THE USE OF THE STATIONARY PHASE METHOD AS A MATHEMATICAL TOOL TO DETERMINE THE PATH OF OPTICAL BEAMS

Abstract. We use the stationary phase method to determine the path of optical beams which propagate through a dielectric block. In the presence of partial internal reflection, we recover the geometrical result obtained by using the Snell law. For total internal reflection, the stationary phase method overreaches the Snell law predicting the Goos-Hänchen shift.

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[11 pages, 2 figures]
I. INTRODUCTION

Optical beams propagating through dielectrics with dimensions greater than the wavelength of light can be described by rays obeying a set of geometrical rules. In this limit, Snell’s law is used to determine the relationship between the incidence and refraction angles and, consequently, the paths of optical beams.

In this paper, we discuss beam propagation through the dielectric block illustrated in Fig. 1 and, as the first step in our analysis (Sec. II), we calculate its geometrical path using Snell’s law.

The analogy between optics and quantum mechanics has been a matter of discussion in recent works. The possibility of linking Maxwell’s equations for photon propagation in the presence of dielectric blocks with the quantum-mechanical equations for the propagation of electrons in the presence of potential steps allows one to determine, in a simple and intuitive way, the reflection and transmission coefficients at the dielectric block interfaces. As the second step in our analysis (Sec. III), we obtain the reflection and transmission coefficients for s-polarized waves transmitted through the dielectric system of Fig. 1.

Once we obtain the transmission coefficient at the exit interface, we use the stationary phase method (SPM) to calculate the optical path by imposing the cancellation (in the oscillatory electric field integral) of sinusoids with rapidly varying phase. The calculation of the position of the maximum of the outgoing beam at the exit of the dielectric block, based on the SPM (Sec. IV), represents an alternative way to obtain the optical path, one that does not require a geometrical analysis. The SPM analysis only requires that we cancel the derivative of the outgoing beam phase. However, the use of the SPM as a mathematical tool to obtain the path of optical beams propagating into dielectric blocks is not simply a matter of taste. For total internal reflection, the SPM also predicts the Goos-Hänchen (GH) shift (see Fig. 2). This shows the importance of the SPM not only to recover Snell’s law but also to obtain a typical quantum-mechanical effect.

The SPM, illustrated in this paper for calculating the path of optical beams, is a mathematical tool easily extended to other fields of physics in which wave packets play an important role.

Finally, after discussing our conclusions, we extend our results to p-polarized waves, suggest how to amplify the GH shift by building a band of dielectric blocks, and propose further theoretical investigations.

II. THE OPTICAL PATH VIA SNELL’S LAW

Let us consider an incoming gaussian beam, with beam waist size $w_0$ and wavenumber $k$, which moves along the z-direction:

$$E_{in}(r) = E_0 e^{ikz} \frac{w_0^2}{w_0^2 + 2iz/k} \exp\left( -\frac{x^2 + y^2}{w_0^2 + 2iz/k} \right),$$

where

$$r = x e_x + y e_y + z e_z.$$  \hspace{1cm} (1)

The plane of incidence is chosen to be the yz-plane. The normals to the left/right and up/down sides of the dielectric block are respectively oriented along the direction of the unit vectors $e_z$ and $e_z^*$, see Fig. 1 (a). The unit vector $e_z$ forms an angle $\theta$ with $e_z$,

$$e_{\hat{y}} = e_y \cos \theta + e_z \sin \theta,$$

$$e_z = -e_y \sin \theta + e_z \cos \theta,$$  \hspace{1cm} (3)

and $e_{z^*}$ an angle $\pi/4$ with $e_z$,

$$e_{y^*} = (e_{\hat{y}} + e_z)/\sqrt{2},$$

$$e_{z^*} = (-e_{\hat{y}} + e_z)/\sqrt{2}.\hspace{1cm} \text{ (5)}$$

In the new coordinates systems, the components of the position vector $r$ are then related by

$$\begin{pmatrix} y^* \\ z^* \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \hat{y} \\ \hat{z} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix},$$

$$= R \left( \frac{\pi}{4} + \theta \right) \begin{pmatrix} y \\ z \end{pmatrix}$$

$$\equiv \begin{pmatrix} y^* \\ z^* \end{pmatrix}$$  \hspace{1cm} (7)
where \( R \) represents the rotation matrix. Observe that \( x = \tilde{x} = x_\ast \). The new coordinates systems, in this Section, will be used to calculate the geometrical path by the Snell law and in the following section to determine the reflection and transmission coefficients of the optical beam.

In order to determine the optical path by the Snell law, we first demonstrate that the outgoing beam is parallel to the incoming one, then we calculate the exit point \( P_{\text{right}} \), and finally we obtain the distance \( d \) between the incoming and outgoing beams, see Fig. 1(a).

When an optical beam falls onto a boundary between two homogeneous media with different refractive indices, it is split into two beams. The refracted (transmitted) beam propagates into the second medium and the reflected beam propagates back into the first medium. Snell’s law states that the ratio of the sines of the angles of incidence and refraction is equal to the reciprocal of the ratio of the refraction indices of the media in which the beams propagate. Thus, for the left air/dielectric boundary we have

\[
\sin \theta = n \sin \psi, \tag{8}
\]

where \( n \) is the index of refraction for the dielectric material, \( \theta \) and \( \psi \) are the angles that the incoming and refracted beams form with \( e_z \). In this case, the second medium is optically denser than the first and consequently the refracted angle is a real quantity for all incident angles. The beam propagates into the dielectric block at an angle \( \psi \) with the \( \tilde{z} \)-axis and an angle \( \pi/4 + \psi \) with the \( z_\ast \)-axis, see Fig. 1(a). Due to the fact that the triangles \( AP_{\text{left}} P_{\text{down}} \) and \( CP_{\text{right}} P_{\text{up}} \) are similar,

\[
A P_{\text{left}} P_{\text{down}} = C P_{\text{right}} P_{\text{up}} = \frac{\pi}{4} - \psi \equiv \alpha. \tag{9}
\]

Consequently, the optical beam forms an angle \( \psi \) with the normal to the right side of the dielectric block. By using the Snell law, we then find that the outgoing beam forms an angle \( \theta \) with \( e_z \). This implies that the outgoing beam is parallel to the incoming one.

Let us not determine the point in which the outgoing beam leaves the block, i.e. \( P_{\text{right}} \). To do it, the better coordinates system choice is represented by the \( \{ e_y, e_z \} \) system. Without loss of generality, we can choose as origin of the coordinates system the incident point on the left (air/dielectric) interface:

\[
P_{\text{left}} = \{ 0, 0 \}. \tag{10}
\]

By simple geometrical considerations, we immediately get

\[
P_{\text{down}} = P_{\text{left}} A \sin \frac{\pi}{4} \left\{ \frac{1}{\tan \alpha}, 1 \right\} = \frac{a}{\sqrt{2}} \left\{ \tan \left( \frac{\pi}{4} + \psi \right), 1 \right\} \tag{11}
\]

and

\[
P_{\text{up}} = P_{\text{down}} + \overline{AB} \sin \frac{\pi}{4} \left\{ \frac{1}{\tan \alpha}, -1 \right\} = \frac{1}{\sqrt{2}} \left\{ (a + b) \tan \left( \frac{\pi}{4} + \psi \right), a - b \right\}. \tag{12}
\]

To obtain the coordinates of \( P_{\text{right}} \) we need to calculate the intersection between the following straight lines \( r(P_{\text{up}} P_{\text{right}}) \) and \( r(P_{\text{right}} C) \),

\[
r \left( P_{\text{up}} P_{\text{right}} \right) \cap r \left( P_{\text{right}} C \right).
\]

In the \( \{ e_y, e_z \} \) system, the straight lines are represented by

\[
r \left( P_{\text{up}} P_{\text{right}} \right) : \quad z_\ast - \frac{a - b}{\sqrt{2}} = \tan \left( \frac{\pi}{4} - \psi \right) \left[ y_\ast - \frac{b + a}{\sqrt{2}} \tan \left( \frac{\pi}{4} + \psi \right) \right], \tag{13}
\]

\[
r \left( P_{\text{right}} C \right) : \quad z_\ast - \frac{a - b}{\sqrt{2}} = - \left[ y_\ast - \left( \frac{b - a}{\sqrt{2}} + c \sqrt{2} \right) \right]. \tag{14}
\]

After simple algebraic manipulations, the previous equation can be simplified to

\[
r \left( P_{\text{up}} P_{\text{right}} \right) : \quad z_\ast = \tan \left( \frac{\pi}{4} - \psi \right) y_\ast - b \sqrt{2}, \tag{15}
\]

\[
r \left( P_{\text{right}} C \right) : \quad z_\ast = - y_\ast + c \sqrt{2}. \tag{16}
\]
Finally, $P_{\text{right}}$ can be determined by solving the system (15) and (16),

$$P_{\text{right}} = \left\{ \frac{(b+c)\sqrt{2}}{1+\tan\left(\frac{\pi}{4}-\psi\right)}, \frac{c\tan\left(\frac{\pi}{4}-\psi\right)-b}{1+\tan\left(\frac{\pi}{4}-\psi\right)} \right\}. \quad (17)$$

Once known $P_{\text{right}}$, observing that

$$r(SP_{\text{left}}) : z_* = \tan\left(\frac{\pi}{4} - \theta\right) y_* \quad (18)$$

we can immediately calculate the shift between the incoming and outgoing beams,

$$d[P_{\text{right}}, r(SP_{\text{left}})] = \left[ \tan\left(\frac{\pi}{4} - \theta\right) y_* (P_{\text{right}}) - z_* (P_{\text{right}}) \right] / \sqrt{1 + \tan^2\left(\frac{\pi}{4} - \theta\right)} = \left[ b + (b+c) \frac{\sin \theta}{\sqrt{n^2 - \sin^2 \theta}} \right] \cos \theta - c \sin \theta. \quad (19)$$

Before concluding this section, we note that, depending on the incidence angle $\theta$, the internal reflections can be partial or total. Let us explain this in detail by calculating the critical angle which characterizes the limit angle between partial and total reflection. As observed at the begin of this section, at the first air/dielectric interface the second medium ($n > 1$) is optically denser than the first one ($n = 1$) and we always find a refracted beam which moves into the dielectric block forming a real angle $\psi = \arcsin(\sin \theta / n)$ with the $\tilde{z}$-axis. At the second interface, $P_{\text{down}}$, we have total internal reflection when

$$n \sin \left(\frac{\pi}{4} + \psi\right) > 1. \quad (20)$$

It is important to note here that for $\psi > \pi/4$ the refracted beam cannot reach the second interface. This represents an additional constraint to be considered in our discussion. By adding this constraint to Eq. (20), we obtain the condition for total internal reflection,

$$\arcsin\left(\frac{1}{n}\right) - \frac{\pi}{4} < \psi < \frac{\pi}{4}. \quad (21)$$

In terms of the incidence angle $\theta$, the previous condition becomes

$$\arcsin\left(\frac{1 - \sqrt{n^2 - 1}}{\sqrt{2}}\right) < \theta < \arcsin\left(\frac{n}{\sqrt{2}}\right). \quad (22)$$

In Fig. 2(a), we plot the critical angle

$$\theta_c = \arcsin\left(\frac{1 - \sqrt{n^2 - 1}}{\sqrt{2}}\right) \quad (23)$$

as a function of the refractive index $n$. This curve separates the partial and total reflection zones. We conclude this section, by observing that for

$$\frac{1 - \sqrt{n^2 - 1}}{\sqrt{2}} \leq -1,$$

which implies

$$n \geq \sqrt{4 + 2\sqrt{2}}, \quad (24)$$

we always find total internal reflection, see Fig. 2(a).
III. MAXWELL EQUATIONS AND TRANSMISSION COEFFICIENT

In this section, by using the Maxwell equations [1][2] we calculate the transmission and reflection coefficient at each interface. The phase of the outgoing beam will be then used to calculate by the stationary phase method [21–23] the path of the optical beam.

The plane wave solution of
\[
\left[\partial_{xx} + \partial_{yy} + \partial_{zz} - \frac{\partial_t}{c^2}\right] E(r, t) = 0 \tag{25}
\]
is given by
\[
\exp\left[i\left(k_xx + k_yy + k_zz - \omega t\right)\right], \tag{26}
\]
providing that
\[
k = \sqrt{k_x^2 + k_y^2 + k_z^2} = \omega/c.
\]

Obviously a convolution of these plane waves is also a solution of Eq.(25). Many lasers emit beams that approximate a gaussian profile [18, 19, 30]
\[
E(r, t) = E_0 \frac{w_0^2}{4\pi} \int dk_xdk_y \exp\left[-\left(k_x^2 + k_y^2\right) w_0^2/4\right] \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)], \tag{27}
\]
where \(w_0\) is the beam waist size. Observe that for \(\sqrt{k_x^2 + k_y^2} \ll k\) (paraxial approximation) the previous integral can be calculated analytically and the integration leads to Eq.(1).

Due to the fact that the first air/dielectric discontinuity is along the \(\hat{z}\)-axis, it is convenient to rewrite the Maxwell equations by using the coordinate system \((x, \hat{y}, \hat{z})\),

\[
\left[\partial_{xx} + \partial_{\hat{y}\hat{y}} + \partial_{\hat{z}\hat{z}} - n^2(\hat{z}) \frac{\partial_t}{c^2}\right] E_{\text{left}}(r, t) = 0, \text{ with } n(\hat{z}) = \begin{cases} 1 \text{ for } \hat{z} < 0, \\ n \text{ for } \hat{z} > 0. \end{cases} \tag{28}
\]

The plane wave solution is now given by
\[
\exp\left[i k_{\hat{y}} \hat{y} - \omega t\right] \times \begin{cases} \exp[i k_{\hat{z}} \hat{z}] + R_{\text{left}} \exp[-i k_{\hat{z}} \hat{z}] & \text{for } \hat{z} < 0, \\ T_{\text{left}} \exp[i q_{\hat{z}} \hat{z}] & \text{for } \hat{z} > 0, \end{cases} \tag{29}
\]
where
\[
\begin{pmatrix} k_{\hat{y}} \\ k_{\hat{z}} \end{pmatrix} = R[\theta] \begin{pmatrix} k_y \\ k_z \end{pmatrix} \quad \text{and} \quad \{ q_{\hat{y}}, q_{\hat{z}} \} = \{ k_{\hat{y}}, \sqrt{n^2k^2 - k_{\hat{z}}^2 - k_{\hat{y}}^2} \}. \]

Observe that the \(\hat{y}\)-component of the wave number does not change because the discontinuity is along the \(\hat{z}\)-axis [31]. By matching the function (29) imposing the continuity of \(E_{\text{left}}\) and \(\partial\hat{z} E_{\text{left}}\) at the point where the refractive index is discontinuous, i.e. \(\hat{z} = 0\), we find
\[
R_{\text{left}} = \frac{k_{\hat{z}} - q_{\hat{z}}}{k_{\hat{z}} + q_{\hat{z}}} \quad \text{and} \quad T_{\text{left}} = \frac{2k_{\hat{z}}}{k_{\hat{z}} + q_{\hat{z}}} \tag{30}
\]

At the second interface it is convenient to use the coordinate system \((x, y_s, z_s)\) and solve the Maxwell equation
\[
\left[\partial_{xx} + \partial_{yy} + \partial_{zz} - n^2(z_s) \frac{\partial_t}{c^2}\right] E_{\text{down}}(r, t) = 0, \text{ with } n(z_s) = \begin{cases} n \text{ for } z_s < a/\sqrt{2}, \\ 1 \text{ for } z_s > a/\sqrt{2}. \end{cases} \tag{31}
\]

The plane wave solution is
\[
\exp\left[i(k_xx + q_y, y_s - \omega t)\right] \times \begin{cases} \exp[i q_z, z_s] + R_{\text{down}} \exp[-i q_z, z_s] & \text{for } z_s < a/\sqrt{2}, \\ T_{\text{down}} \exp[i k_z, z_s] & \text{for } z_s > a/\sqrt{2}, \end{cases} \tag{32}
\]
where
\[
\begin{pmatrix} q_y \\ q_z \end{pmatrix} = R[\pi] \begin{pmatrix} k_{\hat{y}} \\ q_{\hat{z}} \end{pmatrix} \quad \text{and} \quad \{ k_{y_s}, k_{z_s} \} = \{ q_y, \sqrt{k^2 - k_{\hat{z}}^2 - q_{\hat{y}}^2} \}. \]
As happens for the first interface, the \( y \)-component of the wave number is not modified because the discontinuity of the second interface is along the \( z \)-axis. By matching the function \( \tilde{z} = a/\sqrt{2} \), we find

\[
R_{\text{down}} = \frac{q_{z_2} - k_{z_2}}{q_{z_2} + k_{z_2}} \exp \left[ 2 i q_{z_2} \frac{a}{\sqrt{2}} \right] \quad \text{and} \quad T_{\text{down}} = \frac{2 q_{z_2}}{q_{z_2} + k_{z_2}} \exp \left[ i (q_{z_2} - k_{z_2}) \frac{a}{\sqrt{2}} \right]. \tag{33}
\]

For the up interface, we can use the result obtained for the down interface. By replacing

\[
(k_{z_2}, q_{z_2}) \to - (k_{z_2}, q_{z_2}) \quad \text{and} \quad \frac{a}{\sqrt{2}} \to \frac{a - b}{\sqrt{2}},
\]

we find the reflection and transmission coefficients for the up interface,

\[
R_{\text{up}} = \frac{q_{z_2} - k_{z_2}}{q_{z_2} + k_{z_2}} \exp \left[ 2 i q_{z_2} \frac{b - a}{\sqrt{2}} \right] \quad \text{and} \quad T_{\text{up}} = \frac{2 q_{z_2}}{q_{z_2} + k_{z_2}} \exp \left[ i (q_{z_2} - k_{z_2}) \frac{b - a}{\sqrt{2}} \right]. \tag{34}
\]

In a similar way, the reflection and transmission coefficients for the right interface can be directly obtained from the coefficients calculated for the left interface. By replacing

\[k_{z_2} \leftrightarrow q_{z_2}\]

and observing that the discontinuity is now located at \( \tilde{z} = c \), we obtain

\[
R_{\text{right}} = \frac{q_{z_2} - k_{z_2}}{q_{z_2} + k_{z_2}} \exp \left[ 2 i q_{z_2} c \right] \quad \text{and} \quad T_{\text{right}} = \frac{2 q_{z_2}}{q_{z_2} + k_{z_2}} \exp \left[ i (q_{z_2} - k_{z_2}) c \right]. \tag{35}
\]

Finally, the outgoing transmission coefficient is

\[
T_{\text{out}} = T_{\text{left}} R_{\text{down}} R_{\text{up}} T_{\text{right}} = \frac{4 k_{z_2} q_{z_2}}{(q_{z_2} + k_{z_2})^2} \left( \frac{q_{z_2} - k_{z_2}}{q_{z_2} + k_{z_2}} \right)^2 \exp \left[ i q_{z_2} b \sqrt{2} + i (q_{z_2} - k_{z_2}) c \right]. \tag{36}
\]

The spatial phases of the optical beam in the different regions are

- **in**
  \[
k_x x + k_y y + k_z z = k_x x + k_y y + k_z \tilde{z},
\]
- **left \to down**
  \[
k_x x + k_y y + q_{z_2} \tilde{z} = k_x x + q_y y + q_{z_2} z,
\]
- **down \to up**
  \[
k_x x + q_y y + q_{z_2} z = k_x x + q_y y + q_{z_2} z,
\]
- **up \to right**
  \[
k_x x + k_y y + q_{z_2} \tilde{z} = k_x x + k_y y + k_z \tilde{z},
\]
- **out**
  \[
k_x x + k_y y + q_{z_2} \tilde{z} = k_x x + k_y y + k_z z.
\]

The outgoing beam, which as expected is parallel to the incoming one, is then given by \( \text{Eq. } 18 \).

\[
E_{\text{out}}(r, t) = E_0 \frac{w_0^2}{4 \pi} \int dk_x dk_y \frac{4 k_{z_2} q_{z_2}}{(q_{z_2} + k_{z_2})^2} \left( \frac{q_{z_2} - k_{z_2}}{q_{z_2} + k_{z_2}} \right)^2 \exp \left[ - \left( k_{z_2}^2 + k_y^2 \right) w_0^2 / 4 \right] \times \exp \left\{ i (q_{z_2} b \sqrt{2} + (q_{z_2} - k_{z_2}) c + \mathbf{k} \cdot \mathbf{r} - \omega t) \right\}.
\tag{37}
\]

In the next section, by using the SPM, we calculate the position of the maximum of the outgoing beam and consequently the position of the optical beam at the exit of our dielectric system. The calculation based on the SPM thus represents an alternative way (with respect to the standard Snell law) to obtain the optical path. More important, the SPM calculation also allows one to obtain the GH-shift.

**IV. THE OPTICAL PATH VIA THE STATIONARY PHASE METHOD**

The SPM is a basic principle of asymptotic analysis which applies to oscillatory integrals. The main idea of the SPM relies on the cancelation of sinusoids with rapidly varying phase. To illustrate this principle let us consider the incoming beam given in Eq. (27). In order to maximize the integral we impose that

\[
\left[ \frac{\partial}{\partial k_x} (k \cdot r - \omega t) \right]_{(0,0)} = \left[ \frac{\partial}{\partial k_y} (k \cdot r - \omega t) \right]_{(0,0)} = 0.
\]

The derivatives have to be calculated at the maximum value of the convolution function. For the incoming optical beam of Eq. (27), the convolution function is a gaussian distribution, consequently, the maximum of the incoming beam is located at

\[
x = y = 0.
\]

This result, obtained without any integration, is confirmed by Eq. (1).
IV.A PARTIAL INTERNAL REFLECTION

As discussed in section II, for $\theta < \theta_c$, we get partial internal reflection. In this case, the outgoing optical beam has, with respect to the incoming beam, an additional phase given by

$$\phi = q_z, b \sqrt{2} + (q_z - k_z) c. \quad (38)$$

By using the SPM, in order to maximize the outgoing integral, we have now to impose the following constraints

$$\left[ \frac{\partial}{\partial k_x} (\phi + k \cdot r - \omega t) \right]_{(0,0)} = 0,$$

Observing that

$$\frac{\partial q_z}{\partial k_{x,y}} = -k_y \frac{q_z}{q \sqrt{2}} - k_y \frac{q_z}{q \sqrt{2}} \left[ \cos \theta \frac{\partial k_y}{\partial k_{x,y}} + \sin \theta \frac{\partial k_z}{\partial k_{x,y}} \right],$$

$$\frac{\partial q_z}{\partial k_{x,y}} = -k_y \frac{q_z}{q \sqrt{2}} \left[ \cos \theta \frac{\partial k_y}{\partial k_{x,y}} + \sin \theta \frac{\partial k_z}{\partial k_{x,y}} \right], \quad (39)$$

we immediately find

$$x = -\left[ \frac{\partial \phi}{\partial k_x} \right]_{(0,0)} = 0 \quad (40)$$

and

$$y = -\left[ \frac{\partial \phi}{\partial k_y} \right]_{(0,0)} =$$

$$= b \left( \frac{\sin \theta}{\sqrt{n^2 - \sin^2 \theta}} + 1 \right) \cos \theta + c \frac{\sin \theta}{\sqrt{n^2 - \sin^2 \theta}} \cos \theta - c \sin \theta$$

$$= \left[ b + (b + c) \frac{\sin \theta}{\sqrt{n^2 - \sin^2 \theta}} \right] \cos \theta - c \sin \theta. \quad (41)$$

Thus, the SPM, recovering the result (19), represents an alternative way to obtain the geometrical path of optical beams. For partial internal reflection, see Fig. 1 (b), the phase (38) is the only phase which contributes to the SPM calculation, so this shift in the $y$-coordinate is the real shift seen in an optical experiment. As we shall discuss in the next subsection an additional phase appears for total internal reflection ($\theta > \theta_c$) and an additional shift, which cannot be predicted by Snell’s law, has to be considered.

IV.B TOTAL INTERNAL REFLECTION

As anticipated in the previous section, for $\theta > \theta_c$ an additional phase comes from the double internal reflection coefficient

$$\left( \frac{q_z - k_z}{q_z + k_z} \right)^2. \quad (42)$$

Indeed, by observing that

$$k_{z,+}^2 = \left[ 1 - \left( \frac{\sin \theta + \sqrt{n^2 - \sin^2 \theta}}{\sqrt{2}} \right)^2 \right] k^2 + O [k_x, k_y]$$

$$= \left( 1 - \frac{n^2}{2} - \sin \theta \sqrt{n^2 - \sin^2 \theta} \right) k^2 + O [k_x, k_y], \quad (43)$$
and using the constraint (24), we have \( k_z^2 < 0 \). Consequently, the additional phase

\[
\varphi = -4 \arctan \left( \frac{|k_z^*|}{q_z} \right)
\]  

has to be included in our calculation. In order to calculate this new contribution, we start from

\[
\frac{\partial \varphi}{\partial k_{x,y}} = -4 \frac{q_z^2}{q_z^2 + |k_z|^2} \frac{\partial}{\partial k_{x,y}} \left[ \frac{|k_z^*|}{q_z} \right] = -\frac{4}{q_z^2 + |k_z|^2} \left( k_z \frac{\partial |k_z^*|}{\partial k_{x,y}} - |k_z^*| \frac{\partial q_z}{\partial k_{x,y}} \right)
\]

which, by using the relation

\[
q_z^2 + |k_z|^2 = (n^2 - 1)k^2 \Rightarrow \frac{\partial |k_z^*|}{\partial k_{x,y}} = -\frac{q_z}{|k_z^*|} \frac{\partial q_z}{\partial k_{x,y}},
\]

becomes

\[
\frac{\partial \varphi}{\partial k_{x,y}} = \frac{4}{|k_z^*|} \frac{\partial q_z}{\partial k_{x,y}}.
\]

Finally, the additional shift in the \( y \)-axis, also known as GH-shift, is

\[
\delta = -\left[ \frac{\partial \varphi}{\partial k_y} \right]_{(0,0)} = \frac{4 \cos \theta}{k} \frac{\sin \theta + \sqrt{n^2 - \sin^2 \theta}}{\sqrt{n^2 - 2 + 2 \sin \theta \sqrt{n^2 - \sin^2 \theta}}}.
\]

The SPM allows one to obtain the \( y \)-shift of the outgoing beam both for partial and total internal reflection,

\[
\Delta y = \begin{cases} 
  d & \text{for } \theta < \theta_c \text{ [partial internal reflection]}, \\
  d + \delta & \text{for } \theta > \theta_c \text{ [total internal reflection]}. 
\end{cases}
\]

In concluding this section, we recall that the GH shift (\( \delta \)) cannot be obtained by using Snell’s law. This additional shift is due to the presence of evanescent waves in the air zones close to the down and up interfaces and cannot be predicted from geometrical laws [32]. It is similar to the quantum delay time discussed in quantum mechanics [33–35].

V. CONCLUSIONS AND OUTLOOKS

In this article, we have shown the value of the SPM as a mathematical tool in determining the path of optical beams. Our analysis, which is done in section III for s-polarized waves, can be immediately extended to p-polarized waves [1, 18, 19, 36]

\[
T_{\text{out}}^{(p)} = \frac{4 n^2 k_x q_z}{(q_z + n^2 k_x)^2} \left( \frac{q_z - n^2 k_x}{q_z + n^2 k_x} \right)^2 \exp[i q_z b \sqrt{2 + (q_z - k_z c)}].
\]

For partial internal reflections, the SPM reproduces the geometrical result predicted by Snell’s law, i.e. \( \Delta y = d \). For total internal reflections, the SPM takes into account the GH shift, predicting \( \Delta y = d + \delta \). This additional shift is proportional to the wavelength of the incoming beam, see Eq. (47). The order of magnitude of the GH shift for a double total internal reflections is thus relatively small and this makes any experimental observations difficult. Note that red (\( \lambda \approx 0.633 \mu m \)) lasers, whose beam waist is \( w_0 = 1 \) mm, undergo a shift \( \delta \approx 10^{-4} w_0 \). Due to the fact that this shift depends on the number of internal reflections, it is clear that, to make this experimental measurement possible [37], we have to amplify this effect by considering, for example, a band of \( N \) dielectric blocks. In this case, the final GH shift will be given by \( N \delta \).
To guarantee two internal reflections in each block, we impose that the $z_*$-component of the exit point, $P_{\text{right}}$, be the same as the incoming one, $P_{\text{left}}$. This implies, see Eq.(17),

$$c \tan\left[\frac{\pi}{4} - \psi\right] = b$$

which after simple algebraic manipulations leads to

$$c = \frac{n^2 + 2 \sin \theta \sqrt{n^2 - \sin^2 \theta}}{n^2 - 2 \sin^2 \theta} b.$$  \hspace{1cm} (50)

In a forthcoming work, we intend to analytically solve the integral [37] and obtain the outgoing beam profile. This analytical study can be done by approximating the transmission coefficient in view of the result obtained in section IV.

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Figure 1: Geometric layout of the dielectric block used to determine the optical path by the Snell law. The $\tilde{z}$ and $z_*$ axes represent, respectively, the normal to the left/right and up/down interfaces. The origin is chosen at the point in which the incoming beam touches the first interface, $P_{\text{left}}$. In (a), the $\tilde{z}$ axis is obtained from the $z$ axis by a clockwise rotation of angle $\theta$. In (b), the rotation is anticlockwise ($\theta \rightarrow -\theta$).
Figure 2: In (a), the critical angle, $\theta_c$, is plotted as a function of the refractive index, $n$. The forbidden region represents incidence angles for which the refracted beam at the left interface cannot reach the down boundary. For $n > \sqrt{4 + 2\sqrt{2}}$, we always find total internal reflection. In (b), we can see that, for $\theta > \theta_c$, the geometrical path predicted by the Snell law suffers an additional shift, $\delta$. This additional shift, known as Goos-Hänchen shift, can be calculated by using the stationary phase method.