Mean field frozen percolation

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Abstract

We define a modification of the Erdős-Rényi random graph process which can be regarded as the mean field frozen percolation process. We describe the behavior of the process using differential equations and investigate their solutions in order to show the self-organized critical and extremum properties of the critical frozen percolation model. We prove two limit theorems about the distribution of the size of the component of a typical frozen vertex.

1 Statements

The frozen percolation process on a binary tree was defined by D. J. Aldous in [2]: it is a modification of the percolation process which makes the following informal description mathematically rigorous: we only occupy an edge if both end-vertices are in a finite cluster. The self-organized critical property of this model manifests in the fact that for \( t \geq \frac{1}{2} \), which is the critical time of the corresponding percolation process, a typical finite cluster has the distribution of a critical percolation cluster.

I. Benjamini and O. Schramm showed that it is impossible to define a similar modification of the percolation process on \( \mathbb{Z}^2 \). An explanation of this non-existence result can be found in Section 3. of [7].

First we give an informal description of the mean field frozen percolation process: It is a modification of the Erdős-Rényi random graph process: Initially we have a (not necessarily empty) graph on \( \lfloor N \cdot m_0(0) \rfloor \) vertices (one should think about \( N \) as being large, but the initial mass \( m_0(0) \) is fixed), and between every possible pair of vertices, edges appear with rate \( \frac{1}{N} \). Simultaneously lightnings strike vertices with rate \( \lambda(t) \mu(N) \) at time \( t \) and when a vertex is struck, the fire spreads along the edges and burns the connected component of that vertex: that subgraph is removed from the graph, including vertices. Thus the number of vertices of the random graph decreases with time. The expressions "burnt", "frozen", "deleted" and "removed" are treated as synonyms in the sequel.

If \( \nu_{k}^N(t) \) denotes the number of vertices contained in components of size \( k \) in the random graph at time \( t \), then the vector-valued stochastic process \( \underline{\nu}(t) = (\nu_1^N(t), \nu_2^N(t), \ldots) \) also has the Markov property (the main advantage of the mean field model is that the graph structure of the connected components has no effect on the evolution of component sizes). We are interested in the model when \( 1 \ll N \).

Denote by \( \mathbb{N} = \{1, 2, \ldots\} \) and \( \mathbb{N}_0 = \{0, 1, 2, \ldots\} \).
Definition 1.1. We fix $m_0(0) \in \mathbb{R}^+$. The mean field frozen percolation process on $N$ vertices is a continuous time Markov process with state space

$$\Omega_N = \{ \psi \in \mathbb{N}_0^N : \sum_{k \geq 1} \psi_k \leq \lfloor N \cdot m_0(0) \rfloor, \forall k \frac{\psi_k}{N} \in \mathbb{N}_0 \}$$

We define the coagulation and deletion operators

$$q_{k,l}^+ := \begin{cases} (\psi_1, \psi_2, \ldots, \psi_k - k, \ldots, \psi_l - l, \ldots, \psi_{k+l} + k + l, \ldots) & \text{if } k < l \\ (\psi_1, \psi_2, \ldots, \psi_k - 2k, \ldots, \psi_{2k} + 2k, \ldots) & \text{if } k = l \end{cases}$$

$$q_{k}^- := (\psi_1, \ldots, \psi_k - k, \ldots)$$

Let $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ be a positive continuous function and $\mu : \mathbb{N} \to \mathbb{R}^+$. The transition rates of the Markov process are

$$\lambda(\psi \to q_{k,l}^+) = \begin{cases} \frac{1}{N} \cdot \psi_k \cdot \psi_l & \text{if } k < l \\ \frac{1}{N} \cdot \psi_k (\psi_k - k) & \text{if } k = l \end{cases}$$

$$\lambda(\psi \to q_{k}^-) = \lambda(t) \cdot \mu(N) \cdot \psi_k$$

Let $v_k(t) := \frac{\psi_k(t)}{N}$ denote the mass of components of size $k$ at time $t$.

The mean field frozen percolation model is closely related to the mean field forest fire model (discussed in [6]), the only difference in the definition of the Markov process is that in the case of the forest fire model, a burnt component of size $k$ is replaced by $k$ isolated vertices, so that the number of vertices in the random graph remains unchanged. The two models both have the self-organized critical property (and we believe that they are in the same universality class, which means that the theorems of this paper have analogous “forest fire” versions), but the corresponding partial differential equations have an explicit solution in the case of the frozen percolation model which enables us to say more about this model.

$$V := \{ v = (v_k)_{k=1}^{\infty} : v_k \in \mathbb{R}, v_k \geq 0 \text{ and } \sum_{k=1}^{\infty} v_k < \infty \}$$

$$V^* := \{ v : v \in V, \exists K < +\infty \forall k \geq K \ \ v_k = 0 \}$$

Definition 1.2. We consider a sequence of mean field frozen percolation processes with $N \to \infty$, but with the initial state

$$\mathbf{v}(0) = (v_1^N(0), v_2^N(0), \ldots, v_K^N(0), 0, 0, \ldots) = \left( \frac{\psi_1^N(0)}{N}, \frac{\psi_2^N(0)}{N}, \ldots, \frac{\psi_K^N(0)}{N}, 0, 0, \ldots \right) \in V^*$$

and the lightning rate function $\lambda(t)$ fixed (independently of $N$). Such a sequence is called

- subcritical if $\mu(N) \equiv 1$
• critical if $\frac{1}{N} \ll \mu(N) \ll 1$
• alternating if $\mu(N) = \frac{1}{N}$.

If $v_k(0) = I_{\{k=1\}} \cdot m_0(0)$ then the initial state is called monodisperse, otherwise it is polydisperse.

We are going to describe the time evolution of the limit object

$$\lim_{N \to \infty} v_k^N(t) = v_k(t).$$

(5)

We introduce differential equations to characterize the limiting component size distributions $v_k(t)$ where $k \in \mathbb{N}$ and $t \in \mathbb{R}_+$. They are modifications of the Smoluchowski coagulation equation with multiplicative rate kernel:

$$\dot{c}(t) = \frac{1}{2} \sum_{l=1}^{k-1} l \cdot (k - l) \cdot c_l(t) \cdot c_{k-l}(t) - c_k(t) \sum_{l=1}^{\infty} l \cdot c_l(0)$$

Flory’s model

(6)

$$\dot{c}(t) = \frac{1}{2} \sum_{l=1}^{k-1} l \cdot (k - l) \cdot c_l(t) \cdot c_{k-l}(t) - c_k(t) \sum_{l=1}^{\infty} l \cdot c_l(t)$$

Stockmayer’s model

(7)

If we let $v_k(t) = k \cdot c_k(t)$ then (6) becomes

$$\dot{v}_k(t) = \frac{k}{2} \sum_{l=1}^{k-1} v_l(t) v_{k-l}(t) - k \cdot v_k(t) \cdot \sum_{l=1}^{\infty} v_l(0)$$

(8)

We are going to use the formulation (8) rather than the classical (6).

The differential equations (8) describe the time evolution of $(v_k(t))_{k=1}^{\infty}$ defined by (5) for the dynamical Erdős-Rényi random graph process (see [1]). If we only look at the evolution of the component size vector $\mathbf{v}(t)$ in the dynamical Erdős-Rényi random graph model, we get the Marcus-Lushnikov process (see [5]) with multiplicative kernel which is the $\mu(N) \equiv 0$ case of our model (no deletions, only coagulations).

Definition 1.3. If $(v_k)_{k=1}^{\infty} = \mathbf{v} \in \mathbf{V}$ let

$$m_0 := \sum_{k=1}^{\infty} v_k \quad m_1 := \sum_{k=1}^{\infty} k v_k \quad m_2 := \sum_{k=1}^{\infty} k^2 v_k \quad m_3 := \sum_{k=1}^{\infty} k^3 v_k$$

Remark 1. Our definition of the moments $m_n$ differs from the convention of the literature of the Smoluchowski equation by a shift of indices.

If we define

$$w_k^N(t) := \sum_{l=1}^{k} v_l^N(t) \quad \text{and} \quad \Phi^N(t) := \sum_{l=1}^{\infty} v_l^N(0) - \sum_{l=1}^{\infty} v_l^N(t) = m_0^N(0) - m_0^N(t)$$

(9)
then for all \( k \) the random function \( w_k^N(t) \) is decreasing and \( \Phi^N(t) \) (the mass of burnt vertices) is increasing.

It might happen (e.g. in the case of the Erdős-Rényi model) that

\[
\theta(t) := \lim_{k \to \infty} \lim_{N \to \infty} (m_0^N(t) - w_k^N(t)) \neq \lim_{N \to \infty} \lim_{k \to \infty} (m_0^N(t) - w_k^N(t)) = 0.
\]

In this case the mass missing from the small components is contained in a giant component of mass \( 0 < \theta(t) \).

**Definition 1.4.** If \( \psi(t) \) is a solution of (8), we define the gelation time by

\[
T^g := \inf \{ t : m_1(t) = +\infty \}.
\]

It is well-known from the theory of the Smoluchowski coagulation equation that an alternative characterisation of the gelation time is

\[
T^g = \inf \{ t : m_0(t) < m_0(0) \}.
\]

For the solution of (8) the gelation time is \( T^g = \frac{1}{m_0(0)} \), the mass of the giant component is \( \theta(t) = m_0(0) - m_0(t) \). \( \psi(t) \) undergoes a phase transition:

- For \( 0 \leq t < T^g \) the system is subcritical: \( \theta(t) = 0 \) and \( k \mapsto v_k(t) \) decay exponentially with \( k \).
- For \( T^g < t \) the system is supercritical: \( \theta(t) > 0 \) and \( k \mapsto v_k(t) \) decay exponentially with \( k \). Further on: \( t \mapsto \theta(t) \) is smooth and strictly increasing with \( \lim_{t \to \infty} \theta(t) = m_0(0) \).
- Finally, at \( t = T^g \) the system is critical: \( \theta(t) = 0 \) and

\[
\sum_{k=K}^{\infty} v_k(T^g) \asymp K^{-1/2} \quad \text{as} \quad K \to \infty.
\]

Our aim is to understand in similar terms the asymptotic behavior of the system when, beside the Erdős-Rényi coagulation mechanism, deletions due to lightnings also take place.

**Definition 1.5.** We say that \( \psi(t) = (v_k(t))_{k=1}^\infty \in V \) solves the general frozen percolation equation on \([0,T]\) with initial condition \( \psi(0) \in V^\ast \), a continuous nonnegative rate function \( \lambda : \mathbb{R}_+ \to \mathbb{R}_+ \) and control function \( \Phi : \mathbb{R}_+ \to \mathbb{R}_+ \) if

\[
\forall 0 \leq s \leq t \leq T \quad 0 \leq \Phi(0) \leq \Phi(s) \leq \Phi(t) < m_0(0)
\]

and for all \( k = 1, 2, \ldots \) the equations

\[
v_k(t) = v_k(0) + \int_0^t \frac{k-1}{2} \sum_{l=1}^{k-1} v_l(s)v_{k-l}(s) - kv_k(s) ((m_0(0) - \Phi(s)) + \lambda(s)) ds \quad (12)
\]

and the inequality

\[
\forall t \quad 0 \leq \theta(t) := m_0(0) - m_0(t) - \Phi(t)
\]

is satisfied.
It is easy to see by induction that the absolutely continuous functions \( v_1(t), v_2(t), \ldots \) are completely determined by (12), the initial condition \( v(0) \) and the functions \( \lambda \) and \( \Phi \). The only reason why we do not write
\[
\dot{v}_k(t) = \frac{k^{k-1}}{2} \sum_{l=1}^{k-1} v_l(t) v_{k-l}(t) - k v_k(t) \left( (m_0(0) - \Phi(t)) + \lambda(t) \right)
\] instead of (12) is that the increasing function \( \Phi(t) \) might have jumps.

There are three versions of the general frozen percolation equation corresponding to the three regimes on Definition 1.2:

- **The subcritical system of integral equations** are (12) with the extra conditions \( \forall t < \lambda_{\text{inf}} \leq \lambda(t) \) and
\[
\Phi(t) \equiv m_0(0) - m_0(t) .
\] That is \( \theta(t) \equiv 0 \) by (13) (no giant components appear due to frequent lightnings) and the equations take on the form
\[
v_k(t) = v_k(0) + \int_0^t \frac{k-1}{2} \sum_{l=1}^{k-1} v_l(s) v_{k-l}(s) - k \cdot v_k(s) m_0(s) - \lambda(s) k \cdot v_k(s) ds
\] (16)

The term \( -\lambda(s) k \cdot v_k(s) \) indicates that in the subcritical regime even small components are burnt with a rate proportional to their sizes and \( \lambda(s) \).

- **The critical equations** are (12) with the extra conditions \( \lambda(t) \equiv 0 \) and (15):
\[
v_k(t) = v_k(0) + \int_0^t \frac{k-1}{2} \sum_{l=1}^{k-1} v_l(s) v_{k-l}(s) - k \cdot v_k(s) m_0(s) ds
\] (17)

\( \lambda(t) \equiv 0 \) indicates that in the critical regime lightnings are not frequent enough to do any harm to small components, but (15) indicates that they are frequent enough to keep the mass of the giant component at zero.

- Let \( 0 = T_0^b < T_1^b < T_2^b < \ldots \) be a sequence with no accumulation points. Let
\[
M(t) := \max \{ i : T_i^b < t \}
\] (18)

\( \Psi(t) \) solves the **alternating equations** with burning times \( T_1^b, T_2^b, \ldots \) if
\[
\dot{v}_k(t) = \frac{k^{k-1}}{2} \sum_{l=1}^{k-1} v_l(t) v_{k-l}(t) - k \cdot v_k(t) m_0(T^b_{M(t)})
\] (19)

Mind the difference between (8) and (17): in the case of the Erdős-Rényi model the small components are allowed to coagulate with the giant component (which is of size \( \theta(t) = m_0(0) - m_0(t) \) by \( \Phi(t) \equiv 0 \) and (13)), but in the case of the frozen
percolation model the giant components are removed at the time of their birth. Using the terminology of the theory of Smoluchowski coagulation equations we might say that in the case of (8) the gel and the sol do react in the post-gel phase (Flory’s model, (6)), but in the case of (17) they do not react (Stockmayer’s model, (7)). Nevertheless, for \( t \leq T^g \) the solutions of (8) and (17) are identical since \( m_0(t) = m_0(0) \) in this regime.

The intuitive meaning of (19) is that giant components are removed from the system at the burning times. Thus (19) is (12) with

\[
\theta(t) = m_0(T^b_{M(t)}) - m_0(t)
\]

(20)

\[
\Phi(t) = m_0(0) - m_0(T^b_{M(t)}) = m_0(0) - m_0(t) - \sum_{j=1}^{M(t)} \theta(T^b_j)
\]

(21)

Both \( \theta(t) \) and \( \Phi(t) \) are left-continuous functions of \( t \).

Note that in the case of the (sub)critical frozen percolation equations ((16) and (17)) the fact that \( \Phi(t) \) is an increasing function automatically follows by (15):

\[
\Phi(t) - \Phi(s) = m_0(s) - m_0(t) = \sum_{k=1}^{\infty} \int_s^t \left( -\frac{k}{2} \sum_{l=1}^{k-1} \nu_l(u) \nu_{k-l}(u) + k \cdot \nu_k(u) m_0(u) + \lambda(u) \cdot k \cdot \nu_k(u) \right) du = \lim_{N \to \infty} \int_s^t \sum_{k=1}^{N} \sum_{l=1}^{N-k+1} k \cdot \nu_k(u) \nu_l(u) + \lambda(u) \cdot k \cdot \nu_k(u) du \geq 0
\]

**Theorem 1.1.**

- For any \( \nu(0) \in V^* \) and \( 0 < \lambda_{\text{inf}} \leq \lambda(t) \) the equations (16) have a unique solution.
- For any \( \nu(0) \in V^* \) the equations (17) have a unique solution.
- For any \( \nu(0) \in V^* \) and any sequence of burning times the equations (19) have a unique solution.

We prove this theorem in Section 3.

**Definition 1.6.** The solution of the random alternating equations with rate function \( \lambda : \mathbb{R}_+ \to \mathbb{R}_+ \) is a \( V \)-valued continuous-time Markov process: \( \nu(t) \) evolves deterministically, driven by the equations (19), but the sequence of burning times \( T^b_1, T^b_2, \ldots \) is random:

\[
\lim_{dt \to 0} \frac{1}{dt} \mathbb{P}(t \leq T^b_{M(t)+1} \leq t + dt \mid \mathcal{F}_t) = \lambda(t) \theta(t)
\]

(22)

where \( \mathcal{F}_t \) is the natural filtration generated by the process.
In plain words: a lightning strikes and burns the giant component with rate proportional to its size and $\lambda(t)$.

**Definition 1.7.**

$$\mathcal{W} := \{ (w_k)_{k=1}^\infty : 0 \leq w_1 \leq w_2 \leq \cdots < +\infty \}$$

$$\mathcal{W}^* := \{ (w_k)_{k=1}^\infty \in \mathcal{W} : \exists K < +\infty \ \forall k \geq K \ w_k = w_K \}$$

If $w \in \mathcal{W}$ denote by $m_0 := \sup_k w_k$.

We say that $((w_k(\cdot))_{k=1}^\infty, \Phi(\cdot))$ is a frozen percolation evolution on $[0,T]$ with initial condition $(w_k(0))_{k=1}^\infty = w \in \mathcal{W}^*$, or briefly

$$((w_k(\cdot))_{k=1}^\infty, \Phi(\cdot)) \in \mathcal{W}_w[0,T]$$

if for all $0 \leq t \leq T$ we have $(w_k(t))_{k=1}^\infty \in \mathcal{W}$, for all $k$ the functions $w_k : [0,T] \to [0,m_0(0)]$ are left-continuous and decreasing, $\Phi : [0,T] \to [0,m_0(0)]$ is left continuous and increasing with initial condition $\Phi(0) = 0$, moreover for all $t \leq T$ we have (13).

We define convergence on the space $\mathcal{W}_w[0,T]$:

$$((w^\alpha_k(\cdot))_{k=1}^\infty, \Phi^\alpha(\cdot)) \to ((w_k(\cdot))_{k=1}^\infty, \Phi(\cdot))$$

as $n \to \infty$ if for all $k$ we have $w^\alpha_k(t) \to w_k(t)$ for all $t$ which is a point of continuity of $w_k$ and $\Phi^\alpha(t) \to \Phi(t)$ for all $t$ which is a point of continuity of $\Phi$.

With this topology the space $\mathcal{W}_w[0,T]$ is metrizable, complete and compact.

From the frozen percolation process of Definition 1.7 one gets a random element of $\mathcal{W}_w[0,T]$ by (9). Denote the probability measure on $\mathcal{W}_w[0,T]$ corresponding to the process by $\mathbb{P}_N$.

It is easy to check that $((w_k(\cdot))_{k=1}^\infty, \Phi(\cdot)) \in \mathcal{W}_w[0,T]$ where $w_k(t) = \sum_{l=1}^k \xi_l(t)$ and $\xi(t)$ is a solution of the general frozen percolation equation (11) & (12) & (13).

**Theorem 1.2.** We consider a sequence of frozen percolation processes (see Definition 1.7) with initial state $\xi^N(0) = \xi(0) \in V^*$ and $\lambda(t)$ positive and continuous. Define $w_k^N(t)$ and $\Phi^N(t)$ as in (9). Denote the probability measure on $\mathcal{W}_w[0,T]$ corresponding to the process by $\mathbb{P}_N$.

Then $\mathbb{P}_N$ converges with respect to the weak convergence of probability measures on the polish space $\mathcal{W}_w[0,T]$ to a limiting measure $\mathbb{P}$, which depends on the decay rate of $\mu(N)$ in the following way:

- If $\mu(N) \equiv 1$ then $\mathbb{P}$ is concentrated on the unique solution of (16) with rate function $\lambda(t)$.

- If $\frac{1}{N} \ll \mu(N) \ll 1$ then $\mathbb{P}$ is concentrated on the unique solution of (17).

- If $\mu(N) = \frac{1}{N}$ then $\mathbb{P}$ is the law of the solution of the random alternating equation (see Definition 1.6) with rate function $\lambda(t)$. 

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We prove the $\mu(N) \equiv 1$ and the $1 \ll \mu(N) \ll 1$ part of this theorem in Section 4. In fact, these proofs are almost identical to the corresponding convergence results of [6], but we present them here as well for the sake of completeness.

We omit the proof of the $\mu(N) = \frac{1}{N}$ part of Theorem 1.2., but we believe that the methods introduced in Section 4 can be easily generalized for this case as well.

If we formally substitute $\lambda(t) \equiv 0$ into (16) or $T_{M(t)}^b \equiv t$ into (19), we get (17).

Rigorously:

Theorem 1.3. Let $(\psi_n(t))_{n=1}^\infty$ be a sequence of solutions of (16) with the same initial condition $\psi(0) \in V^*$ where $\lambda_n(t) \to 0$ uniformly as $n \to \infty$. Then for all $t$ and $k$ we have

$$\lim_{n \to \infty} \sup_i \{T_{i+1}^b(n) - T_i^b(n)\} = 0.$$

Then for all $t$ and $k$ we have $v_k(t) = \psi_k(t)$ where $\psi(t)$ is the solution of (17) with the same initial data. $\lim_{n \to \infty} \Phi_n(t) = \Phi(t)$ uniformly on $[0, \infty)$.

In plain words: if the rate of lightning is very small in the subcritical equations, then the solution is similar to that of the critical equation. We prove this theorem in Section 6.

Theorem 1.4. Let $(\psi_n(t))_{n=1}^\infty$ be a sequence of solutions of (19) with the same initial condition $\psi(0)$ where the sequence of burning times satisfy

$$\lim_{n \to \infty} v_k^b(t) = v_k(t) \text{ where } v(t) \text{ is the solution of (17) with the same initial data.}$$

$\lim_{n \to \infty} \Phi_n(t) = \Phi(t)$ uniformly on $[0, \infty)$.

In plain words: if the burning times of the alternating equations are very frequent, then the solution is similar to that of the critical equation. We prove this theorem in Section 7.

The solution of (17) has the self-organized critical property: for all $T^g \leq t$ it has the power-law decay of (10):

$$T_g = \frac{1}{m_1^g}, \Phi(t) = \int_{T^g}^t \Phi_{crit}(s)ds \to R_+ \text{ is positive and continuous, and for all } t \geq T^g \text{ we have}$$

$$\lim_{K \to \infty} K^2 \sum_{k=K}^\infty v_k(t) = \sqrt{2\Phi_{crit}(t)}.$$  \hspace{1cm} (23)

Definition 1.8. Let $x^*(t) = \inf\{x : \sum_{k=1}^\infty v_k(t)e^{-kx} < +\infty\}$.

The solutions of our equations have a remarkable rigidity property:

Theorem 1.6. If $\psi(t)$ is the solution of (16) or (19) and $\Psi(t)$ is the solution of (17) with the same initial condition, then for all $t \geq T^g$ and $k \geq 1$ we have

$$\tilde{v}_k(t) = v_k(t)e^{-kx^*}.$$
The solution of (17) with monodisperse initial condition is well-known (see e.g. [8]) and explicit:

Claim 1. If \( y(t) \) is the solution of (17) with \( v_k(0) = \mathbb{I}_{\{k=1\}} \cdot m_0(0) \) then for \( t \geq T^g = \frac{1}{m_1(0)} = \frac{1}{m_0(0)} \) we have

\[
v_k(t) = \frac{1}{t} \frac{k^{k-1}}{k!} e^{-k}. \tag{24}
\]

That is, for all \( T^g \leq t \) in the \( N \to \infty \) limit, the component size of a uniformly chosen (unburnt) vertex in the critical frozen percolation model has Borel distribution, which is the same as that of a vertex in the Erdős-Rényi graph at \( t = T^g \). The Borel distribution \( (v_k(1))_{k=1}^{\infty} \) in (24) is the distribution of the size of a critical Galton-Watson tree with POI(1) offspring distribution (see [1]).

The same self-similarity phenomenon can be observed in Aldous’ frozen percolation model (see [2]) on the binary tree: for \( t \geq \frac{1}{2} \), which is the critical time of the percolation process on the binary tree, a typical finite cluster has the distribution of a critical percolation cluster.

The solutions started from a polydisperse initial state are asymptotically self-similar:

Theorem 1.7. If \( y(t) \) is the solution of the critical equation (17) with \( y(0) \in V^*, \) and \( v_1(0) > 0 \) then

\[
\lim_{t \to \infty} t \cdot v_k(t) = \frac{k^{k-1}}{k!} e^{-k} \quad \text{and} \quad \lim_{t \to \infty} t \cdot m_0(t) = 1. \tag{25}
\]

Theorems 1.5, 1.6 and 1.7 are proved in Section 5 using the method of Laplace transforms, which is classical for the Smoluchowski equation with multiplicative kernel. The results (25) and \( \frac{v_k(t)}{k} = c_k(t) \sim k^{-5/2} \) (which is a variant of (23)) are already present in [8], but we believe that our approach based on the notion of the critical core of \( v(t) \) (defined in Section 2) gives new insight into these results about the solution of (17).

In the frozen percolation model on the binary tree, components are frozen (i.e. removed from the system) when their size becomes infinite. The question may arise:

What is the typical size of a frozen component in the mean field process of Definition 1.1?

In order to precisely formulate this question recall (2) and let

\[
\Phi^N([t_1,t_2],k) := \frac{k}{n} \cdot \left| \left\{ t \in [t_1,t_2] : \nu'(t_+) = \nu'(t_-) \right\} \right|.
\]

Thus \( \Phi^N([t_1,t_2],k) \) is the mass of burnt components of size \( k \) from \( t_1 \) to \( t_2 \). We have

\[
\sum_{k \geq 1} \Phi^N([t_1,t_2],k) = \Phi^N(t_2) - \Phi^N(t_1) =: \Phi([t_1,t_2])
\]
Thus $p^N_k[t_1, t_2] := \frac{\Phi_k^N([t_1, t_2], k)}{\Phi^N([t_1, t_2])}$, $k = 1, 2, \ldots$ is a random probability distribution for all $N$ and $t_1 < t_2$.

Denote by $|C^N_{\text{max}}(t)|$ the size of the largest component at time $t$.

**Conjecture 1.1.** If $\mu(N) = N^{-\alpha}$ in a critical sequence of frozen percolation processes (see Definitions 1.1 and 1.2), where $0 < \alpha < 1$, and if we define

$$\beta(\alpha) := \begin{cases} 2\alpha & \text{if } \alpha \leq \frac{1}{3} \\ \frac{\alpha + 1}{2} & \text{if } \alpha \geq \frac{1}{3} \end{cases} \quad (26)$$

then for every $T^* < t$ we have

$$\lim_{N \to \infty} \frac{\log \left( E \left( m^N_1(t) \right) \right)}{\log(N)} = \alpha \quad (27)$$

$$\lim_{N \to \infty} \frac{\log \left( E \left( m^N_2(t) \right) \right) - \log \left( E \left( m^N_1(t) \right) \right)}{\log(N)} = \beta(\alpha) \quad (28)$$

Moreover for every $v(0), T^* < t_1 < t_2$ and $\alpha$ there exists a non-defective probability distribution function $F : (0, \infty) \to (0, 1)$, $\lim_{x \to 0^+} F(x) = 0$, $\lim_{x \to \infty} F(x) = 1$ such that for all $x \in \mathbb{R}_+$ we have

$$\lim_{N \to \infty} \sum_{k \geq 1} \mathbb{I} \left[ k \leq xN^{\beta(\alpha)} \right] \cdot p^N_k[t_1, t_2] = F(x) \quad (30)$$

In plain words we might say that after gelation the typical component size of a frozen vertex and the size of the largest component is of order $N^{\beta(\alpha)}$. This conjecture is supported by heuristic arguments, computer simulations and Theorems 1.6 and 1.9 below. For $0 < \alpha < \frac{1}{3}$ the model is conjectured to behave similarly to the subcritical case described in Theorem 1.8 whereas for $\frac{1}{3} < \alpha < 1$ it is conjectured to behave similarly to the alternating case described in Theorem 1.9. Note that $\beta(\frac{2}{3}) = \frac{2}{3}$ and $N^{\frac{2}{3}}$ is the order of the size of the largest component in the critical Erdős-Rényi random graph.

**Theorem 1.8.** If $\mathbf{v}(t)$ is the solution of (16) with rate function $\lambda(t) \equiv \lambda$ and $\mathbf{v}(0) = \mathbf{v}(0) \in \mathbf{V}^+$ then there is a constant $C$ that depends only on the initial data and $T$ such that for all $0 < \lambda \leq 1$ and $\frac{1}{m_1(0)} < t \leq T$ we have

$$|\varphi_\lambda(t) - \varphi_{\text{crit}}(t)| \leq C\lambda \quad (31)$$

where

$$\frac{d}{dt} \varphi_\lambda(t) = \lambda m_1^\lambda(t). \quad (32)$$

Moreover if we define the random variable $Y_\lambda(t)$ to have distribution

$$P(Y_\lambda(t) = k) = \frac{\lambda \cdot k \cdot \mathbf{v}_\lambda^k(t)}{\Phi_\lambda(t)} = \frac{k \cdot v_\lambda^k(t)}{m_1^\lambda(t)}$$

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then
\[
\lim_{\lambda \to 0} \mathbf{P} \left( \frac{\lambda^2}{2 \Phi_{ \text{crit}}(t)} Y_\lambda(t) < x \right) = \int_0^x \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{y}} e^{-y^2} dy \quad (33)
\]

In plain words: for any \( t > T^{t^2} \) the distribution of the size-biased sample from the component-size distribution \( Y_\lambda(t) \) rescaled by \( \lambda^{-2} \) converges in distribution to a \( \Gamma(\frac{1}{2}, 1) \) distribution as \( \lambda \to 0 \). We prove this theorem in Section 7.

The relevance of Theorem 1.8 to Conjecture 1.1 is the following: if we consider a sequence of subcritical frozen percolation models (see Definition 1.2) with \( \lambda(t) \equiv \lambda \) then by Theorem 1.2 we get

\[
\lim_{\lambda \to 0} \mathbf{P} \left( \lambda^{-2} Y_\lambda(t) < x \right) = \int_0^x \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{y}} e^{-y^2} dy
\]

In (34) is the "subcritical" version of (27), \( m_\lambda(t) = \mathbf{E}(Y_\lambda(t)) \sim \lambda^{-2} \) corresponds to \( \beta(\alpha) = 2\alpha \) in (28), and (33) is a version of (30).

**Theorem 1.9.** Let \( Y_\lambda(t) \) denote the solution of the random alternating equations (see Definition 1.6) with a constant rate function \( \lambda(t) \equiv \lambda \).

Let \( \delta(\lambda) \) be a function satisfying \( \lambda^{-\frac{1}{2}} \ll \delta(\lambda) \ll 1 \) as \( \lambda \to \infty \).

Recalling (18) and (20) let

\[
\Phi_\lambda(t, x) := \sum_{j=1}^{M(t)} \theta^\lambda(T_j^\lambda) \mathbb{1}[\theta^\lambda(T_j^\lambda) > x]
\]

be the random mass of frozen giants of size at least \( x \). Then

\[
\lim_{\lambda \to \infty} \frac{\Phi_\lambda \left( t + \delta(\lambda), 2 \sqrt{\Phi_{ \text{crit}}(t)} x \right) - \Phi_\lambda \left( t, 2 \sqrt{\Phi_{ \text{crit}}(t)} x \right)}{\delta(\lambda) \Phi_{ \text{crit}}(t)} = \int_x^\infty \frac{4}{\sqrt{\pi}} y^2 e^{-y^2} dy
\]

in probability.

We prove this theorem in Section 7.

The heuristic meaning of this theorem is the following: if we pick a vertex uniformly from all vertices that were frozen between \( t \) and \( t + \delta(\lambda) \) and denote the mass of the giant component of that vertex by \( Z_\lambda(t) \), then the distribution of \( \frac{1}{\lambda} \sqrt{\Phi_{ \text{crit}}(t)} Z_\lambda(t) \) converges to a size-biased Rayleigh distribution (see Definition 7.3) as \( \lambda \to \infty \). Thus the typical mass of a frozen giant is of order \( \lambda^{-\frac{1}{2}} \), which suggests that if \( \mu(N) = \frac{N^\alpha}{N} \) (that is \( \alpha = 1 - \varepsilon \) in Conjecture 1.1) then the typical size of a frozen component is of order \( (N^\varepsilon)^{-\frac{1}{2}} \cdot N = N^{1-\frac{1}{2}\varepsilon} \), that is \( \beta(\alpha) = \frac{\alpha+1}{2} \). (34) is the "alternating" version of (30).
The critical frozen percolation model has an extremum property compared to the subcritical and alternating models (see Definition 1.2): if each burnt/frozen vertex produces profit at a rate \( \frac{1}{N} \) per time unit after it has been frozen, but each lightning (even the ones hitting burnt vertices) costs \( \frac{1}{N m_0(0)} \), then asymptotically (as \( N \to \infty \)) the critical model is the best choice if we want to maximize our profit on \([0, T]\). We reformulate this extremum principle in terms of the differential equations (16), (17), (19).

The asymptotic value of our profit produced by burnt vertices as \( N \to \infty \) is \( \int_0^T \Phi(t) dt \) according to Theorem 1.2. The asymptotic cost of lightnings is \( \int_0^T \lambda(t) dt \) for the solution of (16), but it is zero for (17) and (19), since the price we have to pay for the lightnings vanishes in the case of critical and alternating models as \( N \to \infty \).

**Theorem 1.10.** We fix \( v(0) \in V^* \). Let \( v^{\text{crit}}(t) \) denote the solution of (17) with initial condition \( v(0) \) and let \( v^{\text{sub}} \) denote the solution of (16) with lightning rate function \( \lambda(t) \) and the same initial condition. Then for any \( T > 0 \)

\[
\int_0^T \Phi^{\text{sub}}(t) dt - \int_0^T \lambda(t) dt \leq \int_0^T \Phi^{\text{crit}}(t) dt - \int_0^T 0 dt \quad (35)
\]

If \( v^{\text{alt}}(t) \) denotes the solution of (19) with an arbitrary sequence of burning times and initial condition \( v(0) \) then

\[
\int_0^T \Phi^{\text{alt}}(t) dt \leq \int_0^T \Phi^{\text{crit}}(t) dt \quad (36)
\]

**Remark 2.** Let \( T > T^g = \frac{1}{m_1(0)} \) and \( \varepsilon > 0 \). For a suitable choice of \( \lambda(t) \) we have

\[
\int_0^T \Phi^{\text{sub}}(t) dt - (1 - \varepsilon) \int_0^T \lambda(t) dt > \int_0^T \Phi^{\text{crit}}(t) dt - (1 - \varepsilon) \int_0^T 0 dt \quad (37)
\]

For a suitable choice of burning times

\[
\int_0^T \Phi^{\text{alt}}(t) dt + \varepsilon \Phi^{\text{alt}}(T) > \int_0^T \Phi^{\text{crit}}(t) dt + \varepsilon \Phi^{\text{crit}}(T) \quad (38)
\]

The idea that the critical forest fire model solves a variational problem is already present in [3].

### 2 Definitions, Transformations

We consider a solution of the general frozen percolation equation (see Definition 1.5).

Denote the Laplace transform (generating function) of \( v(t) \) by

\[
V(t,x) := \sum_{k=1}^\infty v_k(t) e^{-kx} \quad (39)
\]
for \( x > 0 \). Then \( V(t, 0) = V(t, 0^+) = m_0(t) \) and by dominated convergence for \( x > 0 \) (16) is transformed into

\[
V(t, x) = V(0, x) + \int_0^t V'(s, x) \left(-V(s, x) + (m_0(0) - \Phi(s)) + \lambda(s)\right) ds \quad (40)
\]

In the sequel we denote the derivative of functions \( f(t, x) \) with respect to the time and space variables by \( f'(t, x) \) and \( f''(t, x) \), respectively.

Let

\[
U(t, x) := V(t, x) - (m_0(0) - \Phi(t)) \quad (41)
\]

Thus (40) is transformed into

\[
U(t, x) = U(0, x) + \int_0^t -U(s, x)U''(s, x)U''(s, x)ds + \Phi(t) \quad (42)
\]

Since \( V(t, \cdot) \) is a Laplace transform we have

\[
U(0, 0) = -\theta(t) \quad U'(0, 0) = -m_1(t) \quad \lim_{x \to \infty} U(x) = -m_0(0) + \Phi(t) \quad (43)
\]

and \( U \) is a monotone decreasing convex function of the variable \( x \) for every \( t \).

**Definition 2.1.** Denote by \( X(t, u) \) the inverse function of \( U(t, x) \) with respect to \( x \), that is \( U(t, X(t, u)) = u \).

The domain of \( X(t, u) \) in the variable \( u \) is \((-m_0(t) + \Phi(t), -\theta(t))\).

\[
X(t, -\theta(t)) = 0 \quad (44)
\]

The notion of \( X(t, \cdot) \) and a version of the following lemma is already present in [8].

**Lemma 2.1.** If \( X(t, u) \) is defined using a solution of the general frozen percolation equation then the following identity holds:

\[
X(t, u) = X(0, u - \Phi(t)) + t \cdot (u - \Phi(t)) - \int_0^t \lambda(s) ds + \int_0^t \Phi(s) ds \quad (45)
\]

**Proof.** We fix an \( x_{\text{min}} > 0 \). For any \( x \geq x_{\text{min}} \) we have

\[
|U(t, x)| \leq m_0(0), \quad |U'(t, x)| \leq \frac{m_0(t)}{x_{\text{min}}}, \quad |U''(t, x)| \leq \frac{m_0(t)}{x_{\text{min}}^2}, \quad (46)
\]

moreover \( \sup_{0 \leq s \leq T} \lambda(s) < +\infty \). For an \( x(0) > x_{\text{min}} \) denote by \( x(t) \) the solution of the integral equation

\[
x(t) = x(0) + \int_0^t U(s, x(s)) - \lambda(s) ds \quad (47)
\]

This equation is well-posed on the domain \( x(t) \geq x_{\text{min}} \), since \( U(s, x) - \lambda(s) \) is bounded and Lipschitz-continuous in \( x \).
Moreover
\[ x(t+dt) - x(t) = O(dt), \quad |U(t,x(t)) - U(t,x(t+dt))| = O\left(\frac{dt}{x_{\min}}\right). \]

If we differentiate (42) w.r.t. \( x \) we get
\[ |U'(t+dt,x) - U'(t,x)| = O\left(\frac{dt}{x_{\min}}\right). \]

Thus
\[ U(t+dt,x(t+dt)) - U(t,x(t)) - (\Phi(t+dt) - \Phi(t)) = \]
\[ (U(t+dt,x(t+dt)) - U(t,x(t+dt))) + \]
\[ (U(t,x(t+dt)) - U(t,x(t))) - (\Phi(t+dt) - \Phi(t)) = \]
\[ \int_t^{t+dt} -U(s,x(t+dt))U'(s,x(t+dt)) + \lambda(s)U''(s,x(t+dt))ds + \]
\[ U'(t,x(t+dt)) \int_t^{t+dt} U(s,x(s)) - \lambda(s)ds + O\left(\frac{dt^2}{x_{\min}}\right) = \]
\[ \int_t^{t+dt} U(s,x(t+dt))(U'(t,x(t+dt)) - U'(s,x(t+dt)))ds + \]
\[ \int_t^{t+dt} U'(t,x(t+dt))(U(s,x(s)) - U(s,x(t+dt)))ds + \]
\[ \int_t^{t+dt} \lambda(s)(U'(s,x(t+dt)) - U'(t,x(t+dt)))ds + O\left(\frac{dt^2}{x_{\min}}\right) = O\left(\frac{dt^2}{x_{\min}}\right) \]

Thus \( U(t,x(t)) = U(0,x(0)) + \Phi(t) \), and if we substitute this back into (47), we get
\[ x(t) = x(0) + tU(0,x(0)) + \int_0^t \Phi(s)ds - \int_0^t \lambda(s)ds \]

By the definition of \( X(t,u) \) we have \( X(t, U(t,x(t))) = x(t) \), and by substituting
\[ u = U(0,x(0)) + \Phi(t) \]
we obtain (45).

Since \( y(0) \in V^* \), \( V(0,x) \) is well-defined and analytic for all \( x \in \mathbb{R} \), thus \( X(0,u) \) can be analytically extended to \( (0,m_0(0),+\infty) \). (45) makes it possible to extend \( X(t,u) \) to \( (0,m_0(0) + \Phi(t),+\infty) \) analytically. The extended \( X(t,u) \) is a strictly convex function of the \( u \) variable. If we differentiate (45) w.r.t. \( u \), we get
\[ X'(t,u) = X'(0,u - \Phi(t)) + t \quad (48) \]

**Definition 2.2.** Define \( F(t,w) \) by the identity
\[ F(t,-X'(t,u)) = -u \quad (49) \]

Thus \(-F(t,w)\) is the inverse function of \(-X'(t,u)\). If \( \hat{X} \) denotes the Legendre-transform of \( X \) w.r.t. the variable \( u \), then
\[ G(t,w) := \hat{X}(t,-w) = -\min_{u} \{wu + X(t,u)\} = wF(t,w) - X(t,-F(t,w)) \quad (50) \]
Let
\[ E(t, w) = G''(t, w) = F'(t, w). \] (51)

We call \( E(t, \cdot) \) the critical core of \( \tilde{\Phi}(t) \). If we use the extended definition of \( X \) then \( G(t, w) \) is well-defined and analytic for all \( w > -t \).

We have
\[
F(t, -\frac{1}{U'(t, x)}) = -U(t, x) \quad \text{and} \quad E(t, -\frac{1}{U'(t, x)}) = \frac{(-U'(t, x))^3}{U''(t, x)} \] (52)

It follows from the properties of the Legendre-transformation and (45) that
\[
G(t, w) = G(0, w + t) - w \cdot \Phi(t) - \int_0^t \Phi(s) ds + \int_0^t \lambda(s) ds \] (53)
\[
F(t, w) = F(0, w + t) - \Phi(t) \] (54)
\[
E(t, w) = E(0, w + t) \] (55)

\( G(t, \cdot) \) is strictly convex and \( G \) determines \( X \) uniquely since the Legendre-transformation is invertible. Define
\[
w^*(t) := -X'(t, 0) \iff F(t, w^*(t)) = 0 \iff \arg\min_w G(t, w) = w^*(t) \] (56)
\[
X(t, 0) = 0 \implies G(t, w^*(t)) = 0 \implies \forall w \ G(t, w) \geq 0 \] (57)
\[
\theta(t) = 0 \implies w^*(t) = \frac{1}{m_1(t)} \geq 0 \] (58)
\[
x^*(t) = \inf\{x : \sum_{k=1}^{\infty} v_k(t)e^{-kx} < +\infty\} = \min_u X(t, u) = X(t, -F(t, 0)) = -G(t, 0) \] (59)

3 The frozen percolation equations are well-posed

Lemma 3.1. The alternating equation (19) is well-posed.

Proof. If we are given the sequence of burning times \( 0 < T^b_1 < T^b_2 < \ldots \) the solution of (19) can be uniquely constructed by using induction on \( i \): if we already have the solution on \( [0, T^b_i] \), then we are given \( m_0(T^b_i) \), so we can uniquely solve the sequence of ordinary differential equations (19) for \( v_1, v_2, \ldots \) on \( [T^b_i, T^b_{i+1}] \) by repeatedly applying the Picard-Lindelöf theorem, since the equation for \( v_k \) only contains \( v_1, \ldots, v_k \) on its right-hand side. \( \square \)

Lemma 3.2. The solution of the integral equations (16) is unique for every initial condition \( \tilde{\nu}(0) \in \mathbb{V}^* \) if \( \lambda(t) \) is nonnegative and continuous.

Remark 3. Choosing \( \lambda(t) \equiv 0 \) implies the uniqueness of the solutions of (17).
Proof. In order to prove the uniqueness of the solution of (16), we only have to prove that given two solutions with the same initial condition, the function \( \Phi(t) = m_0(0) - m_0(t) \) determined by the two solutions is the same, because \( m_0(t) \) and (16) determines \( v_k(t) \) for all \( k \) uniquely. For a solution \( y(t) \) of (16) we can define \( U \) by (41), then \( X \) by Definition 2.1, which satisfies (45) and the \( G \) of Definition 2.2 satisfies (53).

Assume that \( G_1 \) and \( G_2 \) are obtained this way from two solutions of (16) with the same initial condition \( G(0, w) \). Let \( \tilde{G} = G_1 - G_2 \) and \( \tilde{\Phi} = \Phi_1 - \Phi_2 \). Then

\[
\tilde{G}(t, w) = -w \cdot \tilde{\Phi}(t) - \int_0^t \tilde{\Phi}(s)ds
\]

Now by (15) we have \( \theta(t) = 0 \), thus (44) \( \implies X(t, 0) = 0 \), and (57) \( \implies \min_w G_1(t, w) = \min_w G_2(t, w) = 0 \) and (58) \( \implies w'_i(t) := \argmin_w G_1(t, w) \geq 0 \) for \( i = 1, 2 \), thus we have \( \tilde{G}(t, w'_1(t)) \leq 0 \) and \( \tilde{G}(t, w'_2(t)) \geq 0 \). Thus \( \tilde{\Phi}(t) \) and \( \int_0^t \tilde{\Phi}(s)ds \) cannot have the same sign. But if \( (t_1, t_2) \) is a maximal interval such that for \( t_1 < t < t_2 \) we have \( \int_0^t \tilde{\Phi}(s)ds > 0 \) then \( \int_{t_1}^t \tilde{\Phi}(s)ds = 0 \) and

\[
t \in [t_1, t_2] \implies \int_0^t \tilde{\Phi}(s)ds \geq 0 \implies \tilde{\Phi}(t) \leq 0 \implies \int_{t_1}^t \tilde{\Phi}(s)ds \leq 0
\]

which contradicts the definition of \( t_1 \) and \( t_2 \). Thus \( \int_0^t \tilde{\Phi}(s)ds \leq 0 \) for all \( t \) and interchanging the roles of \( G_1 \) and \( G_2 \) we get \( \int_0^t \tilde{\Phi}(s)ds \equiv 0 \), so \( \Phi_1(t) \equiv \Phi_2(t) \). \( \square \)

Lemma 3.3. If we find a function \( \varphi(t) \) such that defining \( \Phi(t) := \int_0^t \varphi(s)ds \) and \( G(t, w) \) by (53) we have

\[
\min_w G(t, w) = 0 \quad \text{and} \quad w^*(t) = \argmin_w G(t, w) \geq 0 \quad (60)
\]

for all \( t \), then the solution of (12) with the same \( \lambda(\cdot) \), \( \Phi(\cdot) \) and initial condition satisfies (16).

Proof. Since the Legendre-transformation is invertible, from (60) we get

\[
X(t, 0) = 0 \quad \text{and} \quad X'(t, 0) \leq 0.
\]

\( X(t, u) \) is strictly decreasing for \( u < 0 \), thus it is the inverse function of an \( U(t, x) \) satisfying \( U(t, 0) = 0 \). If we plug \( \Phi(\cdot) \) into (12) then we get \( \theta(t) = -U(t, 0) = 0 \), therefore (15) is satisfied. \( \square \)

Lemma 3.4. The \( \Phi \) of the unique solution of (17) is

\[
\Phi(T) = \begin{cases} 
0 & \text{if } t \leq T^g \\
F(0, T) & \text{if } t \geq T^g
\end{cases} \quad (61)
\]

where \( T^g = \frac{1}{m_1(0)} \).

\[
\int_0^T \Phi(t)dt = \begin{cases} 
0 & \text{if } T \leq T^g \\
G(0, T) & \text{if } T \geq T^g
\end{cases} \quad (62)
\]
Proof. The solution is unique according to Lemma 3.2, and to prove its existence we only have to find a function $\varphi(t)$ that satisfies the criteria of Lemma 3.3 (with $\lambda(t) \equiv 0$). We will show that

$$\varphi(t) = \mathbb{I}[t \geq \frac{1}{m_1(0)} E(0,t)]$$

(63)

does the job. For $t \leq T^*$ this is trivial by looking at (53): $G(t, w^*(t)) = 0$ and $w^*(t) = \frac{1}{m_1(0)} - t \geq 0$ if $\Phi(t) \equiv 0$.

We will show that for $t \geq T^*$ we have $G(t, 0) \equiv 0$ and $F(t, 0) \equiv 0$, that is $w^*(t) \equiv 0$. $F(0, T^*) = G(0, T^*) = 0$ by (57) and $w^*(0) = \frac{1}{m_1(0)} = T^*$. $F(t, 0) = 0$ follows from (54) and

$$\Phi(t) = \int_0^t \varphi(s)\,ds = \int_{T^*}^t E(0,s)\,ds = F(0,t) - F(0,T^*) = F(0,t)$$

By (53) we have

$$G(t,0) = G(0,t) - \int_0^t \Phi(s)\,ds = \int_{T^*}^t F(0,s)\,ds - \int_{T^*}^t F(0,s)\,ds = 0$$

\[\Box\]

The well-posedness of the integral equation (17) implies that of the corresponding differential equation, since $m_0(0) - \Phi(t) = m_0(t)$ is a continuous function of $t$, thus $v_k(t)$ are differentiable.

We have shown that the solution of (17) has infinite first moment after the gelation time: $\frac{1}{w(t)} = m_1(t) = +\infty$ for all $t \geq T^*$.

**Definition 3.1.** Let $E(0,w)$ denote the critical core of $y(0)$ (see Definition 2.2).

For $\frac{1}{m_1(0)} \leq w_1 \leq w_2$ define

$$E_{\inf}(w_1, w_2) := \min_{w_1 \leq w \leq w_2} E(0, w) \quad \text{and} \quad E_{\sup}(w_1, w_2) := \max_{w_1 \leq w \leq w_2} E(0, w).$$

$$E_{\sup} := E_{\sup}(\frac{1}{m_1(0)}, +\infty), \quad E_{\inf}(w) := E_{\inf}(\frac{1}{m_1(0)}, w)$$

**Lemma 3.5.** If $w \geq \frac{1}{m_1(0)}$ then the inequalities

$$\frac{m_1(0)}{m_2(0)} \frac{1}{w^2} \leq E(0,w) \leq \frac{1}{w^2}$$

(64)

hold. Thus $E_{\sup} \leq m_1(0)^2$ and $E_{\inf}(w) \geq \frac{m_1(0)}{m_2(0)} \frac{1}{w^2}$.

For all $w \geq \frac{1}{m_1(0)}$ we have

$$|E'(0,w)| \leq 4m_2(0)^2 m_3(0) =: D$$

(65)

which implies

$$E_{\sup}(w_1, w_2) - E_{\inf}(w_1, w_2) \leq D \cdot (w_2 - w_1)$$

(66)
Remark 4. If $m_1(0) = m_2(0)$ then the upper and lower bounds in (64) coincide. This can only happen if $v_k(0) = m_1(0) \cdot \mathbb{I}[k = 1]$, this is the case known as the monodisperse initial condition (the initial graph has no edges).

Proof. Let $U(x) := U(0, x)$. Recalling (52) $E \left( 0, -\frac{1}{U'(x)} \right) = \frac{(-U'(x))^3}{U'(x)}$ holds. The upper bound of (64) follows from $-U''(x) \leq U''(x)$, and $-U''(x) \cdot \frac{m_2(0)}{m_1(0)} \geq U''(x)$ holds because $\log (-U'(x))$ is a convex function, thus $\frac{U''(x)}{U'(x)} = \frac{m_2(0)}{m_1(0)} = m_2(0) - m_1(0)$. The bound on the Lipschitz constant (65) follows from

$$\left| E'(0, -\frac{1}{U'}) \right| = \left| \frac{(U'')^2 U''}{(U')^3} \right| \leq 3 \left( (U')^2 U'' \right) \leq 4 \left| (U')^2 U'' \right|.$$  

Now we turn our attention to the subcritical equation (16). We assume $\lambda(t) > 0$ for all $t$. If we substitute $x = 0$ into the differential equation (16) and assume $|U'(t, 0)| < +\infty$ then (formally) we get

$$\Phi(t) = \varphi(t) = -U'(t, 0) \cdot \lambda(t) = m_1(t) \lambda(t) = \frac{\lambda(t)}{w^s(t)}.$$  

Definition 3.2. If $\mathbf{v}(0) \in \mathbf{V}^*$ and $\lambda(t)$ is a positive continuous function then the subcritical control differential equation for $w^s(t)$ is

$$w^s(t) = \frac{\lambda(t)}{w^s(t)E(0, t + w^s(t))} - 1$$  

(67)

with initial condition $w^s(0) = \frac{1}{m_1(0)} = T^s$.

Lemma 3.6. The subcritical control differential equation is well-posed and the function

$$\varphi(t) := \frac{\lambda(t)}{w^s(t)}$$

(where $w^s(t)$ is the solution of (67) with $w^s(0) = \frac{1}{m_1(0)}$) satisfies the criteria of Lemma 3.3 which implies the existence of solutions to (16).

Proof. We prove the statement of the lemma on $[0, T]$. The Picard-Lindelöf theorem and the Lipschitz-continuity property (65) guarantee the existence and uniqueness of the solution of (67) before the graph of the solution exits

$$\{(t, w^s) : 0 \leq t \leq T, \ w^s_{min} \leq w^s \leq w^s_{max}, \ w^s + t \geq w^s(0)\}$$  

(68)

for some $0 < w^s_{min} < w^s_{max} < +\infty$.

Let $\lambda_{inf} := \inf_{0 \leq t \leq T} \lambda(t)$, $\lambda_{sup} := \sup_{0 \leq t \leq T} \lambda(t)$. From (67) and a “forbidden region”-type argument we get that $w^s(t) + t \geq w^s(0)$ and $w^s(t) \geq \min \{\frac{\lambda_{inf}}{\lambda_{sup}}, w^s(0)\}$, since

$$w^s(t) > 0 \implies \frac{d}{dt} (w^s(t) + t) \geq 0$$

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\[ w^*(t) + t \geq w^*(0) \implies E(0, t + w^*(t)) \leq E_{sup}, \]

thus \( w^*(t) < \frac{\lambda_{inf}}{E_{sup}} \implies \dot{w}^*(t) > 0. \)

Now we prove that \( w^*(t) \) cannot grow too fast using the lower bound of (64).

\[ w^*(t) \leq y(t) \text{ where } y(0) = w^*(0) = T^s \text{ and } \]

\[ \dot{y}(t) = \lambda_{sup} \frac{m_2(0)}{m_1(0)} \left( \frac{(y(t) + t)^2}{y(t)} \right) \leq \lambda_{sup} \frac{m_2(0)}{m_1(0)} \left( \frac{T^s + t}{T^s} \right) \cdot (y(t) + t) \]

since \( y(t) \) is increasing. Thus \( \dot{y}(t) \leq a \cdot y(t) + b \) for some \( a \) and \( b \) depending only on the initial data, the function \( \lambda(t) \) and \( T \). Thus

\[ w^*(t) \leq w^*(0) e^{at} + b \cdot (e^{at} - 1). \]

Now we can see that the graph of the solution of (67) indeed doesn’t exit (68) until \( t = T \) if we define

\[ w_{min} = \min \left\{ \frac{\lambda_{inf}}{E_{sup}}, T^s \right\} \quad \text{and} \quad w_{max} := (T^s + \frac{b}{a})e^{aT} \quad (69) \]

Now we prove that \( \phi(t) := \frac{\lambda(t)}{w^*(t)} \) satisfies the criteria of Lemma 3.3 by showing that

\[ G(t, w^*(t)) \equiv 0 \quad \text{and} \quad F(t, w^*(t)) \equiv 0. \]

This holds for \( t = 0 \), so it suffices to check \( \frac{d}{dt} G(t, w^*(t)) \equiv 0 \) and \( \frac{d}{dt} F(t, w^*(t)) \equiv 0. \)

Using (54)

\[ \frac{d}{dt} F(t, w^*(t)) = E(0, t + w^*(t)) \cdot \left( 1 + \frac{\lambda(t)}{w^*(t) E(0, t + w^*(t))} - 1 \right) - \frac{\lambda(t)}{w^*(t)} = 0 \]

If we combine \( F(t, w^*(t)) \equiv 0 \) with (54) we get

\[ F(0, t + w^*(t)) = \Phi(t) \quad (70) \]

It is straightforward to verify \( \frac{d}{dt} G(t, w^*(t)) \equiv 0 \) by using (53) and (70). \( \square \)

This completes the proof of the well-posedness of (16).

4 Proof of Theorem 1.2.

We consider the sequence \( \mathbb{P}_N \) of probability measures on the compact space \( \mathcal{W}_w[0, T] \). From Prokhorov’s theorem it follows that any subsequence of the measures \( \mathbb{P}_N \) contains a sub-subsequence that converges weakly to a limiting measure on \( \mathcal{W}_w[0, T] \).

Lemma 4.1. Any weak limit point of the measures \( \mathbb{P}_N \) is concentrated on the set of solutions of the general frozen percolation equation (12).
If \( \mu(N) \equiv 1 \), then the \( \lambda(t) \) rate function of (12) is equal to the \( \lambda(t) \) of (4).

If \( \mu(N) \ll 1 \), then the \( \lambda(t) \) rate function of (12) is equal to 0.

**Proof.** From (3) and (4) it follows that

\[
LV_N^k(t) := \lim_{dt \to 0} E(v_k^N(t+dt) - v_k^N(t) \mid \mathcal{F}_t) = \frac{1}{N} v_k^N(v_k^N - k) \left( -\frac{2k}{N} \right) + \left( \sum_{l 
eq k} \frac{1}{N} \phi_l \right) \left( -\frac{k}{N} \right) + \left( \sum_{l=1}^{N-1} \frac{1}{N} v_l v_{k-l} + 2|k| \frac{1}{N} \left( \frac{v_k^N}{2} \right)^2 \right) - k \cdot (m_0(0) - \Phi^N(t)) + \lambda(t) \mu(N) \cdot v_k^N + \frac{k-1}{2} \sum_{l=1}^{N-1} v_l v_{k-l} + \frac{1}{N} \left( k^2 v_k^N - 2|k| \cdot \frac{k^2}{4} v_k^N \right)
\]

(71)

\[
M(t) = v_k^N(t) - v_k^N(0) - \int_0^t LV_k^N(s) ds \text{ is a martingale and}
\]

\[
LM^2(t) := \lim_{dt \to 0} E(M^2(t+dt) - M^2(t) \mid \mathcal{F}_t) = \lim_{dt \to 0} E((v_k^N(t+dt) - v_k^N(t))^2 \mid \mathcal{F}_t) \leq \left( \frac{k}{N} \right)^2 \cdot \left( \frac{|m_0(0)|}{2} \right) \frac{1}{N} + |m_0(0)| \lambda(N) = O \left( \frac{k^3}{N} \right)
\]

Thus \( E(M(T)^2) = E \left( \int_0^T LM^2(s) ds \right) = O \left( \frac{1}{N} \right) \) if we fix \( k \). It follows from Doob’s maximal inequality that for all \( \varepsilon > 0 \), \( k \geq 1 \) and \( T < +\infty \) we have

\[
\lim_{N \to \infty} P \left( \sup_{0 \leq s \leq T} \left| v_k^N(t) - v_k^N(0) - \int_0^t LV_k^N(s) ds \right| > \varepsilon \right) = 0
\]

(72)

If we rewrite this equation in terms of the functions \( \omega_k^N(\cdot) \equiv 1 \) the claim of the lemma follows.

**Lemma 4.2.** If \( \frac{1}{N} \ll \mu(N) \), \( 0 < \lambda_{inj} \leq \lambda(t) \) and \( \mathbf{v}(0) \in \mathbb{V}^* \), then for any weak limit point \( \mathbb{P} \) of the sequence of probability measures \( \mathbb{P}_N \) on \( \mathbb{W}_N^0[0,T] \) we have

\[
\mathbb{P} (\mathbf{0}(t) \equiv 0) = 1
\]

(73)

The subcritical and critical parts of Theorem 1.2 follow from Lemma 4.1 and Lemma 4.2: any weak limit point \( \mathbb{P} \) of the sequence \( \mathbb{P}_N \) is concentrated on the set of frozen percolation evolutions satisfying (12) & (15). When \( \mu(N) \equiv 1 \), \( \mathbb{P} \) is concentrated on the unique solution of (16), when \( \frac{1}{N} \ll \mu(N) \ll 1 \) then \( \mathbb{P} \) is concentrated on the solution of (17).

In the rest of this section we discuss the proof of Lemma 4.2.
Lemma 4.3. We consider a solution of the general frozen percolation equation (12) with initial condition $v(0) \in V^*$. If $\lambda(t) \equiv 0$ or $0 < \lambda_{inf} \leq \lambda(t) \leq \lambda_{sup} < +\infty$ then there is a constant $C^*$ such that for all $t_1 \leq t_2$ we have

$$\theta(t_2) - \theta(t_1) \leq C^* \cdot (t_2 - t_1) \quad (74)$$

Proof. First we prove that there exists a constant $C$ depending only on the initial data $v(0)$ and $\lambda_{inf}$ such that

$$m_1(t) \leq C \quad (75)$$

If $V(t,x) = \sum_{k=1}^\infty v_k(t)e^{-kx}$ then by (12) we get

$$\dot{V}(t,x) = V'(t,x) \cdot (m_0(0) - \Phi(t)) + \lambda(t) - V(t,x) \quad (76)$$

$$\ddot{V}(t,x) = V''(t,x) (m_0(0) - \Phi(t) - \lambda(t) - V(t,x)) - V'(t,x)^2 \quad (77)$$

Substituting $V(t,x) - (m_0(0) - \Phi(t)) \leq 0$ and $-\frac{V'(t,x)^2}{\lambda_{sup}} \leq V''(t,x)$ into (77) we get

$$\frac{d}{dt} (-V'(t,x)) \leq V'(t,x)^2 \cdot \left(1 - \frac{\lambda_{inf}}{\lambda_{sup}} (-V'(t,x)) \right)$$

which implies $-V'(t,x) \leq \max\{m_1(0), \frac{\lambda_{inf}}{\lambda_{sup}}\} : C$ for all $x > 0$ and $t$ by a "forbidden region"-argument. Thus by letting $x \to 0_+$ we get (75).

Now we show that for some constant $C_2$ we have

$$(V(t,x) - (m_0(0) - \Phi(t)) V'(t,x) \leq C_2 \quad (78)$$

for all $x > 0$. If $\lambda_{inf} \leq \lambda(t)$, then by (75) and $-m_0(0) \leq V(t,x) - (m_0(0) - \Phi(t)) \leq 0$ we get (78) with $C_2 = m_0(0)C$.

Denote by $U(t,x) := V(t,x) - (m_0(0) - \Phi(t))$. If $\lambda(t) \equiv 0$ then by (76) and (77) we get

$$\frac{d}{dt} (U(t,x)V'(t,x)) = -2V'(t,x)^2 U(t,x) - U(t,x)V''(t,x) + V'(t,x) \frac{d}{dt} \Phi(t) \leq$$

$$\left(-U(t,x)\right)V'(t,x)^2 \left(2 - \frac{1}{\lambda_{sup}} U(t,x)V'(t,x) \right)$$

Thus we have (78) with $C_2 = \max\{m_1(0), 2\lambda_{sup}\}$ again by a "forbidden region"-argument. Substituting the bounds (75) and (78) into (76) we get

$$\frac{d}{dt} (-V(t,x)) \leq C_2 + C \cdot \lambda_{sup} \equiv C^*$$

for all $x$. Thus $V(t_1,x) - V(t_2,x) \leq C^* \cdot (t_2 - t_1)$. Letting $x \to 0_+$ and substituting into (13) the claim of the lemma follows. \qed
We are going to prove Lemma 4.2 by contradiction: in Lemma 4.4 we show that if \( \theta(\cdot) \neq 0 \) in the limit, then there is a positive time interval such that \( \theta(t) \) has a positive lower bound, and that this implies that even in the convergent sequence of finite-volume models, a lot of mass is contained in arbitrarily big components on this interval. Than in subsequent Lemmas we prove that these big components indeed burn, which produces such a big increase in the value of the burnt mass \( \Phi(\cdot) \) that is in contradiction with \( \Phi(\cdot) \leq m_0(0) \).

For any frozen percolation evolution obtained from a frozen percolation Markov process on a finite number of vertices we obviously have \( \theta^N(t) \equiv 0 \) (see (9) and (13)), thus

\[
\forall K \in \mathbb{N} \quad \sum_{k \geq K} v_k^N(t) = m_0(t) - w_k^N(t) = m_0(0) - \Phi^N(t) - w_k^N(t)
\]  

(79)

**Lemma 4.4.** If \( \mathbb{P}_N \Rightarrow \mathbb{P} \) where \( \mathbb{P} \) does not satisfy (73) on \([0, T]\), then there exist \( \varepsilon_1, \varepsilon_2, \varepsilon_3 > 0 \) and a deterministic \( t^* \in [\varepsilon_1, T] \) such that for every \( K < +\infty \), every \( m < +\infty \) and every sequence

\[
t^* - \varepsilon_1 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_m < \beta_m < t^*
\]

there exists an \( N_0 < +\infty \) such that for every \( N \geq N_0 \) and \( 1 \leq i \leq m \) we have

\[
\mathbb{P}_N \left( \max_{\alpha_i \leq t \leq \beta_i} \sum_{k \geq K} v_k^N(t) > \varepsilon_2 \right) > \varepsilon_3.
\]

(80)

**Proof.** First we prove that if \( \mathbb{P} \) does not satisfy (73) then there exist \( \varepsilon_1, \varepsilon_2, \varepsilon_3 > 0 \) and \( \varepsilon_1 \leq t^* \leq T \) such that

\[
\mathbb{P} \left( \inf_{t^* - \varepsilon_1 \leq t \leq t^*} \theta(t) > \varepsilon_2 \right) > \varepsilon_3.
\]

(81)

Since (73) is violated, we have \( \mathbb{P} \left( \sup_{0 \leq t \leq T} \theta(t) > \varepsilon \right) > \varepsilon \) for some \( \varepsilon > 0 \).

Let \( L := \left\lfloor \frac{2C^*}{\varepsilon} \right\rfloor \) and \( t_i := \frac{t}{L} \) for \( 1 \leq i \leq L \) where \( C^* \) is the constant in (74).

By Lemma 4.1 the random frozen percolation evolution obtained as a weak limit point satisfies (12) with a possibly random control function \( \Phi \), so (74) holds \( \mathbb{P} \)-almost surely for the random element of \( \mathcal{W}_\mathcal{W}[0, T] \) obtained as a weak limit point.

Since \( \theta(0) = 0 \) we have

\[
\left\{ \sup_{0 \leq t \leq T} \theta(t) > \varepsilon \right\} \subseteq \bigcup_{i=1}^{L} \left\{ \theta(t_i) > \varepsilon \frac{t}{2} \right\}
\]

almost surely with respect to \( \mathbb{P} \). Thus \( \mathbb{P} \left( \theta(t^*) > \varepsilon \frac{t^*}{2} \right) > \varepsilon \frac{t^*}{2} \) for some \( t^* \in \{t_1, \ldots, t_L\} \).

Using (74) again (81) follows with \( \varepsilon_1 := \frac{t}{L}, \varepsilon_2 := \frac{t^*}{T}, \varepsilon_3 = \frac{t^*}{T} \).

Now given \( K \) and the intervals \([\alpha_i, \beta_i], 1 \leq i \leq m \) we define the continuous functionals \( f_i : \mathcal{W}_\mathcal{W}[0, T] \rightarrow \mathbb{R} \) by

\[
f_i((w_k(\cdot))_{k=1}^\infty, \Phi(\cdot)) := \frac{1}{\beta_i - \alpha_i} \int_{\alpha_i}^{\beta_i} (m_0(0) - w_K(t) - \Phi(t)) dt
\]

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Thus for all $i$

$$H_i := \{((w_k(\cdot))_{k=1}^{\infty}, \Phi(\cdot)) \in \mathcal{W}_m[0, T] : f_i((w_k(\cdot))_{k=1}^{\infty}, \Phi(\cdot)) > \varepsilon_2\}$$

is an open subset of $\mathcal{W}_m[0, T]$ with respect to the topology of Definition 1.7. Thus by the definition of weak convergence of probability measures we have

$$\lim_{N \to \infty} \mathbb{P}_N(H_i) \geq \mathbb{P}(H_i) \geq \mathbb{P}\left(\inf_{t' - \varepsilon_1 \leq t \leq t'} \theta(t) > \varepsilon_2\right) > \varepsilon_3$$

from which the claim of the lemma easily follows by (79).

**Lemma 4.5.** If $\frac{1}{N} \ll \mu(N)$ and $0 < \lambda_{\inf} \leq \lambda(t)$, then for every $\varepsilon_2 > 0$ there is a $\varepsilon_4 > 0$ such that for every $\varepsilon_2 > 0$ there is a $K$ and an $N_1$ such that

$$\forall N \geq N_1 \quad \sum_{k > K} \nu_k^N(0) \geq \varepsilon_2 \implies \mathbb{E}_N (\Phi^N(t)) \geq \varepsilon_4$$

(82)

The proof of Lemma 4.5 will follow as a consequence of the Lemmas 4.6 and 4.7.

**Proof of Lemma 4.2** We are going to show that if there is a sequence $\mathbb{P}_N$ such that the weak limit point $\mathbb{P}$ violates (73) then for some $N$ we have

$$\mathbb{E}_N (\Phi^N(T)) > m_0(0)$$

(83)

which is in contradiction with (13).

We define $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ and $t^*$ using Lemma 4.2. Next, we define $\varepsilon_4$ using this $\varepsilon_2$ and Lemma 4.5. Given these, we choose $\tilde{t}$ be so small that

$$\left|\frac{\varepsilon_1}{2t}\right| \varepsilon_3 \varepsilon_4 > m_0(0).$$

We choose $K$ and $N_1$ big enough so that (82) holds for this $\tilde{t}$. Further on, we fix the intervals $[\alpha_i, \beta_i], 1 \leq i \leq m = \left\lceil \frac{\beta_1}{\varepsilon_1} \right\rceil$ so that $\alpha_{i+1} - \beta_i > \tilde{t}$ holds for all $i$ and also $T - \beta_m > \tilde{t}$ holds. We choose $N_0$ such that (80) holds and let $N := \max\{N_0, N_1\}$.

Finally, we define the stopping times $\tau_1, \tau_2, \ldots, \tau_m$ by

$$\tau_i := \beta_i \wedge \min\{t : t \geq \alpha_i \text{ and } \sum_{k > K} \nu_k^N(t) \geq \varepsilon_2\}.$$

We have $\tau_i + t^* \leq \beta_i + t^* < \alpha_{i+1} \leq \tau_{i+1}$.

Using the strong Markov property, (82) and (80), the inequality (83) follows:

$$\mathbb{E} (\Phi^N(T)) \geq \sum_{i=1}^{m} \mathbb{E} (\Phi^N(\tau_i + t^*) - \Phi^N(\tau_i)) \geq$$

$$\sum_{i=1}^{m} \mathbb{E}\left(\mathbb{E}\left((\Phi^N(\tau_i + t^*) - \Phi^N(\tau_i)) | \mathcal{F}_{\tau_i}\right)\mathbb{P}\left(\sum_{k > K} \nu_k^N(\tau_i) \geq \varepsilon_2\right)\right) \geq$$

$$\sum_{i=1}^{m} \varepsilon_4 \mathbb{P}\left(\sum_{k > K} \nu_k^N(\tau_i) \geq \varepsilon_2\right) \geq m \varepsilon_4 \varepsilon_3 > m_0(0).$$
For a frozen percolation evolution defined by (9) we have

\[ U(t, x) = \sum_{k \geq 1} v_k N(t) e^{-kx} - (m_0(0) - \Phi^N(t)) = V(t, x) - m_0^N(t) = \sum_{k \geq 1} v_k N(t) \left( e^{-kx} - 1 \right) \]  

(84)

We will make use of the following generating function estimates in the proof of Lemma 4.6.

If \( U(x) = \sum_{k \geq 1} v_k \left( e^{-kx} - 1 \right) \) where \( v \in V \) then

\[ \sum_{k > K} v_k \geq \varepsilon \implies U(1/K) \leq (e^{-1} - 1)\varepsilon \]  

(85)

\[ U(1/K) \leq -\varepsilon \implies \sum_{k > \frac{\varepsilon}{2}} v_k \geq \varepsilon / 2. \]  

(86)

**Lemma 4.6.** There are constants \( C_1 < +\infty, \ C_2 > 0, \ C_3 > 0 \) such that if

\[ \sum_{k > K} v_k^N(0) \geq \varepsilon_2 \]  

(87)

for all \( N \) then

\[ \lim_{N \to \infty} \mathbb{P} \left( \sum_{k > C_3 \varepsilon_2 N^{1/3}} v_k^N(T) + \Phi^N(T) \geq C_2 \varepsilon_2 \right) = 1 \]  

(88)

Where \( T = \frac{C_1}{K \varepsilon_2} \).

**Sketch proof.** If we let \( N \to \infty \) immediately, then by Lemma 4.1 we get that the limiting functions \( v_1(t), v_2(t), \ldots \) solve (14) with initial condition \( v(0) \), a possibly random control function \( \Phi(t) \) and some nonnegative rate function \( \lambda(t) \).

The \( N \to \infty \) limit of (88) is

\[ \theta(T) + \Phi(T) \geq C_2 \varepsilon_2 \]  

(89)

Now we prove that if \( v(\cdot) \) is a solution of (14) then \( \sum_{k > K} v_k(0) \geq \varepsilon_2 \) implies (89) with \( C_1 = 4 \) and \( C_2 = \frac{1}{7} \). This proof will also serve as an outline of the proof of Lemma 4.6.

In order to prove (89) define \( V(t, x) \) by (32). Thus \( V(t, x) \) solves

\[ \dot{V}(t, x) = V'(t, x) \cdot (m_0(0) - \Phi(t) + \lambda(t) - V(t, x)) \]  

(90)

Define \( U(t, x) \) by (41). Define the characteristic curve \( x(\cdot) \) by

\[ \dot{x}(t) = V(t, x(t)) - (m_0(0) - \Phi(t) + \lambda(t)) \quad x(0) = \frac{1}{K} \]  

(91)

Let \( v(t) := V(t, x(t)) \). Now by (90) and (91) we get

\[ \dot{v}(t) = V(t, x(t)) + V'(t, x(t)) \dot{x}(t) = 0 \]  

(92)
Thus $v(t) \equiv v(0)$, moreover by (41) we get $U(t,x(t)) - U(0,x(0)) = \Phi(t)$ and by $V(t,x(t)) \equiv V(0,x(0))$, $V(0,x(0)) - m_0(0) = U(0,x(0))$ and (91) we get

$$x(t) = \frac{1}{K} + \int_0^t \Phi(s) ds - \int_0^t \lambda(s) ds + t \cdot U(0, \frac{1}{K})$$

(93)

By (85) we have $U(0, \frac{1}{F}) \leq -\frac{1}{4} \varepsilon_2$. In order to prove that $\theta(\overline{\tau}) + \Phi(\overline{\tau}) \geq \frac{1}{4} \varepsilon_2$ with $\overline{\tau} = \frac{4}{K \varepsilon_2}$ we consider two cases:

If $\Phi(\overline{\tau}) \geq \frac{1}{4} \varepsilon_2$ then we are done. If $\Phi(\overline{\tau}) < \frac{1}{4} \varepsilon_2$ define $\tau := \min \{ t : x(t) = 0 \}$.

By (93) we have

$$x(\overline{\tau}) \leq \frac{1}{K} + \epsilon(\overline{\tau}) \Phi(\overline{\tau}) + \frac{1}{2} \varepsilon_2 \leq \frac{1}{K} + \frac{1}{2} \varepsilon_2 - \frac{1}{2} \varepsilon_2 = 0$$

Thus $\tau \leq \overline{\tau}$.

$$-\theta(\tau) = U(\tau,0) = U(\tau,x(\tau)) = U(0, \frac{1}{K}) + \Phi(\tau) \leq -\varepsilon_2 + \frac{1}{2} \varepsilon_2 = -\frac{1}{4} \varepsilon_2$$

Thus $\frac{1}{4} \varepsilon_2 \leq \theta(\tau) \leq \theta(\tau) + \Phi(\tau) \leq \theta(\overline{\tau}) + \Phi(\overline{\tau})$ because by (13) the function $\theta(t) + \Phi(t)$ is increasing.

To make this proof work for Lemma 4.6 we have to deal with the fluctuations caused by randomness and combinatorial error terms.

**Proof.** Given a frozen percolation evolution obtained from a Markov process by (9) define $U$ and $V$ by (84).

Using (71) a straightforward calculation shows that

$$LV(t,x) := \lim_{h \to 0} \frac{1}{h} E \left( V(t+h,x) - V(t,x) \mid \mathcal{F}_t \right) =$$

$$V'(t,x) \left( (m_0(0) - \Phi^N(t)) + \lambda(t) \mu(N) - V(t,x) \right) + \frac{1}{N} \left( V''(t,x) - V''(t,2x) \right)$$

(94)

Given the random function $V(t,x)$ we define the random characteristic curve $x(t)$ similarly to (91):

$$\dot{x}(t) = V(t,x(t)) - \left( (m_0(0) - \Phi^N(t)) + \lambda(t) \mu(N) \right), \quad x(0) = \frac{1}{K}$$

(95)

This ODE is well-defined although $V(t,x)$ is not continuous in $t$, but almost surely it is a step function with finitely many steps which is a sufficient condition to have well-posedness for the solution of (95). Define $v(t) := V(t,x(t))$.

$$x(t) = \frac{1}{K} + \int_0^t (v(s) - v(0)) ds + \int_0^t \Phi^N(s) ds - \mu(N) \int_0^t \lambda(s) ds + t \cdot U(0, \frac{1}{K})$$

(96)
Putting together (94) and (95) we get
\[
\lim_{h \to 0} \frac{1}{h} \mathbb{E}(\varpi(t+h) - \varpi(t) \mid \mathcal{F}_t) = \frac{1}{N} \left( V''(t,x(t)) - V''(t,2x(t)) \right) = O \left( \frac{1}{N} V''(t,x(t)) \right)
\]  
(97)

Thus \( \bar{\varpi}(t) = \varpi(t) - \int_0^t \frac{1}{N} (V''(s,x(s)) - V''(s,2x(s))) \, ds \) is a martingale and by (4) and (3) we get
\[
\lim_{h \to 0} \frac{1}{h} \mathbb{E}(\bar{\varpi}(t+h)^2 - \bar{\varpi}(t)^2 \mid \mathcal{F}_t) = \lim_{h \to 0} \frac{1}{h} \mathbb{E} \left( (\varpi(t+h,x(t)) - \varpi(t,x(t)))^2 \mid \mathcal{F}_t \right)
\]
\[
\leq \frac{1}{2} \sum_{k,l=1}^N \left( \frac{k+l}{N} e^{-(k+l)x(t)} - \frac{k}{N} e^{-lx(t)} - \frac{l}{N} e^{-lx(t)} \right)^2 \nu_k^N(t) \nu_l^N(t) N
\]
\[
+ \sum_{i=1}^N \left( \frac{l}{N} e^{-lx(t)} \right)^2 \mu(N) \lambda(t) \nu_i^N(t) N = O \left( \frac{1}{N} V''(t,x(t)) \right)
\]  
(98)

Define the stopping time
\[
\tau_N := \min \{ t : x(t) = N^{-1/3} \}.
\]
(99)

(Note that we could replace \( N^{-1/3} \) by \( N^{-\gamma} \), \( 0 < \gamma < 1/2 \) without changing the proof.)

It follows from (46), (97), (98) and Doob’s maximal inequality that
\[
\sup_{0 \leq t \leq T} |\varpi(t \wedge \tau_N \wedge T) - \varpi(0)| \Rightarrow 0 \quad \text{as} \quad N \to \infty
\]  
(100)

By (85) and (87) we have
\[
U(0,x(0)) \leq (e^{-1} - 1) \varepsilon_2 =: -\varepsilon_5
\]  
(101)

Let
\[
A_N := \left\{ \int_0^{\tau_N \wedge T} |\varpi(s) - \varpi(0)| \, ds \leq \frac{1}{K} \right\} \cap \left\{ |\varpi(\tau_N \wedge T) - \varpi(0)| \leq \varepsilon_5/3 \right\},
\]
\[
B_N := \left\{ \Phi^N(\tau_N) \leq \varepsilon_5/3 \right\}.
\]
\[
T := \frac{3}{K |U(0,x(0))|} \leq \frac{3}{K \varepsilon_5}.
\]

We are going to show that there are constants \( C_2, C_3 < +\infty \) such that
\[
A_N \subseteq \left\{ \sum_{k > C_2 \varepsilon_2 N^{1/3}} \nu_k^N(T) + \Phi^N(T) \geq C_2 \varepsilon_2 \right\}
\]  
(102)

which together with (99) implies \( \lim_{N \to \infty} P(A_N) = 1 \) and (88).
First we show that
\[ A_N \cap B_N \subseteq \{ \tau_N \leq \bar{t} \}. \] (102)

If we assume indirectly that \( A_N, B_N \) and \( \tau_N > \bar{t} \) hold then \( \int_0^{\bar{t}} |\nu(s) - \nu(0)| \, ds \leq \frac{1}{K} \), so by (96) we get
\[
x(\bar{t}) \leq \frac{1}{K} + \frac{1}{K} + \int_0^{\bar{t}} \Phi^N(s) \, ds + \bar{t} U(0, x(0)) \leq -\frac{1}{K} + \bar{t} \varepsilon_5 \leq 0.
\]

But \( x(\bar{t}) \leq 0 \) is in contradiction with \( \tau_N > \bar{t} \), thus (102) holds. Assuming \( A_N \) and \( B_N \) we obtain
\[
|U(\tau_N, x) - U(0, x(0))| \leq |\nu(\tau_N) - \nu(0)| + \Phi^N(\tau_N) \leq \varepsilon_5/3 + \varepsilon_5/3
\]
which together with (100) implies \( A_N \cap B_N \subseteq \{ U(\tau_N, N^{-1/3}) \leq -\varepsilon_5/3 \} \)

By (86)
\[
A_N \subseteq (A_N \cap B_N) \cup B_N^c \subseteq \left\{ \sum_{k>N^{1/3} \varepsilon_5/6} \nu^N_k(\tau_N) \geq \varepsilon_5/6 \right\} \cup \left\{ \Phi^N(\tau_N) > \varepsilon_5/3 \right\}
\]
\[
\subseteq \left\{ \sum_{k>C_3 N^{1/3}} \nu^N_k(\tau_N) + \Phi^N(\tau_N) \geq C_2 \varepsilon_2 \right\}
\]
with \( C_2 = C_3 = (1 - e^{-1})/6 \). But \( \sum_{k>C_3 N^{1/3}} \nu^N_k(t) + \Phi^N(t) \) is a monotone increasing function of \( t \), from which (101) follows.

**Lemma 4.7.** There are constants \( C_4 < +\infty, C_5 > 0 \) such that if
\[
\sum_{k>C_4 N^{1/3}} \nu^N_k(0) \geq C_2 \varepsilon_2/2
\]
for all \( N \) then with
\[
\bar{t}_N := C_4 \varepsilon_2^{-2} \left( N^{-1/3} \log(N) + (N \mu(N))^{-1} \right)
\]
we have
\[
\lim_{N \to \infty} \mathbb{E} \left( \Phi^N(\bar{t}_N) \right) \geq C_5 \varepsilon_2.
\] (104)

**Remark 5.** The upper bound (103) is technical: on one hand it is not optimal, on the other hand, for the proof of Lemma 4.5 we only need \( \bar{t}_N \ll 1 \) as \( N \to \infty \).

**Proof.** If \( v \) is a vertex of the graph \( G(N, t) \) let \( C_N(v, t) \) denote the connected component of \( v \) at time \( t \). Denote by \( \tau_b(v) \) the freezing/burning time of \( v \).
\[
\mathcal{H}_N(t) := \{ v : |C_N(v, 0)| \geq C_3 \varepsilon_2 N^{1/3} \text{ and } \tau_b(v) > t \}
\]
We fix a vertex \( v \in \mathcal{H}_N(0) \).

\[
\begin{align*}
c_N(t) &:= \frac{1}{N} |C_N(v,t)| \\
w_N(t) &:= \frac{1}{N} |\mathcal{H}_N(t)| \\
z_N(t) &:= \frac{1}{N} \left| \mathcal{H}_N(0) \setminus \mathcal{H}_N(t) \right| = w_N(0) - w_N(t)
\end{align*}
\]

Thus \( c_N(t) \) is an increasing process until \( \tau_{b}(v) \), \( w_N(t) \) is decreasing, \( z_N(t) \) is increasing. We consider the right-continuous version of the processes \( c_N(t), w_N(t), z_N(t) \).

\[
w_N(0) \geq C_2 \varepsilon_2 / 2 =: \varepsilon_6.
\]

We are going to prove that there are constants \( C_4 < +\infty, C_5 > 0 \) such that

\[
\lim_{N \to \infty} \mathbf{E}(z_N(\bar{\tau}_N)) \geq C_5 \varepsilon_2
\]

(105)

with \( \bar{\tau}_N \) defined as in (103). This implies (104).

We define the stopping times

\[
\begin{align*}
\tau_w &:= \min\{t : w_N(t) < \varepsilon_6 / 2\} \\
\tau_g &:= \min\{t : c_N(t) > \varepsilon_6 / 4\} \\
\tau &:= \tau_{b}(v) \wedge \tau_w \wedge \tau_g
\end{align*}
\]

Let \( \tilde{N} := C_3 \varepsilon_2 N^{\frac{1}{2}} \). Since \( v \in \mathcal{H}_N(0) \) we have

\[
c_N(t) \geq c_N(0) = \frac{|C_N(v,0)|}{N} \geq \frac{\tilde{N}}{N}
\]

If \( C_N(v,t) \) is connected to a vertex in \( \mathcal{H}_N(t) \) by a new edge at time \( t \) then

\[
c_N(t_+) - c_N(t_-) \geq \frac{\tilde{N}}{N}, \quad \log(c_N(t_+)) - \log(c_N(t_-)) \geq \log \left( 1 + \frac{\tilde{N}}{Nc_N(t_-)} \right) \geq \frac{\log(2)\tilde{N}}{Nc_N(t_-)}
\]

\[
\lim_{dt \to 0} \frac{1}{dt} \mathbf{E}(\log(c_N(t + dt)) - \log(c_N(t)) \mid \mathcal{F}_t) \geq
\]

\[
\frac{\log(2)\tilde{N}}{Nc_N(t)} \lim_{dt \to 0} \frac{1}{dt} \mathbf{P}(c_N(t + dt) - c_N(t) \geq \frac{\tilde{N}}{N} \mid \mathcal{F}_t) \geq
\]

\[
\frac{\log(2)\tilde{N}}{Nc_N(t)} \cdot \frac{1}{N} \left| C_N(v,t) \right| \left( \left| \mathcal{H}_N(t) \right| - \left| C_N(v,t) \right| \right) \mathbb{I}_{\{t \leq \tau_{b}(v)\}} \geq
\]

\[
\log(2)\tilde{N} \cdot (w_N(t) - c_N(t)) \mathbb{I}_{\{t \leq \tau_{b}(v)\}} \geq
\]

\[
\log(2)\tilde{N} \varepsilon_6 / 4 \mathbb{I}_{\{t \leq \tau\}} = N^{1/3} \frac{\log(2)}{8} \cdot C_2 \cdot C_3 \cdot (\varepsilon_2)^2 \cdot \mathbb{I}_{\{t \leq \tau\}} =: a \cdot \mathbb{I}_{\{t \leq \tau\}}
\]
Thus \( \log(c_N(t)) - a \cdot (t \wedge \tau) \) is a submartingale. Using the optional sampling theorem we get
\[
\log(m_0(0)) - a \cdot \mathbb{E}(\tau) \geq \mathbb{E}(\log(c_N(\tau))) - a \cdot \mathbb{E}(\tau) \geq \log(c_N(0)) \geq -\log(N)
\]
By Markov’s inequality we obtain that for some constant \( C \leq +\infty \)
\[
P\left( \tau \leq CN^{-1/3} \varepsilon_2^{-2} \log(N) \right) \geq \frac{1}{2}
\]
if \( N \) is sufficiently large.

If \( \tau_g \leq \tau_b(v) \), then \( \hat{c}_N(v, \tau_g) > \frac{\varepsilon_6}{4} N \), so \( \mathbb{E}(\tau_b(v) - \tau_g) \leq (N\mu(N)\lambda_{inf})^{-1} \frac{1}{\varepsilon_6} \), which implies
\[
P\left( \tau_w \wedge \tau_b(v) \leq CN^{-1/3} \varepsilon_2^{-2} \log(N) + C'(N\mu(N))^{-1} \varepsilon_2^{-1} \right) \geq \frac{1}{4}.
\]
for some constant \( C' \). Define \( \tilde{t} \) of (103) with \( C_4 := \max\{C, C'\} \). Using the linearity of expectation we get
\[
\mathbb{E}(z(\tilde{t})) = \mathbb{E}\left( \frac{1}{N} \sum_{w \in \mathcal{H}_N(0)} \mathbb{I}_{\{\tau_b(v) \leq t\}} \right) \geq \frac{\varepsilon_6}{2} \mathbb{P}(\tau_b(v) \leq t).
\]
The inequality \( \mathbb{I}_{\{\tau_w \leq \tilde{t}\}} \frac{\varepsilon_6}{2} \leq z(\tilde{t}) \) follows from the definition of \( \tau_w \).

\[
\frac{1}{4} \leq \mathbb{P}(\tau_w \wedge \tau_b(v) \leq \tilde{t}) \leq \mathbb{P}(\tau_w \leq \tilde{t}) + \mathbb{P}(\tau_b(v) \leq \tilde{t}) \leq \mathbb{E}(z(\tilde{t})) \frac{2}{\varepsilon_6} + \mathbb{E}(z(\tilde{t})) \frac{1}{\varepsilon_6}
\]
From which (105) follows. \( \square \)

Lemma 4.5 is a straightforward consequence of Lemma 4.6 and Lemma 4.7.

5 Properties of the solutions of the frozen percolation equations

Proof of Theorem 4.5 It is clear from (63) and (65) that \( \varphi(t) \) is continuous. In order to prove (23) we need Example (c) of Theorem 4. of chapter XIII.5 of [4]. By (55)
\[
X''(t,0) = \frac{1}{E(t,0)} = \frac{1}{E(0,t)} = \frac{1}{\varphi(t)}
\]
\[
X(t,u) = \frac{1}{2\varphi(t)} u^2 + O(u^3), \quad \lim_{x \to 0} \frac{-U(t,x)}{\sqrt{x}} = \sqrt{2\varphi(t)}
\]
By the Tauberian theorem for any \( t \geq T^k \) each of the relations
\[
-U(t,x) \sim x^{1/2} \sqrt{2\varphi(t)} \quad \text{and} \quad \sum_{k=K}^{\infty} v_k(t) \sim \frac{1}{\Gamma(\frac{1}{2})} k^{1/2-1} \sqrt{2\varphi(t)}
\]
implies the other, that is for any \( t \geq T^* \)

\[
\lim_{x \to 0} -\frac{U(t, x)}{\sqrt{x}} = \sqrt{2\varphi(t)} \quad \iff \quad \lim_{k \to \infty} K^\frac{1}{2} \sum_{k=K}^{\infty} v_k(t) = \sqrt{\frac{2\varphi(t)}{\pi}}
\]

In order to compare the solutions of (19) and (17) we apply the transformations

\[
\psi(t) \to U(t, x) \to X(t, u) \to G(t, w)
\]

(106)

to the solutions of the alternating equations:

The integral equation

\[
U(t, x) = U(0, x) + \int_0^t -U(s, x)U'(s, x)ds + \Phi(t)
\]
holds, but \( \Phi(t) \) is constant between burning times and jumps by \( \theta(T_i^b) \) at \( T_i^b \), which means that the giant component is burnt:

\[
\lim_{t \to 0} -U(T_i^b + \epsilon, 0) = \lim_{t \to 0} \theta(T_i^b + \epsilon) = \theta(T_i^b) = 0
\]

By Lemma 2.1 the formulae (45), (53), (54) and (55) are valid (with rate function \( \lambda(t) \equiv 0 \)).

In between the burning times \( T_i^b < t \leq T_{i+1}^b \) we have

\[
X(t, u) = X(T_{i+1}^b, u) + (t - T_i^b)u \quad \text{and} \quad G(t, w) = G(T_{i+1}^b, w + (t - T_i^b))
\]

If \( t - T_i^b > w^*(T_{i+1}^b) \) then \( \psi(t) \) is supercritical:

\[
X'(t, 0) > 0, \quad \theta(t) > 0, \quad X(t, -\theta(t)) = 0, \quad X'(t, -\theta(t)) < 0.
\]

\[
\min_w G(t, w) = 0 \quad \text{still holds, but } \arg\min_w G(t, w) = w^*(t) < 0 \quad \text{in the supercritical phase. Thus } -X'(t, 0) = w^*(t) \text{ is well-defined for all } t \geq 0 \text{ for the solutions of the equations (16), (17) and (19) as well, moreover (59) holds. For the solutions of (19) } w^*(t) \text{ is left-continuous.}
\]

By \( G(t, w^*(t)) \equiv 0 \), (53) and (70) we get

\[
\int_0^t \Phi(s)ds = G(0, t + w^*(t)) - w^*(t)F(0, t + w^*(t))
\]

(108)

for the solutions of (19).

If \( \psi(t) \) is the solution of (16), (17) or (19) started from \( \psi(0) \in V^* \), then (55) holds: the evolution of the critical core does not depend on the rate of lightnings. One extra parameter is needed to determine \( \psi(t) \) and \( \theta(t) \): if we know \( w^*(t) \), then

\[
F(t, w) = \int_{w^*(t)}^w E(0, t + y)dy \quad \text{and} \quad G(t, w) = \int_{w^*(t)}^w (w - y)E(0, t + y)dy
\]

(109)

has all the information about \( \psi(t) \) and \( \theta(t) \), since the transformations (106) are invertible (using analytic extensions).
Proof of Claim [7] First assume \( m_0(0) = 1 \). As a consequence of Remark [4] we can see that
\[
E(t, w) = E(0, t + w) = \frac{1}{w + t} = \frac{1}{t^2} E(1, \frac{w}{t}),
\]
but this is the critical core of \( \frac{1}{t} \mathcal{W}(1) \), and together with \( w'(t) \equiv 0 \) for \( t \geq T^g = 1 \) the identity \( v_k(t) = \frac{1}{t} v_k(1) \) follows. We get the explicit formula for \( v_k(1) \) in the following way: since \( X(1, u) = X(0, u) + u \), the inverse function of \( V(1, x) \) is 
\[- \log(v) + v - 1, \]
thus
\[ V(1, x) = -W \left( -e^{-(x+1)} \right) = \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} e^{-k} e^{-kx} \]
where \( W \) is the Lambert \( W \) function, the inverse function of \( z \mapsto z e^z \).

If \( m_0(0) \neq 1 \) but we still have a monodisperse initial condition then (110) still holds and for \( t \geq \frac{1}{m_0(0)} = T^g \) we have \( w'(t) = 0 \) thus \( v_k(t) = \frac{1}{t} k^{k-1} e^{-k} \) must hold. \( \square \)

Proof of Theorem [17] Let \( H(w) := F(0, w) - m_0(0) \), thus \( H \left( \frac{1}{V(0,x)} \right) = -V(0,x) \) by (52). Using Lemma 3.4 and (54) we get
\[ F(t, w) = H(t + w) - H(t) \quad \text{and} \quad m_0(t) = F(t, +\infty) = -H(t) \]
for \( t \geq T^g \). \( v_1(0) > 0 \) implies \( \lim_{x \to \infty} \frac{V'(0,x)}{V(0,x)} = 1 \), so \( \lim_{t \to \infty} t \cdot H(tw) = -\frac{1}{w} \), from which \( \lim_{t \to \infty} t \cdot m_0(t) = 1 \) follows. Moreover
\[
1 - \frac{1}{w + 1} = \lim_{t \to \infty} t \cdot \left( H(t \cdot (w + 1)) - H(t) \right) = \lim_{t \to \infty} t \cdot F(t, tw) = \lim_{t \to \infty} \tilde{F}(t, w)
\]
where \( \tilde{v}_k(t) = tv_k(t) \). This implies the pointwise convergence of the monotone functions \( \tilde{X}'(t, u), \tilde{X}(t, u), \tilde{U}(t, x) \) and \( \tilde{V}(t, x) \) to the desired limit as \( t \to \infty \). The convergence of \( \tilde{v}_k(t) \) to \( \frac{k^{k-1}}{k!} e^{-k} \) follows from the continuity theorem of Laplace transforms. \( \square \)

Proof of Theorem [16] It is easy to check that if \( \tilde{v}_k(t) = v_k(t) e^{-kx(t)} \), then \( \tilde{V}(t, x) = V(t, x + x'(t)) \), so \( \tilde{x}'(t) = 0 \) and \( \tilde{w}'(t) = 0 \), but \( \tilde{E}(t, w) = E(0, t + w) = E(t, w) \), so \( \tilde{y}(t) \) is identical to the solution of (17) at time \( t \). \( \square \)

Proof of Theorem [10] If we consider the solution of (16) with given initial data and lightning rate function \( \lambda(t) \geq 0, 0 \leq t \leq T \) then (53) provides us with a relation between our cost \( \int_0^T \lambda(t) dt \) and reward \( \int_0^T \Phi(t) dt \).

We prove (55) by considering the cases \( T \geq T^g \) and \( T < T^g \) separately.

According to (52), for \( T \geq T^g \) we get
\[ 0 \leq G^{\text{sub}}(T, 0) = \int_0^T \Phi^{\text{cri}}(t) dt - \int_0^T \Phi^{\text{sub}}(t) dt + \int_0^T \lambda(t) dt \]
by substituting \( w = 0 \) into (55).
For \( T \leq T^\ast \), we want to prove \( 0 \geq \int_0^T \Phi^{sub}(t)dt - \int_0^T \lambda(t)dt \). Substitute \( w = T^\ast - T \) into \( (53) \). Since \( G(0,T^\ast) = 0 \) and \( (T^\ast - T)\Phi^{sub}(T) \geq 0 \) we get

\[
0 \leq G^{sub}(T,T^\ast - T) \leq - \int_0^T \Phi^{sub}(t)dt + \int_0^T \lambda(t)dt.
\]

The proof of the extremum property \( (50) \) is equally simple.

**Proof of Remark 2.** In order to prove \( (37) \) first pick an arbitrary \( \lambda > 0 \) and solve \( (67) \) with constant \( \lambda(t) = \lambda \). Since \( w^\ast(t) > 0 \) and \( w^\ast(0) = T^\ast \) there is a \( 0 < t^\ast \leq T \) such that \( w^\ast(t^\ast) = T - t^\ast \), and the lightning rate function \( \lambda(t) = \lambda \cdot \mathbb{1}[t \leq t^\ast] \) makes \( T \) a critical time, so \( (111) \) holds, thus \( (37) \).

Now we prove \( (38) \). By using \( (108) \) we have to show that

\[
G(0,T + w^\ast(T)) - (w^\ast(T) - \varepsilon)F(0,T + w^\ast(T)) > G(0,T) + \varepsilon F(0,T)
\]

Using \( G(0,T + w^\ast(T)) - G(0,T) > w^\ast(T)F(0,T) \) it is easy to see that \( 0 < w^\ast(T) \leq \varepsilon \) is sufficient for this to hold. If there is a \( T^\ast < t^\ast \leq T \) such that \( -X'(t^\ast, -\theta(t^\ast)) = T - t^\ast + \varepsilon \), then burning the giant component at time \( t^\ast \) we get \( -X'(t^\ast,0) = T - t^\ast + \varepsilon \) and \( -X'(T,0) = w^\ast(T) = \varepsilon \). If not, then burning at time \( T \) yields \( 0 < -X'(T, -\theta(T)) = w^\ast(T) < \varepsilon \).

### 6 Proof of the subcritical limit theorem

In order to prove Theorem 1.8., we need to know more about the solution of \( (67) \).

**Lemma 6.1.** If \( y(t) \) is the solution of the differential equation \( \dot{y}(t) = \frac{c}{y(t)} - 1 \) with initial condition \( y(0) = T^\ast \) and \( t \geq T^\ast + c \log(T^\ast) \) then \( y(t) \leq 2c \).

**Proof.** The solution of this differential equation is

\[
y(t) = c \cdot \left( 1 + W \left( \exp \left( \frac{T^\ast - t}{c} - 1 \right) \cdot \left( \frac{T^\ast}{c} - 1 \right) \right) \right)
\]

where \( W \) is the Lambert W function. Thus \( W(x) \leq x \) and our claim follows.

**Lemma 6.2.** If \( w^\ast(t) \) is the solution of \( (67) \) with constant \( \lambda(t) \equiv \lambda \leq 1 \) then there exist \( d_1 \) and \( d_2 \) which depend only on \( y(0) \) and \( T \) such that

\[
T^\ast + d_1 \lambda \log \left( \frac{1}{\lambda} \right) \leq t \leq T \quad \implies \quad \left| w^\ast(t) - \frac{\lambda}{E(0,t)} \right| \leq d_2 \lambda^2.
\]
Proof. We have a uniform a priori bound $w^*(t) \leq w_{\max}^*$ for all $\lambda \leq 1$ depending only on the initial data and $T$ by (69). Thus by Lemma 3.3 we have

$$0 < E_{inf} := E_{inf}(T + w_{\max}^*) \leq E(0, t + w^*(t))$$

and substituting this inequality into (67) we get

$$w^*(t) \leq \frac{\lambda}{w^*(t)E_{inf}} - 1$$

(113)

Using Lemma 6.1 we get

$$\hat{t} := T^g + \frac{\lambda}{E_{inf}} \log \left( \frac{E_{inf} T^g}{\lambda} \right) \leq t \leq T \iff w^*(t) \leq 2 \frac{\lambda}{E_{inf}}.$$

Define

$$z(t) := \frac{w^*(t)E(0, t + w^*(t))}{\lambda} - 1$$

Using (67) we get

$$\dot{z}(t) = -\frac{1}{w^*(t)} z(t) + \frac{E'(0, t + w^*(t))}{E(0, t + w^*(t))}$$

(114)

For $\hat{t} \leq t \leq T$ we have

$$-1 \leq z(t) \leq 2 \frac{E_{sup}}{E_{inf}}, \quad \frac{1}{w^*(t)} \geq \frac{1}{2 \lambda}, \quad \left| \frac{E'(0, t + w^*(t))}{E(0, t + w^*(t))} \right| \leq \lambda \frac{2D}{E_{inf}}$$

(115)

with the D of (65). Solving the linear ODE (114) and using the inequalities (115) we get

$$|z(t)| \leq 2 \frac{E_{sup}}{E_{inf}} \exp \left( -\frac{1}{2 \lambda} E_{inf} (t - \hat{t}) \right) + \frac{2D}{E_{inf}^2 \lambda^2}.$$

Thus for $t \geq \hat{t} + \frac{\lambda}{E_{inf}} \log \left( \frac{1}{\lambda} \right)$ we have $|z(t)| = O(\lambda)$, which implies

$$w^*(t) - \frac{\lambda}{E(0, t + w^*(t))} = O(\lambda^2).$$

If we combine this with

$$\left| \frac{\lambda}{E(0, t + w^*(t))} - \frac{\lambda}{E(0, t)} \right| \leq \lambda^2 \frac{2D}{E_{inf}^2}$$

the claim of the Lemma follows.

From this $\lambda m^2 \frac{\lambda}{T^g}(t) - E(0, t) = \Phi_{\lambda}(t) - \Phi_{crit}(t) = O(\lambda)$ follows which proves (31). Now we prove (33) using Laplace transforms:
Lemma 6.3. Let $U_\lambda(t,x)$ be the solution of (42) with a fixed initial condition $U(0,x)$ obtained from $\gamma(0) \in V^*$ and $\lambda(t) \equiv \lambda$. Then for any $t > T^g$ we have

$$\lim_{\lambda \to 0} \frac{U'_\lambda(t, \frac{\lambda^2}{2E(0,t)} x)}{U'_\lambda(t,0)} = \frac{1}{\sqrt{1+ x}}$$

(116)

Proof: Fix $\lambda > 0$ and denote the solution of (42) with $\lambda(t) \equiv \lambda$ by $U(t,x)$. For all $t \geq 0$ we have

$$X''(t,u) \geq \frac{1}{E_{sup}} \quad \implies \quad X(t,u) \geq \frac{1}{2E_{sup}} u^2 \quad \implies \quad |U(t,x)| = O(\sqrt{x}).$$

We use the shorthand notation $E = E(0,t + w^*(t))$.

$$X'(t,u) = -w^*(t) + \frac{u}{E} + O(u^2), \quad X(t,u) = -uw^*(t) + \frac{u^2}{2E} + O(u^3)$$

$$U(t,x) = Ew^*(t) - \sqrt{(Ew^*(t))^2 + 2E(x - O(U(t,x)^3))} = \frac{Ew^*(t) - \sqrt{(Ew^*(t))^2 + 2Ex + O(x)}}{\sqrt{w^*(t)^2 + \frac{2}{E} x}} + O(1)$$

$$U'(t,x) = \frac{1}{X'(t,U(t,x))} = \frac{1}{-w^*(t) + \frac{U(t,x)}{E}} + O(1) = \frac{-1}{\sqrt{w^*(t)^2 + \frac{2}{E} x} + O(x)} + O(1) = \frac{-1}{\sqrt{w^*(t)^2 + \frac{2}{E} x}} + O(1)$$

Because of Lemma 6.2 we have

$$\lim_{\lambda \to 0} \frac{\lambda^2}{E(0,t + w^*_\lambda(t))E(0,t)w^*_\lambda(t)^2} = 1$$

from which the claim of this lemma follows.

The r.h.s. of (116) is the Laplace transform of the $\Gamma(\frac{1}{2},1)$ distribution and the r.h.s. of (33) is the distribution function of the $\Gamma(\frac{1}{2},1)$ distribution, so (33) follows from the continuity theorem of Laplace transforms.

Proof of Theorem 1.3. First observe that instead of proving uniform convergence of $\Phi_n$ to $\Phi_{crit}$ we only need to show convergence on $[0,T]$ for any $T$, because

$$T \geq T^g \implies m_0(T) = \int_{T+w^*(T)}^{\infty} E(0,w)dw \leq \int_{T}^{\infty} \frac{1}{w^2}dw = \frac{1}{T}$$

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by (64), thus 0 ≤ \( \Phi_n(t) - \Phi_{c,\text{crit}}(t) \leq \frac{1}{T} \) for \( t \geq T \). If we prove that \( w^*(t) \) is small for \( t \geq \frac{1}{m_1(0)} \) then we are done by (70) and Lemma 3.4 since

\[
0 \leq \Phi_n(t) - \Phi_{c,\text{crit}}(t) = F(0, t + w^*_n(t)) - F(0, t) \leq w^*_n(t)E_{\text{sup}} \quad t \geq T^g
\]  

(117)

We can give an upper bound on \( w^*(t) \) for \( t \geq T^g \) if we replace \( \lambda(t) \) with \( \lambda_{\text{sup}} \) in (67): using (113) we get \( w^*(t) = O \left( \lambda \log \left( \frac{1}{T} \right) \right) \) if we substitute \( t \geq T^g \) and \( c = \frac{\lambda_{\text{sup}}}{\epsilon_{\text{inf}}} \) into (112), thus \( \lim_{n \to \infty} w^*_n(t) = 0 \) uniformly for \( T^g \leq t \leq T \).

We obtain \( \lim_{n \to \infty} V^*_k(t) = v_k(t) \) for \( k = 1, 2, \ldots \) by the uniform convergence of \( m^*_0(t) \) and \( \lambda^*(t) \) to the critical \( m_0(t) = m_0(0) - \Phi(t) \) and \( \lambda(t) \equiv 0 \) in (16).

\[
\square
\]

7 Proof of the alternating limit theorem

We turn our attention to the proof of Theorem 1.4 and Theorem 1.9.

In this section we assume \( m_0(0) = 1 \) but the results generalize easily to the \( m_0(0) \neq 1 \) case, since if \( \psi(t) \) is the solution of (19) with burning times \( T_1^b, T_2^b, \ldots \), then \( m_0(0)\psi(m_0(0)t) \) is also a solution of (19) with burning times \( \frac{T_1^b}{m_0(0)}, \frac{T_2^b}{m_0(0)}, \ldots \).

**Definition 7.1.** If \( \psi(t) \) is a solution of (19), let \( w^*_+ (t) := \frac{1}{m_1(0)} \).

If \( w^*(t) \geq 0 \) then \( w^*_+(t) = w^*(t) \), but if \( w^*(t) < 0 \) then \( w^*_+(t) = -X'(t, -\theta(t)) \).

If \( t \) is a burning time then \( w^*(t_+) := \lim_{t \to t_+} w^*(t + \varepsilon) = w^*_+(t) \).

**Lemma 7.1.** We consider the solution of (19) on \([0, T]\) with an arbitrary sequence of burning times. If \( T^g \leq t \leq T \) and \( w^*(t) < 0 \) then

\[
\theta(t) \geq \frac{m_1(0)}{m_2(0)} \frac{1}{T^2} |w^*(t)|
\]

(118)

\[
w^*_+(t) \leq 4 \sqrt{\frac{m_2(0)}{m_1(0)}} \exp \left( \frac{m_2(0)}{m_1(0)} T + 1 \right) \cdot |w^*(t)| =: C(T, \psi(0)) |w^*(t)|
\]

(119)

If \( w^*(t) < 0 \) and if

\[
w_1 + |w^*(t)| \leq t \leq w_2 - \sqrt{\frac{E_{\text{sup}}(w_1, w_2)}{E_{\text{inf}}(w_1, w_2)}} |w^*(t)|
\]

(120)

holds then

\[- \sqrt{\frac{E_{\text{inf}}(w_1, w_2)}{E_{\text{sup}}(w_1, w_2)}} w^*(t) \leq w^*_+(t) \leq - \sqrt{\frac{E_{\text{sup}}(w_1, w_2)}{E_{\text{inf}}(w_1, w_2)}} w^*(t)\]

(121)

\[-2E_{\text{inf}}(w_1, w_2) w^*(t) \leq \theta(t) \leq -2E_{\text{sup}}(w_1, w_2) w^*(t)\]

(122)
Proof. By (49), \( w^*_+(t) = -X'(t, -\theta(t)) \), (109) and (64) we get
\[
\theta(t) = F(t, w^*_+(t)) \geq \int_{w^*_+(t)}^{0} \frac{m_1(0)}{2m_2(0)} \left( \frac{1}{t+y} \right)^2 dy \geq \frac{m_1(0)}{m_2(0)} \frac{1}{T^2} |w^*(t)|
\]
Rearranging (50) and using (49) we get that \( w = w^*_+(t) \) is the positive root of the function
\[
f(w) := G(t, w) - F(t, w)w = G(t, 0) + \left( -\int_{0}^{w} yE(t, y) dy \right) = f(0) + (f(w) - f(0))
\]
We prove (119) by considering the cases \( \frac{|w^*(t)|}{t} \leq \frac{1}{4} \sqrt{\frac{m_1(0)}{m_2(0)}} \) and \( \frac{|w^*(t)|}{t} > \frac{1}{4} \sqrt{\frac{m_1(0)}{m_2(0)}} \) separately.
If \( \frac{|w^*(t)|}{t} \leq \frac{1}{4} \sqrt{\frac{m_1(0)}{m_2(0)}} \), then we prove that \( w^*_+(t) \leq 2 \sqrt{\frac{m_2(0)}{m_1(0)}} |w^*(t)| \) by showing that \( f(0) \leq |f(w) - f(0)| \) with \( w = 2 \sqrt{\frac{m_2(0)}{m_1(0)}} |w^*(t)| \).
\[
f(0) = \int_{0}^{w^*_+(t)} (-y)E(0, t+y) dy \leq \int_{0}^{\frac{|w^*(t)|}{t}} \frac{y}{(t-y)^2} dy
\]
by (109) and (64),
\[
|f(w) - f(0)| \geq \int_{0}^{w} \frac{m_1(0)}{m_2(0)} \frac{y}{(t+y)^2} dy = \int_{0}^{\frac{|w^*(t)|}{t}} \frac{m_1(0)}{m_2(0)} \frac{y}{(t+\frac{|w^*(t)|}{t})^2} dy \quad (123)
\]
It is straightforward to check that
\[
0 \leq y \leq |w^*(t)| \text{ and } \frac{|w^*(t)|}{t} \leq \frac{1}{4} \sqrt{\frac{m_1(0)}{m_2(0)}} \implies \frac{y}{(t-y)^2} \leq \frac{m_1(0)}{m_2(0)} \frac{y}{(t+\frac{|w^*(t)|}{t})^2}
\]
which is sufficient for \( f(0) \leq \left| f(2 \sqrt{\frac{m_2(0)}{m_1(0)}} |w^*(t)|) - f(0) \right| \) to hold.
If \( \frac{|w^*(t)|}{t} > \frac{1}{4} \sqrt{\frac{m_1(0)}{m_2(0)}} \), then
\[
f(0) = G(t, 0) + \int_{w^*_+(t)}^{0} F(t, y) dy \leq |w^*(t)| \leq T
\]
since by (52) we have \( F(t, y) \leq m_0(t) \leq m_0(0) = 1 \). Calculating the middle integral in (123) we get that in order to have \( f(0) \leq |f(w) - f(0)| \)
\[
\frac{m_1(0)}{m_2(0)} \left( \log(1 + \frac{w}{t}) - 1 \right) \geq T
\]
is sufficient. Rearranging this and using \( \frac{|w^*(t)|}{t} > \frac{1}{4} \sqrt{\frac{m_1(0)}{m_2(0)}} \) we obtain (119).

The proof of the upper bound of (121) is similar: using (109) we get that \( w_1 \leq \frac{t - |w^*(t)|}{t + w} \leq w_2 \) implies
\[
f(0) \leq \frac{1}{2} E_{\text{sup}}(w_1, w_2)w^*(t)^2, \quad f(w) - f(0) \leq -\frac{1}{2} E_{\text{inf}}(w_1, w_2)w^2
\]
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Using (120) the inequality \( f\left( -\sqrt{\frac{E_{\text{sup}}(w_1,w_2)}{E_{\text{inf}}(w_1,w_2)}}w^*(t) \right) \leq 0 \) follows. The lower bound of (121) is verified similarly.

If \( u \in [-\theta(t),0] \), then

\[
X(t,u) \leq -w^*(t)u + \frac{1}{2} \frac{1}{E_{\text{inf}}(w_1,w_2)} u^2,
\]
since \( X''(t,u) \) with \( u \in [-\theta(t),0] \) is equal to \( \frac{1}{E_{(0,t+y)}} \) for some

\[
y \in [w^*(t),w^*_+(t)] \subseteq [w^*(t),-\sqrt{\frac{E_{\text{sup}}(w_1,w_2)}{E_{\text{inf}}(w_1,w_2)}}w^*(t)],
\]
thus \( t+y \in [w_1,w_2] \) by (120). This implies the lower bound of (122), and the proof of the upper bound is similar.

The proof of Theorem 1.4 is similar to Theorem 1.3: if \( \varepsilon = \sup \{ T^{b+1}_i - T^{b}_i \} \) and \( T^{b}_i < t \leq T^{b+1}_i \) then \( w^*(t) = w^*_+(T^{b}_i) - (t - T^{b}_i) \geq -\varepsilon \) and by (119) we have \( w^*_+(T^{b}_i) = o(|w^*(T^{b}_i)|) = o(\varepsilon) \) on \([0,T]\).

**Lemma 7.2.** We consider the solution of (12) with initial critical core \( E(0,w) \). If \( T^g_1 < T^g_2 \) are two consecutive gelation times, then the unique burning time in between \( T^g_1 \) and \( T^g_2 \) is

\[
T^b(T^g_1,T^g_2) = \frac{\int_{T^g_1}^{T^g_2} yE(0,y)dy}{\int_{T^g_1}^{T^g_2} E(0,y)dy}
\]

Moreover

\[
\theta(T^b(T^g_1,T^g_2)) = \int_{T^g_1}^{T^g_2} E(0,y)dy
\]

**Proof.** \( T^b \) needs to satisfy \( T^g_2 - T^b = w^*_+(T^b) \), but by the proof of Lemma 7.1 \( w^*_+(T^b) \) is the unique positive root of \( G(T^b,0) - \int_0^w yE(0,T^b+y)dy \). \( G(T^b,0) = -\int_{T^g_1}^{T^g_2} yE(0,T^b+y)dy \) by (109), so \( \int_{T^g_1}^{T^g_2} yE(0,T^b+y)dy = 0 \) must hold, from which (124) easily follows.

By (49), \( w^*_+(t) = -X'(t,-\theta(t)) \) and (109) we get

\[
\theta(T^b) = F(T^b,w^*_+(T^b)) = \int_{w^*_+(T^b)}^{w^*_+(T^b)} E(0,T^b+y)dy = \int_{T^g_1}^{T^g_2} E(0,y)dy
\]

**Definition 7.2.** If \( \tilde{y}(t) \) is the solution of the random alternating equations (see Definition 7.6), denote by \( T^b_1 < T^b_2 < \ldots \) the sequence of random burning times...
and by $T^g = T_1^g < T_2^g < \ldots$ the sequence of random gelation times. Indeed $T_1^g < T_1^b < T_2^g < T_2^b < \ldots$

Let $\tau_i := T_{i+1}^g - T_i^g$ be the length of the $i$-th critical interval.

$$N(t) := \max \{ i : T_i^g < t \}, \quad \tau(t, i) := \tau_{N(t)+i}$$

$\tau(t, 0)$ is the length of the critical interval containing $t$.

Let $\theta(t, i) := \theta(T_{N(t)+i}^b)$, thus $\theta(t, 1)$ is the frozen mass of the first giant component born after $t$.

$$w^-(t, i) := T_{N(t)+i}^b - T_i^g = -w^+(T_{N(t)+i}^b)$$
$$w^+(t, i) := T_i^g - T_{N(t)+i}^b = w^+(T_{N(t)+i}^b)$$

**Definition 7.3.** A nonnegative random variable $X$ has Rayleigh distribution with parameter $\sigma$, briefly $X \sim R(\sigma)$, if

$$P(X > x) = \exp(-\frac{1}{2\sigma^2}x^2) = : R(\sigma, x)$$

$$E(X) = \sigma \sqrt{\frac{\pi}{2}}, \quad Y \sim \text{size-biased Rayleigh distribution with parameter } \sigma, \text{ briefly } Y \sim R_{sb}(\sigma)$$

if

$$P(Y > y) = \frac{E(X \cdot 1[X > y])}{E(X)} = R_{sb}(\sigma, y)$$

The scaling identities

$$R(\alpha \sigma, \alpha x) = \frac{R(\sigma, x)}{\alpha}, \quad R_{sb}(\alpha \sigma, \alpha x) = R_{sb}(\sigma, x)$$

are valid for $\alpha > 0$.

The r.h.s of (34) is $R_{sb}(\frac{1}{\sqrt{2\lambda}}, x)$.

The Rayleigh distribution emerges in our setting in the following way: if we consider the solution of the random alternating equations with burning times defined by a homogenous Poisson process with rate $\lambda$, forget about the error terms in (122) by assuming $w_1 = w_2$ then $\theta(t) = 2E \cdot (t - T_i^g)$ if $T_i^g < t \leq T_i^b$, so

$$P\left(T_i^b - T_i^g > w\right) = \exp(-\frac{1}{\sqrt{2E\lambda}}\int_0^w \lambda \cdot \frac{1}{\sqrt{2E\lambda}} \cdot \frac{1}{\sqrt{2E\lambda}} dw).$$

From $\theta(T_i^b) = 2E \cdot (T_i^b - T_i^g)$ and (126) we get $\theta(T_i^b) \sim R\left(\sqrt{\frac{2E}{\lambda}}\right)$. Assuming $w_1 = w_2$ in (121) we get $w^+(T_i^b) = -w^+(T_i^b)$, thus $\tau_i \sim R\left(\sqrt{\frac{2E}{\lambda}}\right)$.

**Lemma 7.3.** If $v(t)$ is the solution of the random alternating equations with constant rate function $\lambda(t) \equiv \lambda$ then for every $T^g \leq t \leq T$ we have

$$E\left(\theta(T_{N(t)}^b) \mid \tau_{N(t)}^g < T\right) = o(\lambda^{-1})$$

$$E\left(T_{N(t)}^g \mid t \right) = o(\lambda^{-1})$$

as $\lambda \to \infty$ where the constant in the $O$ depends only on the initial data and $T$. 

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Proof. Let $\gamma(t) := t - T_{N(t)}^g$. Then

$$\lim_{dt \to 0} \frac{1}{dt} E(\gamma(t + dt) - \gamma(t) \mid \mathcal{F}_t) = 1 - \gamma(t) \lim_{dt \to 0} \frac{1}{dt} P \left( t \leq T_{N(t)}^g + 1 \leq t + dt \mid \mathcal{F}_t \right) =$$

$$1 - \gamma(t) \lim_{dt \to 0} \frac{1}{dt} P \left( b(T_{N(t)}^g, t) \leq T_{N(t)}^b \leq b(T_{N(t)}^g, t + dt) \mid \gamma(t) \right) =$$

$$1 - \gamma(t) \theta(T_{N(t)}^g, t) \frac{d}{ds} T_{N(t)}^b \bigg|_{s=t} =$$

$$1 - \lambda E(0, t) \gamma(t) \left( t - T_{N(t)}^b - \gamma(t), t \right) \leq 1 - \frac{1}{2} \frac{E_{\inf}(T)^2}{E_{\sup}} \gamma(t)^2$$

by Lemma 7.2. Taking the expectation of both sides of the above inequality and applying Jensen’s inequality we get

$$\frac{d}{dt} E(\gamma(t)) \leq 1 - \frac{1}{2} \lambda \frac{E_{\inf}(T)^2}{E_{\sup}} E(\gamma(t))^2.$$  

This differential inequality together with $\gamma(T^g) = 0$ implies

$$E(\gamma(t)) \leq \frac{1}{2} \frac{\sqrt{2E_{\sup}}}{\sqrt{\lambda} E_{\inf}(T)} = O\left( \lambda^{-\frac{1}{2}} \right), \quad T^g \leq t \leq T.$$  

by a "forbidden region"-type argument. Now we prove

$$E \left( T_{N(t)}^g + 1 \wedge T - T_{N(t)}^g \right) = O\left( \lambda^{-\frac{1}{2}} \right)$$  

(129)

from which (128) trivially follows. We obtain (127) using (129) and $\theta(T_{N(t)}^b) \leq 2E_{\sup} \cdot (T_{N(t)}^b - T_{N(t)}^g)$ by the upper bound of (122).

$$T_{N(t)}^g + 1 \wedge T - T_{N(t)}^g = \gamma(t) + \left( T_{N(t)}^g + 1 \wedge T - t \right) \mathbb{1}[t \geq T_{N(t)}^b] +$$

$$\left( T_{N(t)}^g + 1 \wedge T - T_{N(t)}^b \wedge T \right) \mathbb{1}[t < T_{N(t)}^b] + \left( T_{N(t)}^b \wedge T - t \right) \mathbb{1}[t < T_{N(t)}^b]$$

$$\left( T_{N(t)}^g + 1 \wedge T - t \right) \mathbb{1}[t \geq T_{N(t)}^b] \leq w_+^g(t, 0) \mathbb{1}[t \geq T_{N(t)}^b] \leq$$

$$C(T, \mathbf{y}(0)) w_+^g(t, 0) \mathbb{1}[t \geq T_{N(t)}^b] \leq C(T, \mathbf{y}(0)) \gamma(t)$$

where $C(T, \mathbf{y}(0))$ is defined in (119).

$$\left( T_{N(t)}^g + 1 \wedge T - T_{N(t)}^b \wedge T \right) \mathbb{1}[t < T_{N(t)}^b] \leq w_+^g(t, 0) \mathbb{1}[t < T_{N(t)}^b] \leq$$

$$C(T, \mathbf{y}(0)) \gamma(t) + C(T, \mathbf{y}(0)) \left( T_{N(t)}^b \wedge T - t \right) \mathbb{1}[t < T_{N(t)}^b]$$

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By (22) and (118) we have
\[
E \left( \left( T_{N(t)}^b \wedge T - t \right) \mathbb{I}[t < T_{N(t)}^b] \right) = E \left( \left( T_{N(t)}^b \wedge T - t \right) \vee 0 \right) = \\
\int_0^{T-t} \mathbb{P} \left( T_{N(t)}^b - t \geq x \right) dx \leq \int_0^{T-t} \exp \left( -\lambda \int_0^x \frac{m_1(0)}{m_2(0)} \frac{1}{T^2} y dy \right) = O(\lambda^{-\frac{1}{2}})
\]
\[
E \left( T_{N(t)+1}^g \wedge T - T_{N(t)}^g \right) = O(E(\gamma(t))) + O \left( E \left( \left( T_{N(t)}^b \wedge T - t \right) \vee 0 \right) \right) = O(\lambda^{-\frac{1}{2}})
\]
\]
Sketch proof of Theorem 7.9
Our aim is to make the following argument rigorous: Let
\[
n(\lambda) := \lfloor \delta(\lambda) \sqrt{\frac{E(t,0)}{\pi}} \rfloor.
\]
If \( 1 \ll \lambda \) then \( \theta(t,1), \theta(t,2), \ldots, \theta(t,n(\lambda)) \) are "almost" i.i.d. with distribution \( \theta(t,i) \sim R(\sqrt{\frac{2E(t,0)}{\lambda}}) \). \( \tau(t,i) \approx \frac{\theta(t,i)}{E(t,0)} \), so
\[
\sum_{i=1}^{n(\lambda)} \tau(t,i) \approx \delta(\lambda)
\]
by the weak law of large numbers. Substituting \( \hat{x} = 2\sqrt{\frac{E(t,0)}{\lambda}} x \) into
\[
\frac{\Phi(t + \delta(\lambda), \hat{x}) - \Phi(t, \hat{x})}{\delta(\lambda) E(t,0)} \approx \frac{\sum_{i=1}^{n(\lambda)} \theta(t,i) \cdot \mathbb{I}[\theta(t,i) > \hat{x}]}{\sum_{i=1}^{n(\lambda)} \theta(t,i)} \approx \frac{E(\theta(t,1) \mathbb{I}[\theta(t,1) > \hat{x}])}{E(\theta(t,1))}
\]
we get (34).
\]
Proof of Theorem 7.9
We use the notations of Definitions 7.2 and 7.3
\[
E := E(t,0) = E(0,t) = \Phi_{crit}(t)
\]
We fix \( x \geq 0 \) and define
\[
\hat{x} := 2\sqrt{\frac{E}{\lambda}} x, \quad \theta(t,i, \hat{x}) := \theta(t,i) \mathbb{I}[\theta(t,i) > \hat{x}], \quad n(\lambda, z) := \lfloor \delta(\lambda) \sqrt{\frac{E \lambda}{\pi}} (1 + z) \rfloor
\]
By the assumption \( \lambda^{-\frac{1}{2}} \ll \delta(\lambda) \) we have \( \lim_{\lambda \to 0} n(\lambda, z) = +\infty \) for any \( -1 < z \).
Let \( m(\lambda) := N(t + \delta(\lambda)) - N(t) - 1 \).
\[
\Phi(t + \delta(\lambda), \hat{x}) - \Phi(t, \hat{x}) = \theta(t,0, \hat{x}) \mathbb{I}[T_{N(t)}^b > t] + \\
\sum_{i=1}^{m(\lambda)} \theta(t,i, \hat{x}) + \theta(t + \delta(\lambda), 0, \hat{x}) \mathbb{I}[T_{N(t + \delta(\lambda))}^b > t + \delta(\lambda)]
\]
(130)
In order to prove \((34)\) we only need to show that we have \(\lim_{\lambda \to \infty} P(B(\lambda, \varepsilon)) = 1\) for every \(\varepsilon > 0\) where

\[
B(\lambda, \varepsilon) := \left\{ R_{sb} \left( \frac{1}{\sqrt{2}} \right) x - \varepsilon < \frac{\sum_{i=1}^{n(\lambda)} \theta(t, i, \tilde{x})}{E\delta(\lambda)} < R_{sb} \left( \frac{1}{\sqrt{2}} \right) x + \varepsilon \right\}
\]

because the first and the last term on the r.h.s. of \((130)\) divided by \(E\delta(\lambda)\) converge to 0 in probability as \(\lambda \to \infty\) by \((127)\) and \(\lambda^{-\frac{1}{2}} \ll \delta(\lambda)\).

\[
E_{sup}(\lambda) := E_{sup}(t, t + 2\delta(\lambda)), \quad E_{inf}(\lambda) := E_{inf}(t, t + 2\delta(\lambda))
\]

By \((66)\) we have

\[
E_{sup}(\lambda) \leq E + 2D\delta(\lambda) \quad \text{and} \quad E - 2D\delta(\lambda) \leq E_{inf}(\lambda) \quad (131)
\]

\[
C^u(\lambda) := 1 + \sqrt{\frac{E_{sup}(\lambda)}{E_{inf}(\lambda)}} \quad C^l(\lambda) := 1 + \sqrt{\frac{E_{inf}(\lambda)}{E_{sup}(\lambda)}}
\]

\[
\lim_{\lambda \to \infty} C^u(\lambda) = \lim_{\lambda \to \infty} C^l(\lambda) = 2, \text{ since } \delta(\lambda) \ll 1.
\]

We are going to couple the random variables \(T_{N(t)+1}^g, w^+_1(t, 1), w^+_2(t, 2), \ldots\) to

\[
w^l_1(1), w^l_2(2), \ldots \quad \text{and} \quad w^u_1(1), w^u_2(2), \ldots
\]

where \(w^l_1(i) \sim R(\frac{1}{\sqrt{2E_{sup}(\lambda)\lambda}})\) are i.i.d. and \(w^u_1(i) \sim R(\frac{1}{\sqrt{2E_{inf}(\lambda)\lambda}})\) are i.i.d., moreover the auxiliary random variables are independent from \(T_{N(t)+1}^g\). If we define the events

\[
A^u(\lambda, z, z_2) := \left\{ T_{N(t)+1}^g + C^u(\lambda) \cdot \sum_{j=1}^{n(\lambda, z)} w^u_j(j) \leq t + \delta(\lambda) \cdot (1 + z_2) \right\}
\]

\[
A^l(\lambda, z, z_2) := \left\{ T_{N(t)+1}^g + C^l(\lambda) \cdot \sum_{j=1}^{n(\lambda, z)} w^l_j(j) \geq t + \delta(\lambda) \cdot (1 + z_2) \right\}
\]

then it is an easy consequence of \((128)\), \(\lambda^{-\frac{1}{2}} \ll \delta(\lambda)\), and the weak law of large numbers that

\[
-1 < z < z_2 \implies \lim_{\lambda \to \infty} P(A^u(\lambda, z, z_2)) = 1
\]

\[
z > z_2 > -1 \implies \lim_{\lambda \to \infty} P(A^l(\lambda, z, z_2)) = 1
\]

Our coupling is going to satisfy

\[
A^u(\lambda, z, 1) \subseteq \bigcap_{i=1}^{n(\lambda, z)} \{ w^l_1(i) \leq w^+_1(t, i) \leq w^u_1(i) \} \quad (132)
\]

for any \(z\).
The joint construction of \( w_+^j(j), \ w_-^j(t, j) \) and \( w^u(j) \) for \( j = 1, 2, \ldots \) is as follows: given \( T_{N(t)+1}^g \) and \( w^+(t, 1), \ldots, w^+(t, j-1) \) we can determine \( T_{N(t)+j}^g \) by solving (19). For \( s \geq 0 \) let

\[
\mu(s) := \lambda \theta(T_{N(t)+j}^g + s), \quad \mu_t(s) := \lambda 2E_{sup}(\lambda)s, \quad \mu_u(s) := \lambda 2E_{inf}(\lambda)s.
\]

Let \( w^+(t, j), w^-_+(j) \) and \( w^u(j) \) be the horizontal coordinate of the leftmost point below the graphs of \( \mu, \mu_t \) and \( \mu_u \) of the same standard uniform 2-dimensional Poisson process on the first quadrant of the plane. Thus \( w^+(j) \sim R(\frac{1}{2E_{sup}(\lambda)\lambda}), w^u(j) \sim R(\frac{1}{\sqrt{2E_{inf}(\lambda)\lambda}}) \) are independent from everything that was constructed earlier and \( \mathbb{P}(w^+(j) \leq w^u(j)) = 1 \). The joint distribution of \( T_{N(t)+1}^g, w^+(t, 1), \ldots, w^+(t, j) \) agrees with that of the solution of the random alternating equation.

We are going to prove (132) by induction. Assume that \( A^u(\lambda, z, 1) \) holds. If

\[
\bigcap_{i=1}^{j-1} \{ w^+(i) \} \cap \bigcap_{i=1}^{j-1} \{ \tau(t, i) \leq C^u(\lambda) \cdot w^u(i) \}
\]

(133) holds for some \( j \leq n(\lambda, z) \), then

\[
T_{N(t)+j}^g = T_{N(t)+1}^g + \sum_{i=1}^{j-1} \tau(t, i) \leq T_{N(t)+1}^g + C^u(\lambda) \sum_{i=1}^{j-1} w^u(i)
\]

which implies \( \mu_u(s) \leq \mu(s) \leq \mu_t(s) \) for \( 0 \leq s \leq w^u(j) \) by (122) and \( A^u(\lambda, z, 1) \).

From this \( w^+(j) \leq w^+(t, j) \leq w^u(j) \) follows, and (121) can be applied to deduce

\[
\tau(t, j) = w^+(t, j) + w^u(t, j) \leq \left( 1 + \frac{E_{sup}(\lambda)}{E_{inf}(\lambda)} \right) w^u(t, j) \leq C^u(\lambda) w^u(j)
\]

Thus we can replace \( j \) with \( j + 1 \) in (133). This completes the proof of (132). Let

\[
\theta^u(t, i, \hat{x}) := 2E_{sup}(\lambda)w^u(i) \cdot \mathbb{I}[2E_{sup}(\lambda)w^u(i) > \hat{x}]
\]

\[
\theta^l(t, i, \hat{x}) := 2E_{inf}(\lambda)w^l(i) \cdot \mathbb{I}[2E_{inf}(\lambda)w^l(i) > \hat{x}]
\]

(122) and (132) imply

\[
A^u(\lambda, z, 1) \subseteq \bigcap_{i=1}^{n(\lambda, z)} \{ \theta^l(t, i, \hat{x}) \leq \theta(t, i, \hat{x}) \leq \theta^u(t, i, \hat{x}) \}
\]

\[
B^u(\lambda, z, \varepsilon) := \left\{ \frac{\sum_{i=1}^{n(\lambda, z)} \theta^u(t, i, \hat{x})}{E_{sup}(\lambda)} \leq R_{sb}(\frac{1}{\sqrt{2}} \cdot x) + \varepsilon \right\}
\]

\[
B^l(\lambda, z, \varepsilon) := \left\{ \frac{\sum_{i=1}^{n(\lambda, z)} \theta^l(t, i, \hat{x})}{E_{inf}(\lambda)} \geq R_{sb}(\frac{1}{\sqrt{2}} \cdot x) - \varepsilon \right\}
\]

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The law of large numbers, \((126)\) and \((131)\) imply that
\[ z < \varepsilon \implies \lim_{\lambda \to \infty} P\left(B^\mu(\lambda, z, \varepsilon)\right) = 1 \quad \text{and} \quad -\varepsilon < z \implies \lim_{\lambda \to \infty} P\left(B^\mu(\lambda, z, \varepsilon)\right) = 1. \]

We can use \((132)\) and \((121)\) to show
\[ A^\mu(\lambda, z, 1) \subseteq \bigcap_{i=1}^{n(\lambda, z)} \{ C^\mu(\lambda)w^\mu_-(i) \leq \tau(t, i) \leq C^\mu(\lambda)w^\mu_+(i) \}. \]

Since
\[ m(\lambda) = \max\{ j : T^g_{N(t)+1} + \sum_{i=1}^{j} \tau(t, i) < t + \delta(\lambda) \} \]
and \(A^\mu(\lambda, z, 0) \subseteq A^\mu(\lambda, z, 1)\) by definition,
\[ A^\mu(\lambda, z, 0) \subseteq \{ m(\lambda) \geq n(\lambda, z) \}, \quad A^\mu(\lambda, z, 0) \cap A^\mu(\lambda, z, 1) \subseteq \{ m(\lambda) \leq n(\lambda, z) \}, \]
\[ A^\mu(\lambda, z, 0) \cap A^\mu(\lambda, z, 1) \cap B^\mu(\lambda, z, \varepsilon) \cap B^\mu(\lambda, -z, \varepsilon) \cap A^\mu(\lambda, -z, 0) \subseteq B(\lambda, \varepsilon) \]
This completes the proof of \(\lim_{\lambda \to \infty} P\left(B(\lambda, \varepsilon)\right) = 1. \)

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References

[1] David J. Aldous. Deterministic and stochastic models for coalescence (aggregation and coagulation): a review of the mean-field theory for probabilists. *Bernoulli*, 5(1):3–48, 1999.

[2] David J. Aldous. The percolation process on a tree where infinite clusters are frozen. *Math. Proc. Cambridge Philos. Soc.*, 128(3):465–477, 2000.

[3] B. Drossel and F. Schwabl. Self-organized critical forest-fire model. *Phys. Rev. Lett.*, 69(11):1629–1632, Sep 1992.

[4] William Feller. *An introduction to probability theory and its applications. Vol. II.* Second edition. John Wiley & Sons Inc., New York, 1971.

[5] A. Lushnikov. Some new aspects of coagulation theory. *Izv. Akad. Nauk. SSSR, Ser. Fiz. Atmosfer. I Okeana* vol 14, no 10, 738-743, 1978.
[6] Balazs Rath and Balint Toth. Erdos-Renyi random graphs + forest fires = self-organized criticality. arXiv:0808.2116v1, 2008.

[7] J. van den Berg and B. Tóth. A signal-recovery system: asymptotic properties, and construction of an infinite-volume process. *Stochastic Process. Appl.*, 96(2):177–190, 2001.

[8] R. M. Ziff, M. H. Ernst and E. M. Hendriks Kinetics of gelation and universality *J. Phys. A: Math. Gen.* vol 16, 2293–2320, 1983.