Critical Temperature Associated to Symmetry Breaking of Klein–Gordon fields versus Condensation Temperature in a Weakly interacting Bose–Einstein Gas

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We deduce the relation between the critical temperature associated to the $U(1)$ symmetry breaking of scalar fields with one–loop correction potential immersed in a thermal bath, and the condensation temperature of the aforementioned system in the thermodynamic limit, within the semiclassical approximation for a weakly interacting bosonic gas with a positive coupling constant. Additionally, we show that the shift in the condensation temperature caused by the coupling constant is independent of the thermal bath.

I. INTRODUCTION

Since its observation with the help of magnetic traps [1], the phenomenon of Bose–Einstein condensation has spurred an enormous amount of works on the theoretical and experimental realms associated to this topic. The principal interest in the study on Bose–Einstein condensation is its interdisciplinary nature. From the thermodynamic point of view, this phenomenon can be interpreted as a phase transition, and from the quantum mechanical point of view as a matter wave coherence arising from overlapping de Broglie waves of the atoms, in which many of them condense to the ground state of the system. In quantum field theory, this phenomenon is related to the spontaneous symmetry breaking of a gauge symmetry [2].

Symmetry breaking is one of the most essential concepts in particle theory and has been extensively used in the study of the behavior of particle interactions in many theories [3]. The concept with the accompanying wave function describing the condensate, was first introduced in explaining superconductivity and super fluidity [4]. Phase transitions are changes of state, related with changes of symmetries of the system. The analysis of Symmetry breaking mechanisms have turn out to be very helpful in the study of phenomena associated to phase transitions in almost all areas of physics. Bose-Einstein Condensation is one topic of interest that uses in an extensive way the Symmetry breaking mechanisms [2], its phase transition associated with the condensation of atoms in the state of lowest energy and is the consequence of quantum, statistical and thermodynamical effects.

On the other hand, the results from finite temperature quantum field theory [5, 6] raise important challenges about their possible manifestation in condensed matter systems. By investigating the massive Klein–Gordon equation we can be able, in principle, to simulate a condensed matter system. Since many particles have mass, this can be an essential step in building realistic analogue models. Here, our aim is to study model made up of a real scalar field together with the possibility that this SF might undergo a phase transition as the temperature of the system is lowered. To analyze the scalar field undergone by the system we take a model having an $U(1)$ symmetry, considering the easiest case of a double-well interacting potential (Mexican-hat potential) for a real scalar field $\Phi(\vec{x}, t)$ that goes as

$$V(\Phi) = -\frac{m^2 c^2 \Phi^2}{\hbar^2} + \frac{\lambda}{4\hbar^2 c^2} \Phi^4.$$  \hspace{1cm} (1)

Here one important idea, to which we shall refer, is that of identifying the order parameter which characterizes the phase transition with the value of the real scalar quantum field $\Phi$. From quantum field theory we know that the dynamics of a SF is governed by the Klein-Gordon equation, it is the equation of motion of a field composed of spinless particles. In this case we will add a external field that will interact with the SF to first order, such that the KG equation will be given by

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2\right)\Phi + \frac{dV(\Phi)}{d\Phi} + \frac{2m^2 c^2}{\hbar^2} \Phi \phi = 0.$$  \hspace{1cm} (2)
where the external potential here is denoted by $\phi$. Here we will be interested in studying the properties, behavior and interactions of the scalar field with other particles, all of them interacting inside a thermal bath in a reservoir that can have interaction with its surroundings. In this case, the scalar field can be described by the potential extended to one loop

$$V(\Phi) = -\frac{m^2 c^2}{\hbar^2} \Phi^2 + \frac{\lambda}{4\hbar^2 c^2} \Phi^4 + \frac{\lambda}{8\hbar^2 c^2} \kappa_B^2 T^2 \Phi^2 + \frac{\pi^2}{90 \hbar^2 c^2} \kappa_B^4 T^4.$$  \hspace{1cm} (3)

Where $m$ is the mass of the scalar field, $\kappa_B$ is the Boltzmann’s constant, and $\lambda$ is the coupling constant.

On the other hand, the analysis of a Bose–Einstein condensation phenomena in the ideal case, weakly interacting, and with a finite number of particles in different kinds of potentials, shows that the main thermodynamic properties associated to the condensate, and in particular the condensation temperature, depends strongly of the characteristics of the trapped potential in question.

II. CRITICAL TEMPERATURE FROM THE SYMMETRY BREAKING OF KLEIN–GORDON FIELDS

Let us calculate the critical temperature associated with the Klein–Gordon equation, with the corresponding potential extended to one loop and immersed in a thermal bath (see [18]). When the temperature $T$ is high enough, one of the minimums of the potential is $\Phi = 0$. We suppose that the temperature is sufficiently small so that the interaction between the scalar field and the rest of matter decouples. After this moment, the term proportional to $T^4$ is not longer important, as for sufficiently low $T$ the term that goes as $T^4$ can be dropped out. The critical temperature where the minimum of the potential $\Phi = 0$ becomes a maximum is

$$\kappa_B T_c^{SB} = \frac{2 m c^2}{\sqrt{\lambda}}.$$ \hspace{1cm} (4)

being $\kappa_B$ the Boltzmann constant. This temperature is the temperature at which the symmetry of the system is broken. We notice also, that in the case when $\lambda \to 0$, the critical temperature $T_c^{SB} \to \infty$.

Nevertheless, the critical temperature for the break down of the symmetry is not necessarily a sign of condensation, $T_c^{SB}$ could be different to the critical temperature of condensation. The main goal of this work is to compare both temperatures and to give some ideas how to compare them with experiments in the laboratory.

III. CONDENSATION TEMPERATURE IN A WEAKLY INTERACTING BOSE–EINSTEIN GAS

Let us calculated the condensation temperature associated to the aforementioned system and its relation with the critical temperature $T_c^{SB}$. To this goal, let us insert plane waves in equation in order to obtain the dispersion relation between energy and momentum of a single particle, or the semiclassical spectrum associated to the KG equation in a thermal bath, with the result (we considered here the low velocities limit)

$$E_p \simeq m c^2 + \frac{p^2}{2m} + \frac{\lambda}{2mc^2} \Phi^2 + \frac{\lambda}{4mc^2} (\kappa_B T)^2 + mc^2 \phi.$$ \hspace{1cm} (5)

Where we interpreted $\kappa^2 |\Phi(\vec{r}, t)|^2$ as the density $n(\vec{r}, t)$ of the “cloud”, being $\kappa$ the scale of the system to be determined by the experiment, with dimensions of $[Energy^{-2} \text{Length}^{-3}]$. Additionally, we assume that the system is in static thermal equilibrium so that $n(\vec{r}, t) \approx n(\vec{r})$.

Hence

$$|\Phi|^2 \equiv \kappa^{-2} n(\vec{r}).$$ \hspace{1cm} (6)

Using (6) and neglecting the rest mass, we can re–write the semiclassical energy spectrum (5) as follows

$$E_p \simeq \frac{p^2}{2m} + \frac{\lambda}{2mc^2} \kappa^{-2} n(\vec{r}) + \frac{\lambda}{4mc^2} (\kappa_B T)^2 + mc^2 \phi.$$ \hspace{1cm} (7)
Is noteworthy to mention that if we set $\lambda \equiv 16\pi \hbar^2 c^2 \kappa^2 \alpha$ and the trapping potential $\phi \equiv \alpha r^2$, with $\alpha \equiv 1/2(\omega_0/c)^2$, where $\alpha$ is the scattering length and $\omega_0$ is the frequency of an isotropic harmonic oscillator say, we obtain the semiclassical energy spectrum in the Hartree–Fock approximation for a bosonic gas trapped in an external potential (an isotropic harmonic oscillator according to our previous assumptions), but with an extra term due to the contributions of the thermal bath. The Hartree–Fock approximation consists basically in the assumption that the constituents of the gas behave like a non–interacting bosons, moving in a self–consistent mean field and is valid when $E_p >> \mu$ for dilute gases [21, 22].

On the other hand, within the semiclassical approximation, the spatial density $n(\vec{r})$ reads [21, 22]

$$n(\vec{r}) = \frac{1}{(2\pi\hbar)^3} \int d^3\vec{p} \, n(\vec{r}, \vec{p}). \quad (8)$$

Where $n(\vec{r}, \vec{p})$ is the Bose–Einstein distribution function given by [21, 22]

$$n(\vec{r}, \vec{p}) = \frac{1}{e^{E_p/E_\mu} - 1}. \quad (9)$$

In expression (9), $\mu$ is the chemical potential, $\beta = 1/\kappa_B T$ and $E_p$ is the semiclassical energy spectrum.

The number of particles in the 3–dimensional space obey the normalization condition [21, 22],

$$N = \int d^3\vec{r} \, n(\vec{r}). \quad (10)$$

Let us calculate the condensation temperature. Integrating the spatial density (8) in the momentum space, using the semiclassical energy spectrum (7) associated with the KG equation (2), together with the Bose–Einstein distribution function (9) leads us to

$$n(\vec{r}) = \frac{2\pi}{(2\pi\hbar)^3} (2m\kappa_B T)^{3/2} \Gamma\left(\frac{3}{2}\right) g_{3/2}(z(\vec{r})). \quad (11)$$

Where $g_\nu(z)$ is the Bose–Einstein function, defined by [22, 23]

$$g_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^{\infty} z^{\nu-1} e^{-x} \frac{dx}{x^{\nu-1} e^{x} - 1}. \quad (12)$$

and $z(\vec{r})$ is given by

$$z(\vec{r}) = e^{\beta(\mu - \frac{\lambda}{2m^2} \kappa^2 \alpha r^2 - \frac{\lambda}{4m^2} (\kappa_B T)^2 - mc^2 \phi)} - n(\vec{r}) - \frac{\lambda}{2m^2} \kappa^2 \alpha r^2. \quad (13)$$

With the use of the harmonic oscillator–type–potential $\phi$

$$\phi = \alpha r^2. \quad (14)$$

function $z(\vec{r})$ can be written as follows

$$z(\vec{r}) = e^{\beta(\mu - \frac{\lambda}{2m^2} \kappa^2 \alpha \alpha r^2 - \frac{\lambda}{4m^2} (\kappa_B T)^2 - mc^2 \alpha r^2)}. \quad (15)$$

In the case of a free gas in a box with $\lambda = 0$, $z(\vec{r})$ is just the fugacity $z = \exp(\beta\mu)$ [22, 23].

In order to calculate the normalization condition (10), let us expand the spatial density (11) at first order in the coupling constant $\lambda$, using the properties of the Bose–Einstein functions, [23]

$$x \frac{\partial}{\partial x} g_\nu(x) = g_{\nu-1}(x). \quad (16)$$
With these assumptions the density \( n \) can be written as follows:

\[
n \approx n_0(r) = \lambda \left[ \kappa^{-2} \frac{(\kappa_B T)^2}{mc^2} \left( \frac{(2m)^{3/2}2\pi \Gamma(3/2)}{(2\pi h)^3} \right)^2 g_{3/2} \left( e^{\beta(\mu-\alpha mc^2r^2)} \right) g_{1/2} \left( e^{\beta(\mu-\alpha mc^2r^2)} \right) \right]^{1/3} \tag{17}
\]

where,

\[
n_0(r) = \frac{2\pi}{(2\pi h)^3} (2m\kappa_B T)^{3/2} \Gamma \left( \frac{3}{2} \right) g_{3/2} \left( e^{\beta(\mu-\alpha mc^2r^2)} \right), \tag{18}
\]

is the density for the ideal case \( \lambda = 0 \).

Integrating the normalization condition (10), using (17), with the corresponding potential (14) allows us obtain an expression for the number of particles in function of the chemical potential \( \mu \), the coupling constant \( \lambda \), and the temperature:

\[
N \approx \frac{8\pi^2}{(2\pi h)^3} (2m)^{3/2} \frac{\Gamma \left( \frac{3}{2} \right)^2}{2} (\alpha mc^2)^{-3/2} g_{3}(e^{\beta\mu})(\kappa_B T)^3
\]

\[
- \frac{4\pi^2}{2mc^2} \kappa \frac{(\kappa_B T)^{5/2}}{(2\pi h)^3} \left( \frac{(2m)^{3/2}2\pi \Gamma(3/2)}{(2\pi h)^3} \right)^2 G_{3/2}(e^{\beta\mu})(\alpha mc^2)^{-3/2}(\kappa_B T)^{7/2} \tag{19}
\]

\[
+ \frac{8\pi^2}{2mc^2(2\pi h)^3} (2m)^{3/2} \frac{\Gamma \left( \frac{3}{2} \right)^2}{2} (\alpha mc^2)^{-3/2} g_{2}(e^{\beta\mu})(\kappa_B T)^4.
\]

where

\[
G_{3/2}(z) = \sum_{i,j=1}^{\infty} \frac{z^{i+j}}{i! j! \beta^3/2 (i+j)^{3/2}} \tag{20}
\]

being \( z = \exp(\beta\mu) \) the fugacity. If we set \( \lambda = 0 \) in expression (19), we may obtain the expression for the number of particles for the ideal case. At the condensation temperature in the thermodynamic limit in the case of an ideal bosonic gas, \( \mu = 0 \), and assuming that the number of particles in the ground state above the condensation temperature is negligible, allows us obtain an expression for the condensation temperature in the ideal case in the thermodynamic limit \( T_0 \), given by

\[
\kappa_B T_0 = \left( \frac{N}{\Omega(3)} \right)^{1/3} \tag{21}
\]

where

\[
\Omega = \frac{8\pi^2}{(2\pi h)^3} (2m)^{3/2} \Gamma(3/2)^2 (\alpha mc^2)^{-3/2}, \tag{22}
\]

and \( \Omega(3) \) is the Riemann Zeta function.

In order to obtain the leading correction in the shift for the critical temperature respect to \( T_0 \) caused by the coupling constant \( \lambda \) and the thermal bath in our bosonic gas, let us expand the expression (19) at first order in \( T = T_0, \mu = 0 \), and \( \lambda = 0 \), remaining that \( T_0 \) is the critical temperature in the thermodynamic limit given by expression (21), with the result

\[
N = 4\pi\Omega(\alpha mc^2)^{-3/2} \Gamma(3/2) \left[ \zeta(3)(\kappa_B T_0)^3 + \zeta(3)3(\kappa_B T_0)^2 \kappa_B[T - T_0] \tag{23}
\]

\[
+ (\kappa_B T_0)^2 \zeta(2) \mu - \frac{\lambda\kappa^{-2}}{2mc^2} \Omega G_{3/2}(1)(\kappa_B T_0)^{7/2} \right].
\]
At the condensation temperature the chemical potential in the semiclassical approximation is given by

$$\mu_c = \frac{\lambda \kappa^{-2}}{2mc^2} n(r = 0) + \frac{\lambda}{4mc^2} (\kappa_B T_0)^2. \quad (24)$$

As it is suggested from expression (11), (24) basically corresponds to the definition of the chemical potential at the condensation temperature in the usual case [21], except for the extra term contribution due to the thermal bath.

Inserting (24) in (23) at the condensation temperature, this allows us to obtain the shift in the condensation temperature for the corresponding system in function of the number of particles

$$\frac{\Delta T_c}{T_0} \simeq -\lambda \kappa^{-2} \alpha^{1/4} \frac{m^{1/2}}{e^{3/2} \hbar^{3/2} \Theta N^{1/6}}. \quad (25)$$

where

$$\Theta = 2^{1/2} \left( \frac{\zeta(3) \zeta(2) - G_{3/2}(1)}{3(2\pi)^{5/4} \zeta(3)} \right)^{1/6} \frac{1}{8\pi^2 \Gamma(3/2)^2 \zeta(3)}. \quad (26)$$

Surprisingly, the thermal bath does not contribute to the shift in the condensation temperature, because of the definition for the chemical potential given in expression (24) at the condensation temperature. In other words, the condensation temperature is independent of the thermal bath, within the semiclassical approximation.

If we set \( \lambda = 16\pi \hbar^2 c^2 \kappa^2 a \), and \( \alpha = 1/2(\omega_0/c)^2 \), where \( a \) is the scattering length and \( \omega_0 \) is the frequency of trap assuming an isotropic harmonic oscillator, from equation (25) we recover the expression for the shift in the critical temperature for a weakly interacting Bose–Einstein gas given in [9]

$$\frac{\Delta T_c}{T_0} \simeq -1.3 \left( \frac{a}{\alpha_0} \right) N^{1/6}. \quad (27)$$

where we used the usual definition for the characteristic length of the oscillator

$$a_{\alpha_0} = \left( \frac{\hbar}{m \omega_0} \right)^{1/2}. \quad (28)$$

Analyzing the critical temperature caused by the symmetry breaking expression (4), with respect to the shift in the condensation temperature expression (24), assuming from the very beginning that \( \lambda > 0 \), we notice that when \( \lambda \to 0 \), then \( T_c^{SB} \to \infty \) and the shift \( \Delta T_c/T_0 \to 0 \), which means \( T_c \to T_0 \). This implies that \( T_c^{SB} > T_0 > T_c \), assuming \( \lambda > 0 \). In other words, at certain temperature \( T_c^{SB} \), we have first the occurrence of the symmetry breaking and then at the temperature \( T_c \), we have the condensation.

From expression (24) and (25), we obtain a relation between \( T_c^{SB} \) and \( \Delta T_c/T_0 \) given by

$$T_c^{SB} = \left( 1 - T_r \right) \frac{1}{\kappa_B} \frac{mc}{h} \frac{\kappa^{-1/8}}{2\Theta} \left( \frac{2\Theta}{(2\pi)^{5/4}} \right) N^{1/12}. \quad (29)$$

where \( T_r = T_c/T_0 \) and \( \Theta \) is defined in expression (26). In the case of a harmonic oscillator with \( \lambda = 16\pi \hbar^2 c^2 \kappa^2 a \), and \( \alpha = 1/2(\omega_0/c)^2 \) expression (29) becomes

$$T_c^{SB} = \left( 1 - T_r \right) \frac{5.2}{8\pi \alpha_0} \left( \frac{5.2}{8\pi \alpha_0} \right) N^{1/12}. \quad (30)$$

IV. ESTIMATION OF THE SCALE \( \kappa \)

To estimate the order of magnitude of the scale \( \kappa \), let us resort to the definition of the healing length \( \xi \) [22]. The healing length is a crossover between the phonon spectrum and the single-particle spectrum of the Bogoliubov excitations [24]. Additionally, the healing length is related to the chemical potential below the condensation temperature through the next expression [22]
$$\frac{\hbar^2}{2m\xi^2} = \mu. \quad (31)$$

where $\mu$ below the condensation temperature is given by (24)

$$\mu = \frac{\lambda\kappa^{-2}}{2mc^2} n(\vec{r} = 0) + \frac{\lambda}{4mc^2} (\kappa B T_0)^2. \quad (32)$$

Additionally, $n(\vec{r} = 0)$ is the density in the center of the condensate. Using the relation between the healing length and the chemical potential, allows us to obtain the next expression

$$\xi = \sqrt{\frac{\hbar c}{\lambda \left( \kappa^{-2} n + (\kappa B T_0)^2 / 2 \right)}}. \quad (33)$$

Resorting to the relation between $\lambda$ and the scattering length $a$, $\lambda = 16\pi\hbar^2c^2\kappa^2a$, we obtain for the scale $\kappa$,

$$\kappa = \sqrt{\frac{\xi^{-2} - 16\pi an}{8\pi a(\kappa B T_0)^2}}. \quad (34)$$

where $n$ is the density in the center of the condensate. The order of magnitude of the healing length in typical experiments is approximately (or bigger than) one micron [25]. In the case of $^{87}$Rb and $^{23}$Na $\xi \sim 0.4 \mu$m [21], the scattering length $a \approx 5.77$ nm for $^{87}$Rb, and $a \approx 2.74$ nm for $^{23}$Na, and assuming $T_0 \approx 200 \times 10^{-9} K$ say, together with values of the density $n$ given approximately by $10^{13}$ to $10^{15}$ atoms per cm$^{-3}$ [21], allows us to infer the order of magnitude for the scale $\kappa$ in ordinary units

$$\kappa \approx 3 \times 10^{39} \text{Joules}^{-1} \text{Meters}^{-3/2}. \quad (35)$$

V. CONCLUSIONS

Starting from the Klein–Gordon equation and the mechanism given in quantum field theory for the break down of the symmetry of the system, we where able to reduce this Klein–Gordon equation to a generalized Gross-Pitaevskii equation (see [18]). The question that arrises is whether the critical temperature for the break down of the symmetry of the system and the critical temperature of the condensation are somehow related or whether they are of the same order of magnitude. In this work we have given an answer. We have analyzed the relation between the critical temperature associated to the spontaneous symmetry breaking of scalar fields in a thermal bath characterized by a Klein–Gordon equation and the corresponding condensation temperature for a weakly interacting bosonic gas. We obtain the semiclassical energy spectrum associated to this system, from which we deduced the spatial density of the aforementioned system, interpreting $\kappa^2$ as the scale that relates the usual definition of spatial density within the semiclassical approximation and the density associated to the self–interacting part of the potential in the Klein–Gordon equation. We show that the condensation temperature is independent of the thermal bath, with the aforementioned conditions, and we prove that the critical temperature associated to the symmetry breaking is bigger than the shift in the condensation temperature, when the coupling constant is positive. Additionally, we estimate the order of magnitude of the scale, which according to expression (35) is very small for typical experiments in our low energy world.

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