PERIODIC SOLUTIONS FOR A SUPERLINEAR FRACTIONAL PROBLEM WITHOUT THE AMBROSETTI-RABINOWITZ CONDITION

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ABSTRACT. The purpose of this paper is to study $T$-periodic solutions to
\begin{equation}
\begin{cases}
[-(\Delta + m^2)^s - m^2]u = f(x, u) & \text{in } (0, T)^N \\
u(x + Te_i) = u(x) & \text{for all } x \in \mathbb{R}^N, i = 1, \ldots, N
\end{cases}
\end{equation}
where $s \in (0, 1)$, $N > 2s$, $T > 0$, $m > 0$ and $f(x, u)$ is a continuous function, $T$-periodic in $x$ and satisfying a suitable growth assumption weaker than the Ambrosetti-Rabinowitz condition.

The nonlocal operator $-(\Delta + m^2)^s$ can be realized as the Dirichlet to Neumann map for a degenerate elliptic problem posed on the half-cylinder $S_T = (0, T)^N \times (0, \infty)$. By using a variant of the Linking theorem, we show that the extended problem in $S_T$ admits a nontrivial solution $v(x, \xi)$ which is $T$-periodic in $x$. Moreover, by a procedure of limit as $m \to 0$, we prove the existence of a nontrivial solution to (1) with $m = 0$.

1. Introduction. The aim of this paper is to investigate the existence of periodic solutions to the problem
\begin{equation}
\begin{cases}
[-(\Delta + m^2)^s - m^2]u = f(x, u) & \text{in } (0, T)^N \\
u(x + Te_i) = u(x) & \text{for all } x \in \mathbb{R}^N, i = 1, \ldots, N
\end{cases}
\end{equation}
where $s \in (0, 1)$, $N > 2s$, $m \geq 0$, $(e_i)$ is the canonical basis in $\mathbb{R}^N$ and the nonlinearity $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a superlinear continuous function. The non-local operator $-(\Delta + m^2)^s$ appearing in (1), is defined through the spectral decomposition, by using the powers of the eigenvalues of $-\Delta + m^2$ with periodic boundary conditions.

Let $u \in C^\infty(\mathbb{R}^N)$, that is $u$ is infinitely differentiable in $\mathbb{R}^N$ and $T$-periodic in each variable. Then $u$ has a Fourier series expansion
$$u(x) = \sum_{k \in \mathbb{Z}^N} c_k e^{\omega k \cdot x} \quad (x \in \mathbb{R}^N)$$
where
$$\omega = \frac{2\pi}{T} \quad \text{and} \quad c_k = \frac{1}{\sqrt{T^N}} \int_{(0,T)^N} u(x)e^{-\omega k \cdot x} dx \quad (k \in \mathbb{Z}^N)$$

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are the Fourier coefficients of $u$. The operator $(-\Delta_x + m^2)^s$ is defined by setting

$$(-\Delta_x + m^2)^s u = \sum_{k \in \mathbb{Z}^N} c_k (\omega^2 |k|^2 + m^2)^s e^{i\omega k \cdot x} \sqrt{\text{Vol} N}.$$ 

For $u = \sum_{k \in \mathbb{Z}^N} c_k e^{i\omega k \cdot x} \sqrt{\text{Vol} N}$ and $v = \sum_{k \in \mathbb{Z}^N} d_k e^{i\omega k \cdot x} \sqrt{\text{Vol} N}$, we have that

$$Q(u, v) = \sum_{k \in \mathbb{Z}^N} (\omega^2 |k|^2 + m^2)^s c_k d_k$$

can be extended by density to a quadratic form on the Hilbert space

$$H^{s}_{m, T} = \left\{ u = \sum_{k \in \mathbb{Z}^N} c_k e^{i\omega k \cdot x} \in L^2(0, T)^N : \sum_{k \in \mathbb{Z}^N} (\omega^2 |k|^2 + m^2)^s |c_k|^2 < \infty \right\}$$

equipped with the norm

$$|u|_{H^{s}_{m, T}} = \sqrt{\sum_{k \in \mathbb{Z}^N} (\omega^2 |k|^2 + m^2)^s |c_k|^2}.$$ 

When $m = 1$, we set $H^{s}_{1, T} = H^{s}_{1, T}$. 

In the last decade a great attention has been devoted to the study of fractional Sobolev spaces and non-local operators. Indeed, such operators arise in several fields such as optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, minimal surfaces and water waves; see for instance [1], [2], [3], [4], [5], [6], [8], [10], [13], [16], [21], [22], [23] and references therein. Motivated by the interest shared by the mathematical community in this topic, here we are interested in the existence of periodic solutions of problem (2) under the assumptions that the nonlinearity $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ verifies the following conditions:

1. $f(x, t)$ is $T$-periodic in $x \in \mathbb{R}^N$, that is $f(x + Te_i, t) = f(x, t)$ for any $x \in \mathbb{R}^N$, $t \in \mathbb{R}$ and $i = 1, \ldots, N$;

2. $f$ is continuous in $\mathbb{R}^{N+1}$;

3. $f(x, t) = o(t)$ as $t \to 0$ uniformly in $x \in \mathbb{R}^N$;

4. there exists $1 < p < 2^*_s - 1 = \frac{2N}{N-2s} - 1$ and $C > 0$ such that

$$|f(x, t)| \leq C(1 + |t|^p)$$

for any $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$;

5. $\lim_{|t| \to \infty} \frac{F(x, t)}{|t|^2} = +\infty$ uniformly for any $x \in \mathbb{R}^N$. Here $F(x, t) = \int_0^t f(x, \tau)d\tau$;

6. There exists $\gamma \geq 1$ such that for any $x \in \mathbb{R}^N$

$$G(x, \theta t) \leq \gamma G(x, t)$$

for any $x \in \mathbb{R}^N$, $t \in \mathbb{R}$ and $\theta \in [0, 1]$,

where $G(x, t) := f(x, t)t - 2F(x, t)$.

Our first main result is the following one.

**Theorem 1.1.** Let $m > 0$ and assume that $f : \mathbb{R}^{N+1} \to \mathbb{R}$ is a function satisfying the assumptions (f1)-(f6). Then there exists a solution $u \in H^{s}_{m, T}$ to (2). In particular $u \in C^{0, \alpha}([0, T]^N)$ for some $\alpha \in (0, 1)$. 

To study the problem $\lbrack 2 \rbrack$, we will make use of a Caffarelli-Silvestre type-extension in the periodic framework (see $\lbrack 2 \rbrack$ $\lbrack 3 \rbrack$). This method, which has been originally introduced in $\lbrack 7 \rbrack$ to investigate the fractional Laplacian in $\mathbb{R}^N$, allows us to reformulate the non-local problem $\lbrack 2 \rbrack$ in terms of a local degenerate elliptic problem with a Neumann boundary condition in one dimension higher. More precisely, for $u \in \mathbb{H}^s_{m,T}$ one considers the problem

$$
\begin{cases}
- \text{div}(\xi^{1-2s} \nabla v) + m^2 \xi^{1-2s} v = 0 & \text{in } \mathcal{S}_T := (0, T)^N \times (0, \infty) \\
v|_{\{x_i = 0\}} = v|_{\{x_i = T\}} & \text{on } \partial_L \mathcal{S}_T := \partial(0, T)^N \times [0, \infty) \\
v(x, 0) = u(x) & \text{on } \partial^0 \mathcal{S}_T := \{0, T\} \times \{0\}
\end{cases}
$$

from where the operator $-(\Delta_x + m^2)^s$ is obtained as

$$
- \lim_{\xi \to 0} \xi^{1-2s} \frac{\partial v}{\partial \xi} (x, \xi) = \kappa_s (-\Delta_x + m^2)^s u(x),
$$
in distributional sense and $\kappa_s := 2^{1-2s} \frac{\Gamma(1-s)}{\Gamma(s)}$.

We will exploit this fact and we will look for solutions to

$$
\begin{cases}
- \text{div}(\xi^{1-2s} \nabla v) + m^2 \xi^{1-2s} v = 0 & \text{in } \mathcal{S}_T := (0, T)^N \times (0, \infty) \\
v|_{\{x_i = 0\}} = v|_{\{x_i = T\}} & \text{on } \partial_L \mathcal{S}_T := \partial(0, T)^N \times [0, \infty) \\
\frac{\partial v}{\partial \nu^{1-2s}} = \kappa_s [m^{2s} v + f(x, v)] & \text{on } \partial^0 \mathcal{S}_T := (0, T)^N \times \{0\}
\end{cases}
$$

where

$$
\frac{\partial v}{\partial \nu^{1-2s}} := - \lim_{\xi \to 0} \xi^{1-2s} \frac{\partial v}{\partial \xi} (x, \xi)
$$
is the conormal exterior derivative of $v$.

Since $\lbrack 3 \rbrack$ has a variational nature, its solutions can be found as critical points of the functional

$$
\mathcal{J}_m(v) = \frac{1}{2} ||v||_{X^s_{m,T}}^2 - \frac{m^{2s} \kappa_s}{2} ||v(\cdot, 0)||^2_{L^2(0, T)^N} - \kappa_s \int_{\partial^0 \mathcal{S}_T} F(x, v) \, dx
$$
defined on the space $X^s_{m,T}$, which is the closure of the set of smooth and $T$-periodic (in $x$) functions in $\mathbb{R}^{N+1}_+$ with respect to the norm

$$
||v||_{X^s_{m,T}} := \sqrt{\iint_{\mathcal{S}_T} \xi^{1-2s}(|\nabla v|^2 + m^{2s} v^2) \, dx d\xi}.
$$

Under the assumptions $(f1)$-$(f6)$ we are able to prove that for any fixed $m > 0$, the functional $\mathcal{J}_m$ has a Linking geometry. We recall that in $\lbrack 2 \rbrack$ $\lbrack 3 \rbrack$, we proved that when $f$ satisfies $(f1)$-$(f4)$, $t f(x, t) \geq 0$ in $\mathbb{R}^{N+1}$ and the Ambrosetti-Rabinowitz condition $\lbrack 4 \rbrack$, i.e.

$$
\exists \mu > 2, \ R > 0 : 0 < \mu F(x, t) \leq f(x, t) t, \ \forall |t| \geq R, \ \text{(AR)}
$$

then we can obtain a nontrivial solution to $\lbrack 3 \rbrack$ by applying the standard Linking Theorem $\lbrack 19 \rbrack$ $\lbrack 21 \rbrack$ $\lbrack 22 \rbrack$. Roughly speaking, the role of (AR) is to guarantee the boundedness of Palais-Smale sequences for the functional associated with the problem under consideration. However, although (AR) is a quite natural condition when we deal with superlinear elliptic problems, it is somewhat restrictive. In fact, by a direct integration of (AR), we can deduce the existence of $A, B > 0$ such that

$$
F(x, t) \geq A |t|^\mu - B \text{ for any } t \in \mathbb{R},
$$

which implies, being $\mu > 2$, the condition $(f5)$. If we consider the function $f(x, t) = t \log(1 + |t|)$, then it is easy to prove that $f$ verifies $(f5)$ but does not verify (AR).
This means that there are functions which are superlinear at infinity, but do not satisfy (AR). For this reason, in several works concerning superlinear problems of the type
\[
\begin{cases}
  L u = f(x,u) & \text{in } \Omega \\
  u = 0 & \text{on } \partial \Omega ,
\end{cases}
\] (4)
where \( L \) is a second order elliptic operator and \( \Omega \subset \mathbb{R}^N \) is a smooth bounded domain, some authors tried to drop the condition (AR); see for instance \([11, 14, 17, 18, 20]\) and references therein.

In this paper, we aim to study the non-local counterpart of (4) with \( L = -\Delta + m^2 \), in periodic setting, without assuming (AR). Our basic assumptions on the non-linearity \( f \) are (f1)-(f6). We recall that the hypothesis (f6) was introduced for the first time by Jeanjean in \([14]\) to show that a wide class of functionals having a Mountain Pass structure, possess a bounded Palais-Smale sequence at the Mountain-Pass level.

Here, in order to analyze the problem (3), we invoke a variant of the Linking Theorem proved by Li and Wang in \([15]\), in which the Palais-Smale condition is replaced by the Cerami condition \([9]\); namely any Cerami sequence \( \{v_j\} \) in \( X_{m,T} \) at the level \( \alpha \in \mathbb{R} \), that is such that
\[
J_m(v_j) \to \alpha \quad \text{and} \quad \|J'_m(v_j)\|_{(X_{m,T})'}(1 + \|v_j\|_{X_{m,T}}) \to 0 \quad \text{as } j \to \infty ,
\]
admits a convergent subsequence. At this point, to get the existence of a weak solution to (2), it will be sufficient to show that every Cerami sequence is bounded and possesses a convergent subsequence. This step will be the heart of the proof of Theorem 1.1. Finally, we will also consider the problem (2) when \( m = 0 \), that is
\[
\begin{cases}
  (-\Delta_x)^s u = f(x,u) & \text{in } (0,T)^N \\
  u(x + Te_i) = u(x) & \text{for all } x \in \mathbb{R}^N , \quad i = 1, \ldots, N.
\end{cases}
\] (5)
The existence of a nontrivial periodic solution to (5), will be obtained by passing to the limit in (3) as \( m \to 0 \). This procedure of limit is justified by the fact that we are able to estimate from below and from above the critical levels \( \alpha_m \) of the functionals \( J_m \) independently of \( m \), provided that \( m \) is sufficiently small.

Then we can state our second result.

**Theorem 1.2.** Under the assumptions (f1)-(f6) on \( f \), the problem (5) admits a nontrivial solution \( u \in \mathbb{H}_T^s \).

The paper is organized as follows: in Section 2 we give some notations and preliminaries results on the involved Sobolev spaces; in Section 3 we consider the extension problem (3) which localizes the non-local problem (2); in Section 4 we investigate the existence of solutions to (3) via variational methods; finally, by passing to the limit in (3) as \( m \to 0 \), we show that (5) admits a nontrivial solution.

2. Notations and preliminaries. In this section we introduce the notations and we collect some facts which will be used along the paper.

Let
\[
\mathbb{R}^{N+1}_+ = \{(x,\xi) \in \mathbb{R}^{N+1} : x \in \mathbb{R}^N, \xi > 0\}
\]
be the upper half-space in \( \mathbb{R}^{N+1} \).

We denote by \( S_T = (0,T)^N \times (0,\infty) \) the half-cylinder in \( \mathbb{R}^{N+1}_+ \) with basis \( \partial^0 S_T = (0,T)^N \times \{0\} \) and lateral boundary \( \partial_L S_T = \partial(0,T)^N \times [0,\infty) \).
With \( \|v\|_{L^r(S_T)} \) we always denote the norm of \( v \in L^r(S_T) \) and with \( |u|_{L^r(0,T)^N} \) the \( L^r(0,T)^N \) norm of \( u \in L^r(0,T)^N \).

Let \( s \in (0,1) \) and \( m > 0 \). Let \( A \subset \mathbb{R}^N \) be a domain.

We define \( L^2(A \times \mathbb{R}_+, \xi^{1-2s}) \) as the set of measurable functions \( v \) on \( A \times \mathbb{R}_+ \) such that
\[
\int_{A \times \mathbb{R}_+} \xi^{1-2s}v^2 \, dx \, d\xi < \infty.
\]
We say that \( v \in H^1_m(A \times \mathbb{R}_+, \xi^{1-2s}) \) if \( v \) and its weak gradient \( \nabla v \) belong to \( L^2(A \times \mathbb{R}_+, \xi^{1-2s}) \). The norm of \( v \) in \( H^1_m(A \times \mathbb{R}_+, \xi^{1-2s}) \) is given by
\[
\int_{A \times \mathbb{R}_+} \xi^{1-2s}(|\nabla v|^2 + m^2v^2) \, dx \, d\xi < \infty.
\]
It is clear that \( H^1_m(A \times \mathbb{R}_+, \xi^{1-2s}) \) is a Hilbert space with the inner product
\[
\langle \xi^{1-2s}(\nabla v \nabla z + m^2vz) \rangle \, dx \, d\xi.
\]
When \( m = 1 \), we set \( H^1(A \times \mathbb{R}_+, \xi^{1-2s}) = H^1_1(A \times \mathbb{R}_+, \xi^{1-2s}) \).

We denote by \( C_\infty^T(\mathbb{R}^N) \) the space of functions \( u \in C_\infty(\mathbb{R}^N) \) such that \( u \) is \( T \)-periodic in each variable, that is
\[
u(x + Te_i) = u(x) \text{ for all } x \in \mathbb{R}^N, i = 1, \ldots, N.
\]
Let \( u \in C_\infty^T(\mathbb{R}^N) \). Then we know that
\[
u(x) = \sum_{k \in \mathbb{Z}^N} c_k e^{ik \omega \cdot x} \sqrt{\frac{1}{TN}}
\]
for all \( x \in \mathbb{R}^N \), where
\[
\omega = \frac{2\pi}{T} \quad \text{and} \quad c_k = \frac{1}{\sqrt{TN}} \int_{(0,T)^N} u(x)e^{-ik \omega \cdot x} \, dx \quad (k \in \mathbb{Z}^N)
\]
are the Fourier coefficients of \( u \). We define the fractional Sobolev space \( H^m_T \) as the closure of \( C_\infty^T(\mathbb{R}^N) \) under the norm
\[
|u|_{H^m_T} := \sqrt{\sum_{k \in \mathbb{Z}^N} (\omega^2|k|^2 + m^2)^s |c_k|^2}.
\]
We will also use the notation
\[
|u|_{H^m_T} := \sqrt{\sum_{k \in \mathbb{Z}^N} \omega^2|k|^{2s}|c_k|^2}
\]
to denote the Gagliardo semi-norm of \( u \).

When \( m = 1 \), we set \( H^1_T = H^1_1 \) and \( |\cdot|_{H^1_T} = |\cdot|_{H^1_1} \). Finally we introduce the functional space \( X^s_{m,T} \) defined as the completion of
\[
C_\infty^T(\mathbb{R}^{N+1}_+) = \left\{ v \in C_\infty(\mathbb{R}^{N+1}_+) : v(x + Te_i, \xi) = v(x, \xi) \right\}
\]
for every \( (x, \xi) \in \mathbb{R}_+^{N+1}, i = 1, \ldots, N \) under the \( H^1_m(S_T, \xi^{1-2s}) \) norm
\[
|v|_{X^s_{m,T}} := \sqrt{\int_{S_T} \xi^{1-2s}(|\nabla v|^2 + m^2v^2) \, dx \, d\xi}.
\]
If \( m = 1 \), we set \( X^*_m = X^*_{1,T} \) and \( || \cdot ||_{X^*_m} = || \cdot ||_{X^*_{1,T}} \). In order to lighten the notation, it is convenient to omit the indices \( s \) and \( T \) (which are fixed) appearing in the Sobolev spaces and norms just defined. From now on we will write \( X_m, X, H_m, H, \) \( || \cdot ||_{X_m}, || \cdot ||_X, || \cdot ||_H \) and \( || \cdot ||_{H_m} \).

Now we recall that it is possible to define a trace operator from \( X_m \) to \( H_m \):

**Theorem 2.1.** [2, 3] There exists a surjective linear operator \( \text{Tr} : X_m \to H_m \) such that:

(i) \( \text{Tr}(v) = v|_{\partial \mathcal{S}_T} \) for all \( v \in C^0_c(\mathbb{R}^{N+1}_+, T) \cap X_m \);

(ii) \( \text{Tr} \) is bounded and

\[
\kappa_s ||\text{Tr}(v)||^2_{H_m} \leq ||v||^2_{X_m} \text{ for every } v \in X_m. \tag{6}
\]

In particular, equality holds in (7) for some \( v \in X_m \) if and only if \( v \) weakly solves

\[-\text{div}(\xi^{1-2s}\nabla v) + m^2\xi^{1-2s}v = 0 \text{ in } \mathcal{S}_T.\]

Finally we have the following embeddings:

**Theorem 2.2.** [2, 3] Let \( N > 2s \). Then \( \text{Tr}(X_m) \) is continuously embedded in \( L^q(0,T)^N \) for any \( 1 \leq q \leq 2s \). Moreover, \( \text{Tr}(X_m) \) is compactly embedded in \( L^q(0,T)^N \) for any \( 1 \leq q < 2s \).

3. Extension method. As mentioned in the Introduction, crucial to our results is that the non-local operator \((-\Delta_x + m^2)^s\) can be realized through a local problem in \( \mathcal{S}_T \). More precisely it holds the following theorem.

**Theorem 3.1.** [2, 3] Let \( u \in H^s_m \). Then there exists a unique \( v \in X_m \) such that

\[
\begin{cases}
-\text{div}(\xi^{1-2s}\nabla v) + m^2\xi^{1-2s}v = 0 & \text{in } \mathcal{S}_T \\
v(x,0) = v(x,T) & \text{on } \partial L \mathcal{S}_T \\
v(\cdot,0) = u & \text{on } \partial^0 \mathcal{S}_T
\end{cases} \tag{7}
\]

and

\[-\lim_{\xi \to 0} \xi^{1-2s} \frac{\partial v}{\partial \xi} (x, \xi) = \kappa_s (-\Delta_x + m^2)^s u(x) \text{ in } H^s_m, \tag{8}\]

where \( H^s_m \) is the dual of \( H_m \). We call \( v \) the extension of \( u \).

**Remark 1.** In [2, 3] we proved that if \( u = \sum_{k \in \mathbb{Z}^N} c_k \frac{\psi_k(x)}{\sqrt{T^N}} \in H_m \), then its extension is given by

\[ v(x, \xi) = \sum_{k \in \mathbb{Z}^N} c_k \theta_k(\xi) \frac{\psi_k(x)}{\sqrt{T^N}} \in X_m \]

where \( \theta_k(\xi) = \theta(\sqrt{\omega^2|k|^2 + m^2} \xi) \) and \( \theta \in H^1(\mathbb{R}_+, \xi^{1-2s}) \) solves the following ODE

\[
\begin{cases}
\theta'' + \frac{1-2s}{\xi^2} \theta' - \theta = 0 & \text{in } \mathbb{R}_+ \\
\theta(0) = 1 \text{ and } \theta(\infty) = 0
\end{cases} \tag{9}
\]

It is known (see [12]) that \( \theta(\xi) = \frac{2}{\Gamma(s)} \left( \frac{\xi}{2} \right)^s K_s(\xi) \) where \( K_s \) is the Bessel function of second kind with order \( s \), and being \( K'_s = \frac{s}{y} K_s - K_{s-1} \), we can see that

\[ \kappa_s := \int_0^\infty \xi^{1-2s}(|\theta'(\xi)|^2 + |\theta(\xi)|^2) d\xi = -\lim_{\xi \to 0} \xi^{1-2s} \theta'(\xi) = 2^{1-2s} \frac{\Gamma(1-s)}{\Gamma(s)}. \]

By using this fact we can deduce that \( ||v||_{X_m} = \sqrt{\kappa_s} ||v||_{H_m} \).
Now, we take advantage of the previous result to reformulate nonlocal problems with periodic boundary conditions, in a local way.

Let \( g \in H^s \) and consider the following two problems:

\[
\begin{cases}
-\Delta x + m^2)^2 u = g & \text{in } (0, T)^N \\
u(x + T e_i) = u(x) & \text{for all } x \in \mathbb{R}^N
\end{cases}
\]

and

\[
\begin{cases}
-\text{div}(\xi^{1-2s} \nabla v) + m^2 \xi^{1-2s} v = 0 & \text{in } S_T \\
v|_{\{x_i = 0\}} = v|_{\{x_i = T\}} & \text{on } \partial L S_T \\
\frac{\partial v}{\partial \nu} = g(x) & \text{on } \partial^0 S_T
\end{cases}
\]

\[ 
\text{Definition 3.2.} \quad \text{We say that } v \in X_m \text{ is a weak solution to } (11) \text{ if for every } \phi \in X_m \text{ it holds}
\]

\[ 
\int_{S_T} \xi^{1-2s}(\nabla v \nabla \phi + m^2 v \phi) \, dx = \kappa_s \langle g, \text{Tr}(\phi) \rangle_{H^s_m, \overline{H^s_m}}.
\]

\[ 
\text{Definition 3.3.} \quad \text{We say that } u \in H^s \text{ is a weak solution to } (10) \text{ if } u = \text{Tr}(v) \text{ and } v \text{ is a weak solution to } (11).
\]

\[ 
\text{Remark 2.} \quad \text{Later, with abuse of notation, we will denote by } v(\cdot, 0) \text{ the trace } \text{Tr}(v) \text{ of a function } v \in X_m.
\]

4. Periodic solutions in the cylinder \( S_T \). This section is devoted to prove the existence of a nontrivial solution to (2). As explained in the previous section, we know that the study of (2) is equivalent to investigate the existence of weak solutions to

\[
\begin{cases}
-\text{div}(\xi^{1-2s} \nabla v) + m^2 \xi^{1-2s} v = 0 & \text{in } S_T := (0, T)^N \times (0, \infty) \\
v|_{\{x_i = 0\}} = v|_{\{x_i = T\}} & \text{on } \partial L S_T := \partial(0, T)^N \times [0, \infty). \\
\frac{\partial v}{\partial \nu} = \kappa_s [m^2 v + f(x, v)] & \text{on } \partial^0 S_T := (0, T)^N \times \{0\}.
\end{cases}
\]

For simplicity, we will assume that \( \kappa_s = 1 \).

Let us introduce the following functional on \( X_m \)

\[ 
J_m(v) = \frac{1}{2} \int_{X_m} \int_{(0, T)^N} F(x, v) \, dx.
\]

By conditions (f2)-(f4), we know that for any \( \varepsilon > 0 \) there exists \( C_\varepsilon > 0 \) such that

\[ 
|f(x, t)| \leq 2\varepsilon |t| + (p + 1)C_\varepsilon |t|^p & \forall t \in \mathbb{R}, \forall x \in (0, T)^N
\]

and

\[ 
|F(x, t)| \leq \varepsilon |t|^2 + C_\varepsilon |t|^{p+1} & \forall t \in \mathbb{R}, \forall x \in (0, T)^N.
\]

Then, by Theorem 2.2 it follows that \( J_m \) is well defined on \( X_m \) and \( J_m \in C^1 (X_m, \mathbb{R}) \). By using Theorem 2.1 we notice that the quadratic part of \( J_m \) is nonnegative, that is

\[ 
||v||_{X_m}^2 - m^2 |v(\cdot, 0)|_{L^2(0, T)^N}^2 \geq 0,
\]

and the equality holds in (15) if and only if

\[ 
v(x, \xi) = c\theta(m\xi) \text{ for some } c \in \mathbb{R}
\]

(see Theorem 7 in [3]).

As observed in \([2, 3]\), \( X_m \) can be splitted as

\[ 
X_m = \theta(m\xi) > \oplus \left\{ v \in X_m : \int_{(0, T)^N} v(x, 0) \, dx = 0 \right\} =: Y_m \oplus Z_m
\]
where \( \dim \mathbb{Y}_m < \infty \) and \( \mathbb{Z}_m \) is the orthogonal complement of \( \mathbb{Y}_m \) with respect to the inner product in \( \mathbb{X}_m \).

In order to find critical points of \( \mathcal{J}_m \), we will make use of a suitable version of the Linking Theorem due to Li and Wang \[15\].

Firstly we recall the following definitions.

**Definition 4.1.** Let \( (X, \| \cdot \|_X) \) be a real Banach space with its dual space \( (X', \| \cdot \|_{X'}) \), \( J \in \mathcal{C}^1(X, \mathbb{R}) \) and \( c \in \mathbb{R} \). We say that \( \{v_n\} \subset X \) is a Cerami sequence for \( J \) at the level \( c \) if

\[
J(v_n) \to c \quad \text{and} \quad (1 + \|v_n\|_X)\|J'(v_n)\|_{X'} \to 0.
\]

**Definition 4.2.** We say that \( J \) satisfies the \((C)_c\) condition if any Cerami sequence at the level \( c \) has a strongly convergent subsequence.

Now we are ready to state the following theorem.

**Theorem 4.3.** \[15\] Let \( (X, \| \cdot \|_X) \) be a real Banach space with \( X = Y \oplus Z \), where \( Y \) is finite dimensional. Let \( \rho > r > 0 \) and let \( z \in Z \) be a fixed element such that \( \|z\| = r \). Define

\[
M := \{v = y + \lambda z : y \in Y, \|v\|_X \leq \rho, \lambda \geq 0\},
\]

\[
M_0 := \{v = y + \lambda z : y \in Y, \|v\|_X = \rho, \lambda \geq 0 \text{ or } \|v\|_X \leq \rho, \lambda = 0\},
\]

\[
N_r := \{v \in Z : \|v\|_X = r\}.
\]

Let \( J \in \mathcal{C}^1(X, \mathbb{R}) \) be such that

\[
b := \inf_{N_r} J > a := \max_{M_0} J.
\]

If \( J \) satisfies the \((C)_\alpha\) condition with

\[
\alpha := \inf_{\gamma \in \Gamma} \max_{v \in M} J(\gamma(v)) \quad \text{and} \quad \Gamma := \{\gamma \in \mathcal{C}(M, X) : \gamma = \text{Id on } M_0\},
\]

then \( \alpha \) is a critical value of \( J \).

In what follows, we verify that \( \mathcal{J}_m \) satisfies the assumptions of the above Theorem 4.3. We begin proving a series of lemmas, which ensure that \( \mathcal{J}_m \) possesses a Linking geometry.

**Lemma 4.4.** \( \mathcal{J}_m \leq 0 \) on \( \mathbb{Y}_m \).

**Proof.** Firstly we show that \( F(x, t) \geq 0 \) for any \( x \in \mathbb{R}^N \) and \( t \in \mathbb{R} \). By (f6) we deduce that \( f(x, t) - 2F(x, t) \geq 0 \) for any \( t \geq 0 \). Let \( t > 0 \). For \( x \in \mathbb{R}^N \) we have

\[
\frac{\partial}{\partial t} \left( \frac{F(x, t)}{t^2} \right) = \frac{t^2f(x, t) - 2tF(x, t)}{t^4} \geq 0.
\]

By (f2) we know that

\[
\lim_{t \to 0} \frac{F(x, t)}{t^2} = 0
\]

so we deduce that \( F(x, t) \geq 0 \) for \( t \geq 0 \). Arguing similarly for the case \( t \leq 0 \), eventually we obtain that \( F \geq 0 \) in \( \mathbb{R}^N \times \mathbb{R} \). As a consequence, recalling that \( \|v\|^2_{\mathbb{X}_m} = m^2s|v(\cdot, 0)|^2_{L^2((0,T)^N)} \) for any \( v \in \mathbb{Y}_m \) by (16), we can see that

\[
\mathcal{J}_m(v) = -\int_{\partial^0 S_T} F(x, v) \, dx \leq 0.
\]
Lemma 4.5. There exists \( r_m > 0 \) such that
\[
J_m(v) := \inf \{ J_m(v) : v \in Z_m, ||v||_{X_m} = r_m \} > 0.
\]

Proof. By using Lemma 3 in [3] we know that there is a constant \( C_m > 0 \) such that
\[
||v||^2_{X_m} - m^2s ||v||^2_{L^2_{\omega}(0,T)^N} \geq C_m ||v||^2_{X_m} - \rho \]
for any \( v \in Z_m \). Then, taking into account (14), (18) and Theorem 4 we have
\[
J_m(v) \geq C_m ||v||^2_{X_m} - \varepsilon ||v||^2_{L^2_{\omega}(0,T)^N} - C_m ||v||^2_{L_{r+1}(0,T)^N}
\]
for any \( v \in Z_m \). Choosing \( \varepsilon \in (0, mC_m) \), we can find \( r_m > 0 \) such that
\[
b_m := \inf \{ J_m(v) : v \in Z_m \text{ and } ||v||_{X_m} = r_m \} > 0.
\]

\( \square \)

Lemma 4.6. There exists \( \rho_m > r_m \) and \( z \in Z_m \) with \( ||z||_{X_m} = r_m \) such that, denoting by
\[
M^m = \{ v = y + \lambda z : y \in Y_m, ||v||_{X_m} \leq \rho_m, \lambda \geq 0 \}
\]
and
\[
M_0^m = \{ v = y + \lambda z : y \in Y_m, ||v||_{X_m} = \rho_m, \lambda \geq 0 \text{ or } ||v||_{X_m} \leq \rho_m, \lambda = 0 \},
\]
we have
\[
\max_{M_0^m} J_m(v) \leq 0.
\]

Proof. By using (f2) and (f5), we know that for any \( A > 0 \) there exists \( B_A > 0 \) such that
\[
F(x,t) \geq A|t|^2 - B_A \quad \forall t \in \mathbb{R}, \forall x \in [0,T]^N.
\]
(19)

Let us define
\[
w = \prod_{i=1}^{N} \sin(\omega x_i) \frac{1}{\xi + 1}.
\]
(20)

We observe that \( w \in Z_m \) (since \( \int_0^T \sin(\omega x) \, dx = 0 \)) and
\[
||w||^2_{X_m} = N \left( \prod_{i=1}^{N-1} \int_0^T \sin^2(\omega x_i) \, dx_i \right) \omega^2 \left( \int_0^T \cos^2(\omega x) \, dx \right) \left( \int_0^\infty \xi^{1-2s} \frac{d\xi}{(\xi + 1)^2} \right)
\]
\[
+ \left( \prod_{i=1}^{N} \int_0^T \sin^2(\omega x_i) \, dx_i \right) \left( \int_0^\infty \xi^{1-2s} \frac{d\xi}{(\xi + 1)^2} \right)
\]
\[
+ m^2 \left( \prod_{i=1}^{N} \int_0^T \sin^2(\omega x_i) \, dx_i \right) \left( \int_0^\infty \xi^{1-2s} \frac{d\xi}{(\xi + 1)^2} \right)
\]
(21)

Since
\[
||w(\cdot,0)||^2_{L^2(0,T)^N} = \prod_{i=1}^{N} \int_0^T \sin^2(\omega x_i) \, dx_i
\]
and
\[
\int_0^T \sin^2(\omega x) \, dx = \frac{T}{2} = \int_0^T \cos^2(\omega x) \, dx,
\]
and
\[
\int_0^T \sin^2(\omega x) \, dx = \frac{T}{2} = \int_0^T \cos^2(\omega x) \, dx,
\]
by (21) it follows that there exist $C_1, C_2, C_3 > 0$ (independent of $m$) such that
\[ C_1 |w(\cdot, 0)|_{L^2(0,T)}^2 \leq ||w||_{X_m}^2 \leq (C_2 + m^2 C_3) |w(\cdot, 0)|_{L^2(0,T)}^2. \] (22)

Now, set $z = \frac{r_m w}{||w||_{X_m}}$. It is clear that $z \in Z_m$ and $||z||_{X_m} = r_m$.

Moreover, by (22) we obtain
\[ \frac{r_m^2}{C_2 + m^2 C_3} \leq |z(\cdot, 0)|_{L^2(0,T)}^2 \leq \frac{r_m^2}{C_1}. \] (23)

Take $v = y + \lambda z \in Y_m + R_+ z$. We recall that if $y \in Y_m$ then $g(x, \xi) = d_m \theta(m \xi)$, $d_m \in R$ and $||y||_{X_m}^2 = m^{2s} |y(\cdot, 0)|_{L^2(0,T)}^2$.

Then, by using (23), we have
\[
||v||_{X_m}^2 = ||y||_{X_m}^2 + \lambda^2 ||z||_{X_m}^2
\leq \max\{m^{2s}, 1\} ||y(\cdot, 0)||_{L^2(0,T)}^2 + \lambda^2 ||z||_{X_m}^2
\leq \max\{m^{2s}, 1\} \{||y(\cdot, 0)||_{L^2(0,T)}^2 + \lambda^2 [C_2 + m^2 C_3] |z(\cdot, 0)|_{L^2(0,T)}^2\}
\leq \max\{m^{2s}, 1\} \max\{1, C_2 + m^2 C_3\} \{||y(\cdot, 0)||_{L^2(0,T)}^2 + \lambda^2 |z(\cdot, 0)|_{L^2(0,T)}^2\}
= C(m, s) ||v(\cdot, 0)||_{L^2(0,T)}^2.
\] (24)

Fix $A > \frac{C(m, s)}{2}$. In view of (19) and (24), for any $v = y + \lambda z \in Y_m + R_+ z$ we get
\[
J_m(v) = \frac{1}{2} ||v||_{X_m}^2 - \frac{m^{2s}}{2} |v(\cdot, 0)|_{L^2(0,T)}^2 - \int_{\partial_0 S_T} F(x, v) \, dx
\leq \frac{1}{2} ||v||_{X_m}^2 - A |v(\cdot, 0)|_{L^2(0,T)}^2 + B A T^N
\leq \left[ \frac{1}{2} - \frac{A}{C(m, s)} \right] ||v||_{X_m}^2 + B A T^N \to -\infty
\]
as $||v||_{X_m} = \rho_m \to \infty$. By Lemma 4.4 we also know that $J_m(v) \leq 0$ on $Y_m$.

Take $\rho_m$ large enough, $r_m$ small enough with $\rho_m > 1 > r_m > 0$.

Then
\[
\max_{M_0} J_m(v) \leq 0
\]
and
\[ b_m := \inf_{N_{\rho_m}} J_m > a_m := \max_{M_0} J_m. \]

\[ \square \]

In the forthcoming Lemma we check that the functional $J_m$ satisfies the Cerami condition:

**Lemma 4.7.** Let $c \in R$. Let $\{v_j\} \subset X_m$ be a sequence such that
\[
J_m(v_j) \to c
\] (25)
and
\[ (1 + ||v_j||_{X_m}) ||J_m'(v_j)||_{X_m} \to 0 \] (26)
as $j \to \infty$. Then there exist a subsequence $\{v_{j_n}\} \subset \{v_j\}$ and a function $v \in X_m$ such that $v_{j_n} \to v$ in $X_m$. 
Proof. We start by proving that \( \{v_j\} \) is bounded in \( \mathcal{X}_m \). We proceed as in [17]. We argue by contradiction and assume that

\[
|v_j|_{\mathcal{X}_m} \to \infty
\]

as \( j \to +\infty \).

Let us define

\[
z_j = \frac{v_j}{|v_j|_{\mathcal{X}_m}}.
\]

Then \( |z_j|_{\mathcal{X}_m} = 1 \) and by using Theorem 2.2 we can suppose, up to a subsequence, that

\[
z_j \rightharpoonup z \text{ in } \mathcal{X}_m
\]

and there exists \( h \in L^{p+1}(0, T)^N \) such that

\[
|z_j(x, 0)| \leq h(x) \quad \text{a.e. in } x \in (0, T)^N, \quad \text{for all } j \in \mathbb{N}.
\]

Now we distinguish two cases. Firstly we consider the case \( z \equiv 0 \).

As in [14], we can choose \( \{t_j\}_{j \in \mathbb{N}} \subset [0, 1] \) such that

\[
J_m(t_j v_j) = \max_{t \in [0, 1]} J_m(t v_j).
\]

Since \( |v_j|_{\mathcal{X}_m} \to \infty \) we can take \( r_n = 2\sqrt{n} \) such that

\[
r_n|v_j|_{\mathcal{X}_m}^2 \in (0, 1)
\]

provided \( j \) is large enough. By (29) and the continuity of \( F \), we can see

\[
F(x, r_n z_j(x, 0)) \to F(x, r_n z(x, 0)) \text{ a.e. } x \in (0, T)^N
\]

as \( j \to \infty \) and \( n \in \mathbb{N} \). On the other hand, integrating \( (4) \) and exploiting (30) we get

\[
|F(x, r_n z_j(x, 0))| \leq c_1 |r_n z_j(x, 0)| + c_2 |r_n z_j(x, 0)|^{p+1}
\]

\[
\leq c_1 r_n h(x) + c_2 r_n p h(x)^{p+1} \in L^1(0, T)^N,
\]

a.e. \( x \in (0, T)^N \) and \( n, j \in \mathbb{N} \). Then, taking into account (34), (35) and by using the Dominated Convergence Theorem we deduce that

\[
F(x, r_n z_j(x, 0)) \to F(x, r_n z(x, 0)) \text{ in } L^1(0, T)^N.
\]

Since \( F(\cdot, 0) = 0 \) and (31) holds true, (36) yields

\[
\int_{\partial^p S_T} F(x, r_n z_j) \, dx \to 0 \text{ as } j \to \infty
\]

for any \( n \in \mathbb{N} \). Then (29), (32), (33) and (37) imply

\[
J_m(t_j v_j) \geq J_m(r_n z_j) \geq 2n - \int_{\partial^p S_T} F(x, r_n z_j) \, dx \geq n
\]

provided \( j \) is large enough and for any \( n \in \mathbb{N} \). As a consequence

\[
J_m(t_j v_j) \to \infty \text{ as } j \to \infty.
\]
Since $J_m(0) = 0$ and $J_m(v_j) \to c$ we deduce that $t_j \in (0, 1)$. Thus, by (32) we have
\[
\langle J_m'(t_jv_j), t_jv_j \rangle = t_j \frac{d}{dt} \bigg|_{t=t_j} J_m(t_jv_j) = 0.
\]
(39)

Taking into account (f6), (25), (26) and (39) we get
\[
\frac{2}{\gamma} J_m(t_jv_j) = \frac{2}{\gamma} \left( J_m(t_jv_j) - \frac{1}{2} \langle J_m'(t_jv_j), t_jv_j \rangle \right)
\]
\[
= \frac{1}{\gamma} \int \partial^\circ S_T G(x, t_jv_j) \, dx
\]
\[
\leq \int \partial^\circ S_T G(x, v_j) \, dx
\]
\[
= 2[J_m(v_j) - \frac{1}{2} \langle J_m'(v_j), v_j \rangle] \to 2c \quad \text{as } j \to \infty
\]
which contradicts (38).

Secondly, we suppose that $z \not\equiv 0$.

Thus the set $\Omega = \{ x \in (0, T)^N : z(x, 0) \neq 0 \}$ has positive Lebesgue measure and by
using (28), (29) and (40) we get
\[
|v_j(x, 0)| \to \infty \quad \text{a.e.} \quad x \in \Omega \quad \text{as } j \to \infty.
\]
(41)

By (25), (27) and $F \geq 0$ we can easily deduce that
\[
\alpha(1) = \frac{1}{2} - \frac{m^2}{2} \frac{|v_j(\cdot, 0)|^2_{L^2(0,T)^N}}{||v_j||^2_{X_m}} - \int_{\partial^\circ S_T} \frac{F(x, v_j(x, 0))}{||v_j||^2_{X_m}} \, dx
\]
\[
\leq \frac{1}{2} - \int_{\Omega} \frac{F(x, v_j(x, 0))}{||v_j||^2_{X_m}} \, dx \quad \text{as } j \to \infty.
\]
(42)

Now, taking into account (f5), (28), (29), (41) and by using Fatou’s Lemma we obtain
\[
\int_{\Omega} \frac{F(x, v_j(x, 0))}{||v_j||^2_{X_m}} \, dx \to \infty \quad \text{as } j \to \infty.
\]
(43)

Putting together (42) and (43) we get a contradiction.

Thus the sequence $\{v_j\}$ is bounded in $X_m$. By Theorem 2.2 we can assume, up to a subsequence, that
\[
v_j \to v \quad \text{in } X_m
\]
\[
v_j(\cdot, 0) \to v(\cdot, 0) \quad \text{in } L^{p+1}(0,T)^N
\]
\[
v_j(\cdot, 0) \to v(\cdot, 0) \quad \text{a.e. in } (0,T)^N
\]
as $j \to \infty$ and there exists $h \in L^{p+1}(0,T)^N$ such that
\[
|v_j(x, 0)| \leq h(x) \quad \text{a.e.} \quad x \in (0,T)^N, \quad \text{for all } j \in \mathbb{N}.
\]
(45)

Taking into account (f2), (f4), (41), (45) and the Dominated Convergence Theorem we get
\[
\int_{\partial^\circ S_T} f(x, v_j) v_j \, dx \to \int_{\partial^\circ S_T} f(x, v) v \, dx
\]
(46)
and
\[
\int_{\partial^\circ S_T} f(x, v_j) v \, dx \to \int_{\partial^\circ S_T} f(x, v) v \, dx
\]
(47)
as $j \to \infty$.

By using (26) and the boundedness of $\{v_j\}_{j \in \mathbb{N}}$ in $\mathfrak{X}_m$, we deduce that $(J'_m(v_j), v_j) \to 0$, that is

$$\|v_j\|^2_{\mathfrak{X}_m} - m^{2s}|v_j(\cdot,0)|_{L^2(0,T)}^2 - \int_{\partial^s\mathcal{S}_T} f(x,v_j)v_j\,dx \to 0$$

as $j \to \infty$. By (44), (46) and (48) we have

$$\|v_j\|^2_{\mathfrak{X}_m} \to m^{2s}|v(\cdot,0)|_{L^2(0,T)}^2 - \int_{\partial^s\mathcal{S}_T} f(x,v)\,dx.$$  \hfill (49)

Moreover, by (26) and $v \in \mathfrak{X}_m$, we obtain $(J'_m(v_j), v) \to 0$ as $j \to \infty$, that is

$$(v_j, v)_{\mathfrak{X}_m} - m^{2s}(v_j, v)_{L^2(0,T)} - \int_{\partial^s\mathcal{S}_T} f(x,v_j)v\,dx \to 0$$  \hfill (50)

Taking into account (44), (45), (47) and (50) we get

$$\|v\|^2_{\mathfrak{X}_m} = m^{2s}|v(\cdot,0)|_{L^2(0,T)}^2 - \int_{\partial^s\mathcal{S}_T} f(x,v)\,dx.$$  \hfill (51)

Thus, (49) and (51) imply that

$$\|v_j\|^2_{\mathfrak{X}_m} \to \|v\|^2_{\mathfrak{X}_m} \text{ as } j \to \infty.$$  \hfill (52)

Since $\mathfrak{X}_m$ is a Hilbert space, we have

$$\|v_j - v\|^2_{\mathfrak{X}_m} = \|v_j\|^2_{\mathfrak{X}_m} + \|v\|^2_{\mathfrak{X}_m} - 2(v_j, v)_{\mathfrak{X}_m}$$

and using $v_j \to v$ in $\mathfrak{X}_m$ and (52) we can conclude that $v_j \to v$ in $\mathfrak{X}_m$, as $j \to \infty$. \hfill \Box

**Proof of Theorem 4.1.** Putting together Lemma 4.4 - Lemma 4.7 we can see that $J_m$ satisfies the assumptions of Theorem 4.3. Then, we deduce that for any fixed $m > 0$, there exists a function $v_m \in \mathfrak{X}_m$ such that

$$J_m(v_m) = \alpha_m \text{ and } J'_m(v_m) = 0,$$  \hfill (53)

where

$$\alpha_m := \inf_{\gamma \in \Gamma_m} \max_{v \in M^m} J_m(\gamma(v))$$  \hfill (54)

and

$$\Gamma_m := \{ \gamma \in C(M^m, \mathfrak{X}_m) : \gamma = Id \text{ on } M^m \}. $$

Hence $v_m$ is a nontrivial weak solution to (12), and by Theorem 9 in [3], it follows that $v_m(\cdot,0) \in C^{0,\alpha}([0,T]^N)$ for some $\alpha \in (0,1)$.

5. **Periodic solution in the case $m = 0$.** In this last section, we show that it is possible to find a nontrivial weak solution to (12). In Section 4 we proved that for any $m > 0$ there exists $v_m \in \mathfrak{X}_m$ such that

$$J_m(v_m) = \alpha_m \quad \text{and} \quad J'_m(v_m) = 0,$$  \hfill (55)

where $\alpha_m$ is defined as in (54). Now, we prove that we can estimate from below and from above the critical levels of the functional $J_m$ independently of $m$, when $m$ is sufficiently small. This allows us to pass to the limit in (12) as $m \to 0$ and to prove the existence of a nontrivial solution to

$$\begin{cases}
-\text{div}(\xi^{1-2s}\nabla v) = 0 & \text{in } \mathcal{S}_T := (0,T)^N \times (0,\infty) \\
v|_{\{x_t=0\}} = v|_{\{x_t=T\}} & \text{on } \partial_t \mathcal{S}_T := \partial(0,T)^N \times [0,\infty) \\
\frac{\partial v}{\partial x} = f(x,v) & \text{on } \partial^\delta \mathcal{S}_T := (0,T)^N \times \{0\}.
\end{cases}$$  \hfill (56)
Let us assume that $0 < m < m_0 := \frac{\omega^{2s}}{2}$.

Firstly, we prove that there exists $K_1 > 0$ independent of $m$, such that
\[ \alpha_m \geq K_1 \quad \text{for all } 0 < m < m_0. \] (57)

In order to achieve our aim, we estimate the $L^q$-norm of the trace of $v$, with $q \in [2, 2^*_m]$. Let $v \in Z_m$ and we denote by $c_k$ its Fourier coefficients. By $c_0 = 0$ and Theorem 2.1, it follows that
\[
|v(\cdot, 0)|^{2}_L^{(0,T)^N} = \sum_{|k| \geq 1} |c_k|^2 \leq \frac{1}{\omega^{2s}} \sum_{|k| \geq 1} (\omega^2|k|^2 + m^2)^s |c_k|^2
= \frac{1}{\omega^{2s}} |v(\cdot, 0)|^2_{\mathcal{B}_m} \leq \frac{1}{\omega^{2s}} ||v||^2_{\mathcal{B}_m}. \] (58)

Now, fix $2 < q < 2^*_m$ and we denote by $q'$ its conjugate exponent. Taking into account that $c_0 = 0$, and by using Hölder inequality and Theorem 2.1, we get
\[
\left( \sum_{|k| \geq 1} |c_k|^q \right)^{\frac{1}{q}} \leq \left[ |v(\cdot, 0)|_{\mathcal{B}_m} \left( \sum_{|k| \geq 1} \left( (\omega^2|k|^2 + m^2)^s \right)^{-\frac{q'}{2}} \right)^{\frac{2}{2-q'}} \right]^{\frac{1}{q}}
\leq \omega^{-s} \left( \sum_{|k| \geq 1} |k|^{-\frac{2q'}{2-q'}} \right)^{\frac{2}{2-q'}} |v(\cdot, 0)|_{\mathcal{B}_m}
\leq \frac{\omega^{-s}}{\sqrt{K_1}} \left( \sum_{|k| \geq 1} |k|^{-\frac{2q'}{2-q'}} \right)^{\frac{2}{2-q'}} ||v||_{\mathcal{B}_m} < \infty
\]
because of $1 < \frac{2N}{N+2s} < q' < 2$.

As a consequence, by using the Theorem of Hausdorff-Young-Riesz (see pages 101-102 in [26]), it follows that
\[
|v(\cdot, 0)|^{p+1}_L^{(0,T)^N} \leq \left( \frac{1}{\sqrt{\mathcal{F}}} \right)^{\frac{p+1}{p+1} - 1} \left( \sum_{|k| \geq 1} |c_k|^{q'_{p+1}} \right)^{\frac{1}{p+1}}
\]
and taking $q = p + 1$ we have
\[
|v(\cdot, 0)|^{p+1}_L^{(0,T)^N} \leq \left( \frac{1}{\sqrt{\mathcal{F}}} \right)^{\frac{p+1}{p+1} - 1} \left( \sum_{|k| \geq 1} |c_k|^{q'_{p+1}} \right)^{\frac{1}{p+1}}
\leq C'' ||v||_{\mathcal{B}_m} \] (59)

for some $C'' := C''(T, N, s, p) > 0$ independent of $m$.

Then, by using (14) with $0 < \varepsilon < \frac{\omega^{2s}}{4}$ and exploiting (58) and (59), we can see that for every $0 < m < m_0 = \frac{\omega^{2s}}{2}$
\[
\mathcal{J}_m(v) = \frac{1}{2} \int \int_{\mathcal{S}_x} y^{1-2s}((\nabla v)^2 + m^2 v^2) \, dx dy - \frac{m^{2s}}{2} \int_{\partial^0 \mathcal{S}_x} \|v\|^2 \, dx - \int_{\partial^0 \mathcal{S}_x} F(x,v) \, dx
\geq \frac{1}{2} ||v||^2_{\mathcal{B}_m} - \left( \frac{m}{2} + \varepsilon \right) \|v(\cdot, 0)||^2_L(0,T)^N - C_{\varepsilon} ||v(\cdot, 0)||^{p+1}_{L^{p+1}(0,T)^N}
\geq \left[ \frac{1}{2} - \frac{1}{\omega^{2s}} \left( \frac{m}{2} + \varepsilon \right) \right] ||v||^2_{\mathcal{B}_m} - C_{\varepsilon}'' ||v||^{p+1}_{\mathcal{B}_m}
\geq \left( \frac{1}{4} - \frac{\varepsilon}{\omega^{2s}} \right) ||v||^2_{\mathcal{B}_m} - C_{\varepsilon}'' ||v||^{p+1}_{\mathcal{B}_m}.\]
Set \( b := \frac{1}{4} - \frac{\varepsilon}{\omega^2} > 0 \) and \( r := \left( \frac{b}{2C^m_b} \right)^\frac{1}{2r-1} \). Then, for every \( v \in \mathcal{X}_m \) such that 
\[
\|v\|_{\mathcal{X}_m} = r
\]
\[
\mathcal{J}_m(v) \geq br^2 - C''_b r^{p+1} = \frac{b}{2} \left( \frac{b}{2C^m_b} \right)^\frac{2}{2r-1} =: K_1
\]
from which follows \([57]\). Now, we show that there exists a positive constant \( K_2 > 0 \) independent of \( m \) such that
\[
\alpha_m \leq K_2 \quad \text{for all } 0 < m < m_0.
\]
By using \([22]\) and \( 0 < m < m_0 \) we know that
\[
C_1 \|w(\cdot,0)\|^2_{L^2(0,T)^N} \leq \|w\|^2_{\mathcal{X}_m} \leq \left( C_2 + m_0^2C_3 \right) \|w(\cdot,0)\|^2_{L^2(0,T)^N},
\]
where \( w \) is the function defined in \([20]\).

Let
\[
z := \frac{rw}{\|w\|_{\mathcal{X}_m}}.
\]
Recalling that \( 0 < m < m_0 \), we can see that \([24]\) can be replaced by
\[
\|v\|^2_{\mathcal{X}_m} \leq \max\{m_0^2,1\} \max\{1,C_2 + m_0^2C_3\}\|y(\cdot,0)\|^2_{L^2(0,T)^N} + \lambda^2 \|z(\cdot,0)\|^2_{L^2(0,T)^N}
\[
=: \tilde{C}(m_0,\delta)\|v(\cdot,0)\|^2_{L^2(0,T)^N}
\]
for any \( v = y + \lambda z \in \mathbb{Y}_m \oplus \mathbb{R}_+z \).

Now, fix \( A > \frac{\tilde{C}(m_0,\delta)}{2} \). By using \([19]\) and \( 0 < m < m_0 \), for any \( v = y + \lambda z \in \mathbb{Y}_m \oplus \mathbb{R}_+z \) we have
\[
\mathcal{J}_m(v) = \frac{1}{2} \|v\|^2_{\mathcal{X}_m} - \frac{m_0^2}{2} \|v(\cdot,0)\|^2_{L^2(0,T)^N} - \int_{\partial^0 S_T} F(x,v) \, dx
\leq \frac{1}{2} \|v\|^2_{\mathcal{X}_m} - \frac{A}{\tilde{C}(m_0,\delta)} \|v\|^2_{\mathcal{X}_m} + BT^N
\leq BT^N =: K_2.
\]
Therefore, taking into account \([58]\), \([57]\) and \([60]\), we deduce that
\[
K_1 \leq \alpha_m \leq K_2 \quad \text{for every } 0 < m < m_0.
\]

Now, we show the existence of a subsequence of \( \{v_m\} \), that for simplicity we will denote again with \( \{v_m\} \), and a function \( v \) such that
\[
v \in L^2_{loc}(S_T,\xi^{1-2s}) \text{ and } \nabla v \in L^2(S_T,\xi^{1-2s});
\]
\[
v_m \to v \text{ in } L^2_{loc}(S_T,\xi^{1-2s}) \text{ as } m \to 0;
\]
\[
\nabla v_m \to \nabla v \text{ in } L^2(S_T,\xi^{1-2s}) \text{ as } m \to 0;
\]
\[
v_m(\cdot,0) \to v(\cdot,0) \text{ in } L^q(0,T)^N \text{ for any } q \in \left[ 2, \frac{2N}{N - 2s} \right), \text{ as } m \to 0.
\]
These informations, will be useful to pass to the limit in \([3]\) as \( m \to 0 \).

Firstly, we prove that for any \( \delta > 0 \), the following inequality
\[
\|v\|^2_{L^2((0,T)^N \times (0,\delta),\xi^{1-2s})} \leq \frac{\delta^{2-2s}}{1 - s} \|v(\cdot,0)\|^2_{L^2(0,T)^N} + \frac{\delta^2}{2s} \|\partial_t v\|^2_{L^2(S_T,\xi^{1-2s})}
\]
holds true for any \( v \in \mathcal{X}_m \).
Fix $\delta > 0$ and let $v \in C^\infty_T(\mathbb{R}^{N+1})$ such that $\|v\|_{X_m} < \infty$. For any $x \in [0,T]^N$ and $\xi \in [0,\delta]$, we have

$$v(x, \xi) = v(x, 0) + \int_0^\xi \partial_\xi v(x, t) \, dt.$$ 

By using $(a + b)^2 \leq 2a^2 + 2b^2$ for all $a, b \geq 0$, we obtain

$$|v(x, \xi)|^2 \leq 2|v(x, 0)|^2 + 2\left(\int_0^\xi |\partial_\xi v(x, t)| \, dt\right)^2,$$

and applying the Hölder inequality we deduce

$$|v(x, \xi)|^2 \leq 2\left[|v(x, 0)|^2 + \left(\int_0^\xi t^1-2s|\partial_\xi v(x, t)|^2 \, dt\right)\frac{\xi^{2s}}{2s}\right]. \quad (65)$$

Multiplying both members of $(65)$ by $\xi^{1-2s}$ we have

$$\xi^{1-2s}|v(x, \xi)|^2 \leq 2\left[\xi^{1-2s}|v(x, 0)|^2 + \left(\int_0^\xi t^{1-2s}|\partial_\xi v(x, t)|^2 \, dt\right)\frac{\xi}{2s}\right]. \quad (66)$$

Integrating $(66)$ over $(0, T)^N \times (0, \delta)$ we get

$$\|v\|_{L^2((0, T)^N \times (0, \delta), \xi^{1-2s})} \leq \frac{\delta^{2-2s}}{1-s}|v(\cdot, 0)|_{L^2(0, T)^N}^2 + \frac{\delta^2}{2s}\|\partial_\xi v\|_{L^2(S_T, \xi^{1-2s})}^2.$$

Then $(64)$ is obtained by density.

Secondly, we can observe that by the definition of the norm $\|\cdot\|_{X_m}$ and by Theorem 2.1 it follows that

$$\|v_m\|_{X_m} \geq \|\nabla v_m\|_{L^2(S_T, \xi^{1-2s})}^2 \quad (67)$$

and

$$\|v_m\|_{X_m} \geq |v_m(\cdot, 0)|_{H_m} \geq [v_m(\cdot, 0)]_{H}. \quad (68)$$

We also notice that $(72)$ implies

$$2\int_{\partial\Omega S_T} F(x, v_m) \, dx < 2K_1 + m^{2s}|v_m(\cdot, 0)|_{L^2(0, T)^N}^2 + 2\int_{\partial\Omega S_T} F(x, v_m) \, dx \leq \|v_m\|_{X_m}^2$$

and by applying $(19)$ with $A = 1$, we can deduce that

$$2|v_m(\cdot, 0)|_{L^2(0, T)^N}^2 - 2B_1 T^N \leq \|v_m\|_{X_m}^2. \quad (69)$$

Then, taking into account $(64)$, $(67)$, $(68)$, $(69)$, and Theorem 2.2 it is enough to prove

$$\limsup_{m \to 0} \|v_m\|_{X_m} < \infty \quad (70)$$

to show the validity of $(63)$. For our claim, we proceed as in the first part of the proof of Lemma 1.7 in which we demonstrated the boundedness of Cerami sequences.

We assume by contradiction that, up to a subsequence,

$$\|v_m\|_{X_m} \to \infty \text{ as } m \to 0. \quad (71)$$

We set

$$w_m := \frac{v_m}{\|v_m\|_{X_m}}. \quad (72)$$

Then $\|w_m\|_{X_m} = 1$ and by using $(67)$ and $(68)$, it results

$$\|\nabla w_m\|_{L^2(S_T, \xi^{1-2s})}^2 \leq 1 \quad (73)$$
and

\[ 1 = \|w_m\|^2_{L^2} \geq C(T, N, s)[w_m(\cdot, 0)]^2. \]  \tag{74}

Moreover, by using (69), (71) and (72), we have

\[ \limsup_{m \to 0} |w_m(\cdot, 0)|^2_{L^2(0, T)} \leq \frac{1}{2}. \]  \tag{75}

Putting together (74) and (75) we obtain

\[ |w_m(\cdot, 0)|_{H^1} \leq C \]  \tag{76}

for any \( m \) sufficiently small.

Finally, by using (64) with \( v = w_m \) and exploiting (73) and (75), we get

\[ \limsup_{m \to 0} ||w_m||^2_{L^2((0, T) \times (0, \delta \xi_{1-2s}))} \leq \frac{\delta^2 - 2s}{2(1-s)} + \frac{\delta^2}{2s} =: C(\delta, s). \]  \tag{77}

Taking into account (73), (76), (77), and Theorem 2.2, we can deduce the existence of a subsequence, which we will denote again with \( \{w_m\} \), and a function \( w \) such that

\[ w_m(\cdot, 0) \to w(\cdot, 0) \]  \tag{78}

for any \( q \in \mathbb{N} \). As a consequence

\[ \langle J_m'(t_m v_m), t_m v_m \rangle = t_m \frac{d}{dt} \bigg|_{t=t_m} J_m(t_m v_m) = 0. \]  \tag{83}
Taking into account (f6), (55), (62) and (83) we get
\[
\frac{2}{\gamma} J_m(t_m v_m) = \frac{2}{\gamma} \left( J_m(t_m v_m) - \frac{1}{2} \langle J'_m(t_m v_m), t_m v_m \rangle \right) \\
= \frac{1}{\gamma} \int_{\partial S_T} G(x, t_m v_m) \, dx \\
\leq \int_{\partial S_T} G(x, v_m) \, dx \\
= 2J_m(v_m) - \langle J'_m(v_m), v_m \rangle \leq 2K_2
\]
which contradicts (82).

Now we assume that \( w \neq 0 \).

Thus the set \( \Omega = \{ x \in (0, T)^N : w(x, 0) \neq 0 \} \) has positive Lebesgue measure and by using (72), (78) and (84) we get
\[
|v_m(x, 0)| \to \infty \text{ a.e. } x \in \Omega \text{ as } m \to 0.
\]

In particular, by (f5), it follows that
\[
\frac{F(x, v_m(x, 0))}{||v_m||_{X_m}^2} = \frac{F(x, v_m(x, 0)) |v_m(x, 0)|^2}{||v_m(x, 0)||_{X_m}^2} \\
= \frac{F(x, v_m(x, 0))}{|v_m(x, 0)|^2} |w_m(x, 0)|^2 \to +\infty \text{ a.e. } x \in \Omega.
\]

Since
\[
\frac{J_m(v_m)}{||v_m||_{X_m}^2} \to 0 \text{ as } m \to 0,
\]
using (86) and \( F \geq 0 \), we can deduce via the Fatou’s Lemma that
\[
o(1) = \frac{1}{2} \left[ m^{2s} |v_m(\cdot, 0)|_{L^2(0,T)^N}^2 - \int_{\partial S_T} F(x, v_m(x, 0)) \frac{dx}{||v_m||_{X_m}^2} \right] \\
\leq \frac{1}{2} - \int_{\Omega} F(x, v_m(x, 0)) \frac{dx}{||v_m||_{X_m}^2} \to -\infty \text{ as } m \to 0
\]
that is a contradiction. Then we can assume the existence of \( \{ v_m \} \) and \( v \) verifying (63). At this point, we prove that \( v \) is a weak solution to (56). We proceed as in [3]. Fix \( \varphi \in X \). We know that \( v_m \) satisfies
\[
\int_{S_T} \xi^{1-2s} (\nabla v_m \nabla \eta + m^2 v_m \eta) \, dx = \int_{\partial S_T} [m^{2s} v_m + f(x, v_m)] \eta \, dx
\]
for every \( \eta \in X_m \). Now, we introduce \( \psi \in C^\infty([0, \infty)) \) defined by
\[
\left\{\begin{array}{ll}
\psi = 1 & \text{if } 0 \leq \xi \leq 1 \\
0 \leq \psi \leq 1 & \text{if } 1 \leq \xi \leq 2 \\
\psi = 0 & \text{if } \xi \geq 2
\end{array}\right.
\]
We set \( \psi_R(\xi) := \psi \left( \frac{\xi}{R} \right) \) for \( R > 1 \). Then choosing \( \eta = \varphi \psi_R \in X_m \) in (88) and taking the limit as \( m \to 0 \) we have
\[
\int_{S_T} \xi^{1-2s} \nabla v \nabla (\varphi \psi_R) \, dx = \int_{\partial S_T} f(x, v) \varphi \, dx.
\]
By passing to the limit in (90) as $R \to \infty$ we deduce that $v$ verifies
\[
\iint_{S_T} \xi^{1-2s} \nabla \nu \nabla \phi \, dx \, dt - \int_{\partial S_T} f(x, v) \phi \, ds = 0 \quad \forall \phi \in \mathcal{X}.
\]
Finally we show that $v$ is not identically zero. By using (55), (62), $1 < p < \frac{N+2s}{N-2s}$, (13) and (14) with $\varepsilon = \frac{1}{2}$ we can see that
\[
2K_1 \leq 2J_m(v_m) = 2J_m(v_m) - \langle J_m'(v_m), v_m \rangle
\]
\[
= \int_{\partial S_T} [f(x, v_m)v_m - 2F(x, v_m)] \, ds
\]
\[
\leq |v_m(\cdot, 0)|^{2}_{L^2(0, T)^N} + (p + 3)C_1 |v_m(\cdot, 0)|^{p+1}_{L^{p+1}(0, T)^N}
\]
\[
\leq T^\frac{m(p+1)|v_m(\cdot, 0)|^{2}_{L^2(0, T)^N} + (p + 3)C_1 |v_m(\cdot, 0)|^{p+1}_{L^{p+1}(0, T)^N}}{2}\tag{91}
\]
where in the last inequality we have used Hölder inequality. Taking the limit in (91) as $m \to 0$ and recalling that $v_m(\cdot, 0)$ converges strongly to $v(\cdot, 0)$ in $L^{p+1}(0, T)^N$, we can infer that $|v(\cdot, 0)|^{p+1}_{L^{p+1}(0, T)^N} > 0$, that is $v \neq 0$.

**Remark 3.** By using a Brezis-Kato argument for subcritical nonlinear problems, we can prove that $v(\cdot, 0) \in L^q(0, T)^N$ for any $q < \infty$ (see Lemma 7 in [3]). As a consequence, we obtain $v(\cdot, 0) \in C^{0, \alpha}([0, T]^N)$ for some $\alpha \in (0, 1)$.

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