Soliton decay in composite right- and left-handed transmission lines periodically loaded with Schottky varactors

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Abstract: Soliton decay in a composite right- and left-handed transmission line periodically loaded with Schottky varactors, termed a Schottky CRLH line, is characterized to obtain broadband envelope pulses. When the three waves in the line satisfy the wave-number and frequency resonance conditions, they are governed by the three-wave resonant interaction equation, which have solutions that exhibit soliton decay, i.e., the decay of the highest frequency envelope into those carried by the other two frequency solitons. Due to the left-handedness, solitons resulting from the decay can have much smaller width than the incident envelope. This paper discusses the potential of a Schottky CRLH line as a generator of broadband envelope pulses.

Keywords: solitons, left-handed waves, resonant interactions, pulse shortening

Classification: Microwave and millimeter wave devices, circuits, and systems

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1 Introduction

In this paper, we investigate soliton decay in a composite right- and left-handed transmission line periodically loaded with Schottky varactors termed a Schottky CRLH line. A CRLH line is defined as a lumped transmission line containing a series inductor, series capacitor, shunt inductor, and shunt capacitor in each section exhibits left-handed property [1]. There has been recent research into the properties of nonlinear Schrödinger (NS) solitons developed in CRLH lines with appropriately implemented varactors [2, 3, 4, 5]. In this study, each shunt capacitor is replaced by a Schottky varactor. Through the nonlinearity introduced by the varactors, the dispersion of CRLH lines can be compensated for, resulting in an envelope soliton. Moreover, when the frequencies and wave numbers of three nonlinear waves on a Schottky CRLH line satisfy the respective resonance conditions, these waves exhibit resonant phenomena governed by the three-wave resonant interaction (3WRI) equation [6, 7, 8].

Recently, we consider the head-on collision of two NS solitons in a Schottky CRLH line exhibiting a three-wave resonant process to generate pulsed waves whose frequencies and wave numbers are matched to the sum of those of the interacting waves [9]. Furthermore, to guarantee compact interaction length for generating short pulses, such resonant interaction must be significant for oppositely traveling waves. Because the wave vector of the left-moving wave has the opposite sign as that of the right-moving wave, the sum of the wave number becomes smaller than the wave number of incidents. Conversely, the sum frequency of incidents increases. Thus, the head-on collision cannot lead to the generation of sum-frequency waves in an ordinary right-handed line. In contrast, the CRLH line exhibits left-handedness at frequencies lower than the lower resonant frequency of the shunt or series LC pairs, where the wave number decreases as the frequency increases; therefore, the Schottky CRLH line succeeds in sum-frequency generation.

Generically, the group velocity of each wave is not coincident with the other two. It is well known that the incident wave having the middle group velocity contains solitons, it is expected to decay into the fast and slow solitons [7]. In a Schottky CRLH line, the slow soliton(s) occupies the LH branch, such that it is expected to travel to the opposite direction to the incident envelope and fast soliton(s). As discussed later, this property leads to the shortening of the fast
soliton(s). The initial value problem of the 3WRI equation is solved analytically through the inverse scattering transform, where it is proved that the widths of the fast and slow solitons are inversely proportional to the eigenvalue of the Zakharov-Shabat equation relating with the 3WRI equation. By solving the eigenvalue problem, it is found that the fast soliton can become shorter for longer incident wave. Through these observations, we can utilize the soliton decay in the Schottky CRLH line for generating broadband envelope pulses.

2 Soliton decay in a Schottky CRLH line

Fig. 1(a) shows two adjacent cells of a Schottky CRLH line. \( L_R, C_L, \) and \( L_L \) represent the series inductor, series capacitor, and shunt inductor, respectively. Moreover, \( C_R \) represents the shunt Schottky varactor. The capacitance–voltage relationship of a Schottky varactor is given by

\[
C(V) = C_0 \left(1 + \frac{V_0}{V_f} \right)^m \left(1 - \frac{V}{V_f} \right)^{-m},
\]

where \( C_0, V_f \) and \( m \) are the optimizing parameters. Moreover, \( V_0 \) represents the biasing voltage. Note that \( V_0 > 0 \) for reverse bias. In what follows, we define \( \omega_{se} \) and \( \omega_{sh} \) as \( 1/\sqrt{C_L L_R} \) and \( 1/\sqrt{C_0 L_L} \), respectively. The line then exhibits a left-handed (LH) property at frequencies lower than \( \omega_l \equiv \min(\omega_{se}, \omega_{sh}) \) and an ordinary right-handed (RH) property at frequencies higher than \( \omega_u \equiv \max(\omega_{se}, \omega_{sh}) \). The LH branch is continuously connected with the RH one at \( \omega = \omega_u = \omega_l \) in the balanced line. In contrast, a stop band, where all supporting modes become evanescent, appears between \( \omega_l \) and \( \omega_u \) in the unbalanced line. The dispersion relationship \( \omega = \omega_{RH,LH} \) of the modes propagating on a linear CRLH line (\( \omega_{RH} \) is for the RH and \( \omega_{LH} \) for LH) is expressed as follows:

\[
\omega_{RH,LH}(k) = \sqrt{\omega_s^2(k) \pm \left[ \omega_s^2(k) - \frac{1}{C_L C_0 L_L L_R} \right]},
\]

where \( \omega_s(k) \) is defined as

\[
\omega_s(k) = \sqrt{\frac{k^6}{720 C_0 L_R} - \frac{k^4}{24 C_0 L_R} + \frac{k^2}{2 C_0 L_R} + 1}.
\]

In Eq. (2), \( +(-) \) in the right-hand side is for \( \omega_{RH}(\omega_{LH}) \) and the terms up to the sixth order of the wave number \( k \) are taken into account. Note that the discrepancy between the exact and approximated dispersions is negligible at almost whole range of \( k \) except very short wavelengths. The group velocity \( V_g(k) \) is defined by the derivative of \( \omega_{RH,LH} \) with respect to \( k \).

In this study, we consider the three-wave resonant interactions in a Schottky CRLH line. Let \( k_j \) and \( \omega_j \) (\( j = 1, 2, 3 \)) be the wave number and angular frequency of the three waves then the three-wave resonance becomes eminent when the conditions \( k_2 = k_1 + k_3 \) and \( \omega_2 = \omega_1 + \omega_3 \) are satisfied. By applying the derivative expansion method [10] to the transmission line equations, we obtain the evolution equations of slowly varying envelope functions \( A_j(x,t) \) of voltage as [9]

\[
\frac{\partial A_j}{\partial t} + V_g(k_j) \frac{\partial A_j}{\partial x} = G_j A_{j+1} A_{j+2}^*,
\]
where \( j = 1, 2, 3, \text{mod } 3 \) and the coupling coefficients are given by \( G_{1,3} = iD_{1,3} \) and \( G_2 = -iD_2 \) with

\[
D_j = 180m\omega_jC_0L(-1 + C_LL_R\omega_j^2)/(V_0 + V_j)(-C_LL_Lk_j^2(360 - 30k_j^2 + k_j^4)
- 360C_LL_R + 360C_0L(-1 + 2C_LL_R\omega_j^2)).
\]

(5)

In balanced lines, \( |D_j| \) is shown to increase monotonically as frequency increases. By introducing \( Q_j = i\sqrt{|G_{j+1}|G_{j+2}}A_j \), Eq. (4) is transformed into the standard 3WRI equation, i.e.,

\[
\frac{\partial Q_j}{\partial t} + V_g(k_j)\frac{\partial Q_j}{\partial x} = \gamma_1 Q_{j+1}^*Q_{j+2}^*.
\]

(6)

where \( \gamma_{1,3} = 1 \) and \( \gamma_2 = -1 \). In what follows, an envelope having a carrier frequency of \( \omega_j \) is called \( \omega_j \)-envelope for brevity. When a \( \omega_2 \)-envelope is uniquely applied to the line and the group velocities satisfy \( V_g(k_1) < V_g(k_2) < V_g(k_3) \), its evolution is predicted by solving the eigenvalue problem of the following Zakharov-Shabat (ZS) equation in the framework of the inverse scattering transform [7]:

\[
\frac{\partial u_1}{\partial x} + i\lambda u_1 = qu_2,
\]

(7)

\[
\frac{\partial u_2}{\partial x} - i\lambda u_2 = -qu_1,
\]

(8)

where \( \lambda \) and \( (u_1, u_2)^T \) are the eigenvalue and corresponding eigenvector, respectively. In addition, \( q = q(x) \) is defined by

\[
q(x) = -\frac{Q_j^0(x)}{\sqrt{(V_g(k_2) - V_g(k_1))(V_g(k_3) - V_g(k_1))}},
\]

(9)

for the spatial waveform \( Q_j^0(x) \) of the incident \( \omega_2 \)-envelope.
It is well known that the solitons contained in the $\omega_1$- or $\omega_3$-envelope are stable, i.e., the original envelopes never lose the solitons, while the $\omega_2$-envelope always loses its solitons, which are converted to both the slow and fast envelopes. The latter phenomena is called soliton decay [7]. When $Q_2^{(0)}(x)$ evolves into $N$ solitons, the ZS equation must have $N$ pure imaginary eigenvalues in the upper half plane, whose norms are inversely proportional to the spatial width of the corresponding soliton. Let $\tilde{\lambda}_m^{(j)}$ $(m = 1, \cdots, N)$ be such eigenvalues of Eqs. (7) and (8). Then, it is shown that

$$\tilde{\lambda}_m^{(1)} = \frac{V_g(k_3) - V_g(k_2)}{V_g(k_3) - V_g(k_1)} \tilde{\lambda}_m^{(2)},$$

(10)

$$\tilde{\lambda}_m^{(3)} = \left(1 - \frac{V_g(k_3) - V_g(k_2)}{V_g(k_3) - V_g(k_1)}\right) \tilde{\lambda}_m^{(2)},$$

(11)

where $\tilde{\lambda}_m^{(j)}$ $(j = 1, 3)$ defines the eigenvalue corresponding to the soliton in the $\omega_j$-envelope resulting from the decay of the soliton in the $\omega_2$-envelope corresponding to $\tilde{\lambda}_m^{(2)}$ [7]. For illustration, the incident envelope occupies the region in the neighborhood of $P_2$ in Fig. 1(b). Then, due to the resonant conditions, $\omega_{1,3}$-envelope is shown to be around $P_{1,3}$ in Fig. 1(b) uniquely. Notice that group velocities satisfy $V_g(k_1) < V_g(k_2) < V_g(k_3)$ and $P_1$ is on the LH branch. Due to the negative $V_g(k_1)$ in Eqs. (10) and (11), $\tilde{\lambda}_m^{(1)}$ takes a small value, while $\tilde{\lambda}_m^{(3)}$ becomes rather large. As a result, the solitons in $\omega_1$-envelope start to travel backward with a relatively wide width. Conversely, the $\omega_3$-solitons become short.

To examine the relationship between the incident and finally obtained waveforms, we calculate numerically the eigenvalues of Eqs. (7) and (8) for $Q_2^{(0)}(x) = \Psi_0 \operatorname{sech}\left(x/T_{in}V_g(k_2)\right)$ based on the method developed by Kravchenko and Velasco-Garcia [11]. Fig. 2 shows the dependence of eigenvalue norms on the temporal pulse width $T_{in}$ of the incident sech envelope. Only pure imaginary eigenvalues in the upper half plane are shown. The amplitude $\Psi_0$ is set to 0.25 V. It is observed that the number of such eigenvalues increases in a stepwise way and the norms increases monotonically as $T_{in}$ increases; therefore, it is expected that the emitted solitons becomes shorter for wider incident envelope.

![Fig. 2. Dependence of eigenvalues of Eqs. (7) and (8) on temporal pulse width of incident hyperbolic sech envelope.](image-url)
3 Numerical validation

For validation of the above mentioned analysis, we solve numerically the transmission equations of a Schottky CRLH line shown in Fig. 1(a) given by

\[ L_R \frac{d^2 I_n}{dt^2} = - \frac{I_n}{C_L} - \frac{d}{dt} (V_n - V_{n-1}), \]
\[ C_R \frac{d^2 V_n}{dt^2} = - \frac{V_n}{L_L} + \frac{d}{dt} \left( I_n - I_{n+1} \right) - \frac{dC_R}{dV_n} \left( \frac{dV_n}{dt} \right)^2, \]

where \( I_n \) and \( V_n \) are the current and voltage at the \( n \)th cell, respectively. The line parameters \( C_L, L_L, C_0, \) and \( L_R \) are set to 1.0 pF, 2.5 nH, \( 1.69 C_L \), and \( 1.69 L_L \), respectively. The line is then balanced. The parameters values of the Schottky varactor \( m, V_J, \) and \( V_0 \) are set to 2.0, 2.0 V, and 1.0 V, respectively. In addition, \( \omega_2 \) is set to 4.54 GHz. Fig. 3(a) shows waveforms on the line having a total cell number \( N_c \) of \( 10^4 \). A 0.25 V hyperbolic secant envelope with 3.5-ns duration is applied at the left end. Five spatial waveforms recorded at 250-ns increments are plotted. The incident \( \omega_2 \)-envelope decays into a unique pair of the fast and slow solitons, which are labeled at the fourth waveform as \( A \) and \( A' \), respectively. It is well known that the 3WRI system has the following two Manley-Rowe invariants \( I_{1,2} \):

\[ I_1 = \frac{\alpha_2}{|G_2|} + \frac{\alpha_1}{|G_1|}, \]
\[ I_2 = \frac{\alpha_2}{|G_2|} + \frac{\alpha_3}{|G_3|}, \]

where the action \( \alpha_j \) is defined by

\[ \alpha_j = \int_{-\infty}^{\infty} |A_j|^2 dx, \]

for \( j = 1, 2, 3 \) [7]. \( I_{1,2} \) must be kept constant throughout an interaction. Because the line is discrete, we evaluate \( \alpha_j \) by

\[ \beta_j = \sum_{n=1}^{N_c} |A_{jn}|^2, \]

Fig. 3. Spatio-temporal evolution of hyperbolic secant envelope having \( T_{\text{in}} = 3.5 \) ns. (a) The spatial waveforms recorded at five different times and (b) the temporal evolutions of actions and Manley-Rowe invariants. The left and right axes in Fig. 3(b) measure the actions and Manley-Rowe invariants, respectively.
where $A_{jn}$ is the amplitude of the $\omega_j$-envelope at the $n$th cell. To obtain $\beta_j$, each calculated spatial waveform is Fourier-transformed into $k$ space. Because the wave numbers contained in three envelopes are well separated, $\beta_j$ can be easily reconstructed by inverse Fourier transform after appropriately filtering $k$-space amplitude. Fig. 3(b) shows the temporal evolutions of $\beta_{1,2,3}$ and $I_{1,2}$. The left vertical axis measures the actions. It is seen that the action is exchanged between envelopes, i.e., $\beta_2$ is reduced in a stepwise way by the emission of soliton pair, and $\beta_{1,3}$ experience stepwise increase correspondingly. The right vertical axis measures $I_{1,2}$, whose values are observed to be unaltered by the soliton emission. These stepwise behaviors support the fact that the exchange of action is only carried out through emission of solitons [7].

Fig. 4 shows the results of another calculation, where the incident $\omega_2$-envelope has a 0.25 V hyperbolic secant envelope with 10.5-ns duration. Only difference between Figs. 3 and 4 is the width of the incident envelopes, such that it becomes three times wider for Fig. 4 than one for Fig. 3. Five spatial waveforms recorded at 50-ns intervals are plotted in Fig. 4(a). The incident $\omega_2$-envelope decays into three pairs of the fast and slow solitons, which are labeled as $(A, A')$, $(B, B')$, and $(C, C')$. As expected, the widths of the emitted solitons become narrower in Fig. 4(a) than those in Fig. 3(a). In Fig. 4(b), we can see that the action exchange is significantly enhanced, such that the action is almost transferred to the fast and slow solitons through emission of three pairs. Moreover, the decaying process is completed in much shorter duration than that in Fig. 3. The short decaying duration contributes to the reduction in cell size required to obtain broadband envelopes.

In the above calculations, $N_c$ is set large to illustrate the process of soliton decay. In fact, it must be designed as small as possible for avoiding wave attenuation caused by parasitic resistance. Although we have to develop some loss compensation scheme [12, 13, 14], we restrict our discussion in how to reduce $N_c$ at present. As a broadband pulse generator, it suffices for a Schottky CRLH line to succeed in the emission of the first pair of solitons. In addition, the reduction in $N_c$ requires to maximize coupling coefficients through optimized reactances and to

![Fig. 4. Spatio-temporal evolution of hyperbolic secant envelope having $T_{in} = 10.5\text{ ns}$. (a) The spatial waveforms recorded at five different times and (b) the temporal evolutions of actions and Manley-Rowe invariants. The left and right axes in Fig. 4(b) measure the actions and Manley-Rowe invariants, respectively.](image-url)
tune the amplitude/width of incident signal. Fig. 5 shows the calculated result that demonstrates broadband-pulse generation. $N_c$ is reduced to 130. The input is a 1.0 V hyperbolic secant envelope having 4.5-ns duration. $V_0$ is set to 2.0 V. Other parameter values are the same as ones used to obtain Figs. 4 and 5. Although a bandpass filter is practically required at the output to extract frequencies around $\omega_3$, the ends are terminated with matched impedance $(=\sqrt{L_L/C_L})$ for the present demonstration. Fig. 5(b) shows the $\omega_3$-soliton monitored at the last cell. The raw waveform is Fourier-transformed, filtered, and inversely transformed. A unique envelope pulse is obtained that is generated by the first emission of soliton pair. The waveform monitored at the first cell is shown in Fig. 5(a) for comparison. The shortening of the pulse width is clearly observed.

4 Discussion

In the above-mentioned scheme, the carrier frequency of the incident envelope must be higher than that of the broadband output. However, the converse is preferable to relax the requirement in input preparation. Four-wave resonance of colliding pulses can realize it. A Schottky CRLH line allows two frequencies corresponding to any wave number. One is on the LH branch and the other is on the RH branch. In particular, there is a wave number $k_f$, such that $\omega_{RH}(k_f)$ is three times as high as $\omega_{LH}(k_f)$. Based on Eq. (2), $\omega_f \equiv \omega_{LH}(k_f)$ is explicitly given by

$$\omega_f = \sqrt{\frac{1}{3\sqrt{C_0L_LC_LL_R}}}.$$ (18)
We then consider the case where two LH envelopes interact through head-on collision and assume that there is a wave number \( k_1 \) that satisfies the resonance conditions:

\[
\begin{align*}
    k_1 &= 2k_2 + k_3, \quad (19) \\
    \omega_{RH}(k_1) &= 2\omega_{LH}(k_2) + \omega_{LH}(k_3), \quad (20)
\end{align*}
\]

where \( k_2 \) and \( k_3 \) represent the wave number of the right- and left-moving ones. Then, an envelope whose carrier frequency is \( \omega_r \) (\( \equiv \omega_{RH}(k_1) \)) is expected to develop through four-wave resonance. Obviously, these conditions are satisfied when \( k_1 = k_2 = -k_3 = -|k_1| \). This solution satisfies the condition: \(-k_1 = k_2 + 2k_3, \) too. It then turns out to develop a pair of right- and left-moving \( \omega_r \)-envelopes. In summary, the head-on collision of two \( \omega_r \)-pulses results in the development of envelopes whose carrier frequency is \( 3\omega_r \), which can emit soliton pairs via decaying process when they have relatively large wave numbers.

Based on Eqs. (19) and (20), the derivative expansion method leads to the evolution equations of the envelopes:

\[
\begin{align*}
    i\left( \frac{\partial A_1}{\partial t} + V_g(k_1) \frac{\partial A_1}{\partial x} \right) + \frac{1}{2} \frac{\partial V_g(k_1)}{\partial k_1} \frac{\partial^2 A_1}{\partial x^2} + \mu_1|A_1|^2A_1 &= v_{21}|A_2|^2A_1 + v_{31}|A_3|^2A_1 + \sigma_1A_2A_3^*A_3, \\
    i\left( \frac{\partial A_2}{\partial t} + V_g(k_2) \frac{\partial A_2}{\partial x} \right) + \frac{1}{2} \frac{\partial V_g(k_2)}{\partial k_2} \frac{\partial^2 A_2}{\partial x^2} + \mu_2|A_2|^2A_2 &= v_{12}|A_1|^2A_2 + v_{32}|A_3|^2A_2 + \sigma_2A_1A_3^*A_3^*, \\
    i\left( \frac{\partial A_3}{\partial t} + V_g(k_3) \frac{\partial A_3}{\partial x} \right) + \frac{1}{2} \frac{\partial V_g(k_3)}{\partial k_3} \frac{\partial^2 A_3}{\partial x^2} + \mu_3|A_3|^2A_3 &= v_{13}|A_1|^2A_3 + v_{23}|A_2|^2A_3 + \sigma_3A_1A_2^2 ,
\end{align*}
\]

where the amplitude of an envelope having a wave number of \( k_j \) is represented by \( A_j \) (\( j = 1, 2, 3 \)). The explicit forms of coefficients \( \mu_j, v_{ij}, \) and \( \sigma_j \) (\( i, j = 1, 2, 3 \)) are too lengthy. Instead, the dependence of \( |\sigma_1| \) on \( C_L \) and \( L_L \) is shown in Fig. 6(a) as a measure of efficiency of the four-wave resonance. To obtain Fig. 6(a), \( C_0 \) is fixed to 1.0 pF, while \( L_R \) is varied, such that the line exhibits balanced dispersion, i.e., \( L_R \) is varied as \( L_R = C_LL_L/C_0 \) \( |\sigma_1| \) increases as either \( L_L \) or \( C_L \) decreases. In general, the efficiency is expected to be fairly well for whole calculated values of line reactances.

Fig. 6. Soliton decay of envelopes generated by four-wave resonance.
Fig. 6(b) demonstrates soliton decay supported by four-wave resonance. The parameter values are the same as ones used to obtain Figs. 3 and 4. According to Eq. (18), we set the input carrier frequency to 1.41 GHz. Both ends of the line are applied by a 0.3 V hyperbolic secant envelope pulse with 7-ns duration. Five spatial waveforms recorded at 125-ns intervals are plotted. The pulse labeled as $B$ is the right-moving $\omega_f$-pulse that survives the collision. The pulse $C$ is the remnant collision-induced pulse, while the pulses $A$ and $D$ are the slow and fast solitons emitted in the decaying process. The development of collision-induced pulses is successfully seen together with their subsequent decay.

5 Conclusion

In this paper, the generation of broadband envelope pulses in a Schottky CRLH line is analyzed on the basis of the 3WRI equation and relating ZS equation. The dispersive property of a Schottky CRLH line allows for soliton decay, by which the incident envelope emits several pairs of the fast and slow solitons. It is found that the left-handedness of the line guarantees that the slow soliton travels backward and the fast soliton becomes shorter for wider incident envelope. We think that the properly designed Schottky CRLH line has a potential to provide still more schemes for generation and management of short envelope pulses via nonlinear resonant interactions.

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