RATIONAL APPROXIMATIONS ON TORIC VARIETIES

BY

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ABSTRACT
Using the universal torsor method due to Salberger, we study the approximation of a general fixed point by rational points on split toric varieties. We prove that under certain geometric hypothesis the best approximations (in the sense of McKinnon-Roth’s work) can be achieved on rational curves passing through the fixed point of minimal degree, confirming a conjecture of McKinnon. These curves correspond, according to Batyrev’s terminology, to the centred primitive collections of the fan.

CONTENTS
1. Introduction .................................................. 2
2. Geometric preliminaries ....................................... 10
3. Universal torsors and Cox coordinates ...................... 21
4. Approximation constants ..................................... 25
5. Geometry of numbers ......................................... 27
6. Determination of $\alpha$ and locally accumulating subvarieties 28
7. Toric varieties with Picard number 2 ....................... 40
8. Some questions and remarks .................................. 49
References ....................................................... 50
1. Introduction

1.1. Background and motivation. K. Roth’s theorem is one of most outstanding and beautiful result in classical Diophantine approximation. Let $\theta \in \mathbb{R}$ be an algebraic number and $\mu(\theta) > 0$ be the approximation exponent, that is, the supremum of positive real numbers $\mu$ such that the inequality

$$\left| \frac{p}{q} - \theta \right| < \frac{1}{\max(|p|, |q|)^\mu}$$

has infinitely many solutions $\frac{p}{q} \in \mathbb{Q}$. K. Roth’s theorem now states

$$\mu(\theta) = \begin{cases} 1 & \text{if } [\mathbb{Q}(\theta) : \mathbb{Q}] = 1; \\ 2 & \text{if } [\mathbb{Q}(\theta) : \mathbb{Q}] \geq 2. \end{cases}$$

The exponent $\mu(\theta)$ measures how well the algebraic number $\theta$ can be approximated by rational numbers with the error term controlled by their “size”. It was commonly recognized that the main difficulty lies in bounding $\mu(\theta)$ from above. K. Roth’s theorem says that any irrational algebraic number is better approximated than any rational number and gives the exact approximation exponent 2.

Amongst the generalizations of K. Roth’s theorem to higher dimensional cases, there are Schmidt subspace theorem (cf. [Sch80]) and later on Faltings-Wüstholz theorem (cf. [FW94]). Recently, in series of works [McK07], [MR15] and [MR16], McKinnon and M. Roth introduce the notion of approximation constant $\alpha$ (Proposition-Definition 4.1) and formulate a framework of Diophantine approximation of rational points on arbitrary algebraic varieties, integrating all aforementioned theorems. For $X$ a variety defined over a number field $K$ and for every fixed $Q \in X(\bar{K})$, choose some distance function $d_\nu(\cdot, Q)$ with respect to some fixed place $\nu$ of $K$. Choose a height function $H_L$ associated to some fixed line bundle $L$. Then the constant $\alpha_{L,\nu}(Q, X)$ is defined as the infimum of positive real numbers $\gamma$ such that the inequality

$$d_\nu(P, Q)^\gamma H_L(P) \leq 1,$$

has infinitely solutions $P_i \in X(K)$ such that $d_\nu(P_i, Q) \to 0$. It measures the local behaviour via the complexity increasing of rational points when approaching the fixed point on the variety and it also plays a central role in the author’s

\[1\] In this article, we quote contributions from two mathematicians named Roth – Klaus F. Roth and Mike Roth.
recent investigation [Hua17b] [Hua17a] [Hua18] on local distribution of rational points since it helps detect the *locally accumulating* subvarieties (Definition [4.3]). These subvarieties contain rational points that are “closer” to the given point so that when \( \gamma \) is sufficiently close to \( \alpha_{L,\nu}(Q,X) \), almost all solutions of the inequality (2) are located there. As we put the exponent on the distance rather than the height, smaller \( \alpha \) means better approximation. Indeed in the above setting \( \alpha_{\mathcal{O}(1),\infty}(\theta,\mathbb{P}^1) = \mu(\theta)^{-1} \) (Example [4.2]). As pointed out before, bounding \( \alpha \) from below and even computing its value are challenging problems. Inaugurated by Nakamaye, it now becomes a classical fact that the local positivity of a line bundle should govern the its Diophantine approximation quality. McKinnon and M. Roth call this “local Bombieri-Lang phenomena”. Bearing this spirit, they show that the constant \( \alpha \) has deep relationship with the Seshadri constant \( \varepsilon \). One version [MR15, Theorem 6.3] of their main results is that for any rational point \( Q \) and any ample line bundle \( L \),

\[
\alpha_{L,\nu}(Q,X) \geq \frac{1}{2} \varepsilon_L(Q).
\]

Namely, the factor \( \frac{1}{2} \varepsilon_L(Q) \) is the smallest possible exponent when approximating any fixed rational point \( Q \).

According to some heuristic due to Batyrev and Manin, there exist many similarities between the distribution of rational points and that of rational curves. For example Manin’s conjecture [Man93, RCC] on existence of rational curves via the number of rational points of bounded height. At an attempt of formulating a local analogy, McKinnon has made the following conjecture, based on the empirical fact that rational points tend to accumulate on rational curves when approaching a fixed point. In fact (3) is an equality precisely when both \( \alpha \) and \( \varepsilon \) are computed on some rational curve (verifying another extra condition so that the factor \( \frac{1}{2} \) comes from K. Roth’s Theorem (1)).

**Conjecture 1.1** ([McK07] Conjecture 2.7): Let \( X \) be a variety over a number field \( K \) and \( L \) be an ample line bundle. Suppose that \( Q \in X(K) \) and that there exists a rational curve defined over \( K \) passing through \( Q \) on \( X \). Then there exists a rational curve \( C \) on \( X \) passing through \( Q \) achieving the best approximation constant at \( Q \) with respect to \( L \).

The word “achieve” means that the \( \alpha_{L,\nu}(Q,X) \) can actually be computed with respect to a rational curve. Assuming Vojta’s Conjecture (see [Voj87]),
McKinnon [McK07] §4 shows the consistency of Conjecture 1.1 for varieties of general type.

There have been a considerable number of works about the computation of the approximation constant. It turns out that the local behaviour of rational points is very complicated even for simplest varieties other than the projective spaces. See [McK07], [MR15], [MR16], [Hua17b], [Hua17a], [Hua18]. All of them essentially consider (rational) surfaces, more precisely, (weak) del Pezzo surfaces of degree \( \geq 3 \) and they all satisfy Conjecture 1.1. Such surfaces are typical examples of rational varieties and more generally, rationally connected varieties. One big advantage of working with them is that they contain many free rational curves and a fortiori many rational points, on which Conjecture 1.1 may be interpreted as that along well-chosen movable rational curves the heights increase most slowly for rational points tending to a fixed general point. Also it makes sense to look at what happens on open dense subsets where there is no local accumulation. There would be of no difference in magnitude of height increasing when approaching the fixed point through any tangent direction in such open sets. The essential constant \( \alpha_{\text{ess},L}(Q) \) (Definition 4.3) first introduced by Pagelot [Pag08] gives such a characterization which we shall call generic best approximation. It is defined as the supremum of \( \alpha_{L,\nu}(Q;U) \) for \( U \) ranges over all Zariski dense open sets. If \( \alpha_{\text{ess},L}(Q) \) is finite and is achieved on some open set, then in the light of 1.1 a rational curve realizing \( \alpha_{\text{ess},L}(Q) \) should be deformable while fixing \( Q \) (hence very free) on that open set. We might naively enhance Conjecture 1.1 as follows.

**Principle:** Assume the variety is rationally connected and the point to be approximated is general. Then the best (resp. generic) approximation should be achieved on subvarieties swept out by free (resp. very free) rational curves of small degree.

Besides naive constructions such as products of varieties, there are very few higher dimensional cases for which the approximation constants are known to have been computed. A similar difficulty appears for the Seshadri constants, even though they are known for all del Pezzo surfaces (cf. [Bro06], [GP98]).

1.2. Main result. Toric varieties are special kinds of rational varieties admitting generically transitive group actions. Their arithmetic has been intensively studied. Rational points are very well distributed (the Batyrev-Manin-Peyre’s
principle \[BM90, \text{Pey95}\] thanks to the work of Batyrev & Tschinkel \[BT98, BT95\], Salberger \[Sal98\] and later on Frei \[Fre13\] and Pieropan \[Pie16\].

We shall be interested in toric varieties of dimension \(\geq 2\) satisfying the following geometric condition.

\*(*) The cone of pseudo-effective divisors \(\text{Eff}(X)\) is simplicial.

Examples are products of projective spaces, projective bundles over projective spaces and can have arbitrarily large Picard number. We assume that the point to be approximated is general, so that it lies on the open orbit, i.e. the torus \(T = \mathbb{G}_m^{\dim X}\). The (global) accumulating subvariety (boundary divisors) does not play a role here. We shall prove the following which answers affirmatively McKinnon’s Conjecture 1.1 in a more precise way.

**Theorem 1.2**: Let \(X\) be a split smooth projective toric variety defined over \(K\) equipped with a big line bundle \(L\). Let \(Q \in T(K)\).

1. If \(X\) verifies Hypothesis (*), then the best approximations for \(Q\) can be achieved (resp. is achieved) on free rational curves through \(Q\) of minimal \(L\)-degree belonging to minimal components of \(\text{RatCurve}^{n}(X)\) if \(L\) is globally generated (resp. ample);

2. If \(X\) has Picard number \(\leq 2\) (in particular they verify (*)) and \(L\) is globally generated, then the generic best approximations for \(Q\) can be achieved on very free rational curves through \(Q\) of minimal \(L\)-degree.

The precise meaning of “is achieved” and “can be achieved” will be discussed later (Definition 4.4). We refer to Theorems 6.1 & 7.4 for the complete statements. Our result not only gives the precise value of the approximation constant \(\alpha\), but also reveals the exact shape of the locally accumulating subvariety, therefore can be seen as an effective version of the main theorems in \[MR15, \S6\] in the toric setting. All rational curves achieving the best approximation in Theorem 1.2 are smooth. They correspond in fact to the so-called centred primitive collections invented by Batyrev \[Bat91\]. They turn out to link to those components of the space of rational curves \(\text{RatCurve}^{n}(X)\) (cf. \[Kol96, \text{Definition 2.11}\]) which are minimal, whose existence for general varieties (satisfying some extra condition) can now only be proven using the deep theory of Mori (See \[Ara06\]). Also examples of surfaces reveal that both best approximation and generic approximation, especially the latter, show some degeneration-invariance among families of polarized varieties. Studying toric varieties may give some...
evidence on what happens about Diophantine approximation on other varieties (admitting toric degenerations). It would also be interesting to compare our result with [Ito14], which gives an estimation of the Seshadri constant for general toric varieties.

We believe that, by carefully performing the argument in [McK07, §3], we should be able to work out Conjecture 1.1 for a larger class of varieties not necessarily toric but admit birational morphisms to toric ones. This is the first time that the universal torsor method is employed to the problem of local accumulation. It opens a new potential way to study the local distribution of rational points on other varieties. For example, lower degree del Pezzo surfaces [Der14], some Fano threefolds [DHH+15], certain spherical varieties [Bri07, DG18] and eventually T-varieties [HS10].

We remark that in general it is not always true that smooth rational curves of minimal degree contribute to the best or the generic approximation, especially when the pseudo-effective cone has too many generators. McKinnon [McK07, §4] exhibits first examples — smooth cubic surfaces — on which the best approximation for a general point is achieved on a singular cubic curve (see [MR16, Theorem 4.5] for a detailed statement). So does their toric degeneration with 3A2 singularities. Note that (the desingularisation of) this toric variety does not verify Hypothesis (*). See also the surfaces Y3, Y4 in [Hua17a] and [Hua18]. The actual phenomenon is that certain singular curves, whose approximation constant equal their degree divided by some factor coming from K. Roth’s theorem (1) or their singular multiplicity, give better approximation than the smooth ones. Moreover the nodal type and the cuspidal type singularities have different contributions. See [MR15, Theorem 2.16]. The appearance of such singular curves is quite general (see a family of examples in [Hua17a, §5.5]) and merits further investigation. By Theorem 1.2 Hypothesis (*) indicates a sufficient condition for which they do not enter. Nevertheless, all know results share the light of the definition of the Seshadri constant: looking for rational curves whose multiplicity at a fixed point is comparable with their degree.

1.3. OUTLINE OF THE PROOF. We first outline a general strategy on how to compute the approximation constant. It is essentially a comparison between the growth of height and the decreasing of distance. Precisely speaking, note by α_L,ν(Q, Y) the approximation constant computed at a point Q with respect to a line bundle L, a place ν and a subvariety Y. To prove that α_L,ν(Q, Y) ≤ γ
for some $\gamma > 0$ it suffices to find a rational curve $l$ such that $Y$ contains some of its open dense part and that $\alpha_{L,\nu}(Q, l) = \gamma$ (“the curve $l$ achieves the constant $\gamma$”). If $l$ is smooth at $Q$ then $\alpha_{L,\nu}(Q, l)$ is just $\deg_L(l)$. The main difficulty lies frequently on the lower bound. What we are going to show is, by fixing a distance function $d_{\nu}(Q, \cdot)$ locally around the fixed point $Q$ and a height function $H_L(\cdot)$, an estimation of the form $d_{\nu}(Q, P)^\gamma H_L(P) \geq C > 0$ for $P \in Y(K)$, which can be thought as a Liouville-type inequality. This implies that $\alpha_{L,\nu}(Q, Y) \geq \gamma$. Combining the previous upper bound, we get the exact value of $\alpha_{L,\nu}(Q, Y)$. To derive that $Y$ is locally accumulating, we need a stronger estimation such as for some $\delta > 0$,

\begin{equation}
    d_{\nu}(Q, P)^\alpha_{L,\nu}(Q, X) + \delta H_L(P) \geq C' > 0, \tag{4}
\end{equation}

for any $P \in (X \setminus Y)(K)$. This implies that $\alpha_{L,\nu}(Q, X \setminus Y) \geq \alpha_{L,\nu}(Q, X) + \delta > \alpha_{L,\nu}(Q, X)$. (“On the rest of $Y$ the complexity increases faster”.)

To prove inequalities like (4) we need a kind of parametrization of rational points that makes the computation of heights as simple as possible. The work of Salberger [Sal98] gives a first combinatorial way of parametrizing rational points on split toric varieties and he uses it to reprove the Batyrev-Manin-Peyre’s principle over $\mathbb{Q}$. His method, which we call universal torsor method for now on, is later generalized by Pieropan [Pie16] to imaginary quadratic fields. First introduced by Colliot-Thélène and Sansuc [CS87], universal torsors give us an effective way of parametrizing rational points via integral coordinates in some nice affine spaces as well as computing their heights. They have been crucial tools of attacking Batyrev-Manin-Peyre’s principle since the comprehensive investigation of Derenthal [Der14]. Generalizing this to function fields of one variable, Bourqui [Bou09b, Bou16] studies the distribution of rational curves and proves the geometric Batyrev-Manin-Peyre’s principle for many types of toric varieties. Salberger’s method is what we are going to adopt in this article. All required estimation reduces to the combinatorial data of the structural fan.

By the classification due to Kleinschmidt [Kle88], we know all possible fans defining smooth complete toric varieties of Picard number 2. It turns out that they always satisfy Hypothesis (*). With more explicit information we can improve estimation (4) by adapting the exponent on the distance to be the expected essential constant $\alpha_{\text{ess},L}(Q)$. What remains to be done is to find a dominating family of rational curves in $X \setminus Y$ all passing through $Q$ and achieving $\alpha_{\text{ess},L}(Q)$ and the family of general lines will do.
1.4. Example. To better illustrate our strategy and to give an example to Theorem 1.2, we consider $S_7$ – the toric del Pezzo surface of degree 7. For simplicity we work over $\mathbb{Q}$ and focus on $\infty$-absolute value. The anticanonical bundle $\omega_{S_7}^{-1}$ is ample. We can assume that $S_7$ is the blow-up of $\mathbb{P}^2$ (with homogeneous coordinates $[x : y : z]$) in $[1 : 0 : 0]$ and $[0 : 1 : 0]$. It is easy to see that $\text{Eff}(S_7)$ is generated by the class of the line $z = 0$ and those of the two exceptional divisors. Let $Q = [1 : 1 : 1]$. Let $l_1$ (resp. $l_2$) be the proper transform of the line $(x = z)$ (resp. $(y = z)$). They have minimal $\omega_{S_7}^{-1}$-degree 2. They give the upper bound

$$\alpha(Q, S_7) \leq \alpha(Q, l_i) = 2.$$ 

Note that other lines $l$ passing through $Q$ have degree 3 and they cover $S_7 \setminus (l_1 \cup l_2)$. Therefore

$$\alpha_{\text{ess}}(Q) \leq \alpha(Q, l) = 3.$$ 

The universal torsor $\pi : \mathcal{T} \to S_7$ embeds into the Cox ring $\mathbb{A}^5$. Write the coordinates $(X_1, \cdots, X_5)$ for $\mathbb{A}^5$. On the affine chart $(z \neq 0)$, the map $\pi$ is given by

$$\pi : (X_1, \cdots, X_5) \mapsto \left(\frac{x}{z}, \frac{y}{z}\right) = \left(\frac{X_1}{X_3X_5}, \frac{X_2}{X_4X_5}\right).$$

For $P \in \mathbb{G}_m^2(\mathbb{Q})$, let $P_0 = (X_1, \cdots, X_5) \in \mathcal{T}(\mathbb{Z})$ denote one lift into universal torsor. Define the distance function

$$d_\infty(Q, P) = \max \left(\left|\frac{X_1}{X_3X_5} - 1\right|_\infty, \left|\frac{X_2}{X_4X_5} - 1\right|_\infty\right).$$

We have for $P$ close to $Q$,

$$H_{\omega_{S_7}^{-1}}(P) \asymp |X_3^2X_4^2X_5^3|_\infty.$$ 

We easily check that, by comparing the denominators in $d_\infty(Q, P)$ with $H_{\omega_{S_7}^{-1}}(P)$,

$$d_\infty(Q, P)^2H_{\omega_{S_7}^{-1}}(P) \geq 1, \quad \forall P \neq Q,$$

$$d_\infty(Q, P)^3H_{\omega_{S_7}^{-1}}(P) \geq 1, \quad \forall P \notin l_1 \cup l_2.$$ 

Hence we obtain the lower bounds

$$\alpha(Q, S_7) \geq 2, \quad \alpha(Q, S_7 \setminus (l_1 \cup l_2)) \geq 3.$$ 

Gathering together these bounds, we recover Theorem 1.2 for $S_7$: the subvariety $l_1 \cup l_2$ is locally accumulating and contains minimal degree rational curves.
through $Q$ which are free but not very free, and the generic approximation can be achieved on general lines through $Q$, which are very free of minimal degree.

However as the dimension goes up and the structural fan becomes rich and varied, the complexity of the parametrization soon gets complicated, which makes the estimation of lower bound much more involved. We shall explain more in Section 6 how the use of Hypothesis (*) and centred primitive collections simplifies the parametrization by universal torsors and allows to deduce some positivity (Proposition 6.6) of the height.

1.5. Organization of the Article. In Section 2 we shall recall some basic toric geometry and the notion of freeness for rational curves. Our emphasis is the geometry of centred primitive collections. We also derive a criterion of very-freeness of rational curves on toric varieties, itself being of independent interest. The parametrization of rational points on toric varieties via universal torsors due to Salberger is introduced in Section 3. The formula of calculating heights associated to globally generated line bundles is given. We will define the best approximation constant $\alpha$ and the essential constant $\alpha_{\text{ess}}$ in Section 4 and discuss several of their fundamental properties. In Section 5 we recall a few useful classical fact about algebraic number fields. Section 6, the most technical part, is devoted to the proof of the first part of Theorem 1.2. In Section 7 we study toric varieties of Picard number 2 in detail including their geometry, big and ample cones, cone of effective curves and very free curves of minimal degree. And we prove the second part of Theorem 1.2. We end with several questions which remain untouched at the length of the present article.

1.6. Notations. We shall fix a number field $K$. $\mathcal{O}_K$ denotes the ring of integers, $\text{Cl}_K$ the class group and $\mathcal{M}_K$ the set of places of $K$. $\mathcal{M}_K = \mathcal{M}_K^f \sqcup \mathcal{M}_K^\infty$ comprises of finite places and infinite ones. For $\nu \in \mathcal{M}_K^f$, we shall use the valuation for $|\cdot|_\nu$ normalized with respect to $k$. That is, if $p$ is a prime number such that $\nu | p$, $|x|_\nu = |N_{K_\nu/Q_p}(x)|_p$. If $\nu \in \mathcal{M}_K^\infty$, $|x|_\nu$ is the usual absolute value on the completion $K_\nu$ (extending the real one). For example, $|3 + 4\sqrt{-1}|_\nu = 5$. $	ext{Norm}(\cdot)$ denotes the norm function defined for all fractional ideals of $K$. For $p \in \mathcal{O}_K$, $\text{ord}_p(\cdot)$ denotes the valuation order in the ring $\mathcal{O}_{K,p}$. Let $V$ be a vector space over a field $F$ and $P \subset V$. Then $\text{Vect}_F(P)$ denotes the vector subspace of $V$ spanned by elements in $P$. 
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2. Geometric preliminaries

2.1. Toric geometry. We refer the reader to excellent books [Ful93] and [CLS11] for general introduction to toric varieties. Here we will only state several well-known facts mostly without proof and fix notations.

Fix a lattice $N \simeq \mathbb{Z}^n$ and let $M = N^\vee$. We note $\mathcal{T} = \text{Spec}(K[M]) \simeq \mathbb{G}_m^n$ the open orbit. $N$ (resp. $M$) is naturally identified with the set of co-characters (resp. characters) of the torus $\mathcal{T}$. $\Delta \subset \mathbb{N}^\mathbb{R}$ will denote a $n$-dimensional fan, $\Delta_{\text{max}}$ denotes the set of maximal cones. For any $\sigma \in \Delta_{\text{max}}$, $\sigma^\vee \subset M_{\mathbb{R}}$ denotes its dual cone and $U_{\sigma} = \text{Spec}(K[\sigma^\vee \cap M]) \simeq \mathbb{A}^n$ denotes its associated open neighbourhood. The toric variety $X = X(\Delta)$ is constructed by gluing the data $(U_{\sigma}, \sigma \in \Delta_{\text{max}})$. A maximal cone $\sigma$ is called regular if the set $\sigma(1)$ of its generators, that is, the primitive elements in each of its one-dimensional faces (called rays), form a base of the lattice $N$. The variety $X$ is complete and smooth, which we assume from now on, if $\text{Supp}(\Delta) = N_{\mathbb{R}}$ and all maximal cones are regular. Let $\Delta(1) = \{\varrho_1, \ldots, \varrho_{n+r}\}$ denote the set of one-dimensional rays. Each ray $\varrho_i$ corresponds to a boundary divisor $D_{\varrho_i}$. The group $\text{Pic}(X)$ is torsion-free and of rank $r$. We recall that for smooth toric varieties there is no difference between numerical equivalence and rational equivalence between divisors, in other words $\text{Pic}^0(X) = 0$, $\text{Pic}(X) \simeq \text{NS}(X)$ the Néron-Severi group (see [CLS11, Proposition 6.3.15]). So the intersection pairing $\text{Pic}(X) \times \mathbb{A}_1(X) \to \mathbb{Z}$ is non-degenerate, where $\mathbb{A}_1(X)$ denotes the Chow group of 1-cycles modulo numerical equivalence. For every $D_{\varrho}, \varrho \in \Delta(1)$, let $C_\tau$ be the torus invariant curve corresponding to a $(n-1)$-dimensional cone $\tau$ so that $\varrho, \tau(1)$ generate a maximal cone. Then we have $\langle D_{\varrho}, C_\tau \rangle = 1$ ([CLS11, Proposition 6.3.8]). So the above paring is perfect.
Definition 2.1: In this article, we shall call any non-trivial equality $\mathcal{P} : \sum_{i=1}^{n+r} a_i \varrho_i = 0$ a relation between rays, or simply a relation. It is called positive if all coefficients $a_i$ are non-negative. We denote by $\mathcal{P}(1)$ the subset of the rays $\varrho_i \in \Delta(1)$ such that $a_i \neq 0$.

We may identify the set of relations as a subgroup of $\mathbb{Z}^{\Delta(1)}$ with addition on each of their coefficients. The set of positive relations form a semi-group.

Recall the following fundamental exact sequence of $\mathbb{Z}$-modules ([Bat91, Proposition 2.12], [Ful93, §3.4], [CLS11, Proposition 6.4.1]):

\begin{equation}
0 \to M \xrightarrow{h} \mathbb{Z}^{\Delta(1)} \xrightarrow{i} \text{Pic}(X) \to 0,
\end{equation}

Taking dual, we obtain

\begin{equation}
0 \to \text{Pic}(X)^\vee \xrightarrow{f} \mathbb{Z}^{\Delta(1)} \xrightarrow{g} N \to 0,
\end{equation}

where $\text{Pic}(X)^\vee \subset \text{Pic}(X)^\vee_Q = Q'$ is the dual lattice of $\text{Pic}(X)$ identified with $A_1(X)$. The maps $f, g, h, i$ are given as follows. For $m \in M$, the map $h$ is simply

$$h(m) = (\langle m, \varrho_i \rangle)_{i=1}^{n+r} \in \mathbb{Z}^{\Delta(1)}.$$ 

For $(a_i)_{1 \leq i \leq n+r} \in \mathbb{Z}^{\Delta(1)}$, the map $i$ is given by

$$i((a_i)_{1 \leq i \leq n+r}) = \sum_{i=1}^{n+r} a_i [D_{\varrho_i}] \in \text{Pic}(X).$$

For a curve $C \subset X$,

$$f([C]) = (\langle D_{\varrho_i} \cdot C \rangle)_{1 \leq i \leq n+r} \in \mathbb{Z}^{\Delta(1)}.$$

Extending by linearity, we can compute the restriction of $f$ to $A_1(X)$. And for $(a_i)_{1 \leq i \leq n+r} \in \mathbb{Z}^{\Delta(1)}$,

$$g((a_i)_{1 \leq i \leq n+r}) = \sum_{i=1}^{n+r} a_i \varrho_i \in N.$$ 

We may identify the group $A_1(X)$ as a subset of the kernel of $g$, that is, we view a class of curves as its associated relation, whose coefficients being precisely the intersection multiplicity with each boundary divisor.

Theorem 2.2: There is a bijection between

1. the classes of (rational) curves in $A_1(X)$ intersecting the open orbit;
(2) the positive relations \( \sum_{i=1}^{n+r} c_i \varrho_i = 0, c_i \geq 0 \);

(3) the equivalent families of non-zero homogeneous polynomials \( (f_{\varrho_i}(u, v))_{\varrho_i \in \Delta(1)} \)

with coefficients in \( K \) indexed by the rays such that \( \sum_{i=1}^{n+r} \deg(f_{\varrho_i}) \varrho_i = 0 \) satisfying the coprimality condition

\[
\forall I \subset \Delta(1), \bigcap_{\varrho_i \in I} D_{\varrho_i} = \emptyset \implies \gcd(f_{\varrho_i}(u, v)) = 1.
\]

**Proof.** If a curve meets the open orbit \( \mathcal{T} \), then it intersects properly with the boundary and so all coefficients of its associated relation are non-negative. Conversely, let \( \lambda_i : \mathbb{G}_m \to \mathcal{T} \) be the co-character associated to the ray \( \varrho_i \). Given a positive relation \( \sum_{i=1}^{n+r} c_i \varrho_i = 0, c_i \geq 0 \), we check that for any \((n+r)\)-pairwise distinct elements \( b_i \in K, 1 \leq i \leq n + r \), the Zariski closure of the map

\[
\mathbb{G}_m \to \mathcal{T}, \quad x \mapsto \prod_{i=1}^{n+r} (\lambda_i(x - b_i))^{c_i}
\]

is a rational curve whose class has coefficients \((c_i)\).

The third equivalence can be seen as a description of universal torsors for toric varieties (see Section 3 below) over rational function fields of one variable. It is also a particular case the functoriality of toric varieties due to Cox [Cox95]. We refer to [Bou09a §1.2] for a presentation. \( \blacksquare \)

Since every effective divisor is linearly equivalent to one with support on the boundary [Ful93 §3.4 Proposition], the cone of pseudo-effective divisors is generated by boundary divisors:

\[
\text{Eff}(X) = \sum_{i=1}^{n+r} \mathbb{R}_{\geq 0}[D_{\varrho_i}] \subset \text{Pic}(X)_\mathbb{R}.
\]

For every \( \mathcal{T} \)-invariant divisor \( D = \sum_{i=1}^{n+j} a_{\varrho_i} D_{\varrho_i} \), a piecewise affine (i.e. linear on every maximal cone) function \( \phi_D : N_\mathbb{R} \to \mathbb{R} \) defined as follows. Let \( \varrho \in N_\mathbb{R} \) and choose a maximal cone \( \sigma = \sum_{i=1}^{n} \mathbb{R}_{\geq 0} \varrho_i \) containing \( \varrho \). Let \( m_D(\sigma) = \sum_{i=1}^{n} -a_{\varrho_i} \varrho_i^\vee \in M \), i.e. the unique element in \( M \) determined by \( \langle m_D(\sigma), \varrho_i \rangle = -a_{\varrho_i} \) for any \( \varrho_i \in \sigma(1) \). Then \( \phi_D(\varrho) = \langle m_D(\sigma), \varrho \rangle \). The function \( \phi_D \) is called convex, if for all \( \sigma \in \Delta_{\text{max}} \),

\[
\phi_D(\cdot) \leq \langle m_D(\sigma), \cdot \rangle.
\]
It is called \textit{strictly convex} if moreover for every $\varrho \in N_{\mathbb{R}}$ and every $\sigma \in \Delta_{\text{max}}$ such that $\varrho \not\in \sigma(1)$, we have

\begin{equation}
\phi_D(\varrho) < \langle m_D(\sigma), \varrho \rangle.
\end{equation}

Intuitively, the (strict) convexity means that the graph of $\phi_D$ lies (strictly) under that of the linear function $\langle m_D(\sigma), \cdot \rangle$ for every $\sigma \in \Delta_{\text{max}}$.

\textbf{Definition 2.3}: Let $\mathcal{P} : \sum_{i=1}^{n+r} c_i \varrho_i = 0$ be a relation. We define $\deg_{\mathcal{O}_X(D)} \mathcal{P}$, the $\mathcal{O}_X(D)$-degree of $\mathcal{P}$ to be

$$
\deg_{\mathcal{O}_X(D)} \mathcal{P} = -\sum_{i=1}^{n+r} c_i \phi_D(\varrho_i).
$$

If the class of a curve $C$ corresponds to $\mathcal{P}$, then from the definition of $\phi_D$ and the exact sequences (5) and (6), $\deg_{\mathcal{O}_X(D)} \mathcal{P}$ is nothing but the intersection number $\langle D, C \rangle$, i.e. the $\mathcal{O}_X(D)$-degree of the curve $C$.

The following result establishes several equivalences between the positivity of line bundles on smooth toric varieties and convexity of their associated functions.

\textbf{Theorem 2.4} (Demazure): The line bundle $\mathcal{O}_X(D)$ is globally generated (resp. ample) if and only if it is nef (resp. very ample). This holds precisely when the function $\phi_D$ is convex (resp. strictly convex).

\textbf{Proof}. See for example [Ful93, p. 68, p. 70], and [CLS11, Theorem 6.3.12].

For every $\sigma \in \Delta_{\text{max}}$, on the affine open set $U_{\sigma}$ the line bundle $\mathcal{O}_X(D)$ trivializes as $\chi^{m_D(\sigma)} \mathcal{O}_{U_{\sigma}}$. We see that if $\mathcal{O}_X(D)$ is globally generated, then $m_D(\sigma)$ lifts to a global section of $\mathcal{O}_X(D)$.

To $D$ we associate a polyhedron

$$
P_D = \{ m \in M_{\mathbb{R}} : \langle m, \varrho_i \rangle \geq -a_{\varrho_i} \} \subset M_{\mathbb{R}} \simeq \mathbb{R}^n.
$$

Then the divisor $D$ is \textit{big} ([Laz04, §2.2]) if and only if $P_D$ has positive $n$-dimensional volume.

\section*{2.2. Primitive collections.}

This notion is introduced by Batyrev in classifying higher dimensional smooth toric varieties. We refer to [Bat91, Definitions 2.6-2.10] and the book [CLS11, Definition 6.4.10] for more details.
Definition 2.5 (Batyrev): A subset of rays $\mathcal{I} \subset \triangle(1)$ is called a \textit{primitive collection} if the members of $\mathcal{I}$ do not generate a cone of $\triangle$ but those belonging to any of the proper subsets of $\mathcal{I}$ do. Since $\triangle$ is complete, there exists a unique cone $\sigma$ containing $\sum_{\varrho \in \mathcal{I}} \varrho$. If $\sigma = \{0\}$, we call this collection \textit{centred} and the relation $\mathcal{P} : \sum_{\varrho \in \mathcal{I}} \varrho = 0$ a \textit{centred primitive relation} and we usually write $\mathcal{I} = \mathcal{P}(1)$. Its \textit{cardinality} is $\# \mathcal{I}$.

The $L = \mathcal{O}_X(D)$-degree (Definition [2.3]) of a centred primitive collection $\mathcal{I}$ is thus $-\sum_{\varrho \in \mathcal{I}} \phi_D(\varrho)$. The following result gives a sense to this notion.

Theorem 2.6 (Batyrev, Chen-Fu-Hwang): Centred primitive relations exist for smooth projective toric varieties.

Proof. See [Bat91, Proposition 3.2] [Ara06, Theorem 1.1] and [CFH14, Corollary 3.3].

We are going to see that the notion “centred primitive collections (or relations)” links naturally to rational curves which are “minimal” among deformation families. For this we need introduce some more notations and we refer to [Kol96, II.2] for details. Let $\text{RatCurve}^n(X)$ be the normalized space of rational curves on $X$, $p : \text{Univ}^{rc}(X) \to \text{RatCurve}^n(X)$ be the universal family and $q : \text{Univ}^{rc}(X) \to X$ be the cycle map. We say that an irreducible component $\mathcal{K}$ of $\text{RatCurve}^n(X)$ is \textit{minimal} if $q|_{p^{-1}(\mathcal{K})}$ is dominant and $p^{-1}(\mathcal{K}) \times_X \{x\}$ is proper for a general point $x$ of $X$. Members of $\mathcal{K}$ are called \textit{minimal rational curves}. The $(L)$-degree of $\mathcal{K}$ is the $(L)$-degree of any of its members. Generally speaking, a minimal rational curve is free (see Section [2.3] below, i.e. its deformation family covers a general point of $X$) and does not admit deformation that “breaks” into a reducible curve (through that point).

Let us now identify all minimal components on a toric variety. Fix a centred primitive relation $\mathcal{P}$ and write $\mathcal{P}(1) = \{\varrho_1, \ldots, \varrho_{m+1}\}$. For $i \in \{1, \ldots, m+1\}$, the rays in $\mathcal{P}(1) \setminus \{\varrho_i\}$ generate a cone $\sigma_i$ in $\triangle$. Let $\Sigma$ be the subfan of $\triangle$ consisting of all faces of the $\sigma_i$’s. It defines an open toric subvariety $Y$ of $X$ isomorphic to $\mathbf{P}^m \times \mathbb{G}^{n-m}_m$. Every line in fibres of the second projection is of $\omega_{\mathbf{P}^m}$-degree $m+1$ and the class of which corresponds exactly to the relation $\mathcal{P}$. From the above definition, the irreducible component in $\text{RatCurve}^n(X)$ corresponding to $\mathcal{P}$ is thus dominant and minimal since the deformation family of any line contained in each fibre of $\mathbb{G}^{n-m}_m$ covers precisely $\mathbf{P}^m$ so they are
minimal rational curves. The following result says that these are the only minimal components.

**Theorem 2.7** (Chen-Fu-Hwang [CFH14], Proposition 3.2): Let $X$ be a non-singular complete toric variety. Then minimal components (of given anticanonical degree) are in one-to-one correspondence to centred primitive relations (of given cardinality).

As shown by Batyrev, rational curves belonging to minimal components of minimal $L$-degree are indeed those of minimal $L$-degree amongst all curves passing through a general point when $L$ is ample. In general this is not always true (when $L$ is not sufficiently positive).

**Theorem 2.8** (Batyrev): Let $X$ be a smooth projective toric variety. Suppose that $L$ is globally generated. Then the minimal $L$-degree of positive relations equals the minimal $L$-degree of centred primitive relations. If $L$ is ample, then a positive relation has minimal $L$-degree if and only if it is a centred primitive relation of minimal $L$-degree.

**Proof.** If $L$ is ample, the proof of [Bat91, Proposition 3.2] (in a slightly different setting) actually shows that $P \mapsto \deg_L(P)$ attains minimum precisely when $P$ is centred primitive.

When $L = O_X(D)$ is globally generated where $D = \sum_{i=1}^{n+r} a_i D_{\varrho_i}$, or equivalently by Theorem 2.4 that $L$ is nef, since the ample cone is the relative interior of the nef cone, for every $\delta > 0$ sufficiently small, we can choose an ample $\mathbb{Q}$-divisor $D_\delta = \sum_{i=1}^{n+r} a_{i,\delta} D_{\varrho_i}$ such that

$$|a_i - a_{i,\varepsilon}| \leq \delta.$$

For every relation $P : \sum_{i=1}^{n+r} b_i \varrho_i = 0$, let $\|P\| = \max_{1 \leq i \leq n+r} |b_i|$. Pick a positive relation $P_0 : \sum_{i=1}^{n+r} a_{i,0} \varrho_i = 0$ such that

$$\deg_L(P_0) = \min_{\mathcal{P} \text{ positive}} \deg_L(\mathcal{P}).$$
Now we have
\[
\deg_L(P_0) = -\sum_{i=1}^{n+r} a_{i,0} \phi_D(q_i)
\]
\[
\geq -\sum_{i=1}^{n+r} a_{i,0} \phi_D(q_i) - n\|P_0\|\delta
\]
\[
= \min_{\mathcal{P} \text{ positive}} \deg_{O_X(D_\delta)}(\mathcal{P}) - n\|P_0\|\delta
\]
\[
= \min_{\mathcal{P} \text{ centred primitive}} \deg_{O_X(D_\delta)}(\mathcal{P}) - n\|P_0\|\delta
\]
\[
\geq \min_{\mathcal{P} \text{ centred primitive}} \deg_L(\mathcal{P}) - n(\|P_0\| + 1)\delta.
\]
This shows that
\[
\min_{\mathcal{P} \text{ positive}} \deg_L(\mathcal{P}) = \min_{\mathcal{P} \text{ centred primitive}} \deg_L(\mathcal{P}),
\]
as desired. \qed

2.3. Free and very free curves.

**Definition 2.9:** Fix $X$ a smooth variety. Let $f : \mathbb{P}^1 \to X$ be a rational curve and $d \in \mathbb{N}$. We say that $f$ is $d$-free if its image is contained in the smooth locus of $X$ and $f^*T_X \otimes \mathcal{O}(-d)$ is globally generated. We will denote “free” for 0-free and “very free” for 1-free.

By Grothendieck’s theorem [Har77, Ex. V.2.6], any locally free sheaf $\mathcal{F}$ of finite rank on $\mathbb{P}^1$ splits, i.e. there exist integers $a_1 \leq \cdots \leq a_m$ such that
\[
\mathcal{F} \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_m),
\]
we shall write $\mu_{\min}(\mathcal{F}) = \min_{1 \leq i \leq m} a_i = a_1$. In this notation, $\mathcal{F}$ is ample if and only if $\mu_{\min}(\mathcal{F}) \geq 1$, the map $f$ is free (resp. very free) if and only if $\mu_{\min}(f^*T_X) \geq 0$ (resp. $\geq 1$).

**Example 2.10** (Centred primitive collections): Let now $X$ be smooth projective toric. One can show that for a rational curve $f : \mathbb{P}^1 \to X$ corresponding to a centred primitive collection $\mathcal{I}$,
\[
f^*T_X \cong \mathcal{O}(2) \oplus \underbrace{\mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(1)}_{\sharp \mathcal{I} - 2} \oplus \underbrace{\mathcal{O} \oplus \cdots \oplus \mathcal{O}}_{n - \sharp \mathcal{I} + 1}.
\]
As we have seen from the proof of Theorem 6.1, such curve is a line \( l \) in the projective space \( \mathbb{P}^{\sharp I-1} \), which is a fibre of the toric subvariety \( Y = \mathbb{P}^{\sharp I-1} \times \mathbb{G}_m^{n+1-\sharp I} \) via the second projection. Hence

\[
f^*T_X = T_{\mathbb{P}^{\sharp I-1}}|_l \oplus \mathcal{O}_{\mathbb{P}^{\sharp I-1}}^{\oplus n-\sharp I+1}.
\]

So, unless \( X = \mathbb{P}^n \), \( f \) is not very free because \( \sharp I < n+1 \) and thus \( \mu_{\min}(f^*T_X) = 0 \). This gives evidence that \( Y \) is locally accumulating (Definition 4.3) when \( X \neq \mathbb{P}^n \).

We have the following useful criterion for detecting very free curves. It also outlines a general procedure to compute \( f^*T_X \) for rational curves on toric varieties. To the best of author’s knowledge it did not appear before in the literature.

**Theorem 2.11:** A positive relation \( \mathcal{P} \) represents very free rational curves if and only if \( \text{Vect}_\mathbb{Q}\{\mathcal{P}(1)\} = N_\mathbb{Q} \).

We begin with a well-known lemma, whose proof is left to the reader.

**Lemma 2.12:**

(1) Let \( a_1, a_2 \in \mathbb{N} \) and \( f_i \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a_i)) \setminus \{0\} \). Suppose that \( \gcd(f_1, f_2) = 1 \). Then we have the exact sequence:

\[
0 \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(a_1 + a_2) \longrightarrow 0
\]

\[
h \quad \mapsto \quad (hf_1, hf_2) \quad \mapsto \quad (g_1, g_2) \quad \mapsto \quad f_2g_1 - f_1g_2.
\]

(2) Let \( a_1, \ldots, a_n \in \mathbb{N} \) and \( f_i \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a_i)) \setminus \{0\} \) satisfying \( \gcd(f_i, f_j) = 1, \forall 1 \leq i \neq j \leq n \). Let \( \mathcal{G} \) be the quotient bundle (locally free) defined by

\[
\mathcal{O}_{\mathbb{P}^1} \hookrightarrow \mathcal{F} = \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i).
\]

Then we have \( \mu_{\min}(\mathcal{G}) \geq \mu_{\min}(\mathcal{F}) \).

**Proof of Theorem 2.11.** Let us first discuss how to compute \( f^*T_X \) for toric varieties. Consider the generalized Euler exact sequence (see [CLS11, Theorem 8.1.6]) of sheaves of \( \mathcal{O}_X \)-modules:

\[
0 \longrightarrow \Omega^1_X(X) \longrightarrow \bigoplus_{i=1}^{n-r} \mathcal{O}_X(-D_{\psi_i}) \longrightarrow \text{Pic}(X) \otimes \mathbb{Z} \mathcal{O}_X \longrightarrow 0,
\]
whose dual give raise to

$$0 \longrightarrow A_1(X) \otimes \mathcal{O}_X \longrightarrow \bigoplus_{i=1}^{n+r} \mathcal{O}_X(D_{\vartheta_i}) \longrightarrow T_X \longrightarrow 0,$$

Fix a positive relation \( \mathcal{P} \) with \( \mathcal{P} : \sum_{i=1}^{n+r} c_i \vartheta_i = 0 \). By relabelling we may assume that \( \mathcal{P}(1) = \{ \vartheta_i : c_i \neq 0 \} = \{ \vartheta_1, \ldots, \vartheta_m \} \). By Theorem 2.2 let \( f : \mathbb{P}^1 \to X \) be non-constant intersecting \( \mathcal{T} \) with the corresponding positive relation \( \mathcal{P} \) and choose a general lift \((f_i)_{i=1}^{n+r}, f_i \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\vartheta_i)) \) of \( f \) (that is, \( \gcd(f_i, f_j) = 1 \)). Pull back the above exact sequence gives

$$\mathcal{O}_\mathbb{P}^1 \xrightarrow{\phi_P} \bigoplus_{i=1}^{n+r} \mathcal{O}_{\mathbb{P}^1}(\deg f^*(\mathcal{O}_X(D_{\vartheta_i}))) \longrightarrow f^*T_X \longrightarrow 0,$$

and \( f^*(\mathcal{O}_X(D_{\vartheta_i})) \simeq \mathcal{O}_{\mathbb{P}^1}(\vartheta_i) \). Any relation (not necessarily positive) \( \mathcal{Q} : \sum_{i=1}^{n+r} w_i \vartheta_i = 0 \) defines a morphism

$$i_\mathcal{Q} : \mathcal{O}_{\mathbb{P}^1} \to \bigoplus_{c_i \neq 0}^{i \in \{1, \ldots, n+r\}} \mathcal{O}_{\mathbb{P}^1}(\vartheta_i) \hookrightarrow \bigoplus_{i=1}^{n+r} \mathcal{O}_{\mathbb{P}^1}(\vartheta_i);$$

$$h \mapsto (w_i h f_i)_{i; c_i \neq 0}.$$ 

By choosing \( r \) linearly independent relations \( (\mathcal{P}_j)_{1 \leq j \leq r} \) that the sub-bundle generated by \( i_\mathcal{P}_j(\mathcal{O}_{\mathbb{P}^1}), 1 \leq j \leq r \) equals \( \text{Im}(\mathcal{O}_{\mathbb{P}^1}^{\oplus r}) \), the one generated by the image of \( \mathcal{O}_{\mathbb{P}^1}^{\oplus r} \) under \( \phi_P \) in \( \bigoplus_{i=1}^{n+r} \mathcal{O}_{\mathbb{P}^1}(\vartheta_i) \), up to automorphism, we get from (10),

$$f^*T_X \simeq \bigoplus_{i=1}^{n+r} \mathcal{O}_{\mathbb{P}^1}(\vartheta_i)/\text{Im}(\mathcal{O}_{\mathbb{P}^1}^{\oplus r}).$$

We start by proving the sufficiency. Suppose that \( \vartheta_i \)’s generate the ambient space \( N_\mathcal{Q} = \mathbb{Q}^n \). Then \( m \geq n+1 \) and we may suppose that \( \{ \vartheta_1, \ldots, \vartheta_n \} \) is a \( \mathbb{Q} \)-base of \( N_\mathcal{Q} \). So for all \( n+1 \leq k \leq n+r \), there exist integers \( b_k \neq 0 \) and \( a_{j,k} \), \( 1 \leq j \leq n \) such that we have the following relations

$$\mathcal{Q}_k : b_k \vartheta_k - \sum_{i=1}^{n} a_{i,k} \vartheta_i = 0, \quad n+1 \leq k \leq n+r.$$

For each \( k \) we denote by \( i_\mathcal{Q}_k \) the associated morphism as in (11). These \( r \) relations are linearly independent and form a \( \mathbb{Q} \)-base of \( A_1(X)_\mathcal{Q} \). We define

$$\mathcal{F} = \bigoplus_{i=1}^{n+r} \mathcal{O}_{\mathbb{P}^1}(\deg f^*(\mathcal{O}_X(D_{\vartheta_i}))) = \left( \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}^1}(\vartheta_i) \right) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus n+r-m},$$
$M_1$ (resp. $M_2$) to be the sub-bundle of $\mathcal{F}$ generated by images of the morphisms $i_{Q_k}, m + 1 \leq k \leq n + r$ (resp. $n + 1 \leq k \leq n + r$) and

$$\mathcal{G} = \mathcal{F} / M_1, \quad \mathcal{H} = f^* T_X = \mathcal{F} / M_2.$$  

We now show that

$$G \simeq \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}^1}(c_i).$$  

Since $c_i > 0, 1 \leq i \leq m$, $\mathcal{G}$ is thus ample. Since

$$\deg f^*(\mathcal{O}_X(D_{g_i})) = c_i > 0, 1 \leq i \leq m; \quad \deg f^*(\mathcal{O}_X(D_{g_i})) = 0, m + 1 \leq i \leq n + r,$$

the polynomials $f_k, m + 1 \leq k \leq n + r$ are of degree 0, so they are non-zero constants. Define polynomials $g_{i,k}$ via the equalities

$$b_k f_k g_{i,k} = a_{i,k} f_i.$$  

Consider the automorphism of $\mathcal{F}$ defined by

$$\Psi : \mathcal{F} \to \mathcal{F}$$

$$(h_1, \ldots, h_{n+r}) \mapsto (H_1, \ldots, H_n, h_{n+1}, \ldots, h_{n+r}),$$

where $H_i = h_i - \sum_{k=m+1}^{n+r} h_k g_{i,k}$. Let us show that

$$\Psi(M_1) = \{(0, \ldots, 0, F_1, \ldots, F_{n+r-m}), F_i \in \mathcal{O}_{\mathbb{P}^1}\} = 0 \times \mathcal{O}^{\oplus n+r-m}_{\mathbb{P}^1} \subset \mathcal{F},$$

thus the claim (14) reduces to (15). Indeed, for $m+1 \leq k \leq n+r$, the morphism $i_{Q_k}$ factorises as

$$i_{Q_k} : \mathcal{O}_{\mathbb{P}^1} \to \left( \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(c_i) \right) \oplus \mathcal{O}_{\mathbb{P}^1} \to \left( \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}^1}(c_i) \right) \oplus \left( \bigoplus_{k=m+1}^{n+r} \mathcal{O}_{\mathbb{P}^1} \right) = \mathcal{F}.$$  

Composed with the automorphism $\Phi$, it becomes

$$\Psi \circ i_{Q_k} : h \mapsto \Psi(a_{1,k} h f_1, \ldots, a_{n,k} h f_n, 0, \ldots, 0, b_k h f_k, 0, \ldots, 0)$$

$$= (0, \ldots, 0, b_k f_k f, 0, \ldots, 0).$$

Recall that $f_k \in K^*$ and $b_k \neq 0$. Hence

$$\text{Im}(\Psi \circ i_{Q_k}(\mathcal{O}_{\mathbb{P}^1})) = \{(0, \ldots, 0, F, 0, \ldots, 0), F \in \mathcal{O}_{\mathbb{P}^1}\}.$$  

This proves the claim (15).

Now let $M_3$ denote the sub-bundle generated by the image of $M_2$ via the projection $\pi : \mathcal{F} \to \mathcal{G}$. Note that $M_3 \simeq \text{Im}(\mathcal{O}_{\mathbb{P}^1}^{\oplus m-n})$, the one generated by the
images of $i_{\mathcal{Q}_k}, n + 1 \leq k \leq m$ in $\mathcal{G}$. We arrive at, since the lift $(f_i)_{1 \leq i \leq n + r}$ is general,$$
abla = \mathcal{F}/M_2 \simeq \mathcal{G}/M_3 \simeq \left( \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}_1}(c_i) \right) \big/ \text{Im}(\mathcal{O}_{\mathbb{P}_1}^{\oplus m-n}),$$
with $\mathcal{G}$ ample by (14). It remains to apply Lemma 2.12 to conclude that
$$
\mu_{\min}(\mathcal{H}) \geq \mu_{\min}(\mathcal{G}) = \min_{1 \leq i \leq m} (c_i) > 0.
$$

We now come to the Necessity. Suppose that $V = \text{Vect}_Q\{\varrho, \varrho \in \mathcal{P}(1)\} \neq N_Q$. We choose $\{\varrho_1, \cdots, \varrho_q\}, q \leq \min(m = \#\mathcal{P}(1), n - 1)$, a $Q$-basis of $V$ we complete it into $\{\varrho_1, \cdots, \varrho_q, \varrho_{m+1}, \cdots, \varrho_{m+n-q}\}$ a $Q$-basis of $N_Q$. The $r$ relations (where $b_k \neq 0, a_{i,k}, d_{j,k}$ are integers)

$$
R_k : b_k \varrho_k = \sum_{i=1}^{q} a_{i,k} \varrho_i + \sum_{j=1}^{n-q} d_{j,k} \varrho_{m+j}, \quad q+1 \leq k \leq m, m+n-q+1 \leq k \leq n+r.
$$

are linearly independent. Conserving the notation of $F$ (13). Let $M_4$ (resp. $M_5$) be the sub-bundle of $F$ generated by the images of the morphisms $i_{\mathcal{R}_k}, m + n - q + 1 \leq k \leq n + r$ (resp. $k \in \{q + 1, \cdots, m, m + n - q + 1, \cdots, n + r\}$) given by the relations (16) and let $L, K$ be defined as

$$
L = \mathcal{F}/M_4, \quad K = \mathcal{F}/M_5.
$$

Arguing as before on proves that (again since $(f_i)_{1 \leq i \leq n + r}$ is general),

$$
L \simeq \left( \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}_1}(c_i) \right) \oplus \mathcal{O}_{\mathbb{P}_1}^{\oplus n-q}.
$$

Denoting by $M_6$ the sub-bundle of $L$ generated by the images of $i_{\mathcal{R}_k}, k \in \{q + 1, \cdots, m\}$ composed with the quotient $F \to L$, one gets $K \simeq L/M_6$. However, since $\varrho_k \in V$ for $q + 1 \leq k \leq m$, we have $d_{j,k} = 0, \forall 1 \leq j \leq n - q$. The morphisms $i_{\mathcal{R}_k}$ factorise as

$$
i_{\mathcal{R}_k} : \mathcal{O}_{\mathbb{P}_1} \to \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}_1}(c_i) \hookrightarrow F.
$$

so actually $M_6$ is contained (up to automorphism) in the first summation of $L$. Write $M_6$ for it. Consequently,

$$
f^*T_X \simeq K = L/M_6 \simeq \left( \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}_1}(c_i) \big/ M_6 \right) \oplus \mathcal{O}_{\mathbb{P}_1}^{\oplus n-q}
$$

\begin{enumerate}
\item
\item
\item
\item
\item
\end{enumerate}
is not ample since it possesses $n - q > 0$ trivial factors. ■

3. Universal torsors and Cox coordinates

The notion of universal torsors is first introduced by Colliot-Thélène and Sansuc in [CS87]. In this section we present it in the toric setting.

3.1. Construction and parametrization. The exact sequence (5) gives rise to

$$1 \longrightarrow \mathcal{T}_{\text{NS}} \longrightarrow \mathbb{G}_{\text{m}}^{\Delta(1)} \longrightarrow \mathcal{T} \longrightarrow 1$$

between split tori ($\mathcal{T}_{\text{NS}}$ is the Néron-Severi torus). Consider the open subset of the affine space $\mathbb{A}^{\Delta(1)}$ identified with the spectrum of the Cox ring of $X$:

$$\mathcal{S} = \mathbb{A}^{\Delta(1)} \setminus \bigcup_{P \subset \Delta(1)} \left( \bigcap_{\sigma \in P, D_\sigma = \emptyset} (X_\sigma = 0) \right) = \mathbb{A}^{\Delta(1)} \setminus \left( \bigcap_{\sigma \in \Delta_{\text{max}}} \left( \prod_{\sigma \not\in \sigma(1)} X_\sigma = 0 \right) \right).$$

Then we have the geometry quotient $\pi : \mathcal{S} \to X \simeq \mathcal{S} / \mathcal{T}_{\text{NS}}$. We now write this morphism in coordinates. Choose a maximal cone $\sigma \in \Delta_{\text{max}}$ and let $\sigma(1) = \{ \varrho_1, \cdots, \varrho_n \} = \sigma \cap \Delta(1)$. Since $X$ is non-singular, the lattice $N$ is generated by $\varrho_1, \cdots, \varrho_n$. Let $\{ \varrho^\vee_1, \cdots, \varrho^\vee_n \}$ be its dual base. Now $\pi$ is written as

$$\pi : \pi^{-1} U_\sigma \longrightarrow U_\sigma$$

$$\left( X_1, \cdots, X_{n+r} \right) \longmapsto \left( \prod_{j=1}^{n+r} X_j^{\langle \varrho^\vee_i, \varrho_j \rangle} \right)_{1 \leq i \leq n}.$$

Theorem 3.1 (Colliot-Thélène & Sansuc [CS87] §2.3, Salberger [Sa98] §8): $\mathcal{S}$ is a universal torsor (unique up to isomorphism) over $X$ under $\mathcal{T}_{\text{NS}}$.

Remark: Since $\mathcal{T}$ is split, the unicity follows from Hilbert 90:

$$H^1_{\text{ét}}(K, \mathbb{G}_{\text{m}}^r) = 1.$$

The above construction also gives a $\mathcal{O}_K$-morphism between toric schemes $\tilde{\mathcal{S}} \to \tilde{X}$ which is a $\mathcal{O}_K$-model of $\mathcal{T} \to X$. Since in general $H^1_{\text{ét}}(\mathcal{O}_K, \mathbb{G}_{\text{m}}) = \text{Cl}_K$ is non-trivial, in order to parametrize all rational points, it is necessary to introduce “twisted” torsors [Sko01 p. 20] by elements in $\text{Cl}_K^r$. Following
Let \( C \) be a set of representatives of \( \text{Cl}_{K} \). For any \( r \)-tuple \( c = (c_1, \cdots, c_r) \in C^r \), we identify its class \([c]\) in \( H^1_{\text{ét}}(X, \mathbb{G}_m^r) \) via the morphism (cf. \cite[Théorème 1.5.1]{CS87}) \( \text{Cl}_r^k = H^1_{\text{ét}}(\text{Spec}(O_K), \mathbb{G}_m^r) \rightarrow H^1_{\text{ét}}(X, \mathbb{G}_m^r) \). The twisted torsor \( \tilde{c}T \) is a universal torsor of class \([T] - [c]\) in \( H^1_{\text{ét}}(X, \mathbb{G}_m^r) \). To describe explicitly the integral points, we introduce the following notations. Fix a basis \( D = \{[D_1], \cdots, [D_r]\} \) for \( \text{Pic}(X) \) over \( \mathbb{Z} \). For a divisor \( D \), write \([D] = \sum_{j=1}^r b_j[D_j], b_j \in \mathbb{Z}\) in terms of the basis \( D \). We define the fractional ideal \( cD = \prod_{j=1}^r c_j^{b_j} \).

Let \( X = (X_\sigma)_{\sigma \in \Delta(1)} \in K^{\Delta(1)} \) denote a \((n+r)\)-tuple of points in \( K \) indexed by the rays.

**Theorem 3.2:** The set \( \tilde{\Xi}(O_K) \) contains precisely the \((n+r)\)-tuples \( X \in \bigoplus_{\sigma \in \Delta(1)} cD^\sigma \subset K^{\Delta(1)} \) satisfying the coprimality condition

\[
\sum_{\sigma \in \Delta_{\text{max}}} \prod_{\varrho \in \Delta(1) \setminus \sigma(1)} X_\varrho c^{-D^\varrho} = O_K.
\]

Moreover, we have

\[
X(K) = \bigsqcup_{c \in C^r} \tilde{\Xi}(c\Xi(O_K)).
\]

**Proof.** This is a reformulation of \cite[§2.3]{CS87} originally stated for fields. See \cite[p. 15]{Rob98} and also \cite[Theorem 2.7]{FP16}, \cite[§2]{Pie16}. \(\blacksquare\)

So any rational point of \( X \) admits a lift to an \( O_K \)-point in some twist of \( \Xi \), different lifts differing via the action of the Néron-Severi torus.

### 3.2. Heights on Toric Varieties

In this section we follow Salberger \cite{Sal98} and derive height formulas using the combinatorial data of the fan \( \Delta \).

Let \( D = \sum_{i=1}^{n+r} a_{i\sigma} D_{\sigma} \), be a \( \mathcal{T} \)-invariant divisor considered to be an element in \( Z^{\Delta(1)} \) in the exact sequence (5). We suppose that the line bundle \( L = O_X(D) \) is globally generated. For any \( \sigma \in \Delta_{\text{max}} \), recall that \( m_D(\sigma) \in M \) generates \( L \) on \( U_\sigma \). It defines a character \( \chi^{m(\sigma)} : \mathcal{T} \rightarrow \mathbb{G}_m \) and lifts to a global section of \( L \). For \( \nu \in \mathcal{M}_K \) and \( P_\nu \in \mathcal{T}(K_\nu) \), for any \( s \in H^0(X, L) \) does not vanish on \( P_\nu \), we

\[\text{[Rob98] §2} \] and \[\text{[Pie16] §2.1} \], we shall present the explicit construction and formulas for toric varieties.
define
\[ \|s(P_\nu)\|_{D,\nu} = \inf_{\sigma \in \Delta_{\text{max}}} \left| \frac{s(P_\nu)}{\chi^{m(\sigma)}(P_\nu)} \right|_\nu. \]

**Proposition-Definition 3.3** ([Sal98], Propositions 9.2 & 9.8): The function \( H_D : \mathcal{T}(K) \rightarrow \mathbb{R}_{>0} \) defined by the formula
\[ H_D(P) = \prod_{\nu \in \mathcal{M}_K} \|s(P)\|_{D,\nu}^{-1}, \]
is an Arakelov Height. Its equivalence class only depends on the class of \( D \) in \( \text{Pic}(X) \) (i.e. the image of of \( D \) via \( \mathbb{Z}^{\Delta(1)} \rightarrow \text{Pic}(X) \)).

Salberger also gives formula to compute \( H_D \). For this we introduce the following notation. For a point \( P_0 = (X_1, \ldots, X_{n+r}) \in K^{\Delta(1)} \), we note
\[ X(P_0)^D = \prod_{i=1}^{n+r} X_i^{a_{\varphi_i}}, \]
and for any \( \sigma \in \Delta_{\text{max}} \), Let
\[ D(\sigma) = D + \sum_{i=1}^{n+r} (m_D(\sigma), g_i) D_{g_i} = \sum_{\varphi_i \in \Delta(1) \setminus \sigma(1)} (a_{\varphi_i} + \langle m_D(\sigma), g_i \rangle) D_{g_i}. \]
(Also viewed as an element in \( \mathbb{Z}^{\Delta(1)} \).) We remark that since \( L \) is globally generated, \( \langle m(\sigma), g_i \rangle \geq -a_{g_i} \), so \( D(\sigma) \) is an effective divisor with support contained in \( \bigcup_{\varphi \in \Delta(1) \setminus \sigma(1)} D_{\varphi} \) (cf. [Ful93, p.61-68] or [Sal98, Proposition 8.7]).

**Proposition 3.4:** Suppose that \( \mathcal{O}_X(D) \) is globally generated. Let \( P \in \mathcal{T}(K) \) which lifts to \( P_0 \in \widetilde{c}(\mathcal{O}_K) \) for some \( c \in C^r \). Then
\[ H_D(P) = \prod_{\nu \in \mathcal{M}_K} \sup_{\sigma \in \Delta_{\text{max}}} |X^{D(\sigma)}(P_0)|_\nu \cdot \frac{1}{\text{Norm}(c^D)} \prod_{\nu \in \mathcal{M}_K^\infty} \sup_{\sigma \in \Delta_{\text{max}}} |X^{D(\sigma)}(P_0)|_\nu. \]

**Proof.** Write \( P_0 = (X_{\varphi})_{\varphi \in \Delta(1)} \in K^{\Delta(1)} \). We mostly follow the proof of [Pie16, Proposition 2] which gives the formula for \( H_{\omega_X^{-1}} \). Indeed, by [Sal98, Proposition 10.14], combing Proposition-Definition 3.3 we have
\[ H_D(P) = \prod_{\nu \in \mathcal{M}_K} \sup_{\sigma \in \Delta_{\text{max}}} |X(P_0)^D(\sigma)|_\nu. \]
Fix throughout \( p \in \text{Spec}(\mathcal{O}_K) \) and note \( \nu \in \mathcal{M}_K \) the corresponding place. For every \( \varphi \in \Delta(1) \), define \( m_{\varphi,p}, X_{\varphi,p} \in \mathbb{Z} \) such that
\[ m_{\varphi,p} = \text{ord}_p(c^D_{\varphi}), \quad X_{\varphi,p} = \text{ord}_p(X_{\varphi} \mathcal{O}_K). \]
Define also $a_\varrho,\sigma \in \mathbb{Z}$ such that $D(\sigma) = \sum_{\varrho \in \Delta(1)} a_\varrho, \sigma D_\varrho$. We now compute the $\nu$-adic part of the height above

$$\sup_{\sigma \in \Delta_{\text{max}}} |X(P_0)^D(\sigma)|_\nu = \text{Norm}(p)^{-\min_{\sigma \in \Delta_{\text{max}}} a_\varrho, \sigma x_{\varrho, p}}.$$

Since $\mathcal{O}_X(D)$ is globally generated, the divisor $D(\sigma)$ is effective and its support is in $\cup_{\varrho \in \Delta(1)} \setminus \sigma(1) D_\varrho$, we have that $a_\varrho, \sigma \geq 0$ for every $\varrho \in \Delta(1)$ and in particular $a_\varrho, \sigma = 0$ for $\varrho \in \sigma(1)$. Because $[D] = [D(\sigma)]$ in $\text{Pic}(X)$, we have the following equality

$$\text{ord}_p(c^{D(\sigma)}) = \sum_{\varrho \in \Delta(1)} a_\varrho, \sigma m_{\varrho, p} = \sum_{\varrho \in \Delta(1) \setminus \sigma(1)} a_\varrho, \sigma m_{\varrho, p} = \sum_{\varrho \in \Delta(1)} b_\varrho m_{\varrho, p} = \text{ord}_p(c^D).$$

Note by Theorem 3.2 that $X_\varrho \in c^{D_\varrho}$, we have that $X_\varrho c^{-D_\varrho}$ is an ideal. Recall the coprimality condition (20), we have

$$\min_{\sigma \in \Delta_{\text{max}}} \text{ord}_p \left( \prod_{\varrho \in \Delta(1) \setminus \sigma(1)} (X_\varrho c^{-D_\varrho})^{a_\varrho, \sigma} \right) = 0.$$

So there exists $\sigma \in \Delta_{\text{max}}$ such that

$$\text{ord}_p \left( \prod_{\varrho \in \Delta(1) \setminus \sigma(1)} X_\varrho c^{-D_\varrho} \right) = \sum_{\varrho \in \Delta(1) \setminus \sigma(1)} a_\varrho, \sigma (x_{\varrho, p} - m_{\varrho, p}) = 0.$$

We conclude now

$$\min_{\sigma \in \Delta_{\text{max}}} a_\varrho, \sigma x_{\varrho, p} = \min_{\sigma \in \Delta_{\text{max}}} \left( \sum_{\varrho \in \Delta(1) \setminus \sigma(1)} a_\varrho, \sigma (x_{\varrho, p} - m_{\varrho, p}) + \sum_{\varrho \in \Delta(1)} a_\varrho, \sigma m_{\varrho, p} \right)$$

$$= \sum_{\varrho \in \Delta(1)} b_\varrho m_{\varrho, p}.$$

which is exactly the $p$-th order of $c^D$. We get finally

$$\sup_{\sigma \in \Delta_{\text{max}}} |X(P_0)^D(\sigma)|_\nu = \prod_{p \in \text{Spec} \left( \mathcal{O}_K \right)} \text{Norm}(p)^{-\sum_{\varrho \in \Delta(1)} b_\varrho m_{\varrho, p}}$$

$$= \text{Norm}(c^D)^{-1},$$

the desired factor. □
Briefly, the reason why there are only archimedean factors left is due to the co-
primality condition \((20)\) in Theorem 3.2 which implies \(\sup_{\sigma \in \Delta_{\max}} |X(P_0)^{D(\sigma)}|_\nu = 1\) for all finite places \(\nu\).

4. Approximation constants

We recall the notion due to McKinnon and M. Roth. We will be brief and refer the reader to [MR15, §2]. Let \(X\) be an irreducible projective variety. We fix a rational point \(Q \in X(\overline{K})\). Let \(L\) be a line bundle, to which we associate a height function \(H_L\). For \(\nu \in \mathcal{M}_K\), we choose a projective distance function \(d_\nu(\cdot, \cdot)\) defined locally around \(Q \times Q\) on \(X(K_\nu) \times X(K_\nu)\). We shall mainly use the distance function of the following form. For an affine neighbourhood \(U\) of \(X\), choose elements \(u_1, \ldots, u_m \in H^0(U, \mathcal{O}_X)\) that generate the ideal of \(\mathcal{O}_X(U)\) defining \(Q\). We note \(\| \cdot \|_\nu = | \cdot |_\nu\) if \(\nu \in \mathcal{M}_K^f\) or \(\nu\) is real and \(\| \cdot \|_\nu = | \cdot |_\nu^2\) if \(\nu\) is complex. Then

\[
(21) \quad d_\nu(Q, \cdot) = \min(1, \max_{1 \leq i \leq m} (\| u_i(\cdot) \|_\nu))
\]

is a choice of \(\nu\)-adic distance on \(U(K_\nu)\). We can show that the function \(d_\nu(Q, \cdot)^{-1}\) is (equivalent to) a local \(\nu\)-adic Weil height function associated to the exceptional divisor of the blow up of \(X\) at \(Q\) (cf. for example [MR16, Lemma 3.1]).

**Proposition-Definition 4.1** (McKinnon-M. Roth): For any subvariety \(Y \subset X\), we define the (best) approximation constant \(\alpha_{L,\nu}(Q, Y)\) (depending on \(L\) and \(\nu\)) to the infimum of the following set

\[
A(Q, Y) = \{ \gamma > 0 : \exists C > 0, \exists (y_i) \subset Y(K), d_\nu(Q, y_i) \to 0, d_\nu(Q, y_i)^\gamma H_L(y_i) < C \}.
\]

Assume that \(L\) is big and that \(Q\) is not in the base locus \(\text{Bs}(L)\) of \(L\). Then \(\alpha_{L,\nu}(Q, Y)\) can also be computed as the supremum of the set

\[
B(Q, Y) = \{ \gamma > 0 : \exists C > 0, d_\nu(Q, y) H_L(y)^\gamma \geq C, \forall y \in Y(K) \setminus \{Q\} \}.
\]

**Proof.** Since some power of \(L\) defines a rational map \(X \dashrightarrow \mathbf{P}_K^n\) ([Laz04, Corollary 2.2.7]) which is birational on an open set containing \(Q\), \(L\) verifies the Northcott property locally around \(Q\). So the equality \(\sup B(Q, Y) = \inf A(Q, Y)\) follows from [MR15, Proposition 2.11].
Remark: When $X$ is toric and $Q$ is in the open orbit $T$, then for every big $T$-invariant divisor $D$, $Q$ is never in $\text{Bs}(O_X(D))$ by torus action.

Example 4.2: K. Roth’s Theorem\(^\text{[1]}\) can be formulated by using this $\alpha$-constant. From the definition of the approximation exponent $\mu(\theta)$, we have immediately that $\mu(\theta)$ equals the infimum of $\lambda > 0$ such that the inequality
$$\left| \frac{p}{q} - \theta \right| \leq \frac{1}{\max(|p|, |q|)^\lambda}$$
has only finitely many solutions $\frac{p}{q} \in \mathbb{Q}$. Since $H(\frac{p}{q}) = \max(|p|, |q|)$ is an $\mathcal{O}(1)$-height, and that in Definition\(^\text{[1,1]}\) the exponent is on the distance function, we obtain that for $\theta \in \mathbb{P}^1(\bar{Q} \cap \mathbb{R})$,
$$\alpha_{\mathcal{O}(1), \infty}(\theta, \mathbb{P}^1) = \frac{1}{\mu(\theta)} = \begin{cases} 1 & \text{if } \theta \in \mathbb{P}^1(\mathbb{Q}); \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Most of the time we omit the subscripts $L, \nu$ if there they are fixed throughout. The definition of essential (approximation) constant first appears in the work of Pagelot\(^\text{[Pag08]}\) concerning statistic problems of rational points.

The value of $\alpha$ is independent of the choices of the distance $d_\nu$ and the height $H_L$. So the approximation constant $\alpha$ is an intrinsic notion for each rational point and should inherit geometric properties from the ambient variety.

Definition 4.3 (Pagelot\(^\text{[Pag08]}\)): With the notation above, we define the essential constant of $Q$ with respect to $Y$ to be
$$\alpha_{\text{ess},L}(Q, Y) = \sup_{V \subset Y} \alpha_L(Q, V),$$
where $V$ ranges over all Zariski open dense subvarieties of $Y$. We write $\alpha_{\text{ess},L}(Q) = \alpha_{\text{ess},L}(Q, X)$. If there exists a subvariety $Z$ such that $\alpha_{\text{ess},L}(Q, Z) < \alpha_{\text{ess}}(Q)$, we say that $Z$ is locally accumulating.

Definition 4.4: If $\alpha(Q, Z) = \alpha(Q, X)$, we sometimes say that the approximation constant can be achieved on $Z$. If $\alpha_{\text{ess}}(Q, Z) < \alpha(Q, X \setminus Z)$, we say that the approximation constant is achieved on $Z$.

Thus if $Z$ is locally accumulating, then for any $0 < \gamma < \alpha(Q, X \setminus Z) \leq \alpha_{\text{ess}}(Q)$, all but finitely solutions of the inequality\(^\text{[2]}\) must lie in $Z$. Moreover $Z$ itself does not contain locally accumulating subvariety. The last terminology supersedes and differs slightly from the first version introduced in\(^\text{[Hua17b]}\).
This correction avoids the drawback that \( X \) itself being locally accumulating. It is useful to be aware of the difference between “can be achieved” and “is achieved”. The first means on \( Z \) there exists a sequence of rational points attaining \( \alpha(Q, X) \) in Definition-Proposition 4.1. However the second indicates that looking for sequences of rational points to compute \( \alpha(Q, X) \), we have to restrict to \( Z \).

Useful properties of \( \alpha \) are gathered together below.

**Proposition 4.5** ([MR15], Lemma 2.13, Proposition 2.14): We have:

1. Let \( Q \in \mathbb{P}^n(K) \), equipped with an \( \mathcal{O}(1) \)-height. Then \( \alpha(Q, \mathbb{P}^n) = 1 \).
2. For \( m \in \mathbb{N}_{\geq 1} \), we have \( \alpha_{mL,\nu}(Q, Y) = m\alpha_{L,\nu}(Q, Y) \).
3. \( \alpha(Q, Y) \geq \alpha(Q, X) \) for any subvariety \( Y \) of \( X \).

Our way of computing the approximation constant consists of two steps. First use (3) of Proposition 4.5 to bound \( \alpha(Q, X) \) from above. A frequently chosen ideal candidate is the rational curve \( Y \simeq \mathbb{P}^1 \). If \( Y \) is smooth at \( Q \), then by (1) of Proposition, \( \alpha(Q, Y) \) equals \( \deg L \). Next, prove estimation of the shape \( d_{\nu}(Q, P) \gamma H(P) \geq C > 0 \) for any \( P \neq Q \), so that \( \gamma \in B(Q, X) \). So by Proposition-Definition 4.1, \( \alpha(Q, X) = \inf B(Q, X) \leq \gamma \).

### 5. Geometry of numbers

We collect some classical useful facts here about algebraic number fields and we refer to standard textbooks (say [Sam03]) for proofs. Recall that \([K : \mathbb{Q}] = r_1 + 2r_2\), where \( r_1 \) (resp. \( r_2 \)) is the number of real (resp. complex) places of \( K \). The map \( \varsigma = (\sigma_{\nu})_{\nu \in \mathcal{M}_K} \) embeds \( K \) into the \( \mathbb{R} \)-vector space \( \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \). The following simple observation can be generalised to any fractional ideal, at the expense of adding some extra multiples.

**Lemma 5.1:** Let \( x \in \mathcal{O}_K \setminus 0 \). Then

1. For any \( \nu \in \mathcal{M}_K \), there exists a constant \( \lambda_K > 0 \) only depending on the number field \( K \) such that \( |x|_{\nu} \geq \lambda_K \);
2. For any \( \nu \in \mathcal{M}_K \), \( |x|_{\nu}^{-1} \) divides \( \prod_{\nu \in \mathcal{M}_K} |\sigma_{\nu}(x)|_{\nu} \).

**Proof.** (1) This is because \( \varsigma \) maps \( \mathcal{O}_K \) into a lattice of full rank in \( \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \) so 0 must be isolated.

(2) Let \( p \) denote the prime ideal correspond to \( \nu \). Let \( m_x = \text{ord}_p(x\mathcal{O}_K) \). We have that in \( \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \), \( \varsigma(x\mathcal{O}_K) \) is a sublattice of \( \varsigma(p^{m_x}) \). We thus obtain the
following divisibility between their co-volumes

\[ |x|_\nu^{-1} = \text{Norm}(p^{m_x}) \mid \text{Norm}(x\mathcal{O}_K) = |N_{K/Q}(x)|_\infty = \prod_{\nu \in \mathcal{M}_{\mathbb{K}}} |\sigma_\nu(x)|_\nu, \]

as desired. ■

6. Determination of \( \alpha \) and locally accumulating subvarieties

The goal of this section is to prove the following detailed version of Part I of Theorem 1.2. We shall fix \( D = \sum_{i=1}^{n+r} a_i D_i \) a \( \mathcal{T} \)-invariant divisor and suppose that \( L = \mathcal{O}_X(D) \) is nef. By torus action, we can assume that the point to be approximated is the unit element \((1, \cdots , 1)\). Define \( \beta \in \mathbb{N} \) as

\[ (22) \quad \beta = \min_{\mathcal{P}} \deg_L \mathcal{P}. \]

**Theorem 6.1:** Suppose that \( X \) verifies Hypothesis (*) and that \( L \) is globally generated and big. Let \( Q_0 = (1, \cdots , 1) \in \mathcal{T}(K) \). We have that for every place \( \nu \in \mathcal{M}_K \), \( \alpha_{L,\nu}(Q_0, X) = \beta \). Suppose moreover that \( L \) is ample and \( X \neq \mathbb{P}^n \). Then the constant \( \alpha_{L,\nu}(Q_0, X) \) is achieved on a proper closed subvariety \( Y \) which is a finite union of \( Y_i \simeq \mathbb{P}^{N_i-1} \), each one being the fibre \( \mathbb{P}^{N_i-1} \times \{1\} \) of an open toric subvariety of \( X \), isomorphic to \( \mathbb{P}^{N_i-1} \times \mathbb{G}_m^{n-N_i+1} \) and corresponding to a centred primitive collection of \( L \)-degree \( \beta \) and of cardinality \( N_i \). Furthermore, if there exist two different \( Y_i, Y_j \), then \( Y_i \cap Y_j = Q_0 \).

We start by proving several technical lemmas. We first of all translate the Hypothesis (*) in the beginning into a combinatorial one.

**Lemma 6.2:** The hypothesis (*) is equivalent to

(\( ** \)) There exists \( \sigma_0 \in \Delta_{\text{max}} \) such that all members of the set \( \Delta(1) \setminus \sigma_0(1) \) are linear combinations of those in \( \sigma_0(1) \) with negative coefficients.

**Proof.** Recall first of all that \( \text{Pic}(X) \) is generated by the classes of boundary divisors. Note also the following equivalence:

\[ (23) \quad [D_{\varrho_i}] = \sum_{j=1}^{r} a_{i,j} [D_{\varrho_{n+j}}], \quad 1 \leq i \leq n \Leftrightarrow \varrho_{n+j} = -\sum_{i=1}^{n} a_{i,j} \varrho_i, \quad 1 \leq j \leq r. \]

Briefly, the second system of relations is equivalent to

\[ \langle \varrho_i^\vee, \varrho_{n+j} \rangle = -a_{i,j}, \quad 1 \leq i \leq n, 1 \leq j \leq r. \]
The elements $\varrho_1, \ldots, \varrho_n \in M$ now give the first system of equalities from the definition of the maps $f, g, h, i$ in the exact sequences (5) (6).

So once assume Hypothesis (*), that is, $a_{i,j} \geq 0$ for all $1 \leq i \leq n, 1 \leq j \leq r$, then necessarily $\varrho_1, \ldots, \varrho_n$ form the set of generators of a maximal cone and all $a_{i,j} \in \mathbb{N}$ thanks to the completeness and regularity of the fan. We conclude that

\begin{equation}
\sigma_0 = R_{\geq 0}\varrho_1 + \cdots + R_{\geq 0}\varrho_n \quad \text{and} \quad \text{Eff}(X) = \sum_{j=1}^{r} R_{\geq 0}[D_{\varrho_{r+j}}].
\end{equation}

Now the equivalence between (***) and (**) is clear. \hfill $\blacksquare$

**Lemma-Notation 6.3:** Under Hypothesis (**), let $\sigma_0$ be as in (24). Write

$$
P_{n+j} : \varrho_{n+j} + \sum_{i=1}^{n} b_{i,j} \varrho_i = 0, \quad 1 \leq j \leq r.
$$

Then these are positive relations (i.e. $b_{i,j} \in \mathbb{N}$). Suppose that $L$ is globally generated, then we have (recall $\beta$ in (22))

$$
\deg_L(P_{n+j}) = a_{n+j} + \sum_{i=1}^{n} a_i b_{i,j} \geq \beta.
$$

**Proof.** This follows from Theorem 2.8. \hfill $\blacksquare$

We will use the notations $P_{n+j}, 1 \leq j \leq r, \sigma_0, (a_i)_{1 \leq i \leq n+r}, (b_{i,j})_{1 \leq i \leq n, 1 \leq j \leq r}$ throughout the rest of this section. We next prove lemmas about centred primitive collections. The first one seems well-known.

**Lemma 6.4:** Let $I_1, I_2$ be two different centred primitive relations. Then $I_1(1) \cap I_2(1) = \emptyset$.

**Proof.** If there exists $\varrho \in I_1(1) \cap I_2(1)$, then $$-\varrho = \sum_{\varrho_i \in I_1(1) \setminus \{\varrho\}} \varrho_i = \sum_{\varrho_j \in I_2(1) \setminus \{\varrho\}} \varrho_j.$$ These are positive combinations of generators in (maximal) cones, so they should be the same. i.e. $I_1 = I_2$. \hfill $\blacksquare$

The crucial use of Hypothesis (***) will be clear from the next proposition. For $1 \leq i_0 \leq n$, let $\sigma_{i_0}$ denote the maximal cone which is adjacent to $\sigma_0$ so that

\begin{equation}
\sigma_{i_0} \cap \sigma_0 = R_{\geq 0}\rho_1 + \cdots + R_{\geq 0}\rho_{i_0} + \cdots + R_{\geq 0}\rho_n.
\end{equation}
where “ˆ” means this term does not appear in the summation. The existence of exactly \( n \) such maximal cones follows from the completeness and the regularity of \( \Delta \) (and hence each codimension 1 cone is the common face of a unique pair of maximal cones). (cf. also [Sal98, Lemma 8.9].) Write for some \( 1 \leq j_1 \leq r \),

\[
\sigma_{i_0} = R_{\geq 0} \rho_1 + \cdots + \hat{R}_{\geq 0} \rho_{i_0} + \cdots + R_{\geq 0} \rho_n + R_{\geq 0} \rho_{n+j_1}.
\]

Our second lemma says that the maximal cone \( \sigma_0 \) contains most of the elements of any centred primitive collections, so does certain of its adjacent cones.

**Lemma 6.5:** Under Hypothesis (**), for any centred primitive relation \( \mathcal{I} \), we have

\[
\#(\mathcal{I}(1) \setminus \sigma_0(1)) = 1.
\]

Moreover, for \( 1 \leq i_0 \leq n \), recall the index \( j_1 \) in \((26)\), then \( \varrho_{n+j_1} \in \sigma_{i_0}(1) \cap \mathcal{I}(1) \) if and only if \( \varrho_{i_0} \in \mathcal{I}(1) \).

**Proof.** Since \( \mathcal{I}(1) \nsubseteq \sigma(1) \) for any \( \sigma \in \Delta_{\text{max}} \), let \( \varrho_{n+j_0} \in \mathcal{I}(1) \setminus \sigma_0(1) \). We can write

\[
\varrho_{n+j_0} = -\sum_{i=1}^{n} b_{i,j_0} \rho_i = -\sum_{\rho \in \mathcal{I}(1) \setminus \{\rho_{n+j_0}\}} \rho.
\]

As before this also gives two expressions of \( -\varrho_{n+j_0} \) in terms of positive combinations of rays all in the same cones and hence they coincide. Hence \( \mathcal{I}(1) \setminus \{\varrho_{n+j_0}\} \subset \sigma_0(1) \). Recall the positive relation \( \mathcal{P}_{n+j_1} \) in Lemma 6.3. Now fix \( i_0 \). By looking at the lattice transfer matrix \( \mathcal{M} \) between \( \sigma_0 \) and \( \sigma_{i_0} \), we have \( |\text{det} \mathcal{M}| = |b_{i_0,j_1}| \). We then have necessarily \( b_{i_0,j_1} = 1 \) because the fan \( \Delta \) is regular and \( b_{i_0,j_1} \geq 0 \) by assumption. If \( j_0 = j_1 \), then \( b_{i_0,j_0} = 1 \). So the equality above gives \( \varrho_{i_0} \in \mathcal{I}(1) \). Conversely, if \( \varrho_{i_0} \in \mathcal{I}(1) \), by moving terms in \( \mathcal{P}_{n+j_1} \), we get

\[
\varrho_{n+j_1} = \sum_{i \in \{1, \ldots, n\} \setminus \{i_0\}} b_{i,j_1} \varrho_i = -\varrho_1 = \sum_{\varrho \in \mathcal{I}(1) \setminus \{1\}} \varrho,
\]

an equality between two positive combinations of rays as generators of some cones. So \( \varrho_{n+j_1} \in \mathcal{I}(\varrho_1) \). \( \blacksquare \)

**Proposition 6.6:** Suppose that \( L \) is globally generated. Then for all \( 1 \leq i_0 \leq n, 1 \leq j_0 \leq r \), we have, under Hypothesis (**), (recall the notation \( \mathcal{P}_{n+j} \) in Lemma 6.3 and Definition 2.3)

\[
\deg_L(\mathcal{P}_{n+j_0}) = a_{n+j_0} + \sum_{i=1}^{n} a_i b_{i,j_0} \geq b_{i_0,j_0} \beta.
\]
If $L$ is ample, suppose moreover that there exists $i_0 \in \{1, \cdots, n\}$ (resp. $j_0 \in \{1, \cdots, r\}$) such that $\varrho_{i_0}$ (resp. $\varrho_{n+j_0}$) does not belong to any centred primitive collections of $L$-degree $\beta$. Then for all $1 \leq j_0 \leq r$ (resp. for all $1 \leq i_0 \leq n$), we have
\[
\deg_L (\mathcal{P}_{n+j_0}) = a_{n+j_0} + \sum_{i=1}^{n} a_i b_{i,j_0} > b_{i_0,j_0} \beta.
\]

**Proof.** We begin with the first part, i.e. $L$ is globally generated. We fix indices

$i_0 \in \{1, \cdots, n\}, j_0 \in \{1, \cdots, r\}$ and look at the maximal cone $\sigma_i$ defined just before. If $j_0 = j_1$, the desired inequality is nothing but the one in Lemma [6.3] because $b_{i_0,j_1} = 1$. From now on suppose $j_0 \neq j_1$. We write $\varrho_{n+j_0}$ in terms of the generators of the cone $\sigma_i$, namely $\{\varrho_1, \cdots, \varrho_{i_0}, \cdots, \varrho_n, \varrho_{n+j_1}\}$, using the fact that $b_{i_0,j_1} = 1$.

\[
\rho_{n+j_0} = - \sum_{i=1}^{n} b_{i,j_0} \rho_i
\]

\[
= b_{i_0,j_0} \left( \sum_{i \in \{1, \cdots, n\} \setminus \{i_0\}} b_{i,j_1} \rho_i + \rho_{n+j_1} \right) - \sum_{i \in \{1, \cdots, n\} \setminus \{i_0\}} b_{i,j_0} \rho_i
\]

\[
= b_{i_0,j_0} \rho_{n+j_1} - \sum_{i \in \{1, \cdots, n\} \setminus \{i_0\}} (b_{i,j_0} - b_{i_0,j_0} b_{i,j_1}) \rho_i.
\]

Using the assumption that $L$ is globally generated, the piecewise linear function $\phi_D$ is convex. In particular, its graph lies “below” the linear function $\langle m_D(\sigma_i), \cdot \rangle$, where $m_D(\sigma_i) = - (\sum_{i \in \{1, \cdots, n\} \setminus \{i_0\}} a_i \varrho_i^\vee + a_{n+j_1} \varrho_{n+j_1}^\vee).$ Applying $\phi_D$ to the above equality of $\varrho_{n+j_0}$ we get (by (\ref{25}))
\[
- a_{n+j_0} = \phi_D(\varrho_{n+j_0})
\]

(27)
\[
\leq \langle m_D(\sigma_i), \varrho_{n+j_0} \rangle = \sum_{i \in \{1, \cdots, n\} \setminus \{i_0\}} a_i (b_{i,j_0} - b_{i_0,j_0} b_{i,j_1}) - a_{n+j_1} b_{i_0,j_0}.
\]

So again by Lemma [6.3] and that $b_{i_0,j_1} = 1$,
\[
a_{n+j_0} + \sum_{i=1}^{n} a_i b_{i,j_0} > b_{i_0,j_0} \left( a_{n+j_1} + \sum_{i=1}^{n} a_i b_{i,j_1} \right)
\]

(28) \[
= b_{i_0,j_0} \deg_L (\mathcal{P}_{n+j_1}) \geq b_{i_0,j_0} \beta.
\]
Now assume $L$ is ample and let $\varrho_{i_0}$ be as in the assumption. That is, $\varrho_{i_0}$ is not a member of any centred primitive $\mathcal{I}$ with $\deg_L(\mathcal{I}) = \beta$. Recall $\sigma_{i_0}$ and the index $j_1$ \([26]\). Fix $j_0 \in \{1, \ldots, r\}$. If $j_0 = j_1$, then by Lemma \([6.5]\) $\varrho_{n+j_0} \not\in \mathcal{I}(1)$ for any centred primitive $\mathcal{I}$ of $L$-degree $\beta$. Hence the positive relation $\mathcal{P}_{n+j_0}$ has $L$-degree, by Theorem \([2.8]\) $a_{n+j_0} + \sum_{i=1}^n a_i b_{i,j_0} > b_{i_0,j_1} \beta = \beta$. If $j_0 \neq j_1$, then $\varrho_{n+j_0} \not\in \sigma_{i_0}(1)$. So the strict convexity of the function $\phi_D$ \([9]\) yields that the inequality \([27]\) above is strict. Now assume $\varrho_{n+j_0}$ be as in the second assumption. Fix $i_0 \in \{1, \ldots, n\}$. If $j_0 \neq j_1$, that is, $\varrho_{n+j_0} \not\in \sigma_{i_0}(1)$, then as before the inequality \([27]\) is strict. If $j_0 = j_1$, we have $b_{i_0,j_0} = 1$. Since by assumption, the positive relation $\mathcal{P}_{n+j_0}$ is not a centred primitive one of $L$-degree $\beta$, the inequality \([28]\) is now strict by Theorem \([2.8]\) \(\boxed{}\).

With all these preparations, we are going to prove our main theorem.

**Proof of Theorem \([6.1]\)** To ease notations we shall use the simplification $\alpha(Q_0, Y) = \alpha_{L, \nu}(Q_0, Y)$ and “centred primitive collection” will be abbreviated as “CPC”. We will use the parametrization given by the maximal cone $\sigma_0$ \([23]\). It is given by \([19]\)

$$
\pi : \pi^{-1}U_{\sigma_0} \longrightarrow U_{\sigma_0}
$$

\[(29)\]

$$(X_1, \ldots, X_{n+r}) \longmapsto (y_1, \ldots, y_n) = \left( \frac{X_1}{\prod_{j=1}^r X_{n+j}^{b_{1,j}}}, \ldots, \frac{X_n}{\prod_{j=1}^r X_{n+j}^{b_{n,j}}} \right).$$

Recall that $Q_0 = (1, \ldots, 1)$. To fix notations, choose $\mathcal{C}$ a set of entire ideals as representatives of $\text{Cl}_K$ and choose $[D_{\varrho_{n+1}}], \ldots, [D_{\varrho_{n+r}}]$ as a basis for $\text{Pic}(X)$. The choice of the set $\mathcal{C}$ and the equivalent Hypothesis \((**)\) (Lemma \([6.2]\)) guarantee that $c^{D_{\varrho}}$ is an ideal of $\mathcal{O}_K$ for every $\varrho \in \Delta(1)$, so that $\bigoplus_{\varrho_i \in \Delta(1)} c^{D_{\varrho_i}} \subset \mathcal{O}_K^{\Delta(1)}$. Let $P = (y_1, \ldots, y_n) \in U_{\sigma_0}(K)$ and $P_0 = (X_{\varrho_1}, \ldots, X_{\varrho_{n+r}}) \in \mathcal{C}(\mathcal{O}_K)$ denotes one of its lifts for some $c \in \mathcal{C}$. Before going into the long details, let us sketch the main idea behind. In order that $P$ approximates $Q_0$ with respect to a fixed place $\nu$, that is,

$$
\max_{1 \leq i \leq n} \left| \frac{X_i}{\prod_{j=1}^r X_{n+j}^{b_{1,j}}} - 1 \right|_\nu \to 0,
$$

in archimedean case, The denominators and the numerators both have to tend to infinity so should have almost equal heights. However in ultrametric cases things are different. It is their differences $X_i - \prod_{j=1}^r X_{n+j}^{b_{i,j}}$ that should be sufficiently divisible by powers of the prime ideal associated to $\nu$, but both of
them could have very small $p$-adic orders (hence big $\nu$-adic values), however the decreasing can be controlled by the contribution from all archimedean places. The method is to carefully select maximal cones to bound their heights in order to “compensate” the decreasing of the distance.

Let us now put all this into practice. For $P \neq Q_0$, there exists $i_0 \in \{1, \cdots, n\}$ such that $z_{i_0} = \prod_{j=1}^r X_{n+j}^{-b_{i_0,j}} - X_{i_0} \neq 0$. According the height formula (Proposition 3.4), we choose the maximal cone $\sigma_0$ so that viewed as an element in $Z^{\Delta(1)}$,

$$D(\sigma_0) = D + \sum_{\varrho \in \Delta(1)} \langle m_D(\sigma_0), \varrho \rangle D_\varrho$$

$$= \sum_{j=1}^r (a_{n+j} + \sum_{i=1}^n a_i b_{i,j}) D_{\varrho_{n+j}} = \sum_{j=1}^r \deg_L(P_{n+j}) D_{\varrho_{n+j}},$$

so that

$$X(P_0)^D(\sigma) = \prod_{j=1}^{n+r} X_{n+j}^{-\deg_L(P_{n+j}).}$$

Suppose first that $\nu \in M_K^{\infty}$. Recall the constant $\lambda_K$ in Lemma 5.1 and the notations in Section 3. Since $P_0 \in O_K^{\Delta(1)}$ and

$$\text{Norm}(c^D) H_L(P) \geq \prod_{\nu' \in M_K^{\infty}} |X(P_0)^D(\sigma_0)|_{\nu'}$$

$$\geq \lambda_K^{\# M_K^{\infty} - [K_\nu : R]} \left\| X(P_0)^D(\sigma_0) \right\|_{\nu},$$

by Proposition 6.6, we obtain the following estimation

$$H_L(P) d_\nu(P, Q_0)$$

$$\geq \text{Norm}(c^D)^{-1} \lambda_K^{\# M_K^{\infty} - [K_\nu : R]} \left\| X(P_0)^D(\sigma_0) \right\|_{\nu} d_\nu(P, Q_0)^\beta$$

$$\geq \text{Norm}(c^D)^{-1} \lambda_K^{\# M_K^{\infty} - [K_\nu : R]} \left\| \prod_{j=1}^r X_{n+j}^{-\deg_L(P_{n+j})} \cdot \frac{z_{i_0}^\beta}{\prod_{j=1}^r X_{n+j}^{-\beta b_{i_0,j}}} \right\|_{\nu}$$

$$= \text{Norm}(c^D)^{-1} \lambda_K^{\# M_K^{\infty} + [K_\nu : R] \beta - 1 + r}.$$
Next suppose that $\nu \in \mathcal{M}_K^f$. We first observe by Lemma 5.1 (2),

$$\left| \prod_{j=1}^{r} X_{b_{i_0,j}}^{b_{i_0,j}} - X_{i_0} \right|_{\nu}^{-1} \leq \prod_{\nu' \in \mathcal{M}_K^\infty} \left| \prod_{j=1}^{r} X_{b_{i_0,j}}^{b_{i_0,j}} - X_{i_0} \right|_{\nu'} \leq 2^{2\mathcal{M}_K^\infty} \prod_{\nu' \in \mathcal{M}_K^\infty} \max \left( \left| \prod_{j=1}^{r} X_{b_{i_0,j}}^{b_{i_0,j}} \right|_{\nu'}, \left| X_{i_0} \right|_{\nu'} \right).$$

(31)

Recall the maximal cone $\sigma_{i_0}$ (25), the index $j_1$ (26) and the positive relation $P_{n+j_1}$ in Lemma 6.3 let $c_{n+j, j} \neq j_1$ (resp. $c_{i_0}$) denote the coefficient of the term $D_{b_{n+j}}$ (resp. $D_{c_{i_0}}$) in $D(\sigma_{i_0})$, so that

$$X(P_0)^D(\sigma_{i_0}) = X_{i_0}^{c_{i_0}} \prod_{j \in \{1, \ldots, r\} \setminus \{j_1\}} X_{n+j}^{c_{n+j}}.$$  

(32)

Then by (8) and Lemma 6.3

$$c_{i_0} = a_{i_0} + (m_D(\sigma_{i_0}), \varrho_{i_0}) = a_{i_0} + a_{n+j_1} + \sum_{i \in \{1, \ldots, n\} \setminus \{i_0\}} a_i b_{i,j_1} = \deg_L(P_{n+j_1}) \geq \beta,$$

$$c_{n+j} = a_{n+j} + (m_D(\sigma_{i_0}), \varrho_{n+j}) = -\phi_D(\varrho_{n+j}) + (m_D(\sigma_{i_0}), \varrho_{n+j}) \geq 0.$$

By Proposition 3.4 we have the following lower bound

$$\text{Norm}(c^D)H_L(P) \geq \prod_{\nu' \in \mathcal{M}_K^\infty} \max \left( \left| X(P_0)^D(\sigma_0) \right|_{\nu'}, \left| X(P_0)^D(\sigma_{i_0}) \right|_{\nu'} \right).$$

Taking all this into account we finally get

$$\text{Norm}(c^D)H_L(P)d_\nu(P, Q_0) \geq \beta$$

(33)

$$\left| \prod_{j=1}^{r} X_{b_{i_0,j}}^{b_{i_0,j}} - X_{i_0} \right|_{\nu}^{\beta} \leq \prod_{\nu' \in \mathcal{M}_K^\infty} \left| \prod_{j=1}^{r} X_{b_{i_0,j}}^{b_{i_0,j}} - X_{i_0} \right|_{\nu'} \leq 2^{2\mathcal{M}_K^\infty} \prod_{\nu' \in \mathcal{M}_K^\infty} \max \left( \left| \prod_{j=1}^{r} X_{b_{i_0,j}}^{b_{i_0,j}} \right|_{\nu'}, \left| X_{i_0} \right|_{\nu'} \right).$$

$$= \left| X_{n+j}^{b_{i_0,j}} - X_{i_0} \right|_{\nu}^{\beta} \prod_{\nu' \in \mathcal{M}_K^\infty} \max \left( \left| \prod_{j=1}^{r} X_{n+j}^{b_{i_0,j}} \right|_{\nu'}, \left| X_{i_0} \right|_{\nu'} \right).$$
Using (31) and Proposition 6.6, the above can be bounded from below via

\[ \geq 2^{-\beta \sharp M^\infty_K} \prod_{\nu' \in M^\infty_K} \min \left( \prod_{j=1}^{r} X_{P_{n+j}}^{\deg_L(P_{n+j}) - \beta b_{i_0,j}}, \prod_{j \neq j_1} X_{P_{n+j}}^{c_{i_0} - \beta} \right) \]

\[ \geq \left( \frac{\lambda_K}{2^\beta} \right) \sharp M^\infty_K, \]

because \( c_{i_0} - \beta, c_{n+j}, \deg_L(P_{n+j}) - \beta b_{i_0,j}, \forall j \) are non-negative integers. All these lower bounds are uniform for \( P \in T(K) \setminus \{ Q_0 \} \). Hence we have proven that, by Definition-Proposition 4.1,

\[ \alpha(Q_0, X) = \alpha(Q_0, T) \geq \beta. \]

Combining Proposition 4.5 we get the equality since \( \alpha(Q_0, X) \leq \alpha(Q_0, l) = \beta \), where \( l \) is any smooth rational curve whose class is represented by a CPC of \( L \)-degree \( \beta \).

Now we prove the second half of Theorem 6.1 with the additional assumption that \( L \) ample and \( X \neq P^n \). Let \( Y \) be the union of all fibres over \( 1 \in \mathbb{G}^{n-\beta+1}_m \) of toric subvarieties corresponding to CPCs of \( L \)-degree \( \beta \). By Theorem 2.6, \( Y \) is non-empty. We are going to show that for any Zariski open dense subset \( U \) of \( Y \), we have

\[ \alpha(Q_0, X \setminus Y) > \beta = \alpha(Q_0, U). \]

This will imply \( \alpha_{\text{ess}}(Q_0, Y) = \beta < \alpha(Q_0, X \setminus Y) \) and so \( Y \) is locally accumulating. To see that \( \alpha(Q_0, U) = \beta \), it suffices to note that \( U \) is dominated by rational lines \( l \) in each piece of \( P^{\beta-1} \) intersecting \( U \) and passing through \( Q_0 \) of degree \( \beta \) (cf. Section 2.2).

Next fix any CPC \( \mathcal{I} \). By relabelling on can assume that \( \mathcal{I} = \{ \varrho_1, \cdots, \varrho_{\sharp \mathcal{I}} \} \). By Lemma 6.5, \( \sigma_0 \) contains \( \sharp \mathcal{I} - 1 \) elements of \( \mathcal{I} \) so we can assume that \( \mathcal{I} \cap \sigma_0(1) = \{ \varrho_1, \cdots, \varrho_{\sharp \mathcal{I}-1} \} \). Let \( Y_\mathcal{I} \) be fibre \( P^{\sharp \mathcal{I}-1} \times \{ 1 \} \) of the open toric subvariety \( P^{\sharp \mathcal{I}-1} \times \mathbb{G}^{n-\sharp \mathcal{I}+1}_m \) associated to the subfan constructed from \( \mathcal{I} \). In the parametrization (19), we have

\[ (U_{\sigma_0} \cap Y_\mathcal{I})(K) = \{ \ast, \cdots, \ast, 1, \cdots, 1 \}. \]

We conclude from this that

\[ Y(K) = \bigcup_{\mathcal{I} \text{ CPC: } \deg_L(\mathcal{I}) = \beta} \{ (y_1, \cdots, y_n) : \varrho_i \in \sigma_0(1) \setminus \mathcal{I} \Rightarrow y_i = 1 \}. \]
By Lemma 6.4 any two different CPCs $\mathcal{I}, \mathcal{J}$ have no common ray, so $Y_{\mathcal{I}} \cap Y_{\mathcal{J}} = Q_0$. Now take $P = (y_1, \cdots, y_n) \in (U_{\sigma_0} \setminus Y)(K)$. Two possibilities can arise.

**Case (I).** Suppose the set $\mathcal{I}$ of $i \in \{1, \cdots, n\}$ such that $\varrho_i$ is not member of any CPC of $L$-degree $\beta$ is non-empty, and that there exists one such $i_0 \in \mathcal{I}$ so that $y_i \neq 1$. Let us look at the maximal cone $\sigma_{i_0}$ adjacent to $\sigma_0$ (cf. (25)). Then necessarily the unique element $\varrho_{n+j_1} \in \sigma_{i_0}(1) \setminus \sigma_0(1)$ is not a member of any CPC of $L$-degree $\beta$ neither by Lemma 6.5. In particular we have $c_{i_0} = \deg_L(P_{n+j_1}) > \beta$. Also by the strict inequalities in Proposition 6.6 in bounding from below the product of the height and the distance (i.e. (33) & (30)), we can raise the power $\beta$ to $\beta + \delta$ on the distance where $\delta > 0$ is such that

$$c_{i_0} - \beta \geq 2\delta, \quad a_{n+j_0} + \sum_{i=1}^{n} a_i b_{i,j_0} \geq b_{i_0,j_0}(\beta + \delta) + \delta, \quad \forall 1 \leq j_0 \leq r.$$

**Case (II).** Now assume that for every $\varrho_i \in \sigma_0(1)$ that does not belong to any CPCs of $L$-degree $\beta$, we have $y_i = 1$. Since $n \geq 2$, then by (34), necessarily there are at least two such different CPCs $\mathcal{I}_1, \mathcal{I}_2$ containing say $\varrho_1, \varrho_2$ respectively so that $y_1, y_2 \neq 1$ (otherwise $P$ would be in $Y(K)$). For $i = 1, 2$, by relabelling if necessary, write (Lemma 6.4) $\{\varrho_{n+i}\} = \mathcal{I}_i \setminus \sigma_0(1)$ so that $\mathcal{I}_i = P_{n+i}$ and let $\sigma_i$ be the maximal adjacent cones such that $\sigma_0(1) \setminus \sigma_i(1) = \{\varrho_i\} \leftrightarrow \varrho_{n+i} \in \sigma_i(1) \setminus \sigma_0(1)$. Additionally, consider also $\sigma_3$ (resp. $\sigma_4$), the maximal cone adjacent to $\sigma_1$ (resp. $\sigma_2$) such that $\sigma_1(1) \setminus \sigma_3(1) = \{\varrho_2\}$ (resp. $\sigma_2(1) \setminus \sigma_4(1) = \{\varrho_1\}$). We claim that $\varrho_1 \notin \sigma_3(1)$ and $\varrho_2 \notin \sigma_4(1)$. Otherwise for example if $\varrho_1 \in \sigma_3(1)$, since $\varrho_{n+1} \in \sigma_1(1) \cap \sigma_3(1)$ and $\mathcal{I}_1 \neq \mathcal{I}_2$, then we would have $\mathcal{I}_1(1) \subset \sigma_3(1)$, which is a contradiction.

Let us now look at the divisors $D(\sigma_i), i = 1, 2, 3, 4$. Let $c_{i,j}, c_{k,l}, i \in \{1, 2\}, 1 \leq j \leq r, k \in \{3, 4\}, 1 \leq l \leq n + r$ be defined as

$$D(\sigma_1) = c_{1,1} D_{\varrho_1} + \sum_{j \neq 1} c_{1,j} D_{\varrho_{n+j}}, \quad D(\sigma_2) = c_{2,2} D_{\varrho_2} + \sum_{j \neq 2} c_{2,j} D_{\varrho_{n+j}},$$

$$D(\sigma_3) = \sum_{l=1}^{n+r} c_{3,k} D_{\varrho_l}, \quad D(\sigma_4) = \sum_{l=1}^{n+r} c_{4,k} D_{\varrho_l}.$$

We have by the assumption of (II), that $L$ is ample (so (9) holds), and that $\varrho_1, \varrho_2 \notin \sigma_3(1) \cup \sigma_4(1), \varrho_{n+1} \notin \sigma_2(1), \varrho_{n+2} \notin \sigma_1(1)$, so

$$c_{1,1} = \deg_L(P_{n+1}) = c_{2,2} = \deg_L(P_{n+2}) = \beta,$$

$$c_{1,2}, c_{2,1}, c_{3,2}, c_{4,1} \geq 1, \quad c_{i,j} \geq 0 \text{ otherwise.}$$
We now estimate $c_{3,1}$. Note that $b_{1,1} = 1, b_{2,1} = 0$. Write $\varrho_{n+j'_1} \in \sigma_3(1) \setminus \sigma_1(1)$ and $\varrho^\vee_3, \ldots, \varrho^\vee_n, \varrho^\vee_{n+1}, \varrho^\vee_{n+j'_1}$ the dual base for $\sigma_3(1)$. Then

$$
c_{3,1} = a_1 + \langle m_D(\sigma_3), \varrho_1 \rangle
= a_1 + \langle -a_{n+1} \varrho^\vee_{n+1} - a_{n+j'_1} \varrho^\vee_{n+j'_1} - \sum_{i=3}^n a_i \varrho^\vee_i, \varrho_1 \rangle
= a_1 + a_{n+1} b_{1,1} + a_{n+j'_1} b_{2,1} + \sum_{i=3}^n a_i b_{i,1}
= a_1 + a_{n+1} + \sum_{i=3}^n a_i b_{i,1}
= \deg_L(\mathcal{P}_{n+1}) = \beta.
$$

The same argument shows that $c_{4,2} = \deg_L(\mathcal{P}_{n+2}) = \beta$.

Fix a lift $P_0 = (X_1, \ldots, X_{n+r}) \in \mathcal{O}_K^{n+r}$ of the point $P$. Then

$$
y_1 = \frac{X_1}{X_{n+1} \prod_{j=3}^r X_{n+j}^{b_{1,j}}}, \quad y_2 = \frac{X_2}{X_{n+2} \prod_{j=3}^r X_{n+j}^{b_{2,j}}},
$$

where, if $r = 2$, the product $\prod_{j=3}^r$ is understood as 1. By the height formula (Proposition 3.4), we have

$$
\text{Norm}(c^D)H_L(P) \geq \prod_{\nu' \in \mathcal{M}_K^\infty} \max_{0 \leq m \leq 4} \left( |X(P_0)^D(\sigma_m)|_{\nu'} \right),
$$

where we recall

$$
X(P_0)^D(\sigma_0) = X_{n+1}^{\beta} X_{n+2}^{\beta} \prod_{j \neq 1, 2} X_{n+j}^{\deg_L(\mathcal{P}_{n+j})}.
$$

We may assume as before that $\nu \in \mathcal{M}_K^f$. The $\infty$-adic cases are simpler (compare estimations (30) and (33)). Using Lemma 5.1 again, we can bound $d_\nu(P, Q_0)^2$ via (recall that $X_1 \neq X_{n+1} \prod_{j=3}^r X_{n+j}^{b_{1,j}}, X_2 \neq X_{n+2} \prod_{j=3}^r X_{n+j}^{b_{2,j}}$ since
that deg°C

so that

Without loss of generality assume Subcase (II).

38 ZHIZHONG HUANG

We now determine the possible powers to be put on the distance. For this fix 

In this subcase we can put the power 

We have 

The key tool is the inequality 

max(Z_1, Z_2) \leq Z_1 + Z_2 \gg \lambda_1 Z_1^\lambda_2 Z_2^\lambda_2, \quad Z_1, Z_2, \lambda_1, \lambda_2 > 0, \lambda_1 + \lambda_2 = 1.

Subcase (I).

Before pursuing estimation, we analyse the exponents \( b_{i,j} \). Fix \( j \geq 3 \). If \( g_{n+j} \) belongs to some CPC \( \mathcal{I} \), then \( b_{i,j} = 0 \) for \( i = 1, 2 \) since otherwise \( g_i \in \mathcal{I}_i(1) \cap \mathcal{I}(1) \), a contradiction to Lemma 6.4. If \( g_{n+j} \) does not belong to any CPC of \( L \)-degree \( \beta \), then by Proposition 6.6 we conclude that in any case, \( \deg_L(\mathcal{P}_{n+j}) > b_{i,j} \beta \), for \( i = 1, 2 \) and \( j \geq 3 \). It suffices to take \( 0 < \delta < \beta \) such that 

so that 

In this subcase we can put the power \( \beta + \delta \) on \( d_\nu(P_0, Q_0) \).

Subcase (II). Without loss of generality assume 

\[
\max(|X_1|_{\nu'}, |X_{n+1} \prod_{j=3}^r X_{n+j}^{b_{1,j}}|_{\nu'}) = |X_1|_{\nu'}.
\]
satisfying Subcase (III).

We can bound the product as follows

\[ |X_1|_{\nu'}^{-\alpha_1} |X_{n+2} \prod_{j=3}^{r} X_{n+j}^{b_{2,j}}|_{\nu'}^{\alpha_2} \max(|X(P_0)^{D(\sigma_0)}|_{\nu'}, |X(P_0)^{D(\sigma_1)}|_{\nu'}) \]

\[ \gg |X_1|_{\nu'}^{-\alpha_1} |X_{n+2} \prod_{j=3}^{r} X_{n+j}^{b_{2,j}}|_{\nu'}^{-\alpha_2} |X(P_0)^{D(\sigma_0)}|_{\nu'}^{\lambda_2} |X(P_0)^{D(\sigma_1)}|_{\nu'}^{\lambda_1} \]

\[ \gg |X_1|_{\nu'}^{\lambda_1 \beta - \alpha_1} |X_{n+2}^{\lambda_1 + \lambda_2 \beta - \alpha_2} \prod_{j=3}^{r} X_{n+j}^{\deg L(P_{n+j}) - \alpha_2 b_{2,j}}|_{\nu'} \gg K. \]

This shows that the power \( \alpha_1 + \alpha_2 = \beta + \delta \) is admissible.

**Subcase (III).**

\[ \max(|X_i|_{\nu'}, |X_{n+i} \prod_{j=3}^{r} X_{n+j}^{b_{i,j}}|_{\nu'}) = |X_i|_{\nu'}, \quad i = 1, 2. \]

Recall that we have \( c_{3,2}, c_{4,1} \gg 1 \). Choose \( \delta > 0 \) and \( \lambda_1, \lambda_2 > 0, \lambda_1 + \lambda_2 = 1 \)
satisfying

\[ \min(\lambda_1 \beta + \lambda_2 c_{4,1}, \lambda_1 c_{3,2} + \lambda_2 \beta) \geq \frac{1}{2} (\beta + \delta). \]

We can bound the product as follows

\[ |X_1|_{\nu'}^{-\frac{1}{4}(\beta+\delta)} |X_2|_{\nu'}^{-\frac{1}{4}(\beta+\delta)} \max(|X(P_0)^{D(\sigma_3)}|_{\nu'}, |X(P_0)^{D(\sigma_4)}|_{\nu'}) \]

\[ \gg |X_1|_{\nu'}^{-\frac{1}{4}(\beta+\delta)} |X_2|_{\nu'}^{-\frac{1}{4}(\beta+\delta)} |X(P_0)^{D(\sigma_3)}|_{\nu'}^{\lambda_1} |X(P_0)^{D(\sigma_4)}|_{\nu'}^{\lambda_2} \]

\[ \gg K |X_1|_{\nu'}^{\lambda_1 \beta + \lambda_2 c_{4,1} - \frac{1}{2}(\beta+\delta)} |X_2|_{\nu'}^{\lambda_1 c_{3,2} + \lambda_2 \beta - \frac{1}{2}(\beta+\delta)} \]

\[ \gg K. \]

Inserting into the bound (35), we have proven that in all subcases of **Case (I)**, \( \exists \delta > 0 \) such that uniformly for \( P \not\in Y(K), d_{\nu}(P, Q_0)^{\beta + \delta} H_L(P) \geq C_{1,K} > 0. \)
To summarize, we have proven $d_\nu(P, Q_0)^c H_L(P) \geq C_{2, K} > 0$ for every place $\nu$ and every $P \in (T \setminus Y)(K)$ with some uniform $c > \beta$ in Cases I & II. The equivalent definitions of $\alpha$ in Definition-Proposition 4.1 now implies that $\alpha(Q_0, X \setminus Y) > \beta$, as desired.  

7. Toric varieties with Picard number 2

The goal in this section is denoted to studying rational approximations on split toric varieties of small Picard number. Those of Picard number 1 are projective spaces and the conclusion follows easily by Proposition 4.5. We shall be interested in those of Picard number 2 in what follows. We will see that they automatically satisfy Hypothesis (*), so Theorem 6.1 holds. The main interest here is the generic best approximation, i.e. how $\alpha_{\text{ess}}$ is achieved. The main result is Theorem 7.4, the detailed version of the second part of Theorem 1.2. We fix throughout this section a line bundle $L$.

7.1. Classification and geometry. Complete smooth toric varieties whose rank of the Picard group equals 2 are classified by Kleinschmidt [Kle88]. See also [CLS11, §7.3]. They are all projective and in fact projective bundles over projective spaces:

$$P(\mathcal{O}_{P^s} \oplus \mathcal{O}_{P^s}(a_1) \oplus \cdots \oplus \mathcal{O}_{P^s}(a_t)) \to P^s, \quad (a_r \geq \cdots \geq a_1 \geq 0; s, t \geq 1, s+t = n).$$

Kleinschmidt gives explicit description of the structure of the their fans. Recall that $n = \dim X$. Let $(e_i)_{1 \leq i \leq n} \subset \mathbb{Z}^n$ be the canonical base of $\mathbb{R}^n$. Then we can express the set of all rays (here the numbering starts from 0 by convention) $\triangle(1) = \{q_0, \cdots, q_{n+1}\}$ in the following way:

$$q_i = e_i, \quad 1 \leq i \leq t \text{ and } t + 1 \leq i \leq t + s;$$

$$q_0 = - \sum_{i=1}^{t} e_i;$$

(36)  

$$q_{n+1} = - \sum_{j=1}^{s} e_{t+j} + \sum_{i=1}^{t} a_i e_i.$$  

(37)  

And the set of maximal cones $\triangle_{\text{max}} = \{\sigma_{i,j}, 0 \leq i \leq t, 1 \leq j \leq s\}$ where

$$\sigma_{i,t+j} = R_{\geq 0} q_0 + \cdots + \widehat{R_{\geq 0} q_i} + \cdots + R_{\geq 0} q_t + R_{\geq 0} q_{t+1} + R_{\geq 0} q_{t+j} + \cdots + R_{\geq 0} q_n.$$
The fan $\triangle$ is constructed by the cones in $\Delta_{\text{max}}$ and their faces. Note that we have two primitive collections $\{q_0, \ldots, q_t\}$ and $\{q_{t+1}, \ldots, q_{n+1}\}$. Equations (36) and (37) give rise to the following two 1-cycles with corresponding relations

$$C_1 : \sum_{i=0}^{t} q_i = 0;$$

(38)

$$C_2 : \sum_{j=1}^{s+1} q_{r+j} - \sum_{i=1}^{t} a_i q_i = 0,$$

(39)

the first one being positive. The second one is not positive in general, unless all $a_i = 0$, and this turns out to be the easy case where $X = \mathbb{P}^s \times \mathbb{P}^r$.

Our goal now is to show that all such varieties verify Hypotheses (*)\(\iff\)(**). For this we are going to change the indices as follows. Let $$v_0 = q_t, \quad v_i = q_i \ (1 \leq i \leq t - 1 \text{ and } t + 1 \leq i \leq n + 1), \quad v_t = q_0.$$ So that equations (36) and (37) become

$$v_0 = -\sum_{i=1}^{t} v_i,$$

(40)

$$v_{n+1} = -\sum_{i=1}^{t} b_i v_i - \sum_{j=1}^{s} v_{t+j},$$

(41)

where

$$b_t = a_t, \quad b_i = a_t - a_i, \ (1 \leq i \leq t - 1)$$

satisfy $b_t \geq b_1 \geq b_2 \geq \cdots \geq b_{t-1} \geq 0$. Geometrically, this operation is nothing but the isomorphism $[\text{Har77, Lemma 7.9}]$

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^s} \oplus \mathcal{O}_{\mathbb{P}^s}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^s}(a_t))$$

$$\simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^s}(-b_t) \oplus \mathcal{O}_{\mathbb{P}^s}(-b_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^s}(-b_{t-1}) \oplus \mathcal{O}_{\mathbb{P}^s}).$$

In this way we get two combinations of rays with negative coefficients so that the cone

$$\sigma_{t,n+1} = \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_n$$

(42)

satisfies (23), i.e. Hypothesis (**) is verified. Equivalently, with the notation $b_0 = a_t$ (recall that $D_t \leftrightarrow q_t = v_0$),

$$[D_{q_i}] = [D_t] + b_i [D_{n+1}] \quad (0 \leq i \leq t - 1); \quad [D_{t+j}] = [D_{n+1}] \quad (1 \leq j \leq s),$$
so that
\[
\overline{\mathrm{Eff}}(X) = \sum_{i=0}^{n+1} R_{\geq 0}[D_i] = R_{\geq 0}[D_t] + R_{\geq 0}[D_{n+1}] .
\]

Equations (41) furnishes a third relation
\[
C_3 : \sum_{i=1}^{t} b_i v_i + \sum_{j=1}^{s+1} v_{t+j} = b_t q_0 + \sum_{i=1}^{t-1} b_i q_i + \sum_{j=1}^{s+1} q_{t+j} = 0,
\]
which is positive and verifies (viewed as elements in \(\mathbb{Z}^{\Delta(1)}\))
\[
C_3 = a_t C_1 + C_2.
\]

**Proposition 7.1:** The semi-group of effective 1-cycles is generated by \(C_1\) and \(C_2\):
\[
\mathrm{AE}_1(X) = \mathbb{N}_{\geq 0} C_1 + \mathbb{N}_{\geq 0} C_2 \subset A_1(X).
\]

**Proof.** First of all \(C_1\) and \(C_2\) are linearly independent and primitive, so they generate the group \(A_1(X)\):
\[
A_1(X) = \mathbb{Z} C_1 + \mathbb{Z} C_2.
\]

Let \(C\) be the class of an effective curve \(E\). Then \(\exists p, q \in \mathbb{Z}\) such that
\[
C = p C_1 + q C_2.
\]

We want to show that \(p, q \geq 0\). The relation corresponding to \(C\) is
\[
p q_0 + \sum_{i=1}^{t} (p - qa_i) q_i + \sum_{j=1}^{s+1} q q_{t+j} = 0.
\]
If \(q < 0\), one would have \(\langle D_{r+j}, E \rangle = q < 0\) for all \(1 \leq j \leq s + 1\) and so \(E \subset \cap_{j=1}^{s+1} D_{t+j} = \emptyset\) (recall that \(\{q_0, \ldots, q_t\}, \{q_{t+1}, \ldots, q_{n+1}\}\) are primitive collections), which is absurd. So \(q\) must be non-negative. Similarly if \(p < 0\), then \(\langle D_{q_i}, E \rangle = p - qa_i \leq \langle D_0, E \rangle = p < 0\) for any \(1 \leq i \leq t\), which is again impossible since \(\cap_{i=0}^{t} D_{q_i} = \emptyset\).

**Remark 7.2:** Since \(X\) has small Picard number, it is relatively easy to determine when a divisor is big, nef or ample. Let \(D\) be an effective \(T\)-invariant \(\mathbb{Q}\)-divisor. Write \(L = \mathcal{O}_X(D)\) and \([D] = A[D_t] + B[D_{n+1}]\) for \(A, B \in \mathbb{Q}_{\geq 0}\) by (43). By [Kle88 Theorem 2] or Nakai-Moishezon’s criterion [CLS11 Theorem 6.3.13] combined with Proposition 7.1, \(L\) is globally generated (resp. ample) if and only if \(\deg_L C_1 = A, \deg_L C_2 = B - Aa_t\) are non-negative (resp. positive), or
equivalently, \( B \geq Aa_t \) (resp. \( A > 0 \) and \( B > Aa_t \)). Thus \( \text{Nef}(X) \subsetneq \overline{\text{Eff}}(X) \) unless \( a_t = 0 \), i.e. \( X \cong \mathbb{P}^s \times \mathbb{P}^t \) that they coincide. We now quickly show that \( D \) is big if and only if \( AB > 0 \) (the big cone is the relative interior of \( \overline{\text{Eff}}(X) \) the pseudoeffective cone). As a consequence, if this is the case then we always have \( \deg L C_1 = A > 0 \). To this end we want to find \( m = \sum_{i=1}^n c_i v_i^\vee \in M_\mathbb{Q} \) as an interior point of the polyhedron \( P_D \), i.e. such that

\[
\langle m, v_i \rangle = c_i > 0, 1 \leq i \leq n,
\]

\[
\langle m, v_0 \rangle = -\sum_{i=1}^t c_i > -A \Leftrightarrow \sum_{i=1}^n c_i < A,
\]

\[
\langle m, v_{n+1} \rangle = -\sum_{i=1}^t c_i b_i - \sum_{j=1}^s c_{t+j} > -B \Leftrightarrow \sum_{i=1}^t c_i b_i + \sum_{j=1}^s c_{t+j} < B.
\]

Once \( AB > 0 \), such \((c_i)_{1 \leq i \leq n}\) certainly exist (any sufficiently small \( c_i \)'s suffice) and conversely if such \((c_i)_{1 \leq i \leq n}\) exists then necessarily \( AB > 0 \). This proves the claim.

**Proposition 7.3:** Assume that \( L \) is big. Then very free rational curves of minimal \( L \)-degree have class precisely

\[
\begin{cases}
C_3 & \text{if } \forall i \in \{1, \cdots, t\}, b_i \neq 0; \\
C_1 + C_3 & \text{if } \exists i \in \{1, \cdots, t\}, b_i = 0.
\end{cases}
\]

**Proof.** First of all we always have \( \deg L C_1 > 0 \) and \( \deg L C_3 > 0 \) by Remark 7.2. Observe that \( C_1(1) = \{g_0, \cdots, g_t\} \) is a centred primitive collection. It therefore represents rational curves that are not very free, nor does any of its multiples (alternatively by Theorem 2.11), since \( X \) is not the projective space. Consider the relation of an effective 1-cycle (Proposition 7.1),

\[C = pC_1 + qC_2, \quad p, q \in \mathbb{N}.\]

Comparing the coefficients in (45), in order that it is positive, we must have

\[
p \geq q \max_{1 \leq i \leq t} a_i = qa_t.
\]

The preceding discussion tells us \( q > 0 \). If \( b_i \neq 0, \forall i \), recall the relation \( C_3 (44) \), then \( g_i \in C_3(1) \), \( \forall i \neq t \) and so \( \text{Vect}_\mathbb{Q}(C_3(1)) = N_\mathbb{Q} \). Theorem 2.11 tells us that
some curve of class $C_3$ is very free (see also Theorem 2.2). We conclude that if $C$ is a positive relation,
\[
\deg_L C = p \deg_L C_1 + q \deg_L C_2 \geq q (a_t \deg_L C_1 + \deg_L C_2) \geq q \deg_L C_3 \geq \deg_L C_3.
\]
The above inequalities are equalities if and only if $p = a_t$, $q = 1$, which means $C = C_3$. If $\exists b_i = 0 \Leftrightarrow b_{t-1} = 0$, then
\[
\text{Vect}_Q(C_3(1)) \subseteq \text{Vect}_Q\{\varrho_0, \cdots, \varrho_{t-2}, \varrho_{t+1}, \cdots, \varrho_{n+1}\}
\]
so $\dim \text{Vect}_Q(C_3(1)) \leq r + s - 1 = n - 1$. By Theorem 2.11, the class $C_3$ does not represent very free curves any more. Now consider the relation
\[
C_1 + C_3 = (a_r + 1)C_1 + C_2 : (b_r + 1)\varrho_0 + \sum_{i=1}^{r-1} (b_i + 1)\varrho_i + \sum_{j=1}^{s+1} \varrho_{r+j} = 0.
\]
It verifies
\[
\text{Vect}_Q((C_1 + C_3)(1)) = \text{Vect}_Q\{\varrho_0, \cdots, \varrho_{t-1}, \varrho_{t+1}, \cdots, \varrho_{n+1}\} = N_Q,
\]
so it represents very free rational curves by Theorem 2.11. Take a positive relation $C = pC_1 + qC_2$ with parameters $p, q \in \mathbb{N}$ satisfying (47). The case where $qa_t$ corresponds to $C = qC_3$ not very free. So assume $p \geq qa_t + 1$. We have
\[
\deg_L C = p \deg_L C_1 + q \deg_L C_2
\]
\[
\geq (qa_t + 1) \deg_L C_1 + q \deg_L C_2
\]
\[
= q (a_t \deg_L C_1 + \deg_L C_2) + \deg_L C_1
\]
\[
= q \deg_L C_3 + \deg_L C_1
\]
\[
\geq \deg_L C_3 + \deg_L C_1,
\]
which are equalities precisely when $q = 1, p = a_t + 1 \Leftrightarrow C = C_1 + C_3$. \hfill \qed

Remark: Independently of Theorem 2.11, one can show that for $C$ effective, $T_X|_C$ equals
\[
\begin{cases}
\mathcal{O}_{P^1}(2) \bigoplus \mathcal{O}_{P^1}(1)^{\oplus s-1} \bigoplus (\bigoplus_{i=1}^{r-1} \mathcal{O}_{P^1}(b_i)) & \text{if } b_i \neq 0, \forall i, [C] = C_3;
\mathcal{O}_{P^1}(2)^{\oplus 2} \bigoplus \mathcal{O}_{P^1}(1)^{\oplus s-1} \bigoplus (\bigoplus_{i=1}^{r-1} \mathcal{O}_{P^1}(b_i + 1)) & \text{if } \exists b_i = 0, [C] = C_1 + C_3.
\end{cases}
\]
This also reproves Proposition 7.3.
7.2. Generic Diophantine approximation. To state the main result of this section, let us first define what we shall call “general lines”. They will realize the class of minimal degree in Proposition 7.3. As in Section 6 we can assume \( Q = Q_0 = (1, \cdots, 1) \in \mathcal{T}(K) \). For every \( \varrho \in \Delta(1) \), let \( \lambda_{\varrho} \) be the associated co-character. Fix any maximal cone \( \sigma \) and consider the associated affine neighbourhood \( U_\sigma \simeq \mathbb{A}^n_K \). For a \( n \)-tuple \( m = (m_1, \cdots, m_n) \in \mathbb{K}^n \), a line with parameter \( m \) (with respect to the parametrization given by \( \sigma \)) is, by definition, the Zariski closure of the morphism

\[
\mathbb{A}^1 \longrightarrow U_\sigma \\
t \longmapsto \prod_{\varrho_i \in \sigma(1)} \lambda_{\varrho_i}(m_i + 1) = (m_1 t + 1, \cdots, m_n t + 1)
\]

We say that this line is general if the parameter \( m \in \mathbb{K}^n \) satisfies some open condition.

To compute the class as well as the associated relation of such general lines, we impose the open condition \( \prod_{i=1}^n m_i \neq 0 \). Write \( \sigma(1) = \{ \varrho_1, \cdots, \varrho_n \} \) and consider the unique cone \( \tau \) containing \( -\sum_{i=1}^n \varrho_i \) in its interior so that

\[
\sum_{i=1}^n \varrho_i + \sum_{\varrho \in \tau(1)} a_{\varrho} \varrho = 0,
\]

where all \( a_{\varrho} \)'s are (strictly) positive. This is the positive relation for general lines we are looking for. It computes the intersection multiplicities with all boundary divisors (plus the contribution from the point at infinity).

**Theorem 7.4:** Let \( X \) be a projective smooth toric variety with Picard number 2 and assume that \( L \) is big and nef. We continue to use the notation in Section 7.1. Then

\[
\alpha_{\text{ess}, L}(Q_0) = \begin{cases} 
\deg_L C_3 & \text{if } \forall i \in \{1, \cdots, t\}, b_i \neq 0; \\
\deg_L C_1 + \deg_L C_3 & \text{if } \exists i \in \{1, \cdots, t\}, b_i = 0.
\end{cases}
\]

In all cases, outside a Zariski closed subset, the best approximations can be achieved on general lines (with respect to the cone \( \sigma_{t,n+1} \)) passing through \( Q_0 \). Consequently the essential constant equals the minimal \( L \)-degree of very free rational curves.
Proof. Recall the cone $\sigma_{t,n+1}$ (42). Fix a general line $l$ under its parametrization. Consider the element

$$\varrho = \sum_{i=1}^{n} v_i - \sum_{i=0}^{t-1} q_i + \sum_{j=1}^{s} \varrho_{t+j}.$$ 

If $\forall i, b_i > 0$, we have

$$-\varrho = \varrho_{n+1} + \sum_{i=1}^{t-1} (b_i - 1) q_i + (b_t - 1) \varrho_0 \in \sigma_{t,t+j}, \quad \forall 1 \leq j \leq s,$$

i.e. $-\varrho$ belongs to a face of $\bigcap_{j=1}^{s} \sigma_{t,t+j}$. Its corresponding relation is

$$\varrho + (-\varrho) = \left(\sum_{i=0}^{t-1} q_i + \sum_{j=1}^{s} \varrho_{t+j}\right) + \left(\varrho_{n+1} + \sum_{i=1}^{t-1} (b_i - 1) q_i + (b_t - 1) \varrho_0\right) = 0,$$

namely the class of $l$ is that of $C_3$ (44). If $\exists b_i = 0$, then necessarily $b_{t-1} = 0$ and the same computation shows that

$$-\varrho = \varrho_{n+1} + \varrho_t + \sum_{i=1}^{t-1} b_i q_i \in \bigcap_{j=1}^{s} \sigma_{0,t+j}.$$ 

So the class of $l$ is $C_1 + C_3$. Since $l$ is very free by Proposition 7.3, we conclude that

$$\alpha_{\text{ess}}(Q_0) \leq \alpha(Q_0, l) = \begin{cases} \deg_L C_3, & \text{if } \forall i \in \{1, \cdots, t\}, b_i \neq 0; \\ \deg_L C_1 + \deg_L C_3, & \text{if } \exists i \in \{1, \cdots, t\}, b_i = 0. \end{cases}$$

Recall equations (40), (41). We get the parametrization via $\sigma_{t,n+1}$ (19)

$$\pi : \pi^{-1}(U_{\sigma_{t,n+1}}) \to U_{\sigma_{t,n+1}} \simeq A^n$$

$$(X_0, \cdots, X_{n+1}) \mapsto \left(\frac{X_0}{X_t X_{n+1}^{b_t}}, \frac{X_1}{X_t X_{n+1}^{b_t}}, \cdots, \frac{X_{t-1}}{X_t X_{n+1}^{b_t}}, \frac{X_{t+1}}{X_{n+1}}, \cdots, \frac{X_n}{X_{n+1}}\right).$$

We shall write $H_L = H, \alpha_{L,\nu} = \alpha$ for every fixed place $\nu$. We define the Zariski closed subset

$$Z = \bigcup_{i=1}^{n} (y_i = 1)^{\text{Zar}},$$

where as before $(y_i) \subset K^n$ denotes the coordinates of points in $U_{\sigma_{t,n+1}}(K)$. So that for any $P \in (U_{\sigma_{t,n+1}} \setminus Z)(K), y_i \neq 1, 1 \leq i \leq n$. We shall use the distance
(21) defined for all $P$ near $Q_0$:

$$d_\nu(P, Q_0) = \max_{1 \leq i \leq n} (|y_i - 1|_\nu).$$

Let $P_0 = (X_i)_{0 \leq i \leq n+1} \in K^{n+2}$ denotes one integral lift of $P$ in some twisted torsor $\tilde{\mathcal{X}}$ for some $c \in C^r$ (Theorem 3.2). Recall that we can assume that the set $C$ comprises of integral ideals representing the group $\text{Cl}_K$. Observe that $P_0 \in \mathcal{O}_K^{n+2}$ with the choice of the basis $\{[D_t], [D_{n+1}]\} \leftrightarrow \{v_0, v_{n+1}\}$ for Pic$(X)$.

We want to prove that

$$\alpha(Q_0, X \setminus Z) \geq \alpha(Q_0, l).$$

By Salberger’s height formula (Proposition 3.4), we have

$$\text{Norm}(c^D)H(P) \geq \prod_{\nu' \in \mathcal{M}_K} \max(|X(P_0)^{D(\sigma_{t,n+1})}|_{\nu'}, |X(P_0)^{D(\sigma_{t,t+j_0})}|_{\nu'}),$$

where, by equations (40) and (41),

$$X(P_0)^{D(\sigma_{t,n+1})} = X_t^{\deg L} C_1 X_{n+1}^{\deg L} C_3,$$

$$X(P_0)^{D(\sigma_{t,t+j_0})} = X_t^{\deg L} C_1 X_{t+j_0}^{\deg L} C_3.$$}

We shall only consider the cases where $\nu \in \mathcal{M}_k^f$, as the estimations for archimedean places are simpler and almost identical to (30). Since $P \not\in Z(K)$, we have $y_{t+j_0} = \frac{X_{t+j_0}}{X_{n+1}} \neq 1$ for all $1 \leq j_0 \leq s$. By Lemma 5.1, we have

$$d_\nu(P, Q_0)^{-1} \leq \left| \frac{X_{t+j_0} - X_{n+1}}{X_{n+1}} \right|_{\nu}^{-1}$$

$$\leq \left| X_{t+j_0} - X_{n+1} \right|_{\nu}^{-1}$$

$$\leq \prod_{\nu' \in \mathcal{M}_K^\infty} \left| X_{t+j_0} - X_{n+1} \right|_{\nu'}$$

$$\leq 2\#M_\infty \prod_{\nu' \in \mathcal{M}_K^\infty} \max \left( |X_{t+j_0}|_{\nu'}, |X_{n+1}|_{\nu'} \right).$$
These estimations (50), (51) provide us the lower bound for all $P \in (\mathcal{T} \setminus \mathcal{Z})(K)$:

$$\text{Norm}(c^D)H(P)d_{\nu'}(P, Q_0)^{\deg_L C_3} \geq 2^{-(\deg_L C_3)\sharp \mathcal{M}_K^\infty} \prod_{\nu' \in \mathcal{M}_K^\infty} \max\left(\left|X_{t-1}^{\deg_L C_1} X_{n+1}^{\deg_L C_3}\right|_{\nu'}, \left|X_t^{\deg_L C_1} X_{t+j_0}^{\deg_L C_3}\right|_{\nu'}\right) \geq 2^{-(\deg_L C_3)\sharp \mathcal{M}_K^\infty} \prod_{\nu' \in \mathcal{M}_K^\infty} \left|X_t^{\deg_L C_1}\right|_{\nu'} \geq \left(\frac{\lambda_K}{2^{\deg_L C_3}}\right)^{\sharp \mathcal{M}_K^\infty}.$$

So this proves, by Definition-Proposition 4.1

$$\alpha(Q_0, \mathcal{T} \setminus \mathcal{Z}) \geq \deg_L C_3. \tag{52}$$

If $\exists b_i = 0$, more is true. First we have $b_{t-1} = 0$ since it is smallest amongst all $b_i$’s. It leads us to look at additional maximal cones $\sigma_{t-1, n+1}, \sigma_{t-1, t+j_0}$. For this we rewrite (40) and (41) as (also true for $j_0 = s + 1$)

$$v_{t-1} = -\sum_{i=0}^{t-2} b_i v_i - v_t,$$

$$v_{t+j_0} = -\sum_{i \in \{1, \ldots, t\} \setminus \{t-1\}} b_i v_i - \sum_{j \in \{1, \ldots, s+1\} \setminus \{j_0\}} v_{t+j},$$

again thanks to $b_{t-1} = 0$. Therefore

$$X(P_0)^{D(\sigma_{t-1, t+j_0})} = X_{t-1}^{\deg_L C_1} X_{t+j_0}^{\deg_L C_3},$$

$$X(P_0)^{D(\sigma_{t-1, n+1})} = X_{t-1}^{\deg_L C_1} X_{n+1}^{\deg_L C_3}.$$

We bound the height using the four maximal cones:

$$\text{Norm}(c^D)H(P) \geq \prod_{\nu' \in \mathcal{M}_K^\infty} \max(|X(P_0)^{D(\sigma_{t, t+j})}|_{\nu'}, i \in \{t, t-1\}, j \in \{j_0, s+1\}). \tag{53}$$

In the open subset $U_{\sigma_{t,n+1}} \setminus \mathcal{Z}$, by using additionally the ($t$)-th coordinate

$$y_t = \frac{X_{t-1}}{X_t} X_{n+1}^{b_{t-1}} = \frac{X_{t-1}}{X_t},$$

at the same time, one has that similarly to (51), $d_{\nu}(P, Q_0)^{-1}$ can be bounded via (for any $1 \leq j_0 \leq s$)

$$\leq 2^{\sharp \mathcal{M}_K^\infty} \min\left(\prod_{\nu' \in \mathcal{M}_K^\infty} \max(|X_t|_{\nu'}, |X_{t-1}|_{\nu'}), \prod_{\nu' \in \mathcal{M}_K^\infty} \max(|X_{n+1}|_{\nu'}, |X_{t+j_0}|_{\nu'})\right).$$
Integrating this into the height bound (53), one gets similarly for \( P \in (T \setminus Z)(K) \),

\[
\text{Norm}(c^D) H(P) d_\nu(P, Q_0)^{\text{deg}_L C_1 + \text{deg}_L C_3}
\geq \prod_{\nu' \in \mathcal{M}_K^\infty} \max \left( |X(P_0)^{D(\sigma_{t,n+1})}|_{\nu'}, |X(P_0)^{D(\sigma_{t-1,n+1})}|_{\nu'}, |X(P_0)^{D(\sigma_{t,t+1})}|_{\nu'}, |X(P_0)^{D(\sigma_{t-1,t+1})}|_{\nu'} \right)
\times \left( 2^{2^{\mathcal{M}_K^\infty}} \prod_{\nu' \in \mathcal{M}_K^\infty} \max( |X_t|_{\nu'}, |X_{t-1}|_{\nu'} ) \right)^{-(\text{deg}_L C_1)}
\times \left( 2^{2^{\mathcal{M}_K^\infty}} \prod_{\nu' \in \mathcal{M}_K^\infty} \max( |X_{n+1}|_{\nu'}, |X_{t+j_0}|_{\nu'} ) \right)^{-(\text{deg}_L C_3)}
\geq \left( \frac{\lambda_K}{2^{\text{deg}_L C_1 + \text{deg}_L C_3}} \right)^{\#\mathcal{M}_K^\infty}
\]

So under the extra condition \( \exists b_i = 0 \), again by Definition-Proposition 4.1, we have proven

(54) \[ \alpha(Q_0, T \setminus Z) \geq \text{deg}_L C_1 + \text{deg}_L C_3. \]

Finally the content of Theorem 7.4 is nothing but a union of (52), (54), (48) and Proposition 7.3.

8. Some questions and remarks

8.1. The assumption that the point \( Q \) to be approximated is general, as we have already seen, makes the global geometry of \( X \) useful. Though we neglect here the case where \( Q \) lies on the boundary. Several possibilities can happen when one tries to approximate \( Q \). That is, using rational points on the open orbit, on the boundary or mixtures of them. If the sequence of points happens to lie on the boundary we are restricted to a toric subvariety.

8.2. One might also suppose that \( Q \) is a rational point with irrational coordinates. In dimension one this is K. Roth’s theorem (Example 4.2) and a more quantitative distribution of approximates especially for quadratic irrationalities is obtained in [Hua17a]. In higher dimension this looks like a “simultaneous
approximation” problem. Note that however the curves corresponding to centred primitive relations do not seem to give best approximates since they are smooth rational lines and most of the time they do not pass through \( Q \), unless the coordinates of \( Q \) are linearly dependent over the base field.

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