Two Dimensional Gauge Theoretic Supergravities

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Abstract

We investigate two dimensional supergravity theories, which can be built from a topological and gauge invariant action defined on an ordinary surface. We concentrate on four models. The first model is the \( N = 1 \) supersymmetric extension of Jackiw-Teitelboim model presented by Chamseddine in a superspace formalism. We complement the proof of Montano, Aoaki, and Sonnenschein that this extension is topological and gauge invariant, based on the graded de Sitter algebra. Not only do the equation of motions correspond to the supergravity ones and gauge transformations encompass local supersymmetries, but also we identify the \( \int \langle \eta, F \rangle \)-theory with the superfield formalism action written by Chamseddine. Next, we show that the \( N = 1 \) supersymmetric extension of string inspired two dimensional dilaton gravity put forward by Park and Strominger is a theory that satisfies a non-vanishing curvature condition and cannot be written as a \( \int \langle \eta, F \rangle \)-theory. As an alternative, we propose two examples of topological and gauge invariant theories that are based on graded extension of the extended Poincaré algebra and satisfy a vanishing curvature condition. Both models are interpreted as supersymmetric extensions of the string inspired dilaton gravity.

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Résumé

Nous étudions dans cet article les théories de la supergravité à (1+1)-dimensions qui s’écrit comme une théorie topologique et invariante de jauge. Nous nous concentrons sur quatre modèles. Le premier modèle est l’extension supersymétrique $N = 1$ du modèle de Jackiw-Teitelboim développée par Chamseddine dans un formalisme de super-espace. Nous complétons la preuve de Montano, Aoaki, et Sonnenschein qui en montre le caractère topologique et invariant de jauge, basé sur l’algèbre de de Sitter : Non seulement les équations du mouvement correspondent à celles de la supergravité usuelle et les transformations de jauge sont les mêmes que celles de la supersymétrie locale, mais en plus l’action de Chamseddine est identifiée à une action $\int \langle \eta, F \rangle$. Nous prouvons également que l’extension supersymétrique $N = 1$ du modèle de gravité dilatonique, inspiré par la théorie des cordes, proposée par Park et Strominger est une théorie qui satisfait une condition de courbure non nulle et ne peut pas s’écrit comme une théorie $\int \langle \eta, F \rangle$. On propose ensuite deux exemples de théories topologiques et invariantes de jauge basées sur des généralisations de l’algèbre de Poincaré étendue et qui satisfont une condition de courbure nulle. Les deux modèles sont interprétés comme des extensions supersymétriques de théories de gravité dilatonique inspirée par les théories bidimensionnelles de corde.
Introduction

Recently, there has been much interest for two dimensional gravity theories,\(^1,2,3\) such as the Jackiw-Teitelboim (JT) model\(^1\) and the string inspired dilaton gravity (SI) model.\(^4\) The reasons are numerous and among them, it is the presence of black holes that is attractive.\(^4,5\) Their resemblance to Einstein gravity theory does not end with the presence of black holes. Gravitational collapse, propagation of gravitational waves and Newtonian expansion are also present.\(^6\) Two dimensional gravity theories are therefore an interesting arena to explore features of gravity without the difficulties encountered in the four dimensional world.

Both of the above theories (JT/SI) share an interesting property. They both can be written as topological gauge field theories.\(^7,8,9\) It is interesting because gravity in four dimensions has not been successfully written as a gauge theory. In two dimensions, it provides another way to analyse gravity theories. This was fruitful for the three dimensional case: general relativity theory in (2+1) dimensions was shown to be equivalent to a Chern-Simons gauge theory based on the Poincaré algebra\(^10\) and the Bañados, Teitelboim, and Zanelli black hole solution in (2+1) dimensional anti-de Sitter spacetime\(^11\) fits in a gauge formulation.\(^12\) A similar study of Chern-Simons supergravity as a supersymmetric gauge theory is presented in Ref. 13.

The first model of 2d-gravity was proposed some time ago by Jackiw and Teitelboim.\(^1\) It is obtained by dimensionally reducing the usual Einstein-Hilbert action in (2+1) dimensions,

\[
S_{\text{JT}} = \frac{1}{4\pi} \int d^2x \sqrt{-g} \eta (R - \lambda) \tag{1.1}
\]

the scalar curvature \(R\) is equated to a cosmological constant \(\lambda\) through a Lagrange multiplier \(\eta\).

The second model appeared recently and is inspired by string theory (SI) when restricted to a (1+1) dimensional target space\(^4\)

\[
\overline{S}_{\text{SI}} = \frac{1}{4\pi} \int d^2x \sqrt{-\overline{g}} e^{-2\phi} [ \overline{R} + 4(\nabla \phi)^2 - \lambda] \tag{1.2}
\]

where the overbars are there to stress the presence of a differently scaled metric. We can remove the kinetic term for the dilaton field \(\phi\) using the field redefinition \(\overline{g}_{\mu\nu} = e^{2\phi} g_{\mu\nu}\), \(\eta = e^{-2\phi}\), and Eq. (1.2) can be recast in the more appropriate form for the gauge formulation

\[
S_{\text{SI}} = \frac{1}{4\pi} \int d^2x \sqrt{-g} (\eta R - \lambda) \tag{1.3}
\]

The Lagrange multiplier enforces now the scalar curvature to vanish and the cosmological constant \(\lambda\) only appears in the equations of motion for \(\eta\). Both models (1.1) and (1.3) are known to be topological and gauge models based on two-dimensional de Sitter\(^7\) and extended Poincaré groups respectively.\(^9\)

In this paper, we investigate \(N = 1\) supersymmetric extensions of the Jackiw-Teitelboim model and the string inspired dilaton gravity model. Two approaches are possible. The
first one uses a superfield formalism developed by Howe\textsuperscript{14} for two dimensional superspaces and was carried over by Chamseddine\textsuperscript{15} in the JT model and by Park and Strominger\textsuperscript{16} in the SI model. The second approach focuses on topological theories in conventional two-dimensional spacetime, where a vanishing field strength reproduces the standard torsion and supercurvature of supergravity. In this approach, gauge transformations replace diffeomorphisms, Lorentz transformations and supersymmetries. If the topological nature of the supersymmetric extension of the JT model is known,\textsuperscript{17} only recently Rivelles has worked out a topological theory leading to a supersymmetric extension of the second model.\textsuperscript{18}

The structure of the paper goes as follows: In section II, we review the work of Chamseddine, Park and Strominger. We write their $N=1$ supersymmetric extension in component fields. In section III, we show how the Chamseddine action is also a topological and gauge invariant model based on the graded de Sitter algebra $OSP(1,1\mid 1)$. Actually, it is not possible to write the Park and Strominger action as a topological theory of the $\int \langle \eta, F \rangle$-type (without fields redefinitions). Park and Strominger’s model leads to a non-vanishing curvature and represents a challenge to be written as a topological and gauge invariant model of another type. Nevertheless, in section IV, we are able to construct other $N=1$ supersymmetric extensions described by topological and gauge invariant models using two examples of graded extensions of the extended Poincaré algebra. Rivelles’s proposal is one of them.

II. Supersymmetric Extension of the Jackiw-Teitelboim and String Inspired Models.

Chamseddine proposed a $N=1$ supersymmetric extension of the Jackiw-Teitelboim model in a superfield formalism

$$S_{\text{SJT}} = -\frac{i}{8\pi} \int d^2x \ d^2\theta E\Phi \left( S - \lambda' \right).$$

(2.1)

Our conventions are in the appendix. If we integrate out the $\theta$ variable and eliminate the auxiliary fields by their classical equations of motion

$$A = \lambda' \quad F = \frac{1}{2} \lambda' \phi$$

(2.2)

we recast the action in component fields

$$S'_{\text{SJT}} = \frac{1}{4\pi} \int d^2x \sqrt{-g} \left\{ \phi \left( R + \frac{1}{2} \lambda'^2 - \frac{i}{4} \lambda' \varepsilon^{\mu\nu} \chi^\alpha (\gamma^5)^\alpha_{\beta} \chi_{\nu,\beta} \right) ight. \\
\left. - 2i\lambda^\alpha \left( \varepsilon^{\mu\nu} (\gamma^5)^\alpha_{\beta} D_{\mu} \chi_{\nu,\beta} + \frac{1}{4} \lambda' (\gamma^{\mu})^\alpha_{\beta} \chi_{\mu,\beta} \right) \right\}$$

(2.3)

where the prime on $S_{\text{SJT}}$ means that we have suppressed the auxiliary fields. Setting the fermion fields to zero reproduces $S_{\text{JT}}$ with a negative cosmological constant $\lambda = -(\lambda')^2/2$. 
The supersymmetric extension of the Jackiw-Teitelboim model is rather trivial to obtain since one substitutes for the fields the corresponding superfields. However, this is not the case for the string inspired dilaton gravity model. The same strategy would lead one to move the parenthesis to the left of $\Phi$ to construct a $N = 1$ supersymmetric extension of the model (1.3), that is,

$$S = -\frac{i}{8\pi} \int d^2x \ d^2\theta \ E \left(\Phi S - \lambda'\right)$$

(2.4)

Integrating out the $\theta$-variable and eliminating the auxiliary fields gives the action

$$S' = \frac{1}{4\pi} \int d^2x \ \sqrt{-g} \left\{ \phi R - \frac{i\lambda'}{4} \bar{\epsilon}^{\mu\nu} \chi_\mu^\alpha (\gamma^5)_\alpha^\beta \chi_{\nu,\beta} - 2i\Lambda^\alpha \bar{\epsilon}^{\mu\nu} (\gamma^5)_\alpha^\beta D_\mu \chi_{\nu,\beta} \right\}$$

(2.5)

But, setting the fermion field to zero does not reproduce the model (1.3) since there is no cosmological constant present.

Turning back to the original form (1.2) of the string inspired model and replacing the fields by superfields leads indeed to a suitable $N = 1$ supersymmetric extension. This was carried over by Park and Strominger whose goal was to provide a positive energy theorem. Their actions in superspace and in components are

$$\overline{S}_{SSI} = -\frac{i}{8\pi} \int d^2x \ d^2\theta \ E \ e^{-2\phi} \left[ S + 2iD_\alpha \Phi D^\alpha \Phi - \lambda' \right]$$

$$\overline{S}'_{SSI} = \frac{1}{4\pi} \int d^2x \ \sqrt{-g} \ e^{-2\phi} \left\{ R + 4g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{4}(\lambda')^2 \right.$$}

$$+ 4i\bar{\epsilon}^{\mu\nu} \Lambda^\alpha (\gamma^5)_\alpha^\beta D_\mu \chi_{\nu,\beta} - 4i\Lambda^\alpha (\gamma^5)_\alpha^\beta \partial_\mu \Lambda_\beta - i\lambda' \Lambda^\alpha \Lambda_\alpha + \frac{i}{2}(\lambda')^2 \Lambda^\alpha (\gamma^5)_\alpha^\beta \chi_{\mu,\beta}$$

$$- 2i\Lambda^\alpha (\gamma^\nu)_\alpha^\beta (\gamma^\mu)_\beta^\gamma \chi_{\mu,\gamma} \partial_\nu \phi - \frac{i}{4}\bar{\epsilon}^{\mu\nu} (\lambda' - 2i\Lambda^\gamma \Lambda_\gamma) \chi_\mu^\alpha (\gamma^5)_\alpha^\beta \chi_{\nu,\beta} \right\}$$

(2.6)

where the fields should be read with overbars as in Eq. (1.2). Setting the fermion fields to zero reproduces $\overline{S}_S$ of Eq. (1.2) with negative cosmological constant $\lambda = -(\lambda')^2/4$.

Going from $\overline{S}_S$ in Eq. (1.2) to $S_S$ in Eq. (1.3) was achieved by a Weyl transformation $\bar{g}_{\mu\nu} = e^{2\phi} g_{\mu\nu}$. A supersymmetric extension of $S_S$ can therefore be obtained by performing a super-Weyl transformation on the supersymmetric extension of $\overline{S}_S$. These super-transformations are discussed by Howe and they act on the super-Zweibein by

$$\overline{e}_\mu^\alpha = e^\phi e_\mu^\alpha \ \ \ \overline{\chi}_\mu^\alpha = e^{\frac{i}{2}\phi} \left[ \chi_\mu^\alpha + (\gamma_\mu)^{\alpha\beta} \Lambda_\beta \right]$$

(2.7)

and on the scalar field $\Phi$, they do not affect $\phi$ but its superpartner $\Lambda_\alpha$

$$\overline{\Lambda}_\alpha = e^{-\frac{i}{2}\phi} \Lambda_\alpha$$

(2.8)

[Note that the auxiliary fields get also transformed: $\overline{A} = e^{-\phi}(A + 2F)$ and $\overline{F} = e^{-\phi}F$.] The action (2.6) then becomes

$$S'_{SSI} = \frac{1}{4\pi} \int d^2x \sqrt{-g} \left\{ e^{-2\phi} R + \frac{1}{4}(\lambda')^2 + 6i \bar{\epsilon}^{\mu\nu} \Lambda^\alpha (\gamma^5)_\alpha^\beta D_\mu \chi_{\nu,\beta}$$

$$- 8i e^{-2\phi} \Lambda^\alpha (\gamma^\nu)_\alpha^\beta \partial_\nu \phi e^{-2\phi} - \frac{1}{2} e^{-2\phi} \Lambda^\alpha \Lambda_\alpha g^{\mu\nu} \chi_\mu^\beta \chi_{\nu,\beta}$$

$$+ 2ie^{-2\phi} \bar{\epsilon}^{\mu\nu} D_\mu \Lambda^\alpha (\gamma^5)_\alpha^\beta \chi_{\nu,\beta} - \frac{i}{4} e^{-2\phi} \bar{\epsilon}^{\mu\nu} (\lambda' e^\phi - 4i\Lambda^\gamma \Lambda_\gamma) \chi_\mu^\alpha (\gamma^5)_\alpha^\beta \chi_{\nu,\beta} \right\}$$

(2.9)
This action now obviously reduces to Eq. (1.3) if the fermions fields are set to zero.

**III. Gauge Theoretic Formulation of Supergravity Theories.**

The lineal gravities (1.1) and (1.3) share the remarkable property to possess a topological and gauge invariant formulation\(^9\). The main concern of this paper is to see whether this remains true for supersymmetric extensions. For the Jackiw-Teitelboim model, this formulation is based on the two-dimensional de Sitter group. It turns out that the same is true for the Chamseddine extension (2.1) provided one works with the graded de Sitter algebra \(\text{OSP}(1,1|1)\)

\[
[P_a, P_b] = -\frac{1}{4}(\lambda')^2\epsilon_{ab} J \
[P_a, J] = \epsilon^b_a P_b \quad [Q_\alpha, J] = \frac{1}{2}(\gamma^5)_\alpha^\beta Q_\beta
\]

\[
[P_a, Q_\alpha] = \frac{1}{4}\lambda' (\gamma_a)_\alpha^\beta Q_\beta \quad \{Q_\alpha, Q_\beta\} = -2i(\gamma^a)_\alpha^\beta P_a + i\lambda' (\gamma^5)_\alpha^\beta J
\]

(3.1)

In that case, the (graded, see appendix) invariant non-degenerate inner product

\[
\langle P_a, P_b \rangle \equiv h_{ab} \quad \langle J, J \rangle \equiv 4/(\lambda')^2 \quad \langle Q_\alpha, Q_\beta \rangle \equiv -(8i/\lambda')\epsilon_{\alpha\beta}
\]

(3.2)

is used to write the action

\[
S_{\text{GJT}} = \frac{1}{4\pi} \int d^2x \epsilon^{\mu\nu} \langle \eta, F_{\mu\nu} \rangle
\]

(3.3)

where \(F = dA + A^2\) is the strength field associated with the gauge field

\[
A_\mu = e^a_\mu P_a - \omega_\mu J + \frac{1}{2} \chi_\mu^\alpha (\gamma^5)_\alpha^\beta Q_\beta
\]

(3.4)

and \(\eta = \eta^a P_a + \eta^J J + \eta^\alpha Q_\alpha\) is a world scalar with value in the graded algebra (3.1). This action is explicitly topological and gauge invariant. Remark that it is defined on an ordinary two-dimensional surface. In components, it writes

\[
S_{\text{GJT}} = \frac{1}{4\pi} \int d^2x \left\{ \eta_a \epsilon^{\mu\nu} \left( \partial_\mu e^a_\nu - e^a_\nu \partial_\mu e^b_\nu - \frac{i}{4} \chi_\mu^\alpha (\gamma^a)_\alpha^\beta \chi_{\nu,\beta} \right) \right.

- \frac{4}{(\lambda')^2} \eta^J \epsilon^{\mu\nu} \left( \partial_\mu \omega_\nu + \frac{1}{8}(\lambda')^2 \epsilon_{ab} e^a_\mu e^b_\nu + \frac{i}{8} \chi_\mu^\alpha (\gamma^5)_\alpha^\beta \chi_{\nu,\beta} \right)

+ \frac{4i}{\lambda'} \eta_\alpha \left( \epsilon^{\mu\nu} D_\mu \chi_{\nu,\beta} (\gamma^5)_\beta^\alpha + \frac{1}{4} \epsilon \lambda' \chi_\mu^\beta (\gamma^\mu)_\alpha^\beta \right) \left\}ight.

(3.5)

If we solve the constraint enforced by \(\eta_a\), we get a relation between \(\omega_\mu\) and \(e^a_\mu, \chi_\mu^\alpha\)

\[
\omega_\mu = -e^{a}_{\mu,a} \epsilon^{\rho\sigma} \partial_\rho e^a_\sigma + \frac{i}{2} \chi_\mu^\alpha (\gamma^5\gamma^\nu)_\alpha^\beta \chi_{\nu,\beta}
\]

(3.6)

identical to the one between the spin-connection, the Zweibein and the gravitino [see Eq. (A.13)] obtained from the standard kinematic constraints on the supertorsion.
If we use the relation (3.6) and we set \( \eta^J = (\lambda')^2 \phi / 2 \) and \( \eta^\alpha = -\lambda' \Lambda^\alpha / 2 \) in the action (3.5), it reduces to the action of Chamseddine (2.3). In this gauge formulation the supergravity transformations on the fields are replaced by some gauge transformations of the gauge fields. Indeed, following Howe, but replacing the auxiliary fields as in Eq. (2.2), the supergravity transformations [see Eqs (A.14)] are

\[
\delta e^a_\mu = i \tau^\alpha (\gamma^a)_\alpha^\beta \chi_{\mu,\beta} \\
\delta \omega_\mu = -i \lambda' r^\alpha (\gamma^5)_\alpha^\beta \chi_{\mu,\beta} \\
\delta \chi_{\mu}^\alpha = 2 D_\mu r^\alpha + \frac{\lambda'}{2} (\gamma^\mu)^{\alpha\beta} \tau_{\beta}
\]

for the Zweibein, the spin-connection and the gravitino fields respectively. In the gauge theory, the transformations on the gauge fields are obtained with

\[
A' = U^{-1} A U + U^{-1} dU
\]

and the infinitesimal transformation \( U = 1 + \tau^\alpha (\gamma^5)_\alpha^\beta Q_\beta \) reproduces local supersymmetries of Eq. (3.7).

In the SI case, the supersymmetric extension proposed by Park and Strominger – after the super-Weyl transformation (2.7) – possesses quartic interaction terms and thus does not fit with the form (3.3). Moreover, variation of the action \( S'_{\text{SSI}} \) of Eq. (2.9) with respect to \( e^{-2\phi} \) and \( \Lambda_\alpha \) gives two equations: One containing both the curvature \( R \) and the gravitino kinetic term \( D_\mu \chi_\nu \) and another one that gives an expression for \( D_\mu \chi_{\nu} \). Substituting the equation for \( D_\mu \chi_{\nu} \) into the first produces

\[
R = -16 i \Lambda^\alpha (\gamma^\mu)_\alpha^\beta D_\mu \Lambda_\beta - \frac{1}{2} \Lambda^\alpha \Lambda_\alpha [\chi_{\mu,\beta} g^\mu\nu \chi_{\nu,\beta} + 2 \bar{e}^{\mu\nu} \chi_{\mu}^\beta (\gamma^5)_\beta^\alpha \chi_{\nu,\gamma}] \\
+ 2 i \Lambda^\alpha (\gamma^\mu \gamma^\nu)_\alpha^\beta \chi_{\mu,\beta} \bar{e}^\nu \phi - i \partial_\nu \left[ \Lambda^\alpha (\gamma^\mu \gamma^\nu)_\alpha^\beta \chi_{\mu,\beta} \right] \\
- 2 i \frac{e^{2\phi}}{\sqrt{-g}} D_\mu \left[ \Lambda^\alpha \sqrt{-g} \bar{e}^{\mu\nu} e^{-2\phi} (\gamma^5)_\alpha^\beta \chi_{\nu,\beta} \right] - \frac{i}{8} \lambda' \bar{e}^{\mu\nu} \chi_{\mu}^\alpha (\gamma^5)_\alpha^\beta \chi_{\nu,\beta}
\]

Setting the fermion fields to zero gives \( R = 0 \), which was needed to construct a topological and gauge invariant model. However, if the fermions are present, there is no way (without introducing new fields) to absorb the RHS of Eq. (3.9) in a redefinition of the spin-connection in order to get a vanishing curvature condition.

We then turn to the strategy of building supersymmetric extensions by considering topological theories of the \( \int \langle \eta, F \rangle \)-type. We will present in the next section examples of topological and gauge invariant models that are based on the simplest graded extension of the extended Poincaré algebra and we will see that they actually differ from the action (2.9).

**IV.A Graded extension of the extended Poincaré algebra**

Motivated by the contraction of the graded de Sitter algebra (3.1), Rivelles obtained a graded extension of the extended Poincaré algebra. However, the algebra he presents is not
the result of the contraction, but rather, a modification of it that satisfies the Jacobi identity

\[
[P_a, P_b] = \epsilon_{ab} I \quad [P_a, J] = \epsilon_a^b P_b \quad [P_a, Q_\alpha] = \frac{1}{2} (\gamma^a)_\alpha^\beta U_\beta
\]

\[
[Q_\alpha, J] = \frac{1}{2} (\gamma^5)_\alpha^\beta Q_\beta \quad [U_\alpha, J] = \frac{1}{2} (\gamma^5)_\alpha^\beta U_\beta \quad [Q_\alpha, K] = \frac{1}{2} (\gamma^5)_\alpha^\beta U_\beta \quad (4.1)
\]

\[
\{Q_\alpha, Q_\beta\} = -2i (\gamma^a)_\alpha^\beta P_a + 2i (\gamma^5)_\alpha^\beta K \quad \{U_\alpha, Q_\beta\} = -2i (\gamma^5)_\alpha^\beta I
\]

It possesses the (graded) invariant and non-degenerate inner product

\[
\langle P_a, P_b \rangle = h_{ab} \quad \langle J, I \rangle = -1 \quad \langle K, K \rangle = 1 \quad \langle Q_\alpha, U_\beta \rangle = -4i \epsilon_{\alpha\beta} \quad (4.2)
\]

[see conventions in Eq. (A.1)].

A gauge theory based on the algebra (4.1) is built from the strength field \( F = dA + A^2 \) associated to the gauge field

\[
A_\mu = e_\mu^a P_a - \omega_\mu J + a_\mu I + b_\mu K + \frac{1}{2} \chi_\mu^\alpha (\gamma^5)_\alpha^\beta Q_\beta + \xi_\mu^\alpha U_\alpha \quad (4.3)
\]

and a Lagrange multiplier \( \eta = \eta^a P_a + \eta(J) J + \eta(I) I + \eta(K) K + \eta^\alpha (Q) Q_\alpha + \eta^\alpha (U) U_\alpha \), which transforms (like \( F = dA + A^2 \)) under the adjoint representation. The action

\[
S_{GEP} = \frac{1}{4\pi} \int d^2x \ e^{\mu\nu} \langle \eta, F_{\mu\nu} \rangle \quad (4.4)
\]

is obviously topological and invariant under gauge transformations generated by the algebra (4.1). Local supersymmetries are obtained by gauge transformations in the additional directions \( K, Q_\alpha \) and \( U_\alpha \). The infinitesimal transformation \( U = 1 + \zeta K + \tau^\alpha (\gamma^5)_\alpha^\beta Q_\beta + \sigma^\alpha U_\alpha \) generates on \( A_\mu \) the transformations

\[
\delta e_\mu^a = i \tau^\alpha (\gamma^a)_\alpha^\beta \chi_{\mu,\beta} \quad \delta \omega_\mu = 0 \quad \delta \chi_\mu^\alpha = 2D_\mu \tau^\alpha \\
\delta a_\mu = 2i \tau^\alpha \zeta_{\mu,\alpha} - i \sigma^\alpha \chi_\alpha \quad \delta b_\mu = \partial_\mu \zeta + i \tau^\alpha (\gamma^5)_\alpha^\beta \chi_{\mu,\beta} \quad (4.5)
\]

\[
\delta \xi_\mu^\alpha = D_\mu \sigma^\alpha + \frac{1}{2} \tau^\beta (\gamma^5 \gamma_\mu)_\beta^\alpha - \frac{1}{2} \tau^\alpha \zeta_\mu + \frac{1}{2} \zeta \chi_\mu^\alpha
\]

The first three are Howe’s supergravity transformations [see Eq. (A.14)] with the auxiliary field \( A = 0 \) taking a different value than in Park and Strominger’s approach. The \( \eta \)-field also transforms in the usual way, \( \eta' = U^{-1} \eta U \)

\[
\delta \eta^\alpha = 2i \tau^\alpha (\gamma^\alpha \gamma^5)_\alpha^\beta \eta_\beta (Q) \quad \delta \eta(J) = 0 \quad \delta \eta^\alpha (Q) = -\frac{1}{2} \tau^\alpha \eta(J) \\
\delta \eta(I) = 2i \tau^\alpha \eta_\alpha (U) + 2i \sigma^\alpha (\gamma^5)_\alpha^\beta \eta_\beta (Q) \quad \delta \eta(K) = -2i \tau^\alpha \eta_\alpha (Q) \quad (4.6)
\]

\[
\delta \eta^\alpha (U) = \frac{1}{2} \tau^\beta (\gamma^5 \gamma^a)_\beta^\alpha \eta_a - \frac{1}{2} \tau^\alpha \eta (K) + \frac{1}{2} \zeta \eta^\beta (Q) (\gamma^5)_\beta^\alpha - \frac{1}{2} \sigma^\beta (\gamma^5)_\beta^\alpha \eta(J)
\]
The equations of motion are $F = 0$, or in components,

\[
F^a(P) = de^a - e^a_i \xi e^b_i - \frac{i}{4} \chi^\alpha \left( \gamma^5 \right)_\alpha ^\beta \chi_\beta = 0
\]

\[
F(J) = -d\omega = 0
\]

\[
F(I) = da + \frac{1}{2} \epsilon_{ab} e^a_i e^b_i - i\chi^\alpha \xi_\alpha = 0
\]

\[
F(K) = db - \frac{1}{4} \chi^\alpha \left( \gamma^5 \right)_\alpha ^\beta \chi_\beta = 0
\]

\[
F^\alpha(Q) = \left( d\chi^b - \frac{1}{2} \omega \left( \gamma^5 \right)_\alpha ^\beta \chi_\gamma \right) \frac{1}{2} \left( \gamma^5 \right)_\alpha ^\beta = 0
\]

\[
F^\alpha(U) = d\xi^\alpha - \frac{1}{2} \omega \left( \gamma^5 \right)_\alpha ^\beta \xi_\beta + \frac{1}{4} \epsilon_{ab} \left( \gamma^5 \right)_\alpha ^\beta \chi_\beta - \frac{1}{4} b\chi^\alpha = 0
\]

and $(D\eta) \equiv d\eta + [A, \eta] = 0$, or in components,

\[
(D\eta)_a(P) = d\eta_a - \epsilon_{ab} \omega^b \eta^a - \epsilon_{ab} \chi^\alpha \left( \gamma^5 \right)_\alpha ^\beta \eta_\beta(Q) = 0
\]

\[
(D\eta)(J) = d\eta(J) = 0
\]

\[
(D\eta)(I) = d\eta(I) + e^a \epsilon_{ab} \omega^b \eta_a - 2i\xi^\alpha \left( \gamma^5 \right)_\alpha ^\beta \eta_\beta(Q) - i\chi^\alpha \eta_\alpha(U) = 0
\]

\[
(D\eta)(K) = d\eta(K) + i\chi^\alpha \eta_\alpha(Q) = 0
\]

\[
(D\eta)(Q) = d\eta(Q) - \frac{1}{2} \omega \left( \gamma^5 \right)_\alpha ^\beta \eta_\beta(Q) + \frac{1}{4} \chi_\alpha \eta(J) = 0
\]

\[
(D\eta)(U) = d\eta(U) - \frac{1}{2} \omega \left( \gamma^5 \right)_\alpha ^\beta \eta_\beta(U) + \frac{1}{2} \left( b \left( \gamma^5 \right)_\alpha ^\beta - e^a \left( \gamma_a \right)_\alpha ^\beta \right) \eta_\beta(Q)
\]

\[
+ \frac{1}{4} \left( \gamma^5 \right)_\alpha ^\beta \chi_\beta \epsilon_{ab} \eta^b + \frac{1}{4} \chi_\alpha \eta(K) - \frac{1}{2} \left( \gamma^5 \right)_\alpha ^\beta \xi_\beta \eta(J) = 0
\]

Quite remarkably, it is possible to eliminate some of the fields by finding relations between them, which solve part of the differential equations. We get in the Lagrange multiplier

\[
\eta^a = \bar{e}^{\mu \nu} \partial_\mu \eta(I) e^a_\nu + i \bar{e}^{\mu \nu} e^a_\mu \chi^\alpha \eta_\alpha(U)
\]

\[
\eta(J) = -\lambda
\]

\[
\eta(K) = -\frac{2i}{\lambda} \eta^a(Q) \eta_\alpha(Q) - 4\lambda'
\]

and in the gauge field

\[
\omega_\mu = -e_{,\mu} \epsilon^{\rho \sigma} \partial_\rho e^a_\sigma + \frac{i}{2} \chi^a_\mu \left( \gamma^5 \gamma^\nu \right)_\alpha ^\beta \chi_\nu_\beta
\]

\[
b_\mu = -\frac{i}{\lambda} \chi^\alpha \left( \gamma^5 \right)_\alpha ^\beta \eta_\beta(Q)
\]

\[
\chi^a_\mu = \frac{4}{\lambda} \bar{D}_\mu \eta^a(Q)
\]

\[
\xi_\mu^a = -\frac{i}{2\lambda^2} \eta^a(Q) \eta_\gamma(Q) \left( \gamma^5 \right)_\alpha ^\beta \chi_\mu_\beta + \frac{1}{\lambda} \left( \gamma_\mu \gamma^5 \right)_\alpha ^\beta \eta_\beta(Q)
\]
where $\lambda$ and $\lambda'$ are constants of integration. The fields $b_\mu$ and $\xi_\mu^\alpha$ are also determined up to quantities, which can be gauged away by a transformation in the directions $K$ and $U_\alpha$. As a consequence, the resulting transformations of the fields in (4.A9) and (4.A10) are consistent with the ones given in (4.A5) and (4.A6) if we choose $\zeta = -(2i/\lambda)\tau^\alpha(\gamma^5)_{\alpha\beta}\eta_\beta(Q)$ and $\sigma^\alpha = (i/\lambda^2)\eta^\gamma(Q)\eta_\beta(Q)\tau^\beta(\gamma^5)_\beta^\alpha$

$$
\delta e_\mu^\alpha = i\tau^\alpha(\gamma^\prime)_\alpha^\beta\chi_{\mu\beta} \quad \delta \eta(I) = 2i\tau^\alpha\eta_\alpha(U) \quad \delta \eta^\alpha(Q) = \frac{\lambda}{2}\tau^\alpha
$$

$$
\delta \eta^\alpha(U) = -\frac{1}{2}\tau^\beta(\gamma^\mu)_\beta^\alpha \left(\partial_\mu \eta(I) - i\chi_{\mu\beta}\eta_\beta(U)\right) + 2\lambda'\tau^\alpha \left(1 + \frac{i}{\lambda\lambda'}\eta^\beta(Q)\eta_\beta(Q)\right)
$$

Substitution of relations (4.A9) and (4.A10) in the equations of motion (4.A7) and (4.A8) provides a new set of differential equations; they represent the dynamics of the gauge theory (4.A4) where the remaining unconstrained fields of the theory are $e_\mu^\alpha, \eta(I), \eta^\alpha(Q), \eta^\alpha(U)$

$$
R(e_\mu^\alpha, \chi^\alpha_{\nu}) = 0
$$

$$
(\nabla_\mu \partial_\nu - g_{\mu\nu} \nabla_\rho \partial^\rho)\eta(I) + \lambda g_{\mu\nu} = i(\delta^\rho_\mu \delta^\sigma_\nu + \delta^\sigma_\mu \delta^\rho_\nu - 2g_{\mu\nu}g^{\rho\sigma})\chi_{\rho}^\alpha(\gamma^\sigma)_\alpha^\beta\eta_\beta(Q)
$$

$$
+ i\chi_{\mu}^\alpha\chi_{\nu\alpha} \left(\lambda' + i\frac{i}{\lambda}\eta^\beta(Q)\eta_\beta(Q)\right) - \frac{i}{2}g_{\mu\nu} \frac{1}{\sqrt{-g}}D_\rho(\sqrt{-g}g^{\rho\sigma}\chi_\sigma^\alpha\eta_\alpha(U)
$$

$$
\chi_{\mu}^\alpha = \frac{4}{\lambda}D_\mu\eta^\alpha(Q)
$$

$$
D_\mu\eta^\alpha(U) - \chi_{\mu}^\alpha \left(\lambda' + i\frac{i}{\lambda}\eta^\beta(Q)\eta_\beta(Q)\right) - (\gamma_{\mu})^\alpha\beta\eta_\beta(Q) + \frac{1}{4}\chi_\mu^\beta(\gamma^\nu)^\beta_\beta \partial_\nu\eta(I)
$$

$$
- \frac{i}{4}\chi_\mu^\beta(\gamma^\nu)^\beta_\beta \chi_{\nu\gamma}\eta_{\gamma}(U) = 0
$$

[The covariant derivative $\nabla_\mu$ contains the non-symmetric Christoffel symbol defined by $\Gamma^\rho_{\mu\nu} \equiv \partial_\mu e^\rho_\nu - e^a_\beta\omega^\rho_{\mu\nu} - (i/4)\chi_{\mu}^\alpha(\gamma^\alpha)_\alpha^\beta \chi_{\nu\beta}$.] The gauge field $a_\mu$ is not considered here. Its equation of motion (4.A7) can always be locally integrated because $da$ is equated to a two-form, hence closed in two dimensions.

The corresponding bosonic SI theory possesses classical solutions parametrized by the cosmological constant $\lambda$ and a “mass” $M$. The local symmetries we have added will be useful if they turn out to be symmetries of the above configurations. We have already broken the ones associated with $K$ and $U_\alpha$ in the reduction (4.A9), (4.A10). Let us consider the following family of solutions

$$
\eta^\alpha(Q) = \text{constant} \quad (\chi^\alpha_{\mu} = 0)
$$

$$
\partial_\mu\eta^\alpha(U) = (\gamma_{\mu})^\alpha\beta\eta_\beta(Q) = \text{constant}.
$$

$$
g_{\mu\nu} = h_{\mu\nu} \quad (\partial_\mu \partial_\nu - h_{\mu\nu} \partial_\rho \partial^\rho)\eta(I) + \lambda h_{\mu\nu} = 0
$$

(4.A13)

which reproduces the usual bosonic configurations for $g_{\mu\nu}$ and $\eta(I)$ described in Ref. 9, and reduces the action (4.A4) to the form given in Eq. (1.3). It is invariant under
the transformations (4.A11). This means that the solutions (4.A13) possess the Howe’s supersymmetry (4.A11), which allows us to construct a conserved charge like in Park and Strominger's work.

**IV.B Minimal Graded extension of the extended Poincaré algebra**

We now turn to another proposal for topological and supersymmetric extension of the SI model, which looks simpler and also shows a set of bosonic solutions with a non trivial symmetry. There is a minimal graded algebra consistent with the Jacobi identities and containing the bosonic generators of the extended Poincaré algebra (3.1). Introducing the fermionic generator $Q_\alpha$ as the only newly added charge, we define the following algebra

\[
[P_a, P_b] = \epsilon_{ab} I \quad [P_a, J] = \epsilon_a^b P_b \\
\{Q_\alpha, Q_\beta\} = -2i[(1 - \gamma_5)\gamma^a]_{\alpha\beta} P_a + i(\gamma^5)_{\alpha\beta} I \\
\{Q_\alpha, J\} = \frac{1}{2}(\gamma^5)_\alpha^\beta Q_\beta \\
\{Q_\alpha, P_a\} = [(1 - \gamma_5)\gamma_a]_{\alpha\beta} Q_\beta
\]  

(4.B1)

with inner product given by

\[
\langle P_a, P_b \rangle = h_{ab} \quad \langle J, I \rangle = -1 \quad \langle Q_\alpha, Q_\beta \rangle = 2i\epsilon_{\alpha\beta}
\]  

(4.B2)

A gauge theory based on the algebra (4.B1) will be simpler than the theory based on the previous algebra (4.A1) because there are less fields needed to write a topological and gauge invariant action. The gauge field is

\[
A_\mu = e_\mu^a P_a - \omega_\mu J + a_\mu I + \frac{1}{2}\chi_\mu^\alpha (\gamma^5)_\alpha^\beta Q_\beta
\]  

(4.B3)

and the Lagrange multiplier $\eta = \eta^a P_a + \eta(J) J + \eta(I) I + \eta^a Q_\alpha$. The action is again

\[
S_{MGEP} = \frac{1}{4\pi} \int d^2 x \epsilon^{\mu\nu} \langle \eta, F_{\mu\nu} \rangle
\]  

(4.B4)

where $F = dA + A^2$, and it is topological and invariant under gauge transformations. In particular, in the $Q$-direction with $U = 1 + \tau^\alpha (\gamma^5)_\alpha^\beta Q_\beta$

\[
\delta e_\mu^a = i\tau^\alpha [(1 - \gamma_5)\gamma^a]_{\alpha\beta} \chi_\mu^\beta \quad \delta \omega_\mu = 0 \quad \delta a_\mu = \frac{i}{2}\tau^\alpha (\gamma^5)_\alpha^\beta \chi_\mu^\beta
\]

\[
\delta \chi_\mu^\alpha = 2D_\mu \tau^\alpha + 2\tau^\beta [(1 - \gamma_5)\gamma_\mu]_\beta^\alpha
\]  

(4.B5)

and for the $\eta$-fields

\[
\delta \eta^a = -2i\tau^\alpha [(1 - \gamma^5)\gamma^a]_{\alpha\beta} \eta_\beta \quad \delta \eta(J) = 0 \quad \delta \eta(I) = -i\tau^\alpha \eta_\alpha
\]

\[
\delta \eta^\alpha = -\frac{1}{2}\tau^\alpha \eta(J) + \tau^\beta [(1 - \gamma^5)\gamma^\alpha]_\beta^\alpha \eta_a
\]  

(4.B6)
The equations of motion of the model (4.24) are $F = 0$, or in components,

$$F^a(P) = de^a - e^a_b \omega^b - \frac{i}{4} \chi^a[(1 - \gamma_5)\gamma^a]_\alpha^\beta \chi_\beta = 0$$

$$F(J) = -d\omega = 0$$

$$F(I) = da + \frac{1}{2} \epsilon_{abc} e^a_b e^b - \frac{i}{8} \chi^a(\gamma^5)_\alpha^\beta \chi_\beta = 0$$

$$F^\alpha(Q) = \frac{1}{2} (D\chi^\beta - e^a[(1 - \gamma_5)\gamma_a]^\beta_\gamma \chi_\gamma) (\gamma^5)_\beta^\alpha = 0$$

and $(D\eta) \equiv d\eta + [A, \eta] = 0$, or in components,

$$(D\eta)_a(P) = d\eta_a - e^a_b \omega_b - e_{abc} \eta(J) + i \chi^a[(1 - \gamma_5)\gamma^a]_\alpha^\beta \eta_\beta = 0$$

$$(D\eta)(I) = d\eta(J) = 0$$

$$(D\eta)(I) = d\eta(I) + e^a e_a^b \eta_b + \frac{i}{2} \chi^\alpha \eta_\alpha = 0$$

$$(D\eta)_a(Q) = D\eta_a + \frac{1}{4} \chi_\alpha \eta(J) + e^a[(1 - \gamma_5)\gamma_a]_\alpha^\beta \eta_\beta + \frac{1}{2} \eta^a[(1 - \gamma_5)\gamma_a]_\alpha^\beta \chi_\beta = 0$$

We remark the systematic appearance of combinations we will denote $\tilde{\psi}^R, \tilde{\psi}^L = \frac{1}{2}(1 \mp \gamma^5)_\alpha^\beta \psi_\beta$. As in the previous example, it is possible to eliminate some of the fields by replacing part of these differential equations by relations between the fields. In the gauge fields, we can impose

$$\omega_\mu = -e_{\mu, \alpha} e^{\rho \sigma} \partial_\rho e_\sigma^a + \frac{i}{2} \chi^a \chi^R_\nu \chi^R_\beta$$

and in the Lagrange multiplier

$$\eta^a = \tilde{\epsilon}^{\mu \nu} e^a_\nu \left( \partial_\mu \eta(I) + \frac{i}{2} \chi^a \eta_\alpha \right)$$

$$\eta(J) = -\lambda = \text{constant}$$

The equations of motion provide the dynamics for the remaining unconstrained fields $e^a_\mu, \chi^a, \eta(I), \eta^a$ of the gauge theory (4.24) upon substituting the relations (4.29) into

$$R \left( e^a_\mu, \chi^a \right)(\gamma^\nu)_{\alpha}^\beta \chi^R = 0$$

$$D\chi^L - 2e^a \gamma^a_{\nu, \beta} \chi^R = 0$$

$$D\chi^R = 0$$

$$(\nabla_\mu \partial_\nu - g_{\mu \nu} \nabla_\rho \partial^\rho) \eta(I) + \lambda g_{\mu \nu} = \frac{3i}{2} (\delta^a_\mu \delta_\nu^\alpha + \delta^a_\nu \delta_\mu^\alpha - 2g_{\mu \nu} g^{\rho \sigma}) \chi^R_{\rho, \alpha} (\gamma_\sigma)_{\alpha}^\beta \eta^R_{\beta}$$

$$+ i \frac{4}{1 - g} g_{\mu \nu} \frac{1}{\sqrt{-g}} D\rho (\sqrt{-g} g^{\rho \sigma} \chi^R_\alpha) \eta^L_\alpha + i \frac{4}{1 - g} g_{\mu \nu} \frac{1}{\sqrt{-g}} D\rho (\sqrt{-g} g^{\rho \sigma} \chi^L_\alpha) \eta^L_\alpha$$

$$+ i \frac{4}{\sqrt{-g}} \lambda^R \chi^L_{\nu, \alpha} + i \frac{4}{\sqrt{-g}} \lambda^L \chi^R_{\nu, \alpha}$$

$$D\eta^L_\alpha - \frac{\lambda}{4} \chi^L_{\alpha} + 2e^a (\gamma_a)_\alpha^\beta \eta^R_\beta - \chi^R \gamma^\nu (\gamma^\nu)_\beta \eta^R_\beta \left[ \partial_\nu \eta(I) + \frac{i}{2} \chi^R \gamma^L \eta^L_\gamma + \frac{i}{2} \chi^L \gamma^R_\gamma \right] = 0$$

$$D\eta^R_\alpha - \frac{\lambda}{4} \chi^R_\alpha = 0$$
We do not consider the $a_{\mu}$ field for the same reason as in the previous model. These equations are symmetric under

$$
\delta e_{\mu}^a = 2i\tau^{R,\alpha}(\gamma^a)^{\alpha}_{\beta}\chi_{\mu,\beta}^R, \\
\delta \chi^{R,\alpha}_{\mu} = 2D_{\mu}\tau^{R,\alpha}\delta \chi^{L,\alpha}_{\mu} = 2D_{\mu}\tau^{L,\alpha} + 4\tau^{R,\beta}(\gamma_{\mu})^{\beta}_{\alpha} \\
\delta \eta(I) = -i\tau^{R,\alpha}\eta^L_{\alpha} - i\tau^{L,\alpha}\eta^R_{\alpha} \\
\delta \eta^{R,\alpha} = \frac{\lambda}{2}\tau^{R,\alpha}\delta \eta^{L,\alpha} = \frac{\lambda}{2}\tau^{L,\alpha} + 2\tau^{R,\beta}(\gamma^a)^{\beta}_{\alpha}\eta_{\alpha}
$$

(4.B12)

$\eta_{\alpha}$ being given in Eq. (4.B10). We recover Howe’s supergravity transformations for the right chirality sector [see Eq. (A.14) with $A = 0$], but the left chirality sector follows another type of transformations.

The bosonic solutions described in Ref. 9 are still among the solutions

$$
\chi^{R,\alpha}_{\mu} = 0 \quad \eta^L_{\alpha} = \text{constant} \quad \chi^{L,\alpha}_{\mu} = \frac{8}{\lambda}(\gamma_{\mu})^{\beta}_{\alpha}\eta^R_{\beta} = \text{constant} \\
g_{\mu\nu} = h_{\mu\nu} \quad (\partial_{\mu}\partial_{\nu} - h_{\mu\nu}\partial_{\rho}\partial^{\rho})\eta(I) + \lambda h_{\mu\nu} = 0
$$

(4.B13)

and one can verify that the action (4.B4) reduces to the one discussed in Ref. 9 when the gravitino field $\chi_{\mu}^{\alpha}$ is evaluated as above. This subset of solutions is invariant under (4.B12) with $\tau^R_{\alpha} = 0$ and $\tau^L_{\alpha} = \text{constant}$. The condition $\tau^R_{\alpha} = 0$ is necessary to insure consistency of the third equation in (4.B13). This means that the right chirality transformations identified as Howe supergravity transformations are broken by the solutions (4.B13). Nevertheless, the remaining symmetries is still one of (4.B13) and allows us to construct the corresponding conserved charge.

Conclusions.

Four models for two dimensional supergravity theories have been discussed here. The $N = 1$ supersymmetric extension of JT-model given by Chamseddine, and three models that are $N = 1$ supersymmetric extension of the string inspired dilaton gravity model. We emphasized that the first model is topological and a gauge invariant $\int\langle\eta, F\rangle$-theory based on the graded de Sitter algebra OSP(1,1|1), the local supersymmetries being reproduced by some gauge transformations. We showed that the $\int\langle\eta, F\rangle$ action is indeed the superfield action written by Chamseddine in Howe’s superfield formalism for supergravity theories. The second model we analyzed is the Park and Strominger’s proposal. They write a $N = 1$ supersymmetric extension following the lines of Howe. We arrived at the conclusion that it is not a $\int\langle\eta, F\rangle$-theory and has non-vanishing curvature. Although, it does not mean that it is not an other type of topological theory, we believe that it represents a challenge to write it as such. Then, we turn to the strategy of building two supersymmetric extensions of string inspired dilaton gravity models from topological and gauge invariant $\int\langle\eta, F\rangle$ actions based on graded extension of the extended Poincaré algebras. The first of the two topological models was presented first by Rivelles. The algebra he uses to define the GEP-model was inspired by a contraction of the graded de Sitter algebra. However, it is not a contraction.
but rather an algebra that looks like it and closes under the Jacobi identity. The second model is the minimal graded extended Poincaré model. There, only one charge $Q$ is added. The connection to supergravity theories is made by comparing local supersymmetries given by Howe’s formalism for two-dimensional supergravity theories with some gauge transformations of the proposed topological models. Although, the first model present full supersymmetry, the connection for the second model is made possible only for half the supersymmetry; we have a chiral supersymmetry. We conclude that they are then supergravity theories that extend the string inspired dilaton gravity because upon setting the gravitino field to zero, we recover the bosonic model. In the first model, we find field configurations with vanishing gravitino that remain supersymmetric. The second model offers a similar result, however, in this case, the configurations with vanishing right gravitino break half the gauge symmetry. In both models, a conserved charge can be constructed for the bosonic solutions of Ref. 9. Its consequence in the physics of the system deserve another study.

Appendix

We use the following convention for the inner product presented in section III and IV

$$\langle \xi^A Q_A, \eta^B Q_B \rangle = (-)^{\text{deg}(Q_A) \text{deg}(Q_B)} \xi^A \eta^B \langle Q_A, Q_B \rangle$$  \hspace{1cm} (A.1)

$[Q_A$ indicates any generator of the algebra and $\text{deg}(Q_A) = 0$ if $Q_A$ is an even generator and $\text{deg}(Q_A) = 1$ if $Q_A$ is an odd generator$].$

The rest of the appendix deals with various other conventions. The indices are denoted in the paper like this: $a, b$ are tangent space indices, $\mu, \nu$ are spacetime indices and $\alpha, \beta$ are spinor indices. The bosonic metric is

$$h_{ab} = \text{diag}(-1, 1) \quad \epsilon_{ab} = -\epsilon_{ba} \quad \epsilon_{01} = 1$$  \hspace{1cm} (A.2)

and the contraction of two anti-symmetric tensors is given by

$$\epsilon_{ab} \epsilon^{bc} = \delta_a^c$$  \hspace{1cm} (A.3)

The fermionic metric is

$$\epsilon_{\alpha\beta} = \epsilon^{\alpha\beta} \quad \epsilon_{12} = 1 = -\epsilon_{21} \quad \epsilon_{11} = \epsilon_{22} = 0$$  \hspace{1cm} (A.4)

and contractions are given by

$$\epsilon_{\alpha\beta} \epsilon^{\beta\gamma} = -\delta_\alpha^\gamma \quad \epsilon_{\alpha\beta} \epsilon^{\alpha\beta} = 2$$  \hspace{1cm} (A.5)

The spin indices are raised and lowered by the fermionic metric

$$\psi^\alpha = \epsilon^{\alpha\beta} \psi_\beta \quad \psi_\alpha = \psi^\beta \epsilon_{\beta\alpha}$$  \hspace{1cm} (A.6)
and the product of two Grassmann variables reduces to
\[ \theta^\alpha \theta^\beta = \frac{1}{2} \epsilon^{\alpha\beta} \theta^\gamma \theta^\gamma \]  \hspace{1cm} (A.7)

We choose our \( \gamma \)-matrices real and such that they satisfy
\[ (\gamma^a)^\alpha{}_{\beta} (\gamma^b)^\beta{}_{\gamma} = \eta^{ab} \delta^\alpha{}_{\gamma} - \epsilon^{ab} (\gamma^5)^\alpha{}_{\alpha} \]  \hspace{1cm} (A.8)

With \( \gamma^5 = \gamma^0 \gamma^1 \), we deduce the useful relations
\[ [\gamma^a, \gamma^b] = -2\epsilon^{ab} \gamma^5 \quad [\gamma^a, \gamma^5] = 2\epsilon^{ab} \gamma^b \quad \gamma^a \gamma^5 = \epsilon^{ab} \gamma_b \]  \hspace{1cm} (A.9)

An explicit representation is
\[
\begin{align*}
(\gamma^0)^\alpha{}_{\beta} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
(\gamma^1)^\alpha{}_{\beta} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
(\gamma^5)^\alpha{}_{\beta} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{align*}
\]  \hspace{1cm} (A.10)

The superfield formalism is given in the work of Howe. Here are the needed formulae. The scalar superfield is defined by
\[ \Phi = \phi + i \theta^\alpha \Lambda^\alpha + \frac{i}{2} \theta^\alpha \theta^\beta F \]  \hspace{1cm} (A.11)

where the \( \phi \)-field is a scalar field, \( \Lambda^\alpha \) the fermion field associated to the scalar field, and \( F \) an auxiliary field to be determined by the model under investigation. The supersymmetric covariant derivative, \( D_\alpha = \partial_\alpha + i \theta^\beta (\gamma^a)^\beta{}_{\alpha} \partial_a \), is used in Eq. (2.6). The superfields associated to the graviton multiplet are the supervolume element \( E \), and the supercurvature \( S \) given by
\[
\begin{align*}
E &= e [1 + i \frac{2}{\epsilon} \theta^\alpha (\gamma^\mu)^\alpha{}_{\beta} \chi_{\mu,\beta} + \theta^\alpha \theta_\alpha \left( \frac{i}{4} A + \frac{1}{8} \epsilon^{\mu\nu} \chi^\alpha{}_{\alpha} (\gamma^5)^\beta{}_{\alpha} \chi_{\nu,\beta} \right) ] \\
S &= A + \theta^\alpha \Psi^\alpha + \frac{i}{2} \theta^\alpha \theta^\gamma C \\
C &= -R - \frac{1}{2} \chi^\alpha{}_{\alpha} (\gamma^\mu)^\alpha{}_{\beta} \Psi^\beta + \frac{i}{4} \epsilon^{\mu\nu} \chi^\alpha{}_{\alpha} (\gamma^5)^\beta{}_{\alpha} \chi_{\nu,\beta} A - \frac{1}{2} A^2 \\
\Psi^\alpha &= -2i \epsilon^{\mu\nu} (\gamma^5)^\alpha{}_{\beta} D_\mu \chi_{\nu,\beta} - \frac{i}{2} (\gamma^\mu)^\alpha{}_{\beta} \chi_{\mu,\beta} A
\end{align*}
\]  \hspace{1cm} (A.12)

where the superfields \( E \) and \( S \) are expressed in terms of the Zweibein \( e^a_\mu \), the gravitino \( \chi^\alpha{}_{\alpha} \) and an auxiliary field \( A \). From these fields the spacetime covariant derivative, the spin-connection and the curvature are constructed following Howe’s work. We list these relations here
\[
\begin{align*}
g_{\mu\nu} &= e^a_\mu e^b_\nu h_{ab}, \\
\epsilon &= \frac{1}{2} \epsilon^{\mu\nu} e^a_\mu e^b_\nu \epsilon_{ab} = \sqrt{-g} \\
\gamma^\mu &= e^a_\mu \gamma^a, \\
\gamma^\mu &= E^a_\mu \gamma^a = -\frac{1}{\epsilon} \epsilon^{\mu\nu} \epsilon_{ab} e^b_\nu \gamma^a \\
\omega^\mu &= -e_{\mu, a} \epsilon^{\nu\rho} \partial_\nu \epsilon^a_\rho + \frac{i}{2} \chi^\alpha{}_{\alpha} (\gamma^5 \gamma^\nu)^\beta{}_{\alpha} \chi_{\nu,\beta} \\
R &= -2 \frac{1}{\epsilon} \epsilon^{\mu\nu} \partial_\mu \omega_\nu \\
D_\mu \chi_{\nu,\alpha} &= \partial_\mu \chi_{\nu,\alpha} - \frac{1}{2} \omega^\mu \gamma^a_{\alpha} \chi_{\nu,\beta}
\end{align*}
\]  \hspace{1cm} (A.13)
Note the sign convention for the scalar curvature $R$. Howe deduces also the supergravity transformations after having imposed some kinematics constraints on the supertorsion
\[ \delta e^a_\mu = i \tau^\alpha (\gamma^a)_\alpha^{\beta \chi_{\mu,\beta}} \quad \delta \omega_\mu = - \frac{i}{2} A \tau^\alpha (\gamma^5)_\alpha^{\beta \chi_{\mu,\beta}} \quad \delta \chi^\alpha_\mu = 2 D_\mu \tau^\alpha - \frac{1}{2} \tau^\beta (\gamma^\mu)_\beta^{\alpha} A \]
\[ \delta \phi = 0 \]
\[ \delta \Lambda_\alpha = \tau^\beta (\gamma^\mu)_\beta^{\alpha} \left( \partial_\mu \phi + \frac{i}{2} \chi^\mu_\chi \Lambda_\gamma \right) + \tau_\alpha F \]  
(A.14)

Of some interest are also the antisymmetric tensors
\[ \tilde{\epsilon}^{\mu \nu} = \frac{1}{e} \epsilon^{\mu \nu} = e^{ab} E^\mu_a E^\nu_b \quad \tilde{\epsilon}^{\mu \nu} = e e^{\mu \nu} = e e^{ab} e^{\mu \nu} \quad \tilde{\epsilon}^{\mu \nu} \tilde{\epsilon}_{\nu \rho} = \delta_\rho^\mu \]  
(A.15)

and the action of a Weyl transformation on the scalar curvature
\[ g_{\mu \nu} = e^{2\phi} g'_{\mu \nu} \quad \rightarrow \quad \sqrt{-g} R = \sqrt{-g'} R' - 2 \partial_\mu \left( \sqrt{-g'} g^{\mu \nu} \partial_\nu \phi \right) \]  
(A.16)
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