Magnetic Schrödinger operators with radially symmetric magnetic field and radially symmetric electric potential

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Abstract. The aim of the paper is to derive spectral estimates on the eigenvalue moments of the magnetic Schrödinger operators defined on the two-dimensional disk with a radially symmetric magnetic field and radially symmetric electric potential.

Keywords: Eigenvalue bounds, radial magnetic field, Lieb-Thirring inequalities, discrete spectrum.
1. Introduction

Let us consider a particle in a bounded domain $\Omega$ in $\mathbb{R}^2$ in the presence of a magnetic field $B$ and an electric potential $V$. We define the 2-dimensional magnetic Schrödinger operator associated to this particle as follows:

Let $A$ be a magnetic potential associated to $B$, i.e. a smooth real valued-function on $\Omega \subset \mathbb{R}^2$ verifying $\text{rot } (A) = B$ and $V \geq 0$ be a bounded measurable potential defined on $L^2(\Omega)$. The magnetic Schrödinger operator is initially defined on $C^\infty_0(\Omega)$ by

$$H_{\Omega}(A, V) = (i \nabla + A)^2 - V.$$ 

The case when the magnetic field is not constant can be motivated by anisotropic superconductors (see for instance [CDG95]) or the liquid crystal theory.

Assuming some regularity conditions (RC) on $A$, namely : the magnetic field $B$ is $\in L^\infty_{\text{loc}}(\Omega)$ and the corresponding magnetic potential $A$ is $\in L^\infty(\Omega)$, we get that the magnetic Sobolev norm $\| (i \nabla + A) u \|_{L^2(\Omega)}$, $u \in \mathcal{H}^1_0(\Omega)$, is closed and equivalent to the non-magnetic one which means that they both have purely discrete spectrum. Thus using the boundedness of the potential $V$ the self-adjoint Friedrichs extension of $H_{\Omega}(A, V)$ initially defined on $C^\infty_0(\Omega)$ has a purely discrete spectrum.

In the paper we also consider the case when the magnetic field grows to infinity as the variable approaches the boundary and has a non zero infimum

$$B(z) \to \infty \quad \text{as} \quad z \to \partial \Omega \quad \text{and} \quad K := \inf B(z) > 0. \quad (1.1)$$

In view of the lower bound

$$(H_{\Omega}(A, V)(u), u)_{L^2(\Omega)} \geq \int_{\Omega} (B(z) - \|V\|_{L^\infty(\Omega)}) |u|^2(z) \, dz,$$

one again can construct the Friedrichs extension of $H_{\Omega}(A, V)$ initially defined on $C^\infty_0(\Omega)$. Moreover, it still has a purely discrete spectrum—[T12].

For simplicity, we will use for the Friedrichs extension the same symbol $H_{\Omega}(A, V)$, and we shall denote the increasingly ordered sequence of its eigenvalues by $\lambda_k = \lambda_k(\Omega, A, V)$.

The purpose of this paper is to establish bounds of the eigenvalue moments of such operators. Let us recall the following bound which was proved by Berezin, Li and Yau for non-magnetic Dirichlet Laplacians on a domain $\Omega$ in $\mathbb{R}^d$—[Be72a, Be72b, LY83],

$$\sum_k \left( \Lambda - \lambda_k(\Omega, 0, 0) \right)^\sigma_+ \leq L^{\text{d}}_{\sigma, \Lambda} |\Omega| \Lambda^{\sigma + d/2} \quad \text{for any } \sigma \geq 1 \text{ and } \Lambda > 0, \quad (1.2)$$

where $|\Omega|$ is the volume of $\Omega$, and the constant on the right-hand side,

$$L^{\text{d}}_{\sigma, \Lambda} = \frac{\Gamma(\sigma + 1)}{(4\pi)^{d/2} \Gamma(\sigma + 1 + d/2)}, \quad (1.3)$$

is optimal. Moreover, for $0 \leq \sigma < 1$, the bound (1.2) still exists, but with another constant on the right-hand side—[La97]

$$\sum_k \left( \Lambda - \lambda_k(\Omega, 0, 0) \right)^\sigma_+ \leq 2 \left( \frac{\sigma}{\sigma + 1} \right)^\sigma L^{\text{d}}_{\sigma, \Lambda} |\Omega| \Lambda^{\sigma + d/2}, \quad 0 \leq \sigma < 1. \quad (1.4)$$
Magnetic Schrödinger operators with radially symmetric magnetic field

For Schrödinger operators \( H_{\Omega}(0, V) \) with the Dirichlet boundary conditions the following bound was proved by Lieb-Thirring –[LT76]

\[
\sum_{\lambda_k(\Omega, 0, V) \leq 0} |\lambda_k(\Omega, 0, V)|^\sigma \leq L_{\sigma,d}^c \int_{\Omega} V^{\sigma+d/2}(x) \, dx, \quad \sigma \geq 3/2. \tag{1.5}
\]

The similar estimate for the Schrödinger operators \( H_{\Omega}(A, V) \) with the Dirichlet boundary conditions and with non-zero magnetic field takes place –[LW00]

\[
\sum_{\lambda_k(\Omega, A, V) \leq 0} |\lambda_k(\Omega, A, V)|^\sigma \leq L_{\sigma,d}^c \int_{\Omega} V^{\sigma+d/2}(x) \, dx, \quad \sigma \geq 3/2. \tag{1.6}
\]

In the magnetic case, due to the pointwise diamagnetic inequality which means that under rather general assumptions on the magnetic potentials [LL01]

\[|\nabla|u(x)|| \leq |(i\nabla + A)u(x)| \quad \text{for a.a. } x \in \Omega,\]

we get that \( \lambda_1(\Omega, A, 0) \geq \lambda_1(\Omega, 0, 0) \). However, the estimate \( \lambda_j(\Omega, A, 0) \geq \lambda_j(\Omega, 0, 0) \) fails in general if \( j \geq 2 \). Let us mention that, nevertheless, momentum estimates are still valid for some values of the parameters. In particular, it was shown [LW00] that the sharp bound (1.2) holds true for arbitrary magnetic fields provided \( \sigma \geq 3/2 \), and for constant magnetic fields if \( \sigma \geq 1 \)–[ELV00], [KW15]. In the two-dimensional case the bound (1.4) holds true for constant magnetic fields if \( 0 \leq \sigma < 1 \), and the constant on the right-hand side cannot be improved –[FLW09].

In the present work we study the magnetic Schrödinger operators \( H_{\Omega}(A, V) \) defined on the two-dimensional disk \( \Omega \) centered in zero and with radius \( r_0 > 0 \), with a radially symmetric magnetic field \( B(x) = B(|x|) \) and electric potential \( V = V(|x|) \geq 0 \). Our aim is to extend a sufficiently precise Lieb-Thirring type inequality to this situation. A similar problem was studied recently for magnetic Dirichlet Laplacians in [BEKW16], but under very strong restrictions on the growth of the magnetic field.

Let us also mention that some estimates on the counting function of the eigenvalues of the magnetic Dirichlet Laplacian on a disk were established in [T12], in the case where the field is radial and satisfies some growth condition near the boundary.

2. Main Result

Inspired by the weighted one-dimensional Lieb-Thirring type inequalities [EF08] we establish the weighted eigenvalue bound for the operator \( H_{\Omega}(A, V) \) in terms of the magnetic and electric potentials \( B \) and \( V \). The following theorem holds true:

**Theorem 2.1.** Let \( H_{\Omega}(A, V) \) be the magnetic Schrödinger operator with the Dirichlet boundary conditions defined on the disk \( \Omega \) of radius equal to \( r_0 \) centered at the origin with a radial magnetic field \( B(x) = B(|x|) \) and electric potential \( V = V(|x|) \geq 0 \). Let
us assume the validity of the conditions (RC) or the validity of (1.1). Then for any \(0 < \varepsilon \leq 3/4, 0 \leq \alpha < 1\) and \(\sigma \geq (1 - \alpha)/2\), the following inequality holds

\[
\text{tr} (H_\Omega(A,V))_\sigma^\sigma \leq \frac{2r_0 L_{\sigma+1/2,\alpha}}{1 - \varepsilon} \int_{r_0}^{r_0} \left( \frac{1}{\varepsilon} \right) \frac{1}{r^2} \left( \int_0^r s B(s) \, ds \right)^2 + V(r) - \frac{1}{4r^2} \right)^{\frac{\sigma+1+\alpha}{2}} r^\alpha \, dr
\]

+ \frac{L_{\sigma,\alpha}}{\sqrt{1 - \varepsilon}} \int_{r_0}^{r_0} \left( \frac{1}{\varepsilon} \right) \frac{1}{r^2} \left( \int_0^r s B(s) \, ds \right)^2 + V(r) - \frac{1}{4r^2} \right)^{\frac{\sigma+1+\alpha}{2}} r^\alpha \, dr
\]

+ \frac{L_{\sigma,\alpha}}{\sqrt{1 - \varepsilon}} \int_{r_0}^{r_0} \left( V(r) - \frac{1}{r^2} \left( \int_0^r s B(s) \, ds \right)^2 \right)^{\frac{\sigma+1+\alpha}{2}} r^\alpha \, dr, \quad (2.1)

where \(L_{\sigma+1/2,\alpha}\) and \(L_{\sigma,\alpha}\) are some constants.

**Remark 2.1.** If \(0 \leq \sigma < 3/2\) then even for magnetic Laplacians \((1.2)\) type inequality is known only for constant magnetic fields.

**Remark 2.2.** If \(\sup_{r<r_0} (V(r) - \frac{1}{4r^2}) < -A^2/3\), where \(A\) is: \(\sup_{r<r_0} \frac{1}{r} \int_0^r s B(s) \, ds < \infty\) then we can choose \(\varepsilon \geq 3/4\) such that the first two terms of the right hand side of (2.1) be equal to zero. So we decrease the order of the potential \(V\) in Lieb-Thirring bound \((2.1)\) from \(\sigma + 1\) to \(\sigma + (1 + \alpha)/2 < \sigma + 1\).

**Proof.** We begin by recalling the standard partial wave decomposition :–[E96]

\[
L^2(\Omega, dx) = \bigoplus_{m=-\infty}^{\infty} L^2((0, r_0), 2\pi r dr)
\]

\[
f \rightarrow (\ldots, f_1, f_0, f_1, \ldots) \quad \text{with} \quad f(r, \theta) = \sum_{m=-\infty}^{\infty} e^{im\theta} f_m(r).
\]

Choosing the radial gauge \(A(r, \theta) = (-a(r)\sin\theta, a(r)\cos\theta)\) where

\[
a(r) := \frac{1}{r} \int_0^r s B(s) \, ds,
\]

we get that the operator \(H_\Omega(A,V)\) acts on \(\bigoplus_{m=-\infty}^{\infty} L^2((0, r_0)\) as follows

\[
H_\Omega(A,V) = \bigoplus_{m=-\infty}^{\infty} h_m(B, V),
\]

where the operators \(h_m(B, V)\) are the Friedrichs extension of the closures of the quadratic forms

\[
Q(h_m(B, V))[u] = 2\pi \int_0^{r_0} \left( \frac{|du|}{dr} \right)^2 + \left( \frac{m}{r} - a(r) \right) |u|^2 - V|u|^2 \right) r \, dr,
\]

defined originally on \(C_0^\infty(0, r_0)\), and acting on their domain as

\[
h_m(B) = -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \left( \frac{m}{r} - a(r) \right)^2 - V(r).
\]
Employing the mapping $U : C_0^\infty(0, r_0) \to C_0^\infty(0, r_0)$ defined by

$$(Uf)(r) = \frac{1}{\sqrt{2\pi r}} f(r)$$

one gets the unitarily equivalence between the operators $h_m(B)$ and $l_m(B) = -\frac{d^2}{dr^2} - \frac{1}{4r^2} + \left(\frac{m}{r} - a(r)\right)^2 - V(r)$ defined already on $L^2(0, r_0)$. Thus we are going to consider the self-adjoint operators associated to the closures of the quadratic forms

$$Q(l_m(B, V))[v] = \int_0^{r_0} \left(\frac{|dv|^2}{dr} - \frac{1}{4r^2}|v|^2 + \left(\frac{m}{r} - a(r)\right)^2 |v|^2 - V(r)|v|^2\right) dr,$$

defined originally on $C_0^\infty(0, r_0)$.

We have, for any $0 < \varepsilon < 1$ and any $v \in C_0^\infty(0, r_0)$

$$Q(l_m(B, V))[v] = \int_0^{r_0} \left(\frac{|dv|^2}{dr} - \frac{1}{4r^2}|v|^2 + \frac{m^2}{r^2}|v|^2 - \frac{2m}{r}a(r)|v|^2 + a^2(r)|v|^2 - V(r)|v|^2\right) dr \geq \int_0^{r_0} \left(\frac{|dv|^2}{dr} - \frac{1}{4r^2}|v|^2 + \frac{m^2}{r^2}|v|^2 - \frac{m^2\varepsilon}{r^2}|v|^2 - \frac{1}{\varepsilon}a^2(r)|v|^2 + a^2(r)|v|^2 - V(r)|v|^2\right) dr.$$  

It follows from the above inequality that if $m \neq 0$ and $0 < \varepsilon \leq 3/4$

$$l_m(B, V) \geq g_{B,V} + \frac{(1-\varepsilon)m^2 - 1/4}{r_0^2},$$

where the operator $g_{B,V}$ is associated with the closure of the form

$$Q(g_{B,V})[v] = \int_0^{r_0} \left(\frac{|dv|^2}{dr} - \frac{1}{\varepsilon - 1}\right) a^2(r)|v|^2 - V(r)|v|^2\right) dr$$

initially defined on $C_0^\infty(0, r_0)$.

Let $\{\mu_k(B, V)\}_{k=1}^\infty$ be the set of the negative eigenvalues of $g_{B,V}$. Due to the
minimax principle the inequality \((2.2)\) implies

\[
\begin{align*}
\text{tr} \left( \bigoplus_{m=-\infty}^{\infty} h_m(B, V) \right)^\sigma & \leq \sum_{m=-\infty, m\neq 0}^{\infty} \text{tr} \left( g_{B,V} + \frac{(1-\varepsilon)m^2 - 1/4}{r_0^2} \right)^\sigma \\
+ & \text{tr} \left( -\frac{d^2}{dr^2} - \frac{1}{4r^2} + a^2(r) - V(r) \right)^\sigma \\
& \leq \sum_{m=-\infty}^{\infty} \sum_{m'=0}^{\infty, m\neq m'} \left| \mu_k(B, V) + \frac{(1-\varepsilon)m^2 - 1/4}{r_0^2} \right|^\sigma \\
+ & \text{tr} \left( -\frac{d^2}{dr^2} - \frac{1}{4r^2} + a^2(r) - V(r) \right)^\sigma \\
& \leq \sum_{k=1}^{\infty} \left( 2\sqrt{\left| \mu_k(B, V) \right| r_0^2 + 1/4} \right)^\sigma \\
+ & \text{tr} \left( -\frac{d^2}{dr^2} - \frac{1}{4r^2} + a^2(r) - V(r) \right)^\sigma \\
& \leq \frac{2r_0}{\sqrt{1-\varepsilon}} \sum_{k=1}^{\infty} \left| \mu_k(B, V) \right|^{\sigma+1/2} + \frac{1}{\sqrt{1-\varepsilon}} \sum_{k=1}^{\infty} \left| \mu_k(B, V) \right|^\sigma \\
+ & \text{tr} \left( -\frac{d^2}{dr^2} - \frac{1}{4r^2} + a^2(r) - V(r) \right)^\sigma.
\end{align*}
\]

(2.3)

Let us extend the potential \(- \left( \frac{1}{\varepsilon} - 1 \right) a^2(r) - V(r)\) to \(\mathbb{R}_+\) by zero and denote the corresponding one dimensional Schrödinger operator by \(g^*(B, V)\). Since \(C_0^\infty(0, r_0) \subset C_0^\infty(\mathbb{R}_+)\) then by minimax principle for any \(\delta > 0\)

\[
\sum_k |\mu_k(B, V)|^\delta \leq \sum_k |\nu_k(B, V)|^\delta,
\]

(2.4)

where \(\{\nu_k(B, V)\}_{k=1}^{\infty}\) are the negative eigenvalues of \(g^*(B, V)\).

Applying the Lieb-Thirring inequality \([\text{EF08]}\) for any \(\alpha \in [0, 1)\) and \(\sigma \geq (1-\alpha)/2\) we get

\[
\begin{align*}
\sum_{k=1}^{\infty} |\nu_k(B, V)|^{\sigma+1/2} & \leq L_{\sigma + 1/2, \alpha} \int_0^{r_0} \left( \left( \frac{1}{\varepsilon} - 1 \right) a^2(r) + V(r) - \frac{1}{4r^2} \right)^{\sigma+1/2} r^\alpha dr, \\
\sum_{k=1}^{\infty} |\nu_k(B, V)|^{\sigma} & \leq L_{\sigma, \alpha} \int_0^{r_0} \left( \left( \frac{1}{\varepsilon} - 1 \right) a^2(r) + V(r) - \frac{1}{4r^2} \right)^{\sigma+(1+\alpha)/2} r^\alpha dr, \\
\text{tr} \left( -\frac{d^2}{dr^2} - \frac{1}{4r^2} + a^2(r) - V(r) \right)^\sigma & \leq L_{\sigma, \alpha} \int_0^{r_0} \left( V(r) - a^2(r) \right)^{\sigma+(1+\alpha)/2} r^\alpha dr.
\end{align*}
\]

(2.5)
where $L_{\sigma+1/2,\alpha}$ and $L_{\sigma,\alpha}$ are some constants.

This together with the estimates (2.3)–(2.4) means

$$\text{tr} \left( \bigoplus_{m=-\infty}^{\infty} h_m(B, V) \right)^{\sigma}$$

$$\leq \frac{2r_0L_{\sigma+1/2,\alpha}}{\sqrt{1-\varepsilon}} \int_0^{r_0} \left( \left( \frac{1}{\varepsilon} - 1 \right) a^2(r) + V(r) - \frac{1}{4r^2} \right)_+^{\sigma+1+\alpha/2} r^\alpha \, dr$$

$$+ \frac{L_{\sigma,\alpha}}{\sqrt{1-\varepsilon}} \int_0^{r_0} \left( \left( \frac{1}{\varepsilon} - 1 \right) a^2(r) + V(r) - \frac{1}{4r^2} \right)_+^{\sigma+(1+\alpha)/2} r^\alpha \, dr$$

$$+ L_{\sigma,\alpha} \int_0^{r_0} (V(r) - a^2(r))_+^{\sigma+(1+\alpha)/2} r^\alpha \, dr,$$

which proves the theorem.

\[\square\]

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