I-FAVORABLE SPACES: REVISITED

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Abstract. The aim of this paper is to extend the external characterization of I-favorable spaces obtained in [13]. This allows us to obtain a characterization of compact I-favorable spaces in terms of quasi $\kappa$-metrics. We also provide proofs of some author’s results announced in [14].

1. Introduction

The aim of this paper is to extend the external characterization of I-favorable spaces obtained in [13]. We also provide proofs of some author’s results announced in [14]. All topological spaces are Tychonoff and the single-valued maps are continuous.

P. Daniels, K. Kunen and H. Zhou [2] introduced the so called open-open game: Two players take countably many turns, a round consists of player I choosing a non-empty open set $U \subset X$ and II choosing a non-empty open set $V \subset U$. Player I wins if the union of II’s open sets is dense in $X$, otherwise II wins. A space $X$ is called I-favorable if player I has a winning strategy. This means, see [6], there exists a function $\sigma : \bigcup_{n \geq 0} T^n_X \to T_X$ such that the union $\bigcup_{n \geq 0} U_n$ is dense in $X$ for each game

$$(\sigma(\emptyset), U_0, \sigma(U_0), U_1, \sigma(U_0, U_1), U_2, ..., U_n, \sigma(U_0, U_1, ..., U_n), U_{n+1}, ...),$$

where all $U_k$ and $\sigma(\emptyset)$ are non-empty open sets in $X$, $U_0 \subset \sigma(\emptyset)$ and $U_{k+1} \subset \sigma(U_0, U_1, ..., U_k)$ for every $k \geq 0$ (here $T_X$ is the topology of $X$).

Recently A. Kucharski and S. Plewik (see [6], [7]) investigated the connection of I-favorable spaces and skeletal maps. In particular, they proved in [7] that the class of compact I-favorable spaces and the skeletal maps are adequate in the sense of E. Shchepin [9]. Recall that a map $f : X \to Y$ is skeletal if $\text{Int} f(U) \neq \emptyset$ for every open $U \subset X$. On the other hand, the author announced [14, Theorem 3.1] a characterization of the spaces $X$ such that there is an inverse system $S = \{X_\alpha, p^\beta_\alpha, A\}$ of

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separable metric spaces $X_\alpha$ and skeletal surjective bounding maps $p_\alpha^{\beta}$ satisfying the following conditions: (1) the index set $A$ is $\sigma$-complete (every countable chain in $A$ has a supremum in $A$); (2) for every countable chain $\{\alpha_n\}_{n \geq 1} \subset A$ with $\beta = \sup \{\alpha_n\}_{n \geq 1}$ the space $X_\beta$ is a (dense) subset of $\lim \leftarrow \{X_{\alpha_n}, p_{\alpha_n}^{\alpha_{n+1}}\}$; (3) $X$ is embedded in $\lim \leftarrow S$ and $p_\alpha(X) = X_\alpha$ for each $\alpha$, where $p_\alpha : \lim \leftarrow S \to X_\alpha$ is the $\alpha$-th limit projection. An inverse system satisfying (1) and (2) is called almost $\sigma$-continuous. If condition (3) is satisfied, we say that $X$ is the almost limit of $S$, notation $X = a \leftarrow \lim S$. Spaces $X$ such that $X = a \leftarrow \lim S$, where $S$ is almost $\sigma$-continuous inverse system with skeletal bounding maps and second countable spaces, are called skeletally generated [13].

The following theorem is our first main result:

**Theorem 1.1.** For a space $X$ the following conditions are equivalent:

1. $X$ is $I$-favorable;
2. Every embedding of $X$ in another space $Y$ is $\pi$-regular;
3. $X$ is skeletally generated.

Here, we say that a subspace $X \subset Y$ is $\pi$-regularly embedded in $Y$ [14] if there exists a function $e : \mathcal{T}_X \to \mathcal{T}_Y$ such that for every $U, V \in \mathcal{T}_X$ we have: (i) $e(U) \cap e(V) = \emptyset$ provided $U \cap V = \emptyset$; (ii) $e(U) \cap X$ is a dense subset of $U$. If, $e(U) \cap X = U$, we say that $X$ is regularly embedded in $Y$. An external characterization of $\kappa$-metrizable compacta, similar to condition (2), was established in [11].

**Corollary 1.2.** Every $I$-favorable subset of an extremally disconnected space is also extremally disconnected.

**Corollary 1.3.** Every open subset of an $I$-favorable space is $I$-favorable.

A version of Theorem 1.1 was established in [13], but we used a little bit different notions. First, we considered $I$-favorable spaces with respect to the family of co-zero sets. Also, in the definition of skeletally generated spaces we required the system $S$ to be factorizable (i.e. for each continuous function $f$ on $X$ there exists $\alpha \in A$ and a continuous function $h$ on $X_\alpha$ with $f = h \circ p_\alpha$). Moreover, in item (2) $X$ was supposed to be $C^*$-embedded in $Y$. Corollary 1.2 was also established in [13] under the assumption of $C^*$-embedability.

Recall that a $\kappa$-metric [9] on a space $X$ is a non-negative function $\rho(x, C)$ of two variables, a point $x \in X$ and a canonically closed set $C \subset X$, satisfying the following axioms:

K1) $\rho(x, C) = 0$ iff $x \in C$;
K2) If $C \subset C'$, then $\rho(x, C') \leq \rho(x, C)$ for every $x \in X$;
K3) \( \rho(x, C) \) is a continuous function of \( x \) for every \( C \);
K4) \( \rho(x, \bigcup C_\alpha) = \inf_\alpha \rho(x, C_\alpha) \) for every increasing transfinite family \( \{C_\alpha\} \) of canonically closed sets in \( X \).

We say that a function \( \rho(x, C) \) is a quasi \( \kappa \)-metric on \( X \) if it satisfies the axioms K2) – K4) and the following one:

K1*) For any \( C \) there is a dense open subset \( V \) of \( X \setminus C \) such that \( \rho(x, C) = 0 \) iff \( x \in X \setminus V \).

Our second result provides a characterization of compact I-favorable spaces, which is similar to Shchepin’s characterization (\cite{9}, \cite{10}) of openly generated compacta as compact spaces admitting a \( \kappa \)-metric.

**Theorem 1.4.** A compact space \( X \) is I-favorable iff \( X \) is quasi \( \kappa \)-metrizable.

**Corollary 1.5.** Every I-favorable space is quasi \( \kappa \)-metrizable.

The paper is organized as follows: Section 2 contains the proof of Theorem 1.1 and Corollaries 1.2-1.3. The proofs of Theorem 1.4 and Corollary 1.5 are contained in section 3. In section 4 we provide the proof of some results concerning almost continuous inverse systems with nearly open bounding maps, which were announced in \cite{14}.

2. PROOF OF THEOREM 1.1

If follows from the definition of I-favorability that a given space is I-favorable if and only if there are a \( \pi \)-base \( \mathcal{B} \) and a function \( \sigma : \bigcup_{n \geq 0} \mathcal{B}^n \to \mathcal{B} \) such that the union \( \bigcup_{n \geq 0} U_n \) is dense in \( X \) for any sequence

\[ (\sigma(\emptyset), U_0, \sigma(U_0), U_1, \sigma(U_0, U_1), U_2, \ldots, U_n, \sigma(U_0, U_1, \ldots, U_n), U_{n+1}, \ldots), \]

where \( U_k \) and \( \sigma(\emptyset) \) belong to \( \mathcal{B} \), \( U_0 \subset \sigma(\emptyset) \) and \( U_{k+1} \subset \sigma(U_0, U_1, \ldots, U_k) \) for every \( k \geq 0 \). Such a function will be also called a winning strategy. Recall that \( \mathcal{B} \) is a \( \pi \)-base for \( X \) if every open set in \( X \) contains an element from \( \mathcal{B} \).

**Proposition 2.1.** \cite{3} Let \( \mathcal{B} \) and \( \mathcal{P} \) be two \( \pi \)-bases for \( X \). Then there is a winning strategy \( \sigma : \bigcup_{n \geq 0} \mathcal{B}^n \to \mathcal{B} \) if and only if there is a winning strategy \( \mu : \bigcup_{n \geq 0} \mathcal{P}^n \to \mathcal{P} \).

**Proof.** Suppose \( \sigma : \bigcup_{n \geq 0} \mathcal{B}^n \to \mathcal{B} \) is a winning strategy. We define a winning strategy \( \mu : \bigcup_{n \geq 0} \mathcal{P}^n \to \mathcal{P} \) by induction. We choose any open non-empty set \( \mu(\emptyset) \in \mathcal{P} \) such that \( \mu(\emptyset) \subset \sigma(\emptyset) \). If \( V_0 \in \mathcal{P} \) is the answer of player II in the game played on \( \mathcal{P} \) (i.e., \( V_0 \subset \mu(\emptyset) \)), then we choose \( U_0 \in \mathcal{B} \) with \( U_0 \subset V_0 \) (\( U_0 \) can be considered as the answer...
Observe that for every \( U \in B \) that \( \alpha \) consists of open sets such that for each \( k \leq n - 1 \). Then, we choose \( \mu(V_0, \ldots, V_n) \in \mathcal{P} \) such that \( \mu(V_0, \ldots, V_n) \subset \sigma(U_0, \ldots, U_k) \). If \( V_{n+1} \in \mathcal{P} \) is the choice of player II in the game played on \( \mathcal{P} \) such that \( V_{n+1} \subset \mu(V_0, \ldots, V_n) \), we choose \( U_{n+1} \in B \) with \( U_{n+1} \subset V_{n+1} \). This complete the induction. Since \( \sigma \) is a winning strategy and \( U_k \subset V_k \) for each \( k \), the union \( \bigcup_{n \geq 0} V_n \) is dense in \( X \). So, \( \mu \) is also a winning strategy. \( \square \)

In [13] we considered I-favorable spaces \( X \) with respect to the co-zero sets meaning that there is a winning strategy \( \sigma : \bigcup_{n \geq 0} \Sigma^n \rightarrow \Sigma \), where \( \Sigma \) is the family of all co-zero subsets of \( X \). Proposition 2.1 shows that this is equivalent to \( X \) being I-favorable. So, all results from [13] are valid for I-favorable spaces.

According to [2] Corollary 1.4, if \( Y \) is a dense subset of \( X \), then \( X \) is I-favorable if and only \( Y \) is I-favorable. So, every compactification of a space \( X \) is I-favorable provided \( X \) is I-favorable. And conversely, if a compactification of \( X \) is I-favorable, then so is \( X \). Because of that, very often when dealing with I-favorable spaces, we can suppose that they are compact.

Let us introduced few more notations. Suppose \( X \subset \mathbb{I}^A \) is a compact space and \( B \subset A \), where \( \mathbb{I} = [0, 1] \). Let \( \pi_B : \mathbb{I}^A \rightarrow \mathbb{I}^B \) be the natural projection and \( p_B \) be restriction map \( \pi_B|X \). Let also \( X_B = p_B(X) \). If \( U \subset X \) we write \( B \in k(U) \) to denote that \( p_B^{-1}(p_B(U)) = U \). A base \( \mathcal{A} \) for the topology of \( X \subset \mathbb{I}^A \) consisting of open sets is called \textit{special} if for every finite \( B \subset A \) the family \( \{p_B(U) : U \in \mathcal{A}, B \in k(U)\} \) is a base for \( p_B(X) \) and for each \( U \in \mathcal{A} \) there is a finite set \( B \subset A \) with \( B \in k(U) \).

**Proposition 2.2.** Let \( X \) be a compact I-favorable space and \( w(X) = \tau \) is uncountable. Then there exists a continuous inverse system \( S = \{X_\delta, p_\delta^\gamma, \gamma < \delta < \lambda\} \), where \( \lambda = \text{cf}(\tau) \), of compact I-favorable spaces \( X_\delta \) and skeletal bonding maps \( p_\delta^\gamma \) such that \( w(X_\delta) < \tau \) for each \( \delta < \lambda \) and \( X = \lim S \).

**Proof.** We embed \( X \) in a Tychonoff cube \( \mathbb{I}^A \) with \( |A| = \tau \) and fix a special open base \( \mathcal{A} = \{U_\alpha : \alpha \in A\} \) for \( X \) of cardinality \( \tau \) which consists of open sets such that for each \( \alpha \) there exists a finite set \( H_\alpha \subset A \) with \( H_\alpha \in k(U_\alpha) \). Let \( \sigma : \bigcup_{n \geq 0} \mathcal{A}^n \rightarrow \mathcal{A} \) be a winning strategy. We represent \( A \) as the union of an increasing transfinite family \( \{A_\delta : \delta < \lambda\} \) with \( |A_\delta| < \tau \), and let \( \mathcal{A}_\delta = \{U_\alpha : \alpha \in A_\delta\} \) for each \( \delta < \lambda \).

For any finite set \( C \subset A \) let \( \gamma_C \) be a fixed countable base for \( X_C \). Observe that for every \( U \in \mathcal{A} \) there exists a finite set \( B(U) \subset A \) such that \( B(U) \in k(U) \) and \( p_B(U)(U) \) is open in \( X_{B(U)} \). We are going to
Observe also that each cardinal, is said to be stable with respect to $\sigma$, see (4). Hence, by Lemma 9, for every open set $V \subseteq X$ there exists $W \in B_\delta$ such that whenever $U \subset W$ and $U \in B_\delta$ we have $V \cap U \neq \emptyset$. The last statement yields that $p_\delta$ is skeletal. Indeed, let $V \subseteq X$ be open, and $W \in B_\delta$ be as above. Then $p_\delta(W)$ is open in $X_\delta$ because of condition (2). We claim that $p_\delta(W) \subset \overline{p_\delta(V)}$. Indeed, otherwise $p_\delta(W) \setminus \overline{p_\delta(V)}$ would be a non-empty open subset of $X_\delta$. So, $p_\delta(U) \subset p_\delta(W) \setminus \overline{p_\delta(V)}$ for some $U \in B_\delta$ (recall that $p_\delta(B_\delta)$ is a base for $X_\delta$). Since, by (2), $p_\delta^{-1}(p_\delta(U)) = U$ and $p_\delta^{-1}(\overline{p_\delta(W)}) = W$, we obtain $U \subset W$ and $U \cap V = \emptyset$ which is a contradiction.

Finally, since the class of I-favorable spaces is closed with respect to skeletal images, all $X_\delta$ are I-favorable.  

An inverse system $S = \{X_\alpha, p_\alpha^\delta, \alpha < \beta < \tau\}$, where $\tau$ is a given cardinal, is said to be almost continuous provided for every limit cardinal $\gamma$ the space $X_\gamma$ is the almost limit of the inverse system $S_\gamma = \{X_\alpha, p_\alpha, \alpha < \gamma < \tau\}$. If $\gamma$ is limit and $\gamma < \lambda$, where $\lambda$ is a regular cardinal, then we construct by transfinite induction increasing families $\{B_\delta : \delta < \lambda\}$ and $\{B_\delta : \delta < \lambda\} \subset \mathcal{A}$ satisfying the following conditions for every $\delta < \lambda$:

1. $A_\delta \subset B_\delta \subset A$, $A_\delta \in B_\delta$, $|B_\delta| < |\mathcal{B}_\delta| < \tau$;
2. $B_\delta \in k(U)$ for all $U \in B_\delta$;
3. $p_\delta^{-1}(C) \subset B_\delta$ for each finite $C \subset B_\delta$;
4. $\sigma(U_1, \ldots, U_n) \in B_\delta$ for every finite family $\{U_1, \ldots, U_n\} \subset B_\delta$;
5. $B_\delta = \bigcup_{\gamma < \delta} B_\gamma$ and $B_\delta = \bigcup_{\gamma < \delta} B_\gamma$ for all limit cardinals $\delta$.

Suppose all $B_\gamma$ and $B_\gamma$, $\gamma < \delta$, have already been constructed for some $\delta < \lambda$. If $\delta$ is a limit cardinal, we put $B_\delta = \bigcup_{\gamma < \delta} B_\gamma$ and $B_\delta = \bigcup_{\gamma < \delta} B_\gamma$. If $\delta = \gamma + 1$, we construct by induction a sequence $\{C(m)\}_{m \geq 0}$ of subsets of $A$, and a sequence $\{\mathcal{V}_m\}_{m \geq 0}$ of subfamilies of $\mathcal{A}$ such that:

- $C_0 = B_\gamma$ and $\mathcal{V}_0 = B_\gamma$;
- $C(m + 1) = C(m) \bigcup \{B(U) : U \in \mathcal{V}_m\}$;
- $\mathcal{V}_{2m+1} = \mathcal{V}_{2m} \bigcup \{\sigma(U_1, \ldots, U_n) : U_1, \ldots, U_n \in \mathcal{V}_{2m}, s \geq 1\}$;
- $\mathcal{V}_{2m+2} = \mathcal{V}_{2m+1} \bigcup \{p_\delta^{-1}(C) : C \subset C(2m + 1) \text{ is finite}\}$.

Now, we define $B_\delta = \bigcup_{m \geq 0} C(m)$ and $B_\delta = \bigcup_{m \geq 0} \mathcal{V}_m$. It is easily seen that $B_\delta$ and $B_\delta$ satisfy conditions (1)-(5).

For every $\delta < \lambda$ let $X_\delta = X_{B_\delta}$ and $p_\delta = p_{B_\delta}$. Moreover, if $\gamma < \delta$, we have $B_\gamma \subset B_\delta$, and let $p_\delta^\gamma = p_{B_\delta}^B$. Since $A = \bigcup_{\delta < \lambda} B_\delta$, we obtain a continuous inverse system $S = \{X_\delta, p_\delta^\gamma, \gamma < \delta < \lambda\}$ whose limit is $X$. Observe also that each $X_\delta$ is of weight $< \tau$ because $p_\delta(B_\delta)$ is a base for $X_\delta$ (see condition (3)).

Claim 1. All bonding maps $p_\delta^\gamma$ are skeletal.

It suffices to show that all $p_\delta$ are skeletal. And this is really true because each family $B_\delta$ is stable with respect to $\sigma$, see (4). Hence, by Lemma 9, for every open set $V \subset X$ there exists $W \in B_\delta$ such that whenever $U \subset W$ and $U \in B_\delta$ we have $V \cap U \neq \emptyset$. The last statement yields that $p_\delta$ is skeletal. Indeed, let $V \subset X$ be open, and $W \in B_\delta$ be as above. Then $p_\delta(W)$ is open in $X_\delta$ because of condition (2). We claim that $p_\delta(W) \subset \overline{p_\delta(V)}$. Indeed, otherwise $p_\delta(W) \setminus \overline{p_\delta(V)}$ would be a non-empty open subset of $X_\delta$. So, $p_\delta(U) \subset p_\delta(W) \setminus \overline{p_\delta(V)}$ for some $U \in B_\delta$ (recall that $p_\delta(B_\delta)$ is a base for $X_\delta$). Since, by (2), $p_\delta^{-1}(p_\delta(U)) = U$ and $p_\delta^{-1}(p_\delta(W)) = W$, we obtain $U \subset W$ and $U \cap V = \emptyset$ which is a contradiction.

Finally, since the class of I-favorable spaces is closed with respect to skeletal images, all $X_\delta$ are I-favorable. \qed
We present here a simplified proof concerning almost continuous systems.

Let \( X = a - \lim S \) of an almost continuous inverse system \( S \) and \( H \subset X \), the set
\[
q(H) = \{ \alpha : \text{Int}((p_\alpha^\alpha)^{-1}(p_\alpha(H))) \cap p_{\alpha+1}(H)) \neq \emptyset \}
\]
is called a rank of \( H \).

**Lemma 2.3.** [13] Lemma 3.1] Let \( X = a - \lim S \) and \( U \subset X \) be open, where \( S = \{ X_\alpha, p_\alpha^\beta, \alpha < \beta < \tau \} \) is almost continuous inverse system with skeletal bonding maps. Then we have:

1. \( \alpha \notin q(U) \) if and only if \((p_\alpha^\alpha)^{-1}(\text{Int}p_\alpha(U)) \subset p_{\alpha+1}(U)\);
2. \( q(U) \cap [\alpha, \tau) = \emptyset \) provided \( U = p_\alpha^{-1}(V) \) for some open \( V \subset X_\alpha \).

**Lemma 2.4.** Let \( S = \{ X_\alpha, p_\alpha^\beta, 1 \leq \alpha < \beta < \tau \} \) be an almost continuous inverse system with skeletal bonding maps and \( X = a - \lim S \). The following hold for any open \( U \subset X \):

1. If \((p_\alpha^\alpha)^{-1}(\text{Int}p_1(U)) \subset \text{Int}p_\alpha(U)\) for all \( \alpha < \tau \), then \( p_1^{-1}(\text{Int}p_1(U)) \subset U \);
2. If \( \lambda < \tau \) and \( q(U) \cap [\lambda, \tau) = \emptyset \), then \( p_\lambda^{-1}(\text{Int}p_\lambda(U)) \subset \text{Int}U \).

**Proof.** The first item was proved in [13] Lemma 3.2] under the assumption that \( X = \lim S \), but the same arguments work in our situation. Item (2) is equivalent to the inclusion \((p_\lambda)^{-1}(\text{Int}p_\lambda(U)) \subset U\). Let \( A \) be the set of all \( \alpha \in (\lambda, \tau) \) with \((p_\alpha^\alpha)^{-1}(\text{Int}p_\alpha(U)) \setminus p_\alpha(U) \neq \emptyset \). Suppose \( A \) is non-empty and let \( \gamma = \min A \). Observe that \( \gamma \) is a limit cardinal. Indeed, otherwise \( \gamma = \beta + 1 \) with \( \beta \geq \lambda \), so \((p_\beta^\beta)^{-1}(\text{Int}p_\beta(U)) \subset \text{Int}p_\beta(U)\). Since \( \beta \notin q(U) \), according to Lemma 2.3(1), we have \((p_\beta^\beta)^{-1}(\text{Int}p_\beta(U)) \subset p_\beta(U)\). Hence, \((p_\gamma^\gamma)^{-1}(\text{Int}p_\gamma(U)) \subset p_\gamma(U)\), a contradiction.

Since \( S \) is almost continuous and \( \gamma \) is a limit cardinal, we have \( X_\gamma = a - \lim S_\gamma \), where \( S_\gamma \) is the inverse system \( \{ X_\lambda, p_\lambda^\alpha, \lambda \leq \alpha < \beta < \gamma \} \). Because \( p_\gamma \) is skeletal, \( U_\gamma = \text{Int}p_\gamma(U) \neq \emptyset \). So, we can apply item (1) to \( X_\gamma \), the inverse system \( S_\gamma \) and the open set \( U_\gamma \subset X_\gamma \), to conclude that \((p_\gamma^\gamma)^{-1}(\text{Int}p_\gamma(U)) \subset p_\gamma(U)\). So, we obtain again a contradiction, which shows that \((p_\alpha^\alpha)^{-1}(\text{Int}p_\alpha(U)) \subset p_\alpha(U)\) for all \( \alpha \in [\lambda, \tau) \). Finally, because the system \( S_\lambda = \{ X_\alpha, p_\alpha^\beta, \lambda \leq \alpha < \beta < \tau \} \) is almost continuous and \( X = a - \lim S_\lambda \), by item (1) we have \((p_\lambda^\lambda)^{-1}(\text{Int}p_\lambda(U)) \subset \text{Int}U\). \( \square \)

Next lemma was established in [13] for continuous inverse systems. We present here a simplified proof concerning almost continuous systems.
Lemma 2.5. [13] Lemma 3.3] Let $S = \{X_\alpha, p_\alpha^\beta, \alpha < \beta < \tau\}$ be an almost continuous inverse system with skeletal bonding maps and $X = a - \lim S$. Assume $U, V \subset X$ are open with $q(U)$ and $q(V)$ finite and $\overline{U \cap V} = \emptyset$. If $q(U) \cap q(V) \cap [\gamma, \tau) = \emptyset$ for some $\gamma < \tau$, then $\overline{\text{Int}p_\gamma(U)}$ and $\overline{\text{Int}p_\gamma(V)}$ are disjoint.

Proof. Suppose $\overline{\text{Int}p_\gamma(U)} \cap \overline{\text{Int}p_\gamma(V)} \neq \emptyset$. We are going to show by transfinite induction that $\overline{\text{Int}p_\beta(U)} \cap \overline{\text{Int}p_\beta(V)} \neq \emptyset$ for all $\beta \geq \gamma$. Assume this is done for all $\beta \in (\gamma, \alpha)$ with $\alpha < \tau$. If $\alpha$ is not a limit cardinal, then $\alpha - 1$ belongs to at most one of the sets $q(U)$ and $q(V)$. Suppose $\alpha - 1 \notin q(V)$. Hence, $(p_{\alpha-1}^\alpha)^{-1}(\overline{\text{Int}p_{\alpha-1}(V)}) \subset \overline{\text{Int}p_\alpha(V)}$ (see Lemma 2.3(1)). Due to our assumption, $\overline{\text{Int}p_{\alpha-1}(U)} \cap \overline{\text{Int}p_{\alpha-1}(V)} \neq \emptyset$. Moreover, $p_{\alpha-1}^\alpha(p_\alpha(U))$ is dense in $\overline{\text{Int}p_{\alpha-1}(U)}$. Hence, $\overline{\text{Int}p_{\alpha-1}(V)}$ meets $\overline{p_{\alpha-1}^\alpha(p_\alpha(U))}$. This yields $\overline{\text{Int}p_\alpha(U)} \cap p_\alpha(U) \neq \emptyset$. Finally, since $p_\alpha(U)$ is the closure of its interior, $\overline{\text{Int}p_\alpha(U)} \cap \overline{\text{Int}p_\alpha(U)} \neq \emptyset$.

Suppose $\alpha > \gamma$ is a limit cardinal. Since $q(U) \cup q(V)$ is a finite set, there exists $\lambda \in (\gamma, \alpha)$ such that $\beta \notin q(U) \cup q(V)$ for all $\beta \in [\lambda, \alpha)$. Now, we consider the almost continuous inverse system $S_\alpha = \{X_\delta, p_\delta^\beta, \lambda \leq \delta < \beta < \alpha\}$ with $X_\alpha = a - \lim S_\alpha$. Let $U_\alpha = \overline{\text{Int}p_\alpha(U)}$ and $V_\alpha = \overline{\text{Int}p_\alpha(V)}$ and denote by $q_\alpha(U_\alpha)$ and $q_\alpha(V_\alpha)$ the ranks of $U_\alpha$ and $V_\alpha$ with respect to the system $S_\alpha$. The, according to Lemma 2.3(1), $\beta \in [\lambda, \alpha)$ does not belong to $q_\alpha(U_\alpha)$ if and only if $(p_{\beta+1}^{\beta+1})^{-1}(\overline{\text{Int}p_{\beta+1}^\beta(U_\alpha)}) \subset p_{\beta+1}^\beta(U_\alpha)$. Since $p_{\beta+1}^\beta(U_\alpha) = p_\gamma(U) = p_{\beta-1}^\beta(U_\alpha)$ and $p_{\beta+1}^{\beta+1}(U_\alpha) = p_{\beta+1}(U)$, we obtain that $\beta \notin q_\alpha(U_\alpha)$ is equivalent to $\beta \notin q(U)$. Similarly, $\beta \notin q_\alpha(V_\alpha)$ if and only if $q_\alpha(V_\alpha)$ belongs to $q_\alpha(U_\alpha)$ for all $\beta \in [\lambda, \alpha)$. Then, according to Lemma 2.4(2), $(p_{\gamma}^{\lambda})^{-1}(\overline{\text{Int}p_{\lambda}(U)}) \subset \overline{\text{Int}p_{\lambda}(V)}$ and $(p_{\gamma}^{\lambda})^{-1}(\overline{\text{Int}p_{\lambda}(V)}) \subset \overline{\text{Int}p_{\lambda}(U)}$. Because $\overline{\text{Int}p_{\lambda}(U)} \cap \overline{\text{Int}p_{\lambda}(V)} \neq \emptyset$, we finally have $\overline{\text{Int}p_{\lambda}(U)} \cap \overline{\text{Int}p_{\lambda}(V)} \neq \emptyset$. This completes the transfinite induction.

Therefore, $\overline{\text{Int}p_\beta(U)} \cap \overline{\text{Int}p_\beta(V)} \neq \emptyset$ for all $\beta \in [\gamma, \tau)$. To finish the proof of this lemma, take $\lambda(0) \in (\gamma, \tau)$ such that $(q(U) \cup q(V)) \cap [\lambda(0), \tau) = \emptyset$. Then, according to Lemma 2.4(2) we have the following inclusions:

- $p_{\lambda(0)}^{-1}(\overline{\text{Int}p_{\lambda(0)}(U)}) \subset \overline{\text{Int}U}$;
- $p_{\lambda(0)}^{-1}(\overline{\text{Int}p_{\lambda(0)}(V)}) \subset \overline{\text{Int}V}$.

Since $\overline{\text{Int}p_{\lambda(0)}(U)} \cap \overline{\text{Int}p_{\lambda(0)}(V)} \neq \emptyset$, the above inclusions imply $\overline{U \cap V} \neq \emptyset$, a contradiction. Hence, $\overline{\text{Int}p_\gamma(U)} \cap \overline{\text{Int}p_\gamma(V)} = \emptyset$. \qed
Next proposition was announced in [14] Proposition 3.2 and a proof was presented in [13] Proposition 3.4 (see Proposition 3.2 below for a similar statement concerning inverse systems with nearly open projections).

**Proposition 2.6.** [14] Let $S = \{X_\alpha, p^\beta_\alpha, \alpha < \beta < \tau \}$ be an almost continuous inverse system with skeletal bonding maps such that $X = a - \lim S$. Then the family of all open subsets of $X$ having a finite rank is a $\pi$-base for $X$.

**Proposition 2.7.** Let $X$ be a compact I-favorable space. Then every embedding of $X$ in another space is $\pi$-regular.

**Proof.** We are going to prove this proposition by transfinite induction with respect to the weight $w(X)$. This is true if $X$ is metrizable, see for example [8] §21, XI, Theorem 2]. Assume the proposition is true for any compact I-favorable space $Y$ of weight $< \tau$, where $\tau$ is an uncountable cardinal. Suppose $X$ is compact I-favorable with $w(X) = \tau$. Then, by Proposition 2.2, $X$ is the limit space of a continuous inverse system $S = \{X_\alpha, p^\beta_\alpha, \alpha < \beta < \lambda \}$, where $\lambda = cf(\tau)$, such that all $X_\alpha$ are compact I-favorable spaces of weight $< \tau$ and all bonding maps are surjective and skeletal. If suffices to show that there exists a $\pi$-regular embedding of $X$ in a Tychonoff cube $\prod^A$ for some set $A$.

By Proposition 2.6, $X$ has a $\pi$-base $B$ consisting of open sets $U \subset X$ with finite rank. For every $U \in B$ let $\Omega(U) = \{\alpha_0, \alpha, \alpha + 1 : \alpha \in q(U)\}$, where $\alpha_0 < \lambda$ is fixed. Obviously, $X$ is a subset of $\prod\{X_\alpha : \alpha < \lambda\}$. For every $U \in B$ we consider the open set $\Gamma(U) = \prod\{\text{Int}p^\beta_\alpha(U) : \alpha \in \Omega(U)\} \times \prod\{X_\alpha : \alpha \notin \Omega(U)\}$.

**Claim 2.** $\Gamma(U_1) \cap \Gamma(U_2) = \emptyset$ whenever $U_1 \cap U_2 = \emptyset$. Moreover, there exists $\beta \in \Omega(U_1) \cap \Omega(U_2)$ with $\text{p}_{\beta}(U_1) \cap \text{p}_{\beta}(U_2) = \emptyset$.

Let $\beta = \max\{\Omega(U_1) \cap \Omega(U_2)\}$. Then $\beta$ is either $\alpha_0$ or $\max\{q(U_1) \cap q(U_2)\} + 1$. In both cases $q(U_1) \cap q(U_2) \cap [\beta, \lambda) = \emptyset$. According to Lemma 2.5, $\text{Int}p^\beta_\alpha(U_1) \cap \text{Int}p^\beta_\alpha(U_2) = \emptyset$. Since $\beta \in \Omega(U_1) \cap \Omega(U_2)$, $\Gamma(U_1) \cap \Gamma(U_2) = \emptyset$.

For every $U \in B$ and $\alpha$ let $U_\alpha = \text{Int}p^\alpha_\alpha(U)$.

**Claim 3.** $\bigcap_{\alpha \in \Delta} p^{-1}_\alpha(V) \cap U \neq \emptyset$ for every finite set $\Delta \subset \{\alpha : \alpha < \lambda\}$, where each $V_\alpha$ is an open and dense subset of $U_\alpha$.

Obviously, this is true if $|\Delta| = 1$. Suppose it is true for all $\Delta$ with $|\Delta| \leq n$ for some $n$, and let $\{\alpha_1, \ldots, \alpha_n, \alpha_{n+1}\}$ be a finite set of $n + 1$ cardinals $< \tau$. Then $V = \bigcap_{i \leq n} p^{-1}_\alpha(V_\alpha_i) \cap U \neq \emptyset$. Since $p_{\alpha_{n+1}}$ is a closed and skeletal map, $W = \text{Int}p_{\alpha_{n+1}}^{-1}(V)$ is a non-empty subset of $X_{\alpha_{n+1}}$ and
Let $X$ be a $\pi$-regularly embedded subspace of a product of second countable spaces. Then $X$ is skeletally generated.
Proof of Theorem 1.1. To prove implication (1) ⇒ (2), suppose $X$ is I-favorable subspace of a space $Y$. Then $\widetilde{X} = Y^\ast$ is a compactification of $X$. Since $\widetilde{X}$ is also I-favorable, according to Proposition 2.7, $\widetilde{X}$ is $\pi$-regularly embedded in $\beta Y$. This yields that $X$ is $\pi$-regularly embedded in $Y$.

(2) ⇒ (3) Let $X$ be a subset of a Tychonoff cube $I^A$. Then $X$ is $\pi$-regularly embedded in $I^A$, and by Proposition 2.8, $X$ is skeletally generated.

The implication (3) ⇒ (1) follows as follows. If $X$ is skeletally generated, then $X = \lim_{\rightarrow} S$, where $S$ is an almost $\sigma$-continuous inverse system of second countable spaces $X_\alpha$, $\alpha \in A$, and skeletal bounding maps $p_\alpha^\beta$. Because each $X_\alpha$ is I-favorable, it follows from [4, Theorem 3.3] (see also [6, Theorem 13]) that $X$ is I-favorable too. ✷

Proof of Corollary 1.2. Suppose $X$ is an I-favorable subspace of an extremally disconnected space $Y$. Then there exists a $\pi$-regular operator $e: T_X \to T_Y$. We need to show that the closure (in $X$) of every open subset of $X$ is also open. Since $Y$ is extremally disconnected, $e(U)^Y \cap X = \overline{U^X}$ for all $U \in T_X$. Because $e(U) \cap X$ is a dense subset of $U$, we have $\overline{U^X} \subset e(U)^Y \cap X$. Assume $e(U)^Y \cap X \setminus \overline{U^X} \neq \emptyset$ and choose $V \in T_X$ with $V \subset e(U)^Y \setminus \overline{U^X}$. Then $e(V) \cap e(U)^Y \neq \emptyset$, so $e(V) \cap e(U) \neq \emptyset$. The last one contradicts $U \cap V = \emptyset$. ✷

Proof of Corollary 1.3. Suppose $X$ is I-favorable and $W \subset X$ is open. Then there is a $\pi$-regular embedding of $X$ into a product $\Pi$ of lines. Obviously, $W$ is also $\pi$-regularly embedded in $\Pi$, and by Proposition 2.8, $W$ is I-favorable. ✷

3. QUASI $\kappa$-METRIZABLE SPACES

Proof of Theorem 1.4. Suppose $X$ is a compact I-favorable. We embed $X$ in $\mathbb{R}^\tau$ for some cardinal $\tau$, and let $\rho(z, C)$ be a $\kappa$-metric on $\mathbb{R}^\tau$, see [9]. According to Theorem 1.1, there exists a $\pi$-regular function $e: T_X \to T_{\mathbb{R}^\tau}$. We define a new function $e_1: T_X \to T_{\mathbb{R}^\tau}$,

$$e_1(U) = \bigcup\{e(V) : V \in T_X \text{ and } \overline{V} \subset U\}.$$

Obviously $e_1$ is $\pi$-regular and it is also monotone, i.e. $U \subset V$ implies $e_1(U) \subset e_1(V)$. Moreover, for every increasing transfinite family $\gamma = \{U_\alpha\}$ of open sets in $Y$ we have $e_1(\bigcup_\alpha U_\alpha) = \bigcup_\alpha e_1(U_\alpha)$. Indeed, if $z \in e_1(\bigcup_\alpha U_\alpha)$, then there is an open set $V \in T_X$ with $\overline{V} \subset \bigcup_\alpha U_\alpha$ and $z \in e(V)$. Since $\overline{V}$ is compact and the family is increasing, $\overline{V}$
is contained in some \( U_{a_0} \). Hence, \( z \in e(V) \subset e_1(U_{a_0}) \). Consequently, 
\( e_1(\bigcup_{\alpha} U_{\alpha}) \subset \bigcup_{\alpha} e_1(U_{\alpha}) \). The other inclusion follows from monotonicity of \( e_1 \).

Now, for every open \( U \subset X \) and \( x \in X \) we can define the function 
\[ d(x, U) = \rho(x, e_1(U)) \], where \( e_1(U) \) is the closure of \( e_1(U) \) in \( \mathbb{R}^\tau \). It is easily seen that \( d(x, U) \) satisfies axioms \( K2) - K3 \). Let show that it also satisfies \( K4 \) and \( K1^* \). Indeed, assume \( \{C_\alpha\} \) is an increasing transfinite family of regularly closet sets in \( X \). We put \( U_\alpha = \text{Int}C_\alpha \) for every \( \alpha \) and \( U = \bigcup_{\alpha} U_\alpha \). Thus, \( e_1(U) = \bigcup_{\alpha} e_1(U_\alpha) \). Since \( \{\overline{e_1(U_\alpha)}\} \) is an increasing transfinite family of regularly closed sets in \( \mathbb{R}^\tau \),

\[
 d(x, \bigcup_{\alpha} C_\alpha) = \rho(x, \bigcup_{\alpha} e_1(U_\alpha)) = \inf_{\alpha} \rho(x, e_1(U_\alpha)) = \inf_{\alpha} d(x, C_\alpha).
\]

To show that \( K1^* \) also holds, observe that \( d(x, U) = 0 \) if and only if \( x \in X \cap e_1(U) \). Thus, we need to show that there is an open dense subset \( V \) of \( X \setminus U \) such that \( X \cap e_1(U) = X \setminus V \). Because \( e_1(U) \cap X \) is dense in \( U \), \( \overline{V} \subset e_1(U) \). Hence, \( V = X \setminus \overline{\{e_1(U)\}} \) is contained in \( X \setminus U \). To prove \( V \) is dense in \( X \setminus U \), let \( x \in X \setminus U \) and \( W_x \subset X \setminus U \) be an open neighborhood of \( x \). Then \( W \cap U \) is empty, so \( e_1(W) \cap e_1(U) = \emptyset \). This yields \( e_1(W) \cap X \subset V \). On the other hand, \( e_1(W) \cap X \) is a non-empty subset of \( W \), hence \( W \cap V \neq \emptyset \). Therefore, \( d \) is a quasi \( \kappa \)-metric on \( X \).

Suppose \( X \) is a compact space and let \( d(x, U) \) be a quasi \( \kappa \)-metric on \( X \). We are going to show that \( X \) is skeletally generated. To this end we embed \( X \) in \( I^A \) for some \( A \). Following the notations from the proof of Proposition 2.2, for any countable set \( B \subset A \) let \( \mathcal{A}_B \) be the countable base for \( X_B = p_B(X) \) consisting of all open sets in \( X_B \) of the form \( X_B \cap \prod_{\alpha \in B} V_\alpha \), where each \( V_\alpha \) is an open subinterval of \( \mathbb{I} = [0,1] \) with rational end-points and \( V_\alpha \neq \mathbb{I} \) for finitely many \( \alpha \). For any open \( U \subset X \) denote by \( f_U \) the function \( d(x, U) \). We also write \( p_B \prec g \), where \( g \) is a map defined on \( X \), if there is a map \( h : p_B(X) \to g(X) \) such that \( g = h \circ p_B \). Since \( X \) is compact this is equivalent to the following:

if \( p_B(x_1) = p_B(x_2) \) for some \( x_1, x_2 \in X \), then \( g(x_1) = g(x_2) \). We say that a countable set \( B \subset A \) is \( d \)-admissible if \( p_B \prec f_{p_B^{-1}}(V) \) for every \( V \in \mathcal{A}_B \). Denote by \( \mathcal{D} \) the family of all \( d \)-admissible subsets of \( A \). We are going to show that all maps \( p_B : X \to X_B, B \in \mathcal{D} \), are skeletal and the inverse system \( S = \{X_B : p_B^B : C \subset B, C, B \in \mathcal{D}\} \) is \( \sigma \)-continuous with \( X = \lim \downarrow S \).

**Claim 5.** For every countable set \( C \subset A \) there is \( B \in \mathcal{D} \) with \( C \subset B \).

We are going to construct a sequence of countable sets \( B_n \subset A \) such that for every \( n \geq 1 \) we have:
\begin{itemize}
  \item $C \subset B_n \subset B_{n+1}$;
  \item $p_{B_{n+1}} < f_{p_{B_n}^{-1}}(V)$ for all $V \in A_{B_n}$.
\end{itemize}

We show the construction of $B_1$, the other sets $B_n$ can be obtained in a similar way. Every function $f_{p_C^{-1}}(V), V \in A_C$, has a continuous extension $\tilde{f}_{p_C^{-1}}(V)$ on $\mathbb{I}^4$. Moreover, every continuous function $g$ on $\mathbb{I}^4$ depends on countably many coordinates (i.e., there exists a countable set $B_g \subset A$ with $\pi_{B_g} \prec g$). This fact allows us to find a countable set $B_1 \subset A$ containing $C$ such that $p_{B_1} \prec f_{p_C^{-1}}(V)$ for all $V \in A_C$. Next, let $B = \bigcup_{n=1}^{\infty} B_n$. Since $A_B$ is the union of all families \{(p_{B_n}^{B_1})^{-1}(V) : V \in A_{B_n}\},$ $n \geq 1$, for every $W \in A_B$ there is $m$ and $V \in A_{B_m}$ with $p_B^{-1}(W) = p_{B_m}^{-1}(V)$. Then, according to the construction of the sets $B_n$, we have $p_B \prec f_{p_B^{-1}}(W)$. Hence $p_B \prec f_{p_B^{-1}}(W)$ for all $W \in A_B$, which means that $B$ is $d$-admissible.

\textit{Claim 6}. For every $B \in D$ the map $p_B$ is skeletal.

Suppose there is an open set $U \subset X$ such that the interior in $X_B$ of the closure $p_B(U)$ is empty. Then $W = X_B \setminus \overline{p_B(U)}$ is dense in $X_B$. Let \{W_m\}_{m \geq 1} be a countable cover of $W$ with $W_m \in A_B$ for all $m$. Since $A_B$ is finitely additive, we may assume that $W_m \subset W_{m+1}$, $m \geq 1$. Because $B$ is $d$-admissible, $p_B \prec f_{p_B^{-1}}(W_m)$ for all $m$. Hence, there are continuous functions $h_m : X_B \to \mathbb{R}$ with $f_{p_B^{-1}}(W_m) = h_m \circ p_B$, $m \geq 1$. Recall that $f_{p_B^{-1}}(W_m)(x) = d(x, \overline{p_B^{-1}(W_m)})$ and $\overline{p_B^{-1}}(W) = \bigcup_{m \geq 1} \overline{p_B^{-1}}(W_m)$. Therefore, $f_{p_B^{-1}}(W)(x) = d(x, \overline{p_B^{-1}}(W)) = \inf_{m} f_{p_B^{-1}}(W_m)(x)$ for all $x \in X$. Moreover, $f_{p_B^{-1}}(W_{m+1})(x) \leq f_{p_B^{-1}}(W_m)(x)$ because $W_m \subset W_{m+1}$. The last inequalities together with $p_B \prec f_{p_B^{-1}}(W_m)$ yields that $p_B \prec f_{p_B^{-1}}(W)$.

So, there exists a continuous function $h$ on $X_B$ with $d(x, \overline{p_B^{-1}}(W)) = h(p_B(x))$ for all $x \in X$. Since $p_B(\overline{p_B^{-1}}(W)) = \overline{W} = X_B$, we have that $h$ is the constant function zero. Then $d(x, \overline{p_B^{-1}}(W)) = 0$ for all $x \in X$. But $\overline{p_B^{-1}}(W) \cap U = \emptyset$. So, according to $K1^*$, there is a dense open subset $U'$ of $U$ with $d(x, \overline{p_B^{-1}}(W)) > 0$ for each $x \in U'$, a contradiction.

It is easily seen that the union of any increasing sequence of $d$-admissible sets is also $d$-admissible. This fact and Claims 5 yield that the inverse system $S = \{X_B : p_C^B : C \subset B, C, B \in D\}$ is $\sigma$-continuous and $X = \lim_{\leftarrow} S$. Finally, by Claim 6, all maps $p_B, B \in D$, are skeletal. So are the bounding maps $p_C^B$ in $S$. Therefore, $X$ is skeletally generated, and hence I-favorable by Theorem 1.1.

\textit{Proof of Corollary 1.5}. Since $Y = \beta X$ is I-favorable, by Theorem 1.4 there is a quasi $\kappa$-metric $d$ on $Y$. We are going to show that $d_X(x, \overline{U}_X) = d(x, \overline{U}), U \in T_X$, defines a quasi $\kappa$-metric on $X$, where
\( \overline{U}^X \) and \( \overline{U} \) is the closure of \( U \) in \( X \) and \( Y \) respectively. Since \( \overline{U} \) is regularly closed in \( Y \), this definition is correct. It follows directly from the definition that \( d_X \) satisfies axioms \( K2) \) and \( K3) \). Because for any increasing transfinite family \( \{C_\alpha\} \) of regularly closed sets in \( X \) the family \( \{\overline{C_\alpha}\} \) is also increasing and consists of regularly closed sets in \( Y \),

\[
d_X(x, \bigcup_\alpha C_\alpha^X) = d(x, \bigcup_\alpha C_\alpha) = \inf_\alpha d(x, \overline{C_\alpha}) = \inf_\alpha d_X(x, C_\alpha),
\]

\( d_X \) satisfies \( K4) \). Finally, \( d_X \) satisfies also \( K1^* \)). Indeed, for any \( U \in \mathcal{T}_X \) there exists \( V \in \mathcal{T}_Y \) such that \( V \) is dense in \( Y \setminus U \) and \( d(x, \overline{V}) > 0 \) if and only if \( x \in V \). This implies that the set \( V \cap X \) is dense in \( X \setminus \overline{U}^X \) and \( d_X(x, \overline{U}^X) > 0 \) iff \( x \in V \cap X \). So, \( d_X \) is a quasi \( \kappa \)-metric on \( X \).

4. Inverse systems with nearly open bounding maps

In this section we consider almost continuous inverse systems with nearly open bounding maps. Recall that a map \( f : X \to Y \) is nearly open \([1]\) if \( f(U) \subset \text{Int}(f(U)) \) for every open \( U \subset X \). Nearly open maps were considered by Tkachenko \([12]\) under the name \( d \)-open maps. The following properties of ranks were established in Lemmas 2.3-2.5 when consider almost continuous inverse systems with skeletal bounding maps. The same proofs remain valid and for inverse systems with nearly open bounding maps.

**Lemma 4.1.** Let \( X = a - \lim S \), where \( S = \{X_\alpha, p_\alpha^\beta, \alpha < \beta < \tau\} \) is almost continuous with nearly open bonding maps. Then for every open sets \( U, V \subset X \) we have:

1. \( \alpha \not\in q(U) \) if and only if \((p_\alpha^{\alpha+1})^{-1}(\text{Int}p_\alpha(U)) \subset p_{\alpha+1}(U)\);
2. \( q(U) \cap [\alpha, \tau) = \emptyset \) provided \( U = p_\alpha^{-1}(W) \) for some open \( W \subset X_\alpha \);
3. Suppose \( q(U) \) and \( q(V) \) are finite and \( \overline{U} \cap \overline{V} = \emptyset \). If \( q(U) \cap q(V) \cap [\gamma, \tau) = \emptyset \) for some \( \gamma < \tau \), then \( \text{Int}p_\gamma(U) \) and \( \text{Int}p_\gamma(V) \) are disjoint.

Next proposition was announced in \([14]\) Proposition 2.2] without a proof. Note that a similar statement was established in \([9]\) for inverse systems with open bounding maps.

**Proposition 4.2.** \([14]\) Let \( S = \{X_\alpha, p_\alpha^\beta, \alpha < \beta < \tau\} \) be an almost continuous inverse system with nearly open bonding maps such that \( X = a - \lim S \). Then the family of all open subsets of \( X \) having a finite rank is a base for \( X \).
Proof. We are going to show by transfinite induction that for every
\( \alpha < \tau \) the open subsets \( U \subset X \) with \( q(U) \cap [1, \alpha] \) being finite form a base for \( X \). Obviously, this is true for finite \( \alpha \), and it holds for \( \alpha + 1 \) provided it is true for \( \alpha \). So, it remains to prove this statement for a limit cardinal \( \alpha \) if it is true for any \( \beta < \alpha \). Suppose \( G \subset X \) is open and \( x \in G \). Since \( p_\alpha \) is nearly open, \( G_\alpha = \text{Int}_{p_\alpha}(G) \) contains \( p_\alpha(G) \) (here both interior and closure are taken in \( X_\alpha \)). Let \( S_\alpha = \{ X_\gamma, p_\gamma^\beta, \gamma < \beta < \alpha \}, Y_\alpha = \lim S_\alpha \) and \( \tilde{p}_\alpha^\gamma: Y_\alpha \to X_\gamma \) are the limit projections of \( S_\alpha \). Obviously, \( X_\alpha \) is naturally embedded as a dense subset of \( Y_\alpha \) and each \( \tilde{p}_\alpha^\gamma \) restricted on \( X_\alpha \) is \( p_\gamma^\alpha \). So, there exists \( \gamma < \alpha \) and an open set \( U_\gamma \subset X_\gamma \) containing \( x_\gamma = p_\gamma(x) \) such that \( (\tilde{p}_\alpha^\gamma)^{-1}(U_\gamma) \subset \text{Int}_{p_\alpha(G)}X_\gamma \).

Consequently, \((p_\gamma^\alpha)^{-1}(U_\gamma) \subset G_\alpha \). We can suppose that \( U_\gamma = \text{Int}U_\gamma \).

Then, according to the inductive assumption, there is an open set \( W \subset X \) such that \( q(W) \cap [1, \gamma] \) is finite and \( x \in W \subset (p_\gamma^\alpha)^{-1}(U_\gamma) \cap G \). So, \( x_\gamma \in p_\gamma(W) \subset W_\gamma = \text{Int}p_\gamma(W) \) and \( W_\gamma \subset U_\gamma \). Hence, \( x \in (p_\gamma^\alpha)^{-1}(W_\gamma) \cap G \subset G \).

Next claim completes the induction.

Claim 7. \( q((p_\gamma^\alpha)^{-1}(W_\gamma) \cap G) \cap [1, \alpha] = q(W) \cap [1, \gamma] \).

Indeed, for every \( \beta \leq \gamma \) we have \( \overline{p_\gamma((p_\gamma^\alpha)^{-1}(W_\gamma) \cap G)} = \overline{p_\beta(W)} \). This implies

\[
(6) \quad q(W) \cap [1, \gamma] = q((p_\gamma^\alpha)^{-1}(W_\gamma) \cap G) \cap [1, \gamma].
\]

Moreover, since \((p_\gamma^\alpha)^{-1}(W_\gamma) \subset (p_\gamma^\beta)^{-1}(U_\gamma) \subset p_\alpha(G) \), we have

\[
\overline{p_\beta((p_\gamma^\alpha)^{-1}(W_\gamma) \cap G)} = \overline{p_\beta((p_\gamma^\beta)^{-1}(W_\gamma))}
\]

for each \( \beta \in [\gamma, \alpha] \). Hence,

\[
(7) \quad q((p_\gamma^{-1}(W_\gamma) \cap G) \cap [\gamma, \alpha] = q((p_\gamma^{-1}(W_\gamma))) \cap [\gamma, \alpha].
\]

Note that, by Lemma 4.1(2), \( q((p_\gamma^{-1}(W_\gamma))) \cap [\gamma, \alpha] = \emptyset \). Then the combination of (1) and (2) provides the proof of the claim.

Therefore, for every \( \alpha < \tau \) the open sets \( W \subset X \) with \( q(W) \cap [1, \alpha] \) being finite form a base for \( X \). Now, we can finish the proof of the proposition. If \( V \subset X \) is open and \( x \in V \) we find a set \( G \subset V \) with \( x \in G = p_\beta^{-1}(G_\beta) \), where \( G_\beta \) is open in \( X_\beta \) for some \( \beta < \tau \). Then there exists an open set \( W \subset G \) containing \( x \) such that \( q(W) \cap [1, \beta] \) is finite. Let \( W_\beta = \text{Int}p_\beta(W) \) and \( U = p_\beta^{-1}(W_\beta \cap G_\beta) \). It is easily seen that \( x \in U \) and \( \overline{p_\nu(U)} = \overline{p_\nu(W)} \) for all \( \nu \leq \beta \). This yields \( q(U) \cap [1, \beta] = q(W) \cap [1, \beta] \). On the other hand, by Lemma 4.1(2), \( q(U) \cap [\beta, \tau] = \emptyset \). Hence \( U \) is a neighborhood of \( x \) which is contained in \( V \) and \( q(U) \) is finite. \( \square \)
Proposition 4.3. \cite{[14]} Let $S = \{X_\alpha, p_\alpha^\beta, \alpha < \beta < \tau\}$ be an almost continuous inverse system with nearly open bonding maps such that $X = \varprojlim S$. Then:

1. $X$ is regularly embedded in $\prod_{\alpha < \tau} X_\alpha$;
2. If, additionally, each $X_\alpha$ is regularly embedded in a space $Y_\alpha$, then $X$ is regularly embedded in $\prod_{\alpha < \tau} Y_\alpha$.

**Proof.** (1) We consider the embedding of $X$ in $\tilde{X} = \prod_{\alpha < \tau} X_\alpha$ generated by the maps $p_\alpha$. According to Proposition 4.2, $X$ has a base $\mathcal{B}$ consisting of open sets $U \subset X$ with finite rank $q(U)$. As in Proposition 2.7, for every $U \in \mathcal{B}$ let $\Omega(U) = \{\alpha_0, \alpha, \alpha + 1 : \alpha \in q(U)\}$, where $\alpha_0 < \tau$ is fixed. For all $U \in \mathcal{B}$ and $\alpha < \tau$ let $U_\alpha = \text{Int} p_\alpha(U)$ and $\Gamma(U) \subset \prod \{X_\alpha : \alpha < \tau\}$ be defined by

$$\Gamma(U) = \prod \{U_\alpha : \alpha \in \Omega(U)\} \times \prod \{X_\alpha : \alpha \notin \Omega(U)\}.$$ 

Since $p_\alpha(U) \subset U_\alpha$ for each $\alpha$, $U$ is contained in $\Gamma(U)$.

Using the arguments from the proof of Proposition 2.7, one can show that $\Gamma(U) \cap X \subset \overline{U}$. Finally, we define the required regular operator $e : \mathcal{T}_X \to \mathcal{T}_{\tilde{X}}$ by $e(V) = \bigcup \{\Gamma(U) : U \in \mathcal{B}, \overline{U} \subset V\}$.

(2) For each $\alpha < \tau$ let $e_\alpha : \mathcal{T}_{X_\alpha} \to \mathcal{T}_{Y_\alpha}$ be a regular operator. Define a function $\theta_1 : \mathcal{B} \to \mathcal{T}_{\tilde{Y}}$, where $\tilde{Y} = \prod_{\alpha < \tau} Y_\alpha$, by

$$\theta_1(U) = \prod_{\alpha \notin \Omega(U)} e_\alpha(U_\alpha) \times \prod_{\alpha \in \Omega(U)} Y_\alpha.$$ 

Consider $\theta : \mathcal{T}_X \to \mathcal{T}_{\tilde{Y}}$, $\theta(G) = \bigcup \{\theta_1(U) : U \in \mathcal{B} \text{ and } \overline{U} \subset G\}$. Since $\theta_1(U) \cap X = \Gamma(U)$ and $U \subset \Gamma(U) \subset \overline{U}$ for any $U \in \mathcal{B}$, $\theta(G) \cap X = G$. Moreover, Claim 4 implies that $\theta(G_1) \cap \theta(G_2) = \emptyset$ provided $G_1 \cap G_2 = \emptyset$. Thus, $\theta$ is a regular operator. \hfill \Box

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