On the exit probability of the extended Sznajd model and the Kirkwood approximation

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Abstract. The Sznajd model is a spin model, inspired by sociophysics, where 2 aligned spins influence a neighbouring spin to change its orientation. A bit of controversy arose in the literature recently about the properties of its one dimensional case, namely the exit probability, the probability that the system ends up in the “all spins up” state as a function of the proportion of spins pointing up in the initial condition. The Kirkwood approximation, which is a type of mean-field treatment, gives a surprisingly simple expression for this probability that has been verified for simulations with up to $10^7$ sites. Nevertheless, some aspects of the Kirkwood approximation cast doubt upon its validity. In this work, we use an extended version of the Sznajd model due to G. Kondrat, that allows for any interaction between a pair of spins and a neighbouring spin that respects the symmetry between up and down spins. Looking at the Kirkwood approximation for this case we obtain an expression for the exit probability in the cases where the “all up” state is absorbing and compare this probability with simulations, finding cases where the results are indeed not valid.

1. Introduction
The application of tools from physics (mainly statistical mechanics and dynamical systems) to sociology problems has gathered interest in the last decade, under the label of sociophysics. By taking these tools away from their usual domain of validity new questions arise that are themselves interesting to physical theories. This has happened recently with the study of one dimensional spin models inspired by social interactions.

The Sznajd model [1], which is the model in question has the following simple definition

- Consider a linear chain with periodic boundary conditions and sites that can be in one of 2 states: ++ or --.
- At each time step choose a pair of neighbouring sites, $i$ and $i+1$.
- If these 2 sites are in different states, then nothing happens.
- However, if they have the same state, then we choose at random one of their neighbours ($i-1$ or $i+2$) and change the state of this neighbour to match the state of the pair.
- We continue to follow these rules until all the sites in the linear chain have the same state.

It is trivial that this system possesses 2 absorbing states that be reached: “all spins up” and “all spins down”. The question then arises about what is the probability that we end up in each of the absorbing states. This is given by the so called exit probability, more precisely, the exit
probability is the probability $E(\rho)$ that we end up in the “all spins up” state when we start with an initial condition having no spatial correlations and a proportion $\rho$ of up spins.

In recent papers [2, 3], the exit probability of this model was measured in simulations and a remarkably simple expression was found:

$$E(\rho) = \frac{\rho^2}{\rho^2 + (1 - \rho)^2}.$$ 

This expression can be found analytically through a mean field taking into account pair correlations, called Kirkwood approximation in the literature. However a simple change in the way correlations are truncated in this approximation leads to nonsensical results, which cast doubt upon the validity of the approximation.

2. The Kondrat model and a test for the validity of the Kirkwood approximation

We turn now to the extension of the Sznajd model due to G. Kondrat [4]:

- Consider a linear chain with periodic boundary conditions and sites that can be in one of 2 states: + or −.
- At each time step choose $s = \pm 1$ randomly (50% chance for each) and a site $i$. Let $j = i + s$ and $k = i + 2s$.
- Flip the state of site $k$ with probability $p(\sigma_i, \sigma_j, \sigma_k)$, where $\sigma_X$ denotes the state of site $X$.

The probabilities $p$ are the most general possible while still keeping the symmetry between states + and −. These probabilities (together with the social interpretation of the interactions related to them) are:

- $p(\sigma_i, \sigma_j, \sigma_k) = p(-\sigma_i, -\sigma_j, -\sigma_k)$
- $p(+, +, +) = p_1$ (anticonformity)
- $p(+, +, -) = p_2$ (Sznajd-like conformity)
- $p(-, +, -) = p_3$ (voter-like conformity)
- $p(+, -, -) = p_4$ (disagreement propagation)

The Kirkwood approximation can be used in this generalized model and the results concerning the states “all up” and “all down” are as follows:

- If $p_1 \neq 0$ then neither “all up” nor “all down” are absorbing states and so we can’t talk about an exit probability.
- If $p_1 = 0$ but $p_2 \leq p_4$ the approximation predicts that the time to reach one of these states diverges, meaning that neither are reached in the thermodynamical limit (in a finite lattice we should see a 50-50 split between them, with very little dependence on the initial condition and extremely long transients).
- If $p_1 = 0$ and $p_2 > p_4$ then the approximation predicts the following exit probability:

$$E(\rho) = \frac{\rho(1 - \rho)p_3 + \rho^2(p_2 - p_4)}{2\rho(1 - \rho)p_3 + (\rho^2 + (1 - \rho)^2)(p_2 - p_4)}$$

which is a mixture of sorts of the Sznajd and voter exit probabilities.

We now show simulations for 2 parameter combinations that don’t correspond to any model where the approximation has been tested yet. Figure 1 shows a case where the exit probability agrees with the Kirkwood approximation while figure 2 shows a case with disagreement.
Figure 1. Simulations agreeing with the Kirkwood approximation

Figure 2. Simulations disagreeing with the Kirkwood approximation

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