HILBERT’S 16TH PROBLEM.
I. WHEN DIFFERENTIAL SYSTEMS MEET VARIATIONAL METHODS

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Abstract. We provide an upper bound for the number of limit cycles that planar polynomial differential systems of a given degree may have. The bound turns out to be a polynomial of degree three in the degree of the system. The strategy brings together variational and dynamical system techniques by transforming the task of counting limit cycles into counting critical points for a certain smooth, non-negative functional, through Morse inequalities, for which limit cycles are global minimizers. We thus solve the second part of Hilbert’s 16th problem providing a uniform upper bound for the number of limit cycles which only depends on the degree of the polynomial differential system.

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1. Introduction

We deal with polynomial differential systems in $\mathbb{R}^2$ of the form

\begin{equation}
\frac{dx}{dt} = x' = P(x, y), \quad \frac{dy}{dt} = y' = Q(x, y),
\end{equation}

where the maximum degree of the polynomials $P$ and $Q$ is $n$. This $n$ is called the degree of the polynomial differential system (1).

We recall that a limit cycle of the differential system $\text{(1)}$ is a periodic orbit of this system isolated in the set of all periodic orbits of system $\text{(1)}$. As far as we known the notion of limit cycle appeared in the year 1885 in the work of Poincaré [41]. Moreover, he proved that a polynomial differential equation $\text{(1)}$ without saddle connections has finitely many limit cycles, see [40].

In the Second International Congress of Mathematicians, celebrated in Paris in 1900, Hilbert [23] proposed a list of 23 relevant problems to be solved during the XX century. The 16-th problem of this list reads:

*Problem of the topology of algebraic curves and surfaces*

The maximum number of closed and separate branches which a plane algebraic curve of the $n$th order can have has been determined by Harnack. There arises the further question as to the relative position of the branches in the plane. As to curves of the 6th order, I have satisfied myself—by a complicated process, it is true—that of the eleven branches which they can have according to Harnack, by no means all can lie external to one another, but that one branch must exist in whose interior one branch and in whose exterior nine branches lie, or inversely. A thorough investigation of the relative
position of the separate branches when their number is the maximum seems to me to be of very great interest, and not less so the corresponding investigation as to the number, form, and position of the sheets of an algebraic surface in space. Till now, indeed, it is not even known what is the maximum number of sheets which a surface of the 4th order in three dimensional space can really have.

In connection with this purely algebraic problem, I wish to bring forward a question which, it seems to me, may be attacked by the same method of continuous variation of coefficients, and whose answer is of corresponding value for the topology of families of curves defined by differential systems. This is the question as to the maximum number and position of Poincaré’s boundary cycles (cycles limits) for a differential system of the first order and degree of the form

\[ \frac{dy}{dx} = \frac{Y}{X}, \]

where \( X \) and \( Y \) are rational integral functions of the \( n \)th degree in \( x \) and \( y \). Written homogeneously, this is

\[ X \left( y \frac{dz}{dt} - z \frac{dy}{dt} \right) + Y \left( z \frac{dx}{dt} - x \frac{dz}{dt} \right) + Z \left( x \frac{dy}{dt} - y \frac{dx}{dt} \right) = 0, \]

where \( X, Y, \) and \( Z \) are rational integral homogeneous functions of the \( n \)th degree in \( x, y, z \), and the latter are to be determined as functions of the parameter \( t \).

Clearly the 16–th Hilbert problem is formulated in two parts. The first part is about the mutual disposition of the maximal number of separate branches of an algebraic curve, and its extension to nonsingular real algebraic varieties. The second part is dedicated to limit cycles of polynomial differential systems in \( \mathbb{R}^2 \), where Hilbert asked for the maximal number and relative position of limit cycles of polynomial differential systems [1]. Usually the first part of the 16–th Hilbert problem is studied in real algebraic geometry, while the second part is considered in the theory of differential systems. Hilbert also pointed out that there might exist possible connections between these two parts. Some of these connections are described in the survey about the 16–th Hilbert problem written by Jibin Li, see [29].

From now on, when we talk about the 16–th Hilbert problem, we always mean the second part of the 16–th Hilbert problem.

In 1988 Noel Lloyd [35] observed with respect to the 16–th Hilbert problem that the striking aspect is that the hypothesis is algebraic, while the conclusion is topological.

Arnold in 1977 and 1983 (see [1] and [2], respectively) stated the weakened, infinitesimal or tangential 16–th Hilbert problem which we do not consider here, but there are some surveys for this modified problem. See for instance the survey of Ilyashenko [27] on the 16–th Hilbert problem, the already mentioned survey of Jibin Li, the book of Colin Christopher and Chengzhi Li [13], the survey due to Kaloshin [28], the one of Yakovenko [49], or more recently the work of Binyamini, Novikov and Yakovenko [6].

According to Smale [44], except for the Riemann hypothesis, the second part of the 16–th Hilbert problem seems to be the most elusive of Hilbert’s problems. Smale states the following version of the second half of 16–th Hilbert problem with respect to the number of limit cycles:
Consider the polynomial differential system \((1)\) in \(\mathbb{R}^2\). Is there a bound \(K\) on the number of limit cycles of the form \(K \leq n^q\) where \(n\) is the maximum of the degrees of \(P\) and \(Q\), and \(q\) is a universal constant?

The topological configurations or possible distribution of limit cycles mentioned as position by Hilbert has also attracted the attention of many authors. Coleman in his work \([15]\) on the 16–Hilbert problem said: For \(n > 2\) the maximal number of eyes is not known, nor is it known just which complex patterns of eyes within eyes, or eyes enclosing more than a single critical point, can exist. Here “eye” means a nest of limit cycles. We shall see later that some of the questions on the possible topological configurations of limit cycles realized by polynomial differential systems have been solved.

1.1. On the number of limit cycles. Dulac \([17]\) claimed in 1923 that any polynomial differential system \((1)\) always has finitely many limit cycles. Ilyashenko \([25]\) found an error in Dulac’s paper in 1985. Later, two long works have appeared, independently, providing proofs of Dulac’s assertion: one due to Écalle \([18]\) in 1992, and the other to Ilyashenko \([26]\) in 1991. As Smale mentioned in \([44]\), these two books have yet to be thoroughly digested by the mathematical community. We remark that in no case the results of Écalle and Ilyashenko prove that there exists a uniform upper bound for the maximum number of limit cycles of all the polynomial differential systems of a given degree.

Bamon \([3]\) proved in 1986 that any polynomial differential system of degree 2 has finitely many limit cycles. His result uses previous results of Ilyashenko.

Here a homoclinic or heteroclinic loop is formed by \(k = 1\) or \(k > 1\) saddles (eventually some saddles can be repeated), and \(k\) different separatrices connecting these saddles and forming a loop (eventually some points of this loop can be identified in a repeated saddle) in such a way that at least in one of the two sides of the loop a Poincaré return map is defined. Let \(\mu_i < 0 < \lambda_i\) be the eigenvalues of these saddles. If
\[
\prod_{i=1}^{k} \frac{\lambda_i}{\mu_i} \neq 1,
\]
then the loop is called simple. From the work of Poincaré \([41]\) (see Theorem XVII), it follows that if a polynomial differential system \((1)\) has all its saddle connections forming a simple homoclinic or heteroclinic loop, then the system also has finitely many limit cycles, see the nice work of Sotomayor \([45]\) for more details. There are many other results providing upper bounds for the maximum number of limit cycles which can accumulate or bifurcate from different kinds of homoclinic and heteroclinic loops.

In 1957 Petrovskii and Landis \([38]\) claimed that the polynomial differential systems of degree \(n = 2\) have at most 3 limit cycles. Soon (in 1959) a gap was found in the arguments of Petrovskii and Landis, see \([39]\). Later, Lan Sun Chen and Ming Shu Wang \([10]\) in 1979, and Songling Shi \([43]\) in 1982, provided the first polynomial differential systems of degree 2 having 4 limit cycles, and up to now 4 is the maximum number of limit cycles known for a polynomial differential system of degree 2.

Lower bounds for the maximum number of limit cycles that a polynomial differential system of degree \(n\) can have have been given mainly by Christopher and Lloyd \([14]\), Jibin Li \([29]\), and more recently by Maoan Han and Jibin Li \([22]\).
There are also some relevant results about the 16th Hilbert problem restricted to algebraic limit cycles, see Appendix 1.

1.2. Statement of main results. The following theorem provides an upper bound for the maximum number of limit cycles that a polynomial differential system of degree \( n \) can have. So it provides an answer to the second part of the 16-th Hilbert problem, and an answer to the stronger version of it stated by Smale.

**Theorem 1.** An upper bound for the maximum number \( H(n) \) of limit cycles that a polynomial differential system of degree \( n > 1 \) can have is

\[
H(n) \leq \frac{5}{2} n^3 - \frac{13}{2} n^2 + 6n \quad \text{if \( n \) is even, and}
\]

\[
H(n) \leq \frac{5}{2} n^3 - \frac{13}{2} n^2 + 5n \quad \text{if \( n \) is odd.}
\]

The number \( H(n) \) is usually called the *Hilbert number* for the polynomial differential systems of degree \( n \).

This upper bound for \( H(n) \) yields a universal exponent \( q = 3 \) for the stronger version due to Smale.

A more detailed version of Theorem 1 in the generic case in which all the components of the algebraic curve \( P_x + Q_y = 0 \) are homeomorphic to a straight line or to a circle, and the number of contact points of the vector field with the divergence curve \( P_x + Q_y = 0 \) is finite, is the following one. We recall that a point of the divergence curve is of contact if the vector field is either tangent to the curve at this point, or it is a singular point of the vector field.

**Theorem 2.** Consider a polynomial differential system \( 1 \) of degree \( n > 1 \). Assume that

(i) all of the connected components of the curve \( P_x + Q_y = 0 \), are homeomorphic to a straight line or to an oval (i.e. the algebraic curve \( P_x + Q_y = 0 \) has no singular points);

(ii) \( N \) is the finite number of real solutions of the polynomial system

\[
P(P_{xx} + Q_{yx}) + Q(P_{xy} + Q_{yy}) = 0,
\]

\[
P_x + Q_y = 0,
\]

(i.e. the contact points of the vector field \( P, Q \) with the curve \( P_x + Q_y = 0 \));

(iii) \( M \) is the number of connected components of the curve \( P_x + Q_y = 0 \).

Then an upper bound for the number of limit cycles \( H(n) \) that the differential system \( 1 \) may have is

\[
H(n) \leq n(N + M).
\]

The paper is dedicated to proving Theorems 1 and 2. In particular, it is of the utmost importance to understand the role played by the following two pieces of information:

(I) the divergence curve \( \text{Div} = \text{Div}(x, y) \equiv P_x(x, y) + Q_y(x, y) = 0 \) and

(II) the contact points of the vector field \( P, Q \) with the curve \( \text{Div} = 0 \) of system 2.
It is interesting to point out that our basic arguments do not make special use of the polynomial structure of (1). If for such a general differential system, or a specific class of them, one could find upper bounds for the numbers $N$ and $M$ as well as an additional factor coming from multiplicity, as we will see, then one would have a corresponding upper bound for the number of limit cycles for such a system or class.

All of our efforts are concentrated in proving Theorem 2, together with a short but clear discussion about its validity in a non-generic situation in Section 13. Section 2 is devoted to proving Theorem 1 based on Theorem 2 and its version for a non-generic situation, as just indicated. We also explore the particular situations of quadratic and Lienard differential systems.

Before getting into a serious discussion for a rigorous proof of Theorem 2, it is instructive to have an overall description of the principle and main steps that will guide us, as they are quite different from the typical techniques utilized in polynomial differential systems. This is clearly stated in Section 3. We have also written two final appendices with additional information on Hilbert’s 16th problem.

2. Hilbert’s 16th problem in the generic case

Suppose that the polynomials $P$ and $Q$ are coprime, i.e. equilibria are isolated. If the polynomials $P$ and $Q$ have a common factor, it can be removed by doing a change in the independent variable of the polynomial differential system, and the upper bound on the number of limit cycles of the new polynomial system obtained is also an upper bound for the number of limit cycles of the initial polynomial system having $P$ and $Q$ a common factor.

It is well-known that under small perturbations in the coefficients of $P$ and $Q$ the components of the algebraic curve $\text{Div} = 0$ are homeomorphic either to a straight line or to an oval, and that the number of contact points of the vector field $(P, Q)$ with the curve $\text{Div} = 0$ is finite. We refer to this previous case as the generic one. In Section 13 we will show that the upper bound for the number of limit cycles of a polynomial differential system of degree $n$ in the generic case extends to a non-generic polynomial differential system of degree $n$. So here we focus on proving Theorem 1 in the generic case. To do so, we recall two classical theorems:

**Theorem 3** (Bezout Theorem). Let $R(x, y)$ and $S(x, y)$ be two polynomials with coefficients in $\mathbb{R}$. If both polynomials do not share a non-trivial common factor, then the algebraic system of equations

$$R(x, y) = S(x, y) = 0$$

has at most $\text{degree}(R)\text{degree}(S)$ solutions.

For a proof of Theorem 3 see for instance [19].

**Theorem 4** (Harnack Theorem). The maximum number of connected components of an algebraic curve of degree $k$ is

(a) $1 + (k - 1)(k - 2)/2$ if $k$ is even,
(b) $(k - 1)(k - 2)/2$ if $k$ is odd.

For a proof of Theorem 4 see for instance [21].
Proof of Theorem 7 in the generic case assuming Theorem 2. We need to find an upper bound for the number $N$ of the solutions that system (2) may have when $P$ and $Q$ are polynomials of at most degree $n$. By Bezout’s theorem we have that $N \leq 2(n-1)^2$, because in the generic case we discard the possibility that the two equations of system (2) have a non-trivial common factor.

By Theorem 4 the number $M$ of components of $\text{Div} = 0$ satisfies

$$M \leq \binom{n-2}{2} + 1, \text{ if } n \text{ is even, and } M \leq \binom{n-2}{2}, \text{ if } n \text{ is odd.}$$

The final number in the statement of the theorem is then a direct consequence of Theorem 2, i.e.

$$n(N + M) \leq n \left( \frac{1}{2}(n-2)(n-3) + 1 + 2(n-1)^2 \right)$$

$$= \frac{5}{2}n^3 - \frac{13}{2}n^2 + 6n,$$

if $n$ is even, while

$$n(N + M) \leq n \left( \frac{1}{2}(n-2)(n-3) + 2(n-1)^2 \right)$$

$$= \frac{5}{2}n^3 - \frac{13}{2}n^2 + 5n,$$

if $n$ is odd. This yields the numbers in the statement of Theorem 1 in the generic case. □

Two corollaries of Theorem 2 are the following ones.

Corollary 5. If the divergence of a quadratic polynomial differential system (1) is constant or zero, then it has no limit cycles. Otherwise if the straight line $\text{Div} = 0$ of a quadratic polynomial differential system (1) has:

(a) two contact points, then it cannot have more than 6 limit cycles.
(b) one contact point, then it cannot have more than 4 limit cycles.
(c) no contact points, then it has no limit cycles.

Proof. The set $\text{Div} = 0$ for a quadratic polynomial differential system is either empty, or a straight line, or the whole plane. If it is empty, i.e. if $\text{Div}$ is a non-zero constant, then the system has no limit cycles by Bendixon criterium (see for instance Theorem 7.10 of [16]). If it is the whole plane, the system is Hamiltonian and so it has no limit cycles. We assume that $\text{Div} = 0$ is a straight line. So using (3), we have $n = 2$, $M = 1$, $N \in \{0, 1, 2\}$.

If $N = 2$, then $n(M + N) = 2(1 + 2) = 6$.
If $N = 1$, we obtain $n(M + N) = 2(1 + 1) = 4$.
If $N = 0$, there are no contact points and the limit cycles cannot intersect the straight line $\text{Div} = 0$. Again by Bendixon criterium the system has no limit cycles. □

Corollary 6. We consider the Liénard polynomial differential systems

$$\dot{x} = P(x, y) = y - f(x), \quad \dot{y} = Q(x, y) = g(x),$$

where $p$ is the degree of $f$, and $q$ is the degree of $g$. So $n = \max\{p, q\}$. A system (4) cannot have more than $2\max\{p, q\}(p - 1)$ limit cycles.
Proof. It is well known that a system (4) has at most $p-1$ connected components for the curve $\text{Div} = f'(x) = 0$ corresponds to the critical values of the polynomial $f$. Note that each component is a vertical straight line in the $(x,y)$–plane. System (2) becomes

$$(y - f(x))f''(x) = 0, \quad f'(x) = 0,$$

for system (4). If $f''(x) = f'(x) = 0$ has a solution $x_0$, the vertical straight line $x = x_0$ is formed by contact points, so it is invariant and we do need to take it into account, because limit cycles cannot intersect such straight line. Therefore a connected component of the curve $\text{Div} = 0$ has one single contact point $(x_0, f(x_0))$ for each zero $x_0$ of the polynomial $f'(x)$ such that $f''(x_0) \neq 0$. Using again (3), we have $n = \max\{p, q\}$, and $M = N \leq p - 1$. Hence

$$n(M + N) = 2 \max\{p, q\}(p - 1).$$

This completes the proof.  

We now turn to treat rigorous proofs of all the ingredients that we shall use for proving Theorem 2, and its version for a non-generic situation.

3. Overview

Consider the polynomial planar differential system (1). We will be working with the following functional associated with it in a natural way

$$(5) \quad E_0(x, y) = \int_0^1 \frac{1}{2} (P(x, y)y' - Q(x, y)x')^2 \, dt.$$

The functional $E_0$ is some kind of measure of how far a closed path $(x, y) = (x(t), y(t))$, parameterized in $[0, 1]$ and with $x(0) = x(1)$ and $y(0) = y(1)$, is from being a close path formed by orbits of system (1). It is clear that $E_0$ is smooth ($C^\infty$), $E_0 \geq 0$, and $E_0 = 0$ for every limit cycle of the system (re)parameterized in the interval $[0, 1]$ for normalization. Note that $E_0(x, y) = 0$ for a closed path $(x, y)$ formed by orbits of system (1), because $P(x, y)y' - Q(x, y)x' = 0$. There are several other possibilities for which $E_0(x, y)$ vanishes:

(i) $(x, y)$ could be a constant path;
(ii) $(x, y)$ could be a periodic orbit surrounding a center;
(iii) $(x, y)$ could be a limit cycle run counterclockwise or clockwise, or even run several times in either orientation; or could be a reparameterization of a limit cycle, even with a different starting point for $t = 0$.
(iv) $(x, y)$ could be formed by a singular point and a homoclinic orbit.
(v) Other possibilities.

The point is, however, to realize that limit cycles of system (1) are definitely zeroes of the functional $E_0$. Thus, our aim will be accomplished if we can find an upper bound, depending on the degree of the system (1), of the zeroes of $E_0$.

Where may our bound for limit cycles come from? What might our driving idea be? We have already pointed out earlier, albeit very briefly and in an informal way, that limit cycles of (1) are zeros (absolute minimizers) of the functional $E_0(x, y)$. $E_0$ is definitely non-negative, and vanishes on periodic integral curves of (1) parameterized in the unit interval. It is however true that $E_0$ vanishes in many other paths as we have
already pointed out. This fact is something that needs to be addressed and clarified, but let us ignore it for the time being. As a matter of fact, and as we emphasize below, there are many other concerns to be taken care of before the following idea can be applied. The fundamental idea is to try to bound the number of global minima of \( E_0 \) (or of a suitable perturbation of it) through the number of its critical paths other than global minimizers. This is done through Morse inequalities. The matter is to organize the critical points of a functional, that needs to enjoy a number of important properties, in different classes in such a way that certain numerical identities and inequalities with the number of critical points in each class hold. Fundamental concepts like non-degenerate critical point, (Morse) index of such critical points, the Palais-Smale condition, etc, need to be discussed to better understand and appreciate the rigorous statement about Morse inequalities.

Note that the one-dimensional version of Morse inequalities is essentially the classical Rolle’s theorem. Some readers may be familiar with the mountain-pass lemma which is a quite successful tool in non-linear PDEs, see for instance [7]. Morse inequalities are like a big, global mountain pass lemma. We state here the version of it that can be checked in [4]. We have found this version particularly helpful for our purposes in this work.

**Theorem 7.** Let \( E : H \to \mathbb{R} \) be a \( C^2 \)-functional defined over a Hilbert space \( H \), which is bounded from below, coercive, enjoying the Palais-Smale property, and having a finite number of critical points, all of which are non-degenerate and of a finite index. Put \( M_k \) for the (finite) number of critical points of each fixed index \( k \). Then

\[
M_0 \geq 1, \quad M_1 - M_0 \geq -1, \quad M_2 - M_1 + M_0 \geq 1, \quad \ldots, \quad \sum_{k=0}^{\infty} (-1)^k M_k = 1.
\]

We therefore need a suitable functional \( E \), defined over an appropriate Hilbert space \( H \), that needs to comply with a series of properties if it is to be eligible for Morse inequalities to be applied. This will be our first important step.

We will be using below the notation

\[
\Sigma = \sum_{k=0}^{\infty} (-1)^k M_k, \quad \Sigma(S) = \sum_{k=0}^{\infty} (-1)^k M_k(S),
\]

for a given subset \( S \subset H \), where \( M_k(S) \) indicate the number of critical points of \( E \) of index \( k \) contained in \( S \).

Morse inequalities are quite flexible. In particular, we would like to highlight the following interesting properties:

1. **Additivity.** If \( E \) complies with the assumptions in the above theorem, and \( a \) and \( b \) are non-critical values of \( E \) with \( a < b \), then

\[
\Sigma(\{ E \leq a \}) + \Sigma(\{ a < E \leq b \}) = \Sigma(\{ E \leq b \}),
\]

where each \( \Sigma \) is given by (6) for the corresponding critical points in the given subset.

2. **Morse inequality in valleys.** If \( a \) is a non-critical value, Morse inequalities are valid in each connected component of \( \{ E \leq a \} \), informally called valleys of \( E \),
(3) As a consequence of the previous items, and since it is unlikely that one could determine precisely the indexes of specific critical points, if \( a \) and \( b \) are non-critical with \( a < b \), and \( \{ E \leq b \} \) is connected,

\[
\sharp\{ E \leq a \} \leq 1 + \sum_{k=0}^{\infty} M_k \quad \text{where} \quad M_k = M_k(\{ a < E \leq b \}),
\]

where \( \sharp\{ E \leq a \} \) stands for the number of connected components of \( \{ E \leq a \} \).

If we realize that the sum

\[
\sum_{k=0}^{\infty} M_k(\{ a < E \leq b \})
\]

is actually the total number \( C(\{ a < E \leq b \}) \) of critical points of \( E \) in the set \( \{ a < E \leq b \} \), the previous inequality becomes

\[
(7) \quad \sharp\{ E \leq a \} \leq 1 + C(\{ a < E \leq b \}).
\]

It is in this form (7) that we would like to make use of Morse inequalities. Note that all sums are finite under hypotheses implied by Theorem 7. Our intention is to identify every single limit cycle of our differential system (1) with a component of \( \{ E \leq a \} \) for appropriately chosen \( a \), while finding an upper bound \( C(n) \), depending exclusively on the degree \( n \), for the right-hand side of (7) for a suitable value \( b \)

\[
C(\{ a < E \leq b \}) \leq C(n).\]

If we succeed in carrying out this task, and given that there is also a special component of the set \( \{ E \leq a \} \) determined by constant paths, we will have our bound

\[
(8) \quad H(n) \leq C(n).\]

At any rate, according to Theorem 7, there is a number of crucial properties that \( E \) must comply with before we can even make use of inequality (7). In particular, the zero set of \( E \) (if it is to be non-negative) must be finite, but this is not true for \( E_0 \) in (5) as we have stated several times earlier. As a matter of fact, zeros of \( E_0 \) are not even isolated. A number of important changes and steps are to be covered to reach our objective (8).

3.1. Some changes and new difficulties. Our functional \( E_0 \) in (5) misses all of the requirements in Theorem 7. In the natural space \( H^1([0, 1]; \mathbb{R}^2) \) of absolutely continuous paths with square-integrable, (weak) derivatives, where \( E_0 \) is defined, it is not even coercive. The solution we propose to this difficulty is to pass to a smaller Hilbert space of more regular paths, namely, \( H^2([0, 1]; \mathbb{R}^2) \), whose paths have components with a weak derivatives up to order two which are square-integrable, and to modify our first version of the functional \( E_0 \) to the perturbation

\[
E_\epsilon(u) = E_0(u) + \frac{\epsilon}{2} \| u \|_{H^2([0, 1]; \mathbb{R}^2)}^2, \quad u = (x, y).
\]

For each fixed, positive \( \epsilon \), this functional is coercive in \( H^2([0, 1]; \mathbb{R}^2) \). We can make it comply with all of the requirements in Theorem 7 if we perturb it further to

\[
(9) \quad E_\epsilon(u) = E_0(u) + \frac{\epsilon}{2} \| u \|_{H^2}^2 + \langle v_\epsilon, u \rangle + \frac{1}{2\epsilon} \| v_\epsilon \|^2,
\]
where the path $v_\varepsilon$ is to be chosen appropriately in order that $E_\varepsilon$ precisely satisfies all of the necessary requirements of Theorem 7.

On the other hand, the space of admissible paths that we would like to consider need to be restricted as well. They must be 1-periodic

$$u(0) = u(1), \quad u'(0) = u'(1),$$

and have rotation index +1 to avoid redundant and useless multiplicity. Our ambient space $H$ will thus be the subspace of $H^2([0,1];\mathbb{R}^2)$ complying with (10), and generated by those paths with rotation index +1. We will denote by $H^2_{O,+1}([0,1];\mathbb{R}^2)$ this Hilbert space, which is a subspace (with the same inner product) of $H^2([0,1];\mathbb{R}^2)$.

These unavoidable changes give rise to new difficulties. To begin with, a new small positive parameter $\varepsilon$ enters into the process, and so everything depends on $\varepsilon$ except the ambient space: the functional, the critical paths, the Morse indexes and the numbers $M_k$, etc. In particular, absolute minimizers of $E_0$, in particular limit cycles of system (1), may no longer be critical paths for $E_\varepsilon$ though somehow, for small $\varepsilon$, we expect them not to be far from being so. Moreover, functional $E_\varepsilon$ is a singular perturbation of $E_0$. It is well-known that these problems are delicate, and require fine arguments in proofs.

3.2. Steps to be covered. We will be working with the functional $E_\varepsilon$ defined over $H^2_{O,+1}([0,1];\mathbb{R}^2)$ where $\varepsilon$ can be chosen as small as may be convenient. We plan to apply Theorem 7 to this situation. Our first main step is then the following.

- Show that a path $v_\varepsilon \in H^2_{O,+1}([0,1];\mathbb{R}^2)$ can be chosen so that Theorem 7 can be applied to $E_\varepsilon$ given by (9). This is the content of Sections 5 and 6.

It is important, according to the brief discussion after Theorem 7, to clarify the use we pretend to make of Morse inequalities, and how to organize the counting procedure. As indicated, we will accomplish this by considering the restriction and validity of Morse inequalities to sublevel sets of $E_\varepsilon$ of the form $\{E_\varepsilon \leq a\}$, where $a$ could depend on $\varepsilon$.

- Discuss how Morse inequalities extend to subsets of $H^2_{O,+1}([0,1];\mathbb{R}^2)$ of the form $\{E_\varepsilon \leq a_\varepsilon\}$, $\{a_\varepsilon < E_\varepsilon \leq b_\varepsilon\}$, for non-critical values (of $E_\varepsilon$) $a_\varepsilon < b_\varepsilon$. Show how these versions of Morse inequalities can be utilized to find that

$$\sharp \{E_\varepsilon \leq a_\varepsilon\} \leq 1 + C\{a_\varepsilon < E_\varepsilon \leq b_\varepsilon\},$$

where the left-hand side is the number of connected components of $\{E_\varepsilon \leq a_\varepsilon\}$, and the right-hand side is the full set of critical paths contained in $\{a_\varepsilon < E_\varepsilon \leq b_\varepsilon\}$. This discussion can be found in Section 7.

The next important step would be:

- Each limit cycle of (1) identifies, in a one-to-one fashion, a connected component of the sublevel set $\{E_\varepsilon \leq a_\varepsilon\}$ where $a_\varepsilon > 0$ is a suitable, non-critical value. Hence, each limit cycle counts as unity in the left-hand side of (11). Section 8 includes this objective.
Each such component is informally called a valley of $E_\varepsilon$, as we have already indicated earlier, and it may contain smaller valleys inside. There is an additional special component identified with constant paths. It can be ignored at the expense of transforming (11) into
\begin{equation}
H(n) \leq C\{a_\varepsilon < E_\varepsilon \leq b_\varepsilon\}.
\end{equation}

The core of our estimate is the following.

- For $a_\varepsilon$ and $b_\varepsilon$ well chosen, (12) is valid, and one can show an upper bound
\begin{equation}
C\{a_\varepsilon < E_\varepsilon \leq b_\varepsilon\} \leq C(n)
\end{equation}
uniformly for every $\varepsilon$ sufficiently small, where $C(n)$ only depends on the degree of our differential polynomial system (1).

Once we have covered successfully all of our steps above, we would have our upper bound
\begin{equation}
H(n) \leq C(n).
\end{equation}

3.3. Main steps of the upper bound (13). We divide this last step, the upper bound (13), in various phases. For the counting procedure to be valid, it is necessary to restrict attention to a generic situation in which the connected components of the algebraic curve $P_x + Q_y = 0$ are homeomorphic to either a straight line or an oval.

- Argue how small perturbations of the components $P(x, y)$ and $Q(x, y)$ in (1), without changing their degree, produce a similar differential system which is generic in the sense just indicated. Show that the upper bound (14) extends unchanged for every polynomial differential system of the same degree, provided it is true for such generic differential systems. This argument is provided in Section 13.

As a consequence of this fact, we can restrict attention to such generic situations to find the upper bound (13) depending only on the degree $n$.

Since we aim at counting all of critical paths in a set of the form $\{a_\varepsilon < E_\varepsilon \leq b_\varepsilon\}$, we need to understand the defining features of critical paths. This requires to:

- Examine carefully the Euler-Lagrange system of optimality associated with $E_\varepsilon$ over our ambient space $H^2_{O,+1}([0,1];\mathbb{R}^2)$. A full analysis of this can be checked in Section 9.

The counting procedure itself proceeds in two main stages:

- Classify and count all possible different asymptotic behaviors of branches of critical paths of $E_\varepsilon$ as $\varepsilon$ tends to zero (Section 10). The sum $N + M$ in the statement of Theorem 2 provides an upper bound for the number of such asymptotic behaviors.
- For each possible behavior in the previous item, determine how many branches can possibly share the same asymptotic limit. We show that there cannot be more than $n$ of such branches where $n$ is the degree of system (1) (Section 11).
Section 12 pretends to summarize all of our previous conclusions to facilitate the counting procedure itself. It also explores how to select the values of $a_\varepsilon$ and $b_\varepsilon$ so that all of our arguments above are correct.

4. SOME ANALYTICAL PRELIMINARIES

We briefly state here some basic notions about spaces of functions with weak derivatives having suitable integrability properties, as well as recalling concepts like the coercivity of a functional. It may be convenient to do so for some interested readers not familiar with these concepts. We refer to [8] for a main, accessible source in this regard, and many more related information.

The underlying natural Hilbert space for $E_\varepsilon$ is

$$H = \{(x, y) : [0, 1] \to \mathbb{R}^2 : \int_0^1 [x^2 + y^2 + (x')^2 + (y')^2 + (x'')^2 + (y'')^2] \, dt < \infty\}.$$ 

This is nothing but the classical Sobolev space $H^2([0, 1]; \mathbb{R}^2)$ of paths with square-integrable weak derivatives up to order two. The inner product in this space is

$$\langle (x_1, y_1), (x_2, y_2) \rangle = \int_0^1 (x_1 x_2 + y_1 y_2 + x'_1 x'_2 + y'_1 y'_2 + x''_1 x''_2 + y''_1 y''_2) \, dt,$$

and the associated norm

$$\| (x, y) \|^2 = \int_0^1 [x^2 + y^2 + (x')^2 + (y')^2 + (x'')^2 + (y'')^2] \, dt.$$ 

Norms and inner products occurring henceforth are meant to be these. Paths in $H$ have continuous first derivatives. Since limit cycles are $C^\infty$, they belong to this space.

Coercivity for a general functional $E$ defined in a Hilbert space $H$ means that

$$E(u) \to +\infty \quad \text{as} \quad \|u\| \to \infty \text{ with } u \in H.$$ 

If a base functional $E_0$ defined in $H$ is non-negative, the perturbation

$$E_\varepsilon(u) = E_0(u) + \frac{\varepsilon}{2} \|u - u_\varepsilon\|^2$$

automatically becomes coercive for every positive $\varepsilon$, and every fixed element $u_\varepsilon$.

To summarize our analytical framework, we will concentrate on the functional

$$E_\varepsilon : H^2_{O,+1}([0, 1]; \mathbb{R}^2) \to \mathbb{R}^+.$$
where $H^{2}_{O}([0, 1]; \mathbb{R}^{2})$ is the subspace of $H^{2}_{O}([0, 1]; \mathbb{R}^{2})$ generated by paths having rotation index +1, and

$$H^{2}_{O}([0, 1]; \mathbb{R}^{2}) = \{ u \in H^{2}([0, 1]; \mathbb{R}^{2}) : u(0) = u(1), u'(0) = u'(1) \},$$

$$\langle u, v \rangle = \int_{0}^{1} (u(t) \cdot v(t) + u'(t) \cdot v'(t) + u''(t) \cdot v''(t)) dt,$$

$$\| u \|^{2} = \| u \|_{H^{2}([0, 1]; \mathbb{R}^{2})}^{2} = \int_{0}^{1} (|u''(t)|^{2} + |u'(t)|^{2} + |u(t)|^{2}) dt,$$

$$E_{0}(u) = \frac{1}{2} \int_{0}^{1} (F^{\perp}(u) \cdot u')^{2} dt,$$

$$E_{\varepsilon}(u) = E_{0}(u) + \frac{\varepsilon}{2} \| u \|^{2} + \langle u, v_{\varepsilon} \rangle + \frac{1}{2\varepsilon} \| v_{\varepsilon} \|^{2} = E_{0}(u) + \frac{\varepsilon}{2} \| u - \frac{1}{\varepsilon} v_{\varepsilon} \|^{2},$$

and

$$u = (x, y), \quad F(u) = (P(x, y), Q(x, y)), \quad F^{\perp}(u) = (-Q(u), P(u)), \quad v_{\varepsilon} = (X_{\varepsilon}, Y_{\varepsilon}).$$

A fundamental fact for us to bear in mind is that convergence in $H^{2}([0, 1]; \mathbb{R}^{2})$ implies uniform convergence of first derivatives ($\mathbb{R}$).

5. The functional $E_{\varepsilon}$. Finiteness of the number of critical closed paths over finite level sets

We discuss again, now in greater detail, the analytical scenario in which we will be working. As indicated above, our first objective is to specify the underlying ambient space, and the form and properties of the perturbation $E_{\varepsilon}$ of our basic functional $E_{0}$ given by

$$E_{0}(u) = \frac{1}{2} \int_{0}^{1} (F^{\perp}(u) \cdot u')^{2} dt,$$

in such a way that $E_{\varepsilon}$ turns out to be a Morse functional (Definition 21 below), and so it will be eligible for applying all results in Section 7. This essentially amounts to checking assumptions in Theorem 7 stated in Section 3.

The natural initial space for our analysis is the Hilbert space $H^{2}_{O}([0, 1]; \mathbb{R}^{2})$ of 1-periodic paths with square integrable (weak) derivatives up to order two. However, it is evident that there is a useless multiplicity for each element of this space as regards its image set in $\mathbb{R}^{2}$. Each path in $H^{2}_{O}([0, 1]; \mathbb{R}^{2})$ admits infinitely many reparameterizations: many of them can be deformed continuously from each other without changing the image set of the path; but there are others which cannot be deformed continuously into each other without changing the image set. Think about the possibility of running each path in $H^{2}_{O}([0, 1]; \mathbb{R}^{2})$ several times, either counter- or clockwise. It will be very convenient for us to avoid such fruitless multiplicity. After all, for our counting procedure of limit cycles, we would like to identify each one of them through a single element of our ambient space. Our intention is to specify a certain subspace of $H^{2}_{O}([0, 1]; \mathbb{R}^{2})$ which will, essentially, contain a unique representative of each limit cycle.

Let $\mathcal{H}_{+1}$ stand for the subset of $H^{2}_{O}([0, 1]; \mathbb{R}^{2})$ incorporating all paths which are one-to-one (no self-intersections), and have rotation index +1. Recall that, by the classical theorem of H. Hopf 24, the rotation index of such a smooth curve (in particular
paths in $H^2_0([0,1];\mathbb{R}^2)$ are) is either $+1$ or $-1$, depending on whether they are oriented counter- or clock-wise.

**Definition 8.** We will designate by $H^2_{O,+1}([0,1];\mathbb{R}^2)$ the Hilbert subspace of $H^2_0([0,1];\mathbb{R}^2)$ generated by $\mathcal{H}_{+1}$.

Paths in $H^2_{O,+1}([0,1];\mathbb{R}^2)$ are limits (in the norm of the Hilbert space $H^2([0,1];\mathbb{R}^2)$, and uniformly in $\mathcal{C}^1$) of paths in $\mathcal{H}_{+1}$. In particular, elements of $H^2_{O,+1}([0,1];\mathbb{R}^2)$ need not be one-to-one (constant paths belong to $H^2_{O,+1}([0,1];\mathbb{R}^2)$, for instance). On the other hand, paths for which the rotation index is well-defined and is different from $+1$ need not be one-to-one (constant paths belong to $H^2_{O,+1}([0,1];\mathbb{R}^2)$, for instance). There is still the ambiguity of preserving the image set of a path in $H^2_{O,+1}([0,1];\mathbb{R}^2)$ with infinitely many reparameterizations, but these will not pose a particular difficulty as they can be deformed continuously into each other without changing an essential ingredient of our functional as we will see below.

The next important result clearly specifies the form of our perturbations $E_\varepsilon$ and its main properties as concerns the possibility of using Morse inequalities. Its proof is the goal of this and the next sections.

**Theorem 9.** For every positive $\varepsilon$, there is a $C^\infty$-one-to-one path $v_\varepsilon \in H^2_{O,+1}([0,1];\mathbb{R}^2)$ such that $\|v_\varepsilon\| \leq K\varepsilon$, with $K$ independent of $\varepsilon$, positive and sufficiently small (in particular, $\|v_\varepsilon\| \to 0$ as $\varepsilon \searrow 0$), and the perturbed functional

$$E_\varepsilon(u) = E_0(u) + \frac{\varepsilon}{2}\|u\|^2 + \langle u, v_\varepsilon \rangle + \frac{1}{2\varepsilon}\|v_\varepsilon\|^2$$

is non-negative, coercive, $C^2$, complies with the Palais-Smale condition, and has a finite number (possibly depending on $\varepsilon$ and $\alpha$) of non-degenerate critical closed paths in every finite level set of the form $\{E_\varepsilon \leq \alpha\}$ for arbitrary non-critical value $\alpha$.

Theorem 9 will be proved later, after Lemma 15 below.

Note that the terms added to $E_0$ are such that

$$\frac{\varepsilon}{2}\|u\|^2 + \langle u, v_\varepsilon \rangle + \frac{1}{2\varepsilon}\|v_\varepsilon\|^2 = \frac{\varepsilon}{2}\|u + \frac{1}{\varepsilon}v_\varepsilon\|^2.$$

For proving this main result, we need some preliminary abstract definitions and facts, which we state next for the sake of completeness, most of which can be found in the book of Berger [4] among others.

Suppose that $E : \mathcal{H} \to \mathbb{R}$ is a smooth $C^2$-functional defined in a Hilbert space $\mathcal{H}$. We shall use the following definitions.

(i) A critical point $x \in \mathcal{H}$ is called non-degenerate if the self-adjoint operator $E''(x) : \mathcal{H} \to \mathcal{H}$ is invertible. Otherwise, $x$ is said to be degenerate.

(ii) An element $x \in \mathcal{H}$ is called a regular point for a non-linear $C^1$-operator $F : \mathcal{H} \to \mathcal{H}$ if the linear operator $F'(x) : \mathcal{H} \to \mathcal{H}$ is surjective. Otherwise, $x$ is called a singular point for $F$. When $F$ is the derivative of a $C^2$-functional $E : \mathcal{H} \to \mathbb{R}$, then a critical point $x$ of $E$ is degenerate (respectively, non-degenerate) if it is singular (respectively, regular) for $F = E'$. Note that in this case $F' = E''$ is a self-adjoint operator, and so it is surjective if and only if it is bijective, see Section 2.7 in [8], for instance. The image $F(x)$ of a singular point $x$ is called a singular value of $F$. 


(iii) A mapping $F : H \to H$ is a **non-linear Fredholm operator** if its Fréchet derivative $F'(x) : H \to H$ is a linear Fredholm map for each $x \in H$. The **index** of $F$ is defined to be the difference of the dimensions of the kernel and the cokernel of $F'(x)$. This index is independent of $x$.

(iv) The functional $E$ is a **Fredholm functional** if $E' : H \to H$ is a Fredholm mapping, i.e. if $E''(x) : H \to H$ is a linear Fredholm map for each $x \in H$.

We state several interesting facts (page 100 in [4]).

**Proposition 10.** The following statements hold.

a) Any diffeomorphism between Banach spaces is a Fredholm map of index zero.

b) If $F$ is a Fredholm map, and $G$ is a compact operator, then the sum $F + G$ is also Fredholm with the same index as $F$.

We recall two additional classic results. The first one is the Inverse Function Theorem (page 113, [4]) for Banach spaces.

**Theorem 11.** Let $F$ be a $C^1$-mapping defined in a neighborhood of some point $\bar{x}$ of a Banach space $X$, with range in a Banach space $Y$. If $F'(\bar{x})$ is a linear homeomorphism of $X$ onto $Y$, then $F$ is a local homeomorphism of a neighborhood $U(\bar{x})$ of $\bar{x}$ to a neighborhood of $F(\bar{x})$.

The second one is a version of Sard’s theorem for infinite-dimensional spaces (page 125 of [4]).

**Theorem 12.** Let $F$ be a $C^q$-Fredholm mapping of a separable Banach space $X$ into a separable Banach space $Y$. If $q > \max(\text{index } F, 0)$, the set of singular values of $F$ are nowhere dense (its closure has empty interior) in $Y$.

The proof of Theorem 9 will make use of Proposition 10, and Theorems 11 and 12. The use of these general results requires the compactness of $E_0'$ as a main ingredient. Moreover, we will need to show the Palais-Smale condition for $E_\varepsilon$. We treat these two issues in the next two subsections.

5.1. **Compactness of $E_0'$.** For our functional $E_0$, it is easy to find an expression for $\langle E_0'(u), U \rangle$ for $u$ and $U$ in $H^2_{O,1}([0,1]; \mathbb{R}^2)$. Indeed, by definition we have

$$
\langle E_0'(u), U \rangle = \frac{d}{d\tau} E_0(u + \tau U) \bigg|_{\tau=0}.
$$

Since

$$
E_0(u + \tau U) = \frac{1}{2} \int_0^1 \left[ F^\perp(u + \tau U) \cdot (u' + \tau U') \right]^2 dt,
$$

then from (16) we have

$$
\langle E_0'(u), U \rangle = \int_0^1 \left( F^\perp(u) \cdot u' \right) \left[ (D^2 F^\perp(u) U) \cdot u' + F^\perp(u) \cdot U' \right] dt.
$$

We are, therefore, seeking an element $v(= E_0'(u)) \in H^2_{O,1}([0,1]; \mathbb{R}^2)$ such that

$$
v \cdot U = \langle E_0'(u), U \rangle,
$$
that is to say
\[(18) \quad \int_0^1 (v \cdot U + v' \cdot U' + v'' \cdot U'') \, dt = \int_0^1 (F^+(u) \cdot u') \left[ (DF^+(u)U) \cdot u' + F^+(u) \cdot U' \right] \, dt \]
for every \( U \in H^2_{O,+1}([0,1];\mathbb{R}^2) \). There is a unique such \( v \), which turns out to be the minimizer (with respect to \( U \in H^2_{O,+1}([0,1];\mathbb{R}^2) \)) of the augmented functional
\[(19) \quad \int_0^1 \left[ \frac{1}{2} \|U'' \|^2 + \frac{1}{2} \|U' \|^2 + \frac{1}{2} \|U \|^2 - (F^+(u) \cdot u') \right] \left[ (DF^+(u)U) \cdot u' + F^+(u) \cdot U' \right] \, dt.\]
The existence of a unique minimizer for this problem, which is quadratic, is a consequence of the classical Lax-Milgram Theorem (see Corollary 5.8 of [8] for instance).

Therefore the equation for \( v = E'_0(u) \in H^2_{O,+1}([0,1];\mathbb{R}^2) \) will be the associated Euler-Lagrange system for the functional \((18)\) as it is given by this last theorem
\[(20) \quad [v'' - (v' + (F^+(u) \cdot u')F^+(u))]' + v + (F^+(u) \cdot u')u^TDF^+(u) = 0 \text{ in } (0,1).\]
Its weak formulation is exactly \((18)\).

**Lemma 13.** Let \( \{u_j\} \) be a uniformly bounded sequence in \( H^2_{O,+1}([0,1];\mathbb{R}^2) \), and \( \{v_j\} \) the sequence of derivatives \( v_j = E'_0(u_j) \in H^2_{O,+1}([0,1];\mathbb{R}^2) \) which are solutions of \((20)\) for \( u = u_j \). Then \( \{v_j\} \) is relatively compact in \( H^2_{O,+1}([0,1];\mathbb{R}^2) \).

**Proof.** For the sake of brevity, set
\[
G_j \equiv (F^+(u_j) \cdot u_j')F^+(u_j), \quad H_j \equiv (F^+(u_j) \cdot u_j')u^TDF^+(u_j).
\]
If \( \{u_j\} \) is uniformly bounded in \( H^2_{O,+1}([0,1];\mathbb{R}^2) \), we know that a certain subsequence (not relabelled) of \( \{u_j\} \) converges weakly to some \( u \) in \( H^2([0,1];\mathbb{R}^2) \). Set
\[
G \equiv (F^+(u) \cdot u')F^+(u), \quad H \equiv (F^+(u) \cdot u')u^TDF^+(u).
\]
If we put \( v_j = E'_0(u_j) \) and \( v = E'_0(u) \), then \((18)\) implies
\[
\int_0^1 [v'' \cdot U'' + (v_j' + G_j) \cdot U' + (v_j + H_j) \cdot U] \, dt = 0,
\]
\[
\int_0^1 [v'' \cdot U'' + (v' + G) \cdot U' + (v + H) \cdot U] \, dt = 0,
\]
for every \( U \in H^2_{0,+1}([0,1];\mathbb{R}^2) \). By substracting one from the other
\[
\int_0^1 [(v_j'' - v'') \cdot V'' + (v_j' - v') + G_j - G) \cdot V' + (v_j - v + H_j - H) \cdot V] \, dt = 0
\]
for every \( U \in H^2_{0,+1}([0,1];\mathbb{R}^2) \). We can take \( U = v_j - v \) to find that
\[
\int_0^1 [(v_j'' - v'')^2 + (v_j' - v' + G_j - G) \cdot (v_j' - v') + (v_j - v + H_j - H) \cdot (v_j - v)] \, dt = 0.
\]
This equality can be reorganized as
\[
\|v_j - v\|^2_{H^2([0,1];\mathbb{R}^2)} = -\int_0^1 [(G_j - G) \cdot (v_j' - v') + (H_j - H) \cdot (v_j - v)] \, dt,
\]
Hence, by the standard Hölder inequality for integrals, we can also have
\[(21) \quad \|v_j - v\|_{H^2([0,1];\mathbb{R}^2)} \leq \|G_j - G\|_{L^2([0,1];\mathbb{R}^2)} + \|H_j - H\|_{L^2([0,1];\mathbb{R}^2)}.\]
Since the weak convergence of $u_j$ to $u$ in $H^2_{O,+1}([0, 1]; \mathbb{R}^2)$ implies weak convergence up to second derivatives, by the classical Rellich–Kondrachov Theorem, which implies that the injection $W^{2,p} \subset W^{1,p}$ is always compact (see Theorem 9.16 in [8] for the case with first derivatives $W^{1,p} \subset L^p$), we conclude the convergences

$$G_j \to G, \quad H_j \to H$$

strongly in $L^2([0, 1]; \mathbb{R}^2)$. The proof is then a direct consequence of (21). \hfill $\Box$

As a direct consequence of Lemma 13 we have:

**Corollary 14.** The map $E_0' : H^2_{O,+1}([0, 1]; \mathbb{R}^2) \to H^2_{O,+1}([0, 1]; \mathbb{R}^2)$ is compact.

This compactness property is the only reason why the functional $E_0$ has to be perturbed by a norm involving up to second derivatives. If we had just perturbed $E_0$ up to first derivatives, we would not have the strong convergence of the vector fields $G_j$ and $H_j$ to $G$ and $H$ respectively in the proof of Lemma 13.

### 5.2. Palais-Smale property for $E_\varepsilon$

If $E$ is coercive, we can replace the boundedness of $E$ along the sequence $\{u_j\}$ by the uniform boundedness of $\{u_j\}$ in $H$. We need to show that our perturbation $E_\varepsilon$ in (15) below complies with this the Palais-Smale property.

**Lemma 15.** For each positive $\varepsilon$ fixed, the functional $E_\varepsilon$ is bounded from below, coercive, and enjoys the Palais-Smale property.

**Proof.** Note first that the identity

$$\frac{\varepsilon}{2} \|u\|^2 + (u, v_\varepsilon) = \frac{\varepsilon}{2} \|u + 1 + v_\varepsilon\|^2 - \frac{1}{2\varepsilon} \|v_\varepsilon\|^2$$

then together with the fact that $E_0 \geq 0$, implies the coercivity of each $E_\varepsilon$ on $u$. On the other hand, $E_\varepsilon' = E_0' + \varepsilon 1 + v_\varepsilon$, where $1 : H \to H$ is the identity operator.

Suppose $\{u_j\}$ is uniformly bounded. Since $E_0'$ is compact (Corollary 14), there is a subsequence $u_j$ (not relabelled) such that $E_0'(u_j) \to \overline{u}$ for some $\overline{u}$. Under the Palais-Smale conditions, if $E_\varepsilon'(u_j) \to 0$, due to the identity $E_\varepsilon' = E_0' + \varepsilon 1 + v_\varepsilon$, we have

$$\varepsilon u_j = E_\varepsilon'(u_j) - E_0'(u_j) - v_\varepsilon \to -\overline{u} - v_\varepsilon \text{ as } j \to \infty.$$

Hence $\{u_j\}$ converges strongly in $H$. This is just the Palais-Smale property for each $E_\varepsilon$. \hfill $\Box$

### 5.3. Main proof

We are now in a position to prove Theorem 9. Recall that, except for an harmless additive constant,

$$E_\varepsilon(u) = \int_0^1 \left[ \frac{1}{2} (F^+(u(t))) \cdot u'(t))^2 + \frac{\varepsilon}{2} (|u''(t)|^2 + |u'(t)|^2 + |u(t)|^2) 
+ (u(t) \cdot v_\varepsilon(t) + u'(t) \cdot v_\varepsilon'(t) + u''(t) \cdot v_\varepsilon''(t)) \right] dt.$$

As a matter of fact, the proof of Theorem 9 is essentially described in abstract terms in page 355 of [4]. For the sake of completeness we prove it using all the preliminary results in this section.
Proof of Theorem 9. Consider the functional $\tilde{E}_\varepsilon : H^2_{O,+1}([0,1]; \mathbb{R}^2) \to \mathbb{R}$ given by

$$\tilde{E}_\varepsilon(u) = E_0(u) + \frac{\varepsilon}{2} \|u\|^2.$$ 

Its derivative $F(u) = \tilde{E}_\varepsilon'(u)$ is the sum of a diffeomorphism, $\varepsilon 1$, and a compact operator, $E'_0$. By Proposition 10, this derivative is a Fredholm operator of index zero. By Theorem 12, the set of critical values of the derivative $F$, that is

$$\{ F(u) \in H^2_{O,+1}([0,1]; \mathbb{R}^2) : F'(u) = 0 \},$$

is nowhere dense, and consequently, we can choose an element $v_\varepsilon \in H^2_{O,+1}([0,1]; \mathbb{R}^2)$, with the properties claimed in the statement of the theorem, so that every solution $u$ of the equation

$$F(u) + v_\varepsilon = 0$$

is not a singular point for $F$, i.e. $F'(u) = \tilde{E}_\varepsilon''(u)$ is bijective, and so $u$ is non-degenerate. This argument implies indeed that the critical closed paths of $E_\varepsilon$ are non-degenerate, once $v_\varepsilon$ has been chosen in this way and has been added to $\tilde{E}_\varepsilon$ because we have $E'_\varepsilon(u) = \tilde{E}_\varepsilon''(u)$.

The Inverse Function Theorem 11 implies directly that non-degenerate critical closed paths of a $C^2$-functional, $E_\varepsilon$, are isolated.

Finally, we argue why the number of critical paths in sets of the form $\{ E_\varepsilon \leq \alpha \}$ is finite. Indeed, if we let $a$ be a positive real number and assume that there is an infinite number $\{ u_j \}$ of critical closed paths with $E_\varepsilon(u_j) \leq \alpha$, and $E'_\varepsilon(u_j) = 0$, the Palais-Smale condition for $E_\varepsilon$ would ensure the existence of a suitable subsequence converging to some $\tilde{u}$ which would be a critical, non-isolated path. This is a contradiction with the previous statement about the fact that the critical closed paths are isolated, and so the number of such critical closed paths is finite. This completes the proof of Theorem 9. \hfill \Box

6. Finiteness of Morse indexes for critical points of $E_\varepsilon$

Another main issue to apply Morse inequalities demands to have finite Morse indexes for every critical, non-degenerate path of $E_\varepsilon$. To prove this is the main goal of this section.

We need to focus on the Hessian $E''_\varepsilon(u)$ at a critical path $u$, $E'_\varepsilon(u) = 0$, and show that it has a finite number of negative eigenvalues, that could depend of $u$ and on $\varepsilon$. $E_\varepsilon$ has a linear part, which drops out in the Hessian, a quadratic part coming from the norm in $H^2([0,1]; \mathbb{R}^2)$, and a non-linear, lower-order term. Indeed

$$E''_\varepsilon(u) = E''_0(u) + \varepsilon 1.$$ 

Once again, the main fact in which we can support our proof is the compactness of the linear operator

$$E'_0(u) : H^2_{O,+1}([0,1]; \mathbb{R}^2) \to H^2_{O,+1}([0,1]; \mathbb{R}^2)$$

for each fixed $u$.

If, in general, we have a certain smooth, $C^2$-functional $E : H \to \mathbb{R}$ over a Hilbert space $H$ with derivative $E' : H \to H$, there are various ways to deal with the second
derivative, but probably the best suited for our purposes is to consider the derivative
\[ \langle E''(u), (U, \overline{U}) \rangle = \frac{d}{d\delta} \bigg|_{\delta=0} \langle E'(u + \delta U), \overline{U} \rangle, \]
where both vector fields \( U \) and \( \overline{U} \) belong to \( H \). In our situation, and in view of (17), we have
\[ \langle E''_0(u), (U, \overline{U}) \rangle = \frac{d}{d\delta} \bigg|_{\delta=0} \int_0^1 \left( F^\perp(u + \delta U) \cdot (u' + \delta U') \right) \left( \nabla F^\perp(u + \delta U) \cdot (u' + \delta U') \right) + F^\perp(u + \delta U) \cdot \overline{U} \right) \, dt \]
\[ = \int_0^1 \left( F^\perp(u) \cdot u' \right) \left( \nabla^2 F^\perp(u) : (U, U', u') + \nabla F^\perp(u) : (U', U') \right) + \left( F^\perp(u) : (U, U') \right) \right) \, dt \]
\[ + \int_0^1 \left( \nabla F^\perp(u) U \cdot u' + F^\perp(u) \cdot U' \left( \nabla F^\perp(u) U \cdot u' + F^\perp(u) \cdot \overline{U} \right) \right) \, dt \]
Through this long formula, we can understand, for such a \( u \) given and fixed, the linear operator
\[ E''_0(u) : \mathcal{H}^2_{O,+1}(0,1,\mathbb{R}^2) \to \mathcal{H}^2_{O,+1}(0,1,\mathbb{R}^2) \]
Let \( U \in \mathcal{H}^2_{O,+1}(0,1,\mathbb{R}^2) \) be given. The image
\[ V = E''_0(u) U \in \mathcal{H}^2_{O,+1}(0,1,\mathbb{R}^2) \]
of \( U \) under the linear map \( E''_0(u) \) is determined through
\[ \langle E''_0(u), (U, \overline{U}) \rangle = \langle E''_0(u) U, \overline{U} \rangle = \langle V, \overline{U} \rangle \]
for all \( \overline{U} \in \mathcal{H}^2_{O,+1}(0,1,\mathbb{R}^2) \). The element \( V \) defined through (23) is the solution of a standard quadratic variational problem, for which the weak form of its optimality condition is precisely (23). This form is especially suited to show the compactness we are after. Set
\[ F = (F^\perp(u) \cdot u') \nabla^2 F^\perp(u) : u' + \nabla F^\perp(u) \otimes \nabla F^\perp(u) u', \]
\[ G = \nabla F^\perp(u) + F^\perp(u) \otimes \nabla F^\perp(u) u', \]
\[ H = \nabla F^\perp(u) + \nabla F^\perp(u) u' \otimes F^\perp(u), \]
\[ J = F^\perp(u) \otimes F^\perp(u). \]
This choice is dictated so that
\[ \langle E''_0(u), (U, \overline{U}) \rangle = \int_0^1 \left[ F(U, \overline{U}) + G(U', \overline{U}) + H(U, \overline{U}) + J(U', \overline{U}) \right] \, dt. \]
Exactly as in Lemma 13 one can show the following.

**Lemma 16.** For fixed, given \( u \in \mathcal{H}^2_{O,+1}(0,1,\mathbb{R}^2) \), the operator
\[ U \mapsto V = E''_0(u) U \]
is self-adjoint and compact.

**Proof.** Assume \( \{U_j\} \) is bounded in \( \mathcal{H}^2_{O,+1}(0,1,\mathbb{R}^2) \). In particular, and for reasons already pointed out earlier, \( U_j' \to U \) uniformly for some \( U \in \mathcal{H}^2_{O,+1}(0,1,\mathbb{R}^2) \). Let \( V_j \) and \( V \) determined through (23), respectively. Then
\[ \|V_j - V\|^2 = \langle E''_0(u)(U_j - U), V_j - V \rangle, \]
and
\[ \|V_j - V\| \leq \|E''_0(u)(U_j - U)\| . \]

The key point is to realize in the formulas above that in \( E''_0(u)(U_j - U) \) only up to first derivatives of the differences \( U_j - U \) occur, and these converge strongly to zero. Hence
\[ \|V_j - V\| \to 0. \]

\[ \square \]

Theorem 17. For each \( \varepsilon \) positive, every critical point of \( E_\varepsilon \) is non-degenerate, and has a finite Morse index.

Proof. The proof relies on the standard fact that eigenvalues of a linear, self-adjoint, compact operator in a Banach space, like \( E''_0(u) \), always has a sequence of (real) eigenvalues converging to zero (see, for instance, Chapter 6 of [3]). According to (22), eigenvalues of \( E''_\varepsilon(u) \) are eigenvalues of \( E''_0(u) \) plus \( \varepsilon \), and so there cannot be an infinite number of negative eigenvalues. \[ \square \]

7. Morse inequalities

The discussion in this section is well-known to experts in Morse theory. Since most likely many interested readers will not be familiar with this material, we have made an special effort in explaining facts in the most transparent and intuitive way that we have found with precise references to formal sources. For proofs of main results that we are about to state, we refer to our two principal references [4], [9]. Simple proofs of some specific consequences of more general results that we will be employing are indicated.

The statement of Morse inequalities, which is our main tool, involves the notion of Morse index for a non-degenerate critical point of a smooth functional. Suppose \( E : H \to \mathbb{R} \) is a non-negative, coercive, \( C^2 \)-functional defined over a Hilbert space \( H \). Let \( u \in H \) be a critical point of \( E \), i.e. \( E'(u) = 0 \). A real number \( c \in \mathbb{R} \) is a critical value of \( E \) if there is a critical point \( u \), \( E'(u) = 0 \), such that \( c = E(u) \).

Definition 18. We say that the critical point \( u \) of \( E \) is non-degenerate if the linear mapping \( E'' : H \to H \) is non-singular. If the number of negative eigenvalues of \( E''(u) \) is finite, such number is called the (Morse) index of \( u \).

Morse inequalities can be found in several places, for instance, Corollary (6.5.10) of [3] or Theorem 4.3 of Chapter 1 in [9]. A main, indispensable condition for these inequalities to hold is the Palais-Smale property, (see Section 5.2). For a general, smooth \( C^1 \)-functional \( E : H \to \mathbb{R} \), this important compactness property reads:

If for a sequence \( \{u_j\} \) we have that \( E(u_j) \leq K \) for all \( j \) and a fixed positive constant \( K \), and \( E'(u_j) \to 0 \) as \( j \to \infty \), then a certain subsequence of \( \{u_j\} \) converges (strongly) in \( H \).

If \( E \) is coercive, we can replace the boundedness of \( E \) along the sequence \( \{u_j\} \) by the uniform boundedness of \( \{u_j\} \) in \( H \). We state again Theorem 7 which is the version of Morse inequalities in [3] (Corollary (6.5.10) as indicated above).
Theorem 19. Let $E : H \to \mathbb{R}$ be a $C^2$-functional, bounded from below, coercive, enjoying the Palais-Smale property, and having a finite number of critical points, all of which are non-degenerate and of a finite index. Put $M_k$ for the (finite) number of critical points of each fixed index $k$. Then

\begin{equation}
M_0 \geq 1, \quad M_1 - M_0 \geq -1, \quad M_2 - M_1 + M_0 \geq 1, \quad \ldots, \quad \sum_{k=0}^{\infty} (-1)^k M_k = 1.
\end{equation}

Notice that under the assumed hypotheses, all sums involved in this statement are finite sums.

We will show several special situations in which this fundamental result can be used. The first one focuses on the validity of the same inequalities for “valleys” of $E$.

Definition 20. Let $E : H \to \mathbb{R}$ be a $C^2$-functional. A bounded, connected and with $C^1$-boundary subset $H \subset H$ is a valley for $E$, if

$$\langle E'(u), \nu(u) \rangle > 0$$

for every $u$ in the boundary $\partial H$ of $H$, where $\nu(u)$ is the outer normal vector to $H$.

Note that if $a$ is a non-critical value of $E$, then every connected component of the level set $\{E \leq a\}$ is a valley. This observation makes the following definition suitable for our purposes. Note that

$$\{E \leq a\} = \{u \in H : E(u) \leq a\}.$$

Definition 21. A functional $E : H \to \mathbb{R}$ is called a Morse functional if it is $C^2$-, non-negative, coercive, enjoys the Palais-Smale property, and has a finite number of critical points over each level set $\{E \leq a\}$ for each non-critical value $a$, all of which are non-degenerate and with a finite index.

Morse theory is concerned about Morse functionals. Indeed, Corollary (6.5.11) in [4] reads as follows.

Proposition 22. Let $E : H \to \mathbb{R}$ be a Morse functional. Let $H$ be a valley of $E$ (in particular a connected component of the level set $\{E \leq a\}$ for a non-critical value $a$). Then Morse inequalities (24) are valid restricted to critical points of $E$ in $H$.

This result shows that Morse inequalities are valid restricted to every valley of $E$. As a matter of fact, we will be in need of a much more general version of it. As just indicated, Proposition 22 is proved in [4] for balls, but the proof is exactly the same for arbitrary bounded, connected domains $H$, as the sentence before the statement of this corollary in [4] asserts.

Another special situation where Morse inequalities can be used refers to its extension to infinite dimensional manifolds modeled over Hilbert spaces. Such generalization is treated in a formal way in both of our basic references [4] and [9]. All of the main concepts necessary to state and prove them in this more general context (including the Palais-Smale condition) are extended in a natural way. Theorem 7 is also valid in the context of such a manifold, though the statement of Morse inequalities in this case involve Betti numbers of the manifold, and it becomes more technical. Fortunately, we do not need to examine these more complex situations.
7.1. **Strategy to count critical points.** Assume we have \( p \) disjoint valleys \( H_i \) of \( E \) in \( H \), and let \( H \) be another large valley containing all of the \( H_i \)'s in its interior. Proposition 22 can be applied to all and every one of the \( H_i \)'s and to \( H \) too, and so

\[
p + \sum_i (\Sigma(H_i) + \Sigma(H \setminus \cup_i H_i)) = \Sigma(H) = 1.
\]

If we find a way to calculate (or to find a bound from above for) the sum

\[
-\sum_i (H \setminus \cup_i H_i) \leq q
\]

we will have \( p \leq q + 1 \). Note that (25) corresponds to Morse inequality for critical points in \( H \) not contained in the union of the valleys \( H_i \).

We have already emphasized several times that we plan to apply this strategy to a certain family of functionals \( E_\epsilon \), for each \( \epsilon > 0 \) and small, which are intimately related to our initial polynomial differential system. \( H \) will be \( \{E_\epsilon \leq b_\epsilon \} \) for appropriately chosen large constants \( b_\epsilon \) while \( H_i \) will be the connected components of \( \{E_\epsilon \leq a_\epsilon \} \) for small \( a_\epsilon \). It turns out (Section 8) that every limit cycle of the initial polynomial differential system determines one such different connected component of \( \{E_\epsilon \leq a_\epsilon \} \), for every \( \epsilon > 0 \) and small, and so we pretend to bound the number \( p \) of limit cycles through the number \( q(= q_\epsilon) \) in (25). Such a bound would be sharp if we could estimate the Morse sum \( \sum_i (H \setminus \cup_i H_i) \) in (25). However, since in general it does not look plausible to determine exactly the Morse index of critical paths, we will be contented with the upper estimate

\[
-\sum_i (H \setminus \cup_i H_i) \leq C \{ a_\epsilon < E_\epsilon \leq b_\epsilon \}
\]

where the right-hand side is the total number of critical points of \( E_\epsilon \) in the corresponding set all with a plus sign. Hence, we recover (11)

\[
\sharp \{ E_\epsilon \leq a_\epsilon \} \leq 1 + C \{ a_\epsilon < E_\epsilon \leq b_\epsilon \}
\]

as our method to find the desired bound for the number of limit cycles of our planar, polynomial, differential system.

There are many more general results on the validity of Morse inequalities restricted to subsets of \( H \) other than the one in Proposition 22 for valleys. See some of these in Section 6.1 in page 55 of [9].

8. **Identification of limit cycles with low valleys of \( E_\epsilon \), and relevant critical closed paths**

It is elementary to realize that each limit cycle (regardless of how it is parameterized in \([0, 1]\)) identifies one valley of the level set \( \{E_\epsilon \leq a_\epsilon \} \) for \( a_\epsilon \), non-critical and sufficiently small (depending on \( \epsilon \)). The main issue we would like to address, and which is fundamental for our strategy, is to show that two distinct limit cycles cannot lie in the same connected component of \( \{E_\epsilon \leq a_\epsilon \} \) if \( a_\epsilon \) is non-critical and sufficiently small. Keep in mind the possibility, for each limit cycle, of changing the starting point, or of reparameterizing the curve. The possibility of describing limit cycles by reparameterization going around several times counter- or clock-wise has been discarded in our space \( H^2_{O,+1}([0,1]; \mathbb{R}^2) \).

Each path \( u \in H^2_{O,+1}([0,1]; \mathbb{R}^2) \) whose image set \( \{ u(t) : t \in [0, 1] \} \subset \mathbb{R}^2 \) is a certain limit cycle will give rise to a certain low (low because \( a_\epsilon \) will be chosen small) valley of
Corollary 9.13 in \[8\]) to some path \(L\) able sequence (not relabeled) converges weakly in the same space, but strongly in \(\varepsilon(26)\).

Let \(u\) and \(v\) be two elements of our space \(H^2_{O,1+1}([0,1];\mathbb{R}^2)\) representing two different limit cycles of our differential system. For \(a_\varepsilon\) non-critical and sufficiently small, they cannot belong to the same connected component of \(\{E_\varepsilon \leq a_\varepsilon\}\).

**Proof.** It is clear that \(u\) and \(v\) are in different valleys of \(E_0\) because it is not possible to find a family of continuous closed paths which are formed by solutions of the system which connect \(u\) with \(v\). This is due to the fact that near each limit cycle the orbits spiral. Therefore for sufficiently small \(\varepsilon\), valleys for \(E_\varepsilon\) determined by \(u\) and \(v\) persist.

For a more precise proof, we argue by contradiction. If the statement were not true, for every \(\varepsilon\) and every non-critical value \(a_\varepsilon\), we would have

\[E_\varepsilon(u) \leq a_\varepsilon, \quad E_\varepsilon(v) \leq a_\varepsilon,\]

and both in the same connected component of \(\{E_\varepsilon \leq a_\varepsilon\}\). Note that \(u\) and \(v\) could represent two nested limit cycles. Since we can take \(a_\varepsilon\) arbitrarily small, we would be capable of finding an homotopy

\[\sigma_\varepsilon(s,t) : [0,1] \times [0,1] \to \mathbb{R}^2, \quad \sigma_\varepsilon(0,\cdot) = u, \quad \sigma_\varepsilon(1,\cdot) = v, \quad \sigma_\varepsilon(s,\cdot) \in H^2_{O,1+1}([0,1];\mathbb{R}^2),\]

and \(E_\varepsilon(\sigma_\varepsilon(s,\cdot))\) arbitrarily small for every \(s \in [0,1]\) (because \(a_\varepsilon\) can be taken arbitrarily small). Since \(E_\varepsilon\) can be taken so that \(E_0 \leq E_\varepsilon\) (recall the statement of Theorem 9), we also conclude that \(E_0(\sigma_\varepsilon) \to 0\) as \(\varepsilon \to 0\). Put \(r_0 = \|u - v\| > 0\), and take \(r < r_0\). By reparameterization, if necessary, we can assume that

\[\|\sigma_\varepsilon([0,s_0]) - u\|_{H^2_{O,1+1}([0,1];\mathbb{R}^2)} \leq r/r_0,\]

\[\|\sigma_\varepsilon(s_0) - u\|_{H^2_{O,1+1}([0,1];\mathbb{R}^2)} = r/r_0,\]

for some fixed (independent of \(\varepsilon\)) \(s_0 \in [0,1]\). This implies that the sequence \(\{\sigma_\varepsilon\}\) is a uniformly bounded set in the space \(L^\infty([0,s_0];H^2_{O,1+1}([0,1];\mathbb{R}^2))\). Then a suitable sequence (not relabeled) converges weakly in the same space, but strongly in \(L^\infty([0,s_0];H^1([0,1];\mathbb{R}^2))\) (notice the change in the target space, see for more details Corollary 9.13 in [8]) to some path

\[\sigma : [0,s_0] \to H^1([0,1];\mathbb{R}^2)\]

that can be assume continuous, again by reparameterization if necessary, in the same space. In particular \(\sigma(0) = u\), and \(E_0(\sigma(s,\cdot)) = 0\) for every \(s \in [0,s_0]\) with a non-empty set

\[\{\sigma(s,\cdot) : s \in [0,s_0]\} \setminus \{u\},\]

because (26) implies

\[\|\sigma(s_0) - u\|_{H^1([0,1];\mathbb{R}^n)} = r/r_0.\]

This implies, because \(u\) is an isolated close path formed by orbits of system [1], that \(\sigma(s,\cdot)\) is a reparameterization of \(u\), in our space \(H^2_{O,1+1}([0,1];\mathbb{R}^2)\), for every \(s \in [0,s_0]\). Since the roles of \(u\) and \(v\) can be interchanged, we can conclude that both sets

\[I_u \equiv \{s \in [0,1] : \sigma(s) \text{ is a reparameterization of } u\},\]

\[I_v \equiv \{s \in [0,1] : \sigma(s) \text{ is a reparameterization of } v\},\]

are closed, non-empty (0 belongs to the first, and 1, to the second), and disjoint. The connectedness of \([0,1]\) leads the existence of values of \(t\) arbitrarily close to \(I_u\), for
instance, but not belonging to it, in such a way that $E_0(\sigma(t)) = 0$ for such values of $t$, $\sigma(t)$ is arbitrarily close to $u$ but it is not a reparameterization of $u$ if $t \notin I_u$. This is impossible because $E_0$ vanishes only over closed paths formed by orbits of system [1], and $u$ is one such isolated curve.

Notice how the argument in this proof is not valid if $u, v \in H^2O, +1([0, 1]; \mathbb{R}^2)$ represent the same limit cycle with different parameterizations because in this case both sets $I_u$ and $I_v$ would be the same and equal to the full interval $[0, 1]$. In such a situation, these two representations of the same underlying limit cycle are in the same valley (the same connected component) of $\{E_\varepsilon \leq a_\varepsilon\}$ for every $a_\varepsilon$ sufficiently small.

According to the discussion at the end of Subsection 7.1, each limit cycle of our differential system contributes with a unity to the number $p_\varepsilon$ of connected components of the level sets $\{E_\varepsilon \leq a_\varepsilon\}$ for suitable $a_\varepsilon$ sufficiently small and non-critical (what we have identified intuitively as low valleys). Hence we set to ourselves the task of examining Morse inequalities for critical closed paths $u_\varepsilon$ of $E_\varepsilon$ such that $E_\varepsilon(u_\varepsilon)$ stays away from zero, in order to find an upper bound, independent of $\varepsilon$, for the corresponding number $q_\varepsilon$ in Morse inequalities associated with critical closed paths in $H^2O, +1([0, 1]; \mathbb{R}^2)$ and with a value of $E_\varepsilon$ uniformly away from zero. The following proposition very clearly establishes the bound for limit cycles we seek.

**Proposition 24.** Let non-critical values $a_\varepsilon > 0$ be such that $a_\varepsilon \searrow 0$, and suppose that for non-critical, arbitrarily large values $b_\varepsilon$, $b_\varepsilon \to +\infty$, the number of critical paths of $E_\varepsilon$ contained in $C_\varepsilon = \{a_\varepsilon \leq E_\varepsilon \leq b_\varepsilon\}$ is less than or equal to a number $q_n$ independent of $\varepsilon$ and depending only on the degree of our polynomial differential system. Then there cannot be more than $q_n$ limit cycles.

*Proof.* According to Lemma 23, for $a_\varepsilon$ sufficiently small, each limit cycle will eventually determine a different connected component of $\{E_\varepsilon \leq a_\varepsilon\}$. If $p_\varepsilon$ is the number of such connected components, and $H_\varepsilon$ is the number of limit cycles of our differential system associated with some of those connected components, obviously $H_\varepsilon \leq p_\varepsilon$. Our discussion at the end of Subsection 7.1 implies that

$$p_\varepsilon = 1 - \Sigma(C_\varepsilon) \leq 1 + C(C_\varepsilon).$$

On the other hand, note that there is a distinguished connected component of $\{E_\varepsilon \leq a_\varepsilon\}$ which is associated with the vanishing constant path. This valley is also different from all those determined by true limit cycles. Therefore

$$H_\varepsilon + 1 \leq p_\varepsilon \leq 1 + C(C_\varepsilon) \leq q_n + 1.$$

From now on we focus on understanding critical closed paths of $E_\varepsilon$ for arbitrary small $\varepsilon$ for which $E_\varepsilon$ is away from zero, to see if Morse inequalities restricted to these may lead to our desired uniform bound.
9. The equation for critical closed paths

The object of this section is to derive and study the differential equations which must satisfy the critical closed paths of \( E \) in \( H^2_{O,1}([0,1];\mathbb{R}^2) \). The proof uses standard ideas in Calculus of Variations, but we include them for the sake of completeness.

**Theorem 25.** Let \( F(t,u,z,Z) : [0,1] \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \) be a \((C^\infty)\) function with respect to \((t,u,z,Z)\) with partial derivatives \( F_u, F_z, F_Z \). Suppose that the functional

\[
E(u) = \int_0^1 F(t,u(t),u'(t),u''(t)) \, dt
\]

admits a critical closed path \( u : [0,1] \to \mathbb{R}^2 \) in \( H^2_{O,1}([0,1];\mathbb{R}^2) \). Assume that \( F_u(t,v,v',v'') \) and

\[
F_z(t,v,v',v'') - \int_0^t F_u(t,v,v',v'') \, ds
\]

belong to \( L^1((0,1);\mathbb{R}^2) \) for every feasible \( v \in H^2_{O,1}([0,1];\mathbb{R}^2) \). Then the function

\[
\frac{d}{dt} F_Z(t,u,u',u'') - F_Z(t,u,u',u'')
\]

is absolutely continuous in \([0,1]\) and

\[
\frac{d}{dt} \left( \frac{d}{dt} F_Z(t,u,u',u'') - F_Z(t,u,u',u'') \right) + F_u(t,u,u',u'') = 0 \text{ for a.e. } t \text{ in } (0,1).
\]

Moreover

\[
[F_Z(t,u,u',u'')]_{t=0} = 0.
\]

Brackets in \((43)\) indicate the jump of the field inside at the time indicated (difference between \( t = 1 \), and \( t = 0 \)), that is

\[
[F_Z(t,u,u',u'')]_{t=0} = F_Z(t,u(t),u'(t),u''(t))|_{t=1} - F_Z(t,u(t),u'(t),u''(t))|_{t=0+}.
\]

Notice that the integrability demanded on those combinations of partial derivatives of \( F \) in the statement of Theorem 25 is equivalent to having \( F_u(t,v,v',v'') \) and \( F_z(t,v,v',v'') \) integrable for every feasible \( v \in H^2_{O,1}([0,1];\mathbb{R}^2) \). We have however decided to keep the statement as it is for that is exactly the form in which those combinations of partial derivatives will occur in the proof.

**Proof of Theorem 25.** Take \( U \in H^2_{O,1}([0,1];\mathbb{R}^2) \). If \( u \) is a critical closed path of \( E \), then

\[
\frac{d}{d\delta} E(u + \delta U)|_{\delta=0} = 0,
\]

that is to say

\[
\frac{d}{d\delta} \Bigg|_{\delta=0} \int_0^1 F(t,u(t) + \delta U(t),u'(t) + \delta U'(t),u''(t) + \delta U''(t)) \, dt = 0.
\]

This derivative has the form

\[
\int_0^1 \left[ F_u(t,u,u',u'') \cdot U(t) + F_z(t,u,u',u'') \cdot U'(t) + F_Z(t,u,u',u'') \cdot U''(t) \right] \, dt = 0.
\]
We consider the special subspace $\mathbb{L}$ of variations $U$ defined by
\begin{equation}
\mathbb{L} = \{ U \in H^2_{0,+1}([0,1]; \mathbb{R}^2) : U(0) = U(1) = 0, U'' \in \{1, t\}^\perp \},
\end{equation}
where $\{1, t\}^\perp$ is the orthogonal complement, in $L^2([0,1]; \mathbb{R}^2)$, of the subspace generated by $\{1, t\}$. Since these orthogonality conditions mean
\begin{equation}
\int_0^1 U''(t) \, dt = U'(1) - U'(0) = 0,
\end{equation}
we can also put
\begin{equation}
\mathbb{L} = \{ U \in H^2_{0,+1}([0,1]; \mathbb{R}^2) : U(0) = U(1) = U'(0) = U'(1) = 0 \}.
\end{equation}
We also set
\begin{align*}
\Psi(t) &= \int_0^t F_u(s, u(s), u'(s), u''(s)) \, ds, \\
\Phi(t) &= \int_0^t [-\Psi(s) + F_z(s, u(s), u'(s), u''(s))] \, ds,
\end{align*}
two continuous, bounded functions by hypothesis. For $U \in \mathbb{L}$, an integration by parts in the first term of (30) yields
\begin{equation}
\int_0^1 \left[ -\Psi(t) \cdot U'(t) + F_z(t, u, u', u'') \cdot U'(t) + F_z(t, u, u', u'') \cdot U''(t) \right] \, dt = 0,
\end{equation}
because $\Psi(t)U(t)|_0^1 = 0$ for test fields $U \in \mathbb{L}$. A second integration by parts leads to
\begin{equation}
\int_0^1 \left[ -\Phi(t) + F_z(t, u, u', u'') \right] \cdot U''(t) \, dt = 0,
\end{equation}
again because $\Phi(t)U'(t)|_0^1 = 0$ if $U \in \mathbb{L}$. Due to the arbitrariness of $U \in \mathbb{L}$, according to (31) we conclude that
\begin{equation}
F_z(t, u, u', u'') - \Phi(t) = c + Ct \text{ in } (0, 1),
\end{equation}
with $c$ and $C$ constants. In particular, since $\Phi$ is absolutely continuous (it belongs to $W^{1,1}((0,1); \mathbb{R}^2)$), we know that $F_z(t, u, u', u'')$ must be absolutely continuous too in $(0, 1)$, and as such, it cannot have jumps in $(0, 1)$, though it could possibly have at the endpoints. By differentiating once in (30) with respect to $t$,
\begin{equation}
\frac{d}{dt}F_z(t, u, u', u'') - F_z(t, u, u', u'') + \Psi(t) = C \text{ a.e. in } (0, 1),
\end{equation}
and even further
\begin{equation}
\frac{d}{dt} \left( \frac{d}{dt}F_z(t, u, u', u'') - F_z(t, u, u', u'') \right) + F_u(t, u, u', u'') = 0 \text{ a.e. in } (0, 1).
\end{equation}
We take this information back to (30) for a general $U \in H^2_{0,+1}([0,1]; \mathbb{R}^2)$, not necessarily belonging to $\mathbb{L}$. One integration by parts in the second term in (30) yields
\begin{equation}
\int_0^1 F_z(t, u, u', u'') \cdot U'(t) \, dt = - \int_0^1 \frac{d}{dt}F_z(t, u, u', u'') \cdot U(t) \, dt + \left[ F_z(t, u, u', u'') \cdot U(t) \right]_{t=0}^{t=1}.
\end{equation}
Recall the periodicity conditions for $U$. Two such integrations by parts in the third term of (30) leads to

$$
\int_0^1 F_Z(t, u, u', u'') \cdot U''(t) dt = - \int_0^1 \frac{d}{dt} F_Z(t, u, u', u'') \cdot U'(t) dt + [F_Z(t, u, u', u'')]_{t=0} \cdot U'(0)
$$

$$
= \int_0^1 \frac{d^2}{dt^2} F_Z(t, u, u', u'') \cdot U(t) dt - \left[ \frac{d}{dt} F_Z(t, u, u', u'') \right]_{t=0} \cdot U(0) + [F_Z(t, u, u', u'')]_{t=0} \cdot U'(0).
$$

In this way (30) becomes

$$
\int_0^1 \left[ \frac{d}{dt} \left( \frac{d}{dt} F_Z(t, u, u', u'') - F_Z(t, u, u', u'') \right) + F_u(t, u, u', u'') \right] \cdot U(t) dt + \left[ \frac{d}{dt} F_Z(t, u, u', u'') \right]_{t=0} \cdot U(0) + [F_Z(t, u, u', u'')]_{t=0} \cdot U'(0).
$$

The integral here vanishes precisely by (33), and so we are only left with the contributions on the end-points. Hence we obtain

$$
\frac{d}{dt} F_Z(t, u, u', u'') - F_Z(t, u, u', u'') |_{t=0} \cdot U(0) - [F_Z(t, u, u', u'')]_{t=0} \cdot U'(0) = 0.
$$

Since vectors $U'(0)$ and $U(0)$ can be chosen arbitrarily, and independently of each other, because there is always a path $U$ in $H^2_{\mathcal{O},+1}([0, 1]; \mathbb{R}^2)$ starting in a certain arbitrary vector of $\mathbb{R}^2$ and with any preassigned velocity, we conclude that

$$
[F_Z(t, u, u', u'')]_{t=0} = 0,
$$

and

$$
\frac{d}{dt} F_Z(t, u, u', u'') - F_Z(t, u, u', u'') |_{t=0} = 0.
$$

This completes the proof of Theorem 25. Note that this last condition implies that

$$
\frac{d}{dt} F_Z(t, u, u', u'') - F_Z(t, u, u', u'')
$$

is absolutely continuous in the interval $[0, 1]$, including the endpoints. \qed

The application of Theorem 25 to our situation where

$$
F(t, u, z, Z) = \frac{1}{2} (F^\perp(u) \cdot z)^2 + \frac{\varepsilon}{2} (|Z|^2 + |z|^2 + |u|^2) + u \cdot v_\varepsilon(t) + z \cdot v'_\varepsilon(t) + Z \cdot v''_\varepsilon(t)
$$

is our key tool. Note that we have dropped out the constant term $(1/2\varepsilon)||v_\varepsilon||^2$ as it does not play a role in what follows. The partial derivatives required in the statement of this theorem are

$$
F_Z = \varepsilon Z + v''_\varepsilon(t),
$$

$$
F_z = (F^\perp(u) \cdot z) F^\perp(u) + \varepsilon z + v'_\varepsilon(t),
$$

$$
F_u = (F^\perp(u) \cdot z) D F^\perp(u) z + \varepsilon u + v_\varepsilon(t).
$$
Equation (28) for critical closed paths in $H^2_{O,+1}([0,1];\mathbb{R}^2)$ for $E_\varepsilon$ coming from Theorem 25 involves the combination (27) which in our case is

$$d\varepsilon \epsilon_\varepsilon''(t) + \epsilon_\varepsilon''(t)) - (\mathbf{F}^\perp(u_\varepsilon(t)) \cdot u_\varepsilon'(t)) \mathbf{F}^\perp(u_\varepsilon(t)) - \varepsilon u_\varepsilon'(t) - \epsilon_\varepsilon'(t),$$

which must be an absolutely continuous function in $[0,1]$. Its almost everywhere derivative ought to be, according to system (28).

$$- (\mathbf{F}^\perp(u_\varepsilon(t)) \cdot u_\varepsilon'(t)) D\mathbf{F}^\perp(u_\varepsilon(t)) u_\varepsilon'(t) - \varepsilon u_\varepsilon(t) - \epsilon_\varepsilon(t).$$

Here $u_\varepsilon \in H^2_{O,+1}([0,1];\mathbb{R}^2)$ is an arbitrary critical closed path of $E_\varepsilon$. In addition, from (43) we have

$$[\varepsilon u_\varepsilon''(t) + \epsilon_\varepsilon''(t)]|_{t=0} = 0.$$

We need to examine these conditions carefully.

It is also important to stress, for those not used to it, how this result ensures much more regularity for those critical closed paths precisely because they are critical closed paths of a certain functional. Even though paths in our ambient space are just in $H^2([0,1];\mathbb{R}^2)$, critical closed paths of the functional in the statement of this lemma are much more regular. Recall that $H^2_{O,+1}([0,1];\mathbb{R}^2)$ is the completion in $H^2([0,1];\mathbb{R}^2)$ of the subspace of non-self-intersecting paths with rotation index $+1$, i.e. they travel once in counterclockwise sense. By Theorem 25, the expression in (37) is absolutely continuous. Since the last three terms of (37) and $\epsilon_\varepsilon''$ are continuous, we can conclude that $u_\varepsilon$ is $C^3$ in $[0,1]$. Moreover, due to the fact that the derivative of (37) is equal to (38), again by Theorem 25, it follows that $u_\varepsilon$ is even $C^4$ in $[0,1]$ because all terms in (37), when differentiated with respect to $t$, are continuous except possibly the first one $u_\varepsilon''(t)$, and such a derivative is equal to (38) which is continuous. Note how condition (39) is redundant with the above information.

**Proposition 26.** Critical closed paths $u_\varepsilon$ of the functional $E_\varepsilon$ are $C^\infty$ in $[0,1]$, and are solutions of the system

$$\varepsilon(u_\varepsilon''' - u_\varepsilon'' + u_\varepsilon) - \frac{d}{dt}[(\mathbf{F}(u_\varepsilon) \cdot u_\varepsilon') \mathbf{F}(u_\varepsilon)] + (\mathbf{F}^\perp(u_\varepsilon) \cdot u_\varepsilon') (u_\varepsilon')^T D\mathbf{F}^\perp(u_\varepsilon) = -v_\varepsilon''' + v_\varepsilon'' - \epsilon_\varepsilon$$

in the interval $[0,1]$.

**Proof.** Regularity conditions for $u_\varepsilon$ have been stated previously to the statement of the proposition. Equation (40) is a consequence, according to equation (28), expressing the equality of the derivative of (37) with (38). □

Equation (40) is a key point for counting the critical closed paths of the functional $E_\varepsilon$. We are facing a singularly perturbed, fourth-order ODE system (40) with periodic (unknown) boundary conditions. Our plan to count, and eventually find an upper bound for, the number of solutions of (40), which we call branches to stress the dependence of $\varepsilon$, proceeds in two steps:

1. for a fixed such branch (with smooth dependence on $\varepsilon$), understand its asymptotic behavior as $\varepsilon \searrow 0$, to count how many such different asymptotic behavior there might be; and
Let \( y \in \mathbb{R}^2 \) be a given vector. Suppose that \( u \) is a critical path of \( E \) over the class of feasible paths

\[
H^2_{O,+1,y}([0,1]; \mathbb{R}^2) = \{ v \in H^2_{O,+1}([0,1]; \mathbb{R}^2) : v(0) = y, v'(0) = v'(1) \}.
\]

Then the vector field

\[
\frac{d}{dt} F_Z(t, u, u', u'') - F_Z(t, u, u', u'')
\]

is absolutely continuous in \((0,1)\),

\[
\frac{d}{dt} \left( \frac{d}{dt} F_Z(t, u, u', u'') - F_Z(t, u, u', u'') \right) + F_u(t, u, u', u'') = 0 \text{ a.e. } t \text{ in } (0,1),
\]

and

\[
[F_Z(t, u, u', u'')]_{t=0} = 0.
\]

Notice that the class of feasible paths \( H^2_{O,+1,y}([0,1]; \mathbb{R}^2) \) in this statement is always a subset of \( H^2_{O,+1}([0,1]; \mathbb{R}^2) \) for every \( y \). In fact, if we add to \( H^2_{O,+1}([0,1]; \mathbb{R}^2) \) a constraint fixing the starting (and final) vector \( y \), optimality yields a less restrictive set of conditions, which in this situation amounts to just loosing the continuity of the vector field in \( 41 \) across \( t = 0 \).

**Proof of Theorem 27.** The proof is exactly the same, word by word, as that of Theorem 25. The only difference revolves around the discussion of \( 34 \). Under periodic conditions without imposing a particular vector as starting vector (as we are doing here), \( 34 \) leads to the two vanishing jump conditions \( 35 \) and \( 36 \). However, if we insist in that the starting vector for paths is a given, specific vector \( y \), then feasible variations \( \mathbf{U} \) in \( 34 \) must comply with \( \mathbf{U}(0) = 0 \), and so we are left with

\[
[F_Z(t, u, u', u'')]_{t=0} \cdot \mathbf{U}'(0) = 0.
\]

The arbitrariness of \( \mathbf{U}'(0) \) (which can be chosen freely) leads to the second jump condition \( 36 \), but we have no longer \( 35 \). This translates into the continuity of the vector field \( 41 \) in the open interval \((0,1)\) precisely because we cannot rely on the jump condition across the end-points. However the differential system \( 42 \) or \( 28 \) holds in \((0,1)\) in both situations. \(\square\)

Again, the application of this last general statement to our particular situation, leads to the following new version of Proposition 26.

**Proposition 28.** Critical closed paths \( u_\varepsilon \) of the functional \( E_\varepsilon \) over \( H^2_{O,+1,y}([0,1]; \mathbb{R}^2) \) are \( C^2 \) in \([0,1]\), \( C^\infty \) in \((0,1)\), and are solutions of the fourth-order differential system

\[
\varepsilon (u'''' - u''')u_\varepsilon - \frac{d}{dt} \left[ (F^\perp(u_\varepsilon)u_\varepsilon') (F^\perp(u_\varepsilon)) + (F^\perp(u_\varepsilon')u_\varepsilon) (u_\varepsilon')^T D F^\perp(u_\varepsilon) \right] = -v_\varepsilon'''' + v_\varepsilon'' - v_\varepsilon
\]
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in the interval \((0, 1)\).

For the proof, simply notice that not having the vanishing of the jump of \((37)\), we cannot rely on the continuity of the third derivative \(u''\) (the only term which is not guaranteed to be continuous across \(t = 0\) in \((37)\) across \(t = 0\), and so the critical path \(u_\varepsilon\) can only be ensured to belong to \(C^2\) in \([0, 1]\).

10. Asymptotic behavior

For the sake of transparency, and to facilitate a few interesting computations, we recast system \((40)\) in its two components

\[
(ZQ)' + Z(-Qx' + Px') = -\varepsilon\alpha_1,
\]

\[
(ZP)' + Z(Qy' - Py') = \varepsilon\alpha_2.
\]

where

\[
F = (P, Q), \quad u_\varepsilon = (x, y), \quad v_\varepsilon = \varepsilon(X, Y)
\]

\[
Z \equiv F^\perp(u_\varepsilon) \cdot u_\varepsilon' = P(x, y)y' - Q(x, y)x',
\]

\[
W \equiv F(u_\varepsilon) \cdot u_\varepsilon' = P(x, y)x' + Q(x, y)y',
\]

\[
\text{Div} \equiv P_x + Q_y, \quad \alpha_1 = \overline{x}''' - \overline{x}'' + \overline{x}, \quad \alpha_2 = \overline{y}''' - \overline{y}'' + \overline{y},
\]

with \(\overline{x} = x + X\) and \(\overline{y} = y + Y\). Note that \(Z^2/2\) is precisely the integrand for \(E_0\), and recall that all close paths involved are \(1\)-periodic, \(C^\infty\) and belong to \(H^2_{Q, +1}([0, 1]; \mathbb{R}^2)\), so that we can freely differentiate in \(t\) as many times as needed. In particular, the two equations of the system of critical closed paths become

\[
(ZQ)' + Z\text{Div} y' = -\varepsilon\alpha_1, \quad Z'P + Z\text{Div} x' = \varepsilon\alpha_2.
\]

Moreover \(X, Y\) and their derivatives are uniformly bounded with respect to \(\varepsilon\) by choice of \(v_\varepsilon\) (see the definition of \(v_\varepsilon\)). Note that these functions \(X\) and \(Y\) are the components of \((1/\varepsilon)v_\varepsilon\), so that \(X = (1/\varepsilon)X_\varepsilon\) and \(Y = (1/\varepsilon)Y_\varepsilon\) where \(v_\varepsilon\) was introduced as \(v_\varepsilon = (X_\varepsilon, Y_\varepsilon)\) earlier in the paper.

We remind readers that, according to Proposition \(24\), we are after periodic solutions of this system for which \(E_\varepsilon\) is away from zero. We will therefore discard from our consideration those such solutions for which \(E_\varepsilon\) is arbitrarily small. In particular, we do not need to consider asymptotic behaviors reducing to a point.

We manipulate the two equations in \((45)\) in two ways:

(1) multiply the first equation by \(Q\), the second by \(P\), and add up the results to find

\[
Z'(P^2 + Q^2) = -\varepsilon(\alpha_1Q - \alpha_2P) - ZW\text{Div};
\]

(2) then, multiply the first by \(P\), the second by \(Q\), and subtract the results to have

\[
Z^2\text{Div} = -\varepsilon(\alpha_1P + \alpha_2Q).
\]

Bearing in mind that equilibria of our polynomial, differential system are isolated and they could only be associated with critical closed paths of the kind we are not interested
in (recall the discussion at the end of Section 8), we can further multiply (46) by \( Z \) and divide by \( P^2 + Q^2 \), to have, taking into account the other equation,

\[
(Z^2)' = 2\varepsilon(\alpha_1 x' + \alpha_2 y').
\]

Hence system (40) can be written in the simplified, equivalent form

\[
(Z^2)' = 2\varepsilon(\alpha_1 x' + \alpha_2 y'), \quad Z^2 \text{Div} = -\varepsilon(\alpha_1 P + \alpha_2 Q).
\]

The theory of singularly-perturbed differential problems (see for instance [36]) informs us that convergence of solutions of (47) to the limit system

\[
(Z^2)' = 0, \quad Z^2 \text{Div} = 0,
\]

setting \( \varepsilon = 0 \) in (47), takes place pointwise for a.e. \( t \in [0, 1] \), though there can be small sets where large transition layers may occur; or else solutions may escape to infinity along system (48). The choice of our ambient space \( H^{2,1}_\Omega([0,1];\mathbb{R}^2) \) is most important in this regard.

**Lemma 29.** Let \((x_\varepsilon,y_\varepsilon)\) be a family of solutions of (47) in the space \( H^{2,1}_\Omega([0,1];\mathbb{R}^2) \),

\[
(Z^2)'_\varepsilon = 2\varepsilon(\alpha_1 x'_\varepsilon + \alpha_2 y'_\varepsilon), \quad Z^2_\varepsilon \text{Div}_\varepsilon = -\varepsilon(\alpha_1 P_\varepsilon + \alpha_2 Q_\varepsilon).
\]

For a suitable subsequence (not relabeled),

\[
(Z^2)'_\varepsilon \rightarrow 0, \quad Z^2_\varepsilon \text{Div}_\varepsilon \rightarrow 0,
\]

pointwise for a.e. \( t \in [0,1] \).

**Proof.** The main idea is to realize that periodic solutions of (49) for which the terms involving a fourth-order derivative are not negligible, need to have a frequency going to infinity as \( \varepsilon \) tends to zero. But this is not possible for paths in our ambient space \( H^{2,1}_\Omega([0,1];\mathbb{R}^2) \) because they cannot have period less than one. We try to make this argument precise in the sequel.

The basic tool is a typical scaling argument. Define

\[
\begin{align*}
  u_\varepsilon(s) &= x_\varepsilon(\varepsilon^r s), \quad v_\varepsilon(s) = y_\varepsilon(\varepsilon^r s)
\end{align*}
\]

for a positive exponent \( r \) to be determined, and \( s \in [0, +\infty) \) regarding \((x_\varepsilon,y_\varepsilon)\) defined in all of \([0, +\infty)\) by periodicity. Note that the pair \((u_\varepsilon,v_\varepsilon)\) has period \( \varepsilon^{-r} \) going to infinity, as \( \varepsilon \to 0 \). Taking (50) to (47) leads to the system that \((u_\varepsilon,v_\varepsilon)\) must be a solution of

\[
\begin{align*}
  \varepsilon^{-3r}( & (P_\varepsilon v'_\varepsilon - \bar{Q}_\varepsilon u'_\varepsilon)^2)' = 2\varepsilon^{1-r}u'_\varepsilon(\varepsilon^{-4r}(u_\varepsilon + U_\varepsilon)^{mm} - \varepsilon^{-2r}(u_\varepsilon + U_\varepsilon)^{m} + (u_\varepsilon + U_\varepsilon)), \\
  & +2\varepsilon^{1-r}v'_\varepsilon(\varepsilon^{-4r}(v_\varepsilon + V_\varepsilon)^{mm} - \varepsilon^{-2r}(v_\varepsilon + V_\varepsilon)^{m} + (v_\varepsilon + V_\varepsilon)), \\
  \varepsilon^{-2r}( & (P_\varepsilon v'_\varepsilon - \bar{Q}_\varepsilon u'_\varepsilon)^2(P_{x_\varepsilon} + \bar{Q}_{y_\varepsilon}) = -\varepsilon P_\varepsilon(\varepsilon^{-4r}(u_\varepsilon + U_\varepsilon)^{mm} - \varepsilon^{-2r}(u_\varepsilon + U_\varepsilon)^{m} + (u_\varepsilon + U_\varepsilon)), \\
  & -\varepsilon Q_\varepsilon(\varepsilon^{-4r}(v_\varepsilon + V_\varepsilon)^{mm} - \varepsilon^{-2r}(v_\varepsilon + V_\varepsilon)^{m} + (v_\varepsilon + V_\varepsilon)),
\end{align*}
\]

where derivatives in these equations are taken with respect to the variable \( s \),

\[
P_\varepsilon(s) = P(u_\varepsilon(s),v_\varepsilon(s))
\]

and similarly with \( \bar{Q}, P_x, \bar{Q}_y \). The functions

\[
U_\varepsilon(s) = X_\varepsilon(\varepsilon^r s), \quad V_\varepsilon(s) = Y_\varepsilon(\varepsilon^r s)
\]
correspond to the components of our rescaled auxiliary path \((X_\varepsilon, Y_\varepsilon)\). If we take the right scaling \(r = 1/2\), our system for the pair \((u_\varepsilon, v_\varepsilon)\) becomes
\[
((\mathcal{P}_\varepsilon v'_\varepsilon - \mathcal{Q}_\varepsilon u'_\varepsilon)^2)' = 2u'_\varepsilon((u_\varepsilon + U_\varepsilon)''' - \varepsilon(u_\varepsilon + U_\varepsilon)'' + \varepsilon^2(u_\varepsilon + U_\varepsilon)) \\
+ 2v'_\varepsilon((v_\varepsilon + V_\varepsilon)''' - \varepsilon(v_\varepsilon + V_\varepsilon)'' + \varepsilon^2(v_\varepsilon + V_\varepsilon)),
\]
\[
(\mathcal{P}_\varepsilon v'_\varepsilon - \mathcal{Q}_\varepsilon u'_\varepsilon)^2(\mathcal{P}_{x,\varepsilon} + \mathcal{Q}_{y,\varepsilon}) = -\mathcal{P}_\varepsilon((u_\varepsilon + U_\varepsilon)''' - \varepsilon(u_\varepsilon + U_\varepsilon)'' + \varepsilon^2(u_\varepsilon + U_\varepsilon)) \\
- \mathcal{Q}_\varepsilon((v_\varepsilon + V_\varepsilon)''' - \varepsilon(v_\varepsilon + V_\varepsilon)'' + \varepsilon^2(v_\varepsilon + V_\varepsilon)).
\]
This is no longer a singularly-perturbed system, since the small parameter \(\varepsilon\) does not occur in the highest derivatives. The limit system is
\[
((\mathcal{P}v' - \mathcal{Q}u')^2)' = 2u''u''' + 2v''v''', \quad (\mathcal{P}v' - \mathcal{Q}u')^2(\mathcal{P}_x + \mathcal{Q}_y) = -P u''' - Q v'''.
\]
Recall that \(U_\varepsilon\) and \(V_\varepsilon\), together with their derivatives, tend to zero, according to our choice of the auxiliary path \((X_\varepsilon, Y_\varepsilon)\).

Periodic solutions \((x_\varepsilon, y_\varepsilon)\) of \([49]\), in which neither of the two members of those equations vanish, would correspond to pairs \((u_\varepsilon, v_\varepsilon)\), through \([50]\), solutions of \([51]\) and \([52]\) for which none of the terms not involving a power of \(\varepsilon\) vanish, and this in turn would correspond to limits \((u, v)\) of \([53]\) for which the terms of those two limit equations do not vanish. But these limit solutions \((u, v)\) will have a finite period, and so will \((u_\varepsilon, v_\varepsilon)\), given that \([51]\)–\([52]\) is a regular perturbed system. In this case, the pair \((x_\varepsilon, y_\varepsilon)\) coming from \([50]\) will have a period tending to zero with \(\varepsilon\). This is impossible for elements \((x_\varepsilon, y_\varepsilon)\) \(\in H^2_{O,+1}([0, 1]; \mathbb{R}^2)\).

We are therefore entitled to understand all possible asymptotic behaviors of critical closed paths \((x_\varepsilon, y_\varepsilon)\) \(\in H^2_{O,+1}([0, 1]; \mathbb{R}^2)\), through an analysis of the limit system \([48]\). The first equation in \([48]\) implies that \(Z^2 = k^2\), but since we are only interested in the asymptotic behavior for critical closed paths whose value for \(E_0\) stays away from zero, we discard the case \(k = 0\). In this case, the second equation in \([48]\), implies \(\text{Div} = 0\). We would like to understand the nature of the limit system
\[
Z^2 = k^2 > 0, \quad \text{Div} = 0.
\]
We write this system in the form, differentiating the second equation,
\[
\mathbf{F}^\perp(u) \cdot u' = \pm k \neq 0, \quad \nabla \text{Div}(u) \cdot u' = 0.
\]
These are two implicit, first-order systems that become singular when the determinant
\[
\nabla \text{Div}(u) \cdot \mathbf{F}(u)
\]
of the matrix of the system
\[
\begin{pmatrix}
\mathbf{F}^\perp(u) \\
\nabla \text{Div}(u)
\end{pmatrix}
\]
of the systems vanishes. These singular points are precisely the contact points of our differential system over the curve \(\text{Div} = 0\). The fact that \(\mathbf{F}^\perp(u) \cdot u' = \pm k \neq 0\) shows that \(u_\varepsilon\), for \(\varepsilon\) sufficiently small, can only turn around, changing \(+k\) by \(-k\) or viceversa, near those contact points. As a matter of fact, critical closed paths \(u\) have to turn around whenever one such point is found because at such points, where vectors \(\mathbf{F}^\perp\) and \(\nabla \text{Div}\) are parallel, the derivative \(u'\) in \([55]\) is not defined.

One can examine in a more detailed fashion the optimality system around one such contact point \(p\). To this end, notice that, as just indicated, \([55]\) is incompatible at a
contact point \( p \) because at such a point both vectors \( F^\perp \) and \( \nabla \text{Div} \) are parallel and non-vanishing due to the first equation in (55). If \( u'_\varepsilon \) would stay uniformly bounded in a neighborhood of a contact point, then the two equations in (55) would contradict each other. Hence, the only way to reconcile such situation is for the vector derivative \( u'_\varepsilon \) of the approximated system to grow indefinitely around such a point \( p \).

**Lemma 30.** Let \( u_\varepsilon = (x_\varepsilon, y_\varepsilon) \) be a continuous family of critical closed paths of \( E_\varepsilon \) such that \( E_0(u_\varepsilon) \) stays away from zero. Suppose that in a certain subinterval \([t^-_\varepsilon, t^+_\varepsilon] \subset [0, 1]\), \( t^+_\varepsilon - t^-_\varepsilon \to 0 \), we know that

\[
|u'_\varepsilon(t^\pm_\varepsilon)| \to +\infty, \quad u_\varepsilon(t^\pm_\varepsilon) \to p,
\]

as \( \varepsilon \to 0 \) where \( p \) is a contact point of the system. Then \( u_\varepsilon \) must turn around at \( p \) for \( \varepsilon \) sufficiently small, in the sense

\[
\frac{u'_\varepsilon(t^-_\varepsilon)}{|u'_\varepsilon(t^-_\varepsilon)|} + \frac{u'_\varepsilon(t^+_\varepsilon)}{|u'_\varepsilon(t^+_\varepsilon)|} \to 0
\]

as \( \varepsilon \to 0 \).

**Proof.** We retake the scaling argument of the proof of Lemma 29. Playing with the parameterization and in order to simplify notation, we can assume that

\[
t^\pm_\varepsilon = \pm t_\varepsilon, t_\varepsilon \to 0, \quad |u'_\varepsilon(\pm t_\varepsilon)| \to +\infty, \quad u_\varepsilon(\pm t_\varepsilon) \to p.
\]

Through (50), we define the rescaled path

\[
u_\varepsilon(s) = x_\varepsilon(\varepsilon^{1/2}s), \quad v_\varepsilon(s) = y_\varepsilon(\varepsilon^{1/2}s),
\]

restricted to an interval of the form

\[
I = [-s_0, s_0], \quad \frac{t_\varepsilon}{\varepsilon^{1/2}} \to s_0.
\]

for some \( s_0 > 0 \) in such a way that (57) holds. Recall that \( \varepsilon^{1/2} \) is the appropriate scaling in (50), and thus we can suppose that

\[
|u'_\varepsilon(\pm t_\varepsilon)|\varepsilon^{1/2} \to \alpha, \quad \alpha > 0.
\]

The limit system (53) is

\[
((Pv' - Qv')^2)' = 2u''u''' + 2v''v''', \quad (Pv' - Qv')^2(P_x + Q_y) = -Pu''''' - Qv''''.
\]

We know that close to a contact point \( p \), the quantities

\[
(Z^2)' = ((Pv' - Qv')^2)', \quad \text{Div} = (P_x + Q_y),
\]

converge to zero pointwise, but

\[
Z^2 = (Pv' - Qv')^2
\]

does not (it is one of our hypothesis). Hence the limit system (59), restricted to the interval \( I \), becomes

\[
u''u''' + v''v''' = 0, \quad Pu''''' + Qv''''' = 0.
\]

Since the determinant of this linear system is \( Z \), which does not vanish, these equations amount to

\[
u''' = v''' = 0.
\]

We, hence, conclude that the path \((u_\varepsilon, v_\varepsilon)\) tend to a path \((u, v)\), over the symmetric interval \( I \), with components which are third-degree polynomials having the same value at the two end-points of \( I \), and a derivative there which is of the form \( \pm \alpha Q \) for either
sign, where $\alpha$ comes from (69), and $Q$ is the unit tangent to the curve $\text{Div} = 0$ at $p$. The path $(u, v)$ cannot be affine due to the boundary value given by $p$ at both end-points (it cannot be constant for that would imply a vanishing derivative). Hence $u$ and $v$ are either quadratic or cubic. But in either case, the tangent vector $(u', v')$ cannot maintain its direction across the interval $I$. Moreover the nature of the limit system in the form (55) does not permit two consecutive changes of direction for the tangent vector in a vicinity of a contact point $p$ because this system can only change direction once at each contact point. The statement (56) is just a way to express this change of direction.

□

Our above discussion can be summarized in the following statement that classifies all possible asymptotic behaviors for critical closed paths.

**Theorem 31.** Assume that all the components of the curve $\text{Div} = 0$ are topologically straight lines or ovals. The possible limit behaviors as $\varepsilon \to 0$ of branches of critical closed paths of $E_\varepsilon$, not coming from zeroes of $E_0$, can be identified in a one-to-one fashion with arcs of the connected components of the curve $\text{Div} = 0$ in one of the following possibilities:

(a) If the component is homeomorphic to a straight line, then
   (a.1) the limit behavior is an arc whose endpoints are two contact points and no additional contact points in its interior;
   (a.2) the limit behavior is an arc whose endpoints are one contact point and the infinity, and no additional contact point in its interior;
   (a.3) the limit behavior is the whole component without contact points.

(b) If the component is homeomorphic to an oval, then
   (b.1) the limit behavior is an arc whose endpoints are two contact points and no additional contact points in its interior;
   (b.2) the limit behavior is an arc covering the full oval whose endpoints have to be a single contact point, and no additional contact point in the oval;
   (b.3) the limit behavior is the whole oval without contact points.

11. **Multiplicity**

We are concerned in this section about the possibility that various branches of the set of critical closed paths, for $\varepsilon$ positive, may coalesce into the same limit behavior as $\varepsilon \searrow 0$, and how they can possibly contribute to the inequality in Proposition 24.

To this end, let $u(t; y, \varepsilon)$ be the solution of system (44) for positive $\varepsilon$ and $y \in \mathbb{R}^2$ such that

$$u(0; y, \varepsilon) = u(1; y, \varepsilon) = y.$$

As a matter of fact, system (44) might have several solutions for some $y$. However, if $y$ is selected to be $u_\varepsilon(0)$ for some critical closed path $u_\varepsilon$ of those we are interested in, i.e. solutions of (40), then the solution of (44) is unique and equal to $u_\varepsilon$ itself if there is only one such critical path with such starting vector $u_\varepsilon(0)$. Note that the only difference between (40) and (44) lies in the overall regularity. Such regularity holds for some starting vectors $y$, and does not hold for most of them. Due to periodicity conditions and selecting suitably the starting vector $y = u_\varepsilon(t_\varepsilon)$ for some $t_\varepsilon$, we can assume, without loss of generality, that for such choice of starting point $y = u_\varepsilon(t_\varepsilon)$,
the solution of (44) is unique and coincides with the critical path \( u_\varepsilon \) itself, a solution of (40), in addition to having that \( u_\varepsilon(t_\varepsilon) \) converges to a contact point.

There are two situations for the images of a branch of solutions \( u_\varepsilon \) not covered by the previous paragraph in which we assume that \( u_\varepsilon(t_\varepsilon) \) converges to a contact point. They are:

(i) they tend to a full connected component of \( \text{Div} = 0 \) homeomorphic to a line without contact points;

(ii) they tend to a full connected component of \( \text{Div} = 0 \) homeomorphic to an oval without contact points.

Yet, they can be readily adapted to our following discussion as the arguments only require to have a convergence of \( u_\varepsilon(t_\varepsilon) \) to a finite point \( p \), be a contact point or not, for a sequence of values \( t_\varepsilon \to 0 \).

Since system (44) with \( \varepsilon \) positive is analytic in its initial conditions and parameters, though it is only \( C^2 \) on \( t \), its solutions \( u(t; y, \varepsilon) \), selected as indicated in the previous paragraph, depend analytically on end-point conditions \( y \) and parameter \( \varepsilon \). Therefore for \( s \) sufficiently small and for \( \varepsilon \) fixed, positive and sufficiently small, we can write

\[
(60) \quad u(t; y + sy, \varepsilon) = u(t; y, \varepsilon) + sD_y u(t; y, \varepsilon)y + R(t, s, \varepsilon, y, y),
\]

with

\[
\lim_{s \to 0} \frac{R(t, s, \varepsilon, y, y)}{s} = 0
\]

for \( t \in [0, 1] \) and given \( \varepsilon, y, y \). In what follows we use the following notation

\[
V_\varepsilon(t) = D_y u(t; y(\varepsilon), \varepsilon)y(\varepsilon), \quad U_\varepsilon(t) = u(t; y(\varepsilon), \varepsilon)
\]

once \( y \) and \( y \) have been chosen (below) appropriately. Since we can also use the same argument with first and second derivatives with respect to \( t \), we can also write

\[
\lim_{s \to 0} \frac{1}{s} \|u(t; y + sy, \varepsilon) - U_\varepsilon(t) - sV_\varepsilon(t)\|_{H^2([0,1];\mathbb{R}^2)} = 0
\]

for given \( \varepsilon, y, y \).

Suppose we are dealing with the critical closed paths \( u_{\varepsilon, i}, i = 1, 2, \ldots, m_\varepsilon \) having one of the possible asymptotic limits enumerated above. Even though we could have that \( m_\varepsilon \) could increase with \( \varepsilon \), we will focus our attention on a fixed number \( m \), independent of \( \varepsilon \) but otherwise arbitrary, of such branches of critical closed paths of \( E_\varepsilon \). By changing a little bit the starting vector and taking advantage of the periodicity, if necessary, we can put

\[
(61) \quad u_{\varepsilon, i}(t) = u(t; y(\varepsilon) + s_i(\varepsilon)y(\varepsilon), \varepsilon)
\]

for values \( s_i(\varepsilon), i = 1, 2, \ldots, m \), and a certain unique (independent of \( i \)) unit vector \( y(\varepsilon), \|y(\varepsilon)\| = 1 \). (see Figure 1). We restrict attention to asymptotic limits that have, at least, one contact or finite point \( p \) as end-point, so that

\[
y(\varepsilon) + s_i(\varepsilon)y(\varepsilon) \quad \text{and} \quad y(\varepsilon)
\]

converge to \( p \) as \( \varepsilon \downarrow 0 \), and consequently \( s_i(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \). In this way, taking into account (61), we can write

\[
(62) \quad E'_{\varepsilon}(u(t; y(\varepsilon) + s_i(\varepsilon)y(\varepsilon), \varepsilon)) = 0.
\]
By (60) we have

\begin{equation}
\mathbf{u}(t; \mathbf{y}(\varepsilon) + s_i(\varepsilon)\mathbf{y}(\varepsilon), \varepsilon) = \mathbf{U}_\varepsilon(t) + s_i(\varepsilon)\mathbf{V}_\varepsilon(t) + \mathbf{R}(t, s_i(\varepsilon), \varepsilon, \mathbf{y}(\varepsilon), \mathbf{y}(\varepsilon)),
\end{equation}

with

\[
\lim_{\varepsilon \to 0} \frac{\mathbf{R}_{\varepsilon,i}(t)}{s_i(t)} = \lim_{\varepsilon \to 0} \frac{\mathbf{R}(t, s_i(\varepsilon), \varepsilon, \mathbf{y}(\varepsilon), \mathbf{y}(\varepsilon))}{s_i(\varepsilon)} = 0.
\]

Bear in mind that \( \mathbf{y}(\varepsilon) \to \mathbf{p} \), and that \( \mathbf{y}(\varepsilon) \to \mathbf{y} \) (at least for an appropriate sequence of values for \( \varepsilon \)) because these are unitary vectors.

All these preparations are directed to the proof of the following fundamental fact.

**Lemma 32.** For each possible asymptotic limit given in Theorem 31, there cannot be more than \( n \) critical paths \( \mathbf{u}_{\varepsilon,i} \) converging to it.

**Proof.** Recall that even though the number \( m_\varepsilon \) could increase as \( \varepsilon \to 0 \), we focus on a fixed number \( m \) of such critical paths for a fixed \( \varepsilon \) sufficiently small.

We proceed in two steps. Firstly, we will show that \( m \leq 2n + 1 \). After that, we argue that indeed, \( m \leq n \). The arbitrariness of \( m \) will prove our statement.

We would like to consider two families of real functions

\begin{equation}
P_\varepsilon(r) = \frac{1}{w_\varepsilon} \langle E'_\varepsilon(\mathbf{U}_\varepsilon + r\mathbf{W}_\varepsilon), \mathbf{W}_\varepsilon \rangle, \quad \mathbf{W}_\varepsilon = \frac{\mathbf{V}_\varepsilon}{\|\mathbf{V}_\varepsilon\|}, \quad r = \|\mathbf{V}_\varepsilon\|s,
\end{equation}

\[
g_\varepsilon(r) = \frac{1}{w_\varepsilon} \langle E'_\varepsilon(\mathbf{U}_\varepsilon + r\mathbf{W}_\varepsilon + \mathbf{R}_\varepsilon(r)), \mathbf{W}_\varepsilon \rangle, \quad \mathbf{R}_\varepsilon(r) = \mathbf{R}(t, s, \varepsilon, \mathbf{y}(\varepsilon), \mathbf{y}(\varepsilon)),
\]

where numbers \( w_\varepsilon \) and paths \( \mathbf{W}_\varepsilon \) and \( \mathbf{W}_\varepsilon \) in our ambient space \( H^2_{O,+1}([0,1]; \mathbb{R}^2) \) will be chosen appropriately later.

We know that, regardless of the choice of \( w_\varepsilon, \mathbf{W}_\varepsilon, \) and \( \mathbf{W}_\varepsilon, \) each \( P_\varepsilon(r) \) is a polynomial of at most degree \( 2n + 1 \) in the variable \( r \) with derivative

\[
P'_\varepsilon(r) = \frac{1}{w_\varepsilon} \langle E''_\varepsilon(\mathbf{U}_\varepsilon + r\mathbf{W}_\varepsilon), (\mathbf{W}_\varepsilon, \mathbf{W}_\varepsilon) \rangle.
\]

---

**Figure 1.** Critical closed paths and a transversal section. The square dots are two contact points.
The calculation of the derivative is clear after the chain rule. To realize that \( P_{\varepsilon}(r) \) is a polynomial of at most degree \( 2n + 1 \), recall that
\[
E'_{\varepsilon}(u) = E'_0(u) + \varepsilon u + v_{\varepsilon},
\]
and by (17),
\[
\langle E'_0(u), U \rangle = \int_0^1 (F^\varepsilon(u) \cdot u') \left( (DF^\varepsilon(u)U) \cdot u' + F^\varepsilon(u) \cdot U' \right) dt.
\]
If we replace \( u = U_{\varepsilon} + rW_{\varepsilon} \) and \( U = \tilde{W}_{\varepsilon} \) in this identity, we clearly have our claim about the degree of the polynomials.

Each \( r_{i}(\varepsilon) = \|V_{\varepsilon}\|s_i(\varepsilon) \) is a real root of \( g_{\varepsilon} \) because
\[
u_{i,\varepsilon} = U_{\varepsilon} + r_{i}(\varepsilon)W_{\varepsilon} + R_{\varepsilon,i} = U_{\varepsilon} + s_i(\varepsilon)V_{\varepsilon} + R_{\varepsilon,i}
\]
with
\[
R_{\varepsilon,i}(t) = R(t, s_i(\varepsilon), \varepsilon, y, \bar{y}),
\]
is a critical path for \( E_{\varepsilon} \), i.e.
\[
E'_{\varepsilon}(U_{\varepsilon} + r_{i}(\varepsilon)W_{\varepsilon} + R_{\varepsilon,i}) = 0.
\]
Hence,
\[
P_{\varepsilon}(r_{i}(\varepsilon)) = \frac{1}{w_{\varepsilon}} \langle E''_{\varepsilon}(U_{\varepsilon} + r_{i}(\varepsilon)W_{\varepsilon}) - E''_{\varepsilon}(U_{\varepsilon} + r_{i}(\varepsilon)W_{\varepsilon} + R_{\varepsilon,i}), \tilde{W}_{\varepsilon} \rangle.
\]
By the mean-value theorem, there are numbers \( t_{\varepsilon,i} \in [0, 1] \) such that
\[
P_{\varepsilon}(r_{i}(\varepsilon)) \leq \frac{1}{w_{\varepsilon}} \langle E''_{\varepsilon}(U_{\varepsilon} + s_i(\varepsilon)V_{\varepsilon}), (t_{\varepsilon,i}R_{\varepsilon,i}, \tilde{W}_{\varepsilon}) \rangle
\]
if the choice of \( w_{\varepsilon} \) guarantees that
\[
\frac{1}{w_{\varepsilon}} \|E''_{\varepsilon}(U_{\varepsilon} + s_i(\varepsilon)V_{\varepsilon})\| \leq 1
\]
for every \( i \), and \( \tilde{W}_{\varepsilon} \) is taken from a (uniformly) bounded set in \( H^2_{O,1}([0, 1]; \mathbb{R}^2) \). Since
\[
R_{\varepsilon,i} = R(t, s_i(\varepsilon), \varepsilon, y, \bar{y})
\]
converges to zero, uniformly in \( i \), because \( s_i(\varepsilon) \to 0 \) for every \( i \) in a fixed finite set of indices, as \( \varepsilon \to 0 \), we conclude that the values \( P_{\varepsilon}(r_{i}(\varepsilon)) \) can be made arbitrarily small, uniformly in \( i \), as \( \varepsilon \to 0 \).

We next examine the derivatives
\[
g'_{\varepsilon}(r) = \frac{1}{w_{\varepsilon}} \langle E''_{\varepsilon}(U_{\varepsilon} + rW_{\varepsilon} + R_{\varepsilon}(r)), (W_{\varepsilon} + R_{\varepsilon}'(r), W_{\varepsilon}) \rangle.
\]
The Hessians
\[
E''_{\varepsilon}(U_{\varepsilon} + r_{i}(\varepsilon)W_{\varepsilon} + R_{\varepsilon}(r_{i}(\varepsilon))) = E''_{\varepsilon}(u_{i,\varepsilon})
\]
at the critical paths \( u_{i,\varepsilon} \) are non-singular, and so the set of unit vectors where the associated (infinite-dimensional) quadratic form vanishes has a vanishing measure in the unit sphere of the space \( H^2_{O,1}([0, 1]; \mathbb{R}^2) \). This means, bearing in mind that \( R_{\varepsilon}'(r_{i}(\varepsilon)) \to 0 \) as \( \varepsilon \to 0 \) for all \( i \), that by perturbing a little bit the unit vector \( W_{\varepsilon} \) and passing to an
additional unit vector $\mathbf{W}_\varepsilon$ with $\|\mathbf{W}_\varepsilon - \mathbf{W}_\varepsilon\| \to 0$ as $\varepsilon \to 0$ (this is how we select the path $\mathbf{W}_\varepsilon$), we can guarantee that the numbers

$$\langle E''_\varepsilon(U_\varepsilon + r_i(\varepsilon))\mathbf{W}_\varepsilon + R_\varepsilon(r_i(\varepsilon)), (\mathbf{W}_\varepsilon + R'_\varepsilon(r_i(\varepsilon)), \mathbf{W}_\varepsilon) \rangle$$

are non-zero, and obviously, have the same sign as that of $g'_\varepsilon(r_i(\varepsilon))$ in (67). Because we have a fixed, finite number of values for $i$, this unit path $\mathbf{W}_\varepsilon$ can be chosen to be valid for all $i$ at the same time. The signs of the derivatives $g'_\varepsilon(r_i(\varepsilon))$ might already alternate for $i = 1, 2, \ldots, m$; if not, and $g'(r_i(\varepsilon))g'(r_{i+1}(\varepsilon)) > 0$, then there must be, at least, one intermediate value of $r(\varepsilon) = r_{i,i+1}(\varepsilon)$ such that

$$g_\varepsilon(r_{i,i+1}(\varepsilon)) = 0, \quad g'_\varepsilon(r_i(\varepsilon))g'_\varepsilon(r_{i,i+1}(\varepsilon)) < 0, \quad g'_\varepsilon(r_{i,i+1}(\varepsilon))g'_\varepsilon(r_{i+1}(\varepsilon)) < 0,$$

and then the signs of the derivative at the full collection of points $r_i(\varepsilon), r_{i,i+1}(\varepsilon), r_{i+1}(\varepsilon), \ldots, m - 1$, do alternate. We can thus assume, without loss of generality, that the signs of the derivatives $g'_\varepsilon(r_i(\varepsilon))$ alternate, for otherwise we would enlarge the list of our $m$ chosen critical paths, associated with roots of $g_\varepsilon$, with these other additional intermediate roots of the same function.

On the other hand, as already indicated above, we have

$$P'_\varepsilon(r) = \frac{1}{w_\varepsilon}(E''_\varepsilon(U_\varepsilon + r\mathbf{W}_\varepsilon), (\mathbf{W}_\varepsilon, \mathbf{\bar{W}}_\varepsilon)),$$

and we still have complete freedom of choice for $\mathbf{\bar{W}}_\varepsilon$ as long as they remain in a uniformly bounded set. Our intention is to select such path $\mathbf{\bar{W}}_\varepsilon$ in such a way that the signs of the derivatives $P'_\varepsilon(r_i(\varepsilon))$ is the same as that of $g'_\varepsilon(r_i(\varepsilon))$, and hence alternating too.

If we compare these derivatives, we find that

$$P'_\varepsilon(r_i(\varepsilon)) - g'_\varepsilon(r_i(\varepsilon)) = \frac{1}{w_\varepsilon} \langle E''_\varepsilon(U_\varepsilon + r_i(\varepsilon)\mathbf{W}_\varepsilon), (\mathbf{W}_\varepsilon, \mathbf{\bar{W}}_\varepsilon) \rangle$$

$$- \frac{1}{w_\varepsilon} \langle E''_\varepsilon(u_{\varepsilon,i}), (\mathbf{W}_\varepsilon + R'_\varepsilon(r_i(\varepsilon)), \mathbf{\bar{W}}_\varepsilon) \rangle.$$

From this expression, we can write

(68)

$$P'_\varepsilon(r_i(\varepsilon)) - g'_\varepsilon(r_i(\varepsilon)) = \frac{1}{w_\varepsilon} \langle E''_\varepsilon(U_\varepsilon + r_i(\varepsilon)\mathbf{W}_\varepsilon) - E''_\varepsilon(U_\varepsilon + r_i(\varepsilon)\mathbf{W}_\varepsilon + R_{\varepsilon,i}), (\mathbf{W}_\varepsilon, \mathbf{W}_\varepsilon) \rangle$$

$$+ S_1 + S_2 + S_3,$$

where $S_k$ are remainders given by

$$S_1 = \frac{1}{w_\varepsilon} \langle E''_\varepsilon(U_\varepsilon + r_i(\varepsilon)\mathbf{W}_\varepsilon), (\mathbf{W}_\varepsilon, \mathbf{\bar{W}}_\varepsilon - \mathbf{W}_\varepsilon) \rangle,$$

$$S_2 = \frac{1}{w_\varepsilon} \langle E''_\varepsilon(U_\varepsilon + r_i(\varepsilon)\mathbf{W}_\varepsilon + R_{\varepsilon,i}), (\mathbf{W}_\varepsilon, \mathbf{\bar{W}}_\varepsilon - \mathbf{W}_\varepsilon) \rangle,$$

$$S_3 = \frac{1}{w_\varepsilon} \langle E''_\varepsilon(U_\varepsilon + r_i(\varepsilon)\mathbf{W}_\varepsilon + R_{\varepsilon,i}), (R'_\varepsilon(r_i(\varepsilon)), \mathbf{W}_\varepsilon) \rangle.$$

Once again, by the mean-value theorem, numbers $\bar{t}_{i,\varepsilon} \in [0,1]$ can be found so that

$$P'_\varepsilon(r_i(\varepsilon)) - g'_\varepsilon(r_i(\varepsilon)) = \frac{1}{w_\varepsilon} \langle E''_\varepsilon(U_\varepsilon + r_i(\varepsilon)\mathbf{W}_\varepsilon), (\bar{t}_{i,\varepsilon} R_{\varepsilon,i}, \mathbf{W}_\varepsilon, \mathbf{\bar{W}}_\varepsilon) \rangle + S_1 + S_2 + S_3.$$
The values $w_\varepsilon$ can be taken even larger, if necessary, to ensure that
\[ \frac{1}{w_\varepsilon} \| P''_\varepsilon(U_\varepsilon + r_i(\varepsilon)W_\varepsilon) \| \leq 1, \quad \frac{1}{w_\varepsilon} \| P''_\varepsilon(U_\varepsilon + r_i(\varepsilon)W_\varepsilon + R_{\varepsilon,i}) \| \leq 1, \]
and in this fashion conclude, just as we did earlier and given that the norms
\[ \| R_{\varepsilon,i} \|, \| R'_{\varepsilon,i} \|, \| W_\varepsilon - \hat{W}_\varepsilon \|, \]
tend to zero with $\varepsilon$, that all terms in the right-hand side in (68) converge to zero except possibly for $S_1$. The whole point is to choose $\hat{W}_\varepsilon$ so that the two numbers in the left-hand side of (68) have the same sign.

To see that this is always possible, we look with some care at the structure of the remainder $S_1$. Set
\[ S_\varepsilon(r) = \frac{1}{w_\varepsilon} \langle E''_\varepsilon(U_\varepsilon + r W_\varepsilon), (W_\varepsilon, \hat{W}_\varepsilon) \rangle, \]
where $\hat{W}_\varepsilon = \tilde{W}_\varepsilon - W_\varepsilon$. These are polynomials in $r$ of maximum degree $2n$ and coefficients that are integrals over the unit interval $[0,1]$ of functions involving $U_\varepsilon$, $W_\varepsilon$ and, depending linearly on the yet unknown $\tilde{W}_\varepsilon$ (remember the formulas for the Hessian $E''_\varepsilon$ given in Section 6). Let $X^j_\varepsilon$, depending on $U_\varepsilon$ and $W_\varepsilon$, be the corresponding factor so that the inner product (in $H^2_{D,J+1}([0,1];\mathbb{R}^2)$) $X^j_\varepsilon \cdot \tilde{W}_\varepsilon$ is the coefficient of $S_\varepsilon(r)$ corresponding to power $j$. The perturbation argument used earlier to move from $W_\varepsilon$ to $\tilde{W}_\varepsilon$ can be invoked here to guarantee, by perturbing $W_\varepsilon$ in an appropriate way, that none of the coefficients (they are indeed functions defined in $[0,1]$) in the finite set $X^j_\varepsilon$ vanishes identically. We would like to select this path $\tilde{W}_\varepsilon$ so that the values of $S_\varepsilon(r_i(\varepsilon))$ are provided by (68), bearing in mind that we are only interested in that the signs of $P_\varepsilon(r_i(\varepsilon))$ and $g_\varepsilon(r_i(\varepsilon))$ be the same for all $i$ and all $\varepsilon$ sufficiently small, and knowing that the other terms on the right-hand side of (68) converge to zero as $\varepsilon \to 0$. Since fixing the values of a polynomial at certain distinct points, by choosing the coefficients, involves a non-vanishing Vandermonde determinant, such coefficients can always be determined in a unique way once the values to be adjusted are given. In our case such coefficients are of the form $X^j_\varepsilon \cdot \tilde{W}_\varepsilon$, and none of the $X^j_\varepsilon$ is the zero function. We can, thus, choose a corresponding number of subintervals of $[0,1]$ where the integrals of $X^j_\varepsilon$ are not zero, and then $\tilde{W}_\varepsilon$ can be chosen in any way in such subintervals to adjust the product $X^j_\varepsilon \cdot \tilde{W}_\varepsilon$ to the required values. Off the union of those subintervals, $\tilde{W}_\varepsilon$ can be given arbitrary values to keep them in a fixed bounded set in $H^2_{D,J+1}([0,1];\mathbb{R}^2)$. After all, we have an infinite number of degrees of freedom in $\tilde{W}_\varepsilon$ at our disposal to adjust the sign of a fixed, finite set of numbers.

To summarize, we have been able to arrange for a situation in which we have a family of polynomials $P_\varepsilon(r)$ of maximum degree $2n + 1$, a family of functions $g_\varepsilon(r_i)$ with, at least, $m$ real roots $r_i(\varepsilon)$ corresponding to $m$ critical paths $u_{\varepsilon,i}$ from the full set of those with a given asymptotic behavior, such that
\[ P_\varepsilon(r_i(\varepsilon)) - g_\varepsilon(r_i(\varepsilon)) \to 0 \]
uniformly in $i$, as $\varepsilon \to 0$ and the derivatives $P_\varepsilon'(r_i(\varepsilon))$ alternate sign for all $\varepsilon$ sufficiently small. Consequently, by Rolle’s theorem, given that there is at least one root of $P_\varepsilon'$ inside each subinterval $(r_i(\varepsilon), r_{i+1}(\varepsilon))$, we can conclude that $m \leq 2n + 1$ as desired, and the first step is finished.
Improving to $m \leq n$ is not difficult. Suppose that we could find $m$ distinct branches of critical paths having one of the asymptotic limits described in Theorem 31. Another way of focusing on this set of branches of critical paths is by saying that the corresponding constant

$$\pm k_\varepsilon = F^\perp(u) \cdot u'$$

for all such critical paths $u = u_{\varepsilon,i}$ stays away from zero (see (55)) and that each sign in \text{[69]} corresponds asymptotically to half the unit interval $[0,1]$. Without loss of generality, we may assume that

$$F^\perp(u) \cdot u' - k_\varepsilon \to 0 \text{ in } (0,1/2), \quad -F^\perp(u) \cdot u' - k_\varepsilon \to 0 \text{ in } (1/2,1),$$

for constants $k_\varepsilon$ uniformly away from zero. Recall that the polynomial $P_\varepsilon(r)$ from (64) can be written as

$$P_\varepsilon(r) = \frac{1}{w_\varepsilon} \left( E'_0(U_\varepsilon + r W_\varepsilon), \tilde{W}_\varepsilon \right)$$

$$= \frac{1}{w_\varepsilon} \left( \langle E'_0(U_\varepsilon + r W_\varepsilon), \tilde{W}_\varepsilon \rangle + \langle \varepsilon(U_\varepsilon + r W_\varepsilon), \tilde{W}_\varepsilon \rangle + \langle v_\varepsilon, \tilde{W}_\varepsilon \rangle \right),$$

according to (65). If we denote by $u = U_\varepsilon + r W_\varepsilon$, the first term in $P_\varepsilon(r)$ is

$$\frac{1}{w_\varepsilon} (E'_0(u), \tilde{W}_\varepsilon) = \frac{1}{w_\varepsilon} \int_0^1 \left( F^\perp(u) \cdot u' \right) \left( (D F^\perp(u) \tilde{W}_\varepsilon) \cdot u' + F^\perp(u) \cdot \tilde{W}'_\varepsilon \right) dt,$$

which can also be written in the form

$$\frac{1}{w_\varepsilon} \int_0^{1/2} \left( F^\perp(u) \cdot u' \right) \left( (D F^\perp(u) \tilde{W}_\varepsilon) \cdot u' + F^\perp(u) \cdot \tilde{W}'_\varepsilon \right) dt +$$

$$\frac{1}{w_\varepsilon} \int_{1/2}^1 \left( -F^\perp(u) \cdot u' \right) \left( -(D F^\perp(u) \tilde{W}_\varepsilon) \cdot u' - F^\perp(u) \cdot \tilde{W}'_\varepsilon \right) dt,$$

or even

$$\frac{1}{w_\varepsilon} \int_0^{1/2} \left( F^\perp(u(t)) \cdot u'(t) \right) \left\{ \left( [D F^\perp(u(t)) \tilde{W}_\varepsilon(t)] \cdot u'(t) + F^\perp(u(t)) \cdot \tilde{W}'_\varepsilon(t) \right) - \right\}$$

$$-\left( F^\perp(u(t+1/2)) \cdot u'(t+1/2) \right) \left( [D F^\perp(u(t+1/2)) \tilde{W}_\varepsilon(t+1/2)] \cdot u'(t+1/2) \right)$$

$$-\left( F^\perp(u(t+1/2)) \cdot u'(t+1/2) \right) \left( [D F^\perp(u(t+1/2)) \tilde{W}_\varepsilon(t+1/2)] \cdot \tilde{W}'_\varepsilon(t+1/2) \right) \} \ dt.$$

Since the second factors in the three terms in the integrand of this last integral are polynomials in $r$ of at most degree $n$, and the other terms in $P_\varepsilon(r)$

$$\frac{1}{w_\varepsilon} \left( \langle \varepsilon(U_\varepsilon + r W_\varepsilon), \tilde{W}_\varepsilon \rangle + \langle v_\varepsilon, \tilde{W}_\varepsilon \rangle \right)$$

are polynomials in $r$ of degree 2, if we assume, seeking a contradiction, the existence of more than $n$ roots for $P_\varepsilon(r)$ and every sufficiently small $\varepsilon$, then the first factors

$$\left( F^\perp(u(t)) \cdot u'(t), \quad -(F^\perp(u(t+1/2)) \cdot u'(t+1/2) \right)$$

in those same three terms could not be arbitrarily close to one and the same constant $k_\varepsilon$. This contradicts (70). We can conclude that $P_\varepsilon(r)$ cannot have more than $n$ roots under (69). The lemma is proved.
12. Counting the number of critical closed paths of $E_\varepsilon$

We look at Proposition 24 and for $\varepsilon$ positive and sufficiently small, and $0 < a_\varepsilon < b_\varepsilon$, we let $q_\varepsilon$ be the number of critical closed paths in $\{a_\varepsilon \leq E_\varepsilon \leq b_\varepsilon\}$, with $a_\varepsilon$, sufficiently small, and $b_\varepsilon$, sufficiently large, complying with the following conditions:

(i) $a_\varepsilon$ and $b_\varepsilon$ are non-critical values of $E_\varepsilon$;
(ii) each limit cycle of our differential system identifies one, and only one, of the connected components of $\{E_\varepsilon \leq a_\varepsilon\}$;
(iii) if $CP_\varepsilon$ is the set of critical paths of $E_\varepsilon$ for branches having one of the limit behaviors described in Theorem 31, then

$$\max_{u \in CP_\varepsilon} E_\varepsilon(u) \leq b_\varepsilon;$$

(iv) the level set $\{E_\varepsilon \leq b_\varepsilon\}$ has a single connected component.

Condition (ii) is ensured by Lemma 23; conditions (iii) and (iv) are just a requirement on the largeness of $b_\varepsilon$. We can have our upper bound after Proposition 24, provided we show that the number $q_\varepsilon$ can be taken independent of $\varepsilon$. We will express this bound in terms of the following parameters, in addition to the degree $n$ of the system:

- $M$: the number of connected components of the curve $\text{Div} = 0$;
- $N$: the number of contact points of the differential system.

**Theorem 33.** Under the assumptions and notation of Theorem 2 and for every choice of numbers $a_\varepsilon$ and $b_\varepsilon$ as indicated above

$$-\sum(\{a_\varepsilon \leq E_\varepsilon \leq b_\varepsilon\}) \leq n(M + N),$$

and so our differential system cannot have more than $n(M + N)$ limit cycles.

**Proof.** From Theorem 31 we must compute the number of limit behaviors which are contained in the components of the curve $\text{Div} = 0$.

Assume that the connected component $i$, for $i \in \{0, 1, \ldots, M_1\}$, of $\text{Div} = 0$ homeomorphic to a straight line contains $x_i > 0$ contact points. Then it can have at most $x_i + 1$ limit behaviors. Therefore the number of limit critical closed paths contained in the components of $\text{Div} = 0$ homeomorphic to a straight line is at most $M_1 + \sum x_i$.

Suppose $y_j$ is the number of contact points in the $j$th component, for $j \in \{0, 1, \ldots, M_2\}$ of $\text{Div} = 0$ homeomorphic to an oval. Then we can have at most $\sum_j y_j + O$ different limit behaviors, all of which are bounded.

In summary, the number of limit critical closed paths contained in the components of $\text{Div} = 0$ is at most

$$M_1 + \sum_{i=0}^{M_1} x_i + \sum_{j=0}^{M_2} y_j + M_2 = N + M.$$

However, by Lemma 32 each such possible limit behavior must be multiplied by the corresponding multiplicity factor $n$. Hence, we will have at most $n(M + N)$ critical closed paths of $E_\varepsilon$ for $\varepsilon$ sufficiently small. \qed
13. Non-generic situation

In this section we focus on the treatment of polynomial differential systems (1) for which either the curve Div = 0 has singular points, i.e. the system

\begin{align}
P_{xx} + Q_{yx} = P_{xy} + Q_{yy} = 0, \quad P_x + Q_y = 0
\end{align}

has some solutions; or our initial differential system (1) has non-countable, infinitely many contact points, i.e. system (2) has a continuum of solutions. Note that equations (71) and (2) involve the partial derivatives of Div. Our argument revolves around the idea that such systems can be uniformly approximated by a sequence \(F_\delta\) of non-singular polynomial systems without increasing the degree, and in such a way that the divergence curve of \(F_\delta\) has no singularities, and finitely many contact points with system (1).

We can definitely apply our previous results to the family of functionals

\[ E_{\varepsilon,\delta}(u) = \int_0^1 \left[ \frac{1}{2} (F_\delta (u(t)) \cdot u'(t))^2 + \frac{\varepsilon}{2} (|u''(t)|^2 + |u'(t)|^2 + |u(t)|^2) 
+ (u(t) \cdot v_{\varepsilon,\delta}(t) + u'(t) \cdot v'_{\varepsilon,\delta}(t) + u''(t) \cdot v''_{\varepsilon,\delta}(t)) \right] dt, \]

where the dependence of the smooth paths \(v_{\varepsilon,\delta}\) on \(\delta\) is as regular as necessary. The subindex \(\varepsilon\) in this functional is given and fixed, but sufficiently small, in the process of moving \(\delta\). By our previous results applied for each \(\delta\), we conclude that the number of connected components of the level sets

\[ \{E_{\varepsilon,\delta} \leq a_{\varepsilon,\delta}\} \]

is bounded above by a number \(q_n\) independently of \(\varepsilon\) and \(\delta\), and for arbitrary values \(a_{\varepsilon,\delta} > 0\). Given \(a_{\varepsilon} > 0\), take a sequence \(a_{\varepsilon,\delta}\) converging to \(a_{\varepsilon}\) as \(\delta \to 0\). Because \(E_{\varepsilon,\delta} \to E_{\varepsilon}\) as \(\delta \to 0\) uniformly over bounded subset of \(H^2_{O,+1}([0,1]; \mathbb{R}^2)\) (\(\varepsilon\) is kept fixed here), we would have the convergence

\[ \{E_{\varepsilon,\delta} \leq a_{\varepsilon,\delta}\} \to \{E_{\varepsilon} \leq a_{\varepsilon}\}. \]

But since new connected components cannot be created through convergence of sets, we conclude that the number of connected components of the limit set \(\{E_{\varepsilon} \leq a_{\varepsilon}\}\) is also bounded above by \(q_n\) for every choice \(a_{\varepsilon}\). Consequently, we also have the same upper bound \(q_n\), in terms of the degree \(n\) of the initial polynomial differential system, for the singular system corresponding to a non-generic field \(F\).

Notice that we are not claiming anything about the relationship between limit cycles of \(F\) and limit cycles of \(F_\delta\), or of critical closed paths for \(E_{\varepsilon}\) and of \(E_{\varepsilon,\delta}\).

14. Appendix 1. The 16-th Hilbert problem restricted to algebraic limit cycles

Associated with the polynomial differential system (1), there is the polynomial vector field

\begin{align}
F = P(x, y) \frac{\partial}{\partial x} + Q(x, y) \frac{\partial}{\partial y}.
\end{align}
The algebraic curve \( f(x,y) = 0 \) of \( \mathbb{R}^2 \) is called an invariant algebraic curve of the polynomial vector field \( \mathbf{F} \), or of the polynomial differential system (1), if for some polynomial \( K = K(x,y) \), we have
\[
\mathbf{F} f = P \frac{\partial f}{\partial x} + Q \frac{\partial f}{\partial y} = Kf.
\]
The polynomial \( K \) is called the cofactor of the invariant algebraic curve \( f = 0 \).

Since on the points of an algebraic curve \( f = 0 \), the gradient \( \nabla f = (f_x, f_y) \) of the curve is orthogonal to the vector field \( \mathbf{F} \) (see (72)), this vector field is tangent to the curve \( f = 0 \). Hence, the curve \( f = 0 \) is formed by orbits of the vector field \( \mathbf{F} \).

This justifies the name of invariant algebraic curve given to the algebraic curve \( f = 0 \) satisfying (72) for some polynomial \( K \): it is invariant under the flow defined by the vector field \( \mathbf{F} \).

The next well-known result tell us that we can restrict our attention to the irreducible invariant algebraic curves, for a proof see for instance [30]. Here, as it is usual, \( \mathbb{R}[x,y] \) denotes the ring of all polynomials in the variables \( x \) and \( y \), and coefficients in \( \mathbb{R} \).

**Proposition 34.** Let \( f \in \mathbb{R}[x,y] \), and let \( f = f_1^{m_1} \cdots f_r^{m_r} \) be its factorization in irreducible factors over \( \mathbb{R}[x,y] \). Then for a polynomial vector field \( \mathbf{F} \), \( f = 0 \) is an invariant algebraic curve with cofactor \( K_f \) if and only if \( f_i = 0 \) is an invariant algebraic curve for each \( i = 1, \ldots, r \) with cofactor \( K_{f_i} \). Moreover \( K_f = m_1 K_{f_1} + \ldots + m_r K_{f_r} \).

Consider the space \( \Sigma' \) of all real polynomial vector fields (72) of degree \( n \) having real irreducible invariant algebraic curves.

An algebraic limit cycle is an oval of an algebraic curve which is a limit cycle of a polynomial differential system (1).

A simpler version of the second part of the 16th Hilbert’s problem with respect to the number of limit cycles is: Is there an uniform upper bound for the maximal number of algebraic limit cycles of any polynomial vector field of \( \Sigma' \)? We cannot provide an answer to this question for general real algebraic curves, but we give the answer for the following class of algebraic curves.

We say that a set \( f_j = 0 \), for \( j = 1, \ldots, k \), of irreducible algebraic curves is generic if it satisfies the following five conditions:

(i) There are no points at which \( f_j = 0 \) and all its first derivatives vanish (i.e. \( f_j = 0 \) is a non–singular algebraic curve).
(ii) The highest order homogeneous terms of \( f_j \) have no repeated factors.
(iii) If two curves intersect at a point in the affine plane, they are transversal at this point.
(iv) There are no more than two curves \( f_j = 0 \) meeting at any point in the affine plane.
(v) There are no two curves having a common factor in the highest order homogeneous terms.

The next result was proved by Llibre, Ramírez and Sadovskaia [31] in 2010.

**Theorem 35.** For a polynomial vector field \( \mathbf{F} \) of degree \( n \geq 2 \) having all its irreducible invariant algebraic curves generic, the maximum number of algebraic limit cycles is at
most $1 + (n - 1)(n - 2)/2$ if $n$ is even, and $(n - 1)(n - 2)/2$ if $n$ is odd. Moreover these upper bounds are sharp.

For cubic polynomial vector fields having all their irreducible invariant algebraic curves generic, Theorem 35 says that one is the maximum number of algebraic limit cycles, but there are examples of cubic polynomial vector fields having two algebraic limit cycles. Such vector fields have non–generic invariant algebraic curves. For example, the polynomial differential system of degree 3
\[
\dot{x} = 2y(10 + xy), \quad \dot{y} = 20x + y - 20x^3 - 2x^2y + 4y^3,
\]
has two algebraic limit cycles contained into the invariant algebraic curve $2x^4 - 4x^2 + 4y^2 + 1 = 0$, see Proposition 19 of [34].

Until now, all polynomial vector fields having non–generic invariant algebraic curves and more algebraic limit cycles than the upper bounds given in Theorem 35 for the generic case, have odd degree, and at most one more limit cycle than the upper bound of Theorem 35. So, in [31] we conjectured the following.

**Conjecture.** The maximum number of algebraic limit cycles that a polynomial differential system of degree $n$ can have is $1 + (n - 1)(n - 2)/2$.

Note that the conjectures is true when $n$ is even and we restrict the algebraic limit cycles to generic invariant algebraic curves.

After [31], three other paper have appeared related to the algebraic limit cycles of polynomial differential systems. One due to the same authors [32], and other two due to Xiang Zhang [50].

15. **Appendix 2. More on Hilbert’s 16th problem**

15.1. **On the configuration of the limit cycles of the polynomial differential systems.** A topological configuration of limit cycles is a finite set $C = \{C_1, \ldots, C_n\}$ of disjoint simple closed curves of the plane such that $C_i \cap C_j = \emptyset$ for all $i \neq j$.

Given a topological configuration of limit cycles $C = \{C_1, \ldots, C_n\}$, the curve $C_i$ is primary if there is no curve $C_j$ of $C$ contained into the bounded region limited by $C_i$.

Two topological configurations of limit cycles $C = \{C_1, \ldots, C_n\}$ and $C' = \{C'_1, \ldots, C'_m\}$ are (topologically) equivalent if there is a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h(\bigcup_{i=1}^{n} C_i) = \left(\bigcup_{i=1}^{m} C'_i\right)$. Of course, for equivalent configurations of limit cycles $C$ and $C'$, we have that $n = m$.

We say that a polynomial differential system (1) realizes the configuration $C$ of limit cycles if the set of all limit cycles of (1) is equivalent to $C$.

Llibre and Rodriguez [33] proved the following result in 2004.

**Theorem 36.** Let $C = \{C_1, \ldots, C_n\}$ be a topological configuration of limit cycles, and let $r$ be its number of primary curves. Then the following statements hold.

(a) The configuration $C$ is realizable by some polynomial differential system.
(b) The configuration \(C\) is realizable as algebraic limit cycles by a polynomial differential system of degree \(\leq 2(n+r) - 1\). Moreover, such a polynomial differential system has a first integral of Darboux type.

Statement (a) of Theorem 36 follows immediately from statement (b).

Statement (a) of Theorem 36 was solved, for the first time, by Schecter and Singer [42], and Sverdlove [46], but they do not provide an explicit polynomial vector field satisfying the given configuration of limit cycles, as it was provided in the proof of statement (b) of Theorem 36.

If \(f = f(x, y)\) is a polynomial we denote its partial derivatives with respect to the variables \(x\) and \(y\) as \(f_x\) and \(f_y\), respectively. Christopher [12] proved the following result in 2001.

**Theorem 37.** Let \(f = 0\) be a non-singular algebraic curve of degree \(n\), and \(D\), a first degree polynomial chosen so that the straight line \(D = 0\) lies outside all bounded components of \(f = 0\). Choose the constants \(\alpha\) and \(\beta\) so that \(\alpha D_x + \beta D_y \neq 0\), then the polynomial differential system of degree \(n\),

\[
\dot{x} = \alpha f - Df_y, \quad \dot{y} = \beta f + Df_x,
\]

has all the bounded components of \(f = 0\) as hyperbolic limit cycles. Furthermore, the differential system has no other limit cycles.

Theorem 37 improves a similar result due to Winkel [48], but the polynomial differential system obtained by Winkel has degree \(2n - 1\).

Given a topological configuration of \(k\) limit cycles we can consider an equivalent topological configuration formed by circles. Consider then the algebraic curve \(f = 0\) formed by the product of all the circles. Applying Theorem 37 to the curve \(f = 0\), we obtain a polynomial differential system of degree \(n = 2k\), which realizes the given topological configuration of \(k\) limit cycles with algebraic limit cycles. A difference between the polynomial differential systems of Theorems 36 and 37 is that the first always has a first integral, and the second, in general, has no first integrals.

In short, both Theorems 36 and 37 show that any topological configuration of limit cycles is realizable with algebraic limit cycles for some polynomial differential system, and provide the degree of such polynomial differential systems. But there are many questions which remains open, as for instance: what are the possible topological configurations of limit cycles realizable for the polynomial differential systems of a given degree? This question is definitely more difficult than the question to provide a uniform upper bound for the maximum number of limit cycles that the polynomial differential systems of a given degree can have.

15.2. **Limit cycles and the inverse integrating factor.** Another useful result on the limit cycles of a \(C^1\) differential system in the plane is the following one due to Giacomini, Llibre and Viano [20], see an easier proof in [33]. This result has been used in the proofs of Theorems 35 and 36. First we need a definition.

Let \(U\) be an open subset of \(\mathbb{R}^2\). A function \(V : U \to \mathbb{R}\) is an inverse integrating factor of a \(C^1\) vector field \(F\) defined on \(U\) if \(V\) verifies the linear partial differential
system

\[ P \frac{\partial V}{\partial x} + Q \frac{\partial V}{\partial y} = \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) V \]

in \( U \).

**Theorem 38.** Let \( X \) be a \( C^1 \) vector field defined in the open subset \( U \) of \( \mathbb{R}^2 \). Let \( V : U \to \mathbb{R} \) be an inverse integrating factor of \( X \). If \( \gamma \) is a limit cycle of \( X \), then \( \gamma \) is contained in \( \{(x, y) \in U : V(x, y) = 0\} \).

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