FINITE-TEMPERATURE SUPERSYMMETRY AS A CONSTRAINED SUPERGRAVITY

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We introduce thermal superspace and show how it can be used to reconcile the superfield formulation of supersymmetry with finite temperature environments.

1 Thermal Superspace

Immersing a physical system in a heat bath results in the fields acquiring different properties according to their statistics. E.g., finite-temperature bosonic fields obey periodic boundary conditions, while fermionic fields satisfy antiperiodic b.c.’s. Such a distinction can be seen also at the level of the Green’s functions. Depending on the field’s statistics, thermal propagators obey either a bosonic KMS condition, or a fermionic one. Therefore, thermal effects induce a clear and mandatory distinction between bosons and fermions. As a consequence, finite temperature environments are incompatible with $T = 0$ supersymmetry: the supersymmetry transformation is indeed unable to take into account the distinct thermal properties that go along with different statistics.

The thermal superspace approach [1] allows to formulate supersymmetry at finite temperature in a way which respects the different thermal behaviours of bosons and fermions. The parameters of supersymmetry transformations at $T > 0$ being \textit{time-dependent} and \textit{antiperiodic} on the imaginary time interval $[0, \beta]$, thermal supersymmetry takes periodic bosons into antiperiodic fermions, and \textit{vice-versa}. Assuming the corresponding $T > 0$ supersymmetry charges to induce motion in some superspace, we naturally require the Grassmann coordinates of that superspace to be \textit{time-dependent} and \textit{antiperiodic}, like the thermal supersymmetry parameters. A point in thermal superspace has therefore coordinates $[x^\mu, \hat{\theta}^\alpha(t), \hat{\overline{\theta}}^\dot{\alpha}(t)]$, $t = x^0$, where a “hat” is used to denote thermal quantities, and the thermal Grassmann coordinates are subject to the temperature-dependent \textit{antiperiodicity conditions}

$$\hat{\theta}^\alpha(t + i\beta) = -\hat{\theta}^\alpha(t), \quad \hat{\overline{\theta}}^\dot{\alpha}(t + i\beta) = -\hat{\overline{\theta}}^\dot{\alpha}(t), \quad \text{with } \beta = \frac{1}{T}. \quad (1)$$

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So, thermal superspace – in which we shall define thermal superfields, thermal supercharges, etc. (see below) – reconciles supersymmetry with finite temperature. This approach is particularly welcome for, *e.g.*, cosmology, as one can now contemplate formulating supergraph techniques at finite temperature, and deriving from these the $T > 0$ effective potential in superfield form.

Thermal supersymmetry corresponds to a transformation with time-dependent, that is, *local* parameters. It is in this sense a form of supergravity, but a restricted one, as locality is enforced only in the time direction, and as, moreover, the thermal supersymmetry parameters are subject to an antiperiodicity condition identical to that in superfields as expansions in thermal superspace, and

$$\hat{\phi}[^\mu\nu]\equiv \langle \phi[^\mu\nu] \rangle = x[^\mu\nu] + \sqrt{2} \hat{\phi}(t) \psi[^\mu\nu] \tilde{\phi}(t) f[^\mu\nu],$$

(2)

and thermal antichiral ones follow a similar expansion. The thermal (chiral-antichiral) superfield propagator $G_C [\hat{\psi}_1, \hat{\theta}_1, \hat{\theta}_2] \equiv \langle \mathcal{T}_C \hat{\phi}[\hat{\psi}_1, \hat{\theta}_1] \tilde{\phi}[\hat{\theta}_2, \hat{\theta}_2] \rangle$ expands, in analogy to $T=0$, as

$G_C [\hat{\psi}_1, \hat{\theta}_2, \hat{\theta}_1, \hat{\theta}_2] = D_C [\hat{\psi}_1 - \hat{\theta}_2] - 2 \hat{\theta}_1 \hat{\theta}_2 S_{C}[^\alpha^\beta] [\hat{\psi}_1 - \hat{\theta}_2] + \hat{\theta}_1 \hat{\theta}_2 \hat{\psi} \hat{\theta}_2 \mathcal{F}_C [\hat{\psi}_1 - \hat{\theta}_2],$  

where $D_C, S_C$ and $\mathcal{F}_C$ denote the thermal propagators for the component fields $z, \psi$ and $f$, respectively. Thermal field theory requires the propagators of thermal scalar components to obey a bosonic KMS condition,

$$D_C(t_1; x_1, t_2; x_2) = D_C(t_1 + i \beta; x_1, t_2; x_2), \quad (3)$$

$$\mathcal{F}_C(t_1; x_1, t_2; x_2) = \mathcal{F}_C(t_1 + i \beta; x_1, t_2; x_2), \quad (4)$$

while the thermal fermionic component must fulfill a fermionic KMS constraint,

$$S_C[^\alpha^\beta](t_1; x_1, t_2; x_2) = -S_C[^\alpha^\beta](t_1 + i \beta; x_1, t_2; x_2). \quad (5)$$

Thermal superspace allows to write a KMS condition at the level of thermal superfield propagators. Indeed, defining $G_C^\phi = \langle \hat{\phi} \hat{\phi} \rangle[^\beta]$, $G_C^\phi = \langle \hat{\phi} \hat{\phi} \rangle[^\beta]$, a superfield KMS (or super-KMS) condition can be written as:

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3We write $\hat{\gamma}_i, \hat{\theta}_i, \hat{\sigma}_i$ in place of $\hat{\gamma}_i(t_i)$ and $\hat{\theta}_i(t_i), \hat{\sigma}_i(t_i)$.
\[ G_C^C \{ \hat{g}_1(t_1), \hat{g}_2(t_2), \hat{\theta}_1(t_1), \hat{\theta}_2(t_2) \} = G_C^C \{ \hat{g}_1(t_1 + i\beta), \hat{g}_2(t_2), \hat{\theta}_1(t_1 + i\beta), \hat{\theta}_2(t_2) \}, \quad (6) \]

where \( \hat{g}_1(t_1 + i\beta) = \hat{g}_1(t_1) + (i\beta; 0). \) Clearly, the superfield KMS condition (3) is of bosonic type, since chiral and antichiral superfields are bosonic objects. This condition can be proven directly [1] at the thermal superfield level. Also, one easily verifies [1] that the super-KMS condition yields the component KMS conditions (3), (4) and (5). The antiperiodicity (1), which captures the essence of thermal superspace, is an essential ingredient of these proofs.

Our thermal superfield expansion (2) can easily be seen to be consistent also from the point of view of the fields’ boundary conditions. Thermal chiral superfields being bosonic objects, they obey in thermal superspace a periodic superfield boundary condition, \( \hat{\phi}[\hat{g}(t), \hat{\theta}(t)] = \hat{\phi}[\hat{g}(t+i\beta), \hat{\theta}(t+i\beta)], \) which writes, when cast in the variables \((x, \hat{\theta}, \hat{\varnothing}), x = (t; \mathbf{x}), \)

\[ \hat{\phi}[t; \mathbf{x}, \hat{\theta}(t), \hat{\varnothing}(t)] = \hat{\phi}[t + i\beta; \mathbf{x}, \hat{\theta}(t + i\beta), \hat{\varnothing}(t + i\beta)]. \quad (7) \]

Indeed, expanding both sides in thermal superspace along

\[ \hat{\phi}[x, \hat{\theta}, \hat{\varnothing}] = z + \sqrt{2} \hat{\theta} \psi - \hat{\theta} f - i (\hat{\theta} \sigma^\mu \hat{\varnothing}) \partial_\mu z + \frac{i}{\sqrt{2}} \hat{\theta} \hat{\varnothing} ([\partial_\mu \psi] \sigma^\mu \hat{\varnothing}) - \frac{1}{4} \partial_\mu \hat{\varnothing} \hat{\varnothing} \partial \varnothing, \]

we get from (7): (i) for the thermal scalar field \( z(t; \mathbf{x}) = z(t + i\beta; \mathbf{x}), \) (ii) for the thermal fermion \( \psi, \) upon replacing \( \hat{\theta}(t + i\beta) = -\hat{\theta}(t), \) the antiperiodic b.c. \( \psi(t; \mathbf{x}) = -\psi(t + i\beta; \mathbf{x}), \) and (iii) for the thermal scalar field \( f, \) due to \( \hat{\theta}(t + i\beta)\hat{\varnothing}(t + i\beta) = \hat{\varnothing}(t)\hat{\varnothing}(t), \) the periodic b.c. \( f(t; \mathbf{x}) = f(t + i\beta; \mathbf{x}). \)

3 Thermal Covariant Derivatives

Deriving supersymmetry covariant derivatives and supercharges on thermal superspace can be done simply by evaluating the effect of changing variables from \( T = 0 \) superspace \( [x^\mu, \theta, \varnothing] \) to thermal superspace \( [x^\mu, \theta', \varnothing'] = [x^\mu, \hat{\theta}(t), \hat{\varnothing}(t)]. \) The partial derivatives with respect to \( x, \theta \) and \( \varnothing \) transform trivially, while \( \partial_t \to (\partial_{t'} + (\partial_t \theta^a) \hat{\alpha}_a + (\partial_t \varnothing^a) \hat{\alpha}_a). \) Setting \( \partial_{t'} = 1, \) we define the thermal space-time derivative \( \hat{\partial}_\mu \) – the thermal covariantization of \( \partial_\mu \) – as

\[ \hat{\partial}_\mu = (\partial_t - \Delta : \hat{\theta}), \quad \Delta \equiv (\partial_\theta \theta^a) \hat{\alpha}_a + (\partial_\theta \varnothing^a) \hat{\alpha}_a, \quad (8) \]

which obeys \( \hat{\partial}_\mu \hat{\theta}(t) = \hat{\theta}_\mu \hat{\theta}(t) = 0. \) The thermal covariant derivatives are then defined as :

\[ \hat{D}_a = \hat{\partial}_a - i \sigma^a_{\alpha \beta} \hat{\varnothing}^\beta \hat{\partial}_\mu, \quad \hat{\varnothing}_a = \hat{\varnothing}_a - i \varnothing^a \sigma^a_{\alpha \beta} \hat{\partial}_\mu, \]

\[ \hat{D}_\alpha = \hat{\partial}_\alpha - i \sigma^a_{\alpha \beta} \hat{\varnothing}^\beta \hat{\partial}_\mu, \quad \hat{\varnothing}_\alpha = \hat{\varnothing}_\alpha - i \varnothing^a \sigma^a_{\alpha \beta} \hat{\partial}_\mu, \]
with \( \hat{\partial}_a \equiv \partial/\partial \theta^a, \ \hat{\partial}_\dot{a} \equiv \partial/\partial \dot{\theta} \). \( \hat{\partial}_a, \hat{\partial}_\dot{a} \) close on thermal translations and obey the T = 0 ACRs \( \{ \hat{D}_a, \hat{D}_{\dot{a}} \} = -2i\sigma^{\mu}_{a\dot{a}} \hat{\partial}_\mu, \{ \hat{D}_a, \hat{D}_\beta \} = \{ \hat{D}_{\dot{a}}, \hat{D}_\beta \} = 0 \). Furthermore, the thermal covariant derivatives provide a definition of the thermal chiral and antichiral superfields as the solution to \( \hat{D}_{\dot{a}} \hat{\varphi} = 0, \hat{D}_a \hat{\varphi} = 0 \).

4 Thermal Supercharges and the Thermal Supersymmetry Algebra

The thermal supercharges are constructed using the same procedure as for the thermal covariant derivatives, that is, we replace \( \theta, \theta^* \) by \( \hat{\theta}, \hat{\theta}^* \), and \( \partial_\mu, \partial_a, \partial_{\dot{a}} \) by \( \hat{\partial}_\mu \) [eq. (8)], \( \hat{\partial}_a \) and \( \hat{\partial}_{\dot{a}} \). This yields the thermal supercharges:

\[
\hat{Q}_a = -i\hat{\partial}_a + \sigma^a_\alpha \hat{\partial}_\mu - \sigma^0_{a\dot{a}} \hat{\partial}_{\dot{a}} \left( \partial_\mu \hat{\theta}^* \hat{\sigma}_\alpha \hat{\partial}_\mu + \partial_\mu \hat{\theta}^* \hat{\sigma}_{\dot{a}} \hat{\partial}_{\dot{a}} \right) = -i\hat{\partial}_a + \sigma^a_\alpha \hat{\theta}^* \hat{\partial}_\mu , \\
\hat{\bar{Q}}_a = i\hat{\partial}_a - \theta^a \sigma^0_{a\dot{a}} \partial_\mu + \theta^a \sigma^0_{a\dot{a}} \left( \partial_\mu \hat{\theta}^* \hat{\sigma}_\alpha \partial_\mu + \partial_\mu \hat{\theta}^* \hat{\sigma}_{\dot{a}} \partial_{\dot{a}} \right) = i\hat{\partial}_a - \theta^a \sigma^0_{a\dot{a}} \hat{\partial}_\mu ,
\]

for which, as at \( T = 0 \), \( \{ \hat{Q}, \hat{D} \} = \{ \hat{\bar{Q}}, \hat{\bar{D}} \} = \{ \hat{Q}, \hat{\bar{D}} \} = \{ \hat{\bar{Q}}, \hat{D} \} = 0 \). For the super-Poincaré algebra at finite temperature\(^4\), one gets the same structure as at \( T = 0 \), provided one has appropriately covariantized all the generators with respect to thermal superspace. In particular\(^5\)

\[
\{ \hat{Q}_a, \hat{\bar{Q}}_{\dot{b}} \} = -2\sigma^a_{a\dot{b}} \hat{P}_\mu = -2\left( \sigma^0_{a\dot{b}} \hat{P}_0 - \sigma^i_{a\dot{b}} \hat{P}_i \right), \quad \hat{P}_i = P_i .
\]

However, the fact that the algebra is preserved at finite temperature does not mean that supersymmetry is thermally unbroken. This can be understood as follows. Both the thermal fields’ b.c.’s and the KMS conditions carry information that is of global character, in the sense that it relates the values of the field at distant regions in space-time, along the time direction. As a consequence, the thermal superalgebra, which is a local structure, is insensitive to such global conditions and preserves its structure at finite temperature. In particular, the antiperiodicity conditions on the thermal Grassmann coordinates, eq. (11), have no influence on the algebra. It is only the local statement that the superspace Grassmann variables should be allowed to depend on time which makes it necessary to covariantize the algebra generators.

To analyze thermal supersymmetry breaking, we must investigate systems of thermal fields. As thermal bosons and fermions are distinguished by their global boundary and KMS conditions, we shall see thermal supersymmetry breaking in doing so.

\(^4\)The thermal translation and Lorentz generators are defined similarly [1].

\(^5\)The thermal time translation operator \( \hat{P}_0 = -i\hat{\theta}^0 \) can be interpreted as a central charge of the subalgebra one obtains upon removing the thermal Lorentz boosts [1].
E.g., expanding à la Matsubara the $d = 4, T = 0$ Wess-Zumino action, we get a $d = 3$ Euclidean action for the thermal modes in which the mass degeneracy is seen to be broken [1].

5 Field Realizations

Thermal supersymmetry breaking should also be seen at the level of the realizations of the thermal supersymmetry algebra. The next question we ask, therefore, is how thermal supersymmetry transforms the components of thermal superfields. This means translating into component language the thermal supersymmetry transformation

$$\delta \hat{\phi} = i(\hat{\epsilon}^\alpha \hat{Q}_\alpha + \bar{\hat{\epsilon}}^\alpha \hat{\bar{Q}}^\alpha) \hat{\phi}.$$  

We may either derive straightforwardly the component transformations [1] and note the perfect analogy to the $T=0$ case, or invoke the argument [2] that the thermal supersymmetry parameters should become time-dependent and antiperiodic at finite temperature. In both cases, we get

$$\begin{align*}
\hat{\delta} z &= \sqrt{2} \hat{\epsilon}^\alpha \psi_\alpha, \\
\hat{\delta} f &= -i \sqrt{2} (\sigma^\mu \hat{\epsilon})^\alpha (\partial_\mu \psi_\alpha), \\
\hat{\delta} \psi_\alpha &= -\sqrt{2} \hat{\epsilon}_\alpha f - i \sqrt{2} (\sigma^\mu \hat{\epsilon})_\alpha (\partial_\mu z),
\end{align*}$$

where the unique difference with the case of zero temperature is the appearance of the thermal (time-dependent and antiperiodic) spinorial parameter $\hat{\epsilon}, \hat{\bar{\epsilon}}$, with $\hat{\epsilon}(t + i\beta) = -\hat{\epsilon}(t), \hat{\bar{\epsilon}}(t + i\beta) = -\hat{\bar{\epsilon}}(t)$, in place of the constant spinorial parameter $\epsilon, \bar{\epsilon}$ of $T = 0$ supersymmetry. Eqs. (10) can be translated into transformations of the $d = 3$ Matsubara modes [1], upon developing thermally the $t$-dependent parameters $\hat{\epsilon}, \hat{\bar{\epsilon}}$. Neither the $(T = 0)$ Wess-Zumino kinetic action nor the mass action are invariant under the thermal supersymmetry transformations. At the level of the $d=3$ thermal modes, the variation of the total action is seen to be proportional to the fermionic Matsubara frequency $\omega_F \sim T$. In the $T \to 0$ limit, one expects supersymmetry to be restored. The variations of the mass and kinetic actions indeed vanish separately in that limit.

References

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