Charge-2e Skyrmion condensate in a hidden order state

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A higher angular momentum ($\ell = 2$) $d$-density wave, a mixed triplet and a singlet, interestingly, admits skyrmionic textures. The Skyrmions carry charge $2e$ and can condense into a spin-singlet $s$-wave superconducting state. In addition, a charge current can be induced by a time-dependent inhomogeneous spin texture, leading to quantized charge pumping. The quantum phase transition between this mixed triplet $d$-density wave and skyrmionic superconducting condensate likely leads to deconfined quantum critical points. We suggest connections of this exotic state to electronic materials that are strongly correlated, such as the heavy fermion URu$_2$Si$_2$. At the very least, we provide a concrete example in which topological order and broken symmetry are intertwined, which can give rise to non-BCS superconductivity.

I. INTRODUCTION

It has become very much in vogue to argue that topological aspects of condensed matter bear no relation to broken symmetries. In a strict sense this need not be so. One can construct examples where a broken symmetry state has interesting topological properties and can even be protected by the broken symmetry itself. An interesting example of a mixed triplet $d$-density wave and its possible relevance to one of the many competing phases in the high temperature cuprate phase digram was recently demonstrated where it was found that the system exhibits quantized spin Hall effect even without any explicit spin-orbit coupling. In the present paper we show that the system exhibits charge $2e$ skyrmions, which can condense into a remarkable superconducting state. As we shall discuss, such a mixed triplet $d$-density wave system and the resulting superconductivity is potentially relevant to the heavy fermion URu$_2$Si$_2$ with hidden order.

An early attempt at such a non-BCS mechanism of superconductivity was made by Wiegmann as an extension of Fröhlich mechanism to higher dimension. More recently, several interesting papers have led to discussions of superconductivity in single and bilayer graphenes. Grover and Senthil have provided a mechanism in which electrons hopping on a honeycomb lattice can lead to a charge-2e skyrmionic condensate, possibly relevant to single layer graphene. To a certain degree we follow their formalism; see also the earlier work in Ref. of charge-e skyrmions in a quantum Hall ferromagnet. As to bilayer graphene, a charge-4e skyrmionic condensate has been suggested by Lu and Herbut and Moon. In the present paper we base our results, instead, on an unusual spontaneously broken symmetry generated by electron-electron interaction, not by a given non-interacting band structure of a material, namely the mixed triplet-singlet density wave state of angular momentum $\ell = 2$, and we point out its possible implications to the mysterious hidden order state in URu$_2$Si$_2$, in particular to its superconductivity.

It is appropriate to comment on what we mean by “hidden order”. An order parameter can often be inferred from its macroscopic consequences in terms of certain generalized rigidities. Sometimes its direct microscopic signature is difficult to detect: a direct determination of superconducting order, a broken global $U(1)$ gauge symmetry, requires subtle Josephson effect, and even antiferromagnetic order requires microscopic neutron scattering probes. Density wave states of higher angular momentum, such as the mixed triplet $d$-density wave, are even harder to detect. It does not lead to a net charge density wave or spin density wave to be detected by common $s$-wave probes. It is further undetectable because it does not even break time-reversal invariance. A discussion of possible experimental detections of particle-hole condensates of higher angular momentum was given in Ref. Thus, it is fair to conclude that the state we consider here is a good candidate for a hidden order.

It is also necessary to remark on the realization of particle-hole condensates of higher angular momentum. An effective low energy theory of a strongly correlated system is bound to have a multitude of coupling constants, perhaps hierarchically arranged. In such cases, we can generally expect a phase diagram with a multitude of broken symmetry states. It is a profound mystery as to why non-trivial examples are so few and far between. A partial reason could be, as stated above, that these states are unresponsive to common $s$-wave probes employed in condensed matter physics and therefore appear to be hidden.

The next question is: are these low energy effective Hamiltonians contrived? If so, it would be of value to pursue them. However, simple Hartree-Fock analyses have shown that they certainly are not, at least an onsite repulsion $U$, a nearest neighbor interaction $V$, and an exchange interaction $J$ are sufficient in a single band model.

The structure of this paper is as follows: in Sec. IV we construct the low energy effective action of the mixed triplet and singlet $d$-density wave system. In Sec. III we compute the charge and the spin of a skyrmion and verify that the skyrmions in this system are bosons, which can lead to a superconducting phase transition. In Sec. IV we compute the angular momentum of a skyrmion.
In Sec. IV we study the charge pumping due to a time-dependent inhomogeneous spin texture that is interesting in its own right. In Sec. VI we discuss mainly the problem of URTdS2. In the appendices, the derivation of the nonlinear σ-model and the details of computing the Chern-Simons coefficients and charge pumping are provided.

II. EFFECTIVE ACTION

In the momentum space the mixed triplet and singlet d-density wave order parameter is (c and $c^\dagger$ are fermionic annihilation and creation operators, respectively, $Q = (\pi, \pi)$, and the lattice constant is set to unity)

$$\langle c_{k+Q,\alpha}^\dagger c_{k,\beta}\rangle \propto i(\vec{\sigma} \cdot \hat{N})_{\alpha\beta}W_k + \delta_{\alpha\beta}\Delta_k,$$  \hspace{1cm} (1)

where $\hat{N}$ is a unit vector, $\vec{\sigma}$ are the Pauli matrices acting on spin indices, and the form factors

$$W_k \equiv \frac{W_0}{2}(\cos k_x - \cos k_y),$$ \hspace{1cm} (2)

$$\Delta_k \equiv \Delta_0 \sin k_x \sin k_y,$$ \hspace{1cm} (3)

correspond to the $d_{x^2-y^2}$ and $d_{xy}$ density wave, respectively. It is not necessary that $d_{xy}$ and $d_{x^2-y^2}$ transitions be close to each other, nor are they required to be close in energy.

If we choose the spin quantization axis to be $\hat{z}$, the up spins represent circulating spin currents corresponding to the order parameter $d + id$ and the down spins to $d - id$ (in an abbreviated notation). So, there are net circulating spin currents alternating from one plaquette to the next but no circulating charge currents. By the choice of the quantization axis we have explicitly broken $SU(2)$, but not $U(1)$, and the coset space of the order parameter $S^2 \equiv SU(2)/U(1)$. Such a state can admit Skyrmions in two dimensions (ignoring the possibility of hedgehog configurations in (2 + 1) dimensions (cf. below).

The Hamiltonian is

$$\mathcal{H} = \sum_{k,\alpha,\beta} \psi_k^\dagger \left[\delta_{\alpha\beta}((\vec{\tau} \cdot \eta_k) + \tau^z \Delta_k) - (\vec{\sigma} \cdot \hat{N})_{\alpha\beta}\tau^y W_k\right] \psi_k,$$ \hspace{1cm} (4)

where the summation is over the reduced Brillouin Zone (RBZ) bounded by $k_y \pm k_x = \pm \pi$, the spinor is $\psi_{k,\alpha} \equiv (c_{k,\alpha}^\dagger, c_{k+Q,\alpha}^\dagger)$, and $\epsilon_k \equiv -2it(\cos k_x + \cos k_y)$; addition of longer ranged hopping will not change our conclusions. Here $\tau^i$ ($i = x, y, z$) are Pauli matrices acting on the two-component spinor. It is not necessary but convenient to construct a low energy effective field theory. For this we expand around the points $K_1 \equiv (\frac{\pi}{2}, \frac{\pi}{2})$ and $K_2 \equiv (\frac{\pi}{2}, \frac{\pi}{2})$, what would have been the two distinct nodal points in the absence of the $d_{xy}$ term, and $K_3 \equiv (0, \pi)$, what would have been the nodal point in the absence of the $d_{x^2-y^2}$ term. This allows us to develop an effective low energy theory by separating the fast modes from the slow modes. After that we make a sequence of transformations for simplicity: (1) transform the Hamiltonian to the real space, which allows us to formulate the skyrmion problem; (2) perform a $\pi/2$ rotation along the $\tau^y$-direction, which allows us to match to the notation of Ref. 12 for the convenience of the reader; (3) label $\psi_{K_i+\pi,\alpha}$ by $\psi_{i\alpha}$, since $K_i$ is now a redundant notation; (4) construct the imaginary time effective action, with the definition $\tilde{\psi} \equiv -i\psi^\dagger\tau^z$. Finally, after suppressing the spin indices, and with the definitions $\gamma^0 \equiv \tau^z$, $\gamma^x \equiv \tau^y$, and $\gamma^y \equiv -i\tau^x$, we obtain the effective action in a more compact notation:

$$S = \sum_{j=1,2} \int d^3x \tilde{\psi}_j \left[-i\gamma^0 \partial_\tau - 2it\gamma^y (\eta_j \partial_x + \partial_y) + i\frac{W_0}{2}(\vec{\sigma} \cdot \hat{N})\gamma^y (-\eta_j \partial_x + \partial_y) + i\eta_j \Delta_0\right] \psi_j$$

$$+ \int d^3x \tilde{\psi}_3 \left[-i\gamma^0 \partial_\tau - W_0(\vec{\sigma} \cdot \hat{N})\gamma^y \right] \psi_3,$$ \hspace{1cm} (5)

where $\eta_1 = 1$ and $\eta_2 = -1$.

III. THE CHARGE AND SPIN OF A SKYRMION

We will compute the charge of the skyrmions in the system by following Grover and Senthil’s adiabatic argument. First, consider the action around $K_1 = (\frac{\pi}{2}, \frac{\pi}{2})$ when the order parameter is uniform (say, $\hat{N} = \hat{z}$). The results for $K_2 = (-\frac{\pi}{2}, \frac{\pi}{2})$ and $K_3 = (0, \pi)$ follow iden-
\begin{align}
S_1[A^c, A^s] &= \int d^3x \bar{\psi}_1 \left[ -i\gamma^0 \partial_x + \gamma^0 (A^c_x + \frac{\sigma^z}{2} A^s_x) - 2i\tau^x (\partial_x + \partial_y) + 2i\gamma^x (A^c_y + \frac{\sigma^z}{2} A^s_y + A^c_y + \frac{\sigma^z}{2} A^s_y) \\
&\quad + i\frac{W_0}{2} \sigma^z \gamma^y (-\partial_x + \partial_y) - \frac{W_0}{2} \sigma^z \gamma^y (-A^c_x - \frac{\sigma^z}{2} A^s_x + A^c_y + \frac{\sigma^z}{2} A^s_y) + i\Delta_0 \right] \psi_1, \\
\text{where we set } \hbar = 1. \text{ The non-vanishing transverse spin conductance implies that the low energy effective action for the gauge fields is given by} \\
S_{\text{eff}} &= \frac{i}{2\pi} \int d^3x \epsilon^{\mu\nu\lambda} A^c_\mu \partial_\nu A^s_\lambda, \\
\text{and the charge current is induced by the spin gauge field} \\
j^c_\mu &= \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu A^s_\lambda. 
\end{align}

Consider now a static configuration of the \( \hat{N} \) field with unit Pontryagin index in the polar coordinate \((r, \theta)\): points in space such that \( U^\dagger (\vec{\sigma} \cdot \hat{N}) U = \sigma^z \), and defining \( \psi = U \psi' \), and \( \bar{\psi} = \bar{\psi}' U^\dagger \), we obtain

\( \hat{N} (r, \theta) = [\sin \alpha(r) \cos \theta, \sin \alpha(r) \sin \theta, \cos \alpha(r)] \) (10)

with the boundary conditions \( \alpha(r = 0) = 0 \) and \( \alpha(r \to \infty) = \pi \). Performing a unitary transformation at all points in space such that \( U^\dagger (\vec{\sigma} \cdot \hat{N}) U = \sigma^z \), we get

\( \hat{N} (r, \theta) = [\sin \alpha(r) \cos \theta, \sin \alpha(r) \sin \theta, \cos \alpha(r)] \) (10)

To proceed, we write down the explicit form for \( U(r, \theta) \), which is

\begin{equation}
U(r, \theta) = \begin{pmatrix} \cos \alpha(r) e^{i\frac{\pi}{4}} & -\sin \alpha(r) e^{-i\frac{\pi}{4}} \\ \sin \alpha(r) e^{i\frac{\pi}{4}} & \cos \alpha(r) e^{-i\frac{\pi}{4}} \end{pmatrix},
\end{equation}

(12)

In the far field limit, \( U^\dagger \partial_x U = (\frac{-i \sin \theta}{r}) \sigma^z \), and \( U^\dagger \partial_y U = (\frac{i \cos \theta}{r}) \sigma^z \); substituting into Eq. 11 and introducing \( f_\mu = -iU^\dagger \partial_\mu U \), we get

\begin{align}
S_1 &= \int d^3x \bar{\psi}_1 \left[ -i\gamma^0 \partial_x - 2i\tau^x (\partial_x + \partial_y) + i\frac{W_0}{2} \sigma^z \gamma^y (-\partial_x + \partial_y) + i\Delta_0 \right] \psi_1' \\
&\quad + \int d^3x \bar{\psi}_1' \left[ -i\gamma^0 (U^\dagger \partial_x U) - 2i\tau^x (U^\dagger \partial_x U + U^\dagger \partial_y U) + i\frac{W_0}{2} \sigma^z \gamma^y (-U^\dagger \partial_x U + U^\dagger \partial_y U) \right] \psi_1 \\
&\quad + \int d^3x \bar{\psi}_1' \left[ 2i\tau^x (f_x + f_y) + i\frac{W_0}{2} \sigma^z \gamma^y (f_x - f_y) \right] \psi_1'.
\end{align}

(13)

Equating the above equation and Eq. 11, we obtain in the far field limit

\( A^c_x = A^c_y = 0 \); \( A^c_x = -\frac{2 \sin \theta}{r} \); \( A^c_y = \frac{2 \cos \theta}{r} \).

In other words, the process of tuning the order parameter from \( \sigma^z \) to \( \vec{\sigma} \cdot \hat{N}(r, \theta) \) is equivalent to adding an external spin gauge field

\begin{equation}
\tilde{A}^c = \frac{2 \sin \theta}{r} \tilde{x} + \frac{2 \cos \theta}{r} \tilde{y} = \frac{2}{r} \hat{\theta},
\end{equation}

(14)

The total flux of this gauge field is clearly \( 4\pi \). Suppose we adiabatically construct the Skyrmion configuration \( \hat{N}(r, \theta) \) from the ground state \( \hat{x} \) in a very long time period \( \tau_p \to \infty \). During the process, we effectively thread a spin gauge flux of \( 4\pi \). The transverse spin Hall conductance implies that a radial current \( j^c_\ell \) will be induced by the \( 4\pi \) spin gauge flux of \( \tilde{A}^c(t) \), which is now time-dependent: \( \tilde{A}^c(t = 0) = 0 \) and \( \tilde{A}^c(t = \tau_p) = \tilde{A}^c \), that is,

\begin{equation}
 j^c_\ell (t) = -\frac{1}{2\pi} \partial_\ell A^c_\ell (t).
\end{equation}

(15)
As a result, charge will be transferred from the center to the boundary, and the total charge transferred is

\[ Q^c = \int_0^{r_p} dt \int_0^{2\pi} r d\theta j^c_r(t) = -2. \]  

(16)

Therefore, after restoring the unit of charge to e, we obtain a Skyrmion with charge 2e; its spin is 0.

It is important to verify the adiabatic result by a different method. This can be done by a computation of the Chern-Simons coefficients. The charge and spin of the skyrmions are associated with the coefficients of the Chern-Simons terms by the following relations: \( Q_{\text{skyrmion}} = C_2 e \) and \( S_{\text{skyrmion}} = C_1 \frac{\hbar}{\tau} \), where \( C_1 \) and \( C_2 \) are

\[ C_1 = \frac{\epsilon \mu \lambda}{24 \pi^2} \text{Tr} \left[ \int d^3 k G^{-1}_\mu \frac{\partial G^{-1}_\mu}{\partial k_\nu} G^{-1}_\nu \frac{\partial G^{-1}_\nu}{\partial k_\lambda} G^{-1}_\lambda \right], \]  

(17)

\[ C_2 = \frac{\epsilon \mu \lambda}{24 \pi^2} \text{Tr} \left[ \int d^3 k (\hat{\sigma} \cdot \hat{z}) G^{-1}_\mu \frac{\partial G^{-1}_\mu}{\partial k_\nu} G^{-1}_\nu \frac{\partial G^{-1}_\nu}{\partial k_\lambda} G^{-1}_\lambda \right], \]  

(18)

where \( G \) is the matrix Geen’s function and the trace is taken over the spin index \( \sigma \) and other discrete indices.

If the Green’s function matrix is diagonal in the spin index, then the Chern-Simons coefficients for up and down spins can be computed separately.

\[ \mathcal{N}(G_\sigma) = \frac{\epsilon \mu \lambda}{24 \pi^2} \text{Tr} \left[ \int d^3 k G^{-1}_\sigma \frac{\partial G^{-1}_\sigma}{\partial k_\mu} G_\sigma \frac{\partial G^{-1}_\sigma}{\partial k_\nu} G_\sigma \frac{\partial G^{-1}_\sigma}{\partial k_\lambda} G_\sigma \right], \]  

(19)

and \( C_1 = \mathcal{N}(G_\uparrow) + \mathcal{N}(G_\downarrow) \), \( C_2 = \mathcal{N}(G_\uparrow) - \mathcal{N}(G_\downarrow) \). Furthermore, it can be shown (see Appendix [3]) that for

\[ G^{-1}_\sigma = i \omega \hat{I} - \hat{r} \cdot \hat{h}_\sigma \]  

(20)

with \( \hat{h}_\sigma \) being the Anderson’s pseudospin vector\(^4\) of the Hamiltonian, the Chern-Simons coefficient for spin \( \sigma \) can be written as

\[ \mathcal{N}(G_\sigma) = - \int d^2 k \frac{1}{4 \pi} \hat{h}_\sigma \cdot \frac{\partial \hat{h}_\sigma}{\partial k_x} \times \frac{\partial \hat{h}_\sigma}{\partial k_y}, \]  

(21)

where \( \hat{h}_\sigma \equiv \hat{h}_\sigma / |\hat{h}_\sigma| \) is the unit vector of \( \hat{h}_\sigma \). Here \( C_1 \) and \( C_2 \) are the total Chern number and the spin Chern number \( \mathcal{N}_{\text{spin}} \) defined in our previous paper, respectively.\(^2\) For \( \sigma \mathbf{d}_{x^2-y^2} + \mathbf{d}_{xy} \) system, we have \( \hat{h}_\sigma \equiv (\Delta_k, -\sigma W_k, \epsilon_k) \).

Explicitly, \( C_1 = -1 + 1 = 0 \) and \( C_2 = -1 - 1 = -2 \); thus the results are the same as above.

Because a Skyrmion in the system carries integer spin, it obeys bosonic statistics and may undergo Bose-Einstein condensate. As a result, the charge-2e Skyrmion condensate will lead to a superconducting phase transition. But what about its orbital angular momentum? In the following section, we will prove that it is zero resulting in a s-wave singlet state. This is a bit surprising given the original d-wave form factor.

\[ IV. \ \text{THE ANGULAR MOMENTUM OF A SKYRMION} \]

To compute the angular momentum carried by a skyrmion in the system, we consider the angular momentum density due to the electromagnetic field. For a static spin texture it is clearly zero, because \( \vec{E} = 0 \). For a time dependent texture it is little harder to prove. Consider,

\[ N^x(r, \theta, t) = \sin \alpha(r, t) \cos \beta(\theta, t), \]

\[ N^y(r, \theta, t) = \sin \alpha(r, t) \sin \beta(\theta, t), \]

\[ N^z(r, t) = \cos \alpha(r, t), \]

where \( \alpha(r, t) \) and \( \beta(\theta, t) \) are smooth functions, and \( \alpha(r, t) \) satisfies the boundary conditions \( \alpha(r = 0, t) = 0 \) and \( \alpha(r \to \infty, t) = \pi \), for any \( t \), and \( \frac{\partial \alpha(r, t)}{\partial r} |_{r \to \infty} = 0 \) in the far field limit. The unitary matrix is now time dependent. After a little algebra, we obtain the time-dependent gauge fields in the far field limit to be

\[ A^x(r, \theta, t) = -\frac{2 \sin \theta}{r} \frac{\partial \beta(\theta, t)}{\partial \theta}, \]  

(22)

\[ A^y(r, \theta, t) = \frac{2 \cos \theta}{r} \frac{\partial \beta(\theta, t)}{\partial \theta}. \]  

(23)

So, \( \Phi(\theta, t) = A^x(r, \theta, t) = \frac{2 \beta(\theta, t)}{r} \) and \( \vec{A}^\ast(r, \theta, t) = A^x(r, \theta, t) \hat{x} + A^y(r, \theta, t) \hat{y} = A^0(r, \theta, t) \hat{\theta} \), where

\[ A^0(r, \theta, t) = \frac{2}{r} \frac{\partial \beta(\theta, t)}{\partial \theta}. \]  

(24)

Therefore, the electric field will have a non-zero \( \hat{\theta} \)-component, \( \vec{E} = E_\theta \hat{\theta} \), and the magnetic field will have a non-zero \( \hat{z} \)-component, \( \vec{B} = B_z \hat{z} \), where

\[ E_\theta = -\frac{1}{r} \frac{\partial A^0_\theta(r, \theta, t)}{\partial \theta} - \frac{\partial A^0_\theta(r, \theta, t)}{\partial t} = -\frac{4}{r^2} \frac{\partial^2 \beta(\theta, t)}{\partial \theta \partial t}, \]  

(25)

\[ B_z = -\frac{\partial A^0_\theta(r, \theta, t)}{\partial r} = -\frac{2}{r^2} \frac{\partial \beta(\theta, t)}{\partial \theta}. \]  

(26)

As a result, the angular momentum density still vanishes,

\[ \vec{L}^\text{field} = \frac{1}{4 \pi c^2} \vec{r} \times (E_\theta \hat{\theta} \times B_z \hat{z}) = 0. \]  

(27)

It is possible that superconductivity with non-zero angular momentum may be realized when the interaction between skyrmions is included, but we do not know how to prove it. It would be interesting to explore what other kinds of quantum numbers are carried by the topological textures in the model we have studied.

\[ V. \ \text{QUANTIZED CHARGE PUMPING} \]

In Sec[III] we considered a static spin texture and obtained charge-2e skyrmions in the system. If we consider a time-dependent spin texture, which has a slow variation in one spatial direction, say, \( \hat{y} \), and is uniform in the
other, \( \hat{x} \), charge will be pumped from one side of the system to the other along \( \hat{x} \). This charge pumping effect can be understood from the effective gauge action, which is

\[
S_{\text{eff}}[A^c_\mu, A^s_\mu] = \frac{C_2}{4\pi} \int d^4x \epsilon^{\mu\nu\lambda} A^c_\mu \partial_\nu A^s_\lambda,
\]

where \( \partial_\mu \) is the time derivative, and \( \partial_{\nu} \) is the spatial derivative.

After some straightforward algebra (see Appendix C), the pumped charge can be expressed in terms of the \( \hat{N} \)-vector,

\[
Q_{\text{pumped}} = \frac{C_2}{4\pi} \int_0^{\pi} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dy \ j^c_x
\]

After restoring the unit of charge, we have \( Q_{\text{pumped}} = C_2 \epsilon \). So far we have considered the spin texture with unit Pontryagin index. If the spin texture is generalized to a general Pontryagin index, \( N_P \), then the pumped charge will be \( Q_{\text{pumped}} = C_2 N_P \epsilon \).

How could we observe this charge pumping experimentally? We need to control the direction of the \( \hat{N} \)-vector so that it can be the time-dependent inhomogeneous spin texture discussed above. In topological chiral magnets, the \( \hat{N} \)-vector is the net ferromagnetic moment, which aligns along the external magnetic field, so one can apply a time-dependent magnetic field \( \hat{H}(t) = H(t)\hat{x} \) coupling to the \( \hat{N} \)-vector and control the magnitude of the \( \hat{x} \)-component of \( \hat{N} \).

In the mixed triplet \( d \)-density wave, however, the situation is more complicated. In the presence of an external magnetic field, there will be a spin flop transition and the \( \hat{N} \)-vector will lie in the plane perpendicular to the external field. In other words, we cannot fully control the direction of \( \hat{N} \) with a time-dependent magnetic field. Therefore, it would be a challenge to measure the pumped charges in the system.

Nevertheless, the charge pumping effect provides, at least, a different conceptual approach to probe the topological properties of the system in addition to the quantized spin Hall conductance. For the quantum spin Hall effect, a spin current is induced by the external electric field whereas for the charge pumping effect, a charge current will be induced by the spin texture. It would, of course, be interesting if one can manipulate the \( \hat{N} \)-vector experimentally because the charge current is easier to detect than the spin current.

VI. DISCUSSION AND APPLICATION TO THE HIDDEN ORDER STATE IN URu$_2$Si$_2$

There are two points that we have glossed over. The first is rather simple: in the ordered phase at \( T = 0 \), there are also Goldstone modes that can be easily seen by integrating out the fermions resulting in a non-linear \( \sigma \)-model involving \( \hat{N} \), the form of which is entirely determined by symmetry. These do not lead to any interesting physics, such as charge-2\( \epsilon \) skyrmions that condense into a superconducting state. At finite temperatures they could lead to a renormalized classical behavior. The second point is more subtle: we have assumed that the hedgehog configurations are absent. This would require, as pointed out by Grover and Senthil, that the energy of the Skyrmion (especially in the limit \( \Delta_0 \to 0 \)) is smaller than individual pairs of electrons, a question that is likely to be model dependent. If this assumption is correct, however, the transition from the the mixed \( d \)-density wave state to the superconducting state will correspond to a deconfined quantum critical point, which otherwise would have been a first order transition, as in Landau theory.

We suggest that the superconducting phase driven by the skyrmion condensate may be realized in the URu$_2$Si$_2$.
which hosts an exotic hidden order (HO) phase, with broken translational symmetry below $T_{HO} \approx 17.5K$ and a superconducting phase below $T_c \approx 1.5K$. Recently, Fujimoto proposed a triplet $d$-density wave with the order parameter $\langle c_{k,1,0}^\dagger c_{k+Q,0,2,\beta} \rangle = \hat{d}(k) \cdot \mathbf{\sigma}_{\alpha\beta}$ with $\hat{d}(k) = i(\Delta_1 \sin (k_x - k_z) \sin k_z, 0, 0)$. To describe this state, here 1 and 2 refer to two different bands and $Q_0 = (0,0,1)$ is the nesting vector; even the earlier work in Ref. 19 involving circulating spin current is not entirely unrelated. The order parameter considered in Ref. 18 is different but a close cousin of the order parameter considered in our work; the circulating staggered spin currents in Ref. 18 are in the diagonal planes instead and the crucial $d_{xy}$ part is missing there. That the currents are in the diagonal planes instead of being square planar is conceptually not important, but is necessary to explain the nematicity observed in the experiments. We now discuss the role of spin-orbit coupling before making our final comments.

### Spin-orbit coupling

It will be shown below that the order of magnitude of the spin-orbit energy $E_{SO} \approx |(\hat{\mathbf{x}} \cdot \hat{k})^2 - 1|(\hat{\mathbf{z}} \cdot \hat{k})^2|W_0/W|^2[1+O(W_0/W)^2]$, correcting a mistake in Ref. 12. Here $\Lambda_0$ is the strength of the spin-orbit coupling, given by

$$
\mathcal{H}_{SO} = \sum_k c_{k\alpha}^\dagger \mathbf{\hat{\Lambda}}(k) \cdot \mathbf{\sigma}_{\alpha\beta} c_{k\beta},
$$

where $\mathbf{\hat{\Lambda}}(k) = (\Lambda_0/\sqrt{2})[\hat{x} \sin k_y - \hat{y} \sin k_x]$. In the presence of spin-orbit coupling, the Hamiltonian is

$$
\mathcal{H}_{total} = \mathcal{H} + \mathcal{H}_{SO}
$$

$$
= \sum_k \Psi_k^\dagger \left( \begin{array}{cccc}
\epsilon_k & \Delta_k + iN_zW_k & \Lambda_z(k) - i\Lambda_y(k) & \Lambda_x(k) - i\Lambda_y(k) \\
\Delta_k - iN_zW_k & -\epsilon_k & -iW_k(N_x - iN_y) & -iW_k(N_x - iN_y) \\
\Lambda_z(k) + i\Lambda_y(k) & iW_k(N_x + iN_y) & \epsilon_k & \Delta_k - i\Lambda_y(k) \\
-iW_k(N_x + iN_y) & -i\Lambda_z(k) - i\Lambda_y(k) & -\epsilon_k & \Delta_k + iN_zW_k
\end{array} \right) \Psi_k,
$$

where $\Psi_k^\dagger$ is the four-component spinor $(c_{k\uparrow}^\dagger, c_{k+Q,\uparrow}^\dagger, c_{k\downarrow}^\dagger, c_{k+Q,\downarrow}^\dagger)$. In the absence of spin-orbit coupling, the eigenvalues are $\pm E_{0k}$ with $E_{0k} = \sqrt{\epsilon_k^2 + W_k^2 + \Delta_k^2}$. On the other hand when spin-orbit coupling is present, the eigenvalues of the upper and lower bands now become $\lambda_{up,\pm} = E_{k,\pm}, \lambda_{low,\pm} = -E_{k,\pm}$, respectively, where

$$
E_{k,\pm} = \sqrt{\epsilon_k^2 + W_k^2 + \Delta_k^2 + \Lambda_k^2} \pm 2 \left[ (\epsilon_k^2 + W_k^2)\Lambda_k^2 - W_k^2(\hat{\mathbf{N}} \cdot \mathbf{\hat{\Lambda}})^2 \right]^{\frac{1}{2}}
$$

with $\Lambda_k^2 \equiv |\mathbf{\hat{\Lambda}}(k)|^2 = \Lambda_x^2(k) + \Lambda_y^2(k)$. When the $d_{xy}$ component is absent, $\Delta_k = 0$, and the results of Ref. 12 are recovered. Consider the following two cases separately.

1. $\hat{N} \parallel \hat{z}$

Since the chemical potential is at the mid-gap, we can focus on the lower bands. When $\hat{N} = \hat{z}$, we have $\hat{N} \cdot \mathbf{\hat{\Lambda}} = 0$ and

$$
\lambda_{low,\pm} = -\sqrt{E_{0k}^2 + \Lambda_k^2} \pm 2 \left[ E_{0k}^2 \Lambda_k^2 \right]^{\frac{1}{2}} = -E_{0k} \mp |\mathbf{\hat{\Lambda}}(k)|
$$

Assuming that $\Lambda_0 \ll W_0, \Delta_0 \ll W$ with the electronic bandwidth $W = St$, the change in the ground state energy will be

$$
E_{SO} = \sum_k \left[ (\lambda_{low,+} + \lambda_{low,-}) - 2(-E_{0k}) \right]
$$

$$
= \sum_k \left[ (-E_{0k} - |\mathbf{\hat{\Lambda}}|) - E_{0k} + |\mathbf{\hat{\Lambda}}| + 2E_{0k} \right]
$$

$$
= 0
$$

2. $\hat{N} \perp \hat{z}$

When $\hat{N}$ lies in $xy$-plane, we have $\hat{N} \cdot \mathbf{\hat{\Lambda}} = |\mathbf{\hat{\Lambda}}| \cos \phi_k$, where $\phi_k$ is the angle between $\hat{N}$ and $\mathbf{\hat{\Lambda}}$, and

$$
\cos \phi_k = \frac{\hat{N} \cdot \mathbf{\hat{\Lambda}}}{|\mathbf{\hat{\Lambda}}|} = \frac{N_x \Lambda_x(k) + N_y \Lambda_y(k)}{\sqrt{\Lambda_x^2(k) + \Lambda_y^2(k)}}
$$

(41)
The eigenvalues of the lower bands are now
\[ \lambda_{\text{low, } \pm}^{xy} = -\sqrt{E_{0k}^2 + \Lambda_k^2 \pm 2 \left[ E_{0k}^2 \Lambda_k^2 - W_k^2 \cos^2 \phi_k \right]} \]
\[ \approx -E_{0k} \mp \left( \frac{1}{2} \frac{W_k^2}{E_{0k}^2} |\Lambda_k| - \frac{1}{2} \frac{W_k^2}{E_{0k}^2} (1 + O \left( \frac{W_k^2}{E_{0k}^2} \right)) \right), \]
where we have used \( \cos^2 \phi_k \approx O(1) \). Notice that the signs of the second order terms for \( \lambda_{\text{low, } +}^{xy} \) and \( \lambda_{\text{low, } -}^{xy} \) are both negative, leading to the net change in the ground state energy, which is opposite to the \( N = \hat{z} \) case. Assuming that \( \Lambda_0 \ll W_0, \Delta_0 \ll W \), the change in the ground state energy per lattice site will be
\[ E_{\text{SO}} = \sum_k \left[ \left( \lambda_{\text{low, } +}^{xy} + \lambda_{\text{low, } -}^{xy} \right) - 2(-E_{0k}) \right] \]
\[ \approx -\sum_k \frac{\Lambda_0^2 W_k^2}{E_{0k}^3} \left[ 1 + O \left( \frac{W_k^2}{E_{0k}^2} \right) \right] \]
\[ = -\frac{\Lambda_0^2}{W} \left( \frac{W_0}{W} \right)^2 \left[ 1 + O \left( \frac{W_0}{W} \right)^2 \right] < 0, \]
(43)
Therefore, \( \hat{N} \)-vector should lie in the \( xy \)-plane in the presence of spin-orbit interaction and the result stated above follows.

As large as the spin-orbit coupling may be for U atoms, \( E_{\text{SO}} \) is still a small energy scale. However, if other anisotropies are absent, the order parameter would be in the \( XY \)-plane, resulting in vortices; exchange anisotropy can also result in an easy-axis anisotropy, in which case spin textures could be Ising domain walls that can trap electrons. Although skyrmions are finite energy solutions, vortices cost infinite energy unless they are bound in pairs. We speculate that charge 2e-skyrmionic condensation is a more likely scenario, but the crossover in the texture is an interesting topic for further research.

The following remarks about \( \text{URu}_2\text{Si}_2 \) are relevant: in both magnetic field-temperature (\( H - T \)) and pressure-temperature (\( P - T \)) phase diagrams, the superconducting phase is enclosed within the HO phase. It implies that the superconducting phase is closely related to the HO phase, and is probably induced by it. Throughout our calculation, ignoring of course skyrmions, we have assumed that the system is half-filled. The lower band is filled and the upper band is empty, and the topological invariant is quantized. If this is not the case, then there will be no quantized spin Hall conductance, but an induced superconducting phase from charge 2e-skyrmionic condensation; doping will result in conducting mid-gap states, as in polyacetylene. Of course, such a topological superconducting phase is very sensitive to disorder. Indeed, this may be supported by the destruction of the HO and SC phases with 4% Rh substitution on the Ru site. To summarize, we can find a rationale for a hidden order phase enclosing a superconducting phase at lower temperatures.

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**Appendix A: Derivation of the non-linear \( \sigma \)-model**

To derive the non-linear \( \sigma \)-model, we compute the effective action by integrating out fermions. We start with the action \( S = \sum_{j=1}^{3} S_j \), where
\[ S_j = \int d^3 x \bar{\psi}_j \left[ G_{\sigma_j}^{-1} \right] \psi_j, \]
(41)
with \( G_{\sigma_j}^{-1} = G_{\sigma_j}^{-1} + \Sigma_j \).
For \( j = 1, 2 \), we have
\[ G_{0,j}^{-1} = -i \sigma^0 \tau^x \partial_z - 2i t_0 \sigma^0 \tau^y (\eta_j \partial_z + \partial_y), \]
(42)
\[ \Sigma_j = i \eta_j \Delta_0 \sigma^0 \tau^0 - i \frac{W_0}{2} (\vec{\sigma} \cdot \hat{N}) \tau^z (\eta_j \partial_z + \partial_y), \]
(43)
and for \( j = 3 \), we have
\[ G_{0,3}^{-1} = -i \sigma^0 \tau^z \partial_z, \]
(44)
\[ \Sigma_3 = W_0 (\vec{\sigma} \cdot \hat{N}) \tau^z. \]
The effective action will be $S_{\text{eff}} = \sum_{j=1}^{3} S_{j,\text{eff}}$ with

$$S_{j,\text{eff}} = -\ln \left[ \int D\psi_j D\bar{\psi}_j e^{-S_j} \right]$$
$$= -\ln \left[ \det |G_j^{-1}| \right], \quad \text{(A6)}$$

where the fermion operators can be integrated out easily since the Hamiltonian has only bilinear fermion operator terms. Using the mathematical identity $\ln \det |A| = \text{tr} \ln A$ with $\text{tr}$ being the trace, we have

$$S_{j,\text{eff}} = -\text{tr} \ln G_{0,j}^{-1} [1 + G_{0,j} \Sigma_j]$$
$$= -\text{tr} \ln G_{0,j}^{-1} - \text{tr} [G_{0,j} \Sigma_j] + \frac{1}{2} \text{tr} [G_{0,j} \Sigma_j G_{0,j} \Sigma_j] + \cdots, \quad \text{(A7)}$$

where we have used $\ln(1 + x) = x - \frac{x^2}{2} + \cdots$. The zeroth order term is the effective action for free particles and the first order term vanishes, so our goal is to compute the second order terms:

$$S_{j,\text{eff}}^{(2)} \equiv \frac{1}{2} \text{tr} [G_{0,j} \Sigma_j G_{0,j} \Sigma_j]$$
$$= \frac{1}{2} \int d\tau \int d\tau' \int d^2x \int d^2x' \text{Tr} [G_{0,j}(x, \tau; x', \tau') \Sigma_j(x', \tau') G_{0,j}(x', \tau'; x, \tau) \Sigma_j(x, \tau)]$$
$$= \frac{1}{2} \sum_{k, \tilde{q}} \text{Tr} \left[ G_{0,j}^0(k) \Sigma_j(q) G_{0,j}^0(\tilde{k} + \tilde{q}) \Sigma_j(-\tilde{q}) \right], \quad \text{(A8)}$$

where $k \equiv (k_0, k_x, k_y)$, $\tilde{q} \equiv (q_0, q_x, q_y)$, and $G_{0,j}^0(\tilde{k})$ can be obtained by inverting Eq. (A2) and Eq. (A4).

$$S_{j,\text{eff}}^{(2)} \approx 2 \sum_{k, \tilde{q}} \frac{1}{k_0^2 + 4t^2(\eta_j k_x + k_y)^2} \left[ -\Delta_0^2 + \frac{(W_0^2)}{4}(-\eta_j q_x + q_y)^2(\hat{\mathbf{N}} \cdot \hat{\mathbf{N}}_\perp) \right], \quad \text{(A9)}$$

where terms which are odd in $\tilde{k}$ and $\tilde{q}$ are dropped. Using the relation $\sum_{\tilde{q}} f_{\tilde{q}f} = \int d\tau d^2x |f(\bar{x}, \tau)|^2$, we obtain

$$S_{1,\text{eff}}^{(2)} + S_{2,\text{eff}}^{(2)} \approx \frac{1}{g_1} \int d\tau d^2x \left[ |\partial_X \hat{N}|^2 + |\partial_Y \hat{N}|^2 \right], \quad \text{(A10)}$$

where the constant terms are dropped, $(X, Y)$ is the coordinate after a $\pi/4$ rotation, and

$$\frac{1}{g_1} \sum_{k} \frac{W_0^2}{2(k_0^2 + 4t^2(k_x + k_y)^2)} \quad \text{(A11)}$$

Similarly, for $j = 3$, we obtain

$$S_{3,\text{eff}}^{(2)} \approx -2 \sum_{k, \tilde{q}} \frac{W_0^2}{k_0^2} \frac{q_0}{k_0} \eta_j^2(\hat{\mathbf{N}} \cdot \hat{\mathbf{N}}_\perp)$$
$$= \frac{1}{g_3} \int d\tau d^2x |\partial_x \hat{N}|^2, \quad \text{(A12)}$$

where the constant terms and higher order terms are dropped, and it is rescaled in order to obtain a familiar form.

Putting all together, taking long wavelength limit ($\tilde{q} \to 0$) and keeping only terms up to the second order derivative, we have, for $j = 1, 2$,

$$\frac{1}{g_3} = \sum_{k} \frac{2W_0^2}{k_0^3}. \quad \text{(A13)}$$

Therefore, we obtain the non-linear sigma model,

$$S_{\text{eff}} \approx \frac{1}{g} \int d\tau d^2x \left| \partial_\mu \hat{N} \right|^2, \quad \text{(A14)}$$
Appendix B: Chern-Simons coefficients

In this appendix we are going to prove that

$$\mathcal{N}(G_\sigma) = \frac{\epsilon_{\mu\nu\lambda}}{24\pi^2} \text{Tr} \left[ \int d^3k \frac{G_\sigma^{-1}}{\partial k_\mu} G_\sigma \frac{G_\sigma^{-1}}{\partial k_\nu} G_\sigma \frac{G_\sigma^{-1}}{\partial k_\lambda} \right]$$

$$= - \frac{d^2k}{4\pi} \frac{\delta h_\sigma}{\partial k_x} \times \frac{\delta h_\sigma}{\partial k_y}. \quad (B1)$$

We start by taking $\theta(\sigma, \nu, \lambda)$ to be $(0, x, y)$, and obtain

$$G_\sigma \frac{\partial G_\sigma^{-1}}{\partial k_x} = \frac{1}{(i\omega)^2 - |\tilde{h}_\sigma|^2} \left[ (i\omega \hat{I} + \hat{\pi} \cdot \tilde{h}_\sigma) \cdot (\nabla \hat{h}_\sigma) \right]$$

$$= \frac{-1}{(i\omega)^2 - |\tilde{h}_\sigma|^2} \left[ (\tilde{h}_\sigma \cdot \frac{\delta h_\sigma}{\partial k_x}) \hat{I} + i\pi \cdot (\omega \frac{\delta h_\sigma}{\partial k_x} + \tilde{h}_\sigma \times \frac{\delta h_\sigma}{\partial k_y}) \right], \quad (B3)$$

where we have used the matrix identity $(\hat{\pi} \cdot \hat{a})(\hat{\pi} \cdot \hat{b}) = (\hat{a} \cdot \hat{b}) \hat{I} + i\pi \cdot (\hat{a} \times \hat{b})$. Similarly,

$$G_\sigma \frac{\partial G_\sigma^{-1}}{\partial k_y} = \frac{-1}{(i\omega)^2 - |\tilde{h}_\sigma|^2} \left[ (\tilde{h}_\sigma \cdot \frac{\delta h_\sigma}{\partial k_y}) \hat{I} + i\pi \cdot (\omega \frac{\delta h_\sigma}{\partial k_y} + \tilde{h}_\sigma \times \frac{\delta h_\sigma}{\partial k_x}) \right]. \quad (B4)$$

Therefore,

$$G_\sigma \frac{\partial G_\sigma^{-1}}{\partial k_x} G_\sigma \frac{\partial G_\sigma^{-1}}{\partial k_y} = \frac{1}{((i\omega)^2 - |\tilde{h}_\sigma|^2)^2} \left\{ (\tilde{h}_\sigma \cdot \frac{\delta h_\sigma}{\partial k_x})(\tilde{h}_\sigma \cdot \frac{\delta h_\sigma}{\partial k_y}) \hat{I} \right. \right.$$  

$$+ i\pi \cdot \left[ (\tilde{h}_\sigma \cdot \frac{\delta h_\sigma}{\partial k_y}) (\omega \frac{\delta h_\sigma}{\partial k_x} + \tilde{h}_\sigma \times \frac{\delta h_\sigma}{\partial k_y}) + (\tilde{h}_\sigma \cdot \frac{\delta h_\sigma}{\partial k_y}) (\omega \frac{\delta h_\sigma}{\partial k_x} + \tilde{h}_\sigma \times \frac{\delta h_\sigma}{\partial k_x}) \right]$$

$$\left. - i\pi \cdot (\omega \frac{\delta h_\sigma}{\partial k_y} + \tilde{h}_\sigma \times \frac{\delta h_\sigma}{\partial k_x}) \left[ \hat{I} \cdot (\omega \frac{\delta h_\sigma}{\partial k_y} + \tilde{h}_\sigma \times \frac{\delta h_\sigma}{\partial k_y}) \right] \right\}. \quad (B5)$$

Since we are going to multiply it with the antisymmetric tensor $\epsilon_{\mu\nu\lambda}$, the terms which are symmetric under $(x \leftrightarrow y)$ will vanish. Therefore, only the last term in the bracket contributes,

$$i\pi \cdot \left[ \hat{I} \cdot \left( \omega \frac{\delta h_\sigma}{\partial k_y} + \tilde{h}_\sigma \times \frac{\delta h_\sigma}{\partial k_x} \right) \right]$$

$$= \omega \left( \frac{\delta h_\sigma}{\partial k_x} \times \frac{\delta h_\sigma}{\partial k_y} \right) + \omega \tilde{h}_\sigma (\frac{\delta h_\sigma}{\partial k_x} \cdot \frac{\delta h_\sigma}{\partial k_y}) - \omega \frac{\delta h_\sigma}{\partial k_y} (\tilde{h}_\sigma \cdot \frac{\delta h_\sigma}{\partial k_x})$$

$$- \omega \frac{\delta h_\sigma}{\partial k_x} \left( \frac{\delta h_\sigma}{\partial k_y} \right) + \omega \tilde{h}_\sigma (\frac{\delta h_\sigma}{\partial k_x} \cdot \frac{\delta h_\sigma}{\partial k_y}) - (\tilde{h}_\sigma \cdot \frac{\delta h_\sigma}{\partial k_x} \times \frac{\delta h_\sigma}{\partial k_y}) \tilde{h}_\sigma \right], \quad (B6)$$

where we used the following mathematical identities:

$$\tilde{a} \times (\tilde{b} \times \tilde{c}) = \tilde{b}(\tilde{a} \cdot \tilde{c}) - \tilde{c}(\tilde{a} \cdot \tilde{b}),$$

$$\hat{a} = \tilde{b} \times \tilde{c} = (\tilde{a} \times \tilde{b}) \tilde{a}.$$

Therefore, after combining with $\epsilon_{\sigma\nu\lambda}$ and taking the trace, we have
\[ \epsilon_{\nu\gamma} \text{Tr} \left[ G_\sigma \frac{\partial G_\sigma^{-1}}{\partial \omega} G_\sigma \frac{\partial G_\sigma^{-1}}{\partial k_x} G_\sigma \frac{\partial G_\sigma^{-1}}{\partial k_y} \right] \]

\[ = \frac{-1}{((\omega)^2 - \tilde{h}_\sigma^2)^2} \text{Tr} \left\{ -i\omega \bar{E} \left[ \omega^2 \left( \frac{\partial \tilde{h}_\sigma}{\partial k_x} \times \frac{\partial \tilde{h}_\sigma}{\partial k_y} \right) + (\tilde{h}_\sigma \cdot \frac{\partial \tilde{h}_\sigma}{\partial k_x} \times \frac{\partial \tilde{h}_\sigma}{\partial k_y}) \tilde{h}_\sigma \right] \right\} \]

\[ = \frac{-2}{((\omega)^2 - \tilde{h}_\sigma^2)^2} \left( \frac{\partial \tilde{h}_\sigma}{\partial k_x} \times \frac{\partial \tilde{h}_\sigma}{\partial k_y} \right), \quad \text{(B7)} \]

where we have used the fact that Pauli matrices are traceless, so the only contribution will be the term proportional to \( \tilde{I} \).

We have six non-zero terms because of the \( \epsilon_{\mu\nu\lambda} \) tensor, so

\[ \mathcal{N}(G_\sigma) = -\frac{2 \cdot 6}{24\pi^2} \int d^3k \frac{1}{((\omega)^2 - \tilde{h}_\sigma^2)^2} \left( \frac{\partial \tilde{h}_\sigma}{\partial k_x} \times \frac{\partial \tilde{h}_\sigma}{\partial k_y} \right) \]

\[ = -\int \frac{d^3k}{4\pi \tilde{h}_\sigma^3} \left( \frac{\partial \tilde{h}_\sigma}{\partial k_x} \times \frac{\partial \tilde{h}_\sigma}{\partial k_y} \right) \]

\[ = -\int \frac{d^3k}{4\pi \tilde{h}_\sigma} \frac{\partial \tilde{h}_\sigma}{\partial k_x} \times \frac{\partial \tilde{h}_\sigma}{\partial k_y} \], \quad \text{(B8)}

where the energy integral was done by computing the residue of the second order pole.

**Appendix C: Spin gauge flux** \( F^s_{\mu\nu} \) in terms of \( \bar{N} \)

In the main text, we obtain the spin gauge field to be

\[ f_\mu = \frac{\sigma^z}{2} A^s_\mu, \quad \text{(C1)} \]

Assume that the spin texture has the general form:

\[ \bar{N}(\vec{x}, t) = [\sin \theta(\vec{x}, t) \cos \phi(\vec{x}, t), \sin \theta(\vec{x}, t) \sin \phi(\vec{x}, t), \cos \theta(\vec{x}, t)] \],

where \( \theta(\vec{x}, t) \) and \( \phi(\vec{x}, t) \) can be any function of position and time. Then, we have the unitary matrix

\[ U(\vec{x}, t) = \left( \begin{array}{cc} \cos \frac{\theta(\vec{x}, t)}{2} & -\sin \frac{\theta(\vec{x}, t)}{2} e^{-i\phi(\vec{x}, t)} \\ \sin \frac{\theta(\vec{x}, t)}{2} e^{i\phi(\vec{x}, t)} & \cos \frac{\theta(\vec{x}, t)}{2} \end{array} \right) \],

\[ \partial_\mu U^\dagger(\vec{x}, t) = \left( \begin{array}{cc} -\frac{1}{2} \sin \frac{\theta}{2} \partial_\mu \theta & e^{-i\phi} \left( \frac{1}{2} \cos \frac{\theta}{2} \partial_\mu \theta - i \sin \frac{\theta}{2} \partial_\mu \phi \right) \\ e^{i\phi} \left( \frac{1}{2} \cos \frac{\theta}{2} \partial_\mu \theta - i \sin \frac{\theta}{2} \partial_\mu \phi \right) & -\frac{1}{2} \sin \frac{\theta}{2} \partial_\mu \theta \end{array} \right), \quad \text{(C5)} \]

and

\[ \partial_\mu U(\vec{x}, t) = \left( \begin{array}{cc} -\frac{1}{2} \sin \frac{\theta}{2} \partial_\mu \theta & e^{-i\phi} \left( \frac{1}{2} \cos \frac{\theta}{2} \partial_\mu \theta + i \sin \frac{\theta}{2} \partial_\mu \phi \right) \\ e^{i\phi} \left( \frac{1}{2} \cos \frac{\theta}{2} \partial_\mu \theta + i \sin \frac{\theta}{2} \partial_\mu \phi \right) & -\frac{1}{2} \sin \frac{\theta}{2} \partial_\mu \theta \end{array} \right), \quad \text{(C6)} \]
where we have suppressed the arguments of $\theta(\vec{x}, t)$ and $\phi(\vec{x}, t)$. Therefore, we can calculate the product of the last two matrices, and express the spin gauge flux as

$$F_{\mu \nu}^s = -i \left[ \frac{i}{2} \sin \theta (\partial_\mu \theta \partial_\nu \phi - \partial_\nu \theta \partial_\mu \phi) \right] \times 2$$

$$= \sin \theta (\partial_\mu \theta \partial_\nu \phi - \partial_\nu \theta \partial_\mu \phi).$$

(C8)

In addition, we can also write $\hat{N} \cdot (\partial_\mu \hat{N} \times \partial_\nu \hat{N})$ in terms of $\theta(\vec{x}, t)$ and $\phi(\vec{x}, t)$,

$$\hat{N} \cdot (\partial_\mu \hat{N} \times \partial_\nu \hat{N}) = \sin \theta(\vec{x}, t) \cos \phi(\vec{x}, t)$$

$$= \begin{vmatrix}
\cos \theta(\vec{x}, t) \cos \phi(\vec{x}, t) & \cos \theta(\vec{x}, t) \sin \phi(\vec{x}, t) & \cos \theta(\vec{x}, t) \\
- \sin \theta(\vec{x}, t) \sin \phi(\vec{x}, t) & + \sin \theta(\vec{x}, t) \cos \phi(\vec{x}, t) & \cos \theta(\vec{x}, t) \\
- \sin \theta(\vec{x}, t) \sin \phi(\vec{x}, t) & + \sin \theta(\vec{x}, t) \cos \phi(\vec{x}, t) & \cos \theta(\vec{x}, t)
\end{vmatrix}$$

$$= \sin \theta (\partial_\mu \theta \partial_\nu \phi - \partial_\nu \theta \partial_\mu \phi),$$

(C9)

where, again, we suppressed the arguments of $\theta(\vec{x}, t)$ and $\phi(\vec{x}, t)$. Finally, we obtain

$$F_{\mu \nu}^s = \hat{N} \cdot (\partial_\mu \hat{N} \times \partial_\nu \hat{N}).$$

(C10)