Entanglement-asymmetry correspondence for internal quantum reference frames

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In the quantization of gauge theories and quantum gravity, it is crucial to treat reference frames such as rods or clocks not as idealized external classical relata, but as internal quantum subsystems. In the Page-Wootters formalism, for example, evolution of a quantum system \(S\) is described by a stationary joint state of \(S\) and a quantum clock, where time-dependence of \(S\) arises from conditioning on the value of the clock. Here, we consider (possibly imperfect) internal quantum reference frames \(R\) for arbitrary compact symmetry groups, and show that there is an exact quantitative correspondence between the amount of entanglement in the invariant state on \(RS\) and the amount of asymmetry in the corresponding conditional state on \(S\). Surprisingly, this duality holds exactly regardless of the choice of coherent state system used to condition on the reference frame. Averaging asymmetry over all conditional states, we obtain a simple representation-theoretic expression that admits the study of the quality of imperfect quantum reference frames, quantum speed limits for imperfect clocks, and typicality of asymmetry in a unified way. Our results shed light on the role of entanglement for establishing asymmetry in a fully symmetric quantum world.

\[ |(\psi)_{RS}|^2 \gg 0 \quad \mathcal{A}(\psi_{S|R}) \text{ small} \]
\[ |(\psi_{S|R})_{v_g}|^2 \approx 0 \quad \mathcal{A}(\psi_{S|R}) \text{ large} \]

**Upper pane:** a globally invariant (e.g. timeless) state \(|\psi\rangle_{RS}\) induces asymmetry in a subsystem \(S\) by conditioning on the reference frame (e.g. clock) \(R\). **Lower pane:** more induced asymmetry amounts to smaller overlap of the conditional state with its translations, i.e. a larger value of \(\mathcal{A}(\psi_{S|R})\). We prove that the Rényi-2 entanglement entropy of \(\psi_{RS}\) equals the asymmetry of the conditional state \(\psi_{S|R}\).

**Framework.** We consider two quantum systems \(R\) (the reference) and \(S\) (the system) with \(\dim R < \infty\), carrying unitary representations \(U\) and \(V\) of the compact (possi-
We will abbreviate \(|\psi\rangle_{RS}\) that are globally invariant, \(|\psi\rangle_{RS} = U_g \otimes V_g |\psi\rangle_{RS}\) for all \(g\), reflecting that, in a background-independent theory, all physically meaningful properties are purely relational \([47, 50–53, 55, 56]\). Demanding that pure states do not pick up a global phase under the action of \(G\) can be motivated by preserving entanglement with a purifying system \([55]\). Using terminology from constraint quantization, the subspace of all globally invariant states will be called the “physical Hilbert space” \(\mathcal{H}_{phys}\) \([47]\).

In the PWM, \(G\) is the group of time translations \(\mathbb{R}\), represented on \(S\) via \(V_t = \exp(-it\hat{H}_S/h)\), and on the (infinite-dimensional) reference \(R\) via \(U_t|s\rangle_R = |s + t\rangle_R\), where \(|s\rangle_R\) denotes an improper clock eigenstate. Formally, the physical Hilbert space consists of the globally “timeless” states \(|\psi\rangle_{RS} = \int \mathcal{D}t |t\rangle_R \otimes |\psi(t)\rangle_S\).

Now, we are interested in the quantum state of \(S\) conditional on \(R\) being “oriented in some direction” \(g \in G\). To make precise sense of this intuition, we define a coherent state system \([50, 53]\) \(|g\rangle_R\) by choosing some normalized state \(|e\rangle_R\), where \(e\) is the unit element of \(G\), and setting \(|g\rangle_R := U_g|e\rangle.R\). It follows that \(U_g|g'\rangle_R = |gg'\rangle.R\). We demand that \(|e\rangle_R\) is chosen such that we obtain a resolution of the identity, \(\int \mathcal{D}g |g\rangle\langle g| = c \cdot 1_R\) with \(c > 0\) some constant, where \(dg\) denotes the Haar measure on \(G\) (it follows that \(c = 1/d_R\), with \(d_R = \dim R\)). If \(g \rightarrow U_g\) is an irreducible representation (irrep), then this follows automatically from Schur’s lemma; otherwise it imposes some conditions on \(|e\rangle_R\) described in \([50]\).

The coherent state system defines a covariant positive-operator valued measure (POVM) \([44, 63, 65]\) that we can use to measure the “orientation” \(g\) of \(R\). If we do this on a state \(|\psi\rangle_{RS} \in \mathcal{H}_{phys}\) and find outcome \(g\), then the post-measurement state (see Supplemental Material\([1]\) of \(S\) will be

\[
|\psi(g)\rangle_{SR} := \sqrt{d_R} |g\rangle_R \otimes 1_S |\psi\rangle_{RS}.
\]

We will abbreviate \(|\psi\rangle_{SR} := |\psi(e)\rangle_{SR}\), and we will sometimes emphasize the dependence of this state on the choice of \(|e\rangle_R\) by writing \(|\psi^e\rangle_{SR}\). We have \([50]\)

\[
|\psi(g)\rangle_{SR} = \sqrt{d_R} \langle e|_R U^\dagger_g \otimes V_g |\psi(e)\rangle_{SR} = V_g |\psi(e)\rangle_{SR}.
\]

This is analogous to the PWM, where conditioning on the state \(|t\rangle_R\), i.e. on the clock showing time \(t\), gives us the time-evolved state \(|\psi(t)\rangle_S = \langle t|_R V_t |\psi(0)\rangle_{SR}\).

**Conditional asymmetry.** While the initial physical state \(|\psi\rangle_{RS}\) is fully symmetric, we would like the measurement of \(R\) to break the symmetry and to lead to an asymmetric conditional state, i.e. \(|\psi\rangle_{SR} \neq V_g |\psi\rangle_{SR}\) for \(g \neq e\). Intuitively, for a “good” reference frame \(R\), it should be possible to locate the system \(S\) very precisely relative to \(R\). For example, if \(G\) is a finite group and \(S\) carries an orthonormal coherent state system \(|g\rangle_S\)\(_{g \in G}\), this could mean that \(|\psi\rangle_{SR}\) is very strongly peaked on a single \(h \in G\), i.e. \(|\langle h|\psi\rangle_{SR}\)\(_{\approx}\) 1. Then, translating by \(g \neq e\) will lead to a state \(|\psi(g)\rangle_{SR}\) almost orthogonal to \(|\psi\rangle_{SR}\), because 0 \(\approx \langle g^{-1} h|\psi\rangle_{SR} = |\langle h|\psi(g)\rangle_{SR}|\)\(_{\approx}\).

In other words, a “good” reference frame \(R\) should lead to a conditional state \(|\psi\rangle_{SR} = |\psi(e)\rangle_{SR}\) that is well-distinguishable from its “translations” \(|\psi(g)\rangle_{SR} = |\psi(g)\rangle_{SR}\), see Fig.\([1]\). A well-known operational quantifier of distinguishability of quantum states is the *fidelity* \([66]\), \(\mathcal{F}(\rho, \sigma) := \langle \rho | \sigma \rangle^2\). We have 0 \(\leq \mathcal{F}(\rho, \sigma) \leq 1\), where 0 is attained if and only if \(\rho\) and \(\sigma\) are perfectly distinguishable, and 1 if and only if \(\rho = \sigma\). For pure states, it reduces to \(\mathcal{F}(|\psi\rangle, |\phi\rangle) = |\langle \psi | \phi \rangle|^2\). This motivates the following definition.

**Definition 1.** Given any physical state \(|\psi\rangle_{RS} \in \mathcal{H}_{phys}\), the conditional uniformity of the corresponding conditional state \(|\psi\rangle_{SR} \equiv |\psi^e\rangle_{SR}\) (as defined in Eq.\([1]\)) is

\[
\mathcal{U}(\psi_{SR}) := \int_g dg \mathcal{F}(|\psi\rangle_{SR}, V_g |\psi\rangle_{SR}), \tag{2}
\]

and we define its conditional asymmetry as \(A(\psi_{SR}) := -\log \mathcal{U}(\psi_{SR})\).

Intuitively, for “bad” quantum reference frames \(R\), \(|\psi\rangle_{SR} \approx V_g |\psi\rangle_{SR} = |\psi(g)\rangle_{SR}\) for many \(g\), and \(\mathcal{U}(\psi_{SR})\) will be close to unity; and for “good” ones, this quantity will be close to zero. By invariance of the Haar measure, the conditional uniformity is the same for all \(|\psi(g)\rangle_{SR}\) and can also be written

\[
\mathcal{U}(\psi(g)_{SR}) = \int_g dg' \int_g dg'' \langle \psi(g')|\psi(g'')\rangle_{SR}^2. \tag{3}
\]

A priori, conditional uniformity will depend on the choice of coherent state system \{|g\rangle_R\}\(_{g \in G}\) since \(|\psi\rangle_{SR} = |\psi(e)\rangle_{SR}\) does. Intuitively, the choice of covariant POVM that is used to measure the reference frame should have some impact on the quality of its use. Surprisingly, however, this intuition does not hold up in our context. Using the notation \(\mathcal{H}_a(\rho) := \frac{1}{-\alpha} \log \text{tr}(\rho^\alpha)\) for the Rényi-\(\alpha\) entropy of a quantum state \(\rho\), we get:

**Theorem 1.** The conditional asymmetry of \(|\psi\rangle_{SR}\) equals the Rényi-2 entanglement entropy of \(|\psi\rangle_{RS}\):

\[
A(\psi_{SR}) = \mathcal{H}_2(\text{Tr}_R |\psi\rangle \langle \psi|_{RS}).
\]

In particular, \(A(\psi_{SR}) = A(\psi^e_{SR})\) is independent of the choice of coherent state system, i.e. of \(|e\rangle\), and can be understood as a function of the physical state \(|\psi\rangle_{RS}\).

**Proof.** Expanding the definition of \(\mathcal{U}\), we find
where we have used that the $|g\rangle_R$ yield a resolution of the identity. To simplify this further, replace both occurrences of $|\psi\rangle_{RS}$ by $U_{\rho}^\dagger \otimes V^\dagger_{\rho} |\psi\rangle_{RS}$, use that $V_{\rho} Tr_{R}(|\psi\rangle_{RS}) V_{\rho}^\dagger = Tr_{R}(|\psi\rangle_{RS})$, and compute the group average over $g$ of the resulting expression. This yields $\mathcal{U}(\psi_{S|R}) = \langle \psi_{RS}(1_R \otimes Tr_{\rho} |\psi\rangle_{RS}) |\psi\rangle_{RS} = \text{tr}[(Tr_{\rho} |\psi\rangle_{RS})^2]$ which is independent of $|\rho\rangle_R$. Finally, take the negative logarithm of both sides.

In the special case where $G$ is a finite subgroup of time translations, a version of this result has been given in [18].

**Resource-theoretic consequences.** As we have just seen, for conditional asymmetry, any choice of coherent state system $\{(g)|g\rangle_R\}_{g \in \mathcal{G}}$ is as good as any other. $\{(g')_{g'}\}_{g' \in \mathcal{G}}$: we have $\mathcal{A}(\psi^{(g)}_{S|R}) = \mathcal{A}(\psi^{(g')}_{S|R})$ for all $|\psi\rangle_{RS} \in \mathcal{H}_{\text{phys}}$. However, $\mathcal{A}$ is just one possible measure of asymmetry. Independence from $|\rho\rangle_R$ does not hold for all possible measures of asymmetry. For example, taking the 4th power instead of the 2nd in Eq. 3 defines an alternative measure of uniformity that does depend on the choice of $|\rho\rangle_R$, see Supplemental Material IV for an example.

Thus, a more systematic and operational approach is warranted. Such an approach is to study asymmetry in the context of a resource theory [18]. Resource theories provide useful tools to describe and quantify the role that a resource plays in the performance of certain tasks, be it in thermodynamics [19] or entanglement theory [20]. Here, we use the resource theory of asymmetry [21 22] to study the quality of a quantum reference frame.

To this end, let us introduce several relevant notions. Consider any representation $g \mapsto U_g$ of a compact group $\mathcal{G}$. A quantum state $\rho$ is $\mathcal{G}$-invariant if $U_g \rho U_g^\dagger = \rho$ for all $g \in \mathcal{G}$. We say that a quantum operation $\mathcal{E}$ on the operators of the corresponding Hilbert space is $\mathcal{G}$-covariant if $\mathcal{E}(U_g \rho U_g^\dagger) = U_g \mathcal{E}(\rho) U_g^\dagger$ for all $g$ and all $\rho \in \mathcal{G}$. Crucially, $\mathcal{G}$-covariant operations map $\mathcal{G}$-invariant states to $\mathcal{G}$-invariant states — in this sense, they cannot create $\mathcal{G}$-asymmetry. We say that $\rho$ is at least as asymmetric as $\rho'$ if there is a $\mathcal{G}$-covariant operation $\mathcal{E}$ with $\mathcal{E}(\rho) = \rho'$. If also $\rho'$ is at least as asymmetric as $\rho$, we say that $\rho$ and $\rho'$ are equally $\mathcal{G}$-asymmetric. Note that there are also pairs of states with the property that neither one is at least as asymmetric as the other. In this case, we say that $\rho$ and $\rho'$ are incomparably $\mathcal{G}$-asymmetric.

Our result $\mathcal{A}(\psi^{(g)}_{S|R}) = \mathcal{A}(\psi^{(g')}_{S|R})$ does not automatically imply that $\psi^{(g)}_{S|R}$ and $\psi^{(g')}_{S|R}$ are equally $\mathcal{G}$-asymmetric, but we can show the following:

**Theorem 2.** Consider two choices of coherent state system, $\{(g)|g\rangle_R\}_{g \in \mathcal{G}}$ and $\{(g')_{g'}\}_{g' \in \mathcal{G}}$. Then, for every physical state $|\psi\rangle_{RS} \in \mathcal{H}_{\text{phys}}$, the conditional states $|\psi^{(g)}\rangle_{S|R}$ and $|\psi^{(g')}\rangle_{S|R}$ are either equally or incomparably $\mathcal{G}$-asymmetric. That is, in the resource-theoretic sense, no coherent state system induces more asymmetry on $S$ than any other.

The proof is given in Supplemental Material IV. It employs techniques of [21 22] which use characteristic functions $\chi_{\varphi}(g) := \langle \varphi | V_g | \varphi \rangle$ to characterize pure-state interconvertibility under $\mathcal{G}$-covariant operations. They are related to conditional uniformity via $\mathcal{U}(\psi_{S|R}) = \int_{\mathcal{G}} d\varphi \chi_{\varphi} \rho^{(g)}(\varphi)^2$.

**Physical Hilbert space average.** To quantify how much asymmetry the quantum reference frame $R$ is able to induce on $S$, we have to go beyond single conditional states and consider the collection of all $|\psi\rangle_{S|R}$. Theorem 1 allows us to do so in a particularly elegant way: conditional uniformity $\mathcal{U}(\psi_{S|R})$ is independent of the choice of seed coherent state $|\rho\rangle_R$ and can be understood as a function of $|\psi\rangle_{RS}$. We can thus determine an average of this quantity over all conditional states by computing the Hilbert space average of $\mathcal{U}$ over $\mathcal{H}_{\text{phys}}$. Not only can this be done analytically, but the result will be independent of the coherent state system and quantify the quality of the reference frame in terms of simple properties of the representations $g \mapsto U_g$ on $R$ and $g \mapsto V_g$ on $S$. To this end, let us fix some notation. Following [59], the set of (unitarily inequivalent) irreps of $\mathcal{G}$ is denoted $\hat{\mathcal{G}}$. Decomposing the representation on $R$ into irreps, we get $U_g = \bigoplus_{\alpha \in \hat{\mathcal{G}}} n_\alpha^U T_{g}^{(\alpha)}$, where $n_\alpha^U \in \mathbb{N}_0$ is the multiplicity of the irrep $T_{g}^{(\alpha)}$, $g \mapsto T_{g}^{(\alpha)}$. Similarly, $V_g = \bigoplus_{\beta \in \hat{\mathcal{G}}} n_\beta^V T_{g}^{(\beta)}$. The dimension of the irrep $\alpha$ will be denoted $d_\alpha$, and the conjugate representation of $\alpha$ will be denoted $\alpha^\dagger$, i.e. $T_{g}^{(\alpha)} = T_{g^\dagger}^{(\alpha)}$ in some basis. Note that the existence of a coherent state system on $R$ furnishing a resolution of the identity implies that $n_\alpha^U \leq d_\alpha$. The unitarily invariant measure on the unit vectors of $\mathcal{H}_{\text{phys}}$, normalized such that $\int_{\mathcal{H}_{\text{phys}}} d\psi = 1$, will be denoted $d\psi$. Theorem 3. The physical Hilbert space average $\mathcal{U}_{\text{phys}}$ of conditional uniformity (“physical uniformity”) is

$$\int_{\mathcal{H}_{\text{phys}}} d\psi \mathcal{U}(\psi_{S|R}) = \frac{1}{d_{\text{phys}}(d_{\text{phys}} + 1)} \sum_{\alpha \in \hat{\mathcal{G}}} n_\alpha^U n_\alpha^V (n_\alpha^U + n_\alpha^V) d_\alpha,$$

where $d_{\text{phys}} = \dim \mathcal{H}_{\text{phys}} = \sum_{\alpha \in \hat{\mathcal{G}}} n_\alpha^U n_\alpha^V$.

The proof is given in Supplemental Material III. It relies on quantum information techniques to compute
Hilbert space averages of polynomials via the replica trick, together with representation-theoretic orthogonality and convolution identities for the characters of the group $G$.

Like $U(\psi_{S|R})$, the physical uniformity $U_{\text{phys}}$ lies in the interval $[0,1]$, and it is independent of the choice of coherent state system $\{ |g\rangle_R \}_{g \in G}$. Its value, or rather that of the corresponding physical asymmetry $A_{\text{phys}} := -\log U_{\text{phys}}$, quantifies in representation-theoretic terms how much asymmetry $R$ induces on $S$ on average. Since entanglement entropy is upper-bounded by $\log d_R$, Theorem 1 tells us that $U(\psi_{S|R}) \geq 1/d_R$ for all $|\psi_{S|R}\rangle \in \mathcal{H}_R$, and thus $A_{\text{phys}} \leq \log d_R$. This shows directly the necessity of large Hilbert space dimension for a quantum reference frame $R$ to induce large amounts of asymmetry.

**Example: maximum spin-$J$ reference frame.** Consider a quantum reference frame for the group $SU(2)$, where we constrain $R$ to contain only irreps of spin $J$ or less, i.e. irreps labelled by $\alpha = 0, \frac{1}{2}, 1, \ldots, J$. As stated above Theorem 3 the resolution of the identity necessarily implies $n^V_{\alpha} \leq d_\alpha$. Thus, the best we can have is equality, i.e. $n^V_{\alpha} = d_\alpha = 2n_\alpha$. Suppose that the system of interest is $S = L^2(SU(2))$, i.e. the infinite-dimensional system of wave functions on the group. By the Peter-Weyl theorem [69], we have $n^V_{\alpha} = d_\alpha$. The physical Hilbert space has dimension $d_{\text{phys}} = d_R = \sum_{k=0}^{2J} n^V_{k/2} = \sum_{k=0}^{2J} (k+1)^2 \sim 8J^3/3$. The physical uniformity evaluates to $d_{\text{phys}} = 2/(d_{\text{phys}} + 1) \sim 3J^3/4$. In particular, for $J \to \infty$, the conditional states $|\psi\rangle_{S|R}$ become, on average, perfectly distinguishable from their rotated versions $|\psi(g)\rangle_{S|R}$. This indicates that, for increasing $J$, this $R$ resembles more and more a perfect “classical” reference frame, and the above tells us how good this approximation is for finite $J$.

Here, physical asymmetry is close to its maximal value: $A_{\text{phys}} = \log d_R - \log 2 + O(J^{-3})$. This implies that the average asymmetry $\overline{A}_{\text{phys}} := \int_{\mathcal{H}_\text{phys}} d\psi A(\psi_{S|R})$ is also large. The latter follows because $(-\log)$ is convex, and therefore Jensen’s inequality tells us that $\overline{A}_{\text{phys}} = \int_{\mathcal{H}_\text{phys}} d\psi (-\log U(\psi_{S|R})) \geq -\log \int_{\mathcal{H}_\text{phys}} d\psi U(\psi_{S|R}) = A_{\text{phys}}$. However, if the average asymmetry is close to maximal, then most conditional states $|\psi\rangle_{S|R}$ must be almost maximally asymmetric. This is reminiscent of the phenomenon in quantum information theory that almost all pure states are almost maximally entangled [70], and we can exploit this analogy rigorously via the correspondence of Theorem 1.

**Typical asymmetry.** Suppose that we pick a state $|\psi_{RS}\rangle$ at random from $\mathcal{H}_R$ according to the unitarily invariant measure. Then the techniques of [70] (in particular Lemmas III.1 and III.8) together with the Lipschitz bound on entanglement entropy $H_2$ from [22] imply that

$$\text{Prob} \left\{ |A(\psi_{S|R}) - \overline{A}_{\text{phys}}| \geq \varepsilon \right\} \leq 2 \exp \left( -\frac{d_{\text{phys}}^2}{C \sqrt{d_R}} \right)$$

for all $\varepsilon \geq 0$, where $C = 72\pi^3 \log 2$. Using $\overline{A}_{\text{phys}} \geq A_{\text{phys}}$, the right-hand side also upper-bounds the probability that $A(\psi_{S|R}) \leq A_{\text{phys}} - \varepsilon$. In the SU(2)-example above, for every fixed $\varepsilon > 0$, the probability that a random $|\psi_{S|R}\rangle$ has conditional asymmetry less than $\log d_R - \log 2 - \varepsilon$ is exponentially small in $J^3/2$; almost all conditional states are indeed almost maximally asymmetric.

**Example: periodic quantum clock.** Consider a harmonic oscillator $S$ with Hamiltonian $\hat{H}_S = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \hbar\omega \sum_{n=0}^{\infty} (n + \frac{1}{2})(n + 1)$. It evolves periodically in time with period $\tau = 4\pi/\omega$, hence we can measure time with a periodic quantum clock $R$, carrying a representation $U_\tau$ of $G = SU(1) \simeq \{ e^{i\alpha} \alpha : 0 \leq \alpha \leq 2\pi \}$. This generalizes the notion of clocks used in the PWM with $G = \mathbb{R}$ to the periodic case. The representations of $U(1)$ are labelled by integers $\alpha \in \mathbb{Z}$, with $T_{\alpha}^{(n)} = \exp(i\alpha q)$. Then $n^V_{\alpha} = 1$ if $-k \leq \alpha \leq k$ and 0 otherwise. For $k \to \infty$, one would recover a perfect clock, i.e. a periodic version of the PWM, with associated Hilbert space $L^2(U(1))$.

Up to arbitrary changes of phase in the one-dimensional subspaces spanned by $|\alpha\rangle$, the unique choice of coherent state system yielding a resolution of the identity is via $|\epsilon\rangle_R = \frac{1}{\sqrt{\pi}} \sum_{\alpha=-k}^{k} |\alpha\rangle$ with $d_R = 2k + 1$. Interestingly, in a scenario where we are given a quantum clock and would like to determine time as accurately as possible by measuring the clock, this state has been shown by Holevo [71] to generate the POVM which is optimal for a large class of cost functions (see also [72]). However, Holevo’s results are not directly applicable to our scenario, since our clock is in a joint stationary state $|\psi_{RS}\rangle \in \mathcal{H}_R$ with the harmonic oscillator.

Associating time $t$ with the group element $g \in [0,2\pi]$ via $g = 2\pi t/\tau$, we get that $V_g = \exp (-i\frac{2\pi}{\tau \hbar} \hat{H}_S)$ represents time translations on $S$. On the energy eigenstates, we have $V_g |n\rangle = \exp (-i\pi(2n + 1))/|n\rangle$, hence $n^V_{\alpha} = 1$ if $\alpha = -1, -3, -5, \ldots$ and 0 otherwise. We have $\mathcal{H}_R = \text{span} \{ |2n + 1\rangle |n\rangle_S \mid 0 \leq n \leq (k-1)/2 \}$, and so $d_{\text{phys}} = (k + 1)/2$. The physical uniformity becomes $U_{\text{phys}} = 4/(k + 3)$, which is the physical Hilbert space average of $U(\psi_{S|R}) = \int_0^\infty dt \langle \tilde{\psi}_{S|R}(0) | \tilde{\psi}_{S|R}(t) \rangle^2$, with the time-evolved state $\tilde{\psi}(t) := \psi(2\pi t/\tau)$. This can be interpreted as an instance of a time-energy uncertainty relation: the larger the range of energies in the clock $R$ (i.e. the larger $k$), the more distinguishable will the conditional states be from their time-translated versions on average.

Compare this with what we can learn from the the Mandelstam-Tamm quantum speed limit [76]:

$$\langle \langle \psi_{S|R}(0) | \psi_{S|R}(t) \rangle^2 \rangle \geq \cos^2 \left( \frac{\theta H_S}{\hbar} \right) \left( \frac{t}{\tau} \right),$$

where $\text{cost}(x) = \cos x$ if $0 \leq x \leq \pi/2$ and 0 otherwise. This implies

$$U(\psi_{S|R}) = \int_0^\tau dt \langle \tilde{\psi}_{S|R}(0) | \tilde{\psi}_{S|R}(t) \rangle^2 \geq \int_0^{\min(t_\theta, \tau)} dt \cos^2 \left( \frac{t\Delta H_S}{\hbar} \right),$$
where \( t_0 = \pi \hbar / (2 \Delta H_S) \). Due to (4), every \( \psi_{S|R} \) is supported on the subspace spanned by \( |0\rangle_S, \ldots, |(k-1)/2\rangle_S \), hence these states have \( \Delta H_S \leq \hbar \omega k / 2 \). Most states \( \psi_{S|R} \) will have \( \Delta H_S \geq \hbar \omega / 8 \) if \( k \) is large, hence \( t_0 \leq \tau \). For those \( \psi_{S|R} \), the integral evaluates to \( \mathcal{U}(\psi_{S|R}) \geq \omega k / (16 \Delta H_S) \geq 1/(8k) \). Our result on \( \mathcal{U}_{phys} \) gives essentially the same scaling in \( k \), but improves on the Mandelstam-Tamm result by a factor of 32 on average. Hence, Theorem 3 provides a representation-theoretic time-energy trade-off and generalizes it to more general groups than time translations.

Conclusions. We have shown that there is an exact quantitative correspondence between the amount of entanglement in a globally symmetric quantum state and the amount of asymmetry in the conditional state relative to an internal quantum reference frame, leading to several insights on the quality of imperfect reference frames, speed limits, and typicality of asymmetry. We have also begun to explore the resource-theoretic consequences of our duality in Theorem 2 using the close relation between conditional uniformity and characteristic functions. It would be interesting to explore further how resource-theoretic notions can be imported into this “perspective-neutral” scenario. It would also be worthwhile to explore the generalization to non-compact groups [50] such as the Lorentz and Galilei groups. These possible extensions notwithstanding, we believe that our results shed significant light on the quantum information-theoretic and structural foundations of internal quantum reference frames.

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Lemma 1. The “reduction map” \( R(g) : \mathcal{H}_{\text{phys}} \to S, \quad R(g) = \sqrt{\rho_{g R}} \otimes 1_S, \) is an isometry.

Proof. Since \( |\psi(g)\rangle_S = V_g |\psi(e)\rangle_S, \) it is sufficient to consider the case \( g = e. \) This can be done as follows:

\[
(R(e) |\psi(R(e)\phi)\rangle_S = d_R \int_{\mathcal{G}} dg \langle \psi | R S \otimes V_g (|e\rangle_R \otimes 1_S)(|e\rangle_R \otimes 1_S)U_g^\dagger \otimes V_g^\dagger |\phi\rangle_{RS}
\]

\[
= d_R \langle \psi | R S \left( \int_{\mathcal{G}} dg |g\rangle_R \langle g|_R \right) |\phi\rangle_{RS} = \langle \psi | \phi \rangle_{RS},
\]

where we have used the invariance \( |\psi\rangle_{RS} = U_g^\dagger \otimes V_g^\dagger |\psi\rangle_{RS} \) for \( |\psi\rangle_{RS} \in \mathcal{H}_{\text{phys}}. \)

Note that Lemma 1 was already shown to hold in Ref. [50].

Lemma 2. The coherent state system \( \{|g\rangle_R\}_{g \in \mathcal{G}} \) can be used to calculate the partial trace \( \text{Tr}_R \) as follows:

\[
\text{Tr}_R (\rho_{RS}) = d_R \int_{\mathcal{G}} dg \langle g | R \otimes 1_S \rho_{RS} |g\rangle_R \otimes 1_S.
\]
Proof. Let us define $\sigma_S := d_R \int_G dg \langle g | R \otimes 1_S | \rho_{RS} | g \rangle_R \otimes 1_S$. Let $X_S \in B(S)$ be an arbitrary operator, then

$$
(\sigma_S, X_S)_S = \text{Tr}_S \left( \sigma_S^\dagger X_S \right) = d_R \int_G dg \sum_{i=1}^{d_S} \langle g | R \otimes \langle i | S \rho_{RS}^\dagger 1 \otimes X_S | g \rangle_R \otimes \langle i | S \rangle_S \\
= d_R \int_G dg \sum_{i=1}^{d_S} \text{Tr}_R \left( \rho_{RS}^\dagger 1 \otimes X_S | g \rangle_R \otimes \langle i | S \rangle_S \right) \\
= \text{Tr}_S \left( (\text{Tr}_R(\rho_{RS}))^\dagger X_S \right) = (\text{Tr}_R(\rho_{RS}), X_S)_S,
$$

where $\{|i\rangle_S\}_{i=1}^{d_S}$ is an ONB of $S$ and $(\ , \ )_S$ denotes the Hilbert-Schmidt inner product of operators.

Lemma 3. Let $|\psi\rangle_{RS} \in \mathcal{H}_{\text{phys}}$. The reduced state of $S$ is invariant under the group action: for all $g \in G$,

$$
\text{Tr}_R (|\psi\rangle\langle \psi|_{RS}) = V_g^* \text{Tr}_R (|\psi\rangle\langle \psi|_{RS}) V_g.
$$

Proof. Let $X_S \in B(S)$ be an arbitrary operator, then

$$
(V_g \text{Tr}_R (|\psi\rangle\langle \psi|_{RS}) V_g^\dagger, X_S)_S = \text{Tr}_S (V_g \text{Tr}_R (|\psi\rangle\langle \psi|_{RS}) V_g^\dagger X_S) = \text{Tr}_R (|\psi\rangle\langle \psi|_{RS} \cdot 1_R \otimes (V_g^\dagger X_S V_g)) \\
= \text{Tr}_R (U_g^\dagger \cdot V_g^\dagger \cdot |\psi\rangle\langle \psi|_{RS} \cdot 1_R \otimes X_S \cdot U_g \otimes V_g) = \text{Tr}_R (|\psi\rangle\langle \psi|_{RS} \cdot 1_R \otimes X_S) \\
= \text{Tr}_S (\text{Tr}_R (|\psi\rangle\langle \psi|_{RS}) X_S) = (\text{Tr}_R (|\psi\rangle\langle \psi|_{RS}), X_S)_S.
$$

In the third line, we used $U_g^\dagger \cdot V_g^\dagger |\psi\rangle_{RS} = |\psi\rangle_{RS}$.

The flip operator $F$ on a product Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ is defined by linear extension of $F|\varphi\rangle \otimes |\psi\rangle = |\psi\rangle \otimes |\varphi\rangle$.

Lemma 4. For operators $A$ and $B$, we have $\text{tr}(FA \otimes B) = \text{tr}(A \cdot B)$.

Proof. A simple calculation gives

$$
\text{tr}(FA \otimes B) = \text{tr} \left( F \sum_{i,j,s,k} a_{ij} b_{sk} |i\rangle \otimes |j\rangle \langle s| \langle k| \right) = \text{tr} \left( \sum_{i,j,s,k} a_{ij} b_{sk} |j\rangle \otimes |i\rangle \langle k| \right) = \sum_{i,j,s,k} a_{ij} b_{sk} \delta_{ij} \delta_{sk} \\
= \sum_{ij} a_{ij} b_{ji} = \text{tr}(A \cdot B).
$$

Lemma 5. We have $\text{tr}(\rho^2_{SS'}) = \text{tr}(1_{RR'} \otimes F_{SS'} (|\psi\rangle\langle \psi|_{RS} \otimes |\psi\rangle\langle \psi|_{RS'}))$.

Proof. We can easily see that

$$
\text{tr}(\rho^2_{SS'}) = \text{tr} (F_{SS'} \rho_S \otimes \rho_{S'}) = \text{tr} (F_{SS'} \text{Tr}_{RR'} (|\psi\rangle\langle \psi|_{RS} \otimes |\psi\rangle\langle \psi|_{RS'})) = \text{tr}(1_{RR'} \otimes F_{SS'} (|\psi\rangle\langle \psi|_{RS} \otimes |\psi\rangle\langle \psi|_{RS'})).
$$

Lemma 6. The dimension of the physical Hilbert space $\mathcal{H}_{\text{phys}}$ is given by $d_{\text{phys}} = \sum_{\alpha} n^U_{\alpha} n^V_{\alpha}$, where $U$ and $V$ are the representations carried by reference frame $R$ and system $S$ respectively, and $n^U_{\alpha}$ is the multiplicity of the irrep $T^{(\alpha)}$ in $U$ whereas $n^V_{\alpha}$ is the multiplicity of the conjugate irrep $T^{(\overline{\alpha})}$ in $V$.

Proof. Let us decompose the representations $U_g$ and $V_g$ into irreps:

$$
U_g = \bigoplus_{\alpha \in \hat{G}} n^U_{\alpha} T^{(\alpha)}_g, \quad V_g = \bigoplus_{\beta \in \hat{G}} n^V_{\beta} T^{(\beta)}_g.
$$

Then, we can write

$$
U_g \otimes V_g = \left( \bigoplus_{\alpha \in \hat{G}} n^U_{\alpha} T^{(\alpha)}_g \right) \otimes \left( \bigoplus_{\beta \in \hat{G}} n^V_{\beta} T^{(\beta)}_g \right) = \bigoplus_{\alpha, \beta \in \hat{G}} n^U_{\alpha} n^V_{\beta} (T^{(\alpha)}_g \otimes T^{(\beta)}_g) = \bigoplus_{\alpha, \beta, \gamma \in \hat{G}} n^U_{\alpha} n^V_{\beta} c^{\alpha \beta \gamma} T^{(\gamma)}_g,
$$
where the $c^{\alpha\beta}_i$ are the Clebsch-Gordan coefficients. Note that

$$c^{\alpha\beta}_i = \langle \chi^{(1)} | \chi^{(\alpha)} \chi^{(\beta)} \rangle = \int_{\mathcal{G}} dg \chi^{(\alpha)}(g) \chi^{(\beta)}(g) = \overline{(\chi^{(\alpha)})} | \chi^{(\beta)} \rangle = \delta_{\alpha\beta}.$$

Thus, the dimension of the physical Hilbert space $\mathcal{H}_{\text{phys}}$ can be written as

$$d_{\text{phys}} = \dim \mathcal{H}_{\text{phys}} = \sum_{\alpha, \beta \in \mathcal{G}} n^\alpha_{\alpha} n^\beta_{\beta} c^{\alpha\beta}_i = \sum_{\alpha, \beta \in \mathcal{G}} n^\alpha_{\alpha} n^\beta_{\beta} \delta_{\alpha\beta} = \sum_{\alpha, \beta \in \mathcal{G}} n^\alpha_{\alpha} n^\beta_{\beta}.$$

Proof. Since we are interested in pure-state convertibility under covariant operations, we use the techniques of [21, 22].

Lemma 7. The convolution of two irreducible characters yields

$$\chi^{\alpha} \ast \chi^{\beta} = \frac{\delta_{\alpha\beta} \chi^{\alpha}}{d_{\alpha}}.$$

Proof. We use that the matrix elements of complex irreps $T^{(\alpha)}_{ij}$ of a compact Lie group $\mathcal{G}$ are orthogonal, i.e.

$$\langle T^{(\alpha)}_{ij} | T^{(\beta)}_{kl} \rangle = \int_{\mathcal{G}} dg (T^{(\alpha)}_{ij})_{ij} (T^{(\alpha)}_{kl})_{kk} = \delta_{\alpha\beta} \delta_{ik} \delta_{jk} = \frac{\delta_{\alpha\beta} \chi^{\alpha}}{d_{\alpha}}.$$

II. PROOF OF THEOREM [2]

Proof. Since we are interested in pure-state convertibility under covariant operations, we use the techniques of [21, 22] via characteristic functions $\chi_{\varphi}(g) := \langle \varphi | V_g | \varphi \rangle$. Let us write $|\psi\rangle \xrightarrow{\text{cov}} |\psi'\rangle$ if there is a $\mathcal{G}$-covariant operation $\mathcal{E}$ with $\mathcal{E}(|\psi\rangle |\varphi\rangle) = |\psi'\rangle |\varphi\rangle$. Suppose that $|\psi\rangle_{S|R} \xrightarrow{\text{cov}} |\psi'\rangle_{S|R}$, where $|\psi\rangle_{S|R} := |\psi\rangle_{|S|R}$ and $|\psi'\rangle_{S|R} := |\psi'\rangle_{|S|R}$. Then there is a $\mathcal{G}$-covariant operation $\mathcal{E}$ on $S$ with $\mathcal{E}(\rho) = \rho'$, where $\rho = |\psi\rangle \langle \psi |_{S|R}$ and $\rho' := |\psi'\rangle \langle \psi' |_{S|R}$. Hence

$$\int_{\mathcal{G}} dg F(\rho, V_g \rho V_g^{-1}) = U(\rho) = U(\rho') = \int_{\mathcal{G}} dg F(\rho', V_g \rho' V_g^{-1}) = \int_{\mathcal{G}} dg F(\mathcal{E}(\rho), \mathcal{E}(V_g \rho V_g^{-1})).$$

Since the fidelity satisfies the data-processing inequality, $F(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \geq F(\rho, \sigma)$, and $g \mapsto F(\rho, V_g \rho V_g^{-1})$ is continuous, we must have $F(\rho, V_g \rho V_g^{-1}) = F(\rho', V_g \rho' V_g^{-1})$ for all $g \in \mathcal{G}$. But $F(\rho, V_g \rho V_g^{-1}) = |\langle \psi | U_g | \psi \rangle|_{S|R}^2 = |\chi_{\varphi}(g)|^2$, and so $|\chi_{\varphi}(g)|^2 = |\chi_{\varphi'}(g)|^2$ for all $g \in \mathcal{G}$. Furthermore, it follows from $|\psi\rangle_{S|R} \xrightarrow{\text{cov}} |\psi'\rangle_{S|R}$ that there is a positive definite function [21] Theorem 63] $f : \mathcal{G} \to \mathbb{C}$ (see Supplemental Material for a definition) such that $\chi_{\varphi}(g) = \chi_{\varphi'}(g) f(g)$ for all $g \in \mathcal{G}$. Now, for those $g$ with $\chi_{\varphi}(g) \neq 0$, it follows that $|f(g)| = 1$, i.e. $f(g)^{-1} = \overline{f(g)}$. Since $\chi_{\varphi}(g) = 0 \Leftrightarrow \chi_{\varphi'}(g) = 0$, we have

$$\chi_{\varphi'}(g) = \chi_{\varphi}(g) f(g).$$

If $f$ is positive definite, so is $f$. Hence, it follows again from [21] Theorem 63] that $|\psi'\rangle_{S|R} \xrightarrow{\text{cov}} |\psi\rangle_{S|R}$, which implies that $|\psi\rangle_{S|R}$ is as asymmetric as $|\psi\rangle_{S|R}$.
III. PROOF OF THEOREM 3

Proof. Using Lemma 5, we have

\[
U_{\text{phys}} = \int_{\mathcal{H}_{\text{phys}}} d\psi \, \text{tr}[(\text{Tr}|\psi\rangle\langle\psi|_R)\langle\psi|_S]\rangle = \text{tr}\left(1_{RR'} \otimes F_{SS'}\right) = \frac{1}{d_{\text{phys}}(d_{\text{phys}} + 1)} \text{tr}\left((1_{RR'} \otimes F_{SS'}) \Pi^{RS,R'S'}_{\text{phys,sym}} \right)
\]

Let us first consider the expression

\[
(1_{RR'} \otimes F_{SS'}) \Pi^{RS,R'S'}_{\text{sym}} = \frac{1}{2}(1_{RR'} \otimes F_{SS'})(1 + F_{RS,R'S'}).\]

Letting the second term act on a general basis state, we see

\[
(1_{RR'} \otimes F_{SS'})|i\rangle_R|j\rangle_R'|s\rangle_S|k\rangle_k' = 1_{RR'} \otimes F_{SS'}|i\rangle_R|j\rangle_R'|s\rangle_S|k\rangle_k' = F_{RR'} \otimes 1_{SS'}|i\rangle_R|j\rangle_R'|s\rangle_S|k\rangle_k'.
\]

Thus,

\[
(1_{RR'} \otimes F_{SS'}) \Pi^{RS,R'S'}_{\text{sym}} = \frac{1}{2}(1_{RR'} \otimes F_{SS'} + F_{RR'} \otimes 1_{SS'}).
\]

Hence, Eq. (5) can be written as

\[
U_{\text{phys}} = \frac{1}{d_{\text{phys}}(d_{\text{phys}} + 1)} \left[\text{tr}\left((1_{RR'} \otimes F_{SS'}) (\Pi^{RS}_{\text{phys}} \otimes \Pi^{R'S'}_{\text{phys}})\right) + \text{tr}\left((F_{RR'} \otimes 1_{SS'})(\Pi^{RS}_{\text{phys}} \otimes \Pi^{R'S'}_{\text{phys}})\right)\right].
\]

Let us rewrite the first term in the following way:

\[
\text{tr}\left((1_{RR'} \otimes F_{SS'}) (\Pi^{RS}_{\text{phys}} \otimes \Pi^{R'S'}_{\text{phys}})\right) = \int_{\mathcal{G}} dg \int_{\mathcal{G}} dg' \text{tr}\left((1_{RR'} \otimes F_{SS'}) U_g^R \otimes V_g^S \otimes U_{g'}^{R'} \otimes V_{g'}^{S'}\right)
\]

where \(\chi^U(g) = \text{tr}(U_g) = \sum_{\alpha \in \mathcal{G}} n^U_\alpha \text{tr}(T^{(\alpha)}_g) = \sum_{\alpha \in \mathcal{G}} n^{U}_{\alpha} \chi^{(\alpha)}_{\alpha}(g)\). At the first equality sign, we used \(\Pi^{RS}_{\text{phys}} = \int_{\mathcal{G}} dg U_g^R \otimes V_g^S\) and similarly for \(\Pi^{R'S'}_{\text{phys}}\). We used Lemma 4 to go from the second to the third line.

We can proceed similarly with the second term:

\[
\text{tr}\left((F_{RR'} \otimes 1_{SS'}) (\Pi^{RS}_{\text{phys}} \otimes \Pi^{R'S'}_{\text{phys}})\right) = \int_{\mathcal{G}} dg \int_{\mathcal{G}} dg' \text{tr}\left(F_{RR'} (\Pi^{RS}_{\text{phys}} \otimes \Pi^{R'S'}_{\text{phys}})\right)
\]

Note that in general, we can write

\[
\int_{\mathcal{G}} dg \int_{\mathcal{G}} dg' \chi^A(g) \chi^A(g') = \int_{\mathcal{G}} dg \int_{\mathcal{G}} dg' \text{tr}(\chi^A(g) \chi^A(g')).
\]

Finally, we get

\[
U_{\text{phys}} = \frac{1}{d_{\text{phys}}(d_{\text{phys}} + 1)} \left(\int_{\mathcal{G}} dg \int_{\mathcal{G}} dg' \text{tr}(\chi^U(g) \chi^U(g')) + \int_{\mathcal{G}} dg \int_{\mathcal{G}} dg' \text{tr}(\chi^V(g) \chi^V(g'))\right)
\]

(6)
Note that the first term can be written as

\[
\int_{\mathcal{G}} dg \int dg' \chi^{(\alpha)}(g) \chi^{(\gamma)}(h) \chi^{(\gamma)}(g^{-1} h) = \sum_{\alpha, \beta, \gamma \in \mathcal{G}} n_{\alpha}^{U} n_{\beta}^{V} n_{\gamma}^{U} \int_{\mathcal{G}} dg \int dg' \chi^{(\alpha)}(g) \chi^{(\beta)}(h) \chi^{(\gamma)}(g^{-1} h)
\]

\[
= \sum_{\alpha, \beta, \gamma \in \mathcal{G}} n_{\alpha}^{U} n_{\beta}^{V} n_{\gamma}^{U} \int_{\mathcal{G}} dg \chi^{(\beta)}(h) (\chi^{(\gamma)} \ast \chi^{(\alpha)})(h)
\]

\[
= \sum_{\alpha, \beta \in \mathcal{G}} \frac{1}{d_{\alpha}} (n_{\alpha}^{U})^{2} n_{\beta}^{V} (\chi^{(\beta)}|\chi^{(\alpha)}) ,
\]

while the second term simplifies to

\[
\int_{\mathcal{G}} dg \int dg' \chi^{U}(g) \chi^{V}(h) \chi^{V}(h^{-1} g) = \sum_{\alpha, \beta \in \mathcal{G}} \frac{1}{d_{\alpha}} (n_{\alpha}^{V})^{2} n_{\beta}^{U} (\chi^{(\beta)}|\chi^{(\alpha)}) = \sum_{\alpha, \beta \in \mathcal{G}} \frac{1}{d_{\beta}} (n_{\alpha}^{V})^{2} n_{\alpha}^{U} (\chi^{(\alpha)}|\chi^{(\beta)}) .
\]

Then, Eq. (6) gives

\[
U_{\text{phys}} = \frac{1}{d_{\text{phys}} (d_{\text{phys}} + 1)} \sum_{\alpha, \beta \in \mathcal{G}} \left( \frac{1}{d_{\alpha}} (n_{\alpha}^{U})^{2} n_{\beta}^{V} (\chi^{(\beta)}|\chi^{(\alpha)}) + \frac{1}{d_{\beta}} (n_{\beta}^{V})^{2} n_{\alpha}^{U} (\chi^{(\alpha)}|\chi^{(\beta)}) \right)
\]

\[
= \frac{1}{d_{\text{phys}} (d_{\text{phys}} + 1)} \sum_{\alpha, \beta \in \mathcal{G}} \frac{1}{d_{\alpha}} ((n_{\alpha}^{U})^{2} n_{\beta}^{V} (\chi^{(\beta)}|\chi^{(\alpha)}) + (n_{\beta}^{V})^{2} n_{\alpha}^{U} (\chi^{(\alpha)}|\chi^{(\beta)}) )
\]

\[
= \frac{1}{d_{\text{phys}} (d_{\text{phys}} + 1)} \sum_{\alpha, \beta \in \mathcal{G}} \frac{1}{d_{\alpha}} (\chi^{(\beta)}|\chi^{(\alpha)}) ((n_{\alpha}^{U})^{2} n_{\beta}^{V} + (n_{\beta}^{V})^{2} n_{\alpha}^{U}) .
\]

It is not true that no choice of coherent state system induces more asymmetry. For example, for the two-dimensional representation of \(S_{3}\), the coherent state system \(|g\rangle_{R} \in S_{3}\) is generated by the seed state

\[
|e\rangle = \alpha |0\rangle + \beta |1\rangle + \gamma |2\rangle,
\]

IV. (IN)DEPENDENCE OF ASYMMETRY ON THE COHERENT STATE SYSTEM

We have seen that the conditional uniformity \(U(|\psi\rangle|R) = \int_{\mathcal{G}} dg |\langle \psi|S|V_{g}|\psi\rangle|^{2}\), and thus conditional asymmetry \(A = -\log U\), does not depend on the choice of coherent state system \(|g\rangle_{R} \in \mathcal{G}\), even though \(|\psi\rangle_{S|R} = |\psi\rangle_{S|R}^{e}\) does. However, \(A\) provides just one possible measure of asymmetry, or asymmetry monotone, i.e. map \(\mathcal{M}\) from quantum states to real numbers such that

\[
\rho \text{ is at least as asymmetric as } \rho' \Rightarrow \mathcal{M}(\rho) \geq \mathcal{M}(\rho')
\]

(see Lemma 8 below.) Here, we give an example of different asymmetry measures that do depend on the choice of \(|e\rangle\). But we will see that the resulting order, i.e. which of the two given coherent state systems induces more asymmetry, is reversed by changing the asymmetry measure. This confirms the result of Theorem 2, no choice of coherent state system induces more asymmetry in the resource-theoretic sense than any other.

To this end, consider the smallest finite non-Abelian group \(\mathcal{G} = S_{3}\) acting on \(R \otimes S\) with \(R = S = \mathbb{C}^{3}\). We consider the case where \(S_{3}\) is represented as permutations of the three basis vectors \(|i\rangle_{\text{phys}}\), i.e. both \(R\) and \(S\) carry the fundamental representation. We write \(U_{g}\) for the representation of the element \(g \in S_{3}\) on \(R\) and similarly \(V_{g}\) on \(S\). One can easily show that \(U_{g}\) (and similarly \(V_{g} = U_{g}\)) takes the form

\[
U_{g} = T_{g}^{(1)} \oplus T_{g}^{(\text{std})},
\]

where \(T_{g}^{(1)}\) denotes the trivial and \(T_{g}^{(\text{std})}\) the two-dimensional standard representation of \(S_{3}\). The coherent state system \(|g\rangle_{R} \in S_{3}\) is generated by the seed state

\[
|e\rangle = \alpha |0\rangle + \beta |1\rangle + \gamma |2\rangle,
\]
where we have $U_g|e\rangle = |g\rangle$, $g \in S_3$. There are two conditions on the coefficients $\alpha, \beta, \gamma \in \mathbb{C}$. First, for the states to be normalized, we require $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$. Moreover, the coherent state system generated by $|e\rangle$ needs to give rise to a resolution of the identity, i.e. $\frac{1}{|S_3|} \sum_{g \in S_3} |g\rangle \langle g| = c \cdot 1_R$ with $c \in \mathbb{R}$. With $|S_3| = 6$, we find that $c = \frac{1}{3}$ and $(+|e\rangle|^2 = \frac{1}{3}$ with $|+\rangle = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle)$. The latter condition is equivalent to

$$\alpha\beta^* + \alpha^*\beta + \alpha\gamma^* + \alpha^*\gamma + \beta\gamma^* + \beta^*\gamma = 0.$$ 

Let us choose two different seed states

$$|e\rangle_1 = \frac{1}{3} |0\rangle - \frac{2}{3} |1\rangle - \frac{2}{3} |2\rangle, \quad |e\rangle_2 = \frac{1}{\sqrt{2}} |0\rangle - \frac{i}{\sqrt{2}} |2\rangle,$$

that generate two different systems of coherent states. One can easily check that the above conditions are satisfied. As expected, we find that the physical uniformity as defined in the main body takes on the same value for both systems. More precisely, by direct calculation, we find

$$U_{2,\text{phys}}^{(e)} := \int_{H_{\text{phys}}} d\psi \frac{1}{6} \sum_{g \in S_3} | \langle \psi | S(R) V_g | \psi \rangle^{(e)}_S |^2 = \frac{1}{2} = U_{\text{phys}}$$

for $j = 1, 2$, which confirms the result we get from Theorem 3 with $n_1 = n_2 = 1$ and $n_1^{\text{std}} = n_2^{\text{std}} = 1$, we find $U_{\text{phys}} = \frac{1}{2}$. The subscript in the above equation indicates that we take the second power in the definition of the conditional uniformity.

More generally, one can define

$$U_{p,\text{phys}}^{(e)} := \int_{H_{\text{phys}}} d\psi U_p(\psi_S)$$

where $p \geq 0$ and

$$U_p(\varphi_S) = \frac{1}{6} \sum_{g \in S_3} | \langle \varphi | V_g | \varphi \rangle_S |^p.$$ 

Fig. 2 illustrates $U_{p,\text{phys}}^{(e)}$ as a function of $p$ for the two seed states $|e\rangle_1, |e\rangle_2$ above ($j = 1, 2$).

![Fig. 2. $U_{p,\text{phys}}^{(e)}$ as a function of $p$ for $j = 1, 2$.](image)

In particular, numerical and symbolic integration gives us the values

$$U_{1,\text{phys}}^{(e_1)} \approx 0.611, \quad U_{1,\text{phys}}^{(e_2)} \approx 0.658, \quad U_{1,\text{phys}}^{(e_1)} = \frac{17}{40} \approx 0.425, \quad U_{2,\text{phys}}^{(e_2)} = \frac{229}{640} \approx 0.358.$$
As shown in Lemma 8 below, $A_p := - \log \mathcal{U}_p$ defines an asymmetry monotone. The above shows that the value of these monotones does in general depend on the choice of coherent state system. For example, there exist states $|\psi\rangle_{RS}$ for which $A_4(|\psi\rangle_{S|R}) < A_4(|\psi\rangle_{S|R}^2)$.

Furthermore, neither one of the two coherent state systems can be regarded as “better” on average than the other one: we have $A_{1,\text{phys}} := - \log \mathcal{U}_{1,\text{phys}} > A_{1,\text{phys}}^e$, but $A_{1,\text{phys}}^e < A_{1,\text{phys}}$. That is, the question of whether $|e\rangle_1$ or $|e\rangle_2$ induces more asymmetry on $S$ depends on the choice of monotone that is used to quantify the asymmetry.

This is not only true on average, but also on the level of individual states, which can be seen as follows. As apparent from the plot, we have $A_{p,\text{phys}} > A_{p,\text{phys}}^e$ for which $A_{p,\text{phys}} > A_{p,\text{phys}}^e$. For every compact Lie group $G$, and for every $p \geq 0$, the quantity

$$A_p(\rho) := - \log \int_G d\varrho \mathcal{F}(\rho, V_{\varrho}^\dagger V_{\varrho}^\dagger)^{p/2}$$

is an asymmetry monotone. This includes the case $A = A_2$ from the main text.

Proof. Suppose that $\mathcal{E}$ is any $G$-covariant map. Then, using that $x \mapsto x^{p/2}$ is non-decreasing and the data processing inequality $\mathcal{F}(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \geq \mathcal{F}(\rho, \sigma)$, we get

$$A_p(\mathcal{E}(\rho)) = - \log \int_G d\varrho \mathcal{F}(\mathcal{E}(\rho), V_{\varrho}^\dagger V_{\varrho})^{p/2} = - \log \int_G d\varrho \mathcal{F}(\mathcal{E}(\rho), \mathcal{E}(V_{\varrho}^\dagger V_{\varrho}))^{p/2}$$

$$\leq - \log \int_G d\varrho \mathcal{F}(\rho, V_{\varrho}^\dagger V_{\varrho})^{p/2} = A_p(\rho).$$

□