Minimizing under relaxed symmetry constraints:
Triple and \(N\)-junctions.

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Abstract

We consider a nonnegative potential \(W : \mathbb{R}^2 \to \mathbb{R}\) invariant under the action of the rotation group \(C_N\) of the regular polygon with \(N\) sides, \(N \geq 3\). We assume that \(W\) has \(N\) nondegenerate zeros and prove the existence of a \(N\)-junction solution to the vector Allen-Cahn equation. The proof is variational and is based on sharp lower and upper bounds for the energy and on a new pointwise estimate for vector minimizers.

1 Introduction

This note concerns entire solutions \(u : \mathbb{R}^n \to \mathbb{R}^m\) of the elliptic system

(1.1) \[ \Delta u = W_u(u), \]

where \(W : \mathbb{R}^m \to \mathbb{R}\) is a smooth nonnegative function that satisfies

(1.2) \[ 0 = W(a) < W(u), \quad a \in A, \quad u \in \mathbb{R}^m \backslash A, \]

and \(A = \{a_1, \ldots, a_N\} \subset \mathbb{R}^m\) is a set of \(N\) distinct points.

In phase transition theory, a function \(W\) that satisfies (1.2) can be regarded as a model for the bulk free energy of a system that can exist in \(N\) equally preferred phases represented by the zeros \(a_1, \ldots, a_N\) of \(W\).

We focus on minimizers that are solutions \(u : \mathbb{R}^n \to \mathbb{R}^m\) of (1.1) that satisfy

(1.3) \[ J_{\Omega}(u + v) \geq J_{\Omega}(u), \quad J_{\Omega}(u) = \int_{\Omega} \left( \frac{|\nabla u|^2}{2} + W(u) \right) dx. \]

for every bounded domain \(\Omega \subset \mathbb{R}^n\) and any \(C^1(\overline{\Omega}; \mathbb{R}^m)\) map \(v\) that coincides with \(u\) on \(\partial \Omega\).

In the scalar case \(m = 1\) there is a relationship [19] between minimal surfaces and minimizers of (1.1) and many deep interesting results [15], [9] have been obtained in the process of understanding this relationship.

In the vector case \(m \geq 2\) the situation is quite different. The lack of basic tools like the maximum principle makes the description of the set of all bounded solutions of (1.1) an almost impossible task. To our knowledge the asymmetric layered solutions constructed in [16] is probably the only known minimizer that does not assume any symmetry. On

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the other hand, beginning with the triple junction of [6] and the quadruple junction of [14] various symmetric solutions with complex geometric structure were discovered (see [5] and Ch.6 and Ch.7 in [2]).

The simplest case is where $W$ is invariant under a finite reflection group $G$

$W(gu) = W(u), \ u \in \mathbb{R}^m, \ g \in G.$

and $G$ acts both on the domain space $\mathbb{R}^n$ and on the target space $\mathbb{R}^m$. The minimization is on the set of $G$-equivariant maps

$u(gx) = gu(x), \ x \in \mathbb{R}^n, \ g \in G$

that map fundamental region $F$ for the action of $G$ on $\mathbb{R}^n$ into fundamental region $\Phi$ for the action of $G$ on $\mathbb{R}^m$

$u(F) \subset \Phi.$

For the triple junction case $G = Z_3$, the group of the symmetries of the equilateral triangle, for the quadruple junction $G$ is the group of the symmetries of a regular tetrahedron.

Restricting to $G$-equivariant maps that satisfy (1.6) means minimizing under a constraint but the key point here is that it can be shown that the very fact that $G$ is a finite reflection group implies that the constraint (1.6) is inactive and does not affect the Euler-Lagrange equation yielding a solution of (1.1) that satisfies (1.6). This is a basic fact since a main difficulty one has to face when dealing with the non convex minimization of $J_\Omega$ is the lack of a method for determining the regions of the domain where the minimal solution $u$ of (1.1) is near to one or another of the zeros of $W$. Knowing that $u$ satisfies (1.6) allows to overcome this difficulty. Indeed, if $W$ has a unique zero in each $\Phi$, (1.6) implies that, in each fundamental region $F$, the minimizer $u$ remains away from all the zeros of $W$ but one of them. This is a very important fact that, in the analysis of the structure of $u|_F$, allows to regard $W$ has having a unique zero and, in this sense, to reduce a non convex problem to a convex one and (1.6) is the starting point for deriving sharp pointwise estimates that yield precise information on the geometric structure of $u$.

The scope of this notes is to make a first step toward removing the assumption of symmetry. We relax the symmetry requirements and consider the problem of the existence of multi-junction solution in $\mathbb{R}^2$ which are equivariant with respect to the rotation group $C_N \subset Z_N$ of the regular polygon with $N$ sides. $C_N$ has order $N$ and $N = \sharp A$ coincides with the number of the minima of $W$ which is assumed to be invariant with respect to $C_N$.

Working in the context of maps equivariant with respect to $C_N$ still helps a lot with the proof of existence but differently from the full reflection group $Z_N$ of all the symmetries of the regular $N$-gon does not allow to impose (1.6) and consequently to reduce to a convex problem. Therefore a completely new technique must be devised. Here the fact that we work in $\mathbb{R}^2$ plays an important role.

Our precise assumption are the following

$H_1 \ W: \mathbb{R}^2 \to \mathbb{R}$ is invariant under $C_N$

$W(\omega u) = W(u), \ u \in \mathbb{R}^2, \ \omega = \left( \begin{array}{c} \cos \frac{2\pi}{N} \\
\sin \frac{2\pi}{N} \
\end{array} \right),$

where $\omega$ is the generator of $C_N$. 

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$H_2$  $W \geq 0$ and $A = \{a, \omega a, \ldots, a^{N-1}a\}$ for some $a \in \mathbb{R}^2 \setminus \{0\}$ and the Hessian matrix $W_{uu}(a)$ is positive definite. Moreover

$$W_u(u).u \geq 0, \text{ for } |u| \geq M, \text{ some } M > 0.$$ 

These assumptions imply the existence of a minimizer $\bar{u} : \mathbb{R} \to \mathbb{R}^2$ of the problem

$$J_\mathbb{R}(\bar{u}) = \min_{v \in \mathcal{A}_a} J_\mathbb{R}(v), \quad J_\mathbb{R}(v) = \int_\mathbb{R} \left( \frac{1}{2} |\frac{dv}{ds}|^2 + W(v) \right) ds,$$

$$\mathcal{A}_a = \{v \in H^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^2) : \lim_{s \to \pm \infty} v(s) = a_+, a_- = a, a_+ \in A \setminus \{a\} \}.$$ 

The minimizer $\bar{u}$ satisfies

$$\ddot{u} = W_u(u).$$

We can assume that $a_+ = \omega a$:

$$\lim_{s \to -\infty} \bar{u}(s) = a, \quad \lim_{s \to +\infty} \bar{u}(s) = \omega a.$$

Indeed it can be shown that $a_+ \notin \{\omega a, \omega^{-1}a\}$ contradicts the minimality of $\bar{u}$. On the other hand if $a_+ = \omega^{-1}a$ then $\omega \bar{u}(-\cdot)$ connects $a$ to $\omega a$.

We prove

**Theorem 1.1.** Assume that $H_1$ and $H_2$ hold. Then there exists a $C_N$-equivariant solution $U : \mathbb{R}^2 \to \mathbb{R}^m$ of (1.1)

$$U(\omega x) = \omega U(x), \quad x \in \mathbb{R}^2.$$ 

Moreover there are positive constant $\hat{C}, k, K$ and $\hat{r} \geq (\hat{C}N)^2 \frac{\pi^2}{\hat{r}}$ such that

$$|U(x) - a| \leq Ke^{-kd(x,\partial Q)}, \quad x \in Q,$$

where

$$Q = \{x = x(r, \theta) : r > \hat{r}, \frac{\hat{C}}{r^2} < \theta < \frac{2\pi}{N} - \frac{\hat{C}}{r^2} \},$$

($r, \theta$ are the polar coordinates of $x$).

The solution $U$ in Theorem 1.1 is a $C_N$-equivariant minimizer in the sense that satisfies (1.3) for each $C_N$-equivariant $v$. This is a consequence of the fact that $U$ is obtained as limit of a sequence of $G$-equivariant minimizers $u^R : B_R \to \mathbb{R}$ of $J_{B_R}$.

Theorem 1.1 can be generalized to the case where different rotation groups act on the domain space $\mathbb{R}^n$ and on the target space $\mathbb{R}^m$. As before we let $\omega$ be the generator of the group $C_N$, $N \geq 2$ acting on $\mathbb{R}^m$ and let $\omega^h, h = 1, \ldots$ be the generator of the group $C_{hN}$ acting on $\mathbb{R}^n$. Then the same proof of Theorem 1.1 with minor modification, yields

**Theorem 1.2.** Assume that $H_1$ and $H_2$ hold. Then there exists a classical solution $U : \mathbb{R}^n \to \mathbb{R}^m$ of (1.1) with the following properties:

(i) $U$ satisfies the equivariance relation

$$U(\omega^h x) = \omega U(x), \quad x \in \mathbb{R}^n.$$
(ii) There are positive constant $\hat{C}$, $k$, $K$ and $\hat{r} \geq \frac{(\hat{C}hN)^2}{\pi^2}$ such that

$$|U(x) - a| \leq Ke^{-kd(x, \partial Q)}, \; x \in Q,$$

where

$$Q = \{x = x(r, \theta) : r > \hat{r}, \; \frac{\hat{C}}{\sqrt{r}} < \theta < \frac{2\pi}{hN} - \frac{\hat{C}}{\sqrt{r}}\}.$$

This theorem extends the result in [1], [4] to the case of equivariance with respect to the rotation groups $C_N$ and $C_hN$ for general $N$ and $h$.

For the classical bistable potential $W : \mathbb{R} \to \mathbb{R}$, $W(u) = \frac{1}{4}(1 - u^2)^2$ we have $N = 2$ and it is well known that assumptions $H_1 - H_2$ are satisfied. Then the following theorem can be regarded as a particular case of Theorem 1.2.

**Theorem 1.3.** For each $h = 1, \ldots$ the scalar Allen-Cahn equation

$$\Delta u = u^3 - u, \; x \in \mathbb{R}^2,$$

has a classical solution $U : \mathbb{R}^2 \to \mathbb{R}$ which satisfies the equivariance relation

$$|U(x) + 1| \leq Ke^{-kd(x, \partial Q)}, \; x \in Q,$$

where

$$Q = \{x = x(r, \theta) : r > \hat{r}, \; \frac{\hat{C}}{\sqrt{r}} < \theta < \frac{\pi}{hN} - \frac{\hat{C}}{\sqrt{r}}\}.$$

From (1.14) it follows that, if $h = 2$, the solution $U$ in Theorem 1.3 is saddle shaped. Existence of saddle solution of (1.13) in $\mathbb{R}^2$ equivariant with respect to $Z_4$, the reflection group of the symmetries of the square was first established in [8] and generalized to the case of equivariance with respect to $Z_{2N}$ in [1], see also [17]. Existence and stability of saddle shaped solutions of (1.13) in $\mathbb{R}^{2n}$ was discussed in [7]. Theorem 1.3 shows that minimizing in the larger class of maps obtained by relaxing the symmetry constraint to the mere equivariance with respect to the rotation subgroup of $Z_{2N}$ still yields a saddle shaped solution.

We give some sketchy ideas on the complex minimization process that determines a minimizer $u_R$ of $J_{BR}$ and the estimates necessary to define $U$ as limit of $u_R$ for $R \to +\infty$.

We show that, for each $r > 0$ sufficiently large, there exists a minimizer $u_r$ of

$$\min_{J_r(v)} \; J_r(v) = \int_0^{2\pi r} \left(\frac{|\nu|^2}{2} + W(v)\right) ds,$$

in the class of $2\pi r$-periodic $C_N$-equivariant $v$. This and an accurate estimate on $J_r(u_r)$ allow the derivation of sharp lower and upper bounds for the energy $J_{BR}(u_R)$ of $u_R$ (see Section 3). A by product of these bounds is the estimate

$$\int_0^R \int_0^{2\pi} \left| \frac{\partial}{\partial r} u_R \right|^2 r d\theta dr \leq C,$$
with \( C > 0 \) independent of \( R \).

Next, in Section 4 we derive a lower bound for the energy \( J_r(v) \) of a \( 2\pi r \)-periodic map \( v \) which does not satisfy certain structure requirements necessary to be a minimizer of \( J_r(v) \). From this lower bound and the lower and upper bounds for \( J_{BR}(u^R) \) proved in Section 3 we obtain that the measure \( |\Sigma| \) of the set of the \( r \in (0, R) \) such that the restriction of \( u^R \) to the fiber \( C_r \) (the circumference of radius \( r \)) does not have the structure of \( u_r \) is small in the sense that

\[
|\Sigma| \leq C,
\]

for some \( C > 0 \) independent of \( R \).

For \( r \in (0, R) \setminus \Sigma \) the restriction of \( u^R \) to the fiber \( C_r \) is near \( u_r \) and therefore has a layered structure with \( N \) layers that by equivariance are equally spaced. This implies that the layer positions are approximately determined by the value of a single angle \( \theta_r \). From \( (1.15) \) and \( (1.16) \) it follows that the map \( (0, R) \setminus \Sigma \ni r \to \theta_r \) has some kind of regularity, a sort of Lipschitz property with jumps (cfr. Section 6.1). This and the results from Section 4 imply the existence of certain open sets \( S \) such that, see Corollary 6.2,

\[
x \in \mathcal{J} \setminus \tilde{\Sigma}, \quad \tilde{\Sigma} = \{ x : |x| \in \Sigma \}
\]

\[
|u^R(x) - a| \leq c0^\alpha,
\]

where \( c0^\alpha \) is a small quantity the particular expression of which comes from the characterization of \( u_r \) in Section 4 (statements similar to \( (1.17) \) apply to the images of \( \mathcal{J} \) through \( \omega \)).

The key point of the whole proof is to show that \( (1.17) \) implies

\[
|u^R(x) - a| \leq Ke^{-kd(x, \partial \mathcal{J})}, \quad x \in \mathcal{J}.
\]

This is a consequence of a pointwise estimate given by Theorem 5.2 that we prove in Section 5. This is a delicate point. Indeed we have no control on the behaviour of \( u^R(x) \) for \( x \in \mathcal{J} \cap \tilde{\Sigma} \) and a priori we can not even exclude that, for some \( x \in \tilde{\Sigma} \), \( u^R(x) \) coincides with one of the minima of \( W \) different from \( a \). Theorem 5.2 states that, provided \( |x_0| \) and \( l > 0 \) are sufficiently large, if \( u^R(x) \) is near \( a \) or at least remains at a fixed distance from \( A \setminus \{ a \} \) for \( x \in B_l(x_0) \setminus \tilde{\Sigma} \), then the minimality of \( u^R \) and the bound \( (1.16) \) imply that, regardless if \( x_0 \in \tilde{\Sigma} \) or not, \( u^R(x_0) \) must necessarily be near \( a \).

Pointwise estimates like \( (1.18) \) and the pseudo regularity of the map \( (0, R) \setminus \Sigma \ni r \to \theta_r \) allow for the elaborate construction of a set \( \mathcal{J} \), see Section 6.2, that plays the role of a diffuse interface in the sense that

\[
|u^R(x) - a| \leq C \frac{|x|}{p}, \quad x \in \partial \mathcal{J}^+,
\]

\[
|u^R(x) - \omega^{-1}a| \leq C \frac{|x|}{p}, \quad x \in \partial \mathcal{J}^-,
\]

where \( \partial \mathcal{J}^- \cup \partial \mathcal{J}^+ \simeq \partial \mathcal{J} \) and \( p > 1 \).

In Section 6.3 we associate to \( \mathcal{J} \) a curve \( \gamma^m \) of minimal length and using \( (1.19) \) show that the energy of \( u^R \) is mostly contained in \( \mathcal{J} \) and its images under \( \omega \) and proportional to the length \( |\gamma^m| \) of \( \gamma^m \). This and the upper bound for \( J_{BR}(u^R) \) implies an upper bound
for $|\gamma^n|$. We find $|\gamma^n| \leq R + C$ with $C > 0$ independent of $R$. This estimate gives strong control on the shape of $J$ that as a result is contained in some kind of neighborhood of one of rays of $B_R$. This and another application of Theorem 5.2 lead to the exponential estimate in Theorem 1.1.

The paper is organized as follows. In Section 3 we derive sharp lower and upper bounds for $J_{ BR(u^R)}$. In Section 4 we give quantitative estimate for the one dimensional energy $J_v$ of maps near 2$\pi r$-periodic $C_N$-equivariant minimizers $u_\bullet$. In Section 5 we conclude the proof of Theorem 1.1. In Section 6 we derive detailed information on the structure of the minimizer $u^R$ and conclude the proof of Theorem 1.1.

## 2 Basic lemmas

The assumption on $W$ imply

**Lemma 2.1.** There are constants $\delta_W > 0$, and $c_W, C_W > 0$ such that

$$|z - a| = \delta, \quad \delta \leq \delta_W,$$

(2.1)

$$\Rightarrow \quad \frac{1}{2}c^2_W \delta^2 \leq W(z) \leq \frac{1}{2}C^2_W \delta^2.$$ 

Moreover, given $M > 0$, by reducing the value of $\delta_W$ if necessary, we can also assume

$$\delta \in (0, \delta_W] \quad \text{and} \quad |z| \leq M, \quad \min_{j=1}^N |z - \omega^j -1 a| \geq \delta,$$

(2.2)

$$\Rightarrow \quad \frac{1}{2}c^2_W \delta^2 \leq W(z).$$

We continue with a lower bound for a one dimensional problem. Set $a_- = a$, $a_+ = \omega a$ and $\Gamma_0(a_{\pm}) = \{a_{\pm}\}$ and $\Gamma_\delta(a_{\pm}) = \partial B_\delta(a_{\pm})$ for $\delta > 0$. If $(s_-, s_+) \subset \mathbb{R}$ is a bounded or unbounded interval and $v : (s_-, s_+) \rightarrow \mathbb{R}^m$ is a map in $H^1_{loc}$ we set

$$J(v, (s_-, s_+)) = \int_{s_+}^{s_-} \left( \frac{|\dot{v}|^2}{2} + W(v) \right) ds.$$ 

**Lemma 2.2.** Let $\delta_{\pm} \in [0, \delta_W]$ and let $v : (s_-, s_+) \rightarrow \mathbb{R}^m$ a smooth map such that

$$\lim_{s \rightarrow s_{\pm}} d(v(s), \Gamma_{\delta_{\pm}}(a_{\pm})) = 0.$$ 

(2.3)

Then

$$J(v, (s_-, s_+)) \geq \frac{1}{2}C_W (\delta_+^2 + \delta_-^2),$$

(2.4)

where $\sigma = \int_{\mathbb{R}} |\dot{u}|^2$ is the energy of the heteroclinic $\bar{u}$ that connects $a_-$ to $a_+$.

**Proof.** 1. For $\delta_- = \delta_+ = 0$ (2.4) is just the statement of the minimality of $\bar{u}$. Therefore we can assume that either $\delta_-$ or $\delta_+$ or both are positive. From (2.3) and $\delta_+ > 0$, if $s_+ = +\infty$, it follows $\int_{s_-}^{s_+} W(v) ds = +\infty$ and (2.4) holds trivially. The same is true if $\delta_+ > 0$, $s_+ < +\infty$ and $\lim_{s \rightarrow s_+} v(s)$ does not exist. Indeed in this case we have $\int_{s_-}^{s_+} |\dot{v}|^2 ds = +\infty$. It follows that, if $\delta_+ > 0$, we can assume $s_+ < +\infty$ and moreover that

$$\lim_{s \rightarrow s_+} v(s) = v_+,$$

(2.5)
for some \( v_+ \in \Gamma_{\delta_+}(a_+) \). Analogous conclusion applies to the case \( \delta_- > 0 \).

2. If both \( \delta_- \) and \( \delta_+ \) are positive and \( w_+ \) is a test map that connects \( v_\pm \) to \( a_\pm \), the minimality of \( \bar{u} \) implies

\[
J(v, (s_-, s_+)) \geq \sigma - J(w_-) - J(w_+),
\]

where \( J(w_\pm) \) is the energy of \( w_\pm \). This yields (2.4) provided we show that \( w_\pm \) can be chosen so that

\[
J(w_\pm) \leq \frac{1}{2} C_W \delta_\pm^2.
\]

3. We choose

\[
w_+ = (1 - \frac{\gamma(s)}{\delta_+})a_+ + \frac{\gamma(s)}{\delta_+}v_+,
\]

it follows, using also (2.1)

\[
\frac{1}{2} \int_{s_+}^{+\infty} |v_+|^2 ds = \frac{C_W}{2} |v_+ - a_+|^2 \int_{s_+}^{+\infty} e^{-2C_W(s-s_+)} ds = \frac{1}{4} C_W \delta_+^2,
\]

\[
\int_{s_+}^{+\infty} W(w_+) ds \leq \frac{C_W}{2} |v_+ - a_+|^2 \int_{s_+}^{+\infty} e^{-2C_W(s-s_+)} ds = \frac{1}{4} C_W \delta_+^2.
\]

This and the analogous computation for \( J(w_-) \) establish (2.4) for \( \delta_- \) and \( \delta_+ \) positive.

Clearly (2.4) is valid also if \( \delta_- \) or \( \delta_+ \) vanishes. The proof is complete.

\[ \square \]

**Lemma 2.3.** Consider a smooth family of lines that are transversal to two distinct \( n-1 \) dimensional surfaces \( S \) and \( S' \). Consider a point \( p \in S \) and let \( e \) be a unit vector parallel to the line of the family through \( p \). Let \( \nu \) a unit vector normal to \( S \) at \( p \). Let \( dS \) a small neighborhood of \( p \) in \( S \) and let \( dS' \) the set of the intersections of the lines through \( dS \) with \( S' \). Finally let \( d\Omega \) the union of all segments determined on the lines of the family by \( dS \) and \( dS' \). Let \( v : O \subset \mathbb{R}^n \rightarrow \mathbb{R}^n, O \supset d\Omega \) open, a smooth map that satisfies

\[
(2.6)
\]

\[
|v - a_-| \leq \delta, \; x \in S,
\]

\[
|v - a_+| \leq \delta, \; x \in S'.
\]

Then

\[
J_{d\Omega}(v) \geq (\sigma - C_W \delta^2) \min\{\nu \cdot edS, \nu' \cdot edS'\}.
\]

This inequality still holds if (2.6) is replaced by the condition that each segment in \( \Omega \) contains a point where \( |v - a_-| \leq \delta \) and a point where \( |v - a_+| \leq \delta \).

\[ \square \]

**Proof.** Follows from Fig. 1 and Lemma 2.2

\[ \square \]

3 Lower and upper bounds for \( J(u^R) \).

We construct a lower bound for the energy of \( u^R \) by minimizing in each fiber of radius \( r \in (\bar{r}, R) \), for some fixed \( \bar{r} > 0 \), the energy

\[
(3.1) \quad J_r(v) = \int_{(0,2\pi r)} \left( \frac{1}{2} |v|^2 + W(v) \right) ds
\]

in the class \( \mathcal{V}_r \subset H^1 \) of the map \( v \) which are \( 2\pi r \)-periodic and \( C_N \)-equivariant:

\[
(3.2) \quad u_r(s + \frac{2}{N} \pi r) = \omega u_r(s), \; s \in \mathbb{R}.
\]
Proposition 3.1. There exists \( \bar{r} > 0 \) such that \( r \geq \bar{r} \) implies the existence of a minimizer \( u_r \in \mathcal{Y}_r \) of (3.1).

Proof. 1. Given \( z_+ \in \partial B_\delta(a) \), \( z_+ \neq z_- \), set

\[
\tau = \frac{|z_+ - z_-|}{C_W \delta} \leq \frac{2}{C_W},
\]

and define \( \tilde{u}(t) = z_- + \frac{t}{\tau}(z_+ - z_-) \), \( t \in [0, \tau] \). We have

\[
J_{(0, \tau)}(\tilde{u}) = \int_0^\tau \left( \frac{1}{2} \frac{d\tilde{u}}{dt}^2 + W(\tilde{u}) \right) dt \leq \frac{1}{2} \int_0^\tau \left( \frac{1}{\tau^2} |z_+ - z_-|^2 + C_W^2 \delta^2 \right) dt
\]

\[
= \frac{1}{2} \left( \frac{|z_+ - z_-|^2}{\tau} + C_W^2 \delta^2 \tau \right) = C_W \delta |z_+ - z_-| \leq 2C_W \delta^2,
\]

where we have also used Lemma 2.1.

2. The minimality of \( \bar{u} \) implies that, for small \( \delta > 0 \), \( \bar{u}(\mathbb{R}) \cap \partial B_\delta(\omega_j^{-1}a) \), \( j = 1, 2 \), is a singleton (see e.g. Lemmas 2.4 and 2.5 in [2]). It follows that we can choose \( z_\pm \) and determine \( t_\delta, t^\delta \) by setting:

\[
z_+ = \bar{u}(t_\delta) \in \partial B_\delta(a),
\]

\[
z_- = \bar{u}(t^\delta) \in \partial B_\delta(\omega a).
\]

For each \( \delta \in (0, \delta_W] \) let \( u^\delta \) be the map defined by

\[
u^\delta(t) = \begin{cases} 
\tilde{u}(t), & t \in [0, \tau) \\
\bar{u}(t_\delta + t - \tau), & t \in [\tau, t^\delta - t_\delta + \tau].
\end{cases}
\]

Note that the definition of \( z_- \) implies \( \omega\nu^\delta(0) = \omega\bar{u}(0) = \omega z_- = \bar{u}(t^\delta) = u^\delta(t^\delta - t_\delta + \tau) \). It follows that \( u^\delta \) can be extended to a \( C_N \)-equivariant periodic map of period \( N(t^\delta - t_\delta + \tau) \). We have, using also (3.4)

\[
J(u^\delta) = J_{(t_\delta, t^\delta)}(\bar{u}) + J_{(0, \tau)}(\bar{u}) \leq \sigma + 2C_W \delta^2,
\]

3. For \( \delta \in (0, \delta_W] \) we can write \( \bar{u} = a + \delta n \) with \( \delta = |\bar{u} - a| \) and \( n = \frac{\bar{u} - a}{|\bar{u} - a|} \). This and \( \frac{1}{2} \left\| \frac{d\bar{u}}{dt} \right\|^2 = W(\bar{u}) \) imply

\[
\frac{d\delta}{dt} \leq \left\| \frac{d\bar{u}}{dt} \right\| = \sqrt{2W(\bar{u})} \leq C_W \delta,
\]

\[
\Rightarrow \quad t_\delta \leq t_\delta^W - \frac{1}{C_W} \ln \frac{\delta_W}{\delta}.
\]
A similar computation yields $t^\delta \geq t^{\delta_W} + \frac{1}{C_W} \ln \frac{\delta_W}{\delta}$. Hence

$$t^\delta - t_\delta + \tau \geq t^{\delta_W} - t_\delta + \frac{2}{C_W} \ln \frac{\delta_W}{\delta}.$$  

From this and (3.3) it follows that, given $r \geq \bar{r} = \frac{N}{\pi} (t^{\delta_W} - t_\delta + \frac{2}{C_W})$, there is $\delta \in (0, \delta_W]$ such that

$$r = \frac{N}{\pi} (t^\delta - t_\delta + \tau).$$

4. From (3.8) and (3.7) it follows that the set of $2\pi r$-periodic $C_N$-equivariant map with bounded energy is nonempty and therefore that the existence of a minimizer $u_r$ follows by classical arguments of variational calculus. The minimizer $u_r$ is a solution of (1.9). The proof is complete. \(\square\)

**Lemma 3.2.** Given $\delta \in (0, \delta_W]$. Assume $r \geq r_\delta = \frac{4N\sigma}{\pi c_W \delta^2}$ and let $u_r$ be a minimizer of (3.1). Then

$$u_r \in \mathcal{Y}_r^\star,$$

where $\mathcal{Y}_r^\star \subset \mathcal{Y}_r$ is the set of maps that, after a suitable translation of the independent variable, satisfy

$$v(s) \in B_\delta(a), \ s \in (0, \frac{2\pi r}{N} - s_v, \delta), \ \text{some} \ s_v, \delta \in (0, \frac{4\sigma}{c_W \delta^2})$$

$$v(\frac{2\pi r}{N}) \in B_\delta(a'), \ \text{some} \ a' \in \{\omega a, \omega^{-1} a\},$$

$$v(s) \notin \cup_{\tilde{a} \in A} B_\delta(\tilde{a}), \ s \in [\frac{2\pi r}{N} - s_v, \delta, \frac{2\pi r}{N}].$$

**Proof.** 1. Define $\Lambda_{r,\delta} = \{s \in [0, 2\pi r) : \min_{j} \left| u_r(s) - \omega^{j-1} a \right| \geq \delta \}$. From (3.7) and (2.2) it follows

$$\frac{1}{2} c_W^{2} \delta^{2} |\Lambda_{r,\delta}| \leq \int_{0}^{2\pi r} W(u_r) ds < J_r(u_r) < 2N\sigma,$$

$$\Rightarrow |\Lambda_{r,\delta}| \leq \frac{2N(\sigma + 2C_W \delta^2)}{c_W^2 \delta^2} < \frac{4N\sigma}{c_W \delta^2} = \pi r_\delta.$$  

Hence $r \geq r_\delta$ implies the existence of $s$ such that $u_r(s) \in B_\delta(\omega^{j-1} a)$ for some $j \in \{1, \ldots, N\}$. By equivariance and modulus a translation of the independent variable we can assume that $j = 1$ and $s = 0$. Then by equivariance we have $u_r(\frac{2\pi}{N} r) \in B_\delta(a')$ with $a' \in \{\omega a, \omega^{-1} a\}$.

2. It results

$$u_r([0, \frac{2\pi}{N} r]) \cap B_\delta(\tilde{a}) = \emptyset, \ \tilde{a} \notin \{a, a'\}.$$  

Assume instead that $u_r(s) \in B_\delta(\tilde{a})$ for some $\tilde{a} \notin \{a, a'\}$ and for some $s \in (0, \frac{2\pi}{N} r)$. If this is the case, Lemma 2.2 implies

$$J_{(0,s)}(u_r) \geq \sigma - C_W \delta^2, \quad J_{(s, \frac{2\pi}{N} r)}(u_r) \geq \sigma - C_W \delta^2,$$

hence $J_{(0, \frac{2\pi}{N} r)}(u_r) \geq 2(\sigma - C_W \delta^2)$ in contradiction with (3.7).

3. Finally we observe that the arguments in Lemma 2.4 and Lemma 2.5 in [2] imply that the minimizer $u_r$, once has entered the ball $B_\delta(a)$ can not reenter in it before entering the ball $B_\delta(a')$. This and equivariance imply that $\Lambda_{r,\delta}$ is the union of $N$ equal intervals of size $s_\delta < \frac{\sigma}{c_W \delta^2}$. This concludes the proof. \(\square\)
From Lemmas 2.2 and 3.2 it follows that the energy of \( u_r \) has a lower bound

\[
J_r(u_r) \geq N(\sigma - C_W \delta^2),
\]

where we have also used that \( u_r \) is \( C_N \)-equivariant. From Lemma 3.2 and a classical comparison argument (3.9) can be upgraded to a sharp lower bound that depends on \( r \). This is the content of the following lemma.

**Lemma 3.3.** There exist \( r_{\delta} > 0, \bar{k} > 0 \) and \( \bar{K} > 0 \) such that, for \( r \geq r_{\delta} \), it results

\[
J_r(u_r) \geq N\sigma - \bar{K}e^{-kr},
\]

*Proof.* 1. We can assume that the constants \( \delta_W > 0 \) and \( c_W > 0 \) in Lemma 2.1 are such that

\[
|z - a| \leq \delta \in (0, \delta_W) \quad \Rightarrow \quad W_u(z) \cdot (z - a) \geq c_W^2 |z - a|^2.
\]

This follows from \( W_u(z) = W_{uu}(a)(z - a) + o(|z - a|) \) and from the assumption that \( W_{uu}(a) \) is positive definite.

2. Set \( \varrho = |u_r - a| \). Since \( u_r \) is a solution of (1.9) Step 1. implies

\[
\frac{d^2}{ds^2} \varrho^2 = \left| \frac{d}{ds} \varrho \right|^2 + \frac{d^2}{ds^2} u_r \cdot (u_r - a) \geq W_u(u_r) \cdot (u_r - a) \geq c_W^2 \varrho^2.
\]

3. From Lemma 3.2 with \( \delta = \delta_W \), for \( r \geq r_{\delta_W} = \frac{2N\sigma}{\sigma c_W \delta_W^2} \), we have \( \varrho(0) \leq \delta_W^2 \) and

\[
\varrho(\frac{2\pi}{N}r - s_{\delta_W}) \leq \delta_W^2.
\]

This and (3.11) imply

\[
\varrho(s) \leq \delta_W^2 \frac{\cosh c_W(\frac{\pi}{N}r - \frac{s_{\delta_W}}{2})}{\cosh c_W(\frac{\pi}{N}r - \frac{s_{\delta_W}}{2})},
\]

where the right hand side is the solution of \( \frac{d^2}{ds^2} v = c_W^2 v \) that satisfies \( v = \delta_W^2 \) at the extreme of the interval \((0, \frac{2\pi}{N}r - s_{\delta_W})\).

4. For \( s = \frac{\pi}{N}r - \frac{s_{\delta_W}}{2} \), (3.12) yields

\[
\varrho(\frac{\pi}{N}r - \frac{s_{\delta_W}}{2}) \leq \frac{\delta_W^2}{\cosh c_W(\frac{\pi}{N}r - \frac{s_{\delta_W}}{2})} \leq 2\delta_W^2 e^{c_W\frac{s_{\delta_W}}{2}} e^{-cw} \frac{\pi}{N}r.
\]

This and Lemma 2.2 imply

\[
J_r(u_r^*) \geq N\sigma - NC_W 2\delta_W^2 e^{c_W\frac{s_{\delta_W}}{2}} e^{-cw} \frac{\pi}{N}r,
\]

that concludes the proof with \( \bar{k} = c_W \frac{\pi}{N} \) and \( \bar{K} = NC_W 2\delta_W^2 e^{c_W\frac{s_{\delta_W}}{2}} \).

From Lemma 3.3 we immediately get

\[
J_B_R(u_R) \geq \int_0^R J_r(u_R(r, \frac{\pi}{R}))dr \geq \int_{r_{\delta}}^R J_r(u_r)dr
\]

\[
\geq N\sigma R - \bar{K} \int_{r_{\delta}}^R e^{-kr} dr
\]

\[
\geq N\sigma R - \frac{\bar{K}}{k} (e^{-kr_{\delta}} - e^{-kR}) \geq N\sigma R - C_0,
\]

(3.13)
where \( C_0 > 0 \) is a constant independent of \( R \). We denote by \( C, C_0, C_1, \ldots \) generic positive constants that do not depend on \( R \).

To derive an upper bound we choose a suitable \( C_N \)-equivariant test function \( u_{\text{test}} : B_R \to \mathbb{R}^m \) and obtain \( J_{B_R}(u_R) \leq J_{B_{\rho}}(u_{\text{test}}) \).

Set \( \theta_N = \frac{2\pi}{N} \). We first define a map \( \tilde{u}_{\text{test}} \) in the whole of \( \mathbb{R}^2 \) and then we identify \( u_{\text{test}} \) with the restriction of \( \tilde{u}_{\text{test}} \) to \( B_R \). For \( \rho \in (0, +\infty) \) \( \alpha < \beta \) we denote by \( S_\rho(\alpha, \beta) \) the sector defined by
\[
S_\rho(\alpha, \beta) = \{ z(r, \theta) : r \in (0, \rho), \theta \in [\alpha, \beta] \}.
\]
We begin to define \( \tilde{u}_{\text{test}} \) in the sector \( S_\infty(-\frac{1}{4}\theta_N, \frac{1}{4}\theta_N) \). We note that \( \omega^{-1}\tilde{u} \) connects \( \omega^{-1}a \) to \( a \) and set
\[
(3.14) \quad \tilde{u}_{\text{test}}(y) = \omega^{-1}\tilde{u}(y), \quad |y| \leq \tan \frac{\theta_N}{4}, \quad x \geq 0.
\]
Then we extend \( \tilde{u}_{\text{test}} \) to the sector \( S_\infty(\frac{1}{4}\theta_N - \frac{1}{4}\theta_N, \frac{1}{4}\theta_N + \frac{1}{4}\theta_N) \), \( j = 1, \ldots, N - 1 \) by equivariance. We now observe that from the above construction \( \tilde{u}_{\text{test}} \) is already defined on the boundary of the sector \( S_\infty(\frac{1}{4}\theta_N, \frac{3}{4}\theta_N) \) and we have
\[
(3.15) \quad \tilde{u}_{\text{test}}(z(r, \frac{1}{4}\theta_N)) = \omega^{-1}\tilde{u}(\sin \frac{1}{4}\theta_N r),
\]
\[
\tilde{u}_{\text{test}}(z(r, \frac{3}{4}\theta_N)) = \tilde{u}(-\sin \frac{1}{4}\theta_N r).
\]
Based on this we define \( \tilde{u}_{\text{test}} \) in the sector \( S_\infty(\frac{1}{4}\theta_N, \frac{3}{4}\theta_N) \) by setting
\[
\tilde{u}_{\text{test}} = \omega^{-1}\tilde{u}(\sin \frac{\theta_N}{4} r)(\frac{3}{2} - 2\theta_N r + \tilde{u}(-\sin \frac{\theta_N}{4} r)(2\theta_N - 1),
\]
and extend it by equivariance to the sector \( S_\infty(\frac{1}{4}\theta_N + \frac{1}{4}\theta_N, \frac{1}{4}\theta_N + \frac{3}{4}\theta_N) \), \( j = 1, \ldots, N - 1 \). This complete the definition of \( \tilde{u}_{\text{test}} \) as a continuous \( C_N \)-equivariant \( H^1_{\text{loc}} \) map. Since both the functions on the right hand side of \( (3.15) \) converge exponentially to 0 as \( r \to +\infty \) with first derivatives that converge exponentially to 0 we have
\[
J_{S_\infty(\frac{1}{4}\theta_N, \frac{3}{4}\theta_N)}(\tilde{u}_{\text{test}}) \leq C, \quad \text{some } C > 0.
\]
On the other hand from \( (3.14) \) we have
\[
J_{S_R(-\frac{1}{4}\theta_N, \frac{1}{4}\theta_N)}(u_{\text{test}}) < \int_0^R \int_{-\infty}^{+\infty} \left( \frac{1}{2} \left| \tilde{u}'' \right|^2 + W(\tilde{u}) \right) dydx = \sigma R.
\]
and we can conclude
\[
(3.16) \quad J_{B_R}(u_R) \leq J_{B_{\rho}}(u_{\text{test}}) \leq N\sigma R + C_1,
\]
for some constant \( C_1 > 0 \).

We remark that this and \( (3.13) \) imply
\[
(3.17) \quad \int_0^R \int_0^{2\pi} \left| \frac{\partial}{\partial r} u_R \right|^2 r d\theta dr \leq 2(C_0 + C_1).
\]
4 Quantitative estimate of the energy of maps near $u_r$.

Next we derive a lower bounds for the energy of maps that do not have the structure of minimizers described in Lemma 3.2. As before let $\mathcal{V}_r \subset H^1$ the class of $2\pi r$-periodic $C_N$-equivariant maps. In the following, without explicit mention, we characterize a map $v \in \mathcal{V}_r$ by the properties of a suitable translation of it.

**Proposition 4.1.** There is a constant $c > 0$ such that, given $\alpha \in (0,1)$ and a number $\delta \in (0, \delta_W]$ sufficiently small, there are $r_\delta > 0$, and $s_\nu, \delta \in (0, \frac{4\sigma}{cW\delta})$ such that $r \geq r_\delta$ implies

$$v \in \mathcal{V}_r \setminus \mathcal{V}_r^* \Rightarrow J_r(v) \geq N\sigma + \delta^{1+\alpha},$$

where $\mathcal{V}_r^*$ is the set of maps that after a suitable translation satisfy:

\begin{equation}
(4.1)
\quad v(s) \in B_{s_\nu}(a), \; s \in (0, \frac{2\pi r}{N}, s_\nu, \delta)
\end{equation}

**Proof.** We divide $\mathcal{V}_r \setminus \mathcal{V}_r^*$ in four parts $\mathcal{V}_r^i$, $i = 1, 2, 3, 4$ which are defined in sequence and satisfy

$$\mathcal{V}_r = \mathcal{V}_r^{0,c}; \quad \mathcal{V}_r^{4,c} = \mathcal{V}_r^*,$$

$$\mathcal{V}_r^{i,c} = \mathcal{V}_r^{i+1} \cup \mathcal{V}_r^{i+1,c}, \; n = 0, 1, 2, 3,$$

$$\mathcal{V}_r^{i+1,c} = \mathcal{V}_r^{i,c} \setminus \mathcal{V}_r^{i+1}.$$

1. We define

$$\mathcal{V}_r^1 = \{v \in \mathcal{V}_r : v(s) \notin \bigcup_{j=1}^N B_\delta(\omega^{j-1} a), \; s \in [0, 2\pi r]\}.$$

Therefore $v \in \mathcal{V}_r^{1,c}$ if and only if there are $s \in [0, 2\pi r)$ and $1 \leq j \leq N$ such that $v(s) \in B_\delta(\omega^{j-1} a)$. This and equivariance imply

$$\mathcal{V}_r^{1,c} = \{v \in \mathcal{V}_r : v(0) \in \overline{B}_\delta(a)\},$$

where, as remarked before, we actually mean the set of maps that, after a suitable translation, satisfy $v(0) \in B_\delta(a)$.

For $v \in \mathcal{V}_r^1$ from (2.1) we have

\begin{equation}
(4.2)
\quad J_r(v) \geq \int_0^{2\pi r} W(v) ds \geq \pi c_W^2 \delta^2 r, \; v \in \mathcal{V}_r^1.
\end{equation}

2. We define

$$\mathcal{V}_r^2 = \{v : v(0) \in \overline{B}_\delta(a), \; v(s) \in \overline{B}_\delta(\omega^{j-1} a), \text{ for some } \overline{s} \in (0, \frac{2\pi r}{N}), \text{ and some } 3 \leq j \leq N\}.$$

If $v \in \mathcal{V}_r^2$ equivariance and Lemma [2.2] imply

\begin{equation}
(4.3)
\quad J_r(v) > 2N(\sigma - C_W \delta^2).
\end{equation}

Note that

\begin{equation}
(4.4)
\quad \mathcal{V}_r^{2,c} = \{v \in \mathcal{V}_r : v(0) \in \overline{B}_\delta(a), \; v(\frac{2\pi r}{N}) = \omega v(0) \in \overline{B}_\delta(\omega a) \text{ and } v(s) \notin \overline{B}_\delta(\omega^{j-1} a), \; 3 \leq j \leq N, s \in (0, \frac{2\pi r}{N})\}
\end{equation}
3. Let $\delta' \in (\delta, \frac{1}{2}|\omega a - a|)$ a number to be chosen later. Observe that for each $v \in \mathcal{V}_r^{2,c}$, for the same translation of $v$ considered in (4.4), we can define

\begin{equation}
\begin{aligned}
&\bar{s}_- = \inf \{ s \in (0, \frac{2\pi r}{N}) : |v(s) - a| = \delta' \}, \\
&\bar{s}_+ = \sup \{ s \in (0, \frac{2\pi r}{N}) : |v(s) - \omega a| = \delta' \}. 
\end{aligned}
\end{equation}

We set

$\mathcal{V}_r^{3,c} = \{ v \in \mathcal{V}_r^{2,c} : v(\bar{s}) \in \overline{B}_\delta(a) \cup \overline{B}_\delta(\omega a), \text{ for some } \bar{s} \in (s_-, s_+) \}$

To derive a lower bound for the energy of $v \in \mathcal{V}_r^{3,c}$ we suppose (the case $v(\bar{s}) \in \overline{B}_\delta(\omega a)$ is similar) that

\begin{equation}
v(\bar{s}) \in \overline{B}_\delta(a)
\end{equation}

and define

$\bar{s} = \inf \{ s < \bar{s}_- : v(t) \notin \overline{B}_\delta(a), t \in [s, \bar{s}_-) \}.$

A standard computation and (2.1) shows that

\begin{equation}
\begin{aligned}
\int_{\bar{s}}^{\bar{s}_-} (\frac{|v'|^2}{2} + W(v))ds &\geq \frac{|v(\bar{s}_-) - v(\bar{s})|^2}{2(s_- - \bar{s})} + \frac{1}{2}c_W^2 \delta^2 (s_- - \bar{s}) \\
&\geq \frac{(\delta' - \delta)^2}{2(s_- - \bar{s})} + \frac{1}{2}c_W^2 \delta^2 (s_- - \bar{s}) \geq c_W \delta (\delta' - \delta).
\end{aligned}
\end{equation}

This (4.6), $v(\frac{2\pi r}{N}) \in \overline{B}_\delta(\omega a)$ and Lemma 2.2 that imply

$J_r(v, (\bar{s}, \frac{2\pi r}{N})) \geq \sigma - C_W \delta^2,$

yield, provided we choose $\delta' = c\delta^a$ with $c = 1 + \frac{1}{c_W^2 \delta^2} + \frac{C_W}{c_W},$

\begin{equation}
J_r(v) \geq N(\sigma - C_W \delta^2 + c_W \delta (\delta' - \delta)) \geq N_\sigma + \delta^{1+a},
\end{equation}

4. Observe that

$\mathcal{V}_r^{3,c} = \{ v \in \mathcal{V}_r^{2,c} : v(s) \in B_{\delta^s}(a), s \in [0, s_-), v(s) \in B_{\delta^s}(\omega a), s \in [s_-, \frac{2\pi r}{N}),

v(s) \notin \overline{B}_\delta(\omega^{j-1} a), s \in [s_-^+, s_+] \}, 1 \leq j \leq N \},$

and define

$\mathcal{V}_r^4 = \{ v \in \mathcal{V}_r^{3,c} : s_+^+ - s_-^+ \geq \frac{4\sigma}{c_W^2 \delta^2} \}.$

Then (2.2) implies

\begin{equation}
J_r(v) \geq \frac{N}{2} c_W^2 (s_+^+ - s_-^+) \delta^2 = 2N \sigma.
\end{equation}

5. If $r \geq r_\delta = \frac{N\sigma + \delta^{1+a}}{\frac{\pi c_W^2}{\sigma^a}}$ and $v \in \mathcal{V}_r^1,$ (4.2) yields

\begin{equation}
J_r(v) \geq N \sigma + \delta^{1+a}.
\end{equation}
Since $\delta$ is a small number \([4.3]\) and \([4.8]\) imply that \([4.9]\) is satisfied also for $v \in \mathcal{Y}_r^{2} \cup \mathcal{Y}_r^{4}$. From this and \([4.7]\) we conclude that \([4.9]\) holds for $v \in \mathcal{Y}_r^{A,c}$. To complete the proof it remain to show that $\mathcal{Y}_r^{A,c}$ is a subset of $\mathcal{Y}_r^{*}$. By inspecting the expressions of $\mathcal{Y}_r^{j,c}$, $j = 1, 2, 3$ and the definition of $\mathcal{Y}_r^{4}$ we find that, after a suitable translation, $v \in \mathcal{Y}_r^{A,c} \subset \mathcal{Y}_r$ satisfies

$$v(s) \in B_{\varepsilon^{\alpha}}(a), \ s \in \left[0, s_v^{-}\right], \ v(s) \in B_{\varepsilon^{\alpha}}(\omega a), \ s \in \left(s_v^{+}, \frac{2\pi r}{N}\right],$$

\[(4.10)\]

$$s_v^{+} - s_v^{-} \leq \frac{4\sigma}{c_{\Omega}^{\alpha} \varepsilon^{2}}.$$  

By equivariance we have

$$v(s) \in B_{\varepsilon^{\alpha}}(\omega a), \ s \in \left(s_v^{+}, \frac{2\pi r}{N}\right] \iff v(s) \in B_{\varepsilon^{\alpha}}(a), \ s \in \left(s_v^{+} - \frac{2\pi r}{N}, 0\right]$$

That together with \([4.10]\) imply

$$v(s) \in B_{\varepsilon^{\alpha}}(a), \ s \in \left(s_v^{+} - \frac{2\pi r}{N}, s_v^{-}\right).$$

The translation $s \to s + s_0$, $s_0 = \frac{2\pi r}{N} - s_v^{+}$ transforms this equation into \([4.11]\) with $s_{v, \delta} = s_v^{+} - s_v^{-}$. The proof is complete. \(\square\)

Next we show that the measure of the set of the fibers where the profile of a minimizer $u^R$ is not in $\mathcal{Y}_r^{*}$ is bounded independently of $R$

**Lemma 4.2.** Let $\delta \in (0, \delta_W]$, $\alpha \in (0, 1)$, $r_\delta$ and $\mathcal{Y}_r^{*}$ as before. Set $\Sigma = \{r \in [r_\delta, R) : u^R(x(r, \theta)) \not\in \mathcal{Y}_r^{*} \}$. Then

$$|\Sigma| \leq \frac{C_2}{\delta^{1+\alpha}},$$

where $C_2 = 2C_0 + C_1 + N\sigma r_0$.

**Proof.** From \([3.16]\) we have

\[(4.11)\]

$$\int_{(r_0, R)} J_r(u^R(x(r, \hat{\theta})))dr = \int_{(r_0, r_\delta)} J_r(u^R(x(r, \hat{\theta})))dr$$

$$+ \int_{(r_\delta, R)} J_r(u^R(x(r, \hat{\theta})))dr + \int_{\Sigma} J_r(u^R(x(r, \hat{\theta})))dr \leq N\sigma R + C_1.$$  

From \([3.13]\) and Proposition \([4.1]\) we have

$$\int_{(r_0, r_\delta)} J_r(u^R(x(r, \hat{\theta})))dr \leq N\sigma (r_\delta - r_0) - C_0,$$

$$\int_{(r_\delta, R)} \int_{\Sigma} J_r(u^R(x(r, \hat{\theta})))dr \geq N\sigma (R - r_\delta - |\Sigma|) - C_0,$$

$$\int_{\Sigma} J_r(u^R(x(r, \hat{\theta})))dr \geq (N\sigma + \delta^{1+\alpha}) |\Sigma|.$$  

This and \([4.11]\) conclude the proof. \(\square\)

We set

$$\bar{\Sigma} = \{x \in \mathbb{R}^2 : r(x) \in \Sigma\},$$

where we have used the notation $(r(x), \theta(x))$ for the radial coordinates of $x$. 

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5 A pointwise estimate for \( u^R \)

Let \( u^R : B_R \to \mathbb{R}^2 \) be a minimizer of \( (1.3) \). Then the smoothness of \( W \) and elliptic theory imply that we can assume

\[
\| u^R \|_{L^\infty} + \| \nabla u^R \|_{L^\infty} \leq M, \tag{5.1}
\]

for some \( M > 0 \) independent of \( R \).

Observe that the properties of \( W \) imply Lemma 5.1.

Given \( \delta^* \in (0, \delta_W] \) there is \( \delta_0 > 0 \) such that, for each \( n \in S^1 \), \( \delta \in (0, \delta_0] \) and \( d > \delta \) that satisfy

\[
a + dn \in B_M \setminus \bigcup_{j=2}^N B_{\delta^*}(\omega^{j-1}a),
\]

we have

\[
W(a + \delta n) < W(a + dn). \tag{5.2}
\]

Next we present a result which is essential for our analysis and may be interesting in itself.

If \( v : B_R \to \mathbb{R}^2 \) is a \( H^1 \) map we use the polar form of \( v \):

\[
v = a + q^v n^v, \quad \text{on} \quad \{ v \neq a \}, \tag{5.3}
\]

and the identity

\[
|\nabla v|^2 = |\nabla q^v|^2 + (q^v)^2|\nabla n^v|^2. \tag{5.4}
\]

We now choose the number \( \delta \in (0, \delta_W] \) introduced in Lemma 3.2 in such a way that the inequality \( (5.2) \) can be applied. We fix a number \( \delta^* \in (0, \delta_W] \) and assume \( \delta \in (0, \delta_0], \delta_0 \) as in Lemma 5.1.

**Theorem 5.2.** Assume that a minimizer \( u^R : B_R \to \mathbb{R}^2 \) of \( (1.3) \), a ball \( B_l(x_0) \subset B_R \) and a set \( \Sigma \subset (0, R) \) satisfy, for some constants \( c > 0 \) and \( C > 0 \),

\[
|u^R(x) - a| \leq c\delta^\alpha, \quad x \in B_l(x_0) \setminus \tilde{\Sigma}, \quad \text{for some} \quad \alpha \in (0, 1) \tag{5.5}
\]

where \( \tilde{\Sigma} = \{ x \in B_R : |x| \in \Sigma \} \), and

\[
|\Sigma| \leq \frac{C}{\delta^{1+\alpha'}}, \quad \text{for some} \quad \alpha' \in (0, 1). \tag{5.6}
\]

Assume that

\[
2\alpha + \alpha' < 1.
\]

Then there is a constant \( D > 0 \) such that for each \( \delta \leq D^{\frac{1}{1-(2\alpha+\alpha')}} \) there exist \( l_\delta > 0 \) and \( r_\delta > 0 \) which are independent of \( R \) and such that

\[
l \geq l_\delta, \tag{5.7}
\]

\[
r(x_0) \geq r_\delta, \tag{5.8}
\]

imply

\[
|u^R(x_0) - a| \leq 2\delta.
\]
Before presenting the proof some comments are in order. The point of the Theorem is that, if (5.5) holds for sufficiently large \( r(x_0) \) and \( l \), then, in spite of the fact that nothing is a priori known on the behavior of \( u^R \) on \( \Sigma \), the inequality (5.8) is satisfied also when \( x_0 \in \Sigma \), that is when the center of the ball is in \( \Sigma \). Now a comment on the proof. Suppose we knew that

\[
(5.9) \quad \min_{j=2}^N |u^R(x) - \omega^j a| \geq \delta^*, \quad x \in \Sigma \cap B_l(x_0).
\]

Then we could directly invoke Theorem 5.3 in [2] or Theorem 1.2 in [10] and conclude (5.8) provided \( l > 0 \) is sufficiently large. But we can not assume (5.9) since we can not exclude that, at some point \( \bar{x} \in \Sigma \cap B_l(x_0) \) it results \( u^R(\bar{x}) = \omega^j a \) for some \( j > 1 \). In spite of this we will see that the bound on the size of \( \Sigma \) in (5.6) together with (5.5) allow to overcome this difficulty and to extend the proof of the quoted theorems in order to cover the case at hand.

**Proof.** In this proof, to simplify the notation, we simply write \( B_l, B_{l,+}\eta \) instead of \( B_l(x_0), B_{l,+\eta}(x_0) \).

By inspecting the proof of Theorem 5.3 given in Section 5.5 of [2] we see that the following result, analogous of Lemma 5.3 in [2], can be established by exactly the same arguments used in the proof of that lemma.

**Lemma 5.3.** Let \( \bar{u} : B_R \to \mathbb{R}^2 \) a \( C^{0,1} \), \( \mathbb{C}_N \)-equivariant map (not necessarily a minimizer) that satisfies (5.1) .

Given \( l_0 > 0, \eta > 0 \), assume that, for some \( l \geq l_0 + \eta \),

\[
|\bar{u} - a| \leq \delta, \quad \text{on} \quad B_l(x_0),
\]

where \( B_l(x_0) \subset B_R \) satisfies \( B_l(x_0) \cap B_l(\omega x_0) = \emptyset \). Then there exists a \( C^{0,1} \), \( \mathbb{C}_N \)-equivariant map \( v : B_R \to \mathbb{R}^2 \) that coincides with \( \bar{u} \) on \( B_R \setminus \bigcup_{j=1}^{N} B_l(\omega^j x_0) \) and satisfies

\[
(5.10) \quad J_{B_l(x_0)}(\bar{u}) - J_{B_l(x_0)}(v) \geq k|B_{l-\eta}(x_0) \cap \{ q^{\bar{u}} = \delta \}|,
\]

where \( k = k(W,l_0,\eta,\delta) \) is a constant that does not depend on \( l > l_0 + \eta \) and \( R \).

In the following we will assume

\[
(5.11) \quad l_0 \geq \frac{\delta}{M}.
\]

Next we define a deformation of \( u^R \) into a map \( \bar{u} \) that satisfies the assumptions of Lemma 5.3 and derive a quantitative estimate for the energy \( J_{B_l(\pm \eta)}(\bar{u}) - J_{B_l(\pm \eta)}(u^R) \) spent in the deformation. As in the proof of Lemma 5.4 [2], we set \( p^{u^R}(x) = q^{u^R}(x) - (q^{u^R}(x) - \delta)^+ \) and define \( \bar{u} = a + q^{\bar{u}} n^{u^R} \) by, see Figure 2

\[
q^{\bar{u}} = \begin{cases} 
q^{u^R}(x), & \text{for } x \in B_R \setminus \bigcup_{j=1}^{N} \omega^j B_l(\pm \eta), \\
p^{u^R}(x) + g(x)(q^{u^R}(x) - \delta)^+, & \text{for } x \in B_l(\pm \eta),
\end{cases}
\]

(5.12)

\[
g(x) = \begin{cases} 
0, & \text{for } x \in B_l, \\
\frac{|x - x_0| - \delta}{\eta}, & \text{for } x \in \overline{B_l(\pm \eta)} \setminus B_l.
\end{cases}
\]

We remark explicitly that in the definition of \( \bar{u} \) we have not changed the direction vector: \( n^{\bar{u}} = n^{u^R} \).
The difficulty now is that, as observed before, we cannot assume (5.9) and by consequence we cannot use Lemma 5.1 to deduce, as in the proof of Lemma 5.4 [2], that $J_B(\tilde{u}) \leq J_B(u^R)$. To overcome this difficulty we need to treat differently the part of $\tilde{\Sigma}$ where (5.9) holds from the rest. Note that, by definition $\tilde{u} \neq u^R$ only in the subset where $q^u > \delta$. It follows

\[
J_B(l(\tilde{u})) = J_B(l(u^R)) \cap \{ q^u > \delta \} (\tilde{u}) - J_B(l(u^R)) (u^R).
\]

Let $L_\theta = \{ x(r, \theta) : r \in (0, +\infty) \}$ be the ray through $x(r, \theta)$ and define $\Theta$ by

\[
\Theta = \{ \theta : L_\theta \cap \{ x \in B_{l'} : \min_{a \in A \setminus \{ a \}} |u^R(x) - \omega a| < \delta^* \} \neq \emptyset \}.
\]

Set $l' = (l^2 + |\Sigma|^2)^{\frac{1}{2}}$ and define

\[
U = \cup_{\theta \in \Theta} L_\theta \cap \{ x \in B_{l'} : q^u \geq 2\delta^* \}.
\]

We divide the set $B_{l+\eta} \cap \{ q^u > \delta \}$ in three parts:

\[
U, \ V_1 = (B_l \cap \{ q^u > \delta \}) \setminus U, \ V_2 = ((B_{l+\eta} \setminus B_l) \cap \{ q^u > \delta \}) \setminus U
\]

and estimate separately the difference of energy of $\tilde{u}$ and $u^R$ for the three sets. See Figure 3 for an illustration of the set $U$.

**Lemma 5.4.** Assume $2\alpha + \alpha' < 1$. Then there exists $D > 0$ such that, given $\delta \in (0, D^{1-(2\alpha+\alpha')})$, there are $\eta_\delta > 0$ and $\bar{r} = \bar{r}_\delta > 0$ independent of $l$ and $R$ which, provided

\[
r(x_0) \geq \frac{l}{\sin \frac{\pi}{N}} + \bar{r}_\delta, \quad l \geq l_0 = \frac{\delta}{M},
\]

imply

\[
J_U(\tilde{u}) - J_U(u^R) \leq 0.
\]
Figure 3: The construction of $U$. $U$ is the union of the two curvilinear quadrilatera 1 and 2 marked with red dots.

**Proof.** The set $U$ contains all points in $B_l(x_0)$ where condition (5.9) is not satisfied and by (5.5) is a subset of $\tilde{\Sigma}$. Moreover the choice of $l'$ ensures that, if $\xi \in L_{\theta(\xi)} \cap B_l(x_0)$ is one of such points, then $L_{\theta(\xi)} \cap B_l'(x_0)$ contains a point $\xi' \neq \xi$ that satisfies

$$|u^R(\xi') - a| = 2c\delta^\alpha,$$

and is such that the interval with extremes $\xi$ and $\xi'$ is contained in $U$. This, the smallness of $\delta$ and Lemma 2.2 imply

$$J(u_{\text{restr}}, (0, |\xi' - \xi|)) \geq \sigma - \frac{C_W}{2}(4c^2\delta^{2\alpha} + \delta^2) \geq \sigma - C_W\delta^2,$$

where $u_{\text{restr}}(s) = u^R(\xi + s \frac{\xi' - \xi}{|\xi' - \xi|})$, $s \in (0, |\xi' - \xi|)$, is the restriction of $u^R$ to the interval with extreme at $\xi$ and $\xi'$. From (5.16) we have

$$J_U(u^R) \geq \int_{\{x(r, \theta) \in U\}} (\frac{1}{2} \frac{\partial}{\partial r}|u^R|^2 + W(u^R))rdrd\theta$$

$$\geq (r(x_0) - l') \int_{\Theta} J(u_{\text{restr}}, (0, |\xi' - \xi|))d\theta \geq (r(x_0) - l')(\sigma - C_W\delta^2)|\Theta|.$$

We now derive an upper bound for $J_U(\tilde{u})$ and show that, if $\delta$ is sufficiently small and if $\eta$ is sufficiently large, then $J_U(\tilde{u}) \leq J_U(u^R)$. Since $U \subset \{q^{u^R} > \delta\}$, from (5.12) we have

$$q^{u^R} = \left\{ \begin{array}{ll}
\delta, & x \in U \cap B_l, \\
\delta + \frac{|x - x_0| - l}{\eta} (q^{u^R} - \delta), & x \in U \setminus B_l.
\end{array} \right.$$

This (5.4), (5.1) and $q^{u^R} \geq 2c\delta^\alpha$ on $U$ imply $(E(v) = \frac{\|\nabla v\|^2}{2} + W(v))$

$$E(\tilde{u}) = \frac{1}{2} \delta^2 |\nabla q^{u^R}|^2 + W(\tilde{u}) \leq \frac{1}{2} \left( \frac{\delta}{q^{u^R}} \right)^2 (q^{u^R})^2 |\nabla u^R|^2 + \frac{1}{2} C_W^2 \delta^2$$

$$\leq \frac{1}{8c^2} \delta^{2(1-\alpha)} M^2 + \frac{1}{2} C_W^2 \delta^2 \leq C_0 \delta^{2(1-\alpha)}, \text{ } x \in U \cap B_l.$$
To estimate \( E(\tilde{u}) \) on \( U \setminus B_l \) we note that \((5.6)\) and \((5.11)\) imply

\[
(5.20) \quad l' - l = \frac{|\Sigma|^2}{l' + l} + \frac{C^2}{\delta^2(1 + \alpha')^2(l' + l)} \leq \frac{C^2}{\delta^2(1 + \alpha')^2} \frac{M}{2\delta^3} = \delta. 
\]

From this and \((5.18)\) we have

\[
q^\delta \leq \delta + \frac{l' - l}{\eta} (q^{u_R} - \delta) \leq \delta + \frac{M}{\eta} \delta, 
\]

\[
|\nabla q^\delta| \leq \frac{1}{\eta} (q^{u_R} - \delta + \delta|\nabla q^{u_R}|) \leq \frac{M}{\eta} (1 + \delta). 
\]

These inequalities and

\[
(5.22) \quad \eta = \eta_\delta = \frac{M}{\delta} (1 + \delta) 
\]

imply \( q^\delta \leq 2\delta \) and \( |\nabla q^\delta| \leq \delta \) and in turn, proceeding as in the derivation of \((5.19)\)

\[
(5.23) \quad E(\tilde{u}) \leq \frac{1}{2} \left( \delta^2 + \frac{1}{c^2} \delta^2 (1 - \alpha) M^2 + 4C_W^2 \delta^2 \right) \leq C_0 \delta^2 (1 - \alpha), \quad x \in U \setminus B_l. 
\]

From \((5.19), (5.23)\) and \((5.6)\) that implies

\[
|U| = \int_{\Sigma} \frac{r dr d\theta}{(r(\tilde{u}) + l') |\Sigma||\Theta|} \leq (r(x_0) + l') \frac{C_2}{\delta^1 + \alpha'} |\Theta|, 
\]

il follows

\[
(5.24) \quad J_U(\tilde{u}) \leq (r(x_0) + l') C_4 |\Theta| \delta^1 - (2\alpha + \alpha'). 
\]

From this and \((5.17)\) we have

\[
J_U(\tilde{u}) - J_U(u^R) \leq (r(x_0) + l') C_4 |\Theta| \delta^1 - (2\alpha + \alpha') - (r(x_0) - l') (\sigma - C_W \delta^2) |\Theta|, 
\]

which implies \((5.15)\) provided

\[
\delta^1 - (2\alpha + \alpha') \leq \frac{r(x_0) - l' \sigma - C_W \delta^2}{r(x_0) + l' C_4}. 
\]

Note that

\[
(5.25) \quad r(x_0) \geq \frac{l + \eta_\delta}{\sin \frac{\pi}{N}} = \frac{l}{\sin \frac{\pi}{N}} + \bar{\delta}, \quad \bar{\delta} = \frac{\eta_\delta}{\sin \frac{\pi}{N}}. 
\]

Otherwise we have \( B_{l + \eta_\delta}(x_0) \cap B_{l + \eta_\delta}(\omega x_0) \neq \emptyset \).

Since \( l' < l + \eta_\delta, [5.25] \) implies \( \frac{r(x_0) - l'}{r(x_0) + l'} \geq \frac{1 - \sin \frac{\pi}{N}}{1 + \sin \frac{\pi}{N}} \). Therefore, under that condition, we see that we can secure \((5.15)\) if we fix \( \delta \in (0, \delta_0) \) so that

\[
(5.26) \quad \delta \leq D \frac{1}{1 - (2\alpha + \alpha')}, \quad D = \frac{\sigma (1 - \sin \frac{\pi}{N})}{2C_4 (1 + \sin \frac{\pi}{N})}. 
\]

By increasing the value of \( C_4 \) if necessary we can also assume that \( D \frac{1}{1 - (2\alpha + \alpha')} \leq \delta_0 \). Once \( \delta > 0 \) is fixed, \((5.20)\) implies that \( 4(l' - l) \leq \bar{r} = \bar{\delta} = 4\bar{\delta} \) for some constant \( \bar{r} > 0 \) independent of \( l \) and \( R \). The same is true for \( \eta = \eta_\delta \) defined in \((5.22)\). This concludes the proof. \( \square \)
Remark 5.5. Observe that with $\delta > 0$ fixed also the bound for $|\Sigma|$ given by Lemma 4.2 is a constant independent of $l$ and $R$.

To estimate $J_{V_l}(\tilde{u}) - J_{V_l}(u^R)$, $i = 1, 2$ we proceed as in the proof of Lemma 5.4 [2].

The definition (5.12) of $q^\delta$ implies

\[(5.27) \quad q^\delta = \delta, \quad \nabla q^\delta = 0, \text{ on } V_1.\]

From this (5.4) and Lemma 5.1 we deduce

\[J_{V_1}(\tilde{u}) - J_{V_1}(u^R)\]

\[= \int_{V_1} \left( \frac{1}{2} (-|\nabla q^u|^2 + (\delta^2 - (q^uR)^2)|n^uR|^2) + W(a + q^\delta n^uR) - W(a + q^uR n^uR) \right) dx \]

\[\leq 0, \]

which yields

\[(5.28) \quad J_{V_1}(\tilde{u}) - J_{V_1}(u^R) \leq 0.\]

It remains to evaluate the difference of energy on $V_2 \subset B_{l+\eta} \setminus B_l$. From (5.12) we have

\[|\nabla q^\delta| \leq |\nabla g|(q^uR - \delta) + |g|\nabla q^uR | \leq \left( \frac{1}{\eta} + 1 \right) M, \text{ a.e. on } V_2\]

From this and (5.12) that implies $q^\delta \leq q^uR$ on $B_{l+\eta} \setminus B_l$ it follows

\[(5.29) \quad |\nabla \tilde{u}|^2 - |\nabla u^R|^2 \leq \left( \frac{1}{\eta} + 1 \right) 2M^2 + ((q^\delta)^2 - (q^uR)^2)|n^uR|^2 \leq \left( \frac{1}{\eta} + 1 \right)^2 M^2, \text{ a.e. on } V_2.\]

From (5.1) we have $q^\delta \leq q^uR \leq M$ on $V_2$ and in turn

\[W(a + q^\delta n^uR) - W(a + q^uR n^uR) \leq W(a + q^\delta n^uR) \leq C_M, \text{ a.e. on } V_2\]

for some constant $C_M > 0$. This and (5.29) imply

\[(5.30) \quad J_{V_2}(\tilde{u}) - J_{V_2}(u^R) \leq \left( \frac{1}{2} \left( \frac{1}{\eta} + 1 \right) 2M^2 + W_M \right) |V_2|.\]

From the previous analysis we conclude

**Lemma 5.6.** There exists $\bar{\delta} > 0$ such that, given $\delta \in (0, \bar{\delta}]$ there are $\eta = \eta_\delta > 0$, $K = K_\delta > 0$ and $\bar{r} = \bar{r}_\delta > 0$ which, provided

\[(5.31) \quad l \geq l_0 + \eta, \quad l_0 = \frac{\delta}{\bar{M}}, \quad r(x_0) \geq 2l + \bar{r}, \]

imply

\[J_{B_{l+\eta}}(\tilde{u}) - J_{B_{l+\eta}}(u^R) \leq K|B_{l+\eta} \setminus B_{l-\eta}| \cap \{q^uR > \delta\},\]

**Proof.** From (5.28), (5.30), (5.15) in Lemma 5.4 and (5.13) we obtain

\[J_{B_{l+\eta}}(\tilde{u}) - J_{B_{l+\eta}}(u^R) \leq K|V_2|,\]

where $K_\delta = \frac{1}{2} \left( \frac{1}{\eta_\delta} + 1 \right) 2M^2 + W_M$. This and $V_2 \subset B_{l+\eta} \setminus B_l \subset B_{l+\eta} \setminus B_{l-\eta}$ conclude the proof.
The estimate (5.32) corresponds exactly to the statement of Lemma 5.4 in [2]. Once (5.32) is established the proof proceeds exactly as the proof of Theorem 5.3 (pag.165 in [2]): set $l_h = l_0 + (2h-1)\eta$ for $h = 1, \ldots$ and let $\tilde{u}_h$ the map $\tilde{u}$ given by Lemma 5.6 for $l = l_h$, $h = 1, \ldots$. Let $v_h$ the map $v$ given by Lemma 5.3 with $\tilde{u} = \tilde{u}_h$ and $l = l_h$. Then, the minimality of $u^R$ implies

$$0 \geq J_{B_{l_h} + \eta}(u^R) - J_{B_{l_h} + \eta}(v_h) = J_{B_{l_h} + \eta}(u^R) - J_{B_{l_h} + \eta}(\tilde{u}_h) + J_{B_{l_h} + \eta}(\tilde{u}_h) - J_{B_{l_h} + \eta}(v_h) = J_{B_{l_h} + \eta}(u^R) - J_{B_{l_h} + \eta}(\tilde{u}_h) + J_{B_{l_h}}(\tilde{u}_h) - J_{B_{l_h}}(v_h).$$

This (5.32) and (5.10) yield

$$k|B_{l_h-\eta} \cap \{q^R > \delta\}| \leq K|B_{l_h+\eta} \setminus B_{l_h-\eta} \cap \{q^R > \delta\}|, \quad \text{for} \quad h = 1, \ldots$$

(5.33)

It is rather clear that, if $\mu_0 := |B_{l_0} \cap \{q^R > \delta\}| \neq 0$, this inequality cannot hold for large $h$. Indeed, if we set $\mu_h := |B_{l_h+\eta} \cap \{q^R > \delta\}|$, for $h = 1, \ldots$ then (5.33) yields

$$k \frac{1}{K} \mu_{h-1} \leq \mu_h - \mu_{h-1}, \quad \text{for} \quad h = 1, \ldots$$

(5.34)

and therefore

$$\mu_0 \left(1 + \frac{k}{K}\right)^{h-1} \leq \mu_h, \quad \text{for} \quad h = 1, \ldots$$

From this and (5.34) we obtain

$$\mu_0 \frac{k}{K} \left(1 + \frac{k}{K}\right)^{h-1} \leq \mu_h - \mu_{h-1}, \quad \text{for} \quad h = 1, \ldots$$

(5.35)

Now assume that

$$q^R(x_0) = |u^R(x_0) - a| \geq 2\delta.$$ 

Then (5.1) implies

$$\delta < q^R, \quad \text{for} \quad x \in B_{\frac{1}{2\eta}}(x_0).$$

It follows $\mu_0 \geq |B_{\frac{1}{2\eta}}(x_0)|$. Since:

$$\mu_h - \mu_{h-1} \leq |B_{l_h+\eta} \setminus B_{l_h-\eta}| \leq 4\eta l_h = 4\eta(l_0 + h\eta),$$

the right end side of (5.35) grows linearly in $h$. On the other hand the left hand side grows exponentially in $h$. Hence there exists a minimum value $h_m$ of $h$ such that (5.35) is violated for $h \geq h_m$ in contradiction with the minimality of $u^R$. It follows that $l \geq l_{h_m}$ together with the condition $r(x_0) \geq 2l + \bar{r}$ required from Lemma 5.6 imply

$$q^R(x_0) = |u^R(x_0) - a| < 2\delta.$$ 

Finally we observe that $l_{h_m}$ is actually a function of $\delta$ this is a consequence of the fact that $\eta$, $\mu_0$ and $h_m$ are all function of $\delta$. This concludes the proof Theorem 5.2.

From now on we assume that $\delta \in (0, \tilde{\delta}]$ is fixed and treat $l$ and $\bar{r}$ as fixed constants.
Corollary 5.7. Let $\delta \in (0, \bar{\delta}]$, $\bar{l}$ and $\bar{r}$ be as in Theorem 5.2 and let $u^R$ be a minimizer of (1.3). Assume $x_0 \in B_R$ and $d > 0$ are such that

\begin{align}
B_{\bar{l}+d}(x_0) &\subset B_R, \\
r(x_0) &\geq 2\bar{l} + \bar{r} + d, \\
|u^R(x) - a| &\leq c\delta^\alpha, \ x \in B_{\bar{l}+d}(x_0) \setminus \tilde{\Sigma}.
\end{align} 

Then

\begin{equation}
|u^R(x_0) - a| \leq 2\delta e^{-\bar{k}d},
\end{equation}

where $\bar{k} > 0$ is independent of $R$.

Proof. From (5.36) we have that any ball of radius $\bar{l}$ contained in $B_{\bar{l}+d}(x_0)$ satisfies the assumptions of Theorem 5.2. It follows

\begin{equation}
|u^R(x) - a| \leq 2\delta \ x \in B_d(x_0).
\end{equation}

This and a standard argument, see the proof of Lemma 4.4 in [2], imply the result. \qed

Theorem 5.2 is tailored for application to the problem at hand. We state a version of the theorem more appropriate when rectangular coordinates are used. We let $\Omega_R \subset \mathbb{R}^2$ a bounded smooth domain that depends on a parameter $R > 0$. For $\nu \in \mathbb{S}$ we set $\Omega_R \cdot \nu = \{x \cdot \nu : x \in \Omega_R\}$.

Theorem 5.8. Assume that a minimizer $u^R : \Omega_R \to \mathbb{R}^2$ of $J_{\Omega_R}$, a ball $B_l(x_0)$, $\nu \in \mathbb{S}$ and a set $\Sigma \subset \Omega_R \cdot \nu$ satisfy, for some constants $c > 0$ and $C > 0$,

\begin{equation}
|u^R(x) - a| \leq c\delta^\alpha, \ x \in B_l(x_0) \setminus \tilde{\Sigma}, \ for \ some \ \alpha \in (0, 1)
\end{equation}

where $\tilde{\Sigma} = \{x \in \Omega_R : x \cdot \nu \in \Sigma\}$, and

\begin{equation}
|\Sigma| \leq \frac{C}{\delta^{l+\alpha'}}, \ for \ some \ \alpha' \in (0, 1).
\end{equation}

Assume that

\[ 3\alpha + \alpha' < 1. \]

Then there is a constant $D > 0$ such that for each $\delta \leq D^{\frac{1}{1-(3\alpha+\alpha')}}$ there exists $l_\delta > 0$ independent of $R$ and such that

\begin{equation}
l \geq l_\delta,
\end{equation}

implies

\begin{equation}
|u^R(x_0) - a| \leq 2\delta.
\end{equation}
6 Analysis of the geometric structure of \( u^R \).

From Lemma 4.2, \( r \in [r_δ, R) \setminus \Sigma \) implies that, modulo a suitable translation, it results \( u^R(\hat{x}, \hat{z}) \in \mathcal{V}^r, \mathcal{V}^* \) the set defined by (4.1) in Proposition 4.1. This and equivariance imply in particular the existence of \( \theta_r \in [0, 2\pi) \) such that

\[
\begin{align*}
\theta_r^+ = \theta_r + \frac{\nu}{2r}, \\
\nu = \frac{4\sigma}{c_W \delta^2}, \text{ (see Proposition 4.1).}
\end{align*}
\]

6.1 The pseudo regularity of the map \( r \rightarrow \theta_r \).

We plan to show that the minimality of \( u^R \) implies a relationship between the numbers \( \theta_r \) and \( \theta_r^+ \) corresponding to different radii \( r \) and \( r \in [r_δ, R) \setminus \Sigma \). Actually we will show that the map \( r \rightarrow \theta_r \) has a kind of Lipschitz behavior, see Figure 4. If \( |\theta_r - \theta_r^+| \leq \frac{\nu}{\min\{r_r, r\}} \) nothing can be said on the actual value of the difference \( \theta_r - \theta_r^+ \). The situation is different if \( \theta_r^- - \theta_r^+ > 0 \) or \( \theta_r^+ - \theta_r^+ > 0 \). We have indeed

**Lemma 6.1.** There are \( \tilde{c} > 0, \beta \in (0, 1) \) and \( \bar{r} > r_δ \) such that, if \( R > \bar{r}, r_\ast \in [\bar{r}, R) \) and \( r^* = \min\{(1 + \beta)r_\ast, R\} \), then it results

\[
\begin{align*}
\theta_r^+ + \frac{\nu}{2r} + \tilde{c} \ln \frac{r}{r_\ast} &\geq \theta_r^+ > \theta_r^- \geq \theta_r^+ - \frac{\nu}{2r} - \tilde{c} \ln \frac{r}{r_\ast}, \quad r \in [r_\ast, r^*) \setminus \Sigma, \\
\theta_r^+ + \frac{\nu}{2r} + \tilde{c} \ln \frac{r_\ast}{r} &\geq \theta_r^+ > \theta_r^- \geq \theta_r^+ - \frac{\nu}{2r} - \tilde{c} \ln \frac{r_\ast}{r}, \quad r \in (r_\ast(1 - \beta), r_\ast) \setminus \Sigma.
\end{align*}
\]

and

\[
\theta_r^+ < \theta_r^+ + \frac{2\pi}{N}.
\]

Moreover if \( \theta_r^+ \) does not satisfies (6.2) then \( r \in (r_\ast(1 - \beta), r^*) \).

**Proof.** 1. Let \( v : [r_\ast, r] \rightarrow \mathbb{R}^2, r_\ast < r, \) be a smooth function. Then it results

\[
\int_{r_\ast}^{r} |v'|^2 \rho d\rho \geq \frac{|v(r) - v(r_\ast)|^2}{\ln \frac{r}{r_\ast}}.
\]

2. Consider first the case \( \theta_r^- - \theta_r^+ \in (0, \frac{2\pi}{N} - \bar{\theta}) \), where \( \bar{\theta} > 0 \) is fixed and small. In this case we have

\[
\begin{align*}
u^R(x(r_\ast, \theta_r^+ + \phi)) &\in B_{c_\delta^N}(a), \\
u^R(x(r, \theta_r^+ + \phi)) &\in B_{c_\delta^N}(\omega^{N-1}a), \quad \text{for } \phi \in (0, \theta_r^- - \theta_r^+).\end{align*}
\]

It follows

\[
|u^R(x(r, \theta_r^+ + \phi)) - u^R(x(r_\ast, \theta_r^+ + \phi))| \geq |(\omega^{N-1} - I)a| - 2c\delta^N, \quad \text{for } \phi \in (0, \theta_r^- - \theta_r^+).\]
This and (6.4) imply
\[ \int^\theta_r \int_{r_*}^r \frac{\partial}{\partial \rho} u R^2 \rho \, d\rho \, d\theta \geq \frac{1}{\ln r_*}(\theta^+ - \theta^-), \]
which together with the bound (3.17) yields
\[ (6.6) \quad \theta^+ - \theta^- \leq \tilde{c} \ln \frac{r}{r_*} \iff \theta^+ - \theta^- \leq \frac{\nu}{r} + \tilde{c} \ln \frac{r}{r_*}, \]
where we have set \( \tilde{c} = \frac{2(C_0 + C_1)}{N(|(\omega^{N-1} - I)a| - 2c\delta^\alpha)} \). In case \( \theta^- - \theta^+ \in (0, \frac{2\pi}{N} - \bar{\theta}) \), the same argument developed above and \( |(\omega^{N-1} - I)a| = |\omega - I)a| \) leads to
\[ (6.7) \quad \theta^- - \theta^+ \leq \tilde{c} \ln \frac{r}{r_*}. \]
Equation (6.6) follows from (6.6) and (6.7). To prove (6.2) we observe that proceeding as before we see that \( \bar{\theta}^- - \bar{\theta}^+ \in (0, \frac{2\pi}{N} - \bar{\theta}) \) and \( r < r_* \) imply
\[ \theta^- - \theta^+ \leq \tilde{c} \ln \frac{r}{r_*}. \]
This and the corresponding statement valid for \( \theta^- - \theta^+ \in (0, \frac{2\pi}{N} - \bar{\theta}) \) prove (6.2). 3. From (6.2), if \( r > r_* \) and \( r_* \) is sufficiently large,
\[ 2\tilde{c} \ln \frac{r}{r_*} \leq 2\tilde{c} \ln (1 + \beta) \leq \frac{\pi}{N}, \]
is a sufficient condition for (6.3). From this and the analogous condition for the case \( r_* > r \) it follows that \( \beta \leq 1 - e^{-\frac{\pi}{2N}} \) ensures that (6.3) holds.
4. If \( \theta^- - \theta^+ > \frac{2\pi}{N} - \bar{\theta} \) and \( r > r_* \) the same reasoning that proves (6.6) leads to
\[ \ln \frac{r}{r_*} \geq C_5, \]
for some \( C_5 > 0 \). Analogous conclusion holds in the cases \( \theta^- - \theta^+ > \frac{2\pi}{N} - \bar{\theta} \) etc. The last statement of the lemma follows from this after a reduction of the value of \( \beta \) if necessary. The proof is complete.

**Corollary 6.2.** Let \( r_* \in (\bar{r}, R) \), \( r^* \) and \( \beta \in (0, 1) \) be as in Lemma 6.1. Then, assuming that \( \bar{r} \) has been chosen sufficiently large, the sets
\[ \mathcal{J}_{r_*} = \{ x(r, \theta) : \frac{2\pi}{N} - \frac{3\nu}{2r_*} - \tilde{c} \ln \frac{r}{r_*} > \theta - \theta_* > \frac{3\nu}{2r_*} + \tilde{c} \ln \frac{r}{r_*}, \theta \in [r_*, r) \} \]
\[ (6.8) \]
\[ \mathcal{J}_r = \{ x(r, \theta) : \frac{2\pi}{N} - \frac{\nu}{2r_*} - \tilde{c} \ln \frac{r}{r_*} > \theta - \theta_* > \frac{\nu}{2r_*} + \tilde{c} \ln \frac{r}{r_*}, \theta \in (r_* (1 - \beta), r) \}. \]
are well defined, nonempty and it results
\[ (6.9) \quad |u^R(x) - \omega^{j-1}a| \leq c\delta^\alpha, \text{ for } x \in \omega^{j-1}\mathcal{J}_{r_*} \setminus \mathcal{\Sigma}, \quad j = 1, \ldots, N \]
where
\[ \mathcal{J}_{r_*} = \mathcal{J}_{r_*} \cup \mathcal{J}_r. \]
Figure 4: The Lipschitz like property of the map $r \to \theta_r$ and the set $\mathcal{S}_{r*,\beta}$ (N=3).

Proof. The assumption on $\beta$ implies that, provided $r_* \geq \bar{r}$ is sufficiently large, $\theta_r^+ \leq \theta_r^- + \frac{\pi}{N}$ for $r \in (r_*(1-\beta), r^*)$. This means that the set $\mathcal{S}_{r*,\beta}$ and $\mathcal{S}_{r,\beta}$ are well defined and nonempty. From Lemma 4.2 and Proposition 4.1, $r \in (r_*(1-\beta), r^*) \setminus \Sigma$ implies $u^R(x(r_*)) \in \mathcal{V}_r^*$ and the inequalities (6.2) in Lemma 6.1 and equivariance yield (6.9). The proof is complete.

The set $\mathcal{S}_{r*,\beta}$ is illustrated in Figure 4. Denote $C_r$ the circumference of radius $r \in (0, R)$ and set

$$\hat{p}_{r*}^\pm = x(r_*, \theta_{r_*} \pm \frac{3\nu}{2r_*}).$$

Observe that $\hat{p}_{r*}^+$ and $\omega \hat{p}_{r*}^-$ are the extreme of the arc $\hat{\mathcal{S}}_{r*,\beta} \cap \mathcal{S}_{r*,\beta}$.

Next we introduce a representation of $\partial \mathcal{S}_{r*,\beta}$. We focus on the connected component of $\partial \mathcal{S}_{r*,\beta}$ that contains $\hat{p}_{r*}^+$. Analogous definitions apply to the component that contains $\omega \hat{p}_{r*}$.

Set

$$\hat{g}_{r_*}(r) = x(r, \hat{\theta}_{r_*}(r)), \quad \hat{\theta}_{r_*}(r) = \theta_{r_*} + \frac{3\nu}{2r_*} + \hat{c} \ln \frac{r}{r_*}, \quad r \in [r_*, r^*],$$

$$\hat{g}_{r_*}(r) = x(r, \hat{\theta}_{r_*}(r)), \quad \hat{\theta}_{r_*}(r) = \theta_{r_*} + \frac{\nu}{2r_*} + \hat{c} \ln \frac{r}{r_*}, \quad r \in ((1-\beta)r_*, r^*].$$

Note that $\hat{g}_{r_*}(r_*) = \hat{g}_{r_*}(r_*) = \hat{p}_{r*}^+$ and that $\hat{g}_{r_*}([r_*, r^*])$ coincides with the connected component of $\hat{\mathcal{S}}_{r*,\beta} \cap \mathcal{S}_{r*,\beta}$ that contains $\hat{p}_{r*}^+$. Similarly $\hat{g}_{r_*}((1-\beta)r_*, r_*)$ coincides with the connected component of $\hat{\mathcal{S}}_{r*,\beta} \cap \mathcal{S}_{r*,\beta}$ that contains $\hat{p}_{r*}^-$. 

Definition 6.3. Let $\ell_{r*}^+$ be the half line which has origin at $\hat{p}_{r*}^+$, forms equal angles with the two tangents to $\partial \hat{\mathcal{S}}_{r*,\beta}$ at $\hat{p}_{r*}^+$ and points inside $\hat{\mathcal{S}}_{r*,\beta}$. We let $\ell_{r*}^-$ the analogous half line which has origin in $\hat{p}_{r*}^-$ and points inside $\omega^{-1} \hat{\mathcal{S}}_{r*,\beta}$.
For \( x, y \in \mathbb{R}^2 \) we let \([x, y]\) be the segment with extreme at \( x, y \). If \( x \) and \( y \) satisfy \( r(x) = r(y) \), that is: \( x \) and \( y \) belong to the same circumference, we let \( \text{arc}(x, y) \) denote the shortest arc of \( C_r \) with extreme \( x \) and \( y \) and \( d^r(x, y) \) be the length of \( \text{arc}(x, y) \). If \( S \subset \mathbb{R}^2 \) and \( S \cap C_r \neq \emptyset \) we set
\[
(6.12) \quad d^r(x, S) = \inf_{y \in S \cap C_r(x)} d^r(x, y).
\]

**Lemma 6.4.** Let \( \bar{r}, r_\ast \geq \bar{r} \) and \( \beta \) as in Corollary 6.2. Then there exist \( \epsilon \in (0, 1) \), \( \eta \in (0, 1) \) such that, if \( \bar{r} \) is sufficiently large,

(i)

\[
 x \in \hat{\mathcal{J}}_{r_\ast, \beta}, \quad l \in (0, \epsilon r_\ast] \quad \text{and} \quad d(x, \partial \mathcal{J}_{r_\ast, \beta}) \geq l,
\]

\[
 \Rightarrow \quad B_{\eta l}(x) \subset \mathcal{J}_{r_\ast, \beta}.
\]

The same conclusion is valid if the condition \( d(x, \partial \mathcal{J}_{r_\ast, \beta}) \geq l \) is replaced by

\[
r^\ast - r(x) \geq l \quad \text{and} \quad d^r(x, \partial \mathcal{J}_{r_\ast, \beta}) \geq l,
\]

where, as before, \( r^\ast = \min\{(1 + \beta) r_\ast, R\} \).

(ii)

\[
 \rho \in (0, \epsilon r_\ast] \quad \Rightarrow
\]

\[
 B_{\frac{\rho}{2}}(\tilde{p}^+(\rho)) \subset \hat{\mathcal{J}}_{r_\ast, \beta},
\]

\[
 B_{\frac{\rho}{2}}(\tilde{p}^-(\rho)) \subset \omega^{-1} \hat{\mathcal{J}}_{r_\ast, \beta}.
\]

where \( \tilde{p}^\pm(\rho) \in \ell^\pm_{r_\ast} \cap C_{r_\ast - \rho} \) satisfies \( \lim_{\rho \to 0} \tilde{p}^\pm(\rho) = \tilde{p}^\pm_{r_\ast} \).

**Proof.** To prove (ii) we observe that, since, by definition, \( \ell^\pm_{r_\ast} \) forms equal angles with the two tangents to \( \partial \hat{\mathcal{J}}_{r_\ast, \beta} \) at \( \tilde{p}^\pm_{r_\ast} \), we have approximately

\[
 d(\tilde{p}^+(\rho), C_{r_\ast}) \simeq d(\tilde{p}^+(\rho), \partial \hat{\mathcal{J}}_{r_\ast, \beta} \setminus C_{r_\ast}) \simeq \rho.
\]

This and the analogous statement for \( \tilde{p}^-(\rho) \), under the standing assumption that \( \epsilon > 0 \) is small, imply (ii). The proof is complete. \( \square \)

**Remark 6.5.** Let \( \varphi_\ast \) the angle between the two lines tangent to \( \hat{\mathcal{J}}_{r_\ast, \beta} \) at \( \tilde{p}^\pm_{r_\ast} \). Then

\[
(6.13) \quad \sin \varphi_\ast = \frac{1}{\sqrt{1 + (\dot{c} + \frac{\nu}{r_\ast})^2}}.
\]

Fix a small number \( \epsilon > 0 \) and, for \( r \in ((1 - \epsilon) r_\ast, r_\ast] \) set \( \vartheta(r) = \theta - \theta_{r_\ast} = \frac{\nu}{2 r_\ast} + \frac{\nu}{r} + \dot{c} \ln \frac{r_\ast}{r} \). Let \( 0 \xi_1, \xi_2 \) the positive frame determined by the assumption that the \( \xi_1 \) axis coincides with the ray \( L_{\theta_{r_\ast}} \). Then the definition of \( \hat{\mathcal{J}}_{r_\ast, \beta} \) in Corollary 6.2 implies that \( (\xi_1(r), \xi_2(r)) = (r \cos \vartheta, r \sin \vartheta) \), \( r \in ((1 - \epsilon) r_\ast, r_\ast] \) is a local representation of \( \partial \hat{\mathcal{J}}_{r_\ast, \beta} \setminus C_{r_\ast} \) and \( (\xi_1(r_\ast), \xi_2(r_\ast)) = \tilde{p}^\pm_{r_\ast} \). Let \( \varphi(r) \) the angle that the vector \( (\xi'_1(r), \xi'_2(r)) \) tangent to \( \hat{\mathcal{J}}_{r_\ast, \beta} \) at \( (\xi_1(r), \xi_2(r)) \) forms with the vector \( (\sin \vartheta, -\cos \vartheta) \) tangent to \( C_r \) at \( (\xi_1(r), \xi_2(r)) \). It results

\[
\sin \varphi(r) = \frac{1}{\sqrt{1 + (r \vartheta'(r))^2}}.
\]

This, \( \vartheta'(r) = -\frac{\dot{c}}{r} - \frac{\nu}{r^2} \) and \( r = r_\ast \) imply (6.13).
6.2 Construction of the diffuse interface

The properties of the map \( r \to \theta_r \) discussed above allow for the construction of a set that can be regarded as the diffuse interface that divide \( B_R \) in regions where \( u^R \) is near to one or another of the zeros of \( W \). We will be able to define the length of the interface and by means of Theorem 5.2 show that the energy of \( u^R \) is essentially contained in the interface and proportional to its length. This and the upper bound (3.16) for the energy of \( u^R \) allow to control the length and by consequence the shape of the interface leading to the proof of Theorem 1.1.

We now consider a sequences of balls \( B_{R_n} \) defined by

\[
R_n = (n + 1 + c_1)^2,
\]

where \( c_1 > 0 \) is a constant and let \( u^{R_n} \) a \( C_N \)-equivariant minimizer of \( J_{B_{R_n}} \). Let \( \tilde{c} > 0 \) and \( \mu_j \in [0, |\Sigma|] \), \( j = 1, \ldots, n + 1 \) be constants to be chosen later and define

\[
\begin{align*}
 r_j &= (j + c_1)^2 - \mu_j, \\
 \lambda_j &= \tilde{c} \ln r_j,
\end{align*}
\]

where \( c_1 \) is chosen sufficiently large to ensure

\[
 r_1 \geq \max\{\delta, 2\tilde{l} + \tilde{r}\},
\]

\( \delta \) as in Lemma 4.2 and \( \tilde{l} \) and \( \tilde{r} \) as in Theorem 5.2. From Lemma 4.2 we can assume that, for each \( j \), the constant \( \mu_j \in [0, |\Sigma|] \) is such that

\[
 r_j \in (0, R_n) \setminus \Sigma, \quad j = 1, \ldots, n + 1.
\]

Lemma 6.6. The sequences \( \{r_j\}_{j=1}^{n+1} \) and \( \{\lambda_j\}_{j=1}^{n+1} \) satisfy

(i)

\[
\lim_{j \to +\infty} r_{j+1} - r_j = +\infty,
\]

\[
\lim_{j \to +\infty} \frac{r_{j+1} - r_j}{r_j} = 0,
\]

\[
\lim_{r \to +\infty} \frac{\lambda_j}{r_{j+1} - r_j} = 0.
\]

(ii) If \( c_1 > 0 \) is sufficiently large there exist constants \( c_0, C^0 > 0 \) such that

\[
c_0 r_j^{\frac{1}{2}} \leq r_{j+1} - r_j \leq C^0 r_j^{\frac{1}{2}}, \quad j = 1, \ldots, n + 1,
\]

and we can assume

\[
\beta_j = \frac{r_{j+1} - r_j}{r_j} \leq \frac{\beta}{2}, \quad j = 1, \ldots, n + 1,
\]

\[
\frac{2\lambda_j}{r_{j+1} - r_j} < 1.
\]

where \( \beta \) is as in Lemma 6.1.
Figure 5: The points $p^+_j, p^+_j, q^+_j, q^+_{j+1}$ and the set $\mathcal{J}_j$.

**Proof.** The proof is an elementary computation. \(\square\)

Define (see Figure 5):

\[
p^+_j = x(r_j, \theta_{r_j} \pm \frac{3\nu}{2r_j} \pm \frac{\lambda_j}{r_j}), \quad j = 1, \ldots, n+1,
\]

\[
q^+_j = x(r_j, \theta_{r_{j-1}} \pm \frac{3\nu}{2r_{j-1}} \pm \frac{\nu}{r_j} \pm \hat{c} \ln \frac{r_j}{r_{j-1}} \pm \frac{\lambda_j}{r_j}),
\]

and set $\hat{p}^+_j = \hat{p}^+_j$ where $\hat{p}^+_j$ is defined in (6.10). We have

\[
\theta(p^+_j) - \theta(\hat{p}^+_j) = \frac{\lambda_j}{r_j},
\]

\[
\theta(q^+_j) - \theta(p^+_j) \geq \frac{\nu}{r_{j-1}} - \frac{\nu}{r_j} > 0.
\]

The first equation is obvious, (6.19) follows from the definition (6.18) and (6.2) in Lemma 6.1 with $r^* = r_{j-1}$ and $r = r_j$ which implies

\[
\theta_{r_j} + \frac{\nu}{2r_j} \leq \theta_{r_{j-1}} + \frac{\nu}{2r_{j-1}} + \frac{\nu}{r_j} + \hat{c} \ln \frac{r_j}{r_{j-1}}.
\]

**Lemma 6.7.** Set $a_- = \omega^{-1}a$ and $a_+ = a$. Then

(i) There is a constant $\eta \in (0, 1)$ independent of $n$ such that

\[
x \in \text{arc}[p^+_j, q^+_j] \Rightarrow
\]

\[
|u^R_n - a_{\pm}| \leq c\delta^\alpha, \quad \text{on} \ B_{\eta \lambda_j}(x) \setminus \hat{\Sigma}, \quad j = 2, \ldots, n,
\]

(ii)

\[
x \in [p^+_j, q^+_{j+1}] \Rightarrow
\]

\[
|u^R_n - a_{\pm}| \leq c\delta^\alpha, \quad \text{on} \ B_{\eta \lambda_j}(x) \setminus \hat{\Sigma}, \quad j = 1, \ldots, n-1.
\]
Proof. From (6.19) and (6.12) it follows
\begin{equation}
(6.22) \quad d^p(x, \partial \mathcal{S}_{r_j, \beta}) \geq \lambda_j, \quad x \in \text{arc}(p_j^+, q_j^+), \ j = 2, \ldots, n.
\end{equation}
Set \( r_j^* = \min\{(1 + \beta)r_j, R_n\} \). Then we have
\begin{equation}
(6.23) \quad r_j^* \geq r_{j+1}, \ j = 2, \ldots, n.
\end{equation}
This is obvious if \( r_j^* = R_n \) and follows from assumption (6.17)1 if \( r_j^* = (1 + \beta)r_j \). From (6.23) and (6.17)2 it follows
\begin{equation}
(6.24) \quad r_j^* - r(x) \geq r_{j+1} - r_j \geq \lambda_j, \ j = 2, \ldots, n.
\end{equation}
This and (6.22) imply that we can apply Lemma 6.4 and conclude that \( B_{\eta \lambda_j}(x) \subset \mathcal{S}_{r_j, \beta} \)
and (6.20)1 follows from (6.9) in Corollary 6.2. The proof of (6.20)2 is similar.

Proof of (ii). Let \( \tilde{g}_{r_j} \) and \( \partial_{r_j} \) be defined as in (6.11) with \( r_* = r_j \). The curve \( [r_j, r_{j+1}] \ni r \to \tilde{g}_{r_j}(r) \) is a parametrization of the connected component of \( \partial \mathcal{S}_{r_j, \beta} \cap \{x(r) : r \in [r_j, r_{j+1}]\} \) that contains \( \hat{p}_j^+ \). The curve \( [r_j, r_{j+1}] \ni r \to \tilde{g}(r) = x(r, \tilde{g}_{r_j}(r) + \frac{\lambda_j}{r_j}) \) lies on the left of \( \tilde{g}_{r_j}([r_j, r_{j+1}]) \) and satisfies
\begin{equation}
(6.25) \quad d^p(\tilde{g}(r), \tilde{g}_{r_j}(r)) = \frac{r - r_j}{r_j} \lambda_j \geq \lambda_j.
\end{equation}
From (6.19) we see that \( \tilde{g}(r_j) = p_j^+ \) and \( \theta(q_j^+ - 1) - \theta(\tilde{g}(r_{j+1})) > 0 \). This and the fact that the curve \( \tilde{g} \) turns its concavity toward increasing \( \theta \) imply that the whole curve lies on the right of the segment \( [p_j^+, q_j^+] \). This and (6.25) yield
\begin{equation}
(6.26) \quad d^p(x, \partial \mathcal{S}_{r_j, \beta}) \geq \lambda_j, \quad x \in [p_j^+, q_j^+],
\end{equation}
that together with (6.24) allow to complete the proof as in case (i). This concludes the proof of (6.21)1. The same argument proves (6.21)2. \( \square \)

From (6.18) and (6.15) we obtain
\begin{equation}
(6.27) \quad \lim_{j \to \infty} \frac{|p_j^+ - p_j^-|}{r_{j+1} - r_j} = 0,
\end{equation}
\begin{equation}
(6.28) \quad \lim_{j \to \infty} \frac{|q_j^+ - q_j^-|}{r_{j+1} - r_j} = 2\hat{c}.
\end{equation}
This implies that, by choosing \( c_1 > 0 \) sufficiently large in (6.14), we can assume
\begin{equation}
(6.29) \quad \frac{|p_j^+ - p_j^-|}{r_{j+1} - r_j} \leq \epsilon, \quad \frac{|q_j^+ - q_j^-|}{r_{j+1} - r_j} \leq 3\hat{c}, \quad j = 1, \ldots, n.
\end{equation}
From (6.26) it follows
\begin{equation}
(6.30) \quad \lim_{j \to \infty} \tan \psi_j = \hat{c},
\end{equation}
where \( \psi_j \) is the angle that the vector \( q_j^+ - p_j^- \) forms with \( L_{\theta_{r_j}} \) (recall that \( L_{\theta} \) is the ray determined by \( \theta \)). Let \( [x_j, x_{j+1}] \) be a segment that connects a point \( x_j \in \text{arc}(p_j^-, p_j^+) \) with a point \( x_{j+1} \in \text{arc}(q_j^-, q_j^+) \). From (6.27), since \( \epsilon > 0 \) is a small fixed number, it follows
where $\mathcal{I}_j \subset \{x : r(x) \in [r_j, r_{j+1}]\}$ is the set bounded by the union of the segments $[p_j^+, q_{j+1}^+], [p_j^-, q_{j+1}^-]$ and of the arcs $\text{arc}(p_j^+, p_j^+), \text{arc}(q_j^-, q_{j+1}^-)$, see Figure 5. From (6.26) and (6.28) we see that we can also assume

$$\tilde{\psi}_j \leq \tilde{\psi}_0, \quad j = 1, \ldots, n,$$

where $\tilde{\psi}_j$ is the angle between $[x_j, x_{j+1}]$ and the ray $L_{\theta(x_j)}$ and $\tilde{\psi}_0$ a constant independent of $n$ and $1 \leq j \leq n$. By neglecting the first $j_0$ terms of the sequences defined in (6.15) and by renumbering via $j = i + j_0$ we can assume that (6.29) and (6.30) hold and $\mathcal{I}_j$ is well defined for $j = 1, \ldots, n$.

Lemma 6.7 (i) does not apply for $j = n + 1$ and Lemma 6.7 (ii) does not apply for $j = n$. This together with the fact that the length of $\text{arc}(p_n^{+1}, p_n^{-1})$ diverges to $+\infty$ with $n$ can cause a significant error in the derivation of a lower bound for the energy of $u^R_n$ in the the diffuse interface that we define later. Therefore we need a new definition for the set $\mathcal{I}_n$.

**Lemma 6.8.** Assume that $n \in \mathbb{N}$ is sufficiently large. Let $\ell_{r_{n+1}}^\pm, \tilde{p}_{n+1}^\pm = \tilde{p}_{n+1}^\pm$, be as in Definition 6.3 with $r_* = r_{n+1}$. Then

(i) $[p_n^\pm, q_{n+1}^\pm] \cap \ell_{r_{n+1}}^\pm = \{q^\pm\}$ for some $q^\pm$ and there exist $\eta \in (0, 1), k_1 \in (0, 1), i = 1, 2$, independent of $n$ such that

$$k_1 \lambda_n \leq r_{n+1} - r(q^\pm) \leq k_2 (r_{n+1} - r_n).$$

(ii) $x \in [p_n^\pm, q_n^\pm] \Rightarrow |u^{R_n} - a_\pm| \leq c\delta^\alpha$ on $B_{\eta \lambda_n}(x) \setminus \Sigma$.

(iii) $x \in [q^+, \tilde{p}_{n+1}^+]$ and $r_{n+1} - r(x) = \rho \Rightarrow |u^{R_n} - a_\pm| \leq c\delta^\alpha$ on $B_{\xi}(x) \setminus \Sigma$.

**Proof.** Let $\varphi_{n+1} = \varphi_\epsilon$ with $\varphi_\epsilon$ defined as in Remark 6.5 for $r_\epsilon = r_{n+1}$, $\varphi_{n+1}$ is the angle between the two tangents to $\mathcal{I}_{r_{n+1}}$ at $\tilde{p}_{n+1}^\pm$. From (6.13) $\varphi = \lim_{n \to +\infty} \varphi_{n+1}$ has

$$\sin \varphi = \frac{1}{\sqrt{1+\rho^2}}.$$ 

Note that this and (6.28) imply $\varphi = \frac{\pi}{2} - \psi, \psi = \lim_{n \to +\infty} \psi_n$. Note also that (6.26) and $\text{arc}(\tilde{p}_{n+1}^+, \tilde{p}_{n+1}^-) \subset \text{arc}(p_{n+1}^+, p_{n+1}^-)$ imply $\lim_{n \to +\infty} |\tilde{p}_{n+1}^+ - \tilde{p}_{n+1}^-| = 0$.

These observations yield that, for large $n, p_n^\pm \simeq \tilde{p}_n^\pm, q_{n+1}^\pm$ and $q_{n+1}^\pm$ are approximately the vertices of an isosceles triangle $T_n$ with basis $2c(r_{n+1} - r_n)$ and height $r_{n+1} - r_n$ as indicated in Figure 6. In the same approximation $\tilde{p}_{n+1}^+ \simeq \tilde{p}_{n+1}^-$ lies on the basis of $T_n$ and $\ell_{r_{n+1}}^+$ forms an angle of size $\frac{\pi}{2}$ (i.e., $\varphi = \frac{\pi}{2} - \psi$) with the basis of $T_n$. It follows that the points $q^\pm$ claimed in (i) exist and satisfy the right inequality in (6.31) for some $k_2 \in (0, 1)$ independent of $n$.

To complete the proof of (i) we note that $\tilde{p}_{n+1}^+ = x(r_{n+1}, \tilde{\varphi}_{r_n}(r_{n+1}))$ is the extreme position allowed to $\tilde{p}_{n+1}^+$ on $\text{arc}(q_{n+1}^+, q_{n+1}^-)$. Similarly we define $\tilde{p}_{n+1}^-$, the extreme possible
Moreover, the position of \( \hat{p}_{n+1}^- \) on \( \text{arc}(q_{n+1}^-, q_{n+1}^+) \). This follows from Lemma 6.1 with \( r_* = r_n \). The left inequality (6.31) is determined by the limit position \( \hat{q}^\pm \) assumed by \( q^\pm \) when \( \hat{p}_{n+1}^\pm = \hat{p}^\pm \). For \( n \) large \( \hat{q}^\pm, \hat{p}_{n+1}^\pm = \hat{p}^\pm \) and \( q_{n+1}^\pm \) are the vertices of a triangle which, as illustrated in Figure 6, has \( |q_{n+1}^+ - \hat{p}^\pm| \simeq \lambda_{n+1} \) and the angles in \( q_{n+1}^+ \) and in \( \hat{p}^\pm \) approximately equal to \( \varphi \) and \( \frac{\alpha}{2} \) respectively. A similar argument applies to \( \hat{q}^-, \hat{p}_{n+1}^- \) and \( q_{n+1}^- \). The lower bound for \( r_{n+1} - r(q^\pm) \) is a consequence of the geometric properties of the above triangle.

From the proof of (ii) in Lemma 6.7 we have that \( [p_n^-, q_{n+1}^+] \) is contained in \( \mathcal{J}_{r_n, \beta} \). Moreover \( x \in [p_n^-, q_{n+1}^+] \) implies \( B_{p_n, \lambda_n}(x) \subset \mathcal{J}_{r_n, \beta} \) provided \( r_{n+1} - r(x) \geq \eta \lambda_n \). From (6.31) it follows that, by reducing the value of \( \eta \) if necessary, we can make sure that this condition is satisfied for every \( x \in [p_n^-, q_{n+1}^+] \). This proves the first part of (ii). The proof of the second part is similar.

Statement (iii) is a plain consequence of part (ii) of Lemma 6.4 with \( r_* = r_{n+1} \). The proof is complete.

Figure 6: The triangle \( T_n \) and the points \( q^\pm \). The triangle \( \hat{q}^\pm, \hat{p}^\pm, q_{n+1}^\pm \).

We are now in the position to give a suitable definition of the set \( \mathcal{J} \). We define \( \mathcal{J} \) to be the subset of \( \{ x : r(x) \in [r_n, r_{n+1}] \} \) bounded by the union of \( [p_n^-, q^-], [p_n^+, q^+], [q^-, \hat{p}_{n+1}^-], [q^+, \hat{p}_{n+1}^+], \text{arc}(p_n^-, p_n^+) \) and \( \text{arc}(\hat{p}_{n+1}^-, \hat{p}_{n+1}^+) \).

Define (see Figure 7)

\[
\mathcal{J} = \bigcup_{j=1}^{n} \mathcal{J}_j.
\]

The set \( \mathcal{J} \) is a kind of diffuse interface that separate regions where \( u^{R_n} \) is near to \( a \) or to \( \omega^{-1}a \). Indeed let \( \mathcal{R} \) be defined by

\[
B_{r_{n+1}} \setminus B_{r_1} \cup_{j=1}^{n} \omega^{-1} \mathcal{J} = \bigcup_{j=1}^{n} \omega^{-1} \mathcal{R}.
\]

Then

\[
x \in \omega^{-1} \mathcal{R} \quad \Rightarrow \quad |u^{R_n}(x) - \omega^{-1}a| \leq c \delta^\alpha, \quad r(x) \not\in \Sigma, \quad j = 1, \ldots, N - 1.
\]

\[
(6.34)
\]

6.3 An upper bound for the length of \( \mathcal{J} \).

From what we know up to now the structure of \( \mathcal{J} \) can be quite complex, for example we can not exclude that \( \mathcal{J} \) revolves several times around \( B_{r_1} \). We will show that, instead, the shape of \( \mathcal{J} \) can be controlled. We will associate to \( \mathcal{J} \) a kind of length and show that most of the energy of \( u^{R_n} \) is contained in \( \mathcal{J} \) (and in its images under \( \omega \)) and proportional
to its length. Then from the upper bound \(3.16\) we obtain that the the difference from the length of \(\mathcal{J}\) and \(R_n\) is bounded by a constant independent of \(n\). This implies a strong restriction on the geometry of \(\mathcal{J}\) and eventually leads to Theorem \([1.1]\).

If \((u, v), u, v \in \mathbb{R}^2\), is an ordered pair of linearly independent vectors we say that \((u, v)\) is positive if the rotation from \(u\) to \(v\) through an angle \(< \pi\) is counterclockwise, negative otherwise.

Let \(\Gamma\) be the family of rectifiable curves that connect \(x(r_2, \theta_{r_2})\) to \(x(r_{n+1}, \theta_{r_{n+1}})\) and are contained in \(\mathcal{J}\). We let \(|\gamma|\) the length of \(\gamma \in \Gamma\).

**Proposition 6.9.** (i) There exists \(\gamma^m \in \Gamma\) and \(K > 0\) independent of \(n\) such that

\[(6.35) \quad |\gamma^m| = \min_{\gamma \in \Gamma} |\gamma|, \]

and

\[(6.36) \quad \frac{ds}{dr} \leq K, \quad r \in [r_2, r_{n+1}], \]

where \(s: [0, |\gamma^m|] \rightarrow \mathbb{R}\) is the curvilinear abscissa along \(\gamma^m\).

Moreover

\[(6.37) \quad \gamma^m([s_j, s_{j+1}]) = [\gamma^m_j, \gamma^m_{j+1}], \quad j = 2, \ldots, n\]

where \(s_j\) and \(\gamma^m\) are defined by \(\gamma^m_j = \gamma^m(s_j) \in \text{arc}(p^-_j, p^+_j)\).

(ii) Set \(\tau_j = \frac{\gamma^m_j - \gamma^m_{j+1}}{|\gamma^m_j - \gamma^m_{j+1}|}, \quad j = 2, \ldots, n\). Then

\[\tau_{j-1} \neq \tau_j \Rightarrow \gamma^m_j \in \{p^-_j, p^+_j\}\]

and

\[\gamma^m_j = p^-_j \Rightarrow (\tau_{j-1}, \tau_j) \text{ is negative,}\]
\[\gamma^m_j = p^+_j \Rightarrow (\tau_{j-1}, \tau_j) \text{ is positive.}\]

**Proof.** 1. Given \(\gamma \in \Gamma\) set \(s_2 = 0, s_{n+1} = |\gamma|\) (s the curvilinear abscissa along \(\gamma\)). For \(j = 3, \ldots, n\) there exists \(s_j \in (0, |\gamma|)\) such that \(\gamma(s_j) \in \text{arc}(p^-_j, p^+_j)\). We can assume that \(s_j\) is chosen so that \(s_j < s_{j+1}, j = 2, \ldots, n\). From \((6.29)\) we have

\[\gamma(s_j), \gamma(s_{j+1}) \subseteq \mathcal{J}, \quad j = 2, \ldots, n.\]

Therefore the curve \(\hat{\gamma} = \bigcup_{j=2}^n \gamma(s_j), \gamma(s_{j+1})\) belongs to \(\Gamma\) and

\[(6.38) \quad |\hat{\gamma}| \leq |\gamma|, \quad \gamma \in \Gamma.\]

The length \(|\hat{\gamma}|\) of \(\hat{\gamma}\) is a continuous function of the \(n - 2\) points \(\gamma(s_j) \in \text{arc}(p^-_j, p^+_j), \quad j = 3, \ldots, n - 1\). This implies the existence of \(\gamma^m \in \Gamma\) that satisfies \((6.35)\). The bound \((6.36)\) follows from \((6.30)\) with \(K = \frac{1}{\cos \psi_0}\). This completes the proof of (i). To prove (ii) we observe that, if \(\gamma^m_j \notin \{p^-_j, p^+_j\}\) and \(\tau_{j-1} \neq \tau_j\), there exist \(x, y \in [\gamma^m_j, \gamma^m_{j+1}]\) and \(y \in [\gamma^m_j, \gamma^m_{j+1}]\) such that \([x, y] \subseteq \mathcal{J}\). This contradicts the minimality of \(\gamma^m\) since \(|y - x| < |\gamma^m_j - x| + |y - \gamma^m_j|\). This contradiction proves that \(\gamma^m_j \notin \{p^-_j, p^+_j\}\) implies \(\tau_{j-1} = \tau_j\). The same argument applies to the case \(\gamma^m_j \in \{p^-_j, p^+_j\}\). The proof is complete. \(\square\)
Let $v_j \in \mathbb{R}^2$, $j = 2, \ldots, n$ a unit vector orthogonal to $\tau_j$ and such that $(\tau_j, v_j)$ is positive. Given $r \in [r_j, r_{j+1}]$ let $N_{j,r} = \{x = \gamma^m(r) + tv_j : t \in \mathbb{R}\}$ be the line through $\gamma^m(r)$ orthogonal to $\tau_j$. For $r \neq r_j$, $j = 3, \ldots, n$ and $j$ such that $r \in (r_j, r_{j+1})$ let $[\xi^-_r, \xi^+_r]$ be the closure of the connected component of $N_{j,r} \cap \mathcal{I}$ that contains $\gamma^m(r)$. $\mathcal{E}$ denotes the interior of $E$. If $r = r_j$ and $\tau_{j-1} = \tau_j$ then $N_{j-1,r_j} = N_{j,r_j}$ and we define $[\xi^-_{r_j}, \xi^+_{r_j}]$ as before. If instead, $\tau_{j-1} \neq \tau_j$, we denote $[\xi^-_{r_j}, \xi^-_{r_j}]$ and $[\xi^+_{r_j}, \xi^+_{r_j}]$ the two segments that the previous definition yields with respect to $N_{j-1,r_j}$ and $N_{j,r_j}$, respectively. From Proposition 6.9 (ii) it follows that $r = r_j$ and $\tau_{j-1} \neq \tau_j$ imply that one of the following alternatives holds

a) $\gamma^m_j = p^-_j - \xi^-_j$ and $(\xi^+_{r_j} - \xi^-_{r_j}, \xi^+_{r_j} - \xi^-_{r_j})$ is negative,

b) $\gamma^m_j = p^+_j - \xi^+_{r_j}$ and $(\xi^-_{r_j} - \xi^-_{r_j}, \xi^-_{r_j} - \xi^+_{r_j})$ is positive.

A natural consequence of a) and b) is Proposition 6.10 (ii) below that we prove in detail in Section 6.5.

Define

$$\partial \mathcal{I}_j^\pm = [p^\pm_j, q^\pm_{j+1}] \cup \arcc [q^\pm_{j+1}, p^\pm_{j+1}], \quad j = 1, \ldots, n-1,$$

$$\partial \mathcal{I}_n^\pm = [p^\pm_n, q^\pm_n] \cup \arcc [q^\pm_n, p^\pm_{n+1}].$$

For $j = 1, \ldots, n-1, \partial \mathcal{I}_j^-$ and $\partial \mathcal{I}_j^+$ are the connected components of $\partial \mathcal{I}_j \cup \arcc [p^-_{j+1}, p^+_{j+1}]$. $\partial \mathcal{I}_n^-$ and $\partial \mathcal{I}_n^+$ are the connected components of $\partial \mathcal{I}_n \setminus (\arcc [p^-_n, p^+_n] \cup \arcc [p^-_{n+1}, p^+_n])$.

**Proposition 6.10.** It results

$$\xi^\pm_r \in \bigcup_{i=j-1}^{n+1} \partial \mathcal{I}_i^\pm, \quad r \in (r_j, r_{j+1}), \quad j = 2, \ldots, n-1,$$

and

$$[\xi^-_r, \xi^+_r] \cap [\xi^-_{r'}, \xi^+_r] = \emptyset, \quad r \neq r' \in [r_2, r_{n+1}] \setminus \{r_j\}_{j=2}^{n+1}.$$
Proof. See Section 6.5.

Proposition 6.11. Set $a_- = \omega^{-1}a$, $a_+ = a$. Then

(i) $$r \in (r_j, r_{j+1}), \ j = 2, \ldots, n - 1,$$
$$\Rightarrow$$
$$|u_{Rn}^r - a_\pm| \leq c\delta^\alpha, \ x \in B_{\eta\lambda_{j-1}}(\xi_{r_j}^\pm) \setminus \tilde{\Sigma}. \ (6.41)$$

(ii) There exists $\bar{\rho} > 0$, $\bar{\eta} \in (0, 1)$, $0 < k < k'$ independent of $n$ and $\bar{r}$ such that $r_{n+1} - k\lambda_{n-1} \leq \bar{r} \leq r_{n+1} - k\lambda_{n-1}$ and
$$r \in (r_n, \bar{r}) \Rightarrow$$
$$|u_{Rn}^r - a_\pm| \leq c\delta^\alpha, \ x \in B_{\eta\lambda_{n-1}}(\xi_{r}^\pm) \setminus \tilde{\Sigma}, \ (6.42)$$

Proof. See Section 6.5.

We are now in the position to derive a sharp upper bound for the length $|\gamma^m|$ of $\gamma^m$. For each $r \in (r_2, r_{n+1} - \bar{\rho}) \setminus \{r_j\}_{j=3}^n$ let $J_\gamma(r)$ be the one dimensional energy of the restriction of $u_{Rn}^r$ to the segment $[\xi^r_-, \xi^r_+]$. From Proposition 6.10 and the fact that, for $r \in (r_j, r_{j+1})$, $[\xi^r_-, \xi^r_+]$ remains orthogonal to $[\gamma^m_j, \gamma^m_{j+1}]$, it follows via Lemma 2.3

$$J_{B_{Rn}}(u_{Rn}^r) \geq NJ_\gamma(u_{Rn}^r) \geq N \int_0^{|\gamma^m|} J_\gamma(r(s)) ds, \ (6.43)$$

where $s \in (0, |\gamma^m|)$ is the curvilinear abscissa along $\gamma^m$ and $s \to r(s)$ the inverse of $r \to s(r)$ which exists by (6.36). Set

$$\tilde{c} = \frac{1}{k\eta}, \ (6.44)$$

where $\tilde{c}$ is the constant in (6.14) and $\eta$ is defined in Proposition 6.11. We assume

$$\tilde{c}\ln r_1 = \lambda_1 \geq \frac{i}{\eta}, \ (6.45)$$

which is equivalent to the assumption that the constant $c_1$ in (6.14) be sufficiently large. From (6.44), (6.45), Corollary 5.7 and Proposition 6.11 we obtain

$$|u_{Rn}^r(\xi_{r_j}^\pm) - a_\pm| \leq 2\delta e^{-\tilde{k}(\eta\lambda_{j-1}-i)} \left( 2\delta e^{kJ} e^{-k\eta\ln r_{j-1}} \right) \frac{2\delta e^{kJ}}{r_{j-1}}, \ r \in (r_j, r_{j+1}), \ j = 2, \ldots, n - 1, \ (6.46)$$

$$|u_{Rn}^r(\xi_{r}^\pm) - a_\pm| \leq \frac{2\delta e^{kJ}}{r_{n-1}}, \ r \in (r_n, \bar{r}). \ (6.47)$$
Instead, in the interval \((\tilde{r}, r_{n+1} - \bar{\rho})\), assuming also that \(\bar{\rho} \geq \frac{1}{7}\), we obtain

\[ |u^{R_n}(C_{\pm}^* \setminus a_{\pm})| \leq 2\delta e^{\frac{\beta}{2}} e^{-\frac{\delta}{2}(r_{n+1} - r)}, \quad r \in (\tilde{r}, r_{n+1} - \bar{\rho}). \]  

From \((6.46)\) and Lemma \(2.2\) it follows

\[ J^*(r) \geq \sigma - C_W \frac{4\delta^2 e^{2\frac{\beta}{2}}}{r_j^2}, \quad r \in (r_j, r_{j+1}), \]  

\[ J^*(r) \geq \sigma - C_W \frac{4\delta^2 e^{2\frac{\beta}{2}}}{r_{n-1}}, \quad r \in (r_n, \tilde{r}), \]  

and

\[ J^*(r) \geq \sigma - C_W 4\delta^2 e^{2\frac{\beta}{2}} e^{-\frac{\delta}{2}(r_{n+1} - r)} e^{-\frac{\delta}{2}(\tilde{r} - r)}, \quad r \in (\tilde{r}, r_{n+1} - \bar{\rho}). \]  

Recall that from Proposition \((6.9)\) and Lemma \((6.6)\) we have \(\frac{\partial s}{\partial t} \leq K, \frac{r_j}{r_{j-1}} \leq (1 + \frac{\beta}{2})\) and \(r_{j+1} - r_j \leq C_0 r_j^\frac{1}{2}\). This and \((6.48)\) imply

\[ \frac{1}{r_{j-1}} \int_{s(r_j)}^{s(r_{j+1})} \frac{ds}{dr} dr \leq K \frac{r_{j+1} - r_j}{r_j^2}, \]  

\[ \frac{1}{r_{n-1}} \int_{s(r_j)}^{s(\tilde{r})} \frac{ds}{dr} dr \leq K \frac{1 + \frac{\beta}{2}}{2} C_0 \frac{r_n^2}{r_j^2}. \]  

which, via \((6.48)\), yields

\[ \int_{s(r_j)}^{s(r_{j+1})} J^*(r(s)) ds \geq \sigma (s(r_{j+1}) - s(r_j)) - \frac{C^*_j}{r_j^2}, \quad j = 2, \ldots, n-1, \]  

\[ \int_{s(r_j)}^{s(\tilde{r})} J^*(r(s)) ds \geq \sigma (s(\tilde{r}) - s(r_n)) - \frac{C^*_n}{r_n^2}. \]  

where we have set \(C^*_j = 4\delta^2 C_W K C^0 (1 + \frac{\beta}{2})^2 e^{2\frac{\beta}{2}}\). Finally, in a similar way, from \((6.49)\) we get

\[ \int_{s(\tilde{r})}^{s(r_{n+1} - \bar{\rho})} J^*(r(s)) ds \geq \sigma (s(r_{n+1} - \bar{\rho}) - s(\tilde{r})) - \frac{4\delta^2 C_W K e^{2\frac{\beta}{2}}}{2k\eta}, \]  

\[ \geq \sigma (s(r_{n+1}) - s(\tilde{r})) - K\bar{\rho} - \frac{4\delta^2 C_W K e^{2\frac{\beta}{2}}}{2k\eta}. \]  

By adding this estimate with the estimates \((6.50)\) from \(j = 1\) to \(j = n\) we obtain

\[ \int_0^{r_{n+1} - \bar{\rho}} J^*(r(s)) ds \geq \int_{s(r_{j+1})}^{s(r_{n+1} - \bar{\rho})} J^*(r(s)) ds \]  

\[ \geq \sigma (s(r_{n+1}) - s(r_{j+1})) - C^*_1 = \sigma |\gamma^m| - C^*_1, \]  

where \(C^*_1 = C^*_\infty \sum_{j=1}^\infty \frac{1}{r_j^2} + K\bar{\rho} + \frac{4\delta^2 C_W K e^{2\frac{\beta}{2}}}{2k\eta}\) is a constant independent of \(n\). From \((6.52)\), \((6.43)\) and \((3.16)\) we get

\[ |\gamma^m| \leq R_n + \frac{C^*_1 + C^*_2}{\sigma} \leq r_{n+1} + C^*_2, \quad C^*_2 = |\Sigma| + \frac{C^*_1 + C^*_2}{\sigma}. \]  

This is the announced upper bound for the length of the interface.
6.4 Existence of $C_N$-equivariant $N$-junctions

We can now complete the proof of Theorem 1.1.

1. We have

\[
\begin{align*}
|\gamma^m| & \geq |\gamma^m(s(r)) - \gamma^m_2| + |\gamma^m_{n+1} - \gamma^m(s(r))| , \\
|\gamma^m(s(r)) - \gamma^m_2| & \geq r - r_2, \\
|\gamma^m_{n+1} - \gamma^m(s(r))| & = \left( (r_{n+1} - r \cos \vartheta)^2 + r^2 \sin^2 \vartheta \right)^{\frac{1}{2}} , \\
& = \left( (r_{n+1} - r)^2 + 4r r_{n+1} \sin^2 \frac{\vartheta}{2} \right)^{\frac{1}{2}},
\end{align*}
\]

where

\[
\begin{align*}
\vartheta = \theta(\gamma^m(s(r))) - \theta(\gamma^m_{n+1}).
\end{align*}
\]

From (6.54) and (6.53), after some manipulation, we get

\[
4 \sin^2 \frac{\vartheta}{2} \leq 2C^*_3 \frac{r_{n+1} - r}{rr_{n+1}} + \frac{C^*_3}{r} \left( 2 + \frac{C^*_3}{r_{n+1}} \right), \quad C^*_3 = C^*_2 + r_2.
\]

It follows $4 \sin^2 \frac{\vartheta}{2} \leq \frac{4C^*_3}{r}$ for $n$ sufficiently large. From this we conclude

\[
\begin{align*}
\vartheta & \leq \frac{C^*_3}{r}, \quad r \in [\hat{r}, r_{n+1}], \quad n \geq \bar{n},
\end{align*}
\]

where $C^*_3 > 0$ and $\hat{r} \geq r_2$ are constants independent of $n \geq \bar{n}$, for some $\bar{n}$.

2. The estimate (6.56) gives some control of the shape of $\gamma^m$, the spine of the diffuse interface $I$, and allows to show that $I$ lie in a neighborhood of the ray $L_{\theta(\gamma^m_{n+1})}$ in the sense that

\[
\begin{align*}
& I \subset B_{\hat{r}} \cup D, \\
& D = \{ x(r, \theta) : |\theta - \theta(\gamma^m_{n+1})| \leq \frac{\hat{C}}{r^2}, \quad r \in (\hat{r}, r_{n+1}), \quad \hat{r} = \frac{(\hat{C}N)^2}{\pi^2}, \}
\end{align*}
\]

for some constants $\hat{C} > 0$ and $r_0 \geq \hat{r}$ independent of $n \geq \bar{n}$. The condition $r \geq \frac{(\hat{C}N)^2}{\pi^2}$ in (6.57) ensures that

\[
\omega^j D \cap D = \emptyset, \quad j = 1, \ldots, N - 1.
\]

To prove (6.57) we estimate the thickness of $I$. For $r \in [r_j, r_{j+1}]$ let $x^\pm \in I_j \cap C_r$. Then the definition of $I_j$ in the proof of Lemma 6.7 and (6.27) imply

\[
\begin{align*}
\max_{x^\pm} \frac{d^\pm(x_-, x^+)}{r} & \leq \frac{d^\pm(q_{j+1}^-, q_{j+1}^+)}{r_j} \leq 3\hat{c} \frac{r_{j+1} - r_j}{r_j} , \\
& \leq 3\hat{c} C^0 \frac{r_j}{r_j^2} \leq C^*_5 \frac{r_j}{r_j^2}, \quad r \in (r_j, r_{j+1}], \quad j = 1, \ldots, n,
\end{align*}
\]

where we have also used (6.16) and $\frac{r_j}{r_j^2} \leq 1 + \frac{\hat{\beta}}{2}$ and set $C^*_5 = 3\hat{c} C^0 \left( 1 + \frac{\hat{\beta}}{2} \right)^{\frac{3}{2}}$. This and (6.56) imply (6.57) with $\hat{C} = C^*_4 + C^*_5$. 

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3. If \( u^{R_n} \) is a minimizer of \( J_{B_{R_n}} \) the map \( \rho u^{R_n}(\rho) \) is also a minimizer for each rotation \( \rho : \mathbb{R}^2 \to \mathbb{R}^2 \). Therefore we can assume that \( \theta(\gamma^{m}_{j+1}) = 0 \), for \( n \geq \bar{n} \) and define

\[
Q_n = \{ x(r, \theta) : \theta \in \left( \frac{\hat{C}}{2 \pi}, \frac{2 \pi - \hat{C}}{2}, \frac{\hat{C}}{2 \pi}, 2 \pi - \frac{\hat{C}}{2} \right), r \in (\hat{r}, r_{n+1}), \hat{r} = \frac{(\hat{C} N)^2}{\pi^2} \},
\]

Since \( Q_n \subset \mathcal{R} \), (6.34) implies

\[
|u^{R_n}(x) - \omega^j - 1 a| \leq c\delta^\alpha, \quad x \in \omega^j - 1 Q_n \setminus \bar{\Sigma}, \quad j = 1, \ldots, N - 1.
\]

This and Corollary 5.7. yield

\[
|u^{R_n}(x) - \omega^j - 1 a| \leq \tilde{K} e^{-kd(x, \partial \omega^j - 1 Q_n)}, \quad x \in \omega^j - 1 Q_n, \quad n \geq \bar{n},
\]

where \( \tilde{k} \) is as in Corollary 5.7 and \( \tilde{K} > 0 \) some constant independent of \( n \).

4. The family of minimizers \( \{ u^{R_n} \} \) is uniformly bounded in \( C^{2+\alpha}(B_{R_n}; \mathbb{R}) \), for some \( \alpha \in (0, 1) \). It follows the existence of a subsequence still denoted \( \{ u^{R_n} \} \) that converges in compact in the \( C^2 \) sense to a map \( U : \mathbb{R}^2 \to \mathbb{R}^2 \) which is a solution of (1.1). \( U \) is \( C_N \)-equivariant and, since the estimate (6.61) passes to the limit for \( n \to +\infty \), satisfies (1.11). The proof is complete.

### 6.5 Appendix

**Proof.** (of Proposition 6.10)

1. We have

\[
N_{j, r_j} \cap \text{arc}[p_{j+1}^-, p_{j+1}^+] = \emptyset, \quad j = 1, \ldots, n.
\]

\[
N_{j, r_{j+1}} \cap \text{arc}[p_{j}^-, p_{j}^+] = \emptyset,
\]

If \( N_{j, r_{j+1}} \cap C_{r_j} = \emptyset \), 6.62 is trivially true. Assume instead that there is \( \xi \in N_{j, r_{j+1}} \cap C_{r_j} \). Then \( \gamma^{m}_{j}, \gamma^{m}_{j+1} \) and \( \xi \) are the vertices of a triangle rectangle in \( \gamma^{m}_{j+1} \) and it follows

\[
d^\rho(\gamma^{m}_{j}, \xi) \geq |\gamma^{m}_{j} - \xi| \geq |\gamma^{m}_{j+1} - \gamma^{m}_{j}| \geq |r_{j+1} - r_{j}|.
\]

Since \( d^\rho(\gamma^{m}_{j}, p_{j}^+_{j}) \leq d^\rho(\gamma^{m}_{j}, p_{j}^+_{j}) \) and (6.17) yields

\[
|r_{j+1} - r_{j}| > d^\rho(p_{j}^-, p_{j}^+) = 3 \nu + 2 \lambda_j, \quad j = 1, \ldots, n,
\]

(6.62) follows from (6.63). The same argument applies to (6.62)₁.

2. It results

\[
[\xi^-_{j, r_j}, \xi^-_{j, r_j}] \cap (\text{arc}[p_{j-1}^-, p_{j-1}^+] \cup \text{arc}[p_{j+1}^-, p_{j+1}^+]) = \emptyset
\]

Since \( \{ \xi^+_{j, r_j}, \xi^-_{j, r_j} \} \subset N_{j-1, r_j} \) and \( \{ \xi^+_{j, r_j}, \xi^-_{j, r_j} \} \subset N_{j+1, r_j} \) (6.62) implies

\[
[\xi^+_{j, r_j}, \xi^-_{j, r_j}] \cap \text{arc}[p_{j-1}^-, p_{j-1}^+] = \emptyset,
\]

\[
[\xi^+_{j, r_j}, \xi^-_{j, r_j}] \cap \text{arc}[p_{j+1}^-, p_{j+1}^+] = \emptyset.
\]
If $\tau_{j+1} = \tau_j$ we have $[\xi^+, \xi^-] = [\xi_{j+1}^-, \xi_{j+1}^-]$ and (6.64) follows from (6.65). If $\tau_{j+1} \neq \tau_j$ both in case a) and b) it results that $[\xi^+, \xi^-]$ lies $([\xi^+, \xi^-]$ lies) on the half plane determined by $N_{j+1}, r_j$ (by $N_{j-1}, r_j$) that does not contain arc $[p_{j+1}, p_{j+1}^+]$ (arc $[p_{j-1}, p_{j-1}^+]$). This and (6.65) imply (6.64).

3. From 2. we have that $\xi_{j+1}^-$ and $\xi_{j+1}^-$ ($\xi_{j+1}^+$( and $\xi_{j+1}^+$)) are the extreme of a subarc $\mathcal{C}_{j+1}$ $\cup_{i=j+1}^{j+1} \partial \mathcal{S}_i$ (of a subarc $\mathcal{C}_{j+1}^+ \subset \bigcup_{i=j+1}^{j+1} \partial \mathcal{S}_i^+$). This, since, by definition $[\xi_r^+, \xi_r^-]$, $[\xi_{r_{j+1}}^+, \xi_{r_{j+1}}^-]$ and $[\xi_{r_{j+1}}^+, \xi_{r_{j+1}}^-]$ have the same direction, implies

$$\xi_r^+ \in \mathcal{C}_{j+1} \subset \bigcup_{i=j+1}^{j+1} \partial \mathcal{S}_i^+, \ r \in (r_j, r_{j+1}), \ j = 2, \ldots, n - 1,$$

that concludes the proof of (6.39).

4. We have

$$(6.66) \quad [\xi_r^+, \xi_r^-] \cap [\xi_r^+, \xi_r^-] = \emptyset, \ r, r' \in (r_j, r_{j+2}) \setminus \{r_{j+1}\}, \ j = 2, \ldots, n - 1.$$  

This is obvious if $\tau_{j+1} = \tau_j$. If $\tau_{j+1} \neq \tau_j$ and a) holds, (6.66) follows from the fact that $\mathcal{C}_{j+1}^-$ and $\mathcal{C}_{j+1}^-$ have the extreme $\gamma_{j+1}^m = p_{j+1}^+ = \xi_{j+1}^-$ in common. The same argument applies if b) holds.

5. Assume that there are $r \in (r_j, r_{j+1})$, $r' \in (r_j, r_{j+1})$ and $\xi$ such that

$$(6.67) \quad \{\xi\} = [\xi_r^+, \xi_r^-] \cap [\xi_r^+, \xi_r^-].$$

Without loss of generality we can assume that $j \geq i$. From (6.39) it follows $j \leq i + 2$. On the other hand 4. implies $j > i + 1$ and we conclude that $j = i + 2$. This and (6.39) imply

$$(6.68) \quad r(\xi) \in [r_{j-1}, r_j].$$

From (6.67) it follows that $\tau_{j-2} \neq \tau_j$ and therefore that at least one of the following two possibilities holds:

$$\tau_j \neq \tau_{j-1}, \quad \tau_{j-1} \neq \tau_{j-2}.$$  

We discuss the case $(\tau_{j-1}, \tau_j)$ negative. The analysis of the other possibilities is analogous. We have

$$(\tau_{j-1}, \tau_j) \text{ negative} \Rightarrow \gamma_{j-1}^m = p_{j-1}^- = \xi_{j-1}^-.$$  

This and $r' > r_j$ imply $r(\xi_{r'}) > r_j$ which, since $\xi \in [\xi_{r'}^+, \xi_{r'}^-]$, is in contradiction with (6.68) and therefore with the existence of $\xi$. The proof is complete.

\textbf{Proof.} (of Proposition 6.11)

1. From (6.39) and Lemma 6.7 it follows that if $j = 2, \ldots, n - 2$, for each $x \in \bigcup_{i=j-1}^{j+1} \partial \mathcal{S}_i^+$ and, in particular for $\xi_r^+$, it results

$$|u^{R_n} - a_\pm| \leq c \delta^\alpha, \ \text{on } B_{\eta \lambda_{j-1} \bigwedge \tilde{\Sigma}}.$$  

This concludes the proof of (i) for $j = 2, \ldots, n - 2$.  

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2. For $r$ near $r_{n+1}$, one or both the extreme of $[\xi_r^- , \xi_r^+]$ may lie on $\text{arc}[\hat{p}_{n+1}^-, \hat{p}_{n+1}^+]$. Since $|\hat{p}_{n+1}^- - \hat{p}_{n+1}^+| \leq 3\nu$ A sufficient condition to exclude this is $r \leq r_{n+1} - \hat{\rho}$, $\hat{\rho} = 3\nu$.

3. Assume that $\rho = \rho(x) = r_{n+1} - r(x)$ satisfies

\[(6.69) \quad \rho \geq 4\eta \lambda_{n-1}.
\]

Then we have $B_{\eta \lambda_{n-1}}(x) \subset B_{\hat{\tau}}(x)$ and from Lemma 6.8 it follows

\[(6.70) \quad x \in \bigcup_{j=n-1,n} \partial \mathcal{S}_j^\pm \quad \text{and} \quad \rho \geq 4\eta \lambda_{n-1}
\]

\[\Rightarrow \quad |u^{R_n} - a_\pm| \leq c\delta^\alpha, \quad \text{on} \quad B_{\eta \lambda_{n-1}}(x) \setminus \hat{\Sigma}.
\]

Fix $\bar{\eta} = \frac{1}{4}$. Then

\[\rho < 4\eta \lambda_{n-1},\]

implies $B_{\bar{\eta} \rho}(x) \subset B_{\eta \lambda_{n-1}}(x)$. From this, $B_{\bar{\eta} \rho}(x) \subset B_{\hat{\tau}}(x)$ and Lemma 6.8 we have

\[(6.71) \quad x \in \bigcup_{j=n-1,n} \partial \mathcal{S}_j^\pm \quad \text{and} \quad \rho < 4\eta \lambda_{n-1}
\]

\[\Rightarrow \quad |u^{R_n} - a_\pm| \leq c\delta^\alpha, \quad \text{on} \quad B_{\bar{\eta} \rho}(x) \setminus \hat{\Sigma}.
\]

4. Let $w_n$ the direction vector of the ray $L_{\theta(\gamma^m_{n+1})}$ through $\gamma^m_{n+1}$ and let $\chi_n$ the angle between $w_n$ and $\tau_n$ positive if $(w_n, \tau_n)$ is positive. We assume $\chi_n \geq 0$. The same argument with obvious modifications applies to the case $\chi_n < 0$. Recall the definition of $\xi_+^{n+1}$ in Lemma 6.8 and define $\hat{\xi}$ by setting

\[\hat{\xi}^- \in \xi_-^{n+1},
\]

\[r(\hat{\xi}^-) = r_{n+1} - 4\eta \lambda_{n-1}.
\]

Note that $\hat{\xi}^-$ satisfies (6.69) with the equality sign. We define $\hat{r}$ as the value of $r$ such that $\hat{\xi}^- = \xi_r^-$, that is we let $\hat{r}$ be determined by the condition that $\gamma^m(\hat{r})$ coincides with the intersection of $\{x = \gamma^m_{n+1} + t\tau_n, \ t \in \mathbb{R}\}$ with $\{x = \xi^- + tv_n, \ t \in \mathbb{R}\}$. With this choice of $\hat{r}$ we have $N_{n,\hat{r}} = \{x = \xi^- + tv_n, \ t \in \mathbb{R}\}$ and $[\xi^-, \xi^+] = [\xi^-_r, \xi^+_r]$ where $\xi^+$ is the other extreme of the connected component of $N_{n,\hat{r}} \cap \mathcal{S}$ that contains $\gamma^m(\hat{r})$. Since $\xi^-$ satisfies (6.69) with the equality sign, $\xi_r^-$ satisfies (6.70) or (6.71) depending on whether $r \leq \hat{r}$ or $r > \hat{r}$. This and the fact that $\chi_n \geq 0$ implies $r(\xi^-) \geq r(\xi^+_r)$ show the existence of $\hat{r}$ such that (6.41) and (6.42) hold.

To complete the proof we recall that $\lambda_{n-1} \to +\infty$ as $n \to +\infty$ while $|\gamma^m_{n+1} - \hat{p}_{n+1}^+| \leq 2\nu$ with $\nu$ is independent of $n$. It follows that, by accepting an error of $O(\frac{1}{n})$ we can identify $\gamma^m_{n+1}$ with $\hat{p}_{n+1}^+$ and the circumference $C_r$ with a straight line parallel to the tangent $t$ to $C_{r_{n+1}}$ at $\gamma^m_{n+1}$. In the same order of approximation $\gamma^m(\hat{r})$ can be identified with the intersection $\hat{\gamma}^m$ of $\{x = \gamma^m_{n+1} + t\tau_n, \ t \in \mathbb{R}\}$ with the circumference of diameter $[\gamma^m_{n+1}, \xi^-]$ and $N_{n,\hat{r}} \cap \mathcal{S}$ with the line through $\hat{\gamma}^m$ and $\hat{\xi}^-$ (see Figure 8). Under these identifications that are equivalent to pass to the limit for $n \to +\infty$ we see that $\hat{r}$ has an upper bound $\approx \hat{r} \leq r_{n+1} - k\lambda_{n-1}$ ($k \approx 4\eta$) when $\chi_n = 0$ ($r(\xi^+) = r(\xi^-)$) and a lower bound that corresponds to the situation where the line parallel to $t$ through $\hat{\gamma}^m$ is tangent to the circumference with diameter $[\gamma^m_{n+1}, \xi^-]$ and $t$ we find, see Figure 8, that $\hat{r} \geq r_{n+1} - k'\lambda_{n-1}$ with $k' \approx 2(1 + \frac{2}{\sin 2\hat{\tau}})$. This concludes the proof of (ii).

5. It remain to show that (i) is valid also for $j = n - 1$. This follows from (ii) Proposition 6.10. The proof is complete.
\[ \gamma_{n+1} = p_n \]

\[ \xi_n = \tilde{\gamma}_{n+1} \]

\[ \tau_n \]

\[ w_n \]

\[ t \]

\[ 4\alpha \lambda_n - 1 \]

\[ \xi_n = \tilde{\xi} \]

\[ \ell_{n+1} \]

\[ r_{n+1} \]

\[ \ell_{n+1} \]

\[ \gamma_m = \tilde{\gamma}_m \]

\[ k' \lambda_{n-1} \]

\[ \tilde{r}_n \]

\[ k \lambda_{n-1} \]

\[ N_n, \tilde{r} \]

Figure 8: k and k' and \( \tilde{r} \).

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