Non degeneracy for solutions of singularly perturbed nonlinear elliptic problems on symmetric Riemannian manifolds

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Abstract

Given a symmetric Riemannian manifold \((M, g)\), we show some results of genericity for non degenerate sign changing solutions of singularly perturbed nonlinear elliptic problems with respect to the parameters: the positive number \(\varepsilon\) and the symmetric metric \(g\). Using these results we obtain a lower bound on the number of non degenerate solutions which change sign exactly once.

Keywords: symmetric Riemannian manifolds, non degenerate sign changing solutions, singularly perturbed nonlinear elliptic problems

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1 Introduction

Let \((M, g)\) be a smooth connected compact Riemannian manifold of finite dimension \(n \geq 2\) embedded in \(\mathbb{R}^N\). Let us consider the problem

\[
\begin{aligned}
-\varepsilon^2 \Delta_g u + u &= |u|^{p-2}u & \quad & \text{in } M \\
 u &\in H^1_g(M) 
\end{aligned}
\]

Recently there have been some results on the influence of the topology (see [3, 12, 23]) and the geometry (see [3, 4, 16]) of \(M\) on the number of positive solutions of problem (1). This problem has similar features with the Neumann problem on a flat domain, which has been largely studied in literature (see [6, 8, 10, 11, 13, 19, 24, 25, 26]).

Concerning the sign changing solution the first result is contained in [15] where it is showed the existence of a solution with one positive peak and one negative peak when the scalar curvature of \((M, g)\) is non constant.

Moreover in [9] the authors give a multiplicity result for solutions which change sign exactly once when the Riemannian manifold is symmetric with respect to an orthogonal involution \(\tau\) using the equivariant Lusternik Schnirelmann category.

In this paper we are interested in studying the non degeneracy of changing sign solutions when the Riemannian manifold \((M, g)\) is symmetric.
We consider the problem
\[
\begin{cases}
-\varepsilon^2 \Delta_g u + u = |u|^{p-2} u & u \in H^1_g(M) \\
u(\tau x) = -u(x) & \forall x \in M
\end{cases}
\] (2)

where \( \tau : \mathbb{R}^N \to \mathbb{R}^N \) is an orthogonal linear transformation such that \( \tau \neq \text{Id}, \tau^2 = \text{Id} \) (Id being the identity on \( \mathbb{R}^N \)). Here the compact connected Riemannian manifold \((M, g)\) of dimension \( n \geq 2 \) is a regular submanifold of \( \mathbb{R}^N \) invariant with respect to \( \tau \). Let \( M_\tau = \{ x \in M : \tau x = x \} \). In the case \( M_\tau \neq \emptyset \) we assume that \( M_\tau \) is a regular submanifold of \( M \). In the following \( H^1_g = \{ u \in H^1_g(M) : \tau^* u = u \} \) where the linear operator \( \tau^* : H^1_g \to H^1_g \) is \( \tau^* u = -u(\tau(x)) \).

We obtain the following genericity results about the non degeneracy of changing sign solutions of (2) with respect to the parameters: the positive number \( \varepsilon \), and the symmetric metric \( g \) (i.e. \( g(\tau x) = g(x) \)).

**Theorem 1.** Given \( g_0 \in \mathcal{M}^k \), the set
\[
D = \left\{ (\varepsilon, h) \in (0,1) \times \mathcal{B}_\rho : \text{any } u \in H^1_{g_0} \text{ solution of } -\varepsilon^2 \Delta_{g_0+h} u + u = |u|^{p-2} u \text{ is not degenerate} \right\}
\]
is a residual subset of \((0,1) \times \mathcal{B}_\rho \).

**Remark 2.** By the previous result we prove that, given \( g_0 \in \mathcal{M}^k \) and \( \varepsilon_0 > 0 \), the set
\[
D^* = \left\{ h \in \mathcal{B}_\rho : \text{any } u \in H^1_{g_0} \text{ solution of } -\varepsilon^2 \Delta_{g_0+h} u + u = |u|^{p-2} u \text{ is not degenerate} \right\}
\]
is a residual subset of \( \mathcal{B}_\rho \).

In the following we set
\[
m^*_{\varepsilon_0, g_0} = \inf_{u \in N_{\varepsilon_0, g_0}} J_{\varepsilon_0, g_0}(u)
\]
where
\[
J_{\varepsilon_0, g_0} (u) = \frac{1}{\varepsilon_0^2} \int_M \left[ \frac{1}{2} |\nabla_g u|^2 + u^2 \right] - \frac{1}{p} |u|^p \ d\mu_{g_0}
\]
\[
N_{\varepsilon_0, g_0}^* = \{ u \in H^1_{g_0}(M) : J_{\varepsilon_0, g_0}^*(u) = 0 \}.
\]

**Theorem 3.** Given \( g_0 \in \mathcal{M}^k \) and \( \varepsilon_0 > 0 \). If there exists \( \mu > m^*_{\varepsilon_0, g_0} \) which is not a critical level of the functional \( J_{\varepsilon_0, g_0}^* \), then the set
\[
D^\dagger = \left\{ h \in \mathcal{B}_\rho : \text{any } u \in H^1_{g_0+h} \text{ solution of } -\varepsilon^2 \Delta_{g_0+h} u + u = |u|^{p-2} u \text{ with } J_{\varepsilon_0, g_0}^*(u) < \mu \text{ is not degenerate} \right\}
\]
is an open dense subset of \( \mathcal{B}_\rho \).

Here the set \( \mathcal{B}_\rho \) is the ball centered at 0 with radius \( \rho \) in the space \( \mathcal{H}^k \), where \( \rho \) is small enough and \( \mathcal{H}^k \) is the Banach space of all \( C^k, k \geq 3 \), symmetric covariants 2-tensor \( h(x) \) on \( M \) such that \( h(x) = h(\tau x) \) for \( x \in M \). \( \mathcal{M}^k \subset \mathcal{H}^k \) is the set of all \( C^k \) Riemannian metrics \( g \) on \( M \) such that \( g(x) = g(\tau x) \).

These results can be applied to obtain a lower bound for the number of non degenerate solutions of (2) which change sign exactly once when \( M \) is invariant with respect to the involution \( \tau = -\text{Id} \) and \( 0 \notin M \). We get the following propositions.
Proposition 4. Given \( g_0 \in \mathcal{M}^k \), the set
\[
\mathcal{A} = \left\{ (\varepsilon, h) \in (0, \hat{\varepsilon}) \times \mathcal{B}_\rho : \begin{array}{l}
\text{the equation } - \varepsilon^2 \Delta_{g_0 + \varepsilon h} u + u = |u|^{p-2}u \\
\text{has at least } P_t(M/G) \text{ pairs of non-degenerate solutions} \\
(u, -u) \in H^*_\rho \setminus \{0\} \text{ which change sign exactly once}
\end{array} \right\}
\]
is a residual subset of \((0, 1) \times \mathcal{B}_\rho\).

Proposition 5. Given \( g_0 \in \mathcal{M}^k \) and \( \varepsilon_0 > 0 \), if there exists \( \mu > m_{\varepsilon_0, g_0} \) not a critical value of \( J_{\varepsilon_0, g_0} \) in \( H^*_\rho \), then the set
\[
\mathcal{A} = \left\{ h \in \mathcal{B}_\rho : \begin{array}{l}
\text{the equation } - \varepsilon_0^2 \Delta_{g_0 + \varepsilon h} u + u = |u|^{p-2}u \\
\text{has at least } P_t(M/G) \text{ pairs of non-degenerate solutions} \\
(u, -u) \in H^*_\rho \setminus \{0\} \text{ which change sign exactly once}
\end{array} \right\}
\]
is an open dense subset of \( \mathcal{B}_\rho \).

Here \( P_t(M/G) \) is the Poincaré polynomial of the manifold \( M/G \), where \( G = \{1, -1\} \), and \( P_t(M/G) \) is when \( t = 1 \). By definition we have 
\[
P_t(M/G) = \sum_i \dim H_i(M/G) \cdot t^i
\]
where \( H_i(M/G) \) is the \( i \)-th homology group with coefficients in some field.

The paper is organized as follows. In Section 2 we recall some preliminary results. In Section 3 we sketch the proof of the results of genericity (theorems 1 and 2) using some technical lemmas proved in Section 4. In Section 5 we prove propositions 4 and 5.

2 Preliminaries

Given a connected \( n \) dimensional \( C^\infty \) compact manifold \( M \) without boundary endowed with a Riemannian metric \( g \), we define the functional spaces \( L^2_g, L^2_{\varepsilon, g}, H^2_g \) and \( H^1_{\varepsilon, g} \), for \( 2 \leq p < 2^* \) and a given \( \varepsilon \in (0, 1) \). The inner products on \( L^2_g \) and \( H^1_g \) are, respectively
\[
(u, v)_{L^2_g} = \int_M u \varepsilon^2 u v \, \mu_g, \quad
(u, v)_{H^1_g} = \int_M (\nabla u \nabla v + uv) \, \mu_g,
\]
while the inner products on \( L^2_{\varepsilon, g} \) and \( H^1_{\varepsilon, g} \) are, respectively
\[
(u, v)_{L^2_{\varepsilon, g}} = \frac{1}{\varepsilon^n} \int_M u \varepsilon^2 u v \, \mu_g, \quad
(u, v)_{H^1_{\varepsilon, g}} = \frac{1}{\varepsilon^n} \int_M (\varepsilon^2 \nabla u \nabla v + uv) \, \mu_g.
\]

Finally, the norms in \( L^p_{\varepsilon, g} \) and \( L^p_{\varepsilon, g} \) are
\[
\|u\|_{L^p_{\varepsilon, g}} = \int_M |u|^p \, \mu_g, \quad
\|u\|_{L^p_{\varepsilon, g}} = \frac{1}{\varepsilon^n} \int_M |u|^p \, \mu_g.
\]

We define also the space of symmetric \( L^p \) and \( H^1 \) functions as
\[
L^p_{\varepsilon} = \left\{ u \in L^p_g(M) : \tau^* u = u \right\}, \quad
H^1_{\varepsilon} = \left\{ u \in H^1_g(M) : \tau^* u = u \right\}
\]

As defined in the introduction, \( \mathcal{F}^k \) is the space of all \( C^k \) symmetric covariants 2-tensor \( h(x) \) on \( M \) such that \( h(x) = h(\tau x) \) for \( x \in M \). We define a
Moreover the components of $h$ with respect to local coordinates $(x_1, \ldots, x_n)$ on $V_\alpha$, we define

$$\|h\|_{k} = \sum_{\alpha \in L} \sum_{|\beta| \leq k} \sum_{i,j=1}^{n} \sup_{\psi_\alpha(V_\alpha)} \left| \frac{\partial^\beta h_{ij}}{\partial x_1^1 \cdots \partial x_n^n} \right|.$$  

The set $\mathcal{M}^k$ of all $C^k$ Riemannian metrics $g$ on $M$ such that $g(x) = g(\tau x)$ is an open set of $\mathcal{F}^k$.

Given $g_0 \in \mathcal{M}^k$ a symmetric Riemannian metric on $M$, we notice that there exists $\rho > 0$ (which does not depend on $\varepsilon$ if $0 < \varepsilon < 1$) such that, if $h \in \mathcal{B}_\rho$ the sets $H^1_{\varepsilon, g_0+h}$ and $H^1_{\varepsilon, g_0}$ are the same and the two norms $\| \cdot \|_{H^1_{\varepsilon, g_0+h}}$ and $\| \cdot \|_{H^1_{\varepsilon, g_0}}$ are equivalent. The same for $L^2_{\varepsilon, g_0+h}$ and $L^2_{\varepsilon, g_0}$. If $h \in \mathcal{B}_\rho$ and $\varepsilon \in (0,1)$ we set

$$E_\varepsilon^h(u,v) = \langle u,v \rangle_{H^1_{\varepsilon, g_0+h}} \quad \forall u,v \in H^1_{\varepsilon, g_0+h}$$

$$G_\varepsilon^h(u,v) = \langle u,v \rangle_{L^2_{\varepsilon, g_0+h}} \quad \forall u,v \in L^2_{\varepsilon, g_0+h}$$

$$N(\varepsilon,h)(u) = N_\varepsilon^h(u) = \|u\|_{L^p_{\varepsilon, g_0+h}}^p \quad \forall u \in L^p_{\varepsilon, g_0+h}$$

We introduce the map $A_\varepsilon^h$ which will be used in the following section.

**Remark 6.** If $h \in \mathcal{B}_\rho$ and $0 < \varepsilon < 1$, there exists a unique linear operator

$$A(\varepsilon,h) := A_\varepsilon^h : L^p_{g_0,\varepsilon} \rightarrow H^\tau_{g_0}$$

such that $E_\varepsilon^h(A_\varepsilon^h(u),v) = G_\varepsilon^h(u,v)$ for all $u \in L^p_{\varepsilon, g_0}$, $v \in H^\tau_{\varepsilon, g_0}$ with $2 \leq p < 2^*$. Moreover $E_\varepsilon^h(A_\varepsilon^h(u),v) = E_\varepsilon^h(u,A_\varepsilon^h(v))$ for $u,v \in H^1_{\varepsilon, g_0}$.

Also, we have that $A_\varepsilon^h = i_{\varepsilon, g_0}$ where $i_{\varepsilon, g_0}$ is the adjoint of the compact embedding $i_{\varepsilon, g_0} : H^\tau_{\varepsilon, g_0}(M) \rightarrow L^p_{g_0,\varepsilon}(M)$ with $2 \leq p < 2^*$. We recall that, if $h \in \mathcal{B}_\rho$ with $\rho$ small enough and $\varepsilon > 0$, then $H^1_{\varepsilon, g_0}$ and $H^1_{\varepsilon, g_0+h}$ (as well as $L^p_{\varepsilon, g_0}$ and $L^p_{\varepsilon, g_0+h}$) are the same as sets and the norms are equivalent. This is the reason why we can define $A_\varepsilon^h$ on $L^p_{g_0,\varepsilon}$ with values in $H^\tau_{g_0}$. We summarize some technical results contained in lemmas 2.1, 2.2 and 2.3 of [14].

**Lemma 7.** Let $g_0 \in \mathcal{M}^k$ and $\rho$ small enough. We have

1. The map $E : (0,1) \times \mathcal{B}_\rho \rightarrow \mathcal{L}(H^\tau_{g_0}, H^\tau_{g_0})$ defined by $E(\varepsilon,h) := E_\varepsilon^h$ is of class $C^1$ and it holds, for $u,v \in H^\tau_{g_0}(M)$ and $h \in \mathcal{F}^k$

$$E'(\varepsilon_0,h_0)[\varepsilon,h](u,v) = \frac{1}{2\varepsilon_0} \int_M \text{tr}(g^{-1}h)uv d\mu_g + \frac{1}{\varepsilon_0^2} \int_M \langle \nabla_g u, \nabla_g v \rangle b(h) d\mu_g$$

$$- \frac{n\varepsilon}{\varepsilon_0^{n+1}} \int_M uv d\mu_g - \frac{(n-2)\varepsilon}{\varepsilon_0^{n-1}} \int_M \langle \nabla_g u, \nabla_g v \rangle d\mu_g$$

with the 2-tensor $b(h) := \frac{1}{2}\text{tr}(g^{-1}h)g - g^{-1}hg^{-1}$

2. The map $G : (0,1) \times \mathcal{B}_\rho \rightarrow \mathcal{L}(L^p_{g_0,\varepsilon}, H^\tau_{g_0})$ defined by $G(\varepsilon,h) := G_\varepsilon^h$ is of class $C^1$ and it holds, for $u,v \in H^\tau_{g_0}(M)$ and $h \in \mathcal{F}^k$

$$G'(\varepsilon_0,h_0)[\varepsilon,h](u,v) = \frac{1}{2\varepsilon_0} \int_M \text{tr}(g^{-1}h)uv d\mu_g - \frac{n\varepsilon}{\varepsilon_0^{n+1}} \int_M uv d\mu_g$$
3. The map \( A : (0, 1) \times \mathcal{B}_\rho \to \mathcal{L}(H_{g_0}^r \times H_{g_0}^r, \mathbb{R}) \) is of class \( C^1 \) and for any \( u, v \in H_{g_0}^r(M) \) and \( h \in \mathcal{A}^k \), we have

\[
E'(\varepsilon_0, h_0)[\varepsilon, h](A_{g_0}^0(u), v) + E'_{\varepsilon_0}(A'(\varepsilon_0, h_0)[\varepsilon, h](u), v) = G'(\varepsilon, h_0)[\varepsilon, h](u, v)
\]

4. The map \( N : (0, 1) \times \mathcal{B}_\rho \to C^0(H_{g_0}^r, \mathbb{R}) \) defined by \( (\varepsilon, h) \mapsto N_\varepsilon^r(\cdot) \) is of class \( C^1 \) and it holds, for \( u \in H_{g_0}^r(M) \) and \( h \in \mathcal{A}^k \),

\[
N'(\varepsilon_0, h_0)[\varepsilon, h](u) = \frac{1}{2\pi^2} \int_M \text{tr}(g^{-1}h)|u|^p d\mu_g - \frac{ng}{\varepsilon^{n+1}} \int_M |u|^p d\mu_g
\]

In all these formulas \( g = g_0 + h_0 \) with \( h_0 \in \mathcal{B}_\rho \).

We recall two abstract results in transversality theory (see [20, 21, 22]) which will be fundamental for our results.

**Theorem 8.** Let \( X, Y, Z \) be three real Banach spaces and let \( U \subset X \), \( V \subset Y \) be two open subsets. Let \( F \) be a \( C^1 \) map from \( V \times U \) into \( Z \) such that

(i) For any \( y \in V \), \( F(y, \cdot) : x \to F(y, x) \) is a Fredholm map of index 0.

(ii) 0 is a regular value of \( F \), that is \( F'(y_0, x_0) : Y \times X \to Z \) is onto at any point \((y_0, x_0)\) such that \( F(y_0, x_0) = 0 \).

(iii) The map \( \pi \circ i : F^{-1}(0) \to Y \) is proper, where \( i \) is the canonical embedding form \( F^{-1}(0) \) into \( Y \times X \) and \( \pi \) is the projection from \( Y \times X \) onto \( Y \).

Then the set

\[
\theta = \{ y \in V : 0 \text{ is a regular value of } F(y, \cdot) \}
\]

is a dense open subset of \( V \).

**Theorem 9.** If \( F \) satisfies (i) and (ii) and

(iv) The map \( \pi \circ i : F^{-1}(0) \to Y \) is proper, that is \( F'^{-1}(0) = \bigcup_{s=1}^{\infty} C_s \) where \( C_s \) is a closed set and the restriction \( \pi \circ i|_{C_s} \) is proper for any \( s \)

then the set \( \theta \) is a residual subset of \( V \).

3. **Sketch of the proof of theorems [11] and [3].**

Given \( g_0 \in \mathcal{M}^k \), we introduce the map \( F : (0, 1) \times \mathcal{B}_\rho \times H_{g_0}^r \setminus \{0\} \to H_{g_0}^r \), defined by

\[
F(\varepsilon, h, u) = u - A_{\varepsilon}^1(|u|^{p-2}u).
\]

By the regularity of the map \( A \) (see 3 of Lemma [7]) we get the map \( F \) is of class \( C^1 \). We are going to apply transversality Theorem [8] to the map \( F \), in order to prove Theorem [11]. In this case we have \( X = H_{g_0}^r \), \( Y = \mathbb{R} \times \mathcal{A}^k \), \( Z = H_{g_0}^r \), \( U = H_{g_0}^r \setminus \{0\} \) and \( V = (0, 1) \times \mathcal{B}_\rho \subset \mathbb{R} \times \mathcal{A}^k \).

Assumptions (i) and (iv) are verified in Lemma [10] and in Lemma [11]. Using Lemma [12] we can verify (ii).
Indeed, we have to verify that for \((\varepsilon_0, h_0, u_0) \in V \times U\) such that \(F(\varepsilon_0, h_0, u_0) = 0\) and for any \(b \in H^\tau_{g_0}\), there exists \((\varepsilon, h, v) \subset \mathcal{S}^k \times H^\tau_{g_0}\) such that

\[
F'_u(\varepsilon_0, h_0, u_0)[v] + F'_{\varepsilon, h}(\varepsilon_0, h_0, u_0)[\varepsilon, h] = b.
\]

We recall that the operator

\[
\begin{align*}
F'_u(\varepsilon_0, h_0, u_0)[v] &= v - (p - 1)|\varepsilon|^{p-1} u_0 v \\
F'_{\varepsilon, h}(\varepsilon_0, h_0, u_0)[\varepsilon, h] &= \varepsilon - \mu - m - \varepsilon + \gamma - \mu_0
\end{align*}
\]

is selfadjoint in \(H^\tau_{\varepsilon_0, g_0 + h_0}\) and is a Fredholm operator of index 0. Then

\[
\text{Im } F'_u(\varepsilon_0, h_0, u_0) \oplus \ker F'_u(\varepsilon_0, h_0, u_0) = H^\tau_{g_0}.
\]

Let \(\{w_1, \ldots, w_\nu\}\) be a basis of \(\ker F'_u(\varepsilon_0, h_0, u_0)[v]\). We consider the linear functional \(f_i : \mathbb{R} \times \mathcal{S}^k \rightarrow \mathbb{R}\) defined by

\[
f_i(\varepsilon, h) = (F'_u(\varepsilon_0, h_0, u_0)[\varepsilon, h], w_i)_{H^\tau_{\varepsilon_0, g_0 + h_0}} \quad i = 1, \ldots, \nu.
\]

By Lemma 12 we get that the linear functionals \(f_i\) are independent. Therefore assumption (ii) is verified. At this point by transversality theorems we get that the set

\[
\{(\varepsilon, h) \in (0, 1) \times \mathcal{B}_\rho : \text{any } u \in H^\tau_{g_0} \setminus \{0\} \text{ solution of } -\varepsilon^2 \Delta_{g_0+h_0} u + u = |u|^{p-2} u \text{ is not degenerate}\}
\]

is a residual subset of \((0, 1) \times \mathcal{B}_\rho\). On the other hand we observe that 0 is a non degenerate solution of \(-\varepsilon^2 \Delta_{g_0+h_0} u + u = |u|^{p-2} u\), for any \(\varepsilon > 0\) and any \(h \in \mathcal{B}_\rho\). Then, we complete the proof of Theorem 1.

The proof of Remark 2 is analog to the proof of Theorem 1 using Corollary 13.

We now formulate the problem for Theorem 3. Given \(g_0 \in \mathcal{M}^k\) and \(\varepsilon_0 > 0\), we assume that there exists \(\mu > m^\tau_{\varepsilon_0, g_0}\) which is not a critical level for the functional \(J_{\varepsilon_0, g_0}\). It is clear that any \(\mu_0 \in (0, m^\tau_{\varepsilon_0, g_0})\) is not a critical value of \(J_{\varepsilon_0, g_0}\). We set

\[
\mathcal{D} = \{u \in H^\tau_{g_0} : \mu_0 < J_{\varepsilon_0, g_0}(u) < \mu\}.
\]

Now we introduce the \(C^1\) map \(H : \mathcal{B}_\rho \times \mathcal{S} \rightarrow H^\tau_{g_0}\) defined by

\[
H(h, u) = u - A_h^\tau([u]^{p-2} u) = F(\varepsilon_0, h, u).
\]

We are going to apply transversality theorem 9 to the map \(H\). In this case \(X = H^\tau_{g_0}\), \(Y = \mathcal{S}^k\), \(Z = H^1_{g_0}(M)\), \(U = \mathcal{D} \subset H^\tau_{g_0}\) and \(V = \mathcal{B}_\rho \subset \mathcal{S}^k\). It is easy to verify assumptions (i) and (ii) for the map \(H\) using Lemma 10, Lemma 12, and Corollary 13. Using Lemma 14 we can verify assumption (iii) so we are in position to apply Theorem 9 and to get the following statement: the set

\[
\{h \in \mathcal{B}_\rho : \text{any } u \in H^\tau_{g_0} \text{ solution of } -\varepsilon^2 \Delta_{g_0+h_0} u + u = |u|^{p-2} u \text{ such that } \mu_0 < J_{\varepsilon_0, g_0}(u) < \mu \text{ is not degenerate}\}
\]

is an open dense subset of \(\mathcal{B}_\rho\). Nevertheless 0 is a non degenerate solution of \(-\varepsilon^2 \Delta_{g_0+h} u + u = |u|^{p-2} u\) for any \(h\), and there is no solution \(u \neq 0\) with \(J_{\varepsilon_0, g_0}(u) < \mu_0\), so we get the claim.
4 Technical lemmas

In this section we show some lemmas in order to complete the proof of the results of genericity of non degenerate critical points.

**Lemma 10.** For any \((\varepsilon, h) \in (0, 1) \times \mathcal{B}_\rho\) the map \(u \mapsto F(\varepsilon, h, u)\) with \(u \in H^s_{g_0}\) is a Fredholm map of index zero.

**Proof.** By the definition of the map \(A\), we have

\[
F'(\varepsilon_0, h_0, u_0)[v] = v - (p - 1)A^{\varepsilon_0}_{h_0}[u_0]^{p-2}v = v - K v,
\]

where \(K(v) := (p - 1)i^{\varepsilon_0, g_0 + h_0}[u_0]^{p-2}v\). We will verify that \(K : H^s_{g_0} \to H^s_{g_0}\) is compact. Thus \(K : H^s_{g_0} \to H^s_{g_0}\) is compact and the claim follows. In fact, in \(u_n\) is bounded in \(H^s_{g_0}\), \(v_n\) is also bounded in \(H^s_{g_0}\) because \(h_0 \in \mathcal{B}_\rho\).

Then, up to subsequence, \(v_n\) converges to \(v\) in \(L^p_{g_0+g_0}\) for \(2 \leq t < 2^\ast\). So we have

\[
\int_M |u_0|^{p-2}(v_n - v)|^p d\mu_g \leq \left( \int_M |u_0|^p d\mu_g \right) \left( \int_M |v_n - v|^p d\mu_g \right) \to 0.
\]

Therefore \(i^{\varepsilon_0, g_0 + h_0}[u_0]^{p-2}(v_n - v) \to 0\) in \(H^s_{g_0+g_0}\) and also in \(H^s_{g_0}\). \(\square\)

**Lemma 11.** The map \(\pi \circ i : F^{-1}(0) \to \mathbb{R} \times \mathcal{J}^k\) is \(\sigma\)-proper. Here \(i\) is the canonical immersion from \(F^{-1}(0)\) into \(\mathbb{R} \times \mathcal{J}^k \times H^s_{g_0}\) and \(\pi\) is the projection from \(\mathbb{R} \times \mathcal{J}^k \times H^s_{g_0}\) into \(\mathbb{R} \times \mathcal{J}^k\).

**Proof.** Set \(I_{g_0}(u, R)\) the open ball in \(H^s_{g_0}\) centered in \(u\) with radius \(R\). We have

\[
F^{-1}(0) = \bigcup_{s=1}^{\infty} C_s \text{ where } C_s = \left\{ \left[ \frac{1}{s}, 1 - \frac{1}{s} \right] \times \mathcal{B}_{p-\frac{1}{s}} \times \left\{ I_{g_0}(0, s) \setminus I_{g_0}(0, \frac{1}{s}) \right\} \right\} \cap F^{-1}(0).
\]

We had to prove that \(\pi \circ i : C_s \to \mathbb{R} \times \mathcal{J}^k\) is proper, that is if \(h_n \to h_0\) in \(\mathcal{B}_{p-\frac{1}{s}}\), \(\varepsilon_n \to \varepsilon_0\) in \(\left[ \frac{1}{s}, 1 - \frac{1}{s} \right]\), \(u_n \in \left\{ I_{g_0}(0, s) \setminus I_{g_0}(0, \frac{1}{s}) \right\}\), and \(F(\varepsilon_n, h_n, u_n) = 0\), then, up to a subsequence, the sequence \(\{u_n\}\) converges to \(u_0 \in \left\{ I_{g_0}(0, s) \setminus I_{g_0}(0, \frac{1}{s}) \right\}\).

Since \(\{u_n\}\) is bounded in \(H^1_{g_0}\), then it is bounded in \(H^1_{g_0+g_0}\), since the two spaces are equivalent because \(h_0 \in \mathcal{B}_\rho\). Thus \(u_n\) converges, up to subsequence, to \(u_0\) in \(L^p_{g_0+g_0}\) and in \(L^p_{g_0+g_0+g_0}\) for \(2 \leq p < 2^\ast\), so \(|u_n|^{p-2}u_n \to |u_0|^{p-2}u_0\) in \(L^p_{g_0+g_0}\) and, by continuity of \(A^{\varepsilon_0}_{h_0}\),

\[
i^{\varepsilon_0, g_0 + h_0}[|u_n|^{p-2}u_n] = A^{\varepsilon_0}_{h_0}[|u_n|^{p-2}u_n] \to A^{\varepsilon_0}_{h_0}[|u_0|^{p-2}u_0] \text{ in } H^1_{g_0+g_0+g_0} = H^1_{g_0}. \tag{4}
\]

By the regularity of the map \(A\) we have, for some \(\theta \in (0, 1)\),

\[
\|A^{\varepsilon_n}_{h_n}([|u_n|^{p-2}u_n]) - A^{\varepsilon_0}_{h_0}([|u_0|^{p-2}u_0])\|_{L^p_{g_0+g_0}} \leq \|u_n||L^p_{g_0+g_0}\|L^p_{g_0+g_0} \|\varepsilon_n - \varepsilon_0\| + \|h_n - h_0\|_A \times \|A'(\varepsilon_0 + \theta(\varepsilon_0 - \varepsilon_0), h_0 + \theta(h_0 - h_0))\|_{L^p((0,1) \times \mathcal{B}_\rho, L^p_{g_0+g_0}, H^1_{g_0+g_0})}.
\tag{5}
\]

By (4) and (5) we get that \(A^{\varepsilon_n}_{h_n}([|u_n|^{p-2}u_n]) \to A^{\varepsilon_0}_{h_0}([|u_0|^{p-2}u_0])\) in \(H^1_{g_0}\). Since

\[
0 = F(\varepsilon_n, h_n, u_n) = u_n - A^{\varepsilon_n}_{h_n}([|u_n|^{p-2}u_n)
\]

we get the claim. \(\square\)
Lemma 12. For any \((\varepsilon, h_0, u_0) \in (0, 1) \times \mathcal{P} \times H^1_{g_0} \setminus \{0\}\) such that \(F(\varepsilon, h_0, u_0) = 0\), it holds that, if \(w \in \ker F'_u(\varepsilon, h_0, u_0)\) and
\[
\langle F'_u(\varepsilon, h_0, u_0)[\varepsilon, h], w \rangle_{H^1_{r_0, g_0 + h_0}} = 0 \quad \forall \varepsilon \in \mathbb{R}, \ h \in \mathcal{A}^k,
\]
then \(w = 0\).

**Proof.** Step 1. By the definition of \(F\) and Lemma \ref{lemma:previous} we get
\[
F'_u(\varepsilon, h_0, u_0)[\varepsilon, h] = -A'(\varepsilon, h_0)[\varepsilon, h](|u_0|^{p-2}u_0)
\]
and so
\[
\langle F'_u(\varepsilon, h_0, u_0)[\varepsilon, h], w \rangle_{H^1_{r_0, g_0 + h_0}} = -E_{h_0}(A'(\varepsilon, h_0)[\varepsilon, h](|u_0|^{p-2}u_0), w) = -G'(\varepsilon, h_0)[\varepsilon, h](|u_0|^{p-2}u_0, w) + E'(\varepsilon, h_0)[\varepsilon, h](u_0, w) =
\]
\[
-\frac{1}{2\varepsilon^2} \int_M tr(g^{-1}h)|u_0|^{p-2}u_0 w d\mu_g + \frac{\varepsilon}{\varepsilon_0 + 1} \int_M |u_0|^{p-2}u_0 w d\mu_g
\]
\[
+ \frac{1}{2\varepsilon^2} \int_M tr(g^{-1}h)u_0 w d\mu_g + \frac{1}{\varepsilon_0 + 1} \int_M \langle \nabla_g u_0, \nabla_g w \rangle_{h(h)} d\mu_g - \frac{\varepsilon_0}{\varepsilon_0 + 1} \int_M u_0 w d\mu_g - \frac{(n - 2)\varepsilon}{\varepsilon_0 + 1} \int_M \langle \nabla_g u_0, \nabla_g w \rangle_{h(h)} d\mu_g.
\]
Here we use that \(A_{h_0}^\varepsilon(|u_0|^{p-2}u_0) = u_0\). Moreover \(g = g_0 + h_0\) with \(h_0 \in \mathcal{P}\) and \(b(h) := \frac{4}{\varepsilon} tr(g^{-1}h)g - g^{-1}hg^{-1}\).

If we choose \(\varepsilon = 0\), by the previous equation we get
\[
\langle F'_u(\varepsilon, h_0, u_0)[0, h], w \rangle_{H^1_{r_0, g_0 + h_0}} = \frac{1}{2\varepsilon^2} \int_M tr(g^{-1}h)[u_0\ |u_0|^{p-2}u_0] w d\mu_g + \frac{1}{\varepsilon_0 + 1} \int_M \langle \nabla_g u_0, \nabla_g w \rangle_{h(h)} d\mu_g
\]
(7)

Step 2. We prove that, if \(\langle F'_u(\varepsilon, h_0, u_0)[0, h], w \rangle_{H^1_{r_0, g_0 + h_0}} = 0 \quad \forall h \in \mathcal{A}^k\), then it holds
\[
\langle \nabla_g u_0(\xi), \nabla_g w(\xi) \rangle_{h(h)} = 0 \quad \forall \xi \in M.
\]

Given \(\xi_0 \in M\), we consider the normal coordinates at \(\xi_0\) and we set
\[
\tilde{u}_0(x) = u_0(\exp_{\xi_0} x), \quad \tilde{w}(x) = w(\exp_{\xi_0} x), \quad \text{for} \ x \in B(0, R) \subset \mathbb{R}^n.
\]

We will prove that \(\frac{\partial \tilde{u}_0(0)}{\partial x_1} \frac{\partial \tilde{w}(0)}{\partial x_2} + \frac{\partial \tilde{u}_0(0)}{\partial x_2} \frac{\partial \tilde{w}(0)}{\partial x_1} = 0\). Analogously we can get
\[
\frac{\partial \tilde{u}_0(0)}{\partial x_j} \frac{\partial \tilde{w}(0)}{\partial x_k} + \frac{\partial \tilde{u}_0(0)}{\partial x_k} \frac{\partial \tilde{w}(0)}{\partial x_j} = 0.
\]

If \(\xi_0 \neq \tau \xi_0\), we assume that \(B_{\tau}(\xi_0, R) \cap B_{\tau}(\tau \xi_0, R) = 0\). Then choosing \(h \in \mathcal{A}^k\) vanishing outside \(B_{\tau}(\xi_0, R) \cup B_{\tau}(\tau \xi_0, R)\), by the fact that \(h(\tau x) = h(x)\) on \(M\), by \(\ref{lemma:previous}\) and by our assumption we have
\[
\frac{1}{\varepsilon_0} \int_{B(\xi_0, R)} tr(g^{-1}h)[u_0\ |u_0|^{p-2}u_0] w d\mu_g + \frac{1}{\varepsilon_0 + 1} \int_M \langle \nabla_g u_0, \nabla_g w \rangle_{h(h)} d\mu_g = 0.
\]
(8)
Using the normal coordinates at \( \xi_0 \) we choose \( h \) such that the matrix \( \{ h_{ij}(x) \}_{i,j=1,\ldots,n} \) has the form \( h_{12}(x) = h_{21}(x) \in C_0^\infty(B(0,R)) \) and \( h_{ij} \equiv 0 \) otherwise. By (9) we have

\[
0 = \int_{B(0,R)} |g(x)|^{1/2} h_{12}(x) \left\{ -\varepsilon_0^2 b_{12}(x) \left( \frac{\partial \bar{u}_0}{\partial x_1} \frac{\partial \bar{w}}{\partial x_2} + \frac{\partial \bar{u}_0}{\partial x_2} \frac{\partial \bar{w}}{\partial x_1} \right) + \sigma(x) \right\} \, dx
\]

where

\[
\sigma(x) = -\varepsilon_0^2 \sum_{r,s=1,\ldots,n} b_{rs} \left( \frac{\partial \bar{u}_0}{\partial x_r} \frac{\partial \bar{w}}{\partial x_s} \right) + 2g^{12} \left\{ \frac{\varepsilon_0^2}{2} \sum_{i,j=1}^n g^{ij} \left( \frac{\partial \bar{u}_0}{\partial x_i} \frac{\partial \bar{w}}{\partial x_j} \right) + [\bar{u}_0 - |\bar{u}_0|^{-2} \bar{u}_0] \bar{w} \right\}.
\]

Here \( b_{rs}(x) = \left( g^{-1}(x) \Gamma g^{-1}(x) \right)_{rs} \), where \( \Gamma_{12} = \Gamma_{21} = 0, \Gamma_{ij} = \Gamma_{ji} = 0 \) for \( i,j = 1,\ldots,n \), \( (i,j) \neq (1,2) \). Then \( b_{12}(0) = b_{21}(0) = 1, b_{rs}(0) = 0 \) otherwise, so \( \sigma(0) = 0 \). By (9), at this point we have

\[
-\varepsilon_0^2 b_{12}(x) \left( \frac{\partial \bar{u}_0}{\partial x_1} \frac{\partial \bar{w}}{\partial x_2} + \frac{\partial \bar{u}_0}{\partial x_2} \frac{\partial \bar{w}}{\partial x_1} \right) + \sigma(x) \text{ for } x \in B(0,R).
\]

Then

\[
\frac{\partial \bar{u}_0}{\partial x_1}(0) \frac{\partial \bar{w}}{\partial x_2}(0) + \frac{\partial \bar{u}_0}{\partial x_2}(0) \frac{\partial \bar{w}}{\partial x_1}(0) = 0.
\]

If \( \xi_0 = \tau \xi_0 \), we consider the equality (11) when \( h \in \mathcal{S}^k \) vanishes outside \( B_2(\xi_0, R) \), recalling that \( h(\tau(\xi)) = h(\xi) \) for \( \xi \in M \). Arguing as in the previous case, by (9) we get that

\[
\gamma(x) = \varepsilon_0^2 b_{12}(x) \left( \frac{\partial \bar{u}_0}{\partial x_1} \frac{\partial \bar{w}}{\partial x_2} + \frac{\partial \bar{u}_0}{\partial x_2} \frac{\partial \bar{w}}{\partial x_1} \right) + \sigma(x)
\]

is antisymmetric with respect to \( \bar{\tau} = \exp_{\xi_0}^{-1} \tau \exp_{\xi_0} \). Also, we have that \( \gamma \) is symmetric with respect to \( \bar{\tau} \), so \( \gamma(0) = 0 \), and, since \( b_{12}(0) = 1 \) and \( \sigma(0) = 0 \), we have again

\[
\frac{\partial \bar{u}_0}{\partial x_1}(0) \frac{\partial \bar{w}}{\partial x_2}(0) + \frac{\partial \bar{u}_0}{\partial x_2}(0) \frac{\partial \bar{w}}{\partial x_1}(0) = 0.
\]

Now we prove that \( \frac{\partial \bar{u}_0}{\partial x_i}(0) \frac{\partial \bar{w}}{\partial x_i}(0) = 0 \) for all \( i = 1,\ldots,n \).

If \( \xi_0 \neq \tau \xi_0 \), arguing as in the previous case we get (11). This time we choose the matrix \( \{ h_{ij}(x) \}_{i,j} \) such that \( h_{11} \in C_0^\infty(B(0,R)), h_{22} = -h_{11} \) and \( h_{ij} \equiv 0 \)
otherwise. Because \( \text{tr}(g^{-1}h) = (g^{11} - g^{22})h_{11} \), by (8), we get

\[
0 = \int_{B(0,R)} |g(x)|^{1/2} h_{11}(x) \left( [g^{11}(x) - g^{22}(x)] \times \varepsilon_0^2 \sum_{ij} g^{ij}(x) \frac{\partial \tilde{u}_0}{\partial x_i} \frac{\partial \tilde{w}}{\partial x_j} + \tilde{u}_0 \tilde{w} - |\tilde{u}_0|^{p-2} \tilde{u}_0 \tilde{w} \right) \\
- \varepsilon_0^2 [g^{11}(x)g^{12}(x) - g^{12}(x)g^{21}(x)] \left( \frac{\partial \tilde{u}_0}{\partial x_1} \frac{\partial \tilde{w}}{\partial x_2} + \frac{\partial \tilde{u}_0}{\partial x_2} \frac{\partial \tilde{w}}{\partial x_1} \right) (11)
\]

Then, recalling that \( \frac{\partial \tilde{u}_0}{\partial x_1}(0) = 0 \), and \( g^{ij}(0) = \delta_{ij} \), we have

\[
\left( [g^{11}(0)]^2 - [g^{21}(0)]^2 \right) \frac{\partial \tilde{u}_0(0)}{\partial x_1} \frac{\partial \tilde{w}(0)}{\partial x_2} + \left( [g^{12}(0)]^2 - [g^{22}(0)]^2 \right) \frac{\partial \tilde{u}_0(0)}{\partial x_2} \frac{\partial \tilde{w}(0)}{\partial x_1} = 0.
\]

So \( \frac{\partial \tilde{u}_0(0)}{\partial x_1} \frac{\partial \tilde{w}(0)}{\partial x_1} = \frac{\partial \tilde{u}_0(0)}{\partial x_2} \frac{\partial \tilde{w}(0)}{\partial x_2} \) and analogously \( \frac{\partial \tilde{u}_0(0)}{\partial x_i} \frac{\partial \tilde{w}(0)}{\partial x_i} = 0 \) for all \( i \neq j \) we get

\[
\frac{\partial \tilde{u}_0(0)}{\partial x_i} \frac{\partial \tilde{w}(0)}{\partial x_i} = 0 \text{ for all } i = 1, \ldots, n.
\]

If \( \xi_0 = \tau \xi_0 \), since \( h \) is symmetric with respect to \( \tau \), by (11) we get that

\[
\gamma(x) = [g^{11}(x) - g^{22}(x)] \left( \varepsilon_0^2 \sum_{ij} g^{ij}(x) \frac{\partial \tilde{u}_0}{\partial x_i} \frac{\partial \tilde{w}}{\partial x_j} + \tilde{u}_0 \tilde{w} - |\tilde{u}_0|^{p-2} \tilde{u}_0 \tilde{w} \right)
\]

\[- \varepsilon_0^2 [g^{11}(x)g^{12}(x) - g^{12}(x)g^{21}(x)] \left( \frac{\partial \tilde{u}_0}{\partial x_1} \frac{\partial \tilde{w}}{\partial x_2} + \frac{\partial \tilde{u}_0}{\partial x_2} \frac{\partial \tilde{w}}{\partial x_1} \right)
\]

\[- \varepsilon_0^2 \sum_{k=1}^n \left( [g^{1k}(x)]^2 - [g^{2k}(x)]^2 \right) \frac{\partial \tilde{u}_0}{\partial x_k} \frac{\partial \tilde{w}}{\partial x_k} \]

is antisymmetric with respect to \( \bar{\tau} = \exp_{\xi_0}^{-1} \tau \exp_{\xi_0} \). Concluding

\[
0 = \gamma(0) = (g^{11}(0))^2 \frac{\partial \tilde{u}_0(0)}{\partial x_1} \frac{\partial \tilde{w}(0)}{\partial x_1} - (g^{22}(0))^2 \frac{\partial \tilde{u}_0(0)}{\partial x_2} \frac{\partial \tilde{w}(0)}{\partial x_2}.
\]

At this point, arguing as above we have that

\[
\frac{\partial \tilde{u}_0(0)}{\partial x_i} \frac{\partial \tilde{w}(0)}{\partial x_i} = 0 \text{ for all } i = 1, \ldots, n.
\]

and the Step 2 is proved.

**Step 3.** Conclusion of the proof.
By Step 2, we have that, for any \( h \in \mathcal{H}^k \)
\[
0 = \langle F'_{\varepsilon,\mu}(\varepsilon_0, h_0, u_0) [0], w \rangle_{M_{\varepsilon_0,\mu_0} + h_0} = \frac{1}{2\varepsilon_0^2} \int_M \text{tr}(g^{-1}h)u_0 (1 - |u_0|^{p-2}) w \, d\mu_g.
\] (12)
Here \( g = g_0 + h_0 \). Moreover it holds
\[
-\varepsilon_0 \Delta_g w + w = (p-1)|u_0|^{p-2}w \quad w \in H_g^1
\] (13)
We choose \( h(x) = \alpha(x)g(x) \) for any \( \alpha \in C^\infty(M) \) with \( \alpha(x) = \alpha(x) \), so, by (12), the function \( u_0(1 - |u_0|^{p-2}) \) is antisymmetric with respect to the involution \( \tau \). Furthermore \( u_0(1 - |u_0|^{p-2}) \) is also symmetric, so
\[
u_0 (1 - |u_0|^{p-2}) w \equiv 0.
\] (14)
By contradiction we assume that \( w \) does not vanish indentically in \( M \). Since \( u_0 \in H_g^1 \setminus \{0\} \) we can split
\[
M = M^0 \cup M^1 \cup \tau M^1 \cup M^+ \cup \tau M^+
\]
where \( M^0 = \{ x \in M : u_0(x) = 0 \} \), \( M^1 = \{ x \in M : u_0(x) = 1 \} \), and \( M^+ = \{ x \in M : u_0(x) > 0, u_0(x) \neq 1 \} \). By (14) we have that \( w \equiv 0 \) on the open subset \( M^+ \cup \tau M^+ \). Also, we notice that \( M_0 \) and \( M_1 \) are disjoint sets because \( u_0 \) is a continuous funcn. By this, and by (13), we have that \( -\varepsilon_0 \Delta_g w + w = 0 \) on \( M_0 \) and \( w = 0 \) on \( \partial M_0 \). By the maximum principle, we conclude that \( w = 0 \) on \( M_0 \). So we have that, by (13), \( -\varepsilon_0 \Delta_g w + w = (p-1)w \) on the whole \( M \). On the other hand, by (11), we have that \( \mu_g(\{ x \in M : w(x) = 0 \}) = 0 \). A contradiction arises and that concludes the proof

With the same argument we can prove the following corollary.

**Corollary 13.** Given \( \varepsilon_0, \) for any \( (h_0, u_0) \in \mathcal{H}_p \times H_g^\tau \setminus \{0\} \) such that \( F(\varepsilon_0, h_0, u_0) = 0 \), if \( w \in \ker F'_u(\varepsilon_0, h_0, u_0) \) and
\[
\langle F'_u(\varepsilon_0, h_0, u_0) [h], w \rangle_{M_{\varepsilon_0,\mu_0} + h_0} = 0 \quad \forall h \in \mathcal{H}^k,
\]
then \( w = 0 \).

**Lemma 14.** Given \( g_0 \in \mathcal{M}^k \) and \( \varepsilon_0 \), if there exists a number \( \mu > m_{\varepsilon_0, g_0} \) not a critical level of the functional \( J_{\varepsilon_0, g_0} \), then, for \( \rho \) small enough, the map \( \pi \circ i : G^{-1}(0) \rightarrow \mathcal{H}^k \) is proper. Here \( G \) is defined in (4), \( i \) is the canonical embedding from \( G^{-1}(0) \) into \( \mathcal{H}^k \times H_g^\tau \) and \( \pi \) is the projection from \( \mathcal{H}^k \times H_g^\tau \) into \( \mathcal{H}^k \).

**Proof.** Let \( \{ u_n \} \subset \mathcal{D} \), where
\[
\mathcal{D} = \{ u \in H_g^\tau : \mu_0 < J_{\varepsilon_0, g_0}(u) < \mu \},
\]
and \( \mu_0 \) is an arbitrary number in \( (0, m_{\varepsilon_0, g_0}^\tau) \). It is sufficient to prove that if \( u_n \) satisfies \( -\varepsilon_0 \Delta_{g_0 + h_n} u_n + u_n = |u_n|^{p-2} u_n \) with \( h_n \rightarrow h_0 \in \mathcal{H}_p \), then the sequence \( \{ u_n \} \) has a subsequence convergent in \( \mathcal{D} \). First we show that \( \{ u_n \} \) is bounded in \( H_g^\tau \). Since the sets \( H_{g_0 + h_0}^\tau(M) \) and \( H_{g_0}^\tau(M) \) are the same in \( h \in \mathcal{H}_p \),
and the norms $\| \cdot \|_{H^1_{\theta,0}+h}$ and $\| \cdot \|_{H^1_{\theta,0}}$ are equivalent with equivalence constants $c_1$ and $c_2$ not depending on $h$, we have

$$c_1\|u\|_{H^1_{\theta,0}} \leq \|u\|_{H^1_{\theta,0}+h} \leq c_2\|u\|_{H^1_{\theta,0}}.$$ 

By this, and because $u_n \in N_{\theta,0}^{\tau}$, we have

$$\left(\frac{1}{2} - \frac{1}{p}\right) c_1^2\|u_n\|^2_{H^1_{\theta,0}} \leq \left(\frac{1}{2} - \frac{1}{p}\right) \|u_n\|^2_{H^1_{\theta,0}+h} = J_{\theta,0}^{\tau}(u_n) = \frac{1}{2}\|u_n\|^2_{H^1_{\theta,0}+h} - \frac{1}{p}\|u_n\|^p_{L^p_{\theta,0}+h} \leq J_{\theta,0}(u_n) + \epsilon h_n\|u_n\|_{H^1_{\theta,0}}^2 + \|u_n\|^p_{L^p_{\theta,0}} \leq \mu + \epsilon p\|u_n\|^2_{H^1_{\theta,0}} + \epsilon p\|u_n\|^p_{L^p_{\theta,0}}.$$ 

Moreover, since $\mu_0 < J_{\theta,0}(u_n) < \mu$ we get

$$\mu_0 < \left(\frac{1}{2} - \frac{1}{p}\right) \|u_n\|^p_{L^p_{\theta,0}} < \mu$$

by (15) and (16), if $\|u_n\|_{H^1_{\theta,0}} \to +\infty$ we get

$$\left(\frac{1}{2} - \frac{1}{p}\right) \|u_n\|^2_{H^1_{\theta,0}} \leq \mu + \epsilon p\|u_n\|^2_{H^1_{\theta,0}} + \epsilon p,$$

then, choosing $\rho$ small enough, we get the contradiction.

Since the sequence $\{u_n\}$ is bounded in $H^2_{\theta,0}$ and $H^2_{\theta,0}+h$, up to a subsequence $u_n \to u$ in $L^2_{\theta,0}+h_0(M)$ and $L^2_{\theta,0}+h_0(M)$ for $2 \leq t < 2^\star$. Then

$$i_{\epsilon,0}^{\tau} : (u_n|^{p-2}u_n) = A_{h_0}^{\tau}(|u_n|^{p-2}u_n) \to A_{h_0}^{\tau}(|u|^{p-2}u) \text{ in } H^2_{\theta,0}+h_0$$

Since the map $A$ is of class $C^1$ (see Lemma 7) we have, for some $\theta \in (0,1)$

$$\|A_{h_0}^{\tau}(|u_n|^{p-2}u_n) - A_{h_0}^{\tau}(|u_n|^{p-2}u_n)\|_{H^1_{\theta,0}} = \|A(\epsilon_0, h_0 + \theta(h_n - h_0)) [0, h_n - h_0] (|u_n|^{p-2}u_n)\|_{H^1_{\theta,0}} \leq \|u_n\|^{p-1}\|L_{\theta_{\epsilon,0}}\|_{L^p_{\theta_{\epsilon,0}}} \leq \sup_{0 \leq t \leq 1} \|A(t, \epsilon_0, h_0 + \theta(h_n - h_0))\|_{L^p_{\theta_{\epsilon,0}}}$$

By (17) and (18) we get $A_{h_0}^{\tau}(|u_n|^{p-2}u_n) \to A_{h_0}^{\tau}(|u|^{p-2}u) \text{ in } H^2_{\theta,0}$. Since $0 = u_n - A_{h_0}^{\tau}(|u_n|^{p-2}u_n)$ we get that $u_n$ converges to $u$ in $H^2_{\theta,0}$. Moreover $u - A_{h_0}^{\tau}(|u|^{p-2}u) = 0$. Since $\mu_0$ and $\mu$ are not critical values for $J_{\epsilon,0}(u)$, we have that $\mu_0 < J_{\epsilon,0}(u) < \mu$. Then $u \in \mathcal{D}$. \[ \square \]

5 An application

In this section we choose $\tau = -\text{Id}$ and the manifold $M$ invariant with respect to the involution $\tau = -\text{Id}$. We also assume $0 \notin M$, so $M_\tau = \emptyset$. Using the previous results of genericity for non degenerate sign changing solutions of problem (2), we obtain a lower bound on the number of non degenerate solutions which change
sign exactly once. This estimate is formulated also in [17]. In the cited paper this result is proved under an assumption on non degeneracy of critical points that we do not need.

We sketch the proof of propositions 4 and 5 showing how we use the results of genericity for non degeneracy of critical points to obtain the same estimate.

We recall that there exists a unique positive spherically symmetric function $U \in H^1(\mathbb{R}^n)$ such that \begin{equation*} \Delta U + U = U^{p-1} \end{equation*} in $\mathbb{R}^n$. Also, it is well known that for any $\epsilon > 0$, $U_\epsilon(x) := U(\frac{x}{\epsilon})$ is a solution of $-\epsilon^2 \Delta U_\epsilon + U_\epsilon = U_\epsilon^{p-1}$ in $\mathbb{R}^n$.

Let $g_0$ be in $\mathcal{M}_k$ and $h$ be in $\mathcal{B}_p$ for some $p > 0$. Let us define a smooth cut off real function $\chi_R$ such that $\chi_R(t) = 1$ if $0 \leq t \leq R/2$, $\chi_R(t) = 0$ if $t \geq R$ and $|\chi'(t)| < 2/R$. Fixed $q \in M$ and $\epsilon > 0$ we define on $M$ the function

\begin{equation*}
W_{q,\epsilon}^\tau(x) = \begin{cases}
U_\epsilon(\exp_q^{-1}(x))\chi_R(|\exp_q^{-1}(x)|) & \text{if } x \in B_g(q, R) \\
0 & \text{otherwise}
\end{cases}
\end{equation*}

where $B_g(q, R)$ is the geodesic ball of radius $R$ centered at $q$. We choose $R$ smaller than the injectivity radius of $M$ and such that $B_g(q, R) \cap B_g(-q, R) = \emptyset$. Here and in the following we set $g = g_0 + h$.

We can define a map $\Phi_{\epsilon, g} : M \to N_{\epsilon, g}^\tau$ as

\begin{equation*}
\Phi_{\epsilon, g}(q) = t \left( W_{q,\epsilon}^\tau \right) W_{q,\epsilon}^\tau - t \left( W_{-q,\epsilon}^\tau \right) W_{-q,\epsilon}^\tau.
\end{equation*}

Here

\begin{equation*}
[t \left( W_{q,\epsilon}^\tau \right)]^{p-2} = \frac{\int_M \epsilon^2 |\nabla_g W_{q,\epsilon}^\tau|^2 + |W_{q,\epsilon}^\tau|^2 d\mu_g}{\int |W_{q,\epsilon}^\tau|^p d\mu_g},
\end{equation*}

thus $t \left( W_{q,\epsilon}^\tau \right) W_{q,\epsilon}^\tau \in N_{\epsilon, g}$ and we have $\Phi_{\epsilon, g}(q) = -\Phi_{\epsilon, g}(-q)$. Now we can define

\begin{equation*}
\tilde{\Phi}_{\epsilon, g} : M/G \to N_{\epsilon, g}^\tau/\mathbb{Z}_2
\end{equation*}

\begin{equation*}
\tilde{\Phi}_{\epsilon, g}[q] = [\Phi_{\epsilon, g}(q)] = (\Phi_{\epsilon, g}(q), \Phi_{\epsilon, g}(-q))
\end{equation*}

where

\begin{equation*}
M/G = \{[q] = (q, -q) : q \in M \} \quad N_{\epsilon, g}^\tau/\mathbb{Z}_2 = \{(u, -u) : u \in N_{\epsilon, g}^\tau \}.
\end{equation*}

We set $\tilde{J}_{\epsilon, g}[u] = J_{\epsilon, g}(u)$. Obviously, $\tilde{J}_{\epsilon, g} : N_{\epsilon, g}^\tau/\mathbb{Z}_2 \to \mathbb{R}$.

Lemma 15. For any $\delta > 0$ there exists $\epsilon_2 = \epsilon_2(\delta)$ such that, if $\epsilon < \epsilon_2$ then

\begin{equation*}
\tilde{\Phi}_{\epsilon, g_0 + h}([q]) \in N_{\epsilon, g_0 + h}^\tau \cap J_{\epsilon, g_0 + h}^{2(m_\infty + \delta)} \forall h \in \mathcal{B}_p.
\end{equation*}

Moreover we have that

\begin{equation*}
\lim_{\epsilon \to 0} m_{\epsilon, g_0 + h}^\tau = 2m_\infty \text{ uniformly on } h \in \mathcal{B}_p.
\end{equation*}

For a proof of this result we refer to [3].

For any function $u \in N_{\epsilon, g_0 + h}^\tau$ we define

\begin{equation*}
\beta_g(u) = \frac{\int_M x(u^+) \rho d\mu_g}{\int_M (u^+) \rho d\mu_g}
\end{equation*}

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where \( g = g_0 + h \). Also, we define

\[
\tilde{\beta}_g : (N^\tau_{\epsilon, g_0 + h}/\mathbb{Z}_2) \cap \tilde{J}^{2(m_\infty + \delta)} \rightarrow M_d/G
\]

\[
\tilde{\beta}_g([u]) := [\tilde{\beta}_g(u)] = \{\beta_g(u), \beta_g(-u)\} = \{\beta_g(u), -\beta_g(u)\}
\]

where \( M_d = \{u \in M : d(x,M) < d\} \).

**Lemma 16.** There exists \( \tilde{\delta} \) such that \( \forall \delta < \tilde{\delta} \) there exists \( \tilde{\epsilon} = \tilde{\epsilon}(\delta) \) and for any \( \epsilon < \tilde{\epsilon} \) the map

\[
\tilde{\beta}_g \circ \Phi_{\epsilon,g} : M/G \xrightarrow{\Phi_{\epsilon,g}} (N^\tau_{\epsilon, g_0 + h}/\mathbb{Z}_2) \cap \tilde{J}^{2(m_\infty + \delta)} \tilde{\beta}_g : M_d/G
\]

is continuous and homotopic to identity, for all \( g = g_0 + h \) with \( h \in \mathcal{B}_\rho \).

For a proof of this result we refer to [3].

Let us sketch the proof of Proposition[4]. We are going to find an estimate on the number of pairs non degenerate critical points \((u,-u)\) for the functional \( J_{\epsilon,g} : H^\tau_{\epsilon} \rightarrow \mathbb{R} \) with energy close to \( 2m_\infty \) with respect to the parameters \((\epsilon,h)\) in \((0,\tilde{\epsilon}) \times \mathcal{B}_\rho \), for \( \tilde{\epsilon}, \rho \) small enough.

We recall that, by Theorem[5], given the positive numbers \( \tilde{\epsilon}, \rho \), the set

\[
D(\tilde{\epsilon}, \rho) = \left\{ (\epsilon,h) \in (0,\tilde{\epsilon}) \times \mathcal{B}_\rho : \text{any } u \in H^\tau_{\epsilon,h} \text{ solution of } -\epsilon^2 \Delta_{g_0 + h} u + u = |u|^{p-2} u \text{ is not degenerate} \right\}
\]

is a residual subset in \((0,\tilde{\epsilon}) \times \mathcal{B}_\rho \). Since

\[
\lim_{(\epsilon,h) \rightarrow 0} m^\tau_{\epsilon,g_0 + h} = 2m_\infty,
\]

given \( \delta \in (0, \frac{m_\infty}{2}) \), for \((\epsilon,h)\) small enough we have

\[
0 < 2(m_\infty - \delta) < m^\tau_{\epsilon,g_0 + h} < 2(m_\infty + \delta) < 3m_\infty,
\]

thus \( 2(m_\infty - \delta) \) is not a critical value of \( J_{\epsilon,g} \) on \( H^\tau_{\epsilon} \). At this point we take \((\epsilon,h) \in D(\tilde{\epsilon}, \rho)\) with \( \tilde{\epsilon}, \rho \) small enough. Thus the critical points of \( J_{\epsilon,g} \) such that \( J_{\epsilon,g} < 3m_\infty \) are in a finite number by Theorem[4] and then we can assume that \( 2(m_\infty + \delta) \) is not a critical value of \( J_{\epsilon,g} \).

Let \( N^\tau_{\epsilon}/\mathbb{Z}_2 \) be the set obtained by identifying antipodal points of the Nehari manifold \( N^\tau_{\epsilon} \). It is easy to check that the set \( N^\tau_{\epsilon}/\mathbb{Z}_2 \) is homeomorphic to the projective space \( \mathbb{P}^{\infty} = \Sigma/\mathbb{Z}_2 \) obtained by identifying antipodal points in \( \Sigma \), \( \Sigma \) being the unit sphere in \( H^\tau_{\epsilon} \). We are looking for pairs of nontrivial critical points \((u,-u)\) of the functional \( J_{\epsilon} : H^\tau_{\epsilon} \rightarrow \mathbb{R} \), that is searching for critical points of the functional

\[
\tilde{J}_{\epsilon,g} : (H^\tau_{\epsilon} \setminus \{0\})/\mathbb{Z}_2 \rightarrow \mathbb{R};
\]

\[
\tilde{J}_{\epsilon,g} ([u]) := J_{\epsilon,g}(u) = J_{\epsilon,g}(-u).
\]

We use the same Morse theory argument as in [4]. The following result can be found in [2] and Lemma 5.2 of [4]

\[
P_t \left( \tilde{J}^{2(m_\infty + \delta)}_{\epsilon,g} \cap (N^\tau_{\epsilon}/\mathbb{Z}_2) \right) = t P_t \left( \tilde{J}^{2(m_\infty - \delta)}_{\epsilon,g} \cap (N^\tau_{\epsilon}/\mathbb{Z}_2) \right).
\]
By Lemma 15 and Lemma 16 we have that \(\tilde{\beta}_g \circ \tilde{\Phi}_{\varepsilon,g} : M/G \to M_d/G\) is a map homotopic to the identity of \(M/G\) and that \(M_d/G\) is homotopic to \(M/G\). Therefore we have

\[
P_t \left( J^{2(m_\infty + \delta)}_{\varepsilon,g} \cap (N^\varepsilon_\tau / \mathbb{Z}_2) \right) = P_t(M/G) + Z(t)
\]  

were \(Z(t)\) is a polynomial with non negative coefficients. Since the functional \(J_{\varepsilon,g}\) satisfies the Palais Smale condition by the compactness of \(M\), and the critical points of \(J_{\varepsilon,g}\) in \(J^{2m_\infty}_{\varepsilon,g}\) are non degenerate (because \((\varepsilon, h) \in D(\tilde{\varepsilon}, \rho)\)), by Morse Theory and relations (19) and (20) we get at least \(P_t(M/G)\) pairs \((u, -u)\) of non trivial solutions of \(-\varepsilon^2 \Delta_g u + u = |u|^{p-2} u\) with \(J_{\varepsilon,g}(u) = J_{\varepsilon,g}(-u) < 3m_\infty\). So, these solutions change sign exactly once. That concludes the proof of Proposition 4.

Remark 17. In the same way we obtain that, given \(g_0 \in \mathcal{M}^k\) and \(\varepsilon_0 > 0\), the set

\[
A^* = \left\{ h \in \mathcal{B}_\rho : \text{the equation} \ -\varepsilon_0^2 \Delta_{g_0} h + u = |u|^{p-2} u \ \text{has at least} \ P_t(M/G) \ \text{pairs of non degenerate solutions} \ (u, -u) \in H^\varepsilon_{\tau g} \setminus \{0\} \ \text{which change sign exactly once} \right\}
\]

is a residual subset of \(\mathcal{B}_\rho\).

The proof of Proposition 5 can be obtained with similar arguments.

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