USUBA’S PRINCIPLE UB_λ CAN FAIL AT SINGULAR CARDINALS

MOHAMMAD GOLSHANI and SAHARON SHELAH

Abstract. We answer a question of Usuba by showing that the combinatorial principle UB_λ can fail at a singular cardinal. Furthermore, λ can be taken to be ℵ_κ.

§1. Introduction. In [5], Usuba introduced a new combinatorial principle, denoted UB_λ.¹ He showed that UB_λ holds for all regular uncountable cardinals and that for singular cardinals, some very weak assumptions like weak square or even ADS_κ imply it. It is known that ADS_κ can fail for singular cardinals, for example if κ is supercompact and λ > κ is such that cf(λ) < κ. Motivated by this results, Usuba asked the following question:

**Question 1.1.** [5, Question 2.11] Is it consistent that UB_λ fails for some singular cardinal λ?

In this paper we give a positive answer to the above question by showing that Chang’s transfer principle (ℵ_ω+1, ℵ_ω) ↠ (ℵ_1, ℵ_0) implies the failure of UB_ℵ_ω if ℵ_ω is strong limit (see Theorem 3.1), where a stronger result is proved.

The paper is organized as follows. In Section 2, we present some preliminaries and results and then in Section 3, we prove our main result.

§2. Some preliminaries. In this section we present some definitions and results that are needed for the later section of this paper. Let us start by introducing Usuba’s principle.

**Definition 2.1.** Let λ be an uncountable cardinal. The principle UB_λ is the statement: there exists a function f : [λ^+]^{<ω} → λ^+ such that if x, y ⊆ λ^+ are closed under f, x ∩ λ = y ∩ λ and sup(x ∩ λ) = λ, then x ⊆ y or y ⊆ x.

It turned out this principle has many equivalent formulations. To state a few of it, let S = {x ⊆ λ : sup(x) = λ}, θ > λ be large enough regular and let ◊ be a well-ordering of H(θ). Then we have the following.

**Lemma 2.2.** [5] The following are equivalent:

1. UB_λ.
2. If M, N ◊ (H(θ), ∈, ◊, λ, S, ...) are such that M ∩ λ = N ∩ λ ∈ S, then either M ∩ λ^+ ⊆ N ∩ λ^+ or N ∩ λ^+ ⊆ M ∩ λ^+.

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¹See Section 2 for the statement of the principle.

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(3) If $M, N \prec (H(\theta), \in, \in, \lambda, S, \ldots)$ are such that $M \cap \lambda = N \cap \lambda \in S$ and $\text{sup}(M \cap \lambda^+) \leq \text{sup}(N \cap \lambda^+)$, then $M \cap \lambda^+$ is an initial segment of $N \cap \lambda^+$.

The principle UB$_{\kappa}$ has many nice implications. Here we only consider its relation with Chang’s transfer principles which is also related to our work.

**Definition 2.3.** Suppose $\lambda > \mu$ are infinite cardinal. Chang’s transfer principle $(\lambda^+, \lambda) \rightarrow (\mu^+, \mu)$ is the statement: if $L$ is a countable first-order language which contains a unary predicate $U$, then for any $L$-structure $M = (M, U^M, \ldots)$ with $|M| = \lambda^+$ and $|U^M| = \lambda$, there exists an elementary submodel $N = (N, U^N, \ldots)$ of $M$ with $|N| = \mu^+$ and $|U^N| = \mu$.

Given an infinite cardinal $\nu$, the transfer principle $(\lambda^+, \lambda) \rightarrow_{\leq \nu} (\mu^+, \mu)$ is defined similarly, where we allow the language $L$ to have size at most $\nu$.

The next lemma shows the relation between UB$_{\kappa}$ and Chang’s transfer principles.

**Lemma 2.4.** ([5, Corollary 4.2]) Suppose UB$_{\kappa}$ holds. Then the Chang transfer principles $(\kappa_{\omega+1}, \kappa_\omega) \rightarrow (\kappa_{n+1}, \kappa_n)$ fail for all $1 \leq n < \omega$.

**Remark 2.5.** By [4], $(\kappa_{\omega+1}, \kappa_\omega) \rightarrow (\kappa_{n+1}, \kappa_n)$ fails for all $n \geq 3$.

Since the consistency of the transfer principle $(\kappa_{\omega+1}, \kappa_\omega) \rightarrow (\kappa_{n+1}, \kappa_n)$ is open for $n = 1, 2$, one cannot use the above result to get the consistent failure of UB$_{\kappa}$. In the next section we show that if $\kappa_\omega$ is strong limit, then UB$_{\kappa}$ implies the failure of $(\kappa_{\omega+1}, \kappa_\omega) \rightarrow (\kappa_1, \kappa_0)$ as well, and hence by the results of [3] (see also [1, 2], where the consistency of GCH + $(\kappa_{\omega+1}, \kappa_\omega) \rightarrow (\kappa_1, \kappa_0)$ is proved using weaker large cardinal assumptions) UB$_{\kappa}$ can fail. We also need the following notion.

**Definition 2.6.** An uncountable cardinal $\kappa$ is said to be Jonsson, if for every function $f : [\kappa]^{<\omega} \rightarrow \kappa$ there exists a set $H \subseteq \kappa$ of order type $\kappa$ such that for each $n$, $f''[H]^n \neq \kappa$.

**Notation 2.7.** Given a model $M$ and a subset $A$ of $M$, by $cl(A, M)$ we mean the least substructure of $M$ which includes $A$ as a subset.

**Lemma 2.8.** Assume $\lambda$ is a singular strong limit cardinal of cofinality $\kappa$. Then there is a model $M_0$ with vocabulary $\mathcal{L}_0$ such that:

(a) $|\mathcal{L}_0| = \kappa$ and $|M_0| = \lambda^+$.

(b) If $M$ is an $\mathcal{L}$-structure which expands $M_0$, $|\mathcal{L}| = \kappa$, and $M$ has Skolem functions, then for $\alpha_1, \alpha_2 < \lambda^+$, the following statements are equivalent:

1. For some submodels $N_1, N_2$ of $M$ we have:
   $(\alpha_1)N_1 \cap \lambda = N_2 \cap \lambda$ is unbounded in $\lambda$.
   $(\alpha_2)\alpha_1 \in N_1 \setminus N_2$ and $\alpha_2 \in N_2 \setminus N_1$.
   $(\beta)\alpha_1, \alpha_2$ If $V_\ell = cl(\{\alpha_\ell\}, M) \cap \lambda$, $\ell = 1, 2$, and $V = V_1 \cup V_2$, then $\alpha_1 \notin cl(\{\alpha_2\} \cup V, M)$ & $\alpha_2 \notin cl(\{\alpha_1\} \cup V, M)$.

**Proof.** Let $(\lambda_i : i < \kappa)$ be an increasing sequence cofinal in $\lambda$ such that for all $i < \kappa, 2^{\lambda_i} < \lambda_{i+1}$. For each $0 < n < \omega$, let

$(F_{n, \alpha} : \alpha \in [\lambda_i, 2^{\lambda_i}])$

enumerate all functions from $\lambda_i$ into $\lambda_i$. Let $M_0$ be defined as follows:
We show that the model 

\[ \langle M, \in, \alpha, < \rangle : \alpha < \beta < \lambda^+ \].

\[ c^M_\alpha = \lambda_\alpha. \]

\[ P^M_\alpha = \{ \alpha : \alpha < \lambda \}. \]

\[ F^M_n = \text{an (n+1)-ary function such that:} \]

- if \( i < \kappa, \alpha \in [\lambda_i, 2^{\lambda_i}) \) and \( \beta_0, \ldots, \beta_{n-1} \in \lambda_i \), then

\[ F^M_n(\beta_0, \ldots, \beta_{n-1}, \alpha) = F_n(\alpha, \beta_0, \ldots, \beta_{n-1}). \]

- in all other cases, \( F^M_n(\beta_0, \ldots, \beta_{n-1}, \beta_n) = \beta_n. \)

We show that the model \( M_0 \) is as required. Clause (a) clearly holds. To show that clause (b) is satisfied, let \( M \) be an \( \mathcal{L} \)-structure which expands \( M_0, |\mathcal{L}| = \kappa, \) and suppose \( M \) has Skolem functions. Let also \( \alpha_1, \alpha_2 < \lambda^+ \).

First suppose that \((1)_{\alpha_1, \alpha_2}\) holds, and suppose that the models \( N_1, N_2 \) witness it.

Let also \( V_\ell = cl(\{\alpha_\ell\}, M) \cap \lambda, \ell = 1, 2. \) Clearly each \( V_\ell \) is an unbounded subset of \( \lambda. \) Let \( V = cl(V_1 \cup V_2, M) \cap \lambda \) and set \( N^*_\ell = cl(\{\alpha_\ell\} \cup V,M). \)

**Claim 2.9.** \( N^*_\ell \subseteq N_\ell \) for \( \ell = 1, 2. \)

**Proof.** Fix \( \ell. \) Since \( \alpha_\ell \in N_\ell, \)

\[ V_\ell = cl(\{\alpha_\ell\}, M) \cap \lambda \subseteq N_\ell \cap \lambda. \]

On the other hand, \( N_1 \cap \lambda = N_2 \cap \lambda, \) and hence

\[ V_{3-\ell} = cl(\{\alpha_{3-\ell}\}, M) \cap \lambda \subseteq N_{3-\ell} \cap \lambda = N_\ell \cap \lambda. \]

It follows that \( V_1 \cup V_2 \subseteq N_\ell \cap \lambda, \) and hence

\[ V = cl(V_1 \cup V_2, M) \cap \lambda \subseteq N_\ell. \]

Thus, as \( \{\alpha_\ell\} \cup V \subseteq N_\ell, \) we have

\[ N^*_\ell = cl(\{\alpha_\ell\} \cup V,M) \subseteq N_\ell. \]

The result follows.

**Claim 2.10.** \( \alpha_1 \in N^*_1 \setminus N^*_2 \) and \( \alpha_2 \in N^*_2 \setminus N^*_1 \).

**Proof.** Fix \( \ell \in \{1, 2\}. \) Clearly \( \alpha_\ell \in N^*_\ell. \) On the other hand, by our assumption, \( \alpha_\ell \notin N_{3-\ell}, \) and by Claim 2.9, \( N^*_{3-\ell} \subseteq N_{3-\ell}. \) Thus \( \alpha_\ell \notin N^*_{3-\ell}. \)

Thus \((2)_{\alpha_1, \alpha_2}\) is satisfied.

Conversely suppose that \((3)_{\alpha_1, \alpha_2}\) holds, and for \( \ell = 1, 2, \) set \( N_\ell = cl(\{\alpha_\ell\} \cup V,M). \) By our assumption, clause \((4)\) of \((1)_{\alpha_1, \alpha_2}\) holds.

**Claim 2.11.** For \( \ell \in \{1, 2\}, N_\ell \cap \lambda = V. \)

**Proof.** Fix \( \ell \in \{1, 2\}. \) Clearly \( N_\ell \cap \lambda \supseteq V. \) Now suppose towards a contradiction that \( N_\ell \cap \lambda \neq V \) and let \( \gamma \in N_\ell \cap \lambda \setminus V. \) As \( M \) has Skolem functions, there are \( n, \beta_0, \ldots, \beta_{n-1} \in V \) and \( (n+1)-\)ary function symbol \( F \) in \( \mathcal{L} \) such that

\[ \gamma = F^n(\beta_0, \ldots, \beta_{n-1}, \alpha_\ell). \]

As \( \beta_0, \ldots, \beta_{n-1} \in V \subseteq \lambda \) and \( \gamma < \lambda, \) there is \( i < \kappa \) such that \( \beta_0, \ldots, \beta_{n-1}, \gamma < \lambda_i. \) Define an \( n \)-ary function \( G : \lambda_i \to \lambda_i \) as follows:
Furthermore, the statement

\[ G(\xi_0, \ldots, \xi_{n-1}) = \begin{cases} \mathcal{F}^{\mathcal{M}}(\xi_0, \ldots, \xi_{n-1}, \alpha_\ell), & \text{if } \mathcal{F}^{\mathcal{M}}(\xi_0, \ldots, \xi_{n-1}, \alpha_\ell) < \lambda_i, \\ 0, & \text{otherwise}. \end{cases} \]

Note that \( G \in \{F_{n, \xi} : \xi \in [\lambda_i, 2^{\lambda_i}]\} \). Let

\[ \zeta_* = \min\{\zeta : \forall \xi_0, \ldots, \xi_{n-1} < c_i G(\xi_0, \ldots, \xi_{n-1}) = \mathcal{F}^{\mathcal{M}_0}(\xi_0, \ldots, \xi_{n-1}, \xi, \zeta)\}. \]

\( \zeta_* \) is well-defined and is definable in \( \mathcal{M} \) (even in \( \mathcal{M}_0 \)) from \( \alpha_\ell \), so clearly \( \zeta_* \in cl(\{\alpha_\ell\}, \mathcal{M}) \).

As \( \zeta_* \in cl(\{\alpha_\ell\}, \mathcal{M}) \cap \lambda = V_\ell \subseteq V \) and \( \beta_0, \ldots, \beta_{n-1} \in V \), so

\[ \gamma = \mathcal{F}^{\mathcal{M}}(\beta_0, \ldots, \beta_{n-1}, \alpha_\ell) = \mathcal{F}^{\mathcal{M}_0}(\beta_0, \ldots, \beta_{n-1}) \in V. \]

This contradicts our initial assumption that \( \gamma \in N_\ell \cap \lambda \setminus V \). The claim follows. \( \dashv \)

**Claim 2.12.** \( N_1 \cap \lambda = N_2 \cap \lambda \).

**Proof.** By Claim 2.11, we have \( N_1 \cap \lambda = V = N_2 \cap \lambda \), which concludes the result. \( \dashv \)

By Claim 2.12, \( N_1 \cap \lambda = N_2 \cap \lambda \), which implies clause \((\alpha)\) of \((\dagger)_{\alpha_1, \alpha_2}\). Thus \( N_1 \)
and \( N_2 \) are as required in clause \((\dagger)_{\alpha_1, \alpha_2}\).

This completes the proof of the lemma. \( \dashv \)

§3. UB\_2 can fail at singular cardinals. In this section we prove the following theorem which answers Usuba’s Question 1.1.

**Theorem 3.1.** Assume \( \lambda \) is a singular strong limit cardinal. UB\_2 fails if at least one of the following holds:

\( \lambda = \aleph_\omega \) and Chang’s transfer principle \((\lambda^+, \lambda) \rightarrow (\aleph_1, \aleph_0) \) holds.

\( \lambda > \mu \geq \text{cf}(\lambda) \) are such that \((\lambda^+, \lambda) \rightarrow (\alpha, \lambda) \) holds.

\( \lambda > \mu \geq \text{cf}(\lambda) \) and for every model \( M \) with universe \( \lambda^+ \) and vocabulary of cardinality \( \text{cf}(\lambda) \), we can find an increasing sequence \( \vec{\alpha} = \langle \alpha_i : i < \mu^+ \rangle \) of ordinals less than \( \lambda^+ \) such that

\[ S^{M}_\vec{\alpha} = \{ i < \mu^+ : cl(\{\alpha_i\}, M) \cap \lambda \subseteq cl(\{\alpha_j : j < i\}, M) \} \]

is stationary in \( \mu^+ \).

There exists \( \chi \) with \( \lambda > \chi = \text{cf}(\chi) > \text{cf}(\lambda) \) such that for every model \( M \) with universe \( \lambda^+ \) and vocabulary of cardinality \( \text{cf}(\lambda) \), we can find an increasing sequence \( \vec{\alpha} = \langle \alpha_i : i < \chi \rangle \) of ordinals less than \( \lambda^+ \) such that

\[ S^{M}_\vec{\alpha} = \{ i < \chi : cl(\{\alpha_i\}, M) \cap \lambda \subseteq cl(\{\alpha_j : j < i\}, M) \} \]

is stationary in \( \chi \).

There is no sequence \( \vec{X} = \langle U_i : i < \lambda^+ \rangle \) such that each \( U_i \cap \lambda \) is a cofinal subset of \( \lambda \), \( U_i \cap \lambda \) has size \( \text{cf}(\lambda) \), and for every \( i < \lambda^+ \) there is a sequence \( \vec{X}_i = \langle (\alpha_{i,j}, \beta_{i,j}) : j < i \rangle \) such that:

\( \vec{X}_i \) has no repetition,

\( \alpha_{i,j} \in U_i \),

\( \beta_{i,j} \in U_j \cap \lambda \).

Furthermore, the statement \((e)\) is equivalent to \(-\text{UB\_2} \), provided that \( \text{cf}(\lambda) \) is not a Jonsson cardinal.
Remark 3.2. The assumption ‘‘$\lambda$ is a strong limit cardinal’’ is only used in the proof of (e) implies $\neg\text{UB}_2$.

Proof. We prove the theorem by a sequence of claims. First note that:

Claim 3.3. Clause (a) is a special case of clause (b), and clause (c) implies clause (d).

Claim 3.4. (b) implies (c).

Proof. Let $M$ be a model with universe $\lambda^+$ and vocabulary of cardinality at most $\text{cf}(\lambda)$. By (b), there exists an elementary submodel $N \prec M$ such that $|N| = \mu^+$ and $|N \cap \lambda| = \mu$. Let $\bar{\alpha} = \langle \alpha_i : i < \mu^+ \rangle$ list in increasing order the first $\mu^+$ elements of $N$. So for $i < \mu^+$ we have

$$cl(\{\alpha_i\}, M) \cap \lambda \subseteq N \cap \lambda,$$

and since $N \cap \lambda$ has size $\mu$, we can find some $i^*(*) < \mu^+$ such that

$$\forall i < \mu^+, cl(\{\alpha_i\}, M) \cap \lambda \subseteq \bigcup_{j<i^*} cl(\{\alpha_j\}, M).$$

Hence the set $S^M_{\alpha}$ includes $[i^*(*)]^{\mu^+}$ and so is stationary in $\mu^+$, as requested. $\neg\neg$

Claim 3.5. (d) implies (e).

Proof. Suppose towards a contradiction that (d) holds but (e) fails. As (e) fails, we can find sequences $\bar{X} = \langle U_i : i < \lambda^+ \rangle$ and $\bar{X} = \langle \langle \alpha_{i,j}, \beta_{i,j} : j < i \rangle \rangle$ as in clause (e). Let $M$ be a model in a vocabulary $\mathcal{L}$ such that:

1. $|\mathcal{L}| = \text{cf}(\lambda)$.
2. $M$ has universe $\lambda^+$.
3. $M = (\lambda^+, \langle \tau^M_i : i < \text{cf}(\lambda), H^M \rangle)$, where
   a. $\tau^M_i = i$.
   b. $H^M$ is a 2-place function such that for all $i, U_i \cap \lambda = \{H^M(i, \alpha) : \alpha < \text{cf}(\lambda)\}$.

Now by (d) applied to the model $M$, we can find a sequence $\bar{\zeta} = \langle \zeta_i : i < \chi \rangle$ of ordinals less than $\lambda^+$ such that the set $S^M_{\zeta}$ is stationary in $\chi$. Let $\zeta = \sup_{i<\chi} \zeta_i$. Consider the sequence $\bar{X}_\zeta = \langle (\alpha_{\zeta,j}, \beta_{\zeta,j}) : \xi < \zeta \rangle$.

For $i < \chi$, let

$$W_i = cl(\{\zeta_j : j < i\}, M) \cap \lambda.$$

So $\langle W_i : i < \chi \rangle$ is a $\subseteq$-increasing continuous sequence of sets each of cardinality $< \chi$. Note that for each $i \in S^M_{\zeta}$,

$$\beta_{\zeta,\zeta_i} \in U_{\zeta_i} \cap \lambda \subseteq cl(\{\zeta_i\}, M) \cap \lambda \subseteq W_i.$$

(The former inclusion $\subseteq$ holds because $\text{cf}(\lambda) \cup \zeta_i \subseteq cl(\{\zeta_i\}, M)$ and $cl(\{\zeta_i\}, M)$ is closed under $H^M$. The latter inclusion $\subseteq$ holds because $i \in S^M_{\zeta}$.) Then since $S^M_{\zeta}$ is stationary in $\chi$, there is $\beta_*$ such that

$$U = \{i \in S^M_{\zeta} : \beta_{\zeta,\zeta_i} = \beta_*\}$$
is stationary. Moreover, since \(|U_\xi| = \text{cf}(\lambda) < \chi\), we get some \(i_1 < i_2\) in \(U\) such that 
\[\alpha_{\xi i_1} = \alpha_{\xi i_2}.\]
This contradicts that \(X_\xi\) has no repetition.

**Claim 3.6.** \((e)\) implies \(\neg\text{UB}_\lambda.\)

**Proof.** Suppose not. Thus we can assume that both \((e)\) and \(\text{UB}_\lambda\) hold. Let 
\[f : [\lambda^+]^{<\omega} \rightarrow \lambda^+,\]
which witness \(\text{UB}_\lambda\). Choose a vocabulary \(\mathcal{L}\) of size \(\text{cf}(\lambda)\) and an \(\mathcal{L}\)-model \(M\) such that:

1. \(M\) has universe \(\lambda^+\).
2. \(M\) expands the model \(M_0\) of Lemma 2.8, by expanding \(\mathcal{L}_0\) (the vocabulary of \(M_0\)) using the constant symbols \(\langle d^M_i : i < \text{cf}(\lambda)\rangle\) and the function symbols \(\langle F^M_n : n < \omega\rangle, p^M, G^M_1, G^M_2\), where:
   (a) \(d^M_i = i\) for \(i < \text{cf}(\lambda)\),
   (b) \(F^M_n\) is an \(n\)-ary function such that
   \[F^M_n(\alpha_0, \ldots, \alpha_{n-1}) = f(\{\alpha_0, \ldots, \alpha_{n-1}\}),\]
   (c) \(p^M\) is a pairing function on \(\lambda^+\), mapping \(\lambda \times \lambda\) onto \(\lambda\),
   (d) \(G^M_1\) and \(G^M_2\) are 2-place functions such that for every \(\alpha \in [\lambda, \lambda^+)\),
   \[(G^M_1(\beta, \alpha) : \beta < \alpha)\] ennumerates \(\lambda\) and
   \[\beta < \alpha \land \gamma = G^M_1(\beta, \alpha) \Rightarrow \beta = G^M_2(\gamma, \alpha).\]

By expanding \(M\) further, let us suppose that

3. \(M\) contains Skolem functions.

For \(\alpha < \lambda^+\), set \(N_\alpha = \text{cl}(\{\alpha\}, M)\).

\((*)_1\) \(N_\alpha\) belongs to \([\lambda^+]^{\text{cf}(\lambda)}\) and it contains an unbounded subset of \(\lambda\).

**Proof.** As \(\mathcal{L}\) has size \(\text{cf}(\lambda)\), so \(|N_\alpha| \leq \text{cf}(\lambda)\). On the other hand, by clause (2)(a),
\(\text{cf}(\lambda) \subseteq N_\alpha\) and hence \(N_\alpha\) belongs to \([\lambda^+]^{\text{cf}(\lambda)}\). Also as \(\{c^M_i : i < \text{cf}(\lambda)\} \subseteq N_\alpha\) (see the proof of Lemma 2.8) and \(\langle c^M_i : i < \text{cf}(\lambda)\rangle\) is an unbounded sequence in \(\lambda\), we have \(N_\alpha\) contains an unbounded subset of \(\lambda\).

Let
\[E = \{\delta \in (\lambda, \lambda^+) : \delta = \text{cl}(\delta, M)\}.\]

\(E\) is clearly a club of \(\lambda^+\) and \(E \cap \lambda = \emptyset\). By Lemma 2.8, we have

\((*)_2\) Suppose \(\xi < \zeta\) are in \(E\). Then
\[\xi \in \text{cl}\left(\{\xi\} \cup (N_\zeta \cap \lambda) \cup (N_\xi \cap \lambda), M\right).\]

**Proof.** Suppose by the way of contradiction that \(\xi \notin \text{cl}\left(\{\xi\} \cup (N_\zeta \cap \lambda) \cup (N_\xi \cap \lambda), M\right)\). Let \(V_1 = N_\zeta \cap \lambda, V_2 = N_\xi \cap \lambda\) and \(V = V_1 \cup V_2\). By our assumption,
\[\xi \notin \text{cl}\left(\{\xi\} \cup V, M\right);\]
also, it is clear that
\[\zeta \notin \text{cl}\left(\{\xi\} \cup V, M\right).\]
Thus by Lemma 2.8, we can find submodels \(N_1^*, N_2^*\) of \(M\) such that:
(1) $N_1^* \cap \lambda = N_2^* \cap \lambda$ is unbounded in $\lambda$.
(2) $\xi \in N_1^* \setminus N_2^*$ and $\xi \in N_2^* \setminus N_1^*$.

The models $N_1^*$ and $N_2^*$ are clearly $f$-closed, and by clause (1) above and UB, we have $N_1^* \subseteq N_2^*$ or $N_2^* \subseteq N_1^*$, which contradicts clause (2) above.

Let $(\langle \sigma_i(x_0, \ldots, x_{n(i)}), i < \text{cf}(\lambda) \rangle : i < \text{cf}(\lambda))$ list all terms of $L$. By $(\star)_2$, for each $\xi < \zeta$ from $E$, we can choose some $i(\xi, \zeta) < \text{cf}(\lambda)$ together with sequences $\bar{a}_{\xi, \zeta} \in (N_\xi \cap \lambda)^{< \omega}$ and $\bar{b}_{\xi, \zeta} \in (N_\xi \cap \lambda)^{< \omega}$ such that

\[ (\oplus)_1 \quad \xi = \sigma_{i(\xi, \zeta)}(\zeta, \bar{a}_{\xi, \zeta}, \bar{b}_{\xi, \zeta}). \]

For $\xi \in E$ set $U_\xi = N_\xi = \text{cl}(\{\xi\}, M)$. It follows that $U_\xi = \text{cl}(U_\xi, M)$. For $\xi < \zeta$ use the pairing function $p^M$ to find $\alpha_{\xi, \zeta}$ and $\beta_{\xi, \zeta}$ such that $\alpha_{\xi, \zeta}$ codes $(i(\xi, \zeta), -\bar{a}_{\xi, \zeta}$ and $\beta_{\xi, \zeta}$ codes $\bar{b}_{\xi, \zeta}$.

Now the sequences

\[ \bar{X} = \langle U_\xi : \xi \in E \rangle \]

and

\[ \langle ((\alpha_{\xi, \zeta}, \beta_{\xi, \zeta}) : \xi \in \zeta \cap E) : \xi \in E \rangle \]

witness the failure of (e). We get a contradiction and the claim follows.

Thus so far we have shown that

\[ (a) \implies (b) \implies (c) \implies (d) \implies (e) \implies \neg \text{UB}_2. \]

**Claim 3.7.** Suppose that $\text{cf}(\lambda)$ is not a Jonsson cardinal. Then $\neg \text{UB}_2$ implies (e).

**Proof.** Suppose towards a contradiction that (e) fails and let $\bar{X} = \langle U_\xi : \xi < \lambda^+ \rangle$ and $(\bar{X}_i : i < \lambda^+)$, where $\bar{X}_i = \langle (\alpha_{i,j}, \beta_{i,j}) : j < i \rangle$ as in clause (e) witness this failure. Let $\langle \lambda_i : i < \text{cf}(\lambda) \rangle$ be an increasing sequence cofinal in $\lambda$ and define the function $c : \lambda \to \text{cf}(\lambda)$ as

\[ c(\alpha) = \min\{i < \text{cf}(\lambda) : \alpha < \lambda_i\}. \]

For $\xi < \lambda^+$ let $\langle \gamma_{i,j} : i < \text{cf}(\lambda) \rangle$ enumerate $U_\xi$ such that each element of $U_i$ appears cofinally many often. Let $f : [\lambda^+]^{< \omega} \to \lambda^+$ be such that:

1. If $\xi < \zeta < \lambda^+$, then

\[ f(\alpha_{\xi, \zeta}, \beta_{\xi, \zeta}, \zeta) = \xi. \]

2. If $\zeta < \lambda^+$ and $\alpha < \lambda$, then for arbitrary large $j < \text{cf}(\lambda)$, we have

\[ \sup_{i < j} \lambda_i < \alpha < \lambda_j \implies f(\alpha, \zeta) = \gamma_{\zeta,j}. \]

3. If $A \in [\text{cf}(\lambda)]^{\text{cf}(\lambda)}$, $c(\alpha) = i$ for $i \in A$ and $j < \text{cf}(\lambda)$, then for some $n$ and some sequence $\bar{\xi} = (\xi_0, \ldots, \xi_{n-1}) \in A^n$, we have

\[ j = c(f(\alpha_{\xi_0, \ldots, \xi_{n-1}})). \]
Since \( cf(\lambda) \) is not a Jonsson cardinal, we can define such a function \( f \). Let us show that the pair \( (f, c) \) witnesses \( UB_{\langle f, c \rangle} \) holds, which contradicts our assumption. To see this, suppose \( x, y \subseteq \lambda^+ \) are closed under \( f \), \( x \cap \lambda = y \cap \lambda \) and \( \sup(x \cap \lambda) = \lambda \). Assume towards a contradiction that \( x \nsubseteq y \) and \( y \nsubseteq x \). Let \( \zeta = \min(x \setminus y) \) and \( \zeta = \min(y \setminus x) \), and let us suppose that \( \zeta < \zeta \).

By clause (3), \( cf(\lambda) \subseteq y \), and then by clause (2), and since \( y \cap \lambda \) is cofinal in \( \lambda \), we have \( U \zeta \subseteq y \). Similarly \( U \zeta \subseteq x \). As \( x \cap \lambda = y \cap \lambda \) and \( U \zeta \subseteq \lambda \), we conclude that \( U \zeta \subseteq y \) as well. Thus by item (1), and since \( \alpha, \beta, \zeta \in y \) we have \( \zeta \in x \setminus y \). This completes the proof of the claim. \( \dashv \)

The theorem follows.

**Remark 3.8.** The above proof shows that the following are equivalent:

(1) clause (e) of Theorem 3.1,

(2) for each model \( M \) with universe \( \lambda^+ \) and vocabulary of cardinality \( cf(\lambda) \), there are substructures \( N_0, N_1 \) of \( M \) such that \( N_0 \cap \lambda = N_1 \cap \lambda \), \( N_0 \nsubseteq N_1 \), and \( N_1 \nsubseteq N_0 \).

As we noticed earlier, it is consistent relative to the existence of large cardinals that Chang’s transfer principle \( (\aleph_{\alpha+1}, \aleph_\alpha) \rightarrow (\aleph_1, \aleph_0) \) holds with \( \aleph_\alpha \) being strong limit. Hence by our main theorem, we have the following corollary.

**Corollary 3.9.** It is consistent, relative to the existence of large cardinals, that \( UB_{\aleph_\alpha} \) fails.

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\(^2\)This assumption is used to guarantee clause (3) in definition of \( f \) holds.

\(^3\)We can define a function \( \tilde{f} : [\lambda^+]^{<\aleph_0} \rightarrow \lambda^+ \) which codes \( (f, c) \) so that a set is closed under \( \tilde{f} \) if and only if it is closed under both of \( f \) and \( c \).
