}) \cap R_{M_2}
\]

**Proof** We prove only the first assertion, the proof of the second one being similar. It is clear from the definition of \( \mathcal{I}_{L_1} \) that \( R_{L_1} \) is contained in the intersection \( R_{M_1} \cap w(R_{M_2}) \). To prove the converse inclusion, it suffices to show that every positive root \( \alpha \) in this
The Harder–Narasimhan stratification of $\text{Bun}_G$

intersection can be written as a sum of simple roots corresponding to elements $i \in I_{L_1}$. So let $\alpha = \sum_{i \in I_{M_1}} n_i \alpha_i$ with integers $n_i \geq 0$. Then

$$w^{-1}(\alpha) = \sum_{i \in I_{M_1}} n_i w^{-1}(\alpha_i)$$

lies in $R_{M_2}$ by assumption, but at the same time all $w^{-1}(\alpha_i)$ lie in $R_+$ by Lemma 4.2. Hence, all $w^{-1}(\alpha_i)$ have to lie in $R_{M_2} \cap R_+$ and thus have to be simple roots by Lemma 4.3. Applying $w$ again, we obtain a presentation of $\alpha$ as a sum of simple roots of the desired form.

Next, let $U(Q_1)$ and $U(Q_2)$ denote the unipotent radicals of $Q_1$ and $Q_2$. For the proofs of Propositions 4.1 and 4.2 below, we will furthermore need the following technical lemma:

**Lemma 4.5** The scheme-theoretic intersections

$$P_1 \cap (w P_2 w^{-1}), \quad P_2 \cap (w^{-1} P_1 w), \quad Q_1 \cap (w Q_2 w^{-1}), \quad (w^{-1} U(Q_1) w) \cap Q_2$$

are reduced and connected. We furthermore have the following inclusions:

(a) $P_1 \cap (w P_2 w^{-1}) \subseteq Q_1$

(b) $P_2 \cap (w^{-1} P_1 w) \subseteq Q_2$

(c) $(w^{-1} U(Q_1) w) \cap Q_2 \subseteq U(Q_2)$

**Proof** For reducedness, we need to show that in each of the cases the Lie algebra of the intersection has the minimal dimension it can possibly have, namely the dimension of the intersection itself. This is in turn a consequence of the fact that all subgroups under consideration are defined combinatorially, i.e., on the level of roots. Namely, for every root $\alpha$ occurring in the root decomposition of the Lie algebra of an intersection as above, the intersection contains the root subgroup $U_\alpha \cong \mathbb{G}_\alpha$ corresponding to the root $\alpha$. Thus, the claim about the dimension follows from the fact that the multiplication map

$$\prod_{\alpha \in R_+} U_\alpha \times T \times \prod_{\alpha \in R_+} U_{-\alpha} \longrightarrow G$$

is an open immersion, namely an isomorphism onto the open Bruhat cell of $G$.

Next, the intersection of any two parabolics is connected (see, for example, [4, Prop. 14.22]), establishing the claimed connectedness for the first three intersections. The connectedness of the fourth intersection follows from the following general fact about reductive groups (see, for example, [4, Prop. 14.4]), applied to the conjugate Borel subgroup $w^{-1} B w$: If a closed subgroup of the unipotent part of a Borel subgroup is stable under conjugation by a maximal torus of this Borel, then it must be connected.

We now show that the three inclusions (a), (b), (c) hold. Since all subgroups under consideration are connected, the inclusions can be checked on the level of algebras of
distributions. For this, in turn, it suffices to prove the three inclusions on the level of roots. The assertion of part (a) then translates to the claim that

\[(R_{M_1} \cup R_+) \cap w(R_{M_2} \cup R_+) \subset R_+ \cup R_{L_1}.\]

Proving this inclusion reduces to showing that the sets \(R_{M_1} \cap w(R_{M_2})\) and \(R_{M_1} \cap w(R_+)\) are both contained in the right-hand side. But the first set is equal to \(R_{L_1}\) by Lemma 4.4, and the second set is equal to \(R_{M_1} \cap R_+\) by Lemma 4.2, thus completing the proof of part (a). Part (b) is proven analogously.

Similarly, the assertion of part (c) translates on the level of roots to the claim that

\[w^{-1}(R_+ \setminus R_{L_1}) \cap (R_+ \cup R_{L_2}) \subset R_+ \setminus R_{L_2},\]

which in turn follows immediately from the fact that \(w^{-1}(R_{L_1}) = R_{L_2}\).

4.2.5 Construction of deeper reductions

We now turn to the construction of the deeper reductions \(F_{Q_1}\) and \(F_{Q_2}\). We begin by proving:

**Proposition 4.1** The natural map of stacks

\[Q_1 \setminus (Q_1 w Q_2)/Q_2 \longrightarrow P_1 \setminus (P_1 w P_2)/P_2\]

is an isomorphism.

**Proof** The double quotient \(P_1 \setminus (P_1 w P_2)/P_2\) is equal to the classifying stack of the scheme-theoretic stabilizer \(\text{Stab}_{P_1 \times P_2}(w)\) of \(w\) under the action of the product \(P_1 \times P_2\), and similarly the stack \(Q_1 \setminus (Q_1 w Q_2)/Q_2\) is equal to the classifying stack of the scheme-theoretic stabilizer \(\text{Stab}_{Q_1 \times Q_2}(w)\). Furthermore, both stacks are naturally pointed, i.e., equipped with a map from the point Spec(\(k\)), and the natural map between them is in fact a map of pointed stacks. Thus, to show that this map is an isomorphism reduces to showing that the inclusion of stabilizers

\[\text{Stab}_{Q_1 \times Q_2}(w) \hookrightarrow \text{Stab}_{P_1 \times P_2}(w)\]

is an isomorphism. By definition, this is equivalent to showing that the inclusion of intersections

\[Q_1 \cap (wQ_2w^{-1}) \hookrightarrow P_1 \cap (wP_2w^{-1})\]

is an isomorphism. As both intersections are reduced by Lemma 4.5, we only need to show that the inclusion map is surjective on \(k\)-points. But, this follows immediately from parts (a) and (b) of the same lemma. \(\square\)
Next, recall that the groups $L_1$ and $L_2$ are identified with each other under conjugation by $w$. Thus, we occasionally denote both groups simply by $L$. For example, in the following proposition, we consider the fiber product

$\cdot/ Q_1 \times \cdot/ L \cdot/ Q_2$

of the classifying stacks of $Q_1$, $Q_2$, and $L$.

**Proposition 4.2** The forgetful map

$$Q_1 \backslash (Q_1 w Q_2)/Q_2 \longrightarrow \cdot/ Q_1 \times \cdot/ Q_2$$

canonically factors as

$$Q_1 \backslash (Q_1 w Q_2)/Q_2 \twoheadrightarrow \cdot/ Q_1 \times \cdot/ Q_2$$

Proof All three stacks being naturally pointed, we in fact claim that a canonical factorization as above exists as pointed stacks. Since the double quotient $Q_1 \backslash (Q_1 w Q_2)/Q_2$ is equal to the classifying stack of the stabilizer $\text{Stab}_{Q_1 \times Q_2}(w)$, this assertion in turn follows once we show that the following diagram of groups factors:

Here, we are using that the classifying stack of the fiber product of groups $Q_1 \times Q_2$ is equal to the fiber product of their classifying stacks since $Q_1$ and $Q_2$ surject onto $L$.

As all three groups are reduced, it suffices to prove that the diagram of groups factors as claimed on the level of $k$-points. Thus, let $(x_1, x_2)$ be an element of $\text{Stab}_{Q_1 \times Q_2}(w)$, i.e., let $x_1 \in Q_1$ and $x_2 \in Q_2$ such that $x_1 = w x_2 w^{-1}$. Then we need to show that if

$$x_1 = \ell_1 \cdot u_1 \quad \text{and} \quad x_2 = \ell_2 \cdot u_2$$

are the Levi decompositions of $x_1$ in $Q_1 = L_1 \cdot U(Q_1)$ and $x_2$ in $Q_2 = L_2 \cdot U(Q_2)$, then

$$w^{-1} \ell_1 w = \ell_2.$$
To see this, we write the element $x_2 = w^{-1}x_1 w$ as

$$x_2 = w^{-1}\ell_1 w \cdot w^{-1}u_1 w.$$ 

We claim that this is in fact the Levi decomposition of $x_2$ in $Q_2$. Indeed, the first factor lies in $L_2$ and thus also in $Q_2$. Since the product lies in $Q_2$ as well, the same must hold for the second factor $w^{-1}u_1 w$. Hence, $w^{-1}u_1 w$ lies in the intersection $(w^{-1}U(Q_1)w) \cap Q_2$ and therefore in $U(Q_2)$ by part (c) of Lemma 4.5, proving the claim about the Levi decomposition of $x_2$. By uniqueness of the latter, we conclude that $w^{-1}\ell_1 w = \ell_2$ as desired. □

Combining Propositions 4.1 and 4.2 yields the desired construction of the deeper reductions:

**Corollary 4.1** The forgetful map

$$(P_1 \backslash G/P_2)_w = P_1 \backslash (P_1 w P_2)/P_2 \rightarrow \cdot/P_1 \times \cdot/P_2$$

canonically factors as

$$P_1 \backslash (P_1 w P_2)/P_2 \overset{\cong}{\rightarrow} Q_1 \backslash (Q_1 w Q_2)/Q_2$$

$$\downarrow \quad \downarrow$$

$$\cdot/P_1 \times \cdot/P_2 \quad \cdot/Q_1 \times \cdot/Q_2$$

In other words, if $F_{P_1}$ and $F_{P_2}$ are two reductions of the same $G$-bundle on a scheme $S$ such that $F_{P_1}$ is in relative position $w$ with respect to $F_{P_2}$, then there exist naturally defined reductions $F_{Q_1}$ of $F_{P_1}$ and $F_{Q_2}$ of $F_{P_2}$ to $Q_1$ and $Q_2$ such that $F_{Q_1}$ is still in relative position $w$ with respect to $F_{Q_2}$ and such that their Levi bundles $F_{L_1}$ and $F_{L_2}$ are naturally isomorphic when $L_1$ and $L_2$ are identified via conjugation by $w$.

### 4.3 Tannakian interpretation of Bruhat decomposition

In this section, we give a Tannakian description of the Bruhat decomposition of the stack $P_1 \backslash G/P_2$. More precisely, we analyze what the notion of relative position of two reductions $F_{P_1}$ and $F_{P_2}$ translates to on the level of vector bundles associated to $G$-representations of highest weight $\lambda$. If the parabolics $P_1$ and $P_2$ are both equal to the Borel $B$, our result is stated directly in terms of the associated bundles of $F_{P_1}$ and $F_{P_2}$. For two arbitrary parabolics $P_1$ and $P_2$, our result is phrased in terms of the associated bundles of the deeper reductions $F_{Q_1}$ and $F_{Q_2}$ from the previous Sect. 4.2.

We emphasize again that while it is essential to use Weyl modules in Proposition 3.3 above, all assertions in the current section, as well as in its application to the proof of the comparison theorem via Proposition 4.6 below, hold for an arbitrary finite-dimensional $G$-representation of highest weight $\lambda$ and will thus be stated and proved in this generality.
4.3.1 The case $P_1 = B = P_2$

We shall first state the result in the special case $P_1 = B = P_2$, which is significantly simpler since then $Q_1 = B = Q_2$ and $L_1 = T = L_2$ as well as $W_{1,2} = W$, so that the deeper reductions of Sect. 4.2 above do not appear explicitly.

Let $V$ be any finite-dimensional $G$-representation of highest weight $\lambda \in \Lambda_G^+$ and let $V = \bigoplus_v V[v]$ be its weight decomposition. Then the direct sum of weight spaces

$$V[\geq w\lambda] := \bigoplus_{v \geq w\lambda} V[v]$$

is a submodule of $V$ for the algebra of distributions $\text{Dist}(B)$ of the Borel $B$ and is hence a $B$-subrepresentation of $V$. We denote the quotient of $V[\geq w\lambda]$ by the analogously defined $B$-subrepresentation $V[> w\lambda]$ by $V[w\lambda]$, since $w\lambda$ is its unique weight.

In addition, we also consider the highest weight space $V[\lambda] \subset V$. Since the algebra of distributions $\text{Dist}(U(B))$ of the unipotent radical $U(B)$ of $B$ acts trivially on the subspace $V[\lambda]$ and the quotient space $V[w\lambda]$, the same holds true for $U(B)$ itself and thus both $B$-representations descend to representations of the torus $T$.

Let now $F_B$ and $\tilde{F}_B$ be two reductions to the Borel $B$ of the same $G$-bundle on a scheme $S$, and let $F_T$ and $\tilde{F}_T$ denote their corresponding $T$-bundles. Then the above quotient and inclusion maps of $B$-representations induce the following diagram on associated vector bundles:

$$
\begin{array}{ccc}
V[\lambda]_{\tilde{F}_T} & \xrightarrow{\kappa^\lambda} & V_{FG} \\
\uparrow & & \uparrow \\
V[\geq w\lambda]_{F_B} & \xrightarrow{\pi^\lambda} & V[w\lambda]_{F_T}
\end{array}
$$

With this notation, the result is the following:

**Proposition 4.3** Let $F_B$ and $\tilde{F}_B$ be two reductions to the Borel $B$ of the same $G$-bundle $F_G$ on a scheme $S$, and assume that $F_B$ is in relative position $w$ with respect to $\tilde{F}_B$. Then, for any $G$-representation $V$ of highest weight $\lambda \in \Lambda_G^+$, the inclusion $\kappa^\lambda$ factors through the subbundle $V[\geq w\lambda]_{F_B}$ and the composition of $\kappa^\lambda$ with the surjection $\pi^\lambda$ is an isomorphism of vector bundles

$$\pi^\lambda \circ \kappa^\lambda : V[\lambda]_{\tilde{F}_T} \overset{\cong}{\longrightarrow} V[w\lambda]_{F_T}.$$
Diagrammatically:

\[ V[\lambda]_{FT} \overset{\kappa}{\leftarrow} V_{FG} \]

\[ V[\geq w\lambda]_{FB} \overset{\pi^\lambda}{\rightarrow} V[w\lambda]_{FT} \]

4.3.2 The general case

We now let \( P_1 \) and \( P_2 \) be any parabolic subgroups of \( G \), and let \( w \in W_{1,2} \). In this setting, we have already defined the groups \( Q_1, Q_2, L_1, \) and \( L_2 \) in Sect. 4.2.3 above. Let

\[ \mathbb{Z}R_{L_1} := \text{span}_{\mathbb{Z}}(R_{L_1}) \subset \Lambda_G \]

denote the root lattice of \( L_1 \) and \( \mathbb{Z}R_{L_2} \) the root lattice of \( L_2 \). Given any \( G \)-representation \( V \) of highest weight \( \lambda \in \Lambda^+_G \) with weight decomposition \( V = \bigoplus_v V[v] \) as before, we define the following sums of weight spaces:

\[ V[\lambda + ZR_{L_2}] := \bigoplus_{v \in \lambda + ZR_{L_2}} V[v] \]

\[ V[\geq (w\lambda + ZR_{L_1})] := \sum_{v' \in w\lambda + ZR_{L_1}} \bigoplus_{v \geq v'} V[v] \]

\[ V[> (w\lambda + ZR_{L_1})] := \sum_{v' \in w\lambda + ZR_{L_1}} \bigoplus_{v > v'} V[v] \]

We record some immediate facts in the following lemma:

**Lemma 4.6** (a) The subspace \( V[\lambda + ZR_{L_2}] \) is a \( Q_2 \)-subrepresentation of \( V \). Furthermore, the unipotent radical \( U(Q_2) \) acts trivially, and thus the action of \( Q_2 \) descends to an action of \( L_2 \).

(b) The subspaces \( V[\geq (w\lambda + ZR_{L_1})] \) and \( V[> (w\lambda + ZR_{L_1})] \) are \( Q_1 \)-subrepresentations of \( V \).

(c) The unipotent radical \( U(Q_1) \) of \( Q_1 \) acts trivially on the quotient of \( V[\geq (w\lambda + ZR_{L_1})] \) by \( V[> (w\lambda + ZR_{L_1})] \), so that this quotient is naturally an \( L_1 \)-representation. It will be denoted by \( V[w\lambda + ZR_{L_1}] \) since its weight decomposition is

\[ V[w\lambda + ZR_{L_1}] = \bigoplus_{v \in w\lambda + ZR_{L_1}} V[v]. \]
The Harder–Narasimhan stratification of \( \text{Bun}_G \)

**Proof** Using algebras of distributions, all three parts are verified similarly to the analogous statements in Sect. 4.3.1 or in Lemma 3.5 above. \( \square \)

### 4.3.3 The Tannakian interpretation in the general case

Let now \( F_{P_1} \) and \( F_{P_2} \) be two reductions of the same \( G \)-bundle on a scheme \( S \), and assume that \( F_{P_1} \) is in relative position \( w \in W_{1,2} \) with respect to \( F_{P_2} \). Let then \( F_{Q_1} \) and \( F_{Q_2} \) be the reductions yielded by Corollary 4.1 in this situation.

In this setting, we find the following natural maps between associated vector bundles. First, the inclusion of \( Q_2 \)-representations

\[
V[\lambda + \mathbb{Z} R_{L_2}] \hookrightarrow V
\]

yields a subbundle map

\[
\kappa^{\lambda}_{Q_2} : V[\lambda + \mathbb{Z} R_{L_2}]_{F_{L_2}} \hookrightarrow V_{F_{G}}
\]
as in Sect. 3.2.3. Furthermore, the inclusion and quotient maps of \( Q_1 \)-representations

\[
\begin{array}{ccc}
V & \downarrow & \\
\downarrow & & \\
V[\geq (w\lambda + \mathbb{Z} R_{L_1})] & \longrightarrow & V[w\lambda + \mathbb{Z} R_{L_1}]
\end{array}
\]

yield maps of associated vector bundles

\[
\begin{array}{ccc}
V_{F_{G}} & \uparrow & \\
\downarrow & & \\
V[\geq (w\lambda + \mathbb{Z} R_{L_1})]_{F_{Q_1}} & \longrightarrow & V[w\lambda + \mathbb{Z} R_{L_1}]_{F_{L_1}}.
\end{array}
\]

With this notation, we can finally state the result in the general case:

**Proposition 4.4** Let \( F_{P_1} \) and \( F_{P_2} \) be two reductions of the same \( G \)-bundle \( F_{G} \) on a scheme \( S \), and assume that \( F_{P_1} \) is in relative position \( w \) with respect to \( F_{P_2} \). Then, for any \( G \)-representation \( V \) of highest weight \( \lambda \in \Lambda^+_G \), the inclusion \( \kappa^{\lambda}_{Q_2} \) factors through the subbundle \( V[\geq (w\lambda + \mathbb{Z} R_{L_1})]_{F_{Q_1}} \) and the composition of \( \kappa^{\lambda}_{Q_2} \) with the surjection \( \pi^{\lambda}_{Q_1} \) is an isomorphism of vector bundles

\[
\pi^{\lambda}_{Q_1} \circ \kappa^{\lambda}_{Q_2} : V[\lambda + \mathbb{Z} R_{L_2}]_{F_{L_2}} \xrightarrow{\cong} V[w\lambda + \mathbb{Z} R_{L_1}]_{F_{L_1}}.
\]
Diagrammatically:

\[
\begin{align*}
V[\lambda + ZR_{L_2}]_{F_{L_2}} & \xrightarrow{\kappa^\lambda_{Q_2}} V_{F_G} \\
V[\geq (w\lambda + ZR_{L_1})_{F_{Q_1}}] & \xrightarrow{\pi^\lambda_{Q_1}} V[w\lambda + ZR_{L_1} ]_{F_{L_1}} \\
\cong
\end{align*}
\]

Proof By Corollary 4.1, it suffices to prove that the two assertions hold for the universal family \((F_{Q_1}, F_{Q_2})\) on \(Q_1 \setminus G / Q_2\) over the locus \(Q_1 \setminus (Q_1 wQ_2)/Q_2\). As both assertions can be checked locally in the smooth topology, it furthermore suffices to check them after pulling back the universal family along the smooth surjection

\[G / Q_2 \overset{p}{\longrightarrow} Q_1 \setminus G / Q_2.\]

The pullback of \(F_{Q_1}\) along \(p\) possesses a canonical trivialization, which in turn also induces a trivialization of the pullback of \(F_G\). Moreover, the pullback of \(F_{Q_2}\) along \(p\) is canonically isomorphic to the tautological \(Q_2\)-bundle \(P_{Q_2}\) on \(G / Q_2\), i.e., the \(Q_2\)-bundle corresponding to the quotient map \(G \to G / Q_2\).

We thus obtain the diagram of vector bundles

\[
\begin{align*}
V[\lambda + ZR_{L_2}]_{P_{Q_2}} & \xrightarrow{\kappa^\lambda_{Q_2}} V \otimes \mathcal{O}_{G / Q_2} \\
V[\geq (w\lambda + ZR_{L_1}) \otimes \mathcal{O}_{G / Q_2}] & \xrightarrow{\pi^\lambda_{Q_1}} V[w\lambda + ZR_{L_1}] \otimes \mathcal{O}_{G / Q_2} \\
\cong
\end{align*}
\]

on \(G / Q_2\) and need to verify the assertions (see dotted arrows) over the locus \(Q_1 wQ_2 / Q_2\). As \(G / Q_2\) is a variety, it suffices to check the assertions on the fibers of the vector bundles under consideration. Now, if \(\tilde{x}\) is a point of \(G / Q_2\), then by construction of \(P_{Q_2}\) the map \(\kappa^\lambda_{Q_2}\) maps the fiber of \(V[\lambda + ZR_{L_2}]_{P_{Q_2}}\) at \(\tilde{x}\) isomorphically onto the (well-defined) subspace

\[\tilde{x} \cdot V[\lambda + ZR_{L_2}] \subset V.\]

But if \(\tilde{x} =: q_1 \overline{w}\) lies in the locus \(Q_1 wQ_2 / Q_2\), this subspace equals

\[q_1 \cdot V[w\lambda + ZR_{L_1}] \subset V.\]
since $w(ZR_{L_2}) = ZR_{L_1}$. It is thus contained in the subspace $V[\geq (w\lambda + ZR_{L_1})]$ as the latter is a $Q_1$-representation, proving the first claim. Furthermore, in this case, the quotient map of $Q_1$-representations

$$V[\geq (w\lambda + ZR_{L_1})] \rightarrow V[w\lambda + ZR_{L_1}]$$

of course maps the subspace $q_1 \cdot V[w\lambda + ZR_{L_1}]$ isomorphically onto the target space, establishing the second assertion. $\square$

4.4 Relative position in the curve case and a key inequality

We now apply the results of Sect. 4.3 in the case where the base scheme is the curve $X$. In this situation, we prove a technical inequality of certain slopes (Proposition 4.6) which will form the key ingredient in the proof of the comparison theorem, Theorem 4.1 below.

4.4.1 The curve case

For the remainder of this section, we let $F_{P_1}$ and $F_{P_2}$ be two reductions of the same $G$-bundle on the curve $X$. By Lemma 4.1, there exists a unique $w \in W_{1,2}$ such that $F_{P_1}$ is generically in relative position $w$ with respect to $F_{P_2}$, i.e., there exists an open dense subset $U \subset X$ such that the restriction $F_{P_1}|_U$ is in relative position $w$ with respect to $F_{P_2}|_U$. Thus, Corollary 4.1 yields the reductions $F_{Q_1}|_U$ and $F_{Q_2}|_U$ only on the open subset $U$. It is, however, easy to see:

**Proposition 4.5** The reductions $F_{Q_1}|_U$ and $F_{Q_2}|_U$ on $U$ extend uniquely to reductions $F_{Q_1}$ and $F_{Q_2}$ of $F_{P_1}$ and $F_{P_2}$ on the entire curve $X$. Furthermore, in the situation of Proposition 4.4, the factorization of the inclusion $\kappa_{Q_2}^\lambda$ of associated vector bundles

$$V[\lambda + ZR_{L_2}]_{F_{L_2}} \xrightarrow{\kappa_{Q_2}^\lambda} V_{F_G} \xrightarrow{\pi_{Q_1}^\lambda} V[\geq (w\lambda + ZR_{L_1})]_{F_{Q_1}} \xrightarrow{\kappa_{Q_1}^\lambda} V[w\lambda + ZR_{L_1}]_{F_{L_1}}$$

holds not only on $U$ but on the entire curve $X$, for all $\lambda \in \Lambda_G^+$. Finally, although the composition $\pi_{Q_1}^\lambda \circ \kappa_{Q_2}^\lambda$ is in general only an isomorphism on $U$, it is still an injection of locally free sheaves on $X$.

**Proof** The statement about uniquely extending $F_{Q_1}$ and $F_{Q_2}$ to all of $X$ follows via standard arguments from the facts that $X$ is a smooth curve and that the quotient varieties $P_1/Q_1$ and $P_2/Q_2$ are proper. By a similar argument, the relation of containment among subbundles of a vector bundle on a smooth curve can be checked generically, proving the factorization of the inclusion $\kappa_{Q_2}^\lambda$ on all of $X$. Finally, the kernel of the
map $\pi^\lambda_{Q_1} \circ \kappa_{Q_2}^\lambda$ is locally free because $X$ is smooth, but since it vanishes generically, it must vanish entirely.

4.4.2 The key inequality

By Proposition 4.5, the reductions $F_{Q_1}$ of $F_{P_1}$ and $F_{Q_2}$ of $F_{P_2}$ are defined on the entire curve $X$. Let $\check{\lambda}_{P_1}$, $\check{\lambda}_{P_2}$, $\check{\lambda}_{Q_1}$, and $\check{\lambda}_{Q_2}$ denote their respective degrees. Then we have the following technical inequality of slopes:

**Proposition 4.6**

$$w^{-1} \phi_{Q_1}(\check{\lambda}_{Q_1}) \geq \phi_{Q_2}(\check{\lambda}_{Q_2})$$

**Proof** By part (a) of Lemma 3.2, it suffices to prove that

$$\langle \phi_{Q_1}(\check{\lambda}_{Q_1}), w\lambda \rangle \geq \langle \phi_{Q_2}(\check{\lambda}_{Q_2}), \lambda \rangle$$

for every dominant weight $\lambda \in \Lambda^+_G$. To prove it, consider the map of associated vector bundles

$$\pi^\lambda_{Q_1} \circ \kappa^\lambda_{Q_2} : V[\lambda + \mathbb{Z}R_{L_2}]_{F_{L_2}} \longrightarrow V[w\lambda + \mathbb{Z}R_{L_1}]_{F_{L_1}}$$

in Proposition 4.5 above. According to that proposition, the map $\pi^\lambda_{Q_1} \circ \kappa^\lambda_{Q_2}$ is generically an isomorphism and thus injective. The same then also holds for the induced map of top exterior powers, and hence

$$\mu(V[w\lambda + \mathbb{Z}R_{L_1}]_{F_{L_1}}) \geq \mu(V[\lambda + \mathbb{Z}R_{L_2}]_{F_{L_2}}).$$

Now, on the one hand, we have

$$\mu(V[\lambda + \mathbb{Z}R_{L_2}]_{F_{L_2}}) = \langle \phi_{Q_2}(\check{\lambda}_{Q_2}), \lambda \rangle$$

by part (c) of Proposition 3.2. On the other hand, using part (a) of Proposition 3.2, one computes that

$$\mu(V[w\lambda + \mathbb{Z}R_{L_1}]_{F_{L_1}}) = \langle \phi_{Q_1}(\check{\lambda}_{Q_1}), w\lambda \rangle,$$

and the desired inequality follows. \qed

4.5 The comparison theorem

This section is devoted to the proof of the following **comparison theorem**, which should be regarded as the primary application of Sects. 4.1–4.3. It shows that the canonical reduction is not only unique, but that it moreover enjoys a certain extremal property:
Theorem 4.1 Let $F_{P_1} \in \text{Bun}_{P_1, \tilde{\lambda}_{P_1}}$ and $F_{P_2} \in \text{Bun}_{P_2, \tilde{\lambda}_{P_2}}$ be two reductions of the same $G$-bundle on the curve $X$. Assume furthermore that $F_{P_1}$ lies in fact in $\text{Bun}^\text{ss}_{P_1, \tilde{\lambda}_{P_1}}$ and that $\tilde{\lambda}_{P_1}$ is dominant $P_1$-regular. Then we have

$$\phi_{P_1}(\tilde{\lambda}_{P_1}) \geq \phi_{P_2}(\tilde{\lambda}_{P_2}).$$

If this inequality is in fact an equality, then $P_2$ is already contained in $P_1$ and $F_{P_1}$ is obtained from $F_{P_2}$ by extension of structure group along the inclusion $P_2 \subset P_1$.

4.5.1 Strategy of proof

To prove the comparison theorem, we study how the hypotheses of the theorem affect the relative position of the bundles $F_{P_1}$ and $F_{P_2}$, and relate their slopes via the deeper reductions $F_{Q_1}$ and $F_{Q_2}$ from Sect. 4.4 above. More precisely, to prove the asserted inequality, we first relate the slopes $\phi_{P_1}(\tilde{\lambda}_{P_1})$ of $F_{P_1}$ and $\phi_{P_2}(\tilde{\lambda}_{P_2})$ of $F_{P_2}$ to the slopes $\phi_{Q_1}(\tilde{\lambda}_{Q_1})$ of $F_{Q_1}$ and $\phi_{Q_2}(\tilde{\lambda}_{Q_2})$ of $F_{Q_2}$, respectively, and then use the key inequality 4.6 to relate $\phi_{Q_1}(\tilde{\lambda}_{Q_1})$ and $\phi_{Q_2}(\tilde{\lambda}_{Q_2})$ to each other. If the asserted inequality is in fact an equality, we again use an analysis of the relative position to show that this forces $Q_1 = Q_2 = P_2$ and $Q_1 = Q_2 = F_{P_2}$, from which the second claim of the theorem follows.

4.5.2 Three easy lemmas

We first establish the following three easy lemmas needed in the proof of the comparison theorem below. Let the hypotheses and notation be as in Sect. 4.4 above. The first two lemmas simply spell out the implications of the hypotheses of the comparison theorem in this context:

**Lemma 4.7** Assume that $F_{P_1}$ lies in $\text{Bun}^\text{ss}_{P_1}$. Then we have not only

$$\phi_{P_1}(\tilde{\lambda}_{P_1}) \geq \phi_{Q_1}(\tilde{\lambda}_{Q_1})$$

but also

$$w^{-1} \phi_{P_1}(\tilde{\lambda}_{P_1}) \geq w^{-1} \phi_{Q_1}(\tilde{\lambda}_{Q_1}).$$

**Proof** For the first inequality, see Lemma 3.4 above. Combining this inequality with part (b) of Lemma 3.2 applied to $P_1$ and $Q_1$, we see that

$$\phi_{P_1}(\tilde{\lambda}_{P_1}) - \phi_{Q_1}(\tilde{\lambda}_{Q_1}) \in \sum_{i \in \mathcal{M}_1} Q_{\geq 0} \check{\alpha}_i.$$
Applying \( w^{-1} \) to this difference, Lemma 4.2 shows that
\[
\sum_{i \in I} Q_{\geq 0} \alpha_i
\]
as desired.

**Lemma 4.8** Assume that \( \tilde{\lambda}_{P_1} \) is dominant \( P_1 \)-regular. Then
\[
\phi_{P_1}(\tilde{\lambda}_{P_1}) \geq w^{-1} \phi_{P_1}(\tilde{\lambda}_{P_1}),
\]
and equality holds if and only if \( w = 1 \).

**Proof** The inequality holds since \( \phi_{P_1}(\tilde{\lambda}_{P_1}) \) lies in \( \tilde{\Lambda}_G^{Q,+} \). Now, assume that equality holds and assume that \( w \neq 1 \). Then there exists a simple root \( \alpha_i \) such that \( w^{-1} \alpha_i \) lies in the negative part \( -R_+ \) of \( R \), and since \( w \in W_{1,2} \), we conclude that \( i \notin I_{M_1} \). Pairing both sides of the equality with \( w^{-1} \alpha_i \), we obtain the desired contradiction
\[
0 \geq \langle \phi_{P_1}(\tilde{\lambda}_{P_1}), w^{-1} \alpha_i \rangle = \langle \phi_{P_1}(\tilde{\lambda}_{P_1}), \alpha_i \rangle > 0.
\]

In addition to the last two lemmas about slopes, we will also need the following combinatorial lemma, which generalizes the fact that for a semisimple group the dominant cone \( \Lambda_G^{Q,+} \) is contained in the positive cone \( \Lambda_G^{Q,\text{pos}} \).

**Lemma 4.9** The dominant cone \( \Lambda_G^{Q,+} \) is contained in the cone
\[
\Lambda_G^{Q,+} \cap \Lambda_{G,P_2}^{Q} + \sum_{i \in I_{M_2}} Q_{\geq 0} \alpha_i.
\]

**Proof** First note that the subspaces \( \Lambda_{G,P_2}^{Q} \) and \( \sum_{i \in I_{M_2}} Q \alpha_i \) together span \( \Lambda_G^{Q} \). Thus, given an element \( \lambda \in \Lambda_G^{Q,+} \), we write
\[
\lambda = \mu + \sum_{i \in I_{M_2}} c_i \alpha_i
\]
with \( \mu \in \Lambda_{G,P_2}^{Q} \) and \( c_i \in Q \), and prove that \( \mu \in \Lambda_G^{Q,+} \) and that \( c_i \geq 0 \) for all \( i \). To do so, consider the natural map
\[
p : \Lambda_G^{Q} \rightarrow \Lambda_{[M_2,M_2]}^{Q}.
\]
Then, \( p \) maps the dominant cone \( \Lambda_G^{Q,+} \) into the dominant cone \( \Lambda_{[M_2,M_2]}^{Q,+} \), and thus in particular into the positive cone \( \Lambda_{[M_2,M_2]}^{Q,\text{pos}} \), as the group \( [M_2,M_2] \) is semisimple.
Furthermore, by construction the image of $\lambda$ under $p$ is equal to the sum
\[
\sum_{i \in I_{M_2}} c_i \alpha_i \in \Lambda^Q_{[M_2, M_2]}.
\]
Together these two facts yield that $c_i \geq 0$ for all $i$.

To show that $\mu$ is dominant, we only need to evaluate it on simple coroots $\check{\alpha}_j$ for $j \in I \setminus I_{M_2}$ since it by definition already vanishes on the remaining ones. But, for $j \in I \setminus I_{M_2}$, we have
\[
\langle \check{\alpha}_j, \sum_{i \in I_{M_2}} c_i \alpha_i \rangle \leq 0
\]
and hence
\[
\langle \check{\alpha}_j, \mu \rangle = \langle \check{\alpha}_j, \lambda \rangle - \langle \check{\alpha}_j, \sum_{i \in I_{M_2}} c_i \alpha_i \rangle \geq 0
\]
as desired. \hfill \Box

4.5.3 The proof

We can now finally prove the comparison theorem:

Proof of Theorem 4.1 We prove the claimed inequality by showing that
\[
\phi_{p_1}(\check{\lambda}, \check{\rho}_1) \geq w^{-1} \phi_{p_1}(\check{\lambda}, \check{\rho}_1) \geq \phi_{p_2}(\check{\lambda}, \check{\rho}_2).
\]
The first inequality is immediate from the fact that $\phi_{p_1}(\check{\lambda}, \check{\rho}_1)$ is dominant, as was already recorded in Lemma 4.8 above. In proving the second inequality, part (a) of Lemma 3.2 implies that it suffices to show that
\[
\langle w^{-1} \phi_{p_1}(\check{\lambda}, \check{\rho}_1), \lambda \rangle \geq \langle \phi_{p_2}(\check{\lambda}, \check{\rho}_2), \lambda \rangle
\]
for all $\lambda \in \Lambda^Q_G$. Using Lemma 4.9, we let $\lambda = \mu + \tau$ for $\mu \in \Lambda^Q_G \cap \Lambda^Q_{G, P_2}$ and $\tau \in \sum_{i \in I_{M_2}} \mathbb{Q}_{\geq 0} \alpha_i$ and verify this last inequality for $\mu$ and $\tau$ separately.

For $\tau$ Lemma 4.2 implies that $w \tau \in \Lambda^Q_G$ and thus
\[
\langle w^{-1} \phi_{p_1}(\check{\lambda}, \check{\rho}_1), \tau \rangle = \langle \phi_{p_1}(\check{\lambda}, \check{\rho}_1), w \tau \rangle \geq 0 = \langle \phi_{p_2}(\check{\lambda}, \check{\rho}_2), \tau \rangle.
\]
For $\mu$, we find the sequence of inequalities
\[ \langle w^{-1} \phi_{P_1}(\tilde{\lambda}_{P_1}), \mu \rangle \geq \langle w^{-1} \phi_{Q_1}(\tilde{\lambda}_{Q_1}), \mu \rangle \]

\[ \geq \langle \phi_{Q_2}(\tilde{\lambda}_{Q_2}), \mu \rangle \]

\[ \geq \langle \phi_{P_2}(\tilde{\lambda}_{P_2}), \mu \rangle , \]

finishing the proof of the first assertion of the theorem.

To prove the second assertion, we now assume that \( \phi_{P_1}(\tilde{\lambda}_{P_1}) = \phi_{P_2}(\tilde{\lambda}_{P_2}) \). Then the sequence of inequalities

\[ \phi_{P_1}(\tilde{\lambda}_{P_1}) \geq w^{-1} \phi_{P_1}(\tilde{\lambda}_{P_1}) \geq \phi_{P_2}(\tilde{\lambda}_{P_2}) \]

obtained in the first part of the proof forces that \( \phi_{P_1}(\tilde{\lambda}_{P_1}) = w^{-1} \phi_{P_1}(\tilde{\lambda}_{P_1}) \). This implies that \( w = 1 \) by Lemma 4.8 and hence \( Q_1 = Q_2 \) by Definition 4.2.3.

Next, since for \( w = 1 \) and \( Q_1 = Q_2 \) the substack

\[ (Q_1 \setminus (Q_1 w Q_2)/Q_2)_{w=1} \hookrightarrow Q_1 \setminus G/Q_2 \]

is in fact a closed substack, the bundle \( F_{Q_1} \) has relative position \( w = 1 \) with respect to \( F_{Q_2} \) not only generically, but on the entire curve \( X \), and since \( w = 1 \) we therefore obtain that \( F_{Q_1} = F_{Q_2} \) on all of \( X \).

We now claim that \( I_{M_2} \subset I_{M_1} \). Indeed, if \( i \in I_{M_2} \), then

\[ \langle \phi_{P_1}(\tilde{\lambda}_{P_1}), \alpha_i \rangle = \langle \phi_{P_2}(\tilde{\lambda}_{P_2}), \alpha_i \rangle = 0, \]

and since \( \tilde{\lambda}_{P_1} \) is dominant \( P_1 \)-regular this forces \( i \in I_{M_1} \). But from the facts that \( I_{M_2} \subset I_{M_1} \) and that \( w = 1 \) we conclude that \( I_{L_1} = I_{L_2} = I_{M_2} \). Combined with the above, this in turn implies that \( Q_1 = Q_2 = P_2 \) and \( F_{Q_1} = F_{Q_2} = F_{P_2} \), proving the second assertion of the theorem. \( \square \)

5 The case of characteristic 0 and the case \( P = B \)

This section is independent from the previous Sect. 4. Under the hypothesis that the characteristic of the ground field \( k \) is 0, we give a Tannakian interpretation of the notion of canonical reduction, from which all other results of this section are deduced. This Tannakian interpretation is also valid in arbitrary characteristic if the canonical reduction has as its structure group the Borel \( B \) of \( G \) (see Proposition 5.1). Thus, we make no restriction on the characteristic of \( k \) unless otherwise stated.

We then use this Tannakian interpretation to prove that under the same hypotheses as above, the projection map \( \text{Bun}^{ss}_{P_{\tilde{\lambda}_P}} \to \text{Bun}_G \) is a monomorphism, which will in turn immediately imply the assertion of part (b) of Theorem 2.1 in Sect. 6 below. Finally, we give a quick proof of the comparison theorem in characteristic 0, so that a reader only interested in the case of characteristic 0 can skip Sect. 4 altogether.
5.1 Tannakian interpretation of the canonical reduction

In this section, we give an interpretation of the canonical reduction on the level of the associated subbundles $\kappa^\lambda$ from Sect. 3.2.3 above. To do so, we first use the slope maps $\phi_P$ from Sect. 2.1.3 above to construct the following filtrations:

5.1.1 Filtrations on representations of highest weight $\lambda$

Let $V$ be any $G$-representation of highest weight $\lambda \in \Lambda^+_G$ and let $V = \bigoplus_v V[v]$ be its weight decomposition. Furthermore, fix a parabolic $P$ and an element $\hat{\lambda}_P \in \check{\Lambda}_{G,P}$. We define a filtration $V_\bullet$ on the vector space $V$ which depends on the element $\hat{\lambda}_P$ as follows.

For any rational number $q \in \mathbb{Q}$, we define the subspace $V_q$ as the sum of weight spaces $V_q := \bigoplus_{\langle \phi_P(\hat{\lambda}_P), v \rangle \geq q} V[v]$.

Clearly, $V_{q'} \subseteq V_q$ whenever $q' \geq q$. We will consider the subspaces $V_q$ only for the finitely many $q \in \mathbb{Q}$ where a jump occurs, i.e., only for those $q$ such that $V_{q'} \subsetneq V_q$ for all $q' > q$. Let $q_0$ be the smallest and $q_1$ the largest rational number occurring among such $q$. Then $V_{q_1}$ is the smallest nonzero filtration step, and $V_{q_0}$ equals $V$.

We will in fact consider the filtration $V_\bullet$ only in the following case:

Lemma 5.1 Suppose $\hat{\lambda}_P \in \check{\Lambda}_{G,P}$ is dominant $P$-regular. Then the filtration $V_\bullet$ is a filtration of $V$ by $P$-subrepresentations. Furthermore, the unipotent radical $U(P)$ acts trivially on each successive quotient $\text{gr}_q V_\bullet = \bigoplus_{\langle \phi_P(\hat{\lambda}_P), v \rangle = q} V[v]$, and hence the $P$-action on each such quotient descends to an action of the Levi $M$. Finally, the smallest step of the filtration $V_{q_1}$ is equal to the subspace $V[\lambda + \mathbb{Z}R_M]$ of $V$.

Proof It suffices to check the first two assertions on the level of the algebras of distributions $\text{Dist}(P)$ and $\text{Dist}(U(P))$, respectively, in which case they are easily verified using the fact that $\hat{\lambda}_P$ is dominant $P$-regular. As $\lambda \geq v$ for all weights $v$ of $V$, the last assertion of the lemma follows from $\hat{\lambda}_P$ being dominant $P$-regular as well. \hfill $\square$

5.1.2 Filtrations on associated bundles

The filtrations of the previous section induce filtrations on associated bundles in the following setting. Let $S$ be any scheme over $k$ and let $F_G \in \text{Bun}_G(S)$ be a $G$-bundle on $X \times S$. Let furthermore $F_P \in \text{Bun}_{P,\hat{\lambda}_P}(S)$ be a reduction of $F_G$ to $P$ on $X \times S$
such that $\tilde{\lambda}_P$ is dominant $P$-regular, and as before let $F_M$ denote its corresponding Levi bundle. Then, by twisting the $P$-subrepresentations $V_q$ above by $F_P$, we obtain a filtration $V_{\bullet, F_P}$ of the vector bundle $V_{FG}$ by subbundles

$$0 \neq V[\lambda + \mathbb{Z}R_M]_{F_M} = V_{q_1, F_P} \subset \cdots \subset V_{q, F_P} \subset \cdots \subset V_{q_0, F_P} = V_{FG}.$$  

We will use this filtration in the case of an arbitrary base scheme $S$ over $k$ in Sect. 5.2 below. Before doing so, we specialize to the case $S = \text{Spec}(k)$ and prove the above-mentioned Tannakian interpretation of the canonical reduction:

**Proposition 5.1** Assume that the characteristic of the ground field $k$ is 0, or assume that $P=B$. Let $F_P$ be a reduction to $P$ of a $G$-bundle $F_G$ on the curve $X$ and suppose that $F_P$ lies in $\text{Bun}^{ss}_{P, \tilde{\lambda}_P}$ for $\tilde{\lambda}_P$ dominant $P$-regular. Then the filtration $V_{\bullet, F_P}$ of $V_{FG}$ is the Harder–Narasimhan filtration. In particular, the subbundle

$$\kappa_{\tilde{\lambda}} : V[\lambda + \mathbb{Z}R_M]_{F_M} \hookrightarrow V_{FG}$$

from Sect. 3.2.3 is precisely the maximal destabilizing subsheaf of $V_{FG}$.

**Proof** Since twisting by a bundle is an exact operation and since each $\text{gr}_q V_{\bullet}$ is a representation of the Levi quotient $M$ by Lemma 5.1, we see that

$$\text{gr}_q (V_{\bullet, F_P}) = (\text{gr}_q V_{\bullet})_{F_M}.$$ 

To determine its slope, we apply Proposition 3.2(a) to the group $M$, the $M$-representation

$$\text{gr}_q V_{\bullet} = \bigoplus_{\langle \phi_P(\tilde{\lambda}_P), \nu \rangle = q} V[\nu],$$

and the $M$-bundle $F_M$. Namely, we compute

$$\deg((\text{gr}_q V_{\bullet})_{F_M}) = \left\langle \phi_P(\tilde{\lambda}_P), \sum_{\nu} m_{\nu} \nu \right\rangle = q \cdot \sum_{\nu} m_{\nu} \nu.$$ 

It follows that

$$\mu((\text{gr}_q V_{\bullet})_{F_M}) = q$$

and hence in particular

$$\mu((\text{gr}_q V_{\bullet})_{F_M}) > \mu((\text{gr}_q V_{\bullet'})_{F_M})$$

whenever $q > q'$. This proves the first property of the Harder–Narasimhan filtration (see Sect. 2.4.2).
To verify the second property, we need to show that the subquotients \((\text{gr}_q V_\bullet)_{FM}\) of the filtration are semistable vector bundles. To prove this, we will use the assumption that either the characteristic of the base field \(k\) is 0, or that \(P = B\). First, in both cases, the category of finite-dimensional representations of the Levi \(M\) is semisimple, and thus we can decompose the \(M\)-representation \(\text{gr}_q V_\bullet\) into irreducible components \((\text{gr}_q V_\bullet)_i\). Then the exact same computation as above shows that each individual associated vector bundle \(((\text{gr}_q V_\bullet)_i)_{FM}\) has slope \(q\).

Since a direct sum of semistable vector bundles of the same slope is again semistable, it now suffices to prove that each summand \(((\text{gr}_q V_\bullet)_i)_{FM}\) is semistable. In the case \(P = B\), every irreducible representation of \(M = T\) is 1-dimensional. Thus, the summands are line bundles and hence automatically semistable. In the case that the characteristic of \(k\) is 0 the semistability of the summands follows by applying the following well-known proposition to the groups \(M\) and \(\text{GL}((\text{gr}_q V_\bullet)_i)\).

**Proposition 5.2** Let \(H_1\) and \(H_2\) be reductive groups over an algebraically closed field \(k\) of characteristic 0 and let \(H_1 \to H_2\) be a homomorphism of algebraic groups which maps \(Z_0(H_1)\) to \(Z_0(H_2)\). Let \(F_{H_1}\) be a semistable \(H_1\)-bundle on a smooth complete curve \(X\) over \(k\). Then the \(H_2\)-bundle \(F_{H_2}\) obtained from \(F_{H_1}\) by extension of structure group is again semistable.

**Proof** See, for example, [16, Theorem 3.18].

**5.1.3 Remark**

In view of Lemma 3.5(b) and Proposition 3.3, taking the representations \(V\) to be the Weyl modules \(V^\lambda\) in Proposition 5.1 immediately implies the uniqueness of the canonical reduction in characteristic 0. In fact, in Sect. 5.3 below, we deduce from Proposition 5.1 the comparison theorem, a stronger result.

**5.2 The canonical reduction in families**

Under the assumption that either \(P = B\) or that the characteristic of \(k\) is 0 as above, we now prove that the canonical reduction is also unique “in families.” Recall first that a morphism of algebraic stacks is a *monomorphism* if it is representable by algebraic spaces and becomes a monomorphism of algebraic spaces after any base change to an algebraic space. One can check that a morphism of algebraic stacks \(\mathcal{X} \to \mathcal{Y}\) is a monomorphism if and only if for every scheme \(S\) the functor \(\mathcal{X}(S) \to \mathcal{Y}(S)\) is fully faithful. Then the result is:

**Proposition 5.3** Assume that the characteristic of \(k\) is 0 or that \(P = B\), and let \(\breve{\lambda}_P \in \breve{\Lambda}_{G, P}\) be dominant \(P\)-regular. Then the projection map

\[
p_P : \text{Bun}_{ss}^{\mathcal{X}} \to \text{Bun}_G
\]

is a monomorphism.
Proof Let $F_P \in \text{Bun}^{ss}_{P, \lambda_P}(S)$ and $\tilde{F}_P \in \text{Bun}^{ss}_{P, \lambda_P}(S)$ be two reductions to $P$ of the same $G$-bundle $F_G$ on $X \times S$. We need to prove that $F_P = \tilde{F}_P$. We first show that the induced subbundles

$$V[\lambda + \mathbb{Z}R_M]_{F_M} \hookrightarrow V_{F_G} \hookleftarrow V[\lambda + \mathbb{Z}R_M]_{\tilde{F}_M}$$

are equal for all $G$-representations $V$ of highest weight $\lambda \in \Lambda_G^+$, where the notation follows Sect. 3.2.3 above.

To prove the equality of the two subbundles, we prove the stronger assertion that the vector space of homomorphisms from $V[\lambda + \mathbb{Z}R_M]_{\tilde{F}_M}$ to the quotient bundle $V_{F_G}/V[\lambda + \mathbb{Z}R_M]_{F_M}$ is trivial:

$$\text{Hom}_{\mathcal{O}_{X \times S}}\left(V[\lambda + \mathbb{Z}R_M]_{\tilde{F}_M}, V_{F_G}/V[\lambda + \mathbb{Z}R_M]_{F_M}\right) = 0. \ (\star)$$

As this vector space is the space of global sections $H^0(X \times S, E)$ of the vector bundle

$$E := \left(V_{F_G}/V[\lambda + \mathbb{Z}R_M]_{F_M}\right) \otimes \left(V[\lambda + \mathbb{Z}R_M]_{\tilde{F}_M}\right)^*$$

on $X \times S$, we can prove $(\star)$ by showing that the pushforward $p_*(E)$ of $E$ along the projection $p : X \times S \to S$ is equal to 0. By the theorem on cohomology and base change, the latter can be checked on the fibers of the projection $p$, i.e., by showing that $H^0(X \times \tilde{s}, E_{\tilde{s}}) = 0$ for every geometric point $\tilde{s}$ of $S$. We thus only need to prove $(\star)$ for $S = \text{Spec}(k)$.

In the case $S = \text{Spec}(k)$, Proposition 5.1 shows that the subbundles $V[\lambda + \mathbb{Z}R_M]_{F_M}$ and $V[\lambda + \mathbb{Z}R_M]_{\tilde{F}_M}$ agree as they are both equal to the maximal destabilizing subsheaf of $V_{F_G}$. Now $(\star)$ follows from the next lemma, Lemma 5.2 below.

Having established the equality of the two subbundles, the assertion of the proposition follows by taking the representations $V$ to be the Weyl modules $V^\lambda$ and using Proposition 3.3 above. \qed

**Lemma 5.2** Let $F$ be a vector bundle on $X$ and let $D \subset F$ denote the maximal destabilizing subsheaf of $F$. Then the vector space of homomorphisms from $D$ to the quotient $F/D$ is trivial:

$$\text{Hom}_{\mathcal{O}_X}(D, F/D) = 0.$$ 

**Proof** Assume there exists a nonzero map $D \to F/D$ and let $H$ denote its image. Then the maximal destabilizing subsheaf $K$ of the quotient $F/D$ must have strictly smaller slope than $D$, and hence

$$\mu(H) \leq \mu(K) < \mu(D).$$

But since $H$ is also a quotient of $D$, this contradicts the semistability of $D$ and finishes the proof. \qed
5.3 Proof of the comparison theorem in characteristic 0

We conclude this section by demonstrating how in characteristic 0 the comparison theorem, Theorem 4.1, follows directly from Proposition 5.1 above.

**Proof of Theorem 4.1 in characteristic 0**

As in the proof of Proposition 5.3 above, consider the inclusions of subbundles $\kappa_{\lambda,1}, \kappa_{\lambda,2}$ induced by $FP_1$ and $FP_2$, for any $G$-representation $V$ of highest weight $\lambda \in \Lambda_G^+$. Then, since the maximal destabilizing subsheaf of a vector bundle has maximal slope among all subbundles, the inequality

$$\phi_{P_1}(\tilde{\lambda}, P_1) \geq \phi_{P_2}(\tilde{\lambda}, P_2)$$

follows immediately from Proposition 5.1 together with Proposition 3.2(c) and Lemma 3.2(a).

Next, assume that $\phi_{P_1}(\tilde{\lambda}, P_1) = \phi_{P_2}(\tilde{\lambda}, P_2)$. As $\tilde{\lambda}$ is dominant $P_1$-regular this equality forces $I_{M_2} \subset I_{M_1}$ and thus $P_2 \subset P_1$. We need to show that the $P_1$-bundle $\tilde{FP}_1$ obtained from $FP_2$ by extension of structure group along $P_2 \subset P_1$ agrees with the reduction $FP_1$. As before it suffices to show that the subbundles

$$V[\lambda + \mathbb{Z} R_{M_1}]_{\tilde{F}M_1} \hookrightarrow V_{F_G} \hookleftarrow V[\lambda + \mathbb{Z} R_{M_1}]_{F_{M_1}}$$

agree for any $G$-representation $V$ of highest weight $\lambda \in \Lambda_G^+$, since we can then take the representations $V$ to be the Weyl modules $V^\lambda$ and invoke Proposition 3.3 above.

To prove this, note first that the slope of $\tilde{FP}_1$ is again equal to $\phi_{P_1}(\tilde{\lambda}, P_1)$, and let $V_\bullet$ denote the filtration of $V$ from Sect. 5.1.1 corresponding to the element $\phi_{P_1}(\tilde{\lambda}, P_1)$. By Lemma 5.1, the equality of the above subbundles will follow once we show that in fact all terms of the filtrations $(V_\bullet)_{\tilde{F}P_1}$ and $(V_\bullet)_{FP_1}$ of $V_{F_G}$ agree. To see the latter, recall from the proof of Proposition 5.1 that

$$\mu((\text{gr}_q V_\bullet)_{\tilde{F}M_1}) = q = \mu((\text{gr}_q V_\bullet)_{F_{M_1}})$$

for all $q$. This and Proposition 5.1 together show that the two filtrations satisfy the hypotheses of the next lemma and are thus equal.

**Lemma 5.3**

Let $E$ be a vector bundle on $X$ and let

$$E_\bullet = (0 \neq E_1 \subset E_2 \subset \cdots \subset E_m = E)$$

be its Harder–Narasimhan filtration. Let

$$F_\bullet = (0 \neq F_1 \subset F_2 \subset \cdots \subset F_m = E)$$

be another filtration of $E$ by subbundles, with the same numerical data as $E^\bullet$, i.e., of the same length as $E^\bullet$ and such that $\text{rk}(F_i) = \text{rk}(E_i)$ and $\mu(F_i / F_{i+1}) = \mu(E_i / E_{i+1})$ for all $i$. Then $F^\bullet = E^\bullet$.

**Proof**

Follows easily by repeatedly using the properties of the maximal destabilizing subsheaf stated in Sect. 2.4.2.
6 Construction of the strata and conclusion of proof

In this section, we first define the strata $Bun^P, \tilde{\lambda}_P$ using Drinfeld’s compactifications $\overline{Bun}_P$ and then prove the main theorem, Theorem 2.1, using the results of the previous sections. We begin by reviewing:

6.1 Drinfeld’s compactification $\overline{Bun}_P$

6.1.1 Overview

In [6], Gaitsgory and Braverman construct, following Drinfeld, a relative compactification $\overline{Bun}_P$ of $Bun_P$ along the fibers of the projection map

$$p_P : Bun_P \rightarrow Bun_G$$

which will be used in the present article to construct the strata $Bun^P, \tilde{\lambda}_P$.

Below we summarize the properties of $\overline{Bun}_P$ relevant to the proof of Theorem 2.1. A more extensive discussion can be found in [5, 6] and also in Sect. 7 of the present article. There we also show how $\overline{Bun}_P$ can be constructed for an arbitrary reductive group $G$, whereas the definition in [6] yields the “correct” object only in the case when the derived group $[G, G]$ of $G$ is simply connected.

6.1.2 First properties

Drinfeld’s compactification $\overline{Bun}_P$ is an algebraic stack containing $Bun_P$ as an open dense substack and is equipped with a schematic map

$$\overline{p}_P : \overline{Bun}_P \rightarrow Bun_G$$

which extends the projection $p_P$ and which is proper after restriction to any connected component of $\overline{Bun}_P$. Furthermore, the inclusion of $Bun_P$ into $\overline{Bun}_P$ induces a bijection on the level of connected components:

$$\pi_0(\overline{Bun}_P) = \pi_0(Bun_P) = \tilde{\Lambda}_{G,P}$$

6.1.3 The maps $j_{\tilde{\theta}}$

The stack $\overline{Bun}_P$ possesses a natural stratification which we now describe, for a fixed component $\overline{Bun}_{P, \tilde{\lambda}_P}$. Given an element

$$\tilde{\theta} = \sum_{i \in \mathcal{I} \setminus \mathcal{I}_M} n_i \tilde{\alpha}_i$$

of $\tilde{\Lambda}^{pos}_{G,P}$, we define $X^{\tilde{\theta}}$ to be the partially symmetrized power of the curve
The Harder–Narasimhan stratification of $\text{Bun}_G$

\[
X^{\tilde{\theta}} := \prod_{i \in I \setminus I_M} X^{(n_i)}.
\]

Then, for each $\tilde{\theta}$, there exists a naturally defined locally closed immersion

\[
j_{\tilde{\theta}} : X^{\tilde{\theta}} \times \text{Bun}_{P, \hat{\lambda}_P + \tilde{\theta}} \hookrightarrow \overline{\text{Bun}}_{P, \hat{\lambda}_P}
\]

which renders the diagram

\[
\begin{array}{ccc}
X^{\tilde{\theta}} \times \text{Bun}_{P, \hat{\lambda}_P + \tilde{\theta}} & \xrightarrow{j_{\tilde{\theta}}} & \overline{\text{Bun}}_{P, \hat{\lambda}_P} \\
pr_2 & & \overline{p}_P \\
\text{Bun}_{P, \hat{\lambda}_P + \tilde{\theta}} & \xrightarrow{p_P} & \text{Bun}_G
\end{array}
\]

commutative. If $\tilde{\theta} = 0$, the map $j_0$ is just the inclusion $\text{Bun}_{P, \hat{\lambda}_P} \hookrightarrow \overline{\text{Bun}}_{P, \hat{\lambda}_P}$.

6.1.4 The stratification

The collection of locally closed substacks $\overline{\text{Bun}}_{P, \hat{\lambda}_P}$ corresponding to the immersions $j_{\tilde{\theta}}$ defines a stratification of $\overline{\text{Bun}}_{P, \hat{\lambda}_P}$ in the sense that on the level of $k$-points $\overline{\text{Bun}}_{P, \hat{\lambda}_P}$ is equal to the disjoint union

\[
\overline{\text{Bun}}_{P, \hat{\lambda}_P} = \bigcup_{\tilde{\theta} \in \tilde{\Lambda}_{G,P}^{\text{pos}}} \overline{\text{Bun}}_{P, \hat{\lambda}_P}^{\tilde{\theta}}.
\]

6.1.5 The semistable locus

In the next proposition, we demonstrate that Drinfeld’s compactifications are well suited to the study of semistability by giving a quick proof of the well-known fact that the semistable locus in $\text{Bun}_G$ is open. This yields precisely the open strata of the stratification in Theorem 2.1 and will also be used in the construction of all other strata via Drinfeld’s compactifications in the next section.

**Proposition 6.1** The semistable locus $\text{Bun}^{ss}_G$ is open in $\text{Bun}_G$.

**Proof** We show that $\text{Bun}^{ss}_{G, \hat{\lambda}_G}$ is open in $\text{Bun}_{G, \hat{\lambda}_G}$ for a fixed $\hat{\lambda}_G \in \hat{\Lambda}_{G,G}$. For every $i \in I$ let $\Upsilon_i$ denote the subset of $\hat{\Lambda}_{G,P_i}$ consisting of those elements $\hat{\lambda}_{P_i}$ that map to $\hat{\lambda}_G$ under the natural projection and which satisfy that

\[
\phi_{P_i}(\hat{\lambda}_{P_i}) \not\leq \phi_G(\hat{\lambda}_G).
\]
Then, by the definition of semistability in Sect. 2.2.3(c), a \( G \)-bundle in \( \text{Bun}_{G,\lambda} \) is semistable if and only if it does not lie in the infinite union of \( k \)-points

\[
\bigcup_{i \in I} \bigcup_{\lambda_i \in \Upsilon_i} p_{P_i}(\text{Bun}_{P_i,\lambda_i}).
\]

We prove the proposition by showing that this union is the collection of \( k \)-points of a closed substack of \( \text{Bun}_G \).

We first describe the sets \( \Upsilon_i \) in a different way, taking into account the fact that the parabolics \( P_i \) are maximal. Namely, we claim that an element \( \tilde{\lambda}_{P_i} \in \tilde{\Lambda}^1_G, P_i \) that maps to \( \tilde{\lambda}_G \) under the natural projection satisfies the above condition

\[
\phi_{P_i}(\tilde{\lambda}_{P_i}) \not\leq \phi_G(\tilde{\lambda}_G)
\]

in \( \tilde{\Lambda}^\mathbb{Q}_G \) if and only if the inequality

\[
\tilde{\lambda}_{P_i} > \phi_G(\tilde{\lambda}_G)
\]

holds in \( \tilde{\Lambda}^\mathbb{Q}_{G,P_i} \), i.e., if and only if

\[
\tilde{\lambda}_{P_i} \in \left( \phi_G(\tilde{\lambda}_G) + \tilde{\Lambda}^\mathbb{Q}_{G,P_i}^{\text{pos}} \right).
\]

This assertion follows easily from part (a) of Proposition 3.1, and the fact that the positive cone \( \tilde{\Lambda}^\mathbb{Q}_{G,P_i}^{\text{pos}} \) is generated by the single coroot \( \alpha_i \).

The assertion implies that each subset \( \Upsilon_i \subset \tilde{\Lambda}^\mathbb{Q}_{G,P_i} \) contains a unique minimal element \( \tilde{\mu}_{P_i} \) with respect to the partial order \( \leq \) on \( \tilde{\Lambda}^\mathbb{Q}_{G,P_i} \) induced by \( \tilde{\Lambda}^\mathbb{Q}_{G,P_i}^{\text{pos}} \). We can then write the set \( \Upsilon_i \) as

\[
\Upsilon_i = \tilde{\mu}_{P_i} + \tilde{\Lambda}^{\text{pos}}_{G,P_i}.
\]

We now claim that the finite union

\[
\bigcup_{i \in I} \tilde{p}_{P_i}(\text{Bun}_{P_i,\tilde{\mu}_{P_i}})
\]

naturally forms a closed substack of \( \text{Bun}_G \) which exhibits the desired collection of \( k \)-points. Indeed, since the projection map \( \tilde{p}_{P_i} \) is proper when restricted to any connected component of \( \text{Bun}_{P_i} \), the above union naturally carries a closed substack structure. To see that it possesses the desired collection of \( k \)-points, observe first that the stratification result 6.1.4 and the commutativity of diagram 6.1 together imply that

\[
\tilde{p}_{P_i}(\text{Bun}_{P_i,\tilde{\mu}_{P_i}}) = \bigcup_{\tilde{\theta} \in \tilde{\Lambda}^{\text{pos}}_{G,P_i}} p_{P_i}(\text{Bun}_{P_i,\tilde{\mu}_{P_i}+\tilde{\theta}})
\]
on the level of \( k \)-points. Combined with the above fact that \( \Upsilon_i = \tilde{\mu}_{P_i} + \tilde{\lambda}_{G,P_i}^{\text{pos}} \), we conclude that

\[
\bigcup_{i \in I} \tilde{p}_{P_i}(\Bun_{P_i,\tilde{\mu}_{P_i}}) = \bigcup_{i \in I} \bigcup_{\tilde{\lambda}_{P_i} \in \Upsilon_i} p_{P_i}(\Bun_{P_i,\tilde{\lambda}_{P_i}})
\]

on \( k \)-points, finishing the proof. \( \square \)

6.2 Construction of the strata and proof of (a), (b)

In this section, we define the substacks \( \Bun_{P,\tilde{\lambda}_P} \) that form the strata of the stratification in Theorem 2.1, and then complete the proofs of parts (a) and (b) of the theorem.

6.2.1 Definition of the substacks \( \Bun_{P,\tilde{\lambda}_P} \)

Let \( \tilde{\lambda}_P \in \tilde{\Lambda}_{G,P} \) be dominant \( P \)-regular, and consider Drinfeld’s relative compactification

\[
\bar{p}_P : \Bun_P \longrightarrow \Bun_G
\]

from Sect. 6.1. Since the map \( \bar{p}_P \) is proper when restricted to any connected component and since \( \Bun_{P,\tilde{\lambda}_P}^{ss} \) is open in \( \Bun_P \) by Proposition 6.1 applied to the Levi \( M \), the images of \( \Bun_{P,\tilde{\lambda}_P} \) and \( \Bun_P \setminus \Bun_{P,\tilde{\lambda}_P}^{ss} \) under \( \bar{p}_P \) are closed substacks of \( \Bun_G \). We then define \( \Bun_{G,\tilde{\lambda}_P}^{P,\tilde{\lambda}_P} \) to be the locally closed substack

\[
\Bun_{G,\tilde{\lambda}_P}^{P,\tilde{\lambda}_P} := \bar{p}_P \left( \Bun_{P,\tilde{\lambda}_P} \right) \setminus \bar{p}_P \left( \Bun_P \setminus \Bun_{P,\tilde{\lambda}_P}^{ss} \right).
\]

6.2.2 Proof of parts (a) and (b) of the main theorem

We now use the properties of Drinfeld’s compactifications from Sect. 6.1 above and the comparison theorem, Theorem 4.1, to prove part (a) of the main theorem.

**Proof of Theorem 2.1 (a)** Let \( \mathcal{U} \) be the open substack of \( \Bun_{P,\tilde{\lambda}_P} \) defined as the following fiber product:

\[
\begin{array}{ccc}
\mathcal{U} & \longrightarrow & \Bun_{P,\tilde{\lambda}_P} \\
\downarrow & & \downarrow \bar{p}_P \\
\Bun_{G,\tilde{\lambda}_P}^{P,\tilde{\lambda}_P} & \longleftarrow & \bar{p}_P(\Bun_{P,\tilde{\lambda}_P})
\end{array}
\]

We claim that in fact \( \mathcal{U} = \Bun_{P,\tilde{\lambda}_P}^{ss} \). To prove this, note first that both \( \Bun_{P,\tilde{\lambda}_P}^{ss} \) and \( \mathcal{U} \) are open substacks of \( \Bun_{P,\tilde{\lambda}_P} \). Indeed, for \( \Bun_{P,\tilde{\lambda}_P}^{ss} \), this follows from Proposition
applied to the Levi $M$, and, for $\mathcal{U}$, this holds by construction. Hence, to prove the above claim, it suffices to show that $\mathcal{U}$ and $\text{Bun}_{P,\lambda_p}^{ss}$ coincide on the level of $k$-points.

The inclusion $\mathcal{U} \subset \text{Bun}_{P,\lambda_p}^{ss}$ is immediate from the definition of $\text{Bun}_{G,\lambda_p}^{P,\lambda_p}$. We prove the converse by showing that given a $k$-point of $\text{Bun}_{P,\lambda_p}^{ss}$, there exists no other $k$-point of $\text{Bun}_{P,\lambda_p}^{\lambda_p}$ with the same image under $\mathfrak{p}_p$. By the stratification of $\text{Bun}_{P,\lambda_p}^{\lambda_p}$ in Sect. 6.1.4 and the commutative diagram 6.1, the last assertion is equivalent to the following claim: If a $G$-bundle on $X$ admits a reduction to $P$ lying in $\text{Bun}_{P,\lambda_p}^{ss}$, then this reduction is in fact the only reduction to $P$ which lies in any of the connected components $\text{Bun}_{P,\lambda_p + \tilde{\theta}}^{P,\lambda_p}$ for $\tilde{\theta} \in \tilde{\Lambda}_{G,p}^{\text{pos}}$.

To prove this claim, note first that for any nonzero $\tilde{\theta} \in \tilde{\Lambda}_{G,p}^{\text{pos}}$ we have

$$\phi_p(\tilde{\lambda}_p + \tilde{\theta}) > \phi_p(\tilde{\lambda}_p)$$

by part (a) of Proposition 3.1. Thus, the claim follows from the comparison theorem, Theorem 4.1, completing the proof that $\mathcal{U} = \text{Bun}_{P,\lambda_p}^{ss}$.

Next, the fact that $\mathcal{U} = \text{Bun}_{P,\lambda_p}^{ss}$ implies that the map

$$\text{Bun}_{P,\lambda_p}^{ss} \longrightarrow \text{Bun}_{G,\lambda_p}^{P,\lambda_p}$$

is schematic and proper as it is the base change of a schematic and proper map. Furthermore, Theorem 4.1 shows that the canonical reduction is unique, and hence every geometric fiber of this map must consist of precisely one point as a topological space. But since the map is already proper, it has to be finite and is hence an almost-isomorphism.

Again since $\mathcal{U} = \text{Bun}_{P,\lambda_p}^{ss}$ in the fiber square above, the substack $\text{Bun}_{G,\lambda_p}^{P,\lambda_p}$ is the stack-theoretic image of the stack $\text{Bun}_{P,\lambda_p}^{ss}$. As the latter is reduced and quasi-compact, the same holds true for the former. Finally, the uniqueness of the substacks $\text{Bun}_{G,\lambda_p}^{P,\lambda_p}$ is immediate from the required properties. 

Part (b) of the main theorem now follows immediately from part (a) and the results of Sect. 5 above:

Proof of Theorem 2.1 (b) The map is proper by part (a) and a monomorphism by Proposition 5.3, and hence a closed immersion. As both stacks are reduced, this and part (a) together imply that the map is an isomorphism.

6.3 Existence of the canonical reduction and proof of (c)

By Theorem 4.1, the canonical reduction is unique. Next, we prove its existence and thus establish part (c) of the main theorem. In fact, the extremal property of the canonical reduction established in Theorem 4.1 suggests a strategy for its construction, which we now carry out.
As usual let \( (\omega_i)_{i \in \mathcal{I}} \) be the basis of \( \Lambda^Q_{[G,G]} \) which is dual to the basis \( (\check{\alpha}_i)_{i \in \mathcal{I}} \) of \( \check{\Lambda}^Q_{[G,G]} \), and define \( (\check{\omega}_i)_{i \in \mathcal{I}} \) analogously. Furthermore, for each \( i \in \mathcal{I} \), let \( \lambda_i \in \Lambda^+_G \) be some fixed integral multiple of \( \omega_i \).

**Proposition 6.2** Every \( G \)-bundle on \( X \) possesses a unique canonical reduction to a unique parabolic subgroup of \( G \).

**Proof** We only need to prove existence, as uniqueness is immediate from Theorem 4.1. Given a \( G \)-bundle \( F_G \) on \( X \), we associate to every reduction \( F_P \in \text{Bun}_P,\check{\lambda}_P \) of \( F_G \) its slope, i.e., the element \( \phi_P(\check{\lambda}_P, \lambda_i) \in \check{\Lambda}^Q_G \). Ranging over all possible reductions of \( F_G \) to all parabolics of \( G \) we obtain a subset of \( \check{\Lambda}^Q_G \) which we denote by \( \Phi \).

We claim that \( \Phi \) contains maximal elements with respect to the usual partial ordering on \( \check{\Lambda}^Q_G \) given by \( \leq \). To prove this, it suffices to show that for each weight \( \lambda_i \) as defined above the numbers \( \langle \phi_P(\check{\lambda}_P), \lambda_i \rangle \) remain bounded from above as \( \phi_P(\check{\lambda}_P) \) ranges over \( \Phi \). To find such a bound, first recall that for any vector bundle on the curve \( X \) the set of all possible slopes of subbundles is bounded from above. Now, since every reduction \( F_P \) of \( F_G \) induces a subbundle

\[
V^{\lambda_i} \cdot [\lambda_i + \mathbb{Z} \mathcal{R}_M]_{F_M} \hookrightarrow V^{\check{\lambda}_i}_{F_G}
\]

whose slope is

\[
\mu \left( V^{\lambda_i} \cdot [\lambda_i + \mathbb{Z} \mathcal{R}_M]_{F_M} \right) = \left\langle \phi_P(\check{\lambda}_P), \lambda_i \right\rangle
\]

by Proposition 3.2(c), applying the previous remark to the vector bundle \( V^{\lambda_i}_{F_G} \) yields an upper bound on the values of all \( \langle \phi_P(\check{\lambda}_P), \lambda_i \rangle \).

Let now \( \check{\tau} \) be some maximal element of \( \Phi \). Among all reductions \( (P, F_P) \) of \( F_G \) such that \( \phi_P(\check{\lambda}_P) = \check{\tau} \), we choose one with \( P \) of maximal dimension. We then claim that this pair \( (P, F_P) \) has the desired properties.

We first show that \( F_P \) lies in \( \text{Bun}^\Phi_P \). By Lemma 3.4 it suffices to show that if \( F_P \) admits a reduction \( F_{P'} \in \text{Bun}_{P',\check{\lambda}_{P'}} \) to any maximal proper subparabolic \( P' \subset P \), then \( \phi_{P'}(\check{\lambda}_{P'}) \leq \phi_P(\check{\lambda}_P) \). To prove this, let \( \mathcal{I}_M \setminus \{i\} \) be the subset of \( \mathcal{I}_M \) corresponding to \( P' \). Using Lemma 3.1, we find that

\[
\phi_{P'}(\check{\lambda}_{P'}) - \phi_P(\check{\lambda}_P) = b \cdot \phi_P(\check{a}_i)
\]

for some rational number \( b \in \mathbb{Q} \). But, since the element \( \phi_{P'}(\check{a}_i) \) lies in \( \check{\Lambda}^Q_G \) by part (a) of Proposition 3.1, the choice of \( \phi_P(\check{\lambda}_P) \) as maximal implies that \( b \leq 0 \) and thus \( \phi_{P'}(\check{\lambda}_{P'}) \leq \phi_P(\check{\lambda}_P) \) as desired.

Next, we show that \( \check{\lambda}_P \) is dominant \( P \)-regular, i.e., given any \( j \in \mathcal{I} \setminus \mathcal{I}_M \) we need to show that \( \langle \phi_P(\check{\lambda}_P), \check{\alpha}_j \rangle > 0 \). To do so, let \( P'' \) be the parabolic corresponding to
the subset $I_M \cup \{j\}$ and let $F_p'' \in \text{Bun}_p''$ be obtained from $F_p$ by extension of structure group along the inclusion $P \subseteq P''$. Then, similarly to above we find that
\[
\phi_p(\tilde{\lambda}_p) - \phi_p''(\tilde{\lambda}_p'') = c \cdot \phi_p(\tilde{\alpha}_j)
\]
for some $c \in \mathbb{Q}$. We claim that $c > 0$. Indeed, since $\phi_p(\tilde{\alpha}_j)$ lies in $\tilde{\Lambda}_G^{\mathbb{Q}, \text{pos}}$ by part (a) of Proposition 3.1, the choice of $\phi_p(\tilde{\lambda}_p)$ as maximal forces $c \geq 0$, and since $P \subsetneq P''$ the choice of $P$ as of maximal dimension forces $c > 0$.

We now pair both sides of the last equation with the root $\alpha_j$. Since $\langle \phi_p''(\tilde{\lambda}_p''), \alpha_j \rangle = 0$ by construction and since $\langle \phi_p(\tilde{\alpha}_j), \alpha_j \rangle > 0$ by part (b) of Proposition 3.1, we conclude that $\langle \phi_p(\tilde{\lambda}_p), \alpha_j \rangle > 0$, establishing that $\tilde{\lambda}_p$ is dominant $P$-regular. 

Tautologically, this yields:

Proof of Theorem 2.1 (c) Combine part (a) of Theorem 2.1 with Proposition 6.2 above.

6.4 Proof of (d), (e), (f), (g)

We now establish the remaining parts of the main theorem.

Proof of Theorem 2.1 (d) Immediate from part (a) and Theorem 4.1.

Proof of Theorem 2.1 (e) Using the notation from Sect. 6.3.1 above, we construct an open cover $(U_n)_{n \in \mathbb{Z}_{\geq 0}}$ of a fixed connected component $\text{Bun}_{G, \tilde{\lambda}_G}$ as follows. Choose an ample line bundle $O(1)$ on the curve $X$ and fix for each $i \in I$ a $G$-representation of highest weight $\lambda_i$. For notational concreteness only, let us choose the Weyl modules $V^{\lambda_i}$.

We then define the open substack $U_n$ of $\text{Bun}_{G, \tilde{\lambda}_G}$ as the locus where, for every $i \in I$, the $n$th twist $V_{FG}^{\lambda_i}(n)$ of the associated vector bundle $V_{FG}^{\lambda_i}$ is generated by global sections. We now verify that this cover satisfies the desired properties.

First, it is clear that the $U_n$ cover all of $\text{Bun}_{G, \tilde{\lambda}_G}$. Next, fix $n$ and let $\Psi$ be the subset of $\tilde{\Lambda}_G^{\mathbb{Q}, \text{pos}}$ consisting of all elements $\phi_p(\tilde{\lambda}_p)$ such that $\text{Bun}_{G, \tilde{\lambda}_p}$ meets $U_n$. Then, we need to show that the subset $\Psi$ is finite, or equivalently, bounded. Since by part (a) of Lemma 3.2 every element in $\Psi$ is of the form
\[
\phi_p(\tilde{\lambda}_p) = \sum_{i \in I} c_i \tilde{\omega}_i + \phi_G(\tilde{\lambda}_G)
\]
for certain $c_i \in \mathbb{Q}_{\geq 0}$, it is enough to show that the numbers $c_i$ arising in this way remain bounded from above. Equivalently, it suffices to prove that the numbers $\langle \phi_p(\tilde{\lambda}_p), \lambda_i \rangle$ remain bounded from above as $\phi_p(\tilde{\lambda}_p)$ ranges over $\Psi$.

To prove the latter for a fixed $i \in I$, we first claim that there exists a uniform upper bound on all possible slopes of subbundles of the associated bundles $V_{FG}^{\lambda_i}$ as $F_G$ varies in $U_n$. Indeed, if we denote by $H^0(X, V_{FG}^{\lambda_i}(n))$ the space of global sections of
the twisted bundle $V_{FG}^{\lambda_i}(n)$, then by definition of $\mathcal{U}_n$ every $V_{FG}^{\lambda_i}$ comes equipped with a surjection

$$H^0(X, V_{FG}^{\lambda_i}(n)) \otimes \mathcal{O}(-n) \rightarrow V_{FG}^{\lambda_i},$$

and thus every quotient bundle of $V_{FG}^{\lambda_i}$ is also a quotient of $H^0(X, V_{FG}^{\lambda_i}(n)) \otimes \mathcal{O}(-n)$. Since the latter vector bundle is semistable of slope $-n$ for any $FG$, we find that $-n$ is a uniform lower bound on all possible slopes of quotient bundles of the $V_{FG}^{\lambda_i}$. But, since the slopes of the bundles $V_{FG}^{\lambda_i}$ are computed as

$$\mu(V_{FG}^{\lambda_i}) = \langle \phi_G(\tilde{\lambda}_G), \lambda_i \rangle$$

by Proposition 3.2(b) and are thus independent of $FG$, the existence of a uniform lower bound on the slopes of quotients is equivalent to the existence of a uniform upper bound on the slopes of subbundles of the $V_{FG}^{\lambda_i}$, as desired.

Let now $N \in \mathbb{Z}$ be such an upper bound. We claim that then

$$\langle \phi_P(\tilde{\lambda}_P), \lambda_i \rangle \leq N$$

for any given $\phi_P(\tilde{\lambda}_P) \in \Psi$. To see this, let $F_G$ be a $G$-bundle in the intersection of $\text{Bun}_G^{P,\tilde{\lambda}_P}$ and $\mathcal{U}_n$. Then by construction the canonical reduction $F_P$ of $F_G$ lies in $\text{Bun}_{P,\tilde{\lambda}_P}$ and induces the subbundle

$$V_{FG}^{\lambda_i}[\lambda_i + \mathbb{Z}R_M]_{FM} \rightarrow V_{FG}^{\lambda_i}.$$

Since the slope of this subbundle is

$$\mu(V_{FG}^{\lambda_i}[\lambda_i + \mathbb{Z}R_M]_{FM}) = \langle \phi_P(\tilde{\lambda}_P), \lambda_i \rangle$$

by Proposition 3.2(c), the previous discussion shows that

$$\langle \phi_P(\tilde{\lambda}_P), \lambda_i \rangle \leq N,$$

finishing the proof. $\square$

**Proof of Theorem 2.1 (f)** By construction the closure of $\text{Bun}_G^{P,\tilde{\lambda}_P}$ in $\text{Bun}_G$ is equal to the image $\overline{p}_P(\text{Bun}_{P,\tilde{\lambda}_P})$. But in light of the commutative diagram 6.1, the stratification 6.1.4 of $\overline{\text{Bun}}_{P,\tilde{\lambda}_P}$ implies that

$$\overline{p}_P \left( \overline{\text{Bun}}_{P,\tilde{\lambda}_P} \right) = \bigcup_{\tilde{\theta} \in \Lambda^\text{pos}_{G,P}} p_P \left( \text{Bun}_{P,\tilde{\lambda}_P+\tilde{\theta}} \right)$$

on the level of $k$-points.
Proof of Theorem 2.1 (g) A counterexample demonstrating the first assertion is given in Sect. 6.5 below. Regarding the second assertion, the formula for the closure of the stratum $\text{Bun}^P_{G,\tilde{\lambda}^P}$ in part (f) of the theorem shows that the stratum $\text{Bun}^P_{G,\tilde{\lambda}^P}$ meets the image $p_P(\text{Bun}_{P,\tilde{\lambda}+\tilde{\theta}})$ for some element $\tilde{\theta} \in \tilde{\Lambda}_{G,P}^{\text{pos}}$. Therefore, part (d) of the theorem implies that

$$\phi_P(\tilde{\lambda}^P) \geq \phi_P(\tilde{\lambda} + \tilde{\theta})$$

and hence also

$$\phi_P(\tilde{\lambda}^P) \geq \phi_P(\tilde{\lambda})$$

by part (a) of Proposition 3.1. As the elements $\tilde{\lambda}^P$ and $\tilde{\lambda}'^P$ map to the same element of $\tilde{\Lambda}_{G,\hat{G}}$, the last inequality in turn forces that $\tilde{\lambda}'^P \geq \tilde{\lambda}^P$ in $\tilde{\Lambda}_{G,\hat{G}}$, i.e., that $\tilde{\lambda}'^P$ lies in the set $\tilde{\lambda}^P + \tilde{\Lambda}_{G,\hat{G}}^{\text{pos}}$. In view of the formula for the closure of the stratum $\text{Bun}^P_{G,\tilde{\lambda}^P}$ in part (f) of the theorem, this yields the claim. \hfill \qed

6.5 An example of strata closure

We conclude this section with an example showing that the closure of a stratum in Theorem 2.1 need not be a union of strata. We continue to use the notation and conventions from Sect. 2.

Assume that the genus of the curve is at least 2. Let $G = \text{GL}_3$ and let $B$ denote the standard Borel subgroup of $\text{GL}_3$. Consider the stratum $\text{Bun}_{G,\lambda}^{B,3,0}$ of the moduli stack $\text{Bun}_{\text{GL}_3}$ of vector bundles of rank 3 on $X$. This stratum consists precisely of those vector bundles $E$ whose Harder–Narasimhan flag

$$0 \neq L \subsetneq F \subsetneq E$$

is complete and satisfies

$$\deg(L) = 2, \quad \deg(F/L) = 1, \quad \text{and} \quad \deg(E/F) = 0.$$  

We claim that the closure $\overline{\text{Bun}_{G,\lambda}^{B,3,0}}$ of this stratum is not a union of strata. To prove this, we show that there exists another stratum which meets this closure but is not contained in it. Namely, we claim that this holds true for the stratum $\text{Bun}_{\text{GL}_3}^{P,(3,0)}$ consisting of those vector bundles $E$ whose Harder–Narasimhan flag is of the form

$$0 \neq L \subsetneq E$$

for $L$ a line bundle with

$$\deg(L) = 3 \quad \text{and} \quad \deg(E/L) = 0.$$
To see this, first observe that by the formula for the strata closure in part (f) of Theorem 2.1 the closure $\text{Bun}_{GL_3}^{B, (2,1,0)}$ consists of precisely those vector bundles $E$ of degree 3 which admit a complete flag

$$0 \neq L \subsetneq F \subsetneq E$$

such that

$$\text{deg}(L) \geq 2 \quad \text{and} \quad \text{deg}(E/F) \leq 0.$$  

This description of the closure shows that, given a line bundle $L_0$ of degree 3, the closure contains the vector bundle $L_0 \oplus \mathcal{O} \oplus \mathcal{O}$. Since the latter bundle also lies in the stratum $\text{Bun}_{GL_3}^{P, (3,0)}$, we see that $\text{Bun}_{GL_3}^{B, (2,1,0)}$ and $\text{Bun}_{GL_3}^{P, (3,0)}$ indeed intersect non-trivially.

To complete the argument, we now construct a vector bundle which lies in the stratum $\text{Bun}_{GL_3}^{P, (3,0)}$ but not in the closure $\text{Bun}_{GL_3}^{B, (2,1,0)}$. Recall first that a vector bundle is called **stable** if every proper subbundle has strictly smaller slope than the bundle itself.

By our assumption on the genus of the curve $X$, we can choose a stable vector bundle $F_0$ of rank 2 and of degree 0. Let $L_0$ again denote a line bundle of degree 3 and consider the direct sum

$$E_0 := L_0 \oplus F_0.$$ 

Then $E_0$ is clearly contained in the stratum $\text{Bun}_{GL_3}^{P, (3,0)}$. To prove that $E_0$ does not lie in the closure $\text{Bun}_{GL_3}^{B, (2,1,0)}$, we need to show that it does not admit a complete flag

$$0 \neq L \subsetneq F \subsetneq E_0$$

such that

$$\text{deg}(L) \geq 2 \quad \text{and} \quad \text{deg}(E_0/F) \leq 0.$$ 

First, given any complete flag with these properties, one sees easily that the line subbundle $L \subsetneq E_0$ must already be equal to the first summand $L_0$ of $E_0$. We thus obtain the short exact sequence

$$0 \rightarrow F/L_0 \rightarrow F_0 \rightarrow E_0/F \rightarrow 0.$$ 

But then the stability of $F_0$ forces the quotient $E_0/F$ to be of strictly positive slope, in contradiction to the requirement that $\text{deg}(E_0/F) \leq 0$. 
7 Drinfeld’s compactifications for an arbitrary reductive group

This section can be read independently from the rest of the article. Its only purpose toward the proof of the main theorem, Theorem 2.1, is to provide justification for not requiring that the derived group \([G, G]\) of \(G\) is simply connected in the brief discussion of Drinfeld’s compactification \(\text{Bun}_P\) in Sect. 6.1. We continue to use the notation and conventions from Sect. 2.1.

7.1 The original definition

In this section, we recall the definition of Drinfeld’s compactification \(\text{Bun}^\text{or}_P\) from [6], the superscript “or” standing for “original.” Throughout the article [6], the derived group \([G, G]\) of \(G\) is assumed to be simply connected, since otherwise the definition of \(\text{Bun}^\text{or}_P\) given there does not guarantee that \(\text{Bun}_P\) is a dense substack of \(\text{Bun}^\text{or}_P\). This will be remedied by the “new” definition of \(\text{Bun}_P\) in Sect. 7.2 below. However, in that section, we will also make use of the stack \(\text{Bun}^\text{or}_P\) defined in the present section, but in the case of an arbitrary reductive group \(G\). For this reason, we do not assume that the derived group \([G, G]\) is simply connected in this section either, necessitating the superscript “or.”

7.1.1 Mapping stacks

Let \(G\) be a reductive group and let \(P\) be a parabolic subgroup. To define \(\text{Bun}^\text{or}_P\) in a form which will be convenient for the generalization in Sect. 7.2 below, we introduce the following bit of terminology. Recall that given an algebraic stack \(\mathcal{Y}\), the sheaf of groupoids \(\text{Maps}(X, \mathcal{Y})\) of maps from the curve \(X\) to \(\mathcal{Y}\) is defined to have \(S\)-points

\[
\text{Maps}(X, \mathcal{Y})(S) := \mathcal{Y}(X \times S).
\]

Taking \(\mathcal{Y}\) to be the classifying stack \(\cdot / P\), we recover \(\text{Bun}_P\) as

\[
\text{Bun}_P = \text{Maps}(X, \cdot / P).
\]

Let now \(\mathcal{Y}^0 \subset \mathcal{Y}\) be an open substack. We then define another sheaf of groupoids

\[
\text{Maps}_\text{gen}(X, \mathcal{Y} \supset \mathcal{Y}^0)
\]

by associating to a scheme \(S\) the full subgroupoid of \(\text{Maps}(X, \mathcal{Y})(S)\) consisting of those maps \(X \times S \to \mathcal{Y}\) satisfying the following property: We require that for every geometric point \(\tilde{s} \to S\) there exists an open dense subset \(U \subset X \times \tilde{s}\) such that the restriction of the map \(X \times S \to \mathcal{Y}\) to \(U\) factors through the open substack \(\mathcal{Y}^0 \subset \mathcal{Y}\).
It is immediate from the definition that $\text{Maps}_{\text{gen}}(X, Y \supset \overset{\circ}{Y})$ is indeed a subsheaf of $\text{Maps}(X, Y)$. Furthermore, there is a natural inclusion of sheaves

$$\text{Maps}(X, Y) \leftarrow \text{Maps}_{\text{gen}}(X, Y \supset \overset{\circ}{Y}).$$

### 7.1.2 The target stack $\overset{\text{or}}{Y}$

We now construct an algebraic stack $\overset{\text{or}}{Y}$ containing $\cdot / P$ as an open dense substack and will then define

$$\text{Bun}_{\overset{\text{or}}{P}} := \text{Maps}_{\text{gen}}(X, Y \supset \overset{\text{or}}{Y}).$$

To define $\overset{\text{or}}{Y}$, recall that a scheme $Z$ over $k$ is called strongly quasi-affine if its ring of global functions $\Gamma(Z, \mathcal{O}_Z)$ is a finitely generated $k$-algebra and if the natural map $Z \rightarrow \overline{Z} := \text{Spec}(\Gamma(Z, \mathcal{O}_Z))$ is an open immersion. If $Z$ is strongly quasi-affine, we will call $\overline{Z}$ its affine closure.

It is shown in [6, Thm. 1.1.2] that the quotient $G/[P, P]$ of $G$ by the derived group $[P, P]$ of $P$ is strongly quasi-affine. By definition of the affine closure, the left action of $G$ and the right action of $T_M := P/[P, P]$ on $G/[P, P]$ extend to actions on $G/[P, P]$. We then define $\overset{\text{or}}{Y}$ as the double stack quotient

$$\overset{\text{or}}{Y} := G \backslash (G/[P, P])/T_M.$$

It naturally contains

$$\overset{\text{or}}{Y} := \cdot / P = G \backslash (G/[P, P])/T_M$$

as a dense open substack.

### 7.1.3 Definition and first properties of $\text{Bun}_{\overset{\text{or}}{P}}$

We can now define

$$\text{Bun}_{\overset{\text{or}}{P}} := \text{Maps}_{\text{gen}}(X, Y \supset \overset{\text{or}}{Y}).$$

It is proven in [6, Sec. 1.3.2] that $\text{Bun}_{\overset{\text{or}}{P}}$ is indeed an algebraic stack and that the natural map $\text{Bun}_P \leftarrow \text{Bun}_{\overset{\text{or}}{P}}$ realizes $\text{Bun}_P$ as an open substack.

Next, consider the forgetful map

$$\overset{\text{or}}{Y} = G \backslash (G/[P, P])/T_M \rightarrow G \backslash \cdot.$$

It induces a morphism
\[ \tilde{p}_\mathcal{P} : \overline{\text{Bun}_P} \rightarrow \text{Bun}_G \]

whose composition with the natural map \( \text{Bun}_P \hookrightarrow \overline{\text{Bun}_P} \) equals the projection \( p_P \). It is shown in loc. cit. that the map \( \tilde{p}_\mathcal{P} \) is schematic, and is proper when restricted to any connected component of \( \overline{\text{Bun}_P} \).

7.1.4 The case of simply connected derived group

Assume now that the derived group \([G, G]\) of \( G \) is simply connected. Under this hypothesis, it is shown in [6, Prop. 1.3.8] that \( \text{Bun}_P \) is in fact dense in \( \overline{\text{Bun}_P} \) and that the inclusion of \( \text{Bun}_P \) into \( \overline{\text{Bun}_P} \) induces a bijection on the level of connected components:

\[ \pi_0(\overline{\text{Bun}_P}) = \pi_0(\text{Bun}_P) = \tilde{\Lambda}_{G, P} \]

Furthermore, it is proven in [6, Sec. 1.3.3] that under the above hypothesis the stack \( \overline{\text{Bun}_P} \) possesses the stratification which was already described in Sects. 6.1 and 6.1.4 above.

7.2 The case of an arbitrary reductive group

In this section, we define Drinfeld’s compactification \( \overline{\text{Bun}_P} \) for an arbitrary reductive group. As before let \( G \) be any reductive group, and let \( P \) be a parabolic subgroup. To define \( \overline{\text{Bun}_P} \) we will proceed exactly as above, the only difference being that we will use a new target stack \( \mathcal{Y} \), which agrees with the original target stack \( \mathcal{Y}_{\mathcal{P}} \) if \([G, G]\) is simply connected but differs in general.

7.2.1 Two lemmas

We will need the following two facts from the theory of reductive groups. Both can be easily proven on the level of root data.

**Lemma 7.1** Let \( G \) be a reductive group. Then there exists a short exact sequence

\[ 1 \rightarrow Z \rightarrow \tilde{G} \rightarrow G \rightarrow 1 \]

where \( Z \) is a connected torus which is central in \( \tilde{G} \), and \( \tilde{G} \) is a reductive group whose derived group \([\tilde{G}, \tilde{G}]\) is simply connected.

**Lemma 7.2** Let \( G \) be a reductive group. Then the derived group \([G, G]\) of \( G \) is simply connected if and only if every short exact sequence of the form

\[ 1 \rightarrow Z \rightarrow \tilde{G} \rightarrow G \rightarrow 1 \]

splits, where \( Z \) is a connected torus which is central in \( \tilde{G} \).
7.2.2 Definition of the “correct” target stack $\mathcal{Y}$

To define the new target stack $\mathcal{Y}$, choose a short exact sequence

$$1 \rightarrow Z \rightarrow \tilde{G} \rightarrow G \rightarrow 1$$

as in Lemma 7.1. Let $\tilde{P}$ denote the inverse image of $P$ in $\tilde{G}$, and let $T_M$ denote the torus

$$T_M := \tilde{P} / [\tilde{P}, \tilde{P}].$$

Then, as before, the quotient $\tilde{G} / [\tilde{P}, \tilde{P}]$ is a strongly quasi-affine variety, and the left action of $\tilde{G}$ and the right action of $T_M$ on $\tilde{G} / [\tilde{P}, \tilde{P}]$ naturally extend to actions on the affine closure $\tilde{G} / [\tilde{P}, \tilde{P}]$. Furthermore, since the left action of the central torus $Z$ on $\tilde{G} / [\tilde{P}, \tilde{P}]$ agrees with its right action via the map $Z \rightarrow T_M$, the same holds true for the induced left and right actions of $Z$ on the affine closure $\tilde{G} / [\tilde{P}, \tilde{P}]$. We can thus define the new candidate for the target stack as

$$\mathcal{Y} := G \backslash \tilde{G} / [\tilde{P}, \tilde{P}] / T_M.$$ 

As before, the stack $\mathcal{Y}$ naturally contains the classifying stack

$$\cdot / P = G \backslash \left( \tilde{G} / [\tilde{P}, \tilde{P}] \right) / T_M$$

as a dense open substack. We furthermore have:

**Proposition 7.1** The stack $\mathcal{Y}$ and the inclusion $\cdot / P \hookrightarrow \mathcal{Y}$ are canonically independent of the choice of $\tilde{G}$.

**Proof** Given two short exact sequences

$$1 \rightarrow Z_1 \rightarrow G_1 \rightarrow G \rightarrow 1$$

and

$$1 \rightarrow Z_2 \rightarrow G_2 \rightarrow G \rightarrow 1$$

as in Lemma 7.1, we define $\mathcal{Y}_1$ and $\mathcal{Y}_2$ as above and construct a canonical isomorphism $\mathcal{Y}_1 \cong \mathcal{Y}_2$ which restricts to the identity on the open substack $\cdot / P$.

First, let $G_3$ be the fiber product of groups $G_3 := G_1 \times_G G_2$. Then we obtain a third short exact sequence

$$1 \rightarrow Z_1 \times Z_2 \rightarrow G_3 \rightarrow G \rightarrow 1$$
as in Lemma 7.1, and we define $Y_3$ as above. We then claim that the projection maps

$$G_1 \leftarrow G_3 \rightarrow G_2$$

induce isomorphisms

$$Y_1 \leftarrow \cong \ Y_3 \rightarrow \cong \ Y_2.$$ 

To see this, note first that the short exact sequence

$$1 \rightarrow Z_1 \rightarrow G_3 \rightarrow G_2 \rightarrow 1$$

splits by Lemma 7.2, and hence the map

$$G_3/[P_3, P_3] \rightarrow G_2/[P_2, P_2]$$

is a trivial $Z_1$-bundle. But, since the ring of global functions on a product of two varieties is equal to the tensor product of the rings of global functions on each factor, the induced map between the affine closures

$$\overline{G_3/[P_3, P_3]} \rightarrow \overline{G_2/[P_2, P_2]}$$

is again a $Z_1$-bundle. It hence induces an isomorphism

$$G\overline{G_3/[P_3, P_3]}/T_{M_3} = Y_3 \leftarrow \cong Y_2 = G\overline{G_2/[P_2, P_2]}/T_{M_2}$$

which restricts to the identity on $\cdot / P$, as desired. For $Y_1$, one proceeds identically. □

7.2.3 Example

To see that the natural map $Y \rightarrow Y^{or}$ can fail to be an isomorphism if the derived group $[G, G]$ is not simply connected, consider the simplest example $G = \text{PGL}_2$ with $P = B$ and $N := [B, B]$, and let $\tilde{G} = \text{GL}_2$ with $Z = Z_0(\text{GL}_2) = \mathbb{G}_m$. Then the natural map

$$\left(\overline{G/N}\right)/Z \rightarrow \overline{G/N}$$

cannot be an isomorphism since the torus $Z$ does not act freely on the boundary of $\overline{G/N}$. Namely, the quotient $\tilde{G}/\overline{N}$ is isomorphic to the open subset of affine 3-space $\mathbb{A}^3$ obtained by removing a plane and a line not contained in the plane. The boundary of the affine closure is then equal to the missing line with the origin removed, and the torus $Z$ acts on it by a quadratic character.
7.2.4 Definition of $\overline{\text{Bun}}_P$

Using the “corrected” version $\mathcal{Y}$ of the target stack, we define Drinfeld’s compactification $\overline{\text{Bun}}_P$ for an arbitrary reductive group $G$ as the sheaf of groupoids

$$\overline{\text{Bun}}_P := \text{Maps}_{\text{gen}}(X, \mathcal{Y} \supset \cdot / P).$$

As before, the forgetful map $\mathcal{Y} \to G \backslash \cdot$ induces a morphism

$$\overline{p}_P : \overline{\text{Bun}}_P \longrightarrow \text{Bun}_G$$

whose composition with the natural inclusion $\text{Bun}_P \hookrightarrow \overline{\text{Bun}}_P$ is equal to the projection $p_P$.

Below we will show that $\overline{\text{Bun}}_P$ is indeed an algebraic stack, with the properties listed in Sect. 6.1. Before doing so, we can already observe:

**Corollary 7.1** If the derived group $[G, G]$ of $G$ is simply connected, then the new definition of Drinfeld’s compactification agrees with the previous definition in Sect. 7.1:

$$\overline{\text{Bun}}_P = \overline{\text{Bun}}^\text{or}_P.$$

**Proof** Immediate from Proposition 7.1. $\square$

7.3 Verification of the main properties

7.3.1 The setup

The basic idea in proving that $\overline{\text{Bun}}_P$ is an algebraic stack and satisfies the desired properties is as follows. Choose a short exact sequence

$$1 \longrightarrow Z \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1$$

as in Lemma 7.1, and, as before, let $\tilde{P}$ denote the inverse image of $P$ in $\tilde{G}$. Then we will descend the desired properties from $\overline{\text{Bun}}^\text{or}_P$, in which case they are already established, to $\overline{\text{Bun}}_P$. So in addition to the target stack

$$\mathcal{Y} = G \backslash \tilde{G} / [\tilde{P}, \tilde{P}] / T_{\tilde{M}},$$

we also consider the double quotient stack

$$\tilde{\mathcal{Y}}^\text{or} := \tilde{G} \backslash \tilde{G} / [\tilde{P}, \tilde{P}] / T_{\tilde{M}}.$$

By definition, we have

$$\overline{\text{Bun}}^\text{or}_P = \text{Maps}_{\text{gen}}(X, \tilde{\mathcal{Y}}^\text{or} \supset \cdot / \tilde{P}),$$
and the natural map $\tilde{Y} \to Y$ induces a map

$$\overline{\text{Bun}}_{\tilde{P}} \to \overline{\text{Bun}}_P.$$ 

Moreover, the smooth group stack $\cdot/Z$ naturally acts on the classifying stacks $\cdot/\tilde{G}$ and $\cdot/\tilde{P}$ and on the stack $\tilde{Y}^{\text{tor}}$. This in turn induces actions of the smooth group stack $\text{Maps}(X,\cdot/Z) = \text{Bun}_Z$ on the moduli stacks $\text{Bun}_{\tilde{G}}$ and $\text{Bun}_{\tilde{P}}$ and the compactification $\overline{\text{Bun}}_{\tilde{P}}$. Finally, each of the natural maps $\text{Bun}_{\tilde{G}} \to \text{Bun}_G$ and $\text{Bun}_{\tilde{P}} \to \text{Bun}_P$ and $\overline{\text{Bun}}_{\tilde{P}} \to \overline{\text{Bun}}_P$ is invariant under this action.

In this setup, the desired properties of $\overline{\text{Bun}}_P$ will easily follow from the following proposition.

**Proposition 7.2** All three vertical arrows in the commutative diagram

$$\begin{array}{c}
\text{Bun}_{\tilde{P}} & \xleftarrow{\subset} & \overline{\text{Bun}}_{\tilde{P}} & \xrightarrow{\subset} & \text{Bun}_{\tilde{G}} \\
\downarrow & & \downarrow & & \downarrow \\
\text{Bun}_P & \xleftarrow{\subset} & \overline{\text{Bun}}_P & \xrightarrow{\subset} & \text{Bun}_G
\end{array}$$

are torsors in the etale topology for the smooth group stack $\text{Bun}_Z$. Furthermore, both squares are cartesian.

**Proof** We first show that both squares are cartesian by showing that in fact all three squares of the following extended diagram of sheaves of groupoids are cartesian:

$$\begin{array}{c}
\text{Bun}_{\tilde{P}} & \xleftarrow{\subset} & \overline{\text{Bun}}_{\tilde{P}} & \xrightarrow{\subset} & \text{Maps}(X,\tilde{Y}^{\text{tor}}) & \xrightarrow{\subset} & \text{Bun}_{\tilde{G}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Bun}_P & \xleftarrow{\subset} & \overline{\text{Bun}}_P & \xrightarrow{\subset} & \text{Maps}(X,\mathcal{Y}) & \xrightarrow{\subset} & \text{Bun}_G
\end{array}$$

To see that the right square is cartesian, consider first the cartesian square

$$\begin{array}{c}
\tilde{Y}^{\text{tor}} & \to & \tilde{G}\backslash \cdot \\
\downarrow & & \downarrow \\
\mathcal{Y} & \to & G\backslash \cdot
\end{array}$$
Applying the functor
\[
\text{Maps}(X, -) : (\text{sheaves of groupoids}) \longrightarrow (\text{sheaves of groupoids})
\]
\[
\mathcal{Z} \longmapsto \text{Maps}(X, \mathcal{Z})
\]
to this cartesian square yields the right square in the above diagram, and since the functor \(\text{Maps}(X, -)\) commutes with all homotopy limits, the right square is again cartesian, as desired.

Similarly, the fact that the left square in the diagram is cartesian follows easily after applying \(\text{Maps}(X, -)\) to the cartesian square

\[
\begin{array}{ccc}
\cdot / \tilde{P} & \longrightarrow & \tilde{Y}\text{or} \\
\downarrow & & \downarrow \\
\cdot / P & \longrightarrow & Y
\end{array}
\]

Finally, combining the fact that this last square is cartesian with the definitions of \(\text{Bun}_P\) and \(\text{Bun}^\text{or}_P\) yields that the middle square of the above diagram is cartesian as well.

Knowing that both squares in the proposition are cartesian, it now suffices to show that the right vertical arrow \(\text{Bun}_{\tilde{G}} \to \text{Bun}_G\) is a torsor for \(\text{Bun}_Z\). To see this, recall first that the obstruction to the existence of a reduction of a given \(G\)-bundle on \(X\) to \(\tilde{G}\) lies in the second etale cohomology group \(H^2_{\text{et}}(X, \mathcal{Z})\) with values in \(\mathcal{Z}\), which vanishes since \(\mathcal{Z}\) is a torus. This shows that the map under consideration is surjective.

Next, the fact that \(\tilde{G}\) surjects onto \(G\) implies that the map is smooth. It implies furthermore that the diagram of classifying stacks

\[
\begin{array}{ccc}
\cdot / Z & \times & \cdot / \tilde{G} \\
\downarrow & & \downarrow \text{act} \\
\cdot / \tilde{G} & \longrightarrow & \cdot / G
\end{array}
\]
is cartesian. Now, the torsor property of the map \(\text{Bun}_{\tilde{G}} \to \text{Bun}_G\) follows by applying the functor \(\text{Maps}(X, -)\) to this cartesian square, again using that \(\text{Maps}(X, -)\) commutes with all homotopy limits.

\[\square\]

7.3.2 Algebraicity

From Proposition 7.2, we immediately obtain:

**Corollary 7.2** The sheaf \(\text{Bun}_P\) is an algebraic stack.
Proof Proposition 7.2 implies that the map $\overline{p}_P : \overline{Bun}_P \to Bun_G$ is representable by algebraic spaces, since this can be checked smooth-locally on the target and since the map $\overline{Bun}_P \to Bun_\tilde{G}$ is already known to be schematic. But mapping representably to the algebraic stack $Bun_G$, the sheaf $\overline{Bun}_P$ must be an algebraic stack as well. □

In fact, we have:

**Proposition 7.3** The map $\overline{p}_P : \overline{Bun}_P \to Bun_G$ is schematic.

Proof We use the original stack $\overline{Bun}^{or}_P$ from Sect. 7.1 as an intermediate step. Since the map $\overline{Bun}^{or}_P \to Bun_G$ is already known to be schematic, it suffices to show that the natural map $Bun_P \to \overline{Bun}^{or}_P$ is schematic as well. The latter can be deduced from the definitions using standard arguments. □

7.3.3 Remark

One might guess that the map $Bun_P \to \overline{Bun}^{or}_P$ from the “corrected” version to the original version of Drinfeld’s compactification is a closed immersion. This is, however, not the case in general. While one can show that the map is always radicial, it is not hard to construct examples in positive characteristic showing that the map need not be a monomorphism. However, in characteristic 0, the map is always a closed immersion.

7.3.4 Main properties

We now record the main properties of $\overline{Bun}_P$ in the following corollaries to Proposition 7.2:

**Corollary 7.3** The natural inclusion $Bun_P \hookrightarrow \overline{Bun}_P$ realizes $Bun_P$ as a dense open substack of $\overline{Bun}_P$. On the level of connected components, the inclusion induces a bijection

$$\pi_0(\overline{Bun}_P) = \pi_0(Bun_P) = \tilde{\Lambda}_{G,P}.$$  

Proof Since being an open immersion can be checked smooth-locally on the target, the cartesian square of Proposition 7.2 reduces the question to the case of $\overline{Bun}^{or}_P$, in which case the assertion is already established. The statement about being dense is immediate from the surjectivity of the vertical maps in Proposition 7.2. The assertion about the connected components follows from the analogous assertion for $\overline{Bun}^{or}_P$ and the torsor property in Proposition 7.2. □

**Corollary 7.4** The map $\overline{Bun}_P \to Bun_G$ is proper when restricted to any connected component of $\overline{Bun}_P$.

Proof Let $\tilde{\lambda}_P \in \tilde{\Lambda}_{G,P}$ and choose an element $\tilde{\lambda}_P \in \tilde{\Lambda}_{\tilde{G},\tilde{P}}$ in the preimage of $\tilde{\lambda}_P$ under the surjection $\tilde{\Lambda}_{\tilde{G},\tilde{P}} \to \tilde{\Lambda}_{G,P}$. Furthermore, denote by $\tilde{\lambda}_G$ and $\tilde{\lambda}_{\tilde{G}}$ the images of $\tilde{\lambda}_P$ and $\tilde{\lambda}_P$ in $\tilde{\Lambda}_{G,G}$ and $\tilde{\Lambda}_{\tilde{G},\tilde{G}}$, respectively. Then Proposition 7.2 implies that the square
is cartesian and that the vertical arrows are torsors for the identity component \( \text{Bun}_{Z,0} \) of the group stack \( \text{Bun}_Z \); in particular, they are smooth. Since being proper can be checked smooth-locally on the target, the properness of the bottom horizontal arrow follows from the properness of the top horizontal arrow.

7.3.5 The stratification

As in the proof of Corollary 7.4, let \( \tilde{\lambda}_P \in \tilde{\Lambda}_{G,P} \) and choose an element \( \tilde{\lambda}_{\tilde{P}} \in \tilde{\Lambda}_{\tilde{G},\tilde{P}} \) in the preimage of \( \tilde{\lambda}_P \) under the surjection \( \tilde{\Lambda}_{G,\tilde{P}} \twoheadrightarrow \tilde{\Lambda}_{G,P} \). We now deduce the stratification result for the connected component \( \text{Bun}_{P,\tilde{\lambda}_P} \) stated in Sect. 6.1.4 from the corresponding result for \( \text{Bun}_{\tilde{P},\tilde{\lambda}_{\tilde{P}}} \). Namely, in the notation of Sect. 6.1, consider for any \( \tilde{\theta} \in \tilde{\Lambda}_{\tilde{G},\tilde{P}}^{\text{pos}} = \tilde{\Lambda}_{G,P}^{\text{pos}} \) the locally closed immersion

\[
\tilde{j}_{\tilde{\theta}} : X^{\tilde{\theta}} \times \text{Bun}_{\tilde{P},\tilde{\lambda}_{\tilde{P}} + \tilde{\theta}} \hookrightarrow \text{Bun}_{\tilde{P},\tilde{\lambda}_{\tilde{P}}}.
\]

It is apparent from its definition in [6, Sec. 1.3.3] that the map \( \tilde{j}_{\tilde{\theta}} \) is equivariant with respect to the action of the identity component \( \text{Bun}_{Z,0} \) of the group stack \( \text{Bun}_Z \). By Proposition 7.2, it thus descends to a locally closed immersion

\[
j_{\tilde{\theta}} : X^{\tilde{\theta}} \times \text{Bun}_{P,\lambda_P + \tilde{\theta}} \hookrightarrow \text{Bun}_{P,\lambda_P}.
\]

Furthermore, as before, the diagram

\[
\begin{array}{ccc}
X^{\tilde{\theta}} \times \text{Bun}_{P,\lambda_P + \tilde{\theta}} & \xrightarrow{j_{\tilde{\theta}}} & \text{Bun}_{P,\lambda_P} \\
pr^2 \downarrow & & \downarrow \pi_P \\
\text{Bun}_{P,\lambda_P + \tilde{\theta}} & \xrightarrow{p_P} & \text{Bun}_G
\end{array}
\]  

(7.1)

commutes. Moreover, for \( \tilde{\theta} = 0 \), the map \( j_0 \) equals the natural inclusion \( \text{Bun}_{P,\lambda_P} \hookrightarrow \text{Bun}_{P,\lambda_P} \). Finally, since \( \text{Bun}_{P,\lambda_P} \) is a torsor for \( \text{Bun}_{Z,0} \) over \( \text{Bun}_{P,\lambda_P} \) by Proposition 7.2, the stratification of the former descends to the latter:

**Corollary 7.5** The collection of locally closed substacks \( \text{Bun}_{P,\lambda_P} \) corresponding to the immersions \( j_{\tilde{\theta}} \) defines a stratification of \( \text{Bun}_{P,\lambda_P} \) in the sense that on the level of
k-points $\text{Bun}_{P, \tilde{\lambda}, P}$ is equal to the disjoint union

$$\text{Bun}_{P, \tilde{\lambda}, P} = \bigcup_{\tilde{\theta} \in \tilde{\Lambda}^\text{pos}_{G, P}} \text{Bun}_{P, \tilde{\theta}, P}.$$ 

7.4 Drinfeld’s compactification $\tilde{\text{Bun}}_P$

In addition to the stack $\text{Bun}_P$, the authors of [6] also consider another relative compactification $\tilde{\text{Bun}}_P$ of the map $\text{Bun}_P \to \text{Bun}_G$. The stack $\tilde{\text{Bun}}_P$ is defined analogously to $\text{Bun}_P$, using the affine closure of the quotient $G/U(P)$ by the unipotent radical $U(P)$ of $P$ instead of the affine closure of $G/[P, P]$. This compactification is, however, not used in the present article. As in the case of $\text{Bun}_P$, the stack $\tilde{\text{Bun}}_P$ as defined in [6] has the desired properties only under the assumption that the derived group $[G, G]$ of $G$ is simply connected. However, the exact same strategy as in Sect. 7.2 can be carried out in this situation as well. The “corrected” definition of $\tilde{\text{Bun}}_P$ for an arbitrary reductive group $G$ then satisfies the analog of Proposition 7.2, from which all desired properties follow just as in the case of $\text{Bun}_P$.

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