Some new integrable systems of two-component fifth order equations

Daryoush Talati \textsuperscript{a}, * Abdul-Majid Wazwaz \textsuperscript{b}, †

\textsuperscript{a} Department of Engineering Physics, Ankara University 06100 Tandogan, Ankara, Turkey.
\textsuperscript{b} Department of Mathematics, Saint Xavier University, Chicago, IL 60655 USA

August 19, 2016

Abstract

In this work we develop some fifth-order integrable coupled systems of weight 0 and 1 which possess seventh-order symmetry. We establish four new systems, where in some cases, related recursion operator and bi-Hamiltonian formulations are given. We also investigate the integrability of the developed systems.

1 Introduction

Integrable systems of equations, that possess sufficiently large number of conservation laws and give rise to multiple soliton solutions play a major role in theoretical physics and in propagation of waves. The work on integrable systems of equations is flourishing because these systems have richer phenomena in scientific applications than the regular systems.

An evolution equation is defined to be integrable in symmetry sense if it admits infinitely many symmetries. Integrable systems are nonlinear differential equations which can be solved analytically. Exactly solvable models and integrable evolution equations in nonlinear science play an essential role in many branches of science and engineering. The useful findings in integrable systems of equations have stimulated much research activity.

The study of constructing integrable systems of equations by using methods, such as recursion operator, symmetries, bi-Hamiltonian, and others, is an interesting topic of growing interest and has gained large interest recently. Magri \cite{magri} studied the connection between conservation laws and symmetries from the geometric point of view, where he proved that some systems

*Talati@eng.ankara.edu.tr, Talati@eng.ankara.edu.tr
†Wazwaz@sxu.edu
admitted two distinct but compatible Hamiltonian structures, now known as bi-Hamiltonian system.

In recent years studies on fifth-order systems of two-component nonlinear evolution equations have received considerable attention \[2, 10, 8\]. Multi-component generalizations of fifth order Kaup-Kupershmidt equation

\[
u_t = u_{5x} + 10uv_{3x} + 25u_x u_{xx} + 20u^2 u_x, \tag{1}\]

Sawada-Kotera equation

\[
u_t = u_{5x} + 5uu_{3x} + 5u_x u_{xx} + 5u^2 u_x, \tag{2}\]

and Kupershmidt equation

\[
u_t = u_{5x} + 5u_x u_{3x} + 5u_{xx}^2 - 5u^2 u_{3x} - 20uu_x u_{xx} - 5u_x^2 + 5u^4 u_x, \tag{3}\]

have been the subject of systematic integrability study. Among these, only five homogeneous systems of two-component cases have been found \[3, 7, 9\] so far. Here we mention papers pertaining to multi-component generalizations of fifth order systems only. For the other integrable systems and their properties, we refer the readers to the useful papers \[5, 11, 4, 11\] and the some of the references therein.

So far the only known integrable systems of fifth order two-component equations are as follows

\[
\begin{pmatrix}
  u \\
  v
\end{pmatrix}_t = \begin{pmatrix}
  -\frac{5}{3}u_{5x} - 10uv_{3x} + 10uu_{3x} + 25u_x u_{xx} - 15v_x v_{xx} - 12u^2 u_x \\
  + 6v^2 u_x + 12uv v_x - 6v^2 v_x \\
  15v_{5x} - 10v_{3x} - 30uu_{3x} - 35v_x u_{xx} + 30v_x v_{xx} - 45u_x v_{xx} \\
  + 6v^2 u_x - 6v^2 v_x + 12uv u_x + 12v^2 u_x
\end{pmatrix}, \tag{4}\]

\[
\begin{pmatrix}
  u \\
  v
\end{pmatrix}_t = \begin{pmatrix}
  u_{5x} + 10uu_{3x} + 25u_x u_{xx} + 20u^2 u_x + v^2 v_x \\
  u_{3x} v + u_{xx} v_x + 8uv u_x + 4v^2 v_x
\end{pmatrix}, \tag{5}\]

\[
\begin{pmatrix}
  u \\
  v
\end{pmatrix}_t = \begin{pmatrix}
  -\frac{1}{3}u_{5x} - 2uu_{3x} - 2u_x u_{xx} - \frac{32}{9} u^2 u_x + v_x \\
  \frac{4}{9} u_{5x} + 6uv_{3x} + 6u_x v_{xx} + 4u_{xx} v_x + \frac{32}{9} u^2 v_x
\end{pmatrix}, \tag{6}\]

\[
\begin{pmatrix}
  u \\
  v
\end{pmatrix}_t = \begin{pmatrix}
  u_5 + \frac{5}{2}v_5 + 6u_3 u + 18u_3 v + 12v_3 u + 42v_3 v + 12u_2 u_1 + 24u_2 v_1 + 21v_2 u_1 + 42v_2 v_1 + \frac{54}{5} u_1 u^2 + \frac{108}{5} u_1 uv - 18u_1 v^2 + \frac{72}{5} v_1 u^2 - \frac{72}{5} v_1 uv - 144v_1 v^2 \\
  \frac{5}{2} u_5 + \frac{7}{2} v_5 + 3u_3 u + 6v_3 u - 6v_3 v + \frac{3}{2} u_2 u_1 - 6u_2 v_1 - 3v_2 u_1 - 33v_2 v_1 + \frac{26}{5} u_1 v^2 - \frac{16}{5} v_1 u^2 - \frac{36}{5} v_1 uv + \frac{126}{5} v_1 v^2
\end{pmatrix}, \tag{7}\]
and

\[
\begin{pmatrix}
  u \\
  v \\
  t
\end{pmatrix}
= 
\begin{pmatrix}
  u_{5x} - 30vv_{4x} + 5u_xu_{3x} - 5u^2u_{3x} + 15v^2u_{3x} - 75v_xv_{3x} \\
  + 60uvv_{3x} + 90v^2v_{3x} + 5u^2 - 20uv_xu_{xx} + 60v_xu_{xx} - 45v_{xx} \\
  + 90vu_xv_{xx} + 90uv_xv_{xx} + 540v_xv_{xx} + 30v^2v_{xx} - 180uv^2v_{xx} \\
  - 90v^3v_{xx} - 5u_x^2 + 45u_xv_x^2 + 60uvu_xv_x - 180v^2u_xv_x + 5u^4u_x \\
  - 90u^2v^2u_x + 45v^4u_x + 180v^3 + 30u^2v_x^2 - 360uvv_x^2 - 270v^2v_x^2 \\
  - 60u^3vv_x + 180uv^3v_x
\end{pmatrix}.
\]

(8)

Bi-Hamiltonian structures and recursion operators for the aforementioned systems are discussed in [12][13] and in some of the references therein. System (4) and (5) admit a reduction \( v = 0 \) to the Kaup-Kupershmidt eqnarray. By setting \( v = 0 \), system (4) reduces to the Sawada-Kotera equation. By setting \( v = 0 \) the well known Kupershmidt equation is an obvious reduction of system (8).

2 New homogeneous fifth-order integrable Systems

In the literature, all of classified integrable systems are second and third order generalization of the KdV and Burgers equations, or equations related to the KdV and Burgers equations. In the case of fifth-order systems, because of the very big number of arbitrary terms that must be considered, the act of classification of such systems is very complicated. Motivated by some existing examples of bi-Hamiltonian two-component generalization of fifth-order equations, we considered a narrow class of fifth-order two-component systems with specific Jordan matrix for integrability.
From a practical point of view, we observed that in second and third order integrable systems, when there is a 2-homogeneous integrable equation in a specific Jordan form, then there is certainly at least one 1,0-homogeneous system in that Jordan form. Then using the sense of 2-homogeneous fifth order systems introduced in [6,7], we aim to develop new integrable 1,0-homogeneous systems in the same Jordan form. Our analysis found four new integrable systems, where some of these systems allow us to write Miéchi schemes which contain the new systems proving it complete integrability. In what follows, we introduce the new fifth order two-component systems with the form

\[
\begin{pmatrix}
u(t) \\
u(0)
\end{pmatrix} = \begin{pmatrix}
4u_5 + 5v_5 + 20u_4u_1 + 10u_4v_1 + 40u_1v_4 + 20v_4v_1 + 20u_3u_2 - 40u_3u_1^2 + 140u_3u_1v_1 + 70u_3v_2 + 80u_3v_1^2 + 40u_2^2u_1 + 80u_2^2v_1 - 80u_2u_2^2 + 360u_2u_1^2v_1 + 400u_2u_1v_2 + 600u_2u_1v_1^2 + 70u_2v_3 + 260u_2v_2v_1 + 200u_2v_1^3 + 24u_1^2 - 240u_1^2v_1 - 160u_1^2v_2 + 360u_1^2v_1^2 + 40u_1^2v_3 + 720u_1v_2v_1 + 1200u_1v_1^2 + 370u_1v_2^2 + 1200u_1v_1v_2^2 + 600u_1v_1^4 + 110v_3v_2 + 100v_3v_1^2 + 200v_2v_1^3 + 400v_2v_1^2 & \text{)}
\end{pmatrix}, \quad (9)
\]

\[
\begin{pmatrix}
u(t) \\
u(0)
\end{pmatrix} = \begin{pmatrix}
10u_5 + 14v_5 - 40u_4u_1 - 20u_4v_1 - 20u_1v_4 + 20u_3u_2 - 40u_3u_1^2 - 40u_3u_1v_1 + 100u_3v_2 - 100u_3v_1^2 - 200u_2^2u_1 - 40u_2v_1 + 160u_2u_1^3 - 720u_2u_1v_1 - 560u_2u_1v_2 - 1200u_2u_1v_1^2 + 100u_2v_3 - 400u_2v_2v_1 - 400u_2v_2^3 + 600u_1v_1^4 + 80u_1v_1v_2^2 + 1200u_1v_1v_2^2 - 200u_1^2v_3 - 360u_1^2v_2v_1 + 360u_1^2v_1^3 - 380u_1v_3v_1 - 320u_1v_2^3 - 600u_1v_2v_1 - 240u_1^4v_1 - 10v_4v_1 + 140v_3v_2 - 320v_3v_1^2 - 370v_2v_1 - 200v_2v_1^3 + 24v_5 & \text{)}
\end{pmatrix}, \quad (10)
\]
\[
\begin{pmatrix}
u \\
u_t
\end{pmatrix} =
\begin{pmatrix}
\begin{bmatrix}
11 & 7 & 6 & 4 & 4 & 2 & 2 & 2 & 2 & 2 \\
11 & 7 & 6 & 4 & 4 & 2 & 2 & 2 & 2 & 2 \\
11 & 7 & 6 & 4 & 4 & 2 & 2 & 2 & 2 & 2 \\
11 & 7 & 6 & 4 & 4 & 2 & 2 & 2 & 2 & 2 \\
11 & 7 & 6 & 4 & 4 & 2 & 2 & 2 & 2 & 2 \\
11 & 7 & 6 & 4 & 4 & 2 & 2 & 2 & 2 & 2 \\
11 & 7 & 6 & 4 & 4 & 2 & 2 & 2 & 2 & 2 \\
11 & 7 & 6 & 4 & 4 & 2 & 2 & 2 & 2 & 2 \\
11 & 7 & 6 & 4 & 4 & 2 & 2 & 2 & 2 & 2 \\
11 & 7 & 6 & 4 & 4 & 2 & 2 & 2 & 2 & 2
\end{bmatrix}
\end{pmatrix}
\begin{pmatrix}
u \\
u_t
\end{pmatrix},
\]

(11)

and

\[
\begin{pmatrix}
u \\
u_t
\end{pmatrix} =
\begin{pmatrix}
\begin{bmatrix}
11 & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
11 & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
11 & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
11 & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
11 & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
11 & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
11 & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
11 & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
11 & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
11 & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2
\end{bmatrix}
\end{pmatrix}
\begin{pmatrix}
u \\
u_t
\end{pmatrix},
\]

(12)

To find the second set of systems, we use a classification that we restricted to the case \(\lambda = 0\) homogeneous symmetrically coupled systems. We determine all equations of the form as

\[
\begin{pmatrix}
u \\
u_t
\end{pmatrix} =
\begin{pmatrix}
\begin{bmatrix}
A & 0 \\
0 & A
\end{bmatrix}
\begin{pmatrix}
u \\
u_t
\end{pmatrix},
\]

(13)
with The class of two-component 0-homogeneous symmetrically coupled systems with undetermined constant coefficients \( \gamma \) have the form

\[
A = \begin{pmatrix}
\gamma_1 u_{5x} + \gamma_2 v_{5x} + \alpha_1 u_x u_{4x} + \alpha_2 v_x u_{4x} + \alpha_3 u_x v_{4x} + \alpha_4 v_x v_{4x} + \alpha_5 u_{xx} u_{3x} \\
+ \alpha_6 v_{xx} u_{3x} + \alpha_7 u_x^2 u_{3x} + \alpha_8 u_x v_x u_{3x} + \alpha_9 v_x^2 u_{3x} + \alpha_{10} u_{xx} v_{3x} + \alpha_{11} v_{xx} v_{3x} \\
+ \alpha_{12} u_x^2 v_{3x} + \alpha_{13} u_x v_x v_{3x} + \alpha_{14} v_x^2 v_{3x} + \alpha_{15} u_x u_{xx}^2 + \alpha_{16} v_x u_{xx}^2 \\
+ \alpha_{17} u_x v_{xx} u_{xx} + \alpha_{18} v_x u_{xx} u_{xx} + \alpha_{19} u_x^3 u_{xx} + \alpha_{20} u_x^2 v_x u_{xx} + \alpha_{21} u_x v_x^2 u_{xx}
\end{pmatrix}
\]

possessing an admissible generator of form (13) with The main matrix of these systems is

\[
\begin{pmatrix}
\gamma_1 & \gamma_2 \\
\gamma_2 & \gamma_1
\end{pmatrix}
\]

By a linear change of variables, the matrix (13) can be reduced to following canonical Jordan form \( \begin{pmatrix} \gamma_1 + \gamma_2 & 0 \\ 0 & \gamma_1 - \gamma_2 \end{pmatrix} \)

Because of properties of systems, we will restrict our attention to \( \gamma_2 \leq \gamma_1, \gamma_1, \gamma_2 = 0, 1 \). Similarly we will deal with two canonical Jordan form \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \). Imposing compatibility condition among the classes of systems and an arbitrary seventh order 0-homogeneous, we obtain a system of equations among the undetermined constants. If we separate out the coefficients of powers of \( u \) and \( v \) in this equation then in some condition the coefficients of \( u_{nx}^m v_{n'x}^{m'} \) all vanish identically. Solutions of the compatibility condition are given in the following theorems.

**Theorem 2.1** A coupled fifth-order system of two-component evolution equations of the forms (13) and (14) that possesses a seventh-order generalized symmetry of form (13) with \( \gamma_1 = \gamma_2 = 1 \) have a lower order symmetry or can be transformed by a linear change of variables to one of the following two systems (11) and (13).

**Theorem 2.2** Every coupled fifth-order system of two-component evolution equations of form (13) and (14) that possesses a seventh-order generalized symmetry of form (13) with \( \gamma_1 = 1, \gamma_2 = 0 \) have a lower order symmetry.

### 2.1 Integrability of the system (9)

System (9) possesses a symplectic operator as

\[
S = \begin{pmatrix}
2D_x & D_x \\
D_x & 2D_x
\end{pmatrix}
\]

Second Hamiltonian or symplectic operator for this system is an open question for us.
2.2 Integrability of the system \([\text{10}]\)

Our main concern now is to show the integrability of the system (2.2). To achieve this goal, we set

\[
R = \begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix}
\]

where

\[
R_1 = \alpha_1 D_x^6 + \alpha_2 D_x^4 + \alpha_3 D_x^3 + \alpha_4 D_x^2 + \alpha_5 D_x + \alpha_6 + \alpha_0 D_x^{-1} \alpha_0 + \alpha_0 D_x^{-1} \alpha_5
\]

\[
R_2 = \alpha_7 D_x^5 + \alpha_8 D_x^4 + \alpha_9 D_x^3 + \alpha_{10} D_x^2 + \alpha_{11} D_x + \alpha_{12} + \alpha_0 D_x^{-1} \alpha_8 + \alpha_0 D_x^{-1} \alpha_6
\]

\[
R_3 = \alpha_{13} D_x^5 + \alpha_{14} D_x^4 + \alpha_{15} D_x^3 + \alpha_{16} D_x^2 + \alpha_{17} D_x + \alpha_{18} + \alpha_0 D_x^{-1} \alpha_7 + \alpha_0 D_x^{-1} \alpha_5
\]

\[
R_4 = \alpha_{19} D_x^4 + \alpha_{20} D_x^3 + \alpha_{21} D_x^2 + \alpha_{22} D_x + \alpha_{23} + \alpha_0 D_x^{-1} \alpha_8 + \alpha_0 D_x^{-1} \alpha_6
\]

\[
\alpha_1 = 12
\]

\[
\alpha_2 = 72u_1 - 72u^2 - 30v^2
\]

\[
\alpha_3 = 180u_2 - 360u_1 u - 18uv^2 - 60v_1 v
\]

\[
\alpha_4 = 168u_3 - 480u_2 u - 372u_1 u^2 - 72u_1 u^2 - 126u_1 v^2 + 108u^4 + 114u^2 v^2 - 48uv_1 v - 72v_2 v - 72v_1^2 + 12v^4
\]

\[
\alpha_5 = 72u_4 - 360u_3 u - 756u_2 u_1 - 108u_2 u_2 - 126u_2 v^2 - 216u_1 u^2 + 648u_1 u^3 + 342u_1 u v^2 - 180u_1 v_1 v + 18u^3 v^2 + 180u_2 v_1 v - 72uv_2 v - 72uv_1^2 + 9uv^4 - 48v_3 v - 144v_2 v_1 + 54v_1 v^3
\]

\[
\alpha_6 = 12u_5 - 144u_4 u - 276u_3 u_1 - 36u_3 u^2 - 36u_3 v^2 - 180u_2^2 - 456u_2 u_1 + 456u_2 u^3 + 210u_2 v^2 - 96u_2 v_1 v - 72u_1^3 + 888u_1 u^2 + 132u_1 u^2 + 108u_1 u^2 v^2 + 288u_1 u_1 v^2 - 84u_1 v_1 v - 84u_1 v_1^2 + 18u_1 v^4 - 48u^6 - 84u^2 v^2 + 48u^3 v_1 + 156u_2 v_2 v + 156u_2 v_1^2 - 30u^2 v^4 - 72uv_3 v - 216uv_2 v_1 + 78uv_1 v^3 - 12v_4 v - 48v_3 v_1 - 36v_2^2 + 24v_2 v^3 + 36v_1^2 v^2
\]

\[
\alpha_7 = 12 v
\]

\[
\alpha_8 = -24uv + 60v_1
\]

\[
\alpha_9 = -48u_1 v - 24u^2 v - 96uv_1 + 120v_2 - 30v^3
\]
\[\alpha_{10} = -60u_2v - 72u_1uv - 144u_1v_1 + 48u^3v - 72u^2v_1 - 144uv_2 + 42uv^3 + 120v_3 - 150v_1v^2\]

\[\alpha_{11} = -72u_3v - 72u_2uv - 120u_2v_1 - 84u_1v^2v - 144u_1uv_1 - 144u_1v_2 + 54u_1v^3 + 12u_4v + 96u^3v_1 - 72u^2v_2 + 30u^2v^3 - 96uv_3 + 156uv_1v^2 + 60v_4 - 162v_2v^2 - 192v^2v + 12v^5\]

\[\alpha_{12} = -48u_4v - 24u_3uv - 72u_3v_1 - 228u_2uv_1 + 180u_2u^2v - 72u_2uv_1 - 60u_2v_2 + 54u_2v^3 + 240u_1^2uv - 84u^2v_1 + 24u_1u^3v + 216u_1u^2v_1 - 72u_1uv_2 + 66u_1uv^3 - 48u_1v_3 + 114u_1v_1v^2 - 24u_5v + 12u_4v_1 + 48u_3v_2 - 42u_3v^3 - 24u_2v_3 + 66u_2v_1v^2 - 24uv_4 + 114uv_2v^2 + 144uv^3_1v - 15uv^5 + 12v_5 - 78v_3v^2 - 276v_2v_1v - 72v_3^2 + 66v_1v^4\]

\[\alpha_{13} = -6v\]

\[\alpha_{14} = 6uv + 12v\]

\[\alpha_{15} = -36u_1v + 30u^2v - 12uv_1 + 15v^3\]

\[\alpha_{16} = -54u_2v + 150u_1uv + 84u_1v_1 - 30u^3v - 60u^2v_1 + 21uv^3 + 24v_1v^2\]

\[\alpha_{17} = -30u_3v + 144u_2uv + 24u_2v_1 + 78u^2v - 54u_1u^2v - 180u_1uv_1 + 45u_1v^3 - 24u^4v + 60u^3v_1 - 24u^2v^3 + 30uv_1v^2 + 18v_2v^2 - 30u^2v - 6v^5\]

\[\alpha_{18} = -6u_4v + 66u_3uv + 36u_3v_1 + 90u_2u_1v - 36u_2u^2v - 108u_2uv_1 + 18u_2v^3 + 48u^2uv + 24u^2v_1 - 144u_1u^3v - 72u_1u^2v_1 - 72u_1uv^3 + 18u_1v_1v^2 + 24u^5v + 48u^4v_1 - 12u^3v^3 - 48u^2v_1v^2 + 18uv_2v^2 - 42uv^3_1v - 12uv^5 + 6v^3v^2 + 30v_2v_1v + 12v_3^2 - 18v_1v^4\]

\[\alpha_{19} = -6v^2\]

\[\alpha_{20} = 18uv^2 - 12v_1v\]

\[\alpha_{21} = 12u_1v^2 - 6u^2v^2 + 18uv_1v - 36v_2v + 36v^2_1 + 15v^4\]

\[\alpha_{22} = 18u_2v^2 + 18u_1uv + 36u_1v_1v - 18u^3v^2 + 54uv_2v - 72uv^2_1 - 9uv^4 - 24v_3v + 36v_2v_1 + 54v_1v^3\]

\[\alpha_{23} = 18u_3v^2 - 30u_2v_1v + 30u^2v_1^2 - 66u_1u^2v^2 - 42u_1uv_1v + 12u_1v_2v + 12u_1v^2_1 - 15u_1v^4 + 12u^2v^2 + 18u^3v_1v - 6u^2v_2v + 12u^2v^2 - 6u^2v^4 + 18uv^3v - 36uv_2v_1 - 27uv_1v^3 - 6uv_4 + 12v_3v_1 + 33v^2v^3 - 6v^2_1v^2 - 6v^6\]

\[\alpha_{01} = -24u_5 - 120u_3u_2 + 120u_3u_2 - 120u_2 + 480u_2u_1v + 36u_2uv^2 + 48u_2v_1v + 120u_3^2 + 72u_1v^2 - 120u_1u^4 - 144u_1u^2v^2 + 48uv_1v_1v + 72u_1v_2v + 72u_1v^2_1 - 6u_1v^4 - 48u^3v_1v + 24u^2v_2v + \ldots
24u^2v_1^2 + 48uv_3v + 144uv_2v_1 - 60uv_1v^3 - 24v_4v - 96v_3v_1 - 72v_2^2 + 48v_2v^3 + 72v_1^2v^2

\alpha_{02} = -3u_1

\alpha_{03} = 12u_4v - 12u_3uv - 24u_3v_1 + 60u_2u_1v - 48u_2u^2v + 24u_2uv_1 - 24u_2v^3 - 96u_1^2uv - 72u_1^2v_1 + 48u_1u^3v + 96u_1u^2v_1 - 48u_1uv^3 - 36u_1v_1v^2 - 24u_4v_1 - 36uv_2v^2 + 36uv_1^2v + 12v_3v^2 + 12v_2v_1v - 24v_1^3 - 30v_1v^4

\alpha_{04} = -3v_1

\alpha_{05} = 8u_4 + 8v_3v + 40u_2u_1 - 40u_2u^2 - 4u_2v^2 + 24v_2v_1 - 16v_2uv - 40u_1^2u - 8u_1v_1v - 16v_1^2u - 8v_1u^2v - 4v_1v^3 + 8u^5 + 8v^3u^2 + 2uv^4

\alpha_{06} = -8u_3v - 16u_2uv - 8v_2v^2 - 12u_1^2v + 8u_1u^2v + 4u_1v^3 - 8v_1^2v + 4u_4v + 4u_2v^3 + v^5

\alpha_{07} = u

\alpha_{08} = \frac{v}{2}

This shows that the system (10) passes the integrability test.

### 2.3 Integrability of system (11)

We proceed as before to show the integrability of the system (11). By change of dependent variables

\[ u \rightarrow \frac{1}{2} \int (w - z) \, dx \quad (17) \]

\[ v \rightarrow \frac{1}{2} \int (w + z) \, dx \quad (18) \]
system (11) can be written in its canonical form as

\[
\begin{pmatrix}
  u \\
  v \\
  t
\end{pmatrix} =
\begin{pmatrix}
  w_{5x} - 2zz_{4x} - 10w_x w_{3x} - 20w^2 w_{3x} - 2z^2 w_{3x} - 8zz_{3x} \\
  -8wzz_{3x} - 10w_{2xx} - 80ww_x w_{xx} - 8zz_x w_{xx} - 6z_x^2 \\
  -12w_x z_{xx} - 24w z_x z_{xx} + 8w^2 z_z z_{xx} + 4z^3 z_{xx} - 20w_x^3 - 12w_x z_x^2 \\
  +16w x z_x + 80w^4 w_x + 48w^2 z^2 w_x + 4z^4 w_x + 8w_x^2 \\
  +12z^2 z_x^2 + 32w^3 z_x + 16w z_x^3
\end{pmatrix}
\]

(19)

Proposition The infinite hierarchy of the system (19) can be written in two different ways

\[
\begin{pmatrix}
  w_t \\
  z_t
\end{pmatrix} = J \begin{pmatrix}
  \delta_w \\
  \delta_z
\end{pmatrix} \int \rho_1 \, dx = K \begin{pmatrix}
  \delta_w \\
  \delta_z
\end{pmatrix} \int \rho_0 \, dx
\]

(20)

with the compatible pair of Hamiltonian operators

\[
J = \begin{pmatrix}
  D_x & 0 \\
  0 & 2D_x
\end{pmatrix},
K = \begin{pmatrix}
  K_1^1 & K_1^2 \\
  K_3^1 & K_4^1
\end{pmatrix}
\]

(21)

where

\[
K_1^1 = D_x^7 + \omega_1 D_x^5 + D_x^5 \omega_1 + \omega_2 D_x^3 + D_x^3 \omega_2 + \omega_3 D_x + D_x \omega_3 + 8w_x D_x^{-1} w_t + 8w_t D_x^{-1} w_x
\]

\[
K_2^1 = D_x^6 \omega_4 + D_x^5 \omega_5 + D_x^4 \omega_6 + D_x^3 \omega_7 + D_x^2 \omega_8 + D_x \omega_9 + \omega_10 + 8w_x D_x^{-1} z_t + 8w_t D_x^{-1} z_x
\]

\[
K_3^1 = -\omega_4 D_x^6 + \omega_5 D_x^5 - \omega_6 D_x^4 + \omega_7 D_x^3 - \omega_8 D_x^2 + \omega_9 D_x - \omega_10 + 8z_x D_x^{-1} w_t + 8z_t D_x^{-1} w_x
\]

\[
k_4^1 = \omega_{11} D_x^5 + D_x^5 \omega_{11} + \omega_{12} D_x^3 + D_x^3 \omega_{12} + \omega_{13} D_x + D_x \omega_{13} + 8z_x D_x^{-1} z_t + 8z_t D_x^{-1} z_x
\]

(22)
and the coefficients satisfy

\[
\begin{align*}
\omega_1 &= -6w_x - 12w^2 - 2z^2 \\
\omega_2 &= 16w_{3x} + 40ww_{xx} + 8zw_{xx} + 58w_x^2 + 24w^2w_x + 12z^2w_x + 8z_x^2 + 16wzz_x \\
&+ 72w^4 + 40w^2z^2 + 18z^4 \\
\omega_3 &= -10w_{5x} - 24ww_{4x} - 4z_{3z} - 100w_xw_{3x} - 24w^2w_{3x} - 12z^2w_{3x} - 16z_xz_{3x} \\
&- 84w_{3xx} - 64ww_xw_{xx} - 64zw_{zz}w_{xx} - 128w^3w_{xx} - 64w_x^2w_{xx} - 12z_{xx}^2 \\
&+ 56zw_{zz}w_{xx} - 16w^2z_{zz}w_{xx} - 152z^3z_{xx} - 48w_z^3 - 704w_x^2w_x - 80z^2w_x \\
&+ 56w_{zz}^2 - 288wzw_{xx} - 96z^4w_x - 16w^2z_x^2 - 216z_xz^2 \\
&- 64w^3z_{xx} + 96w^3z_x - 128w^6 - 128w^4z^2 - 96w^2z^4 - 16z^6 \\
\omega_4 &= -4z \\
\omega_5 &= 4z_x - 16wz \\
\omega_6 &= +48zw_x + 16wz_x + 32w^2z \\
\omega_7 &= -72zw_{xx} - 32w_xw_x - 160wzw_x - 32w^2z_x + 96z^2z_x + 128w^3z \\
\omega_8 &= +40zw_{3x} + 40zw_xw_x + 192zwzw_{xx} + 208zw^2 - 96w_xw_xz_x - 576w^2w_x \\
&- 320z_{xx}^2 - 128w^3z_x - 64w^4z \\
\omega_9 &= -8zw_x - 128wzw_{xx} - 104z^2z_{3x} - 240zw_xw_{xx} - 96w_xw_{xx} + 480w^2w_{xx} \\
&- 96z^3w_{xx} - 152zw_{xx}w_{xx} + 576w^2z_{xx}w_{xx} - 12w^2z_xw_x + 640w_xw_{xx}z_x \\
&+ 192w^2w_xz_x - 192zw_xz_x + 384w^3z_xw_x + 384w^3w_x + 216z_x + 448wzw_{xx} + 64w^4z_x \\
&- 768w^2z^2z_x - 96z^4z_x - 256w^5z \\
\omega_{10} &= +8zw_{4x} + 32zw_{4x} + 40z^2z_{4x} + 32w_zw_{3x} - 128w^2zw_{3x} + 48z^3w_{3x} \\
&+ 224z^{3}w_{3x} - 288w^2z_{3x} - 80w_xw_{3x}w_x - 320wzw_{3x}w_x - 288w^2z_xw_x \\
&- 128w^3z_{xx}w_{xx} - 192w^3z_{xx}w_{xx} + 304z^2z_{xx}w_{xx} + 120z_{xx}^2 + 64z^2w_xw_{xx} \\
&+ 192z_{xx}^2w_{xx} - 1376w^2zw_{xx}w_{xx} + 384w^2z^2z_{xx} + 48z^4z_{xx} - 256w^5z_{xx} \\
&- 256w^2z_{xx}^2 + 64z^4w_{xx}^2 + 160zw_xz_{xx} - 128w^3z_xz_x - 192w^2w_xz_x \\
&+ 512w^4w_x - 512zw_{xx}^2 + 1216w^2zz_{xx}^2 + 272z^3z_{xx}^2 + 256w^5z_{xx} \\
\omega_{11} &= -12z^2 \\
\omega_{12} &= +100z_{xx} - 72z^2w_x + 80z_x^2 - 96wzz_x + 144w^2z^2 + 36z^4 \\
\omega_{13} &= -68zwz_{4x} + 8z^2w_{3x} - 292z^{2}z_{3x} + 176wzz_{3x} + 232zzz_xw_{xx} + 64w^2w_{xx} \\
&- 264z_{xx}^2 + 512zw_zw_{xx} + 816wzz_{xx}w_{xx} - 896w^2zz_{xx}w_{xx} - 88z^3z_{xx} - 64z^2w_x^2 \\
&+ 280w_xz_x^2 + 1056zw_xz_x + 96z^4w_x - 752w^2z_x^2 - 380z^2z_x^2 \\
&+ 768w^3z_{xx} + 96w^3z_{xx} - 384w^4z^2 - 192w^2z^4 - 32z^6 \\
\rho_0 &= \alpha \\
\rho_1 &= 2w^2 + z^2 \\
\rho_2 &= +3ww_{4x} + 10w^2w_{3x} + 6z^2w_{3x} - 20w^3w_{xx} - 6w_x^2w_{xx} - 18w^2z_{xx}w_{xx} - 4z^3z_{xx} \\
&- 18w^2z_x^2 - 16w^3z_{xx} + 24w^3z_x + 16w^6 + 24w^4z^2 + 12w^2z^4 + 2z^6 \\
\end{align*}
\]
These densities are sufficient to write two Magri schemes with the same Hamiltonian operators such that one of them contains the new system, and this confirms the integrability of the system [27].

2.4 Integrability of system (12)

In a manner parallel to the analysis presented earlier, and to prove the integrability of the system (12), we use the change of dependent variables

\[ u \to \frac{1}{2} \int (w - z) \, dx \]  \hspace{1cm} (25)

\[ v \to \frac{1}{2} \int (w + z) \, dx \]  \hspace{1cm} (26)

which carries the system (12) to its canonical form as

\[
\begin{pmatrix}
  u \\
  v \\
  t
\end{pmatrix} = \begin{pmatrix}
  w_{5x} - zz_{4x} + 10w_x w_{3x} - 20w^2 w_{3x} - 8z^2 w_{3x} - 4z_x z_{3x} - 2w z z_{3x} \\
  + 10w_x^2 - 80w w_x w_{xx} - 32z z_{x} w_{xx} - 3z_x^2 - 18z w_x z_{xx} - 6w z_x z_{xx} \\
  + 16w^2 z_{xx} + 8z^3 z_{xx} - 20w_x^3 - 18w_x z_{x}^2 + 32w z w_x z_x + 80w^4 w_x \\
  + 48w^2 z^2 w_x + 4z^4 w_x + 16w^2 z_x^2 + 24z^2 z_x^2 + 32w^3 z_x + 16z^3 z_x
\end{pmatrix}
\]  \hspace{1cm} (27)

**Proposition** The infinite hierarchy of system (27) can be written in not just one but two different ways

\[
\begin{pmatrix}
  w_t \\
  z_t
\end{pmatrix} = J \begin{pmatrix}
  \delta_w \\
  \delta_z
\end{pmatrix} \int \rho_1 \, dx = K \begin{pmatrix}
  \delta_w \\
  \delta_z
\end{pmatrix} \int \rho_1 \, dx
\]  \hspace{1cm} (28)

with the compatible pair of Hamiltonian operators

\[
J = \begin{pmatrix}
  D_x \\
  0
\end{pmatrix}, \quad K^2 = \begin{pmatrix}
  K_1^2 \\
  K_3^2 \\
  K_4^2
\end{pmatrix}
\]  \hspace{1cm} (29)
where

\[
K_1^2 = D^7_x + \psi_1 D^5_x + D^5_x \psi_1 + \psi_2 D^3_x + D^3_x \psi_2 + \psi_3 D_x + D_x \psi_3 + 8 w_x D^{-1}_x w_t + 8 w_t D^{-1}_x w_x
\]
\[
K_2^2 = D^6_x \psi_4 + D^5_x \psi_5 + D^4_x \psi_6 + D^3_x \psi_7 + D^2_x \psi_8 + D_x \psi_9 + \psi_1 + 8 w_x D^{-1}_x z_t + 8 z_t D^{-1}_x w_x
\]
\[
K_3^2 = -\psi_4 D^6_x + \psi_5 D^5_x - \psi_6 D^4_x + \psi_7 D^3_x - \psi_8 D^2_x + \psi_9 D_x - \psi_1 + 8 z_x D^{-1}_x w_t + 8 z_t D^{-1}_x w_x
\]
\[
K_4^2 = \psi_11 D^5_x + D^5_x \psi_5 + \psi_12 D^3_x + D^3_x \psi_12 + \psi_13 D_x + D_x \psi_13 + 8 z_x D^{-1}_x z_t + 8 z_t D^{-1}_x z_x
\]

where the coefficients satisfy

\[
\psi_1 = 6 w_x - 12 w^2 - 5 z^2
\]
\[
\psi_2 = -16 w_3 + 40 w w_x + 26 z z_x + 58 w^2 - 24 w^2 w_x - 12 z^2 w_x + 26 z^2
\]
\[+ 20 w z z_x + 72 w^4 + 52 w^2 z^2 + 18 z^4\]
The first few conserved densities of the system \( (27) \) are listed as follows

\[
\rho_0 = \alpha \\
\rho_1 = 2w^2 + z^2 \\
\rho_2 = -3w^2w_{xx} - 10w^2w_{3x} + 3z^2w_{3x} - 20w^3w_{xx} - 24w^2w_{wx} - 4wz^2w_{xx} \\
- 4wz_{xx} + 18w^2z^2 + 32w^2z_z + 48wz^3z_x + 16w^6 + 24w^4z^2 + 12w^2z^4 + 2z^6
\]

These densities suffice to write two Magri schemes with same Hamiltonian operators that one of them contains the new system, and this in turn emphasizes the integrability of the system \( (27) \).
3 Discussion

In this work we established four fifth-order integrable coupled systems of weight 0 and 1. We examined the related recursion operator and bi-Hamiltonian formulations for the developed systems. We used the compatible pair of Hamiltonian operators to formally prove the integrability of the developed systems. The obtained results will add valuable findings to the existing integrable systems of fifth order two-component equation. It is expected that other works will be conducted for recovering the scientific features of these systems of equations.

References

[1] Adler V.E., Shabat A.B., and Yamilov R.I., (2000). Symmetry approach to the integrability Problem, Theor. Math. Phys. 125, no.3, 1603-1661.

[2] De Sole, A., Kac, V.G, (2013). Non-local Poisson structures and applications to the theory of integrable systems. Japanese Journal of Mathematics, 8(2), 233-347.

[3] Magri, F., (1980). Lectures Notes in Physics, Vol. 120, Springer, Berlin.

[4] Mikhailov A.V., Novikov V.S., Wang J.P., Symbolic representation and classification of integrable systems, In: Algebraic Theory of Differential Equations, pp. 156-216. London Mathematical Society Lecture Note Series, vol. 357. (Cambridge University Press, Cambridge, 2009).

[5] Mikhailov A.V., Shabat A.V., Sokolov V.V.,(1991). The symmetry approach to classification of integrable equations, in: Zakharov V.E., ed., What is Integrability?, pp. 115-184. Springer-Verlag, New York.

[6] Mikhailov, A.V., Novikov, V.S., and Wang,J.P., (2007). On classification of integrable non-evolutionary equations, Stud. Appl. Math., 118:419-457.

[7] Mikhailov,A.V. Novikov,V.S. and Wang, J. P., (2009). Symbolic representation and classification of integrable systems, in: Algebraic Theory of Differential Equations edited by Mikhailov, A. V. and MacCallum, M. A. H. (Cambridge UniversityPress ), pp. 156-216 ; e-print arXiv:0712.1972.

[8] Talati, D., and Turhan,R., (2011). On a Recently Introduced Fifth-Order Bi-Hamiltonian Equation and Trivially Related Hamiltonian Operators, SIGMA 7, 081.

[9] Talati, D., (2013). A fifth-order bi-Hamiltonian system, arXiv preprint arXiv:1304.1987.

[10] Talati, D., (2016). Trivially related lax pairs of the Sawada-Kotera equation. Bulletin of the Iranian Mathematical Society, 41(1), 201-215.
[11] Talati, D., Turhan, R., (2016). Two-component integrable generalizations of Burgers equations with nondiagonal linearity. Journal of Mathematical Physics, 57(4), 041502.

[12] Talati, D., (2015). Complete integrability of the fifth-order Mikhailov-Novikov-Wang system. Applied Mathematics and Computation, 250, 776-778.

[13] Vojcak, P., (2011). On complete integrability of the Mikhailov-Novikov-Wang system. J. Math. Phys. 52, 043513.