Measurement Theory in the Philosophy of Science

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The philosophy of science is a discipline concerning the metaphysical aspect of science. Recently, I proposed measurement theory, which is characterized as the metaphysical and linguistic interpretation of quantum mechanics. I assert that this theory is one of the most fundamental languages in science, and thus, it is located at the central position in science. This assertion will be examined throughout this preprint, which is written as the draft of my future book (concerning the philosophy of science). Hence, I hope to hear various opinions about this draft.

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Measurement theory is composed of two axioms as follows.

\[
\text{measurement theory (scientific language)} := \begin{array}{c}
\text{[Axiom 1]} \\
\text{measurement (probabilistic interpretation)}
\end{array} + \begin{array}{c}
\text{[Axiom 2]} \\
\text{causality (the Heisenberg picture)}
\end{array}
\]

And it has the following classification:

\[
\begin{align*}
\text{(Y)} \\
\text{Chap. 1 measurement theory (scientific language)}
\end{align*}
\]

\[
\begin{align*}
\text{quantum measurement} & : \text{Chap. 3} \\
\text{classical measurement} & : \\
\text{continuous} & \begin{cases} 
\text{pure type}^1 : \text{(Chaps. 2–9)} \\
\text{mixed type}^2 : \text{Sec.4.4}
\end{cases} \\
\text{bounded} & \begin{cases} 
\text{pure type}^3 : \text{Chaps. 10, 11} \\
\text{mixed type}^4 : \text{Notes 10.4, 11.3}
\end{cases}
\end{align*}
\]
For the section in which each axiom is explained, see the following table: In this preprint, we mainly devote ourselves to classical pure measurement theory (① and ③).

### Measurement Theory born from Quantum Mechanics

Measurement theory is characterized as a language created in order to describe ordinary phenomena numerically, though it is oriented in quantum mechanics. In this chapter, the outline of two axioms (i.e., Axiom 1 (in Chap. 2) and Axiom 2 (in Chap. 5) ) will be introduced.

#### 1.1 Why do we start from quantum mechanics?

We think that readers, who read the abstract, ask the following:

(A1) Why is quantum mechanics (i.e., the physics concerning the microscopic world) applicable to the description of ordinary phenomena (i.e., economics, psychology, electric engineering, etc. )? Or, why is the quantum mechanical approach (i.e., measurement theory) indispensable? That is, why does measurement theory — quantum mechanical language — hold? Further, is there another approach?

Although the problem will be answered here and there in this print, our answers are not sufficient. However, we are convinced that the readers, who read the whole of this print, agree to the following opinion:

(A2) Measurement theory — quantum mechanical approach — is most natural. And, there is no powerful rival against measurement theory.

The worst understanding of measurement theory is to consider measurement theory as "an ass in a lion’s skin". In this preprint, we consistently assert (cf. [13]) that measurement theory is more fundamental than quantum mechanics (cf. Sec.9.3).

#### 1.1.1 The classical mechanical world-view

Before we explain the quantum mechanical world-view (i.e., measurement theory), in this section we discuss the classical mechanical approach and its problems.

The law is most essential in physics, and moreover, there is no physics without the law. However, the language should be prepared before the is declared. In this sense, physics has the following structure:
For example, consider Newtonian mechanics. If we do not know how to use the term "causality" (i.e., "time", "space", "velocity", "acceleration" and so on), then we cannot declare "the law of Newton kinetic equation". That is, Newtonian mechanics has the following structure:

\[ \text{(Newton kinetic equation)} \]

This means that the linguistic aspect is valid without the test of the law. Consider Newton mechanics (C). The time variable differential equation is valid independently whether the law is true or not. That is,

\[ \text{Newton mechanics without verification experiment, linguistic aspect, how to use "causality"} \]

And, applying differential equation to lots of phenomena, you get the belief such that

differential equation is quite applicable to sciences and engineering. (In this preprint, we consider that "sciences = engineering" (cf. Note 1.2, Note 1.11, Sec.8.1(m)).)

If you get this belief, it may be called the causal world-view. The general form of differential equation is represented by the system of differential equation (which is called the state equation):

\[
\begin{align*}
\frac{d\omega_1}{dt}(t) &= v_1(\omega_1(t), \omega_2(t), \ldots, \omega_n(t), t) \\
\frac{d\omega_2}{dt}(t) &= v_2(\omega_1(t), \omega_2(t), \ldots, \omega_n(t), t) \\
&\quad \ldots \\
\frac{d\omega_n}{dt}(t) &= v_n(\omega_1(t), \omega_2(t), \ldots, \omega_n(t), t)
\end{align*}
\]   \tag{1.1}

Therefore,

(E1) the causal world-view (state equation method) is the spirit that every motion phenomena should be represented by state equation (1.1), or the spirit to believe that, if a certain phenomenon is represented by state equation (1.1), it has the causality (cf. Sec.6.1.2(c))

And further, the state equation method acquires the concept of "probability", we get

(E2) dynamical system theory (=statistics)

(Here, we consider that "dynamical system theory = statistics" (cf. Sec.7.1). ) This is the classical mechanical world-view method, whose starting point is Newtonian mechanics.

This classical mechanical world-view method, in which Newtonian mechanics is regarded as the starting point, we have acquired a great success. Since there is no science without statistics (that is, differential equation, probability). the classical mechanical world-view (=statistics-dynamical system theory) is a base supporting present age science

However, there are points of uncertainty in the classical mechanical method. For example, the following problems (F1)–(F5) are considered:
(F_1) What kind of theory is statistics? That is, is it "mathematics", "mathematical method", "applied mathematics", "world-view", "language" (and so on)?

(F_2) Why is the concept of "probability" added to the state equation method? That is, why does classical world-view have two birthplaces (i.e., Newtonian mechanics and gamble)?

(F_3) The concept of "probability" could be added to the state equation method. If it be so, some may want to add another concept (e.g., fuzzy, chaos, etc.). Is it possible?

In other words,

(F_4) Is the development (from the state equation method to statistics) inevitable?

which is outstanding point. And further, we have the following most fundamental question:

(F_5) Why is mathematical theories differential equation and probability applicable to the description of ordinary phenomena?

♠ **Note 1.1** Since language has not fully been prepared yet, this question (F_5) may be a vague expression. But, there is a reason to consider that

(♯_1) the useful mathematical theory that has the reason to be useful. That is, the powerful mechanical world view is hidden behind a useful mathematical theory.

Because mathematic itself is independent of world. Thus, world-description method is indispensable. For example,

| mathematics                | world-description method       |
|----------------------------|--------------------------------|
| differential geometry      | the theory of relativity       |
| differential equation      | Newton mechanics, electromagnetism |
| Hilbert space              | quantum mechanics              |

Problem (F_5) is equivalent to the following:

(♯_2) Behind hat kind of world-description is probability theory? hidden

### 1.1.2 Start from quantum mechanics and not Newtonian mechanics

When we start from Newtonian mechanics, we have nuisances such as (F_1)–(F_5): Thus, we start from quantum mechanics. That is,

(G) measurement theory is a scientific language modeled on quantum mechanics.

Now, let us explain it.

As mentioned later (in Chap. 3), quantum mechanics, which was discovered by Heisenberg, Schrödinger, Born in 1925–1927, is physics for the microscopic world. quantum mechanics is composed of two laws (i.e., "Born’s probabilistic interpretation of quantum mechanics" and "quantum kinetic equation (due to Heisenberg and Schrödinger)”). That is,
As the linguistic turn of quantum mechanics, we get measurement theory — quantum mechanical world-view — as follows:

Quantum mechanics is physics for microscopic phenomena. However, its linguistic turn (=measurement theory) has a power to describe phenomena in our usual world.

Measurement theory is quite simple language, which has two key-words (i.e., "measurement" and "causality").

Writing diagrammatically,

where scientific language must be distinguished from ordinary language and mathematics (=mathematical language).

Here, the quantum mechanical world-view method (=measurement theory) clarifies classical mechanical problems (F1)–(F5) in what follows (F1′)–(F5′):

(F1′) measurement theory is a world-description language (i.e., scientific language)

(F2′) The source of measurement theory is quantum mechanics.

(F3′) If some try to add a basic concept to measurement theory, they must start to add the basic concept to quantum mechanics. Therefore, we can assure that the trial is impossible.

(F4′) As mentioned in Fig. 8.2 in Chap. 8, we assert:

the development from the causal world-view (E1) to classical mechanical world-view (E2) is not inevitable

Also, we can do well without "gamble".

The reason that the problems (F1)–(F4) are solved is due to the fact that quantum mechanics itself possesses the concept "probability".

If it be so, there may be a reason to choose measurement theory — quantum mechanical world-view method —, however, it is a matter of course that
(L₁) in every theory, the most important thing is to determine the starting point.

Thus, an immediate conclusion should be avoided. However, from the above (F₁′)–(F₄′) we assert that

(L₂) statistics-dynamical system theory (classical mechanical world-view) is the abbreviation of measurement theory (quantum mechanical world-view method)

And thus, The question (♯₂) in (F₅) (=Note 1.1) — Why are the mathematical theories (i.e., differential equation and probability theory) are useful? — is answered as follows

(F⁵') measurement theory is hidden behind these mathematical theories

Note 1.2 In this print, we assert that

measurement theory = the language of engineering (or, sciences)

That is, (L₂) — statistics-dynamical system theory is immature, and measurement theory is mature — says that

measurement theory makes engineering (or, sciences) mature

(Note 1.12).

1.1.3 Why does measurement theory hold?

If we believe in the argument mentioned in the previous section (i.e., the quantum mechanical world-view(K) is superior to classical mechanical world-view(E₂) ), increasingly we want to answer (A₁):

(A₁) Why is quantum mechanics (i.e., the physics concerning the microscopic world) applicable to the description of ordinary phenomena (i.e., economics, psychology, electric engineering, etc.)? Or, why is the quantum mechanical approach (i.e., measurement theory) indispensable? That is, why does measurement theory — quantum mechanical language — hold? Further, is there another approach?

Of course, we do not have the absolute answer. Thus we want to add the following discussion: For example, consider

the statement such as "even monkeys fall from trees"

This is the famous proverb in Japan. This is the same as the proverb "Even Homer sometimes nods" or "A good swimmer is not safe against drowning". The statement "Even monkeys fall from trees", which must have described the actual phenomenon, is isolated from reality. And the statement becomes the proverb "Even monkeys fall from trees". Writing diagrammatically,

\[
\text{the spirit:"world is before language"} \quad \text{proverbalizing} \quad \text{the spirit:"language is before world"}
\]

Even monkeys fall from trees

( Wording describing the actual phenomenon )

Even monkeys fall from trees

( Wording separated from reality )
As in the above, "proverbalizing" means that the order of the "world" and "language" is reversed. That is, the proverb "Even monkeys fall from trees" can be applicable to different world (which is not related to "monkey" nor "tree"). This is

\((M_1)\) Wonder of man’s linguistic competence

This cannot but accept as a fact.

Thus, measurement theory is characterize as the linguistic turn. That is, rewriting (I):

\[(M_2)\]

\[(\text{the terms in (J) connect reality})\]
\[\text{quantum mechanics(H)}\]
\[\text{proverbalizing}\]
\[\text{the linguistic turn}\]
\[\text{measurement theory(K)}\]

\[(\text{the terms in (J) have no reality})\]

Quantum mechanics is physics for microscopic world, however, its linguistic turn (i.e., measurement theory) can describe ordinary world.

1.1.4 Two world-description method — realistic vs. linguistic

It is a matter of course that

\[(M_3)\] The essence of the proverb is not “experimental verification” but “usability”

And further,

\[(N)\] measurement theory is based on the spirit of "language is before world"

♠ Note 1.3 metaphysics is an academic discipline concerning the propositions in which empirical validation is impossible. Lord Kelvin(1824–1907) said that

Mathematics is the only good metaphysics.

This is very persuasive saying. However, Our purpose is

(2) to establish metaphysics(called measurement theory) as a discipline which forms the base of science

If it be so, the world-description is classified as follows.

\[(O)\] world-description

\[\begin{align*}
\text{realistic method(In the beginning was the "world")} \\
\text{physical phenomena are directly described in mathematics.} \\
\text{physics is created by this method.} \\
\text{The law is main} \\
\text{"world is before language".}
\end{align*}\]

\[\begin{align*}
\text{linguistic method(In the beginning was the "word")} \\
\text{Phenomena are described in a scientific language.} \\
\text{Various sciences are created by this method} \\
\text{The scientific law and the scientific language are main} \\
\text{As a scientific language, measurement theory is adopted} \\
\text{"language is before world".}
\end{align*}\]
The linguistic method is, for the first time, established my measurement theory, and thus it is a new world-description method. Thus, for completeness, we add the following two notes.

♠ Note 1.4 For example, assume that you want to understand some economical phenomenon \( P \). For this, consider the following four methods (a)–(d):

(a) you exactly measure the economical phenomenon \( P \), and represent it mathematically. Then, you can create a certain economical theory \( T_a \).

(b) First we decide use the mathematical theory (differential equation, probability theory). you exactly measure the economical phenomenon \( P \), and represent it by the above mathematics. Then, you can create a certain economical theory \( T_b \).

(c) First we decide use statistics (=dynamical system theory). you exactly measure the economical language before world \( P \), and represent it by the above mathematics. Then, you can create a certain economical theory \( T_c \).

(d) First we decide use measurement theory. you exactly measure the economical phenomenon \( P \), and describe it by measurement theory. Then, you can create a certain economical theory \( T_d \). This is the linguistic method.

Note that the economical phenomenon \( P \) is common. Thus, if each (a)–(d) is the fully considered theory, we can expect that \( T_a = T_b = T_c = T_d \).

(e) if \( T_a = T_b = T_c = T_d \), then we can consider that it is created by (d).

However, the theory of relativity can not be understood in measurement theory i.e., in the linguistic method). Also, as seen in Chap. 9, it is interesting to see that \( T'_c \neq T'_d \) in equilibrium statistical statistics. In this case, we assert that \( T'_d \) should be adopted.

♠ Note 1.5 If some may regard "realistic method vs. linguistic method" (the world-description classification (O)) as "materialism vs. idealism" in philosophy, they never accept "linguistic method". However, we think that "linguistic method" is acceptable for everyone. As mentioned in Chap. 8, we think that

(2) "idealism = linguistic method"

In this sense, the idealism can not be understood without measurement theory.

1.2 Monism and dualism

1.2.1 Monism[="matter"] and dualism[="mind and matter"]

In the previous section, we discuss the world-description classification (O). In this section, we introduce another world-description classification, i.e., monism and dualism. In monism, we consider that "world" = "matter", and in dualism, "world" = "I(=mind)"+"matter". That is, That is,

\[
(P_1) \quad \text{world-description} \begin{cases} 
\text{monism}=["\text{matter}"] \\
\text{dualism}=["\text{I}(=\text{mind})"+"\text{matter}""]
\end{cases}
\]

Newtonian mechanics and the theory of relativity, which are formulated in monism, acquire a great success.

If we are concerned with the troublesome thing such as "I(=mind)", objectivity is spoiled, and thus, we are without science. However, quantum mechanics, which are formulated in dualism, acquires a great success.

Therefore, the dualism in this not is the dualism inspired from quantum mechanics.
Note 1.6 The world-description mentioned in this print is always quantitative. And thus, mathematics is always fundamental and essential in our world-description. However, it should be noted that

(2) mathematics itself is independent of world. That is, mathematics exists without world.

That is, mathematics is the origin learning (before science, that is, before the world-description classifications (O) and (P)).

Considering (O) and (P_1), we get the following classification:

\[(P_2) \text{ world-description} \left\{ \begin{array}{l} \text{realistic method} \{ \text{monism} \cdots \text{classical mechanics,..} \\
\text{dualism} \cdots \text{quantum mechanics} \\
\text{linguistic method} \{ \text{monism} \cdots \text{(cf. Sec.1.2.2)} \\
\text{dualism} \cdots \text{(measurement theory, Sec.1.2.3)} \end{array} \right. \]

1.2.2 Linguistic world-description in monism (state equation method)

The realistic world-description in monism is well known as physics (i.e., Newtonian mechanics, etc.). Thus, in this section, we devote ourselves to the linguistic world-description in monism. Although this may not be authorized yet, there may be a reason to consider that it is the same as the state equation method, which is characterized as the linguistic turn of Newtonian mechanics (see the (D)). The essence of the state equation method is only "causality". Thus, the state equation method without causality is quite simple. That is,

The key-words are as follows:

(Q) \text{object(=matter) and state(=property)}

And thus, the statements (in linguistic world-description method) are as follows:

\begin{frame}
\textbf{Linguistic world-description method in monism}
\end{frame}

\begin{quote}
An \textbf{object} has a \textbf{state} \(\omega\)
\end{quote}

This can be easily understood in the following simple examples:

(R_1) The temperature of water in this cup is 5 °C
\[\Rightarrow \text{[The object (i.e., the water in this cup)] has a state such as [5 °C]}\]

(R_2) John has 500 dollars in his purchase
\[\Rightarrow \text{[The object (i.e., John’s purchase)] has a state such as [500 dollars]}\]

(R_3) This flower is red
\[\Rightarrow \text{[The object (i.e., this flower)] has a state such as [red]}\]

(R_4) The water in this cup is cold
\[\Rightarrow \text{[The object (i.e., the water in this cup)] has a state such as [cold]}\]
Here, \((R_1)\) and \((R_2)\) are important. That is, \((R_3)\) and \((R_4)\) are regarded as the preparation of Sec. 1.2.3 (The linguistic world-description in dualism).

There is a possibility that a state is \(\omega\) or \(\omega'\). Put \(\Omega = \{\omega, \omega', \ldots\}\), which is a state space. For example,

\(\text{(R') In the case of (R_1), we see that the state space } \Omega = \{\omega \mid \omega \degree C \text{ is the temperature of water }\} = [0, 100].\)

\(\text{(R_2') In the (R_2), we consider the possibility such that 1 cent, 2 cents, \ldots. Therefore, The state space } \Omega = \{0, 1, 2, \ldots\}\)

\[\star \text{ Note 1.7 Most scientists may not be familiar with the following questions:} \]
\[\text{(2) "realistic method" or "linguistic method"? "monism" or dualism"?}\]

As mentioned in (F_1), the reason is due to the fact that these questions have been been kept ambiguous in statistics (=dynamical system theory). That is, as mentioned in (L_2), statistics (=dynamical system theory) does not have a power to clarify the questions (2). Therefore, the argument in this section is not authorized, and thus it should be regarded as the preparation for the following (Sec.1.2.3) (Linguistic method in dualism).

1.2.3 Linguistic world-description method in dualism (= the Copenhagen interpretation of measurement theory)

Quantum mechanics is dualistic physics (i.e., dualistic and realistic world-description), and, measurement theory is the linguistic turn (verbalizing, proverbializing) of quantum mechanics. And therefore, measurement theory is dualistic and linguistic world-description. The monastic and linguistic world-description mentioned in the previous section is too simple, but the dualism and linguistic world-description (i.e., measurement theory) is rather troublesome.

As mentioned in (K) and (J_1), measurement theory has the following structure:

\[
\text{(K) measurement theory := [Axiom 1] measurement (J_1) + [Axiom 2] causality (J_2)}
\]

in which we see two key-words (i.e., "measurement" and "causality"). Thus, even if we omit "causality", we have "measurement" which is composed of the following terms:

\[
\text{(J_1) measurement ( the items: observer, measuring object, state, observable (≈measuring instrument), measured value , probability ).}
\]

Thus, compared with monism, dualism has many key-words.

The concept of "measurement" can be, for the first time, understood in dualism. Let us explain it. The image of "measurement" is as shown in Fig. 1.1.
1 MEASUREMENT THEORY BORN FROM QUANTUM MECHANICS

Figure 1.1: The image of “measurement(=\(a+\bar{b}\))” in dualism

In the above,

(S_1) \(\circ\): it suffices to understand that ”interfere” is, for example, ”apply light”.

\(\bullet\): perceive the reaction.

That is, ”measurement” is characterized as the interaction between ”observer” and ”measuring object”. However,

(S_2) In measurement theory, ”interaction” must not be emphasized.

Therefore, in order to avoid confusion, it might better to omit the interaction ”\(a\) and \(b\)” in Fig. 1.1.

Before we mention several rules [(U_1)–(U_7)] (which is called ”the Copenhagen interpretation”), we must say so called ”the Copenhagen interpretation” in that follows.

(T) The Copenhagen interpretation is one of the earliest and most commonly taught interpretations of quantum mechanics. The essential concepts of the Copenhagen interpretation were devised by Niels Bohr, Werner Heisenberg and others. And finally it was accomplished by von Neumann [32](1932).

This Copenhagen interpretation in (T) is within physics. On the other, our ”Copenhagen interpretation [(U_1)–(U_7)]” is not physics. Thus the two are completely different. In this sense, there may be an opinion that ”the Copenhagen interpretation”, Hence in this book it should have been called ”the linguistic interpretation” or ”the linguistic Copenhagen interpretation”.

The Copenhagen interpretation

(U_1) Consider the dualism composed of “observer” and “system( =measuring object)”. And therefore, “observer” and “system” must be absolutely separated. ((this section, Sec.5.3.2) If it says for a metaphor, we say ”Audience should not be up to the stage”

(U_2) Of course, ”matter(=measuring object)” has the space-time. On the other hand, the observer does not have the space-time. Thus, the question: “When and where is a measured value obtained?” is out of measurement theory. Thus, there is no tense in measurement theory. This implies that there is no tense in science. (Sec. 2.3.3, Sec.6.4.2).
In measurement theory, "interaction" must not be emphasized (this section, Sec.3.4).

Only one measurement is permitted (Sec.2.5). Thus, the state after measurement is meaningless.

There is no probability without measurement (Sec.2.2, Note 4.7).

observable is before state, Or, observable is superior to state (Sec.2.6.2).

State never moves (Sec.6.4.1) and so on. Also, see Sec.3.3.1(e), Sec. 6.4.3(f).

Now we can state the Prototype of Axiom 1(measurement) as follows (for the precise version, see Axiom p 1(page 23)).

---

**Axiom 1(measurement) Prototype**

> When an observer takes a measurement of an observable O (or, by a measuring instrument O) for a measuring object with a state ω, the probability that a measured value \(x\) is obtained is given by \(P\).

---

Here, measurement theory assert that Describe every phenomenon modeled on Axiom 1 under the direction of the Copenhagen interpretation \([(U_1)-(U_7)]\)

\[\blacklozenge\textbf{ Note 1.8} \] Summing up (M2), we see

\[\text{(the terms (J) have reality)} \quad \frac{\text{proveralizing}}{\text{linguistic turn}} \quad \text{(the terms (J) have no reality)} \]

\[
\begin{array}{c}
\text{(M2)} \\
\text{quantum mechanics(H):physics} \\
\text{(the Copenhagen interpretation)}
\end{array}
\rightarrow
\begin{array}{c}
\text{measurement theory(K):language} \\
\text{(the Copenhagen interpretation)}
\end{array}
\]

Therefore,

\[\text{(2) the Copenhagen interpretation\[(U_1)-(U_7)]\ is common in both quantum mechanics(H) and measurement theory(K).}\]

For example, the following is one of the simplest dualistic statements.

(V) When an observer measures by the exact ruler (i.e., measuring instrument) for the pencil (i.e., measuring object) with the state (the length 15.3 cm), the probability that measured value [15.3 cm] is obtained is 1.

Here, the region in which there is a possibility that a state \(ω\) [resp. a measured value \(x\)] exists is called a state space [resp. measured value space], and denoted by \(Ω\) [resp. \(X\)].

In the above (V), we may consider that \(Ω = [5, 30]\). Also, if the rule’s length is 25cm, we may put \(X = [0, 25]\).
The following example is typical:

**Example 1.1 [The water is cold or hot?]** Let testees drink water with various temperature $\omega \degree C$ $(0 \leq \omega \leq 100)$. And you ask them “cold” or “hot” alternatively. Gather the data, (for example, $g_c(\omega)$ persons say “cold”, $g_h(\omega)$ persons say “hot”) and normalize them, that is, get the polygonal lines such that

$$f_c(\omega) = \frac{g_c(\omega)}{\text{the numbers of testees}}$$

$$f_h(\omega) = \frac{g_h(\omega)}{\text{the numbers of testees}}$$

And

$$f_c(\omega) = \begin{cases} 
1 - \frac{\omega - 10}{60} & (0 \leq \omega \leq 10) \\
0 & (10 \leq \omega \leq 70) \\
\frac{70 - \omega}{60} & (70 \leq \omega \leq 100)
\end{cases}, \\
f_h(\omega) = 1 - f_c(\omega)$$

This is described in terms of Axiom 1 in what follows.

Define the state space $\Omega$ such that $\Omega = \text{interval } [0, 100](\subset \mathbb{R})$ and measured value space $X = \{c, h\}$. Here, consider the "[C-H]-thermometer" such that

(W₁) You choose one person from the testees, and you ask him/her “cold” or “hot” alternatively. Then the probability that he/she says "cold" or "hot" is given by $\begin{bmatrix} f_c(55) = 0.25 \\
f_h(55) = 0.75 \end{bmatrix}$

This is described in terms of Axiom 1 in what follows.

(W₂) for water with $\omega \degree C$, [C-H]-thermometer presents $\begin{bmatrix} c \\
h \end{bmatrix}$ with probability $\begin{bmatrix} f_c(\omega) \\
f_h(\omega) \end{bmatrix}$. This [C-H]-thermometer is denoted by $O = (f_c, f_h)$

Note that this [C-H]-thermometer can be easily realized by "random number generator".

Here, we have the following identification:

(W₃) $\iff$ (W₁)

Therefore, the statement (W₁) in ordinary language can be represented in terms of measurement theory as follows.
When an observer takes a measurement by \([\text{[C-H]-instrument]}\) measuring instrument \(O = (f_c, f_h)\) for \([\text{water}]\) \((\text{System (measuring object)})\) with \([55 \, ^\circ C]\) \((\text{state} = \omega \in \Omega)\), the probability that measured value \([c_h]\) is obtained is given by

\[
\begin{bmatrix} f_c(55) \\ f_h(55) \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix}
\]

In the above, note that,

- The terms in \((J_1)\) are used.
- It is due to the Copenhagen interpretation \([U_1]–[U_7]\). (although \((U_2), (U_6), (U_7)\) are not used, these will be used in the relation with Axiom 2(Chap. 6).

This example will be again discussed in the following chapter(Example 2.7).

\[\hat{\text{Note}} \ 1.9\] It may be understandable to consider "observable" ="the partition of word". For example, Fig. 1.2 says that \((f_c, f_h)\) is the partition between "cold" and "hot". Also, "measuring instrument" is the instrument that choose a word among words. In this sense, we consider that "observable" = "measurement instrument". Also, John Locke's famous sayings "primary quality (e.g., length, weight, etc.)" and "secondary quality (e.g., sweet, dark, cold, etc.)" urge us to associate the following correspondence:

\[\text{state} \leftrightarrow \text{primary quality}, \quad \text{observable} \leftrightarrow \text{second quality}\]

These words form the basis of dualism.

### 1.2.4 Monism vs. dualism

The readers may understand that dualistic world-description is rather troublesome. Here, recall Examples \((R_3)\) and \((R_4)\). That is,

\[\begin{align*}
(R_3) \quad & \text{This flower is red} \\
\Rightarrow & \text{[The object (i.e., this flower)] has a state such as [red]} \\
(R_4) \quad & \text{The water in this cup is cold} \\
\Rightarrow & \text{[The object (i.e., the water in this cup)] has a state such as [cold]} 
\end{align*}\]

The readers may feel that these \((R_3)\) and \((R_4)\) are unnatural. Form the dualistic point of view, "red" and "cold" should be regarded as "measured values" and not "states". This is a reason that, in science (i.e., the monastic world-description), we may say "This water is 55", On the other hand, in science, we seldom say "This water is hot". The statement "this water is hot" is not represented in the monism but in the dualism. as seen in \((W_4)\) of Example 1.1). That is, dualistic method has a power to clarify the difference between "state (= the original property of measuring object)" and "measured value".

Compared with monism, the dualistic view has a possibility to extend the range of description.
Thus, we push on to scientific dualism. In spite that we do not consider "the mind cannot be solved by the principle of a substance", we believe that dualism (with a great power of expression) should be adopted.

♠ Note 1.10 Readers may want to know the conclusion of "monism vs. dualism". In physics, this is discussed in Einstein–Bohr debate is characterized as "monism vs. dualism" in physics. This is still unsolved. We think that (1) In the linguistic method, dualism is superior to monism. That is because the monism is not mature in the linguistic method (Note 1.7).

Also, we think that statistics = dynamical system theory is should be regarded as the abbreviation of measurement theory (cf. Chap.4). Summing up the above arguments, we have the following:

| Table 1.1: realistic and linguistic world-description methods \ monism-dualism |
|--------------------------------------------------|
| realistic world-description method | monism | dualism |
| (realistic world-view) | Newtonian mechanics ( electromagnetism , ...) | quantum mechanics |
| linguistic world-description method | state equation method (Note 1.7) | measurement theory |
| (linguistic world-view) |

Therefore, measurement theory has two kinds of "absurdness." That is,

(♯2) the absurdness of measurement theory

{ idealism ···linguistic world-view
  
  dualism ···the Copenhagen interpretation

♠ Note 1.11 Readers may want to ask:

What is science (other than physics) ?

The answer will be presented in Sec.8.1(m). For the time being, think

various sciences (other than physics )=engineering

We add some remark in what follows. For example, consider the question:

a heart transplant operation is medical or engineering?

A heart transplant operation is, of course, most advanced science and technology. However, it is still unripe if we regard a heart transplant operation as engineering. That is because we can not yet create the robot who has the technology of a heart transplant. Such a robot will be realized in the development. The heart transplant operation is still considered to be infancy as one field of engineering.

1.3 Ordinary language— lawless area —

It is sure that

Ordinary language is human beings’ greatest invention.

But, ordinary language is elusive ambiguous and very strange monster languages to take in, and it cannot be said that the framework is clear Thus, let us explain the relation among ordinary language, scientific language and mathematical language.

For example, clearly mathematics is included in the ordinary language statements (R1), (R2) and (W1).

The word Greek "logos" has a meaning of both "logic" and "language." Thus, considering that "mathematics + ordinary language" \ wide ordinary language, we may assert that "wide ordinary language", is the Origin language before world-description(Chap. 1(O)).

If it be so, we may see:
the relation between widely ordinary language and world-description method:

\[
\begin{aligned}
\text{widely ordinary language} & \quad \Rightarrow \quad \text{world-description} \\
\text{(before science)} & \quad \text{(Chap. 1(0))} \\
\end{aligned}
\]

\begin{itemize}
\item \textcircled{1} \text{realistic method} \\
\text{(Newtonian mechanics, etc.)}
\item \textcircled{2} \text{linguistic method} \\
\text{(measurement theory)}
\end{itemize}

And thus, we think:

\((X_2)\) every science holds as a premise of widely ordinary language(0).

But, it is a matter of course that ordinary language is not created for science, and the framework of widely ordinary language is not clear. Even if there exists the rule of ordinary language, it is too complicated to write it. Thus, ordinary language is so-called ”lawless area” or ”disorderly language”.

Although there may be many opinions, it is sure that the following is one of them:

\((X_3)\) ”\(\textcircled{0}\) widely ordinary language” includes the statement \((R_1), (R_2), (W_1)\) in Example 1.1 and further, arithmetical word problems ( (Example 7.6) or (Chap.11 (A2)) and so on), statistics (=dynamical system theory)

That is, there is a reason to consider that these somehow sink into ordinary language.

But, measurement theory is composed of two rules (Axioms 1 and 2). And thus, it has the clear framework. Therefore, as the statement \((W_1)\) in Example 1.1 in ordinary language was written by the measurement theoretical \((W_4)\),

\((X_4)\) As much as possible, we start from ”\(\textcircled{2}\) measurement theory” and not ”\(\textcircled{0}\) widely ordinary language”.

And

\((X_5)\) to establish metaphysics(called measurement theory) as a discipline which forms the base of science, or as a basic language by which sciences are described. Or equivalently,

measurement theory is the special language by which sciences are described. And reversely, sciences hold by the measurement theoretical description.

This is our purpose of this print.

\begin{itemize}
\item Note 1.12 The readers may ask:
\end{itemize}

Why do we need to assert the \((X_5)\)?

This answer was already presented in Note 1.11. That is, measurement theory is the language for engineering (or, sciences), or in other words, for quantification, mechanization, and automation.

There may be several opinion for the following disciplines:

Computational psychology, mathematical economics, financial engineering, cognitive science, management engineering, a medical engineering, educational technology, ergonomics, etc.
However, in order to regard these as sciences, there is no method besides describing these in terms of measurement theory. Also, some may ask:

(A1) Why does measurement theory hold? Is there another scientific language?

However, we have no answer to this question. Let us add "two purposes" to (X1) in what follows.

\[
(X_1) \quad \text{widespread ordinary language} \quad \Rightarrow \quad \text{world-description (Chap. 1)}
\]

Lastly, we add the classification of measurement theory (as mentioned in the guide of this print).

\[
(Y) \quad \begin{aligned}
\text{measurement theory} && \\
\text{scientific language} && \\
\end{aligned}
\]

\[
\begin{aligned}
\text{classical measurement} && \\
\text{quantum measurement} : \text{Chap. 3} & \\
\text{continuous} && \\
\text{pure type} : (\text{Chaps. 2-9}) & \\
\text{mixed type} : \text{Sec.4.4} & \\
\text{bounded} && \\
\text{pure type} : \text{Chaps. 10, 11} & \\
\text{mixed type} : \text{Note 11.3} & \\
\end{aligned}
\]

2 Axiom\textsuperscript{p}\textsubscript{c} 1 — measurement

In Part II, we study "classical (continuous pure type) measurement theory" in the whole picture (Y) of measurement theory mentioned in Section1.3. This is characterized as follows:

\[
\text{measurement theory} = \text{[Axiom}\textsuperscript{p}\textsubscript{c} 1\text{measurement]} + \text{[Axiom}\textsuperscript{p}\textsubscript{c} 2\text{causality]}
\]

measurement theory proclaims that

(2) After the example of the sentences of Axiom\textsuperscript{p}\textsubscript{c} 1 and Axiom\textsuperscript{p}\textsubscript{c} 2, every phenomenon should be described. Or, making a model of the sentences of Axioms 1 and 2, describe every phenomenon.

In this chapter, we are devote ourselves to the mathematical formulation of Axiom\textsuperscript{p}\textsubscript{c} 1 (measurement). Axiom\textsuperscript{p}\textsubscript{c} 2 (causality) will be discussed in Chapter 6 later.

2.1 Classical (continuous pure type) measurement theory

2.1.1 State and Observable — the first quality and the secondary quality

If we expect that many scientists are interested in the philosophy of science (i.e., the studies about the metaphysical aspects of science), we can not do it without mathematics. Therefore, in this section 2.1.1, we prepare some elementary mathematical results.
Throughout this print, put \( \mathbb{N} = \{1, 2, 3, \ldots\} \) (i.e., the set of all natural numbers), \( \mathbb{N}_0 = \{0, 1, 2, \ldots\} \) (i.e., the set of all non-negative integers), \( \mathbb{Z} = \{0, \pm1, \pm2, \ldots\} \) (i.e., the set of all integers), \( \mathbb{R} \) (i.e., the set of all real numbers). Let \( X \) be a set. And let \( \mathcal{P}(X) \) or \( 2^X \) be a set of all subsets of \( X \). That is,
\[
\mathcal{P}(X) = 2^X = \{ \Xi \mid \Xi \subseteq X \}
\]

Let \( \Omega \) be a locally compact space, for example, \( \Omega = \mathbb{N} \) (or, the set of all natural numbers), the real line \( \mathbb{R} \), the interval in \( \mathbb{R} \), the plane (=2-dimensional space) \( \mathbb{R}^2 \) and so on. It is usually assumed that a finite set \( \Omega \) has the discrete metric \( d_D \), where
\[
d_D(\omega, \omega') = \begin{cases} 
1 & (\omega \neq \omega') \\
0 & (\omega = \omega')
\end{cases}
\]

This \((\Omega, d_D)\) is called a discrete metric space.

Let \( \mathcal{C}(\Omega) \) be the set of all complex valued bounded continuous function \( f : \Omega \to \mathbb{C} \), that is, \( \mathcal{C}(\Omega) = \{ f \mid f \text{ is a complex valued continuous function on } \Omega \text{ such that } \| f \|_{\mathcal{C}(\Omega)} < \infty \} \) where the norm \( \| f \|_{\mathcal{C}(\Omega)} \) is defined by
\[
\| f \|_{\mathcal{C}(\Omega)} = \sup_{\omega \in \Omega} | f(\omega) |
\]
It is elementary that the vector space \( \mathcal{C}(\Omega) \) is a Banach space.

Our present purpose is to read several examples in Section 2.3. Thus, we have to add some mathematical preparations.

**Definition 2.1 [Observable, state space, state, measured value, measured value space]** A triplet \( \mathcal{O} = (X, \mathcal{F}, \mathcal{F}) \) is called an observable (or, measuring instrument) in \( \mathcal{C}(\Omega) \) if it satisfies that

(i) [Measurable space]. \( X \) is a set (called a “measured value set”, “sample space”, or “label set”), and \( \mathcal{F} \subseteq \mathcal{P}(X) \) (\( \equiv \{ \Xi : \Xi \subseteq X \} \)) is the field. That is,

(a) \( \emptyset \text{ (empty set)} \in \mathcal{F}, \ X \in \mathcal{F}, \ (b) : \Xi_i \in \mathcal{F} \ (i = 1, 2, \ldots, n) \implies \bigcup_{i=1}^n \Xi_i \in \mathcal{F} \)

(c) \( \Xi \in \mathcal{F} \implies X \setminus \Xi \in \mathcal{F} \)

where \( X \setminus \Xi = \{ x \mid x \in X, x \notin \Xi \} \), i.e., the compliment of \( \Xi \). Also, the pair \( (X, \mathcal{F}) \) is called a (finitely) measurable space.

(ii) [Positivity]. for every \( \Xi \in \mathcal{F} \), \( F(\Xi) \) is an element in \( \mathcal{C}(\Omega) \) such that \( 0 \leq F(\Xi) \leq 1 \) (that is, \( 0 \leq [F(\Xi)](\omega) \leq 1 \ (\forall \omega \in \Omega) \), \( F(\emptyset) = 0 \) and \( F(X) = 1 \) (where 0 is the 0-element in \( \mathcal{C}(\Omega) \)),

(iii) [Complete additivity]. for any countable decomposition \( \{ \Xi_1, \Xi_2, \ldots, \Xi_n, \ldots \} \) of \( \Xi \) (i.e., \( \Xi, \Xi_n \in \mathcal{F} \), \( \bigcup_{n=1}^\infty \Xi_n = \Xi, \ \Xi_n \cap \Xi_m = \emptyset \text{ (if } n \neq m) \)), it holds that

\[
[F(\Xi)](\omega) = \lim_{N \to \infty} \sum_{n=1}^N [F(\Xi_n)](\omega) \quad (\forall \omega \in \Omega).
\]
The \( C(\Omega) \) is called a basic algebra. Also, the \( \Omega \) and its element \( \omega(\in \Omega) \) is respectively called a state space (or, spectrum) and a state. And, the \( X \) and its element \( x(\in X) \) is respectively called a measured value space and a measured value. Let \( \omega \in \Omega \). The triplet \( (X, \mathcal{F}, [F(\cdot])(\omega)) \) is called a sample probability space.

\[ \text{Note 2.1} \] Some may think that the function space \( C(\Omega) \) is too simple. However, the \( C(\Omega) \) or, the \( L^\infty(\Omega, \nu) \) (Chap. 10,11) is an inevitable conclusion from dynamics ( cf. [11, 29]). It is prohibited to consider the other function spaces. If some intend to improve and extend measurement theory, they must start from the improvement of quantum mechanics. In this sense, the improvement of measurement theory may be impossible (\( (F'_3) \) in Chap. 1).

### 2.1.2 Examples of Observables

In what follows, we shall mention several examples of observables.

**Example 2.2 [Existence observable]**

Let \( O \equiv (X, \mathcal{F}, F) \) be any observable in a \( C(\Omega) \). Define the observable \( O^{(\text{exi})} \equiv (X, \{\emptyset, X\}, F^{(\text{exi})}) \) in a basic algebra \( C(\Omega) \) such that:

\[
F^{(\text{exi})}(\emptyset) \equiv 0, \quad F^{(\text{exi})}(X) \equiv I_{C(\Omega)},
\]

which may be called the existence observable (or, null observable). Consider any observable \( O = (X, \mathcal{F}, F) \) in \( C(\Omega) \). Note that \( \{\emptyset, X\} \subseteq \mathcal{F} \). And we see that

\[
[F(\emptyset)](\omega) = 0, \quad [F(X)](\omega) = 1 \quad (\forall \omega \in \Omega)
\]

Thus, we see that \( (X, \{\emptyset, X\}, F^{(\text{exi})}) = (X, \{\emptyset, X\}, F) \), and therefore, we say that any observable \( O = (X, \mathcal{F}, F) \) includes the existence observable \( O^{(\text{exi})} \).

**Example 2.3 [The resolution of the identity \( I \)]**

Also, we may find the similarity between an observable \( O \) and the resolution of the identity \( I \) in what follows. Consider an observable \( O \equiv (X, \mathcal{F}, F) \) in \( C(\Omega) \) such that \( X \) is a countable set (i.e., \( X \equiv \{x_1, x_2, \ldots\} \)) and \( \mathcal{F} = \mathcal{P}(X) \). Then, it is clear that

(i) \( F(\{x_k\}) \geq 0 \) for all \( k = 1, 2, \ldots \)

(ii) \( \sum_{k=1}^{\infty} F(\{x_k\}) = I_{C(\Omega)} \) in the sense of weak topology of \( C(\Omega) \),

which imply that the \( \{F(\{x_k\}) : k = 1, 2, ..., n\} \) can be regarded as the resolution of the identity element \( I_{C(\Omega)} \). Thus we say that

An observable \( O \ (\equiv (X, \mathcal{F}, F)) \) in \( C(\Omega) \) can be regarded as

\[ \text{“the resolution of the identity } I_{C(\Omega)} \]

i.e., \( [F(\{x_k\}) : k = 1, 2, ..., n] \).
In Fig. 2.1, assume that $\Omega = [0, 100]$ is the axis of temperatures (°C), and put $X = \{\text{C("cold")}, \text{L("lukewarm") = "not hot enough"}, \text{H("hot")}\}$. And further, put $f_{x_1} = f_C, f_{x_2} = f_L, f_{x_3} = f_H$. Then, the resolution $\{f_{x_1}, f_{x_2}, f_{x_3}\}$ can be regarded as the word’s partition $\text{C("cold")}, \text{L("lukewarm") = "not hot enough"}, \text{H("hot")}$.

Also, putting $F(=2^X) = \{\emptyset, \{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_2\}, \{x_2, x_3\}, \{x_1, x_3\}, X\}$

and

$$
\begin{align*}
[F(\emptyset)](\omega) &= 0, \\
[F(X)](\omega) &= f_{x_1}(\omega) + f_{x_2}(\omega) + f_{x_3}(\omega) = 1 \\
[F(\{x_1\})](\omega) &= f_{x_1}(\omega), \\
[F(\{x_2\})](\omega) &= f_{x_2}(\omega), \\
[F(\{x_3\})](\omega) &= f_{x_3}(\omega) \\
[F(\{x_1, x_2\})](\omega) &= f_{x_1}(\omega) + f_{x_2}(\omega), \\
[F(\{x_2, x_3\})](\omega) &= f_{x_2}(\omega) + f_{x_3}(\omega) \\
[F(\{x_1, x_3\})](\omega) &= f_{x_1}(\omega) + f_{x_3}(\omega)
\end{align*}
$$

then, we have the observable $(X, F(=2^X), F)$ in $C([0, 100])$.

**Example 2.4 [Triangle observable]** Let testees drink water with various temperature $\omega(0 \leq \omega \leq 100)$. And you ask them “How many degrees (°C) is roughly this water? Gather the data, (for example, $h_n(\omega)$ persons say $n$ ($n \equiv 0, 10, 20, \ldots, 90, 100$). and normalize them, that is, get the polygonal lines. For example, define the state space $\Omega$ by the closed interval $[0, 100]$ ($\subseteq \mathbb{R}$). For each $n \in \mathbb{N}_{10}^{100} = \{0, 10, 20, \ldots, 100\}$, define the (triangle) continuous function $g_n : \Omega \rightarrow [0, 1]$ by

$$
g_n(\omega) = \begin{cases} 
0 & (0 \leq \omega \leq n-10) \\
\omega - n + 10 & (n - 10 \leq \omega \leq n) \\
\frac{-\omega}{10} + n + 10 & (n \leq \omega \leq n+10) \\
0 & (n+10 \leq \omega \leq 100)
\end{cases}
$$
Figure 2.2: Triangle observable

Putting $Y = \mathbb{N}_{10}^{100}$ and define the triangle observable $O^\Delta = (Y, 2^Y, F^\Delta)$ such that

$$[F^\Delta(\emptyset)](\omega) = 0, \quad [F^\Delta(Y)](\omega) = 1$$

$$[F^\Delta(\Gamma)](\omega) = \sum_{n \in \Gamma} g_n(\omega) \quad (\forall \Gamma \in 2^{\mathbb{N}_{10}^{100}})$$

Then, we have the triangle observable $O^\Delta = (Y = \mathbb{N}_{10}^{100}, 2^Y, F^\Delta)$ in $C([0, 100])$.

**Example 2.5 [Exact observable]** Consider a commutative basic algebra $C(\Omega)$. Let $B_\Omega$ be the Borel field, i.e., the smallest $\sigma$-field that contains all open sets. For each $\Xi \in B_\Omega$, define the characteristic function $\chi_\Xi : \Omega \to \mathbb{R}$ such that

$$\chi_\Xi(\omega) = \begin{cases} 1 & \omega \in \Xi \\ 0 & \omega \notin \Xi \end{cases}$$

Put $[F^{(\text{exa})}(\Xi)](\omega) = \chi_\Xi(\omega) (\Xi \in B_\Omega, \omega \in \Omega)$. The triplet $O^{(\text{exa})} = (\Omega, B_\Omega, F^{(\text{exa})})$ may be called an exact observable, if $\chi_\Xi : \Omega \to \mathbb{R}$ is continuous for all $\Xi \in B_\Omega$. For example, when $\Omega = \mathbb{N}, \mathbb{Z}$, the $O^{(\text{exa})}$ is an observable. However, when when $\Omega = \mathbb{R}$, it is not so. Of course, we want to consider the $O^{(\text{exa})}$ is an observable. However, it is not always true, since the exact observable $O^{(\text{exa})}$ does not always exist (i.e., generally, $\chi_\Xi \notin C(\Omega)$) in the basic algebraic formulation. This is a weak point of the basic algebraic formulation. For this, we must prepare the bounded type formulation as mentioned in Chap. 10 and 11. However, it is convenient to consider the exact observable $O^{(\text{exa})}$. Thus, (G₃) in spite of the wrong usage, we sometimes use (G₁) or (G₂).

Of course, when $\Omega$ is finite, or $\Omega = \mathbb{N}$, any function $f : \Omega \to \mathbb{R}$ is continuous, and therefore, we see that $O^{(\text{exa})}$ is an observable in $C(\Omega)$.

**♠ Note 2.2** In usual case such as $\Omega = \mathbb{R}$, the $O^{(\text{exa})}$ can not regarded as the existence observable in $C(\mathbb{R})$. This fact is a weak point in measurement theory (continuous type). This will be improved in measurement theory (bounded type; Chap. 10, 11). However, in spite of the weak point, measurement theory (continuous type) is, of course, fundamental.

**Example 2.6 [Round observable is not observable]** Define the state space $\Omega$ by $\Omega = [0, 100]$. For each $n \in \mathbb{N}_{10}^{100} = \{0, 10, 20, \ldots, 100\}$, define the discontinuous function $g_n : \Omega \to [0, 1]$ such that

$$g_n(\omega) = \begin{cases} 0 & (0 \leq \omega \leq n - 5) \\ 1 & (n - 5 < \omega \leq n + 5) \\ 0 & (n + 5 < \omega \leq 100) \end{cases}$$
Define \( \mathcal{O}_{\text{RND}} = (Y = \{0, 1\}, 2^Y, G_{\text{RND}}) \) such that
\[
G_{\text{RND}}(\emptyset) = 0, \quad G_{\text{RND}}(Y) = 1, \quad G_{\text{RND}}(\Gamma) = \sum_{n \in \Gamma} g_n(\omega) \quad (\forall \Gamma \in 2^Y = 2^{\{0, 1\}})
\]
Recall that \( g_n \) is not continuous. Therefore, the triplet \( \mathcal{O}_{\text{RND}} = (Y, 2^Y, G_{\text{RND}}) \) is not an observable in \( C([0, 100]) \). Of course, we want to regard it as an observable. For this, we must prepare "bounded type measurement theory" (cf. Chaps. 10, 11).

2.2 Axiom \( p_c1 \) — There is no science without measurement

As mentioned in Chap. 1, measurement theory is formulated as follows. That is,

\[
\text{measurement theory} \quad := \quad \begin{cases} 
\text{measurement} \\
\text{probabilistic interpretation}
\end{cases} + \begin{cases} 
\text{causality} \\
\text{the Heisenberg picture}
\end{cases}
\]

In what follows, we shall explain Axiom \( p_c1 \). (For Axiom \( p_c2 \) (causality), see Chap. 6).

With any classical system \( S \), a basic algebra \( C(\Omega) \) can be associated in which measurement theory of that system can be formulated. A state of the system \( S \) is represented by a state \( \omega \in \Omega \), i.e., a state space. Also, an observable is represented by \( \mathcal{O} \equiv (X, F, F) \) in the \( C(\Omega) \).

(a) An observer takes a measurement

\[
\text{of observable } [\mathcal{O}] \quad \text{by measuring instrument} [\mathcal{O}] \quad \text{for a measuring object}
\]

with a state.

and thus, in short, we write:

(a) An observer take a measurement \( \mathcal{M}_{C(\Omega)}(\mathcal{O}, S_\omega) \) ( or, \( \mathcal{M}_{C(\Omega)}(\mathcal{O}, S_{[\delta_\omega]})) \).

where \( \delta_\omega \) is a point measure at \( \omega \). Thus, the value \( [F(\Xi)](\omega) \) is also written by \( \mathcal{M}_{C(\Omega)}(\delta_\omega, F(\Xi)) \).

And further, by measurement \( \mathcal{M}_{C(\Omega)}(\mathcal{O}, S_\omega) \), measured value \( x \in X \) is obtained.

Axiom \( p_c1 \) is the linguistic turn of Born’s quantum measurement as follows (cf. Sec.3.2).

Since Axiom \( p_c1 \) below is similar to [Axiom 1 Prototype] (12 page), all readers can easily understand it.
Consider a measurement $M_{C(\Omega)}(O := (X, \mathcal{F}, F), S_\omega)$ formulated in a basic algebra $C(\Omega)$. Assume that the measured value $x (\in X)$ is obtained by the measurement $M_{C(\Omega)}(O, S_\omega)$. Then, the probability that the $x (\in X)$ belongs to a set $\Xi (\in \mathcal{F})$ is given by $[F(\Xi)](\omega) \equiv \langle \delta_\omega, F(\Xi) \rangle_{C(\Omega)}$.

If it writes without carrying out simple, we see:

(b) When an observer takes a measurement of an observable $O=(X, \mathcal{F}, F)$ (or, by a measuring instrument $O$) for a measuring object with a state $\omega$, the probability that a measured value belongs to $\Xi(\in \mathcal{F})$ is given by $[F(\Xi)](\omega)$.

Measurement theory says that Describe every phenomenon modeled on Axiom\(^p\) 1. That is, the key-words in Axiom\(^p\) 1 is as follows.

(c) measurement, observer, system(measuring object), state, observable , measured value , probability and

Use these key-words modeled on Axiom\(^p\) 1.

The meanings of key-words in (c) are not explained. Therefore, the experimental verification is meaningless. That is,

(d) Axiom\(^p\) 1(measurement) is a metaphysical statement, in the sense that it can not be verified by experiment. This is the remarkable feature of linguistic method.

This should be compared to physics (i.e., realistic method). Many readers may no be familiar to linguistic method. However, several examinations in the following section will promote reader’s understanding.

Note 2.3 If metaphysics has history of failure, this is due to the serious trial to answer the following problem

(2) What is the meaning of the key-words in (c)?

Although this (2) may be attractive, however, it is not productive. What is important is to know how to use the key-words in (c). Of course, this answer is mentioned in Axiom\(^p\) 1.

The following example (under Axiom\(^p\) 1 ) is the same as Example 1.1.

Example 2.7 [Continued from Example 1.1] [The measurement of ”cold or hot” for water in a cup]

In Example 1.1, we consider this [C-H]-thermometer $O = (f_c, f_h)$, where the state space $\Omega = [0, 100]$, the measured value space $X = \{c,h\}$. That is,
Then, we have the (temperature) observable \( O_{ch} = (X, 2^X, F_{ch}) \) in \( C(\Omega) \) such that

\[
\begin{align*}
[F_{ch}(\emptyset)](\omega) &= 0, \\
[F_{ch}(\{c\})](\omega) &= f_c(\omega), \\
[F_{ch}(\{h\})](\omega) &= f_h(\omega)
\end{align*}
\]

Thus, we get a measurement \( M_{C(\Omega)}(O_{ch}, S_{[\varnothing,\omega]}(=55)) \) belongs to set \(
\begin{bmatrix}
\emptyset \\
\{c\} \\
\{h\} \\
\{c,h\}
\end{bmatrix}
\) is given by \(
\begin{bmatrix}
[F_{ch}(\emptyset)](55) = 0 \\
[F_{ch}(\{c\})](55) = 0.25 \\
[F_{ch}(\{h\})](55) = 0.75 \\
[F_{ch}(\{c,h\})](55) = 1
\end{bmatrix}
\)

If it writes without omitting, we see:

(f) When an observer takes a measurement by \([\text{C-H}-\text{instrument}]\) measuring instrument \( O_{ch} = (X, 2^X, F_{ch}) \) for \([\text{water in cup}]\) with \([55 \degree C]\) the probability that measured value \([c, h]\) is obtained is given by \[
\begin{bmatrix}
f_c(55) = 0.25 \\
f_h(55) = 0.75
\end{bmatrix}
\]

Here, note that the key-words in (c) — observer, state, system (=measuring object), observable (=measuring instrument), measurement, measured value, probability — are contained in the above (f).

### 2.3 Simple examples of measurements

#### 2.3.1 linguistic world-view — Wonder of man’s linguistic competence

The applied scope of physics (realistic world-description method) is rather clear. But the applied scope of measurement theory (as well as the proverb "Even monkeys fall from trees") is ambiguous.

As mentioned in Note 2.3, what we can do in measurement theory is

\[\begin{align*}
\text{(a)}: \text{Use the language defined by } \text{Axiom}_p \text{ 1} \\
\text{(a)}: \text{Trust in man’s linguistic competence (Chap. 1(M1))}
\end{align*}\]

Thus, some readers may doubt that

(b) Is it science?

However, the spirit of measurement theory is different from that of physics.
2.3.2 Elementary examples — urn problem, etc.

Since measurement theory is a language, we cannot master it without examinations. Thus, we present simple examples in what follows.

**Example 2.8 [The number of balls in urn]** In a certain urn \( U \), some balls are contained.

(a) Counting the number of the balls contains \( n \) balls, we get "n".

Now we shall represent the obvious statement (a) in terms of measurement theory.

Define the state \( \omega_n \) of the urn \( U \) such that

\[ \omega_n \cdots n \text{ balls are contained in the urn } U. \quad (n = 0, 1, 2, \ldots) \]

Therefore, the state space \( \Omega \) is defined by

\[ \Omega = \{ \omega_0, \omega_1, \omega_2, \ldots \} \]

with the discrete metric.

![Figure 2.4: The number of balls in the urn](image)

Define the measured value space by \( N_0 = \{0, 1, 2, \ldots\} \). And define observable \( O = (N_0, 2^{N_0}, F) \) in \( C(\Omega) \) such that

\[ [F(\Xi)](\omega_n) = \begin{cases} 1 & (n \in \Xi) \\ 0 & (n \notin \Xi) \end{cases} \quad (\forall n \in N_0, \forall \Xi \subseteq N_0) \]

This observable \( O = (N_0, 2^{N_0}, F) \) is called a **counting observable**. Therefore, the above statement (a) in ordinary language can be translated to the following (b) in terms of Axiom\( P \) 1,

(b) The probability the a measured value obtained by measurement \( M_{C(\Omega)}(O, S_{[\omega_n]}) \) belong to \( \Xi(\subseteq N_0) \) is given by

\[ [F(\Xi)](\omega_n) = \begin{bmatrix} 1 & (n \in \Xi) \\ 0 & (n \notin \Xi) \end{bmatrix} \]

That is,

(c) a measured value obtained by measurement \( M_{C(\Omega)}(O, S_{[\omega_n]}) \) is surely \( n \)
Example 2.9 [Continued from Example 2.4 (triangle observable)] Let testees drink water with various temperature $\omega^\circ C$ $(0 \leq \omega \leq 100)$. And you ask them "How many degrees ($^\circ C$) is roughly this water?". Gather the data, (for example, $h_n(\omega)$ persons say $n^\circ C$ ($n = 0, 10, 20, \ldots, 90, 100$). and normalize them, that is, get the polygonal lines. For example, define the state space $\Omega$ by the closed interval $[0, 100]$ ($\subseteq \mathbb{R}$). For each $n \in \mathbb{N}_{100} = \{0, 10, 20, \ldots, 100\}$, define the (triangle) continuous function $g_n : \Omega \to [0, 1]$ by

$$g_n(\omega) = \begin{cases} 
0 & (0 \leq \omega \leq n - 10) \\
\omega - n - 10 & (n - 10 \leq \omega \leq n) \\
-\omega + n + 10 & (n \leq \omega \leq n + 10) \\
0 & (n + 10 \leq \omega \leq 100)
\end{cases}$$

(a) You choose one person from the testees, and you ask him/her "How many degrees ($^\circ C$) is roughly this water?". Then the probability that he/she says "about $40^\circ C""$$ about 50^\circ C" is given by $\begin{bmatrix} g_{40}(47) = 0.25 \\
 f_{50}(47) = 0.75 \end{bmatrix}$

This is described in terms of Axiom $p_1$ in what follows.

Putting $Y = \mathbb{N}_{100}$ and define the triangle observable $O^\Delta = (Y, 2^Y, F^\Delta)$ such that

$$[F^\Delta(\emptyset)](\omega) = 0, \quad [F^\Delta(Y)](\omega) = 1$$

$$[F^\Delta(\Gamma)](\omega) = \sum_{n \in \Gamma} g_n(\omega) \quad (\forall \Gamma \in 2^{\mathbb{N}_{100}})$$

Then, we have the triangle observable $O^\Delta = (Y(= \mathbb{N}_{100}), 2^Y, F^\Delta)$ in $C([0, 100])$. And we get a measurement $M_{C(\Omega)}(O^\Delta, S_{\{\omega\}})$. For example, putting $\omega = 47^\circ C$, we see, by Axiom $p_1$ (page 23), that

(b) the probability that a measured value obtained by the measurement $M_{C(\Omega)}(O^\Delta, S_{\{\omega\}})$ is $\begin{bmatrix} "about 40^\circ C" \\
 "about 50^\circ C" \end{bmatrix}$ is given by $\begin{bmatrix} [F^\Delta(\{40\})](47) = 0.3 \\
 [F^\Delta(\{50\})](47) = 0.7 \end{bmatrix}$

Example 2.10 [The urn problem]. There are two urns $U_1$ and $U_2$. The urn $U_1$ [resp. $U_2$] contains 8 white and 2 black balls [resp. 4 white and 6 black balls] (cf. Fig. 2.5).

| Table 2.1: urn problem |
|------------------------|
| Urn \ w-b | white ball | black ball |
| Urn $U_1$ | 8 | 2 |
| Urn $U_2$ | 4 | 6 |
Here, consider the following “statement (a)”: 

(a) When one ball is picked up from the urn $U_2$, the probability that the ball is white is 0.4.

In measurement theory, the “measurement (a)” is formulated as follows: Assuming

$$U_1 \cdots \text{ “the urn with the state } \omega_1\text{”}$$

$$U_2 \cdots \text{ “the urn with the state } \omega_2\text{”}$$

define the state space $\Omega$ by $\Omega = \{\omega_1, \omega_2\}$. That is, we assume the identification;

$$U_1 \approx \omega_1, \quad U_2 \approx \omega_2,$$

Put “$w$” = “white”, “$b$” = “black”, and put $X = \{w, b\}$. And define the observable $O( \equiv (X \equiv \{w, b\}, 2^{\{w, b\}}, F))$ in $C(\Omega)$ by

$$\begin{align*}
\ [F(\{w\})](\omega_1) &= 0.8, \quad [F(\{b\})](\omega_1) = 0.2, \\
\ [F(\{w\})](\omega_2) &= 0.4, \quad [F(\{b\})](\omega_2) = 0.6.
\end{align*}$$

Thus, we get the measurement $M_{C(\Omega)}(O, S_{\delta_{\omega_1}})$. Here, Axiom$^P_1$ (page 23) says that

(b) the probability that a measured value $b$ is obtained by $M_{C(\Omega)}(O, S_{\delta_{\omega_1}})$ is given by

$$F(\{b\})(\omega_1) = 0.8$$

\begin{itemize}
\item **Note 2.4** Readers may feel that Example 2.7–Example 2.10 are too easy. However, note that

(2) Since how to write in addition to this was not learned, it wrote like this reluctantly.

As mentioned in (a) of Sec.2.3.1, what we can do is

\begin{itemize}
\item to be faithful to Axiom$^P_0$, and to trust in Man’s linguistic competence
\end{itemize}

If some find the other writing, it will be praised as the greatest discovery on history of science. That is because it means the discovery beyond quantum mechanics.

\begin{itemize}
\item **Note 2.5** The statement (a) in Example 2.10 is not necessarily guaranteed:
\end{itemize}
When one ball is picked up from the urn $U_2$, the probability that the ball is white is 0.4.

What we say is that the statement (a) in ordinary language should be written by the measurement theoretical statement (b).

It is a matter of course that “probability” cannot be derived from mathematics itself. For example, the following ($\sharp_1$) and ($\sharp_2$) are not guaranteed.

($\sharp_1$) From the set $\{1, 2, 3, 4, 5\}$, choose one number. Then, the probability that the number is even is given by $2/5$.

($\sharp_2$) From the closed interval $[0, 1]$, choose one number $x$. Then, the probability that $x \in [a, b] \subseteq [0, 1]$ is given by $|b - a|$.

The common sense — “probability” cannot be derived from mathematics, which is independent of our world — is well known as Bertrand’s paradox (cf. [11]). Thus, it is usual to add the term ”at random” to the above ($\sharp_1$) and ($\sharp_2$). In this print, this term ”at random” is frequently omitted.

### 2.3.3 About the space in our world — Leibniz’s relationalism

**Example 2.11** [Approximate measurement of the position of a particle] Let $\Omega$ be an interval of the one-dimensional space $\mathbb{R}$. Consider a particle $P$ with the position $\omega_0 (\in \Omega = \mathbb{R})$. Consider the situation described in Fig. 2.6.

Consider the following measurement (a):

(a) measure a particle’s position roughly.

This (a) will be characterized as follows. Let $\sigma$ be a fixed positive real. Define the normal observable (or Gaussian observable) $O_G^\sigma \equiv (\mathbb{R}, B_{\mathbb{R}}, G^\sigma)$ in $C(\Omega)$ (where $\Omega = \mathbb{R}$) such that:

$$
[G^\sigma(\Xi)](\omega) = \frac{1}{\sqrt{2\pi}\sigma^2} \int_{\Xi} e^{-\frac{(x-\omega)^2}{2\sigma^2}} \, dx \quad (\forall \Xi \in B_{\mathbb{R}}, \forall \omega \in \Omega \equiv [a, b]),
$$

which will be often used in this book. See Fig. 2.6.

![Figure 2.6: Error function](image)
Here, note that \( \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} dx = 1 \) and
\[
\int_{-\sigma}^{\sigma} e^{-\frac{x^2}{2\sigma^2}} dx = 0.683..., \quad \int_{-2\sigma}^{2\sigma} e^{-\frac{x^2}{2\sigma^2}} dx = 0.954...
\]
\[
\int_{-1.96\sigma}^{1.96\sigma} e^{-\frac{x^2}{2\sigma^2}} dx \approx 0.95, \quad \int_{-\infty}^{-1.65\sigma} e^{-\frac{x^2}{2\sigma^2}} dx \approx 0.95 \quad (2.5)
\]
Thus, the (a) is characterized as follows.

(b) the probability that a measured value obtained by the measurement \( M_{C(R)}(O_{G^\sigma}, S_{(\delta_{\omega})}) \) belongs to \( \Xi (\in B_R) \) is given by \( \left[G^\sigma(\Xi)\right](\omega_0) \).

Leibniz’ opinion (metaphysical space-time)

The problem ”What is space-time?” is fundamental in any world-description method. For this problem, measurement theory answers as follows (for time, see Sec.6.4.2):

(a) The position of a matter is represented by a state (that is, a state space \( \mathbb{R}^3 \)). Therefore, the position (Example 2.11), the temperature (Example 2.7, Example 2.9), the number of balls (Example 2.8) and so on are all kinds of states. That is,

This world where we live is regarded as a kind of state space, and thus, it is represented by \( \mathbb{R}^3 \).

(For the general cases (including quantum theory), see [14, 16]) As mentioned above, space-time is not regarded as something special in measurement theory. This idea is considerably different from common sense (i.e., the common sense of the theory of relativity).

The above argument urges us to recall Leibniz-Clarke correspondence as follows.

[Leibniz-Clarke Correspondence]: Leibniz-Clarke Correspondence (1715–1716) is important to know both Leibniz’s and Clarke’s (=Newton’s) ideas concerning space and time.

(b) Newton’s absolutism says that the space-time should be regarded as a receptacle of a ”thing.” Therefore, even if ”thing” does not exits, the space-time exists. On the other hand, Leibniz’s relationalism says that

(b1) Space is a kind of state of ”thing”.

(b2) Time is an order of occurring in succession which changes one after another.

\(^\triangledown\) Note 2.6 Many scientists may think that

Newton’s assertion is understandable, in fact, his idea was inherited by Einstein. On the other, Leibniz’s assertion is incomprehensible and literary. Thus, his idea is not related to science.
However, recall the classification of the world-description:

\[
\text{world-description (Chap. 1(O))} \begin{cases} 
1^\circ : \text{realistic method (i.e., world is before language)} \\
2^\circ : \text{linguistic method (i.e., language is before world)} 
\end{cases}
\]

in which Newton and Leibniz respectively devotes himself to $1^\circ$ and $2^\circ$. Although Leibniz’s assertion is not clear, we believe that Leibniz found the importance of "linguistic space and time" in science, though he did not propose his language.

As mentioned in (a), measurement theory adopts Leibniz’s relationalism. For time, see Sec. 6.4.

\textbf{♠ Note 2.7} Although "What is space and time?" is long time unsolved problem, Newton’s and Leibniz’s ideas are characterized in the following classification of the world-description:

\[
\begin{align*}
(\theta) & : \text{Newotn, Clarke} \\
(\text{realistic world view}) & \quad \text{(successors: Einstein, etc.)} \\
\text{(space-time in physics)} & \quad \text{"What is space-time?"} \\
(\text{O}) & : \text{Leibniz} \\
(\text{linguistic world view}) & \quad \text{(i.e., spectrum, tree)} \\
\text{(space-time in measurement theory)} & \quad \text{"How should space-time be represented?"}
\end{align*}
\]

Measurement theory is in Leibniz’s side. Thus, we consider that:

(\sharp_2) Space should be described as a kind of spectrum (=state space in classical situation)

That is, we think that the Leibniz-Clarke debates should be essentially regarded as "linguistic world view $2^\circ$" v.s. "realistic world view $1^\circ$".

Recall the proverb "Even monkeys fall from trees" in Sec.1.1 such that:

\[
\begin{align*}
(\sharp_1) & : \text{quantum mechanics} \\
(\text{realistic method}) & \quad \text{linguistic turn} \\
\text{proverbalizing} & \quad \text{measurement theory} \\
\text{linguistic [space-time-probability]} & \quad \text{(linguistic method)}
\end{align*}
\]

Although quantum mechanics will be explained in Chap. 3, we can now conclude that

(\sharp_2) It suffices to use the terms (space, time, probability) in measurement theory by an analogy of quantum mechanics. Moreover, the usage is possible even if we do not quantum mechanics. That is because these terms may be used according to our common sense.

In linguistic method, the question "What is space (time, probability)?" is not important (Note 2.3). What is important is

(\sharp_3) How should the term space (time, probability) be used ?

This is respectively answered in (a) in this section, Sec. 6.4.2, Axiom$^p_1$.

\textbf{♠ Note 2.8} The space-time in measuring object is well discussed in the above. However, we have to say something about "observer’s time". We conclude that observer’s time is meaningless in measurement theory as mentioned the Copenhagen interpretation$(U_2)$ in Chap. 1. That is, the following question is nonsense in measurement theory:

\[
\begin{align*}
(\sharp_1) & : \text{When and where does an observer take a measurement} \\
(\sharp_2) & : \text{Therefore, the is no tense (present, past, future) in sciences.}
\end{align*}
\]
2.4 The age of engineering and various sciences

2.4.1 Measurement Theory — help the weak

Measurement Theory sided with the "Leibniz’s relation theory" which may be called "endangered species" for the foregoing paragraph, and again with the John Locke’s "the primary quality and the second quality" in Note 1.9. If it has a worldly way of speaking, measurement theory will be an ally of weak things - what cannot have confidence, a thing currently considered to be suspicious, a thing which is not trusted, a thing like endangered species, a thing which is not respected, a cheap thing -. The concrete image is summarized as notes.

Note 2.9 We consider that

| The strong | vs. | The weak |
|------------|-----|---------|
| ① Newton’s space-time | | Leibniz’s space-time |
| ② materialism | | idealism |
| ③ physical science | | metaphysics |
| ④ monism | | dualism |
| ⑤ physics | | engineering (= various science) (cf. Sec. 8.1(m)) |
| ⑥ Einstein | | Fisher (or, von Neumann) |
| ⑦ theory of relativity | | quantum mechanics |
| ⑧ realistic world-view | | linguistic world-view |

Supposing it decides "by majority", please understand the standard which classifies "a strong thing" and "a weak thing" to such an extent that it says that it will become like this. Since it does not have the science view whose another side settled while one side has a "realistic science view", both are divided. Namely,

"The strong side" has a common language called physics. However, "the weak side" does not have a common language.

In this book, I claim that

a sight in which "the weak side" depends on the authority (mathematics, physics, application, culture) of the other place, and makes a living is caught from another’s eyes, it is not a suitable good thing, and needs to establish independent authority - linguistic science view - . Namely,

(♯1) If "the weak side" is bundled considering Measurement Theory as a common language, it can rank the strong side.

Although it is the same,

(♯2) If the engineering theory - Measurement Theory - which bundles a weak side is made, it can rank the strong side.

This is an opinion of this book. (Chap. 12). See Note 9.7 about upper ⑥, and see Note 9.3 about ⑦. Moreover, ⑧ is a conclusion of this book.

In Chapter 9, equilibrium statistical mechanics is taken from the territory of physics, and it is regarded as one field of engineering and various sciences. If it goes by this flow,

(c) Measurement theory is a good fellow who helps "a weak thing".

However, we should say that this view is superficial. In fact, the position of "a strong side" and "a weak side" is substantially reversed bordering on the second half of the 20th century (as a symbolic incident,
Apollo's landing on the moon (1969) occurs. It is what is called "the end of a big tale", and it is as follows, if the "ultimate purpose" is made to contrast and it writes.

The understanding of God’s rule $\rightarrow$ The manufacture of a robot like man
(The age of physics) Apollo (1969) (The age of engineering)

Now, physics is a storyteller of "the myth of space creation." Although it may turn out that there is an irresistible charm in a myth lover and there may also be an opinion that it becomes a serious problem 1000 years after, I think that the meaning of "the seriousness of 1000-years-after" becomes delicate by lower (e).

All the things that are actually in personal appearance and are moving the world now are the products of engineering - engineering works - construction, a car, an airplane, electrical machinery and apparatus, petroleum products, a computer, etc. - , and engineering is also bearing human beings’ fate (or influences the fate of one country). If this is taken into consideration,

(d) By obtaining a formal language called Measurement theory, engineering and science make the scaffold steadfast, and sees adolescence - fast progressive era -.

This is the greatest message of this book. (as mentioned in (I) of Chap. 13). If it thinks simply that what is necessary is just to leave science to a robot after that if "about the same robot as a scientist" is made, the meaning of what "man contributes to development of engineering and science" will become doubtful.

It follows,

(e) The time of engineering — the best-before date of Measurement theory (up to the day when "the time when a robot does science, and the time when a robot makes a robot" come) — will be at most four hundreds of years from now on.

♠ Note 2.10 About a scientific language, when a conclusion is written previously, it is as follows. (Chap. 12).

\[
\begin{align*}
\circled{0} : \text{language in mathematics} & \quad \cdots \text{set theory} \\
\circled{1} : \text{language in physics} & \quad \text{the theory of relativity, ...} \\
\circled{2} : \text{language in engineering} & \quad \text{measurement theory} \\
\circled{3} & \quad \text{(linguistic world-view:create a robot like a man)} \\
\end{align*}
\]

\(\circled{0}\) and \(\circled{1}\) will be common sense. About \(\circled{2}\), it will argue about this book whole volume. Although Gauss (1777 – 1855) did not know set theory, the mathematical achievements were great, and although neither watt (1736 – 1819) nor Edison (1847 - 1931) knew modern control theory, they did large invention. Moreover, there are various ways to make "about the same robot as a scientist." It is an extreme talk;

\(\circled{3}\) "Man" was made, when neglecting water, air, and a gravel without doing anything for about 4 billion years.

Therefore, large invention may be possible if there is even time (even if there is no control theory(\(\subset\)Measurement theory; Chapter 7)). However, in order "to hurry", control theory is indispensable and this thought is spontaneous generation, but I think that it was popularized by Norbert Wiener’s (1894 year–1964 year) "cybernetics." The reason which we must hurry - the reason the easy thing of "if there is even time" is not said. - is written in the following paragraph.
2.4.2 Measurement Theory - For the further development of engineering and various science

The reason into which engineering must be developed immediately is explained below. Dr. Hawking (1942-), a British theoretical physicist, is warning as follows on the website "Big Think":

(a) The resources of the earth are limited while population increases geometrically. Furthermore, by the time progress of technology changed global environment also often and bad, it resulted. We accomplished the development which should attract attention over the past 100 years. But, it is not a thing which one chance merely remains behind on the earth in order to overcome 100 years of future, but is spread in the universe.

Even if Dr. Hawking does not say, everyone surely feel that "a big misfortune" will surely happen before long if lives on this narrow earth. If Dr. Hawking’s warning (a) is said at a word,

(b) [Human beings’ continuation] ⇐⇒ [The construction of Space Colony]

The story of this book is developed by believing this.

♠ Note 2.11 As mentioned above, I would like to consider

(♯) earth escape (construction and settlement of a space colony) hundreds of years after

as the present target of the human beings, us. There may be some readers puzzled to the "space colony." However, this is a device for fixing the purpose of "realization of a space colony" and keeping an argument from being spread. For achievement of this (♯), since human beings have to mobilize fully all the results (intellectual product) of the engineering and various science accumulated in the past, I think that it is perfect as a target. Moreover, if human beings reside permanently in the universe in 10,000 years, and if they look back upon history then, I will think that "the beginning of agriculture" and "earth escape" are commented as two major events of a history of man. If that is right, we may call that the challenge(♯) is "the greatest battle of human beings."

In this book, a "space colony" is often used as a metaphor. This is because I think as follows.

(c) Since this book is writing of engineering, I should declare the clear purpose.

It is because "the standard of importance" will become diffuse and it will deviate from the meaning of (c) in the argument which is not shown the concrete target.

(d) In order to secure the firm foothold of engineering and to promote development of engineering further,

Measurement theory was made as "a language of engineering."

The author’s "spirit" is as follows.

(e) Measurement theory was made in order to defeat the greatest battle of human beings - Construction and settlement of the space colony of hundreds of years of after -.

♠ Note 2.12 If you are a reader of a theoretical lover, you may ask as follows.
(♯) Although measurement theory includes quantum mechanics (Chap. 1(Y)), why don’t you propose the "measurement theory" include the theory of relativity?

However, this book is "writing of engineering" as mentioned above. Although I think that "refraining from doing too much" is an engineering sense, there is no mind which, of course, stops that the reader of a theoretical lover challenges the above-mentioned (♯).

2.5 The Copenhagen interpretation — Only one measurement is permitted

In this section, we examine the Copenhagen interpretation (Chap. 1(U)), i.e., "Only one measurement is permitted". "Only one measurement" implies that "only one observable" and "only one state". That is, we see:

\[
\text{[only one measurement]} \implies \begin{cases} 
\text{only one observable (=measuring instrument)} \\
\text{only one state}
\end{cases}
\] (2.6)

\[\text{♠ Note 2.13} \quad \text{Although there may be several opinions, the author believes that the standard Copenhagen interpretation says that only one measurement is permitted. This spirit is inherited to measurement theory.}\]

2.5.1 "Observable is only one" and simultaneous measurement

For example, consider the following situation:

(a) There is a cup in which water is filled. Assume that the temperature is \(\omega \text{°C} \ (0 \leq \omega \leq 100)\). Consider two questions "Is this water cold or hot?" and "How many degrees (°C) is roughly the water?". This implies that we take two measurements such that

\[
\begin{align*}
\text{(♯1)}: & \ M_C(\Omega) = (\{c, h\}, 2^{\{c, h\}}, F_{ch}), S_{[\omega]} \text{ in Example 2.7} \\
\text{(♯2)}: & \ M_C(\Omega) = (\{N_{10}^{100}, 2^{N_{10}^{100}}, G_{AB}\}), S_{[\omega]} \text{ in Example 2.9}
\end{align*}
\]

However, as mentioned in the above, "only one observable" must be demanded. Thus, we have the following problem.

**Problem 2.12** Represent two measurements \(M_C(\Omega) = (\{c, h\}, 2^{\{c, h\}}, F_{ch}), S_{[\omega]} \) and \(M_C(\Omega) = (\{N_{10}^{100}, 2^{N_{10}^{100}}, G_{AB}\}), S_{[\omega]} \) by only one measurement.

This will be answered in what follows.

**Definition 2.13 [Product measurable space]** For each \(k = 1, 2, \ldots, n\), consider a measurable \((X_k, \mathcal{F}_k)\). The product space \(\prod_{k=1}^n X_k\) of \(X_k\) \((k = 1, 2, \ldots, n)\) is defined by

\[
\prod_{k=1}^n X_k = \{(x_1, x_2, \ldots, x_n) \mid x_k \in X_k \ (k = 1, 2, \ldots, n)\}
\]

Similarly, define the product \(\prod_{k=1}^n \Xi_k\) of \(\Xi_k(\in \mathcal{F}_k)\) \((k = 1, 2, \ldots, n)\) by

\[
\prod_{k=1}^n \Xi_k = \{(x_1, x_2, \ldots, x_n) \mid x_k \in \Xi_k \ (k = 1, 2, \ldots, n)\}
\]

Further, the \(\sigma\)-field \(\bigotimes_{k=1}^n \mathcal{F}_k\) on the product space \(\prod_{k=1}^n X_k\) by
\( \mathcal{F}_k \) is the smallest field including \( \{ X_k \in \mathcal{F}_k \mid \omega_k \in \mathcal{F}_k \} \)

( \( \bigotimes_{k=1}^n X_k \), \( \bigotimes_{k=1}^n \mathcal{F}_k \)) is called the \{product measurable space. Also, in the case that \( (X, \mathcal{F}) = (X_k, \mathcal{F}_k) \) \((k = 1, 2, \ldots, n)\), the product space \( \bigotimes_{k=1}^n X_k \) is denoted by \( X^n \), and the product measurable space \( ( \bigotimes_{k=1}^n X_k, \bigotimes_{k=1}^n \mathcal{F}_k) \) is denoted by \( (X^n, \mathcal{F}^n) \).

**Definition 2.14 [simultaneous observable, simultaneous measurement]** For \( k = 1, 2, \ldots, n \) consider observable \( O_k = (X_k, F_k, F_k) \) in \( C(\Omega) \). Let \( ( \bigotimes_{k=1}^n X_k, \bigotimes_{k=1}^n \mathcal{F}_k) \) be the product measurable space. An observable \( \hat{O} = ( \bigotimes_{k \in K} X_k, \bigotimes_{k=1}^n \mathcal{F}_k, \hat{F} ) \) in \( C(\Omega) \) is called the simultaneous observable of \( \{ O_k : k = 1, 2, \ldots, n \} \), if it satisfies the following condition:

\[
[F(\bigotimes_{k=1}^n X_k)](\omega) = [F_1(\bigotimes_{k=1}^n X_k)](\omega) \cdot [F_2(\bigotimes_{k=1}^n X_k)](\omega) \cdots [F_n(\bigotimes_{k=1}^n X_k)](\omega)
\]

(\( \forall \omega \in \Omega, \ \forall \bigotimes_{k=1}^n X_k \in \mathcal{F}_k (k = 1, 2, \ldots, n) \))

(2.7)

\( \hat{O} \) is also denoted by \( X_{k=1}^n O_k, \hat{F} = X_{k=1}^n F_k \). Also, the measurement \( M_{C(\Omega)}(X_{k=1}^n O_k, S[\omega]) \) is called the simultaneous measurement.

In what follows, let us explain the simultaneous measurement. We want to take two measurements \( M_{C(\Omega)}(O_1, S[\omega]) \) and measurement \( M_{C(\Omega)}(O_2, S[\omega]) \). That is, it suffices to image the following:

(b) \[
\begin{array}{ccc}
\text{state} & \xrightarrow{\text{observable}} & \text{measured value} \\
\omega(\in \Omega) & \xrightarrow{O_1=(X_1,F_1,F_1)} & x_1(\in X_1) \\
\end{array}
\]

(c) \[
\begin{array}{ccc}
\text{state} & \xrightarrow{\text{simultaneous observable}} & \text{measured value} \\
\omega(\in \Omega) & \xrightarrow{O_1 \times O_2} & (x_1,x_2)(\in X_1 \times X_2) \\
\end{array}
\]

However, the Copenhagen interpretation (Chap. 1(U4)) says that two measurements cannot be taken. Therefore, combining two observables \( O_1 \) and \( O_2 \), we construct the simultaneous observable \( O_1 \times O_2 \), and take the simultaneous measurement \( M_{C(\Omega)}(O_1 \times O_2, S[\omega]) \) in what follows.

Example 2.15 [The answer to Problem] Consider the state space \( \Omega \) such that \( \Omega = [0, 100] \), the closed interval. And consider two observables, that is, ch-observable \( O_{ch} = (X=\{c,h\}, 2^X, F_{ch}) \) (in Example 2.7) and about-observable \( O_{AB} = (Y (=\mathbb{N}^{00}_{10}), 2^Y, G_{AB}) \) (in Example 2.9). Thus, we get the simultaneous observable \( O_{ch} \times O_{AB} = (\{c,h\} \times \mathbb{N}^{00}_{10}, 2^{\{c,h\} \times \mathbb{N}^{00}_{10}}, F_{ch} \times G_{AB}) \), take the simultaneous measurement \( M_{C(\Omega)}(O_{ch} \times O_{AB}, S[\omega]) \). For example, putting \( \omega = 55 \), we see

(d) when the simultaneous measurement \( M_{C(\Omega)}(O_{ch} \times O_{AB}, S[55]) \) is taken, the probability that the measured value

\[
\begin{bmatrix}
(c, \text{about 50 °C}) \\
(c, \text{about 60 °C}) \\
(h, \text{about 50 °C}) \\
(h, \text{about 60 °C})
\end{bmatrix}
\]

is obtained is given by

\[
\begin{bmatrix}
0.125 \\
0.125 \\
0.375 \\
0.375
\end{bmatrix}
\]

(2.8)
That is because
\[
[(F_{ch} \times G_{AB})\{(c, \text{about } 50^\circ \text{C})\}](55) = [F_{ch}\{c\}](55) \cdot [G_{AB}\{\text{about } 50^\circ \text{C}\}](55) = 0.25 \cdot 0.5 = 0.125
\]
and similarly,
\[
[(F_{ch} \times G_{AB})\{(c, \text{about } 60^\circ \text{C})\}](55) = 0.25 \cdot 0.5 = 0.125
\]
\[
[(F_{ch} \times G_{AB})\{(h, \text{about } 50^\circ \text{C})\}](55) = 0.75 \cdot 0.5 = 0.375
\]
\[
[(F_{ch} \times G_{AB})\{(h, \text{about } 60^\circ \text{C})\}](55) = 0.75 \cdot 0.5 = 0.375
\]

\textbf{Note 2.14} The above argument does not have generality. In quantum mechanics, a simultaneous observable \(O_1 \times O_2\) does not always exist (Note 3.3 and Heisenberg's uncertainty principle in Sec.3.4).

\subsection*{2.5.2 "State is only one" and parallel measurement}

For example, consider the following situation:

(a) There are two cups \(A_1\) and \(A_2\) in which water is filled. Assume that the temperature of the water in the cup \(A_k\) \((k = 1, 2)\) is \(\omega_k^\circ\text{C} (0 \leq \omega_k \leq 100)\). Consider two questions "Is the water in the cup \(A_1\) cold or hot?" and "How many degrees (\(^\circ\text{C}\)) is roughly the water in the cup \(A_2\)". This implies that we take two measurements such that

\[
\{\begin{array}{l}
(\sharp_1): M_{C(\Omega)}(O_{ch} = \{(c, h), 2\{c, h\}, F_{ch}, S_{[\omega_1]}\}) \text{ in Example 2.7} \\
(\sharp_2): M_{C(\Omega)}(O_{AB} = [N_{100}^{10}, 2^{N_{100}}G_{AB}, S_{[\omega_2]}]) \text{ in Example 2.9}
\end{array}
\]

However, as mentioned in the above, "only one state" must be demanded. Thus, we have the following problem.

\begin{problem}
Represent two measurements \(M_{C(\Omega)}(O_{ch} = \{(c, h), 2\{c, h\}, F_{ch}, S_{[\omega_1]}\})\) and \(M_{C(\Omega)}(O_{AB} = [N_{100}^{10}, 2^{N_{100}}G_{AB}, S_{[\omega_2]}])\) by only one measurement.
\end{problem}

\begin{definition}[Parallel observable, parallel measurement]
For each \(k = 1, 2, \ldots, n\), consider a measurement \(M_{C(\Omega)}(O_k \equiv (X, F, F_k), S_{[\rho_k]}^n)\) in a \(C^*\)-algebra \(C(\Omega)_k\). Put \(\widehat{C(\Omega)} = \bigotimes_{k=1}^n C(\Omega)_k\), i.e., the tensor product \(C^*\)-algebra of \(\{C(\Omega)_k : k = 1, 2, \ldots, n\}\). Here, consider the tensor product \(C^*\)-observable \(\bigotimes_{k=1}^n O_k \equiv (X^n, \bigotimes_{k=1}^n F_k)\), \(\widehat{F} \equiv \bigotimes_{k=1}^n F_k\) in \(\widehat{C(\Omega)} \equiv \bigotimes_{k=1}^n C(\Omega)_k\) such that:

\[
\widehat{F}(\Xi_1 \times \Xi_2 \times \cdots \times \Xi_n) = F_1(\Xi_1) \otimes F_2(\Xi_2) \otimes \cdots \otimes F_n(\Xi_n) \quad (\forall \Xi_k \in F, k = 1, 2, \ldots, n).
\]

Therefore, we get the measurement \(M_{\otimes C(\Omega)_k}(\bigotimes_{k=1}^n O_k, S_{[\rho_k]}^n)\) in \(\bigotimes_{k=1}^n C(\Omega)_k\), which is also denoted by \(\bigotimes_{k=1}^n M_{C(\Omega)_k}(O_k, S_{[\rho_k]}^n)\) and called the parallel measurement of \(\{M_{C(\Omega)_k}(O_k, S_{[\rho_k]}^n)\}_{k=1}^n\).
\end{definition}
In what follows, let us explain the parallel measurement. We want to take two measurements \( M_{C(\Omega)}(O_1, S_{[\omega_1]}) \) and measurement \( M_{C(\Omega)}(O_2, S_{[\omega_2]}) \). That is, it suffices to image the following:

(b) \[
\begin{align*}
\text{state} \quad \omega_1 & \rightarrow \text{observable} \quad O_1 \rightarrow \text{measured value} \quad x_1 \\
\text{state} \quad \omega_2 & \rightarrow \text{observable} \quad O_2 \rightarrow \text{measured value} \quad x_2
\end{align*}
\]

However, the Copenhagen interpretation (Chap. I(\(U_1\))) says that two measurements can not be taken. Let us regard two states \( \omega_1 \) and \( \omega_2 \) as one state \( (\omega_1, \omega_2) \) \((\in \Omega_1 \times \Omega_2)\). And further, combining two observables \( O_1 \) and \( O_2 \), we construct the simultaneous observable \( O_1 \times O_2 \), and take the simultaneous measurement \( M_{C(\Omega_1 \times \Omega_2)}(O_1 \otimes O_2, S_{[\omega_1, \omega_2]}) \) in what follows.

(c) \[
\begin{align*}
\text{state} \quad (\omega_1, \omega_2) & \rightarrow \text{parallel observable} \quad O_1 \otimes O_2 \rightarrow \text{measured value} \quad x_1, x_2
\end{align*}
\]

Example 2.18 [Answer to Problem 2.16] Put \( \Omega_1 = \Omega_2 = [0, 100] \), and define the state space \( \Omega_1 \times \Omega_2 \). And consider two observables, that is, ch-observable \( O_{ch} = (X = \{c, h\}, 2^X, F_{ch}) \) in \( C(\Omega_1) \) (in Example 2.7) and about-observable \( O_{AB} = (Y(=N^{100}_1), 2^Y, G_{AB}) \) in \( C(\Omega_2) \) (in Example 2.9). Thus, we get the parallel observable \( O_{ch} \times O_{AB} = (\{c, h\} \times N^{100}_1, \{c, h\} \times N^{100}_1, F_{ch} \otimes G_{AB}) \) in \( C(\Omega_1 \times \Omega_2) \), take the parallel measurement \( M_{C(\Omega_1 \times \Omega_2)}(O_{ch} \otimes O_{AB}, S_{[\omega_1, \omega_2]}) \). For example, putting \( (\omega_1, \omega_2) = (25, 55) \), we see the following.

(d) When the parallel measurement \( M_{C(\Omega_1 \times \Omega_2)}(O_{ch} \otimes O_{AB}, S_{[(25, 55)]}) \) is taken, the probability that the measured value \[
\begin{bmatrix}
(c, \text{about 50 °C}) \\
(c, \text{about 60 °C}) \\
h, \text{about 50 °C} \\
h, \text{about 60 °C}
\end{bmatrix}
\]
is obtained is given by \[
\begin{bmatrix}
0.375 \\
0.375 \\
0.125 \\
0.125
\end{bmatrix}
\]

That is because

\[
[(F_{ch} \otimes G_{AB})(\{(c, \text{about 50 °C})\})(25, 55) = [(F_{ch}(\{c\})(25) \cdot [G_{AB}(\{\text{about 50 °C})\])(55) = 0.75 \cdot 0.5 = 0.375
\]

Thus, similarly,

\[
\begin{align*}
[(F_{ch} \otimes G_{AB})(\{(c, \text{about 60 °C})\})(25, 55) &= 0.75 \cdot 0.5 = 0.375 \\
[(F_{ch} \otimes G_{AB})(\{h, \text{about 50 °C})\})(25, 55) &= 0.25 \cdot 0.5 = 0.125 \\
[(F_{ch} \otimes G_{AB})(\{h, \text{about 60 °C})\})(25, 55) &= 0.25 \cdot 0.5 = 0.125
\end{align*}
\]

Remark 2.19 Also, for example, putting \( (\omega_1, \omega_2) = (55, 55) \), we see: 

(e) parallel measurement \( M_{C(\Omega_1 \times \Omega_2)}(O_{ch} \otimes O_{AB}, S_{[(55, 55)]}) \). Therefore,
The probability that a measured value \[
\begin{bmatrix}
(c, \text{about } 50 \degree C) \\
(c, \text{about } 60 \degree C) \\
(h, \text{about } 50 \degree C) \\
(h, \text{about } 60 \degree C)
\end{bmatrix}
\]
is obtained is given by \[
\begin{bmatrix}
0.125 \\
0.125 \\
0.375 \\
0.375
\end{bmatrix}
\tag{2.10}
\]

That is because, we similarly, see
\[
\begin{align*}
[F_{ch}(\{c\})](55) &\cdot [G_{AB}(\{\text{about } 50 \degree C\})](55) = 0.25 \cdot 0.5 = 0.125 \\
[F_{ch}(\{c\})](55) &\cdot [G_{AB}(\{\text{about } 60 \degree C\})](55) = 0.25 \cdot 0.5 = 0.125 \\
[F_{ch}(\{h\})](55) &\cdot [G_{AB}(\{\text{about } 50 \degree C\})](55) = 0.75 \cdot 0.5 = 0.375 \\
[F_{ch}(\{c\})](55) &\cdot [G_{AB}(\{\text{about } 60 \degree C\})](55) = 0.75 \cdot 0.5 = 0.375
\end{align*}
\]

This is the same as Example 2.15 (cf. Note 2.15 later).

The follow is obvious, it is deep.

**Theorem 2.20** The sample probability space of a simultaneous measurement \(M_{C(\Omega)}(X^n_{k=1} O_k, S_\omega)\) is the same as that of a parallel measurement \(M_{C(\Omega^n)}(\bigotimes_{k=1}^n O_k, S_{[\omega]})\).

**Proof.**

\[
(2.7) = "(2.9)" \text{ in the case that } \omega_k = \omega \ (\forall k = 1, 2, \ldots, n).\]

Thus, the proof is immediately follows. \(\square\)

\(\Diamond\) **Note 2.15** Theorem 2.20 is rather deep in the following sense. For example, "To toss a coin 10 times" is a simultaneous measurement. On the other hand, "To toss 10 coins once" is characterized as a parallel measurement. The two have the same sample space. This means that the two are not distinguished by the sample space and not the measurements (i.e., a simultaneous measurement and a parallel measurement). However, this is peculiar to classical pure measurements. It does not hold in classical mixed measurements and quantum measurement.

### 2.5.3 The law of Large Numbers — How to find out the sample space

Let \(O = (X, \mathcal{F}, F)\) be an observable in \(C(\Omega)\). Consider its \(n\)-dimensional parallel observable \(\tilde{O}(= \bigotimes_{k=1}^n O)\) \((= \bigotimes_{k=1}^n F)) \text{ in } C(\Omega^n)\). That is,

\[
[\tilde{F}(\Xi_1 \times \Xi_2 \times \cdots \times \Xi_n)](\omega_1, \omega_2, \ldots, \omega_n) = [F(\Xi_1)](\omega_1)[F(\Xi_2)](\omega_2) \cdots [F(\Xi_n)](\omega_n)
\]

\[(\forall (\omega_1, \omega_2, \ldots, \omega_n) \in \Omega^n, \ \forall \Xi_k \in \mathcal{F} \ (k = 1, 2, \ldots, n))\]

Further, put \(\mathcal{M}_{+1}(X) = \{ \nu : \nu \text{ is a probability measure on } X \}\). Define the map \(w : X^n \to \mathcal{M}_{+1}(X)\) such that,

\[
[w(x_1, x_2, \ldots, x_n)](\Xi) = \frac{\#[\{k : x_k \in \Xi\}]}{n} = \frac{1}{n} \sum_{k=1}^n \chi_\Xi(\pi_k(\tilde{x})) = \frac{1}{n} \sum_{k=1}^n \chi_\Xi(x_k)
\]

\[\forall \Xi \in \mathcal{F}, \ \forall \tilde{x} = (x_1, x_2, \ldots, x_n) \in X^n\]

where \(\#[A] = \text{the number of the elements of a set } A\), \(\chi\) is a characteristic function such that \(\chi_\Xi(x) = 1 \ (x \in \Xi), = 0 \ (x \notin \Xi), \ \pi_k : X^n \to X\) is defined by \(\pi_k(\tilde{x}) = x_k\)

Before we present Theorem 2.21 (the weak law of large numbers in measurement theory), we add the following note.
Note 2.16  The weak law of large numbers in probability theory is as follows.

(2) Let \((X, \mathcal{F}, P)\) be a probability space. Consider its \(n\)-dimensional product probability space \((X^n, \mathcal{F}^n, P^n)\). Let \(f : X \rightarrow \mathbb{R}\) be a measurable function such that \(\int_X f(x)P(dx) = \mu\) and \(\int_X |f(x) - \mu|^2 P(dx) = \sigma^2\). Then it holds that

\[
P^n\left( \{ (x_1, x_2, \ldots, x_n) \in X^n \mid \frac{\sum_{k=1}^n f(x_k)}{n} - \mu > \varepsilon \} \right) \leq \frac{\sigma^2}{\varepsilon^2 n} \quad (\forall \varepsilon > 0, \forall n = 1, 2, \ldots)
\]

This theorem, discovered by Jacob Bernoulli (1654-1705) and was announced in 1713 (after his death), is the most fundamental assertion in science. Recall that mathematics is independent of our world. Thus some may ask that

Why is a mathematical theorem (2) useful?

This is due to the fact that the mathematical theorem (2) sinks into the widely ordinary language \(\Box\) in the following diagram:

\[
\begin{array}{ccc}
(X_1) & \Box \text{ widely ordinary language} & \Rightarrow \text{world-description} \\
\text{(Chap. 1(O))} & \text{(before science)} & \text{(Chap. 1(\Omega))} \\
\end{array}
\]

However, the following theorem 2.21 (the weak law of large numbers in measurement theory) is not in \(\Box\) but \(\Diamond\).

Theorem 2.21 [The weak law of large numbers in measurement theory](cf. [10, 11, 13]) Suppose the above parallel measurement \(M_{\Omega^n}(\bigotimes_{k=1}^n O, S_{\{\omega, \ldots, \omega\}})\) in \(C(\Omega^n)\). For any \(\varepsilon > 0\) and any \(\Xi (\in \mathcal{F})\), define \(\tilde{D}_{\Xi, \varepsilon} (\in \bigotimes_{k=1}^n \mathcal{F})\) by

\[
\tilde{D}_{\Xi, \varepsilon} = \left\{ \tilde{x} = (x_1, x_2, \ldots, x_n) \in X^n : \left| w(\tilde{x})[\Xi] - [F(\Xi)](\omega) \right| < \varepsilon \right\}.
\]

Then we see that

\[
1 - \frac{1}{4\varepsilon^2 n} \leq \left[ F(\tilde{D}_{\Xi, \varepsilon}) \right](\omega, \ldots, \omega) \leq 1, \quad (\forall \omega \in \Omega, \forall \Xi \in \mathcal{F}, \forall \varepsilon > 0, \forall n). \quad (2.11)
\]

Proof. \(\omega \in \Omega, \Xi \in \mathcal{F}\). Define \(\mu\) and \(\sigma\) such that

\[
\mu = \int_X x \chi_{\Xi}(x) [F(dx)](\omega) = [F(\Xi)](\omega)
\]

\[
\sigma^2 = \int_X |x - \mu|^2 [F(dx)](\omega) = [F(\Xi)](\omega)(1 - [F(\Xi)](\omega))
\]

Then, the law of large numbers (Note 2.16) says that

\[
\left[ F(X^n \setminus \tilde{D}_{\Xi, \varepsilon}) \right](\omega, \ldots, \omega) \leq \frac{\sigma^2}{\varepsilon^2 n} = \frac{1}{\varepsilon^2 n} [F(\Xi)](\omega)(1 - [F(\Xi)](\omega))
\]

\[
= \frac{1}{\varepsilon^2 n} \left( \frac{1}{4} - \left( [F(\Xi)](\omega) - \frac{1}{2} \right)^2 \right) \leq \frac{1}{4\varepsilon^2 n}
\]

Thus, we get (2.11) \(\square\)

\[\Box\] Note 2.17 As mentioned in Note 1.1, we believe that:

\(\#_1\) Behind a useful mathematical theory, the powerful world view is always hidden
Because mathematic itself is independent of our world. In fact,

(♯2) In the proof of Theorem 2.21 (The law of large numbers in measurement theory), we can find the law of large numbers in probability theory ((♯1) in Note 2.16).

Therefore, for example,

| mathematics                  | world-description method               |
|------------------------------|----------------------------------------|
| differential geometry        | the theory of relativity               |
| differential equation        | Newton mechanics, electromagnetism     |
| Hilbert space                | quantum mechanics                      |
| probability theory           | measurement theory                     |

♠ Note 2.18 Now we can expect readers to believe in our assertion that

(♯1) There is the metaphysics (called measurement theory) in the center of sciences.

♠ Note 2.19 As mentioned in Chap. 8, measurement theory is deeply related to traditional philosophies. For example, we see:

\[
\begin{aligned}
\text{linguistic method (language is before world)} & \quad \text{... Saussure (Sec.8.1)} \\
\text{state and observable} & \quad \text{... Locke’s primary quantity and secondary quantity} \\
\text{only one measurement} & \quad \text{... only one state, no movement (Parmenides)} \\
\text{observer’s time (Note 2.8)} & \quad \text{... Augustine’s time, McTaggart’s paradox ([27, 14])} \\
\text{primary substance \cdot secondary substance} & \quad \text{... The problem of universals ([12])} \\
\text{observable is before state} & \quad \text{... Recognition constitutes the world (Kant Copernican turn cf. Sec.8.1)}
\end{aligned}
\]

3 From Quantum Mechanics to Measurement Theory

As mention in Chap. 1, we assert that

(♯) \[ \text{quantum mechanics} \xrightarrow{\text{proverbalizing}} \text{measurement theory} \]

Therefore, first we shall review the elementary step of quantum mechanics. And further, we derive Axiom P 1 in Sec. 2.2 from Born’s quantum measurement theory. Discussing EPR-paradox, Schrödinger’s cat, Heisenberg’s uncertainty principle we study the spirit of quantum mechanics. However, our assertion (i.e., the linguistic world-view) is the reverse arrow of the (♯), that is, ”from measurement theory to quantum mechanics”. This will be discussed in Sec. 9.3.

3.1 The quick review on quantum mechanics

quantum mechanics is composed of two axioms (i.e., ”Born’s quantum measurement theory” and ”quantum kinetic equation”). That is,
3 FROM QUANTUM MECHANICS TO MEASUREMENT THEORY

\[ \text{quantum mechanics} := \text{quantum measurement} + \text{kinetic equation} \]

In Sec.3.2, we derive classical measurement theory (Axiom 1) from quantum mechanics (Born’s quantum measurement theory). This is rather concrete. For the abstract argument, see [8, 11].

3.1.1 Born’s quantum measurement theory

Let \( \mathbb{C} \) be the complex field (i.e., the set of all complex numbers). Let \( \mathbb{C}^n \) be the \( n \)-dimensional complex space. That is,

\[ \mathbb{C}^n = \{ \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \mid \alpha_1, \alpha_2, \ldots, \alpha_n \text{is complex number} \} \]

The inner product \( \langle \cdot, \cdot \rangle \) and the norm \( \| \cdot \| \) are respectively defined by

\[ \langle \alpha, \beta \rangle = \sum_{k=1}^{n} \overline{\alpha_k} \cdot \beta_k, \quad \| \alpha \| = \| \langle \alpha, \alpha \rangle \|^{1/2} \quad (\forall \alpha, \forall \beta \in \mathbb{C}^n) \]

(where \( \overline{\alpha_k} \) is the conjugate complex number of \( \alpha_k \)). \( H = \mathbb{C}^n \) is called the \( n \)-dimensional Hilbert space. Also, an infinite dimensional Hilbert space \( \mathbb{C}^\infty \) is defined by \( H = \{ \alpha \in \mathbb{C}^\infty \mid \| \alpha \| = (\sum_{k=1}^{\infty} \overline{\alpha_k} \cdot \alpha_k)^{1/2} < \infty \} \).

Define the state space \( \hat{\Omega} \) such that \( \hat{\Omega} \subset \mathbb{C}^n \). That is,

\[ \hat{\Omega} = \{ \alpha \in \mathbb{C}^n \mid \| \alpha \| = 1 \} \]

where \( \alpha \) and \( \beta \) are identified if \( \alpha = e^{i\theta} \beta \) (for some \( \theta \in \mathbb{R} \)). The \( \omega(\in \hat{\Omega}) \) is called the state.

Let \( B(\mathbb{C}^n) \) be the set of all \( n \times n \)-complex matrices. That is,

\[ B(\mathbb{C}^n) = \{ A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \mid a_{ij} \text{ is complex numbers} \} \]

which is called the basic algebra. The matrix \( A(\in B(\mathbb{C}^n)) \) is said to be non-negative Hermitian matrix if it satisfies the following conditions (i) and (ii):

(i) \( A \) is Hermitian, that is, \( A = A^* \), (where \( A^* \) is the conjugate transposed matrix)

(ii) \( A \succeq 0 \), That is, \( \langle \alpha, A\alpha \rangle \geq 0 \quad (\forall \alpha \in \mathbb{C}^n) \)

For simplicity, assume that the measured value space \( X \) is finite set. The triplet \( O = (X, 2^X, F) \) is called an observable in the basic algebra \( B(\mathbb{C}^n) \), if it satisfies the following (cf. [3]):

(i) The map \( F : 2^X \to B(\mathbb{C}^n) \) satisfies that (a): \( F(\emptyset) = 0(=0 \text{ matrix}), F(X) = I (= \text{ unit matrix}) \) (b): for each \( \Xi \in 2^X \), \( F(\Xi) \) is a non-negative Hermitian matrix.
(ii) for each $\Xi \in 2^X$, it holds that $F(\Xi) = \sum_{x \in \Xi} F\{\{x\}\}$

Also, the observable $O = (X, 2^X, F)$ is said to be a projective observable, if it satisfies that $F(\Xi) = (F(\Xi))^2$ ($\forall \Xi \in 2^X$).

Here, we have the quantum measurement $M_{B(C^n)}(O = (X, 2^X, F), S_{[\omega]})$ (where $\omega \in \hat{\Omega}$). That is, the quantum measurement $M_{B(C^n)}(O, S_{[\omega]})$

= the quantum measurement of the observable $O$ for the measuring object $S$ with a quantum state $\omega(\in \hat{\Omega})$

Under the above preparation, we can introduce "Born’s quantum measurement theory" as follows.

\textbf{Axiom(Q) 1 [Born’s quantum measurement theory]}

Consider a measurement $M_{B(C^n)}(O := (X, 2^X, F), S_{[\omega]})$ formulated in a basic algebra $B(C^n)$. Assume that the measured value $x \in X$ is obtained by the measurement $M_{B(C^n)}(O, S_{[\omega]})$.

Then, the probability that a measured value $x \in X$ is obtained is given by $\langle \omega, F\{\{x\}\}\omega \rangle$.

Also, the parallel quantum measurement $\bigotimes_{k=1}^N M_{B(C^n)}(O, S_{[\omega]})$ is possible, and thus, we can get the sample probability space $(X, F, \langle \omega, F(\cdot)\omega \rangle)$.

\textbf{Note 3.1} Since quantum mechanics is physics, the following terms have reality:

(♯) measurement, observer, measuring object, state, observable (≈ measuring instrument), measured value, probability

Now, we shall show that an Hermitian matrix $A(\in B(C^n))$ can be regarded as a projective observable. For simplicity, this is shown in the case that $n = 3$. We see (for simplicity, assume that $x_j \neq x_k$ (if $j \neq k$) )

\begin{equation}
A = U^* \begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix} U
\end{equation}

where $U (\in B(C^3))$ is the unitary matrix, $x_k \in \mathbb{R})$. Putting $X = \{x_1, x_2, x_3\}$,

$$F_A(\{x_1\}) = U^* \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U, \quad F_A(\{x_2\}) = U^* \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} U,$$

$$F_A(\{x_3\}) = U^* \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} U$$

the, we get the projective observable $O_A = (X, 2^X, F_A)$ in $B(C^3)$. Thus, we have the following identification:

\begin{equation}
A \quad \longleftrightarrow \quad O_A = (X, 2^X, F_A)
\end{equation}
Let $A(\in B(\mathbb{C}^n))$ be an Hermitian matrix. Under the identification (3.2), we have the quantum measurement $M_{B(\mathbb{C}^n)}(O_A, S_\omega)$. Born’s quantum measurement theory say that

The probability that a measured value $x(\in X)$ is obtained by the quantum measurement $M_{B(\mathbb{C}^n)}(O_A, S_\omega)$ is given by $\langle \omega, F_A(\{x\}) \omega \rangle$.

Therefore, the expectation of a measured value is given by

$$\int_X x \langle \omega, F_A(\{dx\}) \omega \rangle = \sum_{x_i \in X} x_i \langle \omega, F_A(\{x_i\}) \omega \rangle = \langle \omega, A \omega \rangle$$

Also, its variance $(\delta_\omega^2)$ is given by

$$(\delta_\omega^2) \equiv \sum_{x_i \in X} (x_i - \langle \omega, A \omega \rangle)^2 \langle \omega, F_A(\{x_i\}) \omega \rangle = \langle A \omega, A \omega \rangle - |\langle \omega, A \omega \rangle|^2 \quad (3.3)$$

Quantum measurement theory (Stern–Gerlach experiment (1922))

Assume that we examine the beam (of silver particles) after passing through the magnetic field. Then, as seen in the following figure, we see that all particles are deflected either equally upwards or equally downwards in a 50:50 ratio. See Fig. 3.1.

Consider the two dimensional Hilbert space $V = \mathbb{C}^2$. And therefore, we get the non-commutative basic algebra $C(\Omega) = B(V)(= B_c(V))$, that is, the algebra composed of all $2 \times 2$ matrices. Note that $C(\Omega) = B(V) = B_c(V)$ since the dimension of $V$ is finite.

The spin state of an electron $e$ is represented by $\omega \in \tilde{\Omega} (\subset \mathbb{C}^2)$. Put $\omega = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$ (where, $||\omega||^2 = |\alpha_1|^2 + |\alpha_2|^2 = 1$).

Define $O_z \equiv (Z, 2^Z, F_z)$, the spin observable concerning the $z$-axis, such that, $Z = \{\uparrow, \downarrow\}$ and

$$F_z(\{\uparrow\}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad F_z(\{\downarrow\}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$F_z(\emptyset) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad F_z(\{\uparrow, \downarrow\}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$
Here, Born’s quantum measurement theory (the probabilistic interpretation of quantum mechanics) says that

When a quantum measurement $M_{B(\mathbb{C}^2)}(\mathbb{C}, S_\omega)$ is taken, the probability that

- a measured value \( \begin{bmatrix} \uparrow \\ \downarrow \end{bmatrix} \) is obtained is given by
  \[ \langle \omega, F^z(\{\uparrow\})\rangle = |\alpha_1|^2 \]
  \[ \langle \omega, F^z(\{\downarrow\})\rangle = |\alpha_2|^2 \]

That is, putting $\omega = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \in \hat{\Omega}$, we says that

When the electron with a spin state $\omega$ progresses in a magnetic field, the probability that the Geiger counter $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ sounds

\[
\begin{bmatrix}
[\bar{\pi}_1 & \bar{\pi}_2]
\end{bmatrix}
\begin{bmatrix}
1 \\ 0
0 \\ 0
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\ \alpha_2
\end{bmatrix}
= |\alpha_1|^2
\]
\[
\begin{bmatrix}
[\bar{\pi}_1 & \bar{\pi}_2]
\end{bmatrix}
\begin{bmatrix}
0 \\ 1
0 \\ 0
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\ \alpha_2
\end{bmatrix}
= |\alpha_2|^2
\]

**EPR-paradox**

Next, let us explain EPR-paradox (Einstein–Podolsky–Rosen) \([4, 30]\). Consider Two electrons $P_1$ and $P_2$ and their spins. The tensor Hilbert space $H = \mathbb{C}^2 \otimes \mathbb{C}^2$ is defined in what follows. That is,

\[
e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

(i.e., the complete orthonormal system \(\{e_1, e_2\}\) in the $\mathbb{C}^2$),

\[
\mathbb{C}^2 \otimes \mathbb{C}^2 = \{ \sum_{i,j=1,2} \alpha_{ij} e_i \otimes e_j \mid \alpha_{ij} \in \mathbb{C}, i,j = 1,2 \}
\]

Put $u = \sum_{i,j=1,2} \alpha_{ij} e_i \otimes e_j$ and $v = \sum_{i,j=1,2} \beta_{ij} e_i \otimes e_j$. And the inner product $\langle u, v \rangle_{\mathbb{C}^2 \otimes \mathbb{C}^2}$ is defined by

\[
\langle u, v \rangle_{\mathbb{C}^2 \otimes \mathbb{C}^2} = \sum_{i,j=1,2} \bar{\alpha}_{ij} \cdot \beta_{ij}
\]

Therefore, we have the tensor Hilbert space $H = \mathbb{C}^2 \otimes \mathbb{C}^2$ with the complete orthonormal system \(\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\}\).

For each $F \in B(\mathbb{C}^2)$ and $G \in B(\mathbb{C}^2)$, define the $F \otimes G \in B(\mathbb{C}^2 \otimes \mathbb{C}^2)$ (i.e., linear operator $F \otimes G : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$) such that

\[
(F \otimes G)(u \otimes v) = Fu \otimes Gv
\]

Let us define the singlet $s \in \mathbb{C}^2 \otimes \mathbb{C}^2$ of two particles $P_1$ and $P_2$ by

\[
s = \frac{1}{\sqrt{2}}(e_1 \otimes e_2 - e_2 \otimes e_1)
\]
Here, we see that \( \langle s, s \rangle_{c^2 \otimes c^2} = \frac{1}{2}(e_1 \otimes e_2 - e_2 \otimes e_1, e_1 \otimes e_2 - e_2 \otimes e_1)_{c^2 \otimes c^2} = \frac{1}{2}(1 + 1) = 1 \), and thus, \( s \) is a state. Also, assume that two particles \( P_1 \) and \( P_2 \) are far.

Let \( O = (X, 2^X, F^z) \) in \( B(\mathbb{C}^2) \) (where \( X = \{\uparrow, \downarrow\} \) ) be the spin observable concerning the \( z \)-axis such that

\[
F^z(\{\uparrow\}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad F^z(\{\downarrow\}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\]

The parallel observable \( O \otimes O = (X^2, 2^X \times 2^X, F^z \otimes F^z) \) in \( B(\mathbb{C}^2 \otimes \mathbb{C}^2) \) is defined by

\[
(F^z \otimes F^z)(\{(\uparrow, \uparrow)\}) = F^z(\{\uparrow\}) \otimes F^z(\{\uparrow\}),
\]

\[
(F^z \otimes F^z)(\{(\downarrow, \uparrow)\}) = F^z(\{\downarrow\}) \otimes F^z(\{\uparrow\}),
\]

\[
(F^z \otimes F^z)(\{(\uparrow, \downarrow)\}) = F^z(\{\uparrow\}) \otimes F^z(\{\downarrow\}),
\]

\[
(F^z \otimes F^z)(\{(\downarrow, \downarrow)\}) = F^z(\{\downarrow\}) \otimes F^z(\{\downarrow\})
\]

Thus, we get the measurement \( M_{B(\mathbb{C}^2 \otimes \mathbb{C}^2)}(O \otimes O, S_{[s]}) \) The Born's quantum measurement theory says that when the parallel measurement \( M_{B(\mathbb{C}^2 \otimes \mathbb{C}^2)}(O \otimes O, S_{[s]}) \) is taken, the probability that the measured value

\[
\begin{bmatrix}
\langle \uparrow, \uparrow \rangle \\
\langle \downarrow, \uparrow \rangle \\
\langle \uparrow, \downarrow \rangle \\
\langle \downarrow, \downarrow \rangle
\end{bmatrix}
\]

is obtained

\[
\begin{bmatrix}
\langle s, (F^z \otimes F^z)(\{(\uparrow, \uparrow)\})s \rangle_{c^2 \otimes c^2} = 0 \\
\langle s, (F^z \otimes F^z)(\{(\downarrow, \uparrow)\})s \rangle_{c^2 \otimes c^2} = 0.5 \\
\langle s, (F^z \otimes F^z)(\{(\uparrow, \downarrow)\})s \rangle_{c^2 \otimes c^2} = 0.5 \\
\langle s, (F^z \otimes F^z)(\{(\downarrow, \downarrow)\})s \rangle_{c^2 \otimes c^2} = 0
\end{bmatrix}
\]

That is because, \( F^z(\{\uparrow\})e_1 = e_1, F^z(\{\downarrow\})e_2 = e_2, F^z(\{\uparrow\})e_2 = F^z(\{\downarrow\})e_1 = 0 \) For example,

\[
\langle s, (F^z \otimes F^z)(\{(\uparrow, \downarrow)\})s \rangle_{c^2 \otimes c^2} = \frac{1}{2}(e_1 \otimes e_2 - e_2 \otimes e_1, (F^z(\{\uparrow\}) \otimes F^z(\{\downarrow\})e_2 \otimes e_2 - e_2 \otimes e_1))_{c^2 \otimes c^2} = \frac{1}{2}
\]

Here, it should be noted that we can assume that the \( x_1 \) and the \( x_2 \) (in \( (x_1, x_2) \in \{ (\uparrow, \uparrow), (\uparrow, \downarrow), (\downarrow, \uparrow), (\downarrow, \downarrow) \} \) are respectively obtained in Tokyo and in New York (or, in the earth and in the polar star).

\[
\begin{array}{c|c|c}
(b) & (probability \frac{1}{2}) \\
\hline
\uparrow_z & \downarrow_z & \text{or} \\\n\hline
\text{Tokyo} & \text{New York} & \\
\hline
\end{array}
\]

\[
\begin{array}{c|c|c}
(c) & (probability \frac{1}{2}) \\
\hline
\downarrow_z & \uparrow_z & \\
\hline
\text{Tokyo} & \text{New York} & \\
\hline
\end{array}
\]

This fact is, figuratively speaking, explained as follows:

- Immediately after the particle in Tokyo is measured and the measured value $\uparrow_z$ [resp. $\downarrow_z$] is observed, the particle in Tokyo informs the particle in New York “Your measured value has to be $\downarrow_z$ [resp. $\uparrow_z$].”

Therefore, the above fact implies that quantum mechanics says that there is something faster than light. This is essentially the same as the de Broglie paradox (cf. [30]. Also see § 9.3.3). That is,

- if we admit quantum mechanics, we must also admit the fact that there is something faster than light (i.e., so called "non-locality").

\[ \text{(3.5)} \]

\[ \boxed{\text{Note 3.2}} \]

As shown and emphasized in [16], quantum syllogism does not generally hold. We believe that this fact was, for the first time, discovered in EPR-paradox [4]. The reason that we think so is as follows. Consider the two-particles system composed of particles $P_1$ and $P_2$, which is formulated in a Hilbert space $L^2(\mathbb{R}^2)$. Let $\rho_s(\in \mathcal{S}(B_c(L^2(\mathbb{R}^2))))$ be the EPR-state in EPR-paradox (or, the singlet state in Bohm’s situation). Here, consider as follows:

\[ (Z_1) \text{ Assume that } (x_1, p_2) \text{ and } p'_2 \text{ are obtained by the simultaneous measurement of } [\text{the position of } P_1, \text{the momentum of } P_2] \text{ and } [\text{the momentum of } P_1]. \text{ Since it is clear that } p_2 = p'_2, \text{ thus, we see that} \]

\[ (x_1, p_2) \implies p_2 \implies [\text{the momentum of } P_1] \]

Here, for the definition of “$\implies$”, see ref. [8].

\[ (Z_2) \text{ Assume that } p_1 \text{ and } p_2 \text{ are obtained by the simultaneous measurement of } [\text{the momentum of } P_1] \text{ and } [\text{the momentum of } P_2]. \text{ Since the state } \rho_s(\in \mathcal{S}(B_c(L^2(\mathbb{R}^2)))) \text{ is the EPR-state, we see that } p_1 = -p_2, \text{ that is, we see that} \]

\[ p_2 \implies -p_2 \implies [\text{the momentum of } P_1] \]

\[ (Z_3) \text{ Therefore, if quantum syllogism holds, } (Z_1) \text{ and } (Z_2) \text{ imply that} \]

\[ -p_2 \implies [\text{the momentum of } P_1] \]

that is, the momentum of $P_1$ is equal to $-p_2$.

Since the above (Z1)- (Z3) is not the approximately simultaneous measurement (cf. the definition (N)), it is not related to Heisenberg’s uncertainty principle (Theorem 3.2 later). Thus, the conclusion (Z3) is not contradictory to Heisenberg’s uncertainty principle. However, now we can say that the conclusion (Z3) is not true. That is because the interpretation (H2) (i.e., only one measurement is permitted) says, as seen in [13], that quantum syllogism does not hold by the non-commutativity of the above three observables, i.e.,

\[ \left\{ \begin{array}{c}
[\text{the position of } P_1, \text{the momentum of } P_2] \\
[\text{the momentum of } P_2] \\
[\text{the momentum of } P_1]
\end{array} \right. \]

Thus we see that EPR-paradox is closely related to the fact that quantum syllogism does not hold in general.

3.1.2 Supplement — Bell’s inequality

Let us have the argument in the previous section develop.

Put $a = (a_1, a_2) \in \mathbb{R}^2, |a| = \sqrt{|a_1|^2 + |a_2|^2} = 1$. Define the observable $O_a = \{X = \{1, -1\}, 2^X, F_a\}$ in $B(\mathbb{C}^2)$ such that
\[ F_a(\{1\}) = \frac{1}{2} \left[ \frac{1}{a_1 + a_2 \sqrt{-1}} a_1 - a_2 \sqrt{-1} \right], \]
\[ F_a(\{-1\}) = \frac{1}{2} \left[ \frac{1}{-a_1 - a_2 \sqrt{-1}} -a_1 + a_2 \sqrt{-1} \right]. \]

Further, put \( b = (b_1, b_2) \in \mathbb{R}^2, |b| = \sqrt{|b_1|^2 + |b_2|^2} = 1 \), and, by the same way, define the observable \( O_b = (X = \{1, -1\}, \alpha^X, F_b) \) in \( B(\mathbb{C}^2) \).

\[ \blacktriangleright \textbf{Note 3.3} \]

For example, assume that \( a = (1, 0), b = (1/\sqrt{2}, 1/\sqrt{2}) \). Then, the simultaneous observable \( O_a \times O_b \) in \( B(\mathbb{C}^2) \) does not exist. The proof is easy, thus, it is omitted.

Of course, we have the parallel observable \( \tilde{O}_{ab} (= O_a \otimes O_b) = (X^2, 2X^2, F_a \otimes F_b) \) in \( B(\mathbb{C}^2 \otimes \mathbb{C}^2) \). And further, we have the measurement \( M_{B(\mathbb{C}^2 \otimes \mathbb{C}^2)}(\tilde{O}_{ab}, S_s) \), where \( s \) is a singlet state. Born’s quantum measurement theory says that

The probability that a measured value \( \tilde{x} \) \( (= (x_1, x_2)) \in X^2 (= \{1, -1\}^2) \) is obtained by the measurement \( M_{B(\mathbb{C}^2 \otimes \mathbb{C}^2)}(\tilde{O}_{ab}, S_s) \) is given by \( \nu_{ab}(\{(x_1, x_2)\}) \), where

\[ \nu_{ab}(\{(x_1, x_2)\}) = \langle s, (F_a \otimes F_b)(\{(x_1, x_2)\}) \rangle s_{\mathbb{C}^2 \otimes \mathbb{C}^2} \]

(3.6)

Now, define the correlation function \( C_{ab} \) by

\[ C_{ab} = \int_{X^2} x_1 \cdot x_2 \ \nu_{ab}(dx_1 dx_2) = \int_{X^2} x_1 \cdot x_2 \ \langle s, (F_a \otimes F_b)(dx_1 dx_2) \rangle s_{\mathbb{C}^2 \otimes \mathbb{C}^2} \]

\[ = \sum_{(x_1, x_2) \in X^2} x_1 \cdot x_2 \langle s, (F_a \otimes F_b)(\{(x_1, x_2)\}) \rangle s_{\mathbb{C}^2 \otimes \mathbb{C}^2} \]

A simple calculation shows (cf. [30]),

\[ = a_1 b_1 + a_2 b_2 \]

Here, put \( a^1 (= (a_1^1, a_1^2)), a^2 (= (a_2^1, a_2^2)), b^1 (= (b_1^1, b_1^2)), b^2 (= (b_2^1, b_2^2)) \). And we have the following four measurements:

\[ M_{B(\mathbb{C}^2 \otimes \mathbb{C}^2)}(\tilde{O}_{a^1 b^1}, S_s), \quad M_{B(\mathbb{C}^2 \otimes \mathbb{C}^2)}(\tilde{O}_{a^1 b^2}, S_s), \]
\[ M_{B(\mathbb{C}^2 \otimes \mathbb{C}^2)}(\tilde{O}_{a^2 b^1}, S_s), \quad M_{B(\mathbb{C}^2 \otimes \mathbb{C}^2)}(\tilde{O}_{a^2 b^2}, S_s) \]

Therefore, we have the parallel measurement \( \bigotimes_{i, j = 1, 2} M_{B(\mathbb{C}^2 \otimes \mathbb{C}^2)}(\tilde{O}_{a^i b^j}, S_s) \). We easily see that, for each \( x \in \{1, -1\}, \)

\[ \nu_{a^1 b^1}(\{x\} \times X) = \nu_{a^1 b^2}(\{x\} \times X), \quad \nu_{a^1 b^1}(X \times \{x\}) = \nu_{a^1 b^2}(X \times \{x\}) \]
\[ \nu_{a^2 b^1}(\{x\} \times X) = \nu_{a^2 b^2}(\{x\} \times X), \quad \nu_{a^2 b^1}(X \times \{x\}) = \nu_{a^2 b^2}(X \times \{x\}) \]

Here, put

\[ a^1 = (0, 1), \quad b^1 = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \quad a^2 = (1, 0), \quad b^2 = \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) \]
then, we see
\[ |C_{a^1b^1} - C_{a^2b^2}| + |C_{a^1b^2} + C_{a^2b^2}| = 2\sqrt{2} \] (3.7)

\[ \star \text{ Note 3.4} \]
The theoretical conclusion (3.7) is completely verified by experiment (cf. [30]). Also, the inequality such as
\[ \text{"formula like the left-hand side of (3.7)" \leq 2} \]
is called "Bell’s inequality". For example, in mathematics, Bell’s inequality is as follows. Let \((Y, \mathcal{G}, \mu)\) be a probability space. Consider each measurable function \(f_k : Y \rightarrow \{-1, 1\}, (k = 1, 2, 3, 4)\). And put
\[ C_{13} = \int_Y f_1(y) \cdot f_3(y) \mu(dy), \quad C_{14} = \int_Y f_1(y) \cdot f_4(y) \mu(dy), \quad C_{23} = \int_Y f_2(y) \cdot f_3(y) \mu(dy), \quad C_{24} = \int_Y f_2(y) \cdot f_4(y) \mu(dy). \]
Then, adding the proof, we can mention Bell's inequality as follows.
\[ |C_{13} - C_{14}| + |C_{23} + C_{24}| = |\int_Y f_1(y) \cdot f_3(y) - f_1(y) \cdot f_4(y) \mu(dy)| + |\int_Y f_2(y) \cdot f_3(y) - f_2(y) \cdot f_4(y) \mu(dy)| \leq 2 \]

### 3.2 The derivation of Axiom² 1 from Born’s quantum measurement

Let us derive Axiom² 1 from Born’s quantum measurement. That is, Let \(H\) be a Hilbert space such that \(H = \mathbb{C}^n\). In this case, we restrict the state space \(\tilde{\Omega}(\subset \tilde{\Omega} \subset \mathbb{C}^n)\) such that
\[ \tilde{\Omega} = \{e_1, e_2, \ldots, e_n\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\} \]

Thus, \(e \in \tilde{\Omega}\) is called a state.

Further, define \(B_D(\mathbb{C}^n)\) by all \(n \times n\)-diagonal matrices, which is called a basic structure.

(a) The above argument is not physical but mathematics. we do not mind it since our interest is the linguistic aspect of quantum mechanics.

Assume that a measured value space \(X\) is finite. The triple \(\tilde{\mathcal{O}} = (X, 2^X, \tilde{F})\) is called an observable in \(B_D(\mathbb{C}^n)\), if it satisfies the following (i) and (ii):

(i) \(\tilde{\mathcal{O}} = (X, 2^X, \tilde{F})\) is an observable in \(B(\mathbb{C}^n)\).

(ii) For each \(\Xi \in 2^X, \tilde{F}(\Xi)\) is a diagonal.
Now we have the (diagonal matrices type) quantum measurement \( M_{BD(C^n)}(\tilde{O}, S[e]) \) (where \( e \in \tilde{\Omega} \)). That is,

\[
M_{BD(C^n)}(\tilde{O}, S[e]) = \text{The measurement of the observable } \tilde{O} \text{ in } BD(C^n) \text{ for the system with the state } e(\in \tilde{\Omega})
\]

And we have the (diagonal matrices type) quantum measurement theory as follows.

**Axiom(D)** 1 quantum measurement diagonal matrix type

Consider a measurement \( M_{BD(C^n)}(\tilde{O} = (X, 2^X, \tilde{F}), S[e]) \) formulated in a basic algebra \( BD(C^n) \).

Assume that the measured value \( x (\in X) \) is obtained by the measurement \( M_{BD(C^n)}(\tilde{O}, S[e]) \).

Then, the probability that a measured value \( x (\in X) \) is obtained is given by \( \langle e, \tilde{F}({\{x}\}})e \rangle \).

Now, let \( \tilde{O} = (X, 2^X, \tilde{F}) \) be an observable in \( BD(C^n) \). Thus, for each \( x(\in X) \), we see that

\[
\tilde{F}({\{x}\}) = \begin{bmatrix}
 f_{11}({\{x}\}) & 0 & \cdots & 0 \\
 0 & f_{22}({\{x}\}) & \cdots & 0 \\
 \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & \cdots & f_{nn}({\{x}\})
\end{bmatrix} \in BD(C^n)
\]

Here, a discrete metric space \( \Omega = \{\omega_1, \omega_2, \ldots, \omega_n\} \) is regarded as a space. For each \( x \in X \), define \( f_x : \Omega \to \mathbb{R} \) such that

\[
f_x(\omega_k) = f_{kk}({\{x}\}) = \langle e_k, \tilde{F}({\{x}\}})e_k \rangle \quad (\forall k = 1, 2, \ldots, n)
\]

Then, we can regard \( \{f_x\}_{x \in X} \) as the resolution of the unity on \( \Omega \). Thus, putting

\[
[F(\Xi)](\omega) = \sum_{x \in \Xi} f_x(\omega) \quad (\forall \omega \in \Omega)
\]

we get the observable \( O = (X, 2^X, F) \) in \( C(\Omega) \). Therefore, under the following identification:

\[
\tilde{\Omega} = \{e_1, e_2, \ldots, e_n\} \ni e_k \quad \overset{\text{idemification}}{\longrightarrow} \quad \Omega = \{\omega_1, \omega_2, \ldots, \omega_n\} \ni \omega_k
\]

we see

\[
\underbrace{\text{(diagonal matrix type) quantum measurement } M_{BD(C^n)}(\tilde{O}, S[e_k])}_{\text{classical measurement } M_{C(\Omega)}(O, S[\omega_k])}
\]
Axiom\textsuperscript{p} 1 (measurement: finite $\Omega$)

Consider a measurement $M_{C(\Omega)}(O=(X, 2^X, F), S_{[\omega]})$ formulated in a basic algebra $C(\Omega)$. Then, the probability that a measured value $x \ (\in X)$ is obtained by the measurement $M_{C(\Omega)}(O=(X, 2^X, F), S_{[\omega]})$ is given by $[F(\{x\})](\omega)$.

If it writes without carrying out simple, we see:

(b) When an observer takes a measurement of an observable $O=(X, 2^X, F)$ (or, by a measuring instrument $O$ ) for a measuring object with a state $\omega$, the probability that a measured value belongs to $\Xi(\in 2^X)$ is given by $[F(\Xi)](\omega)$.

\textbf{Note 3.5} Note that a measurement $M_{B(C^n)}(O := (X, 2^X, F), S_{[\omega]})$ in Axiom\textsuperscript{p} 1 (The probabilistic interpretation of quantum mechanics) is physics, but a measurement $M_{C(\Omega)}(O=(X, 2^X, F), S_{[\omega]})$ in Axiom\textsuperscript{p} 1 (Measurement; finite $\Omega$) is language.

This is the mechanism of the derivation such that

(c) \hspace{2cm} \begin{tabular}{c|c}
Born’s quantum measurement \hline (quantum mechanics) & Axiom\textsuperscript{p} 1 \hline Derivation & \text{measurement theory}
\end{tabular}

However, the above is merely the starting point. Our purpose is to assert as follows.

(d) What is important is to show the power of classical measurement theory, and to declare that it is a scientific language. That is, we have to show that Axiom\textsuperscript{p} 1 is the most powerful proverb in science.

\section{3.3 Schrödinger’s cat}

In what follows, we expain our opinion about the Copenhagen interpretation.

\subsection{3.3.1 The Copenhagen interpretation — measured value is perceived by the brain —}

Schrödinger’s cat is the most famous paradox in quantum science.

[Schrödinger’s cat paradox]. Note that Schrödinger’s cat does not appear in the world of measurement theory. Let us explain it as follows: In 1935 (cf. \cite{1935Schr}) Schrödinger published an essay describing the conceptual problems in quantum mechanics. A brief paragraph in this essay described the cat paradox.

(a) Suppose we put a cat in a cage with a radioactive atom, a Geiger counter, and a poison gas bottle; further suppose that the atom in the cage has a half-life of one hour, a fifty-fifty chance of decaying within the hour. If the atom decays, the Geiger counter will tick; the triggering of the counter will get the lid off the poison gas bottle, which will kill the cat. If the atom does not decay, none of the above things happen, and the cat will be alive. Now the question:

(b) We then ask: What is the state of the cat after one hour?
The answer according to quantum mechanics is that

(c) the cat is in a state which can be thought of as half-alive and half-dead, that is, the state such as

\[ \frac{\text{Fig.3.2}(\uparrow_1) + \text{Fig.3.2}(\uparrow_2)}{\sqrt{2}} \] (or more theoretically, \[ |\text{Fig.3.2}(\uparrow_1)\rangle + |\text{Fig.3.2}(\uparrow_2)\rangle \].)

![Figure 3.2: Schrödinger’s cat](image)

Of course, this answer (A) is curious. This is the so-called Schrödinger’s cat paradox. This paradox is due to the fact that micro mechanics and macro mechanics are mixed in the above situation. On the other hand, as seen in (2.38), micro mechanics (= quantum measurement theory) and macro mechanics (= classical measurement theory) are always separated in measurement theory. Therefore, Schrödinger’s cat does not appear in the world of measurement theory, though this may be a surface solution of Schrödinger’s cat paradox.

(d) At the moment of the measurement (i.e., at the moment of having opened the window of the box and seeing inside), “alive” or “dead” is determined. That is,

half-alive and half-dead \( \xrightarrow{\text{At the moment of having opened the window of the box and seeing inside}} \) \( \{ \text{alive}, \text{dead} \} \)

This is famous Schrödinger cat paradox. For author’s opinion, see Note 3.6. Here, note that

(e\_1) ”At the moment of the measurement” means that ”At the moment that the signal reach observer’s brain”

The readers may doubt it. However, the dualism is to partition ”brain” and ”matter”.

Thus, the dualism says that

(e\_2) **There is no measurement without our brain.**

Some may think, from the point of the realistic view, that the (e\_2) is absurd. However, our proposal is linguistic. We think that this (e\_2) is also one of the (linguistic) Copenhagen interpretation.


\( \clubsuit \) Note 3.6 The cause of "confusion" is because it is caught by realism. According to the linguistic world-view (i.e., "The limits of my language mean the limits of my world." due to Wittgenstein, Chap. 8 (m)),

(2) Schrödinger’s cat is out of the description of measurement theory, and thus, Schrödinger’s cat does not exist (cf. [14, 16]).

3.4 Heisenberg’s uncertainty principle

3.4.1 The thought experiment by \( \gamma \)-rays microscope

Heisenberg’s uncertainty principle[5] is the following Proposition 3.1 [(i) and (ii)]. This is usually said to be the greatest scientific result in the 20-th century. However, as mentioned is the following section, it is doubtful.

Proposition 3.1 [Heisenberg’s uncertainty relation] (cf. [5]).

(i) The particle position \( q \) and momentum \( p \) can be measured “simultaneously”, if the “errors” \( \Delta(q) \) and \( \Delta(p) \) in determining the particle position and momentum are permitted to be non-zero.

(ii) Moreover, for any \( \epsilon > 0 \), we can take the above “approximate simultaneous” measurement of the position \( q \) and momentum \( p \) such that \( \Delta(q) < \epsilon \) (or \( \Delta(p) < \epsilon \)). However, the following Heisenberg’s uncertainty relation holds:

\[
\Delta(q) \cdot \Delta(p) \geq \frac{\hbar}{2},
\]

for all “approximate simultaneous” measurements of the particle position and momentum.

![Figure 3.3: The thought experiment by \( \gamma \)-rays microscope](image)

Heisenberg’s argument (the thought experiment by \( \gamma \)-rays) is formed by the following two approximate equalities (3.10) and (3.11):

\[
\Delta x = \frac{\lambda}{\sin \epsilon} \quad \text{(the optical resolution (concerning } x \text{-axixs), } \lambda \text{: wavelength)}
\]

(3.10)

Also, as seen in Fig. 3.3, the error of the momentum \( p_x \) (\( x \)-axis direction) is, by Compton recoil, estimated as

\[
\Delta p_x = \frac{h \nu}{c} \sin \epsilon \quad \text{(c: light speed, thus, } c = \lambda \cdot \nu \text{))}
\]

(3.11)
Thus, Heisenberg’s uncertainty principle (3.9) is obtained by
\[ \Delta x \cdot \Delta p_x = (3.10) \times (3.11) \approx \hbar \]
For precise argument, see [32], in which my favorite explanation is written.

However, it should be noted that the above is not argument in the framework of quantum mechanics.

\[ \star \textbf{Note 3.7} \]
We think that Heisenberg’s uncertainty principle (Proposition 3.1) is meaningless. That is because 
For example,
(1) The approximate measurement and ”error” in Proposition 3.1 are not defined.

This will be improved in Theorem 3.4 in the framework of quantum mechanics. That is, Heisenberg’s thought 
experiment is an excellent idea before the discovery of quantum mechanics. Some may ask that 
If it be so, why is Heisenberg’s uncertainty principle (Proposition 3.1) famous?

The author thinks that 
Heisenberg’s uncertainty principle (Proposition 3.1) was used as the slogan for advertisement of quantum 
mechanics in order to emphasize the difference between classical mechanics and quantum mechanics.

3.4.2 The quantum mechanical formulation of Heisenberg’s uncertainty principle

We think that the standard Copenhagen interpretation says that

A measurement is not related to an interaction

Therefore, we do not trust Heisenberg’s thought experiment by γ-rays, which clearly is related to a thing 
like an interaction.

Heisenberg’s uncertainty principle is often misunderstood as Robertson’s uncertainty relation as follows.

\[ \textbf{Theorem 3.2 [Robertson’s uncertainty principle]} \]
For simplicity, put \( H = C^n \). Let \( A, B (\in B(C^n)) \) be Hermitian, therefore, by (3.2), its observable representation is \( O_A, O_B \) respectively. Here, consider 
the measurement \( M_{B(C^n)}(O_A, S_{[\omega]}) \) and \( M_{B(C^n)}(O_B, S_{[\omega]}) \). That is, consider a parallel measurement 
\( M_{B(C^n) \otimes B(C^n)}(O_A \otimes O_B, S_{[\omega \otimes \omega]}) \). Then, it holds that:
\[ \delta^\omega_A \cdot \delta^\omega_B \geq \frac{1}{2} |\langle \omega, (AB - BA)\omega \rangle| \quad (\forall \omega \in \hat{\Omega}(\subset C^n)) \]
where, \( \delta^\omega_A \) and \( \delta^\omega_B \) is defined by (3.3). That is,
\[ \begin{align*}
\delta^\omega_A &= \left[ \langle A\omega, A\omega \rangle - |\langle \omega, A\omega \rangle|^2 \right]^{1/2} \\
\delta^\omega_B &= \left[ \langle B\omega, B\omega \rangle - |\langle \omega, B\omega \rangle|^2 \right]^{1/2}
\end{align*} \]

Of course, the above holds in the case of infinite dimensional Hilbert space. For example, when \( Q \) and \( P \) is respectively the position observable and the momentum observable (i.e., when \( [Q, P] = i\hbar \)), it holds that \( \delta^\omega_Q \cdot \delta^\omega_P \geq \frac{1}{2} \hbar \)

As pointed out in “mathematical foundations of quantum mechanics(1932);[32]”, Robertson’s uncertainty principle is not the mathematical representation of Heisenberg’s uncertainty principle.
Let us begin with ”approximately simultaneous observable”.

**Definition 3.3 [Approximately simultaneous observable, error]** For simplicity, put $H = \mathbb{C}^n$. And let $A, B (\in B(\mathbb{C}^n))$ be Ermitian. Let $X$ and $Y (\subset \mathbb{R})$ be finite sets. The observable $O_{AB} = (X \times Y, 2^{X \times Y}, F_{AB})$ in $B(\mathbb{C}^n)$ is called the approximately simultaneous observable of $A$ and $B$, if it satisfies that

$$\begin{align*}
\langle \omega, A\omega \rangle &= \sum_{x \in X} x \langle \omega, F_{AB}(\{x\} \times Y) \omega \rangle \\
\langle \omega, B\omega \rangle &= \sum_{y \in Y} y \langle \omega, F_{AB}(X \times \{y\}) \omega \rangle \\
&\quad \quad \quad (\forall \omega \in \hat{\Omega}(\subset \mathbb{C}^n))
\end{align*}$$

Further, the errors $\Delta_{A\omega}^\omega$ and $\Delta_{B\omega}^\omega$ of the simultaneous measurement $M_{B(\mathbb{C}^n)}(O_{AB}, S_{[\omega]})$ is respectively defined by

$$\begin{align*}
\Delta_{A\omega}^\omega &= \left[ \sum_{x \in X} x^2 \langle \omega, F_{AB}(\{x\} \times Y) \omega \rangle - \langle A\omega, A\omega \rangle \right]^{1/2} \\
\Delta_{B\omega}^\omega &= \left[ \sum_{y \in Y} y^2 \langle \omega, F_{AB}(X \times \{y\}) \omega \rangle - \langle B\omega, B\omega \rangle \right]^{1/2} \\
&\quad \quad \quad (\forall \omega \in \hat{\Omega}(\subset \mathbb{C}^n))
\end{align*}$$

Now, we can present the mathematical representation of Heisenberg’s uncertainty principle as follows.

**Theorem 3.4 [Heisenberg’s uncertainty principle (cf. [7, 11])]** Put $H = \mathbb{C}^n$. Let $A, B (\in B(\mathbb{C}^n))$ be Hermitian. Then it holds that

(i) A simultaneous measurement $M_{B(\mathbb{C}^n)}(O_{AB}, S_{[\omega]})$ of $A$ and $B$ exists.

(ii) Further, Heisenberg’s uncertainty principle holds as follows.

$$\Delta_{A\omega}^\omega \cdot \Delta_{B\omega}^\omega \geq \frac{1}{2} |\langle \omega, (AB - BA)\omega \rangle| \\
(\forall \omega \in \hat{\Omega}(\subset \mathbb{C}^n))$$

(ii)' Of course, the above holds in the case of infinite dimensional Hilbert space. For example, when $Q$ and $P$ is respectively the position observable and the momentum observable (i.e., when $QP - PQ = \hbar \sqrt{-1}$), it holds that

$$\Delta_{Q\omega}^\omega \cdot \Delta_{P\omega}^\omega \geq \frac{1}{2} \hbar \\
(\forall \omega \in \hat{\Omega}(\subset \mathbb{H}))$$

(3.12)

▲ **Note 3.8** As mentioned in Note 3.2, quantum syllogism does not hold. However, it should be noted that Heisenberg’s uncertainty principle and the result concerning syllogism(♯1)–(♯3) in (Theorem 3.1) do not contradict. That is because Heisenberg’s uncertainty principle is related to an approximate measurement.

What we did in this section is

The theory that is not described by quantum mechanics quantum mechanics (i.e., Heisenberg’s uncertainty principle(Proposition 3.1) ) is described in quantum mechanics such as Heisenberg’s uncertainty principle(Theorem 3.4).
Replacing "quantum mechanics" to "measurement theory", we have the spirit such that

**Our standing-point 3.5 [Chap. 1(X)]** Thus, outstanding point is as follows.

(♯) **The theory described in ordinary language should be described in measurement theory.**

In the following chapters, the readers will find that

(♯2) Many ambiguous theories can be automatically solved if they are described in measurement theory.

**Note 3.9** The author proceeded

from quantum mechanics (cf. [7]) to classical measurement theory ([8, 9]).

Thus, I was convinced at the time of the beginning as follows.

(♯) Unless we know quantum mechanics, we can not understand classical measurement theory.

However, it is not true. This fact was taught by the students of my seminar. That is because without the knowledge of quantum mechanics, we can understand and use measurement theory. In fact, the undergraduate students in my seminar can understand measurement theory without the knowledge of quantum mechanics. Thus, the author studied, from my students the following fact (= the main theme of this print = linguistic world view):

(♯2) Even if we do not know "monkey" and "tree", we can use the proverb "Even monkeys fall from trees".

(Continued to Sec. 8.1.2)

## 4 Fisher statistics I

Measurement theory (continuous pure type) is formulated as follows.

\[
\begin{align*}
\text{measurement theory} & = \frac{[\text{Axiom}^P 1]}{\text{measurement}} + \frac{[\text{Axiom}^P 2]}{\text{causality}} \\
& \text{(scientific language) [probabilistic interpretation] [the Heisenberg picture]}
\end{align*}
\]

In Chap. 2, we explained Axiom$^P 1$. In this chapter, Fisher statistics is described in terms of Axiom$^P 1$. The term "Fisher statistics" is used in order to distinguish Bayesian statistics, which will be introduced as mixed measurement theory in Sec. 4.4.

### 4.1 Why is statistics useful in science?

#### 4.1.1 Is the foundations of statistics firm?

Statistics is quite important discipline. Statistics is indispensable for life insurance, DNA identification of a trial. the determination of the economic policy of a country, etc. Therefore, statistics has to be regarded as "the discipline with absolute authority". However, from the view-point of world-description, statistics is not firm. That is because the following question is not yet answered:

What kind of world-view is statistics due?
That is, Our standing-point 3.5(=Chap. 1(X4)) says that

Every engineering (or, science) should be described by measurement theory (i.e., Axiom P 1 and 2).

If it be so, what we have to do is

Statistical methods — Fisher maximum likelihood method, confidence interval, statistical hypothesis testing, Bayes’ method, etc. — are described in terms of measurement theory

This will be done in this chapter. Also, this means to answer the problem:

Why is statistics useful in science?

4.1.2 Trial and measurement

The term ”trial” is studied in mathematics of the high school. However, the following question is not easy.

Is the term ”trial” a mathematical term?

Although it is not easy, in what follows we say something.

Let \((X, \mathcal{F}, P)\) be a probability space. Here, \(X\) is called a sample space, and its element is said to be a sample. Following common sense, we define the ”trial” as follows.

the ”trial” is an experiment repeatable repeatedly such as ”coin-tossing”, ”throwing dice”, etc.

By the trial, a sample \(x \in X\) is obtained. When a sample belongs to \(\Xi \in \mathcal{F}\), an event \(\Xi \in \mathcal{F}\) is said to happen.

Now, we think that the following three sentences (A1)–(A3) are same:

(A1) The probability that an event \(\Xi \in \mathcal{F}\) happens is given by \(P(\Xi)\).

(A2) When a trial is taken, the probability that a sample belongs to an event \(\Xi \in \mathcal{F}\) is given by \(P(\Xi)\).

(A3) When a trial \((X, \mathcal{F}, P)\) is taken, the probability that a sample belongs to an event \(\Xi \in \mathcal{F}\) is given by \(P(\Xi)\).

Since a trial is repeatable, we can get a sample data, and thus, a sample probability space \((X, \mathcal{F}, P)\).

In the statement (A3), a trial \((X, \mathcal{F}, P)\) and a sample probability space \((X, \mathcal{F}, P)\) overlap. Thus, the is not usual, but we adopt often the (A3) in this print.

\(\blacklozenge\) **Note 4.1** (A1) may be mathematical, on the other hand, (A3) may be linguistic. However, these can not be clarified without the measurement theoretical view-point.

Let \(\Omega\) be a set, which is called a parameter space. and , and . For each parameter \(\omega \in \Omega\), define a trial \((X, \mathcal{F}, P_\omega)\) and consider a family of trials \(\{(X, \mathcal{F}, P_\omega)\}_{\omega \in \Omega}\). The, we think that the following three statements (B1)–(B3) are the same.:
(B_1) Let \( \omega_0 \in \Omega \). A trial \((X, \mathcal{F}, P_{\omega_0})\) is taken, the probability that a sample belongs to an event \( \Xi(\in \mathcal{F}) \) is given by \( P_{\omega_0}(\Xi) \).

(B_2) Let \( \omega_0 \in \Omega \). When, for a population \( S_{[\omega_0]} \), a trial \((X, \mathcal{F}, P_{\omega_0})\) is taken, the probability that a sample belongs to an event \( \Xi(\in \mathcal{F}) \) is given by \( P_{\omega_0}(\Xi) \).

(B_3) When a trial \( T(\{(X, \mathcal{F}, P_{\omega})\}_{\omega \in \Omega}, S_{[\omega_0]}) \) is taken, the probability that a sample belongs to an event \( \Xi(\in \mathcal{F}) \) is given by \( P_{\omega_0}(\Xi) \).

Similarly,, a sample probability space \((X, \mathcal{F}, P_{\omega_0})\) is obtained.

In the above (B_1)–(B_3), the term ”trial” is used in confusion, however, we expect readers to read these such as

\[(B_1) = (B_2) = (B_3)\]

Although the statement (B_3) may be familiar, we adopt the (B_3). And we present Axiom\( ^{\text{p}_1} \) 1( trial type) as follows.

**Axiom(T) 1 (trial version)**

When a trial \( T(\{(X, \mathcal{F}, P_{\omega})\}_{\omega \in \Omega}, S_{[\omega_0]}) \) is taken, the probability that a sample belongs to an event \( \Xi(\in \mathcal{F}) \) is given by \( P_{\omega_0}(\Xi) \).

This as well as Axiom\( ^{\text{p}_1} \) 1(23 page) is linguistic. Since we prepare (A_1)–(B_3), readers are expected to understand it as ”linguistic” than ”mathematical”.

The following example make readers understand the delicate difference between ”measurement” and ”trial”.

**Example 4.1 [Example 2.10(urn problem):measurement and trial]** Again consider Example 2.10(urn problem). There are two urns \( U_1 \) and \( U_2 \). The urn \( U_1 \) [resp. \( U_2 \)] contains 8 white and 2 black balls [resp. 4 white and 6 black balls] (Fig. 4.1).

![Figure 4.1: Urn problem(=Fig. 2.5)](image)

Like Example 2.10, consider the following “statement (a)”:

(a) When one ball is picked up from the urn \( U_2 \), the probability that the ball is white is 0.4.
Now, let us describe the (a) in terms of "measurement" and "trial".

[I: Description by measurement]
This was already mentioned in Example 2.10 (urn problem).

[II: Description by trial]
Let \( \Omega = \{\omega_1, \omega_2\} \) be a parameter space. Consider the following identification:

\[
\text{Urn } U_1 \approx \text{parameter } \omega_1, \quad \text{Urn } U_2 \approx \text{parameter } \omega_2
\]

Define a trial \( \{\{(w, b), 2\{w, b\}, P_\omega\}\}_{\omega \in \Omega} \) by

\[
\begin{align*}
P_{\omega_1}(\{w\}) &= 0.8, \\
P_{\omega_1}(\{b\}) &= 0.2, \\
P_{\omega_2}(\{w\}) &= 0.4, \\
P_{\omega_2}(\{b\}) &= 0.6
\end{align*}
\]

Therefore, the statement (a) is, by Axiom 1(trial), described as follows.

(b) When a trial \( T(\{(w, b), 2\{w, b\}, P_\omega\}_{\omega \in \Omega}, S_{\{\omega_2\}}) \) is taken, the probability that a sample \( \begin{pmatrix} w \\ b \end{pmatrix} \) is obtained is given by

\[
\begin{pmatrix}
P_{\omega_2}(\{w\}) \\
P_{\omega_2}(\{b\})
\end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix}
\]

The similarity between [I: Description by measurement] and [II: Description by trial] is due to the following theorem (Theorem 4.2).

**Theorem 4.2** A trial and a classical measurement are mathematically equivalent.

**Proof.** Consider a trial \( T(\{(X, F, P_\omega)\}_{\omega \in \Omega}, S_{\{\omega_2\}}) \) and a measurement \( M_{C(\Omega)}(O=(X, F, S), S_{\{\omega_2\}}) \) such that

\[
P_\omega(\Xi) = [F(\Xi)](\omega) \quad (\forall \Xi \in F, \omega \in \Omega)
\]

This completes the proof. \( \square \)

\( \blacklozenge \) **Note 4.2** Theorem 4.2 says "trial= classical measurement" as a mathematical structure. For example,

(2) A trial is repeatable, but only one measurement is permitted.

Still, the (2) is not problem if we introduce a parallel measurement (Sec. 2.5.2). Further, the spirit of Kolmogorov extension theorem — regarding many trials as one trial — is caused by the Copenhagen interpretation (only one measurement is permitted.) (cf [13, 16]). However, the spirit of measurement theory (linguistic world-view [Chap. 1(I)] and the Copenhagen interpretation [(U_1)-(U_7)] is omitted in a trial. Thus we consider that "trial" is within ordinary language. Again consider the \( (X_1) \) (in Chap. 1 and Note 2.4):

\[
\begin{array}{c}
(X_1) \\
(\text{Chap. 1}) \\
(\text{Ordinary language})
\end{array}
\begin{array}{c}
\oplus \text{widely ordinary language} \\
(\text{Before science})
\end{array}
\Rightarrow \begin{array}{c}
\text{world-description} \\
(\text{Chap. 1 (O)})
\end{array}
\begin{array}{c}
\blacklozenge \text{realistic method} \\
\circ \text{linguistic method}
\end{array}
\]

where the trial is located in \( \oplus \). Also, the following is important:

(2) state and observable are indispensable in measurement theory, and thus, it is connected to quantum mechanics. Since the trial is not connected to quantum mechanics, the problem "monism or dualism?" is neglected in the trial.

Therefore, the overestimation of Theorem 4.2 (mathematical equivalence) must be avoided.
4.2 Fisher consider Born’s reverse

As shown in Theorem 4.2, measurement and statistics (trial) are similar. Therefore, we can expect that statistical methods can be described in terms of measurement theory. In what follows, this will be done.

4.2.1 Inference problem

**Problem 4.3** [The urn problem by Fisher’s maximum likelihood method].

There are two urns \( U_1 \) and \( U_2 \). The urn \( U_1 \) [resp. \( U_2 \)] contains 8 white and 2 black balls [resp. 4 white and 6 black balls].

Here consider the following procedures (i) and (2).

(i) One of the two (i.e., \( U_1 \) or \( U_2 \)) is chosen and is settled behind a curtain. Note, for completeness, that you do not know whether it is \( U_1 \) or \( U_2 \).

(ii) Pick up a ball out of the urn chosen by the procedure \( (A_1) \). And you find that the ball is white.

Here, we have the following problem:

(iii) Infer the probability that the ball obtained in the above \( (A_3) \) is white (or, black)?

You do not know which the urn behind the curtain is, \( U_1 \) or \( U_2 \).

Assume that you pick up a white ball from the urn.

The urn is \( U_1 \) or \( U_2 \)? Which do you think?

![Figure 4.2: Which is the hidden urn, \( U_1 \) or \( U_2 \)?](image)

The answer is easy, that is, thr urn behind the curtain is \( U_1 \). That is because the urn \( U_1 \) has more white balls than \( U_2 \). It is too easy, but it includes the essence of Fisher maximim likelihood method.

4.2.2 Fisher maximim likelihood method in measurement theory

We begin with the following definition.

**Notation 4.4** \( [M_{\mathcal{A}}(O, S_{[\rho^P]}))] \). Consider a measurement \( M_{\mathcal{A}}(O \equiv (X, \mathcal{F}, F), S_{[\rho^P]}) \) formulated in a \( C^* \)-algebra \( \mathcal{A} \). In most measurements, it is usual to think that the state \( \rho^P (\in \mathbb{S}^{P}(A^*)) \) is unknown. That is because the measurement \( M_{\mathcal{A}}(O, S_{[\rho^P]}) \) may be taken in order to know the state \( \rho^P \). Thus, when we want to stress that we do not know the state \( \rho^P \), the measurement \( M_{\mathcal{A}}(O, S_{[\rho^P]}) \) is often denoted by \( M_{\mathcal{A}}(O, S_{[\ast]}) \).
Using this notation, we characterize our problem (i.e., inference) as follows.

(a) Assume that a measured value obtained by a measurement $\mathcal{M}_{\mathcal{O}}(\Omega=(X, \mathcal{F}, F), S_{\omega})$ belongs to $\Xi(\in \mathcal{F})$. Then, infer the unknown state $[s] (\in \Omega)$

Therefore, the measurement is "the view from the front", that is,

(b) \((\text{observable}[\mathcal{O}], \text{state}[\omega(\in \Omega)]) \xrightarrow{\text{measurement}} \text{measured value}[x(\in X)]\)

On the other hand, the inference is "the view from the back", that is,

(c) \((\text{observable}[\mathcal{O}], \text{measured value}[x(\in \Xi(\in \mathcal{F}))]) \xrightarrow{\text{inference}} \text{state}[\omega(\in \Omega)]\)

In this sense, the inference problem is the reverse problem of measurement. Therefore, it suffices to image Fig. 4.3.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure43.png}
\caption{The image of inference}
\end{figure}

In order to answer the above problem (a), we shall describe Fisher maximum likelihood method in terms of measurement theory.

**Theorem 4.5** [Fisher maximum likelihood method (measurement theoretical representation) (cf. [10, 20])] Consider a measurement $\mathcal{M}_{\mathcal{O}}(\Omega=(X, \mathcal{F}, F), S_{\omega})$.

Assume that we know that a measured value obtained by a measurement $\mathcal{M}_{\mathcal{O}}(\Omega=(X, \mathcal{F}, F), S_{\omega})$ belongs to $\Xi (\in \mathcal{F})$. Then, there is a reason to infer that the unknown state state $[s]$ is $\omega_0 (\in \Omega)$ such that

$$[F(\Xi)](\omega_0) = \max_{\omega \in \Omega} [F(\Xi)](\omega)$$

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure44.png}
\caption{Fisher maximum likelihood method}
\end{figure}
Proof. Let $\omega_1$ and $\omega_2$ be elements in $\Omega$ such that $[F(\Xi)](\omega_1) < [F(\Xi)](\omega_2)$. Thus, by Axiom 1 (measurement),

(i) the probability that a measured value obtained by a measurement $M_{C(\Omega)}(O, S_{\omega_1})$ belongs to $\Xi$ is equal to $[F(\Xi)](\omega_1)$

(ii) the probability that a measured value obtained by a measurement $M_{C(\Omega)}(O, S_{\omega_2})$ belongs to $\Xi$ is equal to $[F(\Xi)](\omega_2)$

Since we assume that $[F(\Xi)](\omega_1) < [F(\Xi)](\omega_2)$, we can conclude that "(i) is more rare than (ii)". Thus, there is a reason to infer that $[\ast] = \omega_2$.

\[\square\]

\textbf{Note 4.3} Fisher maximum likelihood method in statistics is easily obtained if a measurement $M_{C(\Omega)}(O = (X, F, \mathcal{P}), S_{\omega_0})$ is replaced by a trial $T(\{(X, F, \mathcal{P})\}_{\omega_0} \in \Omega, S_{\omega_0})$ in Theorem 4.5.

\textbf{Answer 4.6} [The measurement theoretical answer to Problem 4.3] Consider a measurement $M_{C(\Omega)}(O = (\{w, b\}, 2^{\{w, b\}}, F), S_{\ast})$ in Example 2.10. The formula (2.5) says that

\[\max\{[F(\{w\}))(\omega_1), [F(\{w\}))(\omega_2)\} = \max\{0.8, 0.4\} = 0.8 = F(\{w\}))(\omega_1)\]

Therefore, Theorem 4.5 says that the urn behind the curtain is $U_1$.

\[\square\]

\textbf{Note 4.4} As seen in Fig. 4.3, inference (Fisher maximum likelihood method) is the reverse of measurement. Here note that

Born’s discovery "the probabilistic interpretation of quantum mechanics" (42 page) : [2] (1926)

Fisher’s great book "Statistical Methods for Research Workers" (1925)

Thus, it is surprising that Fisher and Born considered the same thing in the different fields in the same age.

\section{4.3 Statistical methods in measurement theory}

\subsection{4.3.1 Examples of Fisher maximum likelihood method}

\textbf{Example 4.7} [Urn problem] Each urn $U_1$, $U_2$, $U_3$ contains white balls and black ball such as:

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
w-b \hspace{1cm} & \textbf{Urn} $U_1$ & \textbf{Urn} $U_2$ & \textbf{Urn} $U_3$
\hline
white ball & 80\% & 40\% & 10\%
\hline
black ball & 20\% & 60\% & 90\%
\hline
\end{tabular}
\end{table}

Here,

(i) one of three urns is chosen, but you do not knot it. Pick up one ball from the unknown urn. And you find that its ball is white. Then, How do you infer the unknow urn, i.e., $U_1$, $U_2$ or $U_3$?

Further,

(ii) And further, you pick up another ball from the unknown urn. And you find that its ball is black. That i, after all, you have one white ball and one black ball. Then, How do you infer the unknw urn, i.e., $U_1$, $U_2$ or $U_3$?
In what follows, we shall answer the above problems (i) and (ii) in terms of measurement theory. Put

\[ \omega_j \longleftrightarrow \text{[the state such that urn } U_j \text{ is chosen]} \quad (j = 1, 2, 3) \]

Thus, we have the state space \( \Omega = \{\omega_1, \omega_2, \omega_3\} \). Further, define the observable \( O = (\{w, b\}, 2^{\{w, b\}}, F) \) in \( C(\Omega) \) such that

\[
\begin{align*}
F(\{w\})(\omega_1) &= 0.8, & F(\{w\})(\omega_2) &= 0.4, & F(\{w\})(\omega_3) &= 0.1 \\
F(\{b\})(\omega_1) &= 0.2, & F(\{b\})(\omega_2) &= 0.6, & F(\{b\})(\omega_3) &= 0.9
\end{align*}
\]

**Answer to (i):** Consider the measurement \( M_{C(\Omega)}(O, S_{[\ast]}) \), by which a measured value “\( w \)” is obtained. Therefore, we see

\[ [F(\{w\})](\omega_1) = 0.8 = \max_{\omega \in \Omega} [F(\{w\})](\omega) = \max\{0.8, 0.4, 0.1\} \]

Thus, by Fisher maximum likelihood method (Theorem 4.5), we see that

\[ [\ast] = \omega_1 \]

Thus, we can infer that the unknown urn is \( U_1 \).

**Answer to (ii):** Next, consider the simultaneous measurement \( M_{C(\Omega)}(\times_{k=1}^2 O, S_{[\ast]}) \), by which a measured value \( (w, b) \) is obtained. Here, we see

\[ [\hat{F}(\{(w, b)\})](\omega_1) = 0.16, \quad [\hat{F}(\{(w, b)\})](\omega_2) = 0.24, \quad [\hat{F}(\{(w, b)\})](\omega_3) = 0.09 \]

Thus, by Fisher maximum likelihood method (Theorem 4.5), we see that

\[ [\ast] = \omega_2 \]

Thus, we can infer that the unknown urn is \( U_2 \).

**Example 4.8 [Normal observable(i)]** As mentioned in Example 2.11, consider the normal observable \( O_{G_\sigma} = (\mathbb{R}, B_\mathbb{R}, G_\sigma) \) in \( C(\mathbb{R}) \) (where \( \Omega = \mathbb{R} \)) such that

\[
G_\sigma(\Xi)(\mu) = \frac{1}{\sqrt{2\pi}\sigma} \int_{\Xi} \exp\left[ -\frac{1}{2\sigma^2}(x - \mu)^2 \right] dx
\]

\[(\forall \Xi \in B_\mathbb{R}, \quad \forall \mu \in \Omega = \mathbb{R})\]

Thus, the simultaneous observable \( \times_{k=1}^3 O_{G_\sigma} \) (in short, \( O_{G_\sigma}^3 \)) = \( (\mathbb{R}^3, B_{\mathbb{R}^3}, G_\sigma^3) \) in \( C(\mathbb{R}) \) is defined by
Thus, we get the measurement $M_C(\mathbb{R}, O, G^3_3, S)$

Now we consider the following problem:

(a) By the measurement $M_C(\mathbb{R}, O, G^3_3, S)$ assume that a measured value $(x_0^1, x_0^2, x_0^3) \in \mathbb{R}^3$ is obtained.

Then, infer the unknown state $[\ast] \in \mathbb{R}$.

**Answer (a)**

Put

$$\Xi_i = \left[x_i^0 - \frac{1}{N}, x_i^0 + \frac{1}{N}\right] \quad (i = 1, 2, 3)$$

Assume that $N$ is sufficiently large. Fisher maximim likelihood method (Theorem 4.5) says that the unknown state $[\ast] = \mu_0$ is found in what follows.

$$[G^3_3(\Xi_1 \times \Xi_2 \times \Xi_3)](\mu_0) = \max_{\mu \in \mathbb{R}} [G^3_3(\Xi_1 \times \Xi_2 \times \Xi_3)](\mu)$$

Since $N$ is sufficiently large, we see

$$\frac{1}{(\sqrt{2\pi} \sigma)^3} \exp\left[ -\frac{(x_0^1 - \mu_0)^2 + (x_0^2 - \mu_0)^2 + (x_0^3 - \mu_0)^2}{2\sigma^2} \right] = \max_{\mu \in \mathbb{R}} \left[ \frac{1}{(\sqrt{2\pi} \sigma)^3} \exp\left[ -\frac{(x_0^1 - \mu)^2 + (x_0^2 - \mu)^2 + (x_0^3 - \mu)^2}{2\sigma^2} \right] \right]$$

That is,

$$(x_0^1 - \mu_0)^2 + (x_0^2 - \mu_0)^2 + (x_0^3 - \mu_0)^2 = \min_{\mu \in \mathbb{R}} \{ (x_0^1 - \mu)^2 + (x_0^2 - \mu)^2 + (x_0^3 - \mu)^2 \}$$

Therefore, solving $\frac{d}{d\mu} \{ \cdots \} = 0$, we conclude that

$$\mu_0 = \frac{x_0^1 + x_0^2 + x_0^3}{3}$$

**[Normal observable (ii)]** Next, consider the case:

we know that the length of the pencil $\mu$ is satisfied that $10 \text{cm} \leq \mu L \text{ cm} \leq 30$.

And we assume that

(2) the length of the pencil $\mu$ and the roughness $\sigma$ of the ruler are unknown.
That is, assume that the state space $\Omega = [10, 30] \times \mathbb{R}_+$ \(\{\mu \in \mathbb{R} \mid 10 \leq \mu \leq 30\} \times \{\sigma \in \mathbb{R} \mid \sigma > 0\}\).

Define the observable $O = (\mathbb{R}, B_{\mathbb{R}}, G)$ in $C([10, 30] \times \mathbb{R}_+)$ such that

\[
G(\Xi)(\mu, \sigma) = [G_\sigma(\Xi)](\mu) \quad (\forall \Xi \in B_{\mathbb{R}}, \forall (\mu, \sigma) \in \Omega = [10, 30] \times \mathbb{R}_+)
\]

Therefore, the simultaneous observable $O^3 = (\mathbb{R}^3, B_{\mathbb{R}^3}, G^3)$ in $C([10, 30] \times \mathbb{R}_+)$ is defined by

\[
\frac{1}{(\sqrt{2\pi}\sigma)^3} \int_{\Xi_1 \times \Xi_2 \times \Xi_3} \exp\left[-\frac{(x_1 - \mu)^2 + (x_2 - \mu)^2 + (x_3 - \mu)^2}{2\sigma^2}\right] dx_1 dx_2 dx_3
\]

(\forall \Xi_k \in B_{\mathbb{R}}, k = 1, 2, 3, \forall (\mu, \sigma) \in \Omega = [10, 30] \times \mathbb{R}_+)

Thus, we get the simultaneous measurement $M_{C([10,30] \times \mathbb{R}_+)}(O^3, S_{[\cdot]})$. Here, we have the following problem:

(b) When a measured value \((x_0^1, x_0^2, x_0^3) \in \mathbb{R}^3\) is obtained by the measurement $M_{C([10,30] \times \mathbb{R}_+)}(O^3, S_{[\cdot]})$,

infer the unknown state \([\cdot] = (\mu_0, \sigma_0) \in [10, 30] \times \mathbb{R}_+\), i.e., the length $\mu_0$ of the pencil and the roughness $\sigma_0$ of the ruler.

**Answer (b)** By the same way of (a), Fisher maximim likelihood method(Theorem 4.5) says that the unknown state \([\cdot] = (\mu_0, \sigma_0)\) such that

\[
\frac{1}{(\sqrt{2\pi}\sigma)^3} \exp\left[-\frac{(x_0^1 - \mu)^2 + (x_0^2 - \mu)^2 + (x_0^3 - \mu)^2}{2\sigma^2}\right] = \max_{(\mu,\sigma)\in[10,30] \times \mathbb{R}_+} \left\{\frac{1}{(\sqrt{2\pi}\sigma)^3} \exp\left[-\frac{(x_0^1 - \mu)^2 + (x_0^2 - \mu)^2 + (x_0^3 - \mu)^2}{2\sigma^2}\right]\right\}
\]

Thus, solving $\frac{\partial}{\partial \mu} \{\cdots\} = 0, \frac{\partial}{\partial \sigma} \{\cdots\} = 0$ we see

\[
\begin{align*}
\mu_0 &= \begin{cases} 
10 & \text{ (when } (x_0^1 + x_0^2 + x_0^3)/3 < 10 \text{)} \\
(x_0^1 + x_0^2 + x_0^3)/3 & \text{ (when } 10 \leq (x_0^1 + x_0^2 + x_0^3)/3 \leq 30 \text{)} \\
30 & \text{ (when } 30 < (x_0^1 + x_0^2 + x_0^3)/3 \text{)} 
\end{cases} \\
\sigma_0 &= \sqrt{(x_0^1 - \bar{\mu})^2 + (x_0^2 - \bar{\mu})^2 + (x_0^3 - \bar{\mu})^2}/3
\end{align*}
\]

where

\[
\bar{\mu} = (x_0^1 + x_0^2 + x_0^3)/3
\]

\(\blacksquare\)

### 4.3.2 Monty Hall problem — High school students’ puzzle

The Monty Hall problem is well-known and elementary. Also it is famous as the problem in which even great mathematician P. Erdős made a mistake (cf. \([6]\)). The Monty Hall problem is as follows:

**Problem 4.9** [Monty Hall problem (cf. \([11, 19]\))] You are on a game show and you are given the choice of three doors. Behind one door is a car, and behind the other two are goats. You choose, say, door
1, and the host, who knows where the car is, opens another door, behind which is a goat. For example, the host says that

(b) the door 3 has a goat.

And further, He now gives you the choice of sticking with door 1 or switching to door 2? What should you do?

![Figure 4.5: Monty Hall problem](image)

**Answer:** Put $\Omega = \{\omega_1, \omega_2, \omega_3\}$ with the discrete topology. Assume that each state $\delta_{\omega_m} (\in \mathfrak{S}^p (C(\Omega)^*))$ means

$$\delta_{\omega_m} \iff \text{the state that the car is behind the door } 1 \quad (m = 1, 2, 3)$$

Define the observable $O_1 \equiv (\{1, 2, 3\}, 2^{\{1, 2, 3\}}, F_1)$ in $C(\Omega)$ such that

$$[F_1(\{1\})](\omega_1) = 0.0, \quad [F_1(\{2\})](\omega_1) = 0.5, \quad [F_1(\{3\})](\omega_1) = 0.5,$$

$$[F_1(\{1\})](\omega_2) = 0.0, \quad [F_1(\{2\})](\omega_2) = 0.0, \quad [F_1(\{3\})](\omega_2) = 1.0,$$

$$[F_1(\{1\})](\omega_3) = 0.0, \quad [F_1(\{2\})](\omega_3) = 1.0, \quad [F_1(\{3\})](\omega_3) = 0.0, \quad (4.2)$$

where it is also possible to assume that $F_1(\{2\})(\omega_1) = \alpha, F_1(\{3\})(\omega_1) = 1 - \alpha (0 < \alpha < 1)$. The fact that you say “the door 1” means that we have a measurement $M_{C(\Omega)}(O_1, S_{[s]})$. Here, we assume that

a) “a measured value 1 is obtained by the measurement $M_{C(\Omega)}(O_1, S_{[s]})$”

$\iff$ The host says “Door 1 has a goat”

b) “measured value 2 is obtained by the measurement $M_{C(\Omega)}(O_1, S_{[s]})$ ”

$\iff$ The host says “Door 2 has a goat”

c) “measured value 3 is obtained by the measurement $M_{C(\Omega)}(O_1, S_{[s]})$ ”

$\iff$ The host says “Door 3 has a goat”

Recall that, in Problem 1, the host said “Door 3 has a goat.” This implies that you get the measured value “3” by the measurement $M_{C(\Omega)}(O_1, S_{[s]})$. Therefore, Theorem 1 (Fisher’s maximum likelihood method) says that you should pick door number 2. That is because we see that

$$[F_1(\{3\})](\omega_2) = 1.0 = \max\{0.5, \quad 1.0, \quad 0.0\} = \max\{[F_1(\{3\})](\omega_1), [F_1(\{3\})](\omega_2), [F_1(\{3\})](\omega_3)\},$$
and thus, there is a reason to infer that $[\ast] = \delta_{\omega_2}$. Thus, you should switch to door 2. This is the first answer to Problem 1 (Monty-Hall problem).

\[ \square \]

\textbf{Note 4.5} The above answer is one of Answers of Monty Hall problem. Of course, the answer based on Bayes’ theorem — Problem 4.16 — is usual. However, it is temporary. Our final answer is presented in Problem 6.20 [Monty Hall problem] in Sec. 6.4.5.

### 4.3.3 Confidence interval

Let $O(\equiv (X, F, F))$ be an observable formulated in a basic algebra $C(\Omega)$. Assume that $X$ has a metric $d_X$. And assume that the state space $\Omega_0$ has the metric $d_{\Omega_0}$. Let $E : X \to \Omega$ be a continuous map, which is called “estimator.” Let $\gamma$ be a real number such that $0 < \gamma < 1$, for example, $\gamma = 0.95$. For any $\omega_0(\in \Omega_0)$, define the positive number $\eta^\gamma_{\omega_0}$ ($> 0$) such that:

$$\eta^\gamma_{\omega_0} = \inf\{\eta > 0 : [F(E^{-1}(B(\omega; \eta)))(\omega) \geq \gamma]\}$$

where $B(\omega; \eta) = \{\omega_1(\in \Omega_1) : d_{\Omega_1}(\omega_1, \omega) \leq \eta\}$. For any $x(\in X)$, put

$$D^\gamma_x = \{\omega(\in \Omega) : d_{\Omega}(E(x), \omega) \leq \eta^\gamma_{\omega_0}\}. \quad (4.3)$$

![Figure 4.6: Inference interval $D^\gamma_{x_0}$](image)

The $D^\gamma_x$ is called the $(\gamma)$-inference interval of the measured value $x$.

The following is clear:

for any $\omega_0(\in \Omega_0)$, the probability, that the measured value $x$ obtained by the measurement $M_{C(\Omega)}(O := (X, F, F), S_{[\omega_0]})$ satisfies the following condition (b), is larger than $\gamma$ (e.g., $\gamma = 0.95$).

(a) $E(x) \in B(\omega_0; \eta^\gamma_{\omega_0})$ or equivalently, $d(E(x), \omega_0) \leq \eta^\gamma_{\omega_0}$.

Assume that we get a measured value $x_0$ by the measurement $M_{C(\Omega)}(O := (X, F, F), S_{[\omega_0]})$. Then, we see the following equivalences:

(b) $\iff d_{\Omega}(E(x_0), \omega_0) \leq \eta^\gamma_{\omega_0} \iff D^\gamma_{x_0} \ni \omega_0$. 
Summing the above argument, we have the following theorem.

**Theorem 4.10 [Inference interval (cf. [15, 20])]** Let \( O := (X, \mathcal{F}, F) \) be an observable in \( C(\Omega) \). Let \( \omega_0 \) be any fixed state, i.e., \( \omega_0 \in \Omega \). Consider a measurement \( M_{C(\Omega)}(O := (X, \mathcal{F}, F), S_{[\omega_0]}) \). Let \( E : X \to \Omega \) be an estimator. Let \( \gamma \) be such as \( 0 \ll \gamma < 1 \) (e.g., \( \gamma = 0.95 \)). For any \( x(\in X) \), define \( D^\gamma_x \) as in (8). Then, we see,

(c) the probability that the measured value \( x_0(\in X) \) obtained by the measurement \( M_{C(\Omega)}(O := (X, \mathcal{F}, F), S_{[\omega_0]}) \) satisfies the condition that

\[ D^\gamma_{x_0} \ni \omega_0, \]

is larger than \( \gamma \).

**Example 4.11 [Urn problem]** Put \( \Omega = [0, 1] \), i.e., the closed interval in \( \mathbb{R} \). We assume that each \( \omega(\in \Omega \equiv [0, 1]) \) represents an urn that contains a lot of black balls and white balls such that:

\[
\text{the number of white balls in the urn } \omega \\
\text{the total number of balls in the urn } \omega
\approx \omega \quad (\forall \omega \in [0, 1] \equiv \Omega).
\]

Define the observable \( O = (X \equiv \{b, w\}, \mathcal{P}(\{b, w\}), F) \) in \( C(\Omega) \) such that

\[
F(\emptyset)(\omega) = 0, 
F(\{b\})(\omega) = \omega, 
F(\{w\})(\omega) = 1 - \omega, 
F(\{b, w\})(\omega) = 1 \\
(\forall \omega \in [0, 1] \equiv \Omega).
\]

Here, consider the following measurement \( M_\omega \):

\[ M_\omega := \text{“Pick out one ball from the urn } \omega, \text{ and recognize the color of the ball”} \]

That is, we consider

\[ M_\omega = M_{C(\Omega)}(O, S_{[\delta_\omega]}). \]

Moreover we define the product observable \( O^N \equiv (X^N, \mathcal{P}(X^N), F^N) \), such that:

\[
[F^N(\Xi_1 \times \Xi_2 \times \cdots \times \Xi_{N-1} \times \Xi_N)](\omega) = [F(\Xi_1)](\omega) \cdot [F(\Xi_2)](\omega) \cdots [F(\Xi_N)](\omega) \\
(\forall \omega \in \Omega \equiv [0, 1], \forall \Xi_1, \Xi_2, \cdots, \Xi_N \subseteq X \equiv \{b, w\}).
\]

Note that

“ take a measurement \( M_\omega \) \( N \) times” \( \Leftrightarrow “ \text{ take a measurement } M_{C(\Omega)}(O^N, S_{[\delta_\omega]})” \)

Define the estimator \( E : X^N(\equiv \{b, w\}^N) \to \Omega(\equiv [0, 1]) \)
\[
E(x_1, x_2, \cdots, x_{N-1}, x_N) = \frac{\sharp\{n \in \{1, 2, \cdots, N\} \mid x_n = b\}}{N}

(\forall x = (x_1, x_2, \cdots, x_{N-1}, x_N) \in X^N \equiv \{b, w\}^N).
\]

For each \(\omega (\in [0, 1] \equiv \Omega)\), define the positive number \(\eta_\omega^\gamma\) such that:

\[
\eta_\omega^\gamma = \inf \left\{ \eta > 0 \mid |F_N^{\omega}\{\{x_1, x_2, \cdots, x_N\} \mid \omega - \eta \leq E(x_1, x_2, \cdots, x_N) \leq \omega + \eta\}\} \right\} > 0.95
\]

Put

\[
D_\omega^\gamma = \{\omega (\in \Omega) : |E(x) - \omega| \leq \eta_\omega^\gamma\}.
\]

For example, assume that \(N\) is sufficiently large and \(\gamma = 0.95\). Then we see, from the property of binomial distribution, that

\[
\eta_\omega^{0.95} \approx 1.96 \sqrt{\frac{\omega(1-\omega)}{N}}
\]

and

\[
D_x^{0.95} = [E(x) - \eta_-, E(x) + \eta_+]
\]

where

\[
\eta_- = \eta_{E(x)-\eta_+}^{0.95}, \quad \eta_+ = \eta_{E(x)+\eta_+}^{0.95}.
\]

Under the assumption that \(N\) is sufficiently large, we can consider that

\[
\eta_- \approx \eta_+ \approx \eta_{E(x)}^{0.95} \approx 1.96 \sqrt{\frac{E(x)(1-E(x))}{N}}.
\]

Then we can conclude that

\[
(d) \text{ for any urn } \omega (\in \Omega \equiv [0, 1]), \text{ the probability, that the measured value } x = (x_1, x_2, \cdots, x_N) \text{ obtained by the measurement } M_{C(\Omega)}(O^N, S_{[\delta_\omega]}), \text{ satisfies the following condition (\sharp), is larger than } \gamma \text{ (e.g., } \gamma = 0.95\).
\]

\[
(\sharp) |\omega - E(x)| \leq 1.96 \sqrt{\frac{E(x)(1-E(x))}{N}} \leq \frac{0.98}{\sqrt{N}}
\]

4.3.4 statistical hypothesis testing — St. Valentine’s Day chocolate

In what follows, we shall describe "statistical hypothesis testing" in terms of measurement theory.

St Valentine’s Day chocolate
You (male) would like to verify the truth of the following hypothesis.

(a') Hypothesis: [Caroline is fond of you]

Here, you want to judge this hypothesis (a'). by the result whether Caroline gives tomorrow’s St Valentine’s Day chocolate.

(b') How do we consider the two cases;

\[
\begin{align*}
\text{Case } \Box & : \text{Caroline presents you a chocolate} \\
\text{Case } \@ & : \text{Caroline presents you a chocolate}
\end{align*}
\]

Everyone guess as follows.

(c') \[
\begin{align*}
\text{Case } \Box & : \text{there is a great possibility that Caroline is not fond of you.} \\
& \quad \text{Thus, the hypothesis (a') is should be rejected} \\
\text{Case } \@ & : \text{This may be an obligatory-gift chocolate,} \\
& \quad \text{Thus, the hypothesis (a') is can not be rejected}
\end{align*}
\]

In what follows we shall study “statistical hypothesis testing.” Consider a measurement \(M_{C(\Omega)}(O \equiv (X, F, F, S_\ast))\) formulated in \(C(\Omega)\).

Here, we assume that \((X, \tau_X)\) is a topological space, where \(\tau_X\) is the set of all open sets. And assume that \(F = B_X\); the Borel field, i.e., the smallest \(\sigma\)-field that contains all open sets in \(X\). Note that we can assume, without loss of generality, that \(F(\Xi) \neq 0\) for any open set \(\Xi(\in \tau_X)\) such that \(\Xi \neq 0\). That is because, if \(F(\Xi) = 0\), it suffices to redefine \(X\) by \(X \setminus \Xi\).

**Problem 4.12** [Statistical hypothesis testing (cf. [15, 20])]

Assume the following hypothesis called “null hypothesis”:

(a) the unknown state \([\ast]\) belongs to a set \(N_H (\subseteq \Omega)\).

Then, our problem is as follows.

(b) Define a proper \([D](\in F)\) such that

if a measured valued obtained by the measurement \(M_{C(\Omega)}(O \equiv (X, F, F, S_{[\ast]}))\) belongs to \([D]\),

then we can deny the null hypothesis (a), that is, \([F([D])](\omega)\) is sufficiently small for any \(\omega \in N_H\).
Answer: Define a function such that $\Lambda_{N_H} : X \to [0, 1]$

$$\Lambda_{N_H}(x) = \lim_{\Xi \to \{x\}} \sup_{\omega \in N_H} [F(\Xi)](\omega) \quad (\forall x \in X)$$

Also, for any $\varepsilon (0 < \varepsilon \leq 1)$, define the $[D]_{N_H}^{\varepsilon}$ ($\in \mathcal{F}$) such that

$$[D]_{N_H}^{\varepsilon} = \{ x \in X \mid \Lambda_{N_H}(x) < \varepsilon \}$$

And define $\varepsilon_{max}^{0.05} (\in [0, 1])$ (i.e., significant level=0.05) such that

$$\varepsilon_{max}^{0.05} = \sup\{ \varepsilon \mid \sup_{\omega_0 \in N_H} [F([D]_{N_H}^{\varepsilon} \mid \omega_0)](\omega_0) \leq 0.05 \}$$

Thus, we can get the rejection region $[D]_{N_H}^{\varepsilon_{max}^{0.05}}$ (depending on $N_H$, $O$, $\varepsilon(=0.05)$). The following is obvious:

if $[s] \in N_H$, then the probability that a measured value obtained by the measurement $M_{C(O)}(O = (X, \mathcal{F}, F), S_{[s]})$ belongs to $[D]_{N_H}^{\varepsilon_{max}^{0.05}} (\in \mathcal{F})$ is less than 0.05.

Since it is quite rare that "[measured value] $\in [D]_{N_H}^{\varepsilon_{max}^{0.05}}$", we can deny the hypothesis (a).

Therefore, we can conclude that

$$\begin{cases} 
\text{if [measured value] } \in [D]_{N_H}^{\varepsilon_{max}^{0.05}}, \text{ then we can deny the hypothesis (a).} \\
\text{if [measured value] } \notin [D]_{N_H}^{\varepsilon_{max}^{0.05}}, \text{ then we can not deny the hypothesis (a).}
\end{cases}$$

\[\square\]
Typical Examples in Classical Measurements

Our argument in the previous section may be too abstract and general. However, it is surely usual. In this section, this will be shown as easy examples in classical measurements.

Put \( \Omega = \mathbb{R}, C(\Omega) = C(\Omega) \). Fix \( \sigma > 0 \). And consider the normal observable \( O_\sigma \equiv (\mathbb{R}, B_\mathbb{R}, F_\sigma) \) in \( C(\Omega) \) such that:

\[
[F_\sigma(\Xi)](\omega) = \frac{1}{\sqrt{2\pi\sigma}} \int_\Xi \exp\left[-\frac{(x-\omega)^2}{2\sigma^2}\right]dx \quad (\forall \Xi \in B_\mathbb{R}, \ \forall \omega \in \Omega = \mathbb{R}).
\]

And further, consider the product observable \( O_\sigma^2 \equiv (\mathbb{R}^2, B_{\mathbb{R}^2}, F_\sigma^2) \) in \( C(\Omega) \). That is,

\[
[F_\sigma^2(\Xi_1 \times \Xi_2)](\omega) = [F_\sigma(\Xi_1)](\omega) \cdot [F_\sigma(\Xi_2)](\omega) = \frac{1}{(2\pi\sigma)^2} \iint_{\Xi_1 \times \Xi_2} \exp\left[-\frac{\sum_{k=1}^2 (x_k - \omega)^2}{2\sigma^2}\right]dx_1dx_2
\]

\((\forall \Xi_k \in B_\mathbb{R}(k = 1, 2), \ \forall \omega \in \Omega = \mathbb{R}).\)

In what follows, we consider the measurement \( M_{C(\Omega)}(O_\sigma^2 = (\mathbb{R}^2, B_{\mathbb{R}^2}, F_\sigma^2), S_{\{4\}}) \).

[Case(I): Two sided test, i.e., \( N_H = \{\omega_0\} \).] Assume that \( N_H = \{\omega_0\}, \omega_0 \in \Omega = \mathbb{R} \). Note the identification (1), i.e., \( \delta_{\omega_0} \approx \omega_0 \). Then, we see that, for any \( (x_1, x_2) \in \mathbb{R}^2 \),

\[
\Lambda_{N_H}(x_1, x_2) = \sup_{\omega \in \{\omega_0\}} L((x_1, x_2), \delta_{\omega}) = \lim_{\Xi_1 \times \Xi_2 \to (x_1, x_2)} \sup_{\omega \in \Omega} \left\{ \frac{[F_\sigma^2(\Xi_1 \times \Xi_2)](\omega)}{[F_\sigma^2(\Xi_1 \times \Xi_2)](\omega_0)} \right\} = \frac{\exp\left[-\frac{(x_1 - \omega_0)^2 + (x_2 - \omega_0)^2}{2\sigma^2}\right]}{\exp\left[-\frac{(x_1 - (x_1 + x_2)/2)^2 + (x_2 - (x_1 + x_2)/2)^2}{2\sigma^2}\right]}
\]

Also, for any \( \epsilon(>0) \), define \( D'_{\{\omega_0\}}(\in B_{\mathbb{R}^2}) \) such that:

\[
D'_{\{\omega_0\}} = \{(x_1, x_2) \in \mathbb{R}^2 | \Lambda_{\{\omega_0\}}(x_1, x_2) \leq \epsilon\}.
\]

Thus we can define \( \epsilon(\alpha) \) such that:

\[
\epsilon(\alpha) = \sup\{\epsilon | \sup_{\omega \in \{\omega_0\}} [F_\sigma^2(D'_{\{\omega_0\}})](\omega) \leq \alpha\}.
\]

Thus, putting \( \alpha = 0.05 \), we see that

\[
\bar{D}_{\{\omega_0\}}^{(0.05)} = D_{\{\omega_0\}}^{(0.05)}
\]

\[
= \{(x_1, x_2) \in \mathbb{R}^2 | (x_1 + x_2)/2 \leq \omega_0 - \frac{1.96\sigma}{\sqrt{2}}\} \bigcup \{(x_1, x_2) \in \mathbb{R}^2 | (x_1 + x_2)/2 \geq \omega_0 + \frac{1.96\sigma}{\sqrt{2}}\}
\]

= “Slash part in Fig. 4.9”
Figure 4.9: Rejection region $\hat{R}_{\{\omega_0\}}^{0.05}$

[Case (II): One sided test, i.e., $\mathcal{N}_H = [\omega_0, \infty]$]. Assume that $\mathcal{N}_H = [\omega_0, \infty)$, $\omega_0 \in \Omega = \mathbb{R}$. Then,

$$
\Lambda_{[\omega_0, \infty)}(x_1, x_2) = \sup_{\omega \in [\omega_0, \infty)} L((x_1, x_2), \delta_\omega) = \sup_{\omega \in [\omega_0, \infty)} \lim_{\omega \in \Omega} \sup_{\xi_1 \times \xi_2 \rightarrow (x_1, x_2)} [F^2_\sigma(\xi_1 \times \xi_2)](\omega) = \sup_{\omega \in \Omega} \exp \left[ - \frac{(x_1 + x_2 - 2\omega)^2}{4\sigma^2} \right] = \begin{cases} 
\exp \left[ - \frac{(x_1 + x_2 - 2\omega)^2}{4\sigma^2} \right] & (x_1 + x_2 < \omega_0) \\
1 & \text{(otherwise)}
\end{cases}
$$

Also, for any $\epsilon (> 0)$, define $D_c^{\epsilon}_{[\omega_0, \infty)} \in \mathcal{B}_{\mathbb{R}^2}$ such that:

$$
D_c^{\epsilon}_{[\omega_0, \infty)} = \{(x_1, x_2) \in \mathbb{R}^2 \mid \Lambda_{[\omega_0, \infty)}(x_1, x_2) \leq \epsilon\} = \{(x_1, x_2) \in \mathbb{R}^2 \mid \frac{x_1 + x_2}{2} - \omega_0 < \sqrt{4\sigma^2 \log \epsilon}\}.
$$

Thus we can define $\epsilon(\alpha)$ such that:

$$
\epsilon(\alpha) = \sup\{\epsilon \mid \sup_{\omega \in [\omega_0, \infty)} [F^2_\sigma(D_c^{\epsilon}_{[\omega_0, \infty)}))](\omega) \leq \alpha\}.
$$

Therefore, putting $\alpha = 0.05$, we see that

$$
\hat{R}_{[\omega_0, \infty)}^{0.05} = D_c^{(0.05)}_{[\omega_0, \infty)} = \{(x_1, x_2) \in \mathbb{R}^2 \mid \frac{x_1 + x_2}{2} \leq \omega_0 - \frac{1.65\sigma}{\sqrt{2}}\} = \text{“Slash part in Fig. 4.10”}
$$
\[ c = 2(\omega_0 - 1.65\sigma/\sqrt{2}) \]

Figure 4.10: Rejection region \( \hat{R}^{0.05}_{[\omega_0, \infty]} \)

\[ \therefore \text{Note 4.6} \]

Although several statistical methods are described in terms of measurements in this section. However, it should be that there are some gap between the statistical description and the measurement theoretical description. Recall that measurement theory says that

(\( \sharp_1 \)) the state after a measurement is meaningless.

Therefore, Theorem 4.5 [Fisher maximum likelihood method (measurement theoretical representation)] should be formally written as follows.

(\( \sharp_2 \)) Theorem 4.5’ [Fisher maximum likelihood method (measurement theoretical representation) (cf. [11, 20])] Let \( O \times O_1 = (X \times Y, F \times G, F \times G) \) be a simultaneous observable in \( C(\Omega) \). Consider a simultaneous measurement \( M_{C(\Omega)}(O \times O_1, S[\star]) \). Assume that a measured value \( (x, y) \) belongs to \( \Xi \times Y \) (\( \in F \subseteq G \)). Then, there is a reason to infer that the probability that \( y \in \Gamma \) is given by \( [G(\Gamma)](\omega_0) \) (\( \forall \gamma \in G \)) where \( [F(\Xi)](\omega_0) = \max_{\omega \in \Omega}[F(\Xi)](\omega) \).

Theorem 4.5 is the abbreviation Theorem 4.5’. Note that (\( \sharp_2 \)) is applicable to quantum mechanics (under the existence of a simultaneous observable \( O \times O_1 \)).

4.4 Bayesian statistics

4.4.1 What is mixed measurement?

Recall the classification of measurement theory (Chap. 1(Y)), that is,

\[
\text{quantum measurement(5)} : \text{Chap. 3}
\]

\[
\text{classical measurement}
\]

(a) measurement theory (scientific language)

\[
\begin{align*}
\text{continuous} & : \text{(Chaps. 2-9)} \\
\text{mixed type(2)} & : \text{Sec. 4.4}
\end{align*}
\]

\[
\begin{align*}
\text{bounded} & : \text{(Chaps. 10, 11)} \\
\text{mixed type(3)} & : \text{Notes 10.4, 11.3}
\end{align*}
\]

In this preprint we mainly devote ourselves to classical pure measurement (\( \text{(1) and (3)} \)) and not classical mixed measurement. However, in what follows, only a few will describe this.

Let \( \Omega \) be a locally compact space, \( B_{\Omega} \) be its Borel field. \( M(\Omega) \) and \( M_{+1}(\Omega) \) are defined as follows.
(b) \[
\begin{align*}
\text{The space of complex valued measures} & : \mathcal{M}(\Omega) \\
& = \{ \nu = \nu_1 - \nu_2 + \sqrt{-1}(\nu_3 - \nu_4) \mid \nu_k (k = 1, 2, 3, 4) \text{ is a finite measure on } \Omega \} \\
\text{The space of probability measures} & : \mathcal{M}_{+1}(\Omega) \\
& = \{ \nu \mid \nu \text{ is a finite measure on } \Omega \text{ such that } \nu(\Omega) = 1 \}
\end{align*}
\]

In measurement theory, the \( \mathcal{M}_{+1}(\Omega) \) is called a mixed state space (or, statistical state space, compound state space), \( \nu(\in \mathcal{M}_{+1}(\Omega)) \) is said to be a mixed state (or, statistical state, compound state).

In what follows, we shall explain mixed measurement.

**Example 4.13 [Urn problem and coin tossing]** In Problem 4.3 [urn problem] (and Answer 4.6), putting \( \Omega = \{ \omega_1, \omega_2 \} \), and consider a measurement \( M_{C(\Omega)}(O=(\{w,b\}, 2^{\{w,b\}}, F), S) \), where \( O = (\{w,b\}, 2^{\{w,b\}}, F) \) is defined by the formula (2.5). That is,

\[
\begin{align*}
F(\{w\})(\omega_1) &= 0.8, & F(\{b\})(\omega_1) &= 0.2 \\
F(\{w\})(\omega_2) &= 0.4, & F(\{b\})(\omega_2) &= 0.6
\end{align*}
\]

Let us write Fig. 4.2 (=Fig. 4.11) again.

You do not know which the urn behind the curtain is, \( U_1 \) or \( U_2 \).

Assume that you pick up a white ball from the urn.

The urn is \( U_1 \) or \( U_2 \)? Which do you think?

A mixed measurement is characterized such as "measurement \( M_{C(\Omega)}(O=(\{w,b\}, 2^{\{w,b\}}, F), S) \)" + "probabilistic property of the unknown state[\( \ast \)]". Let us explain it. Consider the following two procedures (c) and (d)(Fig. 4.12):

(c) Consider an unfair coin-tossing \( (T_{p, 1-p}) \) such that \( (0 \leq p \leq 1) \): That is,

\[
\begin{align*}
\text{the possibility that } "\text{head}" \text{ appears is } 100p\% \\
\text{the possibility that } "\text{tail}" \text{ appears is } 100(1-p)\%
\end{align*}
\]

If "head" [resp. "tail"] appears, put an urn \( U_1(\approx \omega_1) \) [resp. \( U_2(\approx \omega_2) \)] behind the curtain. Assume that you do not know which urn is behind the curtain, \( U_1 \) or \( U_2 \). The unknown urn is denoted by \( [\ast](\in \{\omega_1, \omega_2\}) \).
This situation is represented by

\[ \nu_0 = p\delta_{\omega_1} + (1-p)\delta_{\omega_2} \] (That is, \( \nu_0(\{\omega_1\}) = p, \nu_0(\{\omega_2\}) = 1-p \))

where \( \delta_{\omega} \) is the point measure at \( \omega \). That is, it suffices to consider that \( \nu_0 \) is the distribution of [\(*\)].

(d) Consider the "measurement" such that a ball is picked out from the unknown urn. This "measurement" is denoted by \( M_{C(\Omega)}(O, S_{[\omega]}(\nu_0)) \), and called a mixed measurement.

You do not know which the urn behind the curtain is, \( U_1 \) or \( U_2 \), but the probability: \( p \) and \( 1-p \).

Assume that you pick up a white ball from the urn.

The urn is \( U_1 \) or \( U_2 \)? Which do you think?

You do not know the distribution \( \nu_0 \) of [\(*\)].

Figure 4.12: Which is the hidden urn, \( U_1 \) or \( U_2 \)? (Mixed measurement)

Now consider the following problem:

(e_1) Calculate the probability that a white ball is picked out by the mixed measurement \( M_{C(\Omega)}(O, S_{[\omega]}(\nu_0)) \)!
(e_2) And further, when a white ball is picked out by the mixed measurement \( M_{C(\Omega)}(O, S_{[\omega]}(\nu_0)) \), do you infer the unknown urn \( U_1 \) or \( U_2 \)?

(See Fig. 4.12.)

Answer (e_1) The following is clear:

(i) The possibility that "[\(*\)] = \omega_1" is 100p%. Also, The possibility that "[\(*\)] = \omega_2" is 100(1 - p)%.

Further,

(ii) the probability that a measured value \( x \ (\in \{w, b\}) \) is obtained by a measurement \( M_{C(\Omega)}(O, S_{[\omega_1]}(\nu_0)) \) is

\[ [F(\{x\})](\omega_1) = 0.8 \text{ (when } x = w), \quad 0.2 \text{ (when } x = b) \]

the probability that a measured value \( x \ (\in \{w, b\}) \) is obtained by a measurement \( M_{C(\Omega)}(O, S_{[\omega_2]}(\nu_0)) \) is

\[ [F(\{x\})](\omega_1) = 0.4 \text{ (when } x = w), \quad 0.6 \text{ (when } x = b) \]
Therefore, by (i) and (ii), the probability that a measured value \( x \in \{ w, b \} \) is obtained by a mixed measurement \( M_{C(\Omega)}(O, S_{[\nu]}(\nu)) \) is

\[
P(\{x\}) = \int_\Omega [F(\{x\})](\omega) \nu(\omega) \, d\omega = p[F(\{x\})](\omega_1) + (1-p)[F(\{x\})](\omega_2)
\]

\[
= \begin{cases} 
0.8p + 0.4(1-p) & (x = w \text{ and }) \\
0.2p + 0.6(1-p) & (x = b \text{ and }) 
\end{cases}
\]

This is the answer to Problem (e_1).

**Answer(e_2)** Problem (e_2) will be presented in Note 4.8.

---

**♠ Note 4.7** The following question is natural. That is,

In the above (i) or (c), why is "the possibility that \([*]=\omega_1\) is 100\% \cdots" replaced by "the probability that \([*]=\omega_1\) is 100\% \cdots"?

However, the Copenhagen interpretation says that there is no probability without measurements.

This is the reason why the term "probability" is not used.

---

### 4.4.2 Mixed measurement theory

As seen in the above, we say that

(a) Pure measurement theory is fundamental. Adding the concept of "mixed state", we can construct mixed measurement theory as follows.

\[
\text{mixed measurement theory} \quad M_{C(\Omega)}(O, S_{[\nu]}(\nu)) := \text{pure measurement theory} \quad M_{C(\Omega)}(O, S_{[\nu]}) + \text{mixed state} \quad \nu
\]

Thus we can present the following Axiom\(_C^m\) 1.

**Axiom\(_C^m\) 1 (measurement: continuous mixed type)**

Consider a mixed measurement \( M_{C(\Omega)}(O=(X, \mathcal{F}, F), S_{[\nu]}(\nu)) \) formulated in \( C(\Omega) \). The probability that a measured value \( x \in X \) obtained by the mixed measurement \( M_{C(\Omega)}(O, S_{[\nu]}(\nu)) \) belongs to \( \Xi \in \mathcal{F} \) is given by

\[
\int_\Omega [F(\Xi)](\omega) \nu(\omega) \, d\omega
\]

Thus we see:

(b) a mixed measurement is characterized as the following correspondence:

\[
\begin{array}{ccc}
\text{(mixed state, observable)} & \text{mixed measurement} & \text{measured value} \\
\text{probabilistic} & \text{probe} & \text{measured}
\end{array}
\]
In statistics, both "fluctuation" and "measurement error" are represented by random variable. Thus, we sometimes confuse the two. On the other hand, in measurement theory, the two have different mathematical structures, that is,

\[
\begin{align*}
(i) &: \text{[pure state]} \xrightarrow{\text{fluctuation}} \text{[mixed state]} \\
(ii) &: \text{[exact observable]} \xrightarrow{\text{measurement error}} \text{[non-projective observable]}
\end{align*}
\]

Thus, in classical measurement theory, we can avoid the confusion. However, as seen in Heisenberg’s uncertainty principle (Theorem 3.4), in quantum mechanics, even pure state has a fluctuation.

The following example will promote the understanding of mixed measurements.

**Example 4.14 [Coin-tossing]** The coin-tossing is described in the following three situations:

(i): Let \( \Omega \) be a state space of states of a coin. Let \( \omega_0 \in \Omega \) be a state of a fair coin. Consider a measurement \( M_{C(\{\Omega\})}(O=(\{h, t\}, 2^{\{h, t\}}, F), S_{[\omega_0]}) \) (where, "h" and "t" respectively means "head" and "tail". And \( F(\{h\})(\omega_0) = F(\{t\})(\omega_0) = 1/2 \). Of course, by the measurement, the probability that a measured value "h" is obtained is equal to \( |F(\{h\})(\omega_0)| = 1/2 \).

(ii): Someone (or, a robot) tosses a fair coin. And the measurement is defined by the check of the result (i.e., "h" or "t"). Define the state space \( \Omega \) such that \( \Omega = \{h, t\} \). Then the mixed exact measurement is considered as \( M_{C(\{h, t\})}(O^{\text{exa}}, S_{[\ast]}((\frac{1}{2}(\delta_h + \delta_t)))) \) where \( \delta_{\omega}(\in \mathcal{M}_{+1}(\Omega)) \) is the point measure at \( \omega \). Of course, the probability that a measured value "h" is obtained is given by \( \int_{\{h, t\}} |F^{\text{exa}}(\{h\})(\omega)| \cdot \frac{1}{2}(\delta_h + \delta_t)(d\omega) = 1/2 \).

(iii): Someone intentionally grasped coin with the (right or left) hand, and asked you "right or left?". In this case, you may think that the right or the left is half-and-half. The reason that you think so is due to the principle of equal weight. Here,

(d) the justification of the principle of equal weight --- unless we have sufficient reason to regard one possible case as more probable than another, we treat them as equally probable --- is the most famous problem in statistics.

This will be solved in Sec. 6.4.5 (cf. [11, 19]).

**Remark 4.15 [Bayes’ theorem in measurement theory]** In this book, we are not closely related to mixed measurement theory. But, we have to mention only Bayes’s theorem in mixed measurement theory as follows.

(e) [Bayes’s theorem in mixed measurement theory (cf. [15, 20])] Consider a mixed measurement \( M_{C(\Omega)}(O=(X, F), S_{[\ast]}(\nu)) \). When we know that a measured value belong to \( \Xi \) (\( \in F \)), we can understand that the mixed state \( \nu \) changes to a new mixed state \( \nu_{\text{new}} \) such that

\[
\nu_{\text{new}}(D) = \frac{\int_{[\Omega]} F(\Xi)](\omega)\nu(d\omega)}{\int_{[\Omega]} F(\Xi)](\omega)\nu(d\omega)} \quad (\forall D \in \mathcal{B}_\Omega)
\]
Note 4.8 Now we can answer Problem (e) in Sec. 4.4.2: Since "white ball" is obtained by a statistical measurement $M_{C(\Omega)}(O, S[\nu](\nu_0))$, a new mixed state $\nu_{\text{new}}(\in M_{+1}(\Omega))$ is given by

$$
\nu_{\text{new}}(D) = \frac{\int_{\Omega} F(\{w\})(\omega)\nu_0(d\omega)}{\int_{\Omega} F(\{w\})(\omega)\nu_0(d\omega)} = \begin{cases} 
0.8p & (\text{when } D = \{\omega_1\}) \\
0.8p + 0.2(1 - p) & (\text{when } D = \{\omega_2\}) \\
0.2(1 - p) & (\text{when } D = \{\omega_3\})
\end{cases}
$$

Note 4.9 According to the Copenhagen interpretation, (cf. Note 4.6) the (e) in Remark 4.15 should be described as the following (e)′:

(e)′ Consider a simultaneous observable $O \times O_1 = (X \times Y, F \times G, F \times G)$ in $C(\Omega)$, and a simultaneous measurement $M_{C(\Omega)}(O \times O_1, S[\nu](\nu))$. When we know that a measured value $(x, y)$ belongs to $\Xi \times Y (\in F \times G)$, the probability $P_\Gamma$ (i.e., the probability that $y \in \Gamma (\in G)$) is given by

$$
P_\Gamma = \frac{\int_{\Omega} F(\Xi) \cdot G(\Gamma)(\omega)\nu_0(d\omega)}{\int_{\Omega} F(\Xi)(\omega)\nu_0(d\omega)}
$$

The (e) should be regarded as the abbreviation of the (e)′.

As an easy example, we shall study Monty Hall problem in measurement theory.

Problem 4.16 [Monty Hall problem (Continued from Problem 4.9) (cf. [11, 19])]

Suppose you are on a game show, and you are given the choice of three doors (i.e., “number 1,” “number 2,” “number 3”). Behind one door is a car, behind the others, goats. You pick a door, say number 1. Then, the host, who set a car behind a certain door, says

(♯1) the car was set behind the door decided by the cast of the distorted dice. That is, the host set the car behind the $k$-th door (i.e., “number $k$”) with probability $p_k$ (or, weight such that $p_1 + p_2 + p_3 = 1$, $0 \leq p_1, p_2, p_3 \leq 1$).

And further, the host says, for example,

(♭) the door 3 has a goat.

He says to you, “Do you want to pick door number 2?” Is it to your advantage to switch your choice of doors?

Answer Recalling Problem 4.9 (Monty Hall problem), in what follows we study this problem. Let $\Omega$ and $O_1$ be as in Section 3.1. Under the hypothesis (♯1), define the mixed state $\nu_0 (\in M_{+1}(\Omega))$ such that:

$$
\nu_0(\{\omega_1\}) = p_1, \quad \nu_0(\{\omega_2\}) = p_2, \quad \nu_0(\{\omega_3\}) = p_3
$$

Thus we have a mixed measurement $M_{C(\Omega)}(O_1, S[\nu](\nu_0))$. Note that

a) “measured value 1 is obtained by the mixed measurement $M_{C(\Omega)}(O_1, S[\nu](\nu_0))” \Rightarrow \text{the host says "Door 1 has a goat"}
b) “measured value 2 is obtained by the mixed measurement $M_{C(\Omega)}(O_1, S_{\{\nu_0\}})$”
⇔ the host says “Door 2 has a goat”

c) “measured value 3 is obtained by the mixed measurement $M_{C(\Omega)}(O_1, S_{\{\nu_0\}})$”
⇔ the host says “Door 3 has a goat”

Here, assume that, by the mixed measurement $M_{C(\Omega)}(O_1, S_{\{\nu_0\}})$, you obtain a measured value 3, which corresponds to the fact that the host said “Door 3 has a goat.” Then, Theorem 3 (Bayes’ theorem) says that the posterior state $\nu_{\text{post}} \in M_{+1}(\Omega)$ is given by

$$\nu_{\text{post}} = \frac{F_1(\{3\}) \times \nu_0}{\langle \nu_0, F_1(\{3\}) \rangle}.$$ 

That is,

$$\nu_{\text{post}}(\{\omega_1\}) = \frac{p_1}{p_1 + p_2}, \quad \nu_{\text{post}}(\{\omega_2\}) = \frac{p_2}{p_1 + p_2}, \quad \nu_{\text{post}}(\{\omega_3\}) = 0.$$

Particularly, we see that

(♯2) if $p_1 = p_2 = p_3 = 1/3$, then it holds that $\nu_{\text{post}}(\{\omega_1\}) = 1/3$, $\nu_{\text{post}}(\{\omega_2\}) = 2/3$, $\nu_{\text{post}}(\{\omega_3\}) = 0$, and thus, you should pick Door 2.

♠ Note 4.10 It is not natural to assume the rule (♯1) in Problem 4.16. That is because the host may intentionally set the car behind a certain door. Thus we think that Problem 4.16 is temporary. For our formal proposal, see Sec. 6.4.5.

4.4.3 Entropy — The value of eyewitness information

As one of applications (of Bayes theorem), we now study the “entropy” of the measurement(cf. [31]). Here we have the following definition.

Definition 4.17 [Entropy(cf. [10])] Consider a mixed measurement $M_{C(\Omega)}(O \equiv (X, 2^X, F), S(\rho_0))$ in a commutative basic algebra $C(\Omega)$, where the label set $X$ is assumed to be at most countable, i.e., $X = \{x_1, x_2, ..., x_n, ...\}$. Then, the $H(M)$, the (fuzzy) entropy of $M_{C(\Omega)}(O, S(\rho_0))$, is defined by

$$H\left(M_{C(\Omega)}(O, S(\rho_0))\right) = \sum_{n=1}^{\infty} \left( \int_{\Omega} |F(\{x_n\})|/\rho_0(d\omega) \int_{\Omega} \frac{|F(\{x_n\})|/\rho_0(d\omega)}{\int_{\Omega} |F(\{x_n\})|/\rho_0(d\omega)} \log \frac{|F(\{x_n\})|/\rho_0(d\omega)}{\int_{\Omega} |F(\{x_n\})|/\rho_0(d\omega)} \rho_0(d\omega) \right)$$

$$= \sum_{n=1}^{\infty} P(\{x_n\}) \cdot I(\{x_n\}) \quad (4.7)$$
where, \( P\{x_n\} = \int_\Omega [F\{x_n\}](\omega)\rho_0(d\omega) \)

\( = \) the probability that a measured value \( x_n \) is obtained

\[
I\{x_n\} = \int_\Omega \left[ F\{x_n\}\right](\omega) \log \left[ F\{x_n\}\right](\omega) \rho_0(d\omega)
- \log \left[ F\{x_n\}\right](\omega) \rho_0(d\omega)
= \frac{1}{P\{x_n\}} \int_\Omega [F\{x_n\}](\omega) \log[F\{x_n\}](\omega)\rho_0(d\omega) - \log P\{x_n\}
(= \) the information quantity when a measured value \( x_n \) is obtained

The following is clear:

\[
H(M) = \sum_{n=1}^{\infty} \int_\Omega \left[ F\{x_n\}\right](\omega) \log[F\{x_n\}](\omega)\rho_0(d\omega) - \sum_{n=1}^{\infty} P\{x_n\} \log P\{x_n\}. \tag{4.8}
\]

**Example 4.18** [The offender is man or female? fast or slow?]

Assume that

(a) There are 100 suspected persons such as \( s_1, s_2, \ldots, s_{100} \), in which there is one criminal.

Define the state space \( \Omega = \{\omega_1, \omega_2, \ldots, \omega_{100}\} \) such that

\( \omega_n \) the state such that suspect \( s_n \) is a criminal \( (n = 1, 2, \ldots, 100) \)

Define a male-observable \( O_m = (X = \{y_m, n_m\}, 2^X, M) \) in \( C(\Omega) \) by

\[
|M\{y_m\}\rangle(\omega_n) = m_{y_m}(\omega_n) = \begin{cases} 0 & (n \text{ is odd}) \\ 1 & (n \text{ is even}) \end{cases}
|M\{n_m\}\rangle(\omega_n) = m_{n_m}(\omega_n) = 1 - |M\{y_m\}\rangle(\omega_n)
\]

For example,

Taking a measurement \( M_{C(\Omega)}(O_m, S_{\omega_{17}}) \) — the sex of the criminal \( s_{17} \) —, we get the measured value \( n_m(=) \).

Also, define the fast-observable \( O_f = (Y = \{y_f, n_f\}, 2^Y, F) \) in \( C(\Omega) \) by

\[
[F\{y_f\}](\omega_n) = f_{y_f}(\omega_n) = \frac{n - 1}{99},
[F\{n_f\}](\omega_n) = f_{n_f}(\omega_n) = 1 - [F\{y_f\}](\omega_n)
\]

According to the principle of equal weight (due to Theorem 6.21 later), there is a reason to consider that a mixed state \( \nu_0 \in M_{+1}(\Omega) \) is equal to the state \( \nu_\epsilon \) such that \( \nu_0(\{\omega_n\}) = \nu_\epsilon(\{\omega_n\}) = 1/100 \ (\forall n) \). Thus, consider two mixed measurement \( M_{C(\Omega)}(O_m, S_{\epsilon}[\nu_\epsilon]) \) and \( M_{C(\Omega)}(O_f, S_{\epsilon}[\nu_\epsilon]) \). Then, by (4.8), we see:
\[ H(M_{C}(\Omega)(O_{m}, S_{[\nu]}(\nu_{c}))) = \int_{\Omega} m_{y_{m}}(\omega)\nu_{c}(d\omega) \cdot \log \int_{\Omega} m_{y_{m}}(\omega)\nu_{c}(d\omega) \]
\[ - \int_{\Omega} m_{(n_{m})}(\omega)\nu_{c}(d\omega) \cdot \log \int_{\Omega} m_{n_{m}}(\omega)\nu_{c}(d\omega) \]
\[ = -\frac{1}{2} \log \frac{1}{2} - \frac{1}{2} \log \frac{1}{2} = \log 2 = 1 \text{ (bit)} \]

\[ H(M_{C}(\Omega)(O_{f}, S_{[\nu]}(\nu_{c}))) = \int_{\Omega} f_{y_{f}}(\omega) \cdot \log f_{y_{f}}(\omega) \nu_{c}(d\omega) \]
\[ + \int_{\Omega} f_{n_{f}}(\omega) \cdot \log f_{n_{f}}(\omega)\nu_{c}(d\omega) - \int_{\Omega} f_{y_{f}}(\omega)\nu_{c}(d\omega) \cdot \log \int_{\Omega} f_{y_{f}}(\omega)\nu_{c}(d\omega) \]
\[ - \int_{\Omega} f_{n_{f}}(\omega)\nu_{c}(d\omega) \cdot \log \int_{\Omega} f_{n_{f}}(\omega)\nu_{c}(d\omega) \]
\[ \approx 2 \int_{0}^{1} \lambda \log_{2} \lambda d\lambda + 1 = -\frac{1}{2} \log_{2} e + 1 = 0.278 \ldots \text{ (bit)} \]

Therefore, as eyewitness information, "male of female" has more valuable than "fast or slow".
5 Practical Logic

The term "practical logic" means the logic in measurement theory. It is certain that pure logic is merely a kind of rule in mathematics (or meta-mathematics). If it is so, the logic is not guaranteed to be applicable to our world. For instance, mathematical logic does not assure the following famous statement:

\( \text{(♯1)} \text{ Since Socrates is a man and all men are mortal, it follows that Socrates is mortal.} \)

That is, we think that

\( \text{(♯2)} \text{ the above (♯1) is not clarified yet.} \)

In this chapter, we prove the (♯1) in classical systems. As seen in Note 3.2, it should be recalled that syllogism does not hold in quantum systems.

5.1 Reconsider the logic of ordinary language

Logic has various aspects such as the following (A_1)–(A_3):

(A_1) [Logic in mathematics or mathematical logic]. It is natural to believe that logic in mathematics is most reliable. But, it should be noted that mathematical logic is independent of our world. Thus, mathematical logic can say nothing to our world, unless some interpretation is added. For example, the following syllogism is obvious.

\( \text{(♯)} \text{ "A } \Rightarrow \text{ B, B } \Rightarrow \text{ C" then "A } \Rightarrow \text{ C"} \)

This is a rule in mathematical logic. However, the syllogism(♯) is closed in mathematics, and thus, it is not guaranteed to be related to our world.

(A_2) [Ordinary language]. There is logic buried in ordinary language. This logic is various, for example, "logic in marital dispute", "logic in a court", etc. However, the rule of ordinary language is not clear, and therefore, the following (♯1)–(♯4) have the room for reconsideration (Problem 5.1 and Note 5.1–5.3).

\( \text{(♯1)} \text{ Since Socrates is a man and all men are mortal, it follows that Socrates is mortal.} \)
\( \text{(♯2)} \text{ Flying arrow is not moving (Zeno’s paradoxes)} \)
\( \text{(♯3)} \text{ I think, therefore I am. (Descartes)} \)
\( \text{(♯4)} \text{ "1 (kg) + 1 (kg) =2 (kg)" (Edison; the master of invention) (cf. Sec.11.5)} \)

(A_3) [Logic in the world-descriptions]. As mentioned frequently, we have to say that

\( \text{(♯1)} \text{ mathematics, independent of our world, can not assert anything to the world without the world-description (e.g., Newtonian mechanics, measurement theory, etc. ).} \)

\( \text{(♯2)} \text{ the above (♯1) is not clarified yet.} \)
The spirit of world-description says that,

\((\sharp_2)\) First, describe each phenomenon by a language (induced by world-description). Next, calculate its numerical representation.

As in the above, there is various logic \((A_1)-(A_3)\). However, recall our standing point 3.5(=Chap. 1(X_4))

Describe any theory (which is not yet described by measurement theory) in terms of measurement theory.

\[\text{Note 5.1} \quad \text{As seen in Chap. 1(X_1), we have the following diagram;} \]

\[
\begin{array}{c}
(X_1) \quad \text{wide ordinary language (before science)} \\
\text{(Chap. 1)} \\
\Rightarrow \text{world-description (Chap. 1(O))} \\
\end{array}
\]

\(\begin{cases}
\text{① realistic method (Newtonian mechanics, etc.)} \\
\text{② linguistic method (measurement theory)}
\end{cases}\)

Here, it is natural to see that

\((\sharp_1)\) "wide ordinary language\(\oplus\)" includes

\((b)\) the statement \((\sharp_1) - (\sharp_4)\) in \((A_2)\), statistics (=dynamical system theory) \((\text{Chaps. 4,7,11})\)

However, the framework of ordinary language is not clear. Therefore, according to Our standing point 3.5 (=Chap. 1(X_4)), our purpose is to reconsider this \((b)\) in measurement theory\(\ominus\)

About 2500 years ago, Zeno (BC490-BC430) pointed out the ambiguity of the logic in ordinary language as follows.

**Problem 5.1 [Flying arrow is not moving]**

\((B_1)\) [Problem ]: Is flying arrow is moving or not?

\((B_2)\) [Zeno’s answer] Consider a flying arrow. In any one instant of time, the arrow is not moving.

Therefore, If the arrow is motionless at every instant, and time is entirely composed of instants, then motion is impossible.

If Zeno’s logic \((B_2)\) is not true, it is natural to consider that the \((\sharp_1) - (\sharp_4)\) in \((A_2)\) in ordinary language cannot be believed easily. Hence, Zeno’s paradoxes urge us to ask the following question:

\((B_3)\) By what kind of world-description should the flying arrow be described?

This problem \((B_3)\) may be the most famous unsolved problem in science. This problem \((B_3)\) will be solved in Answer 11.11 in Chap. 11.

As an answer to Problem \((B_3)\), some consider Newtonian mechanics. However, the "flying arrow" is a symbol of various motion-change, e.g., the growth of a tree, economic growth of a country, etc. Thus, Newtonian mechanics is not proper. Of course, some may assert Laplace’s demon. However, we think that even a physical supremacism hesitates to say Laplace’s demon.
\* \textbf{Note 5.2} If we know the present state of the universe and the kinetic equation (=the theory of everything), and if we calculate it, we can know everything (from past to future). There may be a reason to believe this idea. This intellect is often referred to as Laplace’s demon. Laplace’s demon is sometimes discussed as the realistic-view over which the degree passed. Although the discussion about Laplace’s Demon vs. measurement theory is interesting, we do not concerned with Laplace demon in this book.

\* \textbf{Note 5.3} Recall the classification of measurement theory a follows.

\[
\begin{align*}
(Y) & \quad \text{measurement theory} \\
(C_{\text{map},1}) & \quad \left\{ \begin{array}{ll}
\text{quantum measurement theory} \\
\text{classical measurement theory} \end{array} \right.
\end{align*}
\]

If we, from the quantum mechanical point of view, discuss \((\Omega_1)\) — (physicist) — (Example 2.10; Urn problem) — (flying arrow) in \((A_2)\) — (cyclist) — (Example 2.10; Urn problem) in \((A_2)\). these meaning will be clearer.

In what follows, we mention them as our final answers:

\((\Omega_1)\) the syllogism — (physicist) in \((A_2)\) — does not hold in quantum systems (Note 3.8), but it is true in classical systems (Theorem 5.11(i)).

\((\Omega_2)\) The \((\Omega_2)\) (flying arrow) in \((A_2)\) — the trajectories of electron — is meaningless (Note 11.8) in quantum systems, but it is true in classical systems (Sec.11.5).

\((\Omega_3)\) The statement "I think, \cdots" (\((\Omega_4)\) in \((A_2)\)) is nonsense in both classical and quantum systems (Sec.5.3.2).

\((\Omega_4)\) Edison’s problem \((\Omega_4)\) is true (Sec.11.5).

If it be so, the readers may agree that it is worth while studying \((\Omega_1)\) — (physicist) — (Example 2.10; Urn problem) in \((A_2)\).

\subsection{5.2 Quasi-product observable and marginal observable}

As a generalization of a simultaneous observable (Definition 2.14), we introduce the following "quasi-product observable".

\textbf{Definition 5.2 [Quasi-product observable]} For each \(k = 1, 2, \ldots, n\), consider an observable \(O_k = (X_k, F_k, F_k)\) in \(C(\Omega)\). An observable \(O_{1\ldots n} = (\bigtimes_{k=1}^n X_k, \bigotimes_{k=1}^n F_k, F_{1\ldots n})\) in \(C(\Omega)\) is called a quasi-product observable of \(\{O_k\}_{k=1}^n\), and denoted by \(\bigotimes_{k=1,2,\ldots,n} O_k = (\bigtimes_{k=1}^n X_k, \bigotimes_{k=1}^n F_k, F_{1\ldots n})\), if it satisfied that

\[
F_{1\ldots n}(X_1 \times \cdots \times X_{k-1} \times \Xi_k \times X_{k+1} \times \cdots \times X_n) = F_k(\Xi_k)
\]

\((\forall \Xi_k \in F_k, \forall k = 1, 2, \ldots, n)\)

Of course, a simultaneous observable is a kind of quasi-product observable, and thus, a quasi-product observable is not generally determined uniquely.

\* \textbf{Note 5.4} Recall (Example 2.10; Urn problem). Putting \(\Omega = \{\omega_1, \omega_2\}\), we, by (2.5), define an observable \(O_{wb} = \{\{w, b\}, 2^{\{w,b\}}, F\}\) in \(C(\Omega)\). Now we can define the quasi-product observable \(O_{\text{qp}} = O_{12} = \{\{w, b\} \times \{w, b\}, 2^{\{w,b\} \times \{w,b\}}, F_{12}\}\) in \(C(\Omega)\)

\[
\begin{align*}
F_{12}(\{\{w, w\}\}) & = \frac{8 \times 7}{90}, & F_{12}(\{\{w, b\}\}) & = \frac{8 \times 2}{90} \\
F_{12}(\{\{b, w\}\}) & = \frac{2 \times 8}{90}, & F_{12}(\{\{b, b\}\}) & = \frac{2 \times 1}{90} \\
F_{12}(\{\{w, w\}\}) & = \frac{4 \times 3}{90}, & F_{12}(\{\{w, b\}\}) & = \frac{4 \times 6}{90} \\
F_{12}(\{\{b, w\}\}) & = \frac{6 \times 4}{90}, & F_{12}(\{\{b, b\}\}) & = \frac{6 \times 5}{90}
\end{align*}
\]

Here, note that \(O_{12}\) is a quasi-product observable and not a simultaneous observable.
Lemma 5.4 [The condition of quasi-product observables]

Consider an observable $O_{12 \ldots n} = (\bigotimes_{k=1}^{n} X_k, \bigotimes_{k=1}^{n} \mathcal{F}_k, F_{12 \ldots n})$ in $C(\Omega)$. For each $1 \leq j \leq n$, define $F_{12 \ldots n}^{(j)}$ by

$$F_{12 \ldots n}^{(j)}(\Xi_j) = F_{12 \ldots n}(X_1 \times \cdots \times X_{j-1} \times \Xi_j \times X_{j+1} \times \cdots \times X_n) \quad (\forall \Xi_j \in \mathcal{F}_j)$$

Here, $O_{12 \ldots n}^{(j)} = (X_j, \mathcal{F}_j, F_{12 \ldots n}^{(j)})$ is an observable in $C(\Omega)$. The $O_{12 \ldots n}^{(j)}$ is called a marginal observable (precisely, $(j)$-marginal observable) of $O_{12 \ldots n}$. This can be generalized as follows. For example, putting $O_{12 \ldots n}^{(12)} = (X_1 \times X_2, \mathcal{F}_1 \times \mathcal{F}_2, F_{12 \ldots n}^{(12)})$,

$$F_{12 \ldots n}^{(12)}(\Xi_1 \times \Xi_2) = F_{12 \ldots n}(\Xi_1 \times \Xi_2 \times X_3 \times \cdots \times X_n) \quad (\forall \Xi_1 \in \mathcal{F}_1, \forall \Xi_2 \in \mathcal{F}_2)$$

we have the observable $O_{12 \ldots n}^{(12)} = (X_1 \times X_2, \mathcal{F}_1 \times \mathcal{F}_2, F_{12 \ldots n}^{(12)})$. Of course, it holds that $F_{12 \ldots n} = F_{12 \ldots n}^{(12)}$.

An observable $O = (X, \mathcal{F}, F)$ in $C(\Omega)$ has another representation as follows.

$$\text{Rep}_\mathcal{F}[O] = \left[ [F(\Xi)](\omega), [F(\Xi^c)](\omega) \right]$$

$\Xi^c = \{ x \in X \mid x \notin \Xi \}$, i.e., the copmlement of $\Xi$. Similarly, an observable $O_{12} = (X_1 \times X_2, \mathcal{F}_1 \times \mathcal{F}_2, F_{12})$ in $C(\Omega)$ is represented by

$$\text{Rep}_{\mathcal{F}_1 \times \mathcal{F}_2}[O_{12}] = \left[ [F_{12}(\Xi_1 \times \Xi_2)](\omega), [F_{12}(\Xi_1 \times \Xi_2)](\omega) \right]$$

where, it should be noted that

$$[F_{12}(\Xi_1 \times \Xi_2)](\omega) + [F_{12}(\Xi_1 \times \Xi_2)](\omega) = [F_{12}^{(1)}(\Xi_1)](\omega)$$

$$[F_{12}(\Xi_1 \times \Xi_2)](\omega) + [F_{12}(\Xi_1 \times \Xi_2)](\omega) = [F_{12}^{(1)}(\Xi_1)](\omega)$$

$$[F_{12}(\Xi_1 \times \Xi_2)](\omega) + [F_{12}(\Xi_1 \times \Xi_2)](\omega) = [F_{12}^{(2)}(\Xi_2)](\omega)$$

$$[F_{12}(\Xi_1 \times \Xi_2)](\omega) + [F_{12}(\Xi_1 \times \Xi_2)](\omega) = [F_{12}^{(2)}(\Xi_2)](\omega)$$

Lemma 5.4 [The condition of quasi-product observables (cf. [8])]

Let $O_1 = (X_1, \mathcal{F}_1, F_1)$ and $O_2 = (X_2, \mathcal{F}_2, F_2)$ be observables in $C(\Omega)$. Let $O_{12} = (X_1 \times X_2, \mathcal{F}_1 \times \mathcal{F}_2, F_{12}=F_{1 \circ \circ 2})$ be a quasi-product observable of $O_1$ and $O_2$. That is, it holds that

$$F_1 = F_{12}^{(1)}, \quad F_2 = F_{12}^{(2)}$$

Then, putting $\alpha_{\Xi_1 \times \Xi_2}(\omega) = [F_{12}(\Xi_1 \times \Xi_2)](\omega)$, we see

$$\text{Rep}_{\mathcal{F}_1 \times \mathcal{F}_2}[O_{12}] = \left[ [F_{12}(\Xi_1 \times \Xi_2)](\omega), [F_{12}(\Xi_1 \times \Xi_2)](\omega) \right]$$

$$= \left[ \alpha_{\Xi_1 \times \Xi_2}(\omega), \left[ F_1(\Xi_1)](\omega) - \alpha_{\Xi_1 \times \Xi_2}(\omega) \right] + \alpha_{\Xi_1 \times \Xi_2}(\omega) \left[ F_2(\Xi_2)](\omega) - \alpha_{\Xi_1 \times \Xi_2}(\omega) \right] \right]$$

$\left(5.1\right)$
and
\[
\max\{0, [F_1(\Xi_1)](\omega) + [F_2(\Xi_2)](\omega) - 1\} \leq \alpha_{\Xi_1 \times \Xi_2}(\omega) \leq \min\{[F_1(\Xi_1)](\omega), [F_2(\Xi_2)](\omega)\}
\]
\[
(\forall \Xi_1 \in \mathcal{F}_1, \forall \Xi_2 \in \mathcal{F}_2, \forall \omega \in \Omega)
\]
(5.2)

Reversely, for any $\alpha_{\Xi_1 \times \Xi_2} \in C(\Omega)$ such that (5.2), the observable $O_{12}$ defined by (5.1) is a quasi-product observable of $O_1$ and $O_2$. Also, it holds that
\[
[F(\Xi_1 \times \Xi_2)](\omega) = 0 \iff \alpha_{\Xi_1 \times \Xi_2}(\omega) = [F_1(\Xi_1)](\omega) \iff [F_1(\Xi_1)](\omega) \leq [F_2(\Xi_2)](\omega)
\]
(5.3)

Consider yes-no observables $O_1 \equiv (X_1, 2^{X_1}, F_1)$ and $O_2 \equiv (X_2, 2^{X_2}, F_2)$ in $C(\Omega)$ such that:
\[
X_1 = \{y_1, n_1\} \quad \text{and} \quad X_2 = \{y_2, n_2\}.
\]

Let $O_{12} \equiv (X_1 \times X_2, 2^{X_1} \times X_2, F \equiv F_1 \times O_{12} F_2)$ be a quasi-product observable with the marginal observables $O_1$ and $O_2$.

Put
\[
\text{Rep}[O_{12}] = \begin{bmatrix}
[F(\{(y_1, y_2)\})](\omega) & [F(\{(y_1, y_2)\})](\omega) \\
[F(\{(n_1, y_2)\})](\omega) & [F(\{(n_1, n_2)\})](\omega)
\end{bmatrix}
\]
\[
= \begin{bmatrix}
\alpha(\omega) & [F_1(\{y_1\})](\omega) - \alpha(\omega) \\
[F_2(\{y_2\})](\omega) - \alpha(\omega) & 1 + \alpha(\omega) - [F_1(\{y_1\})](\omega) - [F_2(\{y_2\})](\omega)
\end{bmatrix},
\]
where $\alpha \in C(\Omega)$. (Note that $[F(\{(y_1, y_2)\})](\omega) + [F(\{(y_1, n_2)\})](\omega) = [F_1(\{y_1\})](\omega)$ and $[F(\{(n_1, y_2)\})](\omega) + [F(\{(n_1, n_2)\})](\omega) = [F_2(\{y_2\})](\omega)$.)

That is,
\[
\begin{array}{c|c|c}
F_1 \setminus F_2 & [F_2(\{y_2\})](\omega) & [F_2(\{n_2\})](\omega) \\
\hline
[F_1(\{y_1\})](\omega) & \alpha(\omega) & [F_1(\{y_1\})](\omega) - \alpha(\omega) \\
[F_1(\{n_1\})](\omega) & [F_2(\{y_2\})](\omega) - \alpha(\omega) & 1 + \alpha(\omega) - [F_1(\{y_1\})](\omega) - [F_2(\{y_2\})](\omega)
\end{array}
\]

Then, it holds that
\[
\max\{0, [F_1(\{y_1\})](\omega) + [F_2(\{y_2\})](\omega) - 1\} \leq \alpha(\omega) \leq \min\{[F_1(\{y_1\})](\omega), [F_2(\{y_2\})](\omega)\}
\]
(\forall \omega \in \Omega).

Conversely, for any $\alpha \in C(\Omega)$ that satisfies (5.2), the observable $O_{12}$ defined by (5.1) is a quasi-product observable with the marginal observables $O_1$ and $O_2$. Also, note that
\[
[F(\{(y_1, n_2)\})](\omega) = 0 \iff \alpha(\omega) = [F_1(\{y_1\})](\omega) \Rightarrow [F_1(\{y_1\})](\omega) \leq [F_2(\{y_2\})](\omega).
\]

Proof. Though this lemma is easy, we add a brief proof for completeness. Since $0 \leq [F(\{(x_1^\omega, x_2^\omega)\})](\omega) \leq 1$, (5.1) that
\[
0 \leq \alpha(\omega) \leq 1, \quad 0 \leq [F_1(\{y_1\})](\omega) - \alpha(\omega) \leq 1, \quad 0 \leq [F_2(\{y_2\})](\omega) - \alpha(\omega) \leq 1,
\]
\[
0 \leq 1 + \alpha(\omega) - [F_1(\{y_1\})](\omega) - [F_2(\{y_2\})](\omega) \leq 1,
\]

which clearly implies (5.2). Conversely, if \( \alpha \) satisfies (5.2), then we easily see (5.1). Also, (5.2) is obvious. This completes the proof. \( \square \)

Let \( O_{12} = (X_1 \times X_2, F_1 \times F_2, F_{12} = F_1^{\text{op}} \times F_2) \) be a quasi-product observable (in \( C(\omega) \)) of \( O_1 = (X_1, F_1, F_1) \) and \( O_2 = (X_2, F_2, F_2) \). Assume that a measured value \( (x_1, x_2) \) \( \in X_1 \times X_2 \) is obtained by a measurement \( \Lambda_{C(\omega)}(O_{12} = (X_1 \times X_2, F_1 \times F_2, F_{12} = F_1^{\text{op}} \times F_2, S_1)) \). If we know that \( x_1 \in \Xi_1 \), then we can calculate the probability \( P \) that \( x_2 \in \Xi_2 \) (that is, the conditional probability ) is given by

\[
P = \frac{[F_{12}(\Xi_1 \times \Xi_2)](\omega)}{[F_1(\Xi_1)](\omega)} = \frac{[F_{12}(\Xi_1 \times \Xi_2)](\omega)}{[F_{12}(\Xi_1 \times \Xi_2)](\omega) + [F_{12}(\Xi_1 \times \Xi_2^2)](\omega)}
\]

And it is, by (5.2), estimated as follows.

\[
\max\{0, [F_1(\Xi_1)](\omega) - [F_2(\Xi_2)](\omega) - 1\} \leq P \leq \min\{[F_1(\Xi_1)](\omega), [F_2(\Xi_2)](\omega)\}
\]

**Example 5.5 [Tomatoes]** Let \( \Omega = \{\omega_1, \omega_2, \ldots, \omega_N\} \) be a set of tomatoes, which is regarded as a compact Hausdorff space with the discrete topology. Consider yes-no observables \( O_{\text{RD}} \equiv (X_{\text{RD}}, 2^{X_{\text{RD}}}, F_{\text{RD}}) \) and \( O_{\text{SW}} \equiv (X_{\text{SW}}, 2^{X_{\text{SW}}}, F_{\text{SW}}) \) in \( C(\Omega) \) such that:

\[
X_{\text{RD}} = \{y_{\text{RD}}, n_{\text{RD}}\} \quad \text{and} \quad X_{\text{SW}} = \{y_{\text{SW}}, n_{\text{SW}}\},
\]

where we consider that “\( y_{\text{RD}} \)” and “\( n_{\text{RD}} \)” respectively mean “RED” and “NOT RED”. Similarly, “\( y_{\text{SW}} \)” and “\( n_{\text{SW}} \)” respectively mean “SWEET” and “NOT SWEET”.

For example, the \( \omega_1 \) is red and not sweet, the \( \omega_2 \) is red and sweet, etc. as follows.

![Figure 5.1: Tomatoes (Red?, Sweet?)](image)

Next, consider the quasi-product observable as follows.

\[
O_{12} = (X_{\text{RD}} \times X_{\text{SW}}, 2^{X_{\text{RD}}} \times X_{\text{SW}}, F = F_{\text{RD}}^{\text{op}} \times F_{\text{SW}})
\]

That is,

\[
\text{Rep}_{\omega_k}^{(y_{\text{RD}}, y_{\text{SW}})} [O_{12}] = \begin{bmatrix}
F(\{(y_{\text{RD}}, y_{\text{SW}})\})(\omega_k) & F(\{(y_{\text{RD}}, n_{\text{SW}})\})(\omega_k) \\
F(\{(n_{\text{RD}}, y_{\text{SW}})\})(\omega_k) & F(\{(n_{\text{RD}}, n_{\text{SW}})\})(\omega_k)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\alpha_{(y_{\text{RD}}, y_{\text{SW}})} & [F_{\text{RD}}(\{y_{\text{RD}}\})] - \alpha_{(y_{\text{RD}}, n_{\text{SW}})} \\
[F_{\text{SW}}(\{y_{\text{SW}}\})] - \alpha_{(n_{\text{RD}}, y_{\text{SW}})} & 1 + \alpha_{(n_{\text{RD}}, n_{\text{SW}})} - [F_{\text{RD}}(\{y_{\text{RD}}\})] - [F_{\text{SW}}(\{y_{\text{SW}}\})]
\end{bmatrix}
\]
where \( \alpha_{(\text{RD}, \text{SW})}(\omega_k) \) satisfies the (5.2). When we know that a tomato \( \omega_k \) is red, the probability \( P \) that the tomato \( \omega_k \) is sweet is given by

\[
P = \frac{[F((y_{\text{RD}}, y_{\text{SW}}))](\omega_k)}{[F((y_{\text{RD}}, y_{\text{SW}}))](\omega_k) + [F((y_{\text{RD}}, n_{\text{SW}}))](\omega_k)} = \frac{[F((y_{\text{RD}}, y_{\text{SW}}))](\omega_k)}{[F_{\text{RD}}(\{y_{\text{RD}}\})](\omega_k)}
\]

Since \([F((y_{\text{RD}}, y_{\text{SW}}))](\omega_k) = \alpha_{(\text{RD}, \text{SW})}(\omega_k)\), the conditional probability \( P \) is estimated by

\[
\frac{\max\{0, [F_1((y_{\text{RD}}))](\omega_k) + [F_2((y_{\text{SW}}))](\omega_k) - 1\}}{[F_{\text{RD}}(\{y_{\text{RD}}\})](\omega_k)} \leq P \leq \frac{\min\{[F_1((y_{\text{SW}}))](\omega_k), [F_2((y_{\text{SW}}))](\omega_k)\}}{[F_{\text{RD}}(\{y_{\text{RD}}\})](\omega_k)}
\]

5.3 Implication — The definition of "\( \Rightarrow \)"

5.3.1 Implication and contraposition

In Example 5.5, consider the case that \([F((y_{\text{RD}}, n_{\text{SW}}))](\omega) = 0\). In this case, we see

\[
\frac{[F((y_{\text{RD}}, y_{\text{SW}}))](\omega)}{[F((y_{\text{RD}}, y_{\text{SW}}))](\omega) + [F((y_{\text{RD}}, n_{\text{SW}}))](\omega)} = 1
\]

Therefore, when we know that a tomato \( \omega \) is red, the probability that the tomato \( \omega \) is sweet Ig equal to 1. That is,

\[
"[F((y_{\text{RD}}, n_{\text{SW}}))](\omega) = 0" \quad \iff \quad \"\text{Red} \Rightarrow \text{Sweet}\"
\]

Motivated by the above argument, we have the following definition.

**Definition 5.6 [Implication]** Let \( O_{12\ldots n} = (\times_{k=1}^n X_k, \otimes_{k=1}^n F_k) \), \( F_{12\ldots n} = \otimes_{k=1}^n F_k \) be a quasi-product observable in \( C(\Omega) \). Let \( \Xi_1 \in \mathcal{P}(X_1) \) and \( \Xi_2 \in \mathcal{P}(X_2) \). Then, the condition

\[
F_{12\ldots n}^{(ij)}(\Xi_i \times (\Xi_j^c))(\omega) = 0
\]

is denoted by

\[
[O_{12\ldots n}^{(ij)}: \Xi_i]_{M_{C(\Omega)}(O_{12\ldots n}, S_{[\omega]})} \Longrightarrow [O_{12\ldots n}^{(ij)}: \Xi_j]
\]

///

Of course, it should be read as follows.

Assume that a measured value \((x_1, x_2)\in X_1 \times X_2\) is obtained by a measurement \(M_{C(\Omega)}(O_{12}, S_{[\omega]})\).

When we know that \( x_1 \in \Xi_1 \), then we can assure that \( x_2 \in \Xi_2 \).

**Theorem 5.7 [Contraposition]** Let \( O_{12} = (X_1 \times X_2, F_1 \times F_2) \), \( F_{12} = F_1 \otimes F_2 \) be a quasi-product observable in \( C(\Omega) \). Let \( \omega \in \Omega \). Let \( \Xi_1 \in F_1 \) and \( \Xi_2 \in F_2 \). If it holds that

\[
[O_{12}^{(1)}: \Xi_1]_{M_{C(\Omega)}(O_{12}, S_{[\omega]})} \Longrightarrow [O_{12}^{(2)}: \Xi_2]
\]

(5.4)
then we see:

\[
\begin{align*}
\mathbf{O}^{(1)}_{12} : \Xi_1 & \xleftarrow{M_{C(\Omega)}} \mathbf{O}^{(2)}_{12} : \Xi_2
\end{align*}
\]

\textbf{Proof.} The proof is easy, but we add it. Assume the condition (5.4). That is,

\[
[F_{12}(\Xi_1 \times (X_2 \setminus \Xi_2))](\omega) = 0
\]

Since \(\Xi_1 \times \Xi_2^c = (\Xi_1^c) \times \Xi_2^c\) we see

\[
[F_{12}((\Xi_1^c) \times \Xi_2^c)](\omega) = 0
\]

Therefore, we get

\[
\begin{align*}
\mathbf{O}^{(1)}_{12} : \Xi_1 & \xleftarrow{M_{C(\Omega)}} \mathbf{O}^{(2)}_{12} : \Xi_2
\end{align*}
\]

\(\square\)

\textbf{Note 5.5} In what follows, reconsider the statistical hypothesis testing (Sec. 4.3.4). Consider the observable \(O_{Nh} = (Y = \{0, 1\}, 2^{\{0, 1\}}, F_{Nh})\) in \(C(\Omega)\) such that

\[
[F_{Nh}(\{1\})](\omega) = \begin{cases} 1 & (\omega \in Nh) \\ 0 & (\omega \notin Nh) \end{cases}, \quad [F_{Nh}(\{0\})](\omega) = 1 - [F_{Nh}(\{1\})](\omega)
\]

( Even if \(F_{Nh}(\{1\}) \notin C(\Omega)\), we do not mind it (cf. Chap. 10)). And consider the simultaneous measurement \(M_{C(\Omega)}(O_{Nh} \times O = (Y \times X, 2^{\{0, 1\}} \times F, F_{Nh} \times F), S_{[\omega]})\). If the measured value \((y, x)\) belongs to \(\{1\} \times X\), the probability that \(x \notin [D]_{Nh}^{0.05}\) is estimated by

\[
\frac{[(F_{Nh} \times F)((1) \times (X \setminus [D]_{Nh}^{0.05}))](\omega)}{[(F_{Nh} \times F)((1) \times X)](\omega)} = [F(X \setminus [D]_{Nh}^{0.05})](\omega) \geq 0.95
\]

That is, "\(y = 1\)" implies "mostly, \(x \notin [D]_{Nh}^{0.05}\). This contraposition (i.e., if "\(x \in [D]_{Nh}^{0.05}\), then it is rare that \(y = 1\)) is similar to statistical hypothesis testing (cf. [20]).

5.3.2 "I think, therefore I am" is doubtful

The previous example is somewhat unnatural, it may be dispensable for the understanding of dualism.

\textbf{Example 5.8 [Brain death]} Let \(\omega_n (\in \Omega = \{\omega_1, \omega_2, \ldots, \omega_N\})\) be the state of Peter. Let \(O_{12} = (X_1 \times X_2, 2^{X_1 \times X_2}, F_{12} = F_1 \times F_2)\) be the brain death observable in \(C(\Omega)\) such that \(X_1 = \{T, \overline{T}\}\ X_2 = \{L, \overline{L}\},\) where \(T = "\text{think}\", \overline{T} = "\text{not think}\", L = "\text{live}\", \overline{L} = "\text{not live}\"\). For each \(\omega_n (n = 1, 2, \ldots, N), O_{12}\) satisfies the condition in Table 5.1.

| \(F_1 \setminus F_2\) | \(F_2((\{L\}))(\omega_n)\) | \(F_2((\{\overline{L}\}))(\omega_n)\) |
|-----------------|-----------------|-----------------|
| \(F_1((\{T\}))(\omega_n)\) | \((1 + (-1)^n)/2\) | \((= [F_{12}((\{T\}) \times (\{L\})](\omega_n))\) |
| \(F_1((\{\overline{T}\}))(\omega_n)\) | \((= [F_{12}((\{\overline{T}\}) \times (\{L\}])](\omega_n))\) | \((1 - (-1)^n)/2\) |
Since \( F_{12}(\{T\} \times \{\mathcal{T}\})(\omega_n) = 0 \), the following formula holds:

\[
[O_{12}^{(1)}; \{T\}]_{M_{C(13)}(O_{12}; S_{\omega_n})} \rightarrow [O_{12}^{(2)}; \{L\}]
\]

Of course, this implies that

Peter thinks, therefore, Peter lives.

This is the same as the statement concerning brain death. Note that in the above example, we see that

observer \( \leftrightarrow \) doctor, system \( \leftrightarrow \) Peter,

This should not be confused with the following famous Descartes’ saying:

"I think, therefore I am".

in which the following identification may be assumed:

observer \( \leftrightarrow \) I, system \( \leftrightarrow \) I

And thus, the above is not a statement in dualism (=measurement theory). In order to propose Fig. 1.1 (i.e., dualism) (that is, in order to establish the concept “T” in science), he started from the ambiguous statement "I think, therefore I am". Summing up, we want to say the following irony:

Descartes proposed the dualism (i.e., Fig. 1.1) by the statement (\#1) which is not understandable in dualism.

Even in physics, there is a case that a meaningless statement, which is useful to create the theory, becomes famous. This (i.e., Heisenberg’s uncertainty principle (Proposition 3.1)) is already pointed out in Sec.3.4.

\[ \text{Note 5.6} \] It is not true to consider that every phenomena can be describe in terms of measurement theory. Readers may think that the following can be described in measurement theory, but I believe that it is impossible. For example,

(\#1) "observer’s space-time", "tense—past, present, future —", "Heidegger’s saying: “In-der-Welt-sein”", "the measurement of a measurement", "Only the present exists", "subjective time (Note 6.7)"

Although these words cannot not be understood, we think that these are inconsistent with the Copenhagen interpretation [Chap. 1(U_1)–(U_7)].

5.4 Practical syllogism — Is Socrates mortal?

The term: "practical syllogism" means "syllogism in measurement theory. And thus, the syllogism should be proved in measurement theory.
5.4.1 Combined observable — Observable is only one

The Copenhagen interpretation says that observable must be only one. Thus, many observables must be combined.

**Theorem 5.9 [Combined observable (cf. [8])]** Let $O_{12} = (X_1 \times X_2, \mathcal{F}_1 \times \mathcal{F}_2, F_{12})$ and $O_{23} = (X_2 \times X_3, \mathcal{F}_2 \times \mathcal{F}_3, F_{23})$ be observables in $C(\Omega)$. Here, for simplicity, assume that $X_i = \{x_1^i, x_2^i, \ldots, x_n^i\}$ ($i = 1, 2, 3$) is finite. Also, assume that $\mathcal{F}_i = 2^{X_i}$. Further assume that

$$O_{12}^{(2)} = O_{23}^{(2)} \quad \text{(That is, } F_{12}(X_1 \times \Xi_2) = F_{23}(\Xi_2 \times X_3) \quad (\forall \Xi_2 \in 2^{X_2}))$$

Then, we have the observable $O_{123} = (X_1 \times X_2 \times X_3, \mathcal{F}_1 \times \mathcal{F}_2 \times \mathcal{F}_3, F_{123})$ in $C(\Omega)$ such that

$$O_{123}^{(12)} = O_{12}, \quad O_{123}^{(23)} = O_{23}$$

That is,

$$F_{123}^{(12)}(\Xi_1 \times \Xi_2 \times X_3) = F_{12}(\Xi_1 \times \Xi_2), \quad F_{123}^{(23)}(X_1 \times \Xi_2 \times \Xi_3) = F_{23}(\Xi_2 \times \Xi_3)$$

$$\quad (\forall \Xi_1 \in \mathcal{F}_1, \forall \Xi_2 \in \mathcal{F}_2, \forall \Xi_3 \in \mathcal{F}_3) \quad (5.5)$$

The $O_{123}$ is called the combined observable of $O_{12}$ and $O_{23}$.

**Proof.** $O_{123} = (X_1 \times X_2 \times X_3, \mathcal{F}_1 \times \mathcal{F}_2 \times \mathcal{F}_3, F_{123})$ is, for example, defined by

$$[F_{123}((\{x_1, x_2, x_3\}))](\omega) = \begin{cases} [F_{12}((\{x_1, x_2\}))](\omega) \cdot [F_{23}((\{x_2, x_3\}))](\omega) & (|F_{12}(X_1 \times \{x_2\})|(\omega) \neq 0 \text{ and }) \\ 0 & (|F_{12}(X_1 \times \{x_2\})|(\omega) = 0 \text{ and }) \\ (\forall \omega \in \Omega, \forall (x_1, x_2, x_3) \in X_1 \times X_2 \times X_3) \end{cases}$$

This clearly satisfies (5.5). \qed

**Remark 5.10 [Bell’s inequality is useful]** Put $X_1 = X_2 = X_3 = X_4 = \{-1, 1\}$. Let $O_{13} = (X_1 \times X_3, 2^{X_1} \times 2^{X_3}, F_{13}), O_{14} = (X_1 \times X_4, 2^{X_1} \times 2^{X_4}, F_{14}), O_{23} = (X_2 \times X_3, 2^{X_2} \times 2^{X_3}, F_{23})$ and $O_{24} = (X_2 \times X_3, 2^{X_2} \times 2^{X_4}, F_{24})$ be observables in $C(\Omega)$ such that

$$O_{13}^{(1)} = O_{14}^{(1)}, \quad O_{23}^{(2)} = O_{24}^{(2)}, \quad O_{13}^{(3)} = O_{23}^{(3)}, \quad O_{14}^{(4)} = O_{24}^{(4)}$$

Define the probability measure $\nu_{ab}$ on $\{-1, 1\}^2$ by the formula (3.6). Assume that there exists a state
Then, we can easily get the following Bell’s inequality (cf. Note 3.4).

\[\omega_0 \in \Omega \text{ such that}\]
\[\begin{align*}
[F_{13}\{(x_1, x_3)\}](\omega_0) &= \nu_{a_1 b_1}\{(x_1, x_3)\}, \\
[F_{14}\{(x_1, x_4)\}](\omega_0) &= \nu_{a_1 b_2}\{(x_1, x_4)\}, \\
[F_{23}\{(x_2, x_3)\}](\omega_0) &= \nu_{a_2 b_1}\{(x_2, x_3)\}, \\
[F_{24}\{(x_2, x_4)\}](\omega_0) &= \nu_{a_2 b_2}\{(x_2, x_4)\}
\end{align*}\]

Now we have the following problem:

(a) Does the observable \(O_{1234} = (\times_{k=1}^4 X_k, \times_{k=1}^4 F_k, F_{134})\) in \(C(\Omega)\) satisfying the following (\(\ddagger\))?

\(\ddagger\) \(O^{(13)}_{1234} = O_{13}, \; O^{(14)}_{1234} = O_{14}, \; O^{(23)}_{1234} = O_{23}, \; O^{(24)}_{1234} = O_{24}\)

In what follows, we show that the above observable \(O_{1234}\) does not exist.

Assume that the observable \(O_{1234} = (\times_{k=1}^4 X_k, \times_{k=1}^4 F_k, F_{134})\) exists. Then, it suffices to show the contradiction. Define \(C_{13}(\omega_0), C_{14}(\omega_0), C_{23}(\omega_0)\) and \(C_{24}(\omega_0)\) such that

\[\begin{align*}
C_{13}(\omega_0) &= \int_{X_1^4} x_1 \cdot x_3 \; [F_{134}\{(x_1, x_3)\}](\omega_0) \\
&= \int_{X_1 \times X_3} x_1 \cdot x_3 \; \nu_{a_1 b_1}\{(x_1, x_3)\} (dx_1 dx_3) \\
C_{14}(\omega_0) &= \int_{X_1^4} x_1 \cdot x_4 \; [F_{134}\{(x_1, x_4)\}](\omega_0) \\
&= \int_{X_1 \times X_4} x_1 \cdot x_4 \; \nu_{a_1 b_2}\{(x_1, x_4)\} (dx_1 dx_4) \\
C_{23}(\omega_0) &= \int_{X_2^4} x_2 \cdot x_3 \; [F_{134}\{(x_2, x_3)\}](\omega_0) \\
&= \int_{X_2 \times X_3} x_2 \cdot x_3 \; \nu_{a_2 b_1}\{(x_2, x_3)\} (dx_2 dx_3) \\
C_{24}(\omega_0) &= \int_{X_2^4} x_2 \cdot x_4 \; [F_{134}\{(x_2, x_4)\}](\omega_0) \\
&= \int_{X_2 \times X_4} x_2 \cdot x_4 \; \nu_{a_2 b_2}\{(x_2, x_4)\} (dx_2 dx_4)
\end{align*}\]

Then, we can easily get the following Bell’s inequality (cf. Note 3.4).

\[
\begin{align*}
|C_{13}(\omega_0) - C_{14}(\omega_0)| + |C_{23}(\omega_0) + C_{24}(\omega_0)| \\
&\leq \int_{X_1^4} |x_1| \cdot |x_3 - x_4| + |x_2| \cdot |x_3 + x_4| \; [F_{134}\{(x_1, x_3)\}](\omega_0) \\
&\leq 2 \quad \text{(since } x_k \in \{-1, 1\}\)
\end{align*}\]  \(5.6\)

However, the formula (3.7) says that this (5.6) must be \(2\sqrt{2}\). Thus, by contradiction, we says that \(O_{1234}\) satisfying (a) does not exist. Thus we can not take a measurement \(M_{C(\Omega)}(O_{1234}, S_{[\omega_0]})\).

However, it should be noted that

(b) in stead of \(M_{C(\Omega)}(O_{1234}, S_{[\omega_0]})\), we can take a parallel measurement \(M_{C(\Omega^2)}(O_{13} \otimes O_{14} \otimes O_{23} \otimes O_{24}, \\
S_{[(\omega_0, \omega_0, \omega_0, \omega_0)]})\). In this case, we easily see that (5.6) = \(2\sqrt{2}\) as the formula (3.7).

That is,
(c) in the case of a parallel measurement, Bell’s inequality is broken in both quantum and classical systems.

\[ \text{Note 5.7} \] In the above argument, Bell’s inequality is used in the framework of measurement theory. This is of course true. However, since mathematics is of course independent of the world, now we have the following question:

1. In order that mathematical Bell’s inequality asserts something to quantum mechanics, what kind of idea do we prepare?

We can not answer this question.

### 5.4.2 Practical syllogism and its variations

Now we show several theorems of practical syllogisms (i.e., theorems concerning “implication” in Definition 5.6).

**Theorem 5.11** [Practical syllogism (cf. [8])] Let $O_{123} = (X_1 \times X_2 \times X_3, F_1 \times F_2 \times F_3, F_{123} = \prod_{k=1,2,3} F_k)$ be an observable in $C(\Omega)$. Fix $\omega \in \Omega$, $\Xi_1 \in F_1$, $\Xi_2 \in F_2$, $\Xi_3 \in F_3$. Then, we see the following (i) – (iii).

(i). (practical syllogism)

\[ O_{123; \Xi_1}^{(1)} \quad \Rightarrow \quad O_{123; \Xi_2}^{(2)} \quad \Rightarrow \quad O_{123; \Xi_3}^{(3)} \]

implies

\[ \text{Rep}_{\Xi_1 \times \Xi_3} \mathcal{O}_{123}^{(13)} = \begin{bmatrix} [F_{123}^{(13)}(\Xi_1 \times \Xi_3)](\omega) & [F_{123}^{(13)}(\Xi_1 \times \Xi_3^c)](\omega) \\ [F_{123}^{(13)}(\Xi_1^c \times \Xi_3)](\omega) & [F_{123}^{(13)}(\Xi_1^c \times \Xi_3^c)](\omega) \end{bmatrix} \]

That is, it holds:

\[ O_{123; \Xi_1}^{(1)} \quad \Rightarrow \quad O_{123; \Xi_3}^{(3)} \] (5.7)

(ii).

\[ O_{123; \Xi_1}^{(1)} \quad \Leftrightarrow \quad O_{123; \Xi_2}^{(2)} \quad \Rightarrow \quad O_{123; \Xi_3}^{(3)} \]

implies

\[ \text{Rep}_{\Xi_1 \times \Xi_3} \mathcal{O}_{123}^{(13)} = \begin{bmatrix} [F_{123}^{(13)}(\Xi_1 \times \Xi_3)](\omega) & [F_{123}^{(13)}(\Xi_1 \times \Xi_3^c)](\omega) \\ [F_{123}^{(13)}(\Xi_1^c \times \Xi_3)](\omega) & [F_{123}^{(13)}(\Xi_1^c \times \Xi_3^c)](\omega) \end{bmatrix} \]

\[ = \begin{bmatrix} \alpha_{\Xi_1 \times \Xi_3} & [F_{123}^{(1)}(\Xi_1)](\omega) - \alpha_{\Xi_1 \times \Xi_3} \\ [F_{123}^{(3)}(\Xi_3)](\omega) - \alpha_{\Xi_1 \times \Xi_3} & 1 - \alpha_{\Xi_1 \times \Xi_3} - [F_{123}^{(1)}(\Xi_1)] - [F_{123}^{(3)}(\Xi_3)] \end{bmatrix} \]
Thus, we get,

\[ \alpha_{_{E_1 \times E_3}}(\omega) \leq \min\{[F_{123}^{(1)}(\Xi_1)](\omega), [F_{123}^{(3)}(\Xi_3)](\omega)\} \]

(iii).

\[ [O_{123}^{(1)}; \Xi_1] \quad \overset{M_{C'(1)}}{\Rightarrow} \quad [O_{123}^{(2)}; \Xi_2], \quad [O_{123}^{(2)}; \Xi_2] \quad \overset{M_{C'(1)}}{\Leftarrow} \quad [O_{123}^{(3)}; \Xi_3] \]

implies

\[ \text{Rep}_{\omega}^i_{_{E_1 \times E_3}}\{O_{123}^{(13)}\} = \begin{bmatrix} [F_{123}^{(13)}(\Xi_1 \times \Xi_3)](\omega) & [F_{123}^{(13)}(\Xi_1 \times \Xi_3)](\omega) \\ [F_{123}^{(13)}(\Xi_1 \times \Xi_3)](\omega) & [F_{123}^{(13)}(\Xi_1 \times \Xi_3)](\omega) \end{bmatrix} \]

where

\[ \max\{0, [F_{123}^{(1)}(\Xi_1)](\omega) + [F_{123}^{(3)}(\Xi_3)](\omega) - [F_{123}^{(2)}(\Xi_2)](\omega)\} \leq \alpha_{_{E_1 \times E_3}}(\omega) \leq \min\{[F_{123}^{(1)}(\Xi_1)](\omega), [F_{123}^{(3)}(\Xi_3)](\omega)\} \]

**Proof.** (i): By the condition, we see

\[ 0 = [F_{123}^{(12)}(\Xi_1 \times \Xi_2)](\omega) = [F_{123}(\Xi_1 \times \Xi_2 \times \Xi_3)](\omega) + [F_{123}(\Xi_1 \times \Xi_2 \times \Xi_3)](\omega) \]

\[ 0 = [F_{123}^{(23)}(\Xi_2 \times \Xi_3)](\omega) = [F_{123}(\Xi_1 \times \Xi_2 \times \Xi_3)](\omega) + [F_{123}(\Xi_1 \times \Xi_2 \times \Xi_3)](\omega) \]

Therefore,

\[ 0 = [F_{123}(\Xi_1 \times \Xi_2 \times \Xi_3)](\omega) = [F_{123}(\Xi_1 \times \Xi_2 \times \Xi_3)](\omega) \]

\[ 0 = [F_{123}(\Xi_1 \times \Xi_2 \times \Xi_3)](\omega) = [F_{123}(\Xi_1 \times \Xi_2 \times \Xi_3)](\omega) \]

Hence,

\[ [F_{123}^{(13)}(\Xi_1 \times \Xi_3)](\omega) = [F_{123}(\Xi_1 \times \Xi_2 \times \Xi_3)](\omega) + [F_{123}^{(13)}(\Xi_1 \times \Xi_2 \times \Xi_3)](\omega) = 0 \]

Thus, we get, (5.7).

For the proof of (ii) and (iii), see [8]. \( \square \)

**Example 5.12 [Continued from Example 5.5]** Let \( O_1 = O_{SW} = (X_{SW}, 2^{X_{SW}}, F_{SW}) \) and \( O_3 = O_{RD} = (X_{RD}, 2^{X_{RD}}, F_{RD}) \) be as in Example 5.5. Putting \( X_{RP} = \{y_{RP}, n_{RP}\} \), consider the new observable \( O_2 = O_{RP} = (X_{RP}, 2^{X_{RP}}, F_{RP}) \). Here, “\( y_{RP} \)” and “\( n_{RP} \)” respectively means “ripe” and “not ripe”. Put

\[ \text{Rep}[O_1] = \{[F_{SW}(\{y_{SW}\})(\omega_k), [F_{SW}(\{n_{SW}\})(\omega_k)] \]

\[ \text{Rep}[O_2] = \{[F_{RP}(\{y_{RP}\})(\omega_k), [F_{RP}(\{n_{RP}\})(\omega_k)] \]

\[ \text{Rep}[O_3] = \{[F_{RD}(\{y_{RD}\})(\omega_k), [F_{RD}(\{n_{RD}\})(\omega_k)] \]
Consider the following quasi-product observable:

\[
O_{12} = (X_{SW} \times X_{RP}, 2X_{SW} \times X_{RP}, f_{12} = F_{SW}^{qp} F_{RP})
\]
\[
O_{23} = (X_{RP} \times X_{RD}, 2X_{RP} \times X_{RD}, f_{23} = F_{RP}^{qp} F_{RD})
\]

Let \( \omega_k \in \Omega \). And assume that

\[
\begin{align*}
(O_{123}^{(1)}; \{y_{SW}\}) & \quad \xrightarrow{M_{C(\Omega)}(O_{123}, S_{\omega_k})} \quad (O_{123}^{(2)}; \{y_{RP}\}) , \\
(O_{123}^{(2)}; \{y_{RP}\}) & \quad \xrightarrow{M_{C(\Omega)}(O_{123}, S_{\omega_k})} \quad (O_{123}^{(3)}; \{y_{RD}\})
\end{align*}
\]

Then, by Theorem 5.11(i), we get:

\[
\text{Rep}[O_{13}] = \begin{bmatrix}
[F_{13}(\{y_{SW}\} \times \{y_{RD}\})(\omega_k) & [F_{13}(\{y_{SW}\} \times \{n_{RD}\})(\omega_k) \\
[F_{13}(\{n_{SW}\} \times \{y_{RD}\})(\omega_k) & [F_{13}(\{n_{SW}\} \times \{n_{RD}\})(\omega_k)
\end{bmatrix}
\]

Therefore, when we know that the tomato \( \omega_k \) is sweet by measurement \( M_{C(\Omega)}(O_{123}, S_{\omega_k}) \), the probability that \( \omega_k \) is red is given by

\[
\frac{[F_{13}(\{y_{SW}\} \times \{y_{RD}\})(\omega_k)}{[F_{13}(\{y_{SW}\} \times \{y_{RD}\})(\omega_k) + [F_{13}(\{y_{SW}\} \times \{n_{RD}\})(\omega_k)]} = \frac{[F_{13}(\{y_{RD}\})(\omega_k)}{[F_{13}(\{y_{RD}\})(\omega_k) + [F_{13}(\{n_{SW}\} \times \{y_{RD}\})(\omega_k)]} = 1
\]

Of course, (5.9) means

“Sweet” \( \implies \) “Ripe” \quad “Ripe” \( \implies \) “Red”

Therefore, by (5.10), we get the following conclusion.

“Sweet” \( \implies \) “Red”

However, it is not useful in the market. What we want to know is such as

“Sweet” \( \implies \) “Sweet”

This will be discussed in the following example.

**Example 5.13** [Continued from Example 5.12, [8]]. Instead of (5.9), assume that

\[
\begin{align*}
O_1^{(y_1)} & \quad \xleftarrow{M_{C(\Omega)}(O_{123}, S_{\omega_k})} \quad O_2^{(y_2)} , \\
O_2^{(y_2)} & \quad \xrightarrow{M_{C(\Omega)}(O_{123}, S_{\omega_k})} \quad O_3^{(y_3)}
\end{align*}
\]

When we observe that the tomato \( \omega_n \) is “RED”, we can infer, by the fuzzy inference \( M_{C(\Omega)}(O_{13}, S_{\delta_{y_n}}) \), the probability that the tomato \( \omega_n \) is “SWEET” is given by

\[
Q = \frac{[F_{13}(\{y_{SW}\} \times \{y_{RD}\})(\omega_n)}{[F_{13}(\{y_{SW}\} \times \{y_{RD}\})(\omega_n) + [F_{13}(\{n_{SW}\} \times \{y_{RD}\})(\omega_n)]}
\]
which is, by (5.8), estimated as follows:

\[
\max \left\{ \frac{[F_{RP}(\{y_{RP}\})(\omega_n)]}{[F_{RD}(\{y_{RD}\})(\omega_n)]}, \frac{[F_{SW}(\{y_{SW}\})(\omega_n)]}{[F_{RD}(\{y_{RD}\})(\omega_n)]} \right\} \leq Q \leq \min \left\{ \frac{[F_{SW}(\{y_{SW}\})(\omega_n)]}{[F_{RD}(\{y_{RD}\})(\omega_n)]}, 1 \right\}.
\]

(9.12)

Note that (5.11) implies (and is implied by)

“RIPE” \implies “SWEET” and “RIPE” \implies “RED”.

And note that the conclusion (5.12) is somewhat like

“RED” \implies “SWEET”.

Therefore, this conclusion is peculiar to “fuzziness”.

\[\star\] Note 5.8 Recall the (A2) in Sec.5.1, that is,

(♯1) Since Socrates is a man and all men are mortal, it follows that Socrates is mortal.
(♯2) Flying arrow is not moving.
(♯3) I think, therefore I am.
(♯4) Edison’s ”1 + 1 = 2”

In this chapter, the above (♯1) and (♯3) are clarified.

If these four statements are childish, we can not explain the fact that these tales have been transmitted from generation to generation, and these have been continuously passed by many persons with sharp sensibility. However, measurement theory urges us to understand (♯1)–(♯4) without sharp sensibility. Recall Chap. 1 (X1):

(X1) widely ordinary language (before science) \implies world-description (Chap. 1(O)) \begin{align*}
\{ & 1 \text{realistic method} \\
& \text{(realistic world-view)} \\
& 2 \text{linguistic method} \\
& \text{(linguistic world-view)} \end{align*}

If we believe in this, our problem is as follows.

Should each (♯1)–(♯4) be discussed in (0), 1 or 2?

In this chapter, (♯1) and (♯3) are explained. The (♯2) and (♯4) will be discussed in Chap. 11.
6 Axiom$_{PM}$ 2 - causality

Measurement theory is formulated as follows:

\[
\text{measurement theory} = \text{[Axiom 1] measurement [probabilistic interpretation]} + \text{[Axiom 2] causality [the Heisenberg picture]}
\]

Although the preceding chapter was an introduction of the Axiom 1 about measurement, from this chapter, I explain the Axiom 2 about movement and change (causality). If I say definitely roughly,

(2) Science is the learning of causality, i.e., the learning about the phenomenon which can be expressed in the word "causality."

Therefore, in this chapter, we arrived at "causal relationship" in the main question at last. However, in dualism, after understanding "measurement" enough, unless it comes out, we cannot understand "movement and change." In dualism, it is because we understand "movement and change" as a debt of "measurement" and "causality."

6.1 Outstanding-problem -What is causality?

6.1.1 Modern science started from the discovery of "causality."

When a certain thing happens, the cause exists. This is called causality. You should just remember the proverb of "smoke is not located on the place which does not have fire." It is not so simple although you may think that it is natural. For example, if you consider

Is it because that my feeling feels it refreshed this morning went to sleep well last night? or is it because I go to favorite golf from now on?

you may be able to understand the difficulty of how to use the word "causality. In daily conversation, it is used in many cases, mixing up "a cause (past)" , "a reason (connotation)" , and "the purpose and a motive (future)."

It may be supposed that Heraclitus’s (BC.540 -BC.480) "Everything changes." and "Movement does not exist." of Parmenides (born around BC. 515) who is Zeno’s teacher are the beginning of research of movement and change. However, those meanings are not clear. However, these two pioneers - Heraclitus and Parmenides - noticed first that "movement and change" were the primary importance keywords in science(= "world description") , i.e., it is

\[\text{[World description]} = \text{[Description of movement and change]}\]

However, Aristotle (BC384–BC322) further investigated about the essence of movement and change, and he thought that all the movements had the "purpose." For example, supposing a stone falls, that is because there is the purpose that the stone tries to go downward. Supposing smoke rises, that is because there is the purpose that smoke rises upwards. Under the influence of Aristotle, "Purpose" continued remaining as a mainstream idea of "Movement" for a long time of 1500 or more.
Although "the further investigation" of Aristotle was what should be praised, it was not able to be said that "the purpose was to the point." In order to free ourselves from Purpose and for human beings to discover that the essence of movement and change is "causal relationship", we had to wait for the appearance of Galileo, bacon, Descartes, Newton, etc.

Revolution to "Causality" from "Purpose"

is the greatest history-of-science top paradigm shift - It is not an overstatement even if we call it "birth of modern science". - , and determined the "scientific revolution" after it.

6.1.2 Four answers to "what is causality?"

As mentioned above, about "what is an essence of movement and change?", it was once settled with the word "causality." However, not all were solved now. We do not yet understand "causality" fully. In fact,

"What is causality?" is the most important outstanding problems in science.

There may be a reader who is surprised with saying like this although it is the outstanding problems in the present. Below, I arrange the history of the answer to this problem.

(a) [Realistic causality]: Newton advocated the realistic describing method of Newtonian mechanics as a final settlement of accounts of ideas, such as Galileo, bacon, and Descartes, and he thought as follows. :

"Causality" actually exists in the world. The equation of motion of Newton described faithfully this "causality" that exists in fact by the differential equation - the equation of a causal chain -.

This realistic causality may be a very natural idea, and you may think that you cannot think in addition to this. In fact, probably, we may say that the current of the realistic causal relationship which continues like "Newtonian mechanics → Electricity and magnetism → Theory of relativity → ···" is a scientific flower.

However, there is also another idea and there is three "nonexistent causalities" as follows.

(b) [Cognitive causality]: Hume, Kant, etc. who are philosophers thought as follows. :

We can not say that "Causality" actually exists in the world, or that it do not exist in the world. And when we think that "something" in the world is "causality", we should just believe that the it has "causality".

Several readers may regard this as it being "a kind of rhetoric", moreover, several readers may be convinced in "That is right if you say so." Surely, since you are looking through the prejudice "causality", you may
look such. It is Kant’s famous "Copernican revolution" (that is, "recognition constitutes the world.") which is considered that the recognition circuit of causality is installed in the brain, and when it is stimulated by "something" and reacts, "there is causal relationship." (Refer to later Section 8.1.) Probably, many readers doubt about the substantial influence which this (b) had on the science after it. However, in this book (Refer to later Section 8.1), I adopted the friendly story to the utmost to Kant.

(c) [Mathematical causality(Dynamical system theory)]: Since dynamical system theory has developed as the mathematical technique in engineering, they have not investigated "What is causality?" thoroughly. However,

In dynamical system theory, we think that there is mathematics of an equation of state previously (i.e., the time first degree alliance differential equation of the variable (1.1)) , and the phenomenon described with the equation has "causality." (Refer to Chap. 1(E)).

With the ordinary feeling of science, you may tend to understand this (c) because you think somehow "=.” However, you should be cautious of it being a typical example of the form of the mathematics buried into ordinary language. However, for the purpose of "Being helpful", I think that (c) should be evaluated more.

(d) [Linguistic causal relationship (Measurement Theory)]: The causal relationship of measurement theory is decided by the Axiom 2 of this chapter. If I say in detail,:

Although measurement theory consists of the two Axioms 1 and 2, it is the Axiom 2 that is concerned with causal relationship. When describing a certain phenomenon in a language called measurement theory and using the Axiom 2, we think that the phenomenon has causality.

Although it is above, it is the next difference when (a)–(d) is summarized.

(a) World is first  (b) Recognition is first  
(c) Mathematics(buried into ordinary language) is first 
(d) Language (Measurement Theory) is first 

Now, in measurement theory, we assert the next as said repeatedly.:

Measurement theory is a basic language which describes various sciences.

Supposing this is recognized, we can assert the next. Namely,

**In science, causality is claimed in upper (d).**

This is an answer of measurement theory to "What is causality?", and I explain these details after the following paragraph.
Note 6.1 For the question "What is space-time?", there are two answers as follows.

(\(\sharp_1\)) world description (Chap. 1(\(\Omega\)))

\[
\begin{array}{l}
\text{1: realistic method} \cdots \text{Newton's space-time (realistic space-time)} \\
\text{evolution} \quad \text{Einstein's space-time} \quad \text{evolution} \\
\end{array}
\]

\(\Rightarrow\)

\[
\begin{array}{l}
\text{2: linguistuc method} \cdots \text{Leibniz's relationalism (metaphysical space-time)} \\
\text{literary representation} \\
\text{evolution} \quad \text{time-space in measurement theory}
\end{array}
\]

Concerning "What is causality?", a similar argument is possible.

(\(\sharp_2\)) world description (Chap. 1(\(\Omega\)))

\[
\begin{array}{l}
\text{1: realistic method} \cdots \text{Newton's causality (realistic causality)} \\
\text{2: linguistuc method} \cdots \text{causality in measurement theory (metaphysical causality)}
\end{array}
\]

Also,

(\(\sharp_3\)) In Sec. 8.1, we discuss the relation among (b), (c) and (d).

Note 6.2 As one of the by-products of measurement theory, I can reply to the outstanding problems

(\(\sharp_1\)) What are time, space, causality, and probability?

in a metaphysical position (linguistic position of an anti physics supreme principle). In metaphysics, answering to "What is" is defining how to use the language. (Note 2.3, Note 5.6). Therefore, this (\(\sharp_1\)) is equivalent to the following (\(\sharp_2\)).

(\(\sharp_2\)) To propose the linguistic universe describing method containing the word "time, space, causality, and probability."

Of course, in this book, measurement theory (i.e., establishment of the linguistic method) is proposed as this answer.

### 6.2 Causality — No smoke without fire

#### 6.2.1 The Heisenberg picture and the Schrödinger picture

Let \(\Omega\) be a state space. state space. Let \(C(\Omega)\) be a space of all real continuous valued functions on \(\Omega\).

Also, recall \(\mathcal{M}(\Omega)\) and \(\mathcal{M}_{+1}(\Omega)\) in Sec.4.4.1(b).

**Definition 6.2 [Causal operator (causal operator)]** Let \(\Omega_1\) and \(\Omega_2\) be state spaces. a continuous linear operator \(\Phi_{1,2} : C(\Omega_2) \to C(\Omega_1)\) is called a causal operator (or, Markov causal operator) if it satisfies (i)—(iii):

(i) \(f_2 \in C(\Omega_2), \quad f_2 \geq 0 \implies \Phi_{12} f_2 \geq 0\)

(ii) \(\Phi_{12} 1_2 = 1_1\) where, \(1_k(\omega_k) = 1 \quad (\forall \omega_k \in \Omega_k, k = 1, 2)\)

(iii) There exists a continuous linear operator \(\Phi_{1,2}^* : \mathcal{M}(\Omega_1) \to \mathcal{M}(\Omega_2)\) such that :

\[
\int_{\Omega_1} [\Phi_{1,2} f_2](\omega_1) \rho_1(d\omega_1) = \int_{\Omega_2} f_2(\omega_2) (\Phi_{1,2}^* \rho_1)(d\omega_2) \\
(\forall \rho_1 \in \mathcal{M}(\Omega_1), \forall f_2 \in C(\Omega_2))
\]

If \(\Omega\) is compact, this condition is a consequence of the above (i) and (ii).
The causal operator \( \Phi_{1,2} : C(\Omega_2) \to C(\Omega_1) \) is regarded as the Heisenberg picture representation of the "causality". Also, the dual causal operator \( \Phi_{1,2}^* \) called the Schrödinger picture representation of the "causality". In addition, the causal operator \( \Phi_{1,2} \) is called a **deterministic causal operator**, if there exists a continuous map \( \phi_{1,2} : \Omega_1 \to \Omega_2 \) such that

\[
[\Phi_{1,2}f_2](\omega_1) = f_2(\phi_{1,2}(\omega_1)) \quad (\forall f_2 \in C(\Omega_2), \forall \omega_1 \in \Omega_1)
\]  

(6.1)

This continuous map \( \phi_{1,2} : \Omega_1 \to \Omega_2 \) is said to be a **deterministic causal map**.

![Figure 6.1: deterministic causal map \( \phi_{1,2} \) and deterministic causal operator \( \Phi_{1,2} \)]

**Theorem 6.3 [Causal operator and observable]** For any observable \( O_2 = (X, \mathcal{F}, F_2) \) in \( C(\Omega_2) \), the \( (X, \mathcal{F}, \Phi_{1,2}F_2) \) is an observable in \( C(\Omega_1) \). We denote that \( \Phi_{1,2}O_2 = (X, \mathcal{F}, \Phi_{1,2}F_2) \).

**Proof.** For any \( \Xi \in \mathcal{F} \), consider the countable decomposition \( \{ \Xi_1, \Xi_2, \ldots, \Xi_n, \ldots \} \) (that is, \( \Xi = \bigcup_{n=1}^{\infty} \Xi_n, \Xi_n \in \mathcal{F}, (n = 1, 2, \ldots), \Xi_m \cap \Xi_n = \emptyset \ (m \neq n) \)). Recalling the condition: (2.3), we see, for any \( \rho_1 \in \mathcal{M}(\Omega_1) \),

\[
\int_{\Omega_1} [\Phi_{1,2}F_2](\bigcup_{n=1}^{\infty} \Xi_n)(\omega_1) \rho_1(d\omega_1) = \int_{\Omega_2} [F_2(\bigcup_{n=1}^{\infty} \Xi_n)](\omega_2) \Phi_{1,2}^*\rho_1(d\omega_2)
\]

\[
= \sum_{n=1}^{\infty} \int_{\Omega_2} [F_2(\Xi_n)](\omega_2) \Phi_{1,2}^*\rho_1(d\omega_2) = \sum_{n=1}^{\infty} \int_{\Omega_1} [\Phi_{1,2}F_2](\Xi_n)(\omega_1) \rho_1(d\omega_1)
\]

Thus, \( \Phi_{1,2}O_2 = (X, \mathcal{F}, \Phi_{1,2}F_2) \) is an observable in \( C(\Omega_1) \). \( \square \)

**Theorem 6.4** Consider a continuous map \( \phi_{1,2} : \Omega_1 \to \Omega_2 \). The operator \( \Phi_{1,2} : C(\Omega_2) \to C(\Omega_1) \) is defined by the formula (6.1), that is,

\[
[\Phi_{1,2}f_2](\omega_1) = f_2(\phi_{1,2}(\omega_1)) \quad (\forall f_2 \in C(\Omega_2), \forall \omega_1 \in \Omega_1)
\]

(6.1)

Then, the operator \( \Phi_{1,2} : C(\Omega_2) \to C(\Omega_1) \) is a deterministic causal operator. This means that "continuous map =deterministic causal map".

**Proof.** It suffices to show the existence of the dual causal operator \( \Phi_{1,2}^* : \mathcal{M}(\Omega_1) \to \mathcal{M}(\Omega_2) \). This is shown as follows.

\[
[\Phi_{1,2}\rho_1](D_2) = \rho_1(\phi_{1,2}^{-1}(D_2)) \quad (\forall D_2 \in B_{\Omega_2}, \forall \rho_1 \in \mathcal{M}(\Omega_1))
\]

(6.2)
Thus, we get the dual causal operator $\Phi_{1,2}^* : \mathcal{M}(\Omega_1) \to \mathcal{M}(\Omega_2)$.

\begin{proof}
Let $f_2, g_2$ be elements in $C(\Omega_2)$ Let $\phi_{1,2} : \Omega_1 \to \Omega_2$ be a deterministic causal operator of a deterministic causal operator $\Phi_{1,2} : C(\Omega_2) \to C(\Omega_1)$. Then, we see:

\[ (\Phi_{1,2}(f_2 \cdot g_2))(\omega_1) = (f_2 \cdot g_2)(\phi_{1,2}(\omega_1)) = f_2(\phi_{1,2}(\omega_1)) \cdot g_2(\phi_{1,2}(\omega_1)) \]

This completes the proof.
\end{proof}

6.2.2 A simple example — A matrix representation in the case of finite state space

\textbf{Example 6.6 [Deterministic causal operator, deterministic dual causal operator, deterministic causal map]} Define the two states space $\Omega_1$ and $\Omega_2$ such that $\Omega_1 = \Omega_2 = \mathbb{R}$. Define the deterministic causal map $\phi_{1,2} : \Omega_1 \to \Omega_2$ such that

$$\omega_2 = \phi_{1,2}(\omega_1) = 3\omega_1 + 2 \quad (\forall \omega_1 \in \Omega_1 = \mathbb{R})$$

Then, by (6.2), we get the deterministic dual causal operator $\Phi_{1,2}^* : \mathcal{M}(\Omega_1) \to \mathcal{M}(\Omega_2)$ such that

$$\Phi_{1,2}^* \delta_{\omega_1} = \delta_{3\omega_1 + 2} \quad (\forall \omega_1 \in \Omega_1)$$

where $\delta(\cdot)$ is the point measure. Also, the deterministic causal operator $\Phi_{1,2} : C(\Omega_2) \to C(\Omega_1)$ is defined by

$$[\Phi_{1,2}(f_2)](\omega_1) = f_2(3\omega_1 + 2) \quad (\forall f_2 \in C(\Omega_2), \forall \omega_1 \in \Omega_1)$$

\textbf{Example 6.7 [Dual causal operator-causal operator]} Put $\Omega_1 = \{\omega_1^1, \omega_1^2, \omega_1^3\}$ and $\Omega_2 = \{\omega_2^1, \omega_2^2\}$. And define $\rho_1(\in \mathcal{M}_{+1}(\Omega_1))$ such that

$$\rho_1 = a_1 \delta_{\omega_1^1} + a_2 \delta_{\omega_1^2} + a_3 \delta_{\omega_1^3} \quad (0 \leq a_1, a_2, a_3 \leq 1, a_1 + a_2 + a_3 = 1)$$

Then, the dual causal operator $\Phi_{1,2}^* : \mathcal{M}_{+1}(\Omega_1) \to \mathcal{M}_{+1}(\Omega_2)$ is represented by

$$\Phi_{1,2}^*(\rho_1) = (c_{11}a_1 + c_{12}a_2 + c_{13}a_3)\delta_{\omega_1^1} + (c_{21}a_1 + c_{22}a_2 + c_{23}a_3)\delta_{\omega_1^2}$$

$$0 \leq c_{ij} \leq 1, \sum_{i=1}^{2} c_{ij} = 1$$
and. Consider the identification: $\mathcal{M}(\Omega_1) \approx \mathbb{R}^3$, $\mathcal{M}(\Omega_2) \approx \mathbb{R}^2$, That is,

$$
\mathcal{M}(\Omega_1) \ni \alpha_1 \delta_{\omega_1} + \alpha_2 \delta_{\omega_2} + \alpha_3 \delta_{\omega_3} \quad \Longleftrightarrow \quad \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \in \mathbb{R}^3
$$

$$
\mathcal{M}(\Omega_2) \ni \beta_1 \delta_{\omega_1} + \beta_2 \delta_{\omega_2} \quad \Longleftrightarrow \quad \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \in \mathbb{R}^2
$$

Then, putting

$$
\Phi^*_{1,2} (\rho_1) = \beta_1 \delta_{\omega_1} + \beta_2 \delta_{\omega_2} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix},
$$

$$
\rho_1 = \alpha_1 \delta_{\omega_1} + \alpha_2 \delta_{\omega_2} + \alpha_3 \delta_{\omega_3} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}
$$

write, by matrix representation, as follows.

$$
\Phi^*_{1,2} (\rho_1) = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}
$$

Next, from this dual causal operator $\Phi^*_{1,2} : \mathcal{M}(\Omega_1) \to \mathcal{M}(\Omega_2)$, we shall construct a causal operator $\Phi_{1,2} : C(\Omega_2) \to C(\Omega_1)$. Consider the identification: $C(\Omega_1) \approx \mathbb{R}^3$, $C(\Omega_2) \approx \mathbb{R}^2$, that is,

$$
C(\Omega_1) \ni f_1 \quad \Longleftrightarrow \quad \begin{bmatrix} f_1(\omega_1^1) \\ f_1(\omega_2^1) \\ f_1(\omega_3^1) \end{bmatrix} \in \mathbb{R}^3, \quad C(\Omega_2) \ni f_2 \quad \Longleftrightarrow \quad \begin{bmatrix} f_2(\omega_1^2) \\ f_2(\omega_2^2) \end{bmatrix} \in \mathbb{R}^2
$$

Let $f_2 \in C(\Omega_2)$, $f_1 = \Phi_{1,2} f_2$. Then, we see

$$
\begin{bmatrix} f_1(\omega_1^1) \\ f_1(\omega_2^1) \\ f_1(\omega_3^1) \end{bmatrix} = f_1 = \Phi_{1,2}(f_2) = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix} \begin{bmatrix} f_2(\omega_1^2) \\ f_2(\omega_2^2) \end{bmatrix}
$$

Therefore, the relation between the dual causal operator $\Phi^*_{1,2}$ and causal operator $\Phi_{1,2}$ is represented as the transposed matrix.

**Example 6.8** [Deterministic dual causal operator, deterministic causal map, deterministic causal operator] Consider the case that dual causal operator $\Phi^*_{1,2} : \mathcal{M}(\Omega_1)(\approx \mathbb{R}^3) \to \mathcal{M}(\Omega_2)(\approx \mathbb{R}^2)$ has the matrix representation such that

$$
\Phi^*_{1,2} (\rho_1) = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}
$$

In this case, it is the deterministic dual causal operator. This deterministic causal operator $\Phi_{1,2} : C(\Omega_2) \to C(\Omega_1)$ is represented by

$$
\begin{bmatrix} f_1(\omega_1^1) \\ f_1(\omega_2^1) \\ f_1(\omega_3^1) \end{bmatrix} = f_1 = \Phi_{1,2}(f_2) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_2(\omega_1^2) \\ f_2(\omega_2^2) \end{bmatrix}
$$

with the deterministic causal map $\phi_{1,2} : \Omega_1 \to \Omega_2$ such that

$$
\phi_{1,2}(\omega_1^1) = \omega_2^2, \quad \phi_{1,2}(\omega_2^1) = \omega_2^1, \quad \phi_{1,2}(\omega_3^1) = \omega_2^1
$$
6.2.3 Sequential causal operator — A chain of causalities

Let \((T, \leq)\) be a tree, i.e., a partial ordered finite set such that “\(t_1 \leq t_3\) and \(t_2 \leq t_3\)” implies “\(t_1 \leq t_2\) or \(t_2 \leq t_1\)”. Assume that there exists an element \(t_0 \in T\), called the root of \(T\), such that \(t_0 \leq t\) (\(\forall t \in T\)) holds.

Put \(T_a^2 = \{(t_1, t_2) \in T^2 : t_1 \leq t_2\}\). An element \(t_0 \in T\) is called a root if \(t_0 \leq t\) (\(\forall t \in T\)) holds. Since we usually consider the subtree \(T_{t_0}(\subseteq T)\) with the root \(t_0\), we assume that the tree-like ordered set has a root. In this chapter, assume, for simplicity, that \(T\) is finite (though it is sometimes infinite in applications).

For simplicity, assume that \(T\) is finite, or a finite subtree of a whole tree. Let \(T = \{0, 1, ..., N\}\) be a tree with the root 0. Define the parent map \(\pi : T \setminus \{0\} \rightarrow T\) such that \(\pi(t) = \max\{s \in T : s < t\}\). It is clear that the tree \((T \equiv \{0, 1, ..., N\}, \leq)\) can be identified with the pair \((T \equiv \{0, 1, ..., N\}, \pi : T \setminus \{0\} \rightarrow T)\).

Also, note that, for any \(t \in T \setminus \{0\}\), there uniquely exists a natural number \(h(t)\) (called the height of \(t\)) such that \(\pi^{h(t)}(t) = 0\). Here, \(\pi^2(t) = \pi(\pi(t))\), \(\pi^3(t) = \pi(\pi^2(t))\), etc. Also, put \(\{0, 1, ..., N\}_\leq^2 = \{(m, n) | 0 \leq m \leq n \leq N\}\). In Fig. 6.2, see the root \(t_0\), the parent map: \(\pi(t_3) = \pi(t_4) = t_2\), \(\pi(t_2) = \pi(t_5) = t_1\), \(\pi(t_1) = \pi(t_6) = \pi(t_7) = t_0\)

![Figure 6.2: Tree](image)

**Definition 6.9 [Sequential causal operator]** The family

\[\{\Phi_{t_1, t_2} : C(\Omega_{t_2}) \rightarrow C(\Omega_{t_1})\}_{(t_1, t_2) \in T^2_\leq} \quad \text{or,} \quad \{C(\Omega_{t_2}) \xrightarrow{\Phi_{t_1, t_2}} C(\Omega_{t_1})\}_{(t_1, t_2) \in T^2_\leq}\]

is called a **sequential causal operator**, if it satisfies that (Fig. 6.3):

(i) For each \((t_1, t_2) \in T^2_\leq\), a basic algebra \(C(\Omega_{t_1})\) is determined.

(ii) For each \((t_1, t_2) \in T^2_\leq\), a causal operator \(\Phi_{t_1, t_2} : C(\Omega_{t_2}) \rightarrow C(\Omega_{t_1})\) is defined such as \(\Phi_{t_1, t_2}\Phi_{t_2, t_3} = \Phi_{t_1, t_3} \ (\forall (t_1, t_2), \forall (t_2, t_3) \in T^2_\leq)\). Here, \(\Phi_{t, t} : C(\Omega_t) \rightarrow C(\Omega_t)\) is the identity operator.

![Causal operators](image)
The dual family \( \{ \Phi_{t_1,t_2} : \mathcal{M}(\Omega_{t_1}) \rightarrow \mathcal{M}(\Omega_{t_2}) \}_{(t_1,t_2) \in T_\Delta^2} \) is called the **dual sequential causal operator** of a sequential causal operator \( \{ C(\Omega_{t_2}) \ circ \Phi_{t_1,t_2} C(\Omega_{t_1}) \}_{(t_1,t_2) \in T_\Delta^2} \).

### 6.2.4 Sequential causal operator — Simultaneous differential equation of the first order

**Example 6.10 [Pheasants and rabbits problem]** Consider the following situation:

(a) [Pheasants and rabbits problem] A number of \( m \) pheasants and \( n \) rabbits are placed together in the same cage. Then there are \( m+n \) heads and \( 2m+4n \) legs.

In what follows, this statement in ordinary language will be changed to the measurement theoretical statement. Putting \( \mathbb{N}_0 = \{0, 1, 2, \ldots\} \), define state space \( \Omega_0, \Omega_1, \Omega_2 \) such that

\[
\Omega_0 = \mathbb{N}_0 \times \mathbb{N}_0, \quad \Omega_1 = \mathbb{N}_0, \quad \Omega_2 = \mathbb{N}_0
\]

Put \( T(0) = \{0, 1, 2\} \) with the parent map \( \pi : \{1, 2\} \rightarrow \{0, 1, 2\} \) such that

\[
\pi(1) = 0, \quad \pi(2) = 0
\]

Then, the deterministic causal map \( \phi_{0,1} : \Omega_0 \rightarrow \Omega_1 \) and \( \phi_{0,2} : \Omega_0 \rightarrow \Omega_2 \) is respectively represented by

\[
\phi_{0,1}(m,n) = m+n, \quad \phi_{0,2}(m,n) = 2m+4n
\]

Therefore, by Theorem 6.4, the deterministic causal operator \( \Phi_{0,1} : C(\Omega_1) \rightarrow C(\Omega_0) \) and \( \Phi_{0,2} : C(\Omega_2) \rightarrow C(\Omega_0) \) are defined as follows.

\[
\begin{cases}
  [\Phi_{0,1}(f_1)](m,n) = f_1(m+n) & (\forall f_1 \in C(\Omega_1), \forall (m,n) \in \Omega_0) \\
  [\Phi_{0,2}(f_2)](m,n) = f_2(2m+4n) & (\forall f_2 \in C(\Omega_2), \forall (m,n) \in \Omega_0)
\end{cases}
\]

Thus, we get a deterministic sequential causal operator \( \{ C(\Omega_t) \ circ \Phi_{0,t} C(\Omega_0) \}_{t=1,2} \).

**Example 6.11 [State equation]** Let \( T = \mathbb{R} \) be the time axis. For each \( t(\in T) \), consider the state space \( \Omega_t = \mathbb{R}^n \) (\( n \)-dimensional real space). And consider simultaneous differential equation of the first order

\[
\begin{cases}
  \frac{d\omega_1}{dt}(t) = v_1(\omega_1(t), \omega_2(t), \ldots, \omega_n(t), t) \\
  \frac{d\omega_2}{dt}(t) = v_2(\omega_1(t), \omega_2(t), \ldots, \omega_n(t), t) \\
  \quad \cdots \\
  \frac{d\omega_n}{dt}(t) = v_n(\omega_1(t), \omega_2(t), \ldots, \omega_n(t), t)
\end{cases}
\]

(6.3) \((=1.(1))\)

which is called a **state equation** (Chap. 1(1.1)). Let \( \phi_{t_1,t_2} : \Omega_{t_1} \rightarrow \Omega_{t_2}, \ (t_1 \leq t_2) \) be a deterministic causal map induced by the state equation (6.3). It is clear that \( \phi_{t_2,t_3}(\phi_{t_1,t_2}(\omega_{t_1})) = \phi_{t_1,t_3}(\omega_{t_1}) \) \((\omega_{t_1} \in \Omega_{t_1}, t_1 \leq t_2 \leq t_3) \). Therefore, by Theorem 6.4, we have the deterministic sequential causal operator \( \{ \Phi_{t_1,t_2} : C(\Omega_{t_2}) \rightarrow C(\Omega_{t_1}) \}_{(t_1,t_2) \in T_\Delta^2} \).
Definition 6.14 [Sequential observable]  
Consider a sequential causal operator \( \{ \Phi_{t_1, t_2} : C(\Omega_{t_2}) \to C(\Omega_{t_1}) \}_{(t_1, t_2) \in \mathbb{T}_2^\infty} \). Assume that for each \( t \in T \), an observable \( O_t = (X_t, \mathcal{F}_t, F_t) \) in \( C(\Omega_t) \) is determined. Then, the pair \( \{(O_t)_{t \in T}, \{\Phi_{t_1, t_2} : C(\Omega_{t_2}) \to C(\Omega_{t_1})\}_{(t_1, t_2) \in \mathbb{T}_2^\infty}\) is called a sequential observable, and denoted

Example 6.12 [Difference equation of the second order]  
Consider the discrete time \( T = \{0, 1, 2, \ldots \} \) with the parent map \( \pi : T \setminus \{0\} \to T \) such that \( \pi(t) = t - 1 \ (\forall t = 1, 2, \ldots) \). For each \( t \in T \), consider a state space \( \Omega_t \) such that \( \Omega_t = \mathbb{R} \). For example, consider the following difference equation: That is,

\[
\omega_{t+2} = \phi(\omega_t, \omega_{t+1}) = \omega_t + \omega_{t+1} + 2 \quad (\forall t \in T)
\]

Here, note that the state \( \omega_{t+2} \) depends on both \( \omega_{t+1} \) and \( \omega_t \) (i.e., multiple markov property). This must be modified as follows. For each \( t \in T \) consider a new state space \( \Omega_t = \mathbb{R} \times \mathbb{R} \). And define the deterministic causal map \( \tilde{\phi}_{t,t+1} : \tilde{\Omega}_t \to \tilde{\Omega}_{t+1} \) as follows.

\[
(\omega_{t+1}, \omega_{t+2}) = \tilde{\phi}_{t,t+1}(\omega_t, \omega_{t+1}) = (\omega_{t+1}, \omega_t + \omega_{t+1} + 2)
\]

\[ (\forall (\omega_t, \omega_{t+1}) \in \tilde{\Omega}_t, \forall t \in T) \]

Therefore, by Theorem 6.4, the deterministic causal operator \( \tilde{\Phi}_{t,t+1} : C(\tilde{\Omega}_{t+1}) \to C(\tilde{\Omega}_t) \) is defined by

\[
[\tilde{\Phi}_{t,t+1} \tilde{f}_t](\omega_t, \omega_{t+1}) = \tilde{f}_t(\omega_{t+1}, \omega_t + \omega_{t+1} + 2)
\]

\[ (\forall (\omega_t, \omega_{t+1}) \in \tilde{\Omega}_t, \forall \tilde{f}_t \in C(\tilde{\Omega}_{t+1}), \forall t \in T \setminus \{0\}) \]

Thus, we get the deterministic sequential causal operator \( \{ \tilde{\Phi}_{t,t+1} : C(\tilde{\Omega}_{t+1}) \to C(\tilde{\Omega}_t) \}_{t \in T \setminus \{0\}} \).

Note 6.3 In measurement theory, multiple markov process and time-lag process are prohibited.

Example 6.13 [Random walk]  
Put \( \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots \} \). Define the dual causal operator \( \Phi^* : \mathcal{M}_+1(\mathbb{Z}) \to \mathcal{M}_+1(\mathbb{Z}) \) such that

\[
\Phi^*(\delta_i) = \frac{\delta_{i-1} + \delta_{i+1}}{2} \quad (i \in \mathbb{Z})
\]

where \( \delta_{(i)} \in \mathcal{M}_+1(\mathbb{Z}) \) is a point measure. Therefore, the causal operator \( \Phi : C(\mathbb{Z}) \to C(\mathbb{Z}) \) is defined by

\[
[\Phi(f)](i) = \frac{f(i - 1) + f(i + 1)}{2} \quad (\forall f \in C(\mathbb{Z}), \forall i \in \mathbb{Z})
\]

Now, consider the discrete time \( T = \{0, 1, 2, \ldots, N\} \). For each \( t \in T \), a state space \( \Omega_t \) is define by \( \Omega_t = \mathbb{Z} \). Then, we have the sequential causal operator \( \{ \Phi_{\pi(t), t} (= \Phi) : C(\Omega_t) \to C(\Omega_{\pi(t)}) \}_{t \in T \setminus \{0\}} \).

6.3 Realized causal observable — Only one measurement

Let \( (T(t_0), \preceq) \) (or, \( T(t_0) = \{t_0, t_1, \ldots, t_N\} \) ) be a semi-ordered tree with the root \( t_0 \). Mainly consider the parent map representation \( (T = \{t_0, t_1, \ldots, t_N\}, \pi : T \setminus \{t_0\} \to T) \).

Definition 6.14 [Sequential observable]  
Consider a sequential causal operator \( \{ \Phi_{t_1, t_2} : C(\Omega_{t_2}) \to C(\Omega_{t_1}) \}_{(t_1, t_2) \in \mathbb{T}_2^\infty} \). Assume that for each \( t \in T \), an observable \( O_t = (X_t, \mathcal{F}_t, F_t) \) in \( C(\Omega_t) \) is determined. Then, the pair \( \{(O_t)_{t \in T}, \{\Phi_{t_1, t_2} : C(\Omega_{t_2}) \to C(\Omega_{t_1})\}_{(t_1, t_2) \in \mathbb{T}_2^\infty}\} \) is called a sequential observable, and denoted
by \([O_T]\) or \([O_{T(t_0)}]\). That is, \([O_T] = [(O_t)_{t \in T}, \{\Phi_{t_1,t_2} : C(\Omega_{t_2}) \to C(\Omega_{t_1})\}_{(t_1,t_2) \in T^2}]\). Using the parent map \(\pi : T \setminus \{0\} \to T\), we also write that \([O_T] = [(O_t)_{t \in T}, \{C(\Omega_t) \xrightarrow{\Phi_{\pi(t),t}} C(\Omega_{\pi(t)})\}_{t \in T \setminus \{0\}}]\).

According to the Copenhagen interpretation (i.e., only one observable is permitted), we must regard many observables \(\{O_t\}_{t \in T}\) in a sequential observable \([O_T] = [(O_t)_{t \in T}, \{\Phi_{t_1,t_2} : C(\Omega_{t_2}) \to C(\Omega_{t_1})\}_{(t_1,t_2) \in T^2}]\) as only one observable. This is realized as follows.

**Definition 6.15 [Realized causal observable]** Let \([O_{T(t_0)}] = [(O_t)_{t \in T}, \{\Phi_{\pi(t),t} : C(\Omega_t) \to C(\Omega_{\pi(t)})\}_{t \in T \setminus \{t_0\}}]\) be a sequential observable. For each \(s \in T\), put \(T_s = \{t \in T | t \geq s\}\). And define the observable \(\hat{O}_s = (X_{t \in T_s} X_t, \boxdot_{t \in T_s} F_t, \hat{F}_s)\) in \(C(\Omega_s)\) as follows.

\[
\hat{O}_s = \begin{cases} 
O_s & (s \in T \setminus \pi(T) \text{ and } ) \\
O_s \times (X_{t \in \pi^{-1}(s)} \Phi_{\pi(t),t}) \hat{O}_t & (s \in \pi(T) \text{ and } )
\end{cases}
\]  

(6.4)

using this iteratively, after all we get the observable \(\hat{O}_{t_0} = (X_{t \in T} X_t, \boxdot_{t \in T} F_t, \hat{F}_{t_0})\) in \(C(\Omega_{t_0})\). Put \(\hat{O}_{t_0} = \hat{O}_{T(t_0)}\). The \(\hat{O}_{T(t_0)} = (X_{t \in T} X_t, \boxdot_{t \in T} F_t, \hat{F}_{t_0})\) is called the realized causal observable of the sequential observable \([O_{T(t_0)}] = [(O_t)_{t \in T}, \{\Phi_{\pi(t),t} : C(\Omega_t) \to C(\Omega_{\pi(t)})\}_{t \in T \setminus \{t_0\}}]\).

**Example 6.16 [Simple example]** (Continued from Fig. 6.3) Suppose that a tree \((T = \{0,1, \ldots, 6,7\}, \pi)\) has an ordered structure such that \(\pi(1) = \pi(6) = \pi(7) = 0, \pi(2) = \pi(5) = 1, \pi(3) = \pi(4) = 2.\) (See Fig. (6.3).)

Consider a sequential observable \([O_T] = [(O_t)_{t \in T}, \{C(\Omega_t) \xrightarrow{\Phi_{\pi(t),t}} C(\Omega_{\pi(t)})\}_{t \in T \setminus \{0\}}]\). Now, we shall construct its realized causal observable \(\hat{O}_{T(t_0)} = (X_{t \in T} X_t, \boxdot_{t \in T} F_t, \hat{F}_{t_0})\) in what follows.

Put

\[
\hat{O}_t = O_t \quad \text{and thus} \quad \hat{F}_t = F_t \quad (t = 3, 4, 5, 6, 7).
\]

First we construct the product observable \(\hat{O}_2\) in \(C(\Omega_2)\) such as

\[
\hat{O}_2 = (X_2 \times X_3 \times X_4, \boxdot F_3 \boxdot F_4, \hat{F}_2) \quad \text{where} \quad \hat{F}_2 = F_2 \times (X_{t=3,4} \Phi_{2,t} \hat{F}_t),
\]

Iteratively, we construct the following:

\[
\begin{array}{c}
C(\Omega_0) \leftarrow F_0 \times \Phi_{0,6} \hat{F}_6 \times \Phi_{0,7} \hat{F}_7 \\
\downarrow \\
F_0 \times \Phi_{0,6} \hat{F}_6 \times \Phi_{0,7} \hat{F}_7 \\
\downarrow \\
(F_0 \times \Phi_{0,6} \hat{F}_6 \times \Phi_{0,7} \hat{F}_7, \hat{F}_1) \leftarrow F_1 \times \Phi_{1,5} \hat{F}_{5} \\
\downarrow \\
(F_1 \times \Phi_{1,5} \hat{F}_{5}, \hat{F}_2) \leftarrow \Phi_{1,2} \hat{F}_2 \\
\downarrow \\
(F_2 \times \Phi_{2,3} \hat{F}_{3} \times \Phi_{2,4} \hat{F}_{4}, \hat{F}_5)
\end{array}
\]

That is, we get the product observable \(\hat{O}_1 \equiv (X_{t=1} X_t, \boxdot_{t=1} F_t, \hat{F}_1)\) of \(O_1, \Phi_{1,2} \hat{O}_2\) and \(\Phi_{1,5} \hat{O}_5\), and finally, the product observable

\[
\hat{O}_0 \equiv (X_{t=0} X_t, \boxdot_{t=0} F_t, \hat{F}_0 (= F_0 \times (X_{t=1,0,7} \Phi_{0,t} \hat{F}_t)))
\]
of \( O_0, \Phi_{0,1}\hat{O}_1, \Phi_{0,6}\hat{O}_6 \) and \( \Phi_{0,7}\hat{O}_7 \). Then, we get the realization of a sequential observable \([\{O_t\}_{t \in T}, \{C(\Omega_t \cap \Phi_{t(t)})\}_{t \in T \setminus \{0\}}\]. For completeness, \( \hat{F}_0 \) is represented by

\[
\hat{F}_0(\Xi_0 \times \Xi_1 \times \Xi_2 \times \Xi_3 \times \Xi_4 \times \Xi_5 \times \Xi_6 \times \Xi_7) = F_0(\Xi_0) \times \Phi_{0,1}(F_1(\Xi_1) \times \Phi_{1,5}F_5(\Xi_5) \times \Phi_{1,2}(F_2(\Xi_2) \times \Phi_{2,3}F_3(\Xi_3) \times \Phi_{2,4}F_4(\Xi_4))) \times \Phi_{0,6}(F_6(\Xi_6)) \times \Phi_{0,7}(F_7(\Xi_7))
\] (6.5)

**Remark 6.17** In the above example, consider the case that \( O_t (t = 2, 6, 7) \) is not determined. In this case, it suffices to define \( O_4 \) by the existence observable \( O_t^{\text{exi}}\equiv(X_t, \emptyset, X_t, F_t^{(\text{exi})}) \). Then, we see that

\[
\hat{F}_0(\Xi_0 \times \Xi_1 \times \Xi_2 \times \Xi_3 \times \Xi_4 \times \Xi_5 \times \Xi_6 \times \Xi_7) = F_0(\Xi_0) \times \Phi_{0,1}(F_1(\Xi_1) \times \Phi_{1,5}F_5(\Xi_5) \times \Phi_{1,2}(F_2(\Xi_2) \times \Phi_{2,3}F_3(\Xi_3) \times \Phi_{2,4}F_4(\Xi_4)))
\] (6.6)

This is true. However, the following is not wrong. Putting \( T' = \{0, 1, 3, 4, 5\} \), consider the \( [O_{T'}] = [\{O_t\}_{t \in T'}, \{\Phi_{t_1, t_2} : C(\Omega_{t_2}) \rightarrow C(\Omega_{t_1})\}_{(t_1, t_2) \in (T')^2} \] ). Then, the realized causal observable \( \hat{O}_{T'(0)} = (\times_{t \in T'} X_t, \boxtimes_{t \in T'} \mathcal{F}_t, \hat{F}_0) \) is defined by

\[
\hat{F}_0(\Xi_0 \times \Xi_1 \times \Xi_2 \times \Xi_3 \times \Xi_4 \times \Xi_5) = F_0(\Xi_0) \times \Phi_{0,1}(F_1(\Xi_1) \times \Phi_{1,5}F_5(\Xi_5) \times \Phi_{1,4}F_4(\Xi_4) \times \Phi_{1,3}F_3(\Xi_3) \times \Phi_{1,4}F_4(\Xi_4))
\] (6.7)

which is different from the true (6.6). We may sometimes omit "existence observable". However, if we do so, we omit it on the basis of careful cautions.

**Theorem 6.18** [Deterministic sequential observable realized causal observable] Let \((T(t_0), \leq)\) be a tree. Let \([O_T] = [\{O_t\}_{t \in T}, \{\Phi_{t_1, t_2} : C(\Omega_{t_2}) \rightarrow C(\Omega_{t_1})\}_{(t_1, t_2) \in T^2} \] ) be a deterministic causal observable. Then, the realization \( \hat{O}_{t_0} \equiv (\times_{t \in T} X_t, \boxtimes_{t \in T} \mathcal{F}_t, \hat{F}_{t_0}) \) is represented by

\[
\hat{O}_{t_0} = \times_{t \in T} \Phi_{t_0,t} O_t
\]

That is, it holds that

\[
[\hat{F}_{t_0}(\times_{t \in T} \Xi_t)](\omega_{t_0}) = \times_{t \in T} [\Phi_{t_0,t} F_t(\Xi_t)](\omega_{t_0}) = \times_{t \in T} [F_t(\Xi_t)](\phi_{t_0,t} \omega_{t_0})
\]

\((\forall \omega_{t_0} \in \Omega_{t_0}, \forall \Xi_t \in \mathcal{F}_t)\)

**Proof.** It suffices to prove the classical case of Example 4.15. Using Theorem 4.2 repeatedly, we see that

\[
(4.31) = \hat{F}_0 = F_0 \times (\times_{t=1,6,7} \Phi_{0,t} \hat{F}_t)
\]

\[
= F_0 \times (\Phi_{0,1} \hat{F}_1 \times \Phi_{0,6} \hat{F}_6 \times \Phi_{0,7} \hat{F}_7) = F_0 \times (\Phi_{0,1} \hat{F}_1 \times \Phi_{0,6} \hat{F}_6 \times \Phi_{0,7} \hat{F}_7)
\]
\[ \left( \times_{t=0,6,7} \Phi_{0,t}F_t \right) \times (\Phi_{0,1}F_{1}) = \left( \times_{t=0,6,7} \Phi_{0,t}F_t \right) \times \Phi_{0,1}(F_1 \times (\times_{t=2,5} \Phi_{1,t}F_t)) \]

\[ = \left( \times_{t=0,1,6,7} \Phi_{0,t}F_t \right) \times \Phi_{0,1}(\times_{t=2,5} \Phi_{1,t}F_t) = \left( \times_{t=0,1,6,7} \Phi_{0,t}F_t \right) \times \Phi_{0,1}(\Phi_{1,2}F_2 \times \Phi_{1,5}F_5) \]

\[ = \left( \times_{t=0,1,5,6,7} \Phi_{0,t}F_t \right) \times \Phi_{0,1}(\Phi_{1,2}F_2) = \left( \times_{t=0,1,5,6,7} \Phi_{0,t}F_t \right) \times \Phi_{0,1}(\Phi_{1,2}(F_2 \times (\times_{t=3,4} \Phi_{2,t}F_t))) \]

\[ = \times_{t=0} \Phi_{0,t}F_t \]

This completes the proof.

\[ \square \]

\[ \text{Note 6.4} \]
Note that a simultaneous observable (and a parallel observable) can be regarded as a kind of realized causal observable.

6.4 Axiom\textsuperscript{pm} 2 —”No smoke without fire”

@No smoke without fire

6.4.1 The Heisenberg picture

Summing up the arguments in the previous section, we assert Axiom\textsuperscript{pm} 2 as follows.

Axiom\textsuperscript{pm} 2 (causality : continuous type)

(i) A chain of causalities
A chain of causalities is represented by sequential causal operator \( \{ \Phi_{t_1,t_2} : C(\Omega_{t_2}) \to C(\Omega_{t_1}) \}_{(t_1,t_2) \in T_0} \).

(ii) realized causal observable
A sequential observable \( [O_{T(t_0)}] = \{ \{O_t\}_{t \in T}, \{ \Phi_{t_1,t_2} : C(\Omega_{t_2}) \to C(\Omega_{t_1}) \}_{(t_1,t_2) \in T_0} \} \) is realized by its realized causal observable \( \hat{O}_{T(t_0)} = (\times_{t \in T} X_t, \boxtimes_{t \in T} F_t, \hat{F}_{t_0}) \).

Therefore, we get a continuous-pure type classical measurement theory as follows.

\[
\begin{array}{ccc}
\text{pure measurement theory} & := & \text{[Axiom\textsuperscript{P} 1]} \text{ (pure) measurement} + \text{[Axiom\textsuperscript{PM} 2]} \text{ causality} \\
\text{[scientific language]} & + & \text{[probabilistic interpretation]} \text{[the Heisenberg picture]}
\end{array}
\]

Thus, we say that

(a) The probability that a measured value \( (x_t)_{t \in T} \) obtained by the measurement \( M_{C(\Omega_{t_0})}(\hat{O}_{T,S_{\{\omega_{t_0}\}}}) \) belongs to \( \hat{\Xi}(\in \boxtimes_{t \in T} F_t) \) is given by \( [F_{t_0}((\hat{\Xi})\{\omega_{t_0}\})] \).

For completeness, note that

(b) A state \( \omega_0 \) is fixed, and thus, it does not change.
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\*\* Note 6.5 \*\* The (i) in AxiomPM C 2 is fundamental. However, the (ii) may be regarded as is the consequence of "Only one observable is permitted" (i.e., the Copenhagen interpretation).

**Remark 6.19 [Mixed measurement theory]** In mixed measurement theory, AxiomPM C 2 is valid. That is, it is common in pure and mixed measurement theories. Thus, we get mixed measurement theory as follows.

\[
\text{mixed measurement theory} := \begin{bmatrix} \text{[AxiomPM}^P 1] \\ \text{mixed measurement} \end{bmatrix} + \begin{bmatrix} \text{[AxiomPM}^C 2] \\ \text{causality} \end{bmatrix}
\]

That is,

(c) The probability that a measured value \((x_t)_{t \in T}\) obtained by the measurement \(M_{C(\Omega_t)}(\hat{O}_T, S_t[\nu])\) belongs to \(\hat{E} = \bigotimes_{t \in T} F_t\) is given by \(\int_{\Omega_t} [\hat{F}_t(\hat{E})] (\omega_t) \nu_t(d\omega_t)\)

Also, a mixed state \(\nu_t\) is fixed, and thus, it does not change.

**6.4.2 How should time be represented? — Leibniz's relationalism**

In Sec. 2.3.3, we conclude that

(d) The space of our world is described as a kind of state space (or precisely, spectrum).

In , Leibniz-Clarke Correspondence (Sec. 2.3.3), Leibniz says "Time is an order of occurring in succession which changes one after another". Measurement theory agrees to Leibniz's opinion as follows.

(e) Time axis \(\mathbb{R}\) (or, \(\mathbb{Z}\)) is described as a kind of the semi-ordered tree \((T, \leq)\).

\*\* Note 6.6 \*\* After Newton, physical space-time may be superior to metaphysical space-time. The reason may be as follows.

(\(\sharp_1\)) The classification of the world-description(Chap. 1 (O)) — realistic method and linguistic method — is not firm.

(\(\sharp_2\)) Many people investigate "What is space-time?" and not "How should space-time be represented?"

For example, the following two definitions are famous:

(\(\sharp_3\)) [Leibniz]: "Time is an order of occurring in succession which changes one after another".

(\(\sharp_4\)) [Augustinus(354–430)]: Only present exists. Past is in memory. Future is in presentiment.

However, these are rather literary, than scientific.

\*\* Note 6.7 \*\* In this section, the space-time in measuring object is explained. However, measurement theory assert that observer's space-time does not exist. Thus, there is no tense — past, present, future — in science. For example, note that

(\(\sharp\)) In AxiomP C 1, "the probability that a measured value was obtained" and "the probability that measured value will be obtained" are confused.

That is, measurement theory is not concerned with observer's time (or, subjective time).
6.4.3 Why does measurement theory hold?

Now we have several key-words in measurement theory.

(f1) [Axiom\textsuperscript{p} 1]: measurement(observer, measuring object, measurement, observable, state, measured value, probability)

(f2) [Axiom\textsuperscript{pm} 2]: causality (semi-ordered tree, sequential causal operator, realized causal observable)

(f3) [The Copenhagen interpretation]: space as a kind of state space (or, precisely, spectrum), time as a kind of semi-ordered tree.

Measurement theory says that these words should be used according to modeled on Axiom\textsuperscript{pc}s 1 and 2. Thus, the following question is natural:

(g) Why can various sciences be described by the only two axioms?

\textbullet\textbf{Note 6.8} The above question (g) may be deeper than the question such that (F5) Why are two mathematical theories (differential equation and probability theory) useful in science? (Chap. I)

That is because this is solved if the (g) is answered. Thus, what is important is the following.

(2) Why is measurement theory — the language of quantum mechanics — applicable to various sciences? Why is the absurd theory (i.e., dualism and the Copenhagen interpretation) necessary?

We have no answer to this problem.

6.4.4 State change —the Schrödinger picture—

The Copenhagen interpretation — Chap. 1(U1) — says that "only one measurement is permitted", which implies "Only one state and only one observable" Thus, we construct the realized causal observable \(\widehat{O}_T = (X_{t \in T}, X_t, \bigotimes_{t \in T} F_t, \widehat{F}_t)\) from the sequential observable \([O_T] = \{O_t\}_{t \in T}, \{\Phi_{t_1, t_2} : C(\Omega_{t_2}) \to C(\Omega_{t_1})\}_{(t_1, t_2) \in T^2}\). And we take a measurement \(M_{C(\Omega_0)}(\widehat{O}_T, S_{[\omega_0]})\).

\textbullet\textbf{Note 6.9} Summing up the above argument as follows. classical mechanical world-view causality, [only one measurement is permitted] \(\Rightarrow\) [Only one observable] \(\Rightarrow\) [realized causal observable] (the Copenhagen interpretation)\textsuperscript{pm} 2(ii)

However, as a convenient method, we sometimes use the state change due to the Schrödinger picture. This is not general but it may be understandable. Thus, in what follows, we explain this viewpoint (i.e., state change).

We begin with the simplest example. Put \(T = \{0, 1\}\). Consider a deterministic causal operator \(\Phi_{0,1} : C(\Omega_1) \to C(\Omega_0)\) with a deterministic causal map \(\phi_{0,1} : \Omega_0 \to \Omega_1\). Let \(O_1 = (X_1, F_1, F_1)\) be an observable in \(C(\Omega_1)\).

be and . \(O_1 = (X_1, F_1, F_1)\) deterministic causal operator and . \(O_1 = (X_1, F_1, F_1)\) in \(C(\Omega_1)\)

Consider a measurement \(M_{C(\Omega_0)}(\Phi_{0,1}O_1, S_{[\omega_0]})\). Axiom\textsuperscript{p} 1(measurement says that
Theorem 6.18 (6.8), in the particular case of deterministic sequential causal operator, 

\[ M_t \text{ be a state at time } t \]

Here it should be noted that, \( \Phi_t \)

Thus, for each \( t \in F_1 \) is given by \[ F_1(\Xi_t))(\phi_{0,1}(\omega_0)) \]

Here, note that \( [\Phi_{0,1} F_1(\Xi_t)][(\omega_0) = [F_1(\Xi_t)](\phi_{0,1}(\omega_0)) \)

As seen in Sec. 4.4, we see:

\( \text{EAXIOM}_C \)

Consider a mixed measurement \( M_{C(\Omega_t)}(O_1, S(\phi_{0,t}(\omega_0))) \). Then, by Axiom \( \text{EAXIOM} \) 1 (mixed measurement) in Sec. 4.4, we see:

\( \text{Axiom}_C \)

f) The probability that a measured value obtained by a mixed measurement \( M_{C(\Omega_t)}(O_1, S(\phi_{0,t}(\omega_0))) \) belongs to \( \Xi_t(\in F_t) \) is given by \[ \int_{\Omega_t} F_t(\Xi_t)[\phi_{0,t}(\delta_{\omega_0})](d \omega_t) \]

Thus, for each \( t \in T \), we have the following identification:

\[ M_{C(\Omega_t)}(\Phi_{0,t} O_1, S_{\omega_0}) = M_{C(\Omega_t)}(O_1, S_{\omega_0}(\phi_{0,t}) \delta_{\omega_0})) \]

However, it should be noted that

\[ \times \int_{t \in T} F_t(\Xi_t)[\phi_{0,t}(\delta_{\omega_0})](d \omega_t) \neq [\widehat{F}_{0,t}(\times \Xi_t)](\omega_0) \]

As seen in Theorem 6.18, in the particular case of deterministic sequential causal operator, \( \neq \) can be replaced by \( = \) in (6.8).
6.4.5 The principle of equal weight — Famous unsolved problem

Reconsidering

Monty Hall problem (Problem 4.9, Problem 4.16), we present the final answer of Monty Hall problem.

Problem 6.20 [Monty Hall problem (Continued from Problem 4.9, Problem 4.16) (cf. [11, 19])] Suppose you are on a game show, and you are given the choice of three doors (i.e., “number 1,” “number 2,” “number 3”). Behind one door is a car, behind the others, goats.

(1) You choose a door by the cast of the fair dice, i.e., with probability 1/3.

According to the rule (1), you pick a door, say number 1, and the host, who knows where the car is, opens another door, behind which is a goat. For example, the host says that

(b) the door 3 has a goat.

He says to you, “Do you want to pick door number 2?” Is it to your advantage to switch your choice of doors?

Answer As Problem 4.9 (Monty Hall problem), consider a state space \( \Omega = \{ \omega_1, \omega_2, \omega_3 \} \). And the observable \( \mathcal{O} = (X, \mathcal{F}, F) \) is defined by the formula (4.2). The map \( \phi : \Omega \rightarrow \Omega \) is defined by

\[
\phi(\omega_1) = \omega_2, \quad \phi(\omega_2) = \omega_3, \quad \phi(\omega_3) = \omega_1
\]

we get a causal operator \( \Phi : C(\Omega) \rightarrow C(\Omega) \) by \( [\Phi(f)](\omega) = f(\phi(\omega)) \ (\forall f \in C(\Omega), \forall \omega \in \Omega) \). Assume that a car is behind the door \( k \ (k = 1, 2, 3) \). Then, we say that

(a) By the dice-throwing, you get \( \begin{bmatrix} 1, 2 \\ 3, 4 \\ 5, 6 \end{bmatrix} \), then, take a measurement \( \begin{bmatrix} M_{C(\Omega)}(\mathcal{O}, S_{[\omega_1]}) \\ M_{C(\Omega)}(\Phi \mathcal{O}, S_{[\omega_k]}) \\ M_{C(\Omega)}(\Phi^2 \mathcal{O}, S_{[\omega_k]}) \end{bmatrix} \)

We, by Sec.6.4.4(c), see the following identifications: \( M_{C(\Omega)}(\Phi \mathcal{O}, S_{[\omega_k]}) = M_{C(\Omega)}(\mathcal{O}, S_{[\phi(\omega_k)]}) \), \( M_{C(\Omega)}(\Phi^2 \mathcal{O}, S_{[\omega_k]}) = M_{C(\Omega)}(\mathcal{O}, S_{[\Phi^2(\omega_k)]}) \). Thus, the above (a) is equal to

(b) By the dice-throwing, you get \( \begin{bmatrix} 1, 2 \\ 3, 4 \\ 5, 6 \end{bmatrix} \), then, take a measurement \( \begin{bmatrix} M_{C(\Omega)}(\mathcal{O}, S_{[\omega_1]}) \\ M_{C(\Omega)}(\mathcal{O}, S_{[\phi(\omega_k)]}) \\ M_{C(\Omega)}(\mathcal{O}, S_{[\Phi^2(\omega_k)]}) \end{bmatrix} \)

Here, note that \( \frac{1}{3} (\delta_{\omega_k} + \delta_{\phi(\omega_k)} + \delta_{\Phi^2(\omega_k)}) = \frac{1}{3} (\delta_{\omega_1} + \delta_{\omega_2} + \delta_{\omega_3}) \ (\forall k = 1, 2, 3) \). Thus, the (b) is identified with the mixed measurement \( M_{C(\Omega)}(\mathcal{O}, S_{[\nu]}(\nu_e)) \) where \( \nu_e = \frac{1}{3} (\delta_{\omega_1} + \delta_{\omega_2} + \delta_{\omega_3}) \). Therefore, Problem 6.20 is the same as Problem 4.16. Hence, you should choose the door 2.

\( \blacklozenge \) Note 6.10 The above argument is easy. That is, since you have no information, we choose the door by a fair dice throwing. In this sense, the principle of equal weight — unless we have sufficient reason to regard one possible case as more probable than another, we treat them as equally probable — is clear in measurement theory. However, it should be noted that the above argument is based on dualism.

The following is the measurement theoretical formulation of ”the principle of equal weight”. Theorem 6.21 [The principle of equal weight] Consider a finite state space \( \Omega \), that is, \( \Omega = \{ \omega_1, \omega_2, \ldots, \omega_n \} \).
Let \( O = (X, \mathcal{F}, F) \) be an observable in \( C(\Omega) \). Consider a measurement \( M_{\Omega}^{C}(O, S_\omega) \). If the observer has no information for the state \([\ast]\), there is a reason to that this measurement is identified with the mixed measurement \( M_{\Omega}^{C}(O, S_\omega(\nu)) \), where \( \nu = \frac{1}{n} \sum_{i=1}^{n} \delta_{\omega_i} \in \mathcal{M}_{n+1}(\Omega) \).

**Proof.** The proof is a easy consequence of the above Monty Hall problem (or, see [11, 20]).

6.5 Examples in ”Axiom\( _c \) 1(measurement) + Axiom\( _{pm} \) 2(causality )”

In dualism(measurement theory), Axiom\( _{pm} \) 2(causality) is not independently used, but it is used with Axiom\( _c \) 1(measurement).

6.5.1 Parallel structure

Consider a semi-ordered tree \( (T(0)=\{0,1,\ldots,N\}, \pi : T \setminus \{0\} \rightarrow T) \) with parallel structure such that \( \pi(t) = 0 \ (\forall t \in T \setminus \{0\}) \).

![Parallel structure](image)

Consider a sequential observable \( O_T = \{\{O_t\}_{t \in T}, \{\Phi_{\pi(t),t} : C(\Omega_t) \rightarrow C(\Omega_{\pi(t)})\}_{t \in T \setminus \{0\}}\} \) and its realized causal observable \( \hat{O}_T = (\times_{t=0}^{N} X_t, \bigotimes_{t=0}^{N} F_t, \hat{F}_0) \), That is,

\[
\hat{F}_0(\Xi_0 \times \Xi_1 \times \Xi_2 \times \cdots \times \Xi_N) = \times_{t \in T} \Phi_{0,t} F_t(\Xi_t)
\]

Thus, we have the measurement \( M_{C(\Omega)}(\hat{O}_T=(\times_{t \in T} X_t, \bigotimes_{t \in T} F_t, \hat{F}_0), S_{[\omega_0]}). \) Therefore, in the case of parallel structure, note that the equality ”=” holds in (6.8).

That is, we see

the probability that a measured value obtained by a parallel measurement \( M_{C(\Omega)}(\hat{O}_T=(\times_{t \in T} X_t, \bigotimes_{t \in T} F_t, \hat{F}_0), S_{[\omega_0]}). \) belongs to \( \Xi_0 \times \Xi_1 \times \cdots \times \Xi_N \) is given by

\[
[\hat{F}_0(\times_{t \in T} \Xi_t)](\omega_0) = \times_{t \in T} [\Phi_{0,t} F_t(\Xi_t)](\omega_0)
\]

**Example 6.22** [Before pheasants and rabbits problem([Example 6.10]+[measurement])] Like Example 6.10, consider the following problem:
(a) [Pheasants and rabbits problem] A number of $m$ pheasants and $n$ rabbits are placed together in the same cage. Then $m + n$ heads and $2m + 4n$ legs are counted. Find the number of pheasants and rabbits.

ordinary language

In Example 6.10, the statement (a) in ordinary language is understood as

"$m''$, "$n''$, "$m + n''" and "$2m + 4n''" are states

However, here we understand as follows.

"$m''$" and "$n''$" are states, but "$m + n''" and "$2m + 4n''" are measured values.

As mentioned in Example 6.10, put

$$\Omega_0 = \mathbb{N}_0 \times \mathbb{N}_0, \quad \Omega_1 = \mathbb{N}_0, \quad \Omega_2 = \mathbb{N}_0$$

Putting $T = \{0, 1, 2\}$, $\pi(1) = 0$, $\pi(2) = 0$. then we get a sequential causal operator $\{C(\Omega_t) \phi_{\pi(t)}^{-1, t} C(\Omega_{\pi(t)})\}_{t \in T \setminus \{0\}}$. For each $t \in \{1, 2\}$, consider exact observable $O_T^{(exa)} = (\mathbb{N}_0, 2^{\mathbb{N}_0}, F_t^{(exa)})$ in $C(\Omega_t)$. Thus, we get the sequential observable $[O_T] = \{\{\Omega_t\}_{t=1, 2}, \{\Phi_{\pi(t), t} : C(\Omega_t) \rightarrow C(\Omega_{\pi(t)})\}_{t=1, 2}\}$ and its realized causal observable $\tilde{O}_0 = (\mathbb{N}_0 \times \mathbb{N}_0, 2^{\mathbb{N}_0 \times \mathbb{N}_0}, \tilde{F}_0)$ such that

$$[\tilde{F}_0(\Xi_1 \times \Xi_2)](m, n) = [\Phi_{0, 1} F_1^{(exa)}(\Xi_1)](m, n) \cdot [\Phi_{0, 2} F_2^{(exa)}(\Xi_2)](m, n)$$

$$[F_1^{(exa)}(\Xi_1)](m + n) \cdot [F_2^{(exa)}(\Xi_2)](2m + 4n)$$

$(\forall \Xi_1, \forall \Xi_2 \in 2^{\mathbb{N}_0}, \forall (m, n) \in \Omega_0)$

Hence, we get the measurement $M_{C(\Omega_0)}(\tilde{O}_0, S_{[(m, n)\}]})$. It is clear that

(b) By measurement $M_{C(\Omega_0)}(\tilde{O}_0, S_{[(m, n)\]}), we get a measured value $(m + n, 2m + 4n)$ with probability 1.

Here, it should be noted that there are many interpretations (measurement theory).

6.5.2 Series structure — Measurement of time

Assume that the semi-ordered tree $(T =\{0, 1, \ldots, N\}, \pi)$ has the series structure such that That is, $\pi(t) = t - 1$ $(\forall t \in T \setminus \{0\})$. Consider a sequential causal operator $\{C(\Omega_t) \phi_{\pi(t)}^{-1, t} C(\Omega_{\pi(t)})\}_{t \in T \setminus \{0\}}$ such that

$$C(\Omega_0) \phi_{0, 1} \leftarrow C(\Omega_1) \phi_{1, 2} \leftarrow C(\Omega_2) \phi_{2, 3} \leftarrow \ldots \ldots \ldots \leftarrow C(\Omega_{N-1}) \phi_{N-2, N-1} \leftarrow C(\Omega_N)$$

Now, consider a sequential observable $[O_T] = \{\{\Omega_t\}_{t \in T}, \{\Phi_{\pi(t), t} : C(\Omega_t) \rightarrow C(\Omega_{\pi(t)})\}_{t \in T \setminus \{0\}}\}$. And let us construct the realized causal observable $\tilde{O}_T = (X^N_{t=0} X_t, \prod_{t=0}^N F_t, \tilde{F}_0)$.

Firstly, put, $\tilde{O}_N (= (X_N, F_N, \tilde{F}_N)) = O_N (= (X_N, F_N, F_N))$. We have the simultaneous observable $\tilde{O}_{N-1} = O_{N-1} \times \Phi_{N-1, N} O_N = (X_{N-1} \times X_N, F_{N-1} \otimes F_N, \tilde{F}_{N-1})$ in $C(\Omega_{N-1})$. That is,

$$\tilde{F}_{N-1}(\Xi_{N-1} \times \Xi_N) = (F_{N-1} \times (\Phi_{N-1, N} F_N))(\Xi_{N-1} \times \Xi_N)$$
Similarly, we get the simultaneous observable \( \hat{O}_{N-2} = O_{N-2} \times \Phi_{N-2,N-1} \hat{O}_{N-1} = (X_{N-2} \times X_{N-1} \times X_N, \mathcal{F}_{N-2} \otimes \mathcal{F}_{N-1} \otimes \mathcal{F}_n, \hat{F}_{N-2}) \) in \( C(\Omega_{N-2}) \). That is,

\[
\hat{F}_{N-2}(\Xi_{N-2} \times \Xi_{N-1} \times \Xi_N) = (F_{N-2}(\Phi_{N-2,N-1} \hat{F}_{N-1}))(\Xi_{N-2} \times \Xi_{N-1} \times \Xi_N)
\]

Iteratively,

\[
\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
F_0 & F_1 & \ldots & F_{N-1} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
(\hat{F}_0 \times \Phi F_1) & (\hat{F}_1 \times \Phi F_2) & \ldots & (\hat{F}_{N-1} \times \Phi F_N) \\
\end{array}
\]

And finally, we get the simultaneous observable \( \hat{O}_0 = O_0 \times \Phi_{0,1} \hat{O}_1 = (X_{t=0}^N, \otimes_{t=0}^N F_t, \hat{F}_0) \) in \( C(\Omega_0) \). That is,

\[
\hat{F}_0(\Xi_0 \times \Xi_1 \times \Xi_2 \times \cdots \times \Xi_N) = (F_0(\Phi_{0,1} \hat{F}_1))(\Xi_0 \times \Xi_1 \times \Xi_2 \times \cdots \times \Xi_N)
\]

Here, \( \hat{O}_0 \) is the realized causal observable \( \hat{O}_T \) of the sequential observable \( \{\{O_t\}_{t \in T}, \{\Phi_{\tau(t),t} : C(\Omega_t) \to C(\Omega_{\tau(t)})\}_{t \in T \setminus \{0\}}\} \). Thus, we get measurement:

\[
M_{C(\Omega_0)}(\hat{O}_T) = (X_{t \in T}, \otimes_{t \in T} F_t, \hat{F}_0, S_{[\omega_1]})
\]

**Example 6.23 [Measurement of discrete time]** Considering a state space \( \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\} \), define the deterministic causal map \( \Phi : \mathbb{Z} \to \mathbb{Z} \) by

\[
\mathbb{Z} \ni i \mapsto i + 1 \in \mathbb{Z}
\]

and thus, define the deterministic causal operator \( \Phi : C(\mathbb{Z}) \to C(\mathbb{Z}) \) by

\[
[\Phi(f)](i) = f(i - 1) \quad (\forall i \in \mathbb{Z}, \forall f \in C(\mathbb{Z}))
\]

Consider the exact observable \( O^{(\text{exa})} = (\mathbb{Z}, 2^\mathbb{Z}, F^{(\text{exa})}) \) in \( C(\mathbb{Z}) \). Let \( T = \{0, 1, 2, \ldots, N\} \) be the discrete time, that is, \( (T(0), \leq) \). For \( t(\in T) \), consider a state space \( \Omega_t = \mathbb{Z} \). Define the deterministic causal operator \( \Phi_{m,n} : C(\Omega_m) \to C(\Omega_n) \) \( (0 \leq m \leq n \leq N) \) such that

\[
\Phi_{m,n} = \Phi \cdot \Phi \cdot \ldots \cdot \Phi = \Phi^{n-m}
\]

Putting \( O^{(\text{exa})}_t = O^{(\text{exa})} (\forall t \in T) \), we have the sequential deterministic causal exact observable \( \{\{O^{(\text{exa})}_t\}_{t \in T}, \{\Phi : C(\Omega_m) \to C(\Omega_{m-1})\}_{m \in T \setminus \{0\}}\} \). Thus, by Theorem 6.18, we get the realized causal observable \( \hat{O} = (\times_{m \in T} \mathbb{Z}, \otimes_{m \in T} 2^\mathbb{Z}, \hat{F}) \) such that

\[
\hat{F}(\times_{m \in T} \Xi_m) = \times_{m \in T} \Phi^m F^{(\text{exa})}(\Xi_m)
\]
Therefore, for the initial state $\omega_0(\in \Omega_0)$, we get the "time measurement" $M_{C(\Omega_0)}(\hat{O}, S_{[\omega_0]})$. The measured value is clearly

$$(\omega_0, \omega_0 + 1, \omega_0 + 2, \ldots, \omega_0 + N)$$

That is, if the initial state space $\Omega_0$ is assumed to be at time $\omega_0$, then the clock (at time $t$) shows time $\omega_0 + t$.

\textbf{Note 6.11} The above example says that each time $t (\in T = \{0, 1, \ldots, N\})$ is not a state, but a state of the clock is a state, i.e., $\omega_t$. And thus it can be measured.

\textbf{Example 6.24 [Random walk(Continued from Example 6.13)]} \(Z = \{0, \pm 1, \pm 2, \ldots\}\) and . In Example 4.10, put

$$C(\Omega_t) = C(Z) \quad (\forall t \in T = \{0, 1, \ldots, N\})$$

where $Z$ is the set of all integers, i.e., $Z = \{0, \pm 1, \pm 2, \ldots\}$. And define a causal operator $\Phi_{t-1,t}(\equiv \Phi) : C(\Omega_t)(\equiv C(Z)) \rightarrow C(\Omega_{t-1})(\equiv C(Z))$ such that:

$$(\Phi f)(n) = (\Phi_{t-1,t} f)(n) = \frac{f(n+1) + f(n-1)}{2} \quad (\forall f \in C(\Omega_t)(\equiv C(Z)), \forall n \in Z).$$

Further, define the sequence observable $[\{O_t\}_{t \in T}, \{\Phi_{t,t+1} = \Phi\} : C(\Omega_t) \rightarrow C(\Omega_{t+1})]_{t \in T \setminus \{0\}}$ as follows:

Putting $X_t = \Omega_t = Z$, for example, define

$$O_t = \begin{cases} 
\text{exact observable:} O_t^{(\text{exa})} = (Z(= X_t), 2^Z, F_t^{(\text{exa})}) & (t = 2, 4) \\
\text{existence observable:} O_t^{(\text{exi})} = (Z(= X_t), \{\emptyset, Z\}, F_t^{(\text{exi})}) & (\text{otherwise})
\end{cases}$$

Since existence observables can be ignored, we have the realized observable $\hat{O}_0 = (Z^2(= X_2 \times X_4), 2^Z \times 2^Z, \hat{F})$ such that

$$\hat{F}(\Xi_2 \times \Xi_4) = F^2 \left( F^{(\text{exa})}(\Xi_2) \times F^{(\text{exa})}(\Xi_4) \right) \quad (\forall \Xi_2 \in 2^X, \forall \Xi_4 \in 2^X)$$

Let $0 (\in \Omega_0 = Z)$ be a state. Consider the measurement measurement $M_{C(\Omega_0)}(\hat{O}_0, S_{[0]})$. For example, we shall calculate

(F) the probability that the measured value belongs to $\{0, 1\} \times \{-1, 0\} = \Xi_2 \times \Xi_4(\subseteq X_2 \times X_4)$ is equal to $\frac{1}{4}$.

This is easily shown as follows. Using the characteristic function $\chi(\cdot)$, we see

$$[F_2^{(\text{exa})}(\Xi_2)](\omega) = \chi_{\Xi_2}(\omega) = \chi_{\{0\}}(\omega) + \chi_{\{1\}}(\omega) \quad (\forall \omega \in \Omega_2 = Z)$$

$$[F_4^{(\text{exa})}(\Xi_4)](\omega) = \chi_{\Xi_4}(\omega) = \chi_{\{0\}}(\omega) + \chi_{\{-1\}}(\omega) \quad (\forall \omega \in \Omega_4 = Z)$$
And therefore,
\[ (\Phi \chi_{(m)})(\omega) = \frac{1}{2}(\chi_{(m)}(\omega + 1) + \chi_{(m)}(\omega - 1)) = \frac{x_{(m-1)}(\omega) + x_{(m+1)}(\omega)}{2} \]
thus,
\[ \Phi^2(F^{(\text{exa})}(\Xi_4)) = \Phi \left( \frac{x_{(-1)} + x_{(1)}}{2} + \frac{x_{(-2)} + x_{(0)}}{2} \right) = \frac{1}{4}\left( x_{(-3)} + x_{(-2)} + 2x_{(-1)} + 2x_{(0)} + x_{(1)} + x_{(2)} \right). \]

Thus, we conclude that
\[ (F) = \hat{F}((0, 1) \times \{-1, 0\})(0) \]
\[ = \frac{1}{16}\left( 2x_{(-2)}(0) + x_{(-1)}(0) + 4x_{(0)}(0) + 2x_{(1)}(0) + 2x_{(2)}(0) + x_{(3)}(0) \right) = \frac{4}{16} = \frac{1}{4} \]

6.6 Two kinds of absurdness — idealism and dualism

As mentioned in Note 1.10, measurement theory has two kinds of absurdness. That is,

\[ (\sharp_2) \text{Two kinds of absurdness} \begin{cases} \text{idealism} & \cdots \text{linguistic world-view} \\ (\text{Fit feet to shoes}) \end{cases} \begin{cases} \text{dualism} & \cdots \text{the Copenhagen interpretation} \\ (\text{A spectator does not go up to the stage}) \end{cases} \]

In what follows, we explain these.

6.6.1 The Copenhagen interpretation — A spectator does not go up to the stage

Remark 6.25 [ A spectator does not go up to the stage ] Consider the elementary problem with two steps (a) and (b):

(a) Consider an urn, in which 3 white balls and 2 black balls are. Consider the following trial:

Pick out one ball from the urn. If it is black, you return it in the urn. If it is white, you do not return it and have it. Assume that you take three trials.
(b) Then, calculate the probability that you have 2 white balls after (a) (i.e., three trials).

**Answer** Put $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Assume that there are $m$ white balls and $n$ black balls in the urn. This situation is represented by a state $(m, n) \in \mathbb{N}_0^2$. We can define the dual causal operator $\Phi^* : M_{+1}(\mathbb{N}_0^2) \to M_{+1}(\mathbb{N}_0^2)$ such that
\[
\Phi^*(\delta_{(m,n)}) = \begin{cases} 
\frac{m}{m+n} \delta_{(m-1,n)} + \frac{n}{m+n} \delta_{(m,n)} & \text{ (when } m \neq 0 ) \\
\delta_{(0,n)} & \text{ (when } m = 0 )
\end{cases}
\]
where $\delta_{(\cdot)}$ is the point measure.

Let $T = \{0, 1, 2, 3\}$ be discrete time. For each $t \in T$, put $\Omega_t = \mathbb{N}_0^2$. Thus, we see:
\[
[\Phi^*]^3(\delta_{(3,2)}) = [\Phi^*]^2 \left( \frac{3}{5} \delta_{(3,2)} + \frac{2}{5} \delta_{(2,2)} \right) = \Phi^* \left( \frac{3}{5} \left( \frac{2}{3} \delta_{(1,2)} + \frac{2}{4} \delta_{(2,2)} \right) + \frac{2}{5} \left( \frac{3}{5} \delta_{(2,2)} + \frac{2}{3} \delta_{(3,2)} \right) \right)
\]
\[
= \frac{3}{10} \frac{1}{3} \delta_{(0,2)} + \frac{2}{3} \delta_{(1,2)} + \frac{27}{50} \frac{2}{4} \delta_{(1,2)} + \frac{2}{4} \delta_{(2,2)} + \frac{4}{25} \left( \frac{3}{5} \delta_{(2,2)} + \frac{2}{3} \delta_{(3,2)} \right)
\]
\[
= \frac{1}{10} \delta_{(0,2)} + \frac{47}{100} \delta_{(1,2)} + \frac{183}{500} \delta_{(2,2)} + \frac{8}{125} \delta_{(3,2)}
\]
Define the observable $O = (\mathbb{N}_0, 2^{\mathbb{N}_0}, F)$ in $C(\Omega_3)$ such that
\[
[F(\Xi)](m, n) = \begin{cases} 
1 & (m, n) \in \Xi \times \mathbb{N}_0 \subseteq \Omega_3 \\
0 & (m, n) \notin \Xi \times \mathbb{N}_0 \subseteq \Omega_3
\end{cases}
\]
Therefore, the probability that a measured value "2" is obtained by the measurement $M_{C(\mathbb{N}_0^2)}(\Phi^3O, S_{\{3,2\}})$ is given by
\[
[\Phi^3(F(\{2\}))(3, 2)] = \int_{\Omega_3} [F(\{2\})](\omega)([\Phi^*]^3(\delta_{(3,2)}))(d\omega) = \frac{183}{500}
\]
\[
\square
\]
The above may be easy, but we should note that

(c) the part (a) is related to causality, and the part (b) is related to measurement.

Thus, the observer is not in the (a). Figuratively speaking, we say:

A spectator does not go up to the stage

Thus, someone in the (a) should be regarded as "robot".

♠ **Note 6.12** The part (a) is not related to "probability". That is because The spirit of measurement theory says that

there is no probability without measurements.

although something like "probability" in the (a) is called "Markov probability".
6.6.2 linguistic world-view — Fit feet to shoes

Ordinary language has everything, i.e., monism, dualism, tense and so on. Also, in ordinary language, there is no clear rule how to use the terms: "measurement" and "causality".

Remark 6.26 [Confusion of Measurement and causality (Continued from Example 2.7)] Recall Example 2.7 [The measurement of "cold or hot" for water]. Consider the measurement \( M_{C(\Omega)}(O_{ch}, S_{[\omega]}(=5)) \) where \( \omega = 5(\, ^\circ C) \). Then we say that

(a) By the measurement \( M_{C(\Omega)}(O_{ch}, S_{[\omega]}(=5)) \), the probability that a measured value \( x(\in X = \{c, h\}) \) belongs to a set

\[
\begin{bmatrix}
0 (= \text{empty set}) \\
\{c\} \\
\{h\} \\
\{c, h\}
\end{bmatrix}
\]

is equal to

\[
\begin{bmatrix}
0 \\
[F(\{c\})]\!(5) = 1 \\
[F(\{h\})]\!(5) = 0 \\
1
\end{bmatrix}
\]

Here, we should not think:

"5 \, ^\circ C" is the cause and "cold" is a result.

That is, we never consider that

(b) \( 5 \, ^\circ C \) (cause) \( \rightarrow \) cold \ (result)

The reason is that Axiom_\text{PM}^C \text{2} is not used in (a), though the (a) may be sometimes regarded as the causality (b) in ordinary language.

♠ Note 6.13 However, from the different point of view, the above (b) can be justified as follows. Define the dual causal operator \( \Phi^* : M([0,100]) \rightarrow M(\{c, h\}) \) by

\[
[\Phi^* \delta_\omega](D) = f_c(\omega) \cdot \delta_C(D) + f_h(\omega) \cdot \delta_H(D) \quad (\forall \omega \in [0,100], \ \forall D \subseteq \{c, h\})
\]

Then, the (b) can be regarded as "causality". That is,

(2) "measurement or causality" depends on how to describe a phenomenon.

This is the linguistic world-description method.

Remark 6.27 [Confusion of mixed measurement and Markov causality ] Reconsider Example 4.13 (urn problem:mixed measurement). Consider a state space \( \Omega = \{\omega_1, \omega_2\} \), and define the observable \( O = (\{w, b\}, 2^{\{w, b\}}, F) \) in \( C(\Omega) \) by the formula (2.5). Define the mixed state by \( \nu_0 = p \delta_{\omega_1} + (1-p) \delta_{\omega_2} \). Then the probability that a measured value \( x (\in \{w, b\}) \) is obtained by the mixed measurement \( M_{C(\Omega)}(O, S_{[\nu_0]}(t_0)) \) is, by (4.5), given by

\[
P(\{x\}) = \int_{\Omega} [F(\{x\})](\omega) \nu_0(d\omega) = p[F(\{x\})](\omega_1) + (1-p)[F(\{x\})](\omega_2)
\]

\[
= \begin{cases} 
0.8p + 0.4(1-p) & \text{(when } x = w \text{)} \\
0.2p + 0.6(1-p) & \text{(when } x = b \text{)} 
\end{cases} \quad (6.9)
\]
Now, define a new state space $\Omega_0$ by $\Omega_0 = \{\omega_0\}$. And define the dual Markov causal operator $\Phi^* : M_{+1}(\Omega_0) \to M_{+1}(\Omega)$ by $\Phi^*(\delta_{\omega_0}) = p\delta_{\omega_1} + (1-p)\delta_{\omega_2}$. Thus, we have the Markov causal operator $\Phi : C(\Omega) \to C(\Omega_0)$.

Here, consider a pure measurement $M_{C(\Omega_0)}(\Phi^*_O,S_{[\omega_0]}^*)$. Then, the probability that a measured value $x$ ($\in \{w, b\}$) is obtained by the measurement is given by

$$P(\{x\}) = \int_{\Omega} [F(\{x\})](\omega)\nu_0(d\omega)$$

which is equal to (6.9). Therefore, the mixed measurement $M_{C(\Omega)}(\Phi^*_O,S_{[\omega]}^*)$ can be regarded as the pure measurement $M_{C(\Omega_0)}(\Phi^*_O,S_{[\omega_0]}^*)$.

\[\text{Note 6.14} \text{ In the above arguments, we see that}
\]

\[\text{(2) Concept depends on the description}
\]

This is the linguistic world-description method. As mentioned in Note 2.3, we are not concerned with the question "what is $\bigcirc\bigcirc$?". The reason is due to the (2).

\[\text{Note 6.15} \text{ As mentioned in Note 6.13, "Measurement or Causality" depends on the description. Some may recall Nietzsche's famous saying:}
\]

\[\text{There are no facts, only interpretations.}
\]

This is just the linguistic world-description method with the spirit: "Fit feet (=world) to shoes (language)".

### 7 Fisher statistics II

As mentioned before, measurement theory is formulated as follows. That is,

$$\text{measurement theory}_{\text{(scientific language)}} = \text{measurement}_{\text{[probabilistic interpretation]}} + \text{causality}_{\text{[the Heisenberg picture]}}$$

In Chap. 5, we studied Fisher statistics in Axiom$^p_1$. In this chapter, Fisher statistics will be discussed in Axiom$^p_1$ and 2.

#### 7.1 Measurement (= the view from the front), Inference-Control (= the view from the back)

**7.1.1 inference problem(statistics)**

**Problem 7.1 [Inference problem and regression analysis]** Let $\Omega \equiv \{\omega_1, \omega_2, \ldots, \omega_{100}\}$ be a set of all students of a certain high school. Define $h : \Omega \to [0,200]$ and $w : \Omega \to [0,200]$ such that:

$$h(\omega_n) = \text{“the height of a student } \omega_n \text{”} \quad (n = 1, 2, \ldots, 100)$$

$$w(\omega_n) = \text{“the weight of a student } \omega_n \text{”} \quad (n = 1, 2, \ldots, 100) \quad (7.1)$$

For simplicity, put, $N = 5$. For example, see Table 7.1.
Table 7.1: Height and weight

| Height (\(h(\omega)\)) | \(\omega_1\) | \(\omega_2\) | \(\omega_3\) | \(\omega_4\) | \(\omega_5\) |
|------------------------|-------------|-------------|-------------|-------------|-------------|
| Height (\(h(\omega)\)) | 150         | 160         | 165         | 170         | 175         |
| Weight (\(w(\omega)\)) | 65          | 55          | 75          | 60          | 65          |

Assume that:

(a) The principal of this high school knows the both functions \(h\) and \(w\). That is, he knows the exact data of the height and weight concerning all students.

Also, assume that:

(a) Some day, a certain student helped a drowned girl. But, he left without reporting the name. Thus, all information that the principal knows is as follows:

(i) he is a student of his high school.

(ii) his height [resp. weight] is about 170 cm [resp. about 80 kg].

Now we have the following question:

(b) Under the above assumption (a) and (a), how does the principal infer who is he?

This will be answered in Answer 5.4.

7.1.2 control problem (dynamical system theory)

Adding measurement equation \(g : \mathbb{R}^3 \rightarrow \mathbb{R}\) to state equation(6.3), we get dynamical system theory(7.2). That is,

\[
\text{dynamical system theory} = \begin{cases} 
(i) : \frac{d\omega(t)}{dt} = v(\omega(t), t, e_1(t), \beta) & \cdots \text{ (state equation)} \\
\text{(initial at } t=0 \text{ is } \alpha) \\
(ii) : x(t) = g(\omega(t), t, e_2(t)) & \cdots \text{ (measurement)} 
\end{cases}
\]  
(7.2)

where \(\alpha, \beta\) are parameters, \(e_1(t)\) is noise, \(e_2(t)\) is measurement error.

The following example is the simplest problem concerning inference.

Problem 7.2 [Control problem and regression analysis] We have a rectangular water tank filled with water. Assume that the height of water at time \(t\) is given by the following function \(h(t)\):

\[
\frac{dh}{dt} = \beta_0, \text{ then } h(t) = \alpha_0 + \beta_0 t, \tag{7.3}
\]

where \(\alpha_0\) and \(\beta_0\) are unknown fixed parameters such that \(\alpha_0\) is the height of water filling the tank at the beginning and \(\beta_0\) is the increasing height of water per unit time. The measured height \(h_m(t)\) of water at time \(t\) is assumed to be represented by

\[
h_m(t) = \alpha_0 + \beta_0 t + e(t),
\]
where \( e(t) \) represents a noise (or more precisely, a measurement error) with some suitable conditions. And assume that we obtained the measured data of the heights of water at \( t = 1, 2, 3 \) as follows:

\[
    h_m(1) = 0.5, \quad h_m(2) = 1.6, \quad h_m(3) = 3.3.
\]  

(7.4)

\[\text{Figure 7.1: Water tank}\]

Under this setting, we consider the following problem:

\( (c_1) \) [Control]: Settle the state \((\alpha_0, \beta_0)\) such that measured data (7.4) will be obtained.

or, equivalently,

\( (c_2) \) [Inference]: when measured data (7.4) is obtained, infer the unknown state \((\alpha_0, \beta_0)\).

This will be answered in Answer 5.4.

Note that \((c_1) = (c_2)\) from the theoretical point of view. Thus we consider that

\( (d) \) Inference problem and control problem are the same problem. And these are characterized as the reverse problem of measurements.

and Remark (Sec.4.2.2(c)).

### 7.2 Regression analysis — causality + Fisher maximum likelihood method

Combining Axiom\textsuperscript{2}m 2(causality) and Fisher maximum likelihood method(Theorem 4.5)), we can easily prove the following.

**Theorem 7.3 [regression analysis (cf. [15])]** Let \( T = \{t_0, t_1, \ldots, t_N\}, \pi : T \setminus \{t_0\} \to T \) be semi-ordered tree. Let \( \hat{O_T} = (X_{t \in T}, \boxtimes_{t \in T} F_t, \hat{F}_{t_0}) \) be the realized causal observable of a sequential observable \( \{O_t\}_{t \in T}, \{\Phi_{\pi(t)}, t : C(\Omega_t) \to C(\Omega_{\pi(t)})\}_{t \in T \setminus \{t_0\}} \). Consider a measurement

\[
M_{C(\Omega_{t_0})}(\hat{O_T} = (X_{t \in T}, \boxtimes_{t \in T} F_t, \hat{F}_{t_0}), S_{[\ast]})
\]

Assume that a measured value by the measurement belongs to \( \hat{\Xi} (\in \boxtimes_{t \in T} F_t) \). Then, there is a reason to infer that

\[
[\ast] = \omega_{t_0}
\]
where \( \omega_{t_0} (\in \Omega_{t_0}) \) is defined by

\[
[F_{t_0}(\dot{z})](\omega_{t_0}) = \max_{\omega \in \Omega_{t_0}} [F_{t_0}(\dot{z})](\omega)
\]

The proof is a direct consequence of Axiom \( t \) (causality) and Fisher maximum likelihood method (Theorem 4.5). Thus, we omit it. \( \Box \)

Now we can present the answer to Problem 7.1.

**Answer 7.4** [(Continued from Problem 7.1 (inference problem)) regression analysis] For each \( t = 1, 2 \), let \( O_{G_{\sigma_t}} = (\mathbb{R}, \mathcal{B}_R, G_{\sigma_t}) \) be the normal observable with a standard deviation \( \sigma_t > 0 \) in \( C(\Omega_t) \). That is,

\[
[G_{\sigma_t}(\dot{z})](\omega) = \frac{1}{\sqrt{2\pi \sigma_t^2}} \int_{\mathbb{R}} e^{-\frac{(x-\omega)^2}{2\sigma_t^2}} \, dx \quad (\forall \omega \in \Omega_t)
\]

Thus, we have a deterministic sequence observable \( \{O_{G_{\sigma_t}}\}_{t=1,2}, \{\Phi_{0,t} : C(\Omega_t) \to C(\Omega_0)\}_{t=1,2} \). Its realization

\[
\tilde{O}_T = (\mathbb{R}^2, \mathcal{F}_R^2, \tilde{F}_0)
\]

is defined by

\[
[F_0(\dot{z}_1, \dot{z}_2)](\omega) = [G_{\sigma_1}(\dot{z}_1)](\omega) \cdot [G_{\sigma_2}(\dot{z}_2)](\omega) = [G_{\sigma_1}(\omega)](\omega) \cdot [G_{\sigma_2}(\omega)](\omega)
\]

\( (\forall \dot{z}_1, \dot{z}_2 \in \mathcal{F}_R, \forall \omega \in \Omega_0 = \{\omega_1, \omega_2, \ldots, \omega_5\}) \)

Let \( N \) be sufficiently large. Define intervals \( \Xi_1, \Xi_2 \subset \mathbb{R} \) by

\[
\Xi_1 = \left[165 - \frac{1}{N}, 165 + \frac{1}{N}\right], \quad \Xi_2 = \left[65 - \frac{1}{N}, 65 + \frac{1}{N}\right]
\]

The measured data obtained by a measurement \( M_{C(\Omega_0)}(\tilde{O}_T, S_{\sigma_t}) \) is

\( (165, 65) (\in \mathbb{R}^2) \)

Thus, measured value belongs to \( \Xi_1 \times \Xi_2 \). Using Regression analysis ( Theorem 7.3), Problem 7.1(b) is characterized as follows:

(2) Find \( \omega_0 (\in \Omega_0) \) such as

\[
[F_0(\Xi_1, \Xi_2)](\omega_0) = \max_{\omega \in \Omega} [F_0(\Xi_1, \Xi_2)](\omega)
\]

Since \( N \) is sufficiently large,

\[
(2) \implies \max_{\omega \in \Omega_0} \frac{1}{\sqrt{2\pi \sigma_1^2 \sigma_2^2}} \int_{\Xi_1} \int_{\Xi_2} \exp \left[ -\frac{(x_1 - h(\omega))^2}{2\sigma_1^2} - \frac{(x_2 - w(\omega))^2}{2\sigma_2^2} \right] dx_1 dx_2
\]

\[
\implies \max_{\omega \in \Omega_0} \exp \left[ -\frac{(165 - h(\omega))^2}{2\sigma_1^2} - \frac{(65 - w(\omega))^2}{2\sigma_2^2} \right]
\]

\[
\implies \min_{\omega \in \Omega_0} \left( \frac{(165 - h(\omega))^2}{2\sigma_1^2} + \frac{(65 - w(\omega))^2}{2\sigma_2^2} \right) \quad \text{(for simplicity, assume that } \sigma_1 = \sigma_2) \)
\]

\( \implies \text{When } \omega_4, \text{ minimum value } \frac{(165 - 170)^2 + (65 - 60)^2}{2\sigma_1^2} \text{ is obtained} \)

\( \implies \text{The student is } \omega_4 \)
Therefore, we can infer that the student who helps the girl is $\omega_1$.

Next we shall present the answer to Problem 7.2.

Answer 7.5 [(Continued from Problem 7.2(control problem)) regression analysis]

In what follows, from the measurement theoretical point of view, we shall answer Problem (7.2). Let $T = \{0, 1, 2\}$ be a series ordered set such that the parent map $\pi : T \setminus \{0\} \rightarrow T$ is defined by $\pi(t) = t - 1$ ($t = 0, 1, 2$). Put $\Omega_0 = [0, 2] \times [0, 2]$, $\Omega_1 = [0, 4] \times [0, 2]$, $\Omega_2 = [0, 6] \times [0, 2]$. For each $t = 1, 2$, consider a continuous map $\phi_{\pi(t), t} : \Omega_{\pi(t)} \rightarrow \Omega_t$ such that

$$\phi_{0,1}(\alpha, \beta) = (\alpha + \beta, \beta) \quad (\forall \omega_0 = (\alpha, \beta) \in \Omega_0)$$

$$\phi_{1,2}(\alpha, \beta) = (\alpha + \beta, \beta) \quad (\forall \omega_1 = (\alpha, \beta) \in \Omega_1).$$

Then, we get the deterministic causal operators thus, $\{\Phi_{\pi(t), t} : C(\Omega_t) \rightarrow C(\Omega_{\pi(t)})\}_{t \in \{1, 2\}}$ such that

$$(\Phi_{0,1} f_1)(\omega_0) = f_1(\phi_{0,1}(\omega_0)) \quad (\forall f_1 \in C(\Omega_1), \forall \omega_0 \in \Omega_0)$$

$$(\Phi_{1,2} f_2)(\omega_1) = f_2(\phi_{1,2}(\omega_1)) \quad (\forall f_2 \in C(\Omega_2), \forall \omega_1 \in \Omega_1).$$

Thus, we have the causal relation as follows.

$$C(\Omega_0) \xrightarrow{\Phi_{0,1}} C(\Omega_1) \xrightarrow{\Phi_{1,2}} C(\Omega_2).$$

Put $\phi_{0,2}(\omega_0) = \phi_{1,2}(\phi_{0,1}(\omega_0))$, $\Phi_{0,2} = \Phi_{0,1} \cdot \Phi_{1,2}$.

Let $\mathbb{R}$ be the set of real numbers. Fix $\sigma > 0$. For each $t = 0, 1, 2$, define the normal observable $\hat{O}_t = (\mathbb{R}, \mathcal{B}_\mathbb{R}, G^n_t)$ in $C(\Omega_t)$ such that

$$[G^n_t(\Xi)](\omega_t) = \frac{1}{\sqrt{2\pi \sigma^2}} \int_\Xi \exp\left(-\frac{(x-\alpha)^2}{2\sigma^2}\right)dx$$

$$(\forall \Xi \in \mathcal{B}_\mathbb{R}, \forall \omega_t = (\alpha, \beta) \in \Omega_t = [0, 2t + 2] \times [0, 2]).$$

Thus, we get the sequential deterministic causal observable $[\mathcal{D}_T] = \{\hat{O}_t\}_{t = 0, 1, 2}, \{\Phi_{\pi(t), t} : C(\Omega_t) \rightarrow C(\Omega_{\pi(t)})\}_{t = 1, 2}$. Then, from Theorem 6.12, the realized causal observable $\hat{O}_0 \equiv (\mathbb{R}^3, \mathcal{B}_\mathbb{R}^3, \hat{F}_0)$ in $C(\Omega_0)$ is obtained as follows:

$$[\hat{F}_0(\Xi_0 \times \Xi_1 \times \Xi_2)](\omega_0) = [G^n_{\pi(0)}(\Xi_0) \Phi_{0,1}(G^n_{\pi(1)}(\Xi_1) \Phi_{1,2}(G^n_{\pi(2)}(\Xi_2)))](\omega_0)$$

$$= [G^n_{\pi(0)}(\Xi_0)](\omega_0) \cdot [G^n_{\pi(1)}(\Xi_1)](\phi_{0,1}(\omega_0)) \cdot [G^n_{\pi(2)}(\Xi_2)](\phi_{0,2}(\omega_0))$$

$$(\forall \Xi_0, \Xi_1, \Xi_2 \in \mathcal{B}_\mathbb{R}, \forall \omega_0 = (\alpha, \beta) \in \Omega_0).$$

We have the measurement $M_{C(\Omega_0)}(\hat{O}_0, S_{[\alpha]} \cdot)$. We see that the measured value $(x_0, x_1, x_2)$ obtained by the measurement $M_{C(\Omega_0)}(\hat{O}_0, S_{[\alpha]} \cdot)$ is equal to

$$(0.5, 1.6, 3.3) (\in \mathbb{R}^3).$$
Define the closed interval $\Xi_t$ ($t = 0, 2, 3$) such that

$$\Xi_0 = [0.5 - \frac{1}{2N}, 0.5 + \frac{1}{2N}], \quad \Xi_1 = [1.6 - \frac{1}{2N}, 1.6 + \frac{1}{2N}], \quad \Xi_2 = [3.3 - \frac{1}{2N}, 3.3 + \frac{1}{2N}],$$

for sufficiently large $N$. Here, Fisher’s method says that it suffices to solve the problem.

(♯) Find $(\alpha_0, \beta_0)$ such as

$$\max_{(\alpha, \beta) \in \Omega_0} [\hat{F}_0(\Xi_0 \times \Xi_1 \times \Xi_2)(\alpha, \beta)$$

Putting

$$U(x_0, x_1, x_2, \alpha, \beta) = \sum_{k=0}^{2} (x_k - (\alpha + k\beta))^2$$

we have the following problem that is equivalent to (♯):

Calculate

$$\frac{\partial}{\partial \alpha} U(0.5, 1.6, 3.3, \alpha, \beta) = 0, \quad \frac{\partial}{\partial \beta} U(0.5, 1.6, 3.3, \alpha, \beta) = 0,$$

Then, we get

$$(\alpha, \beta) = (0.4, 1.4)$$

Therefore, in order to get the measured value $(1.9, 3.0, 4.7)$, the control state $(\alpha, \beta)$ should be defined by $(0.4, 1.4)$.

Here, again note the equivalence of control problem($c_1$) and inference problem($c_2$).

Example 7.6 [Pheasants and rabbits problem] Consider the following situation:

(a) [Pheasants and rabbits problem] A number of pheasants and rabbits are placed together in the same cage. 7 heads and 22 feet are counted. Find the number $m$ of pheasants and the number $n$ of rabbits.

Answer This problem — Pheasants and rabbits problem — has various aspects. Usually, we consider that,

(b) The statement (a) is in ordinary language.

This aspect (b) may assert that "5", "14", "$(m,n)$" should be states. However, in what follows, we regard the problem (a) as the inference problem. That is,

(c) Regarding $5, 15$ as an exact measured value, infer the state $(m, n)$. 
Put $N_0 = \{0, 1, 2, \ldots\}$, $\Omega_0 = N_0 \times N_0$, $\Omega_1 = N_0$ and $\Omega_2 = N_0$. Define the causal operator $\Phi_{0,1} : C(\Omega_1) \to C(\Omega_0)$ and $\Phi_{0,2} : C(\Omega_2) \to C(\Omega_0)$ such that

$$[\Phi_{0,1}(f_1)](m,n) = f_1(m+n), \quad [\Phi_{0,2}(f_2)](m,n) = f_2(2m+4n)$$

\((\forall f_i \in C(\Omega_i), i = 1, 2, \forall (m,n) \in \Omega_0)\)

For each $t \in \{1, 2\}$, consider the exact observable $O_t^{(\text{exa})}$ in $C(\Omega_t)$. That is, $O_t^{(\text{exa})} = (N_0, 2^{N_0}, F^{(\text{exa})})$ satisfies

$$[F^{(\text{exa})}](\Xi)(n) = \begin{cases} 
1 & (n \in \Xi) \\
0 & (n \notin \Xi)
\end{cases}$$

Hence, we get the sequential deterministic causal exact observable $[O_t^{(\text{exa})}]_{t=1,2}, \{\Phi_{0,t} : C(\Omega_t) \to C(\Omega_0)\}_{t \in \{1,2\}}$. Then, the realized causal observable $\hat{O}_0 = (N_0 \times N_0, 2^{N_0 \times N_0}, \hat{F})$ in $C(\Omega_0)$ is defined by

$$[\hat{F}(\Xi_1 \times \Xi_2)](m,n) = [\Phi_{0,1}F^{(\text{exa})}](m,n) \cdot [\Phi_{0,2}F^{(\text{exa})}](m,n) = [F^{(\text{exa})}(\Xi_1)](m+n) \cdot [F^{(\text{exa})}(\Xi_2)](2m+4n)$$

\((\forall \Xi_1, \Xi_2 \in 2^{N_0}, \forall (m,n) \in \Omega_0)\)

Assume that a measured value $(5, 14) \in N_0 \times N_0$ is obtained by the measurement $M_{C(\Omega_0)}(\hat{O}_0, S_{\{\ast\}})$.

Therefore, Fisher maximum likelihood method (Theorem 4.5) says that

(2) find $(m,n) \in N_0 \times N_0$ such that

$$[\hat{F}([5] \times \{14\})](m,n) = \max_{(m,n) \in N_0 \times N_0} [\hat{F}([5] \times \{14\})](m,n)$$

Therefore,

$$\max_{(m,n) \in N_0 \times N_0} (F^{(\text{exa})}([5])(m+n) \cdot [F^{(\text{exa})}([14])(2m+4n))$$

$$\Rightarrow [F^{(\text{exa})}([5])(m+n) = 1 \quad \text{and} \quad [F^{(\text{exa})}([14])(2m+4n) = 1$$

$$\Rightarrow m + n = 5 \quad 2m + 4n = 14$$

$$\Rightarrow m = 3, n = 2$$

Thus, there is a reason to infer that $(m,n) = (3,2)$. 

\[\Box\]

\[\triangle \text{ Note 7.1} \] Since measurement theory is based on dualism, it has several interpretations of "Pheasants and rabbits problem" as follows.
Table 7.2: Several interpretations of "Pheasants and rabbits problem"

| Variations \ (m, n, (5,14)) | m     | n     | (5,14) |
|-----------------------------|-------|-------|--------|
| usual interpretation in monism (i.e., without measurement) | state | state | state |
| dualism (Example 7.6)       | state | state | measured value |
| dualism (cf. Example 11.11) | measured value | measured value | measured value |

Recall Chap. 1(X₁), that is,

\[(X₁) \text{ widely ordinary language (before science)} \Rightarrow \text{world-description (Chap. 1(O))} \begin{cases} \text{realistic scientific language (Newtonian mechanics, etc.)} \\ \text{linguistic scientific language (measurement theory, etc.)} \end{cases} \]

And recall Chap. 1(X₃), that is,

\[(X₃) \text{ "@ widely ordinary language" includes} \begin{cases} \text{the arithmetical word problems ( (Example 7.6) or (Chap.11 A₂)) and so on), } \text{statistics (≈dynamical system theory)} \end{cases} \]

And our standing point is to reconsider each in the (2) from the measurement theoretical point of view. If fact we can see several "pheasants and rabbits problems as in Table 7.2.

7.3 Measurement theory is valueless if not used

Two world-views (i.e., the realistic world-view and the linguistic world-view) are composed of different principles. That is,

(a) The realistic world-view needs official guarantee of the experts. The number of specialists may be about 100. On the other hand, The linguistic world-view does not need official guarantee of the specialists but popular support.

In other words,

(b) The linguistic world-view is worthless if it is used by many persons.

In chapters 4,7, we assert that

(c) statistics-dynamical system theory is characterized as the abbreviation of measurement theory

If it is so, measurement theory satisfies the condition (b).

8 Reconsideration of traditional philosophies in measurement theory

If we investigate "language", a philosophical domain must be trodden in somewhat inevitably. This chapter explains the genealogy of the idealism shown in Fig. 8.2 later:

\[\text{Descartes \rightarrow Kant} \quad (\text{Recognition constitutes world}) \quad \text{linguist philosophy} \quad (\text{language constitutes world}) \quad \text{measurement theory} \quad (\text{language constitutes world})\]
Although this was the scenery seen from measurement theory, I wrote that the reader of science also understood. Of course, there is no classification of liberal arts and science, and if you do not understand measurement theory, I think that you do not idealism.

8.1 Genealogy of dualism idealism

8.1.1 Descartes and Kant

Although measurement theory is metaphysics (i.e., a learning which cannot decide whether right or wrong by experiments), if you only master measurement theory, it is not necessarily indispensable to understand measurement theory in relation with philosophy.

However, the consideration of the relation of measurement theory and philosophy is indispensable to check a position of measurement theory in the science (i.e., to verify the opinion of this book:

Science is describing by measurement theory.

). The head family of metaphysics is philosophy and philosophers consider many things truly. And measurement theory is subject to influences of some from their work. Although the philosopher has not argued in ordinary language and the author could not necessarily fully understand their ideas, I thought that what was suggested from philosophy should have been written.

(a) The spirit of measurement theory (Copenhagen interpretation) resembles the philosophy made into the main stream to the extent that it will be called surreptitious use of originality, if I do not write.

This chapter explains this. Now, I will put up the image (Fig. 1.1 of Chap. 1) of measurement again.

Moreover, as shown in Fig. 8.2, we consider that

(b) Measurement theory is also the linguistic version of not only quantum mechanics but Descartes-Kant philosophy.”

---

**Figure 8.1 (≡ Fig. 1.1): The image of “measurement(=\(\oplus+\oplus\))” in dualism**
If written by diagram,

\[(\sharp_1) : \text{quantum mechanics} \xrightarrow{\text{proverbalizing \ linguistic turn}} \text{linguistic turn} \rightarrow \text{measurement theory} \]

\[(\sharp_2) : \text{Descartes – Kant philosophy} \xrightarrow{\text{axomatization \ linguistic turn}} \text{linguistic turn} \rightarrow \text{scientific language} \]

Chapter 3 explained the portion of this quantum mechanics of \((\sharp_1)\). This section explains a lower portion\((\sharp_2)\).

If a conclusion is described previously, correspondence of the keyword of Descartes-Kant philosophy and measurement theory (Copenhagen interpretation) will become as it is shown in Table 8.1.

Table 8.1: Descartes–Kant philosophy and measurement theory (cf. [14, 16])

| Descartes–Kant (recognition) | measurement (language) | ① : observer (sense organ, secondary quantity) | ② : observable (measuring instrument) | ③ : measured value (perceive by brain) | ④ : measuring object (state=property of measuring object) |
|-----------------------------|------------------------|------------------------------------------|---------------------------------|---------------------------------|---------------------------------------------------|
| I                           | body                   | secondary quantity                       | observable                     | measured value                  | measuring object                                  |
| 2                           | body                   | sense organ                              | observable                     | measured value                  | measuring object                                  |
| 3                           | mind                   | brain                                    | measured value                 | measured value                  | measuring object                                  |
| 4                           | matter                 | primary quantity                         | measured value                 | measured value                  | measuring object                                  |

If you see this table, you can guess a correspondence-related meaning generally, but I will add some notes.

1. \["I" \leftrightarrow "observer"\] : I think that this does not need to explain.

2. \["body" \leftrightarrow "observable"\]: Probably, this will also be good since Body\leftrightarrow Sense organ \leftrightarrow Observable \leftrightarrow Observable.

3. \["mind" \leftrightarrow "measured value"\]: This may be unexpected. However, measurement is perceiving a signal with an observer’s brain. And the "perceived value" is then called "measured value." For example, if an observer does not look at the value even if the scale of the voltmeter has pointed out 1.5V (namely, if it does not reach to the observer’s brain), we can not call it measurement. The phenomenon "the scale of a voltmeter points out 1.5V" is a perfect physical phenomenon, and if it becomes so much, it can be said that the world of monism (only "thing") is enough. Therefore, if it says in motto, we can say "measured value does not exist without a brain." (Chapter 3.3), and this is "the standard interpretation of quantum mechanics." I think that dualism is to consider that "self (Brain-Heart)" is a special existence.

4. \["matter" \leftrightarrow "measuring object"\] This is also natural and it will not be necessary to explain it.

As mentioned above, there is correspondence of Table 8.1 about the basic keywords of Descartes-Kant philosophy and measurement theory. Therefore, about the basic keywords, we may consider that "Descartes Kantianism = measurement theory." Of course, since both are dualism, the similarity may be natural one.
Since there is the "\(\text{身体(= Observable)}\)" in the middle of "self" and a "thing" so that it may understand, if Fig. 1.1 (image figure of measurement) of Chap. 1 is seen, it is good though it belongs to the direction of a "thing." When thinking so, it may be called "mind-body dualism", but "to which it belonging" and "how to call" are not important. Moreover, if I think "Body\(\rightarrow\text{Observable}\)", there may be quite big "Body." I think that it is so much comfortable even if it considers that "glasses" is a part of body. However, as an example of "the measuring instrument to a measuring object", if it considers "the scale of the voltmeter to voltage should shake", "the jet stream to an airplane", "the Polestar( for checking a direction)" may be sufficient. Since "Body(=Observable)" is "what exists in the middle of a brain and a measuring object ", we can consider that a jet stream is the body(=Observable). However, it should be the same as arguing about "What is a monkey? What is a tree? " in "Even monkeys fall from trees" of an idiom to have such a discussion — namely, "What is the body? " etc. — , and we should notice that a productive argument is not expectable.

If too earnest to a question of "What is \(\bigcirc \bigcirc\)?" as the Note 2.3 described, it will fit into a dead end. It is because a concept is decided in the context in a linguistic science view. (Note 6.14)

Although monism and materialism won a great success in physics. On the other hand, in philosophy, they adhered to dualism and idealism. If it carries out from the common sense feeling of science, we think that the following question is natural.

(\(\sharp 1\)) Why has "strange theories", such as idealism and dualism, clung to many wise people with the talent which is equal to Newton or Einstein?

This question will not be canceled if there is no diagram of the following world description in mind.

\[
\begin{align*}
\text{(X)} & \quad \text{(widely ordinary language)} \\
\text{(Chap. 1)} & \quad \text{world-description} \\
\text{(before science)} & \quad \text{(Chap. 1(O))} \\
\text{① realistic scientific language} & \quad \text{(monism, materialism)} \\
\text{② linguistic scientific language} & \quad \text{(dualism, idealism)} 
\end{align*}
\]

Supposing philosophy has a difficult portion, it is to use ambiguous ordinary language\(\text{①)}\) (Of course, there are also philosophical fields (ethical philosophy etc.) which must be done so plentifully), but I would like to think that it is true about the strong will which refused the realistic world view, and sharp intuition.

Immanuel Kant(1724–1804) is a philosopher with the biggest influence in modernization, and advocated what is called "Copernican revolution" (that is, "recognition constitutes the world.") in epistemology. As general explanation of "pure reason criticism" of Kant [24], explanation of MSN (Encarta encyclopedia ) is quoted below.

(d) [Pure reason criticism] It is "pure reason criticism" that makes the basis of Kant’s critical philosophy, and the target suited seeing and reaching to an extreme of man’s cognitive ability. As a result, it is clarified that man’s cognitive ability is merely passively struck with things of the world, and it does not only take, and that it is working actively in the world rather and completes the object of the recognition itself. Although built, the world is not necessarily completed from nothing like God. The world is a certain form and there is already it, and in order to materialize recognition, the information from this world acquired by pushing in feeling is required as a material. However, this information is only the disorderly confused thing as it is. Man’s cognitive ability must be pushed in a fixed form with which he is originally endowed, and must give orderly order to the information on this confused feeling. Moreover, it is since the object of the first recognition to unify by it is summarized. According to Kant, the form with which man is endowed is as follows.

\[
\begin{align*}
(\text{i)} : \quad & \text{Form of sensitivity(intuition)(Space-time(=R x R^3))} \\
(\text{ii)} : \quad & \text{Form of understanding(thinking) (For example, the concept of a quantity, whether it is single or a large number, the concept of a relation like causality, etc.)}
\end{align*}
\]
If that is right, in spite of being unable to prove the proposition "all the thing is among time and space", and "all follow causal relationship", they will be unconditionally applied to the object of all the experiences experientially. It is because the object will not be constituted without space, time, and the form of causal relationship. It is like it being considered that the utterance "the world is green" is right for all human beings, when all human beings see the world for example, having covered green sunglasses. (MSN – ( the Encarta encyclopedia. 2009 DVD Japanese version(translated by the author)).

Probably, we may consider the following correspondence compared with measurement theory because Kant has said that it related to space, time, or causal relationship in upper (i) of (d), and the portion of (ii).

\[
\begin{align*}
\text{Sensitivity} & \leftrightarrow \text{Axiom}_1^m \\
\text{Understanding} & \leftrightarrow \text{Axiom}_2^{cm} \\
\end{align*}
\]

\[8.1.2 \text{ Linguistic revolution and measurement theory — Idealism which a monkey can not understand}\]

Now, Descartes-Kant philosophy develops for the purpose of "theory of basing of science", and, probably, may conclude that the compilation was made by Kant. Possibly the intention suffered a setback. However, if we may regard it as "Theory of basing of science" = "The basic language which describes science" (Kantianism) = "The basic language which describes science" (Measurement theory )

the purpose of Descartes-Kant philosophy and measurement theory will become the same. However,

(f₁) Although Descartes-Kant philosophy and measurement theory are dualism with the same purpose and the correspondence (Table 8.1 and (e)) mentioned above is among both, even if both are alike, why have not they resembled it closely?

I think that the reason — Although it is having stated repeatedly since Note 2.3 — is the next (f₂),(f₃)).

(f₂) Descartes Kant philosophy investigates in detail about the basic keywords "I" ( "body", "mind", etc.).

On the other hand,

(f₃) Measurement theory tells the world as directions of the Axioms 1 and 2. and not investigates the keywords "observer" ( "observable", "measured value", etc. ).

That is, I think that it is the difference between "Philosophy told about" and "Philosophy told by". In this sense, Newtonian mechanics is also "Philosophy told by".

Although this difference is decisive, Descartes-Kant philosophy and measurement theory (Copenhagen interpretation) are considerably alike. For example, there is the next resemblance. :

(g₁) The importance of "space-time", "causal relationship", and "measurement (≈ recognition)" was observed. (Therefore, it means that Kant was sure of "the miracle of Section 6.4.3 (g)".)
(g2) \[ \text{Recognition constitutes world} \iff \text{Observable is before state} \]
(Copernican turn) (the Copenhagen interpretation (Chap. 1 (U6)))

Moreover, it is as follows, if explanation of the Encarta encyclopedia of (d) is imitated and measurement theory is described.

(h) **Measurement Theory** Measurement Theory is the linguistic describing method about an everyday phenomenon. The description by measurement theory does not simply describe things of the world as it is passively. It is working actively in the world rather, completes the object as a fiction and describes it. Although completed as a fiction, it does not necessarily complete from nothing. Since the world is a certain form and is already there, in order to materialize description, the information from this world acquired by pushing in feeling is required as a material. However, this information is only the disorderly confused thing as it is. The description by measurement theory must be pushed in a fixed form with which measurement theory is originally equipped, must give orderly order to this confused information, and must summarize the first description (fiction) to unify by it. The form with which measurement theory is equipped is as follows.

\[
\begin{align*}
(i) & : \text{Axiom}^p_c 1 \text{ (Measurement)} \\
(ii) & : \text{Axiom}^p_c 2 \text{ (Causal relationship)}
\end{align*}
\]

If that is right, in spite of being unable to prove "all the thing is among time and space", and the proposition "all follow causal relationship", they will be unconditionally applied to the object of all the experiences experientially. Space, time, and causal relationship are because the object will not be described without the form (Axiom$^p_c 1$ and 2). Supposing it has the rule that only the word "green" can be used as a color, it is like what we can only describe "the color in the world is green."

From the above thing, we understand the similarity of a Kantianism and measurement theory.

The differences among both are a "recognition version" and a "language version." Supposing that is right, you will think that you want the proposition which is unconditionally applied to the object of all the experiences to correspond with measurement theory (Axiom$^p_c 1$ and 2) in spite of a priori overall judgment of Kant (That is, a proposition which is unconditionally applied to the object of all the experiences in spite of the ability not to prove experientially (empirical validation cannot be carried out)). I would like to think so, since both aim at establishment of metaphysics.

\[\text{Note 8.3}\] Measurement theory is materialized from the following two beliefs (Section 2.3.1 (a)).

\[
\begin{align*}
\{ & \text{Faithful to Axiom 1 and 2} \\
\{ & \text{Reliance to man’s linguistic competence and cognitive ability}
\end{align*}
\]

In measurement theory, about the portions of "the linguistic competence and cognitive ability of man", we only merely wonder and we do not do investigation beyond it. However, it may be thought that the direction of "wonder of the linguistic competence and cognitive ability of man" was trodden in in the Kantianism. Even if the learning which tells "recognition" was inherited in a modern style to science (= material study (Psychology, cognitive science, brain science, artificial intelligence, etc.)), the direction which Kant aimed at must be metaphysical world description. As analogy of "atomism (Demokritos) to atomism (theory of elementary particles)", there may be some some readers who think "the unripe state of science (= material study) is philosophy."

However, if the Kantianism (pure reason criticism) is considered so, Kant does not rest in peace. Although the metaphysical opinion is carried out, if it is misunderstood and criticized in case of material study, Kant may be embarrassed. Of course, there is no philosopher who is doing confusion of metaphysics and material study.

\[\text{Note 8.4}\] I have heard the opinion said "It is because that Gauss (1777–1855) refrained from the official announcement of non-Euclidean geometry wanted to avoid friction with the Kantists who claim "Space-time (= \( \mathbb{R} \times \mathbb{R}^3 \)) is the sensitivity with which man is endowed." " Although this truthfulness is not certain, I think that it is a fact that the influence of the Kantianism of those days was so greatest that it was not amusing even if there was a talk said like this. If it considers from now on, it is natural to doubt "why generally it was
supported?”, but I would like to make as a fiction the plot in which the Kantianism greatly affected modern science, in the evolution

\[
\text{Kant} \xrightarrow{\text{(linguistic turn)}} \text{the philosophy of language} \xrightarrow{\text{(axiomatization)}} \text{measurement theory}
\]

It is because only a negative answer will be contemporarily thought of to a problem:

What on earth was the dualism idealism (Plato, Descartes, Kant) made into a philosophical mainstream?

if our fiction does not exist.

As mentioned above, though it is \[(d):\text{Pure reason criticism (epistemology)} \vdash \neg [(h):\text{Measurement theory (language)}]\], "[Recognition] \neq [Language]" is also worried too. When becoming it so, "the linguistic turn" — namely, revolution to "linguistic philosophy" from "epistemology" — carried out by philosophers, such as Saussure (1857–1913) and Wittgenstein (1889-1951), after Kant has a meaning important for measurement theory. That is,

from "recognition" to "language"

If it writes diagrammatically,

\[
\begin{align*}
\text{(recognition constitutes world)} & \hspace{1cm} \text{Kant} \xrightarrow{\text{epistemology}} \text{recognition} \xrightarrow{\text{linguistic turn}} \text{language} \\
\text{language constitutes world} & \hspace{1cm} \text{the philosophy of language} \xrightarrow{\text{ordinary language}} \text{scientific language} \xrightarrow{\text{scientification}} \text{measurement theory}
\end{align*}
\]

Note that "Recognition constitutes the world." is used in two meanings((g2),(i)). Although (g2) is still used in the Copenhagen interpretation of measurement theory(Chap. 1(U5)), it was revolved in (i) to "Language constitutes the world."

That is, the phrases:

"Language is before the world" "The limits of my language mean the limits of my world" "Language constitutes the world" "Language game"

of the linguistic philosophy which philosophers like and use is borrowed, and it is only a cut about these at the basic spirit of measurement theory.

We are impressed by the sharp sensitivity of the philosophers who arrived at such a phrase in ordinary language, without having an easy concrete model called measurement theory (i.e., Axioms 1 and 2). Even though that was right, the author was taught the following fact (j) from the students of my seminar and not from Saussure (Refer to Note 3.9):
(j) Even if we do not know a "monkey" and a "tree", we can use the proverb that "Even monkeys fall from trees".

Ordinary language is a monster language taken in vaguely -even in case of a realistic science view and linguistic scientific view. In ordinary language, no clear things can be said in a strict meaning. For example, "Which came first, the world or the language? also has the side like "which came first, the chicken or the egg?" and it is not that "which came first" is so clear. On the other hand, in measurement theory, it can be said completely that "language is first" That is because Axioms 1 and 2 are "perfect mystic words", and it does not exist from the starts, such as the world of corresponding (any fields other than quantum mechanics).

It follows,

(k) Measurement theory presented the meaning of idealism (the spirit of "language is before world") in the form which everyone can understand.

If that is right, idealism should be known also by whom, but regrettably a monkey does not understand it. That is because

(l) measurement theory is based on man's linguistic competence.

But a monkey can count some apples, and there is also computer software comparable as the world champion of chess. However, it will be thought at least for about 50 years from now on that idealism belongs only to man.

Wittgenstein does not have a worldly way of speaking like "fitting feet with shoes(Note 6.15)." He said lucidly as follows more smartly.

(m) **Language constitutes the world.** Therefore, since we decided that measurement theory described science, what it described is the world of many science and engineering. (Note 6.15(♯2)). That is, I understand that all the following (m₁)–(m₃) is the same meaning.

(m₁) Engineering and science are the worlds described by measurement theory.

(m₂) The limit of a language called measurement theory is the limit of the world of engineering and science.

(m₃) Measurement theory = The language of engineering and science (Note 1.2)

I think that there are very beneficial to measurement theory. If we trust Wittgenstein, it means that he had answered by upper (m) about

(n) What is various science?
\*\* Note 8.5 The above (m) is not "an eternal definition" of "science." It is because there is a best-before date in measurement theory as Section 2.4.2 [space colony] described. Moreover, it is because you should propose another linguistic science language and constitute "the world of another science," if you are not pleased with this "world of science (measurement theory constitutes)."

8.2 Where is measurement theory in traditional philosophies

The argument to this chapter is summarized as the following Fig. 8.2.

Here, although the branch (i.e., quantum mechanics $\rightarrow$ {1} $\rightarrow$ {2} $\rightarrow$ ) of quantum mechanics was described in the Note 3.6, it argues in Section 9.3.

\*\* Note 8.6 The Note 3.9 also described,

The author progressed in the order

from "Quantum mechanics(Theorem 3.4 (Formulation within the quantum mechanics of the uncertainty principle of Heisenberg ))" to "Classic measurement theory [8, 9]"

I expected that classic measurement theory became a completely different thing from dynamical system theory and statistics at the beginning. In this meaning,

\[ \text{dynamical system theory statistics} \quad \text{abbreviation} \quad \text{measurement theory} \]

of Fig. 8.2 was too appropriate and a disappointment. Conversely, I got the firm belief "there is only measurement theory for world description."

8.3 Supplement: About ordinary language

It is how to use the word "ordinary language" that I strayed most while writing this book, and I do not still define it clearly. Therefore, I am getting confused and using the word "ordinary language" in whole this book. That is, since I am getting confused and using "Case 1" and "Case 2" like
Case 1: \([\text{ordinary language}] \cap [\text{measurement theory}] = \emptyset\)

Case 2: \([\text{ordinary language}] \supset [\text{measurement theory}]\)

I would like to add some "supplement:ordinary language" here. However, since it does not necessarily become clear in particular in this section, you may skip this section.

Although we do not know in the time of when human beings invented language, language is continuing developing continuously after it. Therefore, we may consider that "development of language" is almost synonymous (that is, proportional) with "development of civilization."

In this sense, we may think that

\((\sharp_1)\) the theme of this book is one of the trials which make power of expression of ordinary language rich.

I explain this below. I persisted in the following diagram in this book.

\[
\begin{array}{ccc}
(X_1) & \text{widely ordinary language} & \implies \text{world-description} \\
(\text{Chap. 1}) & \text{(before science)} & \{ \begin{array}{l}
1 \text{ realistic scientific language} \\
2 \text{ linguistic scientific language}
\end{array} \\
(\text{Chap. 1(O)}) & \text{(world is before language)} & \text{(language is before world)}
\end{array}
\]

And since it could not say that the framework of ordinary language\((\odot)\) in a broad sense was clear, I chose to start from measurement theory\((\mathbb{2})\). However, if it further gives a broad interpretation of ordinary language, it is reasonable also for thinking as follows.

\((\sharp_2)\) \((\odot \cup 1 \cup 2) \subset \text{"more widely ordinary language"}\)

Rather, it is more natural to think like this.

Mathematics and physics may also be "ordinary language" for a mathematician or a physicist. Moreover, I will not deny the opinion that it is an expedient diagram for this \((X_1)\) to make contrast with physics and measurement theory conspicuous. That is, I will not argue about "Ordinary language of \((X_1)\) vs. Ordinary language of \((\sharp_1)\)". It is because avoiding the argument in connection with ordinary language as much as possible must have been the plan which cohered.

If that is right, it is the same as \((\sharp_1)\) to have carried out in this book. If I repeat,

\((\sharp_3)\) Into a lawless area called ordinary language, a steadfast small dualism language area called measurement theory was found out (adding), and power of expression of ordinary language was made somewhat rich.

Of course, it is not only measurement theory to make ordinary language rich.

\((\sharp_4)\) "Mathematics", "physics", "the good", "justice", "love", "freedom", "art", "the theory of evolution", "DNA", "the Internet", "democracy", "economy", "environment" ···

That is, all the things learned in the education of primary schools and junior and senior high schools, and the language and the concept and theory of the special field of study of the extension also make ordinary language rich.
if that is right, we would like to come to compare each of (♯₄) and measurement theory

It is possible for a certain grade to carry out this comparison. As mentioned above, it is because it will be quite fair to consider

(♯₅) all the researches of all learning are estimated by the measure "which made language powerful richly."

if "Development of language $\propto$ Development of civilization" is right

Measurement theory made language rich. However, in addition to it, the language of

Kant, Fischer, Wittgenstein, von Neumann

was swung, for example, and language was made still more powerful.

Moreover, I think that this measure(♯₅) is almost the same as "the standard of the importance of this book" - "Is it how much helpful for construction and settlement of a space colony?" , namely, "Is it how much helpful in order that human beings may survive?" -.

9 Equilibrium statistical mechanics

Our purpose is to establish the following spirit:

(♯₁) Describing ordinary phenomena by a metaphysical language (i.e., measurement theory), we make engineering (or, science).

Following this spirit,

(♯₂) we study equilibrium statistical mechanics in measurement theory

And therefore, we conclude that equilibrium statistical mechanics is not physics. In addition, we assert that

(♯₃) quantum mechanics is the greatest examples of the applications of measurement theory.

9.1 equilibrium statistical mechanics

It is usual to consider that equilibrium statistical mechanics is constructed on the base of dynamical system theory. On the other hand, we construct equilibrium statistical mechanics in measurement theory.

Thus, we have the following problem:

[equilibrium statistical mechanics] (new method due to measurement theory) vs. [equilibrium statistical mechanics] (the conventional method due to dynamical system theory)

This "vs." must be settled in future.
9.1.1 The dynamical aspect of equilibrium statistical mechanics — Ergodic hypothesis

Assume that about \( N (\approx 10^{24}) \) particles (for example, hydrogen molecules) move in a box. It is natural to assume the following phenomena ① – ④:

① Every particle obeys Newtonian mechanics.

② Every particle moves uniformly in the box. For example, a particle does not halt in a corner.

③ Every particle moves with the same statistical behavior concerning time.

④ The motions of particles are (approximately) independent of each other.

In what follows we shall devote ourselves to the problem:

(a) how to describe the above equilibrium statistical mechanical phenomena ① – ④ in terms of measurement theory.

For completeness, again note that measurement theory is a kind of language.

In this preprint, the knowledge of statistical mechanics is not required. Thus, we add the allegory as follows.

♠ Note 9.1 The original idea may be due to L. Boltzmann: (Vorlesungen über Gastheorie, 1895, J Ambrosius Barth, 1923). Let us explain ② – ④ as allegory as follows. 100 kindergarteners are carrying out a swing [SW], a sliding way [SL], and sand play [SN] to the lunch break of 1 hour in the yard of the kindergarten. Then, ② – ④ can be understood as the following allegory:

② Every kindergartner is fickle and changes play one after another. For example, a kindergartner plays as follows.

(♯) SW (5 min.) → SL (3 min.) → SN (6 min.) → SL (7 min.) → SW (9 min.) → SL (8 min.) → SW (9 min.)

③ Every kindergartner has the same palatability. Therefore, the sum total time of each three play is the same. For example, for every kindergartner, we see that

\[
\begin{align*}
\text{the time which played the swing is} & \quad 30 \text{ minutes} \\
\text{the time which played the sliding way is} & \quad 18 \text{ minutes} \\
\text{the time which played the sand play is} & \quad 12 \text{ minutes}
\end{align*}
\]

④ Every kindergartner is independently playing. It is hardly influenced by other kindergarteners’ play. They do not do group action.

Imaging the above ②–④, readers may read as follows.

About ①

In Newtonian mechanics, any state of a system composed of \( N (\approx 10^{24}) \) particles is represented by a point \((q, p)\) (≡ (position, momentum) = \((q_{1n}, q_{2n}, q_{3n}, p_{1n}, p_{2n}, p_{3n})_{n=1}^{N}\)) in a phase (or state) space \(\mathbb{R}^{6N}\). Let \(H : \mathbb{R}^{6N} \to \mathbb{R}\) be a Hamiltonian such that

\[
H((q_{1n}, q_{2n}, q_{3n}, p_{1n}, p_{2n}, p_{3n})_{n=1}^{N}) = \sum_{n=1}^{N} \sum_{k=1, 2, 3} \frac{(p_{kn})^2}{2 \times \text{particle’s mass}} + U((q_{1n}, q_{2n}, q_{3n})_{n=1}^{N}).
\]
Let \( \{\psi^E_t\}_{-\infty < t < \infty} \) be the flow on the energy surface \( \Omega_E \) induced by the Newton equation with the Hamiltonian \( \mathcal{H} \), or equivalently, Hamilton’s equation:

\[
\frac{dq_{kn}}{dt} = \frac{\partial \mathcal{H}}{\partial p_{kn}}, \quad \frac{dp_{kn}}{dt} = -\frac{\partial \mathcal{H}}{\partial q_{kn}} \quad (k = 1, 2, 3, \ n = 1, 2, \ldots, N). \tag{9.1}
\]

Fix \( E > 0 \). And define the measure \( \nu_E \) on the energy surface \( \Omega_E \) (\( \equiv \{(q, p) \in \mathbb{R}^{6N} \mid \mathcal{H}(q, p) = E\} \)) such that

\[
\nu_E(B) = \int_B |\nabla \mathcal{H}(q, p)|^{-1} dm_{6N-1} (\forall B \in \mathcal{B}_{\Omega_E}, \text{ the Borel field of } \Omega_E)
\]

where

\[
|\nabla \mathcal{H}(q, p)| = \left( \sum_{n=1}^{N} \sum_{k=1,2,3} \left( (\frac{\partial \mathcal{H}}{\partial q_{kn}})^2 + (\frac{\partial \mathcal{H}}{\partial p_{kn}})^2 \right) \right)^{1/2}
\]

and \( dm_{6N-1} \) is the usual surface measure on \( \Omega_E \).

Liouville’s theorem says that the measure \( \nu_E \) is invariant concerning the flow \( \{\psi^E_t\}_{-\infty < t < \infty} \). That is, it holds that

\[
\nu_E(S) = \nu_E(\psi^E_t(S)) \quad (0 \leq t < \infty, \ \forall S \in \mathcal{B}_{\Omega_E}) \tag{9.2}
\]

Putting \( C(\Omega) = C_0(\Omega_E) = C(\Omega_E) \) (from the compactness of \( \Omega_E \), \( T = \mathbb{R}, \omega_t = (q(t), p(t)) \), \( \phi_{t_1, t_2} = \psi^E_{t_2-t_1}, \Phi^*_{t_1, t_2} \delta_{\omega_{t_1}} = \delta_{\phi_{t_1, t_2}(\omega_{t_1})} (\forall \omega_{t_1} \in \Omega_E) \), we define the deterministic Markov relation \( \{\Phi_{t_1, t_2} : C(\Omega_E) \to C(\Omega_E)\}_{(t_1, t_2) \in \mathcal{T}_S} \) in Axiom \( \text{pm}^2 \).

About \( 2 \)

Now let us begin with the well-known ergodic theorem. (cf. [33]).

For example, consider one particle \( P_1 \). Put \( S_{P_1} = \{ \omega \in \Omega_E \mid \text{a state } \omega \text{ such that the particle } P_1 \text{ always stays a corner of the box } \} \). Clearly, it holds that \( S_{P_1} \subseteq \Omega_E \). Also, if \( \psi^E_t(S_{P_1}) \subseteq S_{P_1} \) (\( 0 \leq t < \infty \)), then the particle \( P_1 \) must always stay a corner. This contradicts \( 2 \). Therefore, \( 2 \) means the following:

\( 2' \) [Ergodic property]: If a compact set \( S \subseteq \Omega_E \) satisfies \( \psi^E_t(S) \subseteq S \) (\( 0 \leq t < \infty \)), then it holds that \( S = \Omega_E \).

The ergodic theorem says that the above \( 2' \) is equivalent to the following equality:

\[
\int_{\Omega_E} \int_{(\text{state space average})} f(\omega) \mathcal{F}_E (d\omega) = \lim_{T \to \infty} \frac{1}{T} \int_{(\text{time average})} \int_{t=0}^{a+T} f(\psi^E_t(\omega_0)) dt (\forall \omega_0 \in \Omega_E)
\]

where \( \mathcal{F}_E \) is defined by such that \( \mathcal{F}_E = \frac{\nu_E}{\nu_E(\Omega_E)} \).

\[
\begin{align*}
\int_{\Omega_E} f(\omega) \mathcal{F}_E (d\omega) & = \lim_{T \to \infty} \frac{1}{T} \int_{t=0}^{a+T} f(\psi^E_t(\omega_0)) dt (\forall \omega_0 \in \Omega_E) \tag{9.3}
\end{align*}
\]
After all, the ergodic property says that if $T$ is sufficiently large, it holds that

$$
\int_{\Omega_E} f(\omega) \varpi_E(d\omega) \approx \frac{1}{T} \int_0^{a+T} f(\psi^E_t(\omega_0))dt.
$$

Put $\varpi_T(dt) = \frac{dt}{T}$. The probability space $([\alpha, \alpha + T], \mathcal{B}_{[\alpha, \alpha + T]}, \varpi_T)$ (or equivalently, $([0, T], \mathcal{B}_{[0, T]}, \varpi_T)$) is called a (normalized) first staying time space, also, the probability space $(\Omega_E, \mathcal{B}_{\Omega_E}, \varpi_E)$ is called a (normalized) second staying time space. Note that these mathematical probability spaces are not related to "probability".

About (3) and (4)

Put $D_N = \{1, 2, \ldots, N(\approx 10^{24})\}$. For each $k (\in D_N)$, define the coordinate map $X_k : \Omega_E (\subset \mathbb{R}^6) \to \mathbb{R}^6$ such that

$$
X_k(\omega) = X_k(q, p) = X_k((q_{1n}, q_{2n}, q_{3n}, p_{1n}, p_{2n}, p_{3n})_{n=1}^N) = (q_{1k}, q_{2k}, q_{3k}, p_{1k}, p_{2k}, p_{3k})
$$

for all $\omega = (q, p) = (q_{1n}, q_{2n}, q_{3n}, p_{1n}, p_{2n}, p_{3n})_{n=1}^N \in \Omega_E (\subset \mathbb{R}^6N)$.

Also, for any subset $D (\subseteq D_N = \{1, 2, \ldots, N (\approx 10^{24})\})$, define the distribution map $R_D^{(q,p)} : \Omega_E (\subset \mathbb{R}^6N) \to M_{+1}(\mathbb{R}^6)$ such that

$$
R_D^{(q,p)} = \frac{1}{\#D} \sum_{k \in D} \delta_{X_k(q,p)} (\forall (q, p) \in \Omega_E (\subset \mathbb{R}^{6N}))
$$

where $\#D$ is the number of the elements of the set $D$.

Let $\omega_0(\in \Omega_E)$ be a state. For each $n (\in D_N)$, we define the map $Y_n^{\omega_0} : [0, T] \to \mathbb{R}^6$ such that

$$
Y_n^{\omega_0}(t) = X_n(\psi^E_t(\omega_0)) \quad (\forall t \in [0, T]).
$$

And, we regard $\{Y_n^{\omega_0}\}_{n=1}^N$ as random functions on the probability space $([0, T], \mathcal{B}_{[0, T]}, \varpi_T)$. Then, (3) and (4) respectively means

(3') $\{Y_n^{\omega_0}\}_{n=1}^N$ is a sequence with the approximately identical distribution concerning time. In other words, there exists a normalized measure $\rho_E$ on $\mathbb{R}^6$ (i.e., $\rho_E \in M^m_{+1}(\mathbb{R}^6)$) such that:

$$
\varpi_T\{t \in [0, T] : Y_n^{\omega_0}(t) \in \Xi\} \approx \rho_E(\Xi) \quad (\forall \Xi \in \mathcal{B}_{\mathbb{R}^6}, n = 1, 2, \ldots, N)
$$

(4') $\{Y_n^{\omega_0}\}_{n=1}^N$ is approximately independent, in the sense that, for any $D_0 \subset \{1, 2, \ldots, N(\approx 10^{24})\}$ such that $1 \leq \#D_0 \ll N$ (that is, $\frac{\#D_0}{N} \approx 0$), it holds that

$$
\varpi_T\{t \in [0, T] : Y_k^{\omega_0}(t) \in \Xi_k(\in \mathcal{B}_{\mathbb{R}^6}), k \in D_0\}
\approx \times_{k \in D_0} \varpi_T\{t \in [0, T] : Y_k^{\omega_0}(t) \in \Xi_k(\in \mathcal{B}_{\mathbb{R}^6})\}.
$$
The following important remark was missed in [?, ?]. This is the advantage of our method in comparison with Ruelle’s method (cf.[28]).

Thus, the law of large numbers says that, putting $D_0(\subset D_N)$ such that $1 \ll \sharp[D_0] \ll N$ (that is, $\frac{1}{\sharp[D_0]} \equiv 0 \div \frac{\sharp[D_0]}{N}$), for almost time $t \in [0, T]$, we see that

$$\frac{1}{\sharp[D_0]} \sum_{k \in D_0} \delta_{Y_k^{\omega_0}(t)} \overset{\mu_E}{=} \rho_E \quad (\forall \omega_0 \in \Omega_E), \quad (\text{by } \circ \text{ and } \circ) \quad (9.4)$$

Also, we see, by (9.3), that, for $D_0(\subset D_N)$ such that $1 \leq \sharp[D_0] \ll N$,

$$\overline{m}_{\psi}(\{t \in [0, T] : Y_k^{\omega_0}(t) \in \Xi_k(\in B_{\Xi_k}), k \in D_0\}) = \overline{m}_{\psi}(\{t \in [0, T] : X_k(\omega_0) \in \Xi_k(\in B_{\Xi_k}), k \in D_0\})$$

$$= \overline{m}_{\psi}(\{t \in [0, T] : \psi_k^{\omega_0}(\omega_0) \in ((X_k)_k \in D_0)^{-1}(\times \Xi_k))\}) = \overline{m}_{\psi}(((X_k)_k \in D_0)^{-1}(\times \Xi_k))$$

$$\equiv (\nu_\psi \circ ((X_k)_k \in D_0)^{-1})(\times \Xi_k).$$

Particularly, putting $D_0 = \{k\}$, we see:

$$\overline{m}_{\psi}(\{t \in [0, T] : Y_k^{\omega_0}(t) \in \Xi\}) \approx (\nu_\psi \circ X_k^{-1})(\Xi) \quad (\forall \Xi \in B_{\Xi_k}). \quad (9.5)$$

Hence, the sentences $\circ$ and $\circ$ in ordinary language can be translated to measurement theoretical sentence as follows.

**Hypothesis 9.1** $[\circ \text{ and } \circ]$ Put $D_N = \{1, 2, \ldots, N(\approx 10^24)\}$. Let $\mathcal{H}, E, \nu_\psi, \nu_\psi, X_k : \Omega \rightarrow \mathbb{R}^6$ be as in the above. Then, summing up $\circ$ and $\circ$, we say:

(b) $\{X_k : \Omega \rightarrow \mathbb{R}^6\}_{k=1}^N$ is approximately independent random variables with the identical distribution in the sense that there exists $\rho_\psi (\in M_{+1}(\mathbb{R}^6))$ such that

$$\bigotimes_{k \in D_0} \rho_\psi (\text{“product measure”}) \approx \nu_\psi \circ ((X_k)_k \in D_0)^{-1}.$$

for all $D_0 \subset D_N$ and $1 \leq \sharp[D_0] \ll N$.

Also, a state $(q, p)(\in \Omega_E)$ is called an *equilibrium state* if it satisfies $\nu_{(q, p)} \approx \rho_\psi$.

Now, we have the following theorem (cf.[17]):

**Theorem 9.2** *Ergodic hypothesis*(ergodic hypothesis) Assume Hypothesis 9.1 (or equivalently, $\circ$ and $\circ$). Then, for any $\omega_0 = (q(0), p(0)) \in \Omega_E$, it holds that

$$[\nu_{(q(t), p(t))}(\Xi) \approx \overline{m}_{\psi}(\{t \in [0, T] : Y_k^{\omega_0}(t) \in \Xi\}) \quad (\forall \Xi \in B_{\Xi_k}, k = 1, 2, \ldots, N(\approx 10^24)) \quad (9.6)$$

for almost all $t$. That is, $0 \leq \overline{m}_{\psi}(\{t \in [0, T] : (9.6) \text{ does not hold}\}) \ll 1.$
Proof. Let $D_0 \subset D_N$ such that $1 \ll \sharp[D_0] \equiv N_0 \ll N$ (that is, $\frac{1}{\sharp[D_0]} \approx 0 \ll \frac{\sharp[D_0]}{N}$). Then, from Hypothesis A, the law of large numbers (cf. [26]) says that

$$R_{D_0}^{(q(t),p(t))} \approx \nu_E \circ X^{-1}_k (\approx \rho_E)$$

(9.7)

for almost all time $t$. Consider the decomposition $D_N = \{D(1), D(2), \ldots, D(L)\}$. (i.e., $D_N = \bigcup_{l=1}^{L} D(l)$, $D(l) \cap D(l') = \emptyset$ ($l \neq l'$), where $\sharp[D(l)] \approx N_0$ ($l = 1, 2, \ldots, L$). From (9.7), it holds that, for each $k$ ($= 1, 2, \ldots, N$ ($\approx 10^{24}$)),

$$R_{D_N}^{(q(t),p(t))} = \frac{1}{N} \sum_{l=1}^{L} \sharp[D(l)] \times R_{D(l)}^{(q(t),p(t))}$$

$$\approx \frac{1}{N} \sum_{l=1}^{L} \sharp[D(l)] \times \rho_E \approx \nu_E \circ X^{-1}_k (\approx \rho_E),$$

(9.8)

for almost all time $t$. Hence, the proof is completed.

We believe that Theorem 9.2 is just what should be represented by the "ergodic hypothesis" as follows.

**Corollary 9.3 [ergodic hypothesis]**

- "population average of $N$ particles at each $t$"
- "time average of one particle"

Thus, we can assert that the ergodic hypothesis is related to equilibrium statistical mechanics. Here, the ergodic property $\exists'$ and the above ergodic hypothesis should not be confused. Also, it should be noted that the ergodic hypothesis does not hold if the box (containing particles) is too large.

**Note 9.2** Let us explain Corollary 9.3 as allegory of Note 9.1 as follows. We see that, at almost every time

- the number of the kindergartner who is doing the swing is 50 persons
- the number of the kindergartner who is doing the sliding way is 50 persons
- the number of the kindergartner who is doing the sand play is 20 persons

Here, we want to add the remark about the time interval $[0, T]$. For example, as one of typical cases, consider the motion of $10^{24}$ particles in a cubic box (whose long side is 0.3m). It is usual to consider that "averaging velocity" $= 5 \times 10^2$ m/s, "mean free path" $= 10^{-7}$ m. And therefore, the collisions rarely happen among $\sharp[D_0]$ particles in the time interval $[0, T]$, and therefore, the motion is "almost independent". For example, putting $\sharp[D_0] = 10^{10}$, we can calculate the number of times a certain particle collides with $D_0$-particles in $[0, T]$, as $(10^{-7} \times 10^{24})^{-1} \times (5 \times 10^2) \times T \approx 5 \times 10^{-2} \times T$. Hence, in order to expect that $\exists'$ and $\exists'$ hold, it suffices to consider that $T \approx 5$ seconds. Also, note that the term "ergodic" may be used in various meanings. For example, the formula (9.3) is also said to be ergodic. However, here, we use it in the above sense.

**Note 9.3** The entropy $H(q, p)$ of a state $(q, p) \in \Omega_E$ is defined by

$$H(q, p) = k \log[\nu_E((q', p') \in \Omega_E : R_{D_N}^{(q, p)} \approx R_{D_N}^{(q', p')})]$$

where

$$k = \frac{\text{Boltzmann constant}}{\text{[Plank constant]}^{\frac{3}{2}} \times N!}$$

Since almost every state in $\Omega_E$ is equilibrium, the entropy of almost every state is equal $k \log \nu_E(\Omega_E)$. Therefore, it is natural to assume that the law of increasing entropy holds.
9.1.2 The probabilistic aspect of equilibrium statistical mechanics — Where does probability come?

In this section we shall study the probabilistic aspects of equilibrium statistical mechanics. For completeness, note that

(a) the argument in the previous section is not related to “probability” since Axiom $p_1$ does not appear in Section 3.1. Also, recall the (E$_4$), that is, there is no probability without measurement.

Theorem 9.2 says that the equilibrium statistical mechanical system at almost all time $t$ can be regarded as:

(b) a box including about $10^{24}$ particles such as the number of the particles whose states belong to $\Xi \in \mathcal{B}_{\mathbb{R}^6}$ is given by $\rho_E(\Xi) \times 10^{24}$.

Thus, it is natural to assume as follows.

(c) if we, at random, choose a particle from $10^{24}$ particles in the box at time $t$, then the probability that the state $(q_1, q_2, q_3, p_1, p_2, p_3) \in \mathbb{R}^6$ of the particle belongs to $\Xi \in \mathcal{B}_{\mathbb{R}^6}$ is given by $\rho_E(\Xi)$.

In what follows, we shall represent this (J) in terms of measurements. Define the observable $O_0 = (\mathbb{R}^6, \mathcal{B}_{\mathbb{R}^6}, F_0)$ in $C(\Omega_E)$ such that

$$[F_0(\Xi)](q,p) = |R^{(q,p)}_{D_{\mathbb{R}}}[(\Xi)](\equiv |\sum_{k}^{|D_{\mathbb{R}}(\Xi)}\sum_{k}^{\mathbb{R}^6}X_k(q,p) \in \Xi})| \quad \forall \Xi \in \mathcal{B}_{\mathbb{R}^6}, \forall (q,p) \in \Omega_E (\subset \mathbb{R}^{6N}).$$

Thus, we have the measurement $M_{C(\Omega_E)}(O_0 := (\mathbb{R}^6, \mathcal{B}_{\mathbb{R}^6}, F_0), S_{[\delta_{\psi(t)}(q_0, p_0)]})$. Then we say, by Axiom $p_1$, that

(d) the probability that the measured value obtained by the measurement $M_{C(\Omega_E)}(O_0 := (\mathbb{R}^6, \mathcal{B}_{\mathbb{R}^6}, F_0), S_{[\delta_{\psi(t)}(q_0, p_0)]})$ belongs to $\Xi \in \mathcal{B}_{\mathbb{R}^6}$ is given by $\rho_E(\Xi)$. That is because Theorem 9.2 says that $[F_0(\Xi)](\psi(t_0, q_0, p_0)) \approx \rho_E(\Xi)$ (almost every time $t$).

Also, let $\Psi^t : C(\Omega_E) \rightarrow C(\Omega_E)$ be a deterministic Markov operator determined by the continuous map $\psi^t : \Omega_E \rightarrow \Omega_E$ (cf. Section 3.1.2). Then, it clearly holds $\Psi^t O_0 = O_0$. And, we must take a $M_{C(\Omega_E)}(O_0, S_{[\delta(t_{k})q(t_k), \psi(t_k)])}$ for each time $t_1, t_2, \ldots, t_k, \ldots, t_n$. However, Interpretation (E$_2$) says that it suffices to take the simultaneous measurement $M_{C(\Omega_E)}(X^a_{k=1} O_0, S_{[\delta(t_{0}), \psi(t_0)])})$.

\begin{itemize}
  \item Note 9.4 We devote ourselves to the linguistic world-view. We first have a language called measurement theory. And, we tried to describe the phenomena (1)–(3) in Sec. 9.1.1. And we construct equilibrium statistical mechanics. We are faithful to the linguistic spirit "language is before world". In this sense, we think that our equilibrium statistical mechanics is not physics.
\end{itemize}
9.2 "Dynamical system theory" vs. "Measurement theory" in equilibrium statistical mechanics

Equilibrium statistical mechanics is to derive thermo-dynamics from [Newtonian mechanics]+ [α].

Our method (measurement theory) is

(a) \[ \text{equilibrium statistical mechanics} := \text{(measurement theory)} \]

\[ \begin{align*}
&= \text{(Sec. 6.4)} \text{Axiom}_c^\pm 2(\text{causality}) + \text{(Sec. 2.2)} \text{Axiom}_p 1 \\
&\quad \text{("Newton equation 1\textsuperscript{c} and 2\textsuperscript{c} – 3") (urn problem)}
\end{align*} \]

where \( \nu_E \) is the normalized staying time derived from Newtonian mechanics.

On the other hand, the conventional method (dynamical system theory(cf. [28])) is

(b) \[ \text{equilibrium statistical mechanics} \]

\[ \begin{align*}
&\quad \text{(dynamical system theory)} \\
&= \text{Newtonian mechanics} \quad \text{("Newton 1\textsuperscript{c} and 2\textsuperscript{c}")} + \text{equal probability} \\
&\quad \text{(probabilistic interpretation of} \ \nu_E) \n\end{align*} \]

Thus, we have

\[ \text{(a) vs. (b)} \]

Here, recall Note 1.4. This "vs." must be settled in future.

\[ \star \text{ Note 9.5 We do not agree that there are several "equilibrium statistical mechanics". Also, we should hurry up to describe various sciences in terms of measurement theory. For "psychological tests", see [25].} \]

9.3 Is quantum mechanics physics, or engineering?

For the foregoing paragraph, I considered that equilibrium statistical mechanics was one of the science. In this section, I further promote this and argue about

Is quantum mechanics physics, or several sciences (engineering)?

Of course, if I say that quantum mechanics is not physics, you will think that I do not have common sense. However, Einstein was skeptical to quantum mechanics throughout life(cf. [30]). At least, I can be say that quantum mechanics was not "Einstein’s physics." It is more convenient to think "quantum mechanics is one of the science instead of physics" also for measurement theory. Although I explain the reason below, it becomes the argument about von Neumann’s work "mathematical basis of quantum mechanics [32];(1932)" after all. Then, I recommend that you read the following, after you refer to the Internet etc. about von Neumann’s -the man who was the cleverest at the 20-th century- genius.

I think over the classification of the measurement theory of Chap. 1(Y) as follows.
(a) measurement theory (=MT)

\[
\begin{align*}
\text{quantum MT} & \xrightarrow{\text{quantum phenomena}} \text{quantum mechanics (=quantum engineering)} \\
\text{classical MT} & \xrightarrow{\text{ordinary phenomena}} \text{statistical mechanics, economics, \ldots}
\end{align*}
\]

That is, it is a position which we consider is a linguistic science language rather than consider quantum measurement theory to be physics. Namely,

I consider that Born’s quantum measurement theory[Axiom(Q);Chapter 3.1.1] should be regarded as "mystic words", and I assert a linguistic science view.

This is the position of considering that a linguistic science language called measurement theory (= [Quantum measurement theory] + [Classic measurement theory]) occurs first, and considering that the theory (one of the science) which described the law in the micro world with the language is quantum mechanics. In this position, quantum mechanics is one of the science - although it may be a coined word of this book, quantum engineering - . If you may think like this (i.e., if you may think that quantum measurement theory is not "physics" but a "language" as shown in (a)), Copenhagen interpretation ((U₁)–(U₇) of Chap. 1) can also be allowed to be wonderful theory (Section 3.3). Rather, the author thinks as follows (Note 3.6).

(b) Copenhagen interpretation is not the thing of "physics" but a thing for "language (measurement theory)."

Mathematical formulation of quantum mechanics was performed by von Neumann in "The mathematical basis of quantum mechanics [32];(1932)" - formulation of the quantum mechanics by Hilbert space -. Supposing that is right, you should just be going to consider the following (c) in detail. Namely,

(c) Although physics, such as Newtonian mechanics, electromagnetism, and the theory of relativity, is formalized in the mathematics of differential geometry, why is quantum mechanics formalized in the mathematics of Hilbert space? Is quantum mechanics really physics?

Supposing it is formalized in different mathematics, it is natural that we consider that it is the world describing method of a different category. If that is right, I would like to come to think

\[
\begin{align*}
\text{[The realistic method]}: \\
\text{Physics should be formulized in differential geometry.} \\
\text{[The linguistic method]}: \\
\text{Various sciences should be formulized in Hilbert space theory.}
\end{align*}
\]

in world description.
Thinking in this way, in this book, we think that von Neumann proposed in "The mathematical basis of quantum mechanics [32]" not only formulation of quantum mechanics but big work with one another more, i.e.,

(d) Formulation of the linguistic side of quantum mechanics (namely, quantum measurement theory of (a))

I think that [32] is written by me so that it can read also not as the book of physics but as a book of the linguistic describing method. Therefore, although it becomes the upper repetition,

(e) Substantially, measurement theory was advocated in [32]. That is, although Heisenberg, Schrödinger, and Born advocated the physics of quantum mechanics, von Neumann formalized the linguistic side of quantum mechanics.

Just before Chap. 1(B), I wrote "In order to declare a law, the language for describing the law must be prepared before that." However, actual history is somewhat complicated, should be corrected as follows and should be written.

Heisenberg and others described "the law of quantum mechanics" in the imperfect language. And von Neumann made the imperfect language perfect.

If that is right, von Neumann’s opinion is "changing into an idiom [Chap. 1(M2)]" i.e., reverse "Using an idiom" of

\[(f_1)\quad [(M_2) \text{ in Chap. 1}]: \text{quantum mechanics} \xrightarrow{\text{proverbializing}} \text{measurement theory}\]

Namely,

\[(f_2)\quad \text{quantum engineering} \xleftrightarrow{\text{using proverb}} \text{measurement theory}\]

That is, although it becomes re-described of (a),

(g) In a language called Measurement theory (General quantum mechanics), it thinks as follows.

\[(g_1)\quad \text{The theories which describe quantum phenomena are several sciences of quantum mechanics (namely, quantum engineering).}\]

\[(g_2)\quad \text{The theories which describe an everyday phenomenon are ordinary science (for example, equilibrium statistical mechanics).}\]

If it thinks in this way, I can think that quantum mechanics (quantum mechanics based on the Copenhagen interpretation of [32] at least) is one of the various sciences - quantum engineering - instead of physics.
(Note 3.6). Although this \((g_1)\) may not be von Neumann’s \([32]\) intention, I would like to consider it like this in this book.

The above is a reason for "branch of quantum mechanics"

\[
\begin{align*}
& (\text{linguistic}) \\
& \quad \text{QM 1} \quad \text{Copenhagen interpretation} \quad \text{measurement theory} \quad \text{continued to (a)} \\
& (\text{physical}) \\
& \quad \text{QM 2} \quad \text{the theory of everything}
\end{align*}
\]

in Fig 8.2. If that is right,

(i) the quantum mechanics (namely, quantum mechanics currently ordinarily taught at the university) explained in the third chapter is not "quantum mechanics (2)" but quantum engineering. That is, I would like to come to consider

\[
[\text{quantum engineering}] = [\text{quantum mechanics based on the Copenhagen interpretation}].
\]

Supposing that is right, we would like to come to a conclusion with "the language of measurement theory = science(Chapter 8.3(m))" as follows.

(j) **Originally Copenhagen interpretation \([\text{U}_1)-(\text{U}_7)\] is a universal tenet for an understanding of engineering, and science (not being a thing of quantum mechanics).**

♠ **Note 9.6** Although "whether it is \((f_1)\) or \((f_2)\)?" may be an endless dispute, we can assert one side of the following \((g_1)\) or \((g_2)\) anyway.

\((g_1)\) If \((f_2)\) is right, quantum mechanics (= quantum engineering) and equilibrium statistical mechanics are the greatest examples of application of measurement theory.

\((g_2)\) If \((f_1)\) is right, measurement theory is the greatest example of application of quantum mechanics.

Although it is un-physical unique application, considering the extensiveness of science (since most science is science), the "greatest" description will be allowed.

Of course, in the position of this book, we will assert \((g_1)\) — "The measurement theory is more fundamental than quantum mechanics (in the sense of \((f_2)\))." As mentioned before, the worst understanding of measurement theory is to consider measurement theory as an ass in a lion’s skin.

♠ **Note 9.7** von Neumann’s "mathematical-basis of quantum mechanics\([32]\)" is the book which can be read by various methods. More precisely,

\[
\begin{align*}
& (0) \quad \text{: the mathematical book of Hilbert space,} \\
& (1) \quad \text{: the book of quantum mechanics (Copenhagen interpretation)} \\
& (2) \quad \text{: the book of quantum engineering (Copenhagen interpretation)}
\end{align*}
\]

The reading of \((1)\) was reading of the direction which von Neumann probably meant, and though it was natural, it won a great success. Development of a mathematics - Hilbert space theory and the operator algebra - was greatly urged also to the reading of \((0)\). It may be common sense to evaluate \([32]\) from a viewpoint of \((0)\) and \((1)\) (especially \((1)\)), and to conclude it as "great achievements (Bible of quantum mechanics)" under the authority of mathematics and physics. However,
10 Axiom\(p_b^1\) 1 — measurement (bounded type)

Classical measurement theory is classified as follows: (Chap. 1(Y)):

\[
\text{classical measurement theory} \begin{cases} 
\text{continuous type measurement theory (Chap. 2–8)} \\
\text{bounded type measurement theory (Chap. 110.11)}
\end{cases}
\]

Although continuous type measurement theory is fundamental, an exact measurement is not generally defined. Thus, we shall introduce bounded type measurement theory as follows.

\[
\text{bounded pure type measurement theory} := \left(\text{bounded type } \text{Axiom}^p_b 1\right) + \text{causality}
\]

In the chapter, we shall devote ourselves to the above axiom 1. This is similar Axiom\(p_b^1\) 1 (continuous pure type) in Chap. 2. Thus, it is surely understandable.

10.1 State and Observable — Primary quantity and Secondary quantity

In continuous type measurement theory (Chap. 2–8), a continuous function (i.e., the element of \(C(\Omega)\)) plays an important role. On the other hand, a bounded function is main in bounded type measurement theory. Since it is a small change, readers can easily understand the bounded type measurement theory.

Consider a locally compact space \(\Omega\) and measure space \((\Omega, \mathcal{B}_\Omega, \nu)\), where \(\mathcal{B}_\Omega\) is the Borel field of \(\Omega\), that is, the smallest \(\sigma\)-field that contains all open sets. Further assume that
(a) 0 < \nu(U) \leq \infty (\forall \text{ open set } U), \text{ and } \nu \text{ is } \sigma\text{-finite.}

A Banach space \( L^r(\Omega, \nu) \) (where, \( r = 1, \infty \)) is the space composed of all complex valued measurable function \( f : \Omega \to \mathbb{R} \) with the norm \( \|f\|_{L^r(\Omega, \nu)} < \infty \) where the norm \( \|f\|_{L^r(\Omega, \nu)} \) is defined by

\[
\|f\|_{L^r(\Omega, \nu)} = \begin{cases} 
\int_{\Omega} |f(\omega)| \nu(\omega) & \text{ (when } r = 1) \\
\text{ess.sup}_{\omega \in \Omega} |f(\omega)| & \text{ (when } r = \infty) 
\end{cases}
\]

(10.1)

Here,

\[
\text{ess.sup}_{\omega \in \Omega} |f(\omega)| = \sup \{a \in \mathbb{R} | \nu(\{\omega \in \Omega : |f(\omega)| \geq a\}) > 0\}
\]

A function \( f \in L^\infty(\Omega, \nu) \) is said to be essential continuous at \( \omega_0 \in \Omega \) if there exists a function \( g \in L^\infty(\Omega, \nu) \) that satisfies the following (b):

(b) \( g : \Omega \to \mathbb{R} \) is continuous at \( \omega_0 \) and

\[
\nu(\{\omega \in \Omega | f(\omega) \neq g(\omega)\}) = 0.
\]

In this case, the value \( f(\omega_0) \) is defined by \( g(\omega_0) \).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{continuity.png}
\caption{Essentially continuity}
\end{figure}

\textbf{Note 10.1} It, of course, holds that \( C(\Omega) \subseteq L^\infty(\Omega, \nu) \). Also, the space \( L^\infty(\Omega, \nu) \) is necessary as a consequence of quantum mechanics (as mentioned in Note 2.1).

Following Definition 2.1 in \( C(\Omega) \), we have the following definition:

\textbf{Definition 10.1} [Observable, state space, state, measured value space, measured value] The triple \( O=(X, \mathcal{F}, F) \) is called an observable in \( L^\infty(\Omega, \nu) \), it it satisfies:

(i) \( X \) is a set, and \( \mathcal{F} \subseteq \mathcal{P}(X) = \{\Xi | \Xi \subseteq X\} \) is a \( \sigma\)-field.

(ii) The map \( F : \mathcal{F} \to L^\infty(\Omega, \nu) \) satisfies:

(a) \( \Xi \in \mathcal{F} \implies F(\Xi) \geq 0 \ (\nu\text{-a.e.}) \)
(b) \( F(\emptyset) = 0 \) and \( F(X) = 1 \ (\nu\text{-a.e.}) \),
(c) [Completely countability]: For any countable decomposition \( \{\Xi_1, \Xi_2, \ldots, \Xi_n, \ldots\} \) (that is, \( \Xi = \bigcup_{n=1}^{\infty} \Xi_n, \Xi_n \in \mathcal{F}, (n = 1, 2, \ldots), \Xi_m \cap \Xi_n = \emptyset \ (m \neq n) \) ) of any \( \Xi \in \mathcal{F} \), it holds that

\[
\int_{\Omega} [F(\Xi)](\omega) \rho(\omega) \nu(d\omega) = \lim_{N \to \infty} \sum_{n=1}^{N} \int_{\Omega} [F(\Xi_n)](\omega) \rho(\omega) \nu(d\omega) \quad (\forall \rho \in L^1(\Omega, \nu))
\]
Here, $\Omega$ (or, $(\Omega, \mathcal{B}_\Omega, \nu)$) is called a state space, its element $\omega(\in \Omega)$ is called a state. Also, $X$ and $x(\in X)$ is respectively called a measured value space and measured value. In addition, if $F(\Xi) = (F(\Xi))^2$ holds for any $\Xi(\in \mathcal{F})$, the $(X, \mathcal{F}, F)$ is called a projective observable.

The following theorem is clear.

**Theorem 10.2** If $O = (X, F, F)$ is an observable in $C(\Omega)$, the it is the observable in $L^\infty(\Omega, \mathcal{B}_\Omega, \nu)$.

The following examples (Example 10.3–Example 10.5) are similar to the examples in Chap. 2.

**Example 10.3** [(i): Exact observable (cf. Example 2.5)] Define the observable $O_{\text{exa}} = (\Omega, \mathcal{B}_\Omega, F_{\text{exa}})$ in $L^\infty(\Omega, \mathcal{B}_\Omega, \nu)$ such that

$$[F_{\text{exa}}(\Xi)](\omega) = \chi_\Xi(\omega) = \begin{cases} 1 & (\omega \in \Xi(\in \mathcal{B}_\Omega)) \\ 0 & (\omega \notin \Xi(\in \mathcal{B}_\Omega)). \end{cases}$$

where $\chi$ is the characteristic function. This $O_{\text{exa}}$ is called the exact observable. As mentioned in Example 2.5, the $O_{\text{exa}}$ is no necessarily the observable in $C(\Omega)$.

**Example 10.4** [Normal observable (cf. Example 2.11)] Put $\Omega = \mathbb{R}$ (= the real line) or, $\Omega = [a, b]$ (= the interval $\subseteq \mathbb{R}$). Consider the Lebesgue measure $m(d\omega)(= d\omega)$. Assume that $\sigma > 0$ is the standard deviation. Define the normal observable $O_{G_\sigma} = (\mathbb{R}, \mathcal{B}_\mathbb{R}, G_\sigma)$ in $L^\infty(\Omega, m)$ such that

$$[G_\sigma(\Xi)](\omega) = \frac{1}{\sqrt{2\pi \sigma^2}} \int_{\Xi} e^{-\frac{(x-\omega)^2}{2\sigma^2}} dx \quad (\forall \Xi \in \mathcal{B}_\mathbb{R}, \forall \omega \in \Omega)$$

This is also the observable in $C(\Omega)$.

**Example 10.5** [Round observable (Example 2.6)] Consider the state space $\Omega$ by the interval $[0, 100]$ with the Lebesgue measure $d\omega$. For each $n \in \mathbb{N}_{100} = \{0, 10, 20, \ldots, 100\}$, define the function $g_n : \Omega \to [0, 1]$ such that

$$g_n(\omega) = \begin{cases} 0 & (0 \leq \omega \leq n - 5) \\ 1 & (n - 5 < \omega \leq n + 5) \\ 0 & (n + 5 < \omega \leq 100) \end{cases}$$

Here, define the $O_{\text{RND}} = (Y(=\mathbb{N}_{100}^{100}), 2^Y, G_{\text{RND}})$ by

$$[G_{\text{RND}}(\emptyset)](\omega) = 0, \quad [G_{\text{RND}}(Y)](\omega) = 1, \quad [G_{\text{RND}}(\Gamma)](\omega) = \sum_{n \in \Gamma} g_n(\omega) \quad (\forall \Gamma \in 2^Y = 2^{\mathbb{N}_{100}}).$$
Then, the \( O_{\text{RND}} = (Y(=\mathbb{N}^{100})_\mathbb{N}, 2^Y, G_{\text{RND}}) \) is an projective observable in \( L^\infty([0,100]) \). But, it is not an observable in \( C([0,100]) \).

10.2 Axiom_b^P 1 (Bounded type measurement)

10.2.1 Axiom_b^P 1 (bounded type)

With any classical system \( S \), a basic algebra \([C(\Omega), L^\infty(\Omega, \nu)]\) (or in short, \( L^\infty(\Omega, \nu) \) ) can be associated in which measurement theory of that system can be formulated. A state of the system \( S \) is represented by \( \omega(\in \Omega, \text{i.e., a state space } \Omega) \). Also, an observable is represented by \( O \equiv (X, \mathcal{F}, F) \) in the \( L^\infty(\Omega, \nu) \).

The measurement of an observable \( O \) for a system with a state \( \omega \) is represented by \( \mathcal{M}_{L^\infty(\Omega)}(O, S_{[\omega]}) \) (or, \( \mathcal{M}_{L^\infty(\Omega, \nu)}(O, S_{[\omega]}) \) ). Also, by measurement \( \mathcal{M}_{L^\infty(\Omega)}(O, S_{[\omega]}) \), a measured value \( x \ (\in X) \) is obtained.

By the same way of Axiom_b^P 1 [(continuous type) measurement] in Sec.2.2, we can introduce Axiom_b^P 1 [(bounded type) measurement] as follow.

**Axiom_b^P 1 (measurement : bounded pure type)**

\[
\text{Consider a measurement } \mathcal{M}_{L^\infty(\Omega)}(O \equiv (X, \mathcal{F}, F), S_{[\omega]}) \text{ formulated in a basic algebra } [C(\Omega), L^\infty(\Omega, \nu)]. \text{ The probability that a measured value } x (\in X) \text{ obtained by the measurement } \mathcal{M}_{L^\infty(\Omega)}(O, S_{[\omega]}) \text{ belongs to } \Xi (\in \mathcal{F}) \text{ is given by } [F(\Xi)](\omega) \text{, if } F(\Xi) \text{ is essentially continuous at } \omega.
\]

It is a matter of course that Axiom_b^P 1 should be use according to the Copenhagen interpretation [\([U_1]–[U_7]\)] in Chap. 1.

**Remark 10.6** When \( F(\Xi) \) is not essentially continuous at \( \omega(\in \Omega) \), the sample probability space \( (X, \mathcal{F}, [F(\cdot)](\omega)) \) is not a mathematical probability space.

10.2.2 Simple examples — Urn problem and so on

The following examples ( Example 10.7, Example 10.8 ) are similar to the examples in Chap. 2.

The bounded type measurement theory is similar to continuous type. Thus, we introduce only typical examples.

**Example 10.7 [The exact measurement of temperature (Sec.1.2(V))]** Consider the exact measurement for the water with the temperature \( \omega_0 \ ^\circ\text{C} \). Put \( \Omega = X = [0,100] \). Let Lebesgue measure space \( (\Omega, \mathcal{B}_\Omega, m) \) be a state. Consider the exact observable \( O^{(\text{exa})} = (X, \mathcal{B}_X, F^{(\text{exa})}) \) in \( L^\infty(\Omega, m) \). Then, we say that

Assume that a measured value \( x_0 (\in X=\Omega) \) is obtained by the exact measurement \( \mathcal{M}_{L^\infty(\Omega)}(O^{(\text{exa})}, S_{[\omega_0]}) \).

Then, we can surely say that \( x_0 = \omega_0 \).
That is because:

Let $D(\subseteq \Omega = X)$ be any open set such that $\omega_0 \in D$. Then, the probability that $x_0 \in D$ is given by 

\[ [F^{(\text{exa})}(D)](\omega_0) = \chi_D(\omega_0) = 1. \]

Therefore, from the arbitrariness of $D$, we see that $x_0 = \omega_0$.

This completes the proof. $\Box$

The following is a bounded type measurement version of Example 2.10.

**Example 10.8 [Example 2.10] [Bounded type measurement version of the urn problem]** There are two urns $U_1$ and $U_2$. The urn $U_1$ [resp. $U_2$] contains 8 white and 2 black balls [resp. 4 white and 6 black balls] (cf. Fig. 2.5).

[I]: Here, consider the following phenomenon:

(a) Pick out one ball at random from the urn $U_2$. Then the probability that the ball is white is given by 0.4.

This statement in ordinary language will be translated to the statement (b) of measurement theory in what follows.

Consider the state space $\Omega = \{\omega_1, \omega_2\}$ with the discrete metric $d_D$ and with the measure $\nu$ such that

\[ \nu(\{\omega_1\}) = 1, \quad \nu(\{\omega_2\}) = 1. \]

(It is also possible to assume that $\nu(\{\omega_1\}) = 2$ and $\nu(\{\omega_2\}) = 3$). Further, assume that

\[ U_1 \ldots \text{“the urn with the state } \omega_1\text{”} \]
\[ U_2 \ldots \text{“the urn with the state } \omega_2\text{”} \]

And thus, we consider the following identification:

\[ U_1 \approx \omega_1, \quad U_2 \approx \omega_2, \]

Put “$w$” = “white”, “$b$” = “black”, and put $X = \{w, b\}$. And define the observable $O( \equiv (X \equiv \{w, b\}, 2^{\{w,b\}}, F))$ in $L^\infty(\Omega, \nu)$ by

\[ [F(\{w\})](\omega_1) = 0.8, \quad [F(\{b\})](\omega_1) = 0.2, \]
\[ [F(\{w\})](\omega_2) = 0.4, \quad [F(\{b\})](\omega_2) = 0.6. \]

Thus, we have the measurement $M_{L^\infty(\Omega, \nu)}(O, S_{\{\omega_2\}})$.

Here, Axiom$^p_1$ (page 152) says that

(b) the probability that a measured value $b$ is obtained by $M_{C(\Omega)}(O, S_{\{\omega_2\}})$ is given by

\[ F(\{b\})(\omega_1) = 0.6 \]
Now we can present the following theorem.

**Theorem 10.9 [Exact measurement]** Consider the exact observable \( O^{(exa)} = (X, F, F^{(exa)}) \) in \( L^\infty(\Omega, \nu) \). Assume that a measured value \( x \in X \) is obtained by the exact measurement \( M_{L^\infty(\Omega, \nu)}(O^{(exa)}, S_{\omega_0}) \). The probability that \( x = \omega_0 \) is equal to 1.

**Proof.** The proof is the same as the proof in Example 10.7. Let \( D(\subseteq \Omega = X) \) be any open set such that \( \omega_0 \in D \). Then, the probability that \( x_0 \in D \) is given by \( [F^{(exa)}(D)](\omega_0) = \chi_D(\omega_0) = 1 \). Therefore, from the arbitrariness of \( D \), we see that \( x_0 = \omega_0 \). This completes the proof.

### 10.3 System quantity — The origin of observables

In classical mechanics, the term "observable" usually means the continuous real valued function on a state space (that is, physical quantity). An observable in measurement theory is characterized as the natural generalization of the physical quantity. This will be explained in the following examples.

**Example 10.10 [System quantity]** Let \( L^\infty(\Omega, \nu) \) be a basic algebra. A continuous real valued function \( \tilde{f} : \Omega \to \mathbb{R} \) (or generally, a measurable real valued function \( \tilde{f} : \Omega \to \mathbb{R}^n \)) is called a system quantity (or in short, quantity) on \( \Omega \). Define the projective observable \( O = (\mathbb{R}, B_\mathbb{R}, F) \) in \( L^\infty(\Omega, \nu) \) such that

\[
[F(\Xi)](\omega) = \begin{cases} 
1 & \text{when } \omega \in \tilde{f}^{-1}(\Xi) \\
0 & \text{when } \omega \notin \tilde{f}^{-1}(\Xi) 
\end{cases} \quad (\forall \Xi \in B_\mathbb{R})
\]

Here, note that

\[
\tilde{f}(\omega) = \lim_{N \to \infty} \sum_{n=-N^2}^{N^2} \frac{n}{N} \left[ F \left( \left( \frac{n}{N}, \frac{n+1}{N} \right) \right) \right](\omega) = \int_\mathbb{R} \lambda [F(d\lambda)](\omega)
\]

Thus, we have the following identification:

\[ \tilde{f} \quad (\text{system quantity on } \Omega) \longleftrightarrow \quad O = (\mathbb{R}, B_\mathbb{R}, F) \quad (\text{projective observable in } L^\infty(\Omega, \nu)) \]

This \( O \) is called the observable representation of a system quantity \( \tilde{f} \). Therefore, we say that

(a) An observable in measurement theory is characterized as the natural generalization of the physical quantity.

Here, recall the identification (3.2), which is the quantum version of the identification (10.2).

**Example 10.11 [Position observable , momentum observable , energy observable]** Consider Newtonian mechanics in the basic algebra \( L^\infty(\Omega, \nu) \). For simplicity, consider the two dimensional space

\[ \Omega = \mathbb{R}_q \times \mathbb{R}_p = \{ (q, p) = (\text{position, momentum}) \mid q, p \in \mathbb{R} \} \]
The following quantities are fundamental:

\[
\begin{align*}
(\star_1) \quad & \tilde{q} : \Omega \to \mathbb{R}, \\
(\star_2) \quad & \tilde{p} : \Omega \to \mathbb{R}, \\
(\star_3) \quad & \tilde{e} : \Omega \to \mathbb{R},
\end{align*}
\]

\[\tilde{q}(q, p) = q \quad (\forall (q, p) \in \Omega)\]

\[\tilde{p}(q, p) = p \quad (\forall (q, p) \in \Omega)\]

\[\tilde{e}(q, p) = \text{[potential energy]} + \text{[kinetic energy]}\]

\[= U(q) + \frac{p^2}{2m} \quad (\forall (q, p) \in \Omega)\]

where, \(m\) is the mass of a particle. Under the identification (10.2), \((\star_1)\) and \((\star_2)\) is respectively called a position observable and a momentum observable.

In what follows, we shall introduce Schrödinger equation.

\[\blacklozenge \text{ Note 10.2} \quad \text{The Hamiltonian } H(q, p) = \frac{\tilde{p}^2}{2m} + U(q) \text{ produces classical and quantum kinetic equations.}\]

\[(\star_1) \quad \text{Classical case: [Newtonian equation]} \]

\[\frac{\partial u_t(q)}{\partial t} = \frac{\partial H(q, p)}{\partial p} \quad \text{"the simple case of (9.1)"} \quad \text{produces classical and quantum kinetic equations.}\]

\[(\star_2) \quad \text{quantum case: [Schrödinger equation]} \]

\[\hbar \sqrt{-1} \frac{\partial^2 u_t(q)}{\partial t^2} = H(q, p) \text{ produces classical and quantum kinetic equations.}\]

The solution \(\{u_t\} \in \mathbb{R}\) of Schrödinger equation represents the state change.

For each \(k = 1, 2, \ldots, n\), consider an observable \(O_k = (X_k, F_k, F_k)\) in \(L^\infty (\Omega, \nu)\). And consider the simultaneous observable \(X^n_{k=1} O_k = (\otimes_{k \in K} X_k, \bigotimes_{k=1}^n F_k, \bigotimes_{k=1}^n F_k)\), which is defined by the same way of Definition 2.14.

The following theorem is clear.

\[\text{Theorem 10.12 [Exact measurement and system quantity]} \quad \text{Let } O_0^{\text{exa}} = (X, \tilde{F}, F^{\text{exa}}) \text{ (i.e., } (X, \tilde{F}, F^{\text{exa}}) = (\Omega, B_{\Omega}, \chi) \text{ ) be the exact observable in } L^\infty (\Omega, \nu)\). \text{ Let } O_1 = (\mathbb{R}, B_{\mathbb{R}}, G) \text{ be the observable that is induced by a quantity } \tilde{g} : \Omega \to \mathbb{R} \text{ as in Example 9.6. Consider the simultaneous observable } O_0^{\text{exa}} \otimes O_1. \text{ Let } (x, y) \in \mathbb{R} \times \mathbb{R} \text{ be a measured value obtained by the simultaneous measurement } M_{L^\infty (\Omega, \nu)} (O_0^{\text{exa}} \otimes O_1, S_{\{D_0\}}). \text{ Then, we can surely believe that } x = \omega, \text{ and } y = \tilde{g}(\omega)\).

\[\text{Proof.} \quad \text{Let } D_0 (\in B_{\Omega}) \text{ be any open set such that } \omega (\in \Omega = X). \text{ Also, let } D_1 (\in B_{\mathbb{R}}) \text{ be any open set such that } \tilde{g}(\omega) \in D_1. \text{ The probability that a measured value } (x, y) \text{ obtained by the measurement } M_{L^\infty (\Omega, \nu)} (O_0^{\text{exa}} \otimes O_1, S_{\{D_0\}}) \text{ belongs to } D_0 \times D_1 \text{ is given by } \chi_{D_0}(\omega) \cdot \chi_{\tilde{g}^{-1}(D_1)}(\omega) = 1. \text{ Since } D_0 \text{ and } D_1 \text{ are arbitrary, we can surely believe that } x = \omega \text{ and } y = \tilde{g}(\omega). \quad \square\]
10.4 Measurement theoretical Kolmogorov extension theorem

We consider that

(a) The utility of Kolmogorov extension theorem in probability theory is due to Theorem 10.13 (measurement theory version of Kolmogorov extension theorem)

This will be asserted in this section.

In this section we study “Kolmogorov’s extension theorem” in the measurement theory. It is generally said that Kolmogorov’s extension theorem is most fundamental in Kolmogorov’s probability theory. That is because this theorem assures the existence of a probability space (i.e., sample space). On the other hand, our theorem (= Theorem 10.1, i.e., Kolmogorov’s extension theorem in measurement theory) assures the existence of a measurement (or, observable). Recall the our spirit (see Remark (in §2.3(I))):

(♯) there is no probability without measurements.

Thus, in measurement theory, the concept of “measurement” is more fundamental than that of “sample space”. Therefore, this theorem (i.e., Kolmogorov’s extension theorem in measurement theory) is very important in measurement theory. That is, this theorem (= Theorem 10.1) is essential to measurement theory just like Kolmogorov’s extension theorem is so in his probability theory. Using this theorem, we can define “particle’s trajectory” by “the sequence of measured values”. And further we prove:

(i) the existence of “particle’s trajectory” in Newtonian mechanics,

(ii) the existence of Brownian motion.

Thus, we can understand the difference between the concepts of “particle’s trajectory” and “state’s evolution” in both classical and quantum mechanics.

Let \( \hat{\Lambda} \) be an index set. For each \( \lambda \in \hat{\Lambda} \), consider a set \( X_\lambda \). For any subsets \( \Lambda_1 \subseteq \Lambda_2 (\subseteq \hat{\Lambda}) \), \( \pi_{\Lambda_1,\Lambda_2} \) is the natural projection such that:

\[
\pi_{\Lambda_1,\Lambda_2} : \bigotimes_{\lambda \in \Lambda_2} X_\lambda \longrightarrow \bigotimes_{\lambda \in \Lambda_1} X_\lambda.
\]

Especially, put \( \pi_\Lambda = \pi_{\Lambda,\hat{\Lambda}} \). For each \( \lambda \in \hat{\Lambda} \), consider an observable \( (X_\lambda, \mathcal{F}_\lambda, F_\lambda) \) in \( L^\infty(\Omega, \nu) \). Note that the quasi-product observable \( \overline{\mathcal{O}} \equiv (\bigotimes_{\lambda \in \hat{\Lambda}} X_\lambda, \bigotimes_{\lambda \in \hat{\Lambda}} \mathcal{F}_\lambda, F_\hat{\Lambda}) \) of \( \left\{ (X_\lambda, \mathcal{F}_\lambda, F_\lambda) \mid \lambda \in \hat{\Lambda} \right\} \) is characterized as the observable such that:

\[
F_\hat{\Lambda}(\pi^{-1}_\Lambda(\Xi_\lambda)) = F_\lambda(\Xi_\lambda) \quad (\forall \Xi_\lambda \in \mathcal{F}_\lambda, \forall \lambda \in \hat{\Lambda}),
\]

though the existence and the uniqueness of a quasi-product observable are not guaranteed in general. The following theorem says something about the existence and uniqueness of the quasi-product observable.
Let \( \tilde{A} \) be a set. For each \( \lambda \in \tilde{A} \), consider a set \( X_\lambda \). For any subset \( \Lambda_1 \subseteq \Lambda_2 (\subseteq \tilde{A}) \), define the natural map \( \pi_{\Lambda_1, \Lambda_2} : \times_{\lambda \in \Lambda_2} X_\lambda \rightarrow \times_{\lambda \in \Lambda_1} X_\lambda \) by

\[
\times_{\lambda \in \Lambda_2} X_\lambda \ni (x_\lambda)_{\lambda \in \Lambda_2} \mapsto (x_\lambda)_{\lambda \in \Lambda_1} \in \times_{\lambda \in \Lambda_1} X_\lambda \tag{10.3}
\]

The following theorem guarantees the existence and uniqueness of observable. It should be noted that this is due to Chap. 1 the Copenhagen interpretation \((U_d)\), i.e., “only one measurement is permitted”.

**Theorem 10.13** [measurement theoretical version of Kolmogorov extension theorem (cf. [11])] For each \( \lambda \in \tilde{A} \), consider a Borel measurable space \((X_\lambda, \mathcal{F}_\lambda)\), where \( X_\lambda \) is a separable complete metric space. Define the set \( \mathcal{P}_0(\tilde{A}) \) such as \( \mathcal{P}_0(\tilde{A}) \equiv \{ \Lambda \subseteq \tilde{A} \mid \Lambda \text{ is finite} \} \). Assume that the family of the observables \( \{ \mathcal{O}_\lambda \equiv (\times_{\lambda \in \Lambda} X_\lambda, \times_{\lambda \in \Lambda} \mathcal{F}_\lambda, F_\lambda) \mid \Lambda \in \mathcal{P}_0(\tilde{A}) \} \) in \( L^\infty(\Omega, \nu) \) satisfies the following “consistency condition”:

- for any \( \Lambda_1, \Lambda_2 \in \mathcal{P}_0(\tilde{A}) \) such that \( \Lambda_1 \subseteq \Lambda_2 \),
  \[
  F_{\Lambda_2}(\pi_{\Lambda_1, \Lambda_2}(\Xi_{\Lambda_1})) = F_{\Lambda_1}(\Xi_{\Lambda_1}) \quad (\forall \Xi_{\Lambda_1} \in \times_{\lambda \in \Lambda_1} \mathcal{F}_\lambda). \tag{10.4}
  \]

Then, uniquely exists the observable \( \tilde{\mathcal{O}}_{\tilde{A}} \equiv (\times_{\lambda \in \tilde{A}} X_\lambda, \times_{\lambda \in \tilde{A}} \mathcal{F}_\lambda, \tilde{F}_{\tilde{A}}) \) in \( L^\infty(\Omega, \nu) \) such that:

\[
\tilde{F}_{\tilde{A}}(\pi_{\tilde{A}}^{-1}(\Xi_{\tilde{A}})) = F_\Lambda(\Xi_\Lambda) \quad (\forall \Xi_\Lambda \in \times_{\lambda \in \Lambda} \mathcal{F}_\lambda, \forall \Lambda \in \mathcal{P}_0(\tilde{A})). \tag{10.5}
\]

**Proof.** For the proof, see [11]. \( \square \)

**Corollary 10.14** [Finite simultaneous observable] For each \( k \in K \equiv \{1, 2, \ldots, |K|\} \), consider an observable \( \mathcal{O}_k \equiv (X_k, \mathcal{F}_k, F_k) \) in \( L^\infty(\Omega, \nu) \). If the commutativity condition:

\[
F_{k_1}(\Xi_{k_1})F_{k_2}(\Xi_{k_2}) = F_{k_2}(\Xi_{k_2})F_{k_1}(\Xi_{k_1}) \quad (\forall \Xi_{k_1} \in \mathcal{F}_{k_1}, \forall \Xi_{k_2} \in \mathcal{F}_{k_2}, k_1 \neq k_2)
\]

holds, then we can uniquely construct a product observable \( \tilde{\mathcal{O}} \equiv (\times_{k \in K} X_k, \bigotimes_{k \in K} \mathcal{F}_k, \tilde{F} \equiv \times_{k \in K} F_k) \) such that:

\[
\tilde{F}((\times_{\lambda \in \mathcal{A}_0} X_\lambda) \times (\times_{\lambda \in \tilde{A} \setminus \mathcal{A}_0} X_\lambda)) = \times_{\lambda \in \mathcal{A}_0} F_\lambda(\Xi_\lambda) \quad (\forall \Xi_\lambda \in \mathcal{F}_\lambda, \forall \lambda \in \mathcal{A}_0)
\]

The product observable is also called a simultaneous observable.

(ii): Even if \( K \) is infinite, the product observable similarly exists.

Consider a basic structure \([C(\Omega); L^\infty(\Omega, \nu)]\). Let \( A \) be a set. For each \( \lambda \in \tilde{A} \), assume that \( X_\lambda \) is a separable complete metric space, \( \mathcal{F}_\lambda \) is its Borel field. For each \( \lambda \in \tilde{A} \), consider an observable \( \mathcal{O}_\lambda \equiv (X_\lambda, \mathcal{F}_\lambda, F_\lambda) \) in \( L^\infty(\Omega, \nu) \). Then, a simultaneous observable \( \tilde{\mathcal{O}} \equiv (\times_{\lambda \in \tilde{A}} X_\lambda, \bigotimes_{\lambda \in \tilde{A}} \mathcal{F}_\lambda, \tilde{F} \equiv \times_{\lambda \in \tilde{A}} F_\lambda) \) uniquely exists. That is, for any finite set \( \Lambda_0 (\subseteq \tilde{A}) \), it holds that:

\[
\tilde{F}((\times_{\lambda \in \Lambda_0} X_\lambda) \times (\times_{\lambda \in \tilde{A} \setminus \Lambda_0} X_\lambda)) = \times_{\lambda \in \Lambda_0} F_\lambda(\Xi_\lambda) \quad (\forall \Xi_\lambda \in \mathcal{F}_\lambda, \forall \lambda \in \Lambda_0)
\]

**Proof.** The proof is a direct consequence of Theorem 10.13. Thus, it is omitted. \( \square \)
Note 10.3 if "continuous type measurement" is compared with "bounded type measurement", the former may be essential, but the latter is handy from the mathematical point of view.

Note 10.4 In basic algebra $[C(\Omega); L^\infty(\Omega, \nu)]$, a mixed state $\rho (\in L^1(\Omega))$ is defined by

$$\rho \geq 0, \quad \int_\Omega \rho(\omega)\nu(d\omega) = 1$$

Then we have the following.

**Axiom** b1 (measurement (bounded · mixed type ))

Consider a mixed measurement $M_{L^\infty(\Omega, \nu)}(O = (X, F, F), S_{\omega}(\rho))$ in basic algebra $L^\infty(\Omega, \nu)$. Then, the probability $P(\Xi)$ that a measured value obtained by $M_{L^\infty(\Omega, \nu)}(O = (X, F, F), S_{\omega}(\rho))$ belongs to $\Xi(\in F)$ is given by

$$P(\Xi) = \int_\Omega [F(\Xi)](\omega)\rho(\omega)\nu(d\omega)$$

11 **Axiom** pm 2 - causality (bounded type)

Same as continued type measurement theory, bounded type measurement theory is formalized as follows:

Thus, we shall introduce bounded type measurement theory as follows.

\[
\text{bounded pure type measurement theory} = \begin{cases} \text{measurement} & \text{[bounded type Axiom p_1]} \\ \text{probabilistic interpretation} & \text{[bounded type Axiom pm 2]} \end{cases} + \text{causality} \quad \text{[the Heisenberg picture]}
\]

In this chapter, I explain the bounded type Axiom pm 2. Since there is a close resemblance between it and "continuation and pure type measurement theory of Chapter 6 (Axiom pm 2)", you should be able to read easily. However, you should be cautious of a tree semi-ordered set $T$ not being necessarily a finite set. Bounded type measurement theory has a field including a vast domain, and when writing out, there is no end. Therefore, in this chapter, it extracted only to Zeno’s paradox and the argument of the circumference of it.

11.1 Zeno’s paradox — Flying arrow is not moving.

In this section, I explain the meaning of Zeno’s paradox.

11.1.1 Movement function method

In this book, movement function method is considered as follows. Movement function method is the quantitative describing method of "movement and change", and if it is called a time-position function method, you may not have misunderstanding. Namely,

$$(A_1) \text{ In order to express movement and change, } x \text{ of a position of a particle is denoted by function of time } t. \text{ That is, the time-position function } x = q(t) \text{ expresses change of the position of each time (for example, a position, height, academic ability, GDP of a country, etc.)}$$
Of course, we studied the movement function method in elementary school as follows.

\[
\begin{cases}
q(t_2) - q(t_1) = v, & \text{this is called a speed if it does not depend on } (t_1, t_2) \\
q(t_2) - t_1 = a
\end{cases}
\]

The time-position function \(q(t) = vt + a\) can be found from easy calculation. We think

(A2) Movement function method (= time-position function method) is a kind of world-view such as describe movement and change quantitatively with a time-position function

If I have a worldly way of speaking, ”See movement and change through prejudice called movement function method.”



\begin{itemize}
\item[♠] 11.1 For completeness, again recall
\end{itemize}

\[
\begin{align*}
(X_1) & \quad \text{(Chap. 1)} \\
\quad \text{widely ordinary language} \quad \Rightarrow & \quad \text{world-description (Chap. 1(O))}
\end{align*}
\]

Out problem is

Where do we discuss the movement function method, \(\circ\), \(\Box\) or \(\square\)?

Usually, we may argue in \(\circ\) as elementary and junior high school students’ problem. However, if we persist in Position 3.5 of this book, naturally

Movement function method should be described in \(\square\).

11.1.2 Zeno’s paradox — Why is it paradox?—

As mentioned in problem 5.1, Zeno(BC490-BC430) considered the following problem about 2500 years ago.:  

Problem 11.1 [Problem 5.1 (same as the arrow which is flying (movement function method))]  

(B1) [Problem]: Is the flying arrow moving? Here, of course, ”flying arrow” is one symbol of movement and change such as ”running tortoise” ”the growth of rice”, ”the flying migratory birds”, ”the growth of economy of the country” and so on.

(B2) [Zeno’s answer] Assume that the arrow is flying. This arrow has stopped at the time of when at that moment. If it has stopped at the time of when at the moment, it will always have stopped, and the arrow will have stopped, therefore it will not move.

(B2) [The answer using movement function method] In every time \(t\), the position of the arrow becomes settled as a value of a time-position function \(q(t)\). However, we cannot conclude that time-position function \(x = q(t)\) is a constant function (i.e., function with a constant value). After all, we say ”The flying arrow which is moving.”

(B3) [Problem] Of course, the problem which Zeno raised is the next.
Which shall you choose between Zeno’s answer (B₁) and a common sense answer (B₂)? Or what is the basis as which you choose a common sense answer (B₂)? Furthermore, a time-position function method (= movement function method) will not be a physical law, and the empirical validation of it will also be impossible. Although that is right, why do you believe and use a time-position function method? What is the basis?

If it puts in another way,

(2) What is the best linguistic science language which includes a time-position function method (= movement function method)?

I will summarize the above and will write in the style of a fiction.

Zeno’s paradox

(C₁) About 2500 years ago, Zeno showed us “the perfect logic of the arrow which flies (B₁).”

(C₂) We refuted it by ”movement function method (A₂)” like (B₂), and answered with confidence ”It is an easy problem, Mr. Zeno.”

(C₃) At this time, Zeno brought forth a counterargument as follows immediately.

Movement function method is mystic words which cannot carry out empirical validation. Why do you believe and use such random and irresponsible movement function method? Is it science?

(C₄) We have to reply something to Zeno. And, of course, we have continued thinking earnestly 2500 years. However, we cannot yet answer.

Of course, we become silent in (C₄) because we are caught by the realistic method. Moreover, it is because we feel inferior in asserting dignifiedly the mathematics of the form where it was somehow buried into ordinary language - statistics and dynamical system theory -. Namely,

(C₅) It is because we cannot have the confidence to which we retort to Zeno ”We can answer easily in statistics and dynamical system theory (that is, as the easy case of an equation of state (1.1)).”

Probably, it is good although the price for which statistics and dynamical system theory have depended on two authority (Mathematics and Application) has turned.

However, in the position of this book which considers two classifications of world description (Chap. 1 (O)):

world-description (Chap. 1(O))

\begin{align*}
\text{1} & \text{ realistic method} \\
& \text{(world is before language)}
\end{align*}

\begin{align*}
\text{2} & \text{ linguistic method} \\
& \text{(language is before world)}
\end{align*}

it is as follows.
(C_0) We are not in the extravagant situation where we choose the logic of Zeno(B_2) and the common sense answer(B_1). That is, we know only a language called measurement theory as a language which describes science, and we cannot but describe by it. And then, what was described is "the world of the arrow which flies." (Chapter 8.1(m)).

Although I will answer this in Answer 11.11, some preparations of (Section 11.2–Section 11.4) are needed before that.

Note 11.2 Although Zeno’s paradox has some types elsewhere ("Achilles and a tortoise", "dichotomy", "stadium", etc.), "the arrow which flies" expresses the essence of the problem exactly and is the first masterpiece. However, since "Achilles and the tortoise" may be more famous, I will also describe this. The line of argument of Zeno about "the paradox of Achilles and a tortoise" is as follows. :

I consider competition of Achilles and a tortoise. Let the start point of a tortoise (a late runner) be the front from the starting point of Achilles (a quick runner). Suppose that both started simultaneously. If Achilles tries to pass a tortoise, Achilles has to go to the place in which a tortoise is present now. However, then, the tortoise should have gone ahead more. Achilles has to go to the place in which a tortoise is present now further. Even Achilles continues this infinite, he can never catch up with a tortoise.

Generally, you may suppose "Achilles and a tortoise are the problems of infinite geometrical progression." That is, the time-position function of Achilles and a tortoise is set to $x(t) = v t$ and $y(t) = \gamma vt + a$, respectively. ($0 < \gamma v < v, a > 0$). Here, you may think that you can solve by calculating the solution $s_0 = \frac{a}{(1-\gamma)v}$ to $q_1(s_0) = q_2(s_0)$ by the infinite geometrical progression

$s_0 = \frac{a}{v(1 + \gamma + \gamma^2 + \gamma^3 + ...)} = \frac{a}{(1 - \gamma)v}$.

However, if it is a problem finished now, they will say that the philosophers who have continued considering Zeno’s paradox 2500 years are foolish.

11.2 Causal operator, predual causal operator, deterministic causal map

Like Chap. 10, assume a state space $\Omega$ (i.e., locally compact space $\Omega$) and a measure space $(\Omega, \mathcal{B}_\Omega, \nu)$ satisfying the condition (a) in Sec.10.1.

Definition 11.2 [Causal operator, deterministic causal map] Let $[C(\Omega_1), L^\infty(\Omega_1, \nu_1)]$ and $[C(\Omega_2), L^\infty(\Omega_2, \nu_2)]$ be basic structures. A continuous linear operator $\Phi_{1,2}: L^\infty(\Omega_2, \nu_2) \to L^\infty(\Omega_1, \nu_1)$ is called a causal operator, if it satisfies the following (i) and (ii):

(i) for any $f_2(\in C(\Omega_2))$, there exists $\Phi_{1,2}f_2 \in C(\Omega_1)$, and further, the operator $\Phi_{1,2}: C(\Omega_2) \to C(\Omega_1)$ is the causal operator in the sense of Definition 6.2.

(ii) There exists a continuous linear operator $[\Phi_{1,2}]_*: L^1(\Omega_1, \nu_1) \to L^1(\Omega_2, \nu_2)$ such that

$\int_{\Omega_1} [\Phi_{1,2}]_*(\omega_1) \cdot \rho_1(\omega_1) \, \nu_1(d\omega_1) = \int_{\Omega_2} f_2(\omega_2) \cdot ([\Phi_{1,2}]_*\rho_1)(\omega_2) \, \nu_2(d\omega_2)$

$(\forall \rho_1 \in L^1(\Omega_1, \nu_1), \forall f_2 \in L^\infty(\Omega_2, \nu_2))$

This $[\Phi_{1,2}]_*$ is called the pre-dual causal operator.
In addition, the causal operator $\Phi_{1,2} : L^\infty(\Omega_2, \nu_2) \to L^\infty(\Omega_1, \nu_1)$ is called a **deterministic causal operator**, if there exists a continuous map $\phi_{1,2} : \Omega_1 \to \Omega_2$ such that

$$\Phi_{1,2}(f_2)(\omega_1) = f_2(\phi_{1,2}(\omega_1)) \quad (\text{a.e. } \omega_1, \forall f_2 \in L^\infty(\Omega_2))$$

Also, the continuous map $\phi_{1,2} : \Omega_1 \to \Omega_2$ is said to be a **deterministic causal map**.

**Theorem 11.3** [Causal operator and observable] For any observable $(X, \mathcal{F}, F_2)$ in $L^\infty(\Omega_2, \nu_2)$, the causal operator $\Phi_{1,2} : \Omega_1 \to \Omega_2$ is an observable in $L^\infty(\Omega_1, \nu_1)$, which is denoted by $\Phi_{12}O_2$.

**Proof.** It is easy to see that, for any countable decomposition $\{\Xi_j\}_{j=1}^\infty$ of $\Xi$, $(\Xi_j, \Xi \in \mathcal{F})$, it holds that

$$\int_{\Omega_1} \rho_1(\omega_1) \cdot [\Phi_{1,2}F_2(\Xi)](\omega_1) \nu_1(\omega_1) d\omega_1 = \int_{\Omega_2} [(\Phi_{1,2})_*\rho_1](\omega_2) \cdot [F_2(\Xi)](\omega_2) \nu_2(d\omega_2)$$

$$= \int_{\Omega_2} ([\Phi_{1,2}]_*\rho_1)(\omega_2) \cdot [F_2(\bigcup_{i=1}^\infty \Xi_i)](\omega_2) \nu_2(d\omega_2) = \lim_{N \to \infty} \int_{\Omega_2} ([\Phi_{1,2}]_*\rho_1)(\omega_2) \cdot \sum_{i=1}^N [F_2(\Xi_i)](\omega_2) \nu_2(d\omega_2)$$

$$= \lim_{N \to \infty} \int_{\Omega_1} \rho_1(\omega_1) \cdot \sum_{i=1}^N [\Phi_{1,2}F_2(\Xi_i)](\omega_1) \nu_1(\omega_1) d\omega_1$$

Thus, by Definition 10.1, we get the proof. \(\square\)

**Theorem 11.4** Any deterministic causal operator $\Phi_{1,2} : L^\infty(\Omega_2, \nu_2) \to L^\infty(\Omega_1, \nu_1)$ satisfies

$$\Phi_{1,2}(f_2) \cdot \Phi_{1,2}(g_2) = \Phi_{1,2}(f_2 \cdot g_2) \quad (\forall f_2, \forall g_2 \in L^\infty(\Omega_2, \nu_2))$$

**Proof.** The proof is the same as that of Theorem 6.5. We can omit it. \(\square\)

**Theorem 11.5** [Continuous map and deterministic causal map] Let $(\Omega_1, \mathcal{B}_{\Omega_1}, \nu_1)$ and $(\Omega_2, \mathcal{B}_{\Omega_2}, \nu_2)$ be measure spaces. Assume that a continuous map $\phi_{1,2} : \Omega_1 \to \Omega_2$ satisfies:

$$D_2 \in \mathcal{B}_{\Omega_2}, \nu_2(D_2) = 0 \implies \nu_1(\phi_{1,2}^{-1}(D_2)) = 0.$$

Then, the continuous map $\phi_{1,2} : \Omega_1 \to \Omega_2$ is deterministic, that is, the operator $\Phi_{1,2} : L^\infty(\Omega_2, \nu_2) \to L^\infty(\Omega_1, \nu_1)$ defined by (11.5) is a deterministic causal operator.

**Proof.** For each $p_1 \in L^1(\Omega_1, \nu_1)$, define a measure $\mu_2$ on $(\Omega_2, \mathcal{B}_{\Omega_2})$ such that

$$\mu_2(D_2) = \int_{\phi_{1,2}^{-1}(D_2)} p_1(\omega_1) \nu_1(d\omega_1) \quad (\forall D_2 \in \mathcal{B}_{\Omega_2})$$

Then, it suffices to consider the Radon-Nikodym derivative $[\Phi_{1,2}]_* (\nu_1) = d\mu_2/d\nu_2$. That is because

$$D_2 \in \mathcal{B}_{\Omega_2}, \nu_2(D_2) = 0 \implies \nu_1(\phi_{1,2}^{-1}(D_2)) = 0 \implies \mu_2(D_2) = 0$$

Thus, by the Radon-Nikodym theorem, we get a continuous linear operator $[\Phi_{1,2}]_* : L^1(\Omega_1, \nu_1) \to L^1(\Omega_2, \nu_2)$. \(\square\)
11.3 Bounded type Axiom$_b^m$ 2 (causality)

Let $(T, \leq)$ be a tree-like partial ordered set, i.e., a partial ordered set such that

\[ "t_1 \leq t_3 \text{ and } t_2 \leq t_3 \implies "t_1 \leq t_2 \text{ or } t_2 \leq t_1" \]

Put $T_2^2 = \{(t_1, t_2) \in T^2 : t_1 \leq t_2 \}$. An element $t_0 \in T$ is called a root if $t_0 \leq t \ (\forall t \in T)$ holds. If $T$ has the root $t_0$, we sometimes denote $T$ by $T(t_0)$. $T' (\subseteq T)$ is called lower bounded if there exists an element $t_i (\in T)$ such that $t_i \leq t \ (\forall t \in T')$. Therefore, if $T$ has the root, any $T' (\subseteq T)$ is lower bounded. We always assume that $T$ is complete, that is, for any $T' (\subseteq T)$ which is lower bounded, there exists an element $\text{Inf}_T(T') (\in T)$ that satisfies the following (i) and (ii):

(i) $\text{Inf}_T(T') \leq t \quad (\forall t \in T')$

(ii) If $s \leq t \ (\forall t \in T')$, then it holds that $s \leq \text{Inf}_T(T')$

In this book, we are not concerned with the topology (or metric) of the $T$.

**Definition 11.6 [Sequential causal operator, sequential observable]** A family $\Phi_{t_1,t_2} : L^\infty(\Omega, \nu) \to L^\infty(\Omega, \nu)$ is called a causal relation (or, causal relation), if it satisfies the following conditions (i) and (ii).

(i) With each $t \ (\in T)$, a basic algebra $L^\infty(\Omega, \nu)$ is associated.

(ii) For every $(t_1, t_2) \in T_2^2$, a causal operator $\Phi_{t_1,t_2} : L^\infty(\Omega, \nu) \to L^\infty(\Omega, \nu)$ is defined such that $\Phi_{t_1,t_2} \Phi_{t_2,t_3} = \Phi_{t_1,t_3}$ holds for all $(t_1, t_2), (t_2, t_3) \in T_2^2$.

Let an observable $O_t \equiv (X_t, \mathcal{F}_t, F_t)$ in $L^\infty(\Omega, \nu)$ be given for each $t \in T$. The pair $\{\{O_t\}_{t \in T}, \{\Phi_{t_1,t_2} : L^\infty(\Omega, \nu) \to L^\infty(\Omega, \nu)\}_{(t_1, t_2) \in T_2^2} \}$ is called a sequential observable which is denoted by $[O_T]$, i.e., $[O_T] = \{\{O_t\}_{t \in T}, \{\Phi_{t_1,t_2} : L^\infty(\Omega, \nu) \to L^\infty(\Omega, \nu)\}_{(t_1, t_2) \in T_2^2} \}$.

Let $(T(t_0), \leq)$ be a tree with the root $t_0$. For each $t \in T$, define the separable complete metric space $X_t$, and the Borel field $\mathcal{B}_X$, and further, define the observable $O_t=(X_t, \mathcal{F}_t, F_t)$ in $L^\infty(\Omega, \nu)$. That is, we have a sequential observable $[O_T(t_0)] = \{\{O_t\}_{t \in T}, \{\Phi_{t_1,t_2} : L^\infty(\Omega, \nu) \to L^\infty(\Omega, \nu)\}_{(t_1, t_2) \in T_2^2} \}$.

Here, define $\mathcal{P}_0(T) = \mathcal{P}_0(T(t_0)) \subseteq \mathcal{P}(T)$ such that

$\mathcal{P}_0(T(t_0)) = \{ T' \subseteq T \mid T' \text{ is finite, } t_0 \in T' \text{ and satisfies } \text{Inf}_T S = \text{Inf}_T S \ (\forall S \subseteq T') \}$

Let $T'(t_0) \in \mathcal{P}_0(T(t_0))$. Since $(T'(t_0), \leq)$ is finite, we can put $(T'=\{t_0, t_1, \ldots, t_N\}, \pi : T' \setminus \{t_0\} \to T')$, where $\pi$ is a parent map.
Now, consider the sequential observable \[ \{(O_t)_{t \in T}, \{\Phi_\pi(t), t : \mathcal{L}^\infty(\Omega_t, \nu_t) \to \mathcal{L}^\infty(\Omega_{\pi(t)}, \nu_{\pi(t)})\}_{t \in T \setminus \{t_0\}} \]. For each \( s (\in T') \), putting \( T_s = \{ t \in T' | t \geq s \} \), define the observable \( \hat{O}_s = (\times_{t \in T_s} X_t, \times_{t \in T_s} \mathcal{F}_t, \mathcal{F}_s) \) in \( \mathcal{L}^\infty(\Omega_t, \nu_t) \) such that
\[
\hat{O}_s = \begin{cases} O_s & (s \in T' \setminus \pi(T') \text{ and}) \\ 0_s \times (\times_{t \in \pi^{-1}(\pi(s))} \Phi_\pi(t), t \hat{O}_t) & (s \in \pi(T') \text{ and}) \end{cases}
\]

And further, iteratively, we get \( \hat{O}_{t_0} = (\times_{t \in T'} X_t, \times_{t \in T'} \mathcal{F}_t, \mathcal{F}_{t_0}) \), which is also denoted by \( \hat{O}_{T'} = (\times_{t \in T'} X_t, \times_{t \in T'} \mathcal{F}_t, \mathcal{F}_{T'}) \). For any subsets \( T_1 \subseteq T_2 (\subseteq T) \), define the natural projection \( \pi_{T_1, T_2} : \times_{t \in T_2} X_t \to \times_{t \in T_1} X_t \) by
\[
\times_{t \in T_2} X_t \ni (x_t)_{t \in T_2} \mapsto (x_t)_{t \in T_1} \in \times_{t \in T_1} X_t
\]
Assume that the observables \( \{ \hat{O}_{T'} = (\times_{t \in T'} X_t, \times_{t \in T'} \mathcal{F}_t, \mathcal{F}_{T'}) | T' \in \mathcal{F}_0(T) \} \) in \( \mathcal{L}^\infty(\Omega_{t_0}, \nu_{t_0}) \) satisfy the following consistency condition, that is,
for any \( T_1, T_2 (\in \mathcal{F}_0(T)) \) such that \( T_1 \subseteq T_2 \), it holds that
\[
\hat{F}_{T_2}(\pi_{T_1, T_2}^{-1}(\Xi_{T_1})) = \hat{F}_{T_1}(\Xi_{T_1}) \quad (\forall \Xi_{T_1} \in \times_{t \in T_1} \mathcal{F}_t)
\]
Then, by Theorem 10.13[ Kolmogorov extension theorem in measurement theory], there uniquely exists the observable \( \hat{O}_{T} = (\times_{t \in T} X_t, \boxtimes_{t \in T} \mathcal{F}_t, \mathcal{F}_{T}) \) in \( \mathcal{L}^\infty(\Omega_{t_0}, \nu_{t_0}) \) such that:
\[
\hat{F}_{T}(\pi_{T_0, T}^{-1}(\Xi_{T_0})) = \hat{F}_{T_0}(\Xi_{T_0}) \quad (\forall \Xi_{T_0} \in \boxtimes_{t \in T_0} \mathcal{F}_t, \forall T_0 \in \mathcal{F}_0(T))
\]
This observable \( \hat{O}_{T} = (\times_{t \in T} X_t, \boxtimes_{t \in T} \mathcal{F}_t, \mathcal{F}_{T}) \) is called the realization of the sequential observable
\[
[\hat{O}_{T(t_0)}] = \{(O_t)_{t \in T}, \{\Phi_{t_1, t_2} : \mathcal{L}^\infty(\Omega_{t_2}, \nu_{t_2}) \to \mathcal{L}^\infty(\Omega_{t_1}, \nu_{t_1})\}_{(t_1, t_2) \in T^2_S} \}
\]
Summing up the essential part of the above argument, we can propose the following axiom, which corresponds to \( \text{Axiom}_{\text{PM}}^\text{c} \) (Causality: page 109).

\textbf{Axiom}_{\text{PM}}^\text{b} \ (\text{causality : bounded type })

\begin{itemize}
  \item[(i)] A chain of causalities
  A chain of causalities is represented by a, sequential causal operator
  \[ \{\Phi_{t_1, t_2} : \mathcal{L}^\infty(\Omega_{t_2}, \nu_{t_2}) \to \mathcal{L}^\infty(\Omega_{t_1}, \nu_{t_1})\}_{(t_1, t_2) \in T^2_S} \]
  \end{itemize}

\begin{itemize}
  \item[(ii)] Realized causal observable
  A sequential observable
  \[ [O_{T(t_0)}] = \{(O_t)_{t \in T}, \{\Phi_{t_1, t_2} : \mathcal{L}^\infty(\Omega_{t_2}, \nu_{t_2}) \to \mathcal{L}^\infty(\Omega_{t_1}, \nu_{t_1})\}_{(t_1, t_2) \in T^2_S} \]
  is realized by its realized causal observable \( \hat{O}_{T} = (\times_{t \in T} X_t, \boxtimes_{t \in T} \mathcal{F}_t, \mathcal{F}_{T}) \)
Thus, we have the measurement theory (bounded-pure type) as follows.

\[
\text{measurement theory (bounded-pure type)} := \begin{cases} 
[\text{Axiom}^\text{PM}_b 1] & \text{(pure)measurement} \\
[\text{Axiom}^\text{PM}_b 2] & \text{causality} 
\end{cases}
\]

Therefore, we say that

The probability that a measured value \( (x_t)_{t \in T} \) obtained by a measurement \( M_{L^\infty(\Omega, \nu_t_0)}(\hat{O}_T = (X_{t \in T} X_t, \mathbb{X}_{t \in T} F_t, \hat{F}_T), S_{[\omega_0]} ) \) belongs to \( \hat{\Gamma} \) is given by \( [\hat{F}_T(\hat{\Gamma})][\omega_0] \), if \( \hat{F}_T(\hat{\Gamma}) \) is essentially continuous at \( \omega_0 \in (\Omega) \).

\[\text{Note 11.3}\] By an analogy of Remark 6.19, we also get the measurement theory (bounded-mixed type) as follows. That is,

\[
\text{measurement theory (bounded-mixed type)} := \begin{cases} 
[\text{Axiom}^\text{PM}_b 1] & \text{(mixed)measurement} \\
[\text{Axiom}^\text{PM}_b 2] & \text{causality} 
\end{cases}
\]

### 11.4 Is Brownian motion a motion?

It is natural to consider that

(A) Brownian motion should be understood in measurement theory.

Let us explain it as follows.

Let \( (\Lambda, \mathcal{F}_\Lambda, P) \) be a probability space. For each \( \lambda \in \Lambda \), define the real-valued continuous function \( B(\cdot, \lambda) : T = [0, \infty) \to \mathbb{R} \) such that, for any \( t_0 = 0 < t_1 < t_2 < \cdots < t_n \),

\[
P(\{ \lambda \in \Lambda | B(t_k, \lambda) \in \Xi_k \in B_{\mathbb{R}} \ (k = 1, 2, \ldots, n) \}) = \int_{\Xi_1} \cdots (\int_{\Xi_{n-1}} \cdots (\int_{\Xi_{n-1}} G_{t_n-t_{n-1}}^{\lambda}(\omega_k - \omega_{k-1})d\omega_k) \cdots d\omega_1) 
\]

where, \( \omega_0 \in \mathbb{R} \), \( d\omega_k \) is the Lebesgue measure on \( \mathbb{R} \), \( G_{t_n-t_0}(\omega) = \frac{1}{\sqrt{2\pi t}} \exp \left[ -\frac{\omega^2}{2t} \right] \).

Now consider the diffusion equation:

\[
\frac{\partial \rho_t(q)}{\partial t} = \frac{\partial^2 \rho_t(q)}{\partial q^2}, \quad (\forall q \in \mathbb{R}, \forall t \in T = [0, \infty) )
\]

By the solution \( \rho_t \), we get predual operator \( \{ [\Phi_{t_1, t_2}]_* : L^1(\mathbb{R}, dq) \to L^1(\mathbb{R}, dq) \} \) as follows. That is, for each \( \rho_{t_1} \in L^1(\mathbb{R}, m) \), define

\[
([\Phi_{t_1, t_2}]_*(\rho_{t_1}))(q) = \rho_{t_2}(q) = \int_{-\infty}^{\infty} \rho_{t_1}(y)G_{t_2-t_1}(q - y)m(dy) \quad (\forall q \in \mathbb{R}, \forall (t_1, t_2) \in T^2_{\Xi})
\]

For simplicity, we put \( (\Omega, \mathcal{B}, d\omega) = (\mathbb{R}^q, \mathcal{B}(\mathbb{R}^q), dq) \). And therefore, put \( (\mathcal{N}, \mathcal{N}_*) = (L^\infty(\Omega, d\omega), L^1(\Omega, d\omega)) \).

Putting \( \Phi_{t_1, t_2} = ([\Phi_{t_1, t_2}]_*)^* \), we get the causal relation \( \{ \Phi_{t_1, t_2} | (t_1, t_2) \in T^2_{\Xi} \} \). For each \( t \in T \), consider
the exact observable \( O^{(\text{exa})}_t = (\Omega, B, F^{(\text{exa})}) \) in \( L^\infty(\Omega, d\omega) \). Thus, we get the sequential causal exact observable \( [O_T] = \{(O^{(\text{exa})}_t | (t_1, t_2) \in T^2)\} \). The Kolmogorov extension theorem (Theorem 9.21) says that \( O_T \) has the realized causal observable \( \hat{O}_t = (\Omega_T, B(\Omega_T), \hat{F}_t) \) in \( L^\infty(\Omega, d\omega) \).

Assume that a measured value \( \hat{\omega} (= (\omega_t)_{t \in T} \in \Omega^T) \) is obtained by \( M_{L^\infty}(\Omega) \). Let \( D = \{t_1, t_2, \ldots, t_n\} \) be a finite subset of \( T \), where \( t_0 = 0 < t_1 < t_2 < \cdots < t_n \). Put \( \hat{\Xi} = \times_{t \in T} \Xi_t \) where \( \Xi_t = \Omega (\forall t \notin D) \). Then, by Axiom\(^{\text{pm}}\) 2, we see

\[
[\hat{F}_t(\times_{t \in T} \Xi_t)](\omega_0) = \left( F(\Xi_t)\Phi_{t_0, t_1} \left( F(\Xi_{t_1}) \cdots \Phi_{t_{n-2}, t_{n-1}} \left( F(\Xi_{t_{n-1}}) \Phi_{t_{n-1}, t_n} F(\Xi_{t_n}) \right) \right) \cdots \right)(\omega_0) = \int_{\Xi_1} \left( \cdots \left( \int_{\Xi_{n-1}} \left( \int_{\Xi_{t_n}} \times_{k=1}^n G_{t_k - t_{k-1}}(\omega_k - \omega_{k-1}) d\omega_n \right) d\omega_{n-1} \right) \cdots \right) d\omega_1 \tag{11.2}
\]

which is equal to the (11.1).

Thus, we say that

The Brownian motion \( B(t, \lambda) \) is not a motion but a measured value. ( Some may recall Parmenides' saying: There are no “plurality”, but only “one”. And therefore, there is no movement. )

\( \spadesuit \) Note 11.4 The above argument gives an answer to the problem (Chap. 1(F\(^5\))(=Note 1.1(\#2))) , i.e.,

Why is a mathematical theory (i.e., Brownian motion, stochastic process) useful?

That is,

Behind Brownian motion, the world-view (called classical measurement theory) is hidden

In this sense, Nelson’s probabilistic quantization may be the confusion of the order of things. Also, recall Note 2.17 as follows. Therefore, for example,

\[
\begin{array}{c|c}
\text{mathematics} & \text{world-description method} \\
\hline
\text{differential geometry} & \text{the theory of relativity} \\
\text{differential equation} & \text{Newton mechanics, electromagnetism} \\
\text{Hilbert space} & \text{quantum mechanics} \\
\text{probability theory (Hilbert space)} & \text{measurement theory} \\
\end{array}
\]

11.5 Exact measurement of deterministic sequential causal operator and the Schrödinger picture

The Copenhagen interpretation — Chap. 1(U\(^4\)) — says that ”only one measurement is permitted”, which implies ”State does not change”. 
However, as mentioned in Sec.6.4.4, as a convenient method, we sometimes use the state change due to the Schrödinger picture.

**Definition 11.7 [State change — the Schrödinger picture]** Let \( \{(\Phi_{t_1,t_2} : L^\infty(\Omega_{t_2},\nu_{t_2}) \to L^\infty(\Omega_{t_1},\nu_{t_1}))\}_{(t_1,t_2) \in T_2^S} \) be a deterministic causal relation with the deterministic causal maps \( \phi_{t_1,t_2} : \Omega_{t_1} \to \Omega_{t_2} \) (\( \forall (t_1,t_2) \in T_2^S \)). Let \( \omega_{t_0} \in \Omega_{t_0} \) be an initial state. Then, the \( \{\phi_{t_0,t}(\omega_{t_0})\}_{t \in T} \) or \( \{\delta_{\phi_{t_0,t}(\omega_{t_0})}\}_{t \in T} \) is called the Schrödinger picture representation.

The following is similar to Theorem 6.18

**Theorem 11.8 [Deterministic sequential causal operator realized causal observable]** Let \( (T(t_0), \leq) \) be a tree with the root \( t_0 \). Let \( |O_T| = \{|O_t|_{t \in T}, \{\Phi_{t_1,t_2} : L^\infty(\Omega_{t_2},\nu_{t_2}) \to L^\infty(\Omega_{t_1},\nu_{t_1})\}_{(t_1,t_2) \in T_2^S} \} \) be a deterministic sequential observable. Then, the realization \( \hat{O}_{t_0} \equiv (X_{t \in T} X_{t}, \bigotimes_{t \in T} F_{t}, \hat{F}_{t_0}) \) is represented by

\[
\hat{O}_{t_0} = \bigotimes_{t \in T} \Phi_{t_0,t} O_t
\]

That is, it holds that

\[
[\hat{F}_{t_0}(\bigotimes_{t \in T} \Xi_t)](\omega_{t_0}) = \bigotimes_{t \in T} [\Phi_{t_0,t} F_t(\Xi_t)](\omega_{t_0}) = \bigotimes_{t \in T} [F_t(\Xi_t)][\phi_{t_0,t}(\omega_{t_0})]
\]

(\( \forall \omega_{t_0} \in \Omega_{t_0}, \forall \Xi_t \in \mathcal{F}_t \))

**Proof.** The proof is similar to that of Theorem 6.18.

\[
\square
\]

**Theorem 11.9** Let \( |O_{T(t_0)}| = \{|O_t^{(exa)}|_{t \in T}, \{\Phi_{t_1,t_2} : L^\infty(\Omega_{t_2},\nu_{t_2}) \to L^\infty(\Omega_{t_1},\nu_{t_1})\}_{(t_1,t_2) \in T_2^S} \} \) be a deterministic sequential exact observable, which has the deterministic causal maps \( \phi_{t_1,t_2} : \Omega_{t_1} \to \Omega_{t_2} \) (\( \forall (t_1,t_2) \in T_2^S \)). And let \( \hat{O}_{t_0} = (X_{t \in T} X_{t}, X_{t \in T} \mathcal{F}_{t}, \hat{F}_{t_0}) \) be its realized causal observable in \( L^\infty(\Omega_{t_0},\nu_{t_0}) \). Assume that the measured value \( (x_t)_{t \in T} \) is obtained by \( \bigotimes_{t \in T} L^\infty(\Omega_{t_0},\nu_{t_0})(\hat{O}_{T} = (X_{t \in T} X_{t}, X_{t \in T} \mathcal{F}_{t}, \hat{F}_{t_0}), \mathcal{S}_{[\omega_{t_0}]}) \).

Then, we surely believe that

\[
x_t = \phi_{t_0,t}(\omega_{t_0}) \quad (\forall t \in T)
\]

Thus, we say that, as far as a deterministic sequential observable,

exact measured value \( (x_t)_{t \in T} = \) the Schrödinger picture representation \( (\phi_{t_0,t}(\omega_{t_0}))_{t \in T} \)

**Proof.** Let \( D = \{t_1, t_2, \ldots, t_n\} (\subseteq T) \) be any finite subset of \( T \). Put \( \hat{X} = \bigotimes_{t \in T \setminus D} X_t = (X_{t \in D} \Xi_t) \times (X_{t \in T \setminus D} X_t) \), where \( \Xi_t \subseteq X_t (= \Omega_t) \) is an open set such that \( \phi_{t_0,t}(\omega_{t_0}) \in \Xi_t \) (\( \forall t \in D \)). Then, we see that

the probability that the measured value \( (x_t)_{t \in T} \) belongs to \( \hat{X} = \bigotimes_{t \in T} \Xi_t \) is equal to 1.

That is because Theorem 11.8 says that

\[
(\hat{F}_{T}(\hat{X}))(\omega_{t_0}) = \left( \bigotimes_{k=1}^n (\Phi_{t_0,t_k}^{(exa)} F_{t_k}(\Xi_{t_k})) \right)(\omega_{t_0})
\]

\[
= \left( \bigotimes_{k=1}^n F_{t_k}(\phi_{t_0,t_k}^{-1}(\Xi_{t_k})) \right)(\omega_{t_0}) = \bigotimes_{k=1}^n \chi_{\Xi_{t_k}}(\phi_{t_0,t_k}(\omega_{t_0})) = 1
\]
Thus, from the arbitrariness of $\Xi_t$, we surely believe that

$$(c) \ (x_t)_{t \in T} = \phi_{t_0,t}(\omega_{t_0}) \quad (\forall t \in T)$$

$\sqcup$ **Note 11.5** Note that "(b) $\Leftrightarrow (c)$" in the above. That is, (b) is the definition of (c).

The following is a consequence of Theorem 10.12 and Theorem 11.9.

**Corollary 11.10** [Quantity and exact observable]. For each $t \in T(t_0)$, consider the exact observable $O_t^{(exa)} = (X,F_t,F_t^{(exa)}) = (\Omega_t, B_t, \chi)$ in $L^\infty(\Omega_t, \nu_t)$ and a quantity $g_{t} : \Omega_t \to \mathbb{R}$ on $\Omega_t$. Let $O_t' = (\mathbb{R}, B_{R}, G_{t})$ be the observable representation of the quantity $g_{t}$. Assuming the simultaneous observable $O_t^{(exa)} \times O_t'$, define the sequential observable $[O_{T(t_0)}] = \{O_t^{(exa)} \times O_t' \}_{t \in T}, \{\Phi_{t_1,t_2} : L^\infty(\Omega_{t_2}, \nu_{t_2}) \to L^\infty(\Omega_{t_1}, \nu_{t_1})\}_{(t_1,t_2) \in T_2^2}$. Let $\phi_{t_1,t_2} : \Omega_{t_1} \to \Omega_{t_2} (\forall (t_1, t_2) \in T_2^2)$ be the deterministic causal map. Let $\hat{O}_{t_0} = (\times_{t \in T}(X_t \times \mathbb{R}), \times_{t \in T}(F_t \times B_{R}), \hat{F}_{t_0})$ be the realized causal observable. Thus, we have the measurement $\hat{M}_{L^\infty(\Omega_{t_0})}(\hat{O}_{t_0}, S_{[\omega_{t_0}]})$. Let $(x_t,y_t)_{t \in T}$ be the measured value obtained by the measurement $\hat{M}_{L^\infty(\Omega_{t_0})}(\hat{O}_{t_0}, S_{[\omega_{t_0}]})$. Then, we can surely believe that

$$x_t = \phi_{t_0,t}(\omega_{t_0}) \quad y_t = g_{t}(\phi_{t_0,t}(\omega_{t_0})) \quad (\forall t \in T)$$

$\sqcup$ **Note 11.6** As mentioned in Note 1.7, "linguistic monism" is not yet established, or it may not exist. If it exists, it may have the following form:

$$[\text{Sec.1.2.2}] + \text{causality} : \{\phi_{t_1,t_2}\}_{t_1 \leq t_2}$$

However, it may be regarded as the abbreviation of measurement theory. That is, the (2) is absorbed into measurement theory.

Before reading Answer 11.11 (Zeno's paradox(flying arrow)), confirm Standing point 3.5 in Chap. 3. That is,

(d) The theory described in ordinary language should be described in measurement theory. That is because almost ambiguous problems are due to the lack of "world-view".

the fact that

**Answer11.11** [Answer to Problem 11.1: Zeno's paradox(flying arrow)] Let us answer to Problem 11.1(flying arrow)(cf. [18]). As mentioned in Sec.11.1(C6), There is only the method of describing by measurement theory. Therefore, in Corollary 11.10, putting

$$q(t) = g_t(\phi_{t_0,t}(\omega_{t_0}))$$
we get the time-position function \( q(t) \). Thus, it suffices to discuss Problem 11.1(B\(^2\)) by the time-position function.

(c) If Zeno asks "Why do you use measurement theory?", it suffices to answer "We have only measurement theory".

\[ \square \]

\[ \blacklozenge \] **Note 11.7** Thus, we add "Zeno's paradoxes" to Sec.9.3(a), as follows.

\[ \blacklozenge \] **Note 11.8** In quantum measurement theory, the time-position function does not exist (that is, the trajectory of a particle is meaningless. For the further argument, see [21].

**Example 11.12 [Newtonian mechanics in measurement theory]** Let \( T = \mathbb{R} \) be the time axis.

Newtonian equation (9.1) on the state space \( \Omega \) determines the continuous map \( \phi_{t_1,t_2} : \Omega_{t_1}(= \Omega) \rightarrow \Omega_{t_2}(= \Omega) \) \((t_1 \leq t_2)\). The formula (9.2) says that there exists the measure \( \nu(= \nu_t) \) on the state space \( \Omega(= \Omega_t) \) such that

\[
\nu(D_{t_1}) = \nu(\phi^{-1}_{t_1,t_2}(D_{t_1})) \quad (\forall D_{t_1} \in \mathcal{B}_{\Omega_{t_1}}) \quad (11.3)
\]

Therefore, by Theorem 11.5, we get a sequential deterministic causal map \( \{\phi_{t_1,t_2} : \Omega_{t_1}(= \Omega) \rightarrow \Omega_{t_2}(= \Omega)\}_{t_1 \leq t_2} \). Thus, in Newtonian mechanics, Theorem 11.9 says that

the exact measured value sequence = state change due to the Schrödinger picture

Thus we say, from the measurement theoretical point of view, that Newtonian equation does not represent the motion but the time series of exact measured values. In this sense, Newtonian mechanics can be regarded as one of various sciences and not physics.

\[ \blacklozenge \] **Note 11.9** In the above argument, Newtonian equation has two aspects as That is, motion

\[
\begin{cases}
\text{①} : \text{Newtonian equation in physics} & \cdots \text{state change} \\
\text{②} : \text{Newtonian equation in measurement theory} & \cdots \text{exact measured value sequence}
\end{cases}
\]

**Example 11.13 [1 + 1 = 2?]** It is a famous anecdote that, when Thomas Edison, the greatest-ever inventor, was a school child, he made the question: “Why does \( 1 + 1 = 2 \) hold?” on his teacher, and made the teacher embarrassed. Although we do not know his real intention, we consider, from the measurement theoretical point of view, that this question is not so trivial. Consider the following ① – ③:
(1): What is 1 plus 1? Is 1 + 1 = 2 true?

(2): Assume that a particle $A$ with the mass 1 kg and a particle $B$ with the mass 1 kg are combined. Then, how weight the combined particle (i.e., $A+B$)?

(3): Assume that the exact measured value of the mass of a particle $A$ is 1 kg and the exact measured value of the mass of a particle $B$ is 1 kg. Then, the exact measured value of the mass of the combined particle (i.e., $A+B$) is equal to 2 kg. Is it true?

**Answer:** From the mathematical point of view, the equality in (1) is merely a mathematical rule. In physics, the (2) is just the law of conservation of mass. Thus we focus on the measurement theoretical aspect (3), in which the equality is not trivial. The proof is as follows.

[The proof of (3)]: Consider the Lebesgue measure space $(\Omega_1, \mathcal{B}_{\Omega_1}, \nu_1) = (\mathbb{R}, \mathcal{B}_{\mathbb{R}}, m)$ and its product measure space $(\Omega_0, \mathcal{B}_{\Omega_0}, \nu_0) = (\mathbb{R}^2, \mathcal{B}_{\mathbb{R}^2}, m^2)$. Let $\Omega_0$ and $\Omega_1$ be state spaces. Put

$$(\text{the mass of a particle } A, \text{ the mass of a particle } A) \in \Omega_0(=\mathbb{R} \times \mathbb{R})$$

(for simplicity, assume that negative mass is possible). By the law of conservation of mass, define the continuous map $\phi_{0,1}: \Omega_0 \to \Omega_1$ by $\phi_{0,1}(\alpha, \beta) = \alpha + \beta$ $(\forall (\alpha, \beta) \in \mathbb{R} \times \mathbb{R})$. The following is clear:

$$D_1 \in \mathcal{B}_{\Omega_1}(=\mathcal{B}_{\mathbb{R}}), \ m(D_1) = 0 \implies m^2(\phi_{0,1}^{-1}(D_1)) = 0$$

Thus, Theorem 11.5 says that the continuous map $\phi_{0,1}: \Omega_0 \to \Omega_1$ is a deterministic causal map. And thus, the deterministic causal operator $\Phi_{0,1} : L^\infty(\Omega_1, \nu_1) \to L^\infty(\Omega_0, \nu_0)$ is defined by

$$[\Phi_{0,1}(f_1)](\omega_0) = f_1(\phi_{0,1}(\omega_0)) \quad (\forall f_1 \in L^\infty(\Omega_1, \nu_1), \ a.e. \ \omega_0)$$

Consider the exact observable $O_0^{(exa)}$ in $L^\infty(\Omega_0, \nu_0)$ and the exact observable $O_1^{(exa)}$ in $L^\infty(\Omega_1, \nu_1)$. And consider the realized causal observable $\tilde{O}_0 = O_0^{(exa)} \times \Phi_{0,1}O_1^{(exa)}$. Taking the measurement $M_{L^\infty(\Omega_0, \nu_0)}(O_1^{(exa)}) \times \Phi_{0,1}O_1^{(exa)}$, we obtain a measured value $((\alpha_0, \beta_0), \gamma_0)$. Then, by Theorem 11.9, we see, with probability 1,

$$\gamma_0 = \alpha + \beta$$

Therefore, we see that $\alpha_0 + \beta_0 = \gamma_0$. □
12 Realistic world-view and Linguistic world-view

This chapter describes the conclusion of measurement theory in there comparison of mathematics, physics, and various sciences(engineering).

12.1 Mathematics, Physics, Various Sciences·Engineering

12.1.1 Language

If the ecology of various animals is observed, it will be clear that the base of language was due to intimidation · solidarity · reproduction. Language was one of the strongest arms for the survival and breeding. Such a time have continued for millions of years. Of course, the greatest incidents happened in "the history of language", for example, "a rhythm and a song", "logical structure", a "quantity concept", "grammar", "tense", a "character", etc. However, it was too long years ago, we cannot specify the contribution person's name.

12.1.2 Mathematics - The language independent of the world.

When human beings began to have some confidence in "survival and breeding", people who get interested in the "quantity concept" in ordinary language unusually have appeared. Probably, it is good also considering the pioneer as Pythagoras — everything is a number. This flow was inherited to Archimedes (BC287 – BC 212 years), Euler (1707 – 1783), and a gauss (1777 - 1855), and the "quantitative portion" in ordinary language was strengthened rapidly. That is, the mathematical achievement was accumulated rapidly. However, language makes survival and breeding the origin as mentioned above,

(a) Ordinary language was not devised in order to tell mathematics.

It will be Cantor (1845 year–1918 year) that noticed, if it considers from now on. Therefore, Cantor made the special language - set theory (that is, "the mathematical origin is set") - for describing mathematics. Of course, this was inherited to Hilbert (1862 – 1943) or Godel (1906 – 1978), and brought about the prosperity of modern mathematics. However, it should be careful that the axiomatization of mathematics clarifies that mathematics is independent of our real world, as symbolized by Hilbert's words "a point may be a chair and a line may be a desk". Though that is right, the custom to consider that statistics and dynamical system is a part of ordinary language continues to go out of use as a convenient and easy way.

12.1.3 Physics - realistic science view (the world is before language)

If a time is traced back, although Demokritos (around BC370 - BC460 ) — The origin of everything is an atom — may be famous, I will start with modern science, for example. Of course, work of Galileo (1564 –
1642) and Kepler (1571 year–1630 year) is admired. However, language makes survival and breeding the origin,

(b) Ordinary language was not devised in order to tell physical phenomena.

Newton (1642 – 1727) found out it. Therefore, Newton made the languages of special — Newtonian mechanics — for describing a classical mechanics phenomenon. This was inherited to Maxwell (1831 – 1879), Einstein (1879 – 1955), etc., and brought about a great success of the realistic science view. And imprinting, such as ”realistic science view = science”, has become common by the great success.

12.1.4 Various sciences (engineering) - linguistic science view (language is before world.)

When you understand various matters in the world, it is always a given occasion, and if thought individually, naturally you think that it is troublesome. Therefore, the plan to consider is decided previously, and when various matters are faced, I would like to come to depend on the method of considering along with the plan. That is, in order to have understood various matters in the world, when there was ”form of common thinking”, many philosophers must have believed and investigated. The pioneer is Plato (BC 427 – BC 347) - Idealism -, and he advocated the prototype of dualism and idealism. This was inherited to Descartes (1596 – 1650) and Kant (1724 – 1804), and the spirit of ”language is before the world”. - Linguistic science view - was further established very much by linguistic philosophy. It is as having seen in Chapter 8 that these formed the philosophical (= world description) main stream.

However, in respect of the technology of an understanding of various matters in the world, probably, we have to stress the importance of ”the method of the classical mechanics world view”. Although we can not say the name of the founders, Fischer (1890 – 1962), Robert Wiener (1894 – 1964), Kolmogorov (1903 – 1987 year 1), Kalman (1930 –), etc. should be mentioned especially. The two (i.e., ”the form of thinking(philosophy)” and ”the classical mechanics world view (technology)”) have been independent. From this unfortunate fact, various sciences were classified into the category of ”the weak thing” in Section 2.4.1. The reason which has fallen into such a situation is reasonable for thinking that it originates in

(c) Ordinary language was not devised in order to tell usual scientific phenomena

Although I do not know whether von Neumann (1903 – 1957) was conscious of this (c), as a result, von Neumann proposed the quantum measurement theory — at the meaning of Section 9.3, dualism idealism — based on Copenhagen interpretation in ”the mathematical basis of quantum mechanics [32]” And von Neumann’s work was inherited and the languages of special make for describing the scientific phenomena of an everyday scale - Measurement theory - were proposed in this book.
Two things unexpected as follows happened to Measurement theory.

(d₁) Suppose that the method of "considering along with the plan when the plan considered previously is decided and various matters are faced" as mentioned above was adopted. Since various sciences are various, it is a matter of course that even if such a plan exists, it is an non-quantitative plan and the number of the plans increases considerably moreover. However, the plan of Measurement theory was only two quantitative plans (i.e., Axioms 1 and 2).

(d₂) The family line of Measurement theory was quantum mechanics. And measurement theory has been saddled with "two kinds of absurd character" as stated repeatedly. Namely,

(♯) "Absurd character" of Measurement Theory
- Idealism ... Linguistic science view
- Dualism ... Copenhagen interpretation (i.e., dualism)

That such measurement theory is materialized is a mystery which is hard to believe. However, since this is a proposal of this book, the elucidation of this mystery must be left to readers as "homework to readers." The author does not know the thing beyond having stated with Section 9.3.

12.1.5 Mathematics, Physics, Various sciences (engineering)

The above is summarized and the next is obtained. (Note 2.10, Note 9.6).

(e) \begin{align*}
\{ & \begin{align*}
\circ & : \text{Mathematical language} \cdots \text{Set theory} \\
& \quad \quad \text{(The proposal and solution of mathematical outstanding problems )}
\circ & : \text{languages of physics} \cdots \text{Newtonian mechanics, electromagnetism, ...} \\
& \quad \quad \text{(Realistic science view: Theory which explains what God made )}
\circ & : \text{language of engineering} \cdots \text{Measurement theory} \\
& \quad \quad \text{(Linguistic science view: The language for making about the same robot as a scientist )}
\end{align*}
\end{align*}

About (e) and (♯), following some will need to keep in mind that there is probably no complaint about physics(♯). Even if it does not know set theory etc., mathematical outstanding problems may be solved, and even if it does not know Measurement theory (for example, set theory does not seem to have been indispensable in order to solve the four colors problem), "about the same robot as a scientist" may be made(Note 2.10). Even if it says so,

If the foundation is established with languages of special make, "rapid development" should become possible and each language (set theory and measurement theory) of special make must be required also for (e) and (♯).
13 Conclusions

It is a matter of course, ordinary language is the greatest invention of mankind. However, science cannot be ripened only with ordinary language. For that purpose, the world describing method is indispensable. And, it is always continuing evolving and developing toward the direction of "saving of thinking." Supposing that is right, I claimed "evolution and development of world description" of the following figure (= Fig. 8.2).

Figure 13.1: The development of the world-descriptions (= Fig. 8.2)

If this is summarized (i.e., if the "beginning" and the "last" are written), we say that

(X1) (Chap. 1) widely ordinary language
(Chap. 1) before science

⇒ world-description
(Chap. 1(O))

(1) realistic scientific language (the theory of everything)
(Theory which explains what God made)

(2) linguistic scientific language (measurement theory)
(The language for making about the same robot as a scientist)

Although there was 3000 years of history in world description, a major event did not necessarily break out frequently. It has occurred only about at most 10 times, and, as far as about the linguistic describing method, the major event (discovery of = mystic words) has occurred only 5 times. Namely,

(B) motion-change, causality (1), trial, measurement, causality (2).
Even if we have to add the gradually ripe process of dualistic idealism (Sec. 8.1), it can be said that it was quite peaceful history. If Fig. 13.1 is followed, we would like to come to claim a scenario called

(C) the happy end of a big tale

in the sense that both the linguistic world-view and the realistic world-view are compatible.

Of course, the above is only one scenario. In order to make this scenario steadfast (or it overthrows), I would like to come to pursue the following problem:

(D) Why does the metaphysics of Measurement theory hold?

The answers were given to various miscellaneous mistily problems in this book. For example,

(E) [What is space (or, time, causality, probability ?)], [Heisenberg’s uncertainty relation],
    (Sec. 2.3.3, Note 6.2) [Theorem 3.4]

    [syllogism], [the principle of equal weight],
    (Theorem 5.11) (Theorem 6.21)

    [equilibrium statistical mechanics], [1+1=2], [Zeno’s paradoxes]
    (Chap. 9) (Chap. 11) (Chap. 11)

were clarified.

(F) The answer to the question "Why can the problems in (E) be solved?" is clear. That is because these problems are the same problem, i.e., the problem "Propose the language for describing these problems!".

Thus, the problems in (E) are easy exercises in measurement theory. Though that was right, possibly readers had the following comment.

(G) The opinion of this book - establishment of a linguistic science view - was understood once. However, what the author did is that a variety of miscellaneous mistily problems only come back to the one biggest mistily problem such as

Is it possible that metaphysics is introduced as a base of various sciences?

However, it is the same as that of Newtonian mechanics, electromagnetism, or the theory of relativity, that is, we believe that

(H) A scientific theory is returning various miscellaneous mysteries to one big mystery.

Since mathematics and physics are the learning of God (namely, learning common to whole creation people), the reason for the formation is substituted for a word of a "miracle", and the rest may be what
to leave to the high alien. However, measurement theory is man’s learning (namely, learning depending on man’s recognition and linguistic competence), so man may reply to (D).

Although it was above, possibly the author’s interest (namely, about (D)) was written too much. What was necessary was to have said only the following things, supposing that was right.

(I) Measurement theory was not made for ”the happy end (C) of a big tale” (or for the rehabilitation of the Descartes-Kant philosophy). If an author’s spirit is written honestly,

In the time of the engineering which will continue four hundreds years from now on, in order to defeat the battle which risked survival of human beings (Sec. 2.4.2(e), (f)), measurement theory was made as the scripture of engineering and science. (Note 9.7)

I solved outstanding and ambiguous problems (E) since I wanted to have the courage to add this spirit to this final chapter.

If that is right, what should be performed now is the next.

(J) Under the belief (F), in a language called measurement theory, phenomena are described rapidly and engineering and science are developed rapidly.

Since measurement theory is a language,

Measurement theory is valueless if not used.

References

[1] Bohr, N. Can quantum-mechanical description of physical reality be considered complete?, Phys. Rev. (48) 696–702 1935

[2] Born, M. Zur Quantenmechanik der Stoßprozesse (Vorläufige Mitteilung), Z. Phys. (37) 863–867 1926

[3] E. B. Davies, “Quantum Theory of Open Systems,” Academic Press, 1976.

[4] Einstein, A., Podolsky, B. and Rosen, N. Can quantum-mechanical description of reality be considered completely? Physical Review Ser 2(47) 777–780 (1935)

[5] Heisenberg, W. Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik, Z. Phys. 43, 172–198 (1927)

[6] Hoffman, P. The Man Who Loved Only Numbers, The story of Paul Erdös and the search for mathematical truth, Hyperion, New York 1998

[7] S. Ishikawa, “Uncertainty relation in simultaneous measurements for arbitrary observables,” Rep. Math. Phys., Vol. 29, No. 3, pp. 257-273, 1991

[8] S. Ishikawa, “Fuzzy Inferences by Algebraic Method,” Fuzzy Sets and Systems, Vol. 87, No. 2, 1997, pp.181-200. doi: 10.1016/S0165-0114(96)00035-8

[9] S. Ishikawa, “A Quantum Mechanical Approach to Fuzzy Theory,” Fuzzy Sets and Systems, Vol. 90, No. 3, 1997, pp. 277-306. doi: 10.1016/S0165-0114(96)00114-5
[10] S. Ishikawa, “Statistics in measurements,” *Fuzzy sets and systems*, Vol. 116, No. 2, 141-154 (2000). doi:10.1016/S0165-0114(98)00280-2

[11] S. Ishikawa, “Mathematical Foundations of Measurement Theory,” Keio University Press Inc. 335 pages, 2006. [http://www.keio-up.co.jp/kup/mfomt/]

[12] S. Ishikawa, “Fisher’s Method, Bayesf Method and Kalman Filter in Measurement Theory,” *Far East Journal of Theoretical Statistics*, Vol. 29, No. 1, 9-23, 2009

[13] S. Ishikawa, “A New Interpretation of Quantum Mechanics,” *Journal of quantum information science*, Vol. 1, No. 2, 2011, pp.35-42. doi: 10.4236/jqis.2011.12005

[14] S. Ishikawa, “Quantum Mechanics and the Philosophy of Language: Reconsideration of Traditional Philosophies,” *Journal of quantum information science*, Vol. 2, No. 1, 2012, pp.2-9. doi: 10.4236/jqis.2012.21002

[15] S. Ishikawa, “A Measurement Theoretical Foundation of Statistics,” *Applied Mathematics*, Vol. 3, No. 3, 2012, pp. 283-292. doi: 10.4236/am.2012.33044

[16] S. Ishikawa, “The Linguistic Interpretation of Quantum Mechanics,” arXiv:1204.3892v1[physics.hist-ph], 2012.

[17] S. Ishikawa, “Ergodic Hypothesis and Equilibrium Statistical Mechanics in the Quantum Mechanical World View,” *World Journal of Mechanics*, Vol. 2, No. 2, 2012, pp. 125-130. doi:10.4236/wjm.2012.22014

[18] S. Ishikawa, “Zeno’s paradoxes in the Mechanical World View,” arXiv:1205.1290v1 [physics.hist-ph], 2012

[19] S. Ishikawa, “Monty Hall Problem and the Principle of Equal Probability in Measurement Theory,” *Applied Mathematics*, Vol. 3, No. 7, 2012, pp. 788-794. doi:10.4236/am.2012.37117

[20] S. Ishikawa, “What is statistics?; The Answer by Quantum Language,” arXiv:1207.0407v1 [physics.data-an], 2012

[21] Ishikawa, S., Arai, T. and Kawai, T. *Numerical Analysis of trajectories of a quantum particle in two-slit experiment*, Internat. J. Theoret. Phys. 33 1265-1274 (1994)

[22] Ishikawa, S., Arai, T. and Takamura, T. *A dynamical system theoretical approach to Newtonian mechanics*, Far east journal of dynamical system 1, 1-34 (1999)

[23] Ishikawa, S., Kikuchi, K. and Nakamura, M. *Elementary school mathematics in quantitive language*, Far east journal of dynamical system 2 (2), 165-180 (2008)

[24] I. Kant, *Critique of Pure Reason ( Edited by P. Guyer, A. W. Wood ),* Cambridge University Press, 1999

[25] Kikuchi, K., Ishikawa, S. *Psychological tests in measurement theory*, Far east journal of theoretical statistics 32(1), 81–99 (2010)

[26] Kolmogorov, A. *Foundations of probability ( translation ),* Chelsea Publishing Co. 1950

[27] J. M. E. McTaggart, *The Unreality of Time*, Mind (A Quarterly Review of Psychology and Philosophy), Vol. 17, 457-474, 1908

[28] Ruelle, D. *Statistical mechanics, rigorous results*, W.A. Benjamin 1969

[29] Sakai, S. *C*-algebras and W*-algebras*, Ergebnisse der Mathematik und ihrer GreSpringer-Verlag, (1971)

[30] Selleri, F. *Die Debatte um die Quantentheorie,,* Friedr. Vieweg&Sohn Verlagsgesellschaft MBH, Braunschweig (1983)

[31] Shannon, C.E., Weaver. W *A mathematical theory of communication*, Bell Syst. Tech.J. 27 379–423, 623–656, (1948)

[32] von Neumann, J. *Mathematical foundations of quantum mechanics* Springer Verlag, Berlin (1932)

[33] K. Yosida, “Functional Analysis,” Springer-Verlag, 6th edition, 1980.