Projected Three-Pion Correlation Functions

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We propose a new procedure for constructing projected three-pion correlation functions which reduces undesirable artificial momentum dependences resulting from the commonly used procedure and facilitates comparison of three-pion correlation data with theoretical models.

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I. INTRODUCTION

Two-particle intensity interferometry, exploiting Bose-Einstein correlations between pairs of identical bosons (pions, kaons), has been extensively used to extract information about the space-time structure of high-energy hadron-hadron and nucleus-nucleus collisions \cite{1}. Three-particle Bose-Einstein correlations were shown to yield additional information which can not be extracted from two-particle correlations \cite{2,3,4,5}: (i) After subtracting the two-particle correlation contributions and extrapolating to zero relative momentum, the strength of the genuine three-particle correlation term relative to the two-particle correlator provides an unambiguous measure for the degree of chaoticity of the source (or, conversely, it limits the degree of phase-coherence in the particle emission process) \cite{6}. This idea has recently been applied by the STAR Collaboration \cite{7} to three-pion correlations from 130 A GeV Au+Au collisions at RHIC, showing that at these collision energies the data are consistent with a momentum dependence of the “reduced” three-pion correlator \cite{8}.

(ii) For a completely chaotic source, the full momentum dependence of the “reduced” three-pion correlator $r_3(p_1,p_2,p_3)$ can be used to extract spatial asymmetries of the source around its point of maximum emissivity \cite{9}: for a partially coherent source, the momentum dependence of $r_3$ constrains the relative spatial sizes of the coherent and incoherent parts of the emission function \cite{10}. This information is, however, difficult to extract because the required accurate analysis of its full momentum dependence puts statistical demands on the measured three-pion correlator which can not be met by presently available data.

Statistical limitations therefore so far force experimentalists to project the three-pion correlation function onto a single relative momentum variable. The preferred choice \cite{11,12,13,14} for this variable is a Lorentz scalar that is completely symmetric under interchange of the three pions: $Q_3 = \sqrt{Q_{12}^2 + Q_{23}^2 + Q_{31}^2}$ where $Q_{ij} = -p_i - p_j$ are the Lorentz-invariant relative momenta between pairs in the pion triplet. This projection has undesirable, but to some extent unavoidable consequences for the relative momentum dependence of the three-pion correlator. Specifically, it is known \cite{15} that for a completely chaotic source the leading relative momentum dependence of $r_3(K,q_{12},q_{23})$ (where $K = (p_1 + p_2 + p_3)/3$ is the average momentum of the triplet while $q_{12} = p_1 - p_2$ and $q_{23} = p_2 - p_3$ are two linearly independent relative momenta) at small $q_{ij}$ is of 4th order in the components of the relative momenta $q_{ij}$. The projection on $Q_3$ instead introduces a dominant quadratic $q$-dependence which buries the coefficients of the 4th- and higher-order contributions that would allow to extract additional interesting source information, such as the momentum dependence of the point of maximum emissivity and the asymmetry of the source around that point \cite{16}.

In this short note we investigate in some detail the effects of this projection procedure. We show that there are different possibilities to project onto a single variable $Q_3$ of which we can anticipate the above problem far beyond the unavoidable minimum. In particular, the version employed so far in the data analysis has the undesirable feature of introducing a relative momentum dependence in the projected three-pion correlator even for a static Gaussian source for which the underlying unprojected correlator is completely momentum-independent.

We therefore suggest a different projection method which avoids this undesirable feature and furthermore allows for a “staged projection” onto any subset or different combination of the three particle momenta. As a side effect, the new method also facilitates the comparison of the projected three-pion correlator with theoretical models.

II. TWO- AND THREE-PARTICLE CORRELATORS

The two- and three-particle correlation functions are defined as the ratios between the two- and three-particle coincidence cross sections and the products of independent single-particle cross sections:

\begin{equation}
C_2(12) \equiv C_2(p_1,p_2) = \frac{N_2(p_1,p_2)}{N_1(p_1)N_1(p_2)},
\end{equation}

\begin{equation}
C_3(123) \equiv C_3(p_1,p_2,p_3) = \frac{N_3(p_1,p_2,p_3)}{N_1(p_1)N_1(p_2)N_1(p_3)}.
\end{equation}
where
\[ N_1(p_i) = \frac{d^3 N}{d^3 p_i} \quad (i = 1, 2, 3), \]
\[ N_2(p_1, p_2) = \frac{d^6 N}{d^3 p_1 d^3 p_2}, \]
\[ N_3(p_1, p_2, p_3) = \frac{d^9 N}{d^3 p_1 d^3 p_2 d^3 p_3}. \]  
Equations (1) are frame-independent even though they don’t look that way since the energy factors \( E_i \) making the cross sections \( E_i dN/d^3 p_i \) etc. Lorentz invariant cancel between numerators and denominators. In the absence of correlations \( C_2 = C_3 = 1 \). One defines the “true \( n \)-particle correlators” \( R_n \) (cumulants) by subtracting uncorrelated parts and correlations involving fewer than \( n \) particles: \( R_2(ij) = C_2(ij) - 1 \) and \( R_3(123) = C_3(123) - R_2(12) - R_2(23) - R_2(31) - 1 \). The “normalized true three-particle correlator” \( r_3(123) = r_3(p_1, p_2, p_3) \) is obtained by dividing \( R_3(123) \) by the square root of the product of 2-particle correlators:
\[ r_3(123) = \frac{(C_3(123) - 1) - (C_2(12) - 1) - (C_2(23) - 1) - (C_2(31) - 1)}{\sqrt{(C_2(12) - 1)(C_2(23) - 1)(C_2(31) - 1)}}. \]  
For a fully chaotic source, \( r_3 \) approaches the value 2 at zero relative momenta, \( q_{12} = q_{23} = 0 \), for any value of the triplet momentum \( K = (p_1 + p_2 + p_3)/3 \). The leading deviations from this limit in both numerator and denominator are of second order in the components of \( q_{ij} \), but these leading corrections cancel in the ratio, leaving a leading 4th order term in \( r_3 \). It is obvious that a projection of the numerator and denominator onto a smaller number of momentum components will in general destroy the cancellation of the leading 2nd order terms, thus generating an artificial leading quadratic \( q_{ij} \)-dependence of the projected \( r_3 \). This leading quadratic \( q \)-dependence is not interesting because its coefficients depend on properties of the source which can already be extracted from the two-particle correlation function. Unfortunately, this extraction is not completely model-independent, so it is not clear whether, even in principle, it might be possible to subtract the unwanted second order terms, using the measured two-particle correlators, in order to uncover the subleading 4th order terms in the projected \( r_3 \). We will not address this issue here. Instead we will focus on a different question: Is it possible to minimize these projection artefacts by using an optimized projection algorithm?

### III. Shortcomings of the “Standard” Projection Method

In [8] the STAR collaboration constructed a projected version \( \tilde{r}_3(Q_3) \) of the normalized true three-pion correlator using the following procedure: First, they constructed projected two- and three-particle correlation functions, \( \tilde{C}_2(Q_{ij}) \) and \( \tilde{C}_3(Q_3) \), by taking pairs and triplets of pions from the same collision event, summed over all events, binning them in the Lorentz-invariant variables \( Q_{ij} \) and \( Q_3 \), respectively, and dividing the contents of each bin by that of an equivalent bin at the same \( Q \)-value which was filled with uncorrelated pairs or triplets generated from mixed events [9]. This procedure can be formally represented by the following definitions:

\[ \tilde{C}_2(Q_{ij}) = \frac{\int d^3 p_1 d^3 p_2 N_2(p_1, p_2) \delta (Q_{ij} - \sqrt{- (p_1 - p_2)^2})}{\int d^3 p_1 d^3 p_2 N_1(p_1) N_1(p_2) \delta (Q_{ij} - \sqrt{- (p_1 - p_2)^2})}, \quad (i, j = 1, 2, 3) \]  
\[ \tilde{C}_3(Q_3) = \frac{\int d^3 p_1 d^3 p_2 d^3 p_3 N_3(p_1, p_2, p_3) \delta (Q_3 - \sqrt{- (p_1 - p_2)^2 - (p_2 - p_3)^2 - (p_3 - p_1)^2})}{\int d^3 p_1 d^3 p_2 d^3 p_3 N_1(p_1) N_1(p_2) N_1(p_3) \delta (Q_3 - \sqrt{- (p_1 - p_2)^2 - (p_2 - p_3)^2 - (p_3 - p_1)^2})}. \]  

From these projected correlation functions one then constructs the ratio [9] by fixing \( Q_3 \) and summing over all two-particle relative momenta \( Q_{ij} \) whose squares add up to \( Q_3^2 \). Formally
\[ \tilde{r}_3(Q_3) = \left[ \int_0^\infty dQ_{12} dQ_{23} dQ_{31} \delta \left( Q_3 - \sqrt{Q_{12}^2 + Q_{23}^2 + Q_{31}^2} \right) \right]^{-1} \times \]
Note that the four terms in the numerator of the integrand in Eq. (6) involve four different \( \delta \)-functions. This is clearly inconvenient when comparing to theoretical model calculations (for example those by Nakamura and Seki [9]) which are not easily projected in the same way. More seriously, however, the mixing of different projection algorithms for \( \delta \) in Eq. (6) upsets the above-mentioned partial cancellation of the relative momentum dependences in the numerator and denominator of the integrand. It thereby exacerbates the problem of introducing unwanted \( q \)-dependence into \( r_3 \) by projecting it.

We demonstrate this by considering a very simple, spherically symmetric source characterized by the following (unnormalized) emission function:

\[
S(x, p) = E_p \delta(t - t_0) e^{-x^2/2\Delta^2} e^{-p^2/2\Delta^2}.
\]

Particles are emitted instantaneously at global time \( t_0 \), with a Gaussian momentum distribution which is independent of the emission point (no "\( x-p \)-correlations"). The factor \( E_p = \sqrt{m^2 + p^2} \) is introduced for calculational convenience so that it drops out from relations such as \( E_p dN/d^3p = \int d^3x S(x, p) \) which relate the \( n \)-particle spectra \( N_n(p_1, \ldots, p_n) \) in Eq. (2) to the emission function \( S(x, p) \). For simplicity we assume non-relativistic momenta, \( \Delta < m \). General arguments presented in \( [4] \) suggest that for such a source the exact reduced three-particle correlator \( r_3 \) from Eq. (4) should be equal to 2 and completely momentum-independent. The (nonrelativistic) two-particle exchange amplitude \( \rho_{ij} = \langle \hat{a}^\dagger(p_i)\hat{a}(p_j) \rangle \) \( [2, 11, 13] \) for the source \( [7] \) reads

\[
\rho_{ij} = \int d^4x \frac{S(x, K_{ij})}{\sqrt{E_{K_{ij}}}} e^{iq_{ij} \cdot x} = (2\pi R^2)^2 e^{-\frac{k_i^2}{2\Delta^2}} - \frac{q_i^2 R^2}{2} e^{i(E_i - E_j)t_0},
\]

where \( K_{ij} = (p_i + p_j)/2 \) and \( q_{ij} = p_i - p_j \) are the average and relative momentum of the pair \( ij \). It allows to calculate the one-, two- and three-particle cross sections from the relations \( [2, 4] \)

\[
N_1(i) = \rho_{ii}, \quad N_2(1j) = \rho_{ii} \rho_{jj} + |\rho_{ij}|^2,
\]

\[
N_3(123) = 2 \Re(\rho_{12} \rho_{23} \rho_{31}) + |\rho_{12}|^2 |\rho_{31}|^2 + |\rho_{23}|^2 |\rho_{11}|^2 + |\rho_{12} \rho_{23} \rho_{31}|^2
\]

(where \( N_1(i) \) is shorthand for \( N_1(p_i) \), etc.), yielding

\[
N_1(i)N_1(j) = (2\pi R^2)^3 e^{-\frac{k_i^2}{2\Delta^2}} e^{-\frac{q_i^2}{4\Delta^2}},
\]

\[
N_2(1j) = (2\pi R^2)^3 e^{-\frac{k_i^2}{2\Delta^2}} e^{-\frac{q_i^2}{4\Delta^2} + e^{-\frac{q_j^2 + q_{ij}^2}{4\Delta^2}}},
\]

and

\[
N_3(123) = (2\pi R^2)^3 e^{-\frac{k_i^2}{2\Delta^2}} \left( 2 e^{-\frac{q_i^2}{2\Delta^2}} e^{-\frac{q_{ij}^2 + q_{12}^2}{2\Delta^2}} + e^{-\frac{q_{ij}^2 + q_{12}^2}{2\Delta^2}} e^{-\frac{q_{ij}^2 + q_{12}^2}{2\Delta^2}} e^{-\frac{q_{ij}^2 + q_{12}^2}{2\Delta^2}} \right).
\]

After plugging these into equations \( [11] \) and \( [8] \), a bit of algebra confirms that the exact \( r_3 \) is indeed momentum-independent and equal to \( r_3 = 2 \).

Let us now study the projected correlator \( \tilde{r}_3(\Sigma_3) \) defined in Eq. (6). For non-relativistic momenta we can use \(- (p_i - p_j)^2 \approx (p_i - p_j)^2 = q_{ij}^2 \) which allows to do the integrals in Eqs. \( [4] \) analytically. After transforming integration variables according to \( d^3p_i d^3p_j = d^3K_{ij} d^3q_{ij} \) and \( d^3p_i d^3p_j d^3q_{ij} = d^3K d^3q_{ij} d^3q_{jk} \) (where \( \{ij\}, \{jk\} \) indicate two of the three possible pairs \( \{12\}, \{23\}, \{31\} \), as suitable to exploit the cyclic symmetry of the three middle terms in Eq. (12) and further \( d^3q_{ij} d^3q_{jk} = d^3\zeta d^3\xi \) where \( \zeta = q_{ij} - q_{ik} \) and \( \xi = (q_{ij} + q_{jk})/2 \), the integrations are straightforward, and we obtain for the projected correlators \( [4] \)

\[
C_2(Q_{ij}) - 1 = e^{-\frac{1}{2} \Sigma_{ij}},
\]

\[
C_3(Q_{ij}) - 1 = 2 e^{-\frac{1}{2} \Sigma_3} + \frac{48}{\pi} \int_0^1 dz z^2 \sqrt{1-z^2} e^{-\frac{1}{2} \Sigma_3 z^2} = 2 e^{-\frac{1}{2} \Sigma_3} + 6 e^{-\frac{1}{2} \Sigma_3} \frac{1}{3} \frac{1}{3} \Sigma_3.
\]
as the yield of mixed-event triplets multiplied by a weight factor, statistical correlations disappear: Eq. (8) shows that for large $\Sigma_3$ one checks that $\bar{r}_3$ grows exponentially negative: $\bar{r}_3 \rightarrow -\frac{11}{120} \sqrt{\pi} e^{-\Sigma_3}/\Sigma_3$. At small relative momenta its leading $Q_3$-dependence is

$$\bar{r}_3(Q_3) = 2 - \frac{11}{120} \frac{\Sigma_3^2}{\pi} + \mathcal{O}(\Sigma_3^3),$$

i.e. of 4th order in the $q_{ij}$. The continued cancellation of 2nd order terms is probably accidental and due to the particularly simple structure of the emission function $\bar{r}_3$. The coefficient in front of the $\Sigma_3$ term is small, and the deviations of $\bar{r}_3$ from 2 become significant only at $\Sigma_3 \gtrsim 1$. This may explain why Humanic did not see such a deviation in his Monte Carlo model study [12] which explored $\bar{r}_3$ in a restricted $Q_3$ range and with limited statistical accuracy, even though he employed the same projection procedure [6].

### IV. A DIFFERENT PROJECTION METHOD

The $Q_3$-dependence in Eq. (16) is entirely artificial and generated by using different projection procedures in equations (11) and (13) and mixing them in Eq. (16). It can be easily avoided, however. Inserting the definitions $\bar{r}_3$ into Eq. (16) and multiplying both numerator and denominator by $N_1(p_1)N_1(p_2)N_1(p_3)$, $r_3$ can be brought into the form

\[
r_3(123) = \frac{N_3(p_1,p_2,p_3) - N_2(p_1,p_3)N_1(p_2) - N_2(p_2,p_3)N_1(p_1) - N_2(p_3,p_1)N_1(p_2) + 2N_1(p_1)N_1(p_2)N_1(p_3)}{\sqrt{[N_2(p_1,p_2) - N_1(p_1)N_1(p_2)][N_2(p_2,p_3) - N_1(p_2)N_1(p_3)][N_2(p_3,p_1) - N_1(p_3)N_1(p_1)]}}
\]

The numerator contains sums and differences of different kinds of triplet yields: the first term denotes real triplets where all three pions come from the same collision event, the last term mixed event triplets where all three pions come from different events, and the middle three terms subtract mixed-event triplets where two pions come from the same and the third from a different collision. For consistency with Eqs. (13), the distributions of both types of mixed-event triplets must be normalized such that they agree with the real triplets at large relative momenta where all quantum statistical correlations disappear: Eq. (15) shows that for $i \neq j$, $\rho_{ij}$ vanishes in the limit $|q_{ij}| \rightarrow \infty$.

The denominator in (18) involves a product of differences between yields of real pairs and mixed-event pairs (again normalized to the same total number of pairs), with pion momenta as indicated by the arguments. It can be rewritten as the yield of mixed-event triplets multiplied by a weight factor,

\[
\frac{[\text{DEN}(p_1,p_2,p_3)]}{[\text{NUM}(p_1,p_2,p_3)]} = N_1(p_1)N_1(p_2)N_1(p_3) \cdot W(p_1,p_2,p_3).
\]
where the latter is computed from the (Coulomb corrected) two-particle correlation function $C_2(p_1, p_2)$ according to

$$W(p_1, p_2, p_3) = \sqrt{|C_2(p_1, p_2) - 1||C_2(p_2, p_3) - 1||C_2(p_3, p_1) - 1|}. \quad (20)$$

Fully six-dimensional analyses of the two-particle correlation function $C_2(p_i, p_j) = C_2(K_{ij}, q_{ij})$ are available. It can be tabulated and interpolated, allowing for a straightforward calculation of the weight $W(p_1, p_2, p_3)$. (Again the mixed-pair background must be normalized such that $C_2(q_{ij} \to \infty) = 1$.)

We propose to construct $r_3$ directly from the ratio (13) instead of first constructing two- and three-particle correlation functions and then using (20). It is straightforward to project Eq. (13) onto the single relative momentum variable $Q_3$ (or onto any other desired combination of the three pion momenta) by binning the numerator and denominator separately in $Q_3$ and taking the ratio:

$$\langle r_3\rangle(Q_3) = \frac{\int d^3p_1 d^3p_2 d^3p_3 \left[\text{NUM}(p_1, p_2, p_3)\right] \delta \left(Q_3 - \sqrt{-(p_1-p_2)^2 - (p_2-p_3)^2 - (p_3-p_1)^2}\right)}{\int d^3p_1 d^3p_2 d^3p_3 \left[\text{DEN}(p_1, p_2, p_3)\right] \delta \left(Q_3 - \sqrt{-(p_1-p_2)^2 - (p_2-p_3)^2 - (p_3-p_1)^2}\right)}. \quad (21)$$

Note that, by symmetry, the three middle terms in the numerator of Eq. (13) give identical contributions to Eq. (21). The numerator is thus obtained by first running separately through all real triplets, all triplets with two particles from the same event and the third particle from a different event, and all mixed-event triplets, binning each of these in $Q_3$. After normalizing the bin contents of the mixed-event triplets by the appropriate factor which ensures that their distributions agree with that of the real triplets at large $|q_{ij}|$ (note that this does not imply that they necessarily agree at large $Q_3$!), one then subtracts, for each value of $Q_3$, from the bin containing the real triplets three times the bin content from the half-mixed triplets and adds twice the content from the fully mixed triplets. For the denominator one runs over all fully mixed triplets, calculates for each of them the weight $W$ and bins these weights in $Q_3$. At the end one normalizes the bin contents by the same factor as for the fully mixed triplets in the numerator.

While this projection procedure does not, of course, completely sidestep the unavoidable interference with the cancellation of the leading relative momentum dependences between numerator and denominator, at least it preserves the momentum-independence of $r_3$ in the “trivial” case (7) of a spherically symmetric, static source without $x$-$p$-correlations: For the source (7),

$$\langle r_3\rangle(p_1, p_2, p_3) = r_3(p_1, p_2, p_3) = 1.$$

A technical complication for the data analysis arises from the repulsive final state Coulomb interaction between the emitted charged pions which must be corrected for before computing the reduced three-pion correlator $r_3$. This problem poses itself similarly for both projection techniques and can be dealt with in the same way in either approach, by using Coulomb correction factors extracted from a self-consistent Coulomb correction of the two-pion correlation functions. We refer to Refs. [6, 8] for details. Note that in the form given in Eq. (13), the Coulomb correction must be applied to the real pairs and triplets, and not to the mixed-event ones.

V. CONCLUSIONS

We have shown that, for a simple spherically symmetric, non-expanding source with sudden particle emission, the standard projection procedure, as up to now applied in the experimental analysis of three-pion correlations, introduces an artificial momentum dependence into the normalized true three-pion correlator $r_3$, even though the exact result for $r_3$ is completely momentum-independent. This suggests that the observed $Q_3$ dependence of $r_3$ seen by WA98 in Pb+Pb collisions at the SPS [6] and by STAR in Au+Au collisions at RHIC [8] may be significantly affected by artificial projection effects. Even though projection-induced relative momentum dependence of $r_3$ can in general not be completely avoided, it can be minimized by using the new projection procedure suggested in this paper. This procedure computes $r_3$ in a more direct way and applies the same binning to all terms. It preserves the momentum-independence of $r_3$ for a symmetric non-expanding source. The new projection algorithm can also be easily employed in theoretical model calculations, rendering the comparison with data (but not necessarily the interpretation of the extracted source parameters) more straightforward.

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