Fuzzy Reset-Based $H_\infty$ Unknown Input Observer Design for Uncertain Nonlinear Systems with Unmeasurable Premise Variables

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ABSTRACT This paper proposes an $H_\infty$ reset unknown input observer (R-UlO) based on the Takagi-Sugeno (T-S) fuzzy model for the state estimation of nonlinear uncertain systems. Firstly, $H_\infty$ unknown input observer (UIO) is designed for TSFM-based nonlinear systems with measurable and unmeasurable premise variables. Then, according to the importance of observers based on unmeasurable premise variables, the results on UIO is modified to propose R-UlO. The sufficient conditions for the stabilization of the estimation error are derived in terms of linear matrix inequalities (LMIs). The proposed R-UlO benefits from less computation complexity to find the feasible parameters, improvement of the estimation process in viewpoints of convergence speed and overshoot. To verify the effectiveness of the recommended approaches, the methods are applied to a practical system.

INDEX TERMS T-S fuzzy system, Reset mechanism, Unknown inputs (UIs), Unmeasurable premise variables, External disturbance.

I. INTRODUCTION

Takagi-Sugeno (T-S) fuzzy model (TSFM) has attracted serious consideration in recent decades to deal with complicated nonlinear systems [1], [2]. The interest for consideration TSFM is principally that it represent the complex nonlinear systems by blending of local sub models [3]. However, for many practical systems, the states are not easily available or the sensors to measure them are costly or hard to protect. Accordingly, the observer design for state estimation is necessary. Great efforts have been put into observer design for T-S fuzzy systems and excellent results have been reported (see, e.g. [4], [5]).

Moreover, in practical applications, the systems are influenced by unknown inputs (UIs), such as parameter uncertainties [6], [7], external disturbance [8]. As a result, the unknown input observer (UIO) design is a basic concern which attracted special attentions in both theory and application [9]. In [10], a filter is recommended for a class of stochastic nonlinear system to estimate the system states with the presence of external disturbances and no consideration of the uncertainties. In [5], UIO is designed for TSFM subject to UIs and external disturbance affecting both outputs and states of the system and sufficient conditions are derived in terms of linear matrix inequalities (LMIs). Similar results are investigated for switched systems [11], descriptor systems [12] and so on.

Though acceptable researches have been published in designing UIO for TSFM-based nonlinear systems, little research has been focused on UIO for TSFM with nonlinear output equations. The reason is that designing UIOs for TSFM-based nonlinear systems with multiple output matrices $C_i$ is led to a non-convex problem that cannot be easily solved (see Remark 3 in [13]). However, TSFM of many practical systems such as vehicle lateral systems [14], inverted pendulum controlled by DC motor [15] and etc. share different multiple output matrices $C_i$. Accordingly, recent studies are focused on designing UIO for TSFM-based nonlinear systems with nonlinear output equations [13], [15]–[17].

In the same way, if the premise variables of the observer depend on the unavailable states of the system, then the conventional observer with the measurable-based premise variables cannot estimate the system states. A fuzzy observer is designed for a class of T-S fuzzy neural-network systems
in [16] with nonlinear output equations based on the measured premise variables. Similar results are developed for continuous-time T-S fuzzy systems with unmeasurable premise variables of the observer in [17], [18]. In [13], a new structure for UIO is established that can be used for discrete-time TSFM with a nonlinear output equation. A modified UIO for discrete-time TSFM with multiple output matrices is presented in [15]. In this approach, the fuzzy system is reformulated to overcome the non-convex problem raised by designing UIO for TSFM with a nonlinear output equation. However, in [13], [15]–[18], the uncertainties are not considered in the problem formulation.

In the presence of uncertainties, it is difficult to design a fuzzy observer such that the state estimation error converges asymptotically to zero. Meanwhile, some approaches in designing UIO for uncertain TSFM are developed to ensure the asymptotic stability of the state estimation error [19]–[21]. However, the uncertainties in these developed strategies must satisfy a particular condition and be bounded. To solve this issue, in [22], a new method is investigated to design UIO for an uncertain TSFM-based nonlinear system. But, in these studies, the system outputs must be linear.

Moreover, reset theory is a particular type of control strategy focusing to reduce the performance drawbacks such as transient response oscillatory [23]. Based on the fruitful properties of reset control approach, this line of research attracted the researchers’ attentions. Furthermore, the reset method can be implemented to conventional observers to enhance their performance [23]. More specifically, synthesizing the reset method with the linear observers led to reducing the overshoot and the settling-time of the state estimation error, simultaneously [23]. Therefore, a reset observer is safer and more reliable in the industry.

In [24], for the first time, the reset theory was applied to the adaptive observer. In this study, it is shown that synthesizing the reset method to the observer decreases the settling-time and overshooting of the estimation process. The authors in [25] investigate a reset observer for linear time-varying Wing-In-Ground craft. A similar study was extended for linear time-varying delay systems in [26].

Recently, given advantage applications of reset observer for a linear system, the research is continued for nonlinear systems [23], [27]–[29]. A fuzzy reset observer with discrete/continuous measurement is developed in [27] for a class of TSFM-based nonlinear systems. A fuzzy reset observer is designed for a biodiesel production process to provide online estimation in [28]. Compared with the classical fuzzy observer and extended Kalman filter, the performance of the state estimation error by the fuzzy reset observer has been improved. The problem of fault estimation using reset observer for a class of TSFM-based time-delay system is studied in [23]. In [30], reset observer-based event-triggered control is considered for the multiagent system in the presence of disturbance. In comparison with the conventional observer-based event-triggered control, it was demonstrated that the performance of the system significantly improved. However, in [23], [27], [28], [30], the authors were not concerned with unknown input problems.

Recently, according to the importance of UIO, reset strategy is synthesized with UIO and R-UIO is developed to performance enhancement of the conventional UIO (C-UIO). R-UIO is designed for a class of switched Lipschitz nonlinear systems with UIs in [31]. The existence of the recommended R-UIO is given via the Riccati equation in which led to solving complicated equations. An R-UIO for state estimation of linear time-invariant systems with unknown input is presented in [32] to decrease the $L_2$ norm and settling-time of the state estimation error. Extended results are presented in [33] for a class of nonlinear uncertain systems with no external disturbances.

To the best of the author’s knowledge, little research has been focused on designing R-UIO for nonlinear systems, and in view of fruitful applications of TS fuzzy methodologies for complicated nonlinear systems, the problem of R-UIO for TSFM-based nonlinear system has not been addressed in two modes: with measurable premise variables and with unmeasurable premise variables. Also, in the formulation of the previous studies, external disturbance and uncertainties do not exist, together.

Motivated by the above discussions, in this article, a comprehensive fuzzy R-UIO is proposed for a continuous-time uncertain nonlinear system in the presence of external disturbance. At first, we propose a UIO for an uncertain TSFM-based nonlinear system in the presence of external disturbance. The proposed UIO is applicable for both single output matrix and multiple output matrices. To overcome the non-convex conditions in designing UIO for an uncertain TSFM-based nonlinear system with nonlinear output equations, a mathematical transformation is adopted. Besides, to improve the performance of the proposed UIO, the reset mechanism is incorporated to design R-UIO. According to the importance of designing observers with unmeasurable premise variables in practice, the proposed UIO and R-UIO are also modified with unmeasurable premise variables. The main contributions of this paper can be summarized as follows:

- Compared with [5], [34], [35], the proposed UIO is designed with $H_\infty$ performance for uncertain TSFM-based nonlinear systems in the presence of external disturbance. The uncertainties are transformed to unknown inputs for the TSFM which removes the drawbacks of particular constrained conditions of the uncertainties [19]–[21].
- The proposed $H_\infty$ UIO is applicable for any TSFM with both the linear and the nonlinear output equations in comparison with [5], [11], [12], [34].
- Compared to [15]–[22], for the first time, the reset theory is synthesized to UIO for the uncertain TSFM-based nonlinear system. A more practical UIO/ R-UIO are designed in two modes: with measurable and unmeasurable
premise variables which is more reasonable compared to [5], [11]–[13], [16], [34].

Lastly, to verify the effectiveness and superiority of the proposed results obtaining in this article, a practical system is simulated.

The organization of this article is as: Section II concerns with preliminaries and problem formulation. In section III, the main results of UIO/RUIO design procedure are given for uncertain TSFM. Simulation results are brought in Section IV to verify the performance of the estimation process. Finally, section V draws the conclusions along with some recommended future works.

Notation: In this paper, the (∗) indicates the transpose element of the symmetric matrix. I denotes an identity matrix. For brevity, \( h_i \) denotes \( h_i(x(t)) \). \( X^+ \) represent the pseudo-inverse of the matrix \( X \) with \( X^+ = (X^TX)^{-1}X^T \).

## II. PROBLEM FORMULATION

Consider the nonlinear continuous-time system in the presence of external disturbance as follows:

\[
\dot{x}(t) = f(x(t), u(t), w(t))
\]

\[
y(t) = g(x(t), u(t))
\]

where \( f(\cdot) \) and \( g(\cdot) \) are the nonlinear functions and \( x(t) \in \mathbb{R}^n \), \( y(t) \in \mathbb{R}^p \), \( u(t) \in \mathbb{R}^m \) and \( w(t) \in \mathbb{R}^q \) are the state vector, system output, control input and the external disturbance, respectively.

The TSFM based on sector nonlinearity ensures the exact representation of the nonlinear system (1) [3]. The \( i \)-th rule of the TSFM is as follows [3], [15]:

\[
\text{IF } \zeta_i(t) = \kappa_{i2}(t), \ldots, \zeta_p(t) = \kappa_{ip}(t), \text{ THEN }
\]

\[
\begin{align*}
\dot{x}(t) &= (A_i + \Delta A_i)x(t) + (B_i + \Delta B_i)u(t) + D_i w(t) \\
y(t) &= C_i x(t), \; i = 1, 2, \ldots, r
\end{align*}
\]

(2)

where \( x(t) \in \mathbb{R}^n \), \( y(t) \in \mathbb{R}^p \), \( w(t) \in \mathbb{R}^q \) and \( u(t) \in \mathbb{R}^m \) are the state vector, system output, external disturbance and the control input, respectively. \( \zeta_i(t), \zeta_p(t), \ldots, \) and \( \zeta_p(t) \) are the premise variables, \( \kappa_{i1}(t), \kappa_{i2}(t), \ldots, \kappa_{ip}(t) \) are the fuzzy collections, \( r \) is the fuzzy principle numbers, and \( A_i \in \mathbb{R}^{n \times n}, B_i \in \mathbb{R}^{n \times m}, C_i \in \mathbb{R}^{p \times n}, D_i \in \mathbb{R}^{p \times q} \) are known constant matrices. Additionally, \( \Delta A_i \in \mathbb{R}^{n \times n} \) and \( \Delta B_i \in \mathbb{R}^{n \times m} \) are time-varying uncertainties related to the uncertainties of the nominal system which occur from the basic nonlinear function \( f(\cdot) \) [15]. The overall TSFM-based nonlinear system is obtained as:

\[
\begin{cases}
\dot{x}(t) = \sum_{i=1}^{r} h_i(x(t)) (A_i + \Delta A_i)x(t) + (B_i + \Delta B_i)u(t) + D_i w(t) \\
y(t) = \sum_{i=1}^{r} h_i C_i x(t), \; i = 1, 2, \ldots, r
\end{cases}
\]

(3)

where \( h_i(\cdot) \) is such that:

\[
h_i(x(t)) = \frac{\prod_{j=1}^{p} \kappa_{ij}(x(t))}{\sum_{i=1}^{r} \prod_{j=1}^{p} \kappa_{ij}(x(t))} \sum_{i=1}^{r} h_i(x(t)) = 1.
\]

(4)

In order to design fuzzy observers for uncertain systems, an efficient method is to transform them to UIs. There exists matrix \( M \) such that \( \Delta A(t) = MA\Delta A(t) \) and \( \Delta B(t) = MB\Delta B(t) \), where \( M \) is a full column rank known matrix and \( \Delta A(t) \) and \( \Delta B(t) \) are uncertainties.

**Remark 1:** The Luenburger observer cannot handle the uncertainties [15], [19]–[21]. To overcome this issue, several methods were developed [19]–[21], [36], [37]. However, in these studies, the conditions \( \Delta A(t) = M_a\Delta A(t)E_{ai}, \Delta B(t) = M_b\Delta B(t)E_{bi} \) and the bounded conditions \( \Delta A(t)^T \Delta A(t) \leq I \) and \( \Delta B(t)^T \Delta B(t) \leq I \) are needed. But, in our paper more relaxed conditions for the uncertainties are expressed and the bounds of the uncertainties are not required to be known.

Let us define \( a_i(t) = \Delta A_i(t)x(t), \beta_i(t) = \Delta B_i(t)u(t) \) and \( y_i(t) = a_i(t) + \beta_i(t) \). Thus, (2) can be reformulated as:

\[
\begin{align*}
\dot{x}(t) &= \sum_{i=1}^{r} h_i(A_i x(t) + B_i u(t) + M_i y(t) + D_i w(t)) \\
y(t) &= \sum_{i=1}^{r} h_i C_i x(t), \; i = 1, 2, \ldots, r.
\end{align*}
\]

(5)

Define the two following signals:

\[
\begin{align*}
\Gamma_i(t) &= h_i x(t) \\
\Psi_i(t) &= h_i y(t)
\end{align*}
\]

where \( y_i(t) = C_i x(t) \).

Therefore, from (5.a) and (5.b), one obtains:

\[
\begin{align*}
x(t) &= \sum_{i=1}^{r} \Gamma_i(t) \\
y(t) &= \sum_{i=1}^{r} \Psi_i(t)
\end{align*}
\]

(7.a)

(7.b)

**Remark 2:** It should be noted that according to (7.a) and (7.b), the estimated states \( \hat{x}(t) \) and the estimated output \( \hat{y}(t) \) are the sum of \( \Gamma_i(t) \) and \( \Psi_i(t) \), respectively.

**Remark 3:** The existence of the disturbance \( w(t) \) in the practical systems in real applications is unavoidable which leads to degrade the performance of the control systems and observers. Accordingly, designing a proper observer such that minimizes the effect of the disturbance is important. One of the most popular and efficient tools to minimize the effect of the undesired signal is to use \( H_{\infty} \) optimization. The \( H_{\infty} \) performance index, which is considered in this paper, is expressed as follows:

\[
\frac{\|\varphi(t)\|_2}{\|w(t)\|_2} \leq \gamma
\]

where \( \varphi(t) = \hat{y}(t) - y(t) \) and \( \gamma \) \( \text{(H}_{\infty}\text{ index)} \) must be minimized.

## III. MAIN RESULTS

In this section, we firstly propose an \( H_{\infty} \) UIO for an uncertain TSFM-based nonlinear system. Secondly, we incorporate reset method with the designed UIO to improve the performance of the state estimation error process. We discuss
the existence of UIO/R-UIO which is led to the LMI-based optimization problem.

**A. UIO DESIGN WITH MEASURABLE PREMISE VARIABLES**

Consider the following measured-states-based UIO:

\[
\begin{align*}
\hat{z}_i(t) &= N_i x_i(t) + h_i G_i u(t) + L_i \Psi_i(t) \\
\hat{f}_i(t) &= z_i(t) - E_i \Psi_i(t), i = 1, 2, ..., r.
\end{align*}
\]

(8)

where \(z_i(t)\) is the state variable of the observer, \(\Psi_i(t)\) is as defined in (6.b), \(\hat{f}_i(t)\) is the estimated of \(f_i(t)\), the matrices \(N_i, G_i, L_i, E_i\) are the observer matrices with appropriate dimensions and will be designed later.

In the following theorem, the sufficient conditions for the existence of the UIO with \(H_{\infty}\) performance assuming the measurable premise variables are given.

Define the state estimation error as follows:

\[
e(t) = \hat{z}(t) - x(t) = \sum_{i=1}^{r} \hat{f}_i(t) - \Gamma_i(t).
\]

(9)

where \(e_i(t) \triangleq \hat{f}_i(t) - \Gamma_i(t)\).

**Theorem 1.** Consider the system (5), the observer (8), \(F_i = E_i C_i + I, \ M = \sum_{i=1}^{r} h_i F_i\) and \(\varphi(t) = \hat{y}(t) - y(t)\). The matrices \(N_i, G_i, L_i, E_i\) that stabilize the state estimation error (9) and minimize \(\gamma\) in the \(H_{\infty}\) performance of Remark 3 can be obtained if there exists \(X = X^T > 0\) such that the following conditions hold:

\[
\begin{align*}
N_i F_i + L_i C_i - MB A_i &= 0 \quad \text{(10.a)} \\
G_i - MA B_i &= 0 \quad \text{(10.b)} \\
MB M &= 0 \quad \text{(10.c)} \\
\left[N_i^T X N_i + C_i^T C_i - X MB A_i \right]_{*} &\leq 0 \quad \text{(10.d)}
\end{align*}
\]

**Proof.** Substituting (8) in (9) and defining \(F_i \triangleq E_i C_i + I\), yields to:

\[
e(t) = \sum_{i=1}^{r} \left[z_i(t) - E_i \Gamma_i(t) \right] = \sum_{i=1}^{r} z_i(t) - \mathbf{B} \mathbf{x}(t)
\]

(11)

Time-derivative of \(e(t)\) and combining (5) and (8), results in:

\[
\dot{e}(t) = \sum_{i=1}^{r} \left[N_i e_i(t) + N_i F_i \Gamma_i(t) + h_i G_i u(t) + L_i C_i \Gamma_i(t) \right] - \mathbf{B} \dot{\mathbf{x}}(t)
\]

\[
= \sum_{i=1}^{r} \left[N_i e_i(t) + N_i F_i \Gamma_i(t) + h_i G_i u(t) + L_i C_i \Gamma_i(t) \right] - \mathbf{B} \dot{\mathbf{x}}(t)
\]

(12)

Therefore, if the conditions (10.a)-(10.c) hold, (12) becomes as follows:

\[
\dot{e}(t) = \sum_{i=1}^{r} N_i e_i(t) + N_i F_i \Gamma_i(t) + h_i G_i u(t) + L_i C_i \Gamma_i(t) - \mathbf{B} \dot{\mathbf{x}}(t) - h_i MB A_i \mathbf{x}(t) - h_i MB B_i w(t)
\]

\[
= \sum_{i=1}^{r} h_i (N_i e_i(t) - \mathbf{B} A_i \mathbf{x}(t) - \mathbf{B} B_i w(t))
\]

(13)

Meanwhile, the error between \(y(t) = \sum_{i=1}^{r} \Psi_i(t)\) and \(\hat{y}(t) = \sum_{i=1}^{r} \hat{\Psi}_i(t)\) is obtained as:

\[
\varphi(t) = \hat{y}(t) - y(t) = \sum_{i=1}^{r} e_i(t)
\]

(14)

The state estimation error (9) is stabilized and satisfies the \(H_{\infty}\) performance assuming the \(V(t) = e^T(t) X e(t)\), where \(X\) is a positive-definite symmetric matrix such that:

\[
V(t) \leq -\sum_{i=1}^{r} h_i h_j e^T(t) C_i^T C_j e(t) + \gamma^2 w^T(t)w(t).
\]

(15)

From (13) and (15), the following results are obtained.

\[
\begin{align*}
e^T(t) X e(t) + e^T(t) \dot{e}(t) + \sum_{i=1}^{r} h_i h_j e^T(t) C_i^T C_j e(t) \\
- \gamma^2 w^T(t)w(t)
\end{align*}
\]

\[
\leq \sum_{i=1}^{r} h_i h_j e^T(t) C_i^T C_j e(t) + \sum_{j=1}^{r} h_i e^T(t) C_j e(t)
\]

\[
- \gamma^2 w^T(t)w(t)
\]

\[
\leq \sum_{i=1}^{r} h_i e^T(t) w^T(t) U_i \left[ \begin{array}{c} e(t) \\ w(t) \end{array} \right] \leq 0
\]

(16)

where \(U_i = \left[ N_i^T X N_i + C_i^T C_i - X MB A_i \right]_{*} \quad \gamma^2 I \).

So, the inequality (10.d) is sufficient to hold (16) and this completes the proof.

**Remark 4:** The sufficient condition (10.d) is not a convex optimization problem. To transform it into LMI, a systematic approach is proposed in the following sub-section.

**B. PARAMETRIZATION OF THE UIO WITH MEASURABLE PREMISE VARIABLES**

In this sub-section, an efficient procedure is generated to get the matrices gains of (8). To solve the bilinear matrix inequality (BMI) (10.d), the following Lemma is needed.
Lemma 1 [15]: For any matrices $R \in \mathbb{R}^{m \times n}$ and $Q \in \mathbb{R}^{m \times k}$ where $RR^*Q = Q$, the solution of $LR = Q$ is $L = QR^* + Y(I - RR^*)$, where $Y \in \mathbb{R}^{n \times k}$ is an arbitrary matrix. The matrices $N_i$ and $L_i$ can be generated as the following to satisfy the condition (10.a):

$$N_i = \Delta A_i - K_i C_i$$

(17)

$$L_i = K_i(I + C_i E_i) - \Delta A_i E_i$$

(18)

Besides, based on Lemma 1, the general solution of $E_i$ from the condition (10.c) is obtained as:

$$E_i = -M(C_i M)^* + Y(I - (C_i M)(C_i M)^*)$$

(19)

Defining $U_i \triangleq -M(C_i M)^*$ and $V_i \triangleq (I - (C_i M)(C_i M)^*)$, (18) is rewritten as $E_i = U_i + YV_i$. Therefore, defining $\tilde{V} \triangleq \sum_{i=1}^r h_i U_i C_i$ and $\tilde{V} \triangleq \sum_{i=1}^r \hat{h}_i V_i C_i$, (17) can be reformulated as follows:

$$N_i = A_i + \tilde{V} A_i + Y \tilde{V} A_i - K_i C_i$$

(20)

Accordingly, by replacing (20) in (10.d) and rewriting $-\Delta B D_i$, the BMI condition (10.d) is reformed and the following modified condition is obtained:

$$\begin{bmatrix} \Pi_{1i} & \Pi_{2i} \\ -\gamma I \end{bmatrix} \leq 0$$

(21)

where $\Pi_{1i} \triangleq (XA_i + X \tilde{V} A_i + \tilde{V} \tilde{V} A_i - \tilde{K}_i C_i) + (XA_i + X \tilde{V} A_i + \tilde{V} \tilde{V} A_i - \tilde{K}_i C_i)^*$, $\tilde{V} = \sum_{i=1}^r \tilde{K}_i$, and $\Pi_{2i} = (-X + X \tilde{U} - \tilde{\bar{V}} D_i)$. Thus, the problem of obtaining $\tilde{V}$ and $\tilde{K}_i$ is transformed to find $N_i, L_i, F_i, G_i$ and $E_i$.

C. UIO DESIGN WITH UNMEASURABLE PREMISE VARIABLES

In practice, the states of the system are not always measurable. In this case, if the premise variables of the observer (8) depend on the unmeasurable states, it is impossible to follow the procedure in sub-section A. Therefore, the results in subsection A are modified in this sub-section to design a more practical observer with the premise variables based on the estimated states.

Consider the following UIO whose premise variables depend on the estimated states:

$$\dot{z}_i(t) = N_i z_i(t) + h_i G_i u(t) + L_i \Psi_i(t)$$

$$\dot{\Psi}_i(t) = z_i(t) - E_i \Psi_i(t), i = 1, 2, ..., r.$$  

(22)

where $z_i(t)$ is state variable of the observer, $\Psi_i(t)$ as defined in (6.b), $\dot{\Psi}_i(t)$ is the estimated of $\dot{\Psi}_i(t)$, the matrices $N_i$, $G_i$, $L_i$, $E_i$ are the observer matrices with appropriate dimensions and will be designed later. The lemma 2 is necessary to obtain the essential results in Theorems 2 and 3.

Lemma 2 [15]: For any positive constant $\alpha$ and real matrices $R$ and $Q$ with appropriate dimensions, the following inequality holds:

$$R^T Q + Q^T R \leq \alpha R^T R + \alpha^{-1} Q^T Q$$

Theorem 2: Consider the system (5), the observer (22), $F_i = E_i C_i + I$, $\Delta B = \sum_{i=1}^r h_i F_i$ and $\varphi(t) = \tilde{y}(t) - y(t)$. The matrices $N_i, G_i, L_i, E_i$ that stabilize the state estimation error (9) and minimize $\gamma$ in the $H_\infty$ performance of Remark 3 can be obtained if there exists $X = X^T > 0$ such that the following conditions hold:

$$N_i F_i + L_i G_i - \Delta A_i = 0$$

(23.a)

$$G_i - \Delta B F_i = 0$$

(23.b)

$$M = 0$$

(23.c)

$$\begin{bmatrix} -\gamma I & 0 & \Xi \\ \Xi & -\gamma I & 0 \\ 0 & 0 & -X^T \end{bmatrix} < 0$$

(23.d)

where $\Xi_{i1} \triangleq N_i^T X + X N_i + C_i^T C_i + X + \lambda \rho^2 I$ and $\Xi_{i2} \triangleq -X \Xi_{i1}$.

Proof: Define the state estimation error same as in (9), substituting (22) in (9) and according to definition $F_i \triangleq E_i C_i + I$, time-derivative of $e(t)$, yields to:

$$\dot{e}(t) = \sum_{i=1}^r \dot{z}_i(t) - \dot{\Psi}_i(t) = \sum_{i=1}^r \left( N_i e_i(t) + (N_i F_i + L_i G_i - \Delta A_i) e_i(t) \right) + h_i (G_i - \Delta B F_i) u(t) - h_i (\Delta B D_i) w(t)$$

(24)

Therefore, if the conditions (23.a)-(23.c) hold, (24) leads to:

$$\dot{e}(t) = \sum_{i=1}^r \left( h_i (N_i e_i(t) - \Delta B D_i w(t)) + \Xi e_i(t) \right)$$

(25)

Let us define $\xi(t) \triangleq \sum_{i=1}^r (h_i - h_i) B_i u(t)$, so, (25) is reformed as the following:

$$\dot{e}(t) = \sum_{i=1}^r h_i \left( N_i e_i(t) - \Delta B D_i w(t) \right) + \Xi \xi(t).$$

(26)

In definition of $\xi(t)$, assume that there is a positive scalar $\rho$ such that $\|\xi(t)\| \leq \rho \|e(t)\|$. Indeed, $\|\xi(t)\|$ must be Lipschitz in $e(t)$ [5]. Moreover, the error between $y(t) = \sum_{i=1}^r \Psi_i(t)$ and $\tilde{y}(t) = \sum_{i=1}^r \dot{\Psi}_i(t)$ is obtained as in (14).

The dynamic (26) is stabilized and satisfies the $H_\infty$ performance in Remark 3, if there exist a Lyapunov function $V(t) = e(t)^T X e(t)$, where $X$ is a positive-definite symmetric matrix such that:

$$\dot{V}(t) \leq -\sum_{i=1}^r h_i h_i e^T(t) C_i^T C_i e(t) + \gamma^2 w^T(t) w(t).$$

(27)

From (27), (14) and following the procedure in (16), the following results are obtained:

$$e^T(t) X e(t) + e^T(t) \dot{e}(t) + \sum_{i=1}^r h_i h_i e^T(t) C_i^T C_i e(t) - \gamma^2 w^T(t) w(t) \leq 0$$

(28)
Further simplification based on Lemma 2, yields to:
\[
\begin{align*}
\sum_{i=1}^{r} & h_i \left( e^T(t)(N_i^TX + XN_i + C_i^TC_i + X)e(t) ight) \\
& - w(t)D_i^T\Xi_i X \psi(t) - \gamma^2 w(t)(w(t)) \\
& - e^T(t)\Xi_i \Xi_i D_i w(t) + \xi^T(t)\Xi_i^T Y(\xi(t))
\end{align*}
\]

Thus, from (29), one has:
\[
\begin{bmatrix}
\Xi_i X \psi - \Lambda_i & 0 & 0 \\
0 & \Xi_i & 0 \\
0 & 0 & -Y^2 I
\end{bmatrix}
\leq 0
\]  
(30)

where \( \Xi_i = N_i^TX + XN_i + C_i^TC_i + X + \lambda \rho^2 I \).

Applying Schur-complement [38], (30) is equivalent to the following conditions:
\[
\begin{bmatrix}
-\Lambda_i & 0 & 0 \\
0 & \Xi_i & 0 \\
0 & 0 & -Y^2 I
\end{bmatrix}
\leq 0
\]  
(31)

where \( \Xi_i \) and \( \Xi_i \) were defined in Theorem 2. Pre- and post-multiplying (31) by matrix diag \((I, I, I, X)\), the inequality (23.d) is obtained and the proof is completed.

Remark 5: Note that the condition (23.d) is not an LMI. To change the BMI condition (23.d) to LMI one, the following procedure is presented:

### D. PARAMETRIZATION OF THE UIO WITH UNMEASURABLE PREMISE VARIABLES

Referring to sub-section III.B, to satisfy the conditions (23.a)-(23.c), the same as the conditions (10.a)-(10.c), the matrices \( L_i, E_i, N_i \) should be obtained from (18)-(20). By substituting (18)-(20) in (23.d) and according to the definition \( F_i = E_i C_i + I \), (23.d) is reformed as the following:

\[
\begin{bmatrix}
-\Lambda_1 & 0 & 0 \\
0 & \Lambda_2 & 0 \\
0 & 0 & -Y^2 I
\end{bmatrix}
\leq 0
\]  
(32)

where \( \Lambda_1 \triangleq \overline{U}^T X + \overline{V}^T \overline{Y} + X \), \( \Lambda_2 \triangleq (X^2 + XU^2 + YV^2 - \overline{K_1}C) + (X^2 + XU^2 + YV^2 - \overline{K_1}C) + X + \lambda \rho^2 I \), \( \Lambda_3 \triangleq -X^T X^2 + YV^2 \), \( \overline{Y} \triangleq XY \) and \( \overline{K_1} \triangleq \overline{K_1} \). Therefore, generating \( \overline{Y} \) and \( \overline{K_1} \) is transformed to obtain \( N_i, L_i, E_i \) and \( E_i \).

In the following sub-section, reset strategy is incorporated with the proposed C-UIO to develop the R-UIO. R-UIO improves the transient performance of the observer. The formulations are proposed and derived for more general unmeasurable premise variables case. The results can be derived for the measurable premise variables case by substituting \( h_i \) by \( \hat{h}_i \).

### E. RESET UNKNOWN INPUT OBSERVER (R-UIO)

#### DESIGN WITH UNMEASURABLE PREMISE VARIABLES

In this sub-section, we deal with the situation that the premise variables are based on the unmeasurable states. In this case, the R-UIO is presented as follows:

If \( e(t) \in \Theta_1 \), Then,
\[
\begin{align*}
\dot{z}_i(t) &= N_i \hat{h}_i \hat{z}_i(t) + h_i \hat{h}_i \hat{L}_i(t) + L_i \Psi_i(t) \\
\dot{\Psi}_i(t) &= \psi_i(t) - E_i \hat{\Psi}_i(t), \quad i = 1, 2, ..., r
\end{align*}
\]  
(33.a)

If \( e(t) \in \Theta_2 \), Then,
\[
\begin{align*}
\dot{z}_i^*(t) &= (F_i - A_{Ri} E_i C_i) \hat{z}_i(t) - (I - A_{Ri}) F_i E_i \hat{\Psi}_i(t) \\
\dot{\hat{\Psi}}_i(t) &= \hat{\psi}_i(t) - E_i \hat{\Psi}_i(t), \quad i = 1, 2, ..., r
\end{align*}
\]  
(33.b)

where \( A_{Ri} \) is the after reset matrix, \( \Theta_1 = \{ e(t) \in R^n | e(t)Te(t) \geq 0 \} \) is the flow set and \( \Theta_2 = \{ e(t) \in R^n | e(t)Te(t) \leq 0 \} \) is the jump set and as \( e(t) \in \Theta_2 \), jump will occur. The matrices \( A_{Ri} \) and \( T \) will be obtained in the following.

The discrete-error is provided as follows based on (33.b):

\[
e^*(t) = \sum_{i=1}^{r} i_i^*(t) - \sum_{i=1}^{r} n_i(t) = \sum_{i=1}^{r} f_i^*(t) - n_i(t) = \sum_{i=1}^{r} e_i^*(t)
\]  
(34)

Substituting (7.b) and (33.b) into (34) and considering (8), (9) and defining \( F_i = E_i C_i + I \), one has:

\[
e^*(t) = \sum_{i=1}^{r} \left( F_i - A_{Ri} E_i C_i \right) z_i(t) - (I - A_{Ri}) F_i E_i \hat{\Psi}_i(t) - (I + E_i C_i) \hat{\Psi}_i(t)
\]  
(35)

Considering \( H_i \triangleq (F_i + A_{Ri} - A_{Ri} F_i) \), the state estimation error dynamics is achieved as the following:
\[ \dot{e}(t) = \sum_{i=1}^{r} h_i \left( N_i e(t) - \mathcal{B} \mathcal{D} \mathcal{W}(t) \right) + \mathcal{W}(t): \text{if } e(t) \in \Theta_1 \] (36.a)

\[ e^+(t) = \sum_{i=1}^{r} h_i H_i e(t) : \text{if } e(t) \in \Theta_2 \] (36.b)

The following theorem is concerned with the stability analysis of the state estimation error dynamic (36).

**Theorem 3:** For the system (5), the R-UIO (33) that stabilized the state estimation error (36) and minimized \( r \) in the \( H_{\infty} \) performance of Remark 3 can be generated if there exists symmetric matrices \( X = X^T > 0 \), \( T \), \( Q_i \) and positive constants \( \sigma_1, \sigma_2, \sigma_3, \sigma_4 \) and \( 0 < \sigma_5 \leq 1 \) such that:

\[
\begin{bmatrix}
\mathcal{W}^T X \mathcal{W} - \lambda I \\
\mathcal{W}^T X \mathcal{W} - \lambda I \\
\end{bmatrix}
\begin{bmatrix}
\chi_i + \sigma_2 T & -X \mathcal{D} \mathcal{D} \mathcal{W} \\
\chi_i + \sigma_2 T & -X \mathcal{D} \mathcal{D} \mathcal{W} \\
\end{bmatrix}
< 0 \\
(37.a)
\]

\[
\begin{bmatrix}
\mathcal{C}_i^T \mathcal{C}_i - \sigma_3 X - \sigma_4 T & (X F_i + Q_i - Q_i F_i)^T \\
\mathcal{C}_i^T \mathcal{C}_i - \sigma_3 X - \sigma_4 T & (X F_i + Q_i - Q_i F_i)^T \\
\end{bmatrix}
\begin{bmatrix}
\mathcal{C}_i e(t) \\
\mathcal{C}_i e(t) \\
\end{bmatrix}
< 0 \\
(37.b)
\]

\[
H_i^T TH_i + \sigma_5 T > 0 \\
(37.c)
\]

**Proof:** Consider the following Lyapunov function:

\[ V = e^T(t) X e(t) \] (38)

The state estimation error dynamics (35) is stabilized and satisfies the \( H_{\infty} \) performance in Remark 3, if the following conditions

\[
\begin{cases}
V(e) < \sum_{i=1}^{r} h_i h_i e^T(t) C_i^T C_i e(t) : e^T(t) T e(t) \geq 0 \\
V(e^+) \leq \sum_{i=1}^{r} h_i h_i e^T(t) C_i^T C_i e(t) : e^T(t) T e(t) \leq 0 \\
\end{cases} \\
\] (39)

Substituting (36.a) in (39) and follow the procedure in (16) for the continuous error dynamic along with some simplifications, yields to:

\[
\dot{V}(e(t)) + \sum_{i=1}^{r} h_i h_i e^T(t) C_i^T C_i e(t) = - \sigma_3 V(e(t)) \] (40)

where \( \sigma_3 = \chi_i + \sigma_2 T \).

Consequently, from (41), (37.a) is obtained.

Besides, for the discrete error dynamic (36.b), combining (36.b) and (39) and follow the procedure in (16), results:

\[
\begin{multlined}
V(e^+(t)) + \sum_{i=1}^{r} h_i h_i e^T(t) C_i^T C_i e(t) - \gamma^2 w^T(t) w(t) \\
= \sum_{i=1}^{r} h_i \left( e^T(t) H_i^T X H_i e(t) + e^T(t) C_i^T C_i e(t) \right) \\
- \sigma_3 e^T(t) X e(t) - \sigma_4 e^T(t) T e(t) = \\
\sum_{i=1}^{r} h_i [e^T(t) T e(t) - \gamma^2 w^T(t) w(t)] \\
\] (42)

Again with the aid of S-procedure [38] and taking \( e^T(t) T e(t) < 0 \) in (42), the following results can be obtained:

\[
\sum_{i=1}^{r} h_i [e^T(t) T e(t) - \gamma^2 w^T(t) w(t)] < 0 \\
\] (43)

Applying Schur complement [38] for (43), results in:

\[
\begin{bmatrix}
\mathcal{C}_i^T \mathcal{C}_i - \sigma_3 X - \sigma_4 T & 0 & H_i^T X \\
\mathcal{C}_i^T \mathcal{C}_i - \sigma_3 X - \sigma_4 T & 0 & X \\
\end{bmatrix}
< 0 \\
(44)
\]

Multiplying (44) by \( \text{diag}(I, I, X) \) from left and right, the following inequality is achieved:

\[
\begin{bmatrix}
H_i^T X \\
H_i^T X \\
\end{bmatrix}
< 0 \\
\] (45)

Replacing \( H_i \) in (45) and using the variable change \( Q_i = X A R_i \), the inequality (45) can be rewritten as follows:

\[
\begin{bmatrix}
\mathcal{C}_i^T \mathcal{C}_i - \sigma_3 X - \sigma_4 T & 0 & (X F_i + Q_i - Q_i F_i)^T \\
\mathcal{C}_i^T \mathcal{C}_i - \sigma_3 X - \sigma_4 T & 0 & X \\
\end{bmatrix}
< 0 \\
\] (46)

In addition, it is needed after the jump, the error trajectories drop out of the jump set, in other words:

\[
(e^+(t))^T T e^+(t) > 0 \text{ if } e^T(t) T e(t) \leq 0 \\
\] (47)

Using S-procedure [38] in (47), results in:

\[
H_i^T T H_i + \sigma_5 T > 0 \\
\] (48)

Consequently, the state estimation error is stabilized and satisfy \( H_{\infty} \) performance in Remark 3, if the conditions (37.a)-(37.c) are satisfied and this completes the proof.

**Remark 6:** It should be noted that if the output of the system be nonlinear, the reset unknown input observer design is led to a non-convex problem that cannot be solved easily [15]. In our paper, a new mathematical definition in (6.a) and (6.b) was proposed to overcome the corresponding non-convex problem. Furthermore, in the presence of uncertainties it is hard to design an observer such that the
state estimation error converges to zero. In our paper, a transformation is adopted to transform the uncertainties to unknown inputs with no bounded conditions. Besides, a more reasonable reset unknown input observer based on the unmeasurable premise variables.

The pseudo code of the proposed R-UIO is illustrated in algorithm 1.

Algorithm 1: The Pseudo-Code of the Proposed R-UIO

1: Initialize $x(0)$, $\sigma_1 \geq 0, \sigma_2 \geq 0, \sigma_3 \geq 0, \sigma_4 \geq 0, 0 < \sigma_5 \leq 1$.
2: Select the matrix $M$ according to $\Delta A_i(t) = M \Delta A_i(t)$ and $\Delta B_i(t) = M \Delta B_i(t)$.
3: Compute $U_i = M(C_i M) + V_i = I - (M(C_i M)) +$.
4: Obtain $X, Y$ and $K_i$ by solving LMI s (37.a)-(37.c).
5: Compute $E_i = U_i + Y V_i$, $P_i = E_i C_i + I$, $N_i = X A_i - K_i C_i$, $G_i = X B_i$, $L_i = K_i + E_i E_i) - X B_i A_i$.

For $t = 1$ to final time
6: Obtain the signals $I_i(t)$ and $\Psi_i(t)$.
7: Check the reset law:
If $e(t) \leq 0$
Receive the estimated states $\hat{x}(t)$ from (33.a)
else
Receive the estimated states $\hat{x}(t)$ from (33.b)
End If
8: Compute the control law $u(t)$ based on $\tilde{x}(t)$.
9: Implement $u(t)$ to the system (1).
10: Update $I_i(t)$ and $\Psi_i(t)$.

End For

Remark 7: LMI (37.a) and (37.b) are the generalized eigenvalue minimization problem (GEVP) [3] which are used to obtain the minimum $H_{\infty}$ index $\gamma$. The GEVP are solved by the existing solvers such as Sedumi [39] and Mosek [40] compatible with Matlab software.

V. SIMULATION RESULTS

To verify the effectiveness of the proposed observer, a two-degree freedom helicopter system is simulated by using MATLAB software. We revisit the following model of a two degree freedom helicopter [41]:

$$\dot{x}(t) = \sum_{i=1}^{r} \left( (A_i + \Delta A_i) x(t) + (B_i + \Delta B_i) u(t) \right) + D_i w(t) \tag{49}$$

where

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.0798 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2.1714 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 5.1404 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 5.4388 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0.2554 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and the membership functions are given by [41]:

$$h_1 = -\frac{1}{36} (x_1(t) - 4) + 1,$$

$$h_2 = \begin{cases} -\frac{1}{36} (x_1(t) - 4) & \text{if } x_1(t) \leq 40 \\ -\frac{1}{36} (x_1(t) - 40) + 1 & \text{if } x_1(t) > 40 \end{cases},$$

$$h_3 = \frac{1}{36} (x_1(t) - 40).$$

Assume the system (49) is affected by the uncertainties as follows:

$$\Delta A_1 = \begin{bmatrix} -0.1 \zeta_1(t) & 0.02 \zeta_2(t) & -0.04 \zeta_1(t) & 0.04 \zeta_2(t) \\ -0.8 \zeta_1(t) & 0.08 \zeta_1(t) & 0.16 \zeta_1(t) & 0.16 \zeta_2(t) \\ -0.4 \zeta_1(t) & 0.04 \zeta_1(t) & -0.08 \zeta_2(t) & 0.08 \zeta_2(t) \\ -0.2 \zeta_1(t) & -0.1 \zeta_1(t) & -0.4 \zeta_1(t) & -0.2 \zeta_2(t) \end{bmatrix},$$

$$\Delta A_2 = \begin{bmatrix} 0.2 \zeta_2(t) & 0.04 \zeta_2(t) & 0.01 \zeta_2(t) & 0.04 \zeta_2(t) \\ -0.8 \zeta_2(t) & -0.16 \zeta_2(t) & -0.04 \zeta_2(t) & 0.16 \zeta_2(t) \\ 0.4 \zeta_2(t) & 0.4 \zeta_2(t) & 0.02 \zeta_2(t) & 0.02 \zeta_2(t) \\ 0.2 \zeta_2(t) & 0.1 \zeta_2(t) & 0.4 \zeta_2(t) & 0.2 \zeta_2(t) \end{bmatrix}$$

$$\Delta B_1 = \begin{bmatrix} -0.1 \zeta_1(t) \\ -0.8 \zeta_1(t) \\ 0.4 \zeta_1(t) \\ 0.1 \zeta_1(t) \end{bmatrix}, \quad \Delta B_2 = \begin{bmatrix} 0.4 \zeta_2(t) \\ 0.2 \zeta_2(t) \\ 0.05 \zeta_2(t) \end{bmatrix},$$

where $\zeta_1(t)$ and $\zeta_2(t)$ are random noises of a normal distribution on $[-1,1]$. The system (3) is transformed to (5) with $M = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0.1 \\ 0 & 0 & 0 & 0 \\ 0.2 & 0 & 0 & 0 \end{bmatrix}$. In this case, we select $C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, $C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, $C_3 = \begin{bmatrix} 0.8 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \end{bmatrix}$, $D_1 = \begin{bmatrix} -0.8 \\ -0.2 \\ -0.1 \\ 0.2 \end{bmatrix}$, $D_3 = D_2 = D_1$, $w(t) = \sin(t/5)$.

The performance evaluation of the proposed controller is presented in two scenarios. In the first scenario, the performance analysis of the proposed C-UIO and R-UIO is evaluated with the measurable premise variables. In this case, based on Theorems 1 and 3 and taking $\sigma_1 = 1.2, \sigma_2 = 0.01, \sigma_3 = 0.2, \sigma_4 = 0.6, \sigma_5 = 0.5$, the parameters of the observer with measurable premise variables are obtained as follows:

$$N_1 = \begin{bmatrix} -0.5 & 0 & 0.25 & 0.25 \\ 0 & -0.5 & 0 & 0 \\ -0.25 & 0 & -0.5 & 0 \\ -0.25 & 0 & -0.5 & 0 \end{bmatrix},$$

$$N_2 = \begin{bmatrix} -0.5 & 0.25 & 0.25 \\ -0.25 & 0.25 & 0 \end{bmatrix},$$

$$N_3 = \begin{bmatrix} 0.2403 & -0.9736 & -0.6005 & -1.3281 \\ 0.8261 & -3.3868 & -2.0987 & -4.6203 \\ -0.2403 & 0.9736 & 0.6005 & 1.3281 \\ 0.5032 & 2.0744 & -1.2883 & -2.8302 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} 0.3424 & 0.0305 \\ 0.0913 & 0.0153 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0.0096 & 0.0596 \end{bmatrix}, \quad G_3 = \begin{bmatrix} 0.0685 & 0.0764 \end{bmatrix}.$$
\[ G_3 = \begin{bmatrix} -0.0009 & -0.5819 \\ -0.0009 & -0.5819 \\ 0.0009 & 0.5819 \\ 0.001 & 0.0582 \end{bmatrix}. \]

Moreover, the \( H_\infty \) index is obtained as \( \gamma = 0.001 \).

The states estimation by the proposed UIO and R-UIO based on the measurable premise variables are illustrated in Figure 1. Compared our approach to [36], the proposed observers can handle the uncertainties, external disturbance and nonlinear output equation, simultaneously.

Obviously, one can find that in comparison with the suggested UIO in [36], our proposed UIO works well for states estimation and the speed of the convergence is faster and better than [36]. Moreover, by the proposed R-UIO, the performance of the estimated states is improved significantly in both convergence speed and overshoot.

**FIGURE 1.** Real states (−), estimated states with measurable premise variables of the proposed UIO (−−), with UIO in [36] (−∙−) and with the proposed R-UIO (⋯).

Figure 2 exhibits the control input signals applying the proposed R-UIO-based controller.

The \( L_2 \) norm of the state estimation error \( e(t) \) and the convergence time of each state are illustrated in Table 1. Obviously, by the R-UIO, the performance of the observer is improved in both convergence time and \( L_2 \) norm of the state estimation error in comparison with the suggested C-UIO in [36] and the proposed C-UIO. Furthermore, the results show the superiority of the proposed \( H_\infty \) C-UIO compared with the C-UIO in [36].

**TABLE 1.** \( L_2 \) norm and convergence time comparison

| Method                  | C-UIO in [36] | Proposed C-UIO | Proposed R-UIO |
|-------------------------|---------------|----------------|----------------|
| Converging time of \( x_1(t) \) | 21.8          | 9.6            | 5.2            |
| Converging time of \( x_2(t) \) | 22.2          | 10.8           | 8.6            |
| Converging time of \( x_3(t) \) | 14.2          | 8              | 4.4            |
| Converging time of \( x_4(t) \) | 33.3          | 16.6           | 3.4            |
| \( \sqrt{\int_0^\infty e^T(t)e(t)dt} \) | 3.170         | 2.051          | 0.1284         |

In the second scenario, for the unmeasurable premise variables-based UIO/R-UIO, based on Theorems 2 and 3 and taking \( \sigma_1 = 1.5, \sigma_2 = 0.05, \sigma_3 = 0.1, \sigma_4 = 0.6, \sigma_5 = 0.6 \), the parameters of the observer with measurable premise variables are obtained as follows:

\[
N_1 = \begin{bmatrix} -0.7063 & 0.3815 & 1.1742 & 1.7351 \\ 0.1765 & -0.0955 & -0.2935 & -0.4338 \\ 0.2648 & -0.1430 & -0.4404 & -0.6507 \end{bmatrix}, \quad N_2 = \begin{bmatrix} -0.1836 & 1.5408 & 1.0252 & 1.5410 \\ -1.557 & 1.0252 & 1.5410 \\ -0.1375 & 1.556 & 1.1557 \end{bmatrix}, \quad N_3 = \begin{bmatrix} -0.6878 & -0.5778 & -0.3845 & -0.5779 \\ 0.0687 & -0.9212 & -0.6881 & -1.5303 \\ 1.1556 & -0.7689 & -1.1558 \end{bmatrix}, \quad N_4 = \begin{bmatrix} -0.1640 & 1.0149 & 0.6970 & 1.5387 \\ 1.1782 & -0.6357 & -1.3682 \end{bmatrix}
\]

\[
G_1 = \begin{bmatrix} 0.0675 & 0.0199 \\ 0.3424 & 0.0305 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0 & 0.1390 \\ -0.0913 & -0.0153 \end{bmatrix}, \quad G_3 = \begin{bmatrix} 0 & 0.0596 \\ 0.0685 & -0.764 \end{bmatrix}
\]

Besides, the \( H_\infty \) index is obtained as \( \gamma = 0.005 \).

The estimation of the (48) system states are depicted in Figure 3 for the estimated premise variables case. As the simulation results show, the estimation process under the proposed both UIO and R-UIO is satisfactory. For thoroughly checking the property of the proposed UIO, the
results are compared with a newly published work in [18].

Figure 4 illustrates the control input signals under the proposed R-UIO-based controller with unmeasurable premise variables.

For the sake of evaluating the effectiveness of the devised observer, the $L_2$ norm and convergence time are calculated. As the results show in Table 2, the proposed R-UIO outperformed the C-UIO in [18] and our designed C-UIO in both $L_2$ norm and convergence time. Moreover, our suggested $H_{\infty}$ C-UIO demonstrated the improvement performance compared with the one in [18].

**TABLE 2. $L_2$ norm and convergence time comparison**

| Method      | C-UIO in [18] | Proposed C-UIO | Proposed R-UIO |
|-------------|---------------|----------------|----------------|
| Converging time of $x_1(t)$ | 19.6          | 12.8           | 7.8            |
| Converging time of $x_2(t)$ | 24.6          | 14.3           | 9.9            |
| Converging time of $x_3(t)$ | 20.1          | 10.8           | 7.2            |
| Converging time of $x_4(t)$ | 33.6          | 16.8           | 4.1            |
| $\sqrt{\int_0^T e^T(t) e(t) dt}$ | 3.2319       | 2.187          | 0.1753         |

**V. CONCLUSION**

This article was concerned with designing $H_{\infty}$ UIO/R-UIO for uncertain TSFM subject to external disturbance. To overcome the non-convex problem of nonlinear output equation, a new representation was presented. Based on Lyapunov theory, the stability of the state estimation error was ensured and new Off-line LMIs were generated. The performance of the proposed $H_{\infty}$ UIO was improved in both convergence time and $L_2$ norm of the state estimation error by incorporating the reset mechanism. The case of unmeasurable states-based premise variables, i.e. asynchronous premise variables for TSFM-based nonlinear system and observer was also considered for the proposed UIO/R-UIO. A practical simulation example was shown the effectiveness of the proposed scheme. How to design the R-UIO for uncertain discrete-time TSFM-based nonlinear systems, study on the data-driven realization of the proposed method for unknown dynamic systems are interesting future research topics.

**REFERENCES**

[1] H. Doubabi, I. Salhi, M. Chennani, and N. Essounbouli, “High Performance MPPT based on TS Fuzzy–integral backstepping control for PV system under rapid varying irradiance—Experimental validation,” ISA Trans., 2021.
[2] S. Xu, H. Wen, and X. Wang, “Observer-based robust fuzzy control of nonlinear networked systems with actuator saturation,” ISA Trans., 2021.
[3] K. Tanaka and H. O. Wang. Fuzzy control systems design and analysis: a linear matrix inequality approach. John Wiley & Sons, 2004.
[4] M. Naghdi and M. A. Sadnia, “A novel fuzzy extended state observer,” ISA Trans., vol. 102, pp. 1–11, 2020.
[5] M. Chadli and H. R. Karimi, “Robust observer design for unknown inputs Takagi–Sugeno models,” IEEE Trans. Fuzzy Syst., vol. 21, no. 1, pp. 158–164, 2012.
[6] B. Ma, P. Li, and Y. Wang, “Observer-based event-triggered type-2 fuzzy control for uncertain steer-by-wire systems,” ISA Trans., 2021.
[7] T.-G. Park and D. Kim, “Design of unknown input observers for linear systems with unmatched unknown inputs,” Trans. Inst. Meas. Control, vol. 36, no. 3, pp. 399–410, 2014.
[8] R. V Gandhi and D. M. Adhyaru, “Hybrid extended state observer based control for systems with matched and mismatched disturbances,” ISA Trans., vol. 106, pp. 61–73, 2020.
[9] Z. Echreshavi, M. Shasadeghi, and M. H. Asemiani, “$H_{\infty}$ dynamic observer-based fuzzy integral sliding mode control with input magnitude and rate constraints,” J. Franklin Inst., vol. 358, no. 1, pp. 575–605, 2021.
[10] W. Qi, J. H. Park, G. Zong, J. Cao, and J. Cheng, “Filter for positive stochastic nonlinear switching systems with phase-type semi-Markov parameters and application,” IEEE Trans. Syst. Man, Cybern. Syst.,
2021.
[11] J. Han, H. Zhang, Y. Wang, and X. Sun, “Robust fault detection for switched fuzzy systems with unknown input,” IEEE Trans. Cybern., vol. 48, no. 11, pp. 3056–3066, 2017.

[12] B. Marx, D. Koenig, and J. Ragot, “Design of observers for Takagi-Sugeno descriptor systems with unknown inputs and application to fault diagnosis,” IET Control Theory Appl., vol. 1, no. 5, pp. 1487–1495, 2007.

[13] M. Chadli, “An LMI approach to design observer for unknown inputs Takagi-Sugeno fuzzy models,” Asian J. Control, vol. 12, no. 4, pp. 524–530, 2010.

[14] S. Mammar and D. Koenig, “Vehicle handling improvement by active steering,” Veh. Syst. Dyn., vol. 38, no. 3, pp. 211–242, 2002.

[15] V.-P. Vu, W.-J. Wang, J. M. Zurada, H.-C. Chen, and C.-H. Chiu, “Unknown input method based observer synthesis for a discrete time uncertain T-S fuzzy system,” IEEE Trans. Fuzzy Syst., vol. 26, no. 2, pp. 761–770, 2017.

[16] H. Li, C. Li, D. Ouyang, S. K. Nguang, and Z. He, “Observer-based dissipativity control for TS fuzzy neural networks with distributed time-varying delays,” IEEE Trans. Cybern., 2020.

[17] L. Wang and H.-K. Lam, “Further study on observer design for continuous-time Takagi–Sugeno fuzzy model with unknown premise variables via average dwell time,” IEEE Trans. Cybern., vol. 50, no. 11, pp. 4855–4860, 2019.

[18] W. Xie, B. Liu, L. Bu, Y. Wang, and J. Zhang, “A decoupling approach for observer-based controller design of TS fuzzy system with unknown premise variables,” IEEE Trans. Fuzzy Syst., 2020.

[19] A. Golabi, M. Beheshti, and H. M. Asemeni, “H∞ robust fuzzy dynamic observer-based controller for uncertain Takagi-Sugeno fuzzy systems,” IET Control Theory Appl., vol. 6, no. 10, pp. 1434–1444, 2012.

[20] M. H. Asemeni and V. J. Majd, “A robust H∞ observer-based controller design for uncertain T–S fuzzy systems with unknown premise variables via LMI,” Fuzzy Sets Syst., vol. 212, pp. 21–40, 2013.

[21] M. Khachou, M. Souissi, and A. Tourmi, “Robust H2 observer-based control design for discrete-time-delay fuzzy systems,” Int. J. Inf. Syst. Sci., vol. 6, no. 1, pp. 35–48, 2010.

[22] V.-P. Vu and W.-J. Wang, “Observer design for a discrete-time TS fuzzy system with uncertainties,” in 2015 IEEE International Conference on Automation Science and Engineering (CASE), 2015, pp. 1262–1267.

[23] S. Pourdehi and P. Karimaghaei, “Reset observer-based fault tolerant control for a class of fuzzy nonlinear time-delay systems,” J. Process Control, vol. 85, pp. 65–75, 2020.

[24] D. Paesa, C. Franco, S. Llorente, G. Lopez-Nicolas, and C. Saguez, “Reset adaptive observer for a class of nonlinear systems,” IEEE Trans. Automat. Contr., vol. 57, no. 2, pp. 506–511, 2011.

[25] A. Aminzadeh and A. Khayatian, “Reset observer design for time-varying dynamics: Application to WIG crafts,” Aerosp. Sci. Technol., vol. 81, pp. 32–40, 2018.

[26] G. Zhao and J. Wang, “Reset observers for linear time-varying delay systems: Delay-dependent approach,” J. Franklin Inst., vol. 351, no. 11, pp. 5133–5147, 2014.

[27] C. P. Guillén-Flores, B. Castillo-Toledo, J. P. García-Sandoval, S. Di Gennaro, and V. G. Álvarez, “A reset observer with discrete/continuous measurements for a class of fuzzy nonlinear systems,” J. Franklin Inst., vol. 350, no. 8, pp. 1974–1991, 2013.

[28] E. Aguilar-Garriga, J. P. García-Sandoval, and D. Dochain, “Monitoring of a biodiesel production process via reset observer,” J. Process Control, vol. 42, pp. 104–113, 2016.

[29] Z. Echreshavi, M. Farboud, M. Shasadeghi, and M. H. Asemeni, “Reset Method Based Unknown Input Observer Design for Continuous-Time TS Fuzzy System,” in 2020 28th Iranian Conference on Electrical Engineering (ICEE), 2020, pp. 1–5.

[30] G. Zhao, C. Hua, and X. Guan, “Reset Observer-Based Zeno-Free Dynamic Event-Triggered Control Approach To Consensus of Multiagent Systems With Disturbances,” IEEE Trans. Cybern., 2020.

[31] J. Zhang, F. Zhu, X. Zhao, and F. Wang, “Robust impulsive reset observers of a class of switched nonlinear systems with unknown inputs,” J. Franklin Inst., vol. 354, no. 7, pp. 2924–2943, 2017.

[32] I. Hosseini, A. Khayatian, P. Karimaghaei, M. Fiacchini, and M. A. D. Navarro, “LMI-based reset unknown input observer for state estimation of linear uncertain systems,” IET Control Theory Appl., vol. 13, no. 12, pp. 1872–1881, 2019.

[33] S. Zhou and G. Feng, “Generalised H2 controller synthesis for uncertain discrete-time fuzzy systems via basis-dependent Lyapunov functions,” IEE Proceedings-Control Theory Appl., vol. 153, no. 1, pp. 74–80, 2006.

[34] Y. Mu, H. Zhang, H. Su, and H. Ren, “Unknown input observer synthesis for discrete-time T–S fuzzy singular systems with application to actuator fault estimation,” Nonlinear Dyn., vol. 100, pp. 3399–3412, 2020.

[35] A.-T. Nguyen, J. Pan, T.-M. Guerra, and Z. Wang, “Avoiding unmeasured premise variables in designing unknown input observers for Takagi-Sugeno fuzzy systems,” IEEE Control Syst. Lett., vol. 5, no. 1, pp. 79–84, 2020.

[36] Z. Ebrahimii, M. H. Asemeni, and A. A. Safavi, “Observer-based controller design for uncertain disturbed Takagi-Sugeno fuzzy systems: A fuzzy wavelet neural network approach,” Int. J. Adapt. Control Signal Process., vol. 35, no. 1, pp. 122–144, 2021.

[37] W. Qi, X. Gao, C. K. Ahn, J. Cao, and J. Cheng, “Fuzzy integral sliding-mode control for nonlinear semi-Markovian switching systems with application,” IEEE Trans. Syst. Man, Cybern. Syst., 2020.

[38] C. Scherer and S. Weiland, “Linear matrix inequalities in control,” Lect. Notes, Dutch Inst. Syst. Control. Delft, Netherlands, vol. 3, no. 2, 2000.

[39] Y. Labit, D. Peaucelle, and D. Henrion, “ScDuMi interface 1.02: a tool for solving LMI problems with ScDuMi,” in Proceedings. IEEE International Symposium on Computer Aided Control System Design, 2002, pp. 272–277.

[40] M. Aps, “Mosek optimization toolbox for matlab,” User’s Guide. Ref. Manual, Version 4, 2019.

[41] Z. Echreshavi, M. Farboud, and M. Shasadeghi, “Fuzzy Event-Triggered Integral Sliding Mode Control of Nonlinear Continuous-Time Systems,” IEEE Trans. Fuzzy Syst., 2021.

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