The Monte Carlo method as a tool for statistical characterisation of differential and additive phase shifting algorithms

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Abstract. Several metrological applications base their measurement principle in the phase sum or difference between two patterns, one original \( s(r, \phi) \) and another modified \( t(r, \phi + \Delta \phi) \). Additive or differential phase shifting algorithms directly recover the sum \( 2\phi + \Delta \phi \) or the difference \( \Delta \phi \) of phases without requiring prior calculation of the individual phases. These algorithms can be constructed, for example, from a suitable combination of known phase shifting algorithms. Little has been written on the design, analysis and error compensation of these new two-stage algorithms. Previously we have used computer simulation to study, in a linear approach or with a filter process in reciprocal space, the response of several families of them to the main error sources. In this work we present an error analysis that uses Monte Carlo simulation to achieve results in good agreement with those obtained with spatial and temporal methods.

1. Introduction

Many optical techniques \([1,2]\) recover the optical phase \( \phi \) using a series of \( M \) two-dimensional irradiance distributions or patterns \( s(r, \phi) \) with a relative phase shift \( \alpha_m \) between two consecutives of them

\[
s_m(r, \phi, \alpha_m) = \sum_{k=0}^{1} a_k(r) \cos\left[k(\phi(r) + \alpha_m)\right]
\]  

(1)

being \( s_m(r, \phi, \alpha_m) \) each pattern of the series and \( a_k(r) \) the coefficients of the harmonic components of the signal. The most popular decoding processes recover the wrapped phase by means of the so-called phase shifting algorithms (PSAs) combining these irradiance values in the argument of an inverse trigonometric function

\[
\phi(r) = \arctan \frac{N_1 \{s_m(r, \phi, \alpha_m)\}}{D_1 \{s_m(r, \phi, \alpha_m)\}}
\]  

(2)

where \( N_1 \{s_m(r, \phi, \alpha_m)\} \) and \( D_1 \{s_m(r, \phi, \alpha_m)\} \) define the particular combination for each PSA in each case. In many other applications of great scientific and technological interest \([3-12]\) the measurand is linked to the phase sum or the phase difference between the original pattern \( s(r, \phi) \) and another one \( t(r, \phi + \Delta \phi) \), that can be shifted in phase to obtain another series of patterns:
where in this case \( t_p(r, \phi + \Delta \phi, \beta_p) \) is each pattern of the new series, \( \beta_p \) is the phase step between irradiance values and the coefficients \( b_g \) weight the harmonic contribution. So, \( 2\phi + \Delta \phi \) or \( \Delta \phi \) can be retrieved with a duplication of the evaluation and unwrapping processes \([13]\) and with a posterior sum or difference of the obtained phases, employing for the phase calculation of the modified pattern a similar expression to (2)

\[
\phi(r) + \Delta \phi(r) = \arctan \frac{N_1 \{ t_p(r, \phi + \Delta \phi, \beta_p) \}}{D_2 \{ t_p(r, \phi + \Delta \phi, \beta_p) \}}
\]

being \( N_2 \{ t_p(r, \phi + \Delta \phi, \beta_p) \} \) and \( D_2 \{ t_p(r, \phi + \Delta \phi, \beta_p) \} \) combinations of the shifted modified patterns. However, it is possible to obtain directly \( 2\phi + \Delta \phi \) and \( \Delta \phi \) phase values making use of the so-called two-stage phase shifting algorithms (TSPSAs). It can be highlighted that, if the phase difference \( \Delta \phi \) does not reach a complete period in the whole image area, continuous values are provided and so \( \Delta \phi \) can even be recovered if the fringe patterns show noise, have low modulation, show abrupt phase changes or a low sampling density \([13]\). There exists papers that show TSPSAs that do not follow a systematic design method \([10-12]\) and others that show TSPSAs with a design derived from a least square fitting to a sinusoidal function of the modified pattern \([14-16]\):

\[
\phi(r) + \Delta \phi(r) = 2 \arctan \frac{N_2 \{ t_p(r, \phi + \Delta \phi, \beta_p) \}}{D_1 \{ s_m(r, \phi, \alpha_m) \} + D_2 \{ s_m(r, \phi, \alpha_m) \}}
\]

In equations (5) and (6) the upper sign stands for the additive TSPSAs and the lower sign for the differential TSPSA. Equation (5) can build a TSPSA with two different PSAs in each pattern but equations (6) demand that both of them use the same PSA. Likewise, irradiance value invariability of the DC term, \( a_0(r) = b_0(r) \), and the irradiance modulation amplitude, \( a_1(r) = b_1(r) \), must be guaranteed in equations (6). So we call the former asymmetric TSPSAs (ATSPSAs) and the later symmetric TSPSAs (STSPSAs).

In general, the TSPSAs are affected by similar error sources than PSAs. The knowledge of their sensitivities to these errors is fundamental to choose the best algorithm in each particular application. It is for this reason that we have considered of interest the characterization of the TSPSAs by means of numerical simulation, error linearization or an analysis of the process in the Fourier space (see our previous works \([17-21]\)). The aim of the present paper is to show that the Monte Carlo method (MCM) \([22]\) can be a practical alternative to analyze the error sensitivity of algorithms in which any approximation cannot be introduced, especially when it is not possible a linearization of the model or it is too complex. In this work, we study the effectiveness of the MCM to measure the effect of the main systematic and random errors in the ATSPSAs (equation (5)) and STSPSAs (equations (6)).
2. The MCM in two-stage evaluation

The main systematic error sources in TSPPSAs, as in PSAs, are the discrepancies in the nominal value for the additional phase shifts and the presence of undesired harmonics in the signals. Moreover, three inherent random error sources in the measuring process should be considered: quantization noise (\(E_{\text{quant}}\) and \(E_{\text{SNR}}\)), photodetector noise (\(E_{\text{SNR}}\) and \(E_{\text{SNR}}\)), and possible vibrations during the process along with other environment influences (\(E_{\text{vib}}\) and \(E_{\text{vib}}\)) that significantly alter the patterns. So the irradiance distributions involved in equations (1) and (3) have to be changed by the following generic expressions:

\[
E_{s_m}(r, \phi, \alpha_m) = \sum_{k=0}^{\infty} a_k(r) + E_{\text{quant}} + E_{\text{SNR}} \cos \left[ k \left( \phi(r) + \alpha_m + \sum_{q=1}^{\infty} \frac{\alpha^q_m}{q!} \phi^{(q)} + E_{\text{vib}} \right) \right]
\]

\[
E_{t_p}(r, \phi + \Delta \phi, \beta_p) = \sum_{g=0}^{\infty} b_g(r) + E_{\text{quant}} + E_{\text{SNR}} \cos \left[ g \left( \phi(r) + \Delta \phi + \beta_p + \sum_{r=1}^{\infty} \chi_r \theta^r + E_{\text{vib}} \right) \right]
\]

where \(a_k\) and \(b_g\) are the error coefficients in the \(q\) and \(r\)-th order phase shifts for each series, \(E_{\text{quant}}\) and \(E_{\text{SNR}}\) the corresponding variation interval of possible values. The law of propagation of uncertainty transfers these uncertainties through the model to the output quantity. In the two-stage evaluation, a number of \(M+P\) PDFs, labeled \(g_{s_m}\) and \(g_{t_p}\), are generated at each sampled point [23] associating a number \(I\) of pseudorandom values [24] to each error in study. A uniform PDF is usually utilized if the corresponding variation limits are known, whereas a normal PDF is instead used if only their expected value and uncertainty are known [25]. Thus for systematic errors, we considered a uniform PDF and that the \(M\) original patterns and the \(P\) modified ones are affected in each iteration \(I\) by the same pseudorandom error value. For random errors, we employed a normal distribution and considered each irradiance distribution, at each iteration weighted by a different pseudorandom value of the PDF. The phase shifted irradiance values with error, equations (7), are combined a sufficient number of times in two-stage algorithm that it is the non-linear model equation that relates the input quantities with the output quantity: \((\phi + \Delta \phi) \pm \phi^{-}([s_m(r, \phi, \alpha_m), t_p(r, \phi + \Delta \phi, \beta_p)]). In such a way, the PDF of the phase, \(g_{\phi+\Delta \phi}\), at each point provides the expected value \(E[(\phi + \Delta \phi)\pm \phi]\) and the standard deviation \(V[(\phi + \Delta \phi)\pm \phi]\). With the standard deviation at each sampled point, the error curve is drawn [26].

3. Characterisation of the Schwider-Hariharan TSPSA by means of the MMC

The behaviour of the MCM is tested with the Schwider-Hariharan [28,29] differential ATSPSA, that is built using equation (5):

\[
\Delta \phi = \arctan \left( \frac{(2s_3 - s_1 - s_4)(t_2 - t_4) - (s_2 - s_4)(2t_3 - t_1 - t_4)}{(s_2 - s_4)(t_2 - t_4) + (2s_3 - s_1 - s_4)(2t_3 - t_1 - t_4)} \right)
\]

and with the Schwider-Hariharan differential STSPPSA built with equation (6a):

\[
\Delta \phi = 2 \arctan \left( \frac{(t_2 - t_4) - (s_2 - s_4)}{(2s_3 - s_1 - s_4)(2t_3 - t_1 - t_4)} \right)
\]

Since MCM allows quickly test different combinations of error sources we analyzed the sensitivity of both Schwider-Hariharan TSPPSAs to the two named systematic errors simultaneously and to the three random errors. For systematic errors (figure 1) we generated a uniform PDF of the error at each
of the sampled points (220 in our case) that it is propagated to each of the irradiance values in such a way that the form of the PDF of each irradiance value is affected by their different response to each error. Finally, these PDFs were combined in the equations (8) and (9). By way of example we assumed a 10% error in the phase shift to first and second order ($\varepsilon_1=\chi_1=0.1, \ varepsilon_2=\chi_2=0.1$) and in the presence of undesired harmonics of second and third order ($a_2=b_2=0.1$ and $a_3=b_3=0.1$). In Figures 2 and 3 we analysed in Schwider-Hariharan differential ATSPSA and STSPSA respectively the number of samples needed to retrieve a PDF with a large coverage interval. As it can be seen in both TSPSAs $l=10^4$ iterations are enough to recover a well-formed PDF $g_{\Delta\phi}$ but with less iterations the standard deviation $V(\Delta\phi)$ builds the error curve with almost the same agreement with error linearization [18]. Figure 3 shows a bad behaviour of the differential Schwider-Hariharan STSPSA with discontinuities that make difficult the recovery process of $\Delta\phi$.

Figure 1. Subsequent stages of uncertainty evaluation for the Schwider-Hariharan differential ATSPSA and STSPSA considering sensitivity to systematic errors.
Figure 2. Calculation of $\Delta \phi$ with the Schwider-Hariharan differential ATSPSA considering systematic errors in both pattern series for distinct number of iterations: (a) Output PDF at a generic sampled point and (b) error curve in blue built with the standard deviation $V(\Delta \phi)$ at each point and compared with the results obtained by error linearization (dashed red).

\[ E\Delta \phi = \frac{\pi}{4 \cos^2 \phi + 2} \left[ \sin \left( \frac{\Delta \phi}{2} \right) \sin \phi + \chi_x \sin \left( \frac{\Delta \phi}{2} \right) \right] \]

Figure 3. The same as Figure 2 but for the Schwider-Hariharan differential STSPSA.
For random errors we considered signals digitized with 8 bits, a signal/noise ratio of 50dB and vibration amplitude of $\pi/200\,\text{rad}$ [27]. Figure 4 illustrates the three stages of the MMC when the patterns are affected by these random errors. Figures 5 and 6 show the obtained phase error with different number of iterations. As the PDFs for the input irradiance values have a normal distribution the central limit theorem sets that the PDF for the calculated phase is also normal. The fit to a normal distribution of all the obtained PDFs $g_{\Delta\phi}$ at each point is checked with a QQ plot between the resulting data quantiles and the standard normal distribution quantiles. The correlation coefficient R approaches unity as the number of iterations increases. It can be seen that unlike what happens for systematic errors, for random errors a number of trials under $10^4$ produce an output PDF that is not normal so the obtained error curve changes appreciably. For example, the first points of the error curve in figure 6 are very diffuse for 100 iterations. The random error is less significant in Schwider-Hariharan differential ATSPSA (figure 5) than in ATSPSA (figure 6). In both TSPSAs, error curves for systematic errors are more than one order of magnitude higher than those for random errors.

![Figure 4](image-url)
Figure 5. Calculation of $\Delta \phi$ with the Schwider-Hariharan differential ATSPSA considering random errors in both pattern series for different number of iterations: (a) Output PDF at a generic sampled point, (b) QQ plot that measures the degree of fit to the normal PDF and (c) error curve in blue build with the standard deviation $V(\Delta \phi)$ at each point.

Figure 6. The same as Figure 5 but for the Schwider-Hariharan differential STSPSA.
Conclusions
TSPSAs provide a direct calculation of the phase sum/difference between two patterns. This feature can improve the development of a large number of recent applications. In this article we analyzed the sensitivity of these algorithms against known systematic and random errors by means of the MCM. We have demonstrated that the MCM allows the rapid and efficient characterization of the behavior of TSPSAs regardless of the non-linear dependence of the irradiance values or the error models used. The analyzed protocol can be used to study any TSPSA with different combinations of error sources, since once programmed the routine, it can quickly test different combinations of PSAs to obtain different compensatory capabilities.

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