A NOTION OF SYNCHRONIZATION OF SYMBOLIC DYNAMICS AND A CLASS OF $C^*$-ALGEBRAS

WOLFGANG KRIEGER AND KENGO MATSUMOTO

Dedicated to the memory of Ki Hang Kim

Abstract. We discuss a synchronization property for subshifts, that we call $\lambda$-synchronization. Under an irreducibility assumption we associate to a $\lambda$-synchronizing subshift a simple and purely infinite $C^*$-algebra.

Keywords: subshift, synchronization, $\lambda$-graph system, $C^*$-algebra, substitution dynamical systems.

AMS Subject Classification: Primary 37B10; Secondary 46L35.

1. Introduction

Let $\Sigma$ be a finite alphabet, and let $S_\Sigma$ be the left shift on $\Sigma^\mathbb{Z}$,

$$S_\Sigma((x_i)_{i \in \mathbb{Z}}) = (x_{i+1})_{i \in \mathbb{Z}}, \quad (x_i)_{i \in \mathbb{Z}} \in \Sigma^\mathbb{Z}.$$ 

The closed shift-invariant subsystems of the shift $(\Sigma^\mathbb{Z}, S_\Sigma)$ are called subshifts. For an introduction to their theory, which belongs to symbolic dynamics, we refer to [11] and [22]. A finite word in the symbols of $\Sigma$ is called admissible for the subshift $X \subset \Sigma^\mathbb{Z}$ if it appears somewhere in a point of $X$. A subshift is uniquely determined by its language of admissible words that we denote by $\mathcal{L}(X)$. We let $\mathcal{L}_n(X)$ denote the set of words in $\mathcal{L}(X)$ of length $n \in \mathbb{N}$.

We set for a subshift $X \subset \Sigma^\mathbb{Z}$

$$\Gamma^+(a) = \{ c \in \mathcal{L}(X) \mid ac \in \mathcal{L}(X) \}, \quad a \in \mathcal{L}(X),$$

and

$$\omega^+_n(a) = \bigcap_{c \in \Gamma^-(a)} \{ b \in \mathcal{L}_n(X) \mid cab \in \mathcal{L}(X) \}, \quad a \in \mathcal{L}(X).$$

$\Gamma^-$ and $\omega^-$ have the time symmetric meaning. An admissible word $v$ of a subshift $X \subset \Sigma^\mathbb{Z}$ is called a synchronizing word of $X$ if for $u, w \in \mathcal{L}(X)$ such that $uv, vw \in \mathcal{L}(X)$ also $uvw \in \mathcal{L}(X)$. A topological transitive subshift is said to be synchronizing if it has a synchronizing word.

In [15] a property (D) of subshifts was introduced that expresses a quality of synchronization that is weaker than synchronization (For other notions of synchronization see [13],[17], [18]). A subshift $X \subset \Sigma^\mathbb{Z}$ has property (D) if for $\sigma \in \Sigma$ and $b \in \Gamma^-(\sigma)$ there exists a word $a \in \Gamma^+(b)$ such that $\sigma \in \omega^+_1(ab)$.

Whereas synchronization of subshifts is time symmetric, property (D) is not. There exist coded systems [1] with property (D) whose inverse does not have property (D) (see Section 4.3). Because of the occurrence of time unsymmetry it is
advisable to choose a time direction that is to be maintained throughout the exposition. In principle the choice of time direction is arbitrary. For this paper we choose the time direction that is the opposite of the time direction that was chosen in [15]. However, we do not change the definition of property (D). Instead, we introduce a notion of "\(\lambda\)-synchronization" that is equivalent to the time symmetric opposite of property (D). Property (D), and therefore also \(\lambda\)-synchronization, is an invariant of topological conjugacy ([15, Proposition 4.3]).

\(\lambda\)-graph systems were introduced in [23]. We will recall their definition in Section 3. There is a one-to-one correspondence between separated one right resolving \(\lambda\)-graph systems and compact Shannon graphs that present a subshift \(X \subset \Sigma^\mathbb{Z}\) (see [18]). For a subshift \(X \subset \Sigma^\mathbb{Z}\) with property (D) there was constructed in [13] a compact Shannon graph \(\mathcal{G}_D(X)\) that presents \(X\), and that is invariantly associated to \(X\), and that generalizes the right Fischer cover [6]. Bypassing the compact Shannon graph \(\mathcal{G}_D(X)\) we give in Section 3 a direct construction of the \(\lambda\)-graph system that corresponds to \(\mathcal{G}_D(X)\) (or rather, due to the different choice of time direction, of its time symmetric opposite, which generalizes the left Fischer cover, the subshift \(X\) now being assumed to be \(\lambda\)-synchronizing). We also give a direct proof that this \(\lambda\)-graph system, that we call the \(\lambda\)-synchronizing \(\lambda\)-graph system, is invariantly associated to the \(\lambda\)-synchronizing subshift.

In Sections 4.1 and 4.2 we give examples of subshifts that are \(\lambda\)-synchronizing and have property (D).

In Section 5 we consider the \(C^*\)-algebras that are obtained from the \(\lambda\)-synchronizing \(\lambda\)-graph systems of \(\lambda\)-synchronizing subshifts.

Acknowledgements. The insight of the referee leads to a substantial improvement of Section 4.2. This work was supported by Grant-in-Aid for Scientific Research (20540215), Japan Society for the Promotion of Science.

2. \(\lambda\)-synchronization

Let \(X \subset \Sigma^\mathbb{Z}\) be a subshift, and let \(l \in \mathbb{N}\). We say that a word \(v \in \mathcal{L}(X)\) is \(l\)-synchronizing if \(\Gamma_+^l(v) \subset \omega_-(v)\). We denote the set of \(l\)-synchronizing words of \(X\) by \(S_l(X)\). We say that \(X\) is \(\lambda\)-synchronizing if for \(w \in \mathcal{L}(X)\) and \(k \in \mathbb{N}\) there is a word \(v \in S_k(X) \cap \Gamma_+^l(w)\).

Lemma 2.1. For a subshift \(X \subset \Sigma^\mathbb{Z}\), the following are equivalent:

(i) \((X, S_{\Sigma}^{-1})\) has property (D).

(ii) \(X\) is \(\lambda\)-synchronizing.

(iii) For \(b \in \mathcal{L}(X)\) there exists an \(a \in \mathcal{L}(X)\) such that \(b \in \omega^-(a)\).

Proof. (i) \(\Rightarrow\) (iii): Assume (i) and let \(b \in \mathcal{L}(X)\). For \(\sigma \in \Gamma_+^l(b)\) there exists by (i) a \(c \in \mathcal{L}(X)\) such that one has for \(a = \sigma c\) that \(b \in \omega^-(a)\).

(iii) \(\Rightarrow\) (i): Assume (iii) and let \(\sigma \in \Sigma\) and \(a \in \Gamma_+^l(\sigma)\). By (iii) there exists a \(b \in \Gamma_+^l(\sigma a)\) such that \(\sigma a \in \omega^-(b)\), and this implies that \(\sigma \in \omega^-(ab)\).

(iii) \(\Rightarrow\) (ii): Assume (iii) and for \(w \in \mathcal{L}(X)\) choose a \(b_0 \in \mathcal{L}(X)\) such that \(w \in \omega^-(b_0)\). Let then \(k \in \mathbb{N}\), set

\[Q = \text{card}(\Gamma_+^l(b_0))\]

and order the set \(\Gamma_+^l(b_0)\), writing

\[\Gamma_+^l(b_0) = \{c_q : 1 \leq q \leq Q\} \quad \text{and} \quad Q = \text{card}(\Gamma_+^l(b_0)).\]
Applying (ii) and [15, Lemma 2.3], one has an inductive procedure that yields an $R \in \mathbb{N}$, and indices $q_r, 1 \leq r \leq R$, such that

$$1 \leq q_{r-1} < q_r \leq Q, \quad 1 < r \leq R,$$

together with words $b_r \in \mathcal{L}(X), 1 \leq r \leq R$, such that

$$c_{q_r} \in \omega_k^-((b_s)_{0 \leq s \leq r}),$$

and

$$q_r = \min\{q > q_{r-1} : c_q \not\in \omega_k^-((b_s)_{0 \leq s \leq r})\}, \quad 1 < r \leq R,$$

Then

$$\{c_{q_r} : 0 \leq r \leq R\} = \Gamma_k^-((b_r)_{0 \leq r \leq R})) = \omega_k^-((b_r)_{0 \leq r \leq R}),$$

and

$$w \in \Gamma^-((b_r)_{0 \leq r \leq R}),$$

and (ii) is shown.

(ii) $\Rightarrow$ (iii): Assume (ii), let $b \in \mathcal{L}(X)$, and let $K$ be the length of $b$. By (ii) there exists an $a \in S_K(X) \cap \Gamma_k^+(b)$ such that $b \in \omega^-(a)$. \hfill \qed

3. $\lambda$-SYNCHRONIZING $\lambda$-GRAPH SYSTEMS

In this section we recall the description of $\lambda$-graph systems and related invariants to define $\lambda$-synchronizing $\lambda$-graph systems.

Notions of $\lambda$-graph system and symbolic matrix system have been introduced in [23]. They are presentations of subshifts and generalizations of finite labeled graphs and symbolic matrices respectively. A $\lambda$-graph system $\Sigma = (V, E, \lambda, \iota)$ over $\Sigma$ consists of a vertex set $V = V_0 \cup V_1 \cup V_2 \cup \cdots$, an edge set $E = E_{0,1} \cup E_{1,2} \cup E_{2,3} \cup \cdots$, a labeling $\lambda : E \to \Sigma$ and a surjective map $\iota_{l,t+1} : V_{l+1} \to V_l$ for each $l \in \mathbb{Z}_+$. An edge $e \in E_{l,t+1}$ has its source vertex $s(e)$ in $V_l$, its terminal vertex $t(e)$ in $V_{l+1}$ and its label $\lambda(e)$ in $\Sigma$. It is then required that there exists an edge in $E_{l,t+1}$ with label $\alpha$ and its terminal is $v \in V_{l+1}$ if and only if there exists an edge in $E_{l-1,t}$ with label $\alpha$ and its terminal is $\iota(v) \in V_l$. For $u \in V_{l-1}$ and $v \in V_{l+1}$, we put

$$E'(u,v) = \{e \in E_{l,t+1} \mid t(e) = v, \iota(s(e)) = u\},$$

$$E_l(u,v) = \{e \in E_{l-1,t} \mid s(e) = u, t(e) = \iota(v)\}.$$

Then there exists a bijective correspondence between $E'(u,v)$ and $E_l(u,v)$ that preserves labels for all pairs $(u,v) \in V_{l-1} \times V_{l+1}$ of vertices. This property is called the local property of the $\lambda$-graph system.

A symbolic matrix system $(M, I)$ consists of a sequence of pairs $(M_{l,t+2}, I_{l,t+1})$, $l \in \mathbb{Z}_+$, of rectangular symbolic matrices $M_{l,t+1}$ and rectangular $\{0,1\}$-matrices $I_{l,t+1}$, where $\mathbb{Z}_+$ denotes the set of all nonnegative integers. Both the matrices $M_{l,t+1}$ and $I_{l,t+1}$ have the same size for each $l \in \mathbb{Z}_+$. The column size of $M_{l,t+1}$ is the same as the row size of $M_{l,t+1,l+2}$. They satisfy the following commutation relations as symbolic matrices

$$I_{l,t+1}M_{l+1,t+2} = M_{l,t+1}I_{l+1,t+2}, \quad l \in \mathbb{Z}_+. \quad (3.1)$$

We further assume that for $i$ there exists $j$ such that the $(i,j)$-component $I_{l,t+1}(i,j) = 1$, and for $j$ there uniquely exists $i$ such that $I_{l,t+1}(i,j) = 1$.

For a symbolic matrix system $(M, I)$, the labeled edges from a vertex $v^l_i \in V_l$ to a vertex $v^l_{j+1} \in V_{l+1}$ are given by the symbols appearing in the $(i,j)$-component $M_{l,t+1}(i,j)$ of $M_{l,t+1}$. The matrix $I_{l,t+1}$ defines a surjection $\iota_{l,t+1}$ from $V_{l+1}$ to $V_l$. 

for each $l \in \mathbb{Z}_+$. By this observation, the symbolic matrix systems and the $\lambda$-graph systems are the same objects. We say that a $\lambda$-graph system $\mathcal{L}$ presents a subshift $X$ if the set $\mathcal{L}(X)$ of admissible words of $X$ coincides with the set of finite label sequences appearing in the labeled Bratteli diagram for $\mathcal{L}$.

For a symbolic matrix system $(M, I)$, let $M_{l,l+1}$ be the nonnegative rectangular matrix obtained from $M_{l,l+1}$ by setting all the symbols equal to 1 for each $l \in \mathbb{Z}_+$. Then the resulting pair $(M, I)$ satisfies the following relations by (3.1)

$$I_{l,l+1}M_{l+1,l+2} = M_{l,l+1}I_{l+1,l+2}, \quad l \in \mathbb{Z}_+.$$  \hspace{1cm} (3.2)

We call $(M, I)$ the nonnegative matrix system for $(M, I)$.

For topological Markov shifts and sofic shifts, several topological conjugacy invariants and flow equivalence invariants, such as dimension groups ([12], [7]) and Bowen-Franks groups ([2], [7]) have been defined by using underlying matrices (cf. [3], [32], [11], [22]). These invariants have been generalized to nonnegative matrix systems in [23]. For a nonnegative matrix system $(M, I)$, let $m(l)$ be the row size of the matrix $I_{l,l+1}$ for each $l \in \mathbb{Z}_+$. Let $Z_I^l$ be the abelian group defined by the inductive limit $Z_I^l = \lim_l \{I_{l+1,l}^1 : \mathbb{Z}^{m(l)} \to \mathbb{Z}^{m(l+1)}\}$. The sequence $M_{l,l+1}^t, l \in \mathbb{Z}_+$ of the transposes of $M_{l,l+1}$ naturally acts on $Z_I^l$ by the relation (3.2), that is denoted by $\lambda_{(M, I)}$. The K-groups for $(M, I)$ have been defined as:

$$K_0(M, I) = Z_I^1/(\text{id} - \lambda_{(M, I)})Z_I^1, \quad K_1(M, I) = \text{Ker}(\text{id} - \lambda_{(M, I)}) \text{ in } Z_I^1.$$  

Set the inductive limits 

$$Z_I^\pm = \lim_l \{I_{l+1}^1 : \mathbb{Z}_{+}^{m(l)} \to \mathbb{Z}_{+}^{m(l+1)}\} \text{ of positive cones. We put } Z_I^+(k) = Z_I^l, k \in \mathbb{N} \text{ and consider the inductive limits:}$$

$$\Delta_{(M, I)} = \lim_k \{\lambda_{(M, I)} : Z_I^+(k) \to Z_I^+(k+1)\},$$

$$\Delta_+^{(M, I)} = \lim_k \{\lambda_{(M, I)} : Z_I^+(k) \to Z_I^+(k+1)\}.$$  

The ordered group $(\Delta_{(M, I)}, \Delta^{+(M, I)})$ is called the dimension group for $(M, I)$. The map $\delta_{(M, I)} : Z_I^+(k) \to Z_I^l(k+1)$ defined by $\delta_{(M, I)}([X, k]) = ([X, k+1])$ yields an automorphism on $(\Delta_{(M, I)}, \Delta^{+(M, I)})$. The triple $(\Delta_{(M, I)}, \Delta^{+(M, I)}, \delta_{(M, I)})$ is named the dimension triple for $(M, I)$. We set the projective limit of the abelian group as 

$$Z_I = \lim_l \{I_{l,l+1} : \mathbb{Z}^{m(l+1)} \to \mathbb{Z}^{m(l)}\}.$$  

The sequence $M_{l,l+1}, l \in \mathbb{Z}_+$ acts on $Z_I$ as an endomorphism that we denote by $M$. The identity on $Z_I$ is denoted by $I$. The Bowen-Franks groups have been formulated as:

$$BF^0(M, I) = Z_I/(I - M)Z_I, \quad BF^1(M, I) = \text{Ker}(I - M) \text{ in } Z_I.$$  

Both the pairs $K_* (M, I), BF_* (M, I)$ for $* = 0, 1$ are invariant under shift equivalence in nonnegative matrix systems ([23]).

In [26], the $C^*$-algebra $O_{\mathcal{L}}$ associated with a $\lambda$-graph system $\mathcal{L}$ has been introduced. These $C^*$-algebras are generalizations of the Cuntz-Krieger algebras and the $C^*$-algebras associated with subshifts. They are universal unique concrete $C^*$-algebras generated by finite families of partial isometries and sequences of projections subject to certain operator relations encoded by structure of the $\lambda$-graph systems. The $C^*$-algebra $O_{\mathcal{L}}$ has a natural one-parameter group action called gauge action. Its fixed point algebra denoted by $\mathcal{F}_{\mathcal{L}}$ becomes an AF-algebra. Let $(M, I)$
be the nonnegative matrix system for the symbolic matrix system of the \(\lambda\)-graph system \(\mathcal{L}\). The following relations hold:

\[
K_0(\mathcal{O}_\Sigma) = K_0(M, I), \quad K_1(\mathcal{O}_\Sigma) = K_1(M, I), \quad (3.3)
\]

\[
\text{Ext}^1(\mathcal{O}_\Sigma) = BF^0(M, I), \quad \text{Ext}^0(\mathcal{O}_\Sigma) = BF^1(M, I), \quad (3.4)
\]

\[
(K_0(\mathcal{F}_\Sigma), K_0(\mathcal{F}_\Sigma)^+) = (\Delta_{(M, I)}^-, \Delta_{(M, I)}^+) \quad (3.5)
\]

where \(\text{Ext}^1(\mathcal{O}_\Sigma) = \text{Ext}(\mathcal{O}_\Sigma)\) and \(\text{Ext}^0(\mathcal{O}_\Sigma) = \text{Ext}(\mathcal{O}_\Sigma \otimes C_0(\mathbb{R}))\).

All the groups above are invariant under shift equivalence of symbolic matrix systems, so that they yield topological conjugacy invariants of subshifts by taking canonical \(\lambda\)-graph systems.

The \(\lambda\)-entropy \(h_\lambda(\mathcal{L})\) of a \(\lambda\)-graph system \(\mathcal{L}\) was introduced in [20]. The \(\lambda\)-entropy measures the growth rate of the cardinalities \(|V_i|\), \(i \in \mathbb{Z}_+\) of the vertex sets \(\{V_i\}_{i \in \mathbb{Z}_+}\). The volume entropy \(h_{\text{vol}}(\mathcal{L})\) of a \(\lambda\)-graph system \(\mathcal{L}\) was introduced in [29]. Denote by \(P(\mathcal{L})\) the set of all labeled paths starting at a vertex in \(V_0\) and terminating at a vertex in \(V_l\). The volume entropy measures the growth rate of the cardinalities \(|P(\mathcal{L})|\) of the labeled paths \(P(\mathcal{L})\). Both the entropic quantities are invariant under shift equivalence of \(\lambda\)-graph systems, so that they yield a topological conjugacy invariants of subshifts.

For \(\mu, \nu \in \mathcal{L}(X)\), if \(\Gamma^{-}_l(\mu) = \Gamma^{-}_l(\nu)\), we say that \(\mu\) is \(l\)-past equivalent to \(\nu\) and write it as \(\mu \sim_l \nu\).

**Lemma 3.1.** Let \(X\) be a \(\lambda\)-synchronizing subshift. Then we have

(i) For \(l \in \mathbb{N}\) and \(\eta \in \mathcal{L}_0(X)\), there exists \(\mu \in \mathcal{S}_l(X)\) such that \(\eta = \Gamma^{-}_l(\mu)\).

(ii) For \(\mu \in \mathcal{S}_l(X)\), there exists \(\mu' \in \mathcal{S}_{l+1}(X)\) such that \(\mu \sim_l \mu'\).

(iii) For \(\mu \in \mathcal{S}_l(X)\), there exist \(\beta \in \Sigma\) and \(\nu \in \mathcal{S}_{l+1}(X)\) such that \(\mu \sim_l \beta \nu\).

**Proof.** (i) This follows from \(\lambda\)-synchronization.

(ii) For \(\mu \in \mathcal{S}_l(X)\) with \(|\mu| = K\), put \(k = K + l + 1 > K\). As \(X\) is \(\lambda\)-synchronizing, there exists \(\nu \in \mathcal{S}_k(X)\) such that \(\mu \nu \in \mathcal{S}_{k-l}(X)\). Put \(\mu' = \mu \nu \in \mathcal{S}_{l+1}(X)\). As \(\mu \in \mathcal{S}_l(X)\), one sees that \(\Gamma^{-}_l(\mu) = \Gamma^{-}_l(\mu \nu)\) so that \(\mu \sim_l \mu'\).

(iii) For \(\mu \in \mathcal{S}_l(X)\) with \(\mu = \mu_1 \cdots \mu_K\), put \(k = K + l > K\). As \(X\) is \(\lambda\)-synchronizing, there exists \(\omega \in \mathcal{S}_k(X)\) such that \(\mu \omega \in \mathcal{S}_{K-l}(X)\). Set \(\beta = \mu_1\) and \(\nu = \mu_2 \cdots \mu_K \omega\). Since \(\omega \in \mathcal{S}_k(X)\), one has \(\nu \in \mathcal{S}_{K-l}(X)\) so that \(\nu \in \mathcal{S}_{l+1}(X)\). As \(\Gamma^{-}_l(\mu) = \Gamma^{-}_l(\mu \omega)\), one sees that \(\mu \sim_l \beta \nu\). \(\square\)

For a \(\lambda\)-synchronizing subshift \(X\) over \(\Sigma\), we will introduce the \(\lambda\)-synchronizing \(\lambda\)-graph system

\[
\mathcal{L}^{(X)} = (V^{(X)}, E^{(X)}, \lambda^{(X)}, \iota^{(X)})
\]

in the following way. Let \(V^{(X)}\) be the \(l\)-past equivalence classes of \(\mathcal{S}_l(X)\). We denote by \([\mu]_l\) the \(l\)-past equivalence class of \(\mu \in \mathcal{S}_l(X)\). For \(\nu \in \mathcal{S}_{l+1}(X)\) and \(\alpha \in \Gamma^{-}_l(\nu)\), define an edge with label \(\alpha\) from \([\alpha \nu]_l \in V^{(X)}_l\) to \([\nu]_{l+1} \in V^{(X)}_{l+1}\). We denote the set of these edges by \(E^{(X)}\). Since \(\mathcal{S}_l(X) \subset \mathcal{S}_l(X)\), we have a natural map \([\mu]_{l+1} \in V^{(X)}_{l+1} \to [\mu]_l \in V^{(X)}_l\) that we denote by \(\iota^{(X)}\).

**Proposition 3.2.** \(\mathcal{L}^{(X)} = (V^{(X)}, E^{(X)}, \lambda^{(X)}, \iota^{(X)})\) is a \(\lambda\)-graph system that presents \(X\).
\textbf{Proof.} We will first show the local property of $\lambda$-graph systems (cf. [26], [28]).

For $[\mu] \in V^\lambda_1(X)$ and $[\nu] \in V^\lambda_2(X)$ with $\mu \in S_1(X), \nu \in S_2(X)$, suppose that there exists a labeled edge from $[\mu]_l$ to $[\nu]_{l+1}$ labeled $\alpha \in \Sigma$. It follows that $\alpha \nu \sim \lambda_\mu$. Hence there exists an edge from $[\alpha \nu]_{l+1}$ to $[\nu]_{l+2}$ labeled $\alpha$ and a $\iota$-map from $[\alpha \nu]_{l+1}$ to $[\nu]_{l+1}$ labeled $\alpha$. Therefore the local property of $\lambda$-graph systems holds.

By $\lambda$-synchronization, for any admissible word $\eta \in \mathcal{L}(X)$, there exists $\varphi \in S_1(X)$ such that $\eta \varphi \in S_k(X)$. This implies that there exists a path labeled $\eta$ in $\mathcal{L}^\lambda(X)$ from the vertex $[\nu]_{k-1} \in V^\lambda_{k-1}(X)$ to the vertex $[\nu] \in V^\lambda_1(X)$ so that $\mathcal{L}^\lambda(X)$ is a $\lambda$-graph system that presents $X$.

As in [23], there is a canonical construction of a $\lambda$-graph system for an arbitrary subshift $X$. The constructed $\lambda$-graph system is called the canonical $\lambda$-graph system for $X$ and denoted by $\mathcal{L}^\lambda(X)$. For $l \in \mathbb{Z}_+$, the vertex set $V^\lambda_l(X)$ of $\mathcal{L}^\lambda(X)$ is defined by the $l$-past equivalence classes $\Gamma^{-}_l(x)$ of the right infinite sequences $x \in X^\ast$. For a symbol $\alpha \in \Sigma$, an edge labeled $\alpha$ from $\Gamma^{-}_l(\alpha x)$ to $\Gamma^{-}_{l+1}(x)$ is defined if $\alpha x \in X^\ast$. The natural inclusions $\Gamma^{-}_{l+1}(x) \subset \Gamma^{-}_{l}(x)$ give rise to the $\iota$-map.

\textbf{Corollary 3.3.} $\mathcal{L}^\lambda(X)$ is a predecessor-separated, left-resolving $\lambda$-graph subsystem of the canonical $\lambda$-graph system of $X$.

\textbf{Proof.} One checks that $\mathcal{L}^\lambda(X)$ is predecessor-separated and left-resolving. Let $\mathcal{L}^\lambda(X) = (V^\lambda, E^\lambda, \lambda^\lambda, \iota^\lambda)$ be the canonical $\lambda$-graph system for $X$. For $\mu \in S_1(X)$ and $x \in X^\ast$ with $\mu \in \Gamma^{-}_l(x)$, one sees that $\Gamma^{-}_l(\mu) = \Gamma^{-}_l(\mu x)$. Hence $[\mu]_l$ can be regarded as a vertex of $V^\lambda_l(X)$ so that the vertex set $V^\lambda_l(X)$ can be regarded as a subset of $V^\lambda_l(X)$. Similarly, the edge set $E^\lambda_{l,l+1}$ can be regarded as a subset of $E^\lambda_{l,l+1}$. The $\iota$-map $\iota^\lambda$ of $\mathcal{L}^\lambda(X)$ is compatible to that of $\mathcal{L}^\lambda(X)$. It follows that $\mathcal{L}^\lambda(X)$ is a $\lambda$-graph subsystem of $\mathcal{L}^\lambda(X)$.

For a synchronizing subshift $X$, the canonical synchronizing $\lambda$-graph system $\mathcal{L}^S(X)$ of $X$ has been introduced in [18].

\textbf{Proposition 3.4.} Let $X$ be a synchronizing subshift and $\mathcal{L}^S(X)$ the canonical synchronizing $\lambda$-graph system of $X$. Then $\mathcal{L}^S(X)$ is isomorphic to the $\lambda$-synchronizing $\lambda$-graph system $\mathcal{L}^\lambda(X)$ of $X$.

For a $\lambda$-synchronizing subshift $X$, denote by $(\mathcal{M}^\lambda(X), I^\lambda(X))$ the symbolic matrix system for the $\lambda$-graph system $\mathcal{L}^\lambda(X)$ (see [23]).

\textbf{Proposition 3.5.} Let $X, X'$ be $\lambda$-synchronizing subshifts. If $X$ is topologically conjugate to $X'$, then $(\mathcal{M}^\lambda(X), I^\lambda(X))$ is strong shift equivalent to $(\mathcal{M}^\lambda(X'), I^{\lambda}(X'))$.

\textbf{Proof.} By [32], we may assume that the subshifts $X, X'$ are bipartitely related to each other. This means that there exists alphabets $C, D$ and a bipartite subshift $\tilde{X}$ over $C \cup D$ such that $\tilde{X} = C \cup D$ and one block conjugacies $\varphi_1 : X \to X_1$, $\varphi_2 : X' \to X_2$ such that

$$\tilde{X}^{[2]} = X_1 \cup X_2,$$
where \( \hat{X}^{[2]} \) is the 2-block shift of \( X \). We will show that the \( \lambda \)-graph systems \( \mathcal{G}^{\lambda(X)} \) and \( \mathcal{G}^{\lambda(X')} \) are bipartitely related to each other. We note that

\[
S_{2k}(\hat{X}) = \varphi_1(S_k(X)) \cup \varphi_2(S_k(X')).
\]  

(3.6)

For \( \eta = (\eta_1, \ldots, \eta_l) \in L_1(\hat{X}) \), assume that \( \eta_1 \in C \).

Case 1: \( \eta_l \in D \).

Hence \( L \) is odd, \( l = 2m - 1 \) for some \( m \geq 1 \). Take \( d \in D \) such that \( \eta_1 \cdots \eta_l d \in L_{l+1}(\hat{X}) = L_{2m}(\hat{X}) \) so that \( \varphi_1^{-1}(\eta_1 \cdots \eta_l d) \in L_m(X) \). For \( k \geq l \), take \( k' \) such as \( k = 2k' - 1 \) if \( k \) is odd, and \( k = 2k' \) if \( k \) is even. Hence \( k' + 1 \geq m \). Since \( X \) is \( \lambda \)-synchronizing, there exists \( \nu \in L_{k'+1}(X) \) such that \( \varphi_1^{-1}(\eta_1 \cdots \eta_l d) \nu \in S_{k'+1-m}(X). \)

As \( 2(k' + 1 - m) = 2k' - 2(m - 1) \geq k - l \), one has \( \eta_1 \cdots \eta_l d \varphi_1(\nu) \in S_{k-l}(\hat{X}). \)

Case 2: \( \eta_l \in D \).

Hence \( L \) is even, \( l = 2m \) for some \( m \geq 1 \). One sees \( \varphi_1^{-1}(\eta_1 \cdots \eta_l) \in L_m(X) \). For \( k \geq l \), take \( k' \) such as \( k = 2k' - 1 \) if \( k \) is odd, and \( k = 2k' \) if \( k \) is even. Hence \( k' \geq m \). Since \( X \) is \( \lambda \)-synchronizing, there exists \( \omega \in L_{k'}(X) \) such that \( \varphi_1^{-1}(\eta_1 \cdots \eta_l d) \omega \in S_{k'-m}(X). \)

As \( 2k' - 2m \geq k - l \), one has \( \eta_1 \cdots \eta_l d \varphi_1(\omega) \in S_{k-l}(\hat{X}). \)

Thus the bipartite subshift \( \hat{X} \) is \( \lambda \)-synchronizing. The equality (3.6) implies

\[
V_{2k}^{\lambda(X)} = V_k^{\lambda(X)} \cup V_k^{\lambda(X')}.
\]

One then easily sees that the \( \lambda \)-graph systems \( \mathcal{G}^{\lambda(X)} \) and \( \mathcal{G}^{\lambda(X')} \) are bipartitely pair in the sense of [32]. Hence \( (\mathcal{M}^{\lambda(X)}, I^{\lambda(X)}) \) is strong shift equivalent to \( (\mathcal{M}^{\lambda(X')}, I^{\lambda(X')}) \) by [23].

For the symbolic matrix system \( (\mathcal{M}^{\lambda(X)}, I^{\lambda(X)}) \), denote by \( (M^{\lambda(X)}, I^{\lambda(X)}) \), its nonnegative matrix system. Set

\[
K^\lambda(X) = K_i(M^{\lambda(X)}, I^{\lambda(X)}), \quad i = 0, 1
\]

(3.7)

\[
BF^i(X) = BF_i(M^{\lambda(X)}, I^{\lambda(X)}), \quad i = 0, 1
\]

(3.8)

\[
(\Delta^\lambda(X), \Delta^+_\lambda(X)) = (\Delta_{(M^{\lambda(X)}, I^{\lambda(X)})}, \Delta^+_{(M^{\lambda(X)}, I^{\lambda(X)})}),
\]

(3.9)

\[
h_{\lambda}(X) = h_{\lambda}(\mathcal{G}^{\lambda(X)}),
\]

(3.10)

\[
h_{\text{vol}}(X) = h_{\text{vol}}(\mathcal{G}^{\lambda(X)}).
\]

(3.11)

Since the above invariants are all shift equivalence invariants and since strong shift equivalence implies shift equivalence, we have

**Corollary 3.6.** The abelian groups \( K^\lambda(X), BF^i(X), i = 0, 1 \), the ordered abelian group \( \Delta^\lambda(X) \) and the entropic quantities \( h_{\lambda}(X), h_{\text{vol}}(X) \) are all invariant under topological conjugacy of \( \lambda \)-synchronizing subshifts.

4. **Examples**

1. **Dyck shifts and Motzkin shifts.**

Starting from the Dyck shifts and the Motzkin shifts, we describe, in increasing generality, classes of subshifts that have been observed to have property (D). These subshifts are also \( \lambda \)-synchronizing, since these classes are closed under taking inverses. The descriptions involve finite directed graphs. The mapping that assigns to a path in a directed graph its source vertex we denote by \( s \) and the mapping that assigns to a path in the graph its target vertex we denote by \( t \).
First we recall the construction of the Dyck shifts and of the Motzkin shifts. We denote the generators of the Dyck inverse monoid (the polycyclic inverse monoid) by $e_n^-, e_n^+, 1 \leq n \leq N, N > 1$. These generators satisfy the relations

$$e_n^- e_n^+ = 1, \quad 1 \leq n \leq N, \quad e_l^- e_m^+ = 0, \quad 1 \leq l, m \leq N, \ l \neq m.$$  

The Dyck shift $D_N$ is the subshift with alphabet $\{e_n^-, e_n^+: 1 \leq n \leq N\}$ and admissible words $(e_i)_{1 \leq i \leq l}, I \in \mathbb{N}$, given by the condition $\prod_{1 \leq i \leq l} e_i \neq 0$. The Motzkin shift $M_N$ is the subshift with alphabet $\{e_n^-, e_n^+: 1 \leq n \leq N\} \cup \{1\}$ and admissible words $(e_i)_{1 \leq i \leq l}, I \in \mathbb{N}$, also given by the condition $\prod_{1 \leq i \leq l} e_i \neq 0$.

The Dyck shifts belong to the class of Markov-Dyck shifts and the Motzkin shifts belong to the class of Markov-Motzkin shifts. To recall the construction of the Markov-Dyck shifts and of the Markov-Motzkin shifts, let there be given an irreducible finite directed graph with vertex set $V$ and edge set $E$. Let $(V, E^-)$ be a copy of $(V, E)$. Reverse the directions of the edges in $E$ to obtain the reversed graph of the graph $(V, E)$ with vertex set $V$ and edge set $E^+$. With idempotents $P_v, v \in V$, the set $E^- \cup \{P_v: v \in V\} \cup E^+$ is the generating set of the graph inverse semigroup $S_{V, E}$ of the directed graph $(V, E)$, where, besides $P_v^2 = P_v, v \in V$, the relations are

$$P_u P_w = 0, \quad u, w \in V, \ u \neq w,$$

$$f^+ g^+ = \begin{cases} P_{s(f)}, & (f = g), \\ 0, & (f \neq g, f, g \in E), \end{cases}$$

and

$$P_{s(f)} f^- = f^- P_{r(f)}, \quad P_{r(f)} f^+ = f^+ P_{s(f)}, \quad f \in E.$$  

The Markov-Dyck shift of the graph $(V, E)$ is the subshift with alphabet $E^- \cup E^+$ with admissible words $(e_i)_{1 \leq i \leq l}, I \in \mathbb{N}$, given by the condition $\prod_{1 \leq i \leq l} e_i \neq 0$. The Dyck shift $D_N$ arises in this way from the single vertex graph with $N$ loops at its vertex. The Markov-Motzkin shift of the graph $(V, E)$ is the subshift with alphabet $E^- \cup \{P_v: v \in V\} \cup E^+$ with admissible words $(e_i)_{1 \leq i \leq l}, I \in \mathbb{N}$, also given by the condition $\prod_{1 \leq i \leq l} e_i \neq 0$. The Motzkin shift $M_N$ arises in this way from the single vertex graph with $N$ loops at its vertex.

Following [8], in [9] a necessary and sufficient condition was given for the existence of an embedding of an irreducible subshift of finite type into target subshifts that were taken from a class of $\lambda$-synchronizing subshifts with property (D). This class contains the Markov-Dyck and Markov-Motzkin shifts. To recall the construction of this class, let there be given, besides the finite irreducible directed graph $(V, E)$, another finite irreducible directed graph with vertex set $\Omega$ and edge set $\Sigma$. Denote by $S_{V, E}$ (resp. $S_{V, E}^\dagger$) the semigroup that is generated by $\{e^- : e \in E\}$ (resp. $\{e^- : e \in E\}$) and let $\lambda$ be a labeling map that assigns to every edge $\sigma \in \Sigma$ a label $\lambda(\sigma) \in S_{V, E}^\dagger \cup \{P_v : v \in V\} \cup S_{V, E}^\dagger$, and extend the mapping $\lambda$ to all finite paths $(\sigma_i)_{1 \leq i \leq l}$ in the graph $(\Omega, \Sigma)$ by

$$\lambda((\sigma_i)_{1 \leq i \leq l}) = \prod_{1 \leq i \leq l} \lambda(\sigma_i).$$

For $v \in V$ let $\Omega_v$ denote the set of $\omega \in \Omega$ such that there exists a path $a$ in the graph $(\Omega, \Sigma)$ such that $s(a) = t(a) = \omega$ and $\lambda(\omega) = P_v$. Assume that $\Omega_v \neq \emptyset, v \in V$, and that $\{\Omega_v : v \in V\}$ is a partition of $\Omega$. Also assume that for every edge that enters
an \( \omega \in \Omega \), one has \( \lambda(\omega)P_{\omega} \neq 0 \) and for every edge \( \sigma \) that leaves an \( \omega \in \Omega \), one has \( P_{\sigma}\lambda(\omega) \neq 0, \sigma \in \mathcal{V} \). Assume that for \( u, w \in \mathcal{V} \) and \( g \in \mathcal{S}_{\mathcal{V}, \mathcal{E}} \) such that \( P_{u}gP_{w} \neq 0 \) there exists a path \( a \) in the graph such that \( s(a) = u, t(a) = w \) and \( \lambda(a) = g \).

We define a subshift \( X(\mathcal{V}, \mathcal{E}, \lambda) \) as the subsystem of the edge shift of \( (\mathcal{V}, \mathcal{E}) \) with allowed words the finite paths \( b \) in \( G \) such that \( \lambda(b) \neq 0 \). The class of subshifts of the form \( X(\mathcal{V}, \mathcal{E}, \lambda) \) is closed under taking inverses and one checks that the subshifts \( X(\mathcal{V}, \mathcal{E}, \lambda) \) have Property (D). In fact, they have stronger synchronization properties as described in [17]. The \( \lambda \)-synchronizing \( \lambda \)-graph system of a subshift \( X(\mathcal{V}, \mathcal{E}, \lambda) \) is given by its Cantor horizon as described for the Dyck shift in [19] and for the Motzkin shift in [27]. While maintaining synchronization properties one has a more general construction that goes beyond the class of subshifts of the form \( X(\mathcal{V}, \mathcal{E}, \lambda) \), where the graph inverse semigroup is replaced by a more general type of semigroup [14].

2. Substitution dynamical systems.

For the theory of substitution dynamical systems see [33].

In proving the next theorem we follow [4, Example 3.6].

**Theorem 4.1.** The substitution dynamical system of a primitive substitution is \( \lambda \)-synchronizing.

**Proof.** For a substitution minimal system of a primitive substitution \( X \subset \Sigma^{\mathbb{Z}} \) and for \( k \in \mathbb{N} \), let \( A^{+}(k) \) denote the set of \( x^{+} \in X_{[0, \infty)} \) such that \( \text{card}(\Gamma_{k}^{-}(x^{+})) > 1 \). \( A^{+}(1) \) is a finite set [33, Section 5.1.1] and therefore the sets \( A^{+}(k), k \in \mathbb{N} \) are also finite. We can for \( b \in \mathcal{L}_{k}(X) \) choose a \( y^{+} \in X_{[0, \infty)} \setminus A^{+}(k) \) such that \( b \in \Gamma_{k}^{-}(y^{+}) \). There is an \( n \in \mathbb{N} \) such that for \( a = y_{[0, n]}^{+} \), \( \text{card}(\Gamma_{k}^{-}(a)) = 1 \) and therefore \( \{b\} = \omega_{k}^{-}(a) \). Otherwise \( \Gamma_{k}^{-}(y^{+}) \), which is the limit of the decreasing sequence \( \Gamma_{k}^{-}(y_{[0, n]}^{+}), n \in \mathbb{N} \), would contain more than one word, contradicting \( y^{+} \notin A^{+}(k) \).

We have proved that \( X \) satisfies condition (iii) of Lemma 2.1. \( \square \)

Note that we have also proved that the invariant probability measure of the substitution dynamical system of a primitive substitution is a \( g \)-measure in the sense of [21]. (We have used the time direction that is opposite to the one in [31], or in [16] or [21]. However, the situation is time symmetric.)

3. A coded system.

With the alphabet \( \Sigma = \{0, 1, \alpha, \beta, \gamma\} \) we consider a subshift \( X \subset \Sigma^{\mathbb{Z}} \) that has property (D) such that its inverse does not have property (D). We obtain \( X \) as the closure of the union of an increasing sequence \( Y_{n}, n \in \mathbb{N} \) of irreducible subshifts of finite type. This implies that \( X \) is a coded system [13, Theorem 1]. \( Y_{n}, n \in \mathbb{N} \) is defined by excluding from \( \Sigma^{\mathbb{Z}} \) the words

\[ 0^{m}, \ m > n, \]

and the words

\[ \beta\alpha c 0^{k} \gamma, \quad c \in \Sigma^{k}, \ 1 \leq k \leq n, \]

as well as the words

\[ \beta\beta, \ \beta\gamma, \ \beta0, \ \beta1. \]

We prove that \( X \) has property (D). For this let \( a \sigma \in \mathcal{L}(X) \). If \( \sigma \neq \gamma \), then, with \( K \) the length of \( a, \sigma \in \omega^{+}(1^{K}a) \). If \( \sigma = \gamma \), then \( \sigma \in \omega^{+}(aa) \).
Let \( a \in \Gamma^+(\beta a) \), and let \( K \) be the length of \( a \). Then \( \gamma 0^K \gamma \in \Gamma^+(\alpha a) \), but \( \beta a \gamma 0^K \gamma \) is not admissible for \( X \), and therefore \( X \) does not satisfy condition (iii) of Lemma 2.1.

5. \( C^* \)-algebras

Generalizing Condition (I) of [5], in [28] \( \lambda \)-condition (I) was introduced, which says that for a vertex \( v \) in the \( \lambda \)-graph system, there exist two distinct paths \( \pi_1, \pi_2 \) starting at \( v \) such that they have the same terminal vertex but different labels.

For a \( \lambda \)-synchronizing subshift \( X \), we say that \( X \) satisfies \textit{synchronizing condition (I)} if for \( l \in \mathbb{N} \) and \( \mu \in \mathcal{S}(X) \), there exist \( \gamma_1, \gamma_2 \in \mathcal{L}_K(X) \) for some \( K \) and \( \nu \in \mathcal{S}_{l+K}(X) \) such that

\[
\gamma_1 \neq \gamma_2, \quad \gamma_1, \gamma_2 \in \Gamma_K^-(\nu), \quad [\gamma_1 \nu]_l = [\gamma_2 \nu]_l = [\mu]_l. \quad (5.1)
\]

We have the following lemma.

**Lemma 5.1.** Let \( X \) be a \( \lambda \)-synchronizing subshift. Then the following conditions are equivalent:

(i) \( X \) satisfies synchronizing condition (I).

(ii) The \( \lambda \)-synchronizing \( \lambda \)-graph system \( \Sigma^{\lambda}(X) \) satisfies \( \lambda \)-condition (I).

**Proof.** (i)\( \Rightarrow \) (ii): Suppose that \( X \) satisfies synchronizing condition (I). For a vertex \( v \in V_i^{\lambda}(X) \) in \( \Sigma^{\lambda}(X) \), take a \( l \)-synchronizing word \( \mu \in \mathcal{S}(X) \) such that \( v = [\mu]_l \). By the synchronizing condition (I) of \( X \), there exist \( \gamma_1, \gamma_2 \in \mathcal{L}_K(X) \) for some \( K \) and \( \nu \in \mathcal{S}_{l+K}(X) \) satisfying (5.1). This implies that there exist two paths beginning with \( v \) and ending in the vertex \([\nu]_{l+K} \in V_{l+K}^{\lambda}(X)\) whose labels are \( \gamma_1, \gamma_2 \). Hence \( \Sigma^{\lambda}(X) \) satisfies \( \lambda \)-condition (I).

(ii)\( \Rightarrow \) (i): Suppose that \( \Sigma^{\lambda}(X) \) satisfies \( \lambda \)-condition (I). Let \( \mu \in \mathcal{S}(X) \) be a \( l \)-synchronizing word of \( X \). By the \( \lambda \)-condition (I), for the vertex \([\mu]_l \in V_i^{\lambda}(X)\), there exist two distinct paths \( \pi_1, \pi_2 \) in \( \Sigma^{\lambda}(X) \) starting at \([\mu]_l\) such that they have the same terminal vertex but different labels. We denote by \( u \) the terminal vertex. As \( u \) belongs to \( V_{l+K}^{\lambda}(X) \) for some \( K \in \mathbb{N} \), one may find a \( l+K \)-synchronizing word \( \nu \in \mathcal{S}_{l+K}(X) \) such that \([\nu]_{l+K} = u\). Denote by \( \gamma_1, \gamma_2 \) the labels of \( \pi_1, \pi_2 \) respectively. Since \( \pi_1, \pi_2 \) begin with \([\mu]_l\) and end in \( u \), one sees that \( \gamma_1, \gamma_2 \in \Gamma_K^-(\nu) \) and \([\gamma_1 \nu]_l = [\gamma_2 \nu]_l = [\mu]_l \). As \( \gamma_1 \neq \gamma_2 \), one sees that \( X \) satisfies synchronizing condition (I). \( \square \)

If \( \lambda \)-graph systems \( \Sigma \) and \( \Sigma' \) are bipartitely related by a bipartite \( \lambda \)-graph system \( \hat{\Sigma} \), and if one of the \( \lambda \)-graph systems \( \Sigma, \Sigma', \hat{\Sigma} \) satisfies \( \lambda \)-condition (I), then so do the other two. Hence \( \lambda \)-condition (I) is invariant under strong shift equivalence of the symbolic matrix systems that correspond to the \( \lambda \)-graph systems. Therefore by the preceding lemma, one knows that synchronizing condition (I) is an invariant condition under topological conjugacy of \( \lambda \)-synchronizing subshifts.

As a condition under which the \( C^* \)-algebra \( \mathcal{O}_\Sigma \) is simple and purely infinite, \( \lambda \)-irreducibility for \( \lambda \)-graph system \( \Sigma \) has been introduced in [28]. A \( \lambda \)-graph system \( \Sigma \) is said to be \( \lambda \)-irreducible if for an ordered pair of vertices \( v_i, v_j \in V_i \), there exists a number \( L_i(i,j) \in \mathbb{N} \) such that for a vertex \( v_h^{i+L_i(i,j)} \in V_{i+L_i(i,j)} \) with \( t^{L_i(i,j)}(v_h^{i+L_i(i,j)}) = v_i \), there exists a path \( \gamma \) in \( \Sigma \) such that \( s(\gamma) = v_j, t(\gamma) = v_h^{i+L_i(i,j)} \), where \( t^{L_i(i,j)} \) means the \( L_i(i,j) \)-times compositions of \( t \), and \( s(\gamma), t(\gamma) \)
denote the source vertex, the terminal vertex of $\gamma$ respectively. A $\lambda$-synchronizing subshift $X$ is said to be synchronized irreducible if for $\mu, \nu \in \mathcal{S}_i(X)$, there exists $k_{\mu, \nu} \in \mathbb{N}$ such that for $\eta \in \mathcal{S}_{i+k_{\mu, \nu}}(X)$ with $\nu \sim \eta$, there exists $\xi \in \mathcal{L}_{k_{\mu, \nu}}(X)$ such that $\xi \eta \sim \mu$. It is direct to see that a $\lambda$-synchronizing subshift $X$ is synchronized irreducible if and only if $\mathcal{L}^\lambda(X)$ is $\lambda$-irreducible.

**Theorem 5.2.** Let $X$ be a $\lambda$-synchronizing subshift satisfying synchronizing condition (I). Suppose that $X$ is synchronized irreducible. Then the $C^*$-algebra $\mathcal{O}_{\mathcal{L}^\lambda(X)}$ associated with the $\lambda$-synchronizing $\lambda$-graph system $\mathcal{L}^\lambda(X)$ is a simple $C^*$-algebra such that

$$K_i(\mathcal{O}_{\mathcal{L}^\lambda(X)}) = K_i^\lambda(X), \quad i = 0, 1 \quad (5.2)$$

$$\text{Ext}^i(\mathcal{O}_{\mathcal{L}^\lambda(X)}) = BF_i(X), \quad i = 0, 1 \quad (5.3)$$

$$K_0(\mathcal{F}_{\mathcal{L}^\lambda(X)}) = \Delta^\lambda(X), \quad (5.4)$$

where $\mathcal{F}_{\mathcal{L}^\lambda(X)}$ is the AF-algebra defined by the fixed point algebra of $\mathcal{O}_{\mathcal{L}^\lambda(X)}$ under the gauge action.

**Proof.** By Lemma 5.1 the $\lambda$-synchronizing $\lambda$-graph system $\mathcal{L}^\lambda(X)$ of $X$ satisfies $\lambda$-condition (I). Also, if $X$ is synchronized irreducible, then $\mathcal{L}^\lambda(X)$ is $\lambda$-irreducible, and by [28, Theorem 3.9] $\mathcal{O}_{\mathcal{L}^\lambda(X)}$ is simple. The equalities (5.2),(5.3),(5.4) follow from (3.3),(3.4),(3.5) and (3.7),(3.8),(3.9) (see also [24], [25], [26]).

**Corollary 5.3.** Let $X$ be a synchronizing subshift satisfying synchronizing condition (I). Then the $C^*$-algebra $\mathcal{O}_{\mathcal{L}^\lambda(X)}$ associated with the synchronizing $\lambda$-graph system $\mathcal{L}^{\mathcal{L}^\lambda(X)}$ is a simple $C^*$-algebra that is isomorphic to the $C^*$-algebra $\mathcal{O}_{\mathcal{L}^\lambda(X)}$ associated with the $\lambda$-synchronizing $\lambda$-graph system $\mathcal{L}^\lambda(X)$ for $X$.

**Proof.** As a synchronizing subshift is irreducible, it is synchronized irreducible so that the $\lambda$-synchronizing $\lambda$-graph system $\mathcal{L}^\lambda(X)$ of $X$ is $\lambda$-irreducible. By Proposition 3.4, $\mathcal{O}_{\mathcal{L}^\lambda(X)}$ is isomorphic to $\mathcal{O}_{\mathcal{L}^\lambda(X)}$ that is simple by the above theorem.

One can prove that the $C^*$-algebras $\mathcal{O}_\beta, 1 < \beta \in \mathbb{R}$, in [10] for the $\beta$-shifts $\Lambda_\beta$ are isomorphic to the $C^*$-algebras $\mathcal{O}_{\mathcal{L}^\lambda(\Lambda_\beta)}$ associated with the the $\lambda$-synchronizing $\lambda$-graph systems for $\Lambda_\beta$. The $\lambda$-graph systems studied in the paper [30] are also the $\lambda$-synchronizing $\lambda$-graph systems for the subshifts.

**References**

[1] F. Blanchard and G. Hansel, *Systems codés*, Theor. Computer Sci. 44(1986), pp. 17–49.
[2] R. Bowen and J. Franks, *Homology for zero-dimensional nonwandering sets*, Ann. Math. 106(1977), pp. 73–92.
[3] M. Boyle and W. Krieger, *Almost Markov and shift equivalent sofic systems*, Proceedings of Maryland Special Year in Dynamics 1986-87, Springer -Verlag Lecture Notes in Math. 1342(1988), pp. 33–93.
[4] T. M. Carlsen and S. Eilers, *Matsumoto K-groups associated to certain shift spaces*, Doc. Math. 9 (2004) pp. 639–671.
[5] J. Cuntz and W. Krieger, *A class of $C^*$-algebras and topological Markov chains*, Inventions Math. 56(1980), pp. 251–268.
[6] R. Fischer, *Sofic systems and graphs*, Monats. für Math. 80(1975), pp. 179–186.
[7] J. Franks, *Flow equivalence of subshifts of finite type*, Ergodic Theory Dynam. Systems 4(1984), pp. 53–66.
[8] T. Hamachi and K. Inoue, *Embedding of shifts of finite type into the Dyck shift*, Monatshefte Mathematik 145(2005), pp. 107–129.
References

[9] T. Hamachi, K. Inoue and W. Krieger, Subsystems of finite type and semigroup invariants of subshifts, J. reine angew. Math. 632 (2009), pp. 37–61.
[10] Y. Katayama, K. Matsumoto and Y. Watatani, Simple $C^*$-algebras arising from $\beta$-expansion of real numbers, Ergodic Theory Dynam. Systems 18 (1998), pp. 937–962.
[11] B. P. Kitchens, Symbolic dynamics, Springer-Verlag, Berlin, Heidelberg and New York, 1998.
[12] W. Krieger, On dimension functions and topological Markov chains, Invent. Math. 56 (1980), pp. 239–250.
[13] W. Krieger, On shifts and topological Markov chains, Numbers, information and complexity (Bielefeld 1998), Kluwer Acad. Publ. Boston MA (2000) pp. 453 – 472.
[14] W. Krieger, On subshifts and semigroups, Bull. London Math. 38 (2006), pp. 617–624.
[15] W. Krieger, On $g$-functions for subshifts, Numbers, information and complexity (Bielefeld 1998), Kluwer Acad. Publ. Boston MA (2000) pp. 453 – 472.
[16] W. Krieger, On a certain class of $g$-functions for subshifts, preprint, arXiv:math.DS/0612345.
[17] W. Krieger, Presentations of symbolic dynamical systems by directed labeled graphs, SDA 2, Paris 2007.
[18] W. Krieger and K. Matsumoto, Shannon graphs, subshifts and lambda-graph systems, J. Math. Soc. Japan 54 (2002), pp. 877–900.
[19] W. Krieger and K. Matsumoto, A lambda-graph system for the Dyck shift and its $K$-groups, Documenta Math. 8 (2003), pp. 79–96.
[20] W. Krieger and K. Matsumoto, A class of topological conjugacy invariants of subshifts, Ergodic Theory Dynam. Systems 24 (2004), pp. 1155–1172.
[21] W. Krieger and B. Weiss, On $g$-measures in symbolic dynamics, Israel J. Math. 176 (2010), pp. 1–27.
[22] D. Lind and B. Marcus, An introduction to symbolic dynamics and coding, Cambridge University Press, Cambridge, 1995.
[23] K. Matsumoto, Presentations of subshifts and their topological conjugacy invariants, Doc. Math. 4 (1999), pp. 285–340.
[24] K. Matsumoto, Dimension groups for subshifts and simplicity of the associated $C^*$-algebras, J. Math. Soc. Japan 51 (1999), pp. 679-698.
[25] K. Matsumoto, Bowen-Franks groups for subshifts and Ext-groups for $C^*$-algebras, K-Theory 23 (2001), pp. 67–104.
[26] K. Matsumoto, $C^*$-algebras associated with presentations of subshifts, Doc. Math. 7 (2002), pp. 1–30.
[27] K. Matsumoto, A simple purely infinite $C^*$-algebra associated with a lambda-graph system of Motzkin shift, Math. Z. 248 (2004), pp. 369–394.
[28] K. Matsumoto, Construction and pure infiniteness of $C^*$-algebra associated with lambda-graph systems, Math. Scand. 97 (2005), pp. 73–89.
[29] K. Matsumoto, Topological entropy in $C^*$-algebras associated with $\lambda$-graph systems, Ergodic Theory and Dynam. Systems 25 (2005), pp. 1935–1951.
[30] K. Matsumoto, A class of simple $C^*$-algebras arising from certain nonsofic subshifts, to appear in Ergodic Theory and Dynam. Systems.
[31] G. Morvai and B. Weiss, Prediction for discrete time series, Prob. Theory Rel. Fields 132 (2005), pp. 1–12.
[32] M. Nasu, Topological conjugacy for sofic shifts, Ergodic Theory Dynam. Systems 6 (1986), pp. 265–280.
[33] M. Queffélec, Substitution Dynamical Systems-Spectral Analysis, Springer-Verlag, Berlin, Heidelberg and New York, 2010.

Institute for Applied Mathematics, University of Heidelberg, Im Neuenheimer Feld 294, 69120 Heidelberg, Germany

Department of Mathematics, Joetsu University of Education, Joetsu 943-8512 Japan