Exponential Boundary Observers for Pressurized Water Pipe

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Abstract. This paper deals with state estimation on a pressurized water pipe modeled by nonlinear coupled distributed hyperbolic equations for non-conservative laws with three known boundary measures. Our objective is to estimate the fourth boundary variable, which will be useful for leakage detection. Two approaches are studied. Firstly, the distributed hyperbolic equations are discretized through a finite-difference scheme. By using the Lipschitz property of the nonlinear term and a Lyapunov function, the exponential stability of the estimation error is proven by solving Linear Matrix Inequalities (LMIs). Secondly, the distributed hyperbolic system is preserved for state estimation. After state transformations, a Luenberger-like PDE boundary observer based on backstepping mathematical tools is proposed. An exponential Lyapunov function is used to prove the stability of the resulted estimation error. The performance of the two observers are shown on a water pipe prototype simulated example.

1. Introduction

Supervision of physical transport plant has been an active research topic in recent years and water distribution network (WDN) monitoring is a major concern [1], [2], [3]. Indeed a water distribution network is a set of interconnected physical components (pipes, pumps, tanks and valves) with the mission of supplying required water flow under sufficient pressure for various load conditions.

The real-time knowledge of the network state is essential to improve the efficiency rate of the network which is usually around 75% mainly due to leakages [4]. For that purpose, WDN are equipped with flow and pressure sensors at certain nodes. Due to cost reasons and physical constraints, it is not possible to implement sensors in every nodes. Thus, unmeasured quantities should be estimated to provide useful information for network supervision and management. Different kinds of estimators or state observers may be used depending on the quantities to be estimated, the available sensors and the desired performance of the estimation scheme.
The pressurized pipes of a WDN are relatively long to consider a one-dimensional flow movement modeled by nonlinear coupled distributed hyperbolic equations derived from the non-conservative laws [5]. Flow and pressure measures are available at certain boundaries in a WDN. Exact boundary observability for one dimensional first order quasilinear hyperbolic systems has been studied in [6]. The observability is based on the existence of a unique semi-global solution and leads to the possibility to design an exponential boundary observer.

Two approaches may be used to design a boundary observer. 1) The indirect approach where Partial Differential Equations (PDE) operating in a functional infinite dimensional space are approximated by Ordinary Differential Equations (ODE) in a finite-dimensional space through a differentiation operator (finite differences schemes are commonly used [7], [8]). The advantage of such approach is to give access to many techniques well developed for observers design in finite dimension. 2) The direct approach that preserves all the system information for state estimation [9], [10] by designing directly PDE observers. This approach should potentially lead to better estimation results since no model approximation is made, however it is much more complex and needs mathematical background.

This paper presents both observer design approaches. An exponential boundary observer for the discretized system is proposed. The Lipschitz property of the nonlinear term and Linear Matrix Inequalities (LMIs) techniques are used to prove the Lyapunov stability of the estimation error and the exponential convergence of the proposed observer. For the direct approach, a Luenberger-like PDE observer based on backstepping mathematical tools [11], [12] is proposed. An exponential Lyapunov function is used to prove the stability of the resulted dynamic estimation error. The exponential convergence of the PDE observer is guaranteed with the wise choice of a parameter $\mu$ (see section 4). Behaviors of observers based on ODE model and PDE model are shown with a length varying pipe.

This paper is structured as follows. Section 2 describes the mathematical model of a pressurized water pipe. The model is discretized and an exponential boundary observer is proposed in section 3. Section 4 deals with the PDE observer. Simulations of both observers are shown and discussed in section 5. Section 6 gives concluding remarks.

2. Mathematical model

We consider an isothermal flow of a non-compressible fluid in a non-deformable pipeline with constant cross-sectional area. The pipe wall friction is assumed to be constant along the pipe and is supposed to be the only contributor to pressure drop along the pipe according to the Darcy-Weisbach formula [5]. Further we assume that the wave speed is large compared with the flow velocity and the length of the pipe is sufficiently large to consider a uniform flow movement in the transversal direction which is described by a set of two nonlinear hyperbolic PDE obtained by applying the mass and momentum balance conditions on a control volume of length $dx$ [5],

$$
\begin{align*}
\partial_t p(t,x) + \frac{\rho a^2}{A} \partial_x Q(t,x) &= 0 \\
\partial_t Q(t,x) + \frac{A}{\rho} \partial_x p(t,x) + \frac{\zeta Q(t,x)Q(t,x)}{2DA} &= 0
\end{align*}
$$

where $t \in [0, +\infty)$ is the time, $x \in [0, L]$ is the coordinate along the pipe, $p(t,x)$ is the pressure drop ($Pa$) and $Q(t,x)$ is the volumetric fluid flow ($m^3.s^{-1}$). $a$ is the wave speed ($m.s^{-1}$), $A$ is a constant.
cross-section($m^2$), $D$ is the pipe diameter ($m$). $\rho$ is the water density ($kg.m^{-3}$) and $\zeta$ the friction coefficient. $\partial_t$ (resp. $\partial_x$) stands for derivative with respect to time (resp. to $x$ axis).

We suppose that we measure three boundary conditions of (1): $p(t,L)$, $Q(t,L)$ and $p(t,0)$. In the following the inputs $u(t)$ and the output $y(t)$ of the system are

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} p(t,L) \\ Q(t,L) \end{bmatrix}^T$$  \hspace{1cm} (2)

$$y(t) = p(t,0)$$ \hspace{1cm} (3)

Our goal is to design in healthy conditions (without leakage) an exponential boundary observer for both discretized and original PDE system to estimate the non-measured state $Q(t,0)$ which will be noted $\hat{Q}(t,0)$ in order to compare the behavior of both observers.

3. **Exponential boundary observer for the discretized system**

Let us define,

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} p(t,x) \\ Q(t,x) \end{bmatrix}^T$$ \hspace{1cm} (4)

Spatial discretization by finite differences of matched states $p(t,x)$ and $Q(t,x)$ at $x=0$ gives,

$$\partial_x p(t,0) \approx \frac{p(t,L)-p(t,0)}{L}; \quad \partial_x Q(t,0) \approx \frac{Q(t,0)-Q(t,L)}{L}$$ \hspace{1cm} (5)

Substituting (4)-(5) into (1)-(3) gives

$$\dot{x}(t,0) = Cx(t,0) + \phi(x(t,0),u(t))$$ \hspace{1cm} (6)

$$y(t) = E x(t,0)$$ \hspace{1cm} (7)

where

$$C = \frac{1}{L} \begin{bmatrix} 0 & -\frac{\rho a^2}{A} \\ +\frac{A}{\rho} & 0 \end{bmatrix}; \quad \phi(x(t,0),u(t)) = \begin{bmatrix} +\frac{\rho a^2}{AL} u_2(t) \\ -\frac{A}{\rho L} u_1(t) - \tau Q^2(t,0) \end{bmatrix}; \quad E = [1 \ 0]$$ \hspace{1cm} (8)

Since $\tilde{c}_{12} = -\frac{\rho a^2}{A} \neq 0$, $(C,E)$ is an observable pair.

The corresponding exponential boundary observer is,

$$\dot{\hat{x}}(t,0) = C \hat{x}(t,0) + \phi(\hat{x}(t,0),u(t)) + H \left(y(t)-\hat{y}(t)\right) = 0$$ \hspace{1cm} (9)

$$\hat{y}(t) = E \hat{x}(t,0)$$ \hspace{1cm} (10)
This particular observer structure has already been studied and several design methods may be found in the literature. One of them is the non-constructive Thau’s method [13]. Another one is an iterative constructive method proposed in [14]. Also a direct computation of observer gain by solving LMIs was developed in [15]. In the following described method, an exponential term is added to the aforementioned work and the Lyapunov stability of the estimation error is proven using LMIs.

**Theorem 1**

Consider system (6)-(7) and observer (9)-(10). If the observer gain \( \Pi \) is chosen such that the following inequalities holds

\[
\begin{bmatrix}
 C^T N - E^T \Pi^T N + N C - N \Pi E + \alpha P - M & N \\
 - \beta I_2
\end{bmatrix} \leq 0
\]  

with \( \beta, M, N \) such that

\[
\begin{align*}
\beta &> 0, \quad \gamma > 0 \\
N &> 0, \quad N^T = N \\
M &< 0, \quad M^T = M
\end{align*}
\]  

then the estimation error \( e(t, 0) = \chi(t, 0) - \hat{\chi}(t, 0) \) tends exponentially to 0 when \( t \) tends to infinity.

**Proof of theorem 1**

The dynamic state estimation error is

\[
\dot{e}(t, 0) = \hat{\chi}(t, 0) - \hat{\chi}(t, 0) = (\dot{C} - \Pi E)e(t) + \phi\left(\chi(t, 0), u(t)\right) - \phi\left(\hat{\chi}(t, 0), u(t)\right)
\]  

Note \( \Delta \phi = \phi\left(\chi(t, 0), u(t)\right) - \phi\left(\hat{\chi}(t, 0), u(t)\right) \). \( \Delta \phi \) is Lipchitz with respect to \( \chi \) i.e. there exists a constant \( \gamma > 0 \) such that the following inequality is satisfied

\[
\|\phi\left(\chi(t, 0), u(t)\right) - \phi\left(\hat{\chi}(t, 0), u(t)\right)\|^2 \leq \gamma \|\chi(t, 0) - \hat{\chi}(t, 0)\|^2
\]  

\( \| \) represents the 2-norm.

Let \( V \) be a Lyapunov function candidate,

\[
V = e^T Ne : \left( N = N^T > 0 \right)
\]  

According to [16], the dynamic error is exponentially stable if \( \dot{V} + \alpha V \leq 0 \), where \( \alpha > 0 \),

\[
\dot{V} + \alpha V = \dot{e}^T Ne + e^T \dot{Ne} + \alpha e^T Ne = e^T \left( C^T N - E^T \Pi^T N + N C - N \Pi E + \alpha N \right) e + \Delta \phi^T Ne + e^T N \Delta \phi
\]  

Consider the following inequality [17],
\[ X^T Y + Y^T X < \beta X^T X + \beta^{-1} Y^T Y \quad (\beta > 0) \] 

Arranging (16) with (17) we get

\[ \dot{V} + \alpha V = e^T \left( C e - \beta \left( T N + N C - N I T E + \beta^{-1} N^T N + \alpha N \right)e + \beta \gamma \phi^T \phi \right) \leq 0 \] 

Knowing that

\[ \Delta \phi^T \Delta \phi \leq \| \Delta \phi \|^2 \quad \text{and} \quad \| \Delta \phi \|^2 \leq \gamma \| e \|^2, \]

equation (18) turns into

\[ \dot{V} + \alpha V = e^T Re + \beta \gamma \| e \|^2 \leq 0 \]

It can be easily proved [17] that there exists a matrix \( M (M < 0) \); \( M^T = M \) such that,

\[ e^T Me \leq \| e \|^2 \cdot \text{Max} \left( \text{eig} \left( M \right) \right) < 0 \] 

and

\[ \beta \gamma \| e \|^2 \leq \| e \|^2 \cdot \text{Max} \left( \text{eig} \left( M \right) \right) < 0 \]

Applying (21) to (22), we get

\[ e^T Me \leq -\beta \gamma \| e \|^2 \]

From (20) and (23), we obtain:

\[ e^T Re \leq e^T Me \leq -\beta \gamma \| e \|^2 \]

\( IT \) is chosen by solving the following Linear Matrix Inequalities (LMIs)

\[ \begin{cases} R \leq M \\ \beta \gamma I_2 + M < 0 \end{cases} \]

With Schur complement, we have (11)-(12). This proves theorem 1.

As we mentioned in our objectives, it would be interesting to design an observer using the original PDE system. This is the aim of the next section.

4. **PDE exponential boundary observer**

System (1) is a strictly hyperbolic system with two distinct real eigenvalues \( \lambda_1 = +a \) and \( \lambda_2 = -a \). The exact boundary observability of hyperbolic systems is studied in [6]. The observation is carried out from one-sided of the pipe on boundary \( x = 0 \). We first apply changes of coordinates to relax our observer’s design task.
4.1 State transformation

Let us define the nonsingular transition matrix

\[ \begin{bmatrix} l_1 & l_2 \end{bmatrix} = \begin{bmatrix} A & 1 \\ -\frac{1}{\rho a} & 1 \end{bmatrix} \]

and the change of coordinates \( \bar{w} = \left[ l_1 \ l_2 \right]^T \). \( \bar{w}_1 \) and \( \bar{w}_2 \) are called Riemann invariants [6]. \( \bar{w}_1 \) (resp. \( \bar{w}_2 \)) is the constant solution of (1)-(2) along the trajectory \( x = at \) (resp. \( x = -at + L \)).

It gives,

\[ \bar{w}_1(t, x) = -\frac{A}{\rho a} \left[ p(t, x) + Q(t, x) \right] \]

\[ \bar{w}_2(t, x) = -\frac{A}{\rho a} \left[ p(t, x) + Q(t, x) \right] \]

System (1) may be rewritten

\[ \begin{cases} \partial_t \bar{w}_1(t, x) + a \partial_x \bar{w}_1(t, x) + \frac{z}{4} \left( \bar{w}_1(t, x) + \bar{w}_2(t, x) \right)^2 = 0 \\ \partial_t \bar{w}_2(t, x) - a \partial_x \bar{w}_2(t, x) + \frac{z}{4} \left( \bar{w}_1(t, x) + \bar{w}_2(t, x) \right)^2 = 0 \end{cases} \]  

Variables \( \bar{w}_1(t, x) \) and \( \bar{w}_2(t, x) \) remain partially matched. We first search for a change of coordinate to put (27) into the following form:

\[ \partial_t w(t, z) + \Lambda \partial_z w(t, z) + H(z) w(t, z) = 0; \quad (t, z) = [0, +\infty) \times [0, 1] \]

\[ \Lambda = \begin{bmatrix} +\Lambda_1 & 0 \\ 0 & -\Lambda_2 \end{bmatrix}, \quad H(z) = \begin{bmatrix} 0 & H_1(z) \\ H_2(z) & 0 \end{bmatrix} \]

Consider the normalization \( z = \frac{x}{L} \), and define new variables

\[ w_i(t, z) = \bar{w}_i(t, z L) e^{\sigma_i \zeta z}, \sigma_i = \{ \text{sign} \lambda_i, i = 1, 2 \} \]

where \( \mu \) is a constant to be fixed.

\( \mu \) represents the transfer of a volumetric flow rate \( Q(t, x) \), due to the pressure wave speed +a propagation of from 0 to \( L \), along a rough pipe with friction coefficient \( \zeta \).

Original variables \( \left[ p(t, x) \ Q(t, x) \right]^T \) are obtained through inverse transformations of (30) and (26)

\[ p(t, x) = \frac{\rho a}{2A} \left[ w_1(t, \frac{x}{L}) e^{-\mu \zeta t} - w_2(t, \frac{x}{L}) e^{\mu \zeta t} \right], \quad Q(t, x) = \frac{1}{2} \left( w_1(t, \frac{x}{L}) e^{-\mu \zeta t} + w_2(t, \frac{x}{L}) e^{\mu \zeta t} \right) \]

Taking the spatial-derivative of (30), we get

\[ \partial_z w_i(t, z) = L \partial_z \bar{w}_i(t, z L) e^{\sigma_i \zeta z} + \mu \sigma_i w_i(t, z) \]

Taking its time derivative we obtain
\( \partial_i w_i(t,z) = \partial_i \tilde{w}_i(t,zL)e^{\sigma_{i\mu z}} \) \hspace{1cm} (33)

For each \( i \), introducing the corresponding expression of \( \partial_i \tilde{w}_i(t,zL) \) from (27) in (33) and then inserting (32) in this result we get:

\[
\begin{align*}
\partial_i w_1(t,z) + \frac{a}{L} \partial_i \tilde{w}_1(t,z) - \left( \frac{a}{L} \mu - H \right) w_1(t,z) + Hw_1(t,z)e^{2\mu z} &= 0 \\
\partial_i w_2(t,z) - \frac{a}{L} \partial_i \tilde{w}_2(t,z) - \left( \frac{a}{L} \mu - H \right) w_2(t,z) + Hw_1(t,z)e^{-2\mu z} &= 0
\end{align*}
\] \hspace{1cm} (34)

\[
\text{where } H = \frac{z}{4} \left( w_1(t,z)e^{-\mu z} + w_2(t,z)e^{\mu z} \right). \hspace{1cm} \text{From (31), we get a simplified expression, } H = \frac{z}{2} Q(t,zL)
\]

To get the form (28)-(29), let us choose

\[
\mu = \frac{L}{a} H = \frac{L}{2a} Q(t,zL)
\] \hspace{1cm} (36)

Then (28) is obtained with,

\[
\Lambda = \frac{1}{T} \begin{bmatrix} +a & 0 \\ 0 & -a \end{bmatrix}; \hspace{0.5cm} H(z) = \frac{r}{2} Q(t,zL) \begin{bmatrix} 0 & e^{2\mu z} \\ e^{-2\mu z} & 0 \end{bmatrix}
\] \hspace{1cm} (37)

We get the corresponding output \( p(t,0) \) by using (31),

\[
y(t) = \frac{\rho_a}{2A} \left( w_1(t,0) - w_2(t,0) \right)
\] \hspace{1cm} (38)

4.2 Observer setup and stability analysis

A Luenberger-like observer is designed. This observer is built as a copy of the system with a correction function of the output estimation error

\[
\begin{align*}
\partial_i \hat{w}_i(t,z) + \Lambda \partial_i \hat{w}_i(t,z) + H(z) \hat{w}_i(t,z) + \Pi(z) \left( y(t) - \hat{y}(t) \right) &= 0 \\
y(t) = \frac{\rho_a}{2A} \left( \hat{w}_1(t,0) - \hat{w}_2(t,0) \right)
\end{align*}
\] \hspace{1cm} (39)

\[
\text{where } \Pi(z) \text{ is a function to be designed to stabilize the state estimation error}
\]

\[
\hat{w}(t,z) = w(t,z) - \hat{w}(t,z)
\] \hspace{1cm} (41)

From the system and observer equations, we obtain the error equation

\[
\begin{align*}
\partial_i \hat{w}_i(t,z) + \Lambda \partial_i \hat{w}_i(t,z) + H(z) \hat{w}_i(t,z) - \Pi(z) \left( \hat{y}(t) - \hat{\gamma}(t) \right) &= 0
\end{align*}
\] \hspace{1cm} (42)

To prove the stability of (42), the following backstepping transformation is used as in [11]
\[ \hat{w}(t, z) = \xi(t, z) - \int_{0}^{1} k(z, \eta) \xi(t, \eta) d\eta \]  

(43)

for \((z, \eta) \in D; \quad D = \{z, \eta: 0 \leq \eta \leq z \leq 1\}\) where \(k(z, \eta)\) must be determined so that (43) is transformed into the following system

\[ \partial_t \hat{\xi}(t, z) + \Lambda \partial_z \hat{\xi}(t, z) = 0 \]  

(44)

\[ \hat{y}(t) = \frac{\rho a}{2A} \left( \hat{\zeta}_1(t, 0) - \hat{\zeta}_2(t, 0) \right) \]  

(45)

Differentiating (43) with respect to \(t\) yields,

\[ \partial_t \hat{w}(t, z) = \partial_t \hat{\xi}(t, z) - \int_{0}^{1} k(z, \eta) \partial_t \hat{\xi}(t, \eta) d\eta = \partial_t \hat{\xi}(t, z) + \int_{0}^{1} k(z, \eta) \left( \Lambda \partial_z \hat{\xi}(t, \eta) \right) d\eta \]

(46)

and with respect to \(z\),

\[ \partial_z \hat{w}(t, z) = \partial_z \hat{\xi}(t, z) - k(z, z) \hat{\xi}(t, z) - \int_{0}^{1} \partial_z k(z, \eta) \hat{\xi}(t, \eta) d\eta \]  

(47)

Introducing resulted equations (46)-(47) into (42) yields

\[ \partial_t \hat{\xi}(t, z) + k(z, z) \Lambda \hat{\xi}(t, z) - k(z, 0) \Lambda \hat{\xi}(t, 0) - \int_{0}^{1} \partial_{\eta} k(z, \eta) \hat{\xi}(t, \eta) d\eta + \Lambda \left( \hat{\zeta}_1(t, z) - H(z) \right) \hat{\xi}(t, z) + H(z) \left( \hat{\zeta}_2(t, z) - H(z) \right) = 0 \]

(48)

The equivalence of systems (42) and (44)-(45) is guaranteed if the following conditions are satisfied

\[ k(z, z) \Lambda - \Lambda k(z, z) + H(z) \hat{\xi}(t, z) = 0 \]  

(49)

\[ -k(z, 0) \Lambda \hat{\xi}(t, 0) - \Pi(z) \left( y(t) - \hat{y}(t) \right) = 0 \]  

(50)

\[ \int_{0}^{1} \left( \partial_{\eta} k(z, \eta) \Lambda + k(z, \eta) H(z) \right) \hat{\xi}(t, \eta) d\eta = 0 \]  

(51)

Our goal is to estimate \(Q(t, 0)\). Conditions (49)-(50) are sufficient to compute the desired \(\Pi(0)\).

Let \(k(z, \eta) = \begin{bmatrix} k_1(z, \eta) & k_1(z, \eta) \\ k_2(z, \eta) & k_2(z, \eta) \end{bmatrix}\), developing (49) and rearranging the terms gives

\[ k_1(z, z) = \frac{L}{2a} H_1(z), \quad k_2(z, z) = -\frac{L}{2a} H_2(z) \]  

(52)
From equations (50) and (45), we obtain

\[
\Pi_1(z) = -\frac{2A}{\rho} Lk_1(z, 0), \quad \Pi_2(z) = -\frac{2A}{\rho} Lk_2(z, 0)
\]  

(53)

At \( z = 0 \), injecting (52) in (53) with (37), we get

\[
\Pi_1(0) = -\frac{AL^2}{2\rho a} \tau \hat{Q}(t, 0), \quad \Pi_2(0) = +\frac{AL^2}{2\rho a} \tau \hat{Q}(t, 0)
\]  

(54)

\( \Pi(0) \) is a varying nonlinear gain function of the estimated \( \hat{Q}(t, 0) \). The exponential convergence of the proposed PDE observer depends on the parameter \( \mu \).

![Diagram of PDE observer scheme for a pressurized water pipe](image)

Furthermore transformation (43) is invertible [11] and the stability of the state estimation error (42) can be concluded from the stability of the target system (44)-(45).

The following Lyapunov function is introduced

\[
V(t) = \frac{1}{2} \int_0^1 \xi^T(t, z) D(z) \xi(t, z) dz
\]  

(55)

where \( D(z) = \text{diag} [\theta_1 e^{+\mu z}, \theta_2 e^{-\mu z}] ; \quad \theta_1 > 0, \theta_2 > 0 \). Then, \( D(z) \Lambda = \Lambda D(z) \)

Its time derivative along the trajectory is

\[
\dot{V}(t) = \int_0^1 \left( d^T(t, z) D(z) \xi(t, z) + \xi^T(t, z) D(z) d(t, z) \right) dz
\]

\[
= -\xi^T(t, 1) D(1) \Lambda \xi(t, 1) + \xi^T(t, 0) D(0) \Lambda \xi(t, 0) - 2\mu \Lambda \int_0^1 \xi^T(t, x) D(x) \xi(t, x) dx
\]

\[
= -\xi^T(t, 1) D(1) \Lambda \xi(t, 1) + \xi^T(t, 0) D(0) \Lambda \xi(t, 0) - 2\mu \Lambda V(t)
\]

(56)

Boundary conditions are chosen such that \(-\xi^T(t, L) D(L) \Lambda \xi(t, L) + \xi^T(t, 0) D(0) \Lambda \xi(t, 0) \leq 0\) we get

\[
\dot{V}(t) \leq -2\mu \Lambda V(t)
\]

(57)

Thus, the exponential stability of the target system is established. Since transformation (43) is invertible, (42) remains stable if conditions (49)-(51) are fulfilled.
5. Simulations and results

Results are validated in simulation on a pipe with parameters presented in table 1. We have considered outlet pressure and flow rate respectively at $0.656 \times 10^5$ Pa and $0.0082544$ $m^3.s^{-1}$ as input of the system.

The observer gain for the discretized system is computed according to LMIs (11)-(12), we get

$$II = \begin{bmatrix} +20363 \times 10^2 \\ -29 \times 10^2 \end{bmatrix}.$$  

For the PDE boundary observer, $II(0)$ is computed with (54) and illustrated in figure 3a. From equation (36) we get $\mu = 0.074181$.

| Parameter         | Symbol | Value          | Unit         |
|-------------------|--------|----------------|--------------|
| Pipe length       | $L$    | 86.49          | (m)          |
| Internal pipe diameter | $D$    | $6.54 \times 10^{-2}$ | (m)          |
| Wave speed        | $a$    | 375            | (ms$^{-1}$)  |
| Friction coefficient | $\zeta$ | $1.72 \times 10^{-2}$ | /            |
| Density           | $\rho$ | 1000           | (kg$\cdot$m$^{-3}$) |

Source [7]

The estimated inlet pressure and flow rate with the two observers are represented in figures 2a and 2b. Both observers converge exponentially; however, the convergence of the PDE observer is faster than the nonlinear observer based on the discretized system.

Fig. 2a. Estimated inlet pressure with $L_1=L$  
Fig. 2b. Estimated inlet flow rate with $L_1=L$

Fig. 3a. PDE observer gain with $L_1=L$  
Fig. 3b. PDE observer gain with $L_1=2L$

The length of the pipe is changed to $L_1 = 2L$. We get $\mu = 0.1490478$ and the corresponding $II(0)$ illustrated in figure 3b. In parallel, a new value $II = \begin{bmatrix} +16461 \times 10^2 \\ -36 \times 10^2 \end{bmatrix}$ is obtained. The simulation results are shown in Figures 4a and 4b.
We note a relative long convergence time for the discretized observer. Figure 4b shows with a PDE observer, a stabilization of the inlet pressure estimation error around $+100\, Pa = 10^{-3}\, bar$ and an accuracy error of 0.48% (shown in figure 5b). Compared with the sensitivity of pressure sensors on distributed water network, we can conclude that both observers converge exponentially to the desired value. Furthermore, the implementation of PDE boundary observer seems to be suitable for relatively long pipes in order to alleviate the computation of observer gain and to guarantee a real time monitoring of distributed water network.

6. Conclusion

State estimation on a pressurized water pipe with three known measures at boundary has been addressed. Two approaches have been explored. The indirect approach uses the discretized system; an exponential boundary nonlinear observer is designed by using the Lipschitz property of the nonlinear term and by solving LMIs to ensure the stability of estimation error. The direct approach, where some state transformations are used in order to design a Luenberger-like PDE boundary observer based on backstepping mathematical tools. An exponential Lyapunov function is used to prove the stability of the error equation.

Simulations have been carried out for a water pipe prototype. Results have shown the effectiveness of both approaches. The convergence time of the nonlinear observer increases with the length of pipe. The PDE observer seems more efficient. These two observers may be used to detect a leakage in the pipe using a reduced number of sensors. We will study in future work the efficiency of both observers for leakage detection. Early studies and simulations show that the leakage detection is more efficient using a PDE observer. Moreover, the PDE observer may also be easily extended to localize the leakage in the pipe if the four boundary conditions are measured.

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