Reconstructing a Polyhedron between Polygons in Parallel Slices

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Abstract
Given two n-vertex polygons, \( P = (p_1, \ldots, p_n) \) lying in the \( xy \)-plane at \( z = 0 \), and \( P' = (p'_1, \ldots, p'_n) \) lying in the \( xy \)-plane at \( z = 1 \), a banded surface is a triangulated surface homeomorphic to an annulus connecting \( P \) and \( P' \) such that the triangulation’s edge set contains vertex disjoint paths \( \pi_i \) connecting \( p_i \) to \( p'_i \) for all \( i = 1, \ldots, n \). The surface then consists of bands, where the \( i \)th band goes between \( \pi_i \) and \( \pi_{i+1} \). We give a polynomial-time algorithm to find a banded surface without Steiner points if one exists. We explore connections between banded surfaces and linear morphs, where time in the morph corresponds to the \( z \) direction. In particular, we show that if \( P \) and \( P' \) are convex and the linear morph from \( P \) to \( P' \) (which moves the \( i \)th vertex on a straight line from \( p_i \) to \( p'_i \)) remains planar at all times, then there is a banded surface without Steiner points.

1 Introduction
The problem of reconstructing a 3D polyhedral structure between two planar cross-sections has been heavily studied because of its many practical applications, e.g., in medicine, for constructing models of body organs from MRI slices. Most approaches, e.g., [3], separate the problem into two steps, both of which are hard and are tackled via heuristics: (1) choose a correspondence between the two cross-sections; (2) then construct a triangulated surface using extra Steiner points. The problem is considered to be well-solved by these heuristic methods, but many theoretical questions remain open. We focus on the second step, i.e., we assume that the correspondence between the two cross-sections is given. Also, we focus on the case of two polygons, though the case of general planar subdivisions (i.e., planar graph drawings) is also of interest.

There is a close connection between the polyhedron reconstruction problem and the problem of “morphing” or continuously transforming one planar structure to another. This connection is explained in more detail later in the Introduction, and motivates our formulation of the polyhedron reconstruction problem.

Given two simple \( n \)-vertex polygons, \( P = (p_1, \ldots, p_n) \) lying in the \( xy \)-plane at \( z = 0 \), and \( P' = (p'_1, \ldots, p'_n) \) lying in the \( xy \)-plane at \( z = 1 \), we want to interpolate between them by constructing a non-self-intersecting triangulated surface \( S \) homeomorphic to an open-ended cylinder (an annulus), with \( P \) at one end and \( P' \) at the other end. Vertices of \( S \) that are not vertices of \( P \) or \( P' \) are called Steiner points. We want the surface to be monotone, in the sense that any plane \( z = t \) intersects the surface in one simple (non-self-intersecting) polygon. Furthermore, we want to maintain the correspondence between \( p_i \) and \( p'_i \) in the following strong sense: for each \( i \) there is a path \( \pi_i \) of edges in the triangulation of \( S \) from \( p_i \) to \( p'_i \), and these paths are vertex disjoint. The paths then partition the surface \( S \) into interior-disjoint bands \( B_1, \ldots, B_n \), where \( B_i \) is the subset of \( S \) between \( \pi_i \) and \( \pi_{i+1} \). We call \( S \) a banded surface and we call this problem banded surface reconstruction between parallel slices or just “banded surface reconstruction”. Figure 1 shows some examples.

The condition that the surface be homeomorphic to an annulus prevents undesirable “solutions” such as placing one Steiner point \( X \) at \( z = \frac{1}{2} \) and building cones from the configurations at \( z = 0 \) and \( z = 1 \)

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Figure 1: Examples of banded surfaces without Steiner points for two triangles $P$ and $P'$. (a) To construct $P$ and $P'$, start with a triangular prism based on equilateral triangle $P = ABC$, and then rotate the top triangle to obtain $P' = A'B'C'$. (b) The Schönhardt polyhedron is a banded surface formed by bending each original rectangular face inward to form two triangles, using the “right” chords $AB'$, $BC'$, $CA'$. (c) Using the outward or “left” chords, $AC'$, $CB'$, $BA'$ also yields a banded surface (an antiprism when $P'$ is rotated by 60°). (d) An example of a triangulated surface that is not banded due to the lack of a path from $A$ to $A'$ disjoint from $BB'$ and $CC'$.

Our Results. We prove the following:

1. For $P$ and $P'$ on $n$ vertices, there exists a banded surface with $O(n^2)$ Steiner points.

2. There is a polynomial time algorithm (using 2-SAT) to decide the banded surface reconstruction problem when no Steiner points are allowed.

3. The existence of a banded surface without Steiner points is preserved by translating $P'$.

4. If $P$ and $P'$ are convex and the linear morph from $P$ to $P'$ preserves planarity (these terms are defined below) then there is a banded surface without Steiner points between $P$ and $P'$. This no longer holds if $P$ and $P'$ are non-convex.

5. In the other direction, the existence of a banded surface without Steiner points does not imply that the linear morph preserves planarity, not even when $P$ and $P'$ are triangles. See Figure 3(b).
Figure 2: The examples of Figure 1(b,c) in top-view with triangle $A'B'C'$ translated horizontally. (Invariance under translation is proved in Lemma 3.) The cross-section at $z = 1/3$ shows the triangle $A_{1/3}B_{1/3}C_{1/3}$ of the linear morph, together with the inward (solid colour) and outward (dashed colour) choices for each edge. Observe that whereas the linear morph uses the edge $A_{1/3}B_{1/3}$ at $z = 1/3$, the inward banded surface using chord $AB'$ uses two edges (shown in solid red), the first parallel to $A'B'$ and the second parallel to $AB$, and the outward banded surface using chord $A'B$ uses two edges (shown in dashed red), the first parallel to $AB$ and the second parallel to $A'B'$.

**Previous Work.** Gitlin, O’Rourke and Subramanian [9] considered a similar problem of joining two polygons via a triangulated surface without adding Steiner points. However, they did not require disjoint paths from $p_i$ to $p'_i$, which gives a lot more freedom, e.g., the two polygons can have different numbers of vertices. Essentially, every edge of $P$ must be in a triangle with some vertex of $P'$, and vice versa, and these triangles must form a non-self-intersecting surface homeomorphic to an annulus. Their main result was a construction of a pair of polygons on 63 vertices with no triangulated surface between them. Their proof involved a computer search. Barequet and Steiner [5] gave a slightly simpler example on 45 vertices. The problem of testing whether two polygons can be joined via a triangulated surface without Steiner points is not known to be NP-complete (nor in P). There is a surprising upper bound on the number of Steiner points required for a triangulated surface. Geiger [8, Appendix A] proved that it suffices to add two Steiner points, one on an edge of $P$ and one on an edge of $P'$. To do this, he first constructed a degenerate surface consisting of two cones, one with $P$ as a base and the rightmost vertex of $P'$ as its apex, and one with $P'$ as a base, and the leftmost vertex of $P$ as its apex. These two cones share one edge, but by adding the two Steiner points, the shared edge can be pulled apart so that the two cones become a single surface homeomorphic to an annulus. This construction is at the heart of our argument in Section 2 that $O(n^2)$ Steiner points suffice to construct a banded surface.

In more applied work, there is a vast literature about interpolating between two families of nested polygons lying in parallel planes via a triangulated surface, see [3, 4]. Barequet and Sharir [4] write: “The primary concern in the literature has usually been to find fast heuristics for selecting a ‘good’ reconstruction among the many available solutions.” There is little work analyzing the number of Steiner points, or examining when a solution with no Steiner points is possible.

Our problem is related to the problem of finding a piecewise linear embedding of a 2D simplicial complex in 3D, which was recently shown to be NP-hard [7]. (One dimension down this is easy, since it is the problem of finding a (poly-line) planar drawing of a graph.) Specifically, the 2D complex that we want to embed in 3D consists of the quadrilaterals $p_i, p_{i+1}, p'_{i+1}, p'_i$, and we have the further constraint that the embeddings of $P$ and $P'$ are already fixed in the 3D space. Our additional structure ensures that there always is a solution so the interesting problems are to minimize the number of Steiner points and/or to optimize other parameters of the solutions such as the bit complexity of the Steiner points, the lengths of the paths from $p_i$ to $p'_i$, or etc.

**Relationship to Morphing.** A morph is a continuous transformation from one shape to another. In particular, a morph from an initial simple polygon [or planar straight-line graph drawing] $P^0$ to a final one,
$P^1$, with the same labelled vertices, is a continuously changing family of polygons [or graph drawings] $P^t$ indexed by time $t \in [0, 1]$. A morph preserves planarity if all intermediate polygons [drawings] $P^t$ are planar. In a linear morph each vertex moves on a straight line from its initial position to its final position at constant speed (where the speed of a vertex depends on the distance it must travel), and an edge is always drawn as a line segment between its endpoints.

Our problem of reconstructing a 3D polyhedral structure between two planar drawings is closely related to morphing—the $z$ direction corresponds to time $t$ in the morph. In fact, it is claimed (for example, by Surazhsky and Gotsman [13]) that morphing algorithms solve 3D shape reconstruction. We now examine this claim more closely. Figure 4 illustrates the idea. Initialize $P^0$ to $P$ and $P^1$ to $P'$. Given a morph $P^t$, $t \in [0, 1]$ between $P^0$ and $P^1$, take a finite set of “snapshots” at time points $t_1, \ldots, t_k$, and form a quadrilateral “patch” between successive vertices $p_i$ and $p_{i+1}$ at times $t_j$ and $t_{j+1}$. Each patch is a ruled surface, and the union of the patches provides a surface in 3D joining $P^0$ and $P^1$. In order to obtain a piece-wise linear surface we must replace each quadrilateral patch by two triangles. This may cause the surface to self-intersect (if it doesn’t already). It seems intuitive that self-intersections can be avoided by taking sufficiently many snapshots, but such analysis is lacking.

An algorithm by Alamdari et al. [1] finds “piece-wise linear” morphs that consist of a sequence of planarity-preserving linear morphs. This would provide a solution to banded surface reconstruction if we could show how to add Steiner points to turn each linear morph into a triangulated surface.

In the other direction, a banded surface (even one with Steiner points) can be interpreted as a morph between the polygons $P$ and $P'$, albeit a morph in which each edge may become a poly-line. Such “morphs with bent edges” have been investigated [11] and come with small grid guarantees, unlike the piece-wise linear morphs of [1]. A banded surface without Steiner points provides a morph with the interesting property that in any intermediate drawing of the morph, an edge $e$ appears as a path of two line segments, one in the direction of the initial version of $e$ and the other in the direction of the final version of $e$. See Figure 2. Such morphs may be valuable for visualizations. We note that there is work on morphing while maintaining edge directions [6]—this only applies in the restricted situation where the initial and final polygons have corresponding edges with the same directions.

To summarize, it seems worth investigating to what extent linear morphs provide banded surfaces, and to what extent banded surfaces provide morphs.
Figure 4: A morph from a rectangle $P^0$ at time (or z-coordinate) $t = 0$ to the “arch-shaped” polygon $P^1$ at time $t = 1$ yields a 3D interpolation by taking “snapshots” of the morph at intermediate time points $t = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{3}{4}$, and joining corresponding vertices between one snapshot and the next. Note that the resulting quadrilateral patches (one of which is coloured blue) are not planar in general. This figure is loosely based on one by Surazhsky and Gotsman [13, Figure 10].

2 Finding a Banded Surface with/without Steiner Points

In this section we show that $O(n^2)$ Steiner points suffice to construct a banded surface between $n$-vertex polygons $P$ and $P'$, and we give an algorithm to find—if it exists—a banded surface without Steiner points between $P$ and $P'$.

As discussed above, Geiger [8] showed how to add two Steiner points to construct a triangulated surface between $P$ and $P'$. The surface he constructs is monotone, homeomorphic to an annulus, and consists of $O(n)$ triangles. To construct a banded surface, take disjoint paths on the surface from $p_i$ to $p'_i$ for each $i$, and refine the triangulation to include all the line segments of these paths. Since there are $n$ paths each crossing $O(n)$ triangles, the result is a surface of $O(n^3)$ triangles. We note that the same bound $O(n^2)$ can be obtained using a technique from Piecewise Linear (PL) Topology in which ears of the polygon are collapsed in successive steps. (See “elementary contractions” in the classical book [12] or the lecture notes [10, p. 30].) One ear collapse replaces a convex vertex $p_i$ by a vertex on the line segment from $p_{i-1}$ to $p_{i+1}$, resulting in a polygon with one fewer line segments. The new polygon is placed on a slightly higher $z$ plane and the two successive polygons are joined with $n$ triangles. In this way we collapse $P$ (working upwards in 3D) to a triangle $T$, collapse $P'$ (working downwards in 3D) to a triangle $T'$ and finally build a surface joining $T$ and $T'$. This may produce a nicer surface than the one obtained from Geiger’s construction.

We next describe an algorithm using 2SAT to find a banded surface without Steiner points, if one exists. The edges $p_ip'_i$ must be used. For each $i = 1, \ldots, n$ we have the choice of the right chord $p_ip'_{i+1}$ or the left chord $p_{i+1}p'_i$. Let the Boolean variable $R_i$ be 1 if the right chord is chosen and 0 otherwise. Each chord choice determines two triangles of the surface, for example $R_i = 1$ determines triangles $p_ip_{i+1}p'_{i+1}$ and $p_ip'_{i+1}$. We say that chord choices for $i$ and $j$ conflict if the resulting open triangles intersect. Note that this can be tested, for given $i, j$, in constant time. The problem of choosing chords to form a non-self-intersecting surface can be formulated as a Boolean satisfiability problem by adding a clause to prohibit conflicts, e.g., if chord choices $R_i$ and $\neg R_j$ conflict then we add the clause $\neg (R_i \land \neg R_j)$. Note that there are $O(n^2)$ clauses.

There is a banded surface without Steiner points if and only if the resulting clauses are satisfiable. Because all clauses have two variables, the result is a 2-SAT instance. Since 2-SAT can be solved in linear time [2], we have:

**Lemma 1.** There is a quadratic time algorithm that either finds a banded surface without Steiner points, or declares that no such surface exists.
3 Conditions for Existence of a Banded Surface without Steiner Points

One approach to banded surface reconstruction with Steiner points is to subdivide the interval \( z \in [0, 1] \) into \( 0 = z_0, z_1, \ldots, z_k = 1 \) and place an \( n \)-vertex polygon at each \( z_i, 0 < i < k \) so that each successive pair of polygons admits a banded surface without Steiner points. Using this approach, the final solution would have \( nk \) Steiner points.

In order to design the intermediate polygons, it would be good to have conditions for when two polygons admit a banded surface without Steiner points. (Our polynomial-time test from the previous section does not seem helpful when the polygons are not given).

In this section we explore two situations where we can guarantee the existence of a banded surface without Steiner points. We show:

1. Translation of \( P' \) in the \( z = 1 \) plane preserves the existence of a banded surface without Steiner points (Lemma 2).
2. If \( P \) is convex and \( P' \) is a rotation of \( P \) by an angle less than \( \pi \), then a banded surface without Steiner points exists (Lemma 3). The example of Figure 8 shows that this property does not hold more generally, not even for a star-shaped polygon.

We first show how translation of the target-polygon affects the intermediate polygons in a linear morph:

**Lemma 2.** Let \( P \) be an \( n \)-vertex polygon in the \( z = 0 \) plane and \( P' \) be an \( n \)-vertex polygon in the \( z = 1 \) plane. Let \( Q' \) be a translation of \( P' \) within the \( z = 1 \) plane. Let \( Q_i \) be the polygon at time \( t \) during the linear morph from \( P \) to \( P' \), and \( Q_i \) is the polygon at time \( t \) during the linear morph from \( P \) to \( Q' \), then \( Q_i \) is a translation of \( P_i \) within the \( z = t \) plane.

**Proof.** Set \( s = Q' - P' \) to be the translation vector and consider an arbitrary point \( p \) of \( P \) that morphs to point \( p' \) of \( P' \) and \( q' \) of \( Q' \). We have \( q' = p' + s \), and hence

\[
q_i = (1 - t)p + tq' = (1 - t)p + tp' + ts = p_i + t \cdot s
\]

so polygon \( Q_i \) is a translation of \( P_i \) by \( t \cdot s \).

In particular, if the linear morph from \( P \) to \( P' \) preserves planarity, then the same holds for the linear morph from \( P \) to any translation of \( P' \). We can argue the same for banded surfaces:

**Lemma 3.** Assume that \( P \), \( P' \) and \( Q' \) are as in Lemma 2. If there is a banded surface without Steiner points between \( P \) and \( P' \), then the same choice of chords yields a banded surface without Steiner points between \( P \) and \( Q' \).

**Proof.** We show that the banded surface between \( P \) and \( P' \) is the same as the linear morph between two modified polygons \( P_D \) and \( P_D' \), which we now define. Initially start with \( P \) and \( P' \). For each \( i = 1, \ldots, n \), if we chose the right chord \( p_ip_{i+1} \), then duplicate vertex \( p_i \) in \( P_D \) (inserting an edge of length 0) and duplicate vertex \( p_{i+1} \) in \( P_D' \). Proceed symmetrically if we chose the left chord. Now consider the linear morph from \( P_D \) to \( P_D' \), where vertices that have been inserted due to a chord correspond to each other. Say we chose the right chord \( p_ip_{i+1} \). Then the zero-length edge \( p_ip_{i+1} \) in \( P_D \) morphs to edge \( p_ip_{i+1} \) in \( P_D' \), hence forms a triangle. Likewise edge \( p_{i+1}p_i \) in \( P_D \) morphs to zero-length edge \( p_{i+1}p_{i+1} \) in \( P_D' \), and also forms a triangle. The two triangles together form exactly the part of the banded surface between edges \( p_ip_{i+1} \) and \( p_{i+1}p_{i+1} \) in \( P \) and \( P' \).

Since the banded surface is the same as the linear morph from \( P_D \) to \( P_D' \) the result now follows from Lemma 2.

We now turn to rotations, beginning with this result on linear morphs when the target-polygon is a rotation of the source-polygon:
Lemma 4. Let $P$ be a polygon and let $P'$ be a rotation of $P$ about an origin $X$ by an angle $\alpha$. For any $0 < t < 1$ let $P_t$ be the polygon at time $t$ during a linear morph from $P$ to $P'$. If $\alpha \neq \pi$ or $t \neq \frac{1}{2}$ then $P_t$ is a rotated copy of $P$ that has been scaled by $s_i \neq 0$.

Proof. We consider $P$, $P'$ and $P_t$ projected to the $xy$ plane. If $\alpha = \pi$ then every point $p$ of $P$ maps to $-p$ in $P'$, which implies $p_t = (1 - 2t)p$. So $P_t = s_tP$ for $s_t = 1 - 2t$, which is non-zero for $t \neq \frac{1}{2}$.

Now suppose that $\alpha < \pi$ (the case $\alpha > \pi$ is symmetric). For any point $p$ of $P$, consider the triangle $\Delta_p := \Delta pXp'$, where $X$ is the center of the rotation. Note that $\Delta_p$ and $\Delta_q$ are similar for any two points $p$ and $q$ of $P$, since they both have angle $\alpha$ and two equal-length incident sides; in particular $\Delta_q$ is obtained from $\Delta_p$ by scaling by $\frac{|q|}{|p|}$ and (possibly) rotating. Also notice that $p_t$ travels along the side of $\Delta_p$ opposite to angle $\alpha$, and is at the point that divides the side at ratio $t/(1-t)$. We can view $p_t$ as having been rotated by some angle $\theta_t$ and scaled by some $s_t > 0$. Both $\theta_t$ and $s_t$ are independent of the choice of $p$ since all triangles $\Delta_p$ are similar. Therefore $P_t$ is obtained from $P$ by scaling by $s_t$ and rotating by $\theta_t$.

Lemma 5. Let $P$ be a convex polygon and let $P'$ be a rotation of $P$ about an origin $X$ by an angle $\alpha < \pi$. Then there is a banded surface without Steiner points between $P$ and $P'$.

Proof. Observe first that the linear morph from $P$ to $P'$ preserves planarity since, by Lemma 4, each intermediate polygon is a rotated and scaled copy of $P$. By Theorem 6 (forthcoming, but there is no circularity) this implies the existence of a banded surface without Steiner points.

4 Linear Morphing versus Banded Surface Reconstruction

In this section we compare the existence of a planarity-preserving linear morph from $P$ to $P'$ and the existence of a banded surface without Steiner points. In general, these two properties are independent, i.e., neither implies the other. Figure 4(b) shows an example of two triangles that have a banded surface without Steiner points, but the linear morph does not preserve planarity. Figure 5 shows an example of two stars that do not have a banded surface without Steiner points, but the linear morph preserves planarity.

When the polygons $P$ and $P'$ are convex, there is an implication:

Theorem 6. If $P$ and $P'$ are convex and the linear morph from $P$ to $P'$ preserves planarity, then there is a banded surface without Steiner points between $P$ and $P'$.

Proof. Let $p_i^t$ be the position of the $i$th vertex at time $t$ (or time) $t$ during the linear morph. In particular, $p_i^0 = p_i$ and $p_i^1 = p_i$. Let $P^t$ be the polygon at time $t$ during the morph. Note that $P^t$ is not necessarily convex. By our convention of numbering polygons in counterclockwise order, the inside of $P$ is to the left of $p_ip_{i+1}$, and the inside of $P'$ is to the left of $p_i'p_{i+1}'$. Also, because the linear morph preserves planarity, the inside of $P^t$ is to the left of $p_ip_{i+1}$.

We begin by defining the surface $S_i$, i.e., which chords to use. Let $v_i^0$ be the vector $p_{i+1} - p_i$ in the $xy$ plane, and let $v_i^1$ be the vector $p_{i+1}' - p_i'$ projected to the $xy$ plane. Let $\theta_i$ be the angle between $v_i^0$ and $v_i^1$, measured towards the inside of $P$, as shown in Figure 5a. We distinguish 3 cases:

- If $\theta_i < \pi$, use the left chord $p_{i+1}p_i'$. In the cross-section of $S$ at $z$-coordinate (or time) $t$, the edge $p_ip_{i+1}'$ is replaced by a segment in the direction $v_i^0$ followed by a segment in the direction $v_i^1$. We call the resulting triangle $\Delta_i^t$ and refer to it as a 01 triangle. Observe that $\Delta_i^t$ lies to the outside of the edge $p_ip_{i+1}'$. See Figure 5(a).

- If $\theta_i > \pi$, use the right chord $p_ip_{i+1}'$. Then, in the cross-section at $z$-coordinate $t$, the edge $p_ip_{i+1}'$ is replaced by a segment in the direction $v_i^1$ followed by a segment in the direction $v_i^0$. We refer to the resulting triangle $\Delta_i^t$ as a 10 triangle. Again, $\Delta_i^t$ lies to the outside of $p_ip_{i+1}'$, see Figure 5(b).

- If $\theta_i = \pi$, use either chord—in this case the quadrilateral $p_i,p_{i+1},p_{i+1}'p_i'$ is coplanar, and $\Delta_i^t$ collapses to the edge $p_ip_{i+1}'$. 


Figure 5: A top view to illustrate choosing chords in the proof of Theorem 6. In order to show the angles clearly, $P'$ has been translated so that $p_0$ is at the same $xy$-coordinates as $p'_i$. (Lemma 3 justifies this.) Note that $p'_i$ then remains at these same $xy$-coordinates. Hatching indicates the inside of the polygon on that edge. (a) If $\theta_i < \pi$, use a left chord to obtain a $01$ triangle $\Delta_i^t$. (b) If $\theta_i > \pi$, use a right chord to obtain a $10$ triangle $\Delta_i^t$. The segments that replace $p'_i p'_{i+1}$ are shown in red/cyan.

We now prove that $S$, as defined by the above chord choices, is non-self-intersecting, which proves that $S$ is a banded surface without Steiner points. In particular, we will prove that $S^t$, the cross-section of $S$ at $z$-coordinate $t$ is a simple polygon. By assumption, the polygon $P^t$ with vertices $p'_1, p'_2, \ldots, p'_n$ is simple. $S^t$ consists of $P^t$ plus triangles $\Delta_i^t$ added to the outside of each edge. See Figure 6. We will show that no two triangles intersect.

**Claim 7.** Suppose that $\Delta_i^t, \Delta_{i+1}^t, \ldots, \Delta_j^t$ are all $01$ triangles. Let $r_i^0$ be the ray from $p_i^t$ in the direction $v_i^0$. Then none of these triangles cross $r_i^0$ from its left to its right.

**Proof.** It suffices to prove that no triangle crosses the ray of the previous triangle, so consider triangle $\Delta_{i+1}^t$ and $r_i^0$. The apex of $\Delta_{i+1}^t$ lies on $r_{i+1}^0$. Rays $r_i^0$ and $r_{i+1}^0$ emanate from the endpoints of the edge $p_i^t p_{i+1}^t$ and the angle between $r_i^0$ and $r_{i+1}^0$ is positive (counterclockwise). Thus the apex of $\Delta_{i+1}^t$ lies to the left of $r_i^0$. 

Figure 6: Polygon $P^t$ (in blue) with 01 triangles $\Delta_i^t, \ldots, \Delta_j^t$ and 10 triangles $\Delta_k^t, \ldots, \Delta_l^t$. In general, there may be many alternations between 01 and 10 triangles.

Define $r_i^1$ to be the ray from $p_{i+1}^t$ in the direction $-v_i^1$. Thus a 01 triangle $\Delta_i^t$ is bounded by $r_i^0$ and $r_i^1$. Symmetrically, for a 10 triangle, define $s_i^1$ to be the ray from $p_i^t$ in the direction $v_i^1$, and $s_i^0$ to be the ray from $p_{i+1}^t$ in the direction $-v_i^0$. Thus a 10 triangle $\Delta_i^t$ is bounded by $s_i^1$ and $s_i^0$.

From Claim 7 by symmetry, we obtain (see Figure 6):

**Claim 8.** If $\Delta_i^t, \Delta_{i+1}^t, \ldots, \Delta_j^t$ are 01 triangles then none of them cross $r_j^1$ from right to left. If $\Delta_k^t, \Delta_{k+1}^t, \ldots, \Delta_l^t$ are 10 triangles then none of them cross $s_k^1$ from left to right and none of them cross $s_l^0$ from right to left.
These two claims imply that \( S^t \) is simple if all the triangles are the same (all 01 or all 10). It remains to consider the possibility that there are triangles of both types.

**Claim 9.** Suppose \( \Delta_{i-1}^t \) is a 10 triangle and \( \Delta_i^t \) is a 01 triangle. Then \( \Delta_{i-1}^t \) and \( \Delta_i^t \) are disjoint. Furthermore, \( P^t \) is convex at \( p_i^t \).

**Proof.** We analyze the top-view projection with \( P^t \) translated so that \( p_i \) and \( p_i' \) are at the same \( xy \)-coordinates.

Consider the angle \( \alpha_i^t = \angle p_{i+1} p_i p_i'_{i+1} \). Because \( \Delta_i^t \) is a 01 triangle, \( \alpha_i^t \) goes from 0 to \( \theta_i < \pi \). Similarly, because \( \Delta_{i-1}^t \) is a 10 triangle, the angle \( \alpha_{i-1}^t = \angle p_{i-1} p_i p_i'_{i-1} \) goes from 0 to \( 2\pi - \theta_{i-1} < \pi \).

If \( p_i p_i'_{i-1} \) and \( p_i p_i'_{i+1} \) cross over each other, as in Figure 7(a), i.e., \( \theta_i + 2\pi - \theta_{i-1} \geq \angle p_{i-1} p_i p_{i+1} \), then there must be some time \( t \) when \( \alpha_i^t + \alpha_{i-1}^t = \angle p_{i-1} p_i p_{i+1} \), i.e., angle \( \angle p_{i-1} p_i p_{i+1} \) becomes 0. But we assumed that \( P^t \) remains simple, so this cannot happen.

![Figure 7: Illustration for the proof of Claim 9](image)

Thus we must have the situation shown in Figure 7(b), so \( \Delta_{i-1}^t \) and \( \Delta_i^t \) are disjoint and \( P^t \) remains convex at \( p_i^t \).

With these claims in hand, we can complete the proof of the theorem. Divide the circular sequence \( \Delta_1^t, \ldots, \Delta_n^t \) into maximal subsequences all of the same type (all 01 or all 10). If \( D_{i,j} = \Delta_{i,j}^t, \ldots, \Delta_{i,j}^t \) is such a maximal subsequence then by Claims 7 and 8 no two triangles of \( D_{i,j} \) intersect, and all the triangles of \( D_{i,j} \) live in the region \( R_{i,j} \) bounded by \( p_i', \ldots, p_{i,j}' \) and two bounding rays—\( r_{i,j}^0 \) and \( r_{i,j}^1 \) in the case of 01 triangles, as shown in Figure 6. Between one sequence \( D_{i,j} \) and the next, \( D_{j+1,i} \), Claim 9 implies that the regions \( R_{i,j} \) and \( R_{j+1,i} \) are disjoint.

**5 Conclusions**

We have introduced the idea of a banded surface to construct a polyhedron between two polygons in parallel slices and have explored some connections between linear morphs and banded surfaces without Steiner points. Many questions remain, the two main ones being:

1. Is there a bound better than \( O(n^2) \) on the number of Steiner points needed to construct a banded surface between two \( n \)-vertex polygons? What if the polygons are convex?

2. Is it NP-hard to minimize the number of Steiner points needed to construct a banded surface?
Figure 8: A star-shaped polygon $P$ where the linear morph to its $90^\circ$ rotation $P'$ preserves planarity, but there is no banded surface: (a) $P$ and $P'$ and the intermediate position of the linear morph at $t = \frac{1}{2}$ (shaded blue); (b) If we choose the chord for edge $BC$ that produces the “outward” triangle (shown in red) then at $t = \frac{1}{2}$ it intersects one choice for $OA$ and one choice for $BA$; (c) The other choices for $OA$ and $BA$ intersect each other; (d) Thus we are forced to choose the chord for edge $BC$ that produces the “inward” triangle (shown in red), and, by symmetry, the “inward” triangle for $DC$ (shown in cyan)—but these intersect.

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