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Holography Beyond AdS

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We continue our study of string theory in a background that interpolates between $\text{AdS}_3$ in the infrared and a linear dilaton spacetime $\mathbb{R}^{1,1} \times \mathbb{R}_\phi$ in the UV. This background corresponds via holography to a $CFT_2$ deformed by a certain irrelevant operator of dimension $(2,2)$. We show that for two point functions of local operators in the infrared CFT, conformal perturbation theory in this irrelevant operator has a finite radius of convergence in momentum space, and one can use it to flow up the renormalization group. The spectral density develops an imaginary part above a certain critical value of the spectral parameter; this appears to be related to the non-locality of the theory. In position space, conformal perturbation theory has a vanishing radius of convergence; the leading non-perturbative effect is an imaginary part of the two point function.
1. Introduction

In two recent papers [1,2] we studied a string background that interpolates between a three dimensional linear dilaton background in the ultraviolet (UV) and $AdS_3$ in the infrared (IR).\(^1\) From the UV point of view, this background can be interpreted as the bulk description of Little String Theory (LST) in a vacuum with $N$ fundamental strings [3,4]. From the IR point of view, it can be thought of as an irrelevant deformation of the two dimensional conformal field theory ($CFT_2$) dual to the above $AdS_3$ background.

Apriori, one would expect the second point of view not to be useful for studying the theory, since it corresponds to flowing up the renormalization group (RG), a process that is in general highly ambiguous. However, as pointed out in [1,2], there are reasons to believe that in this case the situation is better:

(1) While from the point of view of the $CFT_2$, the (irrelevant) deformation in question is not expected to be under control, in the dual string theory on $AdS_3$ it corresponds to adding to the worldsheet Lagrangian a marginal operator, which moreover has a current-current (null abelian Thirring) form, and thus is expected to be well behaved (in fact the deformed worldsheet theory is exactly solvable).

(2) The deformation in question shares many properties with the $T\bar{T}$ deformation of $CFT_2$, studied recently in [5,6]. The latter was argued to be well behaved and in fact the deformed theory appears to be exactly solvable.

One of the interesting properties of the theories studied in [1,2] and [5,6] is that their spectrum approaches that of a $CFT_2$ in the IR (i.e. the corresponding entropy goes like $S_{IR} \sim \sqrt{E}$), while in the UV one finds a Hagedorn spectrum $S_{UV} \sim E$. Thus, the short distance behavior of these theories is not governed by a UV fixed point – they are not local QFT’s. Hence, if one can study these theories by starting with the infrared $CFT_2$ and flowing up the RG, they provide an example of non-local theories that can be understood non-perturbatively.

A natural question is how the non-locality of these theories is reflected in the structure of correlation functions of operators that are local in the infrared CFT. In this note we will initiate the study of this problem in the model of [1,2]. Since this model corresponds to a well behaved vacuum of weakly coupled string theory, we can calculate such correlation functions in an expansion in an effective string coupling (a $1/N$ expansion, or equivalently

\(^1\) In both cases times a compact space that is a spectator under the deformation.
a large central charge expansion in the IR $CFT_2$), and study their properties. We are primarily interested in the answers to two questions:

(a) To what extent can we understand these correlation functions by flowing up the RG from the infrared fixed point?

(b) How does the non-locality of the deformed theory manifest itself in the structure of these correlation functions?

We will perform the calculations using the bulk description of the theory, but this is likely a technicality. If we understood the $CFT_2$ dual of the infrared $AdS_3$ background sufficiently well, we could perform the same calculations directly in the $CFT_2$, and due to the $AdS/CFT$ correspondence we would expect to get the same results.

A natural question is what properties do we expect the correlation functions of the deformed theory to have. According to [3], the non-locality of the theory is expected to be reflected in the fact that correlation functions in momentum space are well behaved, but their position space counterparts are not. We will discuss to what extent these expectations are realized.

Note added: The correlation functions we study were also considered in a recent paper by G. Giribet [7].

2. The construction

The starting point of our discussion is (type II) string theory in the background $AdS_3 \times \mathcal{N}$, where $\mathcal{N}$ is a compact manifold whose precise properties will not play a role below.\footnote{Well studied examples include $\mathcal{N} = S^3 \times T^4$ and $S^3 \times K_3$, which exhibit (4,4) superconformal symmetry on the boundary, and $\mathcal{N} = S^1 \times \hat{N}$, with $\hat{N}$ a (worldsheet) (2,2) superconformal background, which gives rise to (2,2) superconformal symmetry on the boundary. Supersymmetry will not play a role in our analysis below, but we expect that incorporating it into the discussion will lead to further insights.} As discussed in detail in [1,2], the deformation we are interested in corresponds in the boundary $CFT_2$ to adding to the Lagrangian the term

$$\delta \mathcal{L}_b = \lambda D(x),$$

(2.1)

where $D(x)$ is a certain dimension (2,2) quasi-primary of the spacetime conformal symmetry constructed in [8]. In supersymmetric examples, $D(x)$ is the top component of a superfield, so this deformation preserves SUSY while breaking conformal symmetry.
The above deformation can be described by adding to the worldsheet Lagrangian of string theory on $AdS_3$ the term

$$\delta L_{ws} = \lambda J^- J^+, \quad (2.2)$$

where $J^-$ is the worldsheet $SL(2, \mathbb{R})$ current whose zero mode gives rise to the spacetime (or boundary) Virasoro generator $L_{-1}$.

In the bulk description, $\delta L_{ws}$ (2.2) can be thought of as a supergravity deformation; the deformed geometry takes the form $M_3 \times N$, where $M_3$ is described by the worldsheet Lagrangian \[9,4\] (see also \[10\])

$$L = k \partial \phi \partial \phi + \frac{\lambda}{\lambda + e^{-2\phi}} \partial x^+ \partial x^-, \quad (2.3)$$

with a dilaton that goes like $\Phi \sim -\ln(1 + \lambda e^{2\phi})$. The background (2.3) interpolates between a linear dilaton spacetime $\mathbb{R} \times \mathbb{R}$ in the UV region $\phi \to \infty$, and $AdS_3$ in the IR $\phi \to -\infty$ (where it is natural to rescale $x^\pm$ by a factor of $\sqrt{\lambda/k}$, \[2\]). The coupling $\lambda$ sets the scale at which the transition takes place.

A useful description of the deformed model (2.2) is obtained by starting with the background

$$\mathbb{R}^{1,1} \times AdS_3 \times N, \quad (2.4)$$

and gauging the null current

$$i \partial (y - t) + \epsilon J^-, \quad (2.5)$$

and its right-moving analog $i \partial (y + t) + \epsilon J^-$. Here $(t, y)$ are (canonically normalized) coordinates on $\mathbb{R}^{1,1}$, and $\epsilon$ is related to $\lambda$ as $\lambda \sim \epsilon^2$.

To define observables in the background (2.3) using the above coset description, it is useful to recall the form of the observables in string theory on $AdS_3$ (see e.g. \[8,11\] for more detailed discussions and references). A large class of such observables is given by vertex operators in the (NS,NS) sector, which take the form (in the $(-1,-1)$ picture)

$$\hat{O}(x) = \int d^2 z e^{-\varphi - \overline{\varphi}} \Phi_h(x; z) O(z). \quad (2.6)$$

Here $\varphi, \overline{\varphi}$ are worldsheet fields associated with the superconformal ghosts, that keep track of the picture. $\Phi_h(x; z)$ are natural vertex operators on $AdS_3$, labeled by position on the boundary, $x$, and on the worldsheet, $z$. $O$ is an ($N = 1$ superconformal primary) operator in the worldsheet theory on $N$. The operator (2.6) satisfies the mass-shell condition

$$-\frac{h(h - 1)}{k} + \Delta_O = \frac{1}{2}, \quad (2.7)$$

which relates the scaling dimension of the operator $\hat{O}(x)$ in the spacetime (or boundary) CFT, $h$, to the worldsheet scaling dimension of the operator $O$, $\Delta_O$.\footnote{We take $\overline{\Delta}_O = \Delta_O$, so that $\overline{h} = h.$}
The operator $\hat{O}(x)$ (2.6) is a local operator in the boundary $CFT_2$. When we turn on the perturbation (2.2), it is deformed to an operator in the perturbed theory corresponding to the background (2.3), whose form was discussed in [2]. We next review this construction.

First, to facilitate the gauging, we need to Fourier transform the operators $\Phi_h(x; z)$ from the position $(x, x)$ to the momentum $(p, \vec{p})$ basis on the boundary. This gives rise to operators, which we will denote by $\Phi_h(p; z)$ (in a slight abuse of notation), which are eigenfunctions of the currents $(J^-, J^-)$ with eigenvalues $(p, \vec{p})$. These operators behave like

$$\Phi_h(p) = f_h(\phi)e^{i\vec{p} \cdot \vec{\gamma}}, \quad (2.8)$$

where the coordinates $\vec{\gamma} = (\gamma, \vec{\gamma})$ parametrize the boundary of $AdS_3$, and $\phi$ parametrizes the radial direction. Near the boundary at $\phi \to \infty$, one has $f_h(\phi) \sim e^{\sqrt{2}(h-1)\phi}$. Gauge invariance implies that (the Fourier transforms of) the operators (2.6) must be replaced in the deformed theory by

$$\hat{O}(p) = \int d^2 z e^{-\varphi-\varphi} \Phi_h(p)e^{-i\omega t+p_y y}O. \quad (2.9)$$

The mass-shell condition (2.7) is now deformed to

$$-\frac{h(h-1)}{k} + \frac{\alpha'}{4}(p_y^2 - \omega^2) + \Delta = \frac{1}{2}. \quad (2.10)$$

Moreover, gauge invariance sets $\omega = \epsilon p_0$ and $p_y = \epsilon p_1$, where $(p_0, p_1)$ are components of the vector $\vec{p}$ in (2.8).

The operators (2.9) are natural observables in the background (2.3). In the IR limit $p_y, \omega \ll m_s$, they reduce to those in $AdS_3$ (the Fourier transforms of (2.6)), and their correlation functions reduce to those of the $CFT_2$ dual to string theory in $AdS_3 \times N$. To study these correlation functions away from the infrared limit, we need to compute them for general momenta. In this note we will focus on the two point function, which we turn to in the next section.

### 3. Two point function in momentum space

The goal of this section is to compute the two point function of the operators (2.9), $\langle \hat{O}(p)\hat{O}(-p) \rangle$. We will do this calculation in Euclidean space, i.e. rotate $t \rightarrow i\tau$ and $\omega \rightarrow -ip_y$ in (2.9), (2.10). To do the calculation, we need to specify the normalizations of the operators in (2.9). We will take the operator $O$ from the internal worldsheet CFT,
\[ \langle O(z)O(w) \rangle = \frac{1}{|z - w|^{4\Delta_O}}. \] This choice will not play an important role below, essentially because the internal CFT is a spectator under the deformation (2.2).

A more significant choice (as we will see below) is that of the normalization of the operator \( \Phi_h \) in (2.9). Recall that in conformal field theory on \( AdS_3 \) one has [12]

\[ \langle \Phi_h(x; z)\Phi_h(y; w) \rangle = \delta(h - h') \frac{B(h)}{|z - w|^{4\Delta_h}|x - y|^{4h}}, \tag{3.1} \]

where \( \Delta_h = -\frac{h(h-1)}{k} \) is the dimension of \( \Phi_h \) (see (2.7)), and \( B(h) \) is a function whose precise form depends on the normalization of \( \Phi_h \). In a natural normalization, it takes the form

\[ B(h) = \frac{k}{\pi} \nu^{2h-1}\gamma(1 - \frac{2h - 1}{k}), \quad \gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}, \tag{3.2} \]

where \( \nu \) is a constant whose value can be adjusted by shifting the radial coordinate \( \phi \) in (2.3) or, equivalently, by a scale transformation in the boundary CFT [11].

Consider first the undeformed theory, corresponding to the background \( AdS_3 \times \mathcal{N} \). Using the worldsheet two point functions above, one can compute the two point function of the operators (2.6) in the boundary CFT. One finds [11,13]

\[ \langle \hat{O}(x)\hat{O}(y) \rangle = \frac{D(h)}{|x - y|^{4h}}, \quad D(h) = (2h - 1)B(h). \tag{3.3} \]

The calculation that led to (3.3) was done in a particular normalization of the operators \( \Phi_h \), (3.1), (3.2). One could rescale the operators by an arbitrary smooth function of \( h \), which would change the normalization \( D(h) \). In \( AdS_3 \) such a rescaling is harmless, and does not change the essential physics. As we will see, this is the case in the deformed background (2.3) as well, although less trivially so.

As mentioned above, to study the deformed background \( M_3 \times \mathcal{N} \) we need to Fourier transform the operators (2.6). The Fourier transform of the two point function (3.3) is

\[ D(h) \int d^2x e^{ip\cdot\bar{x}} |x|^{-4h} = \pi D(h)\gamma(1 - 2h) \left( \frac{p^2}{4} \right)^{2h-1}, \tag{3.4} \]

where we used the notation \( p = (p_r, p_y) \). The momentum space two point function (3.4) has poles when \( 2h - 1 = n \in \mathbb{Z}_+ \). These poles have a simple interpretation – they correspond to cases where the two point function behaves like \( p^{2n} \ln p^2 \). One can analytically continue the two point function (3.4) in the \( h \) plane around these poles, and we will do this below.
The momentum space two point function (3.4), with \( h \) given by the solution of the mass-shell condition (2.7), is valid in the undeformed background \( AdS_3 \times \mathcal{N} \). However, looking back at (2.9), we see that the same formula describes the two point function in the deformed background \( \mathcal{M}_3 \times \mathcal{N} \), except now \( h \) is given by a solution of the deformed mass-shell condition (2.10).

The dimensionful coupling \( \lambda \), (2.2), does not appear in this mass-shell condition. One can think of this in the following way. In (2.9) we parametrized the boundary in terms of the coordinates \((t, y)\), and their conjugate momenta \((\omega, p_y)\), or after Wick rotation \((\tau, y)\) and \((p_\tau, p_y)\). These coordinates are related by a factor of \( \epsilon \), (2.5), to the natural coordinates on the boundary of \( AdS_3 \), in terms of which the low energy \( CFT_2 \) is phrased, \((\gamma, \bar{\gamma})\).

As explained in [1,2], one can choose a value for \( \lambda \) so that the rescaling factor between the coordinates on the boundary of \( AdS_3 \) and linear dilaton spacetime is equal to one. In that case, the mass-shell condition (2.10) holds in terms of both. More generally, if we want to express the mass-shell condition in terms of the momenta conjugate to the coordinates on the boundary of \( AdS_3 \), \((\gamma, \bar{\gamma})\), we need to replace \((\omega, p_y)\) by \( \epsilon(p_0, p_1) \) in (2.10), which gives rise to a factor of \( \epsilon^2 \sim \lambda \) in front of the \( p^2 \) term.

Since we will be interested in the dependence of the correlation functions on \( \lambda \), we will find it convenient to exhibit the factors of \( \lambda \) explicitly. Hence, we will write the mass-shell condition (2.10) in the form

\[
-\frac{1}{k} h p^2 (h p^2 - 1) + \frac{1}{2} \lambda p^2 + \Delta O = \frac{1}{2},
\]

where \( p^2 = p_\tau^2 + p_y^2 \) (in Euclidean space). Denoting by \( h \) the value of \( h_{p^2} \) in the undeformed theory, (2.7), \( h = \lim_{p \to 0} h_{p^2} \), one has

\[
2h_{p^2} - 1 = \sqrt{2(2h - 1)^2 + 2\lambda kp^2}.
\]

One can think of (3.6) as describing the dependence of the dimension of the operator \( \hat{O}(p) \) (2.9) on the scale. This kind of dependence is standard in non-conformal theories – it reflects the fact that \( \hat{O} \) has a non-vanishing anomalous dimension.

Combining all the elements of the above discussion, we find the two point function of the operators (2.9),

\[
\langle \hat{O}(p)\hat{O}(-p) \rangle = \pi D(h_{p^2}) \gamma(1 - 2h_{p^2}) \left( \frac{p^2}{4} \right)^{2h_{p^2} - 1}.
\]
In the rest of this section we will discuss this result.

The first question we would like to address is the dependence of the result on the normalization of the operators (2.6) in string theory on $AdS_3$. As we mentioned above, before we perturb the theory by (2.2), we can rescale these operators by any function of $h$ without changing the physics. However, (3.7) seems to suggest that after the deformation the result does depend on this normalization, since now the factor $D(h,p)$ depends on the momentum, and changing $D(h)$ changes the momentum dependence of the two point function.

One way to understand the origin of this dependence from the perspective of the boundary theory is the following. We start with the observable $\hat{O}(x)$ in the IR CFT, which has dimension $(h,h)$ and some particular normalization, e.g. (3.3). We then add to the Lagrangian of the boundary theory the irrelevant perturbation (2.1), and evaluate the correlation function (3.7) in the perturbed theory, e.g. by using conformal perturbation theory. This calculation is sensitive to the contact terms of the perturbing operator $D(x)$ (2.1) with the operators $\hat{O}(x)$. Since $D(x)$ is a dimension $(2,2)$ operator, and $\lambda$ has dimension $(-1,-1)$, these contact terms take the form

$$D(x)O(y) = \delta^2(x - y) F(\lambda \partial_y \partial_\overline{y}) \partial_y \partial_\overline{y} O(y),$$

where $F(z) = F_0 + a_1 z + a_2 z^2 + \cdots$ is an arbitrary smooth function of its argument. These contact terms give rise to a $\lambda$ (or, equivalently, momentum) dependent redefinition of the operator $O(x)$, and they are the origin of the dependence of the two point function (3.7) on the normalization in our bulk construction.

The situation is similar to the study of the geometry of the space of conformal field theories in [14]. There, contact terms provide information about the connection on the space of theories, which is not reparametrization invariant information on this space, however the metric on moduli space, which does depend on the choice of contact terms, still contains invariant information, such as the curvature of the space of theories. Here, contact terms parametrize the freedom to redefine the operators in a smooth way as we move around in the space of theories, but the two point function (3.7) still contains invariant information that does not depend on this freedom.

Thus, the relevant question is what is the invariant information contained in the two point function (3.7)? One issue is the analyticity properties of the two point function in momentum space. As is clear from the expression (3.7), there are three possible sources
of singularities: (a) the two point function in $AdS_3$, (3.3), $D(h)$; (b) the factor $\gamma(1 - 2h)$; (c) the function $h_p^2$ (3.6). We next discuss them in turn.

As mentioned before, in a natural normalization, the two point function $D(h)$ is given by (3.2), (3.3). It has singularities when $2h - 1 \in k\mathbb{Z}$, which give rise via (3.6) to singularities in momentum space. These singularities are poles of the sort discussed in [15], where they were referred to as bulk poles. As we also discussed, one can rescale the operators (2.9) so that the function $D(h)$ changes from (3.2), (3.3). We will assume that such a rescaling does not introduce additional singularities.

The key point about the singularities of $D(h)$ is that they reflect physics present already in the undeformed theory, the CFT dual to string theory on $AdS_3 \times \mathcal{N}$; they are related to the continuum of long strings present in this theory. One can work in a regime in which the singularities of the two point function of the deformed theory due to this factor do not play a role, e.g. by taking the level $k$ to be large and studying operators with $h_p^2$ of order one.

The factor $\gamma(1 - 2h)$ has singularities when $2h - 1 \in \mathbb{Z}$ that were mentioned above. According to [15], they have a similar interpretation, as bulk poles. From our perspective here, these poles have a kinematic origin – they appear as a result of the Fourier transform (3.4). Thus, one expects them not to play an important role in studying the theory. We will make this more precise in the next section, when we show that the spectral density does not include this factor.

Thus, the only non-trivial singularities of the two point function (3.7) are those associated with the behavior of the function $h_p^2$ (3.6) in momentum space. In the complex $p^2$ plane, the function $h_p^2$ has a branch cut starting at $p^2 = -(2h - 1)^2/2\lambda k$. Thus, for real Euclidean momenta, the two point function is smooth everywhere.

In Minkowski spacetime, we have $p^2 = p_y^2 - p_t^2$, so the two point function is smooth for spacelike momenta, and has a branch cut in the timelike domain, starting at the point discussed in the previous paragraph. Conformal perturbation theory gives rise in this case to a well defined two point function in a finite range of timelike momenta. This case involves some subtleties that we will leave to future work.

The position of the branch cut is at the point where the modes (2.9) go from being non-normalizable to delta function normalizable ($h_p^2 = 1/2$). Thus, it is associated with the transition from physics dominated by the $AdS_3$ part of the geometry (2.3) to physics dominated by the UV linear dilaton region. Therefore, this branch cut is directly related to the non-locality of the theory.
Another interesting question is to what extent we can define the theory by doing conformal perturbation theory around the infrared fixed point, i.e., perturbing in the interaction Lagrangian (2.1). We see from the form of the result (3.7) that the expansion has a finite radius of convergence. To be precise, if we strip off the factor $D(h_p^2)\gamma(1 - 2h_p^2)$ in (3.7), for the reasons explained above, we are left with a series in $\lambda$ that has a finite radius of convergence, $|(2h - 1)^2/2kp^2|$. We can compute the coefficients in this series by doing conformal perturbation theory; this defines the two point function for arbitrary spacelike momenta, and in a finite range of timelike momenta. In this sense, one can in this case flow up the RG.

As explained in [1,2] and mentioned above, the deformed theory (2.1), (2.2) is not a local QFT. This must be reflected in the form of the two point functions at large Euclidean momenta. In a theory that is governed by a UV fixed point, two point functions of local operators go, in momentum space, like powers of $p$ (as in (3.4)). It is natural to ask what happens in our case. As mentioned above, this calculation is ambiguous due to the freedom of changing the contact terms (3.8) in the boundary theory, or changing the function $D(h)$ (3.3) in the bulk description. Nevertheless, for a given choice, one can calculate the high momentum behavior. Below we describe the results for the natural definition from the bulk perspective (3.1) – (3.3).

We will take the dimension of the operator in the infrared CFT, $h$ (3.6), to be of order one, and study the large momentum limit of the two point function (3.7). There are two natural regimes to consider. One is the regime where $k \gg h_p^2 \gg 1$. In that regime the two point function behaves like

$$
\frac{k^2 \csc \left( \pi \sqrt{k\lambda p^2} \right)}{\sqrt{k\lambda p^2}} \left( k\lambda \Lambda^2 \right)^{-\sqrt{k\lambda p^2}},
$$

(3.9)

where $\Lambda$ is an arbitrary scale, whose value is related to $\nu$ in (3.2). This function behaves like $\exp \left( -2c\sqrt{\lambda kp^2} \right)$ for some constant $c$ (away from the poles).

The exponential behavior of (3.9) is actually physically insignificant. We mentioned above the freedom of shifting the radial coordinate $\phi$, which has the effect of changing the constant $\nu$ in (3.2), and as a consequence the scale $\Lambda$ in (3.9). Performing such a shift in a general $n$ point function of operators of the form (2.9) rescales the amplitude by a factor that goes at large momenta like $\prod_{j=1}^{n} \exp (\alpha p_j)$. Thus, the exponential factor in (3.9) is
a feature of the particular normalization of the operators, and can be removed, uniformly in all correlation functions, by normalizing the operators appropriately.\footnote{For early discussions of this exponential behavior see \cite{16,17}.}

A second regime that one may consider is the asymptotic high Euclidean momentum regime $h_p^2 \gg k$, where one finds

$$k \csc \left( \pi \sqrt{k\lambda p^2} \right) \csc \left( \frac{\pi \sqrt{\lambda p^2}}{k} \right) \left( k\lambda \Lambda^2 \right)^{-\sqrt{k\lambda p^2}} \left( \frac{\lambda p^2}{k} \right)^{-\sqrt{\frac{\lambda p^2}{k}}}.$$  \hspace{1cm} (3.10)

The exponential factor in $p$ can be removed as before, but now the two point function behaves like $p^{-\alpha p}$, with $\alpha$ a constant that can be read off (3.10). Clearly, the behavior of the two point function in this regime differs significantly from that of a local QFT.

The above discussion is reminiscent of the results of \cite{18} on the high momentum behavior of the scattering phase from Euclidean black holes in string theory. There, as here, this behavior comes in the bulk from classical string theory effects, and is invisible in supergravity. The difference between the two analyses is that \cite{18} considers delta function normalizable wave-functions (scattering states), while here we studied non-normalizable wave-functions, relevant for holographic correlation functions. It would be interesting to investigate the connection between the two further.

4. Spectral density and two point function in position space

4.1. Spectral density

The Kallen-Lehmann representation relates the momentum space two point function to the spectral density $\rho(\mu^2)$ via the relation

$$\langle \hat{O}(p)\hat{O}(-p) \rangle = \int_0^\infty d\mu^2 \frac{\rho(\mu)}{\mu^2 + p^2}. \hspace{1cm} (4.1)$$

We can use the form of the two point function (3.7) to compute the spectral density $\rho$. In fact, we will solve a more general problem; we will assume that the two point function takes the form

$$\langle \hat{O}(p)\hat{O}(-p) \rangle = G(\lambda p^2) \left( \frac{p^2}{4} \right)^{F(\lambda p^2)}, \hspace{1cm} (4.2)$$
where $F$, $G$ are arbitrary functions, smooth near the origin, and compute $\rho$ (4.1). In our case, one has

$$
F(\lambda p^2) = 2h p^2 - 1, \quad G(\lambda p^2) = \pi D(h p^2) \gamma(1 - 2h p^2).
$$

(4.3)

Since we assume that $F, G$ are smooth functions, we can Taylor expand (4.2) in $\lambda p^2$, and write it as

$$
\langle \hat{O}(p) \hat{O}(-p) \rangle = \sum_{n=0}^{\infty} a_n (\lambda p^2)^n / n!;
\quad a_n = \partial^n_{\alpha} \left( \left( \frac{p^2}{4} \right)^{F(\alpha)} G(\alpha) \right) \bigg|_{\alpha=0}.
$$

(4.4)

As discussed in the previous section, the series (4.4) has a finite radius of convergence.

We next show that the spectral density $\rho(\mu)$ takes in this case the form

$$
\rho(\mu) = \sum_{n=0}^{\infty} b_n (-\lambda \mu^2)^n / n!;
\quad b_n = \partial^n_{\beta} \left( \left( \frac{\mu^2}{4} \right)^{F(\beta)} H(\beta) \right) \bigg|_{\beta=0},
$$

(4.5)

where the function $H(\beta)$ needs to be computed. To confirm this ansatz and compute $H(\beta)$, we plug (4.5) into the definition (4.1) and compute the integral over $\mu$. Doing that we find that

$$
H(\beta) = -\frac{1}{\pi} G(\beta) \sin \pi F(\beta).
$$

(4.6)

Plugging in the form of $F$ and $G$ for our case, (4.3), we have

$$
H(\lambda \mu^2) = \pi \frac{D(h \mu^2)}{\Gamma^2(2h \mu^2)}.
$$

(4.7)

Plugging (4.7) into (4.5) and summing the Taylor series, one finds

$$
\rho(\mu) = \pi \frac{D(h - \mu^2)}{\Gamma^2(2h - \mu^2)} \left( \frac{\mu^2}{4} \right)^{2h - \mu^2 - 1}.
$$

(4.8)

We see that the spectral density takes a similar form to the momentum space two point function (4.2), with $h p^2 \rightarrow h - \mu^2$. In particular, the radius of convergence of the perturbative series (4.5) is again finite, and given by\footnote{To be precise, for the $D(h)$ given by (3.2), (3.3), this is the radius of convergence of the series for $h \ll k$, when we do not have to worry about singularities of $D(h)$.} $\lambda \mu^2 = (2h - 1)^2 / 2k$. Beyond that point, there is a branch cut and the spectral density develops an imaginary part.

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This is at first sight puzzling, since unitarity implies that the spectral density must be real (and positive). It is possible that the imaginary part of $\rho$ is related to the non-locality of the theory. Indeed, in a local QFT the reality of $\rho$ follows from the requirement that local operators commute for spacelike separations (see e.g. [19]). The operators $\hat{O}$ (2.6) are indeed local in the infrared CFT, but in the full theory one does not expect them to be local. The imaginary part of $\rho$ might be a sign of this non-locality. This interpretation is reasonable since, as before, the branch cut we find occurs precisely at the threshold for creating the continuum of states living in the linear dilaton region.

We now study the large $\lambda \mu^2$ behavior of the spectral density (4.8). For $k \gg \lambda \mu^2 \gg 1$ we find

$$\rho(\mu) \sim e^{-i \sqrt{k \lambda \mu^2} \ln k \lambda \Lambda^2} \left( \frac{\lambda \mu^2}{k^3} \right)^{-\frac{1}{2}} e^{\pi \sqrt{k \lambda \mu^2}},$$

while for $\lambda \mu^2 \gg k$ we have

$$\rho(\mu) \sim e^{-i \sqrt{k \lambda \mu^2} \ln \left[ \left( \frac{\mu}{\pi \nu^2} \right)^{\frac{1}{2}} k \lambda \Lambda^2 \right]} \left( k \lambda \mu^2 \right)^{\frac{1}{2}} e^{\pi \sqrt{k \lambda \mu^2}}.$$  

The exponential growth of $\rho$ with $\mu$ is likely related to the Hagedorn spectrum of the theory.

4.2. Position space two point function

Our next task is to compute the two point function in position space, which is obtained by taking the Fourier transform of (3.7), or equivalently (4.1). Using the identity

$$\int d^2 p \frac{e^{i \vec{p} \cdot \vec{y}}}{p^2 + \mu^2} = 2\pi K_0 (\mu |y|),$$

the Fourier transform of (4.1) gives

$$\langle \hat{O}(\vec{y}) \hat{O}(0) \rangle = 2\pi \int_0^\infty d\mu^2 \rho(\mu) K_0 (\mu |y|),$$

where $K_0$ is the modified Bessel function of the second kind.

Since the spectral density (4.8) is complex for $\lambda \mu^2 > (2h - 1)^2/2k$, we see from (4.12) that the two point function in position space is complex. The imaginary part of the two point function is given by

$$\text{Im} \langle \hat{O}(\vec{y}) \hat{O}(0) \rangle = 2\pi \int_{\mu_0}^\infty d\mu^2 \text{Im} \rho(\mu) K_0 (\mu |y|),$$
where $\mu_0^2 = \frac{(2h-1)^2}{2\lambda k}$.

In the regime where $\mu_0|y| \gg 1$, using $\lim_{|z| \to \infty} K_0(z) \approx e^{-z}$, we find that the imaginary part of the two point function goes like

$$\text{Im}\langle \hat{O}(y)\hat{O}(0) \rangle \sim e^{-\sqrt{\frac{(2h-1)^2}{2\lambda}}|y|^2 / x},$$

Thus, we see that the imaginary part of the two point function is non–perturbative in the natural dimensionless expansion parameter $\lambda/y^2$.

Since the leading non-perturbative effect goes like $e^{-c_n \sqrt{\lambda}/y}$, one expects on general grounds a perturbative series of the form

$$\langle \hat{O}(y)\hat{O}(0) \rangle = \frac{1}{|y|^{4h}} \sum_{n=0}^{\infty} c_n \left( \frac{\lambda}{y^2} \right)^n,$$

where the (real) coefficients $c_n$ have the large order behavior

$$c_n \sim n^\alpha e^{-\beta n} (2^n)!,$$

with some constants $\alpha, \beta$. One can verify that this is indeed the case in the following way. As mentioned above, the perturbative expansion of the spectral density, (4.5), (4.8), has a finite radius of convergence. Substituting this expansion into (4.12), and using the fact that for large $n$ the integral over $\mu$ is dominated by a saddle point at large $\mu y$, leads to the result (4.16).

5. Discussion

In this note we initiated the study of correlation functions in the model of [1,2], that interpolates between a two dimensional CFT in the IR and a theory with a Hagedorn density of states (Little String Theory) in the UV. We saw that two point functions of a large class of scalar operators exhibit an interesting analytic structure that seems to be related to the non-local nature of the theory. There are many things that remain to be understood. Below we mention a few examples.

We saw that in momentum space, conformal perturbation theory in the coupling $\lambda$, (2.1), (2.2), has a finite radius of convergence, and Euclidean two point functions are well defined for arbitrary momenta. Thus, one can define these correlation functions by starting from the IR and flowing up the RG. In position space the expansion in $\lambda$ is asymptotic,
and the leading non-perturbative effect appears to be an imaginary part of the two point functions. It would be interesting to understand this non-perturbative effect better.

Our discussion was restricted to the case where the coupling $\lambda$ is positive. For negative $\lambda$ the geometry (2.3) exhibits a naked singularity, and it is interesting to see what the consequences of that are in the correlation functions computed here. The momentum space two point function (3.7) has for negative $\lambda$ similar properties to those of the spectral density (4.8) for positive $\lambda$ – it is given by a perturbative expansion with a finite radius of convergence, and there is a branch cut starting at the place where $h_{\mu\nu}$ (3.6) reaches the critical value $1/2$. Beyond that point, the momentum space two point function develops an imaginary part. On the other hand, the spectral density (4.8) is now well defined for all finite $\mu$. It would be interesting to see if one can use these correlation functions to learn something about the physics associated with the naked singularity in the bulk geometry.

In this note we considered the simplest correlation functions – two point functions of scalar operators that in the bulk description correspond to vertex operators of the form (2.6), (2.9). There are many interesting generalizations of this analysis. One can consider higher point functions of these operators, that will probably reveal some additional properties of these theories. Another interesting generalization is to correlation functions of conserved currents. In the infrared CFT one typically has a symmetry algebra generated by holomorphic and anti-holomorphic currents, including the Virasoro generators $T(x)$, $\overline{T(\overline{x})}$, affine Lie algebra generators $K^a(x)$ corresponding to various groups, and super-currents. After the deformation, we expect these currents to give rise to conserved but non-holomorphic currents, such as the conserved stress-tensor $T_{\mu\nu}$. It would be interesting to calculate the two point functions of such currents, and use them to compute the Zamolodchikov $c$ function and other related functions.

Our considerations did not use supersymmetry, but most (perhaps all) consistent examples of the theories we discussed are supersymmetric. It would be interesting to bring the techniques of supersymmetric field theory to bear on the analysis of these theories.

As mentioned in the introduction, part of the motivation for our work, here and in [1,2], is the results of [5,6] on $T\overline{T}$ deformed CFT$_2$. It would be interesting to calculate correlation functions analogous to those considered here for that case. There are reasons to believe that these correlation functions have a lot in common with those computed here. In particular, it would interesting to see if the momentum space two point functions are well defined for arbitrary momenta, to see if conformal perturbation theory has a finite radius of convergence in momentum space and zero radius of convergence in position space,
and whether the position space two point function has a non-perturbative imaginary part, as in our case.

Another setting in string theory where irrelevant deformations were analyzed and found to be under some control is two dimensional string theory. The authors of [20,21] analyzed the $c = 1$ CFT coupled to two dimensional gravity, in the presence of a deformation of the form $\delta L = \lambda \cos pX$. Before coupling to gravity, the coupling $\lambda$ is relevant for $p$ smaller than some critical value (for which the dimension of $\cos pX$ is equal to 2), and irrelevant otherwise. When it is irrelevant, one does not expect to be able to flow up the RG generated by the coupling $\lambda$ in a unique way. After coupling to gravity, the coupling $\lambda$ becomes marginal, and one can compute correlation functions in the model in conformal perturbation theory. This was done, using the dual matrix model, in the above papers, and it was found that the resulting expansion is asymptotic in a fixed area representation, but at fixed cosmological constant the expansion has a finite radius of convergence. In [20,21] it was not clear what the UV behavior of the model is in the case where the coupling $\lambda$ is irrelevant, and whether the model makes sense for all scales in this case. In our case one expects the model to exist for all scales, and the challenge is to understand the way its behavior is reflected in the form of the correlation functions. It would be interesting if the two models shed some light on each other.

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