Ultra-light hierarchical meta-materials on a body-centred cubic lattice

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Abstract – Modern fabrication techniques offer the freedom to design and manufacture structures with complex geometry on many lengthscales, offering many potential advantages. For example, fractal/hierarchical struts have been shown to be exceptionally strong and yet light (Rayneau-Kirkhope D. et al., Phys. Rev. Lett., 109 (2012) 204301). In this letter, we propose a new class of meta-material, constructed from fractal or hierarchical struts linking a specific set of lattice points. We present a mechanical analysis of this meta-material resulting from a body-centred cubic (BCC) lattice. We show that, through the use of hierarchy, the material usage follows an enhanced scaling relation, and both material property and overall efficiency can be optimised for a specific applied stress. Such a design has the potential of providing the next generation of lightweight, buckling-resistant meta-materials.

Introduction. – The design of strong, light structures is an enduring challenge in human history, driven by economic demands as much as technological possibilities. A dramatic increase in the interest in mechanical meta-materials and “designer matter” has been witnessed in recent years [1–5]. This growth has been stimulated by new fabrication techniques, which have relaxed the constraints on lengthscale and geometric complexity to which a designer must adhere. For example, designs with features on the nanometer scale can now be fabricated using techniques such as two-photon lithography [6] and electroless plating [7]. New methods are also being introduced including controlled self-assembly of complex thin-walled micro-structures [8], these novel techniques may offer even greater freedom in design. This new found freedom has allowed the creation of meta-materials with non-trivial structural elements on multiple lengthscales [6,7,9–12], resulting in architectures with close resemblance to some remarkable geometries found in nature [13].

Many natural structures owe their remarkable mechanical properties as much to their geometry as to their constituent material. The presence of non-trivial structural order on many lengthscales is a unifying concept in many such architectures. For example, through hierarchy [13], biological structures offer exceptional performance in adhesion [14], fracture toughness [15], elasticity [16], and strength to weight ratios [17]. Although both theoretical [18–23] and experimental [6,9–12,24–27] work has been undertaken, the role of hierarchy in such structures has not yet been fully elucidated: Such understanding will pave the way for new structures and materials that match and exceed the performance of those found in nature.

In this article, we propose a novel hierarchical meta-material constructed from fractal/hierarchical [28] struts linking a specific set of lattice points. We present a mechanical analysis of the resulting infinite, periodic 3-d, pin jointed lattices under arbitrary uniform stress. For this article, we adopt a body-centred cubic lattice, with struts linking nearest and next-nearest lattice sites. In this lattice, all nodes are equivalent, however, our analysis can be generalised to lattices with more than one lattice site in the unit cell. We find that, through prudent choice of the relative spring constant of the linkages within the structure, one can create an isotropic meta-material. The degree of hierarchy within the linkages may be varied, thus endowing the overall meta-material with a high degree of tailorability in its elastic response. A load can be applied to the meta-material with an arbitrary...
macroscopic stress tensor, $\sigma_{ij}$, which defines the stress vector $\sigma_M \equiv (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12})^T$ [29]. For some magnitude of this loading, the linkages will fail either elastically or through yielding of the material. It is noted that although elastic failure can be utilised to create beneficial material properties [4,30,31], the suppression of elastic instability within a (meta-)material is undoubtedly of great engineering significance [6,11,24,29]. For a given stress direction $\hat{\sigma}_M$, we aim, through hierarchical design, to create meta-materials maintaining elastic stability under a given relative stress $\sigma/\sigma_Y$, while minimising the relative density of material, $\rho/\rho_m$, where $\rho$ is the density of the meta-material, $\sigma = [\sigma_M]$, and $\sigma_Y$ and $\rho_m$ are the yield stress and density of the construction material, respectively. While it is noted that other modes of failure could occur, our interest here is in meta-materials of low relative density, thus linkages with high aspect ratios are assumed throughout, consequently elastic failure is the limiting factor. In order to maximise the robustness of our results, that is, to broaden the spectrum of stress directions over which the hierarchical design shows beneficial results, we use a space frame design considered sub-optimal under simple compressive loading as our linkage element; the structure considered here is a modified version of that presented in ref. [32], where a fractal beam structure was optimised for gentle compressive axial loading.

This paper is divided into the following three sections: In the next section we present the key properties of the hierarchical linkage element used here. In the third section, using a novel methodology, we derive the mechanical properties of our chosen pin jointed lattice, establishing the necessary properties of our linkages to give isotropic material properties, and the consequent effective Young’s modulus and Poisson’s ratio of our meta-material. In the fourth section, we combine the results of the two previous sections to establish the elastic limit of our lattice based meta-material with hierarchical linkages for various directions of applied stress, showing significant gains through hierarchical design. We show that for low relative density of material, the use of hierarchy becomes increasingly beneficial, and that these gains occur over a wide range of stress orientations. Furthermore, we show that for a given stress direction, a series of scaling relationships exist relating the minimum volume fraction of material, $\rho/\rho_m$ and the magnitude of the relative stress that results in elastic failure, $\sigma/\sigma_Y$, which depend on the degree of hierarchy of the linkages. For a wide variety of stress directions, these scaling relations can be manipulated in a systematic and beneficial manner.

**Linkage elements.** — **Simple hollow beams:** To serve as a reference for the hierarchical structures, we will consider the lattice structure fabricated from thin-walled cylinders. Euler buckling [33] provides a first limit for the loading capacity of such a structure under compression. Assuming freely hinged end points, a slender beam of length $L$ buckles under a force $F$,

$$F = \frac{\pi^2 Y I}{L^2},$$  

where $I$ is the second moment of area which, for a thin-walled cylinder, is approximated by $I \approx \pi r^4 t$, where $r$ is the radius of the beam and $t$ is the wall thickness. In a thin-walled structure, a short-wavelength buckling can also occur [34,35], providing a second limit to the compressive loading:

$$F < \frac{2\pi Y t^2}{\sqrt{3}(1-\nu^2)},$$  

where $\nu$ is the Poisson’s ratio of the construction material. Utilising these constraints, it is possible to determine the material required for stability of a simple beam for a given value of compressive loading. When the beam is put under tension, the elastic limit of the material requires that the stress does not exceed the yield stress of the material, $\sigma_Y$. The thin-walled cylinder will be referred to as the generation-1 structure.

**(Hierarchical elements:** The first space frame design (generation-2) that we consider is shown in fig. 1 (left), constructed from thin-walled beams forming $n_{1,1}$ octahedra linking two end tetrahedra. The higher-order structures considered here follow a simple iterative procedure, the generation-$G$ frame is constructed through the replacement of all beams in the generation-($G - 1$) structure with scaled space frames, the number of octahedra at each iteration is allowed to vary. We follow the notation introduced first in [18] that allows us to refer to characteristics of a structure at a given lengthscale: $X_{G,i}$ refers to the parameter $X$ at lengthscale $i$ of $G$ ($i = 1$ being the smallest, $i = G$, the longest). Simple geometry gives

$$L_{G,i} = \sqrt{2/3}(n_{G,i} + 2) L_{G,i-1}.$$  

Elastic stability of a generation-$G$ frame can be lost at one of $G + 1$ lengthscales: on the smallest scale, the thin-walled structure can fail due to Koiter buckling (wavelength of order $r$), or Euler buckling (of order $L_{G,1}$). Alternatively, the whole frame or any subframe can fail due to an Euler buckling, with wavelength of order $L_{G,i}$, for $2 \leq i \leq G$.

When a compressive (tensile) axial force $F_{G,G}$ is applied to the generation-$G$ structure, the maximum load experienced by any substructure at level $i$ in the structure will be [32]

$$F_{G,i} = \frac{F}{2^{i/2}};$$  

$$F_{G,i} = \frac{F_{G,G}}{2^{i/2}},$$

for tension (compression) and compression (tension) bearing beams in the structure, respectively. The mechanical properties of the generation-$G$ will be dependent on its geometry and topology. The spring constant, $K_{G,i}$ and
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Fig. 1: (Colour online) The hierarchical frame: the left panel shows the generation-1 frame, with \( n_{1,1} = 6 \); the right panel shows a generation-3 structure with insets showing details at an increasingly small scale. The particular frame shown is with parameters \( n_{3,i} = 6 \) for \( i = 1, 2, 3 \). In contrast to [32], in this work, at each iterative step, all beams within the structure are replaced by generation-1 frames; while such a frame is less efficient under gentle compressive loading, it results in advantageous scaling results over a broader range of loading directions.

bending stiffness, \( YI_{G,i} \) of a frame and its subframes can be calculated as

\[
K_{G,i} = \frac{36K_{G,i-1}}{11n_{G,i} + 43},
\]

\[
YI_{G,i} = BL_{G,i}^3K_{G,i-1},
\]

where \( B \) is a constant [32]. At all hierarchical levels, there is an Euler buckling mode that must be avoided. The constraint in eq. (1) must be satisfied for all pairs \( (YI = YI_{G,i}, L = L_{G,i}), i \leq G \). Furthermore, \( F_{G,1} \) must not exceed the value given in eq. (2). Given these expressions, and a fixed length of structure we can perform a naive optimisation on the space frame for a given applied load [32]: The parameters \( r \) and \( t \) are set such that the smallest beams in the structure have a coincident bifurcation point due to Koiter and Euler buckling. Utilising these values, we set \( n_{G,i} \) from \( i = 1 \) to \( i = G \) for minimal material cost retaining elastic stability beyond the point of failure on length-scales below.

While other modes of failure are possible, in the limit of interest here (that is lightweight meta-materials), high-aspect-ratio structures of benefit, in this limit elastic stability is likely to be the failure mode of interest. Furthermore, it is noted that for the geometry considered here, when tension is applied to the frame of any generation, each hierarchical level will have a sub-frame that withstands a compressive load, the elastic stability of this element thus introduces the active limit on loading.

**Elastic properties of meta-material.** – Here we present the analysis of the mechanical properties of a particular 3-dimensional lattice. This analysis allows, given a macroscopic stress applied to the meta-material, the computation of the loading on the constituent hierarchical linkage elements. The analysis presented here is readily generalisable to other lattice geometries. Here, we consider an infinite body-centred cubic lattice and place struts with spring constant \( k_1 \) between nearest neighbours and spring constants of \( k_2 \) between next nearest neighbours, see fig. 2. In doing so we create a lattice where all points on the lattice are equivalent. We take \( L \) to be the distance between nearest neighbours. If we take one lattice point to be at the origin of Cartesian coordinates we find that nearest neighbours are at points

\[
x_{1,\ldots,8} = L (\pm 1, \pm 1, \pm 1) / \sqrt{3}.
\]

Then, next nearest neighbours are found at

\[
x_{9,10} = 2L (\pm 1, 0, 0) / \sqrt{3},
\]

\[
x_{11,12} = 2L (0, \pm 1, 0) / \sqrt{3},
\]

\[
x_{13,14} = 2L (0, 0, \pm 1) / \sqrt{3}.
\]

We follow the *Cauchy-Born hypothesis*, and consider affine deformations to the lattice structure represented by a symmetric matrix \( \mathbf{e} \), we see that under this deformation,
a point originally at \( x_i \) is translated to point \( x_i' \) given by,
\[
x_i' = (I_3 + \mathbf{e}) \cdot x_i, \tag{12}
\]
where \( I_3 \) is the identity matrix. The energy per unit volume can be expressed as a sum over the contributions from each linkage in the unit cell:
\[
U = \frac{1}{2V_{uc}} \sum_{i=1}^{14} \frac{1}{2} k_i \left| [(I_3 + \mathbf{e}) \cdot x_i] - L_i \right|^2, \tag{13}
\]
where \( L_i \) is the initial distance between the nodes considered, \( V_{uc} \) is the volume of the unit cell, and the factor of half arises to avoid double counting. Through prudent choice of spring constant \( k_1 \) and \( k_2 \) we aim to create an elastically isotropic material. Through equating the Taylor series of eq. (13), truncated at terms quadratic in \( \mathbf{e} \), with the equivalent expression for an isotropic material [36], we find that the relationship
\[
k_2 = \frac{2k_1}{3}, \tag{14}
\]
ensures that the resulting meta-material will be isotropic. Such a meta-material will have Poisson’s ratio and Young’s modulus given by
\[
\tilde{\nu} = \frac{1}{4}, \quad \tilde{Y} = \frac{5k_1}{2\sqrt{3}}. \tag{15}
\]

Given a macroscopic stress applied to our isotropic material, through the expressions in eq. (15), we are able to obtain the macroscopic material strain. We now define three vectors that describe the periodicity of the lattice:
\[
a_1 = L(1,1,1)^T/\sqrt{3}, \tag{16}
\]
\[
a_2 = L(-1,1,1)^T/\sqrt{3}, \tag{17}
\]
\[
a_3 = L(1,-1,1)^T/\sqrt{3}. \tag{18}
\]
Using these vectors, starting at any point on the lattice a transformation can be found involving integer multiples of these vectors taking us to any equivalent point on the lattice. The strain on the material can then be related to the change in the periodic lattice vectors through [29]:
\[
\Delta a_i = A \left[ \epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \gamma_{23}, \gamma_{13}, \gamma_{12} \right]^T, \tag{19}
\]
where \( A \) is a matrix whose values depend on the vector \( a_i \) [29]. Thus, for a given stress we can obtain the deformation in each of the space frames that make up our material and consequently the macroscopic failure stress of our meta-material can be found through consideration of the elastic failure loadings of the constituent space frames.

Elastic instability. – Having derived the material properties of our meta-material, it is possible to relate macroscopic stress with macroscopic strain. Then, through eq. (19) we can relate macroscopic strain with the strain (and therefore force, through eq. (6)) experienced by the member frames. For a given relative density of material, we can then find the failure stress of the meta-material for different degrees of hierarchy. The increase in load bearing capacity of hierarchical structures relative to the generation-1 design is shown in fig. 3 for various stress directions and relative densities. It is observed that hierarchy is increasingly beneficial for structures of lower relative density.

Scaling laws: We further compare the efficiency of meta-materials of various degrees of hierarchy, by establishing the scaling relationship between the minimal value of the relative density of material required for stability, and the relative strength of the meta-material, the magnitude of the stress. For a given generation and loading direction \( \sigma_{M} \), one can numerically obtain the magnitude of the stress that will result in loss of elastic stability. Scaling laws can then be obtained from these results with a high degree of accuracy.

The scaling relationships observed are dependent on the nature of the stress considered. When compressive load on the linkages causes the dominant mode of failure, for example, the structure is placed under isotropic crush pressure or uniaxial compressive load, the scaling of the relative strength with the relative density will follow:
\[
\frac{\sigma}{\sigma_Y} \sim \left( \frac{\rho}{\rho_{m}} \right)^{\frac{d-2}{d+2}}, \tag{20}
\]
for all values of \( G \). These scalings are shown in fig. 4, where the efficiency of the structures presented here is compared with the existing meta-materials.

If, however, the component frames that make up the material fail under tension (for example isotropic tension), the structure constructed from solid/hollow beams will...
follow the scaling:

\[
\frac{\sigma}{\sigma_y} \sim \begin{cases} 
\frac{\rho}{\rho_m}, & \text{if } G = 0, \\
\left(\frac{\rho}{\rho_m}\right)^{\frac{G+2}{G+1}}, & \text{if } G \geq 1,
\end{cases}
\]

(21)

this is shown in the inset of fig. 5. It is noted that in the limit of gentle loading, under compression, higher-generation frames will be increasingly efficient, while under tension, simple beams will be optimal.

We can also determine the optimal generation of a hierarchical beam for all applied macroscopic stresses. In fig. 5, we show the minimum volume fraction required for stability against a variety of stress directions and magnitudes, alongside the optimal generation number.

**Discussion.** – We have proposed a novel hierarchical meta-material constructed from fractal/hierarchical struts linking a specific set of lattice points, and we have presented a mechanical analysis of this structure. Using our methodology, we have designed an isotropic meta-material from a body-centred cubic lattice with nearest and next nearest neighbours linked with hierarchical beams. Given a component beam of a particular degree of hierarchy, and a general loading stress on the material, we establish the magnitude of loading that will cause elastic instability in the lattice. Through manipulation of hierarchy, we have shown that, for a wide range of loading directions on the material, the fundamental scaling laws defining the efficiency of the meta-material can be manipulated in a beneficial manner. This work illustrates a route to materials with exceptionally high strength-to-weight ratio.

Harnessing the potential of hierarchical design could provide the next generation of lightweight, functional materials. The increased resolution of modern fabrication techniques has made the use of hierarchical structures such as the one presented here a realistic possibility \([6,11,12]\), and as such the designs presented here are of great economic and technological potential. While the analysis presented here is restricted to beams/frames of uniform construction down their long axis, the analysis can be generalised to more general linkage elements offering further potential for optimisation. It is also noted that the post-buckling behaviour of the meta-material has not yet been observed.
elucidated: The understanding of this behaviour may lead to as yet unanticipated technological applications. In the future, it may be possible that single and multi-walled carbon nanotubes [38] or DNA helices [39] could act as the component beams on the smallest structural scale, allowing a huge degree of tailorability of the macroscopic material properties. It has been shown that the instability of both carbon nanotubes [38,40] and DNA helices [39,41,42] structures is broadly similar to those considered here, and as such, the analysis presented here would be expected to hold.

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