TRIANGULINE Lifts of Global Mod $p$ Galois Representations

Najmuddin Fakhruddin, Chandrashekhar Khare, and Stefan Patrikis

Abstract. We show that under a suitable oddness condition, for $p \gg F_n$ irreducible $n$-dimensional mod $p$ representations of the absolute Galois group of an arbitrary number field $F$ have characteristic zero lifts which are unramified outside a finite set of primes and trianguline at all primes of $F$ dividing $p$. We also prove a variant of this result under some extra hypotheses for representations into connected reductive groups.

1. Introduction

For any field $F$ we let $\Gamma_F$ be its absolute Galois group. In the articles [14] and [15] we constructed geometric (in the sense of Fontaine–Mazur) lifts for odd representations $\bar{\rho} : \Gamma_F \to G(k)$, where $F$ is a totally real number field, and $G$ is a split reductive group (possibly disconnected) over the ring of integers $O$ of a $p$-adic field $E$ with residue field $k$, when $\bar{\rho}$ satisfies certain additional conditions. For example, if $G$ is connected, $\bar{\rho}|_{\Gamma_F(\zeta_p)}$ is absolutely irreducible, and lifts exist locally (which are regular de Rham at primes dividing $p$) at all primes of $F$, then global lifts exist whenever $p$ is sufficiently large by [14, Theorem A]; in [15] we constructed lifts for reducible representations under some more technical hypotheses.

The oddness assumption is crucial, but if $G = \text{GL}_n$ and $n > 2$ then no $\bar{\rho}$ is odd in the sense of [14, Definition 1.2], so the results of [14] and [15] cannot be used to construct geometric lifts. In fact, Calegari has proved [7, Theorem 5.1] that without any oddness condition whatsoever geometric lifts need not exist even when $n = 2$ and $F = \mathbb{Q}$. The goal of this note is to show that if we weaken the requirement that the lift be geometric to being unramified outside a finite set of primes and trianguline at all primes above $p$, then the methods of [14] and [15] can be adapted to apply even when $G = \text{GL}_n$ and $F$ is an arbitrary number field if we assume the following weaker oddness condition:

Definition 1.1.

1. An involution $c$ in $\text{GL}_n(K)$, where $K$ is a field of characteristic $\neq 2$, is said to be $\text{GL}_n$-odd if $|n^+(c) - n^-(c)| \leq 1$, where $n^+(c)$ (resp. $n^-(c)$) is the number of eigenvalues of $\rho(c)$ which are equal to +1 (resp. -1).

2. Let $F$ be a number field and $\rho : \Gamma_F \to \text{GL}_n(K)$ a continuous representation, where $K$ is any topological field of characteristic $\neq 2$. We say that $\rho$ is $\text{GL}_n$-odd if for every real place $v$ of $F$ with $c_v \in \Gamma_{F_v}$ the corresponding complex conjugation, the involution $\rho(c_v)$ is odd as in (1).

The results of these papers do apply—and have interesting consequences—when $G = \text{GL}_n$ and $F$ is a global function field, and even for number fields if we do not impose any conditions at primes dividing $p$.

The reader may consult [3] for a survey of trianguline representations; we give a brief sketch of the definition in §2.2.

This is often simply called odd in the literature, but we have added $\text{GL}_n$ here to distinguish this notion from the stronger notion of oddness used in [14] for representations into general reductive groups $G$. 

1The results of these papers do apply—and have interesting consequences—when $G = \text{GL}_n$ and $F$ is a global function field, and even for number fields if we do not impose any conditions at primes dividing $p$.

2The reader may consult [3] for a survey of trianguline representations; we give a brief sketch of the definition in §2.2.

3This is often simply called odd in the literature, but we have added $\text{GL}_n$ here to distinguish this notion from the stronger notion of oddness used in [14] for representations into general reductive groups $G$. 
All Galois representations corresponding to cohomological automorphic forms or the cohomology (possibly with torsion coefficients) of locally symmetric spaces for GL$_n/F$ are GL$_n$-odd when $F$ is totally real by results of Caraiani and Le Hung [8], so there is a plethora of such representations.

Trianguline representations were introduced by Colmez [10], motivated by work of Kisin [18] on Galois representations attached to overconvergent modular forms, and they are closely related to the theory of eigenvarieties. For example, Hansen has conjectured [16], Conjecture 1.2.3 that any (semisimple) GL$_n$-odd representation $\rho: \Gamma_F \to \text{GL}_n(E)$ which is unramified outside a finite set of primes and is trianguline at all primes above $p$ arises as the representation associated (in [17, Theorem B]) to a point on one of the eigenvarieties for GL$_n/F$ constructed in [16], so the condition on the lifts that we impose is a natural weakening of geometricity.

For $F$ any number field and $S$ a finite set of finite places of $F$ we let $\Gamma_{F,S}$ denote the Galois group of the maximal extension of $F$ (inside a fixed algebraic closure) unramified outside all places in $S$ (and the infinite places). The following is a special case of the main result of this note.

**Theorem 1.2.** Let $F$ be an arbitrary number field and $\bar{\rho}: \Gamma_{F,S} \to \text{GL}_n(k)$ a GL$_n$-odd representation. If $p \gg n,F_0$ and $\bar{\rho}|_{\Gamma_F(p)}$ is absolutely irreducible, then there exists a finite set of places $S' \supset S$ and a finite extension $E'$ of $E$ with ring of integers $\mathfrak{O}'$ such that $\bar{\rho}$ lifts to a GL$_n$-odd representation $\rho: \Gamma_{F,S'} \to \text{GL}_n(\mathfrak{O}')$ which is regular trianguline at all primes of $F$ above $p$. Furthermore, one can also ensure that $\rho(\Gamma_F)$ contains an open subgroup of $\text{SL}_n(\mathfrak{O}')$.

In the main text this is Corollary 3.12. It is deduced from Theorem 3.8 which allows one to construct trianguline lifts under weaker hypotheses, e.g., for $\bar{\rho}$ which might not be irreducible.

We recall that for odd irreducible representations over totally real fields the Khare–Wintenberger method allows one to construct geometric lifts (for many $G$) without the need for enlarging $S$. However, this depends on potential modularity results but since (global) trianguline representations do not in general correspond to classical automorphic forms it is not possible to apply this to construct trianguline lifts.

Recently, a definition of trianguline representations with values in general (connected) reductive groups $G$ has been given by Vincent de Daruvar in his thesis [12]; one expects that these are related to eigenvarieties for groups that are not forms of GL$_n$. Using his results we can extend the above theorem to general (connected) reductive groups, see Theorem 3.16, with the caveat that the main result of loc. cit. depends on two assumptions: the first is the existence of a sufficiently general trianguline lift of $\bar{\rho}$ and the second is that $G$ should have no factors of type $G_2$, $F_4$ and $E_8$. One expects that the first assumption always holds and the second is unnecessary.

It is an interesting question to extend the definition of trianguline representations to disconnected groups $G$ and to prove analogues of the results of [12] in this case. If this is done, the methods of [14] and [15] should allow one to construct global trianguline lifts also for such $G$.

We also mention that beginning with the article [1], there has been interest in trianguline representations over $k[[t]]$, where $k$ is a finite field; see in particular [1, §1.2.3]. It is then

---

4Almost: all crystalline, and even semi-stable, representations are trianguline, but not all de Rham representations are trianguline, see [3].
a natural question as to whether a version of Theorem 1.2 holds with \( \mathcal{O} \) above replaced by \( k[[t]] \). We intend to pursue this in the near future.

1.1. Our proofs are based on the methods developed in [14] and [15], and as far as the global arguments go there are no essential changes, except that in Section 3 we axiomatise in §3.1 and §3.2 the conditions under which the methods of loc. cit. lead to the construction of lifts, unramified outside a finite set of primes, of a very general class of Galois representations. The main improvements are in the local arguments, so we describe these briefly here.

In the original lifting arguments of Ramakrishna [20] (which are generalized in [14] and [15]) it is assumed that one is given smooth local conditions at all primes at which \( \bar{\rho} \) is ramified, i.e., smooth quotients of the framed local deformation rings, and these should have sufficiently large dimension. One of the key improvements of this method in [14] was that we were able to dispense with the smoothness condition, but the dimension condition cannot be relaxed if we want our lifts to be geometric. However, if we only require that our local lifts be trianguline at primes dividing \( p \) then the dimension condition can indeed be relaxed, but we face the problem that trianguline lifts are not parametrised by a quotient of the universal framed deformation ring. §2 is devoted to showing how we can avoid this difficulty: in §2.1 we explain how the method of [14, §4] can be modified to deal with more general local conditions, and in §2.2 we show that this can be applied in the setting of trianguline lifts using the properties of the “trianguline variety” of Breuil–Hellmann–Schraen [6, Théorème 2.6] (and the generalisation thereof in [12]). Once these local improvements are in place the proofs of the main results follow by specialising Theorem 3.6 to the case where the local lifts are assumed to be trianguline.

Acknowledgements. We thank Sandeep Varma for making us aware of [12] and Vincent de Daruvar for making it available to us.

2. Cocycle constructions

2.1. A general cocycle construction. Let \( E \) be a finite extension of \( \mathbb{Q}_p \), \( \mathcal{O} \) its ring of integers, \( m_\mathcal{O} = (\varpi) \) its maximal ideal and \( k = \mathcal{O}/m_\mathcal{O} \) its residue field. Let \( G \) be a reductive group scheme over \( \mathcal{O} \) (possibly disconnected), and let \( g \) be its Lie algebra. Let \( G^\text{der} \) be the derived group of the identity component of \( G \) and \( g^\text{der} \) its Lie algebra.

Let \( \Gamma \) be any topologically finitely generated profinite group. Given a continuous homomorphism \( \bar{\rho} : \Gamma \to G(k) \), there exists a complete local Noetherian \( \mathcal{O} \)-algebra \( R^{\mathcal{O},\text{univ}} \) with residue field \( k \) and a homomorphism \( \rho^{\text{univ}} : \Gamma \to G(R^{\mathcal{O},\text{univ}}) \) representing the functor of lifts of \( \bar{\rho} \) on the category of complete local Noetherian \( \mathcal{O} \)-algebras with residue field \( k \). For any such algebra \( A \), we set \( \hat{G}(A) := \ker(G(A) \to G(k)) \). The conjugation action of \( \hat{G}(\mathcal{O}) \) on \( G(R^{\mathcal{O},\text{univ}}) \) and the universal property of \( \rho^{\text{univ}} \) induces an action of \( \hat{G}(\mathcal{O}) \) on \( R^{\mathcal{O},\text{univ}} \).

Let \( R \) be a quotient of \( R^{\mathcal{O},\text{univ}} \) by an ideal which is invariant under this action. We assume that \( R \) is reduced and flat over \( \mathcal{O} \). We also assume that \( R \) corresponds to “fixed-multiplier” liftings of \( \bar{\rho} \), by which we mean that the map \( \Gamma \to G/G^{\text{der}}(R) \) induced from \( \rho^{\mathcal{O},\text{univ}} \) by the maps \( R^{\mathcal{O},\text{univ}} \to R \) and \( G \to G/G^{\text{der}} \), factors through a fixed map \( \Gamma \to G/G^{\text{der}}(\mathcal{O}) \).\(^5\)

For any representation \( \rho : \Gamma \to G(\mathcal{O}) \) we will denote by \( \rho_r \) the reduction of \( \rho \) modulo \( \varpi^r \). Also, for any representation \( \rho : \Gamma \to G(\mathcal{O}/\varpi^r) \), \( \rho_r(g^{\text{der}}) \) will denote \( g^{\text{der}} \otimes_\mathcal{O} \mathcal{O}/\varpi^r \) equipped with the \( \text{Ad} \circ \rho_r \) action.

\(^5\)This is not essential, but it is convenient for our applications to impose this condition. The results below hold in general if one removes all occurrences of the superscript “der”.
The following result is essentially contained in [14, §4].

**Proposition 2.1.** Assume that Spec($R$) has an $O$-valued point $y$ such that the corresponding point of Spec($R[1/\varpi]$) is contained in the smooth locus, and let $\rho : \Gamma \to G(O)$ be the corresponding lift of $\tilde{\rho}$. Then there is an open (in the $p$-adic topology) subset $Y \subset$ Spec($R$)($O$) with $y \in Y$ having the following properties: let $Y_n$ be the image of $Y$ in Spec($R$)($O/\varpi^n$) and for integers $n, r \geq 0$ let $\pi^Y_{n,r} : Y_{n+r} \to Y_n$ be the natural maps.

1. Given $r_0 > 0$ there exists $n_0 > 0$ such that for all $n \geq n_0$ and $0 \leq r \leq r_0$ the fibers of $\pi^Y_{n,r}$ are nonempty principal homogenous spaces over a submodule $Z_r \subset Z^1(\Gamma, \rho_r(^g\text{der}))$ which is free over $O/\varpi^r$ of rank $d$, where $d$ is the dimension of Spec($R[1/\varpi]$) at $y$.
2. $B^1(\Gamma, \rho_r(^g\text{der})) \subset Z_r$.
3. The $O$-module inclusions $O/\varpi^{r-1} \to O/\varpi^r$, mapping 1 to $\varpi$, and the surjections $O/\varpi^r \to O/\varpi^{r-1}$ induce inclusions $Z_{r-1} \to Z_r$ and surjections $Z_r \to Z_{r-1}$.
4. Let $L_r$ be the image of $Z_r$ in $H^1(\Gamma, \rho(\text{der}) \otimes O/\varpi^r)$. The groups $L_r$ are compatible with the maps on cohomology induced by the inclusions $O/\varpi^{r-1} \to O/\varpi^r$ and the surjections $O/\varpi^r \to O/\varpi^{r-1}$.
5. $|L_r| = |O/\varpi|^d \cdot |\rho_r(\text{der})|^1 \cdot |\rho_r(\text{der})|^{-1}$.

**Proof.** This is proved in [14, Proposition 4.7] (see [14, Lemma 4.5] for (2)) for $\Gamma$ the absolute Galois group of a local field and some specific choices of rings $R$ for which the dimension $d$ is also known, but the proof given there works without any changes under the conditions that we have given. In particular, the proof of the formula for $|L_r|$ there extends to give the formula in (5).

The goal of this subsection is to formulate a generalisation of Proposition 2.1 which will be the key to our construction of trianguline lifts in §3.4, allowing us to circumvent the fact that there is no quotient of $R^2,\text{univ}$ parametrizing the trianguline lifts of $\tilde{\rho}$. In the next Proposition, $R$ is as above, but we emphasize that the manifold $U$ will no longer be assumed open in Spec($R[1/\varpi]$). For $m > 0$, let $(G^{\text{der}})^{(m)}(O) = \ker(G^{\text{der}}(O) \to G^{\text{der}}(O/\varpi^m))$. We then have the following:

**Proposition 2.2.** Let $U$ be a compact $E$-adic manifold of some dimension $d$ contained in the smooth locus of Spec($R[1/\varpi]$)($E$) with $v \in U$, and let $\rho$ be the lift of $\tilde{\rho}$ corresponding to $v$. Assume that there exists $m > 0$ such that the $(\widetilde{G^{\text{der}}})^{(m)}(O)$-orbit of $v$ is contained in $U$. Then there is an open (in the $p$-adic topology) subset $V \subset U$ with $v \in V$ having the following properties: let $V_n$ be the image of $V$ in Spec($R$)($O/\varpi^n$) and for integers $n, r \geq 0$ let $\pi^V_{n,r} : V_{n+r} \to V_n$ be the natural maps.

1. Given $r_0 > 0$ there exists $n_0 > 0$ such that for all $n \geq n_0$ and $0 \leq r \leq r_0$ the fibers of $\pi^V_{n,r}$ are nonempty principal homogenous spaces over a submodule $Z_r \subset Z^1(\Gamma, \rho_r(^g\text{der}) \otimes O/\varpi^r)$ which is free over $O/\varpi^r$.
2. $B^1(\Gamma, \rho_r(^g\text{der}) \otimes O/\varpi^r) \subset Z_r$.
3. The $O$-module inclusions $O/\varpi^{r-1} \to O/\varpi^r$, mapping 1 to $\varpi$, and the surjections $O/\varpi^r \to O/\varpi^{r-1}$ induce inclusions $Z_{r-1} \to Z_r$ and surjections $Z_r \to Z_{r-1}$.
4. Let $L_r$ be the image of $Z_r$ in $H^1(\Gamma, \rho(^g\text{der}) \otimes O/\varpi^r)$. The groups $L_r$ are compatible with the maps on cohomology induced by the inclusions $O/\varpi^{r-1} \to O/\varpi^r$ and the surjections $O/\varpi^r \to O/\varpi^{r-1}$.
5. $|L_r| = |O/\varpi|^d \cdot |\rho_r(\text{der})|^1 \cdot |\rho_r(\text{der})|^{-1}$. 
Proof. Fix \( r_0 \). We then apply Proposition 2.1 by taking \( y = u \) there and get an open
set \( Y \subset \text{Spec}(R[1/\varpi])(\mathcal{O}) \) with \( u \in Y \), an integer \( n^Y_0 \) and cocycles \( Z^Y_r \) satisfying all the
conclusions there.

Replacing \( U \) by \( U \cap Y \) we may assume that \( U \subset Y \). By applying the result of Serre [21, Proposition 11] to the inclusion \( v \in U \) in the same way as in the proof of [14, Lemma 4.3] we get an open submanifold \( V \subset U \) with \( v \in V \) and free \( \mathcal{O}/\varpi^r \) submodules \( Z^Y_r \subset Z^V_r \) with the
property in (1), with the proviso that the \( n_0 \) associated to \( r_0 \) might be greater than the \( n^Y_0 \) obtained as an output of Proposition 2.1. (The inclusion \( Z^Y_r \subset Z^V_r \) follows from the definitions since \( U \subset Y \) so each \( U_n \subset Y_n \).

Using the assumption on the \((G^\text{der})^{(m)}(\mathcal{O})\)-orbit of \( v \) we see that (2) holds in the same way as in the proof of [14, Lemma 4.5].

The surjectivity part of (3) is clear from the defining property of the \( Z^r \)’s. To prove the injectivity part, we must examine the proof of [14, Lemma 4.3] (since there is no analogue of [14, Lemma 4.4] in the present setting) from which we obtain the \( Z^r \). That proof proceeds by first choosing a formally smooth complete Noetherian \( \mathcal{O} \)-algebra \( A \) and a surjection of \( \mathcal{O} \)-algebras \( A \rightarrow R \). The \( Z^r \) (corresponding to \( U \)) are then constructed as certain submodules of \( \text{Hom}_\mathcal{O}(\Omega_{A/\mathcal{O}} \otimes_{A,y} \mathcal{O}, \mathcal{O}/\varpi^r) \) by choosing local equations for \( U \) in the \( E \)-adic manifold \( \text{Spec}(A)(\mathcal{O}) \) and reducing to the case that \( U \) corresponds to the \( E \)-valued points of a formally smooth quotient \( R' \) of \( A \). It is an immediate consequence of this construction that the map

\[
\text{Hom}_\mathcal{O}(\Omega_{A/\mathcal{O}} \otimes_{A,y} \mathcal{O}, \mathcal{O}/\varpi^r) \rightarrow \text{Hom}_\mathcal{O}(\Omega_{A/\mathcal{O}} \otimes_{A,y} \mathcal{O}, \mathcal{O}/\varpi^{r-1})
\]

induced by the inclusion \( \mathcal{O}/\varpi^{r-1} \rightarrow \mathcal{O}/\varpi^r \) maps \( Z^r_{r-1} \) injectively into \( Z^r \) (since this is true when \( A \) is replaced by \( R' \)). The map \( A \rightarrow R \) induces compatible (with respect to \( \mathcal{O}/\varpi^{r-1} \rightarrow \mathcal{O}/\varpi^r \)) inclusions \( \text{Hom}_\mathcal{O}(\Omega_{R/\mathcal{O}} \otimes_{R,y} \mathcal{O}, \mathcal{O}/\varpi^r) \rightarrow \text{Hom}_\mathcal{O}(\Omega_{A/\mathcal{O}} \otimes_{A,y} \mathcal{O}, \mathcal{O}/\varpi^r) \) from which (3) follows once we identify \( \text{Hom}_\mathcal{O}(\Omega_{R/\mathcal{O}} \otimes_{A,y} \mathcal{O}, \mathcal{O}/\varpi^r) \) with a submodule of \( Z^1(\Gamma, \rho(g^{\text{der}}) \otimes \mathcal{O}/\varpi^r) \) as in [14, §4.2].

Finally, (4) follows immediately from (3) and the definition of \( L_r \) and (5) follows as in Proposition 2.1.

2.2. Cocycles for trianguline deformations. Let \( F \) be a finite extension of \( \mathbb{Q}_p \) and let \( \Gamma_F = \text{Gal}(\overline{F}/F) \) be its absolute Galois group. In this subsection we recall some facts about trianguline representations of \( \Gamma_F \) and carry out the local analysis needed in order to be able to construct the cocycles used to prove the existence of global lifts of \( \text{GL}_n \)-odd representations which are trianguline at primes above \( p \). For a general introduction to trianguline representations the reader may consult [3] (and also [12] for the \( G \)-valued case), but we will only use the existence and some properties of the rigid-analytic variety of (local) trianguline lifts from [6] (and [12] for the \( G \)-valued case). However, to orient the reader not familiar with this notion, we mention briefly that trianguline representations are defined by embedding the category of \( E \)-adic representations of \( \Gamma_F \) (for \( E/\mathbb{Q}_p \) a finite extension) fully faithfully into a larger category, the category of \( (\phi, \Gamma) \)-modules, which are modules over a (Robba)
ring \( \mathcal{R} \) with some extra structure. A Galois representation is said to be trianguline if the corresponding \( (\phi, \Gamma) \)-module has a filtration by subobjects such that the successive subquotients are free of rank one as \( \mathcal{R} \)-modules; the parameter \( \delta \) occurring below corresponds to the ordered tuple of these subquotients. The condition for a \( G \)-valued representation to be trianguline is formulated in Tannakian terms using the tensor structure on the category of \( (\phi, \Gamma) \)-modules or in the language of principal \( G \)-bundles over \( \mathcal{R} \) (with extra structure).
2.2.1. We will first consider the case of $\text{GL}_n$ since the results for general $G$ are slightly weaker and partly conditional. So let $G = \text{GL}_n$ and let $\bar{\rho} : \Gamma_F \to G(k)$ be as in §2.1. Let $\mathfrak{X}_\bar{\rho}$ be the rigid analytic space over $E$ corresponding to the formal scheme $\text{Spf}(R^\square,\text{univ})$, where $R^\square,\text{univ}$ is the universal lifting ring of $\bar{\rho}$; its $E$-valued points are canonically equal to the $\mathcal{O}$-valued points of $\text{Spec}(R^\square,\text{univ})$ and similarly for all finite extensions of $E$. Let $\mathcal{T}$ denote the rigid analytic space over $\mathbb{Q}_p$ parametrising the continuous characters of $F^\times$. Let $X_{\text{tri}}^\square(\bar{\rho}) \subset \mathfrak{X}_\bar{\rho} \times \mathcal{T}_E$ be the space of trianguline deformations of $\bar{\rho}$ as in [6, Définition 2.4] (but our notation is slightly different). It is a Zariski closed rigid analytic subvariety of $\mathfrak{X}_\bar{\rho} \times \mathcal{T}_E$ defined as the Zariski closure of a subset $U_{\text{tri}}^\square(\bar{\rho})^{\text{reg}}$. The points $(x, \delta)$ of the latter consist of certain trianguline lifts $x$ of $\bar{\rho}$ together with a system of parameters $\delta$ of a triangulation of the associated $(\phi, \Gamma)$-module; the assumption is that $\delta$ is regular in the sense explained in the paragraph before [6, Définition 2.4]. For our purposes what is important is that $x$ corresponds to a trianguline lift, and the precise nature of $\delta$ plays no explicit role. We call the trianguline lifts of $\bar{\rho}$ which correspond to points of $X_{\text{tri}}^\square(\bar{\rho})$ good; we do not know whether all trianguline lifts are good.

The following result of Breuil, Hellmann and Schraen is the key input for the construction of cocyles for trianguline lifts.

Lemma 2.3.

(1) The space $X_{\text{tri}}^\square(\bar{\rho})$ is non-empty and equidimensional of dimension $n^2 + [F : \mathbb{Q}_p]\frac{n(n+1)}{2}$.

(2) $U_{\text{tri}}^\square(\bar{\rho})^{\text{reg}}$ is a smooth Zariski open subvariety of $X_{\text{tri}}^\square(\bar{\rho})$ which is Zariski dense.

(3) The projection map $\pi_1 : X_{\text{tri}}^\square(\bar{\rho}) \to \mathfrak{X}_\bar{\rho}$ is an immersion at all points of $U_{\text{tri}}^\square(\bar{\rho})^{\text{reg}}$.

Proof. This is a part of [6, Théorème 2.6]. The third part is not contained in the statement there but is used in the proof, where it is deduced from results of Bellaïche and Chenevier [2, §2.3]. We note that the proof of non-emptiness of $U_{\text{tri}}^\square(\bar{\rho})^{\text{reg}}$ in [6] uses the existence of regular crystalline lifts of $\bar{\rho}$; this is a highly non-trivial result for general $\bar{\rho}$ and was proved only recently in [13].

The group $\widehat{\text{GL}}_n(\mathcal{O})$ acts on $\mathfrak{X}_\bar{\rho} \times \mathcal{T}_E$ via its action on the first factor and this action preserves $U_{\text{tri}}^\square(\bar{\rho})^{\text{reg}}$ and $X_{\text{tri}}^\square(\bar{\rho})$.

For our application we need to work with lifts of $\bar{\rho}$ which have a fixed determinant. To arrange this we consider the morphism $\text{det} : \mathfrak{X}_\bar{\rho} \to \mathfrak{X}_{\text{det}(\bar{\rho})}$ which sends any lift of $\bar{\rho}$ to its determinant and the induced morphism $\text{det}_1 := \text{det} \circ \pi_1 : X_{\text{tri}}(\bar{\rho}) \to \mathfrak{X}_{\text{det}(\bar{\rho})}$. The space $\mathfrak{X}_{\text{det}(\bar{\rho})}$ has dimension $1 + [F : \mathbb{Q}_p]$ and the morphism $\text{det}_1$ commutes with the action of $\widehat{\text{GL}}_n(\mathcal{O})$, where the action on $\mathfrak{X}_{\text{det}(\bar{\rho})}$ by tensor product with $\chi^{\otimes n}$.

If $\chi$ is a character of $\Gamma_F$ with values in a finite extension $E'$ of $E$ with trivial reduction modulo its maximal ideal, then tensor product with $\chi$ induces an automorphism of $X_{\text{tri}}^\square(\bar{\rho})_{E'}$ which preserves $U_{\text{tri}}^\square(\bar{\rho})^{\text{reg}}$. This is compatible with the morphism $\text{det}_1$ if we let $\chi$ act on $\mathfrak{X}_{\text{det}(\bar{\rho})}$ by tensor product with $\chi^{\otimes n}$.

If $p \nmid n$, then any character $\chi$ as above has an $n$-th root. This implies that in this case all the fibres of $\text{det}_1$ over $\mathfrak{X}_{\text{det}(\bar{\rho})}(E')$ are isomorphic, for any finite extension $E'$ of $E$. It follows that for any point $y$ of $\mathfrak{X}_{\text{det}(\bar{\rho})}$, $U_{\text{tri}}^\square(\bar{\rho})^{\text{reg}} \cap (\text{det}_1)^{-1}(y)$ is Zariski dense in $(\text{det}_1)^{-1}(y)$.

Lemma 2.4. Let $\bar{\rho} : \Gamma_F \to \text{GL}_n(k)$ be a continuous homomorphism, and let $\rho : \Gamma_F \to \text{GL}_n(\mathcal{O})$ be a lift of $\bar{\rho}$ corresponding to a point $(x, \delta) \in X_{\text{tri}}^\square(\bar{\rho})(E)$. Assume $p \nmid n$. Then, after replacing $E$ by a finite extension if necessary, the following holds:
(1) There is a quotient $R$ of $R^{\square, \text{univ}}$ by an ideal invariant under the action of $\widehat{\text{GL}}_n(\mathcal{O})$ which is reduced and flat over $\mathcal{O}$.

(2) There is a $E$-adic submanifold $U$ of $\text{Spec}(R[1/\varpi])(E)$ of dimension $d = n^2 - 1 + [F : \mathbb{Q}_p] \left( \frac{n(n+1)}{2} - 1 \right)$ such that all the points of $U$ correspond to regular trianguline lifts $\rho'$ of $\bar{\rho}$ with $\det(\rho) = \det(\rho')$.

(3) Given any integer $N > 0$, $U$ can be chosen such that all $\rho'$ corresponding to points of $U$ are congruent to $\rho$ modulo $\varpi^N$.

(4) There is a point $v \in U$ and an integer $m > 0$ such that the $(\widehat{\text{GL}}_n)^{(m)}(\mathcal{O})$-orbit of $v$ is contained in $U$.

Proof. By replacing $E$ with a finite extension if necessary, we may (and do) ensure that $U^{\square}_{\text{tri}}(\bar{\rho})^{\text{reg}}(E) \neq \emptyset$, so by Lemma 2.3 it has a natural structure of an $E$-adic manifold of dimension $n^2 + [F : \mathbb{Q}_p] \left( \frac{n(n+1)}{2} \right)$. Since the space $X_{\det(\rho)}$ has dimension $1 + [F : \mathbb{Q}_p]$ and all (non-empty) fibres of $\det_1$ are geometrically isomorphic (since $p \nmid n$), it follows from the results of [22, §III.10] that $\mathcal{T} := (\det_1)^{-1}(\det_1((x, \delta)))$ is of pure dimension $d = n^2 - 1 + [F : \mathbb{Q}_p] \left( \frac{n(n+1)}{2} - 1 \right)$ and has a smooth Zariski dense open subset $\mathcal{T}'$ consisting of points in $U^{\square}_{\text{tri}}(\bar{\rho})^{\text{reg}}$.

Now using [14, Lemma 4.9][6]—this might again require replacing $E$ with a finite extension—we can find a sequence of points $(x'_N, \delta'_N) \in \mathcal{T}'(E)$, $N = 1, 2, \ldots$, such that the lift of $\bar{\rho}$ corresponding to $x'_N$ is congruent to $\rho$ modulo $\varpi^N$. For any such $N$, let $U' \subset \mathcal{T}'(E)$ be an $E$-adic neighbourhood of $(x'_N, \delta'_N)$ such that all the lifts of $\rho$ corresponding to points of $U'$ are congruent to $\rho$ modulo $\varpi^N$. By shrinking $U'$ further using (3) of Lemma 2.3, we may and do assume that $\pi_1$ induces an embedding of $U'$ onto a compact $E$-adic submanifold $U$ of $\text{Spec}(R^{\square, \text{univ}}[1/\varpi])(E)$.

Let $\tilde{U}' = \text{GL}_n(\mathcal{O}) \cdot U' \subset \mathcal{T}'(E)$ and let $R$ be the reduced quotient of $R^{\square, \text{univ}}$ corresponding to the Zariski closure in $\text{Spec}(R^{\square, \text{univ}})$ of $\pi_1(\tilde{U}')$. Since $\pi_1(\tilde{U}')$ is preserved by the action of $\text{GL}_n(\mathcal{O})$, the kernel of the quotient map $R^{\square, \text{univ}} \to R$ is also preserved by this group. By the Zariski density of $\pi_1(\tilde{U}')$ in $\text{Spec}(R)$ and the reducedness of $\text{Spec}(R)$, we may find a point $v \in \pi_1(\tilde{U}')$ which is a smooth point of $\text{Spec}(R[1/\varpi])$. The $\text{GL}_n(\mathcal{O})$-equivariance of $\pi_1$ implies that we may take $v = \pi_1(v')$ for some $v' \in U'$. Replacing $U'$ by a compact open neighbourhood of $v'$ we may assume that $\pi_1(U')$ is contained in the smooth locus of $\text{Spec}(R[1/p])(E)$.

Letting $U = \pi_1(U')$ and $v = \pi_1(v')$, it is clear from the above that (1), (2) and (3) hold. To prove (4), we note that $\text{GL}_n(\mathcal{O})$ acts continuously on $\mathcal{T}'(E)$. Since $U'$ is an open neighbourhood of $v'$ in $\mathcal{T}'(E)$, and the sequence of subgroups $(\text{GL}_n)^{(m)}(\mathcal{O})$ of $\text{GL}_n(\mathcal{O})$ forms a neighbourhood basis of the identity, for all $m \gg 0$ the $(\text{GL}_n)^{(m)}(\mathcal{O})$-orbit of $v'$ is contained in $U'$. The $\text{GL}_n(\mathcal{O})$-equivariance of $\pi_1$ then implies that for all $m \gg 0$ the $(\text{GL}_n)^{(m)}(\mathcal{O})$-orbit of $v$ is contained in $U$.

Remark 2.5. If $\bar{\rho}$ is upper triangular with respect to some basis, then the points of $U^{\square}_{\text{tri}}(\bar{\rho})^{\text{reg}}$ corresponding to pairs $(x, \delta)$ with $x$ being upper triangular with respect to some basis form a non-empty $\text{GL}_n(\mathcal{O})$-invariant open subspace: this is a consequence of the inductive nature of the proof of existence of regular crystalline lifts in [13]. In this case one may arrange that

---

[6] The lemma as stated does not quite apply here, but its proof gives what we claim.
all the points of the $U$ constructed in Lemma 2.4 correspond to lifts of $\tilde{\rho}$ which are upper triangular with respect to some basis.

2.2.2. Now suppose $G$ is a connected split reductive group over a $p$-adic field $E$. Then the notion of trianguline representations of $\Gamma_F$ valued in $G(E)$ is defined in [12] using $(\phi, \Gamma)$-modules with $G$-structure, which are defined using Tannakian methods. We do not recall the details of the construction, but only mention that this definition generalizes the usual definition of trianguline representations in a natural way. The main result of [12], Theorem 6.22, is an analogue for $G$-valued representations of [6, Théorème 2.6], with some modifications and conditions which we briefly explain.

For $\tilde{\rho} : \Gamma_F \to G(k)$, we let $R_{\square, \text{univ}}$ and $\mathcal{X}_\tilde{\rho}^\square$ be as before. For $T$ a maximal split torus of $G$ contained in a Borel subgroup $B$, we let $T^\vee$ be the dual torus and we replace $\mathcal{T}_E^a$ by $\hat{T}^\vee$, the rigid analytic space over $E$ parametrising (continuous) characters of $T^\vee(F)$. Then $X_{\text{tri}}(\tilde{\rho}) \subset \mathcal{X}_\tilde{\rho}^\square \times \hat{T}^\vee$ is defined as the Zariski closure of a subset $U_{\text{tri}}(\tilde{\rho})^{\text{vreg}}$ of very regular lifts. The points of the latter correspond to pairs $(x, \delta)$ where $x$ is a trianguline lift of $\tilde{\rho}$ in the sense of [12, Definition 4.9] with a triangulation such that the associated parameter $\delta$ is very regular in the sense of [12, Definition 6.1]. We then have the following analogue of Lemma 2.3.

**Lemma 2.6.** Assume $G$ has no factors of type $G_2$, $F_4$ and $E_8$ and $\tilde{\rho}$ has a very regular trianguline lift. Then

1. The space $X_{\text{tri}}(\tilde{\rho})$ is non-empty and equidimensional of dimension $\dim(G) + [F : \mathbb{Q}_p] \dim(B)$.
2. $U_{\text{tri}}(\tilde{\rho})^{\text{vreg}}$ is a smooth Zariski open subvariety of $X_{\text{tri}}(\tilde{\rho})$ which is Zariski dense.
3. The projection map $\pi_1 : X_{\text{tri}}(\tilde{\rho}) \to \mathcal{X}_\tilde{\rho}^\square$ is an immersion at all points of $U_{\text{tri}}(\tilde{\rho})^{\text{vreg}}$.

It is expected that the condition on $G$ is unnecessary, the condition on $\tilde{\rho}$ is always satisfied, and “very regular” can be replaced by “regular”.

**Proof.** This is part of [12, Theorem 6.22], the last part being a consequence of the proof using [12, Proposition 6.6].

As in the case of $\text{GL}_n$, we would like to work with “fixed determinant” lifts, where now by “determinant” we mean the map $\det : G \to G/G^\text{der}$, with $G^\text{der}$ being the derived group of $G$. This induces a map $\det : \mathcal{X}_\tilde{\rho}^\square \to \mathcal{X}_\det(\tilde{\rho})$ which sends any lift of $\tilde{\rho}$ to its determinant, and we have an induced morphism $\det_1 := \det \circ \pi_1 : X_{\text{tri}}(\tilde{\rho}) \to \mathcal{X}_\det(\tilde{\rho})$. Let $n$ be the order of the kernel of the isogeny $Z(G) \to G/G^\text{der}$, where $Z(G)$ is the connected centre of $G$.

**Lemma 2.7.** Assume $G$ has no factors of type $G_2$, $F_4$ and $E_8$, and that $p \nmid n$. Let $\tilde{\rho} : \Gamma_F \to G(k)$ be a continuous homomorphism and let $\rho : \Gamma_F \to G(\mathcal{O})$ be a lift of $\tilde{\rho}$ corresponding to a point $(x, \delta) \in X_{\text{tri}}(\tilde{\rho})(E)$. Then, after replacing $E$ by a finite extension if necessary, the following holds:

1. There is a quotient $R$ of $R_{\square, \text{univ}}$ by an ideal invariant under the action of $\widehat{G}(\mathcal{O})$ which is reduced and flat over $\mathcal{O}$.
2. There is a compact $E$-adic submanifold $U$ of $\text{Spec}(R[1/\varpi])(E)$ of dimension $d = \dim(G^\text{der}) + [F : \mathbb{Q}_p](\dim(B) - \dim(Z(G)))$ such that all the points of $U$ correspond to very regular trianguline lifts $\rho'$ of $\tilde{\rho}$ with $\det(\rho') = \det(\rho')$.
3. Given any integer $N > 0$, $U$ can be chosen such that all $\rho'$ corresponding to points of $U$ are congruent to $\rho$ modulo $\varpi^N$. 
(4) There is a point \( v \in U \) and an integer \( m > 0 \) such that the \( \widehat{G}^{(m)}(\mathcal{O}) \)-orbit of \( v \) is contained in \( U \).

Proof. This is proved in essentially the same way as Lemma 2.4 using Lemma 2.6 instead of Lemma 2.3. \( \square \)

3. Trianguline lifting theorems

In this section we formulate a general lifting theorem for representations of global fields.

3.1. Selmer and relative Selmer groups. Let \( F \) be a global field, and let \( E, \mathcal{O}, \varpi \) be as before. Let \( G \) be a reductive group scheme over \( \mathcal{O} \), \( G^{\text{der}} \) the derived group of its identity component and \( g^{\text{der}} \) the Lie algebra of \( G^{\text{der}} \). For some \( n \geq 1 \) let \( \rho_n : \Gamma_F \to G(\mathcal{O}/\varpi^n) \) be a continuous homomorphism. Let \( S \) be a finite set of primes containing all primes at which \( \rho_n \) is ramified and also all primes dividing \( p \). We first recall some definitions and results from [14].

In what follows, when we have an integer \( 0 < r < n \), we will write \( \rho_r \) for the reduction \( \rho_n \mod \varpi^r \). For each prime \( v \in S \) we assume that for \( 0 < r \leq n \) we have subgroups 
\[
Z_{r,v} \subset Z^1(\Gamma_{F,v}, \rho_r(g^{\text{der}}))
\]
such that
- Each \( Z_{r,v} \) contains the group of boundaries \( B^1(\Gamma_{F,v}, \rho_r(g^{\text{der}})) \).
- As \( r \) varies, the inclusion and reduction maps induce short exact sequences
\[
0 \to Z_{a,v} \to Z_{a+b,v} \to Z_{b,v} \to 0
\]
as in Proposition 2.2.

We let \( L_{r,v} \subset H^1(\Gamma_{F,v}, \rho_r(g^{\text{der}})) \) be the image of \( Z_{r,v} \), and we let \( L_{r,v}^\perp \subset H^1(\Gamma_{F,v}, \rho_r(g^{\text{der}})^*) \) be the annihilator of \( L_{r,v} \) under the local duality pairing. Let \( \mathcal{L}_r = \{ L_{r,v} \}_{v \in S}, 0 < r \leq n \), and similarly define \( \mathcal{L}_r^\perp \). The Selmer group \( H^1_{\mathcal{L}_r}(\Gamma_{F,S}, \rho_r(g^{\text{der}})) \) is defined to be
\[
\ker \left( H^1(\Gamma_{F,S}, \rho_r(g^{\text{der}})) \to \bigoplus_{v \in S} H^1(\Gamma_{F,v}, \rho_r(g^{\text{der}})) / L_{r,v} \right)
\]
and the dual Selmer group \( H^1_{\mathcal{L}_r^\perp}(\Gamma_{F,S}, \rho_r(g^{\text{der}})^*) \) is defined analogously.

The definition below is a variant of the definition of balanced in [14, Definition 6.2] (which is the condition when \( a = 0 \)).

**Definition 3.1.** We say that the local conditions \( \mathcal{L}_r \), for \( 0 < r \leq n \), are semi-balanced if there exists a non-negative integer \( a \) such that
\[
|H^1_{\mathcal{L}_r}(\Gamma_{F,S}, \rho_r(g^{\text{der}}))| = |\mathcal{O}/\varpi^r|^a \cdot |H^1_{\mathcal{L}_r^\perp}(\Gamma_{F,S}, \rho_r(g^{\text{der}})^*)|
\]
for all \( 0 < r \leq n \).

The basic objects that we need to control in order to prove lifting results using the methods of [14] and [15] are the relative (dual) Selmer groups of [14, Definition 6.2], so we recall them here:

**Definition 3.2.** For \( 0 < r \leq n \), we define the \( r \)-th relative Selmer group to be
\[
\overline{H}^1_{\mathcal{L}_r}(\Gamma_{F,S}, \rho_r(g^{\text{der}})) := \text{Im} \left( H^1_{\mathcal{L}_r}(\Gamma_{F,S}, \rho_r(g^{\text{der}})) \to H^1_{\mathcal{L}_1}(\Gamma_{F,S}, \rho(g^{\text{der}})) \right)
\]
and the \( r \)-th relative dual Selmer group to be
\[
\overline{H}^1_{\mathcal{L}_r^\perp}(\Gamma_{F,S}, \rho_r(g^{\text{der}})^*) := \text{Im} \left( H^1_{\mathcal{L}_r^\perp}(\Gamma_{F,S}, \rho_r(g^{\text{der}})^*) \to H^1_{\mathcal{L}_1^\perp}(\Gamma_{F,S}, \rho(g^{\text{der}})^*) \right).
\]
3.2. Adequate cocycles and semi-balancedness. We continue with the notation from §3.1, but now assume that $F$ is a number field. We set $\tilde{\rho} = \rho_1$ and for each infinite place $v$ of $F$ we let $c(\tilde{\rho}, v) = \dim(\mathfrak{g}^{\text{der}})_{\Gamma_{F_v}}$.

The definition below is a modification of [15, Definition 1.4]; it is useful for lifting representations which are more general than the odd representations considered there.

**Definition 3.3.** We say that the $O$-submodules $L_{r,v} \subset H^1(\Gamma_{F_v}, \rho_r(\mathfrak{g}^{\text{der}}))$ are adequate if for each finite place $v$ there exists an integer $a_v$ such that $a_v = 0$ for all but finitely many $v$,

$$|L_{r,v}| = |\rho_r(\mathfrak{g}^{\text{der}})_{\Gamma_{F_v}}| \cdot |\mathcal{O}/\mathcal{w}^r|^{a_v},$$

and $\sum_{v \text{ finite}} a_v \geq \sum_{v | \infty} c(\tilde{\rho}, v)$.

The following lemma is a variant of [14, Lemma 6.3].

**Lemma 3.4.** If the collection of $\mathcal{O}/\mathcal{w}^r$-modules $\mathcal{L}_r$ is adequate and the spaces of invariants $\tilde{\rho}(\mathfrak{g}^{\text{der}})_{\Gamma_F}$ and $(\tilde{\rho}(\mathfrak{g}^{\text{der}})^*)_{\Gamma_F}$ are both zero, then the relative Selmer and dual Selmer groups are also semi-balanced in the sense that

$$\dim(H^1_{\mathcal{L}_n}(\Gamma_{F_S}, \rho_n(\mathfrak{g}^{\text{der}}))) \geq \dim(H^1_{\mathcal{L}_n}(\Gamma_{F_S}, \rho_n(\mathfrak{g}^{\text{der}})^*))$$

**Proof.** The hypotheses together with the Greenberg-Wiles formula [11, Theorem 2.18] imply that the Selmer and dual Selmer groups are semi-balanced: the integer $a$ of Definition 3.1 is $\sum_{v \text{ finite}} a_v - \sum_{v | \infty} c(\tilde{\rho}, v)$. From this the proof of semi-balancedness of the relative Selmer and dual Selmer group follows from [14, Lemma 6.1] as in the proof of [14, Lemma 6.3]. $\square$

3.3. The general lifting theorem. As above, let $F$ be a global field and $\tilde{\rho} : \Gamma_F \to G(k)$ be a continuous representation, where $k$ is a finite field of characteristic $p$ and $G$ is a split reductive group over the ring of integers $O$ of a finite extension $E/\mathbb{Q}_p$; the group $G$ need not be connected, and if this is the case we assume that the component group of $G$ has order prime to $p$ and $G$ is a semi-direct product of its identity component $G^0$ and the component group. Let $\tilde{F}$ be the smallest extension of $F$ such that $\tilde{\rho}(\Gamma_{\tilde{F}}) \subset G^0(k)$ and let $K = F(\tilde{\rho}, \mu_p)$.

We let $\bar{\mu} : \Gamma_F \to (G/G^{\text{der}})(k)$ be the map induced by $\tilde{\rho}$ and the quotient map $G \to G/G^{\text{der}}$, and we fix a lift $\mu : \Gamma_F \to G(O)$ of $\bar{\mu}$.

**Assumption A.**

1. $\tilde{\rho}$ and $\mu$ are unramified outside a finite set of places $S$ of $F$ containing all places of $F$ over $p$ if $F$ is a number field.
2. If $F$ is a function field we assume that $p \neq \text{char}(F)$.
3. $H^1(\text{Gal}(K/F), \tilde{\rho}(\mathfrak{g}^{\text{der}})^*) = 0$.
4. $\tilde{\rho}(\mathfrak{g}^{\text{der}})$ and $\tilde{\rho}(\mathfrak{g}^{\text{der}})^*$ do not contain the trivial representation as a submodule.
5. There is no surjection of $F_p[\Gamma_F]$-modules from $\tilde{\rho}(\mathfrak{g}^{\text{der}})$ onto any $F_p[\Gamma_F]$-module subquotient of $\tilde{\rho}(\mathfrak{g}^{\text{der}})^*$.

We also need some local assumptions on $\tilde{\rho}$ which we formulate separately.

**Assumption B.** There is a finite extension $E'$ of $E$ with ring of integers $O'$ such that for each finite place $v \in S$ there exists a lift $\rho_v$ of $\tilde{\rho}|_{\Gamma_{F_v}}$ to a continuous homomorphism $\Gamma_{F_v} \to G(O')$ with fixed determinant $\mu|_{\Gamma_{F_v}}$. Furthermore, $\rho_v$ is an element of an $E$-adic manifold $U_v$ contained in $\text{Spec}(\tilde{H}^\infty_{\tilde{\rho}|_{\Gamma_{F_v}}}(O'))$, with fixed determinant as above, and which is
invariant under conjugation by $\hat{\Gam}(O')$ and of dimension $d_v$, with $d_v = \dim_k(\mathfrak{g}^\text{der})$ for each finite $v \nmid p$. Finally,

\begin{equation}
\sum_{v \text{ finite}} (d_v - \dim(\mathfrak{g}^\text{der})) \geq \sum_{v \mid \infty} c(\bar{\rho}, v).
\end{equation}

**Remark 3.5.**

1. We usually choose $U_v$ to lie in the $\hat{O}'$-points of an irreducible component of $\text{Spec}(R)$, where $R$ is a $\hat{\Gam}(O')$-invariant (or $\hat{\Gam}(\hat{\Gam})$)-invariant quotient of the universal local lifting ring of $\bar{\rho}|_{\Gam_F}$.

2. The question of the existence of $U_v$ is most delicate for primes $v \mid p$, but for arbitrary groups $G$ it is still unknown whether a lift $\rho_v$ always exists even for other (ramified) primes. In most applications, if $v \nmid p$ then we choose $U_v$ so that $d_v = \dim_k(\mathfrak{g}^\text{der})$ (see [14], Proposition 4.7]; this is the maximal possible), in which case $a_v = 0$.

The following theorem is now an easy consequence of the results of [14] and [15].

**Theorem 3.6.** Let $\bar{\rho}: \Gam_F \to G(k)$ satisfy Assumptions A and B and assume that $p \gg G$. Then there exists a finite set of places $S' \supset S$ such that $\bar{\rho}$ lifts to a continuous representation $\rho: \Gam_{F,S'} \to G(O')$ satisfying:

1. The image $\rho(\Gam_{F,S'})$ intersects $G^\text{der}(O')$ in an open subgroup.
2. $\rho|_{\Gam_{F,v}}$ is an element of $U_v$ for all $v \in S$ and $\rho$ can be chosen so that $\rho|_{\Gam_{F,v}}$ is, modulo $G^\text{der}(O')$-conjugation, congruent to $\rho_v$ modulo any pre-specified power of $\infty$.

**Proof.** Assumptions A and the existence of $\rho_v$ in Assumption B imply that all the arguments of Sections 3 and 4 of [15] go through without any changes to construct lifts $\rho_n$ as in Theorem 4.4 of loc. cit.. The point is that neither the archimedean primes, nor the dimensions $d_v$, play any role in the proof of this theorem. To complete the proof we must explain how to carry out the “relative deformation theory” arguments of [15, Section 6].

To do this, we apply Proposition 2.2 to the sets $U_v$ for each $v \in S$ to produce cocycles $Z_{r,v}$ for all $r \leq n$. The condition on $d_v$ in Assumption B and (5) of Proposition 2.2 implies that the $L_{r,v}$ constructed from the $Z_{r,v}$ are adequate in the sense of Definition 3.3. Then by Lemma 3.4, the relative Selmer and dual Selmer groups are semi-balanced. The first part of Assumption 5.1 of [15] is part of (4) of Assumption A above, and this together with the existence of local lifts giving rise to semi-balanced relative Selmer and dual Selmer groups is exactly what is needed for the proof of [15, Theorem 5.1] to go through to produce a lift $\rho$ satisfying all the claimed properties. \hfill \Box

### 3.4. Trianguline lifts of $GL_n$-odd representations of number fields.

Before proving our main theorem we need the following:

**Lemma 3.7.** If $\bar{\rho}: \Gam_F \to GL_n(k)$ is a $GL_n$-odd representation, then for every real place $v$ of $F$,

\begin{equation}
\c(\bar{\rho}, v) = \begin{cases} 2m^2 - 1 & \text{if } n = 2m \text{ is even,} \\ 2m^2 + 2m & \text{if } n = 2m + 1 \text{ is odd.} \end{cases}
\end{equation}

**Proof.** This is an elementary computation. \hfill \Box

We now restate and prove our main theorem.
Theorem 3.8. Let $F$ be a number field and $\bar{\rho} : \Gamma_{F,S} \to \text{GL}_n(k)$ a $\text{GL}_n$-odd representation. If $p \gg_n 0$ and $\bar{\rho}$ satisfies all the conditions in Assumption A then there exists a finite set of places $S' \supset S$ and a finite extension $E'$ of $E$ with ring of integers $\mathcal{O}'$ such that $\bar{\rho}$ lifts to a $\text{GL}_n$-odd representation $\rho : \Gamma_{F,S'} \to \text{GL}_n(\mathcal{O}')$ which is regular trianguline at all primes of $F$ above $p$. Furthermore, one can also ensure that $\rho(\Gamma_{F,S'})$ contains an open subgroup of $\text{SL}_n(\mathcal{O}')$.

Proof. The proof is an application of Theorem 3.6.

We begin by choosing any lift $\mu$ of $\bar{\rho}$. The conditions of Assumption A hold tautologically, so we only need to show that the conditions of Assumption B also hold.

Since the group $G$ here is $\text{GL}_n$, for $v \nmid p$ this holds by [9, §2.4.4] with $d_v = \dim_v(\mathfrak{g}^\text{der})$. The main change that we need to make is in the local arguments for primes $v \mid p$. For each such prime, we apply Lemma 2.4 (with the $F$ there being $F_v$) to obtain a set $U_v$ of trianguline lifts of $\bar{\rho}|_{\Gamma_{F_v}}$. For these $U_v$ we have by (2) of Lemma 2.4 $d_v = n^2 - 1 + [F_v : \mathbb{Q}_p]\left(\frac{n(n+1)}{2} - 1\right)$.

Since $\sum_{v \mid p}[F_v : \mathbb{Q}_p] = [F : \mathbb{Q}]$, we get $\sum_{v \mid p} d_v = \sum_{v \mid p}(n^2 - 1) + [F : \mathbb{Q}]\left(\frac{n(n+1)}{2} - 1\right)$. On the other hand $\sum_{v \mid \infty} c(\bar{\rho}, v) = \sum_v \text{real } c(\bar{\rho}, v) + \sum_v \text{complex } c(\bar{\rho}, v)$. Using Lemma 3.7 we see that if $v$ is a real place then $c(\bar{\rho}, v) \leq \left(\frac{n(n+1)}{2} - 1\right)$ and if $v$ is a complex place then $c(\bar{\rho}, v) = n^2 - 1 - 2\left(\frac{n(n+1)}{2} - 1\right)$. Using this it follows that Equation (3.2) holds, so the proof is complete. 

Remark 3.9. It seems reasonable to expect that Assumption A and the assumption that $p \gg_n 0$ are superfluous and trianguline lifts as in Theorem 3.8 exist as long as $\bar{\rho}$ is $\text{GL}_n$-odd.

Remark 3.10. If $\bar{\rho}|_{\Gamma_{F_v}}$ is upper triangular for a prime $v$ above $p$, using Remark 2.5 one may arrange that the same holds for $\rho|_{\Gamma_{F_v}}$.

Remark 3.11. The proof of Theorem 3.8 actually gives a family of (fixed-determinant) lifts of $\bar{\rho}$ (unramified outside a fixed finite set $S'$ and trianguline at all primes above $p$) parametrised by a $([F : \mathbb{Q}]\left(\frac{n(n+1)}{2} - 1\right) - \sum_{v \mid \infty} c(\bar{\rho}, v))$-dimensional $E$-adic manifold.

Corollary 3.12. Let $F$ be an arbitrary number field and $\bar{\rho} : \Gamma_{F,S} \to \text{GL}_n(k)$ a $\text{GL}_n$-odd representation. If $p \gg_n 0$ and $\bar{\rho}|_{\Gamma_{F,S'}}$ is absolutely irreducible, then there exists a finite set of places $S' \supset S$ and a finite extension $E'$ of $E$ with ring of integers $\mathcal{O}'$ such that $\bar{\rho}$ lifts to a $\text{GL}_n$-odd representation $\rho : \Gamma_{F,S'} \to \text{GL}_n(\mathcal{O}')$ which is regular trianguline at all primes of $F$ above $p$. Furthermore, one can also ensure that $\rho(\Gamma_F)$ contains an open subgroup of $\text{SL}_n(\mathcal{O}')$.

Proof. This follows immediately from Theorem 3.8 once we note that the irreducibility assumption implies that (3), (4) and (5) of Assumption A automatically hold; see [14, Corollary A.7] for a more general statement that holds for arbitrary $G$.

Remark 3.13. The recent results of [4] show that the second bulleted assumption in [14, Theorem 6.21] is always satisfied for $G = \text{GL}_n$. Consequently, if $F$ is an arbitrary number field, and we drop the $\text{GL}_n$-odd assumption in Theorem 3.8, we can construct characteristic zero lifts which are unramified outside a finite set of primes, but without any control at primes dividing $p$.

Remark 3.14. As far as we are aware, it is not known whether or not every $\text{GL}_n$-odd representation has a geometric lift, even when $F$ is $\mathbb{Q}$—the examples of Calegari mentioned in the introduction are not $\text{GL}_n$-odd—and this question is closely related to the question whether
torsion cohomology Hecke eigenclasses of arithmetic subgroups of \( GL_n \) lift to characteristic zero after passing to a finite index subgroup. However, Conjecture 1.2.3 of [16] combined with Theorem 3.8 implies that any irreducible representation \( \bar{\rho} \) satisfying the hypotheses of the theorem is the reduction of the Galois representation associated to the cohomology of an arithmetic subgroup of \( GL_n \) with coefficients in a suitable infinite-dimensional module (see [17, §5]) which is a \( \mathbb{Q}_p \)-vector space.

### 3.5. Trianguline lifts for representations into connected reductive groups

Using Lemma 2.7 (and the results of [14] and [15]) we now prove a generalisation of Theorem 3.8 for representations valued in general split connected reductive groups \( G \) satisfying the assumptions of the lemma. Before we can do this though we need an analogue of the definition of \( GL_n \)-odd used for \( GL_n \). The definition below, motivated by the numerics of the trianguline local condition, is weaker than \( GL_n \)-odd when applied to \( GL_n \), but it still suffices to prove our lifting result.

**Definition 3.15.**

1. Let \( K \) be any field and \( G \) a split reductive group (not necessarily connected) over \( K \). An involution \( c \in G(K) \) is said to be \( t\)-odd if \( \dim(g^{\operatorname{der}})_c \leq \dim(B^{\operatorname{der}}) \), where \( B^{\operatorname{der}} \) is a Borel subgroup of \( G^{\operatorname{der}} \).

2. Let \( F \) be a number field, \( K \) any topological field, and \( G \) as above. We say that a continuous representation \( \bar{\rho} : \Gamma_F \to G(K) \) is \( t\)-odd if for every real place \( v \) of \( F \) and \( c_v \in \Gamma_{F_v} \), the corresponding complex conjugation, the involution \( \bar{\rho}(c_v) \in G(K) \) is \( t\)-odd as in (1).

We then have the following generalisation of Theorem 3.8.

**Theorem 3.16.** Let \( F \) be a number number field, \( G \) a connected and split reductive group, and \( \bar{\rho} : \Gamma_{F,S} \to G(k) \) a \( t\)-odd continuous representation. Assume \( p \gg_G 0 \), \( \bar{\rho} \) satisfies the conditions of Assumption \( A \) and \( G \) has no factor of type \( G_2, F_4 \) or \( E_8 \). Fix a lift \( \mu \) of \( \bar{\rho} \) as in 3.3 and also assume that after possibly replacing \( E \) by a finite extension, for each finite prime \( v \) of \( F \), \( \bar{\rho} \mid_{\Gamma_{F_v}} \) has a lift to \( G(\mathcal{O}) \) with determinant \( \mu \mid_{\Gamma_{F_v}} \) which is trianguline and very regular if \( v \nmid p \). Then there exists a finite set of places \( S' \supset S \) and an extension \( E' \) of \( E \) with ring of integers \( \mathcal{O}' \) such that \( \bar{\rho} \) lifts to a \( t\)-odd representation \( \rho : \Gamma_{F,S'} \to G(\mathcal{O}') \) with determinant \( \mu \) which is very regular trianguline at all primes of \( F \) above \( p \). Furthermore, one can also ensure that \( \rho(\Gamma_{F,S'}) \) contains an open subgroup of \( G^{\operatorname{der}}(\mathcal{O}') \).

**Proof.** The proof is essentially the same as that of Theorem 3.8 after noting that the definition of \( t\)-odd is designed precisely to ensure that the cocycles obtained by applying Proposition 2.2 and Lemma 2.7 for primes \( v \nmid p \) and [14, Proposition 4.7] for other primes are adequate in the sense of Definition 3.3. This ensures, by Lemma 3.4, that the (relative) Selmer and dual Selmer groups are semi-balanced which is the key condition needed for the arguments of [14] to go through as in the proof of Theorem 3.8. \( \square \)

It will be clear to the reader that using this theorem one may also prove an analogue of Corollary 3.12, but we do not formulate it explicitly here. The \( t\)-odd assumption in the theorem can also be replaced by the slightly weaker condition \( \sum_{v|\infty} \dim(g^{\operatorname{der}})_{\Gamma_{F_v}} \leq |F : \mathbb{Q}| \dim(B^{\operatorname{der}}) \).

**Remark 3.17.** The existence of lifts for primes \( v \nmid p \) has been proved for some groups other than \( GL_n \) by Booher [5] and very regular trianguline (in fact, crystalline) lifts for primes \( v \mid p \) are known to exist when \( \bar{\rho} \mid_{\Gamma_{F_v}} \) is absolutely irreducible by recent work of Lin [19].
References

[1] F. Andreatta, A. Iovita, and V. Pilloni, *Le halo spectral*, Ann. Sci. Éc. Norm. Supér. (4), 51 (2018), pp. 603–655.

[2] J. Bellaïche and G. Chenevier, *Families of Galois representations and Selmer groups*, Astérisque, (2009), pp. xii+314.

[3] L. Berger, *Trianguline representations*, Bull. Lond. Math. Soc., 43 (2011), pp. 619–635.

[4] G. Böckle, A. Iyengar, and V. Paškūnas, *On local Galois deformation rings*, 2021. https://arxiv.org/abs/2110.01638.

[5] J. Booher, *Minimally ramified deformations when ℓ ≠ p*, Compos. Math., 155 (2019), pp. 1–37.

[6] C. Breuil, E. Hellmann, and B. Schraen, *Une interprétation modulaire de la variété trianguline*, Math. Ann., 367 (2017), pp. 1587–1645.

[7] F. Calegari, *Even Galois representations and the Fontaine–Mazur conjecture. II*, J. Amer. Math. Soc., 25 (2012), pp. 533–554.

[8] A. Caraiani and B. V. Le Hung, *On the image of complex conjugation in certain Galois representations*, Compos. Math., 152 (2016), pp. 1476–1488.

[9] L. Clozel, M. Harris, and R. Taylor, *Automorphy for some ℓ-adic lifts of automorphic mod ℓ Galois representations*, Publ. Math. Inst. Hautes Études Sci. (2008), pp. 1–181. With Appendix A, summarizing unpublished work of Russ Mann, and Appendix B by Marie-France Vignéras.

[10] P. Colmez, *Représentations triangulines de dimension 2*, no. 319, 2008, pp. 213–258. Représentations p-adiques de groupes p-adiques. I. Représentations galoisienes et (φ, Γ)-modules.

[11] H. Darmon, F. Diamond, and R. Taylor, *Fermat’s last theorem*, in Elliptic curves, modular forms & Fermat’s last theorem (Hong Kong, 1993), Internat. Press, Cambridge, MA, 1997, pp. 2–140.

[12] V. de Daruvar, *Représentations triangulines avec G-structure*, PhD thesis, L’Institut Polytechnique de Paris, 2020.

[13] M. Emerton and T. Gee, *Moduli stacks of étale (φ, Γ)-modules and the existence of crystalline lifts*, https://arxiv.org/abs/1908.07185, 2019.

[14] N. Fakhruddin, C. Khare, and S. Patrikis, *Relative deformation theory, relative Selmer groups, and lifting irreducible Galois representations*, Duke Math. J., 170 (2021), pp. 3505–3599.

[15] ——, *Lifting and automorphy of reducible mod p Galois representations over global fields*, Invent. Math., 228 (2022), pp. 415–492.

[16] D. Hansen, *Universal eigenvarieties, trianguline Galois representations, and p-adic Langlands functoriality*, J. Reine Angew. Math., 730 (2017), pp. 1–64. With an appendix by James Newton.

[17] C. Johansson and J. Newton, *Extended eigenvarieties for overconvergent cohomology*, Algebra Number Theory, 13 (2019), pp. 93–158.

[18] M. Kisin, *Overconvergent modular forms and the Fontaine-Mazur conjecture*, Invent. Math., 153 (2003), pp. 373–454.

[19] Z. Lin, *Crystalline lifts and a variant of the Steinberg–Winter theorem*, https://arxiv.org/abs/2011.08766, 2020.

[20] R. Ramakrishna, *Deforming Galois representations and the conjectures of Serre and Fontaine-Mazur*, Ann. of Math. (2), 156 (2002), pp. 115–154.

[21] J.-P. Serre, *Quelques applications du théorème de densité de Chebotarev*, Inst. Hautes Études Sci. Publ. Math., (1981), pp. 323–401.

[22] ——, *Lie algebras and Lie groups*, vol. 1500 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2006. 1964 lectures given at Harvard University, Corrected fifth printing of the second (1992) edition.
School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, INDIA
Email address: naf@math.tifr.res.in

UCLA Department of Mathematics, Box 951555, Los Angeles, CA 90095, USA
Email address: shekhar@math.ucla.edu

Department of Mathematics, Ohio State University, 100 Math Tower, 231 West 18th Ave., Columbus, OH 43210, USA
Email address: patrikis.1@osu.edu