Computing Plan-Length Bounds Using Lengths of Longest Paths

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Abstract
We devise a method to exactly compute the length of the longest simple path in factored state spaces, like state spaces encountered in classical planning. Although the complexity of this problem is NEXP-Hard, we show that our method can be used to compute practically useful upper-bounds on lengths of plans. We show that the computed upper-bounds are significantly (in many cases, orders of magnitude) better than bounds produced by previous bounding techniques and that they can be used to improve the SAT-based planning.

Introduction
Many techniques for solving problems defined on transition systems, like SAT-based planning (Kautz and Selman 1992) and bounded model checking (Biere et al. 1999), benefit from knowledge of upper bounds on the lengths of solution transition sequences, aka completeness thresholds. If there is no solution, then that solution need not comprise more than N transitions.

In AI planning, upper bounds on plan lengths have two main applications related to SAT-based planning. Firstly, like for bounded model-checking, an upper bound on plan lengths can be used as a completeness threshold, i.e. to prove a planning problem has no solution. Secondly, it can be used to improve the ability of a SAT-based planner to find a solution. Typically, a SAT-based planner works by repetitively querying a SAT solver to search for plans of different lengths, aka horizons. Given the upper bound as a horizon, the SAT-based planner can focus its search to plans whose length is just the given bound. This was shown to help substantially increase the coverage of SAT-based planners.

Biere et al. (1999) identified the diameter (d) and the recurrence diameter (rd), which are topological properties of the state space, as completeness thresholds for bounded model-checking of safety and liveness properties, respectively. d is the longest shortest path between any two states. rd is the length of the longest simple path in the state space, i.e. the length of the longest path that does not traverse any state more than once. Both, d and rd, are upper bounds on the shortest plan’s length, i.e. they are completeness thresholds for SAT-based planning.

A problem with the practical utilisation of either d or rd is the complexity of computing both of them, which is exponential and doubly-exponential in the size of the state space, respectively. This severe complexity can be alleviated by compositionally computing approximate upper bounds on d or rd instead of exactly computing them. Existing compositional bounding methods compute an upper bound on a factored transition system’s diameter by composing together values of topological properties of state spaces of abstract subsystems (Baumgartner, Kuehlmann, and Abraham 2002; Rintanen and Gretton 2013; Abdulaziz, Gretton, and Norrish 2015; Abdulaziz, Gretton, and Norrish 2017; Abdulaziz 2019). Such compositional methods are currently the only practically viable method to compute bounds on plan lengths or the state space diameter. Compositional approaches provide useful approximations of plan bounds using smaller computational effort, since only explicit representations of abstract subsystems have to be constructed. This can lead to an exponentially less computational costs compared to computing d or rd of the given system.

In this work we study the computation of recurrence diameters of state spaces of classical planning problems, which are factored transition systems. The longest path or its properties are fundamental graph properties. Thus, computing its length for state spaces of planning problems is inherently interesting, as it might reveal interesting properties of different planning problems. However, our goal is to devise better compositional methods to compute upper bounds on plan lengths to aid SAT-based planning.

Our first contribution concerns the relationship between rd and the traversal diameter (td), td is a state space topological property. The best existing compositional bounding method is due to Abdulaziz (2019) and it computes traversal diameters of abstractions and composes them into an upper bound on d and on plan-length. We show that rd is a lower bound on the traversal diameter, and that rd can be exponentially smaller than td. This gives an opportunity for substantial improvements in the bounds computed by compositional bounding methods, if the recurrence diameters of abstractions are used instead of their traversal diameters. However, the practical realisation of this improvement is contingent on whether there is an efficient method to compute rd.
Our second, and main, contribution is that we investigate practically useful methods to compute recurrence diameters of factored systems. We firstly implement a method to compute recurrence diameters based on a method by Biere et al. [1999]. Unlike Biere et al., who devised their method and tested it on model-checking problems, we test their method on planning benchmarks show that it is unpractical. We devise a new method to compute $rd$ that substantially dominates the method of Biere et al. in terms of performance, i.e., running time. We combine our new method with the compositional bounding method by Abdulaziz, Gretton, and Norrish [2017] and show that the bounds are, as predicted theoretically, much tighter when $rd$ is used instead of $td$.

However, a challenge is that both methods to compute $rd$ have worst-case running times that are doubly-exponential in the size of the given factored system. This stems from the complexity of the problem of computing $rd$ for succinct digraphs, which is NEXP-hard (Pardalos and Migdalas 2004; Papadimitriou and Yannakakis 1986). This is worse than the complexity of computing $td$, which is singly-exponential in the size of the factored system. Our third contribution is that we investigate different techniques to alleviate the impact of this prohibitive worst-case running time on the overall compositional bounding algorithm.

Lastly, we experimentally show that the improved bounds lead to an improved problem coverage for state-of-the-art SAT-based planner MP [Rintanen 2012], when the bounds are used as horizons for it.

**Background and Notation**

We consider factored transition systems which are purely characterised in terms of a set of actions. From actions we can define a set of valid states, and then approach bounds by considering properties of executions of actions on valid states. Whereas conventional expositions in the planning and model-checking literature would also define initial conditions and goal/safety criteria, here we omit those features from discussion since the notion of diameter, recurrence diameter and other state-space topological properties we consider are independent of those features.

**Definition 1 (States and Actions).** A maplet, $v \rightarrow b$, maps a variable $v$—i.e. a state-characterising proposition—to a Boolean $b$. A state, $x$, is a finite set of maplets. We write $D(x)$ to denote $\{ v \mid (v \rightarrow b) \in x \}$, the domain of $x$. For states $x_1$ and $x_2$, the union, $x_1 \uplus x_2$, is defined as $\{ v \rightarrow b \mid v \in D(x_1) \uplus D(x_2) \land ((v \rightarrow b) \in x_1) \lor (v \rightarrow b) \in x_2 \}$. Note that the state $x_1$ takes precedence. An action is a pair of states, $(p, e)$, where $p$ represents the preconditions and $e$ represents the effects. For action $\pi = (p, e)$, the domain of that action is $D(\pi) = D(p) \uplus D(e)$.

**Definition 2 (Execution).** When an action $\pi = (p, e)$ is executed at state $x$, it produces a successor state $\pi(x)$, formally defined as $\pi(x) = \{ p \not\subseteq x \rightarrow e \} \cup x$. We lift execution to lists of actions $\vec{\pi}$, so $\pi(x)$ denotes the state resulting from successively applying each action from $\vec{\pi}$ in turn, starting at $x$.

We give examples of states and actions using sets of literals. For example, $\{a, b\}$ is a state where state variables $a$ is (maps to) true, $b$ is false, and its domain is $\{a, b\}$. $\{\{a, b\}, \{c\}\}$ is an action that if executed in a state that has $a$ and $b$, it sets $c$ to true. $D(\{\{a, b\}, \{c\}\}) = \{a, b, c\}$. We also give examples of sequences, which we denote by the square brackets, e.g., $[a, b, c]$.

**Definition 3 (Factored Transition System).** A set of actions $\delta$ constitutes a factored transition system. $D(\delta)$ denotes the domain of $\delta$, which is the union of the domains of all the actions it contains. Let $\text{set}(\vec{\pi})$ be the set of elements from $\vec{\pi}$. The set of valid action sequences, $\delta^*$, is $\{ \pi \mid \text{set}(\vec{\pi}) \subseteq \delta \}$. The set of valid states, $\cup(\delta)$, is $\{ x \mid D(x) = D(\delta) \}$. $G(\delta)$ denotes the state space of $\delta$, which is the set of pairs $\{ (x, \pi(x)) \mid x \in \cup(\delta) \}$, corresponding to different transitions in the state space of $\delta$.

**Example 1.** Consider the factored system $\delta = \{ \pi_1 = 0, \{v_1, v_2\}, \pi_2 = \{0, \{v_1, v_2\} \}, \pi_3 = \{0, \{v_1, v_2\} \}, \pi_4 = \{0, \{v_1, v_2\} \} \}$. The digraph in Figure 1a represents the state space of $\delta$, where different states defined on the variables $D(\{\{a, b\}, \{c\}\}) = \{a, b, c\}$. We also give examples of sequences, which we denote by the square brackets, e.g., $[a, b, c]$.

**Figure 1: The state spaces of the systems from Examples 1, 3, and 4**
states. The recurrence diameter is the length of the longest simple path in the state space, formally

$$ rd(\delta) = \max_{x \in U(\delta), \pi \in \delta^* \setminus \text{distinct}(x, \pi)} |\pi| $$

**Example 2.** For the system $\delta$ from Example 2, $d(\delta) = 1$, since every state can be reached with one action from every other state. Nonetheless, $rd(\delta) = 3$ as there are many paths with 3 actions in the state space that traverse distinct states, e.g. executing the action sequence $[\pi_1, \pi_2, \pi_3]$ at the state $\{\pi_1, \pi_2\}$ traverses the distinct states $\{\pi_1, \pi_2\}, \{v_1, v_2\}, \{\pi_1, \pi_2\}$. Note that in general $rd$ is an upper bound on $d$, and that it can be exponentially larger than $d$.

**Theorem 1** (Biere et al. 1999). For any system $\delta$, we have that $d(\delta) \leq rd(\delta)$. Also, there are infinitely many systems for which the recurrence diameter is exponentially (in the number of state variables) larger than the diameter.

Like $d$, $rd$ is a completeness threshold for SAT-based planning and for safety bounded model-checking but, unlike $d$, it is also a completeness threshold for bounded model-checking of liveness properties, which was the original reason for its inception (Biere et al. 1999).

Algorithms have been developed to calculate both properties for digraphs, and those algorithms can be directly applied to state spaces of explicitly represented (e.g. tabular) transition systems. Exact algorithms to compute $d$ have worse than quadratic runtimes in the number of states (Fredman 1976; Alon, Galil, and Margalit 1997; Chan 2010; Yuster 2010), and approximation algorithms have super-linear runtimes (Angworth et al. 1999; Roditty and Vassilevska Williams 2013; Chechik et al. 2014; Abboud, Williams, and Wang 2016). The situation is worse for $rd$, whose computation is NP-Hard (Pardalos and Migdalas 2004). The impracticality of computing $d$ and $rd$ is exacerbated in settings where transition systems are described using factored representations, like in planning and model-checking (Fikes and Nilsson 1971; McMillan 1993). In particular, the worst-case running time is exponentially worse because, in the worst case, all known methods construct an explicit representation of the state space to compute $d$ or $rd$ of the state space of a succinctly represented system. This follows the general pattern of complexity exponentiation of graph problems when graphs are succinctly represented, where, for succinct digraphs, the complexity of computing $d$ is $\Omega_2$-hard (Hemaspaandra et al. 2010) and the complexity of computing $rd$ is NEXP-hard (Papadimitriou and Yannakakis 1986).

**CompositionalBounding of the Diameter**

The prohibitive complexity of computing $d$ or $rd$ suggests they can only be feasibly computed for very small factored systems, systems that are much smaller than those that arise in typical classical planning benchmarks. However, another possibility is to utilise the computation of $d$ or $rd$ within compositional plan length upper bounding techniques. Existing techniques compute an upper bound on $d$, for a given system, by computing topological properties of abstractions of the given system and then composing the abstraction topological properties. Those abstractions are usually much smaller than the concrete system which needs to be bounded, and their state spaces can be exponentially smaller than the given system’s state space. Thus, computing topological properties of abstractions might be feasible.

Currently, the compositional bounding method by [Abdulaziz, Gretton, and Norrish 2017] is the most successful in decomposing a given system into the smallest abstractions. It decompose a given factored system using two kinds of abstraction: projection and snapshotting. Projection (Knoblock 1994; Williams and Nayak 1997) produces an over-approximation of the given system and it was used for bounding by many previous authors. Snapshotting produces an under-approximation of the given system and it was introduced by [Abdulaziz, Gretton, and Norrish 2017]. The compositional method devised by [Abdulaziz, Gretton, and Norrish 2017] recursively interleaves the application of projection and snapshotting until the system is decomposed into subsystems that can no longer be decomposed, to which we refer here as base cases.

After the system is decomposed into base case systems, a topological property, which we call the base case function, of the state space of each of the base case systems is computed, and then its values for base case systems are composed to bound $d$ of the concrete system. Most authors used the base case function $\text{EXP}$, which is one less than the number of valid states for the given base case system (Rintanen and Gretton 2013; Abdulaziz, Gretton, and Norrish 2015; Abdulaziz, Gretton, and Norrish 2017). A notable exception is [Abdulaziz 2019] who used the traversal diameter, which is a topological property of the state space, as a base case function. The traversal diameter is one less than the largest number of states that could be traversed by any path.

**Definition 6** (Traversal Diameter). Let $\text{SS}(x, \pi)$ be the set of states traversed by executing $\pi$ at $x$. The traversal diameter is

$$ td(\delta) = \max_{x \in U(\delta), \pi \in \delta^*} |\text{SS}(x, \pi)| - 1. $$

**Example 3.** Consider the factored system $\delta = \{\pi_1 = \{\pi_1, \pi_2\}, \{v_1, v_2\}, \pi_2 = \{\pi_1, \pi_2\}, \{v_1, v_2\}, \pi_3 = \{\pi_1, \pi_2\}, \{v_1, v_2\}\}$. The digraph in Figure 1B shows the state space of $\delta$. For $\delta$, $\text{EXP}(\delta) = 3$, while $td(\delta) = 1$.

[Abdulaziz 2019] showed that $td$ is an upper bound on $rd$ and a lower bound on $\text{EXP}$. He also showed that $td$ can be exponentially smaller than $\text{EXP}$, as shown in the above example. This is why when [Abdulaziz 2019] used $td$ as a base case function, his method computed bounds substantially tighter than all previous methods.

**The Recurrence Diameter Versus the Traversal Diameter**

We now study the relationship between the recurrence diameter and the traversal diameter. The core insight we make here is that $rd$ can be exponentially smaller than $td$.

**Theorem 2.** There are infinitely many factored systems whose recurrence diameters are exponentially smaller (in
the number of state variables) than their traversal diameters.

Proof. Let, for a natural number $n$, $D_n$ denote the indexed set of state variable $\{v_1, v_2, \ldots, v_{\log n}\}$. Let $x_i^n$ denote the state defined by completely assigning of the state variables $D_n$, s.t. their assignments binary encode the natural number $i$, where the index of each variable from $D_n$ represents its endianess. Note: $x_i^n$ is well defined for $0 \leq i \leq 2^{\log n} - 1$.

Now, for an arbitrary number $n \in \mathbb{N}$, let $I_n$ denote the factored system (i.e. set of actions) $\{(x_i^{n+1}, x_j^{n+1}) | 1 \leq i \leq n \}$. The diameter of the system $I_n$ is 2, regardless of, since any action sequence that traverses more than 3 states will traverse $x_i^{n+1}$ more than once. Now, let $S$ denote the largest connected component in the state space of $I_n$, which has $n + 1$ states in it. Since for any two states $x_i^{n+1}$, $x_j^{n+1} \in S$, there is an action sequence $\pi \in I_n$ s.t. $\pi(x_i^{n+1}) = x_j^{n+1}$, and since $|S| = n + 1$, then the traversal diameter of $I_n$ is $n$. Accordingly, since $2^{|D(I_n)|} - 2 = 2^{|D_n|} - 2 = 2^{\log n} - 1 \leq n$, we have that $2^{|D(I_n)|} - 2 \leq rd(I_n)$. The theorem follows from this and since $rd(I_n) = 2$.

Example 4. The state space of $I_3$ is depicted in Figure [c]. $rd(I_3) = 2$, and $td(I_3) = 3$.

The fact that $rd$ can be exponentially smaller than $td$ gives rise to the possibility of substantial improvements to the bounds computed if we use $rd$ as a base case function for compositional bounding, instead of $td$.

Using the Recurrence Diameter for Compositional Bounding

Before we proceed with describing how to utilise $rd$ for compositional bounding of $d$ we address this question: why should we focus on computing $rd$ instead of $d$? This question is reasonable since, as stated earlier, $d$ can be computed in exponentially less time compared to $rd$, and it is also a lower bound on $rd$ that can be exponentially smaller.

The reason is simple: it has been shown that $d$ cannot be bounded by diameters of projections [Abdulaziz et al. 2017 Chapter 3, Theorem 1]. On the other hand, previous authors showed that, theoretically, that recurrence diameters of projections can be composed to bound the concrete system’s diameter [Baumgartner, Kuehlmann, and Abraham 2002]. Here, we investigate using $rd$ as a base case function for the compositional algorithm by Abdulaziz, Gretton, and Norrish 2017.

For explicitly represented digraphs, the computational complexity of finding the length of the longest path is NP-hard [Pardalos and Migdalas 2004]. Thus, the best known methods to find it cannot have a worst-case running time smaller than a time exponential in the size of the given digraph. Biere et al. 1999 suggested the only method to compute $rd$ of which we are aware. They encode the question of whether a given number $k$ is $rd$ of a given transition system as a SAT formula. $rd$ is found by querying a SAT-solver for different values of $k$, until the SAT-solver answers positively for one $k$. The method terminates since $rd$ cannot be larger than one less the number of states in the given transition system. The size of their encoding grows linearly in $k^2$. The encoding of Biere et al. is based on the following theorem, which we restate in our notation.

Theorem 3 (Biere et al. 1999). For a factored system $\delta$ and a natural number $k$, we have that $\phi_1(\delta, k)$ is true iff $rd(\delta) \leq k$, where $\phi(\delta, k)$ denotes the conjunction of the formulae

(i) $\forall x_1, x_2 \in G(\delta)$, $G(x_1, x_2)$, 
(ii) $\forall x_1, x_2 \in U(\delta)$. if $(x_1, x_2) \notin G(\delta)$, then $\neg G(x_1, x_2)$, and 
(iii) $\forall x_1, x_2, \ldots, x_{k+2}. (\forall 0 \leq i \leq k, G(x_i, x_{i+1}))$ then $(\exists 0 \leq i \leq k, x_i = x_{i+1})$.

Another encoding of the above question was suggested by Kroening and Strichman 2003 and its size grows linearly in $k \log^2(k)$. It is based on sorting networks [Knuth 1998]. However, Kroening and Strichman report that their encoding has hidden constants that cause its size to be smaller than the encoding by Biere et al. only when $50 < k$. This value of $k$ is larger than recurrence diameters that can be practically computed, thus we decided to implement the technique by Biere et al.

In our experiments we use an SMT solver to reason about encoding of $rd$. Thus, for decidability as well as efficiency reasons, we would like to obtain an encoding that is quantifier free. In particular, we want an encoding that fits the theory of quantifier free uninterpreted functions. Since the formula $\phi_1$ above is universally quantified, we reformulate it to its existentially quantified dual, which is below. Firstly, let for predicates $P$ and $Q$ of arity $n$, $\bigwedge Q(a_1, a_2, \ldots, a_n)$. $P(a_1, a_2, \ldots, a_n)$ denote the conjunction of $P(a_1, a_2, \ldots, a_n)$, for all $(a_1, a_2, \ldots, a_n)$, s.t. $Q(a_1, a_2, \ldots, a_n)$ holds. Analogously, let $\bigvee$ denote a finite disjunction. Note: $\bigwedge Q$. $P(a_1, a_2, \ldots, a_n)$ is only well defined if $Q$ is true for only a finite set of tuples. We also will not explicitly bind the tuple $(a_1, a_2, \ldots, a_n)$ when it is clear from context.

Encoding 1. Let for $\delta$, $\Theta(x_1, x_2)$ denote $(x_1, x_2) \in G(\delta)$. For $\delta$ and $0 \leq k$, let $\phi_1(\delta, k)$ denote the conjunction of the formulae

(i) $\bigwedge \Theta(x_1, x_2)$. $G(x_1, x_2)$, 
(ii) $\bigwedge x_1, x_2 \in U(\delta)$. $\neg \Theta(x_1, x_2)$. $\neg G(x_1, x_2)$, and 
(iii) $1 \leq i \leq k$. $(G(y_i, y_{i+1}) \bigwedge i < j \leq k$. $y_i \neq y_j$).

The SMT formula above is defined over the following uninterpreted constants: one constant $x_i$ for every state in $U(\delta)$, one constant $y_i$ for every state in the simple path of length $k+1$ for which we search, and a function $G$ that is true for a pair of constants $(x_i, x_j)$ if there is an edge from $x_i$ to state $x_j$ in the state space of $\delta$. Clearly, the following holds.

Theorem 4. $\phi_1(\delta, k)$ is satisfiable iff $k < rd(\delta)$.

Proof. An SMT formula is defined over a signature $\Sigma$, which is a finite set of symbols that are either constants (aka objects), sorted uninterpreted constants, or the standard logical connectives. A model $M$ for a signature is a function that maps uninterpreted constants to objects. A model $M$ entails a formula $\phi$, denoted $M \models \phi$, iff $\phi$ evaluates to true, under
the standard interpretation of logical connectives, after each uninterpreted constant \( v \) in \( \phi \) is substituted by \( \mathcal{M}(v) \).

**Lemma 1.** If \( \phi'_1(\delta, k) \) is satisfiable, then there is a list of distinct states \( [x_1, x_2, \ldots, x_{k+1}] \), such that \( (x_i, x_{i+1}) \in G(\delta) \), for \( 1 \leq i \leq k \).

**Proof summary.** Firstly, note that by the way we construst the formula \( \phi'_1(\delta, k) \) there is a sorting constraint: there is only one sort, that of states, and all constants in the formula are of that sort. Furthermore, by the way we construct the formula \( \phi'_1(\delta, k) \), that sort is only populated by the constants \( \{ x \mid \exists x'.\{ x, x', (x', x) \} \cap G(\delta) \neq \emptyset \} \). Thus, this sort is the same as the set of valid states \( \mathcal{U}(\delta) \) of the given factored transition system. Now, since the formula \( \phi'_1(\delta, k) \) is satisfiable, there a model \( \mathcal{M} \), s.t. \( \mathcal{M} \models \phi'_1(\delta, k) \). From the definition of entailment and the third conjunct of \( \phi'_1(\delta, k) \), we have that \( G(\mathcal{M}(y_1), \mathcal{M}(y_{i+1})) \) and \( \mathcal{M}(y_i) \neq \mathcal{M}(y_j) \) hold, for all \( 1 \leq i \leq k \) and \( i < j \leq k \). From this, and the first and second conjuncts of \( \phi'_1(\delta, k) \), and the fact that all the uninterpreted constants \( \{ y_i \mid 1 \leq i \leq k+1 \} \) are of the sort states, we have that \( (\mathcal{M}(y_1), \mathcal{M}(y_{i+1})) \in G(\delta) \). This finishes our proof.

**Lemma 2.** If there is a list of distinct states \( [x_1, x_3, \ldots, x_{k+1}] \), such that \( (x_i, x_{i+1}) \in G(\delta) \), for \( 1 \leq i \leq k \), then \( \phi'_1(\delta, k) \) is satisfiable.

**Proof summary.** Consider the model \( \mathcal{M} \) defined as \( \mathcal{M}(y_i) = x_i \), if \( 1 \leq i \leq k + 1 \). Note that it is well-defined for the set of uninterpreted constants in \( \phi'_1(\delta, k) \), i.e. it is well-defined for the set \( \{ y_i, 1 \leq i \leq k + 1 \} \). From the assumptions of this lemma and the definition of \( \phi'_1(\delta, k) \), we have that \( \mathcal{M} \models \phi'_1(\delta, k) \). This finishes our proof.

From the definition of \( rd \), there is a list of actions \( \pi \mathcal{K} \equiv [\pi_1, \pi_2, \ldots, \pi_k] \) and a state \( x_1 \in \mathcal{U}(\delta) \), s.t. \( \pi \mathcal{K} (x_1) \) traverses distinct states, iff \( k < rd(\delta) \). Also, from the definition of \( G(\delta) \), for any states \( x_1 \) and \( x_2 \), there is an action \( \pi \delta \) s.t. \( x_1 = \pi \delta(x_1, x_2) \in G(\delta) \). Accordingly, there is a list \( \{ x_1, x_1 \ldots, x_{k+1} \} \), s.t. \( \{ x_i, x_{i+1} \} \in G(\delta) \), for \( 1 \leq i \leq k \), iff \( k < rd(\delta) \). The theorem follows from this together with Lemmas 1 and 2.

To use the above encoding to compute \( rd \) of a given system \( \delta \), we iteratively query an SMT solver to check for the satisfiability of \( \phi'_1(\delta, k) \) for different values of \( k \), starting at 1, until we have an unsatisfiable formula. The smallest \( k \) for which the formula is unsatisfiable is \( rd(\delta) \).

Observe that, to use Encoding 1 one has to build the entire state space as a part of building the encoding, i.e. one has to build the graph \( G(\delta) \) and include it in the encoding. In fact, this is true for both methods, the one by Biere et al. and the one by Kroening and Strichman as they are both specified in terms of explicitly represented transition systems. This means that the worst-case complexity of computing \( rd \) using either one of those encodings is doubly-exponential.

Indeed, this the best possible worst-case running time for succinct graphs generally, unless the polynomial hierarchy collapses, since computing \( rd \) is NEXP-hard.

**Experimental evaluation** To experimentally test this encoding, we use it as a base case function for the compositional algorithm by Abdulaziz, Gretton, and Norrish 2017 instead of \( td \), which was used as a base case by Abdulaziz 2019 and led to the tightest bounds of any existing method. We use Yices 2.6.1 [Dutertre 2014] as the SMT solver to prove the satisfiability or unsatisfiability of the resulting SMT formulae. We run the bounding algorithm by Abdulaziz, Gretton, and Norrish 2017 on standard planning benchmarks (from previous planning competitions and ones we modified), once with \( td \) as a base case and a second time with \( rd \) as a base case. We perform our experiments on a cluster of 2.3GHz Intel Xeon machines with a timeout of 20 minutes and a memory limit of 4GB. Our experiments show that Encoding 1 is not practical for planning problems when used as a base case function for the algorithm by Abdulaziz, Gretton, and Norrish 2017 where bounds are only computed within the timeout for less than 0.1% of our set of benchmarks. This is because computing \( rd \) can take time that is exponential in the size of the state space, while computing \( td \) can be computed in time that is linear in the state space (Abdulaziz 2019).

### A New Encoding for the Recurrence Diameter

| Benchmark     | Newopen (1440) | logistics (407) | elevators (60) | rover (90) | nonmystery (70) | zeno (50) | hiking (40) | TPP (30) | GED (40) | woodworking (60) | visitall (70) | openstacks (111) | satellite (10) | hyp (33) | scanalyzer (60) | storage (30) | parpc (40) | maintenance (5) | blocksworld (10) | flootline (100) |
|--------------|----------------|----------------|----------------|------------|----------------|----------|------------|--------|--------|----------------|-------------|----------------|-------------|---------|----------------|------------|----------|----------------|----------------|----------|
| \( td \)     | 3e3 / 7e7 (1293) | 7e1 / 4e6 (406) | 3e6 / 1e9 (14) | 1e2 / 1e6 (51) | 9e0 / 6e4 (70) | 4e1 / 5e5 (50) | 1e3 / 1e9 (20) | 2e1 / 8e6 (16) | 7e6 / 7e8 (15) | 2e2 / 6e7 (10) | 7e0 / 2e3 (4) | 3e5 / 3e6 (6) | 6e2 / 6e3 (10) | 3e3 / 2e4 (7) | 4e3 / 4e4 (6) | 5e3 / 6e1 (3) | 1e3 / 6e0 (1) | 5e2 / 5e8 (5) | 3e0 / 7e0 (27) |
| \( rd \)     | 3e3 / 1e7 (1220) | 6e1 / 1e4 (197) | 9e1 / 9e4 (46) | 7e1 / 8e5 (52) | 9e0 / 9e3 (70) | 4e1 / 6e4 (1) | 2e1 / 4e4 (15) | 2e1 / 4e4 (15) | 7e6 / 7e8 (5) |
| \( b1 \)     | 2e3 / 4e7 (1292) | 6e1 / 1e4 (197) | 7e1 / 8e5 (52) | 7e1 / 8e5 (52) | 9e0 / 9e3 (70) | 4e1 / 6e4 (1) | 2e1 / 4e4 (15) | 2e1 / 4e4 (15) | 7e6 / 7e8 (5) |
| \( b2 \)     | 2e3 / 4e7 (1291) | 6e1 / 1e5 (201) | 7e1 / 8e5 (52) | 7e1 / 8e5 (52) | 9e0 / 9e3 (70) | 4e1 / 6e4 (1) | 2e1 / 4e4 (15) | 2e1 / 4e4 (15) | 7e6 / 7e8 (5) |

Table 1: \( td \): the domain name and the number of problems in it. \( rd \): the minimum/maximum bound (below 10^9) computed using \( td \) as a base case function for the domain, and in parentheses: number of instance successfully bounded below 10^9 within 20 minutes. Col. 3, 4, and 5: similar to Col. 2, but when \( rd, b1, \) and \( b2 \), respectively, are used as base case functions.

We now devise an encoding that performs better than Encoding 1. We observe that both, the encodings by Biere et al. and the one by Kroening and Strichman are similar to Col. 2, but when \( rd, b1, \) and \( b2 \), respectively, are used as base case functions.
do not exploit the compactness of factored representations of transition systems, and instead assume explicitly represented transition systems. In this section we devise a new encoding, where we exploit the factored representation in a way that is reminiscent to encodings used for SAT-based planning (Kautz and Selman 1992). In particular, our aim is to avoid constructing the state space in an explicit form, whenever possible. We devise a new encoding that avoids building the state space as a part of the encoding and, effectively, we let the SMT solver build as much of it during its search as needed.

**Encoding 2.** For a state \( x \), let \( x_i \) denote the formula \( (\bigwedge v \in x. v_1) \land (\bigwedge v \in x. \neg v_i) \). For \( \delta \) and \( 0 \leq k \), let \( \phi_2(\delta, k) \) denote the conjunction of the formulae

(i) \( \bigwedge 1 \leq i \leq k. w_i \rightarrow p_{\text{pre}}(\pi)_i \land p_{\text{eff}}(\pi)_{i+1} \land (\bigwedge v \in \mathcal{D}(\delta) \setminus \mathcal{D}(\pi))(v_i \leftrightarrow v_{i+1}) \).

(ii) \( \bigwedge 1 \leq i \leq k. \bigvee \pi \in \delta. p_{\pi} \).

(iii) \( \bigwedge 1 \leq i \leq k. \bigwedge \pi, \pi' \in \delta. \pi = \pi' \lor p_{\pi} \land \neg p_{\pi'} \lor \neg p_{\pi'} \).

(iv) \( \bigwedge 1 \leq i < j \leq k. \bigvee v \in \mathcal{D}(\delta). v_i \neq v_j \).

Briefly, the encoding above states that \( k \) is not \( rd \) if there is a sequence of \( k \) actions that traverses only distinct states if executed at some valid state. In more details, in the above formulae the following are the intuitive meanings of uninterpreted constants: (i) \( w_i \), for all \( 1 \leq i \leq k \) and \( \pi \in \delta \), is a boolean variable that represents whether action \( \pi \) is executed at state \( i \), and (ii) \( v_i \), for all \( 1 \leq i \leq k + 1 \) and \( v \in \delta \), which represents the truth value of state variable \( v \) at state \( i \).

There are four main conjuncts in the encoding. The first conjunct formalises the fact that, if an action is executed at state \( i \), then all of its preconditions hold at state \( i \), all of its effects hold at state \( i + 1 \), and all the variables that are not in the effects will continue to have the same value at state \( i + 1 \) as they did at state \( i \) (i.e. the frame axiom). The second (third) conjunct states that at least (most) one action must execute at state \( i \). The fourth conjunct states that all states are pairwise distinct by stating that for every two states, at least one variable has a different truth value in both states.

**Theorem 5.** \( \phi_2(\delta, k) \) is satisfiable iff \( k < rd(\delta) \).

**Proof summary.** The theorem follows from Theorem 4 and since \( \phi'_2(\delta, k) \leftrightarrow \phi_2(\delta, k) \). The latter fact follows by an induction on \( k \), and from the definition of \( \mathcal{D}(\delta) \) and \( \mathcal{G}(\delta) \).

**Experimental evaluation** We experimentally test the new encoding as a base case function for the algorithm by Abdulaziz, Gretton, and Norrish 2017. Col. 2 and 3 of Table 1 show some data on the bounds computed with both, \( rd \) and \( td \), as base case functions. We note two observations. Firstly, many more planning problems are successfully bounded within the timeout when Encoding 2 is used to compute \( rd \) compared to using Encoding 1. Encoding 2 performs much better than Encoding 1 in practice since our new encoding is represented in terms of the factored representation of the system, while Encoding 1 represents the system as an explicitly represented state space. This leads to exponentially smaller formulae: Encoding 2 grows quadratically with the size of the given factored system, while Encoding 1 grows quadratically in the size of the state space, which can be exponentially larger than the given factored system. Indeed, Encoding 2 delegates the construction of the explicit state space to the SMT solver, which would effectively construct the state space during its search, but lazily. This is better than constructing the state space a priori in when the formula is satisfiable (i.e. when \( k \) is less than \( rd \)) as the SMT solver only needs to find a path of length \( k \). This is done without necessarily traversing the entire state space due to the SMT solver’s search heuristics. When the formula is unsatisfiable, the SMT solver has to perform an exhaustive search to produce a proof of unsatisfiability, which is equivalent to constructing the entire state space explicitly. Since all queries to the SMT solver, except for the last one, are satisfiable, Encoding 2 is more practically efficient than using Encoding 1. However, it is worth noting that Encodings 1 and 2 have the same worst-case running time: it is doubly exponential in the number of state variables in the given factored system.

Secondly, when \( rd \) is the base case function the bounds computed are much tighter than those computed when \( td \) as a base case function. This agrees with the theoretical prediction of Theorem 2. This is shown clearly in Figure 2 and in Table 1. In particular, in the domains TPP, ParcPrinter, NoMystery, Logistics, OpenStacks, Woodworking, Satellites, Scanalyzer, Hyp and NewOpen (a compiled Qualitative Preference rovers domain), we have between two orders of magnitude and 50% smaller bounds when \( rd \) is used as a base case function compared to \( td \). Also, the domain Visitall has twice as many problems whose bounds are less than \( 10^9 \) when \( rd \) is used instead of \( td \). Also, specially interesting domains are Floortile and BlocksWorld, where the recurrence diameter of some of the smaller instances there is successfully computed to be less than 10, but whose bounds using \( td \) are more than \( 10^9 \). In contrast, equal bounds are found using \( rd \) and \( td \) as base case functions in Zeno.

**Further Experiments**

Note that, although Encoding 2 is more efficient than Encoding 1, the number of problems that were successfully bounded is still much less when using \( rd \) as a base case compared to the \( td \), since it can still be exponentially more difficult to compute \( rd \) compared to \( td \). This can be seen in most domains, but is most extreme in Elevators, Zeno, Hiking, WoodWorking and Storage. We now try to improve the running time. One thing we observe during our experiments is that many of the base case systems have traversal diameters that are 1 or 2. We conjecture the following.

**Conjecture 1.** For any factored transition system \( \delta \), if \( td(\delta) \in \{1, 2\} \), then \( td(\delta) = rd(\delta) \).

We exploit that conjecture in the following way: we devise the following base case function that will only invoke the expensive computation of \( rd \) in case \( td \) is greater than 2.

**Definition 7.** \( b_1(\delta) = \text{if } 2 < td(\delta) \text{ then } rd(\delta) \text{ else } td(\delta) \).

Because of Conjecture 1 the value of \( b_1 \) is the same as \( rd \). Limiting the computation of \( rd \) as in the base case function \( b_1 \) significantly reduces the bound computation time. This leads to to more problems being successfully bounded
Table 2: Table showing the number of instances solved from every domain, using different bounding techniques.

| Domain     | b1  | b2  | td |
|------------|-----|-----|----|
| newopen    | 173 | 174 | 129|
| logistics  | 192 | 195 | 170|
| rover      | 44  | 44  | 41 |
| nomystery  | 7   | 6   | 4  |
| zeno       | 1   | 1   | 30 |
| hiking     | 1   | 1   | 1  |
| TPP        | 14  | 9   | 1  |
| woodworking| 1   | 1   | —  |
| visitall   | 6   | 6   | 4  |
| openstacks | 6   | 6   | 6  |
| satellite  | 4   | 4   | 6  |
| scanalyzer | 1   | 1   | 1  |
| storage    | 1   | 4   | 4  |
| parcprinter| 3   | 2   | 1  |
| maintenance| 4   | 4   | 4  |

Table 2: Table showing the number of instances solved from every domain, using different bounding techniques.

as shown in col.4 of Table 1, compared to when rd is used. Figure 5 shows that bounds computed using b₁ are the same as those computed when rd is used, and that bound computation using b₁ as a base case function takes significantly less time than when rd is used.

Note, however, that the number of problems successfully bounded when using b₁ as a base case function is still less than the number of problems bounded using when td is used. This is because the large computation cost of rd on the base cases on which it is invoked is still much more than the cost of computing td. Another technique to improve the bound computation time is to limit the computation of rd to problems whose state spaces’ sizes are bound by a constant. This is done with the following base case function.

Definition 8. \( b₂(δ) = \text{if} 50 < \text{Exp}(δ) \text{ then } b₁(δ) \text{ else } td(δ). \)

We chose the upper limit on the state size by recording the size of the largest state space for which the function \( b₁ \) could successfully terminate.

As shown in col. 5 of Table 1, the number of problems that are successfully bounded within 20 minutes when b₂ is used as a base case function is substantially more than those when b₁ is used, especially in the domains where rd and b₁ were less successful than td. However, the bounds computed when b₂ is used are sometimes worse than those computed when b₁ is used, like in the case of Floortile. Figure 6 shows this bound degradation and bound computation time improvement more clearly. This is because there are abstractions whose recurrence diameter is computable within the timeout and whose state spaces have more than 50 states. For those abstractions, td computed instead of rd, when b₂ is used. An interesting configuration problem is adjusting the threshold in b₂ to maximise the number of abstractions whose recurrence diameter can be computed within the timeout. We do not fully explore this problem here.

Using the bounds for SAT-based planning

The coverage of MP increases in more domains if we use as horizons the bounds computed when rd is the base case, compared to when using td as a base case function. Table 2 shows that. Also Figures 8, 10 show the running time of the planner when different bounds are used as horizon, for problems where planning succeeds with the two compared bounds. It clearly confirms that computing tighter upper bounds mainly pays-off, despite the fact that computing those tighter bounds can take longer.

Figure 2: Top: bounds computed when using td as base case function vs rd. Bot: running time to compute bounds when using td as base case function vs rd.

Conclusions

The recurrence diameter was identified by many earlier authors as an appealing upper bound on transition sequence lengths, both in the area of verification (Baumgartner, Kuehlmann, and Abraham 2002; Kroening and Strichman 2003; Kroening et al. 2011) and AI planning (Abdulaziz, Gretton, and Norrish 2015; Abdulaziz, Gretton, and Norrish 2017; Abdulaziz 2019). However, previous authors noted that computing the recurrence diameter is not practically useful, since, in the worst case, it can take exponentially longer than solving the underlying planning or model-checking problem. Nonetheless, we show that, indeed, computing the recurrence diameter can be practically very useful when used for compositional bounding. We do so by providing a careful SMT encoding that exploits the factored state space representation, and by cleverly combining the recurrence diameter with other easier to compute topological properties, like the traversal diameter.
Figure 3: Top: bounds computed when using \( td \) as base case function vs \( b_1 \). Bot: running time to compute bounds when using \( td \) as base case function vs \( b_1 \).

Figure 4: Top: bounds computed when using \( td \) as base case function vs \( b_2 \). Bot: running time to compute bounds when using \( td \) as base case function vs \( b_2 \).

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Figure 7: Top: bounds computed when using \( b_1 \) as base case function vs \( b_2 \). Bot: running time to compute bounds when using \( b_1 \) as base case function vs \( b_2 \).

Figure 8: Comparison of planning running times when using \( b_1 \) vs \( td \) as base case functions for bounded. Top: including bounding time. Bottom: excluding it.

Figure 9: Comparison of planning running times when using \( b_2 \) vs \( td \) as base case functions for bounded. Top: including bounding time. Bottom: excluding it.

Figure 10: Comparison of planning running times when using \( b_2 \) vs \( b_1 \) as base case functions for bounded. Top: including bounding time. Bottom: excluding it.