Abstract: This paper is devoted to the construction of the Hamiltonian for non-relativistic string in the Newton-Cartan background. We start with the Hamiltonian for relativistic string in general background. Then we perform limiting procedure on the metric that leads to Newton-Cartan background. We determine constraint structure for non-relativistic string and show that these constraints are the first class constraints. Then we determine corresponding Lagrangian and discuss its properties.
1. Introduction

Recently, non-Lorentzian geometry has gained interest in theoretical physics community from many reasons. Firstly, today it is well known that strong correlated systems in condensed matter can be successfully described with the help of non-relativistic holography \[1, 2, 3\], for review see for example \[4\]. This duality is based on the idea that the strongly coupled theory on the boundary can be described by string theory in the bulk. Further, when the curvature of the space-time is small we can use the classical gravity instead of the full string theory machinery. In case of non-relativistic holography the situation is even more interesting since we have basically two possibilities: Either we use Einstein metric with non-relativistic isometries \[3, 4, 5\] or we introduce non-relativistic gravities in the bulk \[6, 7\], like Newton-Cartan gravity \[10\] or Hořava gravity \[11\]. It is also very instructive to analyze extended objects in Newton-Cartan theory \[23, 24\]. In \[23\] the action for non-relativistic string in Newton-Cartan background was proposed that has many interesting properties. For example, in was argued in \[23\] that in order to define correctly an action for non-relativistic string in Newton-Cartan background two longitudinal directions have to be selected and hence we obtain more general form of the Newton-Cartan geometry. The canonical analysis of this string was performed recently in \[25\]. During this analysis we met an obstacle which was an impossibility to derive Hamiltonian constraint for the string with non-zero gauge field \(m^a_\mu\) that will be defined in the next section. For that reason we were forced to restrict to the case of zero gauge field \(m^a_\mu\) and then we were able to determine canonical structure of the non-relativistic string in Newton-Cartan background. In the same way we proceeded with the case of non-relativistic p-brane. We defined it using the limiting procedure introduced in \[16\]. We again found corresponding action for

\[^1\text{For some recent works, see }12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 12, 22.\]

\[^2\text{For the analysis of point particles in this background, see }26, 27.\]
non-relativistic p-brane in Newton-Cartan background and determined canonical structure for this theory on condition that the gauge field $m_\mu^a$ is zero.

The fact that in our previous work we considered the situation when the gauge field $m_\mu^a$ vanishes is rather unsatisfactory since this field is crucial for the invariance of the theory under Milne boost. It would be nice to develop full canonical formalism where this field is non-zero. We suggested in the conclusion of our previous paper [25] that one way how to proceed is to start with the Hamiltonian for the string in general background and then perform the limiting procedure when we generalize the approach introduced in [16] to the case of two longitudinal directions. Exactly this is the goal of our paper. We start with the Hamiltonian for relativistic string in general background, introduce relativistic vierbeins and NSNS two form that are functions of fields that define Newton-Cartan background. These fields also depend on the free parameter that goes to infinity when we define Newton-Cartan gravity [16]. As a result we will be able to find corresponding Hamiltonian for the string in Newton-Cartan background. However this is not certainly the end of the story since we have to perform consistency checks of this proposal. Explicitly, we have to show that constraints, that define this theory, are the first class constraints. It turns out that this is a non-trivial task due to the complicated form of the Hamiltonian. Secondly, we would like to find Lagrangian for this non-relativistic string and investigate how it is related to the Lagrangian density proposed in [23]. To do this we carefully examine an invariance of the Hamiltonian constraint under generalized Milne boost. We show that the Hamiltonian constraint can be rewritten with the help of variables that are manifestly invariant under Milne transformation so that Hamiltonian is invariant too. Then we can proceed to the analysis of corresponding Lagrangian. As a warm up we consider the case of the non-relativistic string in flat background. We show that there is a crucial difference between inverse Legendre transformation in case of the relativistic string and non-relativistic one. Explicitly, we show that in case of non-relativistic string the Lagrange multipliers corresponding to Hamiltonian and spatial diffeomorphism constraints are determined by projections of the equations of motion for $x^\mu$ to longitudinal directions instead of their equations of motion. Then we will be able to find Lagrangian that agrees with the Lagrangian found in [23]. Further we proceed to the most general case of the non-relativistic string in Newton-Cartan background where the analysis is much more complicated. Despite of this fact we find Lagrangian form of the non-relativistic string in Newton-Cartan background which is manifestly diffeomorphism invariant. The crucial result of our analysis is the presence of two additional terms in the action which depend on generalized Newton potential and that were absent in the Lagrangian for non-relativistic string derived in [23, 25]. We mean that possible explanation for the absence of these terms is following. The limiting procedure performed in the canonical form of the action is more sensitive to the terms in the inverse metric that are proportional to higher orders in $\omega^{-1}$. On the other hand we showed in [25] that performing limiting procedure [16] at the Lagrangian for relativistic string all terms of order $\omega^{-1}$ vanish and hence this procedure is not sensitive to higher order terms in the expansion.

This paper is organized as follows. In the next section (2) we introduce canonical form of the relativistic string action and perform limiting procedure that leads to the
Hamiltonian for non-relativistic string in Newton-Cartan background and determine Poisson algebra of constraints. In section (3) we find the Lagrangian for non-relativistic string in Newton-Cartan background. Finally in conclusion (4) we outline our results and suggest possible extension of this work.

2. Canonical Formulation of Non-relativistic String in Newton-Cartan Background

We start with the Nambu-Gotto form of the action for relativistic string in general background
\[
S = -\tilde{\tau}_F \int d\tau d\sigma \sqrt{-\det(\eta_{AB} E^A_\mu E^B_\nu \eta_{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\nu)} + \tilde{\tau}_F \int d\tau d\sigma \partial_\tau x^\mu \partial_\sigma x^\nu ,
\]
where \( E^A_\mu \) is \( d \)-dimensional vierbein so that the metric components have the form
\[
G_{\mu\nu} = E^A_\mu E^B_\nu \eta_{AB} , \eta_{AB} = \text{diag}(-1, \ldots, 1) .
\]
Note that the metric inverse \( G^{\mu\nu} \) is defined with the help of the inverse vierbein \( E^\mu_B \) that obeys the relation
\[
E^A_\mu E^\mu_B = \delta^A_B , \quad E^A_\mu E^\nu_A = \delta^\nu_\mu .
\]
Further, \( B_{\mu\nu} \) is NSNS two form field. Finally \( x^\mu \), \( \mu = 0, \ldots, d-1 \) are embedding coordinates of the string where the two dimensional world-sheet is parameterized by \( \sigma^\alpha \equiv (\tau,\sigma) \) and \( \tilde{\tau}_F \) is the string tension that could be eventually rescaled when we define non-relativistic string.

Our goal is to find Hamiltonian non-relativistic string in Newton-Cartan background with the help of the following procedure. As the first step we determine Hamiltonian from the action (2.1). Explicitly, from (2.1) we find following conjugate momenta
\[
p_\mu = -\tilde{\tau}_F E^A_\mu E^B_\nu \eta_{AB} \partial_\beta x^\nu g^{\beta\tau} \sqrt{-\det g_{\alpha\beta}} + \tilde{\tau}_F B_{\mu\nu} \partial_\sigma x^\nu ,
\]
where
\[
g_{\alpha\beta} \equiv G_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu , \quad g^{\alpha\beta} g^{\beta\gamma} = \delta^\alpha_\gamma .
\]
Using (2.4) we immediately find that the bare Hamiltonian \( H_B = \int d\sigma (p_\mu \partial_\tau x^\mu - \mathcal{L}) \) is zero while we have following two primary constraints
\[
\mathcal{H}_\tau \equiv (p_\mu - \tilde{\tau}_F B_{\mu\nu} \partial_\sigma x^\nu) E^A_\mu E^B_\nu \eta^{AB}(p_\nu - \tilde{\tau}_F B_{\nu\sigma} \partial_\sigma x^\sigma) + \tilde{\tau}_F^2 \partial_\sigma x^\mu E^A_\mu \eta_{AB} E^B_\nu \partial_\sigma x^\nu \approx 0 , \quad \mathcal{H}_\sigma \equiv p_\mu \partial_\sigma x^\mu \approx 0 .
\]
Now we are ready to find Hamiltonian for the string in Newton-Cartan background with the help of the non-relativistic limit of relativistic vierbein \( E^A_\mu \). However as we argued in our recent paper \(^{[23]}\) in order to find correct non-relativistic limit we have to introduce the generalization of Newton-Cartan gravity following \(^{[26]}\). Explicitly, we split
target-space indices $A$ into $A = (a', a)$ where now $a = 0, 1$ and $a' = 2, \ldots, d - 1$. Then we introduce $\tau^{a}_{\mu}$ so that we write

$$
\tau^{a}_{\mu} = \tau^{a}_{\mu} \tau^{b}_{\nu} \eta_{ab}, \quad a, b = 0, 1 .
$$

(2.7)

In the same way we introduce vierbein $e^{a'}_{\mu}, a' = 2, \ldots, d - 1$ and also introduce gauge field $m^{a}_{\mu}$. The $\tau^{a}_{\mu}$ can be interpreted as the gauge fields of the longitudinal translations while $e^{a'}_{\mu}$ as the gauge fields of the transverse translations. Then we can also introduce their inverses with respect to their longitudinal and transverse translations

$$
e^{a'}_{\mu} e^{b}_{\nu} = \delta^{a'}_{b}, \quad e^{a'}_{\mu} e^{a}_{\nu} = \delta^{a'}_{a} - \tau^{a'}_{\mu} \tau^{a}_{\nu}, \quad \tau^{a}_{\mu} \tau^{b}_{\nu} = \delta^{b}_{a}, \quad \tau^{a}_{\mu} e^{a'}_{\nu} = 0, \quad \tau^{a'}_{\mu} e^{a}_{\nu} = 0 .
$$

(2.8)

Now we are ready to introduce following parameterization of the vierbein $E_{\mu}^{A}$

$$
E_{\mu}^{a} = \omega \tau^{a}_{\mu} + \frac{1}{2\omega} m^{a}_{\mu}, \quad E_{\mu}^{a'} = e^{a'}_{\mu},
$$

(2.9)

where $\omega$ is a free parameter that we take to infinity when we define non-relativistic limit. Note that the inverse vierbein to (2.9) has the form (up to terms of order $\omega^{-3}$)

$$
E^{\mu}_{a} = \frac{1}{\omega} \tau^{\mu}_{a} - \frac{1}{2\omega} \tau^{\mu}_{a} m^{b}_{\rho} \tau^{b}_{\rho} a, \quad E^{\mu}_{a'} = e^{\mu}_{a'} - \frac{1}{2\omega} \tau^{\mu}_{a} m^{b}_{\rho} e^{\rho}_{a'} .
$$

(2.10)

Then with the help of (2.9) and (2.10) we obtain following form of the metric $G_{\mu\nu}$ and its inverse

$$
G_{\mu\nu} = E_{\mu}^{a} E_{\nu}^{\eta_{ab}} + E_{\mu}^{a'} E_{\nu}^{b'} \delta_{a'b'} = \\
= \omega^{2} \tau_{\mu\nu} + h_{\mu\nu} + \frac{1}{2\omega} \tau^{a}_{\mu} m^{b}_{\nu} \eta_{ab} + \frac{1}{2} m^{a}_{\mu} \tau^{b}_{\nu} \eta_{ab} + \frac{1}{4\omega^{2}} m^{a}_{\mu} m^{b}_{\nu} \eta_{ab} ,
$$

$$
G^{\mu\nu} = E^{\mu}_{a} E^{\nu}_{b} \eta^{ab} + E^{\mu}_{a'} E^{\nu}_{b'} \delta^{a'b'} = \\
= \frac{1}{\omega^{2}} \tau^{\mu\nu} + h^{\mu\nu} - \frac{1}{2\omega^{2}} (\tau^{a}_{\mu} m^{b}_{\nu} h^{\rho\mu} + \tau^{a}_{\nu} m^{b}_{\rho} h^{\rho\nu}) - \\
- \frac{1}{2\omega^{2}} (\tau^{a}_{\mu} m^{c}_{\rho} \tau^{\rho\nu} + \tau^{a}_{\nu} m^{d}_{\rho} \tau^{\rho\mu}) + \frac{1}{4\omega^{4}} \tau^{a}_{\mu} m^{a}_{\rho} h^{\rho\sigma} \tau^{\nu}_{b} m^{b}_{\sigma} + O(\omega^{-6}) ,
$$

(2.11)

where

$$
h^{\mu\nu} = \epsilon^{a'}_{\mu} \epsilon^{\nu}_{b'} \delta_{a'b'}, \quad h_{\mu\nu} = \epsilon^{a'}_{\mu} \epsilon^{\nu}_{b'} \delta_{a'b'}, \quad \tau^{\mu\nu} = \tau^{a}_{\mu} \tau^{b}_{\nu} \eta^{ab} .
$$

(2.12)

As the next step we have to introduce an appropriate parameterization of NSNS two form. We suggested in \cite{23} that it is natural to consider following form of NSNS two form

$$
B_{\mu\nu} = \left( \omega \tau^{a}_{\mu} - \frac{1}{2\omega} m^{a}_{\mu} \right) \left( \omega \tau^{b}_{\nu} - \frac{1}{2\omega} m^{b}_{\nu} \right) \epsilon_{ab} = \\
= \omega^{2} \tau^{a}_{\mu} \tau^{b}_{\nu} \epsilon_{ab} - \frac{1}{2} \left( m^{a}_{\mu} \tau^{b}_{\nu} + \tau^{a}_{\mu} m^{b}_{\nu} \right) \epsilon_{ab} + \frac{1}{4\omega^{2}} m^{a}_{\mu} m^{b}_{\nu} \epsilon_{ab} ,
$$

(2.13)
where \( \epsilon_{ab} = -\epsilon_{ba} \), \( \epsilon_{01} = 1 \) .

With the help of this definition we easily find

\[
\frac{1}{\omega^2} \tau_F^2 B_{\mu\sigma} \partial_\sigma x^\mu \tau^{\mu\nu} B_{\nu\rho} \partial_\rho x^\rho = -\omega^2 \tau_F^2 \tau_{\mu\nu} \partial_\sigma x^\mu \partial_\sigma x^\nu
\]

and we see that this divergent contribution to the Hamiltonian constraint

\[
\frac{1}{\omega^2} \tau_F^2 B_{\mu\sigma} \partial_\sigma x^\mu \tau^{\mu\nu} B_{\nu\omega} \partial_\omega x^\omega + \tau_F^2 \partial_\sigma x^\mu \tau_{\mu\nu} \partial_\sigma x^\nu
\]

vanishes. Then we obtain that the Hamiltonian constraint has the form in the limit \( \omega \to \infty \)

\[
H_\tau = p_\mu h^{\mu\nu} p_\nu - 2\tau_F p_\mu \tau^{\mu\alpha} \eta^{ab} \epsilon_{bc} \tau_\rho \partial_\sigma x^\rho + 2\tau_F p_\mu h^{\mu\rho} m_\rho^\beta \epsilon_{\beta\rho} \partial_\sigma x^\rho + 2\tau_F \partial_\sigma x^\mu \tau_\sigma \eta_{\alpha\beta} m_\beta^\alpha \partial_\sigma x^\alpha - \tau_F \partial_\sigma x^\mu \tau_\sigma \beta^a m_\alpha^\beta \partial_\sigma x^\alpha + \tau_F^2 \partial_\sigma x^\mu \partial_\mu \partial_\sigma x^\nu + \tau_F^2 \partial_\sigma x^\mu \partial_\mu \partial_\sigma x^\nu
\]

\[
\equiv p_\mu h^{\mu\nu} p_\nu + p_\mu V^\mu + \tau_F^2 \partial_\sigma x^\nu \tilde{H}_\mu \partial_\sigma x^\nu , \quad V_\mu = V_\mu ^\nu \partial_\sigma x^\nu ,
\]

(2.17)

where we identify \( \tilde{\tau}_F \) with \( \tau_F \) since as follows from the analysis above it is not necessary to rescale \( \tau_F \) in order to have finite Hamiltonian in the limit \( \omega \to \infty \).

We see that this form of the Hamiltonian constraint is rather complicated. For that reason it is necessary to check whether it defines consistent theory. Especially we would like to see whether Hamiltonian and spatial diffeomorphism constraints are the first class constraints. To do this we calculate Poisson algebra of constraints. As usually we introduce smeared form of these constraints

\[
T_\tau(N) = \int d\sigma N H_\tau , \quad T_\sigma(N^\sigma) = \int d\sigma N^\sigma H_\sigma
\]

(2.18)

and we easily find

\[
\{ T_\sigma(N^\sigma), T_\sigma(M^\sigma) \} = \int d\sigma (N^\sigma \partial_\sigma M^\sigma - N^\sigma \partial_\sigma M^\sigma) p_\mu \partial_\sigma x^\mu = T_\sigma(N^\sigma \partial_\sigma M^\sigma - N^\sigma \partial_\sigma M^\sigma) .
\]

(2.19)

In case of the calculation of the Poisson brackets of two Hamiltonian constraints the situation is more involved since the explicit calculation gives

\[
\{ T_\tau(N), T_\tau(M) \} = \int d\sigma (N \partial_\sigma M - M \partial_\sigma N) 2\tau_F^2 (p_\mu h^{\mu\nu} \tilde{H}_\nu \partial_\sigma x^\rho + \partial_\sigma x^\rho \tilde{H}_\rho \mu h^{\rho\mu} p_\nu) +
\]

\[
- 2 \int d\sigma \tau_F (N \partial_\sigma M - M \partial_\sigma N) p_\mu V_\mu h^{\rho\omega} p_\omega +
\]

\[
+ \int d\sigma (N \partial_\sigma M - M \partial_\sigma N) p_\rho V_\rho h^{\omega\nu} \tilde{H}_\nu \partial_\sigma x^\omega +
\]

\[
- \tau_F^2 \int d\sigma (N \partial_\sigma M - M \partial_\sigma N) (V_\nu \partial_\sigma x^\nu \tilde{H}_\mu \rho \partial_\sigma x^\rho + \partial_\sigma x^\rho \tilde{H}_\rho \mu \partial_\sigma x^\nu) .
\]

(2.20)
To proceed further we calculate
\[2p_\mu h^{\mu\nu} \bar{H}_\nu \partial_\sigma x^\rho + 2\partial_\sigma x^\rho \bar{H}_\rho h^{\mu\nu} p_\nu =
\begin{align*}
&= 4\eta^{2}p_\mu h^{\mu\nu}h_{\nu \rho} \partial_\sigma x^\rho + 4\eta^{2} \partial_\sigma x^\rho \eta_\mu m_\nu b h^{\mu\rho} p_\rho ,
\end{align*}
\[p_\rho V_\mu^\nu \partial_\sigma x^\nu = 4\eta^{2}p_\mu m_\nu \tau_\nu \partial_\sigma x^\rho - 4\eta^{2} p_\mu h^{\mu\nu} m_\nu a h^{\tau_\rho} \eta_\sigma \partial_\sigma x^\rho ,
\]
\[V_\nu^\mu \partial_\sigma x^\nu \bar{H}_\mu \partial_\sigma x^\rho + \partial_\sigma x^\rho \bar{H}_\rho m_\nu V_\nu^\mu \partial_\sigma x^\nu = 0 ,
\]
\[p_\mu V_\nu^\mu h^{\nu\omega} p_\omega = 0 .
\]
\[ (2.21)
\]
Collecting these results together we finally obtain
\[\{\mathbf{T}_\tau(N), \mathbf{T}_\tau(M)\} = \mathbf{T}_\sigma((N \partial_\sigma M - M \partial_\sigma N)4\eta^{2}) \]
\[ (2.22)
\]
which is the correct form of the Poisson bracket between Hamiltonian constraints. Finally we calculate the Poisson bracket
\[\{\mathbf{T}_\sigma(N^\sigma), \mathbf{T}_\tau(M)\} . \]
\[ (2.23)
\]
Since
\[\{\mathbf{T}_\sigma(N^\sigma), x^\mu\} = -N^\sigma \partial_\sigma x^\mu , \quad \{\mathbf{T}_\sigma(N^\sigma), p_\mu\} = -\partial_\sigma N^\sigma p_\mu - N^\sigma \partial_\sigma p_\mu
\]
\[ (2.24)
\]
we easily find
\[\{\mathbf{T}_\sigma(N^\sigma), \mathcal{H}_\tau\} = -2\partial_\sigma N^\sigma \mathcal{H}_\tau - N^\sigma \partial_\sigma \mathcal{H}_\tau
\]
\[ (2.25)
\]
or alternatively
\[\{\mathbf{T}_\sigma(N^\sigma), \mathbf{T}_\tau(M)\} = \mathbf{T}_\sigma(N^\sigma \partial_\sigma M - \partial_\sigma N^\sigma M) .
\]
\[ (2.26)
\]
We see that all Poisson brackets \((2.19),(2.22)\) and \((2.26)\) vanish on the constraint surface \(\mathcal{H}_\tau \approx 0, \mathcal{H}_\sigma \approx 0\) and hence they are the first class constraints and the non-relativistic string is well defined system from the canonical point of view.

### 3. Lagrangian Form

In this section we focus on the Lagrangian formulation of the proposed Hamiltonian form of non-relativistic string in Newton-Cartan background. Recall that this string is defined with the Hamiltonian constraint \((2.17)\) and the spatial diffeomorphism constraint \(\mathcal{H}_\sigma \approx 0\). In order to understand subtle points in the transformation from the Hamiltonian to Lagrangian description of this system we firstly start with the simpler problem of non-relativistic string in the flat background.

#### 3.1 Flat space-time limit

The non-relativistic string in the flat background has following Hamiltonian
\[H = \int d\sigma (\lambda^\tau \mathcal{H}_\tau + \lambda^\sigma \mathcal{H}_\sigma) ,
\]
\[ (3.1)
\]
where

\[ \mathcal{H}_\tau = -2\tau_F p_a \eta^{0b} e_{bc} \partial_\sigma x^c + p_i h^{ij} p_j + \tau_F^2 h_{ij} \partial_\sigma x^i \partial_\sigma x^j, \quad \mathcal{H}_\sigma = p_i \partial_\sigma x^i + p_a \partial_\sigma x^a, \]  

(3.2)

and \(a, b, c, \ldots = 0, 1\) and where \(h_{ij} = \delta_{ij}, h^{ij} = \delta^{ij}, i, j, \ldots = 2, \ldots, d - 1\). Our goal is to find Lagrangian formulation of the non-relativistic string in flat background. With the help of the Hamiltonian (3.1) we obtain following equations of motion for \(x^0, x^1\) and \(x^i\)

\[
\begin{align*}
    \partial_\tau x^0 &= \{x^0, H\} = -2\tau_F \lambda^\tau \partial_\sigma x^1 + \lambda^\sigma \partial_\sigma x^0, \\
    \partial_\tau x^1 &= \{x^1, H\} = -2\tau_F \lambda^\tau \partial_\sigma x^0 + \lambda^\sigma \partial_\sigma x^1, \\
    \partial_\tau x^i &= \{x^i, H\} = 2\lambda^\tau h^{ij} p_j + \lambda^\sigma \partial_\sigma x^i.
\end{align*}
\]  

(3.3)

Then it is easy to find corresponding Lagrangian density

\[
\mathcal{L} = p_a \partial_\tau x^a + p_i \partial_\tau x^i - \mathcal{L} = \lambda^\tau p_i h^{ij} p_j - \lambda^\sigma \partial_\sigma x^i \partial_\sigma x^j h_{ij} = \\
= \frac{1}{4\lambda^\tau} (\partial_\tau x^i - \lambda^\sigma \partial_\sigma x^i) h_{ij} (\partial_\tau x^j - \lambda^\sigma \partial_\sigma x^j) - \lambda^\tau \tau_F^2 h_{ij} \partial_\sigma x^i \partial_\sigma x^j.
\]  

(3.4)

We see that this Lagrangian does not depend on the variables \(x^a\) which is confusing since if we perform inverse Legendre transformation from (3.4) and determine corresponding Hamiltonian we will find that it does not depend on \(p_a\). We can resolve this problem when we closely examine equations of motion for \(x^0\) and \(x^1\). We firstly consider the first equation in (3.3) and multiply it with \(\partial_\sigma x^0\) while we multiply the second one with \(\partial_\sigma x^1\). Then if we take their difference we obtain

\[
-\partial_\tau x^0 \partial_\sigma x^0 + \partial_\tau x^1 \partial_\sigma x^1 = \lambda^\sigma (\partial_\sigma x^1 \partial_\sigma x^1 - \partial_\sigma x^0 \partial_\sigma x^0)
\]  

(3.5)

that can be solved for \(\lambda^\sigma\) as

\[
\lambda^\sigma = \frac{a_{\tau \sigma}}{a_{\sigma \sigma}}, \quad a_{\alpha \beta} = \partial_\alpha x^a \partial_\beta x^b \eta_{ab}.
\]  

(3.6)

On the other hand from the equations of motion for \(x^0\) and \(x^1\) we obtain

\[
(\partial_\tau x^0 - \lambda^\sigma \partial_\sigma x^0)^2 = 4(\lambda^\tau)^2 \partial_\sigma x^1 \partial_\sigma x^1, \quad (\partial_\tau x^1 - \lambda^\sigma \partial_\sigma x^1)^2 = 4(\lambda^\tau)^2 \partial_\sigma x^0 \partial_\sigma x^0
\]  

(3.7)

that implies

\[
-a_{\tau \tau} + 2\lambda^\sigma a_{\sigma \tau} - (\lambda^\tau)^2 a_{\sigma \sigma} = 4(\lambda^\tau)^2 a_{\sigma \sigma} \tau_F^2.
\]  

(3.8)

Inserting (3.4) into this equation we find that \(\lambda^\tau\) is equal to

\[
\lambda^\tau = \frac{\sqrt{-\det a_{\alpha \beta}}}{2\tau_F a_{\sigma \sigma}}.
\]  

(3.9)
Then if we combine (3.6) together with (3.9) we get
\[
\frac{1}{\lambda^2} = -2\tau F A^{\tau\tau} \sqrt{-\det a} , \quad \frac{2\lambda^\sigma}{\lambda^2} = 4\tau F A^{\tau\sigma} \sqrt{-\det a} , \\
\frac{(\lambda^\sigma)^2}{4(\lambda^2)^2} - \lambda^\tau F \tau^2 = -\tau F A^{\tau\sigma} \sqrt{-\det a} .
\]
(3.10)
Finally inserting (3.10) into (3.4) we obtain
\[
\mathcal{L} = -\frac{\tau F}{2} \sqrt{-\det a} a^{\alpha\beta} h_{\alpha\beta}
\]
(3.11)
which has exactly the same form as the Lagrangian density that was derived in [23].

3.2 Lagrangian for String in Newton-Cartan Background

Now we proceed to the case of non-relativistic string in Newton-Cartan background. As
the first step we formulate the Hamiltonian constraint with the help of the variables that
reflect an invariance of the theory under generalized Galilean boosts that have the form
\[
\delta e_a^{\mu} = \tau^{\mu}_{\alpha} a^{\alpha} , \quad \delta \tau^\mu_a = e^\mu_a \lambda^a , \quad \delta m_a^\mu = e^\mu_a \lambda^a ,
\]
(3.12)
where \(\lambda^a \) are parameters that obey following relations
\[
\eta_{ac} \lambda^c_a + \delta_{a'b'} \lambda^b_a = 0 , \quad \lambda^a_{c'a} + \lambda^b_{a'd'} \delta_{b'a'} = 0 , \\
\lambda^a_a + \lambda^a_a = 0 , \quad \lambda^a_a + \lambda^a_a = 0 .
\]
(3.13)
Now we define boost invariant temporary vierbein as \(\hat{\tau}^\mu_a\)
\[
\hat{\tau}^\mu_a = \tau^\mu_a - h^\mu_{ab} m^b_{\alpha} \eta_{ab} .
\]
(3.14)
This is invariant under (3.12) since
\[
\delta \hat{\tau}^\mu_a = e^\mu_a \lambda^a - e^\mu_a \delta_{a'b'} \lambda^b_{a'} \eta_{ba} = e^\mu_a \lambda^a + e^\mu_a \lambda^a = 0 ,
\]
(3.15)
where in the last step we used (3.13). With the help of \(\hat{\tau}^\mu_a\) we can rewrite \(V^\mu\) into manifestly
invariant form
\[
V^\mu = V^\mu_{\nu} \partial_{\nu} a^\nu , \quad V_{\nu} = -2\tau F \hat{\tau}^\mu_a e^{ab}_{\omega} \hat{\tau}^\omega_{\nu} = V^\mu_{\nu} \tau_{\nu} ,
\]
(3.16)
where \(e_{ab} \equiv \eta^{ac} \eta^{bd} \epsilon_{cd}\) and where \(V^\mu_{\nu} = -V^\nu_{\mu}\). Let us now analyze in more details the object \(\hat{H}_{\mu\nu}\). After some calculations we obtain that it can be rewritten into the form
\[
\hat{H}_{\mu\nu} = -2\tau F \epsilon_{\mu} \phi e^{ab}_{\nu} \phi_{ab} \hat{\tau}_{\nu} + \tau F \hat{h}_{\mu\nu} , \\
\hat{h}_{\mu\nu} = h_{\mu\nu} + m_{\mu} a^b_{\nu} \eta_{ab} + \tau_{\mu} m^a_{\nu} \eta_{ab} , \\
\phi_{ab} = -\tau_{\mu} \eta^{da}_{\nu} m^b_{\mu} - m^a_{\mu} \tau_{\nu} \eta_{ab} + m^a_{\mu} \eta_{\mu\nu} m^b_{\nu} .
\]
(3.17)
An important property of (3.17) is that it is written with the help of the objects that are invariant under (3.12). To see this let us firstly consider variation of $\tilde{h}_{\mu\nu}$

$$
\delta \tilde{h}_{\mu\nu} = \tau^a \lambda^b \delta_a^b e_{\nu}' + e_{\mu}' \delta_a^b \tau_{\nu}^b \lambda_b^a + e_\mu \lambda^b \delta_a^b \eta_{\mu\nu} + \tau_{\mu\nu}^a a_{\nu b} e_{\nu}' \lambda_b^a = 0 ,
$$

(3.18)

where we used (3.13). Finally we consider the variation of $\Phi^{ab} = \Phi^{ba}$. Note that the matrix $\Phi^{ab}$ can be interpreted as the matrix of Newton-potential in generalized Newton-Cartan gravity which is now matrix valued as opposite to the scalar form of $\Phi$ in ordinary Newton-Cartan gravity. On the other hand it is still invariant under (3.12) since

$$
\delta \Phi^{ab} = -e^\mu \lambda^a_{,\epsilon} c_{,\mu} m_{,\mu}^{,b} \eta_{\epsilon} - m_{,\mu} a^a_{,\epsilon} \lambda^b_{,\mu} \eta_{\epsilon} + e_{\nu} \delta^{\epsilon}_{\nu} \lambda_{\epsilon}^{,a} m_{,\mu}^{,b} + m_{,\mu} a^a_{,\epsilon} \delta^{\epsilon}_{\nu} \lambda_{\epsilon}^{,b} =
$$

$$
= -e^\mu \lambda^a_{,\epsilon} c_{,\mu} m_{,\mu}^{,b} \eta_{\epsilon} - m_{,\mu} a^a_{,\epsilon} \lambda^b_{,\mu} \eta_{\epsilon} - e_{\nu} \delta^{\epsilon}_{\nu} \lambda_{\epsilon}^{,a} m_{,\mu}^{,b} - m_{,\mu} a^a_{,\epsilon} \delta^{\epsilon}_{\nu} \lambda_{\epsilon}^{,b} =
$$

$$
= -e^\mu \lambda^a_{,\epsilon} c_{,\mu} m_{,\mu}^{,b} \eta_{\epsilon} - m_{,\mu} a^a_{,\epsilon} \lambda^b_{,\mu} \eta_{\epsilon} + e_{\nu} \lambda^a_{,\epsilon} \eta_{\epsilon} m_{,\mu}^{,b} + m_{,\mu} a^a_{,\epsilon} \lambda^b_{,\epsilon} \eta_{\epsilon} c_{,\mu}^{,b} = 0 ,
$$

(3.19)

where in the first step we used the relation $\lambda_a^a = -\lambda_a^a$ and in the last step we used the fact that

$$
\lambda_c^{,a} = \lambda_c^{,a} + \lambda_c^{,b} \sigma^{,b} \epsilon^c = 0 .
$$

(3.20)

It is also instructive to elaborate more about an expression that contain the potential $\Phi^{ab}$. We find that after some calculations it can be rewritten into the form

$$
- \tau^2 \partial_t \partial_\sigma x^{,\mu} \tau_{\mu} c_{,\mu} \Phi^{,d} e_{,\mu} \tau_{\mu} d_{,\mu} x^{,\nu} =
$$

$$
= \tau^2 \partial_t \partial_\sigma x^{,\mu} \tau_{\mu} c_{,a} \Phi^{,b} \eta_{bd} \tau_{\mu} d_{,\mu} x^{,\nu} - \tau^2 a_{,a} \Phi^{,b} \eta_{ba} + \tau^2 \bar{h}_{\mu\sigma} ,
$$

(3.21)

so that we finally obtain manifestly invariant Hamiltonian in the form

$$
H = \int d\sigma (\lambda^a \mathcal{H}_\sigma + \lambda^\sigma \mathcal{H}_\sigma) , \quad \mathcal{H}_\sigma = p_\mu \partial_\sigma x^{,\mu} ,
$$

$$
\mathcal{H}_\sigma = p_\mu \lambda^{,\mu} \eta_{\epsilon} - 2 p_\mu \lambda^{,\mu} \eta_{\epsilon}^{,b} \tau_{\mu}^b \tau_{\sigma} \partial_\sigma x^{,\nu} + \tau^2 a^{,d} \eta_{bd} \tau_{\mu} d_{,\mu} x^{,\nu} - \tau^2 a_{,a} \Phi^{,b} \eta_{ba} + \tau^2 \bar{h}_{\mu\sigma} ,
$$

(3.22)

where $\tau^a = \partial_\sigma x^{,\mu} \tau_{\mu}^a$, $\bar{h}_{\mu\sigma} = \partial_\sigma x^{,\mu} \bar{h}_{\mu\sigma} \partial_\sigma x^{,\nu}$ and where $\lambda^a$ and $\lambda^\sigma$ are corresponding Lagrange multipliers.

Before we proceed to the Lagrangian formulation of the theory let us introduce vierbein $\tilde{e}_{,\mu}^a$ defined as

$$
\tilde{e}_{,\mu}^a = e_{,\mu}^a + m_{,\mu}^a e_{,\nu}^c \delta^{a}_{,\nu} \tau_{,\mu}^b \eta_{ba} ,
$$

(3.23)

that is again invariant under (3.12)

$$
\delta \tilde{e}_{,\mu}^a = \tau_{,\mu}^a \lambda_a^a + \lambda_a^a \delta^{a}_{,\nu} \tau_{,\mu}^b \eta_{ba} =
$$

$$
= \tau_{,\mu}^a \lambda_a^a - \lambda_b^b \delta^{a}_{,\nu} \delta^{a}_{,\nu} \tau_{,\mu}^b = 0 .
$$

(3.24)
Note that we have following useful identity
\[ \hat{\tau}_\alpha^\nu \hat{e}_\mu^\alpha = 0 \] (3.25)
and also
\[ \hat{e}_\mu^\alpha \hat{e}_\nu^\beta = \delta^\alpha_\beta, \quad \hat{e}_\mu^\alpha h^{\mu\nu} = \epsilon^\nu_{\nu} \delta^{\alpha^\nu}. \] (3.26)
Now we are ready to proceed to the Lagrangian formulation of the theory. We begin with the canonical equations of motion for \( x^\mu \) that follow from the Hamiltonian (3.22)
\[ \partial_\tau x^\mu = \lambda^\tau (2h^{\mu\nu} p_\nu + V^\mu) + \lambda^\sigma \partial_\sigma x^\nu. \] (3.27)
Let us now multiply this equation with \( \hat{e}_\mu^\alpha \). Using the fact that \( \hat{e}_\mu^\alpha x^\mu = 0 \) we find that
\[ \hat{e}_\mu^\alpha V^\mu = 0 \] and from (3.27) we obtain
\[ \hat{e}_\mu^\alpha \partial_\tau x^\mu = 2\lambda^\tau \epsilon^\nu_{\nu} \delta^{\alpha^\nu} \partial_\tau x^\mu + \lambda^\sigma \hat{e}_\mu^\alpha \partial_\sigma x^\mu \] (3.28)
and consequently
\[ (\partial_\tau x^\mu - \lambda^\sigma \partial_\sigma x^\mu) \hat{e}_\mu^\alpha \partial_\alpha x^\nu \hat{e}_\nu^\beta (\partial_\tau x^\nu - \lambda^\sigma \partial_\sigma x^\nu) = 4(\lambda^\tau)^2 p_\mu h^{\mu\nu} p_\nu. \] (3.29)
With the help of this result we easily find the Lagrangian density in the form
\[ \mathcal{L} = p_\mu \partial_\tau x^\mu - \lambda^\tau \mathcal{H}_\tau - \lambda^\sigma \mathcal{H}_\sigma = \]
\[ = \frac{1}{4\lambda^\tau}(\partial_\tau x^\mu - \lambda^\sigma \partial_\sigma x^\mu) \hat{e}_\mu^\alpha \partial_\alpha x^\nu \hat{e}_\nu^\beta (\partial_\tau x^\nu - \lambda^\sigma \partial_\sigma x^\nu) - \tau^2_\tau \lambda^\tau \mathcal{H}_\sigma. \] (3.30)
To proceed further we observe that (3.27) implies
\[ \hat{e}_\mu^\alpha \partial_\alpha x^\nu \hat{e}_\nu^\beta = \bar{h}_\mu^\nu + \partial_\mu \epsilon^\nu_{\nu} \Phi^\alpha_{\beta} \eta_b \tau^d, \] (3.31)
where we also used
\[ e_\mu^\alpha e_\nu^\beta = \delta^\nu_{\nu} - \tau^a_\mu \tau^a_\nu. \] (3.32)
Then we can rewrite the Lagrangian density (3.30) into the form
\[ \mathcal{L} = \frac{1}{4\lambda^\tau}(\bar{h}_{\alpha\beta} - 2\lambda^\sigma \bar{h}_{\sigma\tau} + (\lambda^\sigma)^2 \bar{h}_{\sigma\sigma} + \]
\[ + \partial_\tau x^\mu \tau^c_{\mu} \epsilon_{\alpha} \Phi^\alpha_{\beta} \eta_b \tau^d \partial_\sigma x^\nu - 2\lambda^\sigma \partial_\sigma x^\mu \tau^c_{\mu} \epsilon_{\alpha} \Phi^\alpha_{\beta} \eta_b \tau^d \partial_\sigma x^\nu + (\lambda^\sigma)^2 \partial_\sigma x^\mu \tau^c_{\mu} \epsilon_{\alpha} \Phi^\alpha_{\beta} \eta_b \tau^d \partial_\sigma x^\nu - \lambda^\tau \tau^2_\tau \partial_\sigma x^\mu \tau^c_{\mu} \epsilon_{\alpha} \Phi^\alpha_{\beta} \eta_b \tau^d \partial_\sigma x^\nu + \lambda^\tau \tau^2_\tau \mathcal{A}_{\sigma\tau} \Phi^\alpha_{\beta} \eta_b - \lambda^\tau \tau^2_\tau \mathcal{H}_{\sigma\tau}, \]
(3.33)
where \( \bar{h}_{\alpha\beta} = \bar{h}_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu. \)
Finally we eliminate \( \lambda^\tau \) and \( \lambda^\sigma \) from (3.33). As in case of the flat space-time limit their form is not determined by their equations of motion. Instead they can be determined
using the equations of motion for $x^\mu$. In fact, if we multiply (3.27) by $\tau_{\mu\nu}$ and use the fact that $\tau_{\mu\nu}h^{\nu\rho} = 0$ we obtain

$$\tau_{\mu\nu}(\partial_\tau x^{\nu} - \lambda^{\sigma} \partial_\sigma x^{\nu}) - \lambda^\tau \tau_{\mu\nu}V^{\nu} = 0.$$  

(3.34)

We can multiply this equation with $\partial_\sigma x^\mu$ and we obtain

$$\lambda^\sigma = \frac{a_{\sigma\tau}}{a_{\sigma\sigma}}, \quad a_{\alpha\beta} = \partial_\alpha x^\mu \tau_{\mu\nu} \partial_\beta x^\nu$$

(3.35)

using the fact that

$$\partial_\sigma x^\mu \tau_{\mu\nu}V^{\nu} = 2\tau_F \partial_\sigma x^\mu \tau_\mu^a \epsilon_{ab} \tau_b^b \partial_\sigma x^{\nu} = 0.$$  

(3.36)

In the similar way we obtain

$$(\partial_\tau x^{\mu} - \lambda^{\sigma} \partial_\sigma x^{\mu})\tau_{\mu\nu}(\partial_\tau x^{\nu} - \lambda^{\sigma} \partial_\sigma x^{\nu}) = (\lambda^{\tau})^2 V^\mu \tau_{\mu\nu}V^{\nu}$$

(3.37)

that can be solved for $\lambda^{\tau}$ as

$$\lambda^{\tau} = \frac{\sqrt{-\det a_{\alpha\beta}}}{\sqrt{-V^\mu \tau_{\mu\nu}V^{\nu} \sqrt{a_{\sigma\sigma}}}},$$

(3.38)

where

$$V^\mu \tau_{\mu\nu}V^{\nu} = -4\tau_F^2 a_{\sigma\sigma}.$$  

(3.39)

Now we see that we can proceed as in the case of the non-relativistic string in flat space-time and we obtain the final result

$$L = -\frac{\tau_F}{2} \sqrt{-\det a} \left( a^{\alpha\beta}\eta_{\alpha\beta} + a^{\alpha\beta} \partial_\alpha x^\mu \tau_\mu^c \eta_{ca} \Phi^{ab} \eta_{bd} \tau_d^b \partial_\beta x^{\nu} - \Phi^{ab} \eta_{ba} \right)$$

(3.40)

which is generalization of the action found in [23] since now contains an additional terms proportional to the matrix valued Newton potential. Further, the action $S = \int d^2\sigma L$ is manifestly invariant under generalized Galilean boost and also it is manifestly invariant under world-sheet diffeomorphism.

**4. Conclusion**

Let us outline our results and suggest possible extension of this work. We found Hamiltonian for non-relativistic string in Newton-Cartan background from the Hamiltonian of relativistic string in general background when we used the limiting procedure introduced in [16]. The corresponding Hamiltonian is linear combination of two constraints and we checked that they are the first class constraints which is a consequence of diffeomorphism invariance of world-sheet theory. We also introduced variables that are invariant under
Milne boost and we showed that the Hamiltonian constraint is invariant under this transformation too. Finally we found Lagrangian formulation of the non-relativistic string in Newton-Cartan background where now Lagrangian contains two additional terms with respect to the action proposed in [23]. These additional terms are proportional to the matrix $\Phi^{ab}$ which is related to Newton potential. We can explain the emergence of these new terms in the action as a consequence of the fact that the inverse relativistic vierbein, which is used in the limiting procedure [16], contains corrections of higher order in parameter $\omega^{-1}$ and which give finite contributions to the Hamiltonian constraint. We mean that this is new and interesting result.

This paper can be extended in different directions. It would be possible to perform similar analysis in the case of non-relativistic p-brane in Newton-Cartan background. Secondly, we could also extend this analysis to the case of superstring. We hope to return to some of these problems in future.

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References

[1] M. H. Christensen, J. Hartong, N. A. Obers and B. Rollier, “Torsional Newton-Cartan Geometry and Lifshitz Holography,” Phys. Rev. D 89 (2014) 061901 doi:10.1103/PhysRevD.89.061901 [arXiv:1311.4794 [hep-th]].

[2] M. H. Christensen, J. Hartong, N. A. Obers and B. Rollier, “Boundary Stress-Energy Tensor and Newton-Cartan Geometry in Lifshitz Holography,” JHEP 1401 (2014) 057 doi:10.1007/JHEP01(2014)057 [arXiv:1311.6471 [hep-th]].

[3] J. Hartong, E. Kiritsis and N. A. Obers, "Lifshitz spacetimes for Schrödinger holography," Phys. Lett. B 746 (2015) 318 doi:10.1016/j.physletb.2015.05.010 [arXiv:1409.1519 [hep-th]].

[4] S. A. Hartnoll, A. Lucas and S. Sachdev, “Holographic quantum matter,” arXiv:1612.07324 [hep-th].

[5] D. T. Son, “Toward an AdS/cold atoms correspondence: A Geometric realization of the Schrödinger symmetry,” Phys. Rev. D 78 (2008) 046003 doi:10.1103/PhysRevD.78.046003 [arXiv:0804.3972 [hep-th]].

[6] K. Balasubramanian and J. McGreevy, “Gravity duals for non-relativistic CFTs,” Phys. Rev. Lett. 101 (2008) 061601 doi:10.1103/PhysRevLett.101.061601 [arXiv:0804.4053 [hep-th]].

[7] C. P. Herzog, M. Rangamani and S. F. Ross, “Heating up Galilean holography,” JHEP 0811 (2008) 080 doi:10.1088/1126-6708/2008/11/080 [arXiv:0807.1099 [hep-th]].

[8] D. T. Son, “Newton-Cartan Geometry and the Quantum Hall Effect,” arXiv:1306.0638 [cond-mat.mes-hall].

[9] S. Janiszewski and A. Karch, “Non-relativistic holography from Horava gravity,” JHEP 1302 (2013) 123 doi:10.1007/JHEP02(2013)123 [arXiv:1211.0005 [hep-th]].
[10] E. Cartan, “Sur les varifs connexion affine et la thorie de la relativit gnralse. (premire partie),” Annales Sci. Ecole Norm. Sup. 40 (1923) 325.

[11] P. Horava, “Quantum Gravity at a Lifshitz Point,” Phys. Rev. D 79 (2009) 084008 doi:10.1103/PhysRevD.79.084008 [arXiv:0901.3775 [hep-th]].

[12] E. Bergshoeff, A. Chatzistavrakidis, L. Romano and J. Rosseel, “Newton-Cartan Gravity and Torsion,” JHEP 1710 (2017) 194 [arXiv:1708.05414 [hep-th]].

[13] E. A. Bergshoeff and J. Rosseel, “Three-Dimensional Extended Bargmann Supergravity,” Phys. Rev. Lett. 116 (2016) no.25, 251601 doi:10.1103/PhysRevLett.116.251601 [arXiv:1604.08042 [hep-th]].

[14] H. R. Afshar, E. A. Bergshoeff, A. Mehra, P. Parekh and B. Rollier, “A Schrdinger approach to Newton-Cartan and Hoava-Lifshitz gravities,” JHEP 1604 (2016) 145 doi:10.1007/JHEP04(2016)145 [arXiv:1512.06277 [hep-th]].

[15] E. Bergshoeff, J. Rosseel and T. Zojer, “Newton-Cartan supergravity with torsion and Schrdinger supergravity,” JHEP 1511 (2015) 180 doi:10.1007/JHEP11(2015)180 [arXiv:1509.04527 [hep-th]].

[16] E. Bergshoeff, J. Rosseel and T. Zojer, “Newton-Cartan (super)gravity as a non-relativistic limit,” Class. Quant. Grav. 32 (2015) no.20, 205003 doi:10.1088/0264-9381/32/20/205003 [arXiv:1505.02095 [hep-th]].

[17] E. A. Bergshoeff, J. Hartong and J. Rosseel, “Torsional Newton-Cartan geometry and the Schrdinger algebra,” Class. Quant. Grav. 32 (2015) no.13, 135017 doi:10.1088/0264-9381/32/13/135017 [arXiv:1409.5555 [hep-th]].

[18] R. Andringa, E. Bergshoeff, S. Panda and M. de Roo, “Newtonian Gravity and the Bargmann Algebra,” Class. Quant. Grav. 28 (2011) 105011 doi:10.1088/0264-9381/28/10/105011 [arXiv:1011.1145 [hep-th]].

[19] J. Hartong and N. A. Obers, “Hoava-Lifshitz gravity from dynamical Newton-Cartan geometry,” JHEP 1507 (2015) 155 doi:10.1007/JHEP07(2015)155 [arXiv:1504.07461 [hep-th]].

[20] J. Hartong, Y. Lei and N. A. Obers, “Nonrelativistic Chern-Simons theories and three-dimensional Hoava-Lifshitz gravity,” Phys. Rev. D 94 (2016) no.6, 065027 doi:10.1103/PhysRevD.94.065027 [arXiv:1604.08054 [hep-th]].

[21] E. Bergshoeff, J. Gomis, B. Rollier, J. Rosseel and T. ter Veldhuis, “Carroll versus Galilei Gravity,” JHEP 1703 (2017) 165 doi:10.1007/JHEP03(2017)165 [arXiv:1701.06156 [hep-th]].

[22] K. T. Grosvenor, J. Hartong, C. Keeler and N. A. Obers, “Homogeneous Nonrelativistic Geometries as Coset Spaces,” arXiv:1712.03980 [hep-th].

[23] R. Andringa, E. Bergshoeff, J. Gomis and M. de Roo, “Stringy’ Newton-Cartan Gravity,” Class. Quant. Grav. 29 (2012) 235020 doi:10.1088/0264-9381/29/23/235020 [arXiv:1206.5176 [hep-th]].

[24] T. Harmark, J. Hartong and N. A. Obers, “Nonrelativistic strings and limits of the AdS/CFT correspondence,” Phys. Rev. D 96 (2017) no.8, 086019 doi:10.1103/PhysRevD.96.086019 [arXiv:1705.03535 [hep-th]].

[25] J. Kluson, “Note About Hamiltonian Formalism for Newton-Cartan String and p-Brane,” arXiv:1712.07430 [hep-th].
[26] A. Barducci, R. Casalbuoni and J. Gomis, “Non-relativistic Spinning Particle in a Newton-Cartan Background,” JHEP 1801 (2018) 002 doi:10.1007/JHEP01(2018)002 [arXiv:1710.10970 [hep-th]].

[27] J. Kluson, “Canonical Analysis of Non-Relativistic Particle and Superparticle,” arXiv:1709.09405 [hep-th].