On the symplectic phase space of KdV

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Abstract

We prove that the Birkhoff map \( \Omega \) for KdV constructed on \( H^{-1}_0(T) \) can be interpolated between \( H^{-1}_0(T) \) and \( L^2_0(T) \). In particular, the symplectic phase space \( H^{1/2}_0(T) \) can be described in terms of Birkhoff coordinates.

1 Introduction

In [12] it is shown that the Birkhoff map for the Korteweg - de Vries equation (KdV), on the circle \( T := \mathbb{R}/\mathbb{Z} \), introduced and studied in detail in [9, 6] can be analytically extended to an analytic diffeomorphism

\[
\Omega : H^{-1}_0(T) \to \mathfrak{h}^{1/2}
\]

from the Sobolev space of distributions \( H^{-1}_0(T) \) (dual of \( H^1_0(T) \)) to the Hilbert space of sequences \( \mathfrak{h}^{1/2} \) where for any \( \alpha \in \mathbb{R} \),

\[
\mathfrak{h}^\alpha := \{ z = (x_k, y_k)_{k \geq 1} \mid \| z \|_\alpha < \infty \},
\]

with

\[
\| z \|_\alpha := \left( \sum_{k \geq 1} k^{2\alpha} (x_k^2 + y_k^2) \right)^{1/2}.
\]

In this paper we show that \( \Omega \) can be interpolated between \( H^{-1}_0(T) \) and \( L^2_0(T) \).

**Theorem 1.** For any \( -1 \leq \alpha \leq 0 \),

\[
\Omega|_{H^\alpha_0(T)} : H^\alpha_0(T) \to \mathfrak{h}^{\alpha+1/2}
\]

is a real analytic diffeomorphism.

As an application of Theorem 1 we characterize the regularity of a potential \( q \in H^{-1}(T) \) in terms of the decay of the gap lengths \( (\gamma_k)_{k \geq 1} \) of the periodic spectrum of Hill’s operator \( -\frac{d^2}{dx^2} + q \) on the interval \([0, 2]\). More precisely, recall

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that the periodic spectrum of \(-\frac{d^2}{dx^2} + q\) on the interval \([0, 2]\) is discrete. When listed in increasing order (with multiplicities) the eigenvalues \((\lambda_k)_{k \geq 0}\) satisfy

\[ \lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \ldots \]

The gap lengths \(\gamma_k = \gamma_k(q)\) are then defined by

\[ \gamma_k := \lambda_{2k} - \lambda_{2k-1} \quad (k \geq 1). \]

**Theorem 2.** For any \(q \in H^{-1}(T)\) and any \(-1 \leq \alpha < 0\), the potential \(q\) is in \(H^\alpha(T)\) if and only if \((\gamma_k(q))_{k \geq 1} \in h^\alpha\).

In a subsequent paper we will use Theorem 1 to study the solutions of the KdV equation (see \([2, 3, 14, 21]\)) in the symplectic phase space \(H^{-1/2}_0(T)\) introduced by Kuksin \([16]\).

**Method of proof:** Theorem 2 can be shown to be a consequence of Theorem 1 and formulas relating the \(n\)'th action variable \(I_n\) with the \(n\)'th gap length \(\gamma_n\) and their asymptotics as \(n \to \infty\). In view of results established in \([12]\) the proof of Theorem 1 consists in showing that for any \(-1 < \alpha < 0\) the restriction of \(\Omega\) to \(H_0^\alpha(T)\), \(\Omega|_{H_0^\alpha(T)} : H_0^\alpha(T) \to h^{\alpha+1/2}\), is onto. Our method of proof combines a study of the Birkhoff map at the origin together with a strikingly simple deformation argument to show that the map \(\Omega|_{H_0^\alpha(T)} : H_0^\alpha(T) \to h^{\alpha+1/2}\) is onto. More precisely it uses that (1), \(d_0 \Omega_\alpha : H_0^\alpha(T) \to h^{\alpha+1/2}\) is a linear isomorphism, (2), that the map \(\Omega : H_0^{-1}(T) \to h^{-1/2}\) is a canonical bi-analytic diffeomorphism, and (3), that the Hamiltonian vector field defining the deformation is actually in \(L^2\). The same method could also be used for the proof of analogous results for more general weighted Sobolev spaces. In a subsequent work we plan to apply our technique to the defocusing Nonlinear Schrödinger equation.

**Related work:** Theorem 1 improves on earlier results in \([12]\) where it was shown that \(\Omega|_{H_0^\alpha(T)} : H_0^\alpha(T) \to h^{\alpha+1/2}\) is a bianalytic diffeomorphism onto its image for any \(-1 < \alpha < 0\). For partial results in this direction see also \([15]\). The statement of Theorem 2 adds to numerous results characterizing the regularity of a potential by the decay of the corresponding gap lengths – see e. g. \([4, 7, 15, 17, 19]\) and references therein. However only a few results concern potentials in spaces of distributions – see \([3, 15]\) (cf. also \([12]\) and the references therein). In a first attempt we have tried to apply the most beautiful and most simple approach among all the papers cited, due to Pöschel \([19]\), to our case. However his methods seem to fail if \(\alpha \leq -3/4\).

The idea of using flows to prove that a map is onto is not new in this subject. It has been used e.g. by Pöschel and Trubowitz in their book \([20]\) or, to give a more recent example, in work of Chelkak and Korotyaev \([1]\). More precisely, in \([20, \text{Theorem 2, p. 115}]\), the authors use flows to characterize sequences coming up as sequences of Dirichlet eigenvalues of Schrödinger operators \(-\frac{d^2}{dx^2} + q\) on \([0, 1]\) with an even \(L^2\)-potential \(q\). Note however, that in this paper the
use of flows is of a different nature, best explained by the fact that they are regularizing - in other words, the vector fields describing the deformations are in a higher Sobolev space than the underlying phase space.

2 Proof of Theorem 1

Let $\Omega$ be the Birkhoff map $\Omega : H^{-1}_0(\mathbb{T}) \to \mathfrak{h}^{-1/2}$ constructed in [12] – see also Appendix for a brief summary of the results in [12]. By Theorem 3 in Appendix, the Birkhoff map $\Omega$ is onto and for any given $\alpha > -1$ its restriction to $H^\alpha_0(\mathbb{T})$ is a map

$$\Omega_\alpha := \Omega_{|H^\alpha_0(\mathbb{T})} : H^\alpha_0(\mathbb{T}) \to \mathfrak{h}^{\alpha + 1/2}$$

which is a bianalytic diffeomorphism onto its image. Hence, in order to prove Theorem 1 we need to prove that (1) is onto.

Assume that there exists $-1 \leq \alpha \leq 0$ such that $\Omega_\alpha : H^\alpha_0(\mathbb{T}) \to \mathfrak{h}^{\alpha + 1/2}$ is not onto. As $\Omega : H^{-1}_0(\mathbb{T}) \to \mathfrak{h}^{-1/2}$ is onto it then follows that there exists

$$q_0 \in H^{-1}_0(\mathbb{T}) \setminus H^\alpha_0(\mathbb{T})$$

such that $\Omega(q_0) \in \mathfrak{h}^{\alpha + 1/2}$.

As $\Omega(0) = 0$ and as by Corollary 1 in the Appendix below the differential $d_0\Omega_\alpha : H^\alpha_0(\mathbb{T}) \to \mathfrak{h}^{\alpha + 1/2}$ of (1) at $q = 0$ is a linear isomorphism, one gets from the inverse function theorem that there exist an open neighborhood $U_\alpha$ of zero in $H^\alpha_0(\mathbb{T})$ and an open neighborhood $V_\alpha$ of zero in $\mathfrak{h}^{\alpha + 1/2}$ such that

$$\Omega_{|U_\alpha} : U_\alpha \to V_\alpha$$

is a diffeomorphism.

Recall that for any $k \geq 1$ the angle variable $\theta_k$ constructed in [12] is a real-analytic function on $H^{-1}_0(\mathbb{T}) \setminus D_k$ with values in $\mathbb{R}/2\pi\mathbb{Z}$ where $D_k := \{ q \in H^{-1}_0(\mathbb{T}) | \gamma_k(q) = 0 \}$ is a real-analytic sub-variety in $H^{-1}_0(\mathbb{T})$ (cf. Appendix). As $\theta_k$ is real-analytic, the mapping $H^{-1}_0(\mathbb{T}) \setminus D_k \to H^1_0(\mathbb{T})$, $q \mapsto \partial_k \theta_k(q)$, is real-analytic and therefore,

$$H^{-1}_0(\mathbb{T}) \setminus D_k \to L^2_0(\mathbb{T}), \quad q \mapsto Y_k(q) := \frac{d}{dx} \frac{\partial \theta_k}{\partial q}(q),$$

is real-analytic as well. Then $Y_k$ is a Hamiltonian vector field on $H^{-1}_0(\mathbb{T}) \setminus D_k$, which defines a dynamical system

$$\dot{q} = Y_k(q), \quad q(0) = q_0 \in H^{-1}_0(\mathbb{T}) \setminus D_k.$$ 

Let $q_0 \in H^{-1}_0(\mathbb{T}) \setminus D_k$ and assume that

$$\Omega(q_0) = (z_1^0, z_2^0, ...) \in \mathfrak{h}^{\alpha + 1/2}$$

$^1 \frac{\partial \theta_k}{\partial q}$ denotes the $L^2$-gradient of $\theta_k$.  

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where for any \( n \geq 1 \), \( z_n^0 = (x_n^0, y_n^0) \). Take \( \varepsilon > 0 \) such that the ball
\[
B(2\varepsilon) := \{z \in \mathfrak{h}^{\alpha+1/2} \mid \|z\|_{\alpha+1/2} < 2\varepsilon \}
\]
is contained in the neighborhood \( V_\alpha \) of zero in \( \mathfrak{h}^{\alpha+1/2} \) chosen in (9). Denote by \( I_n = I_n(q) \) the \( n \)th action variable of a potential \( q \) see Appendix. Note that for any \( q \) in \( H_0^{-1}(\mathbb{T}) \)
\[
2 I_n(q) = \|z_n(q)\|^2 = x_n(q)^2 + y_n(q)^2 \quad (6)
\]
where \( \Omega(q) = (z_n(q))_{n \geq 1} \) and \( z_n(q) = (x_n(q), y_n(q)) \). Consider the sequence of potentials \( (q_n)_{n \geq 1} \) in \( H_0^{-1}(\mathbb{T}) \) defined recursively for \( n \geq 1 \) by
\[
q_n := \begin{cases} q_{n-1} & \text{if } 2 I_n(q_{n-1}) < \varepsilon/(n^{1+2\alpha} 2^n) \\ (q_{n-1}),_{n} & \text{otherwise} \end{cases}
\]
where \( (q_{n-1}),_{n} \) is obtained by shifting \( q_{n-1} \) along the flow of the vector field \( Y_n \) such that
\[
2 I_n((q_{n-1}),_{n}) < \varepsilon/(n^{1+2\alpha} 2^n).
\]
The existence of \( (q_{n-1}),_{n} \) follows from Lemma [1] (a) below. Moreover, by the commutator relations (19) in Appendix,
\[
Y_n(I_m) = \{I_m, \theta_n\} = 0 \quad (n \neq m),
\]
the vector field \( Y_n \) preserves the values of the action variables \( I_m \) for any \( m \neq n \). In particular, we get
\[
2 I_j(q_n) \leq \varepsilon/(j^{1+2\alpha} 2^j), \quad \forall 1 \leq j \leq n \quad (7)
\]
and
\[
2 I_j(q_n) = \|z_j^0\|^2, \quad \forall j > n. \quad (8)
\]
One obtains from (7), (8), and \( \|z_j\|^2 = 2 I_j \) (cf. (17)) that
\[
\|\Omega(q_n)\|_{\alpha+1/2}^2 = \sum_{j=1}^{\infty} j^{1+2\alpha} \|z_j(q_n)\|^2 \leq \varepsilon \sum_{1 \leq j \leq n} \frac{1}{2j} + \sum_{j \geq n+1} j^{1+2\alpha} \|z_j^0\|^2. \quad (9)
\]
As \( \sum_{j=1}^{\infty} j^{1+2\alpha} \|z_j^0\|^2 = \|\Omega(q_0)\|_{\alpha+1/2}^2 < \infty \), one gets from (10) that there exists \( N \geq 1 \) such that for any \( n \geq N \), \( \|\Omega(q_n)\|_{\alpha+1/2} < 2\varepsilon \). In particular, \( \Omega(q_N) \in V_\alpha \) and, as \( \Omega|_{U_\alpha} : U_\alpha \rightarrow V_\alpha \) is a diffeomorphism, the bijectivity of the Birkhoff map \( \Omega : H_0^{-1} \rightarrow \mathfrak{h}^{-1/2} \) implies that
\[
q_N \in U_\alpha \subseteq H_0^0(\mathbb{T}). \quad (10)
\]
On the other side, it follows from (2) and Lemma [1] (b) that
\[
(q_n)_{n \geq 1} \subseteq H_0^{-1}(\mathbb{T}) \setminus H_0^0(\mathbb{T})
\]
which implies \( q_N \in H_0^{-1}(\mathbb{T}) \setminus H_0^0(\mathbb{T}) \), contradicting (10). This completes the proof of Theorem [1] \( \square \)

The following Lemma was used in the proof of Theorem [1]
Lemma 1. For any $k \geq 1$ and for any initial data $q_0 \in H_0^{-1}(\mathbb{T}) \setminus D_k$ the initial value problem (5) has a unique solution in $C^1((-I_k^0, \infty), H_0^{-1}(\mathbb{T}))$ where $I_k^0 \geq 0$ is the value of the action variable $I_k$ at $q_0$. The solution has the following additional properties:

(a) $\lim_{t \to -I_k^0+0} I_k(q(t)) = 0$;

(b) $q(t) - q_0 \in L^2_0(\mathbb{T})$.

Proof of Lemma 1. By Theorem 3 in the Appendix, the Birkhoff map $\Omega : H_0^{-1}(\mathbb{T}) \to \mathfrak{h}^{-1/2}$,

$$q \mapsto \Omega(q) = (z_1, z_2, ...), \quad z_n = (x_n, y_n),$$

is a bianalytic diffeomorphism that transforms the Poisson structure $\{ , \}$ into the canonical Poisson structure on $\mathfrak{h}^{-1/2}$ defined by the relations $\{x_m, x_n\} = \{y_m, y_n\} = 0$ and $\{x_m, y_n\} = \delta_{mn}$ that hold for any $m, n \geq 1$. Moreover, it follows from the construction of the Birkhoff map $\Omega$ that $\theta_k$ is the argument of the complex number $x_k + iy_k$. In particular, in Birkhoff coordinates $(z_1, z_2, ...) \in \mathfrak{h}^{-1/2}$, one has for any $q \in H_0^{-1}(\mathbb{T}) \setminus D_k$

$$d\Omega(Y_k) = \frac{x_k}{x_k^2 + y_k^2} \frac{\partial}{\partial x_k} + \frac{y_k}{x_k^2 + y_k^2} \frac{\partial}{\partial y_k}. \quad (11)$$

The dynamical system corresponding to the vector field (11) in $\mathfrak{h}^{-1/2}$ has a unique solution for any initial data $(x_n^0, y_n^0)_{n \geq 1}$ that is defined on the time interval $(-((x_n^0)^2 + (y_n^0)^2)/2, \infty)$. Hence, as $\Omega : H_0^{-1}(\mathbb{T}) \to \mathfrak{h}^{-1/2}$ is a diffeomorphism, the dynamical system (5) has a unique solution $q(t)$ on $H_0^{-1}(\mathbb{T}) \setminus D_k$ defined for $t \in (-I_k(q_0), \infty)$. Moreover, one gets from (11) and (6) that

$$\lim_{t \to -I_k(q_0)+0} I_k(q(t)) = 0.$$

This completes the proof of (a). In order to prove (b) we integrate both sides of (5) in $H_0^{-1}(\mathbb{T})$ and get that for any $t \in (-\infty, I_k(q_0))$,

$$q(t) = q_0 + \int_0^t Y_k(q(s)) \, ds. \quad (12)$$

As the mapping (4) is real-analytic (and hence, continuous) and as the solution $q(t)$ of (5) is a $C^1$-curve $(-\infty, I_k(q_0)) \to H_0^{-1}(\mathbb{T})$, the integrand in (12) is in $C^0((-I_k(q_0), \infty), L^2_0(\mathbb{T}))$. In particular, the integral in (12) converges with respect to the $L^2$-norm, and hence represents an element in $L^2_0(\mathbb{T})$. This proves (b). \qed

\textsuperscript{2}Here $\delta_{mn}$ denotes the Kronecker delta.
3 Proof of Theorem 2

As for any constant $c \in \mathbb{R}$, the potentials $q$ and $q + c$ have the same sequence of gap lengths $(\gamma_k)_{k \geq 1}$, it is enough to prove the statement of the theorem for $q \in H^{-1}_0(\mathbb{T})$.

For $q \in H^{-1}_0(\mathbb{T})$ given let

$$z = (z_1, z_2, \ldots) = \Omega(q),$$

where for any $n \geq 1$, $z_n = (x_n, y_n)$. By Proposition 1 in Appendix, there exist constants $0 < C_1 < C_2 < \infty$ and $n_0 \geq 1$ depending on $q$ such that for any $n \geq n_0$,

$$C_1 \frac{\gamma_n^2}{n} \leq I_n \leq C_2 \frac{\gamma_n^2}{n}$$

where $I_n$ is the $n$-th action variable of the given potential $q$. Using that

$$I_n = \frac{x_n^2 + y_n^2}{2}$$

we get from (13) that for any given $\alpha \geq -1$,

$$(z_n)_{n \geq 1} \in \mathfrak{h}^{\alpha+1/2} \iff (\gamma_n)_{n \geq 1} \in \mathfrak{h}^{\alpha}. \quad (14)$$

On the other side, it follows from Theorem 1 and the injectivity of $\Omega : H^{-1}_0(\mathbb{T}) \rightarrow \mathfrak{h}^{-1/2}$ that

$$(z_n)_{n \geq 1} \in \mathfrak{h}^{\alpha+1/2} \iff q \in H^\alpha_0(\mathbb{T}). \quad (15)$$

Theorem 2 now follows from (14) and (15). \qed

4 Appendix

In this appendix we collect the properties of the Birkhoff map $\Omega : H^{-1}_0(\mathbb{T}) \rightarrow \mathfrak{h}^{-1/2}$ constructed in [12] that were used in the proofs of Theorem 1 and Theorem 2.

The Korteweg - de Vries equation (KdV)

$$q_t - 6qq_x + q_{xxx} = 0$$

on the circle can be viewed as an integrable PDE, i.e. an integrable Hamiltonian system of infinite dimension. As a phase space we consider the Sobolev space $H^\alpha(\mathbb{T})$ ($\alpha \geq -1$) of real valued distributions on the circle. The Poisson bracket

is the one proposed by Gardner,

$$\{F, G\} := \int_{\mathbb{T}} \frac{\partial F}{\partial q} \frac{d}{dx} \left( \frac{\partial G}{\partial q} \right) dx$$

where $F, G$ are $C^1$-functions on $H^\alpha(\mathbb{T})$ and $\frac{\partial F}{\partial q}, \frac{\partial G}{\partial q}$ denote the $L^2$-gradients of $F$ and $G$ respectively which are assumed to be sufficiently smooth so that the
Poisson bracket is well defined. For $q$ sufficiently smooth, i.e. $q \in H^1_0(\mathbb{T})$, the Hamiltonian $\mathcal{H}$ corresponding to KdV is given by

$$\mathcal{H}(q) = \int_{\mathbb{T}} \left( \frac{(\partial_x q)^2}{2} + q^3 \right) dx$$

and the KdV equation can be written in Hamiltonian form

$$q_t = \frac{d}{dx} \frac{\partial \mathcal{H}}{\partial q}.$$  

Note that the Poisson structure is degenerate and admits the average $[q] := \int_{\mathbb{T}} q(x) dx$ as a Casimir function. Moreover, the Poisson structure is regular and induces a trivial foliation whose leaves are given by

$$H_\alpha \mathbb{T} = \left\{ q \in H^1_0(\mathbb{T}) \mid [q] = c \right\}.$$

Introduce the set $D_k := \left\{ q \in H^{-1}_0(\mathbb{T}) \mid \gamma_k(q) = 0 \right\}$.

For any $q \in H^{-1}_0(\mathbb{T}) \setminus D_k$ define

$$z_k(q) := \sqrt{2I_k(q)} \left( \cos(\theta_k(q)), \sin(\theta_k(q)) \right),$$  

where $I_k(q)$ is the $k$'th action variable and $\theta_k(q)$ is the $k$'th angle variable of the KdV equation (cf. § 3, 4 in [12]). It is shown in [12] § 5 that the mapping $H^{-1}_0 \setminus D_k \to \mathbb{R}^2$, $q \mapsto z_k(q)$, extends analytically to $H^{-1}_0(\mathbb{T})$. For any $q \in H^{-1}_0(\mathbb{T})$ the action variables $(I_k)_{k \geq 1}$ of KdV are defined in terms of the periodic spectrum of the Schrödinger operator $-\frac{d^2}{dx^2} + q$ using the same formulas as in [5] (cf. also [9]). For any given $\alpha \geq -1$ and for any $k \geq 1$ the action $I_k$ is a real analytic function on $H^0_0(\mathbb{T})$ (cf. Proposition 3.3 in [12]). The angle $\theta_k$ is defined modulo $2\pi$ and is a real analytic function on $H^{-1}_0(\mathbb{T}) \setminus (D_k \cap H^0_0)$, where $D_k \cap H^0_0 = \left\{ q \in H^0_0(\mathbb{T}) \mid \gamma_k(q) = 0 \right\}$ is a real analytic sub-variety in $H^0_0(\mathbb{T})$ of co-dimension two (cf. Proposition 4.3 in [12]). By § 6 in [12] we have the following commutator relations

$$\{ I_m, I_n \} = 0 \text{ on } H^{-1}_0(\mathbb{T})$$  

and

$$\{ I_m, \theta_n \} = \delta_{mn} \text{ on } H^{-1}_0(\mathbb{T}) \setminus D_n$$

for any $m, n \geq 1$. For any $q \in H^{-1}_0(\mathbb{T})$ define

$$\Theta(q) := (z_1(q), z_2(q), ...)$$

where $z_k = z_k(q)$ is given by (17). It is shown in [12] that $\Theta(q) \in \mathfrak{h}^{-1/2}$. Recall that, for any $\alpha \in \mathbb{R}$, $\mathfrak{h}^\alpha$ denotes the Hilbert space

$$\mathfrak{h}^\alpha = \{ z = (x_k, y_k)_{k \geq 1} \mid \|z\|_{\alpha} < \infty \}. $$
with the norm
\[ \|z\|_\alpha := \left( \sum_{k \geq 1} k^{2\alpha} (x_k^2 + y_k^2) \right)^{1/2}. \]

We supply \( \mathfrak{h}^{-1/2} \) with a Poisson structure defined by the relations \( \{x_m, x_n\} = \{y_m, y_n\} = 0 \) and \( \{x_m, y_n\} = \delta_{mn} \) valid for any \( m, n \geq 1 \). The following result is proved in [12].

**Theorem 3.** The mapping \( \Omega : H_0^{-1}(\mathbb{T}) \to \mathfrak{h}^{-1/2} \) satisfies the following properties:

(i) \( \Omega \) is a bianalytic diffeomorphism that preserves the Poisson bracket;

(ii) for any \( \alpha > -1 \), the restriction \( \Omega_\alpha \equiv \Omega|_{H_0^{\alpha}(\mathbb{T})} \) is a map \( \Omega|_{H_0^{\alpha}(\mathbb{T})} : H_0^{\alpha}(\mathbb{T}) \to \mathfrak{h}^{\alpha+1/2} \) which is one-to-one and bianalytic onto its image. In particular, the image is an open subset in \( \mathfrak{h}^{\alpha+1/2} \).

**Corollary 1.** For any \( \alpha > -1 \),
\[ d_0 \Omega_\alpha : H_0^{\alpha}(\mathbb{T}) \to \mathfrak{h}^{\alpha+1/2}, \]
is a linear isomorphism.

We will also need the following Proposition (cf. [12, §3]).

**Proposition 1.** There exists a complex neighborhood \( W \) of \( H_0^{-1}(\mathbb{T}) \) in the complex space \( H_0^{-1}(\mathbb{T}, \mathbb{C}) \) such that the quotient \( I_n/\gamma_n^2 \), defined on \( H_0^{-1}(\mathbb{T}) \setminus D_n \), extends analytically to \( W \) for all \( n \). Moreover, for any \( \varepsilon > 0 \) and any \( p \in W \) there exists \( n_0 \geq 1 \) and an open neighborhood \( U(p) \) of \( p \) in \( W \) so that
\[ \left| 8\pi n \frac{J_n}{\gamma_n^2} - 1 \right| \leq \varepsilon \]
for any \( n \geq n_0 \) and for any \( q \in U(p) \).

Further we recall that for any \( q \in H_0^{-1}(\mathbb{T}) \) one has that \( I_n(q) = 0 \) if and only if \( \gamma_n(q) = 0 \). In particular, one concludes from [17] and the fact \( \gamma_n(0) = 0 \) \( \forall n \geq 1 \) that \( \Omega(0) = 0 \).

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