Abstract

We consider the type IIB supergravity in ten dimensions compactified on $S^1 \times T^4$, with intersecting one and five D-branes in the compact dimensions. By imposing the spherical symmetry in the resulting five dimensional theory, we further reduce the s-wave sector of the theory to a two dimensional dilaton gravity. Via this construction, the techniques developed for the general two dimensional dilaton gravities are applicable in this context. Specifically, we obtain the bosonic sector general static solutions. In addition to the well-known asymptotically flat black hole solutions, they include solutions with naked singularities and non-asymptotically flat black holes.

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I. INTRODUCTION

One of the most exciting recent developments in black hole physics is the microscopic counting of the black hole entropy from string theory [1]. For a class of extremal and near extremal black holes in supergravity theories, the area law for the entropy and the quantum emission rate from near extremal black holes were derived from the microscopic considerations of string theory [2]. The simplest and illuminating example comes from the type IIB supergravity compactified on $S^1 \times T^4$ with intersecting one and five $D$-branes [3].

The classical supergravity solutions in this case can be related to the microscopic system involving $D$-branes. The electric and magnetic charges of the RR three form field strength can be interpreted as the wrapping number of $D$ one-branes around $S^1$ and the wrapping number of $D$ five-branes around $S^1 \times T^4$, respectively. The Kaluza-Klein charge along the circle $S^1$ corresponds to the massless excitations of $D$-branes represented by the open strings connecting the one and five $D$-branes.

Throughout these lines of investigation, it is of prime importance to obtain the classical black hole solutions of the supergravity theories, and considerable efforts were devoted to this issue [4] [5]. In Ref. [6], a six-parameter class of black hole solutions in the type IIB supergravity compactified on $S^1 \times T^4$ were constructed; the parameters represent the black hole mass, one-brane charge, five-brane charge, Kaluza-Klein momentum, the asymptotic circle size and the asymptotic four-torus size. One important tool in the construction of the supergravity solutions is the solution generating technique; starting from a particular generating solution, the general black holes solutions with no-hair property are obtained by maximal $S$- and $T$-duality transformations [4] [7] [8].

Considering these developments, it is of some interest to find the general static solutions compatible with the intersecting one-brane and five-brane configurations on $S^1 \times T^4$. First, that kind of analysis will show in detail how potential hairs are prevented for black hole solutions. Secondly, if there exist self-dual solutions under the solution generating transformations mentioned above, they can not be obtained by the solution generating techniques.
In this paper, by restricting our attention only to the s-wave sector of the five dimensional theory (obtained by compactifying type IIB theory on $S^1 \times T^4$), we achieve this task.

In Section II, we observe that the s-wave sector of the type IIB supergravity on $S^1 \times T^4$ becomes a particular type of two dimensional dilaton gravity theory [9]. Due to the vastly simplified dynamics in two dimensions, some analytic treatments are possible [10]. For example, the procedures for getting the general static solutions for a large class of two dimensional dilaton gravity theories were given in Ref. [11]. Following that method, in Section III, we obtain the general static solutions labeled by fifteen parameters without any assumptions about the global structure of the space-time. The solution space has very rich structures composed of many sectors. We present a detailed study in a particular sector that, with an appropriate restriction of parameters, includes the solutions of Ref. [6]. We find that unless the charges of modulus fields are set to certain values (determined by gauge charges and the asymptotic values of the modulus fields) we have naked singularities. Thus, if we make a restriction on the parameters to avoid naked singularities, we recover the asymptotically flat black hole solutions of Ref. [6]. In addition, our general solutions in that sector also include non-asymptotically flat black hole type solutions. A similar result is already known in the context of the four dimensional dilaton gravity; in that theory, there are asymptotically flat black hole solutions by Garfinkle, Horowitz and Strominger [12] and non-asymptotically flat black hole type solutions of Ref. [13]. In Section IV, we comment on some more aspects of the general static solutions and our approach.

II. DIMENSIONAL REDUCTION $10 \to 6 \to 5 \to 2$

We start with the bosonic sector of the ten-dimensional type IIB supergravity compactified on $S^1 \times T^4$.

$$I^{(10)} = \int d^{10}x \sqrt{-g^{(10)}} \left(e^{-2\phi} R^{(10)} + 4e^{-2\phi}(D\phi)^2 - \frac{1}{12}H^{(10)2}\right)$$  \hspace{1cm} (1)

We include in our action the metric $g^{(10)}_{\mu\nu}$, the dilaton $\phi$ and the R-R three-form field strength $H^{(10)}$. Among the bosonic fields of the type IIB theory, we set the NS-NS three-form field
strength, the R-R one-form field strength and the R-R self-dual five-form field strength zero, since we consider the configuration of intersecting one and five D-branes in this paper. The signature for the metric and other conventions are explained in the Appendix A.

We first compactify the ten dimensional theory on the four-torus $T^4$. We assume the maximal symmetry on the torus (the Euclidean Poincare symmetry for the covering space $R^4$ and the hypercubic lattice for the discrete action on $R^4$) and, therefore, we include only single modulus $\psi$ for the $T^4$. The ten dimensional metric is thus given in the block diagonal form

$$ds^2 = g^{(6)}_{\alpha\beta} dx^\alpha dx^\beta + e^{\psi} dx^m dx^m$$

(2)

where the Greek indices $\alpha$ and $\beta$ run from zero to five and the Latin index $m$ runs from six to nine with the flat Euclidean metric. The scalar field $\psi$ measures the size of the four-torus. The gauge invariant three form field strength $H^{(10)}$ should also respect the assumed symmetry. Consequently, we set $H^{(10)}_{\alpha\beta m} = 0$, $H^{(10)}_{\alpha mn} = 0$ and $H^{(10)}_{mnl} = 0$ and retain only $H^{(10)}_{\alpha\beta\gamma} = H^{(6)}_{\alpha\beta\gamma}$, which is both a scalar on the $T^4$ and a three form field strength on the six-dimensional transversal manifold. Under the assumed symmetry, we note that the invariant objects are the scalars that do not depend on $x^m$ coordinates and the volume form $\epsilon^{mnpq}$, that is a pseudo-scalar, on the torus. Therefore the metric tensor, the dilaton, the three form field strength and the torus modulus depend only on the $x^a$ coordinates, and they are of the above form. This type of dimensional reduction picks out only the zero modes on the $T^4$ in the Kaluza-Klein reduction process. Using the formulas given in Appendix B for the block diagonal metric, we get the six-dimensional action

$$I^{(6)} = \int d^6 x \sqrt{-g^{(6)}} (e^{2\psi-2\phi} (R^{(6)} + 3(D\psi)^2 - 8D_\alpha \psi D^\alpha \phi + 4(D\phi)^2) - \frac{1}{12} H^{(6)}{}^2),$$

(3)

where the covariant derivative $D$ are with respect to the six-dimensional metric.

We further compactify the six-dimensional theory on a circle $S^1$ to obtain the five-dimensional theory on $M^5$. We again assume the maximal symmetry on the covering space $R^1$ (one dimensional Euclidean Poincare symmetry) of the $S^1$. Thus, we assume the six-dimensional metric of the form
where all the entries are independent of the the position $\theta$ on the circle. The scalar field $\psi_1$ is the circle modulus. The six-dimensional three form field strength $H^{(6)}$ decomposes into the five-dimensional three form field strength $H^{(5)}_{\alpha\beta\gamma} = H^{(6)}_{\alpha\beta\gamma}$ and the two form field strength $F_{2\alpha\beta} = H^{(6)}_{\alpha\beta\theta}$. Using the formulas summarized in the Appendix A, we retain only the zero modes on the $S^1$ to obtain the dimensionally reduced five-dimensional action

$$I^{(5)} = \int d^5x \sqrt{-g^{(5)}} \left( e^{2\psi_1 + \psi_1/2 - 2\phi} \left( R^{(5)} + 2D_{\alpha}\psi D^{\alpha}\psi_1 - 2D_{\alpha}\phi D^{\alpha}\psi_1 \right) - 8D_{\alpha}\phi D^{\alpha}\psi + 3(D\psi)^2 + 4(D\phi)^2 \right) - \frac{1}{12} H'^2 - \frac{1}{4} e^{-\psi_1} F_{2\alpha\beta} F_{2\alpha\beta} - \frac{1}{4} e^{-2\psi_1} F_{\alpha\beta} F^{\alpha\beta}. $$

The covariant derivative $D$ and the summation of the indices, that run from zero to four, are on the five-dimensional manifold. The two form field strength $F = dA$ denotes the Kaluza-Klein $U(1)$ gauge field that originates from the momentum along $S^1$. In Eq. (5), the three form $H'$ is given by

$$H' = H^{(5)} - A \wedge F_2. $$

Via the construction so far, we obtained a five dimensional theory on the external space-time $\mathcal{M}^5$ described by the action (3). The compact submanifold $S^1 \times T^4$ of the ten dimensional space-time plays the role of the internal space-time. For this interpretation to be meaningful, we should require that the circle modulus $\exp(\psi_1)$ and the four-torus modulus $\exp(\psi)$ be bounded from the above for all points on the external manifold $\mathcal{M}^5$.

The main focus of this note is to investigate the s-wave sector of the five-dimensional action Eq. (3). Thus, we impose the rotational symmetry and consider the five dimensional manifold of the form $\mathcal{M}^5 = \mathcal{M}^2 \times S^3$. The compatible metric can be written as

$$ds^2 = g^{(2)}_{\alpha\beta} dx^\alpha dx^\beta + e^{-2\psi_2} d\Omega^{(3)}$$

where the two-dimensional metric $g^{(2)}_{\alpha\beta}$ on the $\mathcal{M}^2$ and the radius of the sphere $\exp(-\psi_2)$ depend only on the coordinates $x^\alpha$ on the $\mathcal{M}^2$. The metric $d\Omega^{(3)}$ is the metric on the unit $S^3$.
with three angle coordinates $\theta_1, \theta_2$ and $\theta_3$. To make the five dimensional manifold $M^5$ non-compact, the radius of the $S^3$ should not have a finite upper bound on the $M^2$. Imposing the rotational symmetry requires that the scalar fields in the five dimensional action (5) depend only on the $x^\alpha$ coordinates. In addition, it is necessary that the gauge invariant field strengths be the scalars on the $S^3$ or be proportional to the pseudo-scalar $\epsilon_{\theta_1\theta_2\theta_3}$, the volume form on the unit three-sphere. Thus, the Kaluza-Klein two form field strength $F$ has non-vanishing components $F_{\alpha\beta}$, and the other components that have indices along the $S^3$ directions vanish. The similar result holds for the two form field strength $F_2$. Thus in the five-dimensional action (5), $H' = H^{(5)}$ since we can always choose the vector potential $A$ in such a way that $A = A_\alpha dx^\alpha$. For the three form field strength $H^{(5)}$, the components $H^{(5)}_{\alpha\beta\gamma}$ on $M^2$ vanish due to the antisymmetry of indices. The only non-vanishing component for $H^{(5)}$ is $H^{(5)}_{\theta_1\theta_2\theta_3} = H^{(2)}_{\theta_1\theta_2\theta_3}$, where $H^{(2)}$ is the zero form field strength on the $M^2$. The five dimensional equations of motion for the $H^{(5)}$ are identically satisfied by the imposition of the rotational symmetry. From the point of view of the $M^2$, the equations of motion for the zero form field strength are vacuous. Still, the five dimensional Bianchi identity $dH^{(5)} = 0$ implies two equations $\partial\alpha H^{(2)} = 0$.

After the dimensional reduction to the $M^2$ using the formulas from the Appendix B while recalling that the $S^3$ has intrinsic curvature, we find the following two dimensional action

$$I = \int d^2x \sqrt{-g} (e^{-2\phi}(R + 6e^{k\phi} + (\frac{16}{3} - 2k)(D\bar{\phi})^2$$

$$- \frac{1}{2}(Df)^2 - \frac{1}{2}(Df_1)^2 - \frac{1}{2}(Df_2)^2 - \frac{1}{2}D_\alpha f_1 D^\alpha f_2)$$

$$- \frac{1}{4}e^{-\sqrt{2}f_1-(k+2/3)\bar{\phi}} F_1^2 - \frac{1}{4}e^{-\sqrt{2}f_2-(k+2/3)\bar{\phi}} F_2^2 - \frac{1}{4}e^{\sqrt{2}(f_1+f_2)-(k+2/3)\bar{\phi}} F^2).$$

The covariant derivatives are taken with respect to the Weyl-rescaled two dimensional metric $g_{\alpha\beta} = \exp(2\psi_2 - k\bar{\phi})g^{(2)}_{\alpha\beta}$. We introduced four scalar fields $\bar{\phi}, f_1, f_2$ and $f$ via

$$-2\bar{\phi} = -2\phi + \frac{1}{2}\psi_1 - 3\psi_2 + 2\psi, \quad f_1 = -\frac{\sqrt{2}}{3}\phi + \frac{\sqrt{2}}{3}\psi_1 + \frac{4\sqrt{2}}{3}\psi$$

$$f_2 = -\frac{\sqrt{2}}{3}\phi - \frac{\sqrt{2}}{3}\psi_1 - \frac{4\sqrt{2}}{3}\psi$$

$$f = \psi_2 - k\bar{\phi}.$$
\begin{equation}
\psi_1 = \frac{3}{2\sqrt{2}}(f_1 + f_2) - \frac{1}{\sqrt{2}}f, \quad \psi_2 = \frac{2}{3} \bar{\phi} + \frac{1}{4\sqrt{2}}(f_1 + f_2) + \frac{1}{2\sqrt{2}}f.
\end{equation}

In addition, we used an equivalent, but more convenient description of the $H^{(2)}$ field by taking the two-dimensional Hodge dual of the zero-form field strength $H^{(2)}$. Namely, we take the dual transformation via

\begin{equation}
F_{1\alpha\beta} = e^{\sqrt{2}f_1 + (k+2/3)\bar{\phi}} g^{(2)} H^{(2)} \varepsilon_{\alpha\beta} \rightarrow H^{(2)} = -\frac{1}{2} e^{-\sqrt{2}f_1 - (k+2/3)\bar{\phi}} \varepsilon^{\alpha\beta} F_{1\alpha\beta}
\end{equation}

to introduce an extra two-form field strength $F_{1\alpha\beta}$, i.e., a $U(1)$ gauge field, which carries five-brane charges. At the action level, the term

\begin{equation}
\int d^2x \sqrt{-g} \frac{(-1)}{2} e^{\sqrt{2}f_1 + (k+2/3)\bar{\phi}} H^{(2)2}
\end{equation}

was changed to

\begin{equation}
I = \int d^2x \sqrt{-g} \frac{(-1)}{4} e^{-\sqrt{2}f_1 - (k+2/3)\bar{\phi}} g^{(2)\alpha\beta} g^{(2)\gamma\delta} F_{1\alpha\gamma} F_{1\beta\delta}
\end{equation}

to implement the Hodge duality. This change ensures that the equations of motion on the $\mathcal{M}^2$ are equivalent in both cases. We notice that the Bianchi identities for the zero-form field strength become the equations of motion in the dual picture. Just as the equations of motion for the zero-form field strength are vacuous, the Bianchi identities for the two-form field strength in two dimensions are vacuous.

We observe that the action Eq. (11) is a two dimensional dilaton gravity with a particular set of field contents. The circle modulus, the four-torus modulus, the $S^3$ radius and the ten-dimensional dilaton field combine to give four scalar fields on $\mathcal{M}^2$. There are three $U(1)$ field strengths, $F_1$, $F_2$ and $F$, that are produced by the five-brane charges, one-brane
charges and the Kaluza-Klein charges, respectively. On $\mathcal{M}^5$, to get the charge of the $H^{(5)}_{\theta_1 \theta_2 \theta_3}$ that gives $F_1$ via Eq. (10), we can directly integrate the field strength on the $S^3$. Thus, its charges are magnetic-like and its ten-dimensional dual field strength has the dual gauge field $\tilde{A}_{\alpha \theta x^6 x^7 x^8 x^9}$. This naturally couples to the five D-branes wrapped on the $S^1 \times T^4$. To obtain the charge of the $F_{2\alpha \beta}$, we have to integrate the five dimensional dual of $F_2$ on the $S^3$. This means that its charges are electric-like. The ten-dimensional gauge field for the $F_2$ field then satisfies

$$F_{2\alpha \beta} = H^{(10)}_{\alpha \beta \theta} = \partial_\alpha A_{\beta \theta} - \partial_\beta A_{\alpha \theta}.$$ 

We see that $A_{\alpha \theta}$ naturally couples to the one D-branes wrapped on the circle $S^1$.

The ten dimensional symmetries are still present as internal symmetries of the action Eq. (7). For example, the ten dimensional Hodge duality that exchanges one-branes and five-branes corresponds to the symmetry of Eq. (7) under the transformation

$$F_1 \rightarrow F_2, \quad F_2 \rightarrow F_1, \quad f_1 \rightarrow f_2, \quad f_2 \rightarrow f_1.$$ 

When $f = 0$, which holds for solutions without naked singularities that we will explain in the later section, Eq. (3) shows that the above transformation inverts the $T^4$ modulus and the sign of the ten-dimensional dilaton field, a characteristic of the $U$-duality.

III. GENERAL STATIC SOLUTIONS

To get the general static solutions of the bosonic $s$-wave sector of the type IIB supergravity compactified on $S^1 \times T^4$, we need to only consider the two dimensional dilaton gravity described by Eq. (7). The general static solutions of the similar dilaton gravity theories were obtained in Ref. [11]. By using a variant of the arguments given in that paper, we will first obtain the general static solutions on a local coordinate patch. In particular, we make no assumptions about the global and asymptotic structure of the space-time. The discussions on the properties of the solutions will follow.
A. Static equations of motion

For the description of the space-time geometry on a local coordinate patch in \( \mathcal{M}^2 \), we choose to use a conformal gauge for the two dimensional metric

\[
ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = -\exp(2\rho) dx^+ dx^-.
\]  

(12)

We also choose the number \( k = 8/3 \) in the action Eq. (7) to cancel the kinetic term for the two dimensional dilaton field \( \bar{\phi} \). A different choice of the value of \( k \) is related to this choice by a Weyl rescaling of the two dimensional metric.

Under this gauge choice, the equations of motion from Eq. (7) are given as follows. By varying the action with respect to the two-dimensional dilaton field \( \bar{\phi} \), we have

\[
4\partial_+\partial_-\rho - e^{2\rho}\Omega^{-4/3} \partial_+ f \partial_- f + \partial_+ f_1 \partial_- f_1 + \partial_+ f_2 \partial_- f_2 + \frac{1}{2} \partial_+ f_1 \partial_- f_2
\]

\[
+ \frac{1}{2} \partial_+ f_2 \partial_- f_1 + \frac{5}{3} e^{-2\rho}\Omega^{2/3}(e^{-\sqrt{2}f_1} F^2_{1+,+} + e^{-\sqrt{2}f_1} F^2_{2+,+} + e^{\sqrt{2}(f_1+f_2)} F^2_{+,-}) = 0
\]

(13)

where we introduce \( \Omega = \exp(-2\bar{\phi}) \). Similarly for the conformal factor \( \rho \) of the metric, we get

\[
2\partial_-\partial_+ \Omega + 3e^{2\rho}\Omega^{-1/3} - e^{-2\rho}\Omega^{5/3}(e^{-\sqrt{2}f_1} F^2_{1+,+} + e^{-\sqrt{2}f_1} F^2_{2+,+} + e^{\sqrt{2}(f_1+f_2)} F^2_{+,-}) = 0.
\]

(14)

The equations of motion for the scalar fields become

\[
\partial_+(\Omega \partial_- f) + \partial_- (\Omega \partial_+ f) = 0,
\]

(15)

\[
2\partial_+(\Omega \partial_- f_1) + 2\partial_- (\Omega \partial_+ f_1) + \partial_+(\Omega \partial_- f_2) + \partial_- (\Omega \partial_+ f_2)
\]

\[
+ 2\sqrt{2} e^{-2\rho}\Omega^{5/3}(e^{-\sqrt{2}f_1} F^2_{1+,+} - e^{\sqrt{2}(f_1+f_2)} F^2_{+,-}) = 0,
\]

(16)

and

\[
2\partial_+(\Omega \partial_- f_2) + 2\partial_- (\Omega \partial_+ f_2) + \partial_+(\Omega \partial_- f_1) + \partial_- (\Omega \partial_+ f_1)
\]

(17)
We also have equations for the three $U(1)$ gauge fields

\[ \partial_{+}(e^{-\sqrt{2}f_1}\Omega^{5/3}F_{1+}) = \partial_{-}(e^{-\sqrt{2}f_1}\Omega^{5/3}F_{1-}) = 0, \quad (18) \]

\[ \partial_{+}(e^{-\sqrt{2}f_2}\Omega^{5/3}F_{2+}) = \partial_{-}(e^{-\sqrt{2}f_2}\Omega^{5/3}F_{2-}) = 0, \quad (19) \]

and

\[ \partial_{+}(e^{\sqrt{2}(f_1+f_2)}\Omega^{5/3}F_{+}) = \partial_{-}(e^{\sqrt{2}(f_1+f_2)}\Omega^{5/3}F_{-}) = 0. \quad (20) \]

These equations of motion, Eqs. (13) - (20), should be supplemented with the gauge constraints produced by the choice of a conformal gauge

\[ T_{++} = T_{--} = 0, \quad (21) \]

where $T_{\pm\pm}$ denotes the $\pm\pm$ components of the stress-energy tensor.

By requiring that all the fields depend only on a space-like coordinate $x \equiv x^{+} - x^{-}$, we restrict our attention only to the static solutions. We also make a convenient gauge choice for the $U(1)$ gauge fields by introducing three functions (in fact, three static potentials) $A(x)$, $A_1(x)$, and $A_2(x)$ by

\[ A_{\pm} = \frac{1}{2}A(x) , \quad A_{1\pm} = \frac{1}{2}A_{1}(x) , \quad A_{2\pm} = \frac{1}{2}A_{2}(x), \quad (22) \]

where the gauge fields give the field strengths via $F_{2\pm} = \partial_{\pm}A_{2} - \partial_{\mp}A_{2}$, $F_{1\pm} = \partial_{\pm}A_{1} - \partial_{\mp}A_{1}$, and $F_{+\pm} = \partial_{\pm}A_{-} - \partial_{\mp}A_{-}$. The static equations of motion are then summarized by a one-dimensional action

\[ I = \int dx(\Omega'\rho' + \frac{3}{4}e^{2\rho}\Omega^{-1/3} - \frac{1}{4}\Omega f^2 - \frac{1}{4}\Omega f_1^2 - \frac{1}{4}\Omega f_2^2 - \frac{1}{4}\Omega f_1 f_2') \quad (23) \]

\[ + \frac{1}{4}e^{-\sqrt{2}f_1-2\rho}\Omega^{5/3}A_1^2 + \frac{1}{4}e^{-\sqrt{2}f_2-2\rho}\Omega^{5/3}A_2^2 + \frac{1}{4}e^{\sqrt{2}(f_1+f_2)-2\rho}\Omega^{5/3}A^2), \]

while the gauge constraints Eq.(21) become a single condition
\[
\Omega'' - 2\rho'\Omega' + \frac{1}{2}\Omega f'^2 + \frac{1}{2}\Omega f'^2_1 + \frac{1}{2}\Omega f'^2_2 + \frac{1}{2}\Omega f'_1 f'_2 = 0. \tag{24}
\]

The prime denotes the differentiation with respect to \(x = x^+ - x^-\). Our task from now on is to solve the equations of motion from the action Eq.(23) under the gauge constraint Eq.(24).

**B. Symmetries and general static solutions**

The main virtue of writing down the one dimensional action Eq. (23) is that it enables us to clearly identify the symmetries of our problem. We find that there are eight symmetries

(a) \( f \rightarrow f + \alpha, \)

(b) \( f_1 \rightarrow f_1 + \alpha, \quad A_1 \rightarrow A_1 e^{\alpha/\sqrt{2}}, \quad A \rightarrow A e^{-\alpha/\sqrt{2}}, \)

(c) \( f_2 \rightarrow f_2 + \alpha, \quad A_2 \rightarrow A_2 e^{\alpha/\sqrt{2}}, \quad A \rightarrow A e^{-\alpha/\sqrt{2}}, \)

(d) \( A_1 \rightarrow A_1 + \alpha, \)

(e) \( A_2 \rightarrow A_2 + \alpha, \)

(f) \( A \rightarrow A + \alpha, \)

(g) \( x \rightarrow x + \alpha, \)

(h) \( x \rightarrow e^{\alpha} x, \quad \Omega \rightarrow e^{\alpha} \Omega, \quad e^{2\rho} \rightarrow e^{-2\alpha/3} e^{2\rho}, \quad A \rightarrow e^{-2\alpha/3} A, \quad A_1 \rightarrow e^{-2\alpha/3} A_1, \quad A_2 \rightarrow e^{-2\alpha/3} A_2, \)

where \(\alpha\) is an arbitrary real parameter of each continuous transformation. Due to these eight symmetries for the eight fields in our problem, it is possible to integrate the coupled second order differential equations once to get coupled first order differential equations. They can be recast in the form of the Noether charge expressions

\[
f_0 = \Omega f', \tag{25}
\]

\[
f_{10} = \Omega f'_1 + \frac{1}{2}\Omega f'_2 - \frac{1}{\sqrt{2}} e^{-\sqrt{2} f_1 - 2\rho} \Omega^{5/3} A_1 A'_1 + \frac{1}{\sqrt{2}} e^{\sqrt{2} f_1 + f_2 - 2\rho} \Omega^{5/3} A A', \tag{26}
\]

\[
f_{20} = \Omega f'_2 + \frac{1}{2}\Omega f'_1 - \frac{1}{\sqrt{2}} e^{-\sqrt{2} f_2 - 2\rho} \Omega^{5/3} A_2 A'_2 + \frac{1}{\sqrt{2}} e^{\sqrt{2} f_1 + f_2 - 2\rho} \Omega^{5/3} A A', \tag{27}
\]

\[
Q_1 = e^{-\sqrt{2} f_1 - 2\rho} \Omega^{5/3} A'_1, \tag{28}
\]
\( Q_2 = e^{-\sqrt{2}f_2 - 2\rho \Omega^{5/3} A'} \),

\( Q = e^{\sqrt{2}(f_1 + f_2) - 2\rho \Omega^{5/3} A'} \),

\[
c_0 = \Omega' \rho' - \frac{1}{4} \Omega f'^2 - \frac{1}{4} \Omega f_1^2 - \frac{1}{4} \Omega f_2^2 - \frac{1}{4} \Omega f_1' f_2' + \frac{1}{4} e^{-\sqrt{2}f_1 - 2\rho \Omega^{5/3} A_1'^2} \]
\[
+ \frac{1}{4} e^{-\sqrt{2}f_2 - 2\rho \Omega^{5/3} A_2'^2} + \frac{1}{4} e^{\sqrt{2}(f_1 + f_2) - 2\rho \Omega^{5/3} A_1'^2} - \frac{3}{4} e^{2\rho \Omega^{-1/3}},
\]

\[
s + c_0 x = -\frac{1}{3} \Omega' + \rho' \Omega - \frac{1}{3} e^{-\sqrt{2}f_1 - 2\rho \Omega^{5/3} A_1 A' + 1}/3 e^{\sqrt{2}(f_1 + f_2) - 2\rho \Omega^{5/3} A A'},
\]

Corresponding to each symmetry. The gauge constraint Eq. (24) implies \( c_0 = 0 \), thereby reducing the number of constants of motion from eight to seven.

In the Appendix C, we exactly solve the above differential equations. The results are as follows.

\[
f(A) = f_0 I(A) + f_1 \]

\[
Q_1 e^{2f_a(A)} = \frac{Q}{P(A) \sinh^2 \left[ \sqrt{c_1}(I(A) + \tilde{c}_1) \right]} \]

\[
Q_2 e^{2f_b(A)} = \frac{Q}{P(A) \sinh^2 \left[ \sqrt{c_2}(I(A) + \tilde{c}_2) \right]} \]

\[
\tilde{\Omega}^{1/3}(A) = \frac{Q}{P(A) \sinh^2 \left[ \sqrt{D_2}(I(A) + \tilde{c}) \right]} \]

\[
Q_1 A_1(A) = -\sqrt{2} f_{10} - \bar{s} - \sqrt{c_1} \coth \left[ \sqrt{c_1}(I(A) + \tilde{c}_1) \right] \]

\[
Q_2 A_2(A) = -\sqrt{2} f_{20} - \bar{s} - \sqrt{c_2} \coth \left[ \sqrt{c_2}(I(A) + \tilde{c}_2) \right] \]

\[
e^{2\bar{\rho}(A)} = \frac{P(A)}{Q} \tilde{\Omega}^{2/3}(A) \]

\[
x - x_0 = \int \frac{\Omega(A)}{P(A)} dA
\]

where

\[
f_a = (2 f_1 + f_2)/\sqrt{2}, \quad f_b = (f_1 + 2 f_2)/\sqrt{2},
\]

\[
e^{2\bar{\rho}} = e^{-(f_a + f_b)/3} e^{2\rho}, \quad \tilde{\Omega} = e^{(f_a + f_b)/2} \Omega,
\]

\[
P(A) = QA^2 + 2\bar{s}A + c,
\]

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\[
\tilde{s} = s - \frac{\sqrt{2}}{3} (f_{10} + f_{20}), \quad D_2 = \frac{1}{3} \left( g_0^2 + \tilde{s}^2 - Qc + c_1 + c_2 \right),
\]

and we introduce a function

\[
I(A) = \int \frac{dA}{P(A)} = \frac{1}{2\sqrt{\tilde{s}^2 - Qc}} \ln \left| \frac{A - A_+}{A - A_-} \right|.
\]

Here \(A_\pm\) denotes the two solutions of the quadratic equation \(P(A) = 0\) given by

\[
A_\pm = \frac{-\tilde{s} \pm \sqrt{\tilde{s}^2 - Qc}}{Q}.
\]

The parameters \(f_1, c_1, c_2, \tilde{c}_1, \tilde{c}_2, \tilde{c}, c\) and \(x_0\) are eight constants of integration that are produced when we solve the coupled first order differential equations involving eight fields. Combined with seven non-vanishing Noether charges, we find that the general static solutions are labeled by fifteen parameters.

C. Properties of general static solutions

The general static solutions we obtained in the previous subsection contain fifteen parameters. Generically, the local structure of the solution space can be understood as follows. We have three gauge fields that are produced by the one-brane charge \(Q_{1 \text{-brane}}\), the five-brane charge \(Q_{5 \text{-brane}}\) and the Kaluza-Klein charge \(n\). The asymptotic values of the gauge potentials provide additional three parameter \(A_\infty, A_{1\infty},\) and \(A_{2\infty}\), the local variation of which near a generic point in the solution space can not be physically observed, since they are not gauge invariant. In addition, we have three scalar fields that correspond to the circle modulus \(\exp(\psi_1)\), the four-torus modulus \(\exp(\psi)\), and the dilaton field \(\phi\). The asymptotic values of these three scalar fields at the spatial infinity (when they exist) specify the asymptotic size of the circle \(\exp(\psi_{1\infty}/2)\), the four-torus volume \(\exp(\psi_{\infty}/2)\), and the the asymptotic dilaton coupling constant \(\exp(\phi_{\infty})\). The three scalar fields also carry charges \(Q_{\psi_1}, Q_\psi\) and \(Q_\phi\). The gravity sector provides additional three parameters; the black hole mass \(M\) (when it exists, or the mass parameter when it does not exist), the length-scale choice, and the
reference time choice. The length-scale choice is fixed when we require that the five dimensional metric become the standard flat Minkowskian when all the other charges are set to zero. The reference time choice is an isometry and it is an arbitrary number, as long as we consider the static solutions.

However, the global structure of the solution space exhibits much richer features. To see this clearly, we write down the expressions for the circle modulus, the four-torus modulus, the radius of the $S^3$, and the ten dimensional dilaton field. For the circle modulus, we have

$$e^{\psi_1} = \left| \frac{Q \sqrt{c_1 c_2}}{Q_1 Q_2} \right|^{1/2} \frac{e^{-(f_0 I(A) + f_1)/\sqrt{2}}}{|P(A) \sinh \sqrt{c_2} (I(A) + \tilde{c}_2) \sinh \sqrt{c_1} (I(A) + \tilde{c}_1)|^{1/2}},$$

(41)

the four-torus modulus is given by

$$e^{\psi} = \left| \frac{Q_2 \sqrt{c_1}}{Q_1 \sqrt{c_2}} \sinh \sqrt{c_2} (I(A) + \tilde{c}_2) \sinh \sqrt{c_1} (I(A) + \tilde{c}_1) \right|^{1/2},$$

(42)

the square of the $S^3$ radius becomes

$$e^{-2\psi_2} = \left| \frac{D_2 Q_1 Q_2 \sinh \sqrt{c_2} (I(A) + \tilde{c}_2) \sinh \sqrt{c_1} (I(A) + \tilde{c}_1)}{\sqrt{c_1 c_2} \sinh^2 \sqrt{D_2} (I(A) + \tilde{c})} \right|^{1/2} e^{-(f_0 I(A) + f_1)/\sqrt{2}},$$

(43)

and we have the ten-dimensional dilaton expression

$$e^{-2\phi} = \left| \frac{Q_1 \sqrt{c_2} \sinh \sqrt{c_1} (I(A) + \tilde{c}_1)}{Q_2 \sqrt{c_1} \sinh \sqrt{c_2} (I(A) + \tilde{c}_2)} \right| e^{\sqrt{2} (f_0 I(A) + f_1)}.$$

(44)

The two dimensional metric becomes

$$d s^2 = g^{(2)}_{\alpha\beta} d x^\alpha d x^\beta = -e^{2\rho - 2\psi_2 + 8\phi/3} d x^+ d x^- = -\frac{P(A)}{Q} e^{\psi_1} d x^+ d x^-.$$

(45)

We immediately see that the qualitative behaviors of the solutions change when we change any sign(s) of $D_2$, $c_1$ or $c_2$, or when we make any of them become zero. Keeping this fact in mind, we restrict our attention only to the case when $D_2 > 0$, $c_1 > 0$ and $c_2 > 0$, which we will see shortly corresponds to the cases considered in Ref. [6]. We furthermore assume that the quadratic equation $P(A) = 0$ for $A$ has two real solutions $A_{\pm}$, i.e., $z^2 - Q c \geq 0$. This condition implies that $3D_2 - c_1 - c_2 \geq 0$. Under these conditions, we look for the five
dimensional solutions. The five dimensional asymptotic spatial infinities are the points on \( \mathcal{M}^2 \), near of which the \( S^3 \) radius Eq. (13) tends to diverge. At the same time, the circle modulus Eq. (11) and the four-torus modulus should remain bounded from the above. Since

\[
I(A) = \frac{1}{2\sqrt{s^2 - Qc}} \ln \left| \frac{A - A_+}{A - A_-} \right|,
\]

the potentially divergent points for the \( S^3 \) radius are \( A = A_\pm \) and the two values \( A = A_\infty^\pm \) satisfying

\[
I(A_\infty^\pm) + \tilde{c} = 0. \tag{46}
\]

Assuming that the circle modulus are bounded from the above near the asymptotic spatial infinities and, further assuming that they form a time-like curve, Eq. (45) shows that \( A = A_\pm \) can not be the asymptotic spatial infinities.

Depending on the sign of \( \tilde{c} \), we have to consider the three cases. If \( \tilde{c} > 0 \), we find that \( A_\pm < A_\infty^\pm \) and \( \min(A_-, A_+) < A_\infty^\infty < \max(A_-, A_+) \). If \( \tilde{c} = 0 \), we find that \( A_\infty^\pm = \pm \infty \). If \( \tilde{c} < 0 \), we have \( A_\infty^- < A_\pm \) and \( \min(A_-, A_+) < A_\infty^+ < \max(A_-, A_+) \). One can verify that the first and the third case describes the identical situation. Thus, we concentrate on the first case and let the value of \( A \) decrease starting from \( A = A_\infty^+ \), i.e., the asymptotic spatial infinities. Since the circle modulus and the four-torus modulus should remain finite until we hit the null curve (outer event horizon), \( A = \max(A_-, A_+) \) as can be seen from Eq. (15), we have to impose \( \tilde{c} > \tilde{c}_1 \) and \( \tilde{c} > \tilde{c}_2 \). The condition that the moduli fields do not blow up near the potential outer horizon gives us the three conditions

\[
f_0 = 0, \quad D_2 = c_1, \quad D_2 = c_2 \quad \rightarrow \quad D_2 = c_1 = c_2 = s^2 - Qc. \tag{47}
\]

If these conditions are not met, we have naked singularities at the would-be outer horizon. Thus, if the charges of the scalar moduli fields are not set to a certain value, as is familiar from the four dimensional stringy black holes [8], we have naked singularities. This behavior is consistent with the no-hair property and the familiar five dimensional black hole solutions [4] [6] share this property.
To better understand our solutions, we recast them in a radial gauge. We focus on the case when $A_- \leq A_+ < 0$ and, furthermore, to get the description of the space-time outside the outer event horizon, we consider the range of $A > A_+$. Under these conditions, we note that $\sqrt{D_2}(I(A) + \tilde{c}) \leq 0$. A convenient radial coordinate $r$ choice is

$$r^2 = -\sqrt{\frac{D_2 Q}{c}} \left[ \coth(\sqrt{D_2}(I(A) + \tilde{c})) - 1 \right],$$

which becomes

$$(1 - \frac{r_0^2}{r^2})^{\frac{\sqrt{s^2 - Qc}}{D_2}} = e^{2\tilde{c}\sqrt{s^2 - Qc}} \left( \frac{A - A_+}{A - A_-} \right),$$

where we define

$$r_0^2 = 2\sqrt{\frac{D_2 Q}{c}}.$$

As can be seen from the above expression, the range of $\tilde{c}$ is restricted to be $\tilde{c} \geq 0$ if we require that the range $A_+ < A < A_+^\infty$ maps into $r_0 < r < \infty$. The natural time coordinate $t$ in our context is $t = (x^+ + x^-)$, for our fields depend only on a space-like coordinate $x = x^+ - x^-$. In terms of these $(t, r)$ coordinates, the ten dimensional metric in the radial gauge and the dilaton field can be written as follows.

$$ds^2 = Z_0 Z_1^{-1/2} Z_5^{-1/2} f_1(r) \left[ -\beta^2 dt^2 + d\theta^2 + f_2(r) (\beta \cosh \sigma dt + \sinh \sigma d\theta)^2 \right] + Z_1^{1/2} Z_5^{-1/2} dy^m dy^m + Z_0 Z_1^{1/2} Z_5^{1/2} \left( Z_0^{-1} dr^2 + r^2 d\Omega^2(3) \right),$$

$$e^{-2\phi} = Z_0^{-1} Z_5,$$

where we introduce six functions

$$f_1(r) = \frac{\sqrt{Qc}}{\tilde{s} + \sqrt{s^2 - Qc}} e^{\tilde{c}\sqrt{s^2 - Qc}} \left( 1 - \frac{r_0^2}{r^2} \right)^{(1-\sqrt{(s^2 - Qc)/D_2})/2},$$

$$f_2(r) = 1 - \frac{\tilde{s} + \sqrt{s^2 - Qc}}{\tilde{s} - \sqrt{s^2 - Qc}} e^{-2\tilde{c}\sqrt{s^2 - Qc}} \left( 1 - \frac{r_0^2}{r^2} \right) \sqrt{(s^2 - Qc)/D_2}.$$
\[ Z_0 = 1 - \frac{r_0^2}{r^2}, \]

\[ Z_1 = \sqrt{\frac{cQ_2^2}{4c_2Q}} \left[ e^{\sqrt{c_2}(\tilde{c}-\tilde{c}_2)} \left( 1 - \frac{r_0^2}{r^2} \right)^{(1-\sqrt{c_2/D_2})/2} - e^{-\sqrt{c_2}(\tilde{c}-\tilde{c}_2)} \left( 1 - \frac{r_0^2}{r^2} \right)^{(1+\sqrt{c_2/D_2})/2} \right], \]

\[ Z_5 = \sqrt{\frac{cQ_1^2}{4c_1Q}} \left[ e^{\sqrt{c_1}(\tilde{c}-\tilde{c}_1)} \left( 1 - \frac{r_0^2}{r^2} \right)^{(1-\sqrt{c_1/D_2})/2} - e^{-\sqrt{c_1}(\tilde{c}-\tilde{c}_1)} \left( 1 - \frac{r_0^2}{r^2} \right)^{(1+\sqrt{c_1/D_2})/2} \right], \]

\[ Z_\phi = e^{(f_0\tilde{c}-f_1)/\sqrt{2}} \left( 1 - \frac{r_0^2}{r^2} \right)^{-f_0/(2\sqrt{2}D_2)}, \]

and two parameters

\[ \beta = \sqrt{\frac{c}{4Q}}, \]

\[ \tanh \sigma = -\left( \frac{\bar{s} - \sqrt{s^2 - Qc}}{\bar{s} + \sqrt{s^2 - Qc}} \right)^{1/2}. \]

where the parameters \( \beta \) and \( \sigma \) represent the time scale and the boost parameter along the circle \( S^1 \), respectively. Generically, the space-time described by the metric Eq. (49) has naked singularities at the would-be outer horizon at \( r = r_0 \). To get the black hole solutions, we have to impose the three conditions in Eq. (47). Under these conditions, the functions introduced above get much simpler. For example, the functions \( Z_1 \) becomes

\[ Z_1 = \frac{|Q_2|}{r_0^2} \left( 2 \sinh(\sqrt{c_2}(\tilde{c} - \tilde{c}_2)) + e^{-\sqrt{c_2}(\tilde{c} - \tilde{c}_2)} \frac{r_0^2}{r^2} \right), \tag{51} \]

which is a familiar harmonic function.

To see that our black hole solutions indeed represent the solutions of Ref. [9], we make further restrictions on parameters (which correspond to the choice of the asymptotic value of the three gauge potentials) such as

\[ \frac{\bar{s} + \sqrt{s^2 - Qc}}{\bar{s} - \sqrt{s^2 - Qc}} = e^{2\tilde{c}\sqrt{s^2 - Qc}} \tag{52} \]

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In addition, we set the asymptotic value of the ten dimensional dilaton as one by setting $f_1 = 0$, which gives $Z_\phi = 1$ under the black hole conditions Eq. (47). In fact, under the conditions Eqs. (47) and (52), we can also straightforwardly verify that $f_1(r) = 1$, $f_2(r) = r_0^2 / r^2$ and

$$Z_1 = 1 + \frac{r_1^2}{r^2}, \quad Z_5 = 1 + \frac{r_5^2}{r^2},$$

where

$$r_1^2 = \sqrt{Q_2^2 + \frac{r_4^2}{4} - \frac{r_0^2}{2}}, \quad r_5^2 = \sqrt{Q_1^2 + \frac{r_4^2}{4} - \frac{r_0^2}{2}}.$$

We now explicitly see that Eq. (49) represents the six parameter solutions of Ref. [6] in the notation of [14].

The solution with $\tilde{c} = 0$ is our next concern. The similar situation in the context of Ref. [11] results non-asymptotically flat solutions of the type first considered in Ref. [13]. A clear way of demonstrating this is to write down the five dimensional metric expression. In ten (or six) dimensional metric expression, the static gauge transformation for the gauge potential $A$, i.e., an addition of an arbitrary number to the potential $A$, is part of the diffeomorphism of the ten dimensional space-time. This diffeomorphism becomes the internal gauge transformation for the Kaluza-Klein gauge field $A$ and, thus, decouples from the diffeomorphism for the five dimensional space-time. An explicit calculation leads to the following five dimensional metric expression

$$ds^2 = -Z_\phi Z_1^{-1/2} Z_5^{-1/2} f_3(r) \beta^2 dt^2 + Z_\phi Z_1^{1/2} Z_5^{1/2} \left( Z_0^{-1} dr^2 + r^2 d\Omega^{(3)} \right),$$

(53)

where the function $f_3(r)$ is given by

$$f_3(r) = 2 \sqrt{\frac{s^2 - Qc}{Qc}} e^{-\tilde{c}\sqrt{s^2 - Qc}} \frac{(1 - \frac{r_5^2}{r^2})^{(1 + \sqrt{(s^2 - Qc)/D_2})/2}}{1 - e^{-2\tilde{c}\sqrt{s^2 - Qc}} (1 - \frac{r_5^2}{r^2})^{\sqrt{(s^2 - Qc)/D_2}}}. $$

(54)
We can see from Eq. (54) that as long as $\tilde{c} > 0$, the function $f_3(r)$ goes to a finite value as $r \to \infty$. On the other hand, if $\tilde{c} = 0$, we find that $f_3(r) \to r^2$ as $r \to \infty$ due to the contribution from the denominator of Eq. (54). This asymptotic behavior at the spatial infinity shows that the solutions with $\tilde{c} = 0$ are the analogues of the Chan-Horne-Mann solutions [13] as explained in [11]. If we indeed impose the conditions Eq. (47) to consider the black-hole-like space-time structure, we get the five dimensional metric expression

$$ds^2 = -U dt_1^2 + U^{-1} d\tilde{r}^2 + \tilde{r} Z_1^{1/2} Z_0^{1/2} d\Omega^3$$

for the $\tilde{c} = 0$ case, after the coordinate change $\tilde{r} = r^2$ and the time rescaling. Here, the function $U$ is given by

$$U(\tilde{r}) = 4\tilde{r} \left( 1 - \frac{r_0^2}{\tilde{r}} \right) \frac{1}{\sqrt{(1 + r_0^2/\tilde{r})(1 + r^2/\tilde{r})}}$$

Near the region $\tilde{r} = r_0^2$, the space-time looks like a black hole. However, the asymptotic behavior near $\tilde{r} = \infty$ is quite different from being asymptotically flat.

**IV. DISCUSSIONS**

Our theme in this paper is to apply the techniques from the lower dimensional gravity theories to higher dimensional supergravities by restricting our attention to the $s$-wave sector. In particular, we obtained the general static solutions of the $s$-wave sector of the type IIB supergravity on $S^1 \times T^4$. In a sector of our solutions that reproduces the five dimensional black hole results, we get naked singularities and non-asymptotically flat solutions in addition. Still, the general static solutions have much richer structure; for example, in the cases when one or more of $D_2$, $c_1$ and $c_2$ become non-positive, the qualitative behaviors of the solutions change. Even when they are all positive, if the inequalities $\tilde{c} > \tilde{c}_1$ or $\tilde{c} > \tilde{c}_2$ get violated, the circle and/or the four-torus moduli blow up (or become zero, thereby becoming strong coupling) before we reach the spatial infinities. Thus, the external (non-compact)
dimension of the space-time depends on the value of the constants of the integration $\tilde c$. It will be an interesting exercise to study the structure of the whole solution space and the mapping between each sector of the solution space via $U$-duality in detail.

On a broader side of our theme, the urgent next step is to study the dynamics (including the gravitational back-reaction) of the supergravities via the two dimensional dilaton gravity theories. Depending on whether one includes the circle $S^1$ in the dynamical considerations, we end up getting (1+1)-dimensional or (2+1)-dimensional description of the $s$-wave dynamics. We expect the techniques accumulated so far in the context of the two dimensional dilaton gravities [9] will be helpful, and the work in this regard is in progress.

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1We note that essentially the same observation was also made in [16], which we received while this paper was written up. In that reference, the three dimensional BTZ solutions are related to the five dimensional solutions by considering the case when $\tilde c_1 = \tilde c$ and/or $\tilde c_2 = \tilde c$ in our language (we note our Eq. (51)).
APPENDIX A: KALUZA-KLEIN REDUCTION ON $S^1$

In this Appendix, we set up the notation for this note and collect some useful formulas for the Kaluza-Klein reduction on a circle. The signature choice for the metric $g_{\mu\nu}^{(d+1)}$ in $(d+1)$-dimensional space-time in this note is $(-++...+)$, i.e., we use -1 for the time-like coordinate $x^0$, and +1 for a space-like coordinate $x^i$ (where $i = 1, ..., d$). In $(d+1)$-dimensional space-time, the curvature tensor is given by

\[ R_{\mu\nu\rho}{}^{\sigma} = \partial_{\nu} \Gamma_{\mu\rho}{}^{\sigma} - \partial_{\mu} \Gamma_{\nu\rho}{}^{\sigma} + \Gamma_{\alpha\mu}{}^{\rho} \Gamma_{\sigma}{}^{\alpha\nu} - \Gamma_{\alpha\nu}{}^{\rho} \Gamma_{\sigma}{}^{\alpha\mu}, \]

(A1)

while the Christoffel connection satisfies

\[ \Gamma_{\mu\nu}^{\rho} = \frac{1}{2} g^{(d+1)\rho\sigma} (\partial_{\nu} g_{\sigma\rho}^{(d+1)} + \partial_{\rho} g_{\nu\sigma}^{(d+1)} - \partial_{\sigma} g_{\mu\nu}^{(d+1)}). \]

(A2)

The space-time indices $\mu, \nu, \rho$ and $\sigma$ run from zero to $d$. For the Kaluza-Klein reduction from $(d+1)$-dimensions to $d$-dimensions, we write the metric $g_{\mu\nu}^{(d+1)}$ as

\[
 g_{\mu\nu}^{(d+1)} = \begin{pmatrix}
 g_{\alpha\beta}^{(d)} + e^{\psi_{1}} A_{\alpha} A_{\beta} & e^{\psi_{1}} A_{\alpha} \\
 e^{\psi_{1}} A_{\beta} & e^{\psi_{1}} \\
 e^{\psi_{1}} A_{\alpha} & e^{\psi_{1}} \\
 \end{pmatrix},
\]

(A3)

where the indices $\alpha$ and $\beta$ run from zero to $(d - 1)$. The factor $\exp(\psi_{1}/2)$ represents the radius of the compact circle. The inverse of this metric is computed to be

\[
 g^{(d+1)\mu\nu} = \begin{pmatrix}
 g^{(d)}_{\alpha\beta} & -A^{\alpha} \\
 -A^{\beta} & e^{-\psi_{1}} + g^{(d)}_{\alpha\beta} A_{\alpha} A_{\beta} \\
 \end{pmatrix},
\]

(A4)

where the $d$-dimensional indices are raised and lowered by the $d$-dimensional metric $g_{\alpha\beta}^{(d)}$. We note that the determinant of the metric $g_{\mu\nu}^{(d+1)}$ is simply $\exp(\psi_{1})$ multiplied by the determinant of $g_{\alpha\beta}^{(d)}$.

For the dimensional reduction, we assume that the $(d+1)$-dimensional metric does not depend on the circle coordinate $\theta \equiv x^{d}$. In other words, we only retain the zero modes along the circle by setting $\partial_{\theta} g_{\mu\nu}^{(d+1)} = 0$. By the direct computation, we verify that the $(d+1)$-dimensional scalar curvature $R^{(d+1)}$ decomposes into
\[ R^{(d+1)} = R^{(d)} - D^{(d)}_{\alpha} D^{(d)}_{\alpha} \psi_1 - \frac{1}{2} D^{(d)}_{\alpha} \psi_1 D^{(d)}_{\alpha} \psi_1 - \frac{1}{4} \psi_1 g^{(d)\alpha\beta} g^{(d)\gamma\delta} F_{\alpha\gamma} F_{\beta\delta}, \quad (A5) \]

where \( D^{(d)}_{\alpha} \) is the \( d \)-dimensional covariant derivative and \( F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \). If we have a \((d + 1)\)-dimensional scalar field \( f \) that satisfies \( \partial_\theta f = 0 \), we simply have

\[ g^{(d+1)\mu\nu} D^{(d+1)}_{\mu} f D^{(d+1)}_{\nu} f = g^{(d)\alpha\beta} D^{(d)}_{\alpha} f D^{(d)}_{\beta} f. \quad (A6) \]

If we have a \((d+1)\)-dimensional three form field strength \( H^{(d+1)} \), the kinetic term decomposes into

\[ H^{(d+1)\mu\nu\rho} H^{(d+1)}_{\mu\nu\rho} = H^{(d)\alpha\beta\gamma} H^{(d)}_{\alpha\beta\gamma} + 3 e^{-\psi_1} H^{(d)\alpha\beta\theta} H^{(d)}_{\alpha\beta\theta}, \quad (A7) \]

where we introduce

\[ H^{(d)}_{\alpha\beta\gamma} = H^{(d)}_{\alpha\beta\gamma} - (A_{\alpha} H^{(d)}_{\beta\gamma\theta} + A_{\beta} H^{(d)}_{\gamma\alpha\theta} + A_{\gamma} H^{(d)}_{\alpha\beta\theta}). \]

We can interpret \( H^{(d)}_{\alpha\beta\theta} \) as a two form field strength in \( d \) dimensions.

**APPENDIX B: DIMENSIONAL REDUCTION WITH BLOCK-DIAGONAL METRIC**

We consider the dimensional reduction where the metric tensor \( g^{(d)}_{\mu\nu} \) is block-diagonal. In other words, \( g^{(d)}_{\mu\nu} \) is given by

\[ g^{(d)}_{\mu\nu} = \begin{pmatrix} g^{(D_l)}_{\alpha\beta} & 0 \\ 0 & h^{(D_t)}_{ij} \end{pmatrix}, \quad (B1) \]

where the longitudinal space-time with the dimension \( d_l \) has the metric \( g^{(d_l)}_{\alpha\beta} \) and the transversal space with the dimension \( d_t \) has the metric \( h^{(d_t)}_{ij} \). Thus, we have \( d = d_l + d_t \). The indices \( \alpha \) and \( \beta \) run from 0 to \( d_l - 1 \), whereas the indices \( i \) and \( j \) run from 1 to \( d_t \). For other conventions, we follow the Appendix A. This type of dimensional reduction was utilized in Ref. [15] for the study of the gravitational scatterings at Planckian energies in four dimensional space-time.
The curvature scalar $R^{(d)}$ decomposes into

$$
R^{(d)} = R^{(d_t)} + R^{(d_t)} \frac{2}{\sqrt{-g^{(d_t)}}} h^{(d_t)}_{ij} D_i D_j \sqrt{-g^{(d_t)}} - \frac{2}{\sqrt{h^{(d_t)}}} g^{(d_t)\alpha\beta} D_\alpha D_\beta \sqrt{h^{(d_t)}}
$$

(B2)

$$
\frac{1}{4} h^{(d_t)ij} D_i g^{(d_t)\alpha\beta} D_j g^{(d_t)\gamma\delta} (g^{(d_t)\alpha\delta} g^{(d_t)\beta\gamma} - g^{(d_t)\alpha\beta} g^{(d_t)\gamma\delta})
$$

$$
- \frac{1}{4} g^{(d_t)\alpha\beta} D_\alpha h^{(d_t)ij} D_\beta h^{(d_t)kl} (h^{(d_t)il} h^{(d_t)jk} - h^{(d_t)ij} h^{(d_t)kl})
$$

where $D_i$ represents the covariant derivative in the transversal space and $D_\alpha$ denotes the covariant derivative in the longitudinal space-time. Here $g^{(d_t)}$ and $h^{(d_t)}$ represent the determinants of the metric.

We consider the $d$-dimensional action

$$
I = \frac{1}{2^d \pi G_d} \int d^d x \sqrt{-g^{(d)}} (e^{a\phi} R^{(d)} + 4 e^{b\phi} (D\phi)^2 - \frac{1}{12} e^{c\phi} H^{(d)2})
$$

(B3)

where we have the dilaton field $\phi$, the three-form field strength $H^{(d)}$ and real parameters $a$, $b$ and $c$. In terms of the antisymmetric two-form gauge field $B^{(d)}$, we have $H^{(d)} = dB^{(d)}$. We assume that the $d$-dimensional metric is of the form

$$
ds^2 = g^{(d_t)}_{\alpha\beta} dx^\alpha dx^\beta + e^\psi \delta^{(d_t)}_{ij} dx^i dx^j,
$$

(B4)

where $R^{(d_t)} = 0$ and

$$
\int_{(d_t)} dy^1 \cdots dy^{d_t} = V
$$

for a constant $V$. We further suppose that all the relevant fields do not depend on the transversal coordinates. By this assumption, we are retaining only the zero modes of the Kaluza-Klein reduction. Using Eq. (B2), we can integrate out the action Eq. (B3) over the transversal space. The result of the integration is

$$
I = \frac{V}{2^d \pi G_d} \int d^d x \sqrt{-g^{(d_t)}} e^{d_t \psi/2} (e^{a\phi} R^{(d_t)} + \frac{d_t(d_t-1)}{4} e^{a\phi} (D\psi)^2)
$$

$$
+ ad_t e^{a\phi} D^\alpha \phi D^\beta \psi + 4 e^{b\phi} (D\phi)^2 - \frac{1}{12} e^{c\phi} H^{(d_t)2}
$$

(B5)
The $d$-dimensional three-form field strength $H^{(d)}_{i\mu\nu}$ decomposes into the $d_t$-dimensional three-form field strength $H^{(d_t)}_{\alpha\beta\gamma}$, $d_t$ $d_t$-dimensional two-form field strength $F^{(d_t)}_{\alpha\beta}$ and $d_t(d_t-1)/2$ one-form field strength $D_{\alpha}B_{ij}$.

**APPENDIX C: DERIVATION OF GENERAL STATIC SOLUTIONS**

The differential equations Eqs. (25) - (32) are quite similar to the ones considered in Ref. [11]. In analogy with that work, we introduce a following set of field redefinitions and a coordinate change

\[
\bar{\Omega} = \exp\left(\frac{3}{2\sqrt{2}}(f_1 + f_2)\right)\Omega, \quad \exp(2\bar{\rho}) = \exp\left(-\frac{1}{\sqrt{2}}(f_1 + f_2)\right)\exp(2\rho),
\]

\[
d\bar{x} = \exp\left(\frac{3}{2\sqrt{2}}(f_1 + f_2)\right)dx,
\]

which simplifies the calculations. In terms of these redefined fields, we can rewrite Eqs. (25) - (31) and (32) as follows.

\[
f_0 = \bar{\Omega}\dot{f},
\]

\[
\sqrt{2}f_{10} = \bar{\Omega}\dot{f}_a + QA - Q_1A_1, \quad Q_1\dot{A} = Qe^{-2f_0}\dot{A}_1,
\]

\[
\sqrt{2}f_{20} = \bar{\Omega}\dot{f}_b + QA - Q_2A_2, \quad Q_2\dot{A} = Qe^{-2f_0}\dot{A}_2,
\]

\[
Q = e^{-2\bar{\rho}\bar{\Omega}^{5/3}}\dot{A},
\]

\[
\bar{s} = -\frac{1}{3}\ddot{\bar{\Omega}} + \dot{\bar{\rho}}\bar{\Omega} - QA,
\]

where the overdot represents the differentiation with respect to $\bar{x}$ and we introduced $\bar{s} = s - (\sqrt{2}/3)(f_{10} + f_{20})$, $f_a = (2f_1 + f_2)/\sqrt{2}$, and $f_b = (2f_2 + f_1)/\sqrt{2}$. Combining Eqs. (C5) and (C6), we find
\[(2s + 2QA) \dot{A} = \frac{d}{dx}(e^{2b\bar{\Omega}^{-2/3}})Q,\]

which, upon integration, becomes

\[Qe^{2b\bar{\Omega}^{-2/3}} = QA^2 + 2sA + c \equiv P(A), \quad \text{(C7)}\]

where we introduced a function \(P(A)\) and \(c\) is a constant of integration. We note that \(\epsilon\), the sign of \(P(A)\), should be the same as the sign of \(Q\). Putting Eq. (C7) into Eq. (C5), we get

\[\bar{\Omega} \frac{dA}{dx} = P(A) \rightarrow \bar{\Omega} \frac{d}{dx} = P(A) \frac{d}{dA}. \quad \text{(C8)}\]

By changing the differentiation variable from \(\bar{x}\) to \(A\) with the help of Eq. (C8), we immediately find that Eq. (C2) can be integrated to yield

\[f = f_0 \int \frac{dA}{P(A)} + f_1 \quad \text{(C9)}\]

where \(f_1\) is the constant of integration. In a similar fashion, we can rewrite Eq. (C3) as

\[Q \frac{dA_1}{dA} = Q_1 e^{2f_a} \quad \text{(C10)}\]

and

\[P(A) \frac{d}{dA} f_a + QA - Q_1 A_1 = \sqrt{2} f_{10}. \quad \text{(C11)}\]

Differentiating Eq. (C11) with respect to \(A\) and using Eq. (C10), we get

\[\frac{d}{dA} (P(A) \frac{d}{dA} f_a) + Q - \frac{Q^2}{Q} e^{2f_a} = 0. \quad \text{(C12)}\]

By setting

\[f_a = -\frac{1}{2} \ln |P(A)| + \hat{f}_a,\]

and introducing a new variable

\[\hat{x} = \int \frac{dA}{P(A)},\]

we find that Eq. (C12) is in fact a one-dimensional classical Liouville equation.
\[
\frac{d^2}{d\hat{x}^2} \hat{f}_a - \frac{Q^2}{\epsilon Q} \exp(2\hat{f}_a) = 0.
\]

This equation can be exactly solved to determine

\[
e^{2\hat{f}_a} = \frac{|c_1 Q|}{Q_1^2} \frac{1}{|P(A)|} \sinh^2 \left[ \frac{1}{\sqrt{c_1} (\int \frac{dA}{P(A)} + \tilde{c}_1)} \right]
\]

which, in turn, yields

\[
Q_1 A_1 = -\sqrt{2} f_{10} - \bar{s} - \sqrt{c_1} \coth \left[ \sqrt{c_1} \left( \int \frac{dA}{P(A)} + \tilde{c}_1 \right) \right]
\]

upon using Eq. (C11). Arbitrary real numbers \(c_1\) and \(\tilde{c}_1\) are the constants of integration. We note that if \(c_1 < 0\), the hyperbolic functions change into the trigonometric functions. For \(c_1 = 0\), only the leading order term in the Taylor expansion of the hyperbolic function survives.

The analysis leading to Eqs. (C13) and (C14) can be repeated for the fields \(f_b\) and \(A_2\) from Eq. (C4). The results are

\[
e^{2\hat{f}_b} = \frac{|c_2 Q|}{Q_2^2} \frac{1}{|P(A)|} \sinh^2 \left[ \frac{1}{\sqrt{c_2} (\int \frac{dA}{P(A)} + \tilde{c}_2)} \right]
\]

and

\[
Q_2 A_2 = -\sqrt{2} f_{20} - \bar{s} - \sqrt{c_2} \coth \left[ \sqrt{c_2} \left( \int \frac{dA}{P(A)} + \tilde{c}_2 \right) \right]
\]

where arbitrary real number \(c_2\) and \(\tilde{c}_2\) are the constants of integration.

We express Eq. (32) in terms of the redefined fields Eq. (C1), change variable from \(x\) to \(\hat{x}\), plug in Eqs. (C13) - (C16), to get

\[
\frac{8}{3} \left( \frac{d\hat{\phi}}{dx} \right)^2 - \frac{1}{2|Q|} e^{-8\hat{\phi}/3} - \frac{3}{2} D_2 = 0
\]

where we introduced

\[
3\Omega^{4/3} = \exp(-\frac{8}{3} \hat{\phi} - \ln |P(A)|)
\]

and

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\[ D_2 \equiv \frac{1}{3}(g_0^2 + s^2 - Qc + c_1 + c_2). \]

The seemingly complicated Eq. (32) drastically simplifies after the field redefinitions. In fact, Eq. (C17) is the first integration of the one-dimensional classical Liouville equation where \( D_2 \) plays the role of the constant of the integration. This equation can be immediately integrated to yield

\[ \Omega^{4/3} = |D_2Q| \frac{1}{|P(A)|} \left[ \frac{1}{\sinh^2 \left[ \sqrt{D_2}(f \frac{dA}{P(A)} + \tilde{c}) \right] } \right] \]  

(C18)

where \( \tilde{c} \) is a constant of integration. Since Eq. (C8) determines \( A \) in terms of \( x \) through

\[ x - x_0 = \int \frac{\Omega}{P(A)} dA \]  

(C19)

where \( x_0 \) is a constant of integration, we have solved all the equations of motion.
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