Disaggregation of Long Memory Processes on $C^\infty$ Class

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Abstract

We prove that a large set of long memory (LM) processes (including classical LM processes and all processes whose spectral densities have a countable number of singularities controlled by exponential functions) are obtained by an aggregation procedure involving short memory (SM) processes whose spectral densities are infinitely differentiable ($C^\infty$). We show that the $C^\infty$ class of spectral densities is the optimal class to get a general result for disaggregation of LM processes in SM processes, in the sense that the result given in $C^\infty$ class cannot be improved taking for instance analytic functions instead of indefinitely derivable functions.

Keywords: Aggregation; disaggregation; long memory process; mixture.

1 Introduction

LM processes, are used in many fields such as economics, finance, hydrology or communication networks. Let $\gamma$ be the covariance function of the process and

$$\|\gamma\|_{SM} = \sum_{k=0}^{\infty} |\gamma(k)|.$$

If $\|\gamma\|_{SM} < \infty$ then we say that the process is SM and if $\|\gamma\|_{SM} = \infty$, we say that the process is LM.

The long memory is generally associated to the singularities of the spectral density $f$. 
Some of these LM processes can be seen as an aggregation, that is a normalized sum of elementary SM processes, see [4, 5, 7, 8, 10].

All important examples of LM in finance, hydrology or communication networks etc. are obtained by a model of aggregation, perhaps more complicated that the model used here. Our result says that one has almost always the possibility to think the LM macroprocess as an aggregation of elementary processes with $C^\infty$ spectral density.

The procedure of aggregation can be developed considering doubly random elementary processes $Z^i = \{Z_i(t, Y^i), t \in T\}$ centered second order and stationary, where $Y = \{Y^i, i \in \mathbb{N}\}$ is a sequence of i.i.d. random variables with distribution $\mu$ on $\mathbb{R}^s$; $T = \mathbb{Z}$ in the case of discrete processes and $T = \mathbb{R}$ in the continuous case, see [2].

For every fixed trajectory of $Y$, we define the sequence of partial aggregations $X^N = \{X^N_t(Y), t \in T\}$, of elementary processes $\{Z^i\}$, by

$$X^N_t(Y) = \frac{1}{B_N} \sum_{i=1}^N Z_i(t, Y^i),$$

where $B_N$ is a normalization sequence of positive numbers. Under some general conditions, for almost every trajectory $Y$, $X^N$ converges in distribution to the same Gaussian process $X$, which is called the aggregation of the elementary processes $\{Z^i\}$.

Equivalently, the aggregation procedure can be also developed using mixtures of spectral densities as main tool. Let $g(\lambda, y), \lambda \in I, y \in \mathbb{R}^s$, be a spectral densities family, where $I = (-\pi, \pi]$ in the case of discrete processes and $I = \mathbb{R}$ in the continuous case. We consider, for $Y^i = y$ fixed, that $g(\lambda, y)$ is the spectral density of the elementary process $Z^i(Y^i)$. Let $\mu$ be a probability on $\mathbb{R}^s$, the $\mu$-mixture of the spectral densities $g(\lambda, y)$, is defined by

$$f(\lambda) = \int_{\mathbb{R}^s} g(\lambda, y)d\mu(y).$$

$f(\lambda)$ is a well defined spectral density iff

$$\int_I f(\lambda)d\lambda < \infty.$$  

The sequences of processes $\{X^N, N \in \mathbb{N}\}$, defined in (1), is now the partial aggregation sequence of the elementary processes $\{Z^i\}$ associated to the mixture $f(\lambda)$. 

Under condition (3) in [2] it is shown that if the elementary processes are independent then the aggregation exists $\mu - a.s.$ and its spectral density function is $f(\lambda)$. Furthermore, the case of non-independent elementary processes is studied considering an interactive correlation between elementary processes.

Disaggregation is the inverse procedure of aggregation. Let $\mathcal{G}$ be a class of spectral densities and let $\mathcal{M}(\mathcal{G})$ denote the set of mixtures of spectral densities belonging to $\mathcal{G}$. We say that a process with spectral density $f(\lambda)$ can be disaggregated into elementary processes with spectral densities in $\mathcal{G}$ if $f \in \mathcal{M}(\mathcal{G})$; equivalently by disaggregation in $\mathcal{G}$, we mean the existence of a representation of a given process as an aggregation of elementary processes belonging to $\mathcal{G}$. The disaggregation problem is then equivalent to the following question: When do we have $f \in \mathcal{M}(\mathcal{G})$ for $f$ and $\mathcal{G}$ given? In this work, we consider the procedure of aggregation only for independent elementary processes. A known example is the disaggregation of the FARIMA($d$) process on $\mathcal{G} = \mathbb{A}\mathbb{R}(1)$ the class of autoregressive processes of order 1, see [8]. A more general development of the disaggregation in the class of $AR(p)$ processes is given in [3] where we also show results of disaggregation, for continuous processes, in the class of $p$-order Ornstein-Uhlenbeck processes, $OU(p)$.

Our purpose is to study the disaggregation procedure in SM processes and on subclasses $\mathcal{G}$ for which the covariances decrease as fast as possible. We prove that a large set of LM processes, whose spectral densities have singularities of different kinds, are obtained by an aggregation procedure involving processes whose spectral densities are infinitely differentiable. There is no unicity for this procedure. Classical LM processes are included in this set.

Conditions for the existence of a disaggregation on $\mathbb{A}\mathbb{R}(p)$, $OU(p)$ classes or on $\mathcal{A}$, set of analytical spectral densities, are very restrictive. They are related, as in the $\mathbb{A}\mathbb{R}(1)$ case, to specific algebraic properties of $f$, for instance to be a Mellin transform, and the elementary processes have a spectral density roughly of the form $\Phi(\gamma f(\lambda))$ where the regularity of $\Phi$ is sufficient in order to destroy the singularity of $f$.

We show that the class $C^\infty$ of indefinitely differentiable spectral densities is the right class to get a general result for disaggregation of LM processes in SM processes. For $f \in C^\infty$, the respective
covariance function decreases faster than $n^{-j}$ for any $j$. This decay can measure the level of SM comparatively to that of $\mathcal{A}$ which is exponential. Disaggregation on $\mathcal{C}^\infty$ is linked to very weak analytical properties much easier to satisfy by $f$ than the algebraic properties required by analytic disaggregation and the elementary densities are here roughly of the form $g(\lambda, y) = f(\lambda)K(\lambda, y)$, where $K$ is a kernel such that at each singularity $\lambda_0$ of $f$ we have $f(\lambda_0)K(\lambda_0, y) = 0$ as well as for all its derivatives. The $\mathcal{C}^\infty$ behavior of $K$ drives the $\mathcal{C}^\infty$ behavior of $g(\lambda, y)$. Then we show examples of $\mathcal{C}^\infty$ functions that cannot be disaggregated on the class $\mathcal{A}$. On the other hand, for very rough densities, we give examples of LM processes which cannot be the aggregation of short memory processes. We shall call such a situation as "hard" long memory. This result is a good illustration of "how much singularity have to be concentrated" in order to generate LM by aggregation.

2 Condition for the Existence of Disaggregation in SM

Let us begin with some remarks about mixtures and LM property giving rough necessary conditions for existence of a disaggregation on SM densities. We use only the fact that every SM density is continuous, (in fact sufficient conditions for SM property are for instance to be $\alpha$-lipchitzian for every $\alpha$, $0 < \alpha < 1$, or to be $\alpha$-lipchitzian for some $\alpha > \frac{1}{2}$ and of bounded variation).

Lemma 1. A spectral density is a mixture of continuous spectral densities iff it is lower semicontinous (l.s.c.).

Proof. If $f$ is such a mixture, applying Fatou’s lemma we see that it is l.s.c. Conversely, if $f$ is l.s.c. and positive, it exists, a sequence $g_n$ of continuous functions such that $f = \sup_n g_n$. Taking the supremum until $n$ we can choose $g_n$ increasing. So $f = \sum_{n \geq 1} 2^{-n} (2^n (2^{n+1} - g_n))$ which is of the form $\int g \, d\mu$. □

From the lemma, we see that every non l.s.c. spectral density has LM property and cannot be disaggregated on SM class. For instance, if $h$ is the function equal to one on a perfect set, $\mathcal{F}$, of strictly positive Lebesgue measure, then $h$ is upper semicontinuous (u.s.c.) but non l.s.c. and it is the density of a absolutely continuous probability with respect to the Lebesgue measure. This provides
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an example of a situation that we can call "hard" LM process which cannot be disaggregated by SM processes. In fact, we are very closed to the case of a non absolutely continuous spectral measure.

3 Disaggregation of LM Processes on $C^\infty$ Class

Long memory of a process with spectral density $f$ is in general associated to singularities of $f$ or of some of its derivatives at a frequency $\lambda_0$. Singularities are often classified as a first order when one side limit exists and second order singularity when the function tends to $\infty$ as $\lambda$ tends to $\lambda_0$ or it does not exist any limit at $\lambda_0$ (the function being bounded or not, with bounded variation or not).

We try to take into account most of these situations. Our main purpose is to obtain for a class as broad as possible, including all classical examples but not limited to more or less explicit densities, a disaggregation on elementary processes with the best possible decay of correlations. We are lead to work mainly with $G = C^\infty$ (resp. $C^H$) the class of all spectral densities belonging to $C^\infty$ (resp. $C^H$); an equivalent property is that the correlation function of $f \in C^\infty$ tend to 0 faster than $1/n^j$ for any $j \in \mathbb{N}$ (resp. faster than $c/n^H$ for some $c > 0$). For $H$, $1 \leq H \leq \infty$, given we get for a function $f \in C^H$ except for a finite or countable set of frequencies, a disaggregation on a $C^H$ class. It seems that disaggregation on the class $C^\infty$ is easier to reach than a disaggregation on the SM class.

In order to extend the qualitative situation analyzed for the case $G = A \mathbb{R}(1)$, in [3], we introduce the following definition.

**Definition 1.** Let $\Lambda \subset I$ be a finite set of frequencies, we define $F^H_\Lambda$ as the set of spectral densities in $C^H$, $1 \leq H \leq \infty$, which are 0 on $\Lambda$ as all their $H$ derivatives. If $\Lambda = \{0\}$ we denote $F^H_\Lambda$ by $F^H_0$.

Let now $K$ be a positive kernel, for instance on $\mathbb{R}^+$, so $\int_0^\infty K(y)dy = 1$ and let $\phi$ be a positive function on $I$. We use the trivial relation

$$f(\lambda) = \int_0^\infty g(\lambda,y)dy$$

where $g(\lambda,y) = f(\lambda)\phi(\lambda)K(\phi(\lambda)y)$. (4)

The pair $[K;\phi]$ will be called a killer kernel (it kills the singularities). The generic situation is given by the formula $f(0)\phi(0)K(\phi(0)y) = \lim_{\lambda \to 0} f(\lambda)\phi(\lambda)K(\phi(\lambda)y) = 0$ for every $y \in \mathbb{R}^+$, even if
Let \( f(0) = \infty \). So \( f \) is the dy mixture of \( g(\lambda, y) \in F_0^H \). We can extend the definition of mixtures of spectral densities, given in (4), taking for \( \mu \) a non bounded measure, as the Lebesgue measure dy, in this case we can consider that we take a strictly positive density of probability \( \sigma(y) \) and then we consider the mixtures of \( \sigma^{-1}(y)g(\lambda, y) \) by the probability \( \sigma(y)dy \).

Let us give some examples of disaggregation of a function \( f \) with a single singularity at \( \lambda_0 \). We consider the standard situation where \( f \) and its derivatives are explicitly controlled by exponential functions. The key to solve our problem is the representation of \( f \) given in (1).

**Example 1.** Let \( f(\lambda) = 1_{(-\lambda_0, \lambda_0)}(\lambda) \) for \( \lambda_0 \in I, \phi(\lambda) = 1/|\lambda^2 - \lambda_0^2|^p \) with \( 0 < p < 1 \), and \( K(y) = e^{-y} \).

In this case is easy to check that \( g(\lambda, y) \in F_{-\lambda_0, \lambda_0}^\infty \), since all derivatives of \( g \) are 0 for \( |\lambda| = \lambda_0 \).

**Example 2.** Let \( f(\lambda) = |1 - \cos(\lambda - \lambda_0)|^{-d}, 0 < d < 1, \lambda \in I = (-\pi, \pi] \). We keep \( \phi(\lambda) = 1/|\lambda - \lambda_0|^p \) with \( 0 < p < 1 \), and \( K(y) = e^{-y} \). All derivatives of \( f \) at \( \lambda = \lambda_0 \) are controlled by a negative power of \( |\lambda - \lambda_0| \) and so \( g(\lambda, y) \in F_{-\lambda_0, \lambda_0, y}^\infty \). The same properties can be easily checked for a strongly oscillating function \( f \) as \( \cos(\pi(\lambda - \lambda_0))/|\lambda - \lambda_0|^q \), for \( 0 < q < 1 \). So for these kinds of controlled singularities, we show that \( f \in M(F_{-\lambda_0, \lambda_0, y}^\infty) \subset M(C^\infty) \).

**Definition 2.** Let \( f \) be a spectral density. We say that \( f \in S^H, 1 \leq H \leq \infty \), iff \( f \) has a continuous \( H \) derivative at every frequency except for a finite set of frequencies \( \Lambda = \{\lambda_j, j \in J\} \) and if it exists \( q, 0 < q < 1, \) and \( a, 0 < a < 1, \) such that for all \( j \in J \) and for all \( l \leq H \)

\[
\lim_{\lambda \to \lambda_j} \exp \left( -\frac{a}{|\lambda - \lambda_j|^q} \right) |f^{(l)}(\lambda)| = 0.
\]

If \( \Lambda \) is a countable infinite set instead of finite and has only a finite number of accumulation points, then we say that \( f \in T^H \).

We state now a theorem for the standard situation.

**Theorem 1.** Let \( f \in S^H, 1 \leq H \leq \infty \), then \( f \in M(F_{-\lambda_0, \lambda_0, y}^\infty) \subset M(C^\infty) \).

**Proof.** Let \( \phi(\lambda) = \left( \prod_{j \in J} |\lambda - \lambda_j|^p \right)^{-1}, K(y) = e^{-y} \) and \( g(\lambda, y) = f(\lambda)\phi(\lambda)K(\phi(\lambda)y) \). Then \( f(\lambda) = \int_0^\infty g(\lambda, y)dy \). We choose \( p \) such that \( 0 < q < p < 1 \). If \( \Psi(\lambda, y) = \phi(\lambda)K(\phi(\lambda)y) \) then for all the \( l \)-derivatives of \( g, l \leq H \), we have that it exists constants \( b_i, C_i \) and \( m_i \) such that

\[
|g^{(l)}(\lambda, y)| = \sum_{k=0}^{l} C_{k,l} f^{(k)}(\lambda, y) \psi^{(l-k)}(\lambda, y) \leq C_l |\Psi(\lambda, y)|^{m_i} e^{-\sum_{j \in J} \frac{a_k}{|\lambda - \lambda_j|^q}}.
\]
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So $g(\lambda, y) \in F^H_\lambda$ and $f \in \mathcal{M}(F^H_\lambda) \subset \mathcal{M}(C^H)$.

**Example 3.** If $f(\lambda) = |\lambda|^{-d}$, $\lambda \in I = \mathbb{R}$, $0 < d < 1$, $f$ is the spectral density of continuous fractional Brownian motion. Then $|f^{(l)}(\lambda)| = O(1/|\lambda|^{d+l})$ and the conditions of Theorem I are satisfied.

Next theorem is an extension of Theorem I but using a localization required if $\Lambda$ is infinite.

**Theorem 2.** Theorem I remains valid if $f \in T^H$.

**Proof.** Suppose, in order to simplify notations, that $\Lambda$ is an infinite countable set with only one accumulation point. The general case can be easily obtained by re-indexing of $\Lambda$ points using the partition of $I$ defined by the points of accumulation of $\Lambda$ and then applying the same proof. So we can suppose $\Lambda = \{\lambda_j, j \geq 1\}$, with $\lambda_j < \lambda_{j+1}$ for every $j \in \mathbb{N}$.

We build in the same spirit than previously a family of functions $g^\Lambda(\lambda, y)$, multiplying $f$ by a killer kernel $[K; \phi^\Lambda](\lambda, y)$ that annihilates the points of discontinuity of $f$.

Let us note $a = \inf I$, $b = \sup I$ and $\lambda_\infty = \lim_{j \to \infty} \lambda_j = \sup_j \lambda_j$. Let be $p$ such that $0 < q < p < 1$, and we consider

\[
[K, \phi^0](\lambda, y) = \frac{1}{|\lambda - \lambda_1|^p} \exp \left(-\frac{y}{|\lambda - \lambda_1|^p}\right) 1_{(a, \lambda_1)}(\lambda),
\]

\[
[K, \phi^j](\lambda, y) = \frac{1}{|\lambda - \lambda_j|^p|\lambda - \lambda_{j+1}|^p} \exp \left(-\frac{y}{|\lambda - \lambda_j|^p|\lambda - \lambda_{j+1}|^p}\right) 1_{(\lambda_j, \lambda_{j+1})}(\lambda),
\]

\[
[K, \phi^\infty](\lambda, y) = \frac{1}{|\lambda - \lambda_\infty|^p} \exp \left(-\frac{y}{|\lambda - \lambda_\infty|^p}\right) 1_{(\lambda_\infty, b)}(\lambda).
\]

Then we define

\[
g^\Lambda(\lambda, y) = f(\lambda)[K; \phi^\Lambda](\lambda, y),
\]

and

\[
g^\Lambda(\lambda, y) = f(\lambda)[K; \phi^\Lambda](\lambda, y).
\]

We have that,

\[
\int_0^\infty [K; \phi^\Lambda](\lambda, y) dy = \int_0^\infty e^{-z} dz \left(1_{(a, \lambda_1)} + \sum_{j \geq 1} 1_{(\lambda_j, \lambda_{j+1})} + 1_{(\lambda_\infty, b)}\right) = 1.
\]

So $V(y) = \int_I g^\Lambda(\lambda, y) d\lambda$ and by applying Fubini’s theorem $\int_0^\infty V(y) dy = \int_I f(\lambda) d\lambda = \gamma_0 < \infty$. 

We can prove, by using the same proof as for Theorem 1, that the $H$ derivatives with respect to $\lambda$ of $g^\lambda(\lambda,y)$ go to 0 when $\lambda \to \lambda_j$, since $q < p$. So $g^\lambda(\lambda,y) \in F^H_\lambda$.

**Remark 1.** Killer kernels $[K;\phi]$ selected to build only the mixtures are never the best ones, for covariances decay. For instance, we can take $\exp(-\exp(y))$ instead of $\exp(-y)$ getting covariances decreasing to 0 slightly faster and so on.

**Remark 2.** We can define mixtures of Wold regular densities, in the Wold Theorem sense, which verify the following condition

$$\int \log g(\lambda,y)d\lambda > -\infty, \quad \mu - a.s. \quad (5)$$

In this case, we say that the processes with spectral densities $g(\lambda,y)$ are regular. If $f(\lambda)$ is the mixture of the densities $g(\lambda,y)$, condition (5) does not imply that $\int \log f(\lambda)d\lambda > -\infty$. But if $f$ is regular then $g(\lambda,y)$ is regular $\mu - a.s.$ by Jensen’s inequality. The main point on this topic is that we can choose the killer kernel $[K;\phi]$ such that if $f$ is Wold regular then all the elementary processes used in the aggregation are also Wold regular.

### 4 Analytic spectral densities

Let us prove, in a certain sense, that the previous results cannot be improved taking analytic functions instead of indefinitely derivable functions.

Disaggregation is a hierarchical procedure: if $f \in M(G)$ and $G \subset M(H)$ then $f \in M(H)$, in fact if $g(\lambda,y) = \int h(\lambda,z)d\nu(y,z)$ then

$$f(\lambda) = \int g(\lambda,y)d\mu(y) = \int \int [h(\lambda,y,z)d\nu(y,z)]d\mu(y).$$

In general we have $G \subset M(G) \subset \overline{M(G)}$ with strict inclusion, the closure being taken in $L^1(d\lambda)$. The obvious exception is $G = MA(q)$, the set of $q$-moving average densities, for which $MA(q) = M(MA(q)) = \overline{M(MA(q))}$.

We use this hierarchical procedure in order to show that our result can not be improved in the following sense: we cannot take analytic functions instead of $C^\infty$. So we have to check that the
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functions we have used in $C^\infty$, as $f(\lambda) = \frac{1}{|\lambda|^p} \exp(-\frac{1}{|\lambda|^p})$, do not belong to $M(A)$, in order to show that our result can not be improved.

Let $f \in C^\infty_{0, +}$, where $C^\infty_{0, +} = \{ f : f \geq 0, f \in C^\infty, \text{it exists } \lambda_0 \text{ such that } f^{(j)}(\lambda_0) = 0 \text{ for every } j \in \mathbb{N} \}$.

Suppose $f(\lambda) = \int_{\mathbb{R}^+} g(\lambda, y) d\mu(y)$ with $g \in A \mu - a.s.$ The Fatou’s Lemma implies

$$f^{(j)}(\lambda_0) \geq \int_{\mathbb{R}^+} g^{(j)}(\lambda_0, y) d\mu(y),$$

and $f^{(j)}(\lambda_0) = 0$ implies that, if $g^{(j)}(\lambda_0, y) \geq 0 \mu - a.s.,$ then $g^{(j)}(\lambda_0, y) = 0 \mu - a.s.$ From $g(\lambda, y) \geq 0 \mu - a.s.$ we get $g(\lambda_0, y) = 0 \mu - a.s.$ and $g^{(1)}(\lambda_0, y) \geq 0 \mu - a.s.$, and so $g^{(1)}(\lambda_0, y) = 0 \mu - a.s.$ and $g^{(2)}(\lambda_0, y) \geq 0 \mu - a.s.$ By induction we have that $g^{(j)}(\lambda_0, y) = 0 \mu - a.s.$ and $\mu \{ y, g(\lambda, y) \in A \} = 0.$

We have proved that $f(\lambda) \not\in M(A)$.

Disaggregation in the class of analytic functions is a very limited possibility. A slight modification of the proof just above shows that spectral densities which are constant or linear or polynomial (of given degree) by pieces cannot be in $M(A)$ except if they are themselves elements of $A$, that is, if they are polynomials.

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