Determinantal Point Processes in the Flat Limit: Extended L-ensembles, Partial-Projection DPPs and Universality Classes

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This paper has now been divided into two parts:

• Part I details extended L-ensembles as a new representation for DPPs, is entitled “Extended L-ensembles: a new representation for Determinantal Point Processes” has now been published here [24]. This first part is independent of the flat limit problem.
• Part II studies the flat limit of L-ensembles (that is best described by extended L-ensembles), is entitled “Determinantal Point Processes in the Flat Limit” has now been published here [5].

In both papers, we have added examples and illustrations, and removed some technical material, in order to clarify our main results.

Abstract: Determinantal point processes (DPPs) are repulsive point processes where the interaction between points depends on the determinant of a positive-semi definite matrix. The contributions of this paper are two-fold.

First of all, we introduce the concept of extended L-ensemble, a novel representation of DPPs. These extended L-ensembles are interesting objects because they fix some pathologies in the usual formalism of DPPs, for instance the fact that projection DPPs are not L-ensembles. Every (fixed-size) DPP is an (fixed-size) extended L-ensemble, including projection DPPs. This new formalism enables to introduce and analyze a subclass of DPPs, called partial-projection DPPs.

Secondly, with these new definitions in hand, we first show that partial-projection DPPs arise as perturbative limits of L-ensembles, that is, limits in $\varepsilon \to 0$ of L-ensembles based on matrices of the form $\varepsilon A + B$ where $B$ is low-rank. We generalise this result by showing that partial-projection DPPs also arise as the limiting process of L-ensembles based on kernel matrices, when the kernel function becomes flat (so that every point interacts with every other point, in a sense). We show that the limiting point process depends mostly on the smoothness of the kernel function. In some cases, the limiting process is even universal, meaning that it does not depend on specifics of the kernel function, but only on its degree of smoothness.
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Introduction

Determinantal point processes are by now perhaps the most famous example of repulsive point processes. They first appeared as a model for the position of fermionic particles in an energy potential [16], but also occur in random matrix theory and graph theory. More recently they have been advocated in machine learning as a way of providing samples with guaranteed diversity [14]. In that framework, one has a set of \( n \) items, and one desires to produce a subset \( \mathcal{X} \) of size \( m \ll n \) such that no two items in \( \mathcal{X} \) are excessively similar. A key aspect of DPPs is that “diversity” is defined relative to a notion of similarity represented by a positive-definite kernel. For instance, if the items are vectors in \( \mathbb{R}^d \), similarity may be defined via the squared-exponential (Gaussian) kernel:

\[
\kappa_\varepsilon(x, y) = \exp\left(-\varepsilon \|x - y\|^2\right)
\]  

(1)

Here \( x \) and \( y \) are two items, and similarity is a decreasing function of distance.

The class of DPPs can be separated into two subclasses: a large subclass called \textit{L-ensembles} grouping the DPPs that can sample the empty set (the probability of sampling the empty set is strictly positive); and a much smaller class grouping DPPs that cannot (the probability is strictly zero). Precise definitions are to be found in section 1.

By definition, an L-ensemble based on the \( n \times n \) kernel matrix \( L = [\kappa_\varepsilon(x_i, x_j)]_{i,j} \) is a distribution over random subsets \( \mathcal{X} \) such that:

\[
P(\mathcal{X}) \propto \det[\kappa_\varepsilon(x_i, x_j)]_{x_i, x_j \in \mathcal{X}^2}
\]

If two or more points in \( \mathcal{X} \) are very similar (in the sense of the kernel function), then the matrix \( L_\mathcal{X} = [\kappa_\varepsilon(x_i, x_j)]_{x_i, x_j \in \mathcal{X}^2} \) has rows that are nearly collinear and the determinant is small (see fig. 1). This in turns makes it unlikely that such a set \( \mathcal{X} \) will be selected by the L-ensemble.

Importantly, how fast similarity decreases with distance is determined by the inverse-scale parameter \( \varepsilon \). Like other kernel methods, L-ensembles are plagued with hyperparameters and finding the “right” value for \( \varepsilon \) is no easy task. Partial answers to this difficulty may be obtained via the study of the so-called “flat limit”, originally studied by Driscoll & Fornberg in Radial Basis Function interpolation, which simply consists in taking \( \varepsilon \to 0 \) in eq. (1) (or similar kernels).

This paper addresses the question of the behaviour of L-ensembles based on similarity kernels for which \( \varepsilon \to 0 \). To this end, we build upon the work in [4], where general results on the spectral properties of kernel matrices are established in the flat limit.

Contributions

Our contributions go beyond a study of the flat limit. As it turns out, the limit processes belong to a specific subclass of DPPs we call “partial-projection...
Fig 1: L-ensembles generate random subsets with probability proportional to the determinant of a kernel matrix. The ground set $\Omega$ represents the items to sample from: in this figure the points in light gray. Two possible subsets of size 3 are represented in blue and red, respectively. An L-ensemble may be defined using the Gaussian kernel (eq. 1), for instance, and $\epsilon$ controls the length-scale of the kernel (the “standard deviation” of the Gaussian kernel equals $\frac{1}{2\sqrt{\epsilon}}$, represented by the two vertical bars on the left). On the right, we show the kernel matrices corresponding to the two sets, for two values of $\epsilon$. The set $X = \{a, b, c\}$ contains points that are much closer together than the set $X' = \{d, e, f\}$: accordingly, the kernel matrix formed from $X'$ is much better conditioned than one formed from $X$, which is reflected in the determinant. An L-ensemble is therefore much more likely to sample $X'$ than $X$. 
Fig 2: In this article, we study the limit of L-ensembles as $\varepsilon \to 0$, meaning that the length-scale of the kernel goes to infinity. Although all kernel matrices are equal to the constant matrix in that limit, and all determinants go to 0, ratios of two determinants go to a fixed quantity. This is what the figure shows: the left-hand part shows the determinants of the two kernel matrices from fig. 1 corresponding to $\mathcal{X}$ and $\mathcal{X}'$, as a function of $\varepsilon$. The right-hand part shows their ratio. The two red dots are for $\varepsilon = 10$ and $\varepsilon = 3/4$. As $\varepsilon \to 0$, set $\mathcal{X}'$ is roughly 100 times more likely than set $\mathcal{X}$ to be sampled.
Fig 3: An overview of the space of DPPs as studied in this article. The whole sphere represents the class of DPPs. The interior of the sphere represents the (large) subclass of L-ensembles. In L-ensembles, the size of the point process is always allowed to be zero. The gray boundary represents DPPs that do not allow the empty set. In such processes, $|\mathcal{A}| \geq p > 0$ almost surely. We call such DPPs “partial-projection DPPs” for reasons explained in section 2. A special case of partial projection DPPs are the projection DPPs, in which $|\mathcal{A}|$ is fixed. While L-ensembles are based on a single matrix $L$, we show in section 2 that partial projection DPPs can be defined based on a pair of matrices $L$ and $V$. This is in fact a valid representation for all DPPs: in L-ensembles the $V$ part of the pair is empty, and in projection DPPs it is $L$ that is empty. We call this generic representation of DPPs “extended L-ensembles”. Sections and 2 introduce these concepts. In the second part of the manuscript (section 3 and onwards), we study limits of L-ensembles $\mathcal{X}_\varepsilon$ as $\varepsilon \to 0$. As illustrated here, many interesting limits of L-ensembles “hit the boundary” and become partial-projection DPPs, which is why we need the extended L-ensemble representation.
DPPs"\textsuperscript{,} which precisely groups all DPPs that are not L-ensembles (thus sampling sets with size always strictly superior to zero). In order to manipulate joint probability mass functions for DPPs in this subclass, we have to introduce our first contribution: extended L-ensembles.

Section 2 is devoted to the definition of extended L-ensembles, a novel representation of DPPs that we believe is interesting in itself. Extended L-ensembles provide a unified description of DPPs: whereas not all DPPs are L-ensembles, all DPPs are extended L-ensembles. In addition, they let us write easy-to-understand, explicit formulas for joint probabilities even in cases where the DPP at hand is not an L-ensemble.

With these definitions in hand, we first study the limiting process of an L-ensemble based on the perturbed matrix \( \varepsilon A + B \) (where \( B \) is low-rank) as \( \varepsilon \) tends to zero. We show that this limiting process is a partial projection DPP; meaning that partial-projection DPPs form in a sense the exterior boundary of the space of L-ensembles. Such perturbative limits form the topic of section 3. Figure 3 summarises some of the main concepts used here.

The next sections are devoted to the flat limit proper, that is: the study of the limiting process of an L-ensemble based on a kernel matrix, as \( \varepsilon \) tends to zero. We show the following results:

- Surprisingly, in the flat limit, such L-ensembles stay well-defined (see fig. 2 for an intuitive explanation of why that occurs)
- The limiting process depends mostly on the smoothness of the kernel function
- In particular cases (depending on the dimension \( d \)), they exhibit universal limits, i.e. all kernels within the same smoothness class lead to the same limiting L-ensemble

As an example of our results, we can prove the following (the notation is made precise later): let \( \Omega \subset \mathbb{R} \) (a finite set of points on the real line), and \( \mathcal{X} \) an L-ensemble on \( \Omega \). Let \( \kappa_\varepsilon \) be a kernel function that is \( C^\infty \) in both \( x \) and \( y \) at 0 and analytic in \( \varepsilon \) (e.g., the Gaussian). Pick an odd integer \( p < 2|\Omega| - 1 \). Then, applying Thm 6.2, as \( \varepsilon \to 0 \) the L-ensemble based on the matrix \( [\varepsilon^{-p}\kappa_\varepsilon(x_i, x_j)]_{x_i, x_j \in \Omega^2} \) has the law:

\[
p(\mathcal{X} = \{x_1, \ldots, x_m\}) = \begin{cases} \frac{1}{Z} \prod_{i<j}(x_i - x_j)^2 & \text{if } m = \frac{p+1}{2}, \\ 0 & \text{otherwise.} \end{cases}
\]

(2)

On the other hand, if the kernel function is only once differentiable at 0, e.g. with \( \kappa_\varepsilon(x, y) = \exp(-\varepsilon|x - y|) \), then taking the limit of the L-ensemble based on the matrix \( [\varepsilon^{-1}\kappa_\varepsilon(x_i, x_j)]_{(x_i, x_j) \in \Omega^2} \) we obtain a different process, with joint probability:

\[
p(\mathcal{X} = \{x_1, \ldots, x_m\}) = \begin{cases} \frac{1}{Z} \gamma^m \prod_{i=1}^{m-1}(x_{i+1} - x_i) & \text{if } m \geq 1, \\ 0 & \text{otherwise,} \end{cases}
\]

where we have ordered the points so that \( x_1 \leq x_2 \leq \ldots \leq x_m \). Whereas the previous limit was completely universal, in the sense that the limiting distribu-
Fig 4: Suppose a (fixed-size) L-ensemble is used to sample 6 of the 7 labelled points shown on the figure. With a Gaussian kernel, as $\varepsilon \to 0$, the set $X = \{1, 2, 3, 4, 5, 6\}$ has a probability 0 of being sampled, while the set $X' = \{2, 3, 4, 5, 6, 7\}$, which is less spread-out, has a small but non-zero probability of being sampled. With an exponential kernel, on the other hand, both sets have a non-zero probability of being sampled, but in this case $X$ is much more likely to be sampled than $X'$. The explanation for that counter-intuitive behaviour is to be found in section 5.3.
and only look at finite DPPs, leaving aside the continuous case. All results should extend to continuous DPPs on a compact subset of $\mathbb{R}^d$, with the appropriate change in notation. The case of continuous DPPs on a non-compact subspace of $\mathbb{R}^d$ appears to us harder to deal with.

**Practical implications**

The practical-minded reader might object to the abstract nature of this work. However, we stress that flat limits are an elegant way of partially answering the questions of hyper-parameter tuning, and, to a lesser extent, the choice of similarity function.

One outcome of this work is that as $\varepsilon \to 0$, DPPs have limits that are sensible, repulsive and so should behave reasonably in applications. One advantage of directly sampling from the limiting DPP is that there is no spatial scaling parameter to choose from. The only one that remains is how many points one wishes to sample. This assumes of course that one has chosen a particular kernel function, which leads us to our second point.

The second conclusion of our work is that what the exact kernel is, matters much less than what its smoothness order is. If one were to speculate based on the results in the unidimensional case, kernels with low regularity lead to mostly local repulsion whereas kernels with high regularity lead to a more global form of repulsion; and this is borne out as well by some numerical evidence. Kernels with high regularity lead to some surprising long-distance repulsiveness properties, as fig. 4 illustrates.

In addition, we suspect that there are computational implications of our results as well, enabling faster sampling of DPPs, but we leave this for future work.

**Structure of the paper**

We begin with some definitions and background in section 1. Section 2 introduces extended L-ensembles and partial-projection DPPs and gives some major properties. Partial-projection DPPs arise as limits of L-ensembles, and section 3 explains how in a simple case of an L-ensemble based on a linearly perturbed matrix. Some of the results proved there should help understand what happens in the flat limit.

For clarity, flat limit results are given in increasing order of complexity. We begin with results on the limits of fixed-size L-ensembles (the “k-DPPs” of [12]), because these results are much easier to state and serve as a building block for the case of variable-size L-ensembles. Thus, section 4 and section 5 study fixed-size L-ensembles in the flat limit. For pedagogical reasons, we begin with univariate results (where the points are a subset of the real line), before giving the results for the multivariate case, which require some background on multivariate polynomials. Limits of varying-size L-ensembles are covered in section 6, which again has a subsection on the univariate case that serves as a warm-up for the more difficult multivariate case.
1. Definitions and background

We briefly recall some definitions. For details we refer the reader to [3] and [12]. All of the results below are classical.

DPPs are based on determinants of kernel matrices, so we begin with some material on kernel functions and determinants. We then introduce DPPs along with fixed-size DPPs, a useful variant (as well as L-ensembles and fixed-size L-ensembles). Our proofs require that we work with asymptotic expansions of probability mass functions, which we do via two lemmas that we introduce. We then give some very simple results from matrix perturbation theory. They are not necessary for our proofs but help build an understanding of the limits we investigate. Finally, we provide the necessary background material on multivariate polynomials, as they are very important for flat limits and appear here or there in our developments.

1.1. Kernels, smoothness orders

We only outline the basic concepts needed to express the results from [4], which our analysis is based on. For more on kernels the reader is invited to consult [22] or [26]. A kernel is a positive definite function \( \kappa : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \). We call the kernel stationary if \( \kappa(x, y) = f(\|x - y\|_2) \) for some function \( f \), i.e. it only depends on the (Euclidean) distance between \( x \) and \( y \). We assume further that \( f \) is analytic at 0, and expand it as:

\[
\begin{align*}
  f(\|x - y\|_2) &= f_0 + f_1 \|x - y\|_2 + f_2 \|x - y\|_2^2 + f_3 \|x - y\|_2^3 + \ldots 
\end{align*}
\]

where \( f_i = \frac{f^{(i)}(0)}{i!} \), i.e. the rescaled derivatives at 0 of \( f \). The smoothness order of the kernel is defined with respect to the odd derivatives of \( f \) at 0. Specifically:

**Definition 1.1.** The smoothness order \( r \) of a stationary kernel \( \kappa(x, y) = f(\|x - y\|_2) \) is defined as:

\[
  r = \min\{r \mid f_{2r-1} \neq 0 \}
\]

i.e., the smallest \( r \) such that the \( r \)-th odd derivative is non-zero.

A kernel like the squared-exponential (eq. (1)) depends on the squared distance and so has \( r = \infty \). We call such kernels completely smooth. Kernels with finite values of \( r \) are called finitely smooth (f.s.). An example of a kernel with \( r = 1 \) is the exponential kernel:

\[
  \kappa_\varepsilon(x, y) = \exp(-\varepsilon \|x - y\|_2)
\]

An example of a kernel with \( r = 2 \) is:

\[
  \kappa_\varepsilon(x, y) = (1 + \varepsilon \|x - y\|_2) \exp(-\varepsilon \|x - y\|_2)
\]

The Matérn kernels [22], popular in spatial statistics, are a generic family of kernels which have \( r \) as a parameter. Other examples of finitely-smooth kernels can be found in our numerical results, for instance in fig. 6.

---

1 We choose this assumption for simplicity, but it can be relaxed to an assumption of differentiability up to a required order.
1.2. Some determinant lemmas

Let \( A \) be a \( n \times n \) matrix, and \( Y, Z \) be two subsets of indices. Then \( A_{Y,Z} \) is the submatrix of \( A \) formed by retaining the rows in \( Y \) and the columns in \( Z \). Furthermore, \( A_{:,Y} \) (resp. \( A_{Y,:) \)) is the matrix made of the full columns (resp. rows) indexed by \( Y \). Finally, we let \( A_Y = A_{Y,Y} \). Also, for a matrix \( V \), by \( \text{span}(V) \) we denote its column span, and by \( \text{orth}(V) \) the orthogonal complement of \( \text{span}(V) \).

We shall need a number of basic results on determinants. The Cauchy-Binet lemma is central to the theory of DPPs and generalises the well-known relationship \( \det(AB) = \det(A) \det(B) \) (for square \( A \) and \( B \)) to rectangular matrices.

**Lemma 1.2 (Cauchy-Binet).** Let \( M = AB \), with \( A \) a \( m \times n \) matrix, \( B \) a \( n \times m \) matrix. Then:

\[
\det(M) = \sum_{Y \subseteq \{1, \ldots, n\}, |Y| = m} \det(A_{:,Y}) \det(B_{Y,:})
\]

where the sum is over all subsets \( Y \subseteq \{1, \ldots, n\} \) of size \( m \).

We will also frequently use the following simple corollary of the Cauchy-Binet lemma.

**Corollary 1.3.** Let \( M = U \Lambda U^\top \), where \( U \) is \( m \times n \), \( n \geq m \) and \( \Lambda \) is a diagonal matrix. Then:

\[
\det(M) = \sum_{Y \subseteq \{1, \ldots, n\}, |Y| = m} (\det(U_{:,Y}))^2 \det(\Lambda_Y).
\]

The next result is a well-known determinantal counterpart of the Sherman-Woodbury-Morrisson lemma:

**Lemma 1.4.** Let \( A \) be an invertible matrix of size \( n \times n \), \( U \) of size \( n \times m \), and \( W \) an invertible matrix of size \( m \times m \). Then it holds that:

\[
\det(A + UWU^\top) = \det(A) \det(W) \det(W^{-1} + U^\top A^{-1} U).
\]

Finally, a related lemma is useful for block matrices:

**Lemma 1.5.** Let \( M = \begin{pmatrix} A & U \\ U^\top & W \end{pmatrix} \), with \( A \) invertible. Then

\[
\det(M) = \det(A) \det(W - U^\top A^{-1} U).
\]

The next two lemmas concern so-called “saddle-point matrices”, and are proved in [4, Appendix A].

**Lemma 1.6 ([4, Lemma 3.10]).** Let \( L \in \mathbb{R}^{n \times n} \), \( V \in \mathbb{R}^{n \times p} \), with \( V \) of full column rank and \( p \leq n \). Let \( Q \in \mathbb{R}^{n \times (n-p)} \) be an orthonormal basis for \( \text{orth}(V) \) (i.e., \( Q^\top V = 0 \), \( \text{rank}(Q) = n - p \)). Then:

\[
\det \begin{pmatrix} L & V \\ V^\top & 0 \end{pmatrix} = (-1)^p \det(V^\top V) \det(Q^\top LQ).
\]
In the next lemma, we use \( t^r \) to denote the coefficient corresponding to \( t^r \) in the power series \( g(t) \). For instance, if \( g(t) = 1 - t^2 + 2t^3 \), then \( [t^0]g(t) = 1 \) and \( [t^3]g(t) = 2 \).

**Lemma 1.7** ([4, Lemma 3.11]). Let \( L \in \mathbb{R}^{n \times n} \) and \( V \in \mathbb{R}^{n \times p} \). Then:

\[
[t^p] \det(L + tVV^\top) = (-1)^p \det \begin{pmatrix} L & V \\ V^\top & 0 \end{pmatrix}.
\]

**Remark 1.8.** The polynomial \( g(t) = \det(L + tVV^\top) \) is of degree at most \( p \), i.e., lemma 1.7 gives the coefficient for the highest possible power of \( t \). While this remark is missing in the original statement of lemma 1.7 (see [4, Lemma 3.11]), it can be easily verified by inspecting the proof of the lemma in [4, Appendix A].

### 1.3. Determinantal processes

#### 1.3.1. DPPs

Let \( \Omega = \{x_1, \ldots, x_n\} \subset \mathbb{R}^d \) be a collection of vectors called the **ground set**. A finite point process \( \mathcal{X} \) is a random subset \( \mathcal{X} \subseteq \Omega \). Abusing notation, we sometimes use \( \mathcal{X} \) to designate the indices of the items, rather than the items themselves. Which one we mean should be clear from context.

**Definition 1.9 (Determinantal Point Process).** Let \( K \in \mathbb{R}^{n \times n} \) be a positive semi-definite matrix verifying \( 0 \preceq K \preceq I \). In this context, \( K \) is called a marginal kernel. Then, \( \mathcal{X} \) is a DPP with marginal kernel \( K \) if

\[
\forall A \subseteq \Omega \quad P(A \subseteq \mathcal{X}) = \det K_A,
\]

where by convention, \( \det K_\emptyset = 1 \).

This definition is the historical one [16] and determines what we will refer to as the **class of DPPs**. However, manipulating inclusion probabilities rather than the joint probability distribution itself is often cumbersome. This usually leads authors to consider a slightly less general class of DPPs: the L-ensembles [6].

**Definition 1.10 (L-ensemble).** Let \( L \in \mathbb{R}^{n \times n} \) designate a positive semi-definite matrix verifying \( 0 \preceq L \preceq I \). An L-ensemble based on \( L \) is a point process \( \mathcal{X} \) defined as

\[
P(\mathcal{X} = A) = \frac{\det L_A}{Z},
\]

where by convention, \( \det L_\emptyset = 1 \). Thus: \( P(\mathcal{X} = \emptyset) = 1/Z > 0 \).

In Eq. (12), \( Z = \sum_{X \subseteq \Omega} \det L_X \) is a normalisation constant and can be shown [14] to equal \( \det(I + L) \).

L-ensembles are indeed a subclass of DPPs:

**Lemma 1.11.** An L-ensemble based on the positive semi-definite matrix \( L \) is a DPP. It is noted \( \mathcal{X} \sim \text{DPP}(L) \) and its marginal kernel verifies

\[
K = L(I + L)^{-1}.
\]
Proof. See, e.g., Thm 2.2 of [14]; or the discussion in Appendix A.

L-ensembles are in fact a strict subset of all DPPs:

**Lemma 1.12.** A DPP with marginal kernel $K$ is an L-ensemble if and only if $K$ verifies $0 \preceq K \prec I$ (note the $\prec$ sign, implying that no eigenvalue of $K$ is allowed to be equal to one). If $\mathcal{X}$ is a DPP with such a marginal kernel, then $\mathcal{X} \sim \text{DPP}(L)$, with $L$ verifying:

$$L = K(I - K)^{-1}.$$  

**Proof.** ($\Leftarrow$) If $K$ does not contain any eigenvalue equal to 1, then Eq. (13) inverts as $L = K(I - K)^{-1}$. ($\Rightarrow$) We show the contraposition. If $\mathcal{X}$ is a DPP with a marginal kernel $K$ containing at least one eigenvalue equal to one, then its size $|\mathcal{X}|$ is necessarily larger than one (see lemma 1.14). Thus, it cannot be an L-ensemble (L-ensembles have a non-null probability of sampling $\emptyset$).

**Remark 1.13.** As a consequence, the class of DPPs can be separated in two: the L-ensembles (all DPPs with marginal kernel verifying $0 \preceq K \prec I$), and the rest (all DPPs with marginal kernel whose spectrum contains at least one eigenvalue equal to one).

In DPPs, the size (cardinal) of $\mathcal{X}$, denoted by $|\mathcal{X}|$, is a random variable. Its distribution is as follows [10]:

**Lemma 1.14.** Let $0 \preceq K \preceq I$ be a marginal kernel with eigenvalues $\mu_1, \ldots, \mu_n$. Let $\mathcal{X}$ be a DPP with this marginal kernel. Then, $|\mathcal{X}|$ has the same distribution as $\sum_{i=1}^n B_i$, where $B_i$ is a Bernoulli random variable with expectation $E(B_i) = \mu_i$, and the $B_i$’s are distributed independently. In particular, the expected size of the DPP, $E(|\mathcal{X}|)$, can be directly deduced from the above to be

$$E(|\mathcal{X}|) = \sum \mu_i = \text{Tr}(K) \quad (14)$$

1.3.2. Fixed-size DPPs

The cardinal of a DPP is thus in general random. Such varying-sized samples are not practical in many applications (one desires a subset of size 50, not something of size 50 on average but which may be of size 35 or 56); which led the authors of [13] to define fixed-size DPPs

**Definition 1.15** (Fixed-size Determinantal Point Process). A fixed size DPP of size $m$ is a DPP $\mathcal{X}$ conditioned on $|\mathcal{X}| = m$.

A subclass of fixed-size DPPs is the class of fixed-size L-ensembles:

\(^2\)They are often called k-DPPs in the literature, but we prefer “fixed-size DPPs” in order not to overload the symbol $k$ too much.
Definition 1.16 (Fixed-size L-ensemble). Let $0 \preceq L$ be a positive semi-definite matrix. A fixed-size L-ensemble is a point process $\mathcal{X}$ defined as:

$$
P(\mathcal{X} = X) = \begin{cases} 
\frac{\det L_X}{Z_m} & \text{if } |X| = m, \\
0 & \text{otherwise.}
\end{cases}$$

(15)

where $Z_m$ is the normalisation constant.

Using the indicator function $I(\cdot)$, we may rewrite Eq. (15) more compactly as:

$$
P(\mathcal{X} = X) = \frac{\det L_X}{Z_m} I(|X| = m).$$

Lemma 1.17. A fixed-size L-ensemble is a fixed-size DPP, and we write it $\mathcal{X} \sim |DPP|_m(L)$.

We use the notation $\mathcal{X} \sim |DPP|_m(L)$ to distinguish from (standard) random-size L-ensembles.

It is important to understand that, in general, fixed-size DPPs are not DPPs, with the exception of projection DPPs (see Sec. 1.3.3). In particular, whereas all DPPs have a marginal kernel, fixed-size DPPs (again with the exception of projection DPPs) do not have marginal kernels: there does not exist a matrix whose principal minors are the marginal probabilities. The question of inclusion probabilities in fixed-size DPPs is treated at length in [3].

The constant $Z_m = \sum_{X, |X| = m} \det L_X$ in Eq. 15 is a normalisation constant and one can show that it equals the $m$-th “elementary symmetric polynomial” of $L$, a quantity that depends only on the spectrum of $L$, and plays an important role in the theory of DPPs.

Lemma 1.18 ([9, Theorem 1.2.12]). Let $L \in \mathbb{R}^{n \times n}$ be a matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. The $m$-th elementary symmetric polynomial of $L$ is defined as:

$$
e_m(L) := \sum_{|X| = m} \prod_{i \in X} \lambda_i,
$$

(16)

i.e., $e_0(L) = 1$, $e_1(L) = \sum \lambda_i = \text{Tr}(L)$, $e_2(L) = \sum_{i<j} \lambda_i \lambda_j$, \ldots, $e_n(L) = \det(L)$. Then:

$$
Z_m = \sum_{X, |X| = m} \det L_X = e_m(L).
$$

(17)

Since $e_m(L)$ is the sum of all the principal minors of fixed size $m$, we immediately obtain the following corollary on the distribution of the size of an L-ensemble:

Corollary 1.19. The probability that $\mathcal{X} \sim \text{DPP}(L)$ has size $m$ is given by:

$$
p(|\mathcal{X}| = m) = \frac{e_m(L)}{e_0(L) + e_1(L) + \ldots + e_n(L)}.
$$

(18)
Remark 1.20. Since a fixed-size L-ensemble is just an L-ensemble conditioned on size, an L-ensemble may also be viewed as a mixture of fixed-size L-ensembles. The size $m$ can be drawn according to its marginal distribution (Eq. (18)), and conditional on $|X| = m$, the fixed-size L-ensemble can be sampled.

1.3.3. Two useful special cases

There are two special cases of (fixed-size) DPPs that are useful to study on their own, both from a practical and theoretical viewpoint. These are the DPPs with diagonal kernels and those with projection kernels.

As it will be shown in section 1.3.4, these two examples are the key components for sampling any DPP using the mixture representation.

Diagonal kernels. Diagonal L-ensembles are in a way the most basic kind of DPPs (although the fixed-size case is surprisingly intricate).

Lemma 1.21. An L-ensemble based on a diagonal positive semi-definite matrix $L$, $\mathcal{Y} \sim \text{DPP}(\mathbf{\Lambda})$ with $\mathbf{\Lambda} = \text{diag}(\lambda_1, \ldots, \lambda_n)$, is a Bernoulli process: each event $i \in \mathcal{Y}$ is independent and occurs with probability $\pi_i = \frac{\lambda_i}{1 + \lambda_i}$.

Proof.

$$P(\mathcal{Y} = \mathcal{Y}) = \frac{\prod_{i \in \mathcal{Y}} \lambda_i}{\det(\mathbf{I} + \mathbf{\Lambda})} = \frac{\prod_{i \in \mathcal{Y}} \lambda_i}{\prod_{j=1}^{n} (1 + \lambda_j)} = \left( \prod_{i \in \mathcal{Y}} \frac{\lambda_i}{1 + \lambda_i} \right) \left( \prod_{j \in \mathcal{Y}^c} \frac{1}{1 + \lambda_j} \right) = \prod_{i=1}^{n} (\pi_i)^{B_i} (1 - \pi_i)^{1 - B_i}$$

where $B_i$ is the Bernoulli variable indicating $i \in \mathcal{Y}$. \hfill $\square$

Remark 1.22. For fixed-size L-ensembles this is no longer true: $\mathcal{Y} \sim |\text{DPP}|_m(\mathbf{\Lambda})$ is not a Bernoulli process, as the events are no longer independent but indeed negatively associated. To see why, note that since the total size is fixed, conditional on $i \in \mathcal{Y}$ other points are less likely to be included.

Remark 1.23. $\mathcal{Y} \sim |\text{DPP}|_m(\mathbf{I})$ is a uniform sample of size $m$ without replacement.

Fixed-size diagonal L-ensembles have been studied at some length in the past, notably in the sampling survey literature. Many important features of these processes were reported in [7].

Projection DPPs. Projection DPPs designate DPPs formed from projection matrices. Projection DPPs have many unique features, for instance that of being both DPPs and fixed-size DPPs. Section 2 will introduce a generalisation called “partial projection DPPs”. The definition of a projection DPP is as follows:

Definition 1.24 (Projection DPP). Let $\mathbf{U}$ be an $n \times m$ matrix with $\mathbf{U}^\top \mathbf{U} = \mathbf{I}_m$. A projection DPP is a DPP with marginal kernel $\mathbf{K} = \mathbf{U} \mathbf{U}^\top$. 
The name “projection DPP” comes from the fact that $UU^\top$ is a projection matrix (its eigenvalues are 1, with multiplicity $m$, and 0 with multiplicity $n-m$). As $K$’s spectrum contains at least an eigenvalue equal to 1, a projection DPP is not an L-ensemble (see lemma 1.12). However, a projection DPP can be equivalently defined as a fixed-size L-ensemble:

**Lemma 1.25** (See e.g., [3, Lemma 1.3]). Let $U$ be an $n \times m$ matrix with $U^\top U = I_m$. A projection DPP with marginal kernel $UU^\top$ is a fixed-size L-ensemble $X \sim |DPP|_m(UU^\top)$.

In fact, the only class of fixed-size DPPs that admit a marginal kernel are the projection DPPs. The next result states that a projection DPP is what one obtains when sampling a fixed-size L-ensemble of size $m$ from a positive semi-definite matrix $L$ of rank $m \leq n$.

**Lemma 1.26** (See [3, result 1].). Let $X \sim |DPP|_m(L)$, with rank($L$) = $m$, and let $U \in \mathbb{R}^{n \times m}$ denote an orthonormal basis for span $L$. Then, equivalently, $X \sim |DPP|_m(UU^\top)$.

**Proof.** Given the assumptions, we may write $L = UMM^\top U^\top$ with $U \in \mathbb{R}^{n \times m}$, and $M \in \mathbb{R}^{m \times m}$. Now, bearing in mind that $|X| = m$, we have:

$$P(X = x) \propto \det(L_x) = \det(U_x M)^2 \propto \det(UU^\top)_x$$

where we used the fact that $U_x$ is square and that $\det(M)$ is independent of $X$. Note that any orthonormal basis works, for instance the eigenvectors of $L$ associated with a non-null eigenvalue, but not only: the Q factor in the QR factorisation of $L$ would work as well.

**Remark 1.27.** Note that lemma 1.26 is valid only for fixed-size L-ensembles with rank of $L$ exactly equal to $m$. In the case rank $L > m$, the fixed-size L-ensemble $X \sim |DPP|_m(L)$ is no longer a projection DPP.

**Remark 1.28.** The normalisation constant is particularly simple in the case of projection DPPs. Let $UU^\top$ denote a projection kernel. Then (trivially), $m$ of its eigenvalues equal 1 and the rest are null. By lemma 1.18,

$$\sum_X \det(UU^\top)_x = e_m(UU^\top) = 1$$

If as above $L = UMM^\top U^\top$ with $U \in \mathbb{R}^{n \times m}$, then by the same reasoning as in the proof of lemma 1.26:

$$\sum_X \det(L_x) = \det(M^\top M) \sum_X \det(UU^\top)_x = \det(M^\top M)$$

### 1.3.4. Mixture representation

Determinantal point processes have a well-known representation as a mixture of projection-DPPs (also sometimes called “elementary DPPs” in the literature).
See [3] for details. The following mixture representation (due to [10]) is fundamental, both for theoretical and computational purposes, since it serves as the basis for exact sampling of DPPs. There are two variants, one for DPPs and one for fixed-size DPPs. For the purposes of this paper, we describe here the mixture representation of L-ensembles only.

**Lemma 1.29** (Mixture representation of fixed-size L-ensembles [12]). Let $X \sim |\text{DPP}|_m(L)$ be an L-ensemble based on $L$, and $L = U\Lambda U^\top$ be the spectral decomposition of $L$. Then, equivalently, $X$ may be obtained from the following mixture process:

1. Sample $m$ indices $Y \sim |\text{DPP}|_m(\Lambda)$
2. Form the projection matrix $M = U_{:,Y}(U_{:,Y})^\top$
3. Sample $X|Y \sim |\text{DPP}|_m(M)$

Equivalently, the probability mass function of $X$ can be written as:

$$P(X = X) = \frac{\mathbb{I}(|X| = m)}{e_m(\Lambda)} \sum_{Y, |Y| = m} \det(U_{X,Y})^2 \prod_{i \in Y} \lambda_i$$  \hspace{1cm} (19)

The mixture representation can be understood as (a) first sample which eigenvectors to use and (b) sample a projection DPP with the selected eigenvectors.

The counterpart for L-ensembles looks highly similar.

**Lemma 1.30** (Mixture representation of L-ensembles, see e.g. [14]). Let $X \sim \text{DPP}(L)$ and $L = U\Lambda U^\top$. Then, equivalently, $X$ may be obtained from the following mixture process:

1. Sample indices $Y \sim \text{DPP}(\Lambda)$
2. Form the projection matrix $M = U_{:,Y}(U_{:,Y})^\top$
3. Sample $X|Y \sim |\text{DPP}|_m(M)$

Equivalently, the probability mass function of $X$ can be written as:

$$P(X = X) = \frac{1}{\det(L + I)} \sum_Y \det(U_{X,Y})^2 \prod_{i \in Y} \lambda_i.$$  \hspace{1cm} (20)

The only step that varies is the first one, where we sample from $\text{DPP}(\Lambda)$ instead of $|\text{DPP}|_m(\Lambda)$.

### 1.4. Convergence of DPPs from asymptotic series

In this section we specify which type of convergence is proved in this paper. Below, we say that a random variable $X_\varepsilon$ converges to a random variable $X_*$ in $\varepsilon \to 0$ if for all outcomes $A$

$$P(X_\varepsilon = A) \to P(X_* = A).$$

Note that since our space of outcomes is finite, this definition coincides with all possible notions of convergence. For example, it is equivalent to convergence in
total variation \( \lim_{\varepsilon \to 0} D_{TV}(X_\varepsilon, X_\star) = 0 \), where for discrete random variables \( X' \) and \( Y \) defined on the same space of outcomes, the total variation distance equals:

\[
D_{TV}(X', Y) = \sum_A |P(X' = A) - P(Y = A)|.
\] (21)

What the results from [4] provide us with are asymptotic expansions of the determinants involved in the probability mass functions. To connect asymptotic expansions with convergence of random variables we shall use the following simple lemma.

**Lemma 1.31.** Let \( X_\varepsilon \) be a family of discrete random variables (e.g., a discrete point process) with values in the finite set \( \Phi \). Let

\[
P(X_\varepsilon = X) = \frac{f_\varepsilon(X)}{\sum_{Y \in \Phi} f_\varepsilon(Y)},
\]

where the following asymptotic expansion holds for \( f_\varepsilon \) and an integer \( p \), possibly negative:

\[
f_\varepsilon(X) = \varepsilon^p (f_0(X) + O(\varepsilon)).
\]

Then \( X_\varepsilon \) converges to the random variable \( X_\star \) (with values in \( \Phi \)), defined as

\[
P(X_\star = X) = \frac{f_0(X)}{\sum_{Y \in \Phi} f_0(Y)}.
\]

**Proof.** By direct inspection, we have

\[
P(X_\varepsilon = X) = \frac{f_\varepsilon(X)}{\sum_{Y \in \Phi} f_\varepsilon(Y)} = \frac{f_0(X) + O(\varepsilon)}{\sum_{Y \in \Phi} (f_0(Y) + O(\varepsilon))} \rightarrow \frac{f_0(X)}{\sum_{Y \in \Phi} f_0(Y)},
\]

where convergence holds everywhere since \( \Phi \) is a finite set.

We will also encounter discrete distributions in which the (unnormalised) probability mass function \( f_\varepsilon \) may involve different powers of \( \varepsilon \). For instance, consider the random variable \( Y_\varepsilon \in \{1, 2, 3\} \) with unnormalised mass function \( f_\varepsilon(Y_\varepsilon = 1) = \alpha_1 \varepsilon, f_\varepsilon(Y_\varepsilon = 2) = \alpha_2, \) and \( f_\varepsilon(Y_\varepsilon = 3) = \alpha_3 \varepsilon^{-1} \). What is the law of \( Y_\varepsilon \) as \( \varepsilon \to 0 \)? After normalisation, we have:

\[
P(Y_\varepsilon = 1) = \frac{\alpha_1 \varepsilon}{\alpha_1 + \alpha_2 + \alpha_3 \varepsilon^{-1}} = \frac{\alpha_1 \varepsilon^2}{\alpha_3 + O(\varepsilon)} = O(\varepsilon^2)
\]
\[
P(Y_\varepsilon = 2) = \frac{\alpha_2}{\alpha_1 + \alpha_2 + \alpha_3 \varepsilon^{-1}} = \frac{\alpha_2 \varepsilon}{\alpha_3 + O(\varepsilon)} = O(\varepsilon)
\]
\[
P(Y_\varepsilon = 3) = \frac{\alpha_3 \varepsilon}{\alpha_1 + \alpha_2 + \alpha_3 \varepsilon^{-1}} = \frac{\alpha_3}{\alpha_3 + O(\varepsilon)} = 1 + O(\varepsilon)
\]

The diverging order wins, and \( Y_\varepsilon \) equals 3 almost surely as \( \varepsilon \to 0 \).

This line of reasoning can be easily generalised to obtain the following lemma, which simply says that the smallest order in \( \varepsilon \) always wins:
Lemma 1.32. Let $\mathcal{X}_\varepsilon$ be a family of discrete random variables with values in the finite set $\Phi$. Let
\[ P(\mathcal{X}_\varepsilon = X) = \frac{f_\varepsilon(X)}{\sum_{Y \in \Phi} f_\varepsilon(Y)}, \]
where the following Laurent series holds for $f$:
\[ f_\varepsilon(X) = \varepsilon^{\eta_X}(f_0(X) + \mathcal{O}(\varepsilon)), \]
for some $\eta_X \in \mathbb{Z}$ which may be negative. Let $\eta_{\min} = \min_{X \in \Phi} \eta_X$ and $\Phi_{\min} = \{ X | \eta_X = \eta_{\min} \}$. Then $\mathcal{X}_\varepsilon \in \Phi_{\min}$ almost surely as $\varepsilon \to 0$. Moreover, $\mathcal{X}_\varepsilon \to X_\star$, where $X_\star$ is the random variable with support in $\Phi_{\min}$, with
\[ P(X_\star = X) = \frac{f_0(X)}{\sum_{Y \in \Phi_{\min}} f_0(Y)}. \]

1.5. Some matrix perturbation theory

In what follows we will be concerned with perturbed matrices. Matrix perturbation theory is often used in statistics, but unfortunately the perturbation problems that appear here are singular (they feature matrices that become non-invertible at $\varepsilon = 0$), and the theoretical tools we need are a bit more exotic. In this section we introduce some basic results, a full treatment can be found in [11].

We are interested in asymptotic expansions for the eigenvalues and eigenvectors of matrices of the form
\[ A(\varepsilon) = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \ldots \]
Each entry in $A(\varepsilon)$ is analytic in $\varepsilon$, and this is therefore known as an “analytic perturbation” (of $A_0$). The simplest case is just the linear perturbation, also called a “matrix pencil”:
\[ A(\varepsilon) = A_0 + \varepsilon A_1 \]
The difficulty comes from the fact that $A_0$ may be singular, in which case some of the eigenvalues will be 0 at $\varepsilon = 0$.

Rellich’s perturbation theorem is very useful here ([19], th. I.1.1):

Lemma 1.33. Let $A(\varepsilon) = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \ldots$ with $A(\varepsilon)$ Hermitian for real $\varepsilon$ in a neighbourhood of 0. The eigenvalues $\lambda_1(\varepsilon) \ldots \lambda_n(\varepsilon)$ and corresponding eigenvectors $u_1(\varepsilon) \ldots u_n(\varepsilon)$ may be chosen analytic in a (complex) neighbourhood of 0.

Armed with Rellich’s theorem, it is easy to prove some results on (singular) linear perturbations by matching orders in series.

Lemma 1.34. Let
\[ A(\varepsilon) = A_0 + \varepsilon A_1 \ldots \]
be an $n \times n$ positive semi-definite matrix, and $\text{rank}(A_0) = p < n$. Then $p$ eigenvalues of $A$ are $\mathcal{O}(1)$ (but not $\mathcal{O}(\varepsilon)$), and the remaining $n - p$ are $\mathcal{O}(\varepsilon)$. 
Proof. This result may also be proved using the Courant-Weyl minimum-maximum principle, as in [25]. Here we rely on a series expansion instead. Let $\lambda, u$ designate an eigenvalue/eigenvector pair of $A$. It verifies:

$$A(\varepsilon)u(\varepsilon) = \lambda(\varepsilon)u(\varepsilon)$$

(22)

which we may expand as:

$$(A_0 + \varepsilon A_1 + \ldots)(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots) = (\lambda_0 + \lambda_1 \varepsilon + \lambda_2 \varepsilon^2 + \ldots)(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots)$$

(23)

by Rellich’s theorem. Matching orders in $\varepsilon$, eq. (23) implies at constant order:

$$A_0 u_0 = \lambda_0 u_0$$

(24)

implying that the first order pair $(\lambda_0, u_0)$ is an eigenpair of $A_0$. By hypothesis, since $A_0$ has rank $p$, there are $p$ eigenvalues of order $O(1)$ (but not $O(\varepsilon)$), and the rest are $O(\varepsilon)$ or less.

Continuing the process further, we have:

Lemma 1.35. Under the same condition as in lemma 1.34, a limiting basis of eigenvectors can be written as $[U_0, \tilde{U}_1]$, where $U_0$ is an $n \times p$ matrix concatenating the $p$ eigenvectors of $A_0$ associated with its non-null eigenvalues, and $\tilde{U}_1$ concatenating the $(n - p)$ eigenvectors associated with the non-null eigenvalues of $\tilde{A}_1 = (I - U_0 U_0^\top)A_1(I - U_0 U_0^\top)$.

Proof. Let $(\lambda, u)$ denote an eigenpair as before. If $\lambda_0$ is non-null, then $u_0$ is an non-null eigenvector of $A_0$. There are $p$ such eigenvectors, which we collect as $U_0$. If $\lambda_0 = 0$, then eq. (24) implies that $u_0$ belongs to the kernel of $A_0$. Define $P = I - U_0 U_0^\top$ the projector on orth($A_0$) : then $Pu_0 = u_0$. The eigenvalue equation (eq. (23)) implies at order $\varepsilon$ that:

$$A_0 u_1 + A_1 Pu_0 = \lambda_1 u_0$$

Multiplying by $P$ on the left, we have:

$$PA_0 u_1 + PA_1 Pu_0 = \lambda_1 u_0$$

which from the definition of $P$ implies

$$PA_1 Pu_0 = \lambda_1 u_0$$

This last expression is an eigenvalue equation for the matrix $\tilde{A}_1 = PA_1 P$, which has at most $n - p$ non-null eigenvalues.

Example. As an example, we take the matrix

$$A(\varepsilon) = \begin{pmatrix} 1 + \varepsilon & 1 \\ 1 & 1 + \varepsilon \end{pmatrix} = 11^t + \varepsilon \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}.$$
The results above imply that as $\varepsilon \to 0$ the eigenvalues should be $O(1)$ and $O(\varepsilon)$, and the associated eigenvectors proportional to \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) and \( \begin{pmatrix} -1 \\ 1 \end{pmatrix} \) in the limit. Indeed, in this case the computations can be done by hand, and we find that

$$\lambda_1(\varepsilon) = \frac{1}{4}(\sqrt{\varepsilon^2 + 16} + 3\varepsilon + 4) = 4 + O(\varepsilon)$$

and

$$\lambda_2(\varepsilon) = \frac{1}{4}(-\sqrt{\varepsilon^2 + 16} + 3\varepsilon + 4) = 0 + O(\varepsilon).$$

The associated eigenvectors are

$$\left( \frac{1}{4}\left(\varepsilon + \sqrt{\varepsilon^2 + 16}\right) \right) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + O(\varepsilon)$$

and

$$\left( \frac{1}{4}\left(\varepsilon - \sqrt{\varepsilon^2 + 16}\right) \right) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + O(\varepsilon)$$

Note that since the square-root terms can be expanded in a power series around 16 the eigenvalues and eigenvectors are indeed analytic at 0.

1.6. Polynomials

Multivariate polynomials. Polynomials will play an important role in the paper, especially when we study the flat limit of DPPs in section 4 and beyond. We recall here the essential facts on multivariate polynomials.

Let $x = (x_1, x_2, \ldots, x_d) \top \in \mathbb{R}^d$. A monomial in $x$ is a function of the form:

$$x^\alpha = \prod_{i=1}^{d} x_i^{\alpha_i}$$

for $\alpha \in \mathbb{N}^d$ (a multi-index). Its total degree (or degree for short) is defined as $|\alpha| = \sum_{i=1}^{d} \alpha_i$. For instance:

$$x^{(2,1)} = x_1^2 x_2$$

and it has degree 3. A multivariate polynomial in $x$ is a weighted sum of monomials in $x$, and its degree is equal to the maximum of the degrees of its component monomials. For instance, the following is a multivariate polynomial of degree 2 in $\mathbb{R}^3$:

$$x^{(0,1,1)} - x^{(1,0,1)} + 2.2 x^{(1,0,0)} - 1.$$

One salient difference between the univariate and the multivariate case is that when $d > 1$, there are several monomials of any given degree, instead of
just one. For instance, with $d = 2$, the first few monomials are (by increasing degree):

\[
\begin{align*}
x^{(0,0)} \\
x^{(1,0)}, x^{(0,1)} \\
x^{(2,0)}, x^{(1,1)}, x^{(0,2)}
\end{align*}
\]

There is a well-known formula for counting monomials of degree $k$ in dimension $d$:

\[
\mathcal{H}_{k,d} = \binom{k + d - 1}{d - 1}.
\] (26)

The notation $\mathcal{H}_{k,d}$ comes from the notion of homogeneous polynomials. A homogeneous polynomial is a polynomial made up of monomials with equal degree. Therefore, the set of homogeneous polynomials of degree $k$ has dimension $\mathcal{H}_{k,d}$.

The set of polynomials of degree $k$ is spanned by the sets of homogenous polynomials up to $k$, and has dimension:

\[
\mathcal{P}_{k,d} = \mathcal{H}_{0,d} + \mathcal{H}_{1,d} + \ldots + \mathcal{H}_{k,d} = \binom{k + d}{d}.
\] (27)

Note for instance that $\mathcal{P}_{0,d} = 1$ and $\mathcal{P}_{1,d} = d + 1$. By convention, we will also set $\mathcal{P}_{-1,d}$ to be equal to 0.

**Multivariate Vandermonde matrices.** We now define the multivariate generalisation of Vandermonde matrices. Monomials are naturally ordered by degree, but monomials of the same degree have no natural ordering. To properly define our matrices, we require (formally) an ordering. For the purposes of this paper which ordering is used is entirely arbitrary. For more on orderings, see [4] and references therein.

For an ordered set of points $\Omega = \{x_1, \ldots, x_n\}$, all in $\mathbb{R}^d$, we define the multivariate Vandermonde matrix as:

\[
V_{\leq k} = [V_0, V_1, \ldots, V_k] \in \mathbb{R}^{n \times \mathcal{P}_{k,d}},
\] (28)

where each block $V_i \in \mathbb{R}^{n \times \mathcal{H}_{i,d}}$ contains the monomials of degree $i$ evaluated on the points in $\Omega$. As an example, consider $n = 3$, $d = 2$ and the ground set

\[
\Omega = \{[[y_1], [z_1]], [[y_2], [z_2]], [[y_3], [z_3]]\}.
\]

One has, for instance for $k = 2$:

\[
V_{\leq 2} = \begin{bmatrix}
1 & y_1 & z_1 & y_1^2 & y_1 z_1 & z_1^2 \\
1 & y_2 & z_2 & y_2^2 & y_2 z_2 & z_2^2 \\
1 & y_3 & z_3 & y_3^2 & y_3 z_3 & z_3^2
\end{bmatrix},
\]

where the ordering within each block is arbitrary.

We will use $V_{\leq k}(\mathcal{X})$ to denote the matrix $V_{\leq k}$ reduced to its lines indexed by the elements in $\mathcal{X}$. As such, $V_{\leq k}(\mathcal{X})$ has $|\mathcal{X}|$ rows and $\mathcal{P}_{k,d}$ columns.
2. Extended L-ensembles

The goal of this section is to introduce extended L-ensembles, a novel way of representing the class of DPPs. This representation has the advantage of giving explicit expressions for the joint probability distribution of all varying and fixed-size DPPs (not only varying and fixed-size L-ensembles).

In particular, the extended L-ensemble viewpoint will provide easy-to-use, explicit formulas for the joint probability of DPPs in cases where the spectrum of the DPP’s marginal kernel contains eigenvalues equal to 1 (that is, in cases where the DPP at hand is not an L-ensemble) \(^3\). According to Lemma 1.14, those are the cases where the size of the DPP is the sum of a deterministic part (the number of such eigenvalues equal to 1) and a random part. Such DPPs, that we will call partial projection DPPs for reasons that will become clear when we study their mixture representation, arise as limits of certain L-ensembles, as we will see in later sections.

2.1. Conditionally positive (semi-)definite matrices

L-ensembles are naturally formed from positive semi-definite matrices, because \(L\) being positive semi-definite is a sufficient condition for \(\det L_X\) being non-negative. Extended L-ensembles, defined below, can accommodate a broader set of matrices called conditionally positive semi-definite (CPD) matrices.

**Definition 2.1.** A matrix \(L \in \mathbb{R}^{n \times n}\) is called conditionally positive (semi-)definite with respect to a rank \(p \geq 0\) matrix \(V \in \mathbb{R}^{n \times p}\) if \(x^\top L x > 0\) (resp., \(x^\top L x \geq 0\)) for all \(x\) such that \(V^\top x = 0\).

**Remark 2.2.** Note that we authorize \(p = 0\) in the definition: in this case, the definition simply boils down to that of positive semi-definite matrices.

The set of vectors such that \(V^\top x = 0\) is the space orthogonal to the span of \(V\), which we note \(\text{orth} V\). The conditionally positive definite requirement may be read as a requirement for \(L\) to be positive definite within \(\text{orth} V\). Positive-definite matrices are therefore also conditionally positive-definite, but matrices with negative eigenvalues may also be conditionally positive-definite.

**Proposition 2.3.** Let \(L\) be conditionally positive (semi-)definite with respect to \(V \in \mathbb{R}^{n \times p}\), that we suppose full column rank. Let \(Q \in \mathbb{R}^{n \times p}\) designate an orthonormal basis for \(\text{span} V\), so that \((I - QQ^\top)\) is a projection on \(\text{orth} V\). Let \(\tilde{L} = (I - QQ^\top)L(I - QQ^\top)\). Then the eigenvalues of \(\tilde{L}\) are all non-negative.

**Proof.** Follows directly from the definition: for all \(x\), \(x^\top (I - QQ^\top)L(I - QQ^\top)x \geq 0\).

The above remark will become important when we define extended L-ensembles. The following example of a conditionally positive definite is classical (but surprising), and is a special case of a class of conditionally positive definite kernels studied in [17]. We take this example because it arises in section 4:

\(^3\)A formula due to [16] exists in this case but it is unwieldy
Proof. Let us write
\[ D^{(1)} = \|x_i - x_j\|_{i,j} \]
the distance matrix between \( n \) points in \( \mathbb{R}^d \). Then \(-D^{(1)}\) is conditionally positive definite with respect to the all-ones vector \( 1_n \).

Some extensions of this example can be found in section 2.8.2.

2.2. Nonnegative Pairs

The central object when defining extended L-ensembles is what we call a Nonnegative Pair (NNP for short).

Definition 2.4. A Nonnegative Pair, noted \((L; V)\) is a pair \( L \in \mathbb{R}^{n \times n}, V \in \mathbb{R}^{n \times p}, 0 \leq p \leq n\), such that \( L \) is symmetric and conditionally positive semi-definite with respect to \( V \), and \( V \) has full column rank. Wherever a NNP \((L; V)\) appears below, we consistently use the following notation:

- \( Q \in \mathbb{R}^{n \times p} \) is an orthonormal basis of span \( V \), such that \( I - QQ^\top \) is a projector on orth \( V \)
- \( \tilde{L} = (I - QQ^\top)L(I - QQ^\top) \in \mathbb{R}^{n \times n} \) is also symmetric and thus diagonalisable. From Proposition 2.3, we know that all eigenvalues are non-negative. We will denote by \( q \) the rank of \( \tilde{L} \). Note that \( q \leq n - p \) as the \( p \) columns of \( Q \) are trivially eigenvectors of \( L \) associated to \( 0 \). We write
  \[ \tilde{L} = \tilde{U} \tilde{\Lambda} \tilde{U}^\top \]
its truncated spectral decomposition; where \( \tilde{\Lambda} = \text{diag}(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_q) \in \mathbb{R}^{q \times q} \) and \( \tilde{U} \in \mathbb{R}^{n \times q} \) are the diagonal matrix of nonzero eigenvalues and the matrix of the corresponding eigenvectors of \( \tilde{L} \), respectively.

Remark 2.5. Again, note that we authorize \( p = 0 \) in the definition: in this case, \( Q = 0 \) and \( \tilde{L} = L \).

Let us first formulate the following lemma, useful for the next section.

Lemma 2.6. Let \((L; V)\) be a NNP. Then, for any subset \( X \subseteq \{1, \ldots, n\} \):

\[ (-1)^p \det \begin{pmatrix} L_X & V_{X^c} \\ (V_{X^c})^\top & 0 \end{pmatrix} = (-1)^p \det \begin{pmatrix} \tilde{L}_X & V_{X^c} \\ (V_{X^c})^\top & 0 \end{pmatrix} \geq 0. \]

Proof. Let us write \( m = |X| \) the size of \( X \). The case rank \( V_{X^c} < p \) is trivial as both sides of the equality are zero. Next, assume that \( V_{X^c} \in \mathbb{R}^{m \times p} \) is full column rank. If \( m = p \), then \( V_{X^c} \) is square and both sides are equal to \((\det V_{X^c})^2\). Now consider the case \( m > p \). Let \( Q \) be as in Definition 2.4, so that \( V = QR \) (with \( R \) nonsingular). Let \( B(X) \in \mathbb{R}^{m \times (m-p)} \) be the basis of orth \( (V_{X^c}) = \text{orth}(Q_{X^c}) \). Then, using lemma 1.6, we have that

\[ (-1)^p \det \begin{pmatrix} L_X & V_{X^c} \\ (V_{X^c})^\top & 0 \end{pmatrix} = \det((V_{X^c})^\top V_{X^c}) \det((B^\top(X) L_X B(X))) = \det((V_{X^c})^\top V_{X^c} \det((B^\top(X) L_X B(X)) = (-1)^p \det \begin{pmatrix} \tilde{L}_X & V_{X^c} \\ (V_{X^c})^\top & 0 \end{pmatrix}, \]
where the last but one equality is from \( \tilde{L} = (I - QQ^\top)L(I - QQ^\top) \) and the \( \tilde{L}^\top(X)Q_{X,:} = 0 \) and hence \( (I - QQ^\top)_{X}B(X) = B(X) \). Finally, \( \det(B^\top(X)\tilde{L_{X}}B(X)) \geq 0 \) due to positive semidefiniteness of \( \tilde{L} \), which completes the proof. \( \square \)

2.3. \textbf{DPPs via extended L-ensembles}

\textbf{Definition 2.7 (Extended L-ensemble).} Let \((L; V)\) be any NNP. An extended L-ensemble \( \mathcal{X} \) based on \((L; V)\) is a point process verifying:

\[
\forall X \subseteq \Omega, \quad P(X = X) \propto (-1)^p \det\left( \begin{pmatrix} L_{X} & V_{X,:} \\ V_{X,:}^\top & 0 \end{pmatrix} \right), \tag{29}
\]

\textbf{Remark 2.8.} We stress that an extended L-ensemble reduces to an L-ensemble only in the case \( p = 0 \). If \( p \geq 1 \), an extended L-ensemble is not an L-ensemble, since the probability mass function of \( \mathcal{X} \) is not expressed as a principal minor of a larger matrix. Also, the right-hand side in eq. (29) is non-negative by Lemma 2.6, and thus defines a valid probability distribution. The normalisation constant is tractable and given later (see section 2.7.1). On a more minor note, the factor \((-1)^p\) arises because of the peculiar properties of saddle-point matrices, see Lemma 1.6.

One shows in fact that the class of extended L-ensembles is identical to the class of DPPs, as the two following theorems demonstrate.

\textbf{Theorem 2.9.} Let \((L; V)\) be any NNP, and \( \mathcal{X} \) be an extended L-ensemble based on \((L; V)\). Then, \( \mathcal{X} \) is a DPP with marginal kernel

\[
K = QQ^\top + \tilde{L}(I + \tilde{L})^{-1}. \tag{30}
\]

\textbf{Proof.} See Appendix C. \( \square \)

Thus, an extended L-ensembles is a DPP. Importantly, the converse is also true: any DPP (not only L-ensembles) is an extended L-ensemble.

\textbf{Theorem 2.10.} Let \( 0 \preceq K \preceq I \) be any marginal kernel and \( \mathcal{X} \) its associated DPP. Denote by \( V \in \mathbb{R}^{n \times p} \) the matrix concatenating the \( p \geq 0 \) orthonormal eigenvectors of \( K \) associated to eigenvalue 1 and \( L = K(I - K)^\dagger \) with \( \dagger \) representing the Moore-Penrose pseudo-inverse. Then, \( \mathcal{X} \) is an extended L-ensemble based on the NNP \((L; V)\).

\textbf{Proof.} See Appendix C. \( \square \)

Recall that, as per definition 1.15, a fixed-size DPP is simply a DPP conditioned on size. As a consequence of the equivalence between extended L-ensembles and DPPs, one thus obtains the following explicit expression of the probability mass function of any fixed-size DPP:
Corollary 2.11. Let $0 \preceq K \preceq I$ be any marginal kernel and $\mathcal{X}$ its associated fixed-size DPP of size $m$. Let $(L; V)$ be the NNP as defined in theorem 2.10. Then

$$\forall X \subseteq \Omega, \quad P(\mathcal{X} = X) \propto (-1)^p \det \left( \begin{array}{c|c} L_X & V_X : \\ \hline \end{array} \right) \mathbb{I}(|X| = m).$$

Remark 2.12. Fixed-size DPPs of size $m$ with marginal kernel $K$ cannot be defined for $m$ smaller the multiplicity of 1 in the spectrum of $K$. In other words, one cannot condition the DPP based on $K$ having fewer samples than its number of eigenvalues equal to one (by lemma 1.14). Consequently, from the extended L-ensemble viewpoint, $m$ should always be larger than or equal to $p$.

2.4. Partial projection DPPs

The previous section made clear that

- any DPP in the class of DPPs may be defined equivalently either via a marginal kernel $0 \preceq K \preceq I$ from the marginal point of view, or via a NNP $(L; V)$ from the point of view of the explicit probability mass function.
- the class of fixed-size DPPs, being in all generality defined as DPPs conditioned on size, are in fact best described with extended L-ensembles. Their probability mass function are given by Eq. (31). Apart from the special case where $m = p$ that implies a projection DPP \(^4\), fixed-size DPPs do not have marginal kernels.

In the following, for the purpose of this work, we differentiate DPPs (both varying-size and fixed-size) defined by NNPs $(L; V)$ for which

- $p = 0$: Eq. 29 (resp. Eq. 31) boils down to Eq. 12 (resp. Eq. 15): we recover the L-ensembles $\mathcal{X} \sim DPP(L)$ (resp. fixed-size L-ensembles $\mathcal{X} \sim |DPP|_m(L)$).
- $p \geq 1$: in this case, the associated DPPs are not L-ensembles; and we will call them partial-projection DPPs (pp-DPPs) for reasons that will become clear in section 2.6. We will denote them $\mathcal{X} \sim DPP(L; V)$ and $\mathcal{X} \sim |DPP|_m(L; V)$ for the varying-size and the fixed-size cases respectively.

2.5. A generalisation of the Cauchy-Binet Formula

The cornerstone of the mixture representation of L-ensembles, discussed in Section 1.3.4, is in fact the Cauchy-Binet formula, recalled in Lemma 1.2 (see for instance [10, 14]). In order to provide a similar spectral understanding of extended L-ensembles, we need the following generalisation of the Cauchy-Binet formula.

\(^4\)If $m = p$, $V_X :$ is square in Eq. (31) and by Lemma 1.6, $P(\mathcal{X} = X) \propto \det(V_X :)^2$, which is the probability mass function of a projection DPP (see lemma 1.26).
Theorem 2.13. Let \((L; V)\) be a NNP, and \(Q, \tilde{U}, \tilde{\Lambda}\) and \(q\) be as in Definition 2.4. Then for any subset \(X \subseteq \{1, \ldots, n\}\) of size \(|X| = m\), \(p \leq m \leq p + q\), it holds that

\[
(-1)^p \det \left( \frac{L_X}{(V_X)_0} ; 0 \right) = \det(V^\top V) \sum_{Y, |Y| = m - p} \det \left( \left[ Q_X : \tilde{U}_{X,Y} \right] \right)^2 \prod_{i \in Y} \tilde{\lambda}_i.
\]

Proof. First of all, writing the \((Q, R)\) decomposition of \(V\) as \(V = QR\) one has:

\[
\det \left( \frac{L_X}{(V_X)_0} ; 0 \right) = (\det(R))^2 \det \left( \frac{L_X}{(Q_X)_0} ; 0 \right).
\]

Noting that \(\det(V^\top V) = (\det(R))^2\), to prove Eq. (32) it is sufficient to show that:

\[
(-1)^p \det \left( \frac{L_X}{(Q_X)_0} ; 0 \right) = \sum_{Y, |Y| = m - p} \det \left( \left[ Q_X : \tilde{U}_{X,Y} \right] \right)^2 \prod_{i \in Y} \tilde{\lambda}_i.
\]

Now, the case \(\text{rank } Q_X < p\) is trivial as both sides in (33) are zero. Next, we assume that \(Q_X\) is full rank. Using first lemma 2.6 and then lemma 1.7, one has:

\[
(-1)^p \det \left( \frac{L_X}{(Q_X)_0} ; 0 \right) = (-1)^p \det \left( \frac{\tilde{L}_X}{(Q_X)_0} ; 0 \right) = [t^p] \det(\tilde{L}_X + tQ_X;(Q_X)_0^\top).
\]

Using the fact that \(\tilde{L} = \tilde{U}\tilde{\Lambda}\tilde{U}^\top\), the right hand side may be re-written:

\[
[t^p] \det(\tilde{L}_X + tQ_X;(Q_X)_0^\top) = [t^p] \det \left( \left[ Q_X : \tilde{U}_{X,Y} \right] \left( tI_p \tilde{\Lambda} \right) [Q_X : \tilde{U}_{X,Y}]^\top \right)

= \sum_{|Y| = m - p} (\det([Q_X : \tilde{U}_{X,Y}]))^2 \det(\tilde{\Lambda}_Y),
\]

where the last equality follows from the Cauchy-Binet lemma.

2.6. Mixture representation

In the mixture representation of L-ensembles (see Sec. 1.3.4), one first samples a set of orthonormal vectors, forms a projective kernel from these eigenvectors, and then samples a projection DPP from that kernel. In that sense, a projection DPP is the trivial mixture in which the same set of eigenvectors is always sampled. In this section, we will see that in partial projection DPPs, a subset of orthogonal vectors is included deterministically (coming from \(V\)), and the rest are subject to sampling, from the part of \(L\) orthogonal to \(V\), hence the name partial projection.

In fact, examining Eq. (32), the kinship with the mixture representation of fixed-size L-ensembles should be clear upon comparison with equation (19). The left-hand side of Eq. (32) is the probability mass function, and on the
right-hand side we recognise a sum (over \( Y \)) of probability mass functions for projection DPPs \( \left( \det \left( \begin{bmatrix} Q_{X,:} & \tilde{U}_{X,Y} \end{bmatrix} \right)^2 \right) \) indexed by \( Y \), weighted by a product of eigenvalues \( \prod_{i \in Y} \tilde{\lambda}_i \). This lets us represent the partial-projection DPP as a probabilistic mixture. Contrary to fixed-size L-ensembles, some eigenvectors appear with probability 1: the ones that originate from \( V \) (represented by \( Q_{X,:} \) in Eq. (32)). The rest are picked randomly according to the law given by the product \( P(Y = Y) \propto \prod_{i \in Y} \tilde{\lambda}_i \).

Seen as a statement about probabilistic mixtures, theorem 2.13 provides a recipe for sampling from \( X \sim \left| \text{DPP} \right|_m (L; V) \). We summarize this recipe in the following statement:

**Corollary 2.14.** Let \( (L; V) \) be a NNP, and \( Q, \tilde{U}, \tilde{\Lambda} \) and \( q \) be as in Definition 2.4. Let \( X \sim |\text{DPP}|_m (L; V) \) with \( p \leq m \leq p + q \). Then, equivalently, \( X \) may be obtained from the following mixture process:

1. Sample \( m - p \) indices \( Y \sim |\text{DPP}|_{m-p}(\tilde{\Lambda}) \)
2. Form the projection matrix \( M = QQ^\top + \tilde{U}_{:,Y}(\tilde{U}_{:,Y})^\top \) (recall that \( Q \) and \( \tilde{U} \) are orthogonal)
3. Sample \( X|Y \sim |\text{DPP}|_m(M) \)

Note that at step 1 we only sample from the optional part, since the eigenvectors from \( V \) need to be included anyways. The total number of eigenvectors to include is \( m \), so \( m - p \) need to be sampled randomly.

Using theorem 2.13, as in the fixed-size case, we arrive easily at the following mixture characterisation for the varying-size case:

**Corollary 2.15.** Let \( (L; V) \) be a NNP, and \( Q, \tilde{U}, \tilde{\Lambda} \) be as in Definition 2.4. Let \( X \sim \text{DPP} (L; V) \). Then, equivalently, \( X \) may be obtained from the following mixture process:

1. Sample indices \( Y \sim \text{DPP}(\tilde{\Lambda}) \)
2. Form the projection matrix \( M = QQ^\top + \tilde{U}_{:,Y}(\tilde{U}_{:,Y})^\top \)
3. Sample \( X|Y \sim |\text{DPP}|_{p+|Y|}(M) \)

The only difference from the fixed-size case is in step 1. Again, we include all eigenvectors from \( V \) (they make up the \( QQ^\top \) part of the projection matrix \( M \)), then the remaining ones are sampled from \( Y \sim \text{DPP}(\tilde{\Lambda}) \), which is equivalent to including the eigenvector \( \tilde{u}_i \) with probability \( \frac{\tilde{\lambda}_i}{1 + \tilde{\lambda}_i} \).

### 2.7. Properties

#### 2.7.1. Normalisation

Using theorem 2.13, the normalisation constant is tractable both in the fixed-size and varying-size cases, as shown by the following corollary (see also [4, Lemma 3.11] for an alternative formulation).
Corollary 2.16. Let \((L; V)\) be a NNP, and \(\tilde{L}\) and \(q\) as in Definition 2.4. For \(m\) such that \(p \leq m \leq n\), one has:

\[
(-1)^p \sum_{|X|=m} \det \left( \begin{bmatrix} L_X & V_X \end{bmatrix}^\top \begin{bmatrix} V_X \end{bmatrix} \right) = e_{m-p}(\tilde{L}) \det(V^\top V) \tag{34}
\]

and

\[
(-1)^p \sum_X \det \left( \begin{bmatrix} L_X & V_X \end{bmatrix}^\top \begin{bmatrix} V_X \end{bmatrix} \right) = \det(I + \tilde{L}) \det(V^\top V) \tag{35}
\]

Proof. If \(m > p + q\), then the right-hand side is zero, as well as the left-hand side (by lemma 1.6). In the case \(m \leq p + q\), from theorem 2.13 we have:

\[
(-1)^p \sum_{|X|=m} \det \left( \begin{bmatrix} L_X & V_X \end{bmatrix}^\top \begin{bmatrix} V_X \end{bmatrix} \right) = \det(V^\top V) \sum_{|X|=m; |Y|=m-p} \det \left( \begin{bmatrix} Q_{X,Y} & \bar{U}_{X,Y} \end{bmatrix} \right) \prod_{i \in Y} \lambda_i
\]

\[
= \det(V^\top V) \sum_{Y; |Y|=m-p} \prod_{i \in Y} \lambda_i
\]

\[
= e_{m-p}(\tilde{L}) \det(V^\top V),
\]

where the sum over \(X\) is just the normalisation constant of a projection DPP (see remark 1.28). The proof for varying size is similar, using: \(\sum_Y \prod_{i \in Y} \lambda_i = \prod_{i=1}^q (1 + \tilde{\lambda}_i)\).

Using these results, we easily obtain the distribution of the size of \(|X'|\) for \(X' \sim DPP (L; V)\). One may check that equivalent results are obtained either using the mixture representation (see corollary 2.15) or the associated marginal kernel (via Eq. 30 and lemma 1.14).

Corollary 2.17. Let \(X' \sim DPP (L; V)\). Then

\[
P(|X'| = m) = \begin{cases} 0, & \text{if } m < p, \\ e_{m-p}(L) / \det(L+I), & \text{otherwise}. \end{cases}
\]

2.7.2. Complements of DPPs

A known (see e.g., [14]) result about DPPs is that the complement of a DPP in \(\Omega\) is also a DPP, i.e., if \(X\) is a DPP, \(X^c = \Omega \setminus X\) is also a DPP. We shall give a short proof and some extensions.

Theorem 2.18. Let \(X\) be a DPP with marginal kernel \(K\). Then the complement of \(X\), noted \(X^c\), is also a DPP, and its marginal kernel is \(I - K\).

Proof. We first prove this for projection DPPs. Let \(A \sim \text{DPP}|_m(UU^\top)\) for orthogonal \(U\) of rank \(m\). Then

\[
P(A^c = A) = P(A = A^c) \propto \det(U_{A_{c,:}})^2.
\]
Note that for the probability to be non null we need $A$ to be of size $n - m$.

Let $V \in \mathbb{R}^{n \times (n - m)}$ so that $I = UU^t + VV^t$. $M = \begin{pmatrix} U & V \end{pmatrix}$ is an orthogonal basis for $\mathbb{R}^n$ which we may partition as $\begin{pmatrix} U_{A^c, \cdot} & V_{A^c, \cdot} \\ U_{A^c, \cdot} & V_{A^c, \cdot} \end{pmatrix}$ By lemma 1.5

$$\det M = \det U_{A^c, \cdot} \det \left( V_{A^c, \cdot} - V_{A^c, \cdot} (U_{A^c, \cdot})^{-1} V_{A^c, \cdot} \right).$$

This gives

$$\mathbb{P}(A = A) \propto \det \left( (V_{A^c, \cdot} - V_{A^c, \cdot} (U_{A^c, \cdot})^{-1} V_{A^c, \cdot})^{-1} \right).$$

By the inversion formula for block matrices this is equal to the lower-right block in $M^{-1} = M^t$, and so:

$$\mathbb{P}(A = A) \propto \det \left( (V_{A^c, \cdot})^2 \right)$$

where we recognise a projection DPP ($A \sim |DPP|_{n - m}(VV^t)$, as claimed). We now use the mixture property to show the general case. In the general case,

$$\mathbb{P}(X = Y) = \sum_{\mathcal{X}} \mathbb{P}(Y) \det(U_{X, Y})^2$$

so that:

$$\mathbb{P}(X^c = A) = \sum_{\mathcal{Y}} \mathbb{P}(Y) \det(U_{A^c, Y})^2 = \sum_{\mathcal{Y}^c} \mathbb{P}(Y^c) \det(V_{A^c, Y^c})^2$$

Since each eigenvector is picked independently in $\mathbb{P}(Y)$ with probability $\pi_i$, picking each eigenvector independently with probability $1 - \pi_i$ produces a draw from $\mathbb{P}(Y^c)$. $\mathbb{P}(X^c = A)$ is therefore a DPP, and its kernel is $\mathbf{I} - \mathbf{K}$. \hfill \Box

Applying the theorem to L-ensembles we obtain:

**Corollary 2.19.** Let $\mathcal{X} \sim \text{DPP}(L)$, with $L$ a rank $p$ matrix and $p \leq n$. Then $\mathcal{X}^c \sim \text{DPP}(L^\perp, V)$ with $V$ a basis for orth $L$. In particular, if $p = n$ ($L$ is full rank), we have $\mathcal{X}^c \sim \text{DPP}(L^{-1})$.

For extended L-ensembles this generalises to:

**Corollary 2.20.** Let $\mathcal{X} \sim \text{DPP} (L; V)$, and let $Z$ be a basis for orth $\mathcal{L} \setminus \text{span} V$. Then $\mathcal{X}^c \sim \text{DPP}(\mathcal{L}^\perp, Z)$.

The following fixed-size variant is new: it states that the complement of a fixed-size DPP is also a fixed-size DPP

**Proposition 2.21.** Let $\mathcal{X} \sim |DPP|_m (L; V)$, and let $Z$ be a basis for orth $\mathcal{L} \setminus \text{span} V$. Then $\mathcal{X}^c \sim |DPP|_{n - m}(\mathcal{L}^\perp, Z)$.

**Proof.** Proof sketch: repeat the proof of th. 2.18 up to the mixture representation, where we note that since $p(Y = Y) \propto \prod_{i \in Y} \lambda_i$, $p(Y^c) \propto \prod_{j \in Y^c} \frac{1}{\lambda_j}$ which is again a diagonal fixed-size DPP. \hfill \Box
2.7.3. Partial Invariance

We parametrise partial-projection DPPs using a pair of matrices (the NNP \((L; V)\)), but this is an over-parameterisation since all that matters is the linear space spanned by \(V\), as the following makes clear:

**Remark 2.22.** Consider a NNP \((L; V)\). Let \(V' = VR\) with \(R \in \mathbb{R}^{p \times p}\) invertible. We have \(\text{span} V' = \text{span} V\). Then \(X \sim DPP(L; V)\) and \(X' \sim DPP(L; V')\) define the same point process. This also holds for \(X \sim |DPP|_m(L; V)\) for any \(m \geq p\).

**Proof.** This is clear from theorem 2.13 or the mixture representation of the partial-projection DPP. Nothing on the right-hand side of equation (32) is affected by replacing \(V\) with a matrix with identical span. In particular, the distribution is invariant to rescaling of \(V\) by any non-zero scalar.

Notice that this generalises a property of projection DPPs given in the introduction (section 1.3.3), which is that \(X \sim \text{DPP}_m(L)\) and \(X \sim \text{DPP}_m(L')\) are the same if \(L\) and \(L'\) have the same column span and rank \(m\).

Another source of invariance in partial projection DPPs lies in \(L\): we can modify \(L\) along the subspace spanned by \(V\) without changing the distribution.

**Remark 2.23.** Consider a NNP \((L; V)\). Let \(L' = L + VX^T + YY^T\) for any two matrices \(X, Y \in \mathbb{R}^{n \times p}\). Then \(X \sim DPP(L; V)\) and \(X' \sim DPP(L'; V)\) define the same random variable.

**Proof.** Indeed, by Definition 2.4, we have \(\tilde{L}' = \tilde{L}\). Therefore, by lemma 2.6, the DPPs defined by \(L'\) and \(L\) coincide.

2.8. Examples

We give here a few examples of partial projection DPPs and their NNPs.

2.8.1. Partial projection DPPs as conditional distributions

A simple example of a partial projection DPP arises when the columns of the matrix \(V\) come from a canonical basis (i.e., each column of \(V\) is a standard unit vector). In this case, partial projection DPPs can be interpreted as a particular conditional of a DPP. For simplicity, assume that \(V = [I_p \ 0]^T\), so that the projected \(L\) matrix becomes

\[
\tilde{L} = \begin{bmatrix}
0 & 0 \\
0 & L_{\{x_{p+1}, \ldots, x_n\}}
\end{bmatrix}.
\]

In this case, the mixture representation for pp-DPPs (resp. fixed-sized pp-DPPs) implies that:

- all the points \(x_1, \ldots, x_p\) are always sampled;
the remaining points are sampled according to the L-ensemble (resp. fixed-size L-ensemble) based on \( L(x_{p+1}, \ldots, x_n) \).

For example, in the varying-size case \( X \sim DPP (L; V) \), the probability of sampling the remaining points is

\[
P(X \cap \{ x_{p+1}, \ldots, x_n \} = X') \propto \det L_{X'},
\]

which is linked to a certain conditional distribution of the ordinary L-ensemble based on \( L \) (see [14, §2.4.3] for more details).

2.8.2. Partial projection DPPs and conditional positive definite functions

An important generalisation of positive definite kernels is the notion of conditional positive definite kernels (see for example [17],[26]), especially in interpolation problems with polynomial regularisation. Conditional positive definite kernels generate conditionally positive definite matrices when evaluated at a finite set of locations, just like positive definite kernels generate positive definite matrices. We will show here that extended L-ensembles let us construct DPPs based on conditional positive definite functions.

Definition 2.24. A function \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) is conditionally positive definite of order \( \ell \) if and only if, for any \( n \in \mathbb{N} \), any \( X = (x_1, \ldots, x_n) \in (\mathbb{R}^d)^n \), any \( \alpha \in \mathbb{R}^n \) satisfying \( \sum_i \alpha_i x_i^\beta = 0 \) for all multi-indices \( \beta \) s.t. \( |\beta| < \ell \), the quadratic form

\[
\sum_{i,j} \alpha_i \alpha_j f(x_i - x_j)
\]

is non-negative.

Suppose now that we introduce “Gram” matrices \( L_X = [f(x_i - x_j)]_{i,j} \), and the multivariate Vandermonde matrix \( V_{\leq \ell - 1}(X) \). Then, an equivalent definition is

Definition 2.25. A function \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) is conditionally positive definite of order \( \ell \) if and only if, for any \( n \in \mathbb{N} \), any \( X = (x_1, \ldots, x_n) \in (\mathbb{R}^d)^n \), the matrix \( L_X \) is conditionally positive definite with respect to \( V_{\leq \ell - 1}(X) \).

This extends the possible functions used to measure diversity in DPP sampling. For example, it can be shown that \( f(x) = \phi(\|x\|^2) \) where \( \phi : \mathbb{R}^+ \rightarrow \mathbb{R} \) is the so-called multiquadrics \( (-1)^{[\beta]}(c^2 + r^2)^\beta; c, \beta > 0, \beta \notin \mathbb{N} \) is conditional positive definite of order \( [\beta] \). To be explicit, we may for instance define a valid extended L-ensemble based on a NNP \( (L; 1) \) with \( L_{ij} = -\sqrt{c^2 + \|x_i - x_j\|^2} \). Likewise, the so-called "thin-plate spline" \( \phi(r) = (-1)^{k+1} r^{2k} \log(r) \) makes \( f(x) = \phi(\|x\|^2) \) a conditional positive definite function of order \( k + 1 \) on \( \mathbb{R}^d \).

A last example of great interest for this paper is the case of \( \phi(r) = (-1)^{[\beta/2]} r^{\beta}; \beta > 0, \beta \notin 2\mathbb{N} \) which makes \( f(x) = (-1)^{[\beta/2]} \|x\|^{2\beta} \) a conditional positive function of order \( [\beta/2] \). Indeed, we will encounter in sections 4 to 6 extended L-ensembles.
of the form \((-1)^r D^{(2r-1)}; V_{\leq r-1}\) where \(D^{(2r-1)} = \left[\|x_i - x_j\|\right]_{i,j}^{2r-1}\), for \(r\) a positive integer, corresponding to \(\beta = r - 1/2\).

We stated above that a link exists to interpolation. To illustrate the link, suppose we want to interpolate points \(((x_1, y_1), \ldots, (x_n, y_n)) \in (\mathbb{R}^d \times \mathbb{R})^n\) using the function \(s(x) = \sum_i \alpha_i f(x_i - x_j) + \sum_{k=1}^{q_d} \beta_k p_k(x)\) where \(f\) is a conditionally positive function of order \(\ell\), and \(p_k, k = 1, \ldots, q_d\) is a basis for the set of polynomials of degree less or equal than \(\ell - 1\). The solution of this interpolation problem is then equivalent to the solution of the linear system

\[
\begin{pmatrix}
L_X & V_{\leq \ell - 1} \\
(V_{\leq \ell - 1})^\top & 0
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix}
= 
\begin{pmatrix}
y \\
0
\end{pmatrix}
\]

where we recover the matrix defining the \(L\)-ensemble in partial projection DPPs. A DPP based on the conditional positive definite kernel \(f\) will sample a good design for interpolation, since the interpolation points are selected such that the interpolation matrix is well-conditioned. This link between DPP sampling and interpolation theory deserves to be further studied, but is beyond the scope of the paper.

### 2.8.3. Roots of trees in uniform spanning random forests are partial projection DPPs

It is known (e.g. [1]) that the roots of the trees in a uniform random spanning forest over a graph with \(n\) nodes and Laplacian \(L\) are distributed according to a DPP with marginal kernel \(K = q(I + L)^{-1}\) for some real parameter \(q > 0\). Figure 5 illustrates what a spanning forest over a graph is. Let us denote as \(\lambda_1 \geq \ldots \geq \lambda_n = 0\) the eigenvalues of the Laplacian, and \(\{u_i\}_i\) the associated set of orthonormal eigenvectors. It is well known that \(\lambda_n = 0\) for any graph; \(K\) thus has at least one eigenvalue equal to 1 and, as such, the associated DPP is not an \(L\)-ensemble. It can however be described by an extended \(L\)-ensemble:

**Proposition 2.26.** The set of roots in a uniform random spanning forest over a connected graph with Laplacian \(L\) is distributed according to a partial projection DPP with NNP \((qL^\dagger; 1)\), where \(\dagger\) stands for the Moore-Penrose inverse.

**Proof.** Applying theorem 2.10, a DPP with marginal kernel \(K\) can be described by an extended \(L\)-ensemble based on the NNP \((L, V)\) with \(V\) and \(L\) verifying:

- the matrix \(V\) concatenates all eigenvectors of \(K\) associated to eigenvalue 1: in a connected graph, there is only one such eigenvalue and it is associated to eigenvector \(u_n = n^{-1/2} 1\)
- the matrix \(L\) is equal to \(K(I - K)^\dagger\), which is equal to \(qL^\dagger\)

**Remark 2.27.** This example also provides a nice illustration for the properties of complements of DPPs (section 2.7.2). Since \(L\) is a positive-definite matrix, we may define \(C \sim DPP(L)\). The complement of \(C\) is a DPP \(C^c \sim DPP(L^\dagger; 1)\),
Fig 5: Roots of uniform random forests over a graph are distributed according to a partial projection DPP. Vertices or nodes are depicted in gray; edges as thin lines. A random forest is depicted: its trees are surrounded by light gray zones; edges of the trees are thick black lines; roots are the black nodes. The forest is spanning the graph as each node of the graph appears once in a tree of the forest.
which from the result above corresponds to the roots process. C therefore samples every node except the roots of a random forest on the graph.

3. Partial projection DPPs as limits

The main goal of this section is to serve as a warm-up for the study of flat limits, and illustrate on a simple case the mathematical tools used later in the paper, as well as some of the peculiarities of limits of L-ensembles (such as dependence on scaling).

As stated above, pp-DPPs arise as limits of certain L-ensembles, and in this section we exhibit one such limit: the L-ensemble based on the linear perturbation of a (low-rank) positive semi-definite matrix; i.e., we consider L-ensembles based on matrices of the form:

\[ L_{\varepsilon} \triangleq \varepsilon A + VV^T \]  

(38)

where \( A \) has full rank\(^5 \) \( n \) and \( V \) has full column rank \( p < n \).

Thus \( L_{\varepsilon} \) defined in (38) is a regular matrix pencil. One should think about this scenario as constructing a kernel as a sum of (a) a few important features contained in \( VV^T \) and (b) a generic kernel in \( A \).

3.1. Limit of fixed-size L-ensembles based on \( \varepsilon A + VV^T \)

We begin with the more straightforward fixed-size case. We seek the limiting process \( \mathcal{X}_{\varepsilon} \) of \( \mathcal{X}_{\varepsilon} \sim |\text{DPP}|_m(L_{\varepsilon}) \) as \( \varepsilon \to 0 \). The following theorem establishes the limiting distribution using asymptotic expansions of the determinants.

3.1.1. Limiting process

**Theorem 3.1.** Let \( \mathcal{X}_{\varepsilon} \sim |\text{DPP}|_m(L_{\varepsilon}) \), with \( L_{\varepsilon} \) as in Eq. (38). Then the limiting process is:

\[ \mathcal{X}_{\varepsilon} \to \mathcal{X}_\ast \sim \begin{cases} |\text{DPP}|_m(VV^T), & m \leq p \\ |\text{DPP}|_m (A; V), & m > p. \end{cases} \]

**Proof.** First, we consider the case \( m \leq p \). Note that the unnormalized probability mass function for the L-ensemble based on \( L_{\varepsilon} \) is

\[ f_{\varepsilon}(X) = \det((\varepsilon A + VV^T)_X) = \det(\varepsilon A_X + V_{X,:}(V_{X,:})^T) = \det(V_{X,:}(V_{X,:})^T) + O(\varepsilon). \]

Since rank \( V = p \geq m \), there exists a subset of rows \( X_0 \) such that

\[ \det(V_{X_0,:}(V_{X_0,:})^T) \neq 0. \]  

(39)

\(^5\)The case where \( A \) is not full rank can also be studied, but it is more burdensome and not much more informative
Therefore, by lemma 1.32, we get that $\mathcal{X}_\varepsilon \rightarrow |DPP|_m(\mathbf{VV}^\top)$.

The case $m > p$ is more delicate, as eq. (39) no longer holds true, and we need to determine the order of $\varepsilon$ in the expansion of $f_\varepsilon(X)$. For this, we can invoke lemma 1.7 and remark 1.8 to get

\[
f_\varepsilon(X) = \det(\varepsilon \mathbf{A}_X + \mathbf{V}_{X,:}(\mathbf{V}_{X,:})^\top) = \varepsilon^m \det(\mathbf{A}_X + \varepsilon^{-1} \mathbf{V}_{X,:}(\mathbf{V}_{X,:})^\top).
\]

By applying lemma 1.31, we get

\[
\mathbf{P}(\mathcal{X}_\varepsilon = X) \propto (-1)^p \det \left( \begin{array}{c} \mathbf{A}_X \\ (\mathbf{V}_{X,:})^\top \end{array} \right) \mathbf{V}_{X,:},
\]

and hence $\mathcal{X}_\varepsilon \rightarrow |DPP|_m(\mathbf{A};\mathbf{V})$.

**Remark 3.2.** Note that if $m = p$ the limiting process is a projection DPP by lemma 1.26.

### 3.2. A spectral view

As we show in this section, the limiting distribution in theorem 3.1 can be obtained using a completely different, and, in our opinion, more interpretable approach.

Recall the mixture representation of L-ensembles and fixed-size L-ensembles described in section 1.3.4. Given a positive semi-definite matrix $\mathbf{L} = \mathbf{L}_\varepsilon$, one first samples some eigenvectors of $\mathbf{L}$, then builds a projection matrix $\mathbf{U} = \mathbf{U}_{\varepsilon,Y}$ from these eigenvectors, then samples a projection DPP from $\mathbf{U}_{\varepsilon,Y}(\mathbf{U}_{\varepsilon,Y})^\top$. We shall now study the asymptotic distribution of $\mathcal{X}_\varepsilon$ from the mixture point of view, using the spectral results of section 1.5.

Lemma 1.34 implies that the spectrum of $\mathbf{L}_\varepsilon$ contains $p$ eigenvalues $\lambda_1(\varepsilon), \ldots, \lambda_p(\varepsilon)$ of order $O(1)$, and $n - p$ eigenvalues $\lambda_{p+1}(\varepsilon), \ldots, \lambda_n(\varepsilon)$ of order $O(\varepsilon)$. In other words, their expansion reads

\[
\lambda_i(\varepsilon) = \lambda_{i,0} + \varepsilon \lambda_{i,1} + O(\varepsilon^2),
\]

where $\lambda_{i,0} \neq 0$ for $i \leq p$, and $\lambda_{i,0}$ is null otherwise.

In the case of fixed-size L-ensembles, in the mixture representation, the eigenvectors are sampled according to the following law ($\mathcal{Y}_\varepsilon$ indexes the sampled eigenvectors):

\[
\mathbf{P}(\mathcal{Y}_\varepsilon = Y) \propto \prod_{i \in Y} \lambda_i(\varepsilon) \cdot \mathbb{I}(|Y| = m),
\]

where $\lambda_i(\varepsilon)$ are as in (41). Intuitively: if $m \leq p$, then all the sets $Y \subseteq \{1, \ldots, p\}$ have probability mass $O(1)$. All other sets $Y$ have probability mass $O(\varepsilon)$ or
smaller. As $\varepsilon \to 0$, the limiting process must then only select $Y \subseteq \{1, \ldots, p\}$. If $m > p$, then the process is forced to select some of the small eigenvalues, but then as few as possible: the lowest possible order in $\varepsilon$ of the probability mass function is $O(\varepsilon^{m-p})$, which is obtained by having $\{1, \ldots, p\} \subset Y$, and selecting the $m - p$ remaining ones at random. This discussion can be summarized as follows.

**Proposition 3.3.** If $m \leq p$, the limiting distribution of $Y_\varepsilon$ is:

$$P(Y_\varepsilon = Y) \propto \prod_{i \in Y} \lambda_i(\varepsilon) \cdot I(|Y| = m \text{ and } Y \subseteq \{1, \ldots, p\})$$

As a special case, if $m = p$ then $Y_\varepsilon = \{1, \ldots, p\}$ with probability 1.

If $m > p$ the limiting distribution of $Y_\varepsilon$ is

$$P(Y_\varepsilon = Y) \propto \prod_{i \in Y \cap \{p+1, \ldots, n\}} \lambda_i(\varepsilon).$$

**Proof.** Let $Z = Y \cap \{1, \ldots, p\}$. We first characterise the limiting distribution of $|Z|$, then the conditional $Y_\varepsilon|Z_\varepsilon$. If $m \leq p$, we see that $P(|Z| = m) = 1 + O(\varepsilon)$, hence the conditional distribution is

$$P(Y = Y| |Z| = m) \propto \prod_{i \in Y \cap \{1, \ldots, m\}} \lambda_i(\varepsilon).$$

If $m > p$, we see that $P(|Z| = p) = 1 + O(\varepsilon)$, the conditional distribution is

$$P(Y = Y| |Z| = p) \propto \prod_{i \in Y \cap \{p+1, \ldots, n\}} \lambda_i(\varepsilon).$$

In both cases, we may invoke lemma 1.31 to complete the proof. \qed

We now know how $Y_\varepsilon$ is sampled in the limit. In parallel, we have conditional distributions $X_\varepsilon|Y_\varepsilon$ that are projection-DPPs. By lemma, 1.35 the eigenvectors of $L_\varepsilon$ converge to $[Q \quad \tilde{U}]$, where $Q$ and $\tilde{U}$ are as in Definition 2.4 for the extended L-ensemble $(A; \tilde{V})$. This establishes the following:

**Proposition 3.4.** Let $X_\varepsilon \sim |DPP|_m(L_\varepsilon)$, $L_\varepsilon$ as in (38). Note $L_\varepsilon = U(\varepsilon)\Lambda(\varepsilon)U(\varepsilon)^T$ the eigendecomposition of $L_\varepsilon$. Then the mixture representation of $X_\varepsilon$, i.e.

1. $Y_\varepsilon \sim |DPP|_m(\Lambda(\varepsilon))$,
2. $X_\varepsilon|Y_\varepsilon \sim |DPP|_m(M(Y_\varepsilon, \varepsilon))$ with $M(Y, \varepsilon) = U(\varepsilon)(U(\varepsilon)^T \Lambda(\varepsilon))^T$,

has the limit:

1. $Y'_\varepsilon \sim |DPP|_{m-p}(\tilde{\Lambda})$,
2. $X_\varepsilon|Y'_\varepsilon \sim |DPP|_{m}(M(Y'_\varepsilon, \varepsilon))$ with $M(Y', \varepsilon) = QQ^T + \tilde{U}(\varepsilon)(\tilde{U}(\varepsilon)^T \Lambda(\varepsilon))^T$.

which is equivalent to corollary 2.14.
Put more plainly, if \( m \geq p \) the limiting fixed-size L-ensembles is a partial projection DPP: the top \( p \) eigenvectors are included with probability 1, and the \( m - p \) others are picked according to the law of a diagonal L-ensemble with diagonal entries equal to the (non-zero) eigenvalues of \((I - QQ^\top)A(I - QQ^\top)\), by lemma 1.35.

### 3.3. Limits of variable-size L-ensembles based on \( A + \varepsilon^{-1}VV^\top \)

The variable-size version of the results requires a bit more care. In fixed-size L-ensembles, the law of \( X \) is invariant to a rescaling of the positive semi-definite matrix it is based on: \( X \sim |DPP|_m(L) \) is equivalent to \( |DPP|_m(\alpha L) \) for any \( \alpha > 0 \). For regular (variable-size) DPPs this is not true. That feature both enriches and complicates a little the asymptotic analysis.

#### 3.3.1. A trivial limit

Let us start with a straightforward limit, namely \( \mathcal{X}_\varepsilon \sim DPP(L_\varepsilon) \) based on the matrix pencil defined in (38). There are several equivalent ways of obtaining the limiting process, but let us use the mixture representation, to contrast with the fixed-size case. In the mixture representation, the only difference between L-ensembles and fixed-size L-ensembles is in how one samples the eigenvectors. In variable-size L-ensembles, by lemma 1.30, these are sampled from a Bernoulli process with inclusion probability

\[
\pi_i(\varepsilon) = \frac{\lambda_i(\varepsilon)}{1 + \lambda_i(\varepsilon)}
\]

Inserting expansions of \( \lambda_i(\varepsilon) \) from (41), we can directly compute the limit of the inclusion probabilities:

\[
\pi_i(\varepsilon) = \frac{\lambda_{i,0}}{1 + \lambda_{i,0}} + O(\varepsilon).
\]

Thus, the probability to sample each of the eigenvectors goes to \( \frac{\lambda_{i,0}}{1 + \lambda_{i,0}} \), which is equal to 0 for the last \( n - p \) eigenvectors. Since these events are independent, this implies that in \( \varepsilon \to 0 \) (with probability 1) we only sample from the top \( p \) eigenvectors of \( L(\varepsilon) \). By lemma 1.35, these top \( p \) eigenvectors themselves tend to the eigenvectors of \( VV^\top \), which is enough to show:

**Proposition 3.5.** Let \( \mathcal{X}_\varepsilon \sim DPP(\varepsilon^{-1}A + VV^\top) \). Then the limiting process \( \mathcal{X}_* \) is \( \mathcal{X}_* \sim DPP(VV^\top) \).

The result is not very surprising. It has a noteworthy consequence, which is that as \( \varepsilon \to 0 \), the expected sample size will be bounded by \( p \) from above:

\[
E(|\mathcal{X}_*|) = \sum_{i=1}^{n} \pi_i(\varepsilon) = \sum_{i=1}^{p} \frac{\lambda_{i,0}}{1 + \lambda_{i,0}} + O(\varepsilon) \leq p + O(\varepsilon).
\]

If we wish to sample a larger number of points on average, then it appears that we are out of luck.
3.3.2. A more interesting limit

We may instead look at a very similar limit: instead of taking $L_\varepsilon$, we will now take $L'_\varepsilon = A + \varepsilon^{-1}VV^\top$, which carries the same intuition of giving more importance to $VV^\top$ than $A$.

Since we know the limiting eigenvalues and eigenvectors of $L_\varepsilon$, we know those of $L'_\varepsilon$: the eigenvectors are unaffected, but the eigenvalues are scaled by $\varepsilon^{-1}$.

The scaling affects the probabilities of including eigenvectors, since we now have:

$$\pi'_i(\varepsilon) = \frac{\varepsilon^{-1}\lambda_i(\varepsilon)}{1 + \varepsilon^{-1}\lambda_i(\varepsilon)} = \begin{cases} 1 + O(\varepsilon), & \text{if } i \leq p, \\ \frac{\lambda_{i,1}}{1 + \lambda_{i,1}}(1 + O(\varepsilon)), & \text{otherwise}. \end{cases}$$

With the new scaling, the probability of being included goes to 1 for the $p$ first eigenvectors, and tends to $\frac{\lambda_{i,1}}{1 + \lambda_{i,1}}$ for the remaining eigenvectors. We have a partial-projection DPP, i.e., we obtain:

**Proposition 3.6.** Let $X_\varepsilon \sim DPP(A + \varepsilon^{-1}VV^\top)$. Then the limiting process is $X_\star \sim DPP(A; V)$.

Importantly, the expected sample size goes to:

$$E(|X_\star|) = \sum_i \pi_i = 1 = p + \sum_{i=p+1}^n \frac{\lambda_{i,1}}{1 + \lambda_{i,1}} \geq p,$$

so the rescaled L-ensemble allows for a larger sample size.

3.4. Scaling L-ensembles to control sample size

To sum up, partial-projection DPPs also arise as limits of L-ensembles. The types of limits we obtain are analogous to the fixed-size case, but some attention has to be paid to scaling, so that $|X|$ is controlled in expectation. The goal of this section is to motivate rescalings of the form $\alpha\varepsilon^{-p}L_\varepsilon$. It is technical and may be skipped on a first reading. Here we shall consider general kernels at an abstract level, and not just the matrix pencils studied in the rest of the section.

In L-ensembles, the natural way of controlling the expected sample size is to multiply the positive semi-definite matrix $L$ it is based on by a scalar. In other words, we need to rescale $L$ to $\beta L$, with $\beta$ such that

$$E(|X_\varepsilon|) = \text{Tr} (\beta L_\varepsilon (\beta L_\varepsilon + I)^{-1}) = m,$$

where $m$ is the average sample size we would like to obtain. Rescaling by a scalar is a natural process if one thinks of the elements of $L$ as representing similarity, which is defined on a ratio scale (i.e. the similarity between $i$ and $j$ is actually $\frac{L_{i,j}}{\sqrt{L_{i,i}L_{j,j}}}$, which is invariant to rescaling by a scalar). The effect of rescaling is
best seen from the point of view of the inclusion probabilities of the eigenvectors (that we noted $\pi_i$ above). For $X_\varepsilon \sim DPP(\beta L_\varepsilon)$, we have
\[
E|X_\varepsilon| = \sum_{i=1}^{n} \pi_i = \sum_{i=1}^{n} \frac{\beta \lambda_i(\varepsilon)}{1 + \beta \lambda_i(\varepsilon)} = s_\varepsilon(\beta) \tag{43}
\]
It is not too hard to see that $s_\varepsilon(\beta)$ is a continuous, monotonic function of $\beta$ and that:
\[
0 = s_\varepsilon(0) \leq s_\varepsilon(\beta) < \text{rank } L_\varepsilon
\]
Because $s$ is monotonic, for every $\varepsilon$ there exists a unique $\beta$ such that $s_\varepsilon(\beta) = m$ for $m < \text{rank } L_\varepsilon$. This value of $\beta$ is an implicit function of $m$ and $\varepsilon$, which we note $\beta_m^*(\varepsilon)$. One may verify using the implicit function theorem that $\beta_m^*(\varepsilon)$ is continuous and differentiable. In addition, it has an expansion in $\varepsilon$ as a Puiseux series. To see why, note that $s_\varepsilon(\beta) = m$ may be rewritten as a polynomial equation:
\[
\sum_{i=1}^{n} \frac{\beta \lambda_i(\varepsilon)}{1 + \beta \lambda_i(\varepsilon)} = m \iff \sum_{i=1}^{n} \beta \lambda_i(\varepsilon) \prod_{j \neq i} (1 + \beta \lambda_j(\varepsilon)) = m \prod_{j=1}^{n} (1 + \beta \lambda_j(\varepsilon)),
\]
which is a polynomial in $\beta$, with coefficients that depend analytically on $\varepsilon$ (via the $\lambda_i$’s). We call the solution $\beta_m^*(\varepsilon)$ a scaling function because it specifies how to rescale the matrix $L$ (as a function of $\varepsilon$) so that $E(|X_\varepsilon|) = m$ for all $\varepsilon$. Because $\beta_m^*(\varepsilon)$ is the solution of a polynomial equation with analytical coefficients, the Newton-Puiseux theorem states that the solution can be written (in an non-empty, punctured neighbourhood of 0, see [18]) as:
\[
\beta_m^*(\varepsilon) = \sum_{i=-s}^{\infty} \alpha_i \varepsilon^{i/c} \tag{44}
\]
where $c$ is some positive integer and $s$ determines the order of the divergence at 0. This Puiseux series is simply a Laurent series in $\varepsilon^{1/c}$. While we could go deeper in the study of scaling functions, it would require introducing quite a bit of background on Newton diagrams (which enable us to show for instance that $c = 1$ in most cases). Instead, for the purposes of this article, we are content to note that scaling functions are asymptotically of the form $\alpha \varepsilon^{-p}$ for some $\alpha$ and $p$ that depend on $m$. In the theorems below (section 6), we study limits of $L$-ensembles rescaled by $\alpha \varepsilon^{-p}$, and describe what happens as $p$ varies.

3.5. A summary

It may be helpful to take a step back and look broadly at the space of DPPs, fixed-size DPPs, partial-projection DPPs and their relationships. Recall figure 3. Partial projection DPPs can be thought of as forming part of the boundary of the space of DPPs. Seen from the point of view of marginal kernels, they are on the boundary of the set $\mathcal{K}$ of positive semi-definite matrices with eigenvalues
between 0 and 1 (since in a partial projection DPP, at least one of the eigenvalues equals 1). Seen from the point of view of L-ensembles, partial projection DPPs can be obtained by taking certain limits. The following facts are useful to keep in mind:

- A projection DPP may be obtained by taking the limit in $\varepsilon \to 0$ of the L-ensemble $L(\varepsilon) = \varepsilon^{-1}VV^T$. The limiting DPP is a projection DPP, $X^* \sim |DPP|_{\text{rank } V}(VV^T)$. It has an L-ensemble as a fixed-size DPP, but not as a DPP (the L-ensemble diverges in the limit).
- A partial projection DPP may be obtained by taking the limit in $\varepsilon \to 0$ of the L-ensemble $L(\varepsilon) = A + \varepsilon^{-1}VV^T$. This is proposition 3.6.
- A partial projection DPP with fixed-size $m$ may be obtained by taking the limit in $\varepsilon \to 0$ of a $|DPP|_m$ with $L(\varepsilon) = \varepsilon A + VV^T$, if $m \geq \text{rank } V$. This is theorem 3.1. If $m = \text{rank } V$, then the limit is a projection DPP.

4. The flat limit of fixed-size L-ensembles (univariate case)

Now that we have introduced partial-projection DPPs, and seen how they arise as limits in the specific case of pencil matrices, we have the requisite tools to deal with flat limits of L-ensembles in general. In this section and the two following ones, we study L-ensembles based on kernel matrices taken in the flat limit. More specifically, Section 4 starts gently with fixed-size L-ensembles in the univariate (the ground set $\Omega$ is a subset of the real line) case. Then, Section 5 extends these results to the multivariate case ($\Omega \subseteq \mathbb{R}^d$, $d \geq 1$), but still in the fixed-size context. Finally, Section 6 deals with the more involved limits of varying-size L-ensembles, again first in the univariate case before extending to the multivariate case.

We begin by defining our objects of study, and summarise a few properties of determinants in the flat limit, taken from [15, 4]. We then apply these results to study the flat limit of fixed-size L-ensembles, which as we will see depends mostly on $r$, the smoothness parameter of the kernel. The section concludes with some numerical results.

4.1. Introduction

We focus on stationary kernels, as defined in section 1.1, where $\varepsilon$ plays the role of an inverse scale parameter. Thus, we consider L-ensembles based on matrices of the form

$$L(\varepsilon) = [\kappa_\varepsilon(x_i, x_j)]_{i=1,j=1}^n$$

for a set of points $\Omega = \{x_1, \ldots, x_n\}$, all on the real line and all different from one another. From stationarity, the kernel function $\kappa_\varepsilon$ may be written as:

$$\kappa_\varepsilon(x_i, x_j) = f(\varepsilon|x_i - x_j|)$$
and we further assume that $f$ is analytic in a neighbourhood of 0. As in equation (3), we expand the kernel in powers of $\varepsilon$ as:

$$\kappa_\varepsilon(x_i, x_j) = f_0 + \varepsilon f_1 |x_i - x_j| + \varepsilon^2 f_2 |x_i - x_j|^2 + \varepsilon^3 f_3 |x_i - x_j|^3 + \ldots$$

The expansion for individual entries may be represented in a more compact and familiar manner in a matrix form:

$$L(\varepsilon) = f_0 D^{(0)} + \varepsilon f_1 D^{(1)} + \varepsilon^2 f_2 D^{(2)} + \ldots$$

where

$$D^{(p)} = [ |x_i - x_j|^p ]_{i,j}$$

Our goal is to characterise the limiting processes that arise from varying-size and fixed-size L-ensembles based on $L(\varepsilon)$ as $\varepsilon \to 0$. One may recognise in Eq. (45) a more complex version of the linearly perturbed matrix studied in section 3. It is indeed useful to think of the terms $\varepsilon^i f_i D^{(i)}$ as containing features that are increasingly down-weighted as $\varepsilon \to 0$. The analysis is more complicated than in the simple case above, notably because the matrices $D^{(i)}$ are rank-deficient for even $i$ (up to some index depending on $n$) but invertible for odd $i$ [4]. The smoothness order of the kernel (see section 1.1) defines how soon in the decomposition the first invertible matrix appears. For instance, if $r = 2$ then $f_1 = 0$ and we get:

$$L(\varepsilon) = f_0 D^{(0)} + \varepsilon^2 f_2 D^{(2)} + \varepsilon^3 f_3 D^{(3)} + \ldots$$

If $n > 2$, the first invertible matrix to appear in the expansion in $\varepsilon$ is $D^{(3)}$, and it will lead to different asymptotic behaviour than if the first invertible matrix had been $D^{(1)}$ ($r = 1$) or $D^{(5)}$ ($r = 3$). If the kernel is completely smooth, then:

$$L(\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^{2i} f_{2i} D^{(2i)}$$

and odd terms never appear. This again has its own asymptotic behaviour. A subtle issue is that if the matrix under consideration is small enough compared to the regularity order, then the asymptotics are the same than in the completely smooth case. We invite the reader to pay attention to the interplay between $m$ (the size of the L-ensemble) and $r$ (the regularity order) in our theorems. For more on the flat asymptotics of kernel matrices, we refer again to [4].

### 4.1.1. Univariate polynomials and Vandermonde matrices

Recall that we define the Vandermonde matrix of order $k$ as:

$$V_{\leq k} = \begin{bmatrix} 1 & x_1 & \cdots & x_1^k \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \cdots & x_n^k \end{bmatrix}, \quad (46)$$
where $x_1, \ldots, x_n$ are the $n$ points of the ground set $\Omega$ (We may sometimes use the notation $V_{<k} = V_{<k-1}$ as well). Note that $V_{<k}$ has $k + 1$ columns. The “classical” Vandermonde matrix has $k = n - 1$, which makes it square. $V_{<n-1}$ is invertible if and only if the points in $\Omega$ are distinct, which can be established from the following well-known determinantal formula:

$$\det V_{\leq n-1} = \prod_{i<j} (x_i - x_j)$$  \hspace{1cm} (47)

As short-hand, we shall use $v_l = (x_1^{l-1}, \ldots, x_n^{l-1})^\top$ to denote the $l$-th column of $V$. Submatrices of $V_{\leq k}$ corresponding to a subset of points $X$ will be denoted $V_{\leq k}(X) \in \mathbb{R}^{|X| \times (k+1)}$.

### 4.1.2. Some results on limiting determinants and spectra

In this section we summarise some of the main results from [4]. These concern the limiting determinants and spectra of kernel matrices. All we need for the proofs are the results on the limiting determinants, but the results on asymptotic spectra may help understand how the limiting process arises.

The statements involve the Wronskian matrix of the kernel, which we now define. The Wronskian is a matrix of derivatives of the kernel at 0, specifically:

$$W_{\leq k} \overset{\text{def}}{=} \begin{bmatrix} \kappa^{(0,0)}(0,0) & \cdots & \kappa^{(0,k)}(0,0) \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \kappa^{(k-1)}(0,0) & \cdots & \kappa^{(k,k)}(0,0) \\ k!0 & \cdots & k!k! \end{bmatrix}.$$  \hspace{1cm} (48)

Thus, $W_{\leq k}$ contains derivatives up to order $k$. It is important to realise that $W$ depends only on the kernel, and is independent of the locations $\Omega$ at which the kernel is evaluated.

The first theorem concerns the limiting determinants in the smooth case, which tie in directly to Vandermonde determinants:

**Theorem 4.1.** Let $\kappa$ be a kernel function and $X$ a set of $m$ points. If the smoothness order $r$ satisfies $r \geq m$ then, for small $\varepsilon$, the determinant of $L_X(\varepsilon) = [\kappa(\varepsilon x_i, \varepsilon x_j)]_{i,j=1}^m$ has the expansion

$$\det(L_X(\varepsilon)) = \varepsilon^{m(m-1)/2} \det(V_{\leq m-1}(X))^2 \det W_{\leq m-1} + O(\varepsilon).$$  \hspace{1cm} (49)

We have made explicit in the notation the quantities that depend on the points $X$ versus those that do not.

This result appeared originally in [15], and can be found in this form in theorem 4.1 of [4]. It can be generalised to cases with lower order of smoothness, leading to:

**Theorem 4.2.** Let $\kappa$ be a kernel function and $X$ a set of $m$ points. If the smoothness order $r$ satisfies $r \leq m$ then, for small $\varepsilon$, the determinant of $L_X(\varepsilon) = [\kappa(\varepsilon x_i, \varepsilon x_j)]_{i,j=1}^m$ has the expansion

$$\det(L_X(\varepsilon)) = \varepsilon^{m(m-1)/2} \det(V_{\leq m-1}(X))^2 \det W_{\leq m-1} + O(\varepsilon).$$  \hspace{1cm} (49)
Determinantal Point Processes in the Flat Limit

$[\kappa(\varepsilon x_i, \varepsilon x_j)]_{i,j=1}^m$ has the expansion

$$\det(L_X(\varepsilon)) = \varepsilon^{m(2r-1)-r^2} \left( \bar{l}(X) + O(\varepsilon) \right), \quad (50)$$

where the main term is given by

$$\bar{l}(X) = (-1)^r \det W_{\leq r-1} \det \begin{bmatrix} f_{2r-1} D^{(2r-1)}(X) & V_{\leq r-1}(X) \ V_{\leq r-1}(X)^\top \end{bmatrix}$$

(51)

**Remark 4.3.** Note that for $r = m$, equations (49) and (50) coincide, since $V_{\leq m-1}(X)$ is square and the determinant in (50) reduces to $(-1)^m \det(V_{\leq m-1}(X))^2$.

**Remark 4.4.** In the introduction (see fig. 1), we stated that while determinants of kernel matrices go to 0 in the flat limit, ratios of determinants go to a finite value. The statement follows as a direct consequence of thm. 4.1 and 4.2: For instance, under the conditions of Theorem 4.1, we have:

$$\frac{\det(L_{X'}(\varepsilon))}{\det(L_X(\varepsilon))} = \frac{\det(V_{\leq m-1}(X'))^2}{\det(V_{\leq m-1}(X))^2} + O(\varepsilon)$$

By itself this observation is almost enough to prove convergence.

### 4.2. Flat limit in the fixed-size case

Consider $X_\varepsilon \sim |DPP|_m(L(\varepsilon))$ with $m \leq n$ and $m$ and $n = |\Omega|$ fixed (no large $n$ asymptotics are involved here). We are interested in the limiting distribution of $X_\varepsilon$ as $\varepsilon \to 0$.

It is not at first blush obvious that the limiting point process exists and is non-trivial. Indeed, as $\varepsilon \to 0$, every entry of the matrix $L(\varepsilon)$ goes to 1, and so $\det(L(\varepsilon)_X)$ goes to 0 for all subsets $X$. What makes the limit non-trivial is, as we shall see in the proofs, that these quantities go to 0 at different speeds.

The first result characterises the smooth case, where the smoothness order of the kernel is larger than $m$.

**Theorem 4.5.** Let $L_\varepsilon = [\kappa_\varepsilon(x_i, x_j)]_{i,j}$ with $\kappa$ a stationary kernel of smoothness order $r \geq m$. Then $X_\varepsilon \sim |DPP|_m(L(\varepsilon))$ converges to $X_\star \sim |DPP|_{m}(V_{\leq m-1}V_{\leq m-1}^\top)$.

**Proof.** The result follows directly from theorem 4.1, applied to minors of $L(\varepsilon)$ of size $m \times m$, and lemma 1.31. To be more explicit, let $L^*_\star = V_{\leq m-1}V_{\leq m-1}^\top$. Theorem 4.1 implies that:

$$P(X_\varepsilon = X) = \frac{\varepsilon^{m(m-1)}(\det W_{\leq m-1} \det L_X^* + O(\varepsilon))}{\varepsilon^{m(m-1)}(\det W_{\leq m-1} \sum_{Y, |Y| = m} \det L_Y^* + O(\varepsilon))}$$

We may apply lemma 1.31 directly. $X_\varepsilon$ tends to $X_\star$, a fixed-size DPP with law:

$$P(X_\star = X) = \frac{\det L_X^*}{\sum_{Y, |Y| = m} \det L_Y^*}$$
Remark 4.6. The result says that as $\epsilon \to 0$ the limiting point process is (a) a fixed-size L-ensemble (and even a projection DPP as $V_{\leq m-1}V_{\leq m-1}^T$ is of rank $m$) and (b) the positive semi-definite matrix it is based on is a Vandermonde matrix of $\Omega$. It is worth studying this matrix in greater detail. Let $M = V_{\leq m-1}V_{\leq m-1}^T$. Then for any subset $X \subset \Omega$ of size $m$, $\det M_X = \det^2(V_{\leq m-1}(X))$, because $V_{\leq m-1}(X)$ is a square matrix. From the Vandermonde determinant formula (eq. (47)), this means that if $\mathcal{X} \sim |DPP|_m(M)$,

$$P(X) = \frac{1}{Z} \prod_{(x,y) \in X^2} (x - y)^2$$  \hspace{1cm} (52)

Remark 4.7. The conditional law $P(\mathcal{X} = \{x\} \cup Y | Y)$ (the conditional law of one of the points when the rest are fixed) tends to:

$$P(\mathcal{X} = \{x\} \cup Y | Y) \propto \prod_{y \in Y} (x - y)^2$$

which is evidently a repulsive point process (since small distances between points are unlikely).

To summarise: if we sample a fixed-size L-ensemble of size $m$, and the kernel is regular enough compared to $m$ (i.e., $r \geq m$), then whatever the kernel the limiting process exists and is the same. The probability of sampling a set $X$ is just proportional to a squared Vandermonde determinant, and that defines a projection DPP.

The next theorem describes what happens when the kernel is less smooth. We obtain a partial projection kernel, where the projective part comes from polynomials, and the non-projective part comes from the first nonzero odd term in the kernel expansion (see Eq. (45)).

Theorem 4.8. Let $L_\epsilon = [\kappa_{\epsilon}(x_i,x_j)]_{i,j}$ with $\kappa$ a stationary kernel of smoothness order $r \leq m$. Then $\mathcal{X} \sim |DPP|_m(L_\epsilon)$ converges to $\mathcal{X}_* \sim |DPP|_m(D^{2r-1}; V_{\leq r-1})$.

Proof. The argument is exactly the same as in theorem 4.5, this time using the limiting form of the determinant given by theorem 4.2. \hfill \Box

Example. In the case of the exponential kernel $\kappa_\epsilon(x,y) = e^{-\epsilon|x-y|}$, $r = 1$, and the theorem states that

$$P(\mathcal{X}_* = X) \propto \det \left( \begin{array}{cc} -D^{(1)} & 1 \\ 1^T & 0 \end{array} \right)$$  \hspace{1cm} (53)

Equivalently, from a mixture point of view, the constant eigenvector $q_0 = \frac{1}{\sqrt{n}} \mathbf{1}$ is sampled with probability 1, and the remaining $m-1$ eigenvectors are sampled from a (diagonal) fixed-size L-ensemble with diagonal entries equal to the eigenvalues of $D = -(I - q_0 q_0^T)D^{(1)}(I - q_0 q_0^T)$.

\footnote{The “whatever the kernel” part becomes more complicated in the multidimensional case, as we shall see.}
Remark 4.9. Some algebra reveals that
\[
\det \begin{pmatrix} -D_{X}^{(1)} & 1 \\ 1^t & 0 \end{pmatrix} = (2)^{m-1} \prod_{i=1}^{m} (x_{i+1} - x_i)
\]  
(54)

where in the last expression we have sorted the points in $X$ so that $x_1 \leq x_2 \leq \ldots \leq x_m$. As in (52) above, the repulsive nature of the limit point process is immediately apparent from eq. (54). Unlike (52), which involves all distances, eq. (54) only involves distances between direct neighbors. We speculate that similar expressions exist for $r > 1$ but we unfortunately have not been able to derive them.

Proof. Eq. (53) may be derived by using a finite difference operator of the form:
\[
F = \begin{pmatrix} 1 & 0 & \ldots \\ \frac{1}{\delta_1} & 0 & \ldots \\ \vdots & \vdots & \ddots \\ \frac{1}{\delta_m} & 0 & \ldots \end{pmatrix}
\]

where $\delta_i = x_{i+1} - x_i$. Since $F$ is lower-triangular, $\det F = \prod_{i=1}^{m-1} \delta_i^{-1}$. Then applying lemma 1.6 to
\[
\det \begin{pmatrix} F & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -D_{X}^{(1)} & 1 \\ 1^t & 0 \end{pmatrix} \begin{pmatrix} F^t & 0 \\ 0 & 1 \end{pmatrix}
\]

and simplifying yields the result. \hfill \Box

4.3. Some numerical illustrations

To illustrate the convergence theorems above, a good visual tool is to examine the convergence of conditional distributions of the form:
\[
P(X = \{x\} \cup Y | Y) \propto \det L_{\{x\} \cup Y} \propto (L_{x,x} - L_{x,Y} L_{Y,Y}^{-1} L_{Y,x})
\]  
(55)

This should be interpreted as the conditional probability of the $m$-th item fixing the first $m-1$. The conditional law $P(X = \{x\} \cup Y | Y)$ tends to that of $P(X = \{x\} \cup Y | Y)$, and in dimension 1 we can depict this, as a function of $x$.

We do so in figure 6, where we assume $X$ is a $m = 5$ fixed-size L-ensemble, and the ground set is a finite subset of $[0,1]$. The conditioning subset $Y$ is chosen to be of size 4, and for the sake of illustration, we let $x$ vary as a continuous parameter in $[0,1]$. The four panels correspond to four different kernel functions. The conditional probability is plotted for different values of $\varepsilon$. In all plots we observe a rapid convergence with $\varepsilon$. In the top panel, the difference between the asymptotics obtained for $r = 1$ and $r = \infty$ are quite striking. In the bottom panel, we have two different kernels with identical smoothness index, and as predicted by Theorem 4.8 the $\varepsilon \to 0$ limits are identical.
Fig 6: Asymptotics of conditional densities of L-ensembles based on four different kernels. Here we plot $P(\mathcal{X}_\varepsilon = \{x\} \cup Y | Y)$, the conditional density of a fixed size L-ensemble (with $m = 5$) where four of the points are fixed ($Y$) and the last is varying ($x$). The points in $Y$ are at 0.1, 0.3, 0.5, 0.9. The curves in blue are the conditional densities for different values of $\varepsilon$: 4, 1.5, 5, 1, in blue. The dotted red line is the asymptotic limit in $\varepsilon \to 0$. Note that the two kernels in the bottom row have the same regularity coefficient $r = 2$, and as predicted by the results the limiting densities are equal.

(a) $k(x, y) = \exp(-|x - y|)$, a kernel with $r = 1$

(b) $k(x, y) = \exp(-(x - y)^2)$, a kernel with $r = \infty$

(c) $k(x, y) = (1 + |x - y|) \exp(-|x - y|)$, a kernel with $r = 2$

(d) $k(x, y) = \sin(|x-y|+\frac{\pi}{4}) \exp(-|x-y|)$, another kernel with $r = 2$
Fig 7: Flat limit of inclusion probabilities of (fixed-size) L-ensembles for three different kernels. Here we plot $P(x \in X_\varepsilon)$, the inclusion probabilities for a fixed size L-ensemble (with $m = 5$), where the ground set $\Omega$ consists in 20 points drawn at random from the unit interval. The dots in blue (joined by lines for clarity) are inclusion probabilities for $\varepsilon = 4, 1.5, .5, .1$. The dots in red correspond to the asymptotic limit in $\varepsilon \to 0$. The three kernels are, from left-to-right, $\exp(-\delta), (1 + \delta) \exp(-\delta), (3 + 3\delta + \delta^2) \exp(-\delta)$, where $\delta = |x - y|$. These kernels have $r = 1, 2$ and 3, respectively.

Another set of quantities that are easy to examine visually are the first order inclusion probabilities ($P(x \in X)$). We refer to [3] for how to compute these quantities in fixed-size L-ensembles. Since $X_\varepsilon$ converges to $X_\star$, so must the inclusion probabilities, and this is shown in figure 7 for three kernels with increasing values of $r$. For these plots, the ground set consists in 20 points drawn at random in the unit interval. We depict the first order inclusion probabilities for four different values of $\varepsilon$. Rapid convergence with $\varepsilon$ is also observed.

5. The flat limit of fixed-size L-ensembles (multivariate case)

The univariate results we stated above have a multivariate generalisation, and in some cases they are almost the same. The only major difference is that in the univariate case, the only aspect of the kernel function that plays a role in determining the limiting process is the smoothness order $r$. Two kernels may look different, but if they have the same smoothness order they have the same limiting DPP.

When $d > 1$ this is no longer always true. The limiting process may sometimes depend on the specific values of the derivatives of the kernel at 0 (not just whether they exist). Sometimes, but not always: for instance, all kernels with $r = 1$ give the same limiting fixed-size DPP. All kernels with $r = 2$ give the same limiting fixed-size ($m$) L-ensemble, as long as $m > d$. The case of infinitely smooth kernels is particularly intriguing: there is a universal limiting process, but only for $m$ in a set of “magic” values $M_d$ to be defined below. When $m$
falls in between these values, then the limiting process depends on the kernel (although perhaps not strongly).

To build a picture of what the final results look like, we state the easiest first:

**Example.** Let \( L_\epsilon = [\kappa_\epsilon(x_i, x_j)]_{i,j} \) with \( \kappa \) a stationary kernel of smoothness order \( r = 1 \). Then \( \mathcal{X}_\epsilon \sim |DPP|_{m}(L_\epsilon) \) converges to \( \mathcal{X}_\star \sim |DPP|_{m}(\mathbb{D};1) \).

A more general statement is given later, but this one has the advantage of being identical to the univariate result.

As the more general statements are also more complicated, we present our results in increasing order of complexity. The general theorem is found at the end of the section, and all results we state first (including the above) are special cases. But before delving into this, we need to recall some aspects of Vandermonde matrices and introduce the magic numbers \( M_d \). Furthermore, we will give in section 5.4 the spectral interpretation for the universal/non universal limits. We will then present the technical results.

### 5.1. Multivariate Vandermonde matrices

We recall for the sake of readability the appropriate generalisations for multivariate Vandermonde matrices presented in the background section 1.6 on polynomials. For an ordered set of points \( \Omega = \{x_1, \ldots, x_n\} \), all in \( \mathbb{R}^d \), the multivariate Vandermonde matrix is defined as:

\[
V_{\leq k} = \begin{bmatrix} V_0 & V_1 & \cdots & V_k \end{bmatrix} \in \mathbb{R}^{n \times \mathcal{P}_{k,d}}
\]  

(56)

where each block \( V_i \in \mathbb{R}^{n \times \mathcal{P}_{i,d}} \) contains the monomials of degree \( i \) evaluated on the points in \( \Omega \).

As in the previous section, we use \( V_{\leq k}(X) \) to denote the matrix \( V_{\leq k} \) reduced to its lines indexed by the elements in \( X \). As such, \( V_{\leq k}(X) \) has \( |X| = m \) rows and \( \mathcal{P}_{k,d} \) columns. For some values of \( m \) and \( k \) it is square and (potentially) invertible. For instance, consider \( V_{\leq k} \) as in Eq. (56), with \( k = 1 \) and \( d = 2 \). Choosing a subset \( X \) of size \( m = 3 \), the matrix \( V_{\leq 1}(X) \) is square. In dimension 2, there exists a square Vandermonde matrix for sets \( X \) of size \( m = 1, 3, 6, 10, 15, 21, \) etc.

In fact, for an arbitrary dimension \( d \), there exists a square Vandermonde matrix for any size \( m \) such that there exists \( k \in \mathbb{N} \) verifying \( \mathcal{P}_{k,d} = m \), that is, any \( m \) included in the set of integers:

\[
M_d = \{ \mathcal{P}_{k,d} | k \in \mathbb{N} \}.
\]

(57)

We will see that these values of \( m \) are in some sense natural sizes for L-ensembles, because they lead to universal limits, and that is the reason for calling them magic numbers.

We note in passing that while we may easily determine whether \( V_{\leq k}(X) \) is square, whether it is invertible is a complicated question that depends on the geometry of the points \( X \), as there are some non-trivial configurations for which it is not [8]. The results below show that such configurations have probability 0 in the flat limit under any L-ensemble with \( r \) sufficiently large compared to \( m \).
5.2. Universal and non-universal limits, a spectral view

To understand why universal limits sometimes arise and sometimes not, it is worth making a small detour to examine the behaviour of the eigenvalues in the flat limit.

Schaback in [20, Theorem 6] showed that eigenvalues of completely smooth kernels have different orders in $\varepsilon$. All but the first go to 0 as $\varepsilon \to 0$, but they do so at different rates. When $d = 2$, the top eigenvalue is $O(1)$, the next two are $O(\varepsilon^2)$, the next three are $O(\varepsilon^4)$, the next four are $O(\varepsilon^6)$, etc. The reader may notice that there are as many eigenvalues of order $O(\varepsilon^{2i})$ as $H_{i,d}$, the number of monomials of degree $i$ in dimension $d = 2$. This is indeed the general case for smooth kernels in any dimension $d$. In [4] the result is extended to finitely smooth kernels, and the main term in the expansion of the eigenvalues as $\varepsilon \to 0$ is given. In finitely smooth kernels of smoothness order $r$, the first $r$ groups of eigenvalues behave as in the completely smooth case, meaning that the first group (of size $H_{0,d} = 1$) has order $O(1)$, the second of size $H_{1,d} = d$ has order $O(\varepsilon^2)$, etc. up to the group of order $O(\varepsilon^{2(r-1)})$ with size $H_{r-1,d}$. Then all the remaining eigenvalues form a single group of order $O(\varepsilon^{2r-1})$ and of size $n - P_{r-1,d}$. For instance, if $r = 2$, and $d = 2$, the top eigenvalue is $O(1)$, the next two are $O(\varepsilon^2)$, and the remaining $n - 3$ eigenvalues are all $O(\varepsilon^3)$. Let us examine this case more closely, in light of the spectral mixture viewpoint on L-ensembles. The asymptotic expansion of the eigenvalues for $r = 2$, and $d = 2$ are as follows:

$$\lambda_0(\varepsilon) = \tilde{\lambda}_0 + O(\varepsilon)$$

Group 1

$$\begin{align*}
\lambda_1(\varepsilon) &= \varepsilon^2(\tilde{\lambda}_1 + O(\varepsilon)) \\
\lambda_2(\varepsilon) &= \varepsilon^2(\tilde{\lambda}_2 + O(\varepsilon)) \\
\lambda_3(\varepsilon) &= \varepsilon^3(\tilde{\lambda}_3 + O(\varepsilon)) \\
&\vdots \\
\lambda_{n-1}(\varepsilon) &= \varepsilon^3(\tilde{\lambda}_{n-1} + O(\varepsilon))
\end{align*}$$

Group 2

Group 3

We highlight the first two groups in blue because they correspond to the smooth part of the spectrum, i.e. the part that behaves in the same way in the completely smooth case. The rest is the non-smooth part. What the precise values of $\tilde{\lambda}_0, \tilde{\lambda}_1, \ldots$ are does not matter here (see Theorem 6.3 in [4] for the expression), but what matters to this explanation is the following: in the smooth part, the eigenvalues depend non-trivially on the Taylor expansion of the kernel at 0. Different kernels with equal order of regularity may have different asymptotic eigenvalues, but they will appear in groups with the same structure. In the non-smooth part, that is not the case, apart from a trivial global scaling that does not matter here. To sum up: in our example of $r = 2$ and $d = 2$, as $\varepsilon \to 0$, $\frac{\lambda_2}{\lambda_1}$ depends on the kernel, while e.g. $\frac{\lambda_3}{\lambda_1}$ does not. Now consider what happens when
we sample a fixed-size L-ensemble, going into the limit \( \varepsilon \to 0 \), and bearing in mind lemma 1.32.

With \( m = 1 \), only the top eigenvector will ever be sampled (its eigenvalue is \( \mathcal{O}(1) \), all the rest are asymptotically smaller). The result is a projection DPP and the limit is universal. With \( m = 2 \), the top one is always sampled, then either of the next two. We have a partial-projection DPP again. The relative probability of sampling the second or third eigenvector depends on \( \lambda \), which in turn depends on the kernel. The limit is here non-universal. With \( m = 3 \), the top three eigenvectors are necessarily sampled, the ratio \( \frac{\lambda_1}{\lambda_2} \) is irrelevant. Again, we find a projection DPP as the universal limit. Finally, with \( m > 4 \), we start hitting the non-smooth part. The first three eigenvectors are necessarily sampled, and then \( m - 3 \) eigenvectors from the remaining ones. In that part of the spectrum the ratios \( \frac{\lambda_1}{\lambda_2} \) do not depend on the kernel, and so the limit is universal (and a partial-projection DPP). In conclusion, with \( r = 2 \) and \( d = 2 \), there is a universal limit for every value of \( m \) except \( m = 2 \). With \( r = 2 \) and \( d = 3 \), and repeating the same reasoning, we find a universal limit for every \( m \) except \( m = 2 \) and \( m = 3 \).

Theorem 5.4 below will describe the general pattern for \( m \leq \mathcal{P}_{r-1,d} \), gives the asymptotic process for non-universal limits (\( m \) non magic) and universal (\( m \) magic). Before presenting it, we will present separately the case of universal limits alone given for the cases \( m > \mathcal{P}_{r-1,d} \) and \( m \in \mathcal{M}_d \).

The statement of the theorems involves derivatives of the kernel. A convenient short-hand notation for higher-order derivatives uses multi-indices:

\[
f^{(\alpha)}(x) = \frac{\partial^{\alpha} f}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}(x)
\]

The Wronskian matrix of the kernel is defined as:

\[
W_{\leq k} = \left[ k^{(\alpha, \beta)}(0,0) \right]_{|\alpha| \leq k, |\beta| \leq k} \in \mathbb{R}^{k \times k}
\]

Here we index the matrix using multi-indices (equivalently, monomials), so that an element of \( W_{\leq k} \) is e.g., \( W_{(0,2),(2,1)} \) which is a scaled derivative of \( k(x, y) \) of order \((0, 2) \) in \( x \) and \((2, 1) \) in \( y \). For example, for \( d = 2 \) and \( k = 2 \) we may write

\[
W_{\leq 2} = \begin{bmatrix}
  k_{((0,0),(0,0))} & k_{((0,0),(1,0))} & k_{((0,0),(0,1))} & k_{((0,0),(2,0))} & k_{((0,0),(1,1))} & k_{((0,0),(0,2))} \\
  k_{((1,0),(0,0))} & k_{((1,0),(1,0))} & k_{((1,0),(0,1))} & k_{((1,0),(2,0))} & k_{((1,0),(1,1))} & k_{((1,0),(0,2))} \\
  k_{((0,1),(0,0))} & k_{((0,1),(1,0))} & k_{((0,1),(0,1))} & k_{((0,1),(2,0))} & k_{((0,1),(1,1))} & k_{((0,1),(0,2))} \\
  k_{((2,0),(0,0))} & k_{((2,0),(1,0))} & k_{((2,0),(0,1))} & k_{((2,0),(2,0))} & k_{((2,0),(1,1))} & k_{((2,0),(0,2))} \\
  k_{((1,1),(0,0))} & k_{((1,1),(1,0))} & k_{((1,1),(0,1))} & k_{((1,1),(2,0))} & k_{((1,1),(1,1))} & k_{((1,1),(0,2))} \\
  k_{((0,2),(0,0))} & k_{((0,2),(1,0))} & k_{((0,2),(0,1))} & k_{((0,2),(2,0))} & k_{((0,2),(1,1))} & k_{((0,2),(0,2))}
\end{bmatrix} \in \mathbb{R}^{2 \times 2}
\]

for a given ordering of the monomials, and where all the derivatives are taken at \( x = 0, y = 0 \).
5.3. Universal (easy) limits

The following result applies when the kernel is sufficiently smooth and the L-ensemble has fixed size \( m \in \mathbb{N}_d \).

**Theorem 5.1.** Let \( d \in \mathbb{N}^\ast \) and \( \mathbf{L}_\varepsilon = [\kappa_\varepsilon(x_i,x_j)]_{i,j} \) for \( \kappa \) a stationary kernel of smoothness order \( r \) and \( x_1, \ldots, x_n \) vectors in \( \mathbb{R}^d \). Then for all \( m \in \{ \mathcal{P}_{k,d} \}_{k \leq r-1} \subset \mathbb{N}_d \), the fixed-size L-ensemble \( \mathcal{X}_\varepsilon \sim |DPP|_m(\mathbf{L}(\varepsilon)) \) has the limiting distribution:

\[
\mathcal{X}_\varepsilon \sim |DPP|_m(\mathbf{V}_{\leq k} \mathbf{V}_{\leq k}^T)
\]

Equivalently, if \( Q \) is an orthonormal basis for \( \mathbf{V}_{\leq k} \), then:

\[
\mathcal{X}_\varepsilon \sim |DPP|_m(Q Q^T)
\]

**Proof.** Case 1 of theorem 6.1 in [4] states the behavior in \( \varepsilon \) of the determinant in this case:

\[
\forall X \text{ s.t. } |X| = m, \quad \det(\mathbf{L}_\varepsilon, X) = \varepsilon^M \left( \det \mathbf{W}_{\leq k} \left( \det \mathbf{V}_{\leq k}(X) \right)^2 + O(\varepsilon) \right)
\]

for some \( M \in \mathbb{N} \) that we do not need to specify in this proof. \( \mathbf{W}_{\leq k} \) is the Wronskian matrix. It is irrelevant here as it does not depend on \( X \). Similarly to the univariate proof (of theorem 4.5), one obtains that the limiting distribution is indeed \( \mathcal{X}_\varepsilon \sim |DPP|_m(\mathbf{V}_{\leq k} \mathbf{V}_{\leq k}^T) \). The equivalence between the two formulations of the limiting process comes from the fact that \( \mathbf{V}_{\leq k} \) has dimension \( n \times m \), and we may apply lemma 1.26. Any orthonormal basis will do. \( \square \)

**Remark 5.2.** Since \( \mathbf{V}_{\leq k} \) is a polynomial basis, \( Q \) is a basis of orthogonal polynomials. The limiting process we see appearing here is the same as the one studied in [23] in the discrete case. A similar theorem can be proved for continuous DPPs, essentially by tediously changing the notation, and leads to the multivariate orthogonal ensembles studied in [2]. What this means is that the properties proved in [2] (good properties for integration) and [23] (asymptotic rebalancing) also hold for any sufficiently smooth kernel in the flat limit, at least for DPPs of size \( m \in \mathbb{N}_d \).

The case of kernels with finite smoothness is simple if \( m \) is greater than \( \mathcal{P}_{r-1,d} \). We then obtain another universal limiting process:

**Theorem 5.3.** Let \( d \in \mathbb{N}^\ast \) and \( \mathbf{L}_\varepsilon = [\kappa_\varepsilon(x_i,x_j)]_{i,j} \) for \( \kappa \) a stationary kernel of smoothness order \( r \) and \( x_1, \ldots, x_n \) vectors in \( \mathbb{R}^d \). Then, for all \( m \geq \mathcal{P}_{r-1,d} \), the limiting distribution of \( \mathcal{X}_\varepsilon \sim |DPP|_m(\mathbf{L}(\varepsilon)) \) is:

\[
\mathcal{X}_\varepsilon \sim |DPP|_m \left( (-1)^r \mathbf{D}^{(2r-1)}; \mathbf{V}_{\leq r-1} \right)
\]

**Proof.** Case 1 of theorem 6.3 in [4] states the behavior in \( \varepsilon \) of the determinant in this case:

\[
\forall X \text{ s.t. } |X| = m \geq \mathcal{P}_{r-1,d}, \quad \det(\mathbf{L}_\varepsilon, X) = \varepsilon^M \left( \bar{l}(X) + O(\varepsilon) \right),
\]
with $\tilde{l}(X)$ as in Eq. (50) (with $D^{2r-1}(X)$, $W_{\leq r-1}$ and $V_{\leq r-1}(X)$ replaced by their multivariate equivalent – see section 5.4 to see how this is done), and $M \in \mathbb{N}$ that we do not need to specify in this proof neither. Similarly to the univariate proof (of theorem 4.8), one obtains that the limiting distribution is indeed $\mathcal{X}_s \sim |DPP|_m (D^{2r-1}; V_{\leq r-1})$. □

With these two theorems in hand, we can go back to the teaser (figure 4) we gave in the introduction. In figure 4, the points 1 to 6 are on a parabolic curve: $x_2 = x_1^2$, while point 7 ($x_1 = 0.5, x_2 = 0.6$) is not. For now let $X = \{1, 2, 3, 4, 5, 6\}$ and $X' = \{2, 3, 4, 5, 6, 7\}$. Applying theorem 5.1 for a $|DPP|_6$ with a Gaussian kernel, we see that $p(X_s = X) \propto \det V_{\leq 2}(X)^2 = 0$ (the matrix is square and has two identical columns). On the other hand, one may check numerically that $\det V_{\leq 2}(X')$ is non-zero, even though $X'$ is less spread-out than $X$. For the case of the exponential kernel, we apply theorem 5.3, and we can verify numerically that $X$ is much more likely than $X'$. In fact, the two theorems tell us more: the case of the Gaussian kernel holds in fact for all kernels with smoothness order $r > 1$, which all give zero probability to set $X$. The more general phenomenon this illustrates is that DPPs defined from smooth kernels avoid non-unisolvent sets, even though they may be acceptably spread-out.

### 5.4. The general case.

Up to here, we have covered all the easy cases which lead to universal limits. To be precise, for a fixed $d \in \mathbb{N}^*$ and $r \in \mathbb{N}^*$:

- Thm. 5.3 covers the case $m \geq \mathcal{P}_{r-1,d}$.
- Out of the remaining cases where $m \leq \mathcal{P}_{r-1,d}$, Thm. 5.1 covers the special cases where $m \in \mathcal{M}_d$: $m = \mathcal{P}_{0,d}$, $m = \mathcal{P}_{1,d}$, ..., $m = \mathcal{P}_{r-1,d}$.

What remains is to cover the not-so-easy cases where $m \leq \mathcal{P}_{r-1,d}$ and $m \notin \mathcal{M}_d$, as provided by:

**Theorem 5.4.** Let $d \in \mathbb{N}^*$ and $L_{\kappa} = [\kappa(z_i, x_j)]_{i,j}$ for $\kappa$ a stationary kernel of smoothness order $r$, and $x_1, \ldots, x_n$ vectors in $\mathbb{R}^d$. Let $m \leq \mathcal{P}_{r-1,d}$ and $k \leq r-1$ the integer such that $\mathcal{P}_{k-1,d} \leq m \leq \mathcal{P}_{k,d}$. Let us partition the Wronskian $W_{\leq k}$ as:

$$W_{\leq k} = \begin{bmatrix} W_{\leq k-1} & W_s \\ \bar{W}_c & W_s \end{bmatrix}.$$

Then, the limiting distribution of $\mathcal{X}_s \sim |DPP|_m (L_{\kappa})$ is:

$$\mathcal{X}_s \sim |DPP|_m (V_k \bar{W} W_s^T ; V_{\leq k-1})$$

where $\bar{W} \in \mathbb{R}^{m \times m}$ is the Schur complement:

$$\bar{W} = W_d - W_c (W_{\leq k-1})^{-1} W_s$$

**Proof.** Let $X \subset \Omega$ be a subset of size $m$. Case 2 of theorem 6.1 in [4] states the behavior in $\varepsilon$ of the determinant in this case:

$$\det L_{\varepsilon, X} = \varepsilon^{2s(k,d)} \det(Y W_{\leq k} Y^T) \det(V_{\leq k-1}(X)^T V_{\leq k-1}(X)) + O(\varepsilon)$$  \hspace{1cm} (59)
with \( s(k, d) = d \binom{k + d}{d+1} - k(\mathcal{P}_{k,d} - m) \) and \( Y \in \mathbb{R}^{m \times \mathcal{P}_{k,d}} \) defined as:

\[
Y = \begin{bmatrix} I_{\mathcal{P}_{k-1,d}} & Q_{\perp}(X)^T V_k(X) \end{bmatrix},
\]

\( I_{\mathcal{P}_{k-1,d}} \) being the identity matrix of dimension \( \mathcal{P}_{k-1,d} \), \( Q_{\perp}(X) \in \mathbb{R}^{m \times (m-\mathcal{P}_{k-1,d})} \) is an orthonormal basis for the space orthogonal to span \( V_{\leq k-1}(X) \).

Expanding the expression:

\[
\det(YW_{\leq k}Y^T) = \det \begin{pmatrix} W_{\leq k-1} & W_k(X)^T Q_{\perp}(X) \\ Q_{\perp}(X)^T V_k(X) & Q_{\perp}(X)^T V_k(X) W_{\leq k-1} W_{\leq k-1} \end{pmatrix}
\]

Applying lemma 1.5:

\[
\det(YW_{\leq k}Y^T) = \det(W_{\leq k-1}) \det \left( \begin{pmatrix} W_{\leq k-1} & W_k(X)^T Q_{\perp}(X) \\ Q_{\perp}(X)^T V_k(X) & Q_{\perp}(X)^T V_k(X) W_{\leq k-1} W_{\leq k-1} \end{pmatrix} \right)
\]

\[
= \det(W_{\leq k-1}) \det \left( Q_{\perp}(X)^T V_k(X) W_k(X)^T Q_{\perp}(X) \right)
\]

Injecting into (59) and applying lemma 1.6, we obtain:

\[
\det L_{\varepsilon,X} = \varepsilon^{2s(k,d)} \left( \det(W_{\leq k-1}) \det \begin{pmatrix} V_k(X) W_k(X)^T V_{\leq k-1}(X)^T \\ V_{\leq k-1}(X)^T \end{pmatrix} \right) + O(\varepsilon)
\]

The rest of the proof is identical to the univariate case.

5.5. Numerical illustrations

We show here some numerical results analogous to those of section 4.3. In figures 8 and 9, we show the convergence of conditional densities for two different kernels. We illustrate the conditional probabilities of \( x \cup Y | Y \) where \( Y \) comprises seven points already sampled. Even if the ground set is finite and for the sake of illustration, \( x \) varies continuously in the unit square. Figure 10 shows the convergence of inclusion probabilities in an example.

6. The flat limit of varying-size L-ensembles

As we saw in section 3.3 a difficulty in studying limits of varying-size L-ensembles is the control of the sample size. Using the interesting fact that L-ensembles are not invariant to a rescaling of the matrix it is based on, we showed how to control the sample size by using appropriate scaling functions. We restrict ourselves to scaling functions that are asymptotically of the form \( \alpha \varepsilon^{-p} \), and we study the limiting process as a function of \( p \) (and \( \alpha \), but \( p \) plays the more important role).

Studying the flat limit of rescaled L-ensembles reveals an intricate interplay between the scaling parameter \( p \) and the smoothness order \( r \) of the kernel. This will be summarized by pictures analogous to phase diagrams featuring phase transitions. Once again, we begin the study with the \( d = 1 \) case before delving into the multivariate case.
Fig 8: Conditional probability density for $x \in [0, 1]^2$ conditional on the 7 nodes in red, for the exponential kernel $\exp(-\|x - y\|)$. The four panels represent the density for different values of $\varepsilon$ (panels are labelled with the value). The top-left panel is the theoretical limit.
Fig 9: Same as in figure 8, but for the kernel \( (1 + \|x - y\|) \exp(-\|x - y\|) \)
Fig 10: Flat limit of inclusion probabilities of (fixed-size) L-ensembles for three different kernels, multivariate case. Here we plot $P(x \in X_\varepsilon)$, the inclusion probabilities for a fixed size L-ensemble (with $m = 7$), where the ground set $\Omega$ consists in 20 points drawn at random from the unit square. To better visualise the convergence, we plot $P(x_i \in X_\varepsilon)$ as a function of the index $i$, and we have ordered the points according to their inclusion probability for the first kernel. Everything else is analogous to fig. 7. The dots in blue (joined by lines for clarity) are inclusion probabilities for $\varepsilon = 4, 1.5, .5, .1$. The dots in red represent the limit in $\varepsilon \to 0$. The three kernels are, from left-to-right, $\exp(-\delta), (1+\delta) \exp(-\delta), (3+3\delta+\delta^2) \exp(-\delta)$, where $\delta = \|x - y\|$. These kernels have $r = 1, 2$ and 3, respectively.
6.1. The univariate case

In the simple case examined in section 3.3, we had to rescale \( L(\varepsilon) \) by \( \varepsilon^{-1} \) in order to have \( E(|X_\varepsilon|) > p \) in the limit. Here we generalise the scaling to \( \alpha \varepsilon^{-p} L \), and the limiting size of the L-ensemble will depend on \( p \). Interestingly, we will see that in some cases, if \( p \) is odd then the limit is a projection DPP, whereas if \( p \) is even the limit is a partial projection DPP. As in the fixed-size case, finitely smooth kernels are indistinguishable from completely smooth kernels if \( |X| \) is small enough, so that a subtle interplay between \( p \) and \( r \) is at work in our result given in theorem 6.2.

This interplay is summed up in figure (11). In a \((p,r)\) plot, we distinguish three different zones in which the limiting behavior is different. If \( p \geq 2n - 1 \) (where we recall that \( n \) is the size of the ground set \( \Omega \)) or \( p > 2r - 1 \), the limit is trivial since the process converges with probability 1 to the ground set. If \( p < 2n - 1 \) and \( p < 2r - 1 \), the limiting process depends on the parity of \( p \) as announced above. An odd \( p \) gives a fixed-size L-ensemble as a limit, whereas an even \( p \) leads to a partial projection DPP with two possible sample size. Finally, on the line defined by \( r = (p + 1)/2 \) for \( p \) varying from 0 to \( 2n - 1 \), the limit process is a partial projection DPP with a wide range of possible sample size, the probability mass of which is explicitly given in Lemma 6.1. The definition of the limit processes are given in Theorem 6.2.

We will prove these results in two steps. The first step is to characterise the distribution of the size of \( |X_\varepsilon| \) in the limit. Once we know how \( |X_\varepsilon| \) is distributed, we use the fact that the conditional law \( X_\varepsilon \mid |X_\varepsilon| = m \) is a fixed size L-ensemble and use the results derived in the previous section to work out the limit of the point process.

6.1.1. Distribution of \(|X_\varepsilon|\) in the flat limit

**Lemma 6.1.** Let \( p \in \mathbb{N}, \alpha > 0, \) and \( \Omega = \{x_1, \ldots, x_n\} \) a set of \( n \) distinct points on the real line. Let \( L_\varepsilon = [\kappa(x_i, x_j)]_{i,j} \) with \( \kappa \) a stationary kernel of smoothness order \( r \in \mathbb{N^*} \). Let \( X_\varepsilon \sim DPP(\alpha \varepsilon^{-p} L_\varepsilon) \). In the limit \( \varepsilon \to 0 \), the distribution of the size of \( X_\varepsilon \) depends on the interplay between \( p, r \) and \( n \). First of all, if \( \frac{p+1}{2} \geq n \) then, for any value of \( r \), as \( \varepsilon \to 0 \), \( |X_\varepsilon| = n \) with probability one. If \( \frac{p+1}{2} \leq n \), there are three scenarios depending on the value of \( r \):

1. if \( r < \frac{p+1}{2} \), then, as \( \varepsilon \to 0 \), \( |X_\varepsilon| = n \) with probability one.
2. if \( r > \frac{p+1}{2} \), the size of \( X_\varepsilon \) has a distribution that depends on the parity of \( p \):
   
   (a) If \( p \) is odd, then, as \( \varepsilon \to 0 \), \( |X_\varepsilon| = \frac{p+1}{2} \) with probability one.
   
   (b) If \( p \) is even then, as \( \varepsilon \to 0 \), and noting \( l = p/2 \)

\[
|X_\varepsilon| = \begin{cases} 
  l & \text{with probability } \frac{1}{1+\alpha}\frac{1}{1+\alpha}\gamma \\
  l+1 & \text{with probability } \frac{\alpha^2}{1+\alpha}\gamma
\end{cases}
\]
Fig 11: Phase transition diagram \((p, r)\) for the scaling \(\varepsilon^p\) in the flat limit of varying size L-ensemble, for kernels with smoothness parameter \(r\). In the light gray zone, the process converges to the whole ground set. On the diagonal line \((r = (p + 1)/2)\), the limit process is a partial projection DPP, with a size distributed over the integers \((p + 1)/2\) up to \(n\). In the dark grey zone, the limit process depends on the parity of \(p\) and is either a partial projection DPP \((p\) even\) of a fixed-size L-ensemble \((p\) odd\). The parameters defining the limit process are given in Lemma 6.1 and Theorem 6.2.

\[ p \text{ even: } \mathcal{X}_* = \text{ppDPP and } |\mathcal{X}_*| = \frac{p}{2} \text{ or } \frac{p}{2} + 1 \]

\[ p \text{ odd: } \mathcal{X}_* = |\text{DPP}| \text{ and } |\mathcal{X}_*| = \frac{p+1}{2} \]

\[ r = \frac{p+1}{2} : \mathcal{X}_* = \text{ppDPP and } \frac{p+1}{2} \leq |\mathcal{X}_*| \leq n \]

\[ X_* = \Omega \text{ and } |X_*| = n \]
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with
\[ \gamma^{-1} = ( (V_{\leq t}^T V_{\leq t})^{-1})_{t+1,t+1} ((W_{\leq t})^{-1})_{t+1,t+1} \]

3. if \( r = \frac{p+1}{2} \), then, as \( \varepsilon \to 0 \), the distribution tends to:

\[ P(|X| = m) = \begin{cases} 
0 & \text{if } m < r \text{ or } m > n \\
\frac{e_m(L)}{e_0(L) + e_1(L) + \ldots + e_n(L)} & \text{otherwise}
\end{cases} \tag{63} \]

where \( D^{(2r-1)} = (I - QQ^T)D^{(2r-1)}(I - QQ^T) \), \( Q \) being an orthonormal basis of \( \text{span}(V_{\leq r-1}) \).

**Proof.** In the following, \( L_{\varepsilon,X} \) stands for the matrix \( L_{\varepsilon} \) reduced to its lines and columns indexed by \( X \). First, recall that if \( X \sim \text{DPP}(L) \), then the marginal distribution of the size \( |X| \) is given by Eq. (18):

\[ P(|X| = m) = e_m(L) \left( e_0(L) + e_1(L) + \ldots + e_n(L) \right)^{-1} \tag{64} \]

where \( e_m(L) \) is the \( m \)-th elementary symmetric polynomial of \( L \) and for consistency \( e_0(L) = 1 \) for all matrices \( L \). Here, we consider the rescaled kernel matrix \( \alpha \varepsilon^{-p} L_{\varepsilon} \). Recall that \( \det(\alpha \varepsilon^{-p} L_{\varepsilon,X}) = \alpha^{|X|} \varepsilon^{-p|X|} \det(L_{\varepsilon,X}) \). One thus has \( \forall i \in \mathbb{N}^* \):

\[ e_i(\alpha \varepsilon^{-p} L_{\varepsilon}) = \alpha^i \varepsilon^{-ip} e_i(L_{\varepsilon}) \]

Let \( r \in \mathbb{N}^* \). In the flat limit, we can apply theorem 4.1 for any set \( X \) of size \( i \leq r \):

\[ \forall i \leq r, \quad e_i(L_{\varepsilon}) = \sum_{|X|=i} \det(L_{\varepsilon,X}) = \varepsilon^{i(i-1)} \left( \sum_{|X|=i} \det(V_{\leq i-1}(X))^2 \det(W_{\leq i-1}) + O(\varepsilon) \right) \]

\[ = \varepsilon^{i(i-1)} \left( \det(V_{\leq i-1}^T V_{\leq i-1}) \det(W_{\leq i-1}) + O(\varepsilon) \right) \]

where we used Cauchy-Binet to write the second line. Denoting

\[ \forall i \leq r, \quad \tilde{e}_i = \det(V_{\leq i-1}^T V_{\leq i-1}) \det(W_{\leq i-1}) \tag{65} \]

one has:

\[ \forall i \leq r, \quad e_i(L_{\varepsilon}) = \varepsilon^{i(i-1)} \left( \tilde{e}_i + O(\varepsilon) \right). \]

Also, we can apply theorem 4.2 for any set \( X \) of size \( i \geq r \):

\[ \forall i \geq r, \quad e_i(L_{\varepsilon}) = \sum_{|X|=i} \det(L_{\varepsilon,X}) = \varepsilon^{i(2r-1)-r^2} \left( \sum_{|X|=i} \tilde{l}(X) + O(\varepsilon) \right) \]

\[ = \varepsilon^{i(2r-1)-r^2} \left( \tilde{e}_i + O(\varepsilon) \right) \]
\[ \forall i > r, \quad \bar{e}_i = \sum_{|X| = i} \tilde{t}(X) = \alpha m \varepsilon^{-p} \varepsilon^{m(m-1)} (\bar{e}_m + \mathcal{O}(\varepsilon)) \]

Now, injecting into Eq. (64), we have:

\[
\text{for } m \leq r : 
\mathbf{P}(|\mathcal{X}| = m) = \frac{\alpha m \varepsilon^{-p} \varepsilon^{m(m-1)} (\bar{e}_m + \mathcal{O}(\varepsilon))}{1 + \sum_{i=1}^r \alpha i \varepsilon^{-p} \varepsilon^i (\bar{e}_i + \mathcal{O}(\varepsilon)) + \sum_{i=r+1}^m \alpha i \varepsilon^{-p} \varepsilon^i (\bar{e}_i + \mathcal{O}(\varepsilon))}
\]

\[
\text{for } m \geq r : 
\mathbf{P}(|\mathcal{X}| = m) = \frac{\alpha m \varepsilon^{-p} \varepsilon^{m(m-1)-r^2} (\bar{e}_m + \mathcal{O}(\varepsilon))}{1 + \sum_{i=1}^r \alpha i \varepsilon^{-p} \varepsilon^i (\bar{e}_i + \mathcal{O}(\varepsilon)) + \sum_{i=r+1}^m \alpha i \varepsilon^{-p} \varepsilon^i (\bar{e}_i + \mathcal{O}(\varepsilon))}
\]

One can re-write these two equations as:

\[ \forall m, \quad \mathbf{P}(|\mathcal{X}| = m) = \frac{\varepsilon^{\eta(m)} (f_0(m) + \mathcal{O}(\varepsilon))}{\sum_{i=0}^{\eta(i)} \varepsilon^{\eta(i)} (f_0(i) + \mathcal{O}(\varepsilon))} \]

where \( \eta(\cdot) \) and \( f_0(\cdot) \) are two \( \varepsilon \)-independent functions verifying:

\[
\eta(i) = \begin{cases} 
\eta_1(i) = i(i - p - 1) & \text{if } i \leq r \\
\eta_2(i) = i(2r - p - 1) - r^2 & \text{if } i \geq r
\end{cases}
\]

and

\[
f_0(i) = \begin{cases} 
\alpha^i \bar{e}_i & \text{if } i \leq r \\
\alpha^i e_i & \text{if } i \geq r.
\end{cases}
\]

Note that we are precisely in the context of lemma 1.32, that we now apply. The question is: what is \( \text{argmin}_{i \in [0, n]} \eta(i) \)? The answer to this question depends on \( p, r \) and \( n \) which explains the different cases of the theorem. Let us make first two simple observations on \( \eta_1 \) and \( \eta_2 \) (refer to figure 12 for an illustration)

- \( \eta_1 \) is a second order polynomial, it is equal to 0 at \( i = 0 \) and then decreases until \( i = \frac{p+1}{2} \), where it reaches its minimum and then increases again.
- \( \eta_2 \) is a linear function with slope \( 2r - p - 1 \). The sign of that slope is equal to the sign of \( r - \frac{p+1}{2} \).

We shall now explore all the cases of the theorem sequentially.

First of all, if \( \frac{p+1}{2} \geq n \), there are two cases: either \( r \geq n \), in which case \( \eta = \eta_1 \) for the whole interval \([0, n]\) and \( \text{argmin} \eta = n \); or \( r \leq n \) in which case \( \eta \) decreases up to \( i = r \) and then continues to decrease (as the slope of \( \eta_2 \) is negative) up to \( i = n \), implying \( \text{argmin} \eta = n \). Thus, whatever the value of \( r \), \( \text{argmin} \eta = n \).

Applying lemma 1.32, for all values of \( r \), as \( \varepsilon \to 0 \), \( |\mathcal{X}| = n \) with probability 1.

Now, if \( \frac{p+1}{2} \leq n \), there are three scenarios depending on the value of \( r \):

1. if \( r < \frac{p+1}{2} \), \( d \) decreases up to \( i = r \) and then continues to decrease (as the slope of \( \eta_2 \) is negative) up to \( i = n \), implying \( \text{argmin} d = n \). Applying lemma 1.32, as \( \varepsilon \to 0 \), \( |\mathcal{X}| = n \) with probability 1.
The behavior of $|X_i|$ is governed by the argument minimizing the function $\eta(m)$ defined in the proof of Lemma 6.1. The curve of $\eta(m)$ is made of a parabola $\eta_1$ up to $m = r$ and then from a line $\eta_2$ with slope $\eta_1'(r)$ (extending integers to reals obviously!). Three cases appear depending of the relative position $(p + 1)/2$ of the minimum of $\eta_1$ with respect to $r$. To study the behavior of $|X_i| \leq n = |\Omega|$, we then have to locate $n$. In the left plot, we observe that the minimum is for $m = n$ if $n < (p + 1)/2$, whereas it is $m = (p + 1)/2$ if $n \geq (p + 1)/2$. Note in this situation that we have either one minimum if $p$ is odd, or two is $p$ is even, since we work with integers. In the right plot, in the case $r < (p + 1)/2$ (thick line), whatever $n$ the minimum is attained at $m = n$ since $\eta$ strictly decreases. If $r = (p + 1)/2$ however (thick dashed horizontal line), the minimum is attained for the range $[(p + 1)/2; n]$ if $n > (p + 1)/2$, otherwise at $n$. 
2. if \( r > \frac{p+1}{2} \), \( \eta \) decreases up to \( \frac{p+1}{2} \), then increases up to \( i = r \), and then continues to increase (as the slope of \( \eta_2 \) is now positive) up to \( i = n \), implying \( \text{argmin} \eta = \frac{p+1}{2} \). Now,

(a) If \( p \) is odd, then \( \frac{p+1}{2} \) is an integer. Applying lemma 1.32, \( |X_\varepsilon| = \frac{p+1}{2} \) with probability 1, as \( \varepsilon \to 0 \).

(b) If \( p \) is even, \( \frac{p+1}{2} \) is not an integer. In that case \( \text{argmin} \eta \) has two integer solutions: \( \frac{p}{2} \) and \( \frac{p}{2} + 1 \). Let \( l = p/2 \). According to lemma 1.32, as \( \varepsilon \to 0 \), \( X_\varepsilon \) will be of size either \( l \) or \( l + 1 \) with probabilities given by

\[
P(|X_\varepsilon| = l) = \frac{\tilde{e}_i}{\tilde{e}_i + \alpha \tilde{e}_{i+1}} \quad \text{and} \quad P(|X_\varepsilon| = l+1) = 1 - P(|X_\varepsilon| = l).
\]

Injecting Eq. 65 and simplifying, we find:

\[
P(|X_\varepsilon| = l) = \frac{1}{1 + \alpha \gamma}
\]

with

\[
\gamma = \frac{\det(V_{\leq l}^T V_{\leq l}) \det W_{\leq l}}{\det(V_{\leq l-1}^T V_{\leq l-1}) \det W_{\leq l-1}} \quad (68)
\]

\[
= \frac{1}{((V_{\leq l}^T V_{\leq l})^{-1})_{l+1,l+1}((W_{\leq l})^{-1})_{l+1,l+1}} \quad (69)
\]

where the last equality follows from Cramer’s rule. Notice that \( \gamma \) depends on the Wronskian of the kernel and not just its order of regularity.

3. the last scenario, \( r = \frac{p+1}{2} \), is the most involved. Indeed, in this case, the function \( \eta \) decreases up to \( i = r \) and then stays constant between \( i = r \) and \( i = n \) (as the slope of \( \eta_2 \) is null). Thus, \( \text{argmin} \eta \) has \( n - r + 1 \) integer solutions: all the integers between \( r \) and \( n \). According to lemma 1.32, as \( \varepsilon \to 0 \), the limiting distribution of \( |X_\varepsilon| \) is:

\[
\forall m \text{ s.t. } r \leq m \leq n, \quad P(|X_\varepsilon| = m) = \frac{\alpha^m \tilde{e}_m}{\sum_{i=r}^{n} \alpha^i \tilde{e}_i}
\]

Now, consider the NNP \( (f_{2r-1} D^{(2r-1)}; V_{\leq r-1}) \) as well as \( f_{2r-1} D^{(2r-1)} \) as defined in Definition 2.4. Note that the rank of \( f_{2r-1} D^{(2r-1)} \) is \( n-r \). One may apply Corollary 2.16 and obtain, for all integer \( i \) such that \( r \leq i \leq n \):

\[
\sum_{|X| = i} \det \begin{bmatrix} f_{2r-1} D^{(2r-1)}(X) & V_{\leq r-1}(X) \\ V_{\leq r-1}^T(X) & 0 \end{bmatrix} = (-1)^r \tilde{e}_{i-r} \left( f_{2r-1} D^{(2r-1)} \right) \det \left( (V_{\leq r-1})^T V_{\leq r-1} \right)
\]

Injecting this in Eq. (66) and simplifying, one re-writes the limiting distribution of $|\mathcal{X}_\epsilon|$ as:

$$\forall m \text{ s.t. } r \leq m \leq n, \quad \mathbb{P}(|\mathcal{X}_\epsilon| = m) = \frac{e_{m-r} \left( \alpha f_{2r-1} \mathbf{D}^{(2r-1)} \right)}{\sum_{i=r}^{n} e_{i-r} \left( \alpha f_{2r-1} \mathbf{D}^{(2r-1)} \right)}.$$  

Noting that $\sum_{i=r}^{n} e_{i-r} \left( \alpha f_{2r-1} \mathbf{D}^{(2r-1)} \right) = \sum_{i=0}^{n} e_{i} \left( \alpha f_{2r-1} \mathbf{D}^{(2r-1)} \right) = \det \left( \mathbf{I} + \alpha f_{2r-1} \mathbf{D}^{(2r-1)} \right)$ finishes the proof.

$\square$

6.1.2. Distribution of $\mathcal{X}_\epsilon$ in the flat limit

Now that we have characterised the distribution of $|\mathcal{X}_\epsilon|$, we can prove the following:

**Theorem 6.2.** Let $p \in \mathbb{N}$, $\alpha > 0$, and $\Omega = \{x_1, \ldots, x_p\}$ a set of $n$ distinct points on the real line. Let $L_x = [\kappa(x_i, x_j)]_{i,j}$ with $\kappa$ a stationary kernel of smoothness order $r \in \mathbb{N}^*$. Let $\mathcal{X}_\epsilon \sim DPP(\alpha \gamma^-p L_x)$. In the limit $\epsilon \to 0$, the distribution of $\mathcal{X}_\epsilon$ depends on the interplay between $p, r$ and $n$. First of all, if $\frac{p+1}{2} \geq n$ then, for any value of $r$, $\mathcal{X}_\epsilon$ has limit $\mathcal{X}_\epsilon = \Omega$ with probability one. If $\frac{p+1}{2} \leq n$, there are three scenarios depending on the value of $r$:

1. if $r < \frac{p+1}{2}$, then $\mathcal{X}_\epsilon$ has limit $\mathcal{X}_\epsilon = \Omega$ with probability one.
2. if $r > \frac{p+1}{2}$, $\mathcal{X}_\epsilon$ has a limiting distribution that depends on the parity of $p$:
   
   (a) If $p$ is odd, then $\mathcal{X}_\epsilon$ has limit $\mathcal{X}_\epsilon \sim |\text{DPP}|_l(V_{\leq l-1} V_{\leq l-1}^T)$ with $l = \frac{p+1}{2}$.
   
   (b) If $p$ is even then $\mathcal{X}_\epsilon$ has limit $\mathcal{X}_\epsilon \sim \text{DPP} \left( \frac{\alpha}{n} \mathbf{w}_{l+1} \mathbf{w}_{l+1}^T, V_{\leq l-1} \right)$ with $l = \frac{p}{2}$ and $\mathbf{w} = (W_{\leq l})^{-1} (l+1).$

3. if $r = \frac{p+1}{2}$, then $\mathcal{X}_\epsilon$ has limit $\mathcal{X}_\epsilon \sim \text{DPP} \left( \alpha f_{2r-1} \mathbf{D}^{(2r-1)}; V_{\leq l-1} \right)$.

**Proof.** We will prove each case sequentially. First of all, for all the cases in Lemma 6.1 for which $|\mathcal{X}_\epsilon| = n$ in the limit $\epsilon \to 0$, the set $\mathcal{X}_\epsilon$ obviously tends to $\Omega$. Let us now focus on scenario number 2.

In the case 2a, we know from Lemma 6.1 that $|\mathcal{X}_\epsilon| = \frac{p+1}{2}$ with probability one. The limiting process is thus a fixed-size $L$-ensemble of size $l = \frac{p+1}{2}$. The fixed-size limit applies and theorem 4.5 implies the result.

Case 2b needs a bit more work. First of all, define the integer $l = \frac{p}{2}$ and consider an orthonormal basis $Q \in \mathbb{R}^{n \times l}$ of span($V_{\leq l-1}$). Also, consider the vector $q_{l+1}$ such that $Q' = [Q, q_{l+1}] \in \mathbb{R}^{n \times (l+1)}$ is an orthonormal basis for span($V_{\leq l}$). From Lemma 6.1 and Theorem 4.5, we know that in the limit $\epsilon \to 0$, $\mathcal{X}_\epsilon$ is a mixture of two fixed-size $L$-ensembles (and hence a partial projection DPP): with probability $\frac{1}{1+\alpha}$, it has size $l$ and distribution $|\text{DPP}|_l(V_{\leq l-1} V_{\leq l-1}^T)$. With
probability \( \frac{\alpha n}{1+\sqrt{\alpha}} \), it has size \( l+1 \) and distribution \( |DPP|_{l+1}(V_{\leq l}V_{l}^T) \). Note that by lemma 1.25, these distributions are equivalent to \( |DPP|_{l}(QQ^\top) \) and \( |DPP|_{l+1}(QQ^\top) \) respectively. Looking at the mixture representation of pp-DPPs described in Corollary 2.15, one observes that this limiting distribution can be succinctly described as a pp-DPP \( X_* \sim DPP(\alpha \gamma Q'Q'^\top; Q) \). Now, by the invariance property of remark 2.23, this is equivalent to \( X_* \sim DPP(\alpha \gamma q_{l+1}q_{l+1}^\top; Q) \).

Also, by the invariance property of remark 2.22, this is in turn equivalent to \( X_* \sim DPP(\alpha \gamma q_{l+1}q_{l+1}^\top; V_{\leq l-1}) \). Finally, noting that

\[
\frac{\det(V_{\leq l}^TV_{\leq l})}{\det(V_{\leq l-1}^TV_{\leq l-1})} q_{l+1}q_{l+1}^\top = v_{l+1}v_{l+1}^\top
\]

and injecting in the expression of \( \gamma \) of Eq. 68, one obtains that \( \gamma q_{l+1}q_{l+1}^\top = \hat{w}^{-1}v_{l+1}v_{l+1}^\top \), finishing the proof that the limit in case 2b is \( X_* \sim DPP(\frac{\alpha}{\hat{w}}v_{l+1}v_{l+1}^\top; V_{\leq l-1}) \).

Let us finish with case 3. From a mixture point of view, the limiting process can be described by:

1. draw the size \( m \) of the set according to Eq. (63) of Lemma 6.1:

\[
P(|X_*| = m) = \begin{cases} 0 & \text{if } m < r \\ \frac{e_{m-r}}{\det(I + \alpha f_{2r-1}D_{(2r-1)})} & \text{if } m \geq r \end{cases}
\]

\( \hat{D}_{(2r-1)} = (I - QQ^\top)D_{(2r-1)}(I - QQ^\top) \), \( Q \) being an orthonormal basis of \( \operatorname{span}(V_{\leq r-1}) \).

2. conditionally on the size, draw a fixed-size pp-DPP, which, according to theorem 4.8, reads \( X_* \sim |DPP|_m(D_{(2r-1)}; V_{\leq r-1}) \).

Noting that \( X_* \sim |DPP|_m(D_{(2r-1)}; V_{\leq r-1}) \) is equivalent to \( X_* \sim |DPP|_m(\alpha f_{2r-1}D_{(2r-1)}; V_{\leq r-1}) \), this mixture is precisely the mixture representation of \( X_* \sim DPP(\alpha f_{2r-1}D_{(2r-1)}; V_{\leq r-1}) \), ending the proof.

### 6.2. The multivariate case

The multivariate case is a mostly straightforward generalisation of the univariate case. The size of \( X_* \) is described in the following lemma, which generalises lemma 6.1

**Lemma 6.3.** Let \( p \in \mathbb{N} \), \( \alpha > 0 \), and \( \Omega = \{ x_1, \ldots, x_n \} \) a set of \( n \) distinct points in \( \mathbb{R}^d \). Let \( L_* = [k_\varepsilon(x_i, x_j)]_{i,j} \) with \( k \) a stationary kernel of smoothness order \( r \in \mathbb{N}^* \). Let \( X_* \sim DPP(\alpha \varepsilon^{-r}L_*) \). In the limit \( \varepsilon \to 0 \), the distribution of the size of \( X_* \) depends on the interplay between \( p, r \) and \( n \). First of all, \( p \) is either even or odd: only one out of the two following values \( \left( \frac{p}{2}, \frac{p+1}{2} \right) \) is an integer. We call that integer \( l \). Now, if \( \mathcal{H}_{-1,d} \geq n \) then, for any value of \( r \), as \( \varepsilon \to 0 \), \( |X_*| = n \) with probability one. Otherwise, there are three scenarios depending on the value of \( r \):
1. if \( r < \frac{p+1}{2} \), then, as \( \varepsilon \to 0 \), \(|X_\varepsilon| = n\) with probability one.
2. if \( r > \frac{p+1}{2} \), the size of \( X_\varepsilon \) has a distribution that depends on the parity of \( p \):
   (a) If \( p \) is odd \((l = \frac{p+1}{2})\), then, as \( \varepsilon \to 0 \), \(|X_\varepsilon| = \mathcal{P}_{l-1,d}\) with probability one.
   (b) If \( p \) is even \((l = \frac{p}{2})\) then, as \( \varepsilon \to 0 \), the distribution tends to:
      \[
P(|X_\varepsilon| = m) = \begin{cases} 0 & \text{if } m < \mathcal{P}_{l-1,d} \text{ or } m > \mathcal{P}_{l,d} \\ \frac{\epsilon_{m-\mathcal{P}_{l-1,d}}(\alpha V_i W V_i^\top)}{\det(I + \alpha V_i W V_i^\top)} & \text{otherwise} \end{cases}
\]
      where \( V_i W V_i^\top \) is as in theorem 5.4, and \( V_i W V_i^\top = (I - Q_i Q_i^\top) V_i W V_i^\top (I - Q_i Q_i^\top) \), \( Q_i \) being an orthonormal basis of \( \text{span}(V_{\leq l-1}) \).
3. if \( r = \frac{p+1}{2} \), then, as \( \varepsilon \to 0 \), the distribution tends to:
      \[
P(|X_\varepsilon| = m) = \begin{cases} 0 & \text{if } m < \mathcal{P}_{l-1,d} \text{ or } m > n \\ \frac{\epsilon_{m-\mathcal{P}_{l-1,d}}(\alpha f_{2r-1} D^{(2r-1)})}{\det(I + \alpha f_{2r-1} D^{(2r-1)})} & \text{otherwise} \end{cases}
\]
      where \( D^{(2r-1)} = (I - Q_r Q_r^\top) D^{(2r-1)} (I - Q_r Q_r^\top) \), \( Q_r \) being an orthonormal basis of \( \text{span}(V_{\leq r-1}) \).

Proof. In appendix, section B \( \square \)

With lemma 6.3 in hand, along with the fixed-size results in section 5.1 we can prove the following:

**Theorem 6.4.** Let \( p \in \mathbb{N}, \alpha > 0 \), and \( \Omega = \{x_1, \ldots, x_n\} \) a set of \( n \) distinct points in \( \mathbb{R}^d \). Let \( L_c = [\kappa_c(x_i, x_j)]_{i,j} \) with \( \kappa \) a stationary kernel of smoothness order \( r \in \mathbb{N}^* \). Let \( X_\varepsilon \sim \text{DPP}(\alpha \varepsilon^{-p} L_c) \). In the limit \( \varepsilon \to 0 \), the distribution of \( X_\varepsilon \) depends on the interplay between \( p, r \) and \( n \). First of all, \( p \) is either even or odd: only one out of the two following values \((\frac{p}{2}, \frac{p+1}{2})\) is an integer. We call that integer \( l \). Now, if \( \mathcal{P}_{l-1,d} \geq n \) then, for any value of \( r \), \( X_\varepsilon \) has limit \( \Omega \) with probability one. Otherwise, there are three scenarios depending on the value of \( r \):

1. if \( r < \frac{p+1}{2} \), then \( X_\varepsilon \) has limit \( X_\varepsilon = \Omega \) with probability one.
2. if \( r > \frac{p+1}{2} \), \( X_\varepsilon \) has a limiting distribution that depends on the parity of \( p \):
   (a) If \( p \) is odd \((l = \frac{p+1}{2})\), then \( X_\varepsilon \) has limit \( \mathcal{X}_\mathcal{P}_{l-1,d}(V_{\leq l-1} V_{\leq l-1}^\top) \)
   (b) If \( p \) is even \((l = \frac{p}{2})\) then \( X_\varepsilon \) has limit \( \mathcal{X}_\mathcal{P}_{l-1,d}(V_{\leq l-1} V_{\leq l-1}^\top) \)
   with \( V_i W V_i^\top \) as in theorem 5.4.
3. if \( r = \frac{p+1}{2} \), then \( X_\varepsilon \) has limit \( \mathcal{X}_\mathcal{P}_{l-1,d}(V_{\leq l-1} V_{\leq l-1}^\top) \).

Proof. Repeats the univariate proof. \( \square \)
Remark 6.5. The following (non-trivial) limit is universal: for odd $p$ and $r > \frac{p+1}{2}$, the limit process is $\{\mathcal{X}_\varepsilon \sim |DPP|_{\mathcal{P}_{r-1,d}}(V_{\leq l-1}V_{\leq l-1}^*)\}$ which does not depend on the Wronskian. This means that L-ensembles in the flat limit tend to exhibit “natural” sizes, the set $\{\mathcal{P}_{1,d}, \mathcal{P}_{2,d}, \ldots\}$.

Another limit exhibits only weak dependency on the Wronskian: if $r = \frac{p+1}{2}$, then $\mathcal{X}_\varepsilon$ has limit $\mathcal{X}_\varepsilon \sim DPP(\alpha f_{2r-1}D_{2^r-1}; V_{\leq r-1})$, where the Wronskian is only present via $f_{2r-1}$, a scaling parameter which can be compensated via $\alpha$.

7. To conclude

The results in this work can be summarised as follows. Two are very general observations, namely that partial-projection DPPs form the closure of the set of DPPs under pertubative limits, and that extended L-ensembles are a natural unifying representation for DPPs and fixed-size DPPs. The rest concern the flat limit: as $\varepsilon \to 0$, L-ensembles formed from stationary kernels stay well-defined (and meaningfully repulsive). In some cases we obtain universal limits where the limit process depends only on $r$ and not the Wronskian of the kernel. In dimension $d > 1$, these universal limits are obtained for certain natural values of $m$ (for fixed-size L-ensembles) or when rescaling with $\varepsilon^{-p}$ for $p$ odd (varying-size L-ensembles).

The question of how fast L-ensembles converge to the limits given here requires expansions to the next order, which we do not yet have. Empirically, we observe that convergence is quite fast in the fixed-size case, but slower in the varying-size case, at least in some instances. This means that the distribution of the size of $\mathcal{X}_\varepsilon$ may converge slowly to its limit. We hope to investigate this further in future work.

In the interests of space we have left some topics aside. Our results on the flat limit should apply as well to D-optimal design, and there is an interesting connection to polyharmonic splines for kernels with finite $r$ (see section 2.8.2, and [21]). We have also entirely skipped the topic of computational applications of these results. Finally, the univariate results point to possible connections with random matrix theory we have yet to explore.

Directions for future work include extending the results to continuous DPPs, and in a related vein letting $n \to \infty$ as $\varepsilon \to 0$ in discrete DPPs. This should let one take advantage of some results from the literature on the asymptotics of Christoffel functions, as in [23]. It would also be worth investigating the flat limit on Riemannian manifolds, rather than on $\mathbb{R}^d$ as we do here.

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Appendix A: Inclusion probabilities in mixtures of projection DPPs

Here, we give formulas for inclusion probabilities valid for mixtures of projection DPPs. These formulas yield the marginal kernels of L-ensembles and partial-projection DPPs as a special case. We give a variant of a calculation in [3], appendix A.2.

Let $U$ be a fixed orthonormal basis of $\mathbb{R}^n$. We assume that $X$ is generated according to the following mixture process:

1. Sample indices $Y \sim \mathbf{P}(Y)$
2. Form the projection matrix $M = U_Y(U_Y)^T$
3. Sample $\mathcal{X}|Y \sim |DPP|_m(M)$

We do not specify $\mathbf{P}(Y)$ for now (it may be an L-ensemble, a fixed-size L-ensemble, etc.).

Since $\mathcal{X}$ is a mixture of projection-DPPs we can write

$$
\mathbf{P}(W \subseteq \mathcal{X}) = \mathbf{E}_Y[\mathbf{P}(W \subseteq \mathcal{X}|Y)]
$$

where the outer expectation is over $Y$, the indices of the columns of $U$ sampled in the mixture process. Since the innermost quantity is an inclusion probability for a projection DPP, we have from lemma 1.25:

$$
\mathbf{P}(W \subseteq \mathcal{X}|Y) = \det (M_W) = \det (U_{W,Y}(U_{W,Y})^T) = \sum_{A \subseteq Y, |A| = |W|} \det (U_{W,A})^2
$$

where the last line follows from the Cauchy-Binet lemma (lemma 1.2). Injecting into 72, we find:

$$
\mathbf{P}(W \subseteq \mathcal{X}) = \mathbf{E}_Y \left[ \sum_{A \subseteq Y, |A| = |W|} \det (U_{W,A})^2 \right]
$$

$$
= \sum_{Y} \mathbf{P}(Y) \sum_{A \subseteq Y, |A| = |W|} \det (U_{W,A})^2
$$

$$
= \sum_{Y,A\subseteq Y, |A| = |W|} \det (U_{W,A})^2 \mathbf{P}(Y) \mathbf{I}\{A \subseteq Y\}
$$

$$
= \sum_{A\subseteq Y, |A| = |W|} \det (U_{W,A})^2 \mathbf{P}(A \subseteq Y).
$$

In the case of L-ensembles and partial projection DPPs, we can go a bit further, since the distribution of $Y$ is a Bernoulli process (meaning that each element $i$ is
included independently with probability $\pi_i$). In that case $P(A \subseteq Y) = \prod_{i \in A} \pi_i$, and using the Binet-Cauchy lemma once again we find:

$$P(W \subseteq X) = \sum_{A/|A|=|W|} \det (U_{W,A})^2 \prod_{i \in A} \pi_i = \det U_{W,:} \text{diag}(\pi_1, \ldots, \pi_n)(U_{W,:})^\top = \det K_W$$  \hspace{1cm} (73)

with $K = U \text{diag}(\pi_1, \ldots, \pi_n)U^\top$.

**Appendix B: Size of $X_\varepsilon$ in the multivariate case**

We prove lemma 6.3. In the following, $L_{\varepsilon,X}$ stands for the matrix $L_{\varepsilon}$ reduced to its lines and columns indexed by $X$. First, recall that if $X \sim \text{DPP}(L)$, then the marginal distribution of the size $|X|$ is given by Eq. (18):

$$P(|X| = m) = \frac{e_m(L)}{e_0(L) + e_1(L) + \ldots + e_n(L)}. \hspace{1cm} (74)$$

where $e_m(L)$ is the $m$-th elementary symmetric polynomial of $L$ and for consistency $e_0(L) = 1$ for all matrices $L$. Here, we consider the $L$-ensemble $\alpha \varepsilon^{-p} L_{\varepsilon}$.

Recall that $\det(\alpha \varepsilon^{-p} L_{\varepsilon,X}) = \alpha^{|X|} \varepsilon^{-p|X|} \det(L_{\varepsilon,X})$. One thus has $\forall i: e_i(\alpha \varepsilon^{-p} L_{\varepsilon}) = \alpha^i \varepsilon^{-ip} e_i(L_{\varepsilon})$.

Let $r \in \mathbb{N}^*$, $d \geq 2$ and consider $i \leq \mathcal{P}_{r-1,d}$. In the flat limit, we can apply theorem 6.1 in [4]. There are two cases: either $i$ is a magic number ($i \in \mathcal{M}_d$) in which case $k \in \mathbb{N}$ will denote the integer verifying $i = \mathcal{P}_{k,d}$, or it is a muggle number ($i \notin \mathcal{M}_d$) in which case $k \in \mathbb{N}$ denotes the smallest integer such that $i \leq \mathcal{P}_{k,d}$.

In both cases, we denote by $M(i)$ the integer $M(i) = d(k+1)$. Combining points 1 and 2 of Theorem 6.1 in [4], one has, $\forall 1 \leq i \leq \mathcal{P}_{r-1,d}$:

$$e_i(L_{\varepsilon}) = \sum_{|X|=i} \det L_{\varepsilon,X}$$

$$= \varepsilon^{2(M(i)+k(\mathcal{P}_{k,d}-i))} \left( \sum_{|X|=i} \det(YW_{\leq k}Y^\top) \det(V_{\leq k-1}(X)^\top V_{\leq k-1}(X)) + \mathcal{O}(\varepsilon) \right)$$

where $Y$ is as in Eq. (60). Denoting

$$\forall 1 \leq i \leq \mathcal{P}_{r-1,d} \quad \tilde{e}_i = \sum_{|X|=i} \det(YW_{\leq k}Y^\top) \det(V_{\leq k-1}(X)^\top V_{\leq k-1}(X)),$$

one has:

$$\forall 1 \leq i \leq \mathcal{P}_{r-1,d} \quad e_i(L_{\varepsilon}) = \varepsilon^{2(M(i)+k(\mathcal{P}_{k,d}-i))} (\tilde{e}_i + \mathcal{O}(\varepsilon)). \hspace{1cm} (75)$$
Also, we can apply theorem 6.3 of [4] for any set $\mathcal{X}$ of size $i \geq \mathcal{P}_{r-1,d}$:

$$
\forall i \geq \mathcal{P}_{r-1,d} \quad e_i(L_\varepsilon) = \sum_{|\mathcal{X}|=i} \det L_{\varepsilon,\mathcal{X}} = \varepsilon^{2d\left(\frac{r}{d+1}\right)} + (2r-1)(i-\mathcal{P}_{r-1,d}) \left( \sum_{|\mathcal{X}|=i} \hat{l}(\mathcal{X}) + O(\varepsilon) \right)
$$

where $\tilde{e}_i$ verifies the same equation than in the univariate case, Eq. (66), replacing $W_{r-1,r-1}$, $V_{\leq r-1}$ and $D^{(2r-1)}$ by their multivariate counterparts. Now, injecting into Eq. (74), and following the proof scheme of the univariate case, one shows that $P(|\mathcal{X}| = m)$ may be written as:

$$
\forall m, \quad P(|\mathcal{X}| = m) = \frac{\varepsilon^{\eta(m)} (f_0(m) + O(\varepsilon))}{\sum_{i=0}^m \varepsilon^{\eta(i)} (f_0(i) + O(\varepsilon))}
$$

where $\eta(\cdot)$ and $f_0(\cdot)$ are two $\varepsilon$-independent functions verifying:

$$
\eta(i) = \begin{cases} 
\eta_0(i) = 0 & \text{if } i = 0 \\
\eta_1(i) = i(2-p) - 2 & \text{if } 0 < i \leq \mathcal{P}_{1,d} \\
\eta_2(i) = i(4-p) - 2d - 4 & \text{if } \mathcal{P}_{1,d} \leq i \leq \mathcal{P}_{2,d} \\
\vdots & \text{if } \mathcal{P}_{1,d} \leq i \leq \mathcal{P}_{r,d} \\
\eta_{r-1}(i) = i(2r - 2 - p) - 2\left(\frac{d+r-1}{d+1}\right) & \text{if } \mathcal{P}_{r-2,d} \leq i \leq \mathcal{P}_{r-1,d} \\
\eta_r(i) = i(2r - 1 - p) - \left(2 + \frac{p}{r-1}\right) \left(\frac{d+r-1}{d+1}\right) & \text{if } i \geq \mathcal{P}_{r-1,d} 
\end{cases}
$$

and

$$
f_0(i) = \begin{cases} 
1 & \text{if } i = 0 \\
\alpha^i \tilde{e}_i & \text{if } 0 < i \leq \mathcal{P}_{r-1,d} \\
\alpha^i \tilde{e}_i & \text{if } i \geq \mathcal{P}_{r-1,d}.
\end{cases}
$$

As in the univariate case, we will make use of lemma 1.32. In order to apply it, one needs to find the integers between 0 and $n$ for which $\eta(\cdot)$ is minimal:

$$
\arg\min_{i \in \mathbb{N}, i \in [0,n]} \eta(i).
$$

The answer to this question depends on $p, r$ and $n$ which explains the different cases of the theorem. Let us make first a few simple observations on the function $\eta : \mathbb{R}^+ \to \mathbb{R}$:

- $\eta(\cdot)$ is continuous (everywhere except in $i = 0$) and piecewise linear,
- the slope of each of the linear pieces of $\eta(\cdot)$ is strictly increasing, starting at $2 - p$ for the first piece $0 < i \leq \mathcal{P}_{1,d}$ and finishing at $2r - 1 - p$ for the last piece $i \geq \mathcal{P}_{r-1,d}$.

We shall now explore all the possible cases sequentially.
1. if \( r < \frac{p+1}{2} \), i.e., \( 2r - 1 - p < 0 \); the slope of all the pieces of \( \eta(\cdot) \) are negative, and \( \eta(\cdot) \) is thus strictly decreasing. In this case, the integer in \([0, n]\) minimizing \( \eta \) is \( i = n \). Applying lemma 1.32, as \( \varepsilon \to 0 \), \( |X_i| = n \) with probability 1.

2. if \( r > \frac{p+1}{2} \):

(a) if \( p \) is odd, then \( \frac{p+1}{2} \) is an integer and \( \mathcal{P}_{\frac{p+1}{2}, d} \) is well defined. Trivially, \( r > \frac{p+1}{2} \) implies \( \mathcal{P}_{\frac{p+1}{2}, d} < \mathcal{P}_{r-1, d} \). Also, note that \( \eta(\cdot) \) decreases strictly between \( 0^+ \) and \( \mathcal{P}_{\frac{p+1}{2}, d} \), and then increases strictly after \( \mathcal{P}_{\frac{p-1}{2}, d} \). The integer in the interval \([0, n]\) minimizing \( \eta(\cdot) \) is thus:

\[
\min \left( \mathcal{P}_{\frac{p+1}{2}, d}, n \right) \, \text{Applying lemma 1.32, as} \, \varepsilon \to 0 \, \text{,} \, |X_i| = \min \left( \mathcal{P}_{\frac{p+1}{2}, d}, n \right) \, \text{with probability 1.}
\]

(b) if \( p \) is even (the case \( p = 0 \) falls into this category, recall that \( \mathcal{P}_{-1, d} \) is by convention set to 0), then \( r > \frac{p+1}{2} \) implies \( \frac{p}{2} \leq r - 1 \) and thus \( \mathcal{P}_{\frac{p}{2}, d} \leq \mathcal{P}_{r-1, d} \). Also, note that \( \eta(\cdot) \) decreases strictly between \( 0^+ \) and \( \mathcal{P}_{\frac{p}{2}, d} \), and then increases strictly after \( \mathcal{P}_{\frac{p}{2}, d} \). The integers in the interval \([0, n]\) minimizing \( \eta(\cdot) \) are:

- \( \{n\} \) if \( n \leq \mathcal{P}_{\frac{p}{2} - 1, d} \). In this case, applying lemma 1.32, as \( \varepsilon \to 0 \), \( |X_i| = n \) with probability 1.
- all the integers contained in the interval \([ \mathcal{P}_{\frac{p}{2} - 1, d}, \min \left( \mathcal{P}_{\frac{p}{2}, d}, n \right) \] \) if \( n \geq \mathcal{P}_{\frac{p}{2} - 1, d} \). In the following \( I_{p, d} \) is the list of these integers. Applying lemma 1.32, as \( \varepsilon \to 0 \):

\[
\forall m \in I_{p, d} \quad P(|X_i| = m) = \frac{\alpha^m \tilde{e}_m}{\sum_{i \in I_{p, d}} \alpha^i \tilde{e}_i} \tag{77}
\]

Now, using the same arguments as in the proof of theorem 5.4, note that Eq. (75) may be re-written as:

\[
\forall i \in I_{p, d} \quad \tilde{e}_i = \det(W_{\leq k-1}) \sum_{|X| = i} \det \left( \begin{array}{c} V_k(X)WV_k(X)^T \ V_{\leq k-1}(X)^T \ V_{\leq k-1}(X) \ 0 \end{array} \right) \tag{78}
\]

where \( W \) is as in theorem 5.4 and \( k = \frac{p}{2} \). Now, consider the NNP \( (V_k \W V_k^T; V_{\leq k-1}) \) as well as \( V_k \W V_k^T \) as defined in Definition 2.4. Note that the rank of \( V_k \W V_k^T \) is \( \min (n - \mathcal{P}_{k-1, d}) \). One may apply Corollary 2.16 and obtain, for all integer \( i \in I_{p, d} \):

\[
\tilde{e}_i = \det(W_{\leq k-1})(-1)^{\mathcal{P}_{k-1, d}} e_{i - \mathcal{P}_{k-1, d}} \left( V_k \W V_k^T \right) \det \left( (V_{\leq k-1})^T V_{\leq k-1} \right) \tag{79}
\]
Simplifying, one obtains:

\[ \forall m \in I_{p,d} \quad P(|\mathcal{X}| = m) = \frac{e_{m-\mathcal{P}_{k-1,d}}(\alpha V_k\tilde{W}V_k^\top)}{\sum_{i \in I_{p,d}} e_{i-\mathcal{P}_{k-1,d}}(\alpha V_k\tilde{W}V_k^\top)}. \]  

(80)

Changing the summing index gives:

\[ \sum_{i \in I_{p,d}} e_{i-\mathcal{P}_{k-1,d}}(\alpha V_k\tilde{W}V_k^\top) = \sum_{i=0}^{\min(\mathcal{H}_{k,d}, n-\mathcal{P}_{k-1,d})} e_i(\alpha V_k\tilde{W}V_k^\top). \]  

(81)

Finally, note that, as rank \((\alpha V_k\tilde{W}V_k^\top)\) = min \((\mathcal{H}_{k,d}, n - \mathcal{P}_{k-1,d})\), all the elementary symmetric polynomials \(e_i\) for \(i > \min(\mathcal{H}_{k,d}, n - \mathcal{P}_{k-1,d})\) are null. The denominator of Eq. (80) is thus \(\sum_{i=0}^{n} e_i(\alpha V_k\tilde{W}V_k^\top)\) = det \((I + \alpha V_k\tilde{W}V_k^\top)\) and one obtains:

\[ \forall m \in I_{p,d} \quad P(|\mathcal{X}| = m) = \frac{e_{m-\mathcal{P}_{k-1,d}}(\alpha V_k\tilde{W}V_k^\top)}{\det(I + \alpha V_k\tilde{W}V_k^\top)}. \]  

(82)

3. if \(r = \frac{p+1}{2}\), \(\eta(\cdot)\) decreases strictly between 0+ and \(\mathcal{P}_{r-1,d}\), and is constant after \(\mathcal{P}_{r-1,d}\). The integers in the interval \([0, n]\) minimizing \(\eta(\cdot)\) are thus:

- \(\{n\}\) if \(n \leq \mathcal{P}_{r-1,d}\). In this case, applying lemma 1.32, as \(\varepsilon \to 0\), \(|\mathcal{X}| = n\) with probability 1.
- all those contained in the interval \([\mathcal{P}_{r-1,d}, n]\) if \(n \geq \mathcal{P}_{r-1,d}\). In the following \(I_{r,d}\) is the list of these integers. Applying lemma 1.32, as \(\varepsilon \to 0\):

\[ \forall m \in I_{r,d} \quad P(|\mathcal{X}| = m) = \frac{\alpha^m \bar{e}_m}{\sum_{i \in I_{r,d}} \alpha^i \bar{e}_i}. \]

Now, using the same line of arguments as in the proof of lemma 6.1, one obtains:

\[ P(|\mathcal{X}| = m) = \begin{cases} 0 & \text{if } m < \mathcal{P}_{r-1,d} \\ \frac{e_{m-\mathcal{P}_{r-1,d}}(\alpha f_{2^{r-1}D^{(2^{r-1})-1}})}{\det(I + \alpha f_{2^{r-1}D^{(2^{r-1})-1}})} & \text{if } m \geq \mathcal{P}_{r-1,d} \end{cases} \]  

(83)
where \( \widetilde{D}^{(2r-1)} = (I - QQ^\top)D^{(2r-1)}(I - QQ^\top) \), \( Q \) being an orthonormal basis of \( \text{span}(V_{\leq r-1}) \).

Finally, one may see that the three cases just described can in fact be equivalently stated in the form of the Lemma, finishing the proof.

Appendix C: Equivalence of extended L-ensembles and DPPs

We prove Th. 2.9 and 2.10.

Proof of Th. 2.9. Let \((L; V)\) be any NNP, and \(\widetilde{L}, Q, \widetilde{U}, \widetilde{\Lambda}\) and \(q\) be as in Definition 2.4. Let \(X \in \Omega\) be drawn according to the distribution:

\[
\forall X \subseteq \Omega, \quad P(X = X) \propto (-1)^p \det \left( \frac{L_X}{(V_{X:})^\top} \right).
\]

Using the generalized Cauchy-Binet formula (theorem 2.13), this can be rewritten as

\[
\forall X \subseteq \Omega, \quad P(X = X) \propto \det(V^\top V) \sum_{Y, |Y| = m-p} \det \left( \left[ Q_{X:} \widetilde{U}_{X,Y} \right] \right)^2 \prod_{i \in Y} \widetilde{\lambda}_i.
\]

As made precise by corollary 2.15, this equation can be interpreted from a mixture point of view. As such, the generic inclusion probability formulas of Appendix A are applicable and one obtains the result.

Proof of Th. 2.10. Given a marginal kernel \(K\), we can always rewrite its spectral factorisation in the form of Eq. (30), by grouping all the eigenvectors corresponding to the eigenvalue 1 in \(Q\); all the remaining eigenvalues can be always represented as \(\tilde{\lambda}_i/(1 + \lambda_i)\).

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