Approximating Holant problems by winding

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Abstract

We give an FPRAS for Holant problems with parity constraints and not-all-equal constraints, a generalisation of the problem of counting sink-free orientations. The approach combines a sampler for near-assignments of “windable” functions – using the cycle-unwinding canonical paths technique of Jerrum and Sinclair – with a bound on the weight of near-assignments. The proof generalises to a larger class of Holant problems; we characterise this class and show that it cannot be extended by expressibility reductions.

We then ask whether windability is equivalent to expressibility by matchings circuits (an analogue of matchgates), and give a positive answer for functions of arity three.

1 Introduction

In this paper we will show that the following problem has an FPRAS (a type of approximation algorithm – see Section 2.2).

Name \#ParityNAE
Instance A multigraph \(G\) in which each vertex is labelled Even, Odd, or NAE
Output The number of subsets \(F \subseteq E(G)\) such that:
- each Even vertex has an even number of incident edges in \(F\)
- each Odd vertex has an odd number of incident edges in \(F\)
- each NAE vertex has at least one incident edge in \(F\) and at least one incident edge in \(E(G) \setminus F\)

Theorem 1. There is an FPRAS for \#ParityNAE.

1.1 Relationships with other counting problems

A sink-free orientation of a graph is a choice of orientation of each edge such that no vertex has out-degree zero. The problem \#SFO is: given a graph, count the number of sink-free orientations. We can also allow “skew” edges, where the ends of the edge must both be oriented outwards or both oriented inwards.

Bubley and Dyer studied \#SFO and gave an FPRAS [4]. They showed as a corollary that there is an FPRAS for counting solutions to a formula in conjunctive normal form in which every variable appears at most twice, which they showed is a \#P-hard problem. The first part of their argument was a standard reduction to sampling - finding a fully polynomial almost uniform sampler (FPAUT) for sink-free orientations. Then, they constructed a Markov chain that converges to the uniform distribution on sink-free orientations, and
bounded its mixing time using a two-stage path coupling argument. Cohn, Pemantle and Propp later gave an exact sampler with $O(|V| \cdot |E|)$ mean running time using a kind of rejection sampling [9].

A simple reduction from \#SFO to \#ParityNAE is illustrated in Figure 1, showing that \#ParityNAE generalises the problem of counting sink-free orientations in a graph (while also allowing parity constraints). Given an instance $G$ of \#SFO, label all the vertices NAE, subdivide each non-skew edge $uv$, label the new vertices (which we will refer to as “$m_{uv}$”) Odd, then attach a degree-one Odd vertex to each NAE vertex. This gives an instance $G'$ of \#ParityNAE. For all orientations $O$ of $G$ define a set $F_O \subseteq E(G')$ by taking all edges attached to degree-one Odd vertices and all “heads”: for non-skew edge $uv$ take $um_{uv} \in F$ if and only if $uv$ is directed towards $u$, and for skew edges $uv$ take $uv \in F$ if and only if $uv$ is oriented outwards. Each degree-two Odd vertex in $G'$ has exactly one incident edge in $F_O$, and each NAE vertex in $F_O$ has at least one incident edge in $F$, and if $O$ is sink-free then each NAE vertex in $F_O$ has at least one incident edge not in $F$. Furthermore, any $F \subseteq E(G')$ satisfying these conditions is $F_O$ for some sink-free orientation $O$. The function $O \mapsto F_O$ therefore gives a bijection from sink-free orientations of $G$ to the set of subsets of $E(G')$ that get counted by \#ParityNAE.

\#ParityNAE, at least when restricted to bounded-degree graphs, is a type of Boolean Holant problem. Holant problems are a quite general type of graphical counting problem. The constraints Odd, Even and NAE in \#ParityNAE are generalised to functions $F: \{0,1\}^k \rightarrow \mathbb{C}$, called (Boolean) signatures in this context. Relations $R \subseteq \{0,1\}^k$ can also be used by considering the function $R: \{0,1\}^k \rightarrow \{0,1\}$ that takes the value 1 exactly on elements of $R$. In this discussion we will take the codomain to be the set of complex numbers, but afterwards we will restrict to non-negative rational-valued signatures, which we call weight-functions.

A Holant instance is a graph $G$ equipped with a function $F_v: \{0,1\}^{J_v} \rightarrow \mathbb{C}$ for each vertex $v$, where $J_v$ is the set of edges incident to $v$. In fact we will want to allow self-loops, which calls for a slightly more complicated definition - see Section 2. We are interested in the total weight

$$\sum_{x \in \{0,1\}^{E(G)}} \prod_{v \in V(G)} F_v(x|J_v).$$

For example, if all $F_v$ take the value 1 on vectors of Hamming weight 1, and take the value 0 otherwise, then the total weight is just the number of perfect matchings of $G$, because a vector $x \in \{0,1\}^{E(G)}$ is the characteristic vector of a perfect matching of $G$ if and only if $\prod_{v \in V(G)} F_v(x|J_v) = 1$.

Given a finite set $\mathcal{F}$ of signatures, Holant($\mathcal{F}$) is the problem of evaluating the total weight

Figure 1: Reduction from \#SFO to \#ParityNAE. The edge with two arrows is a skew edge. A sink-free orientation is illustrated with the corresponding set $F$ draw in thick grey.
where $G$ is given as input, and where we require that each $F_v$ is a copy of some $F \in \mathcal{F}$: for some enumeration $v_1, \ldots, v_k$ of $J_v$ we have $F_v(x) = F(x(v_1), \ldots, x(v_k))$ for all $x \in \{0,1\}^k$. For example, let $F$ be the function defined by $F(0,0,1) = F(0,1,0) = F(1,0,0) = 1$ and $F(i,j,k) = 0$ elsewhere. Then Holant($\{F\}$) is the problem of counting perfect matchings in degree-three graphs.

For all positive integers $k$ define $\text{Even}_k, \text{Odd}_k, \text{NAE}_k : \{0,1\}^k \to \{0,1\}$ by setting $\text{Even}_k(x_1, \ldots, x_k)$ to be 1 if and only if $x_1 + \cdots + x_k$ is even, setting $\text{Odd}_k(x_1, \ldots, x_k)$ to be 1 if and only if $x_1 + \cdots + x_k$ is odd, and setting $\text{NAE}_k(x_1, \ldots, x_k)$ to be 1 if and only if $1 \leq x_1 + \cdots + x_k \leq k-1$. The restriction of $\#\text{ParityNAE}$ to graphs of maximum degree at most $d$ is then equivalent to Holant($\mathcal{F}_d$) where $\mathcal{F}_d = \{\text{Even}_1, \text{Odd}_1, \text{NAE}_1, \ldots, \text{Even}_d, \text{Odd}_d, \text{NAE}_d\}$. By Theorem 1 this problem has an FPRAS for each $d$.

$\#\text{ParityNAE}$ is also a counting constraint satisfaction problem, at least when restricted to bounded-degree graphs. Roughly speaking, a instance of a constraint satisfaction problem is a list of constraints like “$x \lor y$, $y \land z$, $y = z$”, and we are interested in the number of configurations satisfying the constraints. In particular [5], for any finite set of signatures $\mathcal{F}$, an instance of $\#\text{CSP}(\mathcal{F})$ is a set of variables $V$ and a list of formal function applications $F_1(v_{1,1}, \ldots, v_{1,k_1}), \ldots, F_s(v_{s,1}, \ldots, v_{s,k_s})$ where $F_i : \{0,1\}^{k_i} \to \mathbb{C}$ is a function in $\mathcal{F}$ for each $1 \leq i \leq s$, and $v_{i,j} \in V$ is a variable for each $1 \leq i \leq s$ and $1 \leq j \leq k_i$. The value of this instance is

$$\sum_{x \in \{0,1\}^V} \prod_{i=1}^m F_i(x(v_{i,1}), \ldots, x(v_{i,k_i})).$$

For information about the complexity of approximately evaluating $\#\text{CSP}s$, see [3].

A $\#\text{CSP}(\mathcal{F})$ instance can be drawn as a “dual constraint hypergraph”, which has vertices for each of the constraints $c_1, \ldots, c_s$, and a hyperedge $\{c_i \mid v = x_{i,j}\}$ for each variable $v$ (ignoring multiplicities for now). The dual constraint hypergraph is a graph if and only if every variable appears exactly twice. In this way, Holant($\mathcal{F}$) is the restriction of $\#\text{CSP}(\mathcal{F})$ to read-twice instances. Note that the variables of the $\#\text{CSP}$ are the edges of the Holant instance; sometimes a $\#\text{CSP}$ is described in the opposite way, as the primal constraint hypergraph, with variables as vertices and $s$ edges or hyperedges.

We now recall the relationships between $\#\text{CSP}s$ and Holant problems in the Hadamard basis as discussed in [15]. Note that while these equivalences are usually stated in the context of exact evaluation, the reductions just involve preprocessing the input, and so also apply in the context of approximate counting. Firstly, equality constraints can be used to break the read-twice restriction. Letting $\equiv_3$ denote the function $\{0,1\}^3 \to \mathbb{C}$ taking the value 1 on $(0,0,0)$ and $(1,1,1)$, and taking the value 0 elsewhere, if $\equiv_3$ is in $\mathcal{F}$ then Holant($\mathcal{F}$) is equivalent to $\#\text{CSP}(\mathcal{F})$ [7, Proposition 5]. Secondly, let $\widehat{F} : \{0,1\}^k \to \mathbb{C}$ denote the Hadamard transform, defined by

$$\widehat{F}(x_1, \ldots, x_k) = 2^{-k/2} \sum_{y \in \{0,1\}^k} F(y_1, \ldots, y_k)(-1)^{y_1 x_1 + \cdots + y_k x_k}.$$

Holant($\mathcal{F}$) is always equivalent to Holant($\{\widehat{F} \mid F \in \mathcal{F}\}$); see [7] Proposition 1 or [22]. Also, $\widehat{\hat{F}} = F$ for any $F$. So if $\mathcal{F}$ contains $\equiv_3$, then Holant($\mathcal{F}$) is equivalent to $\#\text{CSP}(\{\widehat{F} \mid F \in \mathcal{F}\})$. But $\equiv_3$ is just $\text{Even}_3$ multiplied by a factor of $\sqrt{2}$ (which can be easily accounted for).

Taking $\mathcal{F}$ to be the set $\mathcal{F}_d$ defined above, with $d \geq 3$, we find that the restriction of $\#\text{ParityNAE}$ to instances of degree at most $d$ is equivalent to $\#\text{CSP}(\{\widehat{F} \mid F \in \mathcal{F}_d\})$. By Theorem 1 this problem has an FPRAS for each $d$. In this sense, $\#\text{ParityNAE}$ generalises $\#\text{SFO}$ to a $\#\text{CSP}$. Note that $\text{Odd}_1(0) = 1/\sqrt{2}$ and $\text{Odd}_1(0) = -1/\sqrt{2}$. So we get a class of FPRASes for $\#\text{CSP}s$ using functions with mixed signs.

1.2 Techniques

Like Bubley and Dyer we will use Markov chains, but to bound the mixing time we will instead apply the canonical paths technique. More precisely, we will use a multicommodity
flow with cycle-unwinding as used by Jerrum and Sinclair [19]. They proved the following relevant result: for any polynomial \( p \) we can sample efficiently from the uniform distribution of perfect matchings, in graphs \( G \) satisfying

\[
\frac{\text{number of matchings of order } \frac{1}{2}|V(G)| - 1}{\text{number of matchings of order } \frac{1}{2}|V(G)|} \leq p(|V(G)|).
\]  (1)

Recall that a matching of a graph is a set of edges not sharing any vertices, and a matching is perfect if it has order \(|V(G)|/2\). A perfect matching is a satisfying assignment to a certain system of constraints: each edge is either IN or OUT, and every variable enforces a perfect matchings constraint, that exactly one of its incident edges is IN. From this perspective a natural question is: what weight-functions can we use instead of perfect matchings constraints? We show that Jerrum and Sinclair’s result generalises in a certain sense to windable functions, defined as follows.

**Definition 2.** For any finite set \( J \) and any configuration \( x \in \{0,1\}^J \) define \( \mathcal{M}_x^J \) to be the set of partitions of \( \{i \mid x_i = 1\} \) into pairs and singletons. A function \( F : \{0,1\}^J \rightarrow \mathbb{Q}_{\geq 0} \)

is windable if there exist values \( B(x,y,M) \geq 0 \) for all \( x,y \in \{0,1\}^J \) and all \( M \in \mathcal{M}_{x\oplus y}^J \) satisfying:

1. \( F(x)F(y) = \sum_{M \in \mathcal{M}_{x\oplus y}^J} B(x,y,M) \) for all \( x,y \in \{0,1\}^J \), and
2. \( B(x,y,M) = B(x \oplus S, y \oplus S, M) \) for all \( x,y \in \{0,1\}^J \) and all \( S \in M \in \mathcal{M}_{x\oplus y}^J \).

Here \( x \oplus S \) denotes the vector obtained by changing \( x_i \) to \( 1 - x_i \) for the one or two elements \( i \) in \( S \).

The next question is: what kinds of constraints guarantee a bound like (1)? We give one answer: strictly terraced functions.

**Definition 3.** A function \( F : \{0,1\}^J \rightarrow \mathbb{Q}_{\geq 0} \) is strictly terraced if

\[
F(x) = 0 \implies F(x \oplus e_i) = F(x \oplus e_j) \quad \text{for all } x \in \{0,1\}^J \text{ and all } i,j \in J.
\]

Here \( x \oplus e_i \) denotes the vector obtained by changing \( x_i \) to \( 1 - x_i \).

We will discuss these definitions more throughout the paper. Using properties of these classes, we will establish Theorem 4. A feature of the techniques is that they cannot be extended by expressibility reductions, where we just substitute a constraint by a “circuit”, a gadget gluing together other constraints. The following theorem makes this precise.

**Theorem 4.** Let \( F \) be the class of strictly terraced windable functions. Then

- \( F \) is closed under taking weight-functions of connected circuits
- \( F \) contains \( \text{Even}_k \), \( \text{Odd}_k \), and \( \text{NAE}_k \) for all \( k \geq 1 \)
- for all finite subsets \( F' \subset F \) there is an FPRAS for Holant\((F')\)

The reason to take \( F' \) to be finite is to make sense of the computational problem \( \text{Holant}(F') \). As in Theorem 4 if one is careful about how the input is specified, it is also possible to allow infinite \( F' \) in some cases.

1.3 Matching circuits

In Section 7 we will consider a natural type of gadget for reducing Holant problems to \#\text{PerfMatch}, the problem of counting the number of perfect matchings in a graph. \#\text{PerfMatch} is, famously, \#P-complete even when restricted to bipartite instances [25]. This suggests that there is no efficient exact algorithm, leaving the question of whether there is an approximation algorithm. A major result in this direction is that there is an FPRAS for \#\text{PerfMatch} restricted to bipartite graphs [20]. Our study of matching circuits is an attempt to identify which Holant problems reduce to \#\text{PerfMatch} in the sense of expressibility.

Consider a clique of order four, where at the \( i \)’th vertex we attach an “outgoing” edge \( d_i \). For each of the sixteen possible subsets \( M \subseteq \{d_1,d_2,d_3,d_4\} \) of the outgoing edges,
we can count the number of ways to add internal edges to $F$ to obtain a perfect matching. Because the clique of order four has 3 perfect matchings, we have $F(\emptyset) = 3$, while $F(\{d_1\}) = 0$ and $F(\{d_1, d_2, d_3, d_4\}) = 1$.

We will say that a function $F : \{0, 1\}^J \rightarrow \mathbb{Q}^{\geq 0}$ has a matchings circuit if there is a similar graph fragment, with outgoing edges $J$, matching. Because the clique of order four has 3 perfect matchings containing the outgoing edges $\{i \in J : x_i = 1\}$. Substituting each vertex of a Holant($\{F\}$) instance by the graph fragment gives a reduction from Holant($\{F\}$) to $\#$PerfMatch. Actually, in Section 4 following Jerrum and Sinclair we will allow non-negative edge-weights and a “fugacity” at each vertex, because these do not add any more computational power; the important property is:

**Proposition 5.** If $F$ is a finite set of weight-functions that have matchings circuits, then Holant($F$) $\leq_{AP}$ $\#$PerfMatch.

Here $\leq_{AP}$ denotes a type of approximation-preserving reduction used to study the relative complexity of approximate counting problems - see Section 2.2. In particular, if Holant($F$) $\leq_{AP}$ $\#$PerfMatch and $\#$PerfMatch has an FPRAS then Holant($F$) has an FPRAS. The main result is the following theorem.

**Theorem 6.** Let $F : \{0, 1\}^3 \rightarrow \mathbb{Q}^{\geq 0}$. The following are equivalent:

1. $F$ is windable
2. For all $x_1, x_2, x_3 \in \{0, 1\}$ we have
   \[
   F(x_1, x_2, x_3)F(1 - x_1, 1 - x_2, 1 - x_3) \\
   \leq F(x_1, x_2, 1 - x_3)F(1 - x_1, 1 - x_2, x_3) + F(x_1, 1 - x_2, x_3)F(1 - x_1, x_2, 1 - x_3) + F(x_1, 1 - x_2, 1 - x_3)F(1 - x_1, x_2, x_3)
   \]
3. $F$ has a matchings circuit

Theorem 6 gives a class of problems that reduce to counting perfect matchings. For example, the Holant problem allowing only the relation $\{(0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 0, 1), (0, 1, 1)\}$ reduces to $\#$PerfMatch, but is not known to have an FPRAS.

### 1.4 Related work

A matroid is sbo (strongly basis orderable) if for all bases $A$ and $B$ there is a bijection $\pi : A \setminus B \rightarrow B \setminus A$ such that for all $X \subseteq A \setminus B$ the set $(A \cup \pi(X)) \setminus X$ is a basis. Bouchet and Cunningham generalised the sbo property as linkability for the class of even delta-matroids, and showed that this class is closed under an analogue of circuits [2]. These conditions are just windability over the two-element Boolean semiring $(\mathbb{B} = \{0, 1\}, \max, \min)$, for the set of bases when considered as a function $\{0, 1\}^3 \rightarrow \mathbb{B}$, by taking the characteristic vector of characteristic vectors of bases. Gamblin used the sbo property to approximately count the number of bases in certain matroids [11].

Valiant [20] introduced matchgates and matchcircuits, which are similar to matchings circuits but give efficient exact algorithms. Matchcircuits can be understood as planar graphs with edge-weights, with no restriction to non-negative numbers. Cai and Choudhary characterised the expressibility of matchgates [6]. The name “matchings circuits” used in this paper is meant to suggest a version of matchgates.

The focus on (the negative side of) expressibility for approximate counting problems appears in [5], where logsupermodular functions are shown not to express non-logsupermodular functions in the context of $\#$CSPs.

Yamakami [27] and the current author [23] have given partial classifications for classes of Holant problems. The bulk of these results deal with intractability: reductions from a named problem such as $\#$SAT to a given Holant problem. The focus of the current paper is on tractability: either in the absolute sense of an FPRAS, or by reductions to $\#$PerfMatch.
A related \#CSP with mixed signs appears in the context of the Tutte polynomial. By the proof of [10, Lemma 7], the following problems are equivalent in the sense of approximate counting, for any fixed \( y < -1 \).

- \#PerfMatch
- \#CSP\{\( B_y \)\} where \( B_y : \{0,1\}^2 \to \mathbb{Q}^{\geq 0} \) is defined by \( B_y(0,0) = B_y(1,1) = y \) and \( B_y(0,1) = B_y(1,0) = 1 \)
- evaluating the Tutte polynomial at the point \((x,y)\) where \( (x-1)(y-1) = 2\)

### 1.5 Outline

In Section 3 we adapt the conductance argument of Jerrum and Sinclair to “even-windable” functions, which are a slightly simpler version of windable functions. We study windable functions in Section 4. We study strictly terraced functions in Section 5. In Section 6 we establish Theorem 1 and Theorem 2. Finally, in Section 7 we discuss matchings circuits and establish Proposition 5 and Theorem 3.

### 2 Preliminaries

A configuration of a finite “indexing” set \( J \) is a function \( x \in \{0,1\}^J \). A weight-function is a function \( F : \{0,1\}^J \to \mathbb{Q}^{\geq 0} \). A copy of \( F \) is a function \( G : \{0,1\}^J \to \mathbb{Q}^{\geq 0} \) of the form \( G(x) = F(x \circ \pi) \) for some bijection \( \pi : J \to I \). We will use bold face to distinguish between sets \( S \subseteq J \) and the characteristic vector \( S \). Similarly the bold version of a relation \( R \subseteq \{0,1\}^J \) is the corresponding zero-one-valued weight-function.

We will not distinguish between \( \{0,1\}^{\{1,\ldots,k\}} \) and \( \{0,1\}^k \), or between \( \{0,1\}^1 \) and \( \{0,1\} \).

Also, we will sometimes allow indexing sets to be partially enumerated in a certain way. This is for notational power: the enumerated indices are easy to refer to explicitly, while the unenumerated indices are easy to fix. For all positive integers \( k \) and all finite sets \( J \), when \( k + J \) is used as an indexing set it means the disjoint union of \( \{1,\ldots,k\} \) and \( J \). Elements of \( \{0,1\}^{k+J} \) will be denoted by \((x_1,\ldots,x_k,y)\) where \( x_1,\ldots,x_k \in \{0,1\} \) and \( y \in \{0,1\}^J \).

For all sets \( I \subseteq J \), all configurations \( p \) of \( I \) and all configurations \( x \) of \( J \setminus I \), let \((x,p)\) denote the unique common extension of \( x \) and \( p \) to a configuration of \( J \). The pinning of a weight-function \( F : \{0,1\}^J \to \mathbb{Q}^{\geq 0} \) by \( p \in \{0,1\}^I \) \((I \subseteq J)\) is the weight-function \( F' : \{0,1\}^{J \setminus I} \to \mathbb{Q}^{\geq 0} \) defined by \( F'(x) = F(x,p) \).

The distance \(|\{i \in J \mid x_i \neq y_i\}|\) between two configurations \( x,y \in \{0,1\}^J \) will be denoted \( d(x,y) \). We say \( F \) is even\footnote{This may be confusing terminology - an even function can be non-zero on vectors of odd Hamming weight. But it is a common definition for delta-matroids.} if \( d(x,y) \) is even for all \( x,y \) with \( F(x), F(y) > 0 \). Define \( x \oplus y \in \{0,1\}^J \) by \( (x \oplus y)_i = x_i + y_i \) \( \pmod{2} \). For all \( x \in \{0,1\}^J \) define \( \overline{x} \in \{0,1\}^J \) by \( \overline{x}_i = 1-x_i \), and for all \( F : \{0,1\}^J \to \mathbb{Q}^{\geq 0} \) define \( F^\langle \rangle : \{0,1\}^J \to \mathbb{Q}^{\geq 0} \) by \( F^\langle \rangle (x) = F(x)F(\overline{x}) \).

For all \( F : \{0,1\}^J \to \mathbb{Q}^{\geq 0} \) and \( y \in \{0,1\}^J \) define the flip of \( F \) by \( y \) to be the weight-function \( F^y : \{0,1\}^J \to \mathbb{Q}^{\geq 0} \) defined by \( F^y(x \oplus y) = F(x) \) for all \( x \in \{0,1\}^J \). For all \( i \in J \) define \( e_i \in \{0,1\}^J \) (where \( j \) is implicit) to be the characteristic vector of \( \{i\} \).

For all finite sets \( J \) define

\[
\begin{align*}
\text{Even}_J &= \{ x \in \{0,1\}^J \mid \sum_{i \in J} x_i \text{ is even} \} \\
\text{Odd}_J &= \{ x \in \{0,1\}^J \mid \sum_{i \in J} x_i \text{ is odd} \} \\
\text{NAE}_J &= \{ x \in \{0,1\}^J \mid 1 \leq \sum_{i \in J} x_i \leq |J| - 1 \} \\
\text{EvenNAE}_J &= \text{Even}_J \cap \text{NAE}_J
\end{align*}
\]

\( \text{Even}_J \) and \( \text{Odd}_J \) are parity relations. The last relation \( \text{EvenNAE}_J \) is only used for calculations (and only with \( |J| \) even).
2.1 Circuits

In this paper, circuits are a type of graph equipped with weight-functions at each vertex, and allowing external edges. A little care is needed to allow self-loops and asymmetric weight-functions.

A graph fragment \( G \) is specified by:

- a set \( J^G \) whose elements are called incidences
- a set \( V^G \) of vertices, and sets \( J_v^G, v \in V^G \), that partition \( J^G \)
- a set \( A^G \subseteq J^G \) whose elements are called external edges
- a partition \( E^G \) of \( J^G \setminus A^G \) into pairs called internal edges

See Figure 2.

A circuit \( \phi \) is a graph fragment equipped with a constraint \( F^\phi_v : \{0,1\}^{J^\phi_v} \rightarrow \mathbb{Q} \geq 0 \) for each vertex \( v \). We can also use a relation \( R \subseteq \{0,1\}^{J^\phi_v} \) as a constraint by taking \( F^\phi_v = R \).

\( G \) is closed if it has no external edges. Standard graph-theoretic terminology extends to graph fragments. In particular we will refer to connected graph fragments. An edge is either an internal edge or an external edge. A vertex \( v \) and an internal edge \( e \) are incident if \( J_v \) intersects \( e \). If an internal edge \( e \) is uniquely identified by the vertices \( u, v \) it is incident to, we will denote \( e \) by \( uv \).

Given a circuit \( \phi \), for any configuration \( x \) of \( J \),

- \( x \) is an assignment (with respect to \( E \)) if \( x_i = x_j \) for all \( \{i,j\} \in E \).
- \( x|_{J_v} \) denotes the restriction of \( x \) to \( J_v \).
- The weight of \( x \) is \( \text{wt}_\phi(x) = \prod_{v \in V} F^\phi_v(x|_{J_v}) \).

The weight-function of \( \phi \) is the function \( \llbracket \phi \rrbracket : \{0,1\}^A \rightarrow \mathbb{Q} \geq 0 \) defined by

\[
\llbracket \phi \rrbracket (x) = \sum_{x'} \text{wt}_\phi(x') \quad (x \in \{0,1\}^A)
\]

where the sum is over extensions of \( x \) to assignments \( x' : \{0,1\}^J \rightarrow \mathbb{Q} \geq 0 \) with respect to \( E \).

If a weight-function \( F \) is equal to \( \llbracket \phi \rrbracket \), we will say that \( F \) has the circuit \( \phi \).

Another way to think of a circuit is as a “read-twice pps-formula”, a special case of the pps-formulas of [5]. For example, consider an equation

\[
F(x) = \sum_{y=0}^{1} \sum_{z=0}^{1} G(x, y)G(y, z)H(z) \quad (x \in \{0,1\}).
\]
Note how on the right-hand-side, each bound (summed) variable appears exactly twice, and each free (unsummed) variable appears exactly once. Any equation of this form defines a circuit in a natural way: incidences correspond to the variable occurrences $x, y, z, w$; vertices correspond to terms $G(x, y), G(y, z), H(z)$; the sets $J_v$ are scopes for each term; external edges correspond to free variables; and internal edges correspond to summed variables.

For any partition $E$ of a finite set $J$ into pairs, for all non-negative integers $k$, a $k$-assignment with respect to $E$ is a configuration $x$ of $J$ such that $x_i = x_j$ for all but exactly $k$ pairs $\{i, j\} \in E$. So an assignment is a 0-assignment. For all closed circuits $\phi$ and all integers $k \geq 0$ define
\[
Z_k(\phi) = \sum_{k\text{-assignments } x} \text{wt}_\phi(x).
\]
So $Z_0(\phi)$ is just $[\phi]$ (evaluated on the empty configuration).

### 2.2 Computational definitions

A **counting problem** is a function $f$ taking instances (encoded as strings over a finite alphabet $\Sigma$) to non-negative reals. A **randomised approximation scheme** for $f$ is a randomised algorithm that takes an instance $x$ and error parameter $\epsilon > 0$ and returns an approximation $Z$ to $f(x)$ satisfying
\[
\Pr\left[e^{-\epsilon}f(x) \leq Z \leq e^\epsilon f(x)\right] \geq 3/4.
\]
A fully polynomial randomised approximation scheme (FPRAS) for $f$ is a randomised approximation scheme that runs in polynomial time in $|x|$ and $\epsilon^{-1}$. (To be concrete, we can require the error parameter to be specified by a binary integer $\epsilon^{-1}$.)

Let $f$ and $g$ be counting problems. A randomised oracle algorithm $A$ meeting the following conditions is an approximation-preserving reduction from $f$ to $g$, and if such a reduction exists we write $f \leq_{AP} g$.

$A$ takes inputs $(w, \epsilon)$ where $w$ is in the domain of $f$, and $\epsilon > 0$. The run-time of $A$ is polynomial in $|w|$ and $\epsilon^{-1}$ and the bit-size of the values returned by the oracle (this avoids requiring that the oracle gives concise responses). The oracle calls made by $A$ are of the form $(v, \delta)$, where $v$ is an instance of $g$ and $\delta > 0$ is an error parameter, such that $|v| + \delta^{-1}$ is bounded by a polynomial in $|w|$ and $\epsilon^{-1}$ (depending only on $A$). If the oracle’s outputs meet the specification of a randomised approximation scheme for $g$, then $A$ is a randomised approximation scheme for $f$.

The above definitions are based on [13]; the main difference is that we allow non-integer-valued problems.

For any finite set $F$ of weight-functions define the following counting problem.

**Name** Holant($F$)

**Instance** A closed circuit $\phi$ using copies of weight-functions in $F$

**Output** $[\phi]$

Since $F$ is finite, it is not particularly important how the functions $F^\phi_{i}$ are specified. For concreteness: $F^\phi_{i}$ should be specified by an index $i$ into a fixed enumeration $F = \{F_1, \ldots, F_{|F|}\}$, along with a bijection from $J_0^\phi$ to the indexing set $I_0$ of $F_i : \{0, 1\}^{I_0} \to \mathbb{Q}^{\geq 0}$.

By substituting circuits, if $F$ has a circuit using copies of weight-functions from a finite set $\mathcal{F}$, then Holant($F \cup \{F\}$) $\leq_{AP}$ Holant($\mathcal{F}$). This justifies the focus on expressibility in this paper.

### 3 Even-windable functions

#### 3.1 Idea

Windability is an abstraction of a property of the distribution of perfect matchings in a graph with external edges. We will illustrate the idea briefly by the arity 4 case, where
windability is already used implicitly in [19]. But higher-arity conditions are important for
showing that windability is preserved by circuits.

Consider a graph $G$ with four external edges $e_1, e_2, e_3, e_4$. For all $x_1, x_2, x_3, x_4$, let
$F(x_1, x_2, x_3, x_4)$ be the number of perfect matchings in $G$ that include the outgoing edges
$\{e_i \mid x_i = 1\}$. So $F(0,0,0,0)F(1,1,1,1)$ is the number of pairs of perfect matchings
$(M, M')$ such that $M$ includes all the external edges and $M'$ includes none. But for any
such pair $(M, M')$, the symmetric difference $M \triangle M'$ consists of cycles and paths, and the
path starting at $e_1$ ends at either $e_2, e_3$, or $e_4$, depending on the choice of $(M, M')$. Thus
$F(0,0,0,0)F(1,1,1,1)$ splits into three terms. Denote these by $B((0,0,0,0),(1,1,1,1), M)$
where $M$ is a partition of $\{1, 2, 3, 4\}$ into pairs: either $\{(1, 2), (3, 4)\}$ or $\{(1, 3), (2, 4)\}$
or $\{(1, 4), (2, 3)\}$. We can similarly define $B((1, 1, 0, 0), (0, 0, 1, 1), M)$ for example.

When $M \triangle M'$ contains a path $P$ from $e_1$ to $e_2$, the sets $M \triangle P$ and $M' \triangle P$ are also
perfect matchings - see Figure 3. The only external edges in $M \triangle P$ are $e_3$ and $e_4$, while the
only external edges in $M' \triangle P$ are $e_1$ and $e_2$. Thus $B((0,0,0,0),(1,1,1,1),\{\{1, 2\}, \{3, 4\}\})$
equals $B((1,1,0,0),(0,0,1,1),\{\{1, 2\}, \{3, 4\}\})$.

In this section, for simplicity, we will consider only even functions.

3.2 Definition

For any configuration $\mathbf{x} \in \{0, 1\}^J$ define $M_\mathbf{x}$ to be the set of partitions of $\{i \in J \mid x_i = 1\}$
into pairs. In particular, if $\sum_{i \in J} x_i$ is odd then $M_\mathbf{x} = \emptyset$.

A function $F : \{0, 1\}^J \to \mathbb{Q}^{\geq 0}$ is even-windable (with witness $B$) if there exist values
$\mathbf{B}(\mathbf{x}, \mathbf{y}, M) \geq 0$ for all $\mathbf{x}, \mathbf{y} \in \{0, 1\}^J$ and all $M \in M_{\mathbf{x} \oplus \mathbf{y}}$, i.e. all partitions $M$ of the set
$\{i \in J \mid x_i \neq y_i\}$ into pairs, satisfying:

**EW1.** $F(\mathbf{x})F(\mathbf{y}) = \sum_{M \in M_{\mathbf{x} \oplus \mathbf{y}}} \mathbf{B}(\mathbf{x}, \mathbf{y}, M)$ for all $\mathbf{x}, \mathbf{y} \in \{0, 1\}^J$, and

**EW2.** $\mathbf{B}(\mathbf{x}, \mathbf{y}, M) = \mathbf{B}(\mathbf{x} \oplus \mathbf{S}, \mathbf{y} \oplus \mathbf{S}, M)$ for all $\mathbf{x}, \mathbf{y} \in \{0, 1\}^J$ and all $\mathbf{S} \in M_{\mathbf{x} \oplus \mathbf{y}}$.

Note that in the second condition, $\mathbf{S}$ is a pair $\{i, j\}$ in $M$: we are swapping the values of
$x_i$ and $y_j$, and swapping the values of $x_j$ and $y_j$. By swapping a sequence of pairs, EW2 is
equivalent to

**EW2'.** $\mathbf{B}(\mathbf{x}, \mathbf{y}, M) = \mathbf{B}(\mathbf{x} \oplus \mathbf{S}_1 \oplus \cdots \oplus \mathbf{S}_k, \mathbf{y} \oplus \mathbf{S}_1 \oplus \cdots \oplus \mathbf{S}_k, M)$ for all $\mathbf{x}, \mathbf{y} \in \{0, 1\}^J$ and all
$\mathbf{S}_1, \ldots, \mathbf{S}_k \in M_{\mathbf{x} \oplus \mathbf{y}}$.

3.3 2-decompositions

Using pinnings, the even-windability conditions can be stated in a form that is sometimes
easier to check. A function $H : \{0, 1\}^J \to \mathbb{Q}^{\geq 0}$ has a 2-decomposition if there are values
$D(\mathbf{x}, M) \geq 0$, where $\mathbf{x}$ ranges over $\{0, 1\}^J$ and $M$ ranges over partitions of $J$ into pairs, such
that:

1. $H(\mathbf{x}) = \sum_M D(\mathbf{x}, M)$ for all $\mathbf{x}$, where the sum is over partitions of $J$ into pairs, and
2. $D(\mathbf{x}, M) = D(\mathbf{x} \oplus \mathbf{S}, M)$ for all $\mathbf{x}, M$ and all $\mathbf{S} \in M$.

In particular if $|J|$ is odd then the first condition forces $H$ to be identically zero.

A function $F$ is even-windable if and only if for all pinnings $G$ of $F$ the function $G\mathbf{H}$ has a
2-decomposition. For the forwards direction, given a witness $B$ that $F$ is even-windable, for
each $I \subseteq J$ and each $\mathbf{p} \in \{0, 1\}^I$ define $D_p(\mathbf{x}, M) = B((\mathbf{x}, \mathbf{p}), (\mathbf{x}, \mathbf{p}), M)$ for all $\mathbf{x} \in \{0, 1\}^{J \setminus I}$.
to obtain a 2-decomposition $D_p$ of the pinning of $F$ by $p$. For the backwards direction, for each $I \subseteq J$ and each $p \in \{0,1\}^J$, pick a 2-decomposition $D_p$ of the pinning of $F$ by $p$. For all $x,y \in \{0,1\}^J$, define $B(x,y,M) = D_p(x,M)$ where $p$ is the restriction of $x$ to $\{i \in J \mid x_i = y_i\}$ and $x'$ is the restriction of $x$ to $\{i \in J \mid x_i \neq y_i\}$. Then $B$ witnesses that $F$ is even-windable.

**Lemma 7.** Let $F : \{0,1\}^J \to \mathbb{Q}_{\geq 0}$ with $|J| \leq 3$. If $F$ is even then $F$ is even-windable.

**Proof.** Let $G : \{0,1\}^J \to \mathbb{Q}_{\geq 0}$ be a pinning of $F$.

If $I = \emptyset$ define $D(x,\emptyset) = G(x)$ where $x \in \{0,1\}^0$ is the empty configuration. Then $G(x) = \sum_M D(x,M)$ where $M$ ranges over the set $\{\emptyset\}$ of partitions of $I$ into pairs, so $D$ is a 2-decomposition of $G$.

If $|I| = 2$, let $i,j$ be the elements of $I$ and define $D(x,\{i,j\}) = G(x)$. For all $x \in \{0,1\}^J$ we have $G(x) = \sum_M D(x,M)$ where $M$ ranges over the set $\{\{i,j\}\}$ of partitions of $I$ into pairs, so $D$ is a 2-decomposition of $G$.

If $|I|$ is 1 or 3 then $G(x)$ and $G(\emptyset)$ cannot be simultaneously be non-zero because $G$ is a pinning of the even function $F$, and $\sum_{i \in I} x_i \equiv |I| + \sum_{i \in I} (1 - x_i) \pmod{2}$. Thus $G$ is identically zero. There are also no partitions of $I$ into pairs, so the empty function is a 2-decomposition of $G$.

**Lemma 8.** Even$_J$ and Odd$_J$ have a 2-decomposition whenever $|J|$ is even. Even$_J$ and Odd$_J$ are even-windable for any $J$.

**Proof.** First consider Even$_J$. Fix a partition $N$ of $J$ into pairs. Define

$$D(x,M) = \begin{cases} 1 & \text{if } M = N \text{ and } \sum_{i \in J} x_i \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

Then for all $x \in \{0,1\}^J$ we have Even$_J(x) = \sum_M D(x,M)$ (where the sum ranges over partitions $M$ of $J$ into pairs). Similarly for Odd$_J$, define

$$D(x,M) = \begin{cases} 1 & \text{if } M = N \text{ and } \sum_{i \in J} x_i \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

Then for all $x \in \{0,1\}^J$ we have Odd$_J(x) = \sum_M D(x,M)$.

Now consider a pinning $G : \{0,1\}^K \to \mathbb{Q}_{\geq 0}$ of Even$_J$ or Odd$_J$. If $|K|$ is odd then $G$ is identically zero, by the same argument used in Lemma 7. Otherwise, $G = G \emptyset$ is either Even$_K$ or Odd$_K$, which we showed have 2-decompositions.

The following argument gives a more difficult example of a 2-decomposition. It will be used later (in the proof of Lemma 17) to show that NAE$_J$ is windable.

**Lemma 9.** Let $J$ be a finite set with $|J|$ even. Then EvenNAE$_J$ has a 2-decomposition.

**Proof.** For each subset $I \subseteq J$ of even order fix a partition $M_I$ of $I$ into pairs. Set

$$D(x,M) = 2^{-k+2} \left| \left\{ I \subseteq J \mid |I| \text{ is even, } \sum_{i \in I} x_i \text{ and } \sum_{i \in J \setminus I} x_i \text{ are odd, and } M = M_I \cup M_{J \setminus I} \right\} \right|. $$

$S \in M_I \cup M_{J \setminus I}$ implies $S \subseteq I$ or $S \subseteq J \setminus I$. The conditions that $\sum_{i \in I} x_i$ and $\sum_{i \notin I \cup J} x_i$ are odd are therefore not affected by changing $x$ to $x \oplus S$. Thus $D(x \oplus S,M) = D(x,M)$ for all $S \in M$.

For any $x$, if EvenNAE$_J(x) = 0$ then $D(x,M) = 0$. If EvenNAE$_J(x) = 1$, pick $i,j$ with $x_i = 0$ and $x_j = 1$. For each of the $2^{k-2}$ subsets $I' \subseteq J \setminus \{i,j\}$ there is a unique set $I'' \subseteq \{i,j\}$ such that the order of $I = I' \cup I''$ is even and such that $\sum_{i \in I} x_i$ is odd and $\sum_{i \notin I'} x_i$ are odd. There are thus $2^{k-2}$ such subsets $I$ for each fixed $x$, which gives $\sum_M D(x,M) = 1$. So $D$ is a 2-decomposition of EvenNAE$_J$.

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3.4 Expressibility

We will show that the weight-function of any circuit using even-windable functions is even-windable. We will use a certain graph associated with a choice of matching of incidences.

Let $M$ and $E$ each be a set of disjoint pairs of some set. Define the link graph $L_E(M)$ to be the multigraph on the vertex set $\bigcup_{S \in M} S$ with edge set the disjoint union of $M$ and $\{\{i,j\} \in E \mid i,j \in \bigcup_{S \in M} S\}$ (so edges in $M \cap E$ give pairs of parallel edges in $L_E(M)$).

Note that for each vertex $i$ of $L_E(M)$, the degree of $i$ is two if $\{i,j\} \in E$ for some $j \in \bigcup_{S \in M} S$, and otherwise $i$ has degree one. So $L_E(M)$ consists of paths and cycles.

We will use this graph later for the analysis of the near-assignments chain. For now, consider an assignment $x$ of some circuit with internal edges $E$ and external edges $A$, and let $M \in M_x$. For any $i$ not in $A$ with $x_i = 1$, the unique $j$ with $\{i,j\} \in E$ satisfies $x_j = 1$. This means that $i \in \bigcup_{S \in M} S$ has degree 1 in $L_E(M)$ if and only if $i \in A$. So every path component of $L_E(M)$ ends in $\{i \in A \mid x_i = 1\}$, and every such $i$ is at the end of a path. See Figure 4.

Lemma 10. Let $\phi$ be a circuit using only weight-functions that are even-windable. The weight-function of $\phi$ is even-windable.

Proof. Recall that $V,J,J_v,A,E,F_v$ denote vertices, incidences, vertices’ incidences, external edges, internal edges, and vertices’ weight-functions. For each $v \in V$ pick a function $B_v$ witnessing that $F_v$ is even-windable.

Consider a set $M'$ of disjoint pairs of $J$. We will say that $M'$ induces the set of pairs $\{i,j\} \subseteq A$ such that there is a path from $i$ to $j$ in $L_E(M')$.

For all $x,y \in \{0,1\}^A$ and all $M \in M_{x \oplus y}$ define

$$B(x,y,M) = \sum_{x',y'} \sum_{\{M_v\} \text{ inducing } M} \prod_{v \in V} B_v(x'|J_v,y'|J_v,M_v)$$

where:

- $\sum_{x',y'}$ denotes the sum over assignments $x'$ and $y'$ extending $x$ and $y$ respectively.
- $\sum_{\{M_v\} \text{ inducing } M}$ denotes the sum over all choices of $M_v \in M_{(x' \oplus y')|J_v}$ for each $v \in V$, such that $\bigcup_{v \in V} M_v$ induces $M$. 

Figure 4: A circuit $\phi$ and a link graph $L_E(M)$ for some $M \in M_x$ where $x$ is an assignment of $E^\phi$. (In particular, the $M$ drawn is a union of partitions $M_v \in M_{x|J_v}$.) Circles represent vertices of the circuit. Squares are incidences $i \in J^\phi$ of the circuit, and are filled black where $x_i = 1$. Elements of $M$ are drawn as thick black lines. Elements $\{i,j\} \in E$ are drawn as thin lines.
For all \( x, y \in \{0,1\}^A \) we have
\[
\sum_{M \in \mathcal{M}_{x \oplus y}} B(x, y, M) = \sum_{M \in \mathcal{M}_{x \oplus y}} \sum_{\{M_e\} \text{ inducing } M} \prod_{v \in V} B_v(x'|J_v,y'|J_v,M_v)
= \sum_{\{M_e\} \text{ inducing } M} \prod_{v \in V} B_v(\{x'|J_v\},\{y'|J_v\},M_v)
= \sum_{\{M_e\} \text{ inducing } M} \prod_{v \in V} F_v(x'|J_v)F_v(y'|J_v)
= \sum_{\{M_e\} \text{ inducing } M} \prod_{v \in V} F_v(x'|J_v)F_v(y'|J_v)
= \lbrack \phi \rbrack (x) \lbrack \phi \rbrack (y).
\]
Here \( \sum_{\{M_e\}} \) denotes the sum over all choices of \( M_v \in \mathcal{M}_{\{x'|y'\}|J_v} \) for each \( v \in V \): the sum over \( M \) eliminates the condition that \( \bigcup_{v \in V} M_v \) induces \( M \).

Now fix \( x, y \in \{0,1\}^A \) and \( S = \{i,j\} \in \mathcal{M}_{x \oplus y} \). For any choice of \( \{M_e\} \) inducing \( M \), there is a unique path component \( P_{\{M_e\}} \) (also depending on \( x, y, S \)) from \( i \) to \( j \) in \( L_E(\bigcup_{v \in V} \bigcup_{S \in \mathcal{M}_v} S) \). By construction of the link graph, the vertices of \( P_{\{M_e\}} \) are a union of pairs \( S \in \bigcup_{v \in V} \mathcal{M}_v \). In particular, for each \( v \in V \), the intersection \( P_{\{M_e\}} \cap J_v \) is a union of pairs \( S \in \mathcal{M}_v \). Using EW2, we have
\[
B(x, y, M) = \sum_{\{M_e\} \text{ inducing } M} \prod_{v \in V} B_v(\{x'|J_v\},\{y'|J_v\},M_v)
= \sum_{\{M_e\} \text{ inducing } M} \prod_{v \in V} B_v(\{x' \oplus P_{\{M_e\}}\}|J_v,\{y' \oplus P_{\{M_e\}}\}|J_v,M_v)
= B(x \oplus S,y \oplus S,M).
\]
So \( B \) witnesses that \( \lbrack \phi \rbrack \) is even-windable. \( \square \)

3.5 The near-assignments Markov chain

Throughout this subsection fix an even-windable weight-function \( F : \{0,1\}^J \to \mathbb{Q}^\geq 0 \) and a partition \( E \) of \( J \) into pairs. This can be thought of as a circuit with one vertex. We will define and study the near-assignments Markov chain for \( (F,E) \).

Set \( n = |J| \). For each \( k \geq 0 \) let \( \Omega_k \) denote the set of \( k \)-assignments of \( J \) with respect to \( E \) that satisfy \( F(x) > 0 \). The state-space is \( \Omega = \Omega_0 \cup \Omega_2 \). The transitions are Metropolis updates at state to distance two. More specifically, the transition probability from \( x \) to \( y \) is defined to be
\[
P(x,y) = \begin{cases} 
\frac{1}{n} \min(1, F(y)/F(x)) & \text{if } d(x,y) = 2 \\
1 - \frac{1}{2n} \sum_{y' : d(x,y') = 2} \min(1, F(y')/F(x)) & \text{if } y = x \\
0 & \text{otherwise.}
\end{cases}
\]

(We will not consider the initial state to be part of the Markov chain itself: the Markov chain is completely described by the matrix \( P \in \mathbb{R}^{|\Omega| \times |\Omega|} \).) Define a probability distribution \( \pi \) on \( \Omega \) by
\[
\pi(x) = F(x) \sum_{y \in \Omega} F(y) \quad (x \in \Omega).
\]

By abuse of notation we will also denote \( \sum_{x \in X} \pi(x) \) by \( \pi(X) \) for subsets \( X \subseteq \Omega \). By adapting the arguments of [19], we will show:

**Theorem 11.** For all \( x \in \Omega \) and all non-negative integers \( t \), we have
\[
\frac{1}{2} \sum_{y \in \Omega} |P^t(x,y) - \pi(y)| \leq \frac{1}{2} \pi(x)^{t-1/2} \exp(-t\pi(\Omega_0)^2/n^4)
\]
Figure 5: The link graph $L_E(M)$ for some 2-assignment-matching $M$. Squares are elements of $J$. Elements of $M$ are drawn as thick black lines. Elements $\{i,j\} \in E$ are drawn as thin lines.

Here $P^t$ denotes the $t$’th matrix power. The factor of $\frac{1}{2}$ is convention: the left hand side is called total variation distance.

We will use a congestion argument, with the following definitions. A flow-path is a directed path $\gamma$ in the transition graph $[3]$, equipped with a weight $wt(\gamma)$, and also equipped with a label so that a set of paths can include the same path more than once. A flow $\Gamma$ from $X \subseteq \Omega$ to $Y \subseteq \Omega$ is a set of flow-paths which each start in $X$ and end in $Y$, satisfying

$$\sum_{\text{paths } \gamma \in \Gamma \text{ from } x \text{ to } y} wt(\gamma) = \pi(x)\pi(y) \quad \text{for all } x \in X \text{ and } y \in Y.$$ 

The congestion of a flow $\Gamma$ is defined to be

$$\rho(\Gamma) = \max_{\text{transitions } (x,y) \in \Gamma} \frac{1}{\pi(x)P(x,y)\pi(y)} \sum_{\gamma \in \Gamma \text{ with } \gamma \ni (x,y)} wt(\gamma).$$

Be aware that this is not the same as the definition in [24]: we are using a set of weighted paths rather than an assignment of weights to paths. When applying the results of [24] in the proof of Theorem 11 we will need to sum the total weight along each (unlabelled) path.

In the following arguments we will often use $k$-assignments (with respect to $E$) of the form $x \oplus y$ for $x \in \Omega_k$ and $y \in \Omega_k$. Note that we do not require $F(x) > 0$ for $k$-assignments $x$, though we do require $F(x) > 0$ for $x \in \Omega_k$ ($\Omega_k$ is the set of “satisfying” $k$-assignments). For any non-negative integer $k$, a $k$-assignment-matching (with respect to $E$) is a set $M$ of disjoint pairs of $J$ such that exactly $k$ edges $\{i,j\} \in E$ have exactly one endpoint, $i$ or $j$, in $\bigcup_{S \in M} S$. In other words, the characteristic vector of $\bigcup_{S \in M} S$ is a $k$-assignment.

Consider a $k$-assignment-matching $M$. Recall the definition of the link graph $L_E(M)$ given in Section 3.4, which consists of cycles and paths. For any $i$ with $i \in \bigcup_{S \in M} S$, the unique $j$ with $\{i,j\} \in E$ satisfies $j \in \bigcup_{S \in M} S$, except for exactly $k$ values $i$. Thus $L_E(M)$ has precisely $k/2$ path components. See Figure 5.

For all non-negative integers $k$ define

$$Z_k = \sum_{x \in \Omega_k} F(x).$$

(This is $Z_k(\phi)$ if we consider $F$ as a one-vertex circuit $\phi$.)

**Lemma 12.** $Z_0Z_4 \leq Z_2Z_2$.

**Proof.** We have

$$Z_0Z_4 = \sum_{x \in \Omega_0} F(x)F(y) = \sum_{x \in \Omega_0} \sum_{y \in \Omega_0} \sum_{M \in \mathcal{M}_{x \oplus y}} B(x,y,M).$$

$^2$the directed graph with vertex set $\Omega$ and an arc $(x,y)$ whenever $P(x,y) > 0$. 

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For each 4-assignment-matching \( M \), pick a path component of \( L_E(M) \) and let \( H_M \) be the set of vertices of this component. Let \( B \) be a function witnessing that \( F \) is even-windable. Each \( H_M \) is a union of path components \( M \) so by EW2',

\[
Z_0Z_1 = \sum_{x \in \Omega_0} \sum_{y \in \Omega_2} B(x \oplus H_M, y \oplus H_M, M).
\]

But \((x \oplus H_M, y \oplus H_M, M)\) determines \((x,y,M)\), and \( x \oplus H_M, y \oplus H_M \in \Omega_2 \). So

\[
Z_0Z_1 \leq \sum_{x' \in \Omega_2} \sum_{y' \in \Omega_2} B(x', y', M) = \sum_{x' \in \Omega_2} F(x')F(y') = Z_2Z_2.
\]

\[\square\]

**Lemma 13.** Assume \( Z_0 > 0 \). There is a flow \( \Gamma_0 \) from \( \Omega_0 \) to \( \Omega \), using flow-paths of length at most \( n/2 \), and with congestion at most \( \frac{2}{n}\pi(\Omega_0) \).

**Proof.** We will first construct a “winding” enumeration \( S(M, 1), \ldots, S(M, |M|) \) of each 0- or 2-assignment-matching \( M \). The property we will need is that for each \( 0 \leq k \leq |M| \), the characteristic vector of \( S(M, 1) \cup \cdots \cup S(M, k) \) is a 0- or 2-assignment.

First define the final pair \( T(M) = S(M, |M|) \) for all non-empty 0- or 2-assignment-matchings \( M \) as follows. If \( M \) is a non-empty 0-assignment-matching (so \( L_E(M) \) consists of cycles), pick any vertex \( i \in L_E(M) \). If \( M \) is a non-empty 2-assignment-matching, pick an endpoint \( i \) of the unique path component in \( L_E(M) \). In either case let \( j \) be the unique index with \((i,j) \in M \), and set \( T(M) = \{i,j\} \). In any case \( L_E(M \setminus \{T(M)\}) = L_E(M) \setminus \{i,j\} \) has at most one path component.

So \( M \setminus \{T(M)\} \) is a 0- or 2-assignment-matching. By induction on \( |M| - k \) define \( S(M, k) = T(M \setminus \{S(M, k+1), \ldots, S(M, |M|)\}) \).

So \( M \setminus \{S(M, k+1), \ldots, S(M, |M|)\} \) is always a 0- or 2-assignment-matching. This completes the construction of \( S(M, k) \).

Let \( B \) be a function witnessing that \( F \) is even-windable. Let \( \Gamma_0 \) be the set consisting of a flow-path \( \gamma_{x,y,M} \) for each \( x \in \Omega_0 \) and \( y \in \Omega \) and \( M \in \mathcal{M}_{x,y} \), where \( \gamma_{x,y,M} \) is the flow-path

\[
x = x \oplus J_{M,0} \rightarrow x \oplus J_{M,1} \rightarrow \cdots \rightarrow x \oplus J_{M,|M|} = y
\]

equipped with weight \( B(x, y, M) / (Z_0 + Z_2)^2 \) and label \((x,y,M)\), where \( J_{M,k} \) denotes the characteristic vector of \( S(M, 1) \cup \cdots \cup S(M, k) \).

\( \Gamma_0 \) is a flow from \( \Omega_0 \) to \( \Omega \) because for all \( x \in \Omega_0 \) and \( y \in \Omega_2 \) we have

\[
\sum_{M \in \mathcal{M}_{x,y}} B(x, y, M) / (Z_0 + Z_2)^2 = F(x)F(y) / (Z_0 + Z_2)^2 = \pi(x)\pi(y).
\]

The congestion of \( \Gamma_0 \) is

\[
\rho(\Gamma_0) = \max_{\text{transitions } (x,z') \in \pi(z)P(z,z')} \sum_{\gamma \in \Gamma_0 \text{ with } (x,z') \in \gamma} \text{wt}(\gamma)
\]

But \( \pi(z)P(z,z') = \frac{2}{n^2} \min(\pi(z), \pi(z')) \), so

\[
\rho(\Gamma_0) \leq \max_{x \in \Omega} \frac{n^2}{2} \cdot \pi(z) \sum_{\gamma \in \Gamma_0 \text{ with } x \in \gamma} \text{wt}(\gamma)
\]

\[
= \max_{x \in \Omega} \frac{n^2}{2} F(x) / (Z_0 + Z_2) \sum_{y \in \Omega_2} \sum_{M \in \mathcal{M}_{x,y}} B(x, y, M)
\]

\[\text{14}\]
In the last summation, $z \in \gamma_{x,y,M}$ implies $z = x \oplus J_{M,k}$ for some $k$, so by EW2' we have $B(x,y,M) = B(z,z \oplus w,M)$ where $w = x \oplus y$. Thus,

$$\rho(\Gamma_0) \leq \max_{x \in \Omega} \frac{n^2}{2F(z)(Z_0 + Z_2)} \sum_{g \in \Omega_0} \sum_{x \in \Omega_0} \sum_{M \in M_w} B(z,z \oplus w,M)$$

For each $(z,w,M)$ with $M \in M_w$, the only values of $x$ such that $z \in \gamma_{x,y;w,M}$ are the $|M| + 1$ values $z \oplus J_{M,0}, \ldots, z \oplus J_{M,|M|}$. Thus,

$$\rho(\Omega_0) \leq \max_{x \in \Omega} \frac{n^2}{2F(z)(Z_0 + Z_2)} \sum_{g \in \Omega_0} (|M| + 1) \sum_{M \in M_w} B(z,z \oplus w,M)$$

Using $z \oplus w \in \Omega_0 \cup \Omega_2 \cup \Omega_4$,

$$\rho(\Omega_0) \leq \frac{n^2}{2} \sum_{g \in \Omega_0} (|M| + 1) \sum_{M \in M_w} B(z,z \oplus w,M)$$

If $Z_2 = 0$ then by Lemma 12 we also have $Z_4 = 0$, so the congestion is at most $n^3/2$. Otherwise by Lemma 12 we have $Z_4/Z_2 \leq Z_2/Z_0$ and

$$\frac{Z_0 + Z_2 + Z_4}{Z_0 + Z_2} \leq 1 + \frac{Z_4}{Z_2} \leq 1 + \frac{Z_2}{Z_0} = 1/\frac{Z_0 + Z_2}{Z_0} = 1/\pi(\Omega_0).$$

**Lemma 14.** Assume $Z_0 > 0$. There is a flow $\Gamma$ from $\Omega$ to $\Omega$, using flow-paths of length at most $n$, and with congestion at most $n^3/\pi(\Omega_0)^2$.

**Proof.** As in [20], we will randomly route through $\Omega_0$.

For each pair of flow-paths $g, g'$ starting at the same state $y$, construct a flow-path $\gamma(g,g')$ by appending $g'$ to the reverse of $g$, and assigning a weight of $wt(g)wt(g')/\pi(y)\pi(\Omega_0)$, and assigning the label $(g,g')$. Let $\Gamma_0$ be the set of flow-paths given by Lemma 12. Let $\Gamma_0(y,x)$ denote the set of flow-paths in $\Gamma_0$ starting at $y$ and ending at $x$. Let $\Gamma$ denote the set of flow-paths $\gamma(g,g')$ with $g \in \Gamma_0(y,x)$ and $g' \in \Gamma_0(y,z)$ for some $x, z \in \Omega$ and $y \in \Omega_0$. Then $\Gamma$ is a flow from $\Omega$ to $\Omega$ because for all $x, z \in \Omega$ we have

$$\sum_{y \in \Omega_0} \sum_{g \in \Gamma_0(y,x)} \sum_{g' \in \Gamma_0(y,z)} \frac{wt(g)wt(g')}{\pi(y)\pi(\Omega_0)} = \sum_{y \in \Omega_0} \pi(x)\pi(y)\pi(y)\pi(z)/\pi(y)\pi(\Omega_0) = \pi(x)\pi(z).$$

Letting $(w,w')$ denote an arbitrary transition, the congestion of $\Gamma$ is

$$\rho(\Gamma) = \max_{(w,w')} \frac{1}{\pi(w)P(w,w')} \sum_{x \in \Omega} \sum_{y \in \Omega_0} \sum_{g \in \Gamma_0(y,x)} \sum_{g' \in \Gamma_0(y,z)} \frac{wt(g)wt(g')}{\pi(y)\pi(\Omega_0)}$$

such that $(w,w') \in \gamma(g,g')$.
By symmetry,
\[ \rho(\Gamma) = 2 \max_{(w,w')} \frac{1}{\pi(w)P(w,w')} \sum_{x,z \in \Omega} \sum_{y \in \Omega_0} \frac{\text{wt}(g) \text{wt}(g')}{\pi(y)\pi(\Omega_0)} \]
\[ = 2 \max_{(w,w')} \frac{1}{\pi(w)P(w,w')} \sum_{x,z \in \Omega} \sum_{y \in \Omega_0 \text{ such that } (w,w') \in g'} \frac{\pi(x)\pi(y) \text{wt}(g')}{\pi(y)\pi(\Omega_0)} \]
\[ = 2 \max_{(w,w')} \frac{1}{\pi(w)P(w,w')} \sum_{x,z \in \Omega} \sum_{y \in \Omega_0 \text{ such that } (w,w') \in g'} \text{wt}(g')/\pi(\Omega_0) \]
\[ = 2 \rho(\Gamma_0)/\pi(\Omega_0) \leq n^3/\pi(\Omega_0)^2 \]
by Lemma 13.

The remaining task is to relate the congestion to Markov chain mixing.

**Theorem 11.** For all \( x \in \Omega \) and all non-negative integers \( t \), we have
\[ \frac{1}{2} \sum_{y \in \Omega} |P^t(x,y) - \pi(y)| \leq \frac{1}{2} \pi(x)^{-1/2} \exp(-t\pi(\Omega_0)^2/n^4). \]

**Proof.** The transition matrix \( P \) is reversible relative to \( \pi \): it obeys the detailed balance condition
\[ \pi(y)P(y,z) = \pi(z)P(z,y) \text{ for all } y, z \in \Omega. \]

We have
\[ P(x,x) \geq 1 - \frac{2}{n^2} \left( \frac{n}{2} \right) \geq 1/n \quad \text{for all } x \in \Omega. \]  
(3)

In particular, the Markov chain is aperiodic. Also, by Lemma 14 there exists a flow \( \Gamma \) from \( \Omega \) to \( \Omega \), which implies that the Markov chain is connected. This allows us to use the results from [24] and [12]. \( P \) has eigenvalues
\[ 1 = \lambda_0 > \lambda_1 \geq \ldots \lambda_{|\Omega|-1} \geq -1. \]

By setting \( f(\gamma) \) in [24, Corollary 6'] to be the sum of the weight of flow-paths in \( \Gamma \) whose underlying directed path is \( \gamma \), we have
\[ \lambda_1 \leq 1 - \frac{1}{\rho(\Gamma)n} \leq 1 - \pi(\Omega_0)^2/n^4. \]

using Lemma 13 By (3) and equation 1 of [17] we have
\[ -\lambda_{|\Omega|-1} \leq 1 - 2/n \leq \lambda_1. \]

By [12, Proposition 3],
\[ \frac{1}{2} \sum_{y \in \Omega} |P^t(x,y) - \pi(y)| \leq \frac{1}{2} \left( 1 - \frac{\pi(x)}{\pi(\Omega)} \right) \max(\lambda_1, -\lambda_{|\Omega|-1})^t \]
\[ \leq \frac{1}{2} \pi(x)^{-1/2} \exp(-t\pi(\Omega_0)^2/n^4). \]
4 Windable functions

In this section we extend the analysis of even-windable functions to windable functions. The definition of windability is a natural extension of even-windability, but turns out not to give much extra generality.

For all \( F : \{0,1\}^J \to \mathbb{Q}^{\geq 0} \) define \( F_\oplus : \{0,1\}^{1+J} \to \mathbb{Q}^{\geq 0} \) by

\[
F_\oplus(p;x) = \begin{cases} F(x) & \text{if } p + \sum_{i \in J} x_i \text{ is even} \\ 0 & \text{otherwise} \end{cases} \quad (p \in \{0,1\}, x \in \{0,1\}^J).
\]

Lemma 15. \( F : \{0,1\}^J \to \mathbb{Q}^{\geq 0} \) is windable if and only if \( F_\oplus \) is even-windable.

Proof. \((\Rightarrow)\) Pick an ordering of \( J \). Consider a partition \( M \) of a subset \( I \subseteq J \) into singletons \( \{a_1\}, \ldots, \{a_k\} \) and pairs \( S_1, \ldots, S_t \). Define \( \mu(M) \), when \( |I| \) is even, to be the union of \( \{S_1, \ldots, S_t\} \) with a partition (depending only on \( M \)) of \( \{a_1, \ldots, a_k\} \) into pairs. Define \( \mu(M) \), when \( |I| \) is odd, to be the union of \( \{S_1, \ldots, S_t\} \) with a partition of \( \{1, a_1, \ldots, a_k\} \) into pairs. Let \( B \) be a witness that \( F \) is windable. For all \( (p;x), (q;y) \in \{0,1\}^{1+J} \) and all \( M \in \mathcal{M}_{(p;x) \oplus (q;y)} \), define

\[
B'(p;x), (q;y), M) = \begin{cases} \sum_{M' : \mu(M') = M} B(x,y,M') & \text{if } p + \sum_{i \in J} x_i \text{ and } q + \sum_{i \in J} y_i \text{ are even} \\ 0 & \text{otherwise} \end{cases}
\]

For all \( S \in M = \mu(M') \), if we let \( S' = S \setminus \{1\} \) then \( B(x \oplus S', y \oplus S', M') = B(x,y,M') \). So \( B' \) witnesses that \( F_\oplus \) is even-windable.

\((\Leftarrow)\) For all sets \( M \) of disjoint pairs of \( 1+J \) define \( \nu(M) \) to be \( \{S \setminus \{1\} \mid S \in M \} \). Let \( B \) be a witness that \( F_\oplus \) is windable. For all \( x,y \in \{0,1\}^J \) and all \( M \in \mathcal{M}_{x \oplus y} \), define

\[
B'(x,y,M) = \sum_{p,q=0} \sum_{M' : \nu(M') = M} B(p;x), (q;y), M')
\]

For all \( S \in M = \nu(M') \), let \( S' = S \) if \( |S| = 2 \) and \( S' = S \cup \{1\} \) otherwise. Then \( B(x \oplus S', y \oplus S', M') = B(x,y,M') \). So \( B' \) witnesses that \( F \) is windable. \( \square \)

Lemma 16. Let \( \phi \) be a circuit using only weight-functions that are windable. The weight-function of \( \phi \) is windable.

Proof. Replace each constraint \( F_v \) by \( (F_v)_\oplus \), rename the new incidences \( p_v, v \in V(\phi) \), and add a constraint \( \text{Even}_{1+J} \) where \( P = \{p_v \mid v \in V(\phi)\} \). This produces a circuit \( \phi_\oplus \) with \( [\phi_\oplus] = [\phi]_\oplus \). By Lemmas 15, 10, and 9 we find that \( [\phi_\oplus] \) is even-windable. So by Lemma 15 again, \( [\phi] \) is windable. \( \square \)

Lemma 17. For any \( J \), the weight-functions \( \text{Even}_J, \text{Odd}_J \), and \( \text{NAE}_J \) are windable.

Proof. By Lemma 8 there is a witness \( B \) that \( \text{Even}_J \) is even-windable. Extending \( B \) by setting \( B(x,y,M) = 0 \) for all \( M \in \mathcal{M}_{x \oplus y} \setminus \mathcal{M}_{x \oplus y} \) we get a witness \( B' \) that \( \text{Even}_J \) is windable. Similarly, \( \text{Odd}_J \) is even-windable by Lemma 8 and it is therefore windable.

For \( \text{NAE}_J \), by Lemma 15 it suffices to show that the weight-function \( (\text{NAE}_J)_\oplus \) is even-windable. Let \( I \subseteq 1+J \), let \( p \in \{0,1\}^I \), let \( K = (1+J) \setminus I \) and let \( G : \{0,1\}^K \to \mathbb{Q}^{\geq 0} \) be the pinning of \( (\text{NAE}_J)_\oplus \) by \( p \). We wish to show that \( G_G \) has a 2-decomposition. If \( |K| \) is odd then \( G_G \) is identically zero so has a 2-decomposition. We can therefore assume that \( |K| \) is even.

Let \( c \in \{0,1\} \) be equal to \( |J| \) modulo 2. \( (\text{NAE}_J)_\oplus \) is the weight-function corresponding to the relation \( \text{Even}_{1+J} \setminus X \) where \( X \) consists of two configurations at distance \( |J| + c \). Specifically, \( X \) contains the all-zeros configurations of \( 1+J \), and also contains \( (c;x) \) where \( x \) is the all-ones configuration. We first argue that in all cases, \( G_G \) is either \( \text{Even}_K \) or \( \text{Odd}_K \), or a flip of \( \text{EvenNAE}_K \).
Figure 6: A circuit with weight-function \( F \) with \( F(0,0) = 2 \) and \( F(1,1) = 1 \) and \( F(0,1) = F(1,0) = 0 \). Vertices represent “exact-one” constraints \( \{(1,0,0), (0,1,0), (0,0,1)\} \).

If \( \sum_{i \in I} p_i \) is odd, then \( G \) takes the value 1 precisely on Odd\(_K\) \( \setminus X' \) where \( X' \) consists of at most one configuration \( x \in \text{Odd}_K \). If \( X' = \emptyset \) then \( G \overrightarrow{G} = \text{Odd}_K \). If \( X' = \{x\} \) then \( G \overrightarrow{G} \) is the flip of EvenNAE\(_K\) by \( x \).

If \( \sum_{i \in I} p_i \) is even, then \( G \) takes the value 1 precisely on Even\(_K\) \( \setminus X' \) where \( X' \) consists of at most two configurations in Even\(_K\). If \( |X'| \leq 1 \) we are done by the same argument as the previous paragraph: \( G \overrightarrow{G} \) is either Even\(_K\) or a flip of EvenNAE\(_K\). If \( |X'| = 2 \) then \( X' \) consists of two configurations \( x, y \) with \( d(x, y) = |J| + c \). But \( |J| + c \leq |K| \leq |J| + 1 \), and \( |K| \) and \( |J| + c \) are both even, so \( |K| = |J| + c \). Thus \( y = x \), and again \( G \overrightarrow{G} \) is a flip of EvenNAE\(_K\).

By Lemma 8 the weight-functions Even\(_K\) and Odd\(_K\) have 2-decompositions. So we only need to check the last case where \( G \overrightarrow{G} \) is a flip, by \( z \in \{0,1\}^K \) say, of EvenNAE\(_K\). Let \( D \) be a 2-decomposition of EvenNAE\(_K\) given by Lemma 9. Define \( D'(x, M) = D(x \oplus z, M) \) for all \( x \in \{0,1\}^K \) and for all partitions \( M \) of \( K \) into pairs. For all \( x \in \{0,1\}^K \) we have \( G \overrightarrow{G}(x) = \text{EvenNAE}_{K}(x \oplus z) = \sum_M D(x \oplus z, M) = \sum_M D'(x, M) \), where \( M \) ranges over partitions of \( K \) into pairs. So \( D' \) is a 2-decomposition of \( G \overrightarrow{G} \).

5 Strictly terraced functions

5.1 Idea

To apply Theorem 11 to Holant problems, the challenge is to find a class of circuits for which the ratio of the weight of 2-assignments to the weight of 0-assignments is polynomially bounded in the size of the Holant instance.

The weight of 2-assignments of a closed circuit can be written in terms of the weight-functions obtained by breaking two edges; the challenge then reduces to trying to find a bound on the ratios \( F(x)/F(y) \) between the values in these weight-functions, when \( F(y) \neq 0 \).

It is instructive to consider multiplication of two-by-two matrices. To see the relationship between multiplication of matrices and circuits (in the form of read-twice pps-formulas), for matrices \( M \) with rows and columns indexed by \( \{0,1\} \), define \( F_M : \{0,1\}^2 \to \mathbb{Q}_{\geq 0} \) by \( F_M(i,j) = M_{i,j} \); then \( F_{MN}(i,k) = \sum_j F_M(i,j) F_N(j,k) \).

Matrix multiplication can produce exponentially-large ratios: for any \( x, y > 0 \), we have

\[
\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} x^n & y(x^{n-1} + \cdots + 1) \\ 0 & 1 \end{pmatrix}
\]

and \( x^n/1 \) is exponentially large if \( x > 1 \).

In fact, the matrix \( \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \) corresponds to the circuit depicted in Figure 6 using “exact-one” constraints \( \{(1,0,0), (0,1,0), (0,0,1)\} \), which can be used to construct counterexamples to the bound 1 on nearly perfect matchings 3.

We might guess that exponentially-large ratios can only be produced by matrix multiplication when the zero entry in the matrix is surrounded by values that are different. And indeed this property of being “strictly terraced” turns out to give some control over ratios. For strictly terraced functions, the worst ratio in a weight-function is bounded by the sum of the worst ratios that can be obtained by mixing the individual functions with parity relations.
5.2 Definitions

A function $F : \{0,1\}^J \to \mathbb{Q}^{\geq 0}$ is strictly terraced if

$$F(x) = 0 \implies F(x \oplus e_i) = F(x \oplus e_j)$$

for all $x \in \{0,1\}^J$ and all $i, j \in J$.

For all weight-functions $F$ that are not identically zero, a parity-weight-function of $F$ is a constant multiple of the weight-function of a circuit using one $F$ constraint and such that all other constraints are parity relations. Define

$$\theta(F) = \max \left\{ \frac{F'(0)}{F'(1)} \mid F' : \{0,1\} \to \mathbb{Q}^{\geq 0} \text{ is a parity-weight-function} \right\}.$$

We extend $\theta$ to all weight-functions $F$ by setting $\theta(F) = 0$ if $F$ is identically zero.

We can show that $\theta$ is well-defined using the following operation. For any circuit $\phi$ and any internal edge $e \in E^\phi$ between incidences $i_u \in J_u$ and $i_v \in J_v$, with $u, v \in V$ (not necessarily distinct), define the contraction of $\phi$ by $e$ to be the circuit $\phi'$ obtained by replacing $u$ and $v$ by a vertex $w$ with incidences $(J_u \cup J_v) \setminus \{i_u, i_v\}$ and equipping $w$ with the weight-function of the circuit with constraints $F_u$ and $F_v$, edge $e$, and external edges $(J_u \cup J_v) \setminus \{i_u, i_v\}$.

**Lemma 18.** Let $\phi$ be a connected circuit whose constraints are all parity relations. The weight-function of $\phi$ is a constant multiple of a parity constraint.

**Proof.** By induction on the number of edges of $\phi$, it suffices to show that contracting a single edge of $\phi$ leaves only parity constraints (up to multiplication by constants).

Consider the case that the edge $\{i, j\}$ goes between distinct vertices, which are equipped with a $\text{Even}_{\{i\} \cup J}$ constraint and a $\text{Even}_{\{j\} \cup J}$ constraint. Then contraction gives a copy of $\text{Even}_{\{i,j\}}$, because $\text{Even}_{\{i,j\}}(x, y) = \sum_{t} \text{Even}_{1+J}(t; x) \text{Even}_{1+J}(t; y)$ for all $x \in \{0,1\}^J$ and all $y \in \{0,1\}^J$. Similarly $\text{Odd}_{\{i\} \cup J}$ and $\text{Odd}_{\{j\} \cup J}$ produce $\text{Even}_{\{i,j\}}$, while $\text{Even}_{\{i\} \cup J}$ and $\text{Odd}_{\{i\} \cup J}$ produce $\text{Odd}_{\{i,j\}}$.

If the edge $\{i, j\}$ is a loop on a vertex with constraint $\text{Even}_{\{i,j\}}$, contracting $\{i, j\}$ produces $2\text{Even}_J$. Similarly $\text{Even}_{\{i,j\}} \cup J$ produces $2\text{Odd}_J$.

Note that contracting an edge does not affect the weight-function of a circuit. By contracting edges between parity relations, the circuits appearing in the definition of a parity-weight-function can be rewritten not to use any edges except external edges and edges incident to the $F$ constraint. For fixed $F$ there are therefore a finite number of equivalence classes of parity-weight-functions $F' : \{0,1\} \to \mathbb{Q}^{\geq 0}$ with $F'(1) > 0$, under the equivalence relation of multiplication by constants. Thus the maximum in the definition of $\theta(F)$ is taken over a finite set, which can be seen to be non-empty if $F$ is not identically zero (if $F(x) > 0$ for some $x \in \text{Even}_J$ then the function $F' : \{0,1\}^J \to \mathbb{Q}^{\geq 0}$ defined by $F'(t) = \sum_{x \in \{0,1\}^J} \text{Odd}_1(t)F(x)\text{Even}_J(x)$ satisfies $F'(1) > 0$, and if $F(x) > 0$ for some $x \in \text{Odd}_J$ then the function $F' : \{0,1\}^J \to \mathbb{Q}^{\geq 0}$ defined by $F'(t) = \sum_{x \in \{0,1\}^J} \text{Odd}_1(t)F(x)\text{Odd}_J(x)$ satisfies $F'(1) > 0$).

Note that if $G$ is a parity-weight-function of $F$, then $\theta(G) \leq \theta(F)$. In particular if $G$ is a pinning of $F$ then $\theta(G) \leq \theta(F)$. Also, since the disequality relation $\text{Odd}_2 = \{(0,1), (1,0)\}$ is a parity relation, it is not important that we took $F'(0)/F'(1)$ rather than $F'(1)/F'(0)$ in the definition of $\theta$.

5.3 Examples

A relation $R \subseteq \{0,1\}^J$ is coindependent if for all $x \in \{0,1\}^J \setminus R$ we have $x \oplus e_i \notin R$ for all indices $i$. For example, the disequality relation $\{(0,1), (1,0)\}$ is coindependent. Any coindependent relation $R$ gives an example $R$ of a strictly terraced weight-function.

**Lemma 19.** For all finite sets $J$, the functions $\text{Even}_J$, $\text{Odd}_J$ and $\text{NAE}_J$ are strictly terraced. Also, $\theta(\text{Even}_J) = \theta(\text{Odd}_J) = 0$, and $\theta(\text{NAE}_J) \leq 3$. 

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Proof. The first statement follows from the fact that the corresponding relations are coindependent. To show \( \theta(\text{Even}) = \theta(\text{Odd}) = 0 \), note that by Lemma 13 a parity-weight-function of a parity relation must be even.

Now we will show that \( \theta(\text{NAE}) \leq 3 \). Consider a connected circuit \( \phi \) with one external edge, such that \( \phi \) uses one NAE constraint, and all other constraints are parity relations, with no internal edges between parity relations (this is without loss of generality, because we can contract any such edge). Assume that \( [\phi](0) \) and \( [\phi](1) \) are non-zero. We will show that \( [\phi](0) \leq 3 [\phi](1) \).

We can write
\[
[\phi](t) = \sum_{x \in \text{NAE}_J} R(t; x)
\]
where \( R \) is an affine subspace of \( \mathbb{F}(2)^{1+J} \). Since \( [\phi](0) \) and \( [\phi](1) \) are non-zero, the sets \( R_0 = \{ x \mid (0; x) \in R \} \) and \( R_1 = \{ x \mid (1; x) \in R \} \) are non-empty. Since \( R \) is an affine subspace, \( |R_0| = |R_1| \), so
\[
[\phi](0) \leq |R_0| = |R_1| \leq [\phi](1) + 2 \leq 3 [\phi](1).
\]

\[\square\]

5.4 Properties

An important property we will use is that a strictly terraced function \( F \) is either identically zero or its support \( \{ x \mid F(x) > 0 \} \) is coindependent. (If \( F(x) = 0 \) and \( F(y) > 0 \) for some \( y \), pick such a \( y \) with \( d = d(x, y) \) minimal. If \( d > 1 \), there are distinct indices \( i, j \) such that \( x_i \neq y_i \) and \( x_j \neq y_j \), so \( F(y \oplus e_i) = F(y \oplus e_i \oplus e_j) = 0 \) by minimality of \( d(x, y) \), which means \( F \) is not strictly terraced: \( F(y \oplus e_i) = 0 \) but \( F((y \oplus e_i) \oplus e_j) \neq F((y \oplus e_i) \oplus e_j) \).

The Cartesian product of coindependent relations is in general not coindependent, for example \( \{(0,1), (1,0)\} \times \{(0,1), (1,0)\} \) is not coindependent (set \( x = (0, 0, 0, 0) \) and \( i = 1 \)). Thus the class of strictly terraced functions is not closed under taking weight-functions of disconnected circuits.

Lemma 20. Let \( \phi \) be a connected circuit using strictly terraced weight-functions. Then \( [\phi] \) is strictly terraced.

Proof. We will argue by induction on the number of internal edges of \( \phi \). If there are no internal edges, then \( \phi \) consists of a single constraint using a strictly terraced function \( F \), and \( [\phi] = F \). Otherwise, pick an internal edge \( e \). We wish to argue that the function created by contracting \( e \) is strictly terraced. There are two cases.

(i) \( e \) is loop on a vertex \( v \).

Let \( F : \{0,1\}^{1+J} \rightarrow \mathbb{Q}^{\geq 0} \) be a copy of \( F_v \), indexed so that the ends of \( e \) become enumerated indices. We wish to show that the function \( H : \{0,1\}^J \rightarrow \mathbb{Q}^{\geq 0} \) defined by
\[
H(x) = \sum_{t=0}^{1} F(t; t; x) \quad (x \in \{0,1\}^J)
\]
is strictly terraced. Consider \( x \in \{0,1\}^J \) satisfying \( H(x) = 0 \) and let \( i, j \in J \). Since \( F \) is strictly terraced and \( F(0,0; x) = 1,1; x) = 0 \), we have \( F(0,0; x \oplus e_i) = F(0,0; x \oplus e_j) \) and \( F(1,1; x \oplus e_i) = F(1,1; x \oplus e_j) \). Hence \( H(x \oplus e_i) = H(x \oplus e_j) \).

(ii) \( e \) is incident to distinct vertices \( u \) and \( v \).

Let \( F : \{0,1\}^{1+J} \rightarrow \mathbb{Q}^{\geq 0} \) and \( G : \{0,1\}^{1+J} \rightarrow \mathbb{Q}^{\geq 0} \) be copies of \( F_u \) and \( F_v \), respectively, reindexed so that the ends of \( e \) become the enumerated indices (and with \( J \) disjoint). We wish to show that the function \( H : \{0,1\}^{1+J} \rightarrow \mathbb{Q}^{\geq 0} \) defined by
\[
H(x,y) = \sum_{t=0}^{1} F(t; x)G(t; y) \quad (x \in \{0,1\}^J, y \in \{0,1\}^J)
\]
is strictly terraced. If $F$ or $G$ is identically zero then $H$ is identically zero and therefore strictly terraced.

Otherwise consider $x \in \{0,1\}^J$ and $y \in \{0,1\}^J$ satisfying $H(x,y) = 0$. Since $F$ and $G$ have co-independent support and $F(0;x)G(0;y) + F(1;x)G(1;y) = 0$, there exists $t \in \{0,1\}$ such that $F(t;x)G(1-t;y) = 0$ and $F(1-t;x), G(t;y) > 0$. For all $i \in I$ we have

$$H(x \oplus e_i, y) = F(t; x \oplus e_i) G(t; y) = F(1-t; x) G(t; y).$$

Similarly for $i \in J$ we have

$$H(x, y \oplus e_i) = F(1-t; x) G(1-t; y \oplus e_i) = F(1-t; x) G(t; y).$$

Therefore for all $i, j \in I \cup J$ we have $H((x, y) \oplus e_j) = H((x, y) \oplus e_i)$. $\square$

The following calculations bound ratios produced by certain circuits.

**Lemma 21.** Let $F : \{0,1\}^{1+J}$ and $G : \{0,1\}^J \to \mathbb{Q}_{\geq 0}$. Define $H(0), H(1)$ by

$$H(t) = \sum_{x \in \{0,1\}^J} F(t;x)G(x).$$

Assume that $F$ and $G$ are strictly terraced and that $H(1) > 0$. Then

$$H(0) \leq (\theta(F) + \theta(G))H(1).$$

**Proof.** We will use induction on $|J|$. For the base case $J = \emptyset$ we have $H(0) \leq \theta(F)H(1)$ by definition of $\theta(F)$. So assume that $J$ is non-empty.

For each $i \in J$ and each $c \in \{0,1\}$ define $F_{i,c}$ to be the pinning of $F$ by taking $i$ to $c$, and similarly define $G_{i,c}$ to be the pinning of $g$ by taking $i$ to $c$, and define

$$H_{i,c}(t) = \sum_{x \in \{0,1\}^{J \setminus \{i\}}} F_{i,c}(t;x)G_{i,c}(x) \quad (t \in \{0,1\}).$$

Since pinnings are parity-weight-functions, $\theta(F_{i,c}) \leq \theta(F)$ and $\theta(G_{i,c}) \leq \theta(G)$. If there exists $i \in J$ such that $H_{i,0}(1)$ and $H_{i,1}(1)$ are non-zero, then by the induction hypothesis we have

$$H(0) = H_{i,0}(0) + H_{i,1}(0) \leq (\theta(F) + \theta(G))(H_{i,0}(1) + H_{i,1}(1)) = (\theta(F) + \theta(G))H(1).$$

Taking a choice for each $i$, we may assume that there exists $y \in \{0,1\}^J$ such that for all $i \in J$ we have $H_{i,1-y_i}(1) = 0$. So for each $i \in I$ the sets $R = \{x \mid F_{i,1-y_i}(1;x) > 0\}$ and $S = \{x \mid G_{i,1-y_i}(x) > 0\}$ are disjoint. $R$ and $S$ are pinnings of co-dependent relations, so they are co-dependent. For all $x \in R$ we have $x \notin S$, so $x \oplus e_i \notin S$ for any $i$, and $x \oplus e_i \notin R$. Repeatingly this, we find that $R$ consists of the configurations at even distance from $x$, and $S$ consists of the configurations at odd distance from $x$. In other words, for each $i \in J$ there exists $c_i \in \{0,1\}$ such that

$$F(1,x) > 0 \iff c_i + \sum_{j \in J} x_j \text{ is even},$$

$$G(x) > 0 \iff c_i + \sum_{j \in J} x_j \text{ is odd.} \quad (x \in \{0,1\}^J, x_i \neq y_i) \quad (4)$$

For any $i, j \in J$ there is some choice of $x \in \{0,1\}^J$ with $x_i \neq y_i$ and $x_j \neq y_j$, so $c_i = c_j$. Thus there is a single choice of $c$ such that $4$ holds for all $i$ taking $c_i = c$:

$$F(1,x) > 0 \iff c + \sum_{j \in J} x_j \text{ is even, and}$$

$$G(x) > 0 \iff c + \sum_{j \in J} x_j \text{ is odd.} \quad (x \in \{0,1\}^J \setminus \{y\}) \quad (5)$$

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For any \( x \in \{0, 1\}^J \) with \( c + \sum_{i \in J} x_i \) even, and any distinct \( i, j \in J \), we have \( F(1; x) = F(1; x \oplus e_i \oplus e_j) \) because \( F \) is strictly terraced and either \( F(1; x \oplus e_i) \) or \( F(1; x \oplus e_j) \) is zero. Similarly for any \( x \in \{0, 1\}^J \) with \( c + \sum_{i \in J} x_i \) odd, and any distinct \( i, j \in J \), we have \( G(x) = G(x \oplus e_i \oplus e_j) \). This implies that there are constants \( \lambda, \mu > 0 \) such that

\[
F(1; x) = \lambda \text{ if } c + \sum_{i \in J} x_i \text{ is even, and} \quad G(x) = \mu \text{ if } c + \sum_{i \in J} x_i \text{ is odd.} \tag{6}
\]

If \( c + \sum_{i \in J} y_i \) is odd then by \( \{1\} \) and \( \{3\} \), \( G \) is a \( \mu \text{Even}_J \) (if \( c = 1 \)) or \( \mu \text{Odd}_J \) (if \( c = 0 \)). So \( G \) is a constant multiple of a parity relation. Considering \( H \) as the weight-function of a circuit with constraints \( F \) and \( G \), we get \( H(0) \leq \theta(F)H(1) \) by definition of \( \theta \). We may therefore assume that \( c + \sum_{i \in J} y_i \) is even.

Define \( F'(0), F'(1), G'(0), G'(1) \) by

\[
F'(t) = F(t; y) \\
G'(t) = \sum_{x \in \{0, 1\}^J} \text{Odd}_{2+J}(t, c; x)G(x)
\]

so

\[
F'(0) = F(0; y) \\
F'(1) = F(1; y) = \lambda \\
G'(0) = \mu 2^{|J|-1} \\
G'(1) = G(y)
\]

Since \( F' \) is a parity-weight-function of \( F \) we have \( F(0; y)/\lambda \leq \theta(F) \). Since \( G' \) is a parity-weight-function over \( G \) we have \( \mu 2^{|J|-1}/G(y) \leq \theta(G) \). For all \( x \in \{0, 1\}^J \) with \( c + \sum_{i \in J} x_i \) odd, we have \( F(1; x) = 0 \) and \( F(1; x \oplus e_i) = \lambda \) (for any \( i \in J \)) and therefore \( F(0; x) = \lambda \) because \( F \) is strictly terraced. So

\[
\frac{H(0)}{H(1)} = \frac{F(0, y)G(y) + \lambda \mu 2^{|J|-1}}{\lambda G(y)} \leq \theta(F) + \theta(G).
\]

\[\square\]

**Lemma 22.** Let \( \phi \) be a circuit using strictly terraced weight-functions. Then

\[
\theta([\phi]) \leq \sum_{v \in V^\phi} \theta(F_v^\phi).
\]

**Proof.** We will argue by induction on the number \( k \) of constraints that are not parity relations. The cases \( k = 0 \) and \( k = 1 \) follow from the definition of \( \theta \). Components of a circuit not connected to the external edges simply contribute a constant factor to the weight-function. So for a given \( k \), it suffices to show that \([\phi(0)] / [\phi(1)] \leq \sum_{v \in V^\phi} \theta(F_v^\phi)\) whenever:

- \( \phi \) is a connected circuit with one external edge, with \([\phi(1)] > 0 \), and
- \( \phi \) uses strictly terraced weight-functions, at most \( k \) of which are not parity relations.

For the \( k = 2 \) case, if there is a loop on a vertex \( v \), contract it. This changes the weight-function \( F_v \), but the resulting weight-function is a parity-weight-function of \( F_v \), so this process does not increase \( \sum_{v \in V} \theta(F_v) \). And \( F_v \) is still strictly terraced by Lemma 20.

Similarly, if there is an edge incident to distinct vertices \( u, v \) where \( F_u \) is a parity constraint, contract that edge. The weight-function \( F \) introduced by the contraction is a parity-weight-function of \( F_{uv} \), so this process does not increase \( \sum_{v \in V} \theta(F_v) \). And again, \( F \) is strictly terraced by Lemma 20. Repeating this process we end up with a circuit with at most two
vertices. If there is only one vertex we can appeal to the \( k \leq 1 \) case, and otherwise we are done by Lemma 21.

For \( k > 2 \), contract any internal edge. From the \( k \leq 2 \) case we know that \( \sum_{v \in V} \theta(F_v) \) has not increased. This process does not change the weight-function of \( \phi \), and by Lemma 20 the constraint function introduced by the contraction is strictly terraced.

Lemma 23. Let \( \phi \) be a closed circuit using strictly terraced constraints, and assume that \( Z_0(\phi) > 0 \). Then

\[
\frac{Z_2(\phi)}{Z_0(\phi)} \leq \frac{1}{2} |E^s|^2 \max \left( 1, \sum_{v \in V^s} \theta(F_v^s) \right)^2.
\]

Proof. We will consider a circuit \( \psi \) obtained by attaching \( \text{Even}_3 \) to edges of \( \phi \) as illustrated in Figure 7. In words: let \( J^* \) be a disjoint copy of \( J \), consisting of an element \( i^* \) for each \( i \in J \). Define \( \psi \) to have incidences \( E \cup J \cup J^* \), external edges \( \{i, i^*\} \) for each \( i \in J \), vertex set \( V \cup E \), the same constraints at each \( v \in V \), and \( F_{(i,j)}^\psi = \text{Even}_{(i^*,j^*,(i,j))} \) for all \( \{i,j\} \in E \).

\( Z_k(\phi) \) is the sum of \([\psi]\) over configurations \( x \) of \( E^\phi \) with \( \sum_{e \in E^\phi} x_e = k \). By pinning, \( \theta \) bounds the ratio \( F(x \oplus e_i)/F(x) \) between the weights of neighbouring configurations of non-zero weight. Letting \( 0 \) denote the all-zeros vector, for all \( i \neq j \) such that \([\psi]\) \((e_i) \neq 0, \)

\[
[\psi] (e_i + e_j) \leq \theta(\psi) [\psi] (e_i) \leq \theta(\psi)^2 [\psi] (0) .
\]

If \([\psi]\) \((e_i) = 0 \) we have \([\psi]\) \((e_i + e_j) = [\psi] (0) \) because \([\psi]\) is strictly terraced. Thus

\[
Z_2(\phi) \leq Z_0(\phi) \left( \frac{|E^\phi|}{2} \right) \max(1, \theta(\psi))^2.
\]

The result follows by applying Lemmas 19 and 22.

6 Proofs of Theorem 1 and Theorem 4

Theorem 1. \#ParityNAE has an FPRAS.

Proof. We are given a labelled graph, which is naturally a closed circuit \( \phi \) using constraints of the form \( \text{Even}_J, \text{Odd}_J, \) and \( \text{NAE}_J \).

The decision problem, deciding whether \( Z_0(\phi) > 0 \), can be solved in polynomial time by Cornujoël’s algorithm for the general factor problem [10]. And degree-1 parity relations can
be used to fix edges to take a particular value. This means that \#ParityNAE is self-reducible in the sense of [21, Theorem 6.4]. So it suffices to give a fully polynomial almost uniform sampler (FPAUS): an algorithm that, when given an error parameter \(0 < \delta < 1\) and an instance of \#ParityNAE corresponding to a closed circuit \(\phi\) with \(Z_0(\phi) > 0\) using parity and not-all-equal relations, outputs an assignment \(z\) satisfying

\[
\frac{1}{2} \sum_x |Pr(x = z) - wt_\phi(x)/Z_0(\phi)| \leq \delta
\]

in time polynomial in the size of the input and in \(\log \delta^{-1}\).

We will use the near-assignments chain to sample from assignments of \(\phi\). Define \(F' : \{0,1\}^J \to \mathbb{Q}^{\geq 0}\) by

\[
F'(x) = \prod_{i \in V} F_i(x|_{J_i})
\]

and

\[
F(x) = \begin{cases} 
F'(x) & \text{if } \sum_{i \in J} x_i \text{ is even} \\
0 & \text{otherwise.}
\end{cases}
\]

By Lemma 17, all the constraints of \(\phi\) are windable. By Lemma 10, \(F'\) is windable. By Lemma 15, \((F')_{\ominus}\) is even-windable. But \(F\) is a pinning of \((F')_{\ominus}\) (setting the parity bit to zero). A pinning of an even-windable function is even-windable; this is immediate from the characterisation in terms of 2-decompositions given in Section 3.3.

We will use the notation \(\pi, \Omega, \Omega_0\) from Theorem 11 for the near-assignment chain on the pair \((F, E^\phi)\).

Recall from Lemma 19 that \(\theta(\text{NAE}_k) \leq 3\) and \(\theta(\text{Odd}_k) = 0\), and that all these weight-functions are strictly terraced. Let \(R = 3|V|^2|E|^2\); by Lemma 24 we have \(1/R \leq Z_0(\phi)/Z_2(\phi) \leq Z_0(\phi)/(Z_0(\phi) + Z_2(\phi)) = \pi(\Omega_0)\).

By Cornujois’ algorithm, mentioned above, we get an assignment \(y\) with \(\|\phi\|(y) > 0\) and in particular \(\pi(y) > 2^{-|E|}\). Applying Theorem 11 by simulating the near-assignments Markov chain of \((F, E)\) for \(t \geq (2|E|)^4R^2(\log \frac{2R}{\delta} + |E| \log 2)\) steps we can take a sample \(z\) from near-assignments of \(\phi\) such that

\[
\frac{1}{2} \sum_{x \in \Omega} |Pr(x = z) - \pi(x)| \leq \delta/2R
\]

Thus

\[
\frac{1}{2} \sum_{x \in \Omega_0} |Pr(x = z|z \in \Omega_0) - F(x)/Z_0(\phi)| \leq \delta/2
\]

So we get an FPAUS by rejection sampling: run the simulation at least \(2R \log \frac{2}{\delta} \) times and return the first sample in \(\Omega_0\). The probability that this fails is small (at most \(1 - \frac{1}{2} \pi(\Omega_0)2^R \log \frac{2}{\delta} \leq \delta/2\)).

\[\square\]

**Theorem 4.** Let \(\mathcal{F}\) be the class of strictly terraced windable functions. Then

- \(\mathcal{F}\) is closed under taking weight-functions of connected circuits
- \(\mathcal{F}\) contains Even\(_k\), Odd\(_k\), and NAE\(_k\) for all \(k \geq 1\)
- for all finite subsets \(\mathcal{F}' \subset \mathcal{F}\) there is an FPRAS for Holant\((\mathcal{F}')\)

**Proof.** The first statement is Lemma 16. The second statement is Lemma 17.

For the third statement, given \(\mathcal{F}'\), let \(\mathcal{F}'' = \mathcal{F}' \cup \{\text{Even}_1, \text{Odd}_1\}\). We can use Even\(_1\) and Odd\(_1\) to fix the value an edge takes, so Holant\((\mathcal{F}'')\) is self-reducible. (There is a minor difference from 24: we allow rational-valued functions. But this is not important.) For the decision problem we can use Feder’s algorithm for coindependent relations 14, Theorem 4). Otherwise the argument proceeds as in the previous proof, taking \(R\) to be \(|E|^2|V|^2 \max_{F \in \mathcal{F}} \theta(F)\). We find that Holant\((\mathcal{F}'')\), and therefore Holant\((\mathcal{F}')\), has an FPRAS.

\[\square\]
7 Matchings circuits

Define a **matchings circuit** \( G \) to be a graph fragment equipped with:

- a non-negative rational edge-weight \( w(e) \) for each internal edge \( e \)
- a non-negative rational fugacity \( \lambda(v) \) for each vertex \( v \)

Note that in this definition the external edges are not given weights.

Let \( \deg_F(v) \) denote the number of edges in \( F \) incident to the vertex \( v \). The weight of a set of edges \( F \subseteq A \cup E \) is:

\[
\text{wt}_G(F) = \begin{cases} 
0 & \text{if } \deg_F(v) \geq 2 \text{ for all vertices } v \\
(\prod_{v \in F} \lambda(v)) \left( \prod_{e \in F} w(e) \right) & \text{otherwise}.
\end{cases}
\]

The weight-function of \( G \) is the function \([ [G] ] : \{0, 1\}^A \to \mathbb{Q} \geq 0\) where \( A \) is the set of external edges and

\[
[ [G] ](x) = \sum_{F \subseteq E} \text{wt}_G(F).
\]

As with circuits, if \( F = [ [G] ] \) we will say the \( F \) has the matchings circuit \( G \).

For all \( w \geq 0 \) define \( \text{Edge}^w : \{0, 1\}^2 \to \mathbb{Q} \geq 0 \) by

\[
\text{Edge}^w(i, j) = \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix}_{i,j}
\]

where the matrix rows and columns are indexed by \( \{0, 1\} \). For all \( \lambda \geq 0 \) and all finite sets \( J \) define \( \text{Fugacity}^\lambda_J : \{0, 1\}^J \to \mathbb{Q} \geq 0 \) by

\[
\text{Fugacity}^\lambda_J(x) = \begin{cases} 
\lambda & \text{if } \sum_{i \in J} x_i = 0 \\
1 & \text{if } \sum_{i \in J} x_i = 1 \\
0 & \text{otherwise}.
\end{cases}
\]

Given \( G \), define a circuit by equipping each vertex \( v \) with the function \( \text{Fugacity}^\lambda(v) \), then subdividing each edge \( e \) and equipping the new vertex with the function \( \text{Edge}^w(e) \). The circuit clearly has the same weight-function as the matchings circuit. So matchings circuits are just a special type of circuit. We will use the same notation and terminology.

### 7.1 Example

**Proposition 24.** For all finite sets \( J \) define \( \text{OR}_J = \{ x \in \{0, 1\}^J \mid \sum_{i \in J} x_i > 0 \} \). Then \( \text{OR}_J \) has a matchings circuit.

**Proof.** We may assume \( J = \{1, \ldots, k\} \). The matchings circuit is illustrated in Figure 8.

Define \( F : \{0, 1\}^3 \to \mathbb{Q} \geq 0 \) by

\[
F(i, 0, j) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}_{i,j}
\]

\[
F(i, 1, j) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}_{i,j}
\]

(with rows and columns indexed from zero.) Each of the smaller boxes shown in Figure 8 have the weight-function \( F \) (where external edges are numbered from left to right).
For all $x_1, \ldots, x_k \in \{0, 1\}$,

$$[G](x_1, \ldots, x_k) = \sum_{y_1, \ldots, y_{k-1}} F(1, x_1, y_1) F(y_1, x_2, y_2) \cdots F(y_{k-1}, x_k, 0)$$

$$= \left( \begin{array}{cc}
2 & 0 \\
0 & 2
\end{array} \right)^{k-x_1+\cdots-x_k} \left( \begin{array}{cc}
1 & 1 \\
1 & 1
\end{array} \right)^{x_1+\cdots+x_k}$$

$$= 2^{k-1} \text{OR}_k(x_1, \ldots, x_k).$$

So the weight-function of $G$ is $2^{k-1} \text{OR}_k$. To deal with the scalar multiple, add an isolated vertex with fugacity $1/2^{k-1}$.

In particular, let $k \geq 1$ be odd. Then $(\text{OR}_k)^\oplus$ is a copy of $\text{EvenOR}_{k+1}$ where $\text{EvenOR}_{k+1}$ is defined as $\text{Even}_{k+1} \cap \text{OR}_{k+1}$. By Lemma 15, $\text{EvenOR}_{k+1}$ is even-windable. Thus $\text{EvenOR}_{k+1} \oplus \text{EvenOR}_{k+1} = \text{EvenNAE}_{k+1}$ has a 2-decomposition. This gives an alternate proof of Lemma 9 which, via Lemma 17, shows that $\text{NAE}_J$ is windable for all finite sets $J$. But this argument does not seem to show that $\text{NAE}_J$ has a matchings circuit.

7.2 Approximate counting

Define

Name $\#\text{PerfMatch}$

Instance A simple graph $G$

Output The number of perfect matchings in $G$

The aim of this section is to establish Proposition 5, that $\text{Holant}(F) \leq_{AP} \#\text{PerfMatch}$ for any finite set $F$ of weight-functions of matchings circuits, showing that matchings circuits are a natural choice of circuit for $\#\text{PerfMatch}$. We will reduce via the following problem.

Name $\#\text{FugacityWeightedPM}$

Instance A closed matchings circuit $\phi$ where fugacities and edge-weights are given as ratios of non-negative integers specified in binary

Output $Z_0(\phi)$

The fugacities and edge-weights can both be simulated using matchings circuits. A similar reduction appears in [16].

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Figure 9: \( G_{7,2} \), with one path in the copy of \( G_{7,1} \) labelled. All fugacities are 0, all edge-weights are 1.

**Lemma 25.** There is a polynomial-time algorithm which, given non-negative integers \( p,q \) specified in binary, outputs a matchings circuit \( G_{p,q} \) whose fugacities are all 0 and whose edge-weights are all 1, and with two external edges such that

\[
[G_{p,q}]_{i,j} = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}_{i,j} \quad \text{for all } i,j \in \{0,1\},
\]

where we consider the rows and columns of the matrix to be indexed from zero.

**Proof.** See Figure 9.

For all \( p \geq 0 \) there is a unique binary expansion \( p = 2^{n_1} + \cdots + 2^{n_k} \), with \( 0 \leq n_1 < \cdots < n_k \). Define \( G_{p,1} \) in the following way. Take two vertices \( s \) and \( t \), each with one external edge. For each \( 1 \leq i \leq k \), if \( n_i = 0 \) add an edge between \( s \) and \( t \), and otherwise add a path between \( s \) and \( t \) of length \( 2n_i - 1 \), which we can denote \( s = v_{i,1}, v_{i,2}, \ldots, v_{i,2n_i} = t \), and add a parallel edge in the odd positions: between \( v_{i,2j-1} \) and \( v_{i,2j} \) for each \( 1 \leq j \leq n_i \).

There is a unique perfect matching of \( G_{p,1} \) that includes the external edges: it uses the edges in even position in each path, \( v_{i,2j}v_{i,2j+1} \) for all \( 1 \leq i \leq k \) and all \( 1 \leq j < n_i \). The perfect matchings of \( G_{p,1} \) that do not include the external edges are determined by a choice of \( 1 \leq i \leq k \) such that the \( i \)th path uses edges in odd positions, and a choice of edge in each of the \( n_i \) odd positions in this path; there are \( 2^{n_i} \) choices for each \( i \). So \( G_{p,1} \) has the correct weight-function: \([G_{p,1}]_{i,j} = (p_0)_{i,j} \).

Define \( H \) to be the circuit consisting of one vertex with fugacity zero, with two external edges. For \( q \neq 1 \) define \( G_{p,q} \) to be serial composition of copies of \( G_{p,1}, H, G_{q,1}, \) and \( H \), that is, we identify the second external edge of the \( i \)th circuit with the first external edge of the \((i+1)\)th, for \( i = 1,2,3 \). The weight-function of \( G_{p,q} \) is then given by the matrix

\[
\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}.
\]

\( \square \)

**Lemma 26.** Given a matchings circuit \( G \) for a weight-function \( F \), we can efficiently construct a matchings circuit \( G' \) for \( F_{\oplus} \) (defined in Section 7) in which every vertex has fugacity zero. Conversely, given a matchings circuit \( G \) for \( F_{\oplus} \), we can efficiently construct a matchings circuit \( G' \) for \( F \).

**Proof.** For the first statement, pick an enumeration \( v_1, \ldots, v_n \) of the vertices of \( G \). Form \( G' \) as follows. For each \( 1 \leq i \leq n \), add vertices \( a_i, b_i, c_i \), edges \( a_ib_i, a_ic_i, b_ic_i \) with edge-weight 1, add an edge \( v_iv_i \) with weight \( \lambda(v_i) \), and if \( i < n \) add an edge \( c_ib_{i+1} \) with edge-weight 1. Set all the fugacities to zero and add an external edge at \( b_1 \). See Figure 10. Consider a matching \( M \subseteq E \) of \( G \). We will argue that there is a unique way to extend \( M \) to a perfect matching \( M' \) of \( G' \).

Let \( U = \{ i \mid \deg_M(v_i) = 0 \} \) be the indices of unmatched vertices. Let \( M_1 = \{ a_iv_i \mid i \in U \} \). Note that the extension \( M' \) must include \( M_1 \), and if \( i \notin U \) we cannot have \( b_ic_i \in M' \). Consider the following subset \( P \) of external and internal edges: the external edge at \( b_1 \),
edges $b_i c_i$ for all $i \in U$, and the edges $b_i a_i$ and $a_i c_i$ for all $i \notin U$. So $P$ is a path, except that at one endpoint, $P$ has an external edge $b_1$. Observe that there is a unique choice of perfect matching $M_2 \subseteq P$ along this path: the odd-numbered edges starting from the end of $P$ not incident to the external edge $b_1$. (If $P$ has an odd number of vertices then we get $b_1 \in M_2$, and otherwise $b_1 \notin M_2$.) Define $M' = M \cup M_1 \cup M_2$. Any extension of $M$ to a perfect matching of $G'$ would have to include $M_1$, and hence $M_2$, and so the extension is unique.

This gives a weight-preserving bijection between matchings $M$ of $G$ and perfect matchings $M'$ of $G'$. Since $G'$ has an even number of vertices, the sets $M'$ must include an even number of external edges. Thus $[G'] = F_{\oplus}$.

The converse is easy: given a matchings circuit $G$ for $F_{\oplus}$, add a vertex of fugacity 1 to the external edge 1 to get a matchings circuit for $F$.

Lemma 27. $\#FugacityWeightedPM \leq_{AP} \#\text{PerfMatch}$

Proof. Given a matchings circuit $G_1$ with no external edges, we will construct a simple graph $G$ with $C[G_1]$ perfect matchings where $C$ is an easily computed positive integer.

By Lemma 26 we get a matchings circuit $G_2$ such that $[G_2] = [G_1]_{\oplus}$. Deleting the external edge, we get a circuit $G_3$ with $[G_3] = [G_1]$. At each edge $e$ of $G_3$, we have integers $p_e, q_e$ such that the weight of $e$ is $p_e/q_e$. Insert a copy of the circuit $G_{p,q}$ given by Lemma 25; this produces a circuit $G'$ whose weight-function is $C [G_3]$ where $C = \prod_{e \in E G_3} q_e$, and where all fugacities are 0 and all edge-weights are 1. Forgetting the fugacities and edge-weights we get a multigraph with $C [G_3]$ perfect matchings. To construct $G$, delete any loops and subdivide each edge into a path of length 3; this does not affect the number of perfect matchings.

Proposition 5. If $\mathcal{F}$ is a finite set of weight-functions that have matchings circuits, then $\text{Holant}(\mathcal{F}) \leq_{AP} \#\text{PerfMatch}$.

Proof. Pick a choice of matchings circuit $G_F$ for each $F \in \mathcal{F}$. Given an instance $\psi$ of Holant($\mathcal{F}$), for each vertex $v$ the function $F_v$ is a copy of some $F \in \mathcal{F}$; we can substitute $G_F$ into $\psi$ at $v$. This process gives a matchings circuit $G'$ with the same weight-function as $\psi$. We can then appeal to Lemma 27.

7.3 Expressive power

Lemma 28. The weight-function of any matchings circuit is windable.
Figure 11: A weighted clique. All fugacities are zero, and $w(uv_i) = F(e_i)$ for all $i$. The other edge-weights are to be determined.

**Proof.** By Lemma 26 and Lemma 15 it suffices to show that the weight-function of any matchings circuit where all fugacities are zero is even-windable. For all $w \geq 0$, Lemma 7 implies that $\text{Edge}^w$ is even-windable. For all $\lambda \geq 0$ and all finite sets $J$, consider a pinning $G : \{0,1\}^I \rightarrow \mathbb{Q}^{\geq 0}$ of Fugacity. If $(G_G)(\mathbf{x}) > 0$ for some $\mathbf{x}$ then $\sum_{i \in J} x_i$ and $\sum_{i \in J} (1 - x_i)$ are at most 1, so $|I| \leq 2$. Thus $G_G$ has a 2-decomposition as in Lemma 7.

To give circuits for low-arity functions we will apply linear programming duality in the form of Farkas’ lemma. For a very short proof of Farkas’ lemma, as well as a statement explicitly allowing a general ordered field, see [1]. For all $x, \phi \in \mathbb{Q}^d$ denote the dot product $x_1 \phi_1 + \cdots + x_d \phi_d$ by $\mathbf{x} \cdot \mathbf{\phi}$.

**Lemma 29.** Let $x_1, \ldots, x_d, y \in \mathbb{Q}^d$. The following are equivalent:
1. $y = c_1 x_1 + \cdots + c_k x_k$ for some $c_1, \ldots, c_k \in \mathbb{Q}^{\geq 0}$
2. $y \cdot \phi \geq 0$ for all $\phi \in \mathbb{Q}^d$ that satisfy $x_1 \cdot \phi, \ldots, x_k \cdot \phi \geq 0$

**Lemma 30.** Let $F : \{0,1\}^4 \rightarrow \mathbb{Q}^{\geq 0}$. Assume that $F(e_1), F(e_2), F(e_3), F(e_4)$ are not all zero, and that $F(x_1, x_2, x_3, x_4) = 0$ whenever $x_1 + x_2 + x_3 + x_4$ is even, and that for all $x_1, x_2, x_3, x_4 \in \{0,1\}$ we have

$$
\begin{align*}
F(x_1, x_2, x_3, x_4) & F(1 - x_1, 1 - x_2, 1 - x_3, 1 - x_4) \\
& \leq F(x_1, x_2, 1 - x_3, 1 - x_4) F(1 - x_1, 1 - x_2, x_3, x_4) \\
& \quad + F(x_1, 1 - x_2, x_3, 1 - x_4) F(1 - x_1, x_2, 1 - x_3, x_4) \\
& \quad + F(x_1, 1 - x_2, 1 - x_3, x_4) F(1 - x_1, x_2, x_3, 1 - x_4).
\end{align*}
$$

Then $F$ has a matchings circuit.

**Proof.** We will construct values $w(v_i v_j) \geq 0$ satisfying

$$
\begin{align*}
F(e_1) &= F(e_2) w(v_3 v_4) + F(e_3) w(v_4 v_2) + F(e_4) w(v_2 v_3) \\
F(e_2) &= F(e_3) w(v_4 v_1) + F(e_4) w(v_1 v_3) + F(e_1) w(v_3 v_4) \\
F(e_3) &= F(e_4) w(v_1 v_2) + F(e_1) w(v_2 v_4) + F(e_2) w(v_4 v_1) \\
F(e_4) &= F(e_1) w(v_2 v_3) + F(e_2) w(v_3 v_1) + F(e_3) w(v_1 v_2)
\end{align*}
$$

(and also $w(v_i v_j) = w(v_i v_j)$.) This suffices because then $F = \llbracket G \rrbracket$ where $G$ is the weighted clique illustrated in Figure 11. We need to show that the vector

$$y = (F(e_1), F(e_2), F(e_3), F(e_4))$$

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is a non-negative linear combination of the vectors $e_i F(e_j) + e_j F(e_i)$ with $1 \leq i < j \leq 4$. By Farkas’ lemma (Lemma 29), it suffices to show that $y \cdot \phi \geq 0$ for all $\phi \in \mathbb{Q}^4$ satisfying

$$\phi_i F(e_i) + \phi_j F(e_j) \geq 0$$

for all $1 \leq i < j \leq 4$. (8)

If $\phi_1, \phi_2, \phi_3, \phi_4 \geq 0$ we are done. Otherwise $\phi_i < 0$ for some $i$. By assumption $F(e_j) > 0$ for some $j$. If $j \neq i$, then (8) implies that $\phi_j F(e_i)$ is non-zero. In any case $F(e_i) > 0$. If $i = 1$ then

$$F(e_1)F(e_1) \leq F(e_2)F(e_1) + F(e_3)F(e_1) + F(e_4)F(e_1)$$

$-\phi_1 F(e_1)F(e_1) \leq -\phi_1 F(e_2)F(e_1) - \phi_1 F(e_3)F(e_1) - \phi_1 F(e_4)F(e_1)$

$-\phi_1 F(e_1)F(e_1) \leq \phi_2 F(e_2)F(e_1) + \phi_3 F(e_3)F(e_1) + \phi_4 F(e_4)F(e_1)$

$-\phi_1 F(e_1) \leq \phi_2 F(e_2) + \phi_3 F(e_3) + \phi_4 F(e_4)$

Therefore $y \cdot \phi \geq 0$. By symmetry the other cases, $i \neq 1$, are similar.}

**Theorem 3** Let $F : \{0, 1\}^3 \to \mathbb{Q}^{\geq 0}$. The following are equivalent:

1. $F$ is windable
2. For all $x_1, x_2, x_3 \in \{0, 1\}$ we have

$$F(x_1, x_2, x_3)F(1 - x_1, 1 - x_2, 1 - x_3)$$

$$\leq F(x_1, x_2, 1 - x_3)F(1 - x_1, 1 - x_2, x_3)$$

$$+ F(x_1, 1 - x_2, x_3)F(1 - x_1, x_2, 1 - x_3)$$

3. $F$ has a matchings circuit

**Proof.** For notational convenience, in the following argument we will use a particular copy of $F_\mathbb{Q}$. For all $x_1, x_2, x_3, x_4 \in \{0, 1\}$ define

$$F'(x_1, x_2, x_3, x_4) = \begin{cases} F(x_1, x_2, x_3) & \text{if } x_1 + x_2 + x_3 + x_4 \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

$(1 \iff 2)$ Let $B$ be a witness that $F'$ is even-windable (using Lemma 15). Let $x_1, x_2, x_3 \in \{0, 1\}$. Let $c \in \{0, 1\}$ be the unique value such that $x_1 + x_2 + x_3 + c$ is even. Then $(x_1, x_2, x_3, c) \oplus (1 - x_1, 1 - x_2, 1 - x_3, 1 - c) = (1, 1, 1, 1)$. Note that

$$\mathcal{M}_{(1,1,1,1)} = \{\{1, 2\}, \{3, 4\}, \{1, 3\}, \{2, 4\}, \{1, 4\}, \{2, 3\}\}.$$

We have

$$F(x_1, x_2, x_3)F(1 - x_1, 1 - x_2, 1 - x_3)$$

$$= F'(x_1, x_2, x_3, c)F'(1 - x_1, 1 - x_2, 1 - x_3, 1 - c)$$

$$= B((x_1, x_2, x_3, c), (1 - x_1, 1 - x_2, 1 - x_3, 1 - c), \{\{1, 2\}, \{3, 4\}\})$$

$$+ B((x_1, x_2, x_3, c), (1 - x_1, 1 - x_2, 1 - x_3, 1 - c), \{\{1, 3\}, \{2, 4\}\})$$

$$+ B((x_1, x_2, x_3, c), (1 - x_1, 1 - x_2, 1 - x_3, 1 - c), \{\{1, 4\}, \{2, 3\}\})$$

$$\leq F'(x_1, x_2, 1 - x_3, 1 - c)F'(1 - x_1, 1 - x_2, x_3, c)$$

$$+ F'(x_1, 1 - x_2, x_3, 1 - c)F'(1 - x_1, x_2, 1 - x_3, c)$$

$$+ F'(x_1, 1 - x_2, 1 - x_3, c)F'(1 - x_1, x_2, x_3, 1 - c)$$

$$= F(x_1, x_2, 1 - x_3)F(1 - x_1, 1 - x_2, x_3)$$

$$+ F(x_1, 1 - x_2, x_3)F(1 - x_1, x_2, 1 - x_3)$$

$$+ F(x_1, 1 - x_2, 1 - x_3)F(1 - x_1, x_2, x_3)$$

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We can assume that $F$ is not identically zero (otherwise, take two vertices of fugacity 0, and attach four outgoing edges to one of them - the isolated vertex can never be matched). Pick $x \in \{0, 1\}^4$ with $F'(x_1, x_2, x_3, x_4) > 0$. Lemma 30 implies that the flip $F''$ of $F'$ by $x \oplus (1, 1, 1, 0)$ has a matchings circuit. By subdividing the $i$’th outgoing edge for each $i$ with $x_i = 1$, we get a matchings circuit for $F'$. By Lemma 26 we get a matchings circuit whose weight-function is $F$.

In particular by Theorem 6 and Proposition 5 Holant($\{R\}$) $\leq_{AP}$ #PerfMatch where
\[ R = \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 0, 1), (0, 1, 1)\} .\]

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