Infinite families of $t$-designs from a type of five-weight codes

Cunsheng Ding

Department of Computer Science and Engineering, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, China

Abstract

It has been known for a long time that $t$-designs can be employed to construct both linear and nonlinear codes and that the codewords of a fixed weight in a code may hold a $t$-design. While a lot of progress in the direction of constructing codes from $t$-designs has been made, only a small amount of work on the construction of $t$-designs from codes has been done. The objective of this paper is to construct infinite families of 2-designs and 3-designs from a type of binary linear codes with five-weights. The total number of 2-designs and 3-designs obtained in this paper are exponential in any odd $m$ and the block size of the designs varies in a huge range.

Keywords: Cyclic code, linear code, weight distribution, $t$-design.

2000 MSC: 05B05, 51E10, 94B15

1. Introduction

We start with a brief recall of $t$-designs. Let $\mathcal{P}$ be a set of $v \geq 1$ elements, and let $\mathcal{B}$ be a set of $k$-subsets of $\mathcal{P}$, where $k$ is a positive integer with $1 \leq k \leq v$. Let $t$ be a positive integer with $t \leq k$. The pair $\mathbb{D} = (\mathcal{P}, \mathcal{B})$ is called a $t$-$(v,k,\lambda)$ design, or simply $t$-design, if every $t$-subset of $\mathcal{P}$ is contained in exactly $\lambda$ elements of $\mathcal{B}$. The elements of $\mathcal{P}$ are called points, and those of $\mathcal{B}$ are referred to as blocks. We usually use $b$ to denote the number of blocks in $\mathcal{B}$. A $t$-design is called simple if $\mathcal{B}$ does not contain repeated blocks. In this paper, we consider only simple $t$-designs.

A $t$-design is called symmetric if $v = b$. It is clear that $t$-designs with $k = t$ or $k = v$ always exist. Such $t$-designs are trivial. In this paper, we consider only $t$-designs with $v > k > t$. A $t$-$(v,k,\lambda)$ design is referred to as a Steiner system if $t \geq 2$ and $\lambda = 1$, and is denoted by $S(t,k,v)$.

A necessary condition for the existence of a $t$-$(v,k,\lambda)$ design is that

$$\begin{pmatrix} k-i \\ t-i \end{pmatrix} \lambda \begin{pmatrix} v-i \\ t-i \end{pmatrix}$$

(1)

for all integer $i$ with $0 \leq i \leq t$.

The interplay between codes and $t$-designs goes in two directions. In one direction, the incidence matrix of any $t$-design generates a linear code over any finite field $\text{GF}(q)$. A lot of progress in this direction has been made and documented in the literature (see, for examples, [1], [5], [18, 14]). In the other direction, the codewords of a fixed Hamming weight in a linear or
nonlinear code may hold a $t$-design. Some linear and nonlinear codes were employed to construct $t$-designs \[1, 2, 10, 12, 13, 19\]. Binary and ternary Golay codes of certain parameters give 4-designs and 5-designs. However, the largest $t$ for which an infinite family of $t$-designs is derived directly from codes is $t = 3$. According to the references \[1, 13, 19\] and \[13\], not much progress on the construction of $t$-designs from codes has been made so far, while many other constructions of $t$-designs are documented in the literature \[3, 4, 13, 18\].

The objective of this paper is to construct infinite families of 2-designs and 3-designs from a type of binary linear codes with five weights. The obtained $t$-designs depend only on the weight distribution of the underlying binary codes. The total number of 2-designs and 3-designs presented in this paper are exponential in $m$, where $m \geq 5$ is an odd integer. In addition, the block size of the designs can vary in a huge range.

2. The classical construction of $t$-designs from codes

Let $C$ be a $[v, \kappa, d]$ linear code over GF($q$). Let $A_i := A_i(C)$, which denotes the number of codewords with Hamming weight $i$ in $C$, where $0 \leq i \leq v$. The sequence $(A_0, A_1, \ldots, A_v)$ is called the weight distribution of $C$, and $\sum_{v=0}^{v_i}A_i z^i$ is referred to as the weight enumerator of $C$. For each $k$ with $A_k \neq 0$, let $B_k$ denote the set of the supports of all codewords with Hamming weight $k$ in $C$, where the coordinates of a codeword are indexed by $(0, 1, 2, \cdots, v-1)$. Let $\mathcal{P} = \{0, 1, 2, \cdots, v-1\}$. The pair $(\mathcal{P}, B_k)$ may be a $t$-$(v, k, \lambda)$ design for some positive integer $\lambda$. The following theorems, developed by Assmus and Mattson, show that the pair $(\mathcal{P}, B_k)$ defined by a linear code is a $t$-design under certain conditions.

**Theorem 1.** [Assmus-Mattson Theorem \[3, [12], p. 303\]] Let $C$ be a binary $[v, \kappa, d]$ code. Suppose $C^\perp$ has minimum weight $d^\perp$. Suppose that $A_i = A_i(C)$ and $A_i^\perp = A_i(C^\perp)$, for $0 \leq i \leq v$, are the weight distributions of $C$ and $C^\perp$, respectively. Fix a positive integer $t$ with $t < d$, and let $s$ be the number of $i$ with $A_i^\perp \neq 0$ for $0 < i < v - t$. Suppose that $s \leq d - t$. Then

- the codewords of weight $i$ in $C$ hold a $t$-design provided that $A_i \neq 0$ and $d \leq i \leq v$,
- the codewords of weight $i$ in $C^\perp$ hold a $t$-design provided that $A_i^\perp \neq 0$ and $d^\perp \leq i \leq v$.

To construct $t$-designs via Theorem 1, we will need the following lemma in subsequent sections, which is a variant of the MacWilliam Identity \[20, p. 41\].

**Theorem 2.** Let $C$ be a $[v, \kappa, d]$ code over GF($q$) with weight enumerator $A(z) = \sum_{v=0}^{v_i}A_i z^i$ and let $A^\perp(z)$ be the weight enumerator of $C^\perp$. Then

$$A^\perp(z) = q^{-\kappa}(1 + (q - 1)z)^\kappa A\left(\frac{1 - z}{1 + (q - 1)z}\right).$$

Later in this paper, we will need also the following theorem.

**Theorem 3.** Let $C$ be an $[n, k, d]$ binary linear code, and let $C^\perp$ denote the dual of $C$. Denote by $C^\perp$ the extended code of $C^\perp$, and let $C^\perp\perp$ denote the dual of $C^\perp$. Then we have the following.

1. $C^\perp$ has parameters $[n, n - k, d^\perp]$, where $d^\perp$ denotes the minimum distance of $C^\perp$. 

2.
2. \( \overline{C^⊥} \) has parameters \([n+1,n-k,\overline{d^⊥}]\), where \( \overline{d^⊥} \) denotes the minimum distance of \( \overline{C^⊥} \), and is given by

\[
\overline{d^⊥} = \begin{cases} 
\overline{d^⊥} & \text{if } \overline{d^⊥} \text{ is even}, \\
\overline{d^⊥} + 1 & \text{if } \overline{d^⊥} \text{ is odd}.
\end{cases}
\]

3. \( \overline{C^{⊥⊥}} \) has parameters \([n+1,k+1,\overline{d^{⊥⊥}}]\), where \( \overline{d^{⊥⊥}} \) denotes the minimum distance of \( \overline{C^{⊥⊥}} \). Furthermore, \( \overline{C^{⊥⊥}} \) has only even-weight codewords, and all the nonzero weights in \( \overline{C^{⊥⊥}} \) are the following:

\[
w_1, w_2, \ldots, w_t; n+1-w_1, n+1-w_2, \ldots, n+1-w_t; n+1,
\]

where \( w_1, w_2, \ldots, w_t \) denote all the nonzero weights of \( C \).

**Proof.** The conclusions of the first two parts are straightforward. We prove only the conclusions of the third part below.

Since \( C^⊥ \) has length \( n+1 \) and dimension \( n-k \), the dimension of \( \overline{C^⊥} \) is \( k+1 \). By assumption, all codes under consideration are binary. By definition, \( \overline{C^⊥} \) has only even-weight codewords. Recall that \( \overline{C^⊥} \) is the extended code of \( \overline{C} \). It is known that the generator matrix of \( \overline{C^⊥} \) is given by \( \left[ \begin{array}{cc} \bar{1} & 1 \\ G & \bar{0} \end{array} \right] \),

where \( \bar{1} = (111 \cdots 1) \) is the all-one vector of length \( n \), \( \bar{0} = (000 \cdots 0)^T \), which is a column vector of length \( n \), and \( G \) is the generator matrix of \( C \). Notice again that \( \overline{C^⊥} \) is binary, the desired conclusions on the weights in \( \overline{C^{⊥⊥}} \) follow from the relation between the two generator matrices of the two codes \( \overline{C^⊥} \) and \( C \).

3. A type of binary linear codes with five-weights and related codes

In this section, we first introduce a type of binary linear codes \( C_m \) of length \( n = 2^m - 1 \), which has the weight distribution of Table 1 and then analyze their dual codes \( \overline{C^⊥_m} \), the extended codes \( \overline{C^{⊥⊥}_m} \), and the duals \( \overline{C^{⊥⊥}_m} \). Such codes will be employed to construct \( t \)-designs in Sections 4 and 5. Examples of such codes will be given in Section 6.

| Weight \( w \) | No. of codewords \( \Lambda_w \) |
|--------------|-----------------------------|
| 0            | 1                           |
| \( 2^{m-1} + 2^{(m+1)/2} \) | \( (2^m - 1) \cdot 2^{(m-5)/2} \cdot (2^{(m-3)/2} + 1) \cdot (2^{m-1} - 1)/3 \) |
| \( 2^{m-1} + 2^{(m+1)/2} \) | \( (2^m - 1) \cdot 2^{(m-3)/2} \cdot (2^{(m-1)/2} + 1) \cdot (5 \cdot 2^{m-1} + 4)/3 \) |
| \( 2^{m-1} \) | \( (2^m - 1) \cdot (9 \cdot 2^{2m-4} + 3 \cdot 2^{m-3} + 1) \) |
| \( 2^{m-1} + 2^{(m+1)/2} \) | \( (2^m - 1) \cdot 2^{(m-3)/2} \cdot (2^{(m-1)/2} - 1) \cdot (5 \cdot 2^{m-1} + 4)/3 \) |
| \( 2^{m-1} + 2^{(m+1)/2} \) | \( (2^m - 1) \cdot 2^{(m-5)/2} \cdot (2^{(m-3)/2} - 1) \cdot (2^{m-1} - 1)/3 \) |
Theorem 4. Let \( m \geq 5 \) be an odd integer and let \( C_m \) be a binary code with the weight distribution of Table 1. Then the dual code \( C_m^\perp \) has parameters \( [2^m - 1, 2^m - 1 - 3m, 7] \), and its weight distribution is given by

\[
2^m A_k = \binom{2^m - 1}{k} + aU_a(k) + bU_b(k) + cU_c(k) + dU_d(k) + eU_e(k),
\]

where \( 0 \leq k \leq 2^m - 1 \),

\[
a = (2^m - 1)2^{(m-5)/2}(2^{(m-3)/2} + 1)(2^m - 1)/3,
\]

\[
b = (2^m - 1)2^{(m-3)/2}(2^{(m-1)/2} + 1)(5 \times 2^m + 4)/3,
\]

\[
c = (2^m - 1)(9 \times 2^{m-4} + 3 \times 2^{m-3} + 1),
\]

\[
d = (2^m - 1)2^{(m-3)/2}(2^{(m-1)/2} - 1)(5 \times 2^m + 4)/3,
\]

\[
e = (2^m - 1)2^{(m-5)/2}(2^{(m-3)/2} - 1)(2^m - 1)/3,
\]

and

\[
U_a(k) = \sum_{\substack{0 \leq i \leq 2^{m-1} - 2^{(m+1)/2} \\ 0 \leq j \leq 2^{m-1} + 2^{(m+1)/2} - 1}} (-1)^i \binom{2^m - 1 - 2^{(m+1)/2}}{i} \binom{2^m - 1 + 2^{(m+1)/2} - 1}{j},
\]

\[
U_b(k) = \sum_{\substack{0 \leq i \leq 2^{m-1} - 2^{(m+1)/2} \\ 0 \leq j \leq 2^{m-1} + 2^{(m+1)/2} - 1}} (-1)^i \binom{2^m - 1 - 2^{(m-1)/2}}{i} \binom{2^m - 1 + 2^{(m-1)/2} - 1}{j},
\]

\[
U_c(k) = \sum_{\substack{0 \leq i \leq 2^{m-1} \\ 0 \leq j \leq 2^{m-1} + 2^{(m+1)/2} - 1}} (-1)^i \binom{2^m - 1}{i} \binom{2^m - 1}{j},
\]

\[
U_d(k) = \sum_{\substack{0 \leq i \leq 2^{m-1} - 2^{(m+1)/2} \\ 0 \leq j \leq 2^{m-1} - 2^{(m-1)/2} - 1}} (-1)^i \binom{2^m - 1 + 2^{(m-1)/2}}{i} \binom{2^m - 1 + 2^{(m-1)/2} - 1}{j},
\]

\[
U_e(k) = \sum_{\substack{0 \leq i \leq 2^{m-1} + 2^{(m+1)/2} \\ 0 \leq j \leq 2^{m-1} + 2^{(m+1)/2} - 1}} (-1)^i \binom{2^m - 1 + 2^{(m+1)/2}}{i} \binom{2^m - 1 + 2^{(m+1)/2} - 1}{j}.
\]

Proof. By assumption, the weight enumerator of \( C_m \) is given by

\[
A(z) = 1 + dz^{2^m - 1 - 2^{(m+1)/2}} + bz^{2^m - 1 - 2^{(m-1)/2}} + cz^{2^m - 1} + dz^{2^m - 1 + 2^{(m-1)/2}} + ez^{2^m - 1 + 2^{(m+1)/2}}.
\]
It then follows from Theorem 2 that the weight enumerator of $C_m$ is given by

$$2^{3m} A^\perp(z) = (1 + z)^{2^{m-1}} \left[ 1 + a \left( \frac{1 - z}{1 + z} \right)^{2^{m-1}-2 \frac{m+1}{2}} + b \left( \frac{1 - z}{1 + z} \right)^{2^{m-1}+2 \frac{m+1}{2}} \right] +$$

$$+ (1 + z)^{2^{m-1}} \left[ c \left( \frac{1 - z}{1 + z} \right)^{2^{m-1}} + d \left( \frac{1 - z}{1 + z} \right)^{2^{m-1}+2 \frac{m+1}{2}} + e \left( \frac{1 - z}{1 + z} \right)^{2^{m-1}+2 \frac{m+1}{2}} \right].$$

Hence, we have

$$2^{3m} A^\perp(z) = (1 + z)^{2^{m-1}} +$$

$$a(1 - z)^{2^{m-1}-2^{(m+1)/2}} (1 + z)^{2^{m-1}+2^{(m+1)/2} - 1} +$$

$$b(1 - z)^{2^{m-1}-2^{(m-1)/2}} (1 + z)^{2^{m-1}+2^{(m-1)/2} - 1} +$$

$$c(1 - z)^{2^{m-1}} (1 + z)^{2^{m-1} - 1} +$$

$$d(1 - z)^{2^{m-1}+2^{(m-1)/2}} (1 + z)^{2^{m-1}-2^{(m-1)/2} - 1} +$$

$$e(1 - z)^{2^{m-1}+2^{(m+1)/2}} (1 + z)^{2^{m-1}-2^{(m+1)/2} - 1}.$$

Obviously, we have

$$(1 + z)^{2^{m-1}} = \sum_{k=0}^{2^{m-1}} \binom{2^m - 1}{k} z^k.$$

It is easily seen that

$$(1 - z)^{2^{m-1}-2^{(m+1)/2}} (1 + z)^{2^{m-1}+2^{(m+1)/2} - 1} = \sum_{k=0}^{2^{m-1}} U_a(k) z^k$$

and

$$(1 - z)^{2^{m-1}-2^{(m-1)/2}} (1 + z)^{2^{m-1}+2^{(m-1)/2} - 1} = \sum_{k=0}^{2^{m-1}} U_b(k) z^k.$$

Similarly,

$$(1 - z)^{2^{m-1}+2^{(m-1)/2}} (1 + z)^{2^{m-1}-2^{(m-1)/2} - 1} = \sum_{k=0}^{2^{m-1}} U_d(k) z^k$$

and

$$(1 - z)^{2^{m-1}+2^{(m+1)/2}} (1 + z)^{2^{m-1}-2^{(m+1)/2} - 1} = \sum_{k=0}^{2^{m-1}} U_e(k) z^k.$$

Finally, we have

$$(1 - z)^{2^{m-1}} (1 + z)^{2^{m-1} - 1} = \sum_{k=0}^{2^{m-1}} U_c(k) z^k.$$
Combining these formulas above yields the weight distribution formula for $A_k^\perp$.

The weight distribution in Table 1 tells us that the dimension of $C_m$ is $3m$. Therefore, the dimension of $C_m^\perp$ is equal to $2^m - 1 - 3m$. Finally, we prove that the minimum distance of $C_m^\perp$ equals 7.

We now prove that $A_k^\perp = 0$ for all $k$ with $1 \leq k \leq 6$. Let $x = 2^{(m-1)/2}$. With the weight distribution formula obtained before, we have

\[
\begin{aligned}
\left( \begin{array}{c}
2^m - 1 \\
1
\end{array} \right) &= 2x^2 - 1, \\
aU_a(1) &= 1/3x^7 + 7/12x^6 - 2/3x^5 - 7/8x^4 + 5/12x^3 + 7/24x^2 - 1/12x, \\
bU_b(1) &= 10/3x^7 + 5/3x^6 - 2/3x^5 + 1/2x^4 - 11/6x^3 - 2/3x^2 + 2/3x, \\
cU_c(1) &= -9/2x^8 + 3/4x^6 - 5/4x^4 + 7/4x^2 - 1, \\
dU_d(1) &= -10/3x^7 + 5/3x^6 + 2/3x^5 + 1/2x^4 + 11/6x^3 - 2/3x^2 - 2/3x, \\
eU_e(1) &= -1/3x^7 + 7/12x^6 + 2/3x^5 - 7/8x^4 - 5/12x^3 + 7/24x^2 + 1/12x.
\end{aligned}
\]

Consequently,

\[
2^m A_{1}^\perp = \left( \begin{array}{c}
2^m - 1 \\
1
\end{array} \right) + aU_a(1) + bU_b(1) + cU_c(1) + dU_d(1) + eU_e(1) = 0.
\]

Plugging $k = 2$ into the weight distribution formula above, we get that

\[
\begin{aligned}
\left( \begin{array}{c}
2^m - 1 \\
2
\end{array} \right) &= 2x^4 - 3x^2 + 1, \\
aU_a(2) &= 7/12x^8 + 5/6x^7 - 35/24x^6 - 13/12x^5 + 7/6x^4 + 1/6x^3 - 7/24x^2 + 1/12x, \\
bU_b(2) &= 5/3x^8 - 5/3x^7 - 7/6x^6 + 7/6x^5 - 7/6x^4 + 7/6x^3 + 2/3x^2 - 2/3x, \\
cU_c(2) &= -9/2x^8 + 21/4x^6 - 2x^4 + 9/4x^2 - 1, \\
dU_d(2) &= 5/3x^8 + 5/3x^7 - 7/6x^6 - 7/6x^5 - 7/6x^4 - 7/6x^3 + 2/3x^2 + 2/3x, \\
eU_e(2) &= 7/12x^8 - 5/6x^7 - 35/24x^6 + 13/12x^5 + 7/6x^4 - 1/6x^3 - 7/24x^2 - 1/12x.
\end{aligned}
\]

Consequently,

\[
2^m A_{2}^\perp = \left( \begin{array}{c}
2^m - 1 \\
2
\end{array} \right) + aU_a(2) + bU_b(2) + cU_c(2) + dU_d(2) + eU_e(2) = 0.
\]

After similar computations with the weight distribution formula, one can prove that $A_k^\perp = 0$ for all $k$ with $3 \leq k \leq 6$. Plugging $k = 7$ into the weight distribution formula above, we arrive at

\[
A_7^\perp = \frac{(x^2 - 1)(2x^2 - 1)(x^4 - 5x^2 + 34)}{630}.
\]

Notice that $x^4 - 5x^2 + 34 = (x^2 - 5/2)^2 + 34 - 25/4 > 0$. We have $A_k^\perp > 0$ for all odd $m \geq 5$. This proves the desired conclusion on the minimum distance of $C_m^\perp$. \[\square\]
Theorem 5. Let \( m \geq 5 \) be an odd integer and let \( C_m \) be a binary code with the weight distribution of Table 1. The code \( C_m^\perp \) has parameters
\[
\left[ 2^m, 3m+1, 2^{m-1} - 2^{(m+1)/2} \right],
\]
and its weight enumerator is given by
\[
A_{C_m^\perp} (z) = 1 + u z^{2^{m-1}-2^{m-3}} + v z^{2^{m-1}-2^{m-3}+1} + w z^{2^{m-1}+2} + u z^{2^{m-1}+2^{m-3}} + z^m,
\]
where
\[
\begin{align*}
    u &= \frac{2^{3m-4} - 3 \times 2^{m-4} + 2^{m-3}}{3}, \\
v &= \frac{5 \times 2^{3m-2} + 3 \times 2^{m-2} - 2^{m+1}}{3}, \\
w &= 2(2^m-1)(9 \times 2^{m-4} + 3 \times 2^{m-3} + 1).
\end{align*}
\]

Proof. It follows from Theorem 3 that the code has all the weights given in \( \mathbf{2} \). It remains to determine the frequencies of these weights. The weight distribution of the code \( C_m \) given in Table 1 and the generator matrix of the code \( C_m^\perp \) documented in the proof of Theorem 3 show that
\[
A_{C_m^\perp} = 2c = w,
\]
where \( c \) was defined in Theorem 4.

We now determine \( u \) and \( v \). Recall that \( C_m^\perp \) has minimum distance 7. It then follows from Theorem 3 that \( C_m^\perp \) has minimum distance 8. The first and third Pless power moments say that
\[
\begin{align*}
    \sum_{i=0}^{2^m} A_{C_m^\perp}^i &= 2^{3m+1}, \\
    \sum_{i=0}^{2^m} i^2 A_{C_m^\perp}^i &= 2^{3m-1} 2^m (2^{m+1}).
\end{align*}
\]

These two equations become
\[
\begin{align*}
    1 + u + v + c &= 2^{3m}, \\
    (2^{2m-2} + 2^{m+1}) u + (2^{2m-2} + 2^{m-1}) v + 2^{2m-2} c + 2^{2m-1} = 2^{4m-2} (2^{m+1}).
\end{align*}
\]

Solving this system of equations proves the desired conclusion on the weight enumerator of this code.

Finally, we settle the weight distribution of the code \( C_m^\perp \).

Theorem 6. Let \( m \geq 5 \) be an odd integer and let \( C_m \) be a binary code with the weight distribution of Table 1. The code \( C_m^\perp \) has parameters \( [2^m, 2^{m-1} - 3m, 8] \), and its weight distribution is given by
\[
2^{3m+1} A_{C_m^\perp}^k = \left( 1 + (-1)^k \right) \left( \begin{array}{c} 2^m \\ k \end{array} \right) + w E_0(k) + u E_1(k) + v E_2(k) + v E_3(k) + u E_4(k),
\]

\[
(3)
\]
where \(w, u, v\) are defined in Theorem 5 and

\[
E_0(k) = \frac{1 + (-1)^k}{2} (-1)^{\lfloor k/2 \rfloor} \left( \begin{array}{c} 2^{m-1} \\ i \\ \end{array} \right) \left( \begin{array}{c} 2^{m-1} + 2^{(m+1)/2} \\ j \end{array} \right),
\]

\[
E_1(k) = \sum_{0 \leq i \leq 2^{m-1} - 2^{(m+1)/2}} \sum_{0 \leq j \leq 2^{m-1} + 2^{(m+1)/2} \atop i + j = k} (-1)^i \left( \begin{array}{c} 2^{m-1} - 2^{(m+1)/2} \\ i \end{array} \right) \left( \begin{array}{c} 2^{m-1} + 2^{(m+1)/2} \\ j \end{array} \right),
\]

\[
E_2(k) = \sum_{0 \leq i \leq 2^{m-1} - 2^{(m-1)/2}} \sum_{0 \leq j \leq 2^{m-1} + 2^{(m-1)/2} \atop i + j = k} (-1)^i \left( \begin{array}{c} 2^{m-1} + 2^{(m-1)/2} \\ i \end{array} \right) \left( \begin{array}{c} 2^{m-1} - 2^{(m-1)/2} \\ j \end{array} \right),
\]

\[
E_3(k) = \sum_{0 \leq i \leq 2^{m-1} + 2^{(m-1)/2}} \sum_{0 \leq j \leq 2^{m-1} - 2^{(m-1)/2} \atop i + j = k} (-1)^i \left( \begin{array}{c} 2^{m-1} + 2^{(m-1)/2} \\ i \end{array} \right) \left( \begin{array}{c} 2^{m-1} - 2^{(m-1)/2} \\ j \end{array} \right),
\]

\[
E_4(k) = \sum_{0 \leq i \leq 2^{m-1} + 2^{(m+1)/2}} \sum_{0 \leq j \leq 2^{m-1} - 2^{(m+1)/2} \atop i + j = k} (-1)^i \left( \begin{array}{c} 2^{m-1} + 2^{(m+1)/2} \\ i \end{array} \right) \left( \begin{array}{c} 2^{m-1} - 2^{(m+1)/2} \\ j \end{array} \right).
\]

and \(0 \leq k \leq 2^m\).

**Proof.** By definition,

\[
\dim \left( \overline{C_{m}^{-}} \right) = \dim \left( \overline{C_{m}^{+}} \right) = 2^m - 1 - 3m.
\]

It has been shown in the proof of Theorem 4 that the minimum distance of \(\overline{C_{m}^{-}}\) is equal to 8. We now prove the conclusion on the weight distribution of this code.

By Theorems 2 and 5 the weight enumerator of \(\overline{C_{m}^{-}}\) is given by

\[
2^{3m+1} A_{\overline{C_{m}^{-}}} (z) = (1 + z)^{2^m} \left[ 1 + \left( \frac{1 - z}{1 + z} \right)^{2^m} + w \left( \frac{1 - z}{1 + z} \right)^{2^m-1} \right] +

(1 + z)^{2^m} \left[ u \left( \frac{1 - z}{1 + z} \right)^{2^{m-1} - z^{2^m + 1}} + v \left( \frac{1 - z}{1 + z} \right)^{2^{m-1} - z^{2^m + 1}} \right] +

(1 + z)^{2^m} \left[ v \left( \frac{1 - z}{1 + z} \right)^{2^{m-1} + z^{2^m + 1}} + u \left( \frac{1 - z}{1 + z} \right)^{2^{m-1} + z^{2^m + 1}} \right], \tag{4}
\]
Consequently, we have
\[
2^{3m+1}A(z) = (1+z)^{2m} + (1-z)^{2m} + w(1-z^2)^{2m-1} + u(1-z)^{2m-1}z^2 + v(1+z)^{2m-1}z^2 + v(1-z)^{2m-1}z^2 + u(1+z)^{2m-1}z^2 + (1+z)^{2m-1}z^2 + (1+z)^{2m-1}z^2.
\]

We now treat the terms in (5) one by one. We first have
\[
(1+z)^{2m} + (1-z)^{2m} = \sum_{k=0}^{2m} \binom{2m}{k} 2^m.
\]

One can easily see that
\[
(1-z^2)^{2m-1} = \sum_{i=0}^{2m-1} (-1)^i \binom{2m-1}{i} z^{2i} = \sum_{k=0}^{2m} \frac{1 + (-1)^k}{2} (-1)^{k/2} \binom{2m-1}{k/2} z^k.
\]

Notice that
\[
(1-z)^{2m-1}z^{(m+1)/2} = \sum_{i=0}^{2m-1} \binom{2m-1}{i} (-1)^i z^i
\]
and
\[
(1+z)^{2m-1}z^{(m+1)/2} = \sum_{i=0}^{2m-1} \binom{2m-1}{i} (-1)^i z^i.
\]

We have then
\[
(1-z)^{2m-1}z^{(m+1)/2} (1+z)^{2m-1}z^{(m+1)/2} = \sum_{k=0}^{2m} E_1(k) z^k.
\]

Similarly, we have
\[
(1-z)^{2m-1}z^{(m-1)/2} (1+z)^{2m-1}z^{(m-1)/2} = \sum_{k=0}^{2m} E_2(k) z^k,
\]
\[
(1-z)^{2m-1}z^{(m-1)/2} (1+z)^{2m-1}z^{(m-1)/2} = \sum_{k=0}^{2m} E_3(k) z^k,
\]
\[
(1-z)^{2m-1}z^{(m+1)/2} (1+z)^{2m-1}z^{(m+1)/2} = \sum_{k=0}^{2m} E_4(k) z^k.
\]

Plugging (6), (7), (8), (9), (10), and (11) into (5) proves the desired conclusion.\[\square\]
4. Infinite families of $2$-designs from $\mathbb{C}_m^\perp$ and $\mathbb{C}_m$

**Theorem 7.** Let $m \geq 5$ be an odd integer and let $\mathbb{C}_m$ be a binary code with the weight distribution of Table 1. Let $\mathcal{P} = \{0, 1, 2, \cdots, 2^m - 2\}$, and let $\mathcal{B}$ be the set of the supports of the codewords of $\mathbb{C}_m$ with weight $k$, where $A_k \neq 0$. Then $(\mathcal{P}, \mathcal{B})$ is a $2$-$(2^m - 1, k, \lambda)$ design, where

$$\lambda = \frac{k(k - 1)A_k}{(2^m - 1)(2^m - 2)},$$

where $A_k$ is given in Table 1. Let $\mathcal{P} = \{0, 1, 2, \cdots, 2^m - 2\}$, and let $\mathcal{B}^\perp$ be the set of the supports of the codewords of $\mathbb{C}_m^\perp$ with weight $k$ and $A_k^\perp \neq 0$. Then $(\mathcal{P}, \mathcal{B}^\perp)$ is a $2$-$(2^m - 1, k, \lambda)$ design, where

$$\lambda = \frac{k(k - 1)A_k^\perp}{(2^m - 1)(2^m - 2)},$$

where $A_k^\perp$ is given in Theorem 4.

**Proof.** The weight distribution of $\mathbb{C}_m^\perp$ is given in Theorem 4, and that of $\mathbb{C}_m$ is given in Table 1. By Theorem 4, the minimum distance $d^\perp$ of $\mathbb{C}_m^\perp$ is equal to 7. Put $t = 2$. The number of $i$ with $A_i \neq 0$ and $1 \leq i \leq 2^m - 1 - t$ is $s = 5$. Hence, $s = d^\perp - t$. The desired conclusions then follow from Theorem 4 and the fact that two binary vectors have the same support if and only if they are equal.

**Example 8.** Let $m \geq 5$ be an odd integer and let $\mathbb{C}_m$ be a binary code with the weight distribution of Table 1. Then the BCH code $\mathbb{C}_m$ holds five $2$-designs with the following parameters:

- $(v, k, \lambda) = \left(2^m - 1, 2^{m-1} - 2^\frac{m+1}{2}, \frac{2^m(2^{m+1} - 1)}{6} \left(2^{m-1} - 2^\frac{m+1}{2} - 1\right)\right)$.
- $(v, k, \lambda) = \left(2^m - 1, 2^{m-1} - 2^\frac{m+1}{2}, \frac{2^{m-2}(2^{m-1} - 2^\frac{m+1}{2} - 1)}{6} \left(5 \times 2^{m-1} + 1\right)\right)$.
- $(v, k, \lambda) = \left(2^m - 1, 2^{m-1} + 2^\frac{m+1}{2}, \frac{2^{m-2}(2^{m-1} + 2^\frac{m+1}{2} - 1)}{6} \left(5 \times 2^{m-1} + 1\right)\right)$.
- $(v, k, \lambda) = \left(2^m - 1, 2^{m-1} + 2^\frac{m+1}{2}, \frac{2^m(2^{m+1} - 1)}{6} \left(2^{m-1} + 2^\frac{m+1}{2} - 1\right)\right)$.
- $(v, k, \lambda) = \left(2^m - 1, 2^{m-1} + 2^\frac{m+1}{2}, \frac{2^m(2^{m+1} - 1)}{6} \left(2^{m-1} + 2^\frac{m+1}{2} - 1\right)\right)$.

**Example 9.** Let $m \geq 5$ be an odd integer and let $\mathbb{C}_m$ be a binary code with the weight distribution of Table 1. Then the supports of all codewords of weight 7 in $\mathbb{C}_m$ give a $2$-$(2^m - 1, 7, \lambda)$ design, where

$$\lambda = \frac{2^{(m-1)} - 5 \times 2^{m-1} + 34}{30}.$$
Proof. By Theorem 4, we have

\[ A_7 = \frac{(2^{m-1} - 1)(2^m - 1)(2^{2m-1} - 5 \times 2^{m-1} + 34)}{630}. \]

The desired conclusion on \( \lambda \) follows from Theorem 7.

Example 10. Let \( m \geq 5 \) be an odd integer and let \( C_m \) be a binary code with the weight distribution of Table 1. Then the supports of all codewords of weight 8 in \( C_m^\perp \) give a \( 2-(2^m - 1, 8, \lambda) \) design, where

\[ \lambda = \frac{(2^{m-1} - 4)(2^{2m-1} - 5 \times 2^{m-1} + 34)}{90}. \]

Proof. By Theorem 4, we have

\[ A_8 = \frac{(2^{m-1} - 1)(2^m - 1)(2^{2m-1} - 5 \times 2^{m-1} + 34)}{2520}. \]

The desired conclusion on \( \lambda \) follows from Theorem 7.

Example 11. Let \( m \geq 7 \) be an odd integer and let \( C_m \) be a binary code with the weight distribution of Table 1. Then the supports of all codewords of weight 9 in \( C_m^\perp \) give a \( 2-(2^m - 1, 9, \lambda) \) design, where

\[ \lambda = \frac{(2^{m-1} - 4)(2^{m-1} - 16)(2^{2m-1} - 2^{m-1} + 28)}{315}. \]

Proof. By Theorem 4, we have

\[ A_9 = \frac{(2^{m-1} - 1)(2^m - 4)(2^{m-1} - 16)(2^{m-1} - 1)(2^{2m-1} - 2^{m-1} + 28)}{11340}. \]

The desired conclusion on \( \lambda \) follows from Theorem 7.

5. Infinite families of 3-designs from \( C_m^\perp \) and \( C_m^{\perp\perp} \)

Theorem 12. Let \( m \geq 5 \) be an odd integer and let \( C_m \) be a binary code with the weight distribution of Table 1. Let \( P = \{0, 1, 2, \ldots, 2^m - 1\} \), and let \( B^{\perp\perp} \) be the set of the supports of the codewords of \( C_m^\perp \) with weight \( k \), where \( A^{\perp\perp}_k \neq 0 \). Then \( (P, B^{\perp\perp}) \) is a \( 3-(2^m, k, \lambda) \) design, where

\[ \lambda = \frac{A^{\perp\perp}_k (k)}{\left(\begin{array}{c} k \\ 3 \end{array}\right)}, \]

where \( A^{\perp\perp}_k \) is given in Theorem 5.

Let \( P = \{0, 1, 2, \ldots, 2^m - 1\} \), and let \( B^{\perp} \) be the set of the supports of the codewords of \( C_m^\perp \) with weight \( k \) and \( A^{\perp}_k \neq 0 \). Then \( (P, B^{\perp}) \) is a \( 3-(2^m, k, \lambda) \) design, where

\[ \lambda = \frac{A^{\perp}_k (k)}{\left(\begin{array}{c} k \\ 3 \end{array}\right)}, \]

where \( A^{\perp}_k \) is given in Theorem 6.
Proof. The weight distributions of $\overline{C_m}$ and $\overline{C_m^\perp}$ are described in Theorems 3 and 5. Notice that the minimum distance $d_m$ of $\overline{C_m}$ is equal to 8. Put $t = 3$. The number of $i$ with $A^+i \neq 0$ and $1 \leq i \leq 2^m - t$ is $s = 5$. Hence, $s = d_m - t$. Clearly, two binary vectors have the same support if and only if they are equal. The desired conclusions then follow from Theorem 1.

Example 13. Let $m \geq 5$ be an odd integer and let $C_m$ be a binary code with the weight distribution of Table 1. Then $\overline{C_m}$ holds five 3-designs with the following parameters:

- $(v, k, \lambda) = \left(2^m, 2^{m-1} - 2^{m-1}, \frac{2^{m-2} - 2 - 2}{48}\right)$.
- $(v, k, \lambda) = \left(2^m, 2^{m-1} - 2^{m-1}, \frac{2^{m-2} - 2 - 2}{3}\right)(5 \times 2^{m-3} + 1)$.
- $(v, k, \lambda) = \left(2^m, 2^{m-1}, (2^{m-2} - 1)(9 \times 2^{m-4} + 3 \times 2^{m-3} + 1)\right)$.
- $(v, k, \lambda) = \left(2^m, 2^{m-1} + 2^{m-1}, \frac{2^{m-2} - 2 - 2}{3}\right)(5 \times 2^{m-3} + 1)$.
- $(v, k, \lambda) = \left(2^m, 2^{m-1} + 2^{m-1}, \frac{2^{m-2} - 2 - 2}{3}\right)(5 \times 2^{m-3} + 1)$.

Example 14. Let $m \geq 5$ be an odd integer and let $C_m$ be a binary code with the weight distribution of Table 1. Then the supports of all codewords of weight 8 in $\overline{C_m}$ give a 3-(2$^m$, 8, $\lambda$) design, where $\lambda = \frac{2^{2(m-1)} - 5 \times 2^{m-1} + 34}{30}$.

Proof. By Theorem 5, we have

$$\overline{A^+8} = \frac{2^m(2^{m-1} - 1)(2^{m-1} - 1)(2^{2(m-1)} - 5 \times 2^{m-1} + 34)}{315}.$$

The desired value of $\lambda$ follows from Theorem 12.

Example 15. Let $m \geq 7$ be an odd integer and let $C_m$ be a binary code with the weight distribution of Table 1. Then the supports of all codewords of weight 10 in $\overline{C_m}$ give a 3-(2$^m$, 10, $\lambda$) design, where $\lambda = \frac{(2^{m-1} - 4)(2^{m-1} - 16)(2^{2(m-1)} - 2^{m-1} + 28)}{315}$.

Proof. By Theorem 5, we have

$$\overline{A^+10} = \frac{2^{m-1}(2^{m-1} - 1)(2^{m-1} - 1)(2^{m-1} - 4)(2^{m-1} - 16)(2^{2(m-1)} - 2^{m-1} + 28))}{4 \times 14175}.$$

The desired value of $\lambda$ follows from Theorem 12.
Proof. Since the weight distribution of Table 1 may not be known, it is known that the dual of a BCH code may not be a BCH code. The following theorem describes a family of cyclic codes having the weight distribution of Table 1 which may not be BCH codes.

**Theorem 17.** Let \( m \geq 5 \) be an odd integer and let \( \delta = 2^{m-1} - 1 - 2^{(m+1)/2} \). Then the BCH code \( C_{(2,2^m-1,\delta,0)} \) has length \( n = 2^m - 1 \), dimension \( 3m \), and the weight distribution in Table 1.

**Proof.** A proof can be found in [5].

It is known that the dual of a BCH code may not be a BCH code. The following theorem describes a family of cyclic codes having the weight distribution of Table 1 which may not be BCH codes.

**Theorem 18.** Let \( m \geq 5 \) be an odd integer. Let \( C_m \) be the dual of the narrow-sense primitive BCH code \( C_{(2,2^m-1,7,1)} \). Then \( C_m \) has the weight distribution in Table 1.

**Proof.** A proof can be found in [12].

---

**Example 16.** Let \( m \geq 5 \) be an odd integer and let \( C_m \) be a binary code with the weight distribution of Table 1. Then the supports of all codewords of weight 12 in \( C_m \) give a 3-(\( 2^m \), 12, \( \lambda \)) design, where

\[
\lambda = \frac{(2^h-2)(2^h - 55 \times 2^h + 647 \times 2^h - 2727 \times 2^h + 11541 \times 2^h - 47208)}{2835}
\]

and \( h = m - 1 \).

**Proof.** By Theorem 6, we have

\[
\widetilde{\Lambda}_{12} = \frac{\varepsilon_2(\varepsilon^2 - 4)(\varepsilon^2 - 1)(2\varepsilon^{10} - 55\varepsilon^8 + 647\varepsilon^6 - 2727\varepsilon^4 + 11541\varepsilon^2 - 47208)}{8 \times 467775}
\]

where \( \varepsilon = 2^{(m-1)/2} \). The desired value of \( \lambda \) follows from Theorem 12.

---

**6. Two families of binary cyclic codes with the weight distribution of Table 1**

To prove the existence of the 2-designs in Section 4 and the 3-designs in Section 5, we present two families of binary codes of length \( 2^m - 1 \) with the weight distribution of Table 1.

Let \( n = q^m - 1 \), where \( m \) is a positive integer. Let \( \alpha \) be a generator of GF(\( q^m \)). For any \( i \) with \( 0 \leq i \leq n - 1 \), let \( M_i(x) \) denote the minimal polynomial of \( \alpha^i \) over GF(\( q \)). For any \( 2 \leq \delta \leq n \), define

\[
g_{(q,n,\delta,\delta)}(x) = \text{lcm}(M_0(x), M_{\delta+1}(x), \cdots, M_{n-\delta-2}(x)),
\]

where \( \delta \) is an integer, lcm denotes the least common multiple of these minimal polynomials, and the addition in the subscript \( b = i \) of \( \text{mod} \) always means the integer addition modulo \( n \). Let \( C_{(q,n,\delta,\delta)} \) denote the cyclic code of length \( n \) with generator polynomial \( g_{(q,n,\delta,\delta)}(x) \). \( C_{(q,n,\delta,\delta)} \) is called a primitive BCH code with designed distance \( \delta \). When \( b = 1 \), the set \( C_{(q,n,\delta,\delta)} \) is called a narrow-sense primitive BCH code.

Although primitive BCH codes are not good asymptotically, they are among the best linear codes when the length of the codes is not very large [5, Appendix A]. So far, we have very limited knowledge of BCH codes, as the dimension and minimum distance of BCH codes are in general open, in spite of some recent progress [6, 7]. However, in a few cases the weight distribution of a BCH code can be settled. The following theorem introduces such a case.

**Theorem 17.** Let \( m \geq 5 \) be an odd integer and let \( \delta = 2^{m-1} - 1 - 2^{(m+1)/2} \). Then the BCH code \( C_{(2,2^m-1,\delta,0)} \) has length \( n = 2^m - 1 \), dimension \( 3m \), and the weight distribution in Table 1.

**Proof.** A proof can be found in [5].

---

**Theorem 18.** Let \( m \geq 5 \) be an odd integer. Let \( C_m \) be the dual of the narrow-sense primitive BCH code \( C_{(2,2^m-1,7,1)} \). Then \( C_m \) has the weight distribution in Table 1.

**Proof.** A proof can be found in [12].
7. Summary and concluding remarks

In this paper, with any binary linear code of length $2^m - 1$ and the weight distribution of Table 1, a huge number of infinite families of 2-designs and 3-designs with various block sizes were constructed. These constructions clearly show that the coding theory approach to constructing $t$-designs are in fact promising, and may stimulate further investigations in this direction. It is open if the codewords of a fixed weight in a family of linear codes can hold an infinite family of $t$-designs for some $t \geq 4$.

It is noticed that the technical details of this paper are tedious. However, one has to settle the weight distribution of a linear code and the minimum distance of its dual at the same time, if one would like to employ the Assmus-Mattson Theorem for the construction of $t$-designs. Note that it could be very difficult to prove that a linear code has minimum weight 7. This explains why the proofs of some of the theorems are messy and tedious, but necessary.

Acknowledgments

The research of C. Ding was supported by the Hong Kong Research Grants Council, under Project No. 16300415.

References

[1] E. F. Assmus Jr. and J. D. Key, Designs and Their Codes, Cambridge: Cambridge University Press, 1992.
[2] E. F. Assmus Jr. and H. F. Mattson Jr., Coding and combinatorics, SIAM Rev. 16 (1974) 349–388.
[3] T. Beth, D. Jungnickel and H. Lenz, Design Theory, Cambridge: Cambridge University Press, 1999.
[4] C. J. Colbourn and R. Mathon, Steiner systems, in: C. J. Colbourn and J. Dinitz (Eds.), Handbook of Combinatorial Designs, pp. 102–110, New York: CRC Press, 2007.
[5] C. Ding, Codes from Difference Sets. Singapore: World Scientific, 2015.
[6] C. Ding, Parameters of several classes of BCH codes, IEEE Trans. Inf. Theory 61(10) (2015) 5322–5330.
[7] C. Ding, X. Du and Z. Zhou, The Bose and minimum distance of a class of BCH codes, IEEE Trans. Inf. Theory 61(5) (2015) 2351–2356.
[8] C. Ding, C. Fan and Z. Zhou, The dimension and minimum distance of two classes of primitive BCH codes, arXiv:1603.07007.
[9] M. Harada, M. Kitazume, and A. Munemasa, On a 5-design related to an extremal doubly-even self-dual code of length 72, J. Combin. Theory, Ser. A, 107 (2004) 143–146.
[10] M. Harada, A. Munemasa, V. D. Tonchev, A characterization of designs related to an extremal doubly-even self-dual code of length 48, Annals of Combinatorics 9 (2005) 189–198.
[11] W. C. Huffman and V. Pless, Fundamentals of Error-Correcting Codes, Cambridge: Cambridge University Press, 2003.
[12] T. Kasami, Weight distributions of Bose-Chaudhuri-Hocquenghem codes, in: R. C. Bose and T. A. Dowlings, Eds., Combinatorial Mathematics and Applications, Chapel Hill, NC, Univ. North Carolina Press, 1969, Ch. 20.
[13] G. B. Khosrovshahi and H. Lane, t-designs with $t \geq 3$, in: C. J. Colbourn and J. Dinitz (Eds.), Handbook of Combinatorial Designs, pp. 79–101, New York: CRC Press, 2007.
[14] J. H. Koolen and A. Munemasa, Tight 2-designs and perfect 1-codes in Doob graphs, J. Stat. Planning and Inference 86 (2000) 505-513.
[15] F. J. MacWilliams and N. J. A. Sloane, The Theory of Error-Correcting Codes. Amsterdam: North-Holland, 1977.
[16] C. Reid and A. Rosa, Steiner systems $S(2,4)$ - a survey, The Electronic Journal of Combinatorics (2010), #DS18.
[17] A. Munemasa, V. D. Tonchev, A new quasi-symmetric 2-(56,16,6) design obtained from codes, Discrete Math. 29 (2004) 231–234.
[18] V. D. Tonchev, Codes and designs, in: V.S. Pless and W.C. Huffman (Eds.), Handbook of Coding Theory, Vol. II, pp. 1229–1268, Amsterdam: Elsevier, 1998.
[19] V. D. Tonchev, Codes, in: C.J. Colbourn and J.H. Dinitz (Eds.), Handbook of Combinatorial Designs, 2nd Edition, pp. 677–701, New York: CRC Press, 2007.
[20] J. H. van Lint, Introduction to Coding Theory, Third Edition, New York: Springer Verlag, 1999.