What can we learn from the breaking of the Wandzura–Wilczek relation? 1

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Abstract. We review the study of the Wandzura–Wilczek relation for the structure function $g_2$, with a particular attention on the connection with the framework of Transverse Momentum Dependent factorization. We emphasize that the relation is broken by two distinct twist-3 terms. In the light of these findings, we clarify what can be deduced from the available experimental data on $g_2$, which indicate a breaking of the order 20–40%, and how to individually measure the twist-3 terms.

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At large virtuality $Q^2$, the lepton-nucleon Deep Inelastic Scattering (DIS) cross section scales with $x = Q^2/(2M \nu)$ modulo logarithmic corrections, where $M$ is the target’s mass and $\nu$ the virtual photon energy. At low $Q^2$, power suppressed contributions become important, e.g., target mass corrections of $O(M^2/Q^2)$ and jet mass corrections of order $O(m_j^2/Q^2)$ with $m_j$ the invariant mass of the current jet\cite{[1, 2]}, and higher-twist (HT) corrections of $O(\Lambda_{QCD}^2/Q^2)$ related to quark-gluon correlations inside the nucleon\cite{[3]}. There are many reasons why we need to identify and measure higher-twist terms in experimental data, for example, (i) to verify quark-hadron duality\cite{[4]}, (ii) to measure twist-2 Parton Distribution Functions (PDF) at large fractional momentum $x$ and low-$Q^2$, e.g., the $d/u$ and $\Delta d/d$ quark ratios, sensitive to the nonperturbative structure of the nucleon\cite{[5, 6, 7]}, (iii) to measure multiparton correlations, important to understand the nucleon structure beyond the PDFs\cite{[8]}, (iv) to determine the perturbative QCD evolution of the $g_2$ polarized structure function among others\cite{[9]}, (v) to calculate the high-$k_T$ tails of transverse momentum dependent (TMD) parton distribution functions\cite{[10]}.

The inclusive DIS cross section is determined by the hadronic tensor $W^{\mu\nu}$, defined as the imaginary part of the forward virtual photon Compton scattering amplitude. $W^{\mu\nu}$ can be decomposed in 2 unpolarized structure functions, $F_{1,2}$, which we do not discuss here, and 2 polarized structure functions $g_{1,2}$. Its antisymmetric part reads

$$W^{\mu\nu}(P,q) = \frac{1}{P \cdot q} \varepsilon^{\mu\nu\rho\sigma} q_{\rho} \left[ S_\sigma g_1(x, Q^2) + \left( S_\sigma - \frac{S \cdot q}{P \cdot q} p_\sigma \right) g_2(x, Q^2) \right], \quad (1)$$

where $P,S$ are the target momentum and spin. Among the structure functions, $g_2$ is unique because it is the only one with twist-3 contributions that can be measured in

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inclusive DIS. Furthermore, its higher-twist contribution can be isolated thanks to the Wandzura-Wilczek (WW) relation, first obtained in the Operator Product Expansion (OPE) formalism [11]:

$$g_2(x, Q^2) = g_2^{WW}(x, Q^2) + \Delta(x, Q^2).$$

(2)

Here $g_2^{WW}$ is determined by the leading twist (LT) part of $g_1$, which is rather well known experimentally:

$$g_2^{WW}(x, Q^2) = -g_1^{LT}(x, Q^2) + \int_x^1 \frac{dy}{y} g_1^{LT}(y, Q^2).$$

(3)

Strictly speaking, the WW relation is the LT part of (2). The breaking term $\Delta$ is a pure HT term, meaning that its moments are matrix elements of local operators of twist-3 or higher, with “twist” defined as dimension minus spin of the local operator [12, 13]. In this talk we will limit our analysis to twist-3 operators, and will drop the dependence on $Q^2$ for ease of notation. Details can be found in Ref. [14].

PARTON DISTRIBUTIONS IN PERTURBATIVE QCD

In perturbative QCD the structure functions can be expressed as a convolution of a perturbatively calculable coefficient, and a number of nonperturbative LT parton distributions and HT parton correlations. In particular, the WW relation can be obtained in the framework of collinear factorization [9] or in transverse momentum dependent factorization, as we will shortly describe.

Let’s define the quark-quark correlator

$$\Phi_i^a(x, \tilde{k}_T) = \int \frac{d^2 \xi}{(2\pi)^3} e^{i\tilde{k}_T \cdot \xi} \langle P, S | \bar{\psi}_i^a(0) W(0, \xi | n_-) \psi_j^a(\xi) | P, S \rangle |_{\xi^+ = 0}.$$ (4)

where $i, j$ are Dirac indices, $a$ is the quark flavor index, $k$ its 4-momentum, $x = k \cdot n_- / P \cdot n_-$ its fractional momentum and $\tilde{k}_T$ its transverse momentum relative to the parent nucleon. The “plus” and “minus” components of a 4-vector are defined as $a^\pm = a \cdot n_\pm$ in terms of two orthogonal light-cone vectors $n_+^2 = n_-^2 = 0$ such that $n_- \cdot n_+ = 1$, and $n_\mu$ is proportional to $P^\mu$ up to mass corrections. $W$ is a Wilson line (gauge link) whose precise form depends on the process. The direction of the Wilson line is determined by an additional 4-vector beside $P, S$, which in tree-level analyses such as we pursue here is identified with the light-cone vector $n_-$. In the light-cone gauge $n_- \cdot A = 0$ the Wilson line is identically equal to 1 and

$$\Phi_{ij}^a(x, \tilde{k}_T) \overset{\text{LC}}{=} \int \frac{d^2 \xi}{(2\pi)^3} e^{i\tilde{k}_T \cdot \xi} \langle P, S | \bar{\psi}_i^a(0) \psi_j^a(\xi) | P, S \rangle |_{\xi^+ = 0}. \quad (5)$$

Nonetheless, the dependence on $n_-$ appears explicitly in the gauge field propagators and cannot be in general neglected.
For any Dirac matrix $\Gamma$ we define the projection $\Phi^{a[\Gamma]} = \text{Tr}[\Gamma \Phi^a]/2$. The relevant TMDs are defined as follows:

\[
\Phi^{a[\gamma^+ \gamma_5]}(x, \vec{k}_T) = S_L g^a_1(x, \vec{k}^2_T) + \frac{\vec{k}_T \cdot \vec{S}_T}{M} g^a_T(x, \vec{k}^2_T),
\]

\[
\Phi^{a[\gamma^i \gamma_5]}(x, \vec{k}_T) = \frac{M}{P^+ S_T} g^a_2(x, \vec{k}^2_T) + \ldots
\]

The inclusive DIS is determined by collinear parton distribution functions (PDFs) which are defined by transverse momentum integration of the TMDs: $g^a_2(x) = \int d^2 k_T g^a_2(x, \vec{k}_T)$ and $g^{(1)}_a(x) = \int d^2 k_T \frac{\vec{k}^2_T}{2M} g^a_2(x, \vec{k}_T)$, with $\xi$ indicating any of the above defined TMDs.

**EQUATIONS OF MOTION AND LORENTZ INVARIANCE**

The Dirac equations of motions for the quarks, and the Lorentz invariance of the theory imply the following 2 relations between twist-2 and pure twist-3 functions:

(EOM) \[
g_{1T}^{a(1)}(x) = xg_1^a(x) - x\tilde{g}_1^a(x) + O(m/M) \quad (6)
\]

(LIR) \[
g_1^a(x) = \frac{d}{dx} g_{1T}^{a(1)}(x) + \tilde{g}_1^a(x) \quad (7)
\]

where for light quarks we can neglect the term proportional to the quark mass $m$ compared to a typical hadronic scale $M_\chi$. $\tilde{g}$ and $\tilde{g}$ are pure twist-3 parton correlation functions (PCF) defined in terms of the quark-gluon-quark correlator, which in the light-cone gauge reads

\[
i\Phi_{F ij}^{\alpha}(x, x') = \int \frac{d\xi^- d\eta^-}{(2\pi)^2} e^{ik^-\xi} e^{i(k' - k)^-\eta} \langle P| \bar{\psi}_j(0) ig \partial_\eta^+ A_\eta^T(\eta) \psi_i(\xi)|P\rangle \bigg|_{\xi^+ = \xi_T = 0}, \ \ (8)
\]

where $\alpha$ is a transverse index, $x' = k'_{\perp, \eta}$, and $F$ is the QCD field strength tensor. The Lorentz decomposition of $\Phi_F$ defines the relevant PCFs [15, 16],

\[
i\Phi^{\rho}_F(x, x') = \frac{M}{4} \left[ G_F(x, x')\epsilon^{\rho}_T \epsilon_\alpha \mathcal{S}_{T \alpha} + \tilde{G}_F(x, x') S_T \gamma_5 + \ldots \right] \Phi + \ldots, \ \ (9)
\]

where hermiticity and parity constrain $G_F(x, x') = G_F(x', x)$ and $\tilde{G}_F(x, x') = -\tilde{G}_F(x', x)$. The pure twist-3 functions in Eqs. (6)-(7) are particular projections over $x'$ of $G_F(x, x')$ and $\tilde{G}_F(x, x')(\text{PV denotes the principal value})$:

\[
x\tilde{g}_T^a(x) = \text{PV} \int dx' \frac{G_F(x, x') + \tilde{G}_F(x, x')}{2(x' - x)} \quad (10)
\]

\[
\tilde{g}_T(x)^a = \text{PV} \int dx' \frac{\tilde{G}_F(x, x')/(x' - x')}{x - x'}, \quad (11)
\]

and as such are sensitive to different parts of the quark-gluon-quark correlator. It is very important to find several such quantities, because physically it is only possible
to measure $x$ but the full dependence on $(x, x')$ is needed, e.g., to determine the QCD evolution of $g_2$ or to compute the high-$k_T$ tails of TMDs. Note also that since the integrand in Eq. (11) is antisymmetric in $x, x'$, we obtain the non trivial property
\[
\int_0^1 dx \hat{g}_T^a(x) = 0.
\] (12)

THE WW RELATION

Eliminating $g_{1T}^{a(1)}$ from (6)-(7) one can derive the Wandzura-Wilczek relation (2) for the structure function $g_2 = -g_1 + \frac{1}{2} \sum_a g_T^2$, and explicitly write down its breaking term $\Delta = g_2 - g_{WW}^2$:
\[
g_2(x) - g_{WW}^2(x) = \frac{1}{2} \sum_a g_a^2 \left( \hat{g}_T^a(x) - \int_x^1 \frac{dy}{y} \hat{g}_T^a(y) + \int_x^1 \frac{dy}{y} \hat{g}_T^a(y) \right),
\] (13)

Note that $g_2$ explicitly satisfies the Burkhardt–Cottingham sum rule $\int_0^1 g_2(x) = 0$, which is not in general guaranteed in the OPE [12, 13].

A natural question is: how much is the WW relation broken? Model calculations have been used to repeatedly argue that the pure twist-3 terms are not necessarily small [13, 17]. However, in the recent past, since the LIR-breaking $\hat{g}_T$ term was not considered in Eq. (13) and the quark-mass term with $h_1$ was neglected, the breaking of the WW relation was considered to be a direct measurement of the pure twist-3 term $\hat{g}_T$. Therefore, the presumed experimental validity of the WW relation, which we are presently going to challenge, was taken as evidence that $\hat{g}_T$ is small. This observation was also typically generalized to assume that all pure twist-3 terms are small.

Our present analysis shows instead that, precisely due to the presence of $\hat{g}_T$, the measurement of the breaking of the WW relation does not offer anymore the possibility of measuring a single pure twist-3 matrix element, nor to generically infer its size. On the theory side, the quark-target model of Refs. [17, 18] can be used to determine both $\hat{g}_T$ and $\hat{g}_T$, which are both comparable in size to the other twist-2 functions. On the experimental side, we used data on polarized DIS on proton and neutron targets to fit the WW breaking term $\Delta(x)$ defined as the difference of the experimental data and $g_{WW}^2$:
\[
\Delta(x) = g_2^{ex}(x, Q^2) - g_{WW}^2(x, Q^2).
\] (14)

$g_{WW}^2$ was determined using the LSS06 leading twist $g_1$ parametrization [19], and $\Delta$ fitted to a functional form allowing for a change in sign and satisfying the Burkhardt–Cottingham sum rule. The result is presented in Fig. 1, and Table 1, where the deviation from the WW relation is quantified for a given $[x_{\min}, x_{\max}]$ interval by
\[
r^2 = \frac{\int_{y_{\min}}^{y_{\max}} dy x \Delta_{lh}^2(x)}{\int_{y_{\min}}^{y_{\max}} dy x g_2^2(x)},
\] (15)

with $y = \log(x)$. The value of $r$ is a good approximation to the relative magnitude of $\Delta$ and $g_2$, which are sign-changing functions. For the proton, we considered three intervals:
FIGURE 1. Top panels: the experimental proton and neutron $g_2^p$ structure function compared to $g_2^{WW}$. The crosses are $g_2^{WW}$ computed at the experimental kinematics. The lines are $g_2^{WW}$ computed at the average $Q^2$ of the E155x experiment: the solid (dashed) line is computed with the LSS2006 fits of $g_1$, with (solid) and without (dashed) the HT contribution obtained in the fit. Data points for the proton target [20, 21] have been slightly shifted in $x$ for clarity. For the neutron only the high precision data from [21, 22, 23] have been included. Bottom panels: The WW-breaking term $\Delta_{th}$ for model (I) and (II) compared to the higher-twist contribution to $g_1$. See text for further details.

TABLE 1. Results of the 1-parameter fits of the WW breaking term $\Delta_{th}$ for different choices of its functional form. Value $r$ of the relative size of the breaking term is computed for the whole measured $x$ range, [0.02,1]; the low-$x$ region, [0.02,0.15]; the large-$x$ region, [0.15,1]. See text for further details.

|        | proton          | neutron         |
|--------|-----------------|-----------------|
|        | $\chi^2$/d.o.f | $r_{tot}$       | $r_{low}$      | $r_{hi}$         |
| (I)    | $\Delta_{th} = 0$ | 1.22            |                |                |
| (II)   | $\Delta_{th} = \alpha(1-x)^\beta ((\beta+2)x-1)$ | 1.05 15-32%    | 18-36%         | 14-31%          |
|        | $\alpha = 0.13 \pm 0.05$ |               |                |                |
|        | $\beta = 4.4 \pm 1.0$     |                |                |                |
|        | $\Delta_{th} = 0$ | 1.66            |                |                |
| (II)   | $\Delta_{th} = \alpha(1-x)^\beta ((\beta+2)x-1)$ | 1.11 18-40%    |                |                |
|        | $\alpha = 0.64 \pm 0.92$ |               |                |                |
|        | $\beta = 24 \pm 10$       |                |                |                |

The whole measured $x$ range, [0.02,1]; the low-$x$ region, [0.02,0.15]; the large-$x$ region, [0.15,1]. For the neutron, due to the limited statistical significance of the low-$x$ data, we limit ourselves to quoting the value of $r$ for the large-$x$ region, [0.15,1].

In summary, we have found that the experimental data are compatible with a substantial breaking of the WW relation in the 15-40% range.
A PROPOSAL FOR AN EXPERIMENTAL CAMPAIGN

Figure 1 clearly shows the need for better precision in $g_2$ measurements with both proton and neutron targets. In particular, for the neutron high precision is needed away from $x \approx 0.15 - 0.20$ where JLab E01-012 data almost completely determine the presented fits. But even if in the future the WW approximation is found to be more precise than in our analysis, we would only be able to conclude that

$$
\sum_a e_a^2 \left( \tilde{g}_T^a(x) - \int_x^1 \frac{dy}{y} g_T^a(y) + \int_x^1 \frac{dy}{y} \hat{g}_T^a(y) \right) \approx 0 .
$$

This can clearly happen because either $\hat{g}_T$ and $\tilde{g}_T$ are both small, or because they accidentally cancel each other. Therefore no information can be obtained on the size of the twist-3 quark-gluon-quark term $\tilde{g}_T$ from the experimental data on $g_2$ alone.\(^2\)

However, individually determining the size of $\hat{g}_T$ and $\tilde{g}_T$ is very important to gather information on the $x, x'$ dependence of the quark-gluon-quark correlator. This can be experimentally accomplished by using the EOM (6) and LIR (7) and measuring the $g_1^{(1)}$ function, accessible in semi-inclusive deep inelastic scattering with transversely polarized targets and longitudinally polarized lepton beams (see, e.g., Ref. [25]):

$$
\hat{g}_T^a(x) = g_T^a(x) - g_1^a(x) - \frac{d}{dx} g_1^{a(1)}(x)
$$

$$
\tilde{g}_T^a(x) = g_T^a(x) - \frac{1}{x} g_1^{a(1)}(x) .
$$

In TMD factorization, $g_1^{(1)}$ is the first transverse moment of a twist-2 TMD. Its experimental determination is challenging because it requires measuring a double spin asymmetry in semi-inclusive DIS up to rather large hadron transverse momentum. Furthermore, in the LIR it appears differentiated in $x$, which requires a rather fine $x$ binning. Preliminary data from the E06-014 and SANE (E-07-003) experiments at Jefferson Lab will soon be available, and will demonstrate the feasibility of the proposed measurement of $\hat{g}_T$ and $\tilde{g}_T$.

This measurement is also very important because the EOM (6), LIR (7) and WW relation breaking (13) provide 3 independent measurement for 2 independent quantities. Verifying them will constitute a pretty stringent test of TMD factorization and its connection to collinear factorization.

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\(^2\) Note that these results were essentially already obtained in Ref. [24]. In that work, however, the authors assumed $\hat{g}_T$ small and the WW relation small, obtaining a small $\tilde{g}_T$, which is unjustified as we have just discussed.
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