Equivalence of generator coordinate Brink cluster model and nonlocalized cluster model and supersolidity of $\alpha$ cluster structure in nuclei

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It is found that $\alpha$ cluster structure has the apparently opposing dual property of crystallinity and condensation simultaneously. The mathematical equivalence of the spatially localized Brink $\alpha$ cluster model in the generator coordinate method (GCM) and the nonlocalized cluster model (NCM), which is also called the THSR (Tohsaki-Horiuchi-Schuck-Röpke) wave function based on the condensation of $\alpha$ clusters, is shown. The latter is found to be an equivalent representation of the localized cluster model and it is a natural consequence that the many NCM (THSR) calculations reproduce the proceeding cluster model calculations using the GCM and the resonating group method (RGM). Localized cluster models, which have been successfully used for more than half a century, will continue to be very powerful. The equivalence is a manifestation of the duality of incompatible aspects: crystallinity and coherent wave nature due to condensation of $\alpha$ clusters, i.e. the dual properties of a supersolid. The Pauli principle causes the duality. The evidence for supersolidity, the emergence of a Nambu-Goldstone mode caused by the spontaneous symmetry breaking of the global phase, is discussed.

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The α particle, which is the most tightly bound nucleus, has been playing an important role for more than a century in quantum physics and nuclear physics. The observation of anomalously large backward angle scattering of α particles from a gold target lead Rutherford to the discovery of the nucleus \(^{1}\). The α particle emission from radioactive nuclei was understood by Gamow \(^{2}\) and Gurney and Condon \(^{3, 4}\) for the first time as quantum tunneling, which has been shed light recently from the viewpoint of Lefschetz thimble \(^{5}\). The α particle model, in which the α particle is considered as a constituent unit of the nucleus, was proposed as the first nuclear structure model in 1937 \(^{6, 7, 8}\). While the naive classical α particle model was criticized \(^{9}\) in the advent of the shell model \(^{10, 11}\) and the collective model \(^{12}\), the successful shell model and the collective model \(^{13, 14, 15}\) also encountered difficulty explaining the emergence of very low-lying intruder states in light nuclei such as the mysterious \(^{0+}\) state in \(^{16}\)O \(^{16, 17}\). The developed quantum α cluster model, in which the effect of the Pauli principle is taken into account, has witnessed tremendous success in recent decades in explaining both shell-model like states and α cluster states comprehensively, which are reviewed in light nuclei in Refs. \(^{18, 19, 20}\) and in the medium-weight nuclei in Ref. \(^{21}\).

Cluster models such as the Brink cluster model using the generator coordinate method (GCM) \(^{22}\), the resonating group method (RGM), which is equivalent to the GCM \(^{23}\), the orthogonality condition model (OCM) \(^{24}\), and the local potential model (LPM) \(^{25, 26, 27} 28\), are all based on the picture that the cluster structure has a geometrical configuration. Examples are the two α dumbbell structure of \(^{8}\)Be \(^{29, 30}\), the three α triangle structure of \(^{12}\)C \(^{31, 32, 33}\), and the α+\(^{16}\)O structure in \(^{20}\)Ne \(^{34, 35, 36}\). In recent decades structure studies using cluster models in the bound and quasi-bound energies have all supported the spatially localized cluster picture \(^{18, 19, 20, 21}\). Also the unified understanding of cluster structure in the low energy region and prerainbows and nuclear rainbows in the scattering region, which are confirmed for the systems such as α+\(^{16}\)O and α+\(^{40}\)Ca \(^{28, 37}\), supports the geometrical localized cluster picture.

On the other hand, calculations using the nonlocalized cluster model (NCM), which was originally proposed to explain the dilute gas-like \(^{0+}\) Hoyle state in \(^{12}\)C from the viewpoint of Bose-Einstein condensation (BEC) of α clusters \(^{38}\) and named THSR wave function for the authors (Tohsaki-Horiuchi-Schuck-Röpke) \(^{39}\), reproduced the α cluster structure in \(^{8}\)Be, \(^{12}\)C and \(^{20}\)Ne \(^{40, 41, 42, 43, 44, 45}\), almost as well as the preceding GCM and RGM calculations. In contrary to the traditional geometrical localized cluster picture, the NCM calculations conclude that the most typical nucleus, \(^{8}\)Be, is more of a very dilute gas-like structure of two α cluster rather than a solid dumbbell structure of two α clusters \(^{43}\). The two concepts of gas and solid are opposing. Furthermore because the two pictures are
based on the incompatible concepts, from the viewpoint of the NCM (THSR) it has been summarized in Ref. [42] that the traditional understanding is incorrect, namely, that the localized cluster picture is strongly supported by the energy curve with the Brink wave function which gives the minimum point at a nonzero value of the intercluster distance parameter.

Why the NCM calculations based on the apparently exclusive concept give similar results to the preceding GCM and RGM calculations based on the localized cluster picture has not been understood and has remained puzzling for the last two decades. Very recently Ohkubo et al. used a superfluid $\alpha$ cluster model to report [46] that this puzzle can be solved by noticing that the $\alpha$ cluster structure has a duality of crystallinity (localization) and condensation (nonlocalization), a property of supersolidity. According to this theory, while the former is the view from the particle nature of the cluster structure, the latter is the view from the wave nature due to the coherence of the condensate cluster structure and the two are compatible. It is important to reveal generally and rigorously the relation between the geometrical localized cluster picture and the nonlocalized cluster picture and to deepen the underlying physical meaning of the $\alpha$ cluster structure.

In this paper it is shown that the NCM, namely the THSR wave function, is mathematically rigorously equivalent to the traditionally used geometrical Brink cluster model in the generator coordinate method. The reason why the THSR calculations [39, 40, 41, 42, 43, 44, 45, 47, 48, 49] give very similar results to the preceding cluster model calculations using the GCM and RGM based on the geometrical picture is clarified. It is shown generally that $\alpha$ cluster structure has the duality of apparently exclusive properties of crystallinity (localization) and condensation (nonlocalization), i.e. supersolidity.

The $n\alpha$ cluster model based on the geometrical crystalline picture such as the two $\alpha$ cluster model of $^8$Be and the three $\alpha$ cluster model of $^{12}$C, is given by the following Brink wave function [22]

$$\Phi^B_{n\alpha}(R_1, \ldots, R_n) = \frac{1}{\sqrt{(4n)!}} \det[\phi_0(r_1 - R_1)\chi_{\tau_1,\sigma_1}] \cdots \phi_0(r_{4n} - R_n)\chi_{\tau_{4n},\sigma_{4n}},$$  \hspace{1cm} (1)

where $R_i$ is a parameter that specifies the center of the $i$-th $\alpha$ cluster, $\phi_0(r - R)$ is a 0s harmonic oscillator wave function with a size parameter $b$ around a center $R$,

$$\phi_0(r - R) = \left(\frac{1}{\pi b^2}\right)^{3/4} \exp\left[-\frac{(r - R)^2}{2b^2}\right],$$  \hspace{1cm} (2)

and $\chi_{\sigma,\tau}$ is the spin-isospin wave function of a nucleon. Eq.(1) is rewritten as

$$\Phi^B_{n\alpha}(R_1, \ldots, R_n) = \mathcal{A} \left[ \prod_{i=1}^{n} \exp\left\{-2\frac{(X_i - R_i)^2}{b^2}\right\} \phi(\alpha_i) \right],$$  \hspace{1cm} (3)
where $X_i$ is the center-of-mass coordinate of the $i$-th $\alpha$ cluster and $\phi(\alpha_i)$ represents the internal wave function of the $i$-th $\alpha$ cluster. $\mathcal{A}$ is the antisymmerization operator. The generator coordinate wave function $\Psi_{n\alpha}^{GCM}$ based on the geometrical configuration of the Brink wave function is given by

$$\Psi_{n\alpha}^{GCM} = \int d^3 R_1 \cdots d^3 R_n f(R_1, \ldots, R_n) \Phi_n^B(R_1, \ldots, R_n). \tag{4}$$

We show that the localized cluster model of Eq. (4) and the nonlocalized cluster model are mathematically equivalent. For the sake of simplicity we treat hereafter the simplest two $\alpha$ cluster structure of $^8{\text{Be}}$. The generator coordinates $R_1$ and $R_2$, which specify the position parameters of the two $\alpha$ clusters, are rewritten as follows by using $R_G$ and $R$, which are the center-of-mass and the relative vectors, respectively,

$$R_1 = R_G + \frac{1}{2} R, \quad R_2 = R_G - \frac{1}{2} R. \tag{5}$$

We take $R_G=0$ to remove the spurious center-of-mass motion and use the notation $\Phi_{2\alpha}^B(R)$ for $\Phi_{2\alpha}(\frac{1}{2} R, -\frac{1}{2} R)$. Thus Eq. (4) is written as

$$\Psi_{2\alpha}^{GCM} = \int d^3 R f(R) \Phi_{2\alpha}^B(R). \tag{6}$$

We introduce $g(\mu)$, which is related to $f(R)$ by the Laplace transformation

$$f(R) = \int_0^\infty d\mu_x \int_0^\infty d\mu_y \int_0^\infty d\mu_z \exp \left[ -\left( \mu_x R_x^2 + \mu_y R_y^2 + \mu_z R_z^2 \right) \right] g(\mu), \tag{7}$$

where $\mu = (\mu_x, \mu_y, \mu_z)$. Then Eq. (6) reads

$$\Psi_{2\alpha}^{GCM} = \int d^3 \mu g(\mu) \left[ \int d^3 R \exp \left\{ -\left( \mu_x R_x^2 + \mu_y R_y^2 + \mu_z R_z^2 \right) \right\} \Phi_{2\alpha}^B(R) \right]. \tag{8}$$

The term $[\cdots]$ in the rhs of Eq. (8) is nothing but the definition of the NCM (THSR) wave function $\Phi_{2\alpha}^{NCM}$ for the two $\alpha$ clusters of $^8{\text{Be}}$ [40, 41, 42],

$$\Phi_{2\alpha}^{NCM}(\mu) \equiv \int d^3 R \exp \left\{ -\left( \mu_x R_x^2 + \mu_y R_y^2 + \mu_z R_z^2 \right) \right\} \Phi_{2\alpha}^B(R), \tag{9}$$

$$\propto \mathcal{A} \prod_{i=1}^2 \exp \left\{ -2 \left( \frac{X_{ix}^2}{B^2_x} + \frac{X_{iy}^2}{B^2_y} + \frac{X_{iz}^2}{B^2_z} \right) \right\} \phi(\alpha_i), \tag{10}$$

where

$$B_k = \sqrt{b^2 + \mu_k^{-1}} \quad (k = x, y, z). \tag{11}$$
Eq. (10) is the internal wave function and is independent of the center-of-mass motion. Eq. (8) reads
\[
\Psi_{2\alpha}^{GCM} = \int d^3 \mu g(\mu) \Phi_{2\alpha}^{NCM}(\mu). \tag{12}
\]
While in Eq. (6) the GCM wave function is expressed based on the geometrical picture using the Brink function \(\Phi_{B2\alpha}^\ast(\mathbf{R})\) as a base function, in Eq. (12) the same GCM wave function is expressed using the nonlocalized wave function \(\Phi_{2\alpha}^{NCM}\) as a base function.

The weight function \(f(\mathbf{R})\) in Eq. (6) is determined by the variation principle
\[
\delta \frac{\langle \Psi_{na}^{GCM} | H | \Psi_{na}^{GCM} \rangle}{\langle \Psi_{na}^{GCM} | \Psi_{na}^{GCM} \rangle} = 0. \tag{13}
\]
This leads to the Hill-Wheeler equation,
\[
\int d^3 \mathbf{R'} [\mathcal{H}(\mathbf{R}, \mathbf{R'}) - E\mathcal{B}(\mathbf{R}, \mathbf{R'})] f(\mathbf{R'}) = 0, \tag{14}
\]
where
\[
\mathcal{H}(\mathbf{R}, \mathbf{R'}) = \langle \Phi_{na}^{B}(\mathbf{R}) | H | \Phi_{na}^{B}(\mathbf{R'}) \rangle, \tag{15}
\]
and
\[
\mathcal{B}(\mathbf{R}, \mathbf{R'}) = \langle \Phi_{na}^{B}(\mathbf{R}) | \Phi_{na}^{B}(\mathbf{R'}) \rangle. \tag{16}
\]
\(H\) and \(E\) are the Hamiltonian of the system and the eigenenergy, respectively.

Similarly the weight function \(g(\mu)\) in Eq. (12) is determined by solving the following Hill-Wheeler equation for \(g(\mu)\),
\[
\int d^3 \mu' [\mathcal{H}^{NCM}(\mu, \mu') - E\mathcal{B}^{NCM}(\mu, \mu')] g(\mu') = 0, \tag{17}
\]
where
\[
\mathcal{H}^{NCM}(\mu, \mu') = \langle \Phi_{na}^{NCM}(\mu) | H | \Phi_{na}^{NCM}(\mu') \rangle, \tag{18}
\]
and
\[
\mathcal{B}^{NCM}(\mu, \mu') = \langle \Phi_{na}^{NCM}(\mu) | \Phi_{na}^{NCM}(\mu') \rangle. \tag{19}
\]

We show that the weight function \(g(\mu)\) is a \(\delta\) function. We expand \(f(\mathbf{R})\) using a Gaussian function as follows,
\[
f(\mathbf{R}) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} c_i^{(x)} c_j^{(y)} c_k^{(z)} \exp \left[-(a_{ix} R_x^2 + a_{iy} R_y^2 + a_{iz} R_z^2)\right], \tag{20}
\]
where \(c_i^{(x)}, c_j^{(y)}, c_k^{(z)}\) are the coefficients expanded by the Gaussian function with the width parameters \(a_{ix}, a_{iy}\) and \(a_{iz}\). Putting Eq. (20) into Eq. (7) of the Laplace transformation, we
\[ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} c_i^{(x)} c_j^{(y)} c_k^{(z)} \exp \left[ -\left( a_{ix} R_x^2 + a_{jy} R_y^2 + a_{kz} R_z^2 \right) \right] \]

\[ = \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} d\mu_x d\mu_y d\mu_z \exp \left[ -\left( \mu_x R_x^2 + \mu_y R_y^2 + \mu_z R_z^2 \right) \right] g_x(\mu_x) g_y(\mu_y) g_z(\mu_z). \quad (21) \]

The \( x \) component reads
\[ \sum_{i=0}^{\infty} c_i^{(x)} \exp \left[ -a_{ix} R_x^2 \right] = \int_0^{\infty} d\mu_x \exp \left[ -\mu_x R_x^2 \right] g_x(\mu_x) \quad (22) \]

By using the following Laplace transformation formula with \( s = R_x^2 \), \( \tau = a_{ix} \) and \( t = \mu_x \),
\[ e^{-\tau s} = \int_0^{\infty} e^{-st} \delta(t - \tau) dt, \quad (23) \]
we find
\[ c_i^{(x)} = 1 \text{ and } g_x(\mu_x) = \delta(\mu_x - a_{ix}) \quad \text{for } i = i_0, \]
\[ c_i^{(x)} = 0 \text{ for } i \neq i_0. \quad (24) \]

Putting \( c_{ijk} \equiv c_i^{(x)} c_j^{(y)} c_k^{(z)} \), Eq. (21) holds only when
\[ c_{ijk} = 1 \text{ and } g(\mu) = \delta(\mu_x - a_{i_0 x}) \delta(\mu_y - a_{j_0 y}) \delta(\mu_z - a_{k_0 z}), \]
for \( i = i_0, j = j_0, k = k_0, \]
\[ c_{ijk} = 0 \text{ for } i \neq i_0, j \neq j_0, k \neq k_0. \quad (25) \]

Thus it is found that the weight functions, \( f(\mathbf{R}) \) in Eq. (6) and \( g(\mu) \) in Eq. (12), have the simple functional forms as follows,
\[ f(\mathbf{R}) = \exp \left[ -\left( \alpha_x R_x^2 + \alpha_y R_y^2 + \alpha_z R_z^2 \right) \right], \quad (26) \]
\[ g(\mu) = \delta(\mu - \alpha), \quad (27) \]
where \( \alpha = (\alpha_x, \alpha_y, \alpha_z) \) is defined as \( (\alpha_x, \alpha_y, \alpha_z) \equiv (a_{i_0 x}, a_{j_0 y}, a_{k_0 z}) \). Putting Eq. (25) into Eq. (12), we get
\[ \Psi_{2\alpha}^{GCM} = \int d^3 \mu \delta(\mu - \alpha) \Phi_{2\alpha}^{NCM}(\mu), \]
\[ = \Phi_{2\alpha}^{NCM}(\alpha). \quad (28) \]

Eq. (28) means that \( \Psi_{2\alpha}^{GCM} \) is completely equivalent to a single NCM (THSR) wave function \( \Phi_{2\alpha}^{NCM}(\alpha) \). This means that the cluster wave function Eq. (6) based on the geometrical \( \alpha \) cluster picture can be always represented by the single NCM wave function.
The above discussion for the simplest two $\alpha$ cluster system can be generalized to the $n$-$\alpha$ cluster system. The Laplace transformation relation is generalized to

$$f(R_1, \cdots, R_n) = \int_0^\infty d\mu \exp \left[ -\sum_{i=1}^n (\mu_x R_{ix}^2 + \mu_y R_{iy}^2 + \mu_z R_{iz}^2) \right] g(\mu).$$

(29)

The nonlocalized cluster model wave function for $n$-$\alpha$ clusters is given by

$$\Phi_{NCM}^{n\alpha}(\mu) = \int d^3 R_1 \cdots d^3 R_n \exp \left[ -n \sum_{i=1}^n (\mu_x R_{ix}^2 + \mu_y R_{iy}^2 + \mu_z R_{iz}^2) \right] \Phi_B^{n\alpha}(R_1, \cdots, R_n),$$

(30)

$$\propto \prod_{i=1}^n \exp \left\{ -2 \left( \frac{X_{ix}}{B_x^2} + \frac{X_{iy}}{B_y^2} + \frac{X_{iz}}{B_z^2} \right) \right\} \phi(\alpha_i).$$

(31)

Then Eq.(4) reads

$$\Psi_{GCM}^{n\alpha} = \int d^3 \mu g(\mu) \Phi_{NCM}^{n\alpha}(\mu).$$

(32)

Similar to the two $\alpha$ cluster case, $g(\mu)$ is shown to be a $\delta$ function. Then

$$\Psi_{GCM}^{n\alpha} = \int d^3 \mu (\mu - \alpha) \Phi_{NCM}^{n\alpha}(\mu),$$

$$= \Phi_{NCM}^{n\alpha}(\alpha).$$

(33)

Eq.(33) means $\Psi_{GCM}^{n\alpha}(J\pi) = \Phi_{NCM}^{n\alpha}(J\pi)$ for any $J\pi$ states where $\Psi_{GCM}^{n\alpha}(J\pi) = \hat{P}^J \hat{P}_{MK} \Psi_{GCM}^{n\alpha}$ and $\Phi_{NCM}^{n\alpha}(J\pi) = \hat{P}^J \hat{P}_{MK} \Phi_{NCM}^{n\alpha}(\alpha)$ with $\hat{P}^J$ and $\hat{P}_{MK}$ being the parity projection and the angular momentum projection operators, respectively. Therefore the numerically calculated squared overlap, $\kappa = |\langle \Psi_{GCM}^{n\alpha}(J\pi) | \Phi_{NCM}^{n\alpha}(J\pi) \rangle|^2$, should be unity for any $J\pi$ states if the computations of $\Psi_{GCM}^{n\alpha}(J\pi)$ and $\Phi_{NCM}^{n\alpha}(J\pi)$ are accurate enough. For the two $\alpha$ cluster calculations of $^8$Be $^{43}$, $\kappa=0.9980$ for the ground state $0^+$. For the three $\alpha$ cluster calculations of $^{12}$C, $\kappa=0.93$ for $0^+$, $\kappa=0.90$ for $2^+$ $^{48}$, $\kappa=0.95-0.97$ $^{44}$ $^{45}$ and 0.99 $^{48}$ for $0^+$, $\kappa=0.96$ for $3^-$ $^{49}$ and $\kappa=0.92$ for $4^-$ $^{49}$. For the three $\alpha$ linear chain structure $\kappa=0.987 (0^+), 0.989 (2^+)$ and 0.981 (4$^+$) $^{50}$. For the four $\alpha$ cluster calculations of $^{16}$O $^{48}$ $^{51}$, $\kappa=0.98, 0.98, 0.98$ and 0.96 for $0^+$ (g.s.), $0^+$ (6.05 MeV), $0^+$ (13.6 MeV) and $0^+$ (15.1 MeV), respectively. For the four $\alpha$ linear chain structure of $^{16}$O, $\kappa=0.944 (0^+), 0.942 (2^+)$ and 0.931 (4$^+$) $^{50}$. As for the five $\alpha$ cluster structure of $^{20}$Ne, $\kappa=0.9929, 0.9879$ and 0.9775 for the ground band $0^+$, $2^+$ and $4^+$ states, respectively $^{40}$ $^{41}$, and $\kappa=0.9998$ and 0.9987 for the $1^-$ and $3^-$ states of the $K = 0_1^-$ band, respectively $^{41}$ $^{42}$, were reported by using the $\alpha+^{16}$O
cluster model, which corresponds to \( f(R_1, R_2, R_2, R_2, R_2) \) in Eq. (29), namely a limiting case that the four \( \alpha \) clusters approach the same parameter position \( R_2 = R_3 = R_4 = R_5 \) to form the \(^{16}\text{O}\) shell model wave function centered at \( R_2 \). That the calculated squared overlaps of the single NCM wave function with the GCM wave function give the values, \( \kappa \approx 1 \), for all the cases reported is the natural consequence of the equality of Eq. (33). The calculated values should be \( \kappa = 1 \) for any \( J^\pi \) states described by the GCM in more precise numerical computations.

Eq. (33) tells us that all the physical quantities such as the energy levels, reduced widths, electric transition probabilities, root mean square radii, etc. calculated using the NCM (THSR) wave functions of Eq. (30) and Eq. (31) are identical to those calculated using the Brink localized cluster model in the GCM. Because of the equivalence of the GCM to the RGM, it is a natural consequence that the numerical calculations using the NCM give almost exactly the same wave functions to those of the preceding RGM calculations based on the localized cluster picture. It is naturally expected that the NCM calculations, if the computer power allows, would give similar physical results in medium-weight nuclei such as \(^{40}\text{Ca}\) and \(^{44}\text{Ti}\) where the OCM and LPM calculations based on the localized cluster picture have been successful [21, 52].

We consider that there should be profound underlying physical meaning behind the equivalence of the GCM wave function and its NCM representation. Firstly, there is no doubt that the \( \alpha \) cluster structure has a geometrical crystalline structure as has been evidenced by a number of theoretical and experimental studies [18, 20, 21] of the structure in the bound and quasi-bound energy region, molecular resonances, ALAS (anomalous large angle scattering) or BAA (backward angle anomaly), prerainbows and nuclear rainbows in the scattering energy region. For the bound and quasi-bound state energy region, that the GCM wave function shows a geometrical localized \( \alpha \) cluster can be seen quantitatively and intuitively in the GCM energy surface \( V^{GCM}(J, \pi)(R) = \mathcal{H}^{J\pi}(R, R) = \langle \Phi_{B}^{(J, \pi)}(R) | H | \Phi_{B}^{(J, \pi)}(R) \rangle \). For the two \( \alpha \) cluster structure of \(^{8}\text{Be}\), the GCM energy surface (Fig. 1 of [29]) shows the minimum at \( R \neq 0 \) (\( \approx 3.5 \text{ fm} \)), which corresponds approximately to the relative distance in the coordinate space between the two \( \alpha \) clusters. This shows clearly that \(^{8}\text{Be}\) has a dumbbell structure of the two \( \alpha \) clusters. For the three \( \alpha \) clusters, the GCM energy surfaces (Fig. 2 of [31]) show the minimum at \( R \neq 0 \) favoring a geometrical configuration, equilateral triangle for the ground state \( 0^+ \) and the \( 3^-_1 \) state in \(^{12}\text{C}\). Similarly the GCM energy surfaces calculated using the \( \alpha+^{16}\text{O} \) Brink cluster model (Fig. 17 of [35]) show the energy minimum at \( R \neq 0 \) indicating a geometrical cluster structure for the parity-doublet \( K = 0^+_1 \) and \( K = 0^-_1 \) bands states in \(^{20}\text{Ne}\). In addition, the existence of the higher nodal band states in \(^{20}\text{Ne}\), in which the intercluster relative motion is excited and whose higher spin member states are responsible
Fig. 1: Illustrative figures of crystallinity, condensation and supersolidity of the α clusters (filled circles) in $^8\text{Be}$, $^{12}\text{C}$, $^{16}\text{O}$ and $^{20}\text{Ne}$. As the excitation energy increases vertically, the structure change occurs. In each nucleus (a) crystallinity, (b) condensation associated with a coherent wave and (c) supersolidity with both crystallinity and coherent wave of the α clusters are shown. The original Ikeda diagram based on crystallinity picture corresponds to (a) in each nucleus. In (b) of each nucleus α clusters are sitting in the 0s state of the harmonic oscillator potential with a coherent wave (broad curve). In (c) of each nucleus the α clusters are sitting in the 0s state of the distinct harmonic oscillator potentials separated due to the Pauli repulsion associated with a coherent wave (broad curve).

for the ALAS (BAA) phenomena in α scattering from $^{16}\text{O}$ [26, 27], give strong support to the geometrical α cluster viewpoint. The existence of a higher nodal band with α cluster structure has been also confirmed in medium-weight nuclei such as $^{40}\text{Ca}$ [53, 54, 55, 56] and $^{44}\text{Ti}$ [37, 56, 57, 58, 59, 60, 61]. Secondly, at the same time, it is clear that the NCM representation of Eq.(31) of the GCM wave function shows that the α cluster structure simultaneously has a condensate nature since the α clusters are trapped in the 0s state of the harmonic oscillator potential.

Illustrative pictures based on the above geometrical structure and the condensate structure of the α clusters in $^8\text{Be}$, $^{12}\text{C}$, $^{16}\text{O}$ and $^{20}\text{Ne}$ are displayed in (a) and (b) of each nucleus in Fig. 1. The pictures (a) correspond to the Ikeda diagram [34, 62], which has been a useful guide of cluster structure study in nuclei for more than half a century. The pictures (b) represent the wave aspect of the α cluster structure due to the condensation. Thus it is clear that the GCM wave function $\Psi_{\alpha}^{GCM}$ has the seemingly opposing above two aspects, particle nature and coherent wave nature since $\Psi_{\alpha}^{GCM} = \Psi_{\alpha}^{NCM}$ in Eq.(33). The exclusive two pictures, the duality of crystallinity (localization) and condensate coherent wave (non-localization), can be reconciled in the unified pictures displayed in (c) of each nucleus in Fig. 1 where the α clusters sitting in the 0s state of the distinct potentials due to the Pauli repulsion between the α clusters [63] form a coherent wave. The duality evokes Landau's
two-fluid picture of superfluidity of He II [64]. We divide the total density $\rho_{\text{GCM}}$ into the two components, $\rho_{\text{GCM}} = \rho_{\text{GCM}}^s + \rho_{\text{GCM}}^n$ where $\rho_{\text{GCM}}^s$ is the superfluid density distribution of the condensate, which corresponds to (b) of each nucleus in Fig. 1, and the normal density component $\rho_{\text{GCM}}^n$ is defined in the equation. It should be noted that $\rho_{\text{NCM}}^s = \rho_{\text{GCM}}^s$, $\rho_{\text{NCM}}^n = \rho_{\text{GCM}}^n$ and $\rho_{\text{NCM}} = \rho_{\text{GCM}}$ since $\Psi_{\text{GCM}} = \Phi_{\text{NCM}}$. $\rho_{\text{GCM}}^s$ is considered to correspond to the superfluid density $\rho_{\text{SCM}}^s$ of the superfluid cluster model (SCM) based on effective field theory, in which the order parameter is embedded by rigorously treating the Nambu-Goldstone mode due the spontaneous symmetry breaking (SSB) of the global phase [46]. The geometrical localization, degree of clustering, is characterized by the order parameter $R$ and the condensation is characterized by the order parameter, superfluid density $\rho_{\text{s}}$.

For $^8\text{Be}$, in Fig. 1(c), the two $\alpha$ clusters, which can penetrate by quantum tunneling [2, 3, 4, 5] the intercluster barrier due to the Pauli principle, are trapped in each 0s state of the two local minima of the double-well potential to form a coherent wave. The $\alpha$ cluster structure has the duality of crystallinity and coherent wave due to condensation. In the case that the potential has three, four, $\cdots$, $n$ local minima, it is clear that the three, four, $\cdots$, $n$ $\alpha$ linear chain cluster structure has the duality of crystallinity and condensation. Because of $\Psi_{n\alpha}^{\text{GCM}} = \Psi_{n\alpha}^{\text{NCM}}$ in Eq. (33), whatever the geometrical configuration, number of $\alpha$ clusters, and degree of clustering, the GCM $\alpha$ cluster wave function has the duality.

In more detail for each nucleus, in $^{12}\text{C}$, the dilute gas-like BEC Hoyle state with three $\alpha$ clusters appears near the $\alpha$ threshold. The Hoyle state and the BEC excited states built on it were shown to be reproduced well by the recent SCM calculations [65, 66, 67]. For $^{16}\text{O}$, from the geometrical cluster viewpoint, the ground state with the four $\alpha$ clusters at the vertices of the tetrahedron [68], which has been recently revisited in Refs. [69, 70, 71, 72], makes a structure change to the $\alpha + ^{12}\text{C}(0^+_1)$ cluster structure near the $\alpha$ threshold as revealed in Refs. [73, 74] and to the loosely coupled well-developed four $\alpha$ cluster states with the dilute gas-like $\alpha + ^{12}\text{C}(0^+_2)$ cluster structure near the four $\alpha$ threshold [75]. The four $\alpha$ linear chain structure is considered to appear at much higher energies above the four $\alpha$ threshold energy [76]. The above structure change is consistent with NCM calculations [48] and the BEC four $\alpha$ cluster calculations using the SCM [77]. For $^{20}\text{Ne}$, the ground state with five $\alpha$ clusters at the vertices of a trigonal bipyramid [78, 79, 80, 81, 82] makes a structure change near the $\alpha$ threshold as the excitation energy increases [36] and near the five $\alpha$ threshold energy well-developed gas-like $\alpha$ cluster BEC superfluid states [67] are expected to appear as observed in recent experiments [83, 84] before the five $\alpha$ linear chain structure at higher energies. Fig. 1 may be extended to the Ikeda diagram in medium-weight and heavy nuclei [52].
It should be noted that both the NCM and the GCM wave functions contain the two aspects of crystallinity and condensation. In other words, the nonlocalized wave function does not fully correspond to Fig. 1(b) of each nucleus as the localized cluster GCM wave function does not fully correspond to Fig. 1(a) of each nucleus. To what extent the wave function contains the condensate component illustrated in Fig. 1(b) depends on the degree of clustering of the cluster state. The unified pictures in Fig. 1(c) evoke an optical lattice in cold atom physics [85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95]. The \( \alpha \) cluster structure with crystallinity and condensation, i.e. supersolidity, is a supersolid. A supersolid has been searched for in recent decades in He II [96, 97, 98, 99, 100] and has been observed very recently in an optical lattice [88, 89, 90, 91, 92, 93, 94, 95]. The structure change in Fig. 1 from the ground state with both solidity and superfluidity of \( \rho_s \) to the dilute gas-like BEC state near the three and four \( \alpha \) threshold is considered to be a phase transition from a supersolid to a superfluid.

What is the evidence for the supersolidity? The direct evidence of supersolidity is the observation of a Nambu-Goldstone mode [101, 102, 103] due to SSB of the global phase, which was observed very recently for an optical lattice supersolid [88, 94, 95]. Since the superfluid density \( \rho_s \) is the order parameter of the SSB of the global phase [46], the existence of \( \rho_s \neq 0 \) in the GCM \( \alpha \) cluster wave function of the ground state due to the duality accompanies the Nambu-Goldstone mode states, which are to be very low-lying collective states and difficult to explain in the shell model. This logic is same as the emergence of rotational band states in deformed nuclei, for which the order parameter, deformation parameter, \( \delta \neq 0 \), caused by SSB of rotational invariance due to a quadrupole boson condensation in the ground state [15].

It is known that the very low-lying intruder collective \( 0^+ \) states appear systematically near the \( \alpha \) threshold energy in light and medium-weight nuclei such as the mysterious \( 0_2^+ \) states in \(^{16}\)O and in \(^{40}\)Ca, which are analog of the intruder \( 0_2^+ \) state in \(^{12}\)C. The appearance of such intruder collective states at a very low excitation energy near the \( \alpha \) threshold, which has been understood by the empirical threshold rule of the Ikeda diagram [34, 62], is considered to be understood from the viewpoint the Nambu-Goldstone mode due to SSB of the global phase of the \( \alpha \) cluster structure as discussed for \(^{12}\)C in Ref. [46].

Finally we mention the importance of the Pauli principle for the duality of geometrical localization and nonlocalization due to condensation of \( \alpha \) cluster structure. The geometrical localization of the \( \alpha \) clusters has been known to be caused by the Pauli principle [28, 104]. In (c) of each nucleus in Fig. 1, the coherent wave of the \( \alpha \) cluster structure is the consequence of the geometrical localization. Thus the Pauli principle has the dual role of causing the geometrical clustering and condensation. In this sense the origin of the superfluidity of \( \alpha \)
cluster structure is different from that of the BCS superfluidity in heavy nuclei and cold atoms.

To summarize, we have shown that the Brink $\alpha$ cluster model in the generator coordinate method with crystallinity based on the geometrical picture is mathematically equivalent to the nonlocalized cluster model based on the condensation of $\alpha$ clusters. Thus the apparently opposing nonlocalized cluster model is reconciled with the traditional geometrical localized cluster models such as the Brink cluster model in the generator coordinate method, the resonating group method, the orthogonality condition model, and the local potential model, which has been powerful in understanding cluster structure in nuclei intuitively and quantitatively. The equivalence is a manifestation of the duality of the crystallinity and condensation, particle nature and wave nature, of the geometric cluster structure. The $\alpha$ cluster structure is understood to have crystallinity and condensation simultaneously, that is, supersolidity. The Pauli principle causes the duality. The evidence of supersolidity of $\alpha$ cluster structure is the emergence of the Nambu-Goldstone mode due to the SSB of the global phase, i.e. emergence of a collective motion at very low excitation energy. The emergence of the $\alpha$ cluster states at very low excitation near the $\alpha$ threshold such as the Hoyle state $0^+_2$ of $^{12}\text{C}$ and the mysterious $0^+_2$ state of $^{16}\text{O}$ are considered to be such a member state of the Nambu-Goldstone mode.

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