Non-stationary layered vector fields and their divergence functions

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Abstract. This paper discusses the general properties of non-stationary layered vector fields and their functions of divergence. According to the classification of vector fields proposed in [1], layered vector fields are vortex vector fields that require two scalar functions to be defined. The main attention is paid to the study of a class of non-stationary layered vector fields that have the property of convergence to a stationary layered vector field. This means that these stationary layered vector fields can be considered as the result of the convergence process of non-stationary layered vector fields. It is shown that the condition for convergence of some non-stationary layered vector fields to a stationary layered vector field is that one of the two scalar functions that must be set to define a layered vector field belongs to the set of so-called scalar T-functions. T–functions are divided into two classes and the general properties of T–functions belonging to each of these classes are considered. Scalar characteristics of the mentioned non-stationary layered vector fields with properties of the Lyapunov function [2] are established.

1. Introduction

Three different approaches can be used to study the diffusion phenomenon. These approaches are the well-known physical [3] and hydrodynamic [4] approaches, as well as the so-called mechanical approach. In the mechanical approach, the phenomenon of diffusion is considered as the phenomenon of transfer of the amount of motion. The mechanical approach is related to the kinematic method of dividing a chemically homogeneous substance into two parts that exchange the amount of substance and the amount of motion.

The kinematic method of division makes it possible to divide the components of an inhomogeneous gas-dynamic mixture into two parts, for the names of which in [5] it is proposed to use the terms-normal component and diffusant. A distinctive feature of a diffusant is that its velocity field belongs to a set of layered vector fields.

Naturally, the normal component and the diffusant exchange matter and momentum. For the name of this type of exchange of matter and momentum, it is proposed to use the term – fractional type of exchange. When obtaining a ratio that allows us to quantify the intensity of this fractional exchange, it is important to take into account the general properties of the divergence functions of layered vector fields.
It is essential that to obtain characteristics of the instantaneous fractional exchange of matter and momentum between the diffusant and the normal component, it is sufficient to take into account the general properties of stationary layered vector fields and their divergence functions. While studying the processes of fractional exchange of matter and momentum, it is necessary to take into account the general properties of non-stationary layered vector fields and their divergence functions.

An important special case of fractional exchange processes is the processes that ensure the stability of stationary states to fluctuations in the concentration of matter and temperature [6]. In the study of the stability of thermodynamic states to fluctuations in the concentration of matter and temperature.

The main goal of this paper is to describe the general properties of non-stationary layered vector fields and their divergence functions. Special attention is paid to the study of general properties of non-stationary layered vector fields that have the property of convergence to a stationary layered vector field.

A sufficient condition for convergence of a non-stationary layered vector field to a stationary layered vector field is established. This condition is that one of the two functions used to define this non-stationary layered field [1] belongs to the class of so-called T-functions. In this regard, a significant part of this paper is devoted to the study of General properties of T-functions.

The paper discusses the general properties of two types T-functions of deviation: the macroscopic or global type T-functions of deviation and the local type T-functions of deviation. General properties of macroscopic or global T-functions of deviation are important in the study of physical phenomena and processes occurring in finite volumes of the considered part of space. While the general properties of local T-functions of deviation are used to describe physical phenomena and processes occurring in infinitesimal neighborhoods of the considered point in space.

2. The scalar distinctive hallmark of the convergence of non-stationary potential and layered vector fields

In [1], the scalar – vector approach to the study of general properties of scalar functions and vector fields is widely used. This approach is based on the establishment and use of invariants that remain valid when moving from considering general properties of scalar functions to general properties of vector fields and vice versa. The scalar-vector approach is important in establishing a distinctive hallmark of convergence of non-stationary potential and layered vector fields, since this feature is a scalar hallmark.

The possibility of using the scalar feature as a common distinctive hallmark of convergence non-stationary potential \( a^{\text{pot}}(x,t) \) and layered vector fields of the type \( a^{\text{lay}}(x,t) = f(x)\nabla \phi(x,t) \) (where \( f(x) > 0 \)) is due to the following circumstances.

For each convergent non-stationary layered vector field of type \( a^{\text{lay}}(x,t) = f(x)\nabla \phi(x,t) \) (where \( f(x) > 0 \)), we can assign a convergent sequence of stationary layered vector fields that has the form

\[
a^{\text{lay}}_i(x, t_0 + i\Delta t) = f(x)\nabla \phi(x, t_0 + i\Delta t) \quad (i \in \{1, \ldots, \infty\}) \tag{2.1}
\]

where \( \Delta t \) is the time step. While a convergent non-stationary potential vector field \( a^{\text{pot}}(x,t) \) can be matched to a convergent sequence of scalar functions \( \phi(x,t) \), using an expression of the form

\[
a^{\text{pot}}_i(x, t_0 + i\Delta t) = \nabla \phi(x, t_0 + i\Delta t), \quad (i \in \{1, \ldots, \infty\}) \tag{2.2}
\]

These two types of converging sequences must have a common distinctive hallmark, since formally the set of potential vector fields \( a^{\text{pot}} = \nabla \phi \) is no more than a subset of layered vector fields \( a^{\text{lay}} = f\nabla \phi \) that satisfy the condition \( f = 1 \).

Taking into account expressions (2.2), we can say that the existence of a common distinctive hallmark of convergence of potential vector fields \( a^{\text{pot}}(x,t_0+i\Delta t) \) and their generating sequences of scalar functions \( \phi(x,t_0+i\Delta t) \) is not in doubt.

It follows from the above that in order to establish a distinctive hallmark of convergence of sequences of potential \( a^{\text{pot}}(x,t_0+i\Delta t) \) and sequences of layered vector fields \( a^{\text{lay}}(x,t_0+i\Delta t) \), it is sufficient to establish a distinctive hallmark of convergence of sequences of scalar functions \( \phi(x,t_0+i\Delta t) \).
It can be shown that a distinctive hallmark of convergence of sequences of scalar functions \( \varphi(x, t_0 + i\Delta t) \), and consequently, sequences of potential \( \Delta P(x, t_0 + i\Delta t) \) and sequences of layered \( \Delta P(x, t_0 + i\Delta t) \) vector fields, is the condition that converging scalar functions \( \varphi(x, t) \) belong to the number of T–functions of deviation \( f^0_{T}(x,t) \).

### 3. Definition of T–functions and T-functions of deviation

By definition, T-functions are scalar functions that have a negative sign of the time derivative at the points of the maximum and a positive sign at the points of the minimum.

More precisely, each T-function \( f_{T}(x,t) \) is an element of the set of doubly differentiable scalar functions \( \{f_{T}(x,t)\} \), which have the values of their partial time derivatives \( \partial f_{T}(x,t)/\partial t \) at any internal point of the maximum \( x_{\text{max}} \) less than zero, and at any internal point of the minimum \( x_{\text{min}} \) more than zero

\[
\begin{align*}
\frac{\partial}{\partial t} f_{T}(x_{\text{max}},t) &< 0 \\
\frac{\partial}{\partial t} f_{T}(x_{\text{min}},t) &> 0
\end{align*}
\]

(3.1) (3.2)

By definition, the subset of the T-function of deviation \( f^0_{T}(x,t) \) belongs to T-functions whose value on the boundary \( s \) of the domain of their definition \( v \) is zero. This means that the condition that the T–function \( f_{T}(x,t) \) belongs to the subset of T–functions of deviation \( f^0_{T}(x,t) \) is a condition of the form

\[
f^0_{T}(x,t) \big|_{s_{<0}} = 0.
\]

(3.3)

In the introduction it was already mentioned that in this paper we discuss general properties of two types of T–functions of deviation: macroscopic or global type of T–functions of deviation having finite regions its definition, and the local T–functions of deviation, with infinitely small regions its definition.

The main General properties of the global T–function of the deviation of \( f^0_{T}(x,t) \) are:
- at \( t \to \infty \) values of the T–functions of deviation \( f^0_{T}(x,t) \) in all internal points of the domain of their definition should tend to zero, that is, \( f^0_{T}(x,\infty) = 0 \);
- if at some time \( t = t_0 \) the time derivative \( \partial f^0_{T}(x,t)/\partial t \) of the T–function of deviation \( f^0_{T}(x,t) \) has the constant sign at all internal points of the area of its definition, then the T–function of deviation \( f^0_{T}(x,t) \) at this time also has a constant sign inside this area. In this case, the sign of the T–function of deviation \( f^0_{T}(x,t_0) \) is opposite to the sign of its time derivative \( \partial f^0_{T}(x,t)/\partial t \);
- if the values of the T–function of deviation \( f^0_{T}(x,t) \) have a constant sign, then an inequality of the form must be fulfilled

\[
\left( \frac{\partial}{\partial t} f^0_{T}(x_{\text{min}},t) / \frac{\partial}{\partial t} f^0_{T}(x_{\text{max}},t) \right) \big|_{s=x_{\text{max}}} < 0
\]

(3.4)

where \( x = x_{\text{max}} \) is the inner point of the function definition area.

It is essential that the maximum modulus \( \max |f^0_{T}(x,t)| \) value of the T–function of deviation \( f^0_{T}(x,t) \), considered as a function of time \( f^0_{T}(x,t) = \max |f^0_{T}(x,t)| \), is one of the important invariant integral characteristics of this function. This characteristic and inequality (3.4) are used to establish the existence of a convergence criterion for macroscopic relaxation processes of heat and matter.

Along with the general properties of the T–functions of deviation, which have finite areas of their definition, the general properties of the local T–functions of deviation \( f_{\text{loc}}^0_{T}(x,t) \), whose areas of definition are spheres \( \Omega(\delta r_{n}) \) of infinitesimal radius \( \delta r_{n} \), are of important application.

It can be shown that the local T–functions of deviation \( f_{\text{loc}}^0_{T}(x,t) \) must have the following properties:
- the values of the local T–functions of deviation \( f_{\text{loc}}^0_{T}(x,t) \) have a second order of smallness;
- the local T–function of deviation \( f_{\text{loc}}^0_{T}(x,t) \) reaches its maximum value in modulus at the point \( x=0 \) in the center of its domain of definition;
- the values of modules of the gradients \( |\nabla f_{\text{loc}}^0_{T}(x,t)| \) local T–functions of deviation \( f_{\text{loc}}^0_{T}(x,t) \) have the first order of smallness;
- the sign of the local T–function of deviation \( f_{\text{loc}}^0_{T}(x,t) \) is opposite to the sign of its scalar Laplacian \( \Delta f_{\text{loc}}^0_{T}(x,t) \).
- an inequality of the form must be met

\[ f^0_{Tloc}(x,t)\Delta f^0_{Tloc}(x,t) < 0 \]  \hspace{1cm} (3.5)

Inequality (3.5) is important in the study of microscopic processes of spontaneous conductive relaxation of heat and matter.

4. Proof of belonging of solutions of a second-order linear parabolic equations to the number of T-functions

It is easy to see that the solution of a second-order linear parabolic equation having the form

\[
\frac{\partial}{\partial t} \varphi(x,t) = \text{div}(f(x)\nabla \varphi(x,t)) \quad (f(x) > 0)
\]  \hspace{1cm} (4.1)

belongs to the set of T-functions. To do this, first take into account that at the extremum points of the function \( \varphi(x,t) \), its gradient is zero (\( \nabla \varphi(x,t) = 0 \)). Therefore, in the vicinity of the extremum points, expression (4.1) degenerates into an expression of the form

\[
\frac{\partial}{\partial t} \varphi(x,t) \Bigg|_{\partial t} = f(x)\nabla \varphi(x,t) \quad (f(x) > 0)
\]  \hspace{1cm} (4.2)

Then we must take into account that the sign of the scalar Laplacian \( \Delta \varphi(x,t) \) of the function \( \varphi(x,t) \) is always negative at the points of maximum, and always positive at the points of minimum. It follows that at the points of maximum of the function \( \varphi(x,t) \), the sign of its partial derivative in time \( \frac{\partial \varphi(x,t)}{\partial t} \) is negative, and at the points of minimum it is positive. This was what had to be proved.

5. The amplitude of the global T function of the deviation as a Lyapunov function

It was mentioned above that the maximum modulus \( \max |f^0_T(x,t)| \) value of the global T-function of deviation \( f^0_T(x,t) \), considered as a function of time \( f_{\max}^0(t) = \max |f^0_T(x,t)| \) is one of the important invariant integral characteristics of this function. For the name of the function \( f_{\max}^0(t) \), it is proposed to use the term "function of the amplitude of the T-function of deviation ", and for its designation it is proposed to use the symbol \( A_{\max}^0(t) \).

It is easy to check that the function \( A_{\max}^0(t) \) is a monotonically decreasing function of time, that is, an inequality of the form is valid for this function

\[
\frac{\partial}{\partial t} |A_{\max}^0(t)| \Bigg|_{\partial t} < 0
\]  \hspace{1cm} (5.1)

This means that the function \( A_{\max}^0(t) \) has the main property of the Lyapunov function [2]. Recall that Lyapunov functions are strictly positive functions whose values monotonically decrease in time.

The introduction of the deviation amplitude function \( A_{\max}^0(t) \) allows us to obtain an alternative method for proving the stability of thermodynamic states to fluctuations in the concentration of matter and temperature. This method of proof is fundamentally different from the generally accepted method of proof, in which the second variation of entropy is used as the Lyapunov function [2].

6. Modulus of the divergent component of the Laplace derivative as a Lyapunov function

In the previous paragraph, it was said that the function \( A_{\max}^0_T(t) \), which is an important invariant characteristic of global T – functions of deviation \( f^0_T(x,t) \), has the properties of the Lyapunov function and can be used to prove the property of stability of thermodynamic states to fluctuations in the concentration of matter and temperature.

However, the function \( A_{\max}^0_T(t) \) practically loses its meaning when passing to the consideration of infinitesimal volumes, due to the fact that in this case it has a second order of smallness. In connection with the above, the question naturally arises about finding a function that could be considered as a mathematical analog of the function \( A_{\max}^0_T(t) \), and the values of which would have finite values.
It can be shown that the desired mathematical analog is the value of the modulus $|\varphi''_{\Delta 3 \text{div}}|$ of the divergent component $\varphi''_{\Delta 3 \text{div}}$ of the Laplace derivative $\varphi''_{\Delta 3}$ of the function $\varphi$ present in the expression $a^{\text{div}}_{\varphi}(x,t) = f(x)\nabla \varphi(x,t)$.

It can also be shown that an inequality of the form

$$\frac{\partial}{\partial t} |\varphi''_{\Delta 3 \text{div}}(t)| / \partial_t t < 0 \quad (6.1)$$

is just. This means that the function $|\varphi''_{\Delta 3 \text{div}}(t)|$ has the properties of the Lyapunov function.

In this formulation, the main local thermodynamic state means the thermodynamic state in which the considered thermodynamic system, provided that external influences are preserved, can remain for any length of time. In this case, the concept of a main local thermodynamic state is a generalization of the concepts of an equilibrium thermodynamic state and a non-equilibrium stationary thermodynamic state.

7. The application value of introducing the concepts of the T–function and the T–function of deviation

Twenty years ago, in the proceedings of one of the first conferences, an article was published by the author of this work [6], which formulated the principle of direction of spontaneous processes of establishing local ground states. One of the possible formulations of this principle is as follows: "the processes of local spontaneous fractional transformations of heat and matter are always directed towards the establishment of the main local thermodynamic states".

In this formulation, the main local thermodynamic state means the thermodynamic state in which the considered thermodynamic system, provided that external influences are preserved, can remain for any length of time. In this case, the concept of a main local thermodynamic state is a generalization of the concepts of an equilibrium thermodynamic state and a non-equilibrium stationary thermodynamic state. In [6], we proposed a form for recording the directivity criterion for spontaneous processes of establishing local ground states. In this paper, we obtain inequality (6.1), which is one of the important alternative forms of writing this directivity criterion.

In the previous paragraphs, we introduced the concept of a function of the deviation amplitude $A_{\max 0 T}(t)$ and obtained the inequality (5.1). The introduction of the deviation amplitude function $A_{\max 0 T}(t)$ allows us to obtain an alternative method for proving the stability of thermodynamic states to fluctuations in the concentration of matter and temperature. This method of proof is fundamentally different from the generally accepted method of proof, in which the second variation of entropy is used as the Lyapunov function [2].

An important application of introducing the concept of global T–functions of deviation $f_{\text{glob}}^{0 T}(x,t)$ is that their general properties are of key importance in the derivation of inequality (5.1). While an important application value of introducing local T–functions of deviation $f_{\text{loc}}^{0 T}(x,t)$ is that their general properties are of key importance in the derivation of inequality (6.1).

Conclusions

The main results of this work can be formulated as follows:
- it is shown that stationary layered vector fields can be considered as the result of establishing a certain type of non-stationary layered vector fields;
- a sufficient condition for the convergence of a non-stationary layered vector field to its stationary state is formulated;
- the concepts of "T-function", "global T–function of deviation " and "local T–function of deviation" are introduced and the general properties of these types of scalar functions are considered.
- it is shown that solutions of a linear parabolic equation of the second order belong to the class of T-functions
- it is shown that the value of the modulus of the amplitude of the T–function of deviation and the value of the modulus of the divergent Laplace derivative, considered as functions of time, have the properties of the Lyapunov function.
Inequalities are obtained, which from the physical point of view are forms of recording criteria for the direction of spontaneous processes of establishing local ground states. These inequalities are important in studying the properties of stability of ground states to fluctuations in temperature and concentration of matter.

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