COMPLEX MONGE-AMPE`RE EQUATIONS WITH OBLIQUE BOUNDARY VALUE

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Abstract. The existence and regularity of the classical plurisubharmonic solution for complex Monge-Amp`ere equations subject to the semilinear oblique boundary condition which is $C^1$ perturbation of the Neumann boundary condition, are proved in the certain strictly pseudoconvex domain in $\mathbb{C}^n$.

1. Introduction and main results

Let $\Omega$ be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with $C^\infty$ boundary, $f(z)$ be a positive function on $\Omega$, and $\varphi(z, u)$ be a function on $\partial \Omega \times \mathbb{R}$. We shall study the existence and regularity of plurisubharmonic solutions to the complex Monge-Amp`ere equations:

(1.1) $\det \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} = f(z), \quad z \in \Omega,$

with the oblique boundary condition:

(1.2) $D_\beta u = \varphi(z, u), \quad z \in \partial \Omega.$

Assume $\beta$ is a strictly oblique unit vector field, satisfying

(1.3) $\beta(z) \cdot \nu(z) \geq \beta_0 > 0,$

where $\nu(z)$ is the unit outer normal vector field to $\partial \Omega$. In order to apply the method of continuity for the existence of solutions, it is necessary to obtain a priori estimates. We also need to assume some conditions on $\varphi$ to derive the maximum modulus estimates. We assume the conditions in [11, 13] namely

(1.4) $\varphi(z, u) < 0$ for all $z \in \partial \Omega$ and all $u \geq N_1$,

for some constant $N_1$, and

(1.5) $\varphi(z, u) \to \infty$ as $u \to -\infty$ uniformly for $z \in \partial \Omega$.

Conditions (1.4) and (1.5) are used for the upper bound and the lower bound of the solution, respectively. We also assume that

(1.6) $\inf f(\Omega) > 0.$

The Dirichlet problem for complex Monge-Amp`ere equations (1.1) has been an object of intensive research. In 1976, Bedford and Taylor in [2] considered weak plurisubharmonic solutions for complex Monge-Amp`ere equations: $\det \left( \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} \right) dV = 2000 Mathematics Subject Classification. Primary 32A05, 35J60.

Key words and phrases. complex Monge-Amp`ere equation; oblique boundary; plurisubharmonic.

The first author was supported by the National Natural Science Foundation of China (No.11101132) and foundation of Hubei Pro vincial department of education (No.Q20120105).
\( \mu \), where \( \mu \) was a bounded non-negative Borel measure. Cheng and Yau in [4] solved the Dirichlet problem for the equations \( \det(\frac{\partial^2 u}{\partial z_i \partial z_j}) = f(z, u, \nabla u), \ z \in \Omega \) with \( f = e^u, \ u = \infty \) on \( \partial \Omega \), obtaining a solution \( u \in C^\infty(\Omega) \). In the non-degenerate case \( f > 0 \), the existence and uniqueness of a classical plurisubharmonic solution subject to the Dirichlet boundary condition for complex Monge-Ampère equations, under suitable restrictions on \( f \) and \( \phi \), have been proved by Caffarelli, Kohn, Nirenberg and Spruck in [3].

The methods to deal with the oblique boundary value problems for real Monge-Ampère equations originated from the paper of Lions, Trudinger and Urbas in [11] dealing with the Neumann boundary value problems. In 1994, Li Song-Ying in [7] considered the Neumann problem for complex Monge-Ampère equations in strictly pseudoconvex domain in \( \mathbb{C}^n \) with smooth boundary such that,

\[
\inf_{\partial \Omega}(\gamma_0 + 2\lambda_2(z)) > 0,
\]

where \( \varphi(z, u) = -\gamma_0 u + \varphi(z) \), \( \lambda_2(z) \) denotes the smallest principal curvature of \( \partial \Omega \) at \( z \in \partial \Omega \). He obtained the existence and uniqueness of classical plurisubharmonic solutions. The existence of generalized solutions for

\[
\det D^2 u = f(x, u, Du) \text{ in } \Omega
\]

and

\[
D_{\beta}u + \phi(x, u) = 0 \text{ on } \partial \Omega
\]

has been established in [14] by Wang Xu-jia for general strictly oblique vector field \( \beta \) under relatively weak regularity hypotheses on the data. In [12] examples were constructed showing that one cannot generally expect smoothness of solutions of (1.1) and (1.2), no matter how smooth \( \beta, \phi, f \) and \( \partial \Omega \) are. Some structural condition on \( \beta \) is necessary. The examples in [12] are constructed in such a way that \( \beta \) can be made arbitrarily close to \( \nu \) in the \( C^0 \) norm, but not \( C^1 \) norm. Therefore, Urbas in [13] proved the existence of classical solutions for real Monge-Ampère type equations subject to the semilinear oblique boundary condition which is a \( C^1 \) perturbation of the Neumann boundary condition. In 1999, Li Song-Ying in [8] used different methods than Urbas in [13] to prove the existence, uniqueness and regularity for oblique boundary value problem to real Monge-Ampère equations in a smooth bounded strictly convex domain.

In this article we prove the existence and regularity of the strictly plurisubharmonic solution for the oblique problem (1.1) and (1.2) in strictly pseudoconvex domains under suitable conditions.

In order to prove the existence of plurisubharmonic solutions, we have to deduce a priori estimates for the derivative of such solutions up to second order. In the real case, Lions, Trudinger and Urbas in [11] used the convexity of the solution to obtain the first order derivative bound for the oblique boundary value problems. Unfortunately, there is no convexity for strictly plurisubharmonic function. Inspired by the method in [7] and [13], we employ similar argument to obtain a priori estimates.

Now we can state our theorem as follows:

**Theorem 1.1.** Assume \( \Omega \) is a bounded strictly pseudoconvex domain in \( \mathbb{C}^n \) with \( \partial \Omega \in C^4 \). Suppose \( f \in C^2(\Omega) \), \( \varphi \) and \( \beta \) satisfy the conditions (1.3)-(1.6). In
COMPLEX MONGE-AMPÈRE EQUATIONS WITH OBLIQUE BOUNDARY
VALUE

In addition, we assume that \( \varphi \in C^{2,1}(\partial \Omega \times \mathbb{R}) \) with
\[
\varphi_u(z, u) \leq -\gamma_0 < 0 \text{ on } \partial \Omega \times \mathbb{R}, \quad \inf_{\partial \Omega} (\gamma_0 + 2\lambda_2(z)) > 0,
\]
where \( \lambda_2(z) \) denotes the smallest principal curvature of \( \partial \Omega \) at \( z \in \partial \Omega \). Then there is a positive constant \( \epsilon_0 > 0 \) such that
\[
\| \beta - \nu \|_{C^1(\partial \Omega)} \leq \epsilon_0.
\]

Thus the oblique problem for complex Monge-Ampère equations (1.1) and (1.2) has a unique plurisubharmonic solution \( u \in C^2(\Omega) \) and (1.1) can be improved by the linear elliptic theory in [5] if the data are sufficiently smooth.

The paper is organized as follows: in Section 2, we introduce some terminology, then derive \( C^0 \) estimates for the solutions. In Section 3, we study the first order derivative estimates. In Section 4, we complete the second order derivative estimates. In Section 5, we give the proof of our main theorem by using the method of continuity.

2. MAXIMUM MODULUS ESTIMATES

In this section, we first introduce some terminology. Then we derive the maximum modulus estimate of the solution \( u \) for the oblique boundary value problem (1.1) and (1.2) under some structural conditions on \( \varphi(z, u) \).

Let \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \), where \( z_i = x_i + \sqrt{-1}y_i, \) \( (i = 1, \ldots, n) \). We may write \( z \) in real coordinates as \( z = (t_1, \ldots, t_{2n}) \). Given \( \xi \in \mathbb{R}^{2n}, D_\xi \) denotes the directional derivative of \( u \) along \( \xi \). In particular, \( D_k = \frac{\partial}{\partial x_k} \). For the complex variables, we shall use notations: \( \partial_k = \frac{\partial}{\partial z_k}, \partial_k = \frac{\partial}{\partial \bar{z}_k} \) and \( \partial_{\xi} = \frac{\partial}{\partial \xi} \). \( u^{ij} \) denotes the inverse matrix of \( u_{ij} \). For a compact set \( X \), we let \( |u|_{k,X} \) denote the \( C^k \) norm on \( X \).

Let \( f > 0 \) and \( g = \log f \). Then \( \log[\det(u_{ij})] \) is a concave function in \( u_{ij} \). We use Einstein convention. If \( u \) is a solution of (1.1), and \( \xi \in \mathbb{R}^{2n} \), then
\[
(2.1) \quad u^{ij} \partial_{ij} D_\xi u = D_\xi \log[\det(u_{ij})] = D_\xi g,
\]
\[
(2.2) \quad u^{ij} \partial_{ij} D_\xi u \geq D_\xi g.
\]
Moreover, if we let \( \tilde{f} = f^\frac{1}{n} \) and \( F^{ij} = \frac{1}{n} \tilde{f} u^{ij} \), then
\[
(2.3) \quad F^{ij} \partial_{ij} D_\xi u = D_\xi \tilde{f}, F^{ij} \partial_{ij} D_\xi u \geq D_\xi \tilde{f},
\]
\[
(2.4) \quad \text{tr}(F^{ij}) = \frac{1}{n} \text{tr}(u^{ij}) \geq \frac{1}{n} \text{tr}\tilde{f}^{-1} = 1.
\]

Although the argument in this part is rather standard, for completeness, we give the maximum principle without proof:

**Lemma 2.1.** Suppose that \( L = F^{ij} \partial_{ij} \) is elliptic, and \( LH \geq 0(\leq 0) \) in \( \Omega \) with \( H \in C^2(\Omega) \cap C^0(\overline{\Omega}) \). Then the maximum (minimum) of \( H \) in \( \overline{\Omega} \) is achieved on \( \partial \Omega \), that is,
\[
(2.5) \quad \sup_{\overline{\Omega}} H = \sup_{\partial \Omega} H = (\inf_{\overline{\Omega}} H = \inf_{\partial \Omega} H).
\]
**Lemma 2.2.** Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ with $C^1$ boundary, and $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a plurisubharmonic solution of (1.1) and (1.2). Assume $\varphi$ satisfies assumption (1.2), then $u(z) \leq N_1$ on $\overline{\Omega}$.

**Proof.** Since $u$ is plurisubharmonic, $u$ attains its maximum over $\overline{\Omega}$ on $\partial \Omega$, say at $z_0 \in \partial \Omega$, then $D_\nu u(z_0) \geq 0$. On the other hand, $\beta$ can be decomposed into normal and tangential part, from (1.3) and the vanishing of the tangential derivative at $z_0$, we obtain $D_\nu u(z_0) \geq 0$. If $u(z_0) \geq N_1$, by (1.3) we have $\varphi(z_0, u(z_0)) < 0$, this is contradiction. So $u(z_0) < N_1$ and the proof is complete. □

**Lemma 2.3.** Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a plurisubharmonic solution of (1.1) and (1.2). Assume that $\varphi$ satisfies (1.2), then $u(z) \geq N_2 > -\infty$, $z \in \overline{\Omega}$, where $N_2$ is a constant depending only on $|f|_{0,\overline{\Omega}}, \gamma_0, \varphi$.

**Proof.** We use the auxiliary function

$$h(z) = u(z) - \eta |z|^2, \quad \eta = |f|_{0,\overline{\Omega}}^\frac{1}{2} + 1.$$ 

We assume that $h$ attains its minimum over $\overline{\Omega}$ at $z_0 \in \overline{\Omega}$. Now we want to show $z_0 \in \partial \Omega$. If $z_0 \in \Omega$, then

$$0 \leq Lh(z_0) = F^{ij}(z_0) \partial_{ij} h(z_0) = F^{ij}(z_0) \partial_{ij} u(z_0) - \eta F^{ij}(z_0) \delta_{ij} = f^\frac{1}{n} - \eta tr(F^i_j(z_0)) \leq -1 < 0.$$ 

This inequality leads to a contradiction, so we have shown $z_0 \in \partial \Omega$. Without loss of generality, we can assume $u(z_0) \leq 0$, otherwise we obtain the result. Therefore, by the oblique boundary condition (1.2),

$$0 \geq D_\nu h(z_0) = D_\nu u(z_0) - D_\nu (\eta |z|^2)|_{z=z_0} = \varphi(z_0, u(z_0)) - D_\nu (\eta |z|^2)|_{z=z_0}.$$ 

Then

$$\varphi(z_0, u(z_0)) \leq D_\nu (\eta |z|^2)|_{z=z_0} \leq C.$$ 

Then by the condition (1.3), we have $u(z_0) > \tilde{N}$, where $\tilde{N} > -\infty$ is a constant depending on $n$, $\Omega$, $\varphi$, $|f|_{0,\overline{\Omega}}$. For all $z \in \Omega$, $h(z) \geq h(z_0)$, which leads to,

$$u(z) = h(z) + \eta |z|^2 \geq h(z_0) + \eta |z|^2 = u(z_0) + \eta |z|^2 - \eta |z_0|^2 \geq \tilde{N} + \eta |z|^2 - \eta |z_0|^2 \geq N_2 > -\infty,$$

where $N_2$ depends on $n$, $\Omega$, $\varphi$, $|f|_{0,\overline{\Omega}}$. The proof of Lemma 2.3 is complete. □
Theorem 2.4. Let $\Omega$ be a bounded strictly pseudoconvex domain with $C^1$ boundary. Assume that $f$ is non-negative, and $\varphi$ satisfies $(1.1)-(1.6)$. If $u \in C^2(\Omega) \cap C(\Omega)$ is a plurisubharmonic solution of $(1.1)$ and $(1.2)$, then $|u|_{0,\overline{\Omega}} \leq C$, where $C$ is a constant depending only on $n, \Omega, |f|_{0,\overline{\Omega}}, \gamma_0, \beta$.

Remark 2.5. In the proof of Lemma 2.3, the estimate on $|u|_{0,\overline{\Omega}}$ is independent of the lower bound of $f$. So when $f \geq 0$, we can obtain the same estimate by considering $f_\varepsilon = f + \varepsilon, \varepsilon > 0$ first, then let $\varepsilon \to 0^+$ to complete the proof of theorem 2.4. If a similar argument is needed in the following sections, we will not repeat again.

Remark 2.6. We remark that $\overline{C_l}$ and $C_l (l = 1, 2, \cdots)$ denote the constants depending on the known datas. As usual, constants may change from line to line in the context.

3. Gradient estimates

In this section, we follow the idea in [7] and [12] to derive gradient estimates.

Theorem 3.1. Let $\Omega$ be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with $C^3$ boundary. Assume $\beta, \varphi \in C^{1,1}(\partial\Omega \times \mathbb{R})$ and $f \in C^1(\Omega)$ satisfy $(1.3)-(1.6)$. In addition $\varphi$ satisfies $(1.10)$. Then

$$|u|_{1,\overline{\Omega}} \leq C,$$

where $C$ is a constant depending only on $|f|_{1,\overline{\Omega}}, \gamma_0, \Omega, \beta$ and $|\varphi|_{C^{1,1}(\partial\Omega)}$.

Proof. In order to prove $|u|_{1,\overline{\Omega}} \leq C$, from Theorem 2.4 it suffices to prove

$$D_\xi u(z) \leq C, \quad \forall \xi \in S^{2n-1},$$

where $S^{2n-1}$ is the unit sphere in $\mathbb{R}^{2n}$. We still use $\varphi$ to denote a $C^1$ extension of $\varphi$ from $\partial\Omega$ to $\overline{\Omega}$.

First, we reduce the gradient estimates by choosing the auxiliary function $R(z, \xi) = D_\xi u(z) + \eta_1 |z|^2$, where $\eta_1 = |f|_{1,\overline{\Omega}} + 1$. By calculation, we have $LR > 0$. By the maximum principle,

$$\max_{\overline{\Omega}} D_\xi u(z) \leq \max_{\partial\Omega} (D_\xi u(z) + \eta_1 |z|^2) \leq \max_{\partial\Omega} D_\xi u(z) + \eta_1 \max_{\partial\Omega} |z|^2.$$

(3.3)

So the rest task is the estimation of $D_\xi u(z)$ on $\partial\Omega$. From the condition $(1.2)$, when $z \in \partial\Omega$,

$$|D_\beta u(z)| = |\varphi(z, u(z))| \leq C_1.$$

At any boundary point, any direction $\xi$ can be written in terms of a tangential component $\tau(\xi)$ and a component in the direction $\beta$, namely

$$\xi = \tau(\xi) + \frac{(\nu \cdot \xi)}{(\beta \cdot \nu)} \beta,$$

where $\tau(\xi) = \xi - (\nu \cdot \xi)\nu - \frac{(\nu \cdot \xi)}{(\beta \cdot \nu)} \beta T$ and $\beta T = \beta - (\beta \cdot \nu)\nu$. We compute under the condition $(1.3)$,

$$|\tau(\xi)|^2 = 1 - (1 - \frac{|\beta T|^2}{(\beta \cdot \nu)^2}) \nu \cdot \xi |^2 - 2 \frac{\nu \cdot \xi |(\beta T, \xi)}{(\beta \cdot \nu)} \leq 1 + \frac{4|\beta T|}{\beta \cdot \nu}.$$
Thus, if we get the tangential derivative estimate of $u$ on $\partial\Omega$, from (3.4), for any $\xi \in S^{2n-1}$, we have

$$
D_\xi u(z) = D_{\tau(\xi)} u(z) + \frac{(\nu \cdot \xi)}{|\xi^T\beta|} D_{\beta} u(z)
$$

(3.7)

$$
\leq D_{\tau(\xi)} u(z) + C_2
$$

$$
= |\tau(\xi)| D_{\tau(\xi)} u(z) + C_2
$$

$$
\leq \sqrt{(1 + \frac{2}{\beta_0}|\beta^T|)} C_3 + C_2
$$

$$
\leq C_4.
$$

Next the key task is to get the bound for the tangential derivative of $u$ on $\partial\Omega$. We assume that the maximum tangential first order derivative on $\partial\Omega$ is attained at a boundary point which may take to be the origin, in a tangential direction which may take to be $e_1 = (1, 0, \cdots, 0)$, $x_n$ is the inner normal vector at 0. Thus

$$
D_1 u(0) = \sup_{z \in \partial\Omega, \tau \text{ is unit tangential at } z} D_{\tau} u(z).
$$

(3.8)

And without loss of generality, we can assume $D_1 u(0) > 0$. Otherwise, we finish the proof.

We choose the auxiliary function

$$
Q(z) = \frac{D_1 u(z)}{D_1 u(0)} + G|z|^2 - Ax_n - 1,
$$

(3.9)

and the domain $S_\mu = \{z \in \Omega : x_n \leq \mu\}$, where $G, A$ and $\mu$ are constants to be fixed.

From (2.3) we have

$$
F^{ij} \partial_{\partial_i} \frac{D_1 u(z)}{D_1 u(0)} = \frac{D_1 u(z)}{D_1 u(0)} \geq - \frac{C_5}{D_1 u(0)},
$$

and

$$
F^{ij} \partial_{\partial_i} G|z|^2 = G r(F^{ij}) \geq G.
$$

Thus, if we take

$$
G \geq \frac{C_5}{D_1 u(0)},
$$

we have

$$
LQ = F^{ij} Q_{ij} \geq 0 \quad \text{in } S_\mu.
$$

(3.12)

Then we consider the estimates of $Q(z)$ on the boundary of $S_\mu$. On $\partial\Omega \cap \overline{S_\mu}$ near 0, there is a constant $a > 0$ such that (see in [3], Lemma 1.3)

$$
x_n \geq a|z|^2.
$$

(3.13)

Thus

$$
Q(z) \leq 1 + \frac{C_6|z|^2}{D_1 u(0)} + G|z|^2 - Ax_n - 1
$$

$$
\leq (G + \frac{C_6}{D_1 u(0)} - Aa)|z|^2.
$$

(3.14)

If we take

$$
Aa \geq G + \frac{C_6}{D_1 u(0)}
$$

(3.15)

then $Q(z) \leq 0$ on $\partial\Omega \cap \overline{S_\mu}$ near 0.
On the other hand, from (3.3), (3.7) and (3.8) we have
\begin{align}
D_1 u(z) & \leq \max_{\partial \Omega} D_1 u(z) + \eta_1 \text{diam}(\Omega)^2 \\
& \leq \max_{\partial \Omega} \sqrt{(1 + \frac{2}{\beta_0} |\beta^T|)} D_1 u(0) + C_7.
\end{align}
(3.16)

Then on \( \{x_n = \mu\} \cap S_{\mu} \),
\begin{align}
Q(z) & \leq \sqrt{(1 + \frac{2}{\beta_0} |\beta^T|)} + \frac{C_7}{D_1 u(0)} + G |z|^2 - A \mu - 1 \\
& \leq \sqrt{(1 + \frac{2}{\beta_0} |\beta^T|)} + \frac{C_7}{D_1 u(0)} + G |z|^2 - A \mu - 1 \\
& \leq (1 + \frac{2}{\beta_0} |\beta^T|) + \frac{C_7}{D_1 u(0)} + G |z|^2 - A \mu - 1 \\
& \leq \frac{2}{\beta_0} |\beta^T| + \frac{C_7}{D_1 u(0)} + G \mu - A \mu.
\end{align}
(3.17)

If
\begin{align}
\frac{2}{\beta_0} |\beta^T| + G \mu + \frac{C_7}{D_1 u(0)} & \leq A \mu,
\end{align}
(3.18)
then \( Q \leq 0 \) on \( \{x_n = \mu\} \cap S_{\mu} \).

We now proceed to fix \( G \) and \( \mu \), depending on \( A \) which will be fixed later. First we fix \( G > 0 \) so small that
\begin{align}
CG & \leq \frac{1}{2} A \text{ and } G \leq \frac{1}{2} A a,
\end{align}
(3.19)
then fix \( \mu \in (0, 1) \). Then (3.11), (3.15) and (3.18) will be hold whenever
\begin{align}
\frac{C_7}{D_1 u(0)} & \leq G, \\
\frac{C_7}{D_1 u(0)} & \leq \frac{A a}{2}, \\
\frac{2}{\beta_0} |\beta^T| + \frac{C_7}{D_1 u(0)} & \leq \frac{1}{2} A \mu.
\end{align}
(3.20)
By the maximum principle we have \( Q \leq 0 \) in \( S_{\mu} \), and since \( Q(0) = 0 \), then
\begin{align}
D_\beta Q(0) & \geq 0.
\end{align}
(3.21)

From (1.10)
\begin{align}
D_\beta D_1 u(0) & = D_1 D_\beta u(0) - \sum_{k=1}^{2n} (D_1 \beta_k) D_k u(0) \\
& = D_1 \varphi(z, u(0)) - \sum_{k=1}^{2n} (D_1 \beta_k) D_k u(0) \\
& = \varphi_1(0, u(0)) + \varphi_2(0, u(0)) D_1 u(0) - \sum_{k=1}^{2n} (D_1 \beta_k) D_k u(0) \\
& \leq -\gamma D_1 u(0) - \sum_{k=1}^{2n} (D_1 \beta_k) D_k u(0) + C,
\end{align}
(3.22)
where \( \beta_k \) is the component of \( \beta \). Because \( \Omega \) is a bounded strictly pseudoconvex domain, there is a strongly plurisubharmonic defining function \( r(z) \) for \( \Omega \) satisfying \( |\nabla r| = 1 \) on \( \partial \Omega \) = \( \{z \in C^n : r(z) = 0\} \). And there is an orthogonal matrix \( U \) over \( R^{2n-1} \) such that
\begin{align}
U^t \left[ \frac{\partial^2 r(0)}{\partial k \partial l} \right]_{(2n-1) \times (2n-1)} U = \text{diag}(\lambda_2(0), \ldots, \lambda_{2n}(0)),
\end{align}
(3.23)
where \( \lambda_{2n}(0) \geq \cdots \geq \lambda_2(0) \). Since \( \Omega \) is a bounded domain with \( C^2 \) boundary, there are constants \( \Lambda \) and \( \lambda \) such that \( \Lambda > \sup \{\lambda_{2n}(z) : z \in \partial \Omega\} \) and \( \lambda < \inf \{\lambda_2(z) : z \in \partial \Omega\} \).
To handle the $\sum_{k=1}^{2n}(D_1b_k)D_k u(0)$, we express $\nu$ in terms of $\beta$ and tangential components,

$$D_\nu u(0) = \sum_{k=1}^{2n-1}(-\frac{\beta_k}{\beta_{2n}})D_k u(0) + \frac{1}{\beta_{2n}}D_\beta u(0),$$  

and $\nu_k(z) = \frac{\partial r}{\partial n_k}(z)$, $z \in \partial \Omega$, $\nu_k$ is the component of $\nu$. Then

$$\sum_{k=1}^{2n}(D_1b_k)D_k u(0) = \sum_{k=1}^{2n}(D_1\beta_k - D_1\nu_k)D_k u(0) + \sum_{k=1}^{2n}(D_1\nu_k)D_k u(0)$$

$$= \sum_{k=1}^{2n-1}(-\frac{\partial^2 r}{\partial n_1 \partial n_k}D_k u(0)) + \sum_{k=1}^{2n-1}D_1(\beta_k - \nu_k)D_k u(0)$$

$$+ \sum_{k=1}^{2n-1}D_1(\beta_{2n} - \nu_{2n})[\sum_{k=1}^{2n-1}(-\frac{\beta_k}{\beta_{2n}})D_k u(0) + \frac{1}{\beta_{2n}}D_\beta u(0)].$$

Thus

$$D_\beta D_1 u(0)$$

$$\leq C - \gamma_0 D_1 u(0) - \sum_{k=1}^{2n-1}\frac{\partial^2 r}{\partial n_1 \partial n_k}D_k u(0) - \sum_{k=1}^{2n-1}D_1(\beta_k - \nu_k)D_k u(0)$$

$$- \sum_{k=1}^{2n-1}D_1(\beta_{2n} - \nu_{2n})[\sum_{k=1}^{2n-1}(-\frac{\beta_k}{\beta_{2n}})D_k u(0) + \frac{1}{\beta_{2n}}D_\beta u(0)]$$

$$\leq C - \gamma_0 D_1 u(0) - 3\Lambda D_1 u(0) + \sum_{k=1}^{2n-1}[\Lambda\delta_k - \frac{\partial^2 r}{\partial n_1 \partial n_k}D_k u(0)$$

$$+ \sum_{k=1}^{2n-1}[\Lambda\delta_{k1} - D_1(\beta_k - \nu_k)]D_k u(0)$$

$$+ \sum_{k=1}^{2n-1}[\Lambda\delta_{k1} + \frac{\partial^2 r}{\partial n_1 \partial n_k} + D_1(\beta_{2n} - \nu_{2n})](\frac{\beta_k}{\beta_{2n}})D_k u(0).$$

Let

$$A_1 = \sum_{k=1}^{2n-1}[\Lambda\delta_k - \frac{\partial^2 r}{\partial n_1 \partial n_k}D_k u(0),$$

$$A_2 = \sum_{k=1}^{2n-1}[\Lambda\delta_{k1} - D_1(\beta_k - \nu_k)]D_k u(0),$$

$$A_3 = \sum_{k=1}^{2n-1}[\Lambda\delta_{k1} + \frac{\partial^2 r}{\partial n_1 \partial n_k} + D_1(\beta_{2n} - \nu_{2n})](\frac{\beta_k}{\beta_{2n}})D_k u(0).$$

By using an argument similar to that given in the proof of Theorem 3.1 in [7], we want to obtain the relationship between $A_1$, $A_2$ and $A_3$ with $D_1 u(0)$.

First, by the conclusion in [7] directly, we set

$$\tau_1 = (c'_1[\Lambda I_{2n-1} - \frac{\partial^2 r(0)}{\partial t_1 \partial t_k}]_{k=1}^{2n-1}, 0),$$

where $c'_1 = (1, 0, \cdots, 0)$. Since the matrix $[\Lambda I_{2n-1} - \frac{\partial^2 r(0)}{\partial t_1 \partial t_k}]_{k, l=1}^{2n-1}$ is non-negative definite with maximum eigenvalue $\Lambda - \lambda_2(0)$, we have $|\tau_1| \leq \Lambda - \lambda_2(0)$. Thus by
and $\tau_1$ is the tangential direction at 0,

$$A_1 = D_{\tau_1} u(0) = |\tau_1| D_{\tau_1} u(0) \leq (\Lambda - \lambda_2(0)) D_1 u(0).$$

Then, set

$$M_{2n-1} = \begin{pmatrix} D_1(\beta_1 - \nu_1) & D_1(\beta_2 - \nu_2) & \cdots & D_1(\beta_{2n-1} - \nu_{2n-1}) \\ D_1(\beta_2 - \nu_2) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ D_1(\beta_{2n-1} - \nu_{2n-1}) & 0 & \cdots & 0 \end{pmatrix},$$

$$\tau_2 = (e^i [M_{2n-1}^2 - M_{2n-1}], 0).$$

Thus

$$A_2 = D_{\tau_2} u(0) = |\tau_2| D_{\tau_2} u(0).$$

By calculation, the eigenvalues of matrix $M_{2n-1}$ are

$$\pi_1 = 0, \quad \pi_{2,3} = D_1(\beta_1 - \nu_1) \pm \sqrt{(D_1(\beta_1 - \nu_1))^2 + 4[\sum_{i=2}^{2n-1} (D_1(\beta_i - \nu_i))^2]} \notag$$

where $\pi_3$ is $(2n-3)$ multiple eigenvalues. In the condition (1.11), we can take $\epsilon_0 \leq \frac{\Lambda}{1 + \sqrt{8n - 7}}$ so that the matrix $M_{2n-1}^2 - M_{2n-1}$ is non-negative definite with $|\tau_2| \leq (\Lambda + \frac{1 + \sqrt{8n - 7}}{2} \epsilon_0)$. Thus

$$A_2 \leq (\Lambda + \frac{1 + \sqrt{8n - 7}}{2} \epsilon_0) D_1 u(0).$$

At last, we consider $A_3$.

$$A_3 = \sum_{k=1}^{2n-1} [\Lambda \delta_k + (\frac{\partial^2 r}{\partial t_{2n}} + D_1(\beta_{2n} - \nu_{2n}))\frac{\beta_k}{\beta_{2n}}] D_k u(0).$$

Set

$$N(0) = [(\frac{\partial^2 r}{\partial t_{2n}} + D_1(\beta_{2n} - \nu_{2n}))] (0),$$

here $N(0)$ has uniform upper bound, i.e. $|N(0)| \leq N$, where $N$ is a constant independent of 0. Set

$$G_{2n-1} = \begin{pmatrix} \beta_1 & \beta_2 & \cdots & \beta_{2n-1} \\ \beta_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{2n-1} & 0 & \cdots & 0 \end{pmatrix},$$

$$\tau_3 = (e^i [\Lambda I_{2n-1} + \frac{N(z_0)}{\beta_{2n}} G_{2n-1}], 0).$$

So, if take $\epsilon_0 \leq \frac{\Lambda |\beta_{2n}|}{N(1 + \sqrt{8n - 7})}$, we have

$$A_3 = D_{\tau_3} u(0) = |\tau_3| D_{\tau_3} u(0) \leq (\Lambda + \frac{N}{\beta_{2n}} \frac{1 + \sqrt{8n - 7}}{2} \epsilon_0) D_1 u(0).$$
Thus by inserting (3.31), (3.36) and (3.41) into (3.30) we get
\[ D_\beta D_1 u(0) \leq C - \gamma_0 D_1 u(0) - 3\Lambda D_1 u(0) + (\Lambda - \lambda_2(0)) D_1 u(0) + (\Lambda + |\frac{N}{\beta_2 n} \frac{1}{\sqrt{1 + \sqrt{8n - 7}}} \leq 0 |\frac{N}{\beta_2 n} \frac{1}{\sqrt{1 + \sqrt{8n - 7}}} D_1 u(0), \]
(3.42)

Again, we take \( \epsilon_0 \leq \frac{\gamma_0 + \lambda_2(0)}{[(1 + \sqrt{8n - 7})^2 \sqrt{8n - 7}]}, \)
then
\[ \gamma_0 + \lambda_2(0) \geq (1 + \sqrt{8n - 7}) \epsilon_0 + (1 + \sqrt{8n - 7}) |\frac{N}{\beta_2 n} \epsilon_0. \]
(3.43)

Ultimately, by (1.3), \( \beta_2 n(0) = \beta(0) \cdot \nu(0) \geq \beta_0, \) we choose
\[ \epsilon_0 \leq \min\{ \frac{\Lambda}{1 + \sqrt{8n - 7} \cdot N(1 + \sqrt{8n - 7}) |\frac{N}{\beta_2 n} \epsilon_0, \}
(3.44)

then
\[ \gamma_0 + \lambda_2(0) - \left( \frac{1 + \sqrt{8n - 7}}{2} \right) \epsilon_0 - \left( \frac{1 + \sqrt{8n - 7}}{2} \right) |\frac{N}{\beta_2 n} \epsilon_0 \geq \frac{\sigma_0}{2}, \]
where \( \sigma_0 = \inf_{\partial \Omega} (\gamma_0 + \lambda_2). \) Then by inserting (3.45) into (3.42),
(3.46)

Finally we fix \( A \) so small that
(3.48)

we have for any tangential vector \( D_1 u(0) \leq \frac{a_0}{\bar{a}}. \) We finish the proof of Theorem 4.1.

\[ \square \]

4. Second Order Derivative Estimates

In this section, we aim to derive the second order derivative estimates. First, we reduce the interior second order derivatives to the boundary. Then, we derive the second order derivative estimates on the boundary. Finally, from these estimates we have Theorem 4.8.

**Theorem 4.1.** Let \( \Omega \) be a bounded strictly pseudoconvex domain in \( \mathbb{C}^n \) with \( C^4 \) boundary. Assume \( \beta, \varphi \in C^{2,1}(\partial\Omega \times \mathbb{R}) \) and \( f \in C^2(\bar{\Omega}) \) satisfy (1.3)-(1.6). In addition \( \varphi \) satisfies (1.7). Then
(4.1)

\[ |D^2 u|_{0, \bar{\Omega}} \leq \bar{C}, \]
where \( \bar{C} \) is a constant depending only on \( |u|_{1, \bar{\Omega}}; |f\frac{\partial}{\partial \bar{z}}|_{C^2(\bar{\Omega})}, \gamma_0, \Omega, \beta \) and \( |\varphi|_{C^{2,1}(\partial\Omega)}. \)

First, we reduce the second order derivative estimates to the boundary by choosing the auxiliary function \( R(z, \xi) = D_\xi u(z) + \eta_2 |z|^2, \) where \( \eta_2 = |f\frac{\partial}{\partial \bar{z}}|_{C^2(\bar{\Omega})} + 1. \)

We have
(4.2)
thus $LR_1 = F^{ij} \partial_{ij} R_1 > 0$. By the maximum principle,
\begin{equation}
\max_{\partial \Omega} D_{\xi \xi} u(z) \leq \max_{\partial \Omega} (D_{\xi \xi} u(z) + \eta_2 |z|^2) \\
\leq \max_{\partial \Omega} (D_{\xi \xi} u(z) + \eta_2 |z|^2) \\
\leq \max_{\partial \Omega} (D_{\xi \xi} u(z) + \eta_2 \text{diam}(\Omega))^2.
\end{equation}

Next, we give some lemmas below.

**Lemma 4.2.** We reformulate (4.11) as follow,
\begin{equation}
D_{\xi \xi} u(z) \leq C, \quad \xi \in \mathbb{R}^{2n},
\end{equation}
whenever $u$ is subharmonic.

**Lemma 4.3.** $|D_{\beta \nu} u| \leq C$ on $\partial \Omega$ for any unit tangential vector field $\nu$.

**Proof.** By applying the tangential gradient operator to the boundary condition we obtain $|D_{\beta \nu} u| \leq C$ on $\partial \Omega$ for any tangential vector $\nu$. \qed

**Lemma 4.4.** $|D_{\beta \nu} u| \leq C$ on $\partial \Omega$.

**Proof.** Without loss of generality, we set $0 \in \partial \Omega$, $x_n$ is inner normal vector at $0$.

Case 1: Suppose that $Du(0) = 0$.

Now we use the auxiliary function
\begin{equation}
W = \pm D_{\beta \nu} \varphi(z, u) + |Du|^2 + \bar{K}|z|^2 - \bar{G}x_n.
\end{equation}

Now we compute $LW$.

\begin{equation}
LD_{\beta \nu} u = F^{ij} \partial_i \partial_j \beta_k D_k u + F^{ij} (\partial_{ij} \beta_k) (\partial_i D_k u) + F^{ij} (\partial_i \beta_k) (\partial_j D_k u) + F^{ij} \beta_k \partial_i \partial_j D_k u \\
= F^{ij} \partial_i \partial_j \beta_k D_k u + F^{ij} (\partial_{ij} \beta_k) (\partial_i D_k u) + F^{ij} (\partial_i \beta_k) (\partial_j D_k u) + \beta_k D_k \bar{f}.
\end{equation}

From the Cauchy inequality and the positivity of the matrix $(F^{ij})$, we have
\begin{equation}
|F^{ij} (\partial_{ij} \beta_k) (\partial_i D_k u)| \leq (F^{ij} (\partial_i \beta_k \partial_j \beta_k))^\frac{1}{2} (F^{ij} (\partial_i D_k u \partial_j D_k u))^\frac{1}{2},
\end{equation}
\begin{equation}
|F^{ij} (\partial_i \beta_k) (\partial_j D_k u)| \leq (F^{ij} (\partial_i \beta_k \partial_j \beta_k))^\frac{1}{2} (F^{ij} (\partial_i D_k u \partial_j D_k u))^\frac{1}{2}.
\end{equation}

Hence
\begin{equation}
|LD_{\beta \nu} u| \leq \bar{C}_1 tr(F^{ij}) + F^{ij} (\partial_i D_k u) (\partial_j D_k u) + C,
\end{equation}
where $\bar{C}_1$ depends on $|u|_{1, \Omega}$, $\beta$ and $f^\frac{1}{2}$. We also have
\begin{equation}
L \varphi(z, u) = F^{ij} \partial_i \partial_j \varphi(z, u) \\
= F^{ij} [\varphi_{ij} + \varphi_{ui} u_i + \varphi_{ui} u_j + \varphi_{ui} u_i u_j + \varphi_{ui} u_j] \\
= F^{ij} [\varphi_{ij} + \varphi_{ui} u_i + \varphi_{ui} u_j + \varphi_{ui} u_i u_j] + \bar{f} \varphi u \\
\leq \bar{C}_2 tr(F^{ij}) + C,
\end{equation}
where $\bar{C}_2$ depends on $\varphi$, $f^\frac{1}{2}$ and $|u|_{1, \Omega}$.

\begin{equation}
L |Du|^2 = 2F^{ij} (\partial_i D_k u \partial_j D_k u + D_k u \partial_i \partial_j D_k u) \\
= 2F^{ij} (\partial_i D_k u) (\partial_j D_k u) + 2D_k u \partial_i \partial_j D_k u \\
\geq 2F^{ij} (\partial_i D_k u) (\partial_j D_k u) - C,
\end{equation}
where $C$ depends on $|u|_{1, \Omega}$ and $f^\frac{1}{2}$.

\begin{equation}
L |z|^2 = tr(F^{ij}).
\end{equation}
Therefore, we obtain
\begin{equation}
(4.13) \quad LW \geq (\widetilde{K} - \widetilde{C}_1 - \widetilde{C}_2)tr(F^ij) - \widetilde{C}_3 \geq \tilde{K} - \tilde{C}_4,
\end{equation}
we can choose \( \tilde{K} > \tilde{C}_4 \) such that \( LW \geq 0 \).

Finally, we use a standard barrier argument for \( D_n W \) on \( \partial \Omega \) (see in [3], Lemma 1.3 or [5], Corollary 14.5). Let \( S_{\mu_1} = \{ z \in \Omega | x_n \leq \mu_1 \} \). On \( \partial S_{\mu_1} \cap \Omega \) if \( \tilde{G} \) is sufficiently large, we get \( W \leq 0 \). On \( \partial S_{\mu_1} \cap \partial \Omega \), we have for some \( a > 0 \), \( x_n \geq a|z|^2 \). Then we can choose \( \tilde{G} \) large enough such that \( W \leq 0 \) on \( \partial S_{\mu_1} \cap \partial \Omega \). Since \( W(0) = 0 \), by the maximum principle, we obtain \( W_{x_n}(0) \leq 0 \), then \( |D_{\beta x_n} u| \leq \tilde{G} + |\varphi|_{C^{1,1}(\partial \Omega)} \).

Case 2: If \( Du(0) \neq 0 \), we can take
\begin{equation}
(4.14) \quad W = \pm D_{\beta u} \mp \varphi(z, u) + |Du(z) - Du(0)|^2 + \tilde{K}|z|^2 - \tilde{G}x_n,
\end{equation}
The proof above is still valid. \( \square \)

**Remark 4.5.** Combining Lemma 4.3 with Lemma 4.4, for any direction \( \xi \), we obtain \( |D_{\beta \xi} u| \leq C \) on \( \partial \Omega \). In particular \( |D_{\beta \xi} u| \leq C \) on \( \partial \Omega \).

Since we have the bounds for \( |D_{\tau \beta} u| \) and \( |D_{\beta \beta} u| \) on \( \partial \Omega \), in order to finish the proof of Theorem 4.1, the remaining task is to get the bounds for the tangential second derivatives of \( u \) on \( \partial \Omega \). We now assume that the maximum tangential second order derivative on \( \partial \Omega \) is attained at a boundary point which may take to be the origin, in tangential direction which we may take to be \( e_1 \), and \( x_n \) is the inner normal vector at \( 0 \). Thus
\begin{equation}
(4.15) \quad D_{11} u(0) = \sup_{z \in \partial \Omega, \tau \text{ is unit tangential at } z} D_{\tau \tau} u(z).
\end{equation}
Without loss of generality, we can take \( D_{11} u(0) > 0 \). Otherwise, we finish the proof.

As for any direction \( \xi \), on \( \partial \Omega \),
\begin{equation}
(4.16) \quad D_{\xi \xi} u = D_{\tau (\xi) \tau (\xi)} u + 2 \langle \nu \xi, \nu \rangle D_{\tau (\xi)} \beta u + \langle \nu \xi \rangle^2 D_{\beta \beta} u,
\end{equation}
\( \leq (1 + \frac{1}{n}) D_{11} u(0) + C \) on \( \partial \Omega \).

We introduce the tangential gradient operator \( \delta = (\delta_1, \cdots, \delta_{2n-1}) \), where \( \delta_i = (\delta_{ij} - \nu_i \nu_j) D_j \). Applying this tangential operator to the boundary condition \( (1.2) \), we have
\begin{equation}
(4.17) \quad D_k u \delta_i \delta_k + \beta_k \delta_i D_k u = \delta_i \varphi, \quad \text{on } \partial \Omega,
\end{equation}
then
\begin{equation}
(4.18) \quad D_{\tau \beta} u = -D_k u (\delta_i \delta_k) \tau_i + \delta_i \varphi \tau_i.
\end{equation}

For the case \( \xi = e_1 \) in \( (1.10) \), we have
\begin{equation}
(4.19) \quad D_{11} u(z) \leq (1 - \frac{2\nu_1 \beta^T}{\beta \cdot \nu}) D_{11} u(0) + \frac{2\nu_1 \beta^T}{\beta \cdot \nu} D_{\tau (e_1) \beta} + \frac{\nu_1^2}{(\beta \cdot \nu)^2} D_{\beta \beta} u(z).
\end{equation}

Similarly to the real case in \( (13) \), let \( S_{\mu_2} = \{ z \in \Omega : x_n \leq \mu_2 \} \). We construct a function
\begin{equation}
H(z) = \frac{D_{11} u(z) - V(z)}{D_{11} u(0)} + \frac{2\nu_1 \beta^T}{\beta \cdot \nu} + \hat{B}|Du(z)|^2 + \hat{G}|z|^2 - \hat{A}x_n - 1,
\end{equation}
\( \hat{G}, \hat{A}, \hat{B} \) and \( \mu_2 \) are constants to be fixed. \( V(z) \) is a linear function with respect to \( Du \), such that
\begin{equation}
(4.20) \quad V(z) = a_k(z) D_k u + b(z), \quad \text{in } \Omega,
\end{equation}
where \( D_k u \).
COMPLEX MONGE-AMPERE EQUATIONS WITH OBLIQUE BOUNDARY VALUE

where $a_k(z), b(z)$ are smooth functions and

\[
(4.21) \quad a_k(z) = -2 \frac{\nu, e_1}{\beta, \nu} (\delta_i \beta_k) \tau_i(e_1), \quad b(z) = 2 \frac{\nu, e_1}{\beta, \nu} \delta_i \varphi \tau_i(e_1), \quad \text{on } \partial \Omega.
\]

So $V(0) = 0$ on $\partial \Omega$.

Assume $Du(0) = 0$, or we let

\[
(4.22) \quad H(z) = \frac{D_{11} u(z) - V(z)}{D_{11} u(0)} + \frac{2 \nu_1 \beta_1^T}{\beta \cdot \nu} + \tilde{B} |Du(z) - Du(0)|^2 + \tilde{G} |z|^2 - \tilde{A} x_n - 1.
\]

According to (2.30) and (2.41), we have $LD_{11} u(z) \geq -\tilde{C}_5$.

\[
-LV
= -L(a_k D_k u + b)
= -F^{ij} \partial_i (\partial_j a_k D_k u + a_k \partial_j D_k u + \partial_j b)
= -F^{ij} \partial_j a_k D_k u + \partial_j a_k \partial_i D_k u + a_k \partial_j \partial_i D_k u + \partial_j b
\geq -F^{ij} \partial_j a_k D_k u + a_k \partial_j D_k u - C \text{tr}(F^{ij}) - a_k D_k \tilde{f} - C
\geq -C \partial_i \text{tr}(F^{ij}) - F^{ij} \partial_i D_k u \partial_j D_k u - \tilde{C}_7,
\]

where $\tilde{C}_5$ and $\tilde{C}_7$ depend on $a_k, b, f^T$ and $|u|_{1, \Omega}$.

\[
L(|Du|^2) = F^{ij} (2 \partial_i D_k u \partial_j D_k u + 2 D_k u \partial_i \partial_j D_k u)
\geq 2 F^{ij} \partial_i D_k u \partial_j D_k u - \tilde{C}_8,
\]

where $\tilde{C}_8$ depends on $|u|_{1, \Omega}$ and $f^T$.

\[
L |G| z^2 = \tilde{G} \text{tr}(F^{ij}).
\]

And see (2.26) in [8],

\[
(4.26) \quad L (\frac{2 \nu_1 \beta_1^T}{\beta \cdot \nu}) \geq - (\tilde{C}_9 \sqrt{\mu_2} |\beta|_{2, \Omega} + \tilde{C}_{10} |\beta^T|_{1, \Omega}) \text{tr}(F^{ij}) \quad \text{in } S_{\mu_2}.
\]

Therefore,

\[
(4.27) \quad LH \geq -\tilde{C}_5 \text{tr}(F^{ij}) - \tilde{C}_7 - F^{ij} \partial_i D_k u \partial_j D_k u + 2 B F^{ij} \partial_i D_k u \partial_j D_k u - \tilde{B} \tilde{C}_8 + \tilde{G} \text{tr}(F^{ij})

- (\tilde{C}_9 \sqrt{\mu_2} |\beta|_{2, \Omega} + \tilde{C}_{10} |\beta^T|_{1, \Omega}) \text{tr}(F^{ij})

= \left( \frac{\tilde{G}}{2} \text{tr}(F^{ij}) - \tilde{B} \tilde{C}_8 - \tilde{C}_5 + \tilde{C}_7 \right)

+ (\tilde{C}_5 \sqrt{\mu_2} |\beta|_{2, \Omega} + \tilde{C}_{10} |\beta^T|_{1, \Omega}) \text{tr}(F^{ij})

+ (2 \tilde{B} - \frac{1}{D_{11} u(0)}) F^{ij} \partial_i D_k u \partial_j D_k u

\geq (\tilde{G} \frac{1}{2} - \tilde{B} \tilde{C}_8 - \frac{\tilde{C}_5 + \tilde{C}_7}{D_{11} u(0)})

+ (\tilde{G} \frac{1}{2} - \tilde{C}_9 \sqrt{\mu_2} |\beta|_{2, \Omega} - \tilde{C}_{10} |\beta^T|_{1, \Omega}) \text{tr}(F^{ij})

+ (2 \tilde{B} - \frac{1}{D_{11} u(0)}) F^{ij} \partial_i D_k u \partial_j D_k u, \quad \text{in } S_{\mu_2}.
\]

So if

\[
(4.28) \quad \frac{\tilde{G}}{2} - \tilde{B} \tilde{C}_8 - \frac{\tilde{C}_5 + \tilde{C}_7}{D_{11} u(0)} \geq 0, \quad \frac{\tilde{G}}{2} - \tilde{C}_9 \sqrt{\mu_2} |\beta|_{2, \Omega} - \tilde{C}_{10} |\beta^T|_{1, \Omega} \geq 0, \quad 2 \tilde{B} \geq \frac{1}{D_{11} u(0)},
\]

we have $LH \geq 0$ in $S_{\mu_2}$. 13
On \( \partial \Omega \cap \overline{S_{\mu_2}} \) near 0, from (4.19), Lemma 4.3 and Lemma 4.4,

\[
H(z) \leq 1 + \frac{\tilde{C}_{12}|z|^2}{D_{11}u(0)} + (\tilde{G} - \tilde{A}a)|z|^2 - 1
\]
\[
\leq \frac{\tilde{C}_{12}}{D_{11}u(0)} + \tilde{G} - \tilde{A}a|z|^2,
\]
if
\[
(4.30) \quad \frac{\tilde{C}_{12}}{D_{11}u(0)} + \tilde{G} \leq \tilde{A}a,
\]
we have \( H(z) \leq 0 \) on \( \partial \Omega \cap \overline{S_{\mu}} \) near 0.

On the other hand, from (3.6), (4.3), and Theorem 3.1,

\[
H(z) \leq (1 + 2\beta_0|\beta^T|) + \frac{\tilde{C}_{13}}{D_{11}u(0)} + \tilde{C}_{14}\sqrt{\mu_2}|\beta^T| + \hat{B}C + \tilde{G}|z|^2 - \hat{A}\mu_2 - 1,
\]
\[
\leq \frac{2}{\beta_0}|\beta^T| + \frac{\tilde{C}_{13}}{D_{11}u(0)} + \tilde{C}_{14}\sqrt{\mu_2}|\beta^T| + \tilde{G}\tilde{C}_{16}\mu_2 - \hat{A}\mu_2.
\]
if
\[
(4.32) \quad \frac{2}{\beta_0}|\beta^T| + \frac{\tilde{C}_{13}}{D_{11}u(0)} + \tilde{C}_{14}\sqrt{\mu_2}|\beta^T| + \hat{B}C + \tilde{G}\tilde{C}_{16}\mu_2 \leq \hat{A}\mu_2,
\]
we have \( H(z) \leq 0 \) on \( \{x_n = \mu\} \cap \overline{S_{\mu_2}} \).

We now proceed to fix \( \hat{G}, \mu_2 \) and \( \hat{B} \), depending on \( \hat{A} \) which will be fixed later. We first fix \( \hat{G} > 0 \) so small that

\[
(4.33) \quad \hat{G} \leq \frac{\hat{A}a}{2} \text{ and } \tilde{G}\tilde{C}_{16} \leq \frac{\hat{A}}{2},
\]
and then fix \( \mu_2 \in (0, 1) \) so that

\[
(4.34) \quad \tilde{C}_9\sqrt{\mu_2}|\beta|_{2,\overline{\tau}} \leq \frac{\hat{G}}{4},
\]
and then fix \( \hat{B} \) so that

\[
(4.35) \quad \hat{B}\tilde{C}_8 \leq \frac{\hat{G}}{4} \text{ and } \hat{B}C \leq \frac{\hat{A}\mu_2}{4}.
\]

Then (4.28), (4.30) and (4.32) will be hold whenever

\[
\frac{\tilde{C}_8 + \tilde{C}_9}{D_{11}u(0)} \leq \frac{\hat{G}}{4},
\]
\[
\frac{\tilde{C}_8}{D_{11}u(0)} + \tilde{C}_{10}|\beta^T|_{1,\overline{\tau}} \leq \frac{\hat{G}}{4},
\]
\[
(\frac{2}{\beta_0} + \tilde{C}_{14})|\beta^T| + \frac{\tilde{C}_{13}}{D_{11}u(0)} \leq \frac{\hat{A}\mu_2}{4}.
\]

When these conditions (4.36) are satisfied we have \( Q \leq 0 \) in \( S_{\mu_2} \) by the maximum principle, and since \( H(0) = 0 \), then \( D_\beta H(0) \geq 0 \).
We have

\begin{equation}
D_\beta D_1 u(0) \leq D_{11} \varphi(0, u(0)) - \sum_{k=1}^{2n} (D_{11} \beta_k) D_k u(0) - 2 \sum_{k=1}^{2n} (D_1 \beta_k) D_k D_1 u(0) + C
\end{equation}

\begin{equation}
\leq \varphi_n D_{11} u(0) - \sum_{k=1}^{2n} (D_{11} \beta_k) D_k u(0) - 2 \sum_{k=1}^{2n} (D_1 \beta_k) D_k D_1 u(0) + C
\end{equation}

\begin{equation}
\leq -\gamma_0 D_{11} u(0) - 2 \sum_{k=1}^{2n} (D_1 \beta_k) D_k D_1 u(0) + C
\end{equation}

\begin{equation}
= C - \gamma_0 D_{11} u(0) - 2 \sum_{k=1}^{2n} (D_1 \nu_k) D_k D_1 u(0) - 2 \sum_{k=1}^{2n} D_1 (\beta_k - \nu_k) D_k D_1 u(0)
\end{equation}

\begin{equation}
= C - \gamma_0 D_{11} u(0) - 2 \sum_{k=1}^{2n-1} \frac{\partial^2 \nu}{\partial t_k \partial t_{1}} D_k D_1 u(0)
\end{equation}

\begin{equation}
- 2 \sum_{k=1}^{2n-1} D_1 (\beta_k - \nu_k) D_k D_1 u(0) - 2 \left[ \frac{\partial^2 \nu}{\partial t_k \partial t_{1}} + D_1 (\beta_k - \nu_k) D_k D_1 u(0) \right].
\end{equation}

To handle the last term above, we express \( \nu \) in terms of \( \beta \) and tangential components,

\begin{equation}
D_\beta D_1 u(0) = \sum_{k=1}^{2n-1} \left( -\frac{\beta_k}{\beta_{2n}} \right) D_k D_1 u(0) + \frac{1}{\beta_{2n}} D_\beta D_1 u(0).
\end{equation}

We take (4.38) into the last term of (4.37),

\begin{equation}
D_\beta D_1 u(0) \leq C - \gamma_0 D_{11} u(0) - 2 \sum_{k=1}^{2n-1} \frac{\partial^2 \nu}{\partial t_k \partial t_{1}} D_k D_1 u(0)
- 2 \left[ \frac{\partial^2 \nu}{\partial t_k \partial t_{1}} + D_1 (\beta_k - \nu_k) \right] \left[ \sum_{k=1}^{2n-1} \left( -\frac{\beta_k}{\beta_{2n}} \right) D_k u(0) + \frac{1}{\beta_{2n}} D_1 u(0) \right]
\end{equation}

\begin{equation}
\leq C - \gamma_0 D_{11} u(0) - 2 \sum_{k=1}^{2n-1} \frac{\partial^2 \nu}{\partial t_k \partial t_{1}} D_k D_1 u(0)
+ 2 \left[ \frac{\partial^2 \nu}{\partial t_k \partial t_{1}} + D_1 (\beta_k - \nu_k) \right] \left[ \sum_{k=1}^{2n-1} D_1 (\beta_k - \nu_k) \right] D_1 u(0)
\end{equation}

\begin{equation}
= C - \gamma_0 D_{11} u(0) - 6 \Lambda D_1 u(0) + \sum_{k=1}^{2n-1} 2 \left[ \Lambda \delta_{k1} - \frac{\partial^2 \nu}{\partial t_k \partial t_{1}} \right] D_k D_1 u(0)
\end{equation}

\begin{equation}
+ \sum_{k=1}^{2n-1} 2 \left[ \Lambda \delta_{k1} + \left( \frac{\partial^2 \nu}{\partial t_k \partial t_{1}} + D_1 (\beta_k - \nu_k) \right) \right] D_k D_1 u(0).
\end{equation}

Let

\begin{equation}
A'_1 = \sum_{k=1}^{2n-1} 2 \left[ \Lambda \delta_{k1} - \frac{\partial^2 \nu}{\partial t_k \partial t_{1}} \right] D_k D_1 u(0),
\end{equation}

\begin{equation}
A'_2 = \sum_{k=1}^{2n-1} 2 \left[ \Lambda \delta_{k1} - D_1 (\beta_k - \nu_k) \right] D_k D_1 u(0),
\end{equation}

\begin{equation}
A'_3 = \sum_{k=1}^{2n-1} 2 \left[ \Lambda \delta_{k1} + \left( \frac{\partial^2 \nu}{\partial t_k \partial t_{1}} + D_1 (\beta_k - \nu_k) \right) \right] D_k D_1 u(0).
\end{equation}

By using an argument similar to that given in the proof of Theorem \ref{thm:principal}, we want to obtain the relationship between \( A'_1, A'_2, A'_3 \) with \( D_{11} u(0) \).

First, by the conclusion in \ref{thm:principal} directly, we have

\begin{equation}
A'_1 = (2 \epsilon_1 \left[ \Lambda I_{2n-1} - \left( \frac{\partial^2 \nu}{\partial t_k \partial t_{1}} \right) \Lambda_{k,1} \right]) \cdot (D_1 D_1 u(0), \ldots, D_{2n-1} D_1 u(0))
\end{equation}

\begin{equation}
= (2 \epsilon_1 U^T \left[ \Lambda I_{2n-1} - \left( \frac{\partial^2 \nu}{\partial t_k \partial t_{1}} \right) \Lambda_{k,1} \right] U) \cdot (D_1 D_1 u(0), \ldots, D_{2n-1} D_1 u(0)) U)
\end{equation}

\begin{equation}
\leq 2 \left( \Lambda - \lambda_2 \right) \Lambda_{11} u(0) + C,
\end{equation}

where \( U \) is defined in \ref{eq:U}. Because the orthogonal transformation does not change the distance between two points, the second identity of (4.43) holds. One can find the detail for the inequality of (4.43) in \ref{thm:principal}( Theorem 3.5 and Theorem 4.2).
An argument similar to that given in the proof of Theorem 3.1 gives,

\[ A'_2 \leq 2(\Lambda + \frac{1 + \sqrt{8n - 7}}{2} \epsilon_0)D_{11}u(0) + C, \]

when \( \epsilon_0 \leq \frac{\Lambda}{1 + \sqrt{8n - 7}} \). And, if we take \( \epsilon_0 \leq \frac{\Lambda |\beta_{2n}|}{M (1 + \sqrt{8n - 7})} \), we have

\[ A'_3 \leq 2(\Lambda + \frac{M |\beta_{2n}|}{1 + \sqrt{8n - 7}} \epsilon_0)D_{11}u(0) + C, \]

where

\[ M(0) = \frac{\partial^2 P}{\partial t_2 \partial t_1} + D_1(\beta_{2n} - \nu_{2n}), \]

and \( |M(0)| \leq M, M \) is a constant independent of 0.

Thus by inserting (4.43), (4.44) and (4.45) into (4.39) we get

\[ D_\beta D_{11}u(0) \leq C - \gamma_0 D_{11}u(0) - 6\Lambda D_{11}u(0) + 2(\Lambda - \lambda_3(0))D_{11}u(0) + 2(\Lambda + \frac{M |\beta_{2n}|}{1 + \sqrt{8n - 7}} \epsilon_0)D_{11}u(0)
\]

\[ = C - \gamma_0 + 2\lambda_2(0) - (1 + \sqrt{8n - 7})\epsilon_0 + (1 + \sqrt{8n - 7}) M |\beta_{2n}| \epsilon_0 D_{11}u(0). \]

Again, we take \( \epsilon_0 \leq \frac{\gamma_0 + 2\lambda_2(0)}{2(1 + \sqrt{8n - 7})(1 + |\beta_{2n}|)} \), then

\[ \frac{\gamma_0 + 2\lambda_2(z_0)}{2} \geq (1 + \sqrt{8n - 7})\epsilon_0 + (1 + \sqrt{8n - 7}) M |\beta_{2n}| \epsilon_0. \]

Ultimately, we choose

\[ \epsilon_0 \leq \min\left\{ \frac{\Lambda}{1 + \sqrt{8n - 7}}, \frac{\Lambda |\beta_{2n}|}{M (1 + \sqrt{8n - 7})}, \frac{\gamma_0 + 2\lambda_2(z_0)}{2(1 + \sqrt{8n - 7})(1 + |\beta_{2n}|)} \right\}, \]

then

\[ \gamma_0 + 2\lambda_2(0) - (1 + \sqrt{8n - 7})\epsilon_0 + (1 + \sqrt{8n - 7}) M |\beta_{2n}| \epsilon_0 \geq \frac{\sigma_1}{2}, \]

where \( \sigma_1 = \inf_{\partial \Omega} (\gamma_0 + 2\lambda_2) \). Then

\[ 0 \leq D_\beta H(0) \leq \frac{C - \frac{\sigma_1}{2} D_{11}u(0)}{D_{11}u(0)} - \hat{A} D_\beta x_n(0). \]

\[ \left[ \frac{\sigma_1}{2} - \hat{A} \beta \cdot \nu(0) \right] D_{11}u(0) \leq C. \]

Finally we fix \( \hat{A} \) so small that

\[ \hat{A} \beta \cdot \nu \leq \frac{\sigma_1}{4}, \text{ on } \partial \Omega, \]

we have for any tangential vector \( D_{11}u(0) \leq \frac{\sigma_1}{4} \). We finish the proof of Theorem 4.1.

Remark 4.6. We remark that the condition (1.3) is necessary in the above proof. It can not be extended to the degenerate oblique case \( \beta \cdot \nu \geq 0 \).

Remark 4.7. If \( \gamma_0 \) is sufficiently large, then we do not need any structural assumptions on \( \beta \) and the principal curvature of \( \partial \Omega \), for example, the condition (1.11) can be omitted.
**Theorem 4.8.** Let $\Omega$ be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with $C^4$ boundary. Let $\varphi \in C^{3,1}(\partial \Omega \times \mathbb{R})$, and $f \in C^2(\Omega)$, so that they satisfy (1.3)-(1.6). In addition $\varphi$ satisfies (1.10). Then

\[(4.54) \quad |u|_{2,\Omega} \leq C,\]

where $C$ is a constant depending on $|f|_{C^2(\Omega)}$, $\gamma_0$, $\Omega$, $\beta$ and $|\varphi|_{C^{3,1}(\partial \Omega)}$.

5. The Proof of Theorem

In this subsection, we shall prove Theorem 1.1. Although, the argument in this part is rather standard, we present its sketch here for completeness.

With the $C^0$, $C^1$ and $C^2$ bounds for the solution $u$ formulated in previous sections, the complex Monge-Ampère equation (1.1) is uniformly elliptic. Since the second derivatives are bounded, the bounds of their Hölder norms follow from the uniformly elliptic theory developed by Lieberman and Trudinger [6], that is $\|u\|_{C^{2,\alpha}(\Omega)} \leq C$ for some $\alpha \in (0,1)$. Such a priori $C^{2,\alpha}$ estimation enables us to carry out the method of continuity in [5] and [11], thus we obtain the existence of classical solutions. This completes the proof of Theorem 1.1.

**Remark 5.1.** We remark that the proof for the uniqueness of the solutions is similar to that in [7] for the Neumann boundary case, here we need to use the condition $\varphi_u < 0$.

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