ON CHARACTER SPACE OF THE ALGEBRA OF BSE-FUNCTIONS

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Abstract. Suppose that $A$ is a semi-simple and commutative Banach algebra. In this paper we try to characterize the character space of the Banach algebra $C_{BSE}(\Delta(A))$ consisting of all BSE-functions on $\Delta(A)$ where $\Delta(A)$ denotes the character space of $A$. Indeed, in the case that $A = C_0(X)$ where $X$ is a non-empty locally compact Hausdorff space, we give a complete characterization of $\Delta(C_{BSE}(\Delta(A)))$ and in the general case we give a partial answer.

Also, using the Fourier algebra, we show that $C_{BSE}(\Delta(A))$ is not a $C^*$-algebra in general. Finally for some subsets $E$ of $A^*$, we define the subspace of BSE-like functions on $\Delta(A) \cup E$ and give a nice application of this space related to Goldstine’s theorem.

1. Introduction and Preliminaries

Suppose that $A$ is a semi-simple commutative Banach algebra and $\Delta(A)$ is the character space of $A$, i.e., the space of all non-zero homomorphisms from $A$ into $\mathbb{C}$.

A bounded continuous function $\sigma$ on $\Delta(A)$ is called a BSE-function if there exists a constant $C > 0$ such that for each $\phi_1, \ldots, \phi_n \in \Delta(A)$ and complex numbers $c_1, \ldots, c_n$, the inequality

$$\left| \sum_{i=1}^{n} c_i \sigma(\phi_i) \right| \leq C \left\| \sum_{i=1}^{n} c_i \phi_i \right\|_{A^*}$$

holds. For each $\sigma \in C_{BSE}(\Delta(A))$ we denote by $\|\sigma\|_{BSE}$ the infimum of such $C$. Let $C_{BSE}(\Delta(A))$ be the set of all BSE-functions. We have a good characterization of $C_{BSE}(\Delta(A))$ as follows:

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Theorem 1.1. \( C \text{BSE}(\Delta(A)) \) is equal to the set of all \( \sigma \in C_b(\Delta(A)) \) for which there exists a bounded net \( \{x_\lambda\} \) in \( A \) with \( \lim_\lambda \phi(x_\lambda) = \sigma(\phi) \) for all \( \phi \in \Delta(A) \)

Proof. see [11, Theorem 4 (i)]. \( \square \)

Using the above characterization one can see that \( C \text{BSE}(\Delta(A)) \) is unital if and only if \( A \) has a bounded weak approximate identity in the sense of Lahr and Jones. We recall that a net \( \{x_\alpha\} \) in \( A \) is called a bounded weak approximate identity (b.w.a.i) for \( A \) if \( \{x_\alpha\} \) is bounded in \( A \) and

\[
\lim_\alpha \phi(x_\alpha a) = \phi(a) \quad (\phi \in \Delta(A), a \in A),
\]

or equivalently, \( \lim_\alpha \phi(x_\alpha) = 1 \) for each \( \phi \in \Delta(A) \).

Also, Theorem 1.1 gives the following definition of \( \| \cdot \|_{\text{BSE}} \):

\[
\|\sigma\|_{\text{BSE}} = \inf\{\beta > 0 : \exists\{x_\lambda\} \text{ in } A \text{ with } \|x_\lambda\| \leq \beta, \lim_\lambda \phi(x_\lambda) = \sigma(\phi) \ (\phi \in \Delta(A))\}\).

The theory of BSE-algebras for the first time introduced and investigated by Takahasi and Hatori; see [11] and two other notable works [5, 3]. In [3], the authors answered to a question raised in [11]. Examples of BSE-algebras are the group algebra \( L^1(G) \) of a locally compact abelian group \( G \), the Fourier algebra \( A(G) \) of a locally compact amenable group \( G \), all commutative \( C^* \)-algebras, the disk algebra, and the Hardy algebra on the open unit disk. We recall that a commutative and without order Banach algebra \( A \) is a type I-BSE algebra if \( \hat{M}(A) = C \text{BSE}(\Delta(A)) = C_b(\Delta(A)) \), where \( M(A) \) denotes the multiplier algebra of \( A \) and \( \hat{M}(\hat{A}) \) denotes the space of all \( \hat{T} \) which defined by \( \hat{T}(\varphi)(x) = \hat{T}(x)(\varphi) \) for all \( \varphi \in \Delta(A) \). Note that \( x \in A \) should satisfies \( \varphi(x) \neq 0 \).

In this paper, we give a partial characterization of the character space of \( C \text{BSE}(\Delta(A)) \) where \( A \) is a semi-simple commutative Banach algebra. Indeed, we show that if \( A \) has a b.w.a.i and \( C \text{BSE}(\Delta(A)) \) is an ideal in \( C_b(\Delta(A)) \), then

\[
\Delta(C \text{BSE}(\Delta(A))) = \text{w}^*\Delta(A).
\]

Also, we give a negative answer to this question; Whether \( (C \text{BSE}(\Delta(A)), \| \cdot \|_{\text{BSE}}) \) is a \( C^* \)-algebra? At the final section of this paper we study the space of BSE-like functions on subsets of \( A^* \) which containing \( \Delta(A) \) and as an application of this space we give a nice relation with Goldstine’s theorem.
2. Character space of $C_{\text{BSE}}(\Delta(A))$

In view of [11, Lemma 1], $C_{\text{BSE}}(\Delta(A))$ is a semi-simple commutative Banach algebra. So, the character space of $C_{\text{BSE}}(\Delta(A))$ should be non-empty and one may ask: Is there a characterization of $\Delta(C_{\text{BSE}}(\Delta(A)))$ for an arbitrary Banach algebra $A$?

In the sequel of this section we give a partial answer to this question. Let $X$ be a non-empty locally compact Hausdorff space and put

$$C_{\text{BSE}}(X) := C_{\text{BSE}}(\Delta(C_0(X))).$$

To proceed further we recall some notions. Let $X$ be a non-empty locally compact Hausdorff space. A function algebra (FA) on $X$ is a subalgebra $A$ of $C_b(X)$ that separates strongly the points of $X$, that is, for each $x, y \in X$ with $x \neq y$, there exists $f \in A$ with $f(x) \neq f(y)$ and for each $x \in X$, there exists $f \in A$ with $f(x) \neq 0$. A Banach function algebra (BFA) on $X$ is a function algebra $A$ on $X$ with a norm $\| \cdot \|$ such that $(A, \| \cdot \|)$ is a Banach algebra.

A topological space $X$ is completely regular if every non-empty closed set and every singleton disjoint from it can be separated by continuous functions.

**Theorem 2.1.** Let $X$ be a non-empty locally compact Hausdorff space. Then $C_{\text{BSE}}(X)$ is a unital BFA and its character space is homeomorphic to $X^{\text{w}^*}$, that is, $\Delta(C_{\text{BSE}}(X)) = \{ \phi_x : x \in X \}^{\text{w}^*}$.

**Proof.** By [11, Lemma 1], $C_{\text{BSE}}(X)$ is a subalgebra of $C_b(X)$ and $\| \cdot \|_{\text{BSE}}$ is a complete algebra norm. Since $C_0(X)$ has a bounded approximate identity, $C_{\text{BSE}}(X)$ is unital. So, for each $x \in X$, there exists $f \in C_{\text{BSE}}(X)$ with $f(x) \neq 0$. On the other hand, using the Urysohn lemma, for each $x, y \in X$ with $x \neq y$ one can see that there exists $f \in C_{\text{BSE}}(X)$ such that $f(x) \neq f(y)$.

Finally, since $X$ is a locally compact Hausdorff space, it is completely regular by [1 Corollary 2.74]. On the other hand, by [11 Theorem 3], we know that $C_0(X)$ is a type I-BSE algebra. Therefore, $C_{\text{BSE}}(X) = C_b(\Delta(C_0(X))) = C_b(X)$. Also, for every $f \in C_{\text{BSE}}(X)$, by the remark after Theorem 4 of [11], we have $\|f\|_X \leq \|f\|_{\text{BSE}}$. Also, by the Open mapping theorem there exists a positive constant $M$ such that $\|f\|_{\text{BSE}} \leq M\|f\|_X$. So, $C_{\text{BSE}}(X)$ and $C(X)$ are topologically isomorphic, and so $\Delta(C_{\text{BSE}}(X))$ and $\Delta(C_0(X))$ are homeomorphic. Now, by using [1 Theorem 2.4.12], we have

$$\Delta(C_{\text{BSE}}(X)) = \Delta(C_0(X)) = X^{\text{w}^*} = \{ \phi_x : x \in X \}^{\text{w}^*}.$$ 

$\square$
Remark 2.2. In general for a commutative Banach algebra $A$, we have the following conditions concerning the character space of $C_{\text{BSE}}(\Delta(A))$:

(i) If $C_{\text{BSE}}(\Delta(A)) = C_b(\Delta(A))$, then

$$\Delta(C_{\text{BSE}}(\Delta(A))) = \overline{\Delta(A)^{w^*}}.$$ 

Examples of Banach algebras $A$ satisfying $C_{\text{BSE}}(\Delta(A)) = C_b(\Delta(A))$ are finite dimensional Banach algebras and commutative $C^*$-algebras; see the remark on page 609 of [12]. Also, see [11, Lemma 2] for a characterization of Banach algebras $A$ for which $C_{\text{BSE}}(\Delta(A)) = C_b(\Delta(A))$.

(ii) If $A$ has a b.w.a.i, then $C_{\text{BSE}}(\Delta(A))$ is unital and so $\Delta(C_{\text{BSE}}(\Delta(A)))$ is compact and hence it is $w^*$-closed. On the other hand, we know that

$$\Delta(A) \subseteq \Delta(C_{\text{BSE}}(\Delta(A))),$$

in the sense that for each $\varphi \in \Delta(A)$, $f_\varphi : C_{\text{BSE}}(\Delta(A)) \to \mathbb{C}$ defined by $f_\varphi(\sigma) = \sigma(\varphi)$ is an element of $\Delta(C_{\text{BSE}}(\Delta(A)))$. Note that $f_\varphi \neq 0$, since in this case $C_{\text{BSE}}(\Delta(A))$ is unital and $f_\varphi(1) = 1$. So

$$\overline{\Delta(A)^{w^*}} \subseteq \overline{\Delta(C_{\text{BSE}}(\Delta(A)))^{w^*}} = \Delta(C_{\text{BSE}}(\Delta(A))).$$

(iii) On can see that if $(B, \| \cdot \|_B)$ is a Banach algebra which containing the Banach algebra $(C, \| \cdot \|_C)$ as a two-sided ideal, then every $\varphi \in \Delta(C)$ extends to one $\hat{\varphi} \in \Delta(B)$. Now, let $C = C_{\text{BSE}}(\Delta(A))$ and $B = C_b(\Delta(A))$. If $C$ is an ideal in $B$, then

$$\Delta(C_{\text{BSE}}(\Delta(A))) = \Delta(C) \subseteq \Delta(B) = \overline{\Delta(A)^{w^*}}.$$ 

(iv) Suppose that $B$ is a commutative semi-simple Banach algebra such that $\Delta(B)$ is compact. Then $B$ is unital; see [4, Theorem 3.5.5]. Now, If $A$ has no b.w.a.i, then

$$\Delta(C_{\text{BSE}}(\Delta(A))) \neq \overline{\Delta(A)^{w^*}}.$$ 

Because if $\Delta(C_{\text{BSE}}(\Delta(A))) = \overline{\Delta(A)^{w^*}}$, by using the above assertion, $C_{\text{BSE}}(\Delta(A))$ is unital, since $\overline{\Delta(A)^{w^*}} = \Delta(C_b(\Delta(A)))$ is compact and $C_{\text{BSE}}(\Delta(A))$ is a semi-simple commutative Banach algebra. Therefore, $A$ has a b.w.a.i which is impossible.

(v) If $A$ has a b.w.a.i and $C_{\text{BSE}}(\Delta(A))$ is an ideal of $C_b(\Delta(A))$, then using parts (ii) and (iii), we have

$$\Delta(C_{\text{BSE}}(\Delta(A))) = \overline{\Delta(A)^{w^*}}.$$
3. \((C_{\text{BSE}}(\Delta(A)), \| \cdot \|_{\text{BSE}})\) is not a C*-algebra

The theory of C*-algebras is very fruitful and applied. As an advantage of this theory, especially in Harmonic Analysis, one can see the C*-algebra approach for defining a locally compact group, see [6]. So, verifying the Banach algebras from a C*-algebraic point of view is very helpful. In this section, using a result due to Kaniuth and Ülger in [5], we show that \((C_{\text{BSE}}(\Delta(A)), \| \cdot \|_{\text{BSE}})\) is not a C*-algebra in general. On the other hand, there is a question which left open that, under what conditions on \(A\), \((C_{\text{BSE}}(\Delta(A)), \| \cdot \|_{\text{BSE}})\) is a C*-algebra?

In the sequel for each locally compact group \(G\), let \(A(G)\) denote the Fourier algebra and \(B(G)\) denote the Fourier-Stieltjes algebra introduced by Eymard; see [9], §19. Also, let \(\hat{G}\) denote the dual group of \(G\) and \(M(G)\) denote the Measure algebra; see [2], §3.3. For the convenience of reader we give the definitions of \(A(G)\) and \(B(G)\) as follows:

Let \(G\) be a locally compact group. Suppose that \(A(G)\) denotes the subspace of \(C_0(G)\) consisting of functions of the form \(u = \sum_{i=1}^{\infty} f_i * \tilde{g}_i\), where \(f_i, g_i \in L^2(G)\), \(\sum_{i=1}^{\infty} ||f_i||_2 ||g_i||_2 < \infty\) and \(\tilde{f}(x) = \overline{f(x^{-1})}\) for all \(x \in G\). The space \(A(G)\) with the pointwise operation and the following norm is a Banach algebra,

\[
\|u\|_{A(G)} = \inf \{ \sum_{i=1}^{\infty} ||f_i||_2 ||g_i||_2 : u = \sum_{i=1}^{\infty} f_i * \tilde{g}_i \},
\]

which we call it the Fourier algebra. It is obvious that for each \(u \in A(G)\), \(\|u\| \leq \|u\|_{A(G)}\) where \(\|u\|\) is the norm of \(u\) in \(C_0(G)\).

Now let \(\Sigma\) denote the equivalence class of all irreducible representations of \(G\). Then \(B(G)\) consisting of all functions \(\phi\) of the form \(\phi(x) = \left< \pi(x)\xi, \eta \right>\) where \(\pi \in \Sigma\) and \(\xi, \eta\) are elements of \(H_{\pi}\), the Hilbert space associated to the representation \(\pi\). It is well-known that \(A(G)\) is a closed ideal of \(B(G)\).

Also, recall that an involutive Banach algebra \(A\) is called a C*-algebra if its norm satisfies \(\|aa^*\| = \|a\|^2\) for each \(a \in A\). We refer the reader to [8] to see a complete description of C*-algebras.

In the following remark we give the main result of this section.

\textbf{Remark 3.1.} In general \((C_{\text{BSE}}(\Delta(A)), \| \cdot \|_{\text{BSE}})\) is not a C*-algebra, that is, there is not any involution "**" on \(C_{\text{BSE}}(\Delta(A))\) such that

\[
\|\sigma^*\sigma\|_{\text{BSE}} = \|\sigma\|_{\text{BSE}}^2 \quad \forall \sigma \in C_{\text{BSE}}(\Delta(A)).
\]

Because we know that every commutative C*-algebra is a BSE-algebra. For a non-compact locally compact Abelian group \(G\) take \(A = A(G)\). By [5], Theorem 5.1, we know that \(C_{\text{BSE}}(\Delta(A)) = B(G)\) and for each \(u \in B(G)\), \(\|u\|_{B(G)} = \|u\|_{\text{BSE}}\). But \(B(G) = M(\hat{G})\) and it is shown
in [11] that \( M(\widehat{G}) \) and hence \( B(G) \) is not a BSE-algebra. Therefore, 
\((C_{\text{BSE}}(\Delta(A)), \| \cdot \|_{\text{BSE}})\) is not a \( C^* \)-algebra.

As the second example, let \( G \) be a locally compact Abelian group. It is well-known that \( C_{\text{BSE}}(\Delta(L^1(G))) \) is isometrically isomorphic to \( M(G) \), where \( L^1(G) \) denotes the group algebra; see the last remark on page 151 of [11]. On the other hand, by the Gelfand-Naimark theorem we know that every commutative \( C^* \)-algebra should be symmetric. But in general \( M(G) \) is not symmetric, i.e., the formula \( \mu^*(\xi) = \overline{\mu}(\xi) \) does not hold for every \( \xi \in \Delta(M(G)) \). For example if \( G \) is non-discrete then by [10] Theorem 5.3.4, \( M(G) \) is not symmetric and hence fails to be a \( C^* \)-algebra.

It is a good question to characterize Banach algebras \( A \) for which 
\((C_{\text{BSE}}(\Delta(A)), \| \cdot \|_{\text{BSE}})\) is a \( C^* \)-algebra.

4. BSE-FUNCTIONS ON SUBSETS OF \( A^* \)

Suppose that \( A \) is a Banach algebra and \( E \subseteq A^* \setminus \Delta(A) \). A complex-valued bounded continuous function \( \sigma \) on \( \Delta(A) \cup E \) is called a BSE-like function if there exists an \( M > 0 \) such that for each \( f_1, f_2, f_3, \ldots, f_n \in \Delta(A) \cup E \) and complex numbers \( c_1, c_2, c_3, \ldots, c_n \),

\[
\left\| \sum_{i=1}^{n} c_i \sigma(f_i) \right\| \leq M \left\| \sum_{i=1}^{n} c_i f_i \right\|_{A^*}.
\]

(4.1)

We show the set of all the BSE-like functions on \( \Delta(A) \cup E \) by \( C_{\text{BSE}}(\Delta(A), E) \) and let \( \| \sigma \|_{\text{BSE}} \) be the infimum of all \( M \) satisfying relation (4.1). Obviously, \( C_{\text{BSE}}(\Delta(A), E) \) is a linear subspace of \( C_b(\Delta(A) \cup E) \) and we have 
\[ \{ \sigma|_{\Delta(A)} : \sigma \in C_{\text{BSE}}(\Delta(A), E) \} \subseteq C_{\text{BSE}}(\Delta(A)). \]

Clearly, \( \iota_A(A) \subseteq C_{\text{BSE}}(\Delta(A), E) \) where \( \iota_A : A \to A^{**} \) is the natural embedding. For \( a \in A \), we let \( \widehat{a} = \iota_A(a) \) and \( \widehat{A} = \iota_A(A) \).

To proceed further, we recall the Helly theorem.

**Theorem 4.1.** (Helly) Let \((X, \| \cdot \|)\) be a normed linear space over \( \mathbb{C} \) and let \( M > 0 \). Suppose that \( x_1, \ldots, x_n, c_1, \ldots, c_n \) are in \( X^* \) and \( c_1, \ldots, c_n \) are in \( \mathbb{C} \). Then the following are equivalent:

(i) for all \( \epsilon > 0 \), there exists \( x_\epsilon \in X \) such that \( \| x_\epsilon \| \leq M + \epsilon \) and \( x_\epsilon^*(x_k) = c_k \) for \( k = 1, \ldots, n \).

(ii) for all \( a_1, \ldots, a_n \in \mathbb{C} \),

\[
\left\| \sum_{i=1}^{n} a_i c_i \right\| \leq M \left\| \sum_{i=1}^{n} a_i x_i^* \right\|_{X^*}.
\]

**Proof.** See [7] Theorem 4.10.1. \( \square \)
As an application of Helly’s theorem, we give the following characterization which is similar to [11] Theorem 4 (i).

**Theorem 4.2.** \( C_{BSE}(\Delta(A), E) \) is equal to the set of all \( \sigma \in C_b(\Delta(A) \cup E) \) for which there exists a bounded net \( \{x_\alpha\} \) in \( A \) with \( \lim_\alpha f(x_\alpha) = \sigma(f) \) for all \( f \in \Delta(A) \cup E \).

**Proof.** Suppose that \( \sigma \in C_b(\Delta(A) \cup E) \) is such that there exists \( \beta < \infty \) and a net \( \{x_\alpha\} \subseteq X \) with \( \|x_\alpha\| < \beta \) for all \( \alpha \) and \( \lim_\alpha f(x_\alpha) = \sigma(f) \) for all \( f \in \Delta(A) \cup E \). Let \( f_1, \ldots, f_n \) be in \( \Delta(A) \cup E \) and \( \alpha_1, \ldots, \alpha_n \) be complex numbers. Then we have

\[
\left| \sum_{i=1}^n c_i \sigma(f_i) \right| \leq \left| \sum_{i=1}^n c_i f_i(x_\alpha) \right| + \left| \sum_{i=1}^n c_i (f_i(x_\alpha) - \sigma(f_i)) \right| \\
\leq \beta \left| \sum_{i=1}^n c_i f_i \right| + \sum_{i=1}^n |c_i| \|f_i(x_\alpha) - \sigma(f_i)\|
\]

Taking the limit with respect to \( \alpha \), we conclude that \( \sigma \in C_{BSE}(\Delta(A), E) \).

Conversely, let \( \sigma \in C_{BSE}(\Delta(A), E) \). Suppose that \( \Lambda \) is the net consisting of all finite subsets of \( \Delta(A) \cup E \). By Helly’s theorem, for each \( \epsilon > 0 \) and \( \lambda \in \Lambda \), there exists \( x_{(\lambda, \epsilon)} \in A \) with \( \|x_{(\lambda, \epsilon)}\| \leq \|\sigma\|_{BSE} + \epsilon \) and \( f(x_{(\lambda, \epsilon)}) = \sigma(f) \) for all \( f \in \lambda \). Clearly, \( \{(\lambda, \epsilon) : \lambda \in \Lambda, \epsilon > 0\} \) is a directed set with \( (\lambda_1, \epsilon_1) \preceq (\lambda_2, \epsilon_2) \) iff \( \lambda_1 \subseteq \lambda_2 \) and \( \epsilon_1 \leq \epsilon_2 \). Therefore, we have

\[
\lim_{(\lambda, \epsilon)} f(x_{(\lambda, \epsilon)}) = \sigma(f) \quad (f \in \Delta(A) \cup E).
\]

\( \square \)

**Remark 4.3.** As an application of Theorem 4.2 if \( E = A^* \setminus \Delta(A) \), then one can see that \( A = A^{**} \), i.e., we conclude Goldstine’s theorem. That is, \( \hat{A} \) with the \( w^* \)-topology of \( A^{**} \) is dense in \( A^{**} \).

**Remark 4.4.** We say that \( A \) has a b.w.a.i respect to \( E \) if there exists a bounded net \( \{x_\alpha\} \) in \( A \) with

\[
\lim_\alpha f(x_\alpha) = 1 \quad (f \in \Delta(A) \cup E).
\]

Using Theorem 4.2 one can check that \( 1 \in C_{BSE}(\Delta(A), E) \) if and only if \( A \) has a b.w.a.i respect to \( E \).

We conclude this section with the following question.

**Question** Is \( C_{BSE}(\Delta(A), E) \) a commutative and semi-simple Banach algebra? If it is what is its character space?

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REFERENCES

1. C. D. Aliprantis and K. C. Border, *Infinite Dimensional Analysis*, Springer-Verlag Berlin Heidelberg, edition 3, 2006.
2. H. G. Dales, *Banach Algebras and Automatic Continuity*, Clarendon Press, Oxford, 2000.
3. Z. Kamali and M. L. Bami, *The Bochner-Schoenberg-Eberlein Property for \( L^1(\mathbb{R}^+ \) ), J. Fourier Anal. Appl., 2014, 20, 2, 225–233.
4. E. Kaniuth, *A Course in Commutative Banach Algebras*, Springer Verlag, Graduate texts in mathematics, 2009.
5. E. Kaniuth and A. ¨Ulger, *The Bochner-Schoenberg-Eberlein property for commutative Banach algebras, especially Fourier and Fourier-Stieltjes algebras*, Trans. Amer. Math. Soc., 2010, 362, 4331–4356.
6. J. Kustermans and S. Vaes, *Locally compact quantum groups* Ann. Sci. Ecole Norm. Sup. (4) 33 (2000), 837934.
7. R. Larsen, *Functional Analysis: an introduction*, Marcel Dekker, New York, 1973.
8. G. J. Murphy, *C∗-Algebras and Operator Theory*, Academic Press Inc, 1990.
9. J. P. Pier, *Amenable Locally Compact Groups*, Wiley Interscience, New York, 1984.
10. W. Rudin, *Fourier Analysis on Groups*, Wiley-Interscience, New York, 1962.
11. S.- E. Takahasi and O. Hatori, *Commutative Banach algebras which satisfy a Bochner-Schoenberg-Eberlein-type theorem*, Proc. Amer. Math. Soc., 1990, 110, 149–158.
12. S.- E. Takahasi and O. Hatori, *Commutative Banach algebras and BSE-inequalities*, Math. Japonica, 1992, 37, 4, 607–614.

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