OSCILLATING SHELLS: A MODEL FOR A VARIABLE COSMIC OBJECT

DARIO NÚÑEZ
Instituto de Ciencias Nucleares, UNAM, Circuito Exterior CU, A.P. 70-543, México, D. F. 04510, México; nunez@nuclecu.unam.mx

Received 1996 October 21; accepted 1997 January 13

ABSTRACT

A model for a possible variable cosmic object is presented. The model consists of a massive shell surrounding a compact object. The gravitational and self-gravitational forces tend to collapse the shell, but the internal tangential stresses oppose the collapse. The combined action of the two types of forces is studied, and several cases are presented. In particular, we investigate the spherically symmetric case, in which the shell oscillates radially around a central compact object.

Subject headings: circumstellar matter — dense matter — stars: oscillations — supernovae: general

1. INTRODUCTION

Since the pioneering works of Ostriker & Gunn (1977), the study of the motion of thin clouds surrounding an exploding star has been the basis for understanding the dynamic process occurring after a supernova explosion, the motion of a blast wave (Sato 1993), and several other related phenomena. These studies were done in Newtonian theory, in which an equation of motion derived from Newton’s second law was used, introducing the forces that act on the shell. For the case of a supernova, we have as the interacting forces the dimensional force, the force due to the radiation pressure emitted by the star, and a friction-type force due to the interaction of the shell with the interstellar medium. The deduction is clear and straightforward, and the picture obtained is quite useful and has a good degree of accuracy for several cases in describing the main processes occurring during the motion.

On the other hand, in general relativity, the relativistic version of the equation of motion of a shell has its origins in studies of the conditions needed to join two regions of spacetime, the so-called matching conditions. Initially, the Lichnerowicz matching conditions (Lichnerowicz 1955) gave conditions on the boundary between two regions of the spacetime, where the boundary was just the end of one region and the beginning of the next. Israel (1966, 1967) gave a different interpretation to this boundary, allowing it to be a thin shell of matter; thus, the two regions of the spacetime are matched on this boundary, the thin shell, obtaining new matching conditions, for which, when the shell parameters tend to zero, the Lichnerowicz ones are recovered. Israel’s matching conditions for two spaces can be interpreted as the equations that govern the motion of the shell in order to separate consistently the two given spacetimes, that is, can be read as the equations of motion for the shell, which have had very different applications, such as allowing the study of collisions of shells (Núñez, de Oliveira, & Salim 1993). Recently, Núñez & Oliveira (1996) explicitly obtained the general relativistic version of the Newtonian equation of motion. Thus, both theories are now on the same footing for analyzing the motion of a shell.

Even though the relativistic analysis demands for its formulation a more complicated mathematical machinery, and the results often are negligible corrections to the Newtonian description, the theory has several conceptual advantages and applications that make it worth considering as an alternative approach to analyzing the motion of shells. It is definitely necessary to use it in cases in which the gravitational fields are strong, with high values of the density compared to the pressure, or when the velocities are close to the velocity of light. With respect to the motion of the shell, it is necessary to use the relativistic formalism when the shell surrounds a black hole or when the shell itself is very dense and moves at high speeds. The model presented in this work belongs to this last case. Moreover, using the fact that in general relativity the explicit equations relating the geometrical quantities (the “forces” of the Newtonian theory) with the matter energy distribution in the space are given (the Einstein equations!), the set of equations of motion can be reduced to a set of first-order differential equations; that is, a first integral of the equations of motion can be obtained, which in the Newtonian limit is related to the energy of the system. The direct deduction of this first integral for the case of spherical symmetric spacetimes with a matter shell of perfect fluid was done by Núñez & Oliveira (1996).

In the present work we use this set of first-order differential equations plus an equation of state for the surface density and the tangential pressure of the shell, in order to formulate a model in which the shell moves because of the combined action of the tangential stress and the gravitational attraction. With an appropriate selection of the parameters, different types of motion can be described. We present four cases of motion, an indefinite expansion of the shell starting from rest, which can be seen as an explosion; two cases in which the shell oscillates; and a case in which the shell is static. The oscillation case is particularly interesting because it implies a change in the shell’s surface density, which in turn could be related to a change in temperature, which would look like an object of variable magnitude to a distant observer. In all cases, the values of the parameters of the shell obtained, the tangential pressure, and the surface energy are large. They are the two-dimensional analogy of the values of a neutron star, so this model represents the motion of a sort of “neutron shell.” Finally, we have to call the reader’s attention to the fact that the equation of state used in the derivation of the oscillatory motions and the expansion one is quite unusual, and it is unlikely that it describes a realistic type of matter, so the model should be taken as a first approach. This is not so for the static case, in which the result is valid for all equations of state.

Even though in order to analyze the oscillation of an
object such as a Cepheid star one would need to take into account more parameters, we think that the type of analysis presented here is in the correct direction in obtaining an analytic model for such objects.

The paper is organized as follows. In the next section we present the equations of motion. A detailed discussion of the projections of the geometrical quantities from both spaces on the shell and the deduction of the junction conditions are presented in the Appendix. We also present a brief analysis of the stability of the shells under radial perturbations. In §3 we introduce an expression for the tangential pressure on the shell, obtain the equation of state, and construct the model, presenting and analyzing several cases for different sets of parameters. We also include the static case, which, as we said, is independent of the equation of state. Finally, in §4 we give some conclusions and mention possible directions for further research.

2. EQUATIONS OF MOTION AND STABILITY ANALYSIS

As was mentioned above, in the general relativistic formalism the motion equations of the shell are obtained as matching conditions between the two spacetimes that the shell separates. In the case in which these spacetimes are spherical symmetric and the matter-energy distribution of the shell is described by a perfect fluid, the equations of motion reduce to a set of two coupled first-order differential equations.

The general form of the line element of a spherical symmetric spacetime is given by

\begin{equation}
\text{ds}^2 = -e^\psi \text{d}t^2 \left(1 - \frac{2m}{r}\right)e^\psi \text{d}v - 2 \text{d}r \text{d}v + r^2 \text{d}\Omega^2 + \sin^2 \theta \text{d}\phi^2,
\end{equation}

where \(v\) is the advanced null coordinate, and \(m\) and \(\psi\) are arbitrary functions of \(v\) and \(r\). We are using the usual units in general relativity, where the speed of light and the gravitational constant are equal to unity, but in the concrete examples we will give the quantities in the international system of units as well. For this symmetry the equations of motion are the mass-energy conservation equation:

\begin{equation}
\dot{M} + pA = (T_{\nu^i} n^\nu u^i)_+ - (T_{\nu^i} n^\nu u^i)_-,
\end{equation}

where \(M\) is the mass of the shell (not necessarily constant!), \(p\) is its two-dimensional tangential pressure (it is not the ordinary three-dimensional pressure), \(A = 4\pi R^2(t)\) is the area of the shell, related to the shell’s density \(\sigma\) by \(\sigma = M/A\) (see Appendix), and the subscripts “+” and “-” denote a quantity evaluated outside the shell, +, and inside the shell, respectively; a dot over a variable denotes differentiation with respect to the proper time of the shell, \(\tau\), given in equation (A8).

The second equation is the dynamic equation for the radius of the shell:

\begin{equation}
\dot{R}^2 + V(R) = 0,
\end{equation}

where \(V(R)\) is the potential given by

\begin{equation}
V(R) = 1 - \left(\frac{m_+ - m_-}{M}\right)^2 - \frac{m_+ + m_-}{R} - \left(\frac{M}{2R}\right)^2.
\end{equation}

The Einstein equation for the line element given above are for each spacetime

\begin{align*}
m_{\nu^i} &= 4\pi r^2 T_{\nu^i}^\nu; \\
m_{r^i} &= -4\pi r^2 T_{r^i}^r; \\
\psi_{r^i} &= 4\pi r T_{r^i}.
\end{align*}

To completely determined the motion of the shell, one has to supply an equation of state for the matter of the shell that relates the tangential pressure which the superficial density, \(p = p(\sigma)\). In this way, the dynamic problem for determining the motion of the shell between two given spaces consists of two first-order coupled differential equations.

A typical problem could be posed as follows: first, determine where the shell is going to be moving, that is, give the metric tensors \(g_{\mu^nu^\nu}\) and \(g_{\mu^nu^\nu\nu}\) of the two regions of the spacetime, \(\mu^+\) and \(\mu^-\), respectively, which is equivalent to giving the matter-energy distribution of each region of the spacetime through the Einstein equations. Second, we choose a equation of state for the matter on the shell, and we specify what it is composed of, i.e., \(p = p(\sigma)\). This information completely specifies all the parameters in the equations of motion, so we can proceed to their analysis.

Notice that the formalism to study the motion of the shell has been developed to the point that no more deduction of equations is needed; we need only define the parameters and proceed directly to study the equations of motion.

Finally, it is of interest to say some words about the stability of the shells. We will restrict ourselves to the stability analysis with respect to radial perturbations, which are most important because of the spherical symmetry, and we describe how the shell reacts to the influence of radial deviations.

We start from the dynamic equation for the radius of the shell (eq. [3]) and perform a variation \(R(\tau) \rightarrow R(\tau) + \delta R(\tau)\), which, to first order in \(\delta R(\tau)\), implies

\begin{equation}
\dot{R}^2(\tau) \rightarrow \dot{R}^2(\tau) + 2\dot{R}(\tau)\delta R(\tau) \delta R(\tau)\;
\end{equation}

and for the potential, \(V(R)\), we have that \(V(R) \rightarrow V(R + \delta R)\), and a Taylor expansion around \(V(R)\) to first order in \(\delta R\) gives (in what follows we do not show explicitly the dependence on \(\tau\))

\begin{equation}
V(R) \rightarrow V(R) + \frac{\partial V}{\partial R}\bigg|_R\delta R.
\end{equation}

Substituting these last two equations in equation (3) and using the fact that we are perturbing a solution of this equation, we get

\begin{equation}
2\dot{R}(\delta R)' + \frac{\partial V}{\partial R}\bigg|_R\delta R = 0.
\end{equation}

which is the equation for the evolution of the radial perturbation. Again using equation (3), expressed as \(R = \sqrt{-V}\), yields the following upon integration:

\begin{equation}
\delta R = \exp \int \frac{\partial \sqrt{-V}}{\partial R}\bigg|_R\delta R\;d\tau.
\end{equation}

Now the integrand, using equation (4), has the following form:

\begin{align*}
\frac{\partial \sqrt{-V}}{\partial R}\bigg|_R &= \left[-2(m_+ + m_-)R + M^2/2R^3\right]
\frac{2\sqrt{-V}}{2\sqrt{-V}}
\left[M^2 - 2(m_+ - m_-)R^2/2M^2R^2\right] \partial M/\partial R
\end{align*}

\begin{equation}
+ \frac{[M^4 - 2(m_+ - m_-)2R^2/2M^2R^2] \partial M/\partial R}{2\sqrt{-V}}.
\end{equation}
In order to proceed further, we would need to specify the model chosen and then obtain explicitly $R$ and $M$ as functions of $\tau$, perform the integration, and study how the perturbation behaves. Fortunately, some general remarks can be made about the integrand, so something can be said even without an explicit solution. We will return to this point in the next section.

3. THE MODEL

In the present work we want to show the action of two combined forces acting on the shell and how their interaction produces several types of motion, including oscillatory motion. We will construct a simple model in which there will be vacuum outside the shell and a compact object in vacuum inside of it; that is, we will take the outside and inside spacetimes to be described by the Schwarzschild metric, with constant gravitational masses $m_+$ and $m_-$, respectively, and $T_{\mu\nu} = 0$. This will generate a gravitational attraction toward the center, so that it accounts for the inward force.

With respect to the outward force, consider that as it collapses, the density of the shell grows, which in turn produces an increase in the tangential pressure, which generates a roman arch type of force that opposes the collapse. In the present model, we will represent the growth of the tangential pressure between particles of the shell with dependence on the radius of the shell by the expression

$$p = p_0 e^{-\kappa R},$$

with $p_0$ and $\kappa$ constant parameters, and $R = R(\tau)$ is the radius of the shell at the time $\tau$, the proper time of the shell.

With these suppositions the mass-energy conservation equation of the shell, equation (2), becomes

$$\dot{M} = -8\pi p_0 R e^{-\kappa R} \dot{R},$$

which can be solved in terms of $R$:

$$\dot{M} = \dot{M}_A + \frac{8\pi p_0}{\kappa^2 M_0} (1 + \kappa R_0 \dot{R}) e^{-\kappa R_0 \dot{R}},$$

with $\dot{M}_A = \text{const}$, the “dust mass” of the shell, and we are expressing the masses at multiples of the solar mass, $M_0$, and the radius $R$ as multiples of the solar radius $R_0$, that is, $M = M_0 \dot{M}$, $R = R_0 \dot{R}$. $\dot{M}$ and $\dot{R}$ are unitless. We can go further and obtain an equation of state for the matter of the shell,

$$\sigma = \frac{M_0 \dot{M}_A \kappa^2 + 8\pi [1 - \ln(p/p_0)] p}{4\pi [\ln(p/p_0)]^2}.$$  \hspace{1cm} (14)

This equation of state is rather unusual, but it is well behaved for $p > p_0$, which is always the case in the motion studied, since the shell never reaches a zero radius. Besides, as will be shown below in the examples, it does not violate the energy conditions, namely, $|p| < \sigma$ is always satisfied, so the matter is not of an “exotic” type. Still, we have to agree that it is unlikely that the matter defined by such an equation of state would be realistic, so the model presented here should be taken just as a first approach.

In the dynamic equation for the shell (eq. [3]), the potential (eq. [4]) has the gravitational masses $m_-$ inside and $m_+$ outside, which are constants, and the gravitational mass of the shell is given by equation (13). Thus, in terms of the solar parameters, we have for the potential the following expression:

$$V = 1 - \left(\frac{m_+ - m_-}{M}\right)^2 - \left(\frac{M_0}{R_0}\right) \frac{m_+ + m_-}{R} - \left(\frac{M_0}{R_0}\right)^4 \frac{\dot{M}^2}{2R^2},$$

and with the units we are using, $G = c = 1$, both the solar mass and the solar radius are in length units, $M_0 = 1.473 \times 10^5$ cm, and $R_0 = 6.95 \times 10^{10}$ cm. The constants $\kappa$ and $p_0$ have units of $\text{cm}^{-1}$.

The dynamic equation (eq. [3]), is the expression for the total conserved energy of the shell which, as it is characteristic in general relativity, does not have an arbitrary value but a fixed one (in our case zero), so the kinetic energy is equal to minus the potential energy for all the motion. From this fact we can conclude several properties of the potential term. First, as the kinetic energy always has to be positive, the potential in a well-defined motion has to be negative over all the range of radii. Also, wherever the shell stops, the potential energy term has to be equal to zero at those points and vice versa. A minimum for the potential corresponds to a maximum of the kinetic energy, and of course there can be only one minimum between the turning points and the second derivative, or the potential with respect to the radius has to be positive. From inspection of the potential equation we can also conclude that far from the shell, the leading behavior of the potential is $1 - (m_+ - m_- / M_0)^2$. Finally, from the relation between the solar mass and the solar radius we expect a relation between the values of the radius, $R$, and the values of the masses, $m_+$, $m_-$, and $M$ of the order of $10^6$. Thus, the problem is posed as a five-parameter dynamic problem.

With these general considerations, even though the motion equation cannot be solved analytically, we can search for parameter sets that determine what motions could be described by our model.

As a matter of fact, there is a large range of values of the parameters for which several types of motion can occur. We present three cases: two of oscillatory motion, one with masses of the order of unity and another with radius of the order of unity, and a case with indefinite expansion.

Finally, we present a static case, in which the shell is at rest. This case is particularly interesting because the mass energy conservation equation is satisfied directly, and we do not need to specify an equation of state; also, the results for this model hold for all equations of state.

With respect to the stability analysis, we can also give some general remarks. As mentioned above, the motion is defined for a range of radii in which the potential is negative, so that the integrand for the radial perturbation (eq. [9]) is real for the range in which the motion is taking place, so that the perturbations either increase exponentially or decrease exponentially, but we do not expect oscillatory types of perturbations. For the cases in which the shell oscillates or expands, we would need the explicit solution of the motion to say something more definite. Again, this is not the case for the static model, in which the perturbation analysis has to be taken to the second order of the perturbations, where the perturbation equation can be solved as we show below.

Now we proceed to present the different cases:

1. With values for the masses of the order of solar masses, we take for the Schwarzschild mass outside, $m_+ = $
1.1, that is, \( m_\odot = 2.18 \times 10^{33} \) g for the Schwarzschild mass inside, \( m_\odot = 0.5 \), that is, \( m_\odot = 9.93 \times 10^{32} \) g for the gravitational “dust mass” of the shell, \( \bar{M}_A = 0.603 \), that is, \( \bar{M}_A = 1.198 \times 10^{33} \) g for the pressure constant, \( \rho_0 = 10^{-3} \) cm\(^{-1} \); and for the constant, \( \kappa = 2.8 \times 10^{-6} \) cm\(^{-1} \).

These values generate a potential of the form shown in Figure 1. Notice the region below the R-axis, where the potential takes negative values which, as we explained above, is the region where the motion is allowed. We have two crossing points with the R-axis where the potential equals zero, so the motion is bounded by these two values of the radius, which are \( R_{\min} = 9.0847 \times 10^{-5} \) (\( R_{\min} = 6.313 \times 10^6 \) cm), and \( R_{\max} = 3.414 \times 10^{-4} \) (\( R_{\max} = 2.373 \times 10^7 \) cm). At these extremes the shell stops and starts moving in the opposite direction; in Figure 2 we present a graph of the shell’s velocity, reminding the reader that these velocities are given as factor of the velocity of light, which in these units has a value of \( c = 3 \times 10^{10} \) cm s\(^{-1} \). Notice that the maximum velocity of the shell occurs closer to the minimum radius, where the tangential pressure acts strongly and accelerates the shell, finally stopping it. At that point the velocity reaches a maximum with a value of the order of 0.14. Taking the average velocity to be roughly half this maximum, \( \bar{v} = 0.07 \), and recalling that the one-way distance covered is \( d = R_{\max} - R_{\min} = 17.4 \times 10^6 \) cm, we obtain for the period of the oscillation \( T = 1.65 \times 10^{-2} \) s.

Finally, from the expression for the surface density (eq. [14]) and from that for the tangential pressure (eq. [11]), we can obtain the range of values over which these functions vary:

\[
\sigma \in [1.798 \times 10^{-10}, 1.254 \times 10^{-11}] \text{ cm}^{-1} \\
\quad = [2.426 \times 10^{18}, 1.693 \times 10^{17}] \text{ g cm}^{-2}, \quad (16)
\]

\[
p \in [2.0998 \times 10^{-11}, 1.3758 \times 10^{-32}] \text{ cm}^{-1} \\
\quad = [2.549 \times 10^{38}, 1.670 \times 10^{17}] \text{ dyn cm}^{-1}, \quad (17)
\]

where in the brackets we give the values that the function takes at the minimum radius and at the maximum. We want to stress the fact that in the whole range the strong energy condition, \( \sigma > |p| \), holds, which means that the energy density is positive definite for all observers; thus, as we mentioned, even though the equation of state is unusual, it describes a well-defined type of matter. Finally, we remind the reader that these values of density and pressure are defined on a surface, so the comparison of their magnitudes with quantities defined on volumes is not well posed. Nevertheless, we can say that we are talking about large densities and stresses, the “neutron shell” that we mentioned.

2. As was discussed above, for large radius the potential tends to a constant value given by \( 1 - (\tilde{m}_+ - \tilde{m}_- / \bar{M}_A)^2 \), so choosing for the outside Schwarzschild mass \( \bar{M}_+ = 1.5 \) (\( m_+ = 2.98 \times 10^{33} \) g), for the inside Schwarzschild mass \( \bar{m}_- = 0.73 \) (\( m_- = 1.45 \times 10^{33} \) g), and for the gravitational “dust mass” of the shell \( \bar{M}_A = 0.74 \) (\( M_A = 1.47 \times 10^{33} \) g), the potential tends to the value \( V \rightarrow -4.05 \times 10^{-2} \); that is, it expands indefinitely with a constant velocity of \( \nu = 0.201c = 6 \times 10^7 \) km s\(^{-1} \). This behavior represents an explosion. With the values of \( \rho_0 = 5 \times 10^{-4} \) cm\(^{-1} \), \( \kappa = 2.38 \times 10^{-6} \) cm\(^{-1} \), the potential is shown in Figure 3. The present case can be interpreted as a shell starting from rest at a radius of \( R_0 = 9.34 \times 10^{-5} = 6.473 \times 10^6 \) cm from the center, and the tangential pressure expels it in such a way that the gravitational attraction cannot stop the motion, so the shell continues expanding indefinitely, tending toward a uniform motion with constant velocity.

3. Choosing the parameters in such a way that the shell oscillates within values of the radius of the order of a solar radius, we take \( \bar{m}_+ = 1.3 \times 10^{5} \), \( m_- = 4.45 \times 10^{4} \), \( M_A = 10^5 (m_+ = 2.59 \times 10^{38} \text{ g}, m_- = 8.51 \times 10^{37} \text{ g}, M_A = 1.989 \times 10^{38} \text{ g}) \), and taking \( \rho_0 = 5 \times 10^{14} \) cm\(^{-1} \), \( \kappa = 7.41 \times 10^{-10} \) cm\(^{-1} \). In this way we obtain a potential that again describes oscillatory motion. The region of the potential with negative values is presented in Figure 4. The minimum and maximum radius of the oscillation, the turning points, are \( R_{\min} = 1.2321 \) \( R_0 = 8.56 \times 10^{10} \) cm, and \( R_{\max} = 1.4045 \) \( R_0 = 9.76 \times 10^{10} \) cm, which, as we wanted, are of the order of a solar radius.

A graph of the shell’s velocity is presented in Figure 5. Notice that the maximum velocity of the shell is \( v_{\text{max}} = 0.15c \) (which is of the order of the maximum velocity for the first case presented). Taking the average as \( \bar{v} = 0.07c \), we obtain that the oscillation period for this case is \( T = 31.5 \) s.

Finally, for the surface density and the tangential pressure, we obtain that their values change between the minimum radius and the maximum in the range.
Again, the strong energy condition holds over the whole range, and we obtain large values for the surface density and the tangential stress.

The equations of motion in our model also allow for a static solution. In this case we demand that the gravitational and stress forces cancel each other. This implies a constant gravitational mass of the shell, $M$ (see eq. [2]). Notice that we do not need to specify the equation of state for the matter on the shell. Our only claim is that $M$ is constant, so this model holds for all equations of state.

Since $\ddot{R} = 0$, then by the equation of motion (eq. [3]), the potential must also be zero. Equating to zero the expression for the potential (eq. [15]) with the all masses taken as constants, we obtain an expression for the constant radius that satisfies this equation, that is, the static radius, $R_{st}$:

$$R_{st} = \frac{M_{\odot}M^2}{2R_{\odot}} \left[ \tilde{m}_+ + \tilde{m}_- \pm \sqrt{M^2 + 4\tilde{m}_+ \tilde{m}_-} \right]$$

(20)

Thus, at this radius the shell remains at rest. If we choose a shell of mass $\tilde{M} = 0.1$, $M = 1.98 \times 10^{32}$ g, surrounding a star with a mass such as that of the Sun, $\tilde{m}_+ = 0$, $\tilde{m}_- = 1.98 \times 10^{33}$ g, and taking for the exterior space a gravitational mass $\tilde{m}_+ = 1.05$, $\tilde{m}_- = 2.088 \times 10^{33}$ g, then we obtain the value of $R_{st}$ for which the shell is static surrounding this object,

$$R_{st} = 5.7956 \times 10^{-6} R_{\odot} = 40.27 \text{ km}$$

(21)

For the object in the interior we can choose any radius less than the static one and greater than Schwarzschild horizon of the spacetimes or that of the shell (these radii are, for the values of the gravitational masses chosen, $r_{sch.} = 3.09 \text{ km} > r_{sch.} = 2.945 \text{ km} > r_{schm} = 0.294 \text{ km}$). It could be a neutron star with a typical radius of $R = 15.12 \text{ km}$ (Shapiro & Teukolsky 1983).

The stability analysis for this case has to be carried out to the second-order perturbations, as the first-order one (eq. [9]) evaluated at the static solution implies $\left( \frac{dV}{dR} \right)_{R_{st}} = 0$ and does not give us information about the perturbation. It seems more convenient to analyze the perturbations starting from the second-order differential equation for the motion

$$\ddot{R} = -\frac{\partial V}{2\partial R}$$

(22)

which implies for the perturbation $\delta R$

$$\delta \ddot{R} = -\frac{\partial^2 V}{2\partial R^2} \left. \delta R \right|_{R_{st}}$$

(23)

an harmonic oscillation equation. It is important to notice that, even though for the static case $M = 0$, this does not imply $\partial M/\partial R = 0$. Actually, for each static radius, the mass of the shell changes accordingly. In obtaining this variation of the mass of the shell with respect to the static radius, we use again the static condition, $V(R) = 0$, taking it as an equation for $\dot{M} = \dot{M}(R)$, which results in a fourth-order algebraic equation for $\dot{M}$, with roots

$$\dot{M} = \frac{R_{\odot}}{M_{\odot}} \sqrt{2R_{st}[\tilde{R}_{st} - (\tilde{m}_+ + \tilde{m}_-)(M_{\odot}/R_{\odot}) \pm \sqrt{\Delta}}$$

(24)

where

$$\Delta = [\tilde{R}_{st} - 2\tilde{m}_+(M_{\odot}/R_{\odot})][\tilde{R}_{st} - 2\tilde{m}_-(M_{\odot}/R_{\odot})]$$

and we are ignoring the clearly negative roots. From this
last equation for $\tilde{M}$, we obtain that
\[
\frac{\partial M}{\partial R_{st}} = \pm \frac{M(R_{st} \pm \sqrt{\Delta})}{2R_{st} \sqrt{\Delta}}.
\] (25)

If we substitute this last expression in equation (10), which amounts to evaluating $(\partial V / \partial R)$ at the static radius, then we obtain that $(\partial V / \partial R)|_{R_{st}} = 0$, as expected. Now we proceed to analyze the second derivative of the potential $V(R)$, which is given by
\[
\frac{\partial^2 V}{\partial R^2} = -\frac{2(\dot{m} + \ddot{m})(M_{\odot}/R_{\odot})^2}{M^2} - \frac{3\tilde{M}^2(M_{\odot}/R_{\odot})^2}{2R^2}
\]
\[-\left[\frac{6(\dot{m} - \ddot{m})^2}{M^2} + \tilde{M}^2(M_{\odot}/R_{\odot})^2\right](\frac{\tilde{M}}{M})^2,
\]
\[+ \frac{\tilde{M} \tilde{M}'(M_{\odot}/R_{\odot})^2}{R^2}
\]
\[+ 2\left[\frac{(\dot{m} - \ddot{m})^2}{M^2} - \frac{\tilde{M}^2(M_{\odot}/R_{\odot})^2}{4R^2}\right](\frac{\tilde{M}''}{M^2}),
\] (26)

where a prime symbol stands for the derivative with respect to $\tilde{R}$. Computing the second derivative of $M$ and substituting in this last expression that, as we mentioned above, is equivalent to evaluating the potential at the static solution (the results have been checked using MapleV-4, which has been used also in obtaining the figures), we obtain that
\[
\frac{\partial^2 V}{\partial R^2}|_{R_{st}} = 0.
\] (27)

Thus, the perturbation equation for this case reduces to
\[
(\delta \tilde{R}) = 0,
\] (28)

which in turn implies that the radial perturbations for the static shell are given by
\[
\delta R = a \tau + b,
\] (29)

with $a$ and $b$ being arbitrary constants, so that they grow linearly, and we thus conclude that the static shell is unstable under radial perturbations.

These results allow us to conclude that the potential for the static case has no minimum around the static radius, but an inflection point, so it has the form of a large plateau around that static radius and that under radial perturbations, which will grow linearly with respect to the proper time of the shell, will tend either to collapse or expand, depending on the sign of the initial conditions imposed for the perturbation. If $\delta \tilde{R}(\tau_0) = a > 0$, it expands, and if $\delta \tilde{R}(\tau_0) = a < 0$, it collapses.

4. CONCLUSIONS

In the present work we have presented the analysis of the equations of motion for a shell of perfect fluid using the formalism of general relativity. We have chosen a model in which the effects of the combined action of two types of forces acting on the shell are shown, namely, the gravitational attraction versus a force due to the tangential stresses. Even though the problems have been reduced to a set of two coupled partial differential equations, an analytic solution was not found. Nevertheless, the equations are very tractable, and from the properties of the potential energy we have been able to present several types of motion allowed in our model, which depends on five parameters. We presented cases for oscillatory motion for masses of the order of a solar mass and small radius, and for a radius of the order of solar radius, which implies large masses. With an appropriate selection of the parameters for any value of mass or radius, such a motion can be found. We have also presented two other cases allowed by our model: one which could be interpreted as explosion, an ejection of a shell starting from rest because of the force associated with the tangential pressure, and another in which that force equals the gravitational one, allowing the shell surrounding a cosmic body to be static. The matter in the shell is associated with an unusual equation of state, which does not violate any energy condition, so we can say that it is not an “exotic” type of matter but still unlikely to describe a realistic one; thus, it is actually a new, mathematically correct solution, but the reader should be aware of the type of matter used in this model.

This last remark, however, does not apply to the last case presented, the static one, in which we have obtained the model without any reference to an equation of state, so it holds for all of them. We also showed that the static shell is an unstable configuration under radial perturbations, which grow linearly with respect to the proper time of the shell.

We can incorporate more parameters to describe other types of objects. For instance, we can take into account the radiation emitted from the inner body to the shell and the interaction of the shell with the medium, the radiation pressure and the friction type force, respectively, considered in the classical formalism. This case should be obtained working with the matching conditions on the shell of a Vaidya universe inside (Vaidya 1951), and a Friedmann-Robertson-Walker dust universe outside (see Weinberg 1972). Also, some deviations from sphericity could be considered, so that the matching conditions could be analyzed for an axisymmetric type of spacetime. This case is expected to be highly unstable, so the point would be to study the final state of that model. Finally, it would be of interest to study the motion of shells composed of different types of matter, such as one constructed from a scalar field. These ideas are currently under investigation.

We think that the model presented elucidates several features of the possible types of motion of the shell.

I want to acknowledge the project IN105496 from DGAPA-UNAM for partial support during the development of the present work. Also, I want to thank H. Quevedo as well as M. Salgado, D. Sudarsky, and T. Zannias for fruitful comments and discussions about this work. Finally, I thank the anonymous referee for helpful comments and suggestions.

APPENDIX

In this Appendix we present a review of the description of the deduction of the equations of motion of the shell (eqs. [2] and [3]). The four-dimensional spacetime is taken to be composed of two parts, $M_-$ and $M_+$, separated by a boundary $\Sigma$. The main goal in this analysis consists in showing how the geometry on $\Sigma$ is determined by that of the spacetime in which it is
Recalling that it is not hard to see that the extrinsic curvature is related to the acceleration by the hypersurface:

where $M$ is the gravitational mass of the shell, defined by $M = \frac{1}{2} \gamma_{ab} S \mu$, $S = \int S d\Sigma$ is the surface energy tensor of the shell, as it is the limit of the integral of the stress energy tensor through the thickness of the shell when this thickness tends to zero (Israel 1966). We will take the matter on the shell to be described by a perfect fluid, $\rho$.

One of the most important geometric quantities in the embedding is the extrinsic curvature defined by $K_{ab} = n_\beta \epsilon^\alpha_{(a)b} \epsilon^\nu_{(b)}$. Recalling that $\epsilon^\alpha_{(a)} n_\beta = 0$, it is not hard to see that the extrinsic curvature is related to the acceleration by

$$K_{ab} u^a u^b = n_\alpha \frac{\partial u^\alpha}{\partial \tau} .$$

Thus, the extrinsic curvature projected on the three-dimensional velocity is equal to the projection of the four-dimensional acceleration becomes

$$K_{ab} n^a u^b = \frac{\partial u^\alpha}{\partial \tau} ,$$

where $\epsilon^\alpha_{(a)}$ contains, then demanding continuity of the line elements on $\Sigma$ and analyzing the jump on the curvature due to the presence of the shell, in order to obtain the “matching” conditions or, as we see in the present work, the equations of motion of the shell.

The first condition for joining the two regions is that the line element of the region $\mathcal{M}_-$ should equal the line element of $\mathcal{M}_+$ at $\Sigma$; that is, both must be equal to $ds^2_{\Sigma}$.

The metric on the shell is described by $\gamma_{ab}$ (Latin indexes are 0, 2, 3; Greek are 0, 1, 2, 3). It is related to the metric of the spacetime by $\gamma^a_{(a)} \gamma^b_{(b)} = g^{ab} + n^a n^b$, where $n^a$ is a unit four-dimensional vector normal to $\Sigma$, and the $\epsilon^\alpha_{(a)}$ are the projectors on the hypersurface: $\epsilon^\alpha_{(a)} A^\alpha = A^\alpha$, $\epsilon^\alpha_{(a)} A_\alpha = A_\alpha$, for an arbitrary vector $A$. The four-dimensional velocity is $u^\alpha = dx^\alpha/d\tau$, and the four-dimensional acceleration $(\partial u^\alpha/\partial \tau) = u^\alpha u^a$ (semicolon denotes covariant derivative with respect to $g_{ab}$).

Another important geometric quantity is the symmetric three-tensor $S_{ab}$, defined by the “Lanczos equations,”

$$\mu_{ab} - \gamma_{ab} \mu = 8\pi S_{ab} ,$$

or, equivalently, $\mu_{ab} = 8\pi(S_{ab} - \frac{1}{2}\gamma_{ab} S)$. $S_{ab}$ can be considered to be the surface energy tensor of the shell, as it is the limit of the integral of the stress energy tensor through the thickness of the shell when this thickness tends to zero (Israel 1966). We will take the matter on the shell to be described by a perfect fluid,

$$S_{ab} = (\sigma + p) u^a u^b + p\gamma_{ab} ,$$

where $\sigma$ is the matter-energy density on the shell, $p$ is the tangential pressure of that matter, and $u^a$ stands for the three-dimensional timelike velocity vector, $\gamma_{ab} u^a u^b = -1$. Then, the equation (A3) for the difference of the projections of the acceleration becomes

$$(\sigma + p)\frac{\partial u^a}{\partial \tau} = M R^2 - 8\pi p ,$$

where $M$ (not necessarily constant) is the gravitational mass of the shell, defined by $M = A\sigma$, with $A$ being the area of the shell, $A = 4\pi R^2$. Equation (A5) is the first of the matching equations, which tells us that a discontinuity of the normal component of the acceleration is related to the matter present on the shell.

There are two other matching equations that can be deduced starting from the relations of the components of the Riemann tensor in the spacetime with the one on $\Sigma$. These relations are called the Codazzi-Mainardi equations,

$$4 \mathcal{R}_{abcd} = K_{bc} |d - K_{bd} |c ,$$

and the Gauss-Codazzi equations,

$$4 \mathcal{R}_{abcd} = 3 R_{abcd} + K_{ac} K_{bd} - K_{ad} K_{bc} ,$$

where “$|$” denotes covariant derivative on $\Sigma$, i.e., $\gamma_{ab} |c = 0$. Multiplying the Codazzi-Mainardi equations by $\gamma^c$, and the Gauss-Codazzi by $\gamma^a \gamma^b$, we get a set of four equations, valid on each side of the shell:

$$G_{\mu\nu} n^\alpha \epsilon^\nu_{(a)} |^\pm = (K_{a|b} - K_{ab}) |^\pm ,$$

$$2 G_{\mu\nu} n^\mu n^\nu |^\pm = 3 R + (K^2 - K_{ab} K^{ab}) |^\pm .$$

With these definitions and the Einstein equations, $G_{\mu\nu} = 8\pi T_{\mu\nu}$, the difference of the projections of the stress-energy tensor evaluated on the shell are

$$(T_{\mu\nu} n^\mu u^\nu)_+ - (T_{\mu\nu} n^\mu u^\nu)_- = u^\alpha S_{ab} ,$$

$$2[(T_{\mu\nu} n^\mu n^\nu)_+ - (T_{\mu\nu} n^\mu n^\nu)_-] = - S_{ab}(K_{ab}^+ + K_{ab}^-) ,$$

which are the other two matching equations, relating the projections of the stress energy tensor on the normal and on the velocity and on the normal on both indexes to the matter distribution on the shell. Equation (A6) is the equation of energy balance that tells us how the fluxes of matter-energy distribution from both sides determine the dynamics of the shell. On the
other hand, equation (A7) has no clear physical interpretation (see below).

For the matter-energy distribution of the shell described by a perfect fluid, and taking into account the fact that \( \dot{\sigma} = d\sigma/dt \) and that \( u^\nu_{\nu b} = (1/\sqrt{-\gamma}) (\sqrt{-\gamma}u^\nu)_{\nu b} \), with \( \gamma \equiv \det (g_{ab}) \), as well as that for spherical symmetry, the line element of the shell is given by equation (1). The shell \( \Sigma \) is a three-dimensional manifold, characterized by a line element

\[
 ds^2 = -d\tau^2 + R^2(\tau) d\Omega^2 ,
\]

with \( R(\tau) \) being the radius of the shell. Thus, we can obtain that \( u^b_{\nu b} = \dot{\Lambda}/\Lambda \), with \( A \) being the area of the shell defined above, and the energy balance equation (eq. [A4]) takes the form

\[
 \dot{M} + p\dot{\Lambda} = -4\pi R^2(T_{\mu v}n^v u^\mu)_+ - (T_{\mu v}n^v u^\mu)_- ,
\]

which is the form of the second matching condition (Núñez & Oliveira 1996). The interpretation as the equation of motion for the shell looks particularly clear in this form. Here it is important to stress the fact that with spherical symmetry and a perfect fluid, the first matching condition (eq. [A5]) can be integrated without further assumptions, using equation (A9), to yield a quadratic first order equation for \( R(\tau) \) (eq. [3]). For details on this deduction see Núñez & Oliveira (1966).

Finally, the third matching condition (eq. [A7]), using equation (A1) and \( S_{\mu b} \) as a perfect fluid (eq. [A4]), tells us that the sum of the projections of the acceleration on the normal, from each side of \( \Sigma \), is given by

\[
 \sigma \left( n^\mu \frac{\partial u^\mu}{\partial \tau} \right) = -2[(T_{\mu v}n^v n^\mu)_+ - (T_{\mu v}n^v n^\mu)_-] - p\{K_{ab} u^a u^b + K\} .
\]

As mentioned above, this equation has no clear physical meaning. As a matter of fact, it can be shown that, at least for the spherical case with the matter-energy distribution of the shell described by a perfect fluid (eq. [A4]), the last equation (eq. [A10]) is the same as the matching equation (eq. [A5]), so we do not have to consider it in the analysis of the motion of the shell. For details on the demonstration of this equality see Núñez (1966).

In this way, the motion equations for the shell reduce to a set of two first-order equations, namely, equations (2) and (3). The equation for the change in the advanced time can be obtained from the equation for the change in the radius, using the normalization equation for the cuadrivelocity. The problem is completed with a equation of state for the matter-energy distribution of the shell, \( p = p(\sigma) \).

REFERENCES

Israel, W. 1966, Nuovo Cimento B, 44, 1
Lichnerowicz, A. 1955, Théories Relativistes de la Gravitation et de l'Electromagnetisme (Paris: Masson & Cie)
Núñez D. 1996, in preparation
Núñez, D., & de Oliveira, H. P. 1996, Phys. Lett. A, 214, 227
Núñez, D., de Oliveira, H. P., & Salim, J. 1993, Classical Quantum Gravity, 10, 1117
Ostriker, J. P., & Gunn, J. E. 1971, ApJ, 164, L95
Sato, H. 1993, Prog. Theor. Phys., 90, 841
Shapiro, S., & Teukolsky, S. 1983, Black Holes, White Dwarfs, and Neutron Stars (New York: Wiley)
Vaidya, P. C. 1951, Proc. Indian Acad. Sci. A, 33, 264
Weinberg, S. 1972, Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity (New York: Wiley)