On Fermion Zero-Modes in Instanton $V - A$ Models With Spontaneous Symmetry Breaking

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Abstract

The left and right handed fermion zero-modes are examined. Their behaviour under the variation of the size of the instanton, $\rho_I$, and the size of the Higgs core, $\rho_H$, for a range of Yukawa couplings corresponding to the fermion masses in the electroweak theory are studied. It is shown that the characteristic radii of the zero-modes, in particular those the left handed fermions, are locked to the instanton size, and are not affected by the variation of $\rho_H$, except for fermion masses much larger than those in the standard electroweak theory.

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It was realised in the early studies of instanton physics, that zero modes of fermions play an essential role in theories with topological configurations of gauge fields \([1, 2, 3, 4]\) (for a review on instantons, see ref. \([5]\)). For instance, the amplitude of tunnelling between vacua \(|n\rangle\) and \(|n+1\rangle\) vanishes in presence of any fermion zero modes. The existence of fermion zero modes, also gives rise to fermion number violation in the instanton-induced vertices. This can be seen by considering the matrix element of a product of bilinears of Fermi fields in an instanton background:

\[
< F(\psi) >_{\nu=1} = < \bar{\psi}_1 \psi_1 \cdots \bar{\psi}_j \psi_j >_{\nu=1}
\]

(in general, each factor may be of the form \(\bar{\psi}_i J_i \psi_i\)). If the Euclidean fermion action is

\[
I_F = \int d^4 x \sum_j \bar{\psi}_j M \psi_j \psi_j
\]

with \(j = 1, \ldots, N_F\), then

\[
< F > \propto \int \prod_{\nu=1} D_A \int \prod_{\kappa} d \lambda_\kappa \lambda_\kappa^* F(\psi) e^{-\sum_{\kappa} \lambda_\kappa a_\kappa a_\kappa^* e^{-I_{YM}}},
\]

where we have expanded in the modes \(\phi_\kappa\) of \(M_\psi\), given by \(M_\psi \phi_\kappa = \lambda_\kappa \phi_\kappa\), and

\[
\psi = \sum a_\kappa \phi_\kappa \text{ (the flavour indexes have been suppressed).}
\]

Since the exponential only contains contributions from \(\lambda_\kappa \neq 0\), we write:

\[
< F > \propto \int \prod_{\kappa \neq 0} d a_\kappa d a_\kappa^* \left[ \int \prod_{\lambda_\kappa = 0} N_0 d a_\kappa^0 \right] \left[ \prod_{\lambda_\kappa = 0} N_0 d a_\kappa^0 \right] F(a) e^{-\Sigma_{\kappa \neq 0} \lambda_\kappa a_\kappa a_\kappa^*}.
\]

Indeed, the Grassmannian integrals in the brackets are nonvanishing only if \(F(\psi)\) is a product of precisely \(N_0\) bilinears of different flavours. In fact for winding number \(\nu = 1\), by virtue of the index theorem \([3]\) and the presence of the anomalous chiral fermion current, the number of the fermion zero modes is equal to \(N_F\). So, in the instanton-induced fermion vertices, there is chiral charge violation of \(\Delta Q_5 = 2N_F\). This leads to baryon and lepton number violations (albeit, suppressed by factors of the order \(\sim e^{-8\pi^2/g^2}\)). There has been a considerable recent interest in the possibility that the baryon number in the universe was generated by the above mechanism, at the scale of electroweak interactions (see ref. \([7]\) and the references therein). In such a scenario, the fermion zero-modes are clearly crucial.

The zero modes of fermions in \(V - A\) theories, such as the Standard Model of Weinberg and Salam have previously been studied \([8]\). In this case, due to Higgs symmetry breaking, the fermions are massive. And the \(V - A\) nature of the theory makes it nontrivial to calculate, exactly, the zero-modes except for some simple cases (for example, spherically symmetric \textit{ansatz}, and equal masses of the up-like and down-like quarks \([8]\)). Furthermore, it has been assumed that the size of the instanton and that of the approximate Higgs solution \([10]\) are equal. Indeed, there is no \textit{a priori} reason that this should be the case.
In this paper, we shall investigate the fermion zero modes in the Weinberg-Salam model with gauge field configurations of winding number \( \nu = 1 \), in the case that the Higgs scale, \( \rho_H \), and the instanton scale, \( \rho_I \), are not equal. We address the following question: Are the scales of the (left handed and right handed) zero modes “locked” to \( \rho_I \) or \( \rho_H \)? Or alternatively, which scale determines the characteristic size of the zero-modes? We examine the behaviour the left-handed and right-handed zero mode solutions (the latter do not couple to the instanton, but only to the Higgs) for a wide range of Yukawa coupling, corresponding to fermion masses \( \sim 1 MeV \) to \( \sim 100 GeV \).

Let us start with the Lagrangian for the \( SU(2) \times U(1) \) Standard Model:

\[
L = -\frac{1}{4} \text{tr}(F_{\mu\nu}F^{\mu\nu}) - \frac{1}{4} G_{\mu\nu}G^{\mu\nu} - \bar{L} \partial\!\!\!/L - \bar{R} \partial\!\!\!/R
\]

\[+ (D_\mu \Phi)^\dagger D^\mu \Phi - V(\Phi) - g_d \{ (\bar{L} \Phi) R - \bar{R} (\Phi^\dagger L) \} \]  

(1)

where \( \Phi = \left( \phi^+ \phi^0 \right) \) is the Higgs doublet, \( L = \left( \begin{array}{c} \nu \\ e_L \end{array} \right) \) is the fermion doublet, \( R = e_R \) is the right handed singlet, and \( V(\Phi) \) is the Higgs potential. The part of the Lagrangian that concerns us is the part that involves the fermions, excluding their \( U(1) \) terms. The fermion equations of motion after the symmetry breaking are:

\[
i \bar{s}_\mu \partial_\mu \psi(x) + g_\gamma A_\nu \psi(x) = 0
\]

\[
i \bar{s}_\mu A_\mu \psi(x) + g e_\Phi^0 \psi(x) = 0
\]

\[
i \bar{s}_\mu e_L - g e_\Phi^0 e_L = 0.
\]

(2)

We use the chiral representation of the \( \gamma \)-matrices:

\[
\gamma^\mu = \left( \begin{array}{cc} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{array} \right), \quad \gamma_5 = \left( \begin{array}{cc} -i & 0 \\ 0 & i \end{array} \right)
\]

where \( \sigma^\mu = (\sigma^0, \vec{\sigma}) \), \( \bar{\sigma}^\mu = (\sigma^0, -\vec{\sigma}) \), \( \sigma^0 = i \Pi \), and metric: \( \eta_{\mu\nu} = \text{Diag}(-1, +1, +1, +1) \). To Euclideanize, we simply let

\[
\sigma^\mu \rightarrow s_\mu = (s_4, \vec{s}) = (-i, \vec{s})
\]

\[
\bar{\sigma}^\mu \rightarrow \bar{s}_\mu = (-i, -\vec{s}) = -\bar{s}_\mu^T.
\]

(3)

In this representation, we may express the fields in terms of two component spinors:

\[
\nu_L = \left( \begin{array}{c} \alpha \\ 0 \end{array} \right), \quad e_L = \left( \begin{array}{c} \beta \\ 0 \end{array} \right), \quad e_R = \left( \begin{array}{c} 0 \\ \epsilon \end{array} \right),
\]

(4)

and re-write equations (2) as follows:

\[
i \bar{s}_\mu \partial_\mu \psi(x) + \frac{g}{2} \bar{s}_\mu A_\mu^a \tau^a \psi(x) - g e_\Phi^0(x) \chi(x) = 0
\]

\[
i \bar{s}_\mu \partial_\mu \chi(x) - g e_\Phi^0(x) \psi(x) = 0
\]

(5)

\[^1\text{We choose to work with one lepton family. This simplifies the algebra, since there would be an additional equation for the second right handed fermion, identical in form to that of the first one.}\]
where

\[
\psi_{\alpha s} = \begin{pmatrix} 0 \\ \alpha \\ 0 \\ \beta \end{pmatrix} ; \quad \chi_{s} = \begin{pmatrix} 0 \\ \epsilon \end{pmatrix} = e_R.
\]

The index \( \alpha \) is the gauge group index, and \( s \) is the spinor index.

In Higgs theories, the exact instanton solutions cease to exist, however, approximate solutions can be found \([4, 10]\), and can be shown to have the same winding number as the gauge field \([3]\). These approximate solutions exist at scales \( \rho_H, \rho_I \ll 1/v \), where \( v \) is the Higgs vacuum expectation value. These solutions do not correspond to the minimum of the Yang-Mills plus Higgs action (although they correspond to saddle points of the action). They can be regarded as the minimum action solutions for a more fundamental theory at a higher energy scale than the EW-scale, whose Lagrangian may contain terms of higher order in derivatives suppressed by an inverse power of a large mass \([2]\). In the finite gauge, these solutions (located at the origin) are given by \([3]\):

\[
A_\mu = -\frac{ig}{2} \tau^a A^a_\mu = \frac{\tau^a \eta^a_{\mu\nu} x_\nu}{(x^2 + \rho_i^2)^{1/2}}
\]

\[
\phi^0 = \frac{i s_\mu x_\mu}{(x^2 + \rho_i^2)^{1/2}}
\]

If we insert these into equation (5) and carry out some algebraic steps using identities such as 
\(- (s_\nu x_\nu) (\partial_\mu s) = x_\mu \partial_\mu - 2 \bar{L}_1 \bar{x} \), and 
\(- (s_\nu x_\nu) (\partial_\mu s) = x_\mu \partial_\mu - 2 \bar{L}_2 \bar{x} \), we arrive at \([4]\):

\[
\frac{1}{x^2} (x_\mu \partial_\mu - 2 \bar{L}_1 \bar{x}) \psi + \frac{3}{(x^2 + \rho_i^2)^{1/2}} \psi = \frac{g_v v}{(x^2 + \rho_i^2)^{1/2}} \chi
\]

\[
\frac{1}{x^2} (x_\mu \partial_\mu - 2 \bar{L}_2 \bar{x}) \chi = \frac{g_v v}{(x^2 + \rho_i^2)^{1/2}} \psi.
\]

We further assume that \( \psi \) and \( \chi \) are spherically symmetric (i.e., \( \bar{L}_1 = \bar{L}_2 = 0 \)):

\[
\psi = p(r) u_{0\alpha s} ; \quad \chi = q(r) w_{0s}
\]

where \( u_0 \) and \( w_0 \) are constant vectors. Now the coupled radial equations read:

\[
\frac{dp}{dr^2} + \frac{3p}{2(r^2 + \rho_i^2)} = \frac{g_v v q}{2(r^2 + \rho_i^2)^{1/2}}
\]

\[
\frac{dq}{dr^2} = \frac{g_v v p}{2(r^2 + \rho_i^2)^{1/2}}
\]

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2I wish to thank S. Rajeev for the discussion on this point.

3Here, \( \eta^a_{\mu\nu} \) are the 't Hooft symbols: \( \eta^a_{\mu\nu} = \epsilon_{\mu\nu} \) for \( \mu, \nu = 1, 2, 3 \); \( \eta^a_{44} = -\delta_{44} \); \( \eta^a_{44} = \delta_{44} \); and \( \eta^a_{44} = 0 \).

4The definitions of angular momentum operators \( L_1 \) and \( L_2 \) are also those given by 't Hooft \([2]\):

\[
L_{1a} = -\frac{i}{2} \eta^a_{\mu\nu} x_\mu \partial_\nu, \quad \text{and} \quad L_{2a} = -\frac{i}{2} \eta^a_{\mu\nu} x_\mu \partial_\nu.
\]
The second order equation for \( q \) is:
\[
\frac{d^2q}{d(r^2)^2} + \left( \frac{3}{2} \frac{1}{r^2 + \rho^2_I} + \frac{1}{r^2 + \rho^2_H} \right) \frac{dq}{dr^2} - \frac{b^2}{(r^2 + \rho^2_H)} q = 0 \tag{12}
\]
where \( b = g_e v / 2 \). Notice that in the limit \( \rho_I = \rho_H = \rho \), this equation reduces to \( q'' + (2/y)q' - (b^2/y)q = 0 \), where \( y = r^2 + \rho^2 \). This equation can be solved by standard methods, yielding the following normalisable solutions \[8\]:
\[
q = \frac{-C g_e v \rho}{(r^2 + \rho^2)^{1/2}} K_1(g_e v (r^2 + \rho^2)^{1/2}) \tag{13}
\]
\[
p = \frac{C g_e v \rho}{(r^2 + \rho^2)^{1/2}} K_2(g_e v (r^2 + \rho^2)^{1/2}) \tag{14}
\]
For \( \rho_I \neq \rho_H \), we may get some qualitative features of the solutions by looking at the factorised form of \( q \). We let:
\[
q = h(r^2) f(r^2) \tag{15}
\]
such that
\[
h' + \mathcal{P} h = 0
\]
\[
f'' + (\mathcal{Q} - \frac{1}{2} \mathcal{P}' - \frac{1}{4} \mathcal{P}^2) f = 0 \tag{16}
\]
where \( \mathcal{P} \) and \( \mathcal{Q} \) are given by:
\[
\mathcal{P} = \frac{3}{2} \frac{1}{r^2 + \rho^2_I} + \frac{1}{2} \frac{1}{r^2 + \rho^2_H}
\]
\[
\mathcal{Q} = b^2 / (r^2 + \rho^2_H),
\]
and find that
\[
h = e^{-\frac{1}{2} \int^2 \mathcal{P}(\xi) d\xi} = \frac{1}{(r^2 + \rho^2_H)^{1/4}(r^2 + \rho^2_I)^{3/4}}. \tag{17}
\]
Now, \( f \) satisfies:
\[
f'' + \left[ \frac{3}{16} \left( \frac{1}{r^2 + \rho^2_I} + \frac{1}{r^2 + \rho^2_H} \right)^2 - \frac{b^2}{(r^2 + \rho^2_H)} \right] f = 0. \tag{18}
\]
For small Yukawa couplings, \( \rho_I \) and \( \rho_H \) come in equal footing in the equation for \( f \); i.e., in this limit, \( f \) is equally sensitive to variations in \( \rho_H \) and \( \rho_I \). Furthermore, as the Yukawa coupling gets stronger, due to the contribution of the \( b^2 / (r^2 + \rho^2_H) \)

\[5\] There are also solutions that are not normalisable, namely those obtained by replacing the \( K_\lambda \) Bessel functions with \( I_\lambda \) Bessel functions in equations 13 and 14. These are eliminated by imposing boundary conditions \( p, q \rightarrow 0 \) as \( r \rightarrow 0 \).
the variations of the Higgs core are expected to affect the zero-modes more. But for physically observed fermion masses in the standard model this effect is small compared to the effect due to the quadratic term in equation (18). So, \( f \) is approximately equally sensitive to variations of \( \rho_H \) and \( \rho_I \). But \( h = (r^2 + \rho_H^2)^{-1/4}(r^2 + \rho_I^2)^{-3/4} \) is clearly more sensitive to change in \( \rho_I \). Thus we expect a stronger correlation between the characteristic radii of the zero-modes and the instanton size than between the characteristic radii of zero-modes and the Higgs core. On physical grounds, the variation of the Higgs core is expected to affect the right handed zero-modes more (if any), because the the right handed fermions are coupled to the Higgs and not to the instanton.

These behaviours of the zero mode solutions were studied numerically. The function \( z = \tanh(r^2M_W^2) \) was used to map the radial coordinate \( r \) to the unit interval, and the boundary conditions were imposed near \( z = 1 \). This simplifies the numerical computations, and is advantageous because small values of \( r^2 \) are mapped almost linearly, and , indeed, we are interested in the behaviour of the solution at length scales of the two cores which are typically small due to the restriction for existence of approximate solutions. The corresponding equation for \( q \) in the \( z \)-coordinate is given by:

\[
\frac{d^2q}{dz^2} + \frac{1}{1-z^2} \left[ -2z + \frac{3}{2(tanh^{-1} z + M_W^2\rho_I^2)} + \frac{1}{2(tanh^{-1} z + M_W^2\rho_H^2)} \right] \frac{dq}{dz} - \frac{b^2}{M_W^2(1-z^2)(tanh^{-1} z + M_W^2\rho_H^2)} q = 0. \tag{19}
\]

Fig-1(a) and 1(b) show the solutions of the right handed and left handed fields respectively for fixed \( \rho_I = 2 \times 10^{-4} GeV^{-1} \) and four \( \rho_H \) values in the range:

\[
10^{-2} \rho_I \leq \rho_H \leq \rho_I
\]

for various \( m_F \). Notice that the left handed zero-mode is unchanged as we decrease \( \rho_H \) over two orders of magnitude. For a larger instanton, \( \rho_I = 10^{-3} GeV^{-1} \), the solutions are shown in Figures 2 and 3, with the Higgs core, again, given by \( 10^{-2} \rho_I \leq \rho_H \leq \rho_I \). The left handed fermion is again, unaffected, except for the very heavy fermion of \( m = 500 GeV \) (Fig- 3), in which case the effect is still quite small.

In another set of plots, Fig.4, we fix \( \rho_H \) and vary \( \rho_I \) between values larger than \( \rho_H \) to values much smaller than \( \rho_H \). We see that the bulk of the left handed zero-mode is “pushed” inside the small instanton core (well inside the Higgs core), while the right handed zero-mode spreads over the instanton core, but its characteristic radius is still of the order \( \rho_I \). This behaviour is generic for the values of Yukawa coupling corresponding to masses of fermions in the standard model up to \( \sim 100 GeV \).

In conclusion, let us note that the non-spherical solutions (see eq. (8), (9)) can also be found numerically, but they would not present any new features under the variations of the two scales discussed above.
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**Figure Captions**

**Fig. 1** - (a) Normalized left handed zero modes for $\rho_I = 0.0002 GeV^{-1}$, and $\rho_H = \rho_I, 0.25\rho_I, 10^{-1}\rho_I, 10^{-2}\rho_I$, superimposed. (b) Normalized right handed zero modes of fermions. The curve with lowest amplitude corresponds to $\rho_I = \rho_H$. The same general picture holds for left and right handed fermions of mass $0.01 GeV \leq m_F \leq 100 GeV$.

**Fig. 2** - Normalized left handed (a) and right handed (b) zero modes for $\rho_I = 0.001 GeV^{-1}$, and $\rho_H = \rho_I, 0.25\rho_I, 10^{-1}\rho_I, 10^{-2}\rho_I$.

**Fig. 3** - Normalized left handed (a) and right handed (b) zero modes. $m_F = 500 GeV$. $\rho_I$ and $\rho_H$ are the same as those given in Fig.2.

**Fig. 4** - Normalized left and right handed fermion zero-modes. Here, $\rho_H$ is kept fixed and (a) $\rho_I = 2\rho_H$, (b) $\rho_I = \rho_H$, (c) $\rho_I = \rho_H/5$, (d) $\rho_I = \rho_H/50$. In these plots, the coordinate $z$ is defined slightly differently from that described in the text, namely, $z = \tanh(r^2v^2)$, where $v = 250 GeV$. The vertical line represents the size of the instanton in the $z$-coordinate.