Deep Linear Networks Dynamics: Low-Rank Biases Induced by Initialization Scale and L2 Regularization

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Abstract

For deep linear networks (DLN), various hyperparameters alter the dynamics of training dramatically. We investigate how the rank of the linear map found by gradient descent is affected by (1) the initialization norm and (2) the addition of $L_2$ regularization on the parameters. For (1), we study two regimes: (1a) the linear/lazy regime, for large norm initialization; (1b) a "saddle-to-saddle" regime for small initialization norm. In the (1a) setting, the dynamics of a DLN of any depth is similar to that of a standard linear model, without any low-rank bias. In the (1b) setting, we conjecture that throughout training, gradient descent approaches a sequence of saddles, each corresponding to linear maps of increasing rank, until reaching a minimal rank global minimum. We support this conjecture with a partial proof and some numerical experiments. For (2), we show that adding a $L_2$ regularization on the parameters corresponds to the addition to the cost of a $L_p$-Schatten (quasi)norm on the linear map with $p = \frac{2}{L}$ (for a depth-$L$ network), leading to a stronger low-rank bias as $L$ grows. The effect of $L_2$ regularization on the loss surface depends on the depth: for shallow networks, all critical points are either strict saddles or global minima, whereas for deep networks, some local minima appear. We numerically observe that these local minima can generalize better than global ones in some settings.

1 Introduction

In spite of their widespread usage, the theoretical understanding of Deep Neural Networks (DNN) remain limited. In contrast to more common statistical methods which are built (and proven) to recover specific structure of the data, the development of DNNs techniques has been mostly driven by empirical results. This has lead to a great variety of models which perform consistently well, but without a theory explaining them. In this paper, we provide a theoretical analysis of linear DNNs (DLNs), whose simplicity makes them particularly attractive as a first step towards the development of such a theory.

DLNs have a non-convex loss landscape and the behavior of training dynamics can be subtle. For shallow networks the convergence of gradient descent is guaranteed by the fact that the saddles are strict and that all minima are global [5, 26, 29, 33]. In contrast, the deep case features non-strict saddles [26], nevertheless convergence can be guaranteed with the right initialization [2].

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A recent line of work focuses on the implicit bias of DLNs, and consistently revealing some form of implicit sparsity. Diagonal networks are known to learn minimal $L_1$ solutions \cite{35,41}. With a specific initialization and MSE loss, DLNs learn the singular components of the signal one by one \cite{38,1,39,4,3}. Recently, it has been shown that with losses such as the cross-entropy and the exponential loss, the parameters diverge towards infinity, but end up following the direction of the max margin classifier w.r.t. the $L_p$-Schatten (quasi)norm \cite{16,17,50,23,24,9,32,35,42}.

In parallel, recent works have shown the existence of two regimes in large-width DNNs: a linear/lazy regime where learning is described by the so-called Neural Tangent Kernel (NTK) guaranteeing linear convergence \cite{21,12,7,4,27,18}, and an active regime where the dynamics is nonlinear \cite{8,37,34,33,9}. For DLNs, both regimes can be observed as well, with evidence that while linear regime exhibits no sparsity, the active regime favors solutions with some kind of sparsity \cite{41,35}.

Short before the publication of this paper, we came aware of recent work \cite{30} which describes a similar Saddle-to-Saddle regime, while our results are almost equivalent for shallow networks we discuss the differences for deeper networks in Section 3.2.2.

### 1.1 Contributions

We study deep linear networks (DLNs) $x \mapsto A_\theta x$ of depth $L$, with $A_\theta = W_L \cdots W_1$ where $W_1, \ldots, W_L$ are matrices and investigate the gradient flow minimizing a loss $C(W_L \cdots W_1)$, for a general cost $C$. We particularly focus on (1) the effect of the initialization scale and (2) the effect of an $L_2$ regularization term on the parameters added to the cost $C$.

(1) **Initialization Scale:** Depending on the variance of the parameters at initialization, we observe two regimes:

(a) **Linear Regime:** when the network is sufficiently wide and the variance of the parameters at initialization is sufficiently large, one falls into the so-called linear or lazy regime, which is governed by the Neural Tangent Kernel (NTK). In this regime, the gradient flow trajectory stays far away from saddle points and does not induce any low-rank bias.

(b) **Saddle-to-Saddle Regime:** as the norm of the parameters at initialization goes to zero, we show that the gradient flow trajectory goes from the saddle $0 = \vartheta^0$ to a rank-one saddle $\vartheta^1$. Motivated by this, we propose a conjecture (backed by numerical experiments) describing the full gradient flow, suggesting that it goes from saddle to saddle, visiting a sequence $\vartheta^0, \ldots, \vartheta^K$ of critical points (the first $K$ ones being saddles, the last one being either a global minimum or a point at infinity), corresponding to matrices of increasing ranks.

(2) **$L_2$-regularization:** We first show a correspondence between the minimizers of the two cost functions $C^{(NN)}$ and $C^{(Sch)}$, where $C^{(NN)}$ is $L_2$-regularized cost $C(A_\theta) + \lambda \|\theta\|^2$ and $C^{(Sch)}$ is the $L_p$-Schatten-regularized cost $C(A) + L \lambda \|A\|_p^p$ with $p = \frac{2}{L}$. This favors lower and lower rank solutions as the depth $L$ grows. We prove that for shallow networks ($L = 2$), all critical points are either strict saddles or global minima, guaranteeing convergence to a solution. For deep networks ($L > 2$), non-optimal local minima of $C^{(NN)}$ appear, corresponding to local minima of $C^{(Sch)}$ (which is non-convex, as $p = \frac{2}{L} < 1$); we show that in fact some of these local minima are good approximations to the minimal-rank solution for small enough $\lambda$.

### 2 Preliminaries

A deep linear network (DLN) of depth $L$ and widths $n_0, \ldots, n_L$ is the composition of $L$ matrices

$$A_\theta = W_L \cdots W_1$$

where $W_\ell \in \mathbb{R}^{n_\ell \times n_{\ell-1}}$ and $\theta = (W_L, \ldots, W_1) \in \mathbb{R}^P$ is the vector of parameters made up of the concatenation of all the matrices and $P = \sum_{\ell=1}^{L} n_{\ell-1} n_\ell$. We call $n_0$ the input dimension and $n_L$ the output dimension. For $\ell \geq k$, we write $W_{\ell:k} = W_\ell \cdots W_k$. All parameters are initialized as i.i.d. $\mathcal{N}(0, \tilde{\sigma}^2)$ Gaussian for some $\tilde{\sigma} > 0$.

We will focus on the so-called rectangular networks, when the number of neurons of all hidden layers is the same, i.e. $n_1 = \cdots = n_{L-1} = w$. In this setting, we call $w$ the width of the network. We write
We will mostly consider general convex losses, though in Section 3.1 we instead require the loss to satisfy the Polyak-Łojasiewicz (β-PL) inequality. We study the dynamics of gradient descent on the network’s parameters to stress the distinction with a cost on matrices, usually denoted by $C : \mathbb{R}^P \to \mathbb{R}$ on the network’s parameters to stress the distinction with a cost on matrices, usually denoted by $C : \mathbb{R}^{nL \times n_0} \to \mathbb{R}$.

### 2.1 Costs

We will mostly consider general convex losses, though in Section 3.1 we instead require the loss to satisfy the β-Polyak-Łojasiewicz (β-PL) inequality [25].

**Definition 1.** A differentiable cost $C$ on some space $E$ with global minimum $C_{\min} \in \mathbb{R}$ satisfies the β-Polyak-Łojasiewicz (β-PL) inequality if $\frac{1}{2} \|\nabla C(x)\|^2 \geq \beta (C(x) - C_{\min})$ for all $x \in E$.

One should think of the β-PL inequality as a generalization of β-strong convexity since

1. Any β-strongly convex loss satisfies the β-PL inequality.
2. The β-PL inequality guarantees a $O(e^{-\beta t})$ rate of convergence of gradient descent initialized at point $x_0$ to a closeby global minimum $x^*$: $\|x_0 - x^*\| \leq \sqrt{\frac{2}{\beta}} (C(x_0) - C_{\min})$.

Note however that the β-PL inequality does not imply convexity, but only invexity, i.e. that all critical points of the loss are global minima. For our numerical experiments, we will focus on two convex costs which both satisfy the PL inequality:

1. The Mean-Squared Error (MSE) loss $C(A) = \frac{1}{N} \|AX - Y\|_F^2$ for some inputs $X \in \mathbb{R}^{n_0 \times N}$ and labels $Y \in \mathbb{R}^{n_0 \times N}$, where $\|\cdot\|_F$ denotes the Frobenius norm. The MSE loss satisfies the β-PL inequality with $\beta = \frac{2s_0^2}{N}$ where $s_0$ is the smallest singular value of $X$, it is also β-strongly convex when $\text{Rank}(X) = n_0$.
2. The Matrix Completion (MC) loss $C(A) = \frac{1}{N} \sum_{i=1}^N (A_{k_i,m_i} - A^*_{k_i,m_i})^2$ for some true matrix $A^*$ of which we observe only $N$ entries $(k_1, m_1), \ldots, (k_N, m_N)$. The MC loss is not strongly convex but satisfies the β-PL inequality for $\beta = \frac{2}{N}$ (see Appendix).
2.2 $L_p$-Schatten quasinorm and regularization

In some setting, such as in Matrix Completion, one typically assumes that the linear map that we are trying to fit is low-rank and we want to recover the lowest rank map amongst the solutions $S_C = \{ A : C(A) = C_{\min} \}$. Finding this minimal rank solution can be very difficult (it is in general NP hard [15]), so a common simplification is to instead minimize the nuclear norm $\| A \|_* = \sum_{i=1}^{\min(n,n,L)} \sigma_i$, where the $\sigma_i$ are the singular values of $A$ [11][10]. Thanks to the convexity of the nuclear norm, one can optimize it efficiently, though it requires some special methods because of its non-differentiability. However in some cases, such as in Matrix Completion, the minimal nuclear norm solution does not always coincide with the minimal rank solution [3][36].

More generally, we can consider the $L_p$-Schatten (quasi)norm $\| A \|_p = (\sum_{i=1}^n \sigma_i^p)^{\frac{1}{p}}$ (the nuclear norm $\| A \|_*$ is a special case when $p = 1$). When $p < 1$ it is only a quasinorm (since it is not convex). In the limit $p \to 0$, the $L_p$-Schatten norm converges to the rank function: $\lim_{p \to 0} \| A \|_p = \text{Rank}(A)$, suggesting that the low-rank bias is stronger for smaller $p$. The non-convexity of the $L_p$-Schatten quasinorm when $p < 1$ leads to the apparition of local minima making its optimization difficult.

For a cost $C$ on matrices, we write $C_{(Sch)}(A) = C(A) + \lambda \| A \|_p^p$.

2.3 Symmetries and Invariance

A key tool in this paper is the use of two important symmetries of the parametrization map $\theta \mapsto A_\theta$ in DLNs: rotations of hidden layers and inclusions in larger DLNs.

**Rotations:** A $L-1$ tuple $R = (O_1, \ldots, O_{L-1})$ of orthogonal $w \times w$ matrices is called a $w$-width network rotation, or in short a rotation. A rotation $R$ acts on a parameter vector $\theta = (W_L, \ldots, W_1)$ as $R\theta = (W_L O_{L-1}^T, O_{L-1} W_{L-1} O_{L-2}^T, \ldots, O_1 W_1)$. The space of rotations is an important symmetry of DLN: indeed, for any parameter $\theta$, and any cost $C$, the two following important properties hold:

$$A_{R\theta} = A_\theta, \quad \nabla \theta C(A_{R\theta}) = R \nabla \theta C(A_\theta),$$

where we considered $\nabla \theta C(A_\theta) \in \mathbb{R}^{P_{L,w}}$ as another vector of parameters. These properties imply that if $\theta(t) = \gamma(t, \theta_0)$ is a gradient flow path, then so is $R\theta(t) = \gamma(t, R\theta_0)$.

**Inclusion:** The inclusion $I_{(w \mapsto w')}^{(w \mapsto w')}$ of a network of width $w$ into a network of width $w' > w$ (by adding zero weights on the new neurons) is defined as

$$I_{(w \mapsto w')}^{(w \mapsto w')} \theta = \left( \begin{array}{c} W_L \ 0 \end{array} \right), \left( \begin{array}{c} W_{L-1} \ 0 \ 0 \end{array} \right), \ldots, \left( \begin{array}{c} W_2 \ 0 \ 0 \end{array} \right), \left( \begin{array}{c} W_1 \ 0 \end{array} \right).$$

For any parameters $\theta$ and any cost $C$, we have $A_{I_{(w \mapsto w')}^{(w \mapsto w')}} = A_\theta$ and $\nabla C(A_{I_{(w \mapsto w')}^{(w \mapsto w')}}) = I_{(w \mapsto w')}^{(w \mapsto w')} \nabla C(A_\theta)$: the image of the inclusion $I_{(w \mapsto w')}^{(w \mapsto w')}$ (as well as any rotation $R \text{Im} I_{(w \mapsto w')}^{(w \mapsto w')}$ thereof) is invariant under gradient flow.

3 Initialization Scale

3.1 Linear/Lazy Regime

Recall that all parameters in $\theta$ are initialized as i.i.d. $\mathcal{N}(0, \sigma^2)$ Gaussian for some $\sigma > 0$. The variance of the entries of $A_\theta$ is then $\sigma^2 L w L^{-1}$ and thus if $\tilde{\sigma} = \sigma \sqrt{w^{-\frac{L-1}{L}}}$, the variance of each entry of $A_\theta$ is equal to $\sigma^2$ for all widths $w$. We call this initialization the $\sigma^2$-Large Norm Initialization.

This limit is the same as the linear/lazy limit of non-linear DNNs, whose dynamics is described by the Neural Tangent Kernel (NTK) [21]. In linear networks, the NTK is simply a $n_L n_0 \times n_L n_0$ matrix (i.e. a linear map from $n_L \times n_0$ matrices to $n_L \times n_0$ matrices) defined as $\Theta(\theta) = J A_\theta (J A_\theta)^T$ for the Jacobian $J A_\theta$. According to [21], for sufficiently large widths, the NTK concentrates at initialization around its expectation $\eta I_{n_L \times n_L}$ with $\eta = L \sigma^2 \sqrt{\frac{L-1}{L}} w^{\frac{L-1}{L}}$ and should stay almost constant during training. Note that, up to a scaling which can be balanced out by changing the learning rate, the limiting NTK does not depend on the depth of the DLN. In particular, as $w \to \infty$, the limiting
Furthermore, if the global minimum is initialized with the standard deviation \( \sigma = w^{-1/2} \). The rank of the network matrix increases incrementally as the gradient trajectory follows the paths between the saddles. **Shallow case (left panels):** \( L = 2 \) and \( w = 50 \); in the saddle-to-saddle regime (shown in red), the initialization scale is \( \sigma = w^{-1/2} \). Bigger initialization scales make the plateaus in the loss curve shorter if the same learning rate is used. **Deep case (right panels):** \( L = 4 \) and \( w = 100 \); in the saddle-to-saddle regime (shown in blue), the initialization scale is \( \sigma = w^{-1} \). We observe that the transitions from saddles to saddles are sharper (this is even more explicit in the norm of the gradient, see the extended figure in the Appendix). The input data is standard Gaussian, the outputs are generated by a rank 3 teacher of size \( 10 \times 10 \) corrupted with noise, and the loss is MSE.

Figure 2: Training in linear/lazy vs. saddle-to-saddle regimes in shallow and deep networks when learning a low rank matrix corrupted with noise. Black lines (lazy regime): the parameters are initialized at the global minimum (Appendix). The input data is standard Gaussian, the outputs are generated by a rank 3 teacher of size \( 10 \times 10 \) corrupted with noise, and the loss is MSE.

In the following, we adapt and simplify the arguments of [27, 31], to guarantee linear convergence to a closeby global minimum for large but finite width:

**Proposition 2.** Let \( C \) be a cost on matrices that satisfies the \( \beta \)-PL inequality for some \( \beta > 0 \) and let \( C_{\min} \in \mathbb{R} \) be its minimum value. Consider a gradient flow path \( \theta(t) = \gamma(t, \theta_0) \) on \( C^{NN} : \theta \mapsto C(A(\theta)) \) for a network of depth \( L \), width \( w \), and with \( \sigma^2 \)-Large Norm Initialization. For all \( \delta > 0 \) and any large enough \( w \), it holds with probability at least \( 1 - \delta \) that \( \theta(t) \) converges to a global minimum \( \theta^* \) and for \( \beta' = \frac{\beta}{2} \sigma^2 w^{L-1} \), we have:

\[
C^{NN}(\theta(t)) - C_{\min} \leq (C^{NN}(\theta_0) - C_{\min}) e^{-\beta't}.
\]

The global minimum \( \theta^* \) satisfies \( \|\theta_0 - \theta^*\| \leq r = \sqrt{\frac{2}{\beta'}} (C(A_{\theta_0}) - C_{\min}). \)

Furthermore, if \( C \) is convex, we have the uniform bound

\[
\|A_{\theta(t)} - A(\eta t)\| = O\left(w^{-\frac{1}{2}}\right),
\]

where \( \eta = L \sigma^2 w^{L-1} \) and \( A(t) = \gamma_C(t, A_{\theta_0}) \) is the gradient flow path on the cost \( A \mapsto C(A) \) starting from \( A_{\theta_0} \).

**Sketch of proof.** Using the fact that the cost \( C \) satisfies the \( \beta \)-PL inequality, we show that \( C^{(NN)}_L \) satisfies the (pointwise) \( |\lambda_{\min}(\Theta(\theta))| \beta \)-PL inequality at any \( \theta \). At initialization, with high probability, \( \lambda_{\min}(\Theta(\theta_0)) > \frac{3\eta}{T} \) in a ball \( B(\theta_0, R) \) with \( R \propto \sqrt{w}r \), the variation of the NTK is small and thus \( \lambda_{\min}(\Theta(\theta)) > \frac{\eta}{2} \) inside this ball. This proves the \( |\beta' = \frac{\beta}{2} \beta \)-PL inequality in \( B(\theta_0, R) \) which guarantees convergence to a global minimum at a distance at most \( r \) under the condition that \( R > r \). This condition is fulfilled when \( w \) is large enough since \( R \propto \sqrt{w}r \). Furthermore, we show that that for all times \( t \), \( \|\Theta^{(L)}(\theta(t)) - \eta I_{n,n}w^2\| = O(\eta w^{-\frac{1}{2}}) \) which implies the \( O(w^{-\frac{1}{2}}) \) convergence to the limiting depth-1 dynamics as \( w \to \infty \). \( \square \)

In contrast to the saddle-to-saddle regime, with large norm initialization, gradient descent remains in a ball around initialization where the loss satisfies the PL inequality. This implies that, around the initialization, all critical points are global minima and it guarantees that convergence is fast. The benign non-convexity of the loss around the gradient path in this regime was also observed in [22].

The drawback of this large norm initialization dynamics is that they are \( O(w^{-\frac{1}{2}}) \) close to the dynamics of gradient descent on matrices \( A(t) \) (up to a rescaling of the learning rate). Thus, the depth of the network has almost no effect on the matrix that is learned and the compositional structure of the network is not leveraged in order to obtain a low-rank bias. Hence, there is no reason to use a DLN in this regime: doing so only increases the memory usage and computational cost.
3.2 Saddle-to-Saddle regime

A very interesting phenomenon happens when one initializes the network with a very small norm: the train error goes through a number of plateaus where the cost is almost constant before suddenly dropping to a lower plateau and so on and so forth until reaching a minimum (Figures 1 and 2).

In order to study the small norm initialization asymptotic, we consider the limit as \( \alpha \to 0 \) of the gradient path \( \theta_i(t) = \gamma(t, \alpha \rho(t)) \) initialized at a point \( \alpha \rho_0 \) for some random approach direction \( \rho(t) \) on the sphere \( S_{P^L,w}^{-1} \). In this setting, we observe and conjecture that during training, gradient descent approaches a sequence of saddles one after the other, each of which slows down learning: this leads to the plateaus mentioned above. The rank of the matrix \( A_0 \) increases by one from saddle to saddle starting from the saddle at the origin with rank 0. This behavior is in sharp contrast to the linear regime where the training path remains far away from any saddle and where the training loss decays exponentially fast.

**Conjecture 3.** With probability 1 over the sampling of the direction \( \rho_0 \in S_{P^L,w}^{-1} \), there exists \( K + 1 \) critical points \( \vartheta^0, \ldots, \vartheta^K \in \mathbb{R}^{P_L,w} \) (with \( \vartheta^0 = 0 \)) and \( K \) paths \( \vartheta^1, \ldots, \vartheta^K : \mathbb{R} \to \mathbb{R}^{P_L,w} \) connecting the critical points (i.e. \( \lim_{t \to -\infty} \vartheta^k(t) = \vartheta^k-1 \) and \( \lim_{t \to +\infty} \vartheta^k(t) = \vartheta^k \)) such that the path \( \theta_i(t) \) initialized at \( \alpha \rho_0 \) converges as \( \alpha \to 0 \) to the concatenation of the paths \( \vartheta^1(t), \ldots, \vartheta^K(t) \) in the following sense: for all \( k < K \), there exists times \( t^k_{\alpha} \) (which depends on \( \rho(t) \)) such that

\[
\lim_{\alpha \to 0} \theta_i(t^k_{\alpha} + t) = \vartheta^k(t).
\]

Furthermore, for all \( k < K \), there is a deterministic path \( \vartheta^k(t) \) and a local minimum \( \vartheta^k \) of a width-\( k \) network such that for some rotation \( R \) (which depends on \( \rho_0 \)), \( \vartheta^k(t) = R f(k \to w)(\vartheta^i(t)) \) and \( \vartheta^k = R f(k \to w)(\vartheta^k) \) for all \( k \) and \( t \).

Note that for losses such as the cross-entropy, the gradient descent may diverge towards infinity, as studied in [40][16]. We focus on the case where \( \vartheta^K \) is a finite global minimum. By the invariance under gradient flow of the image of the inclusion \( \text{Im}[f(k \to w)] \), the inclusion of a width-\( k \) local minimum \( \vartheta^k \) into a larger network is a saddle \( \vartheta^k \) (if \( A_{\vartheta^e} \) is not a global minimum of \( C \)). These type of saddles are closely related to the symmetry-induced saddles studied in [43] for non-linear networks.

As \( \alpha \to 0 \) the time spent close to each saddle increases towards infinity, i.e. \( \lim_{\alpha \to 0} t^k_{\alpha} + t^k_{\alpha} = \infty \). In the next subsection, we describe the rates at which the \( t^k_{\alpha} \)'s grow with \( \alpha \).

In the following, we first discuss the setting of diagonal networks where Conjecture 3 follows from some previous work. Then we prove the existence of the first path \( \vartheta^1 \) in the conjecture. Finally, we discuss the implications of this conjecture on the low rank bias of DLNs in this setting.

### 3.2.1 Diagonal Networks

Conjecture 3 is related to the dynamics of DLN observed in [38][1][39][14][3] during training with the MSE loss when, in the simplest case, the input data \( X \) and output labels \( Y \) are diagonal \footnote{Or for non-square matrices, when the off-diagonal entries are equal to zero, i.e. \( X_{ij} = 0 \) when \( i \neq j \).} and when the network is initialized with diagonal weights \footnote{More complex cases, which contain linear autoencoder and deep linear multi-class predictions, can be obtained as rotations and small perturbations of this simplest case.}. In this setting, the dynamics of training along each output coordinate \( i \) is independent. Besides, the larger the value \( X_{ii}Y_{ii} \), the faster the network learns along it, so that it learns each coordinate sequentially when the variance at initialization vanishes.

In fact, we note that this behavior can also be interpreted as a saddle-to-saddle dynamics where the \( k \)-th saddle \( \vartheta^k \) fits only the first \( k \) coordinates with the largest values \( X_{ii}Y_{ii} \), i.e. \( A_{\vartheta^e}X = P_kY \) where \( P_k \) is the projection on these \( k \) coordinates. In the shallow case, following the argument of [14], note that \( t^k_{\alpha} = c_k - \frac{\log(\alpha)}{X_{kk}Y_{kk}} \) for some undetermined constants \( c_k \), whereas for the deep case we show in Appendix C that \( t^k_{\alpha} = c_k + \frac{\alpha^{-k} - 1}{(L-2)X_{kk}Y_{kk}} \).

The papers dealing with the small norm initialization in this simple context mostly emphasize the independence of the dynamics along the coordinates [38][1][39][14][3], but do not take notice of the
saddle-to-saddle aspect. While the independence is very specific to the particular case described above (for example it breaks down when the loss is not the MSE or when the input data and the labels cannot be diagonalized with the same basis), the saddle-to-saddle behavior is much more general. Going further, the saddle-to-saddle regime could even be extended to non-linear networks.

3.2.2 The First Path

In this paper, we prove the existence of the first path $\theta^1$ in Conjecture 3, this path connects the saddle at the origin $\theta^0 = 0$ to a rank 1 saddle $\theta^1$.

**Theorem 4.** Assume that the largest singular value $s_1$ of the gradient of $C$ at the origin $\nabla C(0) \in \mathbb{R}^{n \times n_0}$ has multiplicity 1. With probability 1 over the sampling of the direction $p_0 \in S^{F_L^L-1}$, there is a path $\theta^1(t) \in \mathbb{R}^{F_L^L}$ with $\lim_{t \to -\infty} \theta^1(t) = \partial^1$ and some $t_0$ such that $\lim_{t \to 0} \theta(t) = \theta^1(t)$. The path $\theta^1$ converges to a critical point $\psi$ as $t \to +\infty$.

Furthermore, there is a unique and deterministic path $\theta^1 : \mathbb{R} \to \mathbb{R}^L$ in the space of DLN of width 1 and a random rotation $R$ such that $\theta^1 = RI(1-w)(\hat{\vartheta}^1)$.

**Sketch of proof.** We consider an escape time $t^1_0$ such that $C(\theta(t_0)) = C(0) - \epsilon$ for some small enough $\epsilon > 0$. The limiting path $\theta^1(t) = \lim_{t \to 0} \theta(t_0 + t)$ is what we call an escape path of $\theta^0$, i.e. a path such that $\lim_{t \to -\infty} \theta(t, S) = 0$. We then show that this path escapes the saddle at an optimal speed of $\text{cst} \times e^{-\text{st}}$ for shallow networks and of $\sqrt{L} (\text{cst} + (L-2)s_1(-t))^{-\frac{1}{2}}$ for deep networks. At last, we prove that all optimal escape paths are of the form $\theta^1(t) = RI(1-w)(\hat{\vartheta}^1(t))$ for a deterministic $\hat{\vartheta}^1(t) : \mathbb{R} \to \mathbb{R}^L$ and that all other paths escape at a strictly lower speed.

**Remark:** Shortly before the publication of this article, we came aware of the paper [30] which is very similar to our description of the Saddle-to-Saddle regime (though it does not discuss the contrast with the linear/lazy regime or the effect of $L_2$ regularization).

For shallow networks both results are almost equivalent, though the techniques are different, in particular when dealing with the fact that the escape directions (and escape paths) are only unique up to rotations: while [30] uses a clever trick to study the dynamics of the output matrix $A_\theta(t)$ directly, in which case the dynamics are unique, we instead focus on the dynamics of the parameters $\theta(t)$ and give an identification of all optimal escape paths, allowing us to show that the path followed is unique up to symmetries of the network. Note that [30] also only proves the first step of the Saddle-to-Saddle regime, for the subsequent steps it is assumed that the next saddle is not approached along a ‘bad’ direction (as we discuss in Section 3.2.3).

For deep networks, our results are more general when it comes to the initialization. To avoid the uniqueness problem, their analysis relies heavily on the assumption that the weights of the network are balanced, allowing a description of the dynamics of $\theta$. Because we do not rely on this trick, our analysis does not require a balanced initialization.

3.2.3 Subsequent Paths

Similar arguments could be applied to paths $\theta^1(t) = \gamma(t, \partial^1 + \alpha \partial^0)$ initialized close to the rank 1 saddle $\partial^1$. Yet, in order to prove the existence of the next paths in Conjecture 3, one has to deal with two issues.

First, even though Theorem 4 guarantees that gradient descent will come arbitrarily close to the next saddle $\partial^1$, it may not approach it along a generic direction: it could approach along a ‘bad’ direction. Note that these bad directions have measure zero, hence they are almost surely avoided when the approaching direction is random as it is the case in Theorem 4.

Second, for deep networks ($L > 2$), the saddle $\partial^1$ has a different local structure to $\partial^0$. Indeed, at the origin, the $L - 1$ first derivatives vanish, leading to an (approximately) $L$-homogeneous saddle at the origin. On the contrary, at the rank 1 saddle $\partial^1 = RI(1-w)(\hat{\vartheta}^1)$, if $\hat{\vartheta}^1$ is a local minimum of the width 1 network, the Hessian is positive along the inclusion $\text{Im} \left[RI(1-w)\right]$. This implies that the dynamics can only escape the saddle through the Hessian null-space, along which the first $L - 1$
Algorithm $A_{c,T,\eta}$:

Initialize a width 1 network with weights $W_1 = ev_1^T$, $W_2 = \epsilon$ and $W_L = \epsilon u_1$ for $u_1, v_1$ the singular vectors of the largest singular value of $\nabla C(0)$.

Let $\theta = (W_L, \ldots, W_1)$ be the parameters after $T$ steps of GD with learning rate $\eta$.

While $C(A_{\hat{\theta}}) > C_{\min} + \epsilon$:

- Initialize a width $w+1$ network initialized with $W_1 = \begin{pmatrix} \hat{W}_1 \\ \epsilon v_1^T \end{pmatrix}$ and $W_L(0) = \begin{pmatrix} \hat{W}_L \\ 0 \end{pmatrix}$ and $W_{L+1}(0) = \begin{pmatrix} \hat{W}_{L+1} \\ \epsilon u_1 \end{pmatrix}$ for $u_1, v_1$ the singular vectors of the largest singular value of $\nabla C(A_{\hat{\theta}})$.

- Let $\hat{\theta} = (\hat{W}_L, \ldots, \hat{W}_1)$ be the parameters after $T$ steps of GD with learning rate $\eta$.

Algorithm $A_{c,T,\eta}$ can be viewed as performing a greedy search for a lowest-rank solution: it first tries to fit a width 1 network, then a width 2 network and so on until reaching a solution.

We used Algorithm $A_{c,T,\eta}$ to approximate the paths $\hat{\theta}^k$ and points $\hat{\theta}^k$ in Figure 1 for large $T$, the parameters $\hat{\theta}$ after $T$ steps of GD approximates $\hat{\theta}^k$ (and the gradient descent path up to time $T$ approximates $\hat{\theta}^k$).

### 3.2.4 Low Rank Bias

In the saddle-to-saddle regime, we can identify two different uses of the low-rank bias in the dynamics. With early-stopping, the network learns a low-rank linear map which does not fit the data: this could be useful in a regression setting with a low-rank signal and full-rank noise, as observed in Figure 2. In the absence of noise, early stopping is not needed: the network converges to a global minimum but the saddle-to-saddle trajectory could bias the network towards low-rank linear maps amongst the solutions $S_C$. Indeed, under the assumption that for all $k$, $A_{\hat{\theta}^k}$ minimizes the cost $C$ restricted to rank $k$ matrices, the final global minimizer $A_{\hat{\theta}}$ must be a minimal rank solution, i.e. $A_{\hat{\theta}} = \arg \min_{A \in S_{C}} \text{Rank}(A)$. This bias toward low-rank matrices is useful in matrix completion as observed in Figure 1.

### 4 $L_2$ Regularization

In this Section, we provide connections between $L_2$ regularization on the parameters $\theta$ of a depth-$L$ DLN and $L_p$-Schatten quasi norm regularization on the output matrix $A_\theta$ with $p = \frac{2}{\ell}$. These connections allow us to describe the low-rank bias of $L_2$ regularization in DLNs. The following Proposition relates the global minimum of the two cost functions $C_{L,\lambda}(\theta)$ and $C_{L,\lambda}^{(Sch)}$.

**Proposition 5.** For a network of depth $L$ with large enough widths (i.e. $n_\ell \geq \min\{n_0, n_L\}$) and any cost $C$, we have for $p = \frac{2}{\ell}$:

$$\min_{\theta} C(A_\theta) + \lambda \|\theta\|^2 = \min_A C(A) + \lambda L \|A\|_p^p.$$
As shown in Figure 3, depending on the initialization scale, we can converge to a variety of local minima. The effect of initialization scale on the low rank bias of the gradient descent for L2 regularized loss. Each dot represents a (3, 20)-DLN at convergence where the parameters are initialized with the standard deviation $\sigma$ (shown in x-axis). As the initialization scale decreases, the network converges to a lower rank local minimum. The MSE loss is regularized with $\lambda = 0.05$. Right: Origin is represented as a plus, the orange dot is a rank-1 local minimum and the red dot is a rank-2 local minimum. The loss surface on the plane between these three points is plotted in the log scale.

Since the $L_p$-Schatten quasinorm penalizes high rank matrices, with the right choice of $\lambda$, one could for example recover the low rank signal from the noise in a regression setting. Also, when $\lambda \to 0$, the minimizer $A^*_\lambda = \arg\min A C(A) + \lambda \|A\|_p^2$ converges to the minimal $L_p$-Schatten quasinorm solution, i.e. $\lim_{\lambda \to 0} A^*_\lambda = A^* = \arg\min A, C(A)=0 \|A\|_p^2$, which could be useful in Matrix Completion to find a low rank matrix fitting the entries.

The regularized loss surface takes a very different form depending on the depth $L$.

**Shallow Networks.** In the shallow case ($L = 2$) convergence to a global minimum is guaranteed since:

**Proposition 6.** Let $C$ be a convex and differentiable cost and let $\lambda > 0$. The critical points of the loss $C_{2,\lambda}^{(NN)} : \theta \mapsto C(A_\theta) + \lambda \|\theta\|^2$ are either strict saddles or global minima. Furthermore if $\theta^*$ is a global minimum, then $A_{\theta^*}$ is a global minimum of $C_{1,2\lambda}^{(Sch)} : A \mapsto C(A) + 2\lambda \|A\|_1$.

For such losses, gradient descent converges to a global minimum with probability 1 [29, 28]. Note however that the global minimizers of $C_{1,2\lambda}^{(Sch)}$ may capture a minimal rank solution [3].

**Deep Networks.** In the deep case, i.e. when $L > 2$ or $p = \frac{2}{L} < 1$, the map $A \mapsto \|A\|_2^2/L$ is not convex anymore. The loss $C_{p,\lambda L}^{(Sch)}(A) = C(A) + \lambda L \|A\|_p^p$ can feature many local minima. We find a correspondence between the local minima of $C_{p,\lambda L}^{(Sch)}$ and those of $C_{L,\lambda}^{(NN)}$:

**Proposition 7.** Let $C$ be a convex and differentiable cost on matrices, let $L > 2$, $\lambda > 0$ and $p = \frac{2}{L}$. If $\theta^*$ is a local minimum of $C_{L,\lambda}^{(NN)}$, then $A_{\theta^*}$ is a local minimum of $C_{p,\lambda L}^{(Sch)}$. Conversely, given a local minimum $A^*$ of $C_{p,\lambda L}^{(Sch)}$ with SVD decomposition $A^* = U S V^T$ and rank $k$, for any rotation $R$ of the width-$w$ network parameters

$$\theta^* = R I(k \to w) \left( U \hat{S}^{\frac{1}{2}}, S^{\frac{1}{2}}, ..., S^{\frac{1}{2}} V^T \right),$$

describes a local minimum of $C_{L,\lambda}^{(NN)}$ with $A_{\theta^*} = A^*$.

As shown in Figure [3] depending on the initialization scale, we can converge to a variety of local minima, each with different train and test losses. The following proposition shows that at least some of these local minima are good approximations of a minimal rank solution:

**Proposition 8.** For all $p < 1$, there is a sequence $B_1^*, B_2^*, \ldots$ where $B_k^*$ is a local minimum of $C_{p,1/k}^{(Sch)}$ such that $\lim_{k \to \infty} B_k^*$ is a minimal rank solution.

However, given the NP hardness of recovering a minimal rank solution [15], it should be in general difficult to find these “good” local minima.
4.1 Conclusion

This paper investigates the effects of initialization scale and $L_2$ regularization on dynamics of DLNs and in particular the associated low-rank bias.

For large-width DLNs with large initialization variance, convergence is fast (linear) thanks to benign non-convexity of the loss (the PL inequality holds along the gradient flow path), but the learned linear map exhibits no low-rank bias and depth does not provide any advantage. For DLNs with (vanishingly) small initialization, we encounter the saddle-to-saddle regime. We propose a conjectural description (with theoretical and numerical support) of the gradient flow: it automatically implements a low-rank bias, by performing a greedy search among saddles, incrementally increasing the corresponding ranks. Assuming that the gradient flow does not stop at a saddle, it will yield a low-rank solution.

We then show that from the point of view of the minimizers, adding an $L_2$ regularization term on the parameters to the cost function yields the same linear maps as the addition of an $L_{1/2}$-Schatten term to the cost on the DLN linear map. For shallow networks, the regularized loss surface has only strict saddles and global minima, guaranteeing convergence; for deep networks, multiple local minima appear, some of which approximate minimal rank solutions.

We thus see two interesting departures from the linear regime, which features an almost convex loss along the gradient path and shows no low-rank bias. One resulting from the small initialization norm (the saddle-to-saddle regime), and the other one from the addition of an $L_2$ regularization. Both regimes exhibit low-rank biases, which are related to the non-convexity of the loss surface - the sequence of saddles visited in the first case or the multiple local minima of the second. This suggests that in a number of cases, non-convexity should be embraced as a useful feature, rather than as a nuisance to the optimization and the resulting model quality.

5 Acknowledgements

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We organize the Appendix as follows:

- In Section A, we present the details for the numerical results presented in the main text together with some discussions.
- In Section B, we present the proofs for the Linear/Lazy regime.
- In Section C, we present the proofs for the Saddle-to-Saddle regime.
- In Section D, we present the proofs the results related to the addition of a $L_2$ regularization term.

**Erratum:** Since the submission of the main text, we have seen two errors that will be corrected in the final version:

- In Section 3.2.1 of the main text, we describe the rate at which the escape time $t^*_i$ grow. For the deep case, the exponent is correct but the constant is wrong. The corrected rates can be found in Section C.6.
- Proposition 8 of the main text requires a small correction. The correct statement is found in Section D.2, Proposition 46.

### A Further Experimental Details

![Graphs showing experimental details](image)

Figure 4: *an extension of Figure 2 in the main.* The top row shows the shallow case with $L = 2$, and the bottom row shows the deep case with $L = 4$. Colored lines (saddle-to-saddle regime): we observe that the gradient norm of the parameters is highly non-monotonic; a decrease down to 0 indicates approaching to a saddle, and a following increase indicates escaping it. We note that the peaks of the gradient norm are sharper in the deep case, suggesting a different rate of escape. Black lines (the NTK regime): the gradient norm decreases down to 0 monotonically. In the deep case the GD training is implemented for 1500000 iterations whereas in the shallow case it is only 100000 iterations.

**Experimental details of Figure 2 and 3 in the main:** We created a rank 3 teacher weight matrix $W_T = W_0W_0^T$ of size $10 \times 10$ where $W_0$ is a $10 \times 3$ matrix with all entries independent Gaussian with zero mean and where all entries in $i$-th column has variance $i$ for all $i \in \{1, 2, 3\}$. We corrupted the teacher weight matrix by an addition of a $10 \times 10$ matrix where each entry is i.i.d. centered Gaussian with standard deviation 0.2. Input points are isotropic Gaussians. The training outputs are generated by the noisy teacher, and the test outputs are generated by the noiseless teacher. We generated 100 training and 1000 test data points. Different runs of the same experiment yielded effectively the same figure. The learning rate is 0.001 both for the shallow and the deep case. Tolerance for the rank is set to $10^{-4}$ (eigenvalues smaller than $10^{-4}$ are set to 0 for the rank calculation). For Figure 3, the setup is the same except that $W_T = W_0W_0^T$ is created by using a new distribution for $W_0$ where all entries are centered independent Gaussian with standard deviation 1.5.
B Proofs for the Linear/Lazy regime

Our strategy (closely related to [27, 31]) to prove the convergence of DLNs in the linear/lazy regime is to show that with high probability, the loss $C$ satisfies the PL inequality in a sufficiently large ball around initialization, in order to apply Proposition [11]. We will first bound from below the smallest eigenvalue $\lambda_{\text{min}}$ of the NTK $\Theta^{(L)}(\theta)$ for all $\theta$ in a ball. We will then use that if $A \mapsto C(A)$ satisfies the $\beta$-PL inequality, then $\theta \mapsto C(A)$ satisfies the PL inequality with a different constant:

\[
\frac{1}{2} \| JA_\theta \nabla C(A_\theta) \|^2 = \frac{1}{2} (\nabla C(A_\theta))^T \Theta^{(L)}(\theta) \nabla C(A_\theta) \\
\geq \frac{\lambda_{\text{min}}}{2} (\Theta^{(L)}(\theta)) \| \nabla A_\theta C(A_\theta) \|^2 \\
\geq \lambda_{\text{min}} (\Theta^{(L)}(\theta)) \| \nabla A_\theta C(A_\theta) \|^2.
\]

B.1 Polyak-Łojasiewicz (PL) Inequality

The Polyak-Łojasiewicz (PL) inequality generalizes the notion of strong convexity while keeping convergence guarantees:

**Definition 9.** A cost $C$ with global minimum $C_{\text{min}}$ satisfies the Polyak-Łojasiewicz (PL) inequality if for all $x$

\[
\frac{1}{2} \| \nabla C(x) \|^2 \geq \beta (C(x) - C_{\text{min}}).
\]

Any $\beta$-strongly convex loss satisfies the $\beta$-PL inequality:

**Proposition 10.** If a cost $C$ is $\beta$-strongly convex, then $C$ satisfies the $\beta$-PL inequality.

**Proof.** Strong convexity implies that for all $x$ and all $y$,

\[C(y) \geq C(x) + (y - x)^T \nabla C(x) + \frac{\beta}{2} \| x - y \|^2.\]

Minimizing on both sides w.r.t. $y$ preserves the inequality. The right-hand side being minimized for $y = x - \frac{1}{\beta} \nabla C(x)$, we obtain

\[C_{\text{min}} \geq C(x) - \frac{1}{2\beta} \| \nabla C(x) \|^2.\]

Only the $\beta$-PL inequality is needed to prove linear convergence of gradient flow:

**Proposition 11.** For a loss $C$ and any point $x_0$ such that the $\beta$-PL inequality is satisfied in the ball $B(x_0, r)$ of radius $r = \sqrt{\frac{2}{\beta}} (C(x_0) - C_{\text{min}})$, the gradient descent path $(\gamma_C(t, x_0))_{t \geq 0}$ converges to a global minimum $x^*$ in $B(x_0, r)$. Furthermore, for all $t \geq 0$, it holds that

\[
C(\gamma(t, x_0)) - C_{\text{min}} \leq (C(x_0) - C_{\text{min}}) e^{-\beta t} \\
\| \gamma_C(t, x_0) - x^* \| \leq re^{-\beta t}.
\]
Proof. Let $T = \inf\{ t \geq 0 : \gamma_C(t, x_0) \notin B(x_0, r) \} \in \mathbb{R} \cup \{ \infty \}$ be the first time when gradient descent escapes the ball; we will show at the end of the proof that $T = \infty$.

For any time $t < T$, we have
\[
\partial_t C(\gamma_C(t, x_0), t)) = -\| \nabla C(\gamma_C(t, x_0), t) \|^2 \geq -2\beta (C(\gamma_C(t, x_0), t) - C_{min})
\]
which by Grönwall's Inequality implies that
\[
C(\gamma(t, x_0)) - C_{min} \leq (C(x_0) - C_{min}) e^{-2\beta t}.
\]

We will bound the distance $\| x_0 - \gamma_C(t, x_0) \|$ by the length of the gradient path $\gamma$ connecting the two points. Since the cost is continuously decreasing over $\gamma_C$, we parameterize the path as function of the costs of the two points $c \in [C(\gamma_C(t, x_0)), C(x_0)]$, i.e. we (uniquely) define $\gamma(c)$ such that $C(\gamma(c)) = c$. We have to compute the derivative $\partial_c \gamma(c)$, we know that
\[
\partial_c \gamma(c) \propto \nabla C(\gamma(c))
\]
which implies that $\partial_c \gamma(c) = \frac{\nabla C(\gamma(c))}{\| \nabla C(\gamma(c)) \|^2}$. The length of $\gamma$ can now be bounded by
\[
\int_{C(\gamma_C(t, x_0))}^{C(x_0)} \| \partial_c \gamma(c) \| dc = \int_{C(\gamma_C(t, x_0))}^{C(x_0)} \frac{1}{\| \nabla C(\gamma(c)) \|} dc
\leq \int_{C(\gamma_C(t, x_0))}^{C(x_0)} \frac{1}{\sqrt{2\beta (c - C_{min})}} dc
= \frac{\sqrt{2}}{\sqrt{\beta}} \left[ \sqrt{c - C_{min}} \right]_{C(\gamma_C(t, x_0))}^{C(x_0)}
= \frac{\sqrt{2}}{\beta} (C(x_0) - C_{min}) - \sqrt{\frac{2}{\beta}} (C(\gamma_C(t, x_0)) - C_{min})
= r - \sqrt{\frac{2}{\beta}} (C(\gamma_C(t, x_0)) - C_{min}).
\]

If $T$ is finite, then at time $T$, the gradient path must lie on the border of the ball, i.e. $\| x_0 - \gamma(t, x_0) \| = r$, which implies that either $C(\gamma(T, x_0)) = C_{min}$ or $T = \infty$. Either way, gradient descent converges to a global minimum $x^* = \lim_{t \to \infty} \gamma(t, x_0)$ inside the ball $B(x_0, r)$.

Using the same parameterized path, we can bound the distance between $\gamma(t, x_0)$ and the global minimum $x^*$:
\[
\| \gamma(t, x_0) - x^* \| = \int_{C_{min}}^{C(\gamma_C(t, x_0))} \| \partial_c \gamma(c) \| dc \leq \sqrt{\frac{2}{\beta}} (C(x_0) - C_{min}) \leq r e^{-\beta t},
\]
as claimed. \qed

Note that the converse of the statement “there is a global minimum at most $\sqrt{\frac{2}{\beta}} (C(x_0) - C_{min})$ far from $x_0$” implies that:

**Corollary 12.** For all $x_0$, denoting by $d$ the smaller distance between $x_0$ and a global minimum, we have
\[
C(x_0) \geq C_{min} + \frac{\beta}{2} d^2
\]
\[
\| \nabla C(x_0) \| \geq \beta d.
\]

**B.2 Initialization**

**Lemma 13.** The expected loss $\mathbb{E}_\theta [C(A_0)]$ for i.i.d. $\mathcal{N}(0, \sigma^2)$ parameters (with $\sigma^2 = \beta^2 w^{-\frac{L-1}{2}}$) converges as $w \to \infty$ to the expectation $\mathbb{E}_A [C(A)]$ over a random matrix $A$ with i.i.d. $\mathcal{N}(0, \sigma^2)$ entries.
Proof. Conditioned on the matrices $W_1, \ldots, W_{L-1}, A_\theta$ has i.i.d. Gaussian lines $\mathcal{N}(0, \sigma^2 \Sigma^{(L-1)})$ with $n_0 \times n_0$ covariance matrix $\Sigma^{(L-1)} = W^{T}_{L-1}W_{L-1}$. We have

$$\bar{\sigma}^2 \mathbb{E} \left[ \Sigma^{(L-1)} \right] = \bar{\sigma}^2 L w^{L-1} I_{n_0} = \sigma^2 I_{n_0}$$

$$\mathbb{E} \left[ \left\| \bar{\sigma}^2 \Sigma^{(L-1)} - \sigma^2 I_{n_0} \right\|_F^2 \right] = \mathcal{O} \left( \frac{1}{w} \right).$$

(1)

This implies that $\bar{\sigma}^2 \Sigma^{(L-1)} \rightarrow \sigma^2 I_{n_0}$ as $w \rightarrow \infty$ and hence $\mathbb{E}_\theta \left[ C(A_\theta) \right] = \mathbb{E}_{\Sigma^{(L-1)}} \left[ \mathbb{E}_{\mathcal{N}(0, \sigma^2 I_{n_0} \otimes \Sigma^{(L)})} \left[ C(A) \right] \right] \rightarrow \mathbb{E}_{\mathcal{N}(0, \sigma^2 I_{n_0} \otimes I_{n_0})} \left[ C(A) \right].$

The above lemma allows us to bound the cost $C(A_\theta)$ with high probability at initialization. In particular, there is an upper bound $\mathbb{E} \left[ C(A_\theta) \right] \leq C_0$ uniform in the width $w$. Let $C_{\text{min}}$ be the global minimum of the loss $C$. By Markov’s Inequality, we have for all $a > C_{\text{min}}$ that

$$\mathbb{P} \left[ C(A_\theta) > a \right] \leq \frac{C_0}{a - C_{\text{min}}}.$$ 

We now need to give high probability bounds on the largest and smallest eigenvalues of the NTK at initialization:

**Lemma 14.** For i.i.d. $\mathcal{N}(0, \bar{\sigma}^2)$ parameters $\theta$ (with $\bar{\sigma}^2 = \sigma^2 w^{-\frac{L-1}{w}}$), we have

$$\mathbb{E} \left[ (L^{(L)})(\theta) \right] = L\sigma^2 \bar{\sigma}^{\frac{L-1}{w}} w^{\frac{L-1}{w}} I_{n_0 n_L},$$

$$\mathbb{E} \left[ \left\| (L^{(L)})(\theta) - \mathbb{E} \left[ (L^{(L)})(\theta) \right] \right\|_F^2 \right] \leq c L \sigma^4 \bar{\sigma}^{\frac{L-1}{w}} w^{\frac{L-1}{w}},$$

for some finite constant $c$. Furthermore, for $\kappa = \frac{16}{L^2 c}$, it holds with probability at least $1 - \frac{\kappa}{w}$ that

$$\lambda_{\text{min}} \left( (L^{(L)})(\theta) \right) \geq (1 - \frac{1}{4}) L \sigma^2 \bar{\sigma}^{\frac{L-1}{w}} w^{\frac{L-1}{w}}$$

$$\lambda_{\text{min}} \left( (L^{(L)})(\theta) \right) \leq (1 + \frac{1}{4}) L \sigma^2 \bar{\sigma}^{\frac{L-1}{w}} w^{\frac{L-1}{w}}.$$ 

**Proof.** We postpone the proof of the first two claims to the forthcoming section [B.3]. For the bounds of $\lambda_{\text{min}} \left( (L^{(L)})(\theta) \right)$, by Markov’s Inequality, we have that

$$\left\| (L^{(L)})(\theta) - \mathbb{E} \left[ (L^{(L)})(\theta) \right] \right\|_F \leq \frac{L}{4} \sigma^2 \bar{\sigma}^{\frac{L-1}{w}} w^{\frac{L-1}{w}}$$

with probability at least

$$1 - \frac{16 \sigma^4 \bar{\sigma}^{\frac{L-1}{w}} w^{\frac{L-1}{w}}}{L^2 \sigma^4 \bar{\sigma}^{\frac{L-1}{w}} w^{\frac{L-1}{w}}} = 1 - \frac{16}{L^2 w} = 1 - \frac{\kappa}{w}.$$ 

We conclude using the first two claims.

**Lemma 15.** Let the parameters $\theta_0$ be sampled i.i.d. $\mathcal{N}(0, \bar{\sigma}^2)$ with $\bar{\sigma}^2 = \sigma^2 w^{-\frac{L-1}{w}}$. For any two $\theta, \hat{\theta} \in B(\theta_0, \sigma^2 w^{-\frac{L-1}{w}})$, we have with probability at least $1 - 2L e^{-\frac{a^2}{w^2}}$:

1. $\| J A_\theta \|_{op} \leq c_1 \sigma^{\frac{L-1}{w}} w^{\frac{L-1}{w}}$,
2. $\| J A_\theta - J A_{\hat{\theta}} \|_{op} \leq c_2 \sigma^{\frac{L-2}{w}} w^{\frac{L-2}{w}} \| \theta - \hat{\theta} \|$
3. $\| (L^{(L)})(\hat{\theta}) - (L^{(L)})(\theta) \|_{op} \leq c_3 \sigma^{\frac{2L-3}{w}} w^{\frac{2L-3}{w}} \| \hat{\theta} - \theta \|$

where $c_1 = L A^{L-1}, c_2 = L (L-1) A^{L-2}, c_3 = \frac{L}{2c_1 c_2}$.

Furthermore, with probability at least $1 - 2e^{-\frac{a^2}{w^2}} - \frac{\kappa}{w}$ we have $\lambda_{\text{min}} \left( (L^{(L)})(\theta) \right) \geq \frac{1}{2} \sigma^2 \bar{\sigma}^{\frac{L-1}{w}} w^{\frac{L-1}{w}}$ for all parameters $\theta$ in the ball $B(\theta_0, R)$ of radius $R = c_3 \sigma^{\frac{1}{w}} w^{-\frac{L}{w}}$.  

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Proof. The parameters $\theta = (W_1, \ldots, W_L)$ in the ball $B(\theta_0, \sigma^2 \frac{1}{w} \frac{1}{\pi})$ are of the form $W_\ell = W_{0,\ell} + \tilde{W}_\ell$ where $W_{0,\ell}$ has i.i.d. $\mathcal{N}(0, \sigma^2)$ entries and $\|\tilde{W}_\ell\|_F \leq \|\theta\| \leq \sigma \frac{1}{w} \frac{1}{\pi}$. Using the forthcoming Theorem 20 we have that with probability at least $1 - 2e^{-\frac{w^2}{2}}$, it holds that
\[
\|W_{0,\ell}\|_{op} \leq \tilde{\sigma} \left(\sqrt{4\ell} + \sqrt{4\ell-1} + \sqrt{w}\right) = 3\sigma \frac{1}{w} \frac{1}{\pi}.
\]
This implies that $\|W_{\ell}\|_{op} \leq (3 + c) \sigma \frac{1}{w} \frac{1}{\pi} \leq 4\sigma \frac{1}{w} \frac{1}{\pi}$. This is satisfied for all $W_{\ell}$ with probability at least $1 - 2Le^{-\frac{w^2}{2}}$.

(1) For the first bound, we have for any $\vartheta = (V_L, \ldots, V_1)$
\[
\|JA_\vartheta[\bar{\vartheta}]\|_F \leq \sum_{\ell = 1}^{L} \|W_L\|_{op} \cdots \|W_{\ell+1}\|_{op} \|V_\ell\|_F \|W_{\ell-1}\|_{op} \cdots \|W_1\|_{op}
\leq L^{L-1} \sigma \frac{L-1}{w} \frac{L-1}{\pi} \|\bar{\vartheta}\|
\]
which implies that $\|JA_\vartheta\|_{op} = \max_{\bar{\vartheta}} \frac{\|JA_\vartheta[\bar{\vartheta}]\|_F}{\|\bar{\vartheta}\|} \leq c_1 \sigma \frac{L-1}{w} \frac{L-1}{\pi}$ for $c_1 = L^{L-1}$.

(2) For the second bound we first bound the Hessian
\[
\|H_{\vartheta}[\bar{\vartheta}, \tilde{\vartheta}]\|_F \leq \sum_{\ell \neq k} \|W_L\|_{op} \cdots \|W_{\ell+1}\|_{op} \|V_\ell\|_F \|W_{\ell-1}\|_{op} \cdots \|W_k+1\|_{op} \|\bar{V}_\ell\|_F \|W_{k-1}\|_{op} \cdots \|W_1\|_{op}
\leq L(L - 1)4^{L-2} \sigma \frac{L-2}{w} \frac{L-2}{\pi} \|\tilde{\vartheta}\| \|\bar{\vartheta}\|.
\]
This implies that
\[
\|JA_\vartheta[\bar{\vartheta}] - JA_\vartheta[\tilde{\vartheta}]\|_F \leq \int_0^1 \|H_{\alpha \vartheta + (1 - \alpha) \tilde{\vartheta}}[\bar{\vartheta}, \tilde{\vartheta}]\|_F \, d\alpha \leq L(L - 1)4^{L-2} \sigma \frac{L-2}{w} \frac{L-2}{\pi} \|\tilde{\vartheta} - \bar{\vartheta}\| \|\tilde{\vartheta}\|.
\]
allowing us to bound the operator norm $\|JA_\vartheta - JA_\vartheta\|_{op} \leq c_2 \sigma \frac{L-2}{w} \frac{L-2}{\pi} \|\tilde{\vartheta} - \bar{\vartheta}\|$ for $c_2 = L(L - 1)4^{L-2}$.

(3) The third bound on the NTK $\Theta^{(L)}(\theta) = JA_\theta (JA_\theta)^T$ follows from the two previous bounds:
\[
\|\Theta^{(L)}(\theta_0) - \Theta^{(L)}(\theta)\|_{op} \leq \|JA_{\theta_0} (JA_{\theta_0} - JA_\theta) (JA_\theta)^T\|_{op} + \|JA_\theta - JA_\theta\|_{op} \leq \|JA_{\theta_0}\|_{op} \|JA_{\theta_0} - JA_\theta\|_{op} + \|JA_\theta\|_{op} \|JA_{\theta_0} - JA_\theta\|_{op}
\leq 2c_1c_2 \sigma \frac{L-1}{w} \frac{L-1}{\pi} \sigma \frac{L-2}{w} \frac{L-2}{\pi} \|\theta_0 - \theta\|
\leq 2c_1c_2 \sigma \frac{2L-3}{w} \frac{2L-3}{\pi} \|\theta_0 - \theta\|.
\]
(4) Take $c_3 = \frac{L}{2c_1^2c_2} \leq 1$ then for any $\theta \in B(\theta_0, R)$ with $R = c_3 \sigma \frac{1}{w} \frac{1}{\pi}$, we have
\[
\lambda_{min} \left(\Theta^{(L)}(\theta)\right) \geq \lambda_{min} \left(\Theta^{(L)}(\theta_0)\right) - \||\Theta^{(L)}(\theta_0) - \Theta^{(L)}(\theta)\|_{op}
\geq (1 - \frac{1}{4})L \sigma^2 \frac{L-1}{w} \frac{L-1}{\pi} - 2c_1c_2 \sigma \frac{2L-3}{w} \frac{2L-3}{\pi} \frac{1}{4} R
\geq \frac{L}{2} \sigma^2 \frac{L-1}{w} \frac{L-1}{\pi}.
\]

Finally we prove the convergence for large enough width and bound the rate of change of the NTK:

**Proposition 16** (Proposition 2 of Main Text). Let $C$ be a cost on matrices that satisfies the $\beta$-PL inequality for some $\beta > 0$ and let $C_{min} \in \mathbb{R}$ be its minimum value. Consider a gradient flow path $\theta(t) = (t, \theta(t))$ on $\mathbb{R}^{CN} : \theta \mapsto C(A_\theta)$ for a network of depth $L$, width $w$, and with $\sigma^2$-Large Norm Initialization. For all $\delta > 0$ and $w$ large enough, it holds with probability at least $1 - \delta$ that $\theta(t)$ converges, as $t \to \infty$, to a global minimum $\theta^*$ and for $\beta' = \frac{\beta}{2} L \sigma^2 \frac{L-1}{w} \frac{L-1}{\pi}$, we have:
\[ C^{NN}(\theta(t)) - C_{\text{min}} \leq (C^{NN}(\theta_0) - C_{\text{min}}) e^{-\beta't}, \quad \forall t \geq 0 \]

The global minimum \( \theta^* \) satisfies \( \|\theta_0 - \theta^*\| \leq r := \sqrt{\frac{2}{\beta}(C(A_{\theta_0}) - C_{\text{min}})} \).

Furthermore, if \( C \) is convex, we have the uniform in time bound
\[ \|A_{\theta(t)} - A(\eta t)\| = O\left(\frac{1}{\sqrt{t}}\right) , \]
where \( \eta = L\sigma^2 \frac{4}{\beta L} w \frac{l-1}{\nu} \) and \( A(t) = \gamma_C(t, A_{\theta_0}) \) is the gradient flow path on the cost \( A \mapsto C(A) \) starting from \( A_{\theta_0} \).

Proof. Lemma 15 implies that in a ball \( B(\theta_0, R) \) of radius \( R = c\sigma^2 w \frac{1}{\nu} \) around initialization the loss \( \theta \mapsto C(A_{\theta}) \) satisfies the \( \beta' \)-PL inequality for \( \beta' = \frac{\beta^2 L - 1}{2} \frac{L}{\nu} \). In order to apply Proposition 11 to prove the convergence, the radius \( R \) must be greater than \( r = \sqrt{\frac{2}{\beta}(C(A_{\theta_0}) - C_{\text{min}})} \).Lemma 15 ensures that for all \( \delta > 0 \), there is a constant \( C_1 \) that does not depend on \( w \) such that with probability \( 1 - \delta \), we have \( C(A_{\theta_0}) - C_{\text{min}} \leq C_1 \). Then for \( w \geq \frac{4}{c^2 \beta \sigma^2} C_1 \), we have
\[ r \leq \sqrt{\frac{4}{\beta \sigma^2 \frac{4}{\beta L} w \frac{l-1}{\nu} C_1} \leq c\sigma^2 w \frac{1}{\nu} = R. \]

Our goal is now to bound the difference between the dynamics of gradient descent on the cost \( C \) and \( A(t) = \gamma_C(t, A_{\theta_0}) \) and the evolution of \( A_{\theta(t)} \).

First, observe that for all \( t \geq 0 \), the NTK \( \Theta^{(L)}(\theta(t)) \) is approximately equal to \( L\sigma^2 \frac{4}{\beta L} w \frac{l-1}{\nu} I_{\theta(t)} \):
\[ \left\| \Theta^{(L)}(\theta(t)) - L\sigma^2 \frac{4}{\beta L} w \frac{l-1}{\nu} I_{\theta(t)} \right\| \leq \left\| \Theta^{(L)}(\theta(t)) - \Theta^{(L)}(\theta_0) \right\| + \left\| \Theta^{(L)}(\theta_0) - L\sigma^2 \frac{4}{\beta L} w \frac{l-1}{\nu} I_{\theta(t)} \right\| . \]

In the proof of Proposition 11 we established that \( \theta(t) \in B(\theta_0, r) \), where \( r \) is given by ?? . By Lemma 15 we can thus bound the first term, uniformly in \( t \geq 0 \), by
\[ c_3 \sigma^2 \frac{4}{\beta L} w \frac{l-1}{\nu} \left\| \theta(t) - \theta_0 \right\| \leq c_3 \sigma^2 \frac{4}{\beta L} w \frac{l-1}{\nu} r = \frac{c_3 C}{\sqrt{\beta}} \frac{4}{\beta L} w \frac{l-1}{\nu} = O\left(\frac{L}{\nu} \right). \]

For the second term (which does not depend on time), we use Lemma 14 then Markov’s Inequality to show that for any \( p > 0 \), with probability at least \( 1 - p \) it holds that
\[ \left\| \Theta^{(L)}(\theta_0) - L\sigma^2 \frac{4}{\beta L} w \frac{l-1}{\nu} I_{\theta(t)} \right\| = \left\| \Theta^{(L)}(\theta_0) - \mathbb{E} \left[ \Theta^{(L)}(\theta_0) \right] \right\| \leq \sqrt{p \mathbb{E} \left[ \left\| \Theta^{(L)}(\theta_0) - \mathbb{E} \left[ \Theta^{(L)}(\theta_0) \right] \right\|^2 \right]} = O\left(\frac{L}{\nu} \right). \]

This ensures that with probability \( 1 - p \), it holds that
\[ \sup_{t \geq 0} \left\| \Theta^{(L)}(\theta(t)) - L\sigma^2 \frac{4}{\beta L} w \frac{l-1}{\nu} I_{\theta(t)} \right\| = O\left(\frac{L}{\nu} \right). \]

We now turn to the second part of the proposition. The similarity between the paths \( A_{\theta(t)} \) and \( A(\eta t) \) for \( \eta = L\sigma^2 \frac{4}{\beta L} w \frac{l-1}{\nu} \) follows from the fact that
\[ \partial_t A_{\theta(t)} = -\nabla C(A_{\theta(t)}) \]
and \[ \partial_t A(\eta t) = -\eta \nabla C(A(\eta t)). \]
Assuming that $C$ is convex, we compute
\[
\partial_t \| A_{\theta(t)} - A(\eta t) \| = - \frac{(A_{\theta(t)} - A(\eta t))^T (\Theta^{(L)}(\theta(t)) \nabla C(A_{\theta(t)}) - \eta \nabla C(A(\eta t)))}{\| A_{\theta(t)} - A(\eta t) \|}
\]
\[
= - \frac{(A_{\theta(t)} - A(\eta t))^T (\Theta^{(L)}(\theta(t)) - \eta I_{n_{n L}}) \nabla C(A_{\theta(t)})}{\| A_{\theta(t)} - A(\eta t) \|}
\]
\[
- \frac{\eta (A_{\theta(t)} - A(\eta t))^T (\nabla C(A_{\theta(t)}) - \nabla C(A(\eta t)))}{\| A_{\theta(t)} - A(\eta t) \|}
\]
\[
\leq \left\| (\Theta^{(L)}(\theta(t)) - \eta I_{n_{n L}}) \right\|_{op} \| \nabla C(A_{\theta(t)}) \| - 0
\]
where we used the convexity of $C$ to show that the second term is negative, since for all $A, B$, it ensures that $(A - B)^T (\nabla C(A) - \nabla C(B)) \geq 0$. Since
\[
\int_0^t \| \nabla C(A_{\theta(t)}) \| dt \leq \int_0^t \frac{1}{\sqrt{\lambda_{\min}(\Theta^{(L)}(t))}} \| \nabla C(A_{\theta(t)}) \| dt
\]
\[
\leq \frac{\sqrt{2}}{\sqrt{\eta}} \int_0^t \| \partial_t \theta(t) \| dt
\]
\[
\leq \frac{\sqrt{2}r}{\sqrt{\eta}}
\]
\[
= O(w^{-\frac{1}{4}}),
\]
we obtain a uniform bound
\[
\| A_{\theta(t)} - A(\eta t) \| \leq O\left( w^{-\frac{1}{4}} \right) = O\left( w^{-\frac{1}{2}} \right).
\]

### B.3 Covariances at initialization

In this section we compute and bound the expectations and variance of the NTK.

First, for any $n_0$-dim vectors $a_1, \ldots, a_{2r}$ and $n_L$-dim vectors $b_1, \ldots, b_{2r}$, we compute the moments $\mathbb{E} [a_1^T A_{\theta_1} b_1 \cdots a_{2r}^T A_{\theta_2} b_{2r}]$ for Gaussian parameter vectors $\theta_1, \ldots, \theta_{2r}$ with covariance $\mathbb{E} [\theta_k, \theta_m, j] = \delta_{ij} K_{km}$, where $K$ is a positive $2r \times 2r$ matrix.

To express this expectation, we need a few definitions:

Let $\mathcal{P}(2r)$ be the set of pairings $\rho = \{(i_1, j_1), \ldots, (i_r, j_r)\}$ on $[2r] := 1, \ldots, 2r$.

For any $2r \times 2r$ positive definite matrix $K$ and $\rho \in \mathcal{P}(2r)$, we define $K_\rho := \prod_{(i,j) \in \rho} K_{ij}$.

For any $\rho, \rho' \in \mathcal{P}(2r)$, the union $\rho \cup \rho'$ is defined as the smallest partition of $[2r]$ such that every block of $\rho$ and $\rho'$ is contained in a block of $\rho \cup \rho'$. We write $|\rho \cup \rho'|$ for the number of cycles in $\rho \cup \rho'$.

We can now express the covariance of the output matrices $A_{\theta_1}, \ldots, A_{\theta_{2r}}$ for correlated parameters $\theta_1, \ldots, \theta_{2r}$ using a technique similar to [19][20]. In the rest of the section, $a_1, \ldots, a_{2r}$ denote any $n_0$-dim vectors and $b_1, \ldots, b_{2r}$ any $n_L$-dim vectors.

**Proposition 17.** We have
\[
\mathbb{E} [a_1^T A_{\theta_1} b_1 \cdots a_{2r}^T A_{\theta_2} b_{2r}] = \sum_{\rho^{(1)}, \ldots, \rho^{(L)} \in \mathcal{P}(2r)} K_{\rho^{(1)}} \cdots K_{\rho^{(L)}} \prod_{\ell=1}^{L-1} \left[ a_{\rho^{(\ell)}}^T \prod_{(i,j) \in \rho^{(\ell)}} a_j \right] \prod_{(i,j) \in \rho^{(L-1)}} b_i^T b_j.
\]
Proof. We write
\[ E \left[ a_1^T A_0, b_1 \cdots a_{2r}^T A_{2r}, b_{2r} \right] = E \left[ a_1^T W_{1,L} \cdots W_{1,1} b_1 \cdots a_{2r}^T W_{2r,L} \cdots W_{2r,1} b_{2r} \right] \]
\[ = \sum_{i_1,0 \cdots i_{2r},i_{2r},0=1} \cdots \sum_{i_{1,L} \cdots i_{2r},L=1} E \left[ (a_{1,i_1,L} W_{1,L,i_1,L+1,i_{1,L-1}} \cdots W_{1,1,i_1,1,i_{1,0}} b_{1,i_{1,0}}) \right. \]
\[ \times \cdots \times (a_{2r,i_{2r,L}} W_{2r,L,i_{2r,L+1,i_{2r,L-1}}} \cdots W_{2r,1,i_{2r,1,i_{2r,0}} b_{2r,i_{2r,0}}}) \]
\[ = \sum_{i_1,0 \cdots i_{2r},i_{2r},0=1} \cdots \sum_{i_{1,L} \cdots i_{2r},L=1} a_{1,i_1,L} \cdots a_{2r,i_{2r,L}} b_{1,i_{1,0}} \cdots b_{2r,i_{2r,0}} \prod_{\ell=1}^{L} E \left[ W_{1,\ell,1,1,1,\ell-1} \cdots W_{2r,\ell,1,1,1,\ell-1} \right]. \]

By Wick’s formula
\[ E \left[ W_{1,\ell,1,1,1,\ell-1} \cdots W_{2r,\ell,1,1,1,\ell-1} \right] = \sum_{\rho \in P(2r)} \prod_{(k,m) \in \rho} K_{km} \delta_{i_k,\ell} i_m, \delta_{i_k,\ell-1,i_m,\ell-1}, \]
which yields the claim.

The cases \( r = 1 \) and \( r = 2 \) will be of particular interest:

Corollary 18. We have
\[ E \left[ a_1^T A_0, b_1 a_2^T A_{2}, b_2 \right] = K_{12} w^{L-1} a_1^T a_2 b_1^T b_2 \]
\[ E \left[ a_1^T A_0, b_1 a_2^T A_{2}, b_3 a_3^T A_3, b_4 \right] = \sum_{\rho^{(1)}, \cdots, \rho^{(L-1)} \in \{1,2,3\}} K_{\rho^{(1)}} \cdots K_{\rho^{(L-1)}} \prod_{\ell=1}^{L-1} w^{1+\delta_{\rho(\ell)}} \prod_{(i,j) \in \rho^{(1)}} a_i^T a_j \prod_{(i,j) \in \rho^{(L-1)}} b_i^T b_j \]

Proof. In the case \( r = 1 \), there is only one pairing of \{1, 2\},

In the case \( r = 2 \), there are three pairings of \{1, 2, 3, 4\}: \( \rho_1 = \{\{1, 2\}, \{3, 4\}\} \), \( \rho_2 = \{\{1, 3\}, \{2, 4\}\} \) and \( \rho_3 = \{\{1, 4\}, \{2, 3\}\} \). It suffices to note that \( |\rho_k \cup \rho_m| = 1 + \delta_{km} \) and to apply the previous proposition.

The two first claims of Lemma 14 given by the following corollary.

Corollary 19. We have
\[ E \left[ \Theta^{(L)}(a_1 b_1^T, a_2 b_2^T) \right] = \sigma^{2L-2} n_1 \cdots n_{L-1} a_1^T a_2 b_1^T b_2. \]
The above can be rewritten in terms of matrices: for two \( n_0 \times n_L \) matrices \( A \) and \( B \), we have
\[ E \left[ \Theta^{(L)}(A, B) \right] = \sigma^{2L-2} n_1 \cdots n_{L-1} \text{Tr} \left[ A B^T \right] \]
The covariance can be bounded as follows:
\[ \text{Cov} \left[ \Theta^{(L)}(a_1 b_1^T, a_2 b_2^T), \Theta^{(L)}(a_3 b_3^T, a_4 b_4^T) \right] \]
\[ = \sigma^{4L-4} \sum_{\rho^{(1)}, \cdots, \rho^{(L-1)} \in \{1,2,3\}} \prod_{\ell=1}^{L-1} w^{1+\delta_{\rho(\ell)}} \prod_{(i,j) \in \rho^{(1)}} a_i^T a_j \prod_{(i,j) \in \rho^{(L-1)}} b_i^T b_j \]
\[ \leq \sigma^{4L-4} w^{2L-3} \left[ a_1^T a_2 a_3 a_4^T + a_2^T a_3 a_4^T a_4 + a_1^T a_4 a_3 a_2^T \right] \left[ b_1^T b_2 b_3 b_4^T + b_2^T b_3 b_4^T b_4 + b_1^T b_4 b_3^T b_3 \right] \]
and the equivalent matrix formulation is
\[ \text{Cov} \left[ \Theta^{(L)}(A, B), \Theta^{(L)}(C, D) \right] \]
\[ \leq \sigma^{4L-4} w^{2L-4} \left( \text{Tr} \left[ A B^T C D^T \right] + \text{Tr} \left[ A B^T D C^T \right] + \text{Tr} \left[ A C^T D B^T \right] + \text{Tr} \left[ A D^T C B^T \right] \right) \]
\[ + \sigma^{4L-4} w^{2L-4} \left( \text{Tr} \left[ A B^T \right] \text{Tr} \left[ C D^T \right] + \text{Tr} \left[ A C^T \right] \text{Tr} \left[ B D^T \right] + \text{Tr} \left[ A D^T \right] \text{Tr} \left[ B C^T \right] \right) \]
\[ + \text{Tr} \left[ A D^T B C^T \right] + \text{Tr} \left[ A C^T B D^T \right]. \]
Finally, we have

$$\mathbb{E} \left[ \left\| \Theta^{(L)} - \mathbb{E} \left[ \Theta^{(L)} \right] \right\|_F^2 \right] = \mathbb{E} \left[ \mathbb{E}_{A,B} \left( \left( \Theta^{(L)}(A,B) - \mathbb{E} \left[ \Theta^{(L)}(A,B) \right] \right)^2 \right) \right]$$

$$\leq c \alpha^{4L-4} w^{2L-3} \mathbb{E}_{A,B} \left[ \left( A^T B A + \text{Tr} AB^T \right) + \text{Tr} \left( A A^T B B^T \right) + \text{Tr} \left( B B^T A A^T \right) + \text{Tr} \left( A B^T B^T A^T \right) \right]$$

$$+ c \alpha^{4L-4} w^{2L-4} \mathbb{E}_{A,B} \left[ 2 \text{Tr} \left( A^T \right)^2 + \text{Tr} \left( A A^T \right) \text{Tr} \left( B B^T \right) + \text{Tr} \left( A B^T B^T A^T \right) + \text{Tr} \left( A A^T B B^T \right) \right]$$

$$= c \alpha^{4L-4} w^{2L-3} \left[ 2n_0 n_L + n_0 n_L^2 + n_0^2 n_L \right] + c \alpha^{4L-4} w^{2L-4} \left[ 2n_0 n_L + n_0^2 n_L + n_0 n_L^2 + n_0^2 n_L \right]$$

### B.4 Spectrum bounds

**Theorem 20.** Let $A$ be a $m \times n$ matrix with i.i.d. $\mathcal{N}(0, \sigma^2)$ entries. For all $t \geq 0$, with probability at least $1 - 2e^{-\frac{t^2}{2}}$, it holds that

$$\sigma \left( -\sqrt{m} - \sqrt{n} - t \right) \leq \lambda_{\text{min}} (A) \leq \lambda_{\text{max}} (A) \leq \sigma \left( \sqrt{m} + \sqrt{n} + t \right).$$

**Corollary 21.** For random parameters $\theta = (W_L, \ldots, W_1)$ with independent $W_\ell \sim \mathcal{N}(0, \sigma^2)$, for all $t \geq 0$, it holds with probability at least $1 - 2e^{-\frac{t^2}{2}}$ that

$$\|A_\theta\|_{\text{op}} \leq (1 + t) \sigma^L \left( \sqrt{m_0} + \sqrt{w} \right) \left( 4w \right)^{\frac{L-1}{2}} \left( \sqrt{w} + \sqrt{n_L} \right).$$

**Proof.** By Theorem 20, with probability greater than $\left( 1 - 2e^{-\frac{t^2}{2}} \right)^2$, we have for all $\ell$

$$\|W_\ell\|_{\text{op}} \leq \sigma \left( \sqrt{m_{\ell-1}} + \sqrt{n_{\ell}} + t \right),$$

where $n_\ell = w$ for $\ell \in \{1, \cdots, L - 1\}$. Hence

$$\|A_\theta\|_{\text{op}} \leq \|W_L\|_{\text{op}} \cdots \|W_1\|_{\text{op}} \leq \sigma^L \prod_{\ell=1}^{L} \left( \sqrt{m_{\ell-1}} + \sqrt{n_{\ell}} + t \right) \leq (1 + t) \sigma^L \prod_{\ell=1}^{L} \left( \sqrt{m_{\ell-1}} + \sqrt{n_{\ell}} \right).$$

\[\square\]

## C Proofs for the Saddle-to-Saddle regime

In this section, we prove Theorem 4 of the main. Given a saddle $\theta^* = RI^{(k+w)}(\theta)$ for $\theta$ a local minimum in a width $w$ network, we want to describe the dynamics of gradient descent $\theta_\alpha(t) = \gamma(t, \theta^* + \alpha \theta_0)$, initialized close to $\theta^*$. We shall consider $\theta^* = 0$ for convenience, though the same arguments could be applied for $\theta^* \neq 0$. We first show that as $\alpha \rightarrow 0$, $\theta_\alpha(t)$ converges to an escape path which escapes along specific direction and speed. We then show that the escape paths which escape at this speed are unique in some aspects.

### C.1 Homogeneous Costs

We say that a cost $H$ is $k$-homogeneous for an integer $k \geq 2$ if $H(\alpha \theta) = \alpha^k H(\theta)$ for all $\theta$ and all scalar $\alpha > 0$. In our setting we are only interested in the case where $H$ is a homogeneous polynomial, more precisely we will study the $k = L$ homogeneous polynomial $H(\theta) = \text{Tr} [G A_\theta]$ for a network of depth $L$ and some $n_L \times n_0$ matrix $G$.

A useful property of gradient descent on a homogeneous cost is that

**Lemma 22.** Gradient descent on a $k$-homogeneous cost $H$ satisfies

$$\gamma_H(t, \lambda x_0) = \lambda \gamma_H(\lambda^{k-2} t, x_0)$$

for all $x_0$ and all $t > 0$.

**Proof.** We simply need show that for all $t > 0$, we have $\frac{1}{\lambda} \gamma_H(\lambda^{2-k} t, \lambda x_0) = \gamma_H(t, x_0)$, i.e., that the path $t \mapsto \frac{1}{\lambda} \gamma_H(\lambda^{2-k} t, \lambda x_0)$ is the solution of gradient descent starting at $x_0$. As needed, we have
As \( \theta \)Optimal Escape Direction
An \( \omega \) will remain along this direction (these paths are equal to \( \theta \)).
The Escape Directions
\( H \)The minimum is equal to \( \lambda \). Our proof is in two steps. We show that the set \( \Omega \) points \( \rho \) \( \Omega \) and their derivative take the form
\( \lambda = \gamma_H(\lambda^{2-k}t, \lambda x_0) = -\lambda^{1-k} \nabla H \left( \gamma_H(\lambda^{2-k}t, \lambda x_0) \right) \)
since \( \lambda \nabla H(\lambda x) = \lambda^k \nabla H(x) \).

The Escape Directions of the cost \( H \) are vectors \( \rho \in S^{d-1} \) such that \( \nabla H(\rho) = s \rho \) for some \( s \in \mathbb{R} \) which we call the escape speed. Gradient descent on \( H \) starting along one of these directions will remain along this direction (these paths are equal to \( \theta(t) = \rho e^{-st} \) when \( k = 2 \) and \( \theta(t) = \rho ((k-2)s(-t))^{-\frac{s}{k-2}} \) when \( k > 2 \)). When \( H(\theta) = \theta^T A \theta \) for some matrix \( A \), the escape directions are simply the eigenvectors of the Hessian of \( 2A \).

An Optimal Escape Direction is an escape direction with the most negative speed \( s^\star \). It is also the minimizer of \( H \) restricted to the hypersphere \( S^{d-1} \):
\[ \rho^\star = \arg \min_{\rho \in S^{d-1}} H(\rho). \]
The minimum is equal to \( H(\rho^\star) = \frac{s^\star}{2} \) for a \( k \)-homogeneous \( H \).

Under some conditions on the Hessian along the escape directions, one can guarantee that gradient descent will escape along an optimal escape path:

**Proposition 23.** Assume that the optimal escape speed \( \lambda \) is negative and that for all escape directions \( \rho \) which are not optimal, there is a vector \( v \perp \rho \) such that \( v^T \nabla H(\rho)v < \lambda v^T v \). Let \( \Omega \) be the set of points \( \theta_0 \) such that the direction of gradient descent converges towards an optimal escape direction as \( t \to \infty \), i.e. \( \lim_{t \to \infty} \frac{\gamma(t, \theta_0)}{\|\gamma(t, \theta_0)\|} = \rho \). We have that \( S^{d-1} \setminus \Omega \) has Lebesgue measure zero.

**Proof.** Our proof is in two steps. We show that the set \( \Omega \) contains the set \( \Omega' \) of points \( \theta_0 \neq 0 \) such that gradient descent on the \( k = 1 \) homogeneous cost \( H(\theta) = H \left( \frac{\theta}{\|\theta\|} \right) \) converges to a global minimum, and then show that \( \Omega' \) is dense in \( S^{d-1} \).

Note that both \( \Omega \) and \( \Omega' \) satisfy \( \Omega = \alpha \Omega \) and \( \Omega' = \alpha \Omega' \) for any \( \alpha > 0 \), we therefore only need to show that \( \Omega \cap S^{d-1} = \Omega' \cap S^{d-1} \). Let us then assume that \( \theta_0 \in \Omega' \cap S^{d-1} \) (which implies that \( \theta'(t) = \gamma_H(t, \theta_0) \) converges to an optimal escape direction \( \rho \) as \( t \to \infty \)) and show that the dynamics \( \theta(t) = \gamma_H(t, \theta_0) \) on the original cost aligns with the same escape direction. Let us first observe that for \( t \to \frac{\int_{-\infty}^{t} \|\theta(s)\|^{-k} ds}{\|\theta(t)\|^{-k}} \) the two functions \( \theta'(t) \) and \( \frac{\theta(t)}{\|\theta(t)\|} \) are equal. Indeed we have
\[ \theta'(t) = \theta_0 = \frac{\theta(t)}{\|\theta(t)\|} \]
and their derivative take the form
\[ \partial_t \theta(t) = -\frac{\nabla H(\theta(t))}{\|\theta(t)\|} + \frac{\theta(t) \theta(t)^T \nabla H(\theta(t))}{\|\theta(t)\|^3} \]
\[ = -\frac{1}{\|\theta(t)\|} \left( I - \frac{\theta(t) \theta(t)^T}{\|\theta(t)\|^2} \right) \nabla H(\theta(t)) \]
\[ = -\|\theta(t)\|^{-k-1} \left( I - \frac{\theta(t) \theta(t)^T}{\|\theta(t)\|^2} \right) \nabla H \left( \frac{\theta(t)}{\|\theta(t)\|} \right) \]
and
\[ \partial_t \theta'(t) = -\frac{\nabla H(\theta'(t))}{\|\theta'(t)\|} \|\theta'(t)\|^{-k-1} \]
\[ = -\|\theta(t)\|^{k-1} \left( I - \frac{\theta'(t) \theta'(t)^T}{\|\theta'(t)\|^2} \right) \nabla H \left( \frac{\theta'(t)}{\|\theta'(t)\|} \right) \].

As \( t \to T \) (where \( T \in \mathbb{R} \cup \{\infty\} \) is the time when \( \theta(t) \) diverges to infinity), \( \tau(t) \to \infty \) and hence \( \lim_{t \to T} \frac{\theta(t)}{\|\theta(t)\|} = \lim_{t \to T} \theta'(t) = \rho \).
Since the cost $\tilde{H}(\theta)$ has only global minima and strict saddles, the points that converge to a saddle have measure 0. First note that all spheres $\alpha S^{d-1}$ are invariant under gradient flow of the cost $\tilde{H}(\theta)$ and the critical points are the escape directions $\rho$ (and their scaling $\alpha \rho$ for $\alpha > 1$). The global minima correspond to the fastest escape directions and the saddles corresponds to non-fastest escape directions $\rho$, which are strict, since taking $v$ such that $v^T \mathcal{H}(\rho)v < \lambda v^Tv$, we have $v^T \mathcal{H}(\rho)v = v^T \mathcal{H}(\rho)v - \nabla C(\rho)^T \rho v^Tv < 0$. 

\[
\begin{align*}
\text{C.1.1 Deep Linear Networks} \\
\text{For a depth } L \text{ DLN and the homogeneous cost } H(\theta) = \text{Tr } [G^T A_b] \text{ with SVD decomposition } G = USV^T, \text{ the escape directions } \rho \text{ are of the form } \\
\frac{1}{\sqrt{L}}(\pm u_1 w_{L-1}^T, w_{L-2}^T, \ldots, w_1 v_1^T) \\
\text{with speed } s = \pm \frac{s_1}{L^{\frac{L}{2}}}. \text{ The optimal speed is } -\frac{s_1}{L^{\frac{L}{2}}}, \text{ where } s_1 \text{ is the largest singular value of } G. 
\end{align*}
\]

Furthermore this loss satisfies the property required to ensure convergence along the fastest escape path:

**Lemma 24.** For a network of depth $L$ with $w \geq 1$, for any escape direction of the form $\rho = \frac{1}{\sqrt{L}}(\pm u_1 w_{L-1}^T, w_{L-2}^T, \ldots, w_1 v_1^T)$ with speed $\pm \frac{s_1}{L^{\frac{L}{2}}} > -\frac{s_1}{L^{\frac{L}{2}}}$ the vector $v = (-u_1 w_{L-1}^T, 0, \ldots, 0, w_1 v_1^T)$ satisfies $v^T \mathcal{H}(\rho)v = \pm \frac{s_1}{L^{\frac{L}{2}}} v^Tv.$

**Proof.** We have $v^T \mathcal{H}(\rho)v = -\frac{2s_1}{L^{\frac{L}{2}}} \pm \frac{s_1}{L^{\frac{L}{2}}} v^Tv = \pm \frac{2s_1}{L^{\frac{L}{2}}} \text{ as needed.}$

This guarantees that gradient flow will not escape along a non-optimal direction, but it does not rule out the possibility that it converges to a saddle of the loss $H(\theta) = \text{Tr } [G^T A_b]$. Each non-zero saddle $\theta^* = (W_1, \ldots, W_L)$ is technically proportional to an escape direction $\rho$ with escape speed 0, since $\nabla H(\theta^*) = 0$. For shallow networks these saddles are strict [26] and so they are almost surely avoided, guaranteeing convergence in direction. For depth $L = 3$ we can apply Proposition[23] since we have:

**Lemma 25.** Consider the cost $H(\theta) = \text{Tr } [GA_b]$ for a rank $\{n_0, n_L\}$ matrix $G$ and a network of depth $L = 3$ with $w \geq 1$. For any 0-speed escape direction $\rho$ there is a vector $v$ such that $v^T \mathcal{H}(\rho)v < 0$.

**Proof.** Since $\rho \neq 0$ there must a non-zero $W_1, W_2$ or $W_3$. We separate the case $W_2 \neq 0$ from $W_1$ or $W_3$ is non-zero.

Case $W_2 \neq 0$: let $u_1, v_1$ be the largest singular vectors of $G$ and $\hat{u}_1, \hat{v}_1$ the largest singular vectors of $W_2$, then $v = (-\hat{u}_1 v_1^T, 0, u_1 v_1^T)$ satisfies $v^T \mathcal{H}(\rho)v = -\text{Tr } [G u_1 \hat{v}_1^T W_2 \hat{u}_1 v_1^T] = -s_1 \hat{s}_1 < 0.$

Case $W_1 \neq 0$ (the case $W_3 \neq 0$ is similar): Let $u_1, v_1$ be the largest singular vectors of $W_1 G$ and $b$ be any unitary $w$-dim vector, then the parameters $v = (0, bv_1^T, u_1 b^T)$ satisfy $v^T \mathcal{H}(\rho)v = -\text{Tr } [G u_1 b^T b v_1^T W_1] = -s_1 < 0.$

For $L > 3$ we were not able to prove that the saddles can be avoided with prob. 1, we therefore put it as an assumption:

**Assumption 26.** For $L > 3$, we assume that the set of initializations which converge to a saddle of the cost $H(\theta) = \text{Tr } [GA_b]$ has measure 0.
It can easily be proven that these saddles can be avoided with prob. at least $\frac{1}{2}$ for a Gaussian initialization for example, since $P(H(\theta_0) < 0) = \frac{1}{2}$ at initialization (this follows from the fact that $H((W_1, \ldots, W_L)) = -H((-W_1, \ldots, W_L))$).

Another motivation for this assumption is the fact that if the network is initialized with balanced weights [2,3], i.e. if $W_i^T W_i = W_{i-1}^T W_{i-1}$ for $1 < t < L$. Since the balancedness is conserved during training, if gradient flow converges to a saddle, this saddle must be balanced. However the only balanced saddle of $H$ is the origin, which can only be approached along an escape direction $\rho$ with positive speed $s > 0$, which are avoided with prob. 1 by Proposition [23].

### C.2 Approximately Homogeneous Costs

In the last section we gave a description of the escape paths of homogeneous costs $H$. To prove the required results, we need to extend these results to more general cost functions, which in our setting are only locally homogeneous around a saddle $\vartheta^*$, i.e. the cost is of the form $C(\theta) = H(\theta - \vartheta^*) + e(\theta - \vartheta^*)$ for a $k$-homogeneous cost $H$ and a correction $e$ whose $m-1$ first derivatives vanish at 0, for $m > k$. We call such costs $(k, m)$-approximately homogeneous. Note that in the setting of interest, i.e. for the cost $C(A_0)$ and the saddle at the origin only the $kL$-th derivatives are non-zero for $k = 0, 1, 2, \ldots$, we will hence be typically interested in costs which are $(L, 2L)$-approximately homogeneous.

Since we are only interested in the local behaviour around the saddle $\vartheta^*$, we may do a localization of the cost. For any smooth cut-off function $h$ which satisfies $h(x) = 1$ if $0 \leq x \leq 1$, $0 \leq h(x) \leq 1$ if $1 < x < 2$ and $h(x) = 0$ when $x \geq 2$, we define the localization $C_r$ of the cost $C$ with radius $r$ as

$$C_r(\theta) = H(\theta - \vartheta^*) + e(\theta - \vartheta^*)h\left(\frac{||\theta - \vartheta^*||}{r}\right).$$

As before, for simplicity, we assume that $\vartheta^* = 0$. The correction $e_r(\theta) = e(\theta)h\left(\frac{1}{r}||\theta||\right)$ satisfies:

**Lemma 27.** Let $h, e : \mathbb{R}_+ \to \mathbb{R}$ be as above. We have

$$\|\partial^k_r e_r\|_\infty = O(r^{m-k}), \text{ as } r \to 0.$$

**Proof.** We have

$$\partial^k_r \left[ e(x)h\left(\frac{1}{r}||x||\right) \right] = \sum_{k_1+k_2=k} \partial^{k_1}_e e(x) \partial^{k_2}_r h\left(\frac{1}{r}||x||\right)$$

Since $\partial^{k_2}_r h\left(\frac{1}{r}||x||\right) = 0$ whenever $||x|| > 2R > 2r$ and $\|\partial^{k_2}_r h\left(\frac{1}{r}||x||\right)\|_\infty = O(r^{-k_2})$, while $\|\partial^{k_1}_e e(x)\| = O(||x||^{m-k_1})$ we have

$$\|\partial^k_r [e(x)h\left(\frac{1}{r}||x||\right)]\|_\infty \leq \sum_{k_1+k_2=k} \|\partial^{k_1}_e e(x) \partial^{k_2}_r h\left(\frac{1}{r}||x||\right)\|_\infty$$

$$= O(r^{m-k_1}) = O(r^{m-k}),$$

as claimed. \qed

### C.3 Escape Cones

We will approximate locally homogeneous costs by homogeneous ones using the following approximation:

**Lemma 28.** For a cost $C(x) = H(x) + e(x)$ where $H(\lambda x) = \lambda^k H(x)$ and $\|\nabla e(x)\| < c||x||^k$ for some $c > 0$, it holds that for all $t \geq 0$, there is a finite constant $c_1(t)$ such that

$$\|\gamma_C(\alpha^{2-k} t, \alpha x_0) - \gamma_H(\alpha^{2-k} t, \alpha x_0)\| = \|\gamma_C(\alpha^{2-k} t, \alpha x_0) - \alpha \gamma_H(t, x_0)\| \leq c_1(t) \alpha^2.$$
Proof. For any $\epsilon > 0$ such that $\|\gamma_C(t, x_0)\| < \epsilon$ and $\|\gamma_H(t, x_0)\| < \epsilon$, we can bound how fast the distance between the two paths increases

$$\partial_t \|\gamma_C(t, x_0) - \gamma_H(t, x_0)\| = -\frac{\gamma_C(t, x_0) - \gamma_H(t, x_0)^T}{\|\gamma_C(t, x_0) - \gamma_H(t, x_0)\|} (\nabla H(\gamma_C(t, x_0)) + \nabla e(\gamma_C(t, x_0)) - \nabla H(\gamma_H(t, x_0)))$$

$$\leq \left( \sup_{\|x\| \leq \epsilon} \|\mathcal{H}(x)\|_{op} \right) \|\gamma_C(t, x_0) - \gamma_H(t, x_0)\| + \|\nabla e(\gamma_C(t, x_0))\|$$

$$\leq c_1 e^{-kt} \|\gamma_C(t, x_0) - \gamma_H(t, x_0)\| + ce^k$$

where $c_1 = \sup_{\|x\| \leq 1} \|\mathcal{H}(x)\|_{op}$. Applying Grönwall’s inequality to $A(t) = \|\gamma_C(t, x_0) - \gamma_H(t, x_0)\| + \frac{c}{c_1} e^k$ (such that $\partial_t A(t) \leq c_1 e^{-kt} A(t)$) we obtain

$$\|\gamma_C(t, x_0) - \gamma_H(t, x_0)\| + \frac{c}{c_1} e^k = A(t) \leq A(0) e^{c_1 e^{-kt}} = \frac{c}{c_1} e^{c_1 e^{-kt}}$$

hence $\|\gamma_C(t, x_0) - \gamma_H(t, x_0)\| \leq \frac{c}{c_1} e^{c_1 e^{-kt}}$ for all times $t$ such that for all $s \leq t$, we have $\|\gamma_C(s, x_0)\| < \epsilon$ and $\|\gamma_H(s, x_0)\| < \epsilon$.

We define the $\epsilon$-Escape Cone as the set $C_\epsilon = \left\{ \theta : \frac{H(\theta)}{\|\theta\|} < \frac{s^* + \epsilon}{k} \right\}$ for the optimal escape speed $s^*$.

In the neighbourhood of the origin the escape cone has the following properties

**Proposition 29.** For all $\epsilon$ small enough there is a $r > 0$ such that

1. for any point $\theta$ on the boundary of the cone $C_\epsilon$ with $\|\theta\| < r$, gradient flow at $\theta$ points inside the cone, i.e. denoting by $n$ the normal at $\theta$ pointing outside of the cone, we have $-\nabla C(\theta)^T n \leq 0$.

2. for any point $\theta$ inside the cone with $\|\theta\| < r$, we have $\|\theta\|^{k-1} (s^* - \epsilon) \leq \nabla C(\theta)^T \frac{\theta}{\|\theta\|} \leq \|\theta\|^{k-1} (s^* + 2\epsilon)$.

3. Let $\theta_0 \in C_\epsilon$ and $\|\theta_0\| < r$, let $T$ be the time when $\|\gamma_C(t, \theta_0)\| = r$. When $k = 2$ we have for all time $0 \leq t < T$

$$\|\theta_0\| e^{-(s^* + 2\epsilon)t} \leq \|\gamma(t, \theta_0)\| \leq \|\theta_0\| e^{-(s^* - \epsilon)t}$$

and when $k \neq 2$ we have for all time $0 \leq t < T$

$$\left[\|\theta_0\|^{-(k-2)} + (k-2)(s^* + 2\epsilon)t\right]^{-\frac{1}{k-2}} \leq \|\gamma(t, \theta_0)\| \leq \left[\|\theta_0\|^{-(k-2)} + (k-2)(s^* - \epsilon)t\right]^{-\frac{1}{k-2}}.$$

**Proof.** Choose

$$r = r(\epsilon) = \min \left\{ \frac{\inf_{H(\rho) = s^* + \epsilon} \left\{ \nabla H(\rho)^T P_\rho \nabla H(\rho) \right\}}{c \sup_{H(\rho) = s^* + \epsilon} \left\{ \sqrt{\nabla H(\rho)^T P_\rho \nabla H(\rho)} \right\}}, \sqrt{\frac{\epsilon}{c}} \right\}.$$

(1) Let $\theta$ be on the boundary of the cone, i.e. $\frac{H(\theta)}{\|\theta\|} = H \left( \frac{\theta}{\|\theta\|} \right) = s^* + \epsilon$, the normal pointing inside the cone is equal (up to a positive scaling) to

$$-\nabla_\theta H \left( \frac{\theta}{\|\theta\|} \right) = -\nabla H \left( \frac{\theta}{\|\theta\|} \right) \frac{1}{\|\theta\|} \left[ I_d - \frac{\theta \theta^T}{\|\theta\|^2} \right] = -\nabla H \left( \frac{\theta}{\|\theta\|} \right) \frac{P_\theta}{\|\theta\|}.$$
where \( P_0 = \left[ I_d - \frac{\theta \theta^T}{\|\theta\|^2} \right] \) is the orthogonal projection to the tangent space of \( S^{d-1} \) at \( \frac{\theta}{\|\theta\|} \). We then have that
\[
\left( \nabla_{\theta} H(\theta) \left\| \theta \right\|^2 \right)^T (-\nabla C(\theta)) = -\nabla H \left( \frac{\theta}{\|\theta\|} \right)^T P_0 \left( -\nabla H(\theta) - \nabla e(\theta) \right)
\]
\[
= \left\| \theta \right\|^{k-1}\nabla H \left( \frac{\theta}{\|\theta\|} \right)^T P_0 \nabla H \left( \frac{\theta}{\|\theta\|} \right) + \nabla H \left( \frac{\theta}{\|\theta\|} \right)^T P_0 \nabla e(\theta)
\]
\[
\geq \left\| \theta \right\|^{k-1} \inf_{H(\rho) = s^* + \epsilon} \left\{ \nabla H (\rho) \right\}^T P_0 \nabla H (\rho)
\]
\[
- \frac{1}{\left\| \theta \right\|} \sqrt{\nabla H (\rho)^T P_0 \nabla H (\rho) \left\| \nabla e(\theta) \right\|}
\]
\[
\geq \left\| \theta \right\|^k \inf_{H(\rho) = s^* + \epsilon} \left\{ \sqrt{\nabla H (\rho)^T P_0 \nabla H (\rho)} \right\}
\]
which is positive since \( \left\| \theta \right\| < r(\epsilon) \leq \frac{\inf_{H(\rho) = s^* + \epsilon} \left\{ \sqrt{\nabla H (\rho)^T P_0 \nabla H (\rho)} \right\}}{c \sup_{H(\rho) = s^* + \epsilon} \left\{ \sqrt{\nabla H (\rho)^T P_0 \nabla H (\rho)} \right\}} \).

(2) Let \( \theta \in C_\epsilon \) then
\[
\nabla C(\theta)^T \theta = \partial_\lambda H(\lambda \theta) \bigg|_{\lambda=1} + (\nabla e(\theta))^T \theta
\]
\[
\leq k H(\theta) + c \left\| \theta \right\|^2
\]
\[
= k \left\| \theta \right\|^k H \left( \frac{\theta}{\left\| \theta \right\|} \right) + c \left\| \theta \right\|^2
\]
\[
\leq k \left\| \theta \right\|^k \left( s^* + \epsilon + c \left\| \theta \right\|^2 \right)
\]
\[
\leq k \left\| \theta \right\|^k \left( s^* + 2\epsilon \right)
\]
where we used that \( \left\| \theta \right\|^2 < r^2 \leq \frac{\epsilon}{\nu} \). In the other direction we obtain
\[
\nabla C(\theta)^T \theta = \partial_\lambda H(\lambda \theta) \bigg|_{\lambda=1} + (\nabla e(\theta))^T \theta
\]
\[
\geq k H(\theta) - c \left\| \theta \right\|^2
\]
\[
= k \left\| \theta \right\|^k H \left( \frac{\theta}{\left\| \theta \right\|} \right) - c \left\| \theta \right\|^2
\]
\[
\geq k \left\| \theta \right\|^k \left( s^* + c \left\| \theta \right\|^2 \right)
\]
\[
\geq k \left\| \theta \right\|^k \left( s^* - \epsilon \right).
\]

(3) Applying Grönwall’s inequality (generalized to polynomial bounds), we have
\[
\partial_t \left\| \theta \right\|^2 = -2 \nabla C(\theta)^T \theta \leq c_1 \left( \left\| \theta \right\|^2 \right)^{\frac{1}{2}}.
\]

\( \square \)

Putting it all together, this allows to guarantee that with prob. 1 over the initialization gradient flow escapes along a path \( \theta^t \) which escapes the saddle at a specific speed:

**Proposition 30.** Let \( \theta_\alpha(t) = \gamma C(t, \alpha \theta_0) \) for all \( t \geq 0 \). With prob. 1 over initialization (and under Assumption 25 when \( L > 3 \)) there is a time horizon \( t_\alpha^1 \) that tends to \( \infty \) as \( \alpha \to 0 \) and a path \( (\theta^t(t))_{t \in \mathbb{R}} \) such that for all \( t \in \mathbb{R} \), \( \lim_{\alpha \to 0} \theta_\alpha(t_\alpha^1 + t) = \theta^1(t) \). Furthermore, for all \( \epsilon > 0 \) there exists \( T \) such that:
(1) Shallow networks: \( e^{-(s^* + 2c)(T + t)} \leq \| \theta^1(t) \| \leq e^{-(s^* - \epsilon)(T + t)} \).

(2) Deep networks: \( [(L - 2)(s^* + 2c)(T + t)]^{\frac{1}{2}} \leq \| \theta^1(t) \| \leq [(L - 2)(s^* - \epsilon)(T + t)]^{\frac{1}{2}} \).

Proof. We consider the gradient flow path \( \tilde{\theta}_\alpha(t) = \gamma_H(t, \alpha \theta_0) \) on the \( k \)-homogeneous cost \( H \). With prob. 1 (and under Assumption 26 when \( L > 3 \), we have \( \frac{H(\tilde{\theta}_\alpha(t))}{\| \tilde{\theta}_\alpha(t) \|^r} \to \frac{s^*}{k} \) as \( t \to \infty \), so that for all \( \epsilon > 0 \), there is a \( t_0 \) such that \( \tilde{\theta}_\alpha(t_0) \) is in the inside of the escape cone \( C_\epsilon \), and more generally by Lemma 22 we have \( \tilde{\theta}_\alpha(\alpha^{-(L-2)} t_0) = \alpha \tilde{\theta}_1(t_0) \in C_\epsilon \). Lemma 28 then shows that there exists \( \alpha_0 > 0 \) such that for all \( \alpha < \alpha_0 \), it holds that \( \tilde{\theta}_\alpha(\alpha^{-(L-2)} t_0) \in C_\epsilon \).

By Proposition 29 once the gradient flow path is inside \( C_\epsilon \), it cannot leave the escape cone until the norm \( \| \theta_\alpha(t) \| \) is larger than some radius \( r \). We define the time horizon \( t^1_\alpha \) as the time such that \( \| \tilde{\theta}_\alpha(t^1_\alpha) \| = \frac{s^*}{2} \) and define the escape path \( \theta^1 \) as the limit \( \theta^1(t) = \lim_{\alpha \to 0} \tilde{\theta}_\alpha(t^1_\alpha + t) \). One can see that \( t^1_\alpha + t > \alpha^{-(L-2)} t_0 \) for any \( t < 0 \), thus it holds that \( \theta^1(t) \in C_\epsilon \) since for a small enough \( \alpha \), we have \( \tilde{\theta}(t^1_\alpha + t) \in C_\epsilon \). Proposition 29 then implies the escape rates for deep and shallow networks.

\( \square \)

C.4 Optimal Escape Paths

In this section, we give a description for the optimal escape paths at the origin.

C.4.1 Shallow Networks

For shallow networks, the loss around the saddle at the origin is approximately 2-homogeneous, i.e. of the form \( C(x) = H(x) + e(x) \) for a 2-homogeneous polynomial \( H \) and a \( e(x) \) with vanishing first \( m - 1 \) derivatives, where \( m > 2 \). There is a direct correspondence between the escape paths of the cost \( C \) and its localization \( C_\tau \), we will therefore study the escape paths of \( C_\tau \).

We will need the following norm on paths. Let \( \beta > 0 \) and define the weight function \( w : t \mapsto e^{-2 \beta t} \). We define the norm:

\[ \| y \|^2_w = \int_{-\infty}^{\infty} w(t) \| y(t) \|^2 dt. \]

Before we continue let us state a few useful properties of the scalar product \( \langle \cdot, \cdot \rangle_w \):

**Lemma 31.** For any differentiable \( x, y \) with \( \| x \|_w, \| y \|_w < \infty \), we have

1. \( \langle x, \dot{y} \rangle_w = 2 \beta \langle x, y \rangle_w - \langle \dot{x}, y \rangle_w \).
2. \( \langle x, \dot{x} \rangle_w = \beta \| x \|_w^2 \).
3. \( \| x \|_w \leq \frac{1}{\beta} \| \dot{x} \|_w \).

**Proof.** We obtain 1. by the product rule for integration:

\[
\langle x, \dot{y} \rangle_w = \int_{-\infty}^{\infty} w(t) (x(t))^T \dot{y}(t) dt = - \int_{-\infty}^{\infty} \dot{w}(t) (x(t))^T y(t) dt - \int_{-\infty}^{\infty} w(t) \dot{x}(t) y(t) dt = 2 \beta \langle x, y \rangle_w - \langle \dot{x}, y \rangle_w.
\]

Taking \( x = y \) we get 2. Finally, 3. follows from

\( \| x \|_w^2 = \frac{1}{\beta} \langle x, \dot{x} \rangle_w \leq \frac{1}{\beta} \| x \|_w \| \dot{x} \|_w \).

\( \square \)
Our goal is to describe the escape paths of the critical point at the origin. They correspond to the fixpoints of the map

\[ \Phi_\eta(x) = t \mapsto (1 - \eta)x - \eta \int_{-\infty}^{t} \nabla C_r(x(t)) \, dt \]

for \( 0 < \eta \leq 1 \). With the right choice of \( \beta, r, \) and \( \eta \) the map \( \Phi_\eta \) is a contraction.

**Lemma 32.** Let \( C = H + e \) be a \((2, m)\)-approximately homogeneous loss where \( H \) is a polynomial. For any \( \beta > |s^*| \), and \( r, \eta > 0 \) small enough, the map \( \Phi_\eta \) is a contraction, that is for all paths \( x, y \).

\[
\|\Phi_\eta(x) - \Phi_\eta(y)\|_w \leq \left( 1 - \frac{\eta}{2} \left[ 1 - \frac{-s^* + O(r^{-m})}{\beta} \right] \right) \|x - y\|_w.
\]

**Proof.** We will show that

\[
\|\Phi_\eta(x) - \Phi_\eta(y)\|_w = \left\| (1 - \eta) (x - y) + \eta \int_{-\infty}^{t} [\nabla C_r(x(t)) - \nabla C_r(y(t))] \, dt \right\|_w^2
\]

\[
= (1 - \eta)^2 \|x - y\|_w^2 + 2\eta(1 - \eta) \left\langle x - y, \int_{-\infty}^{t} [\nabla C_r(x(t)) - \nabla C_r(y(t))] \, dt \right\rangle_w + \eta^2 \left\| \int_{-\infty}^{t} [\nabla C_r(x(t)) - \nabla C_r(y(t))] \, dt \right\|_w^2
\]

\[
\leq (1 - \eta)^2 \|x - y\|_w^2 + 2\eta(1 - \eta) \frac{-s^* + \kappa r^{-m}}{\beta} \|x - y\|_w^2 + \eta^2 \frac{\|HC_r\|_w^2}{\beta^2} \|x - y\|_w^2
\]

\[
= \left( 1 - 2\eta \left[ 1 - \frac{-s^* + \kappa r^{-m}}{\beta} \right] \right) + O(\eta^2) \|x - y\|_w^2,
\]

for some \( \kappa > 0 \). Choosing \( \eta \) small enough, we have

\[
\|\Phi_\eta(x) - \Phi_\eta(y)\|_w \leq \left( 1 - \eta \left[ 1 - \frac{-s^* + \kappa r^{-m}}{\beta} \right] \right) \|x - y\|_w^2.
\]

The bound on the last term follows from point (3) of Lemma 31.

\[
\left\| \int_{-\infty}^{t} [\nabla C_r(x(t)) - \nabla C_r(y(t))] \, dt \right\|_w \leq \frac{1}{\beta} \|\nabla C_r(x) - \nabla C_r(y)\|_w \leq \frac{\|HC_r\|_w^2}{\beta^2} \|x - y\|_w.
\]

To bound the second term, we used the fact that for all \( t \cdot \nabla C_r(x(t)) - \nabla C_r(y(t)) = HC_r(z(t)) (x(t) - y(t)) \) for some \( z(t) \) somewhere on the segment \([x(t), y(t)]\). We then have

\[
\left\langle x - y, \int_{-\infty}^{t} [\nabla C_r(x(t)) - \nabla C_r(y(t))] \, dt \right\rangle_w = \left\langle x - y, \int_{-\infty}^{t} HC_r(z(t)) (x(t) - y(t)) \, dt \right\rangle_w
\]

\[
= \left\langle x - y, \int_{-\infty}^{t} (x(t) - y(t)) \, dt \right\rangle_w + \left\langle x - y, \int_{-\infty}^{t} HC_r(z(t)) (x(t) - y(t)) \, dt \right\rangle_w
\]

\[
\geq \beta \left\langle \int_{-\infty}^{t} (x(t) - y(t)) \, dt, \int_{-\infty}^{t} (x(t) - y(t)) \, dt \right\rangle_w - \|x - y\|_w \left\| \int_{-\infty}^{t} HC_r(z(t)) (x(t) - y(t)) \, dt \right\|_w
\]

\[
\geq \frac{s^*}{\beta} \left\| x - y \right\|_w^2 - \frac{\kappa r^{-m}}{\beta} \|x - y\|_w^2
\]

\[
= \frac{s^*}{\beta} \|x - y\|_w^2.
\]
where we used the fact that for any \( z \), point (1) of Lemma \( \text{31} \) tells us that

\[
\left\langle z, \mathcal{H} \int_{-\infty}^{t} z(dt) \right\rangle = 2\beta \left\langle \int_{-\infty}^{t} z(dt), \mathcal{H} \int_{-\infty}^{t} z(dt) \right\rangle - \left\langle \int_{-\infty}^{t} z(dt), Hz \right\rangle
\]

which by the symmetry of \( \mathcal{H} \) implies that

\[
\beta \left\langle \int_{-\infty}^{t} z(dt), H \int_{-\infty}^{t} z(dt) \right\rangle = \beta \left\langle \int_{-\infty}^{t} \cdot dt, H \int_{-\infty}^{t} z(dt) \right\rangle.
\]

We have the following bijection between gradient flow paths on the costs \( C(x) = H(x) + e(x) \) and \( H(x) \):

**Proposition 33.** For \( \beta > |s^*| \), let \( \mathcal{F}_C \left[ \frac{\beta}{m-1} \right] \) (resp. \( \mathcal{F}_H \left[ \frac{\beta}{m-1} \right] \)) be the set of gradient flow paths on the cost \( C(x) = H(x) + e(x) \) (resp. \( H(x) \)) such that \( \|x(t)\| = O(e^{\frac{m}{m-1}t}) \) as \( t \to -\infty \). There is a unique bijection \( \Psi : \mathcal{F}_C \left[ \frac{\beta}{m-1} \right] \mapsto \mathcal{F}_H \left[ \frac{\beta}{m-1} \right] \) such that \( \|x(t) - \Psi(x)(t)\| = O(e^{\beta t}) \).

**Proof.** It is readily seen that there is a bijection \( \Psi_r \) between \( \mathcal{F}_C \left[ \frac{\beta}{m-1} \right] \) and \( \mathcal{F}_C \left( \frac{\beta}{m-1} \right) \) such that \( \|x(t) - \Psi_r(x)(t)\| = O(0) \).

To show the bijection between \( \mathcal{F}_C \left[ \frac{\beta}{m-1} \right] \) and \( \mathcal{F}_H \left[ \frac{\beta}{m-1} \right] \) we iterate \( \Phi_r \) from a path \( x \in \mathcal{F}_H \) to obtain a gradient flow path of \( C_r \). By Lemma \( \text{32} \) we know that for small enough \( r, \eta > 0 \), \( \Phi_r \) is a contraction w.r.t. the norm \( \|\cdot\|_w \). We now need to ensure that \( \|x - \Phi_r(x)\|_w \) is finite:

\[
\|x - \Phi_r(x)\|_w = \eta \left\| x + \int_{-\infty}^{t} \nabla C_r(x(t))dt \right\|_w
\]

\[
\leq \eta \left\| x + \int_{-\infty}^{t} \nabla H(x(t))dt + \eta \int_{-\infty}^{t} \nabla e_r(x(t))dt \right\|_w
\]

\[
\leq 0 + \eta \frac{\eta}{\beta} \left\| \nabla e_r(x) \right\|_w
\]

\[
\leq \eta \frac{\kappa_3}{\beta} \left\| x(t) \right\|^{m-1}_w
\]

which is finite since \( \|x(t)\|^{m-1}_w = O(e^{-s^*t}) \) and \( \beta > |s^*| \).

Let \( R = \frac{2\beta \left\| x + \int_{-\infty}^{t} \nabla C_r(x(t))dt \right\|_w}{\kappa_2 + \kappa_2 m - 2} \) then for any \( y \) with \( \|x - y\|_w \leq R \), we have

\[
\|x - \Phi_r(y)\|_w \leq \|x - \Phi_r(x)\|_w + \|\Phi_r(x) - \Phi_r(y)\|_w
\]

\[
\leq \eta \left\| x - \int_{-\infty}^{t} f(x(t))dt \right\|_w + \left( 1 - \frac{\eta}{2} \left[ 1 - \frac{s^* + \kappa_2 m - 2}{\beta} \right] \right) R
\]

\[
= R.
\]

In other terms, \( \Phi_r \) is a contraction in the ball \( B(x, R) \). There is hence a unique fixpoint \( \tilde{x} \) in \( B(x, R) \) which we define to be the image \( \Psi(x) \), note that we can obtain \( x \) from \( \tilde{x} \) by iterating the map \( \Phi_r \) which maps paths \( x \) to paths \( t \mapsto \left[ 1 - (1 - \eta)t \right] \left[ \eta \int_{-\infty}^{t} \nabla H(x(t))dt \right] \) starting from \( \tilde{x} \) to obtain a unique gradient flow path \( x \) of \( C_r \) at a finite \( \|\cdot\|_w \) distance (ensuring we recover \( x \)).
Finally since \( x \) and \( \tilde{x} = \Psi_r(x) \) are gradient flow paths of costs \( H \) and \( C_r \) respectively, with \( \|x - \tilde{x}\|_w \leq R \), we have

\[
\|x(t) - \tilde{x}(t)\| = \left\| \int_{-\infty}^{t} \nabla H(x(s))ds - \int_{-\infty}^{t} \nabla C_r(\tilde{x}(s))ds \right\|
\]

\[
\leq \left\| \int_{-\infty}^{t} (\nabla C_r(x(s)) - \nabla C_r(\tilde{x}(s)))ds \right\| + \left\| \int_{-\infty}^{t} \nabla e_r(x(s))ds \right\|
\]

\[
\leq \|HC_r\|_\infty \sqrt{\int_{-\infty}^{t} w(s)\|x(s) - \tilde{x}(s)\|^2 ds} + \sqrt{\int_{-\infty}^{t} \frac{1}{w(s)} ds + \kappa_3 \int_{-\infty}^{t} \|x(s)\|^{m-1} ds}
\]

which ensures that \( \|x(t) - \tilde{x}(t)\| = O(e^{\beta t}) \) as \( t \to -\infty \). This also ensures that \( \|\tilde{x}(t)\| = O(e^{\frac{\beta}{m-1} t}) \) so that \( \tilde{x} \in F_{C_r}[\frac{\beta}{m-1}] \).

\[\tag*{□}\]

\subsection*{C.4.2 Deep Networks}

For deep networks, the saddle at the origin has its first \( L - 1 \) derivatives equal to zero, leading to a polynomial escape rate instead of the exponential rate in shallow networks. We can show a similar bijection between fast escaping paths, using a similar technique but with the norm

\[
\|x\|_{(-t)^\beta} = \int_{-\infty}^{0} (-t)^\beta \|x(t)\|^2 dt.
\]

This norm satisfies

\begin{itemize}
\item[1.] \( \langle x, -t\dot{y}\rangle_{(-t)^\beta} = (\beta + 1) \langle x, y\rangle_{(-t)^\beta} - \langle -t\dot{x}, y\rangle_{(-t)^\beta} \).
\item[2.] \( \frac{2}{\beta + 1} \langle x, -t\dot{x}\rangle_{(-t)^\beta} = \|x\|^2_{(-t)^\beta} \).
\item[3.] \( \|x\|_{(-t)^\beta} \leq \frac{2}{\beta + 1} \|t\dot{x}\|_{(-t)^\beta} \).
\end{itemize}

\begin{proof}
We obtain (1) by the product rule for integration:

\[
\langle x, -t\dot{y}\rangle_{\beta} = \int_{-\infty}^{0} (-t)^{\beta+1} x(t)\dot{y}(t)dt
\]

\[
= 0x(0)y(0) + \int_{-\infty}^{0} (\beta + 1)(-t)^\beta x(t)y(t)dt - \int_{-\infty}^{0} (-t)^{\beta+1}\dot{x}(t)y(t)dt
\]

\[
= (\beta + 1) \langle x, y\rangle_{(-t)^\beta} - \langle -t\dot{x}, y\rangle_{(-t)^\beta}.
\]

Taking \( x = y \) we get (2). Finally (3) follows from

\[
\|x\|^2_{(-t)^\beta} = \frac{2}{\beta + 1} \langle x, -t\dot{x}\rangle_{(-t)^\beta} \leq \frac{2}{\beta + 1} \|x\|_{(-t)^\beta} \|t\dot{x}\|_{(-t)^\beta}.
\]

\[\tag*{□}\]

Roughly speaking, the functions with finite \( \|\cdot\|_{(-t)^\beta} \) norm are those that decay faster than \( (-t)^{-\frac{\beta+1}{2}} \), since the function \( y(t) = (-t)^{-\alpha} \) for some \( \alpha > 0 \) has finite norm \( \|y\|_{(-t)^\beta}^2 = \int_{-\infty}^{0} (-t)^{\beta-2\alpha} dt \), whenever \( \alpha > \frac{\beta+1}{2} \).

We will also need the following Lemma:
Lemma 35. Let $H$ be a $k$-homogeneous polynomial, then
\[
\|\nabla H(x) - \nabla H(y)\| \leq k \|H\|_\infty \|x - y\| \sum_{i=0}^{k-2} \|x\|^i \|y\|^{k-2-i}
\]
for $\|H\|_\infty = \max_{z \in S^{d-1}} |H(z)|$.

Proof. This is a consequence of Banach’s Theorem \[6, 13\] for the spectral norm of symmetric tensors. Indeed any $k$-homogeneous polynomial $H$ can be written as
\[
H(x) = T \cdot x^\otimes k = T \cdot (x \otimes \cdots \otimes x)
\]
for some symmetric tensor $T$ and for the tensor product $\otimes$ and scalar product $\cdot$ on tensors. We hence have
\[
\|\nabla H(x) - \nabla H(y)\| = \max_{v \in S^{d-1}} \ kT \cdot \left[ v \otimes \left( x^\otimes (k-1) - y^\otimes (k-1) \right) \right]
\]
\[
= \max_{v \in S^{d-1}} \ kT \cdot \left[ v \otimes (x - y) \otimes \sum_{i=0}^{k-2} x^\otimes i \otimes y^\otimes (k-2-i) \right]
\]
\[
\leq k \|x - y\| \sum_{i=0}^{k-2} \|x\|^i \|y\|^{k-2-i} \left( \max_{v \in S^{d-1}} T \cdot \left[ v \otimes \frac{x - y}{\|x - y\|} \otimes \frac{x^\otimes i}{\|x\|^i} \otimes \frac{y^\otimes (k-2-i)}{\|y\|^{k-2-i}} \right] \right)
\]
\[
\leq k \|x - y\| \sum_{i=0}^{k-2} \|x\|^i \|y\|^{k-2-i} \left( \max_{z \in S^{d-1}} H(z) \right).
\]
Banach’s Theorem \[6, 13\] was applied in the second bound from line 3 to line 4 in the above. \hfill \Box

Let $C(x) = H(x) + e(x)$, and $C_r(x) = H(x) + h(\frac{x}{r})e(x)$. The escape paths of the flow $C_r$ correspond to fixpoints of the map
\[
\Phi(x) = t \mapsto -\int_{-\infty}^{t} \nabla C_r(x(s)) ds
\]

Proposition 36. For a $(k, m)$-approximately homogeneous cost $C(x) = H(x) + e(x)$ with $m \geq 2k - 2$, then for any $\beta$ s.t. $1 < \beta < \frac{2m-3k+4}{k-2}$ and any $c > \frac{2k(k-1)}{\beta+1} \|H\|_\infty$, there is a bijection $\Psi$ between the sets $\mathcal{F}_C[\beta, c]$ and $\mathcal{F}_H[\beta, c]$ of gradient flow paths of the cost $C$ and $H$ such that $\|x(t)\| \leq (ct)^{-\frac{1}{k-2}} \Psi^{-1}$. For each $x \in \mathcal{F}_C[\beta, c], \Psi(x) \in \mathcal{F}_C[\beta, c]$ is the unique $C_r$ path such that $\|x(t) - \Psi(z)(t)\| = O \left( (t)^{-\frac{2+1}{k-2}} \right)$ (and vice-versa).

Proof. Again we can easily obtain a bijection between the fast escaping paths $\mathcal{F}_C$ and $\mathcal{F}_{C_r}$ of the costs $C$ and $C_r$ respectively, we therefore only need to show the bijection between $\mathcal{F}_C$ and $\mathcal{F}_H$.

Let $x \in \mathcal{F}_H$, then $\|x(t)\| \leq (ct)^{-\frac{1}{k-2}}$. There is a small enough $r$ such that $\Phi$ is a contraction over the set of paths $\{y : \|x - y\|_{(-t)^{\beta}} < \infty\}$. 

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We now need to show that

\[ \Phi(x) - \Phi(y) \] 

at a finite distance.

This implies the existence and uniqueness of a gradient flow path \( \tilde{x} \) of \( C_r \) at a finite distance.

This path satisfies

\[ \|x(t) - \tilde{x}(t)\| \leq \int_{-\infty}^{t} \|\nabla C_r(x(s)) - \nabla C_r(\tilde{x}(s))\| ds \]

\[ \leq \frac{2}{\beta + 1} \|t \|^{m-k} \|x(s)\|^{m-1} \] 

\[ \leq \frac{2k(\|H\|_\infty + r \kappa^{m-k})}{(\beta + 1)c} \|x - y\|_{(-t)^\beta} \]

which is finite since \( -\frac{k-m-1}{m-1} < \frac{\beta + 1}{2} \) (we assumed \( \beta < \frac{2m-3k+4}{k-2} \)).

This implies the existence and uniqueness of a gradient flow path \( \tilde{x} \) of \( C_r \) at a finite distance.

This path satisfies

\[ \|x(t) - \tilde{x}(t)\| \leq \int_{-\infty}^{t} \|\nabla C_r(x(s)) - \nabla C_r(\tilde{x}(s))\| ds \]

\[ \leq \frac{2}{\beta + 1} \|t \|^{m-k} \|x(s)\|^{m-1} \] 

\[ \leq \frac{2k(\|H\|_\infty + r \kappa^{m-k})}{(\beta + 1)c} \|x - y\|_{(-t)^\beta} \]

Since \( \beta < \frac{2m-3k+4}{k-2} \), we have \( (-t)^{1-\frac{m-1}{m-2}} = O((-t)^{-\frac{\beta + 1}{2}}) \) as \( t \to -\infty \), which implies that \( \|x(t) - \tilde{x}(t)\| = O((-t)^{-\frac{\beta + 1}{2}}) \).

There cannot be another path \( \tilde{x}' \) of \( C_r \) such that \( \|x(t) - \tilde{x}'(t)\| = O((-t)^{-\frac{\beta + 1}{2}}) \), as then \( \|x - \tilde{x}'\|_{(-t)^\beta} < \infty \) and \( \tilde{x} \) is the unique path of \( C_r \) at a finite distance.
C.5 Proof of the Theorem

We have now all the tools to prove Theorem 4:

**Theorem 37** (Theorem 4 of Main Text). Assume that the largest singular value $s_1$ of the gradient at the origin $\nabla C(0) \in \mathbb{R}^{n_L \times n_0}$ has multiplicity 1, furthermore if $L > 3$ assume that Assumption \[26\] holds. With probability 1 over the sampling of the direction $\rho_0 \in \mathbb{S}^{P_{L-1}}$, there is a path $\theta^1(t) \in \mathbb{R}^{P_{L-1}}$ with $\lim_{t \to -\infty} \theta^1(t) = \emptyset$ and some $t^1_\alpha$ such that $\lim_{\alpha \to 0} \theta(t^1_\alpha + t) = \theta^1(t)$. The path $\theta^1$ converges to a critical point $\emptyset$ as $t \to +\infty$.

Furthermore, there is a unique and deterministic path $\theta^1 : \mathbb{R} \to \mathbb{R}^{P_{L-1}}$ in the space of DLN of width 1 and a random rotation $R$ such that $\theta^1 = RI^{(1-w)}(\theta^1)$.

**Proof.** From Proposition \[30\] we know that with prob. 1 there is a time horizon $t^1_\alpha$ and an escape path such that $\lim_{\alpha \to 0} \theta(t^1_\alpha + t) = \emptyset(t)$ which for any $\epsilon > 0$ escapes the origin at a rate of at least $e^{(s^* + \epsilon)t}$ for shallow networks and $[(k - 2)(s^* + \epsilon)t]^{\frac{1}{k-2}}$ for deep networks.

Since the loss $C^{NN}$ is $(L, 2L)$-approximately homogeneous, we can apply Proposition \[33\] for shallow networks and Proposition \[36\] for deep networks to obtain that $\emptyset^1$ must be in bijection with an escape path of the homogeneous loss $H$ of the same speed. For small enough $\epsilon$ the only escape path of $H$ of at least this speed are of the form $\rho^* e^{s^*(t + T)}$ for shallow networks and $\rho^* ((k - 1)s^*(-t - T))^{-\frac{1}{k-1}}$ for some constant $T$ and an optimal escape direction $\rho^*$. We therefore call $\emptyset^1$ an optimal escape path since it belongs to the unique set of paths which escape at an optimal speed and are in bijection to the optimal escape directions.

Assuming that the largest singular value of $s_1$ of $\nabla C(0)$ has multiplicity 1, with singular vectors $u_1, v_1$, the optimal escape directions are of the form

$$\rho^* = RI^{(1-w)}(\rho^*) = \frac{1}{\sqrt{L}}RI^{(1-w)}(-v_1^T, 1, \ldots, 1, u_1)$$

for any rotation $R$. In the width 1 network, there is an optimal escape path $\emptyset^1$ corresponding to the escape direction $\rho^*$, then by the unicity of the bijection of Proposition \[33\] (shallow) and \[36\] (deep), the escape path $RI^{(1-w)}(\emptyset^1)$ is the unique optimal escape path escaping along $RI^{(1-w)}(\rho^*)$, as a result, we know that $\emptyset^1 = RI^{(1-w)}(\emptyset^1)$ for some rotation $R$. \[\]

C.6 Diagonal Initialization and Escape Rates

Let us consider the MSE cost $C(A) = \|AX - Y\|_F^2$ with diagonal input matrix $X = \text{diag}(x_1, \ldots, x_w)$ and diagonal output matrix $Y = \text{diag}(y_1, \ldots, y_w)$ (more generally one could consider the case where the input and output covariances have the same eigenvectors, but it is the same up to some orthogonal matrices). For diagonal weight matrices of the linear network, the loss is equal to

$$C(A) = \sum_{i=1}^w (W_{L,ii} \cdots W_{1,ii}x_i - y_i)^2.$$

If the network is initialized diagonally, it will remain diagonal throughout training and the dynamics along each diagonal entries $i$ are independent. In other terms, it is equivalent to training $w$ networks of width 1 independently. For width 1 networks, Conjecture 3 follows from Theorem 4 (since there can only be one escape path).

Let us consider the loss of the $i$-th coordinate

$$C(v) = (v_L \cdots v_1X_{ii} - Y_{ii})^2$$

for scalars $v_1, \ldots, v_L$. At the origin, the escape directions are of the form $\rho = (\sigma_1, \ldots, \sigma_L)$ with $\sigma_i \in \{+1, -1\}$ with $\prod \sigma_i = -\text{sign}(X_{ii}Y_{ii})$. The corresponding optimal escape paths are of the form $\theta(t) = d(t) \sigma_1, \ldots, \sigma_L$ for a unique $d(t)$ (up to a time change $d' = d + T + t$), we choose $T$ such that $d(0) = \frac{\epsilon}{\sqrt{L}}$, so that $\|\theta(0)\| = \epsilon$ since the subspace $\mathbb{R}_\theta$ is invariant. If the networks is initialized as $\alpha \rho$, then $\theta_\alpha(t) = d(t - t^1_\alpha) \sigma_1, \ldots, \sigma_L$ where the escape time $t^1_\alpha$ must satisfy $d(-t^1_\alpha) = \alpha$. 

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For shallow networks the optimal escape speed is \( s^* = -X_i Y_i \), we know from Proposition 33 that
\[
d(t) = ce^{-s^* t} + O(e^{\beta t})
\]
for some \( \beta > |s^*| \) which implies that \( t^{1}_o \approx c' + \frac{-\log \alpha}{X_i Y_i} \).

For deep networks the optimal speed is \( s^* = -\frac{X_i Y_i}{L-2} \) and we know from Proposition 36 that
\[
d(t) = \left((L-2)s^*(-t)\right)^{-\frac{1}{L-2}} + O \left((-t)^{-\frac{\beta+1}{L-2}}\right)
\]
which implies that \( t^{1}_o \approx c' + \frac{a - (L-2)}{L-2} s^* = c' + \frac{a - (L-2)}{L-2} X_i Y_i \).

### C.7 Technical Lemmas

We need a generalization of Grönwall’s inequality for polynomial bounds.

**Lemma 38.** Let \( x(t) \) satisfy
\[
\partial_t x(t) \leq c x(t)^\alpha,
\]
for some \( c > 0 \) and \( \alpha > 1 \) then
\[
x(t) \leq \left[ x(0)^{1-\alpha} - c(\alpha - 1)t\right]^{-\frac{1}{\alpha - 1}}.
\]

**Proof.** Note that the function \( y(t) = \left[x(0)^{1-\alpha} + c(\alpha - 1)t\right]^{-\frac{1}{\alpha - 1}} \) satisfies (for all \( t < \frac{x(0)^{1-\alpha}}{c(\alpha - 1)} \))
\[
\partial_t y(t) = c(1-\alpha) \frac{1}{1 - \alpha} \left[x(0)^{1-\alpha} - c(\alpha - 1)t\right]^{-\frac{1}{\alpha - 1} - 1} = cy(t)^\alpha.
\]

Our goal is to show that w.r.t. to the evolution of the pair \((x(t), y(t))\), the region \( \{(x, y) : x \leq y\} \) is forward invariant (i.e. if \( x(t) \leq y(t) \) then \( x(s) \leq y(s) \) for all \( s \geq t \)). This follows from the fact that on the boundary (i.e. when \( x(t) = y(t) \)) we have
\[
\partial_t x(t) - \partial_t y(t) \leq c x(t)^\alpha - cy(t)^\alpha = 0
\]
which implies that the flow points towards the inside of \( \{(x, y) : x \leq y\} \).

### D \( L_2 \) regularization

A DLN defines a map between parameters and matrices: \( \theta := (W_L, \ldots, W_1) \mapsto A_\theta : W_L \ldots W_1 \).

The following lemma defines an important right-inverse of this map, which sends a matrix \( A \) to the minimal norm parameters which encode \( A \). In this section, we always assume that the DNL has large enough widths (i.e. \( n_{1} \geq \min(n_0, n_L) \)).

**Lemma 39.** There is a continuous map \( \Psi : A \mapsto \theta \) such that \( A_{\Psi(A)} = A \) and \( \|\Psi(A)\|^2 = L \|A\|_p^p \)
with \( p = 2/L \). Furthermore for any parameters \( \theta \) s.t. \( A_\theta = A \), we have \( \|\theta\|^2 \geq L \|A\|^p_{\bar{p}} \).

**Proof.** Let \( A \) be a \( n_L \times n_0 \) matrix, and let \( A = U S V^T \) be its singular value decomposition and \( (s_1, \ldots, s_{\min(n_L, n_0)}) \) be its singular values. The parameters \( \Psi(A) \) are given by
\[
\Psi(A) = \left( S^{\frac{2}{\bar{p}}} V^T, S^{\frac{2}{\bar{p}}}, \ldots, S^{\frac{2}{\bar{p}}}, US^{\frac{2}{\bar{p}}} \right).
\]

It is obvious that \( A_{\Psi(A)} = A \) and that \( \Psi \) is a continuous map. Also, note that the \( L_2 \) norm of the parameters in \( \Psi(A) = (W_1, \ldots, W_L) \) is equal to \( \sum_{i=1}^{L} \text{Tr}(W_i W_i^T) \), and that for \( i = 1, \ldots, L \), we have \( \text{Tr}(W_i W_i^T) = \sum_s s_i^{2/\bar{p}} = \|A_i\|^2_{2/L}. \) Hence the map \( \Psi \) satisfies the following equality:
\[
\|\Psi(A)\|^2 = L \|A\|_p^p.
\]

It remains to prove that for any parameters \( \theta \) s.t. \( A_\theta = A \), \( \|\theta\|^2 \geq L \|A\|_p^p \) holds. Let us consider a minimizer of \( \|\theta\|^2 = \sum_{i=1}^{L} \text{Tr}(W_i W_i^T) \) over the set of parameters \( \theta = (W_L, \ldots, W_1) \) s.t. \( A_\theta = A \). Given such minimum \( (W_1, \ldots, W_L) \), if \( dW_1, \ldots, dW_L \) is such that
\[
\sum W_L \cdots W_{\ell+1} dW_{\ell_1} W_{\ell-1} \cdots W_1 = 0
\]
(2)
(i.e. at first order in $\epsilon$, $(W_1 + \epsilon dW_1) \cdots (W_L + \epsilon dW_L) = A$), then one should have

$$\sum_{\ell} \text{Tr} [W_\ell^T dW_\ell] = 0. \quad (3)$$

For any $\ell = 2, \ldots, L$, any square matrix $M$ of size $n_{\ell-1}$, the Equation (2) is satisfied for $dW_L = 0, \ldots, dW_{\ell+1} = 0, dW_\ell = W_\ell M, dW_{\ell-1} = -MW_{\ell-1}, dW_{\ell-2} = 0, \ldots, dW_1 = 0$. Using Equation (3) and the cyclicity of the trace, we obtain

$$\text{Tr} [W_\ell^T W_\ell M] = \text{Tr} [W_{\ell-1} W_{\ell-1}^T M],$$

and thus for any $\ell = 2, \ldots, L$, $W_\ell^T W_\ell = W_{\ell-1} W_{\ell-1}^T$. This implies that the singular value decomposition of $W_\ell = U_\ell S_\ell V_\ell^T$ must satisfy,

$$V_\ell S_\ell^2 V_\ell^T = U_{\ell-1} S_{\ell-1}^2 U_{\ell-1}^T$$

for $\ell = 2, \ldots, L$, $V_\ell = U_{\ell-1}$ and $S_\ell^2 S_{\ell-1} = S_{\ell-1}^2 S_{\ell-1}$. In particular, using the fact that $W_L \cdots W_1 = A$, this implies that the diagonal terms of $S_\ell$ are, up to possibly null terms, equal to those of $S^{1/2}$. In particular, the norm of the parameters is

$$\|\theta\|^2 = \sum_{i=1}^L \text{Tr} (W_i W_i^T) = L \text{Tr} \left( \left( S^{1/2} \right)^T S^{1/2} \right) = L \|A\|_p^p.$$  

This proves that all minimizers have the same norm $L \|A\|_p^p$, hence for any parameters $\theta$ s.t. $A_\theta = A$, we have $\|\theta\|^2 \geq L \|A\|_p^p$. \hfill \Box

Using this lemma, we can now prove the following:

**Proposition 40 (Proposition 5 of Main Text).** *For a network of depth $L$ with large enough widths (i.e. $n_\ell \geq \min\{n_0, n_L\}$) and any cost $C$, we have for $p = \frac{2}{L}$:

$$\min_{\theta} C(A_\theta) + \lambda \|\theta\|^2 = \min_A C(A) + \lambda L \|A\|_p^p.$$  

*Proof.* Consider the minimal value of $C(A_\theta) + \lambda \|\theta\|^2$. Using the fact that $\|\theta\|^2 \geq \|\Psi(A)\|^2$, one can restrict the minimization over the set of parameters $\{\Psi(A), A \in M_{n_L \times n_0}\}$ where $M_{n_L \times n_0}$ is the set of $n_L \times n_0$ matrices. Since $A_{\Psi(A)} = A$, we get:

$$\min_{\theta} C(A_\theta) + \lambda \|\theta\|^2 = \min_A C(A) + \lambda \|\Psi(A)\|^2.$$  

We conclude using the fact that $\|\Psi(A)\|^2 = L \|A\|_p^p$ with $p = \frac{2}{L}$. \hfill \Box

**D.1 Shallow Networks**

The nuclear norm is given by $\|A\|_* = \text{Tr} [S]$ where $S$ is given by the SVD decomposition $A = USV^T$. It is the dual of the operator norm: $\|A\|_* = \max_{\|B\|_{op} \leq 1} \text{Tr} [B^T A]$.  

The nuclear norm is not differentiable everywhere, but it is convex which allows us to study its subgradients. Recall that, given a convex function $F$ on matrices, a matrix $G$ is a subgradient of $F$ at $A$ if and only if for any matrix $A'$ the following inequality holds:

$$F(A') \geq F(A) + \text{Tr} [G^T (A' - A)].$$

This notion generalizes that of gradient for convex functions: if $F$ is differentiable at $A$, $\nabla F$ is the unique subgradient of $F$ at $A$. The notion of subgradient is linear, in the sense that if $F_1$ and $F_2$ are two convex functions and if $G_1$, resp. $G_2$, is a subgradient of $F_1$, resp. $F_2$, at $A$, then $\alpha G_1 + \beta G_2$ is a subgradient of $\alpha F_1 + \beta F_2$. Note that $A$ is a minimum of $F$ if and only if the 0 matrix is a subgradient of $F$ at $A$.  

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Lemma 41. For a convex cost $C$ on $n_L \times n_0$ matrices, the subgradient of $A \mapsto C(A) + \lambda \|A\|_*$ at $A$ are of the form $G = \nabla C(A) + \lambda \hat{G}$ where $\hat{G}$ satisfies

$$\text{Tr} \left[ \hat{G}^T A \right] = \|A\|_* \quad \text{and} \quad \|\hat{G}\|_{\text{op}} \leq 1.$$ 

A point $A$ is a global minimum of $A \mapsto C(A) + \lambda \|A\|_*$ iff

$$\text{Tr} \left[ \nabla C(A)^T A \right] = -\lambda \|A\|_* \quad \text{and} \quad \|\nabla C(A)\|_{\text{op}} \leq \lambda.$$

Proof. (1) By the duality property of the nuclear norm and the operator norm, we have that any $\hat{G}$ which satisfy $\|\hat{G}\|_{\text{op}} \leq 1$ and $\text{Tr} \left[ \hat{G}^T A \right] = \|A\|_*$ is a subgradient since for all $A'$

$$\|A'\|_* = \sup_{\|B\|_{\text{op}} \leq 1} \text{Tr} [BA'] \geq \text{Tr} [\hat{G}A'] = \text{Tr} [\hat{G}(A' - A)] + \|A'\|_*.$$

Now let us show that all subgradients $\hat{G}$ at $A$ must satisfy these two properties. Let us first show that $\|\hat{G}\|_{\text{op}} \leq 1$. Take $u, v$ of norm 1 such that $v^T \hat{G} u = \|\hat{G}\|_{\text{op}}$ then for the matrix $A' = (\|A\|_* + 1)uv^T$ we have

$$1 = \|A'\|_* - \|A\|_*$$

$$\geq \text{Tr} \left[ \hat{G}^T (A' - A) \right]$$

$$= \|\hat{G}\|_{\text{op}} \left( \|A\|_* + 1 - \text{Tr} \left[ \frac{\hat{G}}{\|\hat{G}\|_{\text{op}}} A \right] \right)$$

$$\geq \|\hat{G}\|_{\text{op}} \left( \|A\|_* + 1 - \|A\|_* \right).$$

Knowing $\|\hat{G}\|_{\text{op}} \leq 1$, we have both

$$\text{Tr} \left[ \hat{G} A \right] \leq \sup_{\|B\|_{\text{op}} \leq 1} \text{Tr} [BA] = \|A\|_*$$

and taking $A' = 0$

$$0 \geq \|A\|_* - \text{Tr} \left[ \hat{G} A \right].$$

(2) A matrix $A$ is a global minimum of the regularized cost if and only if the zero matrix 0 is a subgradient at $A$. Equivalently, $-\frac{1}{\lambda} \nabla C(A)$ must be a subgradient of $\|\cdot\|_*$ at $A$, thus we obtain that $A$ is a global minimum if and only if:

$$\text{Tr} \left[ \nabla C(A)^T A \right] = -\lambda \|A\|_* \quad \text{and} \quad \|\nabla C(A)\|_{\text{op}} \leq \lambda.$$

□

We can now show that all critical points of the loss surface of shallow linear networks ($L = 2$) with $L_2$-regularization are either global minimas or strict saddles. Let us consider a shallow linear network for which the size of the hidden layer $n_1$ is larger than $\min\{n_0, n_2\}$.

Proposition 42 (Proposition 6 of Main Text). Let $C$ be a convex and differentiable cost on matrices and let $\lambda > 0$. The critical points of the loss $C_{2,\lambda}^{(NN)}: \theta \mapsto C(A_0) + \lambda \|\theta\|^2$ are either strict saddles or global minima. Furthermore if $\theta^*$ is a global minimum, then $A_{\theta^*}$ is a global minimum of $C_{1,2\lambda}^{(Sh)}: A \mapsto C(A) + 2\lambda \|A\|_1$.

Proof. Let $\theta^*$ be a critical point of the loss $C_{2,\lambda}^{(NN)}: \theta \mapsto C(A_0) + \lambda \|\theta\|^2$. In particular, it should be a critical point of the $L_2$ loss restricted to the manifold of parameters such that $A_\theta = A_{\theta^*}$: these
have been studied in the proof of Lemma 39, where we shown, using the same notations, that for any \( \ell = 2, \ldots, L \), \( W_\ell^T W_{\ell-1} = W_{\ell-1}^T W_\ell \). For shallow networks, the fact that \( W_2^T W_2 = W_1^T W_1 \) implies that \( W_1, W_2 \) and \( A_\theta \) have the same rank \( k \) and that \( W_1 \) and \( W_2 \) have the same singular values, equal to the squared root of the singular value of \( A_\theta \). The critical points \( \theta^* \) of \( C_{2, \lambda}^{(N, N)} \) therefore take the form \( \theta^* = (U_1 S_1^2 V^T, U_1 S_1^2 U_1^T) \) where \( U, V, S \) are given by the (compact) SVD decomposition \( A_\theta = USV^T \) (with \( S \) a \( k \times k \) diagonal matrix) and \( U_1 \) satisfies \( U_1^T U_1 = I_k \).

The gradient of \( C_{2, \lambda}^{(N, N)} \) with respect to \( W_2 \) is computed by considering a small perturbation \( dW_2 \):

\[
C(A(W_1, W_2 + dW_2)) + \lambda \left[ \text{Tr}(W_1^T W_1) + \text{Tr} \left( (W_2 + dW_2)^T (W_2 + dW_2) \right) \right]
\]

is at first order equal to

\[
C(A_\theta) + \lambda \text{Tr} \left[ \nabla C(A_\theta)^T dW_2 W_1 \right] + 2\lambda \text{Tr} \left[ W_2^T dW_2 \right]
\]

where \( \theta = (W_1, W_2) \). Hence the gradient with respect to \( W_2 \) is equal to

\[
\partial_{W_2} C_{2, \lambda}^{(N, N)}(\theta) = \nabla C(A_\theta) W_1^T + 2\lambda W_2.
\]

Using the criticality property of \( \theta^* \) for the loss \( C_{2, \lambda}^{(N, N)} \), \( \partial_{W_2} \left[ C_{2, \lambda}^{(N, N)}(\theta) \right] = 0 \), hence we get:

\[
\nabla C(A_\theta) V \left( S_1^2 \right)^{1/2} U_1^T + 2\lambda U S_1^2 U_1^T = 0.
\]

Multiplying on the right by \( U_1 S_1^2 U_1^T \) and taking the trace, using the cyclicity of the trace, we obtain:

\[
0 = \text{Tr} \left[ \nabla C(A_\theta)^T \theta \right] + 2\lambda \| A_\theta \|_F^2.
\]

We have thus proven that for all critical points \( \theta \), the matrix \( A_\theta \) satisfies the first of the two criteria of Lemma 39 to be a global minimum of \( A \mapsto C(A) + 2\lambda \| A \|_F^2 \). We now proceed to a case-by-case analysis:

1. If the network is full rank, i.e. \( k = \min\{n_0, n_2\} \), we consider the case \( k = n_2 \) so that \( V \) is a \( n_2 \times n_2 \) orthogonal matrix, we therefore have \( \nabla C(A_\theta) = 2\lambda UV^T \) which implies that \( \| \nabla C(A_\theta) \|_{op} = 2\lambda \) so that it is a global minimum. The case \( k = n_1 \) follows from the same argument but starting from \( \partial_{W_1} C_{2, \lambda}^{(N, N)}(\theta) = 0 \) (instead of \( \partial_{W_2} C_{2, \lambda}^{(N, N)}(\theta) = 0 \)).

2. If \( k < \min\{n_0, n_2\} \) and \( \| \nabla C(A_\theta) \|_{op} \leq 2\lambda \), then \( \theta \) is a global minimum and we are done.

3. If \( k < \min\{n_0, n_2\} \) and \( \| \nabla C(A_\theta) \|_{op} > 2\lambda \), we have to show that \( \theta \) is a non-degenerate critical point. Since \( \| \nabla C(A_\theta) \|_{op} > 2\lambda \), there are unit vectors \( u, v \) such that \( u^T \nabla C(A_\theta) v > 2\lambda \), and since \( k < \min\{n_0, n_2\} \leq n_1 \) there is a unit vector \( w \in \mathbb{R}^{n_1} \) s.t. \( U_1^T w = 0 \) which implies that \( w^T W_1 = 0 \) and \( W_2 w = 0 \). We can now identify a direction \( \partial \) along which the Hessian is negative: choosing the parameters \( \partial = (w^T, -u^T) \), we have

\[
\partial^2 C(A_\theta) + \lambda \| \partial \|^2 = \text{Tr} \left[ -2\nabla C(A_\theta) vu^T wu^T \right] + 2\lambda \| \partial \|^2 < 0.
\]

To finish, we need to prove that if \( \theta^* \) is a global minimum, then \( A_{\theta^*} \) is a global minimum of \( C_{1, 2\lambda}^{(Sch)} \). Since \( \theta^* \) is a global minimum of \( C_{2, \lambda}^{N, N} \), it minimize the \( L_2 \) norm among the parameters \( \theta \) such that \( A_\theta = A_{\theta^*} \). By Lemma 39, this implies that \( C_{2, \lambda}^{N, N}(\theta^*) = C_{1, 2\lambda}^{(Sch)}(A_{\theta^*}) \) and thus, using the same lemma, for any matrix \( A \),

\[
C_{1, 2\lambda}^{(Sch)}(A) = C_{1, 2\lambda}^{(Sch)}(A_{\Phi(A)}) = C_{2, \lambda}^{N, N}(\Phi(A)) \geq C_{2, \lambda}^{N, N}(\theta^*) = C_{1, 2\lambda}^{(Sch)}(A_{\theta^*}).
\]

This proves that \( A_{\theta^*} \) is a global minimum of \( C_{1, 2\lambda}^{(Sch)} \). \( \square \)
D.2 Deep Networks

In order to study local minima in the case of DLN, we use both Lemma 39 and the following lemma:

**Lemma 43.** Let \( \psi : \mathbb{R}^n \to \mathbb{R}^m \) and \( C : \mathbb{R}^m \to \mathbb{R} \) be two continuous functions. For any \( x \in \mathbb{R}^n \), if \( \psi(x) \) is a local minimum of \( C \), then \( x \) is a local minimum of \( C \circ \psi \).

**Proof.** Let \( x \in \mathbb{R}^n \) such that \( \psi(x) \) is a local minimum of \( C \); \( \psi(x) \) is a global minimum of \( C \) restricted to a small enough ball \( B(\psi(x), \delta_1) \). By continuity of \( \psi \), there exists a small ball \( B(x, \delta_2) \) such that \( \psi(B(x, \delta_2)) \subset B(\psi(x), \delta_1) \), and thus, \( x \) is a local minimum of \( C \circ \psi \) restricted to \( B(x, \delta_2) \). In particular, \( x \) is a local minimum of \( C \circ \psi \).

Using Lemma 39 and its proof, we have now the following correspondence between local minima of both losses \( C_{L,\lambda}^{(N,N)} \) and \( C_{p,\lambda L}^{(Sch)} \).

**Proposition 44** (Proposition 7 of Main Text). Let \( C \) be a convex and differentiable cost on matrices, let \( L > 2 \), \( \lambda > 0 \) and \( p = \frac{2}{L} \). If \( \theta^* \) is a local minimum of \( C_{L,\lambda}^{(N,N)} \), then \( A_{\theta^*} \) is a local minimum of \( C_{p,\lambda L}^{(Sch)} \). Conversely, given a local minimum \( A^* \) of \( C_{p,\lambda L}^{(Sch)} \) with SVD decomposition \( A^* = USVT \) and rank \( k \), for any rotation \( R \) of the width-\( w \) network parameters

\[
\theta^* = R\text{I}^{(k \to w)} \left( US^{\frac{1}{k}}, S^{\frac{1}{k}}, ..., S^{\frac{1}{k}}VT \right),
\]

describes a local minimum of \( C_{L,\lambda}^{(N,N)} \) with \( A_{\theta^*} = A^* \).

**Proof.** Note that the maps \( F : \theta \mapsto A_{\theta} \) and \( \Psi : A \mapsto \Psi(A) \) are both continuous; as an application of Lemma 39, we obtain that if \( A_{\theta^*} \) is a local minimum of \( C \) then \( \theta^* \) is a local minimum of \( C(A) \), and that if \( \psi(A) \) is a local minimum of \( C(A) \), then \( A \) is a local minimum of \( C \circ A_{\Psi(A)} \). Since \( B \mapsto A_{\Psi(B)} \) is the identity map, we get the following correspondence between local minima: \( A_{\theta^*} \) is a local minimum of \( C \) if and only if \( \theta^* \) is a local minimum of \( C(A) \).

Besides, if \( \theta^* \) is a local minimum of \( C(A) \), as already seen previously, it should be a critical point of the \( L_2 \) loss restricted to the manifold of parameters such that \( A_{\theta} = A_{\theta^*} \); these have been studied in the proof of Lemma 39, where we shown, using the same notations, that for any \( \ell = 2, ..., L \),

\[
W_{\ell}^T W_{\ell} = W_{\ell-1} W_{\ell-1}^T.
\]

This implies the expression of the minima given in the proposition.

Now we will show that at least some of these local minima are good approximations of minimal rank solutions in \( \arg \min_{A \in S_C} \text{Rank}(A) \) where \( S_C = \arg \min_{B} C(B) \) is the space of global minimizers of \( C \). The key lemma is:

**Lemma 45.** Let \( 0 \leq p < 1 \), \( \lambda > 0 \) and \( k \leq \min\{n_0, n_L\} \). If \( A \) is a local minimizer of the cost \( C_{p,\lambda}^{(Sch)} \) restricted to the set of matrices of rank less or equal to \( k \), it is a local minimizer of the non-restricted cost \( C_{p,\lambda}^{(Sch)} \) on the space of all matrices.

**Proof.** This is a simple consequence of the fact that if \( p < 1 \), \( \frac{d}{dx} |_{x=0} x^{p} = +\infty \); adding a singular value, hence increasing the rank of a matrix, can not locally be compensated by the infinitesimal gain in the (differentiable) cost \( C \).

**Proposition 46** (Proposition 8 of Main Text). For all \( p < 1 \), there is a sequence \( B_1^*, B_2^*, \ldots \) where \( B_k^* \) is a local minimum of \( C_{p,\lambda k}^{(Sch)} \) for \( \lambda_k \searrow 0 \) such that \( \lim_{k \to \infty} B_k^* \) is a minimal rank solution.

**Proof.** Let \( S_C^* = \arg \min_{A \in S_C} \text{Rank}(A) \) be the set of minimal rank solutions, and let \( k^* \) their common rank. For any integer \( i \), let \( M_i \) be the set of global minimizers of the cost \( C_{p,\lambda i}^{(Sch)} \) restricted to the set of matrices of rank less or equal to \( k^* \). From the previous lemma, all matrices \( A \in M_i \) are also local minima of the (non-restricted) cost \( C_{p,\lambda i}^{(Sch)} \). Furthermore, for any \( A \in M_i \), \( C_{p,\lambda i}^{(Sch)}(A) \leq \min_{A^* \in S_C^*} C_{p,\lambda i}^{(Sch)}(A^*) \) hence

\[
\|A\|_p \leq i(C_{\min} - C(A)) + \min_{A^* \in S_C^*} \|A^*\|_p \leq \min_{A^* \in S_C^*} \|A^*\|_p
\]

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and thus, all elements of \( \mathcal{M}_i \) have \( L_p \)-Schatten quasinorm less than \( \min_{A^* \in S^*_n} \| A^* \|_{L_p}^p \). Since \( \mathcal{M}_i \) is contained in a compact set, there exists a convergent sequence \( B^*_i \in \mathcal{M}_i \) whose limit \( B^* = \lim_{i \to \infty} B^*_i \) is of rank \( k^* \) and satisfies \( C(B^*) = C_{\min} \). \( \square \)