(q, h)–analogue of Newton’s binomial formula

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Abstract
In this letter, the (q, h)–analogue of Newton’s binomial formula is obtained in the (q, h)–deformed quantum plane which reduces for h = 0 to the q–analogue. For (q = 1, h = 0), this is just the usual one as it should be. Moreover, the h–analogue is recovered for q = 1. Some properties of the (q, h)–binomial coefficients are also given. This result will contribute to an introduction of the (q, h)–analogue of the well–known functions, (q, h)–special functions and (q, h)–deformed analysis.

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The $q$–analysis is an extension of the ordinary analysis, by the addition of an extra parameters $q$. When $q$ tends towards one the usual analysis is recovered. Such $q$–analysis has appeared in the literature long time ago [1]. In particular, a $q$–analogue of the Newton’s binomial formula, well–known functions like $q$–exponential, $q$–logarithm, · · · etc, the special functions arena’s $q$–differentiation and $q$–integration have been introduced and studied intensively.

In [2], I have introduced the $h$–analogue of Newton’s binomial formula leading thefore to a new analysis, called $h$–analysis.

In this letter, I will go a step further by generalizing the work [2]. Indeed, an analogue of Newton’s binomial formula is introduced here in the $(q, h)$–deformed quantum plane ( i.e. $(q, h)$ Newton’s binomial formula which generalizes Schützenberger’s formula [3] with an extra parameter $h$. ) leading therefore to a more generalized analysis, called $(q, h)$–analysis.

With this generalization, the $q$–analysis, $h$–analysis and ordinary analysis are recovered respectively by taking $h = 0$, $q = 1$ and $(q = 1, h = 0)$.

Newton’s binomial formula is defined as follows :

$$ (x + y)^n = \sum_{k=0}^{n} \binom{n}{k} y^k x^{n-k} $$ (1)

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and it is understood here that the coordinate variables $x$ and $y$ commute, i.e. $xy = yx$.

A $q$–analogue of (1), for the $q$–commuting coordinates $x$ and $y$ satisfying $xy = qyx$, first appeared in literature in Schützenberger [3], see also Cigler [4],

$$ (x + y)^n = \sum_{k=0}^{n} \left[ \binom{n}{k} \right]_q y^k x^{n-k} $$ (2)

where the $q$–binomial coefficient is given by :

$$ \left[ \binom{n}{k} \right]_q = \frac{[n]_q!}{[k]_q![n-k]_q} $$
with \[ j_q = \frac{1 - q^j}{1 - q} \]

The \( h \)-analogue has been introduced and defined in \cite{2} as follows:

\[(x + y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_h y^k x^{n-k} \tag{3}\]

provided that \( x \) and \( y \) satisfy to \([x, y] = hy^2\) and the \( h \)-binomial coefficient \[ \begin{bmatrix} n \\ k \end{bmatrix}_h \] is given by:

\[ \begin{bmatrix} n \\ k \end{bmatrix}_h = \begin{bmatrix} n \\ k \end{bmatrix} h^k (h^{-1})_k \tag{4}\]

where \((a)_k = \Gamma(a + k)/\Gamma(a)\) is the shifted factorial.

Now consider Manin’s \( q \)-plane \( x'y' = qy'x' \). By the following linear transformation (see \cite{5} and references therein):

\[ \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & h \\ \frac{h}{q-1} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \]

Manin’s \( q \)-plane changes to:

\[ xy = qyx + hy^2 \tag{5}\]

Even though the linear transformation is singular for \( q = 1 \), the resulting quantum plane is well-defined.

**Proposition 1:**
Let \( x \) and \( y \) be coordinate variables satisfying (5), then the following identities are true:

\[ x^k y = \sum_{r=0}^{k} \frac{[k]_q!}{[k-r]_q!} q^{k-r} h^r y^{r+1} x^{k-r} \]

\[ xy^k = q^k y^k x + h [k]_q y^{k+1} \tag{6}\]
These identities are easily proved by successive use of (5).

**Proposition 2 :** \((q, h)\)-binomial formula

Let \(x\) and \(y\) be coordinate variables satisfying (5), then we have:

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k}_{(q,h)} y^k x^{n-k}
\]

(7)

where \(\binom{n}{k}_{(q,h)}\) are the \((q, h)\)-binomial coefficients given as follows:

\[
\binom{n}{k}_{(q,h)} = \binom{n}{k}_q h^k (h^{-1})^{|k|}.
\]

(8)

with \(\binom{n}{0}_{(q,h)} = 1\) and

\[
(a)_{[k]} = \Pi_{j=0}^{k-1} (a + [j]_q)
\]

(9)

since by definition \([0]_q = 0\).

**Proof :**

This proposition will be proved by recurrence. Indeed for \(n = 1, 2\), it is verified.

Suppose now that the formula is true for \(n - 1\), which means:

\[
(x + y)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k}_{(q,h)} y^k x^{n-1-k},
\]

with \(\binom{n-1}{0}_{(q,h)} = 1\).

To show this for \(n\), let first consider the following expansion:

\[
(x + y)^n = \sum_{k=0}^{n} C_{n,k} y^k x^{n-k}
\]

where \(C_{n,k}\) are coefficients depending on \(q\) and \(h\).

Then, we have:

\[
(x + y)^n = (x + y)(x + y)^{n-1}
\]
\[
(x + y)^n = \sum_{k=0}^{n-1} \binom{n-1}{k} y^k x^{n-1-k} + \sum_{k=1}^{n-1} \binom{n-1}{k} q^k y^k x^{n-k} + \sum_{k=1}^{n-1} \binom{n-1}{k} (1 + h [k]_q) y^{k+1} x^{n-1-k} + \binom{n-1}{0} y x^{n-1}
\]

Using the result of the first proposition, we obtain:

\[
(x + y)^n = \left[ \begin{array}{c} n-1 \\ 0 \end{array} \right] + \sum_{k=1}^{n-1} \left[ \begin{array}{c} n-1 \\ k \end{array} \right] q^k y^k x^{n-k} + \sum_{k=1}^{n-1} \left[ \begin{array}{c} n-1 \\ k \end{array} \right] (1 + h [k]_q) y^{k+1} x^{n-1-k} + \left[ \begin{array}{c} n-1 \\ 0 \end{array} \right] y x^{n-1}
\]

which yields respectively:

\[
C_{n,0} = \left[ \begin{array}{c} n-1 \\ 0 \end{array} \right]_{(q,h)} = 1,
\]

\[
C_{n,1} = q \left[ \begin{array}{c} n-1 \\ 1 \end{array} \right]_{(q,h)} + \left[ \begin{array}{c} n-1 \\ 0 \end{array} \right]_{(q,h)} = \left[ \begin{array}{c} n \\ 1 \end{array} \right]_{(q,h)}
\]

\[
C_{n,k} = q^k \left[ \begin{array}{c} n-1 \\ k \end{array} \right]_{(q,h)} + (1 + h [k-1]_q) \left[ \begin{array}{c} n-1 \\ k-1 \end{array} \right]_{(q,h)} = \left[ \begin{array}{c} n \\ k \end{array} \right]_{(q,h)},
\]

\[
C_{n,n} = (1 + h [n-1]_q) \left[ \begin{array}{c} n-1 \\ n-1 \end{array} \right]_{(q,h)} = \left[ \begin{array}{c} n \\ n \end{array} \right]_{(q,h)}.
\]

This completes the Proof.

Moreover, the \((q, h)\)-binomial coefficients obey to the following properties \(1 < k < n\):

\[
\left[ \begin{array}{c} n+1 \\ k \end{array} \right]_{(q,h)} = q^k \left[ \begin{array}{c} n \\ k \end{array} \right]_{(q,h)} + (1 + h [k-1]_q) \left[ \begin{array}{c} n \\ k-1 \end{array} \right]_{(q,h)}
\]

and

\[
\left[ \begin{array}{c} n+1 \\ k+1 \end{array} \right]_{(q,h)} = (1 + h [k]_q) \frac{[n+1]_q}{[k+1]_q} \left[ \begin{array}{c} n \\ k \end{array} \right]_{(q,h)}.
\]
In fact, these properties follow from the well-known relations of the 
$q$–binomial coefficients:

\[
\begin{align*}
\binom{n+1}{k}_q &= q^k \binom{n}{k}_q + \binom{n}{k-1}_q \\
\binom{n+1}{k}_q &= \frac{[n+1]_q}{[k]_q} \binom{n}{k-1}_q
\end{align*}
\]

and upon using \((a)\lfloor k \rfloor = (a + [k - 1],_q)(a)\lfloor k-1 \rfloor\).

Now, we make the following remarks. For \(h = 0\) this is just the 
$q$–binomial formula as it should be. For \(q = 1\), it reduces to the \(h\)-analogue Newton’s binomial formula (3) and (4) and for \((q = 1, h = 0)\) the usual one is recovered.

To conclude, we have obtained a more general Newton’s binomial for-
mula in \((q, h)\)-deformed quantum plane which reduces to the known one at some limits. This will lead therefore to a more generalized analysis called \((q, h)\)-analysis.

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