S-matrix elements and covariant tachyon action in type 0 theory

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ABSTRACT

We evaluate the sphere level S-matrix element of two tachyons and two massless NS states, the S-matrix element of four tachyons, and the S-matrix element of two tachyons and two Ramon-Ramond vertex operators, in type 0 theory. We then find an expansion for theses amplitudes that their leading order terms correspond to a covariant tachyon action. To the order considered, there are no $T^4$, $T^2(\partial T)^2$, $T^2H^2$, nor $T^2R$ tachyon couplings, whereas, the tachyon couplings $\bar{F}FT$ and $T^2F^2$ are non-zero.
1 Introduction

The discovery in the early days of string theory that string amplitudes of massless states in low energy regimes may be reproduced by Yang-Mills field theory for open strings and gravitational field theory for closed strings, was the beginning of a long, fruitful study into the relation of string theory and field theory in general and in low energy in particular [1].

Recent days, effective actions that include massless as well as tachyon states playing an important role in understanding the dynamics of tachyon condensation in D-branes of bosonic string theory and in non-BPS D-branes of superstring theory [2]-[30]. Beside the S-matrix based approach [1], there are other approaches to find these actions. One is the method introduced in [31] which is based on derivative expansion of partition function [32, 33]. The other one is based on directly integrating out the massive modes of string field theory to find an action that includes tachyon and massless fields [34]. In general, however, the fields in the resulting effective actions are related to each other by some field redefinition. For example, in the cubic string field theory, after integrating out all the massive and tachyon fields, one will find an effective action for D-branes which is not Dirac-Born-Infeld action plus higher derivative terms [34]. While the massless scalar fields have clear interpretation in the cubic string field theory, they have no (geometrical) interpretation in the effective action. However, using nontrivial field redefinition, one will find a field variable in terms of which the effective action rearranges into the DBI action plus higher derivative terms [34]. The new scalar variables have now the geometrical interpretation as the transverse coordinate of the D-brane in the effective theory, whereas, they have no clear interpretation in the original string field theory. Since the mass squared of tachyon and massive modes are of the same order, it is not clear how to extend the work in [34] to find an action which includes massless fields as well as the tachyon.

It has been speculated in [35] that imposing the non-abelian gauge symmetry on the tachyon action, the S-matrix based approach [1] may be used to find the tachyon-gauge field action. In particular, it has been suggested in [35] that there is a unique expansion for the S-matrix elements of tachyon vertex operators that their leading order terms are consistent with the standard non-abelian kinetic term. The S-matrix element of four open string tachyons, and the S-matrix element of two tachyons and two gauge fields have been analyzed in [35] in favor of the proposal. It has been found that while the leading order massless/tachyonic poles in the expansion are consistent with non-abelian kinetic term the next leading contact terms, in the superstring theory, are in fact consistent with the non-abelian extension of the tachyon Born-Infeld action [37, 38].

There are two problems in the above discussion. While the action consistent with the expansion of the S-matrix elements has the non-abelian gauge symmetry manifestly, the physical interpretation of the expansion is not clear. A different expansion for the S-matrix elements in terms of spatial momenta of the external states has been suggested in [32]. This expansion has the physical interpretation as expansion around a marginal state, however,
it is not clear that the action consistent with the leading terms of the expansion have manifestly non-abelian gauge symmetry. One may expect that the action in the two case would be related to each other by some field redefinition. The second problem is that it does not indicate that the resulting action is an effective action. In fact if one applies the above idea for first excited massive scalars of the D-branes, one will find a similar result [35]. However, if one proves by other means that there is a tachyon-gauge field effective action with non-abelian gauge symmetry, then the above action should be the effective action, because there is only one expansion for the S-matrix elements that corresponds to non-abelian gauge symmetry.

In the present paper, we would like to apply the above idea to the closed string tachyon of type 0 theories. That is, we would like to find an expansion for the S-matrix element of closed string tachyon and massless states that their leading order terms correspond to tachyon action with covariant symmetry. We will call this expansion the covariant expansion. If one prove by other means that there is an effective tachyon-massless fields action, then the action should be the effective action.

Spectrum of type 0 theories can be obtained by a diagonal GSO projection on the superstring spectrum or by orbifolding the corresponding type II theories by \((-1)^F_s\), the total target space fermion number [39]. They are represented as

\[
\begin{align*}
type 0A & : (NS_-, NS_-) \oplus (NS_+, NS_+) \oplus (R_+, R_-) \oplus (R_-, R_+) , \\
type 0B & : (NS_-, NS_-) \oplus (NS_+, NS_+) \oplus (R_+, R_+) \oplus (R_-, R_-) ,
\end{align*}
\]

which then consists, at lowest level, of tachyon and, at massless level, of graviton, dilaton, Kalb-Ramond antisymmetric tensor, and two RR states with opposite chirality. As we will see later, an essential part in finding the covariant expansion of a S-matrix element of tachyons is to compare it with the corresponding S-matrix element of a massless scalar state that its covariant expansion is trivial. Hence, we compactify theory on a torus and consider the scalar components of the dimensional reduction of the graviton as the scalars in the massless level. Furthermore, we will assume all closed string states to be independent of the compact directions. This makes it easier to compare the S-matrix element of tachyon with the S-matrix element of the massless states to find the covariant expansion of the tachyon S-matrix elements.

The paper is organized as follows. In the following section we calculate the S-matrix element of two massless NS and two tachyons. In this case it is easy to find the covariant expansion even without comparing it with the corresponding scalar amplitude. We then show that a covariant action of order \(\alpha'^2\) for the tachyon reproduces exactly the leading terms of the covariant expansion of the amplitude. This calculation indicates that there is no \(T^2H^2\) nor \(T^2R\) couplings at this order in the action. In section 3, we repeat the same calculation for the S-matrix element of four tachyon states. The leading terms of the expansion are again fully consistent with the covariant tachyon action. This calculation predicts the tachyon potential \(V(T)\) has no \(T^4\) term, and the tachyon action has no coupling
In section 4, we do the above calculation for two RR and two tachyon states. The details analysis of the first leading term in the covariant expansion of the amplitude fixes the couplings $F\bar{F}T$ and $F^2T^2$. Section 5 is devoted to the discussion and comments on our results. In the Appendix A, we give the result for the integrals that appear in evaluating the above S-matrix elements. In Appendix B, we evaluate the S-matrix element of four RR states with opposite chirality.

Before continuing with our calculations, let us make a comment on conventions. The spacetime is assumed to be orthogonal product of compact torus and non-compact flat space. The non-compact directions are labeled by $a, b, c, \cdots$ and compact directions are labeled by $i, j, k, \cdots$. The closed string states are assumed to have momentum only in the flat directions. The graviton, Kalb-Ramond and RR polarizations are in the non-compact directions, and the polarization of massless scalars are in the compact direction. Our conventions also set $\alpha' = 2$.

## 2 Two tachyon-two graviton amplitude

In this section we analysis in details the S-matrix element of two tachyons and two massless NS states. In string theory side this amplitude is given by the following correlation function:

$$A(\text{NS, NS, T, T}) \sim \langle : V^{\text{NS}}_{(-1, -1)}(p_1, \varepsilon_1) : V^{\text{NS}}_{(-1, -1)}(p_2, \varepsilon_2) : V^{\text{T}}_{(0, 0)}(p_3) : V^{\text{T}}_{(0, 0)}(p_4) : \rangle,$$

where the tachyon and the NS vertex operators are given as

$$V^{\text{T}}_{(0, 0)}(p) = \int d^2 z : ip \cdot \psi(z)e^{ip \cdot X(z)} : e^{ip \cdot \hat{\psi}(\bar{z})}e^{-ip \cdot \hat{\psi}(\bar{z})} :,$$

$$V^{\text{NS}}_{(-1, -1)}(\varepsilon, p) = \varepsilon_{ab}\int d^2 z : e^{-\phi(z)}\psi^a(z)e^{ip \cdot X(z)} : e^{-\hat{\phi}(\bar{z})}\hat{\psi}^b(\bar{z})e^{ip \cdot \hat{X}(\bar{z})} :.$$  \hspace{1cm} (1)

For graviton and dilaton the polarization tensor $\varepsilon_{ab}$ is symmetric, whereas, for the antisymmetric tensor this polarization is antisymmetric. All the correlators above are simple to evaluate. The final result is

$$A(\text{NS, NS, T, T}) \sim \varepsilon_{1ab}\varepsilon_{2cd} \int d^2 z_1 d^2 z_2 d^2 z_3 d^2 z_4 |z_{12}|^{-2} \prod_{i<j}^4 |z_{ij}|^{2p_{ij}^a p_{ij}^b} \left[ \frac{\eta^{ac} p_3 \cdot p_4}{z_{12} z_{34}} - \frac{p_3^c p_4^c}{z_{12} z_{23}} + \frac{p_4^c p_3^c}{z_{12} z_{23}} - \frac{p_3^d p_4^d}{z_{13} z_{24}} + \frac{p_4^d p_3^d}{z_{13} z_{24}} \right].$$

It is easy to check that the integrand has $SL(2, \mathbb{C})$ symmetry. One should fix this symmetry by fixing position of three vertices at $z_1 = \bar{z}_1 = \infty$, $z_2 = \bar{z}_2 = 0$, and $z_3 = \bar{z}_3 = 1$. After this gauge fixing, one ends up with one complex integral in the $z$-plane. The imaginary part is
zero and the real part is the following (see the Appendix A):

\[ A(NS, NS, T, T) \sim 2\pi \left\{ p_3 \cdot \varepsilon_1^T \cdot \varepsilon_2 \cdot p_4 + p_3 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot p_4 \right\} \frac{\Gamma\left(\frac{1}{2} - \frac{s}{2}\right)\Gamma\left(-\frac{s}{2}\right)\Gamma\left(\frac{1}{2} - \frac{u}{2}\right)}{\Gamma\left(\frac{3}{2} + \frac{u}{2}\right)\Gamma\left(1 + \frac{s}{2}\right)\Gamma\left(\frac{3}{2} + \frac{u}{2}\right)}

+ p_3 \cdot \varepsilon_1 \cdot p_4 \cdot \varepsilon_2 \cdot p_4 \frac{\Gamma\left(\frac{1}{2} - \frac{t}{2}\right)\Gamma\left(-\frac{t}{2}\right)\Gamma\left(\frac{1}{2} - \frac{u}{2}\right)}{\Gamma\left(\frac{3}{2} + \frac{u}{2}\right)\Gamma\left(1 + \frac{t}{2}\right)\Gamma\left(\frac{3}{2} + \frac{u}{2}\right)}

- p_3 \cdot \varepsilon_1 \cdot p_4 \cdot \varepsilon_2 \cdot p_3 \frac{\Gamma\left(\frac{1}{2} - \frac{t}{2}\right)\Gamma\left(-\frac{t}{2}\right)\Gamma\left(\frac{1}{2} - \frac{u}{2}\right)}{\Gamma\left(\frac{3}{2} + \frac{u}{2}\right)\Gamma\left(1 + \frac{t}{2}\right)\Gamma\left(\frac{3}{2} + \frac{u}{2}\right)}

- \frac{1}{2} \text{Tr}(\varepsilon_1^T \varepsilon_2) \frac{\Gamma\left(\frac{1}{2} - \frac{s}{2}\right)\Gamma\left(-\frac{s}{2}\right)\Gamma\left(\frac{1}{2} - \frac{u}{2}\right)\Gamma\left(\frac{1}{2} + \frac{s}{2}\right)\Gamma\left(1 + \frac{s}{2}\right)\Gamma\left(\frac{1}{2} + \frac{u}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{u}{2}\right)\Gamma\left(1 + \frac{s}{2}\right)\Gamma\left(\frac{1}{2} + \frac{u}{2}\right)} + (4 \leftrightarrow 3) \right\}, \quad (2)

where the Mandelstam variables are

\[ s = -(p_3 + p_4)^2, \]
\[ t = -(p_1 + p_4)^2, \]
\[ u = -(p_2 + p_4)^2. \quad (3) \]

They satisfy the on-shell relation

\[ s + t + u = -2. \quad (4) \]

Note that under 4 \leftrightarrow 3 the Mandelstam variables change as \((s, t, u) \leftrightarrow (s, u, t)\). The term in the last line of (2) has been also found in [42]. If one of the NS states is graviton and the other one is the antisymmetric two tensor, then the whole amplitude vanishes. Moreover, when both NS states are the antisymmetric two tensor, then the term in the second line of (2) vanishes. This indicates that there is no linear tachyon coupling to two Kalb-Ramond states, because the gamma functions in this term has a tachyonic pole.

Now to find the covariant expansion for the above amplitude, one should note that the covariant kinetic term produces massless pole in \(s\)-channel and tachyonic pole in \(t\)-channel and \(u\)-channel. Hence, one may send \(s \to 0\) and \(t, u \to -1\). Fortunately, this limit is also consistent with the constraint (4). Hence, in this case the covariant limit is simply

\[ \lim_{s \to 0, t, u \to -1} A \tag{5} \]

One may use the constraint (4) to rewrite the amplitude in such a way that the covariant limit becomes \(s, t, u \to 0\). To manage this, consider, for example, the gamma functions in the first line of (2). They can be rewritten as the following:

\[ \lim_{s \to 0, t, u \to -1} \frac{\Gamma\left(\frac{1}{2} - \frac{s}{2}\right)\Gamma\left(-\frac{s}{2}\right)\Gamma\left(\frac{1}{2} - \frac{t}{2}\right)}{\Gamma\left(\frac{3}{2} + \frac{u}{2}\right)\Gamma\left(1 + \frac{s}{2}\right)\Gamma\left(\frac{3}{2} + \frac{u}{2}\right)} = \lim_{s, t, u \to 0} \frac{\Gamma\left(1 + \frac{s+t-u}{4}\right)\Gamma\left(-\frac{s}{2}\right)\Gamma\left(1 + \frac{s+u-t}{4}\right)}{\Gamma\left(1 + \frac{u-s-1}{4}\right)\Gamma\left(1 + \frac{t-u-1}{4}\right)}. \]
Straightforward calculation, like what we have done before, gives the following final result:

\[
A \sim 2\pi \left\{ (p_3 \cdot \varepsilon_1^T \cdot \varepsilon_2 \cdot p_4 + p_3 \cdot \varepsilon_1 \cdot \varepsilon_2^T \cdot p_4) \frac{\Gamma(1 + \frac{s+t-u}{4})\Gamma(-\frac{s}{2})\Gamma(1 + \frac{s+u-t}{4})}{\Gamma(1 + \frac{u-s-t}{4})\Gamma(1 + \frac{s}{2})\Gamma(\frac{t-u-s}{4})} \\
+ p_3 \cdot \varepsilon_1 \cdot p_4 \cdot \varepsilon_2 \cdot p_4 \frac{\Gamma(-\frac{s}{4})\Gamma(-\frac{s}{2})\Gamma(1 + \frac{s+u-t}{4})}{\Gamma(1 + \frac{u-s-t}{4})\Gamma(1 + \frac{s}{2})\Gamma(\frac{t-u-s}{4})} \\
- p_3 \cdot \varepsilon_1 \cdot p_4 \cdot \varepsilon_2 \cdot p_3 \frac{\Gamma(1 + \frac{s+t-u}{4})\Gamma(-\frac{s}{2})\Gamma(1 + \frac{s+u-t}{4})}{\Gamma(1 + \frac{u-s-t}{4})\Gamma(1 + \frac{s}{2})\Gamma(\frac{t-u-s}{4})} \\
- \frac{1}{2} \text{Tr}(\varepsilon_1^T \varepsilon_2) \frac{\Gamma(1 + \frac{s+t-u}{4})\Gamma(-\frac{s}{2})\Gamma(1 + \frac{s+u-t}{4})}{\Gamma(\frac{s}{4})\Gamma(1 + \frac{s}{2})\Gamma(\frac{t-u-s}{4})} \right\} \quad (4 \leftrightarrow 3)
\]

In this form the covariant limit (5) becomes

\[
\lim_{s,t,u \to 0} A
\]

The S-matrix element of two NS and two massless scalar states can also be written in the above form. To see this, consider the latter amplitude which is given by the following correlation function:

\[
A(\text{NS}, \text{NS}, g, g) \sim \langle V_{(-1,-1)}^{\text{NS}}(p_1, \varepsilon_1) : V_{(0,0)}^{\text{NS}}(p_2, \varepsilon_2) : V_{(0,0)}^{g}(p_3, \zeta_3) : V_{(0,0)}^{g}(p_4, \zeta_4) \rangle,
\]

where the NS vertex operator is given in (1), and the scalar vertex operator is given as

\[
V_{(0,0)}^{g} = \zeta_{ij} \int d^2z (\partial X^{i}(z) + i p_{i} \cdot \psi(z) \psi^{j}(z)) e^{ip \cdot X(z)} : (\partial \hat{X}^{j}(\bar{z}) + i p_{j} \cdot \hat{\psi}(\bar{z}) \hat{\psi}^{j}(\bar{z})) e^{ip \cdot \bar{X}(\bar{z})} :.
\]

Straightforward calculation, like what we have done before, gives the following final result:

\[
A(\text{NS}, \text{NS}, g, g) \sim 2\pi \text{Tr}(\zeta_3^T \zeta_4) \left\{ (p_3 \cdot \varepsilon_1^T \cdot \varepsilon_2 \cdot p_4 + p_3 \cdot \varepsilon_1 \cdot \varepsilon_2^T \cdot p_4) \frac{\Gamma(1 - \frac{s}{2})\Gamma(-\frac{s}{2})\Gamma(1 + \frac{s}{2})}{\Gamma(1 + \frac{u-s-t}{4})\Gamma(1 + \frac{s}{2})\Gamma(\frac{t-u-s}{4})} \\
+ p_3 \cdot \varepsilon_1 \cdot p_4 \cdot \varepsilon_2 \cdot p_4 \frac{-\frac{u}{2}\Gamma(-\frac{s}{2})\Gamma(1 + \frac{s}{2})}{\Gamma(1 + \frac{u-s-t}{4})\Gamma(1 + \frac{s}{2})\Gamma(\frac{t-u-s}{4})} \\
- p_3 \cdot \varepsilon_1 \cdot p_4 \cdot \varepsilon_2 \cdot p_3 \frac{\Gamma(1 - \frac{u}{2})\Gamma(-\frac{s}{2})\Gamma(1 + \frac{s}{2})}{\Gamma(1 + \frac{u-s-t}{4})\Gamma(1 + \frac{s}{2})\Gamma(\frac{t-u-s}{4})} \\
- \frac{1}{2} \text{Tr}(\varepsilon_1^T \varepsilon_2) \frac{\Gamma(1 - \frac{u}{2})\Gamma(-\frac{s}{2})\Gamma(1 + \frac{s}{2})}{\Gamma(\frac{s}{4})\Gamma(1 + \frac{s}{2})\Gamma(\frac{t-u-s}{4})} \right\} \quad (4 \leftrightarrow 3)
\]

where the Mandelstam variables satisfy

\[
s + t + u = 0.
\]
The covariant expansion of the amplitude (6) is
\[ A \sim 2\pi \left\{ (p_3 \cdot \varepsilon_1^T \cdot \varepsilon_2 \cdot p_4 + p_3 \cdot \varepsilon_1 \cdot \varepsilon_2^T \cdot p_4) \left( \frac{s + u - t}{2s} + \frac{\zeta(3)}{32} (s + u - t)(s^2 - (t - u)^2) + \cdots \right) 
+ p_3 \cdot \varepsilon_1 \cdot p_4 \cdot p_3 \cdot \varepsilon_2 \cdot p_4 \left( -\frac{2}{s} + \frac{4}{s + t - u} + \frac{\zeta(3)}{8} (s + u - t)^2 + \cdots \right) 
- p_3 \cdot \varepsilon_1 \cdot p_4 \cdot p_4 \cdot \varepsilon_2 \cdot p_3 \left( -\frac{2}{s} - \frac{\zeta(3)}{8} (s^2 - (t - u)^2) + \cdots \right) 
- \frac{1}{2} \text{Tr}(\varepsilon_1^T \varepsilon_2) \left( -\frac{s^2 - (t - u)^2}{8s} - \frac{\zeta(3)}{128} (s^2 - (t - u)^2) + \cdots \right) + (4 \leftrightarrow 3) \right\}. \tag{11} \]

We shall show that the first order terms above which has two momenta, are reproduced in field theory by the action in which the covariant tachyon kinetic term and the tachyon mass term are added to the standard low energy gravity action. The next leading terms have eight momenta in which we are not interested in their field theory couplings.

The S-matrix element of two dilatons and two tachyons can be read from the general amplitude (6) by replacing the dilaton polarization tensor in the amplitude. The dilaton polarization tensor is \( \varepsilon^{ab} = (\eta^{ab} - p^a \ell^b - p^b \ell^a) / \sqrt{D - 2} \) where \( p \cdot \ell = 1 \). In the amplitude the vector \( \ell^a \) has to be canceled. This is a nontrivial check on the amplitude (6). Replacing this polarization tensor in (6), one finds, after some algebra,
\[ A(\Phi', \Phi', T, T) \sim \frac{2\pi}{D - 2} \left\{ 2p_3 \cdot p_4 \frac{\Gamma(1 + \frac{s+t-u}{4}) \Gamma(-\frac{s}{2}) \Gamma(1 + \frac{s+u-t}{4})}{\Gamma(1 + \frac{u-s-t}{4}) \Gamma(1 + \frac{s}{2}) \Gamma(\frac{t-u-s}{4})} \right. 
+ p_3 \cdot p_3 \cdot p_4 \cdot p_4 \frac{\Gamma(\frac{s+t-u}{4}) \Gamma(-\frac{s}{2}) \Gamma(1 + \frac{s+u-t}{4})}{\Gamma(1 + \frac{u-s-t}{4}) \Gamma(1 + \frac{s}{2}) \Gamma(\frac{t-u-s}{4})} 
- p_3 \cdot p_4 \cdot p_4 \cdot p_3 \frac{\Gamma(1 + \frac{s+t-u}{4}) \Gamma(-\frac{s}{2}) \Gamma(1 + \frac{s+u-t}{4})}{\Gamma(1 + \frac{u-s-t}{4}) \Gamma(1 + \frac{s}{2}) \Gamma(\frac{t-u-s}{4})} 
\left. - \frac{(D - 2)}{2} \frac{\Gamma(1 + \frac{s+t-u}{4}) \Gamma(-\frac{s}{2}) \Gamma(1 + \frac{s+u-t}{4})}{\Gamma(\frac{u-s-t}{4}) \Gamma(1 + \frac{s}{2}) \Gamma(\frac{t-u-s}{4})} + (4 \leftrightarrow 3) \right\}. \tag{12} \]

Note that as expected the auxiliary vector \( \ell^a \) does not appear in the amplitude. Expansion at \( s, t, u \to 0 \) gives, after some algebra,
\[ A(\Phi', \Phi', T, T) \sim \frac{2\pi}{D - 2} \left\{ 2p_3 \cdot p_3 + p_3 \cdot p_3 \cdot p_3 \cdot p_4 \left( \frac{4}{s + t - u} \right) 
- \frac{D - 2}{2} \left( -\frac{s^2 - (t - u)^2}{8s} \right) + \cdots + (4 \leftrightarrow 3) \right\}. \tag{13} \]

where dots represent terms that start from \( \zeta(3) \) order terms.
2.1 Field theory analysis

Now in field theory, we start with adding the covariant tachyon kinetic term and an arbitrary
tachyon mass term to the standard low-energy gravity action in D-dimensional space,

\[ S^T_1 = -\int d^Dx \sqrt{G} \left[ e^{-2\Phi} \left(-2R - 8\partial^a\Phi\partial_a\Phi + \frac{3}{2}H_{abc}H^{abc} + \frac{1}{2}\partial^aT\partial_aT + \frac{1}{2}m^2T^2 \right) \right], \tag{14} \]

where \( H_{abc} = (\partial_aB_{bc} + \partial_cB_{ab} + \partial_bB_{ca})/3 \). In the Einstein frame \((G_{ab} = e^{4\Phi/D}g_{ab})\) it becomes

\[ S^T_1 = -\int d^Dx \sqrt{g} \left[ -2R + \frac{1}{2}\partial^a\Phi'\partial_a\Phi' + e^{-2\Phi'/2} \left( \frac{3}{2}H_{abc}H^{abc} \right) + \frac{1}{2}\partial^aT\partial_aT + e^{-\Phi'/2} \left( \frac{1}{2}m^2T^2 \right) \right], \tag{15} \]

where dilaton \( \Phi' = 4\Phi/\sqrt{D-2} \), and graviton \( h_{ab} \) is related to the Einstein metric as
\( g_{ab} = \eta_{ab} + h_{ab} \). In above field theory, we evaluate the S-matrix element of two NS fields and
two tachyons. Using the fact that particle 1, 2 are massless NS fields, and 3, 4 are tachyon
with arbitrary mass \( m \), the Mandelstam variables (3) become:

\[ s = -2p_1 \cdot p_2 , \]
\[ t = -(m^2 + 2p_2 \cdot p_3) , \]
\[ u = -(m^2 + 2p_1 \cdot p_3) . \]

Conservation of momentum constrains them in the relation

\[ s + t + u = 2m^2 . \tag{17} \]

Note that if one restricts the tachyon mass to the on-shell value \( m^2 = -1 \), then above
relation reduces to the on-shell relation (4).

Unlike in the string theory side that the S-matrix element for graviton, Kalb-Ramond
tensor, and dilaton are given by a unique amplitude (6), in field theory side, one has to
evaluate each separately. The \( u \)-channel amplitude for two tachyons and two gravitons is
given by the following Feynman rule:

\[ A'_u(h,h,T,T) = \hat{V}_{h_1T_3T} \hat{G}_T \hat{V}_{TT_4h_2} , \tag{18} \]

where the propagator and vertex function can be read from (15). They are

\[ \hat{G}_T = \frac{i}{u - m^2} = \frac{-2i}{s + t - u} , \]
\[ \hat{V}_{h_1T_3T} = ip_3 \cdot \varepsilon_2 \cdot p_3 . \]

where in the first line we have used the relation (17). Replacing them in (18), one finds

\[ A'_u(h,h,T,T) = ip_3 \cdot \varepsilon_1 \cdot p_3 p_4 \cdot \varepsilon_2 \cdot p_4 \left( \frac{2}{s + t - u} \right) . \]
Note that the $m$-dependence cancels out. Comparing this with the corresponding pole in the string theory amplitude (11), one finds exact agreement if the amplitude (6) is normalized by the factor $i/(4\pi)$. The $t$-channel amplitude is the same as $u$-channel in which $3 \leftrightarrow 4$, which is obviously in agreement with string theory.

The $s$-channel amplitude is given by the following Feynman rule:

$$A'_s(h, h, T, T) = (\hat{V}_{h1h2})^{ab}(\hat{G}_h)^{cd}(\hat{V}_{hT3T4})_{cd},$$ (19)

where the propagator, $\hat{V}_{hT3T4}$, and $\hat{V}_{h1h2}$ can be read from the action (15) (see e.g., [40]),

$$ (\hat{G}_h)^{ab,cd} = \frac{i}{2s} \left( \eta^{ac}\eta^{bd} + \eta^{ad}\eta^{bc} - \frac{2}{D-2} \eta^{ab}\eta^{cd} \right),$$

$$ (\hat{V}_{hT3T4})^{ab} = -\frac{i}{2} \left[ \eta^{ab}(m^2 - p_3\cdot p_4) + p_3^a p_4^b + p_4^a p_3^b \right],$$ (20)

$$ (\hat{V}_{h1h2})^{ab} = -i \left[ \left( \frac{3}{2} p_1\cdot p_2 \eta^{ab} + p_1^{a} p_2^{b} - k^a k^b \right) \text{Tr}(\varepsilon_1 \varepsilon_2) - p_1\cdot \varepsilon_2\cdot \varepsilon_1\cdot p_2 \eta^{ab} + 2 p_1^{a} \varepsilon_2\cdot \varepsilon_1\cdot p_2 \eta^{ab} + 2 p_1^{a} \varepsilon_2\cdot \varepsilon_1\cdot p_2 - 2 p_1\cdot p_2 \varepsilon_1 \varepsilon_2 - p_1\cdot \varepsilon_2\cdot p_1 \varepsilon_1\cdot p_2 - p_2\cdot \varepsilon_1\cdot p_2 \varepsilon_2 \right],$$

where $k = -(p_1 + p_2)$ and $p_1^{a} p_2^{b}$ means $(p_1^a p_2^b + p_2^a p_1^b)/2$. Replacing them in (19), after some algebra, one finds

$$A'_s(h, h, T, T) = i \left[ p_3\cdot \varepsilon_1\cdot \varepsilon_2\cdot p_4 \left( \frac{s + u - t}{2s} \right) - p_3\cdot \varepsilon_1\cdot p_3 p_4\cdot \varepsilon_2\cdot p_4 \left( \frac{1}{s} \right) + p_3\cdot \varepsilon_1\cdot p_4 p_3\cdot \varepsilon_2\cdot p_4 \left( \frac{1}{s} \right) - \text{Tr}(\varepsilon_1 \varepsilon_2) \left( \frac{(u - t)^2 - s^2}{32s} \right) \right] - \frac{1}{8} \text{Tr}(\varepsilon_1 \varepsilon_2) s - \frac{1}{4} p_1\cdot \varepsilon_2\cdot \varepsilon_1\cdot p_2 - \frac{1}{2} p_4\cdot \varepsilon_1\cdot \varepsilon_2\cdot p_4 - \frac{1}{2} p_3\cdot \varepsilon_1\cdot \varepsilon_2\cdot p_4 + (3 \leftrightarrow 4) \right].$$ (21)

Note that the D-dependence and $m$-dependence cancel out. The massless poles are all in full agreement with the string theory amplitude (11). The left over contact terms should be canceled by the $hhTT$ couplings of field theory. Now the $hhTT$ contact terms in (15) has the following terms in momentum space:

$$i \frac{8}{3} \text{Tr}(\varepsilon_1 \varepsilon_2) s + i p_3\cdot \varepsilon_1\cdot \varepsilon_2\cdot p_4 + (3 \leftrightarrow 4).$$

The two gravitons in the first term above results from expanding the square root of determinant of metric, and in the second term from expanding the inverse of metric appearing in the kinetic term of the tachyon. The above contact terms exactly cancel the contact terms in the last line of (21).

The $u$-channel and $t$-channel amplitude for scattering of two tachyons and two Kalb-Ramond fields are zero, because there is no vertex function with two tachyons and one Kalb-Ramond field in the action (15). These vanishing amplitudes are consistent with the
string theory amplitude (11), e.g., \( u \)-channel appears in the second line of (11) which is zero when both NS states are Kalb-Ramond states. The \( s \)-channel amplitude is given by the following Feynman rule:

\[
A'_s(B, B, T, T) = (\hat{V}_{B_1B_2}h)^{ab}(\hat{G}_h)_{ab}^{cd}(\hat{V}_{hT_3T_4})_{cd} + \hat{V}_{B_1B_2}\Phi' \hat{G}_\Phi \hat{V}_\Phi^{T_3T_4}, \tag{22}
\]

where the propagators and vertex functions are given in (20) and in the following:

\[
\hat{G}_\Phi = \frac{i}{s},
\]

\[
\hat{V}_{T_3T_4} = \frac{-im^2}{\sqrt{D-2}},
\]

\[
\hat{V}_{B_1B_2}\Phi' = \frac{-2i}{\sqrt{D-2}} \left( 2p_1 \cdot \varepsilon_2 \cdot \varepsilon_1 \cdot p_2 - p_1 \cdot p_2 \text{Tr}(\varepsilon_1 \varepsilon_2) \right),
\]

\[
(\hat{V}_{B_1B_2}h)^{ab} = -i \left[ \frac{1}{2} \left( p_1 \cdot p_2 \eta^{ab} - 2p_1^{(a)}p_2^{(b)} \right) \text{Tr}(\varepsilon_1 \varepsilon_2) - p_1 \cdot \varepsilon_2 \cdot \varepsilon_1 \cdot p_2 \eta^{ab} + 2p_1^{(a)} \varepsilon_2^{(b)} \cdot \varepsilon_1 \cdot p_2 \\
+ 2p_2^{(a)} \varepsilon_1 \cdot \varepsilon_2 \cdot p_1 + 2p_1 \cdot \varepsilon_2 \cdot p_1 \cdot p_2 - 2p_1 \cdot p_2 \varepsilon_1^{(a)} \cdot \varepsilon_2^{(b)} \right].
\]

Replacing them in (22), after some algebra, one finds

\[
A'_s(B, B, T, T) = -i \left[ p_3 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot p_4 \left( \frac{s + u - t}{2s} \right) \\
+ p_3 \cdot \varepsilon_1 \cdot p_4 \cdot \varepsilon_2 \cdot p_4 \left( \frac{1}{s} \right) - \text{Tr}(\varepsilon_1 \varepsilon_2) \left( \frac{(u - t)^2 - s^2}{32s} \right) + (3 \leftrightarrow 4) \right],
\]

Note that again all D-dependence and \( m \)-dependence cancel out. The above field theory result is in perfect agreement with the string result (11). Since there is no contact terms left over, one concludes that the tachyon action has no coupling \( H^2 T^2 \), as we didn’t include this in the action (14). The absence of the \( H^2 T^2 \) coupling has been mentioned also in [43].

The \( u \)-channel amplitude for scattering of two dilatons and two tachyon in field theory (15) is given by the following Feynman rule:

\[
A'_u = \hat{V}_{T_3T_4} \hat{G}_T \hat{V}_{TT_4}\Phi'_2
\]

\[
= \left( \frac{2m^4}{D-2} \right) \frac{i}{s + t - u},
\]

which is in exact agreement with the string theory result in (13) including the numerical factor. Note that the amplitude (12) has been already normalized by studying the \( u \)-channel of graviton amplitude. The \( t \)-channel amplitude which can be read from the \( u \)-channel amplitude above by interchanging \( 3 \leftrightarrow 4 \) is agree with string theory result for obvious reason. The \( s \)-channel is given by the following amplitude:

\[
A'_s = (\hat{V}_{T_3T_4}h)^{ab}(\hat{G}_h)_{ab}^{cd}(\hat{V}_{hT_3T_4})_{cd}, \tag{24}
\]
where the propagator and vertex functions are given in (20) and

\[(V_{h\Phi'\Phi'}_2)^{ab} = -\frac{i}{2} \left[ \eta^{ab}(-p_1 \cdot p_2) + p_1^a p_2^b + p_1^b p_2^a \right].\]

Replacing them in (24), one finds

\[A'_s = -\frac{i}{4s} \left(-2p_1 \cdot p_2 p_3 \cdot p_4 + 2m^2 p_1 \cdot p_2 + 2p_1 \cdot p_3 p_2 \cdot p_3 + 2p_1 \cdot p_4 p_2 \cdot p_3\right)\]
\[= \frac{i}{16s} \left(s^2 - (t - u)^2\right).\]

Comparing this with the massless pole in (13), one finds exact agreement. Finally, the contact term in (13) is exactly the coupling $TT\Phi'\Phi'$ in the action (15).

The leading terms of the amplitude (6) at the covariant limit (7), is then consistent with the covariant action (14). As we already mentioned in the previous section, the amplitude (6) is also describe the S-matrix element of two massless scalars $g$ and two NS states. Accordingly, the action (14) is also action for the scalar field as well. For the scalar case the mass term is of course zero, i.e.,

\[S^g = -\int d^4 x \sqrt{G} \left[ e^{-2\Phi}(-2R - 8\partial^a \Phi \partial_a \Phi + \frac{3}{2} H_{abc} H^{abc} + \frac{1}{2} \partial^a g \partial_a g) \right], \quad (25)\]

### 3 Four tachyon amplitude

We repeat the same analysis as in previous section for four tachyons. So we begin with the evaluation of the sphere 4-point function of four tachyon vertex operators. This amplitude is given by the following correlation function:

\[A(T, T, T, T) \sim \langle :V_{(0,0)}^T(p_1):V_{(0,0)}^T(p_2):V_{(-1,-1)}^T(p_3):V_{(-1,-1)}^T(p_4): \rangle,\]

where we have used the tachyon vertex operators in different pictures in order to saturate the background supercharge of the sphere. The vertex operator in (0,0) picture is given in (1), and in (-1,-1) picture is given in the following:

\[V_{(-1,-1)}^T(p) = \int d^2 z : e^{-\phi(z)} e^{ip \cdot X(z)} : e^{-\hat{\phi}(\bar{z})} e^{ip \cdot \hat{X}(\bar{z})}.\]

These correlators have been evaluated in [42],

\[A(T, T, T, T) \sim (p_1 \cdot p_2)^2 \int d^2 z |z|^{2p_2 \cdot p_4} |1 - z|^{2p_1 \cdot p_4 - 2},\]
\[\sim \frac{2\pi \Gamma(-\frac{d}{2}) \Gamma(-\frac{d}{2}) \Gamma(-\frac{d}{2})}{\Gamma(1 + \frac{d}{2}) \Gamma(1 + \frac{d}{2}) \Gamma(1 + \frac{d}{2})}, \quad (26)\]
where the Mandelstam variables are those appearing in (3). In this case they satisfy the on-shell relation

\[ s + t + u = -4 . \]

To find the covariant expansion of this amplitude, one should realize that the tachyon covariant kinetic term produce massless poles in all channels. However, the constraint (27) does not allow us to sent all \( s, t, u \) to zero at the same time, i.e., \( s, t, u \to 0 \), to produce the massless poles. Following [35], to find the correct way of expanding the tachyon amplitude, one should compare the tachyon amplitude with the S-matrix element of four massless scalars. In that case it is known how to expand the amplitude to produce the Feynman amplitude resulting from scalar kinetic term. Following the same steps, one can find the covariant expansion of the tachyon amplitude (26).

The S-matrix element of four scalars is given by the following correlation function:

\[ A(g, g, g, g) \sim \langle V_{(0,0)}^g(p_1, \zeta_1) : V_{(0,0)}^g(p_2, \zeta_2) : V_{(-1,-1)}^g(p_3, \zeta_3) : V_{(-1,-1)}^g(p_4, \zeta_4) : \rangle, \]

where the scalar vertex operator in \((0,0)\) picture is given in (8), and in \((-1,-1)\) picture is given by

\[ V_{(-1,-1)}^g = \zeta_{ij} \int d^2 z : e^{-\phi(z)}\hat{\psi}^i(z)\hat{\psi}^j(z) : e^{-\phi(z)}\hat{\psi}^i(z)\hat{\psi}^j(z) : . \]

Straightforward evaluation of the correlators gives the result

\[
A(g, g, g, g) \sim \zeta_{i1} \zeta_{2d} \zeta_{3m} \zeta_{4p} \int d^2 z_1 d^2 z_2 d^2 z_3 d^2 z_4 \prod_{a<b} |z_{ab}|^{2p_a p_b} \\
\times \left[ \frac{\eta^{ik}\eta^{jp}(1 - p_1 \cdot p_2)}{z_1^2 z_3^2} + \frac{\eta^{jm}\eta^{kp}p_1 \cdot p_2}{z_1^2 z_4^2} - \frac{\eta^{ip}\eta^{kn}p_1 \cdot p_2}{z_2^2 z_3^2} \right] \\
\times \left[ \frac{\eta^{il}\eta^{mq}(1 - p_1 \cdot p_2)}{z_1^2 z_3^2} + \frac{\eta^{im}\eta^{lp}p_1 \cdot p_2}{z_1^2 z_4^2} - \frac{\eta^{ip}\eta^{ln}p_1 \cdot p_2}{z_2^2 z_3^2} \right].
\]

It is easy to check that the integrand has \( SL(2, C) \) symmetry. One should fix this symmetry by fixing position of three vertices at \( z_1 = \bar{z}_1 = \infty, z_2 = \bar{z}_2 = 0 \), and \( z_3 = \bar{z}_3 = 1 \). After this gauge fixing, one ends up with one complex integral in the \( z \)-plane. The imaginary part is zero and the real part is the following (see the Appendix A):

\[
A(g, g, g, g) \sim 2\pi \left( \text{Tr}(\zeta_1^T \zeta_2) \text{Tr}(\zeta_3^T \zeta_4) - \Gamma(1 - \frac{\eta}{2}) \Gamma(1 - \frac{\eta}{2}) \Gamma(1 + \frac{\eta}{2}) \right) \\
- \text{Tr}(\zeta_1^T \zeta_2 \zeta_3^T \zeta_4) \Gamma(1 + \frac{\eta}{2}) \Gamma(1 - \frac{\eta}{2}) \Gamma(1 - \frac{\eta}{2}) \\
- \text{Tr}(\zeta_1^T \zeta_2 \zeta_3^T \zeta_4) \Gamma(1 - \frac{\eta}{2}) \Gamma(1 - \frac{\eta}{2}) \Gamma(1 + \frac{\eta}{2}) \\
+ (1234 \to 1324) + (1234 \to 1432),
\]
where the Mandelstam variables satisfy (10). Note that under \((1234) \rightarrow (1324)\) the Mandelstam variables change as \((s,t,u) \rightarrow (u,t,s)\), and under \((1234) \rightarrow (1432)\) change as \((s,t,u) \rightarrow (t,s,u)\).

Now, the gamma functions in the second and third line of (28) has no massless pole at all. This indicates that these terms have no contribution in producing the massless poles. In other words, only the first line produces the massless pole in \(s\)-channel resulting from covariant kinetic term. In this case, the expansion is of course at \(s,t,u \rightarrow 0\). So at the covariant limit the gamma function should be send to

\[
\frac{\Gamma(1 - \frac{s}{2})\Gamma(-\frac{s}{2})\Gamma(1 - \frac{t}{2})}{\Gamma(\frac{t}{2})\Gamma(1 + \frac{s}{2})\Gamma(\frac{1}{2})} \rightarrow \frac{\Gamma(1)\Gamma(0)\Gamma(1)}{\Gamma(0)\Gamma(1)\Gamma(0)}
\]

Similarly for \(t\) and \(u\) channels. Using these results from the scalar amplitude, one realizes immediately that the tachyon amplitude (26) should be rewritten as \(A(T,T,T,T) = (A + A + A)/3\), and the covariant limit is the following:

\[
s-\text{channel} : \lim_{s \to 0, t,u \to -2} A

t-\text{channel} : \lim_{t \to 0, s,u \to -2} A

u-\text{channel} : \lim_{u \to 0, s,t \to -2} A
\]

(29)

It may seems strange that in this limit one should send \(s\), say, once to zero and once to -2. However, this happens only in the particular form of the amplitude (26). Using the constraint (27), one can rewrite the amplitude in the following form:

\[
A \sim \frac{2\pi}{3} \left( \frac{\Gamma(1 + \frac{s+t-u}{4})\Gamma(-\frac{s}{2})\Gamma(1 + \frac{s+u-t}{4})}{\Gamma(\frac{u-s-t}{4})\Gamma(1 + \frac{s}{2})\Gamma(\frac{t-u-s}{4})} + (1234 \rightarrow 1324) + (1234 \rightarrow 1432) \right)
\]

(30)

In this form, the covariant limit is \(s,t,u \rightarrow 0\). It is easy to check that using the constraint (10), the terms in the first line of (28) can also be rewritten in the form appearing in (30). The scalar amplitude (28) has some other terms that the tachyon amplitude (26) does not have them. This indicates that four scalars couplings and the four tachyon couplings that their coefficients are independent of the tachyon mass, are not the same. The scalar couplings has the tachyon couplings as well as some other couplings that result from expansion of the terms in the second and third line of (28). Expansion of (30) at \(s,t,u \rightarrow 0\) gives the following leading terms:

\[
A(T,T,T,T) \sim \frac{2\pi}{3} \left( \frac{(t-u)^2 - s^2}{8s} - \frac{\zeta(3)}{28}(s^2 - (t-u)^2)^2 + \cdots \right)

+ (1234 \rightarrow 1324) + (1234 \rightarrow 1432)
\]

(31)

The first order terms above should be reproduced by the two derivative action (15).
3.1 Field theory analysis

Now in field theory, using the fact that particles are all tachyon with arbitrary mass $m$, the Mandelstam variables (3) become:

\[
\begin{align*}
 s &= -(-2m^2 + 2p_1 \cdot p_2), \\
 t &= -(-2m^2 + 2p_2 \cdot p_3), \\
 u &= -(-2m^2 + 2p_1 \cdot p_3).
\end{align*}
\]

Conservation of momentum constrains them in the relation

\[s + t + u = 4m^2.\]

Here again if one restricts the mass of tachyon to on-shell value $m^2 = -1$, the above relation also reduces to the on-shell relation (27).

The $s$-channel amplitude in field theory (15) is given by the following Feynman rule:

\[
A'_s = (\hat{V}_{T_1 T_2 h})_{ab} (\hat{G}_h)_{ab} \hat{G}_h_{cd} (\hat{V}_{h T_3 T_4})_{cd} + \hat{V}_{T_1 T_2 \Phi} \Phi \hat{G}_h_{cd} \Phi \hat{V}_{h T_3 T_4},
\]

where the vertex functions and propagators appear in (20) and (23). Replacing them in above equation, one finds

\[
A'_s = -\frac{i}{4s} \left(-2(m^2 - p_1 \cdot p_2)(m^2 - p_3 \cdot p_4) + 2p_1 \cdot p_3 p_2 \cdot p_4 + 2p_1 \cdot p_4 p_2 \cdot p_3\right)
\]

\[= -\frac{i}{16s} ((u - t)^2 - s^2).\]

Note that all $D$-dependence and $m$-dependence cancels out. Now comparing this field theory massless pole with the massless pole of string theory amplitude (31), one finds exact agreement if normalizes the string amplitude (30) by factor $-3i/(4\pi)$. The $t$-channel and $u$-channel calculation in field theory can be read from the $s$-channel amplitude by replacing $(1234) \rightarrow (1432)$ and $(1234) \rightarrow (1324)$, respectively. They obviously agree with string theory amplitude (31). Hence, there is no contact term left over at this order. This indicates that there is no $m^4 T^4$ coupling in the type 0 theory\footnote{\textsuperscript{1}One may object that the next leading terms of (31) may have $T^4$ coupling. This is very unlikely because the coefficient of this term is $\zeta(3)$, and the $\alpha'$ order of this term tell us that even if this term produce $T^4$ coupling, its coefficient would be $m^8$.}. This is unlike the open string case [35], that the tachyon potential has $m^4 T^4$ coupling. Therefore the tachyon potential in type 0 theory is

\[
V(T) = \frac{1}{2} m^2 T^2.
\]

When the tachyon mass is on-shell, the potential has a maximum at $T = 0$ ($V(0) = 0$) and unbounded minimum at $T = \pm \infty$ ($V(\pm \infty) = -\infty$). In the sigma model approach [31], on
the other hand, a bounded potential has been found for tachyon [36]. However, one expects that the two approach are not using the same field variables, i.e., the two results should be related to each other by some field redefinition.

The amplitude (30) describes also one part of the S-matrix element of four scalar (28), i.e., the terms in the first line of (28). The terms in the second and third lines of (28) when the polarization of the scalars are replaced by 1, have zero contribution to the leading two derivative terms. Hence, the leading term in (31) describes the leading terms of the four scalar amplitude as well. Accordingly, the action (25) is consistent with the leading terms of four scalar amplitude (28).

4 Two tachyon-two RR amplitude

We start this section by evaluating the sphere 4-point function of two RR and two tachyon vertex operators. This amplitude for the case of RR scalar has been found in [42]. The amplitude for arbitrary RR state may be given by the following correlation function:

\[ A(C, C, T, T) \sim \langle : V_{(-1/2,-1/2)}^{\text{RR}}(p_1, \varepsilon_1) : V_{(-1/2,-1/2)}^{\text{RR}}(p_2, \varepsilon_2) : V_{(-1,0)}^{T}(p_3) : V_{(0,-1)}^{T}(p_4) : \rangle \quad (33) \]

where \( \varepsilon \)'s are polarization of the RR fields and \( p \)'s are momentum of states. The vertex operators are:

\[
\begin{align*}
V_{(-1/2,-1/2)}^{\text{RR}}(p, \varepsilon) &= \int d^2 z (P_\mp \Gamma_{(n)})^{AB} : e^{-\phi(z)/2} S_A(z)e^{ip\cdot X} : e^{-\hat{\phi}(\bar{z})/2} S_B(\bar{z}) e^{ip\cdot \hat{X}(\bar{z})} :, \\
V_{(-1,0)}^{T}(p) &= \int d^2 z : e^{-\phi(z)} e^{ip\cdot X(z)} : ip \cdot \hat{\psi}(z) e^{ip\cdot \hat{X}(z)} :, \\
V_{(0,-1)}^{T}(p) &= \int d^2 z : ip \cdot \psi(z) e^{ip\cdot X(z)} : e^{-\hat{\phi}(\bar{z})} e^{ip\cdot \hat{X}(\bar{z})} :,
\end{align*}
\]

where \( P_\pm \) are the two different chiral projection operators that refer to two different set of RR states. The RR polarization tensor \( \varepsilon_{a_1a_2...a_{n-1}} \) is included in \( \Gamma_{(n)} \). We refer the reader to ref.[40] for this relation and for our other conventions. The on-shell conditions for RR fields are \( p^2 = 0 = \varepsilon \cdot p \) and for tachyon is \( p^2 = 1 \).

In evaluating the correlators in (33), one needs the correlation of two spin operators and one world-sheet fermion that is given by (see, e.g., [41])

\[
\langle S_A(z_1) : S_B(z_2) : \psi^\mu(z_3) \rangle = \frac{1}{\sqrt{2}} (\gamma^\mu)_{AB} \hat{z}_{12}^{-3/4} (z_{13}z_{32})^{-1/2}.
\]

The other correlators in (33) can easily be evaluated, using different world-sheet propagators [41]. Performing these correlations, one finds that the integrand has \( SL(2, C) \) symmetry. One should fix this symmetry by fixing position of three vertices at say \( z_1 = \bar{z}_1 = \infty, \)
\[ z_2 = \bar{z}_2 = 0, \text{ and } z_3 = \bar{z}_3 = 1. \] After this gauge fixing, one ends up with only one real integral in the \( z \)-plane,
\[
A \sim \alpha \int d^2z |z|^{2p_2 p_4 - 1} |1 - z|^{2p_3 p_4},
\]
where \( \alpha \) includes the kinematic factors,
\[
\alpha = \frac{1}{2} (P_\pm \Gamma_{1(n)})^{AB} (P_\pm \Gamma_{2(n)})^{CD} (ip_4 \cdot \gamma)_{AC} (ip_3 \cdot \gamma)_{BD}
\]
\[
= \frac{-8}{n!} (p_3 \cdot p_4 F_1^{a_1 \cdots a_n} F_2^{a_2 \cdots a_n} - n p_3 a F_1^{a_2 \cdots a_n} F_2^{a_3 \cdots a_n} p_4 - n p_4 a F_1^{a_2 \cdots a_n} F_2^{a_3 \cdots a_n} p_3).\]

We refer the reader to [40] for our convention for the gamma matrices.

The integral in amplitude (36) can be performed, and the result in terms of the Mandelstam variables is (see the Appendix A)
\[
A \sim 2\pi \alpha \frac{\Gamma(-\frac{s}{2})\Gamma(-\frac{s}{2})\Gamma(-\frac{t}{2})}{\Gamma(1 + \frac{s}{2})\Gamma(1 + \frac{s}{2})\Gamma(1 + \frac{t}{2})},
\]
where the Mandelstam variables satisfy the constrain (4). The above result for the case that \( n = 1 \) has been found in [42]. Note that the amplitude has symmetry \( 3 \leftrightarrow 4 \) and \( 1 \leftrightarrow 2 \).

The amplitude (38) has massless poles in all channels. However, only the massless pole in the \( s \)-channel can be reproduced by covariant kinetic term. The massless poles in other channels are reproduced by assuming that the field theory has \( F\bar{F}T \) coupling. To find the covariant expansion of the amplitude, one should send \( s \to 0 \) to produce the massless pole resulting from covariant kinetic term, and send \( t \to 0 \) \( u \to 0 \) to produce the massless pole resulting from the covariant coupling \( F\bar{F}T \) in the \( t \)-channel \( u \)-channel). The constraint (4) does not allow all \( s, t, u \) approach zero all at the same time.

Using the fact that the coefficient \( \alpha \) in (38) has four momenta, the expansion of the gamma function at the covariant limit should have constant massless poles. Moreover, the limit should keep the symmetry of the amplitude. Using these, one may rewrite the amplitude as \( A(C, C, T, T) = (A + A)/2 \). Then, the covariant limit is

\[
s, t \text{ - channel : } \lim_{s, t \to 0, u \to -2} A
\]
\[
s, u \text{ - channel : } \lim_{s, u \to 0, t \to -2} A
\]

Again using the constraint (4), one can rewrite the amplitude as
\[
A(C, C, T, T) \sim \pi \alpha \left( \frac{\Gamma(1 + \frac{s+t}{2})\Gamma(-\frac{s}{2})\Gamma(-\frac{t}{2})}{\Gamma(-\frac{s+t}{2})\Gamma(1 + \frac{s}{2})\Gamma(1 + \frac{t}{2})} \right.
\]
\[
+ \frac{\Gamma(-\frac{s}{2})\Gamma(-\frac{t}{2})\Gamma(1 + \frac{s+u}{2})}{\Gamma(1 + \frac{s}{2})\Gamma(1 + \frac{t}{2})\Gamma(-\frac{s+u}{2})}\right).
\]
In this form, the covariant limit is $s, t, u \to 0$. Expansion at this limit is

$$A(C, C, T, T) \sim \pi \alpha \left(-\frac{4}{s} - \frac{2}{t} - \frac{2}{u} + \frac{\zeta(3)}{2}((s + t)^2 + (s + u)^2) + \cdots\right) \quad (41)$$

One may object that there might be other limit than (39) for the amplitude that produces massless poles in all channels. One may write $A = (A + A + A)/3$ and send

$$s, t - \text{channel} : \lim_{s, t \to 0, u \to -2} A$$
$$s, u - \text{channel} : \lim_{s, u \to 0, t \to -2} A$$
$$t, u - \text{channel} : \lim_{t, u \to 0, s \to -2} A$$

or send

$$s - \text{channel} : \lim_{s \to 0, t, u \to -1} A$$
$$u - \text{channel} : \lim_{u \to 0, s, t \to -1} A$$
$$t - \text{channel} : \lim_{t \to 0, s, u \to -1} A$$

They all would produce massless poles in all $s$, $t$- and $u$-channels. However, the coefficient of the massless pole in the $s$-channel would not be the same as (41). On the other hand, we know the coefficient of $s$-channel is fixed because it related to the standard covariant kinetic term. To show that the coefficient of the $s$-channel in (41) is the correct one, accordingly the covariant limit (39) is the only correct limit, we compare the result with the $s$-channel of the scalar amplitude. The $s$-channel pole of the scalar amplitude must be exactly the same as the tachyon amplitude, as both are related to standard kinetic terms.

The S-matrix element of two RR and two massless scalar vertex operators is given by correlation (33) in which the tachyon vertex operators are replaced by the following scalar vertex operators:

$$V_{(-1,0)}(p_3, \zeta_3) = \zeta_{3ij} \int d^2 z : e^{-\phi(z)} \psi^i(z) e^{ip_3 \cdot X(z)} : (\partial \hat{X}^j(z) + ip_3 \cdot \hat{\psi}(z) \hat{\psi}^j(z)) e^{ip_3 \cdot \hat{X}(\bar{z})} : ,$$

$$V_{(0,-1)}(p_4, \zeta_4) = \zeta_{4kl} \int d^2 z : (\partial X^k(z) + ip_4 \cdot \psi(z) \psi^k(z)) e^{ip_4 \cdot X(z)} : e^{-\hat{\phi}(\bar{z})} \hat{\psi}^l(\bar{z}) e^{ip_4 \cdot \hat{X}(\bar{z})} : , (44)$$

The necessary correlation functions between the world-sheet fermions and the spin operators appearing in this amplitude is

$$<: S_A(z_1) : A_B(z_2) : \psi^i(z_3) : ip_4 \cdot \psi(\zeta_4) \psi^k(z_4) : > ,$$

which can be reduced to the correlation (35) using the following relations (see e.g., [41]):

$$: S_A(z_1) : ip_4 \cdot \psi(z_4) \psi^k(z_4) : \sim \frac{1}{4} (ip_4 \cdot \gamma^k - \gamma^k \gamma^l \gamma^2) A' S_{A'}(z_4) z_4^{-1} ,$$

$$: \psi^i(z_3) : ip_4 \cdot \psi(z_4) \psi^k(z_4) : \sim \eta^i \eta^k ip_4 \cdot \psi(z_3) z_3^{-1} ,$$

16
where in the second line we have used the fact that momentum is in non-compact space and the indices $i$ is in the orthogonal compact space, *i.e.*, $\eta^{ai} = 0$. This property simplifies greatly the evaluation of the correlation functions in $A(C, C, g, g)$. The final result is

$$
A(C, C, g, g) \sim \frac{1}{8} \zeta_{3ij} \zeta_{4kl}(P_+ \Gamma_{1(n)})^{AB}(P_+ \Gamma_{2(n)})^{CD} \int d^2 z_1 d^2 z_2 d^2 z_3 d^2 z_4 \prod_{m<n} |z_{nm}|^{2p_n-p_m} \times \left[ (ip_4 \cdot \gamma^k \gamma^i)_{AC}(z_{23}z_{12}z_{34}z_{14})^{-1} + (ip_4 \cdot \gamma^k \gamma^i)_{CA}(z_{13}z_{12}z_{34}z_{24})^{-1} \right] \times \left[ (ip_3 \cdot \gamma^l \gamma^j)_{BD}(\bar{z}_{24}z_{12}\bar{z}_{43}\bar{z}_{13})^{-1} + (ip_3 \cdot \gamma^l \gamma^j)_{DB}(\bar{z}_{14}z_{12}\bar{z}_{43}\bar{z}_{23})^{-1} \right].
$$

The integrand is $SL(2, C)$ invariant. Fixing this symmetry, like in the tachyon case, one finds a complex integral in the $z$-plane. The imaginary part of the integral is zero and the real part is the following (see the Appendix A):

$$
A(C, C, g, g) \sim \frac{\pi \alpha}{2} \left( \text{Tr}(\zeta_4^T \zeta_4) - \text{Tr}(\zeta_3 \zeta_4) + \text{Tr}(\zeta_3) \text{Tr}(\zeta_4) \right) \frac{\Gamma(1 - \frac{s}{2})\Gamma(-\frac{s}{2})\Gamma(-\frac{t}{2})}{\Gamma(\frac{u}{2})\Gamma(1 + \frac{s}{2})\Gamma(1 + \frac{t}{2})},
$$

(45)

$$
+ 2 \left( \text{Tr}(\zeta_3^T \zeta_4) + \text{Tr}(\zeta_3) \zeta_4) - \text{Tr}(\zeta_3) \text{Tr}(\zeta_4) \right) \frac{\Gamma(1 - \frac{s}{2})\Gamma(-\frac{s}{2})\Gamma(1 - \frac{t}{2})}{\Gamma(1 + \frac{s}{2})\Gamma(1 + \frac{t}{2})},
$$

$$
+ \left( \text{Tr}(\zeta_3^T \zeta_4) - \text{Tr}(\zeta_3) \zeta_4) + \text{Tr}(\zeta_3) \text{Tr}(\zeta_4) \right) \frac{\Gamma(-\frac{s}{2})\Gamma(-\frac{s}{2})\Gamma(1 - \frac{t}{2})}{\Gamma(1 + \frac{s}{2})\Gamma(1 + \frac{t}{2})}.
$$

Note that the result has the expected symmetry between 3,4 and between 1,2. The Mandelstam variables are given in (3), and they satisfy the relation (10).

If one considers the case that there is only one scalar, *i.e.*, the compact space is circle, then the polarization factors simplify to 1. Then, expansion of this amplitude at low energy $s, t, u \to 0$ gives the following leading terms:

$$
A(C, C, g, g) \sim \frac{\pi \alpha}{2} \left( -\frac{s}{8} - \frac{2}{t} - \frac{2}{u} + \cdots \right),
$$

(46)

The coefficient of massless pole in the $s$-channel is exactly the same as the massless pole in the tachyon amplitude (41). This confirms that the covariant limit (39) is the only correct covariant limit of the tachyon amplitude (38), *i.e.*, the limits (42), (43) although consistent with the constraint (4), they are not correct covariant limits.

### 4.1 Field theory analysis

Now in field theory, consider adding the following couplings to the action (15):

$$
S_2^T = - \int d^D x \sqrt{G} \left[ \frac{1}{2} (F_{(n)} \cdot F_{(n)} + \bar{F}_{(n)} \cdot \bar{F}_{(n)}) (1 + b_T T^2) + F_{(n)} \cdot \bar{F}_{(n)} (a_T T) \right],
$$

(47)
where \( F_{(n)} : F_{(n)} \equiv \frac{1}{m} F^{a_1 \cdots a_n} F_{a_1 \cdots a_n} \). The above action is parametrized by two constants \( a_T, b_T \). In the Einstein frame it becomes

\[
S_T^T = - \int d^Dx \sqrt{g} \left[ e^{\frac{D-2n}{2D-2}} \left( \frac{1}{2} (F_{(n)} : F_{(n)} + \bar{F}_{(n)} F_{(n)}) (1 + b_T T^2) + F_{(n)} : \bar{F}_{(n)} (a_T T) \right) \right],
\]

In \( S_T^T \) field theory, we evaluate the \( S \)-matrix element of two RR fields and two tachyons. Using the fact that particle 1, 2 are massless RR fields, and 3, 4 are tachyon with arbitrary mass \( m \), the Mandelstam variables (3) turn into (16).

The \( s \)-channel amplitude is given by the following Feynman rule:

\[
A'_s = (\hat{V}_F \bar{F}_h)^{ab} (\hat{G}_h)_{ab}^{cd} (\hat{V}_h T_3 T_4)^{cd} + \hat{V}_F F : \hat{G}_h \phi \hat{V}_h T_3 T_4 ,
\]

the vertex functions and propagators are given in (20), (23), and in the following:

\[
\hat{V}_F F_{12} = -i \frac{D-2n}{2\sqrt{D-2}} F_{1(2)} : F_{2(2)} ,
\]

\[
(\hat{V}_h F_{12})^{ab} = -i \frac{1}{2n!} \left[ \eta^{ab} F_{a_1 \cdots a_n} F_{a_1 \cdots a_n} - n F_{1 a_2 \cdots a_n} F_{2 b a_2 \cdots a_n} - n F_{1 a_2 \cdots a_n} F_{2 b a_2 \cdots a_n} \right] .
\]

Replacing them in (48), one finds, after some simple algebra,

\[
A'_s = -i \frac{1}{2n!} \left( p_3 \cdot p_4 F_{1}^{a_1 \cdots a_n} F_{2 a_1 \cdots a_n} - np_{3a} F_{1}^{a a_2 \cdots a_n} F_{2 b a_2 \cdots a_n} p_4^b - np_{4a} F_{1}^{a a_2 \cdots a_n} F_{2 b a_2 \cdots a_n} p_3^b \right).
\]

Note that all \( D \)-dependence and \( m \)-dependence cancel out. Now comparing this field theory amplitude with the string theory amplitude (41) in which \( \alpha \) is given by (37), one finds exact agreement if normalizes the string theory amplitude (38) by factor \(-i/(64 \pi)\).

The \( t \)-channel amplitude in field theory is given by the following Feynman rule:

\[
A'_t = (\hat{V}_T F_{3 C})^{a_1 \cdots a_{n-1}} (\hat{G}_C)_{a_1 \cdots a_{n-1}} f_{b_1 \cdots b_{n-1}} (\hat{V}_C F_{2 T_3})_{b_1 \cdots b_{n-1}} ,
\]

where the propagator and the vertex function are

\[
(\hat{G}_C)^{a_1 \cdots a_{n-1}}_{b_1 \cdots b_{n-1}} = \frac{i(n-1)!}{t} \eta^{a_1 \eta^{a_2} \cdots \eta^{a_{n-1}}} ,
\]

\[
(\hat{V}_T F_{2 C})^{a_1 \cdots a_{n-1}} = -\frac{a_T}{(n-1)!} p_{3a} F^{a a_1 \cdots a_{n-1}} .
\]

Replacing them in the amplitude (49), one finds

\[
A'_t = \frac{-i a_T^2}{(n-1)! t} p_{3a} F^{a a_1 \cdots a_{n-1}} F_{1 a_1 \cdots a_{n-1}} p_{4b} .
\]
For simplicity, consider only the terms that have $\varepsilon_1 \cdot \varepsilon_2$. Simple algebra reduces above amplitude to the following:

$$A'_t = -\frac{ia_T^2}{2(n-1)!}(-p_1 \cdot p_2 p_3 \cdot p_4 + p_3 \cdot p_2 p_4 \cdot p_1 + p_4 \cdot p_2 p_3 \cdot p_1)\varepsilon_1^{a_1 \cdots a_{n-1}} \varepsilon_{2a_1 \cdots a_{n-1}} \quad (50)$$

Similarly, the $u$-channel in field theory is

$$A'_u = -\frac{ia_T^2}{2(n-1)!}(-p_1 \cdot p_2 p_3 \cdot p_4 + p_3 \cdot p_2 p_4 \cdot p_1 + p_4 \cdot p_2 p_3 \cdot p_1)\varepsilon_1^{a_1 \cdots a_{n-1}} \varepsilon_{2a_1 \cdots a_{n-1}} \quad (51)$$

Comparing the above poles with the corresponding poles in string theory (41), one finds

$$a_T^2 = 1/2 \quad (52)$$

Moreover, imposing the fact the string theory amplitude does not have the above contact terms fixes the constant $b$ to be

$$b_T = a_T^2/2 = 1/4 \quad (53)$$

The next order terms in (41) are related to eight derivative order terms in the action in which we are not interested in the present paper.

For the scalar action, one may add the following couplings to the action (25):

$$S^g_2 = -\int d^D x \sqrt{G} \left[ \frac{1}{2} (F_{(n)} \cdot F_{(n)} + \bar{F}_{(n)} \cdot F_{(n)}) (1 + a_g g + b_g g^2) \right], \quad (54)$$

The $s$-channel amplitude is exactly like the amplitude (48) in which tachyons are replaced by the scalars. Accordingly, one finds exact agreement with the first term in string amplitude (46). The $t$- and $u$-channel again are like in the tachyon case in which the tachyons are replaced by the scalar, and $\bar{C}$ is replaced by $C$, hence, one finds the result (50) and (51), respectively. Comparing them with string theory amplitude (46), one finds

$$a_g = 1/4 \quad ; \quad b_g = a_g^2/2 = 1/8 \quad (55)$$

5 Discussion

In this paper, we have evaluated various sphere level S-matrix elements involving tachyon vertex operators in type 0 theory. We then find an expansion for these amplitudes that their
leading order terms are correspond to covariant tachyon action. The two derivatives order action that we have found, (14), (47), (52), and (53), in terms of $F^\pm_{(n)} = (F_{(n)} \pm \bar{F}_{(n)})/\sqrt{2}$, is

$$S^T = -\int d^p x \sqrt{G} \left[ e^{-2\Phi} \left( -2R - 8\partial^a \Phi \partial_a \Phi + \frac{3}{2} H^2 + \frac{1}{2} \partial^a T \partial_a T + \frac{1}{2} m^2 T^2 \right) 
+ \frac{1}{2} \left( F^+_{(n)} \cdot F^+_{(n)} \right) f(T) + \frac{1}{2} \left( F^-_{(n)} \cdot F^-_{(n)} \right) f(-T) \right] + \cdots ,$$

(56)

where dots represent couplings that are of order eight derivatives and higher, and their coefficients include $\zeta(3), \zeta(4), \cdots$. The function $f(T)$ is

$$f(T) = 1 \pm \frac{1}{\sqrt{2}} T + \frac{1}{4} T^2 + \cdots .$$

The tachyon coupling $F\bar{F}T$ that we have extracted from the S-matrix element of two RR and two tachyons, can also be extracted from the S-matrix element of four RR states with opposite chirality. The coupling $F\bar{F}T$ appears in this amplitude as a tachyonic pole. However, this amplitude can fix the sum of $F\bar{F}T$ and $F\bar{F}F$ couplings. Since we don’t know the coupling $F\bar{F}F$, this study can not fix the coupling $F\bar{F}T$ without ambiguity. We analyze to some extent this S-matrix element in the Appendix B.

The action for the scalar field that we have found, (25), (54) and (55), is the following:

$$S^g = -\int d^p x \sqrt{G} \left[ e^{-2\Phi} \left( -2R - 8\partial^a \Phi \partial_a \Phi + \frac{3}{2} H^2 + \frac{1}{2} \partial^a g \partial_a g \right) 
+ \frac{1}{2} \left( F_{(n)} \cdot F_{(n)} \right) \left( 1 \pm \frac{1}{2} g + \frac{1}{8} g^2 + \cdots \right) \right] ,$$

(57)

This action is the low energy action for the scalar field. This action should be consistent with the dimensional reduction of the following 10-dimensional action:

$$S = -\int d^{10} x \sqrt{G} \left[ e^{-2\Phi} \left( -2R - 8\partial^a \Phi \partial_a \Phi + \frac{3}{2} H^2 + \frac{1}{2} \partial^a (\log k) \partial_a (\log k) \right) 
+ \frac{1}{2} \left( F_{(n)} \cdot F_{(n)} \right) k + \frac{1}{2} \left( \bar{F}_{(n)} \cdot \bar{F}_{(n)} \right) k \right] .$$

Dimensional reduction of this action to 9-dimension is [44]

$$S = -\int d^9 x \sqrt{G} \left[ e^{-2\Phi} \left( -2R - 8\partial^a \Phi \partial_a \Phi + \frac{3}{2} H^2 + 2\partial^a (\log k) \partial_a (\log k) \right) 
+ \frac{1}{2} \left( F_{(n)} \cdot F_{(n)} \right) k + \frac{1}{2} \left( \bar{F}_{(n)} \cdot \bar{F}_{(n)} \right) k \right] ,$$

where the scalar $k$ is related to the $G_{1010}$ component of the metric [44]. Note that we have considered only those Kalb-Ramond and RR fields that have components in the non-compact space, and only $G_{ab}$ and $G_{1010}$ component of metric. These are the fields that we
have studied in our paper. Using the field redefinition $g = \pm 2 \log k$ to write the kinetic term in the standard form, one finds

$$S = - \int d^9 x \sqrt{G} \left[ e^{-2\Phi} \left( -2R - 8 \partial^{\mu} \Phi \partial_{\mu} \Phi + \frac{3}{2} H^2 + \frac{1}{2} \partial^{\mu} g \partial_{\mu} g \right) e^{\pm g/2} + \frac{1}{2} (F(n) \cdot F(n)) e^{\pm g/2} \right].$$

(58)

It is easy to see that the action (57) is consistent with above action as expected. The above action is also consistent with the fact that, due to the vanishing of their world-sheet correlation functions, self-coupling of odd number of scalars and coupling of odd number of scalars to graviton are zero.

We have been assuming throughout the section 2.1 that there is no tachyon coupling $RT^2$. This assumption was also made in [42]. This is consistent with the observation made in section 2 that the S-matrix element of two tachyons and two gravitons, and the S-matrix element of two scalars and two gravitons can be written in the identical form (6). This indicates that apart from those tachyon couplings that the mass $m$ appears as their coefficient, the coupling of two tachyons and two gravitons, and the coupling of two scalars and two gravitons should be the same. Since the scalar action (58) has no coupling $Rg^2$, accordingly, the tachyon action has no coupling $RT^2$ either. Similarly, following the discussion in the last paragraph in section 3.1, on concludes that there is no tachyon coupling $T^2 \partial_{\alpha} T \partial^{\mu} T$ because there is no $g^2 \partial_{\alpha} g \partial^{\mu} g$ coupling in the scalar action (58). This is unlike the open string tachyon case that such a coupling is non-zero [37, 35].

We have seen that our calculation fixes the form of tachyon potential to be (32). This potential has no local minimum. This means the stability of theory can not be reached through condensation of only tachyon field. As pointed out in [42], however, the instability may be cured in the presence of background RR field. In the presence of background flux $F$ and $\bar{F}$, the on-shell tachyon potential, up to quadratic order of background field, is the following

$$V(T) = -\frac{1}{2} T^2 + \frac{1}{2} F^2 f(T) + \frac{1}{2} \bar{F}^2 f(-T).$$

For appropriate background flux, it may have local minimum that the unstable theory at $T = 0$ condenses to it. In principle, however, one may use a field redefinition such that in the new variables the instability would be cured by condensation of only tachyon. In this regard, it was found in [36] that the tachyon potential in the sigma model approach which is expected to be related to S-matrix based approach by field redefinition, has local minimum.

We have seen that the tachyon and scalar have, to the leading order, similar couplings in the bulk. Now it raises a question: does the tachyon and massless scalars have also similar couplings to D-branes? To answer this question, consider the coupling of two tachyons to
D-branes of type 0 theory. This amplitude is the following [46]:

\[
A(T, T) \sim \frac{\Gamma(-t/2)\Gamma(-2s)}{\Gamma(-1 - t/2 - 2s)},
\]

where \( t = -\alpha'(p_1 + p_2)^2/2 \) and \( s = -\alpha'(p_1 \cdot G \cdot p_1)/2 \) where \( G \) here stands for the open string metric. Now compare it with the scattering amplitude of two massless scalars from D-brane. This amplitude can be read from the general result in [40],

\[
A(g, g) \sim \frac{\Gamma(-t/2)\Gamma(-2s)}{\Gamma(1 - t/2 - 2s)} \left( 4s^2 + t(s + t/4) \right),
\]

where we have assumed there is only one scalar, i.e., \( \zeta \to 1 \). Note that the Mandelstam variables in these amplitudes are arbitrary. As it can be seen, they are not identical amplitude. That means the coupling of tachyon and the scalars to D-branes are not similar. In other words, if one expands both in the limit \( s, t \to 0 \), one finds

\[
A(T, T) \sim 2 \left( \frac{1 + 2s}{t} - \frac{p_1 \cdot N \cdot p_2}{4s} + \frac{3}{4} + \cdots \right),
\]

\[
A(g, g) \sim 2 \left( \frac{2s}{t} - \frac{p_1 \cdot N \cdot p_2}{4s} + \frac{1}{4} + \cdots \right),
\]

where \( N \) is the flat metric in the space orthogonal to the D-brane. The massless poles are reproduce by standard covariant action [46, 47]. The contact terms above indicates that the quadratic tachyon coupling to D-branes is different from the quadratic scalar coupling to the D-branes.

Finally, we note that the expansion of the S-matrix elements of massless scalar states (9) and (28) in the limit that the Mandelstam variables approach zero has, in general, undesirable Euler-Mascheroni constant \( \gamma = 0.5772157 \). When one imposes the on-shell constraint on the Mandelstam variables they disappear. However, these S-matrix elements in the form (6) and (30) have no such undesirable terms. They already disappear by imposing the constrains in the amplitude. For example, consider on-shell S-matrix element of four scalars (28). The first term in this equation has the following expansion:

\[
\frac{\Gamma(1 - \frac{u}{2})\Gamma(-\frac{s}{2})\Gamma(1 - \frac{t}{2})}{\Gamma(\frac{u}{2})\Gamma(1 + \frac{s}{2})\Gamma(\frac{t}{2})} = -\frac{ut}{2s} - \frac{\gamma u(s + t + u)t}{2s} - \frac{\gamma^2 u(s + t + u)^2t}{4s} + \cdots,
\]

if one imposes the on-shell constraint \( s + t + u = 0 \), the terms that have \( \gamma \) vanishes. Now the same term in the S-matrix element in the form (30) has the following expansion:

\[
\frac{\Gamma(1 + \frac{s+u-t}{4})\Gamma(-\frac{s}{2})\Gamma(1 + \frac{s+u-t}{4})}{\Gamma(\frac{u-s-t}{4})\Gamma(1 + \frac{s}{2})\Gamma(\frac{u-s-t}{4})} = -\frac{s^2 - (t - u)^2}{8s} - \frac{\zeta(3)}{128} (s^2 - (t - u)^2)^2 + \cdots.
\]

The constant \( \gamma \) does not appear in the amplitude at all. This may indicate that the constraint is imposed in the amplitude correctly.

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Appendix A

In this appendix we give the result for some integrals that appear in the previous sections. Consider the following integral:

$$I = \int d^2 z \, |z|^{2a} |1 - z|^{2b} f(z, \bar{z}).$$  \hspace{1cm} (60)

To evaluate this integral, one should write

$$|z|^{2a} = \frac{1}{\Gamma(-a)} \int_0^\infty dt \, t^{-a-1} e^{-t|z|^2},$$

and similarly for $|1 - z|^{2b}$. This turns the $z$ integration into an elementary integration that can be explicitly carried out. In this way

$$I = \frac{1}{\Gamma(-a) \Gamma(-b)} \int_0^\infty dt \, t^{a-1} u^{-b-1} J(t, u),$$

where

$$J(t, u) = \int d^2 z \, e^{-t|z|^2 - u|1-z|^2} f(z, \bar{z}).$$

This integral is easy to evaluate for some simple function $f(z, \bar{z})$. After performing this integral, one should make the change of variables $t = x/s$ and $u = x/(1 - s)$. Then using the following definitions for gamma and beta functions:

$$\Gamma(\alpha) = \int_0^\infty dx \, x^{\alpha-1} e^{-x},$$

$$B(\alpha, \beta) = \int_0^1 ds \, (1 - s)^\alpha s^\beta,$$

one finds the final result for the integral (60) in terms of gamma functions. For example

$$\int d^2 z \, |z|^{2a} |1 - z|^{2b} f(z, \bar{z}) = 2\pi \frac{\Gamma(a + 1) \Gamma(b + 1) \Gamma(-a - b - 1)}{\Gamma(-a) \Gamma(-b) \Gamma(a + b + 2)},$$

$$\int d^2 z \, |z|^{2a} |1 - z|^{2b} f(z, \bar{z}) = 2\pi \frac{\Gamma(a + 1) \Gamma(b + 1) \Gamma(-a - b - 1)}{\Gamma(-a) \Gamma(-b) \Gamma(a + b + 3)},$$

$$\int d^2 z \, |z|^{2a} |1 - z|^{2b} f(z, \bar{z}) = 2\pi \frac{\Gamma(a + 1) \Gamma(b + 2) \Gamma(-a - b - 1)}{\Gamma(-a) \Gamma(-b) \Gamma(a + b + 3)},$$

$$\int d^2 z \, |z|^{2a} |1 - z|^{2b} f(z, \bar{z}) = 2\pi \frac{\Gamma(a + 1) \Gamma(b + 2) \Gamma(-a - b - 1)}{\Gamma(-a) \Gamma(-b) \Gamma(a + b + 3)},$$

$$\int d^2 z \, |z|^{2a} |1 - z|^{2b} f(z, \bar{z}) = 2\pi \frac{\Gamma(a + 1) \Gamma(b + 2) \Gamma(-a - b - 1)}{\Gamma(-a) \Gamma(-b) \Gamma(a + b + 3)},$$

$$\int d^2 z \, |z|^{2a} |1 - z|^{2b} f(z, \bar{z}) = -2\pi \frac{\Gamma(a + 2) \Gamma(b + 2) \Gamma(-a - b - 2)}{\Gamma(-a) \Gamma(-b) \Gamma(a + b + 3)}.$$
\[ \int d^2 z \, |z|^{2a} |1 - z|^{2b} (1 - \bar{z})^{1} = -2\pi \frac{\Gamma(a + 2)\Gamma(b + 2)\Gamma(-a - b - 2)}{\Gamma(-a)\Gamma(-b)\Gamma(a + b + 3)}, \]
\[ \int d^2 z \, |z|^{2a} |1 - z|^{2b} z^{2} = 2\pi \frac{\Gamma(a + 3)\Gamma(b + 1)\Gamma(-a - b - 1)}{\Gamma(-a)\Gamma(-b)\Gamma(a + b + 4)}, \]
\[ \int d^2 z \, |z|^{2a} |1 - z|^{2b} \bar{z}^{2} = 2\pi \frac{\Gamma(a + 3)\Gamma(b + 1)\Gamma(-a - b - 1)}{\Gamma(-a)\Gamma(-b)\Gamma(a + b + 4)}, \]
\[ \int d^2 z \, |z|^{2a} |1 - z|^{2b} (1 - \bar{z})^{2} = 2\pi \frac{\Gamma(a + 1)\Gamma(b + 1)\Gamma(-a - b - 1)}{\Gamma(-a)\Gamma(-b)\Gamma(a + b + 4)}, \]
\[ \int d^2 z \, |z|^{2a} |1 - z|^{2b} (1 - \bar{z})^{2} \bar{z}^{2} = 2\pi \frac{\Gamma(a + 1)\Gamma(b + 1)\Gamma(-a - b - 1)}{\Gamma(-a)\Gamma(-b)\Gamma(a + b + 4)}, \]
\[ \int d^2 z \, |z|^{2a} |1 - z|^{2b} (1 - \bar{z})^{2} z^{2} = 2\pi \frac{\Gamma(a + 3)\Gamma(b + 3)\Gamma(-a - b - 3)}{\Gamma(-a)\Gamma(-b)\Gamma(a + b + 4)}, \]
\[ \int d^2 z \, |z|^{2a} |1 - z|^{2b} (1 - \bar{z})^{2} \bar{z}^{2} = 2\pi \frac{\Gamma(a + 3)\Gamma(b + 3)\Gamma(-a - b - 3)}{\Gamma(-a)\Gamma(-b)\Gamma(a + b + 4)}. \]
Appendix B

In this appendix, we evaluate the S-matrix element of four RR with opposite chirality, and then compare it with field theory. In world-sheet conformal field theory, the S-matrix element of two CC and two $\bar{C}\bar{C}$ states is given by the following correlation function:

$$A \sim \langle \substack{V_{RR}^{(1/2,-1/2)}(p_1,\varepsilon_1) : V_{RR}^{(1/2,-1/2)}(p_2,\varepsilon_2) : V_{RR}^{(-1/2,-1/2)}(p_3,\varepsilon_3) : V_{RR}^{(-1/2,-1/2)}(p_4,\varepsilon_4) : \rangle,$$

where the RR vertex operators are given in (34). Two of them have positive (negative) chirality and two others negative (positive) chirality. The nontrivial correlation is the correlation between the four spin operator with opposite chirality. It is give by the following relation [48]:

$$<: S_A(z_1) : S_B(z_2) : S_C(z_3) : S_D :> = \frac{1}{2} (\gamma_\mu)_{AB}(\gamma^\mu)_{CD}(z_{13}z_{14}z_{23}z_{24})^{-1/4}(z_{12}z_{34})^{-3/4} + C_{AC}C_{BD}(z_{12}z_{34})^{1/4}(z_{14}z_{23})^{-1/4}(z_{13}z_{24})^{-5/4} - C_{AB}C_{BC}(z_{12}z_{34})^{1/4}(z_{13}z_{24})^{-1/4}(z_{14}z_{23})^{-5/4},$$

where $C$ is the charge conjugation matrix. The other correlators are easy to evaluate. The final result, after fixing its $SL(2,R)$ symmetry and doing some algebra on the gamma matrices, is

$$A \sim 2\pi \left[ \frac{1}{8} \text{Tr}(P_+\Gamma_1(\gamma_\nu\Gamma_2(\gamma_\mu))\text{Tr}(P_+\Gamma_3(\gamma_\nu\Gamma_4(\gamma_\mu)) \frac{\Gamma(\frac{1}{2} - \frac{\mu}{2})\Gamma(\frac{1}{2} - \frac{\lambda}{2})}{\Gamma(\frac{1}{2} + \frac{\mu}{2})\Gamma(\frac{1}{2} + \frac{\lambda}{2})} 
+ \text{Tr}(P_+\Gamma_3(\gamma_\nu\Gamma_1(\gamma_\mu))\text{Tr}(P_+\Gamma_4(\gamma_\nu\Gamma_2(\gamma_\mu)) \frac{\Gamma(\frac{1}{2} - \frac{\mu}{2})\Gamma(\frac{1}{2} - \frac{\lambda}{2})}{\Gamma(\frac{1}{2} + \frac{\mu}{2})\Gamma(\frac{1}{2} + \frac{\lambda}{2})} 
+ \frac{1}{2} \left( \text{Tr}(P_+\gamma_\mu\Gamma_1(\gamma_\nu\Gamma_3(\gamma_\mu\Gamma_4(\gamma_\nu))\Gamma_2(\gamma_\nu)) + \text{Tr}(P_+\gamma_\mu\Gamma_2(\gamma_\nu\Gamma_1(\gamma_\mu\Gamma_3(\gamma_\mu))\Gamma_4(\gamma_\nu)) \right) \times \frac{\Gamma(\frac{1}{2} - \frac{\mu}{2})\Gamma(1 - \frac{\lambda}{2})\Gamma(\frac{1}{2} - \frac{\lambda}{2})}{\Gamma(\frac{1}{2} + \frac{\mu}{2})\Gamma(\frac{1}{2} + \frac{\lambda}{2})} \frac{\Gamma(\frac{1}{2} - \frac{\mu}{2})\Gamma(1 - \frac{\lambda}{2})\Gamma(\frac{1}{2} - \frac{\lambda}{2})}{\Gamma(\frac{1}{2} + \frac{\mu}{2})\Gamma(\frac{1}{2} + \frac{\lambda}{2})} + 3 \leftrightarrow 4 \right] \right. \left. + \frac{1}{2} \right) \frac{\Gamma(\frac{1}{2} - \frac{\mu}{2})\Gamma(1 - \frac{\lambda}{2})\Gamma(\frac{1}{2} - \frac{\lambda}{2})}{\Gamma(\frac{1}{2} + \frac{\mu}{2})\Gamma(\frac{1}{2} + \frac{\lambda}{2})} \frac{\Gamma(\frac{1}{2} - \frac{\mu}{2})\Gamma(1 - \frac{\lambda}{2})\Gamma(\frac{1}{2} - \frac{\lambda}{2})}{\Gamma(\frac{1}{2} + \frac{\mu}{2})\Gamma(\frac{1}{2} + \frac{\lambda}{2})} + 3 \leftrightarrow 4 \right],$$

where the Mandelstam variables are given in (3). Massless pole appears only in the first term, and all other terms have tachyonic or massive poles. They contribute to contact terms in the covariant limit/expansion, i.e., the expansion at $s, t, u \to 0$.

To normalize the amplitude, we consider, for simplicity, $n = 1$. The momentum expansion is

$$A \sim 2\pi (32)^2 \left[ -\frac{1}{16s} \left( 2p_1 \cdot p_3 p_2 \cdot p_4 + (D - 4)p_1 \cdot p_2 p_3 \cdot p_4 + 2p_1 \cdot p_4 p_2 \cdot p_3 \right) + \cdots + 3 \leftrightarrow 4 \right],$$

25
where dots represent contact terms that have at least four momenta. In field theory, the massless pole is given by the following Feynman rule:

\[
A'_s = (\hat{V}_{F_1 F_2})_{ab} (\hat{G}_h)^{ab}_{cd} (\hat{V}_{h F_3 F_4})^{cd} + \hat{V}_{F_1 F_2} \Phi' \hat{G} \Phi' V_{h F_3 F_4}
\]

\[
= -\frac{i}{48} [(D-4) p_1 \cdot p_2 p_3 \cdot p_4 + 2 p_1 \cdot p_4 p_3 \cdot p_2 + 2 p_1 \cdot p_4 p_3 \cdot p_2]
\]

which is exactly the massless pole of string theory provided one normalizes the amplitude (61) by factor \(i/(\pi (32)^2)\). The next order terms correspond to two different terms in field theory. One is the contact term \(FF\bar{F}\bar{F}\), and the other the tachyonic pole resulting from two \(FF\bar{T}\) couplings and tachyon as propagator. The propagator should be replaced by 1. Hence, the next order terms can fix the sum of \(FF\bar{F}\bar{F}\) and \(FF\bar{T}\) couplings.

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