Symplectic versus Prequantum Induction

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Abstract

This paper establishes two basic properties of the symplectic induction construction of Kazhdan, Kostant, Sternberg, and Weinstein: Induction in Stages and Frobenius Reciprocity. It then argues that a prequantum version of the construction, of which we prove the same two properties, is in fact the appropriate framework to geometrically model representation-theoretic phenomena.

Introduction

Beyond the mere parametrization of irreducible unitary representations by coadjoint orbits originating in the work of Borel-Weil and Kirillov [S54, K62], there exists a certain well-known parallelism between representation theory and the symplectic theory of Hamiltonian G-spaces. To capture it with precision, papers like [K78, W78, G82, G83] introduced purely symplectic constructions meant to mirror operations such as Ind (inducing a representation from a subgroup) or HomG (forming the space of intertwining operators between two representations). In that setting, one of course expects basic properties like induction in stages or Frobenius reciprocity to hold in symplectic geometry. A first goal of this paper is to spell out their proofs (§2, §3), fulfilling promises made in [Z96, p. 9] and [M07, p. 105].

A second goal of the paper is to point out that, while these constructions fit their purpose when the correspondence from representations to coadjoint orbits is one-to-one [F15], they fall short when it is many-to-one [S94]. To remedy this, we propose new versions of both constructions in the category of prequantum G-spaces (§5, §6) and establish the stages and Frobenius properties in that setting (§7, §8). Finally, we illustrate the need for our prequantized versions by what we believe is the simplest example (§4, §9).

Notation and conventions

We use a concise notation for the translation of tangent and cotangent vectors to a Lie group: for fixed $g, q \in G$,

\[
T_q G \to T_{q g} G \quad \text{resp.} \quad T_q^* G \to T_{g q}^* G
\]

\[
v \mapsto g v, \quad p \mapsto g p
\]

will denote the derivative of $q \mapsto q g$, respectively its contragredient, i.e., $\langle g p, v \rangle = \langle p, g^{-1} v \rangle$. Likewise, we define $vg$ and $pg$ with $\langle pg, v \rangle = \langle p, vg^{-1} \rangle$.

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By a Hamiltonian G-space we mean the triple \((X, \omega, \Phi)\) of a manifold \(X\) on which \(G\) acts, a \(G\)-invariant symplectic form \(\omega\) on it, and a \(G\)-equivariant momentum map \(\Phi : X \to g^*\). We identify spaces \(X_1, X_2\) which are isomorphic, i.e., related by a \(G\)-equivariant diffeomorphism which transforms \(\omega_1\) into \(\omega_2\) and \(\Phi_1\) into \(\Phi_2\). If several are in play, we also use subscripts like \(\omega_X, \Phi_X\). We recall two cardinal properties of the momentum map:

\[
\begin{align*}
(0.2) & \quad \text{(a) } \ker(D\Phi(x)) = g(x)^o \\
& \quad \text{(b) } \im(D\Phi(x)) = \ann(g_x).
\end{align*}
\]

The first is the orthogonal relative to \(\omega\) of the tangent space \(g(x)\) to the orbit \(G(x), x \in X\); the second is the annihilator in \(g^*\) of the stabilizer Lie subalgebra \(g_x \subset g\).

### 1 Symplectic Induction

Given a closed subgroup \(H \subset G\) and a Hamiltonian \(H\)-space \((Y, \omega_Y, \Psi)\), [K78] constructs an induced Hamiltonian \(G\)-space as follows. Let \(\sigma\) denote the canonical 1-form on \(T^*G\) given by \(\sigma(\delta p) = \langle p, \delta q \rangle\), where \(\delta p \in T_p(T^*G), \delta q = \pi_*(\delta p) \in T_{\pi(p)}G\), and \(\pi : T^*G \to G\) is the canonical projection. Endow \(N := T^*G \times Y\) with the symplectic form \(\omega := d\sigma + \omega_Y\) and the \(G \times H\)-action \((g, h)(p, y) = (gp^{-1}h, h(y))\), where \(h(y)\) denotes the action of \(h \in H\) on \(y \in Y\). This action admits the equivariant momentum map \(\phi \times \psi : N \to g^* \times h^*\):

\[
\begin{align*}
\phi(p, y) &= pq^{-1} \\
\psi(p, y) &= \Psi(y) - q^{-1}p|_h.
\end{align*}
\]

The induced manifold is, by definition, the Marsden-Weinstein reduced space of \(N\) at \(0 \in h^*\), i.e.

\[
(1.2) \quad \Ind^G_1 Y := N/H = \psi^{-1}(0)/H.
\]

In more detail: the action of \(H\) is free and proper (because it is free and proper on the factor \(T^*G\), where it is the right action of \(H\) regarded as a subgroup of the group \(T^*G\) [B72, §III.1.6]); so \(\psi\) is a submersion \((0.2b)\), \(\psi^{-1}(0)\) is a submanifold, and \((1.2)\) is a manifold; moreover \(\omega|_{\psi^{-1}(0)}\) degenerates exactly along the H-orbits \((0.2a)\), so it is the pull-back of a uniquely defined symplectic form, \(\omega_{N/H}\), on the quotient. Furthermore, the \(G\)-action commutes with the \(H\)-action and preserves \(\psi^{-1}(0)\), and its momentum map \(\phi\) is constant on \(H\)-orbits. Passing to the quotient, we obtain the required \(G\)-action on \(\Ind^G_1 Y\) and momentum map \(\Phi_{N/H} : \Ind^G_1 Y \to g^*\). Note that since \(\psi\) is a submersion and \(H\) acts freely, \((1.2)\) has dimension equal to \(\dim(N) - 2 \dim(H)\), i.e.

\[
(1.3) \quad \dim(\Ind^G_1 Y) = 2 \dim(G/H) + \dim(Y).
\]

### 2 Symplectic Induction in Stages

#### (2.1) Theorem (Stages).

If \(H \subset K \subset G\) and \(H, K\) are closed subgroups of the Lie group \(G\), then

\[
\Ind^K_G \Ind^H_K Y = \Ind^G_K Y.
\]

**Proof.** Let \((N, \omega, \phi \times \psi)\) be as in §1 and consider \(M = T^*G \times T^*K \times Y\) with 2-form \(\omega_M = d\sigma_{T^*G} + d\sigma_{T^*K} + \omega_Y\) and \(G \times K \times H\)-action

\[
(2.2) \quad (g, k, h)(p, \bar{p}, y) = (gpk^{-1}, k\bar{p}h^{-1}, h(y)).
\]

This admits the equivariant momentum map \(\phi \times \tilde{\phi} \times \tilde{\psi} : M \to g^* \times t^* \times h^*\):

\[
\begin{align*}
\phi(p, \bar{p}, y) &= pq^{-1} \\
\tilde{\phi}(p, \bar{p}, y) &= \bar{p}q^{-1} - q^{-1}p|_t \\
\tilde{\psi}(p, \bar{p}, y) &= \Psi(y) - q^{-1}\bar{p}|_h.
\end{align*}
\]
for \((p, \tilde{p}) \in T^*_\gamma G \times T^*_\gamma K\). Define \(r : M \to N\) by \(r(p, \tilde{p}, y) = (p \tilde{q}, y)\) and consider the commutative diagram in Fig. 1, where we have written \(j_1, j_2, j_3\) and \(\pi_1, \pi_2, \pi_3\) for the inclusion and projection maps involved in constructing the reduced spaces \(M//H\), \((M//H)//K = \text{Ind}_H^G \text{Ind}_K^H Y\), and \(N//H = \text{Ind}_H^G Y\); also \(j, \pi\) are the obvious inclusion and restriction, and \(\Phi_{M//H}\) is the momentum map for the residual K-action on \(M//H\). The map \(r \circ j_1 \circ j\) satisfies

\[
\psi((r \circ j_1 \circ j)(p, \tilde{p}, y)) = \psi(p \tilde{q}, y) = \Psi(y) - (q \tilde{q})^{-1}p \tilde{q}\bigr|_b = \Psi(y) - \tilde{q}^{-1}(q^{-1}p) \bigr|_b = \Psi(y) - \tilde{q}^{-1}p\bigr|_b \quad \text{since } \tilde{\phi}(p, \tilde{p}, y) = 0 = 0 \quad \text{since } \tilde{\psi}(p, \tilde{p}, y) = 0.
\]

So \(r \circ j_1 \circ j\) takes values in \(\psi^{-1}(0)\), i.e., there is a map \(s\) as indicated in Fig. 1. Moreover, \(s\) is onto since one verifies that \((p, y) \mapsto (p, (q^{-1}p) \bigr|_b, y)\) provides a right inverse. The map \(s\) is equivariant:

\[
s((g, k, h)(p, \tilde{p}, y)) = r(gp^{-1}, k\tilde{q}h^{-1}, h(y)) = (gp\tilde{q}h^{-1}, h(y)) = (g, h)(p \tilde{q}, y) = (g, h)(s(p, \tilde{p}, y)).
\]

Hence \(s\) descends to a G-equivariant surjection \(t\) as indicated in Fig. 1. Furthermore, one checks without trouble that the fibers of \(s\) are precisely the K-orbits in its domain. As \(\pi_2 \circ \pi\) collapses these orbits to points, it follows that \(t\) is bijective, hence a diffeomorphism by \([B67, 5.9.6]\). The relation \(q = r^*\Phi\) implies that \(t\) relates the momentum maps for \(G\): \(\Phi_{(M//H)//K} = t^*\Phi_{N//H}\), so there only remains to see that \(\omega_{(M//H)//K} = t^*\omega_{N//H}\). To this end we compute

\[
(s^*j_3^*\omega_{T^*G})(\delta p, \delta \tilde{p}, \delta q) = \omega_{T^*G}(\delta[p \tilde{q}]) = \langle p \tilde{q}, \delta[q \tilde{q}] \rangle = \langle p, \delta q \rangle + \langle q^{-1}p, [\delta q] \rangle = \langle p, \delta q \rangle + \langle q^{-1}p, \delta q \rangle \quad \text{since } \delta[q \tilde{q}] = [\delta q] \tilde{q} + q[\delta \tilde{q}] = \omega_{T^*G}(\delta p) + \omega_{T^*K}(\delta \tilde{p}).
\]
Taking exterior derivatives and adding $\omega_Y$ we obtain $s^*j^*\omega_N = j^*j^*_1\omega_M$ or, equivalently (by commutativity of the diagram and definition of the reduced 2-forms), $\pi^*\pi^*_2\omega_{N/H} = \pi^*\pi^*_2\omega_{(M/H)/K}$. Since $\pi_2 \circ \pi$ is a submersion, we are done. 

\section{Symplectic Frobenius Reciprocity}

It is quite rare for an induced Hamiltonian $G$-space to be homogeneous or a fortiori a coadjoint orbit (by which we mean that its momentum map is 1-1 onto an orbit). In fact we have the following, where $\Phi_{N/H}$ is the momentum map for the induced space (1.2).

\begin{proposition}
Let $(Y, \omega_Y, \Psi)$ be a Hamiltonian $H$-space.

(a) A coadjoint orbit $O$ of $G$ intersects $\text{Im}(\Phi_{N/H}) \iff O \mid h := \{m \mid h : m \in O\}$ intersects $\text{Im}(\Psi)$.

(b) If $\text{Ind}_H^G Y$ is homogeneous, then $Y$ is homogeneous.

(c) If $\text{Ind}_H^G Y$ is a coadjoint orbit, then $Y$ is a coadjoint orbit.
\end{proposition}

\begin{proof}
(a): This re-expresses $\text{Im}(\Phi_{N/H}) = \phi(\phi^{-1}(0))$ (1.1). (b): Assume $G$ is transitive on $\text{Ind}_H^G Y$ and let $y_1, y_2 \in Y$. Pick $m \in g^*$ such that $\Psi(y_1) = m \mid h$. Then the $H$-orbits $x_i = H(m_i, y_i)$ are points in (1.2). So transitivity says that $x_1 = g(x_2)$, i.e.

\[(m_1, y_1) = (gm_2h^{-1}, h(y_2)) \quad \text{for some } h \in H.\]

(c) Assume further that $\Phi_{N/H}$ is injective and suppose $\Psi(y_1) = \Psi(y_2)$. Then we can pick $m_1 = m_2$ above. Since $\Phi_{N/H}(x_i) = m_i$ it follows, by injectivity, that $x_1 = x_2$, i.e., we have (3.2) with $g = e$. But then $h = e$ and hence $y_1 = y_2$, as claimed. \qed

If $Y$ is a coadjoint orbit, (3.1a) says that $\text{Ind}_H^G Y$ “involves” just those orbits $O$ whose projection in $h^*$ contains $Y$. Guillemin and Sternberg [G82, §6] proposed to measure the “multiplicity” of this involvement by the (possibly empty) space $\text{Hom}_G(O, \text{Ind}_H^G Y)$, where we write suggestively

\[(3.3) \quad \text{Hom}_G(X_1, X_2) := (X_1^\perp \times X_2)/G,\]

i.e., the Marsden-Weinstein reduction of $X_1^\perp \times X_2$ at $0 \in g^*$; here $X_1^\perp$ is the Hamiltonian $G$-space $X_1$ with its 2-form and momentum map replaced by their negatives. Then (3.1a) can be refined by the following analog of Frobenius’s theorem [B85, III.6.2], already found in [G83, Thm 2.2] when both $X$ and $Y$ are coadjoint orbits.

\begin{theorem}[Frobenius reciprocity]
If $X$ is a Hamiltonian $G$-space and $Y$ a Hamiltonian $H$-space, then

\[\text{Hom}_G(X, \text{Ind}_H^G Y) = \text{Hom}_H(\text{Res}_H^G X, Y).\]
\end{theorem}

\begin{Remarks}
Here $\text{Res}_H^G X$ means $X$ regarded as a Hamiltonian $H$-space, and “=” means only a natural bijection as sets. We believe (but haven’t proved) that both sides are automatically isomorphic as diffeological spaces with diffeological 2-forms as discussed in [S85, §2.5], [I13, §6.38], [K16].

Note also that, by the symmetry of (3.3), we may equally write Frobenius reciprocity in the form $\text{Hom}_G(\text{Ind}_H^G Y, Z) = \text{Hom}_H(Y, \text{Res}_H^G Z)$.

\begin{proof}
For bookkeeping reasons, soon to become clear, rename $G$ also as $K$. Consider the spaces $N = X^{-} \times Y$ with $H$-action $h(x, y) = (h(x), h(y))$, and $M = X^{-} \times K^* \times Y$ with $K \times H$-action $(k, h)(x, \tilde{\p}, y) = (k(x), k\tilde{\p}h^{-1}, h(y))$. Their equivariant momentum maps are $\psi : N \to h^*$,

\[(3.6) \quad \psi(x, y) = \Psi(y) - \Phi(x)|_{h},\]

and $\phi : \text{Ind}_H^G Z \to H$.
\end{proof}

4
and $\tilde{\phi} \times \tilde{\psi} : M \to t^* \times h^*$,

$$
\begin{align*}
\tilde{\phi}(x, \tilde{p}, y) &= \tilde{p} \tilde{q}^{-1} - \Phi(x) \\
\tilde{\psi}(x, \tilde{p}, y) &= \Psi(y) - \tilde{q}^{-1} \tilde{p} |_{h}
\end{align*}
$$

(3.7)

where $\Phi$ and $\Psi$ are the equivariant momentum maps of $X$ and $Y$, respectively. Defining $r : M \to N$ by $r(x, \tilde{p}, y) = (\tilde{q}^{-1}(x), y)$ now sets us up for a proof using the same previous diagram (Fig. 1). Indeed, we have again this time

$$
\psi((r \circ j \circ j)(x, \tilde{p}, y)) = \psi(\tilde{q}^{-1}(x), y) = \Psi(y) - \Phi(\tilde{q}^{-1}(x)) |_{h}
$$

(3.8)

$$
= \Psi(y) - \tilde{q}^{-1} \Phi(x) \tilde{q} |_{h} \quad \text{by equivariance}
$$

$$
= \Psi(y) - \tilde{q}^{-1} \tilde{p} |_{h} \quad \text{since } \tilde{\phi}(x, \tilde{p}, y) = 0
$$

$$
= 0 \quad \text{since } \tilde{\psi}(x, \tilde{p}, y) = 0,
$$

so there is a map $s$ as indicated in Fig. 1. Again, $s$ is onto since $(x, y) \mapsto (x, \Phi(x), y)$ provides a right inverse, and $s$ is equivariant:

$$
s((k, h)(x, \tilde{p}, y)) = r(k(x), k\tilde{p}h^{-1}, h(y))
$$

(3.9)

$$
= ((k\tilde{q}h^{-1})^{-1}(k(x)), h(y))
$$

$$
= (h(\tilde{q}^{-1}(x)), h(y))
$$

$$
= h(s(x, \tilde{p}, y)).
$$

So the fibers of $s$ are again the $K$-orbits and $s$ descends again to a bijection $t$ as required and indicated in Fig. 1.

4 An Example

The following example highlights a basic shortcoming in the analogy of (3.4) with representation theory: it cannot mirror cases where more than one representation “quantizes” a given Hamiltonian $G$-space or $H$-space.

In the solvable group $G'$ of all upper triangular matrices of the form

$$
g' = \begin{pmatrix}
e^{ias} & 0 & 0 \\
1 & e & f \\
1 & a & b
\end{pmatrix} \quad a, e, f \in \mathbb{R}
$$

(4.1)

write $G$ for the subgroup in which $e = 0$ and $H$ for the subgroup of $G$ in which $a \in 2\pi\mathbb{Z}$. Identify $g'^*$ with $\mathbb{R} \times \mathbb{C} \times \mathbb{R}^2$ by writing $(p, q, s, t)$ for the value at the identity of the 1-form

$$
pda + \text{Re}(\tilde{q}db) - sde - tdf.
$$

(4.2)

Likewise, identify $g^*$ with triples $(p, q, t)$ and $h^*$ with pairs $(q, t)$ so that the projections $g'^* \to g^* \to h^*$ become $(p, q, s, t) \mapsto (p, q, t) \mapsto (q, t)$. Then the coadjoint orbit $X' = G'(0, 1, 0, 1)$ projects onto the coadjoint orbit $X = G(0, 1, 1)$ and is its universal covering:

$$
X' = \{ (p, e^{ias}, s, 1) : (p, s) \in \mathbb{R}^2 \}, \quad \omega_{X'} = dp \wedge ds,
$$

(4.3)

$$
\downarrow
$$

$$
X = \{ (p, q, 1) : (p, q) \in \mathbb{R} \times T \}, \quad \omega_{X} = dp \wedge dq.
$$
Moreover, one checks (or finds by [Z14] applied to the normal subgroup H°) that \(X = \text{Ind}_H^G Y\), where Y is the point \((1, 1)\). So symplectic Frobenius reciprocity gives

\[
\text{Hom}_G(X, \text{Res}_H^G X') = \text{Hom}_H(Y, \text{Res}_H^G X') = (X' \to \mathfrak{h}^*)^{-1}(1, 1)/H
\]

which is a single point. But this fact is of little use for representation theory, as it fails to discriminate between the circle worth of representations attached to X, according to [A71] (where, we recall, they are parametrized by the characters of the fundamental group \(\pi_1(X)\)). As one knows, this should be fixed by working instead with prequantum spaces in the sense of the next section.

## 5 Prequantum G-spaces

Following [S70], we call prequantum manifold a manifold \(\tilde{X}\) with a contact 1-form \(\sigma\) whose Reeb vector field generates a circle group action. We recall that \(\sigma\) contact means that \(\text{Ker}(d\sigma)\) is 1-dimensional and transverse to \(\text{Ker}(\sigma)\); its Reeb vector field, \(i\), on \(\tilde{X}\) is defined by

\[
i(\tilde{z}) \in \text{Ker}(d\sigma) \quad \text{and} \quad \sigma(i(\tilde{z})) = 1 \quad \forall \tilde{z} \in \tilde{X}.
\]

Then \((\tilde{X}, d\sigma)\) is a prequantum manifold whose null leaves are the orbits of the circle group \(T = U(1)\) acting on \(\tilde{X}\) and \(d\sigma\) descends to a symplectic form \(\omega\) on the leaf space \(X = \tilde{X}/T\). If a Lie group \(G\) acts on \(\tilde{X}\) and preserves \(\sigma\), then it commutes with \(T\) and the equivariant momentum map \(\Phi : \tilde{X} \to \mathfrak{g}^*\),

\[
(\Phi(\tilde{z}), Z) = \sigma(Z(\tilde{z})),
\]

descends to a momentum map \(\tilde{\Phi} : X \to \mathfrak{g}^*\), making \((X, \omega, \tilde{\Phi})\) a Hamiltonian G-space prequantized by the prequantum G-space \((\tilde{X}, \sigma)\).

We do not distinguish between two spaces \(\tilde{X}_1, \tilde{X}_2\) which are isomorphic, i.e., related by a G-equivariant diffeomorphism which transforms \(\sigma_1\) into \(\sigma_2\). (If several are in play, we may also use subscripts like \(\sigma_x, i_x, \Phi_x\), etc.) We recall three basic constructions in the prequantum category:

**Prequantum dual.** ([S70, 18.47].) We write \(\tilde{X}^-\) for the G-space equal to \(\tilde{X}\) but with opposite 1-form \(-\sigma\) (and consequently opposite Reeb field and \(T\)-action). It prequantizes the dual G-space \((X^-, -\omega, -\tilde{\Phi})\).

**Prequantum product.** ([S70, 18.52].) If \(\tilde{X}_1\) and \(\tilde{X}_2\) are prequantum G-spaces, then \(\tilde{X}_1 \times \tilde{X}_2\) (with diagonal \(G\)-action) is a \(T^2\)-space in which the action of the antidiagonal \(\Delta = \{(z^{-1}, z) : z \in T\}\) has as its orbits the characteristic leaves of the 1-form \(\sigma_1 + \sigma_2\). Hence this descends to the quotient \(\tilde{X}_1 \times \tilde{X}_2 \owns (\tilde{x}_1, \tilde{x}_2) : (\tilde{x}_1 \times \tilde{x}_2)/\Delta\) as a 1-form making it a prequantization of the symplectic product \(X_1 \times X_2\). In view of (5.3), the \(\Delta\)-action on \(\tilde{X}^-_1 \times \tilde{X}^-_2\) is \(z(\tilde{z}_1, \tilde{z}_2) = (z(\tilde{z}_1), z(\tilde{z}_2))\).

**Prequantum reduction.** ([L01, Thm 2].) Assume \(G\) acts freely and properly on \(\tilde{X}\), and consider the level \(L := \Phi^{-1}(0)\). By the very definition (5.2) of \(\Phi\) and its being a momentum map, we have \(g(\tilde{z}) \subset \text{Ker}(\sigma|_L) \cap \text{Ker}(d\sigma|_L)\). Since \(\sigma|_L\) is also \(G\)-invariant, it follows (see [S70, 5.21]) that it descends to a contact 1-form on the quotient \(\tilde{X}/G := \Phi^{-1}(0)/G\). This prequantizes the symplectic reduction \(X/G = \tilde{\Phi}^{-1}(0)/G\).

## 6 Prequantum Induction

Given a closed subgroup \(H \subset G\) and a prequantum H-space \((\tilde{Y}, \sigma_H)\) whose momentum map (5.2) we denote \(\Psi\), we propose to construct an induced prequantum G-space \(\text{Ind}_H^G \tilde{Y}\) as follows. Consider
the prequantum \((G \times H)\)-space \(\tilde{N} = T^*G \times \tilde{Y}\) with 1-form \(\omega_{T \times G} + \sigma_{T \times Y}\) and action \((g, h)(p, \tilde{y}) = (gph^{-1}, h(\tilde{y}))\). This action has the equivariant momentum map \(\phi \times \psi : \tilde{N} \to g^* \times h^*\),

\[
\begin{align*}
(6.1) \\
\phi(p, \tilde{y}) &= pq^{-1} \\
\psi(p, \tilde{y}) &= \Psi(\tilde{y}) - q^{-1}p|_h
\end{align*}
\]

The same arguments as with (1.2), then, show that

\[
(6.2) \text{Ind}^G_H \tilde{Y} := \tilde{N}/H = \psi^{-1}(0)/H
\]

(7.1) Theorem. If \(H \subset K \subset G\) and \(H, K\) are closed subgroups of the Lie group \(G\), then

\[
\text{Ind}^G_K \text{Ind}^K_H \tilde{Y} = \text{Ind}^G_H \tilde{Y}.
\]

Proof. The proof is mutatis mutandis the same as for (2.1), only simpler. We just switch to working with restrictions and push-forwards of the 1-form \(\delta p, \delta \tilde{p}, \delta \tilde{y}\) = \(\langle \tilde{x}, \delta q \rangle + \langle \phi \delta \tilde{p}, \delta \tilde{y} \rangle\) on \(\tilde{M} = T^*G \times T^*K \times \tilde{Y}\) instead of the 2-form \(\omega\) on \(M\).

8 Prequantum Frobenius Reciprocity

The three constructions (5.3–5.5) put together furnish us with a notion of the intertwiner space of two prequantum G-spaces,

\[
(8.1) \quad \text{Hom}^G_{\tilde{X}, \tilde{Y}} := (\tilde{X}_1 \times \tilde{X}_2)/G.
\]

Freeness and properness of the last G-action are not assumed and we again regard (8.1) as just a set.

(8.2) Theorem (Frobenius reciprocity). If \(\tilde{X}\) is a prequantum G-space and \(\tilde{Y}\) a prequantum H-space, then

\[
\text{Hom}^G_{\tilde{X}, \text{Ind}^G_H \tilde{Y}} = \text{Hom}^H_{\text{Res}^G_H \tilde{X}, \tilde{Y}}.
\]

Proof. With \(\Delta\) as in (5.4), define \(\tilde{r}\) in the following commutative diagram by \(\tilde{r}(\tilde{x}, p, \tilde{y}) = (q^{-1}(\tilde{x}), \tilde{y})\), where \(p \in T^*_0G\):

\[
\begin{align*}
\tilde{M} := \tilde{X} \times T^*G \times \tilde{Y} &\quad \tilde{N} := \tilde{X} \times \tilde{Y} \\
\tilde{M}/\Delta &\quad \tilde{N}/\Delta
\end{align*}
\]

Then \(\tilde{r}\) descends, as indicated, to a map \(\tilde{r}\) and a map \(r\) which is the one in our proof of (3.4). Now each floor of this diagram supports a horizontal copy of Fig. 1 giving rise to the appropriate tilded versions of \(s\) and \(t\); a straightforward diagram chase checks that \(\tilde{t} : (\tilde{M}/H)/G \to \tilde{N}/H\) is the required bijection.
9 An Example (Reprise)

Recall the coadjoint orbits $X' \cong \mathbb{R}^2$ and $X \cong \mathbb{R} \times T$ of (4.3). Referring to [S70, 18.117, 18.133, 18.134] and performing direct verifications, one finds:

- There are infinitely many prequantum $G$-spaces prequantizing $X$, namely all $\tilde{X}_\lambda = \text{Ind}^G_H T_\lambda$ where $T_\lambda$ is a single circle on which $H$ acts by the character $\chi_\lambda(h) = e^{-ia \cdot e^{i[\Re(b) - f]}}$ (notation (4.1)). Explicitly
  \begin{equation}
  (9.1) \quad \tilde{X}_\lambda = \mathbb{R} \times T^2 \ni (p, q, z) \quad \text{with} \quad \sigma_\lambda = (p + \lambda)\frac{dq}{iq} + \frac{dz}{iz}.
  \end{equation}
  and $\lambda_1, \lambda_2 \in \mathbb{R}$ give equivalent prequantizations iff they differ by an integer [A59, K06].

- There is a unique prequantum $G'$-space over $X'$, namely $\tilde{X}' = \text{Ind}^{G'}_{H'} T$ where $H'$ is the subgroup $a = 0$ of $G'$ and $T$ is a single circle on which $H'$ acts by the character $\chi(h') = e^{i[\Re(b) - f]}$. Explicitly
  \begin{equation}
  (9.2) \quad \tilde{X}' = \mathbb{R}^2 \times T \ni (p, s, z) \quad \text{with} \quad \sigma = ps + \frac{dz}{iz}.
  \end{equation}

Apply now (8.2) which replaces (4.4) in this case, to conclude

\begin{equation}
(9.3) \quad \text{Hom}_G(\tilde{X}_\lambda, \text{Res}^G_{G'} \tilde{X}') = \text{Hom}_H(T_\lambda, \text{Res}^H_{H'} \tilde{X}').
\end{equation}

A direct verification shows that the right-hand side is a single circle for each $\lambda$. This identity also illustrates the power of Frobenius reciprocity: we can obtain the harder left-hand side from the easier right-hand side. Returning to the representation theoretical interpretation, note that (9.3) “predicts” that once restricted to $G$, the irreducible representation $\text{Ind}^{G'}_{H'} \chi$ (which quantizes $X'$) splits into the direct integral over $\lambda \in \mathbb{R}/\mathbb{Z}$ of the irreducible representations $\text{Ind}^G_H \chi_\lambda$ (which all quantize $X$) with multiplicity 1; this prediction is correct and can be checked directly.

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