MODULI OF CURVES AS MODULI OF $A_\infty$-STRUCTURES

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Abstract. We define and study the stack $U_{ns,a}^{g}$ of (possibly singular) projective curves of arithmetic genus $g$ with $g$ smooth marked points forming an ample non-special divisor. We define an explicit closed embedding of a natural $G_m$-torsor $\tilde{U}_{ns,a}^{g}$ over $U_{ns,a}^{g}$ into an affine space and give explicit equations of the universal curve (away from characteristics 2 and 3). This construction can be viewed as a generalization of the Weierstrass cubic and the $j$-invariant of an elliptic curve to the case $g > 1$. Our main result is that in characteristics different from 2 and 3 the moduli space $\tilde{U}_{ns,a}^{g}$ is isomorphic to the moduli space of minimal $A_\infty$-structures on a certain finite-dimensional graded associative algebra $E_g$ (introduced in [12]). We show how to compute explicitly the $A_\infty$-structure associated with a curve $(C, p_1, \ldots, p_g)$ in terms of certain canonical generators of the algebra $O(C \setminus \{p_1, \ldots, p_g\})$ and canonical formal parameters at the marked points. We show that the GIT quotients associated with our representation of $U_{ns,a}^{g}$ as the quotient of an affine scheme by $G_m$ give weakly modular projective birational models of $M_{g, g}$ in the sense of [9]. We also consider an analogous picture for curves of arithmetic genus 0 with $n$ marked points which gives an interpretation of the moduli space of $\psi$-stable curves in terms of $A_\infty$-structures.

Introduction

The idea to study algebraic varieties in terms of their derived categories of coherent sheaves goes back at least to the work of Bondal-Orlov [5]. An example that stimulated the present work is the fact that a smooth projective curve of genus $g \geq 2$ can be recovered from the corresponding derived category. In the subsequent development of the theory of derived categories it has been realized that a more flexible framework is obtained by considering their enhancements to dg-categories or $A_\infty$-categories (see [4], [39]). The main goal of our work, continuing [12], is to get a computable invariant of a curve from the corresponding enhanced derived category. The immediate motivation was the fact that one can recover the $j$-invariant of an elliptic curve by studying the $A_\infty$-algebra associated with the generator $O \oplus L$ of the derived category, where $L$ is a line bundle of degree 1 (see [10], [19], [20]). To get an analogous picture in the higher genus case we considered in [12] the $A_\infty$-algebra coming from a certain generator of the derived category for a curve of genus $g$ that depends on a choice of $g$ generic points (see (0.0.1) below). We showed that already triple products in this $A_\infty$-algebra give a computable invariant which distinguishes generic curves for $g \geq 6$. In the present paper we push this analysis further by considering the entire $A_\infty$-algebra structure, by comparing the corresponding moduli space with the moduli space of curves, and by extending this picture to some singular curves.

The new feature of this paper is the introduction of a more classical algebraic structure (that we call a marked algebra of genus $g$), associated with a curve $C$ of genus $g \geq 1$

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equipped with \( g \) generic points \( p_1, \ldots, p_g \), which turns out to carry the same information as the corresponding \( A_\infty \)-structure. Namely, we consider the algebra \( \mathcal{O}(C \setminus \{p_1, \ldots, p_g\}) \), equipped with the filtration recording the information about the poles at the marked points. We show (assuming the characteristic is not 2 or 3) that this algebra has a canonical set of generators, so that the defining equations can be written explicitly, generalizing the Weierstrass cubic equation in genus 1. Thus, one can view the coefficients of these defining equations (that can be rescaled by elements of \( \mathbb{G}_m^g \)) as the higher genus analog of the \( j \)-invariant. The classical part of this picture (not involving \( A_\infty \)-structures) is reminiscent of Petri’s analysis of the equations of the canonical curve (see [27]), however, we need fewer genericity assumptions and there is fewer coefficients in our defining equations. For example, we allow the curve to be hyperelliptic and singular. The number of coefficients in our picture is a quadratic polynomial in \( g \), whereas the number of coefficients in Petri’s relations is a cubic polynomial in \( g \).

Let us formulate our main result in more detail. We start with a projective curve \( C \) over \( k \) (where \( k \) is a field) of arithmetic genus \( g \) with \( g \) smooth marked points \( p_1, \ldots, p_g \) such that \( h^0(C, \mathcal{O}_C(p_1 + \ldots + p_d)) = 1 \), i.e., the divisor \( D = p_1 + \ldots + p_g \) is non-special. As in [12], we consider the algebra \( \text{Ext}^*(G, G) \) for

\[
G = \mathcal{O}_C \oplus \mathcal{O}_{p_1} \oplus \ldots \oplus \mathcal{O}_{p_g} \tag{0.0.1}
\]

The condition that \( D \) is non-special implies that the algebra \( \text{Ext}^*(G, G) \) can be identified with the graded associative algebra \( E_g \) (in [12] we called this algebra \( E_{g,g} \)) defined as follows:

\[
E_g = k[Q]/J, \tag{0.0.2}
\]

where \( k[Q] \) is the path algebra of the quiver \( Q \) with \( g+1 \) vertices marked as \( \mathcal{O}, \mathcal{O}_{p_1}, \ldots, \mathcal{O}_{p_g} \) that has for each \( i = 1, \ldots, g \) one arrow \( A_i \) of degree 0 going from \( \mathcal{O}_{p_i} \) to \( \mathcal{O} \) and one arrow \( B_i \) of degree 1 going from \( \mathcal{O} \) to \( \mathcal{O}_{p_i} \). The ideal \( J \) is generated by the relations

\[
A_i B_i A_i = B_i A_i B_i = A_i B_j,
\]

where \( i \neq j \). More precisely, the isomorphism \( \text{Ext}^*(G, G) \simeq E_g \) depends on a choice of trivializations of the tangent spaces at the marked points. Thus, the \( A_\infty \)-enhancement of the derived category provides a minimal \( A_\infty \)-structure on \( E_g \) associated with \((C, p_1, \ldots, p_g)\). In fact, as we showed in [12], any such \( A_\infty \)-structure is determined (up to an equivalence) by \( m_i \) with \( i \leq 6 \), provided \( g \geq 2 \).

The construction of \( A_\infty \)-structures associated with curves works well in families over any affine base. This leads us to define the stack of non-special curves \( \mathcal{U}_{g,g}^{\text{ns},a} \) parametrizing (not necessarily nodal) projective curves \( C \) with \( g \) smooth marked points \( p_1, \ldots, p_g \), such that the divisor \( p_1 + \ldots + p_g \) is non-special and ample. Note that this is an algebraic stack but not a Deligne-Mumford stack. On the other hand, we define the moduli stack of minimal \( A_\infty \)-structures on \( E_g \), compatible with its algebra structure, viewed up to a gauge equivalence, and show that it is actually an affine scheme of finite type over \( k \). Our main result compares the moduli of non-special curves with the moduli of \( A_\infty \)-structures on \( E_g \).

**Theorem A.** Assume that the characteristic of \( k \) is not 2 or 3. The algebraic stack \( \mathcal{U}_{g,g}^{\text{ns},a} \) is equivalent to the quotient-stack \( \overline{\mathcal{U}}_{g,g}^{\text{ns},a}/\mathbb{G}_m^{g} \), where \( \overline{\mathcal{U}}_{g,g}^{\text{ns},a} \) is an affine scheme of finite type.
over $k$. Furthermore, the affine scheme $\widetilde{U}^{ns,a}_{g,g}$ is naturally isomorphic to the moduli scheme of minimal $A_\infty$-structures on the algebra $E_g$ up to a gauge equivalence.

Note that in genus 1 this result is very close to Theorem C in [20] (and we use some ideas of [20] in the proof of our result). The ampleness condition in the definition of $U^{ns,a}_{g,g}$ for $g = 1$ becomes equivalent to the irreducibility. For arbitrary $g \geq 1$ the construction of the $A_\infty$-structure on $E_g$ associated with $(C, p_1, \ldots, p_g)$ works even if $p_1 + \ldots + p_g$ is not ample, however, the ampleness condition is needed in order to have an isomorphism of moduli spaces in Theorem A (see Remark 4.3.2).

The proof of Theorem A consists of two parts. First, we construct an explicit embedding of $\widetilde{U}^{ns,a}_{g,g}$ into an affine space, using a canonical basis in the algebra of the form $\mathcal{O}(C \setminus \{p_1, \ldots, p_g\})$. Then we study the formal neighborhood of the point in $\widetilde{U}^{ns,a}_{g,g}$ corresponding to the most singular curve in it, the cuspidal curve of genus $g$, which is the union of $g$ usual cuspidal curves of genus 1, joined at their cusps (see Section 4.4). We prove that the relevant deformation functors (for curves and for $A_\infty$-structures) are isomorphic and then use the $\mathbb{G}_m$-action, where $\mathbb{G}_m \subset \mathbb{G}_m^g$ is the diagonal, to deduce the result.

Thus, any minimal $A_\infty$-structure on $E_g$ corresponds to some (possibly quite singular) curve. Since smoothness can be characterized intrinsically in terms of the dg-category of perfect complexes (see [22, Prop. 3.13]), we get the following compact description of the derived categories associated to smooth curves.

**Corollary B.** Assume that $k$ is a perfect field of characteristic $\neq 2, 3$. Let $(E^\infty, m_\bullet)$ be a minimal $A_\infty$-algebra such that $(E^\infty, m_2) \simeq E_g$. Then $E^\infty$ is smooth as an $A_\infty$-algebra if and only if there exists a smooth projective curve $C$ of genus $g$ such that

$$
D^b(C) \simeq \text{Perf}(E^\infty),
$$

where $\text{Perf}(E^\infty)$ is the perfect derived category of $E^\infty$.

The above characterization of the perfect derived categories associated to smooth curves could be useful in trying to establish the homological mirror symmetry connecting such categories with appropriate Fukaya categories (the other direction of the homological mirror symmetry involving the Fukaya categories of higher genus curves has been established in [34], [7]). However, at present there is no proposal for what should be considered on the symplectic side.

The equivalence of Theorem A is quite explicit and computable: we show how to calculate the higher products of the $A_\infty$-structure associated with a curve using explicit homotopies in some natural dg-models. The answer is given in terms of the coefficients of expansions of the canonical generators of the algebra $\mathcal{O}(C \setminus \{p_1, \ldots, p_g\})$ in terms of certain canonical formal parameters at the points $p_1, \ldots, p_g$ (these same coefficients define an affine embedding of the moduli space $\tilde{U}^{ns,a}_{g,g}$).

The representation of the moduli space $U^{ns,a}_{g,g}$, which is birational to $\mathcal{M}_{g,g}$, in the form $\tilde{U}^{ns,a}_{g,g}/\mathbb{G}_m^g$ allows to construct some interesting birational models of the coarse moduli space $M_{g,g}$ by taking various GIT quotients $\tilde{U}^{ns,a}_{g,g} // \mathbb{G}_m^g$ and considering the component of smoothable curves. We show that these GIT quotients are closed subschemes of certain projective toric varieties and that they are examples of weakly modular birational models of $M_{g,g}$ in the sense of [9] (see Proposition 2.4.1). It would be interesting to find the place...
of these spaces in the general picture of [9] and to study the natural birational maps from $M_{g,g}$ to these quotients.

Of independent interest could be our study of the moduli functor for minimal $A_\infty$-structures up to a gauge equivalence. We prove (see Theorem 4.2.4 and Corollary 4.2.5) that for any finite-dimensional graded associative $k$-algebra $E$ such that $HH^1(E)_{<0} = 0$, where $HH^i(E)$ denotes Hochschild cohomology (see Section 4.1 for our conventions on the bigrading), this moduli functor is representable by an affine scheme over $k$.

Our explicit embedding of the moduli space $\tilde{U}_{g,g}^{ns,a}$ into an affine space leads us to an explicit description of the hyperelliptic locus, which is simply the set of fixed points of the natural involution changing the sign in the trivializations of the tangent spaces at the marked points (see Theorem 2.6.3). Note that the open part of this locus corresponding to smooth curves is a $\mathbb{G}_m$-torsor over the configuration space of $2g + 2$ distinct points in $\mathbb{P}^1$, $g$ of which are ordered.

The paper is organized as follows. In Section 1 we introduce the moduli stacks $U_{g,g}^{ns,a}$ and $\tilde{U}_{g,g}^{ns,a}$ and study their relation with marked algebras of genus $g$ (which arise as algebras of the form $\mathcal{O}(\mathbb{C}\setminus \{p_1, \ldots, p_g\})$). We prove the equivalence of the corresponding moduli problems over $\text{Spec}(\mathbb{Z}[1/6])$ and show that it leads to an explicit embedding of $\tilde{U}_{g,g}^{ns,a}$ into an affine space (see Theorem 1.2.3).

In Section 2 we find a minimal set of generators of the algebra of functions on $\tilde{U}_{g,g}^{ns,a}$ and some relations between them. Note that some parts of Section 2 are quite computational, however, the results of this Section are not used in the proof of Theorem A. Then in 2.4 we study the GIT quotients $\tilde{U}_{g,g}^{ns,a}/\mathbb{G}_m$ and in 2.6 we study the hyperelliptic locus. In Section 2.7 we work out the relation between our approach and Petri’s analysis of equations for the canonical embedding of a non-hyperelliptic curve.

In Section 3 we give a method for computing $A_\infty$-structures associated with curves, using a version of Čech resolutions (where open neighborhoods of $p_i$’s are replaced with the formal neighborhoods). The explicit formulas for the triple products are quite simple. Those for $m_4$ and $m_5$ are more involved and are given in the Appendix.

In Section 4 we establish an isomorphism between $\tilde{U}_{g,g}^{ns,a}$ and the moduli space of $A_\infty$-structures stated in Theorem A. First, in 4.2 we prove the criterion of representability of the moduli functor of minimal $A_\infty$-structures. Then in 4.4 we study the geometry of the cuspidal curve of genus $g$. In 4.5 we compare the deformation theories for our two moduli problems at the point corresponding to the cuspidal curve, which allows us to finish the proof in 4.6. In 4.7 we point out some consequences for the Hochschild cohomology of $E_g$ and for the normal forms of $A_n$-structures on $E_g$ for small $n$.

Finally, in Section 5 we consider an analogous picture for curves of arithmetic genus 0 with $n$ marked points. We consider the moduli stack of $\psi$-prestable curves (this condition means that each component of a curve contains at least one marked point), and give a natural presentation of the moduli space as a quotient of the explicit affine scheme by $\mathbb{G}_m^*$ (see Theorem 5.1.4), as well as an interpretation in terms of the moduli stack of $A_\infty$-structures (in Theorem 5.2.1). We show that one of the corresponding GIT quotients is the projective moduli scheme of $\psi$-stable curves considered in [9, Sec. 4.2.1].
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1. Moduli of non-special curves

1.1. Non-special curves and marked algebras. Consider the moduli stack $U_{g,g}^{ns,a}$ classifying (not necessarily smooth) projective curves $C$ of arithmetic genus $g$ with $g$ marked points $p_1, \ldots, p_g$, such that the following conditions hold:

(i) $C$ is smooth at each marked point;
(ii) $h^0(C, p_1 + \ldots + p_g) = 1$;
(iii) the divisor $D = p_1 + \ldots + p_g$ on $C$ is ample.

Note that since $\chi(O_C(p_1 + \ldots + p_g)) = \chi(O) + g = 1$, the condition (ii) implies that $H^1(C, O_C(p_1 + \ldots + p_g)) = 0$. Also, the condition (iii) ensures that there is at least one marked point on each component of $C$.

Thus, if $R$ is a commutative ring then $U_{g,g}^{ns,a}(R)$ classifies flat proper families $\pi : C \to \text{Spec}(R)$ of curves of arithmetic genus $g$ with $g$ marked points $p_i : \text{Spec}(R) \to C$, such that $\pi$ is smooth near the images of $p_i$, the effective divisor $D = \text{im}(p_1) + \ldots + \text{im}(p_g) \subset C$ is ample and the natural map

$$R \to R\Gamma(C, O_C(D))$$

is an isomorphism.¹

Let also $\tilde{U}_{g,g}^{ns,a} \to U_{g,g}^{ns,a}$ be the $\mathbb{G}_m$-torsor corresponding to choices of nonzero tangent vectors at the marked points. In terms of families over $\text{Spec}(R)$ this corresponds to choices of sections $v_i \in H^0(C, O_C(p_i)/O_C)$ inducing an isomorphism

$$R \simeq H^0(C, O_C(p_i)/O_C)$$

for each $i = 1, \ldots, g$.

We are going to show (away from characteristics 2 and 3) that $\tilde{U}_{g,g}^{ns,a}$ is an affine scheme of finite type by relating it to a different kind of data.

For a base ring $R$ consider the algebra $R^g = R \times \ldots \times R$ (the direct product of $g$ copies of $R$). We denote by $e_i \in R^g$, $i = 1, \ldots, g$, the natural idempotents. For $g \geq 1$ let us consider the subalgebra

$$C_{R,g} \subset R^g[t]$$

consisting of polynomials with coefficients in $R^g$ with no linear term in $t$ and with constant term in $R \subset R^g$. We view $C_{R,g}$ as a graded $R$-algebra, where $\deg t = 1$.

Let $S_g$ denote the semigroup $(\mathbb{Z}_{>0})^g$. We denote elements of $S_g$ as formal effective divisors $n_1p_1 + \ldots + n_gp_g$ with $n_i \in \mathbb{Z}_{>0}$ (where $p_i$ are formal symbols). We also view $S_g$ as a lattice with respect to the partial order

$$n_1p_1 + \ldots + n_gp_g \leq n'_1p_1 + \ldots + n'_gp_g$$

if $n_1 \leq n'_1, \ldots, n_g \leq n'_g$.

¹The subscripts $ns$ and $a$ stand for non-special and ample, since this is what we require of the divisor $p_1 + \ldots + p_g$. 

5
Definition 1.1.1. A marked algebra of genus $g$ over $R$ is a commutative $R$-algebra $A$ equipped with the following data:

(i) an exhaustive $S_g$-valued algebra filtration $(F_* A)$ on $A$ such that $F_{n_1 p_1 + \ldots + n_g p_g} A$ are $R$-submodules, and such that the map

$$n_1 p_1 + \ldots + n_g p_g \mapsto F_{n_1 p_1 + \ldots + n_g p_g} A$$

is a morphism of lattices, i.e.,

$$F_{\min(n,n')} = F_n \cap F_{n'}, \quad F_{\max(n,n')} = F_n + F_{n'}.$$ 

In addition we require that for $i = 1, \ldots, g$ and any $n \in S_g$, the quotient $F_{n+p_i}/F_n$ is a free $R$-module of rank 1.

(ii) Consider an increasing filtration on $A$ given by

$$F_n A = F_{nD} A \quad \text{for } n \geq 1,$$

where $D = p_1 + \ldots + p_g$, and $F_0 A = R$. Then there should be fixed an isomorphism of graded $R$-algebras

$$\text{gr}_F^* A \simeq C_{R,g}$$

(1.1.3)

where $\text{gr}_F^* A$ is the associated graded algebra with respect to $F_* A$.

As the reader might have anticipated, marked algebras of genus $g$ arise from non-special curves.

Lemma 1.1.2. For a flat proper family of curves $\pi : C \to \text{Spec}(R)$ with $g$ marked points $p_i : \text{Spec}(R) \to C$ and trivializations (1.1.2), such that the map (1.1.1) is an isomorphism, consider the algebra $A := H^0(C \setminus D, \mathcal{O})$ with the filtration

$$F_{n_1 p_1 + \ldots + n_g p_g} = H^0(C, \mathcal{O}_C(n_1 p_1 + \ldots + n_g p_g)).$$

Then the isomorphisms (1.1.2) induce an isomorphism (1.1.3) making $A$ into a marked algebra of genus $g$ over $R$.

Proof. The vanishing of $H^1(C, \mathcal{O}_C(D))$ implies the vanishing of $H^1(C, \mathcal{O}_C(n_1 p_1 + \ldots + n_g p_g))$ for any $n_1 \geq 1, \ldots, n_g \geq 1$. Hence we have exact sequences

$$0 \to F_n \to F_{n+p_i} \to H^0(C, \mathcal{O}_C((n_i + 1)p_i)/\mathcal{O}_C(n_i p_i)) \to 0$$

for every $n = n_1 p_1 + \ldots + n_g p_g \in S_g$, where the third term is isomorphic to $R$. In particular, for every $n \in S_g$ with $n_i > 1$ the natural map

$$F_n \to H^0(C, \mathcal{O}_C(n_i p_i)/\mathcal{O}_C((n_i - 1)p_i)) \simeq R$$

is surjective. This easily implies that $n \mapsto F_n A$ is a morphism of lattices. Also, for every $n > 1$ we have an exact sequence

$$0 \to F_{(n-1)D} \to F_{nD} \to H^0(C, \mathcal{O}_C(nD)/\mathcal{O}_C((n-1)D)) \to 0,$$

while $F_D = R$. Together with the isomorphisms (1.1.2) this leads to an identification of $\text{gr}_F^* A$ with $C_{R,g}$. □
1.2. Canonical generators for marked algebras and the moduli spaces.

**Lemma 1.2.1.** Let $A$ be a marked algebra of genus $g$ over $R$. Assume that $6$ is invertible in $R$.

(i) There exist unique elements $f_i \in F_{D+p_i}A$ and $h_i \in F_{D+2p_i}A$, where $i = 1, \ldots, g$, called canonical generators of $A$, such that

\begin{align*}
f_i \mod F_{DA} &= e_it^2, \\
h_i \mod F_{2DA} &= e_it^3,
\end{align*}

\[ h_i^2 - f_i^3 \in F_{3DA}, \quad f_ih_i^2 - f_i^4 \in F_{4DA}. \]

(ii) The elements $(f_i^n : f_i^nh_i)_{i=1,\ldots,g:n \geq 0}$ form an $R$-basis in $A$.

(iii) The algebra $A$ is generated over $R$ by $(f_i)$ and $(h_i)$ and has defining relations of the form

\begin{align*}
f_if_j &= \alpha_{ij}h_i + \alpha_{ij}h_j + \gamma_{ij}f_i + \gamma_{ij}f_j + \sum_{k \neq i,j} c_{ijk}^k f_k + a_{ij}, \\
f_ih_j &= d_{ij}f_j^2 + t_{ij}h_i + v_{ij}h_j + r_{ij}f_i + \delta_{ij}f_j + \sum_{k \neq i,j} e_{ijk}^k f_k + b_{ij}, \\
h_ih_j &= \beta_{ij}f_i^2 + \beta_{ij}f_j^2 + \varepsilon_{ij}h_i + \varepsilon_{ij}h_j + \psi_{ij}f_i + \psi_{ij}f_j + \sum_{k \neq i,j} \gamma_{ijk}^k f_k + u_{ij}, \\
h_i^2 &= f_i^3 + \sum_{j \neq i} g_{ij}^2 h_j + \pi_i f_i + \sum_{j \neq i} k_{ij}^2 f_j + s_i,
\end{align*} \quad (1.2.1, 1.2.2, 1.2.3, 1.2.4)

where $i \neq j$ (the coefficients are some elements of $R$).

**Proof.** (i) Let $\tilde{f}_i \in F_{D+p_i}A$ and $\tilde{h}_i \in F_{D+2p_i}$ be some liftings of $e_it^2$ and of $e_it^3$, respectively. Then we have

\[ \tilde{f}_i^3 \equiv e_it^6 \equiv \tilde{h}_i^2 \mod F_{5DA}. \]

Hence,

\[ \tilde{h}_i^2 - \tilde{f}_i^3 \in F_{5DA} \cap F_{3DA} = F_{3DA+2p_i}. \]

Note that $F_{D+2p_i}A$ (resp., $F_{D+p_i}$) has an $R$-basis

\[ 1, \tilde{f}_i, \tilde{h}_i \ (\text{resp., } 1, \tilde{f}_i). \]

Thus, any other liftings of $e_it^2$ and $e_it^3$ have form

\[ f_i = \tilde{f}_i + a_i, \quad h_i = \tilde{h}_i + b_i \tilde{f}_i + c_i. \]

We have

\[ h_i^2 - f_i^3 \equiv \tilde{h}_i^2 - \tilde{f}_i^3 + 2b_ih_i\tilde{f}_i + (b_i^2 - 3a_i) \tilde{f}_i^2 \mod F_{3DA}. \]

Since $(\tilde{h}_i\tilde{f}_i, \tilde{f}_i^2)$ projects to a basis of $F_{3DA}/F_{3DA}$, it follows that we can choose $a_i$ and $b_i$ uniquely so that $h_i^2 - f_i^3 \in F_{3DA}$. There still remains an ambiguity in adding a constant to $h_i$. We claim that the condition $f_ih_i^2 - f_i^4 \in F_{4DA}$ uniquely fixes this ambiguity. Indeed, if we replace $h_i$ by $h_i + c$ then $f_ih_i^2 - f_i^4 \in F_{4DA}$ gets changed modulo $F_{4DA}$ to

\[ f_ih_i^2 - f_i^4 + 2c_f h_i \mod F_{4DA}. \]
Now our claim follows from the fact that \( f_i h_i \) projects to a generator of \( F_{4D+p_i} A / F_{4D} A \cong R \).

(ii) This follows immediately from the fact that the initial parts of these elements form a basis in \( g_1^* A \).

(iii) By (ii), \( A \) is generated by \((f_i)\) and \((h_i)\). To see that relations of the form (1.2.1), (1.2.2) and (1.2.3) hold for any \( i \neq j \), we just observe that

\[
\begin{align*}
 f_i f_j & \in F_{2D+p_i+p_j} A, \hspace{1em} f_i h_j \in F_{2D+p_i+2p_j}, \hspace{1em} h_i h_j \in F_{2D+2p_i+2p_j} \\
\end{align*}
\]

and use the fact that the \( g + 5 \) elements

\[
1, f_1, \ldots, f_g, h_i, f_j^2, f_j^2
\]
form an \( R \)-basis of \( F_{2D+2p_i+2p_j} \) with the first \( g + 4 \) (resp., \( g + 3 \)) giving an \( R \)-basis of \( F_{2D+p_i+2p_j} \) (resp., \( F_{2D+p_i+p_j} \)). Similarly, the condition \( h_i^2 - f_i^3 \in F_{3D} A \) implies a relation of the form

\[
h_i^2 - f_i^3 = c_i h_i + \sum_{j \neq i} g_{ij}^2 h_j + \pi_i f_i + \sum_{j \neq i} k_{ij} f_j + s_i.
\]

Further, we have

\[
0 \equiv f_i h_i^2 - f_i^4 \equiv c_i f_i h_i \mod F_{4D} A.
\]

Since \( f_i h_i \) projects to a basis element in \( F_{5D} A / F_{4D} A \), this implies that \( c_i = 0 \).

Let \( A' \) be the quotient of the polynomial algebra \( R[f_1, \ldots, f_g, h_1, \ldots, h_g] \) by the relations (1.2.1)–(1.2.4). We claim that the elements of \( \mathcal{B} = \{ f_i^n, h_i^n | \ i = 1, \ldots, g; n \geq 0 \} \) span \( A' \) as an \( R \)-module. Indeed, let us consider the degree lexicographical order on monomials in \((f_i), (h_i)\), given by

\[
\deg f_i = 2, \hspace{1em} \deg h_i = 3, \hspace{1em} h_1 > \cdots > h_g > f_1 > \cdots > f_g.
\]

Then our relations show that any monomial except for elements of \( \mathcal{B} \) can be expressed in \( A' \) in terms of smaller ones, which proves our claim. Since the natural homomorphism \( A' \to A \) sends elements of \( \mathcal{B} \) to an \( R \)-basis of \( A \), it follows that this homomorphism is an isomorphism.

\[\square\]

**Lemma 1.2.2.** Let \( R \) be a commutative ring. An associative \( R \)-algebra \( B \) generated by \( f_1, \ldots, f_g, h_1, \ldots, h_g \) with defining relations (1.2.1)–(1.2.4) has elements \((f_i^n, h_i^n)_{i=1,\ldots,g,n\geq 0}\) as an \( R \)-basis if and only if the coefficients

\[
(\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \delta_{ij}, \varepsilon_{ij}, \psi_{ij}, \pi_i, d_{ij}, t_{ij}, v_{ij}, r_{ij}, e_{ij}^k, e_{ij}^{k+1}, g_i^j, k_i^j, a_{ij}, b_{ij}, u_{ij}, s_i)
\]

(1.2.6)

satisfy some universal polynomial relations with integer coefficients. Such an algebra \( B \) has a unique structure of a marked algebra of genus \( g \), such that \((f_i)\) and \((h_i)\) are the corresponding canonical generators constructed in Lemma 1.2.1(i).

**Proof.** The property in question holds if and only the four types of elements in \( R[f_1, \ldots, f_g, h_1, \ldots, h_g] \) given by the difference of the left and right sides in (1.2.1)–(1.2.4) form a Gröbner basis of the ideal they generate with respect to the degree lexicographical order, defined by (1.2.5). To get the corresponding relations between the coefficients one has to look at two ways to apply the relations to

\[
f_i f_j, f_i f_j h_k, f_i f_j h_j, f_i f_j^2, f_i h_j h_k, f_i h_j, h_i h_j, h_i^2 h_j, h_i^2
\]
and then compare the coefficients of the elements \((f^n_i, f^n_i h_i)\) (see e.g., [8, Ch. 15]).

Let us rename the basis elements of \(B\) (different from 1) as follows:

\[
b_i[n] := \begin{cases} 
  f_i^{n/2}, & n \text{ is even}, \\
  f_i^{(n-3)/2} h_i, & n \text{ is odd},
\end{cases}
\]

where \(n \geq 2, i = 1, \ldots, g\). To equip \(B\) with the structure of a marked algebra let us define for each \(i\) and \(n \geq 1\) the subspace \(F_{D+(n-1)p_i} B \subset B\) as the span of

\[
(1, b_i[2], \ldots, b_i[n]).
\]

Then we set for \((n_1, \ldots, n_g) \in S_g\),

\[
F_{n_1 p_1 + \ldots + n_g p_g} B = \sum_{i=1}^{g} F_{D+(n_i-1)p_i} B.
\]

This clearly satisfies condition (i) of Definition 1.1.1. Furthermore, the associated graded algebra \(\bigoplus_n F_{nD} B/F_{(n-1)D} B\) is simply the algebra with the relations obtained as initial forms:

\[
f_i f_j = 0, \quad f_i h_j = 0, \quad h_i h_j = 0, \quad h_i^2 = f_i^3,
\]

so there is a unique isomorphism from this algebra to \(C_{R,g}\) sending \(f_i\) to \(e_i t^2\) and \(h_i\) to \(e_i t^3\).

Note that the relation (1.2.4) implies that \(h_i^2 - f_i^3 \in F_{3D} B\). Also, we have

\[
f_i(h_i^2 - f_i^3) = \sum_{j \neq i} g_i^j f_i h_j + \pi_i f_i^2 + \sum_{j \neq i} k_i^j f_i f_j + s_i f_i.
\]

The relation (1.2.2) implies that \(f_i h_j \in F_{4D} B\) for \(i \neq j\), so we deduce that \(f_i(h_i^2 - f_i^3) \in F_{4D}\). Thus, \((f_i)\) and \((h_i)\) are the canonical generators of \(B\). \(\square\)

Let us denote by \(S_g\) the affine scheme over \(\mathbb{Z}\) defined by the universal polynomial relations of Lemma 1.2.2 between the coordinates (1.2.6).

**Theorem 1.2.3.** The stack \(\widehat{U}_{g,g}^{ns,a} \times_{\text{Spec}(\mathbb{Z})} \text{Spec}([1/6])\) is isomorphic to the moduli stack of marked algebras of genus \(g\) (over \(\text{Spec}(\mathbb{Z}[1/6])\)). The latter moduli stack is isomorphic to the affine scheme \(\mathcal{S}_g \times_{\text{Spec}(\mathbb{Z})} \text{Spec}([1/6])\).

These isomorphisms are compatible with the \(\mathbb{G}_m\)-actions\(^2\): on \(\widehat{U}_{g,g}^{ns,a}\) — by rescaling the tangent vectors \(v_i\) at the marked points by

\[
v_i \mapsto \lambda_i^{-1} v_i;
\]

on marked algebras—by composing the isomorphism (1.1.3) with the \(R\)-automorphisms of \(C_{R,g}\) sending \(te_i\) to \(\lambda_i^{-1} te_i\); and on \(S_g\) by rescalings of the structure constants induced by the substitutions

\[
f_i \mapsto \lambda_i^{-2} f_i, \quad h_i \mapsto \lambda_i^{-3} h_i. \tag{1.2.7}
\]

\(^2\)We normalize our \(\mathbb{G}_m\)-actions so that the induced action of the diagonal \(\mathbb{G}_m \subset \mathbb{G}_m^g\) has positive weights on the algebra of functions on \(S_g\), where the action on functions is given by the operators \((\lambda^{-1})^*\).
Proof. Let $R$ be a commutative ring such that 6 is invertible in $R$. By Lemma 1.1.2, we have a functor associating with a family of curves $\pi : C \to \text{Spec}(R)$ with $g$ marked points and trivializations (1.1.2), which is a family in $\mathcal{U}^{gni}_{g,a}(R)$, a marked algebra over $R$.

Conversely, let $A$ be a marked algebra over $R$. Consider the Rees algebra associated with the filtration ($F_nA$) on $A$:

$$RA := \bigoplus_{n \geq 0} F_nA.$$  

This is a graded algebra, so we can consider the corresponding projective scheme

$$C := \text{Proj}(RA)$$

over $\text{Spec}(R)$. Let $T$ denote the element $1 \in F_{DA} = (RA)_1$. Then the open subset $T \neq 0$ in $C = \text{Proj}(RA)$ is the affine scheme $\text{Spec}(A)$, while the complementary closed subset is the subscheme $\text{Proj}(\mathfrak{D}^f A)$ which is isomorphic to $\text{Proj}(C_{R,g})$ because of the isomorphism (1.1.3). We have natural sections $p_i : \text{Spec}(R) \to \text{Proj}(C_{R,g})$, $i = 1, \ldots, g$, such that $\text{Proj}(C_{R,g})$ is the disjoint union of the images of $p_1, \ldots, p_g$.

By Lemma 1.2.1, we have unique generators $(f_i)$ and $(h_i)$ satisfying the defining relations of the form (1.2.1)–(1.2.4). Let $F_i$ (resp., $H_i$) be the element $f_i \in F_{3DA} = (RA)_2$ (resp., $h_i \in F_{3DA} = (RA)_3$). The algebra $RA$ is generated by the elements $(F_i, H_i, T)$, where $i = 1, \ldots, g$, with the following defining relations, which are homogeneous versions of relations (1.2.1)–(1.2.4):

$$F_i F_j = \alpha_{ij} H_i T + \alpha_{ij} H_j T + \gamma_{ji} F_i T^2 + \gamma_{ij} F_j T^2 + \sum_{k \neq i, j} e_{ij}^k F_k T^2 + a_{ij} T^4,$$

$$F_i H_j = d_{ij} F_j T^2 + t_{ij} H_i T^2 + v_{ij} H_j T^2 + r_{ij} F_i T^3 + \delta_{ij} F_j T^3 + \sum_{k \neq i, j} v_{ij}^k F_k T^3 + b_{ij} T^5,$$

$$H_i H_j = \beta_{ij} F_i T^2 + \beta_{ij} F_j T^2 + \varepsilon_{ji} H_i T^3 + \varepsilon_{ij} H_j T^3 + \psi_{ji} F_i T^4 + \psi_{ij} F_j T^4 + \sum_{k \neq i, j} l_{ij}^k F_k T^4 + u_{ij} T^6,$$

$$H_i^2 = F_i^3 + \sum_{j \neq i} g_{ij}^i H_j T^3 + \pi_i F_i T^4 + \sum_{j \neq i} k_{ij}^i F_j T^4 + s_i T^6,$$  

(1.2.11)

where $i \neq j$. On the image $p_i$, we have $T = F_j = H_j = 0$ for $j \neq i$ and $F_i \neq 0$, $H_i \neq 0$. Hence, the section $H_i/F_i$ trivializes $\mathcal{O}_C(1)$ near the image of $p_i$. On the other hand, the section $T$ trivializes it on the subset $(T \neq 0)$. Hence, the sheaf $\mathcal{O}_C(1)$ is invertible. Similarly we see that $\mathcal{O}_C(n) \simeq \mathcal{O}_C(1)^{\otimes n}$, so $\mathcal{O}_C(1)$ is ample. Thus, the divisor $(T = 0) = D = \text{im}(p_1) + \ldots + \text{im}(p_g)$ on $C$ is ample.

Next, we want to check that the projection $\pi : C \to \text{Spec}(R)$ is flat of relative dimension 1 and smooth near the images of $p_1, \ldots, p_g$. We have $C \setminus D = \text{Spec}(A)$, and $A$ is a free $R$-module, as follows from Lemma 1.2.1(ii). To see that $\pi$ is flat near $D$ consider the open subset $F_i H_i \neq 0$. Let $A_i$ be the degree 0 part in the localization $(RA)_{F_i H_i}$. On this open subset $D$ is given as the vanishing locus of the function

$$t_i := TF_i/H_i \in A_i,$$
which is not a zero divisor in \( A_i \), since \( T \) is not a zero divisor in \( RA \). Thus, the localization \( (A_i)_{t_i} \) is flat over \( R \). On the other hand, we have an isomorphism \( A_i/(t_i) \simeq R \). This implies that \( A_i \) is flat over \( R \) by Lemma 1.2.4 below. This also shows that \( \pi \) has relative dimension 1 near \( p_1, \ldots, p_g \). Next, let us show that \( \pi : C \to \text{Spec}(R) \) is smooth near \( p_1, \ldots, p_g \).

Consider the open subset \( F, H_1 \neq 0 \) to study the equations of \( C \) near \( p_1 \). Let \( T \) be the ideal sheaf on this open subset corresponding to \( p \) (which is not a zero divisor in \( A \) over the open subset \( F \)). It follows that

\[
\frac{T^2}{F_1} = \left( \frac{TF_1}{H_1} \right)^2 \cdot \frac{H_1^2}{F_1^3} \in T^2, \quad (1.2.12)
\]

\[
\frac{H_j T}{F_1^2} = \frac{H_j H_1}{F_1^2} \cdot \frac{TF_1}{H_1} \in T^2.
\]

Similarly \( F_1^2 T/(F_1 H_1) \in T^3 \), etc. Hence, the equations (1.2.8) and (1.2.9) give for \( i = 1 \) (over the open subset \( F_1 H_1 \neq 0 \))

\[
\frac{F_j}{F_1} = \alpha_{j1} \frac{H_1 T}{F_1^2} \mod T^2. \quad (1.2.13)
\]

\[
\frac{H_j}{H_1} = 0 \mod T^2. \quad (1.2.14)
\]

It follows that \( \mathcal{I}/T^2 \) is generated by the single element \( t_1 = TF_1/H_1 \) in a neighborhood of \( p_1 \). This implies the required smoothness near \( p_1 \) (and similarly, near \( p_i \)).

Next, we claim that viewing elements of the algebra \( A \) as functions on \( C \setminus D \) and considering various polar conditions near \( p_1, \ldots, p_g \), we recover the \( S_g \)-valued filtration \( (F_i A) \). Indeed, as we have seen above we can use \( t_1 = TF_1/H_1 \) as a local parameter near \( p_i \). The equation (1.2.12) shows that \( f_1 = F_1/T^2 \) has a pole of order 2 near \( p_1 \). Since \( h_1^2/f_1^3 = H_1^2/F_1^3 \) is invertible near \( p_1 \), it follows that \( h_1 \) has a pole of order 3 near \( p_1 \). Now the equations (1.2.13) and (1.2.14) imply that \( f_j \) and \( h_j \) for \( j \neq 1 \) have order at most 1 at \( p_1 \). Next, the equation (1.2.4) implies that

\[
\frac{h_1^2}{f_1^3}(p_1) = 1,
\]

so we deduce that the Laurent decompositions of \( f_1 \) and \( h_1 \) in terms of the parameter \( t_1 \) start with \( t_1^{-2} \) and \( t_1^{-3} \), respectively. Of course, the same arguments works for any point \( p_i \). Since \( (f_1^n, f_i h_i) \) form an \( R \)-basis of \( A \), this gives an identification of the filtration \( (F_i A) \) (whose terms are generated by appropriate parts of this basis) with the filtration given by polar conditions at \( p_1, \ldots, p_g \).

Since we can characterize \( H^0(C, \mathcal{O}(nD)) \) inside the algebra \( A \) by polar conditions at \( p_1, \ldots, p_g \), we deduce that \( R = H^0(C, \mathcal{O}) = H^0(C, \mathcal{O}(D)) \) and that \( H^0(C, \mathcal{O}(nD)) \) is a free \( R \)-module of rank \( (n-1)g+1 \) for \( n \geq 1 \). By flatness of the family the analogous assertion is true for geometric fibers \( C_s \) over \( \text{Spec}(k) \) (with \( R \) replaced by \( k \)). It follows that these fibers \( C_s \) are projective curves of arithmetic genus \( g \) (since \( \mathcal{O}(D) \) is ample), and hence they have \( H^1(C_s, \mathcal{O}(D)) = 0 \). Finally, the local parameters \( t_1, \ldots, t_g \) induce the required trivializations (1.1.2), so our family is an object of \( \tilde{U}_{g,a}^{m,a}(R) \). It is clear that the marked algebra associated with this object is the marked algebra \( A \) we started with.
Conversely, if the marked algebra $A$ was associated with a family of curves $C$ in $U_{g,g}^{ns,a}(R)$ then we need to check that the above construction recovers the original family of curves. Note that we have a natural morphism $C \to \text{Proj}(RA)$. But this morphism can be identified with the projective morphism given by the linear system $|3D|$ with the basis $1,(f_i),(h_i)$, so it is an isomorphism (since $D$ is ample).

The isomorphism with $S_g \times_{\text{Spec}(Z)} \text{Spec}(Z[1/6])$ follows from Lemma 1.2.2.

The compatibility with the $\mathbb{G}_m^g$-actions is a straightforward check. □

**Lemma 1.2.4.** Let $A$ be a commutative $R$-algebra, and let $t \in A$ be a nonzero divisor. Assume that $A_t$ and $A/(t)$ are flat over $R$. Then $A$ is also flat over $R$.

**Proof.** Consider the exact sequence

$$0 \to A \xrightarrow{t} A \to A/(t) \to 0.$$ 

Let $M$ be any $R$-module. Since $A/(t)$ is a flat $R$-module, the long exact sequence of $\text{Tor}(?, M)$ shows that the multiplication by $t$ induces an isomorphism $\text{Tor}^R_i(A, M) \to \text{Tor}^R_i(A, M)$ for any $i \geq 1$. Since $A_t = \lim_t(A \xrightarrow{t} A \xrightarrow{t} A\ldots)$, this implies the isomorphism

$$\text{Tor}^R_i(A, M) \simeq \text{Tor}^R_i(A_t, M) = 0$$

for $i \geq 1$ since $A_t$ is flat over $R$. □

**Remarks 1.2.5.** 1. By universality, the $\mathbb{G}_m^g$-action on the moduli spaces considered in Theorem 1.2.3 extends to the $\mathbb{G}_m^g$-action on the universal curve $C$ such that the sections $p_i$ are $\mathbb{G}_m^g$-equivariant. The corresponding action on $C \setminus D$ is determined by (1.2.7).

2. It is easy to see from the above proof that for any $(C, p_1, \ldots, p_g) \in U_{g,g}^{ns,a}(k)$ the divisor $3D$ is actually very ample on $C$.

2. Further analysis of the algebra of functions on $\tilde{U}_{g,g}^{ns,a}$

**2.1. Canonical parameters near marked points and expansions.** Let $(C, p_1, \ldots, p_g)$ be a family of non-special curves in $\tilde{U}_{g,g}^{ns,a}(R)$, where $R$ is a commutative ring, and let $v_i \in H^0(\mathcal{O}_C(p_i)/\mathcal{O}_C)$ be the corresponding trivializations of normal bundles to sections $p_i$. For example, we could take the universal family over $\mathbb{Z}[1/6]$ (see Theorem 1.2.3). By a parameter of order $N \geq 1$ at $p_i$, compatible with $v_i$, we understand an element $t_i \in H^0(C, \mathcal{O}_C(-p_i)/\mathcal{O}_C(-(N + 1)p_i))$ such that $(v_i, t_i \bmod \mathcal{O}_C(-2p_i)) = 1$. Note that since $H^1(C, \mathcal{O}_C(-2p_i)/\mathcal{O}_C(-(N + 1)p_i)) = 0$, we have an exact sequence

$$0 \to H^0(\mathcal{O}_C(-2p_i)/\mathcal{O}_C(-(N + 1)p_i)) \to H^0(C, \mathcal{O}_C(-p_i)/\mathcal{O}_C(-(N + 1)p_i)) \to H^0(C, \mathcal{O}_C(-p_i)/\mathcal{O}_C(-2p_i)) \to 0$$

which implies the existence of parameters of any order $N \geq 1$ at $p_i$. For any such parameter the induced map

$$R[t_i]/(t_i^{N+1}) \to H^0(C, \mathcal{O}_C(2p_i)/\mathcal{O}_C(-p_i))$$

is an isomorphism. We also have the induced isomorphism

$$(t_i^{-N}R[t_i])/t_iR[t_i] \xrightarrow{\sim} H^0(C, \mathcal{O}_C(Np_i)/\mathcal{O}_C(-p_i)),$$
so for a section $f \in H^0(C, \mathcal{O}_C(Np_i)/\mathcal{O}_C(-p_i))$ we can define a polar part at $p_i$ as the corresponding element of $t_i^{-N}R[t_i]/R[t_i]$.

Set $D = p_1 + \ldots + p_g \subset C$, $D_i = D - p_i$. We have the following relative version of [12, Lem. 4.1.3].

**Lemma 2.1.1.** Assume that $N > 1$ is such that $N!$ is invertible in $R$. Then there exist unique parameters $t_i$ of order $N$ at $p_i$, for $i = 1, \ldots, g$, compatible with $v_i$ and such that for every $n, 1 < n \leq N$, there exists a section

$$f_i[n] \in H^0(C, D_i + np_i)$$

with the polar part $t_i^{-n}$ at $p_i$.

**Proof.** The argument is essentially the same as in [12, Lem. 4.1.3]. We start by picking arbitrary parameters $t_i$ at $p_i$ and then improve them step by step. Since $H^1(C, \mathcal{O}(D)) = 0$, we have an exact sequence

$$0 \rightarrow R = H^0(C, \mathcal{O}(D)) \rightarrow H^0(C, \mathcal{O}(D_i + 2p_i)) \rightarrow H^0(C, \mathcal{O}(2p_i)/\mathcal{O}(p_i)) \rightarrow 0.$$

Thus, we can choose $f_i[2] \in H^0(C, \mathcal{O}(D_i + 2p_i))$ with the polar part at $p_i$ of the form $\frac{1}{t_i} + \frac{c}{t_i}$ for some $c \in R$. Thus, replacing $t_i$ by $t_i + \frac{c}{2}t_i^2$ we obtain that the polar part of $f_i[2]$ becomes just $1/t_i^2$. Then we proceed by induction as in [12, Lem. 4.1.3].

Assume that $(n + 2)!$ is invertible in $R$. Then using the parameters of order $n + 2$ constructed in the above lemma we get partial Laurent expansions of sections of $\mathcal{O}_C(np_i)/\mathcal{O}_C(-3p_i)$ in $t_i^{-n}R[t_i]/t_i^3R[t_i]$. In particular, for each $i$ the section $f_i[n] \in H^0(C, D_i + np_i)$ defined in Lemma 2.1.1 has expansions

$$f_i[n] = \frac{1}{t_i^n} + c_i[n] + l_i[n]t_i + q_i[n]t_i^2 \mod t_i^3,$$

$$f_i[n] = \frac{p_{ij}[n]}{t_j} + c_{ij}[n] + l_{ij}[n]t_j + q_{ij}[n]t_j^2 \mod t_j^3$$

at $p_i$ and $p_j$ (where $j \neq i$) respectively, for some constants $p_{ij}[n], c_{ij}[n], l_{ij}[n]$ and $q_{ij}[n]$ in $R$. Note that $f_i[n]$ are determined uniquely up to adding a constant, hence, all of these constants except for $c_{ij}[n]$ are determined uniquely (and $c_{ij}[n]$ can be modified by $c_{ij}[n] \mapsto c_{ij}[n] + c_i$ for some collection of constants $(c_i)$). Note that if we only need expansions up to linear terms in $t_i$ then it is enough to consider the canonical parameters of order $n + 1$, defined whenever $(n + 1)!$ is invertible.

Now assume that $\text{char}(k) \neq 2$ or 3, and let us apply the construction of Lemma 2.1.1 for $N = 4$, which gives canonical parameters $t_i$ of order 4 and the corresponding sections $f_i[2], f_i[3]$ and $f_i[4]$. Let us fix a choice of such $f_i[2], f_i[3], f_i[4]$ (there is a freedom in adding a constant to each). Let us set $f_i = f_i[2], h_i = f_i[3]$. We will see below that $f_i$ and $h_i$ can always be adjusted by adding constants so as to satisfy conditions of Lemma 1.2.1(i) for the marked algebra $A = H^0(C - D, \mathcal{O})$. We are going to express the structure constants in the defining relations (1.2.1)–(1.2.4) of the marked algebra $A$ in terms of some of the constants $(p_{ij}[n], c_{ij}[n], l_{ij}[n], q_{ij}[n])$ (see Proposition 2.3.2 below).
2.2. Some relations. As we have observed above, the expansions of \( f_i = f_i[2] \) up to quadratic terms (resp., of \( h_i = f_i[3] \) up to linear terms, and of \( f_i[4] \) up to constant terms) at all the marked points are well defined, so the following constants are well defined.

\[
\begin{align*}
\alpha_{ij} & := p_{ij}[2], \\
\gamma_{ij} & := c_{ij}[2], \quad \beta_{ij} := p_{ij}[3], \\
\delta_{ij} & := l_{ij}[2], \quad \varepsilon_{ij} := c_{ij}[3], \quad \eta_{ij} := p_{ij}[4], \\
\pi_{ij} & := q_{ij}[2], \quad \theta_{ij} := l_{ij}[3], \quad \zeta_{ij} := c_{ij}[4].
\end{align*}
\]

For \( i \neq j \) we have \( f_i f_j \in H^0(C, \mathcal{O}(2D + p_i + p_j)) \), and analyzing the polar parts at \( p_i \) and \( p_j \) (we only need to look at the terms \( \frac{1}{t_i^j} \) and \( \frac{1}{t_j^i} \) for \( n \geq 2 \)) we get the following more precise version of (1.2.1) (see [12, (4.1.4)]):

\[
f_i f_j = \sum_{k \neq i,j} \alpha_{ik} \alpha_{jk} f_k + \alpha_{ji} h_i + \alpha_{ij} h_j + \gamma_{ji} f_i + \gamma_{ij} f_j + a_{ij}, \tag{2.2.1}
\]

for some constants \( a_{ij} = a_{ji} \). By comparing the polar parts of both sides at \( p_j \) and \( p_k \) with \( k \neq i, j \) we get (see [12, (4.1.5), (4.1.6)])

\[
\delta_{ij} = \sum_{k \neq i,j} \alpha_{ik} \alpha_{jk} + \alpha_{ji} \beta_{ij} + (\gamma_{ji} - \gamma_{jj}) \alpha_{ij}, \tag{2.2.2}
\]

\[
\alpha_{ik}(\gamma_{jk} - \gamma_{ji}) + \alpha_{jk}(\gamma_{ik} - \gamma_{ij}) = \sum_{l \neq i,j,k} \alpha_{il} \alpha_{jl} \alpha_{lk} + \alpha_{ji} \beta_{ik} + \alpha_{ij} \beta_{jk}. \tag{2.2.3}
\]

By comparing the constant terms in (2.2.1) at \( p_k \) with \( k \neq i, j \) we get (see [12, (4.1.7)])

\[
\alpha_{ik} \delta_{jk} + \alpha_{jk} \delta_{ik} + \gamma_{ik} \gamma_{jk} = \sum_{l \neq i,j} \alpha_{il} \alpha_{jl} \gamma_{lk} + \alpha_{ji} \varepsilon_{ik} + \alpha_{ij} \varepsilon_{jk} + \gamma_{ji} \gamma_{ik} + \gamma_{ij} \gamma_{jk} + a_{ij}. \tag{2.2.4}
\]

In addition, comparing the constant terms in (2.2.1) at \( p_j \) we get

\[
\alpha_{ij} \delta_{jj} + \pi_{ij} = \sum_{k \neq i,j} \alpha_{ik} \alpha_{jk} \gamma_{kj} + \alpha_{ji} \varepsilon_{ij} + \alpha_{ij} \varepsilon_{jj} + \gamma_{ji} \gamma_{ij} + a_{ij}. \tag{2.2.5}
\]

Finally, comparing the linear terms in (2.2.1) at \( p_k \), where \( k \neq i, j \), and at \( p_j \) we get

\[
\pi_{ik} \alpha_{jk} + \pi_{jk} \alpha_{ik} + (\gamma_{ik} - \gamma_{ij}) \delta_{jk} + (\gamma_{jk} - \gamma_{ji}) \delta_{ik} = \sum_{l \neq i,j} \alpha_{il} \alpha_{jl} \delta_{lk} + \alpha_{ji} \theta_{ik} + \alpha_{ij} \theta_{jk}. \tag{2.2.6}
\]

\[
\rho_{ij} + \delta_{ij} \gamma_{jj} + \alpha_{ij} \pi_{jj} = \sum_{k \neq i,j} \alpha_{ik} \alpha_{jk} \delta_{kj} + \alpha_{ji} \theta_{ij} + \alpha_{ij} \theta_{jj} + \gamma_{ji} \delta_{ij}. \tag{2.2.7}
\]

where \( \rho_{ij} \) is the coefficient of \( t_j^3 \) in the expansion of \( f_i \) at \( p_j \).

Similarly, for \( i \neq j \) we have \( f_i h_j - \alpha_{ij} f_j[4] \in H^0(C, \mathcal{O}(2D + p_i + p_j)) \), so analyzing the polar parts we deduce the identity

\[
f_i h_j = \alpha_{ij} f_j[4] + \gamma_{ij} h_j + \delta_{ij} f_j + \beta_{ji} h_i + \varepsilon_{ji} f_i + \sum_{k \neq i,j} \alpha_{ik} \beta_{jk} f_k + b_{ij} \tag{2.2.8}
\]
for some constants $b_{ij}$. By comparing the polar parts of both sides at $p_i$ and $p_k$, where $k \neq i, j$, we get
\[ \vartheta_{ji} = \sum_{k \neq i,j} \alpha_{ik} \beta_{jk} \alpha_{ki} + (\gamma_{ij} - \gamma_{ii}) \beta_{ji} + \delta_{ij} \alpha_{ji} + \alpha_{ij} \eta_{ji}, \]  \hspace{1cm} (2.2.9)
\[ \alpha_{ik} (\varepsilon_{jk} - \varepsilon_{ji}) + (\gamma_{ik} - \gamma_{ij}) \beta_{jk} = \sum_{l \neq i,j,k} \alpha_{il} \beta_{jl} \alpha_{lk} + \beta_{ji} \beta_{ik} + \delta_{ij} \alpha_{jk} + \alpha_{ij} \eta_{jk}. \]  \hspace{1cm} (2.2.10)
Comparing the constant terms in (2.2.8) at $p_k$, where $k \neq i, j$, we get
\[ \alpha_{ik} \vartheta_{jk} + \gamma_{ik} \varepsilon_{jk} + \delta_{ik} \beta_{jk} = \sum_{l \neq i,j} \alpha_{il} \beta_{jl} \alpha_{lk} + \beta_{ji} \varepsilon_{ik} + \varepsilon_{ij} \gamma_{ik} + \delta_{ij} \gamma_{jk} + \alpha_{ij} \zeta_{jk} + b_{ij}. \]  \hspace{1cm} (2.2.11)
Comparing the polar and constant terms in (2.2.8) at $p_j$ we get
\[ \pi_{ij} = \beta_{ji} \beta_{ij} + (\varepsilon_{ji} - \varepsilon_{jj}) \alpha_{ij} + \sum_{k \neq i,j} \alpha_{ik} \alpha_{kj} \beta_{jk}, \]  \hspace{1cm} (2.2.12)
\[ \alpha_{ij} \vartheta_{jj} + \rho_{ij} = \alpha_{ij} \zeta_{jj} + \delta_{ij} \gamma_{jj} + \beta_{ji} \varepsilon_{ij} + \varepsilon_{ij} \gamma_{ij} + \sum_{k \neq i,j} \alpha_{ik} \beta_{jk} \gamma_{kj} + b_{ij}. \]  \hspace{1cm} (2.2.13)
Let us denote by $\nu_{ji}$ the coefficient of $t_3^i$ in the expansion of $h_j$ at $p_i$. Looking at the linear terms in (2.2.8) at $p_i$ we get
\[ \nu_{ji} + \gamma_{ii} \vartheta_{ji} + \pi_{ii} \beta_{ji} = \alpha_{ij} l_{ji}[4] + \gamma_{ij} \vartheta_{ji} + \delta_{ij} \delta_{ji} + \beta_{ji} \vartheta_{ji} + \sum_{k \neq i,j} \alpha_{ik} \beta_{jk} \delta_{ki}. \]  \hspace{1cm} (2.2.14)
Let us fix the unique choice of $f_i[4]$ (by adding a constant) so that we have
\[ f_i^2 = f_i[4] + \sum_{j \neq i} \alpha_{ij}^2 f_j + 2 \gamma_{ii} f_i. \]  \hspace{1cm} (2.2.15)
From this, looking at the polar and constant parts at $p_i$ and $p_j$ we get
\[ 2 \delta_{ii} = \sum_{j \neq i} \alpha_{ij}^2 \alpha_{ji}, \]  \hspace{1cm} (2.2.16)
\[ 2 \pi_{ii} + 2 \gamma_{ii} \delta_{ii} = \sum_{j \neq i} \alpha_{ij}^2 \gamma_{ji} + \gamma_{ii}^2 + \zeta_{ii}, \]  \hspace{1cm} (2.2.17)
\[ \eta_{ij} = 2 \alpha_{ij} (\gamma_{ij} - \gamma_{ii}) - \sum_{k \neq i,j} \alpha_{ik}^2 \alpha_{kj}, \]  \hspace{1cm} (2.2.18)
\[ \gamma_{ij}^2 + 2 \alpha_{ij} \delta_{ij} = \zeta_{ij} + \sum_{k \neq i} \alpha_{ik}^2 \gamma_{kj} + 2 \gamma_{ii} \gamma_{ij}. \]  \hspace{1cm} (2.2.19)
In addition, looking at the linear parts at $p_j$ we get
\[ 2 \alpha_{ij} \pi_{ij} + 2 \gamma_{ij} \delta_{ij} = l_{ij}[4] + \sum_{k \neq i} \alpha_{ik}^2 \delta_{kj} + 2 \gamma_{ii} \delta_{ij}. \]  \hspace{1cm} (2.2.20)
Next, comparing the polar parts of $h_i^3$ and $f_i^3$ we get the relation

\[
h_i^3 = f_i^3 - 3\gamma_{ii}f_i[4] + (2\gamma_{ii} - 3\delta_{ii})h_i + (2\theta_{ii} - 3\pi_{ii} - 3\gamma_{ii}^2)f_i - \sum_{j\neq i}^3\alpha_{ij}^3h_j + \sum_{j\neq i}(\beta_{ij}^2 - 3\alpha_{ij}^2\gamma_{ij})f_j + s_i
\]

(2.2.21)

for some constant $s_i$. Considering the polar and constant terms at $p_j$ we get

\[
2\beta_{ij}\varepsilon_{ij} = 3\alpha_{ij}^2\delta_{ij} + 3\alpha_{ij}\gamma_{ij}^2 - 3\gamma_{ii}\eta_{ij} + (2\varepsilon_{ii} - 3\delta_{ii})\beta_{ij} + (2\theta_{ii} - 3\pi_{ii} - 3\gamma_{ii}^2)\alpha_{ij} - \sum_{k\neq i,j}^3\alpha_{ik}^3\beta_{kj} + \sum_{k\neq i,j}(\beta_{ik}^2 - 3\alpha_{ik}^2\gamma_{ik})\alpha_{kj},
\]

(2.2.22)

\[
\varepsilon_{ij}^2 + 2\beta_{ij}\vartheta_{ij} = 3\alpha_{ij}^2\pi_{ij} + 6\alpha_{ij}\gamma_{ij}\eta_{ij} + \gamma_{ij}^3 - 3\gamma_{ii}\zeta_{ij} + (2\varepsilon_{ii} - 3\delta_{ii})\varepsilon_{ij} + (2\theta_{ii} - 3\pi_{ii} - 3\gamma_{ii}^2)\gamma_{ij} - \sum_{k\neq i,j}^3\alpha_{ik}^3\varepsilon_{kj} + \sum_{k\neq i}(\beta_{ik}^2 - 3\alpha_{ik}^2\gamma_{ik})\gamma_{kj} + s_i.
\]

(2.2.23)

Next, considering the product $h_ih_j$ for $i \neq j$ we get

\[
h_ih_j = \beta_{ij}f_j[4] + \beta_{ji}f_i[4] + \varepsilon_{ij}h_j + \varepsilon_{ji}h_i + \vartheta_{ij}f_j + \vartheta_{ji}f_i + \sum_{k\neq i,j}^3\beta_{ik}\beta_{jk}f_k + u_{ij}
\]

(2.2.24)

for some constants $u_{ij} = u_{ji}$. Looking at the polar and constant terms at $p_k$, where $k \neq i, j$, we get

\[
\beta_{ik}(\varepsilon_{jk} - \varepsilon_{ji}) + \beta_{jk}(\varepsilon_{ik} - \varepsilon_{ij}) = \beta_{ij}\eta_{jk} + \beta_{ji}\eta_{ik} + \vartheta_{ij}\alpha_{jk} + \vartheta_{ji}\alpha_{kj} + \sum_{t\neq i,j,k}^3\beta_{it}\beta_{jt}\alpha_{tk},
\]

(2.2.25)

\[
\beta_{ik}\vartheta_{jk} + \beta_{jk}\vartheta_{ik} + \varepsilon_{ik}\varepsilon_{jk} = \beta_{ij}\zeta_{jk} + \beta_{ji}\zeta_{ik} + \varepsilon_{ij}\varepsilon_{jk} + \varepsilon_{ji}\varepsilon_{ik} + \vartheta_{ij}\gamma_{jk} + \vartheta_{ji}\gamma_{ik} + \sum_{t\neq i,j}^3\beta_{it}\beta_{jt}\gamma_{tk} + u_{ij}.
\]

(2.2.26)

Also, comparing the constant terms at $p_j$ in (2.2.24) we get

\[
\nu_{ij} + \varepsilon_{ij}\varepsilon_{jj} + \beta_{ij}\vartheta_{jj} = \beta_{ij}\zeta_{jj} + \beta_{ji}\zeta_{ij} + \varepsilon_{ij}\varepsilon_{jj} + \varepsilon_{ji}\varepsilon_{ij} + \vartheta_{ij}\gamma_{jj} + \vartheta_{ji}\gamma_{ij} + \sum_{k\neq i,j}^3\beta_{ik}\beta_{jk}\gamma_{kj} + u_{ij}
\]

(2.2.27)

(recall that $\nu_{ij}$ is the coefficient of $t_j^3$ in the expansion of $h_i$ at $p_j$).

Finally, assume in addition that char($k$) > 5. Then we can apply the construction of Lemma 2.1.1 for $N = 5$ extending $t_i$’s to parameters of order 5 and obtaining the corresponding sections $f_i[5] \in H^0(C, O(D_i + 5p_i))$. Looking at the products $f_ih_i \in H^0(C, O(5D))$ and adding an appropriate constant to $f_i[5]$, we get

\[
f_ih_i = f_i[5] + \gamma_{ii}h_i + (\delta_{ii} + \varepsilon_{ii})f_i + \sum_{j\neq i}^3\alpha_{ij}\beta_{ij}f_j.
\]

(2.2.28)

Comparing the polar parts at $p_i$ and $p_j$ we get

\[
\pi_{ii} + \vartheta_{ii} = \sum_{j\neq i}^3\alpha_{ij}\alpha_{ji}\beta_{ij}.
\]

(2.2.29)
2.3. Minimal set of generators. We have seen that the constants (elements of \( R \)) introduced in the previous section, such as the constants (2.2.1), as well as \( a_{ij}, b_{ij}, u_{ij} \) and \( s_i \), satisfy many polynomial relations with integer coefficients. Let us denote by \( \text{Con} \) the subset of constants that can be expressed in terms of \( \alpha_{ij}, \beta_{ij}, \gamma_{ij}, \varepsilon_{ij}, \varepsilon_{ii} \) and \( \pi_{ii} \) (where \( 1 \leq i, j \leq g, i \neq j \)), using these relations. Let us also denote by \( \text{Con}_1 \subset \text{Con} \) the similar subset where we do not allow to use \( \pi_{ii} \), and by \( \text{Con}_0 \subset \text{Con}_1 \) the subset of constants that can be expressed only in terms of \( \alpha_{ij}, \beta_{ij} \) and \( \gamma_{ij} \).

**Lemma 2.3.1.** Assume that \( g \geq 2 \) and that 30 is invertible in \( R \). Then we have

\[
\delta_{ij}, \delta_{ii}, \eta_{ij}, \vartheta_{ij}, \zeta_{ij} \in \text{Con}_0, \\
\pi_{ij}, a_{ij} \in \text{Con}_1, \\
\vartheta_{ii}, \zeta_{ii}, b_{ij}, s_i, u_{ij} \in \text{Con}.
\]

If \( g \geq 3 \) then \( u_{ij} \in \text{Con}_1 \).

If we only assume that 6 is invertible in \( R \) then the above assertions are still true provided we replace \( \text{Con} \) by the set \( \tilde{\text{Con}} \) of constants that can be expressed in terms of the same constants as \( \text{Con} \) plus \( \vartheta_{ii} \).

**Proof.** First, relations (2.2.2), (2.2.16) and (2.2.18) give the inclusions

\[
\delta_{ij}, \delta_{ii}, \eta_{ij} \in \text{Con}_0,
\]

while relations (2.2.12) and (2.2.29) give

\[
\pi_{ij} \in \text{Con}_1, \quad \vartheta_{ii} \in \text{Con}.
\]

Next, (2.2.5), (2.2.9), (2.2.17) and (2.2.19) give

\[
a_{ij} \in \text{Con}_1, \quad \vartheta_{ij} \in \text{Con}_0, \quad \zeta_{ii} \in \text{Con}, \quad \zeta_{ij} \in \text{Con}_0.
\]

Now (2.2.23) and (2.2.7) give the inclusions \( s_i, \rho_{ij} \in \text{Con} \). Hence, from (2.2.13) we get \( b_{ij} \in \text{Con} \).

It remains to express \( u_{ij} \). In the case \( g \geq 3 \) using (2.2.26) we get \( u_{ij} \in \text{Con}_1 \). In the case \( g = 2 \) we have to take a longer route. First, (2.2.20) shows that \( l_{ij}[4] \in \text{Con} \). Next, (2.2.14) gives \( \nu_{ij} \in \text{Con} \), and finally, (2.2.27) shows that \( u_{ij} \in \text{Con} \).

The only place where we used invertibility of 5 in \( R \) was the relation (2.2.29). Hence, we can remove this assumption by adding \( \vartheta_{ii} \) to the generators of \( \text{Con} \). \( \square \)

Recall that we have constructed an isomorphism of the moduli space of non-special curves (with tangent vectors at the marked points) \( \tilde{U}_{g,g}^{n_{ss,ka}} \times \text{Spec}(\mathbb{Z}[1/6]) \) with the affine scheme \( S_g \times \text{Spec}(\mathbb{Z}[1/6]) \) parametrizing algebras with defining relations (1.2.1)–(1.2.4) (see Theorem 1.2.3).

**Proposition 2.3.2.** Assume \( g \geq 2 \).

(i) Let us consider the generators (1.2.6) of the algebra of functions on the affine scheme \( S_g \times \text{Spec}(\mathbb{Z}[1/30]) \). Then one has

\[
ad_{ij} = \alpha_{ij}, \quad t_{ji} = \beta_{ji}, \quad v_{ij} = \gamma_{ij}, \quad \epsilon^k_{ij} = \alpha_{ik}\alpha_{jk}, \quad g^i_j = -\alpha^3_{ij}, \quad r_{ji} = \varepsilon_{ji} - \alpha_{ij}\alpha^2_{ji},
\]

\[17\]
\[
\delta_{ij} = \alpha_{ij}\beta_{ij} + \alpha_{ij}\gamma_{ij} + \sum_{k \neq i, j} \alpha_{ik}\alpha_{jk}\alpha_{kj},
\]
\[
e_i^k = \alpha_{ik}\beta_{jk} - \alpha_{ij}\alpha_{jk}^2,
\]
\[
\psi_{ij} = \gamma_{ij}\beta_{ij} + 3\alpha_{ij}\alpha_{ji}\gamma_{ij} + \sum_{k \neq i, j} \beta_{ik}\alpha_{jk}\alpha_{kj} + \sum_{k \neq i, j} \alpha_{ik}(\alpha_{ij}\alpha_{jk}\alpha_{ki} - \alpha_{ji}\alpha_{kj}\alpha_{ik}),
\]
\[
l_{ij}^k = \beta_{ik}\beta_{jk} - \beta_{ij}\alpha_{jk}^2 - \beta_{ji}\alpha_{ik}^2,
\]
\[
k_{ij}^k = \beta_{ij}^2 - 3\alpha_{ij}^2\gamma_{ij}.
\]

Also, the constants \(a_{ij}, b_{ij}, u_{ij}\) and \(s_i\) are equal to some universal polynomials with coefficients in \(\mathbb{Z}[1/5]\) of
\[
\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \varepsilon_{ij}, \pi_i
\]
(\(where i \neq j\)). Thus, the algebra of functions on the scheme \(\tilde{U}_{g,a}^{ns} \times \text{Spec}(\mathbb{Z}[1/30]) \cong S_g \times \text{Spec}(\mathbb{Z}[1/30])\) is generated over \(\mathbb{Z}[1/30]\) by the functions (2.3.1).

(i) Let \(k\) be a field of characteristic \(\neq 2, 3, 5\). Let \(p_0 \in S_g(k)\) be the point at which all the coordinates (1.2.6) vanish. Let \(m\) be the corresponding maximal ideal in the ring of functions on \(S_g \times \text{Spec}(k)\). Then the elements (2.3.1) project to a basis in \(m/m^2\).

**Proof.** (i) Let us apply the setup of Section 2.2 to the universal family of curves \((C, p_1, \ldots, p_g)\) over \(S_g \times \text{Spec}(\mathbb{Z}[1/30])\), (making a choice of parameters of order 5 at \(p_i\) as in Lemma 2.1.1). We obtain that the functions \((f_i, h_i)\) on \(C \setminus D\), where \(f_i\) is chosen so that \(\gamma_{ii} = 0\), satisfy the following equations:
\[
f_if_j = \alpha_{ij}h_i + \alpha_{ij}h_j + \gamma_{ij}f_i + \gamma_{ij}f_j + \sum_{k \neq i, j} \alpha_{ik}\alpha_{jk}f_k + a_{ij},
\]
(2.3.2)
\[
f_ih_j = \alpha_{ij}f_j^2 + \gamma_{ij}h_j + \beta_{ij}h_i + \delta_{ij}f_j + (\varepsilon_{ij} - \alpha_{ij}\alpha_{ij}^2)f_i + \sum_{k \neq i, j} (\alpha_{ik}\beta_{jk} - \alpha_{ij}\alpha_{jk}^2)f_k + b_{ij},
\]
(2.3.3)
\[
h_i^2 = f_i^3 + (2\varepsilon_{ii} - 3\delta_{ii})h_i + (2\varepsilon_{ii} - 3\pi_{ii})f_i - \sum_{j \neq i} \alpha_{ij}^3 h_j + \sum_{j \neq i} (\beta_{ij}^2 - 3\alpha_{ij}^2\gamma_{ij})f_j + s_i,
\]
(2.3.4)
\[
h_ih_j = \beta_{ij}f_j^2 + \beta_{ij}f_i^2 + \varepsilon_{ij}h_j + \varepsilon_{ij}h_i + (\vartheta_{ij} - \beta_{ji}\alpha_{ij}^2)f_j + (\vartheta_{ij} - \beta_{ji}\alpha_{ij}^2)f_i + \sum_{k \neq i, j} (\beta_{ik}\beta_{jk} - \beta_{ij}\alpha_{jk}^2 - \beta_{ji}\alpha_{ik}^2)f_k + u_{ij},
\]
(2.3.5)
where we used the equations (2.2.1), (2.2.8), (2.2.21), (2.2.24) and expressed \(f_i[4]\) using (2.2.15). We have the remaining ambiguity in adding a constant to \(h_i\), which we use to make \(2\varepsilon_{ii} = 3\delta_{ii}\). Then the equations (2.3.2)-(2.3.5) look as in Lemma 1.2.1(iii), hence, the elements \((f_i, h_i)\) are the canonical generators of \(A = \mathcal{O}(C \setminus D)\) (see Lemma 1.2.2). Thus, all the coefficients in these equations should match. From this we get most of the required equations. For \(\psi_{ij}\) we get
\[
\psi_{ij} = \vartheta_{ij} - \beta_{ji}\alpha_{ij}^2
\]
which we have to transform further using (2.2.9) and using the formula (2.2.18) for \(\eta_{ij}\). Also, we get
\[
\pi_i = 2\vartheta_{ii} - 3\pi_{ii}.
\]
Using (2.2.29) we can rewrite this as
\[ 5\pi_{ii} = 2 \sum_{j \neq i} \alpha_{ij} \alpha_{ji} \beta_{ij} - \pi_i. \] (2.3.6)

Therefore, by Lemma 2.3.1, we can express the remaining constants in terms of (2.3.1).

(ii) We have to calculate the fiber of \( S_g(k[u]/(u^2)) \to S_g(k) \) over \( p_0 \in S_g(k) \). In other words, we have to look at the condition of associativity for the relations (1.2.1)–(1.2.4), where all the coordinates (1.2.6) belong to the ideal \( (u) \subset k[u]/(u^2) \). By part (i), we obtain that relations have form
\[
\begin{align*}
  f_i f_j &= \alpha_{ij} h_i + \alpha_{ji} f_j + \gamma_{ji} f_i + \gamma_{ij} f_j, \\
  f_i h_j &= \alpha_{ij} f_j^2 + \gamma_{ij} h_j + \beta_{ij} h_i + \epsilon_{ji} f_i, \\
  h_i^2 &= f_i^3 + \pi_i f_i, \\
  h_i h_j &= \beta_{ij} f_j^2 + \beta_{ji} f_i^2 + \epsilon_{ij} h_j + \epsilon_{ji} h_i.
\end{align*}
\]

It is easy to check that in fact these relations define an associative algebra over \( k[u]/(u^2) \) with the basis \( (f_i^n, h_i f_i^n) \) for arbitrary \( \alpha_{ij}, \beta_{ij}, \gamma_{ij}, \epsilon_{ij}, \pi_i \in (u) \subset k[u]/(u^2) \), which implies the result. \( \square \)

The above Proposition shows that (2.3.1) are minimal generators of the ring of functions on \( S_g \times \text{Spec}(\mathbb{Z}[1/30]) \).

We have the following geometric interpretation of the vanishing of the functions \( \alpha_{ij} \) and \( \beta_{ij} \) on \( S_g \).

**Lemma 2.3.3.** Assume that \( g \geq 2 \). Let \( (C, p_1, \ldots, p_g) \in \widetilde{U}_{g,a}^{n.s.}(k) \), where \( k \) is a field of characteristic \( \neq 2, 3 \). Let us consider the functions \( \alpha_{ij}, \beta_{ij} \) on \( \widetilde{U}_{g,a}^{n.s.}(k) \simeq S_g(k) \).

(i) One has \( \alpha_{ij}(C, p_1, \ldots, p_g) = 0 \) if and only if \( h^0(2p_i + D_i - p_j) = 2 \), where \( D_i = D - p_i \).

(ii) One has \( \alpha_{ij}(C, p_1, \ldots, p_g) = \beta_{ij}(C, p_1, \ldots, p_g) = 0 \) if and only if \( h^0(3p_i + D_i - p_j) = 3 \).

(iii) If \( C \) is smooth then for every \( i \) either there exists \( j \neq i \) such that \( \alpha_{ij} \neq 0 \) or there exists \( j \neq i \) such that \( \beta_{ij} \neq 0 \).

**Proof.** (i) We use the interpretation of \( \alpha_{ij} \) as the coefficient of \( t_j^{-1} \) in the Laurent series of \( f_i \) at \( p_j \) (see Section 2.2). Thus, \( \alpha_{ij} = 0 \) means that \( f_i \) in fact is regular at \( p_j \), i.e.,
\[
H^0(C, 2p_i + D_i) = H^0(C, 2p_i + D_i - p_j) = 0.
\]
Since \( h^0(C, 2p_i + D_i)=2 \), the assertion follows.

(ii) We use in addition the interpretation of \( \beta_{ij} \) as the coefficient of \( t_j^{-1} \) in the Laurent series of \( h_i \) at \( p_j \). Thus, \( \alpha_{ij} = \beta_{ij} = 0 \) if and only if both \( f_i \) and \( h_i \) are regular at \( p_j \). Since
\[
1, f_i, h_i \text{ is a basis of } H^0(C, 3p_i + D_i),
\]

this is equivalent to having
\[
H^0(C, 3p_i + D_i) = H^0(C, 3p_i + D_i - p_j),
\]

and the assertion follows.

(iii) By part (i), if for some \( i \) we have \( \alpha_{ij} = 0 \) for all \( j \neq i \) then \( f_i \) is a nonconstant section of \( H^0(C, 2p_i) \), so \( C \) is hyperelliptic and \( p_i \) is a Weierstrass point. Let \( f : C \to \mathbb{P}^1 \) be the double covering. Then \( f_* O_C(p_i) = O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-g) \), so
\[
H^0(C, O(3p_i)) = H^0(\mathbb{P}^1, (f_* O(p_i))(1)) \simeq H^0(\mathbb{P}^1, O(1) \oplus O(1 - g))
\]
has dimension 2. Hence, the result follows from part (ii). □

Remark 2.3.4. The interpretations of \( \alpha_{ij} \) and \( \beta_{ij} \) from Section 2.2 imply that \( \alpha_{ij} \) are exactly the functions defined in [12, Sec. 2.4] as triple Massey products, while the values of \( \beta_{ij} \), in the situation when \( \alpha_{ij} = 0 \) for all \( j \neq i \), are given by the quadruple Massey products considered in [12, Sec. 2.5].

2.4. GIT quotients and weakly modular compactifications of \( M_{g,n} \). In this section we work over an algebraically closed field \( k \) of characteristic zero, and we denote by \( \widetilde{U}^{n,s,a}_{g,n} \), \( S_g \), etc., the schemes over \( k \) obtained from the schemes considered above by the base change.

We would like to consider GIT quotients with respect to the \( G^\Lambda_m \)-action on the affine scheme \( S_g \) (see Theorem 1.2.3 for the definition of this action). Such a quotient is determined by a nontrivial character \( \chi : G^\Lambda_m \to \mathbb{G}_m \) (as a linearized ample line bundle on \( S_g \) we take the trivial line bundle with the \( G^\Lambda_m \)-action twisted by \( \chi \)) and is given by the scheme

\[
S_g \sslash \chi G^\Lambda_m := \mathrm{Proj} \left( \bigoplus_{n \geq 0} H^0(S_g, \mathcal{O})^{\chi^n} \right),
\]

where the subscript \( \chi^n \) denotes the subset of functions \( f \) such that \( (\lambda^{-1})^* f = \chi(\lambda)^n f \) for \( \lambda \in G^\Lambda_m \).

We have a closed embedding

\[
S_g \hookrightarrow \mathbb{A}^{4g^2-3g}, \quad (2.4.1)
\]

where \( \mathbb{A}^{4g^2-3g} \) is the affine space with coordinates corresponding to the minimal generators \( \alpha_{ij}, \beta_{ij}, \gamma_{ij}, \varepsilon_{ij} \) and \( \pi_i \) of \( \mathcal{O}(S_g) \) (see Proposition 2.3.2).

Note that the minimal generators (2.3.1) are semihomogeneous for the \( G^\Lambda_m \)-action on \( S_g \). Namely, if we define the action of \( \lambda \in G^\Lambda_m \) on \( \mathcal{O}(S_g) \) by \( f \mapsto (\lambda^{-1})^* f \) then the elements (2.3.1) transform under the action of \( \lambda = (\lambda_1, \ldots, \lambda_g) \) as follows:

\[
\begin{align*}
\alpha_{ij} &\mapsto \lambda_i^2 \lambda_j^{-1} \alpha_{ij}, \quad \beta_{ij} \mapsto \lambda_i^3 \lambda_j^{-1} \alpha_{ij}, \quad \gamma_{ij} \mapsto \lambda_i^2 \gamma_{ij}, \quad \varepsilon_{ij} \mapsto \lambda_i^3 \varepsilon_{ij}, \quad \pi_i \mapsto \lambda_i^4 \pi_i.
\end{align*}
\]

Thus, the embedding (2.4.1) is \( G^\Lambda_m \)-equivariant, and for every character \( \chi \), we have the closed embedding of the corresponding GIT quotients

\[
S_g \sslash \chi G^\Lambda_m \hookrightarrow P_{g,\chi} := \mathbb{A}^{4g^2-3g} \sslash \chi G^\Lambda_m.
\]

We identify the lattice of characters of \( G^\Lambda_m \) with the standard lattice \( \mathbb{Z}^g \subset \mathbb{R}^g \), where the basis vector \( e_i \) corresponds to the projection \( G^\Lambda_m \to \mathbb{G}_m \) to the \( i \)th factor. We denote by \( C \subset \mathbb{R}^g \) the cone generated by the vectors \( 2e_i - e_j \), \( i \neq j \). Note that \( C \) contains all the vectors \( e_i \) and \( 3e_i - e_j \), so for every coordinate of \( \mathbb{A}^{4g^2-3g} \) the corresponding character belongs to \( C \).

Let \( M_{g,n} \) denote the coarse moduli space of smooth curves of genus \( g \) with \( n \) marked points. Recall (see [9]) that a a proper birational model \( X \) of \( M_{g,n} \) is called weakly modular if there exists an open substack \( \mathcal{X} \subset \mathcal{U}_{g,n} \) of the stack of all curves of genus \( g \) with \( n \) marked points and a map \( \pi : \mathcal{X} \to X \) which is categorical for maps to algebraic spaces and which is bijective on closed \( k \)-points (such a map \( \pi \) is called a good moduli map).
Proposition 2.4.1. (i) For every rational ray $\mathbb{R}_{\geq 0}\chi \subset C$ the quotient $P_{g,\chi}$ is a projective toric variety of dimension $4(g^2-g)$. The irreducible component in $S_g/\chi \mathbb{G}_m^g$ corresponding to smoothable curves is a weakly modular birational model of $M_{g,g}$.

(ii) Let $C_0 \subset C$ be the subcone generated by all the vectors $e_i$. Assume that $\chi$ is in $C_0$. Then every smooth curve $(C, p_1, \ldots, p_g) \in \mathcal{U}_{g,s,a}$ corresponds to a semistable point of $S_g/\chi \mathbb{G}_m^g$. For $\chi$ in the interior of $C_0$ the quotient $S_g/\chi \mathbb{G}_m^g$ does not depend on $\chi$.

Proof. (i) Since all the vectors $2e_i - e_j$, $e_i$ and $3e_i - e_j$ have positive scalar product with $e_1 + \ldots + e_g$, we see that any nontrivial combination of them with non-negative coefficients is nonzero. Therefore, $H^0(A^{g^2-3g} \mathcal{O})^{\mathbb{G}_m^g} = k$, which immediately implies that $P_{g,\chi}$ is projective for any $\chi$. Hence, $S_g/\chi \mathbb{G}_m^g$ is also projective. The fact that $P_{g,\chi}$ is a toric variety follows from [30, Thm. 2.2].

Recall that by Theorem 1.2.3, we have a $\mathbb{G}_m^g$-equivariant isomorphism of schemes

$$\tilde{U}_{g,g}^{a,s,a} \simeq S_g.$$ 

Let $\tilde{X} \subset \tilde{U}_{g,g}^{a,s,a}$ be the $\mathbb{G}_m^g$-equivariant open subset corresponding to the $\chi$-semistable locus in $S_g$. Then

$$\mathcal{X} = \tilde{X}/\mathbb{G}_m^g \subset \mathcal{U}_{g,g}^{a,s,a}$$

is an open substack in $\mathcal{U}_{g,g}$, such that the projection $\mathcal{X} \to S_g/\chi \mathbb{G}_m^g$ is a good moduli map (see [9, Prop. 2.37] and the proof of [9, Prop. 2.41]). It remains to check that the generic smooth curve $(C, p_1, \ldots, p_g)$ is $\chi$-semistable. But for the generic curve we have $\alpha_{ij} \neq 0$ for every pair $i \neq j$ (this follows from Lemma 2.3.3). Since $\chi \in C$, there exists a monomial in $\alpha_{ij}$ which belongs to $H^0(S_g, \mathcal{O})^\chi$ for some $n$, which gives the desired semistability.

(ii) For a subset $S$ of coordinates of $A^{g^2-3g}$ let us denote by $C_S \subset C$ the subcone spanned by the corresponding characters of $\mathbb{G}_m^g$. Let $(C, p_1, \ldots, p_g) \in \tilde{U}_{g,g}^{a,s,a}$ with $C$ smooth. By Lemma 2.3.3(iii), for any $i$ either there exists $j \neq i$ such that $\alpha_{ij} \neq 0$ or there exists $j \neq i$ such that $\beta_{ij} \neq 0$. Hence, if $S$ is the set of coordinates of $A^{g^2-3g}$ that do not vanish at the point of $S_g$ corresponding to $(C, p_1, \ldots, p_g)$ then the corresponding characters of $\mathbb{G}_m^g$ (suitably rescaled) satisfy the assumptions of Lemma 2.4.2 below. Hence, any $\chi$ in the interior of $C_0$ lies in $C_S$, which implies the $\chi$-semistability of $(C, p_1, \ldots, p_g)$.

For the last statement it is enough to show that if for a subset of coordinates $S$ the subcone $C_S \subset C$ contains a vector $v$ in the interior of $C_0$ then $C_S$ contains the entire cone $C_0$. To this end we observe that for every $i$ there exists a spanning vector with positive $i$th coordinate, i.e., one of the vectors $e_i$, $e_i - 2e_j$ or $e_i - 3e_j$ for some $j \neq i$. Thus, we can again apply Lemma 2.4.2 to get the result. \hfill \Box

Lemma 2.4.2. The subcone $\mathbb{R}_{\geq 0}v_1 + \ldots + \mathbb{R}_{\geq 0}v_n \subset \mathbb{R}^n$ spanned by vectors of the form

$$v_i = e_i - a_i e_{\sigma(i)}, \quad i = 1, \ldots, n,$$

where $0 \leq a_i < 1$, and $\sigma$ is a map from $\{1, \ldots, n\}$ to itself, such that $\sigma(i) = i$ implies $a_i = 0$, contains every vector $e_i$.

Proof. First, assume that all $a_i$ are positive. Then we have $\sigma(i) \neq i$ for every $i$. Renumbering the indices we can assume that $\sigma(1) = 2, \sigma(2) = 3, \ldots, \sigma(m) = 1$ for some $2 \leq m \leq n$. \hfill \Box
Then
\[ v_1 + a_1v_2 + \ldots + a_1\ldots a_{m-1}v_m = (1 - a_1\ldots a_m)e_1, \]
etc., which shows that \( e_1, \ldots, e_m \) belong to \( \mathbb{R}_{\geq 0}v_1 + \ldots + \mathbb{R}_{\geq 0}v_n \). Repeating this procedure if necessary we obtain that for any \( i \) there exists \( r \geq 1 \) such that \( \sigma^r(i) = j \), where \( e_j \in \mathbb{R}_{\geq 0}v_1 + \ldots + \mathbb{R}_{\geq 0}v_n \). This easily implies that \( e_i \in \mathbb{R}_{\geq 0}v_1 + \ldots + \mathbb{R}_{\geq 0}v_n \) (by induction in \( r \)).

If some of \( a_i \) are zero then the corresponding vectors \( v_i = e_i \) are in the cone, so the above argument still works (we first deal with \( i \) such that \( a_{\sigma^r(i)} = 0 \) for some \( r \geq 1 \)). \( \square \)

2.5. Some other consequences of relations. Let us keep the setup of Section 2.2. As in the end of that section, we assume that 30 is invertible, and choose parameters \( t_i \) of order 5 as in Lemma 2.1.1. In addition we assume that \( f_i = f[i]2 \) and \( h_i = f[i]3 \) are chosen (by adding a constant) in such a way that \( \gamma_{ii} = 0 \) and \( 2\varepsilon_{ii} = 3\delta_{ii} \). Then all the identities obtained in Section 2.2 can be viewed as polynomial relations between the minimal generators (2.3.1) of the algebra of functions on \( \mathcal{S}_g \times \text{Spec}(\mathbb{Z}[1/30]) \), where we express \( \pi_{ii} \) in terms of \( \pi_i \) using (2.3.6).

The relation (2.2.12) expresses \( \pi_{ij} \) in terms of \( (\alpha_*, \beta_*, \varepsilon_*) \). Plugging this expression into (2.2.5) we get
\[ a_{ij} = \alpha_{ij}(\varepsilon_{ji} - 2\delta_{ij}) + \beta_{ji}\beta_{ij} - \alpha_{ji}\varepsilon_{ij} - \gamma_{ji}\gamma_{ij} + \sum_{k \neq i,j} \alpha_{ik}\alpha_{kj}\beta_{jk} - \sum_{k \neq i,j} \alpha_{ik}\alpha_{jk}\gamma_{kj}. \] (2.5.1)

Plugging the expression (2.5.1) for \( a_{ij} \) into (2.2.4) we get after simplifying
\[ -\alpha_{ik}\delta_{jk} - \alpha_{jk}\delta_{ik} + \beta_{ij}\beta_{ji} + \alpha_{ij}(\varepsilon_{jk} + \varepsilon_{ji} - 2\delta_{ij}) + \alpha_{ji}(\varepsilon_{ik} - \varepsilon_{ij}) + \sum_{l \neq i,j} \alpha_{il}\alpha_{jl}\beta_{ij} = (\gamma_{ij} - \gamma_{ik})(\gamma_{ji} - \gamma_{jk}) + \sum_{l \neq i,j} \alpha_{il}\alpha_{jl}(\gamma_{lj} - \gamma_{lk}). \] (2.5.2)

On the other hand, using (2.2.18) we can express \( \eta_{ij} \) in terms of \( (\alpha_*, \beta_*, \gamma_*) \). Plugging the result into (2.2.10) and (2.2.22) we get
\[ \alpha_{ik}(\varepsilon_{jk} - \varepsilon_{ij}) = (\gamma_{ij} - \gamma_{ik})\beta_{jk} + 2\alpha_{ij}\alpha_{kj}\gamma_{jk} + \beta_{ji}\beta_{jk} + \delta_{ij}\alpha_{jk} + \sum_{l \neq i,j,k} \alpha_{il}\alpha_{jl}\alpha_{lk} - \sum_{l \neq i,j,k} \alpha_{ij}\alpha_{jk}\alpha_{lk}, \] (2.5.3)
\[ 2\beta_{ij}\varepsilon_{ij} = 3\alpha_{ij}^2\delta_{ij} + 3\alpha_{ij}\gamma_{ij}^2 + (2\delta_{ii} - 3\pi_{ii})\alpha_{ij} - \sum_{k \neq i,j} \alpha_{ik}^3\beta_{kj} + \sum_{k \neq i,j} (\beta_{ik}^2 - 3\alpha_{ik}^2\gamma_{ik})\alpha_{kj}. \]

Using in addition (2.2.29) and (2.3.6) to express \( \vartheta_{ii} \) and \( \pi_{ii} \) in terms of \( \pi_i \), we can rewrite the last equation as
\[ -\alpha_{ij}\pi_i = -2\beta_{ij}\varepsilon_{ij} + 3\alpha_{ij}^2\delta_{ij} + 3\alpha_{ij}\gamma_{ij}^2 - \sum_{k \neq i,j} \alpha_{ik}^3\beta_{kj} + \sum_{k \neq i,j} (\beta_{ik}^2 - 3\alpha_{ik}^2\gamma_{ik})\alpha_{kj}. \] (2.5.4)

Note also that expressing \( \rho_{ij} \) from (2.2.7) and substituting this expression into (2.2.13) we get
\[ 2\alpha_{ij}\vartheta_{jj} + \sum_{k \neq i,j} \alpha_{ik}\alpha_{jk}\delta_{kj} + \alpha_{ji}\vartheta_{ij} + \gamma_{jj}\delta_{ij} = \alpha_{ij}\pi_{jj} + \alpha_{ij}\vartheta_{jj} + \beta_{ji}\varepsilon_{ij} + \gamma_{ij}\delta_{ij} + \sum_{k \neq i,j} \alpha_{ik}\beta_{kj}\gamma_{kj} + b_{ij}. \]
Subtracting (2.2.11) from this, we get the relation
\[ 2\alpha_{ij}\vartheta_{jj} + \alpha_{ij}\vartheta_{ij} - \alpha_{ik}\vartheta_{jk} + \sum_{k\neq i,j} \alpha_{ik}\alpha_{jk}\delta_{kj} - \delta_{ik}\beta_{jk} = \sum_{l\neq i,j} \alpha_{il}\beta_{jl}(\gamma_{lj} - \gamma_{lk}) + \alpha_{ij}\pi_{jj} - \alpha_{ij}(\zeta_{jk} - \zeta_{jj}) - \delta_{ij}(\gamma_{ji} + \gamma_{jk}) + \beta_{ji}(\epsilon_{ij} - \epsilon_{ik}) + (\gamma_{ij} - \gamma_{ik})(\epsilon_{ji} - \epsilon_{jk}). \]

Further, using (2.2.29), (2.2.17), (2.2.19) and (2.3.6) we can rewrite this as
\[ \delta_{ij} = \text{that} \]

Proposition 2.5.1. Assume that \( g \geq 3 \). Let \( U_1 \subset \mathcal{S}_g \times \text{Spec}(\mathbb{Z}[1/30]) \) be the open subset consisting of points \( x \) such that for every \( i \) either there exists \( j \) with \( \alpha_{ij}(x) \neq 0 \) or there exists \( j \) with \( \alpha_{ji}(x) \neq 0 \). Let also \( U_0 \subset U_1 \) be the open subset of \( x \) such that \( \alpha_{ij}(x) \neq 0 \) for all \( i \neq j \). Then the natural maps to the affine spaces
\[ U_1 \to \text{Spec} \mathbb{Z}[1/30][\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \epsilon_{ij}] \quad \text{and} \quad U_0 \to \text{Spec} \mathbb{Z}[1/30][\alpha_{ij}, \beta_{ij}, \gamma_{ij}] \]
are locally closed embeddings (in the right-hand side we treat \( \alpha_{ij}, \beta_{ij}, \gamma_{ij}, \epsilon_{ij} \) as independent variables).

Proof. Recall that by Proposition 2.3.2, the functions on \( \mathcal{S}_g \times \text{Spec}(\mathbb{Z}[1/30]) \) are generated by \( \alpha_{ij}, \beta_{ij}, \gamma_{ij}, \epsilon_{ij} \) and \( \pi_i \). Thus, the first assertion follows from (2.5.4) and (2.5.5) (recall also that \( \delta_{ij} \in \mathcal{C} \) by (2.2.2)). To check the second we use the relations (2.5.2) and (2.5.3) that have the form
\[ \alpha_{ij}(\epsilon_{ji} + \epsilon_{jk}) + \alpha_{ji}(\epsilon_{ik} - \epsilon_{ij}) = A_{ijk}, \]
\[ \alpha_{jk}(\epsilon_{ik} - \epsilon_{ij}) = B_{ijk} \]
with \( A_{ijk}, B_{ijk} \) depending only on \( \alpha_{ij}, \beta_{ij}, \gamma_{ij} \) (we switched \( i \) and \( j \) in (2.5.3)). Expressing \( \epsilon_{ik} - \epsilon_{ij} \) from the second equation and substituting into the first we get
\[ \alpha_{ij}(\epsilon_{ji} + \epsilon_{jk}) = A_{ijk} - \frac{\alpha_{ji}}{\alpha_{jk}}B_{ijk}. \]

Switching \( i \) and \( k \) we get
\[ \epsilon_{ik} + \epsilon_{ij} = (A_{jik} - \frac{\alpha_{ij}}{\alpha_{ik}}B_{ijk})/\alpha_{ji}. \]

Since on the other hand
\[ \epsilon_{ik} - \epsilon_{ij} = B_{ijk}/\alpha_{jk}, \]
we get an expression for \( \epsilon_{ij} \) in terms of \( \alpha_{ij}, \beta_{ij}, \gamma_{ij} \).

Remarks 2.5.2. 1. Recall that the condition \( \alpha_{ij} = 0 \) for a curve \( (C, p_1, \ldots, p_g) \in \mathcal{U}^{ns,a}(k) \) is equivalent to \( h^0(2p_i + D_i - p_j) = 2 \) (see Lemma 2.3.3). In particular, for smooth \( C \) the condition that \( (C, p_1, \ldots, p_g) \) is not in \( U_1 \) means that \( C \) is hyperelliptic and the points \( p_i \) are Weierstrass points.

2. The action of the diagonal subgroup \( \mathbb{G}_m \subset \mathbb{G}_m^a \) gives the algebra of function on \( \mathcal{S}_g \) a grading so that \( \deg \alpha_{ij} = 1, \deg \beta_{ij} = \deg \gamma_{ij} = 2, \deg \epsilon_{ij} = 3 \) and \( \deg \pi_i = 4 \). All the relations between the coordinates (2.3.1) on \( \mathcal{S}_g \times \text{Spec}(\mathbb{Z}[1/30]) \) are homogeneous with
respect to this grading. For example, (2.2.3) is of degree 3, (2.2.5) and (2.5.3) are of degree 4, (2.2.6), (2.2.25), (2.5.4) and (2.5.5) are of degree 5. By varying $j$ in (2.2.23) and eliminating $s_i$ we get relations of degree 6. Similarly, we get relations of degree 6 by eliminating $u_{ij}$ from (2.2.26). It is possible in principle to write out all the defining relations explicitly using the Gröbner basis approach (see the proof of Lemma 1.2.2) but this does not seem to be very illuminating. The relations become more manageable on the hyperelliptic locus (see Theorem 2.6.3 below).

2.6. Hyperelliptic locus. There is a natural hyperelliptic version of the moduli space $\widetilde{H}^{n,s,a}_{g,g}$. 

Definition 2.6.1. Let us define the moduli stack $\widetilde{H}^{n,s,a}_{g,g}$ by considering families $(C, p_1, \ldots, p_g; v_1, \ldots, v_g)$ in $\widetilde{U}^{n,s,a}_{g,g}$ (where $v_i$ is a nonzero tangent vector at $p_i$) equipped with an involution $\tau : C \to C$ such that $\tau \circ p_i = p_i$ and $\tau^* v_i = -v_i$. We denote by $\mathcal{H}^{n,s,a}_{g,g}$ the similar stack with no fixed choice of tangent vectors (but we still require that $\tau$ acts as $-1$ on the tangent spaces at $p_i$).

We will show that the involution with this property is unique and that in the smooth case it is automatically a hyperelliptic involution.

Proposition 2.6.2. (i) The stack $\widetilde{H}^{n,s,a}_{g,g} \times \text{Spec}(\mathbb{Z}[1/6])$ is the closed subscheme of $\widetilde{U}^{n,s,a}_{g,g} \times \text{Spec}(\mathbb{Z}[1/6])$ given as the stable locus of the involution given by the action of the element $(-1, \ldots, -1) \in G^g_m$. The corresponding subscheme of $S_g \times \text{Spec}(\mathbb{Z}[1/6])$ is given in terms of the coefficients (1.2.6) by the equations

$$\alpha_{ij} = d_{ij} = r_{ij} = \delta_{ij} = c_{ij} = b_{ij} = \varepsilon_{ij} = g^i_j = 0.$$

(2.6.1)

(ii) Assume that $(C, p_1, \ldots, p_g)$ is in $\mathcal{H}^{n,s,a}_{g,g}(k)$, where $k$ is a field, and $C$ is smooth. Then $C$ is hyperelliptic and $p_1, \ldots, p_g$ are Weierstrass points.

Proof. (i) Recall that the $G^g_m$-action rescales the tangent vectors at the marked points. Thus, the stable points of $(-1, \ldots, -1) \in G^g_m$ correspond to curves $(C, p_1, \ldots, p_g)$ such that there exists an automorphism $\tau$ of $(C, p_1, \ldots, p_g)$ such that $\tau^* v_i = -v_i$. Then $\tau^2$ stabilizes $v_i$, which implies that $\tau^2 = \text{id}$.

Let $f_i, h_i$ be the canonical generators of the corresponding marked algebra $A = H^0(C \setminus D, \mathcal{O})$ (see Lemma 1.2.1). Note that the involution $\tau$ defines an automorphism of $A$ as a marked algebra. Furthermore, $\tau^* f_i$ and $-\tau^* h_i$ are also canonical generators, hence we have

$$\tau^* f_i = f_i, \quad \tau^* h_i = -h_i.$$

Therefore, looking at the defining relations (1.2.1)–(1.2.4) we deduce the vanishing (2.6.1).

Conversely, if these equations hold then we can define an involution of $A$ which acts on the marking as $(-1, \ldots, -1)$.

(ii) Since $H^0(C, \omega_C(-p_1 - \ldots - p_g)) = H^1(C, \mathcal{O}(p_1 + \ldots + p_g))^* = 0$, the projection

$$H^0(C, \omega_C) \to \bigoplus_{i=1}^g \omega_C|_{p_i}$$

is an isomorphism. Hence, $\tau$ acts as $-\text{id}$ on $H^0(C, \omega_C)$, so it acts trivially on the projectivization of this space. Note that the canonical morphism $C \to \mathbb{P} H^0(C, \omega, C)^*$ is
compatibility with the action of \( \tau \). If \( C \) is not hyperelliptic then we get that \( \tau \) acts trivially on \( C \), which is a contradiction. Hence, \( C \) is hyperelliptic and \( \tau \) is the hyperelliptic involution.

Using Proposition 2.3.2 we get that over \( \text{Spec}(\mathbb{Z}[1/30]) \) the functions on the hyperelliptic locus are generated by \( \beta_{ij}, \gamma_{ij} \) and \( \pi_i \). Furthermore, using the associativity equations (see Lemma 1.2.2) we can get a complete set of relations between these generators.

**Theorem 2.6.3.** Assume \( g \geq 2 \). The algebra of functions on the affine scheme \( \widetilde{HU}_{g,g}^{ns,a} \times \text{Spec}(\mathbb{Z}[1/30]) \) is generated by \( \beta_{ij}, \gamma_{ij} \) and \( \pi_i \) with the defining relations

\[
\begin{align*}
\beta_{ij}\beta_{ji} &= (\gamma_{ij} - \gamma_{ik})(\gamma_{ij} - \gamma_{jk}), \\
(\gamma_{ij} - \gamma_{ik})\beta_{jk} + \beta_{ji}\beta_{ik} &= 0, \\
-(\pi_i + \gamma_{ij}^2)\gamma_{ij} + 2\beta_{ij}\gamma_{ij} - \sum_{t \neq i,j} \beta_{it}^2\gamma_{ij} &= -(\pi_i + \gamma_{ik}^2)\gamma_{ik} + 2\beta_{ik}\gamma_{ki} - \sum_{t \neq i,k} \beta_{it}^2\gamma_{ik}, \\
(\pi_i + \gamma_{ij}^2)\beta_{ji} + \beta_{ij}\gamma_{jk}(\gamma_{ij} + \gamma_{jk}) + \beta_{ji}\gamma_{ik}(\gamma_{ij} + \gamma_{ik}) + \sum_{t \neq i,k} \beta_{it}\beta_{jk}\gamma_{ik} &= \\
\beta_{ik}\beta_{jk}\gamma_{kj} + \beta_{jk}\beta_{ik}\gamma_{ki} + 2\beta_{ij}\gamma_{ij}^2 + \sum_{k \neq i,j} \beta_{ik}\beta_{jk}\gamma_{ki}, \\
(\pi_i + 3\gamma_{ij}^2)\beta_{ji} &= (\pi_j + 3\gamma_{ji}^2)\beta_{ij},
\end{align*}
\]

where different indices are assumed to be distinct. The open part \( C \setminus \{p_1, \ldots, p_g\} \) of the universal curve is given by the equations

\[
\begin{align*}
f_if_j &= \gamma_{ij}f_i + \gamma_{ij}f_j + a_{ij}, \\
f_ih_j &= \gamma_{ij}h_j + \beta_{ji}h_i, \\
h_i^2 &= f_i^2 + \pi_i f_i + \sum_{j \neq i} \beta_{ij}f_j + s_i, \\
h_jh_j &= \beta_{ij}f_i^2 + \beta_{ji}f_i^2 + \gamma_{ij}\beta_{ij}f_j + \gamma_{ij}\beta_{ji}f_i + \sum_{k \neq i,j} \beta_{ik}\beta_{jk}f_k + u_{ij},
\end{align*}
\]

where

\[
\begin{align*}
s_i &= -(\pi_i + \gamma_{ij}^2)\gamma_{ij} + 2\beta_{ij}\gamma_{ij} - \sum_{k \neq i,j} \beta_{ik}\gamma_{kj}, \\
u_{ij} &= (\pi_i + \gamma_{ij}^2)\beta_{ji} - 2\beta_{ij}\gamma_{ij}^2 - \sum_{k \neq i,j} \beta_{ik}\beta_{jk}\gamma_{ki}.
\end{align*}
\]

**Proof.** By Proposition 2.6.2, the defining relations are obtained by looking at the condition that the relations (2.6.3) define an algebra with the basis \( \{f_i^*, h_i f_i^n\} \). Now the relations (2.6.2) are obtained by the standard Gröbner basis technique (see Lemma 1.2.2). \( \square \)

**Example 2.6.4.** In the case of genus 2 the relations of the above Theorem between the generators \( \beta_{12}, \beta_{21}, \gamma_{12}, \gamma_{21}, \pi_1, \pi_2 \) reduce to the single relation

\[
(\pi_2 + 3\gamma_{21}^2)\beta_{12} = (\pi_1 + 3\gamma_{12}^2)\beta_{21}.
\]

Assume that \( g \geq 4 \). Then using the second equation in (2.6.2) we deduce the following set of quartic equations for \( (\beta_{ij}) \) on the hyperelliptic locus:

\[
\beta_{ij}\beta_{ik}\beta_{kl}\beta_{lj} + \beta_{ki}\beta_{il}\beta_{lj}\beta_{jk} + \beta_{li}\beta_{ik}\beta_{jk}\beta_{kl} = 0,
\]

where \( i, j, k, l \) are distinct.

**Remark 2.6.5.** The locus in \( \widetilde{HU}_{g,g}^{ns,a} \) corresponding to smooth curves can be identified with a \( G_m^n \)-torsor over the configuration space of \( 2g + 2 \) distinct points in \( \mathbb{P}^1 \), \( g \) of which are ordered. By Lemma 2.3.3, the coordinates \( \beta_{ij} \) are all nonzero on this locus (and can be
computed explicitly in terms of the positions of the points—see [12, Prop. 2.5.5]). Using the second equation in (2.6.2) we can express $\gamma_{ij} - \gamma_{ik}$ in terms of $(\beta_{ij})$ (then the first equation in (2.6.2) becomes superfluous). Also, for $g \geq 3$, using the fourth equation in (2.6.2) we can express $\pi_i$ in terms of the other coordinates. Hence, for $g \geq 3$, on the smooth locus our relations can be viewed as equations for the coordinates $\beta_{ij}$ only.

2.7. Differentials and the canonical embedding. Throughout this section we work with curves over an algebraically closed field $k$.

We are going to describe the relation of our picture with Petri’s analysis of the defining ideal of the canonical embedding of a smooth non-hyperelliptic curve $C$ (see [27], [26, Lec. I], [32], [2, Ch. III.3]). Recall that the starting point of this analysis is to consider $g$ distinct points $p_1, \ldots, p_g \in C$ and a basis $\omega_1, \ldots, \omega_g$ of $H^0(C, \omega_C)$, where $\omega_C$ is the canonical line bundle, such that $\omega_i$ vanishes at $p_j$ for $i \neq j$. Such a basis exists precisely when $h^0(p_1 + \ldots + p_g) = 1$ and is uniquely determined by the marked points up to rescaling (since by Serre duality $h^0(\omega_C(-p_1, \ldots, p_g)) = h^1(p_1 + \ldots + p_g)$).

First, we observe an important relation between the basis $(\omega_i)$ and the canonical formal parameters $t_i$ at $p_i$ obtained from Lemma 2.1.1. Assume for a moment that the characteristic is zero. Then these formal parameters are uniquely determined by a collection of nonzero tangent vectors $v_i$ at $p_i$ by the condition that $v_i(t_i) = 1$ for each $i$ and that for every $n \geq 2$ and every $i = 1, \ldots, g$ there exists a rational function $f_i[n] \in H^0(C, \mathcal{O}(np_i + D_i))$, where $D_i = \sum_{j \neq i} p_j$. We claim that in fact for each $i = 1, \ldots, g$, one can rescale $\omega_i$ so that $\omega_i = dt_i$ in the formal neighborhood of $p_i$. If the characteristic is positive we consider the parameters $t_i$ up to $N$th order (where $N < \text{char}(k)$) and this statement has to be modified accordingly.

**Proposition 2.7.1.** Assume that $\text{char}(k) > N$ for some $N > 1$. Let $C$ be a smooth projective curve over $k$, $p_1, \ldots, p_g \in C$ distinct points such that $h^0(p_1 + \ldots + p_g) = 1$, and let $v_1, \ldots, v_g$ be nonzero tangent vectors at these points.

(i) The canonical formal parameters $t_i$ of order $N$ at $p_i$’s from Lemma 2.1.1 are characterized by the property that

$$\omega_i \equiv dt_i \bmod \omega_C(-Np_i)$$

in $\omega_C/\omega_C(-Np_i)$, for $i = 1, \ldots, g$, where $(\omega_i)$ is a basis of $H^0(C, \omega_C)$ such that $\omega_i$ vanishes at $p_j$ for $j \neq i$, and $\langle \omega_i, v_i \rangle = 1$.

(ii) For $i \neq j$ and $n \leq N$, let $p_{ij}[n]$ be the coefficient of $t_j^n$ in the Laurent series of $f_i[n]$ at $p_j$. Then The expansion of $\omega_i$ near $p_j$ (where $i \neq j$) has form

$$\omega_i = -\sum_{n=2}^{N} p_{ji}[n]t_j^{n-1}dt_j \bmod(t_j^Ndt_j).$$

**Proof.** (i) For every $n \leq N$ and $i = 1, \ldots, g$, the rational differential $f_i[n]\omega_i$ can have a pole only at $p_i$, so by the residue theorem, we get $\text{Res}_{p_i}(f_i(n)\omega_i) = 0$. Thus, if we write $\omega_i = \phi_i(t_i)dt_i$ at the formal neighborhood of $p_i$ then we deduce that $\phi_i = 1 \bmod(t_i^N)$ as claimed.

(ii) This follows immediately from the residue theorem applied to the rational differentials $f_{ij}[n]\omega_i$ for $i \neq j$, since $\text{Res}_{p_i}(f_{ij}[n]\omega_i) = p_{ji}[n]$ while $\text{Res}_{p_j}(f_{ij}[n]\omega_i)$ is equal to the coefficient of $t_j^{n-1}dt_j$ in the expansion of $\omega_i$ at $p_j$. 26
Recall that Petri proceeds by assuming that the differentials $\omega_1$ and $\omega_2$ have no common zeroes (which is true if the marked points are sufficiently generic) and shows that the following relations hold in $H^0(C, \omega^2)$:

$$\omega_i \omega_j = \sum_{k>2} (\lambda_{ijk} \omega_1 + \mu_{ijk} \omega_2) \omega_k + \nu_{ij} \omega_1 \omega_2,$$

where $i \neq j$, $i, j > 2$, for some constants $\lambda_{ijk}, \mu_{ijk}$ and $\nu_{ij}$. In addition, if for $i > 2$, $\eta_i \in H^0(\omega_C)$ is a linear combination of $\omega_1$ and $\omega_2$ with double zero at $p_i$ then for $i, j > 2$, one has cubic relations in $H^0(C, \omega_C^3)$ of the form

$$\eta_i \omega_i^2 - \eta_j \omega_j^2 = \sum_{k>2} (a_{ijk} \omega_1^2 + b_{ijk} \omega_1 \omega_2 + c_{ijk} \omega_2^2) \omega_k + d_{ij} \omega_1 \omega_2^2 + e_{ij} \omega_1 \omega_2^2,$$

for some constants $a_{ijk}, b_{ijk}, c_{ijk}, d_{ij}$ and $e_{ij}$.

Furthermore, Petri shows that the ideal of $C$ in the canonical embedding is generated by the relations (2.7.1), (2.7.2) for $g \geq 4$ (for generic points $p_1, \ldots, p_g$).

Now let us assume that $\text{char}(k) \neq 2, 3$ and let us choose canonical parameters $t_i$ of order 4 as in Lemma 2.1.1. Assume also that $H^0(C, \omega_C^2(-3D)) = 0$, where $D = p_1 + \ldots + p_g$ (this is true for $p_1, \ldots, p_g$ generic). Then any quadratic differential is uniquely determined by its expansions up to $t_i^2 dt_i^{\otimes 2}$ at $p_i$, for $i = 1, \ldots, g$. Hence, the coefficients $\lambda_{ijk}, \mu_{ijk}$ and $\nu_{ij}$ from (2.7.1) are determined from linear equations obtained by looking at such expansions. Proposition 2.7.1(ii) implies that these linear equations depend only on $\alpha_{ij} = p_{ij}[2]$ and $\beta_{ij} = p_{ij}[3]$.

For generic curve of genus $g \geq 5$ the quadratic relations generate the ideal of $C$, so let us now show how to determine the coefficients of the cubic relation (2.7.2) in the case $g = 4$ (assuming that the marked points are sufficiently generic). Note first that we can take

$$\eta_i = \alpha_{i1} \omega_1 - \alpha_{i1} \omega_2.$$

Next, we observe that for generic $p_1, \ldots, p_4$ we have $H^0(C, \omega_C^2(-4D)) = 0$. Indeed, this is equivalent to the vanishing $H^1(\omega_C^{-2}(-4D)) = 0$ which follows from the fact that the map $D \mapsto \omega_C^{-2}(4D)$ from $S^4 C$ to $\text{Pic}^4(D)$ is dominant. Thus, any element of $H^0(C, \omega_C^2)$ is determined uniquely by expansions up to $t_i^2 dt_i^{\otimes 3}$ at $p_i$ (these expansions are well defined by our parameters of order 4). Using Proposition 2.7.1(ii) we see that such expansions of the terms of (2.7.2) give equations on the coefficients that depend only on $\alpha_{ij}, \beta_{ij}$ and $\eta_{ij} = p_{ij}[4]$.

This leads to the following result, where we use the notation of Section 2.2.

**Proposition 2.7.2.** Assume $\text{char}(k) \neq 2, 3$. Then for $g \geq 4$ the isomorphism class of a generic pointed curve $(C, p_1, \ldots, p_g)$ is determined by the constants $(\alpha_{ij}, \beta_{ij}, \gamma_{ij})$ (viewed up to an action of $\mathbb{G}_m$). For $g \geq 5$ this isomorphism class is determined by the constants $(\alpha_{ij}, \beta_{ij})$.

**Proof.** We use Petri’s theorem that $C$ is cut out in the canonical embedding by the equations (2.7.1) and (2.7.2) (only by (2.7.1) for $g \geq 5$) together with the above considerations on expressing the coefficients of these equations in terms of $\alpha_{ij}, \beta_{ij}$ and $\eta_{ij}$. It remains to observe that that $\eta_{ij}$ is given by some universal polynomial expression in $(\alpha_{ij}, \beta_{ij}, \gamma_{ij})$.
(see Lemma 2.3.1) and that the point \( p_i \) is recovered as intersection of zero divisors of \( \omega_j \) for \( j \neq i \).

**Remark 2.7.3.** By Theorem [12, Thm. 3.2.1], if the characteristic is zero, then for \( g \geq 6 \) the isomorphism class of \((C, p_1, \ldots, p_g)\) is determined by the constants \((\alpha_{ij})\) alone (viewed up to \( \mathbb{G}_m^g \)-action). On the other hand, for \( g = 5 \) the rational map \( \mathcal{M}_{5,5} \to \mathbb{A}^{15} \) given by \( \alpha_{ij} \) modulo the \( \mathbb{G}_m^5 \)-action is dominant and so its generic fiber has dimension 2 (see [12, Thm. 5.2.2]).

Now we are going to present a different way to determine quadratic relations for the canonical embedding of \( C \), independent from Petri’s analysis (see Proposition 2.7.5 below).

We need the following result based on the Serre duality.

**Lemma 2.7.4.** Assume that for some line bundle \( L \) on \( C \) and an effective divisor \( E \subseteq C \) one has \( H^0(C, \omega_C \otimes L^{-1}(-E)) = 0 \). Then the map

\[
H^0(\omega_C \otimes L^{-1}) \to H^0(L(E)/L)^* : \eta \mapsto \left( s \mapsto \sum_{p \in E} \mathrm{Res}_p(\eta s) \right)
\]

is injective.

**Proof.** Recall that for a line bundle \( L \) the Serre duality isomorphism

\[
H^0(\omega_C \otimes L^{-1}) \overset{\sim}{\longrightarrow} H^1(L)^*
\]

is given by \( \eta \mapsto (\alpha \mapsto \mathrm{tr}(\eta \alpha)) \), where \( \mathrm{tr} : H^1(\omega_C) \to k \) is the canonical trace map. Let us consider the commutative diagram

\[
\begin{array}{ccc}
H^0(L(E)/L) & \xrightarrow{\delta_L} & H^1(L) \\
\downarrow{\eta} & & \downarrow{\eta} \\
H^0(\omega_C(E)/\omega_C) & \xrightarrow{\delta_\omega} & H^1(\omega_C)
\end{array}
\]

where the vertical arrows are given by the multiplication with \( \eta \in H^0(\omega_C \otimes L^{-1}) \), while the horizontal arrows are the connecting homomorphisms in the natural long exact sequences. It is well known that for \( \xi \in H^0(\omega_C(E)/\omega_C) \) one has

\[
(\mathrm{tr} \circ \delta_\omega)(\xi) = \sum_{p \in E} \mathrm{Res}_p(\xi).
\]

Hence, by the commutativity of the above diagram we get that the composed map

\[
H^0(\omega_C \otimes L^{-1}) \overset{\sim}{\longrightarrow} H^1(L)^* \overset{\delta_L^*}{\longrightarrow} H^0(L(E)/L)^*
\]

is given by \( \eta \mapsto (s \mapsto \sum_p \mathrm{Res}_p(\eta s)) \). It remains to observe that the condition \( H^0(\omega_C \otimes L^{-1}(-E)) = 0 \) is equivalent to the vanishing of \( H^1(L(E)) \) which implies the surjectivity of the map \( \delta_L \), and hence, the injectivity of the dual map \( \delta_L^* \). \( \square \)
Proposition 2.7.5. Assume $\text{char}(k) \neq 2,3$ and $H^0(C, \omega_C^2(-3D)) = 0$. Let $I_2 \subset S^2 H^0(C, \omega_C)$ be the kernel of the multiplication map

$$S^2 H^0(C, \omega_C) \rightarrow H^2(C, \omega_C^2). \quad (2.7.3)$$

Let us denote by $\Lambda_2 \subset S^2 H^0(C, \omega_C)$ the subspace spanned by $\omega_i \omega_j$ where $i \neq j$. Let also denote by $V$ the $2g$-dimensional vector space with the basis $(v_i, w_i)_{i=1,\ldots,g}$. Then $I_2$ is contained in $\Lambda_2$ and coincides with the kernel of the map

$$\Lambda_2 \rightarrow V : \omega_i \omega_j \mapsto \sum_{k \neq i,j} \alpha_k \alpha_{kj} w_k - \beta_{ji} w_j - \beta_{ij} w_i - \alpha_{ji} v_j - \alpha_{ij} v_i.$$ 

Proof. Applying Lemma 2.7.4 to the line bundle $L = \omega_C^{-1}$ and the divisor $E = 3D$, we derive that the map

$$\kappa : H^0(\omega_C^2) \rightarrow H^0(\omega_C^{-1}(3D)/\omega_C^{-1})^* : \eta \mapsto \left( v \mapsto \sum_{i=1}^g \text{Res}_{p_i}(\eta v) \right)$$

is injective. Thus, the kernel of the multiplication map (2.7.3) coincides with the kernel of the map

$$S^2 H^0(\omega_C) \rightarrow H^0(\omega_C^{-1}(3D)/\omega_C^{-1})^* : \omega' \mapsto \left( v \mapsto \sum_{i=1}^g \text{Res}_{p_i}(\omega' v) \right).$$

Let $(u_i, v_i, w_i)_{i=1,\ldots,g}$ be the basis in $H^0(\omega_C^{-1}(3D)/\omega_C^{-1})^*$, dual to the basis

$$(t_i^{-1} \frac{\partial}{\partial t_i}, t_i^{-2} \frac{\partial}{\partial t_i}, t_i^{-3} \frac{\partial}{\partial t_i})_{i=1,\ldots,g}$$

of $H^0(\omega_C^{-1}(3D)/\omega_C^{-1})$ (where $t_i$ are the canonical parameters of order 4 at $p_i$). Then using Proposition 2.7.1(ii) we find

$$\kappa(\omega_i^2) = u_i + \sum_{j \neq i} \alpha_{ij}^2 w_j,$$

$$\kappa(\omega_i \omega_j) = \sum_{k \neq i,j} \alpha_k \alpha_{kj} w_k - \beta_{ji} w_j - \beta_{ij} w_i - \alpha_{ji} v_j - \alpha_{ij} v_i,$$

for $i \neq j$, where $\alpha_{ij} = p_{ij}[2], \beta_{ij} = p_{ij}[3]$. Thus, we have a commutative diagram

$$\begin{array}{ccc}
S^2 H^0(\omega_C) & \xrightarrow{\kappa} & H^0(\omega_C^{-1}(3D)/\omega_C^{-1})^* \\
\downarrow & & \downarrow \\
\langle \omega_i^2 \mid i = 1,\ldots,g \rangle & \sim & \langle u_i \mid i = 1,\ldots,g \rangle
\end{array}$$

where the vertical arrows are the coordinate projections on the subspaces generated by $(\omega_i^2)_{i=1,\ldots,g}$ and $(u_i)_{i=1,\ldots,g}$, respectively. Hence, the kernel of $\kappa$ coincides with the kernel of its restriction to the subspace spanned by $\omega_i \omega_j$ with $i \neq j$. Now our statement follows from the explicit formula for $\kappa$. \qed
3. Computation of higher products on a curve

3.1. Čech resolutions. Let $\pi : C \to \text{Spec}(R)$ be a flat proper family of curves equipped with a relative effective Cartier divisor $D \subset C$, such that $\pi$ is smooth near $D$ and $U = C \setminus D$ is affine. Then for every quasicoherent sheaf $\mathcal{F}$ on $C$ we can consider the two-term complex $K^\bullet(\mathcal{F}) = K^\bullet_D(\mathcal{F})$ with

$$K^0(\mathcal{F}) = \lim_{\longleftarrow} \, H^0(C, \mathcal{F}/\mathcal{F}(-nD)) \oplus H^0(U, \mathcal{F}),$$

$$K^1(\mathcal{F}) = \lim_{\longleftarrow} \, \lim_{\longleftarrow} \, H^0(C, \mathcal{F}(mD)/\mathcal{F}(-nD))$$

and the differential

$$d(s_0, s) = \kappa(s) - \iota(s_0),$$

where we use natural maps $\kappa : H^0(C, \mathcal{F}/\mathcal{F}(-nD)) \to K^1(\mathcal{F})$ and $\iota : H^0(U, \mathcal{F}) \to K^1(\mathcal{F})$.

The construction of $K^\bullet(\mathcal{F})$ immediately generalizes to the case when $\mathcal{F}$ is a bounded complex of vector bundles (by taking the total complex of the corresponding bicomplex).

Furthermore, if $\mathcal{A}$ is a complex of quasicoherent sheaves equipped with the structure of an $\mathcal{O}$-dg-algebra then we can equip the complex $K^\bullet(\mathcal{A})$ with a structure of dg-algebra by using the natural componentwise multiplication on $K^0(\mathcal{A})$ and using the multiplications

$$K^0(\mathcal{A}) \otimes K^1(\mathcal{A}) \to K^1(\mathcal{A}) : (s_0, s) \cdot u = \iota(s_0) \cdot u,$$

$$K^1(\mathcal{A}) \otimes K^0(\mathcal{A}) \to K^1(\mathcal{A}) : u \cdot (s_0; s) = u \cdot \kappa(s),$$

where in the right-hand side we use the natural product on $K^1(\mathcal{A})$.

**Lemma 3.1.1.** (i) For a quasicoherent sheaf $\mathcal{F}$ on $C$ there is a natural isomorphism in the derived category of $R$-modules $R\Gamma(C, \mathcal{F}) \simeq K^\bullet(\mathcal{F})$.

(ii) Assume that $R$ is a finitely generated $k$-algebra, where $k$ is a field. Let $\mathcal{A}(V) = \text{End}_\mathcal{O}(V)$ be the endomorphism dg-algebra over $\mathcal{O}$ of a bounded complex of vector bundles $V$ on $C$. Then the dg-algebra $K^\bullet(\mathcal{A}(V))$ is quasi-isomorphic to the dg-algebra of endomorphisms of $V$ computed using any dg-enhancement of $D(\text{Qcoh}(C))$.

**Proof.** (i) Let $j : C \to p$ be the natural open embedding. Then $j_*j^* \mathcal{F} = \lim_{\longleftarrow} \mathcal{F}(mD)$.

We have the sheaf version $\mathcal{K}^\bullet$ of the complex $K^\bullet = K^\bullet(\mathcal{F})$ with

$$\mathcal{K}^0 = \lim_{\longleftarrow} \mathcal{F}/\mathcal{F}(-nD) \oplus j_*j^* \mathcal{F},$$

$$\mathcal{K}^1 = \lim_{\longleftarrow} \lim_{\longleftarrow} \mathcal{F}(mD)/\mathcal{F}(-nD).$$

with the differentials induced by the natural maps $\lim_{\longleftarrow} \mathcal{F}/\mathcal{F}(-nD) \to \mathcal{K}^1$ and $j_*j^* V = \lim_{\longleftarrow} \mathcal{F}(mD) \to \mathcal{K}^1$, so that $K^\bullet = H^0(C, \mathcal{K}^\bullet)$. Since $D$ and $U$ are affine, the sheaves $\mathcal{K}^0$ have no higher cohomology. It remains to check that $\mathcal{K}^0 \to \mathcal{K}^1$ is a resolution of $\mathcal{F}$. Indeed, for each $m, n > 0$ we have an exact sequence

$$0 \to \mathcal{F} \to \mathcal{F}/\mathcal{F}(-nD) \oplus \mathcal{F}(mD) \to \mathcal{F}(mD)/\mathcal{F}(-nD) \to 0.$$ 

By passing to the inverse limit in $n$ we get an exact sequence

$$0 \to \mathcal{F} \to \lim_{\longleftarrow} \mathcal{F}/\mathcal{F}(-nD) \oplus \mathcal{F}(mD) \to \lim_{\longleftarrow} \mathcal{F}(mD)/\mathcal{F}(-nD) \to 0.$$
(note that the exactness is preserved since on the left we have a constant inverse system). Then passing to the direct limit in $m$ we get the exactness of

$$0 \to F \to K^0 \to K^1 \to 0.$$  

(ii) Recall that $D(Qcoh(C))$ and the subcategory Perf($C$) of perfect complexes both have unique dg-enhancements, by the results of Lunts-Orlov (see [23, Cor. 2.11, Thm. 2.12]). Now we observe that for a pair of bounded complexes $V$ and $W$ we can consider the complex $K^*(V^\vee \otimes W)$ and get a dg-category structure on these using products similar to (3.1.1). By part (i), this is a dg-enhancement of Perf($C$), and $K^*(A(V))$ is precisely the corresponding dg-endomorphism algebra of $V$.

3.2. dg-endomorphism algebra of $O_C \oplus O_{p_1} \oplus \ldots \oplus O_{p_g}$. Let $\pi : C \to \text{Spec}(R)$, $p_1, \ldots, p_g : \text{Spec}(R) \to C$ be a family of curves in $\tilde{U}^{ms,a}(R)$, where $R$ is a finitely generated $k$-algebra, and let $v_i \in O_C(p_i)/O_C$ be the corresponding trivializations (for example, we can consider the universal family over $\mathbb{Z}[1/6]$). We are going to apply the construction of Section 3.1 for the divisor $D = p_1 + \ldots + p_g$ to get an explicit presentation for dg-endomorphisms of the object

$$G = O_C \oplus O_{p_1} \oplus \ldots \oplus O_{p_g}. \quad (3.2.1)$$

Here we identify each $p_i : \text{Spec}(R) \to C$ with its image, which is a divisor in $C$. The reason we chose this object is the fact that the corresponding Ext-algebra is independent of the curve (this was also the starting point of [12]).

**Lemma 3.2.1.** One has a natural isomorphism of $R$-algebras $\text{Ext}^*(G, G) \simeq E_{g,\mathbb{Z}} \otimes R$, where $E_{g,\mathbb{Z}}$ is the natural $\mathbb{Z}$-form of the algebra (0.0.2).

**Proof.** We have a natural identification $R \simeq \text{Hom}(O, O_{p_i})$, and we denote by $A_i \in \text{Hom}(O_C, O_{p_i})$ the corresponding generator. The resolution $[O_C(-p_i) \to O_C]$ for $O_{p_i}$ gives isomorphisms

$$\text{Ext}^1(O_{p_i}, O) \simeq H^0(O_C(p_i))/H^0(O_C) \simeq H^0(C, O_C(p_i)/O_C),$$

where the last isomorphism follows from the vanishing of $H^1(O_C(p_i))$. Let us denote by $B_i \in \text{Ext}^1(O_{p_i}, O)$ the generator corresponding to the chosen trivializations $R \simeq H^0(C, O_C(p_i)/O_C)$. The same resolution also gives an isomorphism

$$\text{Ext}^1(O_{p_i}, O_{p_i}) \simeq H^0(O_C(p_i)/O(C)) \simeq R$$

with the generator given by the composition $Y_i = A_i B_i$. Finally, the exact sequence

$$0 \to O_C \to O_C(p_1 + \ldots + p_g) \to \bigoplus_{i=1}^g O_C(p_i)/O_C \to 0$$

together with the vanishing of $H^1(C, O)$ gives an isomorphism

$$\bigoplus_{i=1}^g O_C(p_i)/O_C \to H^1(C, O) = \text{Ext}^1(O_C, O_C),$$

with the generators given by the compositions $X_i = B_i A_i$. \hfill \Box
To calculate the dg-endomorphisms $G$, we first use the resolution $P := [\mathcal{O}_C(-D) \to \mathcal{O}_C]$ for the sheaf $\bigoplus_i \mathcal{O}_{p_i}$. Then we consider the bundle of dg-algebras

$$\mathcal{A} = \mathcal{A}(C, p_1, \ldots, p_g) = \text{End}(\mathcal{O}_C \oplus P)$$

and the corresponding dg-algebra

$$E^\text{dg}_{(C, p_1, \ldots, p_g)} = K^*_D(\mathcal{A}).$$

Let us fix some formal parameters $t_i$ at $p_i$, compatible with $v_i$, i.e., sections of $\lim_{\to} H^0(C, \mathcal{O}_C(-p_i)/\mathcal{O}_C(-np_i))$ inducing isomorphisms

$$R[[t_i]] \overset{\sim}{\longrightarrow} \lim_{\to} H^0(C, \mathcal{O}_C/\mathcal{O}_C(-np_i))$$

and such that $\langle v_i, t_i \mod \mathcal{O}_C(-2p_i) \rangle = 1$. Note that we also have the induced isomorphisms

$$R((t_i)) \overset{\sim}{\longrightarrow} \lim_{\to} \lim_{\to} H^0(C, \mathcal{O}_C(mp_i)/\mathcal{O}_C(-np_i)).$$

Hence, for any integer $n$ we have an identification of $K^1(\mathcal{O}_C(nD))$ with $\bigoplus_{i=1}^g R((t_i))$. For an element $a(t_i) \in R((t_i))$ we denote by $[a(t)]$ the corresponding element of $K^1(\mathcal{O}_C(nD))$.

We have a direct sum decomposition

$$E^\text{dg}_{(C, p_1, \ldots, p_g)} = K_\mathcal{O} \oplus K_{\mathcal{O}, P} \oplus K_{P, \mathcal{O}} \oplus K_{P, P},$$

where $K_\mathcal{O} = K^*_D(\mathcal{O})$ and $K_{P, P} = K^*_D(P_2 \otimes P_1^\vee)$. We denote (local) sections of the 0th term of $P$ by $e \cdot f$, where $f \in \mathcal{O}$, and local sections of the $-1$th term of $P$ by $u \cdot f$, where $f \in \mathcal{O}(\mathcal{O})(-D)$.

We denote elements of $K_\mathcal{O}$ as

$$v + f + [a],$$

where $v \in \bigoplus_{i=1}^g t_i R[[t_i]]$, $f \in \mathcal{O}(U)$, $a \in \bigoplus_{i=1}^g R((t_i))$, $v$ and $f$ have degree 0 and $[a]$ has degree 1. The differential on $K_{\mathcal{O}, \mathcal{O}}$ is given by

$$d_{\mathcal{O}}(v + f + [a]) = [f - v],$$

where in the right-hand side we use the projection

$$\mathcal{O}(U) \to K^1(\mathcal{O}_C) \simeq \bigoplus_{i=1}^g R((t_i))$$

to view $f$ as an element of $\bigoplus_{i=1}^g R((t_i))$.

The summand $K_{\mathcal{O}, P}$ decomposes as a graded space as

$$u \cdot \left( \bigoplus_{i=1}^g t_i R[[t_i]] \oplus \mathcal{O}(U) \right) [1] \oplus u \cdot \bigoplus_{i=1}^g R((t_i)) \oplus e \cdot \left( \bigoplus_{i=1}^g R[[t_i]] \oplus \mathcal{O}(U) \right) \oplus \bigoplus_{i=1}^g e \cdot R((t_i))[1],$$

where $U = C - D$. We will write elements of $K_{\mathcal{O}, P}$ as formal sums

$$u \cdot v + u \cdot f + u \cdot [a] + e \cdot w + e \cdot h + e \cdot [b],$$

where $v \in \bigoplus_{i=1}^g t_i R[[t_i]]$, $w \in \bigoplus_{i=1}^g R[[t_i]]$, $a, b \in \bigoplus_{i=1}^g R((t_i))$, $f, h \in \mathcal{O}(U)$. Here we treat $a, b, v, w, f, h$ as having degree 0, and use the convention that $\deg(u) = -1$, $\deg(e) = 0$ and $\deg([x]) = \deg(x) + 1$. 32
Similarly, elements of $K_{P,O}$ are formal sums
\[ v \cdot u^* + f \cdot u^* + [a] \cdot u^* + w \cdot e^* + h \cdot e^* + [b] \cdot e^*, \]
where $\text{deg}(u^*) = 1$, $\text{deg}(e) = 0$, $v \in \bigoplus_{i=1}^{g} t_i^{-1} R[[t_i]]$.

Elements of $K_{P,P}$ are formal sums
\[
\begin{align*}
&\quad u \cdot (v_{uu} + f_{uu} + [a_{uu}]) \cdot u^* + e \cdot (v_{eu} + f_{eu} + [a_{eu}]) \cdot u^* + u \cdot (v_{ue} + f_{ue} + [a_{ue}]) \cdot e^* + \\
&\quad e \cdot (v_{ee} + f_{ee} + [a_{ee}]) \cdot e^*,
\end{align*}
\]
where $v_{uu} \in \bigoplus_{i=1}^{g} R[[t_i]]$, $v_{eu} \in \bigoplus_{i=1}^{g} t_i^{-1} R[[t_i]]$ and $v_{ue} \in \bigoplus_{i=1}^{g} t_i R[[t_i]]$.

The product on $K_{O,O}^0$ is simply that of the direct sum of rings. The remaining products are determined by the rules
\[ u^* u = 1, \quad e^* e = 1, \quad u^* e = e^* u = 0, \]
where the product on $\bigoplus_i R((t_i))$ is componentwise.

The differentials on $K_{O,P}$, $K_{P,O}$ and $K_{P,P}$ are determined by the Leibnitz rule using the formulas
\[ d(u) = e, \quad d(e) = 0, \quad d(e^*) = -u^*, \quad d(u^*) = 0. \]
For example, the differentials on $K_{O,P}$ and $K_{P,O}$ are
\[
\begin{align*}
d(u \cdot x + e \cdot y) &= -u \cdot d_O(x) + e \cdot x + e \cdot d_O(y), \\
d(x \cdot u^* + y \cdot e^*) &= d_O(x) \cdot u^* - (-1)^{\text{deg}(y)} y \cdot u^* + d_O(y) \cdot e^*.
\end{align*}
\]

By Lemmas 3.1.1(i) and 3.2.1, the cohomology algebra of $E_{(C,p_1,\ldots,p_g)}^{dq}$ is the algebra $\text{Ext}^*(G,G) = E_{q,\mathbb{Z}} \otimes R$.

Let us list convenient representatives for the cohomology of our complexes: For $K_{O,P}$ these are
\[ A_i = e \cdot 1_i + u \cdot [1_i] \in K_{O,P}^0, \quad i = 1, \ldots, g, \]
where $1_i$ is $1 \in R[[t_i]] \subset \bigoplus_{i=1}^{g} R[[t_i]]$. For $K_{P,O}$ we choose
\[ B_i = \frac{1}{t_i} \cdot u^* + [\frac{1}{t_i}]e^* \in K_{P,O}^1. \]

We have induced classes
\[ X_i = B_i A_i = [\frac{1}{t_i}] \in K_{O,O}^1, \]
\[ Y_i = A_i B_i = e \cdot \frac{1}{t_i} \cdot u^* + e[\frac{1}{t_i}]e^* \in K_{P,P}^1. \]

To complete the list of representatives we also need the classes
\[ 1_O = \sum_{i=1}^{g} 1_i, \quad 1 \in K_{O,O}^0 \quad \text{and} \]
\[ e_p := e \cdot 1_i \cdot e^* + u \cdot 1_i \cdot u^* + u[1_i]e^* \in K_{P,P}^0. \]

Note that we have the following product formulas
\[ B_i Y_i = X_i B_i = 0, \]
\[ B_i Y_i = X_i B_i = 0, \]
\[ A_i X_i = Y_i A_i = \varepsilon_i \left[ \frac{1}{t_i} \right]. \]

The products involving \( e_{p_i} \) and \( 1_{\mathcal{O}} \) are the same as on the cohomology level.

### 3.3. Homotopy operators and formulas for higher products.

Next, we are define the homotopy operator \( Q \) on \( E^{dg} = E^{dg}_{(C,p_1,\ldots,p_g)} \). More precisely, it is easy to see that cohomology of \( E^{dg} \) and the submodule \( \text{im}(d) \subset E^{dg} \) are free \( R \)-modules, so we can choose complements to \( \text{im}(d) \) in \( \text{ker}(d) \) and to \( \text{ker}(d) \) in \( E^{dg} \). These complements give a homotopy operator \( Q \) that maps both these complements to zero, has the image in the complement to \( \text{ker}(d) \) and is inverse to \( d \) on \( \text{im}(d) \). Note that in this case \( \Pi = \text{id} - Q d - d Q \) is the projector onto the free \( R \)-submodule of cohomology representatives. Also, it is clear from the construction that we have

\[ Q^2 = \Pi Q = Q \Pi = 0. \]

To implement this construction we make a choice for each \( n \geq 2 \) of a section \( f_i[n] \in H^0(C, \mathcal{O}(D + (n-1)p_i)) \) so that the polar part of \( f_i[n] \) at \( p_i \) has form \( t_i^{-n} \mod t_i^{-n+1} R[[t_i]] \) with respect to the formal parameter \( t_i \) at \( p_i \). Furthermore, we can choose such sections \( f_i[n] \) uniquely up to adding a constant (modifying them by \( f_i[m] \) with \( m < n \)), so that for each \( n \geq 2 \) the polar part at \( p_i \) is

\[ f_i[n](t_i) = \frac{1}{t_i^n} + \frac{p_{ii}[n]}{t_i} + \ldots \tag{3.3.1} \]

for some constants \( p_{ii}[n] \in R \). Also, the polar part of \( f_i[n] \) at \( p_j \) has form

\[ f_i[n](t_j) = \frac{p_{ij}[n]}{t_j} + \ldots \]

for some constants \( p_{ij}[n] \in R \). Note that at this point we do not make any special choice of formal parameters \( t_i \) at \( p_i \). Lemma 2.1.1 tells that by choosing these parameters in a special way we may achieve the vanishing of some of the constants \( p_{ii}[n] \) (in characteristic zero of all of them).

Now to define \( Q \) we use cohomology representatives defined above. Also, we use \( f_i[n] \) to define the complement to \( \text{ker}(d) \). For example, for \( K^0_{\mathcal{O},\mathcal{O}} \) this complement is spanned by \( (f_i[n])_{n \geq 2} \) and by \( \bigoplus_{i=1}^g R[[t_i]] \).

Below for a series \( a(t) = \sum a_i t^i \in R((t)) \) we denote \( a(t)_{\geq n} = \sum_{i \geq n} a_i t^i \).

We define the homotopy operator on \( K^0_{\mathcal{O},\mathcal{O}} \) by

\[ Q([1/t_i]) = 0, \quad Q([v]) = -v \]

for \( v \in \bigoplus_{i} R[[t_i]] \) and

\[ Q([1/t_i]) = f_i[n](t_i)_{\geq 0} + \sum_{j \neq i} f_i[n](t_j)_{\geq 0} 1_j + f_i[n]. \]

On \( K_{\mathcal{O},p} \) we define \( Q \) by

\[ Q(e[b]) = u[b], \quad Q(u[a] + e \cdot v + e \cdot f) = u \cdot v_{\geq 0} + u \cdot f. \]
On $K_{P,\mathcal{O}}$ we set
\[ Q([a]\mathbf{u}^*) = [a]\mathbf{e}^*, \quad Q([a]\mathbf{e}^* + v \cdot \mathbf{u}^* + f \cdot \mathbf{u}^*) = -v_{\geq 0}\mathbf{e}^* - f \cdot \mathbf{e}^*. \]
Finally, on $K_{P,P}$ the homotopy operator is defined by
\[ Q(e[a_eu]\mathbf{u}^*) = u[a_eu]\mathbf{u}^*, \]
\[ Q(u[a_{uu}]\mathbf{u}^* + e[a_{ee}]\mathbf{e}^* + e \cdot v_{ee} \cdot \mathbf{u}^* + e \cdot f_{ee} \cdot \mathbf{u}^*) = u[a_{ee}]\mathbf{e}^* + u \cdot (v_{ee})_{\geq 0} \cdot \mathbf{u}^* + u \cdot f_{ee} \cdot \mathbf{u}^*, \]
\[ Q(u[a_{ae}]\mathbf{e}^* + u \cdot v_{uu} \cdot \mathbf{u}^* + u \cdot f_{uu} \cdot \mathbf{u}^* + e \cdot v_{ee} \cdot \mathbf{e}^* + e \cdot f_{ee} \cdot \mathbf{e}^*) = u \cdot (v_{ee})_{\geq 0} \cdot \mathbf{e}^* + u \cdot f_{ee} \cdot \mathbf{e}^*. \]
The corresponding projector $\Pi$ onto the cohomology representatives is given by
\[ \Pi(v) = \Pi([v]) = \Pi(f_i[n]) = 0, \quad \Pi(1) = 1_{\mathcal{O}}, \quad \Pi(\frac{1}{t_i}) = X_i, \]
\[ \Pi(\frac{1}{t_i}) = -\sum_j p_{ij}[n]X_j, \quad \text{where } n \geq 2, \]
\[ \Pi(u[a] + e \cdot v + e \cdot f) = \sum_i v_i(0)A_i, \]
\[ \Pi([a] \cdot \mathbf{e}^* + v \cdot \mathbf{u}^* + f \cdot \mathbf{u}^*) = \sum_i \text{Res}_i(v)B_i, \]
\[ \Pi(u[a_{uu}]\mathbf{u}^* + e[a_{ee}]\mathbf{e}^* + e \cdot v_{ee} \cdot \mathbf{u}^* + e \cdot f_{ee} \cdot \mathbf{u}^*) = \sum_i \text{Res}_i(v_{ee})X_i, \]
\[ \Pi(u[a_{ae}]\mathbf{e}^* + u \cdot v_{uu} \cdot \mathbf{u}^* + u \cdot f_{uu} \cdot \mathbf{u}^* + e \cdot v_{ee} \cdot \mathbf{e}^* + e \cdot f_{ee} \cdot \mathbf{e}^*) = \sum_i (v_{ee})_i(0)e_{p_i}, \]
where $\text{Res}_i(v)$ for $v = (v_i)$ is the coefficient of $\frac{1}{t_i}$ in $v_i$.

Recall (see [25], [18]) that the above choice of the homotopy operator $Q$ gives homological perturbation formulas for the $\mathbb{A}_\infty$-structure on the cohomology of $E_{dg}^\text{rig}$, which we identified with $E_{g,\mathbb{Z}} \otimes R$. Namely, for cohomology representatives $b_1, \ldots, b_n$, we have
\[ m_n(b_1, \ldots, b_n) = \sum_T \pm m_T(b_1, \ldots, b_n), \]
where $T$ runs over all oriented planar rooted 3-valent trees with $n$ leaves (different from the root) marked by $b_1, \ldots, b_n$ left to right, and the root marked by $\Pi$. The expression $m_T(b_1, \ldots, b_n)$ is obtained by going down from leaves to the root, applying the multiplication in $E_{dg}^\text{rig}$ at every vertex and applying the operator $Q$ at every inner edge (see [18, Sec. 6.4], [29, Sec. 2.1] for details).

**Lemma 3.3.1.** All products $m_i$ with $i \geq 3$ involving $e_{p_i}$ or $1_{\mathcal{O}}$ vanish. Thus, the obtained $\mathbb{A}_\infty$-structure on $E_{g,\mathbb{Z}} \otimes R$, viewed as a category with $n + 1$ objects, is strictly unital.

**Proof.** Due to the vanishing $Q^2 = \Pi Q = 0$ it is enough to check that the multiplication by $e_{p_i}$ and by $1_{\mathcal{O}}$ on the left or the right preserves the image of $Q$. With $1_{\mathcal{O}}$ this is clear since it is a unit in $K_{\mathcal{O},\mathcal{O}}^0$. With $e_{p_i}$ this is easy to check. \hfill \Box

Let us apply the above construction to the universal family $(C, p_1, \ldots, p_r)$ over
\[ \tilde{U}_{g,g}^{\text{ns},a} \times \text{Spec}(\mathbb{Z}[1/6]) =: \text{Spec}(\mathbb{R}^\text{univ}). \]
Recall that we have a natural $G_m^d$-action on this moduli space and on the universal curve. Let us consider the induced $G_m$-action for the diagonal $G_m \subset G_m^d$. Note that we have the corresponding $\mathbb{Z}$-grading on $R^{univ}$. We can choose all formal parameters $t_i$ at $p_i$ to be $G_m$-equivariant, i.e., such that
\[ \lambda^* t_i = \lambda^{-1} t_i \]
(recall that we normalize our formal parameters by $(v_i, t_i \text{ mod } \mathcal{O}(-2p_i)) = 1$ and that $\lambda^* v_i = \lambda^{-1} v_i$). Then the corresponding sections $f_i[n]$ satisfying (3.3.1), where $n \geq 2$, can be chosen to be $G_m$-equivariant as well, i.e., such that
\[ \lambda^* f_i[n] = \lambda^n t_i. \]
Therefore, the corresponding homotopy operator $Q$ on $E^{dg}$ is also going to be equivariant with respect to the $G_m$-action induced by the $G_m$-action on the universal curve.

**Proposition 3.3.2.** For a $G_m$-equivariant homotopy operator $Q$, the structure constants of the higher product $m_n$ with respect to the standard basis of $E_{g,\mathbb{Z}} \otimes R^{univ}$ belong to $R^{univ}_{n-2}$, the component of degree $n - 2$ with respect to the $\mathbb{Z}$-grading of $R^{univ}$.

**Proof.** Let denote by wt the degree of an element of $E^{dg}$ with respect to the $G_m$-action induced by the action of $G_m$ on the universal family. Since the differential $d$, the homotopy $Q$ and the projector $\Pi$ are all homogeneous with respect to this grading, the operations $m_n$ are also homogeneous. Now we observe that $w A_i = 0$ while $w B_i = -1$ (recall that $B_i$ comes from the tangent vector at $p_i$). Since these elements generate $E_{g,\mathbb{Z}}$, it follows that $\lambda^{-1} \in G_m$ acts on $E_{g,\mathbb{Z}}$ by $\lambda^{\text{deg}}$, where deg is the cohomological grading on $E_{g,\mathbb{Z}}$. Since $m_n$ lowers the cohomological grading by $n - 2$, we derive that $(\lambda^{-1})^*$ rescales the coefficients of $m_n$ by $\lambda^{n-2}$. \hfill \Box

Using Proposition 2.3.2 we deduce the following result.

**Corollary 3.3.3.** Consider the $A_\infty$-structure on $E_{g,\mathbb{Z}} \otimes R^{univ}[1/30]$ associated with some $G_m$-equivariant homotopy operator $Q$ on $E^{dg}$. Then the structure constants of $m_n$ are polynomials in generators (2.3.1) of degree $n - 2$, where $\deg \alpha_{ij} = 1$, $\deg \beta_{ij} = \deg \gamma_{ij} = 2$, $\deg \varepsilon_{ij} = 3$ and $\deg \pi_i = 4$.

In the hyperelliptic case we can deduce vanishing of $m_n$ with odd $n$.

**Corollary 3.3.4.** Let $(C, p_1, \ldots, p_g)$ be the universal family over the hyperelliptic locus $\mathcal{H}_{g,\mathbb{Z}} \times \text{Spec}(\mathbb{Z}[1/6]) = \text{Spec}(R^{he})$. Then the $A_\infty$-structure on $E_{g,\mathbb{Z}} \otimes R^{he}$, in the equivalence class given by the homological perturbation of $E^{dg}$, can be chosen so that $m_n = 0$ for all odd $n$.

**Proof.** Let $p : R^{univ} \to R^{he}$ be the natural homomorphism. Note that it is compatible with the $\mathbb{Z}$-grading and since $-1 \subset G_m$ acts trivially on $\text{Spec}(R^{he})$ (see Proposition 2.6.2), we obtain that the odd degree components in $R^{he}$ vanish. It remains to use the $A_\infty$-structure on $E_{g,\mathbb{Z}} \otimes R^{he}$ obtained from some $G_m$-equivariant choices of $(t_i), (f_i[n])$. \hfill \Box

**Remark 3.3.5.** Corollary 3.3.4 is a generalization of the fact that for elliptic curves the corresponding $A_\infty$-structure can be chosen so that $m_n = 0$ for odd $n$, which was a consequence of a direct computation in [29].
Now let us make some explicit computations. We set
\[ P_i = A_i X_i = Y_i A_i = e^{1/t_i}. \]
Note that this is the only double product of \( A_i, X_i, Y_i, B_i \) on which \( Q \) is not zero. Thus, when calculating higher products we can restrict the sum over trees to only those trees in which any two leaves joined to the same internal vertex are either \( A_i \) and \( X_i \), or \( Y_i \) and \( A_i \). Note also that
\[ Q(P_i) = u^{1/t_i}, \]
\[ Q(P_i) B_i = Q(P_i) B_j = Q(P_i) X_i = Q(P_i) X_j = 0, \]
\[ (Q \circ Y)^n(Q(P_i)) B = (Q \circ Y)^n(Q(P_i)) X = 0, \]
\[ Q(K^1_{\Lambda, P}) B_i = Q(K^1_{\Lambda, P}) X_i = 0, \]
\[ X_i Q(K^2_{P, \Lambda}) = Q(K^2_{P, \Lambda}) A_i = 0, \]
\[ Q(K^2_{P, \Lambda}) A_i = Q(K^2_{P, \Lambda}) Y_i = Q(K^1_{P, \Lambda}) Y_i = 0, \]
\[ XQ(Q(BQ(P))) X = XQ(Q(BQ(P)) B) = 0. \]
These vanishings further restrict the types of trees contributing to the higher products.

If \( 2 \) is invertible in \( R \) then we can choose parameters \( t_i \) in such a way that \( p_{ii}[2] = 0 \) (by Lemma 2.1.1). Then from the above observations one can easily derive the following formulas for \( m_3 \):
\[ m_3(B_i, Y_i, A_i) = m_3(B_i, A_i, X_i) = -\sum_{j \neq i} \alpha_{ij} X_j, \] (3.3.2)
where \( \alpha_{ij} = p_{ij}[2] \), while all other \( m_3 \) products of the basis elements vanish. Similarly, if \( 6 \) is invertible in \( R \) we can also assume the vanishing of \( p_{ii}[3] \) and \( p_{ii}[4] \). The obtained explicit formulas for \( m_4 \) and \( m_5 \) can be found in the Appendix.

4. FROM MODULI OF CURVES TO MODULI OF \( A_\infty \)-STRUCTURES

4.1. Generalities on \( A_\infty \) and \( A_n \)-structures. For a graded associative \( S \)-algebra \( A \) (where \( S \) is a commutative ring and \( A \) is flat as \( S \)-module), we denote the terms of the Hochschild cochain complex of \( A \) over \( S \) as follows: \( CH^{s+t}(A/S)_t \) denotes the space of \( S \)-multilinear maps \( A^\otimes s \to A \) of degree \( t \) (where tensoring is over \( S \)). We have the induced bigrading \( HH^{s+t}(A/S)_t \) of the Hochschild cohomology. The corresponding grading by the upper index is compatible with the definition of the Hochschild cohomology for \( A_\infty \)-algebras.

Below we use the notion of \( A_n \)-structure which is a truncated version of an \( A_\infty \)-structure defined by Stasheff (see [37, Def. 2.1]). For a moment let \( A \) be a graded \( S \)-module. Recall that an \( S \)-linear \( A_n \)-structure is given by a collection of \( S \)-multilinear maps
\[ (m_1, \ldots, m_n) \in CH^2(A/S)_1 \times \ldots \times CH^2(A/S)_{2-n} \]
satisfying the standard $A_\infty$-identities involving only $m_1, \ldots, m_n$ (see below). Following [37, (2.4)], $A_n$-structures can be described conveniently in terms of truncated bar-construction

$$\text{Bar}_{\leq n}(A) = \bigoplus_{i=1}^{n} T^i_S(A[1]).$$

It has a natural structure of coalgebra over $S$ (without counit), such that it is a sub-coalgebra of the full bar-construction $\text{Bar}(A) = \bigoplus_{i \geq 1} T^i_S(A[1])$. For each cochain $c \in CH^{s+t}(A/S)_t$, where $s \geq 1$, we denote by $D_c$ the corresponding coderivation of $\text{Bar}(A)$ of degree $s+t-1$, preserving each sub-coalgebra $\text{Bar}_{\leq n}(A)$ (we recover $c$ from the component $\text{Bar}_{\leq s}(A) \to A[1]$ of $D_c$). Then the condition for $m = (m_1, \ldots, m_n)$ to define an $A_n$-algebra structure on $A$ is that

$$D^2_m \big|_{\text{Bar}_{\leq n}(A)} = 0.$$ 

We can rewrite this as a collection of identities

$$\sum_{i=1}^{r} D_{m_i} D_{m_{r+1-i}} \big|_{\text{Bar}_{\leq n}(A)} = 0, \quad (4.1.1)$$

where $r = 1, \ldots, n$.

We denote by $[\cdot, \cdot]$ the supercommutator of coderivations. Recall that

$$[D_c, D_{c'}] = D_{[c,c']},$$

where $[c, c']$ is the Gerstenhaber bracket. Also, if $D_c$ has degree 1 then $D_c^2$ is still a coderivation, so it corresponds to some cochain $([c, c]/2$ when $2$ is invertible). Thus, we can view the identity $(4.1.1)$ as the linear equation for coderivations associated with some Hochschild cochains in $CH^3(A/S)_{3-r}$. Since such cochains $c$ are uniquely determined from the restriction $D_c|_{\text{Bar}_{\leq r}(A)}$, we deduce the following result.

**Lemma 4.1.1.** The elements $(m_1, \ldots, m_n) \in CH^2(A/S)_1 \times \ldots \times CH^2(A/S)_{2-n}$ define an $S$-linear $A_n$-structure on $A$ if and only if

$$\sum_{i=1}^{r} D_{m_i} D_{m_{r+1-i}} = 0,$$

for $r = 1, \ldots, n$. If $2$ is invertible in $S$ then this is equivalent to

$$\sum_{i=1}^{r} [m_i, m_{r+1-i}] = 0,$$

$r = 1, \ldots, n$.

We are interested in minimal $A_n$-structures, i.e., $A_n$-structures with $m_1 = 0$. Then for $n \geq 3$ the product $m_2$ is automatically associative, and we’d like to fix it. So when we talk about minimal $S$-linear $A_n$-structures on a graded associative $S$-algebra $A$ we always assume that $m_2$ is the given product on $A$. Note that for a Hochschild cochain $c \in CH^{s+t}(A/S)_t$ we have

$$[D_{m_2}, D_c] = D_{m_2} D_c + (-1)^{s+t} D_c D_{m_2} = D_{\delta(c)},$$

where $\delta(c) = [m_2, c]$ is the Hochschild differential.
Any $A_{n+1}$-structure induces an $A_n$-structure by forgetting $m_{n+1}$. The following well-known result states that an obstacle to extending an $A_n$-structure to an $A_{n+1}$-structure lies in $HH^3(A/S)_{2-n}$ (it is stated without proof as [1, Lem. 2.3]).

**Lemma 4.1.2.** (i) Let $A$ be an associative algebra with generators $D_1, \ldots, D_n$ and defining relations

$$
\sum_{i=1}^{r} D_i D_{r+1-i} = 0,
$$

for $r = 1, \ldots, n$. Set $S = \sum_{i=2}^{n} D_i D_{n+2-i}$. Then

$$
D_1 S - S D_1 = 0.
$$

(ii) For a minimal $S$-linear $A_n$-structure $m = (m_2, \ldots, m_n)$ on $A/S$ there exists a Hochschild cocycle $\phi_n(m) \in CH^3(A/S)_{1-n}$ such that

$$
D_{\phi_n(m)} = \sum_{i=3}^{n} D_{m_i} D_{m_{n+3-i}}.
$$

The $A_n$-structure $m$ is extendable to an $A_{n+1}$-structure $(m_2, \ldots, m_n, m_{n+1})$ if and only $\phi_n(m)$ is a coboundary.

**Proof.** (i) Let us give $A$ the grading by $\deg D_i = 1$ and use the corresponding supercommutator $[?,?]$. Then we have

$$
[D_1, S] = \sum_{i=2}^{n} [D_1, D_i] D_{n+2-i} - \sum_{i=2}^{n} D_i [D_1, D_{n+2-i}].
$$

Applying the relations we can rewrite the sums in the right-hand side as

$$
\sum_{i=2}^{n} [D_1, D_i] D_{n+2-i} = - \sum_{i,j,k} D_i D_j D_{n+3-i-j},
$$

$$
\sum_{i=2}^{n} D_i [D_1, D_{n+2-i}] = \sum_{i,j,k} D_{n+3-i-j} D_i D_j.
$$

Thus, both sums are equal to

$$
\sum_{i,j,k} D_i D_j D_k,
$$

so they cancel out.

(ii) The existence of the Hochschild cochain $\phi_n(m)$ follows from the fact that the expression in the right-hand side of (4.1.2) is a coderivation. The fact that $\phi_n(m)$ is $\delta$-closed follows from (i). By Lemma 4.1.1, the condition on $m_{n+1}$ to extend $m = (m_2, \ldots, m_n)$ to an $A_{n+1}$-structure is

$$
[D_{m_2}, D_{m_{n+1}}] = - \sum_{i=3}^{n} D_{m_i} D_{m_{n+3-i}},
$$

i.e., $\delta(m_{n+1}) = -\phi_n(m)$, which implies the assertion. \qed
Definition 4.1.3. The group of gauge transformations $\mathfrak{G}$ is the group of degree-preserving coalgebra automorphisms $\alpha : \text{Bar}(A) \to \text{Bar}(A)$ such that the component $\text{Bar}(A) \to A[1]$ is given by a collection

$$(f_1 = \text{id}, f_2, \ldots) \in CH^1(A/S)_{-1} \times CH^1(A/S)_{-2} \times \ldots.$$ 

The group of extended gauge transformations is defined similarly by requiring $f_1$ just to be invertible. Note that any such automorphism automatically preserves any sub-coalgebra $\text{Bar}_{\leq n}(A)$ and the condition $f_1 = \text{id}$ is equivalent to the condition that $\alpha$ acts as identity on every quotient $\text{Bar}_{\leq i}(A)/\text{Bar}_{\leq i-1}(A)$.

We usually identify elements of $\mathfrak{G}$ with the corresponding collections $f = (f_1 = \text{id}, f_2, \ldots)$ and denote by $\alpha_f$ the corresponding automorphism of $\text{Bar}(A)$. Note that the group $\mathfrak{G}$ acts on the set of $A_n$-structures for every $n$: for $f \in \mathfrak{G}$ and an $A_n$-structure $m$, the new $A_n$-structure $f * m$ is determined by

$$D_{f * m} = \alpha_f D_m \alpha_f^{-1},$$

where in the right-hand side we restrict $\alpha_f$ to $\text{Bar}_{\leq n}(A)$.

Definition 4.1.4. For each $n$ let us denote by $\mathfrak{G}_{\geq n} \subset \mathfrak{G}$ the subgroup of $f = (f_1 = \text{id}, f_2, \ldots)$ with $f_i = 0$ for $2 \leq i < n$. To see that this is a subgroup we observe that this vanishing condition is equivalent to the condition that $\alpha_f$ acts as identity on all the quotients $\text{Bar}_{\leq i}(A)/\text{Bar}_{\leq i-n+1}(A)$. In particular, $\mathfrak{G}_{\geq 2} = \mathfrak{G}$.

Lemma 4.1.5. The subgroup $\mathfrak{G}_{\geq n+1}$ acts trivially on the set of $A_n$-structures. The subgroup $\mathfrak{G}_{\geq n}$ acts trivially on the set of minimal $A_n$-structures. For

$$f = (f_1 = \text{id}, 0, \ldots, f_{n-1}, \ldots) \in \mathfrak{G}_{\geq n-1}$$

and a minimal $A_n$-structure $m = (m_2, \ldots, m_n)$ one has $f * m = (m_2, \ldots, m_{n-1}, m'_n)$ with

$$m'_n = m_n \pm \delta(f_{n-1}).$$

The proof is straightforward (cf. [28, Lem. 2.2(ii)] for the last assertion).

When we are dealing with an associative algebra $A$ that has an underlying quiver $Q$, so that $A$ is a quotient of the path-algebra $k[Q]$ by a homogeneous ideal (such as $E_{\emptyset}$), we treat $A$ as a category with the objects corresponding to the vertices of the quiver. Furthermore, we always consider strictly unital $A_\infty$ (and $A_n$)-structures, and we always consider the reduced Hochschild cochains of $A$ in the above definitions. Note that this does not change the Hochschild cohomology groups (see [35, (20d)]).

We will also use the notion of a homotopy between gauge transformations (see [16] and [28, Sec. 2.1], where these are called homotopies between strict $A_\infty$-isomorphisms).

Lemma 4.1.6. Let $(E, m_2)$ be an associative flat $S$-algebra such that $HH^1(E/S)_{-i} = 0$ for $i = 1, \ldots, d - 2$, where $d \geq 2$. Suppose $m = (m_\bullet)$ and $m' = (m'_\bullet)$ is a pair of minimal $S$-linear $A_n$-structures on $E$, where $n \geq d$, such that $m_i = m'_i$ for $i \leq d$, and such that there exists a gauge transformation $f$ such that $f * m = m'$. Then there exists a gauge transformation $f'$, homotopic to $f$, such that $f' * m = m'$ and $f'_i = 0$ for $i \leq d - 1$.

Proof. We are going to construct by induction on $j = 1, \ldots, d - 1$ a gauge transformation $f'$, homotopic to $f$, such that $f' * m = m'$ and $f'_i = 0$ for $i \leq j$. The base case $j = 1$ is
clear. Assume that we have \( f' \) homotopic to \( f \) such that \( f' \ast m = m' \) and \( f'_i = 0 \) for \( i < j \). We have to show that \( f' \) can be improved to make \( f'_i = 0 \) for \( i \leq j \) (provided \( j \leq d - 1 \)). Indeed, by Lemma 4.1.5, we have
\[
0 = m'_{j+1} - m_{j+1} = \pm \delta(f'_j),
\]
so \( f'_j \) is a Hochschild cocycle giving a class in \( HH^1(E/S)_{1-j} \). Since this class is zero, there exists \( \phi'_{j-1} \) of degree \( 1 - j \) such that \( f'_j = [m_2, \phi'_{j-1}] \). By [28, Lem. 2.1], this implies the existence of a gauge transformation \( f'' \), homotopic to \( f' \), with \( f''_i = 0 \) for \( i < j \). \( \square \)

4.2. Moduli of \( A_\infty \)-structures. In this section we fix a finite-dimensional graded associative \( k \)-algebra \( E \), where \( k \) is a field. We are going to study the moduli functors of minimal \( A_n \)-structures on \( E \) extending the product \( m_2 \) on \( E \). We set \( CH^i(E)_j = CH^i(E/k)_j \) and denote by \( \delta^i_j : CH^i(E)_j \rightarrow CH^{i+1}(E)_j \) the Hochschild differential.

We will view all the spaces \( CH^i(E)_j \) as affine spaces over \( k \) and the group \( \mathcal{G} \) of gauge transformations as an affine algebraic group over \( k \), which is the projective limit of the affine algebraic groups of finite type over \( k \),
\[
\mathcal{G}[2,n] := \mathcal{G}/\mathcal{G}_{\geq n+1}.
\]
Note that each projection \( \mathcal{G}[2,n] \rightarrow \mathcal{G}[2,n-1] \) has a natural section (not compatible with the group structure). The kernel of this projection is isomorphic to the direct product of several copies of the additive group over \( k \). In particular, we see that the \( k \)-groups \( \mathcal{G} \) and \( \mathcal{G}[2,n] \) are smooth.

**Definition 4.2.1.** (i) The affine \( k \)-scheme \( A_n = A_{E,n} \) is defined as the closed subscheme of the affine space \( CH^2(A)_{-1} \times \ldots \times CH^2(A)_{2-n} \), given by the \( A_\infty \)-constraints \( (4.1.1) \) (where \( m_2 \) is the given product on \( E \)). For a \( k \)-algebra \( R \), the set of points \( A_n(R) \) is simply the set of minimal \( R \)-linear \( A_n \)-structures on \( E \otimes R \).

(ii) For \( n \geq 3 \) we have an action of the algebraic \( k \)-group \( \mathcal{G}[2,n-1] \) of gauge transformations \( (f_2, \ldots, f_{n-1}) \) on \( A_n \). We define \( \mathcal{M}_n = \mathcal{M}_{E,n} \) to be the functor associating with a commutative \( k \)-algebra \( R \) the set of orbits for the corresponding action on the set of \( R \)-points,
\[
\mathcal{M}_n(R) := A_n(R)/\mathcal{G}[2,n-1](R),
\]
which is the set of gauge equivalence classes of minimal \( R \)-linear \( A_n \)-structures on \( E \otimes R \).

For example, \( A_3 \) is the affine space \( ZH^2(E)_{-1} \) of Hochschild cocycles and the group \( \mathcal{G}[2,2] \) is just the product of additive groups corresponding to \( CH^1(E)_{-1} \) acting on \( A_3 \) via the Hochschild differential \( \delta^1_{-1} : CH^1(E)_{-1} \rightarrow ZH^2(E)_{-1} \). Thus, the functor \( \mathcal{M}_3 \) is represented by the affine space \( HH^2(E)_{-1} \). This observation will be generalized to the moduli functors \( \mathcal{M}_n \) in Theorem 4.2.4 below (under some additional assumptions on \( E \)).

Below we work in the category of schemes over \( k \).

**Definition 4.2.2.** Let \( G \) be a group scheme, \( X \) a scheme with \( G \)-action. We say that a closed subscheme \( S \subset X \) is a *nice section for the action of \( G \) on \( X \) if there exists a morphism \( (\rho, \pi) : X \rightarrow G \times S \), such that the morphism \( \pi : X \rightarrow S \) is \( G \)-invariant and \( \rho(x)\pi(x) = x \) (this is an equality of morphisms \( X \rightarrow X \)).
Note that for any scheme $T$ and any pair of points $x, x' \in X(T)$ one has $\pi(x) = \pi(x')$ if and only if there exists a point $g \in G(T)$ such that $x' = gx$. Thus, if $S$ is a nice section for $(G, X)$ then the functor $T \mapsto X(T)/G(T)$ is represented by $S$.

**Lemma 4.2.3.** Let $G$ be a group scheme acting on a scheme $X$. Assume that $G$ fits into an exact sequence of groups

$$1 \to H \to G \to G' \to 1$$

and that the projection $G \to G'$ admits a section $\sigma : G' \to G$ which is a morphism of schemes (not necessarily compatible with the group structures). Suppose we have a scheme $Y$ with an action of $G'$ and a morphism $f : X \to Y$ compatible with the $G$-action via the homomorphism $G \to G'$. Assume that $S_H \subset X$ is a nice section for the $H$-action on $X$ and $S' \subset Y$ is a nice section for the $G'$-action on $Y$. Finally assume that the following condition holds: for any scheme $T$ and any points $x \in X(T), g \in G(T)$ such that $f(gx) = f(x)$ there exists a point $h \in H(T)$ such that $gx = hx$. Then $S := S_H \cap f^{-1}(S')$ is a nice section for the $G$-action on $X$.

**Proof.** Let $(\rho_H, \pi_H) : X \to H \times S_H$ and $(\rho', \pi') : Y \to G' \times S'$ be the morphisms that exist by the definition of a nice section. Let us define a morphism $\rho_f : X \to G$ by

$$\rho_f(x) = \sigma(\rho'(f(x))).$$

Then we have

$$f(\rho_f(x)^{-1}x) = \rho'(f(x))^{-1}f(x) = \pi'(f(x)),$$

so we obtain a morphism

$$\pi_f : X \to f^{-1}(S') : x \mapsto \rho_f(x)^{-1}x,$$

satisfying $f \circ \pi_f = \pi' \circ f$. Since the morphism $f : X \to Y$ is $H$-invariant, the subscheme $f^{-1}(S')$ is $H$-invariant. Hence, $\pi_H$ sends $f^{-1}(S')$ to $S = S_H \cap f^{-1}(S')$. Thus, we can define a morphism

$$\pi = \pi_H \circ \pi_f : X \to S.$$

By definition, we have

$$x = \rho_f(x)\pi_f(x) = \rho_f(x)\rho_H(\pi_f(x)) \cdot \pi(x).$$

It remains to check that $\pi$ is $G$-invariant. Let $x \in X(T), g \in G(T)$. Then we have

$$f(\pi_f(gx)) = \pi'(f(gx)) = \pi'(f(x)) = f(\pi_f(x)).$$

Also, $\pi_f(gx)$ and $\pi_f(x)$ are in the same $G(T)$-orbit:

$$\pi_f(gx) = \rho_f(gx)^{-1}gx = \rho_f(gx)^{-1}g\rho_f(x)x.$$

Hence, by our assumption, there exists $h \in H(T)$ such that $pi_f(gx) = h\pi_f(x)$, so that

$$\pi(gx) = \pi_H(\rho_f(gx)) = \pi_H(h\pi_f(x)) = \pi_H(\pi_f(x)) = \pi(x).$$

\[\square\]

**Theorem 4.2.4.** Assume that $HH^1(E)_{-i} = 0$ for $1 \leq i \leq n - 2$. Then the action of $\Phi[2, n - 1]$ on $\mathcal{A}_n$ admits a nice section $S_n \subset \mathcal{A}_n$. Hence, the affine scheme $S_n$ represents the functor $\mathcal{M}_n$ of gauge equivalence classes of minimal $A_n$-structures on $E$. 

\[42\]
Proof. We use induction on \( n \). For \( n = 3 \) we define the subscheme
\[
S_3 \subset ZH^2(E)_{-1} = \mathcal{A}_3
\]
to be any subspace complementary to \( \delta_{1,1}(CH^1(E)_{-1}) \). This gives a nice section for the action of \( \mathcal{G}[2, 2] = CH^{1}(E)_{-1} \) on \( \mathcal{A}_3 \). Suppose we already have the subscheme \( S_{n-1} \subset \mathcal{A}_{n-1} \) which is a nice section for the action of \( \mathcal{G}[2, n-2] \) on \( \mathcal{A}_{n-1} \). Let us consider the natural projection \( \pi_n : \mathcal{A}_n \to \mathcal{A}_{n-1} \). Note that we have a regular morphism \( \phi_{n-1} : \mathcal{A}_{n-1} \to CH^3(E)_{2-n} \) given by (4.1.2), such that
\[
\mathcal{A}_n = \{(x, m_n) \in \mathcal{A}_{n-1} \times CH^2(E)_{2-n} \mid \delta_{2-n}^2(m_n) = -\phi_{n-1}(x)\}.
\]
Thus, the image of the projection \( \pi_n \) is the subscheme
\[
\mathcal{A}_{n-1,n} = \{x \in \mathcal{A}_n \mid \phi_{n-1}(x) \in \text{im}(\delta_{2-n}^2)\}.
\]
Let us choose a \( k \)-linear map \( Q^3 : CH^i(E)_{2-n} \to CH^{i-1}(E)_{2-n} \) such that \( \delta_{2-n}^2 \circ Q^3 \) acts as identity on the subspace \( \text{im}(\delta_{2-n}^2) \subset CH^2(E)_{2-n} \). Then the map
\[
s_n : \mathcal{A}_{n-1,n} \to \mathcal{A}_n : x \mapsto (x, Q^3(\phi_{n-1}(x)))
\]
is the section of the projection \( \pi_n : \mathcal{A}_n \to \mathcal{A}_{n-1,n} \). Now let us pick a \( k \)-linear subspace \( R_n \subset ZH^2(E)_{2-n} \) which projects isomorphically to the cohomology \( HH^2(E)_{2-n} \). Let us define the subscheme \( \mathcal{A}'_n \subset \mathcal{A}_n \) by
\[
\mathcal{A}'_n = \{y \in \mathcal{A}_n \mid m_{i}(s_n \pi_n(y)) - m_n(y) \in R_n\}
\]
where for \( y \in \mathcal{A}_n \) we denote by \( m_i(y) \) the corresponding \( i \)-th product. The projection \( \mathcal{A}_n \to \mathcal{A}_{n-1,n} \) is compatible with the homomorphism of groups \( \mathcal{G}[2, n-1] \to \mathcal{G}[2, n-2] \) that fits into the exact sequence of groups
\[
0 \to CH^1(E)_{2-n} \to \mathcal{G}[2, n-1] \to \mathcal{G}[2, n-2] \to 0. \tag{4.2.1}
\]
Furthermore, the section \( s_n \) gives an identification
\[
\mathcal{A}_n \cong ZH^2(E)_{2-n} \times \mathcal{A}_{n-1,n},
\]
compatible with the action of \( CH^1(E)_{2-n} \) (via the differential \( \delta_{2-n}^1 \)). Since
\[
ZH^2(E)_{2-n} = R_n \oplus \delta_{2-n}^1(CH^1(E)_{2-n}),
\]
we deduce that the subscheme \( \mathcal{A}'_n \subset \mathcal{A}_n \) is a nice section for the action of \( CH^1(E)_{2-n} \) on \( \mathcal{A}_n \). Now we would like to apply Lemma 4.2.3 to the group \( G = \mathcal{G}[2, n-1] \) acting on \( X = \mathcal{A}_n \), to the exact sequence (4.2.1) and to the projection \( X \to Y = \mathcal{A}_{n-1} \). Note that the assumptions of this Lemma are satisfied by Lemma 4.1.6. Thus, we conclude that
\[
S_n := \mathcal{A}'_n \cap \pi_n^{-1}(S_{n-1})
\]
is a nice section for the action of \( \mathcal{G}[2, n-1] \) on \( \mathcal{A}_n \). \( \square \)

Corollary 4.2.5. Assume that \( HH^1(E)_{<0} = 0 \). Then the functor \( \mathcal{M}_\infty \) associating with a \( k \)-algebra \( R \) the set of gauge equivalence classes of minimal \( \mathcal{A}_\infty \)-structures on \( E \otimes R \) is represented by the affine scheme \( S_\infty \), the inverse limit of the affine schemes \( S_n \) representing \( \mathcal{M}_n \).
Proof. We have to show that $\mathcal{M}_\infty(R)$ is the inverse limit of $\mathcal{M}_n(R) = \mathcal{A}_n(R)/\mathfrak{S}[2,n-1](R)$. Since the set of minimal $A_\infty$-structures on $E \otimes R$ is $\varprojlim_n \mathcal{A}_n(R)$, and $\mathfrak{S}(R) = \varprojlim_n \mathfrak{S}[2,n](R)$, this follows from surjectivity of the projections $\mathfrak{S}[2,n](R) \to \mathfrak{S}[2,n-1](R)$. □

An important particular case is when we have some additional vanishing for components of $HH^3(E)$ and $HH^3(E)$.

Corollary 4.2.6. Assume that $HH^1(E)_{<0} = HH^2(E)_{<2-n} = HH^3(E)_{<2-n} = 0$. Then the natural morphism of functors $\mathcal{M}_\infty \to \mathcal{M}_n$ is an isomorphism. Hence, in this case $\mathcal{M}_\infty$ is represented by an affine scheme of finite type over $k$.

Proof. It is enough to show that under these assumptions the morphism of functors $\mathcal{M}_{n+1} \to \mathcal{M}_n$ is an isomorphism. Note that the assumption on vanishing of certain Hochschild cohomology groups of $E$ implies vanishing of the same components of $HH^*(E \otimes R/R) = H^*(E) \otimes R$. By Lemma 4.1.2, the vanishing of $HH^3(E \otimes R)_{1-n}$ implies that every minimal $R$-linear $A_n$-structure on $E \otimes R$ extends to an $A_{n+1}$-structure. Now suppose $m$ and $m'$ are two $A_{n+1}$-structures such that the corresponding $A_n$-structures $m_{\leq n}$ and $m'_{\leq n}$ are gauge equivalent. Replacing $m'$ by a gauge equivalent structure we can assume that $m'_{\leq n} = m_{\leq n}$. Then $m'_{n+1} - m_{n+1}$ is a Hochschild cocycle. Now the vanishing of $HH^2(E \otimes R)_{1-n}$ implies that this cocycle is a coboundary. Hence, by Lemma 4.1.5, $m'$ is gauge equivalent to $m$. □

4.3. The map from the moduli space of non-special curves to the moduli space of $A_\infty$-structures. Let us denote by $\mathcal{M}_\infty = \mathcal{M}_\infty \mathbb{Z}$ the functor of minimal $A_\infty$-structures on $E_{g,\mathbb{Z}}$ (the natural $\mathbb{Z}$-form of the algebra $E_g$) up to a gauge equivalence. We are going to relate it to the moduli of non-special curves.

Let $S$ be a commutative algebra. Given a family of curves in $\widetilde{\mathcal{U}}_{g,a}^{ns}(S)$, the construction of Section 3 gives a minimal $S$-linear $A_\infty$-structure on $E_{g,\mathbb{Z}} \otimes S$, defined uniquely up to gauge equivalence (recall that the construction depends on a choice of formal parameters at marked points). This gives a morphism of functors on commutative algebras

$$a_\infty : \widetilde{\mathcal{U}}_{g,a}^{ns,a} \to \mathcal{M}_\infty.$$  \hspace{1cm} (4.3.1)

Later we will show that the restriction of the functor $\mathcal{M}_\infty$ to commutative algebras over a field $k$ is actually represented by a $k$-scheme, so from (4.3.1) we will get a morphism of $k$-schemes.

Recall that $\mathbb{G}_m^g$ acts on $\widetilde{\mathcal{U}}_{g,a}^{ns}$ by rescaling the tangent vectors at the marked point by $v_i \mapsto \lambda_i^{-1}v_i$. The morphism (4.3.1) is compatible with the $\mathbb{G}_m^g$-action, where the action on $\mathcal{M}_\infty$ is induced by the $\mathbb{G}_m^g$-action on $E_g$ given on generators by

$$\lambda : A_i \mapsto A_i, \quad B_i \mapsto \lambda_i B_i.$$ \hspace{1cm} (4.3.2)

We are going to show how to recover a curve (and marked points) from the corresponding $A_\infty$-structure. Namely, for a non-special curve $(C,p_1,\ldots, p_g)$ over Spec$(S)$, let $\text{Perf}(C,p_1,\ldots, p_g) \subset D^b(C)$ denote the thick triangulated subcategory generated by
the objects \( \mathcal{O}_C, \mathcal{O}_{p_1}, \ldots, \mathcal{O}_{p_g} \), or equivalently, by their direct sum, \( G \). Then the functor \( R \text{Hom}(G, ?) \) induces an equivalence

\[
\text{Perf}(C, p_1, \ldots, p_g) \to \text{Perf}(E_{g, \infty}), \tag{4.3.3}
\]

where \( E_{g, \infty} \) is the (minimal) \( A_\infty \)-algebra of endomorphisms of \( G \) in an \( A_\infty \)-enhancement of \( D^b(C) \) (so \( E_{g, \infty} = E_{g, \mathbb{Z}} \otimes S \) as an associative \( S \)-algebra) and \( \text{Perf}(E_{g, \infty}) \) is the derived category of perfect right modules over \( E_{g, \infty} \).

**Proposition 4.3.1.** The map \( a_\infty \) is injective.

**Proof.** Suppose we have a gauge equivalence between the \( A_\infty \)-structures on \( E_{g, \mathbb{Z}} \otimes R \) associated with the curves \((C, p_1, \ldots, p_g)\) and \((C', p'_1, \ldots, p'_g)\) over \( \text{Spec}(R) \). Let us denote the corresponding \( A_\infty \)-algebras \( E_{g, \infty} \) and \( E'_{g, \infty} \). Note that with each idempotent in \( E_{g, \mathbb{Z}} \) we have the associated \( A_\infty \)-module over \( E_{g, \infty} \) (resp., \( E'_{g, \infty} \)) that is the image of one of the objects \( \mathcal{O}_C, \mathcal{O}_{p_1}, \ldots, \mathcal{O}_{p_g} \) under the equivalence \( (4.3.3) \) (resp., \( \mathcal{O}_{C'}, \mathcal{O}_{p'_1}, \ldots, \mathcal{O}_{p'_g} \)). The gauge equivalence gives a quasi-isomorphism

\[
E_{g, \infty} \sim \leftarrow E'_{g, \infty}
\]

of \( A_\infty \)-algebras over \( S \), which induces an equivalence between the derived categories of perfect right modules

\[
\text{Perf}(E'_{g, \infty}) \sim \text{Perf}(E_{g, \infty}).
\]

Using \( (4.3.3) \) we get an equivalence

\[
\Phi : \text{Perf}(C', p'_1, \ldots, p'_g) \sim \text{Perf}(C, p_1, \ldots, p_g)
\]

sending \( \mathcal{O}_{p'_i} \) to \( \mathcal{O}_{p_i} \), \( \mathcal{O}_{C'} \) to \( \mathcal{O}_C \), and inducing the identity maps on

\[
\text{Hom}(\mathcal{O}_{C'}, \mathcal{O}_{p'_i}) = \text{Hom}(\mathcal{O}_C, \mathcal{O}_{p_i}) = S
\]

and

\[
\text{Ext}^1(\mathcal{O}_{p'_i}, \mathcal{O}_{C'}) = \text{Ext}^1(\mathcal{O}_{p_i}, \mathcal{O}_C) = S
\]

(the latter identifications are given by the tangent vectors at the marked points). Let \( D = p_1 + \ldots + p_g \) (resp., \( D' = p'_1 + \ldots + p'_g \)). Note that the line bundles \( \mathcal{O}(nD) \) belong to \( \text{Perf}(C, p_1, \ldots, p_g) \). More precisely, we have exact sequences

\[
0 \to \mathcal{O}_C(nD) \to \mathcal{O}_C((n + 1)D) \to \bigoplus_{i=1}^g \mathcal{O}_{p_i} \to 0
\]

and similarly for \( C' \), which give isomorphisms \( \Phi(\mathcal{O}_{C'}(nD')) \simeq \mathcal{O}_C(nD) \) compatible with these exact sequences. Hence, \( \Phi \) gives an isomorphism of graded algebras

\[
\bigoplus_n H^0(C', \mathcal{O}(nD')) \sim \bigoplus_n H^0(C, \mathcal{O}(nD)),
\]

inducing the identity on

\[
H^0(C', \mathcal{O}(nD'))/H^0(C', \mathcal{O}((n - 1)D')) = S^n = H^0(C, \mathcal{O}(nD))/H^0(C, \mathcal{O}((n - 1)D))
\]

for \( n \geq 2 \). Since the divisor \( D \) on \( C \) (resp., \( D' \) on \( C' \)) is ample, passing to Proj we get an isomorphism \( C \sim C' \), sending \( D \) to \( D' \) and compatible with the given trivializations of \( \mathcal{O}_C(D)/\mathcal{O}_C \) (resp., \( \mathcal{O}_{C'}(D')/\mathcal{O}_{C'} \)). \( \square \)
Remarks 4.3.2. 1. When the curve $C$ is smooth (where $S = k$, a field, and $g \geq 2$), it can be recovered just from the derived Morita equivalence class of the $A_\infty$-structure on $E_g$ associated with $(C, p_1, \ldots, p_g)$ for any non-special divisor $p_1 + \ldots + p_g$. Namely, this follows from the fact that $C$ can be recovered from the equivalence class of its derived category $D^b(C)$ (see [5]). There is an extension of this result to singular $C$ provided $C$ is Gorenstein and $\omega_C$ is ample (see [3]). But of course, the gauge equivalence class of an $A_\infty$-structure used in Proposition 4.3.1 above gives more information than the derived Morita equivalence class.

2. The map (4.3.1) extends naturally to the stack of curves with a non-special divisor $p_1 + \ldots + p_g$ which is not necessarily ample. The extended map is not injective: it does not distinguish some curves that become isomorphic after contracting all irreducible components not containing any marked points. For example, suppose $f : C \to \overline{C}$ a contraction of a tree $T$ of $\mathbb{P}^1$'s that is attached transversally to one of the components of $C$. Then $Rf_*\mathcal{O}_C = \mathcal{O}_{\overline{C}}$. Hence, the pull-back functor $f^* : \text{Perf}(\overline{C}) \to \text{Perf}(C)$ is fully faithful. Since this functor lifts to a dg-level, it induces an equivalence between the $A_\infty$-algebra of endomorphisms of $G$ and of $f^*G$ for any $E \in \text{Perf}(\overline{C})$. This implies that if $T$ does not contain any marked points then the $A_\infty$-structure on $E_g$ associated with $(C, p_1, \ldots, p_n)$ is gauge equivalent to the one associated with $(\overline{C}, \overline{p}_1, \ldots, \overline{p}_n)$ (where $\overline{p}_i = f(p_i)$).

4.4. Cuspidal curve of genus $g$. Starting from this section we work over a field $k$.

Let us define the cuspidal curve of genus $g$, $C_g^{\text{cusp}}$ as the union of $g$ usual (projective) cuspidal curves $C_i$ of arithmetic genus 1, glued along their singular points $q_i \in C_i$ in such a way that the local ring of $C_g^{\text{cusp}}$ at the singular point is the subring of $\prod_{i=1}^{g} \mathcal{O}_{C_i, q_i}$ consisting of $(f_i)$ such that $f_i(q_i) = f_j(q_j)$. We will use the following presentation of the cuspidal curve as $\text{Proj}: C_g^{\text{cusp}} = \text{Proj}(\mathcal{R}A_g^{\text{cusp}})$, where $A_g^{\text{cusp}}$ is the commutative $k$-algebra with generators $f_1, \ldots, f_g$, $h_1, \ldots, h_g$ and defining relations

$$h_i^2 = f_i^3, \quad f_if_j = f_jh_j = h_ih_j = 0, \quad i \neq j,$$

and $\mathcal{R}A_g^{\text{cusp}}$ is the Rees algebra associated with the filtration $F_\bullet$, where $F_n$ is spanned by $f_i^m$ with $2m \leq n$ and $h_i f_i^m$ with $3 + 2m \leq n$. In other words, this is the curve that corresponds to the algebra $A_g^{\text{cusp}}$, viewed as a marked algebra of genus $g$ (see Section 1).

In this presentation $C_g^{\text{cusp}}$ comes equipped with $g$ smooth points $p_1, \ldots, p_g$ such that $D = p_1 + \ldots + p_g$ is the zero locus of the section $1 \in F_1$ of $\mathcal{O}(1)$ (in particular, $D$ is ample). Furthermore, we can view $f_i$ and $h_i$ as functions on the complement $C_g^{\text{cusp}} \setminus D$ and $f_i/h_i$ induces a trivialization of the tangent space at $p_i$.

Let us observe also that $C_g^{\text{cusp}}$ is equipped with the natural action of $\mathbb{G}_m^g$ such that for $f_i$ and $h_i$, viewed as functions on $C_g^{\text{cusp}} \setminus D$, we have

$$\lambda^* f_i = \lambda_i^{-2} f_i, \quad \lambda^* h_i = \lambda_i^{-3} h_i$$

for $\lambda = (\lambda_1, \ldots, \lambda_g) \in \mathbb{G}_m^g$ (this action is the opposite of the one induced by the action on the universal curve considered in Section 3.3). The points $p_1, \ldots, p_g \in C_g^{\text{cusp}}$ are stable under this action and $\lambda$ acts on the tangent space at $p_i$ as rescaling by $\lambda_i$. We will also consider below the induced $\mathbb{G}_m$-action coming from the diagonal $\mathbb{G}_m \subset \mathbb{G}_m^g$. 

46
Proposition 4.4.1. The functor

$$\text{Perf}(C_g^{\text{cusp}}) \to \text{Perf}(E_g)$$

associated with the generator $O \oplus O_{p_1} \oplus \ldots \oplus O_{p_g}$. is an equivalence of categories. Hence, it induces a $\mathbb{G}_m$-equivariant isomorphism $HH^*(C_g^{\text{cusp}}) \simeq HH^*(E_g)$. The second grading on $HH^*(E_g)$ corresponds to the weights of the $\mathbb{G}_m$-action on it.

Proof. We claim that the gauge equivalence class of minimal $\mathcal{A}_\infty$-structures on $E_g$ coming from $C_g^{\text{cusp}}$ contains the structure with $m_i = 0$ for $i > 2$. Indeed, we can choose $\mathbb{G}_m$-equivariant formal parameters at $p_i$'s similarly to the construction of Section 3. Then the vanishing of the corresponding products $m_i$ for $i > 2$ follows as in the proof of Proposition 3.3.2. Namely, the $\mathbb{G}_m$-weights on $E_g$ coincide with the cohomological grading (recall that the $\mathbb{G}_m$-action on $C_g^{\text{cusp}}$ is the opposite of the one used in Proposition 3.3.2). The operation $m_i$ lowers this grading by $i - 2$. On the other hand, by the construction, all higher products have weight 0. This implies that $m_i = 0$ for $i > 2$. The identification of the $\mathbb{G}_m$-weights on $E_g$ with the cohomological grading also implies that the second grading on $HH^*(E_g)$ is given by the $\mathbb{G}_m$-weights. \qed

We also can look at the induced $\mathbb{G}_m$-action on other invariants of $C_g^{\text{cusp}}$.

Lemma 4.4.2. Assume that the char($k$) $\neq 2$ or 3. Let us write for brevity $C = C_g^{\text{cusp}}$.

(i) The $g$-dimensional space $H^1(C, \mathcal{O})$ has weight 1 with respect to the $\mathbb{G}_m$-action.

(ii) The space $H^0(C, \mathcal{T})$, where $\mathcal{T}$ is the tangent sheaf, decomposes as a direct sum

$$H^0(C, \mathcal{T}) = H^0(C, \mathcal{T}(-D)) \oplus V,$$

where $V$ is a $g$-dimensional subspace of weight 1 with respect to the $\mathbb{G}_m$-action such that the composition

$$V \to H^0(C, \mathcal{T}) \to H^0(D, \mathcal{T}|_D)$$

is an isomorphism. Furthermore,

$$H^0(C, \mathcal{T}(-D)) = H^0(C, \mathcal{T})^{\mathbb{G}_m}$$

and this space is spanned by $g$ linearly independent derivations coming from the $\mathbb{G}_m$-action on $C_g^{\text{cusp}}$.

(iii) One has $H^0(C, \mathcal{T}(-2D)) = 0$.

(iv) One has $H^1(C, \mathcal{T}) = 0$.

(v) The natural map $H^0(C, \mathcal{T}(nD)) \to H^0(C, \mathcal{T}(nD)|_D)$ is surjective for $n \geq -1$ and is an isomorphism for $n = -1$.

Proof. (i) Set $U = C \setminus D$, $V = C \setminus q$, where $q$ is the singular point of $C$. Then $(U, V)$ is an affine covering of $C$. Furthermore, $V = \sqcup_{i=1}^g V_i$, where $V_i \simeq \mathbb{A}^1$ and $U \cap V_i = V_i \setminus p_i$. Let $(C_i)$ be irreducible components of $C$, where $C_i = \{q\} \cup V_i$. We can view $C_i$ as $\mathbb{P}^1$ with the point 0 $\in \mathbb{A}^1 \subset \mathbb{P}^1$ pinched: if $x_i$ is the coordinate on $\mathbb{A}^1$ then the algebra of functions on $C_i \setminus p_i$ is the subring $k[x_i^2, x_i^3] \subset k[x_i]$. Hence, we can consider $x_i^{-1}$ as a coordinate on $V_i = C_i \setminus q$. Thus, functions on $U \cap V$ are collections of Laurent polynomials $(P_i(x_i, x_i^{-1}))_{i=1,...,g}$. Such a function extends to $V$ if and only if $P_i \in k[x_i^{-1}]$. On the other
hand, for $P_i \in x_i k[x_i]$ the collection $(P_i)$ extends to $U$ if and only if $P_i$ has no linear term in $x_i$. Thus, $H^1(C, \mathcal{O})$, that can be identified with the cokernel of the map 

$$H^0(U, \mathcal{O}) \oplus H^0(V, \mathcal{O}) \to H^0(U \cap V, \mathcal{O}),$$

is represented by the functions $(a_i x_i)$ for $a_i \in k$. It remains to check that $(\lambda^{-1})_i x_i = \lambda x_i$ for $\lambda \in \mathbb{G}_m$. Indeed, we have $x_i = t_i^{-1}$ where $t_i$ is the canonical parameter near $p_i$, so the assertion follows from the fact that $\lambda t_i = \lambda x_i$.

(ii) Since $C$ is smooth near $D$, the restriction map $H^0(C, \mathcal{T}) \to H^0(U, \mathcal{T})$ is injective, and its image consists of vector fields that extend regularly to $D$. We have further embedding $H^0(U, \mathcal{T}) \to \prod_{i=1}^g H^0(U_i, \mathcal{T})$, where $U_i = C_i \setminus p_i$. Thus, using the coordinate $x_i$ on the normalization of $U_i$ as before, each derivation $v$ on $C$ gives rise to derivations $v_i$ of $k[x_i^2, x_i^3]$. It is easy to see that any such a derivation has to satisfy $3x_i v_i(x_i^2) = 2v_i(x_i^3)$.

Hence, $v_i(x_i^3) \subset x_i^2 k[x_i]$, and so, $v_i(x_i^2) \in x_i^2 k[x_i]$. Thus, we can extend $v_i$ uniquely to a derivation of $k[x_i]$ vanishing at 0. Since $x_i^2 \frac{\partial}{\partial x_i}$ has a pole at infinity of order $n - 2$ for $n > 2$, we deduce that each $v_i$ has form $(a_i x_i + b_i x_i^2) \frac{\partial}{\partial x_i}$. Conversely, every such collection $(v_i)$ defines a derivation of $\mathcal{O}(U)$, regular at infinity. Now we note that $v \in H^0(C, \mathcal{T}(-D))$ if and only if $b_i = 0$ and let $V$ be a subspace consisting of $(v_i = b_i x_i^2 \frac{\partial}{\partial x_i})$.

(iii) This follows immediately from (ii), since the derivations spanning $H^0(C, \mathcal{T}(-D))$ project to a basis in $H^0(C, \mathcal{T}(-D))|_{D}$.

(iv) The proof is similar to that of [20, Lem. 4.16]. Consider the affine covering $(U, V)$ of $C$ defined above. Then $H^0(U \cap V, \mathcal{T})$ consists of vector fields $(v_i)$ of the form $v_i = P_i(x_i, x_i^{-1}) \frac{\partial}{\partial x_i}$. Such a vector field extends to a derivation over $V$ provided $P_i$ is a linear combination of $x_i^n$ with $n \leq 2$. On the other hand, if all $P_i \in x_i k[x_i]$ then $(v_i)$ extends to a derivation over $U$.

(v) As we have seen in the proof of (ii), sections of $\mathcal{T}$ on $U$ with poles of order at most $n$ at $p_1, \ldots, p_g$ correspond to derivations $v_i$ of the form $v_i = x_i P_i \frac{\partial}{\partial x_i}$, where $P_i$ is a polynomial of degree of $n + 1$, and the assertion follows from the fact such sections generate $\mathcal{T}(n p_1)|_{p_1} \oplus \ldots \oplus \mathcal{T}(n p_g)|_{p_g}$ for $n \geq -1$.

\[ \square \]

**Lemma 4.4.3.** (i) Let $C$ be a quasiprojective curve, $q \in C$ a point such that $C \setminus q$ is smooth, $U \subset C$ an open affine neighborhood of $q$. Then one has natural exact sequences

$$0 \to H^1(C, \mathcal{O}) \to H H^1(C) \to H^0(C, \mathcal{T}) \to 0,$$

$$0 \to H^1(C, \mathcal{T}) \to H H^2(C) \to H H^2(U) \to 0.$$

(ii) Assume the characteristic of $k$ is not 2 or 3. Let $C = C_{g, \text{cusp}}$, $U = C \setminus D$. Then the natural map

$$H H^2(C) \to H H^2(U)$$

is an isomorphism. Also, $H H^1(C)|_{<0} = 0$.

**Proof.** (i) This is proved in [20, Sec. 4.1.3].

(ii) By Lemma 4.4.2(iv), we have $H^1(C, \mathcal{T}) = 0$, so the first assertion follows from the second exact sequence in (i). On the other hand, the first exact sequence in (i) together with Lemma 4.4.2(i)(ii) imply that the only weights of $\mathbb{G}_m$ that occur on $H H^1(C)$ are 0 and 1.

\[ \square \]
**Corollary 4.4.4.** Assume that \( \text{char}(k) \neq 2 \) or \(3\). Then one has \( HH^1(E_g)_{<0} = 0 \).

*Proof.* Combine the vanishing of \( HH^1(C_g^{\text{cusp}})_{<0} \) with Proposition 4.4.1. \( \square \)

For a \( k \)-scheme \( X \) we denote by \( L_X \) the cotangent complex of \( X \) over \( k \) (see [14]).

**Lemma 4.4.5.** Assume that \( \text{char}(k) \neq 2 \) or \(3\). Let \( C = C_g^{\text{cusp}} \).

(i) The natural maps
\[
\text{Ext}^i(L_C, \mathcal{O}_C(-2D)) \to \text{Ext}^i(L_C, \mathcal{O}_C(-D)) \to \text{Ext}^i(L_C, \mathcal{O}_C)
\]
are isomorphisms for \( i = 1, 2 \), while the natural maps
\[
\text{Hom}(L_C, \mathcal{O}_C(-D)) \to \text{Hom}(L_C, \mathcal{O}_D(-D)) \quad \text{and} \quad \text{Hom}(L_C, \mathcal{O}_C) \to \text{Hom}(L_C, \mathcal{O}_D)
\]
are surjective. Also, one has
\[
\dim \text{Hom}(L_C, \mathcal{O}_C) = 2g, \quad \dim \text{Hom}(L_C, \mathcal{O}_C(-D)) = g,
\]
\[
\text{Hom}(L_C, \mathcal{O}_C(-2D)) = 0.
\]

(ii) Let \( U = C \setminus D \). Then the natural map
\[
\text{Ext}^i(L_C, \mathcal{O}_C(-2D)) \to \text{Ext}^i(L_U, \mathcal{O}_U)
\]
is an isomorphism for \( i = 1, 2 \).

*Proof.* (i) First, we observe that since \( L \) is a locally free sheaf on the smooth part of \( C \), it follows that
\[
\text{Ext}^0(L_C, \mathcal{O}_D) = 0.
\]

Thus, applying the functor \( \text{Ext}^\bullet(L_C, ?) \) to the exact sequences
\[
0 \to \mathcal{O}_C(-2D) \to \mathcal{O}_C(-D) \to \mathcal{O}_D \to 0,
0 \to \mathcal{O}_C(-D) \to \mathcal{O}_C \to \mathcal{O}_D \to 0,
\]
we get the required isomorphisms for \( i = 2 \). Since \( L_C \in D^{\leq 0}(C) \) and \( H^0(L_C) = \Omega_C \), the maps (4.4.1) and (4.4.2) can be identified with the maps
\[
H^0(C, \mathcal{T}(-D)) \to H^0(D, \mathcal{T}(-D)|_D) \quad \text{and} \quad H^0(C, \mathcal{T}) \to H^0(D, \mathcal{T}|_D)
\]
which are surjective by Lemma 4.4.2(v) (and the first is an isomorphism). This implies the required isomorphisms for \( i = 1 \). The computation of dimensions also follows from Lemma 4.4.2.

(ii) Recall that the natural map \( H^0(C, \mathcal{T}(nD)) \to H^0(C, \mathcal{T}(nD)|_D) \) is surjective for \( n \geq -1 \) by Lemma 4.4.2(v). Using the exact sequences
\[
0 \to \mathcal{O}_C(nD) \to \mathcal{O}_C((n+1)D) \to \mathcal{O}_D \to 0
\]
we deduce that the morphisms \( \text{Ext}^i(L_C, \mathcal{O}_C(nD)) \to \text{Ext}^i(L_C, \mathcal{O}_C((n+1)D)) \) are isomorphisms for \( n \geq -2 \). Finally, since \( L_C \) is a perfect complex, we have
\[
\text{Ext}^i_L(L_U, \mathcal{O}_U) \simeq \text{Ext}^i_C(L_C, \mathcal{O}_U) \simeq \lim_{\to} \text{Ext}^i(L_C, \mathcal{O}_C(nD))
\]
and our assertion follows. \( \square \)
We need the following technical lemma only for the cuspidal curve \( C_{\text{cusp}} \) over \( k \). However, we formulate it in a slightly bigger generality, since this could be of independent interest. Let \( L_{X/R} \) denote the cotangent complex of a scheme \( X \) over \( \text{Spec}(R) \).

**Lemma 4.4.6.** Let \( R \) be a Noetherian ring, \( C \) a proper flat \( R \)-scheme of relative dimension 1 such that the natural map \( R \to H^0(C, \mathcal{O}) \) is an isomorphism, \( D \subset C \) a relative effective divisor such that the projection \( \pi : C \to \text{Spec}(R) \) is smooth along \( D \), and such that the scheme \( U = C \setminus D \) is affine. Assume also that the fibers of \( \pi : C \to \text{Spec}(R) \) are generically smooth. Then the natural map

\[
\text{Ext}^i(L_{U/R}, \mathcal{O}_U) \to HH^{i+1}(U/R)
\]

is an isomorphism for \( i = 1 \), and is an injection for \( i = 2 \).

**Proof.** By [31, Thm. 8.1], there is a spectral sequence

\[
E_2^{pq} = \text{Ext}^p(\Lambda^q L_{U/R}, \mathcal{O}_U) \implies HH^{p+q}(U/R)
\]

where \( \Lambda^*(?) \) denotes the exterior power functor on perfect complexes. Since \( L_{U/R} \in D^{\leq 0} \), it follows that \( \Lambda^q L_{U/R} \in D^{\leq 0} \), so \( E_2^{pq} \neq 0 \) only for \( p \geq 0 \) (and \( q \geq 0 \)). Since \( U \) is affine, we also have \( E_2^{pq} = 0 \) for \( p > 0 \). We claim also that \( E_2^{q0} = 0 \) for \( q > 1 \). Indeed, we have

\[
\text{Hom}(\Lambda^q L_{U/R}, \mathcal{O}_U) = \text{Hom}(H^0(\Lambda^q L_{U/R}), \mathcal{O}_U).
\]

Let \( Z \) be the support of the coherent sheaf \( H^0(\Lambda^q L_{C/R}) \) on \( C \). Then \( Z \) is contained in the singular locus of the morphism \( \pi \) (since \( q > 1 \)), so it is finite over \( \text{Spec}(R) \) and is contained in \( U \). Therefore, we have

\[
\text{Hom}(H^0(\Lambda^q L_{U/R}), \mathcal{O}_U) \simeq \text{Hom}(H^0(\Lambda^q L_{C/R}), \mathcal{O}_C).
\]

The image of a nonzero morphism \( H^0(\Lambda^q L_{C/R}) \to \mathcal{O}_C \) would be a nonzero subsheaf \( \mathcal{F} \to \mathcal{O} \) supported on \( Z \). But then we would have an injection \( H^0(C, \mathcal{F}) \to H^0(C, \mathcal{O}) \). Since all global functions on \( C \) are constant on the fibers of \( \pi \), it follows that \( H^0(C, \mathcal{F}) = 0 \). Since \( Z \) is affine, we get that \( \mathcal{F} = 0 \) and our claim follows.

Thus, the spectral sequence implies that the map (4.4.3) is an isomorphism for \( i = 1 \), while for \( i = 2 \) it fits into an exact sequence

\[
0 \to \text{Ext}^2(L_{U/R}, \mathcal{O}_U) \to HH^3(U/R) \to \text{Ext}^1(\Lambda^2 L_{U/R}, \mathcal{O}_U) \to 0.
\]

\( \square \)

**Remark 4.4.7.** Using the main result of [6], one can similarly deduce that in the situation of Lemma 4.4.6, assuming in addition that we work in characteristic zero, one has

- \( HH^3(C/R) \simeq \text{Ext}^2(L_{C/R}, \mathcal{O}) \oplus \text{Ext}^1(\Lambda^2 L_{C/R}, \mathcal{O}) \)
- \( HH^2(C/R) \simeq \text{Ext}^1(L_{C/R}, \mathcal{O}) \)
- \( HH^1(C/R) \simeq H^1(C, \mathcal{O}) \oplus \text{Hom}(L_{C/R}, \mathcal{O}). \)
4.5. Deformation theory. We refer to [24] for the basic notions of deformation theory used below.

Let us define two deformation functors

\[ F_{g.g}, F_\infty : \text{Art}_k \to \text{Sets} \]

from the category of local Artinian \( k \)-algebras with the residue field \( k \) to the category of sets. The functor \( F_{g.g} \) describes deformations of the cuspidal curve \( C_g^{\text{cusp}} \) viewed as an object of \( \tilde{U}_{g.g}^{ns,a}(k) \). More precisely, \( F_{g.g}(R) \) consists of isomorphism classes of flat proper families \( \pi : C \to \text{Spec}(R) \) equipped with sections \( p_1, \ldots, p_g \) and trivializations \( v_i \) of the relative tangent bundle along each \( p_i \), such that the base change of \( (C, p_1, \ldots, p_g, v_1, \ldots, v_g) \) with respect to the reduction \( R \to k \) gives the cuspidal curve \( (C_g^{\text{cusp}}, p_1, \ldots, p_g) \) with the standard trivializations of the relative tangent bundle at \( p_i \). Note that for such a family we automatically get that \( \pi \) is smooth near the sections \( p_i \) and the divisor \( p_1 + \ldots + p_g \) is relatively ample (by \([13, (9.6.4)]\)). Together with semicontinuity this implies that such a family defines an element of \( \tilde{U}_{g.g}^{ns,a}(R) \).

The functor \( F_\infty \) is defined as follows. We consider minimal \( R \)-linear \( A_\infty \)-structures on \( E_g \otimes R \), such that upon the specialization \( R \to k \) we get the trivial \( A_\infty \)-structure (i.e., the one with \( m_i = 0 \) for \( i > 2 \)) on \( E_g \). We consider gauge transformations between such structures that reduce to the identity under the specialization \( R \to k \). We define \( F_\infty(R) \) as the corresponding set of equivalence classes.

As in Section 4.2, let \( \mathcal{M}_\infty \) denote the functor of minimal \( A_\infty \)-structures on \( E_g \) up to gauge equivalence, defined on all commutative \( k \)-algebras.

**Lemma 4.5.1.** (i) We have a natural identification of \( F_\infty(R) \) with the fiber of \( \mathcal{M}_\infty(R) \to \mathcal{M}_\infty(k) \) over the equivalence class of the trivial \( A_\infty \)-structure on \( E_g \).

(ii) The functor \( \mathcal{M}_\infty \) is representable by an affine \( k \)-scheme.

(iii) The functor \( F_\infty \) is homogeneous.

**Proof.** (i) First, we have to check that if a minimal \( R \)-linear \( A_\infty \)-structure \( m \) on \( E_g \otimes R \) reduces to an \( A_\infty \)-structure on \( E_g \) that is gauge equivalent to the trivial one, then there exists a gauge transformation \( \tilde{f} \) over \( R \) such that \( f \cdot m \) reduces to the trivial \( A_\infty \)-structure under \( R \to k \). This immediately follows from the fact that we can lift any gauge transformation defined over \( k \) to a gauge transformation defined over \( R \).

It remains to show that if we have minimal \( R \)-linear \( A_\infty \)-structures \( m \) and \( m' \) on \( E_g \otimes R \), reducing to the trivial one on \( E_g \), and a gauge equivalence \( f \) such that \( f \cdot m = m' \) then there exists a gauge equivalence \( f' \) reducing to the identity on \( E_g \) and such that we still have \( f' \cdot m = m' \). Let \( \overline{f}, \overline{m}, \text{etc.} \), denote the reduction with respect to \( R \to k \). Thus, \( \overline{m} = \overline{m'} \) is the trivial \( A_\infty \)-structure on \( E_g \). Since \( \overline{HH}^1(E_g)_{<0} = 0 \) (see Corollary 4.4.4), by Lemma 4.1.6, there exists a homotopy \( \overline{h} = (\overline{h}_n) \) over \( k \) from the identity to \( \overline{f} \). We can lift \( \overline{h} \) to a homotopy \( h \) over \( R \) from the identity transformation of \( m' \) to some gauge transformation \( f_1 \) with \( f_1 \cdot m' = m' \) and \( \overline{f}_1 = \overline{f} \). Then setting \( f' = f_1^{-1} \circ f \) gives the gauge transformation with the required properties.

(ii) By Corollary 4.4.4, we have \( \overline{HH}^1(E_g)_{<0} = 0 \), so the representability of \( \mathcal{M}_\infty \) follows from Corollary 4.2.5.

(iii) Parts (i) and (ii) imply that \( F_\infty \) is prorepresentable by the completion of the algebra of functions on \( \mathcal{M}_\infty \) at the trivial point. Hence, \( F_\infty \) is homogeneous. \( \square \)
The following result is well known (and in characteristic zero follows easily from the standard description via the Maurer-Cartan equation) but we give a proof for completeness. Note that for any $A_\infty$-algebra $A$ the Hochschild cochains

$$\text{CH}^*(A)_{\leq 0} = \prod_{i \leq 0} \text{CH}^*(A)_i$$

(4.5.1)

form a subcomplex of the Hochschild complex. We denote the corresponding cohomology by $HH^*(A)_{\leq 0}$.

**Lemma 4.5.2.** Let $E$ be a minimal $A_\infty$-algebra over $k$. Let us consider the functor on $\text{Art}_k$ associating with $R \in \text{Art}_k$ the set of deformations of $E$ to a minimal $A_\infty$-algebra structure on $E \otimes R$ up to extended gauge transformations (see Definition 4.1.3). Then the tangent space to this deformation functor is naturally identified with $HH^2(E)_{\leq 0}$. Furthermore, there is a complete obstruction theory for this functor with values in $HH^3(E)_{\leq 0}$. The similar statements hold for deformations of a small minimal $A_\infty$-category.

**Proof.** It is convenient to think of deformations of an $A_\infty$-structure $m$ as that of the coderivation $D_m$ on $\text{Bar}(E)$ such that $D_m^2 = 0$ (see Section 4.1). To compute the tangent space we have to look at $A_\infty$-structures on $E \otimes k[t]/(t^2)$ reducing to the given one modulo $t$. The corresponding coderivation has the form $D_m + tD_c$ with $c \in \text{CH}^2(E)_{\leq 0}$, so the equation $(D_m + tD_c)^2 = 0$ is equivalent to $[D_m, D_c] = 0$, i.e., to $c$ being a Hochschild cocycle. An extended gauge equivalence corresponds to an automorphism of $\text{Bar}(E) \otimes k[t]/(t^2)$ of the form $\text{id} + tD_f$, so it changes $c$ by $\delta(f)$. This gives the required identification of the tangent space.

Given a small extension

$$0 \to I \to \tilde{R} \to R \to 0$$

in $\text{Art}_k$ (recall that this means that $I$ is annihilated by the maximal ideal $M$ of $\tilde{R}$), and an $A_\infty$-structure on $E \otimes R$, we can lift each $m_i$ to some Hochschild cochain $\tilde{m}_i \in \tilde{R} \otimes \text{CH}^2(E)_{2-i}$ (where for $i = 2$ we take the standard product on $E \otimes \tilde{R}$). Let $D_0$ be the coderivation of $\text{Bar}(E \otimes \tilde{R})$ which corresponds to the $\tilde{R}$-linear extension of the original $A_\infty$-algebra structure on $E$. Then the coderivation associated with $\tilde{m}$ has form $D_0 + D$, where $D$ takes values in $M \otimes \text{Bar}(E)$. The $A_\infty$-equations hold modulo $I$, hence

$$(D_0 + D)^2 = [D_0, D] + D^2 = D_\phi$$

for some $\phi \in I \otimes \text{CH}^3(E)_{\leq 0}$. We have

$$[D_0, D_\phi] = [D_0, D^2] = [[D_0, D], D] = [D_\phi - D^2, D] = [D_\phi, D],$$

which vanishes since $MI = 0$. It follows that $\phi$ is a Hochschild cocycle. If we choose different liftings of $m_i$ then $D$ would change to $D + D'$ where $D'$ takes values in $I \otimes \text{Bar}(E)$. Then

$$(D_0 + D + D')^2 = D_\phi + [D_0, D'] + [D, D'] + (D')^2.$$ 

Here $[D, D'] = (D')^2 = 0$ since $MI = I^2 = 0$, so $\phi$ would change by a Hochschild coboundary. Thus, the class of $\phi$ in $I \otimes HH^3(E)_{\leq 0}$ is well defined. Conversely, if this class is zero then we can correct our choice of $\tilde{m}_i$ to make $\phi = 0$, so that $\tilde{m}_i$ define an $A_\infty$-structure. Thus, $\phi$ is a complete obstruction for the functor of $A_\infty$-structures. The
fact that extended gauge equivalences act trivially on this obstruction follows from the
general theory (see [24, Lem. 2.20]) since this group is smooth (as a functor on $\text{Art}_k$).

Lemma 4.5.3. Let $F \xrightarrow{f} G \xrightarrow{g} H$ be morphisms of deformation functors $\text{Art}_k \to \text{Sets}$. Assume that $g \circ f : F \to H$ is smooth and the map of tangent space $t_F \to t_G$ is surjective. Then $f : F \to G$ is smooth.

Proof. Let $O_F \to O_G \to O_H$ be the maps between the universal obstruction spaces induced by $f$ and $g$. Since $F \to G$ is smooth, the composed map $O_F \to O_H$ is injective by [24, Prop. 2.18]. Therefore, the map $O_F \to O_G$ is injective. Applying the cited result again we deduce that $F \to G$ is smooth. \hfill \□

The following result is a key step in the proof of Theorem A.

Proposition 4.5.4. Assume the characteristic of $k$ is not 2 or 3. Then the morphism of deformation functors $a_\infty : F_{g,g} \to F_\infty$ is an isomorphism. The tangent space to $F_\infty$ is naturally isomorphic to $\text{HH}^2(E_g)_{<0} = \text{HH}^2(E_g)$.

Proof. Let $j$ denote the open embedding of $U = \text{Cusp}_g \setminus \{p_1, \ldots, p_g\}$ into $\text{Cusp}_g$, and let $M = \text{Ext}^*(G, j_* \mathcal{O}_U) = \mathcal{O}(U)$ be the corresponding $A_\infty$-module over $E_g$. Note that endomorphisms of $M$ in the derived category of $A_\infty$-modules are identified with the algebra $\mathcal{O}(U)$. Let us define the $A_\infty$-category $\mathcal{C}$ as follows: it has $E_g$ as a full $A_\infty$-subcategory (where we think of $E_g$ as a category with objects $\mathcal{O}_C$, $\mathcal{O}_{p_1}, \ldots, \mathcal{O}_{p_n}$) and one additional object $\mathcal{O}_U$. The only additional morphisms in $\mathcal{C}$ are elements of $M$, viewed as morphisms from $\mathcal{O}_C$ to $\mathcal{O}_U$, and the algebra $\mathcal{O}(U)$ viewed as endomorphisms of the object $\mathcal{O}_U$. The $A_\infty$-structure on $\mathcal{C}$ comes from the products on $E_g$, the $A_\infty$-module structure on $M$, and the identification of $\mathcal{O}(U)$ with $A_\infty$-endomorphisms of $M$. Let $F_\mathcal{C}$ be the functor of deformations of $\mathcal{C}$ as a minimal $A_\infty$-category up to extended gauge transformations. Let also $F_{U,\text{nc}}$ (resp., $F_U$) be the functor of deformations of $\mathcal{O}(U)$ as an associative algebra (resp., as a commutative algebra). More precisely, for $R \in \text{Art}_k$ the set $F_{U,\text{nc}}$ (resp., $F_U$) consists of associative (resp., commutative) $R$-algebra structures on $\mathcal{O}(U) \otimes R$ reducing to the given on under the homomorphism $R \to k$. We have a commutative diagram of natural morphisms of functors

\[
\begin{array}{cccc}
F_U & \xrightarrow{F_{g,g}} & F_\infty \\
\downarrow & & \downarrow \\
F_{U,\text{nc}} & \xrightarrow{F_\mathcal{C}} & \tilde{F}_\infty
\end{array}
\]

(4.5.2)

where $\tilde{F}_\infty$ is the functor of deformations of $E_g$ as a minimal $A_\infty$-category up to extended gauge transformations.

Step 1. The morphism $F_\mathcal{C} \to \tilde{F}_\infty$ is étale, and the tangent spaces to both functors are naturally isomorphic to $\text{HH}^2(E_g)$. To prove that this morphism is étale it is enough to check that it induces an isomorphism on tangent spaces and an embedding on obstruction
spaces (see [24, Prop. 2.17]). By Lemma 4.5.2, these spaces are given by $HH_{\leq 0}^2$ and $HH_{\leq 0}^3$ (applied to $C$ and $E_g$), respectively. Note that our morphism corresponds to the embedding of the full subcategory on the objects $\mathcal{O}, \mathcal{O}_{p_1}, \ldots, \mathcal{O}(p_g)$ into $C$. Since $\mathcal{O}(U)$ is the algebra of endomorphisms of the $A_\infty$-module $M$, it follows that this embedding induces an isomorphism

$$HH^*(C) \xrightarrow{\sim} HH^*(E_g)$$

(see [17], [21, Thm. 4.1.1]). We have an exact sequence

$$HH_1(C) \to H^1(CH(C)_{\geq 1}) \to HH_2(C)_{\leq 0} \to HH_3(C) \to H^2(CH(C)_{\geq 1}) \to$$

$$HH^3(C)_{\leq 0} \to HH^3(C),$$

(4.5.3)

where $CH(?)_{\geq i} := CH(?) / CH(?)_{\leq i-1}$. Since $E_g$ has trivial higher products, we have a canonical decomposition $HH^i(E_g) = \prod_j HH^i(E_g)_j$, so the similar exact sequence for $E_g$ has trivial connecting homomorphisms. We claim that the map

$$HH_2(C)_{\leq 0} \to HH_2(C)$$

is an isomorphism, while the map

$$HH_3(C)_{\leq 0} \to HH_3(C)$$

is injective. Indeed, first we observe that

$$H^2(CH(C)_{\geq 1}) = H^2(CH(E_g)_{\geq 1}) = 0.$$

Indeed, since $C$ has morphisms only of degree 0 and 1, the only possible cochains in $CH^{s+t}(C)_t$ with $s + t = 2$ and $t \geq 1$ correspond to $(s, t) = (1, 1)$. But such cochains should have form $\text{Hom}^0(X, Y) \to \text{Hom}^1(X, Y)$, so they all vanish (since we exclude the identities in the Hochschild complex). The same argument works for $E_g$. Similar considerations with cochains show that

$$H^1(CH(C)_{\geq 1}) = CH^1(C)_{1} = \text{Ext}^1(\mathcal{O}, \mathcal{O}) \oplus \bigoplus_{i=1}^{g} \text{Ext}^1(\mathcal{O}_{p_i}, \mathcal{O}_{p_i}),$$

and the same formula holds for $E_g$. Thus, in the commutative square

$$\begin{array}{ccc}
HH^1(C) & \xrightarrow{\sim} & H^1(CH(C)_{\geq 1}) \\
\downarrow & & \downarrow \\
HH^1(E_g) & \xrightarrow{\sim} & H^1(CH(E_g)_{\geq 1})
\end{array}$$

both vertical arrows are isomorphisms. Since the bottom horizontal arrow is surjective, the top horizontal arrow is surjective too, and our claim follows from the exact sequence (4.5.3). Our argument also shows that the map

$$HH_2(E_g)_{\leq 0} \to HH_2(E_g)$$

is an isomorphism. Now our assertion follows by considering the map from the exact sequence (4.5.3) to the corresponding exact sequence for $E_g$. 

54
Step 2. The morphisms $F_{g,g} \to F_U$ and $F_U \to F_{U,nc}$ are both étale. For the first morphism we can think geometrically, as passing from a deformation of $(C, p_1, \ldots, p_g, v_1, \ldots, v_g)$ to that of $U = C \setminus D$. Thus, the maps of tangent spaces and of obstruction spaces are the natural maps
\[
\text{Ext}^1_C(L_C, \mathcal{O}_C(-2D)) \to \text{Ext}^1_U(L_U, \mathcal{O}_U) \quad \text{and} \quad \text{Ext}^2_C(L_C, \mathcal{O}_C(-2D)) \to \text{Ext}^2_U(L_U, \mathcal{O}_U).
\]
By Lemma 4.4.5(ii), these are isomorphisms.

Similar assertions for the morphism $F_U \to F_{U,nc}$ follow from Lemma 4.4.6.

Step 3. The morphisms $F_{g,g} \to F_C$ and $F_C \to F_{U,nc}$ induce isomorphisms on tangent spaces. Indeed, Steps 1 and 3 plus the commutativity of diagram (4.5.2) imply that the morphism of tangent spaces induced by $F_C \to F_{U,nc}$ is surjective. By Step 1, the tangent space to $F_C$ is identified with $HH^2(E_g)$. On the other hand, the tangent space to $F_{U,nc}$ is $HH^2(U) \cong HH^2(C_{\text{cusp}}) \cong HH^2(E_g)$ (see Lemma 4.4.3(ii) and Proposition 4.4.1), so it is a $k$-vector space of the same dimension, which implies the assertion for $F_C \to F_{U,nc}$. The assertion for the other morphism follows using Step 2 and the commutativity of diagram (4.5.2).

Step 4. The morphisms $F_{g,g} \to F_\infty$ and $F_\infty \to \tilde{F}_\infty$ induce isomorphisms on tangent spaces. Indeed, Steps 1 and 3 plus the commutativity of (4.5.2) imply that the morphism of tangent spaces induced by $F_\infty \to \tilde{F}_\infty$ is surjective. Similarly to Lemma 4.5.2 we can identify the tangent space to $F_\infty$ with $HH^2(E_g)_{<0}$. Thus, the morphism of tangent spaces induced by $F_\infty \to \tilde{F}_\infty$ is the natural embedding
\[
HH^2(E_g)_{<0} \hookrightarrow HH^2(E_g),
\]
hence, it is in fact an isomorphism. The assertion for the other morphism follows using Steps 1 and 3 and the commutativity of diagram (4.5.2).

Step 5. By Step 2 and the commutativity of diagram (4.5.2), the composition $F_{g,g} \to F_C \to F_{U,nc}$ is étale. Applying Lemma 4.5.3, we deduce that the morphism $F_{g,g} \to F_C$ is smooth, hence étale (using Step 3). Therefore, the composition
\[
F_{g,g} \to F_C \to \tilde{F}_\infty
\]
is also étale (using Step 1). Thus, we obtain that the composition
\[
F_{g,g} \to F_\infty \to \tilde{F}_\infty
\]
is étale. Applying Lemma 4.5.3 again and using Step 4, we deduce that the morphism $a_\infty : F_{g,g} \to F_\infty$ is étale. But $F_\infty$ is homogeneous (see Lemma 4.5.1(iii)), so by [24, Cor. 2.11], $a_\infty$ is an isomorphism. 

4.6. Proof of Theorem A. The fact that $\tilde{U}_{g,g}^{ns,a}$ is an affine scheme was proved in Theorem 1.2.3. It remains to prove that the morphism
\[
a_\infty : \tilde{U}_{g,g}^{ns,a} \to \mathcal{M}_\infty
\]
(see Section 4.3) is an isomorphism. We will use the comparison of the deformation theories worked out above and in addition will exploit the compatibility with $\mathbb{G}_m$-actions.
Recall that \( \tilde{U}_{g,g}^{ns,a} \) has a natural \( \mathbb{G}_m \)-action, coming from the diagonal \( \mathbb{G}_m \subset \mathbb{G}_m^g \), rescaling the trivializations of the tangent spaces at \( p_1, \ldots, p_g \). The morphism \( a_\infty \) is compatible with this action and with the \( \mathbb{G}_m \)-action on the functor \( \mathcal{M}_\infty \) of moduli of minimal \( A_\infty \)-structures on \( E_g \), where \( \lambda \in \mathbb{G}_m \) sends \((m_n) \mapsto (\lambda^{-n+2}m_n)\).

Now recall that by Corollaries 4.4.4 and 4.2.5 the functor \( \mathcal{M}_\infty \) of minimal \( A_\infty \)-deformations of \( E_g \) is representable by an affine \( k \)-scheme. Thus, we can view the morphism \( a_\infty \) as a \( \mathbb{G}_m \)-equivariant morphism of affine schemes over \( k \) (where we use an isomorphism \( \tilde{U}_{g,g}^{ns,a} \simeq S_g \) from Theorem 1.2.3). The cuspidal curve is sent by \( a_\infty \) to the trivial \( A_\infty \)-structure and Proposition 4.5.4 implies that \( a_\infty \) induces an isomorphism between the completions of our affine schemes at these points (which are both \( \mathbb{G}_m \)-stable). Recalling the explicit construction of the representable scheme in Theorem 4.2.4 and the description of the affine scheme \( S_g \), we see that in both cases the maximal ideal of the corresponding point is generated by elements of positive weight with respect to the \( G \)-action (note that the coefficients of \( m_n \), viewed as functions on \( M \), we see that in both cases the maximal ideal of the corresponding point is generated by elements of positive weight with respect to the \( \mathbb{G}_m \)-action (note that the coefficients of \( m_n \), viewed as functions on \( M \), are given by \( \lim_{n \to \infty} a_\infty(m_n) \)). Thus, in our situation we have a degree-preserving homomorphism of graded \( k \)-algebras

\[
f : A = \bigoplus_{n \geq 0} A_n \to B = \bigoplus_{n \geq 0} B_n
\]

with \( A_0 = B_0 = k \) which induces an isomorphism between the completions with respect to powers of \( A_{>0} \) (resp., \( B_{>0} \)). Considering the induced isomorphism

\[
A/(A_{>0})^N \to B/(B_{>0})^N
\]

for \( N \gg 0 \) we see that the map \( A_n \to B_n \) is an isomorphism for every \( n \geq 0 \), so in fact, \( f \) is an isomorphism. □

4.7. Hochschild cohomology and normal forms of \( A_\infty \)-structures on \( E_g \). We continue to work over a field \( k \) and we denote the \( k \)-schemes obtained by the base change from \( \tilde{U}_{g,g}^{ns,a} \) and \( S_g \) by the same symbols. From our results we get a geometric way to calculate some Hochschild cohomology spaces of the algebra \( E_g \) (that were first determined in [11] via quite tedious algebraic calculations).

**Proposition 4.7.1.** Assume that char(\( k \)) is not 2, 3 or 5. Then the nonzero values of \( \dim HH^2(E_g)_t \) are given by

\[
\dim HH^2(E_g)_{-1} = g^2 - g, \quad \dim HH^2(E_g)_{-2} = 2g^2 - 2g,
\]

\[
\dim HH^2(E_g)_{-3} = g^2 - g, \quad \dim HH^2(E_g)_{-4} = g.
\]

The nonzero values of \( \dim HH^1(E_g)_t \) are

\[
\dim HH^1(E_g)_0 = g, \quad \dim HH^1(E_g)_1 = 2g.
\]

**Proof.** By Proposition 4.5.4, the space \( HH^2(E_g) \) is the tangent space to \( \mathcal{M}_\infty \simeq \tilde{U}_{g,g}^{ns,a} \) at the point corresponding to the cuspidal curve \( C_{g,cusp} \). Furthermore, the component \( HH^2(E_g)_t \) corresponds to the weight \( t \) subspace with respect to the \( \mathbb{G}_m \)-action (see Proposition 4.4.1). By Theorem 1.2.3, we can identify \( HH^2(E_g) \) with the tangent space to the affine scheme \( S_g \) at the point where all coordinates are zero. By Proposition 2.3.2(ii), the cotangent space at this point has the basis corresponding to the generators \( \alpha_{ij}, \beta_{ij}, \gamma_{ij}, \varepsilon_{ij} \).
and $\pi_i$. Of these $\alpha_{ij}$ have weight 1, $\beta_{ij}$ and $\gamma_{ij}$—weight 2, $\varepsilon_{ij}$—weight 3, and $\pi_i$—weight 4. Passing to the dual space we get the answers for $HH^2(E_g)$. To compute the components $HH^1(E_g)_t$ we use the identification $HH^1(E_g) \simeq HH^1(C_g^{\text{cusp}})$ (see Proposition 4.4.1), the exact sequence

$$0 \to H^1(C, \mathcal{O}) \to HH^1(C_g^{\text{cusp}}) \to H^0(C, \mathcal{T}) \to 0$$

from Lemma 4.4.3(i), and Lemma 4.4.2. \qed

Using the correspondence between non-special curves and $A_\infty$-structures on $E_g$ we can calculate normal forms of $A_n$-structures on $E_g$ for small $n$. Let $M_n$ denote the moduli of minimal $A_n$-structures on $E_g$. We know by Theorem 4.2.4 and Corollary 4.4.4 that these are affine $k$-schemes. Note that since $HH^3(E_g) \simeq HH^3(C_g^{\text{cusp}})$ is finite-dimensional, by Corollary 4.2.6, we have $M_\infty \simeq M_n$ for sufficiently large $n$.

**Proposition 4.7.2.** Assume that $\text{char}(k)$ is not 2, 3 or 5. Let

$$\Theta = \{\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \varepsilon_{ij}, \pi_i\}$$

be the set of minimal generators of the algebra of functions on the affine scheme $S_g$, and let $I_r$ (resp., $I_{\leq r}$) be the component of degree $r$ (resp., the sum of components of degree $\leq r$) in the ideal of relations between these generators (see Section 2.3 and Remark 2.5.2).

(i) For $n \geq 3$ the algebra of functions on $M_n$ is isomorphic to $\text{Spec } k[\Theta_{\leq n-2}]/(I_{\leq n-2})$, where $\Theta_{\leq i}$ denotes the subset of elements in $\Theta$ of degree $\leq i$.

(ii) The schemes $M_3$ and $M_4$ are isomorphic to the affine spaces with coordinates $\alpha_{ij}$ and $\alpha_{ij}, \beta_{ij}, \gamma_{ij}$, respectively (equivalently, $I_{\leq 2} = 0$). The universal $A_3$-structure on $E_g$ is given by (3.3.2). The universal $A_4$-structure on $E_g$ is given by the formulas of the Appendix. The algebra of functions on $M_5$ is isomorphic to $\text{Spec } k[\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \varepsilon_{ij}]/(I_3)$. The universal $A_5$-structure on $E_g$ is given by the formulas of the Appendix.

**Proof.** (i) By Theorem A and Theorem 1.2.3, we have an isomorphism of affine schemes $S_g \simeq M_\infty$, compatible with $G_m$-action. Note that both these schemes have $G_m$-equivariant closed embeddings into affine spaces

$$S_g \subset \mathbb{A}^{4g^2-3g}, \quad M_\infty \subset \mathbb{A}^N,$$

where the former corresponds to the minimal generators (2.3.1), while the latter corresponds to considering structure constants of the form constants $m_i$ (we use Theorem 4.2.4 and the fact that $M_\infty \simeq M_n$ for $n \gg 0$). Furthermore, by Corollary 3.3.3, there exists a closed $G_m$-equivariant embedding $\mathbb{A}^{4g^2-3g} \to \mathbb{A}^N$, which is a section for a $G_m$-equivariant coordinate projection $\mathbb{A}^N \to \mathbb{A}^{4g^2-3g}$, such that we have a commutative diagram

$$\begin{array}{ccc}
S_g & \sim & M_\infty \\
\mathbb{A}^{4g^2-3g} & \overset{\sim}{\longrightarrow} & \mathbb{A}^N \\
57 & \rightarrow & 57
\end{array}$$
By definition, the equations of $M_\infty$ in $\mathbb{A}^N$ are given by the $A_\infty$-constraints

$$[m_2, m_n] + [m_3, m_{n-1}] + \ldots = 0,$$

$n \geq 3$, where the constraint corresponding to $n$ gives equations homogeneous of degree $n - 2$. To get the presentation of $M_n$ from this we take coordinates of degree $\leq n - 2$ (corresponding to $m_i$ with $i \leq 2$) and equations of degree $\leq n - 2$. The assertion follows from this using Lemma 4.7.3 below.

(ii) The formula (3.3.2) gives $m_3$ as a linear combination of certain $g^2 - g$ Hochschild cohomology classes. To prove that this is a universal $A_3$-structure we need to check that these Hochschild cohomology classes form a basis of $HH^2(E_g)_{-2}$. But this follows from the description of the tangent space to $S_g$ at zero given in Proposition 2.3.2(ii) (see the proof of Proposition 4.7.1).

Next, we need to check that every product $m_3$ given by (3.3.2) extends to an $A_4$-structure, i.e., that $[m_3, m_3]$ is a Hochschild coboundary. Indeed, let us set

$$m_4(X_j, B_i, Y_i, A_i) = \sum_{k \neq j} \alpha_{ij} \alpha_{jk} X_k,$$

$$m_4(X_j, B_i, A_i, X_i) = \sum_{k \neq j} \alpha_{ij} \alpha_{jk} X_k,$$

and let all other $m_4$ products of the basis elements be zero. Then one can check by a direct computation that the $A_\infty$-constraint $[m_2, m_4] + [m_3, m_3]/2 = 0$ is satisfied. Note that the general formulas for $m_4$ in the Appendix differ from this one by adding the linear combinations of certain $2(g^2 - g)$ Hochschild cocycles with $\beta_{ij}$ and $\gamma_{ij}$ as coefficients. It remains to use Proposition 2.3.2(ii) again to see that the corresponding Hochschild cohomology classes form a basis in $HH^2(E_g)_{-2}$. The assertion for $A_5$-structures follows similarly using part (i). 

The following Lemma was used in the proof of Proposition 4.7.2(i) (with $S$ being the algebra of functions on $\mathbb{A}^N$, $\overline{S}$ the algebra of functions on $\mathbb{A}^{4g^2 - 3g}$ and $J$ the ideal of relations defining $M_\infty$ in $\mathbb{A}^N$).

**Lemma 4.7.3.** Let $S$ be a $\mathbb{Z}$-graded ring, $\overline{S} \subset S$ a graded subring and $K \subset S$ a homogeneous ideal such that $S = K \oplus \overline{S}$. Let also $J \subset S$ be a homogeneous ideal containing $K$, so that $J = K \oplus \overline{J}$, where $\overline{J}$ is a homogeneous ideal in $\overline{S}$. For an integer $n$ let $S' \subset S$ (resp., $\overline{S}' \subset \overline{S}$) denote the subring generated by $S_{\leq n}$ (resp., $\overline{S}_{\leq n}$), and let $J' \subset S'$ (resp., $\overline{J}' \subset \overline{S}'$) be the ideal generated by $J_{\leq n}$ (resp., $\overline{J}_{\leq n}$). Then the natural homomorphism

$$S'/J' \rightarrow \overline{S}'/\overline{J}'$$

is an isomorphism.

**Proof.** Let $K' \subset S'$ denote the ideal generated by $K_{\leq n}$. Since $\overline{S}$ is a subring of $S$, from the decomposition

$$S_{\leq n} = K_{\leq n} \oplus \overline{S}_{\leq n},$$

one can easily derive that $S' = K' + \overline{S}'$. Since $K' \subset K$ and $\overline{S}' \subset \overline{S}$, we get a direct sum decomposition

$$S' = K' \oplus \overline{S}'.$$
Now the equality $J_{\leq n} = K_{\leq n} \oplus J_{\leq n}$ implies the decomposition
\[ J' = K' \oplus J', \]
and the assertion follows. \qed

Remarks 4.7.4. 1. One can check by a direct but tedious inspection of the equations defining $S_g$ (see the proof of Lemma 1.2.2) that in fact the equations (2.2.3) span $I_3$, so they form a defining set of relations for the scheme $M_5$.

2. The vanishing of $HH^2(E_g)_{<-4}$ means that every minimal $A_\infty$-structure on $E_g$ is determined by the corresponding $A_6$-structure, i.e., the morphism $M_\infty \to M_6$ is a closed embedding. Also, using the isomorphism of Theorem A we can interpret Proposition 2.5.1 as saying that the morphisms $M_\infty \to M_5$ and $M_\infty \to M_4$ become closed embeddings when restricted to the preimages of explicit open subsets given in terms of nonvanishing of some of $\alpha_{ij}$ (all of them, for the second morphism).

5. $\psi$-(pre)stable curves of genus 0 and $A_\infty$-structures

In this section we will analyze what happens with the picture described above if we replace curves of genus $g$ with $g$ marked points by curves of genus 0 with $n$ marked points (where $n \geq 3$).

5.1. The moduli stack of $\psi$-prestable curves of genus 0. First, similarly to Sec. 1 we consider certain normal forms for curves of genus 0 with $n$ marked points, and the resulting presentation of the moduli space by explicit equations.

Definition 5.1.1. (i) Let $(C, p_1, \ldots, p_n)$ be a reduced connected complete curve of arithmetic genus 0 with $n$ smooth marked points over an algebraically closed field. We say that $(C, p_1, \ldots, p_n)$ is $\psi$-prestable if $O_C(p_1 + \ldots + p_n)$ is ample, or equivalently, if every component of $C$ contains at least one marked point.

(ii) A $\psi$-prestable curve is called $\psi$-stable if every component of $C$ contains at least three distinguished points (i.e., singular or marked points).

Recall that the only singular points that can occur for reduced curves of arithmetic genus zero are rational $m$-fold points, i.e., points for which the completion of the local ring has form
\[ O_{C, p} \simeq k[[x_1, \ldots, x_m]]/(x_i x_j \mid 1 \leq i < j \leq m) \]
(see [36, Lem. 1.17] or [38]). It follows that the above definition of a $\psi$-stable curve is equivalent to the one given [9, Def. 2.26].

For $n \geq 3$ let us denote by $U_{0,n}[\psi]$ the stack of $\psi$-prestable curves classifying families (flat, proper, finitely presented morphisms) with $n$ sections, whose geometric fibers are $\psi$-prestable curves. Let also $\tilde{U}_{0,n}[\psi] \to U_{0,n}[\psi]$ denote the $\mathbb{G}_m^n$-torsor corresponding to choices of nonzero tangent vectors at the marked points.

For a $\psi$-prestable curve $(C, p_1, \ldots, p_g)$ with fixed nonzero tangent vectors $v_i$ at $p_i$ we can construct a natural presentation for the algebra $H^0(C \setminus D, \mathcal{O})$, where $D = p_1 + \ldots + p_n$. Namely, for every $i = 1, \ldots, g$ let us pick a rational function $f_i \in H^0(C, \mathcal{O}(p_i))$ such that $f_i \equiv v_i \bmod O_C$ in $O_C(p_i)/O_C \simeq T_{p_i} C$. Note that such $f_i$ is defined uniquely up to an
additive constant. To get rid of this ambiguity we will impose the condition \( f_i(p_{i+1}) = 0 \) for \( i = 1, \ldots, n \) (where \( p_{n+1} = p_1 \)).

A version of this construction works for a family \( \pi : C \to \text{Spec}(R) \) of \( \psi \)-prestable curves with \( n \) marked points \( p_i : \text{Spec}(R) \to C \), where we choose \( v_i \in H^0(C, \mathcal{O}_C(p_i) / \mathcal{O}_C) \) inducing trivializations \( R \cong H^0(C, \mathcal{O}_C(p_i) / \mathcal{O}_C) \). Then the natural map \( R \to H^0(C, \mathcal{O}_C) \) is an isomorphism and \( H^1(C, \mathcal{O}_C) = 0 \), so we get an exact sequence

\[
0 \to R \to H^0(C, \mathcal{O}_C(p_i)) \to H^0(C, \mathcal{O}_C(p_i) / \mathcal{O}_C) \to 0
\]

so we can choose \( f_i \in H^0(C, \mathcal{O}_C(p_i)) \) projecting to \( 1 \in R \cong H^0(C, \mathcal{O}_C(p_i) / \mathcal{O}_C) \) to \( 0 \).

Let us set \( D = p_1(\text{Spec}(R)) + \ldots + p_n(\text{Spec}(R)) \). Also, for \( i \neq j \) we set

\[
\alpha_{ij} = f_i(p_j)
\]

\[\text{Lemma 5.1.2. Assume } n \geq 3. \text{ Let } \pi : C \to \text{Spec}(R) \text{ be a family of } \psi \text{-prestable curves}
\]

with \( n \) marked points \( p_i : \text{Spec}(R) \to C \) and with trivializations \( R \cong H^0(C, \mathcal{O}_C(p_i) / \mathcal{O}_C) \). Then the algebra \( H^0(C \setminus D, \mathcal{O}) \) is generated over \( R \) by the elements \( f_1, \ldots, f_g \) with the defining relations

\[
f_i f_j = \alpha_{ij} f_j + \alpha_{ji} f_i + c_{ij}, \text{ for } i \neq j, \quad (5.1.1)
\]

where

\[
c_{ij} = \alpha_{ik} \alpha_{jk} - \alpha_{ij} \alpha_{jk} - \alpha_{ji} \alpha_{ik} \quad (5.1.2)
\]

for any three distinct indices \( i, j, k \). Furthermore, for every \( N \geq 1 \) the elements \( 1 \) and \( f_i^m \), with \( i = 1, \ldots, n \), \( 1 \leq m \leq N \), form a basis of \( H^0(C, \mathcal{O}_C(ND)) \) over \( R \).

\[\text{Proof. Since } H^1(C, \mathcal{O}) = 0, \text{ for any } i \neq j \text{ we have an exact sequence}
\]

\[
0 \to R = H^0(C, \mathcal{O}) \to H^0(C, \mathcal{O}(p_i + p_j)) \to H^0(C, \mathcal{O}(p_i + p_j) / \mathcal{O}) \to 0,
\]

so \((1, f_i, f_j)\) is an \( R \)-basis of \( H^0(C, \mathcal{O}(p_i + p_j)) \). Since \( f_i f_j \in H^0(C, \mathcal{O}(p_i + p_j)) \), we get the equation \((5.1.1)\) (the coefficients with \( f_i \) and \( f_j \) are determined by looking at polar parts at \( p_i \) and \( p_j \)). Now the equation \((5.1.2)\) is obtained by evaluating both parts of \((5.1.1)\) at \( p_k \). The last equation follows by induction on \( N \) from the exact sequences

\[
0 \to H^0(C, \mathcal{O}(jD)) \to H^0(C, \mathcal{O}((j+1)D)) \to H^0(C, \mathcal{O}((j+1)D) / \mathcal{O}(jD)) \to 0,
\]

where \( j \geq 0 \). \[\square\]

We also have a much simpler version of Lemma 1.2.2 in our situation.

\[\text{Lemma 5.1.3. Let } R \text{ be a commutative ring. An associative } R \text{-algebra generated by }
\]

\( f_1, \ldots, f_g \) with defining relations \((5.1.1)\) has elements \( 1, (f_i^m)_{i=1, \ldots, n, m \geq 1} \) as an \( R \)-basis if and only if the relations \((5.1.2)\) hold for any distinct \( i, j, k \).

\[\text{Proof. This is an immediate application of the Gröbner basis technique: expanding } f_i f_j f_k,
\]

for distinct \( i, j, k \), in two ways using the defining relations \((5.1.1)\) we get the equation \((5.1.2)\) and in addition, the equation

\[
\alpha_{ij} c_{jk} + \alpha_{ji} c_{ik} = \alpha_{ik} c_{kj} + \alpha_{ki} c_{ij}.
\]
It remains to observe that the latter equation follows from (5.1.2) (applied for appropriate permutations of indices $i,j,k$).

Let us consider the subscheme $\mathcal{T}_n$ in the affine space (over $\mathbb{Z}$) with coordinates $\alpha_{ij}$, where $1 \leq i,j \leq n$, $i \neq j$, defined by the equations

\[
\begin{align*}
\alpha_{i,i+1} &= 0, \\
\alpha_{ik} \alpha_{jk} - \alpha_{ij} \alpha_{jk} - \alpha_{ii} \alpha_{jk} &= \alpha_{il} \alpha_{jl} - \alpha_{ij} \alpha_{jl} - \alpha_{ij} \alpha_{il},
\end{align*}
\]

where $i,j,k,l$ are distinct (with the convention $\alpha_{n,n+1} = \alpha_{n,1}$ in the first equation). Note that for $n = 3$ we have $\mathcal{T}_n \cong \mathbb{A}^3$ (we can take $\alpha_{13}, \alpha_{21}$ and $\alpha_{32}$ as coordinates).

**Theorem 5.1.4.** Assume $n \geq 3$. There is an isomorphism of the stack $\tilde{\mathcal{U}}_{0,n}[\psi]$ with the affine scheme $\mathcal{T}_n$, so that the open part $C \setminus D$ of the universal curve over $\mathcal{T}_n$ is given by the equations (5.1.1) in affine coordinates $f_i$, where $c_{ij}$ is given by (5.1.2). Furthermore, the isomorphism $\tilde{\mathcal{U}}_{0,n}[\psi] \cong \mathcal{T}_n$ is compatible with the $\mathbb{G}_m^n$-actions, where $\lambda = (\lambda_i) \in \mathbb{G}_m^n$ acts on $\tilde{\mathcal{U}}_{0,n}[\psi]$ by rescaling the tangent vectors $v_i \mapsto \lambda_i^{-1} v_i$, and on the scheme $\mathcal{T}_n$ so that

\[(\lambda^{-1})^* \alpha_{ij} = \lambda_i \alpha_{ij}.
\]

**Proof.** We mimic the proof of Theorem 1.2.3. Lemma 5.1.2 gives a natural map

\[
\tilde{\mathcal{U}}_{0,n}[\psi](R) \to \mathcal{T}_n(R)
\]

for every commutative ring $R$. To construct the map in the other direction we note that by Lemma 5.1.3, an $R$-point of $\mathcal{T}_n$ gives rise to an associative $R$-algebra $A$ with generators $f_1, \ldots, f_n$ and defining relations of the form (5.1.1) with $c_{ij}$ given by (5.1.2) (where $\alpha_{ij} \in R$ are the coordinates of our point in $\mathcal{T}_n(R)$), such that 1 and $(f_i^m)$ form an $R$-basis of $A$. For $N \geq 1$ let $F_N A \subset A$ denote the $R$-submodule generated by $f_i^m$ with $m \leq N$, and let $F_0 A = R \subset A$. Then we define the projective family $\pi : C \to \text{Spec}(R)$ by setting $C = \text{Proj}(\mathcal{R}A)$, where $\mathcal{R}A = \bigoplus_{j \geq 0} F_j A$ is the Rees algebra associated with the filtration $(F_i A)$. Let $T$ denote the element $1 \in F_1 A \subset \mathcal{R}A$. Then $\mathcal{R}A/(T)$ is isomorphic to the associated graded quotient

\[\text{gr}^\bullet_{F_i} A \simeq R[f_1, \ldots, f_n]/(f_i f_j \mid 1 \leq i < j \leq n).\]

Hence, $\text{Proj}(\mathcal{R}A/(T))$ is the disjoint union of $n$ copies of $\text{Spec}(R)$, which gives $n$ disjoint sections $p_1, \ldots, p_n : \text{Spec}(R) \to C$, so that $D = p_1(\text{Spec}(R)) + \ldots + p_n(\text{Spec}(R))$ is the divisor $(T = 0)$. Let $F_i$ denote the element $f_i \in F_i A \subset \mathcal{R}A$. Then the algebra $\mathcal{R}A$ is defined by the homogeneous versions of the relations (5.1.1):

\[F_i F_j = \alpha_{ij} F_j T + \alpha_{ji} F_i T + c_{ij} T^2.
\]

Since $\text{deg}(F_i) = \text{deg}(T) = 1$, we can think of $C = \text{Proj}(\mathcal{R}A)$ as a closed subscheme in the projective space $\mathbb{P}^n_R$. In particular, $\mathcal{O}_C(D) = \mathcal{O}_C(1)$, so $D$ is ample. Let $U_i \subset C$ denote the open subset $F_i \neq 0$. Then $D \subset U_i$ is the vanishing locus of the function $T/F_i$ and the relations (5.1.4) show that $F_j/F_i$ is divisible by $T/F_i$ on $U_i$. This implies that $p_i(\text{Spec}(R))$ is locally given by one equation $T/F_i = 0$. The same argument as in the proof of Theorem 1.2.3 shows that the morphism $\pi : C \to \text{Spec}(R)$ is flat of relative dimension 1. Also, since $f_i = F_i/T$, we obtain that $f_i$ projects to a generator of $\mathcal{O}_C(p_i)/\mathcal{O}_C$ and is regular near the images of $p_j$ for $j \neq i$. Thus, we deduce that for $m \geq 0$, $H^0(C, \mathcal{O}(mD))$ is a free
$R$-module of rank $mn + 1$ (with the basis given by the appropriate powers of $f_i$). This implies that the geometric fibers of $\pi$ are connected reduced curves of arithmetic genus 0. Together with ampleness of $D$ this shows that they are $\psi$-prestable, so we get a map

$$\mathcal{T}_n(R) \to \tilde{U}_{0,n}[\psi](R). \quad (5.1.5)$$

As in Theorem 1.2.3 one immediately checks that the maps (5.1.3) and (5.1.5) are inverse of each other. The compatibility with the $\mathbb{G}_m^n$-actions is straightforward.

\[\Box\]

**Example 5.1.5.** For $n = 3$ we have four types of $\psi$-prestable curves, characterized by the number of vanishing coordinates $\alpha_{21}, \alpha_{32}, \alpha_{13}$. For a smooth curve none of them vanish. If $\alpha_{32} = 0$ but $\alpha_{21} \alpha_{13} \neq 0$ then $C$ is the union of two (smooth) components $C_1$ and $C_3$, intersecting transversally at one point, with $p_1, p_2 \in C_1$ and $p_3 \in C_3$. If $\alpha_{21} = \alpha_{32} = 0$ then $C$ is the union of three smooth components $C_1, C_2, C_3$ such that $p_i \in C_i$, where $C_2$ and $C_3$ are disjoint and both $C_2$ and $C_3$ intersect $C_1$ transversally (at distinct points). Finally, when all the coordinates are zero we get the union of three components intersecting at one point which is a rational threefold point.

### 5.2. $A_\infty$-structures for curves of arithmetic genus 0.

Given a curve of arithmetic genus 0 with $g$ (smooth) marked points we can consider the corresponding $A_\infty$-structure on the algebra $\text{Ext}^*(G, G)$ for $G = \mathcal{O}_C \oplus \mathcal{O}_{p_1} \oplus \ldots \oplus \mathcal{O}_{p_n}$. As before, a choice of nonzero tangent vectors at all $p_i$ gives a way to fix generators $B_i \in \text{Ext}^1(\mathcal{O}_{p_i}, \mathcal{O}_C)$ so that the algebra $\text{Ext}^*(G, G)$, as a graded associative algebra gets identified with the algebra

$$E_{0,n} = k[Q]/J_0,$$

where $Q = Q_n$ is the same quiver that we used before (see (0.0.2)), while the ideal $J_0$ is generated by the relations

$$B_iA_i = A_iB_j = 0, \text{ where } i \neq j.$$

More precisely, as in Section 3, the construction of the $A_\infty$-structure on $E_{0,n}$ associated with a curve works well in families with affine bases (depending on a choice of relative parameters at the marked points). We only have to make appropriate changes in the choices of cohomology representatives and of the homotopy operator. Namely, since we now have $H^1(C, \mathcal{O}_C) = 0$, we don’t need the classes $X_i$, and the homotopy operator on $K_{\mathcal{O},\mathcal{O}}$ has to be defined by the formulas $Q([v]) = -v$ and

$$Q([\frac{1}{t_n}]) = f_i[n](t_i)_{\geq 0}1_i + \sum_{j \neq i} f_i[n](t_j)_{\geq 0}1_j + f_i[n]$$

for all $n \geq 1$, where $f_i[n]$ are now defined for all $n \geq 1$ (in fact, we can take $f_i[n] = f_i^n$, where $f_i$ are defined as in Lemma 5.1.2).

Let us fix a field $k$. We denote still by $\hat{U}_{0,n}[\psi]$ the moduli scheme of $\psi$-prestable curves over $k$ (isomorphic to the affine scheme $\mathcal{T}_n \times \text{Spec}(k)$). From the above construction of $A_\infty$-structures, as in section 4.3, we get a morphism of functors on commutative algebras over $k$

$$\hat{U}_{0,n}[\psi] \to \mathcal{M}_\infty(E_{0,n}), \quad (5.2.1)$$
where \( \mathcal{M}_\infty(E_{0,n}) \) is the functor of minimal \( A_\infty \)-structures on the algebra \( E_{0,n} \), up to a gauge equivalence. Furthermore, the map (4.3.1) is compatible with the natural \( \mathbb{G}_m^n \)-action, where the action on \( \mathcal{M}_\infty(E_{0,n}) \) is induced by the rescalings \( B_i \mapsto \lambda_i B_i \).

Let us denote by \( C_{0,n} \) the most singular curve appearing in the moduli space \( \tilde{U}_{0,n} \): it is the union of \( n \) projective lines intersecting at one point which is a rational \( n \)-fold point (with one marked point on each component). Under the isomorphism of Theorem 5.1.4 this point in the moduli space corresponds to the origin in the affine scheme \( T_n \) (where all \( \alpha_{ij} \) vanish). Note that as in the case of the cuspidal curve, we have a natural \( \mathbb{G}_m^n \)-action on \( C_{0,n} \), and we will use the induced action of the diagonal \( \mathbb{G}_m^n \). The analog of Proposition 4.4.1 states in our case that we have an equivalence of categories

\[
\text{Perf}(C_{0,n}) \simeq \text{Perf}(E_{0,n})
\]

inducing a \( \mathbb{G}_m^n \)-equivariant isomorphism

\[
HH^*(C_{0,n}) \simeq HH^*(E_{0,n}), \tag{5.2.2}
\]

so that the second grading on these spaces is given by the weights of the \( \mathbb{G}_m \)-action.

Next, we observe that the assertions of Lemma 4.4.2(ii)-(v) hold for the \( n \)-fold rational singularity curve \( C_{0,n} \), with the same proofs but without any assumption on the characteristic. It follows that the analog of Lemma 4.4.3(ii) also holds for \( C_{0,n} \), so

\[
HH^1(C_{0,n})<0 = 0
\]

and the natural map

\[
HH^2(C_{0,n}) \to HH^2(U)
\]

is an isomorphism, where \( U \subset C_{0,n} \) denotes the complement to the \( g \) marked points. Using the isomorphism (5.2.2) we deduce the vanishing

\[
HH^1(E_{0,n})<0 = 0,
\]

which implies in particular, that the functor \( \mathcal{M}_\infty(E_{0,n}) \) is represented by an affine scheme over \( k \) (see Corollary 4.2.6).

Next, similarly to Lemma 4.4.5(ii), we derive that for \( C = C_{0,n} \) the natural maps

\[
\text{Ext}^1_C(L_C, O(-2D)) \to \text{Ext}^1_U(L_U, O_U) \quad \text{and} \quad \text{Ext}^2_C(L_C, O(-2D)) \to \text{Ext}^2_U(L_U, O_U)
\]

are isomorphisms. This allows us to run the same argument as in Proposition 4.5.4 to show that the deformation theory of \( C_{0,n} \) matches with the deformation theory of the trivial \( A_\infty \)-structure on \( E_{0,n} \). Finally, using the \( \mathbb{G}_m \)-actions as in 4.6 we deduce the following result (this time we don’t need any assumptions on the characteristic).

**Theorem 5.2.1.** For any field \( k \) and any \( n \geq 3 \), the morphism (5.2.1) induces an isomorphism of the moduli scheme of \( \psi \)-prestable curves \( \tilde{U}_{0,n}[\psi] \) with the moduli scheme of minimal \( A_\infty \)-structures on \( E_{0,n} \), up to a gauge equivalence. This isomorphism is compatible with the natural \( \mathbb{G}_m^n \)-actions.
Similarly to Proposition 4.7.2 we can interpret the fact that the algebra of functions on the scheme \(\tilde{U}_{0,n}[\chi] \simeq T_n\) is generated in degree 1 with quadratic relations as saying that the natural projection to the moduli space of \(G\) (see section 2.4 for our conventions on this)

\[ \mathcal{M}_\infty(E_{0,n}) \to \mathcal{M}_4(E_{0,n}), \]

is an isomorphism, while the map \(\mathcal{M}_4(E_{0,n}) \to \mathcal{M}_3(E_{0,n})\) is a closed embedding (where \(\mathcal{M}_3(E_{0,n})\) is an affine space). In particular, any minimal \(A_\infty\)-structure on \(E_{0,n}\) is determined up to a gauge equivalence by \(m_3\). Using the formulas from Section 3 one can check that the nonzero values of \(m_3\) on the basis vectors are given by

\[ m_3(A_j, B_i, A_i) = \alpha_{ij} A_j, \quad m_3(B_j, A_j, B_i) = -\alpha_{ji} B_i, \]

where \(i \neq j\).

5.3. \(\psi\)-stability as GIT stability. In this section we continue to work over a field \(k\). We are going to show that the notion of \(\psi\)-stability (see Definition 5.1.1(ii)) appears naturally as a GIT stability for the action of \(G^m\) on the affine scheme \(\tilde{U}_{0,n}[\chi] \simeq T_n\).

Namely, for a nontrivial character \(\chi : G_m \to G_m\) we consider the GIT quotient \(T_n/\chi G_m\) (see section 2.4 for our conventions on this).

Let us identify the character lattice of \(G_m\) with \(Z^n \subset \mathbb{R}^n\) in the standard way, and let \(C_0 \subset \mathbb{R}^n\) be the cone generated by all the basis vectors \(e_i\). Recall that \(T_n\) is a closed subscheme of the affine space \(\prod_{i=1}^n \mathbb{A}^{n-2}\), where the coordinates \((\alpha_{ij})\) on the \(i\)th factor (where \(j \neq i, i + 1\)) satisfy \((\lambda^{-1})^* \alpha_{ij} = \lambda \alpha_{ij}\). Thus, we have a closed embedding of GIT quotients

\[ T_n/\chi G_m \subset \prod_{i=1}^n (\mathbb{A}^{n-2} \sslash_a G_m), \]

where \(\chi = \sum_i a_i e_i\). Thus, for \(\chi \notin C_0\) the GIT quotients will be empty, while for \(\chi\) in the interior of \(C_0\) the ambient GIT quotients will be just the product of \(n\) copies of \(\mathbb{P}^{n-3}\) (for \(\chi\) on the boundary it will be the product of the copies of \(\mathbb{P}^{n-3}\) corresponding to positive coordinates of \(\chi\)).

**Proposition 5.3.1.** Assume \(n \geq 3\). For any \(\chi\) in the interior of \(C_0\) the \(\chi\)-semistable locus in \(\tilde{U}_{0,n}[\chi]\) coincides with the \(\chi\)-stable locus and consists of \(\psi\)-stable curves. Thus, the corresponding GIT quotient is exactly the moduli scheme of \(\psi\)-stable curves \(\overline{M}_{0,n}[\chi]\). The obtained embedding of \(\overline{M}_{0,n}[\chi]\) into the product of \(n\) copies \(\mathbb{P}^{n-3}\) corresponds to the line bundles \(\psi_1, \ldots, \psi_n\) (the cotangent lines at the marked points).

**Proof.** Note that the conditions of stability and semistability for \(\chi\) in the interior of \(C_0\) are equivariant, since this is true for the action on the ambient affine space. The condition of \(\chi\)-semistability is simply that for every \(i\) there exists \(j \neq i\) such that \(\alpha_{ij} \neq 0\). We claim that this condition is equivalent to the \(\psi\)-stability of the corresponding curve. Indeed, recall that \(\alpha_{ij} = f_i(p_j)\), where \(f_i \in H^0(C, O_C(p_i))\). Let \(C_i \subset C\) be the component containing \(p_i\). Then \(f_i\) has a unique zero on \(C_i\). If \((C, p_1, \ldots, p_n)\) is \(\psi\)-stable then \(C_i\) has at least two distinguished points \(q\) and \(q'\) other than \(p_i\). Let \(p_j\) (resp., \(p_{j'}\)) be either \(q\) (resp., \(q'\)) if it is a marked point, or a point on the component attached to \(C_i\) at \(q\) (resp., at \(q'\)). Note that such marked points \(p_j, p_{j'}\) exist since \((C, p_1, \ldots, p_n)\) is \(\psi\)-prestable. Then we have either \(f_i(q) \neq 0\) or \(f_i(q') \neq 0\) which implies that either \(\alpha_{ij} = f_i(p_j) \neq 0\) or
\[ \alpha_{ij} = f_i(p_{ij}) \neq 0 \] (note that \( f_i \) is constant on all components different from \( C_i \)), hence, we get \( \chi \)-semistability of \((C, p_1, \ldots, p_n)\). Conversely, assume there exists a component \( C_i \subset C \) with only two distinguished points and let us show that \((C, p_1, \ldots, p_n)\) is not \( \chi \)-semistable. Let \( p_i \in C_i \) be a distinguished point and let \( q \in C_i \) be another distinguished point (it exists by connectedness of \( C \)). Then our normalization \( f_i(p_{i+1}) = \alpha_{i,i+1} = 0 \) implies that \( f_i(q) = 0 \). Hence, \( f_i \) is zero on all other components of \( C \), and so \( \alpha_{ij} = 0 \) for all \( j \neq i \).

Let us take \( \chi = e_1 + \ldots + e_n \). Then the identification of the pull-back of \( \mathcal{O}(1) \) from the \( i \)th copy of \( \mathbb{P}^{n-3} \) with \( \psi_i \) is immediate since both line bundles become trivial on \( \tilde{U}_{0,n}[\psi] \) and correspond to the same action of \( G_m^n \) on the trivial line bundle (through the \( i \)th factor \( G_m \)). \( \square \)

**Remarks 5.3.2.**

1. Recall (see [9, Rem. 2.30]) that there is a natural morphism

\[ f_\psi : \overline{M}_{0,n} \to \overline{M}_{0,n}[\psi], \]

where \( \psi(C) \) is obtained from \( C \in \overline{M}_{0,n} \) by contracting all components without marked points. In fact, Proposition 5.3.1 gives another proof of the fact that the line bundle \( \psi = \psi_1 + \ldots + \psi_n \) on \( \overline{M}_{0,n} \) is semiample and that \( f_\psi \) is the corresponding contraction (see [9, Sec. 4.2.1]). The composed morphisms \( \overline{M}_{0,n} \to \overline{M}_{0,n}[\psi] \to \mathbb{P}^{n-3} \) given by the linear series \( \psi_i \) were first considered by Kapranov [15]. Note also that we have a commutative diagram

\[
\begin{array}{ccc}
\overline{M}_{0,n} & \xrightarrow{f_\psi} & \overline{M}_{0,n}[\psi] \\
\downarrow & & \downarrow \\
U_{0,n} & \xrightarrow{\psi} & U_{0,n}[\psi]
\end{array}
\]

where \( U_{0,n} \) is the stack of (reduced connected complete) curves of arithmetic genus 0 with \( n \) smooth marked points and the bottom horizontal arrow associates with a curve \( (C, p_1, \ldots, p_n) \) the corresponding \( A_\infty \)-structure on \( E_{0,n} = \text{Ext}^*(G,G) \) (where we identify \( U_{0,n}[\psi] \) with the moduli stack of \( A_\infty \)-structures on \( E_{0,n} \)).

2. If \( (C, p_1, \ldots, p_n) \) is a \( \psi \)-prestable curve then one has also another natural generator of the perfect derived category \( \text{Perf}(C) \), namely

\[ V = \mathcal{O}_C \oplus \bigoplus_{i=1}^n \mathcal{O}_C(p_i). \quad (5.3.1) \]

It is easy to check that \( H^1(C, \mathcal{O}_C(p_i - p_j)) = 0 \), so in fact, \( V \) is a tilting bundle. The coefficients \( \alpha_{ij} \) appear in the structure constants of the algebra \( \text{End}(V) \) and the equations among them correspond to the associativity equations. One can show that in this way one gets an identification of \( U_{0,n}[\psi] \) with an open substack in the moduli stack of \( k^{n+1} \)-algebra structures on \( \text{End}(V) \) (where the embedding \( k^{n+1} \subset \text{End}(V) \) comes from the decomposition (5.3.1)). This open substack is characterized by the condition that for all
$i \neq j$ the compositions

\[ \text{Hom}(O_C(p_i), O_C(p_j)) \xrightarrow{x \mapsto x \otimes 1} \text{Hom}(O_C(p_i), O_C(p_j)) \otimes \text{Hom}(O_C, O_C(p_i)) \xrightarrow{\mu} \text{Hom}(O_C, O_C(p_j)) \rightarrow H^0(O_C(p_j)/O_C) , \]

where $\mu$ comes from a $k^{n+1}$-algebra structure, are surjective.

APPENDIX: Formulas for $m_4$ and $m_5$.

Here we assume that the characteristic is not 2 or 3. We use the setup of Section 3.3, and in addition, use the constants introduced in Section 2.2. By Lemma 2.3.1, all of these constants are some universal polynomials of the generators (2.3.1).

The computation of $m_4$ is based on the following formulas:

\[ B_i Q(P_j) = 0, \quad B_i Q(P_i) = \left[ \frac{1}{t_i^j} \right], \quad Y_i Q(P_j) = 0, \quad Y_i Q(P_i) = e \left[ \frac{1}{t_i^j} \right], \]

\[ X_i Q \left( \left[ \frac{1}{t_i^j} \right] \right) = \left[ \frac{f_i}{t_i^j} \right] \quad \text{and} \quad X_j Q \left( \left[ \frac{1}{t_i^j} \right] \right) = \left[ \frac{f_j}{t_i^j} \right], \quad Q \left( \left[ \frac{1}{t_i^j} \right] \right) X_i = \left[ (f_i - \frac{\alpha_{ij}}{t_j^j}) \frac{1}{t_i^j} \right], \quad Q \left( \left[ \frac{1}{t_i^j} \right] \right) X_j = \left[ (f_j - \frac{\alpha_{ij}}{t_j^j}) \frac{1}{t_i^j} \right], \]

\[ A_i Q \left( \left[ \frac{1}{t_i^j} \right] \right) = e \cdot \left( f_i - \frac{1}{t_i^j} \right) 1_i + u[f_i, 1_i], \quad A_j Q \left( \left[ \frac{1}{t_i^j} \right] \right) = e \cdot \left( f_j - \frac{\alpha_{ij}}{t_j^j} \right) 1_j + u[f_i, 1_j], \]

\[ Q \left( \left[ \frac{1}{t_i^j} \right] \right) B_i = \left( f_i - \frac{1}{t_i^j} \right) \frac{1}{t_i^j} u^* + \left[ (f_i - \frac{1}{t_j^j}) \frac{1}{t_i^j} \right] e^*, \quad Q \left( \left[ \frac{1}{t_i^j} \right] \right) B_j = \left( f_j - \frac{\alpha_{ij}}{t_j^j} \right) \frac{1}{t_j^j} u^* + \left[ (f_j - \frac{\alpha_{ij}}{t_j^j}) \frac{1}{t_j^j} \right] e^*, \]

\[ B_i Q \left( e \left[ \frac{1}{t_i^j} \right] \right) = \left[ \frac{1}{t_i^j} \right], \quad Y_i Q \left( e \left[ \frac{1}{t_i^j} \right] \right) = e \left[ \frac{1}{t_i^j} \right], \quad B_i Q \left( e \left[ \frac{1}{t_i^j} \right] \right) = Y_i Q \left( e \left[ \frac{1}{t_i^j} \right] \right) = 0. \]

In the formulas below we omit commas between the arguments of $m_i$ for brevity. The indices denoted by different letters are assumed to be distinct. Let $P_i$ stand for either $A_i X_i$ or $Y_i A_i$. Then the nonzero $m_4$ products of the basis elements are:

\[ m_4(B_i Y_i A_i) = \sum_{j \neq i} \beta_{ij} X_j, \]

\[ m_4(B_i Y_i A_j B_j) = -\gamma_{ij} B_j, \]

\[ m_4(B_i Y_i A_i X_i) = \sum_{j \neq i} \beta_{ij} X_j, \]

\[ m_4(B_i Y_i A_j X_j) = -\gamma_{ij} X_j, \]

\[ m_4(B_i A_i X_i B_j) = -\gamma_{ij} B_j, \]

\[ m_4(B_i A_j X_i X_j) = -\gamma_{ij} X_j, \]

\[ m_4(A_j B_i Y_i A_i) = -\gamma_{ij} A_j, \]

\[ m_4(A_j B_i A_i X_j) = -\gamma_{ij} A_j, \]

\[ m_4(X_i B_i Y_i A_i) = \sum_{j \neq i} \beta_{ij} X_j, \]

\[ m_4(X_j B_i Y_i A_i) = -\gamma_{ij} X_j + \sum_{k \neq j} \alpha_{ij} \alpha_{jk} X_k, \]
\[ m_4(X_i B_i A_i X_i) = \sum_{j \neq i} \beta_{ij} X_j, \]
\[ m_4(X_j B_i A_i X_i) = -\gamma_{ij} X_j + \sum_{k \neq j} \alpha_{ij} \alpha_{jk} X_k, \]

Let \( Q_i \) stand for either \( B_i Y_i \) or \( X_i B_i \). Then the nonzero values of \( m_5 \) are:

\[ m_5(B_i Y_i Y_i A_i) = \sum_{j \neq i} \eta_{ij} X_j, \]
\[ m_5(B_i Y_i A_i X_i) = \sum_{j \neq i} \eta_{ij} X_j, \]
\[ m_5(B_i P_i Q_i) = \delta_{ji} B_j, \]
\[ m_5(B_i P_i Q_j) = \delta_{ij} B_j, \]
\[ m_5(B_i P_i X_i X_i) = \delta_{ii} X_i, \]
\[ m_5(B_i P_i X_j X_j) = \delta_{ij} X_j, \]
\[ m_5(Q_i A_i X_i X_i) = \delta_{ii} X_i, \]
\[ m_5(Q_i A_i P_i) = -\epsilon_{ij} B_j, \]
\[ m_5(Q_i P_i X_j) = -\epsilon_{ij} X_j, \]
\[ m_5(P_i B_i P_i) = m_5(A_i B_i P_i X_i) = -\delta_{ii} A_i, \]
\[ m_5(A_i B_i P_i) = -\delta_{ii} e_{\sigma_{ii}}, \]
\[ m_5(A_i Q_i P_i) = \epsilon_{ij} A_j, \]
\[ m_5(A_j X_j B_i P_i) = m_5(A_j B_i P_i X_j) = m_5(Y_j A_j B_i P_i) = -\delta_{ij} A_j, \]
\[ m_5(A_k X_j B_i P_i) = \alpha_{ij} \gamma_{jk} A_k, \]
\[ m_5(A_i X_j B_i P_i) = \alpha_{ij} \gamma_{ji} A_i, \]
\[ m_5(A_j B_i P_i B_j) = -\delta_{ij} e_{\sigma_{ij}}, \]
\[ m_5(X_i X_i B_i P_i) = m_5(X_i B_i Y_i Y_i A_i) = \sum_{j \neq i} \eta_{ij} X_j, \]
\[ m_5(X_j B_i P_i B_j) = \delta_{ij} B_j, \]
\[ m_5(X_i B_i Y_i A_i X_i) = \sum_{j \neq i} \eta_{ij} X_j + \delta_{ii} X_i, \]
\[ m_5(X_i B_i P_i B_i) = \delta_{ii} B_i, \]
\[ m_5(X_j B_i P_i B_i) = -\alpha_{ij} \gamma_{ji} B_i, \]
\[ m_5(X_j B_i P_i B_k) = -\alpha_{ij} \gamma_{jk} B_k, \]
\[ m_5(X_j B_i P_i X_j) = \delta_{ij} X_j, \]
\[ m_5(X_k B_i P_i X_j) = -\alpha_{ik} \gamma_{kj} X_j, \]
\[ m_5(X_j B_i Y_i A_i X_i) = \sum_{k \neq j} \beta_{ij} \alpha_{jk} X_k - \epsilon_{ij} X_j - \alpha_{ij} \gamma_{ji} X_i, \]
\[ m_5(X_j B_i A_i X_i X_i) = -\alpha_{ij} \gamma_{ji} X_i, \]
\[ m_5(X_j X_j B_i P_i) = \sum_{k \neq j} \alpha_{ij} \beta_{jk} X_k, \]
\[ m_5(X_j B_i Y_i A_i) = m_5(X_j X_i B_i P_i) = \sum_{k \neq j} \beta_{ij} \alpha_{jk} X_k - \epsilon_{ij} X_j, \]
\[ m_5(X_i X_j B_i P_i) = \sum_{l \neq i} \alpha_{ij} \alpha_{lj} X_l - \alpha_{ij} \gamma_{ji} X_i, \]
\[ m_5(X_k X_j B_i P_i) = \sum_{l \neq k} \alpha_{ij} \alpha_{jk} \alpha_{kl} X_l - \alpha_{ij} \gamma_{jk} X_k. \]

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