Abstract

Suppose that two compact manifolds $X, X'$ are connected by a sequence of Mukai flops. In this paper, we construct a ring isomorphism between cohomology ring of $X$ and $X'$. Using the localization technique, we prove that the quantum corrected products on $X, X'$ are the ordinary intersection products. Furthermore, $X, X'$ have isomorphic Ruan cohomology. i.e. we proved the cohomological minimal model conjecture proposed by Ruan.

Keywords: Mukai Flop, Ruan cohomology, quantum cohomology, localization

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1 Introduction

After the mathematical foundation of quantum cohomology was established during last decade, see [RT1], now the focus of the research is on its computation and application. We think that the fundamental problem in quantum cohomology is the quantum naturality problem[R1, R2, H]: Define “morphism” of symplectic manifolds so that quantum cohomology is natural. Qin and Ruan [QR] showed that the quantum cohomology is not natural for fibrations. Their results also shows that possible “morphism” must be very rigid. The existence of these rigid morphisms will set apart quantum cohomology from ordinary cohomology and gives it its own identity. Although this result let us feel depressed, the result of [LR] discovers some amazing relations between quantum cohomology and birational geometry. Their result said that threefolds which are connected by a sequence of flops have isomorphic quantum cohomology. This gives us some suggestions to look for the suitable “morphism” from some birational transformations.

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In the study of higher dimensional algebraic geometry, the famous “minimal model program” initiated by Mori is one of the main research topics. So far the existence problem of minimal models is still completely open in dimensions higher than three. Moreover, in contrast to the two dimensional case, the minimal model is not unique in higher dimensions. It is then an important question to see what kind of invariants are shared by all the birationally equivalent minimal models, and more generally, are preserved under certain elementary birational transformations.

It is well-known that the crepant resolutions are not unique in dimensions higher than three. But Wang [W] showed that the different crepant resolutions are connected by “K-equivalence”. Two smooth complex manifolds $X, Y$ are K-equivalent if and only if there is a common resolution $\phi : Z \to X$ and $\varphi : Z \to Y$ such that $\phi^*K_X = \varphi^*K_Y$. Batyrev [B] and Wang [W] showed that two K-equivalent projective manifolds have the same betti number. It is natural to ask if they have the same cohomology ring structure. Unfortunately, they usually have different ring structures. About this problem, Wang [W, W1] proposed his following conjecture:

**Wang’s conjecture:** For K-equivalent manifolds under birational map $f : X \cdots \to X'$, there is a naturally attached correspondence $T \in A^\dim X(X \times X')$ of the form $T = \Gamma_f + \sum_i T_i$ with $\Gamma_f \subset X \times X'$ the cycle of graph closure of $f$ and with $T_i$’s being certain degenerate correspondences (i. e. $T_i$ has positive dimensional fibers when projecting to $X$ or $X'$) such that $T$ is an isomorphism of Chow motives.

In other words, Wang’s conjecture implies that for K-equivalent manifolds $X$ and $X'$, the canonical morphism $\varphi_*\phi^* : H^*(X, Q) \to H^*(X', Q)$ gives rise to an isomorphism with some modification in the middle dimension. In the case of hyperkähler manifolds, using Bishop’s theorem [Bi], Huybrechts [Huy1, Huy2] proved this conjecture by showing the existence of the correction cycles $T_i$. In this paper, for arbitrary projective manifolds connected by Mokai flops, we proved that $\varphi_*\phi^*$ gives rise to an isomorphism of cohomology rings of $X, X'$ with an explicit expression of the correction cycles $T_i$ (see the definition of the map $T$ in section 3).

We will concentrated our attention on a special kind of birational transformations—Mukai Flops [Mukai]. Here we first recall the definition of certain known flops. The simplest type of flops are called ordinary flops. An ordinary $\mathbb{P}^r$-flop (or simply $\mathbb{P}^r$-flop) $f : X \to X'$ is a birational map such that the exceptional set $Z \subset X$ has a $\mathbb{P}^r$-bundle structure $\varphi : Z \to S$ over some smooth variety $S$ and the normal bundle $N_{Z/X}$ is isomorphic to $\mathcal{O}(-1)^{r+1}$ when restricting to any fiber of $\varphi$. The map $f$ and the space $X'$ are then obtained by first blowing up $X$ along $Z$ to get $Y$, with exceptional divisor $E$ a $\mathbb{P}^r \times \mathbb{P}^r$-bundle over $S$, then blowing down $E$ along another fiber direction. Ordinary $\mathbb{P}^r$-flops are also called classical flops. Three dimensional classical flops are the most well-known Atiyah flops over $(-1, -1)$ rational curves.

Another important example is the Mukai flops $f : X \to X'$. In this case it is required that the exceptional set $Z \subset X$ is of codimension $r$ and has a $\mathbb{P}^r$-bundle structure $\varphi : Z = \mathbb{P}_S(F) \to S$ (for some rank $r + 1$ vector bundle $F$) over a
smooth base $S$, moreover the normal bundle $N_{Z/X} \equiv T_{Z/S}^*$, the relative cotangent bundle of $\varphi$. To get $f$, one first blows up $X$ along $Z$ to get $\phi : Y \longrightarrow X$ with exceptional divisor $E = P_S(T^*_{Z/S}) \subset P_Z(F) \times P_S(F^*)$ as the incidence variety. The first projection corresponds to $\phi$ and one may contract $E$ through the second projection to get $\phi' : Y \longrightarrow X'$.

In this paper, we will only consider the following simplest Mukai flops:

**Definition:** Let $X$ be a projective manifold of complex dimension $2n$. A Mukai flop from $(X, Z)$ to $(X', Z')$ is the following birational transformation

\[
E \subset Y \quad \overset{\phi}{\leftarrow} \quad \varphi
\]

\[
Z \cong \mathbb{P}^n \subset X \quad \cdots \quad \longrightarrow \quad X' \supset Z' \cong (\mathbb{P}^n)^*\]

where $E$ is the incidence correspondence between $Z$ and $Z'$. We also call $X$ and $X'$ are connected by a **Mukai flop**.

Throughout this paper, we will call this simplest Mukai flops as Mukai flops.

In the study of birational geometry, one of the most important problems is to find that what kind of cohomology is preserved by $K$-equivalent. For this purpose, Ruan [R3] proposed

**Quantum Minimal Model Conjecture:** Two $K$-equivalent projective manifolds have the same quantum cohomology.

In dimensions higher than three, quantum minimal model conjecture seems to be a difficult problem. We think the difficulty comes from the fact we used all quantum information involving the GW-invariants. So Ruan proposed that we should consider another kind of cohomology with a minimal set of quantum information involving the GW-invariants of exceptional rational curves. We call this new cohomology as Ruan Cohomology, and will give its definition in section 2. In section 4, we will also prove Ruan cohomology is invariant under Mukai flops.

Our main theorem in this paper is

**Theorem:** Two compact projective manifolds which are connected by a sequence of Mukai flops have isomorphic cohomology and Ruan cohomology.

We will divide the proof of the theorem into two cases: ordinary cohomology and Ruan cohomology. In section 3, we will prove that $X, X'$ have isomorphic cohomology, see Theorem 3.2. In section 4, we will prove that for $X, X'$ the quantum correction all vanish. So they have isomorphic Ruan cohomology, see Theorem 4.4.

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2 Ruan Cohomology

In [R3], Ruan defined his quantum corrected cohomology with respect to a birational map. Suppose that \( X, X' \) are K-equivalent and \( \pi : X \rightarrow X' \) is the birational map. Let \( A_1, \ldots, A_k \) be an integral basis of the homology classes of exceptional effective curves. We call \( \pi \) nondegenerate if \( A_1, \ldots, A_k \) are linearly independent. Then the homology class of any exceptional effective curve can be written as \( A = \sum_i a_i A_i \) for \( a_i \geq 0 \). For each \( A_i \), we assign a formal variable \( q_i \). Then \( A \) corresponds to \( q_{a_1} \cdots q_{a_k} \). We define a 3-point function

\[
\langle \alpha, \beta, \gamma \rangle_{qc} (q_1, \ldots, q_k) = \sum_{a_1, \ldots, a_k} \Psi_X^A(\alpha, \beta, \gamma) q_1^{a_1} \cdots q_k^{a_k},
\]

(1)

where \( \Psi_X^A(\alpha, \beta, \gamma) \) is Gromov-Witten invariant and \( qc \) stands for the quantum correction and \( \alpha, \beta, \gamma \in H^*X \). We view \( \langle \alpha, \beta, \gamma \rangle_{qc} (q_1, \ldots, q_k) \) as analytic function of \( q_1, \ldots, q_k \) and set \( q_i = -1 \) and let

\[
\langle \alpha, \beta, \gamma \rangle_{qc} = \langle \alpha, \beta, \gamma \rangle_{qc} (-1, \ldots, -1).
\]

(2)

We define a quantum corrected triple intersection

\[
\langle \alpha, \beta, \gamma \rangle_{\pi} = \langle \alpha, \beta, \gamma \rangle + \langle \alpha, \beta, \gamma \rangle_{qc},
\]

where \( \langle \alpha, \beta, \gamma \rangle = \int_X \alpha \cup \beta \cup \gamma \) is the ordinary triple intersection. Then we define the quantum corrected product \( \alpha *_{\pi} \beta \) by the equation

\[
\langle \alpha *_{\pi} \beta, \gamma \rangle = \langle \alpha, \beta, \gamma \rangle_{\pi}
\]

for arbitrary \( \gamma \). Another way to understand \( \alpha *_{\pi} \beta \) is as follows. Define a product as the ordinary intersection product corrected by \( \alpha *_{qc} \beta \). Namely,

\[
\alpha *_{\pi} \beta = \alpha \cup \beta + \alpha *_{qc} \beta.
\]

(3)

It is easy to see that the quantum corrected product gives rise to a ring structure on the cohomology group of \( X \). Denote this cohomology ring as \( RH^*_\pi(X, C) \).

**Definition 2.1:** Define the quantum corrected cohomology ring \( RH^*_\pi(X, C) \) as Ruan cohomology of \( X \).

Ruan computed some examples of his cohomology in [R3, R4]. About this cohomology, Ruan [R3] proposed the following conjecture

**Cohomological Minimal Model Conjecture:** Suppose that \( \pi : X \rightarrow X' \) and its inverse \( \pi^{-1} \) are nondegenerate. Then \( RH^*_\pi(X, C) \) is isomorphic to \( RH^*_{\pi^{-1}}(X', C) \).

**Example 2.2:** The first example is the flop in dimension three. This case has been worked out in great detail by Li-Ruan[LR]. For example, they proved a theorem...
that quantum cohomology rings are isomorphic under the change of the variable \( q \to \frac{1}{q} \). Notes that if we set \( q = -1 \), \( \frac{1}{q} = -1 \). We set other quantum variables zero. Then, the quantum product becomes the quantum corrected product \( \alpha \cup_{\pi} \beta \). Hence, Cohomological Minimal Model conjecture follows from Li-Ruan’s theorem. In fact, it is easy to calculate the quantum corrected product in this case and verify the Cohomological Minimal Model conjecture without using Li-Ruan’s theorem.

3 Isomorphism of ordinary cohomology

In this section, we will consider the cohomology of compact projective manifolds of complex dimension \( 2n \) connected by Mukai flops. Suppose that \( X \) and \( X' \) are compact projective manifolds of complex dimension \( 2n \), and \( (X, \mathbb{P}^n) \) and \( (X', (\mathbb{P}^n)^*) \) are connected by a Mukai flop. Now the normal bundle of \( \mathbb{P}^n \) in \( X \) is its cotangent bundle \( T^*\mathbb{P}^n \). So we have the following Mukai transformation

\[
\begin{align*}
E & \subset \tilde{X} \\
\phi & \searrow \phi' \\
Z & \cong \mathbb{P}^n \subset X \\
\longrightarrow & \\
X' & \supset (\mathbb{P}^n)^* \cong Z'
\end{align*}
\]

where \( \tilde{X} \) is the blowup of \( X \) along \( Z = \mathbb{P}^n \) and \( E \) is the incidence correspondence between \( Z \) and \( Z' \), i.e.

\[
E = \{(P, L) \mid P \in L\} \subset \mathbb{P}^n \times (\mathbb{P}^n)^*
\]

Before we prove our theorem, we want to first introduce some notations and preliminary results. Let \( X \) be a regularly embedded subscheme of a scheme \( Y \) of codimension \( d \) with normal bundle \( N \). Let \( A_k(X) \) be the group of \( k \)-cycles modulo rational equivalence on \( X \). Denote by \( s(X, Y) \in A_*(X) \) the Segre class of \( X \) in \( Y \), for its definition see Section 4.2 of [F], so \( s(X, Y) \) is the cap product of the total inverse Chern class of the normal bundle with \([X]\). Let \( \tilde{Y} \) be the blowup of \( Y \) along \( X \), and let \( \tilde{X} = \mathbb{P}(N) \) be the exceptional divisor. We have a fiber square

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{j} & \tilde{Y} \\
g \downarrow & & \downarrow f \\
X & \xrightarrow{i} & Y.
\end{array}
\]

Since \( N_{\tilde{X}}\tilde{Y} = \mathcal{O}(-1) \), the excess normal bundle \( \xi \) is the universal quotient bundle on \( \mathbb{P}(N) \):

\[
\xi = g^*N/N_{\tilde{X}}\tilde{Y} = g^*N/\mathcal{O}(-1).
\]

Then we have the following Blowup formula, which is the Theorem 6.7, see P. 116, of [F],
Proposition 3.1: Let $V$ be a $k$-dimensional subvariety of $Y$, and let $\tilde{V} \subset \tilde{Y}$ be the proper transform of $V$, i.e. the blow-up of $V$ along $V \cap X$. Then

$$f^*[V] = [\tilde{V}] + j_*\{c(\xi) \cap g^*(V \cap X, V)\}_k$$

in $A_k\tilde{Y}$. In particular, for all $x \in A_kX$,

$$f^*i_*(x) = j_*(c_{d-1}(E) \cap g^*x).$$

In our proof, we will use Borel-Moore homology as a tool. Therefore we first want to briefly introduce some basics of Borel-Moore homology, see [CG, F]. Borel-Moore homology can be defined using singular cohomology. If a space $X$ is imbedded as a closed subspace of $\mathbb{R}^n$, then we define the Borel-Moore homology with rational coefficients

$$H_{BM}^iX := H^{n-i}(\mathbb{R}^n, \mathbb{R}^n - X)$$

where the group on the right is relative singular cohomology with rational coefficients. From the definition, it is easy to know if $X$ is compact then the ordinary homology of $X$ and the Borel-Moore homology of $X$ coincide. In this paper, we will reserve the symbol $H_*$ for the ordinary homology.

If $X$ is the complement of $U$ in $Y$, $i : X \rightarrow Y$ the closed imbedding, there is a long exact sequence

$$\cdots \rightarrow H_{BM}^{i+1}U \rightarrow H_{BM}^iX \rightarrow H_{BM}^iY \rightarrow H_{BM}^iU \rightarrow H_{BM}^{i-1}X \rightarrow \cdots. \quad (6)$$

In this section, we will prove the following theorem

Theorem 3.2: Suppose that non-singular compact projective manifolds $X$ and $X'$ of complex dimension $2n$ are connected by a sequence of Mukai flops. Then $X$ and $X'$ have isomorphic cohomology rings.

Proof: By the Poincare duality, it is sufficient to prove that $X$ and $X'$ have isomorphic intersection rings. In fact, we want to prove the following morphism $T : H_*X \rightarrow H_*X'$ given by

$$T(\alpha) := \begin{cases} \phi^*\phi^*\alpha, & \text{if dim } \alpha \neq 2n \\ \phi^*(\phi^*\alpha + (-1)^n[-\alpha(P^n)[p^{-1}(\mathbb{P}^1)]], & \text{if dim } \alpha = 2n \end{cases}$$

is a ring isomorphism, where $\alpha(P^n)$ is the topological intersection number of $\alpha$ with $\mathbb{P}^n$ and $\mathbb{P}^1$ is a line in $\mathbb{P}^n$. It is obvious that $T$ is a linear map.

First of all, we want to prove that the restriction of $T$ to $i_*H_k(P^n)$ is an isomorphism from $i_*H_k(P^n)$ to $i'_*H_k((P^n)^*)$. By the linearity of $T$, we only need to prove that $T$ maps a basis of $i_*H_*(P^n)$ to a basis of $i'_*H_((P^n)^*)$. Since all elements in $i_*H_*(P^n)$ are algebraic, so we may apply proposition 3.1. In our case, we have the
following blowup fiber square

\[
\begin{array}{ccc}
E & \xrightarrow{j} & \tilde{X} \\
p \downarrow & & \downarrow \phi \\
\mathbb{P}^n & \xrightarrow{i} & X.
\end{array}
\] (7)

where \(i\) embedded \(\mathbb{P}^n\) into \(X\) with its cotangent bundle \(N_{\mathbb{P}^n|X} \cong T^*\mathbb{P}^n\) as the normal bundle and \(E\) is the exceptional divisor. The excess normal bundle \(Q\) is the universal quotient bundle on \(E\)

\[
Q = \frac{p^*T^*\mathbb{P}^n}{\mathcal{O}_E(-1)}
\]

i.e. we have the exact sequence

\[
0 \rightarrow \mathcal{O}_E(-1) \rightarrow p^*T^*\mathbb{P}^n \rightarrow Q \rightarrow 0
\]

According to Proposition 3.1, we need to compute the Chern class \(c_{n-1}(Q)\). Since \(c(p^*T^*\mathbb{P}^n) = c(Q)c(\mathcal{O}_E(-1))\), so we have

\[
c(Q) = \frac{c(p^*T^*\mathbb{P}^n)}{c(\mathcal{O}_E(-1))} = \sum_{k=0}^{2n-1} \sum_{i+j=k} (-1)^i \binom{n+1}{i} (p^*H)^i c_1(\mathcal{O}_E(1))^j
\]

where \(H\) is the hyperplane class of \(\mathbb{P}^n\). Therefore

\[
c_{n-1}(Q) = \sum_{i+j=n-1} (-1)^i \binom{n+1}{i} (p^*H)^i c_1(\mathcal{O}_E(1))^j
\]

\[
= \sum_{i=0}^{n-1} \sum_{j=0}^{n-i-1} (-1)^i \binom{n+1}{i} \binom{n-i-1}{j} (q^*H^*)^{n-i-j-1} (p^*H)^{i+j}
\]

where \(H^*\) is the hyperplane class of \((\mathbb{P}^n)^*\) and we used that \(c_1(\mathcal{O}_E(1)) = p^*H + q^*H^*\).

Choose \(i_*[\mathbb{P}^k], k = 0, \cdots, n\) as a basis of \(i_*H_*(\mathbb{P}^n)\). For arbitrary \(1 \leq k < n\), i.e. \(x = i_*[\mathbb{P}^k] \in i_*H_*(\mathbb{P}^n)\), by Proposition 3.1, we have

\[
\phi^*(x) = j_*\left\{ \sum_{i=0}^{n-1} \sum_{j=0}^{n-i-1} (-1)^i \binom{n+1}{i} \binom{n-i-1}{j} (q^*H^*)^{n-i-j-1} (p^*H)^{i+j} \cap p^*[\mathbb{P}^k] \right\}
\]

\[
= j_*\left\{ \sum_{i=0}^{n-1} \sum_{j=0}^{n-i-1} (-1)^i \binom{n+1}{i} \binom{n-i-1}{j} (q^*H^*)^{n-i-j-1} \cap p^*(H^{i+j} \cap [\mathbb{P}^k]) \right\}
\]

Therefore, we have

\[
\phi_*\phi^*(i_*[\mathbb{P}^k]) = \sum_{i=0}^{k} (-1)^i \binom{n+1}{i} \binom{n-i}{k-i} i_*^\prime([\mathbb{P}^k]^*])
\]

\[
= (-1)^k i_*^\prime([\mathbb{P}^k]^*])
\] (8)
where we used the facts that for any $k \geq 2$ the maps $\phi' : p^*[P^k] \longrightarrow (P^n)^*$ have positive dimensional fibers.

For the case $k = n$, i.e. $x = i_* (P^n) \in i_* H_*(P^n)$, we have

$$\phi^*(i_*[P^n]) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-i-1} (-1)^i \left( \begin{array}{c} n+1 \\ i \end{array} \right) \left( \begin{array}{c} n-i-1 \\ j \end{array} \right) (q^*H^*)_k \otimes (p^*H)^{i+j} \cap p^*[P^n]$$

$$= \sum_{i=0}^{n-1} (-1)^i \left( \begin{array}{c} n+1 \\ i \end{array} \right) p^*[P^1]$$

$$= (-1)^{n+1} np^*[P^1].$$

Therefore,

$$\phi'_* \phi^*(i_*[P^n]) = (-1)^{n+1} n \phi'_* (p^*[P^1]) = (-1)^{n+1} n i'_* (P^n)^*.$$

Furthermore, by the definition of the map $T$, we have

$$T(i_*[P^n]) = (-1)^{n} i'_* (P^n)^*.$$  

Since $i'_* (P^n)^*$, $k = 0, \cdots, n$ is a basis of $i'_* H_*(P^n)^*$, so the restriction of $T$ to $i_* H_*(P^n)$ is an isomorphism from $i_* H_*(P^n)$ to $i'_* H_*(P^n)^*$.

Next we want to prove that $T$ is an isomorphism of additive homology.

Denote $U := X - P^n$ and $U' := X' - (P^n)^*$. Since $H^{BM}_k P^n$ has at most one generator for all $i$, then, from (6), we have the following exact sequences:

$$0 \longrightarrow i_* H^BM_k P^n \longrightarrow H^BM_k P^n \longrightarrow 0 \longrightarrow 0 \longrightarrow i_* H^BM_k (P^n)^* \longrightarrow H^BM_k (P^n)^* \longrightarrow 0.$$  

Since $H^BM_k P^n$, $H^BM_k P^n$, $H^BM_k U$, $H^BM_k (P^n)^*$, $H^BM_k X'$, $H^BM_k U'$ all are free Abelian groups, so we have

$$H^BM_k X \cong i_* H^BM_k P^n \oplus H^BM_k U$$  

$$H^BM_k X' \cong i_* H^BM_k (P^n)^* \oplus H^BM_k U'.$$

Here we used the following elementary fact from extension theory:

**Proposition 3.3:** ([Bott], P. 168) In a short exact sequence of Abelian groups

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

if $A$ and $C$ are free, then $B \cong A \oplus C$.

In fact, the previous proof shows that the restriction of $T$ to $i_* H^BM_k (P^n)$ is an isomorphism from $i_* H^BM_k (P^n)$ to $i'_* H^BM_k ((P^n)^*)$. On the other hand, since $\phi$ and $\phi'$ are the identity map outside $P^n$ and $(P^n)^*$ respectively, i.e. $U \cong U'$, the
restriction of $T$ to $H_{BM}^k U$ is also an isomorphism from $H_{BM}^k U$ to $H_{BM}^k U'$. By the linearity of $T$, from (13) and (14), we have that $T$ is an isomorphism from $H_{BM}^k X$ to $H_{BM}^k X'$ as additive groups. Since $X$ and $X'$ are compact, Therefore, $T$ also gives an isomorphism from the ordinary homology $H_k X$ to $H_k X'$ as additive groups.

Now it remains to prove that $T$ preserves the multiplication, i.e. for any classes $\alpha, \beta \in H_* X$, we have

$$T(\alpha \cdot \beta) = T(\alpha) \cdot T(\beta). \tag{15}$$

By the transversality theorem, for any homology classes $\alpha, \beta$, we may choose their representatives $M(\alpha)$ and $N(\beta)$ respectively such that they transversally intersect, i.e. $\dim(M \cap N) = \dim M + \dim N - 4n$.

In the rest proof of this theorem, we will use the same symbol to denote the homology class and its representatives.

Since $T$ is linear and the intersection product is distributive, we only need to prove (15) holds for generator classes. From the fact that the intersection product is a map from $H_k X \otimes H_1 X$ to $H_{k+1-4n} X$, we know that (15) holds if $\dim \alpha + \dim \beta < 4n$.

Therefore, we may assume that $\dim \beta \geq 2n$. Since $U := X - Z$ is isomorphic to $U' := X' - Z'$, we have that the map $T$ is the identity map on $H_*(X - Z)$. Therefore, if at least one of the supports of $\alpha, \beta$ does not intersect with $P^n$, then (15) holds. Therefore, we only need to consider the following four cases.

\textbf{Case I: dim} $\alpha < 2n$, $\beta$ is an arbitrary class.

In this case, we may choose a representative submanifold $\alpha$ with support away from $P^n$. Therefore, by the construction of the intersection product and the fact that $T$ is an identity map from $H_*(U)$ to $H_*(U')$, we have

$$T(\alpha \cdot \beta) = T(\alpha) \cdot T(\beta).$$

\textbf{Case II: dim} $\alpha = 2n$ and $\dim \beta = 2n$.

From (13) and the distributivity of intersection product, we only need to consider the case: $\alpha = i_*(P^n)$ and $\beta = i_*(P^n)$. In this case, we have

$$T(\alpha \cdot \beta) = T(-(n+1)[pt]) = -(n+1)[pt] = i_*(P^n)^* \cdot i_*(P^n)^* = T(\alpha) \cdot T(\beta).$$

\textbf{Case III: dim} $\alpha > 2n$ and $\dim \beta > 2n$.

Here we first prove the following claim:

\textbf{Claim:} If $\phi : \tilde{X} \to X$ is the blowup of $X$ along a subvariety, then $\phi^* \alpha \cdot \phi^* \beta = \phi^*(\alpha \cdot \beta)$ for any classes $\alpha, \beta \in H_* X$.

In fact, by definition, we have

$$\phi^*(\alpha \cdot \beta) = PD\phi^*PD(\alpha \cdot \beta) = PD\phi^*(PD(\alpha) \cup PD(\beta)) = PD(\phi^*PD(\alpha) \cup \phi^*PD(\beta)) = PD\phi^*PD(\alpha) \cdot PD\phi^*PD(\beta) = \phi^* \alpha \cdot \phi^* \beta$$
where \(PD\) stands for Poincaré dual.

Since \(\phi': \tilde{X}' \to X'\) is the projection of blowup, so we have \(\phi'\phi^*\alpha = \alpha\) for any \(\alpha \in H_*X'\). From the definition of \(T\), we have \(\phi'^*T(\alpha) = \phi^*\alpha + \xi, \phi'^*T(\beta) = \phi^*\beta + \eta\) where \(\phi'_*\xi = \phi'_*\eta = 0\), i.e. \(\phi'\mid_\xi\) and \(\phi'\mid_\eta\) have positive dimensional fiber. Therefore, if \(\dim(\alpha \cdot \beta) \neq 2n\), from the above claim and the projection formula, we have

\[
T(\alpha \cdot \beta) = \phi'_*\phi^*(\alpha \cdot \beta) = \phi'_*\{\phi^*\alpha \cdot \phi^*\beta\}
\]

\[
= \phi'_*\{\phi'^*T(\alpha) \cdot \phi'^*T(\beta) - \phi'^*T(\alpha) \cdot \eta - \phi'^*T(\beta) \cdot \xi + \xi \cdot \eta\}
\]

\[
= T(\alpha) \cdot T(\beta).
\]

If \(\dim(\alpha \cdot \beta) = 2n\), i.e., \(\dim \alpha + \dim \beta = 5n\), without loss of generality, we may assume that \(\alpha \cdot \beta = ki_*P^n\) and \(\dim \beta < 4n\). By the definition of \(T\) and the intersection product, we also may assume that \(T(\alpha) \cdot T(\beta) = mi'_*(P^n)^*\). Choose a \(l\)-dimensional class \(\gamma\) where \(l\) satisfies \(\dim \beta + l - 4n < 2n\) and \(l < 2n\). Then from the associativity of the intersection product and Case I, we have the triple intersection equality.

\[
T(\alpha \cdot \beta \cdot \gamma) = T(\alpha) \cdot T(\beta \cdot \gamma) = T(\alpha) \cdot T(\beta) \cdot T(\gamma).
\]

Since \(T(\alpha \cdot \beta \cdot \gamma) = T((\alpha \cdot \beta) \cdot \gamma) = T(\alpha \cdot \beta) \cdot T(\gamma) = (-1)^n ki'_*(P^n)^* \cdot T(\gamma)\) and \(T(\alpha) \cdot T(\beta) \cdot T(\gamma) = mi'_*(P^n)^* \cdot T(\gamma)\), so we have \(m = (-1)^n k\). Therefore (15) holds.

**Case IV:** \(\alpha = i_*P^n, \dim \beta > 2n\) and \(\beta\) transversely intersects with \(P^n\).

Since all odd-dimensional classes in \(P^n\) are homologous to zero, without loss of generality, we may assume that \(\dim \beta\) is even. Suppose that \(\gamma\) is any \((6n - \dim \beta)\)-dimensional class in \(H_*X\). Then the intersection product \(\beta \cdot \gamma\) is a \(2n\)-dimensional class in \(H_{2n}X\). From the associativity of the intersection product and Case II and III, we have the triple intersection equality

\[
T(\alpha \cdot \beta \cdot \gamma) = T(\alpha) \cdot T(\beta \cdot \gamma) = T(\alpha) \cdot T(\beta) \cdot T(\gamma).
\]

Suppose that \(P^n \cdot \beta = mi_*[P^{\dim \beta - n}]\) and \((P^n)^* \cdot T(\beta) = ki'_*(P^{\dim \beta - n})^*\). Then by Case I we have

\[
T(\alpha \cdot \beta \cdot \gamma) = mT(i_*[P^{\dim \beta - n}] \cdot \gamma)
\]

\[
= mT(i_*[P^{\dim \beta - n}]) \cdot T(\gamma) = (-1)^{\dim \beta - n} mi'_*[P^{\dim \beta - n}]^* \cdot T(\gamma).
\]

On the other hand,

\[
T(\alpha) \cdot T(\beta) \cdot T(\gamma) = ki'_*[P^{\dim \beta - n}]^* \cdot T(\gamma).
\]

Therefore we have \(m = (-1)^{\dim \beta - n} k\). Therefore

\[
T(\alpha \cdot \beta) = mT(i_*[P^{\dim \beta - n}]) = (-1)^{\dim \beta - n} kT(i_*[P^{\dim \beta - n}])
\]

\[
= ki'_*[P^{\dim \beta - n}]^* = (P^n)^* \cdot T(\beta) = T(\alpha) \cdot T(\beta).
\]

So we proved the equality (15). This proves Theorem 3.2.
4 Isomorphism of Ruan Cohomology

In this section, we will study Ruan cohomologies of $X$ and $X'$. From the previous section, we know that in order to prove isomorphism of Ruan cohomology for the pair $X$ and $X'$, we need to calculate the quantum corrected product coming from exceptional effective curves on $X$ and $X'$ respectively. In fact, we will prove vanishing of the exceptional Gromov-Witten invariants appearing in the definition of quantum corrected product by localization technique.

4.1 Introduction to Localization

The calculation of the exceptional quantum product is local in nature, i.e. only a neighborhood of the embedded $P^n$ in $X$ or $X'$ is relevant to the quantum product with base homology being exceptional curves living in the embedded $P^n$. Similar local invariants appeared in the study of local mirror symmetry. As explained in [CKYZ], local mirror symmetry refers to a specialization of mirror symmetry technique to study geometry of Fano surfaces inside Calabi-Yau manifolds.

Following [CKYZ], we first briefly describe the calculation setup. Let $\overline{\mathcal{M}}_{0,0}(P,d)$ be Kontsevich’s moduli space of stable maps of genus 0 (could be of higher genus) with no marked points. Denote a point in the space by $(C,f)$, where $f: C \to P$ ($P$ is some toric variety), and $[f(C)] = d \in H_2(P)$. Let $\overline{\mathcal{M}}_{0,1}(P,d)$ be the same but with one marked point. Consider the following diagram

$$\overline{\mathcal{M}}_{0,0}(P,d) \leftarrow \overline{\mathcal{M}}_{0,1}(P,d) \to P,$$

where the first arrow denotes the forgetting map $\rho: \overline{\mathcal{M}}_{0,1}(P,d) \to \overline{\mathcal{M}}_{0,0}(P,d)$ which forgets the marked point following stabilization of the domain curve and the second arrow denotes the evaluation map $ev: \overline{\mathcal{M}}_{0,1}(P,d) \to P$ sending $(C,f,x_1)$ to $f(x_1)$.

Let $Q$ be Calabi-Yau defined as the zero section of a convex bundle $V$ over $P$ (here convex means $H^1(C,f^*V) = 0$ for any stable map $(C,f)$). Then $U_d$ is the bundle over $\overline{\mathcal{M}}_{0,0}(P,d)$ defined by

$$U_d := \rho_* ev^*(V).$$

The fiber of $U_d$ over a point $(C,f)$ is $H^0(C,f^*V)$. And the Kontsevich numbers (Gromov-Witten type invariant) are defined to be

$$K_d := \int_{\overline{\mathcal{M}}_{0,0}(P,d)} c(U_d)$$

where $c$ is the appropriate Chern class in the context.

In case the bundle $V$ is also concave (meaning $H^0(C,f^*V) = 0$ for any stable map $(C,f)$), there is also an induced bundle over the moduli space of maps whose
fiber over a point \((C,f)\) is given by \(H^1(C,f^*V)\). In particular if \(V\) is the normal bundle of \(P\) with respect to certain embedding of \(P\), the induced bundle is usually called the obstruction bundle.

In the same spirit of the above setup, there is another well known example (the multiple cover contribution) which we now describe.

Let \(C_0 = P^1\) be a smooth \(P^1\) embedded in a Calabi-Yau 3-fold \(M\) with balanced normal bundle \(\mathcal{O}(-1) \oplus \mathcal{O}(-1)\). The moduli space of stable maps \(\overline{\mathcal{M}}_{0,0}(M,d[C_0])\) has a connected component \(\overline{\mathcal{M}}_{C_0}\) isomorphic to \(\overline{\mathcal{M}}\left(P^1,d|P^1\right)\) consisting stable \(d\)-fold covers of \(C_0\). This component has dimension \(2d - 2\) while the virtual dimension is 0. So to correctly count the number of maps (or to define the corresponding Gromov-Witten invariant), we have to consider the obstruction bundle \(U_d\) whose fiber over \((C,f)\) is given by \(H^1(C,f^*N_{C_0}|M) = C^2 \otimes H^1(C,f^*\mathcal{O}(-1))\). Note that the rank of the obstruction bundle is also \(2d - 2\). And the contribution of \(\overline{\mathcal{M}}_{C_0}\) is given by

\[
M_d := \int_{\overline{\mathcal{M}}\left(P^1,d\right)} c_{2d-2}(U_d).
\]

The above definition is proposed by Kontsevich who also derived a graph summation formula for it. And the value is checked by Y. Manin to be \(\frac{1}{d!}\). (there is difficulty in summing up all the contributions from admissible graphs).

The essence in both examples described above is to determine and evaluate certain cohomology class (over the space of stable maps) which come from bundles induced from bundles over the target space. And solutions to both problems come out of application of localization techniques. Since the target space is toric, the moduli space of maps together with the induced bundles inherit torus action (action on space of maps by translating maps). Hence the classes under consideration can be localized to the fixed points loci and become much more accessible.

In [CKYZ], the authors considered the cases where the bundle \(V\) is a direct sum of line bundles, while in this paper we will consider the case where the target space is \(P^n\) and the bundle \(V\) is the cotangent bundle of \(P^n\) which is a natural example of concave bundles. It is of interest also because it demonstrate rather different phenomena from the examples described above. We will describe obstruction bundle induced from cotangent bundle of \(P^n\) and define related Gromov-Witten type invariants. Surprising we will see that all these invariants are 0.

The essential fact used in the proof is the following observation: let \(C\) be a smooth \(P^1\) mapping onto a line \((P^1)\) inside \(P^n\) with degree \(d\). Denote the map by \(f\). Standard torus action (diagonal action) on \(P^n\) naturally lifted to \(T^*P^n\) induces an action on the vector space \(H^1(C,f^*T^*P^n)\). Calculate the weights of the action, we see that there is a 0 weight piece.

This observation of the 0 weight piece also leads to other interesting applications. For instance, by utilizing it, we can calculate all the Gromov-Witten invariant, hence determine the quantum cohomology ring structure of the projective bundle.
\(P(T^*\mathbb{P}^2 \oplus \mathcal{O})\) over \(\mathbb{P}^2\). Again the difficulty lies in how to sum up, granted with the graph summation machinery developed by Kontsevich. And the simple observation we have will greatly simplify the summation procedure.

The rest of this section is organized as follows: In subsection 4.2, we define our invariant and state the vanishing theorem. In subsection 4.3, we introduce the Bott’s residue formula and Kontsevich’s graph summation formula for computing the invariants. In subsection 4.4, we prove our vanishing theorem and our result about isomorphism of Ruan cohomology.

### 4.2 Definition of invariants

In this subsection we define our invariants. Let \(\overline{\mathcal{M}}_{g,k}(\mathbb{P}^n,d)\) be the moduli space of stable maps from genus \(g\) curves with \(k\) marked points into \(\mathbb{P}^n\) which carries the fundamental class \(d[\mathbb{P}^1] \in H_2(\mathbb{P}^n)\). Denote a typical element in \(\overline{\mathcal{M}}_{g,k}(\mathbb{P}^n,d)\) by \((C,f,x_1,\ldots,x_k)\). The cotangent bundle of \(\mathbb{P}^n\) induces an obstruction bundle over \(\overline{\mathcal{M}}_{g,k}(\mathbb{P}^n,d)\) whose fiber at \((C,f,x_1,\ldots,x_k)\) is \(H^1(C,fT^*\mathbb{P}^n)\). Its Euler class \((\text{denoted by } \Phi)\) plays an important role in defining our invariants.

There are also other cohomology classes on \(\overline{\mathcal{M}}_{g,k}(\mathbb{P}^n,d)\). For instance there is the evaluation maps \(ev_i : \overline{\mathcal{M}}_{g,k}(\mathbb{P}^n,d) \to \mathbb{P}^n\), sending \((C,f,x_1,\ldots,x_k)\) to \(f(x_i)\), So we can pull back cohomology classes from \(\mathbb{P}^n\) via the evaluation maps. Also there is the forgetting map \(\overline{\mathcal{M}}_{g,k}(\mathbb{P}^n,d) \to \overline{\mathcal{M}}_{g,k}\) by forgetting the map \(f\) of \((C,f,x_1,\ldots,x_k)\) where \(\overline{\mathcal{M}}_{g,k}\) is the Deligne-Mumford space of stable curves with \(k\) marked points. So we can also pull back classes from \(\overline{\mathcal{M}}_{g,k}\).

Integrating polynomials in these classes over the moduli space \(\overline{\mathcal{M}}_{g,k}(\mathbb{P}^n,d)\), we get numbers.

In particular, if \(\mathbb{P}^n\) is embedded in a variety \(X\) with normal bundle naturally isomorphic to its cotangent bundle, then to correctly define Gromov-Witten invariant out of the moduli space \(\overline{\mathcal{M}}_{g,k}(M,d[\mathbb{P}^1])\), we have to take account of the Euler class of the obstruction bundle as described above.

So we want to consider the integrals where the class \(\Phi\) appears in the integrand. Formally, we have

**Definition 4.1:** \(K_{(k,g,d,\Theta)} := \int_{\overline{\mathcal{M}}_{g,k}(\mathbb{P}^n,d)} \Theta \wedge \Phi\), where \(\Theta\) is a polynomial in Chern classes of certain equivariant vector bundles over \(\overline{\mathcal{M}}_{g,k}(\mathbb{P}^n,d)\).

For example, let us consider the case of mukai flop. It is well known that the normal bundle of the embedded \(\mathbb{P}^n\) is actually naturally isomorphic to its cotangent bundle because of the existence of holomorphic 2-forms.

**Definition 4.2:** \(K_{(3,0,d,\text{ev}^*(\alpha) \wedge \text{ev}^*(\beta) \wedge \text{ev}^*(\gamma))} := \int_{\overline{\mathcal{M}}_{0,3}(\mathbb{P}^n,d)} \text{ev}^*(\alpha) \wedge \text{ev}^*(\beta) \wedge \text{ev}^*(\gamma) \wedge \Phi\), where \(\alpha, \beta, \gamma\) are any cohomology classes of \(\mathbb{P}^n\) with appropriate degrees, i.e.

\[
\deg(\alpha) + \deg(\beta) + \deg(\gamma) + \deg(\Phi) = \dim \mathcal{M}_{0,3}(\mathbb{P}^n,d).
\]
Note that the invariant defined above includes all quantum correction coming from exceptional effective curve in the case of Mukai flop.

About these invariants, we have the following vanishing theorem

**Theorem 4.3:** The invariants $K_{(k,g,d,\Theta)}$ all vanish regardless of the flexibility of $\Theta$.

### 4.3 Bott’s residue formula and normal bundle contribution

In this subsection, we introduce the technique we use to compute the invariants as defined in previous subsections. The basic idea is to consider torus action and use the Bott’s residue formula to reduce the integral to fixed points loci of the action.

Starting from [K], a lot of work has been done towards localization techniques applied to the computation of Gromov-Witten invariants and verification of mirror symmetry predictions. In the most general case, one has to consider localization of virtual classes as done in [GP, LLY]. In [CKYZ], the authors developed effective ways to compute similar invariants involving Euler classes of obstruction bundles. But they mainly treat direct sums of line bundles. For our computation, the machinery introduced by [K] suffices. Here we will follow the presentation in [K] closely. To keep notation simple, we will only consider integration formula in genus zero case. The proof of vanishing of the invariants in higher genus case will be almost identical. We will point out the slight difference later.

Before proving theorem 4.3, we first want to introduce **Bott’s residue formula:**

Let $X$ be a compact complex projective manifold (orbifold allowed) and $E$ a holomorphic vector bundle (or orbibundle) over $X$. Suppose $T := (\mathbb{C}^*)^{n+1}$ a complex torus acts on $(X, E)$. Denote the fixed points loci by $X^T$ and its connected components by $X^\gamma$. Since the irreducible representations of torus are dimensional one, over $X^\gamma$ the bundle $E$ splits into direct sum of line bundles $E_{\gamma,\lambda}$ twisted by character $\lambda : T \longrightarrow \mathbb{C}^*$, $\lambda \in \mathbb{T} = Z \oplus Z \oplus \cdots \oplus Z$. The normal bundle of $X^\lambda$ (denoted by $N^\lambda$) also splits into sum of line bundles $N_{\gamma,\lambda}$ over characters $\lambda \in \mathbb{T} \setminus \{0\}$.

By splitting principle, we suppose the Chern classes of bundle $E$ are given by homogeneous symmetric polynomials in degree 2 generators $e_i$’s as follows:

$$
\sum_{k \geq 0} c_k(E) = \Pi_i(1 + e_i), \quad e_i \in H^2(X, \mathbb{Q}).
$$

(18)

Analogously, we add generators $e_i^{\gamma,\lambda}$ and $n_i^{\gamma,\lambda}$ to $H^2(X^\gamma, \mathbb{Q})$.

Let $P$ be a homogeneous symmetric polynomial. Then the **Bott’s residue formula** reads:

$$
\int_X P(e_i) = \sum_{\gamma} \int_{X^\gamma} \frac{P(e_i^{\gamma,\lambda} + \lambda)}{\Pi(n_i^{\gamma,\lambda})}.
$$

(19)
The right hand side of the above formula is considered as rational function in \( \lambda \)'s. It turns out to have homogeneous degree 0 (actually a constant independent of choice of \( \lambda \)'s). The numerator of r.h.s. is actually the equivariant extension of the pullback of class \( P(e_i) \) to \( X^\gamma \). The denominator is the equivariant Euler class of normal bundle of \( X^\gamma \).

Now, we want to calculate the fixed points in the moduli space of stable maps in order to apply the Bott’s residue formula.

Let \( T = (C^*)^{n+1} \) acts diagonally on \( P^n \) with generic weights \(-\lambda_1, -\lambda_2, \cdots, -\lambda_{n+1}\). The fixed points are projectivization of coordinate lines of \( C^{n+1} \), denoted by \( p_i \). And the only invariant curves are lines connecting the fixed points labeled by \( l_{ij} = l_{ji} \), where \( i \neq j \).

The action of \( T \) on \( P^n \) induces an action of \( T \) on the moduli space of stable maps \( \overline{M}_{g,k}(P^n, d) \) by moving the image of the map. Let \( (C, f, x_1, \cdots, x_k) \) be a fixed point in the stable map space. Then the geometric image of the map is fixed. So we have

1. The contracted components, the marked points, the ramification points, the nodes all are mapped to the fixed points \( p_i \)'s in \( P^n \).
2. A non-contracted component is map onto one of the lines \( l_{ij} \)'s, ramifying over the two fixed points(end points of the line ), thus is forced to be rational and completely determined by its degree.

We associate with each fixed map a marked graph \( \Gamma \) as follows. The vertices of the graph \( v \in Vert(\Gamma) \) correspond to the connected components \( C_v \) of \( f^{-1}(p_1, p_2, \cdots, p_{n+1}) \). Here the component can be either a point or union of irreducible components of the curve \( C \). The edges \( \alpha \in Edge(\Gamma) \) correspond to non-contracted component of \( C^\alpha \) of genus 0 mapping onto the \( l_{ij} \)'s. There are also tails on the vertices coming from the marked points. We also mark the graph by the following labels:

1. Label the vertices numbers \( f_v \) from 1 to \( n+1 \) defined by \( f(C_v) := p_{f_v} \). Also label a vertex by \( g_v \) (the genus of the 1-dimensional part of \( C_v \), for a point the genus is 0) and a set \( S_v \subset \{1, 2, \cdots, k\} \) the indices of the marked points.
2. Label the edges by the mapping degree \( d_\alpha \in \mathbb{N} \)

The claim is that the connected components of \( \overline{M}_{g,k}(P^n, d)^T \) are isomorphic to \( \Pi_{v \in Vert(\Gamma)} \overline{M}_{g_v, S_v}(\Gamma) / \text{Aut}(\Gamma) \) and can be identified as equivalent classes of connected graphs \( \Gamma \) with labeling satisfying the following conditions:

1. For \( \alpha \in Edge(\Gamma) \) connecting vertices \( u, v \in Vert(\Gamma) \), then \( f_u \neq f_v \),
2. \( 1 - \chi(\Gamma) + \sum_{v \in Vert(\Gamma)} g_v = g \),
3. \( \sum_{\alpha \in Edge(\Gamma)} d_\alpha = d \),
(4) $\cup_{v \in \text{Vert}(\Gamma)} S_v = \{1, 2, \cdots, k\}$. 

From now on we only consider the integration formula for genus 0 case. We first want to give some notations:

(1) For a graph, we define an incident pair of vertex and edge $(v, \alpha)$ to be a flag $F = (v, \alpha)$ and denote by $w_F$ the expression $\frac{\lambda_u - \lambda_v}{\alpha}$ where $u \neq v$ is the other vertex of the edge $\alpha$.

(2) Recall that $\overline{\mathcal{M}}_{0,k}$ is the Deligne-Mumford space of marked stable curves. For each marking $i$, there is a line bundle $L_i \to \overline{\mathcal{M}}_{0,k}$ with fiber $T^*_C p_i$ over the moduli point $C$. Define $\psi_i := c_1(L_i)$.

Now we describe the normal bundle of the fixed points components. For an equivariant bundle $E$, denote by $[E]$ its class in the corresponding equivariant K-group. Also we denote $\overline{\mathcal{M}}_{0,k}(\mathbb{P}^n, d)$ by $\overline{\mathcal{M}}$ for simplicity and often denote a bundle by its geometric fiber at a point $(C, f)$.

To keep notation simple, we ignore the marked points as in $[K]$ and explain the difference along the way.

The class of normal bundle for a component $\overline{\mathcal{M}}^\gamma$ having graph type $\Gamma$ is

$$[N_{\overline{\mathcal{M}}^\gamma}] = [T_{\overline{\mathcal{M}}^\gamma}] - [T_{\overline{\mathcal{M}}}].$$

$$[T_{\overline{\mathcal{M}}^\gamma}] = [H^0(C, f^*(T\mathbb{P}^n))] + \sum_{y \in C^\alpha \cap C^\beta} [T_y(C^\alpha) \otimes T_y(C^\beta)] + \sum_{y \in C^\alpha \cap C^\beta \alpha \neq \beta} \left([T_y(C^\alpha)] + [T_y(C^\beta)]\right) - \sum_{\alpha} [H^0(C^\alpha, TC^\alpha)].$$

The first summand corresponds to infinitesimal deformation of the map $f$ from $C$. The second summand corresponds to smoothing of nodes. And the third comes from deformation of the curve $C$ fixing the singular points. If there is a marked point $x$ on $C^\alpha$, it should also be fixed and in the third summand there would be an additional term $\sum_\alpha [H^0(T_x(C^\alpha))]$. (Same remark applies to the formula below).

$$[T_{\overline{\mathcal{M}}^\gamma}] = \sum_{y \in C^\alpha \cap C^\beta \alpha \neq \beta; \alpha, \beta \notin \text{Edge}(\Gamma)} [T_y(C^\alpha) \otimes T_y(C^\beta)] + \sum_{y \in C^\alpha \cap C^\beta \alpha \neq \beta} [T_y(C^\alpha)] - \sum_{\alpha \notin \text{Edge}(\Gamma)} [H^0(C^\alpha, TC^\alpha)].$$

where the first term corresponds to smoothing of nodes which are intersection of two contracted components. The second term and the third come from deformation of the components preserving singular points.
So we have the following formula

\[ [N_{\mathcal{M}^r}] = [H^0(\mathcal{C}, f^*(\mathcal{TP}^n))] + [N_{\mathcal{M}^r}^{abs}] \]  

(23)

where

\[ [N_{\mathcal{M}^r}^{abs}] := \sum_{y \in C^\alpha \cap C^\beta} [T_y(C^\alpha) \otimes T_y(C^\beta)] \]
\[ + \sum_{y \in C^\alpha \cap C^\beta} [T_y(C^\alpha) \otimes T_y(C^\beta)] \]
\[ + \sum_{y \in C^\alpha \cap C^\beta} [T_y(C^\alpha)] - \sum_{\alpha \in \text{Edge}(\Gamma)} [H^0(\mathcal{C}, \mathcal{T}C^\alpha)]. \]  

(24)

In the formula for \([N_{\mathcal{M}^r}^{abs}]\) above the first and third summand are trivial bundles twisted with characters of the torus. The term \([H^0(\mathcal{C}, f^*(\mathcal{TP}^n))]\) and the classes from the bundle \(\mathcal{E}\) restricted to \(X^\gamma\) in our application later have same nature. When we take the Chern classes of these summand, we just get weights of torus action on the fibers of these bundles (expressed in terms of \(\lambda_i\)'s), hence can be pulled out of the integral. In the second summand, the tangent space of the non-contracted component at \(y\) is fixed but twisted, while the tangent space of the contracted component at \(y\) is moving without twisting. Taking equivariant Chern class we get a sum of certain tangential weight and the \(\psi\) class over suitable space of pointed stable curves. This reduce the integral on the right hand side of Bott’s formula integral to integral of \(\psi\) classes over space of pointed curves for which the answer has been conjectured by Witten and verified by Kontsevich rigorously. Thus we have a contribution (as rational function in \(\lambda_i\)'s) from each of the admissible graphs. The invariant is given by a graph summation collecting all these contributions:

\[ \prod_{\alpha \in \text{Edge}(\Gamma); v_1, v_2; \text{vertices of } \alpha} \frac{(-1)^{d_\alpha} (d_\alpha^{d_\alpha})^{2d_\alpha}}{(d_\alpha^{d_\alpha})^2} \times \prod_{\alpha \in \text{Edge}(\Gamma)} \prod_{k \neq f_{v_1}, k \neq f_{v_2}} \prod_{a, b \geq 0; a + b = d_\alpha} \frac{1}{d_\alpha^{f_{v_1}} + d_\alpha^{f_{v_2}} - \lambda_k} \]
\[ \times \prod_{v \in \text{Vert}(\Gamma)} \left\{ \left( \sum_{\text{flags}: F = (v, \alpha)} w_e^{-1} \right)^{\text{val}(v) - 3} \times \prod_{\text{flags}: F = (v, \alpha)} w_e^{-1} \right\} \times \prod_{j \neq f_v} (\lambda_{f_v} - \lambda_j)^{\text{val}(v) - 1}. \]  

(25)

Here the valence of a vertex includes the counts of the number of tails. The detailed calculation of the weights can be found in [K] which we refer the interested readers to.
4.4 Proof of the vanishing theorem.

In this subsection, we prove the vanishing theorem stated in Section 4.2. We show the calculation for the specific example defined in Definition 4.2 with n=2. The proof for the general cases is almost identical. We will briefly explain the difference at the end of the proof.

**Proof of Theorem 4.3**: First of all, note that the invariant is given by

\[
\sum_{\Gamma} \frac{1}{|\text{Aut}(\Gamma)|} \times (\text{contribution from } ev^*(\alpha) \wedge ev^*(\beta) \wedge ev^*(\gamma))
\times (\text{contribution from } \Phi) \times (\text{formula(25)}).
\]

(26)

Here the contribution of the second and the third terms are just a product of the weights of induced torus action on the corresponding vector bundles. We show that the contribution from the Euler class \( \Phi \) of the obstruction bundle restricted to the fixed point component is zero and thus conclude.

To deal with nodal curves, we need the following normalization sequence. First let us consider the simple case where \( \mathcal{C} = C_\alpha \cup C_\beta \). There is an exact sequence of maps of sheaves (of the holomorphic functions):

\[
0 \rightarrow \mathcal{O}_\mathcal{C} \rightarrow \mathcal{O}_{C_\alpha} \oplus \mathcal{O}_{C_\beta} \rightarrow \mathcal{O}_{C_\alpha \cap C_\beta} \rightarrow 0.
\]

(27)

Here all the maps except the last one are obtained from inclusions. And the last one maps \( (f_1, f_2) \) to \( f_1 - f_2 \).

In general we have the normalization sequence resolving all the nodes of \( \mathcal{C} \) which are forced by a graph type \( \Gamma \)

\[
0 \rightarrow \mathcal{O}_\mathcal{C} \rightarrow \left( \bigoplus_{v \in \text{Vert}(\Gamma)} \mathcal{O}_{C_v} \right) \oplus \left( \bigoplus_{\alpha \in \text{Edge}(\Gamma)} \mathcal{O}_{C_\alpha} \right)
\rightarrow \oplus_{F \in \text{Flag}(\Gamma)} \mathcal{O}_{x_F} \rightarrow 0,
\]

(28)

where \( x_F = C_v \cap C_\alpha \) for a flag \( (v, \alpha) \), and the last map sends \( (g|_{C_v}, h|_{C_\alpha}) \) to \( g - h \) on the intersection point.

Twist the above sequence by \( f^*T^*\mathbb{P}^2 \) and take cohomology to get

\[
0 \rightarrow H^0(C, f^*T^*\mathbb{P}^2)
\rightarrow \left( \bigoplus_{v \in \text{Vert}(\Gamma)} H^0(C_v, f^*T^*\mathbb{P}^2) \right) \oplus \left( \bigoplus_{\alpha \in \text{Edge}(\Gamma)} H^0(C_\alpha, f^*T^*\mathbb{P}^2) \right)
\rightarrow \oplus_{F \in \text{Flag}(\Gamma)} T^*_{f(x_F)} \mathbb{P}^2 \rightarrow H^1(C, f^*T^*\mathbb{P}^2)
\rightarrow \left( \bigoplus_{v \in \text{Vert}(\Gamma)} H^1(C_v, f^*T^*\mathbb{P}^2) \right) \oplus \left( \bigoplus_{\alpha \in \text{Edge}(\Gamma)} H^1(C_\alpha, f^*T^*\mathbb{P}^2) \right)
\rightarrow 0.
\]

(29)

The first term in the third line follows since \( x_F \) is a point, which is why the last term in the last line is 0. Note that \( f^*T^*\mathbb{P}^2|_{C_v} \) is trivial since \( C_v \) is mapped to a point ,
hence $H^0(C_v, f^*T^*\mathbb{P}^2) = T^*\mathbb{P}^2|_{\nu_f}$ and $H^1(C_v, f^*T^*\mathbb{P}^2) = H^1(C_v, \mathcal{O}) \otimes f^*T^*\mathbb{P}^2$. Since we are considering genus zero case, $H^1(C_v, f^*T^*\mathbb{P}^2)$ is also zero. In general, it can be expressed in terms of the first Chern class $C_1$ of Hodge bundle over space of pointed curves. Because of the concavity of $T^*\mathbb{P}^2$, $H^0(C, f^*T^*\mathbb{P}^2)$ is zero. And by looking at the maps in the first line of (29), we see $H^0(C, f^*T^*\mathbb{P}^2)$ is also zero. So we have

$$[H^1(C, f^*T^*\mathbb{P}^2)] = \prod_{v \in Vert(\Gamma)} T^*_v(\mathbb{P}^2) + \prod_{\alpha \in \text{Edge}(\Gamma)} H^1(C_\alpha, f^*T^*\mathbb{P}^2).$$

(30)

The contribution from the l.h.s is the product of those of the two terms on the r.h.s.

Since a non-contracted component is rigid, $[H^1(C_\alpha, f^*T^*\mathbb{P}^2)]$ is a trivial bundle when restricted to fixed point components. To compute the weights, we consider the following description of the cotangent bundle of $\mathbb{P}^2$ by an exact sequence of bundles over $\mathbb{P}^2 = \mathbb{P}(V)$ where $V$ is a complex vector space of dimension 3. First we have

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow V \longrightarrow Q \longrightarrow 0,$$

(31)

where $V$ represents the trivial bundle with vector space $V$ as fiber and $\mathcal{O}(-1)$ is the universal bundle. Tensoring with $\mathcal{O}(1)$, we have

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1) \otimes V \longrightarrow \mathcal{O}(1) \otimes Q \longrightarrow 0,$$

(32)

where $\mathcal{O}(1) \otimes Q = T\mathbb{P}^2$. Dualizing we have

$$0 \longrightarrow T^*\mathbb{P}^2 \longrightarrow \mathcal{O}(-1) \otimes V^* \longrightarrow \mathcal{O} \longrightarrow 0.$$

(33)

Pulling back by $f$ over $C_\alpha$ and taking cohomology, we have

$$0 \longrightarrow H^0(C_\alpha, f^*T^*\mathbb{P}^2) \longrightarrow H^0(C_\alpha, \mathcal{O}(-d) \otimes V^*)$$

$$\longrightarrow H^0(C_\alpha, \mathcal{O}) \longrightarrow H^1(C_\alpha, f^*T^*\mathbb{P}^2)$$

$$\longrightarrow H^1(C_\alpha, \mathcal{O}(-d) \otimes V^*) \longrightarrow H^1(C_\alpha, \mathcal{O}) \longrightarrow 0.$$  

(34)

Note that $C_\alpha$ is rational. $H^0(C_\alpha, \mathcal{O}(-d) \otimes V^*)$ and $H^1(C_\alpha, \mathcal{O})$ are both 0. So we have

$$0 \longrightarrow H^0(C_\alpha, \mathcal{O}) \longrightarrow H^1(C_\alpha, f^*T^*\mathbb{P}^2) \longrightarrow H^1(C_\alpha, \mathcal{O}(-d) \otimes V^*) \longrightarrow 0.$$  

(35)

So the contribution of $[H^1(C_\alpha, f^*T^*\mathbb{P}^2)]$ is given by a product of weights on $H^1(C_\alpha, \mathcal{O}(-d) \otimes V^*)$ and weight on $H^0(C_\alpha, \mathcal{O})$. Obviously the weight on $H^0(C_\alpha, \mathcal{O})$ is zero. So the contribution of $[H^1(C_\alpha, f^*T^*\mathbb{P}^2)]$ is zero for each $\alpha \in \text{Edge}(\Gamma)$.

From (30), we see that the total contribution of the Euler class $\Phi$ is zero. Thus we conclude our proof of the genus zero case.

In the general case of higher genus, formula (26) needs to be modified. $\lambda$ classes (coming from the deformation of the complex structures on $C$) and hence Hodge
integrals will appear in the computation of the normal bundle contribution and the details can be found in [GP]. But the point is that the contribution of Euler class $\Phi$ is still zero, since there is a 0 weight coming from $H^1(C_\alpha, f^*T^*P^2)$ for each non-contracted component (necessarily rational as explained earlier). So Theorem 4.3 still holds.

**Theorem 4.4:** Suppose that non-singular projective manifolds $X$ and $X'$ of complex dimension $2n$ are connected by a sequence of Mukai flops. Then $X$ and $X'$ have isomorphic Ruan cohomologies.

**Proof:** By theorem 4.3, we have that all Gromov-Witten invariants appearing in the right hand side of (1) vanish. Therefore, we have that for $X, X'$ their quantum corrections all vanish. Thus their quantum cohomology are the same as their ordinary Chow ring. By theorem 3.2, we know that $X, X'$ have isomorphic Ruan cohomology. This proves the theorem.

**Corollary 4.5:** For Mukai flops, cohomological minimal model conjecture holds.

Finally, we present a well known proposition to point out that local existence of a holomorphic symplectic 2-form implies natural isomorphism of the normal bundle and the cotangent bundle for a embedded $P^n$.

**Proposition 4.6:** (see [Mukai]) Suppose that $P^n$ is embedded in a smooth variety $X$ with a neighborhood $N$ admitting a holomorphic symplectic 2-form $\omega$, then we have the following

1. $\text{codim}_X P^n \geq n$.

2. In case $\text{codim}_X P^n = n$, there is a natural isomorphism $T^*P^n = N_{X\setminus P^n}$.

**Proof:** Since $H^{2,0}(P^n) = 0$, $\omega |_{P^n} = 0$. Thus $T_pP^n \subset (T_pP^n)^\perp$ for any point $p \in P^n$, where $T_pP^n \subset T_pX$ is considered as a subspace of $T_pX$. Hence $\text{codim}_X P^n = \dim(T_pP^n)^\perp \geq \dim T_pP^n = n$. In case equality holds, $T_pP^n = (T_pP^n)^\perp$.

$\omega |_{T_pX}$ is nondegenerate, so there is an isomorphism $\phi : T_pX = (T_pX)^*$. Thus we have $T_pP^n = (T_pP^n)^\perp = \text{Ann}(T_pP^n) = N^*_X \setminus P^n$, where the second isomorphism is via the map $\phi$.

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