Tight local approximation results for max-min linear programs

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Abstract. In a bipartite max-min LP, we are given a bipartite graph $G = (V \cup I \cup K, E)$, where each agent $v \in V$ is adjacent to exactly one constraint $i \in I$ and exactly one objective $k \in K$. Each agent $v$ controls a variable $x_v$. For each $i \in I$ we have a nonnegative linear constraint on the variables of adjacent agents. For each $k \in K$ we have a nonnegative linear objective function of the variables of adjacent agents. The task is to maximise the minimum of the objective functions. We study local algorithms where each agent $v$ must choose $x_v$ based on input within its constant-radius neighbourhood in $G$. We show that for every $\epsilon > 0$ there exists a local algorithm achieving the approximation ratio $\Delta_I(1 - 1/\Delta_K) + \epsilon$. We also show that this result is the best possible – no local algorithm can achieve the approximation ratio $\Delta_I(1 - 1/\Delta_K)$. Here $\Delta_I$ is the maximum degree of a vertex $i \in I$, and $\Delta_K$ is the maximum degree of a vertex $k \in K$. As a methodological contribution, we introduce the technique of graph unfolding for the design of local approximation algorithms.

1 Introduction

As a motivating example, consider the task of data gathering in the following sensor network.

Each open circle is a sensor node $k \in K$, and each box is a relay node $i \in I$. The graph depicts the communication links between sensors and relays. Each sensor produces data which needs to be routed via adjacent relay nodes to a base station (not shown in the figure).

For each pair consisting of a sensor $k$ and an adjacent relay $i$, we need to decide how much data is routed from $k$ via $i$ to the base station. For each such decision, we introduce an agent $v \in V$; these are shown as black dots in the
We write $\omega$ variable $\{\}$. Let 1.1 Max-min linear programs
constant-radius neighbourhood. grams (LPs) such as (1). In a local algorithm, each agent
the value $x_v$. Each relay constitutes a bottleneck: the relay has a limited battery capacity, which sets a limit on the
total amount of data that can be forwarded through it. The task is to maximise the minimum amount of data gathered from a sensor node. In our example, the variable $x_2$ is the amount of data routed from the sensor $k_2$ via the relay $i_1$, the battery capacity of the relay $i_1$ is an upper bound for $x_1 + x_2 + x_3$, and the amount of data gathered from the sensor node $k_2$ is $x_2 + x_4$. Assuming that the
maximum capacity of a relay is 1, the optimisation problem is to

$$\begin{align*}
\text{maximise} & \quad \min \{x_1, x_2 + x_4, x_3 + x_5 + x_7, x_6 + x_8, x_9\} \\
\text{subject to} & \quad x_1 + x_2 + x_3 \leq 1, \\
& \quad x_4 + x_5 + x_6 \leq 1, \\
& \quad x_7 + x_8 + x_9 \leq 1, \\
& \quad x_1, x_2, \ldots, x_9 \geq 0.
\end{align*}$$

(1)

In this work, we study local algorithms [1] for solving max-min linear programs (LPs) such as (1). In a local algorithm, each agent $v \in V$ must choose the value $x_v$ solely based on its constant-radius neighbourhood in the graph $G$. Such algorithms provide an extreme form of scalability in distributed systems; among others, a change in the topology of $G$ affects the values $x_v$ only in a constant-radius neighbourhood.

1.1 Max-min linear programs

Let $G = (V \cup I \cup K, E)$ be a bipartite, undirected communication graph where each edge $e \in E$ is of the form $\{v, j\}$ with $v \in V$ and $j \in I \cup K$. The elements $v \in V$ are called agents, the elements $i \in I$ are called constraints, and the elements $k \in K$ are called objectives; the sets $V$, $I$, and $K$ are disjoint. We define $V_i = \{v \in V : \{v, i\} \in E\}$, $V_k = \{v \in V : \{v, k\} \in E\}$, $I_v = \{i \in I : \{v, i\} \in E\}$, and $K_v = \{k \in K : \{v, k\} \in E\}$ for all $i \in I$, $k \in K$, $v \in V$.

We assume that $G$ is a bounded-degree graph; in particular, we assume that $|V_i| \leq \Delta_I$ and $|V_k| \leq \Delta_K$ for all $i \in I$ and $k \in K$ for some constants $\Delta_I$ and $\Delta_K$.

A max-min linear program associated with $G$ is defined as follows. Associate a variable $x_v$ with each agent $v \in V$, associate a coefficient $a_{iv} \geq 0$ with each edge $\{i, v\} \in E$, $i \in I$, $v \in V$, and associate a coefficient $c_{kv} \geq 0$ with each edge $\{k, v\} \in E$, $k \in K$, $v \in V$. The task is to

$$\begin{align*}
\text{maximise} & \quad \omega = \min_{k \in K} \sum_{v \in V_k} c_{kv} x_v \\
\text{subject to} & \quad \sum_{v \in V_i} a_{iv} x_v \leq 1 \quad \forall i \in I, \\
& \quad x_v \geq 0 \quad \forall v \in V.
\end{align*}$$

(2)

We write $\omega^*$ for the optimum of (2).
1.2 Special cases of max-min LPs

A max-min LP is a generalisation of a packing LP. Namely, in a packing LP there is only one linear nonnegative function to maximise, while in a max-min LP the goal is to maximise the minimum of multiple nonnegative linear functions.

Our main focus is on the bipartite version of the max-min LP problem. In the bipartite version we have $|I_v| = |K_v| = 1$ for each $v \in V$. We also define the 0/1 version [2]. In that case we have $a_{iv} = 1$ and $c_{kv} = 1$ for all $v \in V, i \in I_v, k \in K_v$.

Our example (1) is both a bipartite max-min LP and a 0/1 max-min LP. The distance between a pair of vertices $s, t \in V \cup I \cup K$ in $G$ is the number of edges on a shortest path connecting $s$ and $t$ in $G$. We write $B_G(s, r)$ for the set of vertices within distance at most $r$ from $s$.

We say that $G$ has bounded relative growth $1 + \delta$ beyond radius $R \in \mathbb{N}$ if $|V \cap B_G(v, r + 2)|/|V \cap B_G(v, r)| \leq 1 + \delta$ for all $v \in V, r \geq R$. Any bounded-degree graph $G$ has a constant upper bound for $\delta$. Regular grids are a simple example of a family of graphs where $\delta$ approaches 0 as $R$ increases [3].

1.3 Local algorithms and the model of computation

A local algorithm [1] is a distributed algorithm in which the output of a node is a function of input available within a fixed-radius neighbourhood; put otherwise, the algorithm runs in a constant number of communication rounds. In the context of distributed max-min LPs, the exact definition is as follows.

We say that the local input of a node $v \in V$ consists of the sets $I_v$ and $K_v$ and the coefficients $a_{iv}, c_{kv}$ for all $i \in I_v, k \in K_v$. The local input of a node $i \in I$ consists of $V_i$ and the local input of a node $k \in K$ consists of $V_k$. Furthermore, we assume that either (a) each node has a unique identifier given as part of the local input to the node [1,4]; or, (b) each vertex independently introduces an ordering of the edges incident to it. The latter, strictly weaker, assumption is often called port numbering [5]; in essence, each edge $\{s, t\}$ in $G$ has two natural numbers associated with it: the port number in $s$ and the port number in $t$.

Let $A$ be a deterministic distributed algorithm executed by each of the nodes of $G$ that finds a feasible solution $x$ to any max-min LP (2) given locally as input to the nodes. Let $r \in \mathbb{N}$ be a constant independent of the input. We say that $A$ is a local algorithm with local horizon $r$ if, for every agent $v \in V$, the output $x_v$ is a function of the local input of the nodes in $B_G(v, r)$. Furthermore, we say that $A$ has the approximation ratio $\alpha \geq 1$ if $\sum_{v \in V_k} c_{kv}x_v \geq \omega^* / \alpha$ for all $k \in K$.

1.4 Contributions and prior work

The following local approximability result is the main contribution of this paper.

**Theorem 1.** For any $\Delta_I \geq 2$, $\Delta_K \geq 2$, and $\epsilon > 0$, there exists a local approximation algorithm for the bipartite max-min LP problem with the approximation ratio $\Delta_I(1 - 1 / \Delta_K) + \epsilon$. The algorithm assumes only port numbering.
We also show that the positive result of Theorem 1 is tight. Namely, we prove a matching lower bound on local approximability, which holds even if we assume both 0/1 coefficients and unique node identifiers given as input.

**Theorem 2.** For any \( \Delta_I \geq 2 \) and \( \Delta_K \geq 2 \), there exists no local approximation algorithm for the max-min LP problem with the approximation ratio \( \Delta_I(1 - 1/\Delta_K) \). This holds even in the case of a bipartite, 0/1 max-min LP and with unique node identifiers given as input.

Considering Theorem 1 in light of Theorem 2, we find it somewhat surprising that unique node identifiers are not required to obtain the best possible local approximation algorithm for bipartite max-min LPs.

In terms of earlier work, Theorem 1 is an improvement on the *safe algorithm* \([3, 6]\) which achieves the approximation ratio \( \Delta_I \). Theorem 2 improves upon the earlier lower bound \((\Delta_I + 1)/2 - 1/(2\Delta_K - 2)\) \([3]\); here it should be noted that our definition of the local horizon differs by a constant factor from earlier work \([3]\) due to the fact that we have adopted a more convenient graph representation instead of a hypergraph representation.

In the context of packing and covering LPs, it is known \([7]\) that any approximation ratio \( \alpha > 1 \) can be achieved by a local algorithm, assuming a bounded-degree graph and bounded coefficients. Compared with this, the factor \( \Delta_I(1 - 1/\Delta_K) \) approximation in Theorem 1 sounds somewhat discouraging considering practical applications. However, the constructions that we use in our negative results are arguably far from the structure of, say, a typical real-world wireless network. In prior work \([3]\) we presented a local algorithm that achieves a factor \( 1 + (2 + o(1))\delta \) approximation assuming that \( G \) has bounded relative growth \( 1 + \delta \) beyond some constant radius \( R \); for a small \( \delta \), this is considerably better than \( \Delta_I(1 - 1/\Delta_K) \) for general graphs. We complement this line of research on bounded relative growth graphs with a negative result that matches the prior positive result \([3]\) up to constants.

**Theorem 3.** Let \( \Delta_I \geq 3 \), \( \Delta_K \geq 3 \), and \( 0 < \delta < 1/10 \). There exists no local approximation algorithm for the max-min LP problem with an approximation ratio less than \( 1 + \delta/2 \). This holds even in the case of a bipartite max-min LP where the graph \( G \) has bounded relative growth \( 1 + \delta \) beyond some constant radius \( R \).

From a technical perspective, the proof of Theorem 1 relies on two ideas: *graph unfolding* and the idea of *averaging local solutions* of local LPs.

We introduce the unfolding technique in Sect. 2. In essence, we expand the finite input graph \( G \) into a possibly infinite tree \( T \). Technically, \( T \) is the *universal covering* of \( G \) \([5]\). While such unfolding arguments have been traditionally used to obtain impossibility results \([8]\) in the context of distributed algorithms, here we use such an argument to simplify the design of local algorithms. In retrospect, our earlier approximation algorithm for 0/1 max-min LPs \([2]\) can be interpreted as an application of the unfolding technique.

The idea of averaging local LPs has been used commonly in prior work on distributed algorithms \([3, 7, 9, 10]\). Our algorithm can also be interpreted as a generalisation of the safe algorithm \([6]\) beyond local horizon \( r = 1 \).
To obtain our negative results – Theorems 2 and 3 – we use a construction based on regular high-girth graphs. Such graphs [11–14] have been used in prior work to obtain impossibility results related to local algorithms [4, 7, 15].

2 Graph unfolding

Let \( \mathcal{H} = (V, E) \) be a connected undirected graph and let \( v \in V \). Construct a (possibly infinite) rooted tree \( \mathcal{T}_v = (\bar{V}, \bar{E}) \) and a labelling \( f_v : \bar{V} \to V \) as follows. First, introduce a vertex \( \bar{v} \) as the root of \( \mathcal{T}_v \) and set \( f_v(\bar{v}) = v \). Then, for each vertex \( u \) adjacent to \( v \) in \( \mathcal{H} \), add a new vertex \( \bar{u} \) as a child of \( \bar{v} \) and set \( f_v(\bar{u}) = u \).

Then expand recursively as follows. For each unexpanded \( \bar{t} \neq \bar{v} \) with parent \( \bar{s} \), and each \( u \neq f(\bar{s}) \) adjacent to \( f(\bar{t}) \) in \( \mathcal{H} \), add a new vertex \( \bar{u} \) as a child of \( \bar{t} \) and set \( f_v(\bar{u}) = u \). Mark \( \bar{t} \) as expanded.

This construction is illustrated in Fig. 1. Put simply, we traverse \( \mathcal{H} \) in a breadth-first manner and treat vertices revisited due to a cycle as new vertices; in particular, the tree \( \mathcal{T}_v \) is finite if and only if \( \mathcal{H} \) is acyclic.

The rooted, labelled trees \((\mathcal{T}_v, f_v)\) obtained in this way for different choices of \( v \in V \) are isomorphic viewed as unrooted trees [5]. For example, the infinite labelled trees \((\mathcal{T}_a, f_a)\) and \((\mathcal{T}_c, f_c)\) in Fig. 1 are isomorphic and can be transformed into each other by rotations. Thus, we can define the unfolding of \( \mathcal{H} \) as the labelled tree \((\mathcal{T}, f)\) where \( \mathcal{T} \) is the unrooted version of \( \mathcal{T}_v \) and \( f = f_v \); up to isomorphism, this is independent of the choice of \( v \in V \). Appendix A.1 provides a further discussion on the terminology and concepts related to unfolding.

2.1 Unfolding and local algorithms

Let us now view the graph \( \mathcal{H} \) as the communication graph of a distributed system, and let \((\mathcal{T}, f)\) be the unfolding of \( \mathcal{H} \). Even if \( \mathcal{T} \) in general is countably infinite, a local algorithm \( A \) with local horizon \( r \) can be designed to operate at a node of \( v \in \mathcal{H} \) exactly as if it was a node \( \bar{v} \in f^{-1}(v) \) in the communication graph \( \mathcal{T} \). Indeed, assume that the local input at \( \bar{v} \) is identical to the local input at \( f(\bar{v}) \), and observe that the radius \( r \) neighbourhood of the node \( \bar{v} \) in \( \mathcal{T} \) is equal to the rooted tree \( \mathcal{T}_v \) trimmed to depth \( r \); let us denote this by \( \mathcal{T}_v(r) \). To gather the information in \( \mathcal{T}_v(r) \), it is sufficient to gather information on all walks of
length at most $r$ starting at $v$ in $\mathcal{H}$; using port numbering, the agents can detect and discard walks that consecutively traverse the same edge.

Assuming that only port numbering is available, the information in $T_v(r)$ is in fact all that the agent $v$ can gather. Indeed, to assemble, say, the subgraph of $\mathcal{H}$ induced by $B_{\mathcal{H}}(v, r)$, the agent $v$ in general needs to distinguish between a short cycle and a long path, and these are indistinguishable without node identifiers.

2.2 Unfolding and max-min LPs

Let us now consider a max-min LP associated with a graph $\mathcal{G}$. The unfolding of $\mathcal{G}$ leads in a natural way to the unfolding of the max-min LP. As the unfolding of a max-min LP is, in general, countably infinite, we need minor technical extensions reviewed in Appendix A.2. A formal definition of the unfolding of a max-min LP and the proof of the following lemma is given in Appendix A.3.

**Lemma 1.** Let $\bar{A}$ be a local algorithm for unfoldings of a family of max-min LPs and let $\alpha \geq 1$. Assume that the output $x$ of $\bar{A}$ satisfies $\sum_{v \in V_k} c_{kv} x_v \geq \omega'/\alpha$ for all $k \in K$ if there exists a feasible solution with utility at least $\omega'$. Furthermore, assume that $\bar{A}$ uses port numbering only. Then, there exists a local approximation algorithm $A$ with the approximation ratio $\alpha$ for this family of max-min LPs.

3 Approximability results

We proceed to prove Theorem 1. Let $\Delta_I \geq 2$, $\Delta_K \geq 2$, and $\epsilon > 0$ be fixed. By virtue of Lemma 1, it suffices to consider only bipartite max-min LPs where the graph $\mathcal{G}$ is a (finite or countably infinite) tree.

To ease the analysis, it will be convenient to regularise $\mathcal{G}$ to a countably infinite tree with $|V_i| = \Delta_I$ and $|V_k| = \Delta_K$ for all $i \in I$ and $k \in K$.

To this end, if $|V_i| < \Delta_I$ for some $i \in I$, add $\Delta_I - |V_i|$ new virtual agents as neighbours of $i$. Let $v$ be one of these agents. Set $a_{iv} = 0$ so that no matter what value one assigns to $x_v$, it does not affect the feasibility of the constraint $i$. Then add a new virtual objective $k$ adjacent to $v$ and set, for example, $c_{kv} = 1$. As one can assign an arbitrarily large value to $x_v$, the virtual objective $k$ will not be a bottleneck.

Similarly, if $|V_k| < \Delta_K$ for some $k \in K$, add $\Delta_K - |V_k|$ new virtual agents as neighbours of $k$. Let $v$ be one of these agents. Set $c_{kv} = 0$ so that no matter what value one assigns to $x_v$, it does not affect the value of the objective $k$. Then add a new virtual constraint $i$ adjacent to $v$ and set, for example, $a_{iv} = 1$.

Now repeat these steps and grow virtual trees rooted at the constraints and objectives that had less than $\Delta_I$ or $\Delta_K$ neighbours. The result is a countably infinite tree where $|V_i| = \Delta_I$ and $|V_k| = \Delta_K$ for all $i \in I$ and $k \in K$. Observe also that from the perspective of a local algorithm it suffices to grow the virtual trees only up to depth $r$ because then the radius $r$ neighbourhood of each original node is indistinguishable from the regularised tree. The resulting topology is illustrated in Fig. 2 from the perspective of an original objective $k_0 \in K$ and an original constraint $i_0 \in I$. 
Fig. 2. Radius 6 neighbourhoods of (a) an objective $k_0 \in K$ and (b) a constraint $i_0 \in I$ in the regularised tree $G$, assuming $\Delta_I = 4$ and $\Delta_K = 3$. The black dots represent agents $v \in V$, the open circles represent objectives $k \in K$, and the boxes represent constraints $i \in I$.

3.1 Properties of regularised trees

For each $v \in V$ in a regularised tree $G$, define $K(v, \ell) = K \cap B_G(v, 4\ell + 1)$, that is, the set of objectives $k$ within distance $4\ell + 1$ from $v$. For example, $K(v, 1)$ consists of 1 objective at distance 1, $\Delta_I - 1$ objectives at distance 3, and $(\Delta_K - 1)(\Delta_I - 1)$ objectives at distance 5; see Fig. 2a. In general, we have

$$|K(v, \ell)| = 1 + (\Delta_I - 1)\Delta_K n(\ell),$$

where

$$n(\ell) = \sum_{j=0}^{\ell-1}(\Delta_I - 1)^j(\Delta_K - 1)^j.$$

Let $k \in K$. If $u, v \in V_k$, $u \neq v$, then the objective at distance 1 from $u$ is the same as the objective at distance 1 from $v$: therefore $K(u, 0) = K(v, 0)$. The objectives at distance 3 from $u$ are at distance 5 from $v$, and the objectives at distance 5 from $u$ are at distance 3 or 5 from $v$: therefore $K(u, 1) = K(v, 1)$. By a similar reasoning, we obtain

$$K(u, \ell) = K(v, \ell) \quad \forall \ell \in \mathbb{N}, \ k \in K, \ u, v \in V_k.$$  \hspace{1cm} (4)

Let us then study a constraint $i \in I$. Define $K(i, \ell) = \bigcap_{v \in V_i} K(v, \ell) = K \cap B_G(i, 4\ell) = K \cap B_G(i, 4\ell - 2)$.

For example, $K(i, 2)$ consists of $\Delta_I$ objectives at distance 2 from the constraint $i$, and $\Delta_I(\Delta_K - 1)(\Delta_I - 1)$ objectives at distance 6 from the constraint $i$; see Fig. 2b. In general, we have

$$|K(i, \ell)| = \Delta_I n(\ell).$$

For adjacent $v \in V$ and $i \in I$, we also define $\partial K(v, i, \ell) = K(v, \ell) \setminus K(i, \ell)$. We have by (3) and (5)

$$|\partial K(v, i, \ell)| = 1 + (\Delta_I \Delta_K - \Delta_I - \Delta_K) n(\ell).$$

(6)
3.2 Local approximation on regularised trees

It now suffices to meet Lemma 1 for bipartite max-min LPs in the case when the underlying graph \( G \) is a countably infinite regularised tree. To this end, let \( L \in \mathbb{N} \) be a constant that we choose later; \( L \) depends only on \( \Delta_I, \Delta_K \) and \( \epsilon \).

Each agent \( u \in V \) now executes the following algorithm. First, the agent gathers all objectives \( k \in K \) within distance \( 4L + 1 \), that is, the set \( K(u, L) \). Then, for each \( k \in K(u, L) \), the agent \( u \) gathers the radius \( 4L + 2 \) neighbourhood of \( k \); let \( \tilde{G}(k, L) \) be this subgraph. In total, the agent \( u \) accumulates information from distance \( r = 8L + 3 \) in the tree; this is the local horizon of the algorithm.

The structure of \( \tilde{G}(k, L) \) is a tree similar to the one shown in Fig. 2a. The leaf nodes of the tree \( \tilde{G}(k, L) \) are constraints. For each \( k \in K(u, L) \), the agent \( u \) formulates the constant-size subproblem of (2) restricted to the vertices of \( \tilde{G}(k, L) \) and solves it optimally using a deterministic algorithm; let \( x^{kL} \) be the solution. Once the agent \( u \) has solved the subproblem for every \( k \in K(u, L) \), it sets
\[
q = 1/(\Delta_I + \Delta_I(\Delta_I - 1)(\Delta_K - 1)n(L)),
\]
\[
x_u = q \sum_{k \in K(u, L)} x^{kL}.
\]

This completes the description of the algorithm.

We now show that the computed solution \( x \) is feasible. Because each \( x^{kL} \) is a feasible solution, we have
\[
\sum_{v \in V_i} a_{iv} x^{kL}_v \leq 1 \quad \forall \text{ non-leaf } i \in I \text{ in } \tilde{G}(k, L),
\]
\[
a_{iv} x^{kL}_v \leq 1 \quad \forall \text{ leaf } i, v \in V_i \text{ in } \tilde{G}(k, L).
\]

Let \( i \in I \). For each subproblem \( \tilde{G}(k, L) \) with \( v \in V_i, k \in K(i, L) \), the constraint \( i \) is a non-leaf vertex; therefore
\[
\sum_{v \in V_i} \sum_{k \in K(i, L)} a_{iv} x^{kL}_v = \sum_{k \in K(i, L)} \sum_{v \in V_i} a_{iv} x^{kL}_v \leq \sum_{k \in K(i, L)} 1 \quad \equiv \Delta_I n(L). \tag{11}
\]

For each subproblem \( \tilde{G}(k, L) \) with \( v \in V_i, k \in \partial K(v, i, L) \), the constraint \( i \) is a leaf vertex; therefore
\[
\sum_{v \in V_i} \sum_{k \in \partial K(v, i, L)} a_{iv} x^{kL}_v \leq \sum_{v \in V_i} \sum_{k \in \partial K(v, i, L)} 1 \quad \equiv \Delta_I (1 + (\Delta_I \Delta_K - \Delta_I - \Delta_K)n(L)). \tag{12}
\]

Combining (11) and (12), we can show that the constraint \( i \) is satisfied:
\[
\sum_{v \in V_i} a_{iv} x_v \overset{(8)}{=} q \sum_{v \in V_i} \sum_{k \in K(v, L)} x^{kL}_v
\]
\[
= q \left( \sum_{v \in V_i} \sum_{k \in K(i, L)} a_{iv} x^{kL}_v \right) + q \left( \sum_{v \in V_i} \sum_{k \in \partial K(v, i, L)} a_{iv} x^{kL}_v \right)
\]
\[
\leq q \Delta_I n(L) + q \Delta_I (1 + (\Delta_I \Delta_K - \Delta_I - \Delta_K)n(L)) \overset{(7)}{=} 1.
\]
Next we establish a lower bound on the performance of the algorithm. To this end, consider an arbitrary feasible solution $x'$ of the unrestricted problem (2) with utility at least $\omega'$. This feasible solution is also a feasible solution of each finite subproblem restricted to $\mathcal{G}(k, L)$; therefore

$$\sum_{v \in V_h} c_{hv} x'^{kL}_v \geq \omega' \quad \forall h \in K \text{ in } \mathcal{G}(k, L).$$

Define

$$\alpha = \frac{1}{q(1 + (\Delta I - 1)\Delta_K n(L))} \equiv \Delta I \left(1 - \frac{1}{\Delta_K + 1/((\Delta I - 1)n(L))}\right).$$

Consider an arbitrary $k \in K$ and $u \in V_k$. We have

$$\sum_{v \in V_k} c_{kv} x'_{kv} = q \sum_{v \in V_k} c_{kv} \sum_{h \in K(u, L)} x'^{hL}_v \geq q \sum_{h \in K(u, L)} \sum_{v \in V_k} c_{kv} x'^{hL}_v \geq q(1 + (\Delta I - 1)\Delta_K n(L)) \omega' \equiv \omega'/\alpha.$$

For a sufficiently large $L$, we meet Lemma 1 with $\alpha < \Delta I (1 - 1/\Delta_K) + \epsilon$. This completes the proof of Theorem 1. For a concrete example, see Appendix A.4.

## 4 Inapproximability results

We proceed to prove Theorems 2 and 3. Let $r = 4, 8, \ldots$, $s \in \mathbb{N}$, $D_I \in \mathbb{Z}^+$, and $D_K \in \mathbb{Z}^+$ be constants whose values we choose later. Let $\mathcal{Q} = (I' \cup K', E')$ be a bipartite graph where the degree of each $i \in I'$ is $D_I$, the degree of each $k \in K'$ is $D_K$, and there is no cycle of length less than $g = 2(4s + 2 + r) + 1$. Such graphs exist for all values of the parameters; a simple existence proof can be devised by slightly modifying the proof of a theorem of Hoory [13, Theorem A.2]; see Appendix A.5.

### 4.1 The instance $\mathcal{S}$

Given the graph $\mathcal{Q} = (I' \cup K', E')$, we construct an instance of the max-min LP problem, $\mathcal{S}$. The underlying communication graph $\mathcal{G} = (V \cup I \cup K, E)$ is constructed as shown in the following figure.

![Graph Diagram](image)

Each edge $e = \{i, k\} \in E'$ is replaced by a path of length $4s + 2$: the path begins with the constraint $i \in I'$; then there are $s$ segments of agent–objective–agent–constraint; and finally there is an agent and the objective $k \in K'$. There are no other edges or vertices in $\mathcal{G}$. For example, in the case of $s = 0$, $D_I = 4$, $D_K = 3$,
and sufficiently large \( g \), the graph \( G \) looks locally similar to the trees in Fig. 2, even though there may be long cycles.

The coefficients of the instance \( S \) are chosen as follows. For each objective \( k \in K' \), we set \( c_{kv} = 1 \) for all \( v \in V_k \). For each objective \( k \in K \setminus K' \), we set \( c_{kv} = D_K - 1 \) for all \( v \in V_k \). For each constraint \( i \in I \), we set \( a_{iv} = 1 \). Observe that \( S \) is a bipartite max-min LP; furthermore, in the case \( s = 0 \), this is a 0/1 max-min LP. We can choose the port numbering in \( G \) in an arbitrary manner, and we can assign unique node identifiers to the vertices of \( G \) as well.

Consider a feasible solution \( x \) of \( S \), with utility \( \omega \). We proceed to derive an upper bound for \( \omega \). For each \( j = 0, 1, \ldots, 2s \), let \( V(j) \) consist of agents \( v \in V \) such that the distance to the nearest constraint \( i \in I' \) is \( 2j + 1 \). That is, \( V(0) \) consists of the agents adjacent to an \( i \in I' \) and \( V(2s) \) consists of the agents adjacent to a \( k \in K' \). Let \( m = |E'| \); we observe that \( |V(j)| = m \) for each \( j \).

Let \( X(j) = \sum_{v \in V(j)} x_v/m \). From the constraints \( i \in I' \) we obtain

\[
X(0) = \sum_{v \in V(0)} x_v/m = \sum_{i \in I'} \sum_{v \in V_i} a_{iv}x_v/m \leq \sum_{i \in I'} 1/m = |I'|/m = 1/D_I.
\]

Similarly, from the objectives \( k \in K' \) we obtain \( X(2s) \geq \omega|K'|/m = \omega/D_K \).

From the objectives \( k \in K \setminus K' \), taking into account our choice of the coefficients \( c_{kv} \), we obtain the inequality \( X(2t) + X(2t + 1) \geq \omega/(D_K - 1) \) for \( t = 0, 1, \ldots, s - 1 \). From the constraints \( i \in I \setminus I' \), we obtain the inequality \( X(2t + 1) + X(2t + 2) \leq 1 \) for \( t = 0, 1, \ldots, s - 1 \). Combining inequalities, we have

\[
\omega/D_K - 1/D_I \leq X(2s) - X(0)
= \sum_{t=0}^{s-1} \left( (X(2t + 1) + X(2t + 2)) - (X(2t) + X(2t + 1)) \right)
\leq s \cdot \left( 1 - \omega/(D_K - 1) \right),
\]

which implies

\[
\omega \leq \frac{D_K}{D_I} \cdot \frac{D_K - 1 + D_K D_I s - D_I s}{D_K - 1 + D_K s}.
\]

4.2 The instance \( S_k \)

Let \( k \in K' \). We construct another instance of the max-min LP problem, \( S_k \). The communication graph of \( S_k \) is the subgraph \( G_k \) of \( G \) induced by \( B_G(k, 4s + 2 + r) \). By the choice of \( g \), there is no cycle in \( G_k \). As \( r \) is a multiple of 4, the leaves of the tree \( G_k \) are constraints. For example, in the case of \( s = 0 \), \( D_I = 4 \), \( D_K = 3 \), and \( r = 4 \), the graph \( G_k \) is isomorphic to the tree of Fig. 2a. The coefficients, port numbers and node identifiers are chosen in \( G_k \) exactly as in \( G \). The optimum of \( S_k \) is greater than \( D_K - 1 \) (see Appendix A.6).

4.3 Proof of Theorem 2

Let \( \Delta_I \geq 2 \) and \( \Delta_K \geq 2 \). Assume that \( A \) is a local approximation algorithm with the approximation ratio \( \alpha \). Set \( D_I = \Delta_I \), \( D_K = \Delta_K \) and \( s = 0 \). Let \( r \)
be the local horizon of the algorithm, rounded up to a multiple of 4. Construct the instance $S$ as described in Sect. 4.1; it is a 0/1 bipartite max-min LP, and it satisfies the degree bounds $\Delta_I$ and $\Delta_K$. Apply the algorithm $A$ to $S$. The algorithm produces a feasible solution $x$. By (15) there is a constraint $k$ such that $\sum_{v \in V_k} x_v \leq \Delta_K/\Delta_I$.

Now construct $S_k$ as described in Sect. 4.2; this is another 0/1 bipartite max-min LP. Apply $A$ to $S_k$. The algorithm produces a feasible solution $x'$. The radius $r$ neighbourhoods of the agents $v \in V_k$ are identical in $S$ and $S_k$; therefore the algorithm must make the same decisions for them, and we have $\sum_{v \in V_k} x'_v \leq \Delta_K/\Delta_I$. But there is a feasible solution of $S_k$ with utility greater than $\Delta_K - 1$ (see Appendix A.6); therefore the approximation ratio of $A$ is $\alpha > (\Delta_K - 1)/(\Delta_K/\Delta_I)$. This completes the proof of Theorem 2.

4.4 Proof of Theorem 3

Let $\Delta_I \geq 3$, $\Delta_K \geq 3$, and $0 < \delta < 1/10$. Assume that $A$ is a local approximation algorithm with the approximation ratio $\alpha$. Set $D_I = 3$, $D_K = 3$, and $s = \lceil 4/(7\delta) - 1/2 \rceil$. Let $r$ be the local horizon of the algorithm, rounded up to a multiple of 4.

Again, construct the instance $S$. The relative growth of $G$ is at most $1 + 2^j/(2^j - 1)(2s + 1)$ beyond radius $R = j(4s + 2)$; indeed, each set of $2^j$ new agents can be accounted for $1 + 2 + \cdots + 2^{j-1} = 2^j - 1$ chains with $2s + 1$ agents each. Choosing $j = 3$, the relative growth of $G$ is at most $1 + \delta$ beyond radius $R$.

Apply $A$ to $S$. By (15) we know that there exists an objective $h$ such that $\sum_{v \in V_h} x_v \leq 2 - 2/(3s + 2)$. Choose a $k \in K'$ nearest to $h$. Construct $S_k$ and apply $A$ to $S_k$. The local neighbourhoods of the agents $v \in V_h$ are identical in $S$ and $S_k$. We know that $S_k$ has a feasible solution with utility greater than 2 (see Appendix A.6). Using the assumption $\delta < 1/10$, we obtain

$$\alpha > \frac{2}{2 - 2/(3s + 2)} = 1 + \frac{1}{3s + 1} \geq 1 + \frac{1}{3(4/(7\delta) + 1/2)} + 1 > 1 + \frac{\delta}{2}.$$ 

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Appendix

A.1 Unfolding in graph theory and topology

We briefly summarise the graph theoretic and topological background related to the unfolding \((T, f)\) of \(\mathcal{H}\) as defined in Sect. 2.

From a graph theoretic perspective, using the terminology of Godsil and Royle [16, §6.8], surjection \(f\) is a homomorphism from \(T\) to \(\mathcal{H}\). Moreover, it is a local isomorphism: the neighbours of \(\bar{v} \in \bar{V}\) are in one-to-one correspondence with the neighbours of \(f(\bar{v}) \in V\). A surjective local isomorphism \(f\) is a covering map and \((T, f)\) is a covering graph of \(\mathcal{H}\).

Covering maps in graph theory can be interpreted as a special case of covering maps in topology: \(T\) is a covering space of \(\mathcal{H}\) and \(f\) is, again, a covering map. See, e.g., Hocking and Young [17, §4.8] or Munkres [18, §53].

In topology, a simply connected covering space is called a universal covering space [17, §4.8], [18, §80]. An analogous graph-theoretic concept is a tree: unfolding \(T\) of \(\mathcal{H}\) is equal to the universal covering \(U(\mathcal{H})\) of \(\mathcal{H}\) as defined by Angluin [5].

Unfortunately, the term “covering” is likely to cause confusion in the context of graphs. The term “lift” has been used for a covering graph [13, 19]. We have borrowed the term “unfolding” from the field of model checking; see, e.g., Esparza and Heljanko [20].

A.2 Infinite max-min LPs

Unfolding (Sect. 2) and regularisation (Sect. 3) in general require us to consider max-min LPs where the underlying graph \(\mathcal{G}\) is countably infinite. Observe that \(\mathcal{G}\) is always a bounded-degree graph, however. This allows us to circumvent essentially all of the technicalities otherwise encountered with infinite problem instances; cf. Anderson and Nash [21].

For the purposes of this work, it suffices to define that \(x\) is a feasible solution with utility at least \(\omega\) if \((x, \omega)\) satisfies

\[
\begin{align*}
\sum_{v \in V_k} c_{kv}x_v & \geq \omega \quad \forall k \in K, \\
\sum_{v \in V_i} a_{iv}x_v & \leq 1 \quad \forall i \in I, \\
x_v & \geq 0 \quad \forall v \in V.
\end{align*}
\]

Each of the sums in (16) is finite.

Observe that this definition is compatible with the finite max-min LP defined in Sect. 1.1. Namely, if \(\omega^*\) is the optimum of a finite max-min LP, then there exists a feasible solution \(x^*\) with utility at least \(\omega^*\).

A.3 Proof of Lemma 1

Assume that an arbitrary finite max-min LP from the family under consideration is given as input. Let \(\mathcal{G} = (V \cup I \cup K, E)\) be the underlying communication graph.
Unfold $\mathcal{G}$ to obtain a (possibly infinite) tree $T = (\bar{V} \cup I \cup \bar{K}, \bar{E})$ with a labelling $f$. Extend this to an unfolding of the max-min LP by associating a variable $x_{\bar{v}}$ with each agent $\bar{v} \in \bar{V}$, the coefficient $a_{f(\bar{v}), f(\bar{v})}$ for each edge $\{\bar{v}, \bar{v}\} \in \bar{E}$, and the coefficient $c_{\bar{v}, \bar{v}} = c_{f(\bar{v}), f(\bar{v})}$ for each edge $\{\bar{k}, \bar{v}\} \in \bar{E}, \bar{k} \in \bar{K}, \bar{v} \in \bar{V}$. Furthermore, assume an arbitrary port numbering for the edges incident to each of the nodes in $\mathcal{G}$, and extend this to a port numbering for the edges incident to each of the nodes in $T$ so that the port numbers at the ends of each edge $\{u, v\} \in \bar{E}$ are identical to the port numbers at the ends of $\{f(u), f(v)\}$.

Let $x^*$ be an optimal solution of the original instance, with utility $\omega^*$. Set $x_{\bar{v}} = x_{f(\bar{v})}^*$ to obtain a solution of the unfolding. This is a feasible solution because the variables of the agents adjacent to a constraint $\bar{v}$ in the unfolding have the same values as the variables of the agents adjacent to the constraint $f(\bar{v})$ in the original instance. By similar reasoning, we can show that this is a feasible solution with utility at least $\omega^*$.

Construct the local algorithm $\mathcal{A}$ using the assumed algorithm $\bar{\mathcal{A}}$ as follows. Each node $v \in V$ simply behaves as if it was a node $\bar{v} \in f^{-1}(v)$ in the unfolding $T$ and simulates $\bar{\mathcal{A}}$ for $\bar{v}$ in $T$. By assumption, the solution $x$ computed by $\bar{\mathcal{A}}$ in the unfolding has to satisfy $\sum_{\bar{v} \in \bar{V}_k} c_{\bar{k}, \bar{v}} x_{\bar{v}} \geq \omega^*/\alpha$ for every $\bar{k} \in \bar{K}$ and $\sum_{\bar{v} \in \bar{V}_1} a_{\bar{v}, \bar{v}} x_{\bar{v}} \leq 1$ for every $\bar{v} \in \bar{I}$. Furthermore, if $f(\bar{u}) = f(\bar{v})$ for $\bar{u}, \bar{v} \in \bar{V}$, then the neighbourhoods of $\bar{u}$ and $\bar{v}$ contain precisely the same information (including the port numbering), so the deterministic $\bar{\mathcal{A}}$ must output the same value $x_{\bar{u}} = x_{\bar{v}}$. Giving the output $x_v = x_{\bar{v}}$ for any $\bar{v} \in f^{-1}(v)$ therefore yields a feasible, $\alpha$-approximate solution to the original instance. This completes the proof.

We observe that Lemma 1 generalises beyond max-min LPs; we did not exploit the linearity of the constraints and the objectives.

A.4 The approximation algorithm in practice

In this section, we give a simple example that illustrates the behaviour of the approximation algorithm presented in Sect. 3.2. Consider the case of $\Delta_I = 4$, $\Delta_K = 3$ and $L = 1$. For each $k \in K$, we construct and solve a subproblem; the structure of the subproblem is illustrated in Fig. 2a. Then we simply sum up the optimal solutions of each subproblem. For any $v \in V$, the variable $x_v$ is involved in exactly $|K(v, L)| = 10$ subproblems.

First, consider an objective $k \in K$. The boundary of a subproblem always lies at a constraint, never at an objective. Therefore the objective $k$ and all its adjacent agents $v \in V_k$ are involved in 10 subproblems. We satisfy the objective exactly 10 times, each time at least as well as in the global optimum.

Second, consider a constraint $i \in I$. The constraint may lie in the middle of a subproblem or at the boundary of a subproblem. The former happens in this case $|K(i, L)| = 4$ times; the latter happens $|V_i| \cdot |\partial K(v, i, L)| = 24$ times. In total, we use up the capacity available at the constraint $i$ exactly 28 times. See Fig. 2b for an illustration: there are 28 objectives within distance 6 from the constraint $i_0 \in I$. 
Finally, we scale down the solution by factor \( q = 1/28 \). This way we obtain a solution which is feasible and within factor \( \alpha = 2.8 \) of optimum. This is close to the lower bound \( \alpha > 2.66 \) from Theorem 2.

### A.5 Bipartite high girth graphs

We say that a bipartite graph \( G = (V \cup U, E) \) is \((a, b)\)-regular if the degree of each node in \( V \) is \( a \) and the degree of each node in \( U \) is \( b \). Here we sketch a proof which shows that for any positive integers \( a, b \) and \( g \), there is a \((a, b)\)-regular bipartite graph \( G = (V \cup U, E) \) which has no cycle of length less than \( g \). We slightly adapt a proof of a similar result for \( d \)-regular graphs [13, Theorem A.2] to our needs. We proceed by induction on \( g \), for \( g = 4, 6, 8, \ldots \).

First consider the basis \( g = 4 \). We can simply choose the complete bipartite graph \( K_{b,a} \) as a \((a, b)\)-regular graph \( G \).

Next consider \( g \geq 6 \). Let \( G = (V \cup U, E) \) be an \((a, b)\)-regular bipartite graph where the length of the shortest cycle is \( c \geq g - 2 \). Let \( S \subseteq E \). We construct a graph \( G_S = (V_S \cup U_S, E_S) \) as follows:

\[
V_S = \{0,1\} \times V, \\
U_S = \{0,1\} \times U, \\
E_S = \{(0,v),(0,u)\} \cup \{(1,v),(1,u)\} : \{v,u\} \in E \setminus S
\]

The expected number of cycles of length \( c \) in \( G_S \) is therefore equal to the number of cycles of length \( c \) in \( G \). The choice \( S = E \) doubles the number of such cycles; therefore some other choice necessarily decreases the number of such cycles. This completes the proof.

### A.6 A feasible solution of the instance \( S_k \)

Consider the instance \( S_k \) constructed in Sect. 4.2. We construct a solution \( x \) as follows. Let \( D = \max \{D_I, D_K + 1\} \). If the distance between the agent \( v \) and the objective \( k \) in \( G_k \) is \( 4j + 1 \) for some \( j \), set \( x_v = 1 - 1/D^{2j+1} \). If the distance is \( 4j + 3 \), set \( x_v = 1/D^{2j+2} \).
This is a feasible solution. Feasibility is clear for each leaf constraint \( i \in I \). Then consider a non-leaf constraint \( i \in I \). They have at most \( D_I \) neighbours, and the distance between \( k \) and \( i \) is \( 4j + 2 \) for some \( j \). Thus
\[
\sum_{v \in V_i} a_{iv}x_v \leq 1 - 1/D^{2j+1} + (D_I - 1)/D^{2j+2} < 1.
\]

Let \( \omega_k \) be the utility of this solution. We show that \( \omega_k > D_K - 1 \). First, consider the objective \( k \). We have
\[
\sum_{v \in V_k} c_{kv}x_v = D_K(1 - 1/D) > D_K - 1.
\]
Second, consider an objective \( h \in K' \setminus \{k\} \). It has \( D_K \) neighbours and the distance between \( h \) and \( k \) is \( 4j \) for some \( j \). Thus
\[
\sum_{v \in V_h} c_{hv}x_v = 1/D^{2j} + (D_K - 1)(1 - 1/D^{2j+1}) > D_K - 1.
\]
Finally, consider an objective \( h \in K \setminus K' \). It has 2 neighbours and the distance between \( h \) and \( k \) is \( 4j \) for some \( j \); the coefficients are \( c_{hv} = D_K - 1 \). Thus
\[
\sum_{v \in V_h} c_{hv}x_v = (D_K - 1)(1/D^{2j}/2 + 1 - 1/D^{2j+1}) > D_K - 1.
\]