New parametrization for the $^3\text{He}(d,p)^4\text{He}$ fusion reaction rate and refinement of the Lawson criterion for d-$^3\text{He}$ thermonuclear reactors

I B Alper, A I Godes and V L Shablov

Obninsk Institute for Nuclear Power Engineering, National Research Nuclear University MEPhI, 1 Studgorodok, Obninsk, Kaluga reg., 240040, Russia
E-mail: godes.ai@yandex.ru

Abstract. We present a new parametrization of the $d + ^3\text{He} \rightarrow p + ^4\text{He}$ fusion reaction astrophysical factor based on the effective range approximation, which is an effective theoretical method for describing near-threshold, including resonance, nuclear reactions. In the framework of this approximation we describe experimental data on the energy dependence of the cross section and the astrophysical factor within the experimental uncertainties in the energy range of 0-800 keV. On this basis we calculate the temperature dependence of the Maxwellian-averaged reaction rate in the range of 0-400 keV. In conclusion, we discuss the effect of the calculated reaction rates on the Lawson criterion for thermonuclear reactors based on d-$^3\text{He}$ fuel.

1. Introduction
For a correct description of kinetic processes in thermonuclear fusion systems, it is necessary to know the energy dependence of the fusion cross sections $\sigma$ and the reaction rates $\langle \sigma v \rangle$ with the greatest possible precision (an uncertainty in these values of the order of 5% is desirable). A change in the value of $\langle \sigma v \rangle$, even by several percent, can significantly affect the parameters of future installations [1]. The aim of this work is to determine the thermal reactivity of the reaction rate $d + ^3\text{He} \rightarrow p + ^4\text{He}$ in the range of 0-400 keV. The relevance of this work is due to the fact that to date no satisfactory theoretical description of the cross section for this reaction, consistent with the most reliable experimental data in the energy range 0–1000 keV [2, 3], has been given. For the cross section and rate of this reaction, various parameterizations have been developed [4-6], the data of which disagree significantly, especially in the resonance region of the $^5\text{Li}$ second excited state. These disagreements are mainly due to the fact that the mentioned parameterizations were created at different times and therefore were based on different experimental material. The only parameterization consistent with the available experimental data is the parameterization [6] included in the NACRE database [7]. Note that the parametrization [6] uses a special procedure for approximating experimental data, which is based on the Breit-Wigner approximation. At the same time, there is a simple and effective theoretical method for describing near-threshold, including resonance, nuclear reactions - the effective range approximation [8, 9]. This approximation is a model free approach, which operates with the experimentally observed quantities — the scattering length, the effective range and the potential shape parameter. In the case of reactions with open inelastic channels, these quantities become complex [10, 11]. The effective range approximation was used in [10, 11] to describe the cross section of this reaction. However, these works did not use or partially used the data of [2, 3].
2. $^3\text{He}(d,p)^4\text{He}$ fusion reaction rate

Within the effective range approximation, the reaction cross section is equal to [11,12]

$$\sigma_r(E) = \frac{8\pi}{3k^2} \frac{\beta(k)D(k)}{(\alpha(k) - 2h(\eta))^2 + (\beta(k) + D(k))^2},$$

(1)

where

$$D(k) = \frac{2\pi}{\exp(2\pi\eta) - 1}$$

(2)

is the Coulomb barrier penetrability, $h(k) = \text{Re} \psi(i\eta) - \text{Im} \psi(z)$ is the logarithmic derivative of the gamma function or the digamma function, $\eta = 1 / (k a_c)$ is the Sommerfeld parameter, $a_c$ - the Bohr radius for $d$-$^3\text{He}$ pair,

$$\alpha(k) = \alpha_0 + \alpha_1(k a_c)^2 + \alpha_2(k a_c)^4, \quad \beta(k) = \beta_0 + \beta_1(k a_c)^2 + \beta_2(k a_c)^4.$$ 

(3)

The values $\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2$ are the parameters of the effective range approximation instead of the scattering length, the effective range and the potential shape parameter [11,12]. The astrophysical factor

$$S(E) = E \exp(2\pi\eta) \sigma(E),$$

(4)

which is usually used in the study of processes at low energies, is described by the expression

$$S(E) = 9.11 \frac{1}{1 - \exp(-2\pi\eta)} \frac{\beta(k)}{(\alpha(k) - 2h(\eta))^2 + (\beta(k) + D(k))^2} (\text{MeV} \cdot \text{b}).$$

(5)

In this work 6 parameters mentioned above were found, which describe the data [2, 3] on the energy dependence of the cross section and the astrophysical factor within the experimental uncertainties in the energy range 0-800 keV (Figure 1). These parameters are

$$\alpha_0=0.05431 \quad \alpha_1=0.25077 \quad \alpha_2=-0.02825 \quad \beta_0=0.00205 \quad \beta_1=0.00707 \quad \beta_2=0.00169.$$ 

(6)

![Figure 1](image-url)  

**Figure 1.** $S$-factor for the reaction $d + ^3\text{He} \rightarrow p + ^4\text{He}$. Solid curve - the effective range approximation, ■ - experimental data [3] (Münster accelerator), ▲ - experimental data [3] (Bochum accelerator), • - experimental data [2]. Experimental uncertainties are about 5% (7% near the maximum).
On this basis the temperature dependence of the Maxwellian-averaged reaction rate was calculated in the range of 0-400 keV

$$<\sigma v> = \frac{32\sqrt{2\pi} c}{30^{3/2} m_r c^2} \int_0^{\infty} \sqrt{E} \exp\left(-\left(\frac{E}{\theta} + \frac{B}{\sqrt{E}}\right)\right) k^2 \sigma_1(E) dE,$$

where $c$ is the speed of light, $m_r$ is the reduced mass, the value $\exp(2\pi\eta)$ is presented as $\exp\left(\frac{B}{\sqrt{E}}\right)$ (for the d $+^3$He system the parameter $B$ is equal to 68.7508 $\sqrt{\text{keV}}$), and the function $\sigma_1(E)$ has the form

$$\sigma_1(E) = \frac{1}{1 - \exp\left(-\frac{B}{\sqrt{E}}\right) (\alpha(k) - 2h(\eta))^2 + (\beta(k) + D(k))^2}.$$

In (7) the energy was measured in keV, so that $m_r c^2 = 1124572$ (keV) and $k^2 = E$ (keV) / 17.31 (Fm$^{-2}$). In calculating the integral (6), the integration region was divided into two parts: up to 800 keV (the boundary energy corresponding to the experimental data) and above 800 keV. To eliminate the uncertainty in the value of the second part of the integral, the reaction rate was determined only for those temperatures (0-400 keV) for which the second part of the integral is much less than the first. The calculated rates are in good agreement with the data [6, 7].

![Figure 2](image-url)  
**Figure 2.** Reaction rates of the reaction $^3$He (d, p) $^4$He, calculated on the basis of the effective range approximation (gray curve), and on the basis of the NACRE compilation [7] (blue curve).

NACRE data are given with the high and low estimates for reaction rates (vertical bars), as is customary in this database.

3. The Lawson criterion

The results obtained made it possible to refine the Lawson criterion, which was used in the form given in [13]:
\[ n\tau = \frac{1.5(1 + \bar{Z})T}{E_{12}(Q^{-1} + f_c)\alpha(1 - \alpha)\sigma_n Z < Z^2 > T^{1/2}}, \] (9)

where \( n \) is the concentration of ions, \( Q \) is the ratio of the volume-average energy of the thermonuclear fusion to the volume-average energy supplied to the plasma from external sources, \( E_{12} \) is the energy yield of the fusion reaction, \( \alpha \) is the partial concentration of nuclei of type 1, \( f_c \) is the fraction of thermonuclear energy absorbed in the plasma, \( A_{ff} \) defines the losses due to bremsstrahlung of electrons on ions (without taking into account relativistic effects), \( \bar{Z} \) the average charge of plasma ions, \( < Z^2 > = \alpha Z_1^2 + (1 - \alpha)Z_2^2 \). As indicated in [14, 15], at \( T \geq 70 \text{ keV} \) relativistic effects must be taken into account in the formula for bremsstrahlung. In accordance with [15], the expression for the volume-average power of the bremsstrahlung of electrons on ions should now be written in the form

\[ P^{iso} = A_{ff}\bar{Z}(\bar{Z})^2 T^{1/2} \left\{ \frac{9}{8\sqrt{2}} \sqrt{\pi} \sqrt{\ln(2y) + \frac{1}{2} + \frac{3}{2} - C} + \exp(-2y) \right\}. \] (10)

In (10) \( C = 0.5772 \ldots \) is the Euler-Mascheroni constant, \( y \) is the temperature of the electron in units of the electron rest energy \( y = \theta(\text{keV})/511 \). In addition, the volume-average power of the electron bremsstrahlung on electrons should be added to the expression (10). This power is equal to [15]

\[ P^{iso} = \frac{4C_F}{\sqrt{\pi}} \alpha r_e^2 m_e c^3 n_e^2 y^3 \left( 1 + 1.17y + 0.28y^2 - 0.6y^3 \right), y \leq 1, \] (11)

where \( \alpha = e^2/\hbar c = 1/137 \) is the fine structure constant, \( r_e = e^2/(m_e c^2) = 2.819 \cdot 10^{-13} \text{ cm} \) is the classical electron radius, \( C_F = 5/9 \ (44 - 3\pi^2) \). The plasma confinement parameter determined on the basis of (8-10), taking into account relativistic effects and using the fusion reaction rate calculated within the effective range approximation with parameters (6), is shown in Figure 3. The minimum of the parameter \( n\tau \) is given for each curve.


Figure 3. The dependence of the confinement parameter on the plasma temperature in the effective range approximation.

The parameters of the Lawson criterion determined for a fusion reactor based on d-\(^3\)He fuel are as follows:

\[ n\tau = 5.85 \times 10^{14} \text{ cm}^{-3} \cdot \text{s}, \ T = 104 \text{ keV}, \]

whereas according to [13]

\[ n\tau = 6.5 \times 10^{14} \text{ cm}^{-3} \cdot \text{s}, \ T = 105 \text{ keV}. \]

The presented results show their difference at the level of 11%, which is associated both with taking into account relativistic effects in the expression for bremsstrahlung and using new values for the fusion reaction rates. As in [13], when determining the Lawson criterion, the input of fusion reactions \(d + d\) and \(d + \ ^3\)He + \(^3\)He was neglected.

4. Summary

In this paper we presented the results of the theoretical description of the \(d + \ ^3\)He \(\rightarrow\) \(p + \ ^4\)He fusion reaction, based on the effective range approximation, including new parametrizations of the astrophysical factor and the fusion reaction rates. Our results are in good agreement with the NACRE database [7] results for this reaction in the energy range of 0-450 keV. We also investigated the temperature dependence of the \(d - \ ^3\)He confinement parameter and refined the corresponding Lawson criterion. It was found that this criterion essentially depends on the used parameterizations of the reaction rates and the correct description of the bremsstrahlung power. We plan further to expand the energy range of parameterizations of the astrophysical factor and reaction rates and take into account the input of accompanying fusion reactions to the Lawson criterion.
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