Phase transition in the collisionless damping regime for wave-particle interaction

Marie-Christine Firpo and Yves Elskens
Equipe turbulence plasma de l’UMR 6633, CNRS–Université de Provence,
case 321, Centre de Saint-Jérôme, F-13397 Marseille cedex 20
(preprint TP99.10 - to appear in Physical Review Letters)

Landau damping is a striking phenomenon first evidenced in plasma physics, but also lately in various systems such as nonlinearly coupled oscillators or neutral rarefied gases. In plasma physics, its classical treatment involves a continuous media analysis through Vlasov-Poisson equations. It was shown by O’Neil that (near-)resonant particles play a major role in the time-asymptotic, generically nonlinear regime experienced by kinetic systems submitted to Landau damping. This has been debated recently by several works. In particular, Isichenko argued for algebraic final decay of the electric field in the nonlinear regime, under the crucial assumption of an asymptotically vanishing field, though in numerical investigations by Manfredi, the electric field was seen to damp exponentially in the linear regime and then to oscillate around a finite amplitude indicating a Bernstein-Greene-Kruskal (BGK) equilibrium. Lately, Lancellotti and Dorning have shown that both types of time-asymptotic states were eligible depending on initial conditions.

In this Letter we address this issue from a different viewpoint and relate both possible time-asymptotic states to different phases in a phase transition picture. Rather than using the large viewpoint and relate both possible time-asymptotic on initial conditions.

rewriting the equilibrium statistical mechanics treatment. This analysis provides further insight on the results of the kinetic approach.

As real plasmas are made of a finite number of particles and as these graininess properties, and notably the existence of spontaneous emission, are discarded by their classical vlasovian treatment, a model taking these properties into account has been derived. The first steps in this direction were made by O’Neil et al. in order to treat the cold beam-plasma instability, and a Hamiltonian formalism was first introduced by Mynick and Kaufman. This Hamiltonian system models the interaction between M (bulk) Langmuir waves and N quasiresonant (tail) particles and has already provided insight into linear Landau growth and damping, the cold beam-plasma instability and Van Kampen modes. From the physical viewpoint, this approach has the advantage to treat the waves as dynamical objects rather than subordinate phenomena, i.e. excitations of the bulk plasma. In the kinetic limit N → ∞, the coupled (many-body) wave-particle dynamics approaches the corresponding dynamics generated by the Vlasov equation over any finite time interval.

For simplicity, we concentrate on the single-mode case (M = 1), which is paradigmatic to study Landau damping. Besides, this case is of special interest for devices such as traveling wave tubes where a single mode can be selected. For N identical (tail) particles moving on the interval of length L = 2π with periodic boundary conditions T := R/(2πZ), with unit mass and charge, and respectively position xr and momentum pr, interacting with one wave of natural frequency ω0, unit wave number, phase θ and intensity I, the Hamiltonian reads

\[
H = \sum_{r=1}^{N} \frac{p_r^2}{2} + \omega_0 I - \frac{\eta \omega_0^2}{N} \sum_{r=1}^{N} \sqrt{2I} \cos(x_r - \theta). \tag{1}
\]

The first term describes ballistic motion of the particles, the second the oscillations of the free wave (harmonic oscillator) and the third couples them; here \( \eta \) is a small parameter denoting the ratio of the tail density over the plasma density. The bulk of the plasma is considered as a linear dielectric supporting plasma waves. The wave frequency \( \omega_0 \) (which is the plasma frequency \( \omega_p \)) provides the natural reference time scale, which can be fixed with no loss of generality. Thus \( \omega_0^{-1} \) will be the time unit, so that the phase velocity of the free wave is also 1.

The self-consistent system can be viewed as a set of nonlinear oscillators coupled through a mean-field, as the equation of motion for any particle \( r \) obtained from (1) reads

\[
\ddot{x}_r = -\sqrt{2\eta I/N} \sin(x_r - \theta) \tag{2}
\]
The minimum of \( g \geq T \) corresponds to taking the limit \( N \to \infty \) while the acceleration felt by any particle should remain finite. This induces the natural kinetic scaling for the wave intensity \( I = O(N) \) and gives its kinetic equivalent as the intensive quantity \( \psi := I/N \).

Most importantly, this approach enables an equilibrium statistical mechanics treatment, provided the dynamics is effectively ergodic. In this Letter we derive the point equation \( g \), the wave intensity, remain finite. This induces the natural kinetic scaling for the two usual constants of the motion, total energy \( E = H/N \) and large enough to trap particles. When \( T \to T_c \), the wave is macroscopic and large enough to trap particles. When \( T \to T_c \), the wave exhibits a formal divergence signaling the crossover from extensive to extensive scaling in the kinetic limit of an infinite number of particles–wave system. The canonical measure (constrained by constant momentum \( P \)) reads

\[
d\mu_c = e^{-H/T} \delta \left( P - \sum_{r=1}^{N} p_r - I \right) d\mathbf{x} d\theta \prod_{r=1}^{N} dp_r dx_r
\]

where the phase space variables \((\mathbf{p}, I, \mathbf{x}, \theta)\) evolve in \( \Lambda = \mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{T}^N \times \mathbb{T} \). With the intensive variable \( \sigma := P/N \), the canonical partition function \( Z_c(T, \sigma, N) \) reads

\[
Z_c = \left( 2\pi \right)^{3N+1} T^{N+1} N^{1/2} \int_0^\infty \exp \left[ -\frac{N}{T} f(\sigma, T, \psi) \right] d\psi
\]

where \( f(\sigma, T, \psi) := (\sigma - \psi)^2/2 + \omega_0 \psi - T \ln I_0(\sqrt{2\eta} \psi/T) \); \( I_n \) denotes the modified Bessel function of order \( n \). For large \( N \), with \( \varphi := \sqrt{2\eta} \psi/T \), one finds the minimum of \( g(\sigma, T, \varphi) := f(\sigma, T, \varphi^2/(2\eta)) \), by solving the saddle-point equation

\[
\frac{\partial}{\partial \varphi} g = \varphi T \left( \frac{T}{\eta} \left( \omega_0 - \sigma + \frac{T^2 \varphi^2}{2\eta} \right) - \frac{I_1(\varphi)}{\varphi I_0(\varphi)} \right) = 0.
\]

The minimum of \( g \) is reached at \( \varphi = 0 \) iff \( \sigma < \omega_0 \) and \( T \geq T_c \), where

\[
T_c := \frac{\eta}{2(\omega_0 - \sigma)}.
\]

Consequently, when \( \sigma < \omega_0 \), a second order phase transition [16,17] occurs at \( T_c \), with order parameter \( \psi = I/N \), i.e. the suitable normalized intensity of the wave [3]. Above \( T_c \), the intensity of the wave loses its extensivity: the particles and the wave decouple in the sense that the wave controls just one among \( N + 1 \) degrees of freedom.

Finally we express the equation of state of the particles-wave system in terms of the Gibbs average of the energy density \( h := H/N \), using \( \langle H \rangle_c = T^2 \partial_T \ln Z_c \). Thus, for \( \varphi = 0 \),

\[
\langle h \rangle_c = \sigma^2/2 + T/2,
\]

where \( \sigma^2/2 \) is interpreted as the mean centre-of-mass kinetic energy of the particles-wave system and \( T/2 \) as the thermal agitation energy in this frame. For \( \varphi^* \neq 0 \), in the limit of large \( N \), one obtains similarly

\[
\langle h \rangle_c = g [\varphi^*(\sigma, T)] + T/2.
\]

Before considering the implications of our results, let us discuss the conditions under which they hold. The Gibbs Ansatz assumes that particles explore the available \((x, p)\) space thanks to the nonintegrability of their dynamics. For physical applications, it suffices that they explore a ‘large’ part of the \((x, p)\) space: in our case, that they be able to wander, say, within a velocity range of two standard deviations on either side of their average velocity. The constraint \( \sum_r p_r + I = P \) ensures that the kinetic energy contribution to \( \langle \rangle \) is a Maxwell distribution centered on \( \sigma - \psi \), with standard deviation \( T^{1/2} \) for \( p_r \). One therefore expects some good mixing properties, due to a trapping/detrapping mechanism [22], if \( T^{1/2} \) is of the order of the velocity width of resonance [3], namely \((\eta \psi)^{1/4}\). This equivalently corresponds to \( \varphi \) being of order 1. In discussing numerical simulations, we shall note that relaxation towards equilibrium occurs for initial data roughly realizing this condition.

Now we turn to the manifestation of this phase transition in the wave-particle dynamics. The problem raised by Landau damping is an initial value problem (hence a nonequilibrium one), where a packet of particles, spatially homogeneous with velocities distributed according to a certain \( f_0(\psi) \), interacts with a wave launched with intensity \( \psi_0 > 0 \). This determines \( E/N = \langle \psi^2 \rangle_0/2 + \omega_0 \psi_0 \) and \( \sigma = \langle \psi \rangle_0 + \psi_0 \). The condition \( \sigma < \omega_0 \) implies \( \langle \psi \rangle_0 < \omega_0 \), which corresponds typically to a bunch of...
particles, centered on the phase velocity of the wave, initially distributed according to a decreasing $f_0(v)$ and thus inducing Landau damping on the wave. The initial particles temperature is $T_0 := \langle v^2 \rangle_0 - \langle v \rangle_0^2$.

Given $E/N$ and $\sigma$, the equation of state (1) in the ‘high temperature’ regime expresses the equilibrium temperature $T$ in terms of initial data

$$T = T_0 + 2\psi_0(\omega_0 - \langle v \rangle_0) - \psi_0^2. \quad (11)$$

Using (1), defining $a := \omega_0 - \langle v \rangle_0 > \omega_0 - \sigma > 0$ and putting $\alpha := a \pm \sqrt{\sigma^2 + T_0}$, condition $T > T_c$ reduces to $2(\psi_0 - a) (\psi_0 - \alpha_+)(\psi_0 - \alpha_-) > \eta$. Therefore, provided that the near-resonant (tail) particle distribution is warm enough $(2aT_0 > \eta)$, if $\psi_0 < \psi_{oc}$ in the limit of large $N$, the equilibrium field $\psi_c$ vanishes, whereas for $\psi_0 > \psi_{oc}$, the equilibrium field remains finite. The critical initial wave amplitude $\psi_{oc}$ is defined by

$$2(\psi_{oc} - a) (\psi_{oc} - \alpha_+) (\psi_{oc} - \alpha_-) = \eta. \quad (12)$$

Moreover, whatever the initial conditions, one deduces that if $\psi_0 > a$ then the asymptotic field will always be finite.

These results are illustrated for an initial normalized, warm particle distribution $f_0(v) = C_1 - C_2/(2 - v)$, with $C_1, C_2 > 0$, over the range $v_1 < v < v_2$ (with $v_2 - \omega_0 = \omega_0 - v_1$) and $f_0(v) = 0$ outside of it. The linear Landau rate $\tilde{\gamma}$ for amplitude $\sqrt{\psi}$ is $\sqrt{T/N}$ is $\gamma_L = (\pi/2)(\omega_0 f_0(\omega_0))$. In our dynamical simulations, $\omega_0 = 1$, $\eta = 5.02 \cdot 10^{-4}$, $v_1 = 0.75$, $v_2 = 1.25$ and $\gamma_L = -1/200$ (from which $C_1$ and $C_2$ are deduced). Then $\langle v \rangle_0 = 0.931$ and $\langle v^2 \rangle_0 = 0.882$, thus $T_0 = 1.50 \cdot 10^{-2}$. The resulting critical mean intensity is $\psi_{oc} = 5.59 \cdot 10^{-2}$.

Running simulations for different values of $\psi_0$ and $N$ yields the plot of Fig. 1 for asymptotic values of $\psi$. This figure shows a clear agreement with thermodynamics predictions in the kinetic limit $N \to \infty$ for both quasi-ballistic and trapping regimes. In the critical region where $\psi_0 \lesssim \psi_{oc}$ (dashed lines), one observes strong metastability effects and critical slowing-down, whereas additional finite-$N$ corrections smooth the transition: in this region, initial field intensities are far larger than their equilibrium averages. Moreover, for the range of $\psi_0$ considered on Fig. 1 the analytic expression for $T$ happens to differ only slightly from $T_0$. Therefore the ‘mixing’ condition corresponds to $\sqrt{2\eta\psi_0/T_0}$ being of order 1.

This mixing condition is roughly fulfilled for the runs of Fig. 1 displaying the temporal evolution of $\ln \sqrt{\psi}$ and progress towards equipartition for different values of $N$. Along these runs, the relative energy variation is less than $2 \cdot 10^{-7}$. On the inset, after the initial linear Landau damping at the prescribed rate, one observes trapping oscillations that could suggest the setting up of a BGK equilibrium. However, on longer times, the sweeping of the phase space is sufficient to drive the system to a far different, ‘high’ temperature, regime where $I$ is no longer extensive. Critical slowing-down reflects on the relaxation times diverging with $N$. Analogous pathological relaxation properties in the limit $N \to \infty$ have been reported for a similar mean-field model [13] in the subcritical energy domain where chaos is paradoxically shown to be maximal [14,15].

It is surprising to note the good ergodic behavior of $\psi$ for the initial domain where $\psi_0 \ll \psi_{oc}$, where no mixing can be expected as the system is close to integrability. Here one recovers the effective linear Landau damping and ergodicity is ensured dynamically by a perturbative analysis [16,21].

In the trapping regime, the equilibrium level is quickly reached as a substantial fraction of particles is liable to undergo separatrix crossings [22]. Nevertheless $\psi_0$ is limited by the velocity range chosen for the tail particles [23]. One checks that the trajectories corresponding to the velocity borders of our distribution in the Boltzmann $(x,p)$ space correspond to KAM tori for the effective 1.5 degrees of freedom Hamiltonian dynamics [2] with time dependent parameters $(\theta, I)$ [21].

Finally, denoting by $\tau_{B0} = (2\eta\psi_0)^{-1/4}$ the trapping time corresponding to the initial mean wave intensity and putting $q = |\gamma|/\tau_{B0}$, it turns out that the phase transition occurs here for $q = q_c \simeq 0.06$. This agrees with the qualitative dynamical threshold drawn (in a non self-consistent way) by O’Neil [3] separating a regime dominated by trapping (if $q < q_c$) from a regime where linear predictions are effective (if $q > q_c$). Existing numerical self-consistent simulations of the damping of a single wave are limited to the early stage of wave evolution and would typically infer a larger $q_c$ (e.g. $q_c = 0.77$ for a linear $f_0(v)$ in [24]). As clearly accounted for by Fig. 2 and inset (for which $q = 0.14$), our new approach relates this apparent discrepancy to critical slowing-down. Our intuition is that an algebraic decay of the order parameter $\psi$ (in the limits $N \to \infty$ and $t \to \infty$) may be a dynamical signature of the vicinity of critical point, as in Landau second order phase transitions’ picture. In a parameter regime $q \lesssim 0.06$, Isichenko’s conclusions [4] may then apply here.

In conclusion, using the self-consistent wave-particle Hamiltonian (1) to describe this system has revealed a new phenomenon, the phase transition associated with the Landau damping regime. This phenomenon eludes the usual Vlasov-Poisson description, which actually treats all particles on the same footing, as a Coulomb system. But the wave-particle interaction is effectively very different for near-resonant particles (trapping time scale $(2\eta\psi)^{-1/4}$ and Landau linear scale $|\gamma_L|^{-1}$) and for bulk particles (adiabatic or non-resonant averaging time scales, exponentially longer than resonant ones): our self-consistent Hamiltonian describes the system on the physical time scales for the wave evolution [13].

Moreover, our finite-$N$ approach shows that, in a real plasma, the wave will never damp completely but will
eventually fluctuate around a finite $N$-dependent thermal level due to spontaneous wave emission.

Stimulating discussions with D.F. Escande, F. Doveil, P. Bertrand and M. Poleni, and critical reading of the manuscript by D.F. Escande are gratefully acknowledged. This work is part of the European network *Stability and universality in classical mechanics* and CNRS GdR *Systèmes de particules chargées* SParCh.

* E-mail : x@newsup.univ-mrs.fr (x=firpo, elskens).

[1] S.H. Strogatz, R.E. Mirollo and P.C. Matthews, Phys. Rev. Lett. 68, 2730 (1992).
[2] P. Stubbe and A.I. Sukhorukov, Phys. Plasmas 6, 2976 (1999).
[3] T.M. O’Neil, Phys. Fluids 8, 2255 (1965).
[4] M.B. Isichenko, Phys. Rev. Lett. 78, 2369 (1997).
[5] C. Lancellotti and J.J. Dorning, Phys. Rev. Lett. 80, 5236 (1998); M.B. Isichenko, ibid. 5237.
[6] G. Manfredi, Phys. Rev. Lett. 79, 2815 (1997).
[7] C. Lancellotti and J.J. Dorning, Phys. Rev. Lett. 81, 5137 (1998).
[8] M. Antoni, Y. Elskens and D.F. Escande, Phys. Plasmas 5, 841 (1998).
[9] D.F. Escande, S. Zekri and Y. Elskens, Phys. Plasmas 3, 3534 (1996).
[10] T.M. O’Neil, J.H. Winfrey and J.H. Malmberg, Phys. Fluids 14, 1204 (1971).
[11] H.E. Mynick and A.N. Kaufman, Phys. Fluids 21, 653 (1978).
[12] J.L. Tennyson, J.D. Meiss and P.J. Morrison, Physica D 71, 1 (1994).
[13] M-C. Firpo and Y. Elskens, J. Stat. Phys. 93, 193 (1998); Y. Elskens and M-C. Firpo, Phys. Scripta T75, 169 (1998).
[14] S.I. Tsunoda, F. Doveil and J.H. Malmberg, Phys. Rev. Lett. 59, 2752 (1987).
[15] J.R. Cary and I. Doxas, J. Comput. Phys. 107, 98 (1993).
[16] K. Huang, *Statistical Mechanics*, John Wiley & Sons (New-York, 1987).
[17] It is important to note that the long range nature of the wave-particle interaction makes it possible to observe a phase transition even in a one space-dimension system.
[18] V. Latora, A. Rapisarda and S. Rufo, Phys. Rev. Lett. 80, 692 (1998).
[19] M.-C. Firpo, Phys. Rev. E 57, 6599 (1998).
[20] Ergodicity for the collective variable $\psi$ in this domain can also been accounted for using a result of Kac on the distribution of sums of trigonometric functions [11].
[21] M.-C. Firpo, thèse de doctorat de l’université de Provence (Marseille, 1999).
[22] J.R. Cary, D.F. Escande and J.L. Tennyson, Phys. Rev. A 34, 4256 (1986); Y. Elskens and D.F. Escande, Non-linearity 4, 615 (1991).
[23] An additional motivation for limiting the velocity range of the tail particles distribution is that one does not want the resonant region to extend as far as the bulk of the plasma, as this would also break the validity of the self-consistent Hamiltonian model [19].
[24] R. Sugihara and T. Kamimura, J. Phys. Soc. Japan 33, 206 (1972).
FIG. 1. Time averages of normalized intensity $\psi$ reached for long times versus $\psi_0$ for $N = 16000$ (triangles), $N = 32000$ (squares) and $N = 64000$ (stars). Filled symbols for simulations times of $2 \cdot 10^6 \omega_0^{-1}$. Open symbols for run times of $5 \cdot 10^6 \omega_0^{-1}$. Canonical estimations are drawn in continuous line for $N = 16000$, 32000 and 64000 particles. Near the transition, dashes indicate that estimate (7) is not expected to be accurate.

FIG. 2. Time evolution of normalized amplitude $\sqrt{\psi} = \sqrt{I/N}$ for $N = 16000$ (dots), $N = 32000$ (thick line) and $N = 64000$ (thin line), for $\psi_0 = 1.56 \cdot 10^{-3} < \psi_{ac}$. Inset: initial evolution, including brief initial linear damping stage. On longer times, for $N = 32000$, $\psi$ wanders around half the value associated to $N = 16000$, in agreement with canonical predictions. Relaxation towards equilibrium is much longer for $N = 64000$ due to critical slowing-down near the phase transition.
