Semiclassical Limit of the Dirac Equation
and Spin Precession

Herbert Spohn
Zentrum Mathematik and Physik Department,
TU München, D–80290 München, Germany
spohn@mathematik.tu-muenchen.de

Abstract. We study the Dirac equation with slowly varying external potentials. Using matrix-valued Wigner functions we prove that the electron follows with high precision the classical orbit and that the spin precesses according to the BMT equation with gyromagnetic ratio $g = 2$. 
1 Introduction

The high precision experiments of the anomalous magnetic moment of the electron or the muon use in their interpretation the BMT equation for spin precession. The derivation of Bargmann, Michel, and Telegdi does not even mention quantum mechanics, however. Rather their arguments are based on relativistic covariance together with a classical notion of an intrinsic angular momentum. Thus one might wonder how exactly the classical BMT equation is connected with the quantum mechanical spin precession. Properly speaking, one should start from QED and then deduce a BMT-like equation for the spin motion. Such a program looks rather difficult at present. But even if we take the one-particle relativistic Dirac equation as starting point, as we do here, the problem seems little explored and is thus still of interest.

Let us first recall the BMT equation. One considers a charge, mass $m$, charge $e$, moving in prescribed, time-independent external electromagnetic fields $E = -\nabla \phi, B = \nabla \times A$ and governed by the Lorentz force equation

$$\frac{d}{dt} m \gamma v_t = e \left( E(q_t) + c^{-1} v_t \times B(q_t) \right). \quad (1.1)$$

Here $q_t$ is the position and $v_t = \dot{q}_t$ the velocity of the particle at time $t$ with $\gamma = 1/\sqrt{1 - (v/c)^2}$. According to BMT, for $g = 2$, the precession of the spin $s$ along the given orbit is determined by

$$\frac{d}{dt} s = \frac{e}{mc} s \times \left[ \frac{1}{\gamma} B - \frac{1}{1 + \gamma} c^{-1} v \times E \right]. \quad (1.2)$$

For $g \neq 2$ the prefactors of $B, v \times E$ change and also a term proportional to $v(v \cdot B)$ appears. We refer to the textbook by Jackson, Section 11.11, for details.

The same physical situation can be described by the relativistic one-particle Dirac equation in the semiclassical limit. This means that the potentials $\phi, A$ are slowly varying on the scale set by the typical extension of the electron/positron wave function. On a formal level it is convenient to introduce the dimensionless scale parameter $\varepsilon, \varepsilon \ll 1$, and to take the potentials to be of the form

$$\phi(\varepsilon x), \ A(\varepsilon x). \quad (1.3)$$
The state of the electron/positron is now described by the four component spinor $\psi$ which evolves according to the Dirac equation as

$$i\hbar \frac{\partial}{\partial t} \psi = H \psi \quad (1.4)$$

with the Dirac hamiltonian

$$H = c\gamma_0 \left( \gamma \cdot \left( -i\hbar \nabla_x - \frac{e}{c} A(\varepsilon x) \right) + mc \right) + e\phi(\varepsilon x) \quad (1.5)$$

acting as a linear operator on $\bigoplus_{n=1}^4 L^2(\mathbb{R}^3)$. Here $\gamma_0, \gamma$ are the Dirac gamma matrices, where we follow the conventions in [4]. For the semiclassical limit one starts with a well localized wave function $\psi$, having zero positron component, and evolves it unitarily according to

$$\psi_t = e^{-iHt/\hbar} \psi \quad (1.6)$$

over such a long time that the response to the external forces is visible. With our choice (1.3) this means times of the order $\varepsilon^{-1}$. If the standard picture is correct, then over that time-span $\psi_t$ should have a negligible positron component and should be well concentrated on the classical path determined by (1.1). In addition, the average polarization should satisfy (1.2). This is exactly what we are going to show.

Before entering in the details of our argument, it might be useful to recall the semiclassical limit of the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \psi = \left( -\frac{\hbar^2}{2m} \Delta + V(\varepsilon x) \right) \psi \quad (1.7)$$

with the slowly varying electrostatic potential $V$. If the initial $\psi$ is well localized at the classical phase point $(q, p)$, then $\psi_t$ is well localized at $(q_t, p_t)$ with $\dot{q} = \frac{1}{m}p, \dot{p} = -\nabla V(q)$. This is an extremely well-studied chapter of mathematical physics [5] with higher order corrections and sharp error bounds available [6]. The Schrödinger equation is scalar. Put it differently, if we set the potential to zero, then the energy function of (1.7) is $E(p) = \frac{1}{2m}p^2$ which is a scalar function. In somewhat more complicated cases, like an additional short scale periodic potential [7], internal degrees of freedom, or several particles, for zero external potential the energy will be matrix-valued as a function of a suitable quasimomentum. If this energy matrix has nondegenerate eigenvalues (= energy band functions), then in essence one is back
to the scalar case. Interference between bands is destroyed through rapid oscillations and the evolution in each band separately is semiclassical \cite{7}. In general, energy bands may cross, i.e. the eigenvalues may be degenerate along lower dimensional submanifolds. The wave function can then propagate into neighboring bands giving rise to intricate interference patterns \cite{8}. In the case of the Dirac equation, the energy matrix is a $4 \times 4$ matrix and it has the two energy bands $E_{\pm}(p) = \pm c\sqrt{m^2c^2 + p^2}$, both two-fold degenerate, which is at the origin for a scenario very distinct from the scalar case. In fact, in the semiclassical limit, the position and momentum become classical variables, whereas the spin remains fully quantum mechanical.

It should be noted that from a purely abstract point of view the Dirac equation is the simplest example for a matrix-valued energy function. One would like to have at least two bands each of which should be degenerate. This makes four to be the minimal matrix dimension.

We close the introduction with some brief comments on previous work. The most elementary approach is to consider average values of spin and position and to use what is known as Ehrenfest’s theorem. For the Dirac equation the details have been carried through by Fradkin and Good \cite{9}. The more sophisticated WKB approach to the Dirac equation was initiated by Pauli \cite{10} and completed by Rubinow and Keller \cite{11}. The full semiclassical, van-Vleck-type propagator has been worked out recently by Bolte and Keppeler \cite{12} including a trace formula for the eigenvalues. The WKB method has the drawback of being local in time only. In the case of the Schrödinger equation one understands how to continue beyond the caustics by adapting new coordinates in the so-called Lagrangian manifold \cite{5, 13}. For matrix-valued energy functions such a program has not been attempted yet. Instead, we will use here matrix-valued Wigner functions. The abstract theory for this approach has been developed in \cite{14} and applications to beam physics are discussed in \cite{15}.

2 Semiclassical limit of the Dirac equation

In the Appendix we study abstractly the semiclassical limit for Schrödinger type equations with matrix-valued symbols. In this section we merely transcribe these general results to the Dirac equation. The emergence of the BMT equation will be explained in Section 3.

In the following $p$ denotes the classical momentum and we use $q, p$ as a
pair of canonical coordinates. \( \{ \cdot, \cdot \} \) is the Poisson bracket. We define it also for matrix-valued functions \( A, B \) through

\[
\{ A, B \} = \nabla_p A \cdot \nabla_q B - \nabla_q A \cdot \nabla_p B
\]  

(2.1)

which is again a matrix-valued function. Here \( \cdot \) refers to the scalar product between the gradients. Note that the order of factors must be respected. In particular,

\[
\{ A, A \} \neq 0 \tag{2.2}
\]

in general.

The Dirac matrix reads

\[
H_D = c \gamma_0 (\gamma \cdot (p - e A(q)) + mc) + e\phi(q),
\]

(2.3)

compare with (1.5), which is regarded as a \( 4 \times 4 \) matrix-valued function on the classical phase space, \( (q, p) \in \mathbb{R}^3 \times \mathbb{R}^3 \). \( H_D \) is easy to diagonalize. The eigenvalues are

\[
h_\pm = \pm cp_0 + e\phi
\]

(2.4)

and the corresponding two-dimensional eigenprojections

\[
P_\pm = \frac{1}{2} (1 \pm \frac{1}{p_0} \gamma_0 (\gamma \cdot (p - e A) + mc)),
\]

(2.5)

where \( p_0 = \sqrt{m^2c^2 + (p - e A)^2} \). Thus

\[
H_D = h_+ P_+ + h_- P_-.
\]

(2.6)

\( P_+ \) is the electron and \( P_- \) the positron subspace. In addition, we need the polarization matrix along the orientation \( a, |a| = 1 \), which is given by

\[
a \cdot S = (\gamma \cdot g + \frac{1}{p_0} a \cdot (p - e A)) \gamma_5
\]

(2.7)

with

\[
g = a - \frac{1}{p_0(p_0 + mc)} (a \cdot (p - e A))(p - e A)
\]

(2.8)
\( \mathbf{a} \cdot \mathbf{S} \) has eigenvalues \( \pm 1 \), \((\mathbf{a} \cdot \mathbf{S})^2 = 1\), and commutes with \( H_D \),

\[ [H_D, \mathbf{S}] = 0. \tag{2.9} \]

We discuss only the motion of an electron, the positron subspace being simply its mirror image. In the semiclassical limit the state of the electron in general is described through the matrix-valued Wigner function \( W_t(\mathbf{q}, \mathbf{p}) \) on classical phase space. According to the Appendix, \( W_t \) is governed by

\[
\frac{\partial}{\partial t} W_t(\mathbf{q}, \mathbf{p}) = -\{h_+(\mathbf{q}, \mathbf{p}), P_+W_t(\mathbf{q}, \mathbf{p})\} - i[H_s^{(+)}(\mathbf{q}, \mathbf{p}), W_t(\mathbf{q}, \mathbf{p})] \tag{2.10}
\]

with

\[
H_s^{(+)} = -i([P_+, \{h_+, P_+\}]
+ \frac{1}{2} P_+(h_+\{P_+, P_+\} - h_-\{P_-, P_-\})P_+) = H_{be} + H_{nn}. \tag{2.11}
\]

If the initial wave packet is well localized at \( \mathbf{q}^0, \mathbf{p}^0 \), then one has to solve (2.10) with the initial condition \( W_0(\mathbf{q}, \mathbf{p}) = \delta(\mathbf{q} - \mathbf{q}^0)\delta(\mathbf{p} - \mathbf{p}^0)|\varphi_0\rangle\langle \varphi_0| \) and \( \varphi_0 \in P_+\mathbb{C}^4 \). In the course of time, this structure is preserved,

\[ W_t(\mathbf{q}, \mathbf{p}) = \delta(\mathbf{q} - \mathbf{q}_t)\delta(\mathbf{p} - \mathbf{p}_t)|\varphi_t\rangle\langle \varphi_t|, \tag{2.12} \]

where \( \mathbf{q}_t, \mathbf{p}_t \) is the solution to

\[
\dot{\mathbf{q}} = \{h_+, \mathbf{q}\} = \nabla_{\mathbf{p}} h_+ , \quad \dot{\mathbf{p}} = \{h_+, \mathbf{p}\} = -\nabla_{\mathbf{q}} h_+ \tag{2.13}
\]

with initial conditions \( \mathbf{q}^0, \mathbf{p}^0 \). (2.13) is equivalent to (1.1) with \( p_0 = mc\gamma \) and the velocity defined through

\[ v = \frac{1}{p_0} (cp - eA). \tag{2.14} \]

The spinor \( \varphi_t \) evolves with a time-dependent hamiltonian according to

\[
i\frac{d}{dt}\varphi_t = H_s^{(+)}(\mathbf{q}_t, \mathbf{p}_t)\varphi_t. \tag{2.15} \]

By (2.11) this implies \( \varphi_t \in P_+(\mathbf{q}_t, \mathbf{p}_t)\mathbb{C}^4 \) for all \( t \).

In (2.10) the Poisson bracket \( \{h_+, \cdot\} \) transports the spinor along the classical orbit maintaining its orientation. Therefore, in general, the spinor would
move out of the electron subspace which is precisely compensated for by $H_{be}$. For that reason, the analogue of $H_{be}$ has been baptized Berry term in [16], while the remainder, $H_{nn}$, has “no name” and is referred to also as Poissonian curvature in [17].

It remains to compute $H_s^{(+)}$. $H_{be}$ is actually not needed in explicit form, but we list it for completeness,

$$H_{be} = \frac{e}{2p_0}(\gamma_5\gamma_0\gamma \cdot (c^{-1}v \times F) - i\frac{1}{p_0} \gamma \cdot F) \quad (2.16)$$

with $F = E + c^{-1}v \times B$. For the no name term, we note that

$$\{P_+, P_+\} = \{1 - P_-, 1 - P_-\} = -i\frac{e}{2cp_0^2}\gamma_5\gamma_0\gamma \cdot B \quad (2.17)$$

and hence

$$H_{nn} = \frac{1}{2}iP_+((-cp_0 - e\phi)\{P_+, P_+\} + (-cp_0 + e\phi)\{P_-, P_-\})P_+$$

$$= -\frac{e}{2p_0}P_+\gamma_5\gamma_0\gamma \cdot BP_+. \quad (2.18)$$

### 3 Spin precession

In the semiclassical limit the average electron polarization along $\mathbf{a}$ is given by

$$\mathbf{a} \cdot \mathbf{s}_t = \int d^3q d^3p \text{tr}[W_t(q, p)P_+(q, p)a \cdot S(q, p)P_+(q, p)] \quad (3.1)$$

with $W_t$ the matrix-valued Wigner function. Note that $[P_+, S] = 0$. According to the Appendix, we have

$$\frac{d}{dt} \mathbf{a} \cdot \mathbf{s}_t = \frac{d}{dt} \int d^3q d^3p \text{tr}[W_tP_+a \cdot S]$$

$$= \frac{1}{2} \int d^3q d^3p \text{tr}[P_+\{\{W_t, H_D\} - \{H_D, W_t\}\}P_+a \cdot S]$$

$$= \int d^3q d^3p \text{tr}[P_+\{\{W_t, h_+\} + i[W_t, H_{nn}]\}P_+a \cdot S]$$

$$= \int d^3q d^3p \text{tr}[W_t\{\{h_+, P_+a \cdot SP_+\} + i[H_{nn}, P_+a \cdot SP_+]\}].$$
Using $P_+\{h_+, P_+\}P_+ = 0$ we have

$$\frac{d}{dt}a \cdot s_t = \int d^3q d^3p \text{tr}[WP_+\{\{h_+, a \cdot S\} - i \frac{e}{2p_0}\gamma_5 \gamma \cdot B, a \cdot S\}]P_+ \quad (3.3)$$

and only have to compute the term $P_+\ldots P_+$, which is accomplished using the identities

$$(p_0/mc)P_+v \cdot \gamma_5 P_+ = P_+v \cdot SP_+ = cP_+\gamma_5 P_+. \quad (3.4)$$

For the Berry term we have

$$P_+\{h_+, a \cdot S\}P_+ = P_+a \cdot \dot{SP}_+ = P_+((\gamma \cdot \dot{g} + c^{-1}a \cdot \dot{v})\gamma_5)P_+$$

$$= c^{-2}\left(-\frac{p_0^2}{(mc + p_0)^2}c^{-2}(a \cdot v)(v \cdot \dot{v}) + \frac{p_0}{mc + p_0}(a \cdot \dot{v})\right)P_+v \cdot SP_+$$

$$- c^{-2}\frac{p_0}{mc + p_0}(a \cdot v)P_+\gamma \cdot \dot{v}P_+, \quad (3.5)$$

where we used (3.4). The last term transforms as

$$P_+\gamma \cdot \dot{v}P_+ = P_+\dot{v} \cdot SP_+ - \frac{p_0}{mc + p_0}c^{-2}(v \cdot \dot{v})P_+v \cdot SP_+ \quad (3.6)$$

and therefore

$$P_+a \cdot \dot{SP}_+ = \frac{p_0}{mc + p_0}c^{-2}(-(a \cdot v)P_+\dot{v} \cdot SP_+ + (a \cdot \dot{v})P_+v \cdot SP_+)$$

$$= \frac{p_0}{mc + p_0}c^{-2}P_+(a \times (v \times \dot{v})) \cdot SP_+ \quad (3.7)$$

$$= - \frac{p_0}{mc + p_0}c^{-2}P_+a \cdot (S \times (v \times \dot{v}))P_+$$

$$= - \frac{1}{mc}P_+a \cdot (S \times (c^{-1}v \times E + c^{-2}v \times (v \times B)))P_+. \quad (3.8)$$

For the no name term we obtain, using again (3.4),

$$-i\frac{e}{2p_0}P_+\gamma_5 \gamma_0 \gamma \cdot B, a \cdot S]P_+$$

$$= \frac{e}{p_0}P_+\gamma \cdot (B \times g)\gamma_5 P_+$$

$$= \frac{e}{p_0}(P_+(B \times g) \cdot SP_+ - \frac{p_0}{mc + p_0}c^{-2}v \cdot (B \times g)P_+v \cdot SP_+)$$

$$= \frac{e}{mc}P_+a \cdot \left[\frac{1}{\gamma}(S \times B) + \frac{1}{1 + \gamma}c^{-2}(S \times (v \times (v \times B)))\right]P_+. \quad (3.8)$$
We add (3.7) and (3.8) and insert in Eq. (3.3). Using (2.12) we finally obtain

\[
\frac{d}{dt} a \cdot s = \frac{e}{mc} a \cdot \left[ s \times \left( \frac{1}{\gamma} B - \frac{1}{1 + \gamma} c^{-1} v \times E \right) \right],
\]

which is the BMT equation for \( g = 2 \).

### A Appendix

We want to understand the solution of the Dirac equation with slowly varying potentials as in (1.3). Changing to the rescaled space-time variables, denoted again by the same symbols, the Dirac equation becomes

\[
i \hbar \frac{\partial}{\partial t} \psi = \left[ \gamma_0 \gamma \cdot (-i \hbar \nabla_x - e c A(x)) + mc \gamma_0 + e \phi(x) \right] \psi.
\]

Since only the combination \( \varepsilon \hbar \) appears, we may set \( \hbar = 1 \) at the expense that formulas do not look right dimensionally.

At this point, it is more convenient to abstract from the particular form of the Dirac equation and to consider a general unitary evolution governed by

\[
i \varepsilon \frac{\partial}{\partial t} \psi = H^\varepsilon \psi
\]

with \( \psi \) an \( n \)-component spinor, \( \psi \in \bigoplus_{j=1}^n L^2(\mathbb{R}^d) \). To define \( H^\varepsilon \) we start from a symmetric \( n \times n \) matrix-valued function \( H(q, p) \) on the classical phase space \( \Gamma = \mathbb{R}^d \times \mathbb{R}^d \). Then, in spirit, \( H^\varepsilon = H(x, -i \varepsilon \nabla_x) \). While for the Dirac operator this substitution is unambiguous, in general it is not, and we define \( H^\varepsilon \) through the Weyl quantization

\[
H^\varepsilon \psi(x) = (2\pi)^{-d} \int d^d y \int d^d \xi H \left( \frac{x + y}{2}, \varepsilon \xi \right) e^{i(x-y) \cdot \xi} \psi(y).
\]

Next we have to list the properties of the symmetric matrix-valued Hamiltonian function \( H \). We assume that it has the spectral decomposition

\[
H(q, p) = \sum_{j=1}^m h_j(q, p) P_j(q, p).
\]
$h_j$ are the real band energies. We order them as $h_j \leq h_{j+1}$ and we assume that bands do not cross which means that

$$h_j(q, p) < h_{j+1}(q, p), \quad (A.5)$$

$j = 1, \ldots, m - 1$. If bands cross, then the situation becomes considerably more complicated. We refer to [8] for the case of two nondegenerate bands touching each other at a single point. $P_j(q, p)$ are orthogonal, symmetric projections. Since $H$ is assumed to be smooth in $q, p$ and by (A.5), their degeneracy must be independent of $q, p$.

$$\text{tr} P_j(q, p) = d_j \quad (A.6)$$

with degeneracy $d_j$. We have $\sum_{j=1}^m d_j = n$. $d_j = 1$ is a scalar band. We study (A.2) in the limit $\varepsilon \to 0$ through the evolution of Wigner density matrices, which for the pure state $\psi_t$ is defined by

$$W_t^\varepsilon(q, p) = (2\pi)^{-d} \int d^d\xi \psi_t(x - \frac{\varepsilon}{2}\xi)\psi_t^*(x + \frac{\varepsilon}{2}\xi)e^{ip\cdot\xi} \quad (A.7)$$

and for a general density matrix by the corresponding incoherent superposition. $W_t^\varepsilon$ is a $n \times n$ matrix-valued function on phase space. Using the evolution law (A.2) for $\psi_t$ and expanding the right hand side of (A.7) in $\varepsilon$, one obtains

$$\frac{\partial}{\partial t} W_t^\varepsilon = -\varepsilon^{-1}i[H, W_t^\varepsilon] + \frac{1}{2}(\{W_t^\varepsilon, H\} - \{H, W_t^\varepsilon\}) + R_t^\varepsilon \quad (A.8)$$

with a remainder $R_t^\varepsilon$ which vanishes as $\varepsilon \to 0$ [14]. Note that $H$ is the given $n \times n$ matrix-valued hamiltonian function. In (A.8) the first piece of the generator is rapidly oscillating while the second piece is of order one. Such a situation has been studied abstractly by Davies [15]. He proves that, for any $t \neq 0$, $W_t^\varepsilon \to W_t$ in the limit $\varepsilon \to 0$ and that $W_t$ commutes with $H$, $[H, W_t] = 0$. The order one piece of the generator is projected onto the invariant subspace of the evolution generated by $-i[H, \cdot]$. This means that in the limit $\varepsilon \to 0$ (A.8) goes over to

$$\frac{\partial}{\partial t} W_t = \sum_{j=1}^m P_j \frac{1}{2}(\{W_t, H\} - \{H, W_t\})P_j \quad (A.9)$$
with initial conditions given through \( \lim_{\varepsilon \to 0} \sum_{j=1}^{m} P_j W_0^j P_j = W_0 \). Note that for all \( t \) we have \([P_j, W_t] = 0, j = 1, \ldots, m\). It is instructive to rewrite (A.9) in a somewhat different form. We insert the spectral decomposition of \( H \),

\[
\frac{1}{2} \sum_{j=1}^{m} P_j \{\{W, H\} - \{H, W\}\} P_j
\]

(A.10)

\[
= \frac{1}{2} \sum_{j=1}^{m} \sum_{\ell=1}^{m} P_j \{\{W, h_\ell P_\ell\} - \{h_\ell P_\ell, W\}\} P_j
\]

\[
= - \sum_{j=1}^{m} P_j \{h_j, W\} P_j + \frac{1}{2} \sum_{j=1}^{m} \sum_{\ell=1}^{m} h_\ell P_j \{\{W, P_\ell\} - \{P_\ell, W\}\} P_j.
\]

To continue we need two identities. The first one reads

\[
P_j \{h_j, P_j\} P_j = 0 \quad (A.11)
\]

which follows from \( \{h_j, P_j\} = \{h_j, P_j^2\} = \{h_j, P_j\} P_j + P_j \{h_j, P_j\} \). Therefore, using that \( P_j W = W P_j \),

\[
P_j \{h_j, W\} P_j = \{h_j, P_j W P_j\} - P_j W \{h_j, P_j\} - \{h_j, P_j\} W P_j
\]

\[
= \{h_j, P_j W P_j\} - [P_j W P_j, [P_j, \{h_j, P_j\}]] \quad (A.12)
\]

The second identity is derived from

\[
A\{B, C\} - \{A, B\} C = \{AB, C\} - \{A, BC\}, \quad (A.13)
\]

compare with [14]. Then

\[
P_j \{P_j, W\} - \{P_j, P_j\} W = \{P_j, W\} - \{P_j, P_j W\},
\]

\[
W \{P_j, P_j\} - \{W, P_j\} P_j = \{W P_j, P_j\} - \{W, P_j\} \quad (A.14)
\]

and

\[
P_j \{\{W, P_j\} - \{P_j, W\}\} P_j = WP_j \{P_j, P_j\} P_j - P_j \{P_j, P_j\} P_j W - P_j \{P_j, \{P_j, W\}\} P_j - \{P_j, WP_j\}\{P_j, \{h_j, P_j\}\} \]

\[
= [W, P_j \{P_j, P_j\} P_j] \quad (A.15)
\]

where we use \([W, P_j] = 0\) and again (A.13). In addition, again by (A.13),

\[
P_1 \{P_2, P_2\} = - \{P_1, P_2\}(1 - P_2), \quad \{P_2, P_2\} P_1 = -(1 - P_2)\{P_2, P_1\} \quad (A.16)
\]

10
Then, by (A.13) and $P_1 P_2 = 0$

$$P_1 \{P_2, W\} - \{P_1, P_2\} W = -\{P_1, P_2 W\},$$

$$W \{P_2, P_1\} - \{W, P_2\} P_1 = \{WP_2, P_1\}$$  \hspace{1cm} (A.17)

and

$$P_1 (\{W, P_2\} - \{P_2, W\}) P_1$$

$$= P_1 \{WP_2, P_1\} - \{P_1, P_2\} W + \{P_1, P_2 W\} - \{WP_2, P_1\} P_1$$

$$= WP_1 (1 - P_2) \{P_2, P_1\} P_1 - P_1 \{P_1, P_2\} (1 - P_2) P_1 W$$

$$= - [W, P_1 \{P_2, P_2\} P_1],$$  \hspace{1cm} (A.18)

where used that $P_1 W = P_1 W$, $P_1 (1 - P_2) = P_1$, (A.17), and

$$P_1 \{WP_2, P_1\} - \{P_1, WP_2\} P_1 = \{P_1, P_2 WP_1\} - \{P_1 WP_2, P_1\} = 0,$$  \hspace{1cm} (A.19)

which follows from (A.13), $P_2 W = WP_2$, and $P_1 P_2 = 0$. Let us define

$$H_s^{(j)} = - i [P_j, \{h_j, P_j\}] - \frac{i}{2} h_j P_j \{P_j, P_j\} P_j + \frac{i}{2} \sum_{\ell=1, \ell \neq j}^m h_{\ell} P_j \{P_{\ell}, P_{\ell}\} P_j.$$

(A.20)

Then we have transformed (A.10) into

$$\frac{\partial}{\partial t} W_t = \sum_{j=1}^m (-\{h_j, P_j W_t P_j\} - i [H_s^{(j)}, P_j W_t P_j]).$$  \hspace{1cm} (A.21)

If we assume that the initial $W_0$ is concentrated in band $j$, i.e. $P_j W_0 P_j = W_0$, and solve

$$\frac{\partial}{\partial t} W_t = - \{h_j, W_t\} - i [H_s^{(j)}, W_t],$$  \hspace{1cm} (A.22)

then $P_j W_t P_j = W_t$ at all times. If $W_0$ is a pure state for all $q, p$, $W_0(q, p) = |\psi_0(q, p)\rangle\langle \psi_0(q, p)|$ with $P_j \psi_0 = \psi_0$, then this structure is maintained in the course of time, $P_j \psi_t = \psi_t$, and $\psi_t$ solves

$$\frac{\partial}{\partial t} \psi_t = - \{h_j, \psi_t\} - i H_s^{(j)} \psi_t.$$  \hspace{1cm} (A.23)
In particular, if initially $W_{0}(q, p) = \delta(q - q^{0})\delta(p - p^{0})|\varphi_{0}\rangle\langle\varphi_{0}|$ with $\varphi_{0} \in P_{j}C^{n}$, then at any other time

$$W_{t}(q, p) = \delta(q - q_{-t})\delta(p - p_{-t})|\varphi_{t}\rangle\langle\varphi_{t}|,$$

(A.24)

where $(q_{t}, p_{t})$ is the solution to

$$\dot{q} = \{h_{j}, q\} = \nabla_{p}h_{j}, \quad \dot{p} = \{h_{j}, p\} = -\nabla_{q}h_{j}$$

(A.25)

with initial conditions $q^{0}, p^{0}$. Thus the wave packet propagates along the classical orbit with the band function $h_{j}$ as hamiltonian. $\varphi_{t}$ evolves according to the time-dependent spin hamiltonian $H_{s}^{(j)}(t) = H_{s}^{(j)}(q_{t}, p_{t})$ of the $j$-th band as

$$i\frac{d}{dt}\varphi_{t} = H_{s}^{(j)}(t)\varphi_{t}.$$  

(A.26)

Note that $P_{j}(q_{t}, p_{t})\varphi_{t} = \varphi_{t}$.

The final answer (A.24), (A.26) has one rather surprising feature. We focus on the $j$-th band. To compute the $q, p$ evolution we only have to know the band function $h_{j}$ of the band under consideration. The first term in the spinor hamiltonian just ensures that, as the wave packet propagates, the spinor does not move out of the $j$-th band subspace. Clearly, $[P_{j}, \{h_{j}, P_{j}\}]$ knows only about $h_{j}$ and $P_{j}$, as does the second term of $H_{s}^{(j)}$. However, the third term depends on all the other bands. Thus if we modify $H^{c}$ keeping the $j$-th band fixed, then the spinor motion, and only it, will be modified. Physically, one might have been tempted to argue that the other bands are separated from the band $j$ by a large energy and therefore modifying them should leave the dynamics in the $j$-th band unaffected. Our computation shows that the actual behavior is otherwise.

References

[1] J. Bailey and E. Picasso, *The anomalous magnetic moment of the muon and related topics*, Prog. Nucl. Phys. 12, 43-75 (1970)

[2] V. Bargmann, L. Michel, and V.L. Telegdi, *Precession of the polarisation of particles moving in a homogeneous electromagnetic field*, Phys. Rev. Lett. 2, 435-436 (1959)
[3] J. D. Jackson, Classical Electrodynamics, 3rd edition Wiley, New York, 1999

[4] F. J. Ynduráin, Relativistic Quantum Mechanics and Introduction to Field Theory. Texts and Monographs in Physics, Springer, Berlin, 1996

[5] D. Robert, Autour de l’Approximation Semi-Classique, Birkhäuser, Basel, 1987

[6] G. A. Hagedorn, A. Joye, *Semiclassical dynamics with exponentially small error estimates*, Comm. Math. Phys., to appear

[7] F. Hövermann, H. Spohn, and S. Teufel, *The semiclassical limit for the Schrödinger equation with a short scale periodic potential*, preprint, 1999

[8] G. A. Hagedorn, *Electron energy level crossings in the time-dependent Born-Oppenheimer approximation*, Theor. Chim. Acta **77**, 163-190 (1990)

[9] D. M. Fradkin and R. H. Good, *Electron polarization operators*, Rev. Mod. Phys. **33**, 343-352 (1961)

[10] W. Pauli, Helv. Phys. Acta **5**, 179 (1932)

[11] S. I. Rubinow and J. B. Keller, *Asymptotic solution of the Dirac equation*, Phys. Rev. **131**, 2789-2796 (1963)

[12] J. Bolte and S. Keppeler, *A semiclassical approach to the Dirac equation*, Ann. Phys. **274**, 125-162 (1999)

[13] V. P. Maslov and M. V. Fedoryuk, *Semiclassical Approximation in Quantum Mechanics*. Reidel, Dordrecht, 1981

[14] P. Gérard, P. Markowich, N. Mauser, and F. Poupaud, *Homogenization limits and Wigner transforms*, Comm. Pure Appl. Math. **50**, 323-379 (1997)

[15] K. Heinemann, D.P. Barber The semiclassical Foldy Wouthuysen transformation, preprint, Los Alamos archive [physics/9901044](http://arxiv.org/abs/physics/9901044)

[16] R. G. Littlejohn and W. G. Flynn, *Geometric phases in the asymptotic theory of coupled wave equations*, Phys. Rev. A **44**, 5239-5256 (1991)
[17] C. Emmrich and A. Weinstein, *Geometry of the transport equation in multicomponent WKB approximations*, Comm. Math. Phys. **176**, 701-711 (1996)

[18] E. B. Davies, *Markovian master equations II*, Math. Annalen **219**, 147-158 (1976)