Corrigendum: Comments on the classification of the finite subgroups of SU(3)

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Table 1 does not contain all known finite subgroups of SU(3). It has to be amended by the addition of

| Group | Order | References |
|-------|-------|------------|
| \(A(m, n) \cong \mathbb{Z}_m \times \mathbb{Z}_n\) (Abelian groups), \(n\) divides \(m\) | \(mn\) | [1, 2] |
| \(B\) (finite subgroups of \(U(2)\)) | No general formula | [1, 2] |
| \(\Sigma(60) \times \mathbb{Z}_3\) | 180 | |
| \(\Sigma(168) \times \mathbb{Z}_3\) | 504 | |

Contrary to what has been stated in the paper,

\[D(n, a; b; d, r, s) \cong (\mathbb{Z}_p \times \mathbb{Z}_q) \rtimes S_3\]

holds true for all choices of the parameters \(d, r\) and \(s\). This has been shown in [2].

Two of the generators of \(\mathcal{A}\) as listed in equation (B.9) are redundant. A minimal set of generators of \(\mathcal{A}\) is given by

\[\mathcal{A} = \langle \langle A, E^{-1}AE, B, E^{-1}BE, F, E^{-1}FE, G^{-1}FG, (EG)^{-1}FEG\rangle\rangle.\]

References

[1] Miller G A, Blichfeldt H F and Dickson L E 1916 Theory and Applications of Finite Groups (New York: Wiley)
[2] Grimus W and Ludl P O 2011 Finite flavour groups of fermions arXiv:1110.6376
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Abstract
Many finite subgroups of SU(3) are commonly used in particle physics. The classification of the finite subgroups of SU(3) began with the work of H F Blichfeldt at the beginning of the 20th century. In Blichfeldt’s work the two series (C) and (D) of finite subgroups of SU(3) are defined. While the group series $\Delta(3n^2)$ and $\Delta(6n^2)$ (which are subseries of (C) and (D), respectively) have been intensively studied, there is not much knowledge about the group series (C) and (D). In this work, we show that (C) and (D) have the structures $(C) \cong (\mathbb{Z}_m \times \mathbb{Z}_m') \rtimes \mathbb{Z}_3$ and $(D) \cong ((\mathbb{Z}_n \times \mathbb{Z}_n') \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$, respectively. Furthermore, we show that, while the (C)-groups can be interpreted as irreducible representations of $\Delta(3n^2)$, the (D)-groups can in general not be interpreted as irreducible representations of $\Delta(6n^2)$.

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1. Introduction

Today finite groups are widely used in physics, and particle physics offers a wide range of applications for the theory of finite groups in particular. Especially, the finite subgroups of SU(3) have been intensively studied in the past, and their investigation and application in various fields of particle physics continues unabated.

The systematic analysis of the finite subgroups of SU(3) for application in particle physics started with the work of Fairbairn et al [1] in 1964. Since then many contributions to the systematic analysis of these groups for application in particle physics have been published. The finite subgroups of SU(3) have been used for model building in hadron physics [1], as well as a computational tool in lattice QCD (see e.g. [2]). Today’s most important application is flavour physics: on the one hand, there is an enormous amount of models using the finite subgroups of SU(3) (see e.g. [3, 4] and references therein) trying to solve the fermion mass and mixing problems in the lepton sector as well as the quark sector. On the other hand, the finite subgroups of SU(3) have also been used in the context of minimal flavour violation (see e.g. [5]).
The well-known group series $T_n$, $\Delta(3n^2)$ and $\Delta(6n^2)$, are sub-series of (C) and (D) [6] of finite subgroups of SU(3). However, (C) and (D) also contain other groups, which have been paid much less attention. The aim of this work is to study the structure of the finite subgroups of SU(3) of types (C) and (D) and their relation to other finite subgroups of SU(3).

The classification of the finite subgroups of SU(3)

Here we want to list the main efforts that have been put in the classification of the finite subgroups of SU(3) in chronological order.

- In 1916, Miller, Blichfeldt and Dickson published their book Theory and Applications of Finite Groups [6]. In the part written by Blichfeldt, the finite subgroups of SU(3) are classified in terms of their generators. The series $\Delta(3n^2)$ and $\Delta(6n^2)$ are not explicitly defined but are contained in the series (C) and (D).
- In 1964, the paper ‘Finite and disconnected subgroups of SU(3) and their application to the elementary-particle spectrum’ by Fairbairn, Fulton and Klink was published [1]. It is the first paper which faced the task of analysing a large set of finite subgroups of SU(3) for their use as symmetries in particle physics (hadron physics in this special case). In [1], the group series $\Delta(3n^2)$ and $\Delta(6n^2)$ are already included.
- In their papers ‘Representations and Clebsch–Gordan coefficients of Z-metacyclic groups’ [7] and ‘Finite subgroups of SU(3)’ [8] (both published in 1981), Bovier, Lüling and Wyler defined and analysed the SU(3)-subgroups $T_n$. In particular, they constructed all irreducible representations and calculated all Clebsch–Gordan coefficients for these groups. Furthermore, Bovier et al investigated the group series $\Delta(3n^2)$ and $\Delta(6n^2)$ in detail giving not only the irreducible representations but also the Clebsch–Gordan coefficients for both series.
- Two years later, in their paper ‘Some comments on finite subgroups of SU(3)’ [9], Fairbairn and Fulton proved that some groups of the type $T_n$ given by Bovier et al in [8] are not the subgroups of SU(3).
- In 2007, Luhn, Nasri and Ramond published their work ‘The flavor group $\Delta(3n^2)$’ [10], giving all conjugacy classes, irreducible representations, character tables and Clebsch–Gordan coefficients of $\Delta(3n^2)$.
- In 2008, Escobar and Luhn published their analysis ‘The flavor group $\Delta(6n^2)$’ [11], giving all conjugacy classes, irreducible representations, character tables and Clebsch–Gordan coefficients of $\Delta(6n^2)$.
- In 2009, the work ‘Systematic analysis of finite family symmetry groups and their application to the lepton sector’ [3] was published. It contains an analysis and summary of all finite subgroups of SU(3). With the help of [10] it can be shown that all SU(3)-subgroups of type (C) can be interpreted as three-dimensional irreducible representations of $\Delta(3n^2)$. The generators of the group series (D) are determined explicitly.
- In the same year, Zwicky and Fischbacher showed that every (D)-group is a subgroup of $\Delta(6n^2)$ for a suitable $n$ in their paper ‘On discrete minimal flavour violation’ [5].
- In the work ‘On the finite subgroups of U(3) of order smaller than 512’ [12] (published in 2010), all finite subgroups of U(3) of order smaller than 512 which possessed a faithful three-dimensional irreducible representation are listed. Among these groups there were no SU(3)-subgroup which did not fit into the classification scheme of Blichfeldt [6].
- In their paper ‘Tribimaximal mixing from small groups’ [13], Parattu and Wingerter also began to analyse those finite subgroups of U(3) which possessed a faithful three-dimensional reducible representation but did not possess any faithful irreducible
Table 1. Types of finite subgroups of SU(3) \([1, 6, 8, 9]\). The allowed values for \(n\) and \(p\) in \(T_n\) are products of powers of primes of the form \(3^k + 1\), \(k \in \mathbb{N}\) \([8, 9]\).

| Group               | Order | References |
|---------------------|-------|------------|
| \(C(n, a, b)\)      | No general formula | [3, 6] |
| \(D(n, a, b; d, r, s)\) | No general formula | [3, 5, 6] |
| \(\Delta(3n^2) \cong C(n, 0, 1), \ n \geq 2\) | \(3n^2\) | [1, 3, 7, 10] |
| \(\Delta(6n^2) \cong D(n, 0, 1; 2, 1, 1), \ n \geq 2\) | \(6n^2\) | [1, 3, 7, 11] |
| \(T_n \cong C(n, 1, a), \ \ (1 + a + a^2) \mod n = 0\), or | \(3n\) | [3, 7–9] |
| \(T_n \cong C(3p, 1, a), \ \ (1 + a + a^2) \mod 3p = 0; n = 3p\) | \(3n = 9p\) | |
| \(A_5 \cong \Sigma(60)\) | 60 | [1, 3, 6, 14, 15] |
| \(PSL(2, 7) \cong \Sigma(168)\) | 168 | [1, 3, 6, 14] |
| \(\Sigma(36 \times 3)\) | 108 | [1, 3, 6, 16, 17] |
| \(\Sigma(72 \times 3)\) | 216 | [1, 3, 6, 16, 17] |
| \(\Sigma(216 \times 3)\) | 648 | [1, 3, 6, 16, 17] |
| \(\Sigma(360 \times 3)\) | 1080 | [1, 3, 6] |

representation. Their analysis of all groups up to order 100 showed no finite subgroups of SU(3) which did not fit into Blichfeldt’s classification scheme \([6]\).

Table 1 shows the different types of finite subgroups of SU(3), as they are classified by now.

2. Abelian subgroups of SU(3)

In the following sections, we will frequently deal with Abelian subgroups of SU(3). The remarkably simple theorem 2.1 provides us with all necessary information we will need in our later analysis.

**Theorem 2.1.** Every finite Abelian subgroup \(A\) of SU(3) is isomorphic to \(\mathbb{Z}_m \times \mathbb{Z}_n\), where

\[
m = \max_{a \in A} \text{ord}(a)
\]

and \(n\) is a divisor of \(m\). The proof of this theorem can be found in appendix A.

3. On the SU(3)-subgroups of type (C)

In this section, we investigate the structure of the SU(3)-subgroups of type (C). Knowing the structure of (C) we can easily show that there exist SU(3)-subgroups of type (C) which neither belong to the series \(\Delta(3n^2)\), nor to the groups of type \([7–9]\)

\[
T_n = \mathbb{Z}_n \rtimes \mathbb{Z}_3.
\]

In the following, the symbol \(\langle \{ \cdots \} \rangle\) means ‘generated by’. The group series (C) is generated by the matrices

\[
E := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad F(n, a, b) := \begin{pmatrix} \eta^a & 0 & 0 \\ 0 & \eta^b & 0 \\ 0 & 0 & \eta^{-a-b} \end{pmatrix},
\]
with $\eta = \exp(2\pi i/n)$.

$$C(n, a, b) := \langle \langle E, F(n, a, b) \rangle \rangle, \quad n \in \mathbb{N}\setminus\{0\}, \quad a, b \in \{0, \ldots, n-1\}. \quad (3)$$

Since the irreducible three-dimensional representations of $\Delta(3n^2)$ are [10]

$$\mathfrak{A}_{(a,b)} : G_1 \mapsto E, \quad G_2 \mapsto F(n, b, a), \quad (4)$$

where $G_1$ and $G_2$ denote the generators of $\Delta(3n^2)$, we find

$$C(n, a, b) \cong \mathfrak{A}_{(b,a)}(\Delta(3n^2)). \quad (5)$$

There is no closed formula for the order of $C(n, a, b)$, but we can give a prescription for the calculation of the order of $C(n, a, b)$ for given $n, a, b$. Let us first think about the structure of $C(n, a, b)$. Defining

$$X := F(n, a, b), \quad Y := F(n, b, -a - b), \quad (6)$$

we find the commutation relations

$$XE = EX^{-1}Y^{-1}, \quad YE = EX. \quad (7)$$

Therefore,

- the subgroup $\langle \langle X, Y \rangle \rangle$ of all diagonal matrices is a normal subgroup of $C(n, a, b)$ and
- every element of $C(n, a, b)$ can be written in the form $E^j X^k Y^l$.

Furthermore

$$\langle \langle E \rangle \rangle \cap \langle \langle X, Y \rangle \rangle = \{1\}, \quad (8)$$

thus,

$$C(n, a, b) = \langle \langle X, Y \rangle \rangle \ltimes \langle \langle E \rangle \rangle. \quad (9)$$

From theorem 2.1, we find

$$\langle \langle X, Y \rangle \rangle \cong \mathbb{Z}_m \times \mathbb{Z}_p, \quad (10)$$

where

$$m = \max_{A \in \langle \langle X, Y \rangle \rangle} \text{ord}(A) = \text{ord}(X) = \text{ord}(Y) = \text{lcm}(\text{ord}(\eta^a), \text{ord}(\eta^b)). \quad (11)$$

$lcm(r, s)$ denotes the lowest common multiple of $r, s \in \mathbb{N}$. Defining

$$p := \min\{k \in \{1, \ldots, m\} | Y^k \in \langle \langle X \rangle \rangle\}, \quad (12)$$

we find

$$\langle \langle X, Y \rangle \rangle = \{X^i Y^j | i = 0, \ldots, m-1; \quad j = 0, \ldots, p-1\} \Rightarrow \langle \langle X, Y \rangle \rangle \cong \mathbb{Z}_m \times \mathbb{Z}_p. \quad (13)$$

For the sake of completeness, we also want to find one possible choice of generators of $\mathbb{Z}_m$ and $\mathbb{Z}_p$, respectively. Applying the same argumentation as above on the definition

$$q := \min\{k \in \{1, \ldots, m\} | X^k \in \langle \langle Y \rangle \rangle\}, \quad (14)$$

we find $\langle \langle X, Y \rangle \rangle \cong \mathbb{Z}_m \times \mathbb{Z}_q \Rightarrow q = p$, and therefore

$$\langle \langle X \rangle \rangle \cap \langle \langle Y \rangle \rangle = \langle \langle X^p \rangle \rangle = \langle \langle Y^p \rangle \rangle. \quad (15)$$

Since $\text{ord}(X^p) = \frac{m}{p}$, this leads to

$$\exists t \in \{1, \ldots, \frac{m}{p} - 1\} : \quad X^{pt} = Y^p \Rightarrow (Y X^{-t})^p = \mathbb{1}_3, \quad (16)$$

4
but 

\[(XY^{-i})^a \neq 1_3 \quad \text{for} \quad a < p,\]

because otherwise \(Y^a \in \langle \langle X \rangle \rangle\) for \(a < p\), which would be a contradiction to the definition (12) of \(p\). Therefore

\[
\langle \langle XY^{-i} \rangle \rangle \cong \mathbb{Z}_p.
\]

Noting furthermore that

\[
\langle \langle X \rangle \rangle \cap \langle \langle Y^{-1} \rangle \rangle = \{1_3\},
\]

we finally find

\[
\langle \langle X, Y \rangle \rangle = \langle \langle X \rangle \rangle \times \langle \langle Y^{-1} \rangle \rangle \cong \mathbb{Z}_m \times \mathbb{Z}_p.
\]

Now there are three cases:

1. \(p = 1 \Rightarrow Y \in \langle \langle X \rangle \rangle \Rightarrow \langle \langle X, Y \rangle \rangle = \langle \langle X \rangle \rangle \Rightarrow C(n, a, b) \cong \mathbb{Z}_m \times \mathbb{Z}_3,
\]

2. \(p = m \Rightarrow \langle \langle X \rangle \rangle \cap \langle \langle Y \rangle \rangle = \{1_3\} \Rightarrow \langle \langle X, Y \rangle \rangle \cong \mathbb{Z}_m \times \mathbb{Z}_m
\]

\[
\Rightarrow C(n, a, b) \cong (\mathbb{Z}_m \times \mathbb{Z}_m) \times \mathbb{Z}_3 \cong \Delta(3m^2).
\]

3. \(p \in \{2, \ldots, m - 1\} \Rightarrow C(n, a, b) \cong (\mathbb{Z}_m \times \mathbb{Z}_p) \times \mathbb{Z}_3,
\]

\(m\) is determined by equation (11) and from equations (12) and (15) we can determine \(p\) and \(t\). One finds

\[
Y^p = X^{pt} \Rightarrow \left\{ \begin{array}{l}
[p(b - at) \mod n = 0, \quad p \in \{1, \ldots, m\}, \quad \text{smallest possible} \\
[p(a + b(1 + t)) \mod n = 0, \quad t \in \{1, \ldots, \frac{m}{p} - 1\}.
\end{array} \right.
\]

Let us summarize our results on the structure of the groups of type (C):

- \(C(n, a, b) \cong (\mathbb{Z}_m \times \mathbb{Z}_p) \times \mathbb{Z}_3, \quad \text{where}
\]
- \(m = \text{lcm}(\text{ord}(n^p), \text{ord}(p^m)), \quad \text{and}
\]
- \(\{ [p(b - at)] \mod n = 0, \quad p \in \{1, \ldots, m\}, \quad \text{smallest possible}, \quad [p(a + b(1 + t))] \mod n = 0, \quad t \in \{1, \ldots, \frac{m}{p} - 1\}.
\]

- In terms of generators: \(C(n, a, b) = (\langle \langle X \rangle \rangle \times \langle \langle Y^{-1} \rangle \rangle) \times \langle \langle E \rangle \rangle.
\]
- \(\text{ord}(C(n, a, b)) = 3mp.
\]
- \(p = 1 \Rightarrow C(n, a, b) \cong \mathbb{Z}_m \times \mathbb{Z}_3 \quad (\rightarrow T_m \text{ for appropriate } m \text{ (see [7–9])})).
\]
- \(p = m \Rightarrow C(n, a, b) \cong \Delta(3m^2) \cong (\mathbb{Z}_m \times \mathbb{Z}_m) \times \mathbb{Z}_3.
\]

Note that (C)-groups can be direct products with \(\mathbb{Z}_3\), i.e. the case

\[
C(n, a, b) \cong (\mathbb{Z}_3 \times \mathbb{Z}_p), \quad \text{is possible for some choices of } n, a, b.
\]

Examples for this case are the groups

\[
C(6, 1, 1) \cong (\mathbb{Z}_6 \times \mathbb{Z}_2) \times \mathbb{Z}_3 \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \times (\mathbb{Z}_3 \times \mathbb{Z}_3) \cong A_4 \times \mathbb{Z}_3
\]

and the group \(T_{21}\), which is described in [9]:

\[
T_{21} \cong C(21, 1, 4) \cong Z_{21} \times Z_3 \cong (Z_7 \times Z_3) \times Z_3.
\]

As the last part of our investigation of the group series (C) we want to give an example for a (C)-group which neither belongs to the series \(\Delta(3n^2)\), nor to the groups of type \(T_n\). We already encountered an example in the group \(C(6, 1, 1)\), but we also want to give an example for a ‘new’ SU(3)-subgroup which is not just a direct product with an already well-known group. In [12], all groups of order smaller than 512 which possess a faithful three-dimensional irreducible representation (and are not isomorphic to a direct product with a cyclic group) have been listed. Among these groups \(C(9, 1, 1)\) appears as the smallest (C)-group which is not classified as \(T_n\) or \(\Delta(3n^2)\). Using the tools we have developed in this section we immediately find
• \( n = 9, \, a = b = 1. \)
• \( m = \text{lcm}(\text{ord}(\eta^a), \, \text{ord}(\eta^b)) = \text{lcm}(9, \, 9) = 9. \)

- The equations for \( p \) and \( t \) read:
  \[
  \begin{align*}
  \left[ p(1 - t) \right] \mod 9 &= 0, \quad p \in \{1, \ldots, 9\}, \quad \text{smallest possible}, \\
  \left[ p(2 + t) \right] \mod 9 &= 0, \quad t \in \{1, \ldots, \frac{n}{p} - 1\}.
  \end{align*}
  \]
  with the solution \( p = 3, \, t = 1. \)

- Thus
  \[
  C(9, \, 1, \, 1) \cong (\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3, \tag{22}
  \]
  which coincides with the structure description given in [13], where the group is named by its SmallGroup number\(^1\) \([81, \, 9]\).

4. On the structure of the SU(3)-subgroups of type (D)

According to [6] the groups of type (D) are generated by the matrices

\[
E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad F = F(n, \, a, \, b) := \begin{pmatrix} \eta^a & 0 & 0 \\ 0 & \eta^b & 0 \\ 0 & 0 & \eta^{-a-b} \end{pmatrix}, \tag{23}
\]

\((\eta = \exp(2\pi i/n), \, n \in \mathbb{N}\{0\}, \, a, \, b \in \{0, \ldots, n - 1\})\) of (C) and an additional generator \(\tilde{G}\) of the form

\[
\tilde{G} = \begin{pmatrix} x & 0 & 0 \\ 0 & 0 & y \\ 0 & z & 0 \end{pmatrix}. \tag{24}
\]

The conditions \(\det \tilde{G} = 1\) and \(\text{ord}(\tilde{G}) < \infty\) lead to [3]

\[
\tilde{G} = \tilde{G}(d, \, r, \, s) := \begin{pmatrix} \delta^r & 0 & 0 \\ 0 & \delta^s & 0 \\ 0 & -\delta^{-r-s} & 0 \end{pmatrix} \tag{25}
\]

with

\[
\delta = \exp(2\pi i/d), \quad d \in \mathbb{N}\{0\}, \quad r, \, s \in \{0, \ldots, d - 1\}. \tag{26}
\]

Thus

\[
D(n, \, a, \, b; \, d, \, r, \, s) := \langle \langle E, \, F(n, \, a, \, b), \, \tilde{G}(d, \, r, \, s) \rangle \rangle. \tag{27}
\]

For a better understanding of the structure of (D), it is helpful to reformulate the generators of the group. We define

\[
A := \tilde{G}^2 = \begin{pmatrix} \delta^{2r} & 0 & 0 \\ 0 & -\delta^{-r} & 0 \\ 0 & 0 & -\delta^{-s} \end{pmatrix}, \quad G := E^2 \tilde{G}^2 E \tilde{G} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & \delta^{2r+s} \\ 0 & \delta^{-(2r+s)} & 0 \end{pmatrix}, \tag{28}
\]

which leads to

\[
D(n, \, a, \, b; \, d, \, r, \, s) = \langle \langle A, \, E, \, F, \, G \rangle \rangle. \tag{29}
\]

Our first important observation is that, as in the case of (C), the subgroup \(A\) of all diagonal matrices is a normal subgroup of (D). This gives us hope to find a semidirect product structure

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\(^1\) See [12, 13] for an explanation of the SmallGroup number (also called GAP ID in [13]).
as in the case of (C). The action of the (non-diagonal) generators of (D) on any diagonal matrix is given by

\[
\begin{align*}
G^{-1}\text{diag}(a, b, c)G &= \text{diag}(a, c, b), \\
E^{-1}\text{diag}(a, b, c)E &= \text{diag}(c, a, b), \\
(EG)^{-1}\text{diag}(a, b, c)EG &= \text{diag}(b, c, a), \\
E^{-2}\text{diag}(a, b, c)E^2 &= \text{diag}(b, c, a), \\
(E^2G)^{-1}\text{diag}(a, b, c)E^2G &= \text{diag}(b, a, c).
\end{align*}
\]

This describes an $S_3$-action, which is well known from $\Delta(6n^2)$ [11], so the structure of (D) will be, though not identical in general, very similar to the structure

\[
\Delta(6n^2) \cong (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes S_3 \cong ((\mathbb{Z}_n \times \mathbb{Z}_n) \times \mathbb{Z}_3) \times \mathbb{Z}_2
\]

of $\Delta(6n^2)$. Indeed

\[
\langle \langle A, E, F, G \rangle \rangle \cong (A \rtimes \langle \langle E \rangle \rangle) \rtimes \langle \langle G \rangle \rangle \cong (A \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2,
\]

with $\langle \langle A, F \rangle \rangle \subset A$. Note that in general (D) will not have the structure of a semidirect product with $S_3$. Later we will see that

\[
D(n,a,b; d,r,s) \cong A \rtimes S_3
\]

only for a special choice of the parameters $d, r, s$. For the explicit construction of $A$, we refer the reader to appendix B. Theorem 2.1 and equation (32) lead to

\[
D(n,a,b; d,r,s) \cong (A \rtimes \langle \langle E \rangle \rangle) \rtimes \langle \langle G \rangle \rangle \cong ((\mathbb{Z}_p \times \mathbb{Z}_q) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2
\]

where $p$ and $q$ are functions of $n, a, b, d, r, s$, respectively. Let us dwell a bit longer on the structure of (D). If $\langle \langle E, G \rangle \rangle \cap A = \{1\}$ we have

\[
D(n,a,b; d,r,s) \cong A \rtimes \langle \langle E, G \rangle \rangle.
\]

This is the case if and only if the only diagonal matrix in $\langle \langle E, G \rangle \rangle$ is the identity matrix. A generating set for the diagonal matrices generated by $E$ and $G$ can be found in equation (B.3) in appendix B. From equation (B.3), we find that

\[
A \cap \langle \langle E, G \rangle \rangle = \{1\} \Leftrightarrow \tilde{B} = \text{diag}(\delta_r+2s, -\delta_r-2s, -\delta_r-2s) = 1, \\
\Leftrightarrow \delta^2 = -1 \Leftrightarrow \frac{2r+s}{d} = k + \frac{1}{2}, \quad k \in \mathbb{N} \\
\Leftrightarrow G = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}
\]

(36)

\[
\Leftrightarrow \langle \langle E, G \rangle \rangle \cong S_3 \Leftrightarrow D(n,a,b; d,r,s) \cong A \rtimes S_3.
\]

The simplest non-trivial choices for the parameters $(d, r, s)$ are those where

\[
A = B = 1, \quad \text{and} \quad G = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix},
\]

(37)

i.e. the ones with

\[
\delta' = \delta' = -1 \Leftrightarrow \frac{r}{d} = k + \frac{1}{2}, \quad \frac{s}{d} = k' + \frac{1}{2}, \quad k, k' \in \mathbb{N},
\]

(38)

the simplest solution being $(d, r, s) = (2, 1, 1)$. While every group of type (C) can be interpreted as an irreducible representation of a group of type $\Delta(3n^2)$ (see equation (5)), a similar statement does not hold for (D) and $\Delta(6n^2)$. In the following, we show that there
is at least one SU(3)-subgroup of type (D), which cannot be interpreted as an irreducible representation of some $\Delta(6n^2)$. In [12], the following (D)-group has been found:

$$D(9, 1, 1; 2, 1, 1) \cong [162, 14],$$

which is of order 162. Since $C(9, 1, 1)$ is invariant under the action of the $\mathbb{Z}_2$-generator $G$, we find

$$D(9, 1, 1; 2, 1, 1) \cong C(9, 1, 1) \times (\langle G \rangle) \cong (\mathbb{Z}_9 \times \mathbb{Z}_3) \times \mathbb{Z}_2 \cong (\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes S_3,$$  \hspace{1cm} (39)

where we have used the fact that, as shown before, $(d, r, s) = (2, 1, 1)$ leads to a semidirect product with $S_3$. Equation (39) suggests that there might be an irreducible three-dimensional representation $D$ of $\Delta(6n^2) \cong (\mathbb{Z}_9 \times \mathbb{Z}_n) \rtimes S_3$ such that $D(\Delta(6n^2)) \cong D(9, 1, 1; 2, 1, 1)$. However, this is not the case, which we show in the following. If we can show, that there is no three-dimensional irreducible representation $D$ of $\Delta(6n^2)$ with $\text{ord}(D(\Delta(6n^2))) = 162$, we have found an example for a (D)-group that cannot be interpreted as an irreducible representation of some $\Delta(6n^2)$. This is indeed possible.

**Proposition 4.1.** Let $D$ be a three-dimensional irreducible representation of $\Delta(6n^2)$; then

$$\exists m \in \{1, \ldots, n\} : \text{ord}(D(\Delta(6n^2))) \leq \Delta(6m^2).$$

The proof of this proposition can be found in appendix C. Since $162/6 = 27$ is not a square number, we have proven that $D(9, 1, 1; 2, 1, 1)$ cannot be interpreted as an irreducible representation of some $\Delta(6n^2)$.

5. Conclusions

In this work, we tried to shed some light onto the hitherto not very well-known series (C) and (D) of finite subgroups of SU(3). We were able to show that

$$C(n, a, b) \cong (\mathbb{Z}_m \times \mathbb{Z}_p) \rtimes \mathbb{Z}_3,$$

and we gave a method for the determination of $m$ and $p$ from the parameters $n, a, b$. We could also give a simple example for a (C)-group which is neither of the form $\Delta(3n^2)$ nor of the form $T_n$, thus showing that (C) contains some hitherto unclassified subgroups of SU(3). For the SU(3)-subgroups of type (D), we could determine the structure

$$D(n, a, b; d, r, s) \cong ((\mathbb{Z}_p \times \mathbb{Z}_q) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2,$$

where $p$ and $q$ are the functions of $n, a, b, d, r, s$, respectively. Since every (D)-group is a subgroup of some $\Delta(6m^2) \cong ((\mathbb{Z}_m \times \mathbb{Z}_m) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$, it is tempting to assume that the (D)-groups can be interpreted as irreducible representations of $\Delta(6m^2)$. However, by giving a counterexample, we could show that this is not the case in general.

We hope that the analysis given here can lead us a small step closer towards the goal of the classification of all finite subgroups of SU(3). Furthermore, we hope that some of the ’new’ types of finite subgroups of SU(3) discovered in this work can be useful for application in particle physics.

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Appendix A. Proof of theorem 2.1

Theorem 2.1. Every finite Abelian subgroup $G$ of SU(3) is isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_n$, where

$$m = \max_{a \in G} \text{ord}(a) \quad \text{(A.1)}$$

and $n$ is a divisor of $m$.

Proof. Since $G$ is an Abelian group of $3 \times 3$ matrices, we can choose a basis in which all group elements are diagonal. Then, due to $\det(a) = 1 \ \forall a \in G$, all elements of $G$ are of the form

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha^* \beta^* \end{pmatrix}, \quad \alpha, \beta \in U(1). \quad \text{(A.2)}$$

Let

$$m := \max_{a \in G} \text{ord}(a). \quad \text{(A.3)}$$

Then $a^m = \mathbb{1}_3 \ \forall a \in G$, which we prove by contradiction. Suppose $\exists a \in G: \ a^m \neq \mathbb{1}_3 \Rightarrow \text{ord}(a)$ does not divide $m$. Let

$$g := \gcd(\text{ord}(a), m) < \text{ord}(a) \quad \text{(A.4)}$$

denote the greatest common divisor of $\text{ord}(a)$ and $m$. Then the group $\langle \langle a^g \rangle \rangle$ is a non-trivial cyclic group fulfilling

$$\langle \langle a^g \rangle \rangle \cap \langle \langle b \rangle \rangle = \{\mathbb{1}_3\} \ \forall b \in G: \text{ord}(b) = m. \quad \text{(A.5)}$$

$$\Rightarrow \text{ord}(a^g b) = \text{ord}(a^g) \text{ord}(b) = \frac{\text{ord}(a)}{\gcd(\text{ord}(a), m)} \times m > m \Rightarrow \text{contradiction to (A.3)}.$$

Defining

$$\mu := \exp(2\pi i/m), \quad \text{(A.6)}$$

every element of $G$ has the form

$$\begin{pmatrix} \mu^i & 0 & 0 \\ 0 & \mu^j & 0 \\ 0 & 0 & \mu^{-i-j} \end{pmatrix}, \quad i, j \in \{0, \ldots, m-1\}. \quad \text{(A.7)}$$

Thus $G$ is a subgroup of

$$\langle \langle \begin{pmatrix} \mu & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu^* \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu^* \end{pmatrix} \rangle \rangle \cong \mathbb{Z}_m \times \mathbb{Z}_m. \quad \text{(A.8)}$$

Let

$$m = p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j} \quad \text{(A.8)}$$

be the prime factorization of $m$ ($p_1, \ldots, p_j$ are the distinct prime factors of $m$). Then

$$\mathbb{Z}_m \cong \mathbb{Z}_{p_1^{k_1}} \times \mathbb{Z}_{p_2^{k_2}} \times \cdots \times \mathbb{Z}_{p_j^{k_j}}. \quad \text{(A.9)}$$
$\mathbb{Z}_m$ is a subgroup of $G$, because by definition (A.3) there exists at least one element of order $m$ in $G$. From (A.9) we find that every subgroup $G$ of $\mathbb{Z}_m \times \mathbb{Z}_m$ with $\mathbb{Z}_m \subset G$ has the form
\[
G \cong \mathbb{Z}_{p_1^{i_1}} \times \mathbb{Z}_{p_2^{i_2}} \times \cdots \times \mathbb{Z}_{p_j^{i_j}} \times \mathcal{H}.
\]
By construction $\mathcal{H}$ is a subgroup of $\mathbb{Z}_{p_1^{i_1}} \times \mathbb{Z}_{p_2^{i_2}} \times \cdots \times \mathbb{Z}_{p_j^{i_j}} \cong \mathbb{Z}_m$. So $\mathcal{H}$ is a cyclic group itself and we end up with
\[
G \cong \mathbb{Z}_m \times \mathbb{Z}_n.
\]
(A.11)

Appendix B. The group $\mathcal{A}$ of diagonal matrices in (D)

In this appendix, we want to construct a generating set of the invariant subgroup $\mathcal{A}$ of all diagonal matrices in (D). The generators of $\mathcal{A}$ are

- $A$ and $F$,
- the actions of $E$ and $G$ on $A$ and $F$ (see equation (30))
- and a generating set of all diagonal matrices that can be constructed from $E$ and $G$.

The generating set of all diagonal matrices that can be constructed from $E$ and $G$ can be obtained in the following way. At first we note that, since $\text{ord}(E) = 3$ and $\text{ord}(G) = 2$, every element of $\langle E, G \rangle$ can be written as a product
\[
E^{i_1}G^{i_2}E \cdots E^{i_s}G^{i_{s+1}}G \cdots G E^{i_n}, \quad i_1, \ldots, i_m \in \{0, 1, 2\}.
\]
(B.1)

Since there is no diagonal matrix of the form $E^i G E^j$, $i, j \in \{0, 1, 2\}$, (B.2)

we find that we need an even number of matrices $G$ to construct a diagonal matrix from $E$ and $G$. The simplest non-trivial diagonal matrices constructed from $E$ and $G$ are those with two matrices $G$, i.e.
\[
E G E^2 G = (E G)^2 = \text{diag}(\delta - 2r^2 e, -\delta - 2r^2 e) = B,
\]
\[
G E G = (G E)^2 = \text{diag}(-\delta - 2r^2 e, \delta - 2r^2 e) = G^{-1} E^{-1} B^{-1} E G.
\]
(G.2)

All non-diagonal matrices of the form $E^i G E^j G^k$ are of the form $D E^l$, $l \in \{1, 2\}$, (B.4)

where $D$ is a diagonal matrix of the form (B.3) or $I$. In the following, $D_i$, $i = 1, \ldots, n$, denote diagonal matrices containing two generators $G$, i.e. one of the six matrices from equation (B.3) or $I$. Let us consider an arbitrary diagonal matrix $D$ constructed from $E$ and $G$. Since we must have an even number of generators $G$ to get a diagonal matrix we find
\[
D = E^{i_1} G^{i_2} E \cdots G^{i_{s+1}} = D_1 E^{i_1} G^{i_2} E \cdots G^{i_{s+1}}
\]
\[
= D_1 D_2 E^{i_1} G^{i_2} E \cdots G^{i_{s+1}}
\]
\[
= \cdots = D_1 \cdots D_{n-1} E^{i_{n-1}+i_{n}} G^{i_{n-1}} G \cdots G^{i_{s+1}}
\]
\[
= D_1 \cdots D_n.
\]
(B.5)
Thus the six matrices from equation (B.3) generate all diagonal matrices in $\langle\langle E, G \rangle\rangle$. Taking into account that not all of the matrices defined in equation (B.3) are independent, we find the following generating set of $A$:

$$A = \begin{pmatrix}
\delta^2 r & 0 & 0 \\
0 & -\delta^{-r} & 0 \\
0 & 0 & -\delta^{-r}
\end{pmatrix}, \quad B := A^{-2} \tilde{B} = \begin{pmatrix}
\delta^{2s} & 0 & 0 \\
0 & -\delta^{-s} & 0 \\
0 & 0 & -\delta^{-s}
\end{pmatrix},$$

$$F = \begin{pmatrix}
\eta^a & 0 & 0 \\
0 & \eta^b & 0 \\
0 & 0 & \eta^{-a-b}
\end{pmatrix}.$$  \hfill (B.6)

$A = \langle\langle A, B, F, \rangle\rangle$,

$$G^{-1} AG, E^{-1} AE, (EG)^{-1} AEG, E^{-2} AE^2, (E^2 G)^{-1} AE^2 G,$$

$$G^{-1} BG, E^{-1} BE, (EG)^{-1} BEG, E^{-2} BE^2, (E^2 G)^{-1} BE^2 G, \quad (E^2 G)^{-1} F E^2 G).$$  \hfill (B.7)

For any diagonal phase matrix $D$ of determinant 1 (thus for any element of $A$), we have

$$E^{-2} D E^2 = [D(E^{-1} D E)]^{-1},$$

therefore the generators $E^{-2} A E^2, (E^2 G)^{-1} AE^2 G, E^{-2} B E^2, (E^2 G)^{-1} BE^2 G, E^{-2} F E^2$ and $(E^2 G)^{-1} F E^2 G$ are redundant. Using furthermore $G^{-1} A G = A$ and $G^{-1} B G = B$, we end up with

$$A = \langle\langle A, B, F, \rangle\rangle,$$

$$E^{-1} AE, (EG)^{-1} AEG,$$

$$E^{-1} BE, (EG)^{-1} BEG,$$

$$G^{-1} FG, E^{-1} FE, (EG)^{-1} FEG\rangle.$$  \hfill (B.9)

**Appendix C. Proof of proposition 4.1**

**Proposition 4.1.** Let $D$ be a three-dimensional irreducible representation of $\Delta(6n^2)$; then

$$\exists m \in \{1, \ldots, n\} : \quad D(\Delta(6n^2)) \cong \Delta(6m^2).$$  \hfill (C.1)

**Proof.** Following [11], equation (C.2) comprises a presentation of $\Delta(6m^2)$ in terms of four generators:

$$P^3 = Q^2 = (P Q)^2 = 1 \quad S_3 \ - \ \text{presentation}$$

$$R^n = S^n = 1, \quad R S = S R \quad \mathbb{Z}_n \times \mathbb{Z}_n \ - \ \text{presentation}$$

$$P R P^{-1} = R^{-1} S^{-1}, \quad P S P^{-1} = R \quad \text{action of } S_3$$

$$Q R Q^{-1} = S^{-1}, \quad Q S Q^{-1} = R^{-1} \quad \text{on } \mathbb{Z}_n \times \mathbb{Z}_n$$

(C.2)

There are only two types of three-dimensional irreducible representations of $\Delta(6m^2)$, namely

\[3_{1(l)} : P \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad Q \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \]

\[R \mapsto \begin{pmatrix} \eta & 0 & 0 \\ 0 & \eta^{-l} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & \eta^{-l} \end{pmatrix}, \quad \eta^l = 1.\]
\[ \mathcal{Z}_2(t) : P \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad Q \mapsto \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \]
\[ R \mapsto \begin{pmatrix} \eta^l & 0 & 0 \\ 0 & \eta^{-l} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta^l & 0 \\ 0 & 0 & \eta^{-l} \end{pmatrix}. \]

\[ \eta := \exp(2\pi i / n), \quad n \in \mathbb{N} \setminus \{0, 1\}, \quad l \in \{1, \ldots, n-1\}. \]

The matrix groups defined by the irreducible representations given above fulfill the presentation (C.2) with \( n \) replaced by \( m := \text{ord}(\eta^l) \), (C.3) and thus

\[ D(\Delta(6n^2)) \cong \Delta(6m^2) \] for all three-dimensional irreducible representations of \( \Delta(6m^2) \). \( \square \)

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