The Ihara expression for generalized weighted zeta functions of Bartholdi type on finite digraphs

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Abstract

The Ihara expression of a weighted zeta function for a general finite digraph is given. It unifies all the Ihara expressions obtained for known zeta functions for finite digraphs. Any digraph in this paper permits multi-edges and multi-loops.

1 Introduction

Graph zeta functions are formal power series associated with finite graphs, that describe closed paths, cycles, or prime cycles of finite graphs. The Ihara expression is one of the four expressions that graph zeta functions may have. It is a determinant expression described by the adjacency matrix and the degree matrix of graphs, and have been one of the main interest in the study of graph zeta functions since its origin was discovered for the Ihara zeta function by Y. Ihara[13]. Ihara’s result is given for the case where graphs are regular. For a general finite graph, the Ihara expression is obtained by H. Bass [4], and the theorem is called the Bass-Ihara theorem. Subsequently, the Bass-Ihara theorem is given its proof in various ways (c.f., [8, 17, 24, 29]), and also it is generalized to other graph zeta functions (c.f., [3, 22, 26, 32]).

Recently, a new significance is given to the Ihara expression from an unexpected point of view. In [16], N. Konno and I. Sato show that the Ihara expression for the Sato zeta function [26], a weighted version of the Ihara zeta function, gives a precise description of the characteristic polynomial of the Grover matrix $U_\Gamma$ on a finite simple graph $\Gamma$. The Grover matrix $U_\Gamma$ is the time-evolution matrix of the Grover walk [9], the most extensively studied quantum walk model on finite graphs (c.f., [11, 12, 15]). The spectrum of the time-evolution matrix is significant since it describes fundamental features of the corresponding
quantum walk: mixing time, periodicity, and localization. In [7], Emms, Severini, Hancock and Wilson consider the Grover matrix on finite simple graphs, and describe its spectrum directly considering the eigenvectors of $U_Γ$. Although Emms et al. [7] do not present the characteristic polynomial explicitly, Konno and Sato [16] give its fine description, provided by the viewpoint coming from the fact that $U_Γ$ is realized as the edge matrix for the Sato zeta function of $Γ$. This process corresponds to reformulating the Hashimoto expression, another determinant expression for graph zeta functions, into the Ihara expression. In other words, the Konno-Sato theorem implies that the Hashimoto expression describes the time-evolution of the Grover walk and the Ihara expression the spectral mapping theorem (c.f., [19, 27]). Thus, the Ihara expression of graph zetas progresses its significance from a quantum walk point of view.

Though the construction of the Ihara expression has relied on case-by-case arguments (c.f., [3, 4, 22, 26]), Y. Watanabe and K. Fukumizu [32] provided a new point of view which has a possibility to unify previous studies. Their idea is reformulating the Hashimoto expression into the Ihara expression by linear algebraic method, which essentially coincide with the idea of the Konno-Sato theorem. Founded on this idea, we construct the Ihara expression for the generalized weighted zeta function by reformulating the Hashimoto expression. The generalized weighted zeta function is defined in [23] which unifies the graph zeta functions that appeared in previous studies. Also in [23], it is investigated the condition for the Hashimoto expression to exist and one can see that the condition is satisfied by the generalized weighted zeta function. Thus, the generalized weighted zeta function has the Hashimoto expression, and we can consider the problem to reformulate it into the Ihara expression. The main result gives the Ihara expression for the generalized weighted zeta function which is described by the weighted adjacency matrix and the weighted backtrack matrix for finite digraphs. The weighted adjacency matrix and the backtrack matrix generalize the adjacency matrix and the degree matrix respectively. These generalized matrices coincide with the ordinary ones on the symmetric digraphs of a finite simple graph. We will make one more remark on digraphs here. In our development, the definition of “inverse arcs” is significant. Since our underlying digraph $Δ$ may have multi-arcs and multi-loops, there will be various definitions for inverse arcs. In this article, every arc with the inverse direction to an arc $a$ is defined to be an inverse arc of $a$. This definition of inverse arcs is a natural generalization of the usual definition which we consider in the case where $Δ$ is the symmetric digraph $Δ(Γ)$ for a finite simple graph $Γ$.

The remaining part of this paper is organized as follows. In Section 2, we recall fundamental notation on graphs, digraphs, words, and dynamical systems, which is required to define the combinatorial zeta functions. In Section 3, we review a fundamental theory of combinatorial zeta functions following [23]. We introduce the definition of a graph zeta function in general, and see that it is combinatorial. As a consequence, a graph zeta function has a determinant expression, called the Hashimoto expression. In Section 4, we review the definition of the generalized weighted zeta function for finite digraphs, and see that classical graph zeta functions, including its Bartholdi type, are unified on a single scheme. In Section 5, we prepare some auxiliary facts relating the Schur complement of a matrix, and verify the
Throughout this paper, we use the following notation. The ring of integers is denoted by \( \mathbb{Z} \), and \( \mathbb{Q} \) denotes the rational number field. For a finite set \( X \), the number of elements in \( X \) is denoted by \( |X| \), and the family of subsets in \( X \) by \( 2^X \). The empty set is denoted by \( \emptyset \). For sets \( X \) and \( Y \), \( X \sqcup Y \) denotes the disjoint union of \( X \) and \( Y \). The Kronecker delta is denoted by \( \delta_{xy} \), which gives one if \( x = y \), zero if \( x \neq y \). For a proposition \( P \), we also use the Kronecker product \( \delta_P \), which equals one if \( P \) is true, zero otherwise. For a square matrix \( M \), the determinant of \( M \) is denoted by \( \det M \), and the trace by \( \text{tr} M \). For a ring \( R \), \( M_n(R) \) denotes the set of square matrices of degree \( n \) with entries in \( R \). The adjacency matrix and the degree matrix of a finite graph \( \Gamma \) are denoted by \( A_{\Gamma} \) and \( D_{\Gamma} \) respectively.

2 Preliminaries

2.1 Digraphs

2.1.1 Digraphs and arcs

A digraph \( \Delta = (V, A) \) is a pair of a set \( V \), and a multi-set \( A \) consisting of ordered pair \( (u,v) \) of elements in \( V \). A digraph is finite if both \( V \) and \( A \) are finite (multi-) sets. An element of \( V \) is called a vertex, and \( A \) an arc respectively. An arc \( a = (u,v) \) is depicted by an arrow from \( u \) to \( v \). For an arc \( a = (u,v) \), the vertex \( u \) is called the tail of \( a \), and \( v \) the head of \( a \), denoted by \( t(a) \) and \( h(a) \) respectively. Note that, since \( A \) is a multi-set, it may occur that \( t(a) = t(a') \) and \( h(a) = h(a') \) for distinct \( a, a' \in A \). An arc \( l \in A \) satisfying \( h(l) = t(l) \) is called a loop of \( \Delta \). For a loop \( l \), the vertex \( n = t(l)(= h(l)) \) is called the nest of \( l \). We denote the set of loops by \( L \), which is a subset of \( A \). If we let \( A_{uv} = \{ a \in A \mid t(a) = u, h(a) = v \} \) for \( u,v \in V \), then in general we have \( |A_{uv}| \geq 1 \) if \( A_{uv} \neq \emptyset \). If \( u = v \), then \( A_{uu} \) consists of loops with nest \( u \), and we also allow \( |A_{uu}| \geq 1 \) for each \( u \). A digraph is called simple if \( A_{uv} \neq \emptyset \) implies \( |A_{uv}| = 1 \) for distinct \( u,v \in V \), and \( A_{uu} = \emptyset \) for every \( u \in V \). In general, we do not suppose simplicity for digraphs in the present article. Thus an element \( a \) of \( A_{uv} \) is sometime called a multi-arc, and the cardinality \( |A_{uv}| \) is called the multiplicity of \( a \). Similarly, an element of \( A_{uu} \) is called a multi-loop with multiplicity \( |A_{uu}| \). For \( u,v \in V \), let

\[
A(u,v) = A_{uv} \cup A_{vu}
\]

(1)

denotes the set of arcs lying between the vertices \( u \) and \( v \). If \( u = v \), then we have \( A(u,u) = A_{uu} \). If \( u \) and \( v \) are distinct, then the union \( A(u,v) = A_{uv} \cup A_{vu} \) is disjoint. For \( u,v \in V \), let

\[
A_{u*} = \{ a \in A \mid t(a) = u \},
\]

\[
A_{*v} = \{ a \in A \mid h(a) = v \}.
\]
2.1.2 The inverses for an arc

Let $\Delta = (V, A)$ be a finite digraph, and $a \in A$ an arc. If $a \in A_{uu}$, then any element $a' \in A_{vu}$ is called an inverse of $a$, that is, $A_{vu}$ is the set of inverse arcs for $a \in A_{uu}$. Thus an inverse of an arc $a$ may not be uniquely determined for $a$. We also note that any loop $l \in A_{uu}$ is an inverse of $l \in A_{uu}$, containing $l$ itself. In particular, each loop is self-inverse. We denote $A_{vu}$ by $S(a)$ if an arc $a$ belongs to $A_{uu}$. Note that $S(l) = A_{uu}$ for $l \in A_{uu}$.

This definition for inverse arcs is a natural generalization for the usual one employed in the theory of graph zeta functions in the case where $\Delta$ is the symmetric digraph of a finite simple graph $\Gamma$. To see this, we recall the usual definition of inverse arcs where $\Gamma$ is a finite simple and directed loop $l$. If the multiplicity of an element $e \in E$ is one, then $e$ is called a simple edge, or simply, an edge. A multi-edge $\{u, v\}$ is called a multi-loop with nest $u$. If $e = \{u, u\}$ is a simple edge, then $e$ is called a loop. Let $L$ denote the set of multi-loops of $\Gamma$. If every element of $E$ is simple and $L = \emptyset$, then $\Gamma$ is called a simple graph. Let $\Gamma = (V, E)$ be a finite graph. For each multi-edge $e = \{u, v\} \in E \setminus L$, we assign two distinct arcs $a_e$ and $\overline{a_e}$ for $e$. We set $\overline{\overline{a_e}} = a_e$, and $a_e \neq a_{e'}$ for $e, e' \in E$ with $e \neq e'$. It is assumed that the arc $a_e$ belongs to $A_{uv}$ or $A_{vu}$. If $a_e \in A_{uv}$, then we have $\overline{a_e} \in A_{vu}$, and vice versa. If $e = \{u, u\} \in E$, then we assign a single directed loop $l_e \in A_{uu}$ for $e$. If $e \neq e'$ in $L$, then $l_e \neq l_{e'}$. Let

$$A = \{ a_e, \overline{a_e} \mid e \in E \setminus L \} \cup \{ l_e \mid e \in L \}.$$

The digraph $(V, A)$ is called the symmetric digraph of $\Gamma$, denoted by $\Delta(\Gamma)$. For the symmetric digraph $\Delta = \Delta(\Gamma)$ of a finite graph, the usual definition of inverse arc is as follows (see, e.g., [4, 3, 22, 26]). Let $a \in A$. If $a = a_e$ for some $e \in E \setminus L$, then $\overline{a_e}$ is the unique inverse arc for $a$, and vice versa. If $a = l_e$ is a loop ($e \in L$), then $a = l_e$ itself is the unique inverse arc for $a$.

**Lemma 1** Suppose that $\Delta = (V, A)$ is the symmetric digraph $\Delta(\Gamma)$ of a finite simple graph $\Gamma$. Then we have

1. $A_{uu} = \emptyset, \forall u \in V$;
2. $A_{uv} \neq \emptyset \Rightarrow |A_{uv}| = 1$;
3. $|A_{uv}| = 1 \Leftrightarrow |A_{vu}| = 1$.

By Lemma 1, one can easily see that if $A_{uv} \neq \emptyset$ then $|A_{uv}| = |A_{vu}| = 1$ for distinct $u, v \in V$. This shows that those definitions of inverse arcs coincide with each other on the symmetric digraph of a finite simple graph.

2.1.3 Closed paths

Let $\Delta = (V, A)$ be a digraph. Recall that $S(a) = A_{vu}$ for $a \in A_{uv}$. A sequence $c = (a_1, a_2, \ldots, a_m)$ of arcs is called a path if it satisfies $h(a_i) = t(a_{i+1})$ for each $i = 1, 2, \ldots, m - 1$. If $a_{i+1} \in S(a_i)$ for some $i$, then the pair $(a_i, a_{i+1})$ of arcs $a_i, a_{i+1}$ is called a backtrack of $c$, or sometimes called a backtrack thorough $h(a_i)$. Then $m$ is called the length of $x$, denoted by $l(x)$. A closed path is a path $c = (a_1, a_2, \ldots, a_m)$ with $h(a_m) = t(a_1)$. A closed path
2.1.5 The adjacency matrix and the backtrack matrix

Let \( C_m = C_m(\Delta) \) denotes the set of closed paths of length \( m \) in \( \Delta \). The set of reduced closed paths of length \( m \) is denoted by \( C^o_m = C^o_m(\Delta) \). The set of closed paths in \( \Delta \) is given by \( C = C(\Delta) := \sqcup_{m \geq 1} C_m \). The set of reduced closed paths is given by \( C^o = C^o(\Delta) := \sqcup_{m \geq 1} C^o_m \).

For a positive integer \( k \), the concatenation of \( k \) copies of a closed path \( c = (a_1, a_2, \ldots, a_m) \) is also a closed path of \( \Delta \). This closed path is called the \( k \)-th power of \( c \), denoted by \( c^k \). The length of \( c^k \) is \( km \) if \( l(c) = m \). If there exists no shorter closed path \( c' \) satisfying \( c' = c^k \), then \( c \) is called prime. We denote the set of prime closed paths of length \( m \) by \( P_m = P_m(\Delta) \).

The set of prime reduced closed paths of length \( m \) is denoted by \( P^o_m = P^o_m(\Delta) \). The set of prime (resp. prime reduced) closed paths of \( \Delta \) is given by \( P = P(\Delta) := \sqcup_{m \geq 1} P_m \) (resp. \( P^o = P^o(\Delta) := \sqcup_{m \geq 1} P^o_m \)).

2.1.4 Cycles

The cyclic permutation \( \sigma = (1, 2, \ldots, m) \) acts on \( C_m \) by

\[
(a_1, a_2, \ldots, a_m)\sigma = (a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(m)}).
\]

Two closed paths \( c = (a_i) \), \( c' = (a'_i) \) which belong to \( C_m \) are called cyclically equivalent, or simply equivalent, if there exists an integer \( k \) satisfying \( c' = c\sigma^k \). The equivalence is denoted by \( c \sim c' \). The binary relation \( \sim \) is indeed an equivalence relation. An equivalence class \( [c] = c \mod \sim \) is called a cycle of \( \Delta \). If \( c \sim c' \) for \( c, c' \in C \), then we have \( l(c) = l(c') \). Hence one can define the length \( l([c]) \) of a cycle \( [c] \) (\( c \in C \)) by \( l([c]) := l(c) \). Let \( [C] = C/\sim \) and \( [C^o_m] = C_m \sim (m \geq 1) \). We have \( [C] = \sqcup_{m \geq 1} C_m \). If \( c \in C \) is prime (resp. reduced), then one can easily see that \( c' \in C \) equivalent to \( c \) is also prime (resp. reduced). Hence we can say that a cycle \( [c] \) is prime (resp. reduced) if a representative \( c \) is prime (resp. reduced).

A prime reduced cycle is a cycle \([c]\) with a representative \( c \) which is prime and reduced. The set of prime cycles, reduced cycles, prime reduced cycles are given by

\[
[P] = P/\sim, \quad [C^o] = C^o/\sim, \quad [P^o] = P^o/\sim,
\]

respectively. These also have the decomposition \( [P] = \sqcup_{m \geq 1} [P_m], [C^o] = \sqcup_{m \geq 1} [C^o_m], [P^o] = \sqcup_{m \geq 1} [P^o_m] \), where \( [P_m], [C^o_m], [P^o_m] \) denote \( P_m/\sim, C^o_m/\sim, P^o_m/\sim \) respectively. Note that these are all well-defined.

2.1.5 The adjacency matrix and the backtrack matrix

Let \( \Delta = (V, A) \) be a finite digraph. The matrix \( A_\Delta = (a_{uv})_{u,v \in V} \) with entries \( a_{uv} = |A_{uv}| \) is called the adjacency matrix of \( \Delta \). For each \( u \in V \), let \( d_u \) denote the number of closed paths \( c = (a_1, a_2) \) satisfying i) \( u = t(a_1) = h(a_2) \), ii) \( a_2 \in S(a_1) \), and iii) \( a_1 \notin L \). In other words, \( d_u \) is the number of backtracks though \( u \). The diagonal matrix \( D_\Delta = (\delta_{uv}d_u)_{u,v \in V} \) is called the backtrack matrix of \( \Delta \).
Note that the backtrack matrix $D_\Delta$ depends on the definition of inverse arcs. Let $\Gamma = (V, E)$ be a finite graph, and $\Delta = \Delta(\Gamma) = (V, A)$ the symmetric digraph of $\Gamma$. For each $e = \{u, v\} \in E \setminus L$, we assign two arcs $a_e, \overline{a_e}$, say $a_e \in A_{uv}$. Fix a vertex $u \in V$. Note that if $\{u, v\} \notin E$ for $v \in V$, then $A_{uv} = A_{vu} = \emptyset$. If we employ the definition of inverse arcs as in [3, 1] (see 2.1.2), then there corresponds a unique backtrack $(\overline{a_e}, a_e)$ thorough $u$ for each multi-edge $e$ of the form $\{u, v\}$ for some $v \in V$, $v \neq u$. Thus we have $d_u = |A_{uv}|$ in this case, and one can readily see that this coincides with the degree of the vertex $u$ in $\Gamma$. Therefore, we have $D_{\Delta(\Gamma)} = D_\Gamma$ in this case. On the other hand, if we work on the definition of inverse arc introduced in 2.1.2, then any element of $A_{uv}$ is an inverse of $a_e$. Hence, the number of backtracks through $u$ equals the sum of $|A_{uv}||A_{vu}| = |A_{uv}|^2$ for $v \in V$ with $\{u, v\} \in E$, i.e., $d_u = \sum_{v \in V} |A_{uv}|^2$. If $\Gamma$ is a finite simple graph, then it follows from Lemma 1 that $d_u$ equals the degree of the vertex $u$ in $\Gamma$, and we have $D_{\Delta(\Gamma)} = D_\Gamma$.

2.2 Words

2.2.1 Definitions

Let $\mathcal{A} = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a finite alphabet, and $\mathcal{A}^*$ the free monoid generated by $\mathcal{A}$. An element of $\mathcal{A}^*$ is called a word on $\mathcal{A}$. Let $w = a_1a_2\ldots a_m$ be a word. The integer $m$ is called the length of $w$, denoted by $|w|$. The multiplication on $\mathcal{A}^*$ is defined by the concatenation of words. Given two words $w, w' \in \mathcal{A}^*$, the product of $w$ and $w'$ is denoted by $ww'$. The $k$-th power of a word $w$ is denoted by $w^k$. If a word $w$ can not be written by a power of a shorter word, then $w$ is called a prime word. Given a word $w = a_1a_2\ldots a_m \in \mathcal{A}^*$, the cyclic rearrangement class $\text{Re}(w)$ is the (multi-)set consisting of the following $m$ words

$$a_1a_2\cdots a_{m-1}a_m, a_2a_3\cdots a_ma_1, \ldots, a_ma_1\cdots a_{m-2}a_{m-1}.$$  

If $w$ is a prime word, then its cyclic rearrangement class is a set.

2.2.2 Lyndon words

Let $\mathcal{A} = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a finite alphabet, which is totally ordered by $\alpha_1 < \alpha_2 < \cdots < \alpha_n$. In this case, the free monoid $\mathcal{A}^*$ is also totally ordered by the lexicographical order induced by the total order $<$ on $\mathcal{A}$. We denote the total order on $\mathcal{A}^*$ by the same symbol $\prec$. If a word $w \in \mathcal{A}^*$ is the minimum element in its cyclic rearrangement class $\text{Re}(w)$, then $w$ is called a Lyndon word (see e.g., [18]). The set of Lyndon words on $\mathcal{A}$ is denoted by $\text{Lyn}(\mathcal{A})$. For example, if $\mathcal{A} = \{1 < 2 < 3\}$, then $w = 1212 \notin \text{Lyn}(\mathcal{A})$, since $w$ is not the minimum element in $\text{Re}(w) = \{1212, 2121, 1212, 2121\}$. One can readily see from this example that a Lyndon word is necessarily a prime word. If $w = 1213$, then $w$ is Lyndon. The well-known Lyndon factorization theorem (c.f., [18]) states that the Lyndon words $\text{Lyn}(\mathcal{A})$ gives the primes of the free monoid $\mathcal{A}^*$.
2.2.3 The Foata-Zeilberger theorem

Let $R$ be a commutative ring, and $\mathfrak{A} = \{\alpha_1 < \alpha_2 < \cdots < \alpha_n\}$ a totally ordered finite alphabet. Let $\text{Mat}_{\mathfrak{A}}(R)$ denote the set of $n \times n$ matrices $(m_{aa'})_{a,a' \in \mathfrak{A}}$ with $m_{aa'} \in R$ for each $a,a' \in \mathfrak{A}$. For $w = a_1a_2\cdots a_k \in \mathfrak{A}^*$ and $M = (m_{aa'})_{a,a' \in \mathfrak{A}} \in \text{Mat}_{\mathfrak{A}}(R)$, let $\text{circ}_M(w)$ denote the circular product

$$m_{a_1a_2}m_{a_2a_3}\cdots m_{a_{k-1}a_k}m_{a_ka_1}$$

of entries in $M$ along $w$. Let $I$ denote the identity matrix of degree $n$ and $t$ an indeterminate. The following proposition is called the Foata-Zeilberger theorem [8].

**Proposition 2 (Foata-Zeilberger)** \( \det(I - M) = \prod_{l \in \text{Lyn}(\mathfrak{A})} (1 - \text{circ}_M(l)). \)

It follows from the Foata-Zeilberger theorem that the inverse of the reciprocal characteristic polynomial $1/\det(I - tM)$ is written by

$$\prod_{l \in \text{Lyn}(\mathfrak{A})} \frac{1}{1 - \text{circ}_M(l)t^{\mid l\mid}}.$$

This identity can be viewed as the Euler product expression for $1/\det(I - tM)$ since the set $\text{Lyn}(\mathfrak{A})$ gives the primes in $\mathfrak{A}^*$.

2.3 Dynamical systems

2.3.1 Prime period

A dynamical system is a pair $(\Xi, \lambda)$ of a set $\Xi$ and a bijection $\lambda : \Xi \to \Xi$. For an positive integer $m$, an element $x \in \Xi$ is called an $m$-periodic point of $(\Xi, \lambda)$ if the condition $\lambda^m(x) = x$ holds. The set of $m$-periodic points in $(\Xi, \lambda)$ is denoted by $X_m = X_m(\Xi)$, and the set of all periodic points in $(\Xi, \lambda)$ by $X = X(\Xi)$. We have $X = \cup_{m \geq 1} X_m$. If $x \in X_m$, then the positive integer $m$ is called a period of $x$. Remark that any multiple of a period of $x$ is also a period of $x$. Let $\text{Per}(x)$ denote the set $\{m \mid x \in X_m\}$ of periods of $x \in \Xi$, and we denote $\varpi(x) = \min \text{Per}(x)$, which is called the prime period of $x$. It obviously follows that $x \in X_{\varpi(x)}$ for any $x \in X$. If we regard $x$ as an element of $X_{\varpi(x)}$, then we denote it by $\pi(x)$. We call $\pi(x)$ the prime section of $x$.

If we denote by $Y_m = Y_m(\Xi)$ the set of periodic points with prime period $m$, i.e.,

$$Y_m = \{x \in X \mid \varpi(x) = m\},$$

then we have a disjoint union $X = \cup_{m \geq 1} Y_m$. A standard argument shows that $\varpi(x)$ divides any period of $x \in X$. See, e.g., Lemma 8 in [23] for a proof.

**Lemma 3** Let $x \in X$. We have $\varpi(x)\mid m$ for any $m \in \text{Per}(x)$. 

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2.3.2 Equivalence

Two elements \( x, y \in \Xi \) is called equivalent in \((\Xi, \lambda)\) if there exists an integer \( k \) satisfying \( y = \lambda^k(x) \), where \( \lambda^{-1} \) denotes the inverse map of \( \lambda \). If \( x \) and \( y \) are equivalent, we denote it by \( x \equiv y \). The set \( \Xi/\equiv \) of equivalence classes is denoted by \([\Xi]\), and an equivalence class \( \xi \in [\Xi] \) with representative \( x \in \Xi \) is denoted by \([x]\). Given \( x \in X \), the equivalence class \([x]\) is called the orbit through \( x \). The following lemma is verified in a straightforward way. See e.g., Lemma 4 in [23].

Lemma 4 If \( x \equiv y \) in \((\Xi, \lambda)\), then \( \text{Per}(x) = \text{Per}(y) \). Hence we have \( \varpi(x) = \varpi(y) \).

Lemma [4] shows that the equivalence relation \( \equiv \) is an equivalence relation on \( X \), and also on each \( X_m \). The equivalence classes \( X/\equiv \) (resp. \( X_m/\equiv \)) is denoted by \([X]\) (resp. \([X]_m\)). We have \([X] = \cup_{m \geq 1}[X]_m\). Note that, by Lemma [4], the binary relation \( \equiv \) is also an equivalence relation on \( Y_m \) for each \( m \). We denote \( Y_m/\equiv \) by \([Y_m]\). It also suggests that we can define the prime period \( \varpi([x]) \) of an orbit \([x] \in [X]\) by \( \varpi([x]) = \varpi(x) \).

2.3.3 Ruelle zeta functions

Let \((\Xi, \lambda)\) be a dynamical system, and \( X = \cup_{m \geq 1}X_m \) the set of periodic points in \((\Xi, \lambda)\), where \( X_m \) denotes the set of \( m \)-periodic points as in [2.3.1]. If each \( X_m \) is a finite set, then the dynamical system \((\Xi, \lambda)\) is called quasi-finite. Let \( R \) be a commutative \( \mathbb{Q} \)-algebra, and let \( \chi_m : X_m \to R \) be a map. The symbol \( \chi \) denotes the multi-valuated map \( X \to R \) which sends \( x \in X_m \) to \( \chi_m(x) \). The triple \((\Xi, \lambda, \chi)\) is called a weighted dynamical system with a weight \( \chi \) (c.f., [11]). If a weighted dynamical system \((\Xi, \lambda, \chi)\) is quasi-finite, then the following sum \( N_m(\chi) = \sum_{x \in X_m} \chi(x) \) is well-defined for each \( m \geq 1 \).

Definition 5 (c.f., [25]) Let \( t \) be an indeterminate, and \((\Xi, \lambda, \chi)\) a quasi-finite weighted dynamical system. The Ruelle zeta function for \((\Xi, \lambda, \chi)\) is the formal power series

\[
\exp \left( \sum_{m \geq 1} \frac{N_m(\chi)}{m} t^m \right),
\]

which we denote by \( Z_\Xi(t; \chi) \). The Ruelle zeta function \( Z_\Xi(t; \chi) \) is also simply called the \( \chi \) function for \((\Xi, \lambda, \chi)\).

3 Zeta functions for digraphs

In this section, we briefly review the theory of combinatorial zeta functions, which is the fundamental framework for considering the Hashimoto expression. See [23] for precise information on the development in this section.
3.1 Dynamical systems on digraphs

Let $\Delta = (V, A)$ be a finite digraph. We denote by $A^\mathbb{Z}$ the set $\{(a_i)_{i \in \mathbb{Z}} \mid a_i \in A, \forall i \in \mathbb{Z}\}$ of two-sided infinite sequences with entries in $A$. Consider the left shift operator

$$\varphi : A^\mathbb{Z} \to A^\mathbb{Z} : (a_i) \mapsto (a_{i+1})$$

on $A^\mathbb{Z}$. Let $x = (x_i)_{i \in \mathbb{Z}} \in A^\mathbb{Z}$ and let $m$ be a positive integer. Any finite consecutive segment $(x_i, x_{i+1}, \ldots, x_{i+m-1})$ of $x$ is called an $m$-section. In particular, the $m$-section $(x_0, x_1, \ldots, x_{m-1})$ is called the principal $m$-section of $x$, denoted by $ps_m(x)$. A subset $\Xi \subset A^\mathbb{Z}$ is called $\varphi$-stable if it satisfies the condition $\varphi(\Xi) \subset \Xi$.

Suppose that $\Xi \subset A^\mathbb{Z}$ is $\varphi$-stable. We denote the restriction $\varphi|_\Xi$ by $\lambda$, and consider the dynamical system $(\Xi, \lambda)$. Recall that $S(a)$ denotes $A_{vu}$ for $a \in A_{uv}$. The examples that we mainly bare in mind are the set $\Pi_\Delta$ of two-sided infinite paths of $\Delta$ and the set $\Pi^\Delta_\Delta$ of two-sided infinite reduced paths of $\Delta$, i.e.,

$$\Pi_\Delta = \{(a_i)_{i \in \mathbb{Z}} \in A^\mathbb{Z} \mid h(a_i) = t(a_{i+1}), \forall i\},$$

$$\Pi^\Delta_\Delta = \{(a_i)_{i \in \mathbb{Z}} \in A^\mathbb{Z} \mid h(a_i) = t(a_{i+1}), a_{i+1} \notin S(a_i), \forall i\}.$$

Consider the case where $\Xi = \Pi_\Delta$. Note that the dynamical system $(\Xi, \lambda)$ is quasi-finite since the cardinality $|X_m|$ of $m$-periodic points does not exceed $|A|^m$. Note also that an $m$-section of $x \in X_m$ corresponds to an element of $C_m = C_m(\Delta)$. Since $x \in X_m$ is completely determined by an $m$-section, in particular by the principal $m$-section $ps_m(x)$, the map

$$ps_m : X_m \to C_m : x \mapsto ps_m(x)$$

is bijective. For $c = (a_0, a_1, \ldots, a_{m-1}) \in C_m$, we denote by $c^\natural$ the element $x = (x_i)_{i \in \mathbb{Z}} \in X_m$ defined by $x_i = a_k$ for $i \in \mathbb{Z}$ with the indices $i$ congruent with $k$ modulo $m$. It is clear that the map

$$\sharp : C_m \to X_m : c \mapsto c^\natural$$

gives the inverse for the bijection $ps_m$. Suppose that $x \in \Xi$ belongs to $Y_m$, the set of elements with prime period $m$. Since $m = \min \text{Per}(x)$, the image $ps_m(x)$ belongs to $P_m = P_m(\Delta)$. Thus the restriction $ps_m|_{Y_m}$ gives a bijective correspondence $Y_m \to P_m$. In the case where $\Xi = \Pi^\Delta_\Delta$, we denote the set of $m$-periodic points by $X^\natural_m = X^\natural_m(\Delta)$. The set of points in $X^\natural_m$ with prime period $m$ is denoted by $Y^\natural_m = Y^\natural_m(\Delta)$. In the same manner, one can see that $X^\natural_m$ (resp. $Y^\natural_m$) is mapped bijectively onto $C^\natural_m$ (resp. $P^\natural_m$) by the map $ps_m$.

Let $\Xi \subset A^\mathbb{Z}$ be $\varphi$-stable, and consider the dynamical system $(\Xi, \lambda)$ where $\lambda = \varphi|_\Xi$. Let $\equiv$ denote the equivalence relation on $(\Xi, \lambda)$ introduced in [2,3]. Note that, if $x \equiv x'$ for $x, x' \in \Xi_m$, then it follows that $ps_m(x) \sim ps_m(x')$. This shows that the map $ps_m : X_m \to C_m$ induces a bijection

$$[ps_m] : [X_m] \to [C_m] : [x] \mapsto [ps_m(x)]$$

for each $m \geq 1$. It also follows that the restrictions $[ps_m]|_{[Y_m]} : [Y_m] \to [P_m]$, $[ps_m]|_{[X^\natural_m]} : [X^\natural_m] \to [C^\natural_m]$, $[ps_m]|_{[Y^\natural_m]} : [Y^\natural_m] \to [P^\natural_m]$ are bijective.
Let \([X] := X/\equiv\) and \([X^0] := X^0/\equiv\). Since \(\equiv\) is an equivalence relation on each \(X_m\) and \(X_m^0\), one has \([X] = \cup_m [X_m]\) and \([X^0] = \cup_m [X_m^0]\). Let \([Y] := Y/\equiv\) and \([Y^0] := Y^0/\equiv\). Since \(\equiv\) is an equivalence relation on each \(Y_m\) and \(Y_m^0\), one has \([Y] = \cup_m [Y_m]\) and \([Y^0] = \cup_m [Y_m^0]\). It also follows that \([Y] = \pi([X])\) and \([Y^0] = \pi([X^0])\).

### 3.2 The path condition

Let \(\Delta = (V, \mathcal{A})\) be a finite digraph, \(S(a) = \mathcal{A}_{vu}\) for \(a \in \mathcal{A}_{uv}\), \(\Xi\) a \(\varphi\)-stable subset of \(\mathcal{A}^2\), and \(\theta : \mathcal{A} \times \mathcal{A} \to R\) a map. Recall that \(\hat{\theta}\) is the element \((x_i)_{i \in \mathbb{Z}}\) of \(\mathcal{A}_{\mathbb{Z}}^\mathbb{Z}\) given by \(x_i = a_j\) if \(i\) is congruent to \(j\) modulo \(k\). We give a total order \(\mathcal{A} = \{\alpha_1 < \alpha_2 < \cdots < \alpha_n\}\) on \(\mathcal{A}\), and consider the set \(\text{Lyn}(\mathcal{A})\) of Lyndon words on the alphabet \(\mathcal{A}\). For a word \(w = a_0a_1 \cdots a_k\in \mathcal{A}^*\), we denote the circular product

\[\theta(a_0, a_1)\theta(a_1, a_2) \cdots \theta(a_{k-2}, a_{k-1})\theta(a_{k-1}, a_1)\]

by \(\text{circ}_\theta(w)\). The following “path condition” was introduced in 3.2.1 in [23].

**Definition 6 (The path condition)** We say that a \(\varphi\)-stable subset \(\Xi\) of \(\mathcal{A}^*\) satisfies the path condition with respect to \(\theta\) iff one has \(\text{circ}_\theta(l) \neq 0 \Rightarrow l^\Xi \in \Xi\) for each \(l \in \text{Lyn}(\mathcal{A})\).

In this case, we also say that \((\Xi, \theta)\) satisfies the path condition. We will see some examples. Let \(\theta^{BL}\) and \(\theta^{BL,\circ}\) be maps \(\mathcal{A} \times \mathcal{A} \to R\) given by \(\theta^{BL}(a, a') = \delta_{\theta(a)(a')}\), and \(\theta^{BL,\circ}(a, a') = \delta_{\theta(a)(a')} - \delta_{a' \in S(a)}\). One can see that \((\Pi_\Delta, \theta^{BL})\) satisfies the path condition, since, for \(l = a_0 \cdots a_{m-1} \in \text{Lyn}(\mathcal{A})\), \(\text{circ}_{\theta^{BL}}(l) \neq 0\) implies \(\hat{h}(a_i) = t(a_{i+1})\) for any \(i = 0, \ldots, m-1\) where \(a_m := a_0\), and hence it follows that \(l^\Xi \in \Pi_\Delta\). It can also be shown that \((\Pi_\Delta, \theta^{BL,\circ})\) satisfies the path condition. On the other hand, one can see that \((\Pi_\Delta, \theta^{BL})\) does not satisfy the path condition. In particular, if we consider the case where \(\Xi = \Pi_\Delta\) or \(\Pi_\Delta^0\) as in these examples, then it can be seen that the path condition is implied by the following simple condition. See 4.1 in [23] for precise information.

**Definition 7** A map \(\theta : \mathcal{A} \times \mathcal{A} \to R\) is said to satisfy the adjacency condition iff \(\theta(a, a') \neq 0\) implies \(\hat{h}(a) = t(a')\) for \(a, a' \in \mathcal{A}\). A map \(\theta : \mathcal{A} \times \mathcal{A} \to R\) is said to satisfy the reduced adjacency condition iff \(\theta(a, a') \neq 0\) implies \(\hat{h}(a) = t(a')\) and \(a' \notin S(a)\) for \(a, a' \in \mathcal{A}\).

**Lemma 8** If a map \(\theta : \mathcal{A} \times \mathcal{A} \to R\) satisfies the adjacency condition, then \((\Pi_\Delta, \theta)\) satisfies the path condition. If \(\theta\) satisfies the reduced adjacency condition, then \((\Pi_\Delta^0, \theta)\) satisfies the path condition.

### 3.3 Zeta functions for finite digraphs

Let \(\Delta = (V, \mathcal{A})\) be a finite digraph with \(n\) arcs, \(S(a) = \mathcal{A}_{vu}\) for \(a \in \mathcal{A}_{uv}\), \(R\) a commutative \(\mathbb{Q}\)-algebra, and \(\theta : \mathcal{A} \times \mathcal{A} \to R\) a map. Let \(\varphi\) be the left shift operator on \(\mathcal{A}_{\mathbb{Z}}\), \(\Xi\) a \(\varphi\)-stable subset of \(\mathcal{A}_{\mathbb{Z}}\), and \(\lambda\) the restriction \(\varphi|_\Xi\).
3.3.1 Definition and the exponential expression

Let \( \Delta = (V, A) \) be a finite digraph, and \( \Xi \) a \( \varphi \)-stable subsets of \( A^\mathbb{Z} \). Given an \( m \)-periodic point \( x = (a_i)_i \in X_m = X_m(\Xi) \), \( \chi(x) \) stands for the following product
\[
\theta(a_1, a_2) \cdots \theta(a_{m-1}, a_m) \theta(a_m, a_1),
\]
which is called the circular product of \( \theta \) along \( x \in X_m \), and we denote it by \( \text{circ}_\theta(x) \). Note that \( \chi(x) \) depends on the choice of a period of \( x \), hence \( \text{circ}_\theta \) is not a map in general. This multi-valuated map \( \text{circ}_\theta : X \to R \) is called the circular weight induced by \( \theta \). Let \( \chi = \text{circ}_\theta \), \( N_m(\chi) = \sum_{x \in X_m} \chi(x) \), and \( t \) an indeterminate.

**Definition 9 (Graph zeta functions)** A graph zeta function is the Ruelle zeta function

\[
\exp \left( \sum_{m \geq 1} \frac{N_m(\chi)}{m} t^m \right)
\]
with the circular weight \( \chi = \text{circ}_\theta \), which we denoted by \( Z_\Xi(t; \theta) \).

The fundamental literature on graph zetas is Terras [31]. This defining identity (3) is called the exponential expression. In particular, if \( \Xi = \Pi_\Delta, \Pi_\Delta^* \), then we denote \( Z_\Xi(t; \theta) \) by \( Z_\Delta(t; \theta), Z_\Delta^*(t; \theta) \) respectively. In particular, if \( \Delta \) is the symmetric digraph of a finite graph \( \Gamma \), then \( Z_\Delta(t; \theta), Z_\Delta^*(t; \theta) \) are denoted by \( Z_\Gamma(t; \theta), Z_\Gamma^*(t; \theta) \) respectively. In the following of this paper, \( N_m(\chi) \) is denoted by \( N_m(\theta) \) if \( \chi = \text{circ}_\theta \).

3.3.2 The Euler expression

It is shown in [23] that the exponential expression \( Z_\Xi(t; \theta) \) of a graph zeta can always be reformulated into the “Euler expression”. Recall that \( \varpi(x) \) is the prime period and \( \pi(x) \) the prime section for a periodic element \( x \in X \) (see 2.3.1). We consider the following formal power series

\[
\prod_{[x] \in [X]} \frac{1}{1 - \text{circ}_\theta(\pi(x)) t^{\varpi(x)}},
\]
which is denoted by \( E_\Xi(t; \theta) \). The following proposition is verified in [23].

**Proposition 10 ([23], Theorem 18)** \( Z_\Xi(t; \theta) = E_\Xi(t; \theta) \).

This identity is called the Euler expression of the graph zeta function \( Z_\Xi(t; \theta) \). Note that, as one can see in [23], the Euler expression does not need the path condition for \( (\Xi, \theta) \).

The zeta function \( Z_\Xi(t; \theta) \) is an example of “zeta function associated with a family of finite set”introduced in [23]. By the notation in [23], a graph zeta \( Z_\Xi(t; \theta) \) is denoted by \( Z_\mathcal{F}(t; \chi) \), where \( \mathcal{F} = \{X_m = X_m(\Xi) | m \geq 1\} \), \( \chi = \text{circ}_\theta \). Since \( (\mathcal{F}, \chi, \Xi) \) satisfies the Euler condition (see 3.1.2, [23]), the zeta \( Z_\mathcal{F}(t; \chi) \) has the Euler expression \( E_\mathcal{F}(t; \chi) \) (see 3.1.2 of [23] for \( E_\mathcal{F}(t; \chi) \)), and one can confirm that this coincides with \( E_\Xi(t; \theta) \) above.

In the case where \( \Xi = \Pi_\Delta \) (resp. \( \Pi_\Delta^* \), we denote \( E_\Xi(t; \chi) \) by \( E_\Delta(t; \theta) \) (resp. \( E_\Delta^*(t; \theta) \)) for \( \chi = \text{circ}_\theta \). For the symmetric digraph \( \Delta = \Delta(\Gamma) \) for a finite graph \( \Gamma \), \( E_\Delta(t; \theta) \) (resp. \( E^*_\Delta(t; \theta) \)) is denoted by \( E_\Gamma(t; \theta) \) (resp. \( E^*_\Gamma(t; \theta) \)). See 3.2.2 of [23] for further information.
3.3.3 The Hashimoto expression

Suppose that $(\Xi, \theta)$ satisfies the path condition. Fixing a total order on $\mathcal{A}$, we consider the matrix $M = (\theta(a, a'))_{a, a' \in \mathcal{A}}$ determined by the map $\theta$. Since $(\Xi, \theta)$ satisfies the path condition, it follows that $\text{circ}_{M}(l) \neq 0$ implies $l^\circ \in \Xi$ for $l \in \text{Lyn}(\mathcal{A})$ (c.f., 2.2.3 for $\text{circ}_{M}(l)$).

This suggests that any closed cycle $[c] = [(a_1, \ldots, a_m)]$ of $\Delta$ satisfying $\text{circ}_{\theta}(c) \neq 0$ indeed appears in $\Xi$. We denote the matrix $M$ by $M_{\Xi}(\theta)$, and call it the edge matrix for $(\Xi, \theta)$. Now we denote the following formal power series

$$
\frac{1}{\det(I - tM_{\Xi}(\theta))},
$$

by $H_{\Xi}(t; \theta)$, where $I$ stands for the identity matrix of degree $n = |\mathcal{A}|$.

**Proposition 11 ([23], Theorem 19)** If $(\Xi, \theta)$ satisfies the path condition, then we have $Z_{\Xi}(t; \theta) = H_{\Xi}(t; \theta)$.

This identity in Proposition 11 is called the Hashimoto expression of a graph zeta function $Z_{\Xi}(t; \theta)$. Contrary to the Euler expression, the Hashimoto expression does need the path condition for $(\Xi, \theta)$ (c.f., [23]). In the case where $\Xi = \Pi_{\Delta}$ (resp. $\Pi'_{\Delta}$), $H_{\Xi}(t; \theta)$ is denoted by $H_{\Delta}(t; \theta)$ (resp. $H'_{\Delta}(t; \theta)$). In particular, if $\Delta = \Delta(\Gamma)$ for a finite graph $\Gamma$, then $H_{\Delta}(t; \theta)$ (resp. $H'_{\Delta}(t; \theta)$) is denoted by $H_{\Gamma}(t; \theta)$ (resp. $H'_{\Gamma}(t; \theta)$). See 3.2.3 in [23] for precise information.

4 The generalized weighted zeta function

Let $\Delta = (V, \mathcal{A})$ be a finite digraph, $S(a) = \mathcal{A}_{uv}$ for $a \in \mathcal{A}_{uv}$, and $R$ a commutative $\mathbb{Q}$-algebra with unity 1. We consider two functions $\tau, \upsilon$ from $\mathcal{A}$ to $R$, and let $\theta^{G} : \mathcal{A} \times \mathcal{A} \to R$ be the map defined by

$$
\theta^{G}(a, a') = \tau(a')\delta_{h(a)\iota(a')} - \upsilon(a')\delta_{a' \in S(a)}.
$$

(5)

4.1 Definition and the three expressions

**Definition 12 (The generalized weighted zeta)** Let $\Xi$ be a $\varphi$-stable subset of $\mathcal{A}^{\mathbb{Z}}$. The formal power series $Z_{\Xi}(t; \theta^{G})$ is called the generalized weighted zeta function.

The generalized weighted zeta was introduced in 4.4 of [23]. It follows from Proposition 10 that $Z_{\Xi}(t; \theta^{G}) = E_{\Xi}(t; \theta^{G})$, and the path condition for $(\Xi, \theta^{G})$ implies $E_{\Xi}(t; \theta^{G}) = H_{\Xi}(t; \theta^{G})$ by Proposition 11. Hence we have the three expressions for the generalized weighted zeta function $Z_{\Xi}(t; \theta^{G})$.

**Proposition 13** If $(\Xi, \theta^{G})$ satisfies the path condition, then we have $Z_{\Xi}(t; \theta^{G}) = E_{\Xi}(t; \theta^{G}) = H_{\Xi}(t; \theta^{G})$. 

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In particular for \( \Xi = \Pi_\Delta, \Pi^0_\Delta \), the generalized weighted zeta \( Z_\Xi(t; \theta^G) \) is denoted by \( Z_\Delta(t; \theta^G), Z^*_\Delta(t; \theta^G) \), respectively. The pairs \((\Pi_\Delta, \theta^G)\) and \((\Pi^0_\Delta, \theta^G)\) may, indeed, satisfy the path condition, that is, we have the following lemma, which is verified in 4.4 of [23].

**Lemma 14** The map \( \theta^G \) satisfies the adjacency condition. If \( \tau = \nu \), then \( \theta^G \) satisfies the reduced adjacency condition.

**Corollary 15** For a finite digraph \( \Delta \), we have \( Z_\Delta(t; \theta^G) = E_\Delta(t; \theta^G) = H_\Delta(t; \theta^G) \). In the case where \( \tau = \nu \), we also have \( Z^*_\Delta(t; \theta^G) = E^*_\Delta(t; \theta^G) = H^*_\Delta(t; \theta^G) \).

### 4.2 Unification of graph zeta functions

The notation \( \Delta = (V, A), R, \tau, \nu \) and \( \theta^G \) are inherited. Let \( \Xi \) be a \( \varphi \)-stable subset of \( A^2 \), where \( \varphi \) is the left shift operator on \( A^2 \). The typical examples of \( \Xi \) are \( \Pi_\Delta \) and \( \Pi^0_\Delta \). Varying \( \tau \) and \( \nu \), the generalized weighted zeta \( Z_\Xi(t; \theta^G) \) degenerates to various graph zetas which have been studied in previous research [3, 5, 13, 22, 26, 29] etc.

#### 4.2.1 The Ihara zeta function

In the case where \( \tau = \nu = 1 \), the generalized weighted zeta \( Z^*_\Delta(t; \theta^G) \) is called the **Ihara zeta function** of \( \Delta \). In this case, we denote the map \( \theta^G \) by \( \theta^I \). By Lemma 14, the map \( \theta^I \) satisfies the reduced adjacency condition, and hence the adjacency condition. Thus both \((\Pi^I_\Delta, \theta^I)\) and \((\Pi_\Delta, \theta^I)\) satisfy the path condition by Lemma 8. Hence we have the identities \( Z^*_\Delta(t; \theta^I) = E^*_\Delta(t; \theta^I) = H^*_\Delta(t; \theta^I) \) and \( Z_\Delta(t; \theta^I) = E_\Delta(t; \theta^I) = H_\Delta(t; \theta^I) \). Since the circular weight \( \text{circ}_\varphi \) instinctively excludes the non-reduced paths, that is, \( \text{circ}_\varphi(x) \neq 0 \) impies \( x \in \Pi^I_\Delta \), it follows that \( N_m(\theta^I) = |X^I_m| \) for both \( \Pi_\Delta \) and \( \Pi^0_\Delta \), and we have \( Z_\Delta(t; \theta^I) = Z^*_\Delta(t; \theta^I) \). Thus these six formal power series all equals.

The Ihara zeta was originally defined for the case where \( \Delta = \Delta(\Gamma) \) is the symmetric digraph of a finite graph \( \Gamma = (V, E) \) [4, 13] (see also [10, 28, 30]). The **Bass-Ihara theorem** [4, 13] states that the reciprocal of the Ihara zeta for \( \Delta = \Delta(\Gamma) \) equals the following polynomial

\[
(1 - t^2)^{|E|-|V|} \det(I - tA_\Gamma + t^2(D_\Gamma - I)).
\]

This expression is called the **Ihara expression** for \( Z_\Gamma(t; \theta^I) \), denoted by \( I_\Gamma(t; \theta^I) \). Thus the Ihara zeta has four expressions \( Z_\Gamma(t; \theta^I), E_\Gamma(t; \theta^I), H_\Gamma(t; \theta^I), I_\Gamma(t; \theta^I) \), and this is the case for the other graph zetas which have been appeared in the previous studies, for instance [3, 22, 26], as we will see in the sequel (except the Bowen-Lanford zeta function in 4.2.2). However, our fundamental point of view in the present article is that the graph zeta functions should be defined for digraphs. The reason is that the Ihara expression, one of the main interest of the preceding studies including the present article, can be constructed for any finite digraph as we will see in our main theorem. Therefore, although the original definitions of those traditional graph zetas were for finite graphs, the definitions of them in this paper are described for finite digraphs.
4.2.2 The Bowen-Lanford zeta function

In the case where $\nu = 0$, the generalized weighted zeta $Z_\Delta(t; \theta^G)$ is called the weighted Bowen-Lanford zeta function \[5\] for $\Delta$ (see also \[23\]). We denote the map $\theta^G$ by $\theta^{BL}_\tau$. Note that the original definition in \[5\] is the case where $\tau = 1$. This is an example of the Artin-Mazur zeta function \[2\] corresponding to the dynamical system naturally defined on a finite simple graph. We call this original one the Bowen-Lanford zeta function. In the case where $\tau = 1$, the map $\theta^G$ is denoted by $\theta^{BL}$. We can see that the map $\theta^{BL}_\tau$ satisfies the adjacency condition. Thus $(\Pi_\Delta, \theta^{BL}_\tau)$ satisfy the path condition by Lemma \[8\]. Hence we have the identities $Z_\Delta(t; \theta^{BL}_\tau) = E_\Delta(t; \theta^{BL}_\tau) = H_\Delta(t; \theta^{BL}_\tau)$. Note also that the pair $(\Pi_\Delta, \theta^{BL}_\tau)$ does not satisfy the path condition. Hence, although the identity $Z^b_\Delta(t; \theta^{BL}_\tau) = E^b_\Delta(t; \theta^{BL}_\tau)$ holds, the formal power series $H^b_\Delta(t; \theta^{BL}_\tau)$ is not equal to these. See \[23\] for precise information.

(Remark that, in \[23\], the weighed Bowen-Lanford zeta function is called the Mizuno-Sato zeta function.)

It is also known that the Bowen-Lanford zeta $Z_\Delta(t; \theta^{BL})$ has the Ihara expression. Let $\Gamma$ be a finite simple graph, and $A_\Gamma$ the adjacency matrix of $\Gamma$. We denote the following formal power series

$$\frac{1}{\det(I - tA_\Gamma)}.$$  

by $I_\Gamma(t; \theta^{BL})$. In \[5\], R. Bowen and O. Lanford shows the identity $Z_\Gamma(t; \theta^{BL}) = I_\Gamma(t; \theta^{BL})$, which we call the Ihara expression of the Bowen-Lanford zeta $Z_\Gamma(t; \theta^{BL})$. See also \[20\].

4.2.3 The Mizuno-Sato zeta function

In the case where $\tau = \nu$, the generalized weighted zeta $Z_\Delta(t; \theta^G)$ is called the Mizuno-Sato zeta function of $\Delta$, which was originally introduced in \[21\]. In this case, we denote the map $\theta^G$ by $\theta^{MS}$. The map $\theta^{MS}$ satisfies the reduced adjacency condition by virtue of Lemma \[14\]. Thus both $(\Pi_\Delta, \theta^{MS})$ and $(\Pi'_\Delta, \theta^{MS})$ satisfy the path condition by Lemma \[8\]. Hence we have the identities $Z_\Delta(t; \theta^{MS}) = E_\Delta(t; \theta^{MS}) = H_\Delta(t; \theta^{MS})$, and $Z^b_\Delta(t; \theta^{MS}) = E^b_\Delta(t; \theta^{MS}) = H^b_\Delta(t; \theta^{MS})$. Since $\theta^{MS}$ excludes the non-reduced closed paths, we have $Z_\Delta(t; \theta^{MS}) = Z^b_\Delta(t; \theta^{MS})$, and those six formal power series are all equal to each other. Note that, for both cases $\Pi_\Delta$ and $\Pi'_\Delta$, we have $N_m(\theta^{MS}) = \sum_{x \in X^m} \text{circ}_{\theta^{MS}}(x)$. See 4.3 in \[23\].

The Ihara expression of the Mizuno-Sato zeta is given as follows \[22\]. Let $\Gamma = (V, E)$ be a finite simple graph. Note that, since $\Gamma$ is simple, there exists at most one edge $e$ lying between two distinct vertices $u, v \in V$. We denote this possible single edge $e$ by $\{u, v\}$. Let $\Delta(\Gamma) = (V, A)$ be the symmetric digraph of $\Delta$. The arc set $A$ consists of arcs $a_e, \bar{a}_e$ for $e \in E$. Given a total order $\prec$ on $V$, if $e = \{u, v\}$ with $u \prec v$, then we say $a_e \in A_{uv}$ and denote it by $a_{uv}$, $\bar{a}_e$ by $a_{vu}$. Consider the matrix $W_\Gamma = (\tau(a_{uv}))_{u,v \in V}$, called the weighted matrix of $\Gamma$ \[22\]. Here we understand that $\tau(a_{uv}) = \tau(a_{vu}) = 0$ if $\{u, v\} \notin E$. Let $I_\Gamma(t; \theta^{MS})$ denote the formal power series given by the reciprocal of the following polynomial

$$\det(I - tW_\Gamma + t^2(D_\Gamma - I)).$$
In [22], H. Mizuno and I. Sato shows the identity $Z_{\Gamma}(t; \theta^{MS}) = I_{\Gamma}(t; \theta^{MS})$, the Ihara expression of $Z_{\Gamma}(t; \theta^{MS})$.

### 4.2.4 The Sato zeta function

In the case where $v = 1$, the generalized weighted zeta $Z_{\Delta}(t; \theta^{G})$ is called the Sato zeta function [26] (see also [23], 4.4). In this case, we denote the map $\theta^{G}$ by $\theta^{S}$. The map $\theta^{S}$ satisfies the adjacency condition. Thus $(\Pi_{\Delta}, \theta^{S})$ satisfy the path condition. Hence we have the identities $Z_{\Delta}(t; \theta^{S}) = E_{\Delta}(t; \theta^{S}) = H_{\Delta}(t; \theta^{S})$. Note that the map $\theta^{S}$ does not satisfy the reduced adjacency condition in general (c.f., Lemma 14). In this case, thought the identity $Z_{\Delta}(t; \theta^{S}) = E_{\Delta}(t; \theta^{S})$ holds, these are not equal to $H_{\Delta}(t; \theta^{S})$. Let $\Gamma = (V, E)$ be a finite simple graph, and $\Delta = (V, A)$ the symmetric digraph of $\Gamma$. Let $W_{\Gamma} = (w_{uv})_{u,v \in V}$ denote the weighted matrix of $\Gamma$ as in [4.2.3], and $D_{\Gamma}(w) = (d_{uv'})_{u,u' \in V}$ the diagonal matrix given by $d_{uv} = \delta_{uu'} \sum_{a \in A_{u,v}} w_{uv}$. If $I_{\Delta}(t; \theta^{S})$ denote the reciprocal of the polynomial

$$(1 - t^2)|E|-|V| \det(I - tW_{\Gamma} + t^2(D_{\Gamma}(w) - I)),$$

the Ihara expression of the Sato zeta $Z_{\Gamma}(t; \theta^{S})$ is given by the identity $Z_{\Gamma}(t; \theta^{S}) = I_{\Gamma}(t; \theta^{S})$ [26].

The Sato zeta function $Z_{\Delta}(t; \theta^{S})$ have a relation with quantum walks on finite graphs [16]. Let $\Gamma$ be a finite simple graph, and $\Delta = \Delta(\Gamma)$ the symmetric digraph of $\Gamma$. If we let $\tau(a') = 1/\deg t(a')$, the Sato zeta $Z_{\Gamma}(t; \theta^{S})$ gives the reciprocal characteristic polynomial of the time evolution matrix $U$ of the “Grover walk” on $\Gamma$ [9], a quantum walk model defined on a finite simple graph $\Gamma$ which is extensively studied in the area. Konno and Sato [16] shows that, motivated by Emms et.al. [7], the Ihara expression for the Sato zeta $Z_{\Delta}(t; \theta^{S})$ (see [26]) gives a fine description of the spectrum of $U$. By the Konno-Sato theorem [16], we may understand that the Hashimoto expression of a graph zeta for $\Gamma$ depicts the time evolution of a quantum walk model on $\Gamma$, and the Ihara expression describes the spectral mapping theorem for the model. For the spectral mapping theorem, consult [19, 27] (see also [14]). In particular, the result suggests that the edge matrix $M_{\Delta(t)}(\theta^{G})$ may provide the time evolution matrix of a quantum walk model on $\Gamma$, and may produce a family of quantum walks on $\Gamma$ which contains the Grover walk as an example. A. Ishikawa [14] considers this problem and obtains such a family of quantum walks on finite graphs.

### 4.2.5 The Bartholdi zeta function

Let $q$ be an indeterminate, and consider the polynomial algebra $R[q]$, a commutative $Q$-algebra with unity. Let $\theta^{B}$ be a map $A \times A \rightarrow R[q]$ given by

$$\theta^{B}(a, a') = \delta_{b(a)t(a')} - (1 - q)\delta_{a' \in S(a)}.$$

This is the case where $\tau(a) = 1$ and $\nu(a) = 1 - q$ ($a \in A$) in $\theta^{G}$. The generalized weighted zeta $Z_{\Delta}(t; \theta^{B})$ is called the Bartholdi zeta function [3] of $\Delta$ (see also [21]). The map $\theta^{B}$ satisfies the adjacency condition. Thus $(\Pi_{\Delta}, \theta^{B})$ satisfies the path condition. Hence we have the identities...
\(Z_\Delta(t; \theta^B) = E_\Delta(t; \theta^B) = H_\Delta(t; \theta^B)\). On \(\Pi_\Delta\), though the identity \(Z^\theta_\Delta(t; \theta^B) = E^\theta_\Delta(t; \theta^B)\) holds, these does not equals \(H^\theta_\Delta(t; \theta^B)\) since the map \(\theta^B\) does not satisfy the reduced adjacency condition.

The Euler expression \(E_\Delta(t; \theta^B)\) is described by a combinatorial statistics, called the “cyclic bump count”. Let \(x = (x_i)_{i \in \mathbb{Z}} \in X_m\) be an \(m\)-periodic point in \(\Pi_\Delta\), and let \(\text{cbc}(x)\) denote the cardinality of the set

\[
\{ i = k, k + 1, \ldots, k + m - 1 \mid x_{i+1} \in S(x_i) \},
\]

where \((x_k, x_{k+1}, \ldots, x_{k+m-1})\) is an \(m\)-section of \(x\). Note that \(\text{cbc}(x)\) does not depend on the choice of an \(m\)-section for \(x \in X_m\). If \(a' \in S(a)\), then we have \(\theta^B(a, a') = q\). This lead to the identity \(\text{circ}_{\theta^B}(x) = q^{\text{cbc}(x)}\) for each \(x \in X_m\), and it follows from Proposition \([10]\) that \(E_\Delta(t; \theta^B)\) equals

\[
\prod_{[x] \in \mathcal{X}} \frac{1}{1 - q^{\text{cbc}(\pi(x))t^\varpi(x)}}.
\]

By Proposition \([11]\) the Hashimoto expression is given by \(H_\Delta(t; \theta^B) = 1/\det(I - tM_\Delta(\theta^B))\), where \(M_\Delta(\theta^B) = (\theta^B(a, a'))_{a, a' \in \mathcal{A}}\). The Bartholdi zeta was originally considered for the symmetric digraph of a finite graph, and it is defined by the Euler expression. See \([3]\) for the original definition. Let \(\Gamma = (V, E)\) be a finite graph, and let \(L\) denote the subset of \(E\) consisting of loops in \(\Gamma\). The Ihara expression of the Bartholdi zeta is the identity \(Z_\Gamma(t; \theta^B) = I_\Gamma(t; \theta^B)\), where \(I_\Delta(t; \theta^B) \in R[[t]]\) is the reciprocal of the following polynomial

\[
(1 + (1 - q)t)^{|E|/(1 - (1 - q)^2)}t^{|E| - |L| - |V|} \det(I - tA_\Gamma + (1 - q)t^2(D_\Gamma - (1 - q)I)).
\]

See \([3]\) for precise information.

Since

\[
\theta^B|_{q=0} = \theta^I,
\]

the Bartholdi zeta \(Z_\Delta(t; \theta^B)\) equals the Ihara zeta \(Z_\Delta(t; \theta^I)\) at \(q = 0\). In this sense, we also call \(Z_\Delta(t; \theta^B)\) the Ihara zeta function of Bartholdi type, and denote it by \(Z_\Delta(q,t; \theta^I)\), the map \(\theta^B\) by \(\theta^I_q\). Hence we have \(Z_\Delta(q, t; \theta^I_q) = Z_\Delta(t; \theta^I_q)\). In the same sense, the Euler expression \(E_\Delta(t; \theta^I_q)\) and the Hashimoto expression \(H_\Delta(t; \theta^I_q)\) are denoted by \(E_\Delta(q, t; \theta^I)\) and \(H_\Delta(q, t; \theta^I)\), respectively. If we agree that \(0^0 = 1\), then we have

\[
q^{\text{cbc}(x)} = \begin{cases} 1, & \text{if } x \in X_m^\circ, \\ 0, & \text{if } x \in X_m \setminus X_m^\circ. \end{cases}
\]

This leads to the identity \(\sum_{x \in X_m^\circ} \text{circ}_{\theta^I_q}(x) = \sum_{x \in X_m^\circ} \text{circ}(x)\), i.e., \(N_m(\theta^I_q|_{q=0}) = N_m(\theta^I)\), which directly shows that \(Z_\Delta(0,t; \theta^I) = Z_\Delta(t; \theta^I)\). The identity \(E_\Delta(0,t; \theta^I) = E_\Delta(t; \theta^I)\) also follows directly from \([7]\). The identity \(H_\Delta(0,t; \theta^I) = H_\Delta(t; \theta^I)\) follows trivially from \([6]\). If we let \(q = 1\), then \(Z_\Delta(q, t; \theta^I)\) equals the Bowen-Lanford zeta \(Z_\Delta(t; \theta^{BL})\). Thus the Bartholdi zeta interpolates the Ihara zeta and the Bowen-Lanford zeta.

There exists other classical graph zetas which were not discussed here. For instance, it should be mentioned here about the edge zeta function \([29]\) and the path zeta function (see...
The formal power series

**Definition 16 (Bartholdi type)** A map given by

\[ \theta(E) = \text{expression} \]

Let \( V, A \) be an indeterminate and

\[ \langle a, \theta \rangle \in \mathcal{A}^\mathbb{Z} \]

We have the identities

\[ Z_{G}(t; \theta^{G}) = \text{Euler expression} \]

Thus we denote the map

\[ \theta^{G}(a, a') = \tau(a')\delta_{h(a)\mathbb{N}(a')} - (1 - q)\nu(a')\delta_{a\mathbb{N}(a)}. \]

Let \( \Xi = \{ a \in \mathcal{A} \mid a \in S(\{ a \}) \} \), then we have

\[ Z_{G}(t; \theta^{G}) = H_{G}(t; \theta^{G}), \]

Varying \( \tau \) and \( \nu \), we obtain various graph zetas of Bartholdi type. In the case where \( \tau = \nu = 0 \) for example, we have

\[ \theta^{G}_{q}\mid_{\| = 0} = \theta^{MS}. \]

Thus we denote the map

\[ \theta^{G}_{q}\mid_{\| = 0} = \theta^{MS} \]

This zeta appears in [8] for the case where \( \Delta \) is simple. We also denote the Euler expression \( E_{G}(t; \theta^{MS}) \) and the Hashimoto expression \( H_{G}(t; \theta^{MS}) \), respectively. Let \( x = (a_{i})_{i \in \mathbb{Z}} \in X_{m} \) be an \( m \)-periodic point. If \( a_{i+1} \in S(\{ a_{i} \}) \), then we have

\[ \theta^{G}(a_{i}, a_{i+1}) = \tau(a_{i+1})q; \text{ otherwise, } \theta^{G}(a_{i}, a_{i+1}) = \tau(a_{i+1}). \]

Hence we have

\[ \text{circ}_{\theta^{MS}}(x) = \text{circ}_{\theta^{G}}(x)q^{\text{cbc}(x)}, \]

4.3 Graph zeta functions of Bartholdi type

In [4.2.5] we introduce the Ihara zeta of Bartholdi type. In the same manner, we can define the Bartholdi type for various graph zetas. Let \( \Delta = (V, \mathcal{A}), S, R, \tau, \nu \) and \( \theta^{G} \) be as in [4.1]

Let \( q \) be an indeterminate and

\[ \theta^{G}_{q} : \mathcal{A} \times \mathcal{A} \to R[q] \]

The map

\[ \theta^{G}(a, a') = \tau(a')\delta_{h(a)\mathbb{N}(a')} - (1 - q)\nu(a')\delta_{a\mathbb{N}(a)}. \]

Therefore they have the three expression on the fundamental framework in our development. See 4.5 of [23] for further information.

**Lemma 17** The map \( \theta^{G}_{q} \) satisfies the adjacency condition.

**Proof.** Suppose that \( \theta^{G}_{q}(a, a') \neq 0 \). If \( a' \in S(\{ a \}) \), then we obviously have \( h(a) = t(a') \). If \( a' \notin S(\{ a \}) \), then we have \( \theta^{G}_{q}(a, a') = \tau(a')\delta_{h(a)\mathbb{N}(a')} \), and the assumption forces \( h(a) = t(a') \).

**Proposition 18** We have the identities

\[ Z_{G}(t; \theta^{G}) = E_{\Delta}(q, t; \theta^{G}) = H_{\Delta}(q, t; \theta^{G}). \]

**Proof.** By Proposition 10, it follows that \( Z_{G}(q, t; \theta^{G}) = E_{\Delta}(q, t; \theta^{G}) \). By Proposition 11, it follows that \( E_{\Delta}(q, t; \theta^{G}) = H_{\Delta}(q, t; \theta^{G}) \), since \( (\Pi_{\Delta}, \theta^{G}_{q}) \) satisfies the path condition by Lemma 17.

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and \( N_m(\theta^\text{MS}_q) = \sum_{x \in X_m} \text{circ}_{\theta^\text{BL}_q}(x) q^\text{cbc}(x) \), which equals \( N_m(\theta^\text{BL}_q) \) if \( q = 1 \). The agreement \( 0^0 = 1 \) (c.f., (7)) leads to the identity \( N_m(\theta^\text{MS}_q|_{q=0}) = N_m(\theta^\text{MS}) \). Thus the exponential expression \( Z_\Delta(q,t;\theta^\text{MS}) \) at \( q = 0 \) and \( q = 1 \) are actually identical with \( Z_\Delta(t;\theta^\text{MS}) \) and \( Z_\Delta(t;\theta^\text{BL}_q) \) respectively. It follows from Proposition 10 that the Euler expression \( E_\Delta(q,t;\theta^\text{MS}) \) is given by

\[
\prod_{[x] \in [X]} \frac{1}{1 - q^\text{cbc}(x)\text{circ}_{\theta^\text{BL}_q}(x)t^\text{inv}(x)}.
\]

By Proposition 11 the Hashimoto expression \( H_\Delta(q,t;\theta^\text{MS}) \) is given by \( 1/\det(I - tM_\Delta(\theta^\text{MS})) \), where \( M_\Delta(\theta^\text{MS}) = (\theta^\text{MS}_{a,a'})_{a,a' \in A} \).

On the other hand, the map \( \theta^G_q|_{\nu=1} \) defines another graph zeta of Bartholdi type. In the case of \( \nu = 1 \), we denote \( \theta^G_q \) by \( \theta^S_q \), and the graph zeta \( Z_\Delta(t;\theta^S_q) \) is called the Sato zeta function of Bartholdi type. We also denote \( Z_\Delta(t;\theta^S_q) \) by \( Z_\Delta(q,t;\theta^S) \) by Proposition 10 and Proposition 11 the exponential expression \( Z_\Delta(q,t;\theta^\text{MS}) \) can be reformulated into the Euler expression \( E_\Delta(q,t;\theta^S) \) and the Hashimoto expression \( H_\Delta(q,t;\theta^S) \) in the same manner.

To summarize, as is suggested by these examples, if we vary \( \tau \) and \( \nu \) in \( \theta^G_q \), then we obtain various graph zetas of Bartholdi type. In particular, if we consider the case \( \Xi = \Pi_\Delta \), it follows from Proposition 18 these zetas of Bartholdi type are always combinatorial, that is, they always have the Hashimoto expression.

5 The Ihara expression for the generalized weighted zeta function of Bartholdi type

In this section, the Ihara expression of the generalized weighted zeta of Bartholdi type is given. Since \( \theta^G_q = \theta^G \) at \( q = 0 \), the result for the case of the generalized weighted zeta is included in the main theorem. However, it is enough to show the main theorem for \( \theta = \theta^G \) since the polynomial ring \( R[q] \) is also a commutative \( \mathbb{Q} \)-algebra with unity. If we choose \( R = R[q] \) and replace \( \nu \) by \( (1 - q)\nu \) in the identity (3), then the map \( \theta^G_q \) is obtained. In the following of this section, let \( \Delta = (V,A) \) be a finite digraph, \( R \) a commutative \( \mathbb{Q} \)-algebra with unity, and \( \theta = \theta^G \). Note that the digraph \( \Delta \) allows multi-arcs and multi-loops. Since \( \theta^G \) satisfies the adjacency condition (Lemma 14), the generalized weighted zeta \( Z_\Delta(t;\theta^G) \) has the Hashimoto expression \( H_\Delta(t;\theta^G) \) (Corollary 15). Recall that \( H_\Delta(t;\theta^G) = 1/\det(I - tM_\Delta(\theta^G)) \), where \( M_\Delta(\theta^G) = (\theta^G_{a,a'})_{a,a' \in A} \).

5.1 Auxiliary results

5.1.1 The Schur complement

We recall some auxiliary facts on block matrices. Let

\[
M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}
\]

(9)
be a block matrix, where $A$ and $D$ are square matrices. Suppose that $A$ and $D$ are invertible. The results in 5.1.1 are well-known. However, brief proofs or comments are provided for readers convenience.

**Definition 19 (The Schur complement)** Let

$$M/A := D - CA^{-1}B,$$
$$M/D := A - BD^{-1}C.$$  

The matrix $M/A$ (resp. $M/D$) is called the Schur complement of $A$ (resp. $D$) in $M$.

**Lemma 20** The Schur complement $M/A$ is invertible if and only if $M/D$ is invertible. In this case, we have

$$\begin{align*}
(M/A)^{-1} &= D^{-1} + D^{-1}C(M/D)^{-1}BD^{-1}, \\
(M/D)^{-1} &= A^{-1} + A^{-1}B(M/A)^{-1}CA^{-1}.
\end{align*}$$  

(10) (11)

It is possible to see that $(M/A)(D^{-1} + D^{-1}C(M/D)^{-1}BD^{-1}) = I$ and vice versa. The identity (11) is also similarly verified. These identities (10), (11) are called the Woodbury identity. Carrying out the multiplication of block matrices, one can easily show the following lemma.

**Lemma 21** If $A$ and $D$ are invertible in the block matrix $M$, then we have the following two decompositions:

$$M = \begin{bmatrix} I & O \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & O \\ O & M/A \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ O & I \end{bmatrix}$$  

(12)

$$= \begin{bmatrix} I & BD^{-1} \\ O & I \end{bmatrix} \begin{bmatrix} M/D & O \\ O & D \end{bmatrix} \begin{bmatrix} I & O \\ D^{-1}C & I \end{bmatrix}.$$  

(13)

In particular, we have $\det M = \det A \det M/A = \det M/D \det D$.

If $A$ and $M/A$ are invertible, the inverse of $M$ is described as follows. In the case where $D$ and $M/D$ are invertible, one can obtain a similar formula. Recall that

$$\begin{bmatrix} I & B \\ O & I \end{bmatrix}^{-1} = \begin{bmatrix} I & -B \\ O & I \end{bmatrix}, \quad \begin{bmatrix} I & O \\ C & I \end{bmatrix}^{-1} = \begin{bmatrix} I & O \\ -C & I \end{bmatrix}.$$  

(14)

**Corollary 22** If $A$ and $M/A$ are invertible, then $M$ is invertible and the inverse of the block matrix $M$ is given by

$$M^{-1} = \begin{bmatrix} (M/D)^{-1} & -A^{-1}B(M/A)^{-1} \\ -(M/A)^{-1}CA^{-1} & (M/A)^{-1} \end{bmatrix}.$$
Proof. By Lemma 21 and the identities (14), we have

\[
M^{-1} = \begin{bmatrix} I & -A^{-1}B \\ O & I \end{bmatrix} \begin{bmatrix} A^{-1} & O \\ O & (M/A)^{-1} \end{bmatrix} \begin{bmatrix} I & O \\ -CA^{-1} & I \end{bmatrix}.
\]

The assertion follows from Lemma 20.

In the proof of the main theorem, we concentrate on a block matrix of the form

\[
M = \begin{bmatrix} I & B \\ C & I \end{bmatrix},
\]

(15)

where \(A\) and \(D\) are identity matrices in the block matrix (9). In this case, the Schur complements are given by

\[M/A = I - CB, \quad M/D = I - BC.\]

**Corollary 23** If \(M\) is a matrix of the form (15), then \(I - BC\) is invertible if and only if \(I - CB\) is invertible. In this case, the matrix \(M\) itself is also invertible, and the inverse \(M^{-1}\) is given by

\[
\begin{bmatrix} (I - BC)^{-1} & -B(I - CB)^{-1} \\ -(I - CB)^{-1}C & (I - CB)^{-1} \end{bmatrix}.
\]

(16)

We also have \(\det M = \det(I - BC) = \det(I - CB)\).

**Proof.** The first assertion follows from Lemma 20. The second assertion follows from Corollary 22. The third assertion follows from Lemma 21.

\[\square\]

### 5.1.2 Column constant matrices

If a matrix \(M = (m_{ij})\) satisfies the condition that \(m_{rj} = m_{sj}\) for all \(r, s, j\), then \(M\) is called a *column constant matrix*. For a column constant matrix \(M = (m_{ij})\), the sum \(\sum_{j \geq 1} m_{ij}\) does not depend on \(i\), which we call the *row sum* of \(M\). We denote the row sum of \(M\) by \(\rho_M\). It is clear that the row sum \(\rho_M\) of a column constant matrix \(M\) equals the trace \(\text{tr} M\) of \(M\).

**Lemma 24** If a square matrix \(M\) is column constant, then we have \(M^2 = \rho_M M\).

**Proof.** It follows from a direct calculation.

\[\square\]

**Lemma 25** Let \(M = (m_{ij})\), \(N = (n_{ij})\) be matrices with sizes \(k \times l\) and \(l \times k\) respectively. If \(M\) and \(N\) are column constant, then the products \(MN, NM\) are also column constant. If \(MN, NM\) are square matrices, then \(\text{tr}(MN)\) and \(\text{tr}(NM)\) equal the product \(\rho_{MN} = \rho_M \rho_N\) of the row sums of \(M\) and \(N\).
Proof. It suffices to show the assertion for $MN$. Since $N$ is column constant, the entries $n_{ij}$ belonging to the $j$-th column of $N$ does not depend on the indices $i$, and we denote it by $n_j$, i.e., $n_j = n_{ij}$ for all $i$. The $(i, j)$-entry of $MN$ is given by

$$
\sum_{k \geq 1} m_{ik}n_{kj} = \rho_Mn_j,
$$

(17)

which does not depend on $i$. Thus the product $MN$ is column constant matrix. The identity (17) also shows that $\text{tr} (MN) = \sum_{i=1}^{k} \rho_Mn_i$, which equals $\rho_M\rho_N$. \hfill \Box

If a matrix $M = (m_{ij}) \in M_n(R)$ is column constant, then one can easily see that $\det(I + M) = 1 + (m_{11} + m_{22} + \cdots + m_{nn})$.

**Lemma 26** We have $\det(I + M) = 1 + \rho_M$ for any column constant square matrix $M$.

Let $t$ be an indeterminate, and $R[[t]]$ the ring of formal power series with coefficients in $R$. For a column constant square matrix $M$, by Lemma 26, we have $\det(I + tM) = 1 + t\rho_M$, which is an invertible element in $R[[t]]$.

**Lemma 27** Let $M \in M_n(R)$. If $M$ is column constant, then $I + tM$ is invertible as an element of $M_n(R[[t]])$, and the inverse matrix $(I + tM)^{-1}$ is given by

$$
\det(I + tM)^{-1} \{ I - t(M - \rho_MI) \}.
$$

**Proof.** If we multiply the matrix $I + tM$ by $\det(I + tM)^{-1}(I - t(M - \rho_MI)$ from the left, then it equals

$$(1 + t\rho_M)^{-1}(I + t\rho_MI - t^2M^2 + t^2\rho_MM).$$

By Lemma 24, this equals $(1+t\rho_M)^{-1}(I + t\rho_MI) = I$. The identity $(I + tM)\det(I + tM)^{-1}(I - t(M - \rho_MI) = I$ is similarly verified. \hfill \Box

5.2 The main theorem

5.2.1 The main theorem

Let $\Delta = (V, A)$ be a finite digraph which may have multi-arcs and multi-loops, and $\theta^G$ a map $A \times A \to R$ of the form $\theta^G(a, a') = \tau(a')\delta_{\theta(a)(a')} - v(a')\delta_{\theta(a)S(a)}$ where $\tau$ and $\nu$ are maps form $A$ to $R$. Recall $A(u, v) = A_{uv} \cup A_{vu}$. Fix a total order $\leq$ on $V$. We may assume $u \leq v$ for $A(u, v)$ since the definition of $A(u, v)$ has the symmetry in $u$ and $v$. Consider the set

$$
\Phi_\Delta = \{ (u, v) \in V \times V \mid A(u, v) \neq \emptyset \}.
$$

Thus, if $(u, v) \in \Phi_\Delta$, then one has $u \leq v$. The set $\Phi_\Delta$ divides into three subsets $\Phi_\Delta = \Phi_\Delta^{(1)} \sqcup \Phi_\Delta^{(2)} \sqcup \Phi_\Delta^{(3)}$, where

$$
\Phi_\Delta^{(1)} = \{ (u, v) \in \Phi_\Delta \mid u = v \},
$$

$$
\Phi_\Delta^{(2)} = \{ (u, v) \in \Phi_\Delta \mid u \neq v, A_{uv} = \emptyset \text{ or } A_{vu} = \emptyset \},
$$

$$
\Phi_\Delta^{(3)} = \{ (u, v) \in \Phi_\Delta \mid u \neq v, A_{uv} \neq \emptyset \text{ and } A_{vu} \neq \emptyset \}.
$$
Thus, for \((u, v) \in \Phi^{(1)}_{\Delta}\), \(A(u, v) = A(u, u)(= A_{uv})\) is the set of loops with nest \(u\). For \((u, v) \in V \times V\), we consider the following polynomial

\[
f_{(u,v)}(t) = \begin{cases} 
1 + t \sum_{a \in A_{uv}} v(a), & \text{if } u = v, \\
1 - t^2 \left\{ \sum_{a \in A_{uv}} v(a) \right\} \left\{ \sum_{a \in A_{vu}} v(a) \right\}, & \text{otherwise},
\end{cases}
\]

which is invertible in the ring \(R[[t]]\) of formal power series. The following lemma follows immediately from the definition.

**Lemma 28** For \((u, v) \in V \times V\), we have:

1) \(f_{(u,v)}(t) = f_{(v,u)}(t)\),

2) \(f_{(u,v)}(t) = 1\) if \((u, v) \notin \Phi_{\Delta}\).

Set \(f_{\Delta}(t) = \prod_{(u,v) \in \Phi_{\Delta}} f_{(u,v)}(t)\). Let

\[
A_{\Delta}(\theta^G) = (a_{ww'}(\theta^G))_{w,w' \in V}, \quad D_{\Delta}(\theta^G) = (d_{ww'}(\theta^G))_{w,w' \in V}
\]

be square matrices of degree \(|V|\), whose entries are respectively given by

\[
a_{ww'}(\theta^G) = f_{(w,w')}(t)^{-1}a_{ww'}, \quad a_{ww'} := \sum_{a \in A_{ww'}} \tau(a),
\]

\[
d_{ww'}(\theta^G) = \delta_{ww'} \sum_{(u,v) \in \Phi_{\Delta}^{(3)}} f_{(u,v)}(t)^{-1}(\delta_{uv}d_{uv} + \delta_{vu}d_{vu}), \quad d_{uv} := \sum_{a \in A_{uv}, a' \in A_{vu}} \tau(a)v(a').
\]

**Theorem 29 (The Main Theorem)** For a finite digraph \(\Delta\), the reciprocal \(Z_{\Delta}(t; \theta^G)^{-1}\) of the generalized weighted zeta function is given by the polynomial

\[
f_{\Delta}(t) \det \left( I - tA_{\Delta}(\theta^G) + t^2D_{\Delta}(\theta^G) \right).
\]

*Proof.* By Proposition 13, the inverse of the generalized weighted zeta \(Z_{\Delta}(t; \theta^G)\) is given by \(\det(I - tM_{\Delta}(\theta^G))\), where \(M = M_{\Delta}(\theta^G) = (\theta^G(a, a'))_{a,a' \in A}\). Let \(H = (h_{aa'})_{a,a' \in A}\) and \(J = (j_{aa'})_{a,a' \in A}\) be square matrices with entries \(h_{aa'} = \tau(a')^2\delta_{b(a)(a')}\) and \(j_{aa'} = v(a')^2\delta_{a' \in S(a)}\). Obviously we have \(M = H - J\). Let \(K = (k_{aa'})_{a \in A, v \in V}\) and \(L = (l_{aa'})_{u \in V, a' \in A}\) denote the matrices with entries \(k_{aa} = \delta_{b(a)v}\) and \(l_{aa'} = \tau(a')^2\delta_{a \in S(a')\}}\) respectively. One can easily see that \(H = KL\). Thus we have

\[
\det(I - tM) = \det(I - t(KL - J)) = \det((I + tJ) - tKL).
\]
For each \((u, v) \in \Phi_\Delta\), let \(J(u, v), K(u, v)\) and \(L(u, v)\) denote the submatrices
\[
J(u, v) = (j_{aa'})_{a, a' \in \mathbb{S}(u, v)}, \quad j_{aa'} = \nu(a')\delta_{a' \in \mathbb{S}(a)},
K(u, v) = (k_{aw})_{a \in \mathbb{S}(u, v), w \in V}, \quad k_{aw} = \delta_{b(a)w},
L(u, v) = (l_{wa'})_{w \in V, a' \in \mathbb{S}(u, v)}, \quad l_{wa'} = \tau(a')\delta_{wt(a')}.
\]
Thus we have block partition for the matrices \(J, K, L\) by
\[
J = (J(u, v))_{(u, v) \in \Phi_\Delta}, K = (K(u, v))_{(u, v) \in \Phi_\Delta}, L = (L(u, v))_{(u, v) \in \Phi_\Delta}.
\]
Note that any total order on \(\Phi_\Delta\) makes \(J = (J(u, v))_{(u, v) \in \Phi_\Delta}\) a block diagonal matrix. In what follows, we fix such an order on \(\Phi_\Delta\). Note that any diagonal block \(J(u, v) ((u, v) \in \Phi_\Delta)\) is column constant. Hence, the matrix \(I + tJ\) is invertible, since, by Lemma \(27\), each diagonal block \(I + tJ(u, v)\) is invertible for any \((u, v) \in \Phi_\Delta\). Therefore, \(J\) is written by
\[
\det(I + tJ)\det(I - t(I + tJ)^{-1}KL),
\]
which equals
\[
\det(I + tJ)\det(I - tL(I + tJ)^{-1}K),
\]
(21) since, for two matrices \(A, B\), we have \(\det(I - AB) = \det(I - BA)\) if \(AB\) and \(BA\) are square matrices. The inverse \((I + tJ)^{-1}\) is also block diagonal, i.e,
\[
(I + tJ)^{-1} = \bigoplus_{(u, v) \in \Phi_\Delta} (I + tJ(u, v))^{-1}.
\]
Hence we have \(L(I + tJ)^{-1}K = \sum_{(u, v) \in \Phi_\Delta} L(u, v)(I + tJ(u, v))^{-1}K(u, v)\). We compute
\[
T(u, v) := L(u, v)(I + tJ(u, v))^{-1}K(u, v)
\]
for each \((u, v) \in \Phi_\Delta^{(i)}\), \(i = 1, 2, 3\).

If \((u, v) \in \Phi_\Delta^{(i)}\), then \(\mathbb{A}(u, v)\) consists of all arcs with nest \(u(= v)\), say \(\mathbb{A}(u, u) = \{a_1, \ldots, a_k\}\). In this case, the square matrix \(J(u, u)\) is given by
\[
\begin{bmatrix}
\nu(a_1) & \cdots & \nu(a_k) \\
\vdots & & \vdots \\
\nu(a_1) & \cdots & \nu(a_k)
\end{bmatrix}
\]
which is column constant. Hence, by Lemma \(26\), it follows that \(\det(I + tJ(u, u)) = 1 + t\sum_{i=1}^k \nu(a_i) = f_{uu}(t)\). Thus, by Lemma \(27\), the inverse \((I + tJ(u, u))^{-1}\) is given by \(f_{uu}(t)^{-1}(I - t(J(u, u) - (\text{tr } J(u, u))I))\). For each \(a \in \mathbb{A}(u, u)\) and \(v \in V\), the \((a, v)\)-entry of the product \((J(u, u) - (\text{tr } J(u, u))I)K(u, u)\) is given by
\[
\sum_{a' \in \mathbb{A}(u, u)} (\nu(a') - \text{tr } J(u, u)\delta_{aa'})\delta_{b(a')v} = \sum_{a' \in \mathbb{A}(u, u)} \nu(a')\delta_{uv} - \sum_{a' \in \mathbb{A}(u, u)} \text{tr } J(u, u)\delta_{aa'}\delta_{uv} = \sum_{a' \in \mathbb{A}(u, u)} \nu(a')\delta_{uv} - \text{tr } J(u, u)\delta_{uv} = 0.
\]

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Therefore we have

\[(I + tJ(u, u))^{-1} K(u, u) = f_{uu}(t)^{-1}\{I - t(J(u, u) - \text{tr} J(u, u)I)\} K(u, u)\]

and it follows that \(T(u, v) = f_{(u,v)}(t)^{-1} L(u, v) K(u, v)\) for \((u, v) \in \Phi_1^{(1)}\).

Suppose that \((u, v) \in \Phi_2^{(2)}\). Due to symmetry of the definition of \(A(u, v)\), we may assume \(A(u, v) = A_{uv}\). In this case, we have \(L(u, v) = O\). Hence \((I + tJ(u, v))^{-1} = I\). Note also that \(f_{(u,v)}(t) = 1\) for \((u, v) \in \Phi_2^{(2)}\), i.e., we have \(\det(I + tJ(u, v)) = f_{(u,v)}(t)\) for \((u, v) \in \Phi_2^{(2)}\).

Therefore we have \(T(u, v) = f_{(u,v)}(t)^{-1} L(u, v) K(u, v)\) for \((u, v) \in \Phi_2^{(2)}\).

Let \((u, v) \in \Phi_2^{(3)}\). Hence \(A(u, v) = A_{uv} \cup A_{vu}, A_{uv} \neq \emptyset, A_{vu} \neq \emptyset;\) say \(A_{uv} = \{a_1, \ldots, a_k\}\), \(A_{vu} = \{a_{k+1}, \ldots, a_{k+2}\}\). If we consider the following two column constant matrices \(J_1 = (v(a'))_{a \in A_{uv}, a' \in A_{vu}}, J_2 = (v(a'))_{a \in A_{uv}, a' \in A_{uv}},\) then the matrix \(I + tJ(u, v)\) is written by

\[
\begin{bmatrix}
I & tJ_1 \\
tJ_2 & I
\end{bmatrix}
\]

If we apply Lemma 27 replacing \(t\) by \(-t^2\), we can see that \(I - (tJ_1)(tJ_2) = I - t^2J_1J_2\) is invertible since the matrix \(J_1J_2\) is column constant. Therefore, it follows from Corollary 23 that the matrix \(I + tJ(u, v)\) is invertible, and the inverse \((I + tJ(u, v))^{-1}\) is given by

\[
\begin{bmatrix}
(I - t^2J_1J_2)^{-1} & -tJ_1(I - t^2J_2J_1)^{-1} \\
-t(I - t^2J_2J_1)^{-1}J_2 & (I - t^2J_2J_1)^{-1}
\end{bmatrix}
\]

(22)

It also follows from Corollary 23 that \(\det(I + tJ(u, v)) = \det(I - t^2J_1J_2) = \det(I - t^2J_2J_1)\). Since the matrix \(-t^2J_1J_2\) is column constant, it follows from Lemma 26 that \(\det(I - t^2J_1J_2) = 1 + \rho_{-t^2J_1J_2} = 1 - t^2\rho_{J_1J_2}\), and one can easily see from the definition that this equals \(f_{(u,v)}(t)\). Therefore we have \(\det(I + tJ(u, v)) = f_{(u,v)}(t)\) for \((u, v) \in \Phi_2^{(3)}\), which also shows that \(\det(I - t^2J_1J_2) = \det(I - t^2J_2J_1) = f_{(u,v)}(t)\). By Lemma 27 the \((1, 1)\)-block matrix \((I - t^2J_1J_2)^{-1}\) in (22) is written by

\[
f_{(u,v)}(t)^{-1}\{I + t^2(J_1J_2 - \rho_{J_1J_2}I)\}.
\]

Similarly we have \((I - t^2J_2J_1)^{-1} = f_{(u,v)}(t)^{-1}\{I + t^2(J_2J_1 - \rho_{J_2J_1}I)\}\), which gives the \((2, 2)\)-block in (22). The remaining blocks contain the products \(J_1J_2J_1\) and \(J_2J_1J_2\), and it is easily seen that \(J_1J_2J_1 = \rho_{J_2J_1}J_1J_2\) and \(J_2J_1J_2 = \rho_{J_1J_2}J_1J_2\). Thus, \((I + tJ(u, v))^{-1}\) equals

\[
f_{(u,v)}(t)^{-1}\left\{\begin{bmatrix} I & O \\ O & I \end{bmatrix} - t \begin{bmatrix} O & J_1 \\ J_2 & O \end{bmatrix} + t^2 \begin{bmatrix} J_1J_2 - \rho_{J_1J_2}I & O \\ J_2J_1 - \rho_{J_2J_1}I & O \end{bmatrix}\right\}.
\]

(23)

Hence the matrix \(T(u, v)\) is a linear combination of the following three matrices \(L(u, v)K(u, v), L(u, v)J(u, v)K(u, v)\) and \(L(u, v)\{P_1 \oplus P_2\} K(u, v)\), where \(P_1 = J_1J_2 - \rho_{J_1J_2}I, P_2 = J_2J_1 - \rho_{J_2J_1}I\). For any two vertices \(\xi, \eta \in V\), let \(K_{\xi\eta} = (k_{\xi\eta})_{a \in A_{\xi}, \eta \in V} \). Since the matrix \(K(u, v)\) is partitioned by two blocks \(K_{uv}\) and \(K_{vu}\), the product \(\{P_1 \oplus P_2\} K(u, v)\) is also partitioned
by the following two blocks $P_1K_{uv}, P_2K_{uv}$. Since $J_1J_2 = \rho J_1(v(a'))_{a,a'\in A_{uv}}$, we have $J_1J_2K_{uv} = \rho J_1(v(a'))_{a,a'\in A_{uv}}(\bar{h}(a)w)_{a\in A_{uv}, w\in V}$, the $(a,w)$-entry of which is given by $\sum_{b\in A_{uv}} \rho J_1v(b)\bar{h}(b)w = \rho J_1\rho J_2 \bar{h}(a)w$ for $a \in A_{uv}$ and $w \in V$. Hence we have $J_1J_2K_{uv} = \rho J_1\rho J_2 K_{uv}$, which implies $P_1K_{uv} = O$ since $\rho J_1\rho J_2 = \rho J_1J_2$ by Lemma 25. In the same manner, one can show that $P_2K_{vu} = O$. Thus we have $L(u,v)\{P_1 \oplus P_2\}K(u,v) = O$, and it follows that $T(u,v) = f_{(u,v)}(t)^{-1}\{L(u,v)K(u,v) - tL(u,v)J(u,v)K(u,v)\}$ for $(u,v) \in \Phi_\Delta^{(3)}$. Consequently we have

$$L(I + tJ)^{-1}K = \sum_{i=1}^{3} \sum_{(u,v)\in \Phi_\Delta^{(i)}} L(u,v)(I + tJ(u,v))^{-1}K(u,v)$$

$$= \sum_{(u,v)\in \Phi_\Delta} f_{(u,v)}(t)^{-1}L(u,v)K(u,v) - t \sum_{(u,v)\in \Phi_\Delta^{(3)}} f_{(u,v)}(t)^{-1}L(u,v)J(u,v)K(u,v).$$

For each $(u,v) \in \Phi_\Delta$, let $A(u,v) := L(u,v)K(u,v)$, and $D(u,v) := L(u,v)J(u,v)K(u,v)$, say $A(u,v) = (\alpha_{wuv}(u,v))_{w,w\in V}$ and $D(u,v) = (\beta_{wuv}(u,v))_{w,w\in V}$. Accordingly we have

$$L(I + tJ)^{-1}K = \sum_{(u,v)\in \Phi_\Delta} f_{(u,v)}(t)^{-1}A(u,v) - t \sum_{(u,v)\in \Phi_\Delta^{(3)}} f_{(u,v)}(t)^{-1}D(u,v). \quad (24)$$

Let $w, w' \in V$. The $(w, w')$-entry $\alpha_{wuv}(u,v)$ of $A(u,v)$ is given by

$$\sum_{a\in A(u,v)} \tau(a)\delta_{wt(a)}\bar{h}(a)w'.$$

If $(u,v) = (u,u) \in \Phi_\Delta^{(1)}$, then $A(u,u) = A_{uw}$ and we have $\alpha_{wuv}(u,v) = \sum_{a\in A_{uw}} \tau(a)\delta_{wv}a_{uv}$. This equals $\delta_{wv}\delta_{wv'}\sum_{a\in A_{uw}} \tau(a) = \delta_{wv}\delta_{wv'}a_{uv}$, where $a_{uv}$ is defined in (18). In the case where $(u,v) \in \Phi_\Delta^{(2)}$, then $A_{uv} = \emptyset$ or $A_{uv} = \emptyset$. In this case where $A_{uv} = \emptyset$, one has $\alpha_{wuv}(u,v) = \sum_{a\in A_{uw}} \tau(a)\delta_{wv}\delta_{wv'}$, and this equals $\delta_{wv}\delta_{wv'}a_{uv}$. Similarly, if $A_{uv} = \emptyset$, then one has $\alpha_{wuv}(u,v) = \delta_{wv}\delta_{wv'}a_{uv}$. For $(u,v) \in \Phi_\Delta^{(3)}$, both $A_{uv}$ and $A_{wu}$ are not empty set, and we have $\alpha_{wuv}(u,v) = \delta_{wv}\delta_{wv'}a_{uv} + \delta_{wv}\delta_{wv'}a_{uv}$. Putting all these together, one can see that the $(w, w')$-entry of the matrix $\sum_{(u,v)\in \Phi_\Delta} f_{(u,v)}(t)^{-1}A(u,v)$ is given by $f_{(w,w')}^{-1}a_{wv}$. Therefore we have

$$A_\Delta(\theta_G^d) = \sum_{(u,v)\in \Phi_\Delta} f_{(u,v)}(t)^{-1}A(u,v).$$

On the other hand, for $(u,v) \in \Phi_\Delta^{(3)}$, the $(w, w')$-entry $\beta_{wuv}(u,v)$ of $D(u,v)$ is given by

$$\sum_{a,a'\in A(u,v)} \tau(a)\delta_{wt(a)}v(a')\delta_{w'(a')}w'.$$
One can easily see that this equals $\delta_{wu}\delta_{w'u}d_{uv} + \delta_{wv}\delta_{w'u}d_{vu}$. Thus the $(w, w')$-entry $d_{ww'}(\theta^G)$ of the matrix $\sum_{(u,v)\in \Phi_\Delta} f_{(u,v)}(t)^{-1} D(u, v)$ is $\sum_{(u,v)\in \Phi_\Delta} f_{(u,v)}(t)^{-1}(\delta_{wu}\delta_{w'u}d_{uv} + \delta_{wv}\delta_{w'u}d_{vu})$, and this equals $\delta_{wv}\sum_{(u,v)\in \Phi_\Delta} f_{(u,v)}(t)^{-1}(\delta_{wu}\delta_{u'}d_{uv} + \delta_{wv}\delta_{v'}d_{vu})$, which shows that

$$D_\Delta(\theta^G) = \sum_{(u,v)\in \Phi_\Delta} f_{(u,v)}(t)^{-1} D(u, v).$$

Finally, since we have verified that $\det(I + tJ(u, v)) = f_{(u,v)}(t)$ for each $(u, v) \in \Phi_\Delta$, it follows that $\det(I + tJ) = \prod_{(u,v)\in \Phi_\Delta} f_{(u,v)}(t) = f_\Delta(t)$. Therefore, from (10), we have

$$\det(I - tM) = f_\Delta(t) \det(I - tA_\Delta(\theta^G) + t^2 D_\Delta(\theta^G)).$$

The identity for the generalized weighted zeta $Z_\Delta(t; \theta^G)$ verified in Theorem 29 is called the Ihara expression for $Z_\Delta(t; \theta^G)$, and the matrices $A_\Delta(\theta^G)$ and $D_\Delta(\theta^G)$ are called the weighted adjacency matrix and the weighted backtrack matrix for $Z_\Delta(t; \theta^G)$, respectively.

### 5.2.2 Comments on the adjacency matrix and the backtrack matrix

The weighted adjacency matrix $A_\Delta(\theta^G)$ and the weighted backtrack matrix $D_\Delta(\theta^G)$ are practically discrepant from the ordinary ones. We will see the difference between them for the case of the Ihara zeta $Z_{\Delta(\Gamma)}(t; \theta^1)$ for a finite simple graph $\Gamma$. Let $\Gamma = (V, E)$ be a finite simple graph, where $V$ is the vertex set, $E$ the edge set. Suppose that a digraph $\Delta = \Delta(\Gamma)$ is the symmetric digraph of $\Gamma$. Let $A$ denote the arc set of $\Delta$. $\Phi^{(i)}_\Delta (i = 1, 2, 3)$ is defined as in 5.2.1.

**Lemma 30** Let $\Delta = (V, A)$ be the symmetric digraph of a finite simple graph. Then we have: 1) $\Phi^{(1)}_\Delta = \Phi^{(2)}_\Delta = \emptyset$, 2) $A(u, v) \neq \emptyset$ implies $|A(u, v)| = 2$. In this case, we have $|A_{uv}| = |A_{vu}| = 1$.

It follows from Lemma 30 that $f_{(u,v)}(t) = 1$ for $(u, v) \in \Phi^{(i)}_\Delta (i = 1, 2)$, and $f_{(u,v)}(t) = 1 - t^2$ for $(u, v) \in \Phi^{(3)}_\Delta$. Remark that the coefficients $f_{(u,v)}(t)$ do not depend on $(u, v) \in \Phi^{(3)}_\Delta$ in this case. Hence we have

$$A_\Delta(\theta) = (1 - t^2)^{-1} \sum_{(u,v)\in \Phi^{(3)}_\Delta} A(u, v), \quad D_\Delta(\theta) = (1 - t^2)^{-1} \sum_{(u,v)\in \Phi^{(3)}_\Delta} D(u, v).$$

Let $(u, v) \in \Phi^{(3)}_\Delta$. Recall that $\theta^1$ is obtained by letting $\tau = v = 1$ for $\theta^G$. Thus the $(w, w')$-entry $\alpha_{ww'}(u, v)$ of the matrix $A(u, v)$ is given by $\alpha_{ww'}(u, v) = \sum_{a \in A(u, v)} \tau(a)\delta_{w_a'(a)w'} = \sum_{a \in A_u} \delta_{w_u}\delta_{w'u} + \sum_{a \in A_v} \delta_{w_v}\delta_{w'u} = \delta_{w_u}\delta_{w'u}|A_{uv}| + \delta_{w_v}\delta_{w'u}|A_{vu}| = \delta_{w_u}\delta_{w'u} + \delta_{w_v}\delta_{w'u}$. This shows that the matrix $\sum_{(u,v)\in \Phi^{(3)}_\Delta} A(u, v)$ is nothing but the usual adjacency matrix $A_{\Gamma}$ for $\Gamma$. It also follows from Lemma 30 that $\beta_{ww'}(u, v) = \sum_{a, a' \in A(u, v)} \delta_{w_a}\delta_{a'v} + S(a)\delta_{b(a)w'} = \sum_{a, a' \in A(u, v)} \delta_{w_a}\delta_{a'v} + S(a)\delta_{b(a)w'} = \sum_{a, a' \in A(u, v)} \delta_{w_a}\delta_{a'v} + S(a)\delta_{b(a)w'}$. 

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Figure 1: A digraph $\Delta = (V, A)$

$\sum_{a \in A_{uv}} \delta_{u'v} + \sum_{a \in A_{vu}} \delta_{uv'} = \delta_{u'=u} + \delta_{v'=v}$. This shows that the matrix $\sum_{(u,v) \in \Phi(3)} D(u, v)$ gives the usual degree matrix $D_{\Gamma}$ of $\Gamma$. Therefore we have

$$A_{\Delta}(\theta) = (1 - t^2)^{-1} A_{\Gamma}, \quad D_{\Delta}(\theta) = (1 - t^2)^{-1} D_{\Gamma}.$$ 

Note that the cardinality $|\Phi(3)_{\Delta}|$ equals the number $|E|$ of edges of $\Gamma$. We also have $f_{(u,v)}(t) = 1 - t^2$ for each $(u, v) \in \Phi(3)_{\Delta}$, and it follows that $f_{\Delta}(t) = (1 - t^2)^{|E|}$. Since the degrees of the square matrix $A_{\Gamma}$ and $D_{\Gamma}$ equal the number $|V|$ of vertices in $\Gamma$, it follows from (19) that

$$Z_{\Delta}(t)^{-1} = (1 - t^2)^{|E|-|V|} \det(I - tA_{\Gamma} + t^2(D_{\Gamma} - I)),$$

which is the classical Bass-Ihara theorem [4, 13] for a finite simple graph.

5.3 Example

Let $\Delta = (V, A)$ be a digraph in Figure 1, where $V = \{v_1, v_2, v_3\}$ is the vertex set and $A = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}$ the arc set. The total order for $V$ is given by $v_1 < v_2 < v_3$, and for $A$ by $a_1 < a_2 < a_3 < a_4 < a_5 < a_6 < a_7 < a_8$. Thus we have

$$\Phi_{\Delta} = \{(v_1, v_1), (v_1, v_2), (v_1, v_3), (v_2, v_3)\},$$

where $\Phi_{\Delta}^{(1)} = \{(v_1, v_1)\}, \Phi_{\Delta}^{(2)} = \{(v_1, v_3)\}, \Phi_{\Delta}^{(3)} = \{(v_1, v_2), (v_2, v_3)\}$. The arc sets $A_{uv}$’s are given by $A_{v_1v_1} = \{a_1, a_2\}, A_{v_1v_2} = \{a_3\}, A_{v_1v_3} = \emptyset, A_{v_2v_1} = \{a_4\}, A_{v_2v_3} = \{a_5, a_6\}, A_{v_3v_2} = \{a_7\}$. The matrices $J, K, L$ and $J(u, v), K(u, v), L(u, v)$ for each $(u, v) \in \Phi_{\Delta}$ are given by
The matrices $A$ and $D$ are for example

\[
A(v_1, v_1) = \begin{bmatrix}
\tau(a_1) + \tau(a_2) & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix},
A(v_1, v_2) = \begin{bmatrix}
0 & \tau(a_3) & 0 \\
\tau(a_4) & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix},
A(v_1, v_3) = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\tau(a_8) & 0 & 0 \\
\end{bmatrix},
\]

\[
D(v_1, v_2) = \begin{bmatrix}
\tau(a_3)v(a_4) & 0 & 0 \\
0 & \tau(a_4)v(a_3) & 0 \\
0 & 0 & 0 \\
\end{bmatrix},
D(v_2, v_3) = \begin{bmatrix}
0 & 0 & 0 \\
0 & \tau(a_5) + \tau(a_6)v(a_7) & 0 \\
0 & 0 & \tau(a_7)(v(a_5) + v(a_6)) \\
\end{bmatrix},
\]
etc. The polynomials $f_{(u,v)}(t)$ are given by $f_{(v_1,v_2)}(t) = 1 + t(v(a_1) + v(a_2))$, $f_{(v_1,v_3)}(t) = 1$, $f_{(v_2,v_3)}(t) = 1 - t^2(v(a_5) + v(a_6))v(a_7)$, and we have

$$A_\Delta(\theta^G) = \begin{bmatrix} \frac{\tau(a_1)+\tau(a_2)}{1+t(v(a_1)+v(a_2))} & \frac{\tau(a_3)}{1-t^2v(a_3)v(a_4)} & 0 \\ \frac{\tau(a_4)}{1-t^2v(a_3)v(a_4)} & \frac{\tau(a_5)+\tau(a_6)}{1-t^2(v(a_5)+v(a_6))v(a_7)} & 0 \\ \tau(a_8) & \tau(a_7) & 0 \end{bmatrix}.$$  

$$D_\Delta(\theta^G) = \begin{bmatrix} \frac{\tau(a_2)v(a_4)}{1-t^2v(a_3)v(a_4)} & 0 & 0 \\ 0 & \frac{\tau(a_3)v(a_4)}{1-t^2v(a_3)v(a_4)} + \frac{(\tau(a_5)+\tau(a_6))v(a_7)}{1-t^2v(a_5)+v(a_6)v(a_7)} & 0 \\ 0 & 0 & \frac{\tau(a_7)v(a_5)+v(a_6)}{1-t^2v(a_5)+v(a_6)v(a_7)} \end{bmatrix}.$$  

The main theorem shows that $\det(I - tM_\Delta(\theta^G)) = f_\Delta(t)\det(I - tA_\Delta(\theta^G) + t^2D_\Delta(\theta^G))$, where $f_\Delta(t) = \prod_{(u,v) \in \Phi_\Delta} f_{(u,v)}(t)$ and $M_\Delta(\theta^G) = (\theta^G(a, a'))_{a, a' \in A}$.

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