AbSTRACT. The purpose in this paper is to prove the end point Strichartz estimate for the Schrödinger equation in the exterior domain of a generic non-trapping obstacle in the case \( n \geq 3 \). In the case \( n = 2 \) we have the same range of Strichartz estimates as in the free case.

1. Introduction

Let \( n \geq 2 \) and \( \Omega \) be the exterior domain in \( \mathbb{R}^n \) of a compact obstacle with smooth boundary \( \partial \Omega \). We consider the Schrödinger equation

\[
i\partial_t u(t, x) + \Delta u(t, x) = 0 \quad \text{in } \mathbb{R} \times \Omega
\]  

with the initial condition

\[
u(0, x) = f(x), \quad x \in \Omega,
\]

and the Dirichlet or Neumann boundary condition:

\[
u(t, x) = 0 \quad \text{or} \quad \partial_n u(t, x) = 0, \quad (t, x) \in \mathbb{R} \times \partial \Omega,
\]

where \( \partial_n \) is the normal derivative at the boundary. Our goal is to prove the end point Strichartz estimates for solutions \( u \) to the problem (1.1)–(1.3), when \( n \geq 3 \). Such end point estimate enables one to use the full range of the parameters \( s, p, q \) in the Strichartz estimate

\[
\|u\|_{L^p([-T,T];L^q(\Omega))} \leq C\|f\|_{H^s(\mathbb{R}^n)}.
\]  

Our goal is therefore to cover the range of the exponents \( s, p, q \) such that \( n \geq 3, \ s \geq 0, \ 2 \leq p, q \leq \infty \) satisfy the scaling admissibility condition

\[
\frac{2}{p} + \frac{n}{q} = \frac{n}{2} - s
\]  

and the triple \( (s, p, q) \) is admissible one. In the case \( n = 2 \) it is natural to exclude the case \( q = \infty \). Recall that the \( (s, p, q) \) satisfying (1.5) with \( q \neq \infty \) if \( n = 2 \) are called admissible. If \( s = 0 \), then we shall say simply that the couple \( (p, q) \) is (Schrödinger) admissible.

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Here $\dot{H}^s(\mathcal{H})$ denotes the Sobolev space defined via the spectral decomposition of the Laplace operator $\mathcal{H} = -\Delta|_D$ or $\mathcal{H} = -\Delta|_N$, i.e. the Dirichlet or Neumann Laplace operator on $\Omega$ (see the end of this section for the definition). In this paper we assume that the obstacle $\mathbb{R}^n \setminus \Omega$ is non-trapping which means that any light ray reflecting on the boundary $\partial\Omega$ according to the laws of geometric optics leaves any compact set in finite time. For a precise definition we refer to Melrose and Sjöstrand [25], [26] (see also Burq, Gérard and Tzvetkov [6] and references therein).

Strichartz estimates were already established for the free Schrödinger equation on $\mathbb{R}^n$. The beginning is the pioneer work by Strichartz [28]. It was generalized by Ginibre and Velo [11] to mixed $L^p_t L^q_x$-norms with the full admissible range $(p, q)$ except for endpoints $(p, q) = (2, 2n/(n-2))$ if $n \geq 3$ and $(p, q) = (2, \infty)$ if $n = 2$. The endpoint estimates for $n \geq 3$ were finally proved by Keel and Tao [21], while the endpoint estimates fail in the case $n = 2$ (see Montgomery-Smith [27]). Once the case $s = 0$ is obtained, the case $s > 0$ follows from the Sobolev embedding theorem. These estimates have played a fundamental role in studying well-posedness, scattering and blow-up for nonlinear Schrödinger equations, and in particular the endpoint estimates are crucial in the mass and energy critical cases (see, e.g., [7], [22], [30]).

Our main goal in this work is to obtain the end point Strichartz estimate (when $n \geq 3$) in the exterior of a non-trapping obstacle imposing Dirichlet or Neumann boundary conditions. There is a large number of literature on the study of Strichartz estimates and nonlinear Schrödinger equations in exterior domains (see [1], [2], [3], [4], [5], [6], [14], [15], [20], [23], [24], [31]). These estimates with loss of derivatives were obtained by [6] (see also [1] and [5]). The result without loss of derivatives was later proved by Blair, Smith and Sogge [4] under the additional assumptions $1/p + 1/q \leq 1/2$ if $n \geq 3$ and $3/p + 2/q \leq 1$ if $n = 2$. Due to these additional assumptions, their result does not include the case $s = 0$. This is currently the best known result in the case of general non-trapping exterior domains. When $\Omega$ is the exterior domain of a strictly convex obstacle, which is non-trapping, the sharp estimates were obtained by Ivanovici [14] with full range except for endpoints. Recently, in this case, Ivanovici and Lebeau proved the dispersive estimate for the Schrödinger equation in three dimensional case $n = 3$, which implies the endpoint Strichartz estimate. At the same time, they proved that the dispersive estimate fails in higher dimensional case $n \geq 4$, even if the obstacle is a ball in $\mathbb{R}^n$ (see [15]). To the best of our knowledge, there is no result on the endpoint case when $\Omega$ is an exterior of non trapping obstacle, except for the result by [15]. Further, in more general case than strictly convex obstacles, it seems that the sharp estimates with $s = 0$ are unknown even in the non-endpoint case.

In the present work, we establish the endpoint Strichartz estimate and therefore Strichartz estimates (1.4) for all admissible triplets $(s, p, q)$. We start by introducing suitable extension operator from $H^2(\mathcal{H})$ to $H^2(\mathbb{R}^n)$ satisfying suitable commutative relations involving the perturbed Laplace operator $\mathcal{H}$. The main novelty in our approach is to combine this extension with appropriate estimates for the free Schrödinger equation that involve Strichartz and smoothing norms that we call Strichartz-smoothing estimates (see the estimates of section 2 in [13] and their applications in [10]). These estimates together with the known local smoothing estimates for solutions to (1.1)–(1.3) in [6] give us the possibility to reduce the endpoint
Strichartz estimate for exterior boundary value problem to the proof of some commutator estimates. Therefore, the next novelty is the proof of new commutator estimates between polynomial weights and fractional differential operators. Our result on the endpoint estimates can be applied to establish well-posedness, scattering and blow-up for the mass and energy critical nonlinear Schrödinger equations on non-trapping exterior domains.

Let us introduce the notations used in this paper. For $m \in \mathbb{N}$, $H^m(\Omega)$ is the usual Sobolev space of $L^2$ type, and $C^m(\overline{\Omega})$ is the space of all $f \in C^m(\Omega)$ such that $\partial^\alpha x f$ extends continuously up to the closure $\overline{\Omega}$ for any multi-index $\alpha$ with $0 \leq |\alpha| \leq m$. The space $C_0^\infty(\Omega)$ is the set of all $C^\infty$-functions on $\Omega$ having compact support in $\Omega$. Then we denote by $H^m_0(\Omega)$ the completion of $C_\infty^0(\Omega)$ with respect to $H^m$-norm. We denote by $S(\mathbb{R}^n)$ the Schwartz space, i.e., the space of rapidly decreasing functions on $\mathbb{R}^n$.

For a Banach space $X$ and an interval $I \subset \mathbb{R}$, we denote by $L^p(I; X)$ the Bochner space of vector-functions $f: I \rightarrow X$ with norm $\|f\|_{L^p(I; X)} = \|\|f(\cdot)\|_X\|_{L^p(I)}$. Given two operators $A$ and $B$, their commutator is defined by the operator $[A, B] = AB - BA$. We write $\langle x \rangle = (1 + |x|^2)^{1/2}$, and denote by $D_s$ the Fourier multiplier $D_s = \mathcal{F}^{-1}(|\xi|^s \mathcal{F})$ for $s \in \mathbb{R}$.

We conclude this section by introducing Sobolev spaces and Besov spaces defined via the spectral decomposition of either the Dirichlet or Neumann Laplacian on $\Omega$. Let us denote by $-\Delta|_D$ and $-\Delta|_N$ the Dirichlet and Neumann Laplacians on $L^2(\Omega)$ with domain

$$\mathcal{D}(-\Delta|_D) = H^2(\Omega) \cap H^1_0(\Omega), \quad \mathcal{D}(-\Delta|_N) = \{ u \in H^2(\Omega) : \partial_n u|_{\partial\Omega} = 0 \}$$

respectively, and let $\mathcal{H} = -\Delta|_D$ or $-\Delta|_N$. Note that $\mathcal{H}$ is non-negative and self-adjoint on $L^2(\Omega)$. For a Borel measurable function $\phi$ on $\mathbb{R}$, an operator $\phi(\mathcal{H})$ is defined by

$$\phi(\mathcal{H}) := \int_0^\infty \phi(\lambda) \, dE_\mathcal{H}(\lambda)$$

with the domain

$$\mathcal{D}(\phi(\mathcal{H})) = \left\{ f \in L^2(\Omega) : \int_0^\infty |\phi(\lambda)|^2 d\|E_\mathcal{H}(\lambda) f\|_{L^2(\Omega)}^2 < \infty \right\},$$

where $\{E_\mathcal{H}(\lambda)\}_{\lambda \in \mathbb{R}}$ is the spectral resolution of the identity for $\mathcal{H}$. Let $\phi_0$ be a non-negative and smooth function on $\mathbb{R}$ such that

$$\text{supp } \phi_0 \subset \{ \lambda \in \mathbb{R} : 2^{-1} \leq \lambda \leq 2 \} \quad \text{and} \quad \sum_{j=-\infty}^{\infty} \phi_0(2^{-j} \lambda) = 1 \quad \text{for } \lambda > 0,$$

and $\{\phi_j\}_{j=-\infty}^{\infty}$ is defined by letting

$$\phi_j(\lambda) := \phi_0(2^{-j} \lambda) \quad \text{for } \lambda \in \mathbb{R}.$$
Remark 1.2. Let \( s \in \mathbb{R} \) and \( 1 \leq p, q \leq \infty \). Then the homogeneous Besov space \( \dot{B}^s_{p,q}(\mathcal{H}) \) is defined by
\[
\dot{B}^s_{p,q}(\mathcal{H}) := \{ f \in \mathcal{Z}'(\mathcal{H}) : \| f \|_{\dot{B}^s_{p,q}(\mathcal{H})} < \infty \}
\]
with norm
\[
\| f \|_{\dot{B}^s_{p,q}(\mathcal{H})} := \left\{ \begin{array}{ll}
\sup_{j \in \mathbb{Z}} 2^{sj} \| \phi_j(\sqrt{\mathcal{H}}) f \|_{L^p(\Omega)} & \text{if } 1 \leq q < \infty, \\
\left( \sum_{j=-\infty}^{\infty} 2^{sj} \| \phi_j(\sqrt{\mathcal{H}}) f \|_{L^p(\Omega)}^q \right)^{\frac{1}{q}} & \text{if } q = \infty,
\end{array} \right.
\]
where \( \mathcal{Z}'(\mathcal{H}) \) is the topological dual of \( \mathcal{Z}(\mathcal{H}) \) defined by
\[
\mathcal{Z}(\mathcal{H}) := \left\{ f \in L^1(\Omega) \cap \mathcal{D}(\mathcal{H}) : \sup_{j \in \mathbb{Z}} 2^{M|j|} \| \phi_j(\sqrt{\mathcal{H}}) f \|_{L^1(\Omega)} < \infty \text{ for all } M \in \mathbb{N} \right\}.
\]

Remark 1.2. \( \mathcal{Z}(\mathcal{H}) \) is a Fréchet space equipped with the family of semi-norms \( \{ q_M(\cdot) \}_{M=1}^{\infty} \) given by
\[
q_M(f) := \| f \|_{L^1(\Omega)} + \sup_{j \in \mathbb{Z}} 2^{M|j|} \| \phi_j(\sqrt{\mathcal{H}}) f \|_{L^1(\Omega)}.
\]
We note that \( f \in \mathcal{Z}(\mathcal{H}) \) means \( \mathcal{H}^M f \in L^1(\Omega) \cap \mathcal{D}(\mathcal{H}) \) for all \( M \in \mathbb{Z} \).

The Besov spaces \( \dot{B}^s_{p,q}(\mathcal{H}) \) are Banach spaces, and enjoy
\[
\mathcal{Z}(\mathcal{H}) \hookrightarrow \dot{B}^s_{p,q}(\mathcal{H}) \hookrightarrow \mathcal{Z}'(\mathcal{H}).
\]
Furthermore, we have the following:

Proposition 1.3 (Sections 2 and 3 in \cite{18}, and also \cite{16, 17, 29}). Let \( s \in \mathbb{R} \) and \( 1 \leq p, q, \tilde{q}, r \leq \infty \). Then the following assertions hold:

(i) The homogeneous Besov spaces enjoy the following properties:
\[
\dot{B}^{s+n(\frac{1}{p}-\frac{1}{r})}_{r,q}(\mathcal{H}) \hookrightarrow \dot{B}^s_{p,q}(\mathcal{H}) \quad \text{if } 1 \leq r \leq p \leq \infty \text{ and } q \leq \tilde{q};
\]
\[
L^p(\Omega) \hookrightarrow \dot{B}^0_{p,2}(\mathcal{H}) \quad \text{if } 1 < p \leq 2 \quad \text{and} \quad \dot{B}^0_{p,2}(\mathcal{H}) \hookrightarrow L^p(\Omega) \quad \text{if } 2 \leq p < \infty.
\]

(ii) Let \( 0 < \theta < 1 \), \( s, s_0, s_1 \in \mathbb{R} \) and \( 1 \leq p, q, \tilde{q}, r \leq \infty \). Assume that \( s_0 \neq s_1 \) and \( s = (1-\theta)s_0 + \theta s_1 \). Then
\[
(\dot{B}^{s_0}_{p,q_0}(\mathcal{H}), \dot{B}^{s_1}_{p,q_1}(\mathcal{H}))_{\theta,q} = \dot{B}^{s}_{p,q}(\mathcal{H}),
\]
where \( (\dot{B}^{s_0}_{p,q_0}(\mathcal{H}), \dot{B}^{s_1}_{p,q_1}(\mathcal{H}))_{\theta,q} \) are real interpolation spaces between \( \dot{B}^{s_0}_{p,q_0}(\mathcal{H}) \) and \( \dot{B}^{s_1}_{p,q_1}(\mathcal{H}) \).

The inhomogeneous Besov spaces \( \dot{B}^s_{p,q}(\mathcal{H}) \) are also defined by the usual modification, and these spaces enjoy similar properties to Proposition 1.3. For \( s \in \mathbb{R} \), we define the Sobolev spaces \( \mathcal{H}^s(\mathcal{H}) \) and \( H^s(\mathcal{H}) \) by
\[
\mathcal{H}^s(\mathcal{H}) := \dot{B}^s_{2,2}(\mathcal{H}) \quad \text{and} \quad H^s(\mathcal{H}) := B^s_{2,2}(\mathcal{H}),
\]
whose norms are written as
\[
\| f \|_{\mathcal{H}^s(\mathcal{H})} = \| \mathcal{H}^s f \|_{L^2(\Omega)} \quad \text{and} \quad \| f \|_{H^s(\mathcal{H})} = \| (I + \mathcal{H})^s f \|_{L^2(\Omega)},
\]
respectively, where \( I \) is the identity operator on \( L^2(\Omega) \).
2. Main result

Our main result is the following:

**Theorem 2.1.** Let $n \geq 2$ and $\Omega$ be the exterior domain in $\mathbb{R}^n$ of a compact non-trapping obstacle with smooth boundary, and let $\mathcal{H}$ be the Dirichlet Laplacian $-\Delta|_D$ or Neumann Laplacian $-\Delta|_N$ on $\Omega$. Suppose that $0 \leq s < n/2$, $2 \leq p, q \leq \infty$ and $(s,p,q)$ is admissible, i.e.,

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2} - s$$

without $q = \infty$ if $n = 2$. Then for any $T > 0$ there exists a constant $C > 0$ such that the solution $u = e^{-it\mathcal{H}}f$ to the equation (1.1) with $f \in \dot{H}^s(\mathcal{H})$ satisfies

$$\|u\|_{L^p([-T,T];L^q(\Omega))} \leq C\|f\|_{\dot{H}^s(\mathcal{H})}.$$  \hfill (2.1)

In particular, if $\mathcal{H} = -\Delta|_D$, then the estimates (2.1) hold with $T = \infty$.

**Remark 2.2.** In our argument, it is not clear whether the estimate (2.1) can have time independent constant $C$ in the Neumann case. This depends on the result on local smoothing estimates in Lemma 2.5 (see Remark 2.6).

**Remark 2.3.** We require the non-trapping condition on $\Omega$ to ensure local smoothing estimates, which are one of important tools in proving Theorem 2.1 (see Lemma 2.5).

Once homogeneous Strichartz estimates (2.1) are established, we can apply $TT^*$ argument by Ginibre and Velo [12] to obtain the inhomogeneous estimates.

**Corollary 2.4.** Let $n \geq 2$ and $\Omega$ be the exterior domain in $\mathbb{R}^n$ of a compact non-trapping obstacle with smooth boundary, and let $\mathcal{H}$ be the Dirichlet Laplacian $-\Delta|_D$ or Neumann Laplacian $-\Delta|_N$ on $\Omega$. Suppose that $2 \leq p, q \leq \infty$ and $(p,q)$ is admissible, i.e.

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}.$$

Then for any $T > 0$ there exists a constant $C > 0$ such that

$$\left\| \int_0^t e^{i\tau\mathcal{H}} F(\tau, \cdot) \, d\tau \right\|_{L^\infty([-T,T];L^2(\Omega))} \leq C\|F\|_{L^{p'}([-T,T];L^{q'}(\Omega))},$$

$$\left\| \int_0^t e^{-i(\tau-t)\mathcal{H}} F(\tau, \cdot) \, d\tau \right\|_{L^p([-T,T];L^q(\Omega))} \leq C\|F\|_{L^{p'}([-T,T];L^{q'}(\Omega))},$$

where $p'$ and $q'$ are conjugate exponents of $p$ and $q$, respectively. In particular, if $\mathcal{H} = -\Delta|_D$, then the above estimates hold with $T = \infty$.

Our approach is based on the following smoothing estimate obtained in the nontrapping setting in [6].

**Lemma 2.5** (Proposition 2.7 in [6]). Let $n \geq 2$ and $\Omega$ be the exterior domain in $\mathbb{R}^n$ of a compact non-trapping obstacle with smooth boundary, and let $\mathcal{H}$ be the Dirichlet Laplacian $-\Delta|_D$ or Neumann Laplacian $-\Delta|_N$ on $\Omega$. Then the following assertions hold:
(i) (Inhomogeneous case) Let $T > 0$ and $s \in [-1, 1]$. Then for any $\chi \in C_0^\infty(\mathbb{R}^n)$, there exists a constant $C > 0$ such that

$$u(t, x) = \int_0^t e^{-i(t-\tau)\mathcal{H}}\chi F(\tau, x)\,d\tau$$

satisfies

$$\|\chi u\|_{L^2([-T,T];H^{s+1}(\mathcal{H}))} \leq C\|\chi F\|_{L^2([-T,T];H^s(\mathcal{H}))}. \quad (2.2)$$

(ii) (Homogeneous case) Let $T > 0$ and $s \in [0, 1]$. Then for any $\chi \in C_0^\infty(\mathbb{R}^n)$, there exists a constant $C > 0$ such that $u(t) = e^{-it\mathcal{H}}f$ satisfies

$$\|\chi u\|_{L^2([-T,T];H^{s+\frac{1}{2}}(\mathcal{H}))} \leq C\|f\|_{H^s(\mathcal{H})}. \quad (2.3)$$

Remark 2.6. The constants $C$ in Lemma 2.5 are independent of $T$ in the Dirichlet case, whereas they might depend on $T$ in the Neumann case (see Remarks 2.8 and 2.9 in [4]). Hence we can assert that the estimates (2.2) and (2.3) hold with $T = \infty$ only in the Dirichlet case.

3. Odd and Even Extensions

3.1. The case of half space $\mathbb{R}_+^n$. We introduce appropriate extension operator. For any function $g \in H^2(\mathbb{R}_+^n)$ with support in $y_1$ contained in $[0, 2\pi)$, we can make an even extension $g$ satisfies Neumann boundary condition $\partial_{y_1}g_N = 0$ on $\{y_1 = 0\}$. Similarly, to have odd extension we need Dirichlet boundary condition $g_N = 0$ on $\{y_1 = 0\}$. For this it is natural to use a decomposition of type $g = g_D + g_N$ with $g_D = \partial_{y_1}g_N = 0$ on $\{y_1 = 0\}$ via the Fourier expansion in $y_1 \in (0, 2\pi)$, i.e.

$$g(y_1, y') = \frac{a_0}{2} + \sum_{k=1}^\infty a_k(y') \cos(ky_1) + \sum_{k=1}^\infty b_k(y') \sin(ky_1)$$

with

$$g_N(y_1, y') = \frac{a_0}{2} + \sum_{k=1}^\infty a_k(y') \cos(ky_1), \quad g_D(y_1, y') = \sum_{k=1}^\infty b_k(y') \sin(ky_1).$$

Then we define the extension of $g$ to $\mathbb{R}^n$ by

$$E[g](y) = \tilde{g}(y) := g_{D,odd}(y) + g_{N,even}(y), \quad y \in \mathbb{R}^n, \quad (3.1)$$

where $g_{D,odd}$ and $g_{N,even}$ are odd and even extensions of $g_D$ and $g_N$, respectively, i.e.,

$$g_{D,odd}(y_1, y') := \begin{cases} g_D(y_1, y') & \text{if } y_1 \geq 0, \\ -g_D(-y_1, y') & \text{if } y_1 < 0, \end{cases} \quad (3.2)$$

$$g_{N,even}(y_1, y') := \begin{cases} g_N(y_1, y') & \text{if } y_1 \geq 0, \\ g_N(-y_1, y') & \text{if } y_1 < 0, \end{cases} \quad (3.3)$$

where $y' = (y_2, \ldots, y_n)$. In this subsection for a differential operator

$$\tilde{\Delta}g(y) := \Delta_{y'}g(y) + \alpha(y')\partial_{y_1}^2g(y) + \sum_{j=2}^n \beta_j(y')\partial_{y_j}\partial_{y_1}g(y) + \gamma(y')\partial_{y_1}g(y) \quad (3.4)$$
with smooth coefficients $\alpha$, $\beta_j$ and $\gamma$ with respect to $y'$, we observe the relation
\[
\Delta \hat{g} = \Delta \hat{g}.
\]

Let $g \in H^\kappa(\mathbb{R}_+^n)$ with $\kappa > n/2$. Note that $H^\kappa(\mathbb{R}_+^n) \subset C^m(\mathbb{R}_+^n)$ with $m \in \mathbb{N} \cap \{0\}$ and $m < \kappa - n/2$. Assume the support of $g$ is included in $[0, 2\pi) \times \mathbb{R}^{n-1}$. These assumptions suggest to make the Fourier expansion of $g$ in $H^\kappa([0, 2\pi))$: 
\[
g(y_1, y') = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k(y') \cos(ky_1) + \sum_{k=1}^{\infty} b_k(y') \sin(ky_1)
\]
with uniformly convergent series expansions together with all derivatives up to $m$th-order. Hereafter, the decomposition of a function $g$ into the odd and even parts is defined via the Fourier transform as follow:
\[
g = g_D + g_N
\]
with
\[
g_D(y_1, y') = \sum_{k=1}^{\infty} b_k(y') \sin(ky_1), \quad \quad \quad (3.6)
\]
\[
g_N(y_1, y') = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k(y') \cos(ky_1). \quad \quad \quad (3.7)
\]

Then $g_D$ and $g_N$ satisfy the Dirichlet and Neumann boundary conditions, respectively, and the odd and even extensions of them are also written as
\[
g_{D, \text{odd}}(y_1, y') = \sum_{k=1}^{\infty} b_k(y') \sin(ky_1), \quad \quad \quad (3.8)
\]
\[
g_{N, \text{even}}(y_1, y') = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k(y') \cos(ky_1) \quad \quad \quad (3.9)
\]
for any $y \in \mathbb{R}^n$. Furthermore, if we take $\kappa = 3 + n/2$ and $m = 2$, then
\[
\partial_{y_j} g_{D, \text{odd}}(y) = (\partial_{y_j} g_D)_{\text{odd}}(y), \quad \partial_{y_j} g_{N, \text{even}}(y) = (\partial_{y_j} g_N)_{\text{even}}(y),
\]
\[
\partial_{y_j} \partial_{y_k} g_{D, \text{odd}}(y) = (\partial_{y_j} \partial_{y_k} g_D)_{\text{odd}}(y), \quad \partial_{y_j} \partial_{y_k} g_{N, \text{even}}(y) = (\partial_{y_j} \partial_{y_k} g_N)_{\text{even}}(y),
\]
\[
\partial_{y_j} g_{D, \text{odd}}(y) = (\partial_{y_j}^2 g_D)_{\text{odd}}(y), \quad \partial_{y_j} g_{N, \text{even}}(y) = (\partial_{y_j}^2 g_N)_{\text{even}}(y),
\]
\[
\partial_{y_j} \partial_{y_k} g_{D, \text{odd}}(y) = (\partial_{y_j} \partial_{y_k}^2 g_D)_{\text{even}}(y), \quad \partial_{y_j} \partial_{y_k} g_{N, \text{even}}(y) = (\partial_{y_j} \partial_{y_k} g_N)_{\text{odd}}(y)
\]
for $j, k = 2, \ldots, n$ and for any $y \in \mathbb{R}^n$. All these relations follow from (3.8) and (3.9). Therefore, we obtain
\[
\Delta \hat{g}(y) = [\Delta_y g_D(y) + \alpha(y') \partial_{y_1}^2 g_D(y) + \sum_{j=2}^{n} \beta_j(y') \partial_{y_j} \partial_{y_1} g_N(y) + \gamma(y') \partial_{y_1} g_N(y)]
\]
\[
+ [\Delta_y g_N(y) + \alpha(y') \partial_{y_1}^2 g_N(y) + \sum_{j=2}^{n} \beta_j(y') \partial_{y_j} \partial_{y_1} g_D(y) + \gamma(y') \partial_{y_1} g_D(y)].
\]
Since
\[ \Delta_y g_D(y) + \alpha(y) \partial_{y_1}^2 g_D(y) + \sum_{j=2}^{n} \beta_j(y) \partial_{y_j} \partial_{y_1} g_N(y) + \gamma(y) \partial_{y_1} g_N(y) \]
satisfies Dirichlet boundary condition, while
\[ \Delta_y g_N(y) + \alpha(y) \partial_{y_1}^2 g_N(y) + \sum_{j=2}^{n} \beta_j(y) \partial_{y_j} \partial_{y_1} g_D(y) + \gamma(y) \partial_{y_1} g_D(y) \]
satisfies the Neumann ones, we can use the odd and even extensions respectively and we get
\[ \tilde{\Delta} g(y) = \left[ \Delta_y g_D(y) + \alpha(y) \partial_{y_1}^2 g_D(y) + \sum_{j=2}^{n} \beta_j(y) \partial_{y_j} \partial_{y_1} g_N(y) + \gamma(y) \partial_{y_1} g_N(y) \right]_{\text{odd}} \]
\[ = (\tilde{\Delta} g)_{D,\text{odd}}(y) \]
\[ + \left[ \Delta_y g_N(y) + \alpha(y) \partial_{y_1}^2 g_N(y) + \sum_{j=2}^{n} \beta_j(y) \partial_{y_j} \partial_{y_1} g_D(y) + \gamma(y) \partial_{y_1} g_D(y) \right]_{\text{even}} \]
\[ = (\tilde{\Delta} g)_{N,\text{even}}(y) \]
\[ = \Delta_y g_{D,\text{odd}}(y) + \alpha(y) \partial_{y_1}^2 g_{D,\text{odd}}(y) + \sum_{j=2}^{n} \beta_j(y) \partial_{y_j} \partial_{y_1} g_{N,\text{even}}(y) + \gamma(y) \partial_{y_1} g_{N,\text{even}}(y) \]
\[ + \Delta_y g_{N,\text{even}}(y) + \alpha(y) \partial_{y_1}^2 g_{N,\text{even}}(y) + \sum_{j=2}^{n} \beta_j(y) \partial_{y_j} \partial_{y_1} g_{D,\text{odd}}(y) + \gamma(y) \partial_{y_1} g_{D,\text{odd}}(y) \]
\[ = \hat{\Delta} g_{D,\text{odd}}(y) + \hat{\Delta} g_{N,\text{even}}(y) = \hat{\Delta} \tilde{g}. \]

Summarizing the above observation, we have the following:

**Proposition 3.1.** Let \( \kappa \geq 3 + n/2 \). Then, for any \( g \in H^\kappa(\mathbb{R}^n) \) satisfying \( \text{supp} \ g \subset [0, \pi) \times \mathbb{R}^{n-1} \) and \( (3.3)-(3.7) \), the extension \( \tilde{g} \) defined in (3.1) satisfies \( \tilde{g} \in H^\kappa(\mathbb{R}^n) \) and

\[ \tilde{\Delta} \tilde{g} = \hat{\Delta} \tilde{g}, \]

where \( \hat{\Delta} \) is a differential operator defined in (3.4).

### 3.2. The case of exterior domain.

In this subsection we apply Proposition 3.1 to the case of exterior domain. To begin with, let us define a mapping \( \Phi \) flattening locally the boundary. Since \( \partial \Omega \) is smooth, for each \( x^{(0)} \in \partial \Omega \) there exist a neighborhood \( U \) of \( x^{(0)} \) and a \( C^\infty \)-function \( \phi : \mathbb{R}^{n-1} \rightarrow \mathbb{R} \) such that
\[ \Omega \cap U = \{ x \in U : x_1 > \phi(x') \}, \]
where \( x' = (x_2, \ldots, x_n) \). To reduce the argument into the half space case, we introduce
\[
\begin{cases}
  y_1 = x_1 - \phi(x') =: \Phi_1(x), \\
  y_j = x_j =: \Phi_j(x) \quad (j = 2, \ldots, n),
\end{cases}
\]
and we write \( y = \Phi(x) = (\Phi_1(x), \ldots, \Phi_n(x)) \) and \( V := \Phi(U) \). Then \( \Phi : U \rightarrow V \) is a \( C^\infty \)-diffeomorphism and \( \det \Phi = \det \Phi^{-1} = 1 \) (see, e.g., Evans [9]).

\[
U_+ := U \cap \Omega, \quad U_- := U \setminus \overline{U}_+, \quad V_\pm := \Phi(U_\pm).
\]

Given any function \( f \in H^\kappa(U \cap \Omega) \) with \( \kappa \geq 3 + n/2 \) satisfying
\[
\text{supp} f \subset (U \cap \overline{\Omega}),
\]
by using \((3.1)\), we can extend \( f \) to \( \tilde{f} \in H^\kappa(U) \) as follows:
\[
\tilde{f}(x) := \tilde{g}(\Phi(x)), \quad x \in U.
\]
Then, letting \( g(y) = f(\Phi^{-1}(y)) \), and noting that the pull - back \( \Phi^* \) of \( \Phi \) deforms the Laplace operator into the following operator in \( y \) coordinates
\[
\hat{\Delta} g(y) := \Phi^* (\Delta_x f)(y)
\]
\[
= \Delta_y g(y) + (1 + |\nabla_y \phi(y')|^2) \partial_{y_i}^2 g(y)
\]
\[
- 2 \sum_{j=2}^n (\partial_{y_j} \phi(y')) \partial_{y_j} \partial_{y_i} g(y) - (\Delta_y \phi(y')) \partial_{y_i} g(y),
\]
\[
= \Delta_y g(y) + \alpha(y') \partial_{y_i}^2 g(y) + \sum_{j=2}^n \beta_j(y') \partial_{y_j} \partial_{y_i} g(y) + \gamma(y') \partial_{y_i} g(y)
\]
with the smooth coefficients
\[
\alpha(y') := 1 + |\nabla_y \phi(y')|^2, \quad \beta_j(y') := -2 \partial_{y_j} \phi(y'), \quad \gamma(y') := -\Delta_y \phi(y'),
\]
by Proposition \((3.1)\) we obtain
\[
\tilde{\Delta} f(x) = \Delta \tilde{f}(x), \quad x \in U. \tag{3.10}
\]

Summarizing the above observation, we have the following:

**Proposition 3.2.** Let \( \kappa \geq 3 + n/2 \), and let \( x_0 \in \partial \Omega \). Then there exists \( \delta > 0 \) so that for
\[
U = B(x_0, \delta) = \{ x \in \mathbb{R}^n : |x - x_0| < \delta \} \tag{3.11}
\]
and for any \( f \in H^\kappa(U_+) \) with \( \text{supp} f \subset U \cap \overline{\Omega} \), one can define the extension operator \( f \rightarrow \tilde{f} \) so that \( \tilde{f} \in H^\kappa(U) \) and \((3.10)\) holds. In addition, if \( f \in \mathcal{D}(\mathcal{H}) \), then
\[
\| \tilde{f} \|_{H^2(U)} \sim \| \tilde{\mathcal{H}} f \|_{L^2(U_+)} + \| \tilde{\mathcal{H}} f \|_{L^2(U_+)} \sim \| \mathcal{H} f \|_{L^2(U_+)} + \| f \|_{L^2(U_+)},
\]
where \( \mathcal{H} \) is the Dirichlet Laplacian \(- \Delta|_D\) or Neumann Laplacian \(- \Delta|_N\) on \( \Omega \).

In the end of this subsection, we show the following:

**Proposition 3.3.** Let \( 0 \leq s \leq 2 \), and let \( x_0 \in \partial \Omega \) and \( U \) be a small \( \delta \)-ball centered at \( x_0 \) as in \((3.11)\). Then one can extend the operator \( f \mapsto \tilde{f} \) in Proposition \((3.2)\) to the bounded operator so that
\[
\| \tilde{f} \|_{H^s(U)} \leq C \| f \|_{H^s(\mathcal{H})} \tag{3.12}
\]
for any \( f \in H^s(\mathcal{H}) \) with \( \text{supp} f \subset U \cap \overline{\Omega} \).
Proof. First we prove the estimate (3.12). In the case \( s = 0, 2 \), it is readily seen from Proposition 3.2 that
\[
\| \chi \tilde{f} \|_{L^2(\mathbb{R}^n)} \leq C(\| f \|_{L^2(U_\epsilon)} + \| \tilde{f} \|_{L^2(U_\epsilon)}) \leq C\| f \|_{L^2(\Omega)},
\]
\[
\| \Delta(\chi \tilde{f}) \|_{L^2(\mathbb{R}^n)} \leq C \left( \| \tilde{f} \|_{L^2(U)} + \| \nabla \tilde{f} \|_{L^2(U)} + \| \Delta \tilde{f} \|_{L^2(U)} \right)
\leq C \left( \| f \|_{L^2(U_\epsilon)} + \| \nabla f \|_{L^2(U_\epsilon)} + \| \tilde{f} \|_{L^2(U_\epsilon)} \right)
\leq C \left( \| f \|_{L^2(U_\epsilon)} + \| \tilde{f} \|_{L^2(U_\epsilon)} \right)
\leq C \| f \|_{L^2(U_\epsilon)} + \| \tilde{f} \|_{L^2(U_\epsilon)}
\leq C \| f \|_{H^2(\mathcal{H})},
\]
which show (3.12) for \( s = 0, 2 \).

We consider the case \( 0 < s < 2 \). Let \( f = f_1 + f_2 \) with \( f_1 \in L^2(\Omega) \) and \( f_2 \in H^2(\mathcal{H}) \). Take \( \eta \in C^\infty(\Omega) \) such that
\[
\text{supp } \eta \subset \overline{U_\epsilon} \quad \text{and} \quad \eta \equiv 1 \text{ on } \overline{U_\epsilon},
\]
and we write
\[
f = \eta f = \eta f_1 + \eta f_2.
\]
Then \( \text{supp } (\eta f_1) \subset U \) and \( \text{supp } (\eta f_2) \subset U \). We use the real interpolation space
\[
\| \tilde{f} \|_{H^s(\mathbb{R}^n)} \leq C\| \tilde{f} \|_{(L^2(\mathbb{R}^n), H^2(\mathbb{R}^n))_{\frac{s}{2},2}} = C \left\{ \int_0^\infty \left( \lambda^{-\frac{s}{2}} K(\lambda, \tilde{f}) \right)^2 \frac{d\lambda}{\lambda} \right\}^{\frac{1}{2}},
\]
where \( K(\lambda, \tilde{f}) \) is Peetre’s K-function
\[
K(\lambda, \tilde{f}) = \inf \left\{ \| g_D \|_{L^2(\mathbb{R}^n)} + \lambda \| g_N \|_{H^2(\mathbb{R}^n)} : \tilde{f} = g_D + g_N \in L^2(\mathbb{R}^n) + H^2(\mathbb{R}^n) \right\}.
\]
By (3.12) for \( s = 0, 2 \), we estimate
\[
K(\lambda, \tilde{f}) \leq \| \eta f_1 \|_{L^2(\mathbb{R}^n)} + \lambda \| \eta f_2 \|_{H^2(\mathbb{R}^n)} \leq C \left( \| f_1 \|_{L^2(\Omega)} + \lambda \| f_2 \|_{H^2(\mathcal{H})} \right),
\]
and hence,
\[
\| \tilde{f} \|_{H^s(\mathbb{R}^n)} \leq C \left[ \int_0^\infty \left\{ \lambda^{-\frac{s}{2}} (\| f_1 \|_{L^2(\Omega)} + \lambda \| f_2 \|_{H^2(\mathcal{H})}) \right\}^2 \frac{d\lambda}{\lambda} \right]^{\frac{1}{2}}.
\]
Taking the infimum of the above inequality over \( f = f_1 + f_2 \in L^2(\Omega) + H^2(\mathcal{H}) \), we find from (iii) in Proposition 1.3 that
\[
\| \tilde{f} \|_{H^s(\mathbb{R}^n)} \leq C\| f \|_{(L^2(\Omega), H^2(\mathcal{H}))_{\frac{s}{2},2}} \leq C\| f \|_{H^s(\mathcal{H})}.
\]
The proof of Proposition 3.3 is finished. \( \square \)
4. Key estimates

4.1. Smoothing estimates. We use two kinds of smoothing estimates in the proof of Theorem 2.1. The first one is Lemma 2.5 stated in Section 2. The second one is the following Strichartz-smoothing estimates for free Schrödinger equation.

**Lemma 4.1.** Let \( n \geq 2 \) and

\[
X = \begin{cases} 
L^2(\mathbb{R}; L_{n-2}^{2n}(\mathbb{R}^n)) & \text{if } n \geq 3 \\
L^p(\mathbb{R}; L^{q}(\mathbb{R}^2)) & \text{if } n = 2,
\end{cases}
\]

where \((p, q)\) is admissible. Suppose that \( u \) is a solution to the equation

\[
i\partial_t u(t, x) + \Delta u(t, x) = F(t, x) \quad \text{in } \mathbb{R} \times \mathbb{R}^n
\]

with initial data \( u(0) = 0 \). Then for any \( s_0 > 1/2 \) there exists a constant \( C > 0 \) such that

\[
\|u\|_X \leq C \|\langle x \rangle^{-s_0} D^{-\frac{1}{2}} F\|_{L^2(\mathbb{R}; L^2(\mathbb{R}^n))}.
\]  (4.1)

**Proof.** By Lemma 3 in [13], we have

\[
\sup_{x_1 \in \mathbb{R}^{n-1}} \int_{\mathbb{R}} |D_1^x u(t, x_1, x')|^2 dt dx' \leq C \|F\|_{X'},
\]

where \( X' \) is the dual space of \( X \), which implies that

\[
\|u\|_X \leq C \|D^x_1 \|F\|_{L^1_1(\mathbb{R}; L^2_{x,t}(\mathbb{R}^{n-1} \times \mathbb{R}))}
\]

by the duality argument. Applying Hölder’s inequality to the right hand side, we obtain (4.1). This completes the proof of Lemma 4.1. \( \square \)

4.2. Commutator estimates. In this subsection we prove commutator estimates between polynomial weights and fractional differential operators.

**Proposition 4.2.** Let \( n \geq 1 \). Suppose that \( 1/2 < a < a_0 \leq 1 \) and \( 1 < p, p_0, q < \infty \) satisfy

\[
\frac{1}{q} = \frac{1}{p} - \frac{a - \frac{1}{2}}{n} = \frac{1}{p_0} - \frac{a_0 - \frac{1}{2}}{n}.
\]

Then

\[
\|[(x)^a, D^{-\frac{1}{2}} \nabla] f\|_{L^q(\mathbb{R}^n)} \leq C \left( \|D^{\frac{1}{2}}((x)^{a_0} f)\|_{L^q(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)} + \|f\|_{L^{p_0}(\mathbb{R}^n)} \right)
\]

for any \( f \in \mathcal{S}(\mathbb{R}^n) \).

In order to prove this proposition, we use the following two estimates.

**Lemma 4.3** (Lemma 12 in Janson [19]). Let \( n \geq 1 \) and \( R_j \) be the Riesz transform

\[
R_j g(x) := c_n \text{ P.V.} \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x - y|^{n+1}} g(y) \, dy
= c_n \lim_{\varepsilon \searrow 0} \int_{|x - y| \geq \varepsilon} \frac{x_j - y_j}{|x - y|^{n+1}} g(y) \, dy
\]

for \( j = 1, \ldots, n \), where

\[
c_n := \frac{\Gamma((n + 1)/2)}{\pi^{(n+1)/2}}.
\]
Assume that $0 < \alpha \leq 1$ and $1 < p < q < \infty$ satisfy $1/q = 1/p - \alpha/n$. Then
\[
\| [R_j, f]g \|_{L^q(\mathbb{R}^n)} \leq C \| f \|_{\Lambda_{\alpha}} \| g \|_{L^p(\mathbb{R}^n)}
\]
for any $f \in \Lambda_{\alpha}$ and $g \in L^p(\mathbb{R}^n)$. where $\Lambda_{\alpha}$ is the Lipschitz space, i.e.,
\[
\Lambda_{\alpha} = \{ f \in C(\mathbb{R}^n) : |f(x) - f(y)| \leq C|x - y|^\alpha \}
\]
with norm $\| f \|_{\Lambda_{\alpha}} = \sup_{x \neq y} (|f(x) - f(y)|/|x - y|^\alpha)$.

**Lemma 4.4.** Let $n \geq 1$. Suppose that $0 \leq s < \alpha \leq 1$ and $1 \leq p < q \leq \infty$ satisfy $1/q = 1/p - (\alpha - s)/n$. Then
\[
\| [D^s, f]g \|_{L^q(\mathbb{R}^n)} \leq C \| f \|_{\Lambda_{\alpha}} \| g \|_{L^p(\mathbb{R}^n)}
\]
for any $f \in \Lambda_{\alpha}$ and $g \in \mathcal{S}(\mathbb{R}^n)$.

**Proof.** The proof is based on the explicit representation
\[
(D^s h)(x) = C(n, s) \text{P.V.} \int_{\mathbb{R}^n} \frac{h(x + y) - h(x)}{|y|^{n+s}} dy
\]
\[
= C(n, s) \lim_{\varepsilon \downarrow 0} \int_{|y| \geq \varepsilon} \frac{h(x + y) - h(x)}{|y|^{n+s}} dy,
\]
where
\[
C(n, s) = \left( \int_{\mathbb{R}^n} \frac{1 - \cos(\zeta)}{\zeta^{n+s}} d\zeta \right)^{-1}
\]
(see [3]). Thanks to this representation, we write the function $[D^s, f]g$ as
\[
[D^s, f]g = D^s(fg) - fD^s g
\]
\[
= C(n, s) \text{P.V.} \int_{\mathbb{R}^n} \frac{f(x + y) - f(x)}{|y|^\alpha} \cdot \frac{g(x + y)}{|y|^{n+s-\alpha}} dy,
\]
and hence, taking $L^p$-norm of the both sides, and using the Sobolev embedding theorem, we obtain
\[
\| [D^s, f]g \|_{L^q(\mathbb{R}^n)} \leq C \| f \|_{\Lambda_{\alpha}} \| D^{s-\alpha} g \|_{L^q(\mathbb{R}^n)} \leq C \| f \|_{\Lambda_{\alpha}} \| g \|_{L^p(\mathbb{R}^n)}.
\]
The proof of Lemma 4.4 is finished. \hfill \Box

**Proof of Proposition 4.2.** Writing
\[
[f, D^{-\frac{1}{2}} \nabla] = [\langle x \rangle^\alpha, RD^{\frac{1}{2}}] = [\langle x \rangle^\alpha, R[D^{\frac{1}{2}} + R[\langle x \rangle^\alpha, D^{\frac{1}{2}}]],
\]
where $R = (R_1, \ldots, R_n)$ is the Riesz transform, we have
\[
\| [\langle x \rangle^\alpha, D^{-\frac{1}{2}} \nabla] f \|_{L^q(\mathbb{R}^n)} \leq \| [\langle x \rangle^\alpha, R[D^{\frac{1}{2}}] f \|_{L^q(\mathbb{R}^n)} + \| [\langle x \rangle^\alpha, D^{\frac{1}{2}}] f \|_{L^q(\mathbb{R}^n)}.
\]
As to the first term, it follows from Lemmas 4.3 and 4.4 that
\[
\| [\langle x \rangle^\alpha, R[D^{\frac{1}{2}}] f \|_{L^q(\mathbb{R}^n)} \leq C \| D^{\frac{1}{2}} f \|_{L^q(\mathbb{R}^n)}
\]
\[
\leq C \| \langle x \rangle^\alpha D^{\frac{1}{2}} f \|_{L^q(\mathbb{R}^n)}
\]
\[
\leq C \left( \| D^{\frac{1}{2}} \langle x \rangle^\alpha f \|_{L^q(\mathbb{R}^n)} + \| [\langle x \rangle^\alpha, D^{\frac{1}{2}}] f \|_{L^q(\mathbb{R}^n)} \right)
\]
\[
\leq C \left( \| D^{\frac{1}{2}} \langle x \rangle^\alpha f \|_{L^q(\mathbb{R}^n)} + \| f \|_{L^{p_0}(\mathbb{R}^n)} \right),
\]
where $1/q = 1/r - a/n$ and $a < a_0 \leq 1$. As to the second term, by $L^p$-boundedness of $R$ and Lemma 4.4 we have

$$\|R[\langle x \rangle^a, D^{\frac{1}{2}}]f\|_{L^q(\mathbb{R}^n)} \leq C\|[\langle x \rangle^a, D^{\frac{1}{2}}]f\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}.$$  

By summarizing the estimates obtained now, we conclude Proposition 4.2. □

5. Proof of Theorem 2.1

We consider only the Dirichlet boundary condition case in dimensions $n \geq 3$, since the Neumann boundary condition case and two dimensional case $n = 2$ are proved in a similar way. So we may omit the proofs.

Let $T > 0$ and $u = u(t)$ be a solution to the equation (1.1) with initial data $f \in C^\infty_0(\Omega)$, and we write $u(t) = e^{it\Delta}f$. Then $u \in C^k([-T, T]; D(\Delta^l_D))$ for any $k, l \in \mathbb{N}$. Therefore $u(t, x)$ is solution to the problem

$$i\partial_t u(t, x) + \Delta u(t, x) = 0, \quad t \geq 0, \quad x \in \Omega,$$

$$u(t, x) = 0, \quad t \geq 0, \quad x \in \partial \Omega,$$

$$u(0, x) = f(x).$$

Since $\partial \Omega$ is of $C^\infty$ and compact, by the Sobolev embedding theorem, we have $u \in C^\infty([-T, T] \times \Omega)$.

Since $\partial \Omega$ is compact, for any $\delta > 0$ there exist finitely many points $x^{(k)} \in \partial \Omega$, $k = 1, \cdots, N$ so that

$$U_\delta(x^{(k)}) = \{x \in \mathbb{R}^n, |x - x^{(k)}| < \delta\}$$

is a covering of $\partial \Omega$. We can choose $\delta > 0$ so small that Propositions 3.2 and 3.3 are applicable for $U_k = U_\delta(x^{(k)})$.

Then we need the partition of unity subordinated to this covering. For the purpose we define the compact

$$K = K_\delta = \left\{x \in \mathbb{R}^n; d(x, \partial \Omega) \leq \frac{\delta}{2}\right\}$$

and let $\{\chi_k\}_{k=1}^N$ be associated partitions of unity so that

$$\chi_k \in C^\infty_0(U_k), \quad \chi_k \geq 0 \quad \text{and} \quad \sum_{k=1}^N \chi_k(x) = 1 \text{ for any } x \in K.$$  

Since $\chi_k u$ has support in $U_k \cap \mathring{\Omega}$, we can apply Proposition 3.2 and use the extension operator \* for the symbol $\tilde{f}$ for the extension operator is a little bit misleading, since it depends on $U_k = U_\delta(x^{(k)})$. However fixing $k$ and using the support assumption we can proceed further.
Moreover, we shall need the relation
\[
\Delta \chi_k u = \Delta (\chi_k \eta_k u) = \chi_k \Delta (\eta_k u) + 2 \nabla \chi_k \nabla (\eta_k u) + (\Delta \chi_k)(\eta_k u)
\]
\[
= \langle \chi_k \mathcal{H} u \rangle + 2 \nabla \chi_k \nabla (\eta_k u) + (\Delta \chi_k)(\eta_k u),
\]
where \( \eta_k \in C^0_\infty(U_k) \) is such that \( \eta_k(x) = 1 \) on the support of \( \chi_k \). In this way we find
\[
i \partial_t (\overline{\chi_k u}) + \Delta (\overline{\chi_k u}) = \overline{\chi_k i \partial_t u + \mathcal{H} \chi_k u + 2 \nabla \chi_k \nabla (\eta_k u) + (\Delta \chi_k)(\eta_k u)}
\]
\[
= 2 \nabla \chi_k \nabla (\eta_k u) + (\Delta \chi_k)(\eta_k u)
\]
\[
= [\mathcal{H}, \chi_k] \eta_k u,
\]
for any \( x \in U_k \). Thus we arrive at the following mixed boundary valued problem
\[
i \partial_t (\overline{\chi_k u}) + \Delta (\overline{\chi_k u}) = [\mathcal{H}, \chi_k] u, \quad t \geq 0, \ x \in U_k,
\]
\[
\overline{\chi_k u} = 0, \quad t \geq 0, \ x \in \partial \Omega,
\]
\[
\overline{\chi_k u}(0, x) = \chi_k f(x), \quad x \in U_k.
\]
As conclusion we have the integral equation in \( \mathbb{R}^n \)
\[
\overline{\chi_k u}(t) = e^{it \Delta (\overline{\chi_k f})} + i \int_0^t e^{i(t-\tau) \Delta ([\overline{\chi_k}, \mathcal{H}] u)} d\tau
\]
(5.1)
for \( k = 1, \ldots, N \).

Since we can use the endpoint Strichartz-smoothing estimate in \( \mathbb{R}^n \) and the smoothing estimate of Lemma 2.5, we can deduce
\[
\| \chi_k u \|_{L^2(\mathbb{R}; L^{\frac{2n}{n-2}}(\Omega))} \leq C \| f \|_{L^2(\Omega)}, \quad k = 1, \ldots, N.
\]
(5.2)
Indeed, fixing \( k \) we shall use as extension operator \( f \to \overline{f} \) the one associated with the small ball \( U_k \). Then by using the Duhamel formula (5.1) and endpoint Strichartz estimates for free Schrödinger equation (see [21]), we have
\[
\| \overline{\chi_k u} \|_{L^2(\mathbb{R}; L^{\frac{2n}{n-2}}(\Omega))} \leq C \| f \|_{L^2(\Omega)} + \left\| \int_0^t e^{i(t-\tau) \Delta ([\chi_k, \Delta] u)(\tau)} d\tau \right\|_{L^2(\mathbb{R}; L^{\frac{2n}{n-2}}(\mathbb{R}^n))}.
\]
Hence it suffices to show that
\[
\left\| \int_0^t e^{i(t-\tau) \Delta ([\chi_k, \Delta] u)(\tau)} d\tau \right\|_{L^2(\mathbb{R}; L^{\frac{2n}{n-2}}(\mathbb{R}^n))} \leq C \| f \|_{L^2(\Omega)}.
\]
(5.3)
Since
\[
[\chi_k, \Delta] u = -(\Delta \chi k) u - 2 \nabla \chi k \cdot \nabla u = (\Delta \chi k) u - 2 \nabla \cdot (\nabla \chi k) u,
\]
we estimate
\[
\text{LHS of (5.3)} \leq \left\| \int_0^t e^{i(t-\tau) \Delta ([\chi_k, \Delta] u)(\tau)} d\tau \right\|_{L^2(\mathbb{R}; L^{\frac{2n}{n-2}}(\mathbb{R}^n))}
\]
\[
+ 2 \left\| \int_0^t e^{i(t-\tau) \Delta (\nabla \cdot (\nabla \chi k) u)(\tau)} d\tau \right\|_{L^2(\mathbb{R}; L^{\frac{2n}{n-2}}(\mathbb{R}^n))}
\]
(5.4)
\[=: I + II.
\]
As to the first term $I$, we apply the endpoint Strichartz estimate in $\mathbb{R}^n$, the Hölder inequality and (ii) in Lemma 2.5 to get
\[ I \leq C \| (\Delta \chi_k) u \|_{L^2(\mathbb{R}; L^{\frac{4n}{n+2}}(U_k))} \leq C \| (\Delta \chi_k) u \|_{L^2(\mathbb{R}; L^2(\Omega))} \leq C \| f \|_{L^2(\Omega)}. \tag{5.5} \]
As to the second term $II$, we find from Lemma 4.1 that
\[
II \leq C \| \langle x \rangle^a D^{-\frac{a}{2}} \nabla \cdot (\widehat{\nabla \chi_k} u) \|_{L^2(\mathbb{R}; L^2(\mathbb{R}^n))} \\
\leq C \| D^{-\frac{a}{2}} \nabla \langle x \rangle^a (\widehat{\nabla \chi_k} u) \|_{L^2(\mathbb{R}; L^2(\mathbb{R}^n))} + \| [\langle x \rangle^a, D^{-\frac{a}{2}} \nabla] \cdot (\widehat{\nabla \chi_k} u) \|_{L^2(\mathbb{R}; L^2(\mathbb{R}^n))} \\
=: II_1 + II_2,
\]
where $1/2 < a < 1$. By Plancherel’s theorem and Proposition 3.3 we have
\[ II_1 = \| \langle x \rangle^a (\widehat{\nabla \chi_k} u) \|_{L^2(\mathbb{R}; H^{-\frac{a}{2}}(\mathbb{R}^n))} \leq C \| \langle x \rangle^a (\nabla \chi_k) u \|_{L^2(\mathbb{R}; H^{-\frac{a}{2}}(-\Delta_D))}. \tag{5.6} \]
To this end we note that $\langle x \rangle^a (\nabla \chi_k) u$ is a smooth and compactly supported one, so applying the smoothing estimate of Lemma 2.5 we get
\[ II_1 = \| \langle x \rangle^a (\nabla \chi_k) u \|_{L^2(\mathbb{R}; H^{-\frac{a}{2}}(\mathbb{R}^n))} \lesssim \| f \|_{L^2}. \tag{5.6} \]

Since $\widehat{\nabla \chi_k} u \in C_0^\infty(\mathbb{R}^n)$, by using Proposition 4.2, Hölder’s inequality and Proposition 3.3, we estimate
\[
II_2 \leq C \left( \| D^{\frac{a}{2}} (\langle x \rangle^a (\nabla \chi_k) u) \|_{L^2(\mathbb{R}^n)} + \| (\nabla \chi_k) u \|_{L^r(\mathbb{R}^n) \cap L^{r_0}(\mathbb{R}^n)} \right) \\
\leq C \left( \| D^{\frac{a}{2}} (\langle x \rangle^a (\nabla \chi_k) u) \|_{L^2(\mathbb{R}^n)} + \| (\nabla \chi_k) u \|_{L^2(\mathbb{R}^n)} \right) \\
\leq C \| (\langle x \rangle^a (\nabla \chi_k) u) \|_{L^2(\mathbb{R}; H^{\frac{a}{2}}(-\Delta_D))} + C \| f \|_{L^2},
\]
where $a < a_0 \leq 1$, $1/r = 1/2 + (a - 1/2)/n$ and $1/r_0 = 1/2 + (a_0 - 1/2)/n$. The term
\[ \| (\langle x \rangle^a (\nabla \chi_k) u) \|_{L^2(\mathbb{R}; H^{\frac{a}{2}}(-\Delta_D))} \]
can be estimated in the same way as we did in (5.6).

By summarizing the above three estimates, we obtain
\[ II \leq C \| f \|_{L^2(\Omega)}. \tag{5.7} \]
Therefore, by combining (5.4) - (5.7), we prove (5.3). Thus we conclude the endpoint Strichartz estimate (5.2).

**Final step.** The endpoint estimate (2.1) with $(p, q) = (2, 2n/(n-2))$ is now obtained, and the case that $(p, q) = (\infty, 2)$ is trivial. Hence, by the Riesz-Thorin interpolation theorem, we get (2.1) for all admissible pairs $(p, q)$. Finally, the case $s > 0$ is proved by combining the Sobolev embedding theorem and the case $s = 0$. In fact, let $(s, p, q)$ be an admissible triplet with $0 < s < n/2$. Then, by (ii) in Proposition 1.3 we have
\[ \| u \|_{L^p([-T,T]; L^q(\Omega))} \leq C \| u \|_{L^p([-T,T]; B_{s+2}(\Delta_D))} \]
where $1/q = 1/r - s/n$. We note that the pair $(p, r)$ is admissible. Hence, in a similar way to the above argument, we obtain
\[ \| u \|_{L^p([-T,T]; B_{s+2}(\Delta_D))} \leq \| f \|_{H^r(\Delta_D)}. \]
Therefore (2.1) is also proved for $0 < s < n/2$. Thus we conclude Theorem 2.1. □

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