Improving Lower Bound on Opaque Set for Equilateral Triangle

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An opaque set (or a barrier) for $U \subseteq \mathbb{R}^2$ is a set $B$ of finite-length curves such that any line intersecting $U$ also intersects $B$. In this paper, we consider the lower bound for the shortest barrier when $U$ is the unit equilateral triangle. The known best lower bound for triangles is the classic one by Jones [9], which exhibits that the length of the shortest barrier for any convex polygon is at least the half of its perimeter. That is, for the unit equilateral triangle, it must be at least $3/2$. Very recently, this lower bounds are improved for convex $k$-gons for any $k \geq 4$ [10], but the case of triangles still lack the bound better than Jones’ one. The main result of this paper is to fill this missing piece: We give the lower bound of $3/2 + 5 \cdot 10^{-13}$ for the unit-size equilateral triangle. The proof is based on two new ideas, angle-restricted barriers and a weighted sum of projection-cover conditions, which may be of independently interest.

1 Introduction

An opaque set (or a barrier) for $U \subseteq \mathbb{R}^2$ is a set $B \subseteq \mathbb{R}^2$ such that any line intersecting $U$ also intersects $B$. A simple example is that given any geometric shape (e.g., square, triangle, and so on), its boundary forms a barrier. Note that we do not assume $B$ is contained in $U$. A barrier is called rectifiable if it is a union of countably many finite-length curves which are pairwise disjoint with each other except at the endpoints. The problem considered in this paper is to minimize the length of rectifiable barriers, that is, what is the shortest barrier for given $U$?

This problem is so classic, which is first posed by Mazurkiewicz in 1916 [11]. Surprisingly, even for simple polygons such as squares or triangles, the length of the shortest barrier is still not identified. Currently, only lower bounds, which are probably not tight, are known: A general lower bound has been shown by Jones in 1964 [9], which proves that the shortest barrier for any convex polygon must be longer than the half of its perimeter. That is, the shortest barrier for the unit-size square must be at least two, and for the unit-size equilateral triangle it must be at least $3/2$. After that, the problem was revived in several times [2, 4, 8], and there are a number of papers considering its algorithmic aspects [1, 3, 6, 7, 12]. This paper focuses more on the mathematical aspect: We argue explicit lower bounds for a specific shape $U$.

For explicit lower bounds beyond Jones’ one, very recently, two papers propose improved lower bounds for squares [5, 10]. The result by [5] is conditional, which assumes that any segment in the barrier is not so far from the boundary of the square. The paper by Kawamura et al. [10] gives an unconditional lower bound of $2.0002$ for the unit-size square. Furthermore, they show that any (possibly non-regular) convex $k$-gon for $k \geq 4$ whose perimeter is $2p$, there exists a constant $\epsilon_k$ such that $p + \epsilon_k$ becomes a lower bound for the barrier. Unfortunately, this result assumes $k \geq 4$ and thus does not cover triangles. The best known lower bound for the unit-size equilateral triangle is still $3/2$. In this paper, we improve this lower bound by a small constant. More precisely, we obtain the lower bound of length $3/2 + 5 \cdot 10^{-13}$. While it is still far from the currently best barrier $O$ with length $\sqrt{3}$ (Figure 1), which is conjectured to be optimal, this result is the first nontrivial improvement of Jones’ bound for equilateral triangles.

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Figure 1: The barrier O (bold lines) conjectured to be optimal.

The following part of the paper is organized as follows: In Section 2, we state several notations and our proof ideas, which include the proof of Jones’ bound. Our proof is divided into two subcases. Section 3 and 4 correspond to those cases, and they are integrated in Section 5. Finally, we conclude this paper in Section 6.

2 Preliminaries and Proof Outline

Throughout this paper, we use the term “equilateral triangle” as the meaning of “unit-size equilateral triangle”. We assume that any barrier considered in this paper is a straight barrier, and thus regard \( B \) as a (possibly infinite) set of segments. Note that this assumption is not essential: By Kawamura et al. [10], it is shown that getting a lower bound for straight barriers implies getting the same bound for general (unconditional) barriers. For \( X \subseteq \mathbb{R}^2 \), we define \( X(\alpha) \in \mathbb{R} \) as the image of \( X \) projected onto the line with angle \( \alpha \) passing the origin. Precisely, \( X(\alpha) = \{ x\cos \alpha + y\sin \alpha | (x, y) \in X \} \). For any set \( X \) of segments, we denote by \(|X|\) the sum of the length of all the segments in \( X \), and denote by \(|X(\alpha)|\) the sum of the length of the segments constituting the image \( X(\alpha) \). We have the following necessary condition:

\[
\forall \alpha \in [0, \pi] : |U(\alpha)| \leq |B(\alpha)| \leq \sum_{l \in B} |l(\alpha)|
\]

That is, for any angle \( \alpha \), the projection of \( U \) must be covered by the projection of \( B \). Otherwise, \( B \) cannot be a barrier because there exists a line orthogonal to the plane with angle \( \alpha \) intersecting \( U \) but not intersecting \( B \). We call this inequality the projection-cover condition for \( \alpha \).

The bound by Jones [9] is obtained by summing up the projection-cover condition for all \( \alpha \in [0, \pi] \).

\[
p = \int_0^\pi |U(\alpha)|d\alpha \leq \int_0^\pi |B(\alpha)|d\alpha \leq \sum_{l \in B} |l| \cdot \int_0^\pi |\cos \alpha|d\alpha = 2|B|,
\]

where \( p \) is the perimeter of \( U \). In the case that \( U \) is the equilateral triangle, \( p = 3/2 \). Note that the first equality is obtained by Cauchy’s surface area formula.

Our lower bound proof is based on two new ideas. The first one is to consider angle-restricted barriers: Letting \( A \subseteq [0, \pi] \), we say that \( B \) is \( A \)-restricted if any segment \( l \in B \) has an angle in \( A \). Given an \( A \)-restricted barrier and an angle \( \phi \in A \), we denote the set of segments in \( B \) with angle \( \phi \) by \( B_\phi \).

The next idea is an extension of Jones’ bound to obtain better bounds for angle-restricted barriers. The key observation behind the extension is an interpretation of Jones’ bound in the context of linear programming. Let \( U \) be a convex polygon, \( L(\alpha) \) be the set of lines with angle \( \alpha + \pi/2 \) intersecting \( U \), and \( L = \cup_\alpha L(\alpha) \). Now we define any segment by their two endpoints, that is, we regard a segment as an element in \((\mathbb{R}^2)^2 \). The length of a segment \( s \) is denoted by \(|s|\). For segment \( s \), we also define a 0-1 variable \( x_s \). Letting \( X_l \) be the set of segments intersecting a line \( l \in \mathcal{L} \), the constraint that the line \( l \) is “blocked”
by some segment is described by \( \sum_{s \in X_l} x_s \geq 1 \). That is, we have an integer-programming formulation for the shortest barrier problem:

\[
\begin{align*}
\text{minimize} & \quad \sum_{s \in (\mathbb{R}^2)^2} |s|x_s \\
\text{subject to} & \quad \sum_{s \in X_l} x_s \geq 1 \quad \forall l \in L \\
& \quad x_s \in \{0, 1\} \quad \forall s \in (\mathbb{R}^2)^2
\end{align*}
\]

Now we sum up the constraints for all \( l \in L(\alpha) \). Since the number of lines with angle \( \alpha + \pi/2 \) intersecting to a geometric shape \( F \) is proportional to the volume of the projection \( F(\alpha) \), the number of appearance for variable \( x_s \) in the left side of the summation is proportional to \( |s(\alpha)| \). Similarly the right-side value is proportional to \( |U(\alpha)| \). Consequently, we can write the summation as follows:

\[
\sum_{s \in (\mathbb{R}^2)^2} |s(\alpha)|x_s \geq |U(\alpha)|
\]

Interestingly, when we fix a barrier (i.e., variable assignments) the inequality above is equivalent to the projection-cover condition for angle \( \alpha \), and thus summing up all constraints for \( l \in L \) consists in Jones’ bound. The implication of this observation is that Jones’ bound can be seen as a construction of a dual solution for the LP relaxation of the above IP, i.e., \( 1/2 \cdot 1 \) is a feasible dual solution (where 1 is a all-one vector).

This interpretation yields a natural question: Can we construct a better dual solution for improving lower bounds? If we can construct such a solution for the dual LP, it implies a better lower bound for the shortest barrier. Unfortunately, this approach fails because the integrality gap of the LP relaxation is not sufficiently small to improve Jones’ bound: Actually, there exists a fractional (more precisely, half-integral) solution for that LP relaxation with the cost matching the Jones’ bound [10]: Letting \( X \) be the set of segments forming the boundary of \( U \), and consider the value assignment setting \( x_s = 1/2 \) for any \( s \in X \). This is a feasible solution of the relaxed LP and the value of the objective function is obviously equal to the half of the perimeter.

Our approach for circumventing this issue is to utilize the LP-based argument to obtain the bound for \( \{0, \pi/6, 5\pi/6\} \)-restricted barriers. Obviously, this class contains the barrier \( O \). That is, the upper bound for the shortest barrier in that class is \( \sqrt{3} \). Interestingly, for the IP formulation of finding the optimal of all \( \{0, \pi/6, 5\pi/6\} \)-restricted barriers, its LP relaxation exhibits an integrality gap arbitrarily close to one. That is, the conjecture shown in Figure 1 is the shortest barrier of all \( \{0, \pi/6, 5\pi/6\} \)-restricted barriers.

The final step of our proof is a reduction: We show that most of barriers with length at most \( 3/2 \) induce \( \{0, \pi/6, 5\pi/6\} \)-restricted barriers with length less than \( \sqrt{3} \). Thus we can eliminate the existence of such barriers. Only the exception is the class of \( \{0, \pi/3, 2\pi/3\} \)-restricted barriers. Any barrier of length \( 3/2 \) in that class induces only \( \{0, \pi/6, 5\pi/6\} \)-restricted barriers of length \( \sqrt{3} \), and thus we can not lead a contradiction. To resolve this exceptional case, we also provide an improved lower bound for \( \{0, \pi/3, 2\pi/3\} \)-restricted barriers based on the approach by Kawamura et al. [10]. Putting all results together, we obtain a general lower bound strictly larger than \( 3/2 \).

### 3 Bound for \( \{\pi/6, \pi/2, 5\pi/6\} \)-Restricted Barriers

In what follows, let \( U \) be the unit-size equilateral triangle. This section provides the optimal bound for \( \{\pi/6, \pi/2, 5\pi/6\} \)-restricted barriers for \( U \) is \( \sqrt{3} \). Note that this restricted class contains the barrier \( O \) conjectured to be optimal. That is, our proof implies that \( O \) is the optimal barrier in that class. As we discussed in the previous section, the proof is based on the construction of a better dual solution for the IP/LP formulation of the shortest barrier problem. It can be interpreted as a “weighted sum” of projection-cover conditions. The core of the proof is to provide a nice weight function yielding the optimal
bound. First, we identity the value of $|U(\alpha)|$, which is described as follows:

$$
|U(\alpha)| = \begin{cases} 
\cos \alpha & 0 \leq \alpha \leq \pi/6 \\
\cos(\pi/3 - \alpha) & \pi/6 \leq \alpha \leq \pi/3 \\
|U(\alpha - \pi/3)| & \pi/3 \leq \alpha.
\end{cases}
$$

It is easy to verify the above description by Figure 2. The case of $\alpha \geq \pi/3$ is obtained by the symmetry of $U$. Let $B$ be the optimal $\{\pi/6, \pi/2, 5\pi/6\}$-restricted barrier. Then the right side of the projection-cover condition can be described as follows:

$$
|U(\alpha)| \leq |B_{\pi/6}| \cdot |\cos(\pi/6 - \alpha)| + |B_{\pi/2}| \cdot |\cos(\pi/2 - \alpha)| + |B_{5\pi/6}| \cdot |\cos(5\pi/6 - \alpha)|. 
$$

(2)

Now we introduce a function $z(\alpha)$ over $[0, \pi]$, which is defined by another arbitrary function $z'(\alpha)$ over $[0, \pi/6]$:  

$$
z(\alpha) = \begin{cases}  
z'(\alpha) & 0 \leq \alpha \leq \pi/6 \\
z'(\pi/3 - \alpha) & \pi/6 \leq \alpha \leq \pi/3 \\
z'(\pi - \alpha) & \pi/3 \leq \alpha.
\end{cases}
$$

We use $z(\alpha)$ as a weight function. Assuming $z(\alpha)$ is non-negative, The weighted sum of projection-cover conditions is stated as follows:

$$
\int_0^\pi z(\alpha)|U(\alpha)|d\alpha \leq |B_{\pi/6}| \int_0^{\pi} z(\alpha) \cdot |\cos(\pi/6 - \alpha)|d\alpha \\
+ |B_{\pi/2}| \int_0^\pi z(\alpha) \cdot |\cos(\pi/2 - \alpha)|d\alpha + |B_{5\pi/6}| \int_0^\pi z(\alpha) \cdot |\cos(5\pi/6 - \alpha)|d\alpha. 
$$

(3)

We look at the right side the above inequality. Since $z(\alpha)$ is a periodic function of period $\pi/3$, and $|\cos(\gamma - \alpha)|$ for any $\gamma \in [0, \pi]$ is a periodic function of period $\pi$, we have

$$
\int_0^\pi z(\alpha) \cdot |\cos(\pi/2 - \alpha)|d\alpha = \int_{-\pi/3}^{2\pi/3} z(\alpha + \pi/3) \cdot |\cos(\pi/2 - (\alpha + \pi/3))|d\alpha \\
= \int_0^\pi z(\alpha) \cdot |\cos(\pi/6 - \alpha)|d\alpha.
$$

Similarly, we also have $\int_0^\pi z(\alpha) \cdot |\cos(5\pi/6 - \alpha)|d\alpha = \int_0^\pi z(\alpha) \cdot |\cos(\pi/6 - \alpha)|d\alpha$. Then the inequality (3) is simplified as follows:

$$
\int_0^\pi z(\alpha)|\cos(\alpha)|d\alpha \leq (|B_{\pi/6}| + |B_{\pi/2}| + |B_{5\pi/6}|) \int_0^\pi z(\alpha) \cdot |\cos(\pi/6 - \alpha)|d\alpha.
$$

$$
\iff \frac{\int_0^\pi z(\alpha)|\cos(\alpha)|d\alpha}{\int_0^\pi z(\alpha) \cdot |\cos(\pi/6 - \alpha)|d\alpha} \leq |B|.
$$

The remaining issue is to find the function $z'(\alpha)$ maximizing the left side of the above inequality. We have the following lemma:

**Lemma 1** Letting $c > 0$ and $z'(\alpha) = c|\cos(\pi/6 - \alpha)|$,

$$
\lim_{c \to \infty} \left( \frac{\int_0^\pi z(\alpha)|\cos(\alpha)|d\alpha}{\int_0^\pi z(\alpha) \cdot |\cos(\pi/6 - \alpha)|d\alpha} \right) = \sqrt{3}.
$$

That is, $|B| \geq \sqrt{3}$ holds.

**Proof** Just a calculation suffices. We add in the appendix the calculation result by Mathematica so that readers can believe its correctness without spending their time.
4 Bound for \( \{0, \pi/3, 2\pi/3\} \)-Restricted Barriers

This section considers lower bounds for the other extreme case, that is, \( \{0, \pi/3, 2\pi/3\} \)-restricted barriers. In this case, the proof relies on the lemma by Kawamura et al. [10].

**Lemma 2 (Kawamura et al. [10])** Let \( \lambda \in (0, \pi/2), \kappa \in (0, \lambda) \) and \( l, D > 0 \). Let \( B^- \) and \( B^+ \) be unions of \( n \) line segments of length \( l \) such that

- every segment of \( B^- \cup B^+ \) makes angle \( > \lambda \) with the horizontal axis;
- \( B^- \cup B^+ \) lies entirely in the disk of diameter \( D \) centered at the origin;
- \( B^- \) and \( B^+ \) are separated by bands of angle \( \kappa \) and width \( W := nl \sin(\lambda - \kappa) \) centered at the origin.

Then,

\[
\int_0^\pi |(B^- \cup B^+)(\alpha)|d\alpha \leq 2|B^- \cup B^+|l - \frac{W^2}{D}.
\]

In this section, let \( B \) be the optimal \( \{0, \pi/3, 2\pi/3\} \)-restricted barrier. We assume \( |B| = 3/2 + \delta \) and will bound \( \delta \) as large as possible. We define \( \delta p_1p_2p_3 = U \) (that is, \( p_1 = (0, 0), p_2 = (1, 0), p_3 = (1/2, \sqrt{3}/2) \)).

**Lemma 3** Let \( q_1 = (13/14, 0), q_2 = (27/28, \sqrt{3}/28), T = \triangle q_1q_2p_2, \) and \( Y_1 \) be the zone whose projection for angle \( 5\pi/6 \) is contained in \( T(5\pi/6) \). Furthermore, \( P_1 \) denotes the right half-plane for the line with angle \( \pi/2 \) passing on \( p_2 \), and \( P_2 \) denotes the lower half-plane for the line with angle \( \pi/6 \) passing on \( p_2 \). Finally, let \( X_1 = Y_1 \setminus (P_1 \cup P_2) \) (depicted in Figure 3). Then \( 1/28 - 2.5\delta \leq |B_{2\pi/3} \cap X_1| \) or \( 1/28 - 2.5\delta \leq |B_0 \cap X_1| \) holds.

**Proof** Since \( B_{2\pi/3} \) does not contribute to cover \( T(5\pi/6) \), by the projection-cover condition for \( \alpha = 2\pi/3 \), we have

\[
\frac{\sqrt{3}}{2} \cdot \frac{1}{14} \leq \frac{\sqrt{3}}{2}(|B_{2\pi/3} \cap Y_1| + |B_0 \cap Y_1|) \\
\Rightarrow \frac{1}{14} \leq |(B_{2\pi/3} \cup B_0) \cap X_1| + |(B_{2\pi/3} \cup B_0) \cap (P_1 \cup P_2)| \\
\Rightarrow \frac{1}{14} - |(B_{2\pi/3} \cap P_1)| - |B_{2\pi/3} \cap P_2| - |B_0 \cap P_1| - |B_0 \cap P_2| \leq |(B_{2\pi/3} \cup B_0) \cap X_1|. \tag{4}
\]
Let us consider the projection of $U$ for angles $0$ and $2\pi/3$. Since $B \cap P_1$ and $B \cap P_2$ do not contribute to cover $U(0)$ and $U(2\pi/3)$ respectively, we also have

\[ 1 \leq \frac{1}{2} (|B_{\pi/3} \setminus P_1| + |B_{2\pi/3} \setminus P_1|) + |B_0 \setminus P_1| \]
\[ \Rightarrow 1 + \frac{1}{2} (|B_{\pi/3} \cap P_1| + |B_{2\pi/3} \cap P_1|) + |B_0 \cap P_1| \leq \frac{1}{2} (|B_{\pi/3}| + |B_{2\pi/3}|) + |B_0| \]
\[ \Rightarrow 2 + |B_{2\pi/3} \cap P_1| + |B_0 \cap P_1| \leq |B_{\pi/3}| + |B_{2\pi/3}| + 2|B_0|, \quad (5) \]
\[ 1 \leq \frac{1}{2} (|B_{\pi/3} \setminus P_2| + |B_{2\pi/3} \setminus P_2|) + |B_{2\pi/3} \setminus P_2| \]
\[ \Rightarrow 1 + \frac{1}{2} (|B_0 \cap P_2| + |B_{\pi/3} \cap P_2|) + |B_{2\pi/3} \cap P_2| \leq \frac{1}{2} (|B_0| + |B_{\pi/3}|) + |B_{2\pi/3}| \]
\[ \Rightarrow 2 + |B_0 \cap P_2| + |B_{2\pi/3} \cap P_2| \leq |B_0| + |B_{\pi/3}| + 2|B_{2\pi/3}| \quad (6) \]

We also have the inequality below, which is equivalent to the projection-cover condition for angle $\pi/3$:

\[ 1 \leq \frac{1}{2} (|B_0| + |B_{2\pi/3}|) + |B_{\pi/3}| \]
\[ \Rightarrow 2 \leq |B_0| + |B_{2\pi/3}| + 2|B_{\pi/3}| \quad (7) \]

Summing up inequalities (4), (5), (6), and (7), we obtain

\[ 6 + \frac{1}{14} \leq |(B_{2\pi/3} \cup B_0) \cap X_1| + 4 \left( |B_0| + |B_{\pi/3}| + |B_{2\pi/3}| \right) \]
\[ \Rightarrow 6 + \frac{1}{14} \leq |(B_{2\pi/3} \cup B_0) \cap X_1| + 4 \cdot \left( \frac{3}{2} + \delta \right) \]
\[ \Rightarrow \frac{1}{14} - 4\delta \leq |(B_{2\pi/3} \cup B_0) \cap X_1|. \]

This implies that $|B_{2\pi/3} \cap X_1|$ or $|B_0 \cap X_1|$ is at least $1/28 - 2\delta$. The lemma is proved. \[ \square \]

By symmetry, we assume $1/28 - 2.5\delta \leq |B_{2\pi/3} \cap X_1|$ in the following argument.
Lemma 4 Let \( q_3 = (4/7, 0), q_4 = (1/14, \sqrt{3}/7), T = \triangle p_1 q_3 p_4, \) and \( Y_2 \) be the zone whose projection for angle \( \pi/3 \) is contained in \( T(\pi/3) \). Furthermore, \( P_1 \) denotes the left half-plane for the line with angle \( \pi/2 \) passing on \( p_1 \), and \( P_3 \) denotes the lower half-plane for the line with angle 0 passing on \( p_2 \). Finally, let \( X_2 = Y_2 \setminus (P_3 \cup P_1) \) (see Figure 3). Then we have \( 1/28 - 2.5\delta \leq |(B_{\pi/3} \cup B_{2\pi/3}) \cap X_2| \).

Proof Consider the projection-cover condition of \( T \) for angle \( \pi/3 \). Then we have

\[
\frac{2}{7} \leq |B_{\pi/3} \cap Y_2| + \frac{1}{2}|B_0 \cap Y_2| + \frac{1}{2}|B_{2\pi/3} \cap Y_2|
\]

\[
\Rightarrow \frac{2}{7} \leq |B_{\pi/3} \cap Y_2| + |B_{2\pi/3} \cap Y_2| + \frac{1}{2}|B_0 \cap Y_2|
\]

\[
\Rightarrow \frac{2}{7} \leq |(B_{\pi/3} \cup B_{2\pi/3}) \cap X_2| + |(B_{\pi/3} \cup B_{2\pi/3}) \cap (P_3 \cup P_4)| + \frac{1}{2}|B_0 \cap Y_1|
\]

\[
\Rightarrow \frac{2}{7} = |B_{\pi/3} \cap P_3| - |B_{\pi/3} \cap P_4| - |B_{2\pi/3} \cap P_3| - |B_{2\pi/3} \cap P_4| \leq |(B_{\pi/3} \cup B_{2\pi/3}) \cap X_2| + \frac{1}{2}|B_0|.
\] (8)

Let us consider the projection-cover condition of \( U \) for angles 0 and \( \pi/2 \). Since \( B \cap P_3 \) and \( B \cap P_4 \) do not contribute to cover \( U(0) \) and \( U(\pi/2) \) respectively, we have

\[
1 \leq \frac{1}{2} (|B_{\pi/3} \setminus P_3| + |B_{2\pi/3} \setminus P_3|) + |B_0 \setminus P_3|
\]

\[
\Rightarrow 1 + \frac{1}{2} (|B_{\pi/3} \cap P_3| + |B_{2\pi/3} \cap P_3|) + |B_0 \cap P_3| \leq \frac{1}{2} (|B_{\pi/3}| + |B_{2\pi/3}|) + |B_0|
\]

\[
\Rightarrow 2 + |B_{\pi/3} \cap P_3| + |B_{2\pi/3} \cap P_3| \leq |B_{\pi/3}| + |B_{2\pi/3}| + 2|B_0|,
\] (9)

\[
\frac{\sqrt{3}}{2} \leq \frac{\sqrt{3}}{2} (|B_{\pi/3} \setminus P_4| + |B_{2\pi/3} \setminus P_4|)
\]

\[
\Rightarrow 1 + |B_{\pi/3} \cap P_4| + |B_{2\pi/3} \cap P_4| \leq |B_{\pi/3}| + |B_{2\pi/3}|
\] (10)

Summing up inequalities (8), (9), and (10), we obtain

\[
\frac{2}{7} + 2 + 1 \leq |(B_{\pi/3} \cup B_{2\pi/3}) \cap X_2| + 2(|B_{\pi/3}| + |B_{2\pi/3}| + |B_0|) + \frac{1}{2}|B_0|
\]

\[
\Rightarrow 3 + \frac{2}{7} \leq |(B_{\pi/3} \cup B_{2\pi/3}) \cap X_2| + 3 + 2\delta + \frac{1}{2}|B_0|
\]

\[
\Rightarrow \frac{2}{7} - 2\delta \leq |(B_{\pi/3} \cup B_{2\pi/3}) \cap X_2| + \frac{1}{2}|B_0|
\]

Considering the projection-cover condition of \( U \) for angle \( \pi/2 \), it is easy to show \(|B_{\pi/3}| + |B_{2\pi/3}| \geq 1\) and thus \(|B_0| \leq 1/2 + \delta\) holds. Thus,

\[
\Rightarrow \frac{2}{7} - 2\delta \leq |(B_{\pi/3} \cup B_{2\pi/3}) \cap X_2| + \frac{1}{2} (\frac{1}{2} + \delta)
\]

\[
\Rightarrow \frac{1}{28} - \frac{5\delta}{2} \leq |(B_{\pi/3} \cup B_{2\pi/3}) \cap X_2|
\]

The lemma is proved. \( \square \)

The above two lemmas allow us to apply Lemma 2.

Lemma 5 \(|B| \geq 3/2 + 0.0001\).

Proof Taking \( B^- = |(B_{\pi/3} \cup B_{2\pi/3}) \cap X_2|, B^+ = |B_{2\pi/3} \cap X_1|, \kappa = \pi/6, \lambda = \pi/3, \) and \( D = 2\sqrt{(9/14)^2 + (4/7\sqrt{3})^2} = \sqrt{307/7\sqrt{3}} \), we apply Lemma 2 with an appropriate shifting of the coordinate system (the origin \( O \) is placed at \((9/14, 0)) \). It is easy to verify that we can draw two bands of angle
\( \kappa = \pi/6 \) and width at least \( 1/28 \) (Figure 4). Then

\[
\int_0^\pi |(B^- \cup B^+)(\alpha)| \, d\alpha \leq 2|B^- \cup B^+| - \frac{(1/28 - 2.5\delta) \sin(\pi/6))^2}{\sqrt{307/7\sqrt{3}}}
\]

Installing this inequality into the proof of Jones’ bound, we have

\[
3 \leq \int_0^\pi |B(\alpha)| \, d\alpha \leq \int_0^\pi |(B \setminus (B^- \cup B^+))(\alpha)| \, d\alpha + \int_0^\pi |(B^- \cup B^+)(\alpha)| \, d\alpha \leq 2|B \setminus (B^- \cup B^+)| + 2|B^- \cup B^+| - \frac{(1/56 - 5\delta/4)^2}{\sqrt{307/7\sqrt{3}}}.
\]

Then we have \( |B| \geq 3/2 + \frac{(1/56 - 5\delta/4)^2}{2\sqrt{307/7\sqrt{3}}} \) and thus \( (1/56 - 5\delta/4)^2 \leq \delta \) holds. Solving this inequality, we obtain \( \delta \geq 0.00010865 \ldots \)

\section{5 Bound for General Barriers}

Putting all together, we show a general lower bound larger than \( 3/2 \) in this section. The key idea for the general-barrier case is to utilize a reduction technique.

\textbf{Lemma 6} Let \( s = s_1s_2 \) be a segment with angle \( \gamma \), and \( \phi = \min_{i \in 0,1,2} |\gamma - i\pi/3| \). Then, we have a \( \{0, \pi/3, 2\pi/3\} \)-restricted (sub)barrier \( C \) such that any line intersecting \( s \) also intersects \( C \), and its total length is

\[
|C| = \left( \cos \phi + \frac{\sin \phi}{\sqrt{3}} \right) |s|.
\]

\textbf{Proof} First, we consider the case for \( \gamma \in [0, \pi/6] \). In this case \( \gamma = \phi \) holds. The construction of \( C \) is as follows: Let \( l_1 \) be the horizontal line passing on \( s_2 \), and \( l_2 \) be the line with angle \( \pi/3 \) passing on \( s_1 \), and \( q_1 \) be the intersection point of \( l_1 \) and \( l_2 \). Then we take the set of two segments \( s_1q_1 \) and \( q_1s_2 \) as \( C \).
It is not difficult to verify that any line intersecting $s$ also intersects $C$. Now we calculate the length of $C$. Let $l_1$ be the line orthogonal to $l_1$ and passing on $s_1$, and $q_2$ be the intersection of $l_1$ and $l_2$. Then $|s_1q_2| = |s|\sin\phi$ holds. Since the angle formed by $q_2s_1$ and $q_1s_1$ is $\pi/6$, we obtain $|q_1q_2| = |s|\sin\phi/\sqrt{3}$ and $|q_2s_1| = 2|s|\sin\phi/\sqrt{3}$. We also obtain $|q_2s_2| = |s|\cos\phi$. Consequently,

$$|C| = |s_1q_1| + |q_1s_2| = \frac{2|s|\sin\phi}{\sqrt{3}} + \left(|s|\cos\phi - \frac{|s|\sin\phi}{\sqrt{3}}\right) = \left|\cos\phi + \frac{\sin\phi}{\sqrt{3}}\right| |s|.$$ 

The right side of the above inequality is mirror symmetric in the period of $[0, \pi/3]$, that is, $(\cos\phi - \frac{\sin\phi}{\sqrt{3}}) = (\cos(\pi/3 - \phi) - \frac{\sin((\pi/3 - \phi))}{\sqrt{3}})$. It follows that the inequality also holds for the case of $\gamma \in [\pi/6, \pi/3]$. For the case of $\gamma \geq \pi/3$, we can show the lemma similarly by rotating the coordinate system by $\pi/3$ (or $2\pi/3$). \hfill \Box

**Lemma 7** Let $s = s_1s_2$ be a segment with angle $\gamma$, and $\phi = \min_{i \in 0,1,2} |(\gamma - (i\pi/3 + \pi/6))|$. Then, we have a $\{\pi/6, \pi/3, 5\pi/6\}$-restricted (sub)barrier $C$ such that any line intersecting $s$ also intersects $C$, and its total length is

$$|C| = \left|\cos\phi + \frac{\sin\phi}{\sqrt{3}}\right| |s|.$$ 

**Proof** By rotating the coordinate system by $\pi/6$, the proof of this lemma can be reduced to that of Lemma 6. \hfill \Box

We define $w(\phi) = (\cos\phi + \frac{\sin\phi}{\sqrt{3}})$, and prove the main theorem.

**Theorem 1** Let $B$ be the optimal (unrestricted) barrier for the equilateral triangle $U$. Then $|B| \geq 3/2 + 5 \times 10^{-13}$ holds.

**Proof** For any segment $s$, we define $\gamma(s)$ as its angle and $\phi(s) = \min_{i \in 0,1,2} |\gamma(s) - i\pi/3|$. For any small angle $\beta \ll \pi/6$, let $C = \{s \in B \mid \phi(s) \leq \beta\}$ and $D = B \setminus C$. Letting $\epsilon$ be a small constant, we consider the two cases of $|C| \geq (1 - \epsilon)|D|$ and $|C| < (1 - \epsilon)|D|$. In the first case, we can construct a $\{0, \pi/3, 2\pi/3\}$-restricted barrier $B^0$ by Lemma 6, whose length is bounded as follows:

$$|B^0| \leq (1 - \epsilon)|B| \cdot w(\beta) + \epsilon|B| \cdot w(\pi/6).$$

In the second case, we can construct a $\{\pi/6, \pi/2, 5\pi/6\}$-restricted barrier $B^1$ using Lemma 7. Its length is bounded by

$$|B^1| \leq \epsilon|B| \cdot w(\pi/6 - \beta) + (1 - \epsilon)|B| \cdot w(\pi/6).$$
By Lemmas 1 and 5, $|B^0| \geq 1.50002$ and $|B^1| \geq \sqrt{3}$ holds. Thus we obtain the following bound for any constants $\beta$ and $\epsilon$.

$$
|B| \geq \min \left\{ \frac{1.5002}{(1 - \epsilon)w(\beta) + \epsilon w(\pi/6)}, \frac{\sqrt{3}}{\epsilon w(\pi/6 - \beta) + (1 - \epsilon)w(\pi/6)} \right\}.
$$

Taking $\beta = 10^{-4.1}$ and $\epsilon = 10^{-3.9}$, we obtain $|B| \geq 1.5 + 5 \cdot 10^{-13}$.

\section{Concluding Remarks}

In this paper, we have shown that any barrier for the unit-size equilateral triangle is at least $3/2 + 5 \cdot 10^{-13}$, which is the first improvement of Jones’ bound for equilateral triangles. To obtain it, we newly introduced several techniques inspired by mathematical programming. Understanding the known bound in the context of linear programming is the core idea of our proof, which might open up the new direction of utilizing much more sophisticated tools in mathematical programming to tackle the shortest barrier problem. It is also an interesting direction to apply our technique to lead better bounds for squires or general convex polygons.

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Appendix

Proof of Lemma 1 by Mathematica

In[1]:= \text{zb}[d_, e_] := E^{((\pi/6 - d)*e)}

In[2]:= \text{z}[a_, c_] := \text{Which}[\text{Mod}[a, \pi/3] \leq \pi/6, 
\text{zb}[\text{Mod}[a, \pi/3], c], \text{Mod}[a, \pi/3] > \pi/6, 
\text{zb}[\pi/3 - \text{Mod}[a, \pi/3], c]]

In[3]:= \text{U}[a_] := \text{Which}[\text{Mod}[a, \pi/3] \leq \pi/6, 
\text{Cos}[\text{Mod}[a, \pi/3]], \text{Mod}[a, \pi/3] > \pi/6, 
\text{Cos}[\pi/3 - \text{Mod}[a, \pi/3]]]

In[4]:= \text{R}[a_] := \text{Abs}[\text{Cos}[\pi/6 - a]]

In[5]:= \text{zU}[c_] := \text{Integrate}[\text{U}[a]*\text{z}[a, c], \{a, 0, \pi\}]

In[6]:= \text{zR}[c_] := \text{Integrate}[\text{R}[a]*\text{z}[a, c], \{a, 0, \pi\}]

In[7]:= \text{Limit}[\text{zU}[c]/\text{zR}[c], c \rightarrow \text{Infinity}]

Out[7]= \sqrt{3}