We study solitons in scalar theories with polynomial interactions on the fuzzy sphere. Such solitons are described by projection operators of rank $k$, and hence the moduli space for the solitons is the Grassmannian $Gr(k, 2j + 1)$. The gradient term of the action provides a non-trivial potential on $Gr(k, 2j + 1)$, thus reducing the moduli space. We construct configurations corresponding to well-separated solitons, and show that although the solitons attract each other, the attraction vanishes in the limit of large $j$. In this limit, it is argued that the moduli space is $(\mathbb{C}P^1)^\otimes_k / S_k \simeq \mathbb{C}P^k$. For the $k$-soliton bound state, the moduli space is simply $\mathbb{C}P^1$, all other moduli being lifted. We find that the moduli space of multi-solitons is smooth and that there are no singularities as several solitons coalesce. When the fuzzy $S^2$ is flattened to a noncommutative plane, we find agreement with the known results, modulo some operator-ordering ambiguities. This suggests that the fuzzy sphere is a natural way to regulate the noncommutative plane both in the ultraviolet and infrared.

1 Introduction

Theories on noncommutative spaces have been studied vigorously for a while now. They arise in string theory in a certain corner of moduli space [1]. Noncommutative theories can also be studied independently and in their own right, and provide a variety of interesting new phenomena. Theories on noncommutative compact manifolds often come with an ultraviolet cut-off, and are thus potentially regulated in their short distance behavior. However, the quantum properties of noncommutative theories are considerably subtle: even in a simple scalar theory on noncommutative $R^4$ there is mixing between ultraviolet and infrared degrees of freedom, as was first shown in [2].

Interestingly, noncommutative scalar theories in $(2n + 1)$-dimensions also have stable finite energy classical configurations. This first shown in [3], who constructed these finite energy configurations from projector operators and interpreted them as solitons. In a subsequent paper [4], the multi-soliton moduli space was studied in detail.

Using the techniques first discussed in [4] (see also [3]), we will study solitons in scalar theories defined on $S^2_F \times R$, where $S^2_F$ is a fuzzy sphere. A fuzzy sphere is described by a finite dimensional algebra generated by 3 matrices $X_a$ that satisfy

$$[X_a, X_b] = i \frac{R}{\sqrt{j(j + 1)}} \epsilon_{abc} X_c, \quad X_a X_a = R^2 1, \quad a, b, c = 1, 2, 3 \quad \text{and} \quad j \in \mathbb{Z}/2. \tag{1.1}$$

The $X_a$ are $(2j + 1) \times (2j + 1)$ matrices proportional to the $(2j + 1)$-dimensional representation of the generators $J_a$ of the $SU(2)$ algebra. In this article, we will be interested in two limits: in one case, we take $j \to \infty$ at fixed $R$ to get an ordinary sphere (of radius $R$), while in the other limit we take $j, R \to \infty$ with $R^2/j$ fixed, to get the noncommutative plane. The multi-soliton configurations will be...
constructed using $SU(2)$ coherent states, and we will look in some detail at the case of two solitons. It will be argued that well-separated solitons are labeled by their location on an ordinary sphere. For the class of solitons that we consider, the solitons are found to attract each other, as was also pointed out by [3]. Interestingly, the attraction becomes very weak in the limit of large $j$ even at finite radius, and the solitons behave as free particles. We will also argue that the multi-soliton moduli space is smooth, and that there are no singularities as two or more solitons coalesce.

It must be emphasized that the configurations we call solitons in this article are different from the ones that have been studied earlier in similar contexts [6–9]. The solitons, monopoles and instantons discussed in [8, 9] were based on a discrete version of the non-linear $\sigma$-model on $S^2_F$ and were related to cyclic cohomology [10].

This article is organized as follows. In section 2, we will show how projection operators correspond to solitons and show that in the limit of large $j$, the soliton is really a smeared version of a point particle on an ordinary sphere. In section 3, the geometry of the moduli space of $k$ well-separated solitons is explored. The moduli space is shown to be non-singular even when the solitons coincide. The gradient term in the action lifts most of the moduli, and the remaining moduli form an ordinary $S^2$. This gradient term provides an attractive potential between the solitons, but vanishes in the limit of large $j$. In section 4, the two soliton case is worked out in detail and the interaction potential is calculated. In section 5, we study the limit in which the fuzzy sphere is flattened in to the noncommutative plane, and argue that all our results conform with the known results for solitons on the noncommutative plane, modulo some effects related to operator-ordering ambiguities. This suggests to us that $S^2_F$ is a natural candidate for the ultraviolet regulated version of the noncommutative plane. Our results are summarized in section 6.

There is now a substantial body of work on $S^2_F$, starting from the works of [6, 11]. Solitons and monopoles in non-linear $\sigma$-models on $S^2_F$ were studied by [3] (see also [3]), while topological issues such as instantons, $\theta$-term and derivation of the chiral anomaly were discussed in [3]. (For an alternate derivation of the chiral anomaly, see [12].) The continuum limit of the fuzzy non-linear $\sigma$-model has been discussed in [13]. The phenomenon of UV-IR mixing for scalar theories on $S^2_F$ was first shown in [14], and further studied in [15,16]. Interest in $S^2_F$ has increased since Myers showed that D0-branes in a constant Ramond-Ramond field arrange themselves in the form of a fuzzy sphere [17]. There have been investigations by [18] regarding open string versions of WZW models which naturally lead to $S^2_F$. Gauge theories on $S^2_F$ have been studied by [13], while their continuum limits have been discussed by [20]. Noncommutative solitons on the fuzzy $S^2$ have appeared in [21] in the context of tachyon condensation and string field theory. Further studies of topological as well as other issues for fuzzy $S^2$ may be found in [22]. Other aspects of noncommutative solitons of the type discussed in [3], including connections to string theory, may be found in [23].

## 2 Solitons from Projectors

The action for a single scalar field on fuzzy $S^2$ is given by

$$S = \frac{1}{2j+1} \text{Tr}_{H^{(j)}} \left( \dot{\Phi}^2 - [J_a, \Phi]^2 - m^2 V[\Phi] \right). \quad (2.1)$$

The field $\Phi$ is an arbitrary $(2j+1) \times (2j+1)$ hermitian matrix, and the $J_a$ are the generators of the $SU(2)$ algebra in the $(2j+1)$ dimensional representation, and $\text{Tr}_{H^{(j)}}$ is the trace is over the $(2j+1)$-
dimensional Hilbert space $\mathcal{H}^{(j)}$. We will call the term $\text{Tr}_{\mathcal{H}^{(j)}}[J_a, \Phi]^2$ as the \textit{gradient} term, since in the continuum limit $(j \to \infty)$ it goes over to $\int (\mathcal{L}_a \Phi)^2$, where $\mathcal{L}_a = i \epsilon_{abc} x_a \partial_b$ are the vector fields generating rotations on the 2-sphere.

In the limit $m^2 \to \infty$, the potential term of (2.1) gives the dominant contribution to the action $\mathcal{E}[\Phi]$.

If the potential $V$ is polynomial, the it is minimized by $\Phi = \lambda \mathcal{P}^{(k)}$ where the $\lambda$ is a minimum of $V(x)$ and $\mathcal{P}^{(k)}$ is a hermitian projector of rank $k$:

$$\mathcal{P}^{(k)2} = \mathcal{P}^{(k)} = \mathcal{P}^{(k)\dagger}. \quad (2.2)$$

Since this is a finite-dimensional matrix model, the rank $k$ of non-trivial a projector satisfies $0 < k < 2j + 1$. In an appropriate choice of basis, a projector is simply a diagonal matrix with $k$ entries being 1, the others being 0. The set of all rank $k$ projectors is simply the Grassmannian $Gr(k, 2j + 1)$.

When $m^2$ is large but finite, the gradient term of the action (2.1) must be taken into account as well. This term is the energy of the configuration and provides a potential on the space $Gr(k, 2j + 1)$. This potential lifts most of the moduli, but we will argue that an $S^2$ worth of moduli remain as the lowest energy configurations. The time dependent term $\text{Tr}_{\mathcal{H}^{(j)}} \Phi^2$ of the action provides the dynamics, and thus gives the metric on the lifted moduli space.

For simplicity, let us start with $k = 1$. The set of all rank 1 projectors is the space $\mathbb{C}P^{2j}$, hence a rank 1 projector is characterized by $4j$ moduli. The gradient term provides a non-trivial potential on the moduli space $\mathbb{C}P^{2j}$. As a result, not all configurations are equivalent: some configurations have lower energy than others. The most general rank 1 projector is of the form $\mathcal{P}^{(1)} = \langle Z|Z \rangle$ where

$$|Z\rangle = \sum_{\mu = -j}^{j} z_\mu |\mu\rangle, \quad z_\mu \in \mathbb{C} \quad \text{and} \quad \langle Z|Z \rangle = 1, \quad (2.3)$$

the $\{|\mu\rangle\}$ being the standard angular momentum basis of the $(2j + 1)$-dimensional Hilbert space $\mathcal{H}^{(j)}$.

We reproduce here the argument of [5] to find the set of lowest energy configurations corresponding to the rank 1 projector. Rewriting the energy of the soliton as

$$\frac{\lambda^2}{2j + 1} \text{Tr}_{\mathcal{H}^{(j)}}\mathcal{P}^{(1)}[J_a, [J_a, \mathcal{P}^{(1)}]] = \frac{\lambda^2}{2j + 1} \left( \langle Z|J_a J_a|Z \rangle - \langle Z|J_a|Z \rangle \langle Z|J_a|Z \rangle \right) = \frac{\lambda^2}{2j + 1} \langle Z| (\Delta J)^2 |Z \rangle, \quad (2.4)$$

we see that $|Z\rangle$ minimizes the energy if and only if it minimizes the dispersion of $\Delta J$, forcing it to be either $|j\rangle$ and $|-j\rangle$, and thus value of energy to be $2j\lambda^2/(2j + 1)$.

The most general (rank 1) projector of this energy is obtained by applying rotations to, say, the state $|−j\rangle$ and using this to construct the projector. This is simply $|\zeta\rangle \langle \zeta|/\langle \zeta|\zeta \rangle$ where $|\zeta \rangle = e^{-j|z\rangle} |−j\rangle$ is the (non-normalized) $SU(2)$ coherent state. The coordinate $\zeta$ has a simple interpretation: it is location of the center-of-mass of the soliton, as we show below.

Corresponding to any operator $\mathcal{O}$ acting on the $\mathcal{H}^{(j)}$, one can associate a function $\mathcal{O}(z, \bar{z}) = \langle z|\mathcal{O}|z \rangle/\langle z|z \rangle$ on the sphere, where $z$ is the stereographic coordinate. This is called the \textit{covariant symbol} [24] of the operator $\mathcal{O}$. The covariant symbol of the projector $\mathcal{P}^{(1)}_{(\zeta)} = |\zeta\rangle \langle \zeta|/\langle \zeta|\zeta \rangle$ is the function

$$\mathcal{P}^{(1)}_{(\zeta)}(z, \bar{z}) = \frac{\langle z|\zeta \rangle \langle \zeta|z \rangle}{\langle z|z \rangle} \quad (2.5)$$
For $\zeta = 0$, the function

$$P_{(0)}^{(1)}(z, \bar{z}) = \frac{1}{(1 + |z|^2/R^2)^2} = \cos(\theta/2)^{4j}, \quad \text{where} \quad z = R\tan(\theta/2)e^{i\phi}. \quad (2.6)$$

may be interpreted as the soliton being at the north pole. For $j$ large, this function is strongly peaked around $\theta = 0$, with a spread of $(0, 2\tan^{-1}(1/\sqrt{2}j))$, and almost zero outside this region. It is in fact a regularized version of the $\delta$-function on the sphere. We interpret this as a soliton of angular size $2\tan^{-1}(1/\sqrt{2}j)$ and located at $\theta = 0$. The dynamics of the single soliton is thus that of a smeared point particle on the sphere.

Since it is only angular sizes that matter, we will restrict to $R = 1$ henceforth whenever the fuzzy sphere is under consideration.

### 3 Geometry of the moduli space

Vectors in the $(2j + 1)$-dimensional Hilbert space $\mathcal{H}^{(j)}$ are usually expanded in terms of the basis $\{|-j\rangle, |-j + 1\rangle, \cdots |j\rangle\}$. One can also use a basis of $SU(2)$ coherent states $\{|\xi_1\rangle, |\xi_2\rangle, \cdots |\xi_{2j+1}\rangle\}$, where all the $\xi_i$ are distinct points on the sphere. The coherent states we use are non-normalized: $|\xi_i\rangle = e^{\xi_iJ_+}|j\rangle$. We show here that there exists a non-singular basis even when some of the $\xi_i$ are not distinct.

Let us expand $|\xi_i\rangle$ as

$$|\xi_i\rangle = e^{\xi_iJ_+}|j\rangle = \sum_{\mu=-j}^{j} \left( \frac{(2j)!}{(j+\mu)!(j-\mu)!} \right)^{1/2} \xi_i^{j+\mu} |\mu\rangle \equiv \sum_{\mu=-j}^{j} c_{\mu} |\mu\rangle. \quad (3.1)$$

The basis $\{|\xi_1\rangle, |\xi_2\rangle, \cdots, |\xi_{2j+1}\rangle\}$ can be expressed in terms of the standard basis $\{|-j\rangle, |-j + 1\rangle, \cdots |j\rangle\}$ as

$$\begin{pmatrix}
|\xi_1\rangle \\
|\xi_2\rangle \\
|\xi_{2j+1}\rangle
\end{pmatrix} = \begin{pmatrix}
c_{-j} & c_{-j+1} & c_{-j+2} & \cdots & c_{j} \\
c_{-j} & c_{-j+1} & c_{-j+2} & \cdots & c_{j} \\
c_{-j} & c_{-j+1} & c_{-j+2} & \cdots & c_{j}
\end{pmatrix} \begin{pmatrix}
|j\rangle \\
|j+1\rangle \\
\vdots
\end{pmatrix}. \quad (3.2)$$

The transformation matrix $U$ can be written as $V C$ where $V$ is the Vandermonde matrix

$$V = \begin{pmatrix}
1 & \xi_1 & \xi_1^2 & \cdots & \xi_1^j \\
1 & \xi_2 & \xi_2^2 & \cdots & \xi_2^j \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \xi_{2j+1} & \xi_{2j+1}^2 & \cdots & \xi_{2j+1}^j
\end{pmatrix} \quad \text{and} \quad C = \text{diag}(c_{-j}, \cdots, c_j). \quad (3.3)$$

What happens when say, $\xi_1 \to \xi_2$? One can choose $|\xi_1\rangle$ and $(|\xi_1\rangle - |\xi_2\rangle)/(|\xi_1\rangle - |\xi_2\rangle)$ as basis vectors instead of $|\xi_1\rangle$ and $|\xi_2\rangle$. The new basis has a well-defined limit even when $(\xi_1 - \xi_2) \to 0$. In fact, it is easy to see that $(|\xi_1\rangle - |\xi_2\rangle)/(\xi_1 - \xi_2)$ tends to $J_+|\xi_1\rangle$. The vectors $|\xi_1\rangle$ and $J_+|\xi_1\rangle$ are linearly independent, and so the basis $\{|\xi_1\rangle, J_+|\xi_1\rangle, |\xi_3\rangle, \cdots, |\xi_{2j+1}\rangle\}$ is non-degenerate.

It is easy to see what happens when $\xi_1, \cdots, \xi_m$ coalesce at the point $\xi$. We choose as basis elements the vectors $|\xi\rangle, J_+|\xi\rangle, \cdots, J_m^m|\xi\rangle$. These are linearly independent, and along with the remaining distinct $|\xi_i\rangle$’s form a non-degenerate basis for our vector space.
The rank $k$ projector corresponding to the $k$-soliton configuration is

$$\mathcal{P}^{(k)} = \sum_{i,j=1}^{k} |\psi_i\rangle (h^{-1})_{ij} \langle \psi_j|,$$  

(3.4)

where the $|\psi_i\rangle$ are linearly independent vectors in the Hilbert space $\mathcal{H}$ and the matrix $h$ has entries $h_{ij} = \langle \psi_i | \psi_j \rangle$. This projector projects onto the subspace spanned by the vectors $|\psi_1\rangle, \ldots, |\psi_k\rangle$. For describing well-separated solitons, we can write this projector in terms of the coherent state basis as

$$\mathcal{P}^{(k)}_{(\zeta_1, \ldots, \zeta_k)} = \sum_{i,j=1}^{k} |\zeta_i\rangle (h^{-1})_{ij} \langle \zeta_j|,$$  

(3.5)

where $\zeta_1, \ldots, \zeta_k$ are $k$ points on the two-sphere. Any permutation of these points gives us the same projector, and hence this projector corresponds to a point on the space $\mathcal{M}_k = \mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1 / S_k$, where $S_k$ is the permutation group of $k$ objects. Now, it well-known $\mathcal{M}_k$ is simply $\mathbb{C}P^k$ (for a physicist’s proof, see [25]), so the projector (3.5) is labeled by a point in $\mathbb{C}P^k$. This is a Kähler manifold with the Kähler potential $K(\zeta_a, \bar{\zeta}_b)$ given by

$$K(\zeta_a, \bar{\zeta}_b) = \ln \det h,$$  

(3.6)

The metric $g_{\alpha\bar{\beta}}$ on the moduli space $\mathcal{M}_k$ is calculated from $K(\zeta_a, \bar{\zeta}_b)$ as

$$g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} K(\zeta_a, \bar{\zeta}_b).$$  

(3.7)

Let us look at the rank 1 projector $\mathcal{P}^{(1)}_{(\zeta)} = |\zeta\rangle \langle \zeta| / \langle \zeta | \zeta \rangle$. The metric $g_{\alpha\bar{\beta}}$ on the reduced moduli space comes from the kinetic energy term

$$\frac{\lambda^2}{2j+1} \text{Tr}_{\mathcal{H}}(\dot{\zeta}) \dot{\zeta}^{(1)2} = \lambda^2 g_{\alpha\bar{\beta}} \dot{\zeta}_a \dot{\bar{\zeta}}_{\bar{\beta}} = \lambda^2 \left( \frac{2j}{2j+1} \right) \frac{2|\zeta|^2}{(1 + |\zeta|)^2}.$$  

(3.8)

Thus $g_{\alpha\bar{\beta}}$ is simply the round metric on the sphere, and the motion of the soliton can be thought of as the motion of a free particle of mass $2j\lambda^2/(2j+1)$ on the sphere.

The gradient term in the the action (2.11) gives the energy of the $k$-soliton configuration. For $j$ finite, it leads to attraction between the solitons and makes them clump together on top of each other, as we will explicitly demonstrate for the case of two solitons in the next section. It will be argued here that the attraction becomes extremely weak in the limit of large $j$, and in fact vanishes as $j \to \infty$.

Let us write the gradient term as

$$E[\zeta_a, \bar{\zeta}_b] = \frac{\lambda^2}{2j+1} \text{Tr}_{\mathcal{H}} (\dot{J}_{a} \circ \mathcal{P}^{(k)}_{(\zeta_1, \ldots, \zeta_k)} [J_a, \mathcal{P}^{(k)}_{(\zeta_1, \ldots, \zeta_k)}])$$

(3.9)

Using the identities

$$\langle \zeta_a | J_+ | \zeta_b \rangle = \partial_{\zeta_b} h_{ab}, \quad \langle \zeta_a | J_- | \zeta_b \rangle = \partial_{\bar{\zeta}_b} h_{ab},$$

(3.10)

$$J_{\bar{\beta}} | \zeta_a \rangle = (-j + \zeta_a J_+) | \zeta_a \rangle,$$  

(3.11)
Let us look at the case of the rank 2 projector in detail. The projector is of the form

$$P_{(1,2)} = 2 \sum_{a,b,c,d=1}^{k} [h_{ab}(h^{-1})_{bc} h_{cd}(h^{-1})_{da}] \left( \frac{\bar{C}d\zeta_a + \bar{C}d\zeta_c - 2 \bar{C}d\zeta_c}{1 + \zeta_a\zeta_b} \right)$$ (3.12)

It is not difficult to see that the product $[h_{ab}(h^{-1})_{bc} h_{cd}(h^{-1})_{da}]$ goes to zero exponentially as $j \to \infty$, so $E[\zeta_a, \bar{\zeta}_b] \to 2 j k / (2 j + 1)$. In other words, the energy of the multi-soliton configuration is a constant independent of the locations of the solitons in the limit $j \to \infty$.

For studying the bound state of $k$ solitons (where $k < j$), one uses the basis $\{ |\zeta\rangle, J_+ |\zeta\rangle, \cdots, J_+^{k-1} |\zeta\rangle \}$, and calculates the Kähler potential (3.6). With a little work, it can be shown that

$$K(\zeta_a, \bar{\zeta}_b) = \ln(f(k,j)) + k(2j - (k - 1)) \ln(1 + |\zeta|^2),$$ (3.13)

where $f(k,j)$ is some function that is not relevant for the purpose of calculating the metric. We can calculate $g_{\alpha\beta}$ from the Kähler potential and see that the kinetic energy term

$$\frac{\lambda^2}{2j+1} \text{Tr}_{H^{(j)}} \Phi^{(k)^2} = \frac{\lambda^2 k(2j - (k - 1))}{2j+1} \frac{2|\zeta|^2}{(1 + |\zeta|^2)^2}$$ (3.14)

allows the interpretation of the bound state of $k$ solitons as a free particle of mass $\lambda^2 k(2j - (k - 1)) / (2j+1)$ moving on the $S^2$. For $k$ fixed and $j \to \infty$, the binding energy $\lambda^2 (k-1) / (2j+1)$ vanishes, and the mass of the bound state is the same as the mass of $k$ solitons of rank 1.

When $m^2 \neq \infty$, in general there are corrections to (3.12) coming from the potential term of (2.1) as well, which we have ignored in this discussion. We briefly comment on these leaving the details for future work. When $m^2$ is finite, fluctuations of the scalar field $\Phi$ need not be restricted to constant rank projectors. One expects such fluctuations to change the number of solitons and hence play an important role in the quantum field theory. Furthermore, interplay between the limits $j \to \infty, R \to \infty$ and $m^2 \to \infty$ hints at an interesting phase structure deserving future exploration.

4 Two-soliton case

Let us look at the case of the rank 2 projector in detail. The projector is of the form

$$P_{(1,2)} = \sum_{i,j=1}^{2} |\zeta_i\rangle (h^{-1})_{ij} \langle \zeta_j|$$ (4.1)

Corresponding to this projector is the function $\langle z|P^{(2)}|z\rangle / \langle z|z\rangle$, its covariant symbol. It is given by

$$P_{(1,2)}^{(2)}(z, \bar{z}) = \frac{1}{\text{det}(h)(1 + |\zeta|^2)^{2j}} \left[ (1 + \bar{z} \zeta_1)^{2j} (1 + \bar{\zeta}_1 z)^{2j} (1 + |\zeta|^2)^{2j} 
- (1 + \bar{z} \zeta_1)^{2j} (1 + \bar{\zeta}_1 z)^{2j} (1 + \bar{\zeta}_1 z)^{2j} (1 + \bar{\zeta}_1 z)^{2j} 
+ (1 + \bar{z} \zeta_1)^{2j} (1 + \bar{\zeta}_1 z)^{2j} (1 + |\zeta|^2)^{2j} \right]$$ (4.2)

where $\text{det}(h) = (1 + |\zeta|^2)^{2j} (1 + \bar{\zeta}_2 z)^{2j} (1 + \bar{\zeta}_2 z)^{2j} (1 + \bar{\zeta}_1 z)^{2j}$. A 2-dimensional plot of this function gives us an idea of how the two soliton configuration looks like, which is plotted in the following figure.
One of the solitons is placed at the north pole ($\zeta = 0$), while the other has spherical polar coordinates $(\theta, 0)$.

(a) Angular separation $\theta = \pi/2$.  
(b) Angular separation $\theta = \pi/4$.

(c) Angular separation $\theta = \pi/8$.  
(d) Angular separation $\theta = \pi/32$.

Two solitons on a sphere of radius 5, with $j = 60$. 


To study the energy \( E(\zeta_1, \zeta_2) \) of the configuration with solitons located at \( \zeta_1 \) and \( \zeta_2 \), it should be remembered that since \( P^{(2)}_{(\zeta_1, \zeta_2)} \) is completely symmetric under \( \zeta_1 \leftrightarrow \zeta_2 \), a good local coordinate for studying the coincident limit is \( x = (\zeta_1 - \zeta_2)^2 \). Without loss of generality, one of the solitons can be put at 0 and the other at \( \zeta \) to find \( E \) as a function of \( x \):

\[
E(x) = \lambda^2 \frac{4j^2}{2j+1} \left( 1 - \frac{2j(1 + x^2)}{(1 + x)[(1 + x)^{2j} - 1]^2} \right). \tag{4.3}
\]

This is bounded, and has a smooth limits as \( x \to 0 \) or \( \infty \):

\[
E(0) = \lambda^2 \frac{2(2j - 1)}{2j+1}, \quad E(\infty) = \lambda^2 \frac{4j}{2j+1}. \tag{4.4}
\]

The function \( E(x) \) is interesting because the \( x \to 0 \) and \( j \to \infty \) limits do not commute. As long as \( j \) is finite, \( E(0) \) is the global minimum. The potential is approximately constant everywhere else. As \( j \to \infty \), \( E(0) \) approaches the asymptotic value \( 4j\lambda^2/(2j+1) \) (We have plotted the behavior of the function \( E(x) \) in Fig. 1 for \( \lambda = 1 \)). Hence although the force between the solitons is attractive, it is extremely weak for large values of \( j \). The solitons move about freely on the sphere, oblivious of each other’s existence, unless they happen to pass very close to each other, in which case they attract to form a weak bound state. In fact, the binding energy between the solitons vanishes as \( j \to \infty \).

As observed earlier, the moduli space of two solitons is \( Gr(2, 2j + 1) \) which is reduced to \( \mathbb{C}P^2 \) because of the gradient term of the action. Using (3.7), one can calculate the metric \( g_{\alpha\bar{\beta}} \) on this

![Figure 1: Two-soliton energy \( E(x) \) for j=60, where \( x = |\zeta|^2 \).](image)
moduli space. The coordinates $\zeta_1, \zeta_2$ are unconventional for describing $\mathbb{C}P^2$, but are the natural choice in this context. The explicit form of the metric is not very illuminating, but has a simple form for $j \to \infty$:

$$g_{\zeta_1 \zeta_1} = \frac{1}{(1 + |\zeta_1|^2)^2}, \quad (4.5)$$

$$g_{\zeta_2 \zeta_2} = \frac{1}{(1 + |\zeta_2|^2)^2}, \quad (4.6)$$

$$g_{\zeta_1 \zeta_2} = g_{\zeta_2 \zeta_1} = 0. \quad (4.7)$$

These limits are derived for $\zeta_1 \neq \zeta_2$. To understand the limit of coincident solitons, we make a change of coordinates $y_1 = (\zeta_1 + \zeta_2), y_2 = (\zeta_1 - \zeta_2)^2$, and expand in the neighborhood of $y_2 = 0$. To order $|y_2|^2$, to find that:

$$\begin{align*}
g_{y_1 \bar{y}_1} &= \left(\frac{2j-1}{2j+1}\right) \frac{8}{(4 + |y_1|^2)^2}, \\
g_{y_2 \bar{y}_2} &= \left(\frac{2j-1}{2j+1}\right) \frac{8(3|y_1|^2 + 8j - 2)}{3(4 + |y_1|^2)^2}, \\
g_{y_1 \bar{y}_2} = g_{y_2 \bar{y}_1} &= 0. \quad (4.10)
\end{align*}$$

The $g_{y_1 \bar{y}_1}$ component of the metric simply tells us that the center-of-mass coordinate $y_1$ describes a sphere. The $g_{y_2 \bar{y}_2}$ component is non-vanishing, proving that the moduli space is indeed smooth at the point $y_2 = 0$.

The gradient term of (2.1) lifts the modulus $y_2$, and the bound state of two solitons is characterized by a single coordinate $\zeta$, and the corresponding projector $P^{(2)}_{(\zeta)}$ can be read off from (3.4) with $|\psi_1\rangle = |\zeta\rangle, |\psi_2\rangle = J_+|\zeta\rangle$. For this projector, one finds that the metric on the moduli space is

$$\frac{\lambda^2}{2j + 1} \text{Tr}_{H(s)} P^{(2)}_{(\zeta)} = \lambda^2 \left(\frac{2(2j-1)}{2j+1}\right) \frac{2|\zeta|^2}{(1 + |\zeta|^2)^2}. \quad (4.11)$$

This is a free particle of mass $2(2j - 1)\lambda^2/(2j + 1)$ on the sphere. The behavior of the many body interaction potential (3.12) is more intricate, but follows certain general features. First of all, the interaction between the solitons is weak for large $j$, and vanishes in the limit $j \to \infty$. For finite $j$, in addition to the global minimum corresponding to all the solitons being on top of each other, there are other extrema corresponding to the solitons clumping at two antipodal points on the sphere. More precisely, if there are $k$ solitons, then $k_1$ of these form a bound state at, say, the north pole, while the remaining $k - k_1$ form a bound state at the south pole. The global minimum of the potential function corresponds to all $k$ solitons sitting on top of each other to form a single bound state. The energy of this configuration is $k(2j - (k-1))\lambda^2/(2j+1)$. When $k_1$ are at the north pole and $k - k_1$ are at the south pole, the energy of the configuration is $k_1(2j - (k_1-1))\lambda^2/(2j+1) + (k-k_1)(2j - (k-k_1-1))\lambda^2/(2j+1)$. We also conjecture that there are extrema when the solitons are placed at the corners of regular solids that can be inscribed by a sphere.
5 The flattening limit

In the coherent state picture, it is very easy to flatten the fuzzy sphere into the noncommutative plane. Following [26], we simply perform the replacements

\[ J_+ = \sqrt{2ja}, \quad J_- = \sqrt{2ja}, \quad \zeta = (2j)^{-1/2} \alpha, \]

and take \( j \to \infty \). Here \( a, a^\dagger \) are the annihilation and creation operators of the harmonic oscillator algebra. Then, for example,

\[ |\zeta\rangle = e^{J_+ \zeta} | -j \rangle \rightarrow |\alpha\rangle = e^{\alpha a^\dagger} |0\rangle \]

Various quantities of interest can now be calculated in this limit. In particular, we get

\[ g_{\alpha_1 \bar{\alpha}_1} = \frac{1 - e^{-|\alpha_1 - \alpha_2|^2}(1 + |\alpha_1 - \alpha_2|^2)}{(1 - e^{-|\alpha_1 - \alpha_2|^2})^2}, \]

\[ g_{\alpha_1 \bar{\alpha}_2} = \frac{-e^{-2|\alpha_1 - \alpha_2|^2}(1 - e^{-|\alpha_1 - \alpha_2|^2}(1 - |\alpha_1 - \alpha_2|^2))}{(e^{-|\alpha_1 - \alpha_2|^2} - 1)^2}, \]

\[ g_{\alpha_2 \bar{\alpha}_2} = \frac{(1 - e^{-|\alpha_1 - \alpha_2|^2}(1 + |\alpha_1 - \alpha_2|^2))}{(1 - e^{-|\alpha_1 - \alpha_2|^2})^2} \]

Again, it is straightforward to see that the singularity at \( \alpha_1 = \alpha_2 \) is a fake one, and that the various components of the metric \( g \) have a smooth limit as \( \alpha_1 \to \alpha_2 \).

The \( |\alpha\rangle \) are the (non-normalized) standard coherent states of the Heisenberg-Weyl algebra that are used to study noncommutative solitons on the plane. It is thus obvious that in this limit, the results on \( S^2_F \) go over to the results on the noncommutative \( \mathbb{R}^2 \) as discussed in [4].

A small puzzle arises when one studies the two-soliton configuration in the limit of the noncommutative plane. In this limit, the corresponding function is

\[ \mathcal{P}^{(2)}_{(\alpha_1, \alpha_2)}(z, \bar{z}) = \frac{e^{-|z - \alpha_1|^2} + e^{-|z - \alpha_2|^2}}{1 - e^{-|\alpha_1 - \alpha_2|^2}} - \frac{e^{-(z - \alpha_1)(\bar{z} - \alpha_2)} + e^{-(z - \alpha_2)(\bar{z} - \alpha_1)}}{e^{|\alpha_1 - \alpha_2|^2} - 1}. \]

In particular for \( \alpha_1, \alpha_2 \to 0 \), \( \mathcal{P}^{(2)}_{(0,0)}(z, \bar{z}) \) has the form

\[ \mathcal{P}^{(2)}_{(0,0)}(z, \bar{z}) = (1 + |z|^2)e^{-|z|^2} \]

which is different (for small \( z \)) from the one obtained by [4]. The resolution is not difficult: the Moyal-Weyl transformation used in [4] corresponds to Weyl ordering, whereas the covariant symbol corresponds to normal ordering. It is well-known that functions corresponding to different ordering of operators match at large distances (which is also true in this case), but can differ for small distances (see for eg [24]). Hence we see that while operator ordering issues are not important while working at finite \( j \), they certainly become relevant in the noncommutative plane limit.

6 Outlook

Scalar theories on fuzzy \( S^2 \) admit finite energy configurations constructed from rank \( k \) projectors, which are localized lumps (i.e. solitons) of size \( \tan^{-1}(1/\sqrt{2j}) \). The low energy dynamics of these
solitons is described by motion on the Grassmannian $Gr(k, 2j + 1)$. Because the gradient term of the action provides a non-trivial potential on $Gr(k, 2j + 1)$, this moduli space is reduced. The solitons attract each other in general and the lowest energy configuration for finite $j$ corresponds to all the solitons being on top of each other. However, the attraction between the solitons vanishes in the limit of large $j$, as does the binding energy, suggesting that the solitons are like BPS particles. The reduced moduli space corresponding to the $k$-soliton configuration is $\mathbb{C}P^k$ and is smooth: the apparent singularity corresponding to the coalescence of several solitons is smoothed out by a different choice of basis in the Hilbert space $\mathcal{H}^{(j)}$.

The limit corresponding to the noncommutative plane reproduces the known results for the solitons on the noncommutative plane (modulo considerations related to various kinds of operator ordering). With hindsight, it should not be surprising that we have reproduced the results for the noncommutative plane starting from the fuzzy sphere. After all, $\mathbb{C}P^1$ looks locally like the complex plane $\mathbb{C}$. The moduli space of $k$ solitons on the noncommutative plane is $C^\otimes k / S_k \simeq C^k$ whereas that for solitons on the fuzzy sphere is $\mathbb{C}P^k$, which looks locally like $\mathbb{C}^k$. It thus seems that the noncommutative sphere is an excellent candidate for the ultraviolet (and simultaneously infrared!) regularization of the noncommutative plane.

The construction of multi-soliton configurations on other fuzzy manifolds (like the ones discussed in [3, 27, 28]) remains an open question. In particular, it would be interesting to see if some of the features that we have discovered continue to hold; in particular, whether the force between solitons vanishes in the continuum limit.

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