ITERATED VANISHING CYCLES, CONVOLUTION, AND A
MOTIVIC ANALOGUE OF A CONJECTURE OF STEENBRINK

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1. Introduction

Let us start by recalling the statement of Steenbrink’s conjecture. Let $f : X \to \mathbb{A}^1$ be a function on a smooth complex algebraic variety. Let $x$ be a closed point of $f^{-1}(0)$. Steenbrink introduced the notion of the spectrum $\text{Sp}(f, x)$ of $f$ at $x$. It is a fractional Laurent polynomial $\sum_{\alpha \in \mathbb{Q}} n_\alpha t^{\alpha}$, $n_\alpha$ in $\mathbb{Z}$, which is constructed using the action of the monodromy on the mixed Hodge structure on the cohomology of the Milnor fiber at $x$. When $f$ has an isolated singularity at $x$, all $n_\alpha$ are in $\mathbb{N}$, and the exponents of $f$, counted with multiplicity $n_\alpha$, are exactly the rational numbers $\alpha$ with $n_\alpha$ not zero.

Let us assume now that the singular locus of $f$ is a curve $\Gamma$, having $r$ local components $\Gamma_\ell$, $1 \leq \ell \leq r$, in a neighborhood of $x$. We denote by $m_\ell$ the multiplicity of $\Gamma_\ell$. Let $g$ be a generic linear form vanishing at $x$ (that is, a function $g$ vanishing at $x$ whose differential at $x$ is a generic linear form). For $N$ large enough, the function $f + g^N$ has an isolated singularity at $x$. In a neighborhood of the complement $\Gamma_\ell^c$ to $\{x\}$ in $\Gamma_\ell$, we may view $f$ as a family of isolated hypersurface singularities parametrized by $\Gamma_\ell^c$. The cohomology of the Milnor fiber of this hypersurface singularity is naturally endowed with the action of two commuting monodromies: the monodromy of the function and the monodromy of a generator of the local fundamental group of $\Gamma_\ell^c$. We denote by $\alpha_{\ell,j}$ the exponents of that isolated hypersurface singularity and by $\beta_{\ell,j}$ the corresponding rational numbers in $[0, 1)$ such that the complex numbers $\exp(2\pi i \beta_{\ell,j})$ are the eigenvalues of the monodromy along $\Gamma_\ell^c$.

1.1. Conjecture (Steenbrink [25]). For $N \gg 0$,

$$\text{Sp}(f + g^N, x) - \text{Sp}(f, x) = \sum_{\ell,j} t^{\alpha_{\ell,j} + (\beta_{\ell,j}/m_\ell)N} \frac{1 - t}{1 - t^{1/m_\ell N}}. \quad (1.1.1)$$

The conjecture of Steenbrink has been proved by M. Saito in [21], using his theory of mixed Hodge modules [18][20]. Later, A. Némethi and J. Steenbrink [17] gave another proof, still relying on the theory of mixed Hodge modules. Also, forgetting the integer part of the exponents of the spectrum, (1.1.1) has been proved by D. Siersma [23] in terms of zeta functions of the monodromy. Notice that, taking ordinary Euler characteristics, (1.1.1) specializes to a result of I. Iomdin [14] who was the first to compare vanishing cohomologies of $f$ and $f + g^N$. The convention we use here, cf. (6.6.2), to define $\text{Sp}(f, x)$ slightly differs from the original one and corresponds to what is denoted by $\text{Sp}'(f, x)$ in [21].
Recently, using motivic integration, Denef and Loeser introduced the motivic Milnor fiber $S_{f,x}$. It is a virtual variety endowed with an action of the group scheme $\hat{\mu}$ of roots of unity and the Hodge spectrum $\text{Sp}(f,x)$ can be retrieved from $S_{f,x}$, cf. [8]. They also showed that an analogue of the Thom-Sebastiani Theorem holds for the motivic Milnor fiber. This result was first stated in a (completed) Grothendieck ring [7] of Chow motives and then extended to a Grothendieck ring of virtual varieties endowed with a $\hat{\mu}$-action in [16] and [8], using a convolution product $*$ introduced in [16]. It is also convenient to slightly modify the virtual varieties $S_{f,x}$, which correspond to nearby cycles, into virtual varieties $S^\phi_{f,x}$ corresponding to vanishing cycles.

It is then quite natural to ask for a motivic analogue of Steenbrink’s conjecture in terms of motivic Milnor fibers. The present paper is devoted to give a complete answer to that question. Our main result, Theorem 5.7 expresses (in its local version 5.16), for $x$ a closed point where $f$ and $g$ both vanish and for $N \gg 0$, the difference $S^\phi_{f,x} - S^\phi_{f+g,x}$ as $\Psi_\Sigma(S_{g,x}^\phi(S_f^\phi))$, where $S_{g,x}^\phi(S_f^\phi)$ corresponds to iterated motivic vanishing cycles and $\Psi_\Sigma$ is a generalization of the convolution product $*$. In fact, in Theorem 5.7 we no longer assume any condition on the singular locus of $f$; also $g$ is not assumed anymore to be a generic linear form and can be any function vanishing at $x$. Formula (1.1.1) may be deduced from Theorem 5.7 by considering the Hodge spectrum.

The plan of the paper is the following. In section 2 we introduce the basic Grothendieck rings we shall use. Then, in section 3 we recall the definition of the Motivic Milnor fiber and we extend it to the whole Grothendieck ring. Such an extension has also been done by F. Bittner in [3], using the weak factorization Theorem and her work [2]; the construction we present here, based on motivic integration, is quite different. We then extend the construction to the equivariant setting, in order to define iterated vanishing cycles in the motivic framework in section 4. In section 5 we first define our generalized convolution operator $\Psi_\Sigma$ and explain its relation with the convolution product $*$. This gives us the opportunity to prove the associativity of the convolution product $*$, a fact already mentioned in [8]. Then comes the heart of the paper, that is the proof of Theorem 5.7. We conclude the section by explaining how one recovers the motivic Thom-Sebastiani Theorem of [7], [16] and [8] from Theorem 5.7. The final section 6 is devoted to applications to the Hodge-Steenbrink spectrum, in particular, we deduce Steenbrink’s conjecture 1.1 from Theorem 5.7.

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2. Grothendieck rings

2.1. By a variety over of field $k$, we mean a separated and reduced scheme of finite type over $k$. If $X$ is a scheme, we denote by $|X|$ the corresponding reduced scheme.
If an algebraic group $G$ acts on a variety $X$, we say the action is good if every $G$-orbit is contained in an affine open subset of $X$. Let $Y$ be a variety over $k$ and let $p : A \to Y$ be an affine bundle for the Zariski topology (the fibers of $p$ are affine spaces and the transition morphisms between trivializing charts are affine). In particular the fibers of $p$ have the structure of affine spaces. Let $G$ be a linear algebraic group. A good action of $G$ on $A$ is said to be affine if it is a lifting of a good action on $Y$ and its restriction to all fibers is affine. Note that affine actions on an affine bundle extend to its relative projective bundle compactification.

If $G$ is finite and $X$ and $Y$ are two varieties with good $G$-action, we denote by $X \times^G Y$ the quotient of the product $X \times Y$ by the equivalence relation $(gx, y) \equiv (x, gy)$. The action of $G$ on, say, the first factor of $X \times Y$ induces a good $G$-action on $X \times^G Y$.

For $n \geq 1$, we denote by $\mu_n$ the group scheme of $n$-th roots of unity and by $\hat{\mu}$ the projective limit $\lim_{\mu_n}$ of the projective system with transition morphisms $\mu_{nd} \to \mu_n$ given by $x \mapsto x^d$. In this paper all $\hat{\mu}$-actions, and more generally all $\hat{\mu}^r$-actions, will be assumed to factorize through a finite quotient.

2.2. Throughout the paper $k$ will be a field of characteristic zero. For $S$ a variety over $k$, we denote by $K_0(\text{Var}_S)$ the Grothendieck ring of varieties over $S$, cf. 

Let us recall it is generated by classes of morphisms of varieties $X \to S$ and that it is also generated by classes of such morphisms with $X$ smooth over $k$ and it suffices to consider relations for smooth varieties. We denote by $L = L_S$ the class of the trivial line bundle over $S$ and set $M_S$ for the localization $K_0(\text{Var}_S)[L^{-1}]$. As in 

similarly, using the category $\text{Var}_S^\hat{\mu}$ of varieties with good $\hat{\mu}$-action over $S$, but adding the additional relation

$$[Y \times A^n_k, \sigma] = [Y \times A^n_k, \sigma']$$

if $\sigma$ and $\sigma'$ are two liftings of the same $\hat{\mu}$-action on $Y$ to an affine action on $Y \times A^n_k$. We shall denote them by $K_0(\text{Var}_S^\hat{\mu})$ and $M_S^\hat{\mu}$. One can more generally replace $\hat{\mu}$ by $\hat{\mu}^r$ in these definitions and define $K_0(\text{Var}_S^\hat{\mu}^r)$ and $M_S^\hat{\mu}^r$. In 

Bittner considers similar equivariant rings, but with an additional relation a priori coarser than the one we use here.

2.3. In the present paper, instead of varieties with $\hat{\mu}$-action over $S$, we choose to work in the equivalent setting of varieties with $G_m$-action with some additional structure.

Let $Y$ be a variety with good $G_m$-action. We say a morphism $\pi : Y \to G_m^r$ is diagonally monomial of weight $n$ in $N_{>0}$, if $\pi(\lambda x) = \lambda^n \pi(x)$ for all $\lambda$ in $G_m^r$ and $x$ in $Y$. Fix $n$ in $\mathbb{N}_{>0}$. We denote by $\text{Var}_{S \times G_m}^{G_m \times G_m}$ the category of varieties $Y \to S \times G_m^r$ over $S \times G_m^r$ with good $G_m$-action such that furthermore, the fibers of the projection $\pi_1 : Y \to S$ are $G_m^r$-invariant and the projection $\pi_2 : Y \to G_m^r$ is diagonally monomial of weight $n$. We define the Grothendieck group $K_0(\text{Var}_{S \times G_m}^{G_m \times G_m})$ as the free abelian group on isomorphism classes of objects $Y \to S \times G_m^r$ in $\text{Var}_{S \times G_m}^{G_m \times G_m}$, modulo
the relations
\[(2.3.1) \quad [Y \to S \times G^r_m] = [Y' \to S \times G^r_m] + [Y \setminus Y' \to S \times G^r_m]\]
for \(Y'\) closed \(G^r_m\)-invariant in \(Y\) and, for \(f : Y \to S \times G^r_m\) in \(\text{Var}^G_{S \times G^r_m}\),
\[(2.3.2) \quad [Y \times A^n_k \to S \times G^r_m, \sigma] = [Y \times A^n_k \to S \times G^r_m, \sigma']\]
if \(\sigma\) and \(\sigma'\) are two liftings of the same \(G^r_m\)-action on \(Y\) to affine actions, the morphism \(Y \times A^n_k \to S \times G^r_m\) being the composition of \(f\) with the projection on the first factor. Of course in (2.3.2), instead of the trivial affine bundle we could have considered any affine bundle over \(Y\).

Fiber product over \(S \times G^r_m\) with diagonal action induces a product in the category \(\text{Var}^G_{S \times G^r_m}\), which allows to endow \(K_0(\text{Var}^G_{S \times G^r_m})\) with a natural ring structure. Note that the unit \(1_{S \times G^r_m}\) for the product is the class of the identity morphism on \(S \times G^r_m\); the \(G^r_m\)-action on \(S \times G^r_m\) being the trivial one on \(S\) and standard multiplicative translation on \(G^r_m\). There is a natural structure of \(K_0(\text{Var}_k)\)-module on \(K_0(\text{Var}^G_{S \times G^r_m})\). We denote by \(L_{S \times G^r_m} = L\) the element \(L \cdot 1_{S \times G^r_m}\) in this module, and we set \(M^G_{S \times G^r_m} = K_0(\text{Var}^G_{S \times G^r_m})[L^{-1}]\).

If \(f : S \to S'\) is a morphism of varieties, composition with \(f\) leads to a push-forward morphism \(f_! : M^G_{S \times G^r_m} \to M^G_{S' \times G^r_m}\), while fiber product leads to a pull-back morphism \(f^* : M^G_{S' \times G^r_m} \to M^G_{S \times G^r_m}\) (these morphisms may already be defined at the \(K_0\)-level).

2.4. For \(n\) in \(\mathbb{N}^r_{>0}\), we denote by \(\mu_n\) the group \(\mu_{n_1} \times \cdots \times \mu_{n_r}\). We consider the functor
\[(2.4.1) \quad G_n : \text{Var}^G_{S \times G^r_m} \to \text{Var}^G_S\]
assigning to \(p : Y \to S \times G^r_m\) the fiber at 1 of the morphism \(Y \to G^r_m\) obtained by composition with projection on the second factor. Note that this fiber carries a natural \(\mu_n\)-action by the monomiality assumption.

On the other side, if \(f : X \to S\) is a variety over \(S\) with good \(\mu_n\)-action, we may consider the variety \(F_n(X) := X \times G^r_m\) and view it as a variety over \(S \times G^r_m\) by sending the class of \((x, \lambda)\) to \((f(x), \lambda^n)\). The standard \(G^r_m\)-action by multiplicative translation on \(G^r_m\) induces a \(G^r_m\)-action on \(F_n(X)\). Note that the second projection is diagonally monomial of weight \(n\), hence \(F_n\) is in fact a functor
\[(2.4.2) \quad F_n : \text{Var}^G_S \to \text{Var}^G_{S \times G^r_m}\]

2.5. Lemma. The functors \(F_n\) and \(G_n\) are mutually quasi-inverse, so that the categories \(\text{Var}^G_S\) and \(\text{Var}^G_{S \times G^r_m}\) are equivalent.

Proof. It is quite clear that \(G_n(F_n(X))\) is isomorphic to \(X\), for \(X\) in \(\text{Var}^G_S\). For \(X\) in \(\text{Var}^G_{S \times G^r_m}\), set \(Y := G_n(X)\). We have a natural morphism \(Y \times G^r_m \to X\) sending \((y, \lambda)\) to \(\lambda y\). Clearly this morphism induces an isomorphism between \(Y \times \mu_n G^r_m\) and \(X\) in \(\text{Var}^G_{S \times G^r_m}\). \(\square\)
We consider the partial order $\mathbf{n} \prec \mathbf{m}$ on $\mathbb{N}^r_0$ given by divisibility of each coordinate, that is, $\mathbf{n} \prec \mathbf{m}$ if $\mathbf{n} = k \mathbf{m}$ for some $k$ in $\mathbb{N}^r_0$. If $\mathbf{n} = k \mathbf{m}$, we have a natural functor

$$\theta^n_m : \text{Var}_{S \times G_m}^{G_r^{m \times n}} \rightarrow \text{Var}_{S \times G_m}^{G_r^{n \times m}},$$

sending $X \rightarrow S \times G_m$ to the same object, but with the action $\lambda \mapsto \lambda x$ on $X$ replaced by $\lambda \mapsto \lambda^k x$. We define the category $\text{Var}_{S \times G_m}^{G_r^{m \times n}}$ as the colimit of the inductive system of categories $\text{Var}_{S \times G_m}^{G_r^{n \times m}}$. We define $K_0(\text{Var}_{S \times G_m}^{G_r^{n \times m}})$ and $\mathcal{M}_{S \times G_m}^{G_r^{n \times m}}$ as in 2.6. Clearly, $K_0(\text{Var}_{S \times G_m}^{G_r^{n \times m}})$ and $\mathcal{M}_{S \times G_m}^{G_r^{n \times m}}$ are respectively the colimits of the rings $K_0(\text{Var}_{S \times G_m}^{G_r^{m \times n}})$ and $\mathcal{M}_{S \times G_m}^{G_r^{m \times n}}$. Since the category $\text{Var}_S^{G_r^n}$ is the colimit of the categories $\text{Var}_S^{G_r^n}$, we have the following statement:

2.6. Proposition. There is a unique pair of functors

$$G : \text{Var}_{S \times G_m}^{G_r^{m \times n}} \rightarrow \text{Var}_S^{G_r^n}$$

and

$$F : \text{Var}_S^{G_r^n} \rightarrow \text{Var}_{S \times G_m}^{G_r^{m \times n}}$$

that restrict to $G_n$ and $F_n$ for every $n$. They are mutually quasi-inverse. In particular $G$ induces canonical isomorphisms

$$(2.6.3) \quad K_0(\text{Var}_{S \times G_m}^{G_r^{m \times n}}) \simeq K_0(\text{Var}_S^{G_r^n}) \quad \text{and} \quad \mathcal{M}_{S \times G_m}^{G_r^{m \times n}} \simeq \mathcal{M}_S^{G_r^n}$$

compatible with the operations $f_!$ and $f^!$. \]

2.7. Let $Y$ be a variety with good $G_r^n$-action. We say a morphism $\pi : Y \rightarrow G_r^n$ is monomial if it is equivariant with respect to some transitive $G_r^n$-action on $G_r^n$ (see section 4.6 for monomial morphisms that are not diagonally monomial morphisms). More generally consider a variety $(\text{section } \S_{\text{4.6}} \text{ for monomial morphisms that are not diagonally monomial morphisms)\text{.}}$

More generally if $W$ is a constructible subset of $Y$ stable by the $G_r^n$-action, we shall call a morphism $\pi : W \rightarrow G_r^n$ piecewise monomial if there is a finite partition of $W$ into locally closed $G_r^n$-invariant subsets on which the restriction of $\pi$ is a monomial morphism. To such a $W$ endowed with a morphism $(p, \pi) : W \rightarrow S \times G_r^n$ such that the fibers of $p : W \rightarrow S$ are $G_r^n$-invariant and $\pi : W \rightarrow G_r^n$ is piecewise monomial, we assign by additivity a class $[(p, \pi) : W \rightarrow S \times G_r^n]$ in $\mathcal{M}_{S \times G_m}^{G_r^{m \times n}}$. 

2.8. **Rational series.** Let $A$ be one of the rings $\mathbb{Z}[L, L^{-1}], \mathbb{Z}[L, L^{-1} \cdot (\frac{1}{L})_{i>0}]$, $\mathcal{M}_{S \times G_m}^{G_m}$. We denote by $A[[T]]_{sr}$ the $A$-submodule of $A[[T]]$ generated by 1 and by finite products of terms $p_{e,i}(T) = \frac{L^T}{1 - L^{-TR}}$, with $e$ in $\mathbb{Z}$ and $i$ in $\mathbb{N}_{>0}$. There is a unique $A$-linear morphism

$$\lim_{T \to \infty} : A[[T]]_{sr} \to A$$

such that

$$\lim_{T \to \infty} \left( \prod_{i \in I} p_{e_i,j_i}(T) \right) = (-1)^{|I|},$$

for every family $((e_i, j_i))_{i \in I}$ in $\mathbb{Z} \times \mathbb{N}_{>0}$, with $I$ finite, maybe empty.

2.9. Let $I$ be a finite set. We shall consider rational polyhedral convex cones in $\mathbb{R}_{>0}^I$. By this we mean a convex subset of $\mathbb{R}_{>0}^I$ defined by a finite number of integral linear inequalities of type $a \geq 0$ or $b > 0$, and stable by multiplication by $\mathbb{R}_{>0}$. Let $\Delta$ be such a cone in $\mathbb{R}_{>0}^I$. We denote by $\bar{\Delta}$ its closure in $\mathbb{R}^I_{\geq 0}$.

Let $\ell$ and $\nu$ be integral linear forms on $\mathbb{Z}^I$ which are positive on $\bar{\Delta} \setminus \{0\}$. Let us consider the series

$$S_{\Delta,\ell,\nu}(T) := \sum_{k \in \Delta \cap \mathbb{N}_{>0}^I} T^{\ell(k)} L^{-\nu(k)}$$

in $\mathbb{Z}[L, L^{-1}][[T]]$.

In the special case when $\Delta$ is open in its linear span and $\bar{\Delta}$ is generated by vectors $(e_1, \ldots, e_m)$ which are part of a $\mathbb{Z}$-basis of the $\mathbb{Z}$-module $\mathbb{Z}^I$, the series $S_{\Delta,\ell,\nu}$ lies in $\mathbb{Z}[L, L^{-1}][[T]]_{sr}$ and $\lim_{T \to \infty} S_{\Delta,\ell,\nu}(T)$ is equal to $(-1)^{\dim(\Delta)}$. By additivity with respect to disjoint union of cones with the positivity assumption, one deduces that, in general, $S_{\Delta,\ell,\nu}$ lies in $\mathbb{Z}[L, L^{-1}][[T]]_{sr}$ and $\lim_{T \to \infty} S_{\Delta,\ell,\nu}(T)$ is equal to $\chi(\Delta)$, the Euler characteristic with compact supports of $\Delta$.

In particular we get the following lemma (compare with Lemma 2.1.5 in [13] and [4] pp. 1006-1007):

2.10. **Lemma.** Let $\Delta$ be a rational polyhedral convex cone in $\mathbb{R}_{>0}^I$ defined by

$$\sum_{i \in K} a_i x_i \leq \sum_{i \in I \setminus K} a_i x_i,$$

with $a_i$ in $\mathbb{N}$ and $a_i > 0$, for $i$ in $K$, and with $K$ and $I \setminus K$ non empty. If $\ell$ and $\nu$ are integral linear forms positive on $\Delta \setminus \{0\}$, then $\lim_{T \to \infty} S_{\Delta,\ell,\nu}(T) = 0$. □

3. **Motivic vanishing cycles**

3.1. **Arc spaces.** We denote as usual by $\mathcal{L}_n(X)$ the space of arcs of order $n$, also known as the $n$-th jet space on $X$. It is a $k$-scheme whose $K$-points, for $K$ a field containing $k$, is the set of morphisms $\varphi : \text{Spec } K[t]/t^{n+1} \to X$. There are canonical morphisms $\mathcal{L}_{n+1}(X) \to \mathcal{L}_n(X)$ which are $\mathbb{A}_{k}^{d}$-bundles when $X$ is smooth of pure dimension $d$. The arc space $\mathcal{L}(X)$ is defined as the projective limit of this system.
We denote by $\pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X)$ the canonical morphism. There is a canonical $\mathbb{G}_m$-action on $\mathcal{L}_n(X)$ and on $\mathcal{L}(X)$ given by $a \cdot \varphi(t) = \varphi(at)$.

For an element $\varphi$ in $K[[t]]$ or in $K[t]/t^{n+1}$, we denote by $\text{ord}_t(\varphi)$ the valuation of $\varphi$ and by $\text{ac}(\varphi)$ its first non-zero coefficient, with the convention $\text{ac}(0) = 0$.

3.2. Motivic zeta function and Motivic Milnor fiber. Let us start by recalling some basic constructions introduced by Denef and Loeser in [5], [9] and [8].

Let $X$ be a smooth variety over $k$ of pure dimension $d$ and $g : X \to \mathbb{A}^1_k$. We set $X_0(g)$ for the zero locus of $g$, and consider, for $n \geq 1$, the variety

\begin{equation}
\mathcal{X}_n(g) := \{ \varphi \in \mathcal{L}_n(X) \mid \text{ord}_t \varphi = n \}.
\end{equation}

Note that $\mathcal{X}_n(g)$ is invariant by the $\mathbb{G}_m$-action on $\mathcal{L}_n(X)$. Furthermore $g$ induces a morphism $g_n : \mathcal{X}_n(g) \to \mathbb{G}_m$, assigning to a point $\varphi$ in $\mathcal{L}_n(X)$ the coefficient $\text{ac}(g(\varphi))$ of $t^n$ in $g(\varphi)$, which we shall also denote by $\text{ac}(g)(\varphi)$. This morphism is diagonally monomial of weight $n$ with respect to the $\mathbb{G}_m$-action on $\mathcal{X}_n(g)$ since $g_n(a \cdot \varphi) = a^n g_n(\varphi)$, so we can consider the class $[\mathcal{X}_n(g)]$ of $\mathcal{X}_n(g)$ in $\mathcal{M}^\mathbb{G}_m_{\mathcal{X}_0(g) \times \mathbb{G}_m}$.

We now consider the motivic zeta function

\begin{equation}
Z_g(T) := \sum_{n \geq 1} [\mathcal{X}_n(g)] \mathbf{L}^{-nd} T^n
\end{equation}

in $\mathcal{M}^\mathbb{G}_m_{\mathcal{X}_0(g) \times \mathbb{G}_m}[[T]]$. Note that $Z_g = 0$ if $g = 0$ on $X$.

Denef and Loeser showed in [5] and [8] (see also [9]) that $Z_g(T)$ is a rational series by giving a formula for $Z_g(T)$ in terms of a resolution of $f$.

3.3. Resolutions. Let us introduce some notation and terminology. Let $X$ be a smooth variety of pure dimension $d$ and let $Z$ a closed subset of $X$ of codimension everywhere $\geq 1$. By a log-resolution $h : Y \to X$ of $(X, Z)$, we mean a proper morphism $h : Y \to X$ with $Y$ smooth such that the restriction of $h : Y \setminus h^{-1}(Z) \to X \setminus Z$ is an isomorphism, and $h^{-1}(Z)$ is a divisor with normal crossings. We denote by $E_i$, $i \in A$, the set of irreducible components of the divisor $h^{-1}(Z)$. For $I \subset A$, we set

\begin{equation}
E_I := \bigcap_{i \in I} E_i
\end{equation}

and

\begin{equation}
E_I^o := E_I \setminus \bigcup_{j \notin I} E_j.
\end{equation}

We denote by $\nu_{E_i}$ the normal bundle of $E_i$ in $Y$, by $\nu_{E_i}^J$, the fiber product, for $J$ contained in $I$, of the restrictions to $E_I$ of the bundles $\nu_{E_i}$, $i \in J$, and by $\pi_I^J : \nu_{E_i}^J \to E_I$ the canonical projections. For any of these vector bundles $\nu$ we will denote by $\mathcal{V}$ the projective bundle associated to the sum of $\nu$ with the trivial line bundle.

We will denote by $U_{E_i}$ the complement of the zero section in $\nu_{E_i}$ and by $U_I^J$ (resp. $U_{E_i}^J$) the fiber product, for $J$ contained in $I$, of the restrictions of the spaces $U_{E_i}$, $i$
in $J$, to $E_0^\gamma$ (resp. $E_1$). We will still denote by $\pi_i^J$ the induced projection from $U_i^J$ (resp. $U_i^J$) onto $E_0^\gamma$ (resp. $E_1$).

When $J = I$ we will simply write $\nu_{E_i}$ (resp. $\nu_{E_i}$, $\pi_I$, $U_I$, $U_{E_i}$) for $\nu_{E_i}^J$ (resp. $\nu_{E_i}^J$, $\pi_i^J$, $U_i^J$, $U_i^J$).

If $\mathcal{I}$ is a sheaf of ideals defining a closed subscheme $Z$ and $h^{-1}(\mathcal{I})\mathcal{O}_Y$ is locally principal, we define $N_i(\mathcal{I})$, the multiplicity of $\mathcal{I}$ along $E_i$, by the equality of divisors

$$h^{-1}(Z) = \sum_{i \in A} N_i(\mathcal{I})E_i.$$  

If $\mathcal{I}$ is principal generated by a function $g$ we write $N_i(g)$ for $N_i(\mathcal{I})$. Similarly, we define integers $\nu_i$ by the equality of divisors

$$K_Y = h^*K_X + \sum_{i \in A} (\nu_i - 1)E_i.$$  

Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two sheaves of ideals on $X$ whose associated reduced closed subschemes $Z_1$ and $Z_2$ have codimension at least one. Let $h : Y \to X$ be a log-resolution of $(X, Z_1 \cup Z_2)$ such that $h^*(\mathcal{I}_1)$ and $h^*(\mathcal{I}_2)$ are locally principal. Then we set

$$\gamma_h(\mathcal{I}_1, \mathcal{I}_2) := \sup_{\{i \in A, |N_i(\mathcal{I}_2) > 0\}} \frac{N_i(\mathcal{I}_1)}{N_i(\mathcal{I}_2)}.$$  

If $x$ is a closed point of $Z_2$, we set

$$\gamma_h(x, \mathcal{I}_1, \mathcal{I}_2) = \sup_{\{i \in A, |N_i(\mathcal{I}_2) > 0\}} \frac{N_i(\mathcal{I}_1)}{N_i(\mathcal{I}_2)},$$  

with $A_x$ the set of $i$ in $A$ such that $|h^{-1}(x)| \cap E_i \neq \emptyset$. Finally we define $\gamma(\mathcal{I}_1, \mathcal{I}_2)$, resp. $\gamma(x, \mathcal{I}_1, \mathcal{I}_2)$ as the infimum of all $\gamma_h(\mathcal{I}_1, \mathcal{I}_2)$, resp. $\gamma_h(x, \mathcal{I}_1, \mathcal{I}_2)$, for $h$ a log-resolution of $(X, Z_1 \cup Z_2)$ such that $h^*(\mathcal{I}_1)$ and $h^*(\mathcal{I}_2)$ are locally principal.

3.4. Let $g$ be a function on a smooth variety $X$ of pure dimension $d$. Assume $X_0(g)$ is nowhere dense in $X$. Let $F$ a reduced divisor containing $X_0(g)$ and let $h : Y \to X$ be a log-resolution of $(X, F)$. We fix $I$ such that there exists $i$ in $I$ with $N_i(g) > 0$. Let us explain how $g$ induces a morphism $g_I : U_I \to G_m$. Note that the function $g \circ h$ induces a function

$$\bigotimes_{i \in I} \nu_{E_i}^{\cdot N_i(g)} |_{E_i} \to A^1_k.$$  

We define $g_I : \nu_{E_I} \to A^1_k$ as the composition of this last function with the natural morphism $\nu_{E_I} \to \bigotimes_{i \in I} \nu_{E_i}^{\cdot N_i(g)} |_{E_i}$, sending $(y_i)$ to $\bigotimes_{i \in I} y_i^{\cdot N_i(g)}$. We still denote by $g_I$ the induced morphism from $U_I$ (resp. $U_{E_I}$) to $G_m$ (resp. $A^1_k$).

We view $U_I$ as a variety over $X_0(g) \times G_m$ via the morphism $(h \circ \pi_I, g_I)$. The group $G_m$ has a natural action on each $U_{E_I}$, so the diagonal action induces a $G_m$-action on $U_I$. Furthermore, the morphism $g_I$ is monomial, so $U_I \to X_0(g) \times G_m$ has a class in $M_{X_0(g) \times G_m}$ which we will denote by $[U_I]$. 


3.5. The morphism \( g_I \) may be described in terms of the following variant of the deformation to the normal cone to \( E_I \) in \( Y \), cf. \([12]\). We consider the affine space \( A^I_k = \text{Spec} \, k[u_i]_{i \in I} \) and the subsheaf

\[
A_I := \sum_{n \in \mathbb{N}^I} \mathcal{O}_Y \times A^I_k \left( -\sum_{i \in I} n_i (E_i \times A^I_k) \right) \prod_{i \in I} u_i^{-n_i},
\]

of \( \mathcal{O}_Y \times A^I_k \). It is a sheaf of rings and we set \( CY_I := \text{Spec} \, A_I \). The natural inclusion \( \mathcal{O}_Y \times A^I_k \to A_I \) induces a morphism \( \pi : CY_I \to Y \times A^I_k \), hence a morphism \( p : CY_I \to A^I_k \). The ring \( A_I \) being a graded subring of the ring \( \mathcal{O}_Y[u_i, u_i^{-1}]_{i \in I} \), we consider the corresponding \( G^I_m \)-action \( \sigma_I \) on \( CY_I \), leaving sections of \( \mathcal{O}_Y \) invariant and acting by \( (\lambda_i, u_i) \mapsto \lambda_i u_i \) on \( u_i \). We may then identify equivariantly \( \nu_{E_I} \) with the fiber \( p^{-1}(0) \). The image by the inclusion \( \mathcal{O}_Y \times A^I_k \to A_I \) of the function \( g \circ h \) is divisible by \( \prod_{i \in I} u_i^{N_i(g)} \) in \( A_I \), so we may consider the quotient \( \tilde{g}_I \) in \( A_I \). The restriction of \( \tilde{g}_I \) to the fiber \( p^{-1}(0) \cong \nu_{E_I} \) is nothing else than \( g_I \). As \( g \) may vanish only on the divisors \( E_i, i \in A \), the function \( g_I \) does not vanish on \( U_I \) and induces a monomial morphism \( g_I : U_I \to G_m \).

Let us note the following "transitivity" property. If we write \( I \) as a disjoint union \( K \sqcup J \), one notices that \( p^{-1}(0 \times G^J_m) \) is equivariantly isomorphic to \( \nu_{E_K} \times G^J_m \).

Hence, restricting \( p : CY_I \to A^I_k \) to \( p^{-1}(0 \times A^J_k) \), the function \( g_J : U_J \to G_m \) can be obtained from \( g_K \) by the same process as we obtained it from \( g \), replacing \( Y \) by \( \nu_{E_K} \), \( I \) by \( J \) and \( g \) by \( g_K \to A^J_k \).

3.6. We now assume that \( F = X_0(g) \), that is \( h : Y \to X \) is a log-resolution of \((X, X_0(g))\). In this case, \( h \) induces a bijection between \( \mathcal{L}(Y) \setminus \mathcal{L}((h^{-1}(X_0(g)))) \) and \( \mathcal{L}(X) \setminus \mathcal{L}(X_0(g)) \). One deduces by using the change of variable formula, in a way completely similar to \([3] \) and \([8] \), the equality

\[
Z_g(T) = \sum_{\emptyset \neq I \subset A} [U_I] \prod_{i \in I} \frac{1}{T - N_i(g)^{\nu_i} - 1},
\]

in \( \mathcal{M}_{X_0(g) \times G_m}[[T]] \).

In particular, the function \( Z_g(T) \) is rational and belongs to \( \mathcal{M}^{G^m}_{X_0(g) \times G_m}[[T]]_{sr} \), with the notation of \([2,8]\) hence we can consider \( \lim_{T \to \infty} Z_g(T) \) in \( \mathcal{M}^{G^m}_{X_0(g) \times G_m} \).

We set

\[
S_g := - \lim_{T \to \infty} Z_g(T),
\]

which by (3.6.1) may be expressed on a resolution \( h \) as

\[
S_g = - \sum_{\emptyset \neq I \subset A} (-1)^{|I|} [U_I],
\]

in \( \mathcal{M}_{X_0(g) \times G_m} \).

We shall also consider in this paper the motivic vanishing cycles defined as

\[
S_g^\phi := (-1)^{d-1} (S_g - [G_m \times X_0(g)])
\]
in \( \mathcal{M}_{X_0(g) \times G_m}^G \). Here \( d \) denotes the dimension of \( X \) and \( G_m \times X_0(g) \) is endowed with the standard \( G_m \)-action on the first factor and the trivial \( G_m \)-action on the second factor.

### 3.7. A modified zeta function

We now explain how to extend \( S_g \) to the whole Grothendieck group \( \mathcal{M}_X \) in such a way that \( S_g([X \to X]) \) is equal to \( S_g \). A similar result has been obtained by F. Bittner in \([3]\). We present here a somewhat different approach that avoids the use of the weak factorization Theorem, by constructing directly \( S_g([Y \to X]) \) for generators of \( \mathcal{M}_X \) of the form \( Y \to X \) with \( Y \) smooth.

Let \( X \) be a smooth variety of pure dimension \( d \) and let \( U \) be a dense open in \( X \). Consider again a function \( g : X \to \mathbb{A}^1_k \). We denote by \( F \) the closed subset \( X \setminus U \) and by \( I_F \) the ideal of functions vanishing on \( F \). We start by defining \( S_g([U \to X]) \).

Fix \( \gamma \geq 1 \) a positive integer. We will consider the modified zeta function \( Z_{g,U}^\gamma(T) \) defined as follows. For \( n \geq 1 \), we consider the constructible set

\[
(3.7.1) \quad X_n^\gamma(g, U) := \{ \varphi \in \mathcal{L}_n(X) \mid \text{ord}_g(\varphi) = n, \text{ord}_g \varphi^*(I_F) \leq n \gamma \}
\]

As in \([3.2]\), we consider the morphism \( X_n^\gamma(g, U) \to G_m \) induced by \( \varphi \mapsto ac(\varphi) \). It is piecewise monomial, so we can consider the class \([X_n^\gamma(g, U)]\) in \( \mathcal{M}_{X_0(g) \times G_m}^G \) by \([2.7]\).

We set

\[
(3.7.2) \quad Z_{g,U}^\gamma(T) := \sum_{n \geq 1} [X_n^\gamma(g, U)] L^{-\gamma nd} T^n
\]

in \( \mathcal{M}_{X_0(g) \times G_m}^G[[T]] \). Note that for \( U = X \), \( Z_{g,U}^\gamma(T) \) is equal to \( Z_g(T) \) for every \( \gamma \), since in this case, \([X_n^\gamma(g, U)]L^{-\gamma nd} = [X_n(g)]L^{-nd} \). Note also that \( Z_{g,u}^\gamma(T) = 0 \) if \( g \) is identically zero on \( X \).

If \( X_0(g) \) is nowhere dense in \( X \) and \( h : Y \to X \) is a log-resolution of \((X, F \cup X_0(g))\), we denote by \( C \) the set \( \{ i \in A \mid N_i(g) \neq 0 \} \).

### 3.8. Proposition

Let \( U \) be a dense open in the smooth variety \( X \) of pure dimension \( d \) with a function \( g : X \to \mathbb{A}^1_k \). There exists \( \gamma_0 \) such that for every \( \gamma > \gamma_0 \) the series \( Z_{g,U}^\gamma(T) \) lies in \( \mathcal{M}_{X_0(g) \times G_m}^G[[T]] \) and \( \lim_{T \to \infty} Z_{g,U}^\gamma(T) \) is independent of \( \gamma > \gamma_0 \). We set \( S_{g,U} = -\lim_{T \to \infty} Z_{g,U}^\gamma(T) \). Furthermore, if \( X_0(g) \) is nowhere dense in \( X \) and \( h : Y \to X \) is a log-resolution of \((X, F \cup X_0(g))\),

\[
(3.8.1) \quad S_{g,U} = -\sum_{\substack{i \neq \emptyset \in C \subset \emptyset}} (-1)^{|I|} [U_I] = h_1 \left( S_{goh,h^{-1}(U)} \right)
\]

in \( \mathcal{M}_{X_0(g) \times G_m}^G \).

**Proof.** We may assume \( X_0(g) \) is nowhere dense in \( X \). Let \( h : Y \to X \) be a log-resolution of \((X, F \cup X_0(g))\). As in the proof of Theorem 2.4 of \([3]\), we deduce from the change of variable formula, or more precisely from Lemma 3.4 in \([3]\), that

\[
(3.8.2) \quad Z_{g,U}^\gamma(T) = \sum_{I \cap C \neq \emptyset} [U_I] S_I(T)
\]
It follows from Lemma 2.10 that in this case
\[ I(3.8.4) \]
\[ \text{smooth and } U \text{ and every such } N \text{ with coordinates in } N_{>0} \text{ with } k \text{ and } l \text{ in } I. \] 
It follows that \( I(T) \) lies in \( M_{X_0(g) \times G_m}[[T]]_{sr} \) and \( \lim_{T \to \infty} I(T) = (-1)^{|I|} \), as soon as \( \gamma \geq \sup_{i \in I} \frac{N_i(I_F)}{N_i(g)}. \)

Now assume \( \emptyset \neq I \setminus C = K. \) For \( \gamma \geq \sup_{i \in I \setminus K} \frac{N_i(I_F)}{N_i(g)} \), the sum runs over the points with coordinates in \( N_{>0} \) of the cone \( \Delta_I \) in \( R^I_{>0} \) defined by the single inequality
\[ (3.9.4) \]
\[ \sum_{i \in K} a_i x_i \leq \sum_{i \notin K} a_i x_i, \]
with \( a_i \) in \( N \) and \( a_i > 0 \) for \( i \) in \( K. \) Note that both \( K \) and \( I \setminus K \) are non empty. It follows from Lemma 2.10 that in this case \( I(T) \) lies in \( M_{X_0(g) \times G_m}[[T]]_{sr} \) and \( \lim_{T \to \infty} I(T) = 0. \) The statement we have to prove then holds if we set \( \gamma_0 = \sup_{i \in C} \frac{N_i(I_F)}{N_i(g)} = \gamma_h(I_F, (g)). \) Note that since this holds for any \( h, \) we could also take \( \gamma_0 = \gamma(I_F, (g)). \)

3.9. Theorem (Extension to the Grothendieck group). Let \( X \) be a variety with a function \( g : X \to \mathbb{A}_k^1. \) There exists a unique \( M_k \)-linear group morphism
\[ (3.9.1) \]
\[ S_g : M_X \longrightarrow M_{X_0(g) \times G_m} \]
such that, for every proper morphism \( p : Z \to X, \) with \( Z \) smooth, and every dense open subset \( U \) in \( Z, \)
\[ (3.9.2) \]
\[ S_g([(U \to X)]) = p_i(S_{g\circ p, U}). \]

Proof. Since \( K_0(\text{Var}_X) \) is generated by classes \( [U \to X] \) with \( U \) smooth connected and every such \( U \to X \) may be embedded in a proper morphism \( Z \to X \) with \( Z \) smooth and \( U \) dense in \( Z, \) uniqueness is clear. For existence let us first note that if we define \( S_g([(U \to X)]) = S_g([(U)]) \) by the right hand side of (3.9.2), the result is independent from the choice of the embedding in a proper morphism \( p : Z \to X. \) Indeed, this is clear if \( g \circ p \) vanishes identically on \( U, \) so we may assume \( (g \circ p)^{-1}(0) \) is of codimension > 0. In this case, if we have another such morphism \( p' : Z' \to X, \) there exists a smooth variety \( W \) with proper morphisms \( h : W \to Z \) and \( h' : W \to Z', \) such that \( p \circ h = p' \circ h' \) and \( h \) and \( h' \) are respectively log-resolutions of \( (Z, (Z \setminus U) \cup (g \circ p)^{-1}(0)) \) and \( (Z', (Z' \setminus U) \cup (g \circ p')^{-1}(0)), \) so the statement follows from (3.8.1).

Let us now prove the following additivity statement: if \( \kappa : U \to X \) is a morphism with \( U \) smooth and \( W \) is a smooth closed subset of \( U, \) then
\[ (3.9.3) \]
\[ S_g([U \to X]) = S_g([W \to X]) + S_g([U \setminus W \to X]). \]

We may assume \( U \) and \( W \) are connected and \( U \setminus W \) is dense in \( U. \) The result being trivial if \( g \circ \kappa \) vanishes identically, we may assume this is not the case. By
Hironaka’s strong resolution of singularities, we may embed $U$ in a smooth variety $Z$ with $p : Z \to X$ a proper morphism extending $\kappa$ such that $Z \setminus U$ is a normal crossings divisor and the closure $\tilde{W}$ of $W$ in $Z$ is smooth. Again by Hironaka’s strong resolution of singularities, there exists a log-resolution $h : \tilde{Z} \to Z$ of $(Z, (Z \setminus U) \cup (g \circ p)^{-1}(0))$ such that the closure $\tilde{W}$ of $h^{-1}(W)$ in $\tilde{Z}$ is smooth and intersects the divisor $D := h^{-1}((Z \setminus U) \cup (g \circ p)^{-1}(0))$ transversally. We denote by $E_i$, $i \in A$, the irreducible components of the divisor $D$ and use the notations of $3.8.1$. It follows from the definition and $(3.8.1)$ that

\[(3.9.4)\]
\[S_g([U \to X]) = - \sum_{i \neq \emptyset \atop i \subset C} (-1)^{|i|} [U_i],\]

in $\mathcal{M}^{G_m}_{X_0(g) \times G_m}$. Note that if $W$ is contained in $(g \circ p)^{-1}(0)$, the above discussion still holds for $U$ replaced by $U \setminus W$, so we have $S_g([U \to X]) = S_g([U \setminus W \to X])$, and $(3.9.3)$ follows, since $S_g([W \to X]) = 0$ in this case.

Now we assume $W$ is not contained in $(g \circ p)^{-1}(0)$. Note that the morphism $h_0 : \tilde{W} \to \overline{W}$ induced by $h$ is a log-resolution of $(\overline{W}, (\overline{W} \setminus W) \cup (g \circ p)^{-1}(0))$. Furthermore, the irreducible components of the normal crossings divisor $h_0^{-1}((\overline{W} \setminus W) \cup (g \circ p)^{-1}(0))$ are exactly those amongst the $E_i \cap \tilde{W}$ which are non empty. Hence, denoting by $U_I|_{E_i \cap \tilde{W}}$ the restriction of the bundle $U_I$ to $E_i \cap \tilde{W}$, it follows from the definition and $(3.8.1)$ that

\[(3.9.5)\]
\[S_g([W \to X]) = - \sum_{i \neq \emptyset \atop i \subset C} (-1)^{|i|} [U_I|_{E_i \cap \tilde{W}}],\]

in $\mathcal{M}^{G_m}_{X_0(g) \times G_m}$. Let us now consider the blowing up $h' : Z' \to \tilde{Z}$ of $\tilde{Z}$ along $\tilde{W}$. The exceptional divisor $W'$ of $\tilde{W}$ is smooth. Furthermore $h \circ h' : Z' \to Z$ is a log-resolution of $(Z, (Z \setminus (U \setminus W)) \cup (g \circ p)^{-1}(0))$, and $D' := (h \circ h')^{-1}((Z \setminus (U \setminus W)) \cup (g \circ p)^{-1}(0))$ is a normal crossings divisor whose irreducible components are the strict transforms $E'_i$ of $E_i$ in $Z'$, $i \in A$ together with $W'$. We set $A' := A \cup \{0\}$ and $E'_0 := W'$ in order to use the notations of $3.3$ in this setting, adding everywhere $'$ as an exponent. Again, it follows from the definition and $(3.8.1)$ that

\[(3.9.6)\]
\[S_g([(U \setminus W) \to X]) = - \sum_{i \neq \emptyset \atop i \subset C'} (-1)^{|i|} [U'_i],\]

in $\mathcal{M}^{G_m}_{X_0(g) \times G_m}$, where $C' = \{i \in A'| N_i(g \circ p \circ h \circ h') \neq 0\}$. The hypothesis made on $W$ insures that $C' = C$. So it is enough to prove that for $I$ non empty and contained in $C$,

\[(3.9.7)\]
\[[U_I] = [U_I|_{E_i \cap \tilde{W}}] + [U'_I]\]
in $\mathcal{M}^{G_m}_{X_0(x) \times G_m}$, which follows from the fact that the restriction $U \mid_{E'_i \setminus (E'_i \cap \tilde{W})}$ of the bundle $U_i$ to $E'_i \setminus (E'_i \cap \tilde{W})$ and the bundle $U'_i$ have the same class in $\mathcal{M}^{G_m}_{X_0(x) \times G_m}$, since $h'$ is an isomorphism outside $W'$. This concludes the proof of (3.9.3).

Let again $U \to X$ be in $\text{Var}_X$ with $U$ smooth and connected. Let $W$ be a smooth proper variety over $k$. Note that

\begin{equation}
(3.9.8) 
\mathcal{S}_g([W \times U \to X]) = [W] \mathcal{S}_g([U \to X])
\end{equation}

in $\mathcal{M}^{G_m}_{X_0(x) \times G_m}$. Indeed, let us embed $U \to X$ in $p : Z \to X$ with $Z$ smooth and proper and $U$ dense in $Z$. We may assume $g \circ p$ is not identically zero. If $h : Y \to Z$ is a log-resolution of $(Z, (Z \setminus U) \cup (g \circ p)^{-1}(0))$, then $W \times U \to X$ may be embedded in $W \times Z \to X$, $\text{id} \times h : W \times Y \to W \times Z$ is a log-resolution of $(W \times Z, ((W \times Z) \setminus (W \times U)) \cup (W \times (g \circ p)^{-1}(0)))$, hence (3.9.8) follows from (3.8.1) and (3.9.2). By the additivity statement we already proved, relation (3.9.8) in fact holds for every variety $W$ over $k$, so our construction of $\mathcal{S}_g$ may be extended uniquely by $\mathcal{M}_k$-linearity to a $\mathcal{M}_k$-linear group morphism $\mathcal{M}_X \to \mathcal{M}^{G_m}_{X_0(x) \times G_m}$, which finishes the proof. \qed

3.10. The equivariant setting. Let $X$ be a variety with a function $g : X \to A^1_k$. By Theorem 3.9 there is a canonical morphism

\begin{equation}
(3.10.1) 
\mathcal{S}_g : \mathcal{M}_X \longrightarrow \mathcal{M}^{G_m}_{X_0(x) \times G_m}.
\end{equation}

We want to lift this morphism to a morphism, still denoted by $\mathcal{S}_g$,

\begin{equation}
(3.10.2) 
\mathcal{S}_g : \mathcal{M}^{G_m}_{X \times G_m} \longrightarrow \mathcal{M}^{G_m \times G_m}_{X_0(x) \times G_m \times G_m}
\end{equation}

such that the diagram

\begin{equation}
(3.10.3) 
\begin{array}{ccc}
\mathcal{M}^{G_m}_{X \times G_m} & \xrightarrow{\mathcal{S}_g} & \mathcal{M}^{G_m \times G_m}_{X_0(x) \times G_m \times G_m} \\
\downarrow & & \downarrow \\
\mathcal{M}_X & \xrightarrow{\mathcal{S}_g} & \mathcal{M}^{G_m}_{X_0(x) \times G_m}
\end{array}
\end{equation}

is commutative, the vertical arrows being given by forgetting the $G_m$-action and taking the fiber over $1$ in $G_m$.

Let us start with some basic facts we shall use without further mention. We fix the variety $X$ which we shall consider as endowed with the trivial $G_m$-action. Let $Z$ be a smooth variety of pure dimension $d$ endowed with a good $G_m$-action and an equivariant morphism $p : Z \to X$. The induced action on the affine bundles $\mathcal{L}_n(Z) \to \mathcal{L}_n(Z)$ is affine. In particular, by relation (2.3.2), $[\mathcal{L}_n(Z) \to X] = L^d[\mathcal{L}_n(Z) \to X]$ in $\mathcal{M}^{G_m}_{X \times G_m}$. Similarly, if $h : Y \to Z$ is a proper birational $G_m$-equivariant morphism with $Y$ smooth with a good $G_m$-action, the fibrations occurring in Lemma 3.4 of [6] are (piecewise) affine bundles and the induced $G_m$-action is affine, hence, by relation (2.3.2), one does not see the action on the fibers in the Grothendieck ring $\mathcal{M}^{G_m}_{X \times G_m}$.
We now assume $X$ is endowed with a morphism $g : X \to A^1_k$ and that $Z$ is endowed with a monomial morphism $f = (f_1, \ldots, f_r) : Z \to G^r_m$ such that $(p, f) : Z \to X \times G^r_m$ is proper. We consider an open dense subset $U$ of $Z$ stable under the $G^r_m$-action. Similarly as in (3.7.1), we set

\begin{equation}
X_n^\gamma(g \circ p, U) := \left\{ \varphi \in L^\gamma_n(Z) \mid \text{ord}_t(g \circ p)(\varphi) = n, \text{ord}_t\varphi^*(\mathcal{I}_F) \leq \gamma n \right\},
\end{equation}

with $F := Z \setminus U$. The $G^r_m$-action on $Z$ induces a $G^r_m$-action on $X_n^\gamma(g \circ p, U)$ via its induced action on the arc space. On the other side, the standard $G_m$-action on arcs

\begin{equation}
(\lambda \cdot \varphi)(t) := \varphi(\lambda t)
\end{equation}

induces a $G_m$-action on $X_n^\gamma(g \circ p, U)$. In this way we get a $G^r_m \times G_m$-action on $X_n^\gamma(g \circ p, U)$. The morphism

\begin{equation}
(f \circ \pi_0, ac(g \circ p)) : X_n^\gamma(g \circ p, U) \to G^r_m \times G_m
\end{equation}

is piecewise monomial, hence, proceeding as in 2.7, we may assign to

\begin{equation}
X_n^\gamma(g \circ p, U) \longrightarrow X_0(g) \times G^r_m \times G_m
\end{equation}

a class $[X_n^\gamma(g \circ p, U)]$ in $\mathcal{M}_X^G \times G_m \times G_m$.

Similarly as in (3.7.2), we consider the corresponding series

\begin{equation}
Z_{g \circ p, U}(T) := \sum_{n \geq 1} [X_n^\gamma(g \circ p, U)] L^{-\gamma nd} T^n
\end{equation}

in $\mathcal{M}_X^G \times G_m \times G_m[[T]]$.

Proceeding as in the proof of Proposition 3.8, one proves that there exists a $\gamma_0$ such that for every $\gamma > \gamma_0$ the series $Z_{g \circ p, U}(T)$ belongs to $\mathcal{M}_X^G \times G_m \times G_m[[T]]_{sr}$ and that $\lim_{T \to \infty} Z_{g \circ p, U}(T)$ is independent of $\gamma > \gamma_0$. Indeed, we may assume that the zero locus $Z_0(g \circ p)$ of $g \circ p$ is nowhere dense in $Z$ and in this case we now use a $G^r_m$-equivariant log-resolution of $(Z, (Z \setminus U) \cup Z_0(g \circ p))$. (For the existence of equivariant resolutions, see [11] [10] [11] [28] [29].) We now define $S_{g \circ p, U}$ in $\mathcal{M}_X^G \times G_m \times G_m$ as

\begin{equation}
\lim_{T \to \infty} Z_{g \circ p, U}(T)
\end{equation}

for $\gamma > \gamma_0$.

Still assuming $Z_0(g \circ p)$ is nowhere dense in $Z$, let $h : Y \to Z$ be such a $G^r_m$-equivariant log-resolution. We shall use again the notations introduced in 3.3. By connectedness of $G^r_m$, the $G^r_m$-action on $Z$ induces the trivial action on the set of strata $E_i$, for $I$ subset of $A$. The $G^r_m$-action on $Y$ induces an action on the normal bundles to the divisors $E_i$, for $i$ in $A$, hence on $U_I$, for $I$ subset of $A$. We also consider the $G_m$-action on $U_I$ which is the diagonal action induced by the canonical $G^r_m$-action on $U_I$. In this way we get a $G^r_m \times G_m$-action on $U_I$. Furthermore, with the notation of 3.4, the morphisms $f$ and $g$ induce morphisms $f_I : U_I \to G^r_m$ and $g_I : U_I \to G_m$. Note that the morphism $(f_I, g_I) : U_I \to G^r_m \times G_m$ is monomial with respect to the $G^r_m \times G_m$-action, since $g_I$ is invariant by the $G^r_m$-action and monomial with respect to the $G_m$-action and the morphism $f_I : U_I \to G^r_m$ is induced from $f$
via the projection $U_I \to Z$. We can then consider the class $[U_I]$ in $\mathcal{M}^{G_m \times G_m}_{X_0(g) \times G_m \times G_m}$ of the morphism

$$\tag{3.10.9} (p \circ h \circ \pi_I, f_I, g_I) : U_I \longrightarrow X_0(g) \times G_m^r \times G_m.$$  

Similarly as in Proposition 3.8 we get that the equality

$$\tag{3.10.10} S_{gop,U} = \sum_{I \neq \emptyset, I \subset C} (-1)^{|I|}[U_I]$$

holds in $\mathcal{M}^{G_m \times G_m}_{X_0(g) \times G_m \times G_m}$.

3.11. Remark. When $r = 0$, what is denoted here by $[\mathcal{X}_n^m(g \circ p, U)]$, $Z_{gop,U}^r(T)$ and $S_{gop,U}$ corresponds to what was denoted by $p_n([\mathcal{X}_n^m(g \circ p, U)])$, $p_n(Z_{gop,U}^r(T))$ and $p_n(S_{gop,U})$ in the non equivariant setting. This slight conflict of notation should lead to no confusion.

We can now state the following equivariant analogue of Theorem 3.9.

3.12. Theorem. Let $X$ be a variety with a function $g : X \to \mathbb{A}_k^1$. We consider $X$ endowed with the trivial $G_m^r$-action. There exists a unique $\mathcal{M}_k$-linear group morphism

$$\tag{3.12.1} S_g : \mathcal{M}^{G_m^r \times G_m}_{X \times G_m} \longrightarrow \mathcal{M}^{G_m^r \times G_m}_{X_0(g) \times G_m \times G_m}$$

such that, for every smooth variety $Z$ with good $G_m^r$-action, endowed with an equivariant morphism $p : Z \to X$ and a monomial morphism $f : Z \to G_m^r$ such that the morphism $p, f : Z \to X \times G_m^r$ is proper, and for every open dense subset $U$ of $Z$ which is stable under the $G_m^r$-action

$$\tag{3.12.2} S_g([U \to X \times G_m^r]) = S_{gop,U},$$

in $\mathcal{M}^{G_m^r \times G_m}_{X_0(g) \times G_m \times G_m}$.

Proof. Let us denote by $K'_0(\text{Var}_{X \times G_m})$ the Grothendieck ring defined similarly as $K_0(\text{Var}_{X \times G_m})$, but without relation $\mathcal{2.3.2}$. Let $U$ be a smooth variety over $k$ with a good $G_m^r$-action endowed with an equivariant morphism $\kappa : U \to X$, and with a monomial morphism $f_U : U \to G_m^r$. Note that $U$ may be embedded equivariantly as an open dense subset of a smooth variety $Z$ with good $G_m^r$-action, endowed with an equivariant morphism $p : Z \to X$ extending $\kappa$ and a monomial morphism $f : Z \to G_m^r$ extending $f_U$, such that $(p, f) : Z \to X \times G_m^r$ is proper. Indeed, using the equivalence of categories of Proposition 2.6 and 2.7 it is enough to know that every smooth variety $U_0$ endowed with a good $\hat{\mu}^r$-action and with an equivariant morphism $\kappa_0 : U_0 \to X$, with $X$ endowed with the trivial $\hat{\mu}^r$-action, may be embedded equivariantly as an open dense subset in a smooth variety $Z_0$ with good $\hat{\mu}^r$-action, endowed with a proper equivariant morphism $Z_0 \to X$ extending $\kappa_0$, which follows from the appendix of 5 and also from Sumihiro’s equivariant completion result 26. Hence we can proceed exactly like in the proof of Theorem 3.9 in an equivariant way, getting existence and unicity of a $K_0(\text{Var}_k)$-linear morphism

$$\tag{3.12.3} S_g : K'_0(\text{Var}_{X \times G_m}) \longrightarrow \mathcal{M}^{G_m^r \times G_m}_{X_0(g) \times G_m \times G_m}.$$
such that, for every smooth variety $Z$ with good $G^r_m$-action, endowed with an equivariant morphism $p : Z \to X$ and a monomial morphism $f : Z \to G^r_m$ such that $(p, f) : Z \to X \times G^r_m$ is proper and for every open dense subset $U$ of $Z$ which is stable under the $G^r_m$-action

$$S_g([U \to X \times G^r_m]) = S_{g_{op,U}}$$

in $\mathcal{M}_{G^r_m \times G^r_m \times G^r_m}$. Let us now prove the compatibility of the morphism $S_g$ with the additional relation (2.3.2). Let $U$ be a smooth variety over $k$ endowed with a good $G^r_m$-action, with an equivariant morphism $\kappa : U \to X$, and with a monomial morphism $f_U : U \to G^r_m$. Let $q : B \to U$ be an affine bundle everywhere of rank $s$ with a good affine $G^r_m$-action over the action on $U$. We claim that $U$ may be embedded equivariantly as an open dense subset in a smooth variety $Z$ with good $G^r_m$-action, endowed with an equivariant morphism $p : Z \to X$ extending $\kappa$ and with a monomial morphism $f : Z \to G^r_m$ extending $f_U$, such that $(p, f) : Z \to X \times G^r_m$ is proper and such that, furthermore, the affine bundle $B$ with its affine $G^r_m$-action extends to an affine bundle $\tilde{B} \to Z$ with an affine $G^r_m$-action over the action on $Z$ extending the previous one. Indeed, this follows, using again the equivalence of categories of Proposition 2.1 and 2.7 from Lemma 3.14. To prove that $S_g([B \to X \times G^r_m])$ does not depend on the affine $G^r_m$-action on $B$ over the action on $U$, it is enough to check that

$$S_g([B \to X \times G^r_m]) = L^s S_g([U \to X \times G^r_m]).$$

We may assume $(g \circ p)^{-1}(0)$ is nowhere dense in $Z$. Let $h : Y \to Z$ be a $G^r_m$-log-resolution of $(Z, (Z \setminus U) \cup (g \circ p)^{-1}(0))$. We denote by $E_i$, $i$ in $A$, the irreducible components of $h^{-1}((Z \setminus U) \cup (g \circ p)^{-1}(0))$ and it follows from (3.10.10) that, with the notations of 3.3 and 3.10

$$S_g([U \to X \times G^r_m]) = - \sum_{\emptyset \neq I \subset C} (-1)^{|I|}[U_I]$$

in $\mathcal{M}_{G^r_m \times G^r_m \times G^r_m}$. Let us consider the projective bundle $\lambda : Z' \to Z$ on $Z$, which is the relative projective completion of the bundle $\tilde{B}$. In particular $Z'$ is endowed with a (projective) $G^r_m$-action. We consider the pull-back $Y' \to Y$ of the bundle $Z'$ along the morphism $h$. We get a proper morphism $h' : Y' \to Z'$ which is an equivariant log-resolution of $(Z', (Z' \setminus B) \cup (g \circ p \circ \lambda)^{-1}(0))$. The set of irreducible components of $h'^{-1}((Z' \setminus B) \cup (g \circ p \circ \lambda)^{-1}(0))$ consists of the restriction $E_i'$ of $Y'$ to $E_i$, for $i$ in $A$, together with $H_\infty$, the divisor at infinity of the projective bundle $Y'$. We set $A' := A \cup \{0\}$ and $E'_0 := H_\infty$ in order to use the notations of 3.3 and 3.10 in this setting, adding everywhere $'$ as an exponent. In particular for every non empty subset $I$ of $C'$, we denote by $U_I'$ the corresponding variety with $G^r_m \times G^r_m$-action and with a monomial morphism $(f'_{|I}, g'_{|I}) : U'_I \to G^r_m \times G^r_m$. Since $g \circ p \circ \lambda$ is not identically zero on $H_\infty$, we have $C' = C$. It follows again from (3.10.10) that

$$S_g([B \to X \times G^r_m]) = - \sum_{\emptyset \neq I \subset C} (-1)^{|I|}[U'_I]$$
in \( M_{X_0(g) \times \mathbb{G}_m^r \times \mathbb{G}_m} \). Now remark that the natural morphism \( p_I : U_I' \to U_I \) is an affine bundle of rank \( s \), with an affine \( \mathbb{G}_m^r \times \mathbb{G}_m \)-action over the one on \( U_I \). Furthermore, the monomial morphism \( U_I' \to \mathbb{G}_m^r \times \mathbb{G}_m \) is the composition of the monomial morphism \( U_I \to \mathbb{G}_m^r \times \mathbb{G}_m \) with \( p_I \). One deduces that \( [U_I'] = \mathcal{L}^s [U_I] \) in \( M_{X_0(g) \times \mathbb{G}_m^r \times \mathbb{G}_m} \) and (3.14.5) follows.

One then extends \( S_g \) by \( M_k \)-linearity to a \( M_k \)-linear group morphism

\[
S_g : M^G_{X \times \mathbb{G}_m^r} \to M^G_{X_0(g) \times \mathbb{G}_m^r \times \mathbb{G}_m}
\]

similarly as in the non equivariant case.

3.13. Remark. It follows from our constructions that the morphism \( S_g^\mu_r : M^\mu_r \to M^\mu_{X_0(g)} \) deduced from (3.12.1) via the canonical isomorphism (2.6.3) is compatible with the one constructed by Bittner in [3], modulo the fact that our additional relation is finer than hers. Indeed, they are easily checked to coincide on classes of \( \hat{\mu} \)-equivariant morphisms \( Z \to X \) with \( Z \) smooth and proper. Note also that diagram (3.10.3) is indeed commutative, by construction.

3.14. Lemma. Let \( n \) be in \( \mathbb{N}_{>0} \). Let \( X \) be a \( k \)-variety with trivial \( \mu_n \)-action and let \( U \) be a smooth variety with a good \( \mu_n \)-action and an equivariant morphism \( \kappa : U \to X \). Consider an affine bundle \( B \to U \) with a good affine \( \mu_n \)-action over the action on \( U \). Then there exists an equivariant embedding of \( U \) as a dense open set in a smooth variety \( Z \) with good \( \mu_n \)-action such that \( \kappa \) extends to a proper equivariant morphism \( p : Z \to X \) and the affine bundle \( B \) with its affine \( \mu_n \)-action extends to an affine bundle \( \hat{B} \) on \( Z \) with an affine \( \mu_n \)-action over the action on \( Z \) extending the previous one.

Proof. Set \( G = \mu_n \) and embed \( U \) equivariantly in \( V \) with a good \( G \)-action with \( V \to X \) proper equivariant extending \( \kappa \). The affine bundle \( B \to U \) corresponds to an exact sequence of vector bundles

\[
\begin{align*}
0 \to E & \to F \\
& \to \mathcal{O}_U \\
& \to 0
\end{align*}
\]

on \( U \), such that the sheaf of local sections of the affine bundle is the preimage of 1 in \( F \). The action of \( G \) on \( U \) gives a \( G \)-action on the exact sequence (3.14.1). (By a \( G \)-action on an \( \mathcal{O}_U \)-module \( F \), we mean an isomorphism \( a^*F \to p^*F \) satisfying the cocycle condition, with \( a : G \times U \to U \) the action and \( p : G \times U \to U \) the projection on the second factor.) By blowing up the coherent ideal defining \( V \setminus U \) with the reduced structure we reduce to the case where the inclusion \( j : U \to V \) is affine. By applying \( j_* \) to the exact sequence (3.14.1) and pulling back along \( \mathcal{O}_V \to j_*\mathcal{O}_U \), we extend (3.14.1) to an exact sequence of quasi-coherent sheaves with \( G \)-action on \( V \):

\[
\begin{align*}
0 \to E' & \to F' \\
& \to \mathcal{O}_V \\
& \to 0.
\end{align*}
\]

Let us note that \( F' \) is the direct limit of its \( G \)-invariant coherent subsheaves. Indeed, this follows from Proposition 15.4 of [15], since (quasi-)coherent sheaves on the quotient stack \([V/G]\) correspond to (quasi-)coherent sheaves with \( G \)-action on \( V \).
It follows we may assume the sheaves in (3.14.2) are coherent. By restricting to a \( G \)-stable union of connected components of \( U \), we may also assume the vector bundle \( E \) is of constant rank \( s \) on \( U \). Let \( q : Z \to V \) be obtained by taking an equivariant resolution of the blow up of the \( s \)-th Fitting ideal \( F_s \) of \( E' \), which is also the \( s + 1 \)-th Fitting ideal \( F_{s+1} \) of \( F' \). Applying \( q^* \) to (3.14.2) and modding out by torsion, we get an exact sequence of coherent sheaves with \( G \)-action

\[
0 \to \tilde{E} \to \tilde{F} \to \mathcal{O}_Z \to 0
\]
on \( Z \). Let us note that \( \tilde{E} \) and \( \tilde{F} \) are in fact locally free. Indeed, \( Z \) being normal, \( \tilde{E} \) and \( \tilde{F} \) are locally free outside a closed subvariety of codimension at least 2, but, by construction, the Fitting ideals \( F_s(\tilde{E}) \) and \( F_{s+1}(\tilde{F}) \) are invertible, hence they should be equal to \( \mathcal{O}_Z \). The preimage of 1 in \( \tilde{F} \) is the sheaf of local sections of an affine bundle with \( G \)-action \( B \) on \( Z \) satisfying the required properties. \( \square \)

3.15. **Remark.** The above proof of Lemma 3.14 was explained to us by Ofer Gabber and works in fact for any linear algebraic group \( G \) over \( k \). See also Lemma 7.4 of [2] for a similar, but different, extension lemma.

3.16. **Compatibility with Hodge realization.** We suppose here that \( k = \mathbb{C} \). If \( X \) is a complex algebraic variety, we denote by \( \text{MHM}_X \) the category of mixed Hodge modules on \( X \), as defined in [20]. We denote by \( K_0(\text{MHM}_X) \) the corresponding Grothendieck ring. By additivity, there is a unique \( \mathcal{M}_k \)-linear morphism

\[
(3.16.1) \quad H : \mathcal{M}_X \to K_0(\text{MHM}_X)
\]

such that, for any \( p : Z \to X \) with \( Z \) smooth, \( H([Z]) \) is the class of the full direct image \( Rp_!(\mathbb{Q}_Z) \) in \( K_0(\text{MHM}_X) \), with \( \mathbb{Q}_Z \) the trivial Hodge module on \( Z \). Here we consider \( K_0(\text{MHM}_X) \) as a \( \mathcal{M}_k \)-module via its \( K_0(\text{MHM}_{\text{Spec}}\mathbb{C}) \)-module structure and the Hodge realization map \( H : \mathcal{M}_k \to K_0(\text{MHM}_{\text{Spec}}\mathbb{C}) \). Note that \( H(L) = [\mathcal{Q}_X(-1)] \). If \( \mu_n = \mu_{n_1} \times \cdots \times \mu_{n_r} \) acts on \( Z \), we may consider the automorphisms \( T_1, \ldots, T_r \) on the cohomology objects \( R^ip_!(\mathbb{Q}_Z) \) associated respectively to the action of the element with \( j \)-component \( \exp(2\pi i/n_j) \) and other components 1. If we denote by \( \text{MHM}^{r\text{-mon}}_X \) the category of mixed Hodge modules on \( X \) with \( r \)-commuting automorphism of finite order, we get in this way a morphism

\[
(3.16.2) \quad H : \mathcal{M}_X^{\mu_n} \to K_0(\text{MHM}^{r\text{-mon}}_X).
\]

(That the morphism \( H \) is compatible with the additional relation (2.2.1), follows from the fact that for every affine bundle \( p : A \to Y \) of rank \( s \) with an affine \( \mu_n \)-action above a \( \mu_n \)-action action on \( Y \), there is a canonical equivariant isomorphism \( R^p_!(\mathcal{Q}_A)[2s](s) \simeq \mathcal{Q}_Y \).) If \( g : X \to \mathbb{A}^1 \) is a function, there is a nearby cycle functor \( \psi_g : \text{MHM}_X \to \text{MHM}_{X_0(g)}^{\text{mon}} \), cf. [20] [21], which induces a morphism \( \psi_g : K_0(\text{MHM}_X) \to K_0(\text{MHM}_{X_0(g)}^{\text{mon}}) \). By functoriality the construction extends to morphisms \( \psi_g : K_0(\text{MHM}^{r\text{-mon}}_X) \to K_0(\text{MHM}^{r+1\text{-mon}}_{X_0(g)}) \).
3.17. **Proposition.** For every $r \geq 0$, with the notations from Remark \[\text{3.17} \] the diagram

\[
\begin{array}{ccc}
\mathcal{M}^{\hat{\mu}^r} & \xrightarrow{S^\hat{\mu}^r} & \mathcal{M}^{\hat{\mu}^r+1} \\
\downarrow H & & \downarrow H \\
K_0(\text{MHM}_{X_0}^{r-\text{mon}}) & \xrightarrow{\psi_g} & K_0(\text{MHM}_{X_0(g)}^{r+1-\text{mon}})
\end{array}
\]

is commutative.

**Proof.** It is enough to prove that $H(S^\hat{\mu}^r([Z \to X])) = \psi_g(H([Z \to X]))$ for $p : Z \to X$ proper and $Z$ smooth with $\hat{\mu}^r$-action. We can further reduce to the case $(g \circ p)^{-1}(0)$ is a divisor with normal crossings stable by the $\hat{\mu}^r$-action. In that case, when $r = 0$, the statement is proved in \[\text{3.16} \] Theorem 4.2.1 and Proposition 4.2.3, in a somewhat different language, when $X$ is a point, but the proof carries over with no change to general $X$. Since the constructions in loc. cit. may be performed in an equivariant way in the case of a $\hat{\mu}^r$-action, the proof extends directly to the case $r > 0$. \[\square\]

4. **Iterated vanishing cycles**

4.1. Let $X$ be a variety endowed with the trivial $G_m^r$-action and with a function $g : X \to A^1_k$. Let $U$ be a smooth $k$-variety of pure dimension $d$ with good $G_m^r$-action endowed with an equivariant morphism $\kappa : U \to X$ and with a monomial morphism $\mathbf{f} = (f_1, \ldots, f_r) : U \to G_m^r$. Let $U \to Y$ be an equivariant embedding as a dense open subset of a smooth variety $Y$ with a good $G_m^r$-action and with a proper equivariant morphism $p : Y \to X$. We assume $(g \circ \kappa)^{-1}(0)$ is nowhere dense in $U$. Let $h : W \to Y$ be a $G_m^r$-equivariant log-resolution of $(Y, (Y \setminus U) \cup (g \circ p)^{-1}(0))$. We shall now explain how to compute $\mathcal{S}_g([U \to X \times G_m^r])$ in terms of $W$. Note that the present set-up is different from the one in Theorem \[\text{3.12} \].

We denote by $E_i, i \in A$, the irreducible components of $h^{-1}((Y \setminus U) \cup (g \circ p)^{-1}(0))$. We shall use again the notation \[\text{3.3} \] and \[\text{3.10} \] whenever possible. Let us assume $I \cap C \neq \emptyset$. We can still consider the spaces $U_I$ and the corresponding monomial morphism $g_I : U_I \to G_m$. We denote by $h' : U' \to U$ the preimage of $U$ in $W$ and we set $F := W \setminus U'$. The morphism $\mathbf{f} : U \to G_m^r$ extends to a rational map $\tilde{\mathbf{f}} : Y \dasharrow (P^1_k)^r$. Furthermore, for $i \in A$, there exists integers $N_i(f_j)$ in $\mathbb{Z}$, such that locally on $W$, each component $\tilde{f}_j \circ h$ of $\tilde{\mathbf{f}} \circ h$ may be written as $u \prod_{i \in A} x_i^{N_i(f_j)}$, with $u$ a unit, $x_i$ a local equation of $E_i$. Similarly as what we did for $g_I$, for every $j, 1 \leq j \leq r$, we may define a rational map $f_{j,I} : V_{E_i} \dasharrow P^1_k$, replacing $N_i(g)$ by $N_i(f_j)$, and we still denote by $f_{j,I}$ the induced morphism from $U_I$ to $G_m$. Finally we get a morphism $f_I : U_I \to G_m^r$ which is monomial for the $G_m^r$-action by Lemma \[\text{4.2} \].

Similarly as we already observed in \[\text{3.10} \], this is enough to get that the morphism $(f_I, g_I) : U_I \to G_m^r \times G_m$ is monomial for the $G_m^r \times G_m$-action. We then denote by $[U_I]$ the corresponding class in $\mathcal{M}_G^r \times G_m \times G_m$. 
4.2. Lemma. Let \( W \) be a smooth variety with a good \( \mathbf{G}_m^r \)-action and let \( U \) be a dense open stable by the \( \mathbf{G}_m^r \)-action. We assume \( F := W \setminus U \) is a divisor with normal crossings and we denote by \( E_i, \ i \) in \( A \), its irreducible components. We consider a monomial morphism \( f = (f_1, \ldots, f_r) : U \to \mathbf{G}_m^r \) and we denote by \( \tilde{f} : W \longrightarrow (\mathbb{P}_k^1)^r \) its extension as a rational map. For any non empty subset \( I \) of \( A \), the morphism \( f_I : U_I \to \mathbf{G}_m^r \) defined similarly as above is monomial for the \( \mathbf{G}_m^r \)-action on \( U_I \).

Proof. Consider the deformation \( CW_I \) to the normal cone to \( E_I \) in \( W \) described in (4.3.3). Hence, \( CW_I := \text{Spec} \, A_I \) with

\[
(4.2.1) \quad A_I := \sum_{n \in \mathbb{N}^r} \mathcal{O}_W \times A_k^I \left( -\sum_{i \in I} n_i (E_i \times A_k^I) \right) \prod_{i \in I} u_i^{-n_i}.
\]

Letting \( \mathbf{G}_m^r \) act trivially on each \( u_i \), the \( \mathbf{G}_m^r \)-action on \( \mathcal{O}_W \) induces a \( \mathbf{G}_m^r \)-action on \( A_I \) and on \( CW_I \). For \( i \) in \( A \), we denote by \( J_i \) the ideal of \( A_I \) generated by \( u_i^{-1} \mathcal{O}_W (-E_i) \), resp. \( \mathcal{O}_W (-E_i) \), if \( i \in I \), resp. \( i \notin I \), and we set \( J := \prod_{i \in A} J_i \). We denote by \( CW_I \) the complement in \( CW_I \) of the closed subset defined by \( J \). The fiber \( CW_I \cap p^{-1}(0) \) may be identified equivariantly with \( U_I \) and \( CW_I \) with \( p^{-1}(\mathbf{G}_m^r) \simeq U \times \mathbf{G}_m^r \) (letting \( \mathbf{G}_m^r \) act trivially on \( U \)).

On \( U \times \mathbf{G}_m^r \) we may consider the function \( (x, u_i) \mapsto f_j(x) \prod_{i \in I} u_i^{-N_i(f_j)} \). Similarly as in (3.3) it extends to a morphism \( F_I : CW_I \to \mathbf{G}_m^r \) whose restriction to \( U_I \) coincides with \( f_{j,t,I} \). Let us consider the morphism \( F = (F_1, \ldots, F_r) : CW_I \to \mathbf{G}_m^r \). Since \( f \) is monomial and \( \mathbf{G}_m^r \) acts trivially on \( u_i \), the restriction of \( F \) to the dense open set \( U \times \mathbf{G}_m^r \) is monomial, hence \( F \) is monomial and so is its restriction to \( U_I \).

4.3. Let \( \gamma \) and \( n \) be in \( \mathbb{N}_{>0} \). We keep the notations from (4.1). In particular \( F = h^{-1}(Y \setminus U) \). Let \( \varphi \) be in \( \mathcal{L}_\gamma(W) \) with \( \text{ord}_t \varphi^*(\mathbb{L}_F) \leq \gamma n \) and \( \text{ord}_g(\varphi) = n \). Let \( D \) denote the set consisting of all \( i \) in \( A \) such that \( \varphi(0) \) lies in \( E_i \) and consider a local equation \( x_i = 0 \) of \( E_i \) at \( \varphi(0) \). By hypothesis, \( x_i(\varphi) \) is non zero in \( \mathcal{L}_\gamma(A_k^1) \), so it has a well defined order \( \text{ord}_t(x_i(\varphi)) \) and angular component \( \text{ac}(x_i(\varphi)) \). Writing the component \( \tilde{f}_j \circ h \) of \( \tilde{f} \circ h \) as \( u \prod_{i \in D} x_i^{N_i(f_j)} \), with \( u \) a unit at \( \varphi(0) \), we set

\[
(4.3.1) \quad \text{ord}_t(\tilde{f}_j \circ h)(\varphi) := \sum_{i \in D} N_i(f_j) \text{ord}_t(x_i(\varphi))
\]

and

\[
(4.3.2) \quad \text{ac}(\tilde{f}_j \circ h)(\varphi) := u(\varphi(0)) \prod_{i \in D} \text{ac}(x_i(\varphi))^{N_i(f_j)}.
\]

By abuse of notation, we write \( (\tilde{f} \circ h)(\varphi) \in \mathbf{G}_m^r \) to mean \( \text{ord}_t(\tilde{f}_j \circ h)(\varphi) = 0 \) for every \( 1 \leq j \leq r \).

Now we consider the constructible set

\[
(4.3.3) \quad \mathcal{W}_n^\gamma := \left\{ \varphi \in \mathcal{L}_\gamma(W) \mid \text{ord}_g(\varphi) = n, \text{ord}_t \varphi^*(\mathbb{L}_F) \leq \gamma n, (\tilde{f} \circ h)(\varphi(0)) \in \mathbf{G}_m^r \right\}.
\]

Similarly as the set in (3.1.3), \( \mathcal{W}_n^\gamma \) is endowed with a \( \mathbf{G}_m^r \times \mathbf{G}_m \)-action and furthermore the morphism \( (\text{ac}(\tilde{f}_j \circ h), \text{ac}(g)) : \mathcal{W}_n^\gamma \to \mathbf{G}_m^r \times \mathbf{G}_m \) is piecewise monomial.
We denote by \([W_n^\gamma]\) the corresponding class in \(M_{X_0(g) \times G_m \times G_m}^{G_n^{\gamma}}\). Let us consider the series
\[
W^\gamma(T) := \sum_{n \geq 1} [W_n^\gamma] L^{-\gamma n} T^n
\]
in \(M_{X_0(g) \times G_m \times G_m}^{G_m^{\gamma}}[[T]]\).

For \(I\) a non empty subset of \(A\), we consider the cone
\[
\Gamma(I) := \left\{ x \in R^I_{>0} \mid \forall j \in \{1, \ldots, r\}, \sum_{i \in I} x_i N_i(f_j) = 0 \right\}
\]
and we denote by \(d(I)\) its dimension. We shall also consider the cone
\[
M_\gamma := \left\{ x \in R^I_{>0} \mid \sum_{i \in I} x_i N_i(T_F) \leq \gamma \sum_{i \in I \cap C} x_i N_i(g) \right\}.
\]

We denote by \(\Delta\) the set of non empty subsets \(I\) of \(A\) such that \(\Gamma(I)\) is non empty and is contained in \(M_\gamma\) for \(\gamma \gg 0\).

4.4. Proposition. Let \(X\) be a variety with trivial \(G_m^r\)-action and with a function \(g : X \to A_1^r\). Let \(U\) be a smooth \(k\)-variety of pure dimension \(d\) with good \(G_m^r\)-action endowed with an equivariant morphism \(\kappa : U \to X\), and with a monomial morphism \(f = (f_1, \ldots, f_r) : U \to G_m^r\). Let \(U \to Y\) be an equivariant embedding as a dense open subvariety of a smooth variety \(Y\) with good \(G_m^r\)-action and with a proper equivariant morphism \(p : Y \to X\). We assume \((g \circ \kappa)^{-1}(0) = \text{nowhere dense in } U\). Let \(h : W \to Y\) be a \(G_m^r\)-equivariant log-resolution of \((Y, (Y \setminus U) \cup (g \circ \kappa)^{-1}(0))\). There exists \(\gamma_0\) such that for every \(\gamma > \gamma_0\) the series \(W^\gamma(T)\) lies in \(M_{X_0(g) \times G_m}^{G_m^{\gamma}}[[T]][s]\) and \(\lim_{T \to \infty} W^\gamma(T)\) is independent of \(\gamma > \gamma_0\). Furthermore, if one sets \(W = -\lim_{T \to \infty} W^\gamma(T)\), the following holds
\[
W = -\sum_{I \in \Delta} (-1)^{d(I)} [U_I]
\]
in \(M_{X_0(g) \times G_m}^{G_m}[[T]]\).

Proof. Similarly as in the proof of Proposition 3.8 we have
\[
W^\gamma(T) = \sum_{I \cap C \neq \emptyset} [U_I] S_I(T)
\]
with
\[
S_I(T) = \sum_{k \in \Gamma(I) \cap M_\gamma \cap N_{I_{>0}}^I} \prod_{i \in I} (T^{N_i(g)} L^{-1})^{k_i}.
\]
The proof now goes on as the proof of Proposition 3.8 with \(N_{I_{>0}}^I\) replaced by \(\Gamma(I) \cap N^I_{>0}\). Indeed, note that the linear form \(\sum_{i \in I \cap C} k_i N_i(g)\) is positive on \(M_\gamma \setminus \{0\}\) and that \(M_\gamma\) is empty if \(I \cap C = \emptyset\). Assume first \(I\) lies in \(\Delta\) and \(I \cap C \neq \emptyset\). Then it follows from 2.9 that \(\lim_{T \to \infty} S_I(T) = (-1)^{d(I)}\) for \(\gamma \gg 0\). Assume now \(I \cap C \neq \emptyset\) and \(I \notin \Delta\). In this case, necessarily, for \(\gamma > 0\), the hyperplane \(\sum_{i \in I} k_i N_i(T_F) = \gamma \sum_{i \in I \cap C} k_i N_i(g)\)
has a non empty intersection with $\Gamma(I)$. It follows that the Euler characteristic $\chi(\Gamma(I) \cap M_r)$ is equal to zero.

4.5. Proposition. Let $X$ be a variety with trivial $G_m^r$-action and with a function $g : X \to \mathbb{A}^1_k$. Let $U$ be a smooth $k$-variety of pure dimension $d$ with good $G_m^r$-action endowed with an equivariant morphism $\kappa : U \to X$, and with a monomial morphism $f = (f_1, \ldots, f_r) : U \to G_m^r$. Let $U \to Y$ be an equivariant embedding as a dense open subvariety of a smooth variety $Y$ with good $G_m^r$-action and with a proper equivariant morphism $p : Y \to X$. We assume $(g \circ \kappa)^{-1}(0)$ is nowhere dense in $U$.

Let $h : W \to Y$ be a $G_m^r$-equivariant log-resolution of $(Y, (Y \setminus U) \cup (g \circ p)^{-1}(0))$. Then, with the previous notation, we have

$$\mathcal{S}_g([U \to X \times G_m^r]) = -\sum_{I \in \Delta} (-1)^{d(I)} [U_I]$$

in $\mathcal{M}_{X_0(0) \times G_m^r \times G_m}$.

Proof. We may reduce to the case where the morphism $f : U \to G_m^r$ extends to a morphism $\tilde{f} : Y \to (P_k^1)^r$. Indeed, there exists an equivariant embedding $U \to Y'$ of $U$ as a dense open subvariety of a smooth variety $Y'$ with a good $G_m^r$-action and with a proper equivariant morphism $p' : Y' \to X$ such that $f$ extends to a morphism $\tilde{f}' : Y' \to (P_k^1)^r$. We may furthermore assume there is a $G_m^r$-equivariant proper morphism $Y' \to Y$ which restricts to the identity on $U$. Let $h' : W' \to Y'$ be a $G_m^r$-equivariant log-resolution of $(Y', (Y' \setminus U) \cup (g \circ p')^{-1}(0))$. We may also assume there is a $G_m^r$-equivariant proper morphism $W' \to W$ such that the diagram

$$\begin{array}{ccc}
W' & \xrightarrow{h'} & Y' \\
\downarrow & & \downarrow \\
W & \xrightarrow{h} & Y
\end{array}$$

is commutative.

Consider $W'$ defined as $W$ but using $W'$ instead of $W$. Since, temporarily, we shall work on $W'$ and not on $W$, we denote by $E_i$, $i \in A$, the irreducible components of $h'^{-1}((Y' \setminus U) \cup (g \circ p')^{-1}(0))$, and keep the previous notation, but for $W'$ instead of $W$. We have

$$W' = \lim_{T \to \infty} \sum_{I \cap C \neq 0} [U_I] S_I(T),$$

while, computing $W^\gamma(T)$ on $W'$ using the change of variable formula, or more precisely Lemma 3.4 in [6], one gets

$$W = \lim_{T \to \infty} \sum_{I \cap C \neq 0} [U_I] \tilde{S}_I(T)$$

with

$$\tilde{S}_I(T) = \sum_{k \in \Gamma(I) \cap M_r} \prod_{i \in I} (T^{N_i(g)} L^{-m_i}) k_i,$$
with $m_i \geq 1$. It follows that $W' = W$, and by Proposition 4.4 we can assume $Y = Y'$ and $W = W'$.

Consider $Z := (\tilde{f} \circ h)^{-1}(G_{m}^{r})$ in $W$. Note that the image of the morphism $Z \to W \times G_{m}^{r}$ given by the inclusion on the first factor and by the restriction of $\tilde{f} \circ h$ on the second factor is the closure of the image of the inclusion $U' \to W \times G_{m}^{r}$, with $U'$ the preimage of $U$ in $W$. It follows that the morphism $(q, \tilde{f} \circ h_{|Z}) : Z \to X \times G_{m}^{r}$ given by composition with $p \circ h$ on the first factor is proper. Since $Z$ is smooth and the morphism $Z \to X \times G_{m}^{r}$ is proper, it follows from (3.12.2) that

\begin{equation}
S_{g}([U \to X \times G_{m}^{r}]) = S_{g \circ U'}
\end{equation}

in $\mathcal{M}_{X_{0}(g) \times G_{m}^{r} \times G_{m}}^{G_{m}^{r} \times G_{m}^{r}}$. Note also that, since $f : U \to G_{m}^{r}$ extends to a morphism $\tilde{f} : Y \to (P_{0}^{k})^{r}$, for a subset $I$ of $A$ with $I \cap C \neq \emptyset$, $\Gamma(I)$ is non empty if and only if $E_{i}^{r}$ is contained in $(\tilde{f} \circ h)^{-1}(G_{m}^{r})$. Furthermore if these conditions hold, $\Gamma(I) = B^{I}_{>0}$. It follows that $\Delta$ consists exactly of those non empty subsets of $C$ for which $E_{i}^{r}$ is contained in $(\tilde{f} \circ h)^{-1}(G_{m}^{r})$, hence the right hand side of (4.5.1) may be rewritten as

\begin{equation}
\sum_{\emptyset \neq I \subseteq C \cap (\tilde{f} \circ h)^{-1}(G_{m}^{r})} (-1)^{|I|} [U_{I}],
\end{equation}

and the required equation (4.5.1) follows now from (4.5.6) and (3.10.10). \qed

4.6. Iterated vanishing cycles. Now we consider a smooth variety $X$ of pure dimension $d$ with two functions $f : X \to \mathbb{A}_{k}^{1}$ and $g : X \to \mathbb{A}_{k}^{1}$. The motivic Milnor fiber $S_{f}$ lies in $\mathcal{M}_{X_{0}(f) \times G_{m}}^{G_{m}^{r}}$. We still denote by $g$ the function $X_{0}(f) \times G_{m} \to \mathbb{A}_{k}^{1}$ obtained by composition of $g$ with the projection $X_{0}(f) \times G_{m} \to X$. Hence, thanks to 3.10 we may consider the image

\begin{equation}
S_{g}(S_{f}) = S_{g}(S_{f}([X \to X]))
\end{equation}

of $S_{f} = S_{f}([X \to X])$ by the nearby cycles morphism

\begin{equation}
S_{g} : \mathcal{M}_{X_{0}(f) \times G_{m}}^{G_{m}^{r}} \to \mathcal{M}_{X_{0}(f) \times X_{0}(g)}^{G_{m}^{r} \times G_{m}^{r}}
\end{equation}

which lies in $\mathcal{M}_{X_{0}(f) \times X_{0}(g) \times G_{m}^{r}}^{G_{m}^{r} \times G_{m}^{r}}$.

We shall now give an explicit description of $S_{g}(S_{f})$ in terms of a log-resolution $h : Y \to X$ of $(X, X_{0}(f) \cup X_{0}(g))$. We shall denote by $E_{i}$, $i$ in $A$, the irreducible components of $h^{-1}(X_{0}(f) \cup X_{0}(g))$ and we shall consider the sets

\begin{equation}
B = \left\{ i \mid N_{i}(f) > 0 \right\} \quad \text{and} \quad C = \left\{ i \mid N_{i}(g) > 0 \right\}.
\end{equation}

Recall, cf. 3.3 that we denoted by $U_{I}^{j}$, for $J \subseteq I$, the fiber product of the restrictions of the $G_{m}$-bundles $U_{E_{i}}$, for $i$ in $J$, to $E_{i}^{r}$. Assume $J := I \cap C$ and $K := I \setminus C$ are both non empty. We now consider the fiber product $U_{K,J} := U_{K}^{K} \times_{E_{i}^{r}} U_{I}^{J}$ which has the same underlying variety than $U_{I}$. There is a natural $G_{m}^{2}$-action on $U_{K,J}$, the first, resp. second, $G_{m}$-action being the diagonal action on $U_{K}^{K}$, resp. $U_{I}^{J}$, and the trivial one on the other factor. The morphism $(f_{I}, g_{I}) : U_{I} = U_{K,J} \to G_{m}^{2}$ being
monomial, the morphism \((h \circ \pi, f_t, g_t) : U_t \to (X_0(f) \cap X_0(g)) \times G_m^2\) has a class in \(\mathcal{M}_{(X_0(f) \cap X_0(g)) \times G_m^2}\) that we denote by \([U_{K,J}]\).

**4.7. Theorem.** With the previous notations, we have

\[
S_g(S_f) = \sum_{K \cap C = \emptyset \atop K \neq \emptyset} (-1)^{|K|} [U_K \to X_0(f) \times G_m] \tag{4.7.1}
\]

in \(\mathcal{M}_{(X_0(f) \cap X_0(g)) \times G_m^2}\).

**Proof.** Consider the inclusions \(i : X_0(g) \times G_m \hookrightarrow X \times G_m\) and \(j : (X \setminus X_0(g)) \times G_m \hookrightarrow X \times G_m\). Note that \(S_f - j_!(S_{f(X \setminus X_0(g))})\) is supported by \(X_0(g) \times G_m\); that is, is of the form \(i_!(\mathcal{A})\). Since \(Y \setminus Y_0(g \cdot \bar{h})\) is a log-resolution of \((X \setminus X_0(g), X_0(f) \setminus X_0(g))\), one deduces from the proof of (4.6.3) that

\[
S_f - \left(- \sum_{K \cap C = \emptyset \atop K \neq \emptyset} (-1)^{|K|} [U_K \to X_0(f) \times G_m] \right) \tag{4.7.2}
\]

is supported by \(X_0(g) \times G_m\); hence, since \(S_g\) is zero on objects of the form \(i_!(\mathcal{A})\), we deduce that

\[
S_g(S_f) = S_g \left(- \sum_{K \cap C = \emptyset \atop K \neq \emptyset} (-1)^{|K|} [U_K \to X_0(f) \times G_m] \right). \tag{4.7.3}
\]

To conclude it is enough to check the following equality in \(\mathcal{M}_{(X_0(f) \cap X_0(g)) \times G_m^2}\), for every non empty subset \(K\) of \(A\) such that \(K \cap C = \emptyset\):

\[
S_g([U_K \to X_0(f) \times G_m]) = - \sum_{\emptyset \neq J \subset C} (-1)^{|J|}[U_{K,J}] \tag{4.7.4}
\]

This will follow from Proposition 4.3. Indeed, let us consider the projective bundle \(\pi_K : \mathcal{V}_{E_K} \to E_K\) with the \(G_m\)-action extending the diagonal one on \(\nu_{E_K}\). Let us set \(A' := A \cup \{\infty\}\). The complement of \(U_K\) in \(\mathcal{V}_{E_K}\) is a divisor with normal crossings whose irreducible components are:

- the divisors \(E'_j := \pi_K^{-1}(E_K \cup \{j\})\), for \(j \notin K\) such that \(E_{K \cup \{j\}} \neq \emptyset\)
- the divisor at infinity \(E'_\infty := \mathcal{V}_{E_K} \setminus \nu_{E_K}\)
- the divisors \(E'_i\), for \(i \in K\), defined as the closure of the fiber product, above \(E_K\), of the zero section of \(\nu_{E_i}\) with the \(\nu_{E_i}\), \(\ell \in K\), \(\ell \neq i\).

Note that all these divisors are stable by the \(G_m\)-action. We shall use the notations of 3.3 and 3.4 with an exponent \(\ell\).

We now determine the set \(\Delta\) of non empty subsets \(J'\) of \(A'\) such that \(\Gamma(J')\) is non empty and is contained in \(M_{\gamma}\) for \(\gamma >> 0\), with the notation of (4.3.5) and (4.3.6).

Note that for \(\Gamma(J')\) to be non empty it is necessary that if \(N_i(f_K) > 0\) (resp. \(N_i(f_K) < 0\)) for some \(i \in J'\), then for some \(i' \in J'\), \(N_{i'}(f_K) < 0\) (resp. \(N_{i'}(f_K) > 0\)). This forces \(J'\) to be either of the form \(J \cup \{\infty\}\) with \(J \cap B \neq \emptyset\) or of the form \(J\) with \(J \cap B = \emptyset\). In each case, the condition that \(\Gamma(J')\) is contained in \(M_{\gamma}\) for \(\gamma >> 0\) implies that \(J \subset C\) and furthermore that \(d(J') = |J|\). We deduce that \(J \cup \{\infty\}\)
belongs to $\Delta$ if and only $J \cap B \neq \emptyset$ and $J \subset C$ and that $J$ belongs to $\Delta$ if and only $J \cap B = \emptyset$, $J \subset C$ and $J \neq \emptyset$. It follows from Proposition 4.5 that

\[(4.7.5) \quad S_g([U_K \to X_0(f) \times G_m]) = - \sum_{\emptyset \neq J \subset C} (-1)^{|J|} [U'_J] - \sum_{J \subset C} (-1)^{|J|} [U'_{J \cup \{\infty\}}]\]

in $\mathcal{M}_{\mathcal{X}_0(g) \times G_m \times G_m}^{G_m \times G_m}$. To conclude it is enough to note that if $\emptyset \neq J \subset C$ and $J \cap B = \emptyset$, then $[U'_J] = [U_{K,J}]$ and that if $J \subset C$ and $J \cap B \neq \emptyset$ we have $[U'_{J \cup \{\infty\}}] = [U_{K,J}]$.

Let us prove the second equality. We consider the image $P(U_{E_K})$ of $U_{E_K}$ in $P(\nu_{E_K})$ and note that the canonical morphism $U_{E_K} \to P(U_{E_K})$ is a $G_m$-bundle, namely the restriction to $U_{E_K}$ of the tautological line bundle on $P(\nu_{E_K})$. We identify $E''_\infty := E'_\infty \setminus \cup_{i \in K} E'_i$ with $P(U_{E_K})$. The restriction of the tautological line bundle to $P(U_{E_K})$ is dual to the restriction to $E''_\infty$ of the normal bundle to $E'_\infty$ in $\mathfrak{t}_{E_K}$.

We have now two $G_m$-bundles on $E''_\infty = P(U_{E_K})$, namely $U_{E_K}$ and the restriction, we shall denote by $U''_\infty$, to $E''_\infty$ of the complement $U_{E'_\infty}$ of the zero section in the normal bundle $\nu_{E'_\infty}$. Let us denote by $U_{E_K}$ the $G_m$-bundle $U_{E_K}$ endowed with the inverse $G_m$-action. The antipody $a : U_{E_K} \to U_{E_K}^g$ whose restriction to the fibers is given by $t \mapsto t^{-1}$ is an isomorphism of $G_m$-bundles with $G_m$-action. By the above description, $U''_\infty$ may be identified, as a $G_m$-bundle with $G_m$-action, with the $G_m$-bundle $U_{E_K}^g$.

Now consider the function $f_K$ on $\nu_{E_K}$. It induces a rational map $\tilde{f}_K$ on $\mathfrak{t}_{E_K}$. Let us check that under the above isomorphism, the restriction $f_K : U_{E_K} \to A^1_k$ composed with the automorphism $a$ corresponds to the morphism $f''_\infty : U''_\infty \to A^1_k$ obtained from $\tilde{f}_K$ by the construction of . Indeed, let $U$ be an open subset of $E_K$ above which the bundle $\nu_{E_K}$ is trivial, isomorphic to $U \times A^1_K$. Denote by $w_i$, for $i \in K$, the coordinates on $A^1_K$. Fix $\ell$ in $K$. The restriction of $U_{E_K}$ to $U$ may be identified, equivariantly, with $U \times P(G^K_m) \times G_m$ by $(u,(w_i)_{i \in K}) \mapsto (u,(x_i = \frac{w_i}{u})_{i \in K \setminus \ell}, t = w_\ell)$, where $G_m$ acts trivially on the first two factors and by multiplicative translation on the last one, with $(x_i)_{i \in K \setminus \ell}$ the standard coordinates on $P(G^K_m) \simeq G^K_m \setminus \ell$. If the restriction of $f_K$ to $U \times G^K_m$ is given by $v(u) \prod_{i \in K \setminus \ell} w_i^{N_i}$, it may be rewritten, under the above identification, as $v(u) \prod_{i \in K \setminus \ell} x_i^{N_i} t^{-\sum_{i \in K \setminus \ell} N_i}$. Composing with the antipody $a$ we get the function $v(u) \prod_{i \in K \setminus \ell} x_i^{N_i} t^{t^{-\sum_{i \in K \setminus \ell} N_i}}$ which corresponds to the restriction of the function $f''_\infty$ on the corresponding open subset.

If $J$ is a subset of $C$ such that $E_{K \cup J} \neq \emptyset$, it follows from the “transitivity” property described in \ref{transitivity} that $f_{K \cup J}$ can be retrieved directly from $f_K : \nu_{E_K} \to A^1_k$ and similarly, the rational map $f'_{J \cup \{\infty\}}$ can be retrieved directly from the rational map $f''_\infty$ (obtained from $\tilde{f}_K$ by the construction of \ref{transitivity} on $\nu_{E'_\infty}$). It follows that, under the isomorphism between $U_{K \cup J}$ and $U'_{J \cup \{\infty\}}$ induced by $\varphi$, $f_{K \cup J}$ corresponds to $f'_{J \cup \{\infty\}}$. The same argument works for the functions induced by $g$ on $U_{K \cup J}$ and $U_{J \cup \{\infty\}}$ (note that in fact $N_i(g) = 0$ for all $i \in K$ and $N_\infty(g) = 0$).

The first equality, which is easier, is checked similarly using $E'_j = \pi_K^{-1}(E_{K \cup J})$ and the canonical isomorphism of bundles $\nu_{E'_j} \simeq (\pi_K|E'_j)^*(\nu_{E_j}|E_{K \cup J})$, for $J \subset C$. \hfill $\square$
5. Convolution and the main result

5.1. Convolution. Let us denote by $a$ and $b$ the coordinates on each factor of $G_m^2$. Let $X$ be a variety. We denote by $i : X \times (a+b)^{-1}(0) \to X \times G_m^2$ the inclusion of the antidiagonal and by $j$ the inclusion of its complement. We consider the morphism

\[(5.1.1) \quad a + b : X \times G_m^2 \setminus (a+b)^{-1}(0) \to X \times G_m\]

which is the identity on the $X$-factor and is equal to $a+b$ on the $G_m^2 \setminus (a+b)^{-1}(0)$-factor. We denote by $pr_1$ and $pr_2$ the projection of $X \times G_m \times (a+b)^{-1}(0)$ on $X \times G_m$ and $X \times (a+b)^{-1}(0)$, respectively.

If $A$ is an object in $\mathcal{M}_{X \times G_m^2}$, the object

\[(5.1.2) \quad \Psi^0_A(A) := -(a+b)j^*(A) + pr_1pr_2i^*(A)\]

lives in $\mathcal{M}_{X \times G_m}$. We now explain how to lift $\Psi^0$ to a $\mathcal{M}_k$-linear group morphism $\Psi^0 : \mathcal{M}_{X \times G_m^2}^G \to \mathcal{M}_{X \times G_m}^G$.

Let $A$ be an object in $\text{Var}_{X \times G_m^2}^G$ with class $[A]$ in $\mathcal{M}_{X \times G_m^2}^G$. It is endowed with a $G_m^2$-action $\alpha$ for which the morphism to $G_m^2$ is diagonally monomial of weight $(n, m)$. We may consider the $G_m$-action $\tilde{\alpha}$ on $A$ given by $\tilde{\alpha}(\lambda)x = \alpha(\lambda^n, \lambda^m)x$. With some obvious abuse of notations, $(a+b)^{j^*(A)}$ is the class of $a+b : A_{\{a+b\neq0\}} \to X \times G_m$. If we endow $A_{\{a+b\neq0\}}$ with the $G_m$-action induced by $\tilde{\alpha}$, the morphism $(a+b) : A_{\{a+b\neq0\}} \to G_m$ is diagonally monomial of weight $nm$. The term $pr_1pr_2i^*(A)$ is the class of $A_{\{a+b\neq0\}} \times G_m \to X \times G_m$, the morphism to $G_m$ being the projection on the $G_m$-factor. We endow $A_{\{a+b\neq0\}} \times G_m$ with the $G_m$-action induced by $\tilde{\alpha}$ on the first factor and the action $(\lambda, z) \mapsto \lambda^{nm}z$ on the second factor. Hence we may set

\[(5.1.3) \quad \Psi^m_A([A]) = -[a+b : A_{\{a+b\neq0\}} \to X \times G_m] + [A_{\{a+b\neq0\}} \times G_m \to X \times G_m] \text{ in } \mathcal{M}_{X \times G_m}^{G_m^2 \times G_m}\]

and extend this construction in a unique way to a $\mathcal{M}_k$-linear group morphism $\Psi^m$. These morphisms being compatible with the morphisms induced by the transition morphisms of (2.5.1), we get after passing to the colimit a $\mathcal{M}_k$-linear group morphism

\[(5.1.4) \quad \Psi^m : \mathcal{M}_{X \times G_m^2}^G \to \mathcal{M}_{X \times G_m}^G\]

Let us now explain the relation of $\Psi^m$ with the convolution product as considered in [7, 10, and 8]. There is a canonical morphism

\[(5.1.5) \quad \mathcal{M}_{X \times G_m}^G \times \mathcal{M}_{X \times G_m}^G \to \mathcal{M}_{X \times G_m^2}^G\]

sending $(A, B)$ to $A \boxtimes B$, the fiber product over $X$ of $A$ and $B$, therefore we may define

\[(5.1.6) \quad * : \mathcal{M}_{X \times G_m}^G \times \mathcal{M}_{X \times G_m}^G \to \mathcal{M}_{X \times G_m}^G\]

by

\[(5.1.7) \quad A * B = \Psi^m(A \boxtimes B).\]
If $S$ is in $\text{Var}_X^\mu$, resp. in $\text{Var}_X^{\mu_1 \times \mu_2}$, we denote by $[S]$ the corresponding class in $\mathcal{M}_X^G$, resp. in $\mathcal{M}_X^{G_2 \times G_2}$, via the isomorphism (2.6.3). Consider the Fermat curves $F^n_0$ and $F^n_1$ defined respectively by $x^n + y^n = 1$ and $x^n + y^n = 0$ in $G_2^m$ with their standard $\mu_n \times \mu_n$-action. If $A$ is a variety in $\text{Var}_X^{\mu_1 \times \mu_2}$, we have

$$\Psi_S([A]) = -[F^n_1 \times_{\mu_n \times \mu_n} A] + [F^n_0 \times_{\mu_n \times \mu_n} A],$$

the $\mu_n$-action on each term in the right hand side of (5.1.8) being the diagonal one. In particular, if $A$ and $B$ are two varieties in $\text{Var}_X^{\mu_1}$, the convolution product $[A] \ast [B]$ is given by

$$[A] \ast [B] = -[F^n_1 \times_{\mu_n \times \mu_n} (A \times_X B)] + [F^n_0 \times_{\mu_n \times \mu_n} (A \times_X B)].$$

The convolution product in [16] and [8] was defined when $k$ contains all roots of unity. Since as soon as $k$ contains a $n$-th root of $-1$ we have $[F^n_0 \times_{\mu_n \times \mu_n} (A \times_X B)] = (L - 1)[(A \times_X B)/\mu_n]$, one gets that the convolution product in [16] and [8], when defined, coincides with the one in (5.1.9).

5.2. Proposition. The convolution product on $\mathcal{M}_X^{G_m}$ is commutative and associative. The unit element for the convolution product is $1$, the class of the identity $X \times G_m \to X \times G_m$ with the standard $G_m$-action on the $G_m$-factor.

Proof. Commutativity being clear, let us prove the statement concerning associativity and unit element. For simplicity of notation we shall assume $X$ is a point and we shall first ignore the $G_m$-actions, that is we shall prove the corresponding statements for $\mathcal{M}_G$. Consider $a : A \to G_m$, $b : B \to G_m$, $c : C \to G_m$. By definition the convolution product $A \ast B$ (with some abuse of notation, we shall denote by the same symbol varieties over $G_m$ and their class in $\mathcal{M}_G$) is equal to

$$-[a + b : (A \times B)|_{a+b \neq 0} \to G_m] + [z : (A \times B \times G_m)|_{a+b=0} \to G_m],$$

with $z$ the standard coordinate on $G_m$.

Associativity follows from the following claim: $(A \ast B) \ast C$ is equal to

$$[a + b + c : (A \times B \times C)|_{a+b+c \neq 0} \to G_m] - [z : (A \times B \times C \times G_m)|_{a+b+c=0} \to G_m].$$

Indeed, $(A \ast B) \ast C$ may be written as a sum of four terms. The first one,

$$[a + b + c : (A \times B \times C)|_{a+b+c \neq 0} \to G_m]$$

may be rewritten as

$$[a + b + c : (A \times B \times C)|_{a+b+c \neq 0} \to G_m] - [c : (A \times B \times C)|_{a+b=0} \to G_m].$$

The second one,

$$-[z : (A \times B \times C \times G_m)|_{a+b+c=0} \to G_m],$$

may be rewritten as

$$-[z : (A \times B \times C \times G_m)|_{a+b+c=0} \to G_m].$$

The third one

$$-[c + z : (A \times B \times C \times G_m)|_{a+b=0} \to G_m].$$
may be rewritten as
\begin{equation}
(5.2.8) \quad -[u : (A \times B \times C \times G_m)_{a+b=0} \to G_m],
\end{equation}
since the corresponding spaces are isomorphic via \((\alpha, \beta, \gamma, z) \mapsto (\alpha, \beta, \gamma, u = c(\gamma) + z)\). Here \(u\) is a coordinate on some other copy of \(G_m\). The fourth term,
\begin{equation}
(5.2.9) \quad [u : (A \times B \times C \times G_m \times G_m)_{a+b=0} \to G_m]
\end{equation}
may be rewritten as
\begin{equation}
(5.2.10) \quad [u : (A \times B \times C \times G_m)_{a+b=0} \to G_m].
\end{equation}
One deduces (5.2.2) by summing up (5.2.4), (5.2.6), (5.2.8), and (5.2.10).

For the statement concerning the unit element, one writes \(A \ast G_m\) as
\begin{equation}
(5.2.11) \quad -[a + z : (A \times G_m)_{a+z \neq 0} \to G_m] + [u : (A \times G_m \times G_m)_{a+z=0} \to G_m].
\end{equation}
Since the first term may be rewritten as
\begin{equation}
(5.2.12) \quad -[u : (A \times G_m)_{a \neq u} \to G_m]
\end{equation}
and the second term as
\begin{equation}
(5.2.13) \quad [u : (A \times G_m) \to G_m],
\end{equation}

it follows that \(A \ast G_m\) is equal to (the class of) \(A\) in \(\mathcal{M}_{G_m}\). The proofs for general \(X\) are just the same. As for \(G_m\)-actions, since by the very constructions they are diagonally monomial of the same weight on each factor, all identifications we made are compatible with the \(G_m\)-actions, and all statements still hold in \(\mathcal{M}_{X \times G_m}^{G_m}\). \(\square\)

5.3. **Remark.** Proposition 5.2, modulo the isomorphism \((2.6.3)\), is already stated in \(\S\).

5.4. In fact, associativity already holds at the \(\Psi]\Sigma\)-level. To formulate this, we need to introduce some more notation.

Let us denote by \(a, b\) and \(c\) the coordinates on each factor of \(G_m^3\). For \(X\) a variety, we denote by \(i\) the inclusion \(X \times (a + b + c)^{-1}(0) \hookrightarrow X \times G_m^3\) and by \(j\) the inclusion of the complement. We consider the morphism
\begin{equation}
(5.4.1) \quad a + b + c : X \times G_m^3 \setminus (a + b + c)^{-1}(0) \longrightarrow X \times G_m
\end{equation}
which is the identity on the \(X\)-factor and is equal to \(a + b + c\) on the \(G_m^3 \setminus (a + b + c)^{-1}(0)\)-factor. We denote by \(pr_1\) and \(pr_2\) the projection of \(X \times G_m \times (a + b + c)^{-1}(0)\) on \(X \times G_m\) and \(X \times (a + b + c)^{-1}(0)\), respectively.

If \(A\) is an object in \(\mathcal{M}_{X \times G_m}^{G_m}\), we consider the object
\begin{equation}
(5.4.2) \quad \Psi_{\Sigma 123}(A) := (a + b + c)_{ij}^*(A) - pr_1^*pr_2^*i^*(A),
\end{equation}
in \(\mathcal{M}_{X \times G_m}\). Similarly as in \(5.1\) we extend \(\Psi_{\Sigma 123}\) to a \(\mathcal{M}_k\)-linear group morphism
\(\Psi_{\Sigma 123} : \mathcal{M}_{X \times G_m}^{G_m} \to \mathcal{M}_{X \times G_m}^{G_m}\). We denote by \(A_{ijk}\) the object \(A\) viewed as an element in \(\mathcal{M}_{X \times G_m}^{G_m}\) by forgetting the projection and the action corresponding to the \(k\)-th \(G_m\)-factor, with \(\{i, j, k\} = \{1, 2, 3\}\). The object \(\Psi_\Sigma(A_{ijk})\) may now be endowed with
a second projection to $G_m$ and a second $G_m$-action, namely those corresponding to the $k$-th $G_m$-factor, so we get in fact an element in $\mathcal{M}_{X \times \mathbb{G}_m}^{G_m}$; we denote by $\Psi_{\Sigma_{ij}}(A)$.

5.5. **Proposition.** Let $A$ be an object in $\mathcal{M}_{X \times \mathbb{G}_m}^{G_m}$. For every $1 \leq i < j \leq 3$, we have

$$\Psi_{\Sigma_{123}}(A) = \Psi_{\Sigma}(\Psi_{\Sigma_{ij}}(A)).$$

**Proof.** The proof is the same as the one for associativity in Proposition 5.2. Indeed, one just has to replace everywhere $A \times B \times C$ by $A$ in the proof, and to remark that (5.2.2) then becomes nothing else than $\Psi_{\Sigma_{123}}(A)$. □

5.6. Let us consider again a smooth variety $X$ of pure dimension $d$ with two functions $f$ and $g$ from $X$ to $A_k$. Let us denote by $i_1$ and $i_2$ the inclusion of $(X_0(f) \cap X_0(g)) \times \mathbb{G}_m$ in $X_0(f) \times \mathbb{G}_m$ and $X_0(f + g^N) \times \mathbb{G}_m$, respectively.

We can now state the main result of this paper.

5.7. **Theorem.** Let $X$ be a smooth variety of pure dimension $d$, and $f$ and $g$ be two functions from $X$ to $A_k$. For every $N > \gamma((f), (g))$, the equality

$$i_1^*S_f^\phi - i_2^*S_{f+g^N}^\phi = \Psi_{\Sigma}(S_{g^N}(\mathcal{S}_f^\phi))$$

holds in $\mathcal{M}_{(X_0(f) \cap X_0(g)) \times \mathbb{G}_m}^{G_m}$.

**Proof.** Let $\varphi$ be in $\mathcal{L}(X)$. A basic observation is that when the inequality $\text{ord}_t f(\varphi) < \text{Ord}_t g(\varphi)$ holds, $f(\varphi)$ and $(f + g^N)(\varphi)$ have same order $\text{ord}_t$ and same angular coefficient ac. If $A$ is a subset of $\mathcal{L}_n(X)$, we denote by $A^+$, resp. $A^0$, the intersection of $A$ with the set of arcs in $\mathcal{L}_0(X)$ such that $\text{ord}_t f(\varphi) > \text{Ord}_t g(\varphi)$, resp. $\text{ord}_t f(\varphi) = \text{Ord}_t g(\varphi)$. In this way one defines series

$$Z_f^+(T) = \sum_{n \geq 1} [\mathcal{X}_n^+(f)] L^{-ndT^n}$$

and

$$Z_f^0(T) = \sum_{n \geq 1} [\mathcal{X}_n^0(f)] L^{-ndT^n}$$

in $\mathcal{M}_{X_0(f) \times \mathbb{G}_m}^{G_m} [[T]]$ and similarly series $Z_{f+g^N}^+(T)$ and $Z_{f+g^N}^0(T)$ in $\mathcal{M}_{X_0(f+g^N) \times \mathbb{G}_m}^{G_m} [[T]]$.

It follows from the previous remark that

$$i_1^*Z_f(T) - i_2^*Z_{f+g^N}(T) = i_1^*(Z_f^+(T) + Z_f^0(T)) - i_2^*(Z_{f+g^N}^+(T) + Z_{f+g^N}^0(T)),$$

where we extend $i_1^*$ and $i_2^*$ to series componentwise.

Let $N$ be a positive integer. For any integer $r$, we denote by $\pi_N$ the morphism $X_0(g) \times \mathbb{G}_m^r \times \mathbb{G}_m \rightarrow X_0(g) \times \mathbb{G}_m^r \times \mathbb{G}_m$ mapping $(x, \mu, \lambda)$ to $(x, \mu, \lambda^N)$. Then we have

5.8. **Lemma.** Given a map $g : X \rightarrow A_k$ and the induced nearby cycles morphism $\mathcal{S}_g$ from $\mathcal{M}_{X_0(g) \times \mathbb{G}_m}^{G_m}$ to $\mathcal{M}_{X_0(g) \times \mathbb{G}_m^r \times \mathbb{G}_m}$, then, for any positive integer $N$, the following equality holds:

$$\mathcal{S}_{g^N} = \pi_N \circ \mathcal{S}_g.$$
Proof. Let $Z$ be a smooth variety with good $G^r_m$-action, endowed with an equivariant morphism $p : Z \to X$ and a monomial morphism $f : Z \to G^r_m$ such that the morphism $(p, f) : Z \to X \times G^r_m$ is proper, and let $U = Z \setminus F$ be an open dense subset of $Z$ which is stable under the $G^r_m$-action. For a positive integer $\gamma$, with the notations of 3.10, let us consider the modified zeta function of $g$ on $U$

\[
Z_{g^{N \circ p, U}}^\gamma(T) := \sum_{n \geq 1} [\mathcal{X}_n^\gamma(g^N \circ p, U)] L^{-\gamma nd T^n}
\]

in $\mathcal{M}_{X_0(g) \times G^r_m \times G_m} T]$. Since $\mathcal{X}_n^\gamma(g^N \circ p, U)$, is empty unless $N$ divides $n$ and

\[
[X_m^\gamma \circ (g^N \circ p, U)] = \pi_{N!}([X_m^\gamma \circ (g \circ p, U)]),
\]

we get that $Z_{g^{N \circ p, U}}^\gamma(T)$ is equal to $\pi_{N!}(Z_{g^{N \circ p, U}}^\gamma(T^N))$, the limit of which, as $T$ goes to infinity, is equal, for $\gamma$ big enough, to $\pi_{N!}(\mathcal{S}_{g^{N \circ p, U}})$. The result follows from Theorem 3.12. \hfill \Box

5.9. Lemma. Assume $N > \gamma((f), (g))$.

Then the series $i_2^*(Z_{f + g^N}^+(T))$ lies in $\mathcal{M}_{(X_0(f) \times X_0(g)) \times G_m} [[T]]_{sr}$ and

\[
\lim_{T \to \infty} i_2^*(Z_{f + g^N}^+(T)) = -\mathcal{S}_{g^N}(\{X_0(f)\}).
\]

Proof. Note that $X_n^+(f + g^N)$ is non empty only if $n$ is a multiple of $N$ and that

\[
[X_n^+(f + g^N)] = \pi_{N!}(\{ \varphi \in L_m(X) | \text{ord}_i g(\varphi) = m, \text{ord}_f (\varphi) > Nm \}),
\]

the variety on the right hand side being endowed with the morphism to $G_m$ induced by $ac(g)$. Summing up, we may write by 5.7.2 and the proof of Lemma 5.8

\[
Z_{f + g^N}^+(T) = \pi_{N!}(Z_g(T^N) - Z_{g, X \setminus X_0}^N(T^N)).
\]

By Proposition 3.8 and its proof, for $N > \gamma((f), (g))$, the series $Z_{g, X \setminus X_0}^N(T)$ lies in $\mathcal{M}_{X_0(f) \times X_0(g)} [[T]]_{sr}$ and its limit $T \to \infty$ is equal to $-\mathcal{S}_{g, X \setminus X_0}$. The same holds for $Z_{g, X \setminus X_0}^N(T^N)$ and the result follows since $\pi_{N!}(\mathcal{S}_g - \mathcal{S}_{g, X \setminus X_0})$ is equal to $\mathcal{S}_{g^N}(\{X_0(f)\})$ by Lemma 5.8. \hfill \Box

5.10. We fix an integer $N$ such that $N > \gamma((f), (g))$ and a log-resolution $h : Y \to X$ of $(X, X_0(f) \cup X_0(g))$ such that $N > \gamma_h((f), (g))$. We keep the notations used in 3.3 and 4.0. In particular, $NN_i(g) > N_i(f)$ for $i \in C$. Note that the stratum $E_i^0$ is contained in $(g \circ h)^{-1}(0)$ if and only if $J = I \cap C$ is non empty.

Fix a non empty stratum $E_i^0$ in $Y$.

We consider the cones $\Delta_i^+$ and $\Delta_i^0$ in $\mathbb{R}_{\geq 0}$ defined respectively by

\[
\sum_{i \in I} N_i(g) x_i > N \sum_{j \in J} N_j(g) x_j
\]

and

\[
\sum_{i \in I} N_i(f) x_i = N \sum_{j \in J} N_j(g) x_j.
\]
Note that when $K = I \setminus C$ is empty, $I = J$ and $N_i(f) < NN_i(g)$ for all $i$, hence the cones $\Delta^+_i$ and $\Delta^0_i$ are both empty.

As in (3.8.2), we have

\[
(5.10.3) \quad i^*_1(Z^+_f(T) + Z^0_f(T)) = \sum_{\substack{I \cap B \neq \emptyset \atop I \cap C \neq \emptyset}} [U_I] \Psi_I(T)
\]

with

\[
(5.10.4) \quad \Psi_I(T) = \sum_{k \in (\Delta^+_I \cup \Delta^0_I) \cap \mathbb{N}^I_{>0}} L^{-\sum_{i \in I} \nu_i k_i} T^{\sum_{i \in I} N_i(f) k_i}.
\]

Since

\[
(5.10.5) \quad \lim_{T \to \infty} \Psi_I(T) = \chi(\Delta^+_I \cup \Delta^0_I) = 0,
\]

we deduce that

\[
(5.10.6) \quad \lim_{T \to \infty} i^*_1(Z^+_f(T) + Z^0_f(T)) = 0.
\]

5.11. We now want to compute the zeta function $Z^0_{f+g^N}(T)$. Fix $k$ in $\Delta^0_I \cap \mathbb{N}^I_{>0}$ and denote by $\phi$ the finite morphism from $\mathbb{A}^1_k$ to $\mathbb{A}^1_k$ sending $u$ to $(u^k)$. We shall still denote by $\phi$ its restriction as a group morphism from $\mathbb{G}_m$ to $\mathbb{G}_m$. Taking the pullback along $\phi$ of the deformation to the normal cone to $E_I$ in $Y$, $p_I : CY_I \to \mathbb{A}^1_k$, introduced in 3.51, one gets a morphism $p : CY_k \to \mathbb{A}^1_k$ having the following description. The scheme $CY_k$ may be identified with $\text{Spec } \mathcal{A}_k$ where

\[
(5.11.1) \quad \mathcal{A}_k := \sum_{n \in \mathbb{N}^I} \mathcal{O}_{Y \times \mathbb{A}^1_k} \left(-\sum_{i \in I} n_i (E_i \times \mathbb{A}^1_k)\right) u^{-\sum_{i \in I} k_i n_i}
\]

is a subsheaf of $\mathcal{O}_{Y \times \mathbb{A}^1_k}[u^{-1}]$, and the natural inclusion $\mathcal{O}_{Y \times \mathbb{A}^1_k} \to \mathcal{A}_k$ induces a morphism $\pi : CY_k \to Y \times \mathbb{A}^1_k$ from which $p$ is derived. Via the same inclusion, the functions $f \circ h$, $g^N \circ h$ and $(f + g^N) \circ h$ are divisible by $u^{\sum_{i \in I} k_i N_i(f)}$ in $\mathcal{A}_k$. We denote the quotients by $\tilde{f}_k$, $\tilde{g}_k$ and $\tilde{F}_k$, respectively.

We denote by $\tilde{E}_i$ the pullback of the divisor $E_i \times \mathbb{A}^1_k$ by $\pi$, by $D$ the divisor globally defined on $CY_k$ by $u = 0$, and by $CE_i$, the divisors $\tilde{E}_i - k_i D$, $i$ in $I$ (resp. $\tilde{E}_i$, $i$ not in $I$). We denote by $CY_k^\circ$ the complement in $CY_k$ of the union of the $CE_i$, $i$ in $A$, and by $Y^\circ$ the complement in $Y$ of the union of the $E_i$, $i$ in $A$. We denote by $F_I$ the function $f_I + g^N_I : U_I \to \mathbb{A}^1_k$.

5.12. Lemma. The scheme $CY_k$ is smooth, the morphism $\pi$ induces an isomorphism above $\mathbb{A}^1_k \setminus \{0\}$, the morphism $p$ is a smooth morphism and its fiber $p^{-1}(0)$ may be naturally identified with the bundle $\nu_{E_I}$. When restricted to $CY_k^\circ$, the fiber of $p$ above 0 is naturally identified with $U_I$ and $\pi$ induces an isomorphism between $CY_k^\circ \setminus p^{-1}(0)$ and $Y^\circ \times \mathbb{A}^1_k \setminus \{0\}$. The restriction of $\tilde{f}_k$ (resp. $\tilde{g}_k$, $\tilde{F}_k$) to the fiber $U_I \subset p^{-1}(0)$ is equal to $f_I$ (resp. $g_I$, $F_I$).
Proof. Since $CY_k$ is covered by open subsets of the form $\text{Spec } O_U[y_i,u]/(z_i - u^{k_i}y_i)$ with $U$ open subset on which the divisors $E_i$ are defined by equations $z_i = 0$, the smoothness of $CY_k$ is clear. The remaining properties are checked directly. \hfill \Box

The $G'_m$-action $\sigma_I$ on $CY_I$ induces via $\phi$ a $G_m$-action on $CY_k$ we denote by $\sigma$, leaving sections of $O_Y$ invariant and acting on $u$ by $\sigma(\lambda) : u \mapsto \lambda^{-1}u$. Note that in coordinate charts such as in the proof of Lemma 5.12, $\sigma$ leaves $z_i$ invariant and $\sigma(\lambda)$ maps $y_i$ to $\lambda^{k_i}y_i$. We have now two different $G_m$-actions on $\mathcal{L}_n(CY_k^o)$: the one induced by the standard $G_m$-action on arc spaces and the one induced by $\sigma$. We denote by $\tilde{\sigma}$ the action given by the composition of these two (commuting) actions.

For $\varphi$ in $\mathcal{L}_n(Y)$ with $\varphi(0)$ in $E_i$, we set $\text{ord}_{E_i} \varphi := \text{ord}_iz_i(\varphi)$, for $z_i$ any local equation of $E_i$ at $\varphi(0)$.

Let us denote by $\tilde{\mathcal{L}}_n(CY_k^o)$ the set of arcs $\varphi$ in $\mathcal{L}_n(CY_k^o)$ such that $p(\varphi(t)) = t$ (in particular $\varphi(0)$ is in $U_I$). For such an arc $\varphi$, composition with $\pi$ sends $\varphi$ to an arc in $\mathcal{L}_n(Y \times A_k^1)$ which is the graph of an arc in $\mathcal{L}_n(Y)$ not contained in the union of the divisors $E_i$, $i$ in $I$. Note that $\tilde{\mathcal{L}}_n(CY_k^o)$ is stable by $\tilde{\sigma}$.

We will consider $\mathcal{X}_n,k$, the set of arcs $\varphi$ in $\mathcal{L}_n(Y)$ such that $\varphi(0)$ is in $E_I^o$ and $\text{ord}_{E_i} \varphi = k_i$ for $i$ in $I$.

5.13. Lemma. Assume $n \geq k_i$ for $i$ in $I$. The morphism $\tilde{\pi} : \tilde{\mathcal{L}}_n(CY_k^o) \to \mathcal{X}_n,k$ induced by the projection $CY_k^o \to Y$ is an affine bundle with fiber $A_k^{\sum_i k_i}$. Furthermore if $\tilde{\mathcal{L}}_n(CY_k^o)$ is endowed with the $G_m$-action induced by $\tilde{\sigma}$ and $\mathcal{X}_n,k$ with the standard $G_m$-action, $\tilde{\pi}$ is $G_m$-equivariant and the action of $G_m$ on the affine bundle is affine.

If $n \geq \sum_i k_iN_i(f)$, the composed maps $ac(\psi \circ h)(\tilde{\pi}(\varphi))$ and $ac(g \circ h)(\tilde{\pi}(\varphi))$ are equal respectively to $f_I(\varphi(0))$ and $g_I(\varphi(0))$, whereas

\begin{equation}
(5.13.1) \quad ac((f + g^N) \circ h)(\tilde{\pi}(\varphi)) = ac(\tilde{F}_k(\varphi)).
\end{equation}

Furthermore, when $F_I(\varphi(0)) \neq 0$, hence $(\text{ord}_i(f + g^N) \circ h)(\tilde{\pi}(\varphi)) = \sum_i k_iN_i(f)$, we have

\begin{equation}
(5.13.2) \quad ac((f + g^N) \circ h)(\tilde{\pi}(\varphi)) = F_I(\varphi(0)).
\end{equation}

Proof. Every point in $E_i^o$ is contained in a open subset $U$ of $Y$ such that the divisors $E_i$, $i$ in $I$ are defined by equations $z_i = 0$ in $U$ and such that there exists furthermore $d-|I|$ functions $w_j$ on $U$ such that the family $(z_i, w_j)$ gives rise to an étale morphism $U \to A_k^{d}$. This morphism induces an isomorphism $\mathcal{L}_n(U) \simeq U \times A_k^d \mathcal{L}_n(A_k^d)$, cf. Lemma 4.2 of [6]. Adding further the coordinate $u$, gives an isomorphism $\mathcal{L}_n(U \times A_k^1) \simeq (U \times A_k^1) \times A_k^{d+1} \mathcal{L}_n(A_k^{d+1})$. The family $(y_i, w_j, u)$, with $z_i = y_iu^{k_i}$, induces an étale morphism $\pi^{-1}(U \times A_k^d) \to A_k^{d+1}$, hence an isomorphism $\mathcal{L}_n(\pi^{-1}(U \times A_k^d)) \simeq (\pi^{-1}(U \times A_k^d)) \times A_k^{d+1} \mathcal{L}_n(A_k^{d+1})$. Under these isomorphisms $\tilde{\pi}$ just corresponds to multiplying each $y_i$-component of an arc by $t^{k_i}$. Note in particular that in that description the action of $\tilde{\sigma}(\lambda)$ on a component $y_i(t)$ is given by $y_i(t) \mapsto \lambda^{k_i}y_i(\lambda t)$, hence $\tilde{\pi}$ is $G_m$-equivariant. The rest of the statement follows also directly from that description. \hfill \Box
We define \( \mathcal{Y}_{n,k} \) as the subset of \( \mathcal{X}_{n,k} \) consisting of those arcs \( \varphi \) such that \( \text{ord}_i((f + g^N) \circ h)(\varphi) = n \). The constructible set \( \mathcal{Y}_{n,k} \) is stable by the usual \( \mathbb{G}_m \)-action on \( \mathcal{L}_n(Y) \) and the morphism \( \text{ac}(f + g^N) \) defines a class \([\mathcal{Y}_{n,k}]\) in \( \mathcal{M}_{(X_0(f) \times X_0(g)) \times \mathbb{G}_m}^G \). By definition \( \mathcal{Y}_{n,k} = \emptyset \) if \( n < \sum_j k_i N_i(f) \).

We then define \( \tilde{\mathcal{Y}}_{n,k} \) as the preimage of \( \mathcal{Y}_{n,k} \) by the fibration \( \tilde{\pi} \) of Lemma 5.13. It consists of arcs \( \varphi \in \mathcal{L}_n(CY_k^\infty) \) such that \( \text{ord}_i(\tilde{F}_k)(\varphi) = n - \sum_j k_i N_i(f) \). We denote by \([\tilde{\mathcal{Y}}_{n,k}]\) the class of \( \tilde{\mathcal{Y}}_{n,k} \) in \( \mathcal{M}_{(X_0(f) \times X_0(g)) \times \mathbb{G}_m}^G \), the morphism \( \tilde{\mathcal{Y}}_{n,k} \to \mathbb{G}_m \) being \( \text{ac}(\tilde{F}_k) \) and the \( \mathbb{G}_m \)-action being induced by \( \tilde{\sigma} \). We denote by \([U_I \setminus (F_I^{-1}(0))]\) the class of \( U_I \setminus (F_I^{-1}(0)) \) in \( \mathcal{M}_{(X_0(f) \times X_0(g)) \times \mathbb{G}_m}^G \), the \( \mathbb{G}_m \)-action being the natural diagonal action of weight \( k \) on \( U_I \setminus (F_I^{-1}(0)) \) and the morphism to \( \mathbb{G}_m \) being the restriction of \( F_I \).

We also consider the class \([\mathbb{G}_m \times F_I^{-1}(0)]\) of \( \mathbb{G}_m \times F_I^{-1}(0) \) in \( \mathcal{M}_{(X_0(f) \times X_0(g)) \times \mathbb{G}_m}^G \), the \( \mathbb{G}_m \)-action on the second factor being the diagonal one and the morphism to \( \mathbb{G}_m \) being the first projection.

5.14. Lemma. The following equalities hold in \( \mathcal{M}_{(X_0(f) \times X_0(g)) \times \mathbb{G}_m}^G \):

1. \([\tilde{\mathcal{Y}}_{n,k}] = L^{nd} [U_I \setminus (F_I^{-1}(0))]\), if \( n = \sum_j k_i N_i(f) \),
2. \([\tilde{\mathcal{Y}}_{n,k}] = L^{nd-m} [\mathbb{G}_m \times F_I^{-1}(0)]\), if \( n - \sum_j k_i N_i(f) = m > 0 \).

Proof. If \( n = \sum_j k_i N_i(f) \), \( \tilde{\mathcal{Y}}_{n,k} \) is the set of arcs \( \varphi(t) \) in \( \mathcal{L}_n(CY_k^\infty) \) such that \( \varphi(0) \) lies in \( U_I \setminus (F_I^{-1}(0)) \) and \( u(\varphi(t)) = t \), and (1) follows.

If \( n - \sum_j k_i N_i(f) = m > 0 \), \( \tilde{\mathcal{Y}}_{n,k} \) is the set of arcs \( \varphi \) in \( \mathcal{L}_n(CY_k^\infty) \) such that \( \text{ord}_i(\tilde{F}_k)(\varphi) = m \) and \( u(\varphi(t)) = t \). Now let us observe that the morphism \( (\tilde{F}_k, u) : CY_k^\infty \to \mathbb{A}_k^\infty \) is smooth on a neighborhood of \( U_I \) in \( CY_k^\infty \), since \( u \) is a smooth function on \( CY_k^\infty \) and the restriction of \( \tilde{F}_k \) to the divisor \( u = 0 \), identified with \( U_I \), is \( F_I = f_I + g_I^N \) which is a smooth function on \( U_I \). The fact that \( F_I = f_I + g_I^N \) is a smooth function on \( U_I \) is checked locally as follows: for \( i \) in \( I \setminus C \) (recall \( I \setminus C \) is non empty), and with local coordinates as above,

\[
(y_i \frac{\partial}{\partial y_i} (f_I + g_I^N) = y_i \frac{\partial f_I}{\partial y_i} = N_i(f) f_I
\]

do not vanish on \( U_I \). \( \square \)

5.15. Lemma. We have

\[
i^*_2 [\mathcal{A}_n^G(\varphi + g^N)] = \sum_{n > 0} \sum_{\cap C = J \neq \emptyset} \sum_{k \in \Delta^J_0 \cap \mathcal{N}_{>0}^\ell} [\mathcal{Y}_{n,k}] L^{-\sum_{i \in I} (n_i - 1) k_i}.
\]

Proof. This is a standard application of the change of variable formula, or more precisely of Lemma 3.4 in [6]. The proof is completely similar to the proof of Theorem 2.4 of [6]. (Recall that \( \Delta^J_0 \) is empty if \( K \) is empty.) \( \square \)

It follows from Lemma 5.15 and Lemma 5.13 that

\[
i^*_2 Z^0_{f + g^N}(T) = \sum_{n > 0} \sum_{\cap C = J \neq \emptyset} \sum_{k \in \Delta^J_0 \cap \mathcal{N}_{>0}^\ell} [\tilde{\mathcal{Y}}_{n,k}] L^{-\sum_{i \in I} n_i k_i L^{-nd} T^n}.
\]
Using Lemma 5.14 we deduce

\[(5.15.3) \quad i_2^* \mathcal{Z}_{f+g}^0(T) = \sum_{\substack{I \subseteq \Gamma \setminus J \neq \emptyset \\ I \subseteq \Gamma \setminus K \neq \emptyset}} \left( [U_I \setminus (F^{-1}_I(0))] + [G_m \times F^{-1}_I(0)] \right) \frac{L^{-1}T}{1-L^{-1}T} \Phi_I(T), \]

with

\[(5.15.4) \quad \Phi_I(T) = \sum_{k \in \Delta_i \cap N_{I_0}} L^{-\sum_i \nu_i k_i} T^{\sum_i N_i(f)k_i}. \]

Since \(\Phi_I(T)\) lies in \(\mathbb{Z}[L, L^{-1}][[T]]_{sr}\) and

\[(5.15.5) \quad \lim_{T \rightarrow \infty} \Phi_I(T) = \chi(\Delta_I) = (-1)^{|I|-1}, \]

\(i_2^* \mathcal{Z}_{f+g}^0(T)\) lies in \(\mathcal{M}^{G_m}_{(X_0(f) \cap X_0(g)) \times G_m}[[T]]_{sr}\). Using Theorem 4.17 we deduce from \(5.15.3\) and \(5.15.5\) that

\[(5.15.6) \quad \lim_{T \rightarrow \infty} i_2^* \mathcal{Z}_{f+g}^0(T) = \Psi_{\Sigma}(S_{g \times}(S_f)). \]

We deduce from \(5.7.4, 5.9.1, 5.15.6\) and \(5.10.6\) that

\[(5.15.7) \quad i_2^* S_f - i_2^* S_{f+g} = \Psi_{\Sigma}(S_{g \times}(S_f)) - S_{g \times}([X_0(f)]). \]

By Proposition 5.2 \(S_{g \times}([X_0(f)]) = \Psi_{\Sigma}(S_{g \times}([G_m \times X_0(f)])),\) hence the statement of the Theorem follows from \(5.15.7\), since \(i_2^* S_f - i_2^* S_{f+g} = (-1)^{d-1} (i_2^* S_f - i_2^* S_{f+g}^\phi). \)

\[\square\]

If \(f\) is a function on the smooth variety \(X\) of pure dimension \(d\) and \(x\) is a closed point of \(X_0(f)\), we write \(S_{f,x}\) for \(i_2^* S_f\), and \(S_{f,x}^\phi\) for \(i_2^* S_f^\phi\), where \(i_x\) stands for the inclusion of \(x\) in \(X_0(f)\). Note that \(S_{f,x}^\phi = (-1)^{d-1}(S_{f,x} - [G_m \times \{x\}]).\)

Theorem 5.7 has the following local corollary:

**5.16. Corollary.** Let \(X\) be a smooth variety of pure dimension \(d\), and \(f\) and \(g\) be two functions from \(X\) to \(A^1_k\). Let \(x\) be a closed point of \(X_0(f) \cap X_0(g)\). For every \(N > \gamma_x((f),(g))\), the equality

\[(5.16.1) \quad S_{f,x}^\phi - S_{f+g,x}^\phi = \Psi_{\Sigma}(S_{g \times,x}(S_f)) \]

holds in \(\mathcal{M}^{G_m}_{G_m}\).

**Proof.** The only point to be checked is that \(\gamma((f),(g))\) may be replaced by the local invariant \(\gamma_x((f),(g))\), which is clear from the proof of Theorem 5.7 \(\square\)

**5.17.** Let us now explain how to deduce from Theorem 5.7 the motivic Thom-Sebastiani Theorem of [7], [16] and [8].

Let \(X\) and \(Y\) be two varieties over \(k\). For \(r\) and \(s\) in \(N\), cartesian product gives rise to an external product

\[(5.17.1) \quad \boxtimes : \mathcal{M}_{X \times G_m}^{G_m} \times \mathcal{M}_{Y \times G_m}^{G_m} \longrightarrow \mathcal{M}_{X \times Y \times G_m}^{G_m+r+s}\]

(not to be confused with the one in 5.1.5).
Let $X_1$ and $X_2$ be smooth varieties and consider functions $f_1 : X_1 \to \mathbb{A}^1_k$ and $f_2 : X_2 \to \mathbb{A}^1_k$. We set $X_0 = f_1^{-1}(0) \times f_2^{-1}(0)$ and, for any $Y \subset X_1 \times X_2$ containing $X_0$ we denote by $i$ the inclusion of $X_0 \times G_m$ in $Y \times G_m$.

5.18. **Theorem.** Let $X_1$ and $X_2$ be smooth varieties of pure dimension $d_1$ and $d_2$ and consider functions $f_1 : X_1 \to \mathbb{A}^1_k$ and $f_2 : X_2 \to \mathbb{A}^1_k$. Denote by $f_1 \oplus f_2$ the function on $X_1 \times X_2$ sending $(x_1, x_2)$ to $f_1(x_1) + f_2(x_2)$. Then

$$i^* S^\phi_{f_1 \oplus f_2} = \Psi_\Sigma(S^\phi_{f_1} \boxtimes S^\phi_{f_2})$$

in $\mathcal{M}_{X_0 \times G_m}^{G_m}$. 

**Proof.** We set $X = X_1 \times X_2$ and we denote by $f$ and $g$ the functions on $X$ induced by $f_1$ and $f_2$, respectively. In particular $f_1 \oplus f_2 = f + g$. If $Y_1 \to X_1$ is a log-resolution of $(X_1, f_1^{-1}(0))$ and $Y_2 \to X_2$ is a log-resolution of $(X_2, f_2^{-1}(0))$, $h : Y := Y_1 \times Y_2 \to X$ is a log-resolution of $(X, f^{-1}(0) \cup g^{-1}(0))$. Using such a log-resolution it is easily checked that $\gamma((f), (g)) = 0$. By (5.15.7),

$$i^* S_f - i^* S_{f+g} = \Psi_\Sigma(S_g(S_f)) - i^* S_g([X_0(f)]).$$

Using the log-resolution $h$ one checks that $i^* S_f = S_{f_1} \boxtimes [f_2^{-1}(0)], S_g(S_f) = S_{f_1} \boxtimes S_{f_2}$ and $i^* S_g([X_0(f)]) = [f_1^{-1}(0)] \boxtimes S_{f_2}$. Hence (5.18.2) may be rewritten as

$$\Psi_\Sigma(S_{f_1} \boxtimes S_{f_2}) = S_{f_1} \boxtimes [f_2^{-1}(0)] + [f_1^{-1}(0)] \boxtimes S_{f_2} - i^* S_{f_1 \oplus f_2}.$$

Since $S_{f_1} \boxtimes [f_2^{-1}(0)] = \Psi_\Sigma(S_{f_1} \boxtimes [f_2^{-1}(0) \times G_m])$ and $[f_1^{-1}(0)] \boxtimes S_{f_2} = \Psi_\Sigma([f_1^{-1}(0) \times G_m] \boxtimes S_{f_2})$ (cf. the proof of the statement concerning the unit element in Proposition 5.2), (5.18.1) directly follows, by definition of $S^\phi$. \hfill \Box

6. **Spectrum and the Steenbrink conjecture**

6.1. We now assume $k = \mathbb{C}$. We denote by HS the abelian category of Hodge structures and by $K_0(\text{HS})$ the corresponding Grothendieck ring (see, eg, [8] for definitions). Note that any mixed Hodge structure has a canonical class in $K_0(\text{HS})$. Recall there is a canonical morphism

$$\chi_h : \mathcal{M}_C \to K_0(\text{HS}),$$

which assigns to the class of a variety $X$ the element $\sum_i (-1)^i [H^i_c(X, \mathbb{Q})]$ in $K_0(\text{HS})$, where $[H^i_c(X, \mathbb{Q})]$ stands for the class of the mixed Hodge structure on $H^i_c(X, \mathbb{Q})$. Let us denote by $\text{HS}^{\text{mon}}$ the abelian category of Hodge structures endowed with an automorphism of finite order and by $K_0(\text{HS}^{\text{mon}})$ the corresponding Grothendieck ring. Let us consider the ring morphism

$$\chi_h : \mathcal{M}_C^{\text{spec}} \to K_0(\text{HS}^{\text{mon}})$$

deduced from (5.16.2) via (2.6.3) and composition with $K_0(\text{MHM}_{\text{spec}}^\text{mon}) \to K_0(\text{HS}^{\text{mon}})$. It is described as follows: if $[X]$ is the class of $f : X \to G_m$ in $\mathcal{M}_C^{\text{spec}}$ with $X$ connected, since $f$ is monomial with respect to the $G_m$-action, $f$ is a locally trivial fibration for the complex topology. Furthermore, if the weight is, say, $n$, $x \mapsto \exp(2\pi it/n)x$ is a geometric monodromy of finite order along the origin. It
follows that $X_1$, the fiber of $f$ at $1$, is endowed with an automorphism of finite order $T_f$, and we have

$$\chi_h([f : X \to G_m]) = \left(\sum_i (-1)^i[H^i_c(X_1, \mathbb{Q})], T_f\right).$$

There is a natural linear map, called the Hodge spectrum,

$$\text{hsp} : K_0(\text{HS}^{\text{mon}}) \to \mathbb{Z}[\mathbb{Q}],$$

such that

$$\text{hsp}([H]) = \sum_{\alpha \in \mathbb{Q} \cap (0,1)} t^{\alpha}(\sum_{p,q} \dim(H^{p,q}_\alpha)t^p),$$

for any Hodge structure $H$ with an automorphism of finite order, where $H^{p,q}_\alpha$ is the eigenspace of $H^{p,q}$ with respect to the eigenvalue $\exp(2\pi i \alpha)$. We identify here $\mathbb{Z}[\mathbb{Q}]$ with $\bigcup_{n \geq 1} \mathbb{Z}[t^{1/n}, t^{-1/n}]$.

We shall consider the composite morphism

$$\text{Sp} := (\text{hsp} \circ \chi_h) : \mathcal{M}_{G_m}^{\mathbb{Q}} \to \mathbb{Z}[\mathbb{Q}].$$

Note that $\text{Sp}$ is a ring morphism for the convolution product $*$ on $\mathcal{M}_{G_m}^{\mathbb{Q}}$, by Lemma 6.3.

Denoting by $\text{HS}^{2-\text{mon}}$ the abelian category of Hodge structures endowed with two commuting automorphisms of finite order and by $K_0(\text{HS}^{2-\text{mon}})$ the corresponding Grothendieck ring, one deduce from (3.16.2) via (2.6.3) a ring morphism

$$\chi_h : \mathcal{M}_{G_m}^{\mathbb{Q}} \to K_0(\text{HS}^{2-\text{mon}})$$

having a description similar to (6.1.3).

Also we can define a Hodge spectrum on $K_0(\text{HS}^{2-\text{mon}})$ as follows. Denote by $\pi : [0,1) \cap \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ the restriction of the projection $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ and by $s : \mathbb{Q}/\mathbb{Z} \to [0,1) \cap \mathbb{Q}$ its inverse. The bijection $\mathbb{Q}/\mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$ sending $(a,b)$ to $s(a)+b$ induces an isomorphism of abelian groups between $\mathbb{Z}[\mathbb{Q}/\mathbb{Z} \times \mathbb{Z}]$ and $\mathbb{Z}[\mathbb{Q}]$. We define the spectrum

$$\text{hsp} : K_0(\text{HS}^{2-\text{mon}}) \to \mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^2 \times \mathbb{Z}]$$

by

$$\text{hsp}([H]) = \sum_{\alpha \in \mathbb{Q} \cap (0,1)} \sum_{\beta \in \mathbb{Q} \cap (0,1)} \sum_{p,q} \left(\dim H^{p,q}_{\alpha,\beta}\right)t^{\pi(\alpha)}u^{\pi(\beta)}v^p,$$

with $H^{p,q}_{\alpha,\beta}$ the eigenspace of $H^{p,q}$ with respect to the eigenvalue $\exp(2\pi i \alpha)$ for the the first automorphism and $\exp(2\pi i \beta)$ for the the second automorphism. We shall denote by $\text{Sp}$ the morphism of abelian groups

$$\text{Sp} := (\text{hsp} \circ \chi_h) : \mathcal{M}_{G_m}^{\mathbb{Q}} \to \mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^2 \times \mathbb{Z}].$$

We denote by $\delta$ the morphism of abelian groups

$$\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^2 \times \mathbb{Z}] \to \mathbb{Z}[\mathbb{Q}]$$

sending $t^a u^b v^c$ to $t^{s(a)+s(b)+c}$. 

Let $A$ be an element of $\mathcal{M}_{G_m}^{G_m}$. The relation between the spectrum of $A$ and the spectrum of $\Psi(A)$ is given by the following proposition.

6.2. Proposition. Let $A$ be an element of $\mathcal{M}_{G_m}^{G_m}$. We have

$$\text{Sp}(\Psi(A)) = \delta(\text{Sp}(A)).$$

Proof. Let $A$ be a smooth variety with a good $G_m$-action and with a morphism to $G_m^2$ which is diagonally monomial of weight $(n, n)$, $n$ in $\mathbb{N}_0$. Let us denote by $A_1$ the fiber of $A$ above $(1, 1)$. By (5.1.8) and using the notation therein, we have

$$\Psi([A]) = -[F_1^n \times \mu \times \mu A_1] + [F_0^n \times \mu \times \mu A_1].$$

The result follows from the following well-known computation of the cohomology of Fermat varieties (cf. [22] and Lemma 7.1 in [16]).

6.3. Lemma. Let $(\alpha, \beta)$ be in $(\mathbb{Q}/\mathbb{Z})^2$. For every common denominator $n$ of $\alpha$ and $\beta$, the Hodge type of the eigenspaces $H^i_c(F^n_1, C)(\alpha, \beta)$ and $H^i_c(F^n_0, C)(\alpha, \beta)$ of $\mu \times \mu$ in $H^i_c(F^n_1, C)$ and $H^i_c(F^n_0, C)$ respectively with character $(\alpha, \beta) \in (n^{-1}\mathbb{Z}/\mathbb{Z})^2$ is independent of $n$ and is computed as follows:

1. $H^1_c(F^n_1, C)(\alpha, \beta)$ is of rank 1 for $(\alpha, \beta) \neq (0, 0)$ and of rank 2 for $(\alpha, \beta) = (0, 0)$; $H^1_c(F^n_1, C)(\alpha, \beta)$ is of Hodge type $(0, 1)$ if $\alpha \neq 0 \neq \beta$ and $0 < s(\alpha) + s(\beta) < 1$, $(1, 0)$ if $1 < s(\alpha) + s(\beta) < 2$ and $(0, 0)$ otherwise, that is, if $\alpha = 0$ or $\beta = 0$ or $\alpha + \beta = 0$; $H^1_c(F^n_0, C)(0, 0)$ is of rank 1 and Hodge type $(1, 1)$; all other cohomology groups are zero.

2. $H^2_c(F^n_0, C)(\alpha, -\alpha)$, resp. $H^2_c(F^n_0, C)(\alpha, -\alpha)$, is of rank 1 and Hodge type $(0, 0)$, resp. $(1, 1)$, for any $\alpha$ in $\mathbb{Q}/\mathbb{Z}$, and all other cohomology groups are zero.

We shall also need the following obvious statement:

6.4. Lemma. For $N \geq 1$, consider the morphism $\pi_N : G_m^2 \rightarrow G_m^2$ given by $(a, b) \mapsto (a, b^N)$. For every $A$ in $\mathcal{M}_{G_m}^{G_m}$,

$$\text{Sp}(\pi_N(A)) = \frac{1-u}{1-u^N} \text{Sp}(A)(t, u^\pi, v).$$

6.5. Let $X$ be a smooth complex algebraic variety of dimension $d$ and let $f$ be a function $X \rightarrow \mathbb{A}^1$. Fix a closed point $x$ of $X$ at which $f$ vanishes. Denote by $F_x$ the Milnor fiber of $f$ at $x$. The cohomology groups $H^i(F_x, \mathbb{Q})$ carry a natural mixed Hodge structure ([24], [27], [18], [20]), which is compatible with the semi-simplification of the monodromy operator $T_{f,x}$. Hence we can define the Hodge characteristic $\chi_h(F_x)$ of $F_x$ in $K_0(\text{HS}_{\text{mon}})$. The following statement follows from [5] and [3] (it is also a consequence of Proposition 8.17):

6.6. Theorem. Assuming the previous notations, the following equality holds in $K_0(\text{HS}_{\text{mon}})$:

$$\chi_h(F_x) = \chi_h(S_{f,x}).$$
In particular, if we define the Hodge spectrum of $f$ at $x$ as

\[(6.6.2) \quad \Sp(f, x) := (-1)^{d-1} \hsp(\chi_h(F_x) - 1),\]

it follows from Theorem 6.6 that

\[(6.6.3) \quad \Sp(f, x) = \Sp(S_{\phi f, x}).\]

Now if $g : X \to \mathbb{A}^1$ is another function vanishing at $x$, we shall set, by analogy with (6.6.3),

\[(6.6.4) \quad \Sp(f, g, x) := \Sp(S_{g, x}(S_{\phi f})).\]

Let us denote by $\delta_N$ the morphism of abelian groups $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^2 \times \mathbb{Z}] \to \mathbb{Z}[(\mathbb{Q})$ sending $t^a u^b v^c$ to $t^{(a)+s(b)/N+c}$.

6.7. Propostion. For every positive integer $N$, the spectrum of $\Psi_\Sigma(S_{g_N, x}(S_{\phi f}))$ is equal to

\[(6.7.1) \quad \Sp(\Psi_\Sigma(S_{g_N, x}(S_{\phi f}))) = \frac{1 - t}{1 - t^N} \delta_N(\Sp(f, g, x)).\]

Proof. Follows directly from Proposition 6.2 and Lemma 6.4.

Hence, we deduce immediately the following statement from Corollary 5.16.

6.8. Theorem. Let $X$ be a smooth variety of pure dimension $d$, and $f$ and $g$ be two functions from $X$ to $\mathbb{A}^1$. Let $x$ be a closed point of $X_0(f) \cap X_0(g)$. Then, for $N > \gamma_x((f), (g))$,

\[(6.8.1) \quad \Sp(f, x) - \Sp(f + g_N, x) = \frac{1 - t}{1 - t^N} \delta_N(\Sp(f, g, x)).\]

6.9. Application to Steenbrink’s conjecture. Let us assume now that the function $g$ vanishes on all local components at $x$ of the singular locus of $f$ but a finite number of locally irreducible curves $\Gamma_\ell, 1 \leq \ell \leq r$. We denote by $e_\ell$ the order of $g$ on $\Gamma_\ell$.

As in the introduction, along the complement of $\{x\}$ in $\Gamma_\ell$, we may view $f$ as a family of isolated hypersurface singularities parametrized by $\Gamma_\ell$. We denote by $\alpha_{\ell, j}$ the exponents of that isolated hypersurface singularity and we note that there are two commuting monodromy actions on the cohomology of its Milnor fiber: the first one denoted by $T_f$ is induced transversally by the monodromy action of $f$ and the second one denoted by $T_\tau$ is the monodromy around $x$ in $\Gamma_\ell$. Since the semi-simplifications of $T_f$ and $T_\tau$ can be simultaneously diagonalized, we may define rational numbers $\beta_{\ell, j}$ in $[0, 1)$ so that each $\exp(2\pi i \beta_{\ell, j})$ is the eigenvalue of the semi-simplification of $T_\tau$ on the eigenspace of the semi-simplification of $T_f$ associated to $\alpha_{\ell, j}$.

We may now deduce from Theorem 5.7 the following statement, first proved by M. Saito in [21], and later given another proof by A. Némethi and J. Steenbrink in [17].
6.10. Theorem. For $N > \gamma_x((f),(g))$, we have
\begin{equation}
\Sp(f + g^N, x) - \Sp(f, x) = \sum_{\ell,j} t^{\alpha_{\ell,j} + (\beta_{\ell,j}/\ell_\ell N)} \frac{1 - t}{1 - t^{1/\ell_\ell N}}.
\end{equation}

Proof. For every $\ell$, we set $S_{f,\ell}^\phi := i_\ell^*(S_f^\phi)$, with $i_\ell$ the inclusion of $\Gamma_\ell$ in $X_0(f)$. Since $S_f^\phi - \sum_\ell i_\ell^!(S_{f,\ell}^\phi)$ has support in $X_0(g) \times G_m$, \begin{equation}
S_{g^N,x}(S_f^\phi) = \sum_\ell S_{g^N,x}(i_\ell^!(S_{f,\ell}^\phi)).
\end{equation}
Now consider the normalization $n_\ell : \tilde{\Gamma}_\ell \to \Gamma_\ell$. Let us choose a uniformizing parameter $\tau_\ell$ at the preimage $x_\ell$ of $x$ in $\tilde{\Gamma}_\ell$. We may write $g \circ n_\ell = \eta \tau_\ell e^{L_\ell}$, with $\eta$ a local unit. We have \begin{equation}
S_{g^N,x}(i_\ell^!(S_{f,\ell}^\phi)) = S_{g^N|_{\Gamma_\ell},x}(S_{f,\ell}^\phi) = S_{\eta \tau_\ell e^{L_\ell},x}(S_{f,\ell}^\phi),
\end{equation}
where in the last term we view $S_{f,\ell}^\phi$ as lying in $\mathcal{M}^{G_m}_{\Gamma_\ell \times G_m}$. By Proposition 3.17 \begin{equation}
\Sp(S_{\eta \tau_\ell e^{L_\ell},x}(S_{f,\ell}^\phi)) = \Sp(S_{\tau_\ell e^{L_\ell},x}(S_{f,\ell}^\phi))
\end{equation}
and \begin{equation}
\Sp(S_{\tau_\ell,x}(S_{f,\ell}^\phi)) = -\sum_j t^{\pi(\alpha_{\ell,j})} u^{\pi(\beta_{\ell,j})} v^{[\alpha_{\ell,j}]},
\end{equation}
where $[\alpha]$ denotes the integer part of $\alpha$. Indeed, note that if $H$ is the mixed Hodge module corresponding to a variation of mixed Hodge structure on a neighborhood of $x_\ell$, the fiber at $x_\ell$ of $\psi_{\tau_\ell}(H)$ is nothing but the generic fiber of the variation endowed with the monodromy around $x_\ell$. The sign in [6.10.3] results from the fact that the numbers $\alpha_{\ell,j}$ occuring in its right hand side are the exponents of an isolated hypersurface singularity in an ambient space of dimension $d - 1$ and not $d$. The result follows now from [6.10.1], [6.10.2], [6.10.3], [6.10.4] and [6.10.5] by plugging together Corollary 5.16 and Proposition 6.7. \hfill \Box

References

1. E. Bierstone, P. Milman, Canonical resolution of singularities in characteristic zero by blowing up the maximal strata of a local invariant, Invent. Math. 128 (1997), 207–230.
2. F. Bittner, The universal Euler characteristic for varieties of characteristic zero, Compositio Math. 140 (2004), 1011–1032.
3. F. Bittner, On motivic zeta functions and the motivic nearby fiber, Math. Z. 249 (2005), 63–83.
4. J. Denef, On the degree of Igusa’s local zeta function, Amer. J. Math. 109 (1987), 991–1008.
5. J. Denef, F. Loeser, Motivic Igusa zeta functions, J. Algebraic Geom. 7 (1998), 505–537.
6. J. Denef, F. Loeser, Germs of arcs on singular algebraic varieties and motivic integration, Invent. Math. 135 (1999), 201–232.
7. J. Denef, F. Loeser, Motivic exponential integrals and a motivic Thom-Sebastiani Theorem, Duke Math. J. 99 (1999), 285–309.
8. J. Denef, F. Loeser, Geometry on arc spaces of algebraic varieties, Proceedings of 3rd European Congress of Mathematics, Barcelona 2000, Progress in Mathematics 201 (2001), 327–348, Birkhaüser.
9. J. Denef, F. Loeser, *Lefschetz numbers of iterates of the monodromy and truncated arcs*, Topology **41** (2002), 1031–1040.
10. S. Encinas, H. Hauser, *Strong resolution of singularities in characteristic zero*, Comment. Math. Helv. **77** (2002), 821–845.
11. S. Encinas, O. Villamayor, *Good points and constructive resolution of singularities*, Acta Math. **181** (1998), 109–158.
12. W. Fulton, *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin, 1984.
13. G. Guibert, *Espaces d’arcs et invariants d’Alexander*, Comment. Math. Helv. **77** (2002), 783–820.
14. I.N. Iomdin, *Complex surfaces with a one-dimensional set of singularities* (Russian), Sibirsk. Mat. Ž. **15** (1974), 1061–1082, 1181, English translation: Siberian Math. J. **15** (1974), no. 5, 748–762 (1975).
15. G. Laumon, L. Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin, 2000.
16. E. Looijenga, *Motivic Measures*, Astérisque **276** (2002), 267–297, Séminaire Bourbaki, exposé 874.
17. A. Némethi, J. Steenbrink, *Spectral pairs, mixed Hodge modules, and series of plane curve singularities*, New York J. Math. **1** (1994/95), 149–177.
18. M. Saito, *Modules de Hodge polarisables*, Publ. Res. Inst. Math. Sci. **24** (1988), 849–995.
19. M. Saito, *Duality for vanishing cycle functors*, Publ. Res. Inst. Math. Sci. **25** (1989), 889–921.
20. M. Saito, *Mixed Hodge modules*, Publ. Res. Inst. Math. Sci. **26** (1990), 221–333.
21. M. Saito, *On Steenbrink’s conjecture*, Math. Ann. **289** (1991), 703–716.
22. T. Shioda, T. Katsura, *On Fermat varieties*, Tôhoku Math. J. **31** (1979), 97–115.
23. D. Siersma, *The monodromy of a series of hypersurface singularities*, Comment. Math. Helv. **65** (1990), 181–197.
24. J. Steenbrink, *Mixed Hodge structures on the vanishing cohomology*, in Real and Complex Singularities, Sijthoff and Noordhoff, Alphen aan den Rijn, 1977, 525–563.
25. J. Steenbrink, *The spectrum of hypersurface singularities*, Actes du Colloque de Théorie de Hodge (Luminy, 1987). Astérisque No. 179-180, (1989), 11, 163–184.
26. H. Sumihiro, *Equivariant completion II*, J. Math. Kyoto Univ., (1975) **15**, 573–605.
27. A. Varchenko, *Asymptotic Hodge structure in the vanishing cohomology*, Math. USSR Izvestija **18** (1982), 469–512.
28. O. Villamayor, *Constructiveness of Hironaka’s resolution*, Ann. Sci. Éc. Norm. Sup. Paris, **22** (1989), 1–32.
29. O. Villamayor, *Patching local uniformizations*, Ann. Sci. Éc. Norm. Sup. Paris, **25** (1992), 629–677.

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