On Recursion Operators for Symmetries of the Pavlov–Mikhalev Equation

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Abstract—In geometry of nonlinear partial differential equations, recursion operators that act on symmetries of an equation $\mathcal{E}$ are understood as Bäcklund auto-transformations of the equation $T\mathcal{E}$ tangent to $\mathcal{E}$. We apply this approach to a natural two-component extension of the 3D Pavlov–Mikhalev equation $u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy}$. We describe the Lie algebra of symmetries for this extension, construct two recursion operators (one of them was known earlier) and find their action. We also establish the hereditary property of these operators as well as their compatibility (in the sense of the Frölicher–Nijenhuis bracket). We find also twelve additional operators which are degenerate in a sense (we call them queer) and discuss their properties. In the concluding part, a geometrical background of two-component conservation laws for multi-dimensional equations is exposed together with its relations to differential coverings.

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INTRODUCTION

In a recent paper [10], we studied a two-component Lagrangian extension of the so-called 3D rdDym equation and showed how the geometric approach to nonlinear PDEs (see [4]) facilitates efficient construction of recursion operators. Below we apply similar techniques to the the Pavlov–Mikhalev equation

$$u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy},$$

see [15, 16], as well as [5]. Equation (1) belongs to the same class as the rdDym one: it is linearly degenerate [6] and admits a Lax pair with non-removable parameter [5, 15, 16]. A recursion operator for symmetries of (1) was found by O. Morozov and is described in [2].

To be more precise, we deal with the system $\mathcal{E}$

$$v_{yy} = v_{xt} - u_x v_{xy} - 2v_y u_{xx} + 2v_x u_{xy} + u_y v_{xx}, \quad u_{yy} = u_{xt} - u_x u_{xy} + u_y u_{xx},$$

where the first equation is obtained by applying the adjoint linearization of (1) to a new variable $v$. The system may be considered as a Lagrangian deformation of the initial equation, see [1], and coincides with cotangent equation of the latter (see details in [8, 11]).

The algorithm of construction and analysis of recursion operators that we use below is based on the interpretation of these operators as Bäcklund auto-transformations of the tangent equation, see [13], and essentially uses the theory of differential coverings [12]. It should be noted that the tangent equation as an efficient instrument in the theory of recursion operators was intensively used in [9].

The algorithm itself can be shortly described as follows1):

1) The exact meaning and details will be explained in Section 1.
Consider the infinite prolongation of the equation under the study as a submanifold $E \subset J^\infty(\pi)$ in an appropriate jet space.

Choose convenient internal coordinates in $E$ for particular computations, cf. [14].

Compute symmetries of $E$ solving the equation $\ell_E(\varphi) = 0$, where $\ell_E$ is the linearization operator of $E$.

Construct the tangent equation $\mathcal{T}E$ by adding the equations $\ell_E(q) = 0$ to the initial ones. Here $q$ is a new odd (and this is essential) dependent variable.

Find differential coverings over $\mathcal{T}E$ linear in $q$ and its derivatives. Usually (but not necessary) these coverings are associated with two-component conservation laws of $\mathcal{T}E$. Let $w$ denote the corresponding nonlocal variables.

Find nonlocal shadows of symmetries linear both in $q$ and $w$. A pair (covering, shadow) provides a Bäcklund auto-transformation of $\mathcal{T}E$, i.e., a recursion operator.

Study the action of the operator on symmetries.

For hereditary and compatibility properties, find nonlocal symmetries that correspond to the shadows, if the former exist. Their super-commutators are exactly the needed Frölicher-Nijenhuis brackets.

All these steps are accomplished in Sections 2–5. Section 1 contains the theoretical background which is necessary for the subsequent exposition. It should be noted that our results comprise two types of recursion operators: two of them are quite “conventional”, while twelve are degenerate, “queer”. Nevertheless, existence of these operators reflect specific properties of the equation under study. We discuss them in Section 6.

Some of our computational results are rather voluminous and to make the reading more comfortable we place them in Appendices A–C.

1. BASIC CONSTRUCTIONS AND NOTATION

In our exposition here we follow the books [4, 11] and the paper [12]. By a number of reasons, we stick to the coordinate version.

1.1. Jets

Let $\pi: \mathbb{R}^m \times \mathbb{R}^n = E \to \mathbb{R}^n = M$ be the trivial bundle with the coordinates $x = (x^1, \ldots, x^n)$ in $\mathbb{R}^n$ (independent variables) and $u = (u^1, \ldots, u^m)$ in $\mathbb{R}^m$ (“unknown functions”). The space of $k$-jets $J^k(\pi)$ is, in addition to $x$ and $u$, endowed with the coordinates $u^j_\sigma$, where $\sigma$ is a symmetric multi-index of length $\leq k$. Variables $u^j_\sigma$ correspond to the partial derivatives $\partial u^j / \partial x^\sigma$. Natural projections

$$
\pi_k: J^k(\pi) \to M, \quad \pi_{k,l}: J^k(\pi) \to J^l(\pi), \quad k \geq l,
$$

are defined, and their inverse limit $J^\infty(\pi)$ is called the space of infinite jets. These projections define also the bundles

$$
\pi_\infty: J^\infty(\pi) \to M, \quad \pi_{\infty,k}: J^\infty(\pi) \to J^k(\pi)
$$

and the embeddings

$$
\mathcal{F}_l(\pi) \subset \mathcal{F}_k(\pi), \quad \Lambda^i_l(\pi) \subset \Lambda^i_k(\varphi),
$$

where $\mathcal{F}_l(\pi) = C^\infty(J^l(\pi))$, $\Lambda^i_l(\pi) = \Lambda^i(J^l(\pi))$ are the $\mathbb{R}$-algebra of smooth functions and the $\mathcal{F}_l(\pi)$-module of differential $i$-forms on $J^l(\pi)$, respectively. The corresponding objects on $J^\infty(\pi)$ are defined by

$$
\mathcal{F}(\pi) = \bigcup_{l \geq 0} \mathcal{F}_l(\pi), \quad \Lambda^i(\pi) = \bigcup_{l \geq 0} \Lambda^i_l(\pi).
$$
A vector field on $J^\infty(\pi)$ is by definition an $\mathbb{R}$-linear derivation $X: \mathcal{F}(\pi) \to \mathcal{F}(\pi)$. The module of vector fields is denoted by $D(\pi)$.

Let $f \in \Gamma(\pi)$ be a (local) section of $\pi$. Then, the section $j_\infty(f) \in \Gamma(\pi_\infty)$ defined by

$$w_\sigma^j = \frac{\partial^\sigma j^f}{\partial x^\sigma}, \quad j = 1, \ldots, m, \quad |\sigma| \geq 0,$$

is called the infinite jet of $f$. Graphs of infinite jets passing through a given point $\theta \in J^\infty(\pi)$ are tangent to each other and their common $n$-dimensional $\pi_\infty$-horizontal tangent plane is denoted by $\mathcal{C}_\theta$. The distribution $\mathcal{C}: \theta \mapsto \mathcal{C}_\theta$ is called the Cartan distribution. It is spanned by the vector fields

$$D_{x^i} = \frac{\partial}{\partial x^i} + \sum_{j,\sigma} u_{\sigma}^{j} \frac{\partial}{\partial u_\sigma^j} \in D(\pi),$$

or dually, annihilates 1-forms

$$\omega_\sigma^j = du_\sigma^j - \sum_i u_{\sigma}^{j} dx^i \in \Lambda^1(\pi).$$

The field $D_{x^i}$ is called the total derivative with respect to $x^i$, while $\omega_\sigma^j$ are Cartan (or vertical) forms on $J^\infty(\pi)$. The Cartan distribution is Frobenius integrable, i.e., $[X, Y] \in \mathcal{C}$ whenever $X, Y \in \mathcal{C}$.

Finally, one has two natural splittings

$$D(\pi) = \mathcal{C} \oplus D^v(\pi), \quad \Lambda^1(\pi) = \Lambda^1_v(\pi) \oplus \Lambda^1_h(\pi),$$

where $D^v(\pi)$ is the submodule of $\pi_\infty$-vertical fields and

$$\Lambda^1_h(\pi) = \left\{ \sum_i a_i dx^i \mid i \in \mathcal{F}(\pi) \right\}, \quad \Lambda^1_v(\pi) = \left\{ \sum_{j,\sigma} b_{\sigma}^{j} \omega_\sigma^j \mid b_{\sigma}^{j} \in \mathcal{F}(\pi) \right\}$$

consist of horizontal and Cartan forms, respectively. Consequently,

$$\Lambda^i(\pi) = \bigoplus_{p+q=i} \Lambda^q_h(\pi) \wedge \Lambda^p_v(\pi),$$

where

$$\Lambda^q_h(\pi) = \Lambda^1_h(\pi) \wedge \cdots \wedge \Lambda^1_h(\pi), \quad \Lambda^p_v(\pi) = \Lambda^1_v(\pi) \wedge \cdots \wedge \Lambda^1_v(\pi)$$

(2)

with the corresponding splitting of the de Rham differential in the horizontal and vertical (Cartan) parts:

$$d_h = \sum_i dx^i \wedge D_{x^i}, \quad d_v = \sum_{j,\sigma} \omega_\sigma^j \wedge \frac{\partial}{\partial u_\sigma^j}.$$

(3)

### 1.2. Equations

Consider an $\mathcal{F}(\pi)$-module $P$ of rank $r$ and its element $F = (F^1, \ldots, F^r)$. An (infinitely prolonged) partial differential equation associated with $F$ is

$$\mathcal{E} = \{ \theta \in J^\infty(\pi) \mid D_{\sigma}(F^j)|_{\theta} = 0, \quad j = 1, \ldots, r, \quad |\sigma| \geq 0 \},$$

(4)

where $D_{\sigma}$ denotes the composition of total derivatives corresponding to the multi-index $\sigma$. The restriction $\pi_\infty|_{\mathcal{E}}$ will be also denoted by $\pi_\infty$. Solutions of $\mathcal{E}$ are sections of $\pi$ such that the graphs of their infinite jets lie in $\mathcal{E}$. Functions and forms on $\mathcal{E}$ are by definition

$$\mathcal{F}(\mathcal{E}) = \mathcal{F}(\pi)|_{\mathcal{E}}, \quad \Lambda^i(\mathcal{E}) = \Lambda^i(\pi)|_{\mathcal{E}},$$

respectively. Vector fields on $\mathcal{E}$ are derivations $X: \mathcal{F}(\mathcal{E}) \to \mathcal{F}(\mathcal{E})$; the $\mathcal{F}(\mathcal{E})$-module of these fields is denoted by $D(\mathcal{E})$. 

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The Cartan plane at \( \theta \in \mathcal{E} \) is defined as \( \mathcal{C}_\theta \cap T_\theta \mathcal{E} \subset T_\theta J^\infty(\pi) \). In this way, we obtain an integrable distribution almost everywhere on \( \mathcal{E} \). We also have splittings similar to (2), as well as the vertical and Cartan differentials defined exactly like in (3).

It follows from (4) that the total derivatives can be restricted to \( \mathcal{E} \). These restrictions will be also denoted by \( D_{x_i} \) and we shall always assume that the only solutions of the system \( D_{x_i}(f) = 0, \ i = 1, \ldots, n \), are constants. Such equations are called differentially connected.

### 1.3. Symmetries

Everywhere below the word “symmetry” means an infinitesimal symmetry.

A symmetry of the Cartan distribution on \( J^\infty(\pi) \) is a vector field \( X \in D^u(\pi) \) such that \([X, \mathcal{C}] \subset \mathcal{C}\). The set of symmetries is a Lie algebra over \( \mathbb{R} \) denoted by \( \text{sym} (\pi) \).

**Theorem 1.** There is a one-to-one correspondence between \( \text{sym} (\pi) \) and the module \( \mathcal{X} = \Gamma(\pi^*_\infty(\pi)) \), where \( \pi^*_\infty(\pi) \) denotes the pull-back. This correspondence is given by the formula

\[
\mathcal{X} \ni \varphi \mapsto E_\varphi = \sum_{j, \sigma} D_\sigma (\varphi^j) \frac{\partial}{\partial u_\sigma^j} \in D^u(\pi).
\]

The field \( E_\varphi \) is said to be evolutionary, \( \varphi \) being generating section. In what follows, we do not distinguish between evolutionary fields and their generating sections when possible.

Since \( \text{sym} (\pi) \) is closed with respect to the commutator, Theorem 1 implies that \([E_\varphi, E_{\varphi'}] = E_{(\varphi, \varphi')}\) for some element \( \{\varphi, \varphi'\} \in \mathcal{X}(\pi) \) which is called the Jacobi bracket of \( \varphi \) and \( \varphi' \). To describe it explicitly, recall the following fact.

**Proposition 1.** Let \( \xi \) be a vector bundle over \( M \). Then, the component-wise action

\[ E_\xi^fi : \Gamma(\pi^*_\infty(\xi)) \to \Gamma(\pi^*_\infty(\xi)) \]

is well defined and

\[ E_\xi^fi(aF) = E_\varphi(aF) + aE_\xi^fi(F) \]

for all \( a \in \mathcal{F}(\pi) \) and \( F \in \Gamma(\pi^*_\infty(\xi)) \). Then,

\[ \{\varphi, \varphi'\} = E_\xi^fi(\varphi') - E_\xi^fi(\varphi). \]

Let now \( \mathcal{E} \subset J^\infty(\pi) \) be an equation. Its symmetry is a symmetry of the Cartan distribution on \( \mathcal{E} \). To describe the algebra \( \text{sym} (\mathcal{E}) \), let us, using Proposition 1, define the operator

\[ \ell_F : \xi \to \Gamma(\pi^*_\infty(\xi)), \quad \ell_F(\varphi) = E_\xi^fi(F). \]

In coordinates, one has

\[ \ell_F = \left( \sum_\sigma \frac{\partial F^\alpha}{\partial u_\sigma^\beta} D_\sigma \right). \]

**Remark 1.** An element \( F \in \Gamma(\pi^*_\infty(\xi)) \) is a nonlinear differential operator acting from sections of \( \pi \) to those of \( \xi \). Hence, \( \ell_F \) is its linearization. Note that \( \ell_F \) is an operator in total derivatives. Such operators (we call them \( \mathcal{C} \)-differential) admit restrictions to graphs of infinite jets and infinite prolongations.

**Theorem 2.** Let \( \mathcal{E} \subset J^\infty(\pi) \) be an equation associated with an element \( F \in \mathcal{P} = \Gamma(\pi^*_\infty(\xi)) \) and such that \( \pi_{\infty,0}(\mathcal{E}) = J^0(\pi) \). Then, \( \text{sym} (\mathcal{E}) = \ker \ell_\xi \), where \( \ell_\xi \) is the restriction of \( \ell_F \) to \( \mathcal{E} \).
1.4. Conservation Laws and Cosymmetries

A conservation law of $\mathcal{E}$ is a $d_h$-closed differential form $\omega \in \Lambda^n_h(\mathcal{E})$. It is trivial if $\omega = d_h(\rho)$ for some $\rho \in \Lambda^{n-2}_h(\mathcal{E})$. Two conservation laws are equivalent if they differ by a trivial one. In coordinates, if

$$\omega = \sum_{i=1}^n a_i x^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n,$$

then, $\omega$ is a conservation law if and only if

$$\sum_{i=1}^n (-1)^{i+1} D_{xi}(a_i) = 0. \quad (5)$$

Direct computation of conservation laws using formula (5) is not simple and is complicated by the existence of trivial laws, which are of no interest. For the “majority” of equations, the procedure can be facilitated.

Let $\mathcal{E}$ be given by some $F \in P$ and consider $G = Q = \Gamma(\pi^*_\omega(\eta))$, where $\eta$ is another vector bundle over $M$. Assume $\mathcal{E}$ to enjoy the following regularity property

$$G|_E = 0 \text{ implies } G = \Delta(F)$$

for some $\mathcal{C}$-differential operator $\Delta = \left( \sum_\sigma a^\sigma_\beta_\alpha D_\sigma \right): P \to Q$. For any such an operator define its adjoint $\Delta^*: \hat{Q} \to \hat{P}$, where $\hat{\bullet} = \text{hom}_{\mathcal{E}(\hat{\bullet})}(\bullet, \Lambda^n_h(\mathcal{E}))$, by $\Delta^* = \left( \sum_\sigma (-1)^{\sigma_\beta_\alpha} D_\sigma \circ a^\beta_\alpha_\sigma \right)$ and consider a form $\bar{\omega} \in \Lambda^n_h(\pi)$ such that $\bar{\omega}|_E = \omega$ for a conservation law $\omega$ of the equation $\mathcal{E}$. Then,

$$d_h \bar{\omega} = \Delta(F), \quad \Delta: P \to \Lambda^n_h(\pi). \quad (6)$$

Set $\psi_\omega = \Delta^*(1)|_E$.

**Theorem 3.** Let $\mathcal{E}$ be a 2-line equation in the sense of [17]. Then:

1. The element $\psi_\omega \in \hat{P}$ is well defined, i.e., does not depend on the choice of $\bar{\omega}$.
2. Conservation laws $\omega$ and $\omega'$ are equivalent if and only if $\psi_\omega = \psi_{\omega'}$. In particular, $\omega$ is trivial if and only if $\psi_\omega = 0$.
3. The element $\psi_\omega$ enjoys the equation

$$\ell^*_\mathcal{E}(\psi_\omega) = 0, \quad (7)$$

where $\ell^*_\mathcal{E}: \hat{P} \to \mathcal{Z}$.

We say that $\psi_\omega$ is the generating section of $\omega$, while elements $\psi \in \hat{P}$ satisfying Equation (7) are called cosymmetries of $\mathcal{E}$.

1.5. Differential Coverings

Let a smooth fiber bundle $\tau: \hat{\mathcal{E}} \to \mathcal{E}$ be such that the restriction $d\tau|_{\hat{\mathcal{C}}_\theta}$, $\hat{\theta} \in \hat{\mathcal{E}}$, is an isomorphism to $\mathcal{C}_{\tau(\theta)}$, where $\hat{\mathcal{C}}_\theta$ and $\mathcal{C}_{\tau(\theta)}$ are the Cartan planes at the corresponding points. Such maps are called (differential) coverings.

Let $w^1, \ldots, w^\alpha, \ldots$ be coordinates in fibers of $\tau$ in the vicinity of some point $\theta \in \mathcal{E}$ (nonlocal variables). Then, the condition that $\tau$ is a covering means that the total derivatives on $\hat{\mathcal{E}}$ are of the form $\hat{D}_x = D_x + X_i$, where

$$X_i = X^1_i \frac{\partial}{\partial w^1} + \cdots + X^n_i \frac{\partial}{\partial w^n} + \cdots$$
are $\tau$-vertical fields such that
\[ D_{x^i}(X_j) - D_{x^j}(X_i) + [X_i, X_j] = 0, \quad i < j. \]
This also means that $\tilde{E}$ may be understood as the overdetermined system
\[ \frac{\partial w^\alpha}{\partial x^i} = X_i^\alpha, \]
whose compatibility conditions coincide with $E$.

Two coverings $\tau_i: \tilde{E}_i \to E, i = 1, 2$, are equivalent if there exists an isomorphism of bundles $f: \tilde{E}_1 \to \tilde{E}_2$ such that $df(C_\theta) = C_f(\theta), \theta \in \tilde{E}_1$. A covering is irreducible if the covering equation is differentially connected and is trivial if it is equivalent to $\tau_0: E_0 \times E \to E$ with $\tilde{D}_{x^i} = D_{x^i}$. For any two coverings over $E$, the Whitney product $\tau_1 \times \tau_2: \tilde{E}_1 \times_E \tilde{E}_2 \to \tilde{E}$ is naturally endowed with the structure of a covering and it can be shown that any covering is locally equivalent to the Whitney product of a trivial and irreducible ones.

A covering $\tau$ given, we treat all the objects on $\tilde{E}$ to be nonlocal with respect to $E$. In particular, a symmetry $\tilde{X}$ of $\tilde{E}$ is a nonlocal symmetry of $E$. The defining equation for nonlocal symmetries is $\ell_{\tilde{E}}(\tilde{\varphi}) = 0$. On the other hand, the restriction
\[ \tilde{\varphi} = \tilde{\varphi}_1 \bigg|_{\tilde{F}(\tilde{E})}: \tilde{F}(\tilde{E}) \to \tilde{F}(\tilde{E}) \]
is an $\tilde{F}(\tilde{E})$-valued derivation that preserves the Cartan distributions. Derivations of such a type are called shadows. We say that a shadow lifts to $\tilde{E}$ if there exists an $\tilde{X}$ such that (8) fulfills. Any shadow is an evolutionary derivation on $E$ taking values in $\tilde{F}(\tilde{E})$ with the generating section $\tilde{\varphi}$ living on $\tilde{E}$ and satisfying the defining equation
\[ \tilde{\ell}_{\tilde{E}}(\tilde{\varphi}) = 0. \]

**Remark.** The notation $\tilde{\ell}$ in (9) means the lift to $\tilde{E}$. Such a lift is possible for any $C$-differential operator just by changing $D_{x^i}$ to $\tilde{D}_{x^i}$.

### 1.6. Bäcklund Transformations

A Bäcklund transformation $B(E_1, E_2)$ between equations $E_1$ and $E_2$ is a diagram of the form
\[ \begin{array}{ccc}
\tau_1 & \tau_2 \\
E_1 & & E_2 \\
\end{array} \]
where $\tau_i$ are coverings. If the equation $E_i, i = 1, 2$, is imposed on unknowns $u_i$, then $E$ is an equation both on $u_1$ and $u_2$ possessing the following characteristic property: if $(u_1, u_2)$ solves $E$ and $u_1$ solves $E_1$, then $u_2$ solves $E_2$ and vice versa. When $E_1 = E_2, B$ is called an auto-transformation.

Consider Bäcklund transformations $B_{12}(E_1, E_2)$ and $B_{23}(E_2, E_3)$. Then, the diagram
\[ \begin{array}{ccc}
\tau_1 & \tau_2 & \tau_3 \\
E_1 & E_2 & E_3 \\
\tau_5(\tau_1) & \tau_6(\tau_2) & \tau_7(\tau_3) \\
E_{12} & E_{23} & E_{33} \\
\end{array} \]
provides a Bäcklund transformation between $E_1$ and $E_3$, which is called the composition of $B_{12}$ and $B_{23}$.

**Remark 3.** It may happen that the top equation is not differentially closed. Then, one should restrict the considerations onto irreducible leaves of the Cartan distribution.
ON RECURSION OPERATORS FOR SYMMETRIES

1.7. The Tangent Covering and Recursion Operators

Consider an equation $\mathcal{E}$ and its tangent bundle $T\mathcal{E} \to \mathcal{E}$. Take the quotient bundle

$$ t: T\mathcal{E} = T\mathcal{E}/\mathcal{C} \to \mathcal{E} \quad (10) $$

and assume that its fibers are odd. Then, (10) is called the tangent covering to $\mathcal{E}$ and $T\mathcal{E}$ is called the tangent equation. Locally, sections of $t$ may be understood as $\pi_\infty$-vertical vector fields on $\mathcal{E}$.

**Theorem 4.** The tangent covering possesses the following properties:

1. Sections of $t$ that preserve the Cartan distributions are identified with symmetries of $\mathcal{E}$.
2. The superalgebra of functions on $T\mathcal{E}$ is canonically isomorphic to the Grassmann algebra $\Lambda^{\ast}_\varepsilon(\mathcal{E})$.
3. If $\mathcal{E}$ is given by $F(u) = 0$, then $T\mathcal{E}$ is given by the system $\{F(u) = 0, \ell_F(q) = 0\}$.
4. The Cartan differential defines an odd nilpotent vector field $X$ on $T\mathcal{E}$, such that $X(u^\sigma_\alpha) = q^j_\sigma$ and $X(q^{j}_{\sigma}) = 0$.

Statement (2) of Theorem 4 implies that fiber-wise linear functions on $T\mathcal{E}$ are identified with $\mathcal{C}$-differential operators $\kappa \to \mathcal{F}(\mathcal{E})$:

$$ \varphi \mapsto i_{E_\varphi}(\Upsilon), \quad \Upsilon = \sum a^{i}_{\sigma} q^i_{\sigma}, $$

where $i$ denotes the inner product. It also follows from Statement (1) that Bäcklund auto-transformations of $T\mathcal{E}$ relate symmetries of $\mathcal{E}$ to each other, i.e., are interpreted as recursion operators. Below, we construct these operators as follows.

Choose internal coordinated $x^i$, $w^i_\sigma$ in $\mathcal{E}$ and let $q^j_\sigma$ be the corresponding coordinates in fibers of $t$. Let also

$$ \omega^{\alpha} = (X^{\alpha}_{i_1} dx^{i_1} + X^{\alpha}_{i_2} dx^{i_2}) \wedge dx^1 \wedge \ldots \wedge dx^{i_1-1} \wedge dx^{i_1+1} \wedge \ldots \wedge dx^{i_2-1} \wedge dx^{i_2+1} \wedge \ldots \wedge dx^n, \quad \alpha = 1, \ldots, s, $$

be two-component conservation laws of $T\mathcal{E}$ linear in the variables $q^j_\rho$. Then, the system of relations

$$ w^{\alpha}_{\rho, i_1} = D_\rho (X^{\alpha}_{i_1}), \quad l = i_1, i_2, $$

$$ w^{\alpha}_{\rho, i_2} = w^{\alpha}_{l}, \quad l \neq i_1, i_2, \quad (11) $$

defines a covering $\tau(\omega): W(\omega) \to T\mathcal{E}$ over $T\mathcal{E}$. Here $\rho$ is a symmetric multi-index that does not contain $i_1$, $i_2$ and the nonlocal variables $w^{\alpha}_{\rho}$ are odd; in the case $n = 2$ the second group of relations is void.

Let now

$$ \varphi^j = \sum a^{j}_{\sigma, \beta} q^\beta_{\sigma} + \sum b^{j}_{\rho, \alpha} w^{\alpha}_{\rho}, \quad j = 1, \ldots, m, $$

be a shadow in $\tau(\omega)$. Consider a second copy of $T\mathcal{E}$ with fiber-wise coordinates $\bar{q}^j_\sigma$ and the map $\tau_\varphi: W(\omega) \to T\mathcal{E}$ given by

$$ \bar{q}^j_\sigma = D_\sigma (\varphi^j) \quad (12) $$

This map provides another covering $\tau^\varphi: W(\omega) \to T\mathcal{E}$ and the resulting Bäcklund transformation $\mathcal{R}$, described by Equations (11) and (12), is the desired recursion operator: substituting a known symmetry instead of $q$ and solving the system with respect to $\bar{q}$, we get the action.

**Remark 4.** A “tradition” prescribes to eliminate nonlocal variables from (11) and (12) and present a recursion operator as a $\mathcal{C}$-differential relation between $q$ and $\bar{q}$. This may convenient in simple cases, but in more complicated ones leads to practically unreadable formulas (see Appendix C).
Let $\mathcal{R}$ be a recursion operator and $\phi$ be the corresponding shadow. Denote by $\bar{\phi}$ the lift of the latter to $W(\omega)$ if it exists. Given another liftable shadow $\phi'$, we have $[E_\phi, E_{\phi'}] = E_{[\phi, \phi']}$ for some nonlocal symmetry $[\phi, \phi']$ of parity 2 (since $E_\phi$ and $E_{\phi'}$ are odd vector fields of parity 1, their commutator is $E_{\phi} \circ E_{\phi'} + E_{\phi'} \circ E_{\phi}$). We say that $[\phi, \phi']$ is the Frölicher-Nijenhuis bracket of $\phi$ and $\phi'$. A recursion operator is said to be hereditary if $[\bar{\phi}, \bar{\phi}] = 0$; two operators are compatible if $[\bar{\phi}, \bar{\phi}'] = 0$.

2. THE EQUATION AND ITS SYMMETRIES

The two-component Pavlov–Mikhalev equation $\mathcal{E}$ reads

$$
\begin{align*}
u_{yy} &= u_{xx} - u_x u_{xy} + u_y u_{xx}, \\
v_{yy} &= v_{xx} - u_x v_{xy} - 2v_y u_{xx} + 2v_x u_{xy} + u_y v_{xx}.
\end{align*}
$$

We choose the functions

$$
\begin{align*}
u_{0,k,l} &= u_{x^{k} x^{l}}, \\
v_{1,k,l} &= u_{y^{k} x^{l}}, \\
v_{0,k,l} &= v_{x^{k} x^{l}}, \\
v_{1,k,l} &= v_{y^{k} x^{l}}
\end{align*}
$$

for internal coordinates on $\mathcal{E}$. The total derivatives acquire the form

$$
\begin{align*}
D_x &= \sum_{k,l \geq 0} \left( u_{0,k,l} \frac{\partial}{\partial u_{0,k,l}} + u_{1,k,l} \frac{\partial}{\partial u_{1,k,l}} + v_{0,k,l} \frac{\partial}{\partial v_{0,k,l}} + v_{1,k,l} \frac{\partial}{\partial v_{1,k,l}} \right), \\
D_t &= \sum_{k,l \geq 0} \left( u_{0,k,l} \frac{\partial}{\partial u_{0,k,l}} + u_{1,k,l} \frac{\partial}{\partial u_{1,k,l}} + v_{0,k,l} \frac{\partial}{\partial v_{0,k,l}} + v_{1,k,l} \frac{\partial}{\partial v_{1,k,l}} \right), \\
D_y &= \sum_{k,l \geq 0} \left( u_{1,k,l} \frac{\partial}{\partial u_{0,k,l}} + D_x D_t (U) \frac{\partial}{\partial u_{1,k,l}} + D_x D_t (V) \frac{\partial}{\partial v_{1,k,l}} \right)
\end{align*}
$$

in these coordinates, where $U$ and $V$ are the right-hand sides of (13).

Equation (9) acquires the form

$$
\begin{align*}
D_y^2(\phi^u) &= D_x D_t (\phi^u) + u_{xx} D_y (\phi^u) + u_y D_x^2 (\phi^u) - u_x D_x D_y (\phi^u) - u_{xy} D_x (\phi^u), \\
D_y^2(\phi^v) &= 2v_x D_y (\phi^v) - v_{xy} D_x (\phi^v) + u_{xx} D_y (\phi^v) - 2v_y D_x (\phi^v) + u_x D_x^2 (\phi^v) - u_{xy} D_x (\phi^v).
\end{align*}
$$

Solving this system, one sees that its space of solutions $\text{sym} (\mathcal{E})$ is generated over $\mathcal{R}$ by the functions

$$
\varphi_1, \varphi_2, \varphi_3[\vartheta], \ldots, \varphi_6[\vartheta], \varphi_7, \ldots, \varphi_{14}, \varphi_{15}[\vartheta], \ldots, \varphi_{18}[\vartheta],
$$

where $\varphi_i = (\phi^u_i, \phi^v_i)$ and $\vartheta = \vartheta (t)$ is an arbitrary smooth function in $t$. The Lie algebra structure of $\text{sym} (\mathcal{E})$ is described in

**Proposition 2.** Let $\mathfrak{g}(2) = \langle \varphi_1, \varphi_2 \rangle$, $\mathfrak{a}(1) = \langle \varphi_7 \rangle$, $\mathfrak{a}(7) = \langle \varphi_8, \ldots, \varphi_{14} \rangle$, $\mathfrak{at}(4) = \langle \varphi_{15}[\vartheta], \ldots, \varphi_{18}[\vartheta] \rangle$, $\mathfrak{gt}(4) = \langle \varphi_3[\vartheta], \ldots, \varphi_6[\vartheta] \rangle$. Then,

$$
\text{sym} (\mathcal{E}) = (\mathfrak{a}(2) \oplus \mathfrak{a}(1)) \oplus \mathfrak{gt}(4) \oplus (\mathfrak{at}(4) \oplus \mathfrak{a}(7)),
$$

where $\oplus$ denotes a semidirect product. The first summand is the 2-dimensional solvable Lie algebra with the commutator $\{\varphi_2, \varphi_1\} = \varphi_1$, the algebras $\mathfrak{a}(1)$, $\mathfrak{at}(4)$ and $\mathfrak{a}(7)$ are Abelian. The structure of $\mathfrak{gt}(4)$ is given by

$$
\begin{align*}
\{\varphi_3[\vartheta], \varphi_6[\vartheta]\} &= \varphi_3[\vartheta_1, \vartheta - \vartheta \vartheta], \\
\{\varphi_4[\vartheta], \varphi_5[\vartheta]\} &= \varphi_3[\vartheta \vartheta_1 - \vartheta_1 \vartheta], \\
\{\varphi_4[\vartheta], \varphi_6[\vartheta]\} &= -\varphi_4[\vartheta \vartheta_1 + \vartheta_1 \vartheta], \\
\{\varphi_5[\vartheta], \varphi_5[\vartheta]\} &= \varphi_3[\vartheta \vartheta_1 - \vartheta_1 \vartheta].
\end{align*}
$$

*2* Explicit presentation of the generators see in Appendix A.
The actions of $a(2)$ and $a(1)$ are

$$
\{\varphi_1, \varphi_4[\vartheta]\} = -2\varphi_3[\vartheta], \quad \{\varphi_1, \varphi_5[\vartheta]\} = \varphi_4[\vartheta], \quad \{\varphi_1, \varphi_9\} = -\varphi_8,
$$

$$
\{\varphi_1, \varphi_{10}\} = -2\varphi_9, \quad \{\varphi_1, \varphi_{11}\} = -3\varphi_{10}, \quad \{\varphi_1, \varphi_{12}\} = -4\varphi_{11},
$$

$$
\{\varphi_1, \varphi_{13}\} = -5\varphi_{12}, \quad \{\varphi_1, \varphi_{14}\} = -6\varphi_{13}, \quad \{\varphi_1, \varphi_{16}\} = -2\varphi_{15},
$$

$$
\{\varphi_1, \varphi_{17}[\vartheta]\} = -5\varphi_{16}[\vartheta], \quad \{\varphi_1, \varphi_{18}[\vartheta]\} = -6\varphi_{17}[\vartheta],
$$

$$
\{\varphi_2, \varphi_{3}[\vartheta]\} = 3\varphi_3[\vartheta], \quad \{\varphi_2, \varphi_4[\vartheta]\} = 2\varphi_4[\vartheta], \quad \{\varphi_2, \varphi_5[\vartheta]\} = 5\varphi_5[\vartheta],
$$

$$
\{\varphi_2, \varphi_9\} = 2\varphi_9, \quad \{\varphi_2, \varphi_{10}\} = 10, \quad \{\varphi_2, \varphi_{12}\} = -12,
$$

$$
\{\varphi_2, \varphi_{13}\} = -2\varphi_{13}, \quad \{\varphi_2, \varphi_{14}\} = -3\varphi_{14}, \quad \{\varphi_2, \varphi_{16}[\vartheta]\} = -\varphi_{16},
$$

$$
\{\varphi_2, \varphi_{17}[\vartheta]\} = -2\varphi_{17}[\vartheta], \quad \{\varphi_2, \varphi_{18}[\vartheta]\} = -3\varphi_{18}[\vartheta]
$$

and $\{\varphi_7, \varphi_1\} = -\varphi_1$ for all $i = 8, \ldots, 18$. Finally, one has the following action of $gt(4)$ on $a(7) \oplus at(4)$:

$$
\{\varphi_3[\vartheta], \varphi_{14}\} = \varphi_{15}[\vartheta_{tt}], \quad \{\varphi_3[\vartheta], \varphi_{18}[\vartheta]\} = \varphi_{15}[\vartheta_{tt}/2 + \vartheta_{t}],
$$

$$
\{\varphi_4[\vartheta], \varphi_{13}\} = -\varphi_{15}[\vartheta_{tt}], \quad \{\varphi_4[\vartheta], \varphi_{14}\} = -2\varphi_{16}[\vartheta_{tt}],
$$

$$
\{\varphi_4[\vartheta], \varphi_{17}[\vartheta]\} = -\varphi_{15}[\vartheta_{tt}/2 + \vartheta], \quad \{\varphi_4[\vartheta], \varphi_{18}[\vartheta]\} = -\varphi_{16}[\vartheta_{tt}],
$$

$$
\{\varphi_5[\vartheta], \varphi_{12}\} = -\varphi_{15}[\vartheta_{tt}], \quad \{\varphi_5[\vartheta], \varphi_{13}\} = -2\varphi_{16}[\vartheta_{tt}],
$$

$$
\{\varphi_5[\vartheta], \varphi_{17}[\vartheta]\} = -\varphi_{15}[\vartheta_{tt}/2 + \vartheta], \quad \{\varphi_5[\vartheta], \varphi_{18}[\vartheta]\} = -\varphi_{16}[\vartheta_{tt}],
$$

$$
\{\varphi_6[\vartheta], \varphi_{11}\} = -\varphi_{15}[\vartheta_{tt}], \quad \{\varphi_6[\vartheta], \varphi_{12}\} = -2\varphi_{16}[\vartheta_{tt}],
$$

$$
\{\varphi_6[\vartheta], \varphi_{13}\} = -\varphi_{15}[\vartheta_{tt}/2 + \vartheta], \quad \{\varphi_6[\vartheta], \varphi_{14}\} = -2\varphi_{16}[\vartheta_{tt}],
$$

$$
\{\varphi_6[\vartheta], \varphi_{17}[\vartheta]\} = -\varphi_{15}[\vartheta_{tt}/2 + \vartheta], \quad \{\varphi_6[\vartheta], \varphi_{18}[\vartheta]\} = -2\varphi_{16}[\vartheta_{tt}],
$$

All the rest brackets vanish.

3. THE TANGENT EQUATION: CONSERVATION LAWS AND COVERINGS

The tangent covering is obtained by adding the equations

$$
p_{yy} = u_y p_{xx} - u_{xy} p_x - u_{xx} p_y + u_{xx} p + p_t,
$$

$$
q_{yy} = 2v_x p_{xy} + v_{xx} p_y - v_{xy} p_x - 2v_{xx} p + u_{xy} q_x + u_y q_{xx} - u_{xx} q_{xy} - 2u_{xx} q_y + q_{xt} \quad (14)
$$

to the initial system, where $p$ and $q$ are odd coordinate functions in the fibers of the covering. We were looking for two-component conservation laws of second order in all jet variables \(^3\) and linear in $p_\sigma$ and $q_\sigma$. This led us to four conservation laws $\omega_i = (X_i dx + Y_i dy) \wedge dt, i = 1, \ldots, 4$, with the following components:

$$
X_1 = 2u_x p_x + p_y, \quad Y_1 = u_y p_x + u_x p_y + p_t,
$$

$$
X_2 = q_y - v_x p_x - u_x q_x, \quad Y_2 = q_t - 2v_y p_x + v_x p_y + u_y q_x - 2u_x q_y,
$$

$$
X_3 = \left(\frac{3}{2} u_x^2 - y u_{xt}\right) p_x + \left(y u_{xx} + \frac{1}{2}\right) p_t - u_{xy} p + u_x p_y - \frac{1}{2} y p_{yt},
$$

$$
Y_3 = (u_x u_{xy} - u_y u_{xx} - u_x) p + \left(u_x u_y - \frac{1}{2} y u_{yt}\right) p_x + \frac{1}{2} (u_x^2 - y u_{xt}) p_y
$$

$$
+ (y u_{yy} + u_x) p_t - \frac{1}{2} y (u_y p_x - u_x p_y + p_t),
$$

$$
X_4 = y(v_x p_t - v_t p_x + u_x q_t - u_x q_x + q_{yt}) - v_x p + v_x p_y - u_x q + u_x q_y - q_t,
$$

\(^3\)An attempt to raise the order makes computations unrealistically time-consuming.
\[ Y_4 = (u_x v_{xy} + 2 v_y u_{xx} - 2 v_x u_{xy} - u_y v_{xx} - v_{xt})p + (u_y v_x - 2 u_x v_y - 2 y v_{yt})p_x \\
+ (u_x v_y + y v_{xt})p_y + (y v_{xy} + v_x)p_t + (u_x u_{xy} - u_y u_{xx} - u_{xt})q \\
+ (u_x u_y + y u_{yt})q_x - (u^2 + 2 y u_{xt})q_y + (y u_{xy} + u_x)q_t \\
+ y(2 v_x p_{yt} - 2 v_y p_{xt} + u_y q_{xt} - u_x q_{yt} + q_t). \]

**Remark 5.** Actually, all the coefficients may be multiplied by an arbitrary function \( f(t) \), but this does not influence the final results.

The generating sections of these conservation laws are

\[ \psi_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \psi_3 = \begin{pmatrix} p_x \\ 0 \\ u_x \end{pmatrix}, \quad \psi_4 = \begin{pmatrix} q_x \\ p_x \\ v_x \end{pmatrix}, \]

so they are all nontrivial.

We shall consider below the coverings

\[ w_{i,x} = X_i, \quad w_{i,y} = Y_i, \quad i = 1, \ldots, 4, \tag{15} \]

with the nonlocal variables \( w_i, \) \( w_{i,t}, \) \( w_{i,t,t}, \ldots \) and their Whitney product \( \tau: W \to TE. \)

### 4. SHADOWS AND LIFTS

We now lift the linearization operator \( \ell_E \) to the covering (15) and solve the equation \( \tilde{\ell}_E(\Phi) = 0, \) where \( \Phi = (\Phi^u, \Phi^v) \) is of second jet order and linear in all the odd variables \( (p_\sigma, q_\sigma, \) and \( w_{i,\sigma}). \) Under these assumptions, the equation has three solutions

\[ \Phi_0^u = p, \quad \Phi_0^v = q; \tag{16} \]
\[ \Phi_1^u = u_x p - w_1, \quad \Phi_1^v = v_x p - 2 u_x q - w_2; \tag{17} \]
\[ \Phi_2^u = u_y p - 2 y u_x p_t - u_x w_1 + y w_{1,t} + 2 w_3, \]
\[ \Phi_2^v = -2 v_y p + y v_x p_t + (3 u^2 + u_y) q + y u_x q_t - v_x w_1 + 2 u_x w_2 + y w_{2,t} - w_4. \tag{18} \]

The solution \( \Phi_0 \) corresponds to the identical recursion operators and thus is of no interest, while \( \Phi_1 \) and \( \Phi_2 \) will be studied in more detail.

**Proposition 3.** The shadows \( \Phi_1 \) and \( \Phi_2 \) can be lifted both to the tangent covering \( T \mathcal{E} \to \mathcal{E}, \) and to \( \tau: \mathcal{E} \to T \mathcal{E}. \)

**Proof.** The proof is accomplished in two technically different steps.

**Step 1** consists in lifting to \( T \mathcal{E} \) and is based on Statements 2 and 4 of Theorem 4. Namely, the equalities

\[ E_\Phi(p_\sigma) = L_{E_\Phi}(X(u_\sigma)) = -X(L_{E_\Phi}(u_\sigma)) = -X(D_\sigma(\Phi^u)) \]

and similar for \( E_\Phi(q_\sigma) \) are to be satisfied (since both \( E_\Phi \) and \( X \) are odd fields, they anticommute). To compute the last term, one needs to find the action of \( X \) on \( w_i. \) To this end, we solve the equations

\[ D_x(X(w_i)) = X(X_i), \quad D_y(X(w_i)) = X(Y_i), \quad i = 1, \ldots, 4, \]

and obtain \( X(w_1) = X(w_2) = 0 \) and

\[ X(w_3) = y p_x p_t + p y, \quad X(w_4) = y(p_x q_t - p_t q_x) + p q_y - p y. \]

Consequently,

\[ \Phi_1^u = p x, \quad \Phi_2^u = 2 p_x q + p q_x; \quad \Phi_1^v = p_x w_1 - p y, \quad \Phi_2^v = -6 u_x p_x q - 2 p_x w_2 - 2 p_y q + q_x w_1 - p q_y, \]

and this finishes the first step.

**Step 2.** To accomplish the second step, we solve the equations

\[ D_x(E_{\Phi_j}(w_i)) = E_{\Phi_{j,1}}(X_i), \quad D_y(E_{\Phi_j}(w_i)) = E_{\Phi_{j}}(Y_i), \quad i = 1, \ldots, 4, \]

for \( j = 1, 2. \) The results are presented in Appendix B. \( \square \)
5. Recursion Operators and Their Action

Thus, we have two recursion operators \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) that correspond to the found shadows: the first is described by Equations (17) combined with the defining equations of the coverings; similarly, the second one is obtained from Equations (18). But such a presentation, as it was indicated above, does not comply with the existing tradition. To obtain the conventional form, we get rid of nonlocal variables and for the first operator obtain the system

\[
D_x(\tilde{x}) = u_{xx}x + u_xD_x(\phi_x) - D_t(\phi_x), \\
D_y(\tilde{y}) = u_{yy}y + u_yD_y(\phi_y) - D_t(\phi_y), \\
D_x(\tilde{v}) = v_{xx}x + 2v_xD_x(\phi_x) - 2u_{xx}x + u_xD_x(\phi_x) - D_y(\phi_x), \\
D_y(\tilde{w}) = v_{xy}y + 2v_yD_y(\phi_y) - 2u_{xy}y + u_yD_y(\phi_y) - D_x(\phi_y).
\]

Note that the first two equations provide the known recursion operator of the one-component Pavlov–Mikhailov equation, see [2]. A similar form for the second operator is quite complicated and we present it in Appendix C.

**Remark 6.** Oleg Morozov found a simpler presentation of this operator.

**Proposition 4.** The operators \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) are hereditary and compatible, i.e.,

\[
[\Phi_1, \Phi_1] = [\Phi_2, \Phi_2] = [\Phi_1, \Phi_2] = 0.
\]

**Proof.** The result is proved by tiresome, but quite straightforward computations. \( \square \)

Let us now describe the action of the constructed operators on symmetries of \( \mathcal{E} \). Note first that the operator \( \mathcal{R}_1 \) relates the trivial symmetry \( \phi = 0 \) with symmetries of the form \( \phi_3[\phi] + \phi_15[\phi] \). In a similar way, \( \mathcal{R}_2 \) transforms zero to

\[
\phi_3[\phi] + \phi_15[\phi] + \phi_4[\phi] + \phi_16[\phi].
\]

All the actions below will be presented modulo these sets of symmetries. Note also that in a number of cases the resulting action is a nonlocal shadow; the latter will be denoted by \( \nu \) (their exact form is presented in Appendix D).

We have

\[
\mathcal{R}_1: \phi_1 \mapsto -\phi_2, \phi_2 \mapsto \phi_1, \\
\phi_3[\phi] \mapsto \phi_4[\phi], \phi_4[\phi] \mapsto -\phi_5[\phi], \phi_5[\phi] \mapsto -\phi_6[\phi], \phi_6[\phi] \mapsto \phi_2[\phi], \\
\phi_7 \mapsto \phi_3, \phi_8 \mapsto -\phi_9, \phi_9 \mapsto -\phi_{10}, \ldots, \phi_{13} \mapsto -\phi_{14}, \phi_{14} \mapsto \phi_4, \\
\phi_{15}[\phi] \mapsto -2\phi_{16}[\phi], \phi_{16}[\phi] \mapsto -\phi_{17}[\phi], \phi_{17}[\phi] \mapsto -\phi_{18}[\phi], \phi_{18}[\phi] \mapsto \nu_5[\phi],
\]

and

\[
\mathcal{R}_2: \phi_1 \mapsto -\phi_1, \phi_2 \mapsto \phi_6, \\
\phi_3[\phi] \mapsto -\phi_5[\phi], \phi_4[\phi] \mapsto \phi_6[\phi], \phi_5[\phi] \mapsto -\nu_2[\phi], \phi_6[\phi] \mapsto \nu_7[\phi], \\
\phi_7 \mapsto \phi_8, \phi_8 \mapsto \phi_10, \phi_{12} \mapsto \phi_{14}, \phi_{13} \mapsto -\nu_4, \phi_{14} \mapsto \nu_9, \\
\phi_{15}[\phi] \mapsto \phi_{16}[\phi], \phi_{16}[\phi] \mapsto \phi_{18}[\phi], \phi_{17}[\phi] \mapsto -\nu_5[\phi], \phi_{18}[\phi] \mapsto \nu_10[\phi].
\]

**Remark 7.** We see that the action of \( \mathcal{R}_2 \) is equivalent to that of \( \mathcal{R}_1 \circ \mathcal{R}_1 \).

But this is not the end of a story.

6. Discussion

There exist yet another two nontrivial coverings over \( \mathcal{T} \mathcal{E} \) linear in \( p_\sigma \) and \( q_\sigma \). They are associated with the nonlocal variables

\[
w_{5,x} = 2u_{xx}p_t - 2u_xt p_x - p_yt, \quad w_{5,y} = 2u_{xy}p_t - u_{yt}p_x - u_xt p_y + u_xp_yt - ptt
\]

and

\[
w_{6,x} = v_{xx}p_t - v_xt P_x + u_{xx}q_t - u_xt q_x + q_yt, \\
w_{6,y} = v_{xy}p_t - 2v_yt p_x - 2v_xt p_y + 2v_xt p_y + 2v_xp_yt + u_{xy}q_t + u_{yt}q_x + u_{y}q_xt - 2u_xt q_y - u_xq_yt + qtt.
\]
The canonical nilpotent field lifts to these coverings by the formulas
\[ \mathbf{X}(w_5) = 2p_x p_t, \quad \mathbf{X}(w_6) = p_x q_t - p_t q_x. \] (19)

The equation \( \tilde{\ell}_c(\Phi) = 0 \), where \( \Phi \) may depend on all the nonlocal variables \( w_1, \ldots, w_6 \) delivers twelve additional solutions \( \Phi_3, \ldots, \Phi_{14} \) with the components
\[ \Phi_3^v = f(v_x p_t + v_x p_t + u_xt q_t + u_x q_t + w_{2,t,t} - w_6,t), \quad \Phi_3^w = 0, \]
\[ \Phi_4^v = f \left( u_xt p_t + u_x q_t - \frac{1}{2} w_{1,t,t} - \frac{1}{2} w_{5,t} \right), \quad \Phi_4^w = 0, \]
\[ \Phi_5^v = -f(y w_{2,t} + y v_x p_t + y u_xt q_t + y u_x q_t + y v_t p_t - u_x^2 q_t - u_x(v_x p_t + w_{2,t} - w_6) - y \tilde{\ell}(u_x q_t + v_x p_t + w_{2,t} - w_6), \]
\[ \Phi_5^w = f v_x(v_x p_t + u_x q_t + w_{2,t} - w_6), \]
\[ \Phi_6^v = f \left( y u_x p_t + y u_xt q_t + u_x^2 p_t + \frac{1}{2} u_x(1 + w_{1,t,t} + w_5) - \frac{1}{2} y w_{5,t} - \frac{1}{2} y w_{1,t,t} \right) \]
\[ + y \tilde{\ell} \left( u_x p_t - \frac{1}{2} (w_{1,t} + w_5) \right), \]
\[ \Phi_6^w = \frac{1}{2} f(y w_{2,t} + y v_x p_t + y u_xt q_t + y u_x q_t + y v_t p_t + \frac{1}{2} u_x^2 q_t \]
\[ - 2 u_x(w_6 - w_{2,t}) + u_x(w_t + w_5) - y w_{6,t}) + \frac{1}{2} y \tilde{\ell}(u_x q_t + v_x p_t - w_6 + w_{2,t}), \]
\[ \Phi_7^v = f(v_x p_t + u_x q_t + w_{2,t} - w_6), \quad \Phi_7^w = 0, \quad \Phi_8^v = 0, \]
\[ \Phi_8^w = f(v_x p_t + v_x p_t + u_x q_t + u_x q_t + w_{2,t} - w_6,t), \]
\[ \Phi_9^w = 0, \quad \Phi_9^v = f \left( u_xt p_t + u_x q_t - \frac{1}{2} (w_{1,t,t} + w_{5,t}) \right), \]
\[ \Phi_{10}^v = 0, \quad \Phi_{10}^w = f \left( u_x p_t - \frac{1}{2} (w_{1,t} + w_5) \right), \]
\[ \Phi_{11}^v = f \left( u_x p_t - \frac{1}{2} (w_{1,t} + w_5) \right), \quad \Phi_{11}^w = 0, \quad \Phi_{12}^v = 0, \]
\[ \Phi_{12}^w = \frac{1}{2} f \left( y u_x p_t + y u_xt p_t + 2 u_x^2 p_t - u_x(w_{1,t} + w_5) - \frac{1}{2} y(w_{1,t} + w_{5,t}) \right), \]
\[ \Phi_{13}^w = 0, \quad \Phi_{13}^v = f(v_x p_t + u_x q_t + w_{2,t} - w_6), \quad \Phi_{14}^v = 0, \]
\[ \Phi_{14}^w = \frac{1}{2} f(y w_{2,t} + y v_x p_t + y u_xt q_t + y u_x q_t + y v_t p_t + u_x^2 q_t \]
\[ + 2 u_x(v_x p_t + w_{2,t} - w_6) - y w_{6,t}) + \frac{1}{2} y \tilde{\ell}(u_x q_t + v_x p_t + w_{2,t} - w_6), \]

where \( f = f(t) \) and \( \tilde{\ell} \) denotes the \( t \)-derivative.

Due to Equation (19), these shadows are lifted to the tangent covering \( \mathcal{T} \mathcal{E} \rightarrow \mathcal{E} \). The result is presented in Appendix E. Moreover, the following result is valid.

**Proposition 5.** All the shadows \( \Phi_3, \ldots, \Phi_{11}, \Phi_{13}, \Phi_{14} \) are lifted to the covering over \( \mathcal{\tilde{T}} \): \( \mathcal{\tilde{T}} \mathcal{E} \rightarrow \mathcal{T} \mathcal{E} \) with the nonlocal variables \( w_1, \ldots, w_6 \). The shadow \( \Phi_{12} \) can be lifted if and only if \( f = \text{const} \). The nonlocal symmetries \( \Phi_1 \) and \( \Phi_2 \) are also lifted to \( \mathcal{\tilde{T}} \). One has
\[ [\Phi_i, \Phi_j] = 0, \quad i, j = 1, \ldots, 14, \] (20)

for these lifts.

**Proof.** The explicit expressions for the lifts are given in Appendix E. The proof of (20) is a straightforward computation.

\[ \square \]
6.1. “Queer” Operators

The recursion operators associated with the shadows \( \Phi_3, \ldots, \Phi_{14} \) are extremely degenerate. Namely, their action is as follows: the operators \( R_3, R_4, R_7, \) and \( R_{10} \) take the entire algebra \( \text{sym}(E) \) to the symmetry \( \varphi_\theta \). In a similar way,

\[
R_5: \text{sym}(E) \to \varphi_4[\theta], \quad R_6: \text{sym}(E) \to \varphi_4[\theta] + \varphi_{16}[\theta],
\]

\[
R_8, R_9, R_{11}, R_{13}: \text{sym}(E) \to \varphi_{15}[\theta], \quad R_{12}, R_{14}: \text{sym}(E) \to \varphi_{16}[\theta].
\]

Perhaps, this phenomenon is partially explained by the nature of the nonlocal variable \( w_5 \) and \( w_6 \), which we, in particular, discuss below.

6.2. On the Theory of Two-Component Conservation Laws

Let us study the coverings \( \tau_5 \) and \( \tau_6 \) in more detail. First of all, easy computations show that their covering equations, as well as the one for the Whitney product \( \tau_5 \oplus \tau_6 \), are differentially connected, i.e., all the three coverings are irreducible.

On the other hand, applying formula (6) to the conservation laws corresponding to the coverings at hand, one sees that

\[
d_k(\omega_5) = -D_t(F_p) dx \wedge dy \wedge dt, \quad d_k(\omega_6) = D_t(F_q) dx \wedge dy \wedge dt, \quad \text{on } J^\infty(\pi),
\]

where \( F_p \) and \( F_q \) are the 1st and 2nd equations in (14), respectively, which means that \( \psi_{\omega_5} = \psi_{\omega_6} = 0 \), i.e., our conservation laws are trivial.

Thus, we see that in the multi-dimensional case relations between two-component conservation laws and the corresponding coverings are more complicated, than in the case \( \text{dim } M = 2 \) (cf. [12]). The following construction is to explain the difference.

Recall that all the conservation laws above were of the form

\[
\omega = (X dx + Y dy) \wedge dt
\]

and consider the subdistribution \( Z \) in the Cartan distribution on \( E \) spanned by the total derivatives \( D_x \) and \( D_y \). Note that \( Z \) is obviously Frobenius integrable. Consequently, one can literally repeat Vinogradov’s construction of the \( C \)-spectral sequence, see [17], using the distribution \( Z \) instead of \( C \). Denote this spectral sequence by \( \{E^n_k(\omega, Z)\} \). The following statement is actually a reformulation of Vinogradov’s results for the case of \( Z \):

Theorem 5. Consider the system \( E \) consisting of Equations (13) and (14). Then:

1. Equivalence classes of coverings associated with the form (21) are in one-to-one correspondence with elements of the group \( E^{0,1}_1(Z) \).

2. Define the generating element \( \psi_\theta \) of a form \( \theta \in \ker d_0 \subset E^{0,1}_0(Z) \) as the image of its coset under the differential \( d_1: E^{0,1}_1(Z) \to E^{1,1}_1(Z) \). Then, two forms \( \theta \) and \( \theta' \) define equivalent coverings if and only if \( \psi_\theta = \psi_{\theta'} \).

Remark 8. To clarify the structure of the spectral sequence at hand, let us discuss the following construction. Consider the infinite set of unknowns \( u_0 = u, u_k = u_{t \cdot t} \), \( k \geq 1 \), and similar for \( v, p, \) and \( q \). Consider also the infinite system of equations

\[
u_{k,yy} = D^k_t(u_{1,x} - u_xu_{xy} + u_yu_{xx}),
\]

\[
v_{k,yy} = D^k_t(v_{1,x} - u_xv_{xy} - 2v_yu_{xx} + 2v_xu_{xy} + u_yv_{xx}),
\]

\[
p_{k,yy} = D^k_t(p_{1,x} - u_xp_{xy} - u_xp_{y} - u_yp_{xx} + u_xp_{y} + p_{1,x}),
\]

\[
q_{k,yy} = D^k_t(2v_xp_{xy} + v_{xxp_y} - v_{xy}p_{xx} - 2v_yp_{xx} + 2u_{xy}q_x + u_yq_{xx} - u_xq_{xy} - 2u_{xx}q_y + q_{1,x})
\]

evidently obtained from (13) and (14) and add to it two formal relations

\[
t_x = 0, \quad t_y = 0.
\]
This is a two-dimensional system; denote its infinite prolongation by \( \hat{\mathcal{E}} \). Then, the Vinogradov \( \mathcal{C} \)-spectral sequence of \( \hat{\mathcal{E}} \) coincides with \( \{ E_s^{p,q}(\mathcal{Z}) \} \). In particular, conservation laws of \( \hat{\mathcal{E}} \) coincide with two-component conservation laws of the initial equation \( \mathcal{E} \).

Let us compute the generating elements of \( \omega_5 \) and \( \omega_6 \). Note first that cosymmetries of \( \hat{\mathcal{E}} \) (elements of \( E_1^{1,1}(\mathcal{Z}) \)) are infinite-dimensional covectors. Nevertheless, it is more convenient to present them as infinite matrices

\[
\psi = \begin{pmatrix}
\psi_0^0 & \psi_0^p & \psi_0^q \\
\psi_1^0 & \psi_1^p & \psi_1^q \\
\vdots & \vdots & \vdots \\
\psi_k^0 & \psi_k^p & \psi_k^q \\
\end{pmatrix},
\]

where the lines correspond to equations in (22), while the superscripts indicate the “type” of equation (obviously, the components corresponding to Equations (23) always vanish). Then, we have

\[
\psi_{\omega_5} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}, \quad \psi_{\omega_6} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
\end{pmatrix},
\]

so, the corresponding coverings are nontrivial indeed.

The above observations lead to the following construction. Consider an \( n \)-dimensional equation \( \mathcal{E} \) and a horizontal form \( \rho \in \Lambda^{n-2}_h(\mathcal{E}) \). Assume that

1. \( d_h(\rho) = 0 \),
2. \( \text{rank} \ker \rho = 2 \), where \( \rho \) is understood as a map \( \mathcal{C}(\mathcal{E}) \to \Lambda^{n-3}_h(\mathcal{E}) \), \( X \mapsto i_X(\rho) \).

Then, \( \mathcal{Z} = \mathcal{Z}_\rho = \ker \rho \) is a Frobenius integrable 2-dimensional sub-distribution in \( \mathcal{C}(\mathcal{E}) \). Indeed, let \( X, Y \in \mathcal{Z} \). Then,

\[
i_{[X,Y]}(\rho) = (i_X \circ L_Y - L_Y \circ i_X)(\rho) = (i_X \circ L_Y)(\rho)
= i_X \circ (d \circ i_Y + i_Y \circ d)(\rho) = (i_X \circ d)(i_Y(\rho)) + (i_X \circ i_Y)d(\rho) = 0,
\]

where \( i \) and \( L \) denote the inner product and Lie derivative, respectively.

Similar to the example above, consider the spectral sequence \( \{ E_s^{p,q}(\mathcal{Z}_\rho) \} \) associated with \( \mathcal{Z}_\rho \). Elements of \( \{ E_1^{0,1}(\mathcal{Z}_\rho) \} \) are called two-component conservation laws of type \( \rho \). To any of them a covering over \( \hat{\mathcal{E}} \) is associated and this covering is nontrivial if and only if \( \rho \neq 0 \).

**Appendix A**

**EXPLICIT FORMULAS FOR SYMMETRIES**

Note that System (13) covers the first equation by \( (u, v) \mapsto u \). The symmetries \( \varphi_1, \ldots, \varphi_6 \) are the lifts from \( u_{yy} = u_{xt} - u_x u_{xy} + u_y u_{xx} \) and are of the form

\[
\varphi_1^u = yu_x - 2x, \quad \varphi_1^v = yv_x, \quad \varphi_2^u = 2xu_x + yu_y - 3u, \quad \varphi_2^v = 2xv_x + yv_y, \\
\varphi_3^u[\vartheta] = \vartheta, \quad \varphi_3^v[\vartheta] = 0, \quad \varphi_4^u[\vartheta] = u_x \vartheta - g \vartheta_t, \quad \varphi_4^v[\vartheta] = v_x \vartheta, \\
\varphi_5^u[\vartheta] = u_y \vartheta + (yu_x - x) \vartheta_t - \frac{1}{2} y^2 \vartheta_{tt}, \quad \varphi_5^v[\vartheta] = v_y \vartheta + yv_x \vartheta_t,
\]

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\( \varphi^v_0[\vartheta] = u_t \vartheta + (xu_x + yu_y - u)\vartheta_t + \frac{1}{3}(y^2u_x - 2xy)\vartheta_{tt} - \frac{1}{6}y^3\vartheta_{tttt}, \)

\( \varphi^v_{11}[\vartheta] = v_t \vartheta + (v_x x + v_y y + 2v)\vartheta_t + \frac{1}{2}y^2v_x \vartheta_{tt}. \)

The symmetries \( \varphi_7, \ldots, \varphi_{18} \) have zero shadows: their \( u \)-components vanish, while the \( v \)-components are

\[
\begin{align*}
\varphi^v_7 &= v, \quad \varphi^v_8 = u_{xxx}, \quad \varphi^v_9 = xu_{xxx} + u_{xy} + \frac{1}{2}u_{xx}, \\
\varphi^v_{10} &= u_{xx} + u_xu_{xxy} + u_xu_{xx} + u_{xy}u_{xx} + (u_x^2 + u_y)u_{xxx}, \\
\varphi^v_{11} &= u_x(u_x^2 + 2u_y)u_{xxx} + (u_x^2 + u_y)u_{xxy} + 2u_xu_{xxt} + \frac{1}{2}(3u_x^2 + 2u_y)u_{xx} \\
&+ (2u_xu_{xy} + u_{xt})u_{xx} + \frac{1}{2}u_{xy} + u_{xyt}, \\
\varphi^v_{12} &= (3u_x^2 + 2u_y)u_{xxt} + u_x(u_x^2 + 2u_y)u_{xxy} + u_{xt} + (2u_x^3 + 3u_xu_y)u_{xx} \\
&+ u_{xx}(u_{yt} + 3u_xu_xt) + (u_x^2 + 2u_y)u_{xy} + (u_x^4 + 3u_x^2u_y + u_x^2)u_{xxx} \\
&+ u_xu_{xy} + u_{xt}u_{xy} + 2u_xu_{xyt}, \\
\varphi^v_{13} &= (u_x^4 + 3u_x^2u_y + u_x^2)u_{xxy} + (4u_x^3 + 6u_xu_y)u_{xxt} + 3u_xu_{xxt} + u_{ytt} \\
&+ \frac{1}{2}(5u_x^4 + 12u_x^2u_y + 3u_x^2)u_{xxx} + ((4u_x^3 + 6u_xu_y)u_{xy} + u_{xt}(6u_x^2 + 3u_y) + 3u_{yt}u_x)u_{xx} \\
&+ (u_x^5 + 4u_x^3u_y + 3u_x^2u_y^2)u_{xxx} + (3u_x^2 + 2u_y)u_{xy} + \frac{1}{2}(3u_x^2 + 2u_y)u_{xy} \\
&+ u_{xy}(3u_xu_{xt} + u_{yt}) + \frac{3}{2}u_x^2, \\
\varphi^v_{14} &= (5u_x^4 + 12u_x^2u_y + 3u_x^2)u_{xxt} + u_x(u_x^2 + u_y)(u_x^2 + 3u_y)u_{xxy} \\
&+ (6u_x^2 + 3u_y)u_{xt} + 3u_xu_{yt} + (3u_x^5 + 10u_x^3u_y + 6u_xu_y^2)u_{xx} \\
&+ ((5u_x^4 + 12u_x^2u_y + 3u_x^2)u_{xy} + (10u_x^3 + 12u_xu_y)u_{xt} + 3(2u_x^2 + u_y)u_{yt})u_{xx} \\
&+ (u_x^6 + 5u_x^4u_y + 6u_x^2u_y^2 + u_x^3)u_{xxx} + (4u_x^3 + 6u_xu_y)u_{xyt} + u_{xtt} \\
&+ (2u_x^3 + 3u_xu_y)u_{xy} + u_{xy}(3u_xu_t + u_x(6u_x^2 + 3u_y)) + 6u_xu_{xt} + 3u_{yt}u_{xt}, \\
\varphi^v_{15}[\vartheta] &= \vartheta, \quad \varphi^v_{16}[\vartheta] = u_x\vartheta + \frac{1}{2}y\vartheta_t, \quad \varphi^v_{17}[\vartheta] = \left( \frac{3}{2}u_x^2 + u_y \right) \vartheta + \left( yu_x + \frac{1}{2}x \right) \vartheta_t + \frac{1}{4}y^2 \vartheta_{tt}, \\
\varphi^v_{18}[\vartheta] &= (2u_x^3 + 3u_xu_y + u)t \vartheta + \left( \frac{3}{2}yu_x^2 + xu_x + yu_y + \frac{1}{2}u \right) \vartheta_t + \frac{1}{2}y(yu_x + x)\vartheta_{tt} + \frac{1}{12}y^3 \vartheta_{tttt},
\end{align*}
\]

where \( \vartheta \) is an arbitrary function in \( t \).

**Appendix B**

**Explicit Formulas for Lifts**

The \( w_i \)-components, \( i = 1, \ldots, 4 \), of \( \mathbf{E}_\varphi \), are

\[
\begin{align*}
\varphi^w_1 &= 2u_x pp_x + pp_y, \quad \varphi^w_2 = -v_x pp_x - u_x pq_x - 2u_x p_x q + pq_y + 2p_y q, \\
\varphi^w_3 &= -\frac{1}{2}gpp_yt + \frac{1}{2}(2yu_{xx} - 1)pp_t + \frac{1}{2}(3u_x^2 - 2yu_{xt} - 2u_y)p p_x + yu_x p_x p_x + u_x p p_y, \\
\varphi^w_4 &= -(yu_{xt} - 2v_y)pp_x + (yu_{xx} - 2)p p_t - (yu_{xt} + u_y)pq_x + (2yu_{xx} - 1)p q \\
&- (2yu_{xt} - u_y)p_x q + yu_x p q_x - yu_x p_x q_t - 2yu_x p q p_x + yu_{xx} p p_t.
\end{align*}
\]
The $w_1$-components of $\mathbf{E}_{\varphi}$ are

$$\varphi_2^{w_1} = -u_ypx - u_xppx + 2u_xpxw_1 - pp_t + p_yw_1,$$

$$\varphi_2^{w_2} = 2(3u_x^2 - u_y)p_x + 2u_xpq - 2u_xpyq + 2u_xpxw_2 - u_xq_w - u_ypx - vppx - px_y - 2p_p - 2p_yw_2 + q_yw_1,$$

$$\varphi_2^{w_3} = -\frac{1}{2}(u_x^2 - uy_xt - 2u_y)p_yy - \frac{1}{2}y(2u_x^2 + u_y)p_p + \frac{1}{2}(2yu_{xx} + 1)p_tw_1$$

$$+ \frac{1}{3}(3u_x^2 - 2yu_xt)p_xw_1 - (yu_{xy} + u_x)p_p + \frac{1}{2}yu_{yy}p_p + \frac{1}{2}yu_{yy}p_xt$$

$$- \frac{1}{2}yu_{xx}p_p - \frac{3}{2}yu_{xx}p_p - u_{xy}p_{w_1} + \frac{1}{2}ypp_t - \frac{1}{2}ypp_t - \frac{1}{2}ypp_t + \frac{1}{2}yu_{yy}p_{w_1} + \frac{1}{2}yu_{yy}p_{w_1},$$

$$\varphi_2^{w_4} = u_xq_yw_1 + yq_yw_{1,t} - 2(yu_{xx} - 1)p_{t} + 3(uy_{xx} + 2yu_{xy} + u_x)p_q$$

$$- (3u_y + 2yu_{yt})p_x - (4u_x^2 + 2yu_{xt} + u_y)p_yq - (yu_{xy} + v_x)p_t$$

$$- (2u_x - u_{xx} + u_y)p_x - (u_x + u_{xx} + 2u_y)p_p - (yu_{xx} + u_yq_yw_1 - \frac{1}{2}yu_{yy}p_{w_1} + \frac{1}{2}yu_{yy}p_{w_1},$$

$$+ (u^2 + 2yu_{xt} + u_y)p_qy + 3(u_xu_y + u_yu_x + u_xt)pq$$

$$+ (6yu_{xt} - 3u_xu_y - 2yu_{yt}p_q - 2yu_{yt}w_2 - v_{xy}p_{w_1} + 2uy_{xy}p_{y_2}$$

$$- 2ypp_yw_2 - u_xqw_1 + yq_yw_1 - ypp_t + v_xp_yw_1 - 2yu_{xt} - 3yu_{yt}p_{q}$$

$$- 2yu_{xt}p_yw_2 + v_{xx}p_xw_1 - 2yu_{yt}p_{q} - yu_{yt}p_{q} - y(u_x^2 + 2u_y)p_tq_x$$

$$- yu_{xt}q_xw_1 + 2yu_{xt}p_xw_2 - 4yu_{xt}p_{q} - yu_{xt}p_{w_1} + y(u^2 - u_y)p_{q}$$

$$+ yu_{xt}p_{q} - 2yu_{xt}p_{q} - yu_{yt}p_{q} + qu_{1,t} + 2p_{w_2,t} - 3yu_{xy}p_{q} + 4y(u_xv_x + v_y)p_p.$$
\[-2(u_x^2 + u_y)v_{xxx}u_{xxx})D_x(\varphi^u) + (3v_{xxx}u_{xx}^2 + (u_xv_{xxx} - 2v_xu_{xxx})u_{xx} - 2u_xv_{xx}u_{xxx})D_y(\varphi^u) + (u_xv_{xxx} - 2v_xu_{xxx})D_t(\varphi^u)\]

\[+ (u_x^2v_{xxx} - v_{xxx}u_{xx}u_{xx} - (v_{yy}u_{xxx} + u_{xy}v_{xxx})u_{xx} + 2v_{xx}u_{xy}u_{xxx})\varphi^u - (u_x^2 + u_y)v_{xxx}^2D_x(\varphi^u) - u_x^2v_{xxx}D_y(\varphi^u) - u_x^2D_xD_y(\varphi^u)\]

\[+ \left((u_x^2 + u_y)u_{xxx}u_{xxx} - 6u_xu_x^3 - 3u_{xx}u_{xy}D_x^2(\varphi^u) + (u_xu_{xxx} - 3u_{xx}^3)D_y(\varphi^u) + u_xu_{xxx}D_t(\varphi^u)\right)\]

\[+ (2u_xu_yu_{xxx} - 6u_x^4 - 2u_x^2u_{xx}D_x(\varphi^u),\]

\[u_{xxx}v_{xxx}D_xD_y(\varphi^u) - u_x^2D_xD_y(\varphi^u) + (u_xu_{xxx}v_{xxx} - 2v_{xx}u_{xy}u_{xxx})D_x(\varphi^u) + u_xu_{xxx}D_x(\varphi^u)\]

\[= u_{xxx}(2(u_xv_y + uyv_x)u_{xx} + u_xu_yv_{xxx})D_x(\varphi^u) + u_{xxx}(2u_yu_{xx} + uyv_x)D_xD_y(\varphi^u) + u_xu_{xxx}D_t(\varphi^u) + u_{xxx}v_{xxx}D_y(\varphi^u)\]

\[+ (u_xu_{xxx}v_{xxx} - 3u_{xx}^2u_{xx} - u_{xx}u_yv_{xxx} + v_{xx}u_{xx})D_x(\varphi^u) - u_yu_{xxx}u_{xxx}D_xD_y(\varphi^u) - u_x^2D_xD_yD_t(\varphi^u)\]

\[+ \left((u_x^2 + uy)u_{xxx}u_{xxx} - (3u_yu_x^2 + 3u_{xx}u_{xx} + u_{xx}u_{xx})D_x(\varphi^u) + (u_xu_{xxx} - 2u_{xx}^3)D_t(\varphi^u)\right)\]

\[+ (2u_xu_{xx}^2 + 2u_xu_{xx}u_{xxx} - u_{xxx}(u_{xx}(6u_{xx}u_{xy} + 2u_yu_{xxx} + u_{xxx}u_{xx} + u_{xx}u_{xx}))\varphi^u,\]

\[u_{xxx}v_{xxx}D_xD_y(\varphi^u) - 2v_{xx}u_{xy} + 2u_yu_{xx} - 2u_{xx}v_{xx} - v_{xx}u_{xx})D_x(\varphi^u)\]

\[= u_{xxx}(4u_xv_x + 2v_y)u_{xx} + (u_x^2 + uy)v_{xx})D_xD_y(\varphi^u) + u_{xxx}u_{xx}D_yD_t(\varphi^u) + u_xu_{xxx}v_{xxx}u_{xxx}D_x(\varphi^u)\]

\[+ (u_xu_{xxx}v_{xxx} - 2(2u_xv_x + v_y)u_{xx} + (u_x^2 + uy)v_{xx})u_{xxx} - 4(uyv_y + uyv_x)+u_{xxx}^2 + 2v_xu_{xx} + v_{xx}u_{xx} + v_{xx}u_{xx})u_{xx}D_x(\varphi^u)\]

\[+ (u_xu_{xxx}v_{xxx} - 2(u_xv_x + v_y)u_{xx} + 4u_yv_{xx})u_{xxx}^2 + 2(2u_xv_x + v_y)u_{xx} + v_{xx}u_{xx} + v_{xx}u_{xx})^2D_x(\varphi^u)\]

\[+ (u_{xx}u_{xxx}u_{xx} + u_{xx}u_{xx}u_{xx})u_{xx}D_y(\varphi^u) + (u_xv_{xxx} - 2v_xu_{xxx} - u_{xx}u_{xx}u_{xx}D_x(\varphi^u) + u_{xx}v_{xxx}u_{xxx}D_t(\varphi^u)\]

\[+ (u_{yy}u_{xxx} + v_{xx}u_{xx})(u_{xx})\varphi^u - u_x^2(u_x^2 + uy)v_{xxx}^2D_xD_y(\varphi^u) - u_{xx}u_{xxx}D_yD_t(\varphi^u)\]

\[= u_{xxx}(u_{xx}^2 + uy)v_{xxx}^2D_xD_y(\varphi^u) + ((u_x^2 + uy)u_{xxx}u_{xxx} + 2u_xu_{xx}u_{xxx})D_t(\varphi^u)\]

\[+ (2u_xu_{xx}u_{xx} + 2u_xu_{xx}u_{xxx} + u_{xxx}(4u_yv_x + uyv_y))u_{xxx}^2 + 2(2u_xv_x + v_y)u_{xx} + v_{xx}u_{xx} + v_{xx}u_{xx})^2D_x(\varphi^u)\]

\[+ (u_xu_{xxx}v_{xxx} - 2(u_xv_x + v_y)u_{xx} + 4u_yv_{xx})u_{xxx}^2 + 2(2u_xv_x + v_y)u_{xx} + v_{xx}u_{xx} + v_{xx}u_{xx})^2D_x(\varphi^u)\]
NONLOCAL SHADOWS GENERATED BY $\mathcal{R}_1$ AND $\mathcal{R}_2$

Consider the covering over $\mathcal{E}$ with the nonlocal variables $z_1, \ldots, z_4$ defined by

$$
\begin{align*}
  z_{1,x} &= u_x^2 + u_y, & z_{1,y} &= u_x u_y + u_t, \\
  z_{2,x} &= u_x v_x - v_y, & z_{2,y} &= 2u_x v_y - u_y v_x - v_t; \\
  z_{3,x} &= (2u_x u_{xt} + u_{yt})y - u_y^3 - 2u_x u_y - u_t, & z_{3,y} &= (u_x u_y + u_{xt} u_y + u_{tt})y - u_x^2 u_y - u_y^2; \\
  z_{4,x} &= (u_x v_{xt} + u_{xt} v_x - v_y) - u_x v_y - u_y v_x + v_t, \\
  z_{4,y} &= u_x^2 v_y - u_y v_y - u_x u_y v_x + (2u_x v_y + 2u_{xt} v_y - u_y v_x - u_{yt} v_x - v_t)y.
\end{align*}
$$

Then, the nonlocal shadows mentioned in Section 5 are

$$
\begin{align*}
  \nu_1^v &= -3uu_x - yu_t - 2uxy + 4z_1, & \nu_1^v &= 3uv_x - yv_t - 2xvy - z_2; \\
  \nu_2^v[\vartheta] &= \frac{1}{24} y^4 \vartheta_{tttt} - \frac{1}{6} y^2 (yu_x - 3x) \vartheta_{ttt} - \frac{1}{2} (2u_x xy + u_y y^2 - 2uy - x^2) \vartheta_{tt} \\
  &\quad - (uu_x + yu_t + xu_y - z_1) \vartheta_t + (u_x u_t - z_{1,t}) \vartheta, \\
  \nu_3^v[\vartheta] &= -\frac{1}{6} y^3 v_x \vartheta_{ttt} - \frac{1}{2} y(2v_x v_y + yv_y + 4v) \vartheta_t \\
  &\quad - (uv_x + 4uv_x + yv_t + xvy - 2z_2) \vartheta_t + (u_t v_x - 2u_x v_t + z_{2,t}) \vartheta; \\
  \nu_3^u &= 0, & \nu_3^v &= 2uv_x + z_2; & \nu_4^v &= 0, \\
  \nu_5^u[\vartheta] &= -(u_x^7 + 6u_x^5 u_y + 10u_x^3 u_y^2 + 4u_x u_y^3)u_{xxx} \\
  &\quad - (u_x^6 + 5u_x^4 u_y + 6u_x^2 u_y^2 + u_y^3)u_{xxy} - (6u_x^5 + 20u_x^3 u_y + 12u_x u_y^2)u_{xxt} \\
  &\quad - (5u_x^4 + 12u_x^2 u_y + 3u_y^2)u_{xyt} - (10u_x^3 + 12u_x u_y)u_{xtt} \\
  &\quad - (6u_x^2 + 3u_y)u_{ytt} - (4u_x - 2u_{xx})u_{ttt} - \left(\frac{7}{2} u_x^6 + 15u_x^4 u_y + 15u_x^2 u_y^2 + 2u_y^3\right)u_{xx} \\
  &\quad - (6u_x^5 + 20u_x^3 u_y + 12u_x u_y^2)u_{xxy} \\
  &\quad - (15u_x^4 u_{xt} + 10u_x^3 u_y + 30u_x^2 u_y u_{xt} - 12u_x u_y u_{yt} - 6u_y^2 u_{xt})u_{xx} \\
  &\quad - \left(\frac{5}{2} u_x^4 + 6u_x^2 u_y + \frac{3}{2} u_y^2\right)u_{xy} - (10u_x^3 u_{xt} + 6u_x^2 u_{yt} - 12u_x u_y u_{xt} - 3u_y u_{yt})u_{xy} \\
  &\quad - 15u_x^2 u_{xxt}^2 - 12u_x u_{yt}^2 u_{xt} - 6u_y u_{xt}^2 - \frac{3}{2} u_y^2 - z_{1,ttt}; \\
  \nu_5^v[\vartheta] &= 0, & \nu_5^u[\vartheta] &= -\frac{1}{48} y^4 \vartheta_{tttt} - \left(\frac{1}{6} y^3 u_x + \frac{1}{4} y^2 u_y\right) \vartheta_{tt} \\
  &\quad - \left(\frac{3}{4} y^2 u_x^2 - xy u_x - \frac{1}{2} y^2 u_y - \frac{1}{2} y - \frac{1}{4} x^2\right) \vartheta_t \\
  &\quad - (2yu_x^3 + \frac{3}{2} xu_x + (3u_y y + u) u_x + yu_t + xu_y + \frac{1}{2} z_1) \vartheta_i \\
  &\quad - \left(\frac{5}{2} u_x^4 + 6u_x^2 u_y + 2u_t u_x + \frac{3}{2} u_y^2 + z_{1,t}\right) \vartheta; \\
  \nu_6^u &= 3uy_x + 2uxu_t + (2ux - 8)z_1 - y(u_x u_t + 2z_{1,t}), & \nu_6^v &= 3uy_y + 2vx u_t + 2vx z_1 + (4ux - 1)z_2 + y(2vu_x - u_t v_x + 2z_{2,t}); \\
  \nu_7[\vartheta] &= \frac{1}{120} y^5 \vartheta_{ttttt} + \frac{1}{24} y^3 (yu_x - 4x) \vartheta_{tttt} + \frac{1}{2} (xy^2 u_x + \frac{1}{3} y^3 u_y - y^2 u - yx^2) \vartheta_{ttt} \\
  &\quad + \left(\frac{1}{2} (2yu + x^2) u_x + \frac{1}{2} y^2 u_t + x(yu_y - u)\right) \vartheta_t \\
  &\quad + (yu_y + u_t - xyu_t u_x - z_1) \vartheta_t - (uy u_t - z_{1,t}) \vartheta + \frac{5}{2} y z_{2,t} - \frac{5}{2} u_x z_3.
\end{align*}
$$
\[\nu_7[\vartheta] = \frac{1}{24} y^4 v_x \vartheta_{utt} + y^2 (\frac{1}{2} v_x x + \frac{1}{6} y v_y + v) \vartheta_{tt} + (y u + \frac{1}{2} x^2) v_x + 4 y v u_x + \frac{1}{2} y^2 v_t + x y v_y) \vartheta_t + (6 u v_x - y u_t v_x + 2 y u_x v_t + u v_y + 4 v u_y + x v_t - 2 z_2) \vartheta_t \]

\[+ (y v_x u_{tt} + 3 u_x^2 v_t + y u_x v_{tt} - u_t v_y - 2 u_y v_t - z_{2,t}) \vartheta - \frac{5}{2} v_x z_3 + 4 u_x z_4 + 2 y z_4 t; \]

\[\nu_8^u = 0, \quad \nu_8^v = 3 u v_x^2 + 2 u v_y - 2 u_x z_2 - y z_2 t + z_4; \quad \nu_9^u = 0, \]

\[\nu_9^v = -4 y u_x u_{ttt} + (u_x^8 + 7 u_y^8 u_y + 15 u_x^4 u_y^2 + 10 u_x^2 u_y^2 + u_y^4) u_{xxx} + (u_x^7 + 6 u_x^5 u_y + 10 u_x^3 u_y^2 + 4 u_x u_y^3) u_{xx} + (u_x^5 + 30 u_x^3 u_y + 30 u_x^2 u_y^2 + 4 u_x u_y^3) u_{xxt} + (6 u_x^5 + 20 u_x^3 u_y + 12 u_x^2 u_y^2 + 6 u_x u_y^3) u_{yxt} + (10 u_x^5 + 12 u_x^2 u_y u_{ytt} + (3 u_x^2 + 2 u_y) u_{tttt} + (4 u_x^5 + 21 u_x^3 u_y + 30 u_x^2 u_y^2 + 10 u_x u_y^3) u_{xxt} + (7 u_x^5 + 30 u_x^3 u_y + 30 u_x^2 u_y^2 + 4 u_x u_y^3) u_{xx} + (21 u_x^5 - 60 u_x^3 u_y + 30 u_x u_y^2)) u_{xxxt} + (15 u_x^4 + 30 u_x^3 u_y + 6 u_x^2 u_y^2) u_{xxxt} + (15 u_x^4 + 30 u_x^3 u_y + 6 u_x^2 u_y^2) u_{xxxt} + (15 u_x^4 + 30 u_x^3 u_y + 6 u_x^2 u_y^2) u_{xxxt} + (15 u_x^4 + 30 u_x^3 u_y + 6 u_x^2 u_y^2) u_{xxxt}
\]

\[+ (30 u_x^4 + 30 u_x^2 u_y) u_{yxt} + (30 u_x^2 + 12 u_y) u_{yxt} + 6 u_x u_{yxt} + y z_1 u_{ttt} + 2 u_x z_1 u_{ttt} - z_{3,ttt}; \]

\[\nu_{10}^u = 0, \quad \nu_{10}^v = \frac{1}{240} y^5 v_x \vartheta_{tttt} + \frac{1}{24} y^3 (y u_x + 2 x) \vartheta_{tt} + \frac{1}{4} y((u_x^2 + 3 u_y + u_t) y^2 + (3 u_x^2 + 2 u u_x + 2 u v_y + z_1) y + x(x u_x + u)) \vartheta_{tt} + \frac{3}{2} y z_{1,t} + \frac{5}{2} y u_x + 2 x u_x + (6 y u + \frac{3}{2} u) u_x^2 + (2 y u_t + 3 x u_y + z_1) u_x + \frac{3}{2} y u_x^2 + u u_y + x u_t - \frac{1}{2} z_3) \vartheta_t + (3 u_x^6 + 10 u_x^3 u_y + 3 u_x u_y^2 + 6 u_x u_y^2 + 2 u_x z_1,t + y z_{1,tt} + 2 u v_y - z_{3,tt} \vartheta_t).
\]

**Appendix E**

**LIFTS OF \(\varphi_3, \ldots, \varphi_{14}\)**

The components \(\varphi_i^p, \varphi_i^q, i = 3, \ldots, 14\), are

\[\varphi_3^p = 0, \quad \varphi_3^q = 0, \quad \varphi_4^p = 0, \quad \varphi_4^q = 0, \quad \varphi_5^p = f(v_x p_t p_x - u_x p_x q_t - p_x w_2, \varphi_5^q = f(u_x q_t q_x + v_x p_t q_x + q_x w_6 - q_x w_2,t), \quad p_x w_6), \]

\[\varphi_6^p = -\frac{1}{2} f(2 u_x p_t p_x + p_x w_5 + p_x w_1,t), \quad \varphi_6^q = f(v_x p_t p_x - u_x p_t q_x - u_x p_q t + p_x w_6 - \frac{1}{2} q_x w_5 - p_x w_2,t - \frac{1}{2} q_x w_1,t), \]

\[\varphi_7^p = 0, \quad \varphi_7^q = 0, \quad \varphi_8^p = 0, \quad \varphi_8^q = 0, \quad \varphi_9^p = 0, \quad \varphi_9^q = 0, \quad \varphi_{10}^p = 0, \quad \varphi_{10}^q = 0.
\]
\[
\varphi_{11}^p = 0, \varphi_{11}^q = 0,
\]
\[
\varphi_{12}^p = 0, \varphi_{12}^q = \frac{1}{2} f(2u_x p_t p_x + p_x w_5 + p_x w_{1,t}),
\]
\[
\varphi_{13}^p = 0, \varphi_{13}^q = 0,
\]
\[
\varphi_{14}^p = 0, \varphi_{14}^q = f(v_x p_t p_x - u_x p_t q_t - p_x w_{2,t} + p_x w_6).
\]

The lifts of \( \varphi_1 \) and \( \varphi_2 \) to \( \tilde{\tau} \) are
\[
\varphi_{15}^{w_5} = -pp_y t + pt p_y + 2uxxpp_t - 2uxtppt - 2uxp_1 p_x,
\]
\[
\varphi_{16}^{w_6} = uxxp_1 q_t - uxxp_2 q_t - 2v_x p_1 p_x - v_x xpp_t + v_x xpp_t + uxtq_p q_x + 2u_x xpp_t + 
\]
\[
+ pq_y t - 2u_x xpp_t + 2pt q_t + 2pt q_t + p_y q_t + u_x xpt q_x;
\]
\[
\varphi_{25}^{w_5} = -(u_x^2 + u_y)pt p_x + u_x xpp_q - 2u_x xpp_t w_1 + 2uxxpt w_1 - 2uxp_1 p_t
\]
\[
+ ugtpp_x + u_y pgtpp_t - u_x xpp_q - u_x xpp_t - 3uxxpt q_y + ppt - p_y w_1 - p_y w_{1,t},
\]
\[
\varphi_{26}^{w_6} = -4uxpp_t q - v_x xpp_y - 2v_y xpp_x - 2v_x xpp_y - 2ux pp_y - u_x xpp_t - gy w_{1,t}
\]
\[
+ 2v_y xpp_t + u_x xpp_y - u_x xpp_t + 2(3uxxux - ugt)px q - u_x xqw_{1,t} + gy w_{1,t}
\]
\[
- 2p_y w_2 + q_t w_1 - ppt + u_x xq v + v_y p p - (u_x^2 + 2u_y)pp q -
\]
\[
- v_xt p_x w_1 + (u_x^2 - u_y)px q + 2uxxpt w_2 - u_x xqw_{1,t} + v_x xpt w_1
\]
\[
- 2uy ypp_t q - 3uxxpt q - v_x ypp t - v_x ypp t;
\]

Finally, the lifts of \( \varphi_3, \ldots, \varphi_{14} \) to \( \tilde{\tau} \) are
\[
\psi_{3}^{w_i} = 0, \quad i = 1, \ldots, 6;
\]
\[
\psi_{4}^{w_i} = 0, \quad i = 1, \ldots, 6;
\]
\[
\varphi_{5}^{w_1} = f(2ux v_x p_t p_x - 2u_x^2 p_x w_2, t + 2uxp_x w_6 + v_x p_t p_y - u_x p_y q_t + p_y w_6 - p_y w_{2,t}),
\]
\[
\varphi_{5}^{w_2} = -f(v_x u_x p_1 p_x q_t + u_x^2 q_t q_x + v_x^2 p_t p_x - v_x u_x p_x q_t - u_x xqw_{1,t} - v_x xq w_{2,t}
\]
\[
+ (u_x q_x + v_x xq - q)w_6 - u_x q_x q_y - v_x xq p_t + q_y w_{2,t},
\]
\[
\varphi_{5}^{w_3} = \frac{1}{2} f((y u_x x p_y q t - v_x x p_y q_y - p_y w_6 + p_y w_{2,t}) - v_x x p_t - u_x p_t q + p_w_6 - w_{2,t})
\]
\[
- f \left( \frac{1}{2} y (v_x x p_1 p_y + p_y w_6) - u_x p_y w_6 + (u_x x p_y + u_x y p) w_{2,t}
\]
\[
\right)
\]
\[
- \left( y u_x x + \frac{3}{2} u_x^2 \right) x w_6 - \left( \frac{1}{2} y u_x x \right) p t w_6 + \left( \frac{3}{2} u_x^2 - y u_x x \right) p_x w_{2,t}
\]
\[
\left( y u_x x + \frac{1}{2} p t w_{2,t} + \left( \frac{3}{2} u_x^2 - y u_x x \right) u_x x w_6 \right) p_t q_t + \left( u_x^2 - \frac{1}{2} y u_x x \right) p_y q_t
\]
\[
+ \left( \frac{1}{2} y v_x x - u_x x \right) p_t q_t + \left( \frac{1}{2} u_x x - u_x x y \right) p_t q_t + \left( \frac{1}{2} v_x x - u_x x y \right) p_t q_t + \frac{1}{2} u_x x p_t q_t
\]
\[
+ u_x y w_6 + \frac{1}{2} v_x x p_t q + u_x x \left( y u_x x + \frac{1}{2} p_t q_t + v_x \left( y u_x x - \frac{3}{2} u_x^2 \right) p_t q_t - \frac{1}{2} y u_x x p_t q_t
\]
\[
\right),
\]
\[
\varphi_{5}^{w_4} = f(y u_x x q_y + y v_x x p_t q_y + u_x x q_t - v_x x q_t + y_q w_6 + (q - y_q) w_{2,t} - q w_6)
\]
\[
+ f(t)(u_x x u_x y + u_x x q_t - y u_x x x - y u_x x v_x y + v_x x p_t q_t - (u_x x q_t + v_x x) p_t q
\]
\[
- v_x x p_t q_t + u_x x q_t q_t + v_x x q_t w_{2,t} - v_x y p w_6 - u_x x y q_t w_{2,t} - (u_x x - 1) q_t w_{2,t}
\]
\[
+ u_x y q_t w_6 + (y u_x x q_y - y q_y) w_{2,t} + y q_y w_6 + (u_x x y + v_x x) p_t q_y
\]
\[
+ v_x x x p_t q_y + (u_x^2 + y u_x x) q_t q_y + v_x x p w_{2,t} + y q_y w_6 + v_x x p y w_6 + (y u_x x - 1) q_t w_6
\]}.
\[ - y g y w_{2,t} + y u x t q x w_{2,t} + q w_{2,t} + y v x x p t w_{6} - v x u x p y q t + v x y u x p q t + v x y v x p p t \\
- q w_{6,t} - u x y g u w_{6} - y v x x p t p x - y u x u x t q t q x - y u x t v x x p t q t + y u v x x t p x q t \\
y - u x t q x w_{6} + y v x x p x w_{2,t} - y v x x p x w_{6} - y v x x p t w_{2,t} + y v x x p t q y t \\
+ y u x q q y t + v u x x t q t q y + y u x t p t q y , \\
\varphi_{5}^{w_{5}} = \frac{1}{2}(v_{x} p_{y} q g + u x y p g - p y w_{6} + p y w_{2,t}) \\
- 2 f(u x t v x x p t p x - u x u x t p q t + u x x u x t p q t + u x t p x w_{6} - \frac{1}{2} u x t p y q t - u x t p x w_{2,t} \\
- u x x p t w_{6} + u x x p t w_{2,t} + \frac{1}{2} v x t p t p y + \frac{1}{2} v x p t p y + \frac{1}{2} v x p t u p y - \frac{1}{2} u x p y q t t \\
- \frac{1}{2} u x p y q t + \frac{1}{2} p g y w_{6} + \frac{1}{2} p g y w_{6} + \frac{1}{2} p g w_{6} - \frac{1}{2} p g w_{2,t} - \frac{1}{2} p g w_{2,t} , \\
\varphi_{6}^{w_{6}} = \frac{1}{2}(u x q t q y + v x p t q y + g y w_{6} - q y w_{2,t}) \\
- f((u x v x - u x x v x) p t q t + u x t v x x p t q t + u x u x t q t q x + v x v x t p t x - u x v x t p x q t + u x t q x w_{6} \\
u x t q t g y - u x t x w_{2,t} - u x x q t w_{6} + u x x q t w_{2,t} + v x t p x w_{6} - v x t (p q x - p x w_{2,t}) \\
v x x p t w_{6} + v x x p t w_{2,t} - v x p t q y t - v x p t q g y - u x q q g y t \\
u x q t q y - q y t w_{6} + g y w_{6} + q y w_{2,t} + q y w_{2,t} ; \\
\varphi_{6}^{w_{1}} = -\frac{1}{2}(2 u x p t q p x + u x p t p y + u x x p x w_{5} + u x x p x w_{1,t} + \frac{1}{2} p g y w_{5} + \frac{1}{2} p g w_{1,t} , \\
\varphi_{6}^{w_{2}} = f(u x p t q x + u x x p x q t - u x x p x w_{6} + \frac{1}{2} u x q x w_{5} - u x p t q y - u x p x w_{2,t} \\
+ \frac{1}{2} u x q x w_{1,t} + \frac{1}{2} u x p x w_{5} + v x p t q y + \frac{1}{2} v x p x w_{1,t} + p g w_{6} - \frac{1}{2} p g w_{5} - p g w_{2,t} - \frac{1}{2} q g w_{1,t} , \\
\varphi_{6}^{w_{3}} = \frac{1}{4} f(2 y u x p t p y + 2 u x p p t + y p y w_{5} + y p y w_{1,t} - p w_{5} - p w_{1,t} \\
+ \frac{1}{2} f(t)(-2 u t x - y u x t) p t p y - (2 u x u x y - u x t) p p t - \frac{1}{2}(2 y u x x + 1) p r w_{1,t} \\
- \frac{1}{2}(3 u x^{2} - 2 y u x t) p x w_{1,t} - u x y p y w_{1,t} - u x (3 u x^{2} - 2 y u x t) p x p x + u x y p w_{1,t} \\
- \frac{1}{2}(2 y u x x + 1) p r w_{5} - \frac{1}{2}(3 u x^{2} - 2 y u x t) p x w_{5} - \frac{1}{2} p w_{1,t} + \frac{1}{2} p w_{5}, t + y u x p t p y + u x y p w_{5} + u x x p t t - u x p y w_{5} + \frac{1}{2} y p y w_{5}, t + \frac{1}{2} y p y w_{1,t} + \frac{1}{2} y p y w_{5} , \\
\varphi_{6}^{w_{4}} = \frac{1}{2} f(-2 y u x p y q y - 2 y u x y p q t + 2 y v x p y q y + 2 u x q t q + 2 u x p t q + 2 v x p p t + 2 y p y w_{5} \\
y y q y w_{5} - 2 y p y w_{2,t} - q y w_{1,t} - 2 p w_{6} + q w_{5} + 2 p w_{2,t} + q w_{1,t} \\
f(t)(y p y w_{2,t} + y p y w_{6}, t - y p y w_{6}, t - y v x x p t p y - y p y w_{6} - u x p y w_{6} + u x p y w_{2,t} \\
u x y p w_{2,t} - u x y p q t + y v x x p t p x - v y x x p t p x - y u x t u x x p x q - y u x t x u x t p t q t + y u x y p q t t \\
y u x p t p q t - y v x x p t p q t - p w_{2,t} + p w_{6}, t + y u x t p t q y - (u x y u x y + u x t) p t q \\
y + (u x^{2} + y u x t) p t q y - \frac{1}{2} y u x t q x w_{5} - \frac{1}{2} y v x x p t w_{1,t} + \frac{1}{2} y v x x p t w_{5} \\
- \frac{1}{2} y v x x p t w_{1,t} - \frac{1}{2} y v x x p t w_{5} - (u x y v x - u x t v x) p x p x + \frac{1}{2} y q t w_{5} + \frac{1}{2} y q t w_{1,t} \\
- \frac{1}{2} v x y p w_{5} - \frac{1}{2} v x y p w_{1,t} - (u x y u x y + u x t) p t q + (y u x x - 1) p t w_{2,t} - (y u x x - 1) p t w_{6} 
\]
\[
+ \frac{1}{2}u_xq_yw_5 - \frac{1}{2}qw_{1,t} + \frac{1}{2}u_xq_yw_{1,t} + (u_xv_{xy} - u_xv_x - v_{xt})p_{yt} + \frac{1}{2}(yu_{xx} - 1)w_{1,t},
\]
\[
+ \frac{1}{2}u_xp_yw_{1,t} + \frac{1}{2}g_{yy}w_{1,t} + (u_x^2 + yu_{xt})p_yq_t + \frac{1}{2}yq_yw_{5,t} - \frac{1}{2}u_xq_yw_5 - u_xp_utq
\]
\[
- \frac{1}{2}u_xq_yw_{1,t} + \frac{1}{2}u_xp_yw_5 - yu_{xt}p_yq_y + yu_xp_yq_y - yu_{xt}p_xw_{2,t} + yu_{xt}p_xw_6
\]
\[
+ 2u_x(yu_{xx} - 1)p_yq_t - \frac{1}{2}qw_{5,t} + \frac{1}{2}(yu_{xx} - 1)q_tw_{1,t},
\]
\[
\frac{\varphi_{w_6}}{u_6} = \frac{1}{2}f(2u_xp_y + p_yw_5 + p_yw_{1,t})
\]
\[
- f(-2u_xu_xtp_{1}p_x + u_xu_xtp_{1}w_5 + u_xu_xtp_{1}w_{1,t} - u_xu_xtp_{1}p_y - u_xu_xtp_{1}p_y - u_xu_xtp_{1}p_y - u_xu_xtp_{1}p_y)
\]
\[
- u_xu_xtw_{1,t} - \frac{1}{2}p_tw_{5} - \frac{1}{2}p_yw_{5,t} - \frac{1}{2}p_yw_{1,t} - \frac{1}{2}p_yw_{1,t},
\]
\[
\frac{\varphi_{w_6}}{6} = \frac{1}{2}f(-2u_xu_xtq_y - 2u_xu_xtq_t + 2u_xu_xtq_y + 2u_xu_xtq_t - q_yw_5 - 2p_yw_{2,t} - q_yw_{1,t})
\]
\[
+ f(-u_xp_yq_t - u_xp_yq_t - p_yw_2,t - p_yw_2,t - p_yw_6,t + p_yw_6,t + \frac{1}{2}v_xtw_{1,t}
\]
\[
+ u_xu_xtp_tq_x - u_xu_xtw_6 - u_xu_xtq_t + u_xu_xtw_{2,t} + u_xu_xtw_{1,t} - u_xu_xtw_2,t
\]
\[
+ v_xtw_{1,t} + v_xtw_{1,t} + v_xtw_{1,t} - 2u_xu_xtw_{1,t} + u_xu_xtq_y - u_xu_xtq_y + \frac{1}{2}v_xtw_{x_5}
\]
\[
- \frac{1}{2}u_xu_xtq_5 + (u_xu_xt - u_xu_xt)v_xtw_{1,t} - u_xu_xtq_y - u_xu_xtq_y - u_xu_xtq_y + \frac{1}{2}v_xtw_{x_5}
\]
\[
- \frac{1}{2}v_xtw_{1,t} + \frac{1}{2}u_xtw_{5} - u_xp_yq_t - \frac{1}{2}v_xtw_{1,t} - \frac{1}{2}u_xq_yw_{1,t} - \frac{1}{2}qw_5
\]
\[
- \frac{1}{2}q_tw_{1,t} - \frac{1}{2}g_yw_{1,t} - \frac{1}{2}q_yw_5); 
\]
\[
\varphi_{w_i} = 0, \quad i = 1, \ldots, 6; \quad \varphi_{w_i} = 0 \quad i = 1, \ldots, 6; \quad \varphi_{w_i} = 0 \quad i = 1, \ldots, 6; \quad \varphi_{w_i} = 0 \quad i = 1, \ldots, 6; \quad \varphi_{w_i} = 0 \quad i = 1, \ldots, 6,
\]
\[
\varphi_{w_i} = \frac{1}{2}f(-u_xp_yw_5 + p_yw_5 + p_yw_{1,t} - p_yw_5 - p_yw_{1,t})
\]
\[
+ f((u_x^2 + yu_{xt})p_yq_y + (u_xu_x + u_xu_x + u_xu_x)p_{yt} + \frac{1}{2}(yu_{xx} - 1)p_tw_{1,t} + \frac{1}{2}u_xp_yw_{1,t} - \frac{1}{2}u_xp_yw_{1,t}
\]
\[
+ \frac{1}{2}(yu_{xx} - 1)p_tw_5 - yu_xu_xtw_{p_x} - \frac{1}{2}yu_xtw_{x_5} - \frac{1}{2}yu_xtw_{x_5} + yu_xp_yq_y + yu_xp_yq_y
\]
\[
- \frac{1}{2}u_xp_yw_5 + u_xp_yw_5 + \frac{1}{2}u_xp_yw_5 + \frac{1}{2}yq_yw_5 + \frac{1}{2}yq_yw_5 + \frac{1}{2}yq_yw_5 + \frac{1}{2}yq_yw_5
\]
\[
+ \frac{1}{2}p_tw_{1,t} - \frac{1}{2}p_tw_{1,t} - \frac{1}{2}p_tw_{1,t}), \quad \varphi_{w_i} = 0,
\]
\[
\varphi_{w_i} = \frac{1}{2}f(2u_xp_y + p_yw_5 + p_yw_{1,t})
\]
\[
+ \frac{1}{2}f(-2u_xu_xtp_{1}p_x + u_xu_xtp_{1}w_5 + u_xu_xtp_{1}w_{1,t} - u_xu_xtp_{1}p_y + 2u_xu_xtp_{1}p_y + u_xu_xtp_{1}p_y + u_xu_xtp_{1}p_y)
\]
\[
- u_xu_xtw_{1,t} + p_yw_5 + p_yw_{5,t} + p_yw_{1,t} + p_yw_{1,t};
\]
\[
\varphi_{w_i} = 0 \quad i = 1, \ldots, 6; \quad \varphi_{w_i} = 0 \quad i = 1, \ldots, 6,
\]
\[
\varphi_{w_i} = f(u_xp_yq_t - u_xu_xtp_{1}p_x + u_xu_xtw_2,t - u_xu_xtw_6 + u_xu_xtw_{2,t} - u_xu_xtw_{6},
\]
\[
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\[ \varphi_{14}^{\mu_3} = 0, \]
\[ \varphi_{14}^{\nu_4} = \hat{f}(yv_x p_t p_y - yu_x p_y q_t + v_x p p_t + u_x q q_t + yp_y w_6 - yp_y w_2,t - pw_6 + pw_2,t) \]
\[ - f(yu_x v_x p_x + yp_y w_2,t + yp_y w_6,t - yp_y w_6 + yu_x p_y q_t + u_x q q_t - v_x p p_t + u_x y w_6 - yv_x p y p_y \]
\[ - yu_x u_x p_x q_t - pw_2,t + pw_6,t - (u_x v_x + yv_x) p_t p_y - (u_x u_x + yu_x) p q_t \]
\[ + (yu_{xx} - 1)p_t w_2,t - (yu_{xx} - 1)p_t w_6 - yu_{xt} p_x w_2,t - (u_{xy} v_x + yv_x) p p_t + (u_x^2 + yu_x) p y q_t \]
\[ + yu_{xt} p_x w_6 + u_x(yu_{xx} - 1)p_t q_t), \quad \varphi_{14}^{\nu_4} = 0, \]
\[ \varphi_{14}^{\nu_6} = \hat{f}(v_x p_t p_y - u_x p_y q_t + p_y w_6 - p y w_2,t) \]
\[ - f(-v_x p_t p_y - p_y w_6,t - v_x p_t p_y + u_x p x w_6 - u_{xx} p_t w_6 - p_y w_6 + u_x p y q_t - u_x p x w_2,t \]
\[ + p_y w_2,t - u_x u_x p_x q_t - u_x v_x p_t p_x + u_x u_x u_x q_t + p_y w_2,t \]
\[ - v_x p_t p_y + u_x p_x q_t + u_x p t w_2,t). \]

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**REFERENCES**

1. H. Baran, I. S. Krasil’shchik, O. I. Morozov, and P. Vojčák, “Higher symmetries of cotangent coverings for Lax-integrable multi-dimensional partial differential equations and Lagrangian deformations,” J. Phys.: Conf. Ser. 482, 012002 (2013); arXiv: 1309.7435.
2. H. Baran, I. S. Krasil’shchik, O. I. Morozov, and P. Vojčák, “Nonlocal symmetries of integrable linearly degenerate equations: A comparative study,” Theor. Math. Phys. 196, 1089–1110 (2018); arXiv: 1611.04938. https://doi.org/10.1134/S0040577918100019
3. H. Baran and M. Marvan, Jets. A Software for Differential Calculus on Jet Spaces and Diffieties. http://jets.math.slu.cz.
4. A. V. Bocharov et al., Symmetries of Differential Equations in Mathematical Physics and Natural Sciences, Ed. by A. M. Vinogradov and I. S. Krasil’shchik (Faktorial, Moscow, 2005; Am. Math. Soc., Providence, RI, 1999).
5. M. Dunajski, “A class of Einstein-Weyl spaces associated to an integrable system of hydrodynamic type,” J. Geom. Phys. 51, 126–137 (2004); arXiv: nlin/0311024. https://doi.org/10.1016/j.geomphys.2004.01.004
6. E. V. Ferapontov and J. Moss, “Linearly degenerate partial differential equations and quadratic line complexes,” Commun. Anal. Geom. 23, 91–127 (2015); arXiv: 1204.2777. https://dx.doi.org/10.4310/CAG.2015.v23.n1.a3
7. B. Fuchssteiner and A. S. Fokas, “Symplectic structures, their Bäcklund transformations and hereditary symmetries,” Phys. D (Amsterdam, Neth.) 4, 47–66 (1981).
8. I. S. Krasil’shchik and A. Verbovetsky, “Geometry of jet spaces and integrable systems,” J. Geom. Phys. 61, 1633–1674 (2011); arXiv: 1002.0077.
9. P. H. M. Kersten and I. S. Krasil’shchik, Symmetries and Recursion Operators for Classical and Supersymmetric Differential Equations (Kluwer Academic, Dordrecht, 2000).
10. I. S. Krasil’shchik and A. M. Verbovetsky, “Recursion operators in the cotangent covering of the rdDym equation,” Anal. Math. Phys. 12 (1) (2022). https://doi.org/10.1007/s13324-021-00611-3
11. I. S. Krasil’shchik, A. M. Verbovetsky, and R. Vitolo, The Symbolic Computation of Integrability Structures for Partial Differential Equations, Texts and Monographs in Symbolic Computation (Springer, Cham, 2017).
12. I. S. Krasil’schik and A. M. Vinogradov, “Nonlocal trends in the geometry of differential equations: Symmetries, conservation laws, and Bäcklund transformations,” Acta Appl. Math. 15, 161–209 (1989). https://doi.org/10.1007/BF00131935

13. M. Marvan, “Another look on recursion operators,” in Differential Geometry and Applications, Proceedings of the Conference, Brno, 1995 (Masaryk Univ., Brno, 1996), pp. 393–402.

14. M. Marvan, “Sufficient set of integrability conditions of an orthonomic system,” Found. Comput. Math. 9, 651–674 (2009).

15. V. G. Mikhalev, “On the Hamiltonian formalism for Korteweg-de Vries type hierarchies,” Funct. Anal. Appl. 26, 140–142 (1992).

16. M. V. Pavlov, “Integrable hydrodynamic chains,” J. Math. Phys. 44, 4134–4156 (2003).

17. A. M. Vinogradov, Cohomological Analysis of Partial Differential Equations and Secondary Calculus, Vol. 204 of Translations of Mathematical Monographs (Am. Math. Soc., Providence, RI, 2001).