I. INTRODUCTION

In the last decade the possibility of a superfluid alkali atom Fermi gas has attracted much attention both theoretically and experimentally because this phenomenon opens a new opportunity to study strongly correlated quantum many-particle systems and to emulate high-temperature superconductors. Optical lattices are made with lasers, and therefore, the lattice geometry is easy to modify by changing the wavelength of the intersecting laser beams. Near the Feshbach resonance the atom-atom interaction can be manipulated in a controllable way because the scattering length $a_s$ can be changed from the BCS side (negative values) to the BEC side (positive values) reaching very large values close to resonance. We focus our attention on the BCS transition (negative scattering length) of degenerate fermionic gases to a superfluid state analogous to superconductivity. In particular, we consider an equal mixture of $^6$Li atomic Fermi gas of two hyperfine states loaded into a cubic three-dimensional optical lattice studied assuming a negative scattering length (BCS side of the Feshbach resonance). When the interaction is attractive, fermionic atoms can pair and form a superfluid. The dispersion of the phonon-like mode and the speed of sound in the long-wavelength limit are obtained by solving the Bethe-Salpeter equations for the collective modes of the attractive Hubbard Hamiltonian.

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equations for the collective mode $\omega(Q)$ and corresponding BS amplitudes $G^\pm(k, Q)$:

\[
\begin{align*}
[\omega(Q) - \varepsilon(k, Q)]G^+(k, Q) &= \\
&= \frac{U}{2N} \sum_q [\gamma_{k,q} \gamma_{q,k} - l_{k,q}l_{q,k}] G^+(q, Q) - \frac{U}{2N} \sum_q [\gamma_{k,q} \gamma_{q,l} - l_{k,q}l_{q,l}] G^-(q, Q) \\
&= \frac{U}{2N} \sum_q [\gamma_{k,q} \gamma_{q,k} - l_{k,q}l_{q,k}] G^+(q, Q) - \frac{U}{2N} \sum_q [\gamma_{k,q} \gamma_{q,l} - l_{k,q}l_{q,l}] G^-(q, Q),
\end{align*}
\]

where $\gamma_{k,q} = u_k u_{k+q} + v_k v_{k+q}$, $l_{k,q} = u_k u_{k+q} - v_k v_{k+q}$, $\tilde{\gamma}_{k,q} = u_k v_{k+q} - v_k u_{k+q}$, $m_{k,q} = u_k v_{k+q} + v_k u_{k+q}$ where $u_k^2 = 1 - v_k^2 = [1 + \pi(k)/E(k)]/2$. The quantity $\varepsilon(k, Q) = E(k + Q) - E(k)$, where $E(k) = \sqrt{\varepsilon_k^2 + \Delta^2}$ depends on the gap function $\Delta$ and the mean-field electron energy $\varepsilon_k$. We use a tight-binding form of the mean-field electron energy: $\varepsilon_k = 2J (\cos k_x d + \cos k_y d + \cos k_z d) - \mu$, where $\mu$ is the chemical potential. The gap function and the chemical potential have to be determined by the BCS number and gap equations:

\[
1 - f = \frac{1}{N} \sum_k \frac{\varepsilon_k^2}{E(k)}, \quad 1 = \frac{1}{N} \sum_k \frac{1}{2E(k)},
\]

where $f = M/N$ is the filling factor, and we have $M$ atoms distributed along $N$ sites.

The BS equations for the collective modes can be reduced to a set of four coupled linear homogeneous equations. The existence of a non-trivial solution requires that the secular determinant $\det \tilde{\chi}^{-1} - \tilde{V}$ is equal to zero, where the bare mean-field-quasiparticle response function $\tilde{\chi}$ and the interaction $\tilde{V} = \text{diag}(-U, -U, U, U)$ are $4 \times 4$ matrices:

\[
\tilde{\chi} = \begin{pmatrix}
I_{\gamma,\gamma} & J_{\gamma,\iota} & I_{\gamma,\tilde{\gamma}} & J_{\gamma,m} \\
J_{\gamma,\iota} & I_{\iota,\iota} & J_{\iota,\tilde{\gamma}} & I_{\iota,m} \\
I_{\gamma,\tilde{\gamma}} & J_{\iota,\tilde{\gamma}} & I_{\tilde{\gamma},\tilde{\gamma}} & J_{\tilde{\gamma},m} \\
J_{\gamma,m} & I_{\iota,m} & J_{\iota,m} & I_{m,m}
\end{pmatrix}.
\]

Here we have introduced symbols $I_{a,b} = F_{a,b}(\varepsilon(k, Q))$ and $J_{a,b} = F_{a,b}(\omega)$, where $F_{a,b}(x)$ is defined as follows (the quantities $a(k, Q)$ and $b(k, Q) = l_{k,q}m_{k,q}$, $\gamma_{k, Q}$ or $\tilde{\gamma}_{k, Q}$):

\[
F_{a,b}(x) = \frac{1}{N} \sum_k \frac{x a(k, Q)b(k, Q)}{\omega^2 - \varepsilon^2(k, Q)}.
\]

It is worth mentioning that the GRPA equations for the collective mode derived by Belkhir and Randeria can be obtained if we neglect in all elements with index $\tilde{\gamma}$. In this case $\tilde{\chi}$ and $\tilde{V}$ are $3 \times 3$ matrices.

\section{II. SPEED OF SOUND IN A CUBIC LATTICE}

The velocity of sound is important because it tells us how fast the sound propagates in the system, but more importantly, it is intimately related to the normal (phonon) part of the liquid according to Landau’s theory of superfluidity.

In our numerical calculations, the sum over $k$ is replaced by a triple integral over the first Brillouin zone: $-\pi \leq k_x d \leq \pi$, $-\pi \leq k_y d \leq \pi$ and $-\pi \leq k_z d \leq \pi$. After that, we applied the substitutions $x = \tan k_x d/4, y = \tan k_y d/4$ and $z = \tan k_z d/4$ to rewrite the integrals in the form of Gaussian quadrature $\int_{-1}^{1} dx \int_{-1}^{1} dy f(x, y, z)/(1 + x)(1 + y)/(1 + z)$. The corresponding integrals are numerically evaluated using $49 \times 49 \times 49$ $(x_i, y_j, z_k)$ points:
FIG. 1: The dispersion $\omega/\Delta$ of the phonon-like collective mode. For filling factor $f = 0.5$, lattice height $s = 2.5$, and scattering length $a_s = -1000a_B$ ($a_B$ is the Bohr radius of hydrogen), the chemical potential $\mu = 0.326E_R$ and the gap energy $\Delta = 0.05E_R$ are obtained by solving the number and gap equations [1]. The speed of sound in the long-wavelength limit is $8.1$ mm/s. The puncture curve represents the dispersion calculated in Ref. [6].

\[ \int_{-1}^{1} dx \int_{-1}^{1} dy \int_{-1}^{1} dz f(x, y, z)/(1 + x)(1 + y)(1 + z) = \sum_{i=1}^{49} \sum_{j=1}^{49} \sum_{k=1}^{49} w_i w_j w_k f(x_i, y_j, z_k), \]

where $w_i$ is the corresponding weight. It can be checked that there is no difference between the approximation by integrals and the case when the sums over $k$ are taken explicitly assuming 128 sites per dimension.

In Fig. 1 and Fig. 2 we present the results of our calculations of the dispersion of the phonon-like mode and the speed of sound as a function of the scattering length assuming that the filling factor and the lattice height are $f = 0.5$ and $s = 2.5$, respectively. The long-wavelength part of the dispersion is linear with sound velocity of about $8.1$ mm/s. For higher momenta the dispersion saturates to $2\Delta$. As it is expected, when the interaction between the atoms is increased by increasing the scattering length, the compressibility of the system increases, and therefore, the speed of sound decreases, as can be seen in Fig. 2. In both figures, there exists a difference of about 10 -15 percents between the BS approach and the response-function calculations presented in Ref. [6].

III. CONCLUSION

In this paper, we have used the BS equations in the GRPA to obtain the dispersion of the phonon-like collective mode and the corresponding sound velocity in the long-wavelength limit in the system of equal mixture of $^6Li$ atomic Fermi gas of two hyperfine states loaded into a cubic three-dimensional optical lattice. It is shown that the previous calculations, which have been obtained by studying the poles of the density response functions, are not in accordance
with our results derived by means of the BS equations in the GRPA.