On the question of symmetries in non-relativistic diffeomorphism invariant theories

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Abstract

Nonrelativistic diffeomorphism invariance has recently emerged as a powerful tool for investigating various phenomena. The flat limit of such an invariance which should yield the Galilean invariance is, surprisingly, riddled with ambiguities and anomalies. We show that our approach, based on Galilean gauge theory, resolves these shortcomings. As a spin-off, we provide a systematic and unique way of interpreting nonrelativistic diffeomorphism invariance and Galilean invariance as appropriate nonrelativistic limits of relativistic invariances in curved and flat backgrounds, respectively. The complementary role of flat and nonrelativistic limits is highlighted.

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1 Introduction

Nonrelativistic diffeomorphism invariance (NRDI) has recently received considerable attention because of its myriad applications. These are as diverse as mesoscopic physics, nonrelativistic gravity, Newton-Cartan geometry, Horava Lifshitz gravity, to name a few. NRDI provides a way of coupling nonrelativistic field theories to background space and the consequent effective field theory becomes a powerful tool for these applications.

Since Galilean symmetry is a subset of the bigger nonrelativistic diffeomorphism symmetry, it is not just desirable, but essential, that the flat limit of NRDI should yield Galilean invariance. But this appears to be riddled with ambiguities and problems. To appreciate this issue, we recall that, lacking a definite prescription, NRDI was originally introduced in an ad-hoc fashion in [1]. An outcome of such an approach was the occurrence of an unusual transformation for the space component of the vector field. It was unusual in the sense that, going to the flat limit, the standard Galilean result (under boosts) could not be reproduced. To overcome this situation, a particular relation between the gauge parameter and boost parameter was suggested. While this cancelled the anomalous boost term it simultaneously created another problem; namely, the original gauge freedom was lost. This is hardly surprising since gauge transformation and general coordinate (diffeomorphism) transformation are independent and any relation between them is bound to affect one or the other. Here the problem merely gets shifted from the boost sector to the gauge sector.

Though nonrelativistic difeomorphism was introduced long back by Cartan when he formulated Newtonian gravity as a geometric theory in what is now called Newton-Cartan (NC) spacetime, the question of coupling a Galilean symmetric theory with curved spacetime is a difficult one. There is a spate of papers obtainable in the literature that proposes one or another point of view. Recently we have tackled this issue by developing a systematic algorithm in [6, 24, 7, 10, 25]. We gauged the Galilean symmetry of a dynamical model for the purpose. This Galilean gauge theory when applied to the model of [1] gives a diffeomorphism invariant theory in space [7, 10] which is satisfactory in all respects. This has been amply demonstrated [7, 10] including the points of departure from [1]. Since a large number of recent works in the theory of fractional Hall effect is based on [1], these points of departure assume quite a lot of significance.

One of the most attractive features of Galilean gauge theory is the complete ease with which one can go to the flat limit. This is not accidental. We have shown that this theory is amenable to a geometric interpretation. The new fields introduced during localization can be identified with the elements of the NC geometry in the vielbein approach. Thus we can construct the standard [24, 10] as well as torsional Newton-Cartan [25] geometry from the

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1 In this paper we shall be strictly confined to spatial NRDI.
vielbein approach without any additional input. Moreover the coordinate system for the curved spacetime automatically emerges as the Galilean frame.

With this perspective in mind let us elaborate the distinctive feature of the present work. When we localize the symmetry of the action corresponding to the motion of a 2 + 1 dimensional trapped electron, the geometry dictates an extra term in the action depending linearly on a new field. Such a field is non-existant in the usual approach [1]. This is obvious because an ad-hoc method cannot provide any justification for this field. While the original fields retain their usual transformation properties, the new field has an anomalous transformation. Their simultaneous effect ensures NRDI. Taking the flat limit poses no problems. The new fields are set to zero and the original Galilean invariant model where the fields have usual transformations is reproduced.

A related issue is the interpretation of models with NRDI as derived by taking the non-relativistic limit of some relativistic model. Here also there is lack of any systematic methodology. Indeed different relativistic models leading to the same nonrelativistic diffeomorphism invariant model have been reported [27, 28]. As we show, taking the nonrelativistic limit and the flat limit can be done in a unified and systematic manner which is unique. There is no room for any arbitrariness.

It may be mentioned that the original motivation for studying nonrelativistic limits was to provide a semblance of justification for the ad-hoc and unusual construction of NRDI models. But, as shown here, this also fails the test of a correct flat limit. Once again this problem is resolved here through the crucial role of the new fields that appear in the Galilean gauge theory scenario.

In section 2 we state the problem providing necessary technical details. Section 3 analyses the free theory. NRDI in this case was not discussed earlier. Neither can it be obtained by simply setting the vector (gauge) field to zero in the action of [1]. Hence this is an important example. Already in this example the new field is introduced. The flat limit is examined and the relativistic origin of NRDI is also clarified. The calculations are then extended to the interacting model in section 4. Both the flat limit and relativistic origin are discussed. Results are compared with the existing ones in the literature [1, 27, 28]. An alternative action whose nonrelativistic limit also yields the same NRDI model is given. This is obtained by an ad-hoc process without the systematic basis that pervades this paper. Expectedly, the flat limit poses problems and once again illustrates the vagaries of an ad-hoc procedure. In section 5 we show that the relativistic metric leading to NRDI with the correct flat limit admits an ADM decomposition. Finally, section 6 contains our conclusions.


2 Statement of the problem

Let us begin with a statement of the problem. A consistent way to couple nonrelativistic particles to an external gauge field and metric tensor such that it manifests a nonrelativistic version of general coordinate invariance has, as already mentioned, generated considerable interest. This invariance has deep consequences and nontrivial applications, particularly in the context of unitary Fermi gas, FQHE and Newton-Cartan geometry.

There are, however, subtle traps and pitfalls in a consistent formulation of nonrelativistic diffeomorphism invariance. There are two issues involved. First, a smooth flat limit should exist that recovers the original Galilean symmetry of the nonrelativistic model. Secondly, the relativistic origin of nonrelativistic diffeomorphism invariance should be clarified. On both these counts the discussions in the existing literature [1, 27, 28, 29] are rather controversial and dubious. These issues are highlighted by taking a specific example.

Consider a system of nonrelativistic particles coupled to an external gauge field $A\mu$ in flat space [1],

$$S = \int dt \; dx \left[ \frac{i}{2} \psi^\dagger \gamma^\mu \partial_\mu \psi - \frac{1}{2m} (D_i \psi)^\dagger D_i \psi \right]. \quad (1)$$

where $\psi^\dagger \gamma^\mu \partial_\mu \psi = \psi^\dagger \partial_\mu \psi - \partial_\mu \psi^\dagger \psi$ and $D_i \psi = \partial_i \psi + i A_i \psi$.

This action has a local gauge invariance

$$\psi \rightarrow \psi' = e^{i \alpha} \psi , \quad A_0 \rightarrow A_0' = A_0 - \dot{\alpha} , \quad A_i \rightarrow A_i' = A_i - \partial_i \alpha \quad (2)$$

where the gauge parameter is space time dependent, $\alpha(x,t)$.

The above action also has a Galilean invariance where the coordinates transform as

$$t \rightarrow t' = t - \varepsilon$$

$$x^i \rightarrow x'^i = x^i + \varepsilon^i + \lambda^i_j x^j - v^i t \quad (3)$$

under which the infinitesimal transformations of the fields are given by

$$\delta \psi = \psi'(x,t) - \psi(x,t) = -\xi^\mu \partial_\mu \psi = \varepsilon \dot{\psi} - (\eta^i - v^i t) \partial_i \psi$$

$$\delta A_0 = A_0'(x,t) - A_0(x,t) = \varepsilon \dot{A}_0 - (\eta^i - v^i t) \partial_i A_0 + v^i A_i$$

$$\delta A_i = A_i'(x,t) - A_i(x,t) = \varepsilon \dot{A}_i - (\eta^j - v^j t) \partial_j A_i + \lambda^i_j x^j. \quad (4)$$

Here $\eta^i = \varepsilon^i + \lambda^i_j x^j$ and $\xi^i = \eta^i - v^i t$. The Galilean parameters corresponding to spatial translations ($\varepsilon^i$), rotations ($\lambda^i_j$) and boosts ($v^i$) are global. This completes the standard description of Galilean symmetry in a nonrelativistic model.

It is now possible to consider a system analogous to eq. (1) but defined in a curved three dimensional manifold with the spatial line element

$$ds^2 = g_{ij}(t, x) \; dx^i \; dx^j. \quad (5)$$
The action of the system is given by

\[ S = \int dt \, dx \, \sqrt{\tilde{g}} \left[ \frac{i}{2} \psi^{†} \partial_t \psi - A_0 \psi^{†} \psi - \frac{g^{ij}}{2m} (\partial_i \psi^{†} - iA_i \psi^{†})(\partial^j \psi + iA^j \psi) \right] \tag{6} \]

where the determinant of \( g_{ij} \) is denoted as \( \tilde{g} = \text{det}g_{ij} \). This action is invariant under the following set of infinitesimal transformations \[1\]

\[ \delta \psi = i \alpha \psi - \xi^k \partial_k \psi \tag{7a} \]
\[ \delta A_0 = -\dot{\alpha} - \xi^k \partial_k A_0 - A_k \dot{\xi}^k \tag{7b} \]
\[ \delta A_i = -\partial_i \alpha - \xi^k \partial_k A_i - A_k \partial_i \xi^k + mg_{ik} \dot{\xi}^k \tag{7c} \]
\[ \delta g_{ij} = -\xi^k \partial_k g_{ij} - g_{ik} \partial_j \xi^k - g_{kj} \partial_i \xi^k. \tag{7d} \]

Here both the gauge parameter \( \alpha \) and the diffeomorphism parameter \( \xi^i \) (corresponding to the shift \( x^i \to x^i + \xi^i \)) are space time dependent. The above transformations (involving \( \xi^i \)) are referred as nonrelativistic spatial diffeomorphisms. Now, for consistency, the Galilean set \[4\] under which eq.(1) is invariant should be reproduced as a restricted class of eq.(7) under which eq.(6) is invariant. The standard replacement of \( g_{ij} \) by the flat metric \( \eta_{ij} \) followed by taking \( \xi^i \) in eq.(7) as space time independent should reproduce eq.(s)(1) and (4) with \( \varepsilon = 0 \) (since only spatial transformations are considered). Let us specifically concentrate on the boosts, which is the only nontrivial sector, by choosing \( \xi^i = -v^i t \) in eq.(7). The passage from eq.(6) to eq.(1) is smooth when \( g_{ij} \to \eta_{ij} \). However, the same cannot be said about eq.(7) to eq.(4). Specifically, eq.(7c) yields

\[ \delta A_i = -\partial_i \alpha + tv^k \partial_k A_i - mv_i \tag{8} \]

which clashes with eq.(4) due to the presence of the last term.

It has been suggested in \[1\] that by choosing \( \alpha = -mv_i x^i \), the extra piece in eq.(8) gets cancelled. However, this does not solve the problem. If this is done then the original gauge symmetry (2) of \( A_i \) is lost. This is bound to happen since gauge and Galilean symmetries are independent. Any identification of gauge parameters with Galilean parameters will destroy this property. This shows that a consistent flat space limit of eq.(7) does not exist.

A possible reason for this failure is that the action (6) for the nonrelativistic diffeomorphism symmetry (7) was written on a purely ad-hoc basis without any systematic approach.

To mitigate a lack of systematic development it was suggested in \[1\] that there was a relativistic origin of eq.(6). The idea was to consider a relativistic field theory of a free complex scalar field \( \phi \) in an external four dimensional metric \( g_{\mu \nu} \). With \( x^\mu = (ct, \mathbf{x}) \) and a mostly positive metric signature \((-+++))\), the invariant length is given by

\[ ds^2 = g_{\mu \nu}(t, \mathbf{x}) \, dx^\mu \, dx^\nu. \tag{9} \]
The relevant action is given by

\[ S = -\int d^4x \sqrt{-g(4)} \left( g^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi + m^2 c^2 \phi^* \phi \right) \]  

(10)

which is invariant under the infinitesimal general coordinate transformations

\[ \delta \phi = -\xi^\lambda \partial_\lambda \phi \]  

(11)

\[ \delta g_{\mu\nu} = -\xi^\lambda \partial_\lambda g_{\mu\nu} - g_{\lambda\nu} \partial_\mu \xi^\lambda - g_{\mu\lambda} \partial_\nu \xi^\lambda. \]  

(12)

Making a change of variables

\[ \phi = e^{-imc^2t} \frac{\psi}{\sqrt{2mc}} \quad \psi = e^{-imc^2t} \frac{\psi}{\sqrt{2mc}} \]  

(13)

and choosing a metric

\[ g_{\mu\nu} = \begin{pmatrix} -1 - \frac{2A_0}{mc^2} + O(c^{-4}) & - \frac{A_i}{mc} + O(c^{-3}) \\ - \frac{A_i}{mc} + O(c^{-3}) & g_{ij} + O(c^{-2}) \end{pmatrix} \]  

(14)

and its inverse

\[ g^{\mu\nu} = \begin{pmatrix} -1 + \frac{2A_0}{mc^2} + \frac{A_i^2}{mc^2} + O(c^{-4}) & - \frac{A_i}{mc} + O(c^{-3}) \\ - \frac{A_i}{mc} + O(c^{-3}) & g^{ij} + O(c^{-2}) \end{pmatrix} \]  

(15)

it is possible to reproduce action (6) in the \( c \to \infty \) limit. Also, the transformations (7) are obtained from eqs.(11)-(13) provided the following identification is used

\[ \xi^\mu = \left( \frac{\alpha}{mc}, \xi^i \right). \]  

(16)

One may therefore interpret eqs.(6) and (7) as suitable nonrelativistic limits of eqs.(10), (11) and (12).

This line of reasoning also fails to provide a satisfactory resolution of the flat limit problem. To obtain this limit, we replace \( g_{\mu\nu} \to \eta_{\mu\nu} \) in eq.(10) which corresponds to setting \( A_\mu = 0 \) and \( g_{ij} = \delta_{ij} \) in eq.(14). This ensures that the action (10) has a viable flat limit which yields the usual theory of complex scalars. The corresponding situation in eq.(6) is, however, problematic. Taking \( A_\mu = 0 \) trivialises the flat limit of the action (6) and fails to reproduce the expected result (1). This failure is hardly surprising since the lack of a proper flat limit is an intrinsic defect and cannot be bypassed by merely algebraic gymnastics.
3 Free Theory

The problem of formulating a theory having NRDI has been clearly posed - how to consistently take the flat limit and recover the standard Galilean symmetry? A related issue is to understand the NRDI as a nonrelativistic limit of the diffeomorphism invariance of some relativistic theory. As will be shown, these are connected issues.

The shortcomings and pitfalls highlighted in the previous section are a consequence of the ad-hoc approach to NRDI. The first thing, therefore, is to develop a systematic algorithm for NRDI. This was earlier presented in a series of papers involving two of the present authors [6, 7, 10, 24]. We shall now use this algorithm to present a resolution of the problems.

Let us first recall that NRDI was initially discussed in the context of the model (6) to eventually analyse the trapping of electrons on a two dimensional plane that is the forerunner of the FQHE problem. A simpler theory would be to consider the noninteracting or free theory. Unfortunately this cannot be obtained by just putting \( A_0 = A_i = 0 \) in eq.(s)(6) and (7). It is easy to check that NRDI does not hold. Once again this points to the arbitrary nature of the construction since results for noninteracting case cannot be simply obtained by switching off the interaction.

We now apply the formalism [6, 7, 10, 24] to the free theory. First, the free nonrelativistic theory on flat background is written

\[
S = \int dt dx \left[ i \frac{\psi^\dagger \partial_t \psi}{2} - \frac{1}{2m} \partial_i \psi^\dagger \partial_i \psi \right]
\]

which is invariant under the Galilean transformations (3) and (4).

The next step is to gauge the galilean symmetry. This is done by taking the parameters of Galilean transformations to be functions of space time. Naturally, the original invariance under (global) Galilean transformations is destroyed. The symmetry is recovered by replacing the ordinary derivatives by suitable covariant derivatives. This entails introduction of new fields. Also, the measure gets altered due to a nontrivial Jacobian. A geometrical interpretation of the gauged theory is possible. Taking into account all considerations, we finally obtain [7, 10]

\[
S = \int dt dx \sqrt{g} \left[ i \frac{\psi^\dagger \partial_t \psi}{2} - B_0 \psi^\dagger \psi - \frac{g^{ij}}{2m} (\partial_i \psi^\dagger - iB_i \psi^\dagger)(\partial_j \psi + iB_j \psi) \right]
\]

\[
+ \int dt dx \sqrt{g} \frac{i}{2} \Delta^k \left[ \psi^\dagger (\partial_k \psi + iB_k \psi) - (\partial_k \psi^\dagger - iB_k \psi^\dagger) \psi \right].
\]

The \( B \) and \( \Delta \) variables are new (external) fields introduced by the gauging process. The
above action is invariant under the following infinitesimal transformations \[ \delta\psi = -\xi^k \partial_k \psi \] \[ \delta B_0 = -\xi^k \partial_k B_0 - B_k \dot{\xi}^k \] \[ \delta B_i = -\xi^k \partial_k B_i - B_k \partial_i \xi^k \] \[ \delta \Delta_i = \dot{\xi}_i - \xi^k \partial_k \Delta_i - \Delta_k \partial_i \xi^k \]

together with the transformation for \( g_{ij} \). The flat limit is easily implemented. The new fields \( B \) and \( \Delta \) are set to zero so that the covariant derivatives become ordinary derivatives and the metric is taken to be flat (\( g_{ij} \rightarrow \delta_{ij} \)). This immediately reproduces the Galilean theory (17) invariant under the transformation (4).

3.1 Relativistic origin of nonrelativistic diffeomorphism invariance

We now show the obtention of eq.(18) by taking an appropriate nonrelativistic limit of a relativistic action. Once again, contrary to earlier studies [1, 27, 28], our approach will be guided by a systematic algorithm. This is now elaborated.

Before considering the abstraction of eq.(18), we take the simpler and familiar theory (17). Indeed this free theory may be derived from its relativistic version

\[ S = -\int d^4x \left( \eta^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi + m^2 c^2 \phi^* \phi \right) \] (20)

by making the change of variables (13) and finally taking the \( c \rightarrow \infty \) limit.

It is now expected that, by taking the curved space generalization of eq.(20), the action (18) should be derivable. Thus, we begin from the action (10) which is invariant under the general coordinate transformations

\[ x^\mu \rightarrow x^\mu + \xi^\mu (x,t) \] (21)

where the field \( \phi \) and the metric \( g_{\mu\nu} \) transform as eq.(11).

Once again making the change of variables (13) and taking the \( c \rightarrow \infty \) limit (keeping \( g_{\mu\nu} \) fixed), yields

\[ S = \int dt dx \sqrt{\bar{g}} \left[ \frac{i}{2} \psi^\dagger \partial_t \psi + \frac{mc^2(g^{00} + 1)}{2} \psi^\dagger \psi + \frac{i}{2} c g^{0i} [\psi^\dagger \partial_i \psi - (\partial_i \psi^\dagger) \psi] + \frac{g^{ij}}{2m} \partial_i \psi^\dagger \partial_j \psi \right] . \] (22)

Requiring that the above equation reproduces eq.(18) leads to the following relations

\[ g^{00} = -1 + \frac{2}{mc^2} \left[ B_0 + \frac{B^i B_i}{2m} + \Delta^i B_i \right] \] (23)

\[ g^{0i} = -\frac{1}{c} \left[ \frac{B^i}{m} + \Delta^i \right] . \] (24)

8
Hence the inverse metric is given by

\begin{equation}
\begin{aligned}
g^{\mu\nu} = & \begin{pmatrix}
-1 + \frac{2B_0}{mc^2} + \frac{B^iB_i}{m^2c^2} + \frac{2\Delta^iB_i}{mc^2} + O(c^{-4}) & - \left[ \frac{B_i}{mc} + \frac{\Delta^i}{c} \right] + O(c^{-3}) \\
- \left[ \frac{B_i}{mc} + \frac{\Delta^i}{c} \right] + O(c^{-3}) & g^{ij} + O(c^{-2})
\end{pmatrix}
\end{aligned}
\end{equation}

where \( B^i \equiv g^{ij}B_j \). The metric therefore reads

\begin{equation}
\begin{aligned}
g^{\mu\nu} = & \begin{pmatrix}
-1 - \frac{2B_0}{mc^2} + \frac{\Delta^i\Delta_i}{c^2} + O(c^{-4}) & - \left[ \frac{B_i}{mc} + \frac{\Delta^i}{c} \right] + O(c^{-3}) \\
- \left[ \frac{B_i}{mc} + \frac{\Delta^i}{c} \right] + O(c^{-3}) & g^{ij} + O(c^{-2})
\end{pmatrix}
\end{aligned}
\end{equation}

where \( \Delta_i = g_{ij}\Delta^j \).

With this form of the relativistic metric, the theory for the relativistic complex scalar field given by the action (10) reduces to the action (18) in the nonrelativistic limit \( c \to \infty \).

We now proceed to take the nonrelativistic limit of the relativistic infinitesimal transformations (11). Taking \( \xi^\mu \) of the form

\begin{equation}
\xi^\mu = (0, \xi^i) \end{equation}

it is easy to see that the infinitesimal transformation for the relativistic field \( \phi \) (11) yields the infinitesimal transformation for the nonrelativistic field \( \psi \) (19a). Now taking the 0i component of the metric in eq.(12) and keeping terms of the \( O(c^{-1}) \) on both sides yields the infinitesimal transformation for the fields \( B_i \) and \( \Delta_i \) (19c, 19d). Taking the 00 component of the metric in eq.(12) and keeping terms of the \( O(c^{-2}) \) on both sides yields the infinitesimal transformation for the field \( B_0 \) (19b) and the following equation for the infinitesimal transformation of \( \Delta^i \)

\begin{equation}
(\delta \Delta^i)\Delta_i + \Delta^i\delta \Delta_i = \Delta_i[\dot{\xi}^i - \xi^k\partial_k\Delta^i - \Delta^k\partial_k\xi^i].
\end{equation}

Substituting the form for the infinitesimal transformation of \( \Delta_i \) (19d) in the above equation leads to

\begin{equation}
\delta \Delta^i = \dot{\xi}^i - \xi^k\partial_k\Delta^i + \Delta^k\partial_k\xi^i.
\end{equation}

The same result also follows on using \( \Delta_j = g_{ji}\Delta^i \) and computing the \( \delta \)-variation on both sides

\begin{equation}
\delta \Delta^i = g^{ij}\delta \Delta_j - g^{ij}\delta g_{jk}\Delta^k.
\end{equation}

Thus one can understand the set of infinitesimal transformations (19a)-(7d) as a nonrelativistic limit of the relativistic general coordinate invariance.

The last point is to demonstrate the consistency of the flat limit. Previously it was mentioned that this limit is accomplished by setting to zero the new fields \( B \) and \( \Delta \), besides
taking \( g_{ij} \rightarrow \delta_{ij} \). That the action (18) and the transformations (19) correctly pass to the flat expressions (17) and (4) was already seen. The new point is that this can be complemented with the flat limit taken here. The structure of the metric (26) is such that it indeed reduces to the flat metric \( \eta_{\mu\nu} \) when \( B \) and \( \Delta \) are set to zero. This is a nontrivial point in discussing flat limits from a nonrelativistic reduction since the metric \( g_{\mu\nu} \) is now endowed with a specific form (like eq.(26)). We mentioned it earlier and will return to it later. Now the theory (10) passes over to eq.(20) whose nonrelativistic version is eq.(17), the flat limit of eq.(18). Everything falls into place like a jigsaw puzzle.

Let us recapitulate the sequence of arguments that yields a closed chain. This is best expressed diagrammatically:

![Diagram](image)

We started from the box (A) to reach (B) by following the single headed arrow. To obtain (C) whose NR limit would yield (B), we use our knowledge of (A) to abstract (D). Knowing (D) it is straightforward to find (C) by following the double headed arrow.

4 Interacting Theory

The important example of particles interacting with a gauge field is now taken up. The analysis will follow exactly the same route as for the free theory succintly depicted in the diagram. Following the same algorithm we first consider the flat space (Euclidean) action already defined in eq.(11). This action is invariant under global Galilean transformations.
as can be checked explicitly [7]. Note that the theory is also invariant under $U(1)$ gauge transformation (2). Naturally these two types of symmetry are mutually exclusive, the former is a symmetry under space time transformations while the latter corresponds to phase rotation in the internal space.

Once again by localising the Galilean symmetry of the action and geometrical reinterpretation we can reformulate the theory in Newton-Cartan manifold on the unique spatial foliation corresponding to the Galilean frame with the spatial line element (5). The action of the system invariant under diffeomorphism of the manifold is given by [7, 10]

$$S = \int dt dx \sqrt{\tilde{g}} \left[ \frac{i}{2} \psi^{\dagger} \partial_t \psi - (A_0 + B_0) \psi^{\dagger} \psi - \frac{g^{ij}}{2m} [\partial_i \psi^{\dagger} - i(A_i + B_i) \psi^{\dagger}] [\partial_j \psi + i(A_j + B_j) \psi] \right]$$

$$+ \int dt dx \sqrt{\tilde{g}} \frac{i}{2} \Delta^k [\psi^{\dagger} [\partial_k \psi + i(A_k + B_k) \psi] - [\partial_k \psi^{\dagger} - i(A_k + B_k) \psi^{\dagger}] \psi] .$$

(31)

The action (31) is invariant under the following infinitesimal transformation

$$\delta A_i = -\partial_i \alpha - \xi^k \partial_k A_i - A_k \partial_i \xi^k$$

(32)

together with the infinitesimal transformations for $\psi$, $A_0$, $B_0$, $B_i$, $\Delta_i$ and $g_{ij}$ given in eqs.(7a, 7b, 19b, 19c, 19d, 7d).

Observe that the fields $A$ and $B$ appear in the combination $(A + B)$. However we do not rename it since $A$ and $B$ have distinct transformation properties. They have different roles when the flat limit is taken.

The above transformation rule (32) for the field $A_i$ is the usual transformation for the vector potential under gauge transformation together with diffeomorphism and differs from that in eq.(7c). There the absence of the new field $\Delta^i$ required an anomalous transformation law for the field $A_i$. However, no such modification in the transformation laws is required once the field $\Delta^i$ is present which was introduced by the gauging prescription.

Taking the flat limit poses no problems. Set the new fields $B_0$, $B_i$ and $\Delta_i$ to zero and replace $g_{ij} \rightarrow \delta_{ij}$ which immediately yields eq.(1). Also, there is no anomalous transformation as happened earlier (since eq.(7c) is now replaced by eq.(32)). Results for the free theory are reproduced by putting the external gauge field $A_\mu$ to zero. This completes the passage from box B to the box A.

### 4.1 Relativistic origin

To reinterpret the action (31) as a nonrelativistic limit of some relativistic theory, the earlier procedure is adopted. We initially construct the box D. This is given by

$$S = -\int d^4x \left[ \eta^{\mu\nu} (D_\mu \phi)^* D^\nu \phi + m^2 c^2 \phi^* \phi \right]$$

(33)
where
\[ D_\mu \phi = \partial_\mu \phi + i A_\mu \phi. \] (34)

This action has a U(1) gauge invariance given by eq.(2) (replace \( \psi \) by \( \phi \)). The passage from box D to box A is now discussed. Substituting the form of the field \( \phi \) in terms of the nonrelativistic field \( \psi \) by eq.(13) and making the identification
\[ A_0 = \frac{A_0}{c}, \quad A_i = A_i \] (35)
yields
\[ S = - \int d^3x \, dt \left[ \frac{i}{2} \bar{\psi} \gamma^i \partial_i \psi - \frac{1}{2m} \partial_i \psi^* \partial_i \psi + \frac{1}{2m} \left( iA^i (\partial_i \psi^* \psi - \psi^* \partial_i \psi) + \left( -\frac{A_0^2}{c^2} + A_i^2 \right) \psi^* \psi \right) ight. \\
\left. - \frac{i}{c} A_0 \left[ \psi^* (imc \dot{\psi} - \frac{1}{c} \dot{\psi}) + \left( imc \dot{\psi}^* + \frac{1}{c} \dot{\psi}^* \right) \psi \right] \right]. \] (36)

Taking the \( c \to \infty \) limit, the action (1) is obtained, thereby completing the passage from box D to box A.

It is now straightforward to construct the box C and study its limit to box B. The appropriate theory pertaining to box C is given by first lifting eq.(33) from a flat to a curved background
\[ S = - \int d^4x \sqrt{-g} \left[ g^{\mu\nu} (D_\mu \phi)^* D_\nu \phi + m^2 c^2 \phi^* \phi \right]. \] (37)

Apart from the U(1) gauge invariance this action is also invariant under general coordinate transformations (21) with eq.(s)(11), (12) and
\[ \delta A_\mu = -\xi^\nu \partial_\nu A_\mu - A_\nu \partial_\mu \xi^\nu. \] (38)

Now taking eq.(s)(13, 35) and the form of the metric (25), (26) it is possible to reproduce the action (31) in the \( c \to \infty \) limit.

It is important to note that the structure of the metric (25), (26) that effects the nonrelativistic passage is the same for the free theory and the interacting theory. This is indicative of the geometrical nature of the NRDI. Exactly the same analysis (from eq.(28) till eq.(30)) gets repeated to correctly reproduce the transformations of \( B_0, B_i, \Delta_i \) and \( g_{ij} \). The transformations of \( A_0 \) eq.(7b), \( A_i \) eq.(32) and \( \psi \) eq.(7a) are trivially obtained from eq.(11) and eq.(38), recalling eq.(28).

As happened for the free theory, the flat limit is consistently implemented since the metric is unchanged. Setting the new fields \( (B_0, B_i, \Delta_i) \) to zero also ensures transition of eq.(s)(25, 26) to their flat versions. The action (37) reduces to eq.(33) whose nonrelativistic limit was seen to be eq.(11). Thus the complete chain indicated by the box diagram is completed.
4.2 Relativistic origin: an alternative action

It is possible to discuss an alternative route for the nonrelativistic reduction. From our previous analysis, it is possible to identify such a path. Start from the action (10) with the metric

\[
g_{\mu\nu} = \begin{pmatrix}
-1 - \frac{2(A_0 + B_0)}{mc^2} + \frac{\Delta^i(A_i + B_i)}{c^2} + O(c^{-4}) & - \left[ \frac{(A_i + B_i)}{mc} + \frac{\Delta_i}{c} \right] + O(c^{-3}) \\
- \left[ \frac{(A_i + B_i)}{mc} + \frac{\Delta_i}{c} \right] + O(c^{-3}) & g_{ij} + O(c^{-2})
\end{pmatrix}
\]

and its inverse

\[
g^{\mu\nu} = \begin{pmatrix}
-1 + \frac{2C_0}{mc^2} + \frac{C^iC_i}{m^2c^2} + \frac{2\Delta^iC_i}{mc^2} + O(c^{-4}) & - \left[ \frac{C_i}{mc} + \frac{\Delta_i}{c} \right] + O(c^{-3}) \\
- \left[ \frac{C_i}{mc} + \frac{\Delta_i}{c} \right] + O(c^{-3}) & g^{ij} + O(c^{-2})
\end{pmatrix}
\]

where \( C_0 = A_0 + B_0 \), \( C_i = A_i + B_i \) and \( C^i \equiv g^{ij}C_j \). In the limit \( c \to \infty \) the action (10) reproduces eq. (31).

We now proceed to take the nonrelativistic limit of the relativistic infinitesimal transformations (11). Taking \( \xi^\mu \) of the form in eq. (16) where \( \alpha \) is fixed in the limit \( c \to \infty \), it is easy to see that the infinitesimal transformation for the relativistic field \( \phi \) (11) yields the infinitesimal transformation for the nonrelativistic field \( \psi \) (1a). Now taking the 0i component of the metric in eq. (12) and keeping terms of the \( O(c^{-1}) \) on both sides yields the infinitesimal transformation for the fields \( A_i, B_i \) and \( \Delta_i \) (32), 19c, 19d. Taking the 00 component of the metric in eq. (12) and keeping terms of the \( O(c^{-2}) \) on both sides yields the infinitesimal transformation for the fields \( A_0, B_0 \) given in eq. (7b, 19b).

Expectedly, the flat limit cannot be consistently implemented. As pointed out earlier, the flat limit corresponds to setting the newly introduced fields \( (B_0, B_i, \Delta_i) \) to zero and putting the curved spacetime metric \( g_{\mu\nu} = \eta_{\mu\nu} \). As seen from eq. (39, 40), the two are not compatible.

5 Connection of the relativistic metric with ADM decomposition

It is interesting to note that the metric (and its inverse) found during the nonrelativistic reduction from the relativistic actions admit an ADM-decomposition. Let us recall that the ADM construction is given by

\[
ds^2 = -N^2c^2dt^2 + g_{ij}(dx^i + N^icdt)(dx^j + N^jcdt)
\]
where $N$ and $N^i$ are the lapse and shift variables. In this case the metric and its inverse are given by

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + g_{ij}N^iN^j & g_{ij}N^j \\ g_{ij}N^j & g_{ij} \end{pmatrix}$$

and

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^i}{N^2} \\ \frac{N^i}{N^2} & \left(g_{ij} - \frac{N^iN^j}{N^2}\right) \end{pmatrix}.$$  

Comparing with the expressions (25, 26) for $g^{\mu\nu}$ and $g_{\mu\nu}$, we find

$$N = \sqrt{1 + 2B_0 mc^2 + \frac{B^iB_i}{m^2 c^2} + \frac{2\Delta iB_i}{mc^2} + O(c^{-4})}$$

$$N_i = -\frac{B_i}{mc} - \frac{\Delta_i}{c} + O(c^{-3}).$$

Computing the variations of the above equations, we get

$$\delta N = -\xi^i \partial_j N$$

$$\delta N_i = -\partial_i \xi^j N_j - \xi^i \partial_j N_i - \frac{\dot{\xi}^j}{c} g_{ij}$$

which is in agreement with the variations of the ADM variables [26].

### 6 Conclusion

In this work we have elaborated on the flat space limit of nonrelativistic diffeomorphism invariance (NRDI) which is expected to yield the Galilean invariance. Usual constructions have not provided a detailed analysis on this aspect which is fundamental to a consistent formulation of NRDI. The transformation rule for boosts becomes anomalous for which there is no satisfactory explanation. Linking the boost parameter with the gauge parameter, as suggested in [1], cannot save the day because then the original gauge symmetry is lost. An alternative attempt based on interpreting NRDI as a limit of nonrelativistic invariance also fails. As we have discussed, the origin of these failures essentially lies in the ad-hoc formulation of obtaining NRDI.

In a set of papers involving two of us [6, 7, 10, 24, 25], a systematic algorithm to discuss NRDI was developed by gauging the Galilean symmetry. The galilean gauge theory

2To make a comparison with [26] one has to put $\xi^0 = 0$ there which is the case examined here.
formulation of NRDI required the introduction of new fields. The transformations of the new fields is spelled out from the approach itself. This, together with the transformations of the original fields, ensures NRDI. This theory has a smooth flat limit. The new fields are set to zero in this limit and the diffeomorphism invariant theory goes over to the Galilean invariant theory where the fields have usual transformation properties.

Our approach also provides a consistent interpretation of NRDI from a relativistic origin. The appropriate relativistic theory was obtained systematically by following the flow chart at the end of section 3. We started with the complex Klein-Gordon field minimally coupled to external gravity. By devising a foliation of the relativistic space time we expand the metric components in appropriate powers of $c^{-1}$. The metric components are obtained by term by term comparison with the nonrelativistic model in question, and extracting the rest energy in the dominant phase factor, complete equivalence with the nonrelativistic model along with its invariances is shown. From the reduction procedure we also show the physical necessity of the new field emerging from the Galilean gauge theory algorithm in yet another way. Finally we have shown that in the reduction process we actually perform an ADM decomposition of the metric. This justifies our identification of the spatial slice in taking the relativistic limit. It is important to point out that here also taking the flat limit has no problems. The new fields introduced in the Galilean gauge theory enter naturally in the relativistic metric as a correction to the flat metric. The important point, however, is that this correction does not involve the original fields, as happens in the other approaches[1, 27, 28]. Consequently, the flat limit is consistently implemented. The new fields have to be set equal to zero which automatically ensure the passage of the curved metric to the flat metric. If the metric involved the original fields, obviously the flat metric cannot be reproduced.

The genesis of the pitfalls and/or shortcomings of earlier approaches is clearly illuminated by the free theory which was analysed here in considerable details. We may note in passing that NRDI of a free model was never discussed in the literature because the focus was on a nonrelativistic charged particle under an external electromagnetic field in curved space which proved to be useful in discussing fractional quantum Hall effect as well as formulating the Newton-Cartan geometry. Indeed, as shown here, if we simply set the external gauge field to zero, we do not recover NRDI in a free theory. The introduction of new fields is mandatory to achieve this invariance. These fields naturally emerge in our approach based on Galilean gauge theory. Indeed, in the present approach, NRDI for the free case is obtained by simply setting the external gauge field to zero. This is a very desirable feature.

It is clear that the Galilean gauge theory approach to NRDI, as advocated in [6, 7, 10], satisfies all consistency checks. The important point is that it is a completely systematic algorithm leaving no room for any ambiguities or arbitrariness. Since NRDI is derived from the Galilean invariance, it is obvious that passing to the flat limit is smooth. The flow chart depicted here, on the other hand, provides a systematic (and unique) way to discuss the
relativistic origin of NRDI. The fact that the metric \([25, 26]\) in such a discussion, remains unchanged irrespective of the theory being free or interacting, highlights the geometrical nature of NRDI.

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