 ITERATES OF THE SCHUR CLASS OPERATOR-VALUED FUNCTION AND THEIR CONSERVATIVE REALIZATIONS

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Abstract. Let $\mathcal{M}$ and $\mathcal{N}$ be separable Hilbert spaces and let $\Theta(\lambda)$ be a function from the Schur class $\mathcal{S}(\mathcal{M}, \mathcal{N})$ of contractive functions holomorphic on the unit disk. The operator generalization of the classical Schur algorithm associates with $\Theta$ the sequence of contractions (the Schur parameters of $\Theta$) $\Gamma_0 = \Theta(0) \in \mathcal{L}(\mathcal{M}, \mathcal{N})$, $\Gamma_n \in \mathcal{L}(\mathcal{D}_{\Gamma_n}, \mathcal{D}_{\Gamma_n}^*)$ and the sequence of functions $\Theta_0 = \Theta$, $\Theta_n \in \mathcal{S}(\mathcal{D}_{\Gamma_n}, \mathcal{D}_{\Gamma_n}^*)$ $n = 1, \ldots$ (the Schur iterates of $\Theta$) connected by the relations

$$\Gamma_n(\lambda) = \Theta_n(\lambda) = \Gamma_n + \lambda D_{\Gamma_n}^* \Theta_{n+1}(\lambda)(I + \lambda \Gamma_n^* \Theta_{n+1}(\lambda))^{-1} D_{\Gamma_n}, \quad |\lambda| < 1.$$ 

The function $\Theta(\lambda) \in \mathcal{S}(\mathcal{M}, \mathcal{N})$ can be realized as the transfer function

$$\Theta(\lambda) = D + \lambda C(I - \lambda A)^{-1} B$$

of a linear conservative and simple discrete-time system $\tau = \left\{ \begin{bmatrix} D & C \\ B & A \end{bmatrix} : \mathcal{M}, \mathcal{N}, \mathcal{H} \right\}$ with the state space $\mathcal{H}$ and the input and output spaces $\mathcal{M}$ and $\mathcal{N}$, respectively.

In this paper we give a construction of conservative and simple realizations of the Schur iterates $\Theta_n$ by means of the conservative and simple realization of $\Theta$.

Contents

1. Introduction 2
2. Completely non-unitary contractions 5
3. Contractions generated by a contraction 8
4. Passive discrete-time linear systems and their transfer functions 12
4.1. Basic definitions 12
4.2. Defect functions of the Schur class functions 14
4.3. Parametrization of contractive block-operator matrices 14
5. The Möbius representations 19
6. Realizations of the Schur iterates 23
6.1. Realizations of the first Schur iterate 23
6.2. Schur iterates of the characteristic function 26
6.3. Conservative realizations of the Schur iterates 28
References 30

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1. Introduction

The Schur class $S$ of scalar analytic functions and bounded by one in the unit disc $\mathbb{D} = \{ \lambda \in \mathbb{C} : |\lambda| < 1 \}$ plays a prominent role in complex analysis and operator theory as well in their applications in linear system theory and mathematical engineering. Given a Schur function $f(\lambda)$, which is not a finite Blaschke product, define inductively

$$f_0(\lambda) = f(\lambda), \quad f_{n+1}(\lambda) = \frac{f_n(\lambda) - f_n(0)}{\lambda(1 - f_n(0)f_n(\lambda))}, \quad n \geq 0.$$ 

It is clear that $\{f_n\}$ is an infinite sequence of Schur functions called the $n$-th Schur iterates and neither of its terms is a finite Blaschke product. The numbers $\gamma_n := f_n(0)$ are called the Schur parameters: $Sf = \{\gamma_0, \gamma_1, \ldots\}$.

Note that

$$f_n(\lambda) = \frac{\gamma_n + \lambda f_{n+1}(\lambda)}{1 + \gamma_n \lambda f_{n+1}(\lambda)} = \gamma_n + (1 - |\gamma_n|^2) \frac{\lambda f_{n+1}(\lambda)}{1 + \gamma_n \lambda f_{n+1}(\lambda)}, \quad n \geq 0.$$ 

The method of labeling $f \in S$ by its Schur parameters is known as the Schur algorithm and is due to I. Schur [33]. In the case when

$$f(\lambda) = e^{i\phi} \prod_{k=1}^{N} \frac{\lambda - \lambda_k}{1 - \bar{\lambda}_k \lambda}$$

is a finite Blaschke product of order $N$, the Schur algorithm terminates at the $N$-th step. The sequence of Schur parameters $\{\gamma_n\}_{n=0}^{N}$ is finite, $|\gamma_n| < 1$ for $n = 0, 1, \ldots, N - 1$, and $|\gamma_N| = 1$.

The Schur algorithm for matrix valued Schur class functions has been considered in the paper of Delsarte, Genin, and Kamp [27] and in the book of Dubovoj, Fritzsche, and Kirstein [28]. An operator extension of the Schur algorithm was developed by T. Constantinescu in [25] and with numerous applications is presented in the book of Bakonyi and Constantinescu [17].

In what follows the class of all continuous linear operators defined on a complex Hilbert space $\mathcal{H}_1$ and taking values in a complex Hilbert space $\mathcal{H}_2$ is denoted by $L(\mathcal{H}_1, \mathcal{H}_2)$ and $L(\mathcal{H}) := L(\mathcal{H}_1, \mathcal{H}_2)$. The domain, the range, and the null-space of a linear operator $T$ are denoted by $\text{dom} T$, $\text{ran} T$, and $\ker T$, respectively. The set of all regular points of a closed operator $T$ is denoted by $\rho(T)$. We denote by $I_H$ the identity operator in a Hilbert space $\mathcal{H}$ and by $P_L$ the orthogonal projection onto the subspace (the closed linear manifold) $L$. The notation $T\upharpoonright L$ means the restriction of a linear operator $T$ on the set $L$. The positive integers will be denoted by $\mathbb{N}$. An operator $T \in L(\mathcal{H}_1, \mathcal{H}_2)$ is said to be

(a) contractive if $\|T\| \leq 1$;

(b) isometric if $\|Tf\| = \|f\|$ for all $f \in \mathcal{H}_1 \iff T^*T = I_{\mathcal{H}_1}$;

(c) co-isometric if $T^*$ is isometric $\iff TT^* = I_{\mathcal{H}_2}$;

(d) unitary if it is both isometric and co-isometric.

Given a contraction $T \in L(\mathcal{H}_1, \mathcal{H}_2)$. The operators

$$D_T := (I - T^*T)^{1/2}, \quad D_{T^*} := (I - TT^*)^{1/2}$$
are called the defect operators of \( T \), and the subspaces \( \mathfrak{D}_T = \overline{\text{ran } D_T} \), \( \mathfrak{D}_{T^*} = \overline{\text{ran } D_{T^*}} \) the defect subspaces of \( T \). The dimensions \( \dim \mathfrak{D}_T \), \( \dim \mathfrak{D}_{T^*} \) are known as the defect numbers of \( T \). The defect operators satisfy the following intertwining relations

\[
TD_T = D_{T^*}T, \quad T^*D_{T^*} = D_T T^*.
\]

It follows from (1.1) that \( T \mathfrak{D}_T \subset \mathfrak{D}_{T^*} \), \( T^* \mathfrak{D}_{T^*} \subset \mathfrak{D}_T \), and \( T(\ker D_T) = \ker D_{T^*}, T^*(\ker D_{T^*}) = \ker D_T \). Moreover, the operators \( T| \ker D_T \) and \( T^*| \ker D_{T^*} \) are isometries and \( T| \mathfrak{D}_T \) and \( T^*| \mathfrak{D}_{T^*} \) are pure contractions, i.e., \( \|Tf\| < \|f\| \) for \( f \in \mathbb{D} \setminus \{0\} \).

The Schur class \( S(\mathfrak{D}_1, \mathfrak{D}_2) \) is the set of all function \( \Theta(\lambda) \) analytic on the unit disk \( \mathbb{D} \) with values in \( L(\mathfrak{D}_1, \mathfrak{D}_2) \) and such that \( \|\Theta(\lambda)\| \leq 1 \) for all \( \lambda \in \mathbb{D} \). The following theorem takes place.

**Theorem 1.1.** [25], [17]. Let \( \mathfrak{M} \) and \( \mathfrak{N} \) be separable Hilbert spaces and let the function \( \Theta(\lambda) \) be from the Schur class \( S(\mathfrak{M}, \mathfrak{N}) \). Then there exists a function \( Z(\lambda) \) from the Schur class \( S(\mathfrak{D}_0, \mathfrak{D}_{\Theta^*}(0)) \) such that

\[
\Theta(\lambda) = \Theta(0) + D_{\Theta^*}(0) Z(\lambda) (I + \Theta^*(0) Z(\lambda))^{-1} D_{\Theta(0)}, \quad \lambda \in \mathbb{D}.
\]

In what follows we will call the representation (1.2) of a function \( \Theta(\lambda) \) from the Schur class the Möbius representation of \( \Theta(\lambda) \) and the function \( Z(\lambda) \) we will call the Möbius parameter of \( \Theta(\lambda) \). Clearly, \( Z(0) = 0 \) and by Schwartz’s lemma we obtain that

\[
\|Z(\lambda)\| \leq |\lambda|, \quad \lambda \in \mathbb{D}.
\]

The operator Schur’s algorithm [17]. Fix \( \Theta(\lambda) \in S(\mathfrak{M}, \mathfrak{N}) \), put \( \Theta_0(\lambda) = \Theta(\lambda) \) and let \( Z_0(\lambda) \) be the Möbius parameter of \( \Theta \). Define

\[
\Gamma_0 = \Theta(0), \quad \Theta_1(\lambda) = \frac{Z_0(\lambda)}{\lambda} \in S(\mathfrak{D}_{\Gamma_0}, \mathfrak{D}_{\Gamma_0^*}), \quad \Gamma_1 = \Theta_1(0) = Z_0'(0).
\]

If \( \Theta_0(\lambda), \ldots, \Theta_n(\lambda) \) and \( \Gamma_0, \ldots, \Gamma_n \) have been chosen, then let \( Z_{n+1}(\lambda) \in S(\mathfrak{D}_{\Gamma_n}, \mathfrak{D}_{\Gamma_n^*}) \) be the Möbius parameter of \( \Theta_n \). Put

\[
\Theta_{n+1}(\lambda) = \frac{Z_{n+1}(\lambda)}{\lambda}, \quad \Gamma_{n+1} = \Theta_{n+1}(0).
\]

The contractions \( \Gamma_0 \in L(\mathfrak{M}, \mathfrak{N}) \), \( \Gamma_n \in L(\mathfrak{D}_{\Gamma_n-1}, \mathfrak{D}_{\Gamma_n^*-1}) \), \( n = 1, 2, \ldots \) are called the Schur parameters of \( \Theta(\lambda) \) and the function \( \Theta_n(\lambda) \in S(\mathfrak{D}_{\Gamma_n}, \mathfrak{D}_{\Gamma_n^*}) \) we will call the \( n \)-th Schur iterate of \( \Theta(\lambda) \).

Formally we have

\[
\Theta_{n+1}(\lambda) \mid \text{ran } D_{\Gamma_n} = \frac{1}{\lambda} D_{\Gamma_n}(I - \Theta_n(\lambda) \Gamma_n^*-1(\Theta_n(\lambda) - \Gamma_n)) D_{\Gamma_n}^{-1} \mid \text{ran } D_{\Gamma_n}.
\]

Clearly, the sequence of Schur parameters \( \{\Gamma_n\} \) is infinite if and only if the operators \( \Gamma_n \) are non-unitary. The sequence of Schur parameters consists of finite number operators \( \Gamma_0, \Gamma_1, \ldots, \Gamma_N \) if and only if \( \Gamma_N \in L(\mathfrak{D}_{\Gamma_{N-1}}, \mathfrak{D}_{\Gamma_{N-1}^*}) \) is unitary. If \( \Gamma_N \) is isometric (co-isometric) then \( \Gamma_n = 0 \) for all \( n > N \).

The following theorem is the operator generalization of Schur’s result.

**Theorem 1.2.** [25], [17]. There is a one-to-one correspondence between the Schur class functions \( S(\mathfrak{M}, \mathfrak{N}) \) and the set of all sequences of contractions \( \{\Gamma_n\}_{n \geq 0} \) such that

\[
(1.3) \quad \Gamma_0 \in L(\mathfrak{M}, \mathfrak{N}), \quad \Gamma_n \in L(\mathfrak{D}_{\Gamma_{n-1}}, \mathfrak{D}_{\Gamma_{n-1}^*}), \quad n \geq 1.
\]
Notice that a sequence of contractions of the form (1.3) is called the choice sequence [24].

It is known [23, 11] that every \( \Theta(\lambda) \in \mathbf{S}(\mathcal{M}, \mathcal{N}) \) can be realized as the transfer function

\[
\Theta(\lambda) = D + \lambda C (I_{\mathcal{H}} - \lambda A)^{-1} B
\]

of a linear conservative and simple discrete-time system (see Section 4)

\[
\tau = \left\{ \begin{bmatrix} D & C \\ B & A \end{bmatrix} ; \mathcal{M}, \mathcal{N}, \mathcal{H} \right\}
\]

with the state space \( \mathcal{H} \) and input and output spaces \( \mathcal{M} \) and \( \mathcal{N} \), respectively. In this paper we study the problem of the conservative realizations of the Schur iterates of the function \( \Theta(\lambda) \in \mathbf{S}(\mathcal{M}, \mathcal{N}) \) by means of the the conservative realization of \( \Theta \).

In this connection it should be pointed out that the similar problem for a scalar generalized Schur class function has been studied in papers [1], [2], [3], [4].

Here we describe our main results. Let \( A \) be a completely non-unitary contraction [38] in a separable Hilbert space \( \mathcal{H} \). Define the subspaces and operators

\[
\begin{align*}
\mathcal{H}_{m,0} &= \ker D_{A^m}, \mathcal{H}_{0,l} = \ker D_{A^l}, \\
\mathcal{H}_{m,l} &= \ker D_{A^m} \cap \ker D_{A^l}, m, l \in \mathbb{N}, \\
A_{m,l} &= P_{m,l} A | \mathcal{H}_{m,l},
\end{align*}
\]

where \( P_{m,l} \) is the orthogonal projection in \( \mathcal{H} \) onto \( \mathcal{H}_{m,l} \).

We prove that

1) if \( A \) is a completely non-unitary contraction in a Hilbert space then for every \( n \in \mathbb{N} \) the operators

\[
A_{n,0}, A_{n-1,1}, \ldots, A_{0,n}
\]

are unitary equivalent completely non-unitary contractions and their Sz.-Nagy–Foias characteristic functions [38] coincide with the pure contractive part [38], [17] for the \( n \)-th Schur iterate \( \Phi_n(\lambda) \) of the characteristic function \( \Phi(\lambda) \) of \( A \);

2) if \( \Theta(\lambda) \in \mathbf{S}(\mathcal{M}, \mathcal{N}) \) is the transfer function of a simple conservative system

\[
\tau = \left\{ \begin{bmatrix} \Gamma_0 & C \\ B & A \end{bmatrix} ; \mathcal{M}, \mathcal{N}, \mathcal{H} \right\}
\]

then the Schur parameters of \( \Theta \) take the form

\[
\begin{align*}
\Gamma_1 &= D_{\Gamma_0}^{-1} C (D_{\Gamma_0}^{-1} B^*)^*, \\
\Gamma_2 &= D_{\Gamma_1}^{-1} D_{\Gamma_0}^{-1} CA \left( D_{\Gamma_1}^{-1} D_{\Gamma_0}^{-1} (B^* | \mathcal{H}_{1,0}) \right)^*, \\
\Gamma_n &= D_{\Gamma_n}^{-1} \cdots D_{\Gamma_0}^{-1} CA^{n-1} \left( D_{\Gamma_n}^{-1} \cdots D_{\Gamma_0}^{-1} (B^* | \mathcal{H}_{n-1,0}) \right)^*, \\
\end{align*}
\]

and the \( n \)-th Schur iterate \( \Theta_n(\lambda) \) of \( \Theta \) is the transfer function of the simple conservative and unitarily equivalent systems

\[
\tau_n^{(k)} = \left\{ \begin{bmatrix} \Gamma_n & D_{\Gamma_n}^{-1} \cdots D_{\Gamma_0}^{-1} (CA^{n-k}) \\ A_{n-k,k} \end{bmatrix} ; \mathcal{D}_{\Gamma_n} \right\}
\]

for \( k = 0, \ldots, n \). Here \( D_{\Gamma_m}^{-1} \) and \( D_{\Gamma_m}^{-1} \) are the Moore–Penrose pseudo-inverses. For a completely non-unitary contraction \( A \) with rank one defect operators it was proved in [10] that the characteristic functions of the operators \( A_{1,0} = P_{\ker D_A} A | \ker D_A \) and \( A_{0,1} = P_{\ker D_A^*} A | \ker D_A^* \) coincide with the first Schur iterate of the characteristic function of \( A \). This result has been established using the model of \( A \) given by a truncated CMV matrix.
Here we use another approach based on the parametrization of a contractive block-operator matrix

\[
T = \begin{bmatrix} D & C \\ B & A \end{bmatrix} : \mathfrak{M} \oplus \mathfrak{H} \rightarrow \mathfrak{N} \oplus \mathfrak{K}
\]

established in [16], [26], [36], and the construction of the passive realization for the Möbius parameter of \( \Theta(\lambda) \) obtained in [8] by means of a passive realization of \( \Theta \).

2. Completely non-unitary contractions

Let \( S \) be an isometry in a separable Hilbert space \( H \). A subspace \( \Omega \) in \( H \) is called wandering for \( V \) if \( S^p\Omega \perp S^q\Omega \) for all \( p, q \in \mathbb{Z}_+ \), \( p \neq q \). Since \( S \) is an isometry, the latter is equivalent to \( S^p\Omega \perp \Omega \) for all \( n \in \mathbb{N} \). If \( H = \sum_{n=0}^{\infty} S^n\Omega \) then \( S \) is called a unilateral shift and \( \Omega \) is called the generating subspace. The dimension of \( \Omega \) is called the multiplicity of the unilateral shift \( S \). It is well known [38, Theorem I.1.1] that \( S \) is a unilateral shift if and only if \( \bigcap_{n=0}^{\infty} S^nH = \{0\} \). Clearly, if an isometry \( V \) is the unilateral shift in \( H \) then \( \Omega = H \ominus S\Omega \) is the generating subspace for \( S \). An operator is called co-shift if its adjoint is a unilateral shift.

A contraction \( A \) acting in a Hilbert space \( \mathfrak{H} \) is called completely non-unitary if there is no nontrivial reducing subspace of \( A \), on which \( A \) generates a unitary operator. Given a contraction \( A \) in \( \mathfrak{H} \) then there is a canonical orthogonal decomposition [38, Theorem I.3.2]

\[
\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1, \quad A = A_0 \oplus A_1, \quad A_j = A|\mathfrak{H}_j, \quad j = 0, 1,
\]

where \( \mathfrak{H}_0 \) and \( \mathfrak{H}_1 \) reduce \( A \), the operator \( A_0 \) is a completely non-unitary contraction, and \( A_1 \) is a unitary operator. Moreover,

\[
\mathfrak{H}_1 = \left( \bigcap_{n \geq 1} \ker D_{A^n} \right) \cap \left( \bigcap_{n \geq 1} \ker D_{A^*n} \right).
\]

Since

\[
\bigcap_{k=0}^{n-1} \ker(D_AA^k) = \ker D_{A^n}, \quad \bigcap_{k=0}^{n-1} \ker(D_{A^*}A^*k) = \ker D_{A^*n},
\]

we get

\[
\bigcap_{n \geq 1} \ker D_{A^n} = \mathfrak{H} \ominus \text{span} \{ A^n D_A \mathfrak{H}, \ n = 0, 1, \ldots \},
\]

(2.1)

\[
\bigcap_{n \geq 1} \ker D_{A^*n} = \mathfrak{H} \ominus \text{span} \{ A^n D_{A^*} \mathfrak{H}, \ n = 0, 1, \ldots \}.
\]

It follows that

(2.2) \( A \) is completely non-unitary \( \iff \left( \bigcap_{n \geq 1} \ker D_{A^n} \right) \cap \left( \bigcap_{n \geq 1} \ker D_{A^*n} \right) = \{0\} \iff \text{span} \{ A^n D_A, A^m D_{A^*}, \ n, m \geq 0 \} = \mathfrak{H} \).

Note that

\[
\ker D_A \supset \ker D_{A^2} \supset \cdots \supset \ker D_{A^n} \supset \cdots,
\]

\[
A \ker D_{A^n} \subset \ker D_{A^{n-1}}, \ n = 2, 3, \ldots.
\]
From (2.1) we get that the subspaces \( \bigcap_{n \geq 1} \ker D_{A^n} \) and \( \bigcap_{n \geq 1} \ker D_{A^{\ast n}} \) are invariant with respect to \( A \) and \( A^\ast \), respectively, and \( A\big| \bigcap_{n \geq 1} \ker D_{A^n} \) and \( A^\ast \big| \bigcap_{n \geq 1} \ker D_{A^{\ast n}} \) are unilateral shifts, moreover, these operators are the maximal unilateral shifts contained in \( A \) and \( A^\ast \), respectively [29 Theorem 1.1, Corollary 1]. Thus, for a completely non-unitary contraction \( A \) we have
\[
\bigcap_{n \geq 1} \ker D_{A^n} = \{0\} \iff A \text{ does not contain a unilateral shift,}
\]
\[
\bigcap_{n \geq 1} \ker D_{A^{\ast n}} = \{0\} \iff A^\ast \text{ does not contain a unilateral shift.}
\]
(2.3)

By definition [29] the operator \( A \) contains a co-shift \( V \) if the operator \( A^\ast \) contains the unilateral shift \( V^\ast \).

The function (see [38] Chapter VI)
\[
\Phi_A(\lambda) = (-A + \lambda D_A^\ast (I - \lambda A^\ast)^{-1} D_A) \upharpoonright \mathfrak{D}_A
\]
is known as the Sz.-Nagy – Foias characteristic function of a contraction \( A \) [38]. This function belongs to the Schur class \( S(\mathfrak{D}_A, \mathfrak{D}_A^\ast) \) and \( \Theta_A(0) \) is a pure contraction. The characteristic functions of \( A \) and \( A^\ast \) are connected by the relation
\[
\Phi_{A^\ast}(\lambda) = \Phi_A^\ast(\bar{\lambda}), \quad \lambda \in \mathbb{D}.
\]

Two operator-valued functions \( \Theta_1 \in S(\mathfrak{M}_1, \mathfrak{N}_1) \) and \( \Theta_2 \in S(\mathfrak{M}_2, \mathfrak{N}_2) \) coincide [38] if there are two unitary operators \( V : \mathfrak{N}_1 \rightarrow \mathfrak{N}_2 \) and \( W : \mathfrak{M}_2 \rightarrow \mathfrak{M}_1 \) such that
\[
V \Theta_1(\lambda) W = \Theta_2(\lambda), \quad \lambda \in \mathbb{D}.
\]
The result of Sz.-Nagy–Foias [38] Theorem VI.3.4 states that two completely non-unitary contractions \( A_1 \) and \( A_2 \) are unitary equivalent if and only if their characteristic functions \( \Phi_{A_1} \) and \( \Phi_{A_2} \) coincide.

It is well known that a function \( \Theta(\lambda) \) from the Schur class \( S(\mathfrak{M}, \mathfrak{N}) \) has almost everywhere non-tangential strong limit values \( \Theta(\xi), \xi \in \mathbb{T}, \) where \( \mathbb{T} = \{\xi \in \mathbb{C} : |\xi| = 1\} \) stands for the unit circle; cf. [38]. A function \( \Theta \in S(\mathfrak{M}, \mathfrak{N}) \) is called inner if \( \Theta^\ast(\xi) \Theta(\xi) = I_{\mathfrak{M}} \) and co-inner if \( \Theta(\xi) \Theta^\ast(\xi) = I_{\mathfrak{N}} \) almost everywhere on \( \xi \in \mathbb{T} \). A function \( \Theta \in S(\mathfrak{M}, \mathfrak{N}) \) is called bi-inner, if it is both inner and co-inner. A contraction \( T \) on a Hilbert space \( \mathcal{H} \) belongs to the class \( C_0. (C_0) \), if
\[
s - \lim_{n \rightarrow \infty} A^n = 0 \quad (s - \lim_{n \rightarrow \infty} A^{\ast n} = 0),
\]
respectively. By definition \( C_{00} := C_0 \cap C_0 \). A completely non-unitary contraction \( A \) belongs to the class \( C_0, C_0, \) or \( C_{00} \) if and only if its characteristic function \( \Phi_A(\lambda) \) is inner, co-inner, or bi-inner, respectively (cf. [38] Section VI.2)). Note that for a completely non-unitary contraction \( A \) the equality \( \ker D_A = \ker D_{A^\ast} \neq \{0\} \) is impossible because otherwise the subspace \( \ker D_A \) reduces \( A \) and \( A^\ast \) is a unitary operator.

We complete this section by a description of completely non-unitary contractions with constant characteristic functions. Note that \( \Phi_A(\lambda) = 0 \in S(\{0\}, \mathfrak{D}_A^\ast) \iff A \) is a unilateral shift, and \( \Phi_A(\lambda) = 0 \in S(\mathfrak{D}_A, \{0\}) \iff A \) is a co-shift.

**Theorem 2.1.** Let \( \mathcal{H} \) be a separable Hilbert space. A completely non-unitary contraction \( A \) with nonzero defect operators has a constant characteristic function if and only if \( \mathcal{H} \) is the orthogonal sum
\[
\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2
\]
and $A$ takes the operator matrix form

$$A = \begin{bmatrix} S_1 & \Gamma \\ 0 & S_2^* \end{bmatrix}: \mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow \mathcal{H}_1 \oplus \mathcal{H}_2,$$

where $S_1$ and $S_2$ are unilateral shifts in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and $\Gamma$ is a contraction such that

$$\begin{aligned}
\text{ran } \Gamma &\subset \mathcal{D}_{S_1^*}, \text{ran } \Gamma^* \subset \mathcal{D}_{S_2^*}, \\
||\Gamma f|| < ||f||, &\quad f \in \mathcal{D}_{S_2^*} \setminus \{0\}, \\
||\Gamma^* h|| < ||h||, &\quad h \in \mathcal{D}_{S_1^*} \setminus \{0\}.
\end{aligned}$$

In particular, the characteristic function of $A$ is identically equal zero if and only if $A$ is the orthogonal sum of a shift and co-shift.

**Proof.** Suppose that the contraction $A$ takes the form (2.5) with unilateral shifts $S_1$ and $S_2$, and the contraction $\Gamma$ with the properties (2.6). Then

$$D_A^2 = \begin{bmatrix} 0 & 0 \\ 0 & D_{S_2^*} - \Gamma^* \Gamma \end{bmatrix}: \mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow \mathcal{H}_1 \oplus \mathcal{H}_2,$$

and

$$D_{A^*}^2 = \begin{bmatrix} D_{S_1^*} - \Gamma \Gamma^* & 0 \\ 0 & 0 \end{bmatrix}: \mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow \mathcal{H}_1 \oplus \mathcal{H}_2.$$

Since $\mathcal{D}_{S_1^*} = \ker S_1^*$, $\mathcal{D}_{S_2^*} = \ker S_2^*$, and $D_{S_1^*}$ and $D_{S_2^*}$ are the orthogonal projections in $\mathcal{H}$ onto $\mathcal{D}_{S_1^*}$ and $\mathcal{D}_{S_2^*}$, respectively, we get from (2.6) the relations

$$\mathcal{D}_A = \mathcal{D}_{S_2^*}, \mathcal{D}_{A^*} = \mathcal{D}_{S_1^*}.$$

Taking into account that $\mathcal{H}_2$ is an invariant subspace for $A^*$, we have

$$D_{A^*}(I_{\mathcal{H}} - \lambda A^*)^{-1}D_A = 0.$$

Hence $\Phi_A(\lambda) = \Gamma|\mathcal{D}_{S_2^*} = \text{const.}$

Because $S_1$ and $S_2$ are unilateral shifts, we get

$$\mathcal{H}_1 = \sum_{n \geq 0} \oplus S_1^n \mathcal{D}_{S_1^*}, \mathcal{H}_2 = \sum_{n \geq 0} \oplus S_2^n \mathcal{D}_{S_2^*}.$$

Since $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, the operator $A$ is completely non-unitary. If $\Gamma = 0$ then $A$ is the orthogonal sum of a shift and co-shift.

Now suppose that the characteristic function of $A$ is a constant. From (2.4) we get

$$D_{A^*}A^*D_A = 0, \quad D_A A^n D_A = 0, \quad n = 0, 1, 2, \ldots.$$

It follows

$$\text{span} \{D_{A^n} \mathcal{D}_A, n = 0, 1, \ldots\} \subset \ker D_{A^*} \iff \bigcap_{n \geq 1} \ker D_A^n \supset \mathcal{D}_{A^*},$$

$$\text{span} \{D_{A^n} \mathcal{D}_{A^*}, n = 0, 1, \ldots\} \subset \ker D_A \iff \bigcap_{n \geq 1} \ker D_{A^n} \supset \mathcal{D}_A.$$
Let
\[ \mathcal{H}_1 = \bigcap_{n \geq 1} \ker D_{A^n}, \quad \mathcal{H}_2 = \bigcap_{n \geq 1} \ker D_{A^*n}. \]

Since
\[ A\mathcal{H}_1 \subset \mathcal{H}_1 \quad \text{and} \quad A\mathcal{H}_1 \perp \mathcal{D}_{A^*}, \]
we get \( \mathcal{H}_1 \ominus A\mathcal{H}_1 \supset \mathcal{D}_{A^*} \) and similarly \( \mathcal{H}_2 \ominus A^*\mathcal{H}_2 \supset \mathcal{D}_A \). Let \( h \in \mathcal{H}_1 \) and \( h \perp \mathcal{D}_{A^*} \). It follows
\[ h \in \ker D_{A^*} \bigcap \left( \bigcap_{n \geq 1} \ker D_{A^n} \right). \]
Then \( h = Ag, \ g \in \ker D_A \). Hence \( g \in \bigcap_{n \geq 1} \ker D_{A^n} = \mathcal{H}_1 \), i.e., \( \mathcal{H}_1 \ominus A\mathcal{H}_1 \supset D_{A^*} \) and similarly \( \mathcal{H}_2 \ominus A^*\mathcal{H}_2 \supset D_A \).

Since \( A \) is completely non-unitary contraction, the operators \( A|\mathcal{H}_1 \) and \( A^*|\mathcal{H}_2 \) are unilateral shifts. Therefore
\[ (2.10) \quad \mathcal{H}_1 = \sum_{n=0}^{\infty} \oplus A^n \mathcal{D}_{A^*}, \quad \mathcal{H}_2 = \sum_{n=0}^{\infty} \oplus A^n \mathcal{D}_A. \]

Note that for all \( \varphi, \psi \in \mathcal{H} \)
\[ (A^n D_A \varphi, A^k D_A \psi) = (D_A A^{m+k} D_A \varphi, \psi) = 0, \ m, k = 0, 1, 2 \ldots \]
Hence \( \mathcal{H}_1 \perp \mathcal{H}_2 \). Taking into account (2.10) and the relation
\[ \mathcal{H} \ominus \mathcal{H}_1 = \operatorname{span} \{ A^n \mathcal{D}_A, \ n = 0, 1, 2 \ldots \}, \]
we get \( \mathcal{H} \ominus \mathcal{H}_1 = \mathcal{H}_2 \). Because \( \mathcal{H}_1 \) is invariant with respect to \( A \), the matrix form of \( A \) is of the form (2.5) with unilateral shifts
\[ S_1 := A|\mathcal{H}_1, \ S_2 := A^*|\mathcal{H}_2, \]
and some operator \( \Gamma \in (\mathcal{H}_2, \mathcal{H}_1) \). Since \( A \) is a contraction, we have
\[ ||\Gamma f||^2 \leq ||D_{S_1^*^-} f||^2, \ f \in \mathcal{H}_2, \]
\[ ||\Gamma^* h||^2 \leq ||D_{S_2^-} h||^2, \ h \in \mathcal{H}_1. \]
From (2.7) and (2.8) we get
\[ \begin{aligned} &\operatorname{ran} (D_{S_2^-} - \Gamma^* \Gamma) = \mathcal{D}_A, \ \operatorname{ran} (D_{S_1^*^-} - \Gamma \Gamma^*) = \mathcal{D}_{A^*}. \end{aligned} \]
It follows that (2.6) holds true and \( \Phi_A(\lambda) = \Gamma \).

If \( A \) is the orthogonal sum of a shift and co-shift then clearly the characteristic function of \( A \) is identically zero. \qed

3. Contractions generated by a contraction

In this section we define and study the subspaces and the corresponding operators obtained from a completely non-unitary contraction \( A \) in a separable Hilbert space \( \mathcal{H} \).

Suppose \( \ker D_A \neq \{0\} \). Define the subspaces
\[ (3.1) \quad \begin{cases} \mathcal{H}_{0,0} := \mathcal{H} \\ \mathcal{H}_{n,0} := \ker D_{A^n}, \ \mathcal{H}_{0,m} := \ker D_{A^m}, \\ \mathcal{H}_{n,m} := \ker D_{A^n} \cap \ker D_{A^m}, \ m, n \in \mathbb{N} \end{cases} \]
Let $P_{n,m}$ be the orthogonal projection in $\mathcal{H}$ onto $\mathcal{H}_{n,m}$. Define the contractions

\[ A_{n,m} := P_{n,m}A|_{\mathcal{H}_{n,m}} \in \mathcal{L}(\mathcal{H}_{n,m}) \]

and

\[ A_{n,m} := A_{n,m}P_{n+1,m}|_{\mathcal{H}_{n,m}} \in \mathcal{L}(\mathcal{H}_{n,m}). \]

In the next theorem we establish the main properties of $A_{n,m}$ and $\mathcal{A}_{n,m}$.

**Theorem 3.1.**

1. Hold the relations

\[ \begin{cases} \ker D_{A_{n,m}}^k = \mathcal{H}_{n+k,m}, & k = 1, 2, \ldots, \\ \ker D_{A_{n,m}^*}^k = \mathcal{H}_{n,m+k} \end{cases} \]

2. \[ \begin{cases} \mathcal{D}_{A_{n,m}} = \text{ran } (P_{n,m}D_{A_{n+1,m}}), \\ \mathcal{D}_{A_{n,m}^*} = \text{ran } (P_{n,m}D_{A_{n+1,m}^*}) \end{cases} \]

3. \[ \begin{cases} A_{n+1,m} = \mathcal{H}_{n-1,m+1}, & n \geq 1, \\ A_{n,m}^* = \mathcal{H}_{n+1,m-1}, & m \geq 1 \end{cases} \]

4. \[ \begin{cases} \ker D_{A_{n,m}^k} = \mathcal{H}_{n+k,m}, & k = 1, 2, \ldots, \\ \ker D_{A_{n,m}^*}^k = \mathcal{H}_{n,m+k} \end{cases} \]

5. \[ \begin{cases} \mathcal{D}_{A_{n,m}} = \mathcal{D}_{A_{n+1,m}}, \\ \mathcal{D}_{A_{n,m}^*} = \mathcal{D}_{A_{n+1,m}^*} \end{cases} \]

6. \[ (A_{n,m})_{k,l} = A_{n+k,m+l}. \]

2. The operators $\{A_{n,m}\}$ and $\{A_{n,m}\}$ are completely non-unitary contractions.

3. The operators

\[ A_{n,0}, A_{n-1,1}, \ldots, A_{n-k,k}, \ldots, A_{0,n} \]

are unitarily equivalent and

\[ A_{n-1,m+1}A f = A A_{n,m}f, \quad f \in \mathcal{H}_{n,m}, \quad n \geq 1. \]

4. The operators

\[ A_{n,0}, A_{n-1,1}, \ldots, A_{n-k,k}, \ldots, A_{0,n} \]

are unitarily equivalent and

\[ A_{n-1,m+1}A f = A A_{n,m}f, \quad f \in \mathcal{H}_{n,m}, \quad n \geq 1. \]

5. The following statements are equivalent

(a) $A_{n,0} \in C_0$ ($A_{n,0} \in C_0$) for some $n$,

(b) $A_{n+1-k,k} \in C_0$ ($A_{n+1-k,k} \in C_0$) for all $k = 0, 1, \ldots, n+1$.

**Proof.** It is sufficient to prove the first equality from (3.4). From (3.1) and (3.2) we have

\[ f \in \mathcal{H}_{n,m}, \quad f \in \ker D_{A_{n,m}^k} \quad \iff \quad \|f\| = \|A^nf\| = \|A^{m}f\| \]

\[ \iff \quad A f, \ldots, A^k f \in \mathcal{H}_{n,m} \quad \iff \quad f \in \mathcal{H}_{n+k,m}. \]
This proves (3.4). Hence
\[
\mathcal{D}_{A_{n,m}} = \mathfrak{H}_{n,m} \ominus \mathfrak{H}_{n+1,m} = \mathfrak{H}_{n,m} \ominus (\ker D_{A^{n+1}} \cap \ker D_{A^m}) = \\
= \mathfrak{H}_{n,m} \cap (\mathcal{D}_{A^{n+1}} + \mathcal{D}_{A^m}) = \overline{\text{ran}} (P_{n,m} D_{A^{n+1}}), \\
\mathcal{D}_{A_{n,m}^*} = \mathfrak{H}_{n,m} \ominus \mathfrak{H}_{n+1,m} = \mathfrak{H}_{n,m} \ominus (\ker D_\mathfrak{H} \cap \ker D_{A^{m+1}}) = \\
= \mathfrak{H}_{n,m} \cap (\mathcal{D}_{A^n} + \mathcal{D}_{A^{m+1}}) = \overline{\text{ran}} (P_{n,m} D_{A^{m+1}}),
\]
i.e., relations (3.5) are valid. Furthermore if \( n \geq 2 \) then
\[
f \in \mathfrak{H}_{n,m} \iff \begin{cases} 
A \psi \in \ker D_{A^{n-1}}, \\
A^* A \psi \in \ker D_\mathfrak{H}, \\
f \in \ker D_{A^m} \text{ (for } m \geq 1) 
\end{cases} \iff A \psi \in \ker D_{A^{m+1}} = \mathfrak{H}_{n-1,m+1}.
\]
If \( n = 1 \) then
\[
f \in \mathfrak{H}_{1,m} \iff \begin{cases} 
A^* A \psi \in \ker D_\mathfrak{H}, \\
f \in \ker D_{A^m} \iff A \psi \in \ker D_{A^{m+1}} = \mathfrak{H}_{0,m+1}.
\end{cases}
\]
Similarly \( A^* \mathfrak{H}_{n,m} = \mathfrak{H}_{n+1,m-1} \), \( m \geq 1 \). Therefore relations (3.6) hold true.
Let \( \varphi \in \mathfrak{H}, \psi \in \mathfrak{H}_{n-1,m+1} \). Then \( A^* \psi \in \mathfrak{H}_{n,m} \) and
\[
(\mathcal{A} P_{n,m} \varphi, \psi) = (P_{n,m} \varphi, A^* \psi) = (\varphi, A^* \psi) = (A \varphi, \psi) = (P_{n-1,m+1} A \varphi, \psi).
\]
Hence
\[
(3.12) \quad \mathcal{A} P_{n,m} = P_{n-1,m+1} A.
\]
Taking into account (3.6), we get
\[
\mathcal{A} P_{n,m} A h = P_{n-1,m+1} A Ah, \quad h \in \mathfrak{H}_{n,m}.
\]
This proves (3.10). Since \( A \) isometrically maps \( \mathfrak{H}_{n,m} \) onto \( \mathfrak{H}_{n-1,m+1} \) for \( n \geq 1 \), the operators \( A_{n-1,m+1} \) and \( A_{n,m} \) are unitarily equivalent, and therefore the operators
\[
A_{n,0}, A_{n-1,1}, \ldots, A_{n-k,k}, \ldots, A_{0,n}
\]
are unitarily equivalent.

Note that (3.4) and (3.6) yield the equalities
\[
(3.13) \quad \cap_{k \geq 1} \ker D_{A^k_{n,m}} = \ker D_{A^m} \cap \left( \cap_{j \geq 1} \ker D_{A^j} \right) = A^m \left( \cap_{j \geq 1} \ker D_{A^j} \right),
\]
\[
\cap_{k \geq 1} \ker D_{A^k_{n,m}} = \ker D_{A^n} \cap \left( \cap_{j \geq 1} \ker D_{A^{*j}} \right) = A^n \left( \cap_{j \geq 1} \ker D_{A^{*j}} \right).
\]
Since \( A \) is a completely non-unitary contraction, we get
\[
\left( \cap_{k \geq 1} \ker D_{A^k_{n,m}} \right) \cap \left( \cap_{k \geq 1} \ker D_{A^{*k}_{n,m}} \right) = \{0\}.
\]
It follows that the contractions \( A_{n,m} \) are completely non-unitary.

Note that \( \mathfrak{H}_{n-1,m+1} \subset \mathfrak{H}_{n-1,m} \) and \( \mathfrak{H}_{n+1,m} \subset \mathfrak{H}_{n,m} \). Using (3.6) we get
\[
A_{n-1,m+1} P_{n,m+1} = P_{n-1,m+1} A P_{n,m+1} = A P_{n,m+1},
\]
\[
A_{n,m} P_{n+1,m} = P_{n,m} A P_{n+1,m} = A P_{n+1,m}.
\]
In particular, it follows that the operators $A_{n,m}P_{n+1,m}$ are partial isometries. From (3.12) we obtain

$$AP_{n,m+1} = A^2P_{n+1,m},$$

i.e.,

$$A_{n-1,m+1}P_{n+1,m+1}Af = AA_{n,m}P_{n+1,m}f \quad \text{for all} \quad f \in \mathfrak{f}_{n,m}.$$  

Because $A$ is unitary operator from $\mathfrak{f}_{n,m}$ onto $\mathfrak{f}_{n-1,m+1}$, we get (3.11) and so the operators $A_{n-1,m+1}$ and $A_{n,m}$ are unitarily equivalent.

By induction it can be easily proved that for every $k \in \mathbb{N}$ holds the equality

$$A^k_{n,m}f = (AP_{n+1,m})^k f = AA_{n+1,m}P_{n+1,m}f, \quad f \in \mathfrak{f}_{n,m}.$$  

Since $A|\mathfrak{f}_{n+1,m}$ is isometric, relations (3.11) imply

$$||A^k_{n,m}f|| = ||A_{n+1,m}P_{n+1,m}f||, \quad f \in \mathfrak{f}_{n,m}, \ k \in \mathbb{N}.$$  

It follows the equivalence of the statements (a) and (b) and

$$\ker DA^k_{n,m} = \ker DA_{n+1,m} = \mathfrak{f}_{n+k,m}. $$

Similarly, since $\left(A_{n,m}P_{n+1,m}\right)^* = A_{n,m}^*P_{n,m+1}$, we get

$$\ker DA^*_n = \ker DA_{n,m+1} = \mathfrak{f}_{n,m+k}. $$

Thus, relations (3.7) are valid.

Now we get that the operators $A_{n,m}P_{n+1,m}$ are completely non-unitary. From (3.4) it follows that

$$\ker DA^k_{n,m} \cap \ker DA^*_n = \mathfrak{f}_{n+k,m} \cap \mathfrak{f}_{n,m+l} = \ker DA^k_{n+k} \cap \ker DA^*_n = \ker DA^k_{n,m+l} = \mathfrak{f}_{n+k,m+l}. $$

Hence

$$(A_{n,m})_{k,l} = P_{n+k,m+l}P_{n,m}A|\mathfrak{f}_{n+k,m+l} = A_{n+k,m+l}. $$

The relation (3.9) yields the following picture for the creation of the operators $A_{n,m}$:

\[
\begin{array}{c}
\text{A} \\
\text{A}_{1,0} \quad \text{A}_{0,1} \\
\text{A}_{2,0} \quad \text{A}_{1,1} \quad \text{A}_{0,2} \\
\text{A}_{3,0} \quad \text{A}_{2,1} \quad \text{A}_{1,2} \quad \text{A}_{0,3}
\end{array}
\]

The process terminates on the $N$-th step if and only if

$$\ker DA^N = \{0\} \iff \ker DA^N \cap \ker DA^* = \{0\} \iff \ldots$$

$$\ker DA^{N-k} \cap \ker DA^k = \{0\} \iff \ldots \ker DA^{*N} = \{0\}.$$
Note that from (2.3), (3.7), and (3.13) we get

**Proposition 3.2.** Let $A$ be a completely non-unitary contraction. If $A$ does not contain a unilateral shift (co-shift) then the same is true for the operators $A_{n,m}$ and $A_{n,m}$ for all $n$ and $m$. Conversely, if for some $n$ and $m$ the operator $A_{n,m}$ or $A_{n,m}$ does not contain a unilateral shift (co-shift) then the same is valid for $A$.

Let $\delta_A = \dim \mathcal{D}_A$, $\delta_{A^*} = \dim \mathcal{D}_{A^*}$ be the defect numbers of a completely non-unitary contraction $A$. For $n = 1, \ldots$ denote by $\delta_n$ and $\delta^*_n$ the defect numbers of unitarily equivalent operators $\{A_{n-m,m}\}_{m=0}^n$. From the relations (3.5) it follows that

\[
\delta_n = \dim \mathcal{D}_{A_0,n} = \dim \left( \overline{\text{ran}} \left( P_{0,n}D_A \right) \right) = \dim \left( \mathcal{D}_A \ominus \mathcal{D}_A \cap \mathcal{D}_{A^n} \right),
\]

\[
\delta^*_n = \dim \mathcal{D}_{A^*_0,n} = \dim \left( \overline{\text{ran}} \left( P_{n,0}D_{A^*} \right) \right) = \dim \left( \mathcal{D}_{A^*} \ominus \mathcal{D}_{A^*} \cap \mathcal{D}_{A^*} \right).
\]

Thus

\[
\delta_A \geq \delta_1 \geq \cdots \geq \delta_n \geq \cdots,
\]

\[
\delta^*_A \geq \delta^*_1 \geq \cdots \geq \delta^*_n \geq \cdots.
\]

Observe also that

\[
\delta_1 = \dim \left( \mathcal{D}_A \ominus \mathcal{D}_A \cap \mathcal{D}_{A^*} \right), \quad \delta^*_1 = \dim \left( \mathcal{D}_{A^*} \ominus \mathcal{D}_A \cap \mathcal{D}_{A^*} \right),
\]

and by induction

\[
\delta_n = \dim \left( \mathcal{D}_{A_{n-1,0}} \ominus \mathcal{D}_{A_{n-1,0}} \cap \mathcal{D}_{A^*_{n-1,0}} \right), \quad \delta^*_n = \dim \left( \mathcal{D}^*_{A^*_{n-1,0}} \ominus \mathcal{D}_{A_{n-1,0}} \cap \mathcal{D}^*_{A^*_{n-1,0}} \right).
\]

4. **Passive discrete-time linear systems and their transfer functions**

4.1. **Basic definitions.** Let $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{H}$ be separable Hilbert spaces. A linear system

\[
\tau = \left\{ \begin{bmatrix} D & C \\ B & A \end{bmatrix} ; \mathcal{M}, \mathcal{N}, \mathcal{H} \right\}
\]

with bounded linear operators $A$, $B$, $C$, $D$ of the form

\[
(4.1) \quad \begin{cases} h_{k+1} = Ah_k + B\xi_k, \\ \sigma_k = Ch_k + D\xi_k, \quad k \geq 0,
\end{cases}
\]

where $\{h_k\} \subset \mathcal{H}$, $\{\xi_k\} \subset \mathcal{M}$, $\{\sigma_k\} \subset \mathcal{N}$, is called a discrete-time system. The Hilbert spaces $\mathcal{M}$ and $\mathcal{N}$ are called the input and the output spaces, respectively, and the Hilbert space $\mathcal{H}$ is called the state space. The operators $A$, $B$, $C$, and $D$ are called the state space operator, the control operator, the observation operator, and the feedthrough operator of $\tau$, respectively. If the linear operator $T_\tau$ defined by the block form

\[
(4.2) \quad T_\tau = \begin{bmatrix} D & C \\ B & A \end{bmatrix} : \mathcal{M} \oplus \mathcal{N} \to \mathcal{H} \oplus \mathcal{H}
\]

is contractive, then the corresponding discrete-time system is said to be passive. If the block operator matrix $T_\tau$ is isometric (co-isometric, unitary), then the system is said to be isometric (co-isometric, conservative). Isometric and co-isometric systems were studied by L. de Branges and J. Rovnyak (see [21], [22]) and by T. Ando (see [6]), conservative systems have been investigated by B. Sz.-Nagy and C. Foiaş (see [38]) and M.S. Brodski (see [23]). Passive systems have been studied by D.Z. Arov et al (see [11], [12], [13], [14], [15]).

The subspaces

\[
(4.3) \quad \mathcal{H}^C := \overline{\text{span}} \{A^nB\mathcal{M} : n = 0, 1, \ldots \} \quad \text{and} \quad \mathcal{H}^O = \overline{\text{span}} \{A^nC^*\mathcal{N} : n = 0, 1, \ldots \}
\]
are said to be the \textit{controllable} and \textit{observable} subspaces of the system $\tau$, respectively. The system $\tau$ is said to be \textit{controllable (observable)} if $\mathcal{H}^c = \mathcal{H}$ ($\mathcal{H}^o = \mathcal{H}$), and it is called \textit{minimal} if $\tau$ is both controllable and observable. The system $\tau$ is said to be \textit{simple} if $\mathcal{H} = \text{clos} \{ \mathcal{H}^c + \mathcal{H}^o \}$ (the closure of the span). It follows from (4.3) that

\begin{equation}
(\mathcal{H}^c)^\perp = \bigcap_{n=0}^{\infty} \ker(B^*A_n^n), \quad (\mathcal{H}^o)^\perp = \bigcap_{n=0}^{\infty} \ker(CA_n^n),
\end{equation}

and therefore there are the following alternative characterizations:

\begin{enumerate}
\item \(\tau\) is controllable \iff \(\bigcap_{n=0}^{\infty} \ker(B^*A_n^n) = \{0\}\);
\item \(\tau\) is observable \iff \(\bigcap_{n=0}^{\infty} \ker(CA_n^n) = \{0\}\);
\item \(\tau\) is simple \iff \(\left(\bigcap_{n=0}^{\infty} \ker(B^*A_n^n)\right) \cap \left(\bigcap_{n=0}^{\infty} \ker(CA_n^n)\right) = \{0\}\).
\end{enumerate}

The \textit{transfer function} \(\Theta_\tau(\lambda) := D + \lambda C(I_{\mathcal{H}} - \lambda A)^{-1}B, \quad \lambda \in \mathbb{D}\),

of the passive system $\tau$ belongs to the Schur class $S(\mathcal{M}, \mathcal{N})$. Conservative systems are also called the \textit{unitary colligations} and their transfer functions are called the characteristic functions [23].

The examples of conservative systems are given by

\[
\Sigma = \left\{ \begin{bmatrix} -A & D_A^* \\ D_A & A^* \end{bmatrix} ; \mathcal{D}_A, \mathcal{D}_A^*, \mathcal{H} \right\}, \quad \Sigma_* = \left\{ \begin{bmatrix} -A^* & D_A \\ D_A^* & A \end{bmatrix} ; \mathcal{D}_A, \mathcal{D}_A^*, \mathcal{H} \right\}.
\]

The transfer functions of these systems

\[
\Phi_\Sigma(\lambda) = (-A + \lambda D_A (I_{\mathcal{H}} - \lambda A^*)^{-1}D_A) \mid \mathcal{D}_A, \quad \lambda \in \mathbb{D}
\]

and

\[
\Phi_{\Sigma_*}(\lambda) = (-A^* + \lambda D_A^* (I_{\mathcal{H}} - \lambda A)^{-1}D_A^*) \mid \mathcal{D}_A^*, \quad \lambda \in \mathbb{D}
\]

are exactly characteristic functions of $A$ and $A^*$, correspondingly.

It is well known that every operator-valued function $\Theta(\lambda)$ from the Schur class $S(\mathcal{M}, \mathcal{N})$ can be realized as the transfer function of some passive system, which can be chosen as controllable isometric (observable co-isometric, simple conservative, minimal passive); cf. [22], [38], [6], [11], [13], [5]. Moreover, two controllable isometric (observable co-isometric, simple conservative) systems with the same transfer function are unitarily similar: two discrete-time systems

\[
\tau_1 = \left\{ \begin{bmatrix} D & C_1 \\ B_1 & A_1 \end{bmatrix} ; \mathcal{M}, \mathcal{N}, \mathcal{H}_1 \right\} \quad \text{and} \quad \tau_2 = \left\{ \begin{bmatrix} D & C_2 \\ B_2 & A_2 \end{bmatrix} ; \mathcal{M}, \mathcal{N}, \mathcal{H}_2 \right\}
\]

are said to be \textit{unitarily similar} if there exists a unitary operator $U$ from $\mathcal{H}_1$ onto $\mathcal{H}_2$ such that

\[
A_1 = U^{-1}A_2U, \quad B_1 = U^{-1}B_2, \quad C_1 = C_2U;
\]

cf. [21], [22], [6], [23], [5]. However, a result of D.Z. Arov [11] states that two minimal passive systems $\tau_1$ and $\tau_2$ with the same transfer function $\Theta(\lambda)$ are only \textit{weakly similar}, i.e., there
is a closed densely defined operator $Z : \mathcal{H}_1 \to \mathcal{H}_2$ such that $Z$ is invertible, $Z^{-1}$ is densely defined, and

$$ZA_1f = A_2Zf, \quad C_1f = C_2Zf, \quad f \in \text{dom } Z, \quad \text{and} \quad ZB_1 = B_2.$$ 

4.2. Defect functions of the Schur class functions. The following result \cite[Proposition V.4.2]{38} is needed in the sequel.

**Theorem 4.1.** Let $\mathcal{M}$ be a separable Hilbert space and let $N(\xi), \xi \in \mathbb{T}$, be an $\mathbf{L}(\mathcal{M})$-valued measurable function such that $0 \leq N(\xi) \leq I_{\mathcal{M}}$. Then there exist a Hilbert space $\mathcal{K}$ and an outer function $\varphi(\lambda) \in \mathbf{S}(\mathcal{M}, \mathcal{K})$ satisfying the following conditions:

(i) $\varphi^*(\xi)\varphi(\xi) \leq N^2(\xi)$ a.e. on $\mathbb{T}$;

(ii) if $\mathcal{K}$ is a Hilbert space and $\bar{\varphi}(\lambda) \in \mathbf{S}(\mathcal{M}, \bar{\mathcal{K}})$ is such that $\bar{\varphi}^*(\xi)\bar{\varphi}(\xi) \leq N^2(\xi)$ a.e. on $\mathbb{T}$, then $\bar{\varphi}^*(\xi)\bar{\varphi}(\xi) \leq \varphi^*(\xi)\varphi(\xi)$ a.e. on $\mathbb{T}$.

Moreover, the function $\varphi(\lambda)$ is uniquely defined up to a left constant unitary factor.

Assume that $\Theta(\lambda) \in \mathbf{S}(\mathcal{M}, \mathcal{K})$ and denote by $\varphi_\Theta(\xi)$ and $\psi_\Theta(\xi), \xi \in \mathbb{T}$ the outer functions which are solutions of the factorization problem described in Theorem 4.1 for $N^2(\xi) = I_{\mathcal{M}} - \Theta^*(\xi)\Theta(\xi)$ and $N^2(\xi) = I_{\mathcal{M}} - \Theta(\xi)\Theta^*(\xi)$, respectively. Clearly, if $\Theta(\lambda)$ is inner or co-inner, then $\varphi_\Theta = 0$ or $\psi_\Theta = 0$, respectively. The functions $\varphi_\Theta(\lambda)$ and $\psi_\Theta(\lambda)$ are called the right and left defect functions (or the spectral factors), respectively, associated with $\Theta(\lambda)$; cf. \cite{17, 18, 19, 20, 29}. The following result has been established in \cite[Theorem 1.1, Corollary 1]{29} (see also \cite[Theorem 3]{19} and \cite[Theorem 1.5]{20}).

**Theorem 4.2.** Let $\Theta(\lambda) \in \mathbf{S}(\mathcal{M}, \mathcal{K})$ and let

$$\tau = \left\{ \begin{bmatrix} D & C \\ B & A \end{bmatrix} : \mathcal{M}, \mathcal{N}, \mathcal{H} \right\}$$

be a simple conservative system with transfer function $\Theta$. Then

(1) the functions $\varphi_\Theta(\lambda)$ and $\psi_\Theta(\lambda)$ take the form

$$\varphi_\Theta(\lambda) = P_\Omega(I_{\mathcal{H}} - \lambda A)^{-1}B,$$

$$\psi_\Theta(\lambda) = C(I_{\mathcal{H}} - \lambda A)^{-1} \Omega_*,$$

where

$$\Omega = (\mathcal{H}^o)^\perp \ominus A(\mathcal{H}^o)^\perp, \quad \Omega_* = (\mathcal{H}^e)^\perp \ominus A^*(\mathcal{H}^e)^\perp$$

and $P_\Omega$ is the orthogonal projector from $\mathcal{H}$ onto $\Omega$;

(2) $\varphi_\Theta(\lambda) = 0$ ($\psi_\Theta(\lambda) = 0$) if and only if the system $\tau$ is observable (controllable).

The defect functions play an essential role in the problems of the system theory, in particular, in the problem of similarity and unitary similarity of the minimal passive systems with equal transfer functions \cite{14, 15} and in the problem of optimal and ($*$) optimal realizations of the Schur function \cite{12, 13}.

4.3. Parametrization of contractive block-operator matrices. Let $\mathcal{H}, \mathcal{K}, \mathcal{M}$ and $\mathcal{N}$ be Hilbert spaces. The following theorem goes back to \cite{16, 26, 36}; other proofs of the theorem can be found in \cite{31, 32, 7, 9}.
Theorem 4.3. Let \( A \in \mathbf{L}(\mathcal{H}, \mathcal{K}), B \in \mathbf{L}(\mathcal{M}, \mathcal{K}), C \in \mathbf{L}(\mathcal{H}, \mathcal{N}), \) and \( D \in \mathbf{L}(\mathcal{M}, \mathcal{N}). \) The operator matrix
\[
T = \begin{bmatrix}
D & C \\
B & A
\end{bmatrix} : \mathcal{M} \oplus \mathcal{N} \rightarrow \mathcal{H} \oplus \mathcal{K}
\]
is a contraction if and only if \( T \) is of the form
\[
(4.6) \quad T = \begin{bmatrix}
-KA^*M + D_{K^*}XD_M & KD_A \\
D_{A^*}M & A
\end{bmatrix},
\]
where \( A \in \mathbf{L}(\mathcal{H}, \mathcal{K}), M \in \mathbf{L}(\mathcal{M}, \mathcal{D}_{A^*}), K \in \mathbf{L}(\mathcal{D}_A, \mathcal{N}), \) and \( X \in \mathbf{L}(\mathcal{D}_M, \mathcal{D}_{K^*}) \) are contractions, all uniquely determined by \( T. \) Furthermore, the following equality holds for all \( h \in \mathcal{M}, f \in \mathcal{H}: \)
\[
(4.7) \quad \| [h] \|^2 - \left\| \begin{bmatrix}
-KA^*M + D_{K^*}XD_M & KD_A \\
D_{A^*}M & A
\end{bmatrix} [f] \right\|^2 = \| D_K(D_Af - A^*Mh) - K^*XD_Mh \|^2 + \| D_XD_Mh \|^2.
\]

Corollary 4.4. Let
\[
T = \begin{bmatrix}
-KA^*M + D_{K^*}XD_M & KD_A \\
D_{A^*}M & A
\end{bmatrix} : \mathcal{M} \oplus \mathcal{N} \rightarrow \mathcal{H} \oplus \mathcal{K}
\]
be a contraction. Then
(1) \( T \) is isometric if and only if \( D_KD_A = 0, D_XD_M = 0, \)
(2) \( T \) is co-isometric if and only if \( D_M^*, D_{A^*} = 0, D_X^*, D_{K^*} = 0. \)

Note that the relation \( D_YD_Z = 0 \) for contractions \( Y \) and \( Z \) means that either \( Z \) is an isometry and \( Y = 0 \) or \( \mathcal{D}_Z \neq \{0\} \) and \( Y \) is an isometry. From (4.7) we get the following statement
If \( T \) given by (4.6) is unitary then \( D_{K^*} = 0 \iff D_M = 0. \)

Let \( \tau = \left\{ \begin{bmatrix}
D & C \\
B & A
\end{bmatrix} ; \mathcal{M}, \mathcal{N}, \mathcal{H} \right\} \) be a conservative system. Then from Corollary 4.3 we get
\[
(\mathcal{H})^\perp = \bigcap_{n \geq 0} \ker(D_{A^*}A^n) = \bigcup_{n \geq 1} \ker(D_{A^n}),
\]
\[
(\mathcal{M})^\perp = \bigcap_{n \geq 0} \ker(D_AA^n) = \bigcup_{n \geq 1} \ker(D_{A^n}).
\]
\( \tau \) is controllable \( \iff \bigcap_{n \geq 1} \ker(D_{A^n}) = \{0\} \iff \) the operator \( A^* \) does not contain a shift,
\( \tau \) is observable \( \iff \bigcap_{n \geq 1} \ker(D_{A^n}) = \{0\} \iff \) the operator \( A \) does not contain a shift.

It follows that a conservative system is simple if and only if the state space operator is completely non-unitary [23].
In \cite{9} we used Theorem 4.3 for connections between the passive system
\[ \tau = \left\{ \begin{bmatrix} D & C \\ B & A \end{bmatrix} ; \mathcal{M}, \mathcal{H} \right\}, \]
its transfer function \( \Theta_\tau(\lambda) \), and the characteristic function of \( A \). In particular, an immediate consequence of (4.6) is the following relation
\[ (4.9) \quad \Theta_\tau(\lambda) = K \Phi_{A^*}(\lambda) M + D K^* X D M, \quad \lambda \in \mathbb{D}, \]
where \( \Phi_{A^*}(\lambda) \) is the characteristic function of \( A^* \).

Recall that if \( \Theta(\lambda) \in \mathcal{S}(\mathfrak{H}_1, \mathfrak{H}_2) \) then there is a uniquely determined decomposition \cite{38, Proposition V.2.1}
\[ \Theta(\lambda) = \begin{bmatrix} \Theta_p(\lambda) & 0 \\ 0 & \Theta_u \end{bmatrix} : \mathcal{D}_{\Theta(0)} \oplus \ker D_{\Theta(0)} \to \mathcal{D}_{\Theta^*(0)} \oplus \ker D_{\Theta^*(0)}, \]
where \( \Theta_p(\lambda) \in \mathcal{S}(\mathcal{D}_{\Theta(0)}, \mathcal{D}_{\Theta^*(0)}) \), \( \Theta_p(0) \) is a pure contraction and \( \Theta_u \) is a unitary constant.

The function \( \Theta_p(\lambda) \) is called the pure part of \( \Theta(\lambda) \) (see \cite{17}). If \( \Theta(0) \) is isometric (co-isometric) then the pure part is of the form \( \Theta_p(\lambda) = 0 \in \mathcal{S}(\{0\}, \mathcal{D}_{\Theta^*(0)}) \) (\( \Theta_p(\lambda) = 0 \in \mathcal{S}(\mathcal{D}_{\Theta(0)}, \{0\}) \)).

From (4.6) and (4.9) we get the following statement.

**Proposition 4.5.** Let
\[ \tau = \left\{ \begin{bmatrix} D & C \\ B & A \end{bmatrix} ; \mathcal{M}, \mathcal{H} \right\} \]
be a a simple conservative system and let \( \Theta(\lambda) \) be its transfer function. Then
\[ (4.10) \quad \dim \mathcal{D}_A = \dim \mathcal{D}_{\Theta^*(0)} = \dim (\mathcal{M} \oplus \ker C^*), \]
\[ \dim \mathcal{D}_{A^*} = \dim \mathcal{D}_{\Theta(0)} = \dim (\mathcal{M} \oplus \ker B), \]
and the pure part of \( \Theta \) coincides with the Sz.-Nagy–Foias characteristic function of \( A^* \).

In addition
1) if \( \Theta(0) \) is isometric then \( B = 0 \), \( A \) is a co-shift of multiplicity \( \dim \mathcal{D}_{\Theta^*(0)} \), and the system \( \tau \) is observable;
2) if \( \Theta(0) \) is co-isometric then \( C = 0 \), \( A \) is a unilateral shift of multiplicity \( \dim \mathcal{D}_{\Theta(0)} \), and the system \( \tau \) is controllable.

**Proof.** According to Theorem 4.3 the operator
\[ T = \begin{bmatrix} D & C \\ B & A \end{bmatrix} : \mathcal{M} \oplus \mathcal{H} \to \mathcal{M} \oplus \mathcal{H} \]
takes the form (4.6). Since \( T \) is unitary, from (4.12) we get that the operators \( K \in \mathcal{L}(\mathcal{D}_A, \mathcal{M}) \) and \( M^* \in \mathcal{L}(\mathcal{D}_{A^*}, \mathcal{M}) \) are isometries and the operator \( X \in \mathcal{L}(\mathcal{D}_M, \mathcal{D}_{K^*}) \) is unitary. From (4.9) it follows that the pure part of \( \Theta \) is given by
\[ \Theta(\lambda) \upharpoonright \text{ran} M^* = K \Phi_{A^*}(\lambda) M \upharpoonright \text{ran} M^* : \text{ran} M^* \to \text{ran} K. \]
Thus, the pure part of $\Theta$ coincides with $\Phi$. Since $\text{ran} \, M^* = \mathcal{D}_{A^*}$, $\text{ran} \, K^* = \mathcal{D}_A$,
\[
D = \Theta(0) = K \Phi A^*(0) M^* = -K A^* M^*, \quad D^* = \Theta^*(0) = -M A K^*,
\]
we get (4.10).

Suppose $D = \Theta(0)$ is an isometry. Then the pure part of $\Theta$ is $0 \in \mathcal{S}(\{0\}, \mathcal{D}_{D^*})$. It follows
that $M = B = 0$ and $\mathcal{D}_{A^*} = \{0\}$. Hence, $A$ is co-isometric and since $A$ is a completely
non-unitary contraction, it is a co-shift of multiplicity $\dim \mathcal{D}_A = \dim \mathcal{D}_{\Theta^*(0)}$, and the system
$\tau$ is observable. Similarly the statement 2) holds.

In this paper we will use a parametrization of a contractive block-operator matrix based on a fixed upper left block $D \in \mathcal{L}(\mathcal{M}, \mathcal{N})$. With this aim we reformulate Theorem 4.3 and Corollary 4.4.

**Theorem 4.6.** The operator matrix
\[
T = \begin{bmatrix} D & C \\ B & A \end{bmatrix} : \mathcal{M} \oplus \mathcal{N} \to \mathcal{H} \oplus \mathcal{K}
\]
is a contraction if and only if $D \in \mathcal{L}(\mathcal{M}, \mathcal{N})$ is a contraction and the entries $A, B,$ and $C$
take the form
\[
B = F D_D, \quad C = D_{D^*} G, \\
A = -F D^* G + D_{F^*} L D_G,
\]
where the operators $F \in \mathcal{L}(\mathcal{D}_D, \mathcal{K})$, $G \in \mathcal{L}(\mathcal{H}, \mathcal{D}_{D^*})$ and $L \in \mathcal{L}(\mathcal{D}_G, \mathcal{D}_{F^*})$ are contractions.
Moreover, operators $F, G,$ and $L$ are uniquely determined. Furthermore, the following equality holds
\[
\left\| D_T \begin{bmatrix} h \\ f \end{bmatrix} \right\|^2 = \left\| D_F (D_D h - D^* G f) - F^* L D_G f \right\|^2 + \left\| D_L D_G f \right\|^2,
\]
and
\[
\left\| D_{T^*} \begin{bmatrix} \varphi \\ g \end{bmatrix} \right\|^2 = \left\| D_{G^*} (D_{D^*} \varphi - D F^* g) - G L^* D_{F^*} g \right\|^2 + \left\| D_{L^*} D_{F^*} g \right\|^2,
\]

(1) the operator $T$ is isometric if and only if
\[
D_F D_D = 0, \quad D_L D_G = 0,
\]
(2) the operator $T$ is co-isometric if and only if
\[
D_G^* D_{D^*} = 0, \quad D_{L^*} D_{F^*} = 0,
\]
(3) if $T$ is unitary then $D_{F^*} = 0 \iff D_G = 0$.

Let us give connections between the parametrization of a unitary block-operator matrix $T$ given by (4.6) and (4.11).
Proposition 4.7. Let

\[ T = \begin{bmatrix} -KA^*M + D_K XD_M & KD_A \\ D_A^*M & A \end{bmatrix} = \begin{bmatrix} D & D_D^*G \\ FD_D & -FD^*G + D_{F^*}LD_G \end{bmatrix} : \mathcal{M} \to \mathcal{N} \]

be a unitary operator matrix. Then

\[ \mathcal{D}_D = \text{ran } M^*, \mathcal{D}_{D^*} = \text{ran } K, \]

\[ F^* = M^* P_{\mathcal{D}_A^*}, \ G = K P_{\mathcal{D}_A}, \]

\[ GFf = KP_{\mathcal{D}_A} Mf, \ f \in \mathcal{D}_D, \]

\[ L = A| \ker D_A. \]

Proof. Since \( D = -KA^*M + D_K XD_M \), we have

\[ ||D_D f||^2 = ||D_A^*Mf||^2 + ||(K A^*M - K^* XD_M)f||^2 + ||DXD_Mf||^2, \ f \in \mathcal{M}, \]

\[ ||D_D^* g||^2 = ||D_A K^*g||^2 + ||(D_M^*AK^* - MX^*DK^*)g||^2 + ||DXDK^*g||^2, \ g \in \mathcal{N}. \]

By Corollary 4.4 the operators \( K \) and \( M^* \) are isometries and \( X \in \mathcal{L}(\mathcal{D}_M, \mathcal{D}_K^*) \) is unitary operator. It follows that

\[ ||D_D f||^2 = ||D_A^*Mf||^2, \ f \in \mathcal{M}, \]

\[ ||D_D^* g||^2 = ||D_A K^*g||^2, \ g \in \mathcal{N}. \]

Hence, \( D_D^2 = M^* D_A^2 M, \ D_{D^*}^2 = K D_A^2 K^* \). Since \( K \) and \( M^* \) are isometries, we obtain

\[ D_D = M^* D_A^* M, \ D_{D^*} = K D_A K^*. \]

It follows that \( \mathcal{D}_D = \text{ran } M^*, \mathcal{D}_{D^*} = \text{ran } K, D_A^* M = FM^* D_A^* M, \) and \( D_A K^* = G^* KD_A K^* \). Therefore,

\[ FM^* = I_{\mathcal{D}_A^*}, \ G^* K = I_{\mathcal{D}_A}. \]

It follows

\[ F = M| \mathcal{D}_D, \ G^* = K^*| \mathcal{D}_{D^*}. \]

Hence, \( F^* = M^* P_{\mathcal{D}_A^*} \) and \( G = K P_{\mathcal{D}_A} \). In addition

\[ D_{F^*}^2 = I_{\mathcal{H}} - MM^* P_{\mathcal{D}_A^*} = P_{ker D_{A^*}}, \ D_G^2 = I_{\mathcal{H}} - K^* KP_{\mathcal{D}_A} = P_{ker D_A}, \]

\[ -FD^*G = -F(-M^* AK^* + D_M^* X^* DK^*) KP_{\mathcal{D}_A} = AP_{\mathcal{D}_A}, \]

\[ A = -FD^*G + D_{F^*} LD_G = AP_{\mathcal{D}_A} + P_{ker D_{A^*}} LP_{ker D_A}. \]

On the other hand

\[ A = AP_{\mathcal{D}_A} + AP_{ker D_A}. \]

Hence \( L = A| \ker D_A. \)

Let \( D : \mathcal{M} \to \mathcal{N} \) be a contraction with nonzero defect operators and let

\[ Q = \begin{bmatrix} 0 & G \\ F & S \end{bmatrix} : \mathcal{D}_D \to \mathcal{D}_{D^*} \]
be a bounded operator. Define the transformation (see [8])

\[ (4.14) \quad \mathcal{M}_D(Q) = \begin{bmatrix} D & 0 \\ 0 & -FD^*G \end{bmatrix} + \begin{bmatrix} D_{D^*}^* \\ I \end{bmatrix} \begin{bmatrix} 0 & G \\ F & S \end{bmatrix} \begin{bmatrix} D_{D^*} \\ 0 \\ I \end{bmatrix}. \]

Clearly, the operator \( T = \mathcal{M}_D(Q) \) has the following matrix form

\[ T = \begin{bmatrix} D & D_{D^*}^*G \\ FD_D & S - FD^*G \end{bmatrix} : \mathcal{M} \to \mathcal{M}. \]

**Proposition 4.8.** [8] Let \( \mathcal{H}, \mathcal{M}, \mathcal{N} \) be separable Hilbert spaces and let \( D : \mathcal{M} \to \mathcal{N} \) be a contraction with nonzero defect operators. Let \( Q = \begin{bmatrix} 0 & G \\ F & S \end{bmatrix} : \mathcal{M} \to \mathcal{M} \) be a bounded operator. Then

(1) \( T = \mathcal{M}_D(Q) \) is a contraction if and only if \( Q \) is a contraction. \( T \) is isometric (co-isometric) if and only if \( Q \) is isometric (co-isometric);

(2) holds the relations

\[ (4.15) \quad \begin{cases} \bigcap_{n=0}^{\infty} \ker (B^*A^n) = \bigcap_{n=0}^{\infty} \ker (F^*S^n), \\ \bigcap_{n=0}^{\infty} \ker (CA^n) = \bigcap_{n=0}^{\infty} \ker (GS^n). \end{cases} \]

5. The Möbius representations

Let \( T : \mathcal{H}_1 \to \mathcal{H}_2 \) be a contraction. In [37] and [34] were studied the fractional-linear transformations of the form

\[ (5.1) \quad Z \to Q = T + D_T \cdot Z(I_{\mathcal{D}_T} + T^*Z)^{-1}D_T = T + D_T \cdot (I_{\mathcal{D}_T^*} + ZT^*)^{-1}ZD_T \]

defined on the set \( \mathcal{V}_{T^*} \) of all contractions \( Z \in \mathcal{L}(\mathcal{D}_T, \mathcal{D}_T) \) such that \(-1 \in \rho(T^*Z)\). The following result holds.

**Theorem 5.1.** [34] Let the \( T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2) \) be a contraction and let \( Z \in \mathcal{V}_{T^*}. \) Then \( Q = T + D_T \cdot Z(I_{\mathcal{D}_T} + T^*Z)^{-1}D_T \) is a contraction,

\[ (5.2) \quad \|D_Qf\|^2 = \|D_Z(I_{\mathcal{D}_T} + T^*Z)^{-1}D_Tf\|^2, \quad f \in \mathcal{H}_1, \]

\( \text{ran } D_Q \subseteq \text{ran } D_T, \) and \( \text{ran } D_Q = \text{ran } D_T \) if and only if \( \|Z\| < 1. \) Moreover, if \( Q \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2) \) is a contraction and \( Q = T + D_T \cdot X^*D_T \), where \( X \in \mathcal{L}(\mathcal{D}_T, \mathcal{D}_T) \) then \( X \in \mathcal{V}_{T^*}, \)

\[ Z = X(I_{\mathcal{D}_T} - T^*X)^{-1} \in \mathcal{V}_{T^*}, \]

and the operator \( Q \) takes the form \( Q = T + D_T \cdot Z(I_{\mathcal{D}_T} + T^*Z)^{-1}D_T \).
Observe that from (5.1) one can derive the equalities
\[ I_{H_2} - QT^* = D_T^*(I_{D_T^*} + ZT^*)^{-1}D_T^*, \]
\[ Z|\text{ran}\, D_T = D_T^*(I_{H_2} - QT^*)^{-1}(Q - T)D_T^{-1}. \]

The transformation (5.1) is called in [34] the unitary linear-fractional transformation. It is easy to see that if \(||T|| < 1\) then the closed unit operator ball in \(L(\mathcal{H}_1, \mathcal{H}_2)\) belongs to the set \(V_T^\ast\) and, moreover
\[ T + D_T^*Z(I_{D_T} + T^*Z)^{-1}D_T = D_T^*(I_{D_T^*} + ZT^*)^{-1}(Z + T)D_T^{-1} \]
for all \(Z \in L(\mathcal{H}_1, \mathcal{H}_2)\), \(||Z|| \leq 1\). Thus, the transformation (5.1) is an operator analog of a well known Möbius transformation of the complex plane
\[ z \to \frac{z + t}{1 + tz}, \quad |t| \leq 1. \]

The next theorem is a version of a more general result established by Yu.L. Shmul’yan in [35].

**Theorem 5.2.** [35] Let \(M\) and \(N\) be Hilbert spaces and let the function \(\Theta(\lambda)\) be from the Schur class \(S(M, N)\). Then

1. the linear manifolds \(\text{ran}\, D\Theta(\lambda)\) and \(\text{ran}\, D\Theta^\ast(\lambda)\) do not depend on \(\lambda \in \mathbb{D}\),
2. for arbitrary \(\lambda_1, \lambda_2, \lambda_3\) in \(\mathbb{D}\) the function \(\Theta(\lambda)\) admits the representation
   \[ \Theta(\lambda) = \Theta(\lambda_1) + D\Theta^\ast(\lambda_2)\Psi(\lambda)D\Theta(\lambda_3); \]
where \(\Psi(\lambda)\) is a holomorphic in \(\mathbb{D}\) and \(L(D\Theta(\lambda_3), D\Theta^\ast(\lambda_2))\)-valued function.

Now using Theorems 5.1 and 5.2 we get Theorem 1.1. Recall that the representation (1.2) of a function \(\Theta(\lambda) \in S(M, N)\) is called the Möbius representation of \(\Theta\) and the function \(Z(\lambda) \in S(D\Theta(0), D\Theta^\ast(0))\) is called the Möbius parameter of \(\Theta\).

The next result established in [8] provides connections between the realizations of \(\Theta(\lambda)\) and \(Z(\lambda)\) as transfer functions of passive systems.

**Theorem 5.3.** [8]

1. Let \(\tau = \left\{ \begin{bmatrix} D & C \\ B & A \end{bmatrix}; M, N, \mathcal{H} \right\}\) be a passive system and let
   \[ T = \begin{bmatrix} D & C \\ B & A \end{bmatrix} = \begin{bmatrix} D & D_D^*G \\ FD_D & -FD^*_G + D_F^*LD_G \end{bmatrix}: M \oplus \mathcal{H} \to N \oplus \mathcal{H}. \]
Let \(\Theta(\lambda)\) be the transfer function of \(\tau\). Then
   (a) the Möbius parameter \(Z(\lambda)\) of the function \(\Theta(\lambda)\) is the transfer function of the passive system
   \[ \nu = \left\{ \begin{bmatrix} 0 & G \\ F & D_F^*LD_G \end{bmatrix}; L_D, L_D^*, \mathcal{H} \right\}; \]
   (b) the system \(\tau\) isometric (co-isometric) \(\Rightarrow\) the system \(\nu\) isometric (co-isometric);
(c) the equalities $\mathcal{H}_\nu^c = \mathcal{H}_\nu^s$, $\mathcal{H}_\nu^o = \mathcal{H}_\nu^c$ hold and hence the system $\tau$ is controllable (observable) $\Rightarrow$ the system $\nu$ is controllable (observable), the system $\tau$ is simple (minimal) $\Rightarrow$ the system $\nu$ is simple (minimal).

(2) Let $\Theta(\lambda) \in \mathcal{S}(\mathcal{M}, \mathcal{N})$ and let $Z(\lambda)$ be the M"obius parameter of $\Theta(\lambda)$. Suppose that the transfer function of the linear system

$$\nu' = \left\{ \begin{bmatrix} 0 & G \\ F & S \end{bmatrix}; \mathcal{D}_\Theta(0), \mathcal{D}_\Theta^*(0), \mathcal{H} \right\}$$

coincides with $Z(\lambda)$ in a neighborhood of the origin. Then the transfer function of the linear system

$$\tau' = \left\{ \begin{bmatrix} \Theta(0) & D_\Theta(0)G \\ FD_\Theta(0) & -F\Theta^*(0)G + S \end{bmatrix}; \mathcal{M}, \mathcal{N}, \mathcal{H} \right\}$$

coincides with $\Theta(\lambda)$ in a neighborhood of the origin. Moreover

(a) the equalities $\mathcal{H}_\tau^c = \mathcal{H}_\tau^s$, $\mathcal{H}_\tau^o = \mathcal{H}_\tau^c$ hold, and hence the system $\nu'$ is controllable (observable) $\Rightarrow$ the system $\tau'$ is controllable (observable), the system $\nu'$ is simple $\Rightarrow$ the system $\tau'$ is simple (minimal),

(b) the system $\nu'$ is passive $\Rightarrow$ the system $\tau'$ is passive (minimal),

(c) the system $\nu'$ isometric (co-isometric) $\Rightarrow$ the system $\tau'$ isometric (co-isometric).

**Corollary 5.4.** 1) The equivalences

$$\varphi_\Theta(\lambda) = 0 \iff \varphi_Z(\lambda) = 0,$$

$$\psi_\Theta(\lambda) = 0 \iff \psi_Z(\lambda) = 0$$

hold.

2) Let $||\Theta(0)|| \mathcal{D}_\Theta(0)|| < 1$. Suppose $\varphi(\lambda) \in \mathcal{S}(\mathcal{M}, \mathcal{L})$ ($\psi(\lambda) \in \mathcal{S}(\mathcal{K}, \mathcal{N})$) and

$$\varphi^*(\xi)\varphi(\xi) = D_\Theta(\xi)^2 \text{ for almost all } \xi \in \mathbb{T}$$

$$\left(\psi(\xi)\psi^*(\xi) = D_\Theta^*(\xi)^2 \right. \text{ for almost all } \xi \in \mathbb{T} \right).$$

Then

$$\tilde{\varphi}(\lambda) := \varphi(\lambda)D^{-1}_\Theta(0)(I - \Theta(0)Z(\lambda)) \in \mathcal{S}(\mathcal{D}_\Theta(0), \mathcal{L})$$

$$\tilde{\psi}(\lambda) := (I - \Theta^*(0)Z(\lambda))D^{-1}_\Theta(0)P_\Theta^*(0)\psi(\lambda) \in \mathcal{S}(\mathcal{K}, \mathcal{D}_\Theta^*(0))$$

and

$$\tilde{\varphi}^*(\xi)\tilde{\varphi}(\xi) = D_{Z(\xi)}^2 \text{ for almost all } \xi \in \mathbb{T}$$

$$\tilde{\psi}(\xi)\tilde{\psi}^*(\xi) = D_{Z^*(\xi)}^2 \text{ for almost all } \xi \in \mathbb{T}.\right.$$

In particular,

$$\Theta(\lambda) \text{ is inner (co-inner) } \iff Z(\lambda) \text{ is inner (co-inner)}.$$

**Proof.** 1) Let $\varphi_\Theta(\lambda) = 0$ ($\psi_\Theta(\lambda) = 0$) and let $\tau = \left\{ \begin{bmatrix} D & C \\ B & A \end{bmatrix}; \mathcal{M}, \mathcal{N}, \mathcal{H} \right\}$ be a simple conservative system with transfer $\Theta(\lambda)$. By Theorem 4.2 the system $\tau$ is observable (controllable). As it is proved above the corresponding system $\nu$ with transfer function $Z(\lambda)$ is conservative and observable (controllable). Theorem 4.2 yields that $\varphi_Z(\lambda) = 0$ ($\psi_Z(\lambda) = 0$).

Conversely. Let $\varphi_Z(\lambda) = 0$ ($\psi_Z(\lambda) = 0$) and let $\nu'$ be a simple conservative system with transfer function $Z(\lambda)$. Again by Theorem 4.2 the system $\nu'$ is observable (controllable). As
it is already proved the corresponding system $\tau'$ with transfer function $\Theta(\lambda)$ is conservative and observable (controllable) as well. Now Theorem 4.2 yields that $\varphi_\Theta(\lambda) = 0$ ($\psi_\Theta(\lambda) = 0$).

2) Let $||\Theta(0)|D_{\Theta(0)}|| < 1$. Since

$$\Theta^*(0)|D_{\Theta^*(0)} = (\Theta(0)|D_{\Theta(0)})^*,$$

we get $||\Theta^*(0)|D_{\Theta^*(0)}|| < 1$. It follows that the operators $D_{\Theta(0)}|D_{\Theta(0)}$ and $D_{\Theta^*(0)}|D_{\Theta^*(0)}$ have bounded inverses. From (5.2) we obtain the relation

$$||D_{\Theta(0)}^{-1}(I_{D_{\Theta(0)}} + \Theta^*(0)Z(\lambda)Zf)||^2 = ||D_{\Theta(0)}f||^2, \lambda \in \mathbb{D}, f \in D_{\Theta(0)}.$$ 

The non-tangential limits $\Theta(\xi)$ and $Z(\xi)$ exist for almost all $\xi \in \mathbb{T}$. It follows that the relation

$$||D_{\Theta(\xi)}D_{\Theta^*(0)}^{-1}(I_{D_{\Theta(0)}} + \Theta^*(0)Z(\xi))f||^2 = ||D_{\Theta(\xi)}f||^2, f \in D_{\Theta(0)}.$$ 

for almost all $\xi \in \mathbb{T}$. This completes the proof. \hfill \Box

**Theorem 5.5.** Let $A$ be a completely non-unitary contraction in the Hilbert space $\mathcal{H}$ and let $Z(\lambda)$ be the M"obius parameter of the Sz.Nagy–Foias characteristic function of $A$. Then $Z(\lambda)$ is the characteristic function of the operator $A_{1,0} = AP_{\ker D_A}$ (see (3.2) and (3.3)). Moreover, the following statements are equivalent

(i) the unitary equivalent operators $A_{1,0}$ and $A_{0,1}$ are unilateral shifts (co-shifts),

(ii) $\mathcal{D}_A \subset \mathcal{D}_{A^*}$ ($\mathcal{D}_{A^*} \subset \mathcal{D}_A$),

(iii) the M"obius parameter takes the form $Z(\lambda) = \lambda I_{\mathcal{D}_A}$ ($Z^*(\lambda) = \lambda I_{\mathcal{D}_{A^*}}$).

**Proof.** The system

$$\Sigma = \left\{ \begin{bmatrix} -A & D_{A^*} \\ D_A & A^* \end{bmatrix} \right\} : \mathcal{D}_A, \mathcal{D}_{A^*}, \mathcal{H}$$

is conservative and simple and its transfer function

$$\Phi(\lambda) = (-A + \lambda D_{A^*} (I_\mathcal{H} - \lambda A^*)^{-1} D_A)|D_A$$

is the characteristic function of $A$. Let $F$ and $G^*$ be the embedding of the subspaces $\mathcal{D}_A$ and $\mathcal{D}_{A^*}$ into $\mathcal{H}$, respectively. It follows that

$$D_{F^*} = P_{\ker D_A}, D_G = P_{\ker D_{A^*}}.$$ 

Let $L = A^*|\ker D_{A^*}$. Then

$$A^* = A^* P_{\mathcal{D}_{A^*}} + A^* P_{\ker D_{A^*}} = -F(-A^*)G + D_{F^*}LD_G$$ 

Let

$$\Phi(\lambda) = \Phi(0) + D_{\Phi^*(0)}Z(\lambda)(I + \Phi^*(0)Z(\lambda))^{-1}D_{\Phi(0)}, \lambda \in \mathbb{D}$$

be the M"obius representation of the function $\Phi(\lambda)$. By Theorem 5.3 the system

$$\nu = \left\{ \begin{bmatrix} 0 & P_{\mathcal{D}_{A^*}} \\ I_{\mathcal{D}_A} & A^* P_{\ker D_{A^*}} \end{bmatrix} \right\} : \mathcal{D}_A, \mathcal{D}_{A^*}, \mathcal{H}$$

is conservative and simple and its transfer function is the function $Z(\lambda)$, i.e.,

$$Z(\lambda) = \lambda P_{\mathcal{D}_{A^*}} (I_{\mathcal{H}} - \lambda A^* P_{\ker D_{A^*}})^{-1} |\mathcal{D}_A, |\lambda| < 1.$$ 

This function is exactly the Sz.-Nagy–Foias characteristic function of the partial isometry $A_{1,0} = AP_{\ker \mathcal{D}_A}$. 

YURY ARLINSKIİ
Suppose \( A_{1,0} = P_{\ker D_A} A^* \) ker \( D_A \) is a unilateral shift. Since \( A \ker D_A = \ker D_A^* \), we have \( \ker D_A^* \subset \ker D_A \). Equivalently \( \mathcal{D}_A \subset \mathcal{D}_A^* \). Hence,

\[
P_{\ker D_A^*} | \mathcal{D}_A = 0 \quad \text{and} \quad (A^* P_{\ker D_A^*})^n | \mathcal{D}_A = 0 \quad \text{for all} \quad n \in \mathbb{N}.
\]

Therefore,

\[
Z(\lambda) = \lambda P_{\mathcal{D}_A^*} | \mathcal{D}_A = \lambda I_{\mathcal{D}_A}.
\]

Conversely, suppose \( Z(\lambda) = \lambda I_{\mathcal{D}_A} \). Then \( \mathcal{D}_A \subset \mathcal{D}_A^* \Rightarrow \ker D_A \supset \ker D_A^* \). It follows \( A \ker D_A \subset \ker D_A \Rightarrow A_{1,0} \) is isometry.

Since the operator \( A_{1,0} \) is completely non-unitary, it is a unilateral shift. \( \square \)

**Corollary 5.6.** Let \( A \) be a completely non-unitary contraction in a separable Hilbert space \( \mathcal{H} \) and let \( ||A| | \mathcal{D}_A|| < 1 \) (\( \iff \) ran \( D_A = \overline{\text{ran}} D_A \)). Then the following statements are equivalent

(i) \( A \in C_0 \) (respect., \( A \in C_0 \)),

(ii) \( A_{1,0} \) \( \in C_0 \) (respect., \( A_{1,0} \in C_0 \)).

**Proof.** By (6.4) we have \( \Phi_A(0) = -A | \mathcal{D}_A \). Then in accordance with [38], Corollary 5.4 and Theorem 5.5 we get the equivalences

\[
A \in C_0 \ (C_0) \iff \Phi_A(\lambda) \text{ is inner (co-inner)} \iff Z(\lambda) \text{ is inner (co-inner)} \\
\iff A_{1,0} \in C_0 \ (C_0).
\]

\( \square \)

### 6. Realizations of the Schur iterates

#### 6.1. Realizations of the first Schur iterate.

**Proposition 6.1.** Let \( \mathcal{H}, \mathcal{L}, \mathcal{R} \) be Hilbert spaces and let \( F \in \mathcal{L}(\mathcal{L}, \mathcal{H}) \), \( G \in \mathcal{L}(\mathcal{H}, \mathcal{R}) \) and \( L \in \mathcal{L}(\mathcal{D}_G, \mathcal{D}_F^*) \) be contractions. Let \( Z_\nu(\lambda) \) be the transfer function of the system

\[
\nu = \begin{bmatrix} 0 & G \\ F & D_F L D_G \end{bmatrix} ; \mathcal{L}, \mathcal{R}, \mathcal{H}.
\]

Then the function \( \Gamma(\lambda) = \lambda^{-1} Z_\nu(\lambda) \) is the transfer function of the passive systems

\[
\eta_1 = \begin{bmatrix} GF & GD_F^* \\ LD_G F & LD_G D_F \end{bmatrix} ; \mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{H}_G \quad \text{and} \quad \eta_2 = \begin{bmatrix} GF & GD_F \tilde{L} \\ D_G F & D_G D_F \tilde{L} \end{bmatrix} ; \mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{H}_G,
\]

where \( \tilde{L} = LP_{D_G} \).

Suppose that the subspaces \( \mathcal{H}_{G_1} = D_F^* \) and \( \mathcal{H}_{G_2} = D_G \) are nontrivial. Then the transfer functions of the passive systems

\[
\zeta_1 = \begin{bmatrix} GF & GD_F \\ LD_G F & LD_G D_F \end{bmatrix} ; \mathcal{L}, \mathcal{R}, \mathcal{H}_{G_1} \quad \text{and} \quad \zeta_2 = \begin{bmatrix} GF & GD_F \tilde{L} \\ D_G F & D_G D_F \tilde{L} \end{bmatrix} ; \mathcal{L}, \mathcal{R}, \mathcal{H}_{G_2}
\]

are equal to \( \Gamma(\lambda) \). Moreover, for the orthogonal complements to the controllable and observable subspaces of the systems \( \nu, \zeta_1, \) and \( \zeta_2 \) hold the following relations

\[
(\mathcal{H}_\nu^c)^\perp = (\mathcal{H}_{G_1}^c)^\perp \cap \ker F^*, \quad (\mathcal{H}_\nu^c)^\perp = (\mathcal{H}_{G_2}^c)^\perp \cap \ker G,
\]

\[
D_G (\mathcal{H}_\nu^c)^\perp \subset (\mathcal{H}_{G_1}^c)^\perp, \quad D_F (\mathcal{H}_\nu^c)^\perp \subset (\mathcal{H}_{G_2}^c)^\perp.
\]
If the operators $G^*$ and $F$ are isometries, then

$$
(\mathcal{F}_{G^*})^\perp = (\mathcal{F}_{F^*})^\perp \cap \ker F^*,
(\mathcal{F}_{G})^\perp = (\mathcal{F}_{F})^\perp \cap \ker G.
$$

**Proof.** We have

$$Z_\nu(\lambda) = \lambda G(I_G - \lambda D_F L D_G)^{-1} F.$$ 

Hence

$$\Gamma(\lambda) = \frac{Z_\nu(\lambda)}{\lambda} = G(I_G - \lambda D_F L D_G)^{-1} F$$

and $\Gamma(0) = GF$. It follows that

$$\Gamma(\lambda) - \Gamma(0) = G(I_G - \lambda D_F L D_G)^{-1} F - GF = \lambda GD_F L D_G (I_G - \lambda D_F L D_G)^{-1} F$$

$$= \lambda GD_F (I_G - \lambda D G D F^*)^{-1} L D G F = \lambda GD_F (I_G - \lambda D G F^*)^{-1} L D G F$$

$$= \lambda GD_F (I_G - \lambda D G D F^*)^{-1} L D G F,$$

$$\Gamma(\lambda) = GF + \lambda GD_F (I_G - \lambda D G D F^*)^{-1} L D G F$$

$$= GF + \lambda GD_F (I_G - \lambda D G D F^*)^{-1} L D G F.$$ 

The operators

$$K_1 = \begin{bmatrix} GF & GD_F L D G F \\ L D G F & L D G L \end{bmatrix} : \mathcal{L} \rightarrow \mathcal{H}$$

and

$$K_2 = \begin{bmatrix} GF & GD_F L \\ D G F & D G L \end{bmatrix} : \mathcal{L} \rightarrow \mathcal{H}$$

are contraction. Actually, let $f, g \in \mathcal{L}$ and $h \in \mathcal{K}$ then one can check that

$$\left\| \begin{bmatrix} f \\ h \end{bmatrix} \right\|^2 - \left\| K_1 \begin{bmatrix} f \\ h \end{bmatrix} \right\|^2 = \left\| F^* f - D F h \right\|^2 + \left\| D L D G (D F^* f + F h) \right\|^2 \geq 0,$$

$$\left\| \begin{bmatrix} f \\ h \end{bmatrix} \right\|^2 - \left\| K_2 \begin{bmatrix} f \\ h \end{bmatrix} \right\|^2 = \left\| F^* L f - D F h \right\|^2 + \left\| D L f \right\|^2 \geq 0.$$ 

Thus, the systems $\eta_1$, $\eta_2$, $\zeta_1$, and $\zeta_2$ are passive and their transfer functions are precisely $\Gamma(\lambda)$.

Since $\tilde{L}^* \ker D_F^* = 0$ and $F^* f = 0 \iff D_F^* f = f$, $G h = 0 \iff D_G h = h$, by induction one can derive the following equalities

$$\begin{cases}
\bigcap_{n \geq 0} \ker (F^* (D_G L^* D_F)^n) = \bigcap_{n \geq 0} \ker \left( F^* (D_G L^*)^n \right), \\
\bigcap_{n \geq 0} \ker (G(D_F^* L D G)^n) = \bigcap_{n \geq 0} \ker \left( G(D_F^* \tilde{L})^n \right), \\
\bigcap_{n \geq 0} \ker \left( F^* D_G L^* (D_F^* D_G \tilde{L}^*)^n \right) = \bigcap_{n \geq 1} \ker \left( F^* (D_G \tilde{L}^*)^n \right), \\
\bigcap_{n \geq 0} \ker \left( G D F^* (L D G D F)^n \right) = \bigcap_{n \geq 0} \ker \left( G(D_F^* \tilde{L})^n \right), \\
\bigcap_{n \geq 0} \ker \left( F^* D_G (L^* D_F^* D_G)^n \right) = \bigcap_{n \geq 0} \ker \left( F^* (D_G \tilde{L}^*)^n D_G \right), \\
\bigcap_{n \geq 0} \ker \left( G D F^* \tilde{L} (D_G D F^* \tilde{L})^n \right) = \bigcap_{n \geq 1} \ker \left( G(D_F^* \tilde{L})^n \right).
\end{cases}
$$
From (6.6) follow the relations (6.3) and (6.4).

**Theorem 6.2.** Let the system
\[
\tau = \left\{ \begin{bmatrix} D & D_G \cr FD_D & -FD_D^*G + D_FLD_G \end{bmatrix} ; \mathcal{M}, \mathcal{N}, \mathcal{H} \right\}
\]
be conservative and simple and let \( \Theta(\lambda) \) be its transfer function. Suppose that the first Schur iterate \( \Theta_1(\lambda) \) of \( \Theta \) is non-unitary constant. Then the systems
\[
\zeta_1 = \left\{ \begin{bmatrix} GF & G \\
LD_GF & LD_G \end{bmatrix} ; \mathcal{D}_D, \mathcal{D}_D^*, \mathcal{D}_F^* \right\},
\]
\[
(6.7)
\]
\[
\zeta_2 = \left\{ \begin{bmatrix} GF & GL \\
DG_F & DG_L \end{bmatrix} ; \mathcal{D}_D, \mathcal{D}_D^*, \mathcal{D}_G \right\}
\]
are conservative and simple and their transfer functions are equal to \( \Theta_1(\lambda) \).

**Proof.** Because the system \( \nu \) is conservative, the operators \( F \) and \( G^* \) are isometries. Since \( \Theta_1(\lambda) \) is non-unitary constant, from (6.3) it follows that the operator \( GF \) is non-unitary. Hence by Theorem 4.6 the subspaces \( \mathcal{D}_F^* \) and \( \mathcal{D}_G \) are nontrivial, and the operator \( L \in \mathcal{L}(\mathcal{D}_G, \mathcal{D}_F^*) \) is unitary. In addition, \( \ker F^* = \mathcal{D}_F^* \), \( \ker G = \mathcal{D}_G \), and the operators \( D_F^* \) and \( D_G \) are orthogonal projections in \( \mathcal{H} \) onto \( \ker F^* \) and \( \ker G \), respectively. One can directly check that the operators
\[
\begin{bmatrix} GF & G \\
LD_GF & LD_G \end{bmatrix} : \mathcal{D}_D^* \rightarrow \mathcal{D}_D^*, \quad \begin{bmatrix} GF & GL \\
DG_F & DG_L \end{bmatrix} : \mathcal{D}_D \rightarrow \mathcal{D}_G
\]
are unitary. Hence, the systems \( \zeta_1 \) and \( \zeta_2 \) are conservative. Relation (6.3) yields in our case that
\[
(\mathcal{H}_\nu^\perp) = (\mathcal{H}_\zeta_1^\perp) \subseteq (\mathcal{H}_\zeta_2^\perp).
\]
Taking into account (6.4) and the simplicity of \( \nu \) we get that the systems \( \zeta_1 \) and \( \zeta_2 \) are simple. \( \square \)

**Theorem 6.3.** Let \( \Theta(\lambda) \in \mathcal{S}([\mathcal{M}, \mathcal{N}], \Gamma_0 = \Theta(0)) \) and let \( \Theta_1(\lambda) \) be the first Schur iterate of \( \Theta \). Suppose
\[
\tau = \left\{ \begin{bmatrix} \Gamma_0 & C \\
B & A \end{bmatrix} ; \mathcal{M}, \mathcal{N}, \mathcal{H} \right\}
\]
is a simple conservative system with transfer function \( \Theta \). Then the simple conservative system
\[
\nu = \left\{ \begin{bmatrix} 0 & D_{\Gamma_0}^{-1}C \\
D_{\Gamma_0}^{-1}B & AP_{ker D_A}^{-1} \end{bmatrix} ; \mathcal{D}_{\Gamma_0}, \mathcal{D}_{\Gamma_0}^*, \mathcal{H} \right\}
\]
has the transfer function \( \lambda \Theta_1(\lambda) \) while the simple conservative systems
\[
\zeta_1 = \left\{ \begin{bmatrix} D_{\Gamma_0}^{-1}C(D_{\Gamma_0}^{-1}B)^* & D_{\Gamma_0}^{-1}C \mid \ker D_A^* \\
AP_{ker D_A}D_{\Gamma_0}^{-1}B & P_{ker D_A}A \mid \ker D_A^* \end{bmatrix} ; \mathcal{D}_{\Gamma_0}, \mathcal{D}_{\Gamma_0}^*, \ker D_A^* \right\},
\]
\[
(6.8)
\]
\[
\zeta_2 = \left\{ \begin{bmatrix} D_{\Gamma_0}^{-1}C(D_{\Gamma_0}^{-1}B)^* & D_{\Gamma_0}^{-1}C \mid \ker D_A^* \\
AP_{ker D_A}D_{\Gamma_0}^{-1}B & P_{ker D_A}A \mid \ker D_A \end{bmatrix} ; \mathcal{D}_{\Gamma_0}, \mathcal{D}_{\Gamma_0}^*, \ker D_A \right\}
\]
have transfer functions \( \Theta_1(\lambda) \). Here the operators \( D_{\Gamma_0}^{-1}, D_{\Gamma_0}^{-1}, \) and \( D_A^{-1} \) are the Moore–Penrose pseudo-inverses.
Proof. Let
\[
T = \begin{bmatrix} \Gamma_0 & C \\ B & A \end{bmatrix} = \begin{bmatrix} \Gamma_0 & D_{\Gamma_0}G \\ F\Gamma_0 & -F\Gamma_0^*G + D_F, LD_G \end{bmatrix} = \begin{bmatrix} -KA^*M + DK^*XD_M & KD_A \\ DA^*M & A \end{bmatrix} : \mathcal{H} \to \mathcal{H}.
\]
Then \( G = D_{\Gamma_0}^{-1}C, \ F^* = D_{\Gamma_0}^{-1}B^* \), \( F = M|\mathcal{D}_{\Gamma_0}, \ M = D_{A^*}^{-1}B \). According to Proposition 4.7 we have
\[
D_{F^*} = P_{\ker D_A^*}, \ D_G = P_{\ker D_A}, \ L = A|\ker D_A.
\]
Hence
\[
GF = D_{\Gamma_0}^{-1}C(D_{\Gamma_0}^{-1}B^*)^*, \ D_GD_{F^*}L = P_{\ker D_A}A|\ker D_A, \\
DG = D_{\Gamma_0}^{-1}C(D_{\Gamma_0}^{-1}B^*)^*, \ GD_{F^*}L = D_{\Gamma_0}^{-1}CP_{\ker D_A}A|\ker D_A, \\
LD_G|\ker D_A^* = AP_{\ker D_A}A|\ker D_A^*, \ LDF = AP_{\ker D_A}A|\ker D_A^*B.
\]
Note that if \( f \in \ker D_A^* \) then
\[
AP_{\ker D_A}f = P_{\ker D_A^*}AP_{\ker D_A}f = P_{\ker D_A^*}Af - P_{\ker D_A^*}AP_{\ker D_A}f = P_{\ker D_A^*}Af.
\]
Now the statement of theorem follows from Theorem 5.3 and Theorem 6.2.\( \square \)

Remark 6.4. Since \( F^* = D_{\Gamma_0}^{-1}B^* \), we get \( F = (D_{\Gamma_0}^{-1}B^*)^* \in L(\mathcal{D}_{\Gamma_0}, \mathcal{H}) \). Hence
\[
D_{A^*}^{-1}B|\mathcal{D}_{\Gamma_0} = (D_{\Gamma_0}^{-1}B^*)^*.
\]
Using the Hilbert spaces and operators defined by (3.1) and (3.2), we get
\[
P_{\ker D_A}D_{A^*}^{-1}B|\mathcal{D}_{\Gamma_0} = P_{1,0}D_{A^*}^{-1}B|\mathcal{D}_{\Gamma_0} = (D_{\Gamma_0}^{-1}(B^*|\mathcal{H}_{1,0}))^* \in L(\mathcal{D}_{\Gamma_0}, \mathcal{H}_{1,0}).
\]
In addition
\[
D_{\Gamma_0}^{-1}C(D_{\Gamma_0}^{-1}B^*)^* = \Gamma_1 \in L(\mathcal{D}_{\Gamma_0}, \mathcal{D}_{\Gamma_0}^*).
\]
So,
\[
\zeta_1 = \left\{ \begin{array}{c} \Gamma_1 \\ A(D_{\Gamma_0}^{-1}(B^*|\mathcal{H}_{1,0}))^* \\ D_{\Gamma_0}^{-1}C \\ A_{0,1} \end{array} \right\} : \mathcal{D}_{\Gamma_0}, \mathcal{D}_{\Gamma_0}^*, \mathcal{H}_{1,0}, \\
\zeta_2 = \left\{ \begin{array}{c} \Gamma_1 \\ D_{\Gamma_0}^{-1}(B^*|\mathcal{H}_{1,0}) \\ C \\ A_{1,0} \end{array} \right\} : \mathcal{D}_{\Gamma_0}, \mathcal{D}_{\Gamma_0}^*, \mathcal{H}_{1,0}.
\]
It follows that
\[
\text{ran} \left( D_{\Gamma_0}^{-1}C|\mathcal{H}_{1,0} \right) \subset \text{ran} \Gamma_1,
\]
\[
\text{ran} \left( D_{\Gamma_0}^{-1}B^*|\mathcal{H}_{1,0} \right) \subset \text{ran} \Gamma_1.
\]

6.2. Schur iterates of the characteristic function.

Theorem 6.5. Let \( A \) be a completely non-unitary contraction in a separable Hilbert space \( \mathcal{H} \). Assume \( \ker D_A \neq \{0\} \) and let the contractions \( A_{n,m} \) be defined by (3.1) and (3.2). Then the characteristic functions of the operators
\[
A_{n,0}, A_{n-1,1}, \ldots, A_{n-m,m}, \ldots, A_{1,n-1}, A_{0,n}
\]
coincide with the pure part of the \( n \)-th Schur iterate of the characteristic function \( \Phi(\lambda) \) of \( A \). Moreover, each operator from the set \( \{A_{n-k,k}\}_{k=0}^n \) is
(1) a unilateral shift (co-shift) if and only if the $n$-th Schur parameter $\Gamma_n$ of $\Phi$ is isometric (co-isometric),

(2) the orthogonal sum of a unilateral shift and co-shift if and only if

$$\mathcal{D}_{\Gamma_{n-1}} \neq \{0\}, \mathcal{D}_{\Gamma_{n-1}} \neq \{0\} \quad \text{and} \quad \Gamma_m = 0 \quad \text{for all} \quad m \geq n.$$ 

Each subspace from the set $\{\mathcal{H}_{n-k,k}\}_{k=0}^{n}$ is trivial if and only if $\Gamma_n$ is unitary.

**Proof.** We will prove by induction. The system

$$\Sigma = \left\{ \begin{bmatrix} -A & D_A^* \\ D_A & A^* \end{bmatrix}; \mathcal{D}_A, \mathcal{D}_A^*, \mathcal{H} \right\}$$

is conservative and simple and its transfer function $\Phi(\lambda)$ is Sz.-Nagy–Foias characteristic function of $A$. As in Theorem 5.5 let $F$ and $G^*$ be the embedding of the subspaces $\mathcal{D}_A$ and $\mathcal{D}_A^*$ into $\mathcal{H}$, respectively. Then $D_{F^*} = P_{ker D_A} = P_{1,0}$, $D_G = P_{ker D_A^*} = P_{0,1}$, and $L = A^* \upharpoonright ker D_A^* \in \mathcal{L}(\mathcal{D}_A^*, \mathcal{D}_A)$ is unitary operator. The system

$$\nu = \left\{ \begin{bmatrix} 0 & P_{\mathcal{D}_A^*} \\ I_{\mathcal{D}_A} & A^* P_{ker D_A^*} \end{bmatrix}; \mathcal{D}_A, \mathcal{D}_A^*, \mathcal{H} \right\}$$

is conservative and simple and its transfer function $Z(\lambda)$ is the Möbius parameter of $\Phi(\lambda)$. Constructing the systems given by (6.7) in Theorem 6.2 we get

$$\zeta_1 = \left\{ \begin{bmatrix} P_{\mathcal{D}_A^*} \upharpoonright \mathcal{D}_A & P_{\mathcal{D}_A^*} \upharpoonright \ker D_A \\ A^* P_{ker D_A^*} \upharpoonright \mathcal{D}_A & A^* P_{ker D_A^*} \upharpoonright \ker D_A \end{bmatrix}; \mathcal{D}_A, \mathcal{D}_A^*, \ker D_A \right\}$$

and

$$\zeta_2 = \left\{ \begin{bmatrix} P_{\mathcal{D}_A^*} \upharpoonright \mathcal{D}_A & P_{\mathcal{D}_A^*} \upharpoonright \ker D_A^* \\ P_{ker D_A^*} \upharpoonright \mathcal{D}_A & A^* P_{ker D_A^*} \upharpoonright \ker D_A^* \end{bmatrix}; \mathcal{D}_A, \mathcal{D}_A^*, \ker D_A^* \right\}.$$ 

By Theorem 6.2 the systems $\zeta_1$ and $\zeta_2$ are conservative and simple and their transfer functions are exactly the first Schur iterate $\Phi_1(\lambda)$ of $\Phi(\lambda)$. Note (see (3.1) and (3.2)) that

$$A^* P_{ker D_A^*} \upharpoonright \ker D_A = A^*_{1,0}; \quad P_{ker D_A^*} A^* \upharpoonright \ker D_A^* = A^*_{0,1}.$$ 

Applying Proposition 4.5 we get that the pure part of $\Phi_1(\lambda)$ coincides with the characteristic functions of the operators $A_{1,0}$ and $A_{0,1}$.

By Theorem 3.1 completely non-unitary contractions $\{A_{n-k,k}\}_{k=0}^{n}$ are unitarily equivalent. Assume that their characteristic functions coincide with the pure part of the $n$-th Schur iterate $\Phi_n(\lambda)$ of $\Phi$. The first Schur iterate of $\Phi_n$ is the function $\Phi_{n+1}(\lambda)$. As is already proved above the pure part of $\Phi_{n+1}$ coincides with the characteristic function of the operators $(A_{n-k,k})_{1,0}$ and $(A_{n-k,k})_{0,1}$. From (3.9) it follows

$$(A_{n-k,k})_{1,0} = A_{n+1-k,k}, \quad (A_{n-k,k})_{0,1} = A_{n-k,k+1} = A_{n+1-(k+1),k+1}.$$ 

Thus, characteristic functions of the unitarily equivalent completely non-unitary contractions $\{A_{n-k,k}\}_{k=0}^{n+1}$ coincide with $\Phi_{n+1}$.

Note that the Möbius parameter of the $n - 1$-th Schur iterate of $\Phi_{n-1}$ is $\lambda \Phi_{n}(\lambda)$ and by Theorem 5.5 this function coincides with the characteristic function of the operator $A_{n,0} = A_{n,0} P_{ker D_{A_{n,0}}}$. Applying Theorem 5.5 once again, we get that $A_{n,0}$ is a unilateral shift if and only if $\Gamma_n$ is an isometry.

The function $\Phi^*(\lambda)$ is the characteristic function of the operator $A^*$ and its Schur parameters are adjoint to the corresponding Schur parameters of $\Phi$. In addition if $B = A^*$ then
\[ B_{n,n} = A_{n,n}^* \]. Therefore, \( A_{0,n}^* \) is a unilateral shift if and only if \( \Gamma^*_n \) is isometric. But \( A_{0,n}^* \) is unitarily equivalent to \( A_{n,0}^* \). Hence, \( A_{n,0} \) is a co-shift if and only if \( \Gamma_n \) is a co-isometry.

It follows that \( \Gamma_n \) is a unitary if and only if \( A_{n,0} \) is a unilateral shift and co-shift in \( \mathcal{H}_{n,0} \leftrightarrow \mathcal{H}_{n,0} = \{ 0 \} \).

Condition (6.10) holds true if and only if \( \Phi_n \) is identically equal zero. This is equivalent to the condition that \( A_{n,0} \) (as well and \( A_{n-1,1}, A_{n-2,2}, \ldots A_{0,n} \)) is the orthogonal sum of a shift and co-shift. \( \square \)

**Remark 6.6.** It is proved that

\[ \Gamma_n \text{ is isometry } \iff \ker D_{A_{n+1}} = \ker D_{A_n} \iff \ker D_{A_n} \cap \ker D_{A^*} = \ker D_{A_{n-1}} \cap \ker D_{A^*} \iff \ldots \iff \ker D_{A_{n-1-k}} \cap \ker D_{A^*k} = \ker D_{A_{n-k}} \cap \ker D_{A^*k} \iff \ldots \iff \ker D_{A_n} \subset \ker D_{A^*n} \]

6.3. Conservative realizations of the Schur iterates.

**Theorem 6.7.** Let \( \Theta(\lambda) \in \mathcal{S}(\mathcal{M}, \mathcal{N}) \) and let

\[ \tau_0 = \left\{ \begin{bmatrix} \Gamma_0 & C \\ B & A \end{bmatrix} ; \mathcal{M}, \mathcal{N}, \mathcal{H} \right\} \]

be a simple conservative realization of \( \Theta \). Then the Schur parameters \( \{ \Gamma_n \}_{n \geq 1} \) of \( \Theta \) can be calculated as follows

\[ \Gamma_1 = D_{\Gamma_0}^{-1} C \left( D_{\Gamma_0}^{-1} B^* \right)^*, \quad \Gamma_2 = D_{\Gamma_1}^{-1} D_{\Gamma_0}^{-1} C A \left( D_{\Gamma_1}^{-1} D_{\Gamma_0}^{-1} \left( B^* \mid \mathcal{H}_{1,0} \right) \right)^*, \ldots, \]

\[ \Gamma_n = D_{\Gamma_{n-1}}^{-1} \cdots D_{\Gamma_0}^{-1} C A^{n-1} \left( D_{\Gamma_{n-1}}^{-1} \cdots D_{\Gamma_0}^{-1} \left( B^* \mid \mathcal{H}_{n-1,0} \right) \right)^*, \ldots. \]

Here the operator

\[ \left( D_{\Gamma_{n-1}}^{-1} \cdots D_{\Gamma_0}^{-1} \left( B^* \mid \mathcal{H}_{n-1,0} \right) \right)^* \in \mathcal{L}(\mathcal{D}_{\Gamma_{n-1}}, \mathcal{H}_{n-1,0}) \]

is the adjoint to the operator

\[ D_{\Gamma_{n-1}}^{-1} \cdots D_{\Gamma_0}^{-1} \left( B^* \mid \mathcal{H}_{n-1,0} \right) \in \mathcal{L}(\mathcal{H}_{n-1,0}, \mathcal{D}_{\Gamma_{n-1}}), \]

and

\[ \text{ran} \left( D_{\Gamma_{n-1}}^{-1} \cdots D_{\Gamma_0}^{-1} \left( B^* \mid \mathcal{H}_{n,0} \right) \right) \subset \text{ran} \left( D_{\Gamma_n} \right), \]

\[ \text{ran} \left( D_{\Gamma_{n-1}}^{-1} \cdots D_{\Gamma_0}^{-1} \left( C \mid \mathcal{H}_{0,n} \right) \right) \subset \text{ran} \left( D_{\Gamma_n} \right). \]
for every \( n \geq 1 \). Moreover, for each \( n \geq 1 \) the unitarily equivalent simple conservative systems
\[
(6.12)
\tau^{(k)}_n = \left\{ \begin{bmatrix} \Gamma_n & D^{-1}_{\Gamma_n} \cdots D^{-1}_{\Gamma_0} (CA^{n-k})^{*} \\ A_{n-k,k} & \end{bmatrix} ; \mathcal{D}_{\Gamma_n}, \mathcal{D}_{\Gamma_n}^*, \mathcal{H}_{n-k,k} \right\}
\]
\( k = 0, 1, \ldots, n \)
are realizations of the \( n \)-th Schur iterate \( \Theta_n \) of \( \Theta \). Here the operator
\[
B_n = \left( D^{-1}_{\Gamma_{n-1}} \cdots D^{-1}_{\Gamma_0} (B^* | \mathcal{H}_{n,0}) \right)^* \in \mathcal{L}(\mathcal{D}_{\Gamma_{n-1}}, \mathcal{H}_{n,0})
\]
is the adjoint to the operator
\[
D^{-1}_{\Gamma_{n-1}} \cdots D^{-1}_{\Gamma_0} (B^* | \mathcal{H}_{n,0}) \in \mathcal{L}(\mathcal{H}_{n,0}, \mathcal{D}_{\Gamma_{n-1}}).
\]

**Proof.** We will prove by induction. For \( n = 1 \) it is already established (see Remark 6.4, 6.8, and (6.9)) that
\[
\Gamma_1 = D^{-1}_{\Gamma_0} C \left( D^{-1}_{\Gamma_0} B^* \right)^*
\]
and the systems
\[
\tau^{(0)}_1 = \left\{ \begin{bmatrix} \Gamma_1 & D^{-1}_{\Gamma_0} (CA) \\ (D^{-1}_{\Gamma_0} (B^* | \mathcal{H}_{1,0}))^* & A_{1,0} \end{bmatrix} ; \mathcal{D}_{\Gamma_0}, \mathcal{D}_{\Gamma_0}^*, \mathcal{H}_{1,0} \right\}
\]
and
\[
\tau^{(1)}_1 = \left\{ \begin{bmatrix} \Gamma_1 & D^{-1}_{\Gamma_0} (C) \\ A (D^{-1}_{\Gamma_0} (B^* | \mathcal{H}_{1,0}))^* & A_{0,1} \end{bmatrix} ; \mathcal{D}_{\Gamma_0}, \mathcal{D}_{\Gamma_0}^*, \mathcal{H}_{0,1} \right\}
\]
are conservative and simple realizations of \( \Theta_1 \). Suppose
\[
\tau^{(0)}_m = \left\{ \begin{bmatrix} \Gamma_m & D^{-1}_{\Gamma_{m-1}} \cdots D^{-1}_{\Gamma_0} (CA^m) \\ (D^{-1}_{\Gamma_{m-1}} \cdots D^{-1}_{\Gamma_0} (B^* | \mathcal{H}_{m,0}))^* & A_{m,0} \end{bmatrix} ; \mathcal{D}_{\Gamma_{m-1}}, \mathcal{D}_{\Gamma_{m-1}}^*, \mathcal{H}_{m,0} \right\}
\]
is a simple conservative realization of \( \Theta_m \). Then
\[
B_m = \left( D^{-1}_{\Gamma_{m-1}} \cdots D^{-1}_{\Gamma_0} (B^* | \mathcal{H}_{m,0}) \right)^* \in \mathcal{L}(\mathcal{D}_{\Gamma_{m-1}}, \mathcal{H}_{m,0}),
\]
\[
C_m = D^{-1}_{\Gamma_{m-1}} \cdots D^{-1}_{\Gamma_0} (CA^m) \in \mathcal{L}(\mathcal{H}_{m,0}, \mathcal{D}_{\Gamma_{m-1}}), \ A_{m,0} \in \mathcal{L}(\mathcal{H}_{m,0}, \mathcal{H}_{m,0}).
\]
Hence
\[
B_m^* = D^{-1}_{\Gamma_{m-1}} \cdots D^{-1}_{\Gamma_0} (B^* | \mathcal{H}_{m,0}) \in \mathcal{L}(\mathcal{H}_{m,0}, \mathcal{D}_{\Gamma_{m-1}}),
\]
The first Schur iterate of \( \Theta_m(\lambda) \) is the function \( \Theta_{m+1}(\lambda) \in \mathcal{S}(\mathcal{D}_m, \mathcal{D}_m^*) \) and the first Schur parameter of \( \Theta_m \) is \( \Gamma_{m+1} \). From (3.4) and (3.9) it follows that
\[
\ker D_{A_{m,0}} = \mathcal{H}_{m+1,0}, \ (A_{m,0})_{1,0} = A_{m+1,0} \in \mathcal{L}(\mathcal{H}_{m+1,0}, \mathcal{H}_{m+1,0}).
\]
Hence by (6.8), and (6.9)
\[
\Gamma_{m+1} = D^{-1}_{\Gamma_m} C_m \left( D^{-1}_{\Gamma_m} B_m^* \right)^* = D^{-1}_{\Gamma_m} \cdots D^{-1}_{\Gamma_0} C A^m \left( D^{-1}_{\Gamma_m} \cdots D^{-1}_{\Gamma_0} (B^* | \mathcal{H}_{m,0}) \right)^*
\]
and the system
\[
\tau^{(0)}_{m+1} = \left\{ \begin{bmatrix} \Gamma_{m+1} & D^{-1}_{\Gamma_{m}} \cdots D^{-1}_{\Gamma_0} (CA^{m+1}) \\ (D^{-1}_{\Gamma_{m}} \cdots D^{-1}_{\Gamma_0} (B^* | \mathcal{H}_{m+1,0}))^* & A_{m+1,0} \end{bmatrix} ; \mathcal{D}_{\Gamma_m}, \mathcal{D}_{\Gamma_m}^*, \mathcal{H}_{m+1,0} \right\}
\]
is a simple conservative realization of $\Theta_{m+1}$. From Proposition 5.3 it follows that the system

$$\tau_{m+1}^{(k)} = \left\{ \begin{array}{c}
A^k \left( D_{\Gamma_m}^{-1} \cdots D_{\Gamma_0}^{-1} (B^* \mid \mathcal{H}_{m+1,0}) \right)^* A_{m+1-k,k} \\
D_{\Gamma_m}^{-1} \cdots D_{\Gamma_0}^{-1}(CA^{m+1-k}) \end{array} \right\}; \mathcal{D}_{\Gamma_m}, \mathcal{D}_{\Gamma_n}, \mathcal{H}_{m+1-k,k}\right.$$

is unitarily equivalent to the system $\tau_{m+1}$ for $k = 1, \ldots, m + 1$ and hence have transfer functions equal to $\Theta_{m+1}$. This completes the proof. \(\square\)

Let us make a few remarks which follow from (4.9), Proposition 4.5, and Theorem 6.5.

If $D_{\Gamma_N} = 0$ and $D_{\Gamma_N} \neq 0$ then $\mathcal{D}_{\Gamma_n} = 0$, $\Gamma^*_n = 0 \in L(\mathcal{D}_{\Gamma_N}, \{0\})$, $\mathcal{D}_{\Gamma_n} = \mathcal{D}_{\Gamma_N}$, and $\mathcal{H}_{0,n} = \mathcal{H}_{0,N}$ for $n \geq N$. The unitarily equivalent observable conservative systems

$$\tau_N^{(k)} = \left\{ \begin{array}{c}
[\Gamma_N \ D_{\Gamma_N}^{-1} \cdots D_{\Gamma_0}^{-1}(CA^{N-k})] \\
0 \\
D_{\Gamma_N-1}^{-1} \cdots D_{\Gamma_0}^{-1}(CA^{N-k}) \\
A_{N-k,k}
\end{array} \right\}; \mathcal{D}_{\Gamma_N-1}, \mathcal{D}_{\Gamma_N-1}, \mathcal{H}_{N-k,k}\right.$$

have transfer functions $\Theta_N(\lambda) = \Gamma_N$ and the operators $A_{N-k,k}$ are unitarily equivalent co-shifting of multiplicity $\dim \mathcal{D}_{\Gamma_N}$, the Schur iterates $\Theta_n$ are null operators from $L(\{0\}, \mathcal{D}_{\Gamma_N})$ for $n \geq N + 1$ and are transfer functions of the conservative observable system

$$\tau_{N+1} = \left\{ \begin{array}{c}
0 \\
0 \\
D_{\Gamma_N-1}^{-1} \cdots D_{\Gamma_0}^{-1}(CA^{N-k}) \\
A_{N-k,k}
\end{array} \right\}; \{0\}, \mathcal{D}_{\Gamma_N}, \mathcal{H}_{0,N}\right.$$

If $D_{\Gamma_N} = 0$ and $D_{\Gamma_N} \neq 0$ then $\mathcal{D}_{\Gamma_n} = 0$, $\mathcal{D}_{\Gamma_n} = \mathcal{D}_{\Gamma_N}$, and $\Gamma_n = 0 \in L(\mathcal{D}_{\Gamma_N}, \{0\})$, $\mathcal{H}_{n,0} = \mathcal{H}_{N,0}$ for $n \geq N$. The unitarily equivalent controllable conservative systems

$$\tau_N^{(k)} = \left\{ \begin{array}{c}
[\Gamma_N \ D_{\Gamma_N-1}^{-1} \cdots D_{\Gamma_0}^{-1}(CA^{N-k})] \\
0 \\
A_{N-k,k}
\end{array} \right\}; \mathcal{D}_{\Gamma_N-1}, \mathcal{D}_{\Gamma_N-1}, \math{H}_{N-k,k}\right.$$

have transfer functions $\Theta_N(\lambda) = \Gamma_N$ and the operators $A_{N-k,k}$ are unitarily equivalent unilateral shifts of multiplicity $\dim \mathcal{D}_{\Gamma_N}$, the Schur iterates $\Theta_n$ are null operators from $L(\mathcal{D}_{\Gamma_N}, \{0\})$ for $n \geq N + 1$ and are transfer functions of the conservative controllable system

$$\tau_{N+1} = \left\{ \begin{array}{c}
0 \\
(D_{\Gamma_N-1}^{-1} \cdots D_{\Gamma_0}^{-1}(B^* \mid \mathcal{H}_{N+1,0}))^* A_{N-k,k} \\
0 \\
A_{N-k,k}
\end{array} \right\}; \mathcal{D}_{\Gamma_N}, \{0\}, \mathcal{H}_{N,0}\right.$$

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