SOME DYNAMICAL PROPERTIES
OF PSEUDO-AUTOMORPHISMS IN DIMENSION 3

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ABSTRACT. Let $X$ be a compact Kähler manifold of dimension 3 and let $f : X \to X$ be a pseudo-automorphism. Under the mild condition that $\lambda_1(f)^2 > \lambda_2(f)$, we prove the existence of invariant positive closed $(1,1)$ and $(2,2)$ currents, and we also discuss the (still open) problem of intersection of such currents. We prove a weak equi-distribution result for Green $(1,1)$ currents of meromorphic selfmaps, not necessarily algebraic 1-stable, of a compact Kähler manifold of arbitrary dimension and discuss how a stronger equidistribution result may be proved for pseudo-automorphisms in dimension 3. As a byproduct, we show that the intersection of some dynamically related currents is well-defined with respect to our definition here, even though not obviously to be seen so using the usual criteria.

1. INTRODUCTION

This paper studies the dynamics of pseudo-automorphisms in dimension 3. Let $X$ be a compact Kähler manifold of dimension 3. A map $f : X \to X$ is a pseudo-automorphism if it is a bimeromorphic map so that both $f$ and $f^{-1}$ have no exceptional hypersurfaces (see Dolgachev-Ortland [28]). Note that (see Lemma 1) if $f$ is a pseudo-automorphism, then so are the iterates $f^n (n \in \mathbb{Z})$. Recent constructions by Bedford-Kim [9], Perroni-Zhang [42], Oguiso [41], [40], and Blanc [4] provide many interesting examples of such maps. Among bimeromorphic selfmaps, it may be argued that the class of pseudo-automorphisms is the second best after that of automorphisms. In dimension 2, pseudo-automorphisms are automorphisms.

One of the difficulties when studying dynamics of meromorphic maps in dimension $> 2$ is that in general we cannot pull back positive closed currents of bidegree $> (1,1)$. Our first main result shows that it is possible to do so for pseudo-automorphisms in dimension 3. For $0 \leq p \leq 3$, let $\mathcal{D}^p(X)$ be the real vector space generated by positive closed $(p,p)$ currents on $X$, and let $DSH^p(X)$ be the space of $DSH (p,p)$ currents on $X$ (for precise definitions of these classes and their properties, see Section 2). By definition it follows that $\mathcal{D}^p(X) \subset DSH^p(X)$. For a closed current $T$ we let $\{T\}$ denote its cohomology class.

Theorem 1 below shows the possibility of pulling back or pushing forward currents by $f$ and proves the compatibility of such operators with the iteration. The definition of pulling back or pushing forward current we use here was developed in our previous paper [45] and is refined in the current paper.
**Theorem 1.** Let $X$ be a compact Kähler manifold of dimension 3 and let $f : X \to X$ be a pseudo-automorphism. Then, with respect to Definitions [7] and [8] for any $n \in \mathbb{Z}$ there are well-defined pullback and pushforward operators $(f^n)^*$ and $(f^n)_*$ from each of the spaces $\mathcal{D}^1(X), \text{DSH}^1(X)$, and $\mathcal{D}^2(X)$ into itself. These operators are continuous with respect to the topologies on the corresponding spaces, and hence are compatible with the pullbacks or pushforwards on cohomology groups. Moreover, these operators are compatible with iteration in the sense that $(f^n)^* = (f^*)^n$ and $(f^n)_* = (f_*)^n$ for any $n \in \mathbb{Z}$.

**Remarks.** 1) In the case $X = \mathbb{P}^k$ a projective space, results similar to Theorem 1 were proved by Dinh-Sibony [25] and de Thélin-Vigny [16] using super-potential theory. For other manifolds, previously there were no such results.

2) The compatibility with the pullbacks or pushforwards on cohomology groups in Theorem 1 follows from the possibility to regularize positive closed currents by smooth closed forms. The latter is proved by Dinh-Sibony [21], [22].

Next we discuss the existence of invariant positive closed currents for $f$. Since the map $f^* : \mathcal{D}^1(X) \to \mathcal{D}^1(X)$ preserves the cone of positive closed $(1, 1)$ currents, it follows by a Perron-Frobenius type argument that $\lambda_1(f)$ is an eigenvalue of $f^* : H^{1,1}(X) \to H^{1,1}(X)$. We have the following:

**Theorem 2.** Let $X$ be a compact Kähler manifold of dimension 3, and $f : X \to X$ a pseudo-automorphism. Assume that $\lambda_1(f)^2 > \lambda_2(f)$. Then

a) There is a non-zero positive closed $(1, 1)$ current $T^+$ such that $f^*(T^+) = \lambda_1(f)T^+$. Moreover, $T^+$ has no mass on hypersurfaces.

b) There is a non-zero positive closed $(2, 2)$ current $T^-$ such that $f_*(T^-) = \lambda_1(f)T^-$. 

c) We can choose $T^+$ and $T^-$ such that in cohomology $\{T^+\} \cdot \{T^-\} = 1$.

**Remarks.** 1) The assumption that $\lambda_1(f)^2 > \lambda_2(f)$ is not a real restriction. In fact, when $\lambda_1(f) > 1$ this condition is satisfied for either the map $f$ or its inverse $f^{-1}$.

2) Part a) of Theorem 2 is already known in the literature; however its refinement in Theorem 4 below seems to be new.

Compared with the results for meromorphic maps in dimension 2 (see e.g. Diller-Favre [17], Diller-Dujardin-Guedj [19]), for automorphisms in any dimension (see e.g. Cantat [11], Dinh-Sibony [26], [25], [24]), for Green $(1, 1)$ currents of meromorphic maps whose invariant cohomology class satisfying several conditions (see Sibony [44], Diller-Guedj [20], Guedj [34], Bayraktar [3]) and for linear fractional maps (see Bedford-Kim [9]) we are led to the following natural questions:

**Question 1.** Does $T^+$ in Theorem 2 satisfy an equi-distribution property; i.e. for every smooth closed $(1, 1)$ form $\theta$ of the same cohomology class as $T^+$ we have

$$\lim_{n \to \infty} \frac{(f^n)^*(\theta)}{\lambda_1(f)^n} = T^+?$$

If this equi-distribution property holds, does it also hold for any smooth closed $(1, 1)$ form $\theta$ for which $f_X^* \theta \wedge T^- = 1$? What about similar questions for $T^-$?

We can answer Question 1 in the affirmative under an additional condition. We say that a meromorphic map $f : X \to Y$ is holomorphic-like if it satisfies the following two conditions: (i) for the eigenvector $\{\theta\} \in H^{1,1}(X)$ of eigenvalue $\lambda_1(f)$ we have $\{\theta\} \cdot \{\theta\} = 0$ and (ii) there is a desingularization $Z$ of the graph of $f$ such
that the induced projection to the first factor \( \pi : Z \to X \) is a composition of blowups along smooth centers for which if \( E \subset Z \) is a hypersurface, then \( \dim(\pi(E)) \geq \dim(X) - 2 \). The proof of Theorem 3 below will be given in Section 6 where we also discuss some other cases where the same idea may apply.

**Theorem 3.** Let \( X \) be a projective manifold, and let \( f : X \to X \) be a dominant meromorphic map which is algebraically 1-stable such that \( \lambda_1(f)^2 > \lambda_2(f) \). Assume that \( f \) is holomorphic-like. Then for any smooth closed \((1,1)\) form \( \theta \) in the cohomology class of \( \{ \theta \} \) the limit

\[
\lim_{n \to \infty} \frac{(f^*)^n(\theta)}{\lambda_1(f)^n}
\]

exists and is the positive closed current given in Theorem 4 below.

Note that in the case \( f \) is a holomorphism the two conditions (i) and (ii) are automatically satisfied, and in this case the result is known in the literature. While condition (i) or some variant of it seems essential we feel that condition (ii) is not needed; see Section 6 for more discussion on this.

In general Question 1 is still open and seems a difficult one. The examples in [8] show that the usual criteria used to prove the equi-distribution property for the Green \((1,1)\) currents (see e.g. [20], [34], [3]) are not applicable to a general pseudo-automorphism in dimension 3. In fact, in the examples in [8], the psef eigenvector \( \alpha \in H^{1,1}(X) \) with eigenvalue \( \lambda_1(f) \) of \( f^* : H^{1,1}(X) \to H^{1,1}(X) \) is not nef, and moreover \( \alpha \cdot C < 0 \) for some curve \( C \subset f(I_f) \) where \( I_f \) is the indeterminacy set of \( f \).

In this aspect, the following result, which provides a canonical Green \((1,1)\) current and which whenever the equi-distribution property is satisfied is the same as the limit \( T^+ \) in Question 1, seems relevant. The canonical Green current \( T \) constructed in Theorem 4 is also extremal among invariant currents, in the sense that if \( S \) is a positive closed \((1,1)\) current such that \( f^*(S) = \lambda_1(f)S \) and \( S \leq T \), then \( S = cT \) for some constant \( c \).

**Theorem 4.** Let \( X \) be a compact Kähler manifold, and \( f : X \to X \) a dominant meromorphic map. Let \( \lambda \geq 1 \) be an eigenvalue of \( f^* : H^{1,1}(X) \to H^{1,1}(X) \). Assume that the eigenvector \( \theta \) corresponding to \( \lambda \) is a psef class. Then there is a positive closed \((1,1)\) current \( T \) such that \( \{ T \} = \theta \), \( f^*(T) = \lambda T \), and \( T \) is extremal among invariant currents. Moreover

\[
T = \lim_{n \to \infty} \frac{1}{\lambda^n} (f^*)^n(T_{\theta}^{\text{min}}),
\]

where \( T_{\theta}^{\text{min}} \) is a positive closed \((1,1)\) current with minimal singularities whose cohomology class is \( \theta \) (see the proof for precise definition of currents with minimal singularities).

Note that in Theorem 4 we do not require that \( f \) be algebraic 1-stable or any additional condition on the eigenvector \( \theta \) (such as Kähler, nef,...). Theorem 4 was first proved by Sibony [44] in the case \( X = \mathbb{P}^k \) a projective space. His proof was then adapted by Guedj [34] and Bayraktar [8] to other situations. Our proof is almost identical to that given for Theorem 1.2 in [8] (the latter in turn followed closely that given for Theorem 2.2 in [34]); however it appears from the comments in those papers (e.g. Remark 4.1 after the proof of Theorem 1.2 in [8]) that these authors were not aware of this. Since the theorem in the general setting has not yet appeared anywhere in the literature, we include it here for completeness.
Part c) of Theorem 2 provides Green (1,1) and (2,2) currents $T^+$ and $T^-$ of a pseudo-automorphism with $\{T^+\}, \{T^-\} > 0$. Hence, if we can make sense of the wedge product of the currents $T^+$ and $T^-$, then the (signed) measure $\mu = T^+ \wedge T^-$ is a good candidate for an invariant measure of $f$. Since the currents $T^+$ and $T^-$ are limits of Cesaro’s means of currents of the form $(f^n)^*(\theta)/\lambda_1(f)^n$ and $(f^n)_*(\eta)/\lambda_1(f)^n$ for positive closed smooth $(1,1)$ and $(2,2)$ forms $\theta$ and $\eta$, the following result is relevant. We recall that if $f : X \to X$ is a dominant meromorphic map, then there is the smallest analytic subvariety $C_f$ of the graph $\Gamma_f$ such that the map $\pi_2 : \Gamma_f - C_f \to X$ is locally finite-to-one. The first indeterminacy set of $f$ is $I_1(f) = \pi_1(C_f)$ (see Section 2 for more on this).

**Theorem 5.** Let $X$ be a compact Kähler manifold of dimension $k$, and $f : X \to X$ a dominant meromorphic map. Assume that the first indeterminacy set $I_1(f)$ has codimension $\geq p + 1$. Let $T$ be a positive closed $(p,p)$ current on $X$ and let $\eta$ be a smooth closed $(k-p,k-p)$ form. Then the wedge product $T \wedge f_*(\eta)$ is well-defined with respect to Definition 12. If moreover $\eta$ is strongly positive, then $T \wedge f_*(\eta)$ is a positive measure.

**Remarks.** 1) Since the current $f_*(\eta)$ may not be smooth on some curves where $T$ may have positive Lelong numbers, it is not obvious that we can define the wedge product $T \wedge f_*(\eta)$ intrinsically on $X$ in a reasonable way.

2) If $f : X \to X$ is a pseudo-automorphism in dimension 3, then we can apply the above with $k = 3$ and $p = 1$.

3) In the situation of 2), if instead $\theta$ is a positive closed smooth $(1,1)$ form and $S$ is a positive closed $(2,2)$ current, then $f^*(\theta) \wedge S$ may not be defined, or even when it can be defined the result may not be a positive measure. For example, consider $X$ the blowup of $\mathbb{P}^3$ at four points $[1 : 0 : 0 : 0], [0 : 1 : 0 : 0], [0 : 0 : 1 : 0]$ and $[0 : 0 : 0 : 1]$. Let $f : \mathbb{P}^3 \to \mathbb{P}^3$ be the map $f[x_0 : x_1 : x_2 : x_3] = [1/x_0 : 1/x_1 : 1/x_2 : 1/x_3]$, and let $F : X \to X$ be the lifting of $f$. Then $F$ is a pseudo-automorphism. If $C \subset X$ is the strict transform of the line $x_0 = x_1 = 0$ and $D$ is the strict transform of the line $x_2 = x_3 = 0$, then in cohomology $F_*(C) = -\{D\}$ (in [15] it was proved that in fact the equality also holds on the level of currents: $F_*[C] = -[D]$). Now let $\theta$ be a Kähler form on $X$. Then even if we may define the wedge product $F^*(\theta) \wedge [C]$, the resulting current cannot be a positive measure because in cohomology $F^*[\theta] \cdot [C] = [\theta] \cdot F_*[C] = -[\theta] \cdot [D] < 0$.

In a forthcoming paper we will study some further properties of pseudo-automorphisms in dimension 3.

The rest of this paper is organized as follows. In Section 2 we recall definitions and results on positive and DSH currents, dynamical degrees, and known results on pseudo-automorphisms in dimension 3. In Section 3 we prove a property of quasi-potentials of positive closed currents and a compatibility with wedge product of the kernels of Dinh and Sibony. In Section 4 we recall the definition of pullback of currents by meromorphic maps from [15], give the definition of intersection of currents, and prove several general results. In Section 5 we apply the previous results to obtain results about pseudo-automorphisms in dimension 3. In the last section we prove Theorem 3 and also discuss how Question 1 may be answered in the affirmative.
2. Preliminaries

In this section we present briefly definitions and previously known results on positive closed and DSH currents, dynamical degrees, and pseudo-automorphisms in dimension 3.

In this section only, let $X$ be a compact Kähler manifold of arbitrary dimension $k$ with a Kähler $(1, 1)$ form $\omega_X$.

2.1. Positive currents, DSH currents. For more details on positive currents the reader is referred to Lelong’s book [36] and Demailly’s book [13], and for more details on DSH currents the reader is referred to the paper Dinh-Sibony [22].

Given $0 \leq p \leq k$, a smooth $(p, p)$ form $\varphi$ on $X$ is called strongly positive if locally it can be written as a convex combination of smooth forms of the type $i^1 \gamma_1 \wedge \overline{\gamma_1} \wedge \ldots \wedge i^p \gamma_p \wedge \overline{\gamma_p}$.

A smooth $(p, p)$ form $\varphi$ is called (weakly) positive if for any strongly positive smooth $(k - p, k - p)$ form $\psi$, $\varphi \wedge \psi$ is a positive measure.

A smooth $(p, p)$ form $\varphi$ is called strictly positive if locally $\varphi \geq \omega^p$, where $\omega$ is a Kähler $(1, 1)$ form.

A $(p, p)$ current $T$ is (weakly) positive if for any strongly positive smooth $(k - p, k - p)$ form $\psi$, $T \wedge \psi$ is a positive measure.

A $(p, p)$ current $T$ is strongly positive if for any weakly positive smooth $(k - p, k - p)$ form $\psi$, $T \wedge \psi$ is a positive measure.

Note that strong and weak positivity coincide for currents of bidegree $(0, 0)$, $(1, 1)$, $(k - 1, k - 1)$ and $(k, k)$. Therefore, if $\dim(X) = 3$, then strong and weak positivity coincide.

For a positive $(p, p)$ current $T$, we define its mass by $||T|| = \langle T, \omega^{k-p}_X \rangle$.

A current $T$ is called positive closed if it is both positive and closed. For a positive closed current $T$, its mass depends only on its cohomology class. We denote by $D^p$ the real vector space generated by positive closed currents. Hence each current $T$ in $D^p(X)$ can be written as $T = T^+ - T^-$ for some positive closed $(p, p)$ currents $T^\pm$. We define the $D^p$ norm of such a $T$ as follows: $||T||_s = \min \{ ||T^+|| + ||T^-|| \}$, where the minimum is taken on all positive closed $(p, p)$ currents $T^\pm$ for which $T = T^+ - T^-$. We define convergence on $D^p$ as follows: If $T_n$ and $T$ are in $D^p$, we say that $T_n$ converges in $D^p$ if $T_n$ weakly converges to $T$ in the sense of currents, and moreover $||T_n||_s$ is bounded.

A $(p, p)$ current $T$ is called DSH if we can find positive $(p, p)$ currents $T_1, T_2$, and positive closed $(p-1, p-1)$ currents $\Omega_1^\pm, \Omega_2^\pm$ for which $T = T_1 - T_2, dd^c T_i = \Omega_i^+ - \Omega_i^-$ for $i = 1, 2$. Denote by $DSH^p(X)$ the set of DSH $(p, p)$ currents on $X$. We define a norm on $DSH^p(X)$ as follows: If $T$ is in $DSH^p(X)$, then

$$||T||_{DSH} = \min \{ ||T_1|| + ||T_2|| + ||\Omega_1^+|| + ||\Omega_2^+|| \},$$

where the minimum is taken on all decompositions $T = T_1 - T_2, dd^c T_i = \Omega_i^+ - \Omega_i^-$ of $T$. We define the convergence in $DSH$ as follows: If $T_n$ and $T$ are in $DSH^p$, we say that $T_n$ converges in $DSH^p$ if $T_n$ weakly converges to $T$ in the sense of currents, and moreover $||T_n||_{DSH}$ is bounded.

2.2. Regularization of DSH currents. In [22], Dinh and Sibony obtained a good regularization of DSH currents on compact Kähler manifolds, which gives for any DSH $(p, p)$ current $T$ a sequence of positive smooth DSH currents $T_n^\pm$ with uniformly bounded masses so that $T_n^+ - T_n^-$ weakly converges to $T$ (see also
Combining the results in [45] and Lemma 3 in Section 3, we obtain the existence of good approximation schemes by $C^s$ forms for DSH currents, whose definitions are given below.

**Definition 6.** Let $s \geq 0$ be an integer. We define a good approximation scheme by $C^s$ forms for DSH currents on $X$ to be an assignment that for a DSH current, $T$ gives two sequences $\mathcal{K}_n^\pm(T)$ (here $n = 1, 2, \ldots$) where $\mathcal{K}_n^\pm(T)$ are $C^s$ forms of the same bidegrees as $T$, so that $\mathcal{K}_n(T) = \mathcal{K}_n^+(T) - \mathcal{K}_n^-(T)$ weakly converges to $T$, and moreover the following properties are satisfied:

1) **Boundedness:** If $T$ is DSH, then the DSH norms of $\mathcal{K}_n^\pm(T)$ are uniformly bounded.

2) **Positivity:** If $T$ is positive, then $\mathcal{K}_n^+(T)$ are positive, and $||\mathcal{K}_n^\pm(T)||$ are uniformly bounded with respect to $n$.

3) **Closedness:** If $T$ is positive closed, then $\mathcal{K}_n^\pm(T)$ are positive closed.

4) **Continuity:** If $U \subset X$ is an open set so that $T|_U$ is a continuous form, then $\mathcal{K}_n^\pm(T)$ converges locally uniformly on $U$.

5) **Linearity:** For any pair of currents $T_1$ and $T_2$, we have $\mathcal{K}_n^\pm(T_1 + T_2) = \mathcal{K}_n^\pm(T_1) + \mathcal{K}_n^\pm(T_2)$.

6) **Self-Adjointness:** If $T$ and $S$ are of complement bidegrees, then

$$\int_X \mathcal{K}_n(T) \wedge S = \int_Y T \wedge \mathcal{K}_n(S),$$

for any $n \in \mathbb{N}$.

7) **Compatibility with the differentials:** $dd^c \mathcal{K}_n^\pm(T) = \mathcal{K}_n^\pm(dd^c T)$.

8) **Convergence of supports:** If $A$ is compact and $U$ is an open neighborhood of $A$, then there is $n_0 = n_0(U, A)$ such that if the support of $T$ is contained in $A$ and $n \geq n_0$, then support of $\mathcal{K}_n(U)$ is contained in $U$.

9) **Compatibility with wedge product:** Let $T$ be a DSH $(p, p)$ current and let $\theta$ be a continuous $(q, q)$ form on $X$. Assume that there is a positive $dd^c$-closed current $R$ so that $-R \leq T \leq R$. Then there are positive $dd^c$-closed $(p + q, p + q)$ currents $R_n$ so that $\lim_{n \to \infty} ||R_n|| = 0$ and $-R_n \leq \mathcal{K}_n(T \wedge \theta) - \mathcal{K}_n(T) \wedge \theta \leq R_n$, for all $n$.

If $R$ is strongly positive or closed, then we can choose $R_n$ to be so.

In fact, let $K_n$ be the weak regularization for the diagonal $\Delta_Y$ as in Section 3. Let $l$ be a large integer dependent on $s$, and let $(m_1)_n, \ldots, (m_l)_n$ be sequences of positive integers satisfying $(m_i)_n = (m_{i+1} \ldots)_n$ and $\lim_{n \to \infty}(m_i)_n = \infty$ for any $1 \leq i \leq l$. In [45] we showed that if we choose $\mathcal{K}_n = K_{(m_1)_n} \circ K_{(m_2)_n} \circ \ldots \circ K_{(m_l)_n}$, then it satisfies conditions 1)-8). Remark 2 in Section 3 shows that it also satisfies condition 9).

### 2.3. Dynamical degrees and algebraic stability

Let $f : X \to X$ be a dominant meromorphic map. It is well-known that we can define the pullback $f^*$ on smooth forms and on cohomology groups (see Section 4 for more detail). For $0 \leq p \leq k = \dim(X)$, the $p$-th dynamical degree of $f$ is defined by

$$\lambda_p(f) = \lim_{n \to \infty} ||(f^n)^*(\omega_X^p)||^{1/n}.$$

Dinh and Sibony ([21] and [22]) showed that the dynamical degrees are well-defined (i.e. the limits in the definition exist) and are bimeromorphic invariants.
Some properties of dynamical degrees: \( \lambda_p(f) \geq 1 \) for all \( p \geq 1 \), and \( \lambda_p(f)^2 \geq \lambda_{p-1}(f)\lambda_{p+1}(f) \) (log-concavity).

When \( f \) is holomorphic, the results by Gromov [33] and Yomdin [47] prove that the topological entropy \( h_{top}(f) \) of \( f \) equals \( \max_{0 \leq p \leq k} \log \lambda_p(f) \). For a general meromorphic map, we can still define its topological entropy. Dinh and Sibony [22] proved that \( h_{top}(f) \leq \max_{0 \leq p \leq k} \log \lambda_p(f) \).

Given \( 0 \leq p \leq k \). Following Fornaess-Sibony [31], we say that \( f^* \) is algebraically \( p \)-stable if for any \( n \in \mathbb{N} \), \( (f^n)^* = (f^*)^n \) as linear maps on \( H^{p,p}(X) \). We can define a similar notion for the pushforward \( f_* \).

### 2.4. Pseudo-automorphisms in dimension 3

Let now \( X \) be a compact Kähler manifold of dimension 3. Let \( f : X \to X \) be a pseudo-automorphism with the graph \( \Gamma_f \subset X \times X \). Let \( \mathcal{I}(f) \) and \( \mathcal{I}(f^{-1}) \) be the indeterminacy sets of \( f \) and \( f^{-1} \). Then it follows that \( f : X - \mathcal{I}(f) \to X - \mathcal{I}(f^{-1}) \) is biholomorphic. Recall that \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) are the critical sets for the projections \( \pi_1, \pi_2 : \Gamma_f \to Y \), i.e. smallest analytic subsets of \( \Gamma_f \) so that the restrictions \( \pi_1 : \Gamma_f - \mathcal{C}_1 \to X \) and \( \pi_2 : \Gamma_f - \mathcal{C}_2 \to X \) are finite-to-one maps.

**Lemma 1.** a) The sets \( \pi_i(\mathcal{C}_j) \) have dimensions \( \leq 1 \) for \( i,j = 1,2 \).

b) For any analytic set \( C \) in \( X \) of dimension \( \leq 1 \), \( f(C) \) and \( f^{-1}(C) \) are analytic sets of dimensions \( \leq 1 \).

c) For any \( n \in \mathbb{Z} \), the maps \( f^n \) are also pseudo-automorphisms.

**Proof.** a) We prove the claim e.g. for \( i = 1 \) and \( j = 2 \). Since \( f : X - \mathcal{I}(f) \to X - \mathcal{I}(f^{-1}) \) is biholomorphic, it follows that \( \pi_1(\mathcal{C}_2) \) is contained in \( \mathcal{I}(f) \), and the latter has dimension \( \leq 1 \).

b) This also follows from the fact that \( f : X - \mathcal{I}(f) \to X - \mathcal{I}(f^{-1}) \) is biholomorphic.

c) Since \( f^{-1} \) is also a pseudo-automorphism, it suffices to prove the claim for \( n \in \mathbb{N} \). Given \( n \in \mathbb{N} \), we define \( I_n(f) = \bigcup_{j=0}^n (f^{-1})^j(\mathcal{I}(f)) \). By b), \( I_n(f) \) is an analytic set of dimension \( \leq 1 \). We have \( f : X - I_n(f) \to f(X - I_n(f)) \) is biholomorphic, and then by induction \( f^j : X - I_n(f) \to f^j(X - I_n(f)) \) is biholomorphic (for \( 2 \leq j \leq n \)). The sets \( f^j(X - I_n(f)) \) are complements of analytic sets of dimensions \( \leq 1 \), by b). Hence \( f^n \) is a pseudo-automorphism. \( \square \)

The following result was given in Propositions 1.3 and 1.4 in [9]. For completeness we give a proof of it here.

**Lemma 2.** 1) The maps \( f_* \) and \( f^* \) are both algebraically 1- and 2-stable.

2) \( f_*f^* = Id \) on \( H^{1,1}(X) \) and \( H^{2,2}(X) \). In particular, \( (f^{-1})^* = f_* : H^{1,1}(X) \to H^{1,1}(X) \) is the inverse of \( f^* : H^{1,1}(X) \to H^{1,1}(X) \).

3) For \( \theta \in H^{1,1}(X) \) and \( \eta \in H^{2,2}(X) \), \( f^*\theta . f^*\eta = \theta . \eta \).

**Proof.** 1) Let \( \theta \) be a smooth closed (1,1) form. Then \( (f^n)^*(\theta) \) and \( (f^n)^*(\theta) \) differ only on the set \( I_n(f) = \bigcup_{j=0}^n (f^{-1})^j(\mathcal{I}(f)) \). Since the latter set is analytic of dimension \( \leq 1 \) by Lemma 1, it cannot contain mass for the normal current \( (f^n)^*(\theta) - (f^n)^*(\theta) \). Therefore the two currents \( (f^n)^*(\theta) \) and \( (f^n)^*(\theta) \) are the same. Passing to cohomology we obtain that \( f^* \) is algebraically 1-stable.

Since \( f_* : H^{1,1}(X) \to H^{1,1}(X) \) and \( f_* : H^{2,2}(X) \to H^{2,2}(X) \) are conjugates, it follows that \( f_* \) is 2-stable. Similarly, \( f^* \) is 2-stable.
2) The proof is similar to that of 1). Let \( \theta \) be a closed smooth \((1, 1)\) form. Then \( f, f^*(\theta) \) and \( \theta \) differ only on an analytic set of dimension \( \leq 1 \), and hence must be the same. Passing to cohomology we obtain the claim for \( H^{1,1}(X) \). By the conjugate property and 3) we obtain the claim for \( H^{2,2}(X) \).

3) From the conjugate property and 2) we have \( f^*\theta. f^*\eta = f_* f^* \theta. \eta = \theta. \eta. \)

3. QUASI-POTENTIALS AND REGULARIZATION KERNELS FOR DSH CURRENTS

Let \( Y \) be a compact Kähler manifold of dimension \( k \). Let \( \pi_1, \pi_2 : Y \times Y \to Y \) be the two projections, and let \( \Delta_Y \subset Y \times Y \) be the diagonal. Let \( \omega_Y \) be a Kähler \((1, 1)\) form on \( Y \). As before, let \( DSH^p(Y) \) be the space of \( DSH \) \((p, p)\) currents.

Fix a Kähler form \( \theta \). A function \( \varphi \) is a \( \theta \)-plurisubharmonic function if it is upper semi-continuous, belongs to \( L^1 \), and \( \theta + dd^c(\varphi) \) is a positive closed \((1, 1)\) current.

**Remark 1.** The following consideration from [5] and [24] is used in both the proof of Lemma 3 and the construction of the kernels \( K_n \) in Lemma 4. Let \( k \) be dimension of \( Y \). Let \( \pi : \tilde{Y} \times Y \to Y \times Y \) be the blowup of \( Y \times Y \) at \( \Delta_Y \). Let \( \tilde{\Delta}_Y = \pi^{-1}(\Delta_Y) \) be the exceptional divisor. Then there is a closed smooth \((1, 1)\) form \( \gamma \) and a negative quasi-plurisubharmonic function \( \varphi \) so that \( dd^c \varphi = [\tilde{\Delta}_Y] - \gamma \). We choose a strictly positive closed smooth \((k-1, k-1)\) form \( \eta \) so that \( \pi_*([\Delta_Y] \wedge \eta) = [\Delta_Y] \).

A useful tool in proving the results in Section 4 is the following, concerning the quasi-potentials of a positive closed \((p, p)\) current \( T \) on a compact Kähler manifold \( Y \). It is known that (see Dinh and Sibony [24], Bost, Gillet and Soulé [5]) there is a \( DSH \) \((p-1, p-1)\) current \( S \) and a closed smooth form \( \alpha \) so that \( T = \alpha + dd^c S \). Here \( S \) is a difference of two negative currents. When \( p = 1 \) or when \( Y \) is a projective space, we can choose \( S \) to be negative. However, in general we cannot choose \( S \) to be negative (see [5]). The following weaker conclusion is sufficient for the purpose of this paper.

**Lemma 3.** Let \( T \) be a positive closed \((p, p)\) current on a compact Kähler manifold \( Y \). Then there is a closed smooth \((p, p)\) form \( \alpha \) and a negative \( DSH \) \((p-1, p-1)\) current \( S \) so that

\[
T \leq \alpha + dd^c S.
\]

Moreover, there is a constant \( C > 0 \) independent of \( T \) so that \( ||\alpha||_{L^\infty} \leq C ||T|| \) and \( ||S|| \leq C ||T|| \). If \( T \) is strongly positive, then we can choose \( S \) to be strongly negative.

Here \( ||.||_{L^\infty} \) is the maximum norm of a continuous form and \( ||.|| \) is the mass of a positive or negative current.

**Proof of Lemma 3** Notation is as in Remark 1. Define \( H = \pi_*(\varphi \eta) \). Then \( H \) is a negative \((k-1, k-1)\) current on \( Y \times Y \).

We write \( \gamma = \gamma^+ - \gamma^- \) for strictly positive closed smooth \((1, 1)\) forms \( \gamma^\pm \). If we define \( \Phi^\pm = \pi_*(\gamma^\pm \wedge \eta) \), then \( \Phi^\pm \) are positive closed \((k, k)\) currents with \( L^1 \) coefficients. In fact (see [22]) \( \Phi^\pm \) are smooth away from the diagonal \( \Delta_Y \), and the singularities of \( \Phi^\pm(y_1, y_2) \) and their derivatives are bounded by \( |y_1 - y_2|^{-(2k-2)} \) and \( |y_1 - y_2|^{-(2k-1)} \). Moreover

\[
\begin{align*}
dd^c H &= \pi_*(dd^c \varphi \wedge \eta) = \pi_*([\Delta_Y] \wedge \eta - (\gamma^+ - \gamma^-) \wedge \eta) = [\Delta_Y] - (\Phi^+ - \Phi^-).
\end{align*}
\]
Consider $S_1 = (\pi_1)_*(H \land \pi_2^*(T))$ and $R_1^\pm = (\pi_1)_*(\Phi^\pm \land T)$. Then $S_1$ is a negative current, and $R_1^\pm$ are positive closed currents. Moreover

$$dd^c S_1 = (\pi_1)_*(dd^c H \land \pi_2^*(T)) = T - R_1^+ + R_1^-.$$ 

Therefore $T \leq R_1^+ + dd^c S_1$. Moreover $R_1^+$ is a current with $L^1$ coefficients, and there is a constant $C_1 > 0$ independent of $T$ so that $||S_1||, ||R_1||_{L^1} \leq C_1 ||T||$ (see e.g. Lemma 2.1 in [22]).

If we apply this process for $R_1^+$ instead of $T$ we find a positive closed current $R_2^+$ with coefficients in $L^{1+1/(2k+2)}$ and a negative current $S_2$ so that $R_1^+ \leq R_2^+ + dd^c S_2$. Moreover

$$||R_2^+||_{L^{1+1/(2k+2)}}, ||S_2|| \leq C_2 ||R_1^+||_{L^1} \leq C_1 C_2 ||T||$$

for some constant $C_2 > 0$ independent of $T$. After iterating this process a finite number of times we find a continuous form $R$ and a negative current $S$ so that $T \leq R + dd^c S$. Moreover, $||R||_{L^\infty}, ||S|| \leq C ||T||$ for some constant $C > 0$ independent of $T$. Since we can bound $R$ by $\omega^n$ up to a multiple constant of size $||R||_{L^\infty}$, we are done.

Next we recall the construction of the kernels $K_n$ from Section 3 in [22]. Notation is as in Remark 1. Observe that $\varphi$ is smooth out of $[\Delta_Y]$ and $\varphi^{-1}(-\infty) = \Delta_Y$. Let $\chi : \mathbb{R} \cup \{-\infty\} \to \mathbb{R}$ be a smooth increasing convex function such that $\chi(x) = 0$ on $[-\infty, -1]$, $\chi(x) = x$ on $[1, +\infty]$, and $0 \leq \chi' \leq 1$. Define $\chi_n(x) = \chi(x + n) - n$ and $\varphi_n = \chi_n \circ \varphi$. The functions $\varphi_n$ are smooth decreasing to $\varphi$, and $dd^c \varphi_n \geq -\Theta$ for every $n$, where $\Theta$ is a strictly positive closed smooth $(1, 1)$ form so that $\Theta - \gamma$ is strictly positive. Then we define $\Theta_n^+ = dd^c \varphi_n + \Theta$ and $\Theta_n^- = \Theta - \gamma$. Finally $K_n^\pm = \pi_*(\Theta_n^\pm \land \eta)$, and $K_n = K_n^+ - K_n^-$. If $K$ is a current on $Y \times Y$ and $T$ is a current on $Y$, we define $K(T) = (\pi_1)_* (K \land \pi_2^*(T))$ whenever the wedge product $K \land \pi_2^*(T)$ makes sense.

**Lemma 4.** Let $Y$ be a compact Kähler manifold. Let $K_n$ be a weak regularization of the diagonal $\Delta_Y$ defined in [22] (see Section 2 for more detail). Let $T$ be a DSH $(p, p)$ current and let $\theta$ be a continuous $(q, q)$ form on $Y$. Assume that there is a positive $dd^c$-closed current $R$ so that $-R \leq T \leq R$. Then there are positive $dd^c$-closed $(p + q, p + q)$ currents $R_n$ so that $\lim_{n \to \infty} ||R_n|| = 0$ and

$$-R_n \leq K_n(T \land \theta) - K_n(T) \land \theta \leq R_n,$$

for all $n$.

If $R$ is strongly positive or closed, then we can choose $R_n$ to be so.

**Proof of Lemma 4** Let us define $H_n = K_n(T \land \theta) - K_n(T) \land \theta$. Since $T$ and $\theta$ may not be either positive or $dd^c$-closed, a priori $H_n$ is neither. However, we will show that there are positive $dd^c$-closed currents $R_n$ such that $\lim_{n \to \infty} ||R_n|| = 0$ and $-R_n \leq H_n \leq R_n$.

By definition we have

$$H_n(y) = \int_{z \in Y} K_n(y, z) \land (\pi_1^* \theta - \pi_2^* \theta) \land \pi_2^* T.$$ 

Fix a number $\delta > 0$. Then by the construction of $K_n$, there is an integer $n_\delta$ so that if $n \geq n_\delta$ and $|y - z| \geq \delta$, then $K_n(y, z) = 0$. Thus

$$H_n(y) = \int_{z \in Y, |z - y| < \delta} K_n(y, z) \land (\pi_1^* \theta - \pi_2^* \theta) \land \pi_2^* T.$$
We define \( h(\delta) = \max_{y,z \in Y} |y - z| \delta \mid \pi_1^* \theta - \pi_2^* \theta \mid \). We now show that

\[
\lim_{\delta \to 0} h(\delta) = 0. \tag{3.1}
\]

Let \( \iota : \Delta \subset Z \times Z \) be the embedding of the diagonal \( \Delta \) into \( Z \times Z \). Since the \((q, q)\) form \( \pi_1^* \theta - \pi_2^* \theta \) is smooth on \( Z \times Z \), and since \( Z \times Z \) (and hence \( \Delta \)) is compact, it suffices to show that the restriction of \( \pi_1^* \theta - \pi_2^* \theta \) to \( \Delta \) is 0. But the latter is clear, since

\[
\pi_1^* \theta - \pi_2^* \theta \mid_{\Delta} = \iota^* (\pi_1^* \theta - \pi_2^* \theta) = (\pi_1 \circ \iota)^* (\theta) - (\pi_2 \circ \iota)^* (\theta),
\]

and the last expression is 0 because the two maps \( \pi_1 \circ \iota, \pi_2 \circ \iota : \Delta \to Z \) are the same map \((z, z) \mapsto z\).

By (3.1), because \( Y \times Y \) is compact, there is a constant \( C > 0 \) independent of \( \theta \) and \( \delta \) so that

\[
-h(\delta) C (\omega_Y (y) + \omega_Y (z))^q \leq \theta (z) - \theta (y) \leq h(\delta) C (\omega_Y (y) + \omega_Y (z))^q
\]

for all \( \delta \leq 1 \) and for all \( |y - z| \leq \delta \). Since \( K_n^\pm (y, z) \) are strongly positive closed and \( -R \leq T \leq R \), it follows that

\[
H_n (y) = \int_{z \in Y, |z - y| < \delta} K_n (y, z) \wedge (\pi_1^* \theta - \pi_2^* \theta) \wedge \pi_2^* T
\]

\[
\leq h(\delta) C \int_{z \in Y, |z - y| < \delta} (K_n^+ (y, z) + K_n^- (y, z)) \wedge (\omega_Y (y) + \omega_Y (z))^q \wedge R(z)
\]

\[
\leq h(\delta) C \int_{z \in Y} (K_n^+ (y, z) + K_n^- (y, z)) \wedge (\omega_Y (y) + \omega_Y (z))^q \wedge R(z).
\]

Thus \( H_n (y) \leq R_n (y) \) where

\[
R_n (y) = h(\delta) C \int_{z \in Y} (K_n^+ (y, z) + K_n^- (y, z)) \wedge (\omega_Y (y) + \omega_Y (z))^q \wedge R(z),
\]

for \( n_\delta \leq n < n_\delta / 2 \). Similarly we have \( H_n (y) \geq -R_n (y) \). It can be checked that \( R_n (y) \) is positive \( dd^c \)-closed. Moreover, there is a constant \( C_1 > 0 \) independent of \( n, \delta, R \) and \( \theta \) so that

\[
||R_n|| \leq h(\delta) C_1 ||R||,
\]

for \( n \geq n_\delta \). This shows that \( ||R_n|| \to 0 \) as \( n \to \infty \). \( \square \)

**Remark 2.** By the estimate (3.2) and by iterating we obtain the following result: Let \( T, R \) and \( \theta \) be as in Lemma 4. Then there are positive \( dd^c \)-closed \( (p + q, p + q) \) currents \( R_{n_1, n_2, \ldots, n_l} \) so that

\[-R_{n_1, n_2, \ldots, n_l} \leq K_{n_1} \circ K_{n_2} \circ \ldots \circ K_{n_l} (T \wedge \theta) - K_{n_1} \circ K_{n_2} \circ \ldots \circ K_{n_l} (T) \wedge \theta \]

\[
\leq R_{n_1, n_2, \ldots, n_l}
\]

and

\[
\lim_{n_1, n_2, \ldots, n_l \to \infty} ||R_{n_1, n_2, \ldots, n_l}|| = 0.
\]

We give the proof of this claim for example when \( l = 2 \). We will write the \( R_n \) in Lemma 4 by \( R_n (R) \) to emphasize its dependence on \( R \). Writing

\[
K_{n_1} \circ K_{n_2} (T \wedge \theta) - K_{n_1} \circ K_{n_2} (T) \wedge \theta
\]

\[
= [K_{n_1} (K_{n_2} (T \wedge \theta) - K_{n_2} (T) \wedge \theta)] + [K_{n_1} (K_{n_2} (T) \wedge \theta) - K_{n_1} (K_{n_2} (T)) \wedge \theta]
\]
and choosing
\[ R_{n_1,n_2} = K_{n_1}^+ (R_{n_2}(R)) + K_{n_1}^- (R_{n_2}(R)) + R_{n_1} (K_{n_2}^+ (R)) + R_{n_1} (K_{n_2}^- (R)), \]
we see that
\[ -R_{n_1,n_2} \leq K_{n_1} \circ K_{n_2} (T \wedge \theta) - K_{n_1} \circ K_{n_2} (T) \wedge \theta \leq R_{n_1,n_2}. \]
That \( R_{n_1,n_2} \) are positive \( d\bar{d} \)-closed follows from the properties of the kernels \( K_n \).
It remains to bound the masses of \( R_{n_1,n_2} \). By (3.2) we have
\[
\| R_{n_1,n_2} \| \leq C_1 (\| R_{n_2}(R) \| + \| R_{n_1} (K_{n_2}^+(R)) \| + \| R_{n_2} (K_{n_2}^-(R)) \|)
\leq C_2 h(\delta) (\| R \| + \| K_{n_2}^+(R) \| + \| K_{n_2}^-(R) \|)
\leq C_3 h(\delta) \| R \|,
\]
for constants \( C_1, C_2, C_3 \) and for all \( n_1, n_2 \geq n_\delta \); here \( n_\delta \) is the constant in the proof of Lemma 4.

4. Pullback of currents by meromorphic maps and intersection of currents

4.1. Pullback of currents. Let \( Y \) be another compact Kähler manifold, and let \( f : X \to Y \) be a dominant meromorphic map. Let \( \Gamma_f \subset X \times Y \) be the graph of \( f \), and let \( \pi_X, \pi_Y : X \times Y \to X, Y \) be the projections. (When \( X = Y \) we denote these two maps by \( \pi_1 \) and \( \pi_2 \).) We denote by \( C_Y \) the critical set of \( \pi_Y \), i.e. the smallest analytic subvariety of \( \Gamma_f \) so that the restriction of \( \pi_Y \) to \( \Gamma_f - C_Y \) has fibers of dimension \( \dim(X) - \dim(Y) \). We have a similar notation \( C_X \) for the map \( \pi_X \).

When \( X = Y \) we denote \( C_X, C_Y \) by \( C_1 \) and \( C_2 \). Hence the set \( \pi_X (C_Y) \) may be regarded as the critical set of the map \( f \). For a set \( A \subset X \), we define its (total) image by \( f(A) = \pi_Y (\pi_X^{-1} (A) \cap \Gamma_f) \), and for a set \( B \subset Y \) we define its (total) pre-image by \( f^{-1}(B) = \pi_X (\pi_Y^{-1} (B) \cap \Gamma_f) \).

If \( T \) is a smooth form on \( Y \), then it is standard to define \( f^* (T) \) as a current on \( X \) by the formula \( f^* (T) = (\pi_X)_* (\pi_Y^* (T) \wedge [\Gamma_f]) \). This definition descends to cohomology classes: If \( T_1 \) and \( T_2 \) are two closed smooth forms on \( Y \) having the same cohomology classes, then \( f^* (T_1) \) and \( f^* (T_2) \) have the same cohomology class in \( X \). This allows us to define a pullback operator on cohomology classes. These considerations apply equally to continuous forms. However, it is not known how to define the pullback of an arbitrary current in general.

Méo [37] defined the pullback of a positive closed \((1,1)\) current in the following way: If \( T \) is a positive closed \((1,1)\) current on \( Y \), then locally we can write \( T = d\bar{d} \varphi \) where \( \varphi \) is a pluri-subharmonic function, and we define \( f^* (T) = d\bar{d} (\varphi \circ f) \). There are extensions of this to the case of positive \( d\bar{d} \)-closed \((1,1)\) currents (see Alessandrini- Bassanelli [1] and Dinh-Sibony [23]).

For a measure \( \mu \) having no mass on the indeterminacy set \( \mathcal{I}(f) \), we can define its pushforward by \( f \) as follows (see e.g. [16]): \((f_*)(\mu)(B) = \mu(f^{-1}(B) \cap X \setminus \mathcal{I}(f))\).

For a holomorphic map whose fibers are either empty or of dimension \( \dim(X) - \dim(Y) \), Dinh-Sibony [23] defined pullback of positive closed currents of any bidegrees. For meromorphic selfmaps of \( \mathbb{P}^k \), they gave a satisfying pullback operator using super-potentials (see [25]). For general meromorphic maps on compact Kähler manifolds, they defined a “strict pullback” on positive closed currents of any bidegrees. However, this “strict pullback” is not compatible with the pullback on cohomology.
Using the good approximation schemes (see Definition 6), we defined in a pullback operator which is compatible with the pullback on cohomology and is compatible with the previous definitions. Moreover if a positive closed current $T$ can be pulled back by the map $f$, then $f^*(T)$ is an extension of the “strict pullback” of Dinh and Sibony.

We now recall the definition from [45], where it was not checked that the kernels $K_n$ satisfy condition 9 in Definition 6.

**Definition 7.** Let $T$ be a $DSH^p(Y)$ current on $Y$. We say that $f^*(T)$ is well-defined if there is a number $s \geq 0$ and a current $S$ on $X$ so that

$$\lim_{n \to \infty} f^*(K_n(T)) = S,$$

for any good approximation scheme by $C^{s+2}$ forms $K_n^\pm$. Then we write $f^*(T) = S$.

The definition for a general current on $Y$ (not necessarily $DSH$) is more complicated. We recall it here and will use it for currents of the form defined if there is a number $s \geq 0$ and a current $S$ on $X$ so that

$$\lim_{n \to \infty} \int_Y T \wedge K_n(f_*(\alpha)) = \int_X S \wedge \alpha,$$

for any smooth form $\alpha$ on $X$ and any good approximation scheme by $C^{s+2}$ forms $K_n$. Then we write $f^*(T) = S$.

By the self-adjointness in Definition 6, we see that Definitions 7 and 8 coincide for $DSH$ currents.

We recall some results from [45] for using later (see Theorems 6 and 9 in [45]):

**Theorem 9.** Let $X$ and $Y$ be two compact Kähler manifolds. Let $f : X \to Y$ be a dominant meromorphic map. Assume that $\pi_X(C_Y)$ is of codimension $\geq p$. Then the pullbacks $f^* : DSH^{p-1}(Y) \to DSH^{p-1}(X)$ and $f^* : D^p(Y) \to D^p(X)$ are well-defined. Moreover these pullbacks are continuous with respect to the topologies on the corresponding spaces.

In fact, even though the statement of Theorem 6 in [45] concerns only the claim for the map $f^* : D^p(Y) \to D^p(X)$ in Theorem 9 its proof confirms the claim for the map $f^* : DSH^{p-1}(Y) \to DSH^{p-1}(X)$ in Theorem 10.

**Theorem 10.** Let $X$ and $Y$ be two compact Kähler manifolds. Let $f : X \to Y$ be a dominant meromorphic map. Let $T$ be a positive measure having no mass on $\pi_Y(C_Y)$. Then $f^*(T)$ is well-defined and coincides with the usual definition (see for example [16]). Moreover, if $T$ has no mass on proper analytic subvarieties of $Y$, then $f^*(T)$ has no mass on proper analytic subvarieties of $X$.

We can define a similar notion $f_*$ of pushforward of currents and obtain similar results to that of Theorems 9 and 10 for the pushforward operator. Note that when $f$ is a bimeromorphic map $f_* = (f^{-1})^*$.

We now prove several additional properties. A current $\tau$ is called quasi-PSH if there is a smooth form $\gamma$ so that $dd^c \tau \geq -\gamma$. We have the following result.
Theorem 11. Let $X$ and $Y$ be compact Kähler manifolds and let $f : X \to Y$ be a dominant meromorphic map. Let $T$ be a DSH $(p,p)$ current and let $\theta$ be a smooth $(q,q)$ form on $Y$. Assume that there is a positive quasi-PSH current $\tau$ so that $-\tau \leq T \leq \tau$.

a) If $f$ is holomorphic and $f^*(T)$ is well-defined, then $f^*(T \wedge \theta)$ is well-defined. Moreover, $f^*(T \wedge \theta) = f^*(T) \wedge f^*(\theta)$.

b) More generally, assume that there is a number $s \geq 0$ and a $(p,p)$ current $(\pi_Y|\Gamma_f)^*(T)$ on $X \times Y$ such that for any good approximation by $C^{s+2}$ forms $K_n$,

$$\lim_{n \to \infty} \pi_Y^*(K_n(T)) \wedge [\Gamma_f] = (\pi_Y|\Gamma_f)^*(T).$$

Then $f^*(T \wedge \theta)$ is well-defined, and moreover $f^*(T \wedge \theta) = (\pi_X)_*(\pi_Y|\Gamma_f)^*(T) \wedge \pi_Y^*(\theta))$.

Roughly speaking, the result b) of Theorem 11 says that under some natural conditions if we can pull back $T$, then we can do it locally.

Proof of Theorem 11. a) We let $s \geq 0$ be a number so that for any good approximation scheme by $C^{s+2}$ forms $K_n$ and for any smooth form $\alpha$ on $X$,

$$\int_X f^*(T) \wedge \alpha = \lim_{n \to \infty} \int_Y T \wedge K_n(f_*(\alpha)).$$

Then for the proof of a) it suffices to show that for any smooth form $\beta$ on $X$,

$$\lim_{n \to \infty} \int_Y T \wedge \theta \wedge K_n(f_*(\beta)) = \int_X f^*(T) \wedge f^*(\theta) \wedge \beta.$$

If we can show that

$$\lim_{n \to \infty} \int_Y T \wedge (\theta \wedge K_n(f_*(\beta)) - K_n(\theta \wedge f_*(\beta))) = 0,$$

then we are done, since we have $\theta \wedge (f_*(\beta)) = f_*(f^*(\theta) \wedge \beta)$ because $f$ is holomorphic, and hence

$$\lim_{n \to \infty} \int_Y T \wedge K_n(\theta \wedge f_*(\beta)) = \lim_{n \to \infty} \int_Y T \wedge K_n(f_*(f^*(\theta) \wedge \beta)) = \int_Y f^2(T) \wedge (f^*(\theta) \wedge \beta).$$

Now we proceed to prove 11.1. For a fixed $n$ we have

$$\int_Y T \wedge (\theta \wedge K_n(f_*(\beta)) - K_n(\theta \wedge f_*(\beta)) = \lim_{m \to \infty} \int_Y K_m(T) \wedge (\theta \wedge K_n(f_*(\beta)) - K_n(\theta \wedge f_*(\beta))).$$

The advantage of this is that $K_n(T)$ are continuous forms, hence if we have bounds of $\theta \wedge K_n(f_*(\beta)) - K_n(\theta \wedge f_*(\beta))$ by currents of order zero, we can use them in the integral and then take the limit when $m \to \infty$.

Because $f_*(\beta)$ is bound by a multiple of $\omega^d(X) - p - q$ and the latter is strongly positive closed, by condition 9) of Definition 6 there are strongly positive closed currents $R_n$ with $\|R_n\| \to 0$ and $-R_n \leq \theta \wedge K_n(f_*(\beta)) - K_n(\theta \wedge f_*(\beta)) \leq R_n$. 

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for all \( n \). Since \( -\tau \leq T \leq \tau \), we have 
\[-(K^+_m(\tau) + K^-_m(\tau)) \leq K_m(T) \leq K^+_m(\tau) + K^-_m(\tau)\].
Since \( K^+_m(\tau) + K^-_m(\tau) \) are positive \( C^2 \) forms, from the above estimates we obtain
\[
-\int_Y (K^+_m(\tau) + K^-_m(\tau)) \wedge R_n \leq \int_Y K_m(T) \wedge (\theta \wedge K_n(f_*(\beta)) - K_n(\theta \wedge f_*(\beta))) \\
\leq \int_Y (K^+_m(\tau) + K^-_m(\tau)) \wedge R_n.
\]
Hence (4.1) follows if we can show that
\[
\lim_{n \to \infty} \lim_{m \to \infty} \int_Y (K^+_m(\tau) + K^-_m(\tau)) \wedge R_n = 0.
\]
By Lemma 3 there are a smooth closed form \( \alpha_n \) and a strongly negative current \( S_n \) for which \( R_n \leq \alpha_n + dd^c S_n \) and \( ||\alpha_n||_{L^\infty}, ||S_n|| \to 0 \). Therefore
\[
0 \leq \int_Y (K^+_m(\tau) + K^-_m(\tau)) \wedge R_n \\
\leq \int_Y (K^+_m(\tau) + K^-_m(\tau)) \wedge \alpha_n + \int_Y (K^+_m(\tau) + K^-_m(\tau)) \wedge dd^c S_n.
\]
Since the currents \( K^+_m(\tau) \) are positive whose masses are uniformly bounded, it follows from \( ||\alpha_n||_{L^\infty} \to 0 \) that
\[
\lim_{n \to \infty} \lim_{m \to \infty} \int_Y (K^+_m(\tau) + K^-_m(\tau)) \wedge \alpha_n = 0.
\]
Now we estimate the other term. We have
\[
\int_Y (K^+_m(\tau) + K^-_m(\tau)) \wedge dd^c S_n = \int_Y (K^+_m(dd^c_\tau) + K^-_m(dd^c_\tau)) \wedge S_n.
\]
Because \( S_n \) is strongly negative and \( dd^c_\tau \geq -\gamma \), the last integral can be bound from above by
\[
\int_Y (K^+_m(dd^c_\tau) + K^-_m(dd^c_\tau)) \wedge S_n \leq \int_Y (K^+_m(-\gamma) + K^-_m(-\gamma)) \wedge S_n.
\]
Since \( \gamma \) is smooth, by condition 4) of Definition \( \text{[8]} \) and the fact that \( ||S_n|| \to 0 \), we obtain
\[
\lim_{n \to \infty} \lim_{m \to \infty} \int_Y (K^+_m(-\gamma) + K^-_m(-\gamma)) \wedge S_n = 0.
\]
Thus, whatever the limit of
\[
\int_Y (K^+_m(\tau) + K^-_m(\tau)) \wedge dd^c S_n
\]
is, it is non-positive. The proof of (4.1) and hence of a) is finished.

b) The proof of b) is similar to that of a).

As some consequences we obtain the following two results, which were known previously using other definitions of pullbacks (see Diller [18], Russakovskii-Shiffman [43] and Dinh-Sibony [23]). The novelty here is that we treat these cases in a unified manner.

**Proposition 1.** Let \( X \) and \( Y \) be compact Kähler manifolds and let \( f : X \to Y \) be a dominant meromorphic map. Let \( \psi \) be a function on \( Y \) bounded by a quasi-PSH function \( \varphi \). Then \( f^*(\varphi) \) is well-defined with respect to Definition \( \text{[8]} \).
We define linear functionals by desingularizing the graph $\Gamma_f$. By subtracting a constant from $\varphi$ if needed, we can assume that $\varphi \leq 0$. By the assumptions, we have $0 \geq \psi \geq \varphi$. To prove that $f^*(\psi)$ is well-defined with respect to Definition 8, we need to show the existence of a current $S$ so that for any smooth form $\alpha$ and any good approximation scheme by $C^2$ forms $K_n$,

$$\lim_{n \to \infty} \int_Y \psi \wedge K_n(f_*(\alpha)) = \int_X S \wedge \alpha.$$  

We define linear functionals $S_n$ and $S_n^\pm$ on top forms on $X$ by the formulas

$$\langle S_n, \alpha \rangle = \int_Y \psi \wedge K_n(f_*(\alpha)),$$

$$\langle S_n^\pm, \alpha \rangle = \int_Y \psi \wedge K_n^\pm(f_*(\alpha)).$$

Then $S_n = S_n^+ - S_n^-$, and it can be checked that $S_n^\pm$ are negative $(0,0)$ currents, and hence $S_n$ is a current of order 0. Moreover, if $\alpha$ is a positive smooth measure, then

$$0 \geq \langle S_n^\pm, \alpha \rangle = \int_Y \psi \wedge K_n^\pm(f_*(\alpha)) \geq \int_Y \varphi \wedge K_n^\pm(f_*(\alpha)) = \int_X f^*(K_n^\pm(\varphi)) \wedge \alpha.$$  

Thus $0 \geq S_n^\pm \geq f^*(K_n^\pm(\varphi))$ for all $n$.

Let us write $dd^c(\varphi) = T - \theta$ where $T$ is a positive closed $(1,1)$ current, and $\theta$ is a smooth closed $(1,1)$ form. By property 4) of Definition 6 there is a strictly positive closed smooth $(1,1)$ form $\Theta$ so that $\Theta \geq K_n^\pm(\theta)$ for any $n$. Then $f^*(K_n^\pm(\varphi))$ are negative $C^2$ forms so that

$$dd^c f^*(K_n^\pm(\varphi)) = f^*(K_n^\pm(dd^c \varphi)) = f^*(K_n^\pm(T - \theta)) \geq f^*(K_n^\pm(-\theta)) \geq -f^*(\Theta)$$

for any $n$; i.e they are negative $f^*(\Theta)$-plurisubharmonic functions. Moreover the sequence of currents $f^*(K_n^\pm(\varphi))$ has uniformly bounded mass (see the proof of Theorem 6 in [45]). Therefore, by the compactness of this class of functions (see Chapter 1 in [13]), after passing to a subsequence if needed, we can assume that $f^*(K_n^\pm(\varphi))$ converges in $L^1$ to negative functions denoted by $f^*(\varphi^\pm)$. Let $S^\pm$ be any cluster points of $S_n^\pm$. Then $0 \geq S^\pm \geq f^*(\varphi^\pm)$, which shows that any cluster point $S = S^+ - S^-$ of $S_n$ has no mass on sets of Lebesgue measure zero. Hence to show that $S$ is uniquely defined, it suffices to show that $S$ is uniquely defined outside a proper analytic subset of $Y$.

Let $E$ be a proper analytic subset of $Y$ so that $f : X - f^{-1}(E) \to Y - E$ is a holomorphic submersion. If $\alpha$ is a smooth measure whose support is compactly contained in $X - f^{-1}(E)$, then $f_*(\alpha)$ is a smooth measure on $Y$. Hence by condition 4) of Definition 6 $K_n(f_*(\alpha))$ uniformly converges to the smooth measure $f_*(\alpha)$. Then it follows from the definition of $S$ that

$$\langle S, \alpha \rangle = \int_Y \psi \wedge f_*(\alpha).$$
Hence $S$ is uniquely defined on $X - f^{-1}(E)$, and thus it is uniquely defined on the whole $X$, as wanted. \hfill \Box

**Proposition 2.** Let $X$ and $Y$ be compact Kähler manifolds and let $f : X \to Y$ be a dominant meromorphic map. Let $\pi_X, \pi_Y : X \times Y \to Y$ be the projections, and let $\Gamma_f \subset X \times Y$ be the graph of $f$. Let $V \subset Y$ be an irreducible variety. If $\pi_Y^{-1}(V) \cap \Gamma_f$ has codimension $\geq$ the codimension of $V$, then for any smooth $(q,q)$ form $\theta$ on $Y$ the pullback $f^*(\theta \wedge \lbrack V \rbrack)$ is well-defined. If moreover $\theta$ is a positive form, then $f^*(\theta \wedge \lbrack V \rbrack)$ is also positive.

**Proof.** By Lemma 5 below we have that the intersection $\pi_Y^*(\lbrack V \rbrack) \wedge \Gamma_f$ is well-defined and is moreover positive. Applying part b) of Theorem 11 we obtain Proposition 2. \hfill \Box

4.2. **Intersection of currents.** We give the following definition of intersection of currents. It corresponds to the definition of pullback of currents for the identity map. (There are many different approaches of intersection of currents in the literature. For some references please see Bedford-Taylor [10], Fornaess-Sibony [30], Demailly [13], and Dinh-Sibony [25, 26, 27].)

**Definition 12.** Let $Y$ be a compact Kähler manifold. Let $T_1$ be a DSH current and let $T_2$ be a $(q,q)$ current of order $s$ on $Y$. Let $s_0$ be the order of $T_2$. We say that $T_1 \wedge T_2$ is well-defined if there is $s \geq s_0$ and a current $S$ so that for any good approximation scheme by $C^{s+2}$ forms $K_n$, $\lim_{n \to \infty} K_n(T_1) \wedge T_2 = S$. Then we write $T_1 \wedge T_2 = S$.

We now prove some properties of this intersection.

**Proposition 3.** Let $T_1$ and $T_2$ be positive $d\bar{d}$-closed currents. Assume that $T_1 \wedge T_2$ is well-defined. Let $\theta$ be a smooth $(q,q)$ form.

- a) $\theta \wedge T_2$ and $T_2 \wedge \theta$ are well-defined and are the same as the usual definition.
- b) $T_2 \wedge T_1$ is also well-defined. Moreover, $T_1 \wedge T_2 = T_2 \wedge T_1$.
- c) $T_1 \wedge (\theta \wedge T_2)$ is also well-defined. Moreover $T_1 \wedge (\theta \wedge T_2) = (T_1 \wedge T_2) \wedge \theta$.

**Proof of Proposition 3** Proof of a): Let $K_n$ be a good approximation scheme by $C^2$ forms. Then $K_n(\theta)$ uniformly converges to $\theta$, and hence $K_n(\theta) \wedge T_2$ converges to the usual intersection $\theta \wedge T_2$.

Let $\alpha$ be a smooth form. Then by conditions 9), 6) and 4) of Definition 6 we have

$$\lim_{n \to \infty} \int_Y K_n(T_2) \wedge \theta \wedge \alpha = \lim_{n \to \infty} \int_Y K_n(T_2 \wedge \theta) \wedge \alpha = \lim_{n \to \infty} \int_Y T_2 \wedge \theta \wedge K_n(\alpha) = \int_Y T_2 \wedge \theta \wedge \alpha.$$  

The proofs of b) and c) are similar. \hfill \Box

**Lemma 5.** Let $T_1$ and $T_2$ be positive closed $(p,p)$ and $(q,q)$ currents of $Y$. Assume that there are closed sets $A_1 \subset Y$ and $A_2 \subset Y$ so that $T_i$ is continuous on $Y - A_i$ for $i = 1, 2$. Assume moreover that $A_1 \cap A_2$ is contained in an analytic set of codimension $\geq p + q$ of $Y$. Then $T_1 \wedge T_2$ is well-defined. If moreover one of $T_1$ and $T_2$ is strongly positive, then $T_1 \wedge T_2$ is positive.
Proof of Lemma 5 Let $\theta$ be a smooth $(p, p)$ form having the same cohomology class as that of $T_1$. Then by Proposition 2.1 in [21], there are positive $(p - 1, p - 1)$ currents $R^\pm$ so that $T_1 - \theta = dd^c (R^+ - R^-)$. Moreover, $R^\pm$ are $DSH$ and we can choose so that $R^\pm$ are continuous outside $A_1$. To prove Lemma 5 it suffices to show that there is a current $S$ so that for any good approximation scheme by $C^2$ forms $K_n$,
\[
\lim_{n \to \infty} K_n (R^+ - R^-) \wedge T_2 = S.
\]

The sequence $K_n^\pm (R^\pm) \wedge T_2$ converges on $Y - A_1 \cap A_2$. In fact, outside of $A_2$, $T_2$ is continuous, hence $\lim_{n \to \infty} K_n^\pm (R^\pm) \wedge T_2 = R^\pm \wedge T_2$, and outside of $V_1$, $K_n^\pm (R^\pm)$ converges locally uniformly (by condition 4) of Definition 6 to a continuous form and hence $K_n^\pm (R^\pm) \wedge T_2$ converges. Then by an argument as in the proof of Theorem 6 in [15] using the Federer-type support theorem in Bassanelli [2], the limit current is the trivial extension of $(R^+ - R^-)|_{X - A_1 \cap A_2} \wedge T_2$. In particular, we see that our definition coincides with the local definition. Since locally we can choose a local potential $H$ of $\theta$ so that the sum of $H$ and $R^+ - R^-$ gives a negative current continuous out of $A_1$ which can be well approximated by smooth negative forms whose $dd^c$ are strictly positive, Oka’s principle in [30] implies that $T_1 \wedge T_2$ is positive (here we used the assumption that one of $T_1$ or $T_2$ is strongly positive). This completes the proof of Lemma 5. \hfill $\square$

5. PROOFS OF THE MAIN RESULTS

5.1. Proof of Theorem 1 That the operators $f_*$, $f^*$ are well-defined on spaces $\mathcal{D}^1$, $DSH^1$ and $\mathcal{D}^2$ and are continuous with respect to the topologies on these spaces follows from Theorem 3 and Lemma 1.

Now we show the compatibility of these operators with iterations.

a) First we show that if $T \in DSH^1(X)$, then $(f^n)_*(T) = (f^*)^n(T)$ for any $n \in \mathbb{N}$. Since all the operators are continuous in the topology on $DSH^1(X)$, it suffices to prove this when $T$ is a smooth form. In this case we can proceed as in the proof of Lemma 2. The two currents $(f^n)_*(T)$ and $(f^*)^n(T)$ differ only on an analytic set of dimensions $\leq 1$. Therefore, the current $(f^n)_*(T) - (f^*)^n(T)$ is a $DSH(1, 1)$ current with support on an analytic set of dimensions $\leq 1$. Since $DSH$ currents are $C$-normal in the sense of Bassanelli [2], the Federer-type support theorem for $C$-normal currents implies that $(f^n)_*(T) - (f^*)^n(T)$ is the zero current, i.e. $(f^n)_*(T) = (f^*)^n(T)$.

b) To extend a) to all $n \in \mathbb{Z}$ we need only show that $(f^{-1})^* = (f^*)^{-1}$. Because $(f^{-1})^* = f_*$, it suffices to check that $f_* f^* = Id$ on $DSH^{1,1}(X)$ currents. To this end we can proceed as in a).

c) Since $\mathcal{D}^1 \subset DSH^{1,1}(X)$, we obtain the compatibility of $f^*$, $f_*$ for $\mathcal{D}^1$ as well.

d) Since the operators considered are continuous on $\mathcal{D}^p$, to prove the compatibility for $\mathcal{D}^2$, it suffices to prove the claim for smooth closed $(2, 2)$ forms. Hence we need to show the following: let $\eta$ be a smooth closed $(2, 2)$ form and let $\theta$ be a smooth $(1, 1)$ form (not necessarily closed); then
\[
\int_X (f^*)^n(\eta) \wedge \theta = \int_X (f^n)_*(\eta) \wedge \theta,
\]
for any \( n \in \mathbb{N} \). By definition
\[
\int_X (f^n)\ast(\eta) \wedge \theta = \int_X \eta \wedge (f^n)\ast(\theta),
\]
and the latter equals
\[
\int_X \eta \wedge (f)\ast^n(\theta),
\]
since \( f \) is compatible with iteration on \( DSH^{1,1}(X) \). Therefore, we need to show only that
\[
\int_X (f^n)\ast(\eta) \wedge \theta = \int_X \eta \wedge (f)\ast^n(\theta),
\]
for any \( n \in \mathbb{N} \).

We prove this by induction on \( n \). When \( n = 1 \), the equality follows from the definitions of \( f \ast \) and \( f \ast \). Assume that we already have
\[
\int_X (f^n)\ast(\eta) \wedge \theta = \int_X \eta \wedge (f^n)\ast(\theta),
\]
for some number \( m \in \mathbb{N} \). Then we will show that
\[
\int_X (f^{n+1})\ast(\eta) \wedge \theta = \int_X \eta \wedge (f^{n+1})\ast(\theta).
\]
Let \( K_j \) be a good approximation of \( DSH \) currents by \( C^2 \) forms. Then \((f^n)\ast(\eta) = \lim_{j \to \infty} f^n(K_j(f^n)\ast(\eta)) \) by the continuity of \( f^n \) on \( D \). Therefore
\[
\int_X (f^n)\ast(\eta) \wedge \theta = \lim_{j \to \infty} \int_X f^n(K_j(f^n)\ast(\eta)) \wedge \theta
= \lim_{j \to \infty} \int_X K_j(f^n)\ast(\eta) \wedge f\ast(\theta).
\]
By property 6) in Definition 6, we have for any \( j \in \mathbb{N} \),
\[
\int_X K_j(f^n)\ast(\eta) \wedge f\ast(\theta) = \int_X (f^n)\ast(\eta) \wedge K_j f\ast(\theta).
\]
The currents \( K_j f\ast(\theta) \) are \( C^2 \) forms by definition of \( K_j \); hence they can be approximated uniformly by smooth \((1,1)\) forms. Therefore the induction assumption implies
\[
\int_X (f^n)\ast(\eta) \wedge K_j f\ast(\theta) = \int_X \eta \wedge (f^n)\ast(K_j f\ast(\theta)),
\]
for any \( j \in \mathbb{N} \). Since \( f\ast \) is continuous on \( DSH^{1,1}(X) \), \( \lim_{j \to \infty} (f^n)\ast(K_j f\ast(\theta)) = (f^n)\ast(\theta) \). Therefore, we obtain
\[
\int_X (f^n)\ast(\eta) \wedge \theta = \int_X \eta \wedge (f^n)\ast(\theta),
\]
and complete the induction step, and also the proof of Theorem 1.
5.2. **Proof of Theorem 2** Theorem 1 in [36] shows that under assumptions of Theorem 2 the growth of \( \| (f^n)_{H^{1,1}(X)} \| = \| (f^n)_{H^{1,1}(X)} \| \sim \lambda_1(f)^n \).

a) Let \( \theta \) be a smooth closed \((1,1)\) form. We write \( \theta = \theta^+ - \theta^- \) where \( \theta^\pm \) are positive closed smooth \((1,1)\) forms. Since \( f \) is algebraically 1-stable and since the growth of \( \| (f^n)_{H^{1,1}(X)} \| = \| (f^n)_{H^{1,1}(X)} \| \sim \lambda_1(f)^n \), it follows that there is a constant \( C > 0 \) so that \( \| (f^n)(\theta^\pm) \| \leq C\lambda_1(f)^n \) for any \( n \in \mathbb{N} \). Hence for any \( N \in \mathbb{N} \), the Cesàro’s means

\[
T^+_N = \frac{1}{N} \sum_{j=1}^N (f^*)^j(\theta^\pm),
\]

are positive closed \((1,1)\) currents of mass \( \leq C \). Therefore we can find a subsequence \( N_j \) so that the sequences \( T^+_N \) weakly converge to positive closed \((1,1)\) currents \( T^\pm \).

We define \( T^+_\theta = T^+ - T^- \). Then it is easy to check that \( (f^*)(T^\pm) = \lambda_1(f)T^\pm \), and hence \( f^*(T^+_\theta) = \lambda_1(f)T^+_\theta \).

If the cohomology class \( \{\theta\} \in H^{1,1}(X) \) is so that \( f^*\{\theta\} = \lambda_1(f)\{\theta\} \), then \( (f^*)^j\{\theta^+ - \theta^-\} = \lambda_1(f)^j\{\theta\} \) for any \( j \). Hence \( \{T^+_N - T^-_N\} = \{\theta\} \) for all \( N \), and therefore the cohomology class of \( T^+_\theta \) is \( \{\theta\} \).

If we choose \( \theta \) to be a Kähler form, then \( T^+ \) is also positive, and because the growth of \( \| (f^n)_{H^{1,1}} \| \sim \lambda_1(f)^n \), \( T^+ \) is non-zero. In this case, we show that \( T^+ \) has no mass on hypersurfaces. This follows from the following claim:

**Claim.** Let \( T \) be a positive closed \((1,1)\) current such that \( f^*(T) = \lambda T \) for some \( \lambda > 1 \). Then \( T \) has no mass on hypersurfaces.

**Proof of the Claim.** This claim follows from standard arguments (see Theorem 2.4 in [19]). We prove by contradiction. Assume otherwise that \( T \) charges hypersurfaces. Then there is a hypersurface \( V \) and a number \( c > 0 \) so that the Lelong number of \( T \) along \( V \) is \( c \). Since \( X \) is compact, by Siu’s decomposition theory there is a number \( M > 0 \) so that \( \nu(T, x) \leq M \) for all \( x \in X \). Let \( n \) be a positive integer number so that \( c > M/\lambda^n \).

Let \( \mathcal{E}_f = \pi_1(C_2) \) be the critical set of \( f \) and let \( \mathcal{I}_f \) be the indeterminacy set of \( f \). By Lemma 1 the set \( A = \{x \in X : f(x) \in \mathcal{I}_f\} \cup \{x \in X : f(x) \in \mathcal{E}_f\} \) is an analytic subset of dimensions \( \leq 1 \). Then by results of Demailly [14] and Favre [29], for all \( x \in X - A \),

\[
\nu(T, x) = \frac{1}{\lambda^n} \nu((f^n)^*T, x) \leq \frac{1}{\lambda^n} \nu(T, f^n(x)) \leq \frac{M}{\lambda^n} < c.
\]

Therefore the contradiction assumption is false, which means that \( T \) has no mass on hypersurfaces.

b) The proof of b) is similar, using Proposition 4 below.

c) For any \( n \in \mathbb{N} \) and any smooth closed \((2,2)\) form, we have by Lemma 2

\[
\{T^+_\theta\}.\{(f^n)_*(\eta)\} = \{(f^n)^*(T^+_\theta).\eta\} = \lambda_1(f)^n\{T^+_\theta.\eta\},
\]

because \( T^+_\theta \) is \( f^* \) invariant. Since \( T^-_\eta \) is a Cesàro mean of the currents \( (f^n)_*(\eta) \), we have that \( \{T^+_\theta\}.\{T^-_\eta\} = \{\theta\}.\{\eta\} \).

Similarly we get \( \{T^+_\eta\}.\{T^-_\eta\} = \{\theta\}.\{T^+_\eta\} \).

If we choose \( \theta \) and \( \eta \) to be strictly positive closed smooth forms, then \( \{T^+\}, \{T^-\} > 0 \).
5.3. **Analytic stability.** We give here a result needed in the proof of Theorem 2. This result was given as a remark without proof in [15]. Let $X$ be a compact Kähler manifold of dimension $k$, and let $f : X \to X$ be a dominant meromorphic map. We recall (see Section 2) that the map $f$ is called algebraically $p$-stable if $(f^*)^n = (f^n)^*$ as linear maps on $H^{p,p}(X)$ for all $n = 1, 2, \ldots$. When this condition is satisfied, it follows that $\lambda_p(f) = r_p(f)$; thus it helps in determining the $p$-th dynamical degree of $f$.

There is also the related condition of analytically $p$-stable (implicitly used in [25] in the case $X$ is the projective space $\mathbb{P}^k$), which requires that
1) $(f^n)^*(T)$ is well-defined for any positive closed $(p,p)$ current $T$ and any $n \geq 1$.
2) Moreover, $(f^n)^*(T) = (f^n)^*(T)$ for any positive closed $(p,p)$ current $T$ and any $n \geq 2$.

Since $H^{p,p}(X)$ is generated by classes of positive closed smooth $(p,p)$ forms, analytic $p$-stability implies algebraic $p$-stability. For the converse of this, we have the following observation.

**Proposition 4.** Let $X$ be a compact Kähler manifold and $f : X \to X$ a dominant meromorphic map. If $\pi_1(C_f)$ has codimension $\geq p$, then $f$ is analytically $p$-stable iff it is algebraically $p$-stable and satisfies condition 1) above so that $(f^*)^n(\alpha)$ is positive closed for any positive closed smooth $(p,p)$ form $\alpha$ and for any $n \geq 1$. Hence algebraically 1-stability is the same as analytically 1-stability.

**Proof.** First, let $\alpha$ be a positive closed smooth $(p,p)$ form. Then $(f^n)^*(\alpha)$ is a current with $L^1$ coefficients. Then the assumption that $(f^*)^n(\alpha)$ is a positive closed current and the fact that $(f^*)^n(\alpha) = (f^n)^*(\alpha)$ outside a proper analytic set imply that $(f^n)^*(\alpha) \geq (f^n)^*(\alpha)$. But by the algebraic $p$-stability, these currents have the same cohomology class and hence must be the same. Hence the conclusion of Remark 4 holds for positive closed smooth $(p,p)$ forms.

Now let $T$ be a positive closed $(p,p)$ current and let $n$ be a positive integer. By Definition 8 there are positive closed smooth $(p,p)$ forms $T_j^\pm$ so that $||T_j^\pm||$ is uniformly bounded, $T_j^+ - T_j^-$ weakly converges to $T$, and

$$(f^n)^*(T) = \lim_{j \to \infty} (f^n)^*(T_j^+ - T_j^-).$$

By the first paragraph of the proof, $(f^n)^*(T_j^+ - T_j^-) = (f^n)^*(T_j^+ - T_j^-)$ for any $n$ and $j$. Because $\pi_1(C_f)$ has codimension $\geq p$, the continuity property in Theorem 9 implies that

$$\lim_{j \to \infty} (f^n)^*(T_j^+ - T_j^-) = (f^n)^*(T).$$

Therefore $(f^n)^*(T) = (f^n)^*(T)$ as wanted. \hfill \Box

5.4. **Proof of Theorem 4.** We follow the proof of Theorem 1.2 in [3], which in turn follows closely the proof of Theorem 2.2 in [34]. Our proof is almost identical to that of [3], but we will include the complete proof here for convenience. We will clearly indicate below where our proof differs from that of [3].

First, we recall the definition of currents with minimal singularities in a psef $(1,1)$ cohomology class. Let $\theta \in H^{1,1}(X)$ be psef, and let’s choose a smooth closed $(1,1)$ form representing $\theta$, which we still denote by $\theta$ for convenience. Following Demailly-Peternell-Schneider (see the proof of Theorem 1.5 in [15]), we define

$$v_\theta^{min} = \sup \{ \varphi \leq 0 : \theta + dd^c \varphi \geq 0 \}$$
and $T_{\theta}^{\text{min}} = \theta + dd^c v_{\theta}^{\text{min}}$. We will show that the limit

$$T = \lim_{n \to \infty} \frac{1}{\lambda^n} (f^*)^n (T_{\theta}^{\text{min}})$$

exists, and $T$ is what’s needed.

Since $f^*\{\theta\} = \lambda \{\theta\}$ in cohomology, we have by the $dd^c$ lemma for compact Kähler manifolds that

$$\frac{1}{\lambda} f^* (T_{\theta}^{\text{min}}) = \theta + dd^c \phi_1,$$

where $\phi_1$ is a quasi-PSH function. Hence we can assume that $\phi_1 \leq 0$, and from the definition of $v_{\theta}^{\text{min}}$ we get $\phi_1 \leq v_{\theta}^{\text{min}}$.

Applying $\frac{1}{\lambda} f^*$ to the above equality we find that

$$\frac{1}{\lambda^2} (f^*)^2 (T_{\theta}^{\text{min}}) = \theta + dd^c \phi_2,$$

where

$$\phi_2 = \phi_1 + \frac{1}{\lambda} (\phi_1 - v_{\theta}^{\text{min}}) \circ f \leq \phi_1.$$

Iterating this we obtain

$$\frac{1}{\lambda^n} (f^*)^n (T_{\theta}^{\text{min}}) = \theta + \phi_n,$$

where

$$\phi_n = \phi_1 + \sum_{j=1}^{n-1} \frac{1}{\lambda^j} (\phi_1 - v_{\theta}^{\text{min}}) \leq \phi_{n-1}.$$

(Here is the first place where our proof differs from that in [3]: We don’t need $f$ to be algebraically 1-stable here.)

$\phi_n$ is therefore a decreasing sequence of quasi-PSH functions. By Hartogs principle, $\phi_n$ either converges uniformly to $-\infty$ or converges to a quasi-PSH function $\phi$. We now use a trick of Sibony [44] to rule out the first possibility.

Let $R$ be a positive closed $(1,1)$ current whose cohomology class is $\{\theta\}$. We consider Cesàro’s means

$$R_N = \frac{1}{N} \sum_{j=1}^{N-1} \frac{1}{\lambda^j} (f^*)^j (R).$$

(Here is the second place where our proof differs from that in [3]: Again, we don’t need $f$ to be algebraically 1-stable.)

Notice that $R_N$ are positive closed $(1,1)$ currents having the same cohomology class $\{\theta\}$; hence they have uniformly bounded masses. We can then extract a cluster point $S$. From the definition, it is easy to see that $f^* (S) = \lambda S$ and the cohomology class of $S$ is $\{\theta\}$. Therefore, by the $dd^c$ lemma we can write

$$S = \theta + dd^c u,$$

where $u$ is a quasi-PSH function. By the invariance of $S$, after adding a constant to $u$ we can assume that

$$\phi_1 - \frac{1}{\lambda} v_{\theta}^{\text{min}} \circ f = u - \frac{1}{\lambda} u \circ f.$$
From this, it is easy to obtain

\[ \phi_n = u + \frac{1}{\lambda_n} v_{\theta}^{\text{min}} \circ f^n - \frac{1}{\lambda_n} u \circ f^n. \]

Here is the last and main difference between our proof and that in [3]: By definition of \( v_{\theta}^{\text{min}} \), there is a constant \( C \) such that \( u \leq v_{\theta}^{\text{min}} + C \). Therefore

\[ \phi_n \geq u - \frac{C}{\lambda_n}. \]

Thus \( \phi_n \) have uniformly bounded \( L^1 \) norms and thus converge to a quasi-PSH function \( g_\theta \). Therefore the limit

\[ \lim_{n \to \infty} \frac{1}{\lambda^n} (f^*)^n(T_{\theta}^{\text{min}}) = T \]

exists, where \( T = \theta + dd^c g_\theta \). By standard arguments (see Sibony’s paper [44]), \( T \) is what’s needed.

5.5. Proof of Theorem 5. Let \( T \) be a positive closed \((p, p)\) current and let \( \eta \) be a closed smooth \((k - p, k - p)\) form. We will show that \( T \wedge f_\ast(\eta) \) is well-defined with respect to Definition 12. We may assume without loss of generality that \( \eta \) is positive. Hence we need to show that there is a \((k, k)\) current \( S \) so that for any good approximation of \( DSH \) currents by \( C^2 \) forms \( K_j \),

\[ \lim_{j \to \infty} K_j(T) \wedge f_\ast(\eta) = S. \]

Note that \( K_j(T) = K_j^+(T) - K_j^-(T) \), where \( K_j^\pm \) are positive closed \((1, 1)\) forms of uniformly bounded masses. Define \( \mu_j^\pm = K_j^\pm \wedge f_\ast(\eta) \); then \( \mu_j^\pm \) are positive measures of uniformly bounded masses. Therefore, there are cluster points \( \mu^\pm \) of \( \mu_j^\pm \). To finish the proof of Theorem 5, it is therefore sufficient to show that \( \mu = \mu^+ - \mu^- \) is a (signed) measure independent of the choice of the good approximation \( K_j \) and the subsequence defining \( \mu^\pm \). To this end, we will show that if \( \beta \) is a smooth function on \( X \), then

\[ \langle \mu, \beta \rangle = \int_X f^*(\beta T) \wedge \eta. \]

(5.1)

Since \( T \) is a positive closed \((p, p)\) current and \( \beta \) is a smooth function, the current \( \beta T \) is a \( DSH \) \((p, p)\) current. Hence by Theorem 4, the \( f^*(\beta T) \) in the integral in the RHS of (5.1) is well-defined and is independent of either the choice of \( K_j \) or the subsequences defining \( \mu^\pm \).

We now proceed to prove (5.1). By definition

\[ \langle \mu, \beta \rangle = \lim_{j \to \infty} \langle \mu_j^+ - \mu_j^-, \beta \rangle = \lim_{j \to \infty} \int_X \beta K_j(T) \wedge f_\ast(\eta). \]

For each \( j \in \mathbb{N} \), by definition we have

\[ \int_X \beta K_j(T) \wedge f_\ast(\eta) = \int_X f^*(\beta K_j(T)) \wedge \eta. \]

It is easy to check that the \( DSH \) currents \( \beta K_j(T) \) converge in \( DSH \) to the current \( \beta T \). Hence by the continuity of \( f^* : DSH^{p,p}(X) \to DSH^{p,p}(X) \), we have \( \lim_{j \to \infty} f^*(\beta K_j(T)) = f^*(\beta T) \). Thus \( T \wedge f_\ast(\eta) \) is well-defined.

Note that if \( \beta \) is positive, then \( (f^*)^\ast(\beta T) \) is positive. In fact, the proof of Theorem 9 in [45] shows that the current \( (f^*)^\ast(\beta T) \) is the extension by zero to \( I_1(f) \) of the
current \((\pi_1)_* (\pi_2)^* (\gamma_{I^*} - C_1) (\beta T)\), and hence is a positive current. Therefore \(\langle \mu, \beta \rangle = \int_X f^* (\beta T) \wedge \eta \geq 0\), which means that \(\mu\) is a positive measure.

6. A Discussion on Question 1

As stated in the introduction, the usual criteria used to prove the equi-distribution property for the Green \((1,1)\) currents (see e.g. \([20, 31, 3]\)) are not applicable to the examples in \([8]\). Hence a complete answer to Question 1 will require developing new tools. In this section we discuss some cases where Question 1, for the Green \((1,1)\) currents, may be answered in the affirmative.

We first state the criteria used in \([20]\) (see Lemma 2.5 therein) and \([3]\) (see Lemma 5.4 and Proposition 5.5 therein). The following lemma is Lemma 5.4 in \([3]\).

**Lemma 6.** Let \(f : X \to X\) be a dominant meromorphic map of a projective manifold of dimension \(\geq 2\). Let \(Z\) be a desingularization of the graph of \(f\), and let \(\pi, g : Z \to X\) be the induced holomorphic maps (here \(\pi\) is the projection to the first factor and \(g\) is the projection to the second factor). Let \(\theta\) be a smooth closed \((1,1)\) form on \(X\). If \(\{g^*(\theta)\}.\{C\} \geq 0\) for any \(\pi\)-exceptional curve \(C\) (i.e. a curve \(C\) for which \(\pi(C)\) is a point), then the quasi-potentials of \(f^*(\theta)\) are bounded from above.

Applying this criterion, we now give a proof of Theorem \([3]\).

**Proof.** Let us denote by \(\theta\) a closed smooth \((1,1)\) form whose cohomology class is \(\{\theta\}\). If we can show that \(\{g^*(\theta)\}.\{C\} = 0\) for all \(\pi\)-exceptional curves \(C\), then Lemma 6 implies that potentials of \(f^*(\theta)\) are bounded from above. From the latter, it follows from the trick of Sibony (see e.g. Step 2 of the proof of Theorem 2.4 in \([20]\)) that the limit

\[
\lim_{n \to \infty} \frac{(f^*)^n(\theta)}{\lambda_1^n}
\]

exists and is the same for any \(\theta\). We can also check easily that this limit is the same as the limit in Theorem \([4]\) see e.g. Proposition 4.3 in \([4]\).

It remains to check that for all \(\pi\)-exceptional curves \(C\), \(\{g^*(\theta)\}.\{C\} = 0\). Let \(D_1, D_2, \ldots, D_m \subset X\) be the irreducible subvarieties of dimension \(\text{dim}(X) - 2\) in the images of the exceptional divisors of \(\pi : Z \to X\). Then by Lemma 4 in \([16]\), we have in cohomology

\[
f^*\{\theta\}.f^*\{\theta\} = \pi_*(g^*\{\theta\}).\pi_*(g^*\{\theta\}) = \pi_*(g^*\{\{\theta\}\}) + \sum_j l(g^*(\theta), D_j)\{D_j\}.
\]

Here \(l(g^*(\theta), C_j)\) are non-negative numbers depending on the intersections between \(g^*(\theta)\) and the \(\pi\)-exceptional curves belonging to \(\pi^{-1}(D_j)\). Since \(f^*\{\{\theta\}\} = \lambda_1\{\theta\}\), the assumption (i) that \(\{\theta\}.\{\theta\} = 0\) implies that \(\sum_j l(g^*(\theta), D_j)\{D_j\} = 0\), which means that each individual term \(l(g^*(\theta), D_j)\) is 0. The latter, when combined with the proof of Lemma 4 in \([16]\) and our assumption (ii), implies that \(\{g^*\theta\}.\{C\} = 0\) for all \(\pi\)-exceptional curves \(C\). The proof is completed.

There are several other classes of pseudo-automorphisms in dimension 3 where we expect that Question 1 has an affirmative answer. Let \(f : X \to X\) be a pseudo-automorphism in dimension 3 such that \(\lambda_1(f)^2 > \lambda_2(f)\). Let \(T^+\) be a non-zero positive closed \((1,1)\) current such that \(f^*(T^+) = \lambda_1(f)T^+\). In fact, it seems that...
Case 2 below should be true for any such pseudo-automorphism. Let \( Z \) be a desingularization of the graph of \( f \) and let \( \pi, g : Z \to X \) be the induced maps, where \( \pi : Z \to X \) is a finite composition of blowups along smooth centers.

**Case 1.** \( \{T^+\}, \{T^+\} = 0 \). This condition is the same as condition (i) in Theorem \( \ref{thm:main} \) and we think that condition (i) alone is enough to prove the conclusions of Theorem \( \ref{thm:main} \). The following result justifies this expectation that only exceptional divisors \( E \) of \( \pi \) whose image \( \pi(E) \) is a curve in \( X \) should be an obstruction to attaining an affirmative answer for Question 1 in the Green (1, 1) currents.

**Lemma 7.** Let \( f : X \to X \) be a pseudo-automorphism, where \( X \) is a projective 3-fold. Assume that there is a desingularization \( Z \) of the graph of \( \Gamma_f \), such that the projection to the first factor \( \pi : Z \to X \) is a finite composition of blowups along smooth centers for which if \( E \subset Z \) is an exceptional divisor, then \( \pi(E) \) is a point in \( X \). Then \( f^* : H^{1,1}(X) \to H^{1,1}(X) \) preserves the cone of nef classes \( H^{1,1}_{\text{nef}}(X) \).

In particular, there is a non-zero eigenvector \( \xi \in H^{1,1}_{\text{nef}}(X) \) with eigenvalue \( \lambda_1(f) \) of \( f^* : H^{1,1}(X) \to H^{1,1}(X) \); therefore, Question 1 has an affirmative answer in this case.

**Proof.** From the assumption, it is easy to check that if \( C \subset X \) is a curve, then in cohomology \( \pi^* \{C\} \) is an effective 1-cycle; i.e. we can write \( \pi^* \{C\} = \sum \lambda_i \{C_i\} \) where \( \lambda_i > 0 \) and \( C_i \) are curves on \( X \). In particular, \( f_* \{C\} = g_* \pi^* \{C\} \) is an effective 1-cycle.

Hence, if \( \{\theta\} \in H^{1,1}_{\text{nef}}(X) \) we find that for any curve \( C \): \( f^* \{\theta\}, \{C\} = \{\theta\} \) \( f_* \{C\} \geq 0 \). Therefore \( f^* \{\theta\} \) is nef by Kleiman’s criterion. Therefore \( f^* : H^{1,1}(X) \to H^{1,1}(X) \) preserves the cone of nef classes.

Finally, using the above and the results in \( \ref{thm:main} \) we find that Question 1 has an affirmative answer for Green (1, 1) currents.

**Case 2.** There is a Kähler form \( \omega \) such that the cluster points of the sequence

\[
\frac{1}{\lambda_1(f)^n} f^* ((f^n)^* (\omega)) \wedge f^* (\omega) - \frac{1}{\lambda_1(f)^n} f^* ((f^n)^* (\omega) \wedge \omega)
\]

contain a positive closed current. In this case taking the limit when \( n \to \infty \) we have by Lemma 4 in \( \ref{thm:main} \)

\[
f^* \{T^+\}, f^* \{\omega\} - f^* \{T^+ \wedge \omega\} = \sum_j l(g^* T^+, g^* (\omega), D_j) \{D_j\},
\]

where \( l(g^* T^+, g^* (\omega), D_j) \) are non-negative numbers depending bilinearly on \( g^* T^+ \) and \( g^* (\omega) \). More precisely, \( l(g^* T^+, g^* (\omega), D_j) \) depends on the products of numbers \( \{g^* T^+\}, \{C\} \) and \( \{g^* (\omega)\}, \{C\} \) where \( C \) are \( \pi \)-exceptional curves belonging to \( \pi^{-1}(D_j) \). Since \( \{g^* (\omega)\}, \{C\} \geq 0 \) for all curves \( C \) for which \( g(C) \) is not a point, we expect that \( \{g^* T^+\}, \{C\} \geq 0 \) for all \( \pi \)-exceptional curves. The expectation is true in the simplest case where \( \pi : Z \to X \) is a blowup along a finite number of pairwise disjoint curves in \( X \). In general, this expectation may not be true; however a modification of the usual criteria, in particular Lemma \( \ref{thm:main} \) may be proved to include the condition in \( \ref{thm:main} \) (note also Lemma \( \ref{thm:main} \) above, which shows that the exceptional divisors \( E \subset Z \) whose image in \( X \) are curves are essentially not an obstruction to proving Question 1 in the affirmative).
We end this section by explaining how (6.1) may be proved for all pseudo-automorphisms in dimension 3. Let $\omega$ be a Kähler form on $X$ and define $T_n = (f^n)^*(\omega)/\lambda_1(f^n)$. Then $T_n$ is a positive closed $(1, 1)$ current which is smooth outside a curve. Both $f^*(T_n)$ and $f^*(\omega)$ are positive closed $(1, 1)$ currents smooth out of a curve, and hence $f^*(T_n) \wedge f^*(\omega)$ is a well-defined positive closed $(2, 2)$ current. The two currents $f^*(T_n) \wedge f^*(\omega)$ and $f^*(T_n \wedge \omega)$ are the same out of a curve. If we can show that the latter current $f^*(T_n \wedge \omega)$ has no mass on curves, then (6.1) is proved.

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References

[1] Lucia Alessandrini and Giovanni Bassanelli, Transforms of currents by modifications and 1-convex manifolds, Osaka J. Math. 40 (2003), no. 3, 717–740. MR2003745 (2004f:32046)
[2] Giovanni Bassanelli, A cut-off theorem for plurisubharmonic currents, Forum Math. 6 (1994), no. 5, 567–595, DOI 10.1515/form.1994.6.567. MR1295153 (95f:32020)
[3] Turgay Bayraktar, Green currents for meromorphic maps of compact Kähler manifolds, J. Geom. Anal. 23 (2013), no. 2, 970–998, DOI 10.1007/s12220-012-9315-3. MR3023864
[4] Jérémie Blanc, Dynamical degrees of (pseudo)-automorphisms fixing cubic hypersurfaces, Indiana Univ. Math. J. 62 (2013), no. 4, 1143–1164. MR3179687
[5] J.-B. Bost, H. Gillet, and C. Soulé, Heights of projective varieties and positive Green forms, J. Amer. Math. Soc. 7 (1994), no. 4, 903–1027, DOI 10.2307/2152736. MR1260106 (95j:14025)
[6] Eric Bedford, S. Cantat, and K.-H. Kim, work in progress, March 2013.
[7] Eric Bedford, J. Diller, and K.-H. Kim, work in progress.
[8] Eric Bedford and K.-H. Kim, Pseudo-automorphisms without dimension-reducing factors, Manuscript.
[9] Eric Bedford and K.-H. Kim, Pseudo-automorphisms of 3-space: periodicities and positive entropy in linear fractional recurrences, arXiv: 1101.1614.
[10] Eric Bedford and B. A. Taylor, A new capacity for plurisubharmonic functions, Acta Math. 149 (1982), no. 1-2, 1–40, DOI 10.1007/BF02392348. MR674165 (84d:32024)
[11] Serge Cantat, Dynamique des automorphismes des surfaces $K3$ (French), Acta Math. 187 (2001), no. 1, 1–57, DOI 10.1007/BF02392831. MR1864630 (2003h:32020)
[12] Serge Cantat and Igor Dolgachev, Rational surfaces with a large group of automorphisms, J. Amer. Math. Soc. 25 (2012), no. 3, 863–905, DOI 10.1090/S0894-0347-2012-00732-2. MR2904576
[13] Jean-Pierre Demailly, Complex analytic and differential geometry, online book, version of Thursday, 10 September 2009.
[14] Jean-Pierre Demailly, Monge-Ampère operators, Lelong numbers and intersection theory, Complex analysis and geometry, Univ. Ser. Math., Plenum, New York, 1993, pp. 115–193. MR1211880 (94k:32009)
[15] Jean-Pierre Demailly, Thomas Peternell, and Michael Schneider, Pseudo-effective line bundles on compact Kähler manifolds, Internat. J. Math. 12 (2001), no. 6, 689–741, DOI 10.1142/S0129167X01000861. MR1875649 (2003a:32032)
[16] H. de Thélin and G. de Vigny, *Entropy of meromorphic maps and dynamics of birational maps*, Memoire de la SMF **122** (2010). MR2752759 (2011m:37075)

[17] Jeffrey Diller and Charles Favre, *Dynamics of bimeromorphic maps of surfaces*, Amer. J. Math. **123** (2001), no. 6, 1135–1169. MR1867314 (2002k:32028)

[18] Jeffrey Diller, *Birational maps, positive currents, and dynamics*, Michigan Math. J. **46** (1999), no. 2, 361–375, DOI 10.1307/mmj/1030132416. MR1704197 (2000e:37052)

[19] Jeffrey Diller, Romain Dujardin, and Vincent Guedj, *Dynamics of meromorphic maps with small topological degree I: from cohomology to currents*, Indiana Univ. Math. J. **59** (2010), no. 2, 521–561, DOI 10.1512/iumj.2010.59.4023. MR2648077 (2012j:32016)

[20] Jeffrey Diller and Vincent Guedj, *Regularity of dynamical Green's functions*, Trans. Amer. Math. Soc. **361** (2009), no. 9, 4783–4805, DOI 10.1090/S0002-9947-09-04740-0. MR2506427 (2010h:32016)

[21] Tien-Cuong Dinh and Nessim Sibony, *Une borne supérieure pour l'entropie topologique d'une application rationnelle* (French, with English summary), Ann. of Math. (2) **161** (2005), no. 3, 1637–1644, DOI 10.4007/annals.2005.161.1637. MR2180409 (2006f:32026)

[22] Tien-Cuong Dinh and Nessim Sibony, *Regularization of currents and entropy* (English, with English and French summaries), Ann. Sci. Ecole Norm. Sup. (4) **37** (2004), no. 6, 959–971, DOI 10.1016/j.ansens.2004.09.002. MR2119243 (2006c:32045)

[23] Tien-Cuong Dinh and Nessim Sibony, *Pull-back currents by holomorphic maps*, Manuscripta Math. **123** (2007), no. 3, 357–371, DOI 10.1007/s00229-007-0103-5. MR2314090 (2008i:32057)

[24] Tien-Cuong Dinh and Nessim Sibony, *Green currents for holomorphic automorphisms of compact Kähler manifolds*, J. Amer. Math. Soc. **18** (2005), no. 2, 291–312 (electronic), DOI 10.1090/S0894-0347-04-00474-6. MR2137979 (2006b:32015)

[25] Tien-Cuong Dinh and Nessim Sibony, *Super-potentials of positive closed currents, intersection theory and dynamics*, Acta Math. **203** (2009), no. 1, 1–82, DOI 10.1007/s11511-009-0038-7. MR2548825 (2011b:32052)

[26] Tien-Cuong Dinh and Nessim Sibony, *Super-potentials for currents on compact Kähler manifolds and dynamics of automorphisms*, J. Algebraic Geom. **19** (2010), no. 3, 473–529, DOI 10.1090/S1056-3911-10-00549-7. MR2629598 (2011f:32072)

[27] Tien-Cuong Dinh and Nessim Sibony, *Density of positive closed currents and dynamics of Hénon-type automorphisms of $C^k$ (part I)*, arXiv:1203.5810.

[28] Igor Dolgachev and David Ortland, *Point sets in projective spaces and theta functions* (English, with English and French summaries), Asterisque **30** (1999), no. 3, 131–186. Notes partially written by Estela A. Gavosto. MR1707353 (99d:14060)

[29] Charles Favre, *Note on pull-back and Lelong number of currents* (English, with English and French summaries), Bull. Soc. Math. France **127** (1999), no. 3, 445–458. MR1724401 (2000j:32057)

[30] John Erik Fornæss and Nessim Sibony, *Oka’s inequality for currents and applications*, Math. Ann. **301** (1995), no. 3, 399–419, DOI 10.1007/BF01446636. MR1324517 (96k:32013)

[31] John Erik Fornæss and Nessim Sibony, *Complex dynamics in higher dimensions* (Montreal, PQ, 1993), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 439, Kluwer Acad. Publ., Dordrecht, 1994, pp. 131–186. Notes partially written by Estela A. Gavosto. MR1332961 (96k:32057)

[32] Phillip Griffiths and Joseph Harris, *Principles of algebraic geometry*, Pure and Applied Mathematics, Wiley-Interscience [John Wiley & Sons], New York, 1978. MR507725 (80b:14001)

[33] Mikhail Gromov, *On the entropy of holomorphic maps*, Enseign. Math. (2) **49** (2003), no. 3-4, 217–235. MR2026895 (2005h:37097)

[34] Vincent Guedj, *Decay of volumes under iteration of meromorphic mappings* (English, with English and French summaries), Ann. Inst. Fourier (Grenoble) **54** (2004), no. 7, 2369–2386 (2005). MR2136979 (2005m:32035)

[35] Vincent Guedj, *Propriétés ergodiques des applications rationnelles* (French, with English and French summaries), Quelques aspects des systèmes dynamiques polynomiaux, Panor. Synthèses, vol. 30, Soc. Math. France, Paris, 2010, pp. 97–292. MR2932434

[36] P. Lelong, *Fonctions plurisousharmoniques et formes différentielles positives* (French), Gordon & Breach, Paris, 1968. MR0243112 (39 #4436)
[37] Michel Meo, *Image inverse d’un courant positif fermé par une application analytique surjective* (French, with English and French summaries), C. R. Acad. Sci. Paris Sér. I Math. **322** (1996), no. 12, 1141–1144. MR1396655 (97d:32013)

[38] Keiji Oguiso and Fabio Perroni, *Automorphisms of rational manifolds of positive entropy with Siegel disks*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **22** (2011), no. 4, 487–504. MR2904995

[39] Keiji Oguiso, *Bimeromorphic automorphism groups of non-projective hyperkähler manifolds—a note inspired by C. T. McMullen*, J. Differential Geom. **78** (2008), no. 1, 163–191. MR2406267 (2009m:32021)

[40] Keiji Oguiso, *Automorphism groups of Calabi-Yau manifolds of Picard number 2*, J. Algebraic Geom. **23** (2014), no. 4, 775–795. MR3263669

[41] Keiji Oguiso, *A remark on dynamical degrees of automorphisms of hyperkähler manifolds*, Manuscripta Math. **130** (2009), no. 1, 101–111, DOI 10.1007/s00229-009-0271-6. MR2533769 (2010m:32021)

[42] Fabio Perroni and De-Qi Zhang, *Pseudo-automorphisms of positive entropy on the blowups of products of projective spaces*, Math. Ann. **359** (2014), no. 1-2, 189–209, DOI 10.1007/s00208-013-0992-4. MR3201898

[43] Alexander Russakovskii and Bernard Shiffman, *Value distribution for sequences of rational mappings and complex dynamics*, Indiana Univ. Math. J. **46** (1997), no. 3, 897–932, DOI 10.1512/iumj.1997.46.1441. MR1488341 (98h:32046)

[44] Nessim Sibony, *Dynamique des applications rationnelles de $\mathbb{P}^k$* (French, with English and French summaries), Dynamique et géométrie complexes (Lyon, 1997), Panor. Synthèses, vol. 8, Soc. Math. France, Paris, 1999, pp. ix–x, xi–xii, 97–185. MR1760844 (2001e:32026)

[45] Tuyen Trung Truong, *The simplicity of the first spectral radius of a meromorphic map*, Michigan Math. J. **63** (2014), no. 3, 623–633, DOI 10.1307/mmj/1409932635. MR3255693

[46] Tuyen Trung Truong, *Pullback of currents by meromorphic maps*, ProQuest LLC, Ann Arbor, MI, Thesis (Ph.D.)–Indiana University, 2012. MR3054986

[47] Y. Yomdin, *Volume growth and entropy*, Israel J. Math. **57** (1987), no. 3, 285–300, DOI 10.1007/BF02766215. MR899797 (90g:58008)

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