CLASSIFYING RESOLVING SUBCATEGORIES

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Abstract. We use the theory of Auslander Buchweitz approximations to classify certain resolving subcategories containing a semidualizing or a dualizing module. In particular, we show that if the ring has a dualizing module, then the resolving subcategories containing maximal Cohen-Macaulay modules are in bijection with grade consistent functions and thus are the precisely the dominant resolving subcategories.

1. Introduction

Classifying various types of subcategories of mod(\(R\)) and \(D(\mathcal{R})\) for a commutative ring \(R\) has been the subject of much recent research. These classifications are intrinsically connected to \(\text{spec} \ R\) or some other topological space. For instance, the Hopkins Neeman Theorem in [14] and [18] and Gabriel's Theorem in [11] give a bijection between the Serre subcategories of \(\text{mod}(\mathcal{R})\), the thick subcategories of perfect complexes, and the specialization closed subsets of \(\text{spec} \ R\). Another example is the work in classifying thick subcategories of \(\text{mod} \ R\) such as in [23] and [20]. Recently, there has been research in classifying the resolving subcategories of \(\text{mod}(\mathcal{R})\). The study of resolving subcategories began with Auslander and Bridger’s influential work in [2] where they define the category of Gorenstein dimension zero modules, which we will denote by GDZ. Also, they generalize the notion of projective dimension by defining Gorenstein dimension through approximations of Gorenstein dimension zero modules. In their paper, they also prove that GDZ has certain homological closure properties which cause Gorenstein dimension to behave similarly to projective dimension. They then take these homological closure properties of GDZ as the definition of resolving subcategories. We can take dimension with respect to a resolving subcategory, and, as in the case of GDZ, these homological closure properties force this dimension function to also behave similarly to projective dimension. See Section 2 for further exposition.

The classification of resolving subcategories was advanced by Dao and Takahashi in [9], where they give a bijection between the set of resolving subcategories of the category of finite projective dimension modules and the set of grade consistent functions. A function \(f : \text{spec} \ R \rightarrow \mathbb{N}\) is called grade consistent if it is increasing (as a morphism of posets) and \(f(p) \leq \text{grade}(p)\) for all \(p \in \text{spec}(\mathcal{R})\). This result motivated the author to find other situations where a similar bijection exists, furthering the use of grade consistent functions in classifying resolving subcategories. Before the work of Dao and Takahashi, Takahashi classifies, over Cohen-Macaulay rings, resolving subcategories closed under tensor products and Auslander transposes in [25], and in [24] he classifies the contravariantly finite resolving subcategories of a Henselian local Gorenstein ring. In [22], Takahashi also studies resolving subcategories which are free on the punctured spectrum. In [3], Auslander and Reiten discover a connection between resolving subcategories and tilting theory, and they classify all the contravariantly finite resolving subcategories using cotilting bundles. Also, the result in [9] was later reproved in [16] by classifying all the tilting classes, an approach which is very different from Dao and Takahashi’s.

In this paper, we assume that \(\mathcal{R}\) is commutative and Noetherian, and we consider only finitely generated modules. Let \(\mathcal{P}\) denote the category of projective modules and \(\Gamma\) the set of grade consistent functions. For categories \(\mathcal{C}, \mathcal{X} \subseteq \text{mod}(\mathcal{R})\) and \(f \in \Gamma\), we define

\[
\Lambda(\mathcal{C})(f) = \{ M \in \text{mod}(\mathcal{R}) \mid \text{add}\mathcal{C}_p \text{-dim} M_p \leq f(p) \quad \forall p \in \text{spec} \mathcal{R} \}
\]

\[
\Phi_{\mathcal{C}}(\mathcal{X}) : \text{spec} \mathcal{R} \rightarrow \mathbb{N} \quad p \mapsto \sup \{ \text{add}\mathcal{C}_p \text{-dim} X_p \mid X \in \mathcal{X} \}.
\]

Let \(\mathcal{R}\) denote all the resolving subcategories of \(\text{mod}(\mathcal{R})\), and for any \(\mathcal{C} \subseteq \text{mod}(\mathcal{R})\) let \(\mathcal{R}(\mathcal{C})\) be all the resolving subcategories that contain \(\mathcal{C}\) and whose objects all have finite dimension with respect to \(\mathcal{C}\). Using our new notation, we can restate Dao and Takahashi’s result from [9].

Theorem 1.1. When \(\mathcal{R}\) is Noetherian, the following is a bijection

\[
\mathcal{R}(\mathcal{P}) \xrightarrow{\Lambda(\mathcal{P})} \Phi_{\mathcal{P}} : \Gamma
\]
where $\Lambda(P)$ and $\Phi_P$ are inverses of each other.

Our first main result is Theorem 4.2 which is the following. Note that through out this paper, all thick subcategories contain $R$.

**Theorem (A).** Let $\Psi$ be a set of increasing functions from $\text{spec} R$ to $\mathbb{N}$. Suppose $C \subseteq D$ such that $C$ cogenerates $D$ and $\text{add} C_p$ is thick in $\text{add} D_p$ for all $p \in \text{spec} R$. Define $\eta_C^D : \mathcal{R}(C) \to \mathcal{R}(D)$ by $\eta_C^D(x) = \text{res}(x \cup D)$ and $\rho_C^D : \mathcal{R}(D) \to \mathcal{R}(C)$ by letting $\rho_C^D(x)$ be the subcategory of modules in $x$ of finite dimension with respect to $C$. If $\Phi_C$ and $\Lambda(C)$ are inverses of each other giving a bijection between $\mathcal{R}(C)$ and $\Psi$, then we have the following commutative diagram.

$$
\begin{array}{ccc}
\mathcal{R}(D) & \xrightarrow{\Phi_D} & \Psi \\
\eta_C^D & \| & \\
\mathcal{R}(C) & \xrightarrow{\Phi_C} & \Psi
\end{array}
$$

Furthermore, $\Phi_D$ and $\Lambda(D)$, and also $\eta_C^D$ and $\rho_C^D$, are pairs of inverse functions.

This result allows us to extend the bijection from $\mathbb{P}$ to a plethora of categories. We use it to prove the following results which are Theorem 8.8 and essentially Corollary 8.6.

**Theorem (B).** Let $C$ be a semidualizing module. For any thick subcategory $C$ of $\mathcal{G}_C$ containing $C$, $\Lambda(C)$ and $\Phi_C$ give a bijection between $\mathcal{R}(C)$ and $\Gamma$.

**Theorem (C).** Let $C$ be a semidualizing module, and let $\mathcal{G}(C)$ be the collection of thick subcategories of totally $C$-reflexive modules containing $C$. Then the following is a bijection.

$$
\Lambda : \mathcal{G}(C) \times \Gamma \to \bigcup_{C \in \mathcal{G}(C)} \mathcal{R}(C) \subseteq \mathcal{R}
$$

Theorem C is really just the bijections of Theorem B patched together. These theorems show that the classification of resolving subcategories is intrinsically linked to the classification of thick subcategories of totally $C$-reflexive modules and hence to the classification of thick subcategories of $\text{mod}(R)$, a topic currently being studied, as mentioned earlier. See for instance [23] or [18]. Applying these results in the Gorenstein case yields Theorem 4.1 which, letting $\text{MCM}$ denote the category of maximal Cohen-Macaulay modules, states

**Theorem (D).** If $R$ is Gorenstein, then we have the following bijections which commute

$$
\begin{array}{ccc}
\{\text{Thick subcategories of MCM}\} \times \Gamma & \xrightarrow{\Lambda} & \{C \in \mathcal{R} \mid C \cap \text{MCM} \text{ is thick in MCM}\} \\
\text{id} \times \Lambda(P) & \downarrow & \\
\{\text{Thick subcategories of MCM}\} \times \mathcal{R}(P) & \xrightarrow{\Xi} & \{C \in \mathcal{R} \mid C \cap \text{MCM} \text{ is thick in MCM}\}
\end{array}
$$

where $\Xi(x, y) = \text{res}(x \cup y)$.

Of independent interest, using semidualizing modules, we generalize the famed Auslander transpose. This generalization is similar, but different, to the generalizations of Geng and Huang in [12] and [13].

This paper is organized as follows: Section 2 gives general information about resolving subcategories, and Section 3 gives pertinent background regarding semidualizing modules. We prove Theorem A in Section 4. In Section 5, we give the generalization of the Auslander transpose, which we use in Section 6 to prove a theorem about resolving subcategories that are locally Maximal Cohen-Macaulay. This result is used in Section 7 to prove that Theorem B holds for certain thick subcategories containing a semidualizing module $C$. Section 8 then proves Theorems B and C by examining thick subcategories of maximal Cohen-Macaulay modules that contain $C$, and then by applying Theorem A. In the last section, these results are applied to the Gorenstein case, and Theorem D and several other results are proven.

2. Resolving Preliminaries

We proceed with an overview of resolving subcategories. All subcategories considered are full and closed under isomorphisms.

**Definition 2.1.** Given a ring $R$, a full subcategory $C \subseteq \text{mod}(R)$ is resolving if the following hold.

- $C$ is full and closed under isomorphisms.
- Every thick subcategory of $C$ that contains $C$ is itself in $C$.
- If $C$ is resolving, then $\text{mod}(C)$ is isomorphic to $\text{mod}(R)$.
- If $D$ is a thick subcategory of $\text{mod}(R)$ that contains $C$, then there is an isomorphism $\text{mod}(C) \cong \text{mod}(D)$.

This definition captures the notion of a resolving subcategory, which is crucial for understanding the classification of thick subcategories.
Lemma 2.7. If $\Omega$ show that with each $F$

Corollary 2.6. C with each $C$

Proof. If $\Omega$ is resolving, then $C$-dim $X = \inf\{\Omega^n M \in C\}$.

Proposition 2.5. If $C$ is resolving and $C$-dim $X \leq n$, then for any exact sequence

with each $C_i \in C$, $U$ is in $C$.

This proposition allows us to prove the following results.

Example 2.4. The following categories are easily seen to be thick subcategories.

Definition 2.3. Let $X \subseteq \text{mod}(R)$. A full subcategory $C \subseteq X$ is a thick subcategory of $X$ (or $C$ is thick in $X$) if it is resolving and for any exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ with $X, Y, Z \in X$, if $X$ and $Y$ are in $C$, then $Z$ is in $C$ too. A thick subcategory refers to a thick subcategory of $\text{mod}(R)$.

For any $X \subseteq \text{mod}(R)$, we let $\text{Thick}(X)$ be the smallest thick subcategory of $\text{mod}(R)$ containing $X$.

Example 2.2. The following categories are easily seen to be resolving.

(1) $\mathcal{P}$

(2) $\text{mod}(R)$

(3) The set of Gorenstein dimension zero modules

(4) For any $X \subseteq \text{Mod}(R)$ and any $n \geq 0$, $\{Y \mid \text{Ext}^n(Y, X) = 0 \ \forall X \in X\}$

(5) For any $X \subseteq \text{Mod}(R)$ and any $n \geq 0$, $\{Y \mid \text{Tor}^n(Y, X) = 0 \ \forall X \in X\}$

(6) When $R$ is Cohen-Macaulay, the set of maximal Cohen-Macaulay modules

A special class of resolving subcategories are thick subcategories.

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For any $X \subseteq \text{mod}(R)$, we let $\text{Thick}(X)$ be the smallest thick subcategory of $\text{mod}(R)$ containing $X$.

Proposition 2.5. If $C$ is resolving and $C$-dim $X \leq n$, then for any exact sequence

with each $C_i \in C$, $U$ is in $C$.

This proposition allows us to prove the following results.

Corollary 2.6. If $C$ is resolving, then $C$-dim $X = \inf\{\Omega^n M \in C\}$.

Proof. If $\Omega^n M \in C$, then we have

with each $F_i$ projective. This shows that $C$-dim $M \leq n$. If $n \leq C$-dim $M$, the same sequence and Proposition 2.0 show that $\Omega^n M$ is in $C$.

Lemma 2.7. If $C$ is resolving, then $C$-dim $X \oplus Y = \max\{C$-dim $X, C$-dim $Y\}$. 

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(1) $R$ is in $C$

(2) $X \oplus Y$ is in $C$ if any only if $X$ and $Y$ are in $C$

(3) If $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ is exact, then $Z \in C$ implies that $Y \in C$ if and only if $X \in C$. 

By [28] Lemma 3.2], this is equivalent to saying these conditions hold.

(1) All projectives are in $C$

(2) If $X \in C$, then add $X \subseteq C$

(3) $C$ is closed under extensions

(4) $C$ is closed under syzygies For a subset $C \subseteq \text{mod}(R)$, we denote by $\text{res}(X)$ the smallest resolving subcategory containing $X$. Let $\mathcal{P}$ be the category of finitely generated projective $R$ modules.

Example 2.4. The following categories are easily seen to be thick subcategories.
Proof. We have \( \Omega^n(X \oplus Y) = \Omega^n X \oplus \Omega^n Y \) for a suitable choice of syzygies. Since \( \Omega^n(X \oplus Y) \) is in \( C \) if and only if \( \Omega^n X \) and \( \Omega^n Y \) are in \( C \), the result follows from Corollary 2.6. Parts (1) and (2) are essentially proved in [17] Theorem 18.

Lemma 2.8. If \( C \) is a resolving category, and \( 0 \to K \to L \to M \to 0 \) is exact, then the following inequalities hold.

(1) \( \text{C-dim} \ K \leq \max \{ \text{C-dim} \ L, \text{C-dim} \ M - 1 \} \)

(2) \( \text{C-dim} \ L \leq \max \{ \text{C-dim} \ K, \text{C-dim} \ M \} \)

(3) \( \text{C-dim} \ M \leq \max \{ \text{C-dim} \ K, \text{C-dim} \ L \} + 1 \)

Proof. For suitable choices of syzygies, we have the following.

\[
0 \to \Omega^k K \to \Omega^k L \to \Omega^k M \to 0
\]

If \( k = \max \{ \text{C-dim} \ K, \text{C-dim} \ M \} \), then, by Corollary 2.6, \( \Omega^k K \) and \( \Omega^k M \) are in \( C \), and thus, so is \( \Omega^k L \), giving us (1). If \( k = \max \{ \text{C-dim} \ K, \text{C-dim} \ L \} \), then, again by Corollary 2.6, \( \Omega^k K \) and \( \Omega^k L \) will be in \( C \). Therefore \( \text{C-dim} \Omega^k M \leq 1 \), and so \( \Omega^{k+1} M \) will be in \( C \). Thus by Corollary 2.6, \( \text{C-dim} \ M \leq k + 1 \), proving (3).

Now take \( k = \max \{ \text{C-dim} \ L, \text{C-dim} \ M - 1 \} \). Then \( L_k \) and \( M_{k+1} \) are in \( C \). We take the pushout diagram

\[
\begin{array}{ccc}
0 & \to & \Omega^k K \\
\downarrow & & \downarrow \\
\Omega^{k+1} M & \longrightarrow & \Omega^{k+1} M \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \Omega^k K \\
\Vert & & \Vert \\
0 & \longrightarrow & \Omega^k L \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \Omega^k M \\
\downarrow & & \downarrow \\
0 & \longrightarrow & 0
\end{array}
\]

with \( F \) free and hence in \( C \). Since, by Corollary 2.6, \( \Omega^{k+1} M \) and \( \Omega^k L \) are in \( C \), so is \( Z \). Since \( F \in C \), \( \Omega^k K \) has to also be in \( C \). Hence \( \text{C-dim} \ K \leq k \), and we have (1). \( \square \)

For a subset \( C \subseteq \text{mod}(R) \), let \( \Delta(C) \) denote the category of modules \( M \) such that \( \text{C-dim} \ M \) is finite. If \( C \) is resolving, then by Corollary 2.6, \( \Delta(C) = \{ M \in \text{mod}(R) \mid \Omega^{>>0} M \in \text{C} \} \). The next result easily follows from the previous lemma.

Corollary 2.9. Let \( C \) be resolving. For any \( n \), the set \( \{ M \in \text{mod}(R) \mid \text{C-dim} \ M \leq n \} \) is resolving. Furthermore, \( \Delta(C) \) is thick, and \( \text{Thick}(C) = \Delta(C) \).

Through these results, we may construct many resolving and thick subcategories. It is easy to show that the intersection of a collection of resolving subcategories and the intersection of a collection of thick subcategories are resolving and thick respectively. The following lemma allows us to construct even more resolving subcategories. For \( C \subseteq \text{mod}(R) \), we say \( C_p = \{ C_p \mid C \in C \} \).

Lemma 2.10. Let \( R \) and \( S \) be rings and \( F : \text{mod}(R) \to \text{mod}(S) \) be an exact functor with \( F(R) = S \). Then for any resolving subcategory \( C \subseteq \text{mod}(S) \), \( F^{-1}(C) \) is a resolving subcategory of \( \text{mod}(R) \).

The proof is elementary and is left to the reader. Applying this lemma to the localization functor, for any \( V \subseteq \text{spec} R \), the category of all \( M \in \text{mod}(R) \) with \( M_p \) free for all \( p \in V \) is also resolving. The following lemmas give insight into the behavior of resolving categories under localization. The first lemma is from [23] Lemma 4.8 and [3] Lemma 3.2(1), and the second is from [9] Proposition 3.3.

Lemma 2.11. If \( \mathcal{X} \) is a resolving subcategory, then so is \( \mathcal{X}_p \) for all \( p \in \text{spec} R \).

Lemma 2.12. The following is equivalent for a resolving subcategory \( \mathcal{X} \) and a module \( M \in \text{mod}(R) \).

(1) \( M \in \mathcal{X} \)
Recall the definition of $\Lambda$ from the introduction. These lemmas show that if $C$ is resolving, then for all $f \in \Gamma$, $\Lambda(C)(f)$ is a resolving subcategory.

**Corollary 2.13.** Set

$$\Lambda(C)(f) = \{ M \in \text{mod}(R) \mid \text{add} C_p - \dim M_p \leq f(p) \quad \forall p \in \text{spec} R \}.$$ 

If $C$ is resolving, then for all $f \in \Gamma$, $\Lambda(C)(f)$ is a resolving subcategory.

Let MCM denote the category of maximal Cohen-Macaulay modules. As noted earlier, when $R$ is Cohen-Macaulay, MCM is resolving. Furthermore, letting $d = \dim R$, $\Omega^d M$ is in MCM for every $M \in \text{mod}(R)$. Hence, $\Delta(\text{MCM}) = \text{mod}(R)$. Dimension with respect to MCM is very computable.

**Lemma 2.14.** Suppose $C$ is a thick subcategory of a resolving subcategory $D$. Then for any module $M \in \Delta(C)$, we have $C \cdot \dim M = D \cdot \dim M$. Furthermore, if $R$ is Cohen-Macaulay, $C$ is a thick subcategory of MCM if and only if dimension with respect to $C$ satisfies the Auslander Buchsbaum Formula, i.e. for all $M \in \Delta(C)$ we have

$$\dim C \cdot \dim M + \text{depth} M = \text{depth} R.$$ 

**Proof.** Suppose $M \in \Delta(C)$. Then we may write $0 \to C_d \to \cdots \to C_0 \to M \to 0$ with $C_i \in C$ and $d = C \cdot \dim M$. Since each $C_i$ is also in $D$, we have $D \cdot \dim M \leq d$. Setting $e = D \cdot \dim M \leq d$, by Corollary 2.6 there exists a $D \in D$ such that

$$0 \to D \to C_{e-1} \to \cdots \to C_0 \to M \to 0 \quad 0 \to C_d \to \cdots \to C_e \to D \to 0$$

are exact. However, since $C$ is thick in $D$, $D$ is also in $C$, which implies that $e = d$, proving the second statement.

Assume $R$ is Cohen-Macaulay. Let $C$ be a resolving subcategory whose dimension satisfies the Auslander Buchsbaum formula. Then for any module $M \in \Delta(C) \cap \text{MCM}$, we have

$$\text{depth} R = C \cdot \dim M + \text{depth} M = C \cdot \dim M + \text{depth} R.$$ 

Thus $C \cdot \dim M = 0$ forcing $M$ to be in $C$. Hence $C$ is thick in MCM.

Now we prove the converse. By what we have proved so far, it suffices to show that dimension with respect to MCM satisfies the Auslander Buchsbaum formula. Take $M \in \Delta(\text{MCM})$. We will show that $C \cdot \dim M = d - \text{depth} M$ by induction on $d - \text{depth} M$. Suppose $d - \text{depth} M = 0$. Then $M \in \text{MCM}$. Now suppose $d - \text{depth} M = n > 0$. Then $\text{depth} M < d$ and so $\text{depth} \Omega M = \text{depth} M + 1$. Therefore $d - \text{depth} M > d - \text{depth} \Omega M$. So by induction, we have

$$C \cdot \dim M = C \cdot \dim \Omega M + 1 = d - \text{depth} \Omega M + 1 = d - \text{depth} M.$$ 

Recall the definition of $\Phi$ from the introduction. If dimension with respect to $\text{add} C_p$ satisfies the Auslander Buchsbaum formula for all $p \in \text{spec} R$, then for all $X \subseteq \Delta(C)$, $\Phi_C(X)$ is in $\Gamma$. Before proceeding, we need one more definition and a result.

**Definition 2.15.** Let $A \subseteq C$. We say $A$ cogenerates $C$, if for every $C \in C$, there exists an exact sequence $0 \to C \to A \to C' \to 0$ with $C' \in C$ and $A \in A$.

The following is an important theorem from [4, Theorem 1.1].

**Theorem 2.16.** Suppose $X$ and $A$ are resolving with $A \subseteq X$. If $A$ cogenerates $X$, then for every $X \in \Delta(X')$ with $\text{dim}_X X = n$, there exists a $Y \in \Delta(A)$ with $A \cdot \dim Y = n$ and $Z \in X$ such that $0 \to X \to Y \to Z \to 0$ is exact.

### 3. Preliminaries: Semidualizing Modules

We fix a module $C \in \text{mod}(R)$ and write $M^\dagger = \text{Hom}(M, C)$.

**Definition 3.1.** A finitely generated module $M$ is totally $C$-reflexive if it satisfies the following.

1. $\text{Ext}^{>0}(M, C) = 0$
2. $\text{Ext}^{>0}(M^\dagger, C) = 0$
3. The natural homothety map $\eta_M : M \to M^\dagger$ defined by $\mu \mapsto (\varphi \mapsto \varphi(\mu))$ is an isomorphism.
Let \( \mathcal{G}_C \) denote the category of totally \( C \)-reflexive modules.

The set \( \mathcal{G}_C \) is essentially the subcategory over which \( \dagger \) is a dualizing functor. The notion of totally \( C \)-reflexivity generalizes Gorenstein dimension zero. In fact, when \( C = R \), \( \mathcal{G}_R \) is simply the category of Gorenstein dimension zero modules, which are also known as totally reflexive modules. See [17] for further information on the subject. The following proposition shows us that \( \mathcal{G}_C \) is almost resolving.

**Lemma 3.2.** The set \( \mathcal{G}_C \) is closed under direct sums, summands, and extensions.

**Proof.** It is easy to show that \( \mathcal{G}_C \) is closed under direct sums and direct summands. Suppose we have

\[
0 \to X \to Y \to Z \to 0
\]

with \( X, Z \in \mathcal{G}_C \). It is easy to check that \( Y \) satisfies condition 1 of Definition 3.1. We have

\[
0 \to Z^\dagger \to Y^\dagger \to X^\dagger \to 0 \quad 0 \to X^{\dagger \dagger} \to Y^{\dagger \dagger} \to Z^{\dagger \dagger} \to 0.
\]

From the first exact sequence, it is easy to see that \( Y \) satisfies condition 2 of Definition 3.1. We can then use the five lemma to show that \( Y \) satisfies condition 3 of Definition 3.1. □

In general, \( \mathcal{G}_C \) will not be resolving. For example, if \( C = R/xR \) for a regular element \( x \in R \), \( \text{Ext}^1(R/xR, R/xR) = R/xR \neq 0 \). So \( R \) cannot be in \( \mathcal{G}_{R/xR} \), and thus \( \mathcal{G}_{R/xR} \) cannot be resolving. It is clear from the definition that \( R \in \mathcal{G}_C \) is a necessary condition for \( \mathcal{G}_C \) to be resolving. In fact, this condition is sufficient.

**Proposition 3.3.** The set \( \mathcal{G}_C \) will be resolving if and only if \( \mathcal{G}_C \) contains \( R \).

**Proof.** If \( \mathcal{G}_C \) is resolving, by definition it contains \( R \), so we prove the converse. So suppose \( R \) is in \( \mathcal{G}_C \). In light of the last lemma, we need only to prove that if \( 0 \to X \to Y \to Z \to 0 \) is exact with \( Y, Z \in \mathcal{G}_C \), then \( X \) is in \( \mathcal{G}_C \) as well. Since \( Y \) and \( Z \) satisfy condition 1 of Definition 3.1 it is easy to show that \( X \) does too. Also, since \( \text{Ext}^1(Z, C) = 0 \), we have

\[
0 \to Z^\dagger \to Y^\dagger \to X^\dagger \to 0.
\]

Hence, we have the following commutative diagram with exact rows.

\[
\begin{array}{cccccc}
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\
\eta_X & & \eta_Y & & \eta_Z & & & & \\
0 & \longrightarrow & X^{\dagger \dagger} & \longrightarrow & Y^{\dagger \dagger} & \longrightarrow & Z^{\dagger \dagger} & \longrightarrow & \text{Ext}^1(X^\dagger, C) & \longrightarrow & 0 \\
\end{array}
\]

Since \( \eta_Y \) and \( \eta_Z \) are isomorphisms, the five lemma shows that \( \eta_X \) is too, and that \( \text{Ext}^1(X^\dagger, C) = 0 \). Thus \( X \) satisfies condition 2 of Definition 3.1. It is easy to check using the first exact sequence that \( \text{Ext}^{\geq 1}(X^\dagger, C) = 0 \), showing that \( X \) satisfies condition 2 of Definition 3.1. □

Motivated by this proposition, we say a module, \( C \), is semidualizing if \( R \) is in \( \mathcal{G}_C \). This is easily seen to be equivalent to the following definition which is standard in the literature.

**Definition 3.4.** A module \( C \) is semidualizing if \( \text{Ext}^{>0}(C, C) = 0 \) and \( R \cong \text{Hom}(C, C) \) via the map \( r \mapsto (c \mapsto rc) \).

For the remainder of the paper, we will let \( C \) denote a semidualizing module. Semidualizing modules were first discovered by Foxby in [10] and were later rediscovered in different guises by various authors, including Vasoncles in [26], who called them spherical modules, and Golod, who called them suitable modules. For an excellent treatment of the general theory of semidualizing modules, see [19]. Examples of semidualizing modules include \( R \) and dualizing modules. If \( R \) is Cohen-Macaulay and \( D \) is a dualizing module, then \( \mathcal{G}_D \) is simply MCM. Dimension with respect to \( \mathcal{G}_C \) is often called Gorenstein \( C \)-dimension, or \( \text{G}_C \)-dimension for short, since it is a generalization of Gorenstein dimension. We would expect \( \mathcal{G}_C \) and Gorenstein dimension to have similar properties. Thus we have the following lemma, which is an easy exercise, and proposition, which is from [13, Theorem 1.22].

**Lemma 3.5.** If \( M \in \Delta(\mathcal{G}_C) \), then \( \mathcal{G}_C \)-dim \( M = \min\{n \mid \text{Ext}^{>n}(M, C) = 0 \} \).

**Proposition 3.6.** For any semidualizing module \( C \), \( \mathcal{G}_C \)-dimension satisfies the Auslander Buchsbaum formula, i.e. for any module \( M \in \Delta(\mathcal{G}_C) \), we have

\[
\mathcal{G}_C \text{-dim} \ M + \text{depth} \ M = \text{depth} \ R.
\]
In light of Lemma 2.14 when \( R \) is Cohen-Macaulay this means that \( \mathcal{G}_C \) is a thick subcategory of MCM. Interest in understanding \( \mathcal{G}_C \)-dimension and the structure of \( \mathcal{G}_C \) is not new. The following conjecture by Gerko from [13] Conjecture 1.23 is equivalent to saying that \( \mathcal{G}_R \) is a thick subcategory of \( \mathcal{G}_C \).

**Conjecture 3.7.** If \( C \) is semidualizing, then for any module \( M \), \( \mathcal{G}_C \)-dim \( M \leq \mathcal{G}_R \)-dim \( M \), and equality holds when both are finite.

We give one more construction in this section. Take any \( M \in \mathcal{G}_C \). Then we have \( 0 \to \Omega M^\dagger \to R^n \to M^\dagger \to 0 \) is exact. Since \( R^\dagger \cong C \), applying \( \dagger \) yields the exact sequence

\[
0 \to M \to C^n \to (\Omega M^\dagger)^\dagger \to 0.
\]

Hence \( \mathcal{G}_C \) is cogenerated by \( \text{add} C \). Furthermore, if \( F_\bullet \) is a projective resolution of \( M^\dagger \) with \( M \in \mathcal{G}_C \), then \( F^\dagger_\bullet \) is an add \( C \) coresolution of \( M \). Splicing this together with a free resolution \( G_\bullet \) of \( M \), we get what is called a complete \( PP_C \) or a complete \( P_C \) resolution of \( M \). See [27] or [19] for more on the matter.

Before proceeding, we summarize the notations of this paper.

1. \( R \) is a commutative noetherian ring
2. \( P \) is the projective \( R \) modules
3. \( \Gamma \) is the set of grade consistent functions
4. \( C \)-dim \( M \) is the dimension of \( M \) with respect to the category \( C \subseteq \text{mod}(R) \)
5. \( \text{add} C \) is the smallest category closed under direct sums and summands containing \( C \subseteq \text{mod}(R) \)
6. \( \Lambda(C)(f) = \{ M \in \text{mod} R \mid \text{add} C \text{-dim } M_p \leq f(p) \ \forall p \in \text{spec } R \} \) with \( f \in \Gamma \)
7. \( \Phi_C(X)(p) = \text{sup} \{ \text{add} C \text{-dim } X_p \mid X \in X \} \) with \( C, X \subseteq \text{mod}(R) \) categories
8. \( \Delta(C) = \{ M \in \text{mod} R \mid C \text{-dim } M < \infty \} \) with \( C \subseteq \text{mod}(R) \) a category
9. \( \mathcal{R}(C) = \{ X \subseteq \text{mod}(R) \mid C \subseteq X \subseteq \Delta(C) \} \)
10. \( \mathcal{R} \) the collection of resolving subcategories
11. \( \text{res} X \) the smallest resolving subcategory of \( \text{mod}(R) \) containing \( X \subseteq \text{mod}(R) \)
12. \( \text{Thick}_C(X) \) the smallest thick subcategory of \( C \) containing \( X \) with \( C, X \subseteq \text{mod}(R) \)
13. \( \mathcal{G}_C \) the collection of totally \( C \)-reflexive modules
14. \( M^\dagger = \text{Hom}(M, C) \) where \( C \) is a semidualizing module

4. Comparing Resolving Subcategories

For the entirety of this section, let \( C \) and \( D \) be resolving subcategories. Recall that \( \mathcal{R}(C) \) is the collection of resolving subcategories \( X \) such that \( C \subseteq X \subseteq \Delta(C) \). In this section, we compare \( \mathcal{R}(C) \) and \( \mathcal{R}(D) \) when \( C \) is contained in \( D \). If \( C \subseteq D \), we may define \( \eta^D_C : \mathcal{R}(C) \to \mathcal{R}(D) \) by \( X \mapsto \text{res}(X \cup D) \) and \( \rho^D_C : \mathcal{R}(D) \to \mathcal{R}(C) \) by \( X \mapsto X \cap \Delta(C) \). Note that if \( C \subseteq D \subseteq \mathfrak{E} \), then \( \eta^D_C \circ \eta^C_C = \eta^D_C \circ \eta^D_D \) and \( \rho^D_C \circ \rho^C_C = \rho^D_D \circ \rho^C_C \).

**Proposition 4.1.** If \( C \) cogenerated \( D \), then the map \( \rho^D_C \) is injective.

**Proof.** Suppose that for \( X, Y \in \mathcal{R}(C) \), we have \( \rho^D_C(X) = \rho^D_C(Y) \), i.e. \( X \cap \Delta(C) = Y \cap \Delta(C) \). Take any \( X \in X \). Since \( X \in \Delta(D) \) and \( C \) cogenerated \( D \), by Theorem 2.16 there exists \( M \in \Delta(C) \) and \( D \in D \) such that \( 0 \to X \to M \to D \to 0 \) is exact. Since \( D \subseteq D \subseteq X \), we know that \( M \) is also in \( X \). But then \( M \) is in \( X \cap \Delta(C) = Y \cap \Delta(C) \) and thus also in \( Y \). Since also \( D \subseteq D \subseteq Y \), we know that \( X \) is also also in \( Y \). Hence \( X \subseteq Y \), and, by symmetry, we have equality. Therefore, \( \rho^D_C \) is injective. \( \square \)

In certain circumstances, this map is a bijection. The following is Theorem A from the introduction.

**Theorem 4.2.** Let \( \Psi \) be a set of increasing functions from spec \( R \) to \( \mathbb{N} \). Suppose, \( C \subseteq D \) such that \( C \) cogenerates \( D \) and add \( C_p \) is thick in add \( D_p \) for all \( p \in \text{spec } R \). If \( \Phi_C \) and \( \Lambda(C) \) are inverse functions giving a bijection between \( \mathcal{R}(C) \) and \( \Psi \), then the following diagram commutes.

\[
\begin{array}{ccc}
\mathcal{R}(D) & \xrightarrow{\Phi_D} & \Psi \\
\eta^D_C & \parallel & \\
\mathcal{R}(C) & \xrightarrow{\Phi_C} & \Psi
\end{array}
\]

Furthermore, \( \Phi_D \) and \( \Lambda(D) \) and also \( \eta^D_C \) and \( \rho^D_C \) are pairs of inverse functions.

The proof of this proposition will be given after this brief lemma.
Lemma 4.3. If $\mathcal{X}$ and $\mathcal{Y}$ are subcategories and $\mathcal{C}$ is resolving, then $\Phi_C(\text{res}(\mathcal{X} \cup \mathcal{Y})) = \Phi_C(\mathcal{X}) \cup \Phi_C(\mathcal{Y})$.

Proof. Since every element in $\text{res}(\mathcal{X} \cup \mathcal{Y})$ is obtained by taking extensions, syzygies, and direct summands a finite number of times, and since these operations never increase the $\mathcal{C}$ dimension, we have $\Phi_C(\text{res}(\mathcal{X} \cup \mathcal{Y})) \leq \Phi_C(\mathcal{X}) \cup \Phi_C(\mathcal{Y})$. However, since $\mathcal{X}, \mathcal{Y} \subseteq \text{res}(\mathcal{X} \cup \mathcal{Y})$, we actually have equality. \qed

Proof of Theorem 4.2. First, we will show that $\rho_C^D$ and $\eta_C^D$ are inverse functions and are thus both bijections. Proposition [4.1] shows that $\rho_C^D$ is injective. Let $Z = \rho_C^D(\mathcal{X}) = \text{res}(\mathcal{X} \cup D) \cap \Delta(\mathcal{C})$. It suffices to show that $Z = \mathcal{X}$. Setting $f = \Phi_C(\mathcal{X})$, since $\Phi_C$ and $\Lambda(\mathcal{C})$ are inverse functions, this is equivalent to showing that $\Phi_C(Z) = f$. Since $\mathcal{X} \subseteq Z$, we know that $\Phi_C(Z) \geq f$. From Lemma 4.3 we have

$$\Phi_D(\text{res}(\mathcal{X} \cup D)) = \Phi_D(\mathcal{X}) \cup \Phi_D(D) = \Phi_D(\mathcal{X})$$

Furthermore, since $\text{add} \mathcal{C}_p$ is thick in $\text{add} D_p$ for all $p \in \text{spec } R$, by Lemma 2.14 add $\mathcal{C}_p$-dim $M$ and add $D_p$-dim $M$ are the same for all $p \in \text{spec } R$ and $M \in \Delta(\mathcal{C})$. Hence $\Phi_C(W)$ equals $\Phi_D(W)$ for all $W \subseteq \Delta(\mathcal{C})$. Therefore,

$$f \leq \Phi_C(Z) = \Phi_D(Z) \leq \Phi_D(\text{res}(\mathcal{X} \cup D)) = \Phi_D(\mathcal{X}) = \Phi_C(\mathcal{X}) = f$$

and so, $\Phi_C(Z) = f$. Hence, $\rho_C^D$ and $\eta_C^D$ are inverse functions. Also, this argument shows that $\Phi_C(\mathcal{X}) = \Phi_D(\text{res}(\mathcal{X} \cup D)) = \Phi_D(\eta_C^D(\mathcal{X}))$, showing that the diagram commutes and hence $\Phi_D$ also gives a bijection.

Furthermore, for any $\mathcal{C}, D \in \mathcal{S}(A)$ the following diagram commutes.

$$\begin{array}{ccc}
\mathcal{R}(\mathcal{D}) & \xrightarrow{\Phi_D} & \Psi \\
\eta_C^D \uparrow & \downarrow & \downarrow \\
\mathcal{R}(\mathcal{C}) & \xrightarrow{\Phi_C} & \Psi \\
\sigma_A^C \uparrow & \downarrow & \downarrow \\
\mathcal{R}(A) & \xrightarrow{\Phi_A} & \Psi \\
\end{array}$$

Furthermore, $\rho_C^D$ and $\eta_C^D$ are inverse functions.

Since $\Lambda$ is an increasing function and both $\Phi_C$ and $\Lambda(\mathcal{C})$ are inverse functions, we have

$$f = \Phi_C(\Lambda(\mathcal{C})(f)) = \Phi_D(\eta_C^D(\Lambda(\mathcal{C})(f))) \leq \Phi_D(\Lambda(\mathcal{D})(f)) = f.$$
with \( C \in \mathcal{C}, \) \( D \in \mathcal{D}, \) and \( A, A' \in \mathcal{A}. \) Consider the following pushout diagram.

\[
\begin{array}{cccccc}
0 & 0 \\
\downarrow & \downarrow \\
0 & \rightarrow & X & \rightarrow & A & \rightarrow & C & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\downarrow & & & & & & & & \downarrow \\
0 & \rightarrow & A' & \rightarrow & T & \rightarrow & C & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
D & \rightarrow & D & \rightarrow & C & \rightarrow & D & \rightarrow & 0 \\
\end{array}
\]

It is easy to see \( T \in C \cap D. \) We also have the exact sequence

\[
0 \rightarrow X \rightarrow A \oplus A' \rightarrow T \rightarrow 0
\]

Since \( A \oplus A' \in \mathcal{A}, \) this completes the proof. \( \square \)

**Proof of Theorem 4.4.** Suppose \( C, D \in \mathfrak{S} \) with \( C \subseteq D. \) From Theorem 4.2, the following diagrams commute.

\[
\begin{array}{cccccc}
\mathfrak{R}(D) & \xrightarrow{\Phi_D} & \Psi \\
\downarrow & & \downarrow \\
\mathfrak{R}(C) & \xrightarrow{\Phi_C} & \Psi \\
\end{array}
\]

From here, it is easy to show that (1) commutes and \( \Phi_D \) and \( \eta_D^P \) are bijections with \( (\eta_C^P)^{-1} = \rho_C^P. \)

Also, Theorem 4.2 shows that \( \text{Im}(\Lambda) = \bigcup_{C \in \mathfrak{S}} \mathfrak{R}(C). \) It remains to show that \( \Lambda \) is injective. Suppose \( Z = \Lambda(C)(f) = \Lambda(D)(g). \) Then we have \( C \subseteq Z \) and \( D \subseteq Z, \) and hence \( C \cap D \subseteq Z. \) For any \( Z \) and any \( n \) greater than \( C \)-dim \( Z \) and \( D \)-dim \( Z, \) \( \Omega^n Z \) is in \( C \cap D \) by Corollary 4.4. Therefore, \( Z \) is contained in \( \Delta(C \cap D) \) and thus \( Z \in \mathfrak{R}(C \cap D). \) Since by the last lemma \( C \cap D \) is in \( \mathfrak{S}(\mathcal{A}), \) by Theorem 4.2, \( \Lambda(C \cap D) : \Psi \rightarrow \mathfrak{R}(C \cap D) \) is a bijection. So there exists an \( h \) such that \( \Lambda(C \cap D)(h) = Z = \Lambda(C)(f) = \Lambda(D)(g). \) Therefore, we may assume that \( C \) is contained in \( D. \)

Then, by assumption, we have \( D \subseteq Z \subseteq \Delta(C). \) Thus, because \( A \subseteq D, \) we have the following.

\[
D = D \cap \Delta(C) = \rho_C^D(D) = \eta_C^A \rho_A^C \rho_C^D(D) = \eta_C^A \rho_A^D = \eta_C^A = C
\]

Since \( \Lambda(C) \) is injective, we then also have \( f = g. \) \( \square \)

As mentioned earlier, in \( [9], \) we have \( \Lambda(P) \) is a bijection from \( \Gamma \) to \( \mathfrak{R}(P). \) In Section 8 and Section 9 we apply Theorem 4.4 when \( A = P, \) and show that \( \mathfrak{S}(P) \) is simply the collection of thick subcategories of \( \mathcal{G}_R. \) The following results gives an alternative way of viewing Theorem 4.4.

**Proposition 4.6.** In the situation of Theorem 4.4, if \( \Psi = \Gamma \) and \( P \) is thick in \( \mathcal{A}, \) then the following diagram commutes.

\[
\begin{array}{cccc}
\mathfrak{S}(A) \times \Gamma & \xrightarrow{\Lambda} & \mathfrak{R} \\
\downarrow & & \downarrow \\
\mathfrak{S}(A) \times \mathfrak{R}(P) & \xrightarrow{\Xi} & \mathfrak{R} \\
\end{array}
\]

where \( \Xi(C, \mathcal{X}) = \text{res}(C \cup \mathcal{X}). \) Furthermore, \( \text{id}_{\mathfrak{S}(A) \times \Lambda(P)} \) is bijective.

**Proof.** Since \( \Lambda(P) \) is bijective, \( \text{id}_{\mathfrak{S}(A) \times \Lambda(P)} \) is too. It suffices to show that for any \( (C, f) \in \mathfrak{S} \times \Gamma \) we have \( \Xi(C, \Lambda(P)(f)) = \Lambda(C)(f). \) First note that \( \mathcal{X} \) is in \( \mathfrak{R}(C). \) Since \( P \) is thick in \( \mathcal{A} \) and hence in \( C, \) by Lemma 4.4, we have

\[
\Phi_C(\mathcal{X}) = \Phi_C(\text{res}(C \cup \Lambda(P)(f))) = \Phi_C(C) \cup \Phi_C(\Lambda(P)(f)) = \Phi_P(\Lambda(P)(f)) = f
\]

and thus \( \Lambda(C)(f) = \mathcal{X}, \) proving the claim. \( \square \)
5. A generalization of the Auslander transpose

Let $C$ be a semidualizing module, and set $-\dagger = \text{Hom}(-, C)$. For the entirety of this section, $\mathcal{A}$ denotes a thick subcategory of $\mathcal{G}_C$ that is closed under $\dagger$. Recalling Proposition 3.6, $\text{A-dim}$ satisfies the Auslander Buchsbaum formula.

The Auslander transpose has been an invaluable tool in both representation theory and commutative algebra. In this section we generalize the notion of the Auslander transpose using semidualizing modules and list some properties which we will use. The Auslander transpose has previously been generalized in [12] and [15], but the construction here is different.

**Definition 5.1.** An $\mathcal{A}$-presentation of $M$ is an exact sequence $G \xrightarrow{\varphi} F \to M \to 0$ with $F, G \in \mathcal{A}$. We set $\text{Tr}_\mathcal{A}M = \text{coker} \varphi$ and $\tilde{\Omega} \text{Tr}_\mathcal{A}M = \text{Im} \varphi$.

These "functors" are not well defined up to isomorphism, motivating a new equivalence relation. For modules $A$ and $B$, we write $A \sim'B$ and $B \sim'A$ if there exists a $K \in \mathcal{A}$ such that $0 \to A \to B \to K \to 0$ is exact. Let $\mathcal{A}$-equivalence, denoted by $\sim'$, be the transitive closure of the relation $\sim'$. Since $\sim'$ is symmetric and reflexive, $\sim$ is an equivalence relation. Stable equivalence implies $\mathcal{A}$-equivalence, and when $\mathcal{A} = \mathcal{P}$, they are the same.

**Proposition 5.2.** The functors $\text{Tr}_\mathcal{A}$ and $\tilde{\Omega} \text{Tr}_\mathcal{A}$ are unique up to $\mathcal{A}$-equivalence.

**Proof.** We say that an $\mathcal{A}$-presentation of $M$, $\pi$, dominates another $\mathcal{A}$-presentation, $\rho$, if there is an epimorphism from $\pi$ to $\rho$. Suppose that $\pi$ is the projective presentation $P_1 \to P_0 \to M \to 0$ and $\rho$ is the $\mathcal{A}$-presentation $G \to F \to M \to 0$. Furthermore, suppose that $\pi$ dominates $\rho$. Then we have the following exact commutative diagram.

\[\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\downarrow & & & & & & & & \\
0 & \to & K & \to & K_1 & \to & K_0 & \to & 0 \\
\downarrow & & & & & & & & \\
P_1 & \to & P_0 & \to & M & \to & 0 \\
\downarrow & & & & & & & & \\
G & \to & F & \to & M & \to & 0 \\
\downarrow & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}\]

(2)

The map $K_1 \to K_0$ is surjective by the snake lemma. Note that $K$, $K_1$, and $K_0$ are in $\mathcal{A}$. Applying $\dagger$ to the diagram yields the following.

\[\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\downarrow & & & & & & & & \\
0 & \to & M^\dagger & \to & F^\dagger & \to & G^\dagger & \to & \text{Tr}_\mathcal{A}^\dagger M & \to & 0 \\
\downarrow & | & | & & & & & & \\
0 & \to & M^\dagger & \to & P_0^\dagger & \to & P_1^\dagger & \to & \text{Tr}_\mathcal{A}^\dagger M & \to & 0 \\
\downarrow & & & & & & & & \\
0 & \to & K_0^\dagger & \to & K_1^\dagger & \to & K^\dagger & \to & 0 \\
\downarrow & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}\]
Since $\mathcal{A}$ is closed under $\dagger$, $K^\dagger$ is in $\mathcal{A}$, and so $\text{Tr}_A^\rho M \sim \text{Tr}_A^\pi M$. Applying the snake lemma to the second two columns gives us

$$0 \to \tilde{\Omega} \text{Tr}_A^\rho M \to \tilde{\Omega} \text{Tr}_A^\pi M \to K^\dagger_0 \to 0.$$ 

Since $K^\dagger \in \mathcal{A}$, we have $\tilde{\Omega} \text{Tr}_A^\rho M \sim \tilde{\Omega} \text{Tr}_A^\pi M$.

It suffices to show that for any two $\mathcal{A}$-presentations, $\rho$ and $\rho'$, there exists a projective presentation that dominates both of them. It is easy to construct projective presentations $\psi$ and $\psi'$ which dominate $\rho$ and $\rho'$ respectively, then the proof of [17][Proposition 4] shows that there is a projective presentation of $\pi$ which dominates both $\psi$ and $\psi'$. But then that $\pi$ will also dominate $\rho$ and $\rho'$.

One can easily use this lemma to show that $\text{res}_A(\text{Tr}_A X) = A$ for any $X \in A$. The following will show that $\mathcal{A}$-equivalence is well behaved under many important operations that will be used in the remainder of this paper.

**Lemma 5.3.** For any $A, B \in \text{mod}(R)$ such that $A \sim B$, the following are true.

1. $\text{res}_A A = \text{res}_A B$
2. $\Omega A \sim \Omega B$
3. $\text{Tr}_A A \sim \text{Tr}_A B$
4. $\tilde{\Omega} \text{Tr}_A A \sim \tilde{\Omega} \text{Tr}_A B$
5. $\text{Tr}_A \text{Tr}_A A \sim A$

**Proof.** For statements (1)-(4), it suffices to assume that $0 \to A \to B \to K \to 0$ with $K \in \mathcal{A}$. Proving (1) is trivial. For suitable choices of syzygies, we have $0 \to \Omega A \to \Omega B \oplus P \to \Omega K \to 0$. Since $\Omega K$ is in $\mathcal{A}$, and since syzygies are unique up to stable, and hence $\mathcal{A}$-equivalence, this implies (2).

Now we wish to show (3). There exist projective modules $P, Q_K, Q_B$ such that we may write the following.

$$\begin{array}{cccccc}
P_B & \longrightarrow & Q_B & \longrightarrow & B & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Omega K & \longrightarrow & Q_K & \longrightarrow & K & \longrightarrow & 0
\end{array}$$

where the rows are exact and the vertical maps surjective. The snake lemma yields the following diagram.

$$\begin{array}{cccccccc}
0 & 0 & 0 & 0 & \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
P_A & \longrightarrow & Q_A & \longrightarrow & A & \longrightarrow & 0 & \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
P_B & \longrightarrow & Q_B & \longrightarrow & B & \longrightarrow & 0 & \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
0 & \longrightarrow & \Omega K & \longrightarrow & Q_K & \longrightarrow & K & \longrightarrow & 0 & \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & \\
\end{array}$$
Since $K$ is in $\mathcal{A}$, so are $\Omega K$, $P_A$, and $Q_A$. Thus, for suitable choices of $\text{Tr}_A A$ and $\text{Tr}_A B$, applying $\dag$ to this diagram gives the following.

\[
\begin{array}{cccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & (Q_K)^\dag & (\Omega K)^\dag \\
\downarrow & \downarrow & \downarrow \\
0 & (Q_B)^\dag & P_B \\
\downarrow & \downarrow & \downarrow \\
0 & (Q_A)^\dag & P_A \\
\downarrow & \downarrow & \downarrow \\
\text{Ext}^1(K, C) & 0 & 0 \\
\end{array}
\]

Since $\text{Ext}^1(K, C) = \text{Ext}^1(K, C) = 0$, we have $\text{Tr}_A A \cong \text{Tr}_A B$ by applying the snake lemma to the middle two columns. This shows (3). Applying the snake lemma to the first two columns gives us (4).

Because of (3), we know $\text{Tr}_A \text{Tr}_A M$ is well defined up to $\mathcal{A}$-equivalence. With $F, G \in \mathcal{A}$, consider the sequence $G \xrightarrow{L} F \rightarrow M \rightarrow 0$. Then we have

\[
F^\dag \xrightarrow{f^\dag} G^\dag \rightarrow \text{Tr}_A M \rightarrow 0
\]

Since $F$ and $G$ are totally $C$ reflexive, we have $\text{Tr}_A \text{Tr}_A M \sim \text{coker} f^{\dag\dag} \cong \text{coker} f = M$. Thus we have shown (5).

Because syzygies are unique up to stable equivalence, and hence $\mathcal{A}$-equivalence, this lemma shows us that the main characters of our proofs, $\text{Tr}_A \Omega^i M$, $\text{Tr}_A \Omega^i \text{Tr}_A \Omega^i M$, and $\text{Tr}_A \Omega^i \Omega \text{Tr}_A \Omega^{i+1} M$ are all well defined up to $\mathcal{A}$ equivalence. It also shows that $\text{Tr}_A \Omega^i \Omega \text{Tr}_A M \sim \text{Tr}_A \Omega^{i+1} \text{Tr}_A M$. We close this section with an example of a property shared by $\text{Tr}_A$ and $\text{Tr}$.

**Lemma 5.4.** Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be an exact sequence in $\text{mod } R$. For suitable choices of $\text{Tr}_A$, we have the exact sequence

\[
0 \rightarrow N^\dag \rightarrow M^\dag \rightarrow L^\dag \rightarrow \text{Tr}_A N \rightarrow \text{Tr}_A M \rightarrow \text{Tr}_A L \rightarrow 0.
\]

Furthermore, if $\text{Ext}^1(L, C) = 0$, then we have

\[
0 \rightarrow \text{Tr}_A \Omega^i N \rightarrow \text{Tr}_A \Omega^i M \rightarrow \text{Tr}_A \Omega^i L \rightarrow 0.
\]

**Proof.** Let $\theta$ denote the map from $M$ to $N$. We have the short exact sequence $0 \rightarrow \Omega^i L \rightarrow \Omega^i M \rightarrow \Omega^i N \rightarrow 0$. Consider the following diagram where each $Q_j$ projective.

\[
\begin{array}{cccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
Q_0^i & Q_0^i & \Omega^i L \\
\downarrow & \downarrow & \downarrow \\
Q_1^i & Q_1^i & \Omega^i M \\
\downarrow & \downarrow & \downarrow \\
Q_2^i & Q_2^i & \Omega^i N \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0
\end{array}
\]
Applying † gives the following.

\[
\begin{array}{ccccccccc}
0 & \rightarrow & (\Omega^i N)^\dagger & \rightarrow & (Q_0^i)^\dagger & \rightarrow & (Q_1^i)^\dagger & \rightarrow & \text{Tr}_A \Omega^i N & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & (\Omega^i M)^\dagger & \rightarrow & (Q_0^i)^\dagger & \rightarrow & (Q_1^i)^\dagger & \rightarrow & \text{Tr}_A \Omega^i M & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \Omega^i L & \rightarrow & (Q_0^i)^\dagger & \rightarrow & (Q_1^i)^\dagger & \rightarrow & \text{Tr}_A \Omega^i L & \rightarrow & 0.
\end{array}
\]

The snake lemma yields

\[
0 \rightarrow (\Omega^i N)^\dagger \xrightarrow{\delta} \Omega^i L \xrightarrow{\delta} \text{Tr}_A \Omega^i N \xrightarrow{\eta} \text{Tr}_A \Omega^i M \rightarrow \text{Tr}_A \Omega^i L \rightarrow 0.
\]

Setting \(i = 0\) at this stage gives us the first claim. The short exact sequence \(0 \rightarrow \Omega^i L \rightarrow \Omega^i M \rightarrow \Omega^i N \rightarrow 0\) gives the following long exact sequence of \(\text{Ext}\) modules.

\[
0 \rightarrow (\Omega^i M)\dagger \xrightarrow{\delta} \Omega^i L \xrightarrow{\delta} \text{Ext}^1(\Omega^i N, C) \xrightarrow{\text{Ext}^1(\Omega^i \theta, C)} \text{Ext}^1(\Omega^i M, C) \rightarrow \cdots
\]

We also have

\[
\cdots \rightarrow \text{Ext}^i(L, C) \rightarrow \text{Ext}^{i+1}(N, C) \xrightarrow{\text{Ext}^{i+1}(\theta, C)} \text{Ext}^{i+1}(M, C) \rightarrow \cdots.
\]

Since, by assumption, \(\text{Ext}^i(L, C) = 0\), \(\text{Ext}^{i+1}(\theta, C)\) and thus \(\text{Ext}^1(\Omega^i \theta, C)\) are injective, forcing \(\delta\) to be zero. Hence \(\lambda\) is surjective. Then the first long exact sequence shows that \(\varepsilon\) is zero, and so \(\eta\) is injective, giving the desired result. \(\square\)

6. Resolving Subcategories which are Maximal Cohen Macaulay On the Punctured Spectrum

For the entirety of this section \((R, m, k)\) is a Noetherian local ring, and \(A\) denotes a thick subcategory of \(G_C\) that is closed under \(\dagger\). Recall that according to Proposition 5.6, dimension with respect to \(A\) satisfies the Auslander Buchsbaum formula. Set \(\text{res}_A M = \text{res}(M \cup A)_0\), \(\Delta(A)_0 = \{ M \in \Delta(A) \mid M_p \in \text{add} A_p \ \forall p \in \text{spec } R(m) \}\), and \(\Delta(A)_0 = \{ M \in \Delta(A) \mid A\text{-dim } M \leq i \}\). This section is devoted to proving the following.

**Theorem 6.1.** If \((R, m, k)\) is a local ring with \(\dim R = d\), the filtration

\[A = \Delta(A)_0 \subseteq \Delta(A)_1 \subseteq \cdots \Delta(A)_d = \Delta(A)_0\]

gives all the resolving subcategories of \(\Delta(A)_0\) containing \(A\).

This theorem and its proof is a generalization of [9, Theorem 2.1]. We now use our new "functors" from the previous section to make the building blocks of the proof of Theorem 6.1.

**Lemma 6.2.** For any module \(M \in \text{mod } R\), for suitable choices of \(\text{Tr}_A M\) and \(\tilde{\Omega} \text{Tr}_A M\), we have

\[0 \rightarrow \text{Ext}^1(M, C) \rightarrow \text{Tr}_A M \rightarrow \tilde{\Omega} \text{Tr}_A \Omega M \rightarrow 0.\]

**Proof.** With \(F_0, F_1, F_2\) projective, consider the sequence

\[F_2 \xrightarrow{\delta} F_1 \xrightarrow{\eta} F_0 \rightarrow M \rightarrow 0.\]

We have \(\text{coker } g^\dagger = \text{Tr}_A M\) and \(\text{Im } f^\dagger = \tilde{\Omega} \text{Tr}_A \Omega M\). By the universal property of kernel and cokernel, we have the following commutative diagram.

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \text{Im } g^\dagger & \rightarrow & F^\dagger & \rightarrow & \text{coker } g^\dagger & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \text{ker } f^\dagger & \rightarrow & F^\dagger & \rightarrow & \text{Im } f^\dagger & \rightarrow & 0.
\end{array}
\]
The snake lemma yields the exact sequence

$$0 \to \ker \iota \to 0 \to \ker \varepsilon \to \text{Ext}^1(M, C) \to 0 \to \coker \varepsilon \to 0.$$ 

Thus $\varepsilon$ is surjective and $\ker \varepsilon \cong \text{Ext}^1(M, C)$. The result follows.

**Proposition 6.3.** If $M \in \Delta(A)_0$, for all $0 \leq i < \text{depth } C$, for suitable choices of $\text{Tr}_A$ and $\bar{\Omega} \text{Tr}_A$, the following is exact.

$$0 \to \text{Tr}_A \Omega^{i+1} \bar{\Omega} \text{Tr}_A \Omega^{i+1} M \to \text{Tr}_A \Omega^{i} \text{Tr}_A \Omega^{i} M \to \text{Tr}_A \Omega^{i} \text{Ext}^{i+1}(M, C) \to 0$$

**Proof.** Using Lemma 6.2 we have

$$0 \to \text{Ext}^{i+1}(M, C) \to \text{Tr}_A \Omega^{i} M \to \bar{\Omega} \text{Tr}_A \Omega^{i+1} M \to 0.$$ 

Since $M \in \Delta(A)_0$, $\text{Ext}^{i+1}(M, C)_p = 0$ for every nonmaximal prime $p$. Thus $\text{Ext}^{i+1}(M, C)$ has finite length, and so $\text{Ext}^{i}(\text{Ext}^{i+1}(M, C), C) = 0$ for all $0 \leq i < \text{depth } C$. Thus, we can apply Lemma 6.4.
Taking the pullback diagram with our last exact sequence yields the following.

\[
\begin{array}{ccccccccc}
0 & 0 & \\
\downarrow & \downarrow & \\
\ker \partial_{L'}^{n} & \ker \partial_{L'}^{n} & \\
\downarrow & \downarrow & \\
0 \rightarrow \text{Tr}_A \Omega^n k & \rightarrow & T & \rightarrow & G_0 & \rightarrow & 0 & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 \rightarrow \text{Tr}_A \Omega^n k & \rightarrow & \text{Tr}_A \Omega^n L & \rightarrow & \text{Tr}_A \Omega^n L' & \rightarrow & 0. & \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & \\
\end{array}
\]

It is now easy to see that it suffices to show that \( \ker \partial_{L'}^{n} \) is in \( \text{res}_A \text{Tr}_A \Omega^n L \). When \( n = 0 \), \((G_0, \partial_{L'}^{n})\) is the resolution

\[
0 \rightarrow G_1 \xrightarrow{\partial_{L'}^{0}} G_0 \rightarrow \text{Tr}_A L' \rightarrow 0,
\]

and we are done since \( \ker \partial_{L'}^{0} = G_1 \in A \subseteq \text{res}_A \text{Tr}_A L \). So suppose \( n > 0 \). We have \( \ker \partial_{L'}^{n} = \text{Tr}_A \Omega^{n-1} L' \), by Lemma 6.3. By induction, \( \text{res}_A \text{Tr}_A \Omega^{n-1} L \) and \( \text{res}_A \text{Tr}_A \Omega^{n-1} L' \) are the same as \( \text{res}_A \text{Tr}_A \Omega^{n-1} k \). So we have \( \ker \partial_{L'}^{n} \in \text{res}_A \text{Tr}_A \Omega^{n-1} L \subseteq \text{res}_A \text{Tr}_A \Omega^n L \), where the inclusion follows from Lemma 6.4 and we are done. Note that this argument works for any choice of \( \text{res}_A \text{Tr}_A \Omega^n L \) or \( \text{res}_A \text{Tr}_A \Omega^n L' \).

These next proofs are essentially identical to the proofs in [9] with the appropriate changes. They are included here for the sake of completeness.

**Proposition 6.6.** For every \( 1 \leq n \leq \text{depth} R \), we have \( \Delta(A)_0^n = \text{res}_A \text{Tr}_A \Omega^{n-1} L \) for every nonzero finite length module \( L \).

**Proof.** By Lemma 6.3 we may assume that \( L = k \). By Lemma 6.4 we know that \( \dim(\text{Tr} \Omega^n k) = n \). Since localization commutes with cokernels, duals and syzygies, we have \( \text{Tr} \Omega^n k \) is in \( \Delta(A) \) and hence in \( \Delta(A)_0^n \).

Suppose \( M \in \Delta(A)_0^n \). Proposition 6.3 tells us that for each \( 0 \leq i < n \), we have

\[
\text{Tr}_A \Omega^i \text{Tr}_A \Omega^i M \in \text{res}_A(\text{Tr}_A \Omega^i \text{Tr}_A \Omega^{i+1} M, \text{Tr}_A \Omega^i \text{Ext}^{i+1}(M, C)).
\]

Lemma 6.5 says that \( \text{Tr}_A \Omega^i \text{Tr}_A \Omega^{i+1} M \subseteq \text{res}_A \text{Tr}_A \Omega^i \text{Tr}_A \Omega^{i+1} M \). Furthermore, because \( \text{Ext}^{i+1}(M, C) \) has finite length, Lemma 6.5 implies that \( \text{Tr}_A \Omega^i \text{Ext}^{i+1}(M, C) \) is in \( \text{res}_A \text{Tr}_A \Omega^i k \subseteq \text{res}_A \text{Tr}_A \Omega^{n-1} k \), where the inclusion follows from Lemma 6.4. Hence we have

\[
\text{Tr}_A \Omega^i \text{Tr}_A \Omega^i M \in \text{res}_A(\text{Tr}_A \Omega^i \text{Tr}_A \Omega^{i+1} M, \text{Tr}_A \Omega^i \text{Tr}_A \Omega^{n-1} k).
\]

It follows by induction that \( \text{Tr}_A \text{Tr}_A M \sim M \) is in

\[
\text{res}_A(\text{Tr}_A \Omega^n M, \text{Tr}_A \Omega^{n-1} k).
\]

However, \( \Omega^n M \in A \) and thus so is \( \text{Tr}_A \Omega^n \text{Tr}_A \Omega^n M \), and we have \( M \in \text{Tr}_A \Omega^{n-1} k \), which completes the proof.

We now prove the main result of this section.

**Proof of Theorem 6.7.** We clearly have the chain \( A = \Delta(A)_0^0 \subseteq \Delta(A)_0^1 \subseteq \cdots \subseteq \Delta(A)_0^d = \Delta(A) \). Take \( X \in \Delta(A)_0^n \setminus \Delta(A)_0^{n-1} \) for \( d \geq n \geq 1 \). We need to show that \( \text{res}_A X = \Delta(A)_0^n \), and we have \( \text{res}_A X \subseteq \Delta(A)_0^n \). For the reverse inclusion, by Proposition 6.6 it suffices to show \( \text{Tr}_A \Omega^{n-1} L \in \text{res}_A X \) for some finite length module \( L \).

Since \( \text{Ext}^n(X, C) \) is not zero and its localization is zero at every prime not equal to \( m \), \( \text{Ext}^n(X, C) \) has finite length, and for every finite length module \( L \), \( \text{Tr}_A \Omega^{n-1} L \in \text{res}_A \text{Tr}_A \Omega^{n-1} \text{Ext}^n(X, C) \). Using the \( A \)-resolution of \( X \)

\[
0 \rightarrow F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow X \rightarrow 0
\]
to compute \( \text{Ext}^i(X,C) \) and the \( \mathcal{A} \)-presentation \( 0 \to F_n \to F_{n-1} \to \Omega^{n-1}X \to 0 \) to compute \( \text{Tr}_A \Omega^{n-1}X \), we have \( \text{Ext}^n(X,C) \sim \text{Tr}_A \Omega^{n-1}X \) (this is why the generalization of the Auslander transpose in [12] and [15] was insufficient). So, for any finite length module \( L \), we have

\[
(3) \quad \text{Tr}_A \Omega^{n-1}L \in \text{res}_A \text{Tr}_A \Omega^{n-1} \text{Ext}^n(X,C) = \text{res}_A \text{Tr}_A \Omega^{n-1} \text{Tr}_A \Omega^{n-1}X.
\]

Therefore it suffices to show that \( \text{Tr}_A \Omega^{n-1} \text{Tr}_A \Omega^{n-1}X \) is in \( \text{res}_A X \) for some (and hence every) choice of \( \text{Tr}_A \Omega^{n-1} \text{Tr}_A \Omega^{n-1}X \). We will show this by induction on \( n \). When \( n = 1 \), we are done by \( \text{Tr}_A \text{Tr}_A X \sim X \). Now assume \( n > 1 \). By induction, (3), and Lemma 6.3 for all finite length \( L \) we have

\[
(4) \quad \text{Tr}_A \Omega^{n-2}L \in \text{res}_A \text{Tr}_A \Omega^{n-2} \text{Tr}_A \Omega^{n-2}(\Omega X) \subseteq \text{res}_A \Omega X \subseteq \text{res}_A X.
\]

Then Proposition 6.3 gives us the exact sequence

\[
0 \to \text{Tr}_A \Omega^{n-2} \text{Tr}_A \Omega^{n-2} X \to \text{Tr}_A \Omega^{n-2} \text{Tr}_A \Omega^{n-2} X \to \text{Tr}_A \Omega^{n-2} \text{Ext}^{n-1}(X,C) \to 0.
\]

But since \( \text{Ext}^{n-1}(X,C) \) has finite length, (3) tells us that \( \text{Tr}_A \Omega^{n-2} \text{Tr}_A \Omega^{n-2}X \) and \( \text{Tr}_A \Omega^{n-2} \text{Ext}^{n-1}(X,C) \) are in \( \text{res} X \). Therefore \( \text{Tr}_A \Omega^{n-2} \text{Tr}_A \Omega^{n-1}X \) is in \( \text{res}_A X \), and so, by Lemma 6.3, \( \text{Tr}_A \Omega^{n-1} \text{Tr}_A \Omega^{n-1}X \) is too. Thus, we are done.

The following corollary is immediate from Theorem 6.1.

**Corollary 6.7.** If \( M \in \Delta(A)_0 \bigsetminus \Delta(A)_0^{n-1} \), then \( \text{res}_A M = \Delta(A)_0^n \).

### 7. Resolving Subcategories and Semidualizing Modules

In this section, we keep the same notations and conventions as the previous section, except we will not assume that \( R \) is local. In this section, we prove Theorem 7.1 which is a critical step towards proving Corollary 8.6 and Theorem 8.8. Note that it is easy to check that \( C_p \) is a semidualizing \( R_p \)-module for all \( p \in \text{spec} R \).

Using Lemma 8.2, it is also easy to show that for all \( p \in \text{spec} R \), add \( A_p \) is a thick subcategory of \( \mathcal{G}_{C_p} \) closed under duals and contains \( C_p \). The following is a modified version of [8][Lemma 4.6], which is a generalization of [22][Proposition 4.2]. For a module \( X \), let \( \text{NA}(X) = \{ p \in \text{spec} R \mid X_p \notin \text{add} A_p \} \).

**Proposition 7.1.** Suppose \( X \in \Delta(A) \). For every \( p \in \text{NA}(X) \), there is a \( Y \in \text{res}_A X \) such that \( \text{NA}(Y) = V(p) \) and add \( A_p \text{-dim} Y_p = \text{add} A_p \text{-dim} X_p \) for all \( p \in V(p) \).

**Proof.** If \( \text{NA}(X) = V(p) \) we are done. So fix a \( q \in \text{NA}(X) \setminus V(p) \). As in the proof of [8][Lemma 4.6], choose an \( x \in p \setminus q \) and consider the following pushout diagram.

\[
\begin{array}{cccccc}
0 & \longrightarrow & \Omega X & \longrightarrow & F & \longrightarrow & X & \longrightarrow & 0 \\
\downarrow x & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega X & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & 0 \\
\end{array}
\]

with \( F \) projective. Immediately, we have \( Y \in \text{res}_A X \). Therefore, for all \( p \in \text{spec} R \), if \( X_p \) is in a resolving subcategory, then so is \( Y_p \), and thus we have \( \text{NA}(Y) \subseteq \text{NA}(X) \). The proof of [8][Lemma 4.6] tells us that

\[
\text{depth}(Y_p) = \min\{\text{depth}(X_p), \text{depth}(R_p)\}
\]

for all \( p \in V(p) \). Thus, by Proposition 8.6 add \( A_p \text{-dim} Y_p = \text{add} A_p \text{-dim} X_p \), for all \( p \in V(p) \). In particular, this shows that \( V(p) \) is contained in \( \text{NA}(Y) \).

Localizing at \( q \), yields the following.

\[
\begin{array}{cccccc}
0 & \longrightarrow & \Omega X_q & \longrightarrow & F_q & \longrightarrow & X_q & \longrightarrow & 0 \\
\downarrow x & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega X_q & \longrightarrow & Y_q & \longrightarrow & X_q & \longrightarrow & 0 \\
\end{array}
\]

Note \( x \) is a unit in \( R_q \). Thus, by the five lemma, \( Y_q \) is isomorphic to \( F_p \) and therefore is projective. So we have \( q \notin \text{NA}(Y) \) and hence \( \text{NA}(Y) \subseteq \text{NA}(X) \).

If \( \text{NA}(Y) \neq V(p) \), then we may repeat this process and construct a \( Y' \) that, like \( Y \), satisfies all the desired properties except \( V(p) \subseteq \text{NA}(Y') \subseteq \text{NA}(Y) \subseteq \text{NA}(X) \). Since spec \( R \) is Noetherian, this process must stabilize after some iteration, producing the desired module.
Lemma 7.2. Let $Z$ be a nonempty finite subset of $\text{spec} R$. Let $M$ be a module and $\mathcal{X}$ a resolving subcategory such that $M_p \in \mathcal{X}_p$ for some $p \in \text{spec} R$. Then there exists exact sequences

$$0 \to K \to X \to M \to 0 \quad 0 \to L \to M \oplus K \oplus R^t \to X \to 0$$

with $X \in \mathcal{X}$ and $\text{NA}(L) \subseteq \text{NA}(M)$ and $\text{NA}(L) \cap Z = \emptyset$.

Proof. The result is essentially contained in the proof of [23][Proposition 4.7]. It shows the existence of the exact sequences and shows that $Z$ is contained in the free locus of $L$ and thus $\text{NA}(L) \cap Z = \emptyset$. Furthermore, the last exact sequence in that proof shows that for any $p \in \text{spec} R$, $L_p$ is in $\text{res} M_p$. Hence, if $L_p$ is not in a resolving subcategory, then $M_p$ cannot be in that category as well, giving us $\text{NA}(L) \subseteq \text{NA}(M)$. □

These lemmas help prove the following proposition which is a key component of the proof of Theorem 7.4. This next result is also where we use Corollary 6.7 of the last section.

Proposition 7.3. For a module $M \in \text{mod}(R)$ and a category $\mathcal{X} \in \mathcal{R}(A)$, if for every $p \in \text{spec} R$, there exists an $X \in \mathcal{X}$ such that $\text{add} A_p \cdot \text{dim} M_p \leq \text{add} A_p \cdot \text{dim} X_p$, then $M$ is in $\mathcal{X}$.

Proof. Because of Lemma 2.12, we may assume $(R, m, k)$ is local. We proceed by induction on $\text{dim} \text{NA} M$. If $\text{dim} \text{NA} M = -\infty$, then $M$ is in $\mathcal{A}$ and we are done. Suppose $\text{dim} \text{NA} M = 0$. Then $M$ is in $\Delta(A)_0$ where $t = \Delta(A).$ By Proposition 7.3 there exists a $Y \in \text{res}_A X \subseteq \mathcal{X}$ with $\mathcal{A} \cdot \text{dim} Y = t$ and $Y \in \Delta(A)_0$, and thus $Y \in \Delta(A)_0 \setminus \Delta(A)_0^{-1}$. By Corollary 6.7 $\text{res}_A Y = \Delta(A)_0$, and thus $M \in \text{res}_A(Y) \subseteq \mathcal{X}$. The rest of the proof uses Lemma 7.3 and is identical to [9][Theorem 3.5], except one replaces the nonfree locus of $M$ with $\text{NA}(M)$ and replaces projective dimension with $\mathcal{A} \cdot \text{dim}$. □

We come to the main theorem of this section. Recall that $\Gamma$ is the set of grade consistent functions.

Theorem 7.4. Assume $R$ is Noetherian. If $\mathcal{A}$ is a thick subcategory of $\mathcal{G}_C$ containing $C$ and is closed under duals, then $\Lambda(\mathcal{A})$ and $\Phi_A$ are inverse functions and give a bijection between $\Gamma$ and $\mathcal{R}(A)$.

Proof. The previous proposition shows that $\Lambda(\mathcal{A})\Phi_A$ is the identity on $\mathcal{R}(A)$. Let $f \in \Gamma$ and $p \in \text{spec} R$. Since $\text{add} A_p \cdot \text{dim} X_p \leq f(p)$ for every $X \in \Lambda(\mathcal{A})(f)$, we have $\Phi_A(\Lambda(\mathcal{A})(f))(p) \leq f(p)$. However, by [9][Lemma 5.1] there is an $M \in \Delta(P) \subseteq \Delta(A)$ such that $\text{pd}_{A_p} M_p = f(p)$ and $\text{pd}_{A_p} M_q \leq f(q)$ for all $q \in \text{spec} R$. Since for all $q \in \text{spec} R \text{pd}_{A_p} M_q = \text{add} A_q \cdot \text{dim} M_q$, $M$ is in $\Lambda(\mathcal{A})(f)$, and we have $\Phi_A(\Lambda(\mathcal{A})(f))(p) = f(p)$. Thus $\Phi_A \Lambda(\mathcal{A})$ is the identity on $\Gamma$. □

8. Resolving Subcategories That Are Closed Under Duals

We wish to expand upon Theorem 7.3 using the results in Section 4. However, to use Theorem 7.3 we need to understand which thick subcategories of $\mathcal{G}_C$ are closed under duals and contain $C$. In this section, $C$ will be a semidualizing module. Since $\mathcal{G}_C$ is cogenerated by $\text{add} C$, as seen at the end of Section 3, it stands to reason that the results of Section 3 are applicable.

Lemma 8.1. Suppose $\mathcal{X} \subseteq \mathcal{G}_C$ is resolving with $C \in \mathcal{X}$. Then $\mathcal{X}$ is thick in $\mathcal{G}_C$ if and only if for every $X \in \mathcal{X}$, $(\Omega X^\dagger)^\dagger$ is in $\mathcal{X}$. In particular, $\mathcal{X}$ is thick in $\mathcal{G}_C$ if and only if it is cogenerated by $\text{add} C$.

When $R = C$, this is equivalent to saying that a resolving subcategory $\mathcal{X}$ of $\mathcal{G}_R$ is thick if and only if it is closed under cosyzygies. Also, since syzygies are unique up to projective summands, $(\Omega X^\dagger)^\dagger$ is unique up to add $C$ summands. Thus, for our purposes, our choice of syzygy is inconsequential.

Proof. We have the following exact sequence.

$$0 \to \Omega X^\dagger \to R^n \to X^\dagger \to 0$$

Applying $\dagger$ yields

$$0 \to X \to C^n \to (\Omega X^\dagger)^\dagger \to 0.$$ 

Since $C \in \mathcal{X}$, if $\mathcal{X}$ is thick in $\mathcal{G}_C$, $(\Omega X^\dagger)^\dagger$ is in $\mathcal{X}$. With regards to the second statement, add $C$ cogenerated $\mathcal{X}$ in this case.

Conversely, suppose for every $X \in \mathcal{X}$, $(\Omega X^\dagger)^\dagger$ is in $\mathcal{X}$. First we show that $\mathcal{X}^\dagger$ is resolving. Since $C \in \mathcal{X}$, $R$ is in $\mathcal{X}^\dagger$. For every $X^\dagger \in \mathcal{X}^\dagger$, we have $(\Omega X^\dagger)^\dagger \cong \Omega X^\dagger \in \mathcal{X}^\dagger$, and so $\mathcal{X}^\dagger$ is closed under syzygies. Given a short exact sequence $0 \to X^\dagger \to Y \to Z^\dagger \to 0$. With $X^\dagger, Z^\dagger \in \mathcal{X}^\dagger$, we have $0 \to Z \to Y^\dagger \to X^\dagger \to 0$. Thus $Y^\dagger \in \mathcal{X}$, and so $Y$ is in $\mathcal{X}^\dagger$. Showing $\mathcal{X}^\dagger$ is closed under direct summands is easy, and so $\mathcal{X}^\dagger$ is resolving.
Given a short exact sequence $0 \to M \to N \to L \to 0$ with $M, N \in \mathcal{A}$, then we have $0 \to L^\dagger \to N^\dagger \to M^\dagger \to 0$ with $N^\dagger, M^\dagger \in \mathcal{A}^\dagger$. So $L^\dagger$ is in $\mathcal{A}^\dagger$ since $\mathcal{A}^\dagger$ is resolving, and $L \cong L^\dagger$ is in $\mathcal{A} = \mathcal{A}^{\dagger\dagger}$. Since $\mathcal{A}$ is resolving, this completes the proof of the first statement.

To prove the second statement, suppose $\mathcal{A}$ is cogenerated by add $C$. Then we have $0 \to X \to C^m \to Y \to 0$ for any $X \in \mathcal{A}$ which yields $0 \to Y^\dagger \to R^m \to X^\dagger \to 0$. So $\Omega X^\dagger$ is stably equivalent to $Y$, and thus $(\Omega X^\dagger)^\dagger \sim Y$. So $(\Omega X^\dagger)^\dagger$ is in $\mathcal{A}$.

The following corollary, although intuitive, is not obvious, and it is not clear if it holds for other subcategories besides $\mathcal{G}_C$.

**Corollary 8.2.** If $\mathcal{A}$ is thick in $\mathcal{G}_C$, then add $\mathcal{A}_p$ is thick in $\mathcal{G}_{C_p}$ for all $p \in \text{spec } R$.

**Proof.** Take $p \in \text{spec } R$. From Lemma 2.11 we know that add $\mathcal{A}_p$ is resolving. By the previous lemma, it suffices to show that for all $X \in \text{add } \mathcal{A}_p$, $(\Omega_{R_p}X^\dagger)^\dagger = \text{Hom}(\Omega_{R_p} \text{Hom}(X, C_p), C_p)$ is in add $\mathcal{A}_p$. For every $X \in \text{add } \mathcal{A}_p$, there exists a $Y$ such that $X \oplus Y \sim Z_p$ for some $Z \in \mathcal{A}$. Consider the following.

$$(\Omega Z^\dagger)^\dagger_p = \text{Hom}(\Omega_R \text{Hom}(Z, C), C)^\dagger_p = \text{Hom}(\Omega_{R_p} \text{Hom}(Z_p, C_p), C_p)$$

$$= \text{Hom}(\Omega_{R_p} \text{Hom}(X \oplus Y, C_p), C_p) = \text{Hom}(\Omega_{R_p} \text{Hom}(X, C_p), C_p) \oplus \text{Hom}(\Omega_{R_p} \text{Hom}(Y, C_p), C_p)$$

By the previous lemma, $(\Omega Z^\dagger)^\dagger$ is in $\mathcal{A}$, and so $(\Omega_{R_p}X^\dagger)^\dagger$ is in add $\mathcal{A}_p$. $\Box$

The following lemmas show how to construct thick subcategories of $\mathcal{G}_C$ closed under duality.

**Lemma 8.3.** For a subset $\mathcal{W} \subseteq \mathcal{G}_C$ containing $C$, if for every $X \in \mathcal{W}$, $(\Omega X^\dagger)^\dagger$ is in $\mathcal{W}$ and $(\Omega C)^\dagger$ is in $\mathcal{W}$, then res $\mathcal{W}$ is thick in $\mathcal{G}_C$.

**Proof.** In light of the last lemma, we only need to show that for every $X \in \text{res } \mathcal{W}$, $(\Omega X^\dagger)^\dagger$ is in res $\mathcal{W}$. We proceed by the number of steps it takes to construct an element in res $\mathcal{A}$. See [22] for a precise definition of the notion of steps with regards to a resolving subcategory. The elements in res $\mathcal{A}$ that take zero steps are $R \cup \mathcal{W}$, and these satisfy our claim by assumption. Suppose $X$ is constructed in $n > 0$ steps. Then there exists a $Y$ and a $Z$ that can be constructed in $n - 1$ steps that satisfy one of the following situations.

1. $0 \to X \to Y \to Z \to 0$
2. $0 \to Y \to X \to Z \to 0$
3. $Z = X \oplus W$

Therefore one of the following is true.

1. $0 \to (\Omega X^\dagger)^\dagger \to (\Omega Y^\dagger)^\dagger \to (\Omega Z^\dagger)^\dagger \to 0$
2. $0 \to (\Omega Y^\dagger)^\dagger \to (\Omega X^\dagger)^\dagger \to (\Omega Z^\dagger)^\dagger \to 0$
3. $(\Omega Z^\dagger)^\dagger = (\Omega X^\dagger)^\dagger \oplus (\Omega W^\dagger)^\dagger$

By induction, $(\Omega Y^\dagger)^\dagger$ and $(\Omega Z^\dagger)^\dagger$ are in res $\mathcal{W}$, and the result follows. $\Box$

**Lemma 8.4.** Suppose $\mathcal{W} \subseteq \mathcal{G}_C$ is a subset containing $C$ and $(\Omega C)^\dagger$. If for every $X \in \mathcal{W}$, $X^\dagger$ and $(\Omega X)^\dagger$ are in $\mathcal{W}$, then res $\mathcal{W}$ is a thick subcategory of $\mathcal{G}_C$ that is closed under $\dagger$.

**Proof.** The previous lemma shows that res $\mathcal{W}$ is thick in $\mathcal{G}_C$. We will show that for every $X \in \text{res } \mathcal{W}$, $X^\dagger$ is in res $\mathcal{W}$ by induction on the number of steps it takes to construct an element in the thick subcategory res $\mathcal{W}$. The step zero modules are $\mathcal{W}$ and hence the claim is true. Now assume that the statement is true for step $n$. Given $0 \to X \to Y \to Z \to 0$, we have $0 \to Z^\dagger \to Y^\dagger \to X^\dagger \to 0$. Hence if any two of $X, Y, Z$ are in step $n$, then two of $X^\dagger, Y^\dagger, Z^\dagger$ are in res $\mathcal{W}$, and so the third is too. Since $\dagger$ splits across direct sums, the result follows. $\Box$

Let $\mathcal{W}$ be the set of all modules obtained by applying $\dagger$ and $\Omega$ to $C$ successive times. The following motivates us to set $\mathcal{A} = \text{res } \{\mathcal{W}\}$.

**Proposition 8.5.** The category $\mathcal{A}$ is the smallest thick category in $\mathcal{G}_C$ containing $C$ which is closed under $\dagger$.

**Proof.** The previous Lemma shows that $\mathcal{A}$ is thick and closed under duals. It is also easy to see that any thick subcategory containing $C$ must contain $\mathcal{W}$. $\Box$

In the notation of Section 3 set $\mathcal{G}(C) = \mathcal{G}(\mathcal{A})$. We may now apply our results from the beginning of the paper.
Corollary 8.6. The following is a bijection.

\[ \Lambda : \mathcal{S}(C) \times \Gamma \to \bigcup_{C \in \mathcal{S}} \mathcal{R}(C) \subseteq \mathcal{R} \]

Furthermore, for any \( C, D \in \mathcal{S}(C) \), then the following diagram commutes.

\[
\begin{array}{cccc}
\mathcal{R}(D) & \xrightarrow{\Phi_D} & \Gamma \\
\eta_D & \uparrow & \\
\mathcal{R}(C) & \xrightarrow{\Phi_C} & \Gamma \\
\eta_A & \uparrow & \\
\mathcal{R}(A) & \xrightarrow{\Phi_A} & \Gamma
\end{array}
\]

In particular, \( \rho_D^C \) and \( \eta_C^D \) are inverse functions.

Proof. The previous proposition tells us that \( \mathcal{A} \) is a thick subcategory of \( \mathcal{G}_C \) which contains \( C \) and is closed under \( \uparrow \). Therefore, by Theorem \( 7.4 \) \( \Lambda(\mathcal{A}) \) and \( \Phi_\mathcal{A} \) give a bijection between \( \mathcal{R}(\mathcal{A}) \) and \( \Gamma \). The result then follows immediately from Theorem \( 4.4 \). \( \square \)

Lemma 8.7. The collection of all the thick subcategories of \( \mathcal{G}_C \) containing \( C \) is contained in \( \mathcal{S}(C) \). Furthermore, when \( R \) is Cohen-Macaulay, every element in \( \mathcal{S}(C) \) is contained in MCM. In particular, when \( C = D \) is a dualizing module, \( \mathcal{S}(D) \) is the collection of thick subcategories of MCM containing \( D \).

Proof. Suppose \( \mathcal{X} \) is thick in \( \mathcal{G}_C \) and contains \( C \). Then \( \mathcal{X} \) must contain \( \mathcal{W} \) and hence contains \( \mathcal{A} \). By Lemma \( 8.1 \) \( \mathcal{X} \) is cogenerated by \( \text{add} \mathcal{C} \) and hence by \( \mathcal{A} \). Since \( \mathcal{A} \) is thick in \( \mathcal{G}_C \), \( \mathcal{A} \) is thick in \( \mathcal{X} \) as well, since any short exact sequence in \( \mathcal{X} \) is a short exact sequence in \( \mathcal{G}_C \). By Corollary \( 8.2 \) \( \text{add} \mathcal{A}_p \) is thick in \( \mathcal{G}_{C_p} \) and thus thick in \( \text{add} \mathcal{X}_p \) for all \( p \in \text{spec} R \). Hence \( \mathcal{X} \) is in \( \mathcal{S}(C) \).

Now suppose that \( R \) is Cohen-Macaulay and \( \mathcal{X} \in \mathcal{S}(C) \). Since \( \mathcal{A} \) cogenerates \( \mathcal{X} \), for any \( X \in \mathcal{X} \) there exists \( 0 \to X \to A_0 \to \cdots \to A_d \to X' \to 0 \) with \( A \in \mathcal{A} \) and \( d = \text{depth} R \). Since \( \mathcal{A} \subseteq \text{MCM} \), \( X \) is in \( \text{MCM} \). The last statement is now clear, since in that case \( \mathcal{G}_D = \text{MCM} \). \( \square \)

We now come to one of the main results of the paper.

Theorem 8.8. For any thick subcategory \( \mathcal{C} \) of \( \mathcal{G}_C \) containing \( C \), \( \Lambda(\mathcal{C}) \) and \( \Phi_\mathcal{C} \) give a bijection between \( \mathcal{R}(\mathcal{C}) \) and \( \Gamma \).

Proof. The previous lemma shows that \( \mathcal{C} \) is in \( \mathcal{S}(C) \). The rest follows from Theorem \( 4.2 \) and Theorem \( 7.4 \). \( \square \)

A resolving subcategory \( \mathcal{X} \) is dominant if for every \( p \in \text{spec} R \), there is an \( n \in \mathbb{N} \) such that \( \Omega^n_{\mathcal{R}_p} R_p / p R_p \in \text{add} \mathcal{X}_p \).

Corollary 8.9. Suppose \( R \) is Cohen-Macaulay and has a canonical module. Then there is a bijection between resolving subcategories containing MCM and grade consistent functions. Furthermore, the following are equivalent for a resolving subcategory \( \mathcal{X} \).

1. \( \mathcal{X} \) is dominant
2. \( \text{MCM} \subseteq \mathcal{X} \)
3. \( \text{Thick} \mathcal{X} = \text{mod}(R) \)

Proof. Letting \( D \) be the dualizing modules of \( R \), MCM is the same as \( \mathcal{G}_D \). Hence, by the previous theorem, \( \Lambda(\text{MCM}) : \Gamma \to \mathcal{R}(\text{MCM}) \) is a bijection, showing the first statement. From \( [9] \) Theorem 1.3, the following is a bijection.

\[ \xi : \Gamma \to \{ \text{Dominant Resolving subcategories of mod}(R) \} \]

\[ \xi(f) = \{ M \in \text{mod}(R) \mid \text{depth} M_p \geq \text{ht} p - f(p) \} \]

It is clear that \( \xi(0) = \text{MCM} \), hence every dominant subcategory contains MCM. Furthermore, we have \( \text{mod}(R) = \Delta(\text{MCM}) \), and hence every dominant resolving subcategory is an element of \( \mathcal{R}(\text{MCM}) \). Then for any \( f \in \Gamma \), we have

\[ \xi(f) = \{ M \in \text{mod}(R) \mid \text{depth} M_p \geq \text{ht} p - f(p) \} = \{ M \in \text{mod}(R) \mid \text{add MCM}_p - \text{dim} M_p \leq f(p) \} = \Lambda(\text{MCM})(f). \]
Thus $\xi$ equals $\Lambda(MCM)$, showing the equivalence of 1 and 2.

It is clear that 2 implies 3. Assume 3. Take a $p \in \text{spec } R$. Then we have $\mathcal{X}$-dim $R/p < \infty$. This implies that $\Omega^n R/p \notin \mathcal{X}$ for some $n$. Hence $\Omega^n R/p, pR_p \in \text{add } \mathcal{X}_p$, and so $\mathcal{X}$ is dominant.

\section{Gorenstein Rings and Vanishing of Ext}  

In this section, $(R, m, k)$ is a local Gorenstein ring. In this case, MCM is the same as $\mathcal{G}_R$, and Lemma 8.7 implies that $\mathcal{S}(R)$ is merely the collection of thick subcategories of MCM. This gives us the following which recovers \cite{9}[Theorem 7.4].

\textbf{Theorem 9.1.} If $R$ is Gorenstein, then we have the following commutative diagram of bijections.

\[ \begin{array}{ccc} 
\{\text{Thick subcategories of MCM}\} \times \Gamma & \overset{\Lambda}{\longrightarrow} & \{\mathcal{C} \in \mathcal{R} \mid \mathcal{C} \cap \text{MCM is thick in MCM}\} \\
\downarrow{id \times \Lambda(\mathcal{P})} & & \downarrow{id} \\
\{\text{Thick subcategories of MCM}\} \times \mathcal{R}(\mathcal{P}) & \overset{\Xi}{\longrightarrow} & \{\mathcal{C} \in \mathcal{R} \mid \mathcal{C} \cap \text{MCM is thick in MCM}\} 
\end{array} \]

\textbf{Proof.} Let $\mathcal{T}$ be the collection of resolving subcategories whose intersection with MCM is thick in MCM. As observed before the Theorem, $\mathcal{S}(R)$ is simply the thick subcategories of MCM. Since for any $\mathcal{C} \in \mathcal{S}(R)$, $\Delta(\mathcal{C}) \cap \text{MCM}$ is $\mathcal{C}$, the image of $\Lambda$ lies in $\mathcal{T}$. Furthermore, for any $\mathcal{X} \in \mathcal{T}$, $\mathcal{X}$ is in $\mathcal{S}(\mathcal{X} \cap \text{MCM})$, thus the result follows from Proposition 1.6 and Theorem 8.8. \hfill $\square$

It is natural to ask when the image $\Lambda$ is all of $\mathcal{R}(R)$. This will happen precisely when every resolving subcategory of MCM is thick. This occurs, by \cite{9}[Theorem 6.4], when $R$ is a complete intersection. We will give a necessary condition for $\text{Im } \Lambda = \mathcal{R}(R)$ by examining the resolving subcategories of the form

$\mathcal{C}_B = \{M \in \text{mod}(R) \mid \text{Ext}^{i>0}(M, N) = 0 \quad \forall N \in B\}$

where $B \subseteq \text{mod}(R)$. Dimension with respect to this category can be calculated in the following manner.

\textbf{Lemma 9.2.} For all $B \subseteq \text{mod}(R)$, we have the following.

$\mathcal{C}_B \text{-dim } M = \inf \{n \mid \text{Ext}^{i>n}(M, N) = 0 \quad \forall N \in B\}$

\textbf{Proof.} Let $M \in \text{mod}(R)$. For all $i > 0$ and $j \geq 0$ and each $N \in B$, we have $\text{Ext}^{i+j}(M, N) = \text{Ext}^{i}(\Omega^j M, N)$. So $\text{Ext}^{i+n}(M, N) = 0$ for all $i \geq 0$ if and only if $\Omega^n M$ is in $\mathcal{C}_B$. \hfill $\square$

\textbf{Lemma 9.3.} For any $B \subseteq \text{mod}(R)$, we have $\mathcal{C}_B \cap \mathcal{D}(\mathcal{P}) = \mathcal{P}$.

\textbf{Proof.} To prove this, it suffices to show that if $\text{pd}(X) = n > 0$, then $\text{Ext}^n(X, M) \neq 0$. Take a minimal free resolution

$0 \rightarrow F_n \xrightarrow{d_n} F_{n-1} \rightarrow \cdots \rightarrow F_0 \rightarrow X \rightarrow 0$.

Note that $\text{Im}(d) \subseteq mF_{n-1}$. We then get the complex

$0 \rightarrow \text{Hom}(X, M) \rightarrow \text{Hom}(F_0, M) \rightarrow \cdots \rightarrow \text{Hom}(F_{n-1}, M) \xrightarrow{d^*} \text{Hom}(F_n, M) \rightarrow 0$.

Now $\text{Im}(d^*)$ will still lie in $m\text{Hom}(F_n, M)$, and thus by Nakayama, $d^*$ cannot be surjective. Hence we have $\text{Ext}^n(X, M) = \text{coker } d^* \neq 0$. \hfill $\square$

Araya in \cite{1} defined AB dimension by $\text{AB-dim } M = \max \{b_m, \mathcal{G}_R \text{-dim } M\}$ where

$b_M = \min \{i \mid \text{Ext}^{i>0}(M, X) = 0 \Rightarrow \text{Ext}^{i>0}(M, X) = 0\}$.

Note that AB dimension satisfies the Auslander Buchsbaum formula. Also, a ring is AB if and only if every module has finite AB dimension.

\textbf{Lemma 9.4.} Taking $B \subseteq \text{mod}(R)$, if $\text{AB-dim } M < \infty$ for all $M \in \Delta(\mathcal{C}_B)$, then $\mathcal{C}_B$ is a thick subcategory of MCM.

\textbf{Proof.} Suppose $\text{AB-dim } \Delta(\mathcal{C}) < \infty$. First, we show that $\mathcal{C}_B$ is contained in MCM. Take any $M \in \mathcal{C}_B$. There is an exact sequence $0 \rightarrow M \rightarrow Y \rightarrow X \rightarrow 0$ with $\text{pd}(Y) < \infty$ and $X \in \text{MCM}$. We claim that $X$ has AB dimension zero. Suppose $\text{Ext}^{i>0}(X, Z) = 0$. Then $\text{Ext}^{i>0}(Y, Z) = 0$ and since $\text{pd } Y = \text{AB-dim } Y$, $\text{Ext}^{i \geq \text{pd } Y}(Y, Z)$ is zero. Then we have $\text{Ext}^i(X, Z) = 0$ and thus $\text{Ext}^{i>0}(M, Z) = 0$. Therefore $\text{Ext}^i(X, Z) = 0$ for all $i > \max \{\text{pd}(Y), b_M\} + 1$. Since $R$ is Gorenstein, that means that $X$ has finite $\mathcal{G}_R$ dimension, and thus $X$ has
finite AB dimension. But since AB dimension satisfies the Auslander Buchsbaum formula, AB-dim \( X \) must be zero.

Since \( Y \in \Delta(C_B) \), we have \( X \in \Delta(C_B) \). So \( \Ext^{>0}(X,N) = 0 \) for all \( N \in B \), and we have \( \Ext^{>0}(X,N) = 0 \) for all \( N \in B \). Hence \( X \) is in \( C_B \). Therefore, \( Y \) is also in \( C_B \), which, by Lemma 9.3, means that \( Y \) is projective and hence in MCM, forcing \( M \) to be in MCM as well.

Now to show that \( C_B \) is thick in MCM, it suffices to show that \( C_B \) is closed under cokernels of surjections in MCM. So take \( 0 \to A \to B \to C \to 0 \) with \( A,B,C \in \text{MCM} \) and \( A,B \in C_B \). Then \( C \in \Delta(C_B) \) and so \( \Ext^{>0}(C,N) = 0 \) for all \( N \in B \). But then \( C \) has finite AB dimension by assumption. Since AB dimension satisfies the Auslander Buchsbaum formula, AB-dim \( C \) is zero. So we have \( \Ext^{>0}(C,N) = 0 \) for all \( N \in B \), and hence, \( C \) is in \( C_B \).

Now let \( d = \dim R \).

**Theorem 9.5.** If \( R \) is Gorenstein, then the following are equivalent.

1. \( R \) is AB
2. \( C_R \) is a thick subcategory of MCM for all \( B \subseteq \text{mod}(R) \)
3. MCM \( \cap C_B \) is thick in MCM for every \( B \subseteq \text{mod}(R) \)
4. \( \Lambda(C_B) \) gives a bijection between \( \mathfrak{R}(C_B) \) and \( \Gamma \)
5. For all \( B \subseteq \text{mod}(R) \) and \( M \in C_B \), \( \Gamma \) contains the function \( f : \text{spec} R \to \mathbb{N} \) defined by the following:
   \[
   f(p) = \min \{ i \mid \Ext^{>0}(M,B) = 0 \}
   \]

**Proof.** The previous lemma shows that 1 implies 2 and 2 implies 3 is trivial. Assuming 3 we will show 1. Suppose \( \Ext^{>0}(M,N) = 0 \). Then \( M \) is in \( \Delta(C_N) \). Letting \( R = d \), we have \( \Omega^dM \in \Delta(C_N) \cap \text{MCM} \). For some \( n \geq d \) we have \( \Omega^nM \in \mathfrak{R}(C_B) \cap \text{MCM} \). But then we have

\[
0 \to \Omega^nM \to F_{n-1} \to \cdots \to F_d \to \Omega^dM \to 0
\]

where each \( F_i \) is projective. By 3, \( \Omega^dM \) is in \( C_B \). So we have \( -\dim_{C_N} M \leq d \), and so \( \Ext^{>d}(M,N) = 0 \).

Theorem 1.2 shows that 2 implies 3. Lemma 9.2 shows that 4 implies 5. Since \( R \) is local, evaluating \( f \) at the maximal ideal shows that 5 implies 1. \( \square \)

**Corollary 9.6.** Set \( r = d - \text{depth} M \). If \( R \) is AB and \( \Ext^{>0}(M,N) = 0 \), then \( \Ext^r(M,N) \neq 0 \). Furthermore, if \( \Ext^r(M,N) = 0 \) or \( \Ext^i(M,N) \neq 0 \) for \( i > r \), then \( \Ext^i(M,N) \neq 0 \) for arbitrarily large \( j \).

**Proof.** Suppose \( R \) is AB. Then 2 holds and \( C_N \)-dim satisfies the Auslander Buchsbaum formula. If \( \Ext^{>0}(M,N) = 0 \) then \( r = C_N \)-dim \( M \) is \( \max \{ i \mid \Ext^i(M,N) \neq 0 \} \). The second statement is just the contrapositive of the first statement. \( \square \)

**Corollary 9.7.** If \( R \) is Gorenstein and every resolving subcategory of MCM is thick, then \( R \) is AB.

**Proof.** The assumption implies 2 in Theorem 9.5. \( \square \)

Thus if \( \Lambda \) in Theorem 5.8 is a bijection from \( \mathfrak{S}(R) \times \Gamma \) to \( \mathfrak{R}(R) \), then \( R \) is AB. In [21], Stevenson shows that when \( R \) is a complete intersection, every resolving subcategory of MCM is closed under duals. The following gives a necessary condition for this property.

**Corollary 9.8.** If \( R \) is Gorenstein and every resolving subcategory of MCM is closed under duals, then \( R \) is AB.

**Proof.** Suppose every resolving subcategory of MCM is closed under duals. Let \( \mathcal{X} \subseteq \text{MCM} \) be resolving. Then for every \( X \in \mathcal{X} \), \( (\Omega X^*)^* \) is in \( \mathcal{X} \). By Lemma 5.1, \( \mathcal{X} \) is thick. The result follows from the previous corollary. \( \square \)

One of the consequences of Theorem 5.8 is that \( R \) is AB if and only if \( \{ C_B \mid B \subseteq \text{mod}(R) \} \subseteq \mathfrak{S}(R) \). The following shows that this is sometimes equality. We let \( \mathcal{X}^+ = \{ M \in \text{mod}(R) \mid \Ext^{>0}(M,X) = 0 \ \forall X \in \mathcal{X} \} \). Recall from [5], that \( X \in Y^+ \) if and only if \( Y \in \mathcal{X}^+ \). Hence we do not need to define \( \mathcal{X}^+ \).

**Proposition 9.9.** If \( R \) is a local complete intersection with an isolated singularity, then \( \mathcal{E}^{+\perp} = \mathcal{E} \) for every thick subcategory. In particular, \( \mathfrak{S}(R) \) is the same as \( \{ C_B \mid B \subseteq \text{mod}(R) \} \).
Proof. First, it is clear that $\mathcal{E} \subseteq \mathcal{E}^{\perp \perp}$, so we show reverse containment. Take any $M \in \mathcal{E}^{\perp \perp}$. Using the theory of support varieties of $[5]$, set $V = \bigcup_{E \in \mathcal{E}} V^*(E) \subseteq \mathbb{P}^n_k$ where $k$ is the algebraic closure of the residue field and we think of each $V^*(E)$ as a closed set in projective space instead of a cone in affine space. Now take any $p \not\in \mathbb{P}^n_k \setminus V$. By [5][Theorem 2], there exists an $N \in \text{mod } R$ such that $V^*(N) = \{p\}$. Since $V^*(N) \cap V^*(E) = \emptyset$ for all $E \in \mathcal{E}$, $N$ is in $\mathcal{E}^{\perp}$ by [5] Theorem 1. So, by assumption and [5] Theorem 1, $V^*(N) = \{p\}$ and $V^*(M)$ must be disjoint. Thus $V^*(M)$ is contained in $V$. Thus, using the classification of thick subcategories of a complete intersection ring in [20] or [7][Remark 5.12], $M$ is in $\mathcal{E}$ as desired. \hfill $\square$

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References

1. T. Araya, A homological dimension related to ab rings, (2012), arxiv:1204.4513v1.
2. M. Auslander and M. Bridger, Stable module theory, (1969), American Mathematical Society, Providence, R.I. MR 0269685 (42 #4580)
3. M. Auslander and I. Reiten, Applications of contravariantly finite subcategories, Adv. Math. 86 (1991), no. 1, 111–152. MR 1097029 (92e:16009)
4. R.-O. Auslander, M. Buchweitz, The homological theory of maximal Cohen-Macaulay approximations, Mém. Soc. Math. France (N.S.) no. 1, 1–20. MR 2346182 (2008j:13036)
5. L. L. Avramov and R.-O. Buchweitz, Support varieties and cohomology over complete intersections, Invent. Math. 142 (2000), no. 2, 285–318. MR 1794064 (2001j:13010)
6. L. L. Avramov and Srikanth B. I., Constructing modules with prescribed cohomological support, Illinois J. Math. 51 (2007), no. 1, 1–20. MR 2346182 (2008j:13036)
7. J. Carlson and S. B. Iyengar, Thick subcategories of the bounded derived category of a finite group, (2012), arXiv:1201.6536v1.
8. H. Dao and R. Takahashi, The radius of a subcategory of modules, (2012), arxiv:1111.2902v3.
9. H.-B. Fuchs, The homological theory of maximal Cohen-Macaulay approximations, (1989), no. 38, 5–37, Colloque en l’honneur de Pierre Samuel (Orsay, 1987). MR 1044344 (91h:13010)
10. H. Dao and R. Takahashi, Duality for bounded derived categories of complete intersections, Adv. Math. 241 (2010), no. 4, 2076–2116. MR 2690200 (2011h:13014)
11. P. Gabriel, Des catégories abéliennes, Bull. Soc. Math. France 90 (1962), 323–448. MR 0232821 (38 #1144)
12. Y. Geng, A generalization of the Auslander transpose and the generalized Gorenstein dimension, Czechoslovak Math. J. 63(138) (2013), no. 1, 143–156. MR 3035502
13. A. A. Gerko, On homological dimensions, Mat. Sh. 192 (2001), no. 8, 79–94. MR 1862245 (2002h:13024)
14. M. J. Hopkins, Global methods in homotopy theory, Homotopy theory (Durham, 1985), London Math. Soc. Lecture Note Ser., vol. 117, Cambridge Univ. Press, Cambridge, 1987, pp. 73–96. MR 932260 (89g:55022)
15. Z. Huang, On a generalization of the Auslander-Bridger transpose, Comm. Algebra 27 (1999), no. 12, 5791–5812. MR 1726277 (2000m:16023)
16. L. A. Hagel, D. Pospisil, J. Stovicek, and J. Trlifaj, Tilting, cotilting, and spectra of commutative noetherian rings, (2012), arXiv:1203.0907v3.
17. V. Masek, Gorenstein dimension of modules, (1999), arxiv:9809121v2.
18. A. Neeman, The chromatic tower for $D(R)$, Topology 31 (1992), no. 3, 519–532, With an appendix by Marcel Bökstedt. MR 1174255 (93h:18018)
19. S. Sather-Wagstaff, Semidualizing modules, (2009), http://www.ndsu.edu/pubweb/ ssatherw/DOCS/han.pdf.
20. G. Stevenson, Subcategories of singularity categories via tensor actions, (2012), arXiv:1105.4698v3.
21. , Duality for bounded derived categories of complete intersections, (2013), arXiv:1206.2724v2.
22. R. Takahashi, Modules in resolving subcategories which are free on the punctured spectrum, Pacific J. Math. 241 (2009), no. 2, 347–367. MR 2507582 (2010b:13027)
23. , Classifying thick subcategories of the stable category of Cohen-Macaulay modules, Adv. Math. 225 (2010), no. 4, 2076–2116. MR 2690200 (2011h:13014)
24. , Contravariantly finite resolving subcategories over commutative rings, Amer. J. Math. 133 (2011), no. 2, 417–436. MR 2797352 (2012h:13027)
25. , Classifying resolving subcategories over a Cohen-Macaulay local ring, Math. Z. 273 (2013), no. 1-2, 569–587. MR 3010176
26. W. V. Vasconcelos, Divisor theory in module categories, North-Holland Publishing Co., Amsterdam, 1974, North-Holland Mathematics Studies, No. 14, Notas de Matemática No. 53. [Notes on Mathematics, No. 53]. MR 0498530 (58 #16637)
27. D. White, Gorenstein projective dimension with respect to a semidualizing module, J. Commut. Algebra 2 (2010), no. 1, 111–137. MR 2607104 (2011d:13013)
28. Y. Yoshino, A functorial approach to modules of G-dimension zero, Illinois J. Math. 49 (2005), no. 2, 345–367 (electronic). MR 2163939 (2006e:13014)