Upper bounds for the piercing number of families of pairwise intersecting convex polygons

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Abstract. A convex polygon \( A \) is related to a convex \( m \)-gon \( K = \bigcap_{i=1}^{m} k_i^+ \), where \( k_1^+, \ldots, k_m^+ \) are the \( m \) halfplanes whose intersection is equal to \( K \), if \( A \) is the intersection of halfplanes \( a_1^+, \ldots, a_l \), each of which is a translate of one of the \( k_i^+ \)-s. The planar family \( A \) is related to \( K \) if each \( A \in A \) is related to \( K \). We prove that any family of pairwise intersecting convex sets related to a given \( n \)-gon has a finite piercing number which depends on \( n \). In the general case we show \( O(3n^3) \), while for a certain class of families, we decrease the bound to \( 4(n-2) \), and for \( n = 3, 4 \) the bound is 3 and 6 respectively.

Definition 1 A convex polygon \( P \) is related to a convex \( m \)-gon \( K = \bigcap_{i=1}^{m} k_i^+ \), where \( k_1^+, \ldots, k_m^+ \) are the \( m \) halfplanes whose intersection is equal to \( K \), if \( P \) is the intersection of halfplanes \( a_1^+, \ldots, a_l^+ \), each of which is a translate of one of the \( k_i^+ \)-s. We use the convention that the line \( l \) is the boundary of the halfplane \( l^+ \) and that \( l^- \) is the halfplane with boundary \( l \) so that \( l^+ \cap l^- = l \). The family \( P \) is related to \( K \) if each \( P \in P \) is related to \( K \).

Theorem 1 A convex family of pairwise intersecting sets related to an \( n \)-gon is \( 3(3) \) pierceable.
Theorem 2 Let $F$ be a family of pairwise intersecting sets related to an $n$-gon $F$ with edges $\overline{h}, \overline{v}, \overline{a}_1, \ldots, \overline{a}_{n-2}$, such that $\overline{h} = \left[-1, 0\right]$, $\overline{v} = \left[0, y\right]$ for any $y > 0$, and the edges $\overline{a}_1, \ldots, \overline{a}_{n-2}$ have positive slopes. Then $F$ is $4(n-2)$ pierceable. If $n = 3, 4$ then the family is 3 and 6 pierceable respectively.

Definition 2 Let $P$ be a family related to $m$-gon $K = \bigcap_{i=1}^{m} k_i^+$. A triangle $T$ is called empty or negative, if $T = \bigcap_{i=1}^{3} l_i^-$, where $l_1^-, l_2^-, l_3^-$ are minimal half-planes that are translates of some $k_j^-$ ($j = 1, \ldots, m$) such that $\bigcap_{i=1}^{3} l_i^+ = \emptyset$.

Proof of theorem 1. Let $F$ be a family of pairwise intersecting polygons related to a convex $n$-gon. Observe the set of $n$ minimal halfplanes. Let $E$ be the family of all empty triangles created by them and let $N = e(n) = |E|$. We will prove the theorem by induction on $N$. It’s obviously true for $N = 0$ since then the intersection of any three minimal halfplanes is not empty, hence the intersection of any three halfplanes is not empty, hence by Helly’s theorem $\bigcap F \neq \emptyset$.

Suppose $N > 0$ and Let $E$ be an arbitrary triangle in $E$. Observe the edges of $E$. Each of them comes from a line through an edge of some $F \in F$. Let $E = e_1^- \cap e_2^- \cap e_3^-$, let $M_1, M_2, M_3$ be the midpoints of the edges of $E$, and let $M = m_1^+ \cap m_2^+ \cap m_3^+$ be the triangle created by the midpoints, where $m_i$ is parallel to $e_i$ for $i = 1, 2, 3$.

Since any two of sets intersect, for any $F \in F$, $F = \bigcap_{i=1}^{3} f_i^+$ contains at least one of the points $M_1, M_2, M_3$. Otherwise, for $i, j, k = 1, 2, 3; i \neq j \neq k$, there would exist an angle $f_i^+ \cap f_j^+$ which is strictly contained in the angle $m_i^+ \cap m_j^+$, hence disjoint from the halfplane $e_k^+$, thus disjoint from a member of $F$. It follows that $F$ can be divided into to three subfamilies, as follows:

$$\mathcal{F}_1 = \{F \in F | M_1 \in \bigcap_{i=1}^{3} f_i^+ \}$$

$$\mathcal{F}_2 = \{F \in F | M_2 \in \bigcap_{i=1}^{3} f_i^+, M_1 \notin \bigcap_{i=1}^{3} f_i^+ \}$$

$$\mathcal{F}_3 = \{F \in F | M_3 \in \bigcap_{i=1}^{3} f_i^+, M_1 \notin \bigcap_{i=1}^{3} f_i^+, M_2 \notin \bigcap_{i=1}^{3} f_i^+ \}$$
where each of these subfamilies contains no empty triangle of type \( E \), hence having at most \( N - 1 \) empty triangles. Since by Helly’s theorem, a family with no empty triangles is 1-pierceable, we get the following recursive inequality for the piercing number \( f(N) \):

\[
f(N) \leq 3f(N - 1)
\]

Hence:

\[
f(N) \leq 3^N
\]

Since the number of maximal empty triangles \( N = e(n) < \binom{n}{3} \) we finally get:

\[
f(n) < 3\binom{n}{3}, \quad \square
\]

More detailed explanations and drawings to be added later...

**Proof of theorem 2.** Let \( \mathcal{F} \) be a family of sets related to the convex \( n - \text{gon} \) \( F \) with edges \( h, v, a_1, \ldots, a_{n-2} \), such that \( h = [-1, 0], v = [0, y] \) for any \( y > 0 \), and the edges \( a_1, \ldots, a_{n-2} \) have positive slopes.

Observe the set of minimal halfplanes \( h^+, v^+, a_i^+ (i = 1, \ldots, n-2) \) and choose \( 1 \leq s \leq n - 2 \), so that \( \Delta E \) is the empty triangle \( h^- \cap v^- \cap a_s^- \neq \emptyset \).

Let \( M_h, M_v \) and \( M_s \) be the midpoints of \( \Delta E \), and let \( \Delta M \) be the triangle \( \Delta M_h, M_v, M_s \) with edges \( m_h, m_v, m_s \). First, note that \( \Delta M \) has the following:

**Two Edges Outside (TEO) property.** Let \( F \in \mathcal{F} \) be a polygon and let \( L = \{l_1, l_2, l_3\} \) be a subset of its edges so that \( l_1 \parallel h, l_2 \parallel v, l_3 \parallel a_s \). Then at most one member of \( L \) intersects \( \Delta M \).

To prove the TEO property it is enough to notice that the pairwise intersection implies that no polygon can have a vertex inside \( \Delta M \), hence no two edges of a polygon can intersect inside \( \Delta M \), hence if \( l_i \in L \) intersects \( \Delta M \), the other two edges must lie outside \( \Delta M \).

Now, proceed by choosing \( a_s \) as the line with the smallest positive slope (with respect to the \( x - \text{axis} \)), such that \( \Delta E = h^- \cap v^- \cap a_s^- \neq \emptyset \). Note that all polygons in \( \mathcal{F} \) have edges \( h', v' \) (\( h' \parallel h, v' \parallel v \)), but there might exist...
ones that do not have edge \( a'_s \parallel a_s \). Let \( A_s \subset F \) be the subfamily of all polygons that have edge \( a'_s \) and do not intersect with \( \{M_h, M_v, M_s\} \). We examine two cases.

**Case 1.** \( M_s \in a'_i \) for \( i = 1, \ldots, n - 2 \).

We note that in this case, any polygon in \( A_s \) has the following properties:

\[
a'_s \text{ is outside } \Delta M
\]

\[
h' \text{ is outside } \Delta M
\]

To establish those properties, note that if property 1 does not hold, then by TEO both \( h' \) and \( v' \) are outside the triangle, hence, both \( h'^+ \) and \( v'^+ \) contain \( M_s \), and since \( M_s \in a'_i \) for \( i = 1, \ldots, n - 2 \), it implies that the polygon itself contains \( M_s \) - a contradiction.

As for property 2, note that if it does not hold, then \( h' \) intersects \( \Delta M \), and by TEO both \( v'^+ \) and \( a'^+_s \) contain \( M_h \). Further more, pairwise intersection implies that the intersection point \( P = h' \cap v \) belongs to the polygon, hence the intersection point \( P_i = a'_i \cap h' \) for any \( i = 1, 2, \ldots, n - 2 \) lies to the left of \( P \). Let \( \alpha = \angle(a_s, h) \) and \( \beta = \angle(a'_i, h) \) for \( i \neq s \). If \( a'^-_i \) does not create an empty triangle with \( h^- \) and \( v^- \) then \( a'^+_i \) contains \( M_h \). If, on the other hand, \( a'^-_i \cap h^- \cap v^- \neq \emptyset \), then since \( a_s \) has the smallest slope among all \( a_i \)'s that create an empty triangle with \( h^- \) and \( v^- \), it follows that \( \beta > \alpha \), hence again, \( M_h \in a'^+_i \) - a contradiction. See figure 1.

It follows that the members of \( A_s \) do not create empty triangles similar to \( h^- \cap v^- \cap a'^-_s \).

**Case 2.** There exits a halfplane \( a^-_i \) such that \( M_s \in a^-_i \).

Let \( A^-(M_s) = \{a^-_i | M_s \in a^-_i \} \). Let \( H = a_s \cap v \), let \( h_s \) be the horizontal line through \( H \), let \( v_s \) be the vertical line through \( M_s \) and let \( P = v_s \cap h_s \). Choose an arbitrary \( a_i \) from \( A^-(M_s) \) and construct a new auxiliary triangle
\[ \triangle T_i = HXY \] as follows:

- if \( P_i = a_i \cap h_s \) lies to the right of \( P \), then \( Y = P \) and \( X = M_s \).

- if \( P_i = a_i \cap h_s \) lies to the left of \( P \), then \( Y = P_i \) and \( X = v_i \cap a_s \) where \( v_i \) is the vertical line through \( P_i \).

See figure 2.

Denoting by \( \mathcal{A}_i \subset \mathcal{F} \) the subfamily of all polygons that have edge \( a'_s \) and do not contain the vertex \( X \), we see \( \triangle T_i \) has properties similar to those of \( \triangle M_h, M_v, M_s \) we examined in case 1, i.e. any polygon in \( \mathcal{A}_i \) has its \( a'_s, h' \) edges outside \( \triangle T_i \).

Hence if \( \mathcal{A}_i \subset \mathcal{F} \) is the subfamily of all polygons that have edge \( a'_s \) and do not intersect with \( \{ M_h, M_v, M_s, X \} \) then the members of \( \mathcal{A} \) do not create
empty triangles similar to $h^- \cap v^- \cap a_s^-$.  

Thus, we get the following recursive inequality for the piercing number $f(N)$ where $N$ is the number of maximal empty triangles:  

$$f(N) < f(N - 1) + 4$$  

hence:  

$$f(N) \leq 4N$$  

and since $N \leq n - 2$ we finally have  

$$f(n) \leq 4(n - 2). \square$$