Fixed point actions and on-shell tree-level Symanzik improvement

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Abstract

In this paper it is argued that the properties of the fixed point action of a renormalization group transformation can be used to implement the on-shell tree-level Symanzik improvement of lattice actions to any given order in the expansion in the lattice spacing, in a way which does not involve any perturbative calculations. In particular, a well-known technique for the lowest order improvement of SU(N) lattice gauge theories is revisited from the point of view of fixed point actions, which allows to shed light on some subtle points.

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1 Introduction

The lattice regularization of a field theory provides a unique tool to study its non-perturbative properties since it allows to perform numerical simulations by Monte Carlo techniques. Being the lattice spacing finite in numerical simulations, the determination of physical quantities from the lattice is plagued by systematic uncertainties. The naïve way to circumvent this problem is to adopt in numerical simulations the simplest possible lattice action (the so called “standard action”) and to make the lattice spacing smaller and smaller until a safe extrapolation to the continuum limit is viable. In practice, this amounts to consider very fine lattices and, consequently, requires very time-consuming and memory-demanding computations, in spite of the simplicity of the discretized action. An alternative way consists in improving the discretization procedure according to some theoretical prescription: the loss of simplicity in the form of the lattice action can be largely compensated by the possibility to extract physics from numerical simulations on relatively coarse lattices.

The improvement method due to Symanzik \[1\] consists in adding irrelevant terms to the standard action with appropriately chosen coefficients to cancel the lattice artifacts in the \(n\)-point Green functions up to a given order in \(a^2\) and up to a given order in perturbation theory \[1\]. For gauge theories this program has to be limited to the “on-shell” improvement, i.e. to the perturbative improvement of the physical quantities \[2, 3\].

A more radical improvement can be obtained according to the Wilson Renormalization Group (RG) theory \[4\]. It is well-established that lattice actions lying in the space of the couplings of the theory on the fixed point (FP) and on the renormalized trajectory (RT) of a given renormalization group transformation are perfect actions \[5\] in the sense that their spectral properties are completely free of cut-off effects. Lattice actions lying on the trajectory which originates at the FP and leaves the critical surface in the orthogonal direction are called FP actions \[6\]. Their spectral properties are free of lattice artifacts only in the classical limit, i.e. FP actions are classical perfect actions. One could say equivalently that FP actions are on-shell tree-level Symanzik improved actions to all orders in \(a^2\).

In this paper, it is argued that the properties of FP actions and of their classical solutions can be used to understand and to implement the on-shell tree-level Symanzik improvement up to any fixed order in \(a^2\) in a way which

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1In the presence of fermions, lattice artifacts appear as power series in \(a\), instead of \(a^2\). In the following, we will restrict for simplicity to bosonic theories.

2It is implicitly assumed here that there is only one (weakly) relevant direction, as in the case of 4d SU(N) gauge theories and of the 2d O(3) \(\sigma\)-model.
requires no perturbative calculations of on-shell Green functions. In particular, a technique for the $O(a^2)$ on-shell tree-level Symanzik improvement, which dates back to the arguments given by Lüscher and Weisz at the end of Sect. 5 in Ref. [6] and has been exploited for the 4d SU(N) lattice gauge theory in Ref. [7] and in Ref. [8], is revisited here from the point of view of FP actions, which allows to shed light on some subtle points (see e.g. footnote (9) of Ref. [6]). This paper develops some statements contained in the Sect. 2 of Ref. [9] where the relation between FP actions and on-shell tree-level Symanzik improvement was already clearly evidenced.

The paper is organized as follows: in Sect. 2, the $O(a^2)$ on-shell tree level Symanzik improvement for 4d SU(N) gauge theories according to the technique of Refs. [6, 7] is briefly reviewed; in Sect. 3, the relevant properties of perfect actions are recalled; in Sect. 4, the on-shell tree-level Symanzik improvement is considered from the point of view of FP actions, the lowest order results of Sect. 1 are revisited and a recipe is given to implement the improvement to any fixed order in $a^2$.

2 On-shell tree-level Symanzik improvement at $O(a^2)$

The result of the tree-level Symanzik improvement is the vanishing of the lattice corrections, up to a given order in $a^2$, in the $n$-point Green functions of a theory. Since the classical action coincides with the tree-level generating functional of the proper (one-particle-irreducible) Green functions of a theory, one expects that lattice corrections up to that given order should be absent also in the lattice action. This suggests that if a lattice action in its general form is somehow expanded in a power series of $a^2$, the tree-level Symanzik conditions on the coupling constants can be immediately written by imposing the vanishing of the irrelevant terms to a given order in $a^2$. This argument has been indeed exploited in 4d SU(N) gauge theories [7, 8] for the on-shell tree-level Symanzik improvement at the lowest order in $a^2$.

Following the notation of the Appendix of Ref. [8], the lattice action for SU(N) can be written in a general form

$$S_L(U) = \beta A_L(U) \ , \quad A_L(U) = \frac{1}{N} \sum_C c(C) [N - \text{Re} \ Tr(U_C)] \ , \quad \beta \equiv 2N/g^2 \ ,$$

(1)

where $C$ denotes any closed path, $U_C$ stands for the product of the link variables $U_\mu(n) \in \text{SU}(N)$ along the path $C$ and $c(C)$ is the coupling associated to the loop
The lattice action can be re-expressed in terms of a symmetrized action density

$$A_L = \sum_n A_L(n) ; \quad A_L(n) = \frac{1}{N} \sum_{C \ni n} c(C) \frac{[N - \text{Re} \text{Tr}(U_C)]}{\text{perimeter}(C)}.$$  \hspace{1cm} (2)

Now, in order to expand $A_L(n)$ in a power series of $a^2$, it is necessary to state the rule according to which the continuum gauge fields $A_{\mu}^a(x)$, $a = 1, \ldots, N^2 - 1$, are approximated by the link variables $U_{\mu}(n)$. This interpolation rule is somewhat arbitrary, although it is important that gauge covariance holds exactly, i.e. not only in the continuum limit $a \to 0$. A convenient choice is then

$$U_{\mu}(n) = \text{P} \exp \int_0^a A_{\mu}(na + s\hat{\mu}) \, ds ,$$  \hspace{1cm} (3)

where $A_{\mu}(x) = i \sum_{a} A_{\mu}^a(x) \lambda^a / 2$ are anti-Hermitian gauge fields (the $\lambda^a$ matrices, $a = 1, \ldots, N^2 - 1$, are the generators of SU(N) in the fundamental representation). It can be easily proven, indeed, that the local gauge transformation of the link variables $U_{\mu}'(n) \equiv g(na)U_{\mu}(n)g(na + \hat{\mu}a)$ corresponds to the exact gauge transformation of the continuum fields $A_{\mu}^a(x)$. Using (3), it is possible to expand $A_L(n)$ in the form

$$A_L(n) = a^4 O_0(na) \sum_C r_0(C)c(C) + a^6 \left[ O_1(na) \sum_C r_1(C)c(C) \right. \left. + O_2(na) \sum_C r_2(C)c(C) \right. \left. + O_3(na) \sum_C r_3(C)c(C) \right] + O(a^8) .$$  \hspace{1cm} (4)

with

$$O_0(x) = -\frac{1}{2} \sum_{\mu, \nu} \text{Tr}(F_{\mu \nu}^2(x)) ,$$  \hspace{1cm} (5)

$$O_1(x) = \frac{1}{12} \sum_{\mu, \nu} \text{Tr}(\mathcal{D}_\mu F_{\mu \nu}(x))^2 ,$$  \hspace{1cm} (6)

$$O_2(x) = \frac{1}{12} \sum_{\mu, \nu, \lambda} \text{Tr}(\mathcal{D}_\mu F_{\mu \lambda}(x) \mathcal{D}_\nu F_{\nu \lambda}(x))^2 ,$$  \hspace{1cm} (7)

$$O_3(x) = \frac{1}{12} \sum_{\mu, \nu, \lambda} \text{Tr}(\mathcal{D}_\mu F_{\mu \lambda}(x)\mathcal{D}_\nu F_{\nu \lambda}(x)) ,$$  \hspace{1cm} (8)

Since we are interested at the lowest order in $a^2$, it is not necessary to consider higher powers of the trace in Eq. (4).
being \( D_\mu F_{\nu\lambda} \equiv \partial_\mu F_{\nu\lambda} + [A_\mu, F_{\nu\lambda}] \). The coefficients \( r_0, r_1, r_2 \) and \( r_3 \) are given in Table 2 of Ref. [8] for all the loops which live on a hypercube \( 2^4 \). The normalization condition \( \sum_C r_0(C)c(C) = 1 \) has to be satisfied in order to ensure the correct continuum limit. For the traditional choice of loops with perimeter not larger than six, i.e. the plaquette (pl), the \( 2 \times 1 \) rectangle (rt), the bent rectangle (br) and the twisted loop (tw) (see Fig. 1), the expansion for \( A_L \) is

\[
A_L = \int d^4x \left\{ (c_{pl} + 8c_{rt} + 16c_{br} + 8c_{tw}) \mathcal{O}_0(x) \right. \\
+ a^2 [(c_{pl} + 20c_{rt} + 4c_{br} - 4c_{tw}) \mathcal{O}_1(x) \\
+ 4c_{tw} \mathcal{O}_2(x) + (12c_{br} + 4c_{tw}) \mathcal{O}_3(x)] + O(a^4) \right\} ,
\]

from which one reads immediately the normalization condition \( c_{pl} + 8c_{rt} + 16c_{br} + 8c_{tw} = 1 \).

\[\text{plaquette}\]

\[\text{rectangle}\]

\[\text{twisted}\]

\[\text{bent rectangle}\]

Figure 1: Loops considered in the \( O(a^2) \) on-shell tree-level Symanzik improvement of the 4d SU(N) lattice gauge theory: plaquette, \( 2 \times 1 \) rectangle, twisted perimeter-six loop and bent rectangle.

Lüscher and Weisz observed in Ref. [8] that the \( O(a^2) \) on-shell tree-level Symanzik conditions determined through the calculation of on-shell scattering amplitudes [3],

\[
c_{tw} = 0 , \quad c_{pl} + 20c_{rt} + 4c_{br} = 0 ,
\]

(10)
can be read off directly from Eq. (9), by imposing that the coefficients of the operators \( \mathcal{O}_1(x) \) and \( \mathcal{O}_2(x) \) vanish. The fact that the term with \( \mathcal{O}_3(x) \) plays no role, led them to observe that the \emph{on-shell} improvement means the improvement of the action for \emph{classical solutions} only. They also declared in the footnote (9)
of Ref. [6] that they could not invert the reasoning because they did not know of an independent argument to this effect.

Another point which deserves a deeper understanding is the interpolation rule (3): Why and to what extent is it arbitrary? Could one determine the $O(a^2)$ on-shell tree-level Symanzik conditions by just making the lattice action more general and expanding in $a^2$, with the same interpolation rule, to the next order?

In the next Sections, it will be argued that FP actions can provide a key to clarify these issues.

### 3 Perfect actions

In the following, the notation will be referred to the case of an unconstrained scalar field theory, in order to make the discussion more compact. The special cases of the 2d O(3) $\sigma$-model and of the 4d SU(N) gauge theories will be treated shortly later on.

A RG transformation with scale factor 2 can be defined in the following way

$$e^{-\beta' A_{L'}(\Phi')} = \int D\Phi \ e^{-\beta [A_L(\Phi) + \kappa \sum_n T(\Phi'(n), f_{\Phi}(n))]} ,$$

(11)

where $\Phi$ is the lattice field living on the original lattice with spacing $a$, $\Phi'$ is the blocked lattice field living on a lattice with spacing $a' = 2a$, $T(\Phi'(n), f_{\Phi}(n))$ is the blocking kernel, positive definite and normalized in order to keep the partition function invariant under the transformation, $\kappa$ is an arbitrary parameter. The blocking kernel is chosen to be zero when $\Phi'(n) = f_{\Phi}(n)$: this relation, when applied to all the sites $n$ of the blocked lattice, defines a blocking step. The functional $f_{\Phi}(n)$ is an averaging of the original field $\Phi$ in the surroundings of the site $n$ of the blocked lattice. Examples of RG transformation with scale factor 2 are those defined by

$$e^{-\kappa T(\Phi'(n), f_{\Phi}(n))} \to \delta(\Phi'(n) - f_{\Phi}(n)) ,$$

(13)

with

$$f_{\Phi}(n) = \frac{2 \tilde{d}^d}{2a} \sum_{\lambda_1 = \pm 1/2} \cdots \sum_{\lambda_d = \pm 1/2} \Phi(2n + \lambda_1 \hat{1} + \cdots + \lambda_d \hat{d}) .$$

(14)
In the factor $2^{(d-2)/2}/2^d$, the denominator is the normalization of the average; the numerator appears because a scalar field in a $d$ space-time has naive mass dimension equal to $(d-2)/2$, and we are using instead fields in lattice units. In the cases in which the blocking step can be iterated analytically, it is more convenient to perform one single RG transformation with scale factor equal to infinity between the continuum and a given lattice, namely

$$e^{-\beta' A_L(\Phi)} = \int D\varphi \, e^{-\beta [A_{\text{cont}}(\varphi) + \kappa \sum_n T(\Phi(n), f_\varphi(n))]}, \quad (15)$$

where now $A_{\text{cont}}(\varphi)$ is the continuum action, functional of the continuum field $\varphi$, and the field $\Phi$ is taken in physical units. The RG transformations with scale factor 2 defined by Eqs. (12) and (13) correspond to the following RG transformations with scale factor equal to infinity

$$T(\Phi(n), f_\varphi(n)) = (\Phi(n) - f_\varphi(n))^2, \quad (16)$$

and, in the $\kappa \to \infty$ limit,

$$e^{-\kappa T(\Phi(n), f_\varphi(n))} \to \delta (\Phi(n) - f_\varphi(n)), \quad (17)$$

with

$$f_\varphi(n) = \int_{-1/2}^{1/2} \cdots \int_{-1/2}^{1/2} d^d t \, \varphi(na + ta). \quad (18)$$

In the following, we will restrict for simplicity to RG transformations with scale factor equal to infinity. The results can be easily extended to all RG transformations with finite scale factor, by applying some trivial iteration arguments.

An important point to observe is that the integral in (15) is not definite for any RG transformation. To have an intuition of this fact, let us consider the massless free scalar theory with the RG transformation defined in Eq. (17): in Fourier transform, $\Phi(n) - f_\varphi(n)$ can be written as

$$\Phi(p) - \sum_{l \in Z^d} \varphi(p + 2\pi l) \Pi(p + 2\pi l), \quad \Pi(p) = \frac{2 \sin (p_\mu a/2)}{p_\mu}, \quad (19)$$

with $p$ in the first Brillouin zone $[-\pi/a, \pi/a]^d$. The integral (15) is quadratic and can be performed analytically giving

$$A_L(\Phi) = \frac{1}{2} \sum_{n,r} \Phi(n)\Phi(n + r) \rho(r), \quad \beta' = \beta, \quad (20)$$

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\textsuperscript{4}The continuum theory implies some regularization: the coupling $\beta$ in Eq. (17) is the bare coupling at the regularization scale.
with
\[
\frac{1}{\rho(p)} = \sum_{l \in \mathbb{Z}^d} \frac{\Pi(p + 2\pi l)^2}{(p + 2\pi l)^2} + \frac{1}{2\kappa}.
\] (21)

If the RG transformation were the “decimation”, i.e. \( f_{\varphi}(n) = \varphi(na) \), then everything would go the same way with the replacement \( \Pi(p) \rightarrow 1 \), except that the summation involved in \( 1/\rho(p) \) would be no more convergent as soon as \( d > 1 \).

In all the cases of “convergent” RG transformations, the long distance (in lattice units) properties of the lattice action \( A_L(\Phi) \) defined by Eq. (15) are the same of the continuum action, without any cut-off dependence. The lattice action \( A_L(\Phi) \) is therefore a perfect action.

For asymptotically free theories, since \( \beta' = \beta - O(1) \) in the \( \beta \rightarrow \infty \) limit, Eq. (15) can be solved by the saddle point approximation, giving
\[
A_{FP}^L(\Phi) = \min_{\{\varphi\}} \left[ A_{\text{cont}}(\varphi) + \kappa \sum_n T(\Phi(n), f_{\varphi}(n)) \right].
\] (22)

This equation defines the FP action of any lattice configuration \( \{\Phi\} \). It is perfect only in the classical limit, i.e. at the tree-level.

Let us concentrate now on the classical solutions of a perfect lattice action. A classical solution \( \{\Phi_{cl}\} \) is defined by the set of equations \( \delta A_L(\Phi)/\delta \Phi(m) = 0 \) or, equivalently, by
\[
\delta \frac{\delta}{\delta \Phi(m)} e^{-\beta' A_L(\Phi)} = \beta \int D\varphi \left[ \delta \frac{\delta}{\delta \Phi(m)} T(\Phi(m), f_{\varphi}(m)) \right] e^{-\beta [A_{\text{cont}}(\varphi) + \kappa \sum_n T(\Phi(n), f_{\varphi}(n))]} = 0.
\] (23)

In the classical limit \( \beta \rightarrow \infty \), i.e. in the case of the FP action, the above equation can be easily solved. Indeed, for any fixed \( \{\Phi\} \) there is a continuum configuration \( \{\bar{\varphi}\} \) such that the integral in (23) can be approximated by the value of the integrand at \( \{\varphi\} = \{\bar{\varphi}\} \). The configuration \( \{\bar{\varphi}\} \) is the minimum of the quantity in square brackets at the exponent. In this limit, Eq. (23) reduces to \( \frac{\delta}{\delta \Phi(m)} T(\Phi(m), f_{\varphi}(m)) = 0 \). Being \( T(\Phi(m), f_{\varphi}(m)) \) positive definite with a local minimum at \( \Phi(m) = f_{\varphi}(m) \) where it is equal to zero, the solution of Eq. (23) is \( \Phi(m) = f_{\varphi}(m) \). Since \( T(\Phi, f_{\varphi}) = 0 \), \( \{\bar{\varphi}\} \) is a minimum of \( A_{\text{cont}}(\varphi) \) and is therefore a classical solution \( \{\varphi_{cl}\} \) of the continuum equations of motion. Summarizing, for any classical solution \( \{\varphi_{cl}\} \) of the continuum theory there is a corresponding classical solution \( \{\Phi_{cl}\} \) of the FP action, related to the former by a blocking transformation. It is easy to convince ourselves that \( A_{FP}^L(\Phi_{cl}) = \)
\( \mathcal{A}_{\text{cont}}(\varphi_{\text{cl}}) \) also holds\(^5\). These results concerning FP classical solutions were obtained in \([9, 11]\) in a slightly different notation.

## 4 FP actions and on-shell tree-level Symanzik improvement

In the previous Section, it was shown that for any averaging functional \( f_{\varphi}(n) \) which defines a “convergent” RG transformation there is a related perfect lattice action, through Eq. (15). It was also shown that the lattice solutions of the equations of motion in the classical limit can be put in correspondence with the continuum solutions through \( \{ \Phi_{\text{cl}} \} = \{ f_{\varphi_{\text{cl}}} \} \), and that

\[
\mathcal{A}_{\text{FP}}^{L}(\Phi_{\text{cl}}) = \mathcal{A}_{\text{FP}}^{L}(f_{\varphi_{\text{cl}}}) = \mathcal{A}_{\text{cont}}(\varphi_{\text{cl}}).
\]

The latter chain of equations says that if we calculate a FP lattice action on a classical solution \( \{ \Phi_{\text{cl}} \} = \{ f_{\varphi_{\text{cl}}} \} \) and expand each \( f_{\varphi_{\text{cl}}}(n) \) in power series of \( a \), we must obtain that the leading term of this expansion reproduces exactly the continuum action calculated on \( \{ \varphi_{\text{cl}} \} \) and that all the irrelevant terms in \( a^2 \) vanish, at least by virtue of the continuum equations of motion.

Now, let us assume that we have built the FP action \( \mathcal{A}_{\text{FP}}^{L}(\Phi) \) related to a certain averaging functional \( f_{\varphi}(n) \) and we calculate it on the lattice field \( \Phi(n) = f'_{\varphi_{\text{cl}}}(n) \), where \( f'_{\varphi}(n) \) is a different averaging functional, which is even allowed to define a non-convergent RG transformation. Since

\[
\mathcal{A}_{\text{FP}}^{L}(f'_{\varphi_{\text{cl}}}) = \mathcal{A}_{\text{cont}}(\varphi_{\text{cl}}) + \sum_{n} \frac{\delta^2 \mathcal{A}_{\text{FP}}^{L}(\Phi)}{\delta \Phi(n)^2} \bigg|_{\{ \Phi \} = \{ f_{\varphi_{\text{cl}}} \}} (f'_{\varphi_{\text{cl}}}(n) - f_{\varphi_{\text{cl}}}(n))^2 + \cdots ,
\]

\( \mathcal{A}_{\text{FP}}^{L}(f'_{\varphi_{\text{cl}}}) \) differs from the continuum action by terms which are quadratic in the differences \( f'_{\varphi_{\text{cl}}}(n) - f_{\varphi_{\text{cl}}}(n) \). By the very nature of average functional, both \( f'_{\varphi}(n) \) and \( f_{\varphi}(n) \) can be expanded as \( \varphi(na) + O(a) \), so the sum of the differences \( f'_{\varphi}(n) - f_{\varphi}(n) \) represents a lattice correction to the continuum action of a classical solution. This correction will be \( O(a^2k) \), if the averaging functionals \( f'_{\varphi}(n) \) and \( f_{\varphi}(n) \) differ by terms \( O(a^k) \). This obvious result can be interpreted in the following sense: if one misses the correct rule which puts into correspondence the classical solutions of a FP action with the classical solutions of the continuum theory, cut-off effects come up at a certain predictable order in \( a^2 \). In the cases where a FP lattice action of a theory is known, this result has no practical use, except as a check. However, when no FP lattice actions of a theory have been built, but it is known by some arguments that a certain averaging functional \( f_{\varphi} \) exists which could define a FP action, the above result ensures that there

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\(^5\)For instanton classical solutions in 2d models and in 4d \( SU(N) \), this is true up to a critical size of the order of the lattice spacing (see, for instance, Ref. [11]).
exists at least one lattice action which is on-shell tree-level Symanzik improved at $O(a^{2k})$ for any “wrong” functional $f'_\varphi$ which differs from $f_\varphi$ at the order $a^{k+1}$.

A heuristic way to find this lattice action is the following:

1) start with a general form for the lattice action $A_L(\Phi, c)$ dictated by the symmetry; $c$ is a collective index for the couplings, constrained only to guarantee the correct continuum limit;

2) take an averaging function $f'_\varphi$ (i.e. what we called the “interpolation rule” in the Sect. 2) which differs at the order $a^{k+1}$ from an arbitrary averaging function $f_\varphi$ which defines a “convergent” RG transformation;

3) make the replacement $\Phi(n) = f'_\varphi cl(n)$ and expand in powers of $a$ up to the order $a^{2k}$ included, so that

$$ A_L(\Phi, c) = A_{cont}(\varphi_{cl}) + \int d^d x \left[ \sum_{n=1}^{k} \sum_{m=1}^{M(n)} a^{2n} v_m(n)(c) O_m^{(n)}(\varphi(x)) + O(a^{2k+2}) \right], $$

where $O_m^{(n)}(\varphi)$, $m = 1, \ldots, M(n)$, are continuum operators with na"ive dimension $d + 2n$;

4) use the continuum equations of motion to reduce the number of the independent operators at any order $n$;

5) impose $v_m(n)(c) = 0$, for $n = 1, \ldots, k$ and $m = 1, \ldots, M(n)$: these equations, together with the condition for the continuum limit, form the set of the Symanzik conditions for the on-shell tree-level improvement to $O(a^{2k})$.

The only non-trivial task in the above recipe is to find an averaging functional $f_\varphi$ which is known from independent arguments to define a convergent RG transformation\textsuperscript{6}.

As an illustration of the above procedure, let us consider the 2d O(3) $\sigma$-model, for which the FP action has been built in Ref. \textsuperscript{5} by iterating the RG transformation with scale factor 2 defined by the average functional

$$ f_S(n) = \frac{\sum_{\lambda_1 = \pm 1/2} \sum_{\lambda_2 = \pm 1/2} S_{2n+\lambda_1, 1+\lambda_2}}{\sum_{\lambda_1 = \pm 1/2} \sum_{\lambda_2 = \pm 1/2} S_{2n+\lambda_1, 1+\lambda_2}^2}. $$(26)

The lattice action can be written in a general form\textsuperscript{7}

$$ S_L = \beta A_L, \quad \beta \equiv 1/g $$

$$ A_L = -\frac{1}{2} \sum_{n_1, n_2} \rho(n_1 - n_2)(1 - S_{n_1} \cdot S_{n_2}) $$

\textsuperscript{6}In many cases, it is sufficient to check that there is convergence for the free theory.

\textsuperscript{7}The notation is that of Ref. \textsuperscript{1}. 

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\[ + \sum_{n_1,n_2,n_3,n_4} c(n_1,n_2,n_3,n_4)(1 - S_{n_1} \cdot S_{n_2})(1 - S_{n_3} \cdot S_{n_4}) + \ldots , \]

with the constraint \( S_n \cdot S_n = 1 \). If we are interested in the on-shell Symanzik improvement at \( O(a^2) \), it is sufficient to choose \( f'_S(n) = S_n \),

\[ \text{(28)} \]

since \( f_S \) and \( f'_S \) differ at the order \( a^2 \). The analog for scale factor equal to infinity would be \( f'_S(n) = S(na) \), where \( S(x) \) at the r.h.s. is the continuum field. Replacing now \( S_n \) with \( S(na) \) in \( A_L \) and expanding in powers of \( a^2 \), one finds

\[ A_L = \int d^2x \left\{ \frac{1}{2} \partial_\mu S \cdot \partial_\mu S \\
+ a^2 \left[ \frac{R_1}{16} (\partial^2 S) \cdot (\partial^2 S) + \frac{R_2}{48} \sum_\mu (S \cdot \partial_\mu^4 S) + \frac{C_1}{4} (S \cdot \partial^2 S)^2 \right. \\
+ \left. \frac{C_2}{2} \sum_{\mu,\nu} (\partial_\mu S \cdot \partial_\nu S)^2 + \frac{C_3}{4} \sum_\mu (\partial_\mu S \cdot \partial_\nu S)^2 \right] + O(a^4) \right\} , \]

having defined

\[ \sum_n \rho(n)n_\mu n_\nu n_\alpha n_\beta = R_1 (\delta_{\mu\nu}\delta_{\alpha\beta} + \delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha}) + R_2 \delta_{\mu\nu\alpha\beta} \]

\[ \text{(30)} \]

and

\[ \frac{1}{V} \sum_{n_1,n_2,n_3,n_4} c(n_1,n_2,n_3,n_4) \Delta_\mu \Delta_\nu \Delta'_\alpha \Delta'_\beta = \\
C_1 \delta_{\mu\nu}\delta_{\alpha\beta} + C_2 (\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha}) + C_3 \delta_{\mu\nu\alpha\beta} , \]

\[ \text{(31)} \]

where \( \Delta = n_1 - n_2 \), \( \Delta' = n_3 - n_4 \) and \( \delta_{\mu\nu\alpha\beta} \) is 1 when all its indices are equal, otherwise it is zero. Using the equations of motion \( \partial^2 S = S(S \cdot \partial^2 S) \), the \( O(a^2) \) on-shell tree-level Symanzik conditions can be read off from Eq. (29)

\[ R_2 = 0 \quad , \quad C_1 + \frac{1}{4} R_1 = 0 \quad , \quad C_2 = 0 \quad , \quad C_3 = 0 . \]

To improve the action at \( O(a^4) \), \( f'_S \) should be chosen to differ from \( f_S \) at \( O(a^3) \), and so on. Of course, the new \( f'_S \) should be chosen simple enough that it can be analytically iterated an infinite number of times. Alternatively, one

\[ ^8 \text{This averaging would define the non-convergent RG transformation of decimation.} \]
could work with scale factor equal to infinity from the beginning and take for instance

\[ f_S(n) = \frac{\int_{-1/2}^{1/2} \int_{-1/2}^{1/2} d^2t S(na + ta)}{\int_{-1/2}^{1/2} \int_{-1/2}^{1/2} d^2t S(na + ta)} , \]  

and choose \( f'_S \) consequently. Of course, the averaging functional \( f_S \) would define a different FP action, which however needs not to be determined in the context of the Symanzik improvement.

In view of the above considerations, the procedure followed in Sect. 2 for the \( O(a^2) \) improvement in 4d SU(N) gauge theories should be fully under control. In that case, the interpolation rule adopted in Refs. \[7, 8\] corresponds to the iteration of the scale factor 2 blocking transformation defined by

\[ f'_U(\mu)(n) = U_\mu(2n) U_\mu(2n + \hat{\mu}) . \]  

In four dimensions this averaging functional does not define a “convergent” RG transformation. We know, however, that there exist averaging functionals which allow to build FP actions for SU(N) \[11, 12\]. The functional \( f'_U \) differs from those averaging functionals at the \( O(a^2) \), at the level of the gauge fields \( A_\mu(n) \).

So, the same arguments apply as in the case of the O(3) \( \sigma \)-model \[9\].

A point which should be remarked is that, both in the case of the 2d O(3) \( \sigma \) model and of 4d SU(N) gauge theories, the \( O(a^2) \) on-shell tree-level Symanzik improvement results in an infinity of possible lattice actions. The argument described in this Section ensures that for any averaging functionals \( f_\varphi \) and \( f'_\varphi \), there is only one on-shell tree-level Symanzik improved action to a given order in \( a^2 \). This could either reflect the fact that for any fixed \( f'_\varphi \) there can be an infinite number of “good” functionals \( f_\varphi \) defining FP lattice actions, or that FP actions are a sub-class of all the lattice actions which satisfy \( A_L(f_\varphi) = A_{cont}(\varphi_{cl}) \) and that this last condition is sufficient to ensure the on-shell tree-level Symanzik improvement to all orders in \( a^2 \). Although something remain to be understood, FP actions provide anyway an interesting approach to the on-shell tree-level Symanzik improvement.

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\[9\] Going to the next order in \( a^2 \) in gauge theories with this method, however, involves serious technical problems owing to the limitations posed by gauge invariance.
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References

[1] K. Symanzik, Nucl. Phys. B 226 (1983) 187 and 205.

[2] P. Weisz, Nucl. Phys. B 212 (1983) 1.

[3] M. Lüscher and P. Weisz, Phys. Lett. B 158 (1985) 250.

[4] K.G. Wilson and J. Kogut, Phys. Rep. C 12 (1974) 75;
K.G. Wilson, Rev. Mod. Phys. 47 (1975) 773; ibid. 55 (1983) 583.

[5] P. Hasenfratz and F. Niedermayer, Nucl. Phys. B 414 (1994) 785;
see also: P. Hasenfratz, Nucl. Phys. B (Proc. Suppl.) 34 (1994) 3; F. Nieder-

dermayer, ibid. 513; P. Hasenfratz, Prospects for the Perfect Actions, hep-


cb/9709110, Proceedings of the XVth International Symposium on Lattice
Field Theory (Edinburgh, July 1997), ed. C.T.H. Davies et al., to appear on
Nucl. Phys. B (Proc. Suppl.).

[6] M. Lüscher and P. Weisz, Comm. Math. Phys. 97 (1985) 59.

[7] M. García Pérez, A. González-Arroyo, J. Snippe and P. van Baal, Nucl.
Phys. B 413 (1994) 535.

[8] F. Farchioni and A. Papa, “Instanton classical solutions of SU(3) fixed point
actions on open lattices”, Bern-preprint, BUTP-97/28, hep-lat/9711030.

[9] P. Hasenfratz and F. Niedermayer, Nucl. Phys. B 507 (1997) 399.

[10] M. Blatter, R. Burkhalter, P. Hasenfratz and F. Niedermayer, Phys. Rev.
D 53 (1996) 923.

[11] T. DeGrand, A. Hasenfratz, P. Hasenfratz and F. Niedermayer, Nucl. Phys.
B 454 (1995) 587.

[12] M. Blatter and F. Niedermayer, Nucl. Phys. B 482 (1996) 286.