The Reshetikhin - Turaev invariants arising from the quantum groups associated with the exceptional Lie algebras $G_2$, $F_4$ and $E_8$ at odd roots of unity are constructed and explicitly computed for all the lens spaces.

1 Introduction

A pressing problem in the field of ‘quantum topology’ [1] [2] is to understand the topological information embodied in the quantum invariants of 3-manifolds [3] - [5] constructed in recent years, and to use the information to settle geometric questions. A direct way to tackle the problem is to compute these invariants for 3-manifolds of interest, then try to extract topological information from the explicit results. This approach was followed by a number of authors, and extensive studies were carried out for the $su(2)$ invariants of Witten - Reshetikhin - Turaev, though only for classes of 3-manifolds of a very simple nature [6] - [11].

Even in the case of the smallest Lie algebra $su(2)$, the computations of such invariants is a rather difficult task, as it involves the evaluation of link invariants arising from all the irreducible representations of the $sl(2)$ quantum group at roots of unity. For bigger Lie algebras, the level of difficulty increases considerably. Nevertheless, we succeeded in computing the invariants for the lens spaces using all the classical Lie algebras [12] in an earlier publication [13]. It is the aim of this letter to extend the
results of \[13\] to the exceptional Lie algebras. We will be mainly concerned with \(G_2\), \(F_4\) and \(E_8\). The reason we work with these Lie algebras is that the fundamental groups of their root systems are all trivial. This fact greatly simplifies the computations. The main result of this letter is an explicit formula for the Reshetikhin-Turaev invariants arising from the quantum groups associated with the above mentioned Lie algebras at odd roots of unity for the lens spaces. We also present a simple construction of these invariants, which enables us to carry out the computations relatively easily.

Recall that the Reshetikhin-Turaev construction of 3-manifold invariants using quantum groups is usually formulated in terms of modular Hopf algebras \[4\] \[2\]. A modular Hopf algebra is a ribbon Hopf algebra admitting a distinguished set of representations, which satisfy some very rigid conditions. It is in general a very difficult problem to show that a Hopf algebra satisfies the requirements of a modular Hopf algebra. In \[14\], we proposed a slightly different construction, which did not depend on the notion of modular Hopf algebras. Instead, it relied on the analysis of eigenvalues of certain central elements of the quantum groups under consideration. The construction worked rather simply for the quantum supergroups \(U_q(osp(1|2))\) and \(U_q(gl(1|2))\), and the classical series of quantum groups. As we will see below, it can be easily implemented for the exceptional quantum groups as well.

The organization of this letter is as follows. In section 2 we present some results on the representation theory and the central algebra of the quantum groups. In section 3 we construct the topological invariants then compute them for the lens spaces.

2 Exceptional Quantum Groups

We will consider only the exceptional Lie algebras \(G_2\), \(F_4\) and \(E_8\). Let \(g\) be any of these Lie algebras, and denote the quantum group associated with \(g\) by \(U_q(g)\). Denote by \(\Phi^+\) the set of the positive roots of \(g\) relative to a base \(\Pi = \{\alpha_1, ..., \alpha_r\}\). Define \(H^* = \bigoplus_{i=1}^r C\alpha_i, \ E = \bigoplus_{j=0}^\infty R\alpha_j\). Let \((\ ,\ ) : \mathcal{E} \times \mathcal{E} \to R\) be an inner product of \(\mathcal{E}\) such that the Cartan matrix \(A\) of \(g\) is given by

\[
A = (a_{ij})_{ij=1}^r, \quad a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}.
\]

We will normalize this inner product such that, for the highest root \(\theta\) of \(g\),

\[
(\theta, \theta) = \begin{cases} 
6, & \text{for } G_2, \\
4, & \text{for } F_4, \\
2, & \text{for } E_8.
\end{cases}
\]

We assume that \(q\) is a primitive \(N\)-th root of unity, where \(N\) is a positive odd integer, which is greater than the dual Coxeter number of \(g\), and is not divisible by 3 in the case of \(G_2\).

The quantum group \[13\] \(U_q(g)\) is defined to be the unital associative algebra generated by \(\{k_i, k_i^{-1}, e_i, f_i \mid i = 1, ..., r\}\), \(r\) being the rank of \(g\), with the following relations

\[
k_i k_j = k_j k_i, \quad k_i k_i^{-1} = 1, \\
k_i e_j k_i^{-1} = q_i^2 e_j, \quad k_i f_j k_i^{-1} = q_i^{-2} f_j,
\] (1)
\[ [e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}, \]

\[
\sum_{t=0}^{1-a_{ij}} (-1)^t \left[ \begin{array}{c} 1 \\ t \end{array} \right] \left( \frac{a_{ij}}{q_i} \right)^t = 0, \quad i \neq j,
\]

\[
\sum_{t=0}^{1-a_{ij}} (-1)^t \left[ \begin{array}{c} 1 \\ t \end{array} \right] \left( \frac{a_{ij}}{q_i} \right)^t = 0, \quad i \neq j,
\]

where \[ \left[ \begin{array}{c} s \\ t \end{array} \right] \] is the Gauss polynomial, and \( q_i = q^{(\alpha_i, \alpha_i)/2} \).

It is well known that the quantum group \( U_q(g) \) has the structure of a Hopf algebra. We will take the following co-multiplication

\[
\Delta(k_i^\pm) = k_i^\pm \otimes k_i^\pm,
\]

\[
\Delta(e_i) = e_i \otimes k_i + 1 \otimes e_i,
\]

\[
\Delta(f_i) = f_i \otimes 1 + k_i^{-1} \otimes f_i.
\]

A co-unit and an antipode can also be defined, but we will not spell them out explicitly here.

As it stands, \( U_q(g) \) does not admit a universal \( R \)-matrix, but an appropriate Hopf quotient of it does. To explain this quotient, we follow [17] to define the elements of \( U_q(g) \): \( e^{(n)}_\alpha, f^{(n)}_\alpha, \alpha \in \Phi^+, n = 0, 1, ..., \) where \( e^{(1)}_\alpha \) and \( f^{(1)}_\alpha \) have properties similar to that of the root spaces of the Lie algebra \( g \). In particular, if \( \alpha = \alpha_i \) for a given \( i \), then \( e^{(1)}_\alpha = e_i \), and \( f^{(1)}_\alpha = f_i \). Then the elements \( e^{(N)}_\alpha, f^{(N)}_\alpha, k_i^N - 1, \alpha \in \Phi^+, i = 1, 2, ..., r \), generate a Hopf ideal \( J \) of \( U_q(g) \). The quotient algebra \( U_q(g)/J \) is again Hopf, and is known to be quasi triangular, that is, it admits a universal \( R \)-matrix. By an abuse of notation, we still denote this quotient algebra by \( U_q(g) \), noting that it is clearly finite dimensional over the complex field.

Express the universal \( R \) matrix of \( U_q(g) \) as \( R = \sum_t a_t \otimes b_t \), and define

\[ u = \sum_t S(b_t) a_t. \]

Set

\[ K_{2\rho} = \prod_{i=1}^{r} (k_i)^{2\rho_i}. \]

where \( \rho_i \in \mathbb{Z}_+ \) are specified by \( \sum_{i=1}^{r} \rho_i \alpha_i = \sum_{\alpha \in \Phi^+} \alpha/2 = \rho \). Then

\[ v = u K_{2\rho}^{-1}, \]

belongs to the center of \( U_q(g) \), and satisfies

\[ \Delta(v) = (v \otimes v)(R^T R)^{-1}. \]

Let \( V \) be a finite dimensional \( U_q(g) \) module. We denote the corresponding representation by \( \pi \). Define

\[ C_V = \text{tr}_V [(\pi \otimes \text{id})(K_{2\rho}^{-1} \otimes 1) R^T R], \]

(3)
where \( tr_V \) represents the trace taken over \( V \). Then \( C_V \) is also central.

Since the quantum group \( U_q(g) \) is finite dimensional over \( \mathbb{C} \), all its irreducible representations are finite dimensional as well, following the textbook result that every irreducible left module over an associative algebra is isomorphic to the quotient of the algebra itself by a maximal left ideal. It can also be shown that every irreducible \( U_q(g) \) module admits a unique (up to scalar multiples) highest and lowest weight vector. The highest weight vector \( v_+ \) of an irreducible \( U_q(g) \) module \( V \) is defined by

\[
\begin{align*}
    k_i v_+ &= q_i^l v_+, \\
    e_i v_+ &= 0, \quad i = 1, 2, \ldots, r,
\end{align*}
\]

where \( l_i \in \{0, 1, \ldots, N - 1\} \) as required by \( k_i^N = 1 \).

Let \( X \) denote the weight lattice of \( g \). For the Lie algebras under consideration, \( X \) coincides with the root lattice of \( g \). Now from the above discussion we easily see that each irreducible \( U_q(g) \) module is uniquely characterized by an element of the set \( X^\sim = X/NX \). We denote the canonical projection \( X \to X^\sim \) by \( p \).

Let \( F = \{x \in X| 0 < \frac{2(\lambda + \rho, \alpha)}{\langle \alpha, \alpha \rangle} < N, \quad \forall \alpha \in \Phi^+ \} \),

\[
F = \{x \in X| 0 < \frac{2(\lambda + \rho, \alpha)}{\langle \alpha, \alpha \rangle} < N, \quad \forall \alpha \in \Phi^+ \},
\]

\[
\Lambda_N^+ = p(F),
\]

\[
\bar{\Lambda}_N^+ = p(\overline{F}).
\]

Define the set

\[
\mathcal{V}(\Lambda_N^+) = \{V(\lambda) \mid \lambda \in \Lambda_N^+\},
\]

of all the irreducible \( U_q(g) \) modules with highest weights belonging to \( \Lambda_N^+ \). A recent result of Andersen and Paradowski\([16]\) asserts that the tensor product of any finite number of irreducible \( U_q(g) \) modules \( V^{\lambda^t} \) with highest weights \( \lambda^t \in \Lambda_N^+ \) can be decomposed into

\[
\bigotimes_t V^{\lambda^t} = \bigoplus_{\lambda \in \Lambda_N^+} V^{\lambda(\oplus m(\lambda))} \oplus \mathcal{N},
\]

where \( m(\lambda) \) is the multiplicity of \( V(\lambda) \) appearing in the tensor product. The module \( \mathcal{N} \) is a direct sum of indecomposable \( U_q(g) \) modules, which has the property that for any module homomorphism \( f : \mathcal{N} \to \mathcal{N} \), the quantum trace of \( f \) vanishes identically, i.e.,

\[
tr_{\mathcal{N}}(K_{2p}f) = 0.
\]

A further useful property of the set \( \Lambda_N^+ \) is the following. For any \( \lambda \in \Lambda_N^+ \), we define \( \lambda^* \) by taking any representative of \( \tilde{\lambda} \in X \), and setting \( \lambda^* = p(-\tau(\tilde{\lambda})) \), where \( \tau \) is the maximal element of the Weyl group of \( g \). Clearly \( \lambda^* \) is independent of the representative \( \tilde{\lambda} \) chosen. It is easy to prove that \( \lambda^* \in \Lambda_N^+ \), and the dual module \( V(\lambda)^* \) of \( V(\lambda) \) is
isomorphic to \( V(\lambda^*) \).

Define a function \( q(\cdot, \cdot) : X_N \times X_N \to \mathbb{C} \) by requiring that its pullback by the canonical projection \( X \to X_N \) be given by the function \( X \times X \to \mathbb{C} \), \( \{ \tilde{\lambda}, \tilde{\mu} \} \mapsto q(\tilde{\lambda}, \tilde{\mu}) \). Acting on any \( V(\lambda) \in \mathcal{V}(\mathbb{C}_N^+) \), the central elements \( v \) and \( C_\mu := C_{V(\mu)}, \mu \in \Lambda_N^+ \), take the following eigenvalues respectively

\[
\chi_{\lambda}(v) = q^{-(\lambda + 2\rho, \lambda)}, \\
\chi_{\lambda}(C_\mu) = S_{\lambda\mu}/Q(\mu),
\]

where

\[
Q(\mu) = \sum_{\sigma \in \mathcal{W}} \det(\sigma) q^{2(\sigma(\rho), \mu + \rho)}, \\
S_{\lambda\mu} = \sum_{\sigma \in \mathcal{W}} \det(\sigma) q^{2(\sigma(\lambda + \rho), \mu + \rho)},
\]

In these equations, \( \rho \) should be interpreted as \( p(\rho) \). The \( \mathcal{W} \) represents the Weyl group of the Lie algebra \( g \), which acts on \( X_N \) by

\[
\sigma(\tilde{\lambda} + NX) = \sigma(\tilde{\lambda}) + NX, \quad \tilde{\lambda} \in X, \quad \sigma \in \mathcal{W}.
\]

Set

\[
d_\lambda = \Omega Q(\lambda), \quad \lambda \in \Lambda_N^+, \\
\Omega = (-1)^{(|\Phi| + q^{3(\rho, \rho)})} G_1(q; g),
\]

where \( G_1(q; g) \) is the \( k = 1 \) case of

\[
G_k(q; g) = \sum_{\lambda \in X_N} (q^{(\lambda, \lambda)})^k.
\]

It is clear that \( d_\lambda = d_{\lambda^*} \).

Define

\[
\delta = v - \sum_{\lambda \in \Lambda_N^+} d_\lambda \chi_{\lambda}(v^{-1}) C_{\lambda}.
\]

Then \( \delta \) acts as the zero map on any element of \( \mathcal{V}(\mathbb{C}_N^+) \), i.e., in any irreducible \( U_q(g) \) module with highest weight belonging to \( \Lambda_N^+ \).

To understand the above assertion, observe that \( \Lambda_N^+ \) furnishes a fundamental domain for the action of \( \mathcal{W} \) on \( X_N \), a fact which can be easily proven by studying the action of the affine Weyl group \( \mathcal{W}_N \) on \( X \). Therefore, for any \( \lambda, \mu \in \Lambda_N^+ \) and \( \sigma, \omega \in \mathcal{W} \),

\[
\sigma(\lambda + \rho) - \rho = \omega(\mu + \rho) - \rho \quad \text{iff} \quad \sigma = \omega, \quad \lambda = \mu.
\]

Also note that \( S_{\lambda\nu} = 0 \) if \( \lambda \in \Lambda_N^+ - \Lambda_N^\pm \). These properties immediately lead to

\[
\sum_{\lambda \in \Lambda_N^+} d_\lambda q^{(\lambda + 2\rho, \lambda)} S_{\lambda\mu} = \sum_{\nu \in X_N} x_{\nu} q^{(\nu + 2\rho, \nu)} S_{\nu\mu},
\]

\[
x_\lambda = \frac{q^{3(\rho, \rho) - 2(\lambda + \rho, \rho)}}{G_1(q; g)}.
\]
Now our assertion follows directly from
\[
\sum_{\nu \in X_N} x_{\nu}q^{(\nu+2\rho,\nu)}S_{\nu\mu} = Q(\mu)q^{-(\mu+2\rho,\mu)}.
\]

By using the Andersen-Paradowski result and the properties of the central element \(\delta\) we can show that if \(Y\) is a \(U_q(g)\) module which is the tensor product of a finite number of elements (not necessarily distinct) of \(\mathcal{V}(\Lambda_N^+)\), then for any module homomorphism \(f : Y \to Y\),
\[
tr_Y(K_{2\rho}\delta f) = 0,
\]
where \(tr_Y\) represents the trace taken over \(Y\).

Define
\[
z = \sum_{\lambda \in \Lambda_N^+} d_{\lambda}q^{-(\lambda+2\rho,\lambda)}D_q(\lambda),
\]
where \(D_q(\lambda)\) denotes the quantum dimension of the irreducible \(U_q(g)\) module \(V(\lambda) \in \mathcal{V}(\Lambda_N^+)\), which is given by \(D_q(\lambda) = Q(\lambda)/Q(0)\). Then we have the following useful formulae
\[
z = (-1)^{\mid \Phi^+ \mid}q^{6(\rho, \rho)}G_{N-1}(q; g)/G_1(q; g),
\]
\[|z| = 1.\]

3 3 - Manifold Invariants

In order to construct 3-manifold invariants using the quantum group \(U_q(g)\), we need to employ the Reshtikhin-Turaev \(F\) functor from the category of coloured ribbon graphs to the category of finite dimensional representations of this quantum group. A detailed discussion of coloured ribbon graphs and this functor can be found in [2]; and we will not repeat it here. Instead, we consider as examples the ribbon \((k, k)\) graphs depicted in Figure 1 to provide some concrete intuition about the functor.

Figure 1. Figures are available on request.

We colour the ribbons of both Figure 1.a and 1.b by \(\{V(\lambda_1), V(\lambda_2), ..., V(\lambda_k)\}\), while we colour the annulus of Figure 1.a by the irreducible module \(V(\mu)\). We denote the resultant coloured ribbon graphs by \(\phi^{(k)}_\mu\) and \(\zeta^{(k)}\) respectively. It is straightforward to obtain,
\[
F(\phi^{(k)}_\mu) = \chi_j(v^{-1})\Delta^{(k-1)}(C_\mu),
\]
\[
F(\zeta^{(k)}) = \Delta^{(k-1)}(v),
\]
(8)
which map $V(\lambda_1) \otimes V(\lambda_2) \otimes \ldots \otimes V(\lambda_k)$ to itself.

The Reshetikhin-Turaev construction makes use of two fundamental theorems in 3-manifold theory, due to Lickorish [18] and Wallace, and Kirby [19] and Craggs respectively. The Lickorish-Wallace theorem states that each framed link in $S^3$ determines a closed, orientable 3-manifold, and every such 3-manifold is obtainable by surgery along a framed link in $S^3$. The disadvantage of this description of 3-manifolds is that different framed links may yield homeomorphic 3-manifolds upon surgery. This problem was resolved by Kirby [19] and Craggs, and Fenn and Rourke [20]. These authors proved that orientation preserving homeomorphism classes of closed, orientable 3-manifolds correspond bijectively to equivalence classes of framed links in $S^3$, where the equivalence relation is generated by the Kirby moves. The essential idea of [4] is to make appropriate combinations of isotopy invariants of a framed link embedded in $S^3$, such that they will be intact under the Kirby moves, and thus qualify as topological invariants of the 3-manifold obtained by surgery along this link.

Let $M_L$ be a closed, oriented 3-manifold, which is given a surgery description, namely, represented by surgery along a framed link $L$ embedded in $S^3$. Assume that the framed link $L$(in blackboard framing) consists of $m$ components $L_i$, $i = 1, \ldots, m$. It gives rise to a unique ribbon graph by extending each component $L_i$ to an annulus, which has $L_i$ itself and an $L'_i$ as its edges, where $L'_i$ is a parallel copy of $L_i$ such that the linking number between the two is equal to the framing number of the latter. We denote this ribbon graph by $\Gamma(L)$.

We colour $\Gamma(L)$ by associating with each component $L_i$ with a $V(\lambda_i) \in V(\Lambda^+_N)$. Set $c = \{\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(m)}\}$, where some $\lambda^{(j)}$’s may be equal, and denote by $C(L)$ the set of all the distinct $c$’s. The ribbon graph coloured by modules associated with $c$ will be denoted by $\Gamma_c(L)$. The Reshetikhin-Turaev functor applied to $\Gamma_c(L)$ yields $F(\Gamma_c)$, which is a homomorphism of the trivial $U_q(g)$ module to itself, i.e., a complex number.

Define

$$\Sigma(L) = \sum_{c \in C(L)} \prod_{i=1}^m d_{\lambda^{(i)}} F(\Gamma_c(L)). \quad (9)$$

Since $d_{\lambda} = d_{\lambda^*}, \forall \lambda \in \Lambda^+_N$, $\Sigma(L)$ is independent of the orientation chosen for $L$. It follows from (4) and (8) that $\Sigma(L)$ is invariant under the positive Kirby moves depicted in Figure 2. a and Figure 2. b.

On the other hand, $\Sigma$ is not invariant under the Kirby (−) moves given in Figure 2.c and 2.d. In particular, if $L'$ is the framed link obtained by applying once the special Kirby (−) move Figure 2.c, namely, adding a framing $-1$ unknot, to the framed link $L$, then

$$\Sigma(L') = z\Sigma(L).$$
Since \(|z| = 1\), the norm of \(\Sigma\) remains intact under both the Kirby (+) moves and the special Kirby (−) move. In view of the fact that these moves together generate the entire Kirby calculus, we conclude that \(|\Sigma(L)|\) is a topological invariant of \(M_L\).

Let \(A_L = (a_{ij})_{m \times m}\) be the intersection form on the second homology group \(H_2(W_L, \mathbb{Z})\) of \(W_L\), where \(W_L\) is the 4-manifold bounded by \(M_L\). Denote by \(\text{sign}(A_L)\) the number of nonpositive eigenvalues of \(A_L\). Under the special Kirby (−) move, \(\text{sign}(A_L) = \text{sign}(A_{L'}) - 1\), while the positive Kirby moves leave \(\text{sign}(A_L)\) unchanged. Therefore \(z^{-\text{sign}(A_L)}\Sigma(L)\) is invariant under the positive Kirby moves and the special negative Kirby move. Hence, the following quantities

\[
\nabla(M_L) = |\Sigma(L)|^2, \quad (10)
\]

\[
\mathcal{F}(M_L) = z^{-\text{sign}(A_L)}\Sigma(L). \quad (11)
\]

are topological invariant of the 3-manifold \(M_L\). Note that \(\nabla(M_L)\) should be closely related to the Turaev-Viro invariant \([21]\).

Now we compute these invariants for the lens spaces \(L(m, n)\), where \(m, n \in \mathbb{Z}\) are co-prime. We assume that \(0 < n < m\). This exhausts all the possible lens spaces apart from \(S^3\) and \(S^2 \times S^1\). These two degenerate cases can be easily treated; we have

\[
\mathcal{F}(S^3) = 1,
\]

\[
\mathcal{F}(S^2 \times S^1) = \frac{1}{\Omega Q(0)}.
\]

For each \(L(m, n)\), there always exists a unique set of integers \(\{a_1, ..., a_s\}\) for some \(s\) with \(2 \leq a_i \in \mathbb{Z}\), such that the ratio \(m/n\) can be expressed as a continued fraction

\[
\frac{m}{n} = a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \frac{1}{\ddots - \frac{1}{a_s - \frac{1}{1}}}}}.
\]

Then the manifold can be obtained by surgery along the framed link, Figure 3,

Figure 3.

with framing numbers of the respective components of the link being \(a_1, a_2, ..., a_s\).

In order to compute the quantum invariants for \(L(m, n)\), we define, for \(\lambda \in X_N\),

\[
\varpi_\lambda = \begin{cases} 
\frac{d_\lambda}{Q(\lambda)}, & \lambda \in \Lambda_N^+, \\
0, & \lambda \notin \Lambda_N^+.
\end{cases}
\]

It can be proved that

\[
\sum_{\sigma \in \mathcal{W}}\varpi_{\sigma(\lambda + \rho) - \rho} = \begin{cases} 
\Omega, & 2\frac{(\lambda + \rho, \alpha)}{(\alpha, \alpha)} \neq 0 \text{ mod } N, \quad \forall \alpha \in \Phi^+,
\\
0, & \text{otherwise}.
\end{cases} \quad (12)
\]
Consider a ribbon graph of the form Figure 3 with \(k+1\) components. We temporarily assume that the all framing numbers are arbitrary integers, except that \(a_{k+1} = 0\). Cutting the left most component open results in a new ribbon graph, which has \(k\) annuli and one ribbon. Order the annuli from right to left, and colour the \(i\) - th annulus by the irreducible \(U_q(g)\) module \(V(\mu_i)\), with \(\mu_i \in \Lambda_N^+\). Colour the ribbon by \(V(\lambda)\), \(\lambda \in \Lambda_N^+\). We denote the resultant coloured ribbon graph by \(\Gamma_\lambda(a_1, ..., a_k; \mu_1, ..., \mu_k)\), and define

\[
h^{(k)}_\lambda(a_1, ..., a_k) = Q(\lambda) \sum_{\mu_1, ..., \mu_k \in \Lambda_N^+} \prod_{i=1}^k d_{\mu_i} F(\Gamma_\lambda(a_1, ..., a_k; \mu_1, ..., \mu_k)).
\]

It is easy to establish the following recursive formula

\[
h^{(k)}_\lambda(a_1, ..., a_k) = \sum_{\mu \in X_N} \sum_{\mu_1, ..., \mu_k \in \Lambda_N^+} \det \sigma q^{a_k(\mu+2\rho, \mu)} q^{2(\sigma(\mu+\rho), \lambda+\rho)} h^{(k-1)}_{\mu_1}(a_1, ..., a_{k-1}),
\]

with

\[
h^{(1)}_\lambda(a_1) = \Omega \sum_{\mu \in X_N} Q(\mu) q^{a_1(\mu+2\rho, \mu)+2(\mu+\rho, \lambda+\rho)}.
\]

Observe that the \(h^{(t)}_\lambda(a_1, ..., a_t)\) are well defined for all \(\lambda \in X_N\), and more importantly,

\[
h^{(t)}_\lambda(a_1, ..., a_t) = \det \sigma h^{(t)}_{\sigma(\lambda+\rho)-\rho}(a_1, ..., a_t), \quad \forall \sigma \in W.
\]

Using this property and equation (12) we obtain

\[
h^{(k)}_\lambda(a_1, ..., a_k) = \Omega \sum_{\mu \in X_N} q^{a_k(\mu+2\rho, \mu)+2(\mu+\rho, \lambda+\rho)} h^{(k-1)}_{\mu}(a_1, ..., a_{k-1}).
\]

Since

\[
\mathcal{F}(L(m, n)) = \frac{1}{Q(0)} h^{(s)}_0(a_1, ..., a_s),
\]

we immediately arrive at

\[
\mathcal{F}(L(m, n)) = \frac{\Omega^s}{Q(0)} \sum_{\mu_1, ..., \mu_s \in \Lambda_N^+} Q(\mu_1) q^{\sum_{i=1}^s [a_i(\mu_i+2\rho, \mu_i)+2(\mu_i+\rho, \mu_i+\rho)]},
\]

where \(\mu_{s+1} = 0\). Also,

\[
\nabla(L(m, n)) = |\mathcal{F}(L(m, n))|^2.
\]

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