ON THE GLOBAL WELL-POSEDNESS TO THE 3-D
INCOMPRESSIBLE ANISOTROPIC
MAGNETOHYDRODYNAMICS EQUATIONS

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Abstract. The present paper is devoted to the well-posedness issue of solutions to the 3-D incompressible magnetohydrodynamic(MHD) equations with horizontal dissipation and horizontal magnetic diffusion. By means of anisotropic Littlewood-Paley analysis we prove the global well-posedness of solutions in the anisotropic Sobolev spaces of type $H^{0,s_0}(\mathbb{R}^3)$ with $s_0 > \frac{1}{2}$ provided the norm of initial data is small enough in the sense that

$$\left(\|u_0^h(0)\|_{H^{0,s_0}}^2 + \|B_0^h(0)\|_{H^{0,s_0}}^2\right) \exp\left\{C_1\left(\|u_0^3\|_{H^{0,s_0}}^4 + \|B_0^3\|_{H^{0,s_0}}^4\right)\right\} \leq \varepsilon_0,$$

for some sufficiently small constant $\varepsilon_0$.

1. Introduction. In this paper, we study the global well-posedness of the following three-dimensional incompressible Magnetohydrodynamics system(MHD) with horizontal dissipation and horizontal magnetic diffusion:

$$\begin{cases}
\partial_t u + (u \cdot \nabla)u + \nabla p - \mu \Delta_h u = (\nabla \times B) \times B, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\
\partial_t B - \nabla \times (u \times B) - \nu \Delta_h B = 0, \\
\nabla \cdot u = 0, & \nabla \cdot B = 0, \\
u|_{t=0} = u_0, & B|_{t=0} = B_0,
\end{cases}$$

(1.1)

where $u = u(t, x)$ and $B = B(t, x)$ are the fluid velocity and magnetic field, depending on the spatial position $x$ and the time $t$. The scalar functions $p = p(t, x)$ denote the pressure. The usual Laplace operator $\Delta$ is replaced by the Laplace operator $\Delta_h$ in the horizontal variables, namely $\Delta_h \overset{\text{def}}{=} \partial_1^2 + \partial_2^2$. The positive constants $\nu$ and $\mu$ are the viscosity and the resistivity coefficients. Without loss of generality we set

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\[ \mu = \nu = 1. \]

Using vector identity, we can rewrite (1.1) as follows:

\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u + \nabla \left( p + \frac{|B|^2}{2} \right) - \Delta_h u &= (B \cdot \nabla) B, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\
\partial_t B + (u \cdot \nabla) B - \Delta_h B &= (B \cdot \nabla) u, \\
\nabla \cdot u &= 0, \quad \nabla \cdot B = 0, \\
u_{t=0} = u_0, \quad B|_{t=0} = B_0.
\end{align*}
\]

The MHD equations govern the dynamics of the velocity and the magnetic field in electrically conducting fluids such as plasmas and reflect the basic physics conservation laws. There have been a lot of studies on MHD by physicists and mathematicians because of their prominent roles in modeling many phenomena, see [17] and the references therein. For the following 2-D MHD equations with partial dissipation and magnetic diffusion,

\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u + \nabla \left( p + \frac{|B|^2}{2} \right) - \frac{\partial^2}{\partial x^2} u &= (B \cdot \nabla) B, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\
\partial_t B + (u \cdot \nabla) B - \frac{\partial^2}{\partial x^2} B &= (B \cdot \nabla) u, \\
\nabla \cdot u &= 0, \quad \nabla \cdot B = 0, \\
u_{t=0} = u_0, \quad B|_{t=0} = B_0,
\end{align*}
\]

the global regularity has been obtained by Cao, Regmi and Wu [2]. For the generalized magnetohydrodynamics equations cases, see also [19]. The global existence in the whole domain in \( \mathbb{R}^2 \) as well as the decay estimates of small smooth solution is proved in [16]. Wu [20] studied the global well-posedness of the incompressible magnetohydrodynamic (MHD) system with a velocity damping term.

For the 3-D MHD equations, Chen, Miao and Zhang [6] proved a Beale-Kato-Majda type blow-up criterion of smooth solutions via the vorticity of velocity, and the authors [7] investigated regularity criterion of the weak solution for the 3-D viscous MHD equations by means of the Fourier localization technique and Bonys para-product decomposition. In a recent remarkable paper [12], by using the Lagrangian coordinates system, Lin, Xu and Zhang proved the global existence of smooth solution of MHD system without magnetic diffusion around the trivial solution \( (x_2, 0) \) (see [13, 18] for 3-D case). Later on, Xu and Zhang [21] proved the global well-posedness of three-dimensional incompressible magnetohydrodynamical (MHD) system with small and smooth initial data by using the algebraic structure of the Lagrangian formulation. Lei and Zhou [11] proved a blow-up criterion for classical solutions to the incompressible magnetohydrodynamic equations with zero viscosity and positive resistivity in \( \mathbb{R}^3 \) and established global weak solutions to 2-D the magnetohydrodynamic equations with zero viscosity and positive resistivity for initial data in Sobolev space \( H^1(\mathbb{R}^2) \). Chemin et al. proved in [5] the existence of solutions to the viscous, non-resistive magnetohydrodynamics (MHD) equations on the whole of \( \mathbb{R}^n \), \( n = 2, 3 \), for divergence-free initial data in Besov spaces \( B^{1/2-1}_2(\mathbb{R}^n) \times B^{1/2}_2(\mathbb{R}^n) \).

Our goal of this paper is to prove that the system (1.1) is globally well-posedness when the initial data are sufficient small. The main difficulty of this work lies in the fact that we will have to deal with the nonlinear coupling between the Navier-Stokes equations with a forcing induced by the magnetic field and the induction equation. Toward this, we will have to work in the framework of anisotropic type Besov spaces here. We should mention that the tool of anisotropic Littlewood-Paley
theory introduced by Chemin-Zhang in [8] and Paicu in [14] will play a crucial role in our work. This method has been used in the study of the global well-posedness to 3-D anisotropic incompressible Navier-Stokes equations [4, 8, 9, 10, 14, 15].

We can now state the main result of this paper.

**Theorem 1.1.** Let $s_0 > \frac{1}{2}$ be a real number, and $(u_0, B_0)$ be a divergence free vector field in $H^{0,s_0}(\mathbb{R}^3) \times H^{0,s_0}(\mathbb{R}^3)$. Then there exists a small enough constant $\varepsilon_0$ and some positive constant $C_0$ such that if

$$\eta_0 \overset{def}{=} (\| u_0^h(0) \|_{H^{0,s_0}}^2 + \| B_0^h(0) \|_{H^{0,s_0}}^2) \exp \left\{ C_0 (\| u_0^3 \|_{H^{0,s_0}}^4 + \| B_0^3 \|_{H^{0,s_0}}^4) \right\}$$

$$\leq \varepsilon_0,$$

the system (1.2) has a unique global solution $(u, B) \in C([0, \infty); H^{0,s_0}(\mathbb{R}^3)) \times C([0, \infty); H^{0,s_0}(\mathbb{R}^3))$ with $(\nabla_h u, \nabla_h B) \in L^2(\mathbb{R}^+; H^{0,s_0}(\mathbb{R}^3)) \times L^2(\mathbb{R}^+; H^{0,s_0}(\mathbb{R}^3))$. Moreover, there holds

$$\| u^h \|_{L^\infty(\mathbb{R}^+, H^{0,s_0})}^2 + \| \nabla_h u^h \|_{L^2(\mathbb{R}^+, H^{0,s_0})}^2$$

$$+ \| B^h \|_{L^\infty(\mathbb{R}^+, H^{0,s_0})}^2 + \| \nabla_h B^h \|_{L^2(\mathbb{R}^+, H^{0,s_0})}^2 \leq C \eta_0,$$

and

$$\| u^3 \|_{L^\infty(\mathbb{R}^+, H^{0,s_0})}^2 + \| \nabla_h u^3 \|_{L^2(\mathbb{R}^+, H^{0,s_0})}^2 + \| B^3 \|_{L^\infty(\mathbb{R}^+, H^{0,s_0})}^2$$

$$+ \| \nabla_h B^3 \|_{L^2(\mathbb{R}^+, H^{0,s_0})}^2 \leq 2 (\| u_0^3 \|_{H^{0,s_0}}^2 + \| B_0^3 \|_{H^{0,s_0}}^2) + \eta_0.$$

The rest of the paper unfolds as follows. In the next section, we introduce the main tool for the proof—the anisotropic Littlewood-Paley decomposition—and some related functional spaces. In Section 3, we focus on the proof of the existence and uniqueness of a solution of (1.1). Let us end this section with the notations we are going to use in this paper.

- Throughout this paper, $C$ represents some “harmless” constant, which can be understood from the context. In some places, we shall alternately use the notation $A \lesssim B$ instead of $A \leq CB$, and $A \approx B$ means $A \lesssim B$ and $B \lesssim A$.

- If $p \in [1, +\infty]$ then we denote by $p'$ the conjugated exponent of $p$ defined by $1/p + 1/p' = 1$.

- If $X$ is a Banach space, $T > 0$ and $p \in [1, +\infty]$ then $L^p_T(X)$ stands for the set of Lebesgue measurable functions $f$ from $[0, T)$ to $X$ such that $t \mapsto \| f(t) \|_X$ belongs to $L^p([0, T))$. If $T = +\infty$, then the space is merely denoted by $L^p(X)$. Finally, if $I$ is some interval of $\mathbb{R}$ then the notation $C(I; X)$ stands for the set of continuous functions from $I$ to $X$. We denote $L^p_T(L^q((0,T)))$ the space $L^p((0,T); L^q(\mathbb{R}^3) \times \mathbb{R}^3; L^q(\mathbb{R}^3)))$.

- Let $A, B$ be two operators, we denote $[A, B] = AB - BA$, the commutator between $A$ and $B$.

- Throughout this paper, $(c_q)_{q \in \mathbb{Z}}$ (resp. $(c_q(t))_{q \in \mathbb{Z}}$) denotes a generic element of the space of $\ell^2(\mathbb{Z})$ with norm 1.

2. **Basic results on Besov spaces.** In order to define Besov space, we need the following anisotropic version of dyadic decomposition of the Fourier space in the case $x \in \mathbb{R}^3$, see [1, 8]. Let $\varphi(\tau)$ and $\chi(\tau)$ be smooth functions such that

$$\text{Supp} \subset \left\{ \tau \in \mathbb{R} \mid \frac{3}{4} \leq |\tau| \leq \frac{8}{3} \right\} \quad \text{and} \quad \forall \tau > 0, \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\tau) = 1,$$
dyadic operators satisfy the property of almost orthogonality:

\[
F(a) \quad \text{(respectively, } C(a) \text{)}
\]

For \( a \in \mathcal{S}'(\mathbb{R}^3) \), we set

\[
\Delta_j^a \equiv \mathcal{F}^{-1}(\varphi(2^{-j}|\xi|)\hat{a}), \quad S_j^a \equiv \mathcal{F}^{-1}(\chi(2^{-j}|\xi|)\hat{a}),
\]

where \( \mathcal{F}^{-1}a \) being the inverse Fourier transform of the distribution \( a \). Then the dyadic operators satisfy the property of almost orthogonality:

\[
\Delta_k^a \Delta_j^a = 0 \quad \text{if } |k - j| \geq 2 \quad \text{and} \quad \Delta_k^a (S_{j-1}^a \Delta_j^a) = 0 \quad \text{if } |k - j| \geq 5. \tag{2.1}
\]

**Definition 2.2.**

\[
H^{0,s}(\mathbb{R}^3) \equiv \left\{ u \in L^2(\mathbb{R}^3) : \|u\|_{H^{0,s}}^2 = \int_{\mathbb{R}^3} |\xi|^s |\hat{u}(\xi)|^2 d\xi < +\infty \right\}
\]

with

\[
\|u\|^2_{H^{0,s}} \equiv \|u\|_{L^2}^2 + \|\partial_x^s u\|_{L^2}^2.
\]

As we shall repeatedly use the anisotropic Littlewood-Paley theory in what follows, we list some basic facts here. Bernstein’s inequality is fundamental in the analysis involving Besov spaces. Please see the details in [8, 14].

**Lemma 2.2.** Let \( B_h \) (respectively, \( B_v \)) be a ball of \( \mathbb{R}^2_h \) (respectively, \( \mathbb{R}^2_v \)), and \( C_h \) (respectively, \( C_v \)) a ring of \( \mathbb{R}^2_h \) (respectively, \( \mathbb{R}^2_v \)); let \( 1 \leq p_2 \leq p_1 \leq \infty \) and \( 1 \leq q_2 \leq q_1 \leq \infty \). Then the following hold:

**If the support of \( \hat{a} \) is included in \( 2^k B_h \), then**

\[
\|\partial_{x_i}^\alpha a\|_{L^{p_1}_h(L^{q_1}_h)} \lesssim 2^{k(|\alpha| + 2(\frac{1}{p_2} - \frac{1}{p_1}))} \|a\|_{L^{p_1}_h(L^{q_1}_h)}.
\]

**If the support of \( \hat{a} \) is included in \( 2^k B_v \), then**

\[
\|\partial_{x_i}^\alpha a\|_{L^{p_1}_v(L^{q_1}_v)} \lesssim 2^{(|\beta| + (\frac{1}{q_2} - \frac{1}{q_1}))} \|a\|_{L^{p_1}_v(L^{q_1}_v)}.
\]

**If the support of \( \hat{a} \) is included in \( 2^k C_h \), then**

\[
\|a\|_{L^{p_1}_h(L^{q_1}_h)} \lesssim 2^{-kN} \sup_{|\alpha|=N} \|\partial_{x_i}^\alpha a\|_{L^{p_1}_h(L^{q_1}_h)}.
\]

**If the support of \( \hat{a} \) is included in \( 2^k C_v \), then**

\[
\|a\|_{L^{p_1}_v(L^{q_1}_v)} \lesssim 2^{-kN} \|\partial_{x_i}^\alpha a\|_{L^{p_1}_v(L^{q_1}_v)}.
\]

We shall use the following anisotropic version of Bony’s decomposition for the vertical variables:

\[
ab = T^v(a, b) + R^v(a, b) \quad \text{or} \quad ab = T^v(a, b) + T^v(a, b) + R^v(a, b), \tag{2.3}
\]

where

\[
T^v(a, b) = \sum_{k \in \mathbb{Z}} S_k^v a \Delta_k^b, \quad T^v(a, b) = T^v(b, a),
\]

\[
R^v(a, b) = \sum_{k \in \mathbb{Z}} \Delta_k^v a S_{k+2}^v b, \quad R^v(a, b) = \sum_{k \in \mathbb{Z}} \Delta_k^v a \Delta_k^v b.
\]
\[ \Delta^*_k b = \sum_{\ell=k-1}^{k+1} \Delta^\ell b. \]

Before giving some properties of the anisotropic Besov spaces, we recall the Hölder and Young’s inequalities in the framework of anisotropic Lebesgue spaces.

**Lemma 2.3.** 1) Let \( f \in L^p_k(L^r_{\nu \ell}(\nu \ell)) \), \( g \in L^q_k(L^r_{\nu \ell}(\nu \ell)) \) for \( 1 \leq r, r_1, r_2, p, p_1, p_2 \leq \infty \). Then \( fg \in L^k(L^r_{\nu \ell}(\nu \ell)) \) and satisfies
\[ \|fg\|_{L^k(L^r_{\nu \ell}(\nu \ell))} \lesssim \|f\|_{L^p_k(L^r_{\nu \ell}(\nu \ell))} \|g\|_{L^q_k(L^r_{\nu \ell}(\nu \ell))}, \] (2.4)

where \( \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \) and \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \).

2) Let \( f \in L^p_k(L^r_{\nu \ell}(\nu \ell)), g \in L^q_k(L^r_{\nu \ell}(\nu \ell)) \) for \( 1 \leq r, r_1, r_2, p, p_1, p_2 \leq \infty \). Then \( f \ast g \in L^k(L^r_{\nu \ell}(\nu \ell)) \) and satisfies
\[ \|f \ast g\|_{L^k(L^r_{\nu \ell}(\nu \ell))} \lesssim \|f\|_{L^p_k(L^r_{\nu \ell}(\nu \ell))} \|g\|_{L^q_k(L^r_{\nu \ell}(\nu \ell))}, \] (2.5)

where \( 1 + \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \) and \( 1 + \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \).

We first present the following property of the anisotropic Besov space, which is crucial to obtain the a priori estimate of the solution of (1.2).

**Lemma 2.4.** (See [9].) Let \( s > \frac{1}{2} \) and \( a, b \in H^{0,s}(\mathbb{R}^3) \). Then there holds
\[ \|\Delta^*_q (a \nabla b)\|_{L^2(\mathbb{R}^3)} \lesssim c_q 2^{-qs} \left( \|a\|_{H^{0,s}} \|\nabla \|_{H^{0,s}} \|\nabla b\|_{H^{0,s}} \right) \] (2.6)

and
\[ \|\Delta^*_q (ab)\|_{L^2(\mathbb{R}^3)} \lesssim c_q 2^{-qs} \left( \|a\|_{H^{0,s}} \|\nabla \|_{H^{0,s}} \|b\|_{H^{0,s}} \|\nabla b\|_{H^{0,s}} \right) \] (2.7)

**Lemma 2.5.** (See [9].) Let \( s > \frac{1}{2} \) and \( a = (a^h, a^3) \in H^{0,s}(\mathbb{R}^3), b \in H^{0,s}(\mathbb{R}^3) \) with \( \text{div} a = 0 \). Then there holds
\[ \|\Delta^*_q (a \cdot \nabla b)\|_{L^2(\mathbb{R}^3)} \lesssim c_q 2^{-2qs} \left( \|a\|_{H^{0,s}} \|\nabla b\|_{H^{0,s}} \right) \] (2.8)

**Lemma 2.6.** (See [9].) Let \( s > \frac{1}{2} \) and \( a = (a^h, a^3) \in H^{0,s}(\mathbb{R}^3) \) with \( \text{div} a = 0 \). Then there holds
\[ \left\| \sum_{\ell, k=1}^{3} \Delta^*_q (-\Delta)^{-1} \partial_k (a^h \nabla a^h) \right\|_{L^2(\mathbb{R}^3)} \lesssim c_2 2^{-2qs} \left( \|\nabla a^3\|_{H^{0,s}} \|a^3\|_{H^{0,s}} \|\nabla b\|_{H^{0,s}} \right) \]
Corollary 2.7. (See [9].) Under the same assumptions of Lemma 2.6, one has
\[
\left| \sum_{\ell,k=1}^{3} ((-\Delta)^{-1} \partial_\ell \partial_k (a^\ell a^k) |\text{div}_h b^h)_{L^2} \right|
\lesssim \|a^h\|_{H^{0,\ast}} \|\nabla_h b^h\|_{H^{0,\ast}} (\|\nabla_h a^h\|_{H^{0,\ast}} + \|\nabla_h a^3\|_{L^2}).
\]

In order to deal with the nonlinear coupling between the Navier-Stokes equations with a forcing induced by the magnetic field and the induction equation, we need the following lemma.

Lemma 2.8. Let \( s > \frac{1}{2} \) and \( a = (a^h, a^3) \in H^{0,\ast}(\mathbb{R}^3), b = (b^h, b^3) \in H^{0,\ast}(\mathbb{R}^3) \) with \( \text{div} a = 0 \). Then there holds
\[
|((\Delta^v_q(a \cdot \nabla b)|\Delta^v_q c)_{L^2}| \leq c^2 q^{2-2q^0} \|c\|_{H^{0,\ast}} \|\nabla_h c\|_{H^{0,\ast}}^\frac{1}{2} \|\Delta^v_q c\|_{L^2}^\frac{1}{2}
\times \left[ (\|a^h\|_{H^{0,\ast}}^\frac{1}{2} \|\nabla_h b\|_{H^{0,\ast}}^\frac{1}{2} + \|\nabla_h a^h\|_{H^{0,\ast}} \|b\|_{H^{0,\ast}}^\frac{1}{2} \|\nabla_h a^h\|_{H^{0,\ast}}^\frac{1}{2} \|\nabla_h b\|_{H^{0,\ast}}^\frac{1}{2}
+ (\|a^h\|_{H^{0,\ast}}^\frac{1}{2} \|\nabla_h b\|_{H^{0,\ast}}^\frac{1}{2} + \|\nabla_h a^h\|_{H^{0,\ast}} \|b\|_{H^{0,\ast}}^\frac{1}{2} \|\nabla_h a^h\|_{H^{0,\ast}}^\frac{1}{2} \|\nabla_h b\|_{H^{0,\ast}}^\frac{1}{2}) \right].
\]

Proof. The main idea of the proof to this lemma essentially follows from that of Lemma 3 of [4] and Proposition 3.3 of [8]. Due to the right hand side of (2.9) does not contain term with \( \partial_3 u \) and \( \partial_3 B \), we distinguish the terms with horizontal derivatives from the terms with vertical ones so that
\[
I_q^{\text{def}} = (\Delta^v_q(a \cdot \nabla b)|\Delta^v_q c)_{L^2} \equiv I_q^h + I_q^v,
\]
with
\[
I_q^h \equiv ((\Delta^v_q(a^h \cdot \nabla_h b)|\Delta^v_q c)_{L^2} \quad \text{and} \quad I_q^v \equiv ((\Delta^v_q(a^3 \cdot \partial_3 b)|\Delta^v_q c)_{L^2}.
\]

Thanks to (2.6) and Lemma 2.2, one can deduce from (2.4) that
\[
I_q^h \leq \|\Delta^v_q(a^h \cdot \nabla_h b)\|_{L^2(I_L^2)} \|\Delta^v_q c\|_{L^2(I_L^2)} \leq c^2 q^{2-2q^0} \left( (\|a^h\|_{H^{0,\ast}}^\frac{1}{2} \|\nabla_h a^h\|_{H^{0,\ast}} \|b\|_{H^{0,\ast}}^\frac{1}{2} \|\nabla_h b\|_{H^{0,\ast}}^\frac{1}{2} + (\|a^h\|_{H^{0,\ast}}^\frac{1}{2} \|\nabla_h b\|_{H^{0,\ast}}^\frac{1}{2} + \|\nabla_h a^h\|_{H^{0,\ast}} \|b\|_{H^{0,\ast}}^\frac{1}{2} \|\nabla_h a^h\|_{H^{0,\ast}}^\frac{1}{2} \|\nabla_h b\|_{H^{0,\ast}}^\frac{1}{2}) \right).
\]

On the other hand, to deal with \( I_q^v \), we use Bony’s decomposition (2.3) \( a^3 \partial_3 b \) and then a commutator process from [4, 8] for \( \Delta^v_q(T_a \partial_3 b) \) so that
\[
I_q^v = (S_{q-1}^v a^3 \partial_3 \Delta^v_q b |\Delta^v_q c) + \sum_{|q' - q| \leq 5} ((\Delta^v_{q'} S_{q'}^v a^3) \partial_3 \Delta^v_q b |\Delta^v_q c)
+ \sum_{|q' - q| \leq 5} ((S_{q-1}^v a^3 - S_{q-1}^v a^3) \partial_3 \Delta^v_q b |\Delta^v_q c)
+ \sum_{|q' - q| \leq 5} (\Delta^v_q(\Delta^v_{q'} a^3 S_{q'-2}^v \partial_3 b) |\Delta^v_q c)
= I_{q,v}^{1,v} + I_{q,v}^{2,v} + I_{q,v}^{3,v} + I_{q,v}^{4,v}.
\]

In what follows, we shall successively estimate all the terms above. Firstly as \( \text{div} a = 0 \), we get by integration by parts that
\[
I_{q,v}^{1,v} = \int_{\mathbb{R}^3} S_{q-1}^v a^3 \div_h a^h \Delta^v_q b |\Delta^v_q c \, dx - \int_{\mathbb{R}^3} S_{q-1}^v a^3 \Delta^v_q b \partial_3 \Delta^v_q c \, dx
\]
from which and Lemma 2.2, we deduce that
\[ |I_q^{1,v}| \leq \|S_{q'}^{-1}\|_{L^\infty(L^q)} \|\Delta_q^v b\|_{L^q(L^2)} \|\Delta_q^v c\|_{L^2(L^2)} + 2^q \|S_{q'}^{-1}\|_{L^\infty(L^q)} \|\Delta_q^v b\|_{L^q(L^2)} \|\Delta_q^v c\|_{L^2(L^2)} \]
\[ \lesssim c_q 2^{-q s} \|\Div_h a^h\|_{H^{q,s}} |b|_{L^{\infty}} \|\nabla_h b\|_{H^{q,s}} \|\nabla_h c\|_{H^{q,s}} \]

To deal with the commutator in \(I_q^{2,v}\), we first use Taylor’s formula. Writing \(\bar{h}(x_3) = x_3 h(x_3)\) and integrating by parts, we find that
\[ I_q^{2,v} = \sum_{q', q'' \leq 5} \int_{\mathbb{R}^3} \int_{\mathbb{R}} \bar{h}(2^q(x_3 - y_3)) \int_0^1 S_{q'}^{-1} \partial_3 a^3(x, \tau y_3 + (1 - \tau) x_3) d\tau \]
\[ \times \partial_3 \Delta_q^v b(x, y_3) dy_3 \Delta_q^v c(x) dx. \]

Young’s inequality, together with Lemma 2.2 and the estimate
\[ \|\Delta_q^v a\|_{L^q(L^q)} \lesssim \|\Delta_q^v a\|_{L^q(L^q)} \|\nabla_h a\|_{L^q(L^q)} \lesssim c_q 2^{-q s} \|a\|_{H^{q,s}} \|\nabla_h a\|_{H^{q,s}}, \]
(2.12)
then yields from (2.5)
\[ |I_q^{2,v}| \lesssim \sum_{q', q'' \leq 5} 2^{-q} \|S_{q'}^{-1}\|_{L^\infty(L^q)} \|\Delta_q^v b\|_{L^q(L^q)} \|\Delta_q^v c\|_{L^2(L^2)} |\bar{h}|_{L^1} \]
\[ \lesssim c_q 2^{-q s} \|\Div_h a^h\|_{H^{q,s}} |b|_{L^{\infty}} \|\nabla_h b\|_{H^{q,s}} \|\nabla_h c\|_{H^{q,s}} \]

Note that
\[ \|\Delta_q^v a^3\|_{L^q(L^q)} \lesssim 2^{-q} \|\Delta_q^v \partial_3 a^3\|_{L^2} \lesssim c_q 2^{-q(s + \frac{1}{2})} \|\Div_h a^h\|_{H^{q,s}} \]
and
\[ \|\Delta_q^v a^3\|_{L^q(L^q)} \lesssim 2^{-q} \|\Delta_q^v \partial_3 a^3\|_{L^2} \lesssim c_q 2^{-q} \|\Div_h a^h\|_{L^2} \quad \text{for} \quad q \leq 0, \]
which together with (2.12) ensures that
\[ |I_q^{3,v}| \lesssim \sum_{q', q'' \leq 5} \|\Delta_q^v a^3\|_{L^q(L^q)} \|\partial_3 \Delta_q^v b\|_{L^q(L^q)} \|\Delta_q^v c\|_{L^2(L^2)} \]
\[ \lesssim c_q 2^{-q s} \|\Div_h a^h\|_{H^{q,s}} |b|_{L^{\infty}} \|\nabla_h b\|_{H^{q,s}} \|\nabla_h c\|_{H^{q,s}} \]
as \( s > \frac{1}{2} \). Finally again thanks to Lemma 2.2, we obtain
\[ |I_q^{4,v}| \lesssim \sum_{q', q'' \geq q - N_0} 2^q \|\Delta_q^v a^3\|_{L^q(L^q)} \|S_{q'}^{-1} b\|_{L^q(L^q)} \|\Delta_q^v c\|_{L^2(L^2)} \]
\[ \lesssim c_q 2^{-q s} \|\Div_h a^h\|_{H^{q,s}} |b|_{L^{\infty}} \|\nabla_h b\|_{H^{q,s}} \|\nabla_h c\|_{H^{q,s}} \]
This gives the estimate of \(I_q^v\). Combining (2.10) and (2.11), we complete the proof of Lemma 2.8. \(\square\)

3. The proof of Theorem 1.1. Motivated by [22], we first rewrite (1.2) as follows:
\[ \begin{cases}
\partial_t u^h + u \cdot \nabla u^h - \Delta_h u^h + \nabla_h \left( p + \frac{|B|^2}{2} \right) = B \cdot \nabla B^h, \\
\partial_t u^3 + u \cdot \nabla u^3 - \Delta_h u^3 + \nabla_3 \left( p + \frac{|B|^2}{2} \right) = B \cdot \nabla B^3, \\
\Div_h u^h + \partial_3 u^3 = 0, \\
(u^h, u^3)|_{t=0} = (u_0^h, u_0^3),
\end{cases} \]
(3.1)
The cut-off operator defined on $L^2$ is given by

$$
E_n u \overset{\text{def}}{=} F^{-1}(1_{B(0,n)} \hat{u}).
$$

The approximate system we consider is of the form

$$
\begin{align*}
\partial_t B^h + u \cdot \nabla B^h - \Delta_h B^h &= B \cdot \nabla u^h, \\
\partial_t B^3 + u \cdot \nabla B^3 - \Delta_h B^3 &= B \cdot \nabla u^3, \\
\partial_t B^1 + \partial_x B^2 + \partial_3 B^3 &= 0,
\end{align*}
$$

(3.2)

Solving system (3.1)-(3.2) will be based on the Friedrichs regularization method: we construct the approximate solutions to (3.1)-(3.2). In order to do so, let $E_n$ be the cut-off operator defined on $L^2$

$$
E_n u \overset{\text{def}}{=} F^{-1}(1_{B(0,n)} \hat{u}).
$$

The approximate system we consider is of the form

$$
\begin{align*}
\partial_t u_n^h + E_n(u_n \cdot \nabla u_n^h) - \Delta_h u_n^h + \sum_{\ell,k=1}^3 E_n \nabla_h (-\Delta)^{-1} \partial_\ell \partial_k (u_n^\ell u_n^k - B_n^\ell B_n^k) \\
&= E_n(B_n \cdot \nabla B_n^h), \\
\partial_t u_n^3 + E_n(u_n \cdot \nabla u_n^3) - \Delta_h u_n^3 + \sum_{\ell,k=1}^3 E_n \partial_3 (-\Delta)^{-1} \partial_\ell \partial_k (u_n^\ell u_n^k - B_n^\ell B_n^k) \\
&= E_n(B_n \cdot \nabla B_n^3),
\end{align*}
$$

(3.3)

and

$$
\begin{align*}
\partial_t B_{n}^h + E_n(u_n \cdot \nabla B_n^h) - \Delta_h B_n^h &= E_n(B_n \cdot \nabla u_n^h), \\
\partial_t B_{n}^3 + E_n(u_n \cdot \nabla B_n^3) - \Delta_h B_n^3 &= E_n(B_n \cdot \nabla u_n^3), \\
\text{div}_h B_n^h + \partial_3 B_n^3 &= 0,
\end{align*}
$$

(3.4)

The system (3.3)-(3.4) appears to be an ordinary differential equation in the space

$$
L^2 \times L^2 \overset{\text{def}}{=} \{(a, b) \in L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) | \text{Supp} \hat{a} \times \text{Supp} \hat{b} \subset B(0, n) \times B(0, n)\}.
$$

This ordinary differential equation is globally wellposed because

$$
\begin{align*}
\|u_n(t)\|_{L^2}^2 + \|B_n(t)\|_{L^2}^2 + 2 \int_0^t (\|\nabla_h u_n(s)\|_{L^2}^2 + \|\nabla_h B_n(s)\|_{L^2}^2) ds \\
= \|E_n u_0\|_{L^2}^2 + \|E_n B_0\|_{L^2}^2 \leq \|u_0\|_{L^2}^2 + \|B_0\|_{L^2}^2.
\end{align*}
$$

We refer to [3] and [14] for the details.

**Conclusion of the proof of Theorem 1.1.** Applying the operator $\Delta^\nu_h$ to the horizontal equations in (3.3)-(3.4) and then taking the $L^2$ inner product of the resulting equation with $\Delta^\nu_h u_n^h$ and $\Delta^\nu_h B_n^h = (\Delta^\nu_h B_n^1, \Delta^\nu_h B_n^3)$, respectively, we have

$$
\begin{align*}
\frac{d}{dt} (\|\Delta^\nu_h u_n^h(t)\|_{L^2}^2 + \|\Delta^\nu_h B_n^h(t)\|_{L^2}^2) \\
&+ 2 \|\nabla_h \Delta^\nu_h u_n^h(t)\|_{L^2}^2 + 2 \|\nabla_h \Delta^\nu_h B_n^h(t)\|_{L^2}^2 \\
&= -2(\Delta^\nu_h (u_n \cdot \nabla u_n^h))_{L^2} + 2(\Delta^\nu_h (B_n \cdot \nabla B_n^h))_{L^2}
\end{align*}
$$

(3.5)
where we used the fact that $\mathbb{E}_n u_n = u_n$ and $\mathbb{E}_n B_n = B_n$.

Applying Lemma 2.5 and Lemma 2.8 gives

$$\begin{align*}
|W_k^1| &\leq c_2^2 2^{-2qs} \left[ u_n^{\frac{3}{2}} \|u_n\|_{\dot{H}^s(\mathbb{R}^n)} \|\nabla_h u_n\|_{\dot{H}^s(\mathbb{R}^n)} \|\nabla_h u_n\|_{\dot{H}^s(\mathbb{R}^n)} \right] (3.6) \\
&\quad \times \left( \|\nabla_h u_n\|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{1}{2}} \|u_n\|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{1}{2}} + \|u_n\|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{1}{2}} \|\nabla_h u_n\|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{1}{2}} \right),
\end{align*}$$

and

$$\begin{align*}
|W_k^2| &\leq c_2^2 2^{-2qs} \left[ \|u_n^{\frac{3}{2}} \|_{\dot{H}^s(\mathbb{R}^n)} \|\nabla_h u_n\|_{\dot{H}^s(\mathbb{R}^n)} \|\nabla_h u_n\|_{\dot{H}^s(\mathbb{R}^n)} \right] (3.7) \\
&\quad \times \left( \|\nabla_h u_n\|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{1}{2}} \|u_n\|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{1}{2}} + \|u_n\|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{1}{2}} \|\nabla_h u_n\|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{1}{2}} \right).
\end{align*}$$

Thanks to Lemma 2.6, we get

$$\begin{align*}
|W_k^3| &\leq c_2^2 2^{-2qs} \left[ \|u_n^{\frac{3}{2}} \|_{\dot{H}^s(\mathbb{R}^n)} \|\nabla_h u_n\|_{\dot{H}^s(\mathbb{R}^n)} \|\nabla_h u_n\|_{\dot{H}^s(\mathbb{R}^n)} \right] (3.8) \\
&\quad \times \left( \|\nabla_h u_n\|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{1}{2}} \|u_n\|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{1}{2}} + \|u_n\|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{1}{2}} \|\nabla_h u_n\|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{1}{2}} \right).
\end{align*}$$

From Lemma 2.5, we deduce that

$$\begin{align*}
|W_k^4| &\leq c_2^2 2^{-2qs} \left[ \|B_n^{\frac{3}{2}} \|_{\dot{H}^s(\mathbb{R}^n)} \|\nabla_h B_n^{\frac{3}{2}}\|_{\dot{H}^s(\mathbb{R}^n)} \right] (3.9) \\
&\quad \times \left( \|\nabla_h B_n^{\frac{3}{2}}\|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{1}{2}} \|B_n^{\frac{3}{2}}\|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{1}{2}} + \|B_n^{\frac{3}{2}}\|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{1}{2}} \|\nabla_h B_n^{\frac{3}{2}}\|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{1}{2}} \right),
\end{align*}$$

and

$$\begin{align*}
|W_k^5| &\leq c_2^2 2^{-2qs} \left[ \|B_n^{\frac{3}{2}} \|_{\dot{H}^s(\mathbb{R}^n)} \|\nabla_h B_n^{\frac{3}{2}}\|_{\dot{H}^s(\mathbb{R}^n)} \right] (3.10) \\
&\quad \times \left( \|\nabla_h B_n^{\frac{3}{2}}\|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{1}{2}} \|B_n^{\frac{3}{2}}\|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{1}{2}} + \|B_n^{\frac{3}{2}}\|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{1}{2}} \|\nabla_h B_n^{\frac{3}{2}}\|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{1}{2}} \right).
\end{align*}$$

Plugging the estimates (3.6)-(3.10) into (3.5) gives

$$\begin{align*}
\frac{d}{dt} \left( \|\Delta_h u_n^0(t)\|_{L^2}^2 + \|\Delta_h B_n^0(t)\|_{L^2}^2 \right) + 2\|\nabla_h \Delta_h u_n^0(t)\|_{L^2}^2 + 2\|\nabla_h \Delta_h B_n^0(t)\|_{L^2}^2 \\
\leq c_2^2 2^{-2qs} \left[ \|u_n^{\frac{3}{2}} \|_{\dot{H}^s(\mathbb{R}^n)} \|\nabla_h u_n^{\frac{3}{2}}\|_{\dot{H}^s(\mathbb{R}^n)} \right] (3.11)
\quad \times \left( \|\nabla_h u_n^{\frac{3}{2}}\|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{1}{2}} \|u_n^{\frac{3}{2}}\|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{1}{2}} + \|u_n^{\frac{3}{2}}\|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{1}{2}} \|\nabla_h u_n^{\frac{3}{2}}\|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{1}{2}} \right).
\end{align*}$$
Therefore, thanks to (3.11), (3.12) and Young’s inequality, we obtain

\[
\begin{align*}
&\left\| B_h^h \right\|_{H^0,\alpha} \left\| \nabla_h B_n^h \right\|_{H^0,\alpha} \left\| \nabla_h u_n^h \right\|_{H^0,\alpha} + \left\| u_n^h \right\|_{H^0,\alpha} \left( \left\| \nabla_h u_n^h \right\|_{H^0,\alpha} + \left\| \nabla_h u_n^3 \right\|_{H^0,\alpha} \right) \\
&+ \left\| B_n^h \right\|_{\tilde{H}^0,\alpha} \left\| \nabla_h B_n^h \right\|_{\tilde{H}^0,\alpha} \left\| u_n^h \right\|_{H^0,\alpha} \left\| \nabla_h u_n^h \right\|_{\tilde{H}^0,\alpha} + \left\| u_n^h \right\|_{H^0,\alpha} \left( \left\| \nabla_h u_n^h \right\|_{\tilde{H}^0,\alpha} + \left\| \nabla_h u_n^3 \right\|_{\tilde{H}^0,\alpha} \right) \\
&+ \left\| B_n^h \right\|_{\tilde{H}^0,\alpha} \left\| \nabla_h B_n^h \right\|_{\tilde{H}^0,\alpha} \left\| u_n^h \right\|_{H^0,\alpha} \left\| \nabla_h u_n^h \right\|_{\tilde{H}^0,\alpha} + \left\| u_n^h \right\|_{H^0,\alpha} \left( \left\| \nabla_h u_n^h \right\|_{\tilde{H}^0,\alpha} + \left\| \nabla_h u_n^3 \right\|_{\tilde{H}^0,\alpha} \right) \\
\end{align*}
\]

Multiplying $2^{2q_{\alpha_0}}$ to the above inequality and summing up for $q \in \mathbb{Z}$, then integrating the resulting equation on $[0, t]$ and using Cauchy’s inequality, we reach

\[
\begin{align*}
\left\| u_n^h(0) \right\|_{H^0,\alpha}^2 + \left\| B_n^h(0) \right\|_{\tilde{H}^0,\alpha}^2 + C \int_0^t \left( \left\| u_n^h \right\|_{H^0,\alpha}^2 + \left\| u_n^3 \right\|_{H^0,\alpha}^2 \right) \left\| \nabla_h u_n^h \right\|_{H^0,\alpha}^2 \\
+ \left( \left\| B_n^h \right\|_{\tilde{H}^0,\alpha}^2 + \left\| B_n^h \right\|_{\tilde{H}^0,\alpha}^2 \right) \left\| \nabla_h u_n^h \right\|_{\tilde{H}^0,\alpha}^2 + \left\| u_n^h \right\|_{H^0,\alpha}^2 \left\| \nabla_h u_n^h \right\|_{\tilde{H}^0,\alpha}^2 \\
+ \left( \left\| u_n^h \right\|_{H^0,\alpha}^2 + \left\| B_n^h \right\|_{\tilde{H}^0,\alpha}^2 \right) \left\| \nabla_h u_n^h \right\|_{\tilde{H}^0,\alpha}^2 + \left( \left\| B_n^h \right\|_{\tilde{H}^0,\alpha}^2 + \left\| B_n^h \right\|_{\tilde{H}^0,\alpha}^2 \right) \left\| \nabla_h u_n^h \right\|_{\tilde{H}^0,\alpha}^2 \\
+ \left( \left\| u_n^h \right\|_{H^0,\alpha}^2 + \left\| \nabla_h u_n^h \right\|_{\tilde{H}^0,\alpha}^2 \right) \left\| u_n^h \right\|_{H^0,\alpha}^2 \left( t' \right) dt' \\
\end{align*}
\]

Whereas a standard energy estimate applied to the $u_h$ and $B_n$ equations of (3.3) and (3.4) gives

\[
\frac{d}{dt} \left( \left\| u_n^h(t) \right\|_{L^2}^2 \right) + \left\| B_n^h(t) \right\|_{L^2}^2 + \left\| \nabla_h u_n^h(t) \right\|_{L^2}^2 + \left\| \nabla_h B_n^h(t) \right\|_{L^2}^2 = 2 \sum_{\ell, k=1}^3 \left( (\Delta)^{-1} \partial_t \partial_k (u_n^\ell u_n^k - B_n^\ell B_n^k) \right) \left( \nabla_h u_n^h \right)_{L^2} + 2 \left( B_n \cdot \nabla B_n^h u_n^h \right)_{L^2} \\
+ 2 \left( B_n \cdot \nabla u_n^h \right) \left( B_n^h \right)_{L^2} = 2 \sum_{\ell, k=1}^3 \left( (\Delta)^{-1} \partial_t \partial_k (u_n^\ell u_n^k - B_n^\ell B_n^k) \right) \left( \nabla_h u_n^h \right)_{L^2} \\
\leq \left\| u_n^h(0) \right\|_{L^2}^2 + \left\| B_n^h(0) \right\|_{L^2}^2 + C \int_0^t \left( \left\| u_n^h \right\|_{H^0,\alpha} \left\| \nabla_h u_n^h \right\|_{\tilde{H}^0,\alpha} + \left\| u_n^h \right\|_{\tilde{H}^0,\alpha} \left\| \nabla_h u_n^3 \right\|_{\tilde{H}^0,\alpha} \\
+ \left\| B_n^h \right\|_{\tilde{H}^0,\alpha} \left\| \nabla_h u_n^h \right\|_{\tilde{H}^0,\alpha} \right) \left( t' \right) dt'.
\]

Therefore, thanks to (3.11), (3.12) and Young’s inequality, we obtain

\[
\begin{align*}
\left\| u_n^h \right\|_{L^2(\tilde{H}^0,\alpha)}^2 + \left\| B_n^h \right\|_{L^2(\tilde{H}^0,\alpha)}^2 + \left\| \nabla_h u_n^h \right\|_{L^2(\tilde{H}^0,\alpha)}^2 + \left\| \nabla_h B_n^h \right\|_{L^2(\tilde{H}^0,\alpha)}^2 \leq \\
\left\| u_n^h(0) \right\|_{\tilde{H}^0,\alpha}^2 + \left\| B_n^h(0) \right\|_{\tilde{H}^0,\alpha}^2 + C \left( \left\| u_n^h \right\|_{L^2(\tilde{H}^0,\alpha)} + \left\| \nabla_h u_n^3 \right\|_{L^2(\tilde{H}^0,\alpha)} \right) \left( \nabla_h u_n^h \right)_{L^2(\tilde{H}^0,\alpha)} \\
+ C \left( \left\| B_n^h \right\|_{\tilde{H}^0,\alpha}^2 + \left\| B_n^h \right\|_{\tilde{H}^0,\alpha}^2 \right) \left( \nabla_h u_n^h \right)_{L^2(\tilde{H}^0,\alpha)} + C \int_0^t \left( \left(1 + \left\| u_n^3 \right\|_{H^0,\alpha} \right) \left\| \nabla_h B_n^h \right\|_{H^0,\alpha} + \left\| u_n^h \right\|_{H^0,\alpha} \right) \left( t' \right) dt'.
\end{align*}
\]
Then if \( C(||u^h_n||_{L^2(\Omega^{0,\omega})}^2 + ||B^h_n||_{L^2(\Omega^{0,\omega})}^2) \leq \frac{1}{2} \), we get by using Gronwall inequality that

\[
||u^h_n||_{L^2(\Omega^{0,\omega})}^2 + ||B^h_n||_{L^2(\Omega^{0,\omega})}^2 + ||\nabla_h u^h_n||_{L^2(\Omega^{0,\omega})}^2 + ||\nabla_h B^h_n||_{L^2(\Omega^{0,\omega})}^2 \\
\leq (||u^h_n(0)||_{L^2(\Omega^{0,\omega})}^2 + ||B^h_n(0)||_{L^2(\Omega^{0,\omega})}^2) \\
\times \exp \left\{ C_1 \left[ (1 + ||u^3_n||_{H^{0,\omega}}^2) ||\nabla_h u^3_n||_{H^{0,\omega}}^2 + (1 + ||B^3_n||_{H^{0,\omega}}^2) ||\nabla_h B^3_n||_{H^{0,\omega}}^2 \right] \right\}.
\]

Similarly, applying \( \Delta^\omega \) to the \( u^3_n \) and \( B^3_n \) equations in (3.3), (3.4) and taking the \( L^2 \) inner product of the resulting equations with \( \Delta^\omega u^3_n \) and \( \Delta^\omega B^3_n \), respectively, we have

\[
\frac{d}{dt} (||\Delta^\omega u^3_n(t)||_{L^2}^2 + ||\Delta^\omega B^3_n(t)||_{L^2}^2) + 2||\nabla_h \Delta^\omega u^3_n(t)||_{L^2}^2 \\
+ 2||\nabla_h \Delta^\omega B^3_n(t)||_{L^2}^2 = -2(\Delta^\omega (u_n \cdot \nabla u_n^3) \Delta^\omega u^3_n)_{L^2} + 2(\Delta^\omega (B_n \cdot \nabla B^3_n) \Delta^\omega u^3_n)_{L^2} \\
+ 2 \sum_{l,k=1}^3 \left( \Delta^\omega (-\Delta)^{-1} \partial_\ell \partial_k (u_n \cdot \nabla u_n^3) (B_n \cdot \nabla B^3_n) \right)_{L^2} \\
- 2(\Delta^\omega (u_n \cdot \nabla B^3_n) \Delta^\omega u^3_n)_{L^2} + 2(\Delta^\omega (B_n \cdot \nabla u_n^3) \Delta^\omega B^3_n)_{L^2}
\]

def V^1 \equiv V^1 + V^2 + V^3 + V^4 + V^5,

where we used the fact that \( \partial_3 u^3_n = -\nabla_h u_n^h \).

Applying Lemma 2.5 and Lemma 2.8 gives

\[
||V^1|| \leq c_2 2^{-2q_\omega} ||u^3||_{H^{0,\omega}}^\frac{3}{2} \left( ||\nabla_h u^3_n||_{H^{0,\omega}}^\frac{1}{2} \right)
\]

times \left[ \left( ||u^3_n||_{H^{0,\omega}}^\frac{1}{2} \cdot ||\nabla_h u^3_n||_{H^{0,\omega}}^\frac{1}{2} + ||\nabla_h u^3_n||_{H^{0,\omega}}^\frac{1}{2} \cdot ||u^3_n||_{H^{0,\omega}}^\frac{1}{2} \right) \right],
\]

and

\[
||V^2|| \leq c_2 2^{-2q_\omega} ||u^3||_{H^{0,\omega}}^\frac{3}{2} \left( ||\nabla_h u^3_n||_{H^{0,\omega}}^\frac{1}{2} \right)
\]

times \left[ \left( ||B^3_n||_{H^{0,\omega}}^\frac{3}{2} \cdot ||\nabla_h B^3_n||_{H^{0,\omega}}^\frac{1}{2} + ||\nabla_h B^3_n||_{H^{0,\omega}}^\frac{3}{2} \cdot ||B^3_n||_{H^{0,\omega}}^\frac{1}{2} \right) \right],
\]

Thanks to Lemma 2.6, we get

\[
||V^3|| \leq c_2 2^{-2q_\omega} \left[ ||u^h_n||_{H^{0,\omega}} \cdot ||\nabla_h u^h_n||_{H^{0,\omega}} \cdot ||\nabla_h u^h_n||_{H^{0,\omega}} \right]
\]

times \left[ \left( ||\nabla_h u^h_n||_{H^{0,\omega}}^\frac{1}{2} \cdot ||u^h_n||_{H^{0,\omega}}^\frac{3}{2} + ||u^h_n||_{H^{0,\omega}}^\frac{1}{2} \cdot ||\nabla_h u^h_n||_{H^{0,\omega}}^\frac{3}{2} \right) \right],
\]
From Lemma 2.5, we deduce that

\[
|V_k^t| \leq c_2^2 2^{-2q_0} \| B_n^3 \|_{H^{0,0}} \frac{1}{2} \| \nabla_h B_n^3 \|_{H^{0,0}} \frac{1}{2} \tag{3.18}
\]

\[
\times \left( \| \nabla_h B_n^3 \|_{H^{0,0}} \| \nabla_h u_n^3 \|_{H^{0,0}} \| u_n^3 \|_{H^{0,0}} \cdot \frac{1}{2} \| \nabla_h u_n^3 \|_{H^{0,0}} \frac{1}{2} \| \nabla_h B_n^3 \|_{H^{0,0}} \frac{1}{2} \right)
\]

and

\[
|V_k^3| \leq c_2^2 2^{-2q_0} \| B_n^3 \|_{H^{0,0}} \frac{1}{2} \| \nabla_h B_n^3 \|_{H^{0,0}} \frac{1}{2} \tag{3.19}
\]

\[
\times \left( \| \nabla_h u_n^3 \|_{H^{0,0}} \| \nabla_h B_n^3 \|_{H^{0,0}} \| u_n^3 \|_{H^{0,0}} \cdot \frac{1}{2} \| \nabla_h u_n^3 \|_{H^{0,0}} \frac{1}{2} \| \nabla_h B_n^3 \|_{H^{0,0}} \frac{1}{2} \right)
\]

Plugging the estimates (3.15)-(3.19) into (3.14) gives

\[
\frac{d}{dt} \left( \| \Delta_y u_n^3(t) \|_{L^2}^2 + \| \Delta_y B_n^3(t) \|_{H^2}^2 \right) + 2\| \nabla_h \Delta_y u_n^3(t) \|_{L^2}^2 + 2\| \nabla_h \Delta_y B_n^3(t) \|_{H^2}^2
\]

\[
\leq c_2^2 2^{-2q_0} \left[ \| u_n^3 \|_{H^{0,0}} \frac{1}{2} \| \nabla_h u_n^3 \|_{H^{0,0}} \right.
\]

\[
\left. \left( \| \nabla_h u_n^3 \|_{H^{0,0}} \| \nabla_h B_n^3 \|_{H^{0,0}} \| u_n^3 \|_{H^{0,0}} \cdot \frac{1}{2} \| \nabla_h B_n^3 \|_{H^{0,0}} \frac{1}{2} \right)
\]

\[
+ \| B_n^3 \|_{H^{0,0}} \| \nabla_h B_n^3 \|_{H^{0,0}} \| u_n^3 \|_{H^{0,0}} \cdot \frac{1}{2} \| \nabla_h B_n^3 \|_{H^{0,0}} \frac{1}{2} \right)
\]

Multiplying $2^{2q_0}$ to the above inequality and summing up for $q \in \mathbb{Z}$, then integrating the resulting equation on $[0, t]$ and using Cauchy's inequality, we reach

\[
\| u_n^3 \|_{L^4(H^{0,0})}^2 + \| B_n^3 \|_{L^4(H^{0,0})}^2 + \| \nabla_h u_n^3 \|_{L^2(H^{0,0})}^2 + \| \nabla_h B_n^3 \|_{L^2(H^{0,0})}^2 \leq \| u_n^3(0) \|_{H^{0,0}}^2 + \| B_n^3(0) \|_{H^{0,0}}^2 + \left( \| B_n^3 \|_{H^{0,0}}^2 + \| B_n^3 \|_{H^{0,0}}^2 \right) \| \nabla_h B_n^3 \|_{H^{0,0}}^2
\]

\[
+ C \int_0^t \left( \left( \| u_n^3 \|_{H^{0,0}} + \| u_n^3 \|_{H^{0,0}}^2 + \| B_n^3 \|_{H^{0,0}}^2 \right) \| \nabla_h B_n^3 \|_{H^{0,0}}^2
\]

\[
+ \left( \| \nabla_h B_n^3 \|_{H^{0,0}}^2 + \| \nabla_h B_n^3 \|_{H^{0,0}}^2 \right) \| \nabla_h B_n^3 \|_{H^{0,0}}^2 + \| B_n^3 \|_{H^{0,0}}^2 \| \nabla_h B_n^3 \|_{H^{0,0}}^2
\]

\[
+ \| \nabla_h B_n^3 \|_{H^{0,0}}^2 + \| \nabla_h B_n^3 \|_{H^{0,0}}^2 \right)
\]

\[
\| \nabla_h B_n^3 \|_{H^{0,0}}^2 + \| \nabla_h B_n^3 \|_{H^{0,0}}^2
\]

\[
+ \| \nabla_h B_n^3 \|_{H^{0,0}}^2 + \| \nabla_h B_n^3 \|_{H^{0,0}}^2 \right)
\]

\[
+ \| \nabla_h B_n^3 \|_{H^{0,0}}^2 + \| \nabla_h B_n^3 \|_{H^{0,0}}^2 + \| B_n^3 \|_{H^{0,0}}^2 \| \nabla_h B_n^3 \|_{H^{0,0}}^2
\]

\[
+ \| \nabla_h B_n^3 \|_{H^{0,0}}^2 + \| \nabla_h B_n^3 \|_{H^{0,0}}^2 + \| B_n^3 \|_{H^{0,0}}^2 \| \nabla_h B_n^3 \|_{H^{0,0}}^2
\]

\[
+ \| \nabla_h B_n^3 \|_{H^{0,0}}^2 + \| \nabla_h B_n^3 \|_{H^{0,0}}^2 + \| B_n^3 \|_{H^{0,0}}^2 \| \nabla_h B_n^3 \|_{H^{0,0}}^2
\]
\[ u_n^h \| _{H^{0,0}}^2 \| \nabla h u_n^h \|_{H^{0,0}}^2 + \| \nabla h u_n^h \|_{H^{0,0}}^2 \| B_n^h \|_{H^{0,0}}^2 \right) (t') dt' \]

Whereas a standard energy estimate applied to the \( u_n^3 \) and \( B_n^3 \) equations of (3.3) and (3.4) gives
\[
\frac{d}{dt} \left( \| u_n^3 \|_{L^2}^2 + \| B_n^3 \|_{L^2}^2 + \| \nabla h u_n^3 \|_{L^2}^2 \right) + \| \nabla h B_n^3 \|_{L^2}^2 + 2 \left( \| B_n \cdot \nabla u_n^3 \|_{L^2}^2 \right)
\]
\[
= \sum_{\ell,k=1}^3 \left( (-\Delta)^{-1} \partial_t \partial_k (u_n^\ell u_n^k - B_n^\ell B_n^k) \right) + \mathrm{div} u_n^3 + 2 \left( \| B_n \cdot \nabla u_n^3 \|_{L^2}^2 \right)
\]
\[
\leq \| u_n^3 \|_{H^{0,0}}^2 \| \nabla h u_n^3 \|_{H^{0,0}}^2 \left( \| \nabla h u_n^3 \|_{H^{0,0}}^2 + \| \nabla h \|_{H^{0,0}} \right) + 2 \| B_n^3 \|_{H^{0,0}} \| \nabla h u_n^3 \|_{H^{0,0}}^2 \left( \| \nabla h B_n^3 \|_{H^{0,0}} + \| \nabla h B_n^3 \|_{L^2} \right)
\]

where we used Corollary 2.7 and divergence-free conditions of the vector fields \( u \) and \( B \).

Integrating the above inequality on \([0,t]\) and using Cauchy's inequality, we get
\[
\| u_n^3 \|_{L^2(t)}^2 + \| B_n^3 \|_{L^2(t)}^2 + \| \nabla h u_n^3 \|_{L^2(t)}^2 + \| \nabla h B_n^3 \|_{L^2(t)}^2 \right) \leq \| u_n^3(0) \|_{L^2(0)}^2 + \| B_n^3(0) \|_{L^2(0)}^2 + C \int_0^t \left( \| u_n^3 \|_{H^{0,0}}^2 + \| u_n^3 \|_{L^2}^2 \right)
\]
\[
+ \| B_n^3 \|_{H^{0,0}} \| \nabla h u_n^3 \|_{H^{0,0}} + \| B_n^3 \|_{L^2} \right) \right) dt'.
\]

Therefore, thanks to (3.20), (3.21) and Young's inequality, we obtain
\[
\| u_n^3 \|_{L^\infty(0)}^2 + \| B_n^3 \|_{L^\infty(0)}^2 + \| \nabla h u_n^3 \|_{L^\infty(0)}^2 + \| \nabla h B_n^3 \|_{L^\infty(0)}^2 \right) \leq \| u_n^3(0) \|_{L^\infty(0)}^2 + \| B_n^3(0) \|_{L^\infty(0)}^2 + C \left( \| u_n^3 \|_{L^\infty(0)}^2 + \| u_n^3 \|_{L^\infty(0)}^2 \right)
\]
\[
+ \| B_n^3 \|_{L^\infty(0)} \| \nabla h u_n^3 \|_{L^\infty(0)} + \| B_n^3 \|_{L^\infty(0)} + \| \nabla h u_n^3 \|_{L^\infty(0)}^2 \right) + \| \nabla h B_n^3 \|_{L^\infty(0)} + \| \nabla h B_n^3 \|_{L^\infty(0)}^2 \right)
\]
\[
+ \| \nabla h B_n^3 \|_{L^\infty(0)}^2 \right) \right). \]

Now let us define
\[
T_n^* \overset{df}{=} \sup \{ T_n > 0 : \left( u_n^3 \|_{L^\infty(T_n)}^2 + \| \nabla h u_n^3 \|_{L^2(T_n)}^2 + \| B_n^3 \|_{L^2(T_n)}^2 \right) \leq 2 \eta_0, \quad \text{and} \quad \left( u_n^3 \|_{L^\infty(T_n)}^2 + \| \nabla h u_n^3 \|_{L^2(T_n)}^2 + \| B_n^3 \|_{L^2(T_n)}^2 \right) \leq 2 \eta_0 \}
\]
\[
+ \| \nabla h B_n^3 \|_{L^2(T_n)}^2 \right) \leq 2 \| u_n^3 \|_{L^2(T_n)}^2 + \| B_n^3 \|_{L^2(T_n)}^2 + \eta_0, \gamma_0 \}
\]

for \( \eta_0 \) given by (1.3). We claim that if \( \eta_0 \) is small enough, then, \( T_n^* = +\infty \). In fact, if \( T_n^* < \infty \), (3.23) implies that for every \( T_n < T_n^* \), there holds
\[
\| u_n^3 \|_{L^2(T_n)}^2 + \| \nabla h u_n^3 \|_{L^2(T_n)}^2 + \| B_n^3 \|_{L^2(T_n)}^2 \right) \leq 2 \| u_n^3 \|_{L^2(T_n)}^2 + \| B_n^3 \|_{L^2(T_n)}^2 + \eta_0.
Corollary 3.1. Under the assumption of Theorem 1.1, the following 3-D viscous Magneto-hydrodynamics equations

\[
\begin{align*}
&\partial_t u + (u \cdot \nabla) u + \nabla p - \Delta u = (\nabla \times B) \times B, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\
&\partial_t B - \nabla \times (u \times B) - \Delta B = 0, \\
&\nabla \cdot u = 0, \quad \nabla \cdot B = 0, \\
&u|_{t=0} = u_0, \quad B|_{t=0} = B_0,
\end{align*}
\]

has a unique global solution \((u, B) \in C([0, \infty); H^0; \mathbb{R}^3)) \times C([0, \infty); H^0; \mathbb{R}^3))\) with \((\nabla_h u, \nabla_h B) \in L^2(\mathbb{R}^+; H^0; \mathbb{R}^3)) \times L^2(\mathbb{R}^+; H^0; \mathbb{R}^3)).\) Moreover, there holds

\[
\begin{align*}
&\|u^h\|^2_{L^\infty(\mathbb{R}^+; H^0; \mathbb{R}^3)} + \|\nabla u^h\|^2_{L^2(\mathbb{R}^+; H^0; \mathbb{R}^3)} \\
&+ \|B^h\|^2_{L^\infty(\mathbb{R}^+; H^0; \mathbb{R}^3)} + \|\nabla B^h\|^2_{L^2(\mathbb{R}^+; H^0; \mathbb{R}^3)} \leq C\eta_0,
\end{align*}
\]

and

\[
\begin{align*}
&\|u^3\|^2_{L^\infty(\mathbb{R}^+; H^0; \mathbb{R}^3)} + \|\nabla u^3\|^2_{L^2(\mathbb{R}^+; H^0; \mathbb{R}^3)} + \|B^3\|^2_{L^\infty(\mathbb{R}^+; H^0; \mathbb{R}^3)} \\
&+ \|\nabla B^3\|^2_{L^2(\mathbb{R}^+; H^0; \mathbb{R}^3)} \leq 2(\|u_0^3\|^2_{H^0; \mathbb{R}^3} + \|B_0^3\|^2_{H^0; \mathbb{R}^3}) + \eta_0,
\end{align*}
\]

with

\[
\eta_0 \overset{\text{def}}{=} \left(\|u^h_0(0)\|^2_{H^0; \mathbb{R}^3} + \|B^h_0(0)\|^2_{H^0; \mathbb{R}^3}\right) \exp \left\{C_1(\|u_0^3\|_{H^0; \mathbb{R}^3}^4 + \|B_0^3\|_{H^0; \mathbb{R}^3}^4)\right\}.
\]

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