Alternative solution of the bohr-mottelson equation in minimal length effect for cotangent potential using the hypergeometric method

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Abstract. The rigid deformed nucleus of minimal length effect is investigated using the Bohr-Mottelson equation that influenced by Cotangent potential. The Bohr-Mottelson equation of minimal length effect is solved by Hypergeometric method to obtain the energy spectrum and wave function. The energy spectrum is calculated by using Matlab software. The wave function is expressed in the Hypergeometric term. The result shown that, the energy spectrum is increased caused existence of minimal length effect.

1. Introduction
The Bohr-Mottelson equation is used to explain the behavior of nucleus such as the rigid deformed nucleus [1-3]. The rigid deformed nucleus occur nucleus rotation motion at low excitation energy [1-3]. The Bohr-Mottelson for various potential has been solved using Super Symmetric Quantum (SUSY QM) for Davidson potential [4], Nikiforov-Uvarov for Eckart potential [5] and Asymptotic Iteration Method (AIM) for Hulthen and Ring Shape potential [1].

The concept of minimal length is connected by commutation relations between position and momentum operators in Heisenberg Uncertainty Principle. The minimal length occur at Heisenberg Uncertainty Principle is influenced by gravity quantum which is called General Uncertainty Principle (GUP) [1-3,6-7]. The General Uncertainty Principle modify Heisenberg Uncertainty Principle with additional a small constant [3].

In this paper, we solved Bohr-Mottelson equation in minimal length effect for Cotangent potential using Hypergeometric method. By using Hypergeometric method, we get energy spectrum and wave function. The Cotangent potential is used to explain nucleon excitation [8-9]. This paper include of 4 sections, there are Bohr-Mottelson Equation in minimal length effect and hypergeometric method in section 2, the result and discussion in section 3 and the last section is conclusion.

2. Experimental
2.1. The Bohr-Mottelson equation in minimal length effect
The general canonical commutation between position and momentum is expressed [6-7]

$$[X, P] \geq i\hbar$$

(1)

where $X$ is a position, $P$ is a corresponding momentum. Then, the general canonical commutation between position and momentum is influenced by quantum gravity, it is becomes [1-3,6-7],

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\[ [X, P] \geq i\hbar \left( P^2 \right) \] (2)

The equation (2) is called General Uncertainty Principle, where \( \alpha \) was a minimal length parameter that has very small positive values. The uncertainty relation is caused by commutation relation. The equation (2) can be reduced becomes [1-3,6-7]

\[ \dot{X}_i = \dot{\hat{x}}_i \] (3)

\[ \dot{P}_i = \left( 1 + \alpha \hat{p}_i^2 \right) \hat{p}_i \] (4)

Then, equation (4) can be written [2,3]

\[ P_i^2 = \frac{-\hbar^2}{2B_m} \left( 1 - \alpha \xi \Delta \right) \Delta \] (5)

where \( \Delta \) is Laplacian operator for nucleus that has three degrees of freedom: \( q_1 = \phi, q_2 = \Theta, q_3 = \beta \), the Laplacian operator as follow [2,3]

\[ \Delta = \frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial \sqrt{g}}{\partial q_i} \frac{\partial \sqrt{g}}{\partial q_j} \] (6)

with \( g \) and \( g^{-1} \) are determinant and inverse of the matrix \( g_{ij} \), respectively. We get Laplacian operator, is given as [2,3]

\[ \Delta = \left[ \frac{1}{\beta^2} \frac{\partial^2}{\partial \beta^2} + \frac{1}{3\beta^2} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) \right] \] (7)

Then equation (5) is inserted in Hamiltonian equation which is expressed,

\[ H = T + V(\beta) = \frac{P_i^2}{2B_m} + V(\beta) \] (8)

where \( P_i \) is momentum operator, \( V(\beta) \) is potential energy in \( \beta \) function and \( B_m \) is a mass parameter.

We obtain,

\[ \left[ -\frac{\hbar^2}{2B_m} \Delta + \frac{\hbar}{B_m} \Delta + r \left( \mu, \Theta, \phi \right) - E \right] \Psi(\beta, \Theta, \phi) = 0 \] (9)

The equation (9) is Bohr-Mottelson equation in minimal length effect. In the case of Bohr-Mottelson equation without the minimal length effect with \( \alpha_{ML} = 0 \) [12] for equation (9), so yields square term is given as [2],

\[ \Delta^2 = \frac{4B_m^2}{\hbar} (V(\beta) - E_0)^2 \] (10)

Equations (7) and (10) are inserted in equation (9) and multiplied by \( -\frac{2B_m}{\hbar} \) and \( \hbar \) (natural unit), is yields

\[ \left[ \frac{1}{\beta^2} \frac{\partial^2}{\partial \beta^2} + \frac{1}{3\beta^2} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) \right] + 2B_m \left( \frac{4\alpha_{ML}B_m}{\hbar} (V(\beta) - E_0)^2 \right) \Psi(\beta, \Theta, \phi) = 0 \] (11)
By setting \( \Psi(\beta, \theta, \phi) = R(\beta) \Theta(\theta) \Phi(\phi) \) which is the separation variable method that used to solve equation (11), we have Euler angles part of Bohr-Mottelson Hamiltonian with minimal length,

\[
- \left( \frac{1}{\Phi(\phi) \sin^2 \theta} \frac{\partial^2 \Phi(\phi)}{\partial \phi^2} + \frac{1}{\Theta(\theta) \sin \theta} \frac{\partial \Theta(\theta)}{\partial \theta} \right) = \lambda
\]

and \( \beta \) - part of Bohr-Mottelson Hamiltonian with minimal length,

\[
\frac{1}{\beta^2} \frac{\partial R(\beta)}{\partial \beta} \left( \frac{\partial R(\beta)}{\partial \beta} + 2B_m \left( E - V(\beta) \right) R(\beta) \right) - 8B_m^2 \alpha_{ML} \left( E_o^2 - 2E_o V(\beta) + V^2(\beta) \right) R(\beta) = \frac{\lambda}{3} R(\beta)
\]

By applying \( R(\beta) = F(\beta) / \beta \) and \( \lambda = L(L+1) \) in equation (13) so we have,

\[
\left\{ \frac{d^2 F(\beta)}{d \beta^2} - \frac{L(L+1)}{3 \beta^2} F(\beta) + 2B_m \left( E - V(\beta) \right) F(\beta) - 8B_m^2 \alpha_{ML} \left( E^2 - 2E V(\beta) + V^2(\beta) \right) F(\beta) \right\} = 0
\]

The Bohr-Mottelson equation for a \( \beta \) - part in a minimal length effect for rigid deformed nucleus case is expressed by equation (14).

2.2. Hypergeometric method
The second-order differential equation of hypergeometric function as follow [10,11],

\[
y(1-y) \frac{d^2 \Phi}{dy^2} + \left( c - (a + b + 1) y \right) \frac{d \Phi}{dy} - ab \Phi = 0
\]

The energy eigenvalue is obtained from the condition in equation (15), [10,11]

\[ a = -n \text{ or } b = -n \]

where \( n=0,1,2,3, \ldots \). Equation (16) can be finite series of polynomials of rank \( n \) by equation (15). The solution of a wave function is given as

\[
\Phi(y) = \sum_{m=0}^{\infty} \left( \frac{a}{c} \right)_m \left( \frac{b}{c} \right)_m \frac{y^m}{n!} = 1 + \frac{ab}{c} y + \frac{a(a+1)b(b+1)}{c(c+1)} y^2 + \ldots
\]

By applying the suitable variable change in equation and reduced to standard hypergeometric equation, we get energy eigenvalue and wave function [10,11].

3. Results and Discussion
The Bohr-Mottelson in equation (14) is inserted by Cotangent potential as follow [8-10],

\[
V(\beta) = V_o \cot \delta \beta + V_1
\]

where \( V_o \) and \( V_1 \) are potential constants, and \( \delta \) is a range of potential. The visualization of Cotangent potential is given figure 1.
Figure 1. The visualization of cotangent potential

Figure 1 is a visualization of cotangent potential in $r$ function that approximately from $0 \text{ fm}$ until $14 \text{ fm}$. We can see, The Cotangent potential comes to infinity value in a very small value of $r$, while for the higher value of $r$, the cotangent potential incline to be constant. Then, The Bohr-Mottelson in equation (14) becomes,

$$F(\beta) = 0$$  \hspace{1cm} (19)$$

by setting,

$$v(v+1)=\left(\frac{\delta^2 L(L+1)}{3}+8B_m^2\alpha V_o^2\right)$$

$$2q = \left(16B_m^2\alpha V_o E_o - 16B_m^2\alpha V_2 - 2B_m V_o\right)$$

$$\kappa^2 = \left(16B_m^2\alpha V_1 E_o + 8B_m^2\alpha V_o^2ight)$$

$$-8B_m^2\alpha V_1^2 - 8B_m^2\alpha E_o^2 - 2B_m V_1 + 2B_m E$$

Equation (19) is reduced becomes

$$\frac{d^2 F(\beta)}{d\beta^2} - \left[\frac{v(v+1)}{\sin^2\delta\beta} - 2q \cot \delta\beta + \kappa^2\right]F(\beta) = 0$$

Equation (23) is differential equation has reduced to the hypergeometric differential equation type by using $\cot (\delta\beta) = i \left(1 - 2z\right)$, we get

$$z(1-z)\frac{d^2 F(\beta)}{dz^2} + (1-2z)\frac{dF(\beta)}{dz} + \left[v'(v+1) - \frac{4\beta_H^2}{4z} - \frac{4\beta_H^2}{4(1-z)}\right]F(\beta) = 0$$

with,
\[
\frac{2qi - \kappa^2}{\delta^2} = 4\alpha_{H_H}, \quad \frac{-2qi - \kappa^2}{\delta^2} = 4\beta_{H_H}, \quad v_i(v_i + 1) = \frac{v(v+1)}{\Delta^2},
\]

by applying the new wavefunction as follow,

\[
F(\beta) = z^{\alpha_{H_H}}(1-z)^{\beta_{H_H}} g(z)
\]

in equation (24) we obtain

\[
\left[\alpha(1-z)\frac{g^2(z)}{dz^2} + \left(2\alpha_{H_H} + 1\right)\frac{1}{2} - \left(2\alpha_{H_H} + 2\beta_{H_H} + 2\right) z \right] \frac{g(z)}{dz} = 0
\]

The equation (27) is hypergeometric differential equation that obtain parametric hypergeometric as follow

\[
a = \alpha_{H_H}, \quad b = \alpha_{H_H} + \beta_{H_H}, \quad c = 2\alpha_{H_H} + 1
\]

By inserting equations (20)-(22), (25), and (28), we get

\[
E = \frac{\delta^2}{2B_m} \left[ \sqrt{\left( \frac{L(L+1)}{3} + \frac{8B_m^2\alpha V_o^2}{\Delta^2} + \frac{1}{4} \right) \left( 1 - \frac{1}{2} - n \right)} \right]^2 + \left[ \frac{4B_m\alpha V_1^2 + 4B_m\alpha E_0^2}{E_1 - 8B_m\alpha V_1 E_0 - 4B_m\alpha V_o^2} \right]
\]

The equation (29) is energy spectrum equation of Bohr-Mottelson in minimal length effect for Cotangent potential. To get energy spectrum, we calculated numerically equation by using Matlab software. The result is shown in Table 1.

Table 1. The energy spectrum of Bohr-Mottelson in minimal length effect for \(^{152}\text{Sm}\), \(\delta = 0.1\), \(V_o = 0.07\) and \(V_i = 0.1\)

| \(n\) | \(l\) | \(E (\text{eV})\) | \(\alpha = 0.001\) | \(\alpha = 0.005\) | \(\alpha = 0.01\) |
|---|---|---|---|---|---|
| 0 | 0 | -6.50609 | -2.87751 | -2.78404 |
| 2 | 0 | -6.15050 | -2.82617 | -2.75838 |
| 4 | 0 | -5.11246 | -2.7128 | -2.70016 |
| 1 | 0 | -10.0042 | -3.51843 | -3.21467 |
| 2 | 0 | -9.09471 | -3.45048 | -3.18338 |
| 4 | 0 | -7.48489 | -3.30104 | -3.11250 |
| 2 | 0 | -17.1663 | -4.38043 | -3.74380 |
| 2 | 0 | -15.1886 | -4.28786 | -3.70510 |
| 4 | 0 | -11.8874 | -4.08523 | -3.61758 |
The table 1 shows that the energy spectrum is increased due to increase of minimal length parameter and angular momentum quantum number ($l$). And the other hand, energy spectrum decreased by increase of quantum number ($n$).

Then, to obtain the wave function use equations (17), (26), and \( \cot (\delta \beta) = i(1-2z) \), we get

\[
F(\beta) = \frac{(1+i\cot(\delta \beta))^{n_i}}{2} \left( 1-\frac{i\cot(\delta \beta)}{2} \right)^{\delta \beta_m} z F_i(a,b,c,z)
\]

And by applying equation (28) in equation (30), we obtain

\[
F(\beta) = \frac{(1+i\cot(\delta \beta))^{n_i}}{2} \left( 1-\frac{i\cot(\delta \beta)}{2} \right)^{\delta \beta_m} \left[ \frac{(-n)(\alpha_m + \beta_m + v+1)}{1+\frac{1+i\cot(\delta \beta)}{2}} \right]^{\delta \beta_m + 1}
\]

The equations (31) and (32) are wave function of Bohr-Mottelson equation in minimal effect for \( n=0 \) and \( n=1 \), respectively. The value of wave function depends on the value of the hypergeometric parameter.

### 4. Conclusion

The Bohr-Mottelson equation is investigated in minimal length effect for Cotangent potential by using Hypergeometric method. The Hypergeometric method is used to obtain energy spectrum and wave function of Bohr-Mottelson equation in minimal length effect. The result is shown the energy spectrum increase due to increase minimal length parameter.

### Acknowledgement

This research was partly supported by Ministry of Research, Technology and Higher Education with contract no. 089/SP2H/LT/DRPM/2018.

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