General Theory of Overmeasurement
of Discrete Quantum Observables
and Application to Simultaneous Measurement

Fedor Herbut

Abstract. A complete theory of overmeasurement by measuring refinements of observables is presented. It encompasses a wider set of functions of observables (coarsenings). Thus the theory has a broad potential application. It is applied to a thorough investigation of simultaneous measurements. In particular, the set of all simultaneous measurements for a given pair of compatible observables is determined.

Keywords Measurement. Functions of observables. Compatible observables.

1 Introduction

It is a textbook claim that two compatible discrete observables, i.e., ones of which the Hermitian operators representing them commute and have no continuous parts in their spectra, can be simultaneously measured. And this is done, so it is further claimed, by finding a common eigenbasis of the two operators and by measuring in which of the basis states the system is. The eigenvalues of the two operators that correspond to the measured basis state are then the simultaneous results of the measurement. This is a superficial and incomplete but typical presentation of simultaneous measurement.

The two discrete observables are overmeasured by a common overmeasurement, though this term is usually not used. In this article a complete theory of overmeasurement is expounded together with a complete theory of simultaneous measurements as an application.

F. Herbut
Serbian Academy of Sciences and Arts, Knez Mihajlova 35, 11000 Belgrade, Serbia
e-mail: fedorh@sanu.ac.rs
If one defines general exact measurement, following [1], by the *calibration condition* (see relation (4) below), then overmeasurement is the *most general exact measurement*. The opposite of overmeasurement is undermeasurement. Since the points of a continuous spectrum cannot be measured, they must be undermeasured. Von Neumann in his famous book [2] (cf chapter III, section 3. p. 220 there) explains this, though he does not use the term "undermeasurement". (His term for undermeasurement is "measurement with only limited accuracy".)

It is hoped that the complete theory of overmeasurement that is to be presented will not only give a deeper conceptual insight in measurement theory, but also find new applications.

The investigation is restricted to *discrete observables* in this article. They will always be given in their *unique spectral form* (unless otherwise stated), which means, by definition, that there is no repetition in the eigenvalues \{o_k : \forall k\} that are displayed in the spectral form:

\[ O_A = \sum_k o_k E^k_A, \quad \text{(1a)} \]

so that \{E^k_A : \forall k\} are the corresponding eigen-projectors. The index A denotes the measured subsystem. The spectral form is accompanied by the spectral (orthogonal projector) decomposition of the identity operator \( I_A \) (also called the "completeness relation")

\[ \sum_k E^k_A = I_A. \quad \text{(1b)} \]

When \( O_A \) is measured in a suitable interaction with a measuring instrument B, then an initial or ready-to-measure state \( |\phi^i_B\rangle \) together with a so-called *pointer observable*

\[ P_B = \sum_k p_k F^k_B \]

are given. The eigen-projectors \{F^k_B : \forall k\} are metaphorically called —pointer positions". Also they satisfy the completeness relation \( \sum_k F^k_B = I_B \). Notice the co-indexing in (2) and (1a) based on a one-to-one relation between the possible measurement results \{o_k : \forall k\} and all possible pointer positions.

The suitable measurement interaction is assumed to be incorporated in a unitary operator \( U_{AB} \), which maps the composite initial state to the final state \( |\Phi^f_{AB}\rangle \)

\[ |\Phi^f_{AB}\rangle = U_{AB} \left( |\phi^i_A\rangle |\phi^i_B\rangle \right), \quad \text{(3)} \]

where \( |\phi^i_A\rangle \) is an arbitrary initial state of the object subsystem.

This is the basic formalism of unitary measurement theory, or premeasurement theory or measurement theory short of collapse [1], [3], [4]. The general unitary (also called "exact") measurements of discrete observables are defined by the *calibration condition*, which requires that if the object has a sharp value \( o_k \) of the measured observable in the initial state, then the final composite state has the corresponding sharp pointer position \( F^k_B \):

\[ E^k_A |\phi^i_A\rangle = |\phi^i_A\rangle \Rightarrow F^k_B |\Phi^f_{AB}\rangle = |\Phi^f_{AB}\rangle. \quad \text{(4)} \]
(Note that the mutually equivalent eigenvalue equations \( E_{A}^{k} |\phi_{A}^{i}\rangle = |\phi_{A}^{i}\rangle \) and \( O_{A} |\phi_{A}^{i}\rangle = o_{A} |\phi_{A}^{i}\rangle \) are the standard way to express certainty in quantum mechanics, and "⇒" stands for logical implication.)

In this study we will not treat the important special case of nondemolition (synonyms: repeatable, predictive, first-kind) measurements, nor the much used even more special special case of ideal measurements [5].

It is known from von Neumann’s book [2] that an observable \( O_{A} \) given by (1a) can be measured by measuring a complete observable, i.e., one with no degeneracy in any of its eigenvalues,

\[
O_{A}^{r} = \sum_{k} \sum_{n_{k}} o_{k,n_{k}}^{r} |k,n_{k}\rangle \langle k,n_{k}|, (5a)
\]

\[
(k,n_{k}) \neq (k',n'_{k'}) \Rightarrow o_{k,n_{k}}^{r} \neq o_{k',n'_{k'}}^{r}, (5b)
\]

which is a so-called refinement of \( O_{A} \), i.e., for which

\[
\forall k : \sum_{n_{k}} |k,n_{k}\rangle \langle k,n_{k}| = E_{A}^{k} (5c)
\]

is valid.

One is dealing with overmeasurement of \( O_{A} \), where actually \( O_{A}^{r} \) is measured, and, if e.g., \( o_{k,n_{k}}^{r} \) is the result of measurement, then by quantum-logical implication, due to \( |k,n_{k}\rangle \langle k,n_{k}| \leq E_{A}^{k} \) (symbolic for \( |k,n_{k}\rangle \langle k,n_{k}| E_{A}^{k} = |k,n_{k}\rangle \langle k,n_{k}| \)), also the pointer position \( E_{A}^{k} \) has occurred or the result \( o_{k} \) of \( O_{A} \) \((= \sum_{k'} o_{k'} E_{A}^{k'}) \) is obtained.

### 2 General Theory of Overmeasurement

The unitary quantum formalism is restricted to unitary evolutions, and, as well known, it cannot in general derive the (unknown) final state \((|\Psi_{AB}\rangle)^{f}_{k} \) of complete measurement, which includes collapse to the definite result \( p_{k} \) or, equivalently, the occurrence of the pointer position \( F_{B}^{k} \). But the very fact that it contains the information of a definite \( o_{k} \) result, i.e., due to \((|\Phi_{AB}\rangle)^{f}_{k} F_{B}^{k} (|\Phi_{AB}\rangle)^{f}_{k} = 1 \), one must have equivalently,

\[
(|\Phi_{AB}\rangle)^{f}_{k} F_{B}^{k} (|\Phi_{AB}\rangle)^{f}_{k}.
\]

The final state (6) of complete measurement might even be mixed. For simplicity we restrict it to a pure state.

#### 2.1 Overmeasurement - The formal part

Overmeasurement is usually defined in a more narrow sense by any single-valued function \( f(\ldots) \) on the real axis. It determines an observable \( \bar{O}_{A} \) that is the corresponding
function of the given observable $O_A \left( = \sum_k o_k E^k_A \right)$:

$$\bar{O}_A \equiv f(O_A) \equiv \sum_k f(o_k) E^k_A = \sum_l \bar{o}_l E^l_A, \quad l \neq l' \Rightarrow \bar{o}_l \neq \bar{o}_{l'}.$$  \hfill (7a)

Note that the first spectral form in (7a), unlike the second one, is, in general, non-unique.

In the context of overmeasurement, $\bar{O}_A$ is called a coarsening or a coarser observable, and $O_A$ is said to be a refinement or a finer observable. (These terms are meant in the improper sense. For instance, "finer" is actually "properly finer" or equal.)

The indices $l$ are defined so as to make the spectral form of the coarser observable $\bar{O}_A$ unique. This implies that the index set $\{\forall k\}$ in the unique spectral form of the finer observable $O$ is broken up into equivalence classes: $\{\forall k\} = \sum_l C_l$. In other words, it can be viewed as the union of non-intersecting subsets (classes) $C_l$. Belonging to the same class $C_l$ is defined as follows.

$$\forall l : k, k' \in C_l \iff f(o_k) = f(o_{k'}) = \bar{o}_l. \quad (7b)$$

Thus $f(\ldots)$, primarily given as a function on the real axis, determines a function, we denote it by the same symbol $f$, mapping the index set $\{\forall k\}$ onto the new index set $\{\forall l\}$. Note that the inverse multivalued function $f^{-1}$ takes the latter index set onto the former and its images are precisely the mentioned equivalence classes:

$$\forall l : k, k' \in C_l \text{ if and only if } k, k' \in f^{-1}(l). \quad (7c)$$

It is sometimes useful to define overmeasurement in a broader sense by an (arbitrary) single-valued map $f$ taking the index set $\{\forall k\}$ of the finer observable $O$ onto the index set $\{\forall l\}$ of the coarser observable $\bar{O}$. But always the essential thing is the relation

$$\forall l : E^l_A = \sum_{k, f(k) = l} E^k_A. \quad (8a)$$

Relation (8a) follows from (7a) if one has the narrower definition, and it is the most important part of the definition of overmeasurement in the broader definition. In the latter case the eigenvalues $\{\bar{o}_l\}$ of the coarser observable need not be related to those of the finer observable.

As a consequence of (8a), the orthogonality of the projectors $\{E^k_A : \forall k\}$ leads to

$$E^l_A E^k_A = 0 \quad \text{if} \quad f(k) \neq l. \quad (8b)$$

Parallelly with the unique spectral form of the coarser observable, also the unique spectral form

$$\bar{P}_B = \sum_l \bar{p}_l F^l_B, \quad (9)$$
of the pointer observable of the coarsening is going to play an important role. Note that the eigenvalues \( \{\bar{p}_l : \forall l\} \) can be arbitrary distinct real numbers. Further, by definition

\[ \forall l : \quad F_B^l = \sum_{k,f(k)=l} F_B^k, \quad (10a) \]

where the function \( f : \{\forall k\} \to \{\forall l\} \) is the one that determines the coarsening \( O \to \bar{O} \). Relations (10a) are symmetrical to (8a).

One has also

\[ F_B^l F_B^k = F_B^k \quad (\Leftrightarrow F_B^l \geq F_B^k) \quad \text{if} \quad f(k) = l \quad (10b) \]

Naturally, the eigen-projectors of the finer and of the coarser observable and the eigen-projectors of the corresponding pointer observables satisfy symmetrical relations. But we have written down only those that we shall make use of.

### 2.2 Overmeasurement - The physical part

Now we make the first physical step showing that any unitary measurement of an observable \( O_A \) is by this very fact a unitary measurement also of any coarser observable \( \bar{O}_A \) related to the finer observable \( O_A \) by a given map of the index set of the latter onto that of the coarser observable. In particular, we shall demonstrate that the calibration condition, which is by definition valid for the measurement of the finer observable, implies that also the calibration condition for the coarser observable is satisfied.

We assume that the initial state \( |\phi\rangle_A^i \) of the object has a sharp value \( \bar{q}_l \) of the coarser observable:

\[ |\phi\rangle_A^i = E_A^\bar{l} |\phi\rangle_A^i. \quad (11) \]

Utilizing the completeness relation \( I_A = \sum_k E_A^k \) in the decomposition \( |\phi\rangle_A^i = \sum_k E_A^k |\phi\rangle_A^i \) and (11), \( |\Phi\rangle_{AB}^f \), which is defined by (3), becomes equal to

\[ \sum_k \|E_A^k \ |\phi\rangle_A^i\| U_{AB} \left[ \left( E_A^k E_A^l |\phi\rangle_A^i / \|E_A^k \ |\phi\rangle_A^i\| \right) |\phi\rangle_B^i \right]. \quad (12) \]

Since the sum can be broken up \( \sum_k \cdots = \sum_{k,f(k)=\bar{l}} \cdots + \sum_{k,f(k)=\bar{l}} \cdots \) (8b) makes the first sum zero. Hence, making use of the assumption that the measurement of the finer observable satisfies the calibration condition in the form of inserting \( F_B^k \), and using (11) again to suppress \( E_A^k \), \( |\Phi\rangle_{AB}^f \) is further equal to

\[ \sum_{k,f(k)=\bar{l}} \|E_A^k \ |\phi\rangle_A^i\| F_B^k U_{AB} \left[ \left( E_A^k E_A^l |\phi\rangle_A^i / \|E_A^k \ |\phi\rangle_A^i\| \right) |\phi\rangle_B^i \right]. \quad (13) \]

Finally, taking into account (10b), we obtain

\[ F_B^\bar{l} |\Phi\rangle_{AB}^f = |\Phi\rangle_{AB}^f, \quad (14) \]
which expresses certainty. This proves the claim. Thus, in view of (11) and (14), the calibration condition is valid for the overmeasurement of the coarser observable.

Naturally, due to the usual convention, if \( E^k_A | \phi_i^j_A \rangle = 0 \) in some term, then the expression that follows in the same term need not be defined; the term is by definition zero.

Now we can make the second physical step concerning the result of complete measurement. The claim is that if the complete measurement of the finer observable produces the result \( o_k \), then this same process of measurement gives the result \( \bar{o}_{f(k)} \) for the coarser observable. The proof is an immediate consequence of (1ob). Namely, putting \( l \equiv f(k) \), one obtains

\[
F^l_B \{|\Phi\rangle^f_{AB}\}^k = F^l_B \left( F^k_A \{|\Phi\rangle^f_{AB}\}^k \right) = \left( F^l_B F^k_A \right) \{|\Phi\rangle^f_{AB}\}^k = \\
F^k_A \{|\Phi\rangle^f_{AB}\}^k = \{|\Phi\rangle^f_{AB}\}^k.
\]

We have thus proved that the final state \( \{|\Phi\rangle^f_{AB}\}^k \) of complete measurement has the definite result \( \bar{o}_{f(k)} \) of the coarser observable. If this final state is mixed, the proof is analogous, but it requires certain generalizations of the formalism. Hence it is omitted for simplicity.

If a coarser observable \( \bar{O}_A \) in the proper sense is given first, there exist various refinements; there can even be refinements of refinements. And one has transitivity: a refinement of a refinement is a refinement of the coarsest observable \( \bar{O}_A \). Therefore one can speak of degrees of overmeasurement of the given observable \( \bar{O}_A \).

The two extreme degrees are: minimal measurement, when there is actually no refinement, and maximal overmeasurement, when the measured finer observable \( O_A \) is a complete observable, i.e., one all eigenvalues of which are non-degenerate (cf the end of the Introduction).

The best known example of minimal measurement is ideal measurement, also called Lüders or von Neumann-Lüders measurement (cf section 7 in [3]).

Minimal measurement in a general sense was introduced by the present author [6]. Maximal overmeasurement is also called measurement in a given basis (having in mind the eigen-basis of the complete observable; its eigenvalues anyway play no role in measurement theory).

One should note that, if minimal measurement is included in overmeasurement (as the trivial, improper extreme), then every measurement is an overmeasurement.

### 3 Simultaneous measurement

This section is devoted to an illustration of application of overmeasurement to a topic that is well known but not well proved and not well understood in its fine details.
To begin with, let us define that by simultaneous measurement of two observables
\[ O'_A \left( = \sum_m o_m E^m_A \right) \] and \[ O''_A \left( = \sum_n o_n E^n_A \right) \] is understood measurement of one observable \[ O_A \left( = \sum_k o_k E_k^A \right) \] that is so chosen that any result \( o_k \) implies (by quantum-logical implication) a result \( o_{m(k)} \) of \( O'_A \) and simultaneously a result \( o_{n(k)} \) of \( O''_A \). Besides, each possible result \( o_m \) of \( O'_A \) and \( o_n \) of \( O''_A \) must be thus obtainable for some initial state \( |\phi_i^A\rangle \).

### 3.1 Common overmeasurement and compatibility

The very definition of simultaneous measurement implies that, by necessity, there must exist two functions \( f' \) and \( f'' \) mapping the set of all indices \( \{\forall k\} \) onto the sets of all indices \( \{\forall m\} \) and \( \{\forall n\} \) respectively so that using the notation
\[
\forall k : \quad f'(k) = m(k), \quad f''(k) = n(k),
\]
one has
\[
\forall k : \quad E^k_A \leq E^{m(k)}_A \left( E^k_A E^{m(k)}_A = E^k_A \right) \quad \text{and} \quad E^k_A \leq E^{n(k)}_A \left( E^k_A E^{n(k)}_A = E^k_A \right). \tag{16b}
\]

It is seen that \( O_A \) must be a common refinement of \( O'_A \) and \( O''_A \) and hence the measurement of \( O_A \) a common overmeasurement of the latter two observable. Thus, necessity of the common refinement claim is proved.

As to proving sufficiency of the stated claim, it clearly follows from the definition of simultaneous measurement that any common overmeasurement will achieve it. \(\square\)

Furthermore, relations (16a,b) imply
\[
\forall m : \quad E^m_A = \sum_{k \in (f')^{-1}(m)} E^k_A
\]
and
\[
\forall n : \quad E^n_A = \sum_{k \in (f'')^{-1}(n)} E^k_A, \tag{17}
\]
which, in turn, has
\[
\forall m, n : \quad [E^m_A, E^n_A] = 0 \tag{18}
\]
as its consequence.

Two observables that satisfy the commutativity condition (18) are said to be compatible. In this way it is proved that for simultaneous measurability compatibility is necessary. We now prove that it is also sufficient.

Assuming the validity of (18), each product \( E^m_A E^n_A \) is a projector, and
\[
(E^m_A E^n_A)(E^{m'}_A E^{n'}_A) = (E^m_A E^{m'}_A)(E^n_A E^{n'}_A) = \]

7
\[ \delta_{m,m} \delta_{n,n}' E^m_A E^n_A, \]
i.e., any two projectors in the set \( \{ E^m_A E^n_A : \forall m, \forall n \} \) are orthogonal. Finally, multiplying the two completeness relations \( \sum_m E^m_A = O_A \) and \( \sum_n E^n_A = I_n \), one obtains the completeness relation \( \sum_m \sum_n E^m_A E^n_A = I_A \).

Let us enumerate by \( k \) all non-zero distinct projectors

\[ E^k_A \equiv E^m_A E^n_A \not= 0, \quad (19a) \]

and take an arbitrary set \( \{ o_k : \forall k \} \) of distinct real numbers. Then, it is obvious from the arguments above, that

\[ O_A \equiv \sum_k o_k E^k_A \quad (19b) \]
is a common refinement of \( O'_A \) and \( O''_A \). Hence its measurement is a common over-measurement of these two given observables.

If two observables \( O'_A \) and \( O''_A \) are bounded, then they are compatible if and only if they commute \( [O'_A, O''_A] = 0 \). Also this claim is, unlike its proof, well known. (For the reader’s convenience we prove it in Appendix B.)

Incidentally, it is known in linear analysis, or rather from the theory of at most countably infinite complex Hilbert spaces \[2\], that if the spectrum \( \{ o_m : \forall m \} \) of a given observable \( O'_A \equiv \sum_m o_m E^m_A \) is known, then a useful necessary and sufficient condition for boundedness of \( O'_A \) is that the spectrum belongs to a finite closed interval:

\[ \{ o_m : \forall m \} \subset [a, b], \quad a < b, \quad a, b \text{ real numbers.} \quad (20) \]

### 3.2 The set of all simultaneous measurements

In this subsection we prove the following claim. Let two compatible observables \( O_A \left( = \sum_m o_m E^m_A \right) \) and \( O'_A \left( = \sum_n o_n E^n_A \right) \) (cf definition in relation (18)) be given, and let us understand the concept of “refinement” in the improper sense (cf last passage in section 2). Then an observable \( \hat{O}_A \left( = \sum_l o_l G^l_A \right) \) is their common refinement if and only if it is a refinement of the observable \( O'^*_A \left( = \sum_k o_k E^k_A \right) \) defined by relations (19a) and (19b). The latter observable is thus the maximal common refinement of the given two compatible observables \( O_A \) and \( O'_A \).

Since any refinement of a refinement is a refinement, also any refinement of a common refinement is a common refinement. Thus sufficiency easily follows.

To prove necessity, we assume that an observable \( \hat{O}_A \left( = \sum_l o_l G^l_A \right) \) is a common refinement of the given two observables \( O_A \) and \( O'_A \). This implies that there are two
surjections (onto maps)\[ \bar{f}^l : \{\forall l\} \rightarrow \{\forall m\}, \quad \bar{f}''' : \{\forall l\} \rightarrow \{\forall n\} \] such that
\[ \forall l : G_A^l \leq E_A^{m \equiv \bar{f}^l(l)}, \quad G_A^n \leq E_A^{n \equiv \bar{f}'''(l)}. \tag{21} \]

Note that \[ \forall l : E_A^{m \equiv \bar{f}^l(l)}E_A^{n \equiv \bar{f}'''(l)} \neq 0 \] because \[ G_A^l \] is a non-zero common lower bound of the two factors. Hence we can define an injection (into map) of the index set \( \{\forall l\} \) into the index set \( \{\forall k\} \) :
\[ f \equiv \bar{f}^l, \quad f''' : \forall l : k(l) \equiv f(l) \equiv [m \equiv \bar{f}^l(l)], [n \equiv \bar{f}'''(l)]. \tag{22} \]

In section 2 we have seen that overmeasurement is based on measuring a refinement. Relation (22) would prove \( \bar{O}_A \) to be a refinement of \( O_A^M \) if it were a surjection of \( \{\forall l\} \) onto \( \{\forall k\} \).

In Appendix A it is shown that \[ \sum_l G_A^l \leq \sum_k E_A^k \] (cf relation (A.5)). Since the observable \( \bar{O}_A \) has its completeness relation \[ \sum_l G_A^l = I_A, \] we have \[ I_A \leq \sum_k E_A^k. \] Since \( I_A \) is an upper bound of all projectors, we have \[ I_A \leq \sum_k E_A^k \leq I_A \] implying \[ \sum_k E_A^k = I_A. \] Hence, after all, we are dealing with a surjection and the necessity of the claim \( \bar{O}_A \) being a refinement of \( O_A^M \) is proved. \( \square \)

Incidentally, the products \( E_A^m E_A^n \) outside the image \( \bar{f}(\{\forall l\}) \) in \( \{\forall m,n\} \) must be all zero on account of the orthogonality of the eigen-projectors.

Every complete observable \( O_A^C \) (cf (5a-c)) that is a refinement of the maximal common refinement \( O_A^M \) given by (19a) and (19b) for two given compatible observables is a local minimum in the set of all common refinements. By definition this means that \( O_A^C \) has no refinement. This is in contrast with \( O_A^M \), which is a global maximum.

### 3.3 Corollaries

**COROLLARY 1** Let \( \{O_A^q = \sum_n o_{n,q} E_A^{n,q} : q = 1,2,\ldots,Q\} \) be an arbitrary set of \( Q \) (a natural number) pairwise compatible discrete observables in their unique spectral forms. The maximal common refinement \( O_A^M \) is defined in its unique spectral form as follows.
\[ O_A^M \equiv \sum_{q_1} \sum_{q_2} \cdots \sum_{q_Q} o_{n_1\ldots n_Q} \prod_{q \in Q} E_A^{n,q}, \tag{23} \]

where it is understood that all terms in which the projectors multiply into zero are omitted and all eigenvalues are arbitrary but distinct.

Simultaneous measurement of all observables from the set is performed if and only if \( O_A^M \) or any of its refinements is measured.
PROOF For \( Q = 2 \) the claim has been proved in the preceding two subsections. Let us assume that it is valid for \( R \) observables, where \( R \) is a natural number. Then we know, again from the preceding two subsections, that for \( R + 1 \) observables the claim of Corollary 1 is valid. Hence, by total induction we conclude that the claim is valid for any natural number \( Q \). \( \square \)

**COROLLARY 2** Let \( O_A \left( = \sum_k o_k E_A^k \right) \) be any discrete observable given in its unique spectral form. Further, let

\[
\{ \forall k \} = \sum_l C_l
\]

be any breaking up the index set into classes, i.e., writing it as the union of non-intersecting subsets \( C_l \). Then, defining

\[
\forall l: \quad E_A^l = \sum_{k \in C_l} E_A^k
\]

any observable

\[
\hat{O}_A = \sum_l o_l E_A^l
\]

with arbitrary but distinct eigenvalues is a coarsening of \( O_A \) and any measurement of \( O_A \) is, at the same time, also a measurement, or rather an overmeasurement of the latter coarsened observable.

No careful reader of section 2 will need proof of Corollary 2.

4 **Summing Up**

The investigation in this article began with von Neumann’s treatment of the measurement of any discrete observable via a suitably chosen complete one (cf relations (5a-c)). It was pointed out that the latter observable is a refinement, and its measurement is overmeasurement of the initially given observable.

Then a general and detailed theory of overmeasurement was presented in the hope that it will find applications.

Next, the study turned to simultaneous measurement, as to an important application of the concept of a refinement of an observable and of overmeasurement as a procedure. It turned out that simultaneous measurement is the same thing as common overmeasurement. To illustrate the power of overmeasurement theory, some fine points of simultaneous measurement, especially finding the set of all simultaneous measurements for a given pair of observables, have been worked out.

**Appendix A: Some helpful projector relations**

We assume that it is known that the set of all projectors in an at most countably-infinite
dimensional complex Hilbert space (state space of a quantum system) is a partially ordered set with the quantum-logical implication \( E \leq F \) \( \equiv EF = E \). Besides it is a complete lattice, i.e., each non-empty subset has both a greatest lower bound (glb) and a least upper bound (lub). We now prove algebraically a few (more or less well known) claims that we make use of in subsection 3B.

If the reader knows that there exists a natural isomorphism between the partially ordered set of all projectors and that of all subspaces of the state space, then he may find it easier to supply the proofs in terms of subspaces. (This isomorphism maps a projector \( E \) into its range \( \mathcal{R}(E) \). The inverse of this isomorphism takes any subspace \( S \) into the projector that makes \( S \) its range.)

**Proof** of the claim

\[ EF = FE \Rightarrow EF = \text{glb}(E,F). \]  

Let \( G \) be any common lower bound of \( E \) and \( F \) : \( GE = GF = G \). Then \( G(EF) = GF = G \). Thus, \( G \) is a lower bound also of \( EF \) as claimed.

**Proof** of the claim that if \( \{E_l : l = 1, 2, \ldots, L\} \) , where \( L \) may even be the power of a countably infinite set, is a set of pairwise orthogonal projectors, then

\[ S \equiv \sum_{l=1}^{L} E_l = \text{lub}\{E_l : \forall l\}. \]  

The projector \( S \) is a common upper bound of the projectors in the given set because

\[ \forall l : E_l S = \sum_{l'=1}^{L} E_l E_{l'} = E_l. \]

Let \( F \) be any common upper bound for all projectors \( E_l \). Then it is also an upper bound of \( S \) because

\[ SF = (\sum_{l=1}^{L} E_l)F = \sum_{l=1}^{L} E_l = S. \]

**Proof** of the claim that if two projectors \( E, F \) are orthogonal, and a third projector \( G \) implies (quantum-logically) one of them, then also \( G \) is orthogonal to the other projector:

\[ EF = 0, \quad G \leq E, \quad \Rightarrow \quad GF = 0. \]  

This is so because

\[ GF = (GE)F = G(EF) = 0. \]
**Proof** of the claim that if one has two finite or infinite sums of pairwise orthogonal projectors, \((\sum_l G_l, \sum_k E_k)\) such that a map \(f\) is given that takes the index set \(\{l\}\) into the index set \(\{k\}\) so that
\[
\forall l : \quad G_l \leq E_{k=f(l)},
\] (A.4)
then the former sum is a lower bound of the latter
\[
\sum_l G_l \leq \sum_k E_k.
\] (A.5)

To begin the proof, we single out the subset \(\{\bar{k} = f(l) : \forall l\} \subseteq \{\forall k\}\) that is the image of \(\{\forall l\}\) regarding the map \(f\). Next we break up \(\{\forall l\}\) into classes \(C_{\bar{k}} \equiv f^{-1}(\bar{k}) : \{\forall l\} = \sum_{\bar{k}} C_{\bar{k}}\). Then, we claim that
\[
\forall \bar{k} : \quad C_{\bar{k}} E_{\bar{k}} = \delta_{\bar{k},k} C_{\bar{k}}.
\] (A.6)
To prove the step (A.6), one has \(C_{\bar{k}} E_{\bar{k}} = \sum_{l \in f^{-1}(\bar{k})} G_l E_{\bar{k}} = C_{\bar{k}}\) due to (A.4). As to the claimed (logical) implication, \((\bar{k} \neq \bar{k}') \Rightarrow C_{\bar{k}} E_{\bar{k}'} = 0\), it follows from (A.3) because \(E_{\bar{k}} E_{\bar{k}'} = 0\), and \(C_{\bar{k}} \leq E_{\bar{k}'}\).

Relation (A.6) implies
\[
\sum_l G_l = \sum_k C_{\bar{k}} \leq \sum_k E_k
\] (A.7)
because \((\sum_l G_l)(\sum_k E_k) = \sum_{k,k'} C_{\bar{k}} E_{\bar{k}} = \sum_k C_{\bar{k}} = \sum_l G_l\).

Next, the orthogonality of the projectors \(E_{\bar{k}}\) implies
\[
\sum_k E_k \leq \sum_{\bar{k}} E_{\bar{k}} = \sum_k E_{\bar{k}} + \ldots.
\] (A.8)
Hence, the transitivity of quantum-logical implication supplies the final proof of (A.5) \(\sum_l G_l \leq \sum_k E_k\).

**Appendix B: On compatibility of bounded observables**
The main claim of Appendix B is a consequence of the following more general claim:

**Claim 1.** Let \(O = \sum_k o_k E_k\) be a bounded discrete observable in its unique spectral form and let \(\bar{O}\) be a bounded linear operator. Then the following three relations are equivalent:
\[
(1) \quad [O, \bar{O}] = 0 \quad \Leftrightarrow \quad (2) \quad \bar{O} = \sum_k E_k \bar{O} E_k
\]
\[
\Leftrightarrow \quad (3) \quad \forall k : \quad [E_k, \bar{O}] = 0.
\] (B.1)

**Proof** of the claimed (logical) implications in (B.1) will be given in an in-circle way as follows: \((1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)\).
(1) ⇒ (2) : One can write (1) in (B.1), on account of $I = \sum_k E_k$, as follows, and, multiplying out the factors in the commutator, one obtains

$$\left[\sum_k o_k E_k, \sum_k \sum_{k'} E_k \tilde{O} E_{k'}\right] = 0 \Rightarrow \sum_k \sum_{k'} o_k E_k \tilde{O} E_{k'} - \sum_k \sum_{k'} o_{k'} E_k \tilde{O} E_{k'} = 0 \Rightarrow \sum_{k \neq k'} (o_k - o_{k'}) E_k \tilde{O} E_{k'} = 0.$$ (B.2)

Taking fixed $k$ and $k'$, we multiply the last relation by $E_k$ from the left and by $E_{k'}$ from the right to obtain $(o_k - o_{k'}) E_k \tilde{O} E_{k'} = 0$, and finally $\forall (k \neq k') : E_k \tilde{O} E_{k'} = 0$. Thus, (2) follows from (1).

(2) ⇒ (3) : Multiplying $\tilde{O} = \sum_k E_k \tilde{O} E_k$ from the left or alternatively from the right by the same arbitrary fixed $E_k$, one obtains the same term $E_k \tilde{O} E_k$. Hence (3) is a consequence of (2) in (B.1).

(3) ⇒ (1) : The third relation implies

$$[O, \tilde{O}] = \sum_k o_k [E_k, \tilde{O}] = 0.$$ (B.3)

This ends the proof.

Claim 1 implies the claim that we actually want to prove in this appendix.

CLAIM 2. Let $O = \sum_m o_m E_m$ and $O' = \sum_n o_n E_n$ be two discrete Hermitian operators given in their unique spectral forms. Then the two operators commute, $[O, O'] = 0$, if and only if each eigen-projector of the former commutes with each eigen-projector of the latter $\forall m, n : [\hat{E}_m, E_n] = 0$.

PROOF. Sufficiency. Assuming $\forall m, n : [E_m, E_n] = 0$, one obtains $[O, O'] = 0$ as seen by substituting the unique spectral forms for both operators and utilizing the bilinearity of the commutator.

Necessity. According to the above proposition, (1) in (B.1) implies (3) in (B.1). Hence, $[O, O'] = 0$ implies $\forall m : [\hat{E}_m, O'] = 0$. A repeated application of the mentioned claim in the Proposition can now be written as

$$\forall m : [O', E_m] = 0 \Rightarrow \forall n : [E_n, E_m] = 0$$.
References

[1] Busch P., Lahti P. J., and Mittelstaedt P.: The Quantum Theory of Measurement. Second edition. Springer, Berlin (1996)

[2] Von Neumann, J.: Mathematical Foundations of Quantum Mechanics. Princeton University Press, Princeton (1955)

[3] Herbut F.: A Review of Unitary Quantum Premeasurement Theory. An Algebraic Study of Basic Kinds of Premeasurements. arXiv:1412.7862

[4] Herbut, F.: Subsystem Measurement in Unitary Quantum Measurement Theory with Redundant Entanglement. Int. J. Quant. Inf. 12, 1450032 (16 pages). arXiv:1302.2250 [quant. phys.] (2014)

[5] Lüders G.: Über die Zustandsänderung durch den Messprozess. (About the Change of State in the Measurement Process. In German) Ann. der Physik. 8, 322-328 (1951).

[6] Herbut F.: Derivation of the Change of State in Measurement from the Concept of Minimal Measurement. Ann. Phys. (N. Y.) 55, 271-300 (1969)