CHERN-WEIL THEORY FOR PRINCIPAL BUNDLES OVER LIE GROUPOIDS

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Abstract. Let \( \mathcal{X} = \left[ X_1 \Rightarrow X_0 \right] \) be a Lie groupoid equipped with a connection, given by a smooth distribution \( \mathcal{H} \subset T X_1 \) transversal to the fibers of the source map. Under the assumption that the distribution \( \mathcal{H} \) is integrable, we define an analog of de Rham cohomology for the pair \( (\mathcal{X}, \mathcal{H}) \) and study connections on principal \( G \)-bundles over \( (\mathcal{X}, \mathcal{H}) \) in terms of the associated Atiyah sequence of vector bundles. Finally, we develop the corresponding Chern-Weil theory and study characteristic classes.

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Acknowledgements
References

1. Introduction

The geometry of principal bundles over Lie groupoids has been quite an active area of research in recent years. In particular, the structure of a connection on Lie groupoids and for principal bundles over Lie groupoids as well as its associated geometric and algebraic properties have been discussed by several authors. Among them [LGTX2, BX, B, CM, CLW] are particularly relevant for this article. For example, the articles [LGTX2] and [CM] study Chern-Weil theory on Lie groupoids via the de Rham cohomology defined using simplicial manifolds associated to the groupoid nerves. Whereas for Behrend in [B] the main ingredient for the construction of connections is the theory of cofoliations on a differentiable stack. Furthermore, Tang [Ta] defined in a similar fashion flat connections for Lie groupoids, which he called étalizations. More recently, Arias Abad-Crainic [AC] introduced general Ehresmann connections for Lie groupoids. Herrera-Ortiz [HO] have

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also informed us that they are currently developing a similar theory for principal 2-bundles over Lie groupoids involving Atiyah LA-groupoids. For a general discussion and recent results on the geometry of Lie groupoids, we also refer to \cite{dH, MM, Ma1, Ma2}.

The objective of this article is to study general connections on principal bundles over Lie groupoids in terms of Atiyah sequences of vector bundles associated to transversal tangential distribution inspired by the classical work of Atiyah \cite{At} on connections for fiber bundles in complex geometry. We will also describe the Chern-Weil theory and associated characteristic classes for such principal bundles. In the course of this paper, the necessary framework for de Rham cohomology on a Lie groupoid with integrable connection is also developed. It is worthwhile to explore the possible relations between these seemingly different approaches to the theory of connections in more detail, which we aim to pursue in a follow-up article. In fact, just before Subsection 5.1 we do give a brief comparison between our approach here with that of Gengoux et al. in \cite{LGT2}. Other variations of the theory of connection on Lie groupoids have appeared in papers such as \cite{B}, \cite{LGT2}, \cite{Ta}, \cite{GSX} in different contexts. Another reason for the interest in the geometry of Lie groupoids is of course its association with the geometry of differentiable stacks, which are basically Morita equivalence classes of Lie groupoids (see for example \cite{He, GSX, Me, CK, Ca, Gi}). Here we have not discussed much about the possible extension of our constructions to stacks. In \cite{BCKN}, some of the issues relevant to differentiable stacks are explored in more detail. Our definition of connection on a Lie groupoid is, in fact, also closely related to that of \cite{B} (compare \cite{BCKN}).

**Outline and organisation of the paper.** In the first section (Section 2) we will recall standard notions and constructions, such as the definitions of principal $G$-bundles and vector bundles over a Lie groupoid $X = [X_1 \rightrightarrows X_0]$ and set up our notations. For a Lie group $G$, a principal $G$-bundle $(E_G \to X_0, [X_1 \rightrightarrows X_0])$ over a Lie groupoid $X = [X_1 \rightrightarrows X_0]$ is a $G$-bundle $E_G \to X_0$ with an action (compatible with the action of $G$ on $E_G$) of $X$ on $E_G$ (Definition 2.5). Likewise, a vector bundle over $X = [X_1 \rightrightarrows X_0]$ is a vector bundle $E \to X_0$ with an action of $X$ on $E$ inducing a linear map between fibers. In the following section (Section 3) we introduce the notation of a connection on a Lie groupoid $X = [X_1 \rightrightarrows X_0]$ as a smooth distribution $\mathcal{H} \subset TX_1$ complementing the kernel of the source map $s_*$. A connection is called flat or integrable if the corresponding distribution $\mathcal{H}$ is integrable. This definition was originally introduced in \cite{BN} (compare also \cite{BCKN}). Assuming our Lie groupoid admits a connection, then a differential form on $X_0$ is said to be a differential form on the Lie groupoid $[X_1 \rightrightarrows X_0]$ if it satisfies certain compatibility condition with respect to the source-target maps and the connection on the Lie groupoid (see Definition 3.1). We show that for an integrable connection, the exterior derivative of a differential form $[X_1 \rightrightarrows X_0]$ is well defined. In turn, we obtain the graded de Rham cohomology algebra $H^{\dr}_d(X, \mathcal{H})$ of the pair $(X, \mathcal{H})$. It should be noted that the integrability condition on $\mathcal{H}$ will be crucial here. Given a principal $G$-bundle $(E_G \to X_0, [X_1 \rightrightarrows X_0])$ over a Lie groupoid $X = [X_1 \rightrightarrows X_0]$, one obtains the associated Atiyah sequence

$$0 \to (E_G \times \mathfrak{g})/G \to (TE_G)/G \to TX_0 \to 0,$$

of vector bundles over the manifold $X_0$ associated to the principal bundle $E_G \to X_0$. In Section 4 we prove a key result, namely that for a Lie groupoid $X$ with a connection $\mathcal{H}$ one can define actions of $X$ on $(E_G \times \mathfrak{g})/G, (TE_G)/G$ and $TX_0$, turning the sequence above into a sequence of vector bundles over the Lie groupoid $X = [X_1 \rightrightarrows X_0]$. A
connection on \((E_G \to X_0, [X_1 \rightrightarrows X_0])\) is then defined as the splitting of this sequence of vector bundles over the Lie groupoid \(X\). In the following sections \(5\) and \(6\) we assume connections on a Lie groupoid exist and to be integrable. In Section \(5\) we characterize connections on a principal \(G\)-bundle \((E_G \to X_0, [X_1 \rightrightarrows X_0])\) over a given Lie groupoid \(X = [X_1 \rightrightarrows X_0]\) in terms of the Lie algebra of \(g\)-valued differential forms on the Lie groupoid. Theorem \(6.1\) in Section \(6\) presents the main result of this article, namely the existence of a well defined Chern-Weil map for a principal bundle \((E_G \to X_0, [X_1 \rightrightarrows X_0])\) over a Lie groupoid \(X = [X_1 \rightrightarrows X_0]\) and its independence from the choice of a connection on \((E_G \to X_0, [X_1 \rightrightarrows X_0])\).

2. Principal bundles over Lie groupoids

In this section, we will recall the notion and basic properties of a principal bundle over a Lie groupoid. We refer also to \([MM]\), \([BX]\) \([He]\) and \([LGTX2]\) for some of the material presented here.

All the manifolds considered here will be second countable, but not necessarily Hausdorff spaces.

Let \(\text{Man} \) be the category of smooth manifolds. A Lie groupoid is a groupoid object in \(\text{Man}\), such that all maps are smooth and both the source and target maps are submersions.

For a Lie groupoid \(X = [X_1 \rightrightarrows X_0]\), \(X_0\) and \(X_1\) are the set of objects and the set of morphisms respectively.

**Definition 2.1** (Action of a Lie groupoid on a manifold). Let \(X = [X_1 \rightrightarrows X_0]\) be a Lie groupoid. Let \(P\) be a smooth manifold. A left action of the Lie groupoid \(X = [X_1 \rightrightarrows X_0]\) on the manifold \(P\) is given by a pair of smooth maps \(a: P \to X_0\) and \(\mu: X_1 \times_{s,X_0,\pi} P \to P\) satisfying the following conditions:

1. \(1_a(p) \cdot p = p\) for all \(p \in P\),
2. \(a(\gamma \cdot p) = t(\gamma)\) for all \((\gamma, p) \in X_1 \times_{s,X_0,a} P\), and
3. \(\gamma' \cdot (\gamma \cdot p) = (\gamma' \circ \gamma) \cdot p\) for all \((\gamma', \gamma, p) \in X_1 \times_{s,X_0,\pi} X_1 \times_{s,X_0,\pi} P\).

A right action of a Lie groupoid on a manifold is defined likewise, with the action map given by

\[ \mu: P \times_{a,X_0,t} X_1 \to P \] (2.1)

**Example 2.2.** Consider the Lie groupoid \([G \rightrightarrows \ast]\) with a singleton object set for a Lie group \(G\). The action of \([G \rightrightarrows \ast]\) on a smooth manifold \(P\) is the same as an action of the Lie group \(G\) on \(P\).

**Example 2.3.** Given a smooth manifold \(M\), we have a Lie groupoid \([M \rightrightarrows M]\). An action of \([M \rightrightarrows M]\) on a smooth manifold \(P\) is the same as a smooth map \(\pi: P \to M\).

**Example 2.4.** Suppose a Lie group \(G\) is acting on a smooth manifold \(M\). Then we have a Lie groupoid \([M \times G \rightrightarrows M]\), with source and target maps given by \((m, g) \mapsto m\) and \((m, g) \mapsto m \cdot g\) respectively. An action of \([M \times G \rightrightarrows M]\) on a manifold \(P\) is an action of the Lie group \(G\) on the manifold \(P\) together with a \(G\)-equivariant smooth map \(\pi: P \to M\).
Definition 2.5 (Principal G-bundle over a Lie groupoid). Let $G$ be a Lie group, and let $\mathbb{X} = [X_1 \rightrightarrows X_0]$ be a Lie groupoid. A (right) principal $G$-bundle over $\mathbb{X} = [X_1 \rightrightarrows X_0]$ is a (right) principal $G$-bundle $\pi : E_G \to X_0$ over the manifold $X_0$ together with a left action
\[(\pi : E_G \to X_0, \mu : X_1 \times_{s,X_0,\pi} E_G \to E_G),\]
of the Lie groupoid $\mathbb{X}$ on the manifold $E_G$, such that $(\gamma \cdot p) \cdot g = \gamma \cdot (p \cdot g)$ for all $(\gamma, p, g) \in X_1 \times_{X_0} E_G \times G$. Henceforth, we shall often write $\gamma \cdot p$ instead of $\mu(\gamma, p)$.

A principal $G$-bundle over $\mathbb{X}$ will be denoted as $(E_G \to X_0, [X_1 \rightrightarrows X_0])$. A morphism of principal $G$-bundles from $(E_G \to X_0, [X_1 \rightrightarrows X_0])$ to $(E'_G \to X_0, [X_1 \rightrightarrows X_0])$ is a morphism
\[\psi : (E_G, \pi, X_0) \to (E'_G, \pi', X_0)\]
of the underlying principal bundles over the manifold $X_0$, such that $\psi(\mu(\gamma, e)) = \mu'(\gamma, \psi(e))$ for all $(\gamma, e) \in X_1 \times_{s,X_0,a} E_G$.

Example 2.6. Let $G, H$ be a pair of Lie groups. A principal $G$-bundle over the Lie groupoid $[H \rightrightarrows \ast]$ is same as a left-action of $H$ on $G$ that commutes with the right-translation action of $G$ on itself, meaning that $h \cdot (gg') = (h \cdot g)g'$ for all $h \in H$ and $g, g' \in G$.

Example 2.7. Let $G$ be a Lie group. Let $[M \rightrightarrows M]$ be the Lie groupoid associated to a manifold $M$. Then, a principal $G$-bundle over $[M \rightrightarrows M]$ is the same as a principal $G$-bundle $\pi : P \to M$ over the manifold $M$.

Example 2.8. Let $G, H$ be a pair of Lie groups. Then an $H$-equivariant smooth principal $G$-bundle $P \to M$ over a smooth manifold $M$ defines a principal $G$-bundle over the Lie groupoid $[M \times H \rightrightarrows M]$.

Following the definition of principal $G$-bundles over a Lie groupoid in Definition 2.5, we define vector bundles over a Lie groupoid.

Definition 2.9 (Vector bundle over a Lie groupoid). Let $\mathbb{X} = [X_1 \rightrightarrows X_0]$ be a Lie groupoid. A (rank $k$) vector bundle over the Lie groupoid $\mathbb{X} = [X_1 \rightrightarrows X_0]$ is a (rank $k$) vector bundle $\pi : E \to X_0$ over the manifold $X_0$ together with a left action
\[(\pi : E \to X_0, \mu : X_1 \times_{s,X_0,\pi} E \to E)\]
of the Lie groupoid $\mathbb{X}$ on $E$ such that the map $\mu_\gamma : E_{s(\gamma)} \to E_{t(\gamma)}$, $a \mapsto \gamma \cdot a$, is linear for every $\gamma \in X_1$.

Morphisms of vector bundles over Lie groupoids are defined by imitating the definition of morphisms of principal bundles.

We also will use the following general notion of a principal bundle (compare also [BX] and [LGSX]).

Definition 2.10 (Principal $\mathbb{X}$-bundle over a manifold). Let $\mathbb{X} = [X_1 \rightrightarrows X_0]$ be a Lie groupoid. Let $M$ be a smooth manifold. A principal (right) $\mathbb{X}$-bundle over $M$ is a surjective submersion $\pi : P \to M$ together with a right action
\[(a : P \to X_0, \mu : P \times_{a,X_0,t} X_1 \to P)\]
of the Lie groupoid $\mathbb{X}$ on the manifold $P$, such that following two conditions are satisfied:
(1) \( \pi(p \cdot h) = \pi(p) \) for all \( (p, h) \in P \times_{X_0} X_1 \), and
(2) the map \( P \times_{a, X_0, t} X_1 \longrightarrow P \times_{\pi, M, \pi} P \) given by \( (p, h) \mapsto (p, p \cdot h) \) is a diffeomorphism.

We will now recall the definition of differentiable stacks, and also that of differentiable stacks associated to Lie groupoids. For that, we first need the following general definitions [LGSX, Vi].

**Definition 2.11.** Let \( C \) and \( D \) be categories and \( \pi : D \longrightarrow C \) be a functor. An arrow \( \theta : \xi \longrightarrow \eta \) in \( D \) is said to be a **Cartesian arrow in** \( D \) if for every morphism \( \theta' : \xi' \longrightarrow \eta \) in \( D \), and a morphism \( h : \pi_D(\xi') \longrightarrow \pi_D(\xi) \) in \( C \) with \( \pi(\theta) \circ h = \pi(\theta') \), there exists a unique morphism \( \Phi : \xi' \longrightarrow \xi \) in \( D \) such that \( \theta \circ \Phi = \theta' \) in \( D \) and \( \pi_D(\Phi) = h \) in \( C \). We visually represent a Cartesian arrow by the following diagram:

\[
\begin{array}{ccc}
\xi' & \xrightarrow{\theta'} & \pi(\xi') \\
\downarrow{\Phi} & & \downarrow{\pi(\Phi)} \\
\xi & \xrightarrow{\theta} & \pi(\xi)
\end{array}
\]

**Definition 2.12 (Fibered category).** A functor \( \pi_D : (D \longrightarrow C) \) is a **fibered category** over \( C \) if for every \( (\eta, f) \in D_0 \times_{C_0, t} C_1 \) there exists a Cartesian arrow \( \theta : \xi \longrightarrow \eta \) with \( \pi_D(\theta) = f \). We call such \( \xi \) to be the **pullback of** \( \eta \) **along** \( f \).

A morphism of fibered categories over \( C \) from \( (D, \pi_D, C) \) to \( (D', \pi_D', C) \) is a functor \( F : D \longrightarrow D' \) that maps a Cartesian arrow in \( D \) to a Cartesian arrow in \( D' \) satisfying \( \pi_{D'} \circ F = \pi_D \).

**Example 2.13.** Consider the category \( BX \) of principal \( X \)-bundles. The functor \( \pi_X : BX \longrightarrow \text{Man} \) that sends a principal \( X \)-bundle \( P \longrightarrow M \) to the manifold \( M \) is a fibered category.

Let \( \pi_D : D \longrightarrow C \), be a fibered category. To an object \( U \in C \) we associate a category \( D(U) \) with

\[
\text{Obj}(D(U)) = \{ \eta \in \text{Obj}(D) \mid \pi_D(\eta) = U \},
\]

\[
\text{Mor}_{D(U)}(\eta, \eta') = \{ f \in \text{Mor}(D) \mid \pi_D(f) = 1_U \}.
\]

The category \( D(U) \) is called the **fiber of** \( U \). In this fashion actually we have a pseudo-functor associated to \( \pi_D : D \longrightarrow C \),

\[
D : C \text{op} \longrightarrow \text{Cat}
\]

that sends \( U \) to \( D(U) \). It should be clarified that by abuse of notation the pseudo-functor is being denoted by \( D \).

Let \( C \) be a category with a specified Grothendieck topology. We refer to such a category as a site. In particular, by the site \( \text{Man} \) we mean the category \( \text{Man} \) together with the étale topology.

For the ease of exposition we simplify our notation here: \( U_{ij} \) will denote \( U_i \times_U U_j \) and so on, while \( pr_i, pr_{ij} \) etc will denote various projection maps obtained from the pullback
diagrams; for an arrow $f : U \to V$ in $\mathcal{C}$ the functor $\mathcal{D}(f) : \mathcal{D}(V) \to \mathcal{D}(U)$ will be denoted by $f^*$.

Let $(\mathcal{D}, \pi_\mathcal{D}, \mathcal{C})$ be a fibered category over a site $\mathcal{C}$, and let $\{\sigma_i : U_i \to U\}$ be a covering of an object $U$ in $\mathcal{C}$. With $\{U_i \to U\}$ we associate the following descent category $\mathcal{D}(\{U_i \to U\})$. An object in $\mathcal{D}(\{U_i \to U\})$ is a family of pairs $(\{\xi_i\}, \{\phi_{ij}\})$, where each $\xi_i$ is an object of $\mathcal{D}(U_i)$ and each $\phi_{ij} : \mathcal{D}(\xi_i) \to \mathcal{D}(\xi_j)$ is an isomorphism in $\mathcal{D}(U_{ij})$ satisfying the co-cycle relation $\mathcal{D}(\phi_{ij}) = \mathcal{D}(\phi_{ik}) \circ \mathcal{D}(\phi_{kj})$ in $\mathcal{D}(U_{ijk})$. An arrow

$$(\{\xi_i\}, \{\phi_{ij}\}) \to (\{\eta_i\}, \{\psi_{ij}\})$$

is a collection $\{\alpha_i : \xi_i \to \eta_i\}$ satisfying $\psi_{ij} \circ \alpha_{ij} = \alpha_{ij} \circ \phi_{ij}$ for every pair of indices $i, j$.

Moreover we have a functor

$$\mathcal{D}(U) \to \mathcal{D}(\{U_i \to U\}) \tag{2.3}$$

that sends an object $\xi$ in $\mathcal{D}(U)$ to the object $\{\sigma_i^*(\xi), \phi_{ij}\}$, where

$$\phi_{ij} : \mathcal{D}(\sigma_i^*(\xi)) \to \mathcal{D}(\sigma_j^*(\xi))$$

are the isomorphisms given by the universal property of a pullback.

**Definition 2.14** (Stack). Let $\mathcal{C}$ be a site. A fibered category $(\mathcal{D}, \pi_\mathcal{D}, \mathcal{C})$ is said to be a stack over the site $\mathcal{C}$ if for each object $U$ of $\mathcal{C}$ and a covering $\{U_i \to U\}$ of $U$, the functor defined in (2.3) is an equivalence of categories. A morphism of stacks is a morphism of the underline fibered categories.

We are particularly interested in categories fibered in groupoids; that is, a fibered category $\mathcal{D} \to \mathcal{C}$ such that each fiber $\mathcal{D}(U)$ is a groupoid. Similarly, for stacks, we assume the underline fibered category to be category fibered in groupoids.

The following standard and useful example will be referred to time and again.

**Example 2.15.** Let $X = [X_1 \to X_0]$ be a Lie groupoid. Then the fibered category $\mathcal{B}X \to \text{Man}$ in Example 2.13 is a stack over the site Man with respect to the étale topology. This is also known as the classifying stack associated to the Lie groupoid $X = [X_1 \to X_0]$.

**Definition 2.16** (Differentiable stack). Let Man be the site with étale topology. A stack $\mathcal{D} \to \text{Man}$ is called a differentiable stack if there exists a Lie groupoid $X$ such that there is an isomorphism of stacks $\mathcal{B}X \cong \mathcal{D}$.

It can be shown that the classifying stack $\mathcal{B}X$ is a differentiable stack (see [BX], [He]).

3. **Integrable connections and de Rham cohomology**

In this section, we recall the notion of an integrable connection $\mathcal{H}$ on a Lie groupoid $X = [X_1 \to X_0]$ and introduce the de Rham cohomology ring of the pair $(X, \mathcal{H})$. We begin by recalling the notion of the Atiyah sequence in the classical set-up.

Let $M$ be a smooth manifold and $\pi : P \to M$ be a principal $G$-bundle. The Lie algebra of $G$ will be denoted by $\mathfrak{g}$. Consider the action of $G$ on $P \times \mathfrak{g}$ given by the adjoint
action of $G$ on $\mathfrak{g}$. For the corresponding diagonal action of $G$ on $P \times \mathfrak{g}$, the quotient

$$\text{ad}(P) := (P \times \mathfrak{g})/G \rightarrow P/G = M$$

is a vector bundle over $M$; it is known as the adjoint bundle of $P$. On the other hand, the action of $G$ on $P$ induces an action of $G$ on the tangent bundle $TP$ of $P$. The quotient

$$\text{At}(P) := (TP)/G \rightarrow P/G = M$$

is a vector bundle, which is known as the Atiyah bundle (see [At]). The action of $G$ on $P$ identifies the trivial bundle $P \times \mathfrak{g} \rightarrow P$ with the relative tangent bundle for the projection $\pi$, and hence $\text{ad}(P)$ is a subbundle of $\text{At}(P)$. Consequently, we obtain a short exact sequence of vector bundles on $M$

$$0 \rightarrow \text{ad}(P) \xrightarrow{j} \text{At}(P) \xrightarrow{d\pi} TM \rightarrow 0 \quad (3.1)$$

which is called the Atiyah exact sequence associated to the principal $G$-bundle $P$ (see [At]). Note that that projection $d\pi$ in (3.1) is the quotient by $G$ of the differential $d\pi : TP \rightarrow \pi^*TM$ of the map $\pi$.

A connection on the above principal $G$-bundle $P$ is then a splitting of the Atiyah sequence. In other words, a connection on $P$ is a homomorphism

$$D : TM \rightarrow \text{At}(P)$$

such that $(d\pi) \circ D = \text{Id}_{TM}$, where $d\pi$ is the homomorphism in (3.1).

Now consider a principal $G$-bundle $(E_G \rightarrow X_0, [X_1 \rightrightarrows X_0])$ over a Lie groupoid $[X_1 \rightrightarrows X_0]$. The underlying (classical) principal $G$-bundle $E_G \rightarrow X_0$ yields the Atiyah sequence of vector bundles over $X_0$,

$$0 \rightarrow \text{ad}(E_G) \xrightarrow{j} \text{At}(E_G) \xrightarrow{d\pi} TX_0 \rightarrow 0 \quad (3.2)$$

(see (3.1)). Our aim is to interpret (3.2) as an exact sequence of vector bundles over the Lie groupoid $\mathcal{X} = [X_1 \rightrightarrows X_0]$. For that we need an action of the Lie groupoid $\mathcal{X}$ on the tangent space manifold $TX_0$. We already have the tangent bundle projection map $TX_0 \rightarrow X_0$. It remains to provide the smooth map

$$\mu : X_1 \times_{X_0} TX_0 \rightarrow TX_0 \quad (3.3)$$

satisfying the conditions in Definition 2.1. As $TX_0 \rightarrow X_0$ is surjective, (by condition (2) in Definition 2.1) the map $\mu$ in (3.3) should produce a linear map $T_{s(\gamma)}X_0 \rightarrow T_{t(\gamma)}X_0$ for every $\gamma \in X_1$. Precisely for this purpose we introduce below the notion of a connection on the Lie groupoid.

Let $\mathcal{X} = [X_1 \rightrightarrows X_0]$ be a Lie groupoid. Let $\mathcal{H} \subset TX_1$ be a distribution on the manifold $X_1$ which is a complement of the kernel of the differential $ds : TX_1 \rightarrow TX_0$ of the source map $s$; so

$$\mathcal{H}_\gamma \oplus \ker(ds)_\gamma = T_{t(\gamma)}X_1 \quad (3.4)$$

for every $\gamma \in X_1$. Let

$$P_{\mathcal{H}_\gamma} : T_{t(\gamma)}X_1 \rightarrow \mathcal{H}_\gamma \subset T_{t(\gamma)}X_1 \quad (3.5)$$

be the projection constructed using the decomposition in (3.4). For any $v \in T_{t(\gamma)}X_1$,

$$P_{\mathcal{H}_\gamma}(v) \in \mathcal{H}_\gamma$$

is called the horizontal component of $v$. 

Since $s$ is a submersion, the decomposition in (3.4) yields an isomorphism $\mathcal{H}_\gamma \sim \sim T_{s(\gamma)}X_0$. In fact, we have an isomorphism of vector bundles

$$\mathcal{H} \sim \sim s^*TX_0$$

over $X_1$. The following compositions of homomorphisms

$$s^*TX_0 \sim \sim \mathcal{H} \hookrightarrow TX_1 \xrightarrow{dt} t^*TX_0$$

(3.6)

over $X_1$ will be denoted by $\theta$, where $dt : TX_1 \rightarrow t^*TX_0$ is the differential of the map $t$.

We now recall the definition of a connection on Lie groupoids from [BN, Def. 3.1] (see also [BCKN, 4.1.])

**Definition 3.1** (Connection on a Lie groupoid). A connection on a Lie groupoid $X = [X_1 \rightrightarrows X_0]$ is a distribution $\mathcal{H} \subset TX_1$ satisfying the following three conditions:

1. $\mathcal{H}$ is a complement of $ds$,
2. $(de)_x(T_xX_0) = \mathcal{H}_{e(x)}X_0$ for all $x \in X_0$, and
3. $\theta_\gamma \circ \theta_{\gamma'} = \theta_{\gamma \circ \gamma'}$ for every composable pair $\gamma, \gamma' \in X_1$, where $\theta$ is the homomorphism in (3.6).

A connection $\mathcal{H} \subset TX_1$ on $X = [X_1 \rightrightarrows X_0]$ is said to be integrable (or flat) if the distribution $\mathcal{H}$ is integrable.

Let $\mathcal{H} \subset TX_1$ be a connection on the Lie groupoid $[X_1 \rightrightarrows X_0]$. Consider a principal $G$-bundle $(E_G \rightarrow X_0, [X_1 \rightrightarrows X_0])$ over $[X_1 \rightrightarrows X_0]$. Then

$$\text{pr}_1 : s^*E_G = X_1 \times_s E_G \rightarrow X_1$$

is a principal $G$-bundle over $X_1$. Then we have the associated Lie groupoid $[s^*E_G \rightrightarrows E_G]$ whose source map $s^*E_G \rightarrow E_G$ is the second projection $(a, \gamma) \mapsto a$ and the target map $s^*E_G \rightarrow E_G$ is the action $(a, \gamma) \mapsto a \cdot \gamma$ (the map $\mu$ in Definition 2.5). Let

$$d\text{pr}_1 : Ts^*E_G \rightarrow TX_1$$

(3.7)

be the differential of the projection $\text{pr}_1 : s^*E_G \rightarrow X_1, (a, \gamma) \mapsto \gamma$.

The following lemma is evident.

**Lemma 3.2.** Let $\mathcal{H} \subset TX_1$ be a connection on the Lie groupoid $X = [X_1 \rightrightarrows X_0]$. Then

$$\widetilde{\mathcal{H}} := (d\text{pr}_1)^{-1}(\mathcal{H}) \subset Ts^*E_G$$

is a connection on the Lie groupoid $[s^*E_G \rightrightarrows E_G]$, where $d\text{pr}_1$ is the map in (3.7). If the connection $\mathcal{H}$ is integrable, then so is $\widetilde{\mathcal{H}}$.

The connection $\widetilde{\mathcal{H}}$ in Lemma 3.2 will be called the pullback of $\mathcal{H}$. For the purpose of future reference we note here the map $\widetilde{\theta}$ (see (3.6)) given by the following compositions of homomorphisms

$$\text{pr}_2^*TE_G \sim \sim \widetilde{\mathcal{H}} \hookrightarrow Ts^*E_G \xrightarrow{d\mu} \mu^*TE_G,$$

(3.8)

for any $v \in T_{\gamma}X_1, w \in T_aE_G$ satisfying $ds(\gamma, v) = d\pi(a, w)$; in other words for any $(v, w) \in T_{(\gamma, a)}s^*E_G$. It is immediate that the map is well defined, since $P_{\mathcal{H}, \gamma}$ is trivial on $\ker(ds)$. 

Example 3.3. Let \([M \rightrightarrows M]\) be the Lie groupoid associated to a smooth manifold \(M\). Then \(\ker(ds) = \{0\}\) and the map \(m \mapsto \mathcal{H}_m : T_m M, m \in M\), defines an integrable connection \(\mathcal{H} \subset TM\) on \([M \rightrightarrows M]\).

Example 3.4. Let \(M\) be a smooth manifold equipped with an action of a Lie group \(G\). Let \([M \times G \rightrightarrows M]\) be the associated action Lie groupoid. As the source map \(s : M \times G \longrightarrow M\) is the projection map, the differential \((ds)_{(m, g)} : T_{(m, g)}(M \times G) \longrightarrow T_m M\) is given by \((v, A) \mapsto v\) for all \(v \in T_m M, A \in T_g G\). Thus,

\[
\ker((ds)_{(m, g)}) = \{(v, A) \in T_m M \times T_g G \mid v = 0\} = T_g G
\]

for each \((m, g) \in M \times G\). Now defining \(\mathcal{H}_{(m, g)}(M \times G) = T_m M \subset T_{(m, g)}(M \times G)\), we obtain a connection \(\mathcal{H} \subset T(M \times G)\) on the action groupoid \([M \times G \rightrightarrows M]\). This connection is clearly integrable.

Example 3.5. Let \(X = [X_1 \rightrightarrows X_0]\) be an étale Lie groupoid, meaning the map \(s : X_1 \longrightarrow X_0\) is a local diffeomorphism. Then the differential \(ds : TX_1 \longrightarrow s^*TX_0\) is an isomorphism. Then the distribution \(\mathcal{H} = TX_1\) is a connection on the Lie groupoid \(X\). This connection is clearly integrable.

Example 3.6. Let \(\pi : E \longrightarrow M\) be a finite rank vector bundle over the manifold \(M\). We can view \([E \rightrightarrows M]\) as a groupoid with both the source and target maps both being \(\pi\) (thus, a pair of composable morphisms will belong to the same fiber), and composition is simply the addition of vectors on the corresponding fiber. Then a connection on the vector bundle \(\pi : E \longrightarrow M\) defines a connection on \([E \rightrightarrows M]\) by smoothly splitting \(E\) into horizontal component complementing \(\ker d\pi\). Obviously, if one connection is integrable, then so is the other.

Example 3.7. Let \(\pi : P \longrightarrow M\) be a principal \(G\)-bundle over a manifold \(M\). Consider the groupoid \([P \times P \rightrightarrows P]\) with source and target maps as first and second projections respectively and the obvious multiplication. The Lie group \(G\) acts on \(P \times P\) by \(g \cdot (p_1, p_2) = (g \cdot p_1, g \cdot p_2)\). Under the quotient by the action of \(G\), the groupoid \([P \times P \rightrightarrows P]\) descends to a Lie groupoid \(P_{\text{Gauge}} \rightrightarrows P\). This groupoid is often called the gauge groupoid or Atiyah groupoid in the literature and it plays an important role in gauge theory and the theory of Lie algebroids (see [RLF] [KoS]). Let \(A\) be a connection on the principal bundle \(\pi : P \longrightarrow M\). The horizontal distribution \(\mathcal{H} \subset TP\) defined by the connection \(A\) induces a connection on the Lie groupoid \(P_{\text{Gauge}}\) as follows. Define

\[
\mathcal{H}_{[p, q]} := \frac{\mathcal{H}_p \oplus T_q P}{T_{p, q}^G \cdot (p, q)} \subset \frac{P \times P}{G},
\]

where \(G \cdot (p, q) \subset P \times P\) is the orbit of the element \((p, q) \in P \times P\). Now since \(\mathcal{H}\) complements \(\ker d\pi\) and it is \(G\) invariant, \(\mathcal{H}_{[p, q]}\) defines a connection on \(P_{\text{Gauge}}\). If the connection \(A\) is integrable, then so is the induced connection on the gauge groupoid \(P_{\text{Gauge}}\).

We refer to [B] Section 2.3 for other interesting examples of connections on Lie groupoids.

3.1. Differential forms and de Rham cohomology of a pair \((X, \mathcal{H})\). Let us fix a pair \((X, \mathcal{H})\) given by a Lie groupoid \(X = [X_1 \rightrightarrows X_0]\) together with an existing connection \(\mathcal{H} \subset TX_1\) on \(X\). We recall from [BN] the notion of a differential \(k\)-form on \(X\).
Definition 3.8 (Horizontal component of a differential form). Let \( \varphi : X_1 \longrightarrow \bigwedge^k T^*X_1 \) be a differential \( k \)-form on the manifold \( X_1 \). The horizontal component of \( \varphi \) (with respect to the connection \( \mathcal{H} \)) is the differential \( k \)-form

\[
\hat{\mathcal{H}}(\varphi)(\gamma)(v_1, \ldots, v_k) = \varphi(\gamma)(P_{\mathcal{H}_\gamma}(v_1), \ldots, P_{\mathcal{H}_\gamma}(v_k))
\]

for all \( \gamma \in X_1 \) and \( v_i \in T_\gamma X_1 \), \( 1 \leq i \leq k \), where \( P_{\mathcal{H}_\gamma} \) is the projection in (3.5).

Definition 3.9 (Differential forms on a Lie groupoid). A differential \( k \)-form \( \varphi : X_0 \longrightarrow \bigwedge^k T^*X_0 \) on the manifold \( X_0 \) is said to be a differential \( k \)-form on \( \mathcal{X} \) (with respect to the connection \( \mathcal{H} \)) if

\[
\hat{\mathcal{H}}(s^*\varphi) = \hat{\mathcal{H}}(t^*\varphi),
\]

where \( \hat{\mathcal{H}} \) is constructed as in Definition 3.8.

The following lemma, which was stated in [BN] without proof, shows that differential forms on a Lie groupoid with connection are closed with respect to exterior derivation under the assumption that the connection on the Lie groupoid is integrable.

Lemma 3.10. Let \( \mathcal{H} \subset TX_1 \) be an integrable connection on a Lie groupoid \( \mathcal{X} = [X_1 \rightrightarrows X_0] \). Let \( \varphi : X_0 \longrightarrow \bigwedge^k T^*X_0 \) be a differential \( k \)-form on \( \mathcal{X} \). Then \( d\varphi : X_0 \longrightarrow \bigwedge^{k+1} T^*X_0 \) is a differential \((k+1)\)-form on \( \mathcal{X} \).

Proof. It is enough to prove the case \( k = 1 \). Indeed, for an arbitrary \( k \), the proof runs almost verbatim.

Let \( \varphi \) be a differential 1-form on \( \mathcal{X} = [X_1 \rightrightarrows X_0] \). That means that we have

\[
(s^*\varphi)(\gamma)(P_{\mathcal{H}_\gamma}(v)) = (t^*\varphi)(\gamma)(P_{\mathcal{H}_\gamma}(v))
\]

for all \( \gamma \in X_1 \) and \( v \in T_\gamma X_1 \). This implies that

\[
(s^*\varphi)(\gamma)(v) = (t^*\varphi)(\gamma)(v)
\]

(3.9)

for all \( \gamma \in X_1 \) and \( v \in \mathcal{H}_\gamma \).

To prove the lemma we need to show that

\[
d(s^*\varphi)(\gamma)(P_{\mathcal{H}_\gamma}(v_1), P_{\mathcal{H}_\gamma}(v_2)) = d(t^*\varphi)(\gamma)(P_{\mathcal{H}_\gamma}(v_1), P_{\mathcal{H}_\gamma}(v_2))
\]

(3.10)

for all \( \gamma \in X_1 \) and \( v_1, v_2 \in T_\gamma X_1 \).

Let \( v_1, v_2 \in \mathcal{H}_\gamma \). Let \( Z_1 \) (respectively, \( Z_2 \)) be a section of \( \mathcal{H} \) defined around \( \gamma \) such that \( Z_1(\gamma) = v_1 \) (respectively, \( Z_2(\gamma) = v_2 \)). We have

\[
d(s^*\varphi)(\gamma)(Z_1, Z_2) = Z_1((s^*\varphi)(Z_2)) - Z_2((s^*\varphi)(Z_1)) - (s^*\varphi)([Z_1, Z_2])
\]

\[
d(t^*\varphi)(\gamma)(Z_1, Z_2) = Z_1((t^*\varphi)(Z_2)) - Z_2((t^*\varphi)(Z_1)) - (t^*\varphi)([Z_1, Z_2])
\]

Now, since \([Z_1, Z_2]\) is a section of \( \mathcal{H} \), using this and (3.9) we conclude that (3.10) holds. \( \square \)

Now let us introduce the following concept.

Definition 3.11. Let \( \mathcal{X} = [X_1 \rightrightarrows X_0], \mathcal{H}^\mathcal{X} \) and \( \mathcal{Y} = [Y_1 \rightrightarrows Y_0], \mathcal{H}^\mathcal{Y} \) be a pair of Lie groupoids equipped with connection. A morphism of Lie groupoids \((F, f) : [X_1 \rightrightarrows X_0] \longrightarrow [Y_1 \rightrightarrows Y_0]\) will be called a morphism of Lie groupoids with connections if

\[
dF(\mathcal{H}^\mathcal{X}) = \mathcal{H}^\mathcal{Y},
\]
where $dF$ is the differential of the map $F$. We shall employ the notation

$$(F, f) : ([X_1 \Rightarrow X_0], \mathcal{H}^X) \longrightarrow ([Y_1 \Rightarrow Y_0], \mathcal{H}^Y)$$

for such a morphism.

The following lemma follows straight-forward from the definition.

**Lemma 3.12.** Let $(F, f) : (X = [X_1 \Rightarrow X_0], \mathcal{H}^X) \longrightarrow (Y = [Y_1 \Rightarrow Y_0], \mathcal{H}^Y)$ be a morphism of Lie groupoids with connection. Let $\varphi : Y_0 \longrightarrow \wedge^k T^*Y_0$ be a differential form on the Lie groupoid $Y$. Then $f^*\varphi : X_0 \longrightarrow \wedge^k T^*X_0$ is a differential form on the Lie groupoid $X$.

When the morphism $F$ in Lemma 3.12 is a surjective submersion, the following converse also holds.

**Lemma 3.13.** Let $(F, f) : (X = [X_1 \Rightarrow X_0], \mathcal{H}^X) \longrightarrow (Y = [Y_1 \Rightarrow Y_0], \mathcal{H}^Y)$ be a morphism of Lie groupoids with connection such that $F$ is a surjective submersion. Let $\varphi$ be a differential $k$-form on $Y_0$ such that $f^*\varphi$ is a differential $k$-form on the Lie groupoid $X$. Then $\varphi$ is a differential form on the Lie groupoid $Y$.

**Proof.** Since $F$ is a surjective submersion, for any two differential $k$-forms $\omega_1$ and $\omega_2$ on $Y_1$, if $F^*\omega_1 = F^*\omega_2$, then $\omega_1 = \omega_2$. The lemma now follows from this. \qed

Let $X = [X_1 \Rightarrow X_0]$ now be a Lie groupoid equipped with an integrable connection $\mathcal{H} \subset T^*X_1$. Let $\Omega^k(X, \mathcal{H})$ denote the vector space of differential $k$-forms on the Lie groupoid $X$, with respect to $\mathcal{H}$. In view of Lemma 3.10 we have a cochain complex

$$\cdots \longrightarrow \Omega^{k-1}(X, \mathcal{H}) \longrightarrow \Omega^k(X, \mathcal{H}) \longrightarrow \Omega^{k+1}(X, \mathcal{H}) \longrightarrow \cdots.$$ 

The $k$th-cohomology group of this cochain complex, denoted by $H^k_{dR}(X, \mathcal{H})$, is called the $k$th-de Rham cohomology of the pair $(X, \mathcal{H})$ and we have the de Rham cohomology ring of the pair $(X, \mathcal{H})$ by setting

$$H^*_d(X, \mathcal{H}) := \bigoplus_{k=0}^{\dim X_0} H^k_{dR}(X, \mathcal{H}).$$

(3.11)

Also, there is a natural homomorphism $H^*_d(X, \mathcal{H}) \longrightarrow H^*_d(X_0, \mathbb{R})$ which is, in general, neither injective nor surjective.

**Remark 3.14.** It may be noted that differential forms are defined on any Lie groupoid equipped with a connection. However, the connection needs to be integrable in order to be able to define the exterior derivative. An alternative definition of the de Rham cohomology of a Lie groupoid was introduced by Laurent-Gengoux, Tu and Xu, [LGTX2], by using the simplicial manifold given by the nerve associated to the Lie groupoid (compare also [EN]).

**Example 3.15.** Let $M$ be a smooth manifold and $[M \rightrightarrows M]$ be the associated Lie groupoid. Fix the connection $\mathcal{H}_m M = T_m M$ as in Example 33. Then the $k$th de Rham cohomology group $H^k_{dR}(M)$ of $M$ coincides with the $k$-th de Rham cohomology group of the pair $([M \rightrightarrows M], \mathcal{H})$. 

Also Lemma 3.10 implies that \( df \) is \( G \)-invariant. Thus, Proposition 3.18. Let \( \pi: P \to M \) be a principal bundle and \( A \) a connection on it. Consider the Lie groupoid \( \mathbb{P}_{\text{Gauge}} = [P \times 
abla G \to M] \) and the connection \( \mathcal{H} \) on it, as described in Example 3.17. It is not difficult to verify that an \( \alpha \in \Omega^k(M), 0 \leq k \), is a differential form on \( \mathbb{P}_{\text{Gauge}} \) if it satisfies the condition \( \pi^*\alpha(p) = \pi^*\alpha(q) \) for all \( p, q \in P \). In particular if \( \alpha \in \Omega^0(M) \), the condition says that \( \pi^*\alpha = \alpha \circ \pi \) is a constant function (and hence by surjectivity of \( \pi, \alpha \) as well). For \( k > 0 \) we have \( \Omega^k(\mathbb{P}_{\text{Gauge}}, \mathcal{H}) = \{ \alpha \in \Omega^k(M) \mid \pi^*\alpha = 0 \} \). In turn when the connection is integrable, we get

\[
H^0_{\text{dr}}(\mathbb{P}_{\text{Gauge}}, \mathcal{H}) = \mathbb{R},
\]

\[
H^1_{\text{dr}}(\mathbb{P}_{\text{Gauge}}, \mathcal{H}) = \{ \alpha \in \Omega^1(M) \mid \pi^*\alpha = 0, \ d\alpha = 0 \}
\]

and so on.

We end this section with a result regarding the cohomology ring homomorphism associated with a morphism of Lie groupoids with connection. The following proposition is deduced using Lemma 3.10.

**Proposition 3.18.** Let \( \mathcal{H}^X \) and \( \mathcal{H}^Y \) be integrable connections on the Lie groupoids \( X = [X_1 \rightrightarrows X_0] \) and \( Y = [Y_1 \rightrightarrows Y_0] \) respectively. Then a morphism of Lie groupoids with connections

\[
(F, f): (X = [X_1 \rightrightarrows X_0], \mathcal{H}^X) \to (Y = [Y_1 \rightrightarrows Y_0], \mathcal{H}^Y)
\]

induces a homomorphism of de Rham cohomology groups

\[
H^*_\text{dr}(Y, \mathcal{H}^Y) \to H^*_\text{dr}(X, \mathcal{H}^X)
\]

that sends any \( [\alpha] \) to \( [f^*\alpha] \). These maps produce a morphism of graded \( \mathbb{R} \)-algebras

\[
H^*_\text{dr}(Y, \mathcal{H}^Y) \to H^*_\text{dr}(X, \mathcal{H}^X).
\]

**Proof.** Let \( [\alpha] \in H^k_{\text{dr}}(Y, \mathcal{H}^Y) \). In particular, that means that \( \alpha \in \Omega^k(Y) \) and \( 0 = d\alpha \in \Omega^{k+1}(Y) \). Then, using Lemma 3.12 we have that \( f^*\alpha \in \Omega^k(X) \) and \( 0 = df^*\alpha \in \Omega^{k+1}(X) \). Also Lemma 3.10 implies that \( df^*\theta \in \Omega^k(Y) \) for any \( \theta \in \Omega^{k-1}(X) \). Thus we have

\[
[f^*\alpha] = [f^*(\alpha + d\theta)].
\]

4. **Connections on principal bundles over Lie groupoids**

Let \( X = [X_1 \rightrightarrows X_0] \) be a Lie groupoid and \((E_G \to X_0, [X_1 \rightrightarrows X_0])\) a principal \( G \)-bundle over \( X \). Let

\[
0 \to \text{ad}(E_G) \to \text{At}(E_G) \to TX_0 \to 0
\] (4.1)
be the associated Atiyah exact sequence corresponding to the principal $G$-bundle $E_G$ (see (3.1)). In this section we will interpret (4.1) as a short exact sequence of vector bundles over $X$. This will facilitate the definition of a connection on $(E_G \rightarrow X_0, [X_1 \rightrightarrows X_0])$.

First we explain the action of the Lie groupoid $X$ on $TX_0$, $\text{At}(E_G) = (TE_G)/G$ and $\text{ad}(E_G) = (E_G \times \mathfrak{g})/G$. For this fix a connection $\mathcal{H} \subset TX_1$ on $X$.

4.1. Action of $X$ on $TX_0$. Consider the homomorphism $\theta$ in (3.6). Using it define the map

$$\mu_{\text{tan}} : X_1 \times_{X_0} TX_0 \rightarrow TX_0$$

$$(\gamma, v) \mapsto (t(\gamma), \theta_\gamma(v)).$$

(4.2)

Let $\tau : TX_0 \rightarrow X_0$ be the natural projection. Then the pair $$(\tau : TX_0 \rightarrow X_0, \mu_{\text{tan}})$$ defines an action of $X$ on $TX_0$ (see Definition 2.1).

4.2. Action of $X$ on $\text{At}(E_G)$. As seen in Lemma 3.2 the connection $\mathcal{H}$ induces a connection $\tilde{\mathcal{H}} := (\text{pr}_1)^{-1}(\mathcal{H}) \subset Ts^*E_G$ on $[s^*E_G \rightrightarrows E_G]$. Substituting $([s^*E_G \rightrightarrows E_G], \tilde{\mathcal{H}})$ in place of $(X, \mathcal{H})$ in (4.2), we get a map

$$\tilde{\mu} : s^*E_G \times_{E_G} TE_G \rightarrow TE_G.$$ Taking the quotient with respect to the action of $G$, we get a map

$$\mu_{\text{at}} : X_1 \times_{X_0} (TE_G)/G = X_1 \times_{X_0} \text{At}(E_G) \rightarrow (TE_G)/G = \text{At}(E_G).$$

(4.3)

Now $\mu_{\text{at}}$ and the natural projection $\pi_{\text{at}} : \text{At}(E_G) \rightarrow X_0$ together define an action of $X$ on $\text{At}(E_G)$.

4.3. Action of $X$ on $\text{ad}(E_G)$. Consider the restriction of the map $\tilde{\mu}$ in Section 4.2 to $s^*E_G \times_{E_G} \ker(d\pi) \subset s^*E_G \times_{E_G} TE_G$, where $d\pi$ is the differential of the natural projection $\pi : E_G \rightarrow X_0$. The image of this restricted map is clearly $\ker(d\pi) \subset TE_G$. We recall that $\text{ad}(E_G) = \ker(d\pi)/G \subset (TE_G)/G = \text{At}(E_G)$. Consequently, the action $(\mu_{\text{at}}, \pi_{\text{at}})$ of $X$ on $\text{At}(E_G)$ in Section 4.2 produces an action of $X$ on $\text{ad}(E_G)$.

Then the exactness of the Atiyah sequence in (4.1) implies the following:

Proposition 4.1. Let $\mathcal{H}$ be a connection on the Lie groupoid $X = [X_1 \rightrightarrows X_0]$. Then $TX_0$, $\text{At}(E_G)$ and $\text{ad}(E_G)$ are vector bundles over $X$. Moreover, the homomorphisms in the short exact sequence in (4.1) are compatible with the actions of $X$ on $TX_0$, $\text{At}(E_G)$ and $\text{ad}(E_G)$.

Now we define connections on a principal bundle over a given Lie groupoid.

Definition 4.2 (Connections on principal bundles over Lie groupoids). Let $\mathcal{H} \subset TX_1$ be a connection on the Lie groupoid $X = [X_1 \rightrightarrows X_0]$. Let $(E_G \rightarrow X_0, X)$ be a principal $G$-bundle over $X$. A connection on $(E_G \rightarrow X_0, X)$ is a splitting of the Atiyah exact sequence in (4.1) of vector bundles over $X$. 
Remark 4.3. Note that a connection on \((E_G \to X_0, \mathbb{X})\) automatically gives a connection on the underlying principal \(G\)-bundle \(E_G \to X_0\); however, the converse is not true in general.

5. Differential forms associated to a connection

Let \(G\) be a Lie group and \(\mathfrak{g}\) its Lie algebra. The vector space of \(\mathfrak{g}\) valued differential \(k\)-forms on a manifold \(Y\) will be denoted as \(\Omega^k(Y, \mathfrak{g})\).

Let \(P\) be a principal \(G\)-bundle over a manifold \(M\). Then, there is a bijective correspondence between the following two sets:

1. the set of splittings of the Atiyah sequence for \(P\) (see (3.1)).
2. the set of \(\mathfrak{g}\)-valued 1-forms \(\omega\) on \(P\) satisfying the following conditions:
   - the map \(\omega : TP \to \mathfrak{g}\) is \(G\)-equivariant for the adjoint action of \(G\) on \(\mathfrak{g}\), and
   - the restriction of \(\omega\) to every fiber of the projection \(P \to M\) coincides with the Maurer-Cartan form for the action of \(G\) on the fiber.

Given a splitting homomorphism \(\rho : TM \to \text{At}(P)\) for the Atiyah sequence for \(P\), the corresponding \(\mathfrak{g}\)-valued 1-forms \(\omega\) on \(P\) is uniquely determined by the condition that the quotient by \(G\) of the kernel of \(\omega\) coincides with the image of \(\rho\). Conversely, given a form \(TP \to P \times \mathfrak{g}\) of the above type, after taking the quotient by the action of \(G\) we get a homomorphism \(\text{At}(P) \to \text{ad}(P)\). This homomorphism gives a splitting of the Atiyah sequence for \(P\).

The above bijective correspondence provides an alternative definition of a connection on \(P\). For more details on the above correspondence, and for the Atiyah sequence approach to connections on principal bundles, we refer to [Ma2, Appendix A].

It was observed in Proposition 4.1 that given a Lie groupoid \(\mathbb{X}\) equipped with a connection \(\mathcal{H}\), and a principal \(G\)-bundle \(E_G\) on \(\mathbb{X}\), there is an Atiyah exact sequence on \(\mathbb{X}\) whose splittings correspond to connections on \(E_G\). On the other hand, in Section 3.1 we developed the associated theory of differential forms on a Lie groupoid \(\mathbb{X}\) with connection \(\mathcal{H}\) and the corresponding de Rham cohomology for the pair \((\mathbb{X}, \mathcal{H})\). In this section, our aim is to describe a connection on a principal bundle \((E_G \to X_0, \mathbb{X})\) as a differential 1-form. Throughout this section, we will always assume the connection \(\mathcal{H}\) to be integrable.

Let \(\mathcal{D} : TX_0 \to \text{At}(E_G)\) be a connection on the principal \(G\)-bundle \((E_G \to X_0, \mathbb{X} = [X_1 \rightrightarrows X_0])\) given by a splitting of the Atiyah sequence of vector bundles

\[
0 \to \text{ad}(E_G) \xrightarrow{i} \text{At}(E_G) \xrightarrow{d\pi} TX_0 \to 0
\]

over \(\mathbb{X}\). Consider the connection on \(E_G \to X_0\) given by \(\mathcal{D}\) (see Remark 4.3). As observed at the beginning of this section, this gives a \(\mathfrak{g}\)-valued differential 1-form \(\omega\) on the manifold \(E_G\). In Lemma 5.2 we will see that this differential 1-form \(\omega \in \Omega^1(E_G, \mathfrak{g})\) on the manifold \(E_G\) is in fact a differential 1-form on the Lie groupoid \([s^*E_G \rightrightarrows E_G]\). For that, we first we make a note of the following property of differential forms on the Lie groupoid \([s^*E_G \rightrightarrows E_G]\) and \([X_1 \rightrightarrows X_0]\).

**Lemma 5.1.** Let \((E_G \xrightarrow{s} X_0, [X_1 \rightrightarrows X_0])\) be a principal \(G\)-bundle over a Lie groupoid \(\mathbb{X} = [X_1 \rightrightarrows X_0]\). Let \(\mathcal{H} \subset TX_1\) be a connection on \(\mathbb{X}\), and let \(\tilde{\mathcal{H}} \subset T(s^*E_G)\) be the pulled
back connection on the Lie groupoid $[s^*E_G \rightrightarrows E_G]$ (see Lemma 3.2). Let $\tau$ be a differential $k$-form on the manifold $X_0$, and let $\pi^*\tau$ be the pulled back form on the manifold $E_G$. Then, $\tau$ is a differential form on the Lie groupoid $\mathcal{X}$ if and only if $\pi^*\tau$ is a differential $k$-form on the Lie groupoid $[s^*E_G \rightrightarrows E_G]$.

Proof. Consider the morphism of Lie groupoids

$$(pr_1, \pi) : [s^*E_G \rightrightarrows E_G] \to \mathcal{X}.$$ 

As $\pi : E_G \to X_0$ is a surjective submersion, its pullback $pr_1 : s^*E_G \to X_1$ is a surjective submersion as well. Take any $(\gamma, a) \in s^*E_G$ and $(v, w) \in T_{(\gamma,a)}s^*E_G$. We have

$$(pr_1)_{\ast, (\gamma,a)}(P_{\tilde{H}_{(\gamma,a)}}(v, w)) = (pr_1)_{\ast, (\gamma,a)}(P_{H_{\gamma}}(v, w)) = P_{H_{\gamma}}(v) = P_{H_{\gamma,a}}((pr_1)_{\ast, (\gamma,a)}(v, w)).$$

Now the lemma follows immediately from Lemmas 3.12 and 3.13.

The following lemma classifies connections on the pair $(E_G \xrightarrow{\pi} X_0, \mathcal{X})$ in terms of $\mathfrak{g}$-valued differential 1-forms on the Lie groupoid $[s^*E_G \rightrightarrows E_G]$.

**Lemma 5.2.** Let $(E_G \xrightarrow{\pi} X_0, \mathcal{X})$ be a principal $G$-bundle over a Lie groupoid with integrable connection $(\mathcal{X} = [X_1 \rightrightarrows X_0], \mathcal{H})$. Let $\omega$ be a connection 1-form on the principal $G$-bundle $E_G \to X_0$. Let $\mathcal{D} : \operatorname{At}(E_G) \to \operatorname{ad}(E_G)$ be the corresponding homomorphism of vector bundles over $X_0$. Then $\mathcal{D}$ defines a connection on $(E_G \xrightarrow{\pi} X_0, \mathcal{X})$ if and only if $\omega$ is a $\mathfrak{g}$-valued 1-form on the Lie groupoid $[s^*E_G \rightrightarrows E_G]$.

Proof. Let $\overline{\mathcal{D}} : TE_G \to E_G \times \mathfrak{g}$ be the homomorphism of vector bundles given by the 1-form $\omega \in \Omega^1(E_G, \mathfrak{g})$. Then, $\overline{\mathcal{D}}$ descends to a homomorphism $\mathcal{D} : \operatorname{At}(E_G) \to \operatorname{ad}(E_G)$ of vector bundles over $X_0$.

Suppose that the above homomorphism $\mathcal{D}$ is a connection on the principal $G$-bundle $(E_G \xrightarrow{\pi} X_0, \mathcal{X})$. We have to show that $\omega$ is a differential form on $[s^*E_G \rightrightarrows E_G]$.

Since $\mathcal{D}$ is a connection on $(E_G \xrightarrow{\pi} X_0, \mathcal{X})$, we know that $\mathcal{D}$ is a morphism of vector bundles over the Lie groupoid $\mathcal{X}$, and hence $\mathcal{D}(\gamma \cdot [r]) = \gamma \cdot \mathcal{D}([r])$ for all $(\gamma, [r]) \in X_1 \times X_0$ $(TE_G)/G$; equivalently, the homomorphism $\overline{\mathcal{D}}$ satisfies the condition that $\overline{\mathcal{D}}(\gamma \cdot (a, v)) = \gamma \cdot \overline{\mathcal{D}}(a, v)$ for all $(\gamma, (a, v)) \in X_1 \times X_0 TE_G = s^*E_G$.

Let $\tilde{\theta}$ be the map in (3.6) corresponding to the pulled back connection $\tilde{H}$ on $s^*E_G \rightrightarrows E_G$. Recall that $\gamma \cdot (a, v) = \tilde{\theta}_{(\gamma,a)}(v)$ and $\overline{\mathcal{D}}((a, v)) = (a, \omega(a)(v))$ (see (3.8)). Thus we conclude that

$$(\gamma \cdot a, \omega(\gamma \cdot a)(\tilde{\theta}_{(\gamma,a)}(v))) = (\gamma \cdot a, \omega(a)(v))$$

$$\Rightarrow \omega(\gamma \cdot a)(\tilde{\theta}_{(\gamma,a)}(v)) = \omega(a)(v).$$
Consequently, for any \((v, r) \in T_{(\gamma, a)} s^*E_G\),
\[
(\hat{\mathcal{H}}(\mu^*\omega)(\gamma, a))(v, r) = \omega(\gamma a)(\Hat{\theta}_{(\gamma, a)}(v)) = \omega(a)(v) = (\hat{\mathcal{H}}(pr_2^*\omega)(\gamma, a))(v, r).
\]

The last equality establishes that \(\omega\) is a differential 1-form on the Lie groupoid \([s^*E_G \Rightarrow E_G]\).

To prove the converse, suppose that \(\omega \in \Omega^1(E_G, \mathfrak{g})\) is a connection 1-form on the underlying principal \(G\)-bundle \(E_G \rightarrow X_0\). Assume \(\omega\) is a \(\mathfrak{g}\)-valued 1-form on the Lie groupoid \([s^*E_G \Rightarrow E_G]\).

Let \(\mathcal{D} : \text{At}(E_G) \rightarrow \text{ad}(E_G)\) be the homomorphism of vector given by \(\omega\). To prove the converse it suffices to show that the morphism \(\mathcal{D}\) is a morphism of vector bundles over the Lie groupoid \(\mathbb{X} = [X_1 \Rightarrow X_0]\).

Since \(\omega\) is a differential 1-form on \([s^*E_G \Rightarrow E_G]\), we have,
\[
(\hat{\mathcal{H}}(\mu^*\omega)(\gamma, a))(v, r) = (\hat{\mathcal{H}}(pr_2^*\omega)(\gamma, a))(v, r)
\]
for \((\gamma, a) \in s^*E_G\) and \((v, r) \in T_{(\gamma, a)}(s^*E_G)\). Observe that the two equations
\[
(\hat{\mathcal{H}}(\mu^*\omega)(\gamma, a))(v, r) = \omega(\gamma a)(\Hat{\theta}_{(\gamma, a)}(v)), \quad (\hat{\mathcal{H}}(pr_2^*\omega)(\gamma, a))(v, r) = (\omega(a))(v)
\]
together imply that
\[
(\omega(\gamma a))(\Hat{\theta}_{(\gamma, a)}(v)) = (\omega(a))(v).
\]

After plugging the last equality into the equation
\[
\mathcal{D}(\gamma \cdot (a, v)) = (\gamma \cdot a, \omega(\gamma \cdot a)(\Hat{\theta}_{(\gamma, a)}(v))),
\]
and comparing with \(\gamma \cdot \mathcal{D}((a, v)) = (\gamma \cdot a, \omega(a)(v))\), we obtain that
\[
\{\mathcal{D}(\gamma \cdot r) = \gamma \cdot \mathcal{D}(r)\} \implies \{\mathcal{D}(\gamma \cdot [r]) = \gamma \cdot \mathcal{D}([r])\}.
\]
Thus, \(\mathcal{D}\) gives a connection on the principal \(G\)-bundle \((E_G \rightarrow X_0, \mathbb{X})\). This completes the proof.

We now recall the Lie bracket operation on the Lie algebra valued differential forms on a smooth manifold \(Y\). Let \(\mathfrak{g}\) be a Lie algebra. Let \(\omega\) and \(\eta\) be \(\mathfrak{g}\)-valued differential \(k\)-form and \(l\)-form respectively on \(Y\). Then, the Lie bracket \([\omega, \eta]\) is a \(\mathfrak{g}\)-valued differential \((k + l)\)-form on \(Y\) constructed as follows:
\[
[\omega, \eta](a)(v_1, \ldots, v_{k+l}) = \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) [\omega(a)(v_{\sigma(1)}, \ldots, v_{\sigma(k)}), \omega(a)(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)})]
\]
for all \(a \in Y\) and \(v_i \in T_a Y\), \(1 \leq i \leq k + l\).

We then have the following basic property of connection forms.
Lemma 5.3. Let \((X = [X_1 \rightrightarrows X_0], \mathcal{H})\) be a Lie groupoid with an integrable connection, and let \((E_G \xrightarrow{\pi} X_0, [X_1 \rightrightarrows X_0])\) be a principal \(G\)-bundle over \(X\). Let \(\mathcal{H} \subset T(s^*E_G)\) be the pullback of the connection \(\mathcal{H}\) to the Lie groupoid \([s^*E_G \rightrightarrows E_G]\). Let \(\mathcal{D}\) be a connection on the principal \(G\)-bundle \(E_G\) on \(X\), and let \(\omega : E_G \to T^*E_G \otimes \mathfrak{g}\) be the associated connection 1-form (see Lemma 5.2). Then the Lie bracket \([\omega, \omega] \in \Omega^2(E_G, \mathfrak{g})\) is a differential 2-form on the Lie groupoid with connection \(([s^*E_G \rightrightarrows E_G], \mathcal{H})\).

Proof. By definition, \([\omega, \omega](a)(v_1, v_2) = [\omega(a)(v_1), \omega(a)(v_2)]\) for all \(a \in E_G\) and \(v_1, v_2 \in T_aE_G\).

For \((\gamma, a) \in s^*E_G\) and \((v_1, r_1)(v_2, r_2) \in T_{(\gamma, a)}(s^*E_G)\), we observe the following

\[
\widetilde{\mathcal{H}}(\text{pr}_2^*([\omega, \omega]))(\gamma, a)((v_1, r_1), (v_2, r_2)) = [\omega(\gamma \cdot a)(\widetilde{\theta}_{(\gamma, a)}(v_1)), \omega(\gamma \cdot a)(\widetilde{\theta}_{(\gamma, a)}(v_2))].
\]

As we saw in the proof of Lemma 5.2, \((\omega(\gamma \cdot a))((\widetilde{\theta}_{(\gamma, a)}(v)), (\omega(a))(v)\) for all \((\gamma, a) \in s^*E_G\) and for all \((v, r) \in T_{(\gamma, a)}(s^*E_G)\). In particular, that means

\[
\widetilde{\mathcal{H}}(\mu^*([\omega, \omega]))(\gamma, a)((v_1, r_1), (v_2, r_2)) = [\omega(\gamma \cdot a)(\widetilde{\theta}_{(\gamma, a)}(v_1)), \omega(\gamma \cdot a)(\widetilde{\theta}_{(\gamma, a)}(v_2))]
\]

Thus, we have established

\[
\widetilde{\mathcal{H}}(\mu^*([\omega, \omega])) = \widetilde{\mathcal{H}}(\text{pr}_2^*([\omega, \omega]));
\]

that is, \([\omega, \omega] \in \Omega^2(E_G, \mathfrak{g})\) is a differential 2-form on the Lie groupoid \(([s^*E_G \rightrightarrows E_G], \mathcal{H})\). \(\square\)

Let \((E_G \to X_0, [X_1 \rightrightarrows X_0])\) be a principal \(G\)-bundle over the Lie groupoid with connection \((X = [X_1 \rightrightarrows X_0], \mathcal{H})\) and

\[
0 \to \text{ad}(E_G) \to \text{At}(E_G) \to TX_0 \to 0
\]

be the associated Atiyah sequence of vector bundles. Let \(\mathcal{D} : TX_0 \to \text{At}(E_G)\) be a connection on the principal \(G\)-bundle \((E_G \to X_0, [X_1 \rightrightarrows X_0])\). Then \(\mathcal{D}\) as a connection on the underline principal \(G\)-bundle \(E_G \to X_0\) defines the \(\mathfrak{g}\)-valued curvature 2-form,

\[
\mathcal{K}_\mathcal{D} : X_0 \to \Lambda^2_T X_0
\]

on the manifold \(X_0\) satisfying the Maurer-Cartan formula

\[
\pi^*\mathcal{K}_\mathcal{D} = d\omega + [\omega, \omega].
\]

We will study the curvature form and related forms in more detail in Section 6.

The following lemma was also stated without proof in the note [BN] and we will give a proof here.
Lemma 5.4. Let \((E_G \to X_0, [X_1 \rightrightarrows X_0])\) be a principal \(G\)-bundle over the Lie groupoid with integrable connection \((X = [X_1 \rightrightarrows X_0], \mathcal{H})\) and \(\mathcal{H} \subset T(s^*E_G)\) be the pullback connection on the Lie groupoid \([s^*E_G \rightrightarrows E_G]\). Let
\[
0 \to \text{ad}(E_G) \to \text{At}(E_G) \to TX_0 \to 0
\]
be the Atiyah sequence associated to \((E_G \to X_0, [X_1 \rightrightarrows X_0])\) and \(\mathcal{D} : TX_0 \to \text{At}(E_G)\) be a connection on the principal \(G\)-bundle \((E_G \to X_0, [X_1 \rightrightarrows X_0])\). Let \(\mathcal{K}_D\) be the curvature 2-form of the connection \(\mathcal{D}\) on the underline principal \(G\)-bundle \(E_G \to X_0\). Then \(\mathcal{K}_D\) is a differential 2-form on the Lie groupoid \(X = [X_1 \rightrightarrows X_0]\).

**Proof.** By Lemma 5.1 it suffices to prove that \(\pi^*\mathcal{K}_D : E_G \to \Lambda^2 T^*E_G\) is a differential 2-form on the Lie groupoid \([s^*E_G \rightrightarrows E_G]\).

As we are assuming that the connection \(\mathcal{H}\) is integrable, it follows that the connection \(\tilde{\mathcal{H}}\) is integrable as well (see Lemma 5.2). By Lemma 5.2 \(\omega\) is a differential 1-form on the Lie groupoid \([s^*E_G \rightrightarrows E_G]\). Integrability of \(\tilde{\mathcal{H}}\) implies that \(d\omega\) is a differential 2-form on the Lie groupoid \([s^*E_G \rightrightarrows E_G]\) (see Lemma 3.10). Whereas Lemma 5.3 implies the same for \([\omega, \omega]\). Thus \(d\omega + [\omega, \omega] = \pi^*\mathcal{K}_D\) is a differential 2-form on the Lie groupoid \([s^*E_G \rightrightarrows E_G]\).

We make a brief digression here to compare our definition of a connection on a principal bundle over a Lie groupoid to that of [LGTX2]. While the definition of a principal \(G\)-bundle \((E_G \to X_0, [X_1 \rightrightarrows X_0])\) in [LGTX2] is the same as ours, the definition of a connection \((E_G \to X_0, [X_1 \rightrightarrows X_0])\) is given as follows (see [LGTX2] Definition 3.5)

**Definition 5.5.** Let \((E_G \to X_0, [X_1 \rightrightarrows X_0])\) be a principal \(G\)-bundle over a Lie groupoid \(X = [X_1 \rightrightarrows X_0]\). A connection on \((E_G \to X_0, [X_1 \rightrightarrows X_0])\) is a connection \(\omega \in \Omega^1(E_G, \mathfrak{g})\) on the underline principal \(G\)-bundle \(E_G \to X_0\) such that, \(pr^*_2 \omega - \mu^* \omega = 0\).

Now let us consider an étale Lie groupoid \(X = [X_1 \rightrightarrows X_0]\) with integrable connection \(\mathcal{H} = TX_1\), as in Example 4.5 Then using Lemma 5.2 it is straightforward to see that both definitions coincide in the case of principal \(G\)-bundles over étale Lie groupoids and a connection always exist in this case.

**Example 5.6.** Let \(G\) be a Lie group an \([G \rightrightarrows \ast]\) its associated Lie groupoid. Then \(\pi : G \to \ast\) is a principal \(G\)-bundle over \([G \rightrightarrows \ast]\), where the action is given by (left) translation. A connection can only exist if \(G\) is discrete.

In addition, it is shown in [LGTX2] Proposition 3.13 that the latter definition of a connection is indeed Morita invariant and therefore gives a notion of connection and integrable connection of principal \(G\)-bundles over differentiable stacks. In particular, it follows that principal \(G\)-bundles over Deligne-Mumford stacks and therefore also over orbifolds always admit a connection (see also [BCKN] for more details on the relation with differentiable stacks). In light of the last example, it follows that the universal principal \(G\)-bundle \(\ast \longrightarrow \mathcal{B}G\) over the classifying stack \(\mathcal{B}G\) can only admit a connection if \(G\) is a discrete group.

### 5.1. Pullback connection on the pullback principal \(G\)-bundle along morphisms of Lie groupoids.

Let \((F, f) : (X = [X_1 \rightrightarrows X_0], \mathcal{H}^X) \to (Y = [Y_1 \rightrightarrows Y_0], \mathcal{H}^Y)\) be a morphism of Lie groupoids with fixed connections.
Let \((E_G \xrightarrow{π} Y_0, [Y_1 \rightrightarrows Y_0])\) be a principal \(G\)-bundle over the Lie groupoid \([Y_1 \rightrightarrows Y_0]\). Consider the Atiyah sequence of vector bundles over the Lie groupoid \(\mathcal{Y} = [Y_1 \rightrightarrows Y_0]\),

\[
0 \to (E_G \times \mathfrak{g})/G \xrightarrow{j^G} (TE_G)/G \xrightarrow{\pi^G} TY_0 \to 0.
\]

(5.2)

associated to the principal \(G\)-bundle \((E_G \xrightarrow{π} Y_0, [Y_1 \rightrightarrows Y_0])\).

Let \((X_0 \times Y_0 \xrightarrow{pr_1} X_0, [X_1 \rightrightarrows X_0])\) be the principal \(G\)-bundle obtained by pulling back the principal \(G\)-bundle \((E_G \to Y_0, [Y_1 \rightrightarrows Y_0])\) over the Lie groupoid \([Y_1 \rightrightarrows Y_0]\), along the morphism of Lie groupoids \((F, f) : [X_1 \rightrightarrows X_0] \to [Y_1 \rightrightarrows Y_0]\). Consider the Atiyah sequence

\[
0 \to ((X_0 \times Y_0 \times E_G) \times \mathfrak{g})/G \xrightarrow{j^G} (T(X_0 \times Y_0 \times E_G))/G \xrightarrow{(pr_1)^G} TX_0 \to 0
\]

(5.3)

associated to the principal \(G\)-bundle \((X_0 \times Y_0 \times E_G \xrightarrow{pr_1} X_0, [X_1 \rightrightarrows X_0])\) over the Lie groupoid \([X_1 \rightrightarrows X_0]\).

Let \(\mathcal{D}\) be a connection on the principal \(G\)-bundle \((E_G \to Y_0, [Y_1 \rightrightarrows Y_0])\) over the Lie groupoid \(\mathcal{Y} = [Y_1 \rightrightarrows Y_0]\). Let \(\omega \in Ω^1(E_G, \mathfrak{g})\) be the associated connection 1-form on the Lie groupoid \([s^*E_G \rightrightarrows E_G]\); i.e.; it satisfies

\[
\tilde{\mathcal{H}}^X((pr_2^Y)^*ω) = \tilde{\mathcal{H}}^X((μ^Y)^*ω).
\]

(5.4)

The pullback \(pr_2^*\omega : (X_0 \times Y_0 \times E_G) \to \Lambda^I_0(T^*(X_0 \times Y_0 \times E_G))\) defines a connection 1-form on the principal \(G\)-bundle \(pr_1 : X_0 \times Y_0 \times E_G \to X_0\) over the manifold \(X_0\). We show that this differential form \(pr_2^*\omega : (X_0 \times Y_0 \times E_G) \to \Lambda^I_0(T^*(X_0 \times Y_0 \times E_G))\) is in fact a differential form on the Lie groupoid \([X_0 \times Y_0 \times E_G \times E_G (Y_1 \rightrightarrows Y_0 \times E_G) \rightrightarrows (X_0 \times Y_0 \times E_G)]\). Then, by the Lemma [5.2], \(pr_2\omega\) defines a connection \(f^*\mathcal{D}\) on the principal \(G\)-bundle \((X_0 \times Y_0 \times E_G \to X_0, [X_1 \rightrightarrows X_0])\).

Let \((γ, (a, e)) \in X_1 \times X_0 (X_0 \times Y_0 \times E_G)\) and \((v, (w, l)) \in T(γ(a,e))(X_1 \times X_0 (X_0 \times Y_0 \times E_G))\). It is straightforward verification that,

\[
\tilde{\mathcal{H}}^X((pr_2^Y)^*(pr_2^*\omega)) \gamma(a, e)(v, (w, l)) = \tilde{\mathcal{H}}^Y((pr_2^Y)^*ω)(F(γ), F(γ) \cdot e)(F_e γ(v), l)
\]

and

\[
\tilde{\mathcal{H}}^X((μ^X)^*(pr_2^*\omega)) \gamma(a, e)(v, (w, l)) = \tilde{\mathcal{H}}^Y((μ^Y)^*ω)(F(γ), F(γ) \cdot e)(F_e γ(v), l).
\]

Using [5.4] we conclude,

\[
\tilde{\mathcal{H}}^X((pr_2^Y)^*(pr_2^*\omega)) = \tilde{\mathcal{H}}^X((μ^X)^*(pr_2^*\omega)).
\]

Therefore, we have established the following

**Proposition 5.7.** Let \((F, f) : (X = [X_1 \rightrightarrows X_0], \mathcal{H}^X) \to (Y = [Y_1 \rightrightarrows Y_0], \mathcal{H}^Y)\) be a morphism of Lie groupoids with connections. Let \((E_G \xrightarrow{π} Y_0, [Y_1 \rightrightarrows Y_0])\) be a principal \(G\)-bundle over the Lie groupoid \(\mathcal{Y} = [Y_1 \rightrightarrows Y_0]\). Let \(ω\) be the differential connection form corresponding to the splitting of the Atiyah sequence

\[
0 \to (E_G \times \mathfrak{g})/G \xrightarrow{j^G} (TE_G)/G \xrightarrow{π^G} TY_0 \to 0
\]
associated to the principal $G$-bundle $(E_G \to Y_0, [Y_1 \rightrightarrows Y_0])$. Then $pr_2^* \omega$ defines a splitting of the Atiyah sequence of vector bundles over the Lie groupoid $\mathcal{X}$

$$0 \to ((X_0 \times_{Y_0} E_G) \times \mathfrak{g})/G \xrightarrow{j_G} (T(X_0 \times_{Y_0} E_G))/G \xrightarrow{(pr_1)_G} TX_0 \to 0$$

associated to the principal $G$-bundle $(X_0 \times_{Y_0} E_G \xrightarrow{pr_1} X_0, [X_1 \rightrightarrows X_0])$.

We will call this connection the pullback connection on the pulled back principal bundle.

6. Chern-Weil map and characteristic classes for principal bundles over Lie groupoids with connections

Let $\mathcal{X} = [X_1 \rightrightarrows X_0]$ be a Lie groupoid equipped with an integrable connection $\mathcal{H} \subset TX_1$. Let $(E_G \to X_0, [X_1 \rightrightarrows X_0])$ be a principal $G$-bundle. Let $\mathcal{H} \subset T(s^*E_G)$ be the pullback connection on the Lie groupoid $[s^*E_G \rightrightarrows E_G]$. Let

$$0 \to \text{ad}(E_G) \to \text{At}(E_G) \to TX_0 \to 0$$

be the Atiyah sequence associated to the principal $G$-bundle $(E_G \to X_0, [X_1 \rightrightarrows X_0])$.

Let $\mathcal{D}$ be a connection on the principal $G$-bundle $(E_G \to X_0, [X_1 \rightrightarrows X_0])$, given as a section $\mathcal{D}: \text{At}(E_G) \to \text{ad}(E_G)$, or, equivalently, as a section $\mathcal{D}: TX_0 \to \text{At}(E_G)$. Let $\omega \in \Omega^1(E_G, \mathfrak{g})$ be the corresponding connection 1-form on the Lie groupoid $[s^*E_G \rightrightarrows E_G]$. Let $\mathcal{K}_\mathcal{D} \in \Omega^2(X_0, \mathfrak{g})$ be the associated curvature 2-form on the Lie groupoid $[X_1 \rightrightarrows X_0]$. Furthermore, let

$$\Omega = \pi^* \mathcal{K}_\mathcal{D} \in \Omega^2(E_G, \mathfrak{g})$$

be the pulled back curvature form on the groupoid $[s^*E_G \rightrightarrows E_G]$.

We briefly recall the construction of the Chern-Weil morphism of the pair $(\omega, \Omega)$ for the underlying principal $G$-bundle $E_G \to X_0$ (see for example [KN]). Let $\text{Sym}^k(\mathfrak{g})$ be the set of all symmetric $k$-linear mappings $\mathfrak{g} \times \cdots \times \mathfrak{g} \to \mathbb{R}$ on the Lie algebra $\mathfrak{g}$. Define the adjoint action of the Lie group $G$ on $\text{Sym}^k(\mathfrak{g})$ by

$$(\text{Ad}_g f)(x_1, \cdots, x_k) := f(\text{Ad}_g x_1, \cdots, \text{Ad}_g x_k)$$

for any $f \in \text{Sym}^k(\mathfrak{g})$. Let us define the $\text{Ad}(G)$-invariant forms as

$$\text{Sym}^k(\mathfrak{g})^G := \{ f \in \text{Sym}^k(\mathfrak{g}) | \text{Ad}_g f = f \text{ for all } g \in G \}.$$ 

For a given $f \in \text{Sym}^k(\mathfrak{g})^G$ we assign a closed 2$k$-form $f(\Omega)$ on $E_G$ given by

$$f(\Omega)(v_1, \cdots, v_{2k}) := \frac{1}{2^k!} \sum_{\sigma} \epsilon_\sigma f(\Omega(v_{\sigma(1)}, v_{\sigma(2)}), \cdots, \Omega(v_{\sigma(2k-1)}, v_{\sigma(2k)})),$$ (6.1)

where $\sigma$ is an element of the symmetric group $\Sigma_{2k}$ of $2k$ elements and $\epsilon_\sigma$ denotes the sign of the permutation $\sigma \in \Sigma_{2k}$. Consider the map $\text{Sym}^k(\mathfrak{g})^G \to \Omega^{2k}(E_G)$ given by $f \mapsto f(\Omega)$. Then this map has the following properties:

(1) There exists a unique closed 2$k$-form $\widetilde{f(\Omega)}_\omega \in \Omega^{2k}(X_0)$ such that $\pi^* \widetilde{f(\Omega)}_\omega = f(\Omega)$.

(2) The map $\text{Sym}^k(\mathfrak{g})^G \to H_{dR}^{2k}(X_0, \mathbb{R})$ given by $f \mapsto \widetilde{f(\Omega)}_\omega$ is independent of the choice of connection $\omega$. 

By linear extension the map in item (2) above naturally defines a homomorphism of algebras, namely

\[
\Sym(g)^G = \sum_{k=0}^{\infty} \Sym^k(g)^G \to H^*_{dR}(X_0, \mathbb{R})
\]

\[
f \mapsto [f(\Omega)].
\]

This homomorphism is called the Chern-Weil homomorphism.

Note that \( f(\Omega) = f(\pi^*K_D) = \pi^*f(K_D) = \pi^*\tilde{f}(\Omega) \). Thus, \( \pi : E_G \to X_0 \) being a surjective submersion implies that \( \pi^*f(K_D) \) is closed and then the uniqueness property (1) above of the Chern-Weil map implies

\[
f(\Omega) = f(K_D).
\]

(6.3)

In our discussion we have so far treated the pair \((\omega, \Omega)\) only as connection and curvature of the underlying principal \(G\)-bundle \(E_G \to X_0\). In particular, this means if \(\omega'\) is any other connection 1-form on the principal \(G\)-bundle \(E_G \to X_0\), then \(f(\Omega) - f(\Omega)' = d\Phi\), for some \((2k - 1)\)-form \(\Phi\) on \(X_0\). However in order to make sense of Chern-Weil theory for a principal bundle over a Lie groupoid \(\mathbb{X} = [X_1 \rightrightarrows X_0]\), we have to show

1. \(\pi^*\tilde{f}(\Omega)_w = f(K_D)\) is a closed 2k-form on \(\mathbb{X} = [X_1 \rightrightarrows X_0]\), and
2. the \((2k - 1)\)-form \(\Phi\) on \(X_0\) is actually a \((2k - 1)\)-form on \(\mathbb{X} = [X_1 \rightrightarrows X_0]\), when both \(\omega\) and \(\omega'\) are connection forms on \(s^*E_G \to G\) for the principal \(G\)-bundle \((E_G \to X_0, [X_1 \rightrightarrows X_0])\).

In turn we will get a homomorphism

\[
\Sym(g)^G \to H^*_{dR}(\mathbb{X}, \mathcal{H})
\]

\[
f \mapsto [f(K_D)]
\]

(6.4)

The part (1) of the above statements was suggested in the note [BN]. We are going to prove both of the statements here.

**Theorem 6.1.** Let \(\mathbb{X} = [X_1 \rightrightarrows X_0]\) be a Lie groupoid equipped with an integrable connection \(\mathcal{H} \subset TX_1\). Let \((E_G \to X_0, [X_1 \rightrightarrows X_0])\) be a principal \(G\)-bundle. Let \(D\) be a connection on the principal \(G\)-bundle \((E_G \to X_0, [X_1 \rightrightarrows X_0])\) and \(K_D\) be the associated curvature 2-form on the Lie groupoid \(\mathbb{X} = [X_1 \rightrightarrows X_0]\). Then we have

1. \(f(K_D)\) is a differential 2k-form on the Lie groupoid \([X_1 \rightrightarrows X_0]\).
2. The map \(\Sym(g)^G \to H^*_{dR}(\mathbb{X}, \mathcal{H})\) defined as \(f \mapsto [f(K_D)]\) does not depend on the connection \(D\).

**Proof.** Part (1): We prove the statement for the case \(k = 1\). The general case follows along the same lines of argument.

We prove that if \(f : g \to \mathbb{R}\) is a linear map satisfying \(f(x) = f(Ad_g x)\) for all \(g \in G, x \in g\), then \(\tilde{H}(s^*(f(K_D))) = \tilde{H}(t^*(f(K_D)))\).
Let \( \gamma \in X_1 \) and \( v_1, v_2 \in T_\gamma X_1 \). We have
\[
\left( \tilde{\mathcal{H}}(s^*(f(\mathcal{K}_D)))(\gamma) \right)(v_1, v_2) = \left( s^*(f(\mathcal{K}_D))(\gamma) \right)(P_{H_\gamma}(v_1), P_{H_\gamma}(v_2))
\]
\[
= \left( f(\mathcal{K}_D)(s(\gamma)) \right)(s_{*\gamma}(P_{H_\gamma}(v_1)), s_{*\gamma}(P_{H_\gamma}(v_2))).
\]

Recall that by Lemma \( \ref{lem:5.4} \), \( \mathcal{K}_D \) is a differential 2-form on the Lie groupoid \([X_1 \rightrightarrows X_0]\). Then for each \( \gamma \in X_1 \) and \( v_1, v_2 \in T_\gamma X_1 \), we have
\[
\left( \mathcal{K}_D(s(\gamma)) \right)(s_{*\gamma}(P_{H_\gamma}(v_1)), s_{*\gamma}(P_{H_\gamma}(v_2))) = \left( \mathcal{K}_D(t(\gamma)) \right)(t_{*\gamma}(P_{H_\gamma}(v_1)), t_{*\gamma}(P_{H_\gamma}(v_2))).
\]

Thus, for \( \tilde{\mathcal{H}}(s^*(f(\mathcal{K}_D))) \in \Omega^2(X_1, g) \), we have
\[
\left( \tilde{\mathcal{H}}(s^*(f(\mathcal{K}_D)))(\gamma) \right)(v_1, v_2) = f(\mathcal{K}_D(s(\gamma)))(s_{*\gamma}(P_{H_\gamma}(v_1)), s_{*\gamma}(P_{H_\gamma}(v_2)))
\]
\[
= f(\mathcal{K}_D(t(\gamma)))(t_{*\gamma}(P_{H_\gamma}(v_1)), t_{*\gamma}(P_{H_\gamma}(v_2)))
\]
\[
= \tilde{\mathcal{H}}(t^*(f(\mathcal{K}_D)))(\gamma)(v_1, v_2).
\]

Hence \( f(\mathcal{K}_D) \) is a differential 2-form on the Lie groupoid \( \mathbb{X} = [X_1 \rightrightarrows X_0] \).

**Part (2):** Let \( \mathcal{D}, \mathcal{D}' \) be a pair of connections on the principal \( G \)-bundle \( (E_G \to X_0, [X_1 \rightrightarrows X_0]) \) and \( \mathcal{K}_D, \mathcal{K}_{D'} \) be the respective curvature 2-forms on the Lie groupoid \( \mathbb{X} = [X_1 \rightrightarrows X_0] \). Given an element \( f \in \text{Sym}^k(g)^G \), we prove that \( [f(\mathcal{K}_D)] = [f(\mathcal{K}_{D'})] \in H^{2k}(\mathbb{X}, \mathcal{H}) \) i.e., there exists a \((2k - 1)\)-form \( \tilde{\Phi} \) on the Lie groupoid \( \mathbb{X} = [X_1 \rightrightarrows X_0] \), such that \( f(\mathcal{K}_D) - f(\mathcal{K}_{D'}) = d\tilde{\Phi} \).

Let \( \omega, \omega' \) be the associated connection 1-forms for connections \( \mathcal{D} \) and \( \mathcal{D}' \) respectively. Let \( \alpha = \omega - \omega' \). We define a family of connection 1-forms \( \alpha_t = \omega' + t\alpha \) for \( 0 \leq t \leq 1 \) on the principal \( G \)-bundle \( (E_G \to X_0, [X_1 \rightrightarrows X_0]) \).

Let \( \Omega_t \) be the associated curvature 2-forms. Consider the differential \((2k - 1)\)-form
\[
\Phi = k \int_0^1 f(\alpha_t, \Omega_t, \Omega_t, \ldots, \Omega_t) dt
\]
on \( E_G \). As per classical Chern-Weil theory (see \( \text{[KN]} \)), this \((2k - 1)\)-form \( \Phi \) on \( E_G \) uniquely projects to a \((2k - 1)\)-form \( \tilde{\Phi} \) on \( X_0 \) as \( \Phi = \pi^*(\tilde{\Phi}) \), and then \( f(\Omega) - f(\Omega') = d\Phi \). So, we have
\[
\pi^*(f(\mathcal{K}_D) - f(\mathcal{K}_{D'})) = \pi^*(d(\tilde{\Phi})).
\]

As \( \pi : E_G \to X_0 \) is a surjective submersion, that means \( f(\mathcal{K}_D) - f(\mathcal{K}_{D'}) = d\tilde{\Phi} \).

In order to prove that \( \tilde{\Phi} \) is a differential form on the Lie groupoid \([X_1 \rightrightarrows X_0]\), it suffices (see Lemma \( \ref{lem:5.1} \)) to prove that \( \Phi \) is a differential form on the Lie groupoid \([s^*E_G \rightrightarrows E_G]\).
Observe that
\[
pr_2^* \Phi = pr_2^* \left( k \int_0^1 f(\alpha_t, \Omega_t, \Omega_t, \ldots, \Omega_t) dt \right)
\]
\[
= \left( k \int_0^1 f(pr_2^*(\alpha_t), pr_2^*(\Omega_t), \ldots, pr_2^*(\Omega_t)) dt \right)
\]
and
\[
\widetilde{H}(pr_2^* \Phi) = k \int_0^1 f(\widetilde{H}(pr_2^*(\alpha_t)), \ldots, \widetilde{H}(pr_2^*(\Omega_t))) dt.
\]
On the other hand, the differential forms \(\alpha_t, \Omega_t\) on the Lie groupoid \([s^*E_G \Rightarrow E_G]\) respectively satisfy \(\widetilde{H}(pr_2^*(\alpha_t)) = \widetilde{H}(\mu^* \alpha_t)\) and \(\widetilde{H}(pr_2^*(\Omega_t)) = \widetilde{H}(\mu^*(\Omega_t))\). Plugging these relations into the last equation, we conclude
\[
\widetilde{H}(pr_2^* \Phi) = \widetilde{H}(\mu^* \Phi).
\]
Thus, \(\Phi\) is a differential \((2k - 1)\)-form on the Lie groupoid \([s^*E_G \Rightarrow E_G]\) and so is \(\Phi\).  

We summarize the discussion of this section as follows. Let \(X = [X_1 \Rightarrow X_0]\) be a Lie groupoid with an integrable connection \(H\). Let \((E_G \rightarrow X_0, [X_1 \Rightarrow X_0])\) be a principal \(G\)-bundle over \(X\). Assume \((E_G \rightarrow X_0, [X_1 \Rightarrow X_0])\) admits a connection. Then we have a well-defined Chern-Weil map
\[
\text{Sym}(g)^G \rightarrow H^*_d(X, H)
\]
given by (6.3) and this map does not depend on the choice of the connection on the principal bundle \((E_G \rightarrow X_0, [X_1 \Rightarrow X_0])\). Along with Proposition 3.18, this allows us to construct characteristic classes for principal \(G\)-bundles over Lie groupoids with connections (compare [Du2]).

6.1. Characteristic classes. Let \(X = [X_1 \Rightarrow X_0]\) be a Lie groupoid equipped with an integrable connection \(H \subset TX_1\) and \((E_G \rightarrow X_0, [X_1 \Rightarrow X_0])\) a principal \(G\)-bundle over \(X\). Let
\[
\text{Ch}_{E_G} : \text{Sym}(g)^G \rightarrow H^*_d(X, H),
\]
\[
f \mapsto [f(K_D)]
\]
(6.5) be the map in Theorem 6.1. We call \(\text{Ch}_{E_G}(f)\) the characteristic class of \(f\). Let \((\Phi, \phi) : (X = [X_1 \Rightarrow X_0], H^X) \rightarrow (Y = [Y_1 \Rightarrow Y_0], H^Y)\) be a morphism of Lie groupoids with connections and \((E_G \xrightarrow{\pi} Y_0, [Y_1 \Rightarrow Y_0])\) a principal \(G\)-bundle over the Lie groupoid \(Y = [Y_1 \Rightarrow Y_0]\). We have seen in Proposition 6.7 that the pullback of the principal \(G\)-bundle \(E_G \xrightarrow{\pi} Y_0\) along \(\phi\) defines a principal \(G\)-bundle \((\phi^*E_G \rightarrow X_0, [X_1 \Rightarrow X_0])\) over the Lie groupoid \(X\). Moreover, any connection \(D\) on \((E_G \xrightarrow{\pi} Y_0, [Y_1 \Rightarrow Y_0])\) pulls back to a connection \(\phi^*D\) on \((\phi^*E_G \xrightarrow{\pi} X_0, [X_1 \Rightarrow X_0])\). Obviously the associated curvature 2-form \(K_D \in \Omega^2(Y_0, g)\) on the Lie groupoid \(Y = [Y_1 \Rightarrow Y_0]\) pulls back to the curvature \(\phi^*K_D\) of \(\phi^*D\) on the Lie groupoid \(X = [X_1 \Rightarrow X_0]\), namely we have
\[
\phi^*K_D = K_{\phi^*D}.
\]
Now for any \( f \in \text{Sym}(\mathfrak{g})^G \) it is immediate from (6.1) that,
\[
f(\mathcal{K}_{\phi^*D}) = \phi^* f(\mathcal{K}_D).
\]
Then by Proposition 3.18 we conclude \([f(\mathcal{K}_{\phi^*D})] = [\phi^* f(\mathcal{K}_D)] = \phi^* [f(\mathcal{K}_D)]\). We arrive at the following ‘naturality’ condition of the Chern-Weil map for principal \( G \)-bundles on Lie groupoids with integrable connections.

**Proposition 6.2.** Let \((\Phi, \phi) : (X = [X_1 \rightrightarrows X_0], \mathcal{H}^X) \to (Y = [Y_1 \rightrightarrows Y_0], \mathcal{H}^Y)\) be a morphism of Lie groupoids with integrable connections and \((E_G, \pi_Y \to Y_0, [Y_1 \rightrightarrows Y_0])\) a principal \( G \)-bundle over the Lie groupoid \( Y = [Y_1 \rightrightarrows Y_0] \). Let \((\phi^* E_G \to X_0, [X_1 \rightrightarrows X_0])\) be the pullback principal \( G \)-bundle over the Lie groupoid \( X \). Then,
\[
\text{Ch}_{\phi^* E_G} = \phi^* \circ \text{Ch}_{E_G}, \tag{6.6}
\]
where \( \phi^* : H^*_\text{dr}(Y, \mathcal{H}) \to H^*_\text{dr}(X, \mathcal{H}) \) is the algebra homomorphism in Proposition 3.18.

Let \((E \to X_0, [X_1 \rightrightarrows X_0])\) be a rank \( r \) vector bundle over the Lie groupoid \( X \) (see Definition 2.9). Consider the underlying vector bundle \( E \xrightarrow{\pi} X_0 \) on \( X_0 \). Let
\[
\text{Fr}(E) = \bigsqcup_{x \in X_0} \text{Iso}(\mathbb{F}^r \to E_x),
\]
where \( E_x = \pi^{-1}(x) \) denotes the fibre over \( x \in X_0 \), \( \mathbb{F} \) is either the field of complex or real numbers and \( \text{Iso}(\mathbb{F}^r \to E_x) \) is the set of linear isomorphisms. The right action of \( GL(r, \mathbb{F}) \) on \( \text{Fr}(E) \) given by \( (x, \sigma) \cdot g = (x, \sigma \circ g) \) defines a (right) principal \( GL(r, \mathbb{F}) \)-bundle \( \text{Fr}(E) \to X_0, (x, \sigma) \mapsto x \), called the frame bundle. Now since \((E \to X_0, [X_1 \rightrightarrows X_0])\) is a vector bundle over the Lie groupoid \( X \), we get a left action \( \mu : X_1 \times X_0 \to E \) of \( X \) on \( E \) such that the restriction for each \( \gamma \in X_1 \) defines a linear map \( \mu_\gamma : E_{\mu(\gamma)} \to E_{\mu(\gamma)} \). This induces a left action \( X_1 \times X_0, \text{Fr}(E) \to \text{Fr}(E) \) by \( \gamma \cdot (x, \sigma) = (y, \mu_\gamma \circ \sigma) \), for \( x \rightrightarrows y \). The compatibility condition \( \gamma \cdot ((x, \sigma) \cdot g) = (\gamma \cdot (x, \sigma)) \cdot g \) is immediate. That is to say, the frame bundle \( \text{Fr}(E) \to X_0 \) is in fact a principal \( GL(r, \mathbb{F}) \)-bundle over the Lie groupoid \( X = [X_1 \rightrightarrows X_0] \). Following our notation, we write
\[
(\text{Fr}(E) \to X_0, [X_1 \rightrightarrows X_0]).
\]
For a given rank \( r \)-vector bundle \((E \to X_0, [X_1 \rightrightarrows X_0])\) over the Lie groupoid \( X \), we define the Chern-Weil homomorphism as
\[
\text{Ch}_E := \text{Ch}_{\text{Fr}(E)}. \tag{6.7}
\]
Now given an element \( \mathfrak{A} \) of the Lie algebra \( \mathfrak{gl}(r, \mathbb{F}) \) of \( GL(r, \mathbb{F}) \), we find the coefficients \( c_i(\mathfrak{A}) \) of the characteristic polynomial of \( \mathfrak{A} \) from the following expansion,
\[
\text{Det}(\mathfrak{A} + tI) = \sum_{i=0}^{k} c_i(\mathfrak{A}) t^{n-k}.
\]
Each \( c_i : \mathfrak{gl}(r, \mathbb{F}) \to \mathbb{R} \) is in fact a degree \( i \) homogeneous polynomial invariant under the adjoint action of \( GL(r, \mathbb{F}) \). Thus \( c_i \) can be identified with an \( \text{Ad}(GL(r, \mathbb{F})) \)-invariant, multilinear, symmetric mapping (see [KN])
\[
\tilde{c}_i : \text{Sym}^i(\mathfrak{g})^G \to \mathbb{R}.
\]
Then various characteristic classes of a vector bundle \((E \to X_0, [X_1 \rightrightarrows X_0])\) are given as the images of the classes \( \tilde{c}_i \) under the homomorphism \( \text{Ch}_{\text{Fr}(E)} \).
In this paper, we have studied the Chern-Weil theory of a principal $G$-bundle over a Lie groupoid $X$ (with an integrable connection $\mathcal{H}$). A natural direction of enquiry would be to study the behaviour of the constructions described in this paper under Morita equivalences and, therefore, whether these structures can be further extended to differentiable stacks. While a conclusive answer to this question for the most general case still seems to be elusive, in [BCKN] we have discussed the problem in certain important cases of interest.

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