Integration without integration: new Killing spinor spacetimes

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Abstract. Non-conformally flat spacetimes admitting a non-null two-index Killing spinor are investigated by means of the Geroch-Held-Penrose formalism. Claims appearing in the literature that such spacetimes are all explicitly known are incorrect. This was shown in [5] for the family where, in the canonical frame, the spin coefficients $\rho$ or $\mu$ vanish. Here the general case with non-vanishing $\rho, \mu, \pi$ and $\tau$ is re-considered. It is shown that the construction in [4] hinges on the tacit assumption that certain integrability conditions hold, implying two algebraic relations for the spin coefficients and the components of the Ricci spinor. All (conformal classes of) spacetimes, in which one of these conditions is violated, are obtained by invariant integration. The resulting classes are each other’s Sachs transform and are characterised by one free function. They admit in general no Killing vectors, but still admit a conformal gauge (different from the trivial unitary gauge) in which a Killing tensor exists.

1. Introduction

The history of the subject which I present goes back to Carter’s work [1] on the complete integration of charged particle orbits in the Kerr metric, making use of what he called "the unexpected separability of the Hamilton-Jacobi equation ..." and to the subsequent construction of all Abelian $G_2$ Hamilton-Jacobi separable spacetimes. The relationship between separability and the existence of Killing tensors led Walker and Penrose [2] to investigate vacuum type D spacetimes admitting quadratic first integrals of the null geodesic equations. They showed that the separability property was linked to the existence of an object which they called a Killing spinor, which is a symmetric valence 2 spinor $X_{AB} = X_{(AB)}$, satisfying the twistor equation

$$\nabla_{A'}(AX_{BC}) = 0. \quad (1)$$

The tensor analogue of a Killing spinor is a tensor $D_{ab} = D_{(ab)} = X_{AB} \epsilon_{A'B'} + \bar{X}_{A'B'} \epsilon_{AB}$, which nowadays usually is called a conformal Killing-Yano (CKY) tensor and which is a two-form satisfying the CKY equation

$$\nabla_{(a}D_{b)c} = \frac{1}{3} \nabla_d(g_{ab}D^d_{c} - D^d_{(a}g_{b)c}). \quad (2)$$

The Killing spinor $X_{AB}$ was used by Walker and Penrose to construct a second rank so called conformal Killing tensor, namely a symmetric tensor $P_{ab}$ obeying the equation

$$\nabla_{(a}P_{bc)} = g_{(ab}P_{c)} \quad (3)$$
and on which depends the construction of a full Killing tensor in all but the Kinnersley case III metrics (namely the C and C-NUT metric).

In the wake of these investigations a lot of effort was devoted to the characterisation and construction of spacetimes admitting Killing spinors, conformal Killing tensors, Killing-Yano tensors etc. (see [3] for references). For Killing spinors the situation is at first sight fairly easy: for a non-null Killing spinor $X_\alpha^{(A\ell B)}$ and a non-conformally flat spacetime, the Petrov type is necessarily D, with the principal spinors of $X$ and of the Weyl spinor being aligned, geodesic and shearfree. Adapting a Geroch-Held-Penrose tetrad to the latter, one obtains

$$\kappa = \sigma = \kappa' = \sigma' = 0$$

(4)

and the equations (1) become

$$\mathcal{D}X = -\rho X, \quad \mathcal{D}'X = -\rho' X$$

(5)

$$\mathcal{D}X = -\tau X, \quad \mathcal{D}'X = -\tau' X$$

(6)

(recall that in the GHP formalism the $'$-operator interchanges the real null vectors $k$ and $l$, as well as $m$ and $\overline{m}$, hence $\rho' = -\mu$ and $\tau' = -\pi$).

Obviously, as these equations are conformally invariant, the spacetime is only defined up to a conformal factor. However one can always construct a unique conformal representant $(\mathcal{M}, \hat{g})$ in which $|\hat{X}| = 1$. In this representant one has

$$\hat{\rho} + \hat{\rho} = \hat{\rho}' + \hat{\rho}' = \hat{\tau} + \hat{\tau}' = 0$$

(7)

and a general representant is now related to $(\mathcal{M}, \hat{g})$ by $g = \Omega^2 \hat{g}$, $X = \Omega \hat{X}$. In $(\mathcal{M}, \hat{g})$ the trace of the conformal Killing tensor $P_{ab}$ associated to $X$ is constant and $P_{ab}$ is a Killing tensor with constant eigenvalues. In the invariant construction of Killing spinor spacetimes these eigenvalues can play a significant role (turning up as possibly independent coordinates) and therefore one tries to construct representants in which the existence of a Killing tensor is preserved under conformal transformations. This requires that the conformal Killing tensor $P_{ab}$ should be the trace-free part of a Killing tensor itself: a necessary and sufficient condition is that a solution exists of the equation

$$\nabla_a P^a_{\, c} + \frac{3}{4} \nabla_c K = 0,$$

(8)

or, in terms of the eigenvalues of the Killing tensor $K_{cd} = 2am_{(c|[md]} + 2b\ell_{(c|[n]d)}$,

$$da = -(a + b)[(\rho + \overline{\rho})\omega^4 + (\rho' + \overline{\rho}')\omega^3]$$

$$db = -(a + b)[(\tau + \overline{\tau})\omega^4 + (\tau' + \overline{\tau}')\omega^3],$$

(9)

(10)

with $a = (\Omega^2 - K)/4$ and $b = (\Omega^2 + K)/4$. In order that this system should have non-constant solutions, the following integrability conditions must be satisfied:

$$\rho' \mathcal{D}a - \rho \mathcal{D}'a = 0$$

(11)

$$\rho' \mathcal{D}b + \rho \mathcal{D}'b = 0$$

(12)

and

$$\tau' \mathcal{D}b - \tau \mathcal{D}'b = 0$$

(13)

$$\tau' \mathcal{D}b + \tau \mathcal{D}'b = 0.$$  

(14)
with \( \phi \) and \( \phi' \) defined by
\[
\Phi_{01} = -3\rho \tau - 2\frac{\rho}{\tau} \phi, \quad \Phi_{21} = -3\rho' \tau' - 2\rho' \phi' / \tau.
\] (15)

It follows that when \( \tau \tau' = 0 \) the classification given in [4] is essentially correct. When \( \tau \tau' \neq 0 \) and \( \rho \rho' = 0 \) an extra spacetime arises [5], admitting a \( G_3 \) (but without Abelian subgroup) and in which the \( b \)-equations only have constant solutions. In the general case \( (\tau \tau' \rho \rho' \neq 0) \) a Killing tensor with non-constant eigenvalues can only exist when the following conditions hold, which I henceforth call \( KS_1 \) and \( KS_2 \):
\[
KS_1: \exists \text{ non-constant } b \iff \phi + \phi = 0 = \phi' + \phi'.
\] (16)
\[
KS_2: \exists \text{ non-constant } a \iff \phi + \phi' = 0.
\] (17)

In this contribution all spacetimes in the sets \( KS_1 \setminus KS_2 \) and \( KS_2 \setminus KS_1 \) are presented.

2. Construction of \( KS_1 \) and \( KS_2 \)

In the \( KS_1 \) spacetimes one first deduces from the GHP commutator equations, Ricci equations and Bianchi equations the following integrable system:
\[
\begin{align*}
\Phi &= 0 \\
\Phi' &= 2\rho \rho'/|\tau|^2(\phi + \phi') \\
\Phi &= \rho/\tau'(|\tau|^2 - 2\phi),
\end{align*}
\] (18)
together with similar equations for \( \tau, \phi, \phi' \) and \( \Psi_2 = E + iH \). These systems provide one immediately with two coordinate-candidates, \( r \) and \( m \), defined by \( r^2 = Q \rho \rho', \ m = |\tau| \ \ (Q = \pm 1) \).

Herewith \( \phi, \phi' \) become functions of \( m \) and \( r \):
\[
\phi = m^2 r^{-2} \phi_0, \phi' = m^2 r^{-2} \chi_0,
\] (19)
with \( \chi_0 = \phi_0 - 2i \chi_1 \) and \( (r^2 - c_2)^2 - (\phi_0 - ic_1)^2 = c_1^2 + c_2^2 \) \( (c_1 \neq 0 \text{ and } |r^2 - c_2| < \sqrt{c_1^2 + c_2^2}) \).

This leads to the following explicit expressions for the exterior derivatives of \( r, m \) and \( b \):
\[
\begin{align*}
dr &= (\phi_0 + \chi_0)\left[\frac{m}{r} (\frac{\tau}{m} \omega^1 - \frac{m}{\tau} \omega^2) + Q \frac{\tau}{r} \omega^3 + \frac{\rho}{r} \omega^4\right] \\
\frac{dm}{dB} &= i(\frac{\tau}{m} \omega^1 - \frac{m}{\tau} \omega^2) \\
db &= i(\frac{\tau}{m} \omega^1 - \frac{m}{\tau} \omega^2)
\end{align*}
\] (20)
(21)
(22)

Calculating the triple wedge product
\[
dr \wedge dm \wedge db = 2iQ \frac{m(\phi_0^2 - \chi_0^2)}{r} (\frac{\tau}{m} \omega^3 \wedge \omega^4 \wedge \omega^1 - \frac{m}{\tau} \omega^3 \wedge \omega^4 \wedge \omega^2) \neq 0.
\] (23)

shows that \( r, m \) and \( b \) are functionally independent scalars. Although one could in principle use the magnetic part \( H \) of \( \Psi_2 \) as a fourth coordinate (and hence obtain a complete invariant integration of the Cartan equations), it is more practical to define a fourth coordinate \( x \) by
\[
\frac{1}{2} (\frac{\tau}{m} \omega^1 + \frac{m}{\tau} \omega^2) = h_1 db + h_2 dr + h_3 dm + dx,
\] (24)

leading one to the following expression for the metric:
that the Sachs asterisk operation \[9\] \(\ast\) situates into each other. As a consequence it is possible to obtain the at the time of writing at least one example has been found of a Killing spinor spacetime in which 1 constant eigenvalues only, the authors claim that all results remain valid also when one of the considered in \[8\]. Although the title of \[8\] gives the impression that the paper deals with non-

\[\begin{align*}
\tau \omega^1 &= dx + \frac{iQmr}{4c_1(\phi_0 - ic_1)}dr + \left( \frac{H_0x}{2} - 4Qc_1x^2 - \frac{2ic_1 + Qm^2}{4c_1} \right)db \\
\frac{H}{\rho} \omega^3 &= \frac{iQ}{4c_1} \left( \frac{dm}{m} - \frac{\phi_0}{\phi_0 - ic_1} \frac{dr}{r} \right) + i(Q \frac{r^2H_0 - 4mi\phi_0}{8r^2c_1} - 2x)db \\
\rho \omega^4 &= - \frac{ir^2}{4c_1} \left( \frac{dm}{m} + \frac{\chi_0}{\chi_0 + ic_1} \frac{dr}{r} \right) - i \frac{r^2H_0 - 4mi\phi_0}{8c_1} - m - 2Qr^2x)db.
\end{align*}\]

The corresponding curvature components are \(\Phi_{00} = -\rho^2; \Phi_{22} = -\frac{\tau^4}{\rho^2}; \Phi_{02} = -\tau^2\) and

\[\begin{align*}
R &= -12(Qr^2 + m^2 + E), \\
E &= -2m^2 - \frac{8}{3}Qr^2 + 32\epsilon^2x^2 + \frac{1}{12}H_0^2 - \frac{1}{6}db - 4Qc_1xH_0 - \frac{4Q}{3r^2\phi_0\chi_0}, \\
H &= (H_0 - 16Qc_1x)m + 2iQ(\phi_0 + \chi_0), \\
\Phi_{11} &= -\frac{9}{2}Qr^2 - \frac{7}{2}m^2 + \frac{3}{2}E - \frac{4Q\phi_0\chi_0}{\tau^2}, \\
\Phi_{01} &= \frac{\rho^2}{\tau^2}(2\phi_0 - 3\rho^2), \\
\Phi_{12} &= -\frac{Q\tau}{\rho}(2\chi_0 + 3\rho^2).
\end{align*}\]

For \(KS_2\) one can proceed along exactly the same lines, but it is more instructive to note that the Sachs asterisk operation \([9]\) \(\ast\), \(\ast^\prime = -\partial; \partial^\prime = -\partial, \partial^\prime = -\partial; \partial^\prime = -\partial\) transforms the two situations into each other. As a consequence it is possible to obtain the \(KS_2\) metric via a complex transformation of \(KS_1\) (see \([3]\)).

A remaining problem stems from a corollary in \([7]\), showing that all Petrov type D spacetimes admitting a conformal Killing tensor of Segre type \([11](11)\) necessarily are conformally Killing-Yano. As a consequence the \(KS_1\) and \(KS_2\) spacetimes ought to belong to the classes of spacetimes considered in \([8]\). Although the title of \([8]\) gives the impression that the paper deals with non-constant eigenvalues only, the authors claim that all results remain valid also when one of the eigenvalues is constant. However the latter metrics appear not to be equivalent to the \(KS_1\) and \(KS_2\) metrics! A further problem is to determine all possible Killing spinor spacetimes in the complement of \(KS_1\) and \(KS_2\) 1.

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1 at the time of writing at least one example has been found of a Killing spinor spacetime in which both conditions \(KS_1\) and \(KS_2\) are violated