Compressive Sensing Reconstruction for Sparse 2D Data

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Abstract—In this paper we study the compressive sensing effects on 2D signals exhibiting sparsity in 2D DFT domain. A simple algorithm for reconstruction of randomly under-sampled data is proposed. It is based on the analytically determined threshold that precisely separates signal and non-signal components in the 2D DFT domain. The proposed solution shows promising results in ISAR imaging, where the reconstruction is achieved even in the case when less than 10% of data is available.

Index Terms—Compressive sensing, signal reconstruction, missing data, ISAR imaging, sparsity

I. INTRODUCTION

COMPRESSIVE sensing (CS) has been introduced recently, as a fundamentally new approach to data acquisition, that overcomes the common approach having large amount of redundant information [1]-[3]. In many of practical applications traditional signal acquisition is complex, resource-demanding and time-consuming. In the case of CS, the acquisition process is much simpler, while full signal reconstruction is done afterwards.

CS assumes that the signal of interest is sparse. This is generally the case in various applications, where the signals, represented by suitable transform basis, have only few nonzero coefficients. Therefore, one of the two important tasks is to determine the domain of signal sparsity. The other requirement is related to the incoherent measurements. In that case, the CS reconstruction can be done by applying efficient minimization algorithms [3]-[6]. Since the CS deals with the undersampled signals, the system of linear equations describing the relationship between the measurements and transform domain, is underdetermined and has infinitely many solutions [1],[2]. Here, the sparsity property brings a major contribution, since it helps to reconstruct the original vector by searching for the sparsest solution that is consistent with the linear measurements. In order to avoid the $\ell_0$-norm minimization, which is generally NP-hard, two alternative approaches were developed: convex relaxation based on $\ell_1$-norm minimization and greedy algorithms.

In this letter we focus on a simple and fast algorithm for the reconstruction of sparse 2D signals, which belongs to the group of greedy algorithm. The proposed solution is completely driven by the extensive analysis and derivations of missing samples phenomena in the 2D DFT domain. Namely, it has been shown that the main difficulty in signal reconstruction is spectral dispersion that appears as a consequence of vast majority of missing samples [4]; If we are able to estimate the spectral noise parameters induced by CS, we can distinguish between signal and non-signal/noise components [7],[8], which is primarily the aim of this work. Thus, the proposed solution does not use any a priori information about the signal and the expected number of components (as it is common in the case of other greedy algorithms). In most applications the proposed algorithm will provide a high precision reconstruction in a single step. For more complex cases when the algorithm requirements are not met, the procedure includes just a couple of iterations.

This work is motivated by the inverse synthetic aperture radar (ISAR) images of a target, obtained using the 2D DFT of the received signal. Such a signal is usually sparse and consists of several pulses at the range and cross-range positions produced by reflecting points [9],[10]. According to the popular CS approaches, radar image can be perfectly reconstructed using far fewer samples than it was done so far (only 9% of samples are used in the experiments).

The letter is organized as follows. The modeling of the missing data phenomenon is done in Section II. An efficient reconstruction algorithm is proposed in Section III, while the experimental evaluation is done in Section IV.

II. MISSING DATA MODELING IN THE CASE OF COMPRESSIVE SENSED 2D SIGNALS

A. Spectral noise in the 2D DFT induced by the incomplete set signals

Consider a two-dimensional multicomponent signal in the form:

$$s(x,y) = \sum_{i=1}^{K} A_i e^{j2\pi k_{ix} x/N_x} e^{j2\pi k_{iy} y/N_y} \quad (1)$$

where $A_i$ denotes the $i$-th component amplitude, while $k_{ix}$ and $k_{iy}$ denote frequencies of the $i$-th component. The two-dimensional (2D) DFT of such a signal can be written as:
\[ S(k_x, k_y) = N_x N_y \sum_{x=1}^{N_x} \sum_{y=1}^{N_y} A e^{-j2\pi(k_x-k_{x_i})x/N_x} e^{-j2\pi(k_y-k_{y_i})y/N_y} \]  

Furthermore, we may observe the following (full) set of samples \((N_x \text{ samples in x direction, and } N_y \text{ along the y dimension})\) given in the form:

\[ Z = \{(x, y) : x \in \{1, ..., N_x\}, \ y \in \{1, ..., N_y\}\}, \tag{3} \]

where \(z(x, y) = A e^{-j2\pi(p_1 x/N_x + q_1 y/N_y)}\), \(p_1 = k_x - k_{x_i}\), \(q_1 = k_y - k_{y_i}\).

In the case of compressed sensed signal, instead of the full set of samples \(Z\), we observe the partial set \(W\):

\[ W = \{(w(x, y) : x \in \{1, ..., M_x\}, \ y \in \{1, ..., M_y\}\) \subset \mathbb{Z} \}, \tag{4} \]

having \(M_x\) samples available along \(x\) direction and \(M_y\) samples available along \(y\) direction.

Obviously, we need to deal with missing samples and consequently, an appropriate model of the missing samples occurrence is required [4]. In that sense, let us observe the missing samples as zero values occurring due to an additive noise influence given in the form:

\[ \epsilon(x, y) = \begin{cases} -z(x, y) = e^{-j2\pi p_1 x/N_x} e^{-j2\pi q_1 y/N_y} & \text{for } (x, y) \text{ belonging to available samples} \\ 0 & \text{otherwise} \end{cases} \]

The 2D DFT of the available samples \(W\) can be written as:

\[ F(k_x, k_y) = \sum_{x=0}^{M_y-1} \sum_{y=0}^{M_y-1} z(x, y) + \epsilon(x, y). \tag{5} \]

In order to characterize the influence of missing samples, in the sequel we will observe the following cases:

a) If \(k_x = k_{x_i}, k_y = k_{y_i}\) the expectation is given as:

\[ E\{F(k_x, k_y)\} = A e^{-j2\pi(k_x-k_{x_i})x/N_x} e^{-j2\pi(k_y-k_{y_i})y/N_y} \sum_{x=1}^{M_x} \sum_{y=1}^{M_y} e^{-j2\pi(k_{x_i}-k_x)x/N_x} e^{-j2\pi(k_{y_i}-k_y)y/N_y} \]  
\[ + A e^{-j2\pi(k_x-k_{x_i})x/N_x} e^{-j2\pi(k_y-k_{y_i})y/N_y} \sum_{x=1}^{M_x} \sum_{y=1}^{M_y} e^{-j2\pi(k_{x_i}-k_x)x/N_x} e^{-j2\pi(k_{y_i}-k_y)y/N_y} \]  
\[ + \ldots + A e^{-j2\pi(k_x-k_{x_i})x/N_x} e^{-j2\pi(k_y-k_{y_i})y/N_y} \sum_{x=1}^{M_x} \sum_{y=1}^{M_y} e^{-j2\pi(k_{x_i}-k_x)x/N_x} e^{-j2\pi(k_{y_i}-k_y)y/N_y} \]  
\[ = A M_x M_y. \]

b) If \(k_x \neq k_{x_i} \quad \text{or} \quad k_y \neq k_{y_i}\), the expectation is given as:

\[ E\{F(k_x, k_y)\} = 0, \tag{7} \]

while \(E\{F(k_x, k_y)\} = 0\) for \(k_x = k_{x_i}\) (which also holds for the coordinate \(y\) and \(k_y = k_{y_i}\)).

Now, we can determine the variance of noise introduced in \(F(k_x, k_y)\):

\[ \sigma^2 \{F(k_x \neq k_{x_i}, k_y \neq k_{y_i})\} = M_x M_y \cdot E[z(x, y)z^*(x, y)] + M_x M_y (M_x M_y - 1) \cdot E[z(x, y)z^*(m, n)] = A M_x M_y + A M_x M_y (M_x M_y - 1) \pi \frac{-1}{N_x N_y - 1}. \tag{8} \]

Finally, in the compact form, the variance that appear in the 2D DFT as a reflection of missing samples influence, can be calculated according to (for \(K\) components):

\[ \sigma^2 = \sum_{i=1}^{K} A_i M_i M_y \frac{N_x N_y - M_x M_y}{N_x N_y - 1}. \tag{9} \]

It is important to emphasize that the same analysis holds for radar signal model, when continuous wave radar transmits a signal in the form of series of chirps and the missing samples appear in one or different chirps. In analogy, the received signal will consist of \(K\) components reflected from different points. After the distance compensation, the received signal can be written as in (1):

\[ R(x, y) = \sum_{i=1}^{N} e^{(2\pi f_0 t/N_y) y}, \tag{9} \]

where the \(A_i r_i\) is the \(i\)-th reflection coefficient of the target, while \(k_x = \alpha_i\) and \(k_y = \beta_y\) correspond to the parameters proportional to the velocity and range, respectively.

III. SIMPLE AND FAST ALGORITHM FOR RECONSTRUCTION OF 2D SIGNALS – SFAR-2D

The estimated noise parameters such as noise variance derived in the previous Section, can be efficiently used to distinguish between signal components, i.e., useful information, and noise components. Here, we can note that the noise components can originate from missing samples, while the noise variance directly depends on the missing samples number. Also, the noise may appear from an external source, making the extraction of signal components even more difficult. On the other side, if we are able to estimate level of noise and to detect positions of useful signal components above the noise, then it would be possible to reconstruct the original signal, as it will presented in the sequel.

Let us observe the noise model introduced in Section II. According to the central limit theorem, the real and imaginary parts of the 2D DFT values at the position where only noise exists (none of the \(K\) signal components occur) can be described by the Gaussian distribution. The corresponding probability density function (pdf) for the absolute values of 2D DFT is modeled by the Rayleigh function. The probability that all DFT values at noise positions are lower than the certain threshold \(\chi\) is given by:

$$ P(|Z| < \chi) = \frac{\chi}{\bar{\chi}} e^{-\frac{\chi^2}{2\bar{\chi}^2}} $$

where \(\bar{\chi}\) is the average of noise values.
the following minimization problem:

\[ P(\chi) = \left( 1 - \frac{2p}{\chi} e^{-\frac{\chi^2}{2\sigma^2}} \right)^{N_y N_x} \]

where \( N_y N_x \) is the total number of samples in 2D DFT of size \( N_y \times N_x \), while \( K \) is the number of signal components and \( K = N_y N_x \). For instance, we can set a fixed probability of error \( P_{\text{Rx}} = 0.99 \) and calculate the threshold as follows:

\[ \chi = \sigma \sqrt{-\log(1 - P_{\text{Rx}})} . \]

It means that with the probability \( P_{\text{Rx}} \) all noise components will be below the signal components. Therefore, if we observe the set of positions representing the support of available samples within the 2D signal:

\[ (x, y) \in \Omega = \{(x_1, y_1), (x_2, y_2), ..., (x_M, y_M)\} , \]

(12)

Then the vector of measurements containing only \( M \) columns out of \( N_y N_x \) samples of original signal can be written in the form:

\[ y = \chi(\Omega) . \]

(13)

Furthermore, we can denote the Fourier transform of the full signal set using the 2D DFT matrix \( \Psi(N_y N_x) \), obtained as Kronecker product of two DFT matrices:

\[ \Psi = \text{DFT}_{N_y N_x} \otimes \text{DFT}_{N_y N_x} . \]

Consequently, the initial Fourier transform of available measurements can be written as follows:

\[ Y = \Psi(\Omega)y \]

(14)

where \( y \) is of size \((M, M, 1)\), while \( \Psi(\Omega) \) contains only \( M \) columns from \( \Psi \) that correspond to available time instants \( \Omega \). Vector \( Y \) contains the 2D DFT coefficients of the incomplete set of measurements \( y \), and thus contains the noise due to the missing samples. In order to select the signal components among the noise, we can apply threshold \( \chi \) to obtain the frequency support of the signal:

\[ k = \text{arg}\{ |Y| > \chi \} . \]

(15)

Finally, we can obtain the reconstructed signal by solving the following minimization problem:

\[ S = (\Psi(\tilde{k}, \Omega)^\dagger \Psi(\tilde{k}, \Omega))^{-1} \Psi(\tilde{k}, \Omega)^\dagger y \]

(16)

where \( S \) is the recovered full 2D DFT of signal, while \( \Psi(\tilde{k}, \Omega) \) is a Compressive sensing matrix that contains rows defined by selected frequency set \( \tilde{k} \) and the columns defined by the positions of measurements \( \Omega \). Based on the previous discussion, we can define a simple and fast algorithm that can be easily applied to the sparse 2D signals in the 2D DFT domain. Fig. 1 summarizes the proposed SFAR-2D algorithm in the pseudo-code format.

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In the case of large number of missing samples, some DFT components will be masked by a strong noise. Therefore, we need slightly modified form, which is summarized in Fig. 2. This is actually an iterative form of the previous algorithm. Namely, in order to reveal weaker components it is important to remove the contribution of stronger ones. Thus, in each iteration \( i \), a set of components on positions \( k_i \) is detected and removed. After just a couple of iterations, we are able to reconstruct all signal components.

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Noisy signal case: If the observed compressive sensed signal is corrupted by external noise, this influence needs to be included in the algorithm as well. Namely, assuming that the variance of external noise is \( \sigma^2 \), the step 1 of both algorithm versions need to include this parameters as follows:
\[ \sigma^2 = \sigma_x^2 + \sum_{i=1}^{K} \frac{M_x M_y (N_x N_y - M_x M_y)}{(N_x N_y - 1)}. \]

IV. EXAMPLES

**Example 1:** Consider a signal that consists of 12 reflecting points:

\[ R(x, y) = \sum_{i=1}^{K} \frac{e^{j2\pi x_i/N_x} e^{j2\pi y_i/N_y}}, \]

with reflecting coefficients varying in the range \(2 < r < 3\). Only 9% of the total number of samples are available (91% of samples are missing). In order to illustrate the desired and expected result of the reconstruction procedure, the 2D DFT of the original signal (ISAR image) with full set of samples is shown in Fig. 3 (a and b).

![Fig. 3. 2DDFT of the considered radar signal, a) 2D view, b) 3D view](image)

Apart from the noise that is produced in the 2D DFT as a consequence of a large number of missing samples, the signal is also corrupted by the external noise (Sig_noise ili SNR). The signal reconstruction procedure is done according to the proposed **SFAR-2D algorithm – single step version**. Fig. 4.a illustrates the result of applying the threshold \(\chi\), which detects all 12 components from the 2D DFT obtained using only 9% of available samples. Based on the detected parameters of components, the signal is successfully reconstructed. The final result of reconstruction is shown in Fig. 4.b.

![Fig. 4. Result of applying the proposed algorithms: a) Components of two-dimensional signal detected all in a single step, b) Resulting recovered 2D DFT of the considered signal](image)

**Example 2:** In this example, we consider the case when the reflecting coefficients notably differ, such that the stronger components are above the noise, but weaker components are immersed into noise. The signal consists of 12 components, where some of the components are significantly smaller than the others. Also, we deal with low percent of available samples (9%). In this case, for the full signal reconstruction, it is necessary to apply the second version of the algorithm, since it is not possible to detect all components at once. After detecting the first set of components, we eliminate their influence in the spectral representation and continue with procedure with new signal parameters (according to Fig. 2). The results of applying the threshold are presented in Fig. 5a (first iteration), and Fig. 5b (second iteration). It is important to emphasize that the result are achieved after just couple of iterations (Fig 5.c), which is an advantage over exhaustive iterative procedure used in CS reconstruction applications.
In this paper we derived the analytical expression describing the spectral dispersion that appears in the transform domain as a consequence of missing data. The analysis is motivated by the ISAR applications and accordingly the 2D DFT is assumed as a domain of signal sparsity. The proposed analysis reveals an important feature: the number of missing samples can be used to define the threshold which precisely separates signal and non-signals components, and facilitates signal reconstruction when dealing with CS signals. Two versions of the algorithm are proposed: the first one is based on the simple single step solution, while the second one performs additional iteration/s when some components are much weaker than the others.

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