TANGENTS TO SUBSOLUTIONS
EXISTENCE AND UNIQUENESS, I

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ABSTRACT

There is an interesting potential theory associated to each degenerate elliptic, fully nonlinear equation $f(D^2 u) = 0$. These include all the potential theories attached to calibrated geometries. This paper begins the study of tangents to the subsolutions in these theories, a topic inspired by the results of Kiselman in the classical plurisubharmonic case. Fundamental to this study is a new invariant of the equation, called the \textit{Riesz characteristic}, which governs asymptotic structures. The existence of tangents to subsolutions is established in general, as is the existence of an upper semi-continuous density function. Two theorems establishing the strong uniqueness of tangents (which means every tangent is a Riesz kernel) are proved. They cover all $O(n)$-invariant convex cone equations and their complex and quaternionic analogues, with the exception of the homogeneous Monge-Ampère equations, where uniqueness fails. They also cover a large class of geometrically defined subequations which includes those coming from calibrations. A discreteness result for the sets where the density is $\geq c > 0$ is also established in any case where strong uniqueness holds. A further result (which is sharp) asserts the Hölder continuity of subsolutions when the Riesz characteristic $p$ satisfies $1 \leq p < 2$. Many explicit examples are examined.

The second part of this paper is devoted to the “geometric cases”. A Homogeneity Theorem and a Second Strong Uniqueness Theorem are proved, and the tangents in the Monge-Ampère case are completely classified.

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1. Introduction.

The point of this paper is to introduce and study tangents for a wide class of degenerate elliptic, fully nonlinear equations of the form $F(D^2) = 0$ in $\mathbb{R}^n$. It was inspired by Kieselman’s study [K1] (cf. [K2]) of tangents to plurisubharmonic functions in classical pluripotential theory. The aim is to develop techniques for studying the behavior, in particular the singular behavior, of subsolutions – the upper semi-continuous functions $u$ which satisfy $F(D^2u) \geq 0$ in the viscosity sense. A number of quite general results are obtained. These include existence, uniqueness and “harmonicity” of tangents for a wide range of equations. Densities for subsolutions are defined and shown to be upper semi-continuous, and a structure theorem is proved for the sets where the density is $\geq c > 0$. A key to the analysis is the notion of the Riesz characteristic of the equation. This invariant is a real number $p \geq 1$ which governs the asymptotic behavior of singularities.

For this study we focus on the closed set $F = \{ A \in \text{Sym}^2(\mathbb{R}^n) : F(A) \geq 0 \}$ (cf. [Kr], [HL4]), and the operator $F$ will play no role. This set is always assumed to have the following three properties:

(i) (Positivity) $F + \mathcal{P} \subset F$ where $\mathcal{P} \equiv \{ A \geq 0 \}$.

(ii) (ST-Invariance) $F$ is invariant under a subgroup $G \subset O(n)$ which acts transitively on the sphere $S^{n-1} \subset \mathbb{R}^n$.

(iii) (Cone Property) $tF \subset F$ for all $t \geq 0$.

A closed set $F$ satisfying Positivity is called a subequation, and the viscosity $F$-subsolutions are called $F$-subharmonic functions. Each subequation $F$ has its own potential theory ([DD], [DDR]). For some of the results here, in addition to these three conditions, $F$ is also assumed to be convex. In this case distribution theory provides an alternate but equivalent foundation (Theorem 7.5) for subsolutions. The equations covered here include many classical examples coming from real, complex and calibrated geometry, such as the Monge-Ampère and Hessian equations. The reader is encouraged to glance at Section 4 for some basic examples.

We begin the paper by introducing the algebraically defined Riesz characteristic $p_F$ for $F$, which determines much of the behavior of subsolutions examined here. The name comes from the fact that when $p \equiv p_F$ is finite, the classical $p^{th}$ Riesz kernel $K_p(|x|)$, where

$$K_p(t) = \begin{cases} t^{2-p} & \text{if } 1 \leq p < 2 \\ \log t & \text{if } p = 2 \\ -\frac{1}{p-2} & \text{if } 2 < p < \infty. \end{cases}$$

is a solution of the non-linear equation $F$. In fact, every increasing radial solution is of the form $\Theta K_p(|x|) + C$ for constants $\Theta \geq 0$ and $C$.

When $p$ is finite, there is an associated tangential $p$-flow on $F$-subharmonic functions $u$ at each point $x_0$, given for $x_0 = 0$ by

$$u_r(x) = \begin{cases} r^{p-2}u(rx) & \text{if } p \neq 2, \text{ and} \\ u(rx) - M(u,r) & \text{if } p = 2. \end{cases}$$

where

$$M(u,r) \equiv \sup_{|x| \leq r} u.$$
The **tangents** to \( u \) at \( 0 \in \mathbb{R}^n \) are defined to be the set \( T_0(u) \) cluster points of the flow (1.2). When \( F \) is convex, these cluster points are taken in \( L^1_{\text{loc}}(\mathbb{R}^n) \). When \( 1 \leq p_F < 2 \) (but \( F \) not necessarily convex), they can be taken in the local \( \beta \)-Hölder norm for \( \beta < 2 - p \). In either case, \( U \in T_0(u) \) if and only if there exists a sequence \( r_j \downarrow 0 \) such that \( u_{r_j} \rightarrow U \) (in the appropriate space). It is a basic result that tangents are always entire \( F \)-subharmonic functions on \( \mathbb{R}^n \). In particular, the \( L^1_{\text{loc}} \)-limits have unique upper semi-continuous representatives which are viscosity \( F \)-subsolutions (see Theorem 7.5(b)). A fundamental result is the following (see Sections 8 and 12).

**THEOREM 1.1. (Existence).** If \( F \) is convex or if \( p_F < 2 \), then tangents always exist.

A natural question is whether tangents are actually solutions (as opposed to subsolutions). This is essentially so. An \( F \)-subharmonic function on \( X^{\text{open}} \subset \mathbb{R}^n \) is called \( F \)-maximal if for each \( F \)-subharmonic function \( v \) on \( X \) and each compact subset \( K \subset X \),

\[
v \leq u \quad \text{on} \quad X - K \quad \Rightarrow \quad v \leq u \quad \text{on} \quad X.
\]

If \( u \) is \( F \)-maximal on \( X \), then on any subdomain \( Y \subset X \) where \( u \) is continuous, it is a viscosity solution (or “\( F \)-harmonic”). In particular, it is always the Perron function for its boundary values on any ball. However, examples from classical pluripotential theory show that tangents may have poles. A second fundamental result is the following (see Section 6).

**THEOREM 1.2. (Maximality).** If \( F \) is convex, then tangents are always maximal. If \( p_F < 2 \), then tangents are \( F \)-harmonic (maximal and continuous) outside the origin.

The “Basic Workhorse Result” in this paper is the following (see Section 5).

**THEOREM 1.3. (Monotonicity).** Let \( u \) be \( F \)-subharmonic in a neighborhood of the origin in \( \mathbb{R}^n \), and let \( M(u, r) \) be defined by (1.3). Then

\[
\frac{M(u, r) - M(u, s)}{K(r) - K(s)}
\]

is increasing in \( r \) and \( s \).

for all \( 0 < s < r \) where \( M \) is defined.

Furthermore, if \( F \) is convex, the same statement holds with \( M(u, r) \) replaced by either

\[
S(u, r) \equiv \int_S u(r \sigma) d\sigma \quad \text{or} \quad V(u, r) \equiv \int_B u(rx) dx
\]

(\( \text{the spherical or volume average} \)) where \( B \equiv \{|x| \leq 1\} \) is the unit ball, \( S \equiv \partial B \) is the unit sphere, and \( \int_S = \frac{1}{|S|} \int_S \) denotes the average or “normalized” integral.

This theorem has several immediate consequences for the functions \( \Psi(u, r) \) for \( \Psi = M, S, V \) (e.g. Corollary 5.5). In particular, it leads to the concept of densities.

**Definition 1.4.** Suppose \( u \) is \( F \)-subharmonic in a neighborhood of \( 0 \in \mathbb{R}^n \). Then the **\( M \)-density** of \( u \) at 0 is the decreasing limit

\[
\Theta^M(u, 0) \equiv \lim_{s \downarrow 0} \frac{M(u, r) - M(u, s)}{K(r) - K(s)}.
\]
When $F$ is convex, there are also $\Psi$-densities
\[
\Theta^\Psi(u,0) \equiv \lim_{s \to 0^+} \frac{\Psi(u,r) - \Psi(u,s)}{K(r) - K(s)}.
\]
for $\Psi = S$ and $V$ as in (1.5).

Elementary results concerning these densities are established in Proposition 5.7.

When $F$ is convex, each $F$ subharmonic function is classically $\Delta$-subharmonic, and so $\Delta u = \mu \geq 0$ (a positive measure). Thus we also have the standard “mass density”
\[
\Theta^\theta(\mu,0) \equiv \lim_{r \to 0^+} \frac{\mu(B_r(0))}{\alpha(q)r^q}
\]
where $q = n - p$.

In this convex case all of the densities for $M,S,V$ and $\mu$ are universally related, and when $p = 2$ we have the further result that $\Theta^M = \Theta^S = \Theta^V$ (see Propositions 5.11 and 5.12).

The existence and maximality of tangents (Theorems 1.1 and 1.2) are a consequence of understanding the averages of tangents determined in Section 7.

**THEOREM 1.5.** (Averages of Tangents). Suppose $F$ is convex and $u$ is an $F$-subharmonic function defined in a neighborhood of the origin in $\mathbb{R}^n$. Let $p = p_F$ be the Riesz characteristic of $F$. If $p \neq 2$, then each tangent $U$ to $u$ at 0 has averages
\[
M(r) = \sup_{S} U(r\sigma) = \Theta^M(u)K(r), \quad S(r) = 2\int_{S} U(r\sigma) d\sigma = \Theta^S(u)K(r),
\]
and
\[
V(r) = \int_{B} U(rx) dx = \Theta^V(u)K(r).
\]
(1.6)

In particular,
\[
\Theta^\Psi(U) = \Theta^\Psi(u) \quad \text{for} \quad \Psi = M, S, \text{or} \ V
\]
(1.7)

When $p = 2$, all the densities of $u$ and any tangent $U$ to $u$ at 0, agree, and will be simply denoted by $\Theta = \Theta(u)$. Specifically, we have
\[
\Theta(u) = \Theta^M(U) = \Theta^S(U) = \Theta^V(U) = \Theta^M(u) = \Theta^S(u) = \Theta^V(u).
\]
(1.8)

Moreover, the averages of a tangent $U$ to $u$ are given by
\[
M(r) = \Theta \log r, \quad S(r) = \Theta \log r + \int_{S} U, \quad \text{and} \quad V(r) = \Theta \log r + \int_{B} U
\]
(1.9)

In the classical case of pluripotential theory the Riesz characteristic is 2, and our results here for $p = 2$ are an extension of the work of Kiselman [K1].

A basic result concerning densities is the following.

**THEOREM 1.6.** (Upper Semi-Continuity of Density). Suppose $u$ is $F$-subharmonic on an open set $X \subset \mathbb{R}^n$. Then each of the densities
\[
\Theta^M(u,x), \quad \Theta^S(u,x), \quad \Theta^V(u,x)
\]
considered above is an upper semi-continuous function of $x$. Equivalently, for all $c \geq 0$ and each $\Theta$ as above, the sets

$$E_c \equiv \{ x : \Theta(u, x) \geq c \}$$

are closed.

We also note that by standard geometric measure theory

$$c\mathcal{H}^{n-p}(E_c) \leq \mu(X).$$

This brings us to the natural question of uniqueness. Here there are two important concepts.

**Definition 1.7.** We say that **uniqueness of tangents** holds for the subequation $F$ if for every $F$-subharmonic function $u$ defined in a neighborhood of 0, there is exactly one tangent to $u$ at 0.

We say that **strong uniqueness of tangents** holds for $F$ if for every such $u$, the unique tangent is $\Theta(u, 0)K_p(|x|)$.

We say that **homogeneity of tangents** holds for $F$ if every tangent to an $F$-subharmonic is fixed by the tangential $p$-flow (1.2).

Since the flow takes a tangent to $u$ to another tangent to $u$, uniqueness of tangents implies homogeneity of tangents.

Several basic cases where uniqueness holds are discussed in Section 9. For example, zero density implies strong uniqueness (Proposition 9.5).

One of the main results of this paper is the Strong Uniqueness Theorem in Section 10. Note that there is a natural action of the group $O(n)$ on $\text{Sym}^2(R^n)$. The subequations $F \subset \text{Sym}^2(R^n)$ which are $O(n)$-invariant are exactly those which are defined in terms of the eigenvalues of the matrices $A \in \text{Sym}^2(R^n)$. Every such subequation has a complex and quaternionic counterpart defined on $\mathbb{C}^n$ and $\mathbb{H}^n$ by applying the same eigenvalue constraints to the complex or quaternionic hermitian symmetric part of $A$.

**Theorem 1.8. (Strong Uniqueness I).** Suppose $F$ is a convex $O(n)$-invariant subequation, or the complex or quaternionic counterpart of such an equation. Then, except for the three basic cases $\mathcal{P}, \mathcal{P}^C, \mathcal{P}^H$, strong uniqueness of tangents holds for $F$.

There do exist **non-convex** $O(n)$-invariant subequations of every Riesz characteristic for which strong uniqueness fails. See Example 10.14.

Theorem 1.8 establishes strong uniqueness for a wide range of equations. These include the $k$th Hessian equations ($k < n$) and $p$-convexity equations ($p$ real, $1 \leq p \leq n$), the trace powers of the Hessian, equations coming from Gårding polynomials, and much more. It does not cover many equations which arise from calibrations and Lagrangian geometry. However, there are also results in these cases.

Suppose $F = F(G)$ is a subequation defined by a compact subset $G \subset G(p, R^n)$ of the Grassmannian of $p$-planes in $R^n$ (see Example 4.4).
THEOREM 1.9. (Strong Uniqueness II). Fix $p \geq 2$ and $n \geq 3$. Then strong uniqueness of tangents to $F(G)$-subharmonic functions holds for:

(a) Every compact $SU(n)$-invariant subset $G \subset G^R(p, C^n)$ except $P^C$, 
(b) Every compact $Sp(n)\cdot Sp(1)$-invariant subset $G \subset G^R(p, H^n)$ with three exceptions, namely the sets of real $p$-planes which lie in a quaternion line for $p = 2, 3, 4$ (when $p = 4$ this is $P^H$),
(c) For $p \geq 5$, every compact $Sp(n)$-invariant subset $G \subset G^R(p, H^n)$.

This result is based on a companion theorem which has further applications. Given $G \subset G(p, R^n)$ as above, we say that $G$ has the transitivity property if for any two vectors $x, y \in R^n$ there exist $W_1, ..., W_k \in G$ with $x \in W_1, y \in W_k$ and $\dim(W_i \cap W_{i+1}) > 0$ for all $i = 1, ..., k - 1$. The subequations attached to Lagrangian, Special Lagrangian, Associative, Coassociative, and Cayley geometries all have this property.

THEOREM 1.10. (Strong Uniqueness III). If $G$ has the transitivity property, then strong uniqueness of tangents holds for all $F(G)$-subharmonic functions.

Theorems 1.9 and 1.10 will be proved in Part II of the paper.

We now return to the subject of the high-density sets. Suppose $F$ is convex and $u$ is an $F$-subharmonic function on an open set $X \subset R^n$. Let $\Theta = \Theta^S(u, \cdot) : X \rightarrow R$ be the area (or volume) density function of $u$, and for $c > 0$ consider the sets

$$E_c = \{ x : \Theta(u, x) \geq c \}$$

For classical plurisubharmonic functions in $C^n$ a deep theorem, due to L. Hörmander, E. Bombieri and in its final form by Siu ([Ho], [B], [Siu]), states that $E_c$ is a complex analytic subvariety. One straightforwardly deduces from this result that for the 2-convexity subequation $P_2$ in $R^{2n}$ the set $E_c$ is discrete, since $P^C(J) \subset P_2$ for all parallel complex structures $J$ on $R^{2n}$.

This very restrictive corollary has a quite general extension.

THEOREM 1.11. (Structure of High Density Sets). Suppose strong uniqueness of tangents holds for $F$. Then for any $F$-subharmonic function $u$, the set $E_c(u)$ is discrete.

Theorem 1.11 is essentially sharp. Suppose $\Omega$ is a domain with strictly convex boundary. Given any finite subset $E = \{ x_j \}_{j=1}^N \subset \Omega$, any set of numbers $\Theta_j > 0$, $j = 1, ..., N$, and any $\varphi \in C(\partial \Omega)$, there exists a unique continuous $u : \overline{\Omega} \rightarrow [-\infty, \infty)$ such that

(1) $u$ is $F$-harmonic on $\Omega - E$,
(2) $u|_{\partial \Omega} = \varphi$, and
(3) $\Theta(u, x_j) = \Theta_j$ for $j = 1, ..., N$.

See Remark 11.2 for more details.

The subequations with characteristic $1 \leq p < 2$ are very different in nature from those where $p \geq 2$. They are discussed in detail in Section 12. In particular, the following is proved.
THEOREM 1.12. (Hölder Continuity $1 \leq p < 2$). Suppose $F$ is a (not necessarily convex) subequation with Riesz characteristic $1 \leq p < 2$. Then each $F$-subharmonic function is locally Hölder continuous with exponent $\alpha \equiv 2 - p$.

Furthermore, if $u$ is an $F$-subharmonic defined in a neighborhood of $0 \in \mathbb{R}^n$, then every sequence $\{u_{r_j}\}_{j=1}^{\infty}$ with $r_j \downarrow 0$, has a subsequence which converges locally uniformly to an $F$-subharmonic function $U$ on $\mathbb{R}^n$. In fact for each $0 < \beta < 2 - p$ there exists a subsequence which converges locally in $\beta$-Hölder norm. Finally, this limit $U$ is $F$-harmonic on $\mathbb{R}^n - \{0\}$.

For the $k^\text{th}$ Hessian equation $p = n/k$, and for $k > n/2$, the Hölder continuity result was proved in [TW1]. Their proof can be carried over to more general convex equations. However, convexity is not required in Theorem 1.12.

It is important to note that uniqueness of tangents does not always hold. In the basic case of convex functions ($F = \mathcal{P}$) we have uniqueness, but strong uniqueness fails. For classical plurisubharmonic functions (the complex counterpart: $F = \mathcal{P}^\mathbb{C}$), the uniqueness question was raised in [H] and answered in the negative by Kiselman [K1], who characterized the sets which can arise as $T_0(u)$ for a plurisubharmonic function $u$ in $\mathbb{C}^n$. In Part II of this paper we analyze the structure of individual tangents in this case. An interesting feature is that each such tangent corresponds to a quasi-plurisubharmonic function on complex projective space $\mathbb{P}^{n-1}(\mathbb{C})$.

A similar result is obtained for the quaternionic counterpart $\mathcal{P}^\mathbb{H}$. The $\mathcal{P}^\mathbb{H}$-subharmonic functions are quaternionic plurisubharmonic functions on $\mathbb{H}^n$ (cf. [A1,2], [AV]). Here each tangent function corresponds to a function on quaternionic projective space $g : \mathbb{P}^{n-1}(\mathbb{H}) \to \mathbb{R}$ which satisfies $\Delta g - 2gI \geq 0$. We leave it to the reader to explore adapting Kiselman’s characterization of the possible tangent sets $T_0(u)$ in the quaternionic case.

In Section 2 we fill a gap in the literature by characterizing the radial viscosity subsolutions.

In Appendix A we examine the radial subequation for $\bar{\mathcal{P}} \equiv \{\lambda_{\text{max}} \geq 0\}$ and establish a basic dichotomy – the Increasing/Decreasing Lemma.

In Appendix B we show that the subequation $\mathcal{P}(\delta) \equiv \{A + \delta \text{tr}(A) \geq 0\}$ is uniformly elliptic in the conventional sense.

While in Section 4 we give a number of examples to which our theory applies, many more examples are given in the appendix to Part II. That appendix also constructs the maximal and minimal subequations of Riesz characteristic $p$ (showing, in particular, that these largest and smallest subequations exist). There is a companion result describing the largest and smallest convex subequations of characteristic $p$. The largest is given in Proposition 10.9. The smallest is given in Lemma A.1 of Part II.

Historical Reflections

In 1982 the authors showed that for each calibration on a riemannian manifold there is an associated family of minimal subvarieties – forming a calibrated geometry [HL1]. More recently [HL2] it was discovered that the calibration also determines a potential theory of functions whose restrictions to each of the distinguished submanifolds are subharmonic. Although there is an analogue in this setting of the $i\partial \bar{\partial}$ operator from complex geometry,
that operator does not play a critical role in the development of the potential theory [HL4]. In fact, somewhat surprisingly, a corresponding potential theory can be established for any collection of submanifolds determined by requiring their tangent spaces to be in an arbitrary given closed subset of the Grassmannian. Even more generally one has the potential theory associated to an elliptic (possibly degenerate) nonlinear inequality $F(D^2u) \geq 0$, provided by viscosity subsolutions ([CIL]).

This raises the possibility of cross-fertilization between two well established and deep fields, pluripotential theory (in several complex variables) and nonlinear elliptic theory. This paper, although not the first, can be viewed as an example of this phenomenon. The authors believe there are many more.
2. The Radial Subequation Associated to a Subequation $F$.

In this section we examine the ordinary differential inequality which governs $C^2$ radial (i.e., spherically symmetric) $F$-subharmonic functions. Our main result is the extension of this characterization from $C^2$ radial $F$-subharmonics to general upper semi-continuous $F$-subharmonics.

Suppose $\psi$ is of class $C^2$ on an interval contained in the positive real numbers, and consider $\psi(|x|)$ as a function on the corresponding annular region in $\mathbb{R}^n$.

**Lemma 2.1.**

\[
D_x^2 \psi = \frac{\psi'(|x|)}{|x|} P_{[x]} + \psi''(|x|) P_{[x]}.
\] (2.1)

where $P_{[x]} = \frac{x}{|x|}$ denotes orthogonal projection onto the line $[x]$ through $x \neq 0$ and $P_{[x]} = I - P_{[x]}$ denotes orthogonal projection onto the hyperplane with normal $[x]$.

**Proof.** First note that $D(|x|) = \frac{x}{|x|}$ and therefore $D^2(|x|) = D\left(\frac{x}{|x|}\right) = \frac{1}{|x|} I - \frac{x}{|x|^2} \circ \frac{x}{|x|} = \frac{1}{|x|}(I - P_{[x]}) = \frac{1}{|x|} P_{[x]}$. Hence,

\[
D_x \psi = \psi'(|x|) \frac{x}{|x|} \quad \text{and}
\]

\[
D_x^2 \psi = \psi'(|x|)D\left(\frac{x}{|x|}\right) + \psi''(|x|) \frac{x}{|x|} \circ \frac{x}{|x|} = \frac{\psi'(|x|)}{|x|} P_{[x]} + \psi''(|x|) P_{[x]}.
\]

**Corollary 2.2.** The second derivative $D_x^2 \psi$ has eigenvalues $\frac{\psi'(|x|)}{|x|}$ with multiplicity $n - 1$ and $\psi''(|x|)$ with multiplicity 1.

Let $F \subset \text{Sym}^2(\mathbb{R}^n)$ be a pure second-order constant coefficient subequation. The by Lemma 2.1 a radial $C^2$-function $u(x) = \psi(|x|)$ is $F$-subharmonic on an annular region in $\mathbb{R}^n$ if and only if

\[
D_x^2 u = \frac{\psi'(t)}{t} P_{[e]} + \psi''(t) P_{[e]} \in F,
\] (2.2)

for $t = |x|$ in the corresponding interval in $(0, \infty)$. We use $\lambda = \psi'(t)$ and $a = \psi''(t)$ as one-variable jet coordinates. Then the basic one-variable subequation associated with $F$ is defined as follows.

**Definition 2.3.** The **radial subequation associated with** $F$ is the reduced variable coefficient subequation $R_F$ on $(0, \infty)$ whose fibre at $t$ is

\[ (R_F)_t \equiv \left\{ (\lambda, a) \in \mathbb{R}^2 : \frac{\lambda}{t} P_{[e]} + a P_{[e]} \in F, \forall |e| = 1 \right\}. \]

Thus for $C^2$-functions we have that

\[
u(x) \equiv \psi(|x|) \text{ is } F \text{ subharmonic} \iff \psi(t) \text{ is } R_F \text{ subharmonic} \quad (2.3)
\]
This can be extended to the viscosity setting where $F$-subharmonic functions are just upper semi-continuous (see [C], [CIL], [HL4,6] for definitions). The proof given below of the implication \( \Rightarrow \) is elementary, whereas the proof of \( \Leftarrow \) will require a lemma. Note that the equivalence: \( u(x) = \psi(|x|) \) is upper semicontinuous \( \iff \psi(t) \) is upper semicontinuous, is obvious.

**THEOREM 2.4. (Radial Subharmonics).** The function \( u(x) \equiv \psi(|x|) \) is $F$-subharmonic on an annular region in $\mathbb{R}^n$ if and only if \( \psi(t) \) is $R_F$-subharmonic on the corresponding open sub-interval of \((0, \infty)\).

**Remark 2.5.** In all but this section of the paper, the subequations $F$ will be assumed to be cones, unless explicitly stated to the contrary. For such subequations the maximum principle holds, i.e., it holds for each $F$-subharmonic function \( u(x) \) (cf. Appendix A). Consequently, if \( u(x) = \psi(|x|) \) is a radial $F$-subharmonic on a ball about 0, then \( \psi(t) \) must be increasing in $t$. This motivates focusing on an “increasing” version of Theorem 2.4.

We will use the fact, which is elementary to establish, that for an upper semi-continuous function \( \psi(t) \),

\[
\psi(t) \text{ is increasing } \iff \psi \text{ is } \{ \lambda \geq 0 \} \text{ subharmonic.} \quad (2.4)
\]

**Definition 2.6.** The increasing radial subharmonic equation \( R_F^\uparrow \) on \((0, \infty)\) is defined by

\[
R_F^\uparrow = R_F \cap \{ \lambda \geq 0 \}. \quad (2.5)
\]

In light of (2.3), it is obvious that for $C^2$-functions \( \psi(t) \):

\[
\psi(t) \text{ is } R_F^\uparrow \text{ subharmonic } \iff \psi(|x|) \text{ is } F \cap \{ x \cdot p \geq 0 \} \text{ subharmonic} \quad (2.6)
\]

where the variable coefficient first-order subequation \( \{ x \cdot p \geq 0 \} \) is the constraint \( x \cdot D_x u \geq 0 \) on $C^2$-functions. The equivalence (2.6) can be extended as in Theorem 2.4.

**THEOREM 2.7. (Increasing Radial Subharmonics).** The function \( u(x) \equiv \psi(|x|) \) is an increasing, radial $F$-subharmonic function if and only if \( \psi(t) \) is $R_F^\uparrow$-subharmonic.

**Remark 2.8.** We will sometimes blur the distinction between \( \psi(t) \) and \( u(x) = \psi(|x|) \) by calling \( \psi(t) \) a radial (or increasing radial) $F$-subharmonic.

**Remark 2.9.** The statement and proof of a theorem analogous to 2.7 for decreasing radial subharmonics is left to the reader.

**Proof of Theorem 2.4.** \((\Rightarrow)\): Suppose \( u(x) \equiv \psi(|x|) \) is $F$-subharmonic. If \( \varphi(t) \) is a test function for \( \psi(t) \) at \( t_0 \), then \( \varphi(|x|) \) is a test function for \( \psi(|x|) \) at any point on the \( t_0 \)-sphere in \( \mathbb{R}^n \). Therefore \( D_{t_0}^2 \varphi \in F \). Applying the formula for \( D_{t_0}^2 \varphi \) in terms of \( \varphi'(t_0) \) and \( \varphi''(t_0) \), the equivalence (2.3), and the definition of \( (R_F)_{t_0} \), we have \( J_{t_0}^2 \varphi(t) \in R_F \). This proves that \( \psi(t) \) is $R_F$-subharmonic.
Lemma 2.10 below ensures the existence of a unique minimum point at the critical point \(z\) in \(z\) near \(u\) and let \(\varphi\) be an upper semi-continuous function (of \(t\)) near \(t, y\) that is a \(C^2\)-function, both defined in a neighborhood of \(z_0\).

**Lemma 2.10.** Suppose \(u(t) < \varphi(z)\) for \(z \neq z_0\) with equality at \(z_0\). If \(\varphi(z)\) is a polynomial of degree \(\leq 2\), then there exists a polynomial \(\varphi(t)\) of degree \(\leq 2\) with

\[
    u(t) \leq \varphi(t) \leq \varphi(z) \quad \text{near } z_0.
\]

**Proof.** We may assume \(z_0 = 0\) and \(u(0) = \varphi(0) = 0\). Then

\[
    \varphi(z) = \langle p, t \rangle + \langle q, y \rangle + \langle At, t \rangle + 2\langle Bt, y \rangle + \langle Cy, y \rangle.
\]

We assume \(u(t) < \varphi(t, y)\) for \(|t| \leq \epsilon\) and \(|y| \leq \delta\) with \((t, y) \neq (0, 0)\).

Setting \(t = 0\), we have \(0 = u(0) < \langle q, y \rangle + \langle Cy, y \rangle\) for \(y \neq 0\) sufficiently small. Therefore, \(q = 0\) and \(C > 0\) (positive definite). Now define

\[
    \varphi(t) = \langle p, t \rangle + \langle (A - B^t C^{-1} B)t, t \rangle.
\]

The inequalities in (2.8) follow from the fact that for \(t\) sufficiently small,

\[
    \varphi(t) = \inf_{|y| \leq \delta} \varphi(z) = \langle p, t \rangle + \langle At, t \rangle + \inf_{|y| \leq \delta} \{2\langle Bt, y \rangle + \langle Cy, y \rangle\}.
\]

To prove (2.10) fix \(t\) and consider the function \(2\langle Bt, y \rangle + \langle Cy, y \rangle\). Since \(C > 0\), it has a unique minimum point at the critical point \(y = -C^{-1} Bt\). The minimum value is \(-\langle B^t C^{-1} Bt, t \rangle\). If \(t\) is sufficiently small, the critical point \(y\) satisfies \(|y| < \delta\), which proves (2.7).
Proof of Theorem 2.7. The arguments given for Theorem 2.4 along with the following missing steps provides the proof. If $\varphi(t)$ is a test function for $\psi(t)$ at a point $t_0$, then $\varphi(|x|)$ is a test function for $\psi(|x|)$ at $x_0$ whenever $|x_0| = t_0$. Now

$$D_{x_0}\varphi = \varphi'(|x_0|)\frac{x_0}{|x_0|} \quad \text{and hence} \quad x_0 \cdot D_{x_0}\varphi = |x_0|\varphi'(|x_0|). \quad (2.11)$$

Hence, if $\psi(|x|)$ is $\{p \cdot x \geq 0\}$-subharmonic, then $\psi(t)$ is $\{\lambda \geq 0\}$-subharmonic, and thus increasing. Conversely, if $\psi(t)$ is increasing and $\varphi(x)$ is a test function for $\psi(|x|)$ at $x_0$, then $\overline{\varphi}(t) \equiv \varphi(\frac{t x_0}{|x_0|})$ is a test function for $\psi(t)$ at $t_0 = |x_0|$. Hence, $\overline{\varphi}(t_0) \geq 0$. However, $\overline{\varphi}(t_0) = (D_{x_0}\varphi) \cdot x_0$. \hfill ■

3. Invariant Cone Subequations – The Riesz Characteristic

Although the radial $F$-subharmonics are determined by the radial subequation $R_F$, this is not the case in general for $F$-harmonics. We now restrict attention to an important class of subequations for which it is the case. Some of the multitude of examples are discussed in the next section.

Recall from the introduction that a subequation $F \subset \text{Sym}^2(\mathbb{R}^n)$ is said to be $\text{ST-invariant}$ if there exists a subgroup $G \subset O(n)$ which acts transitively on the sphere $S^{n-1} \subset \mathbb{R}^n$ and leaves $F$ invariant under the induced action of $G$ on $\text{Sym}^2(\mathbb{R}^n)$.

For an ST-invariant cone subequation $F$,

the slices $F \cap \text{span} \{P_{e\perp}, P_e\}$ for $e \in S^{n-1}$ are all isomorphic. \hfill (3.1)

Note that $\text{span} \{P_{e\perp}, P_e\} = \text{span} \{I, P_e\}$ and that the induced action on $\text{Sym}^2(\mathbb{R}^n)$ sends $P_e$ to $P_{g(e)}$. In particular,

$$\lambda P_{e\perp} + \mu P_e \in F \quad \text{for one } e \in S^{n-1} \quad \iff \quad \lambda P_{e\perp} + \mu P_e \in F \quad \text{for all } e \in S^{n-1}. \quad (3.2)$$

The Riesz Characteristics

Although there is an abundance of interesting ST-invariant cone subequations in dimensions $\geq 3$, there are not many increasing radial subequations. In fact they are described by a single “characteristic” number $p$ between 1 and $\infty$.

**Definition 3.1.** For each $p$ with $1 \leq p < \infty$, the increasing radial subequation $R_p^\uparrow$ is defined by

$$R_p^\uparrow : \quad a + \frac{(p-1)}{t} \lambda \geq 0 \quad \text{and} \quad \lambda \geq 0, \quad (3.3)$$

while for $p = \infty$, the subequation $R_\infty^\uparrow$ is first-order and defined by $R_\infty^\uparrow = \{\lambda \geq 0\}$.

**Definition 3.2.** (The Increasing Riesz Characteristic). Suppose $F$ is an ST-invariant cone subequation. The increasing characteristic $p_F$ of $F$ is defined to be

$$p_F \equiv \sup \{q : P_{e\perp} - (q-1)P_e \in F\}. \quad (3.4a)$$
Equivalently, for finite Riesz characteristic, \( p_F \) is the unique number \( p \) such that
\[
P_{e^\perp} - (p - 1)P_e \in \partial F.
\]

**Proposition 3.3.** Suppose that \( F \) is an ST-invariant cone subequation. Then the increasing radial subequation \( R_F^\dagger \) equals \( R_p^\dagger \) where \( p = p_F \) is the Riesz characteristic of \( F \).

**Proof.** Using Definitions 2.3, 2.6, 3.1 and 3.2, we must show that for \( \lambda \geq 0 \)
\[
\frac{\lambda}{t}P_{e^\perp} + aP_e \in F \iff a + \frac{p-1}{t}\lambda \geq 0.
\]
Set \(- (q - 1) \equiv at/\lambda\), so that \( \frac{\lambda}{t}P_{e^\perp} + aP_e \in F \iff P_{e^\perp} - (q - 1)P_e \in F \). Then \( q \leq p \iff -\frac{at}{\lambda} \leq p - 1 \iff a + \frac{p-1}{t}\lambda \geq 0. \)

Note that by Definition 3.2, the positivity condition for \( F \), and the fact that \( 0 \in F \), we have that \( p_F \geq 1 \). Thus \( 1 \leq p_F \leq \infty \).

The only equation with \( p_F = 1 \) is \( \mathcal{P} \). At the other extreme we have \( p_F = \infty \). Here there is a test which is very simple to apply in all the ST-invariant examples, namely: \( p_F = \infty \) iff \(-P_e \notin F \). Hence, determining when \( p_F < \infty \) is also simple, namely: \( p_F < \infty \) iff \(-P_e \notin F \).

**Lemma 3.4.** For ST-invariant cone subequations \( F \)
\[
\begin{align*}
(a) & \quad p_F = 1 \iff P_{e^\perp} \in \partial F \iff F = \mathcal{P} \\
(b) & \quad p_F = \infty \iff -P_e \notin F \iff -P_e \in \partial F. \\
(c) & \quad p_F < \infty \iff P_e \in \text{Int}\tilde{F} \iff \mathcal{P} \subset \text{Int}\tilde{F}.
\end{align*}
\]

Actually, as we show in Section 4, it is easy to compute the exact value of \( p_F \) in all the examples.

**Proof of (a).** Note first that \( p_F > 1 \iff P_{e^\perp} - \epsilon P_e \in F \) for all small \( \epsilon > 0 \). Now if \( F \) contains an element \( A \) with at least one eigenvalue strictly negative, then by positivity and the cone property there is an element \( A' = P_{e^\perp} - \epsilon P_e \in F \). Hence \( F \neq \mathcal{P} \Rightarrow p_F > 1 \).

**Proof of (b).** Note first that \(-P_e \in F \Rightarrow \alpha P_{e^\perp} - P_e \in F \forall \alpha \geq 0 \Rightarrow P_{e^\perp} - (p - 1)P_e \in F \forall p \geq 1 \Rightarrow p_F = \infty \). On the other hand \(-P_e \notin F \Rightarrow \epsilon P_{e^\perp} - P_e \notin F \forall \epsilon \geq 0 \) small \( \Rightarrow P_{e^\perp} - (p - 1)P_e \notin F \forall p \) large \( \Rightarrow p_F < \infty \). To complete the proof of (b) note that \(-P_e \notin \text{Int}F \) cannot occur unless \( F = \text{Sym}^2(\mathbb{R}^n) \) since \(-P_e \in \text{Int}F \Rightarrow 0 \in \text{Int}F \).

**Proof of (c).** Since \( \sim (F) = \text{Int}\tilde{F} \), the first part of (c) follows from the first part of (b).

For any subequation \( G \) (such as \( \tilde{F} \)), \( A \in \text{Int}G \Rightarrow A + \mathcal{P} \subset \text{Int}G \). Finally, \( P_e + \mathcal{P} = \mathcal{P} \), proving that \( P_e \in \text{Int}\tilde{F} \Rightarrow \mathcal{P} \subset \text{Int}\tilde{F} \).

The primary application of the Riesz characteristics (and the reason for choosing the name) is the fact that the solutions of the associated increasing radial equation \( R_p^\dagger \) are given by the Riesz kernels.
Proposition 3.5. If \( F \) has finite Riesz characteristic \( p = p_F \), then the increasing radial harmonics for \( F \) are:

\[
\Theta K_p(|x|) + C
\]

where \( \Theta \geq 0 \), \( C \in \mathbb{R} \), and \( K_p(t) \) is the \( p \)th Riesz function defined on \( 1 \leq t < \infty \) by

\[
K_p(t) = \begin{cases} 
t^{2-p} & \text{if } 1 \leq p < 2 \\
\log t & \text{if } p = 2 \\
-\frac{1}{t^{p-2}} & \text{if } 2 < p < \infty.
\end{cases}
\]

Proof. From (3.4b) it is easy to see that \( u(x) \equiv \psi(|x|) \) is \( F \)-subharmonic if and only if \( \psi(t) \) is \( R^1_p \)-subharmonic. The ordinary differential equation given by equality in (3.3) is easily solved, and \( \Theta K_p(t) + C \) are the viscosity solutions. One can check directly using Lemma 2.1 that

\[
\frac{D^2 K_p(|x|)}{|x|^p} = \frac{1}{|x|^p} \left( P_{[x]^+} - (p-1)P_{[x]} \right) \quad \text{and} \quad \frac{D K_p}{|x|^p} = \frac{x}{|x|^p}
\]

where \( K_p \) has been renormalized to

\[
\overline{K}_p \equiv \frac{1}{|p-2|} K_p \quad \text{if } p \neq 2 \quad \text{and} \quad \overline{K}_2 = K_2 \quad \blacksquare
\]

The normalization in (3.6) is simpler when the focus is on the function \( u \), while the normalization in (3.8) is simpler when the focus is on the first and second derivatives of \( u \).

Remark 3.6. (Dimension \( n = 2 \)). When \( n = 2 \), the ST-invariant cone subequations are very limited. The subequation defined by

\[
F(p) : \lambda_{\min}(A) + (p-1)\lambda_{\max}(A) \geq 0
\]

is the only subequation with characteristic \( p \). The dual subequation \( \widetilde{F}(p) \) is defined by

\[
\widetilde{F}(p) : (p-1)\lambda_{\min}(A) + \lambda_{\max}(A) \geq 0
\]

It has characteristic \( \frac{1}{p-1} \). That is, \( \widetilde{F}(p) = F(\frac{1}{p-1}) \). Note that the invariance must be full \( O(2) \)-invariance.

Finally, \( F \) is convex in the range \( 1 \leq p \leq 2 \), but not in the range \( 2 < p < \infty \).

We note that the subequation (3.9) makes sense in any dimension \( n \), and it plays a special role. It is the largest conical ST-invariant subequation of characteristic \( p \). See Appendix A in Part II.

Remark 3.7. (The Decreasing Riesz Characteristic). For each ST-invariant cone subequation \( F \), this characteristic, denoted \( q_F \), is defined by

\[
q_F = \sup \{ \bar{q} : -P_{e^\perp} + (\bar{q} - 1)P_e \notin F \}
\]
Note that $1 \leq q_F \leq \infty$. For each $1 \leq q < \infty$ set
\[
R^↓_q : \ a + \frac{q-1}{t} \lambda \geq 0 \quad \text{and} \quad \lambda \leq 0.
\] (3.12)

Then the decreasing version of Proposition 3.3 states that
\[
R^↓_F = R^↓_q \quad \text{with} \quad q \equiv q_F.
\] (3.13)

That is, the decreasing radial $F$-subharmonics $u(x) = \psi(|x|)$ are determined by this decreasing Riesz characteristic $q_F$ via $R^↓_q$.

From the definition of $p_F$ in (3.4b) it is easy to see that
\[
q_F = p^-_F
\] (3.14)

since $\partial \tilde{F} = -\partial F$. Thus the decreasing characteristic of $F$ might also be called the dual characteristic of $F$. These characteristics $(p, q)$ satisfy
\[
(p - 1)(q - 1) \geq 1.
\] (3.15)

**Proof.** Suppose $-P_{e_1} + (q - 1)P_{e_1} \in F$, and $e_1, ..., e_n$ are orthonormal. Adding the form $qP_{e_2} + \cdots + qP_{e_{n-1}}$ to this yields $-P_{e_n} + (q - 1)P_{e_n} \in F$. Hence, $P_{e_n} - \frac{1}{q-1}P_{e_n} \in F$, so that $p - 1 \geq 1/(q - 1)$.

As we shall see in the next section, if $n \geq 3$, there exists an ST-invariant cone subequation $F$ with characteristics $(p, q)$ for any pair $(p, q)$ satisfying (3.15).

**Boundary Convexity and the Riesz Characteristic**

The results of this subsection will not be used in later sections. One example of the use of the Riesz characteristic involves $F$ boundary convexity.

The increasing Riesz characteristics can be used to characterize the the subequations $F$ with the property that all boundaries are strictly $\tilde{F}$-convex for the dual subequation $\tilde{F}$, and as we shall see in the next section the Riesz characteristics are relatively simple to compute for a given subequation.

**Proposition 3.8.** The boundary $\partial \Omega$ of every smoothly bounded domain $\Omega \subset \subset \mathbb{R}^n$ is strictly $\tilde{F}$-convex if and only if the Riesz characteristic $p_F < \infty$.

**Proof.** By Lemma 5.3(ii) in [HL4], $\partial \Omega$ is strictly $\tilde{F}$-convex at $x \in \partial \Omega$ for all domains $\Omega$ if and only if
\[
\forall B \in \text{Sym}^2(W), \ B + tP_e \in \text{Int}\tilde{F} \quad \text{for all} \ t \geq \text{some} \ t_0.
\] (3.16)

where $|e| = 1$ and $W = e_⊥$. Now (3.16) $\Rightarrow$ $P_e \in \text{Int}\tilde{F} \Rightarrow \frac{1}{t}B + P_e \in \text{Int}\tilde{F}$ for all $t \geq \text{some} \ t_0 \Rightarrow$ (3.16). Thus (3.16) is equivalent to $p_F < \infty$ by Lemma 3.4(c).

(See [HL6, §11] for discussion of boundary convexity for general subequations.)
For future reference regarding the inner boundary of an annulus, note that

\[-\partial B \text{ is strictly } \tilde{F} \text{ convex } \iff p_F < \infty. \tag{3.17}\]

Applying Proposition 3.8 to \(\tilde{F}\) instead of \(F\) we have:

**Proposition 3.9.** The boundary \(\partial \Omega\) of every smoothly bounded domain \(\Omega \subset \subset \mathbb{R}^n\) is strictly \(F\)-convex if and only if \(q_F < \infty\).

**Proof.** We have \(p_{\tilde{F}} = q_F\) by (3.14).

The joint condition that both Riesz characteristics be finite is equivalent to:

\[P_e \in \text{Int}F \quad \text{and} \quad -P_e \notin F, \tag{3.18}\]

which is easy to ascertain for a given \(F\).

Results in [HL4] immediately imply the following.

**THEOREM 3.10. (Universal Solvability of the Dirichlet Problem).** Suppose that \(F\) is an ST-invariant cone subequation for which both Riesz characteristics \(p_F\) and \(q_F\) are finite (or equivalently for which the simple condition (3.18) holds). Then for every domain \(\Omega \subset \subset \mathbb{R}^n\) with smooth boundary \(\partial \Omega\), and for every \(\varphi \in C(\partial \Omega)\), there exists a unique \(h \in C(\Omega)\) such that

1. \(h\) is \(F\)-harmonic on \(\Omega\), and
2. \(h|_{\partial \Omega} = \varphi\).

**Remark 3.11.** In fact Theorem 3.10 holds for any constant coefficient second-order subequation \(F\) if and only if its asymptotic cone subequation \(\overrightarrow{F}\) satisfies (3.18) for all \(|e| = 1\).
4. Some Illustrative Examples.

For the basic subequations the Riesz characteristic is quite easy to compute. We shall illustrate this with a selection of examples of differing types. A detailed discussion of subequations of characteristic \( p \), and further results, are given in Appendix A of Part II.

For \( A \in \text{Sym}^2(\mathbb{R}^n) \) we let
\[
\lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A)
\] (4.1)
denote the ordered eigenvalues of \( A \).

**Example 4.1. (The \( p \)-Convexity Equation).** For each real number \( p \) with \( 1 \leq p \leq n \), the smallest (see Lemma A.2 in Part II) convex cone subequation with characteristic \( p \) is also one of the most basic:
\[
\mathcal{P}_p \equiv \{ A : \lambda_1(A) + \cdots + \lambda_\lfloor p \rfloor(A) + (p - \lfloor p \rfloor)\lambda_{\lfloor p \rfloor+1}(A) \geq 0 \}.
\] (4.2)

For \( p \) an integer the \( \mathcal{P}_p \)-subharmonic functions are characterized by the fact that their restrictions to minimal submanifolds of dimension \( p \) are intrinsically subharmonic. For this and a discussion of the geometry associated with this equation, see [HL10]. (Results for integer \( p \) go back to H. Wu [Wu], [Sh].) Note, by the way, that \( \mathcal{P}_1 = \mathcal{P} \) is the homogeneous Monge-Ampère subequation and \( \mathcal{P}_n = \Delta \) is the standard Laplacian.

There are complex and quaternionic analogues \( \mathcal{P}_p^C \) and \( \mathcal{P}_p^H \) defined by (4.2) but using the eigenvalues of the complex (respectively quaternionic) hermitian symmetric part of \( A = D^2u \). When \( p = 1 \) this yields the homogeneous complex and quaternionic Monge-Ampère subequations. The \( \mathcal{P}_p^C \)-subharmonic functions are characterized by the fact that their restrictions to complex \( p \)-dimensional submanifolds are \( \Delta \)-subharmonic. The Riesz characteristics of \( \mathcal{P}_p^C \) and \( \mathcal{P}_p^H \) are \( 2p \) and \( 4p \) respectively. See Lemma 4.8 below.

**Example 4.2. (The Elementary Symmetric or Hessian Equations).** For each integer \( k \), \( 1 \leq k \leq n \), let \( \sigma_k(A) \) denote the \( k \)th elementary symmetric function of the eigenvalues of \( A \in \text{Sym}^2(\mathbb{R}^n) \). The convex cone subequation
\[
\Sigma_k = \{ A : \sigma_1(A) \geq 0, \sigma_2(A) \geq 0, \ldots, \sigma_k(A) \geq 0 \}
\] (4.3)
has Riesz characteristic
\[
p_{\Sigma_k} \equiv \frac{n}{k}
\] (4.4)

These subequations, often called hessian equations, have been the focus of much study (e.g., [TW*], [La*]). There are again complex and quaternionic analogues \( \Sigma_k^C \) and \( \Sigma_k^H \) with Riesz characteristics \( 2n/k \) and \( 4n/k \) respectively.

**Example 4.3. (The \( \delta \)-Uniformly Elliptic Equations).** The \( \delta \)-uniformly elliptic regularization of the basic subequation \( \mathcal{P} \equiv \{ A \geq 0 \} \) (cf. Example 4.10) is
\[
\mathcal{P}(\delta) \equiv \{ A : A + \frac{\delta}{n} \text{tr}(A)I \geq 0 \}.
\] (4.5)
These are convex cone subequations with Riesz characteristic \( p = n(1 + \delta)/(n + \delta) \). Given 
\( p \) with \( 1 \leq p \leq n \) and setting
\[
\delta = \frac{n(p - 1)}{n - p}
\] (4.6)

Lemma A.2 states that \( P(\delta) \) is the largest convex cone subequation with Riesz characteristic \( p \). There are again complex and quaternionic analogues described in Example 4.7 below.

**Example 4.4. (Geometrically Defined Subequations).** These important examples account for our choice of normalization in defining the Riesz characteristic. Fix any compact subset \( G \subset G(p, R^n) \) in the Grassmannian of \( p \)-planes in \( R^n \), and define

\[
F(G) \equiv \{ A : \text{tr}_W(A) \geq 0 \text{ for all } W \in G \}
\] (4.7)

where \( \text{tr}_W(A) \) denotes the trace of \( A|_W \). Then the Riesz characteristic is always
\[
p_{F(G)} = p.
\] (4.8)

Many interesting subequations arise this way. When \( G = G(p, R^n), G^C(p, C^n) \) and \( G^H(p, H^n) \) we retrieve the integer cases in 4.1 above. There are many other interesting examples. One such is \( \text{LAG} \subset G^R(n, C^n) \), the set of Lagrangian \( n \)-planes in \( C^n \). Closely related are the sets of isotropic \( p \)-planes, and \( p \)-planes satisfying certain CR (Cauchy-Riemann) conditions. Also of interest is \( \text{SLAG} \subset \text{LAG} \), the special Lagrangian planes (cf. [HL1]). This latter is an example of a subequation associated to a calibration [HL2]. Other particularly interesting examples come from the associative and coassociative calibrations in \( R^7 \) and the Cayley calibration in \( R^8 \). All the specific subequations in this paragraph have the property that they are ST-invariant, i.e., invariant under a subgroup \( G \subset O(n) \) which acts transitively on the sphere \( S^{n-1} \subset R^n \).

These geometrically defined subequations will be the sole focus of Part II of this paper.

**Example 4.5. (Branches of Gårding Operators).** In many of the cases above, one can associate a homogeneous polynomial operator \( \Phi(D^2u) \). When the polynomial \( \Phi \) is Gårding hyperbolic with respect to the identity \( I \) (which is typically the case), the equation has many branches [G], [HL7], [HL8].

The simplest case is \( P = P_1 \) where the operator is \( \Phi(A) = \det_{R}(A) \). Here the branches are given by \( \{ \lambda_k(A) \geq 0 \} \) (see (4.1)). Unfortunately, in this case the branches for \( k > 1 \) have infinite characteristic.

For the general Gårding polynomial \( \Phi(A) \) of degree \( m \), there are ordered eigenvalues,

\[
\Lambda_1(A) \leq \Lambda_2(A) \leq \cdots \leq \Lambda_m(A), \quad \text{and } \Phi(A) = \Lambda_1(A) \cdots \Lambda_m(A).
\] (4.1)’

Just as with \( \det_{R}(A) \), the \( k \)th branch is defined by \( \{ \Lambda_k(A) \geq 0 \} \) for \( k = 1, \ldots, m \). In many examples these branches have finite characteristic; however, only the smallest branch is convex. Gårding operators are plentiful. For instance, in each of our first three examples
there is an associated Gårding operator, and hence each comes equipped with branches. To illustrate, for the case where \( p \) is an integer in Example 4.1, we have

\[
\Phi(A) = \prod_{i_1 < \cdots < i_p} (\lambda_{i_1}(A) + \cdots + \lambda_{i_p}(A)) = \det(D_A : \Lambda^p \mathbb{R}^n \rightarrow \Lambda^p \mathbb{R}^n). \tag{4.9}
\]

Said differently, \( \Lambda_k(A) = \lambda_{i_1}(A) + \cdots + \lambda_{i_p}(A) \) are the eigenvalues. Here \( D_A \) is the extension of \( A \) as a derivation. The \( k \)-th branch is given by requiring that the \( k \)-th ordered \( p \)-fold sum of the \( \lambda_i \)'s be \( \geq 0 \). One easily computes that the first \( \binom{n-1}{p-1} \) branches have Riesz characteristic \( p \) and the remaining branches have infinite characteristic.

In Example 4.2 the Gårding operator is \( \Phi(A) = \sigma_k(A) \). Although the eigenvalues \( \Lambda_k(A) \) of \( \Phi \) do not have an explicit formula in terms of the standard eigenvalues of \( A \), the \( \binom{n}{k} \)-branches are interesting subequations.

In Example 4.3

\[
\Lambda_k(A) = \lambda_k(A) + \frac{\delta}{n} \mathrm{tr}(A) \geq 0, \quad k = 1, \ldots, n
\]
define the branches, and each of the \( k \)-th branches \( \{\Lambda_k(A) \geq 0\} \), for \( k \geq 2 \), has the same Riesz characteristic \( p = n(1 + \frac{1}{\delta}) \), which is finite but larger than \( n \).

**Example 4.6. (Trace Powers of the Hessian).** Consider the non-convex cone subequation

\[
F \equiv \{ A : \mathrm{tr}(A^q) \geq 0 \}
\]

where \( q > 1 \) is an odd integer. More generally one could define \( A^q \) for any \( q > 0 \) by using the function \( t^q \) for \( t \geq 0 \) and \( -|t|^q \) for \( t < 0 \). In all cases one computes that the Riesz characteristic is

\[
p_F = 1 + (n-1)\frac{1}{q}
\]

More generally, for \( k \in [1, n] \) and \( q > 0 \) real numbers, there is the subequation

\[
F \equiv \{ A : \lambda_1^q(A) + \cdots + \lambda_k^q(A) + (k-[k])\lambda_{[k]+1}^q \geq 0 \}
\]

with \( t^q \) defined as above. Here the Riesz characteristic is

\[
p_F = 1 + (k-1)\frac{1}{q}
\]

**Example 4.7. (Complex and Quaternionic Analogues).** Suppose \( F \subset \mathrm{Sym}^2(\mathbb{R}^n) \) is an \( O(n) \)-invariant subequation. Then \( F \) can be defined by the constraint set \( E \subset \mathbb{R}^n \) imposed by \( F \) on the eigenvalues \( \lambda(A) = (\lambda_1(A), \ldots, \lambda_n(A)) \). Thus \( A \in F \iff \lambda(A) \in E \). The equation \( F \) has complex and quaternionic analogues \( F^C \) and \( F^H \), defined on \( \mathbb{C}^n = (\mathbb{R}^{2n}, J) \) and \( \mathbb{H}^n = (\mathbb{R}^{4n}, I, J, K) \) respectively, as follows. For \( A \in \mathrm{Sym}^2(\mathbb{R}^{2n}) \) consider the hermitian symmetric part

\[
A_C \equiv \frac{1}{2}(A - JAJ)
\]
whose eigenspaces are complex lines with ordered eigenvalues \( \lambda_1(A_C) \leq \cdots \leq \lambda_n(A_C) \). One now defines \( F^C \) by applying the eigenvalue constraints \( E \) of \( F \) to these eigenvalues of \( A_C \). The story in the quaternionic case is parallel and uses the quaternionic hermitian symmetric part \( A_H \equiv \frac{1}{4}(A - IAI - AJ - KAK) \) and eigenvalues \( \lambda_k(A_H) \).

**Lemma 4.8.** If \( F \) is an \( O(n) \)-invariant cone subequation with Riesz characteristic \( p \), then the Riesz characteristics of \( F^C \) and \( F^H \) are

\[
p_{F^C} = 2p \quad \text{and} \quad p_{F^H} = 4p.
\]

**Proof.** We consider the complex case. If \( A = P_{e^*} - (p - 1)P_e \in \text{Sym}^2(\mathbb{R}^{2n}) \), then one computes that

\[
A_C = P_{C_e^*} - \left( \frac{p}{2} - 1 \right) P_{C_e} \quad \text{and} \quad A_H = P_{H_e^*} - \left( \frac{p}{4} - 1 \right) P_{H_e}
\]

which displays the eigenvalues of \( A_C \) and \( A_H \). \hfill \Box

**Example 4.9.** (The Subequation Determined by a Gårding Operator and a Universal Eigenvalue Constraint). The procedures above can be greatly generalized. Note, to begin, that given an \( O(m) \)-invariant subequation \( F \), the eigenvalue set \( E \equiv \lambda(F) \) is closed, invariant under permutation of coordinates and \( \mathbb{R}^m_+ \)-monotone. Conversely, any such eigenvalue set \( E \) determines an \( O(m) \)-invariant subequation \( F = \lambda^{-1}(E) \). Each such \( E \) is a universal eigenvalue subequation in the sense that, for each degree-\( m \) Gårding operator \( \Phi \) on \( \text{Sym}^2(\mathbb{R}^n) \), the set \( F \equiv \lambda_{\Phi}^{-1}(E) \) is a subequation on \( \mathbb{R}^n \), where \( \lambda_{\Phi} : \text{Sym}^2(\mathbb{R}^n) \to \mathbb{R}^m \) is the eigenvalue map associated to \( \Phi \). See Appendix A for the details and further discussion.

**Example 4.10.** (The \( \delta \)-Uniformly Elliptic Regularization of a Subequation). Given a cone subequation \( F \subset \text{Sym}^2(\mathbb{R}^n) \) and \( \delta > 0 \), define

\[
F(\delta) \equiv \{ A : A + \frac{\delta}{n} \text{tr}(A)I \in F \}.
\]

This equation satisfies the uniformly elliptic condition:

\[
F(\delta) + \mathcal{P}(\delta) \subset F(\delta).
\]

One computes that

\[
F \text{ has Riesz characteristic } p \quad \iff \quad F(\delta) \text{ has Riesz characteristic } \frac{pm(1+\delta)}{n+\delta p}.
\]
5. Notions of Density for F-Subharmonic Functions.

In this section we discuss and relate various notions of density for an $F$-subharmonic function $u$ at a point $x_0$. We assume as always that the subequation $F$ is an ST-invariant cone with invariance group $G \subset O(n)$. We further assume that the Riesz characteristic $p$ of $F$ is finite. This is because when $p = \infty$, the increasing radial subequation $R_F^+$ is simply $g'(r) \geq 0$ (Proposition 3.3 and Definition 3.1). Thus, when $p = \infty$, all increasing functions $g(t)$ determine increasing radial subharmonics $g(|x|)$, and no sensible notion of density is possible.

To begin we set some notation. Let $B_r(x_0) = \{ x : |x - x_0| \leq r \}$ denote the ball of radius $r$ about $x_0$, and set $S_r(x_0) = \partial B_r(x_0)$. Here and elsewhere, when $x_0 = 0$, reference to it will be dropped from the notation. Thus, $B_r = B_r(0)$ and $S_r = \partial B_r$. Similarly we set $B = B_1$ and $S = \partial B$.

Each notion of density depends on an averaging process near $x_0$. We start with the extreme case of the density based on the maximum of $u$. Since $F$ is an ST-invariant cone (not necessarily convex), the maximum principle holds for $u$, and we define

$$M(u, x_0; r) \equiv \sup_{B} u(x_0 + rx) = \sup_{S} u(x_0 + rx), \quad (5.1)$$

which is increasing in $r$. By the ST-invariance of $F$

$$M(u, x_0; |x|) \equiv \sup_{g \in G} u(x_0 + gx). \quad (5.2)$$

We now simplify by setting $x_0 = 0$ and using the abbreviated notation $M(r) \equiv M(u; r) = M(u, 0; r)$ when the meaning is obvious.

**Lemma 5.1.** If $u$ is $F$-subharmonic on $B_R$, then $M(|x|)$ is an increasing radial $F$-subharmonic function on $B_R$, and $M(0) = u(0)$.

**Proof.** Let $u_g(x) \equiv u(gx)$ with $g \in G$. Then $M(|x|) = \sup_{g \in G} u_g(x)$. Since $F$ is $G$-invariant, each $u_g$ is $F$-subharmonic. Therefore, by the standard "families locally bounded above" property for $F$, it suffices to prove that

$$M(t)$$

is upper semi-continuous.

This is done as follows. For each $\delta > 0$, $N_\delta \equiv \{ x : u(x) < M(t) + \delta \}$ is an open set containing $\partial B_t$. Hence the annulus $B_{t+\epsilon} - B_{t-\epsilon}$ is contained in $N_\delta$ for $\epsilon > 0$ small. Thus $M(r) < M(t) + \delta$ if $t - \epsilon \leq r \leq t + \epsilon$, proving that $M(t)$ is upper semi-continuous, and hence $M(|x|)$ is $F$-subharmonic. \qed

For the other averages we make the further standing assumption that $F$ is convex. In this case we note the following.

$$F \text{ is an ST - invariant convex cone } \Rightarrow F \subset \Delta \equiv \{ \text{tr} A = 0 \} \quad (5.3)$$

**Proof.** If $F \cap \{ \text{tr} A = c < 0 \}$ is non-empty, then invariance plus convexity implies that $-\frac{c}{n} I \in F$. Now by the cone property, $-\lambda I \in F$ for all $\lambda > 0$. This along with positivity
implies that \( F = \text{Sym}^2(\mathbb{R}^n) \). Since \( \text{tr}(P_{e^p} - (p-1)P_e) = n-p \), the condition \( F \subset \{ \text{tr} A \geq 0 \} \) implies \( p_F \leq n \). Therefore,

\[
F \text{ is an ST } - \text{invariant convex cone} \quad \Rightarrow \quad 1 \leq p_F \leq n. \tag{5.4}
\]

We now define the spherical and volume averages of \( u \) at \( x_0 \) by

\[
S(u, x_0; r) \equiv \frac{1}{|S|} \int_{\sigma \in S} u(x_0 + r\sigma) d\sigma \equiv \int_{S} u(x_0 + r\sigma) d\sigma, \tag{5.5a}
\]

\[
V(u, x_0; r) \equiv \frac{1}{|B|} \int_{x \in B} u(x_0 +rx) dx \equiv \int_{B} u(x_0 +rx) dx. \tag{5.5b}
\]

Note that for any upper semi-continuous function \( u \), each of these functions is jointly upper semi-continuous in \( (x_0, r) \) since \( u(x_0 + rx) \) is the infimum of \( \varphi(x_0 + rx) \) taken over continuous functions \( \varphi \geq u \).

**Lemma 5.2.** Suppose that \( u \) is \( F \)-subharmonic on the ball \( B_R \) of radius \( R \) about the origin. Then both \( S(u; |x|) \) and \( V(u; |x|) \) are increasing radial \( F \)-subharmonic functions on \( B_R \) (with limiting values \( S(u, 0) = V(u, 0) = u(0) \) at \( x = 0 \)).

**Proof.** As noted above \( S(u; r) \) and \( V(u; r) \) are upper semi-continuous in \( r \), and hence so are the functions \( S(u; |x|) \) and \( V(u; |x|) \) of \( x \) defined on \( B_R \). The statement about their limiting values at \( x = 0 \) is a standard fact about \( \Delta \)-subharmonic functions. It remains to show that \( S(u; |x|) \) and \( V(u; |x|) \) are \( F \)-subharmonic on \( B_R \). Note that

\[
S(|x|) = \int_{G} u(gx) dg \tag{5.6}
\]

for a suitably normalized invariant measure \( dg \) on \( G \), and that

\[
V(|x|) = n \int_{0}^{1} S(\rho|x|)\rho^{n-1} d\rho \quad \text{since} \quad |B| = \frac{1}{n}|S|. \tag{5.7}
\]

To prove (5.7), set \( |x| = r \) and compute \( V(r) = \frac{1}{|B|} \int_{B} u(ry) dy \) using polar coordinates. Now since \( F \) is a convex cone subequation, averages such as (5.6) and (5.7) preserve \( F \)-subharmonicity. This is explained further in the next section by Theorem 7.4 and the Remark that follows it.

Since \( S(|x|) \) and \( V(|x|) \) are \( F \)-subharmonic and the maximum principle holds for the cone subequation \( F \), both \( S(r) \) and \( V(r) \) are increasing.

Note that by Theorem 2.7 the lemmas above could have been restated by concluding that the functions \( M(r), S(r) \) and \( V(r) \) are \( R^1_F \)-subharmonic on \( (0, R) \).

It is an important fact that each of these averages is stable under limits in \( L^1 \). This basic classical fact can be found in [Ho$_2$, Sec III.3.2]. We state it here in slightly different form needed later for tangents.

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Lemma 5.3. (Stability of Averages). Suppose \( u_j \) is a sequence of \( \Delta \)-subharmonic functions on \( B_R \) converging in \( L^1(B_R) \) to a \( \Delta \)-subharmonic function \( U \). Then for \( 0 < r < R \),

\[
\begin{align*}
(1) & \quad M(U,r) = \lim_{j \to \infty} M(u_j,r), \\
(2) & \quad S(U,r) = \lim_{j \to \infty} S(u_j,r), \\
(3) & \quad V(U,r) = \lim_{j \to \infty} V(u_j,r),
\end{align*}
\]

Proof. Taking \( K \equiv B_r \) in (3.2.7) of Theorem 3.2.1 in [Ho2] gives us that

\[
\limsup_{j \to \infty} M(u_j,r) \leq M(u,r).
\]

Suppose there exists \( C < M(u,r) \) such that \( M(u_j,r) \leq C \) for all \( j \) sufficiently large. Then in the \( L^1 \)-limit we would have \( u - C \leq 0 \) a.e. on \( B_r \). However, for \( \Delta \)-subharmonic functions, this implies that \( u - C \leq 0 \) everywhere on \( B_r \), contrary to the definition of \( M(u,r) \). We conclude that \( \limsup_{j \to \infty} M(u_j,r) = M(u,r) \). The fact that this is also true for all subsequences proves (1).

As discussed in the paragraph prior to Proposition 3.2.14 in [Ho2], the Theorem 3.2.13 can be applied to spherical measure \( \sigma_r \) on \( \partial B_r \). Thus \( u_j \sigma_r \) converges to \( U \sigma_r \) in the weak topology of measures, yielding (2). Finally, (3) is implied directly by the hypothesis of \( L^1(B_r) \)-convergence.

Quotient Monotonicity for Radial Subharmonics

As stated in Proposition 3.3, for an ST-invariant cone subequation the associated increasing radial subequation \( R^\uparrow_p = R^\uparrow_p \) is uniquely determined by the Riesz characteristic \( p = p_F \) of \( F \). Restricting attention to the finite case \( p < \infty \), it is given by:

\[
R^\uparrow_p : \quad \psi''(t) + \frac{p-1}{t} \psi'(t) \geq 0 \quad \text{and} \quad \psi'(t) \geq 0. \tag{5.8}
\]

The following classical fact underlies many of the results of this paper. For completeness we present two proofs. The first uses comparison. The second uses a change of variables and is described in Remark 5.8. We always assume that \( \psi(t) \) is not identically \(-\infty\).

THEOREM 5.4. (The Fundamental Monotonicity Property). Fix \( 1 \leq p < \infty \). Suppose \( \psi(t) \) is \( R^\uparrow_p \)-subharmonic on the interval \((0,R)\). Then, for \( 0 < r < t < R \), the non-negative quantity

\[
\frac{\psi(t) - \psi(r)}{K(t) - K(r)}
\]

is increasing in \( r \) and \( t \),

where \( K = K_p \) is the \( p \)-th Riesz function (3.6).

There are many consequences of this theorem. The following elementary ones are proved below.
Corollary 5.5. Let \( \psi \) be as in Theorem 5.4. Then:

(a) \( \psi \) is finite-valued on \((0, R)\).

(b) If \( 1 \leq p < 2 \), then \( \psi(0) \equiv \lim_{r \downarrow 0} \psi(r) \) is finite. (Note that \( K(0) = 0 \) is finite.)

(c) The function \( \psi \) is locally Lipschitz.

(d) The left and right hand derivatives \( \psi'_\pm(t) \) exist on \((0, R)\).

First Proof of Theorem 5.4. The function

\[
 h(t) \equiv \psi(R) + \frac{K(R) - K(t)}{K(R) - K(r)}(\psi(r) - \psi(R))
\]

is an (increasing radial) harmonic for \( R \uparrow \) by Proposition 3.5 since \( \Theta = \frac{\psi(R) - \psi(r)}{K(R) - K(r)} \geq 0 \). It has boundary values \( \psi(r) \) at \( t = r \) and \( \psi(R) \) at \( t = R \). Hence, by comparison,

\[
 \psi(t) \leq h(t) \quad \text{on} \quad r \leq t \leq R. \tag{5.9}
\]

This can be written in two ways which, when combined, prove the increasing statements in the Theorem.

\[
 \frac{\psi(R) - \psi(r)}{K(R) - K(r)} \leq \frac{\psi(R) - \psi(t)}{K(R) - K(t)}, \quad 0 < r < t < R \tag{5.10a}
\]

or

\[
 \frac{\psi(t) - \psi(r)}{K(t) - K(r)} \leq \frac{\psi(R) - \psi(r)}{K(R) - K(r)}, \quad 0 < r < t < R. \tag{5.10b}
\]

\[
 \square
\]

Densities.

Proposition 5.6. (Existence of Densities). The decreasing limit

\[
 \Theta^\psi = \lim_{r, t \to 0, t > r > 0} \frac{\psi(t) - \psi(r)}{K(t) - K(r)} \tag{5.11}
\]

exists and defines the \( \psi \)-density \( 0 \leq \Theta^\psi < \infty \).

In particular, the three densities of an \( F \)-subharmonic function \( u \), denoted by \( \Theta^M(u, x) \), \( \Theta^S(u, x) \) and \( \Theta^V(u, x) \), are defined as the limit in (5.11) by taking \( \psi(t) = M(u, x; t), S(u, x; t) \) and \( V(u, x; t) \) respectively.

The following auxiliary expressions for the densities are useful. Since \( \psi \) is increasing on \((0, R)\), we can define \( \psi(0) = \lim_{r \downarrow 0} \psi(t) \).

Proposition 5.7. Suppose that \( \Psi(t) \) is any \( R^\uparrow_p \)-subharmonic function on \((0, R)\). Then if \( 1 \leq p < 2 \),

\[
 \Theta^\Psi = \lim_{r \downarrow 0} \frac{\Psi(r) - \Psi(0)}{K(r)} \quad \text{where} \quad K(r) = r^{2-p}. \tag{5.12}
\]
If \(2 \leq p < \infty\), then

\[
\Theta^\Psi = \lim_{r \downarrow 0} \frac{\Psi(r)}{K(r)} \quad \text{where} \quad K(r) = \begin{cases} 
\frac{1}{r^{p-1}} & \text{if } p > 2 \\
\log r & \text{if } p = 2
\end{cases}
\] (5.13)

Moreover, in all cases \(1 \leq p < \infty\), the functions \(\Psi'_\pm(r)/K'(r)\) are increasing in \(r\) and

\[
\Theta^\Psi = \lim_{r \downarrow 0} \frac{\Psi'_\pm(r)}{K'(r)}.
\] (5.14)

**Proof.** If \(1 \leq p < 2\), then \(K_p(0) = 0\). Hence, taking the limit in (5.11) as \(r \downarrow 0\) we get

\[
\frac{\Psi(t) - \Psi(0)}{K(t) - K(0)} = \frac{\Psi(r) - \Psi(0)}{K(r) - K(0)} \quad (5.15)
\]

we see that both \(1 - \frac{\Psi(t)}{\Psi(r)}\) and \(1 - \frac{K(t)}{K(r)}\) converge to 1 as \(r \downarrow 0\). Hence,

\[
\lim_{r \downarrow 0} \frac{\Psi(r)}{K(r)} = \lim_{r \downarrow 0} \frac{\Psi(t) - \Psi(0)}{K(t) - K(0)} = \Theta^\Psi
\] (5.16)

independent of \(t\).

Finally, since

\[
\frac{\Psi(r + \delta) - \Psi(r)}{K(r + \delta) - K(r)} \quad \text{is increasing in } \delta
\]

and \(K\) is differentiable, the right and left hand derivatives \(\Psi'_\pm(r)\) exist. Dividing the numerator and denominator by \(\delta\) and applying the monotonicity in Theorem 5.4 completes the proof.

**Proof of Corollary 5.5.** We first prove (a). Since \(\Psi\) is increasing and not \(\equiv -\infty\), it must be finite-valued for \(t \geq \) some \(t_0\). Fix \(t_1 > t_0\) and consider the quotient \(\sigma(t) = (\Psi(t_1) - \Psi(t))/(K(t_1) - K(t))\), which is increasing in \(t\) for \(t < t_1\) and bounded above by \(\sigma(t_0)\) for \(t \leq t_0\). It follows immediately that \(\Psi(t)\) is locally bounded below on \((0, t_0]\). Parts (b) and (c) are equally straightforward. Part (d) was noted at the end of the proof of Proposition 5.7.

**Remark 5.8.** (Second Proof of Theorem 5.4). A second, perhaps easier, proof of Theorem 5.3 uses the change of variables \(s \equiv K_p(t)\). Set \(f(s) \equiv \Psi(t(s))\), or \(\Psi(t) = f(s(t))\). Note that \(\Psi'(t) = f'(s) \frac{ds}{dt}\) and that

\[
\Psi''(t) + \frac{p - 1}{t} \Psi'(t) = f''(s) \left(\frac{ds}{dt}\right)^2 + f'(s) \left(\frac{d^2 s}{dt^2} + \frac{p - 1}{t} \frac{ds}{dt}\right)
\]

\[
= f''(s) \left(\frac{ds}{dt}\right)^2
\]

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since \( s(t) \equiv K_p(t) \) is \( R^p \)-harmonic. It is now straightforward to check that \( \Psi(t) \) is \( R^p \)-subharmonic in the viscosity sense if and only if \( f(s) \) satisfies \( f'(s) \geq 0 \) and \( f''(s) \geq 0 \) in the viscosity sense. Since viscosity convexity is equivalent to classical convexity (cf. [HL4, Prop 2.6]), we have the following.

**Lemma 5.9.**

\( \Psi(t) \equiv f(K(t)) \) is \( R^p \)-subharmonic \( \iff \) \( f(s) \equiv f(K^{-1}(s)) \) is convex and increasing.

Now observe that if \( y = K(r) \) and \( s = K(t) \), then

\[
\frac{\Psi(r) - \Psi(t)}{K(r) - K(t)} = \frac{f(y) - f(s)}{y - s}.
\]

The remainder of the proof is left to the reader. Functions \( \Psi(t) \) of the form \( f(K(t)) \) with \( f \) convex, are called \( K \)-convex (log-convex when \( p = 2 \)).

**Remark 5.10.** (The Mass Density). In addition to the three densities \( \Theta^M(u,x) \), \( \Theta^S(u,x) \), and \( \Theta^V(u,x) \) guaranteed by Proposition 5.6, there is a fourth, even more classical, mass density. By (5.3) \( u \) is classically \( \Delta \)-subharmonic. Thus \( \Delta u \) is a measure \( \mu \geq 0 \), which means \( \Delta u \) has a “mass density”. Given a measure \( \mu \geq 0 \) defined in a neighborhood of a point \( x_0 \in \mathbb{R}^n \), and \( 0 < k \leq n \), the limit

\[
\Theta^k(\mu,x_0) \equiv \lim_{r \downarrow 0} \frac{\mu(B_r(x_0))}{\alpha(k)r^k},
\]

if it exists, is called the \( k \)-dimensional mass density of \( \mu \) at \( x_0 \). (See, for example, [F, 2.10.19] for discussion and definition of the constants \( \alpha(k) \).) When \( k \) is an integer, \( \alpha(k) = |B^k| \), the volume of the unit ball in \( \mathbb{R}^k \). Suppose \( \Theta^k(\mu,x) \) exists everywhere or replace \( \lim \) by \( \lim \sup \) in (5.17). Fix an open set \( X \), a constant \( c > 0 \), and define \( E_c \equiv \{ x \in X : \Theta^k(\mu,x) \geq c \} \). Then the Hausdorff \( k \)-measure satisfies (cf. [Si, page 11])

\[
c \mathcal{H}^k(E_c) \leq \mu(X).
\]

**Comparing Densities**

The next proposition states that: All densities but \( \Theta^M \) “agree”, where “agree” means “are equal up to universal factors”.

**Proposition 5.11.** Suppose that \( u \) is \( F \)-subharmonic near \( x_0 \) where \( F \) is convex with characteristic \( p \), and set \( \mu = \Delta u \). Then when \( p \neq 2 \),

\[
\Theta^S(u,x_0) = \frac{n - p + 2}{n} \Theta^V(u,x_0) = \frac{\alpha(n - p)}{n|p - 2||\alpha(n)|} \Theta^{n-p}(\mu,x_0),
\]

and when \( p = 2 \) we have that

\[
\Theta^S(u,x_0) = \Theta^V(u,x_0) = \frac{\alpha(n - 2)}{n\alpha(n)} \Theta^{n-2}(\mu,x_0).
\]

The discussion of all densities is completed by showing that the maximum density and the spherical density are in general “comparable”, and in fact equal when \( p = 2 \).
Proposition 5.12. Suppose that \(u\) is \(F\)-subharmonic near \(x_0\) where \(F\) is convex and of characteristic \(p\). Then there exists a constant \(C = C(p,n) > 1\) such that

\[
\Theta^M(u,x_0) \leq \Theta^S(u,x_0) \leq C \Theta^M(u,x_0) \quad \text{if } 2 < p < \infty, \quad \text{and} \quad (5.20)
\]

\[
\Theta^S(u,x_0) \leq \Theta^M(u,x_0) \leq C \Theta^S(u,x_0) \quad \text{if } 1 < p < 2, \quad \text{while} \quad (5.21)
\]

\[
\Theta^M(u,x_0) = \Theta^S(u,x_0) \quad \text{if } p = 2. \quad (5.22)
\]

Remark 5.13. Kiselman proved the first equality in (5.21) in the plurisubharmonic case where \(F = \mathcal{P}^C\) on \(\mathbb{C}^n\) (see page 161, line 2 ff. in \([K_1]\)) by using Harnak’s Inequality for \(\Delta\)-subharmonic functions. The same proof works for any convex \(F\) of characteristic \(p = 2\).

Note that for \(p = 1\) the left inequality in (5.21) holds but the right inequality fails, even for linear functions.

Proof of Proposition 5.11. We give the proof of the first equality for all \(p\) using (5.13). Set \(x_0 = 0\) and drop \(u\) and \(x_0\) from the notation. Recall (5.7) that

\[
V(r) = n \int_0^1 S(rt)t^{n-1} dt
\]

(5.23)

Hence, we have

\[
\frac{V(r)}{K(r)} = n \int_0^1 \frac{S(rt)K(rt)}{K(r)} t^{n-1} dt.
\]

When \(p \neq 2\), \(K(rt)/K(r) = 1/t^{p-2}\), so that letting \(r \downarrow 0\) and integrating yields the first equality. When \(p = 2\),

\[
\frac{K(rt)}{K(r)} = 1 + \frac{\log t}{\log r};
\]

so letting \(r \downarrow 0\) and integrating yields \(\Theta^V(u) = \Theta^S(u)\).

For the proof of the second inequality we show that the mass density \(\Theta^{n-p}(\mu)\) (\(\mu = \Delta u\)) is the desired multiple of the spherical density \(\Theta^S(u)\). Recall the classical fact that

\[
\mu(B_r) = (n - 2)|S| \frac{S'(r)}{K'_n(r)}. \quad (5.24)
\]

(See (3.2.13)' in \([H_2\), Thm. 3.2.16] for a proof.) Since

\[
\frac{n - 2}{K'_n(r)} = r^{n-1} = \frac{|p - 2| r^{n-p}}{K'_n(r)} \quad \text{when } p \neq 2,
\]

we have

\[
r^{p-n} \mu(B_r) = |p - 2||S| \frac{S'(r)}{K'_n(r)} \quad \text{when } p \neq 2. \quad (5.24)'
\]

If \(p = 2\), this holds with \(|p - 2|\) replaced by 1. Finally, letting \(r \downarrow 0\) and using (5.18) completes the proof.
Proof of Proposition 5.12 For simplicity let $x_0 = 0$. Note that for all $p$ and $r$ we have $S(u, r) \leq M(u, r)$. On the other hand, $K(r) < 0$ when $p \geq 2$ and $K(r) > 0$ when $p < 2$. Dividing by $K(r)$ and letting $r \downarrow 0$ then gives the inequalities on the left as well as the inequality $\Theta^M(u) \leq \Theta^S(u)$ when $p = 2$ (since $u$ and $u + c$ have the same density, we can assume that $u(0) = 0$ when $p < 2$.)

The remainder of the proof is a consequence of Harnak’s inequality. The standard form of this inequality is for a function $v \leq 0$ which is $\Delta$-subharmonic on $B_\rho$. It says, with $\varphi$ defined by

$$
\varphi(\lambda) \equiv \frac{1 - \lambda}{(1 + \lambda)^{n-1}} \quad \text{for } 0 < \lambda < 1,
$$

that

$$
M(v, \lambda r) \leq \varphi(\lambda) S(v, r) \quad \text{for all } 0 < r \leq \rho. \tag{5.25}
$$

(See, for example, Prop. 4.2.2 in [D].) For an arbitrary $\Delta$-subharmonic function $v$, the function $v - M(v, r)$ is $\leq 0$ on $B_r$. Hence, (5.25) gives the following more general form of Harnak’s inequality

$$
M(v, \lambda r) - M(v, r) \leq \varphi(\lambda)(S(v, r) - M(v, r)) \quad \text{for all } 0 < r \leq \rho. \tag{5.26}
$$

for functions not necessarily $\leq 0$.

Suppose first that $p > 2$. We may assume $u(0) = -\infty$ since otherwise the assertion is trivial. Then $u$ is negative near 0, and we can apply the standard form (5.25) of Harnak’s inequality to obtain

$$
\frac{M(u, \lambda r)}{K(\lambda r)} \geq \lambda^{p-2} \varphi(\lambda) \frac{S(u, r)}{K(r)}.
$$

Letting $r \downarrow 0$ gives $\Theta^M(u, 0) \geq c \Theta^S(u, 0)$ where $c = \lambda^{p-2} \varphi(\lambda) > 0$. This gives (5.20) with $C = 1/c$. (Note that $c \equiv \sup_\lambda \lambda^{p-2} \varphi(\lambda)$ provides the best constant $C$.)

Suppose now that $1 < p < 2$. Replace $u$ by $u(x) - u(0)$ so that $u(0) = 0$. Since densities are unchanged by adding a constant, we have $\Theta^M(u, 0) = \lim_{r \downarrow 0} M(u, r)/K(r)$ and $\Theta^S(u, 0) = \lim_{r \downarrow 0} S(u, r)/K(r)$ by (5.12). Since $u$ may not be $\leq 0$, we must use the general form (5.26) of Harnak. Dividing by $K(r)$ gives

$$
\frac{(1 + \lambda)^{n-1}}{1 - \lambda} \left( \frac{M(u, \lambda r)}{K(r)} - \frac{M(u, r)}{K(r)} \right) \leq \frac{S(u, r)}{K(r)} - \frac{M(u, r)}{K(r)}. \tag{5.27}
$$

Using the fact that $K(\lambda r) = \lambda^{2-p} K(r)$ and letting $r \downarrow 0$ gives

$$
\psi(\lambda) \Theta^M(u, 0) \leq \Theta^S(u, 0) \quad \text{with } \psi(\lambda) = 1 + \frac{(1 + \lambda)^{n-1}}{1 - \lambda} (\lambda^{2-p} - 1).
$$

Now direct calculation shows that $\lim_{\lambda \downarrow 0} \psi'(\lambda) = \infty$, and so $c \equiv \sup_{0 < \lambda < 1} \psi(\lambda) > 0$. This gives the desired result with $C = 1/c$.

It remains to prove that $\Theta^S(u) \leq \Theta^M(u)$ when $p = 2$. Set $\lambda = 1/e$ in (5.27) and note the fact that $K(r) = \log r = \log \frac{r}{e} + 1 = K(\lambda r) + 1 = K(\lambda r)(1 + o(r))$. Then taking the
limit as $r \to 0$ in (5.27) yields $0 = \Theta^M(u) - \Theta^M(u) \leq \Theta^S(u) - \Theta^M(u)$ by (5.13). This completes the proof of Proposition 5.12.

The Upper Semi-Continuity of Density.

THEOREM 5.14. Each of the densities $\Theta^M(u, x)$, $\Theta^S(u, x)$, and $\Theta^V(u, x)$ considered above is an upper semi-continuous function of $x$.

Proof. Because of Proposition 5.11 there are only two cases to consider. We must show that

$$
\limsup_{x \to x_0} \Theta(u, x) \leq \Theta(u, x_0).
$$

(5.28)

Set $x_0 = 0$. Assume $0 < |x| < r < t$. Then

$$
\Theta^\Psi(u, x) \leq \frac{\Psi(u, x, t) - \Psi(u, x, r)}{K(t) - K(r)}.
$$

(5.29)

Case 1. $\Psi = M$. By using the facts that $B_t(x) \subset B_{t+|x|}(0)$ and $B_{r-|x|}(0) \subset B_r(x)$, we see that the last quantity above is

$$
\leq \frac{\sup_{B_{t+|x|}(0)} u - \sup_{B_{r-|x|}(0)} u}{K(t) - K(r)}.
$$

The function $M(u, 0, r) \equiv \sup_{B_r(u)} u$ is continuous (see the end of the proof of Proposition 5.7 or Remark 5.8) and increasing. Therefore,

$$
\limsup_{x \to 0} \Theta^M(u, x) \leq \frac{\sup_{B_r(u)} u - \sup_{B_r(u)} u}{K(t) - K(r)} 0 < r < t.
$$

Finally, the limit of the RHS as $r, t \to 0$ equals $\Theta^M(u, 0)$. This proves the first case.

Case 2. $\Psi = V$. It suffices to note that $\lim_{x \to 0} V(u, x, t) = V(u, 0, t)$, which follows since $V(u, x, t) = \int_B u(x + ty) dy$ and $u$ converges in $L^1(B)$ to $u(ty)$ as $x \to 0$. (Similarly, by using Theorem 3.2.13 in [Ho2], we have that $u(x + t\sigma) d\sigma$ converges weakly in measure to $u(t\sigma) d\sigma$ as $x \to 0$, directly proving that $S(u, x, t)$ is continuous in $x$ at $x = 0$ without using Proposition 5.11).

Corollary 5.15. For all $c > 0$, the set

$$
E_c \equiv \{ x : \Theta(u, x) \geq c \}
$$

is closed.

Note 5.16. When $p = 1$ the set where $\Theta(u) = 0$ is just the set of differentiability points of $u$ (see (5.5) in Part II).
6. Maximality of Subharmonics with Harmonic Averages.

In this section we extend the standard notion of maximality in pluripotential theory to \( F \)-potential theory. (An excellent reference for pluripotential theory is [Kl].) Being \( F \)-maximal is close to being \( F \)-harmonic. In fact, a maximal function is harmonic if and only if it is continuous. Our main result, key for the study of tangents, is that an \( F \)-subharmonic whose averages are \( F \)-harmonic, is maximal.

**Definition 6.1.** An \( F \)-subharmonic function \( u \) on an open set \( X \subset \mathbb{R}^n \) is said to be \( F \)-maximal on \( X \) if for each \( F \)-subharmonic function \( v \) on \( X \) and each compact subset \( K \subset X \),

\[
v \leq u \text{ on } X - K \implies v \leq u \text{ on } X
\]  

(6.1)

Note that by replacing \( v \) with \( \max\{u, v\} \), condition (6.1) is equivalent to

\[
v \geq u \text{ on } X \text{ and } v = u \text{ on } X - K \implies v = u \text{ on } X
\]  

(6.1)’

**THEOREM 6.2.** Suppose that \( F \) is an \( ST \)-invariant convex cone subequation, and \( U \) is an \( F \)-subharmonic function on the annulus \( A = \{ x : a < |x| < b \} \). If the spherical average

\[
S(U, t) \equiv \int_S U(t\sigma) \, d\sigma \text{ is increasing } F \text{ harmonic on } (a, b),
\]  

(6.2)

then the function

\[
U \text{ is } F \text{ maximal on } A,
\]  

(6.3)

**Proof.** The hypothesis on \( U \) can be restated as the condition

\[
S(U, t) \text{ is } R^F \text{ harmonic on } (a, b),
\]  

(6.2)’

or equivalently by Proposition 3.5

\[
S(U, t) = \Theta K(t) + c \text{ on } (a, b)
\]  

(6.2)’’

for constants \( \Theta \geq 0 \) and \( c \in \mathbb{R} \). Now this is equivalent to

\[
\frac{S(U, t) - S(U, r)}{K(t) - K(r)} = \Theta \geq 0 \text{ for all } r < t \text{ in } (a, b)
\]  

(6.2)’’’

for some constant \( \Theta \geq 0 \).

As in (6.1)’ assume that \( v \) is \( F \)-subharmonic on \( A \) with \( v \geq U \) and that outside a compact subset \( K \subset A \) we have \( v = U \). By the Fundamental Monotonicity Theorem 5.4 we have that for \( a < r < t < b \),

\[
\frac{S(v, t) - S(v, r)}{K(t) - K(r)} \text{ is increasing in } r \text{ and } t.
\]  

(6.3)

Since \( v = U \) outside \( K \), this quotient equals \( \Theta \) if both \( r \) and \( t \) are sufficiently close to \( a \) or sufficiently close to \( b \). Hence, this quotient equals \( \Theta \) for all \( r < t \) in \( (a, b) \). That is, \( S(v, t) \) satisfies (6.2)’’’). It follows that \( S(v, t) \), in addition to \( S(U, t) \), satisfies (6.2)’’). Therefore,

\[
S(v, t) = S(U, t) + c \quad \forall t \in (a, b)
\]  

(6.4)

Taking \( t \) close to \( a \) shows that \( c = 0 \). Now the fact that \( S(v, t) = S(U, t) \) for all \( t \in (a, b) \) combined with the inequality \( U \leq v \) implies that \( U = v \) on \( A \), thus proving that \( U \) is \( F \)-maximal on \( A \).
The following additional facts about $F$-maximal functions are established as in the basic case (pluripotential theory) where $F = \mathcal{P}^C$. For the convenience of the reader we indicate the proofs. Throughout the remainder of this section $F$ is an arbitrary subequation, i.e., a closed set $F \subset \text{Sym}^2(\mathbb{R}^n)$ which satisfies $F + \mathcal{P} \subset F$.

**Fact 1.**

If $u$ is $F$ harmonic on $X$, then $u$ is $F$ maximal on $X$ \hfill (6.5)

This is immediate since comparison holds for $F$ (cf. [HL13, Thm. 7.1]). The only thing standing in the way of the converse is the continuity of $u$.

**Example 6.3.** The subequation $F = \mathcal{P}^C$ of pluripotential theory has many functions, such as $\log |z|_2$ on $\mathbb{C}^2$, which are maximal but not $F$-harmonic. In fact any function $u(z_1)$, which is $\Delta$-subharmonic on a domain $X_0 \subset \mathbb{C}$, when considered as a function $w(z) \equiv u(z_1)$ on $X = X_0 \times \mathbb{C}^{n-1}$ with $n \geq 2$, is $\mathcal{P}^C$-maximal. (If $v(z) \leq u(z_1)$ on $X - K$, then by applying the maximum principle to $v$ on slices $z_1 = \text{constant}$, we get $v(z) \leq \overline{w}(z)$ on $X$.) Now $\overline{w}(z) \equiv u(z_1)$ is $\mathcal{P}^C$-harmonic if and only if $u$ is continuous, however, $u$ is not necessarily continuous even if it is bounded.

**Fact 2.**

If $u$ is $F$-maximal and continuous on $X$, then $u$ is $F$-harmonic on $X$.

**Proof.** This is the standard “bump-function” argument which occurs for example as far back as [BT] or in [I]. It goes as follows. Suppose $u$ is not $F$-harmonic but is $F$-maximal, and therefore $F$-subharmonic. Then $v \equiv -u$ is not $\bar{F}$-subharmonic. Therefore, by Lemma 2.4 in [HL6], there exist $x \in X$, $\epsilon > 0$ and a quadratic polynomial $Q(y)$ such that $v(y) < Q(y) - \epsilon |y - x|^2$ on $\overline{B}_r(x) - \{x\}$ with equality at $y = x$, but $D^2_y Q \notin \bar{F}$, i.e., $-D^2_y Q \in \text{Int} F$. Thus, $w \equiv -Q + \delta$ is strictly $F$-subharmonic at $x$, and hence in a neighborhood $B_r(x)$. Pick $\delta > 0$ sufficiently small that $v(y) < Q(y) - \delta = -w(y)$ on $\partial B_r(x)$. Then $w(y) < u(y)$ on $\partial B_r(x)$, but $w(x) = u(x) + \delta$. This proves that $u$ is not maximal. \hfill \qed

$F$-harmonic functions may not be closed under decreasing limits. For instance in Example 6.3 each $u(z_1)$ which is $\Delta$-subharmonic is the decreasing limit of functions $u_j(z_1)$ which are smooth and $\Delta$-subharmonic. The extensions $\overline{u}_j \to \overline{u}$ to $\mathbb{C}^n$ give an example for the case $F = \mathcal{P}^C$.

This defect is corrected by enlarging the space of $F$-harmonic functions to the space of $F$-maximal functions. (This is the smallest such enlargement by Fact 4 below.)

**Fact 3.**

If $u$ is the decreasing limit of a sequence of $F$-maximal functions, then $u$ is $F$-maximal.

**Proof.** Suppose $\{u_j\}$ are $F$-maximal and $u_j \downarrow u$ on an open set $X$. Fix a compact set $K \subset X$. Then $v \leq u$ on $X - K \Rightarrow v \leq u_j$ on $X - K \Rightarrow v \leq u_j$ on $X \Rightarrow v \leq u$ on $X$. \hfill \qed

This fact has a strong converse.
Fact 4.

**THEOREM 6.4.**

If \( u \) is locally \( F \)-maximal, then \( u \) is locally the decreasing limit
\[
\lim_{j \to \infty} u_j = \lim_{j \to \infty} u_j
\]
of \( F \)-harmonic functions \( u_j \).

The proof of this fact requires a lemma.

**Lemma 6.5.** Suppose \( u \) is \( F \)-subharmonic on \( X \), \( \Omega^{\text{open}} \subseteq X \), and \( v \in \text{USC} \) is \( F \)-subharmonic on \( \Omega \). If \( v \leq u \) on \( \partial \Omega \), then
\[
\overline{v} \equiv \left\{ \begin{array}{ll}
\max\{u, v\} & \text{on } \Omega \\
u & \text{on } X - \overline{\Omega}
\end{array} \right.
\]
is \( F \)-subharmonic on \( X \).

**Proof.** Sup-convolution provides a decreasing sequence \( u^\epsilon \downarrow u \) of continuous \( F \)-subharmonic functions which are defined on subdomains which contain \( \overline{\Omega} \) and increase to \( X \). Set
\[
v^\epsilon_\delta \equiv \left\{ \begin{array}{ll}
\max\{u^\epsilon + \delta, v\} & \text{on } \Omega \\
u^\epsilon + \delta & \text{on } X - \overline{\Omega}.
\end{array} \right.
\]
Since \( \{v < u^\epsilon + \delta\} \) is a relatively open subset of \( \overline{\Omega} \) containing \( \partial \Omega \), the function \( v^\epsilon_\delta \) is \( F \)-subharmonic on domains containing \( \overline{\Omega} \) which increase to \( X \) as \( \epsilon \downarrow 0 \). Finally, \( v^\epsilon_\delta \downarrow \overline{v} \) as \( \epsilon, \delta \downarrow 0 \), proving that \( \overline{v} \) is \( F \)-subharmonic on \( X \). ■

Using this Lemma 6.5 the definitions (6.1) and (6.1)’ of \( F \)-maximality on \( X \) can be further refined as follows:

For each domain \( \Omega \subset X \) and \( v \in \text{USC} \) which is \( F \)-subharmonic on \( \Omega \),
\[
v \leq u \text{ on } \partial \Omega \quad \Rightarrow \quad v \leq u \text{ on } \overline{\Omega}
\]

(6.1)"

Using this definition of \( F \)-maximality together with the fact that on balls \( B \subset \mathbb{R}^n \) the Dirichlet problem is uniquely solvable by the Perron function, it is easy to prove Theorem 6.4.

**Proof of Theorem 6.4.** Suppose \( u \) is maximal on \( X \) and \( \overline{B} \subset X \) is a closed ball. Choose \( \varphi_j \in C(\partial B) \) such that \( \varphi_j \downarrow u \mid_{\partial \Omega} \). Let \( u_j \in C(\overline{\Omega}) \) denote the solution to the Dirichlet Problem on \( \overline{B} \) with \( u_j \mid_{\partial \Omega} = \varphi_j \) and \( u_j \) \( F \)-harmonic on \( B \). Since \( u_j \) is the Perron function for boundary values \( \varphi_j \), we have \( u \leq u_j \) for all \( j \) and \( u_j \downarrow v \) is decreasing. Thus \( u \leq v \). Also \( v \mid_{\partial B} = \lim u_j \mid_{\partial B} = \lim \varphi_j = u \mid_{\partial B} \), and \( v \) is \( F \)-subharmonic on \( B \). Thus, by (6.1)” above, \( v \leq u \) on \( \overline{B} \). Hence, \( u = v = \lim u_j \). ■
7. Tangents to Subharmonics.

In this section the ST-invariant cone subequation $F$ on $\mathbb{R}^n$ is assumed to be convex. We shall work at a fixed point, which for simplicity is assumed to be the origin. We begin by introducing the notion of tangents at $0$ to $F$-subharmonics. That is, given an $F$-subharmonic function $u$ defined in a neighborhood of $0$, we define the notion of tangent functions to $u$ at $0$, with a required clarification given by the first proposition. Then in Theorems 7.5 and 7.6 the basic properties of a tangent $U$ to $u$ at $0$ are established.

**Definition 7.1.** Suppose that $u$ is $F$-subharmonic on the ball about the origin of radius $\rho$.

The tangential $p$-flow (or $p$-homothety) determined by the Riesz characteristic $p = p_F$ of $F$ is defined as follows.

(a) $u_r(x) = r^{p-2}u(rx)$ if $p > 2$,

(b) $u_r(x) = \frac{1}{r^{\frac{2}{p}}} [u(rx) - u(0)]$ if $1 \leq p < 2$, and

(c) $u_r(x) = u(rx) - M(u, r)$ if $p = 2$

**Remark 7.2.** If $p < 2$, then Corollary 5.5(b) says that $u(0) = M(u, 0)$ is finite. Some readers may prefer to assume at this point that $u(0) = 0$ so that the $p$-flows for all $p \neq 2$ are the same, namely

$$u_r(x) = r^{p-2}u(rx) \quad \text{if } p \neq 2. \quad (7.1)$$

Others may wish to make this assumption in the proofs.

Note that

The functions $u_r$ are $F$ subharmonic on $B_{p/r}$, and as $r \to 0$, these balls expand to $\mathbb{R}^n$.

An upper semi-continuous function $U(x)$ on $\mathbb{R}^n$ taking values in $[−\infty, \infty)$ is invariant under this flow if and only if there exists an u.s.c. function $g$ on the unit sphere $S$ such that

$$U(x) = |x|^{p-2}g \left( \frac{x}{|x|} \right) \quad \text{in the cases where } p \neq 2,$$

while in the case where $p = 2$, we leave it to the reader to prove that

$$U(x) = \Theta \log|x| + g \left( \frac{x}{|x|} \right) \quad \text{with } \sup_{S^{n-1}} g = 0 \quad \text{and} \quad \Theta \geq 0 \text{ a constant.}$$

Functions of this form will be said to have Riesz homogeneity $p$.

Under our assumptions on $F$ each $F$-subharmonic function $u$ is $L^1_{loc}$ since it is $\Delta$-subharmonic by (5.3).

**Definition 7.3.** Suppose that $u$ is an $F$-subharmonic function defined in a neighborhood of the origin. For each sequence $r_j \searrow 0$ such that

$${\mathcal U} \equiv \lim_{j \to \infty} u_{r_j} \text{ converges in } L^1_{loc}(\mathbb{R}^n),$$

(7.2)
the point-wise defined function

\[ U(x) \equiv \lim_{r \to 0} \text{ess sup}_{B_r(x)} U \]  

(7.3)

is called a tangent to \( U \) at 0. We let \( T_0(u) \) denote the set of all such tangents \( U \). (We will refer to \( U \), satisfying (7.2), as an \( L^1_{\text{loc}} \)-tangent when the distinction between the function \( U \) and the equivalence class of functions \( U \) is important.)

Our first result clarifies this Definition.

**Proposition 7.4.** Each tangent \( U \) to \( u \) at 0 is an entire \( F \)-subharmonic function on \( \mathbb{R}^n \). Moreover, \( U \) belongs to the \( L^1_{\text{loc}} \)-class \( U \in L^1_{\text{loc}}(\mathbb{R}^n) \) and is the unique \( F \)-subharmonic function in this \( L^1_{\text{loc}} \)-class.

To prove Proposition 7.4 we use the following result established in [HL5, Cor. 5.4] (see [HL112] for generalizations.) We say that a subequation \( F \) can be defined using fewer of the variables in \( \mathbb{R}^n \) if there exist an \((n-1)\)-dimensional subspace \( W \subset \mathbb{R}^n \) and a subequation \( F' \subset \text{Sym}^2(W) \) which determines \( F \) by: \( A \in F \iff A|_W \in F' \).

An important point here is that the same representative \( u \) of the \( L^1_{\text{loc}} \)-class \( u \) (given by (7.4)) is the correct representative, no matter which subequation \( F \) is being considered.

**THEOREM 7.5. (Distributional versus Viscosity Subharmonics).** Suppose \( F \) is a convex cone subequation which cannot be defined using fewer of the variables in \( \mathbb{R}^n \).

(a) If \( u \) is \( F \)-subharmonic in the viscosity sense, then \( u \) is \( L^1_{\text{loc}} \) and \( F \)-subharmonic in the distributional sense.

(b) If \( \overline{u} \) is an \( F \)-subharmonic distribution, then \( \overline{u} \in L^1_{\text{loc}} \) and the limit

\[ u(x) = \lim_{r \to 0} \text{ess sup}_{B_r(x)} \overline{u} \]  

exists at each point

(7.4)

and defines an upper semi-continuous function \( u \) in the \( L^1_{\text{loc}} \)-class \( \overline{u} \) which is \( F \)-subharmonic in the viscosity sense. Moreover, \( u \) is the unique such representative of \( \overline{u} \).

**Remark 7.6.** We refer the reader to Sections 3, 4, and 5 of [HL5] for a fuller discussion of this result and the definition of an \( F \)-subharmonic distribution (Definition 4.1 and Proposition 4.3). However, the terminology used in [HL5] is somewhat different. Here we use the terminology employed in [HL112]. In [HL5] a convex cone subequation \( F \) is called a “positive cone” and denoted \( \mathcal{P}^+ \). The polar cone is denoted by \( \mathcal{P}_+ \). A convex cone subequation which cannot be defined using fewer of the variables in \( \mathbb{R}^n \) is called an elliptic cone”.

From the distributional point of view it is straightforward to see that averages, or more generally convolution, of an \( F \)-subharmonic function \( u \) with any non-negative measure is again \( F \)-subharmonic.

**Proof of Proposition 7.4.** We use these facts about the ST-invariant convex cone subequation \( F \):

\begin{align*}
(1) & \quad F \subset \Delta \\
(2) & \quad 1 \leq p_F \leq n
\end{align*}

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(3) $F$ cannot be defined using fewer of the variables in $\mathbb{R}^n$.

**Proof.** Properties (1) and (2) have already been noted in (5.3) and (5.4). For Property (3) note that the ST-invariance of $F$ rules out the possibility that $F$ could be defined using fewer of the variables in $\mathbb{R}^n$. Now one can apply Theorem 7.5.

Suppose $\overline{U} = \lim_{j \to \infty} u_{r_j}$ in $L^1_{\text{loc}}(\mathbb{R}^n)$ is an $L^1_{\text{loc}}$ tangent to $u$ at 0. Since $F$ is a cone, each $u_r$ is viscosity $F$-subharmonic, and hence in $L^1_{\text{loc}}$ and distributionally $F$-subharmonic by Part (a) of Theorem 7.5. Hence, in the limit, $\overline{U}$ is distributionally $F$-subharmonic. Now apply Part (b) of Theorem 7.5 to $\overline{U}$ to complete the proof.

In light of Proposition 7.4 we frequently drop the distinction between $U$ and $\overline{U}$.

**Averages of Tangents**

Most of the properties of tangents can be deduced from the following result concerning their averages.

**THEOREM 7.7.** Suppose that $u$ is an $F$-subharmonic function defined in a neighborhood of the origin in $\mathbb{R}^n$. Let $p = p_F$ be the Riesz characteristic of $F$.

If $p \neq 2$, then each tangent $U$ to $u$ at 0 has averages

$$M(r) = \sup_S U(r\sigma) = \Theta^M(u)K(r), \quad S(r) = \int_S U(r\sigma) \, d\sigma = \Theta^S(u)K(r),$$

and

$$V(r) = \int_B U(rx) \, dx = \Theta^V(u)K(r).$$

In particular,

$$\Theta^\Psi(U) = \Theta^\Psi(u) \quad \text{for} \quad \Psi = M, S, \text{ or } V. \quad (7.5)$$

When $p = 2$, all the densities of $u$ and any tangent $U$ to $u$ at 0, agree, and will be simply denoted by $\Theta = \Theta(u)$. Specifically, we have

$$\Theta(u) = \Theta^M(U) = \Theta^S(U) = \Theta^V(U) = \Theta^M(u) = \Theta^S(u) = \Theta^V(u). \quad (7.6)$$

Moreover, the averages of a tangent $U$ to $u$ are given by

$$M(r) = \Theta \log r, \quad S(r) = \Theta \log r + \int_S U, \quad \text{and} \quad V(r) = \Theta \log r + \int_B U, \quad (7.7)$$

with

$$-C\Theta \leq \int_S U \leq 0 \quad \text{and} \quad -(C+1)\Theta \leq \int_B U, \quad \text{where} \quad C = \frac{1}{\varphi(\frac{1}{e})} > 1. \quad (7.8)$$

When $p \neq 2$, these formulas show that any two tangents have the same maxima $M(r)$ and the same spherical averages $S(r)$ and volume averages $V(r)$, all being the appropriate density times $K(r)$. When $p = 2$, $M(r)$, $S(r)$ and $V(r)$ all agree with $\Theta \log r$ modulo an additive constant, but the constant depends on the tangent $U$, not just on $u$. 

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In all cases, for each tangent $U$, the function $S(U, |x|)$ is $F$-harmonic on $\mathbb{R}^n - \{0\}$ since $\Theta K(|x|) + C$ is $F$-harmonic there (Proposition 3.5).

Now by Theorem 6.2 and Fact 2 in Section 6 we have the following corollary.

**THEOREM 7.8.** Every tangent to an $F$-subharmonic function is $F$-maximal, and every continuous tangent is $F$-harmonic.

The proof of Theorem 7.7 follows from the stability of averages (Lemma 5.3) and the determination of the averages of a flow.

**Averages of Flows**

We may assume that $u(0) = 0$ if $1 \leq p < 2$ (Remark 7.2), and that $u(0) = -\infty$ if $2 \leq p < \infty$.

**THEOREM 7.9.** For $p \neq 2$ and $\Psi = M, S$ or $V$,

$$
\lim_{s \downarrow 0} \Psi(u_s, r) = \Theta^\Psi(u) K(r). \tag{7.10}
$$

For $p = 2$, if $\Psi = M$ we also have

$$
\lim_{s \downarrow 0} M(u_s, r) = \Theta^M(u) K(r) = \Theta^M(u) \log r. \tag{7.10a}
$$

In this case the limit is decreasing and uniform in $r \leq R$. For $\Psi = S$ or $V$ we have

$$
\liminf_{s \downarrow 0} S(u_s, r) \geq \Theta^M(u)(\log r - C), \quad \text{and} \quad \liminf_{s \downarrow 0} V(u_s, r) \geq \Theta^M(u)(\log r - C - 1) \tag{7.10b}
$$

with $C$ as in (7.9).

Direct calculations from the definitions of the flow and the averages establish the next result.

**Lemma 7.10.** For $\Psi = M, S$, or $V$:

$$
\Psi(u_s, r) = s^{p-2} \Psi(u, sr) = \frac{\Psi(u, sr)}{K(sr)} K(r) \quad \text{when } p \neq 2, \quad \text{and} \tag{7.11}
$$

$$
\Psi(u_s, r) = \Psi(u, sr) - M(u, s) = \frac{\Psi(u, sr) - M(u, s)}{K(sr) - K(s)} K(r) \quad \text{when } p = 2 \tag{7.12}
$$

**Proof.** For example, when $\Psi$ is the volume average $V$ and $p \neq 2$, we have

$$
V(u_s, r) = \frac{1}{|B|} \int_B u_s(rx) \, dx = \frac{s^{p-2}}{|B|} \int_B u(rsx) \, dx = s^{p-2} V(u, rs).
$$
The remaining calculations are left to the reader.

**Proof of Theorem 7.9.** Proposition 5.7 implies (7.10) when $p \neq 2$. In the case where $p = 2$ the limit (7.10a) for the maximum follows from monotonicity (Proposition 5.6). The limit (7.10c) for $V$ follows from the limit (7.10b) for $S$ since $V(u_s, r) = n \int_0^1 S(u_s, t) t^{n-1} dt$ by (5.23), and $n \int_0^1 (\log rt - C) t^{n-1} dt = \log r - C - 1$.

It remains to prove (7.10b). Harnak’s inequality in the form (5.26) with $v = u_s$ and $\lambda = 1/e$ states that

$$C \left( M \left( u_s, \frac{r}{e} \right) - M(u_s, r) \right) + M(u_s, r) \leq S(u_s, r).$$

We know the limit of the terms involving $M$ as $s \downarrow 0$. This gives

$$C \Theta^M (u) \left( \log \frac{r}{e} - \log r \right) + \Theta^M (u) \log r \leq \lim inf_{s \downarrow 0} S(u_s, r)$$

as desired.

**Proof of Theorem 7.7.** The density statements for $u$ are contained in Propositions 5.1 and 5.12. The density statements for $U$ follow from the formulas in Theorem 7.7 and the density statements for $u$. The formulas in Theorem 7.7 follow immediately from the formulas in Theorem 7.9 for the averages of flows and the stability of averages (Lemma 5.3), with the exception of (7.8) for $S$ and $V$, and the estimates in (7.9).

The estimates (7.10b) and (7.10c) and the Stability Lemma 5.3 show that for any tangent $U$ to $u$ at 0,

$$\Theta^M (u) \left( \log r - C \right) \leq S(U, r) \quad \text{and} \quad \Theta^M (u) \left( \log r - C - 1 \right) \leq V(U, r)$$

for all $0 < r < \infty$. Also we have that $V(U, r) \leq S(U, r) \leq M(U, r) = \Theta^M (u) \log r$.

Since $V(U, e^t)$ and $S(U, e^t)$ are entire convex functions of $t$, the linear inequalities

$$\Theta(t - C) \leq S(U, e^t) \leq \Theta t \quad \text{and} \quad \Theta(t - C - 1) \leq V(U, e^t) \leq \Theta t$$

imply that $S(U, e^t) = \Theta(t + k)$ and $V(U, e^t) = \Theta(t + k' - 1)$ where $k$ and $k'$ satisfy $-C \leq k, k' \leq 0$.
8. Existence of Tangents.

We now address the basic existence question. Again $F$ is assumed here to be convex. However, in the case where $1 \leq p < 2$ much stronger results are true even if $F$ is just a cone and not necessarily convex. These stronger results are established in Section 12.

**THEOREM 8.1. (Existence of Tangents).** Suppose that $u$ is $F$-subharmonic on a ball $B_\rho$. For each $R > 0$ there exists $\delta > 0$ such that the family $\{u_r\}_{0 < r \leq \delta}$ is uniformly bounded above and bounded in norm in $L^1(B_R)$. In particular, the set $\{u_r\}_{0 < r \leq \delta}$ is precompact in $L^1(B_R)$.

**Proof.** The upper bound can be chosen to be any number greater than $\Theta^M(u)K(R)$ by (7.10) if $p \neq 2$ and by (7.10a) if $p = 2$. Consequently the boundedness in $L^1(B_R)$ is equivalent to a lower bound for $V(u_s,R)$ which is uniform in $s$. This lower bound can be chosen to be any number less than $\Theta^V(u)K(R)$ if $p \neq 2$, or $\Theta^M(u)(\log R - C - 1)$ if $p = 2$, by (7.10) and (7.10c) respectively in Theorem 7.9.

The basic properties of the tangent set $T_0(u)$ are contained in the following theorem. Again see Section 12 for the stronger versions of parts (2) and (4) when $1 \leq p < 2$.

**THEOREM 8.2.** Suppose that $u$ is an $F$-subharmonic function defined in a neighborhood of the origin in $\mathbb{R}^n$. Then the tangent set $T_0(u)$ to $u$ at 0 satisfies:

1. $T_0(u)$ is non-empty.
2. $T_0(u)$ is a compact subset of $L^1_{\text{loc}}(\mathbb{R}^n)$.
3. $T_0(u)$ is invariant under the homothety $U \rightarrow Ur$.
4. $T_0(u)$ is a connected subset of $L^1_{\text{loc}}(\mathbb{R}^n)$.

**Proof.** Parts (1) and (2) are immediate from Theorem 8.1. The arguments for parts (3) and (4) are given in [S, Proposition 1.1.1]. We include them here for completeness. To prove (3) note that $U(x) = \lim_{r_j \downarrow 0} u_{r_j}(x)$ implies $U_r(x) = \lim_{s_j \downarrow 0} u_{s_j}(x)$ with $s_j = rr_j$. To prove (4) suppose $u_{r_j} \rightarrow U_0$ and $u_{t_j} \rightarrow U_1$ with $U_0$ and $U_1$ elements of disjoint open sets $N_0$ and $N_1$ which cover $T_0(u)$. We can assume $r_j < t_j$ for all $j$ and choose $s_j$ between $r_j$ and $t_j$ with $u_{s_j} \notin N_0 \cup N_1$. (Note that $s \mapsto u_s$ is a continuous map into $L^1_{\text{loc}}$.) By Theorem 8.1 the sequence $u_{s_j}$ has a convergent subsequence, and its limit is in neither $N_0$ nor $N_1$, a contradiction.
9. Uniqueness of Tangents.

In this section we discuss some basic situations where tangents are unique. Our main uniqueness results are are stated and proved in subsequent sections. As in Section 7 we assume that \( F \) is convex with finite Riesz characteristic \( p \).

**Definition 9.1.** Suppose \( u \) is an \( F \)-subharmonic function defined in a neighborhood of the origin.

(a) If \( T_0(u) = \{ U \} \) is a singleton, then we say that **uniqueness of tangents holds for** \( u \). If uniqueness of tangents holds for all such \( u \), we say the that **uniqueness of tangents holds for** \( F \).

(b) If \( T_0(u) = \{ \Theta K(|x|) \} \) with \( \Theta \geq 0 \) a constant, then we say that **strong uniqueness of tangents holds for** \( u \). If strong uniqueness of tangents holds for all such \( u \), then we say that **strong uniqueness of tangents holds for** \( F \).

(c) If every tangent \( U \) to \( u \) satisfies \( U_r = U \ \forall r \), then we say that **homogeneity of tangents holds for** \( u \). If homogeneity of tangents holds for all such \( u \), then we say that **homogeneity of tangents holds for** \( F \).

Note that (a) can be rephrased since

\[ T_0(u) = \{ U \} \iff \lim_{r \to 0} u_r \text{ exists in } L^1_{\text{loc}}(\mathbb{R}^n) \text{ and equals } U. \quad (9.1) \]

Note also that (a) \( \Rightarrow \) (c) since if \( u_{r_j} \) has limit \( U \), then \( u_{r_r} \) has limit \( U_r \).

In general, \( S(u, r) \leq M(u, r) \). Therefore,

For \( 2 \leq p \leq n \), \( \Theta^M(u) \leq \Theta^S(u) \), and for \( 1 \leq p < 2 \), \( \Theta^S(u) \leq \Theta^M(u) \) \quad (9.2)

because of (5.13) and (5.12) respectively. However, if strong uniqueness holds for \( u \), then all densities “agree” because of Proposition 5.11 and the following.

If for some \( \Theta \geq 0 \), \( T_0(u) = \{ \Theta K \} \), then \( \Theta^M(u) = \Theta^S(u) = \Theta \). \quad (9.3)

This follows from Theorem 7.7 and the fact that \( \Theta^M(K) = \Theta^S(K) = 1 \).

There are two classical cases where strong uniqueness holds, that will prove useful later. For the sake of completeness we include proofs.

**Proposition 9.2. (Radial Subharmonics).** Suppose that \( u(x) = f(|x|) \) is a radial \( F \)-subharmonic function defined on a neighborhood of 0. Then

\[ \lim_{r \to 0} u_r = \Theta(u)K_p(|x|) \]

in \( L^1_{\text{loc}}(\mathbb{R}^n) \) and uniformly on compact subsets in \( \mathbb{R}^n - \{0\} \). Thus, \( T_0(u) = \{ \Theta K_p \} \).

**Proof.** Since \( u \) is radial, we have that \( u_r(x) = M(u_r, |x|) \), but by Theorem 7.9 we know that \( \lim_{r \to 0} M(u_r, |x|) = \Theta K_p(|x|) \) uniformly in \( 0 < |x| \leq R \). \hfill \blacksquare
Remark 9.3. The conclusion of convergence in \( C(\mathbb{R}^n - \{0\}) \) only requires \( F \) to be an ST-invariant cone subequation with finite characteristic. It does not require convexity.

Proposition 9.4. (Newtonian Case). Suppose \( u \) is a \( \Delta \)-subharmonic function defined on a neighborhood of 0. Then

\[
\lim_{r \to 0} u_r(x) = -\frac{\Theta(u)}{|x|^{n-2}} \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^n) \text{ when } n \geq 3, \text{ and }
\lim_{r \to 0} u_r(x) = \Theta(u) \log |x| \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^n) \text{ when } n = 2
\]

Proof. Each such \( u \) is of the form \( u = v + h \) where \( v = K * v \) is a Newtonian potential and \( h \) is harmonic near the origin. (Take the measure \( \nu \) to be a cut-off of the measure \( \mu = \Delta u \) in a small ball about the origin.) This reduces the proof to the case \( v \equiv K * \nu \). (In the \( n = 2 \) case \( u_r \) and \( v_r + h_r \) differ by \( M(v, r) + M(h, r) - M(u, r) \), but this error has limit zero.)

Now one checks that: for \( n \geq 3 \), \( (K * \nu)_r = K * (\frac{1}{r})_* \nu \) and for \( n = 2 \), \( (K * \nu)(r x) = K * (\frac{1}{r})_* \nu)(x) + \nu(1) \log r \), so that \( M(K * \nu, r) = M(K * (\frac{1}{r})_* \nu, 1) + \nu(1) \log r \). Now \( \lim_{r \to 0} (\frac{1}{r})_* \nu \) always exists weakly in the space of measures and equals \( \Theta[0] \), where \( \Theta = \lim_{r \to 0} \nu(B_r) \) is the zero-dimensional density of \( \nu \) at 0. Since \( K \in L^1_{\text{loc}}(\mathbb{R}^n) \), the limit of \( (K * \nu)_r \) exists in \( L^1_{\text{loc}}(\mathbb{R}^n) \) and equals \( K * (\Theta[0]) = \Theta K \). (Note that for \( n = 2 \), \( M(K * (\frac{1}{r})_* \nu, 1) \) has limit \( M(\Theta \log |x|, 1) = 0 \).)

In the \( n = 2 \) case there is a different proof following Kiselman [K1]. Note that by (7.8) we have \( M(U, r) = \Theta \log r \) for any tangent \( U \) to \( u \) at 0. In particular, \( U(x) - \Theta \log |x| \) is \( \leq 0 \) on \( \mathbb{R}^2 \) and \( \Delta \)-subharmonic on \( \mathbb{R}^2 - \{0\} \). Hence, it can be extended to \( \mathbb{R}^2 \) as a subharmonic function, and then by Liouville’s Theorem it must be constant. Since \( M(u_r, 1) = 0 \) for all \( r \) small, \( M(U, r) = 0 \), proving that the constant is zero.

Proposition 9.4 can be partly generalized.

Proposition 9.4’. (Riesz Potentials, \( p > 2 \)). Suppose \( u = K_p * \nu \) where \( \nu \geq 0 \) is a compactly supported measure. Then

\[
\lim_{r \to 0} u_r = -\frac{\Theta(\nu)}{|x|^{p-2}} \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^n)
\]

where, up to a universal constant, \( \Theta(\nu) = \lim_{r \to 0} \nu(B_r) \)

Proof. Ignoring constants, we have (cf. [L])

\[
\Delta u = (\Delta K_p) * \nu = K_{p+2} * \nu \equiv \mu.
\]

Note that

\[
K_n * \mu = K_n * K_{p+2} * \nu = K_p * \nu = u.
\]

We compute that

\[
u_r(x) = r^{p-2} u(r x) = r^{p-2}(K_p * \nu)(r x) \text{ is equal to } K_p * \left\{ \left( \frac{1}{r} \right)_* \nu \right\},
\]
and observe that \( \lim_{r \downarrow 0} (\frac{1}{r})_{*} \nu = \Theta(\nu)[0] \).

We complete this section with a final case where strong uniqueness holds.

**Proposition 9.5. (Zero Density).** Suppose that \( u \) is \( F \)-subharmonic in a neighborhood of the origin and \( F \) is convex with \( p > 1 \). If any of the densities of \( u \) is zero at 0, then all the densities of \( u \) vanish at 0, and in this case

\[
\lim_{r \to 0} u_{r} = 0 \quad \text{in } L^{1}_{\text{loc}}(\mathbb{R}^n). \tag{9.4}
\]

If \( F \) is not convex but \( 1 \leq p < 2 \), then \( \Theta^{M}(u, 0) = 0 \) implies that

\[
\lim_{r \to 0} u_{r} = 0 \quad \text{locally in } \alpha \text{ Holder norm, } \alpha = 2 - p. \tag{9.5}
\]

**Proof.** The first assertion is a direct consequence of Propositions 5.11 and 5.12, while (9.4) follows from Theorem 7.9.

The proof of the final assertion of Proposition 9.5 is postponed as it follows immediately from (12.2).

10. The Strong Uniqueness Theorem.

In this section we prove one of our two main results concerning strong uniqueness. Recall that every \( O(n) \)-invariant subequation \( F \) has complex and quaternionic analogues \( F^{C} \) and \( F^{H} \), which are invariant under \( U(n) \) and \( \text{Sp}(n) \) respectively (see Example 4.7).

**Theorem 10.1.** Suppose that \( F \) is \( O(n) \)-invariant and convex with finite Riesz characteristic \( p \). Then, except for the case \( F = \mathcal{P} \), strong uniqueness of tangents holds for \( F \). Furthermore, except for the cases \( \mathcal{P}^{C} \) and \( \mathcal{P}^{H} \), strong uniqueness of tangents also holds for the complex and quaternionic analogues \( F^{C} \) and \( F^{H} \) of \( F \).

**Remark 10.2.** For the subequations \( \mathcal{P}, \mathcal{P}^{C} \) and \( \mathcal{P}^{H} \), strong uniqueness fails dramatically. Nonetheless, tangents are classified in these cases. This is discussed in the second part of this paper.

**Proof.** Let \( u \) be \( F \)-subharmonic in a neighborhood of the origin and choose \( U \in T_{0}(u) \). Then

\[
U(x) = \lim_{j \to \infty} u_{r_{j}}(x)
\]

for a sequence \( r_{j} \downarrow 0 \), where the flow \( u_{r_{j}}(x) \), given in Definition 7.1, depends on \( p \).

Now Theorem 7.8 states that

\[
U \text{ is maximal on } \mathbb{R}^n - \{0\}, \quad \text{and} \tag{10.1}
\]

\[
\text{If } U \in C(\mathbb{R}^n - \{0\}), \text{ then } U \text{ is } F \text{ harmonic on } \mathbb{R}^n - \{0\}. \tag{10.2}
\]

We first prove the theorem under the additional assumption that \( F \) is uniformly elliptic. (Note, however, from Section 4 that there many examples of subequations \( F \) which are not uniformly elliptic, but for which the theorem still applies.)
Proposition 10.3. If, in addition to the hypotheses of Theorem 10.1, \( F \) is uniformly elliptic, then strong uniqueness of tangents holds for \( F \).

Proof. The two regularity results needed for \( F \) can be stated as follows.

Fact 10.4. A sequence \( \{u_j\} \) of \( F \)-harmonics on \( X^{\text{open}} \subset \mathbb{R}^n \), which is bounded in \( L^\infty(K) \) for each compact \( K \subset X \), is precompact in \( C(X) \).

Fact 10.5. Each \( F \)-harmonic function is \( C^1 \).

The reader is referred to [CC] and [T] for these results. Also for Fact 10.4 one can use the Krylov-Safanov Hölder Estimate 4 in [E] which holds with \( f = 0 \) because of the First Linearization on page 107.

Lemma 10.6.

(a) Suppose \( U \in T_0(u) \). Then \( g^*U \in T_0(g^*u) \) for each \( g \in G \), and the densities \( \Theta^S(g^*U) = \Theta^S(U) = \Theta^S(u) = \Theta^S(g^*u) \) are all equal.

(b) If \( U \in T_0(u) \) and \( g \in G \), then \( \max\{U, g^*U\} \in T_0(\max\{u, g^*u\}) \).

The straightforward proofs are omitted.

The proof of Proposition 10.3 will progress in three stages. First we establish strong for continuous tangents, then for tangents which are locally bounded, and finally for general tangents.

The proof that \( U = \Theta K_p \) for \( U \in C(\mathbb{R}^n - \{0\}) \) is completed as follows. First by Fact 10.4 we conclude that \( U \), and therefore also \( \max\{U, g^*U\} \), are continuous. Therefore by (10.2) and Lemma 10.6 we have that

\[
\max\{U, g^*U\} \text{ is } F \text{-harmonic on } \mathbb{R}^n - \{0\} \text{ for each } g \in G.
\] (10.3)

By the \( C^1 \)-regularity result 10.5 we have that

\[
\max\{U, g^*U\} \text{ is } C^1 \text{ on } \mathbb{R}^n - \{0\} \text{ for each } g \in G.
\] (10.4)

Although the maximum of two \( F \)-subharmonics is always subharmonic, it is unusual for the maximum of two distinct \( F \)-harmonics to be \( F \)-harmonic. In fact we have the following.

Lemma 10.7. Let \( f \) be a function on the unit sphere in \( S \subset \mathbb{R}^n \) with the property that \( \max\{f, g^*f\} \in C^1(S) \) for all \( g \in G \). Then \( f = \text{constant} \).

Proof. We begin with the case \( G = O(n) \). If we can prove constancy on every great circle in \( S^{n-1} \), we are done. So we are immediately reduced to the case \( n = 2 \). Lifting to the covering \( \mathbb{R} \rightarrow S^1 \) we are then reduced to the following elementary fact:

Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a \( 2\pi \)-periodic function with the property that for all \( a \in \mathbb{R} \),

the function \( F_a(x) \equiv \max\{f(x), f(x + a)\} \) is differentiable. Then \( f \equiv \text{constant} \).

We see this as follows. If \( f \) is not constant, there exists a point \( x \) with \( f'(x) > 0 \). Since it is periodic, there must also exist a point \( y \) with \( f'(y) < 0 \). Set \( a = y - x \). Then the left
hand derivative of $F$ is $< 0$ (if it exists), and the right hand one is $> 0$. This completes the argument for $G = O(n)$.

For the general case one considers the closures of 1-parameter subgroups of $G$ acting on the sphere. These are tori, and one can argue as in the proof of Fact 10.4 by considering the (dense) family of closed 1-parameter subgroups. This shows that $f$ is constant on every orbit of every such subgroup. Now for $x \in S^{n-1}$ the value of $f$ must be the same for all orbits of $x$. Since the differential of the action at the identity $T_e(G) \to T_x(S^{n-1})$ is surjective, we see that $f$ is locally constant, and hence constant, on $S^{n-1}$.

This completes the proof of Proposition 10.3 if $U \in C(R^n - \{0\})$.

Combining the regularity result Fact 10.5 with Theorem 6.4 one can strengthen Fact 10.4.

**Lemma 10.8.** Suppose $F$ in uniformly elliptic. Then each locally bounded $F$-maximal function is $F$-harmonic.

This completes the proof of Proposition 10.3 if $U \in L^\infty_{loc}(R^n - \{0\})$.

By Lemma 10.6(b), for each $N > 0$ we have $U^N \equiv \max\{U, NK_p\} \in T_0(\max\{u, NK_p\})$. Since $U^N \in L^\infty_{loc}(R^n - \{0\})$, $U^N$ is a multiple of $K_p$. We now observe that $U^N$ decreases down to $U$ as $N \to \infty$. Hence, if each $U^N$ is a multiple of the Riesz kernel, then so is $U$. This completes the proof of Proposition 10.3.

The last result needed for the proof of Theorem 10.1 in the $O(n)$-invariant case is the following.

**Proposition 10.9.** The subequation

$$P_{p}^{\text{largest}} \overset{\text{def}}{=} \left\{ A : A + \frac{p - 1}{n - p} (\text{tr} A) I \geq 0 \right\}$$

contains all the $O(n)$-invariant convex cone subequations $F$ of Riesz characteristic $p$, and has Riesz characteristic $p$ itself. Since

$$P_{p}^{\text{largest}} = P(\delta) \quad \text{with} \quad \delta = \frac{(p - 1)n}{n - p}$$

(see Example 4.3), the subequation $P_{p}^{\text{largest}}$ is uniformly elliptic when $p > 1$.

**Proof.** Suppose $A = \lambda_1 P_{e_1} + \cdots + \lambda_n P_{e_n}$ is in diagonal form with $\lambda_1 \leq \cdots \leq \lambda_n$. Then by Definition 4.3 we know that

$$A \notin P(\delta) \iff \lambda_1 + \frac{\delta}{n} (\lambda_1 + \cdots + \lambda_n) < 0.$$

If $\mu' = \pi(\lambda')$ is a permutation of $\lambda' = (\lambda_2, \ldots, \lambda_n)$, then $A_\pi = \lambda_1 P_{e_1} + \mu_2 P_{e_2} + \cdots + \mu_n P_{e_n}$ also belongs to the open half-space $H$ defined by $\langle A_\pi, P_{e_1} + \frac{\delta}{n} I \rangle < 0$, and $H$ is disjoint from $P(\delta)$. Averaging $A$ over these permutations yields $B \equiv \lambda_1 P_{e_1} + \frac{\Sigma}{n - 1} P_{e_i}$ where $\Sigma \equiv \lambda_2 + \cdots + \lambda_n$. Since $B \in H$ we have $B \notin P(\delta)$. Hence setting $e \equiv e_1$ and using the
fact that $\mathcal{P}(\delta)$ is a cone, we can rescale to obtain $B' \equiv P_{e^\perp} - (q - 1)P_e \notin \mathcal{P}(\delta)$. Since the characteristic of $\mathcal{P}(\delta)$ is equal to $p = \inf \{q' : P_{e^\perp} - (q' - 1)P_e \notin \mathcal{P}(d)\}$, and we know by (4.6) that $\delta = (p - 1)n/(n - p)$, this proves that $q > p$.

Now if $A \in F$, then since $F$ is $O(n)$-invariant and convex, the average $B \in F$. Finally since $F$ is a cone, $B' \in F$. Since $q > p$, this proves that $F$ has Riesz characteristic $> p$, contrary to assumption. □

Proposition 10.9 says that if $U$ is a tangent to an $F$-subharmonic function where $F$ has characteristic $p$, then $U$ is $\mathcal{P}_p^\text{largest}$-harmonic. Since the subequation $\mathcal{P}_p^\text{largest}$ is uniformly elliptic, the proof of Theorem 10.1 is complete in the orthogonally invariant case.

**Note 10.10.** Some (in fact, many) readers may be uncomfortable with the assertion that $\mathcal{P}(\delta)$-harmonics have the regularity of viscosity solutions to equations which are convex and uniformly elliptic in the conventional sense. A discussion of this point is given in Appendix B.

Consider now the complex analogue $F^\mathbb{C}$ of $F$ on $\mathbb{C}^n$. Then we have $F^\mathbb{C} \subset \mathcal{P}^\mathbb{C}(\delta)$, the complex analogue of the subequation (10.1) above. Now for any $A \in \text{Sym}^2\mathbb{R}(\mathbb{C}^n)$ one has that $\text{tr}(A) = 2\text{tr}_\mathbb{C}(A\mathbb{C})$ and $\lambda_1(A) \leq \lambda_\mathbb{C}^1(A\mathbb{C})$. Hence, $\mathcal{P}^\mathbb{C}(\delta) \subset \mathcal{P}^\mathbb{C}(\delta)$ as subsets of $\text{Sym}^2\mathbb{R}^{2n} = \text{Sym}^2\mathbb{R}(\mathbb{C}^n)$. It follows that $\mathcal{P}^\mathbb{C}(\delta)$ is uniformly elliptic (for $p > 1$). The arguments given above now go through to establish the theorem in this case. Of course Lemma 10.6 must be established with $O(n)$ replaced with $U(n)$. To do this one considers the closures of 1-parameter subgroups of $U(n)$ acting on the sphere. These are tori, and one can argue as in the proof of Fact 10.4 by considering the (dense) family of closed 1-parameter subgroups.

The case of the quaternionic analogue $F^\mathbb{H}$ is proved in exactly the same way. This completes the proof of Theorem 10.1. □

For the interested reader we present a second approach to proving Theorem 10.1 based on regularization via the group $G$ (as for example in [HS]).

**Second Proof of Theorem 10.1** Let $u$ be $F$-subharmonic in a neighborhood of the origin and choose $U \in T_0(u)$. For clarity of exposition we work in the case $p > 2$. Then

$$U(x) = \lim_{j \to \infty} r_j^{p-2}u(r_jx)$$

for a sequence $r_j \downarrow 0$. Let $\chi = \chi_\epsilon : G \to [0, \infty)$ be a family of smooth functions converging to the $\delta$-function at the identity in $G$, and for any function $f$ which is $L^1_{\text{loc}}$ in $\mathbb{R}^n - \{0\}$ and in $L^1(S^{n-1}(r))$ for all $r$, define

$$f^\epsilon(x) \equiv \int_G f(gx)\chi(g)\,dg$$

where $dg$ is Haar measure with unit volume on $G$. The following lemma is proved below.

**Lemma 10.11.**

$$U^\epsilon(x) = \lim_{j \to \infty} r_j^{p-2}u^\epsilon(r_jx)$$

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Now by the Fubini Theorem, $U^\varepsilon$ satisfies

$$S(U^\varepsilon, r) = \int_{|x|=1} U^\varepsilon(rx) \, dx = \int_{|x|=1} \left\{ \int_G U(grx) \chi(g) \, dg \right\} \, dx$$

$$= \int_G \left\{ \int_{|x|=1} U(rgx) \, dx \right\} \chi(g) \, dg = \int_G S(U, r) \chi(g) \, dg$$

$$= S(U, r) = \Theta^S K(r).$$

From this we conclude that $U^\varepsilon$ is maximal. The next lemma is also proved below.

**Lemma 10.12.** $U^\varepsilon$ is continuous and converges to $U$ in $L^1_{loc}(\mathbb{R}^n - \{0\})$ as $\varepsilon \to 0$.

Note that the continuity of $U^\varepsilon$ implies that it is $F$-harmonic (Fact 2 in Section 6).

We now fix $g_0 \in G$ and define

$$V^\varepsilon(x) \equiv U^\varepsilon(g_0x) = \lim_{j \to \infty} r_j^{p-2} u^\varepsilon(r_j g_0 x)$$

where the second equality comes from Lemma 10.11. Clearly $V^\varepsilon$ is a tangent, and it satisfies $S(V^\varepsilon, r) = S(U^\varepsilon, r) = \Theta^S K(r)$. In particular, $V^\varepsilon$ is also maximal. Furthermore, note that

$$\max\{U^\varepsilon(x), V^\varepsilon(x)\} = \lim_{j \to \infty} r_j^{p-2} \max\{u^\varepsilon(r_j x), u^\varepsilon(r_j g_0 x)\}$$

is also a tangent and hence maximal. We have proved the following.

**Proposition 10.13.** For all $g \in G$ and all $\varepsilon > 0$ the function $\max\{U^\varepsilon, g^* U^\varepsilon\}$ is $F$-harmonic.

As in the first proof we now apply elliptic regularity and Lemma 10.7 to conclude that each function $\max\{U^\varepsilon, g^* U^\varepsilon\}$ is $C^1$, and therefore $U^\varepsilon$ is constant on each sphere. Then by Theorem 7.8 $U^\varepsilon$ is an increasing radial harmonic and therefore a multiple of the Riesz kernel. Since $U^\varepsilon \to U$ in $L^1_{loc}$, we conclude that $U = \Theta^S(u) K(|x|)$. This completes the second proof in the orthogonally invariant case. Arguments for the complex and quaternionic analogous proceed as above.

**Proof of Lemma 10.11.** Let $U_j(x) \equiv r_j^{p-2} u(r_j x)$, so that $U_j \to U$ in $L^1_{loc}(\mathbb{R}^n - \{0\})$. Set $A = \{r \leq |x| \leq R\}$. Then

$$\|U_j^\varepsilon - U^\varepsilon\|_{L^1(A)} = \int_A \left| \int_G \{U_j(gx) \chi(g) - U(gx) \chi(g)\} \, dg \right| \, dx$$

$$\leq \int_G \int_A |U_j(gx) - U(gx)| \, dx \chi(g) \, dg$$

$$= \int_G \|g^* U_j - g^* U\|_{L^1(A)} \chi(g) \, dg$$

$$= \int_G \|U_j - U\|_{L^1(A)} \chi(g) \, dg = \|U_j - U\|_{L^1(A)}$$

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Thus \( \lim_{j \to \infty} U^\epsilon_j = \{ \lim_{j \to \infty} U_j \}^\epsilon \) as claimed.

**Proof of Lemma 10.12.** It is standard that the restriction of \( U^\epsilon \) to each sphere \( \{|x| = r\} \) is continuous (in fact, smooth). We see this as follows. Suppose \( x_j \to x \) in \( \{|x| = r\} \). By transitivity we can write \( x_j = g_j x \) where \( g_j \to 1 \) in \( G \). Then

\[
|U^\epsilon(x_j) - U^\epsilon(x)| = \left| \int_G U(gx_j) \chi(g) \, dg - \int_G U(gx) \chi(g) \, dg \right|
\]

\[
= \left| \int_G U(gg_j x) \chi(g) \, dg - \int_G U(gx) \chi(g) \, dg \right|
\]

\[
= \left| \int_G U(hx) \chi(hg_j^{-1}) \, dh - \int_G U(gx) \chi(g) \, dg \right|
\]

\[
\leq \int_G |U(gx)| \left| \chi(gg_j^{-1}) - \chi(g) \right| \, dg
\]

\[
\leq \left\{ \int_{\{|x| = r\}} |U(x)| \, dg \right\} \sup_{g \in G} \left| \chi(gg_j^{-1}) - \chi(g) \right| \to 0
\]

We also know that \( U^\epsilon \) is maximal, and in particular upper semi-continuous with \( S(U^\epsilon, t) \equiv \Theta K(t) \) for all \( t \).

Now for \( |x| = r \), \( g_0 \in G \), and any \( r_1 < r < r_2 \), the calculation above also shows that

\[
|U^\epsilon(g_0 x) - U^\epsilon(x)| \leq \sup_{r_1 \leq t \leq r_2} \left\{ \int_{\{|x| = t\}} |U(x)| \, dg \right\} \sup_{g \in G} \left| \chi(gg_0^{-1}) - \chi(g) \right|
\]

Now every \( y \) with \( |y| = t \) and \( |y - x| < \delta \) can be written as \( y = g_0 x \) with \( d(g_0, 1) < \epsilon(\delta) \) where \( \epsilon(\delta) \to 0 \) as \( \delta \to 0 \). Thus we have

\[
|U^\epsilon(y) - U^\epsilon(x)| \leq \sup_{r_1 \leq t \leq r_2} \left\{ \int_{\{|x| = t\}} |U(x)| \, dg \right\} \sup_{d(g_0, 1) < \epsilon(\delta)} \sup_{g \in G} \left| \chi(gg_0^{-1}) - \chi(g) \right| \leq C \varphi(\delta)
\]

for all \( |x| = t, |y| = t, |y - x| < \delta \) and \( r_1 \leq t \leq r_2 \). This shows that the family of functions

\[
V_t \equiv U^\epsilon(tx) \text{ is uniformly equicontinuous on the sphere } S_{n-1} = \{|x| = 1\}
\]

**Claim:** \( \lim_{t \to t_0} \sup_{S_{n-1}} |V_t - V_{t_0}| = 0 \).

**Proof.** Let \( t_j \to t_0 \) be any sequence. Then by the equicontinuity above, there is a subsequence such that \( V_{t_j} \) converges uniformly to a limit \( \tilde{V} \) on \( S_{n-1} \). We are done if we show that \( \tilde{V} = V_{t_0} \).
Now by the upper semi-continuity of $U^\epsilon$ we have

$$\tilde{V}(x) = \lim_{j \to \infty} V_{t_j}(x) = \lim_{j \to \infty} U^\epsilon(t_j x) \leq U^\epsilon(t_0 x).$$

However, we also have that

$$\int_{S^{n-1}} \tilde{V}(x) \, dx = \lim_{j \to \infty} \int_{S^{n-1}} V_{t_j}(x) \, dx = \lim_{j \to \infty} \int_{S^{n-1}} U^\epsilon(t_j x) \, dx = \int_{S^{n-1}} U^\epsilon(t_0 x) \, dx.$$

since the last two terms are just the averages $S(U^\epsilon, t_j) = \Theta K(t_j) \to \Theta K(t_0) = S(U^\epsilon, t_0).$

By the inequality (2) we conclude that $\tilde{V}(x) = U^\epsilon(t_0 x) = V_{t_0}(x)$ for all $x \in S^{n-1}$. Thus we have shown that $U^\epsilon$ is continuous for all $\epsilon$.

Now it is a general fact that $f^\epsilon \to f$ in $L^1_{\text{loc}}$. The proof is easy and the convergence is uniform when $f \in C_0^\infty$. The general case follows from the fact that $C_0^\infty$ is dense in $L^1$ on compact domains. This completes the proof of Lemma 10.12.

**Example 10.14.** If one drops the convexity hypothesis in Theorem 10.1, then in dimensions $n \geq 3$ there are orthogonally invariant subequations of every finite Riesz characteristic for which strong uniqueness fails. To see this we consider the largest such subequation of characteristic $p$:

$$F_p^{\min/\max} \equiv \{ A : \lambda_{\min}(A) + (p-1)\lambda_{\max}(A) \geq 0 \}.$$

(See Appendix A in Part II for a proof that there exists a largest and it is the one above.) To see that strong uniqueness fails for $F_p^{\min/\max}$ we consider the following functions. Write $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$, $m < n$ with coordinates $z = (x, y)$, and consider the function

$$u(x, y) \equiv K_p(|x|)$$

where $K_p$ is given by (3.8). Then $D^2 u = \frac{1}{|x|^p} (P_x + (p-1)P_x)$ has ordered eigenvalues

$$-(p-1) \frac{1}{|x|^p}, 0, \ldots, 0, \frac{1}{|x|^p}, \ldots, \frac{1}{|x|^p},$$

from which it is clear that $u$ is $F_p^{\min/\max}$-subharmonic on $\mathbb{R}^n$ and, in fact, $F_p^{\min/\max}$-harmonic for $x \neq 0$. Note that $u$ has Riesz homogeneity $p$ and is therefore its own tangent at points of the form $(0, y)$. Hence strong uniqueness fails for $F_p^{\min/\max}$.

Straightforward calculation shows, however, that these “partial Riesz kernels” are not subharmonic for the largest convex subequation of characteristic $p$ given in Proposition 10.9 above.
11. The Structure of the Sets $E_c$ where the Density is $\geq c$.

In this section we assume the subequation $F$ on $\mathbb{R}^n$ is convex with finite Riesz characteristic $p \geq 2$. Fix $u \in F(X)$ where $X$ is an open subset in $\mathbb{R}^n$. Let $\Theta = \Theta^V : X \to \mathbb{R}$ be the density function (for the volume function). For $c > 0$ define

$$E_c(u) \equiv \{x \in X : \Theta(x) \geq c\}.$$

For classical plurisubharmonic functions in $\mathbb{C}^n$ (where $F = \mathcal{P}^\mathbb{C}$), these sets have been of central importance. A deep theorem, due to L. Hörmander, E. Bombieri and in its final form by Siu ([Ho_1], [B], [Siu]), states that in this case $E_c$ is a complex analytic subvariety.

This strong corollary has a quite general extension.

**THEOREM 11.1.** Suppose strong uniqueness of tangents holds for $F$ (e.g., $F = \mathcal{P}_p$). Then for any $F$-subharmonic function $u$ the set $E_c(u)$ is discrete.

This result is essentially sharp. See Remark 11.2 below.

We will prove Theorem 11.1 in the following equivalent form. Consider an $F$-subharmonic function $u$ where $F$ has Riesz characteristic $p$ with $2 < p < \infty$.

**THEOREM 11.1’.** Suppose strong uniqueness of tangents holds for $u$ at a point $x_0$, that is, suppose that the $p$-flow of $u$ has limit

$$\lim_{r\downarrow 0} u_r(x_0; x) = \Theta K(|x - x_0|) \text{ in } L^1_{\text{loc}}(\mathbb{R}^n), \text{ for some } \Theta \geq 0. \quad (11.1)$$

Then

$$\lim_{x \to x_0} \Theta(u, x) = 0,$$

$$x \neq x_0$$

**Proof.** Suppose the conclusion fails. Then there exists a sequence $x_j \to x_0$ with $\Theta(u, x_j) \geq c > 0$ for all $j$. Assume $x_0 = 0$, and set $x_j = r_j \sigma_j$ with $r_j = |x_j|$. Then $r_j \to 0$, and passing to a subsequence we can assume that $\sigma_j \to \sigma \in S^{n-1}$. The idea now is to apply the sequence of $r_j$-homotheties to $u$. This will give a sequence $u_{r_j}$ of $F$-subharmonics with $\Theta(u_{r_j}, \sigma_j) \geq c$. With appropriate estimates from monotonicity, this will contradict (11.1).

To begin pick $\rho > 0$ small, and note that

$$V(u_{r_j}, \sigma_j, \rho) = \frac{V(u, x_j, r_j \rho)}{K(r_j \rho)} \quad (11.2)$$

since

$$V(u_{r_j}, \sigma_j, \rho) = \int_B u_{r_j}(\sigma_j + \rho x) \, dx = r_j^{p-2} \int_B u(x_j + r_j \rho x) \, dx$$

and

$$r_j^{p-2} \frac{1}{K(\rho)} = \frac{1}{K(r_j \rho)}.$$
Next we show that for all $j$

$$\frac{V(u, x_j, r_j \rho)}{K(r_j \rho)} \geq \frac{c}{2}. \quad (11.3)$$

In fact, this uniform bound from below, on the convergence of $\frac{V(u, x_j, t)}{K(t)}$ to $\Theta(u, x_j)$, independent of $x_j$, is obtained from the monotonicity property (Theorem 5.4) as follows. Set $\alpha \equiv 2^{\frac{1}{p-2}}$. Fix $x_j$ and abbreviate notation by setting $t = r_j \rho$ and $V(t) = V(u, x_j, t) = V(u, x_j, r_j \rho)$. We now apply the identity (cf. (5.15))

$$\frac{V(t)}{K(t)} = \left[ \frac{V(\alpha t) - V(t)}{K(\alpha t) - K(t)} \right] \left( \frac{1 - K(\alpha t)}{K(t)} \right), \quad (11.4)$$

with the constant $\alpha > 0$ chosen so that $\frac{K(\alpha t)}{K(t)} = \alpha^{-(p-2)} = \frac{1}{2}$. We assume $u$ and hence $V(t)$ is $\leq 0$ which can be obtained by subtracting a constant, or noting that $\lim_{x \to 0^+} u(x) = -\infty$ since $\Theta(u, 0) \geq c$ by Theorem 5.14.) Then $V(t) \leq V(\alpha t) \leq 0$ since $V(t)$ is increasing in $t$, which implies that the reciprocal of $1 - \frac{V(\alpha t)}{V(t)}$ is $\geq 1$.

By Theorem 5.4 this proves that, as desired,

$$\frac{V(t)}{K(t)} \geq \frac{c}{2}. \quad (11.3)'$$

Combining (11.2) and (11.3) we have

$$\frac{V(u_{r_j}, \sigma_j, \rho)}{K(\rho)} \geq \frac{c}{2}. \quad (11.5)$$

By the hypothesis (11.1) we have

$$\lim_{r_j \downarrow 0} V(u_{r_j}, \sigma_j, \rho) = \lim_{r_j \downarrow 0} \int_{B_{\rho}(\sigma_j)} u_{r_j} = \Theta \int_{B_{\rho}(\sigma)} K(|y|) \, dy. \quad (11.6)$$

Therefore, by (11.5)

$$-\rho^{p-2} \Theta \int_{B_{\rho}(\sigma)} K(|y|) \, dy \geq \frac{c}{2}. \quad (11.7)$$

Since

$$\lim_{\rho \to 0} \int_{B_{\rho}(\sigma)} K(|y|) \, dy = K(1) = -1,$$

this implies that $c = 0$, a contradiction. \hfill \blacksquare

**Remark 11.2.** For $F$ as above, any finite set can occur as the set $E_c$ for an $F$-subharmonic function. In fact, more is true. In a separate paper [HL14] we construct $F$-subharmonics with prescribed asymptotics at a finite set of points and prescribed boundary values.

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THEOREM 11.3. [HL_{14}]. Let $\Omega \subset \mathbb{R}^n$ be a domain with smooth boundary $\partial \Omega$ which is strictly convex (or more generally strictly $F$-convex (cf. [HL_{14}])). Let $E = \{x_j\}_{j=1}^N \subset \Omega$ be a finite subset, and $\{\Theta_j\}_{j=1}^N$ any set of positive real numbers. Then given any $\varphi \in C(\partial \Omega)$, there exists a unique $u \in \text{USC}(\overline{\Omega})$ such that:

1. $u$ is $F$-harmonic in $\Omega - E$,
2. $u|_{\partial \Omega} = \varphi$, and
3. $\Theta(u, x_j) = \Theta_j$ for $j = 1, \ldots, N$. 

12. Subequations with Riesz characteristic $1 \leq p < 2$.

When the Riesz characteristic satisfies $1 \leq p < 2$, the behavior and study of $F$-subharmonics differs greatly from the case $p \geq 2$. To begin, all $F$-subharmonics (not just the $F$-harmonics) are regular.

**THEOREM 12.1.** Suppose that $F$ is a (not necessarily convex) subequation with characteristic $p$, $1 \leq p < 2$. Then each $F$-subharmonic function is locally Hölder continuous with exponent $\alpha \equiv 2 - p$.

More specifically, suppose that $u$ is $F$-subharmonic on $B_{3r}(0)$. Then $u \in C^{0,\alpha}(B_r)$ with $\alpha$-Hölder norm

$$\|u\|_{C^{0,\alpha}(B_r)} \leq c_\alpha \frac{\sup_{B_{3r}(0)} u - u(0)}{(3r)^\alpha} \quad \text{where } c_\alpha \equiv \frac{1}{(\frac{2}{3})^\alpha - (\frac{1}{3})^\alpha} \quad (12.1)$$

(In particular, $u > -\infty$, i.e., $u$ is finite-valued.)

**Proof.** Suppose $|x| \leq r$ and $|y| \leq r$. Then $|x - y| \leq 2r$. Set $\delta = |x - y|$ and $s = 2r$. Then

$$\frac{u(x) - u(y)}{|x - y|^\alpha} \leq \frac{\sup_{B_\delta(y)} u - u(y)}{\delta^\alpha} \leq \frac{\sup_{B_s(y)} u - u(y)}{s^\alpha},$$

where the second inequality follows from the Monotonicity Theorem 5.3. Applying Monotonicity again with $0 < r < s$ gives

$$\frac{\sup_{B_s(y)} u - u(y)}{s^\alpha} \leq \frac{\sup_{B_s(y)} - \sup_{B_r(y)}}{s^\alpha - r^\alpha}.$$

Now since $|y| \leq r$, we have $B_{2r}(y) \subset B_{3r}(0)$ and $0 \in B_r(y)$. The fact that $0 \in B_r(y)$ together with the maximum principle implies that $u(0) \leq \sup_{B_r(y)} u$, i.e., $-\sup_{B_r(y)} u \leq -u(0)$. Therefore,

$$\frac{\sup_{B_s(y)} - \sup_{B_r(y)}}{s^\alpha - r^\alpha} \leq \frac{\sup_{B_{3r}(0)} - u(0)}{(2r)^\alpha - r^\alpha} = c_\alpha \frac{\sup_{B_{3r}(0)} u - u(0)}{(3r)^\alpha} \quad \blacksquare.$$

Lemma A.1 in part II states that $\mathcal{P}_p^{\min/\max} \equiv \{A : \lambda_{\min}(A) + (p - 1)\lambda_{\max}(A) \geq 0\}$ contains every subequation $F$ of characteristic $p$. Thus Theorem 12.1 can be restated as follows. Let $X \subset \mathbb{R}^n$ be open.

**THEOREM 12.1’.** $(0 < \alpha \leq 1)$. Suppose $u \in \text{USC}(X)$ satisfies the subequation

$$\lambda_{\min}(D^2 u) + (1 - \alpha)\lambda_{\max}(D^2 u) \geq 0 \quad \text{on } X,$$

in the viscosity sense. Then $u$ is locally $\alpha$-Hölder continuous on $X$.

**Remark 12.2.** The subequations $\mathcal{P}_p^{\min/\max}$ are never convex unless $p = 1$. In addition we have

$$\mathcal{P}_p^{\min/\max} \subset \Delta \quad \iff \quad 1 \leq 1 + \frac{1}{n-1} \quad \iff \quad \frac{n - 2}{n - 1} \leq \alpha \leq 1.$$

To see this note that $\lambda_1 + (p - 1)\lambda_n \geq 0 \Rightarrow \lambda_1 + \cdots + \lambda_n \geq 0$ if and only if $p - 1 \leq \frac{1}{n-1}$ since $\lambda_1 + \cdots + \lambda_n \geq (n-1)\lambda_1 + \lambda_n = (n-1)(\lambda_1 + \frac{1}{n-1}\lambda_n)$.  

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In the range $1 \leq p < 2$ the arguments for the existence and structure of tangents have a different flavor from the case $p \geq 2$. Suppose $F$ is an ST-invariant cone subequation (not necessarily convex) with characteristic $p$, $1 \leq p < 2$. Set $\alpha = 2 - p$. Recall by Definition 7.1(b) that $u_r(x) = \frac{1}{r^\alpha}(u(rx) - u(0))$.

**THEOREM 12.3.** Suppose $u$ is $F$-subharmonic on the ball $B_\rho$. Then for each $R > 0$ there exists a $\delta > 0$ such that the family $\{u_r\}_{0 < r \leq \delta}$ is bounded in norm on $C^{0,\alpha}(B_R)$. In fact,

$$\limsup_{r \downarrow 0} \|u_r\|_{C^{0,\alpha}(B_R)} \leq c_\alpha \Theta^M(u, 0).$$  \hfill (12.2)

**Proof.** Note that $u_r(0) = 0$ so that Theorem 12.1 states that the $\alpha$-Hölder norm of $u_r$ satisfies

$$\|u_r\|_{C^{0,\alpha}(B_R)} \leq c_\alpha \frac{\sup_{B_{3R}(0)} u_r - u_r(0)}{(3R)^\alpha},$$

and the RHS equals

$$c_\alpha \frac{\frac{1}{r^\alpha} \sup_{B_{3rR}(0)} u - \frac{1}{r^\alpha} u(0)}{(3R)^\alpha} = c_\alpha \frac{\sup_{B_{3rR}(0)} u - u(0)}{(3rR)^\alpha}.$$  

By Monotonicity this is increasing in $r$, and hence on an interval $0 < r \leq \delta$ it is maximized at $r = \delta$. That is,

$$\|u_r\|_{C^{0,\alpha}(B_R)} \leq c_\alpha \frac{\sup_{B_{3\delta R}(0)} u - u(0)}{(3\delta R)^\alpha} \quad \text{if } 0 < r \leq \delta. \hfill (12.3)$$

Finally, the RHS in (12.3) converges downward to $c_\alpha \Theta^M(u, 0)$ as $\delta \downarrow 0$.

**THEOREM 12.4.** Let $F$ and $u$ be as above. Then for every sequence $\{u_{r_j}\}_{j=1}^\infty$ with $r_j \downarrow 0$, there exists a subsequence which converges uniformly to an $F$-subharmonic function $U$ on $B_R$.

In fact by the standard compact embedding theorem, for each $\beta < \alpha$ there exists a subsequence which converges in the $\beta$-Hölder norm.

**F-Harmonicity of Tangents** ($1 \leq p < 2$).

By Theorem 6.2 every tangent is maximal, and by Fact 2 in Section 6, every continuous maximal function is $F$-harmonic. Thus the regularity result Theorem 12.1 implies the following.

**THEOREM 12.5.** Let $F$ and $u$ be as above. Then every tangent $U \in T_0(u)$ is $F$-harmonic in $\mathbb{R}^n - \{0\}$.

**Note 12.6.** Of course by Theorem 10.1 Strong Uniqueness of Tangents holds for every $O(n)$-invariant subequation with $p$ in this range, and for their complex and quaternionic counterparts. This includes the equation $P_p$.  

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Appendix A. Subaffine Functions and a Dichotomy.

For punctured radial subharmonics, i.e., a radial $F$-subharmonic function defined on a ball, there is a useful dichotomy between those which are increasing and those which are decreasing, which we now discuss. The subaffine equation $\tilde{\mathcal{P}} = \{ \lambda_{\text{max}} \geq 0 \}$ is an important special case, since it contains every subequation $F$ (including itself) for which the maximum principle holds. It is also a special case in that the radial subequation $R_{\tilde{\mathcal{P}}}$ on $(0, \infty)$ is constant coefficient. Using the jet variables $(\lambda, a)$, we have

$$R_{\tilde{\mathcal{P}}} = R_+ \times R_+ \equiv \{ (\lambda, a) : \text{ either } \lambda \geq 0 \text{ or } a \geq 0 \}. \tag{A.1}$$

It is important to note that the maximum principle holds for this one-variable subequation.

This subequation $R_+ \times R_+$ is more restrictive that one might guess. The next result shows that near the left endpoint of $(a, b)$ there is a dichotomy for a subharmonic. It is either increasing or it is convex and decreasing.

**Lemma A.1. (Increasing/Decreasing).** Suppose that $\psi$ is a general upper semi-continuous $R_+ \times R_+$-subharmonic function on an open interval $(a, b)$. Then either

1. $\psi$ is increasing on $(a, b)$, or
2. $\psi$ is decreasing and convex on $(a, b)$, or
3. $\exists c \in (a, b)$ such that $\psi$ is decreasing and convex on $(a, c)$ and increasing on $(c, b)$.

**Proof.** Suppose that $\psi$ is not increasing on all of $(a, b)$, that is, $\psi(r) > \psi(s)$ for some $a < r < s < b$. We claim that $\psi$ is decreasing on $(a, r)$. If not, there exist $r_1, r_2$ with $a < r_1 < r_2 < r$ and $\psi(r_1) < \psi(r_2)$. If $\psi(r_2) < \psi(r)$, then since $\psi(r) > \psi(s)$, $\psi$ has a strict maximum on $(r_2, s)$. Thus $\psi(r_2) \geq \psi(r) > \psi(s)$, and since $\psi(r_1) < \psi(r_2)$, we must have a strict maximum on $(r_1, s)$.

Suppose further that $\psi$ is not decreasing on all of $(a, b)$, that is, $\psi(s) < \psi(t)$ for some $r < s < t < b$. The argument above shows that there exists a maximal $c \in (s, t)$ so that $\psi$ is decreasing on $(a, c)$. Now $\psi$ must be increasing on $(c, b)$ for if not, it would have a strict interior maximum on that interval.

When $\psi$ is decreasing on $(a, c)$, it must be convex there. To see this let $\varphi$ be a test function for $\psi$ at $t_0 \in (a, c)$. Then $\varphi'(t_0) \leq 0$. If $\varphi'(t_0) = 0$, then $\psi(t) = \psi(t_0)$ for all $t \leq t_0$ near $t_0$. Therefore, $\varphi''(t_0) \geq 0$. On the other hand, if $\varphi'(t_0) < 0$, then $\varphi''(t_0) \geq 0$ because $\psi$ is $R_+ \times R_+$-subharmonic.

We say that the maximum principle (MP) holds for a subequation $F$ if it holds for all $F$-subharmonic functions.

**Theorem A.2.** The following conditions on a subequation $F \subset \text{Sym}^2(\mathbb{R}^n)$ are equivalent.

1. The maximum principle holds for $F$.
2. $F \subset \tilde{\mathcal{P}}$ (i.e., the subequation $\tilde{\mathcal{P}}$ is universal for (MP)).
3. $0 \notin \text{Int}F$. 

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(4) \( R_F \subseteq \mathbb{R}_+ \times \mathbb{R}_+ \).

**Proof.** Parts (1) – (3) were proved in [HL4, Lemma 2.2 and Proposition 4.8]. For part (4) note that \( F \subset \tilde{P} \Rightarrow (R_F)_{t} \subset (R_{\tilde{P}})_{t} = \mathbb{R}_+ \times \mathbb{R}_+ \). If \( F \) is not contained in \( \tilde{P} \equiv \{ A : \lambda_{\text{max}}(A) \geq 0 \} \), then there exists \( B < 0 \) with \( B \in F \). By positivity \( -\epsilon I \in F \) for some \( \epsilon > 0 \), which implies that \( (R_{\tilde{P}})_{t} \) is not contained in \( \mathbb{R}_+ \times \mathbb{R}_+ \). \( \blacksquare \)

These two results can be combined as follows.

**Corollary A.3.** If the (MP) holds for \( F \), then the conclusions (1), (2) and (3) of the Increasing/Decreasing Lemma A.1 hold for any radial \( F \)-subharmonic function \( u(x) = \psi(|x|) \) defined on an annulus. (In particular, if \( u \) is \( F \)-subharmonic on a ball, then \( \psi(t) \) must be increasing.)

**Proof.** By Theorem 2.4 and Theorem A.2, \( \psi \) is \( \mathbb{R}_+ \times \mathbb{R}_+ \)-subharmonic, and hence Lemma A.1 applies to \( \psi \). \( \blacksquare \)

**Appendix B. Uniform Ellipticity and \( \mathcal{P}(\delta) \).**

The point of this section is to make clear that viscosity harmonics for the subequation

\[
\mathcal{P}(\delta') = \{ A \in \text{Sym}^2(\mathbb{R}^n) : A + \delta \text{tr}(A) \geq 0 \} \quad \delta = \frac{\delta'}{n}
\]

are solutions to a uniformly elliptic equation \( F(D^2u) = 0 \) as defined in [CC], [T], [CIL], etc. We define the operator

\[
F : \text{Sym}^2(\mathbb{R}^n) \to \mathbb{R} \quad \text{by} \quad F(A) \equiv \lambda_{\text{min}}(A) + \delta \text{tr}(A).
\]

It is straightforward to verify that for all \( P \geq 0 \) one has

\[
\delta \text{tr}(P) \leq F(A + P) - F(A) \leq (1 + \delta) \text{tr}(P).
\]

which is one of the standard equivalent versions of uniform ellipticity for the operator \( F \) appearing in the sources above.

Now since \( \mathcal{P}(\delta') = \{ A : F(A) \geq 0 \} \),

it is completely straightforward to verify that a continuous function \( u \) is a viscosity solution of \( F(D^2u) = 0 \) if and only if \( u \) is \( \mathcal{P}(\delta') \)-harmonic.
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