A semidefinite representation for some minimum cardinality problems

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November 16, 2018

Abstract

Using techniques developed in [Las02], we show that some minimum cardinality problems subject to linear inequalities can be represented as finite sequences of semidefinite programs. In particular, we provide a semidefinite representation of the minimum rank problem on positive semidefinite matrices. We also use this technique to cast the problem of finding convex lower bounds on the objective as a semidefinite program.

Keywords: semidefinite programming, sum of squares, rank minimization, K-moment problem, Lagrangian relaxation.

Notation

We note $\mathbb{R}[x_1, \ldots, x_n]$ (or $\mathbb{R}[x]$ when there is no ambiguity) the ring of multivariate polynomials $p(x) = p(x_1, \ldots, x_n)$ on a variable $x \in \mathbb{R}^n$. We say that $p(x) \in \mathbb{R}[x]$ is SOS when $p(x)$ is a sum of squares of polynomials in $\mathbb{R}[x]$. For $x \in \mathbb{R}^n$, $\text{Card}(x)$ will be the cardinal of the set \{i : x_i \neq 0\}. We note $\mathbb{S}^n$ the set of $n \times n$ symmetric matrices. For multivariate polynomials, we adopt the multiindex notation $p(x) = \sum_{\alpha} p_{\alpha} x_{\alpha}$, where $x_{\alpha} := x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$, and we note $d = \sum_{i=1}^n \alpha_i$ the degree of $p(x)$. $\mathbb{R}_d[x]$ is the set of polynomials of degree at most $d$. Finally, $C(p)$ will be the Newton polytope of the polynomial $p(x)$, with $C(p) = \text{Co}\{\alpha : p_{\alpha} \neq 0\}$.

1 Introduction

Given a convex set $C \subset \mathbb{R}^n$, we are interested in solving the following problem:

\[
\begin{align*}
\text{minimize} & \quad \text{Card}(x) \\
\text{subject to} & \quad x \in C,
\end{align*}
\]

in the particular case where $C$ is described by a set of linear inequalities. Except in certain rare instances, this problem is very hard to solve (see [VB96]). Excellent heuristics exist however, a classical one (see [HHB99] for example) replacing the function $\text{Card}(x)$ by $\|x\|_1$, its largest convex lower bound on the unit cube.

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A related problem is that of minimizing the rank of a p.s.d. matrix subject to LMI constraints:

$$\begin{align*}
\text{minimize} & \quad \text{Rank}(X) \\
\text{subject to} & \quad X \in \mathcal{C},
\end{align*}$$

where $\mathcal{C}$ is here an affine subset of the semidefinite cone (a LMI). In this case also, minimizing the nuclear norm $\|X\|_*$ of $X$ will produce excellent approximate solutions (see [FHB00]).

In this paper, using results by [Cas84], [Sho87], [Put93], [CLR95], [Nes00], [Las01], [PS01] and [Las02], we show that the $\text{Min Card}(x)$ and $\text{Min Rank}(X)$ problems in (1) and (2) are equivalent to large scale semidefinite programs (see [NN94]). To be precise, based on a reformulation à la [Sho87] of problems (1) and (2), we use the technique in [Las02] to produce a finite (possibly exponential) sequence of increasingly tighter semidefinite relaxations.

The rest of the paper is organized as follows. In section 2 we recall some key definitions and properties on semidefinite representability and the sum of squares representation of positive polynomials. We also summarize the application of these representations to semialgebraic problems. In section 3 we show that both the $\text{Min Card}(x)$ and the $\text{Min Rank}(X)$ problems are equivalent to large scale semidefinite programs. Based on the work by [Put93], [Nes00] and [Las02] we explicitly construct in section 4 a sequence of semidefinite programs solving problems (1) and (2). We also show how the problem of finding optimal convex lower bounds on the objective function can be represented in a similar way. Finally, in section 5 we discuss the complexity of these techniques.

### 2 Sums of squares and semidefinite programming

We quickly recall here some key definitions and properties linking semidefinite and semialgebraic problems.

Hilbert’s 17th problem (see [Rez96] for an overview), which asked if all positive polynomials could be written as sums of squares of other polynomials, has a positive answer in dimension one. [Nes00] provides an efficient way of computing the SOS representation of a given positive univariate polynomial in the following result from [Nes00].

**Proposition 1** Let $p(x) \in \mathbb{R}[x]$ be a univariate polynomial of degree $d$. Then $p(x)$ for all $x \in \mathbb{R}$ iff there exists a matrix $X \in S^v$ such that:

$$p(x) = y_x^T X y_x, \text{ with } X \succeq 0, \text{ for all } x \in \mathbb{R}$$

with $v = \lceil d/2 \rceil$ and $y_x = (1, x, x^2, \ldots, x^v)$ is the list of univariate monomials up to degree $v$.

The coefficients of the polynomials in the representation are then computed as the eigenvectors of the matrix $X$.

In the general multivariate case, that representation property of positive polynomials is lost. It can be shown (see [Ber80]) that the set of multivariate SOS polynomials is dense in the set of positive polynomials, but there are simple examples of positive polynomials that are not $SOS$. However, recent results in semialgebraic geometry (see [Cas84], [Sch91], [Put93] or [PV99]) bridge the gap between positive and SOS polynomials on compact semialgebraic sets. We cite here the result in [Put93]. Let $g_k(x) \in \mathbb{R}[x_1, \ldots, x_n]$ for $j = 1, \ldots, r$, and we note
$K$, the semialgebraic set defined by $K = \{ x \in \mathbb{R}^n : g_k(x) \geq 0, k = 1, ..., r \}$. We suppose that $K$ is compact and that there exists $u(x) \in \mathbb{R}[x_1, ..., x_n]$ such that $\{ u(x) \geq 0 \}$ is compact with

$$u(x) = u_0(x) + \sum_{k=1}^{r} g_k(x)u_k(x), \quad \text{for all } x \in \mathbb{R}^n$$

(4)

where the polynomials $u_k(x) \in \mathbb{R}[x_1, ..., x_n]$ are SOS for $k = 1, ..., r$. Under this assumption, we can represent all polynomials positive on $K$ using SOS polynomials as in [Put93] or [PV99].

**Proposition 2** Suppose [3] holds. A polynomial $p(x) \in \mathbb{R}[x_1, ..., x_n]$ is positive on $K$ iff:

$$p(x) = g_0(x) + \sum_{k=1}^{r} g_k(x)q_k(x), \quad \text{for all } x \in \mathbb{R}^n$$

(5)

where the polynomials $q_k(x) \in \mathbb{R}[x_1, ..., x_n]$ are SOS for $k = 0, ..., r$.

Now, as in [Par00] or [Las01], we can write the multivariate version of the relation [3] mapping SOS polynomials to the semidefinite cone.

**Proposition 3** Let $p(x) \in \mathbb{R}[x_1, ..., x_n]$ be a polynomial and $K$ a semialgebraic set defined by $K = \{ x \in \mathbb{R}^n : g_k(x) \geq 0, k = 1, ..., r \}$, satisfying assumption [4]. Then $p(x) \geq 0$ on $K$ iff there is an integer $m \in \mathbb{Z}_+$ and matrices $X_k \in \mathbb{S}^N$, with $X_k \succeq 0$ for $k = 0, ..., r$ such that:

$$p(x) = y_x^T X_0 y_x + \sum_{k=1}^{r} (y_x^T X_k y_x) g_k(x), \quad \text{for all } x \in \mathbb{R}^n,$$

(6)

where $N = \left\lfloor \frac{(n+m-1)}{2} \right\rfloor$ and $y_x = (1, x_1, ..., x_n, x_1^2, x_1x_2, ..., x_1x_n, x_2x_3, ..., x_n^2, ..., x_1^m, ..., x_n^m)$ is the vector of all monomials in $\mathbb{R}[x_1, ..., x_n]$, up to degree $m$, listed in graded lexicographic order.

The result on polynomials above shows that testing the positivity of a multivariate polynomial on a semialgebraic set $K$ satisfying the assumption [4] can be cast as a semidefinite program. In general, the result in [Las01] shows that all compact semialgebraic problems, i.e. problems seeking to minimize a polynomial over a compact semialgebraic set, are equivalent to large-scale semidefinite programs. This provides a positive answer in the compact multivariate case to all the open questions in §4.10.2 in [BTN01]. A converse result is also true (and much simpler). Because the positive semidefiniteness of a matrix is equivalent to that of all its principal minors, all semidefinite programs are semialgebraic programs, with additional convexity and invariance properties.

The central result of moment theory exploited in [Las01] sets polynomial positivity problems and moment problems as duals (see e.g. [Ber80]). Let $s$ be a positive semidefinite sequence $s \in \mathbb{R}^N$, we have

$$s \text{ is p.s.d.} \quad \Downarrow$$

$$\langle s, p \rangle \geq 0, \quad \text{for all } p(x) \in \mathbb{R}_m[x] \text{ with } p(x) \text{ SOS},$$

3
and

\[ s \text{ is a moment sequence} \]

\[ \langle s, p_\alpha \rangle \geq 0, \quad \text{for all } p(x) \in \mathbb{R}_m[x] \text{ with } p(x) \geq 0 \text{ on } \mathbb{R}^n, \]

hence the cone of coefficients of \( \text{SOS} \) polynomials and that of p.s.d. sequences are polar, and so are the cones of moment sequences and positive polynomials.

From \cite{Put93} then, we know that the problem of testing if a sequence \( y \) is the moment sequence of some measure \( \mu \) with support in a compact semialgebraic set

\[ K = \{ x \in \mathbb{R}^n : g_k(x) \geq 0, k = 1, ..., r \} \]

and the problem of representing positive polynomials on \( K \) as \( g_k(x) \) weighted sums of \( \text{SOS} \) polynomials are dual of each other and both representable as linear matrix inequalities.

### 3 Semidefinite representation of the \text{MinCard}(x) \text{ and MinRank}(X) \text{ problems.}

As above \( K \) is the semialgebraic set defined by \( K = \{ x \in \mathbb{R}^n : g_k(x) \geq 0, k = 1, ..., r \} \) and we assume that \( \text{[4]} \) holds. Let \( p(x) \in \mathbb{R}[x_1, ..., x_n] \) and as \( K \) is compact, we also note \( t^* = \min_K p(x) \). Of course, \( p(x) - t^* \geq 0 \) on \( K \), hence there are \( \text{SOS} \) polynomials \( q_k(x) \in \mathbb{R}[x_1, ..., x_n] \) for \( k = 1, ..., r \) such that \( \text{[5]} \) holds for \( p(x) - t^* \) on \( K \). We first show that the \text{Min Card}(x) \text{ problem can be cast as a semialgebraic program, hence a semidefinite program, using the results from section \text{[2]} \).

**Proposition 4** Let \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \). There are polynomials \( g_k(x) \in \mathbb{R}[x_1, ..., x_n] \), for \( k = 0, ..., r \) such that the optimum values of:

\[
\begin{align*}
\text{minimize} & \quad \text{Card}(x) \\
\text{subject to} & \quad Ax \geq b,
\end{align*}
\]

and

\[
\begin{align*}
\text{minimize} & \quad g_0(x) \\
\text{subject to} & \quad g_k(x) \geq 0, \quad \text{for } i = 1, ..., r,
\end{align*}
\]

are equal.

**Proof.** First, as in \cite{Sho87} we notice that:

\[
\begin{align*}
\text{Card}(x) = \min & \quad \sum_{i=1}^n v_i \\
\text{s.t.} & \quad (v_i - 1)x_i = 0 \\
& \quad v_i \geq 0, \quad \text{for } i = 1, ..., n,
\end{align*}
\]

hence the \text{Min Card}(x) \text{ problem in \text{[7]} \ can be written:

\[
\begin{align*}
\text{Min Card}(x) \equiv & \quad \min & \quad \sum_{i=1}^n v_i \\
& \text{s.t.} & \quad (v_i - 1)x_i = 0 \\
& & \quad v_i \geq 0, \quad \text{for } i = 1, ..., n \\
& & \quad Ax \geq b.
\end{align*}
\]

which is a semialgebraic problem. □
We now show a similar result on the \textbf{Min Rank}(X), a minimum cardinality problem on the eigenvalues of the matrix $X$.

\textbf{Proposition 5} Let $A_i \in S^n$, for $i = 1, \ldots, p$ and $b \in \mathbb{R}^p$. There are polynomials $g_k(x) \in \mathbb{R}[x_1, \ldots, x_n]$, for $k = 0, \ldots, r$ such that the optimum values of:

\begin{equation}
\begin{aligned}
\text{minimize} \quad & \text{Rank}(X) \\
\text{subject to} \quad & \text{Tr}(A_i X) = b_i, \quad \text{for } i = 1, \ldots, p \\
& X \succeq 0
\end{aligned}
\end{equation}

and

\begin{equation}
\begin{aligned}
\text{minimize} \quad & g_0(x) \\
\text{subject to} \quad & g_k(x) \succeq 0, \quad \text{for } i = 1, \ldots, M
\end{aligned}
\end{equation}

are equal.

\textbf{Proof.} We note $(\lambda_1, \ldots, \lambda_n)$ the eigenvalues of the matrix $X \succeq 0$ and

$$\sigma_k(X) = \sum_{\alpha \subset \{1, \ldots, n\}, |\alpha| = k} \lambda_\alpha$$

the symmetric functions. We note $\chi_t(X)$ the characteristic polynomial of the matrix $X$, with $\chi_t(X) = \sum_{i=1}^n (-1)^i \sigma_i(X)x^i$. Because the matrix $X$ is semidefinite positive, we have $\sigma_k(X) = 0$ iff $\text{Rank}(X) < k$, hence the \textbf{Min Rank}(X) problem can be expressed as a minimum cardinality problem on the coefficients of the characteristic polynomial. With

$$\text{Rank}(X) = \min \quad v_i$$

s.t.  
\begin{align*}
\sigma_i(X)(v_i - 1) &= 0 \\
v_i &\geq 0, \quad \text{for } i = 1, \ldots, n,
\end{align*}

we enforce the remaining constraints to get the following representation of \textbf{Min Rank}(X):

\begin{equation}
\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^n v_i \\
\text{s.t.} \quad & (v_i - 1)\sigma_i(X) = 0 \\
& v_i \geq 0, \quad \text{for } i = 1, \ldots, n \\
& \text{Tr}(A_i X) = b_i, \quad \text{for } i = 1, \ldots, p \\
& d_I(X) \geq 0, \quad \text{for } I \subset \{1, \ldots, n\},
\end{aligned}
\end{equation}

where $d_I(X)$ is the principal minor with index set $I \subset \{1, \ldots, n\}$. This is a semialgebraic program in the coefficients of the matrix $X$. $\blacksquare$

These two results together with the results cited in section 2 show that the two problems considered are equivalent to very large scale semidefinite programs.

\section{Semidefinite relaxations}

In practice, the exact representations obtained in the last section can be exponentially large and in general, we cannot expect these problems to be tractable. Hence, the central contribution of these representations is not to reduce the complexity of these problems, but to
provide a sequence of successively sharper relaxations covering the entire complexity spectrum, thus allowing the complexity/sharpness tradeoff to be tuned. This is what we intend to describe in this section.

We begin by recalling the construction of moment matrices as detailed in [CF00], [Las01] and [Las02]. Again, we let $y_x = (1, x_1, ..., x_n, x_1^2, x_1 x_2, ..., x_1 x_n, x_2 x_3, ..., x_n^{m_n})$ be the vector of all monomials in $R[x_1, ..., x_n]$, up to degree $m$, listed in increasing graded lexicographic order. We note $s(m)$ the size of the vector $y_x$. Let $y \in R^{s(2m)}$ be the vector of moments (indexed according to $y_x$) of some probability measure $\mu$ with support $K = \{x \in R^n : g(x) \geq 0\}$, we note $M_m(y) \in S^{s(m)}$, for the moment matrix defined by

$$M_m(y)_{i,j} = \int_K (y_x)_i (y_x)_j \mu(dx), \quad \text{for } i, j = 1, ..., s(m)$$

(9)

i.e. the (symmetric) matrix of moments with rows and columns indexed as in $y_x$. We note $\beta(i)$ the exponent of the monomial $(y_x)_i$, and conversely, we note $i(\beta)$ the index of the monomial $x^\beta$ in $y_x$. For a given moment vector $\{y \in R^{s(2m)}\}$ ordered as in (4), the first row and columns of the matrix $M_m(y)$ are then equal to $y$. The rest of the matrix is then constructed following:

$M_m(y)_{i,j} = y_{\alpha + \beta}$ if $M_m(y)_{1,i} = y_\alpha$ and $M_m(y)_{j,1} = y_\beta$.

Similarly, let $g(x) \in R[x_1, ..., x_n]$, we derive the moment matrix for the measure $g(x) \mu(dx)$ on $K$ (called the localizing matrix), noted $M_m(gy) \in S^{s(m)}$, from the matrix of moments $M_m(y)$ by:

$$M_m(gy)_{i,j} = \int_K (gy)_i (gy)_j g(x) \mu(dx)$$

(10)

for $i, j = 1, ..., s(m)$. The coefficients of the matrix $M_m(gy)$ are then given by:

$$M_m(gy)_{i,j} = \sum_\alpha g_\alpha M_m(y)_{i(\beta(i)+\beta(j)+\alpha)}$$

(11)

We can remark as in [Las01] that if the measure $\mu$ has its support included in $K = \{x \in R^n : g(x) \geq 0\}$, then for all coefficient vectors $v \in R^{s(2m)}$:

$$\langle v, M_m(gy)v \rangle = \int_K v(x)^2 g(x) \mu(dx) \geq 0$$

hence $M_m(gy) \succeq 0$.

In dimension one, for a given vector $y \in R^{s(2n)}$, $M_m(y) \succeq 0$ (which is a LMI) is also a sufficient condition in order for $y$ to the moment sequence of a probability measure. In $R^n$, this equivalence does not hold in general. The compact semialgebraic case is called the K-moment problem and is dual to the compact SOS problem in $R$. Following [Las01], we now exploit this duality to compute a sequence of semidefinite relaxations for the $\text{Min Card}(x)$ and $\text{Min Rank}(X)$ problems.
4.1 The MinCard(x) problem

In section 3, we saw that the optimum value of the MinCard(x) problem can be computed as the optimum value of the semialgebraic program:

\[
\begin{align*}
\min \quad & \sum_{i=1}^{n} v_i \\
\text{s.t.} \quad & (v_i - 1)x_i = 0 \\
& v_i \geq 0, \quad \text{for } i = 1, \ldots, n \\
& a_j^T x \geq b_j, \quad \text{for } j = 1, \ldots, m.
\end{align*}
\] (12)

As in [Las01], to ensure compactness, we impose the additional constraint \(x_1^2 + \ldots + x_n^2 \leq \alpha\) for some constant \(\alpha > 1\). It is easy to check that the program above, together with this additional bound on the feasible set, satisfies the constraints qualification assumption (4).

For \(N \geq 1\), a lower bound \(l_N\) on the optimal value of the above problem is then computed as:

\[
l_N := \inf \sum_{i=1}^{n} y_i \\
\text{s.t.} \quad M_N(y) \succeq 0 \\
M_{N-1}(x_i(y_{i-1})y) = 0 \\
M_{N-1}((\alpha - x^T x - v^T v)y) \succeq 0 \\
M_{N-1}(v_i y) \geq 0, \quad \text{for } i = 1, \ldots, n \\
M_{N-1}(a_j^T x - b_j y) \geq 0, \quad \text{for } j = 1, \ldots, m,
\] (13)

in the variable \(y \in \mathbb{R}^{2n}\). Theorem 3.2 in [Las02] then states that there exists some \(N^*\) such that

\[l_N = \text{Min Card}(x), \quad \text{for all } N \geq N^*,\]

and the optimum is achieved whenever the rank of the matrices \(M_N(\ldots) y\) stabilizes.

4.2 The MinRank(X) problem

In section 3, for \(X \in \mathbb{S}^n\), we saw that the optimum of the MinRank(X) problem can be computed as the optimum value of the semialgebraic program:

\[
\begin{align*}
\min \quad & \sum_{i=1}^{n} v_i \\
\text{s.t.} \quad & (v_i - 1)\sigma_i(X) = 0 \\
& v_i \geq 0, \quad \text{for } i = 1, \ldots, n \\
& \text{Tr} (A_j X) = b_j, \quad \text{for } j = 1, \ldots, p \\
& d_I(X) \geq 0, \quad \text{for } I \subset [1, n],
\end{align*}
\]

To further simplify this program, we can substitute to the \(2^n\) constraints on the principal minors a more economical semialgebraic constraint. The modified program then reads:

\[
\begin{align*}
\min \quad & \sum_{i=1}^{n} v_i \\
\text{s.t.} \quad & (v_i - 1)\sigma_i(X) = 0 \\
& v_i \geq 0, \quad \text{for } i = 1, \ldots, n \\
& \text{Tr} (A_i X) = b_i, \quad \text{for } i = 1, \ldots, p \\
& u^T X u \geq 0, \quad \text{for } u \in \mathbb{R}^n,
\end{align*}
\]
and again, to ensure compactness, we impose $X^T X + v^T v + u^T u \leq \alpha$ for some constant $\alpha > 1$. If we set the variable $x = (u, X, v)$, for $N \geq \left\lceil \frac{n+1}{2} \right\rceil$, a lower bound $l_N$ on the optimal value of the above problem is computed as:

$$l_N := \inf_{y} \sum_{i=1}^{n} u_i \quad \text{s.t.} \quad M_N(y) \geq 0$$

$$M_{N-\left\lceil \frac{n+1}{2} \right\rceil}((v_i - 1)\sigma_i(X)y) = 0$$

$$M_{N-1}(v_iy) \geq 0$$

$$M_{N-1}((Tr(A_i X) - b_i)y) = 0$$

$$M_{N-1}((\alpha - X^T X + v^T v + u^T u)y) \geq 0$$

$$M_{N-2}(u^T X u) y \geq 0$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, p$, in the variable $y \in \mathbb{R}^{2n}$, where the matrices $M(q(x)y)$ are computed as in (11). Theorem 3.2 in [Las02] then states that there exists some $N^*$ such that

$$l_N = \text{Min Rank}(X), \quad \text{for all } N \geq N^*,$$

and the optimum is reached whenever the rank of the matrices $M_N(...y)$ stabilizes. Alternatively, one could use the fact that if we note $\chi_t(X)$ the characteristic polynomial of $X$, then $X \geq 0$ is equivalent to $\chi_{t \cdot t}(X)$ being SOS as a univariate polynomial in $t$.

### 4.3 Convex envelope

Suppose that instead of having only one $\text{Min Card}(x)$ or $\text{Min Rank}(X)$ problem to solve, we need to solve a (long) sequence of these problems with only some variation in the constraints. Here, instead of computing an exact relaxation for every instance of the problem, we are interested in finding an efficient heuristic method for approximating the solution to all the problems to be solved. The complexity of the first “bound design” program will be high, but that of the subsequent programs will then be much lower. The heuristics in [FHB00] replaced the $\text{Card}(x)$ (resp. $\text{Rank}(X)$) functions by their convex envelope on the sets $0 \leq x \leq 1$ (resp. $0 \leq X \leq I$), i.e. the largest convex function $f(x)$ such that $f(x) \leq \text{Card}(x)$ if $0 \leq x \leq 1$ (resp. $f(X) \leq \text{Rank}(X)$ if $0 \leq X \leq I$). In this section, we extend these bounds to semialgebraic sets with more complex shapes.

Of course, a function and its convex envelope share the same global minimum, so solving for this optimal lower bound is at least as hard as finding the global minimum. Here however we look for a convex lower bound for the problem in (14) inside the set of polynomials of degree at most $d$. This becomes a semialgebraic program:

$$\text{maximize} \quad \int_{[0,1]^n} p(x) dx$$

$$\text{subject to} \quad t - p(x) \geq 0 \text{ on } K$$

$$\sum_{i=1}^{n} v_i - p(x) \geq 0 \text{ on } K$$

$$u^T \nabla p(x) u \geq 0 \text{ on } \|u\|^2 = 1$$

in the coefficients $p_x$ of the polynomial $p(x) \in \mathbb{R}_d[x_1, ..., x_n]$, where $K$ is the compact semi-algebraic set given by:

$$K = (v, x) \in \mathbb{R}^{2n} : \begin{cases} (v_i - 1) x_i = 0 \\ Ax - b \geq 0 \\ x, v \geq 0. \end{cases}$$
Again, this can be cast as a LMI using the technique in [Las02]. We notice that the $l_1$ heuristic is a particular case when the constraint $Ax - b \geq 0$ is dropped.

5 Complexity

Of course, the two semidefinite programs detailed in the last section are far from tractable if the dimension $n$ and the relaxation order $N$ grow beyond textbook example sizes. The Min Card$(x)$ problem is equivalent to solving $2^n$ linear programs, so it is right to ask whether the programs above provide any benefit over, for example, branch-and-bound methods?

Even if these two methodologies have similar worst-case complexities, the semidefinite relaxations in (13) and (14) do sometimes produce the global optimum for low order $N$ (see [Las02]) and because the objective is integer valued here, they only need to be solved up to an absolute precision of $1/2$. We quickly detail below some other possible simplifications.

But the results above have to be considered first as representations, providing an insight of the relative complexity of minimum cardinality problems versus that of tractable convex optimization problems.

5.1 Structure, sparsity and symmetry

The first element that can be used to simplify the programs in (13) and (14) is structure. In [Las02] for example, the constraints $x \in \{-1, 1\}$, that translate into $x^2 - x = 0$ and $M_n((x^2 - x) y) = 0$ also imply that the variables $y_\alpha$, for $\alpha \in \mathbb{Z}_n^+$, can be replaced by $y_{\text{Card}(\alpha)}$ in the program. Secondly, if the constraints in (13) and (14) include some symmetry, we could use the results in [Gat00] and [GP02] to preprocess and simplify the original semialgebraic program. The simplest example of these symmetries is of course when the constraints are invariant with respect to a change of basis, in which case Min Rank$(X)$ reduces to a Min Card$(x)$ problem and as in [GP02] p. 31, the asymptotic complexity goes from being exponential in $n$ to being exponential in $\sqrt{n}$.

In general, the complexity of algorithms in semialgebraic geometry grows at least exponentially with the dimension. Efficiency is then very often measured by the ability of a method to maintain sparsity. Some results on Newton polytopes and SOS polynomials can then be used to efficiently handle sparsity in the SOS representations. In particular, a result of [Rez78] shows that if $p(x), h_i(x) \in \mathbb{R}[x]$, for $i = 1, \ldots, r$, with $p(x) = \sum_{i=1}^r h_i^2(x)$, then $C(h_i) \subseteq \frac{1}{2}C(p)$. That result however does not hold as is for the representation in (3).

Finally, a lower bound on the optimal value can be obtained by simply dropping some of the constraints in (13) and (14).

6 Conclusion

One of the central contributions of semidefinite programs to the optimization toolbox is their ability to efficiently solve a wide class of convex eigenvalue problems. In this work, we have illustrated how the method described in [Las01] for solving semialgebraic programs, by lifting them to semidefinite programs, can also be used to represent some semialgebraic eigenvalue problems and convex envelope relaxations. This contribution is centered around
semidefinite representations and the insight they can provide on the theoretical complexity of these problems. Whether or not they also improve the practical complexity of computing relaxations to these problems remains to be explored.

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