A GEOMETRIC PERSPECTIVE ON LANDAU’S PROBLEM
OVER FUNCTION FIELDS

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Abstract. In this paper we improve on results obtained by Lior Bary-Soroker, Yotam Smilanski, and Adva Wolf in [2] which deals with finding asymptotics for a function field version on sums of two squares in the large degree limit or large \( q \). An asymptotic in the \( q^n \to \infty \) has been recently obtained by Ofir Gorodetsky in [10]. The main tool is a new twisted Grothendieck Lefschetz trace formula, inspired by the paper [4]. Using a combinatorial description of the cohomology we obtain a precise quantitative result, a new type of homological stability phenomena and we recover the \( q^n \to \infty \) asymptotic result.

1. Introduction

The present paper is motivated by the results obtained in [2] in which the authors state and prove a function field analogue of Landau’s theorem about sums of two squares.

Let \( \mathbb{F}_q \) be the finite field with \( q \) elements where \( q \) is an odd integer. We denote with \( \mathcal{M}_{n,q} \) the set of monic polynomials of degree \( n \) in \( \mathbb{F}_q[T] \) and also we shall need the following definition

**Definition.** Let \( q \) be an odd prime power. For a polynomials \( f \in \mathcal{M}_{n,q} \) we define the characteristic function:

\[
\chi_q(f) = \begin{cases} 
1, & \text{if } f(T) = A^2 + TB^2 \text{ for } A, B \in \mathbb{F}_q[T] \\
0, & \text{otherwise.}
\end{cases}
\]

and the counting function \( B_q(n) = \sum_{f \in \mathcal{M}_{n,q}} \chi_q(f) \)

The following two theorems about the asymptotic of \( B_q(n) \) are obtained by Lior Bary-Soroker, Yotam Smilansky and Adva Wolf in [2]

**Theorem (SSW).**

\[
B_q(n) = \frac{1}{4^n} \binom{2n}{n} q^n + \left( \frac{1}{2} \cdot \frac{1}{4^{n-1}} \binom{2(n-1)}{(n-1)} + \frac{1}{4^{n-1}} \binom{2(n-2)}{(n-2)} \right) q^{n-1} + \mathcal{O}_q(q^{n-2}), \quad q \to \infty
\]

**Theorem (SSW).**

\[
B_q(n) = K_n \cdot q^n + \mathcal{O}_q \left( \frac{q^n}{n^{3/2}} \right), \quad n \to \infty
\]

where

\[
K_n = (1 - q^{-1})^{-\frac{1}{2}} \prod_{(\frac{P}{q}) = -1} (1 - |P|^{-2})^{-\frac{1}{2}}
\]

Here \( \binom{a}{b} \) is the Legendre symbol.
Recently in [10] the dependency on \( q \) in the error term in this second theorem was removed by using a generating functions technique.

What we shall prove is an expansion on the first theorem above, namely

**Theorem 1.** For every \( n \geq 2 \) we can write
\[
B_q(n) = \sum_{k=0}^{n} b_{k,n} q^{n-k}
\]
such that
\[
a) \quad b_{k,n} = \sum_{j=k}^{2k} \delta_{k,j,n} \frac{(2(n-j))}{4^{n-j}};  \\
b) \quad We \ have \ that \ \delta_{k,j,n} = \delta_{k,j,n+1} \ for \ n \geq 2k \ and \ \\
|\delta_{k,j,n}| \leq C(1.1)^k
\]
for some absolute constant \( C \).

Next let us make some remarks connecting the theorem we stated with previous results.

**Remarks**

- Part a) of our theorem is a generalization and gives a complete description of the statement of the first theorem stated above.
- Our theorem also could be generalized to count norms in the extension \( \mathbb{F}_q[\sqrt[d]{T}]/\mathbb{F}_q[T] \) provided \( q \equiv 1 \ (\text{mod } d) \). Henderson’s result, [11], covers all groups \((\mathbb{Z}/d\mathbb{Z})^n \rtimes S_n\). We shall only use it for \( d = 2 \) in our proof. The only changes in the combinatorics that need to be made are in propositions 10,11 and 12. The binomials we have in the above theorem will be replaced by \( \frac{(-1/d)}{n} \). This will be an analogue of the result of Odoni, [14], about which positive integers are a norm in the number field \( K \); in our case it will be the \( d \)-th cyclotomic field.
- In connection with section 5.2 in [3], the polynomials which can be written as \( g^d - Th^d \) will be a subvariety of this space of norms.
- In the course of the proof of the theorem we shall give a geometric interpretation to the binomial coefficients appearing in the expansion.
- The stabilization of the coefficients \( \delta_{p,j,n} \) as \( n \) gets large with respect to \( p \) is explained by the stabilization of the multiplicity of a character paired against the cohomology of a certain space and thus can be viewed thus a homological stabilization result. As remarked in [3], the class function which guides this statistic is not a character polynomial so it would be an interesting combinatorial problem to classify all types of possible stabilizations that could arise.
- Our result is in direct connection with Remark 4.1 in [10], namely
\[
B_q(n) = \sum_{i=0}^{2d-2} q^{n-i} \left( \frac{n-i-1}{2} \right) [x^i] \exp \left( \sum_{j \geq 1} \frac{e_j x^j}{j} \right) + O_n(q^{n-d})
\]
where \([x^i]\) represents the coefficient of \( x^i \) in the taylor series expansion of the exponential and \( e_n = \frac{1}{2} + \sum_{i=1}^{v_2(n)} \frac{q^{n/2^i} - 1}{2} \) where for a given natural number \( x \), \( v_2(x) \) is the valuation of 2 in \( x \).
As the reader can notice it would require some combinatorial manipulations to obtain the same form as the result in theorem 1 since every exponential also involves \( q \).

- Another interesting connection between our theorem 1 and all the other results that can be explored further would be to make use of the following binomial expansion from Yudell L. Luke’s book ([18]):

\[
\binom{x}{n} = (-1)^n \cdot n^{-(x+1)} \sum_{k=0}^{\infty} \frac{(x + 1)_k B_k^{(-x)}}{k! n^k}
\]

Here we denote with \( \Gamma(y) \) the usual gamma function, \((y)_k\) is the lower factorial and \( B_k^{(-x)} \) are generalized Bernoulli numbers. If we set \( x = -\frac{1}{2} \) then we obtain

\[
\binom{2n}{n} = \frac{4^n}{\sqrt{\pi n}} \left( 1 + \frac{c_1}{n} + \frac{c_2}{n^2} + \ldots + \frac{c_r}{n^r} + O \left( \frac{1}{n^{r+1}} \right) \right)
\]

where \( c_1, \ldots, c_r \) which can be computed from the above expansion of the binomial.

The difficulty relies on the fact that we need to obtain an asymptotic expansion of sums of central binomial coefficients with exponentially decaying coefficients, i.e

\[
\sum_{i=0}^{k} a_i \binom{2(n-i)}{n-i},
\]

where \( k \) might also have growth with respect to \( n \).

This theorem will be a consequence of theorem 2, but before stating it properly we need to make some definitions.

**Definition.** Let \( q \) an odd prime power. For a polynomials \( f \in \mathcal{M}_{n,q} \) we define the characteristic function:

\[
s_q(f) = \begin{cases} 
1, & \text{if } f(T) = A^2 - TB^2 \text{ for } A, B \in F_q[T] \\
0, & \text{otherwise.}
\end{cases}
\]

Let \( S_{n,q} \subset \mathcal{M}_{n,q} \) be the set of monic square free polynomials.

**Definition.** Define the counting functions \( S_q(n) = \sum_{f \in S_{n,q}} s_q(f) \) and \( S_q^0(n) = \sum_{f \in S_{n,q}, f(0) \neq 0} s_q(f) \).

Equivalently we could say that \( s_q(f) = 1 \) if \( f \) is a norm in the function field extension \( F_q[\sqrt{T}]/F_q[T] \). Also note the obvious relation

\[
S_q(n) = S_q^0(n) + S_q^0(n - 1)
\]

since any monic squarefree either is non zero at zero or vanishes with order 1, namely \( f/T \) would be monic squarefree polynomial of degree \( n - 1 \) and nonvanishing at zero.

Our next theorem will concern finding an asymptotic for \( S_q^0(n) \) as \( q^n \to \infty \).

We thus have the following
Theorem 2. For every \( n \geq 2 \) we can write \( S_0^q(n) = \sum_{i=0}^{n} (-1)^i c_{i,n} q^{n-i} \). Moreover

a) \( c_{i,n} = \sum_{j=0}^{i} \Gamma_{i,j,n} \left( \frac{2(n-j-i)}{n-j-i} \right) \) where \( \Gamma_{i,j,n} \) are rational numbers.

b) \( \Gamma_{i,j,n} = \Gamma_{i,j,n+1} \) for \( n \geq 2i \) and we have the following bound

\[ |\Gamma_{i,j,n}| \leq B \cdot (1.1)^j \]

for some absolute constant \( B \).

Our strategy for the proof of the above theorem is a twisted Grothendieck Lefschetz formula inspired by [4]. Recently this formula was independently in [15] and generalized in [3]. In a different direction a similar formula was proved for ramified coverings [8].

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2. Preliminaries

For a squarefree polynomial \( f \in \mathcal{M}_{n,q} \) with \( f(0) \neq 0 \) let the unordered \( n \) tuple of its roots be \( \{z_1, \ldots, z_n\} \) where \( z_i \neq z_j \) and \( z_i \neq 0 \). Since \( F_q \) fixed the coefficients of the polynomial this induces a permutation on the roots of \( f \). For each \( z_i \) pick an \( x_i \in \overline{F}_q \) such that \( x_i^2 = z_i \). It follows that \( \text{Frob}_q \) induces a signed permutation on these representatives. Call \( R_n \) the space of all tuples \( \{x_1, \ldots, x_n\} \) and we shall also denote with \( G_n \) the signed permutation group.

Now we can restate \( b_q(f) = 1 \) if the two roots of \( x^2 = z_i \) lie in different orbits of the \( \text{Frob}_q \) action on the space \( R_n \), thought as the space of points on the tuples \((x_i, -x_i)\) where these are the roots of \( x^2 = z_i \).

Definition. Let \( L_n \) be the subset of \( G_n \), consisting of all signed permutation \( \pi \) such that \( x_i \) and \( -x_i \) lie in different orbits under \( \pi \).

It is now time to relate the geometry of the space of roots and our counting problem. Note the fact that \( z_i \neq z_j \) imposes that \( x_i \neq \pm x_j \).

I claim that now we can identify \( R_n \) as a hyperplane complement in affine \( n \) space,

\[ R_n = \{(a_1, \ldots, a_n)| a_i = \pm a_j, a_i \neq 0 \} \]. This is because for each \( f \in \mathcal{M}_{n,q} \) considering the tuple \( \{x_1, \ldots, x_n\} \) we can see this is a point of \( R_n \) over \( \overline{F}_q \). The representation theory and homological stability properties of this hyperplane arrangement are well understood; the interested reader can look at [17].

Definition 3. Let \( \chi_n \) be the characteristic function of \( L_n \) as a subset of \( G_n \).

We shall prove a theorem which relates the geometry of our space and the counting problem, which is the same spirit as theorem 3.7 in [3]. The main difference is to state and prove an analogous result for \( G_n \) Galois covers instead of \( S_n \) covers.

To make it more explicit, if we consider a class function \( \chi : G_n \rightarrow \mathbb{Q} \) then we can define its action on a squarefree polynomial \( f \) in the following way: set \( R(f) = \{z_1, z_2, \ldots, z_n\} \) to be sets of roots and we have an induced action of \( \text{Frob}_q \) on
the set of squareroots of these, namely \( \{x_1, x_2, \ldots, x_n\} \) as above and this will give us a signed permutation \( \sigma_f \). We define \( \chi(f) = \chi(\sigma_f) \) and we need to argue this is well defined. By forgetting the signs on the signed permutation, we recover the action of Frob\( _q \) on \( R(f) \) and this has invariant cycle structure since cycles correspond to irreducible factors of \( f \). Since conjugation preserves the cycle structure we are done.

**Theorem 4.** Let \( G_q(n) = \sum_{f \in S_n, f(0) \neq 0} \chi(f) \). Then
\[
G_q(n) = \sum_{i} (-1)^{i}q^{n-i}(\chi, H_{q}^{i}(R_{n}; \mathbb{Q}))_{G_n}
\]

Here \( \langle \cdot , \cdot \rangle \) denotes the standard product of class functions and the subscript \( G_n \) denote the groups where the respective class functions live.

As a corollary for our problem

**Corollary** Applying the above theorem to our special case we have
\[
S_q^0(n) = \sum_{i} (-1)^{i}q^{n-i}(\chi, H_{q}^{i}(R_{n}; \mathbb{Q}))_{G_n}
\]

### 3. Proof of Theorem 4

This is the main technical machinery to setup and prove for our problem. Let \( \text{Conf}_n^0 \) be the affine complement \( U_n = \{ (z_1, z_2, \ldots, z_n) | z_i \neq z_j, z_i \neq 0 \} \) modulo \( S_n \). First let us argue that \( R_n \) is a étale Galois cover of \( \text{Conf}_n^0 \) with Galois group \( G_n \).

Let \( P_n \) be the set of monic degree \( n \) polynomials which are split in the extension \( F_q[\sqrt{T}] / F_q[T] \) and which do not vanish at 0. Consider the map
\[
\pi : \mathbb{A}^n \to P_n
\]
defined by
\[
\pi : (x_1, x_2, \ldots, x_n) \to f(T) = (T - x_1^2)(T - x_2^2) \cdots (T - x_n^2)
\]

The map is well defined using again theorem 2.5 in [2]. Note that the map is invariant under the \( G_n \) action on the points \( (x_1, x_2, \ldots, x_n) \) thus it factors through the scheme theoretic quotient \( \mathbb{A}^n / G_n \). We prove moreover that actually the map
\[
\pi : \mathbb{A}^n / G_n \to P_n
\]
is an isomorphism. The \( G_n \) invariant functions on \( \mathbb{A}^n \) form a ring, namely \( \mathbb{Z}[x_1, x_2, \ldots, x_n]^{G_n} \). First note that if such a function is invariant to switching signs on the \( x_i \)’s then it has to be a polynomial in \( x_i^2 \)’s. Thus \( \mathbb{Z}[x_1, x_2, \ldots, x_n]^{G_n} = \mathbb{Z}[x_1^2, x_2^2, \ldots, x_n^2]^{S_n} \). As a function of \( x_i \), the coefficient \( a_i \) in \( f \) is \( \pm \) the \( i \)th symmetric polynomial \( e_i(x_1^2, x_2^2, \ldots, x_n^2) \). The fundamental theorem of symmetric polynomials states
\[
\mathbb{Z}[x_1^2, x_2^2, \ldots, x_n^2]^{S_n} = \mathbb{Z}[e_1, \ldots, e_n] = \mathbb{Z}[a_1, a_2, \ldots, a_n]
\]
thus giving the desired isomorphism.

Under this map we can look thus at the preimage of \( \text{Conf}_n^0 \) and since this space can be identified with monic squarefree degree \( n \) polynomials which do not vanish at zero, it can be easily seen that this preimage is \( R_n \). Since we can define \( R_n \) in \( \mathbb{A}^n \) as nonvanishing of integral polynomials, \( R_n \) is a smooth \( n \) dimensional scheme over \( \mathbb{Z} \).
Since $G_n$ acts freely on $R_n$ by definition, restricting $\pi$ to a map $R_n \to \text{Conf}^{0}_n$ gives an étale Galois cover with Galois group $G_n$.

Now moving further note the fact that the Galois cover $R_n \to \text{Conf}^{0}_n$, gives a natural correspondence between finite-dimensional representations of $G_n$ and finite-dimensional local systems (locally constant sheaves) on $\text{Conf}^{0}_n$ that become trivial when restricted to $R_n$. Given $V$ a representation of $G_n$, let the $\chi_V$ be the associated character to it and let $V$ be the corresponding local system on $\text{Conf}^{0}_n$. Initially this construction is done over $\mathbb{C}$ but since since every irreducible representation of $G_n$ can be defined over $\mathbb{Z}$ (see [9,5]), the local system $V$ determines an $l$-adic sheaf and we shall not make a distinction between the two objects.

If $f(T) \in \text{Conf}^{0}_n$, and is a fixed point for the action of Frob$_q$ on $\text{Conf}^{0}_n(\mathbb{F}_q)$ then Frob$_q$ acts on the stalk $V_f$ over $f$. To give a concrete description, the roots of $f(T)$ are permuted by the action of Frobenius on $\mathbb{F}_q$, and moreover this induces a signed permutation on the square roots of the roots of the polynomial $f$, $\sigma_f$ which is defined up to conjugacy. The stalk $V_f$ is isomorphic to $V$, and by choosing an appropriate basis the automorphism Frob$_q$ acts according to $\sigma_f$. Thus we can conclude

$$\text{tr}(\text{Frob}_q : V_f) = \chi_V(\sigma_f) \quad (1)$$

The next ingredient we need is a version of the Grothendieck-Lefschetz trace formula with twisted coefficients. Namely for an appropriate system of coefficients $\mathcal{F}$ on a smooth projective variety $X$ defined over $\mathbb{F}_q$ (more precisely terminology is a $l$-adic sheaf), we have:

$$\sum_{x \in X(\mathbb{F}_q)} \text{tr}(\text{Frob}_q : |F_x|) = \sum_{i} (-1)^i \text{tr}(\text{Frob}_q : H^i_\text{et}(X ; \mathcal{F})).$$

This also holds for non-projective $X$, but we need to correct it by either using compactly supported cohomology or via Poincaré duality.

If we apply to our case using compactly supported cohomology we have that

$$\sum_{f \in \text{Conf}^{0}_n(\mathbb{F}_q)} \text{tr}(\text{Frob}_q : |V_f|) = q^n \sum_{i} (-1)^i \text{tr}(\text{Frob}_q : H^i_c(\text{Conf}^{0}_n ; V)) \quad (2)$$

Notice that the left hand side is exactly the statistical count on polynomials we need using (1). The only thing left to unravel is the right hand side of the equality.

First let us make some remarks about the setup. If $V$ is an $G_n$ representation, we denote by $\langle \chi, V \rangle$ the standard inner product of $\chi$ with the character of $V$; we can name this the multiplicity of $\chi$ in $V$, since this is true when $\chi$ is irreducible, by Schur’s lemma. Also note that for any class function on $G_n$ we can decompose it into a sum of irreducible characters and since both sides in (2) are linear in $\chi$, it follows that we can reduce to the case of an irreducible character $\chi$ of $G_n$.

Let $\hat{V}$ denote the pullback of $V$ to $R_n$. Transfer gives us the isomorphism $H^i_c(\text{Conf}^{0}_n ; \hat{V}) \approx (H^i_c(R_n ; \hat{V}))^{G_n}$. Now we know that $\hat{V}$ is trivial on $R_n$, so we have

$$H^i_c(R_n ; \hat{V}) \approx H^i_c(R_n ; \mathbb{Q}_l) \otimes V$$

as $G_n$ representations. Putting it together

$$H^i_c(\text{Conf}^{0}_n ; V) \approx (H^i_c(R_n ; \mathbb{Q}_l) \otimes V)^{G_n} \approx H^i_c(R_n ; \mathbb{Q}_l) \otimes \mathbb{Q}[G_n] V$$
Now this gives the immediate consequence that dim($H^j_{\text{et}}(\text{Conf}^n_\ell ; V)) =$ dim($H^j_{\text{et}}(R_n; \mathbb{Q}_\ell) \otimes_{\mathbb{Q}[G_n]} V$).

Since $V$ is self-dual as an $S_n$ representation, $H^j_{\text{et}}(R_n; \mathbb{Q}_\ell) \otimes_{\mathbb{Q}[G_n]} V$ is isomorphic to Hom$_{\mathbb{Q}[G_n]}(V; H^j_{\text{et}}(R_n; \mathbb{Q}_\ell))$, whose dimension is computed using the inner product $\langle \chi, H^j_{\text{et}}(R_n; \mathbb{Q}_\ell) \rangle$.

Since $R_n$ is smooth of dimension $n$, applying Poincaré duality gives

$$H^{2n-i}_{\text{et}}(R_n; \mathbb{Q}_\ell) \cong \text{Hom}(H^i_{\text{et}}(R_n; \mathbb{Q}_\ell); \mathbb{Q}_\ell(-n)).$$

Since the action of $G_n$ on $\mathbb{Q}_\ell(-n)$ is trivial (this is the constant sheaf), we have that $\langle \chi, H^{2n-i}_{\text{et}}(R_n; \mathbb{Q}_\ell) \rangle = \langle \chi, H^i_{\text{et}}(R_n; \mathbb{Q}_\ell) \rangle$. The last layer to uncover is the action on Frob$_\ell$.

We shall need the following theorem proved by Kim for a general field in [12] (see also Lehrer [13]).

**Theorem.** Let $k$ be a field, and fix $l$ a prime different from the characteristic of $k$. Give a finite set of hyperplanes $H_1, \ldots, H_m$ in $\mathbb{A}^n$ defined over $k$, let $\mathcal{A}$ the complement: $\mathcal{A} = \mathbb{A}^n - \bigcup_{i=1}^m H_i$. Then:

(i) $H^i_{\text{et}}(\mathcal{A}; \mathbb{Q}_\ell)$ is spanned by the images of the $m$ maps:

$$H^1_{\text{et}}(\mathcal{A}^n - H_j; \mathbb{Q}_\ell) \rightarrow H^1_{\text{et}}(\mathcal{A}; \mathbb{Q}_\ell)$$

induced by the inclusion of $\mathcal{A}$ into $\mathbb{A}^n - H_j$ for $j = 1, \ldots, m$.

(ii) $H^i_{\text{et}}(\mathcal{A}; \mathbb{Q}_\ell)$ is generated by $H^1_{\text{et}}(\mathcal{A}; \mathbb{Q}_\ell)$ under cup product.

As a consequence this theorem will give that the action of Frob$_\ell$ on $H^i_{\text{et}}(R_n; \mathbb{Q}_\ell)$ is scalar multiplication by $q^i$. The action of Frob$_\ell$ on $\mathbb{Q}_\ell(-n)$ is scalar multiplication by $q^n$ so the action of Frob$_\ell$ on $H^{2n-i}_{\text{et}}(\text{Conf}^n_\ell ; V))$ is scalar multiplication by $q^{n-i}$.

Putting it all together we obtain that

$$\text{tr}(\text{Frob}_\ell : H^{2n-i}_{\text{et}}(\text{Conf}^n_\ell ; V)) = q^{n-i} \langle \chi, H^i_{\text{et}}(R_n; \mathbb{Q}_\ell) \rangle$$

4. **Computation of the inner products**

To finish to proof of Theorem 2, note that according to Theorem 4 we just need to understand the inner product $\langle \chi_n, H^i_{\text{et}}(R_n; \mathbb{Q}_\ell) \rangle_{G_n}$

Note that we can use instead of the etale cohomology singular cohomology over $\mathbb{C}$, since for hyperplane arrangements, the cohomology depends only on the lattice of intersection of the hyperplane arrangement.

To proceed to actual computations we shall need the following result in [11] which gives a description of $H^i(R_n; \mathbb{C})$ as a $G_n$ representation. A similar result was obtained by [17].

**Theorem (Henderson).** As a representation of $G_n$, $H^p(R_n; \mathbb{C})$ is equal to $\bigoplus_{0 \leq i \leq p} A^i(R_n)$ where $\varepsilon_n \otimes A^i(R_n)$ is isomorphic to the following direct sum:

$$\bigoplus_{\lambda_1, \lambda_2 \vdash |\lambda_1| + |\lambda_2| = n \atop l(\lambda_1) = n-p \atop l(\lambda_2) = l} \text{Ind}_{G_1}^{G_n}((\mu_2 \times \mu_{\lambda_2}) \times \cdots \times (\mu_2 \times \mu_{\lambda_2-p})) \times (S_{\mu_1(\lambda_1)} \times S_{\mu_2(\lambda_2)} \times \cdots) \langle \varepsilon \psi \rangle \times (S_{\mu_1(\lambda_2)} \times S_{\mu_2(\lambda_2)} \times \cdots)$$
where, $\lambda^1 = (\lambda_1^1, \ldots, \lambda_{n-p}^1)$, $\lambda^2 = (\lambda_1^2, \ldots, \lambda_{n-p}^2)$, $|\lambda^1| = \lambda_1^1 + \ldots + \lambda_{n-p}^1$ and similarly for $\lambda^2$. $\psi$ is the product of the standard inclusion characters $\mu_{\lambda_a} \rightarrow \mathbb{C}^\times$ and $\varepsilon$ is the product of the sign characters of the $S_{m_n(\lambda^1)}$ components.

The space $G_n$ can be thought of generalized permutation matrices where in each entry we replace the usual 1 with now a $\pm v$ and its negative. Note that it’s actually sufficient for we can take as generators the $v$ group.

Definition 5. A cell is a factor of the type $\mu_2 \times \mu_v$.

For constructing a matrix representative of the group $\mu_2 \times \mu_v = C_v$ note that we can take as generators the $v \times v$ matrix

$\mathfrak{g}_v = \begin{cases} g_{i+1, i} = 1 & \text{for } 1 \leq i \leq v \text{ where index is taken modulo } v \\ 0, & \text{otherwise} \end{cases}$

and it’s negative. Note that it’s actually sufficient for $v$ odd to take $-\mathfrak{g}_v$ since the group $\mu_2 \times \mu_v$ is cyclic.

Definition 6. A block is a factor of the type $(\mu_2 \times \mu_v)^{m_v} \times S_{m_v}$.

Obviously we can think of blocks as a generalized permutation group on its cells. To construct $H_{\lambda^1, \lambda^2}$ we first arrange the blocks in descending order along the diagonal, first those for $\lambda^1$ by reading for each $1 \leq v \leq n$ it’s multiplicity, say it is $m_{1,v}$, in the partition $\lambda^1$ and putting the block $(\mu_2 \times \mu_v)^{m_{1,v}} \times S_{m_{1,v}}$, and then proceed in a similar fashion for $\lambda^2$.

Now we proceed to the actual computation of the inner products $\langle \chi_n, H_p(X_n; \mathbb{C}) \rangle_{G_n}$.

By theorem 5 and Frobenius reciprocity it is equivalent to computing

$$\langle \text{Res}_{H_{\lambda^1, \lambda^2}} G_n \chi_n, \text{Res}_{H_{\lambda^1, \lambda^2}} G_n \varepsilon_n \otimes \varepsilon \psi \rangle$$

for each $\lambda^1$, $\lambda^2$ subject to the constraints in theorem 5.

First let us look closely at $\text{Res}_{H_{\lambda^1, \lambda^2}} G_n \varepsilon_n$.

Suppose the the block structure of $H_{\lambda^1, \lambda^2}$ is given by the blocks $B_1$, $B_2$, $\ldots$, $B_j$. Then noting that there is natural identification of $\varepsilon_n$ with the determinant of the corresponding permutation matrix

Proposition 7. We have that $\varepsilon_n = \det(B_1) \otimes \det(B_2) \otimes \ldots \otimes \det(B_j)$.

We can restate this proposition also for our inner product

Proposition 8.

$$\langle \text{Res}_{H_{\lambda^1, \lambda^2}} G_n \chi_n, \text{Res}_{H_{\lambda^1, \lambda^2}} G_n \varepsilon_n \otimes \varepsilon \psi \rangle = \langle \chi_{B_1}, \det \otimes (\varepsilon) |_{B_1} (\psi) |_{B_1} \rangle \cdots \langle \chi_{B_j}, \det \otimes (\varepsilon) |_{B_j} (\psi) |_{B_j} \rangle |_{B_j}$$

Remark Here by abuse of notation we denote with $\chi_B$ denotes the set of allowable signed permutations induced by the action of $\text{Frob}_B$.

Further let us say a given block $B$ is given by the factor of the type $(\mu_2 \times \mu_v)^{m_v} \times S_{m_v}$. Let us denote with $C_1$, $C_2$, $\ldots$, $C_{m_v}$ the cells composing this block. Also we ignore the signs on each cell. Then we have

Proposition 9. $\det(\mathcal{B}) = \det(C_1) \otimes \det(C_2) \otimes \ldots \otimes \det(C_{m_v}) \otimes (\varepsilon_{m_v})^v$
Suppose that Proposition 11.

We just have to note that to bring to diagonal form the block if we just think of cells as a unit we would need an even or an odd number of moves to diagonalize according to the sign of \( \varepsilon_{m_v} \). Since cells are \( v \times v \) dimensional, to switch places of cells requires \( v \) moves. Thus the total number of moves needed to bring each cell on the diagonal is multiplied by \( v \) and thus it agrees with \( \varepsilon_{m_v}^v \). \( \square \)

We will work at block level, since actually the blocks will correspond to factors of our polynomial, and the cycle decomposition of each block will determine the degrees of the irreducible factors. This cycle decomposition is influenced by the cells and the cycle structure of the permutation of the cells in the block. We sum this observation in the following proposition

**Proposition 10.** Consider a block \((\mu_2 \times \mu_v)^{m_v} \times S_{m_v}\). Let \( \sigma \) an element of \( S_{m_v} \) and let \( C \) be an arbitrary cycle of \( \sigma \). Ignoring the sign component \( \mu_2 \), let the order of the cells on the cycle be \( a_1, \ldots, a_l(\sigma) \) modulo \( v \). Then this arrangement will correspond to \( \gamma_v(a_1 + \ldots + a_l(\sigma)) \) cycles of length \( l(C) \gamma_v(a_1 + \ldots + a_l(\sigma)) \) in the block structure, where we denote with \( \gamma_v(x) \) denotes the order of the element in the additive group \( \mathbb{Z}/v\mathbb{Z} \).

The next proposition will give the inner product for \( v \geq 2 \)

**Proposition 11.** Suppose that \( v \geq 2 \). Consider the block \( \mathcal{B} = (\mu_2 \times \mu_v) \times S_{m_v} \). We have that

\[
(\chi_{\mathcal{B}} \otimes (\varepsilon)_{\mathcal{B}}(\psi)_{\mathcal{B}})_{\mathcal{B}} = \begin{cases} 
(-1)^{m_v+1} (\frac{1}{l(C)} \ldots (\frac{1}{l(C)} m_v)_{m_v}), & v = 2 \quad \mathcal{B} \in \lambda^1 \\
\frac{2^{l(C)-1} (\frac{1}{l(C)} \ldots (\frac{1}{l(C)} m_v)_{m_v})}{m_v}, & v = 2 \quad \mathcal{B} \in \lambda^2 \\
\frac{(-1)^{m_v} (\frac{1}{l(C)} \ldots (\frac{1}{l(C)} m_v)_{m_v})}{m_v}, & v = 2^k \quad \mathcal{B} \in \lambda^1 \\
\frac{2^{l(C)-1} (\frac{1}{l(C)} \ldots (\frac{1}{l(C)} m_v)_{m_v})}{m_v}, & v = 2^k \quad \mathcal{B} \in \lambda^2 \\
0, & \text{otherwise} 
\end{cases}
\]

**Proof.** Let \( \omega \) be a primitive root of unity of order \( v \). Also note that we can consider for each individual permutation in \( S_{m_v} \), what the inner product is and moreover for a give permutation the inner product is multiplicative on cycles. Thus let \( \sigma \in S_{m_v} \) and let \( C \) be a cycle in it’s decomposition. We will use the same notations as we go through the subcases.

First suppose that \( v \) is odd. Note that the determinant evaluated on each cell is 1, since every cell is an odd cycle. For \( \chi_{\mathcal{B}} \) to be nonzero we must have an even number of minuses on each cycle in \( \mathcal{B} \). This means that for our cycle \( C \) we must an even number of cells with a minus using proposition 10. Thus we obtain that our sum for \( C \) is equal to

\[
\sum_{0 \leq a_1 \leq v-1} 2^{l(C)-1} \omega^{a_1 + \ldots + a_l(\sigma)} = 2^{l(C)-1} \left( \sum_{i=0}^{v-1} \omega^i \right)^{l(C)} = 0
\]

It follows that for each \( \sigma \in S_{m_v} \), the product is zero, and thus we obtain that inner product is zero.

Next suppose that \( v = 2^a b \) where \( a \geq 1 \) and \( b > 1 \) is odd. Using proposition 10 we see that we have to split into two subcases; namely according to \( \gamma_v(a_1 + \ldots + a_l(\sigma)) \)
being odd or even. If this order is even, then we can take arbitrary signs on our cycle and otherwise we need to take an even number of minuses. Secondly the determinant of each cell is \((-1)^{a_j}\). Thus in this case we obtain

\[
\sum_{0 \leq a_1 \leq v-1} 2^{l(C)}(-\omega)^{a_1 + \ldots + a_{l(C)}} + \sum_{0 \leq a_1 \leq v-1} 2^{l(C)-1}(-\omega)^{a_1 + \ldots + a_{l(C)}}
\]

Now obviously the sums \(a_1 + \ldots + a_{l(C)}\) modulo \(v\) are distributed the same, namely for each \(k \in \mathbb{Z}/v\mathbb{Z}\) there are \(v^{l(C)-1}\) with sum \(k\) modulo \(v\).

Thus our sums simplify to

\[
2^{l(C)}v^{l(C)-1} \sum_{0 \leq j \leq v-1} (-\omega)^j + 2^{l(C)-1}v^{l(C)-1} \sum_{0 \leq j \leq b-1} \omega^{2^a j} = 0
\]

since both sums are zero.

All that is that is left is to deal with the case \(v = 2^a\). We need to consider \(a = 1\) separately. We can start from the last line above. What will modify is that the last sum \(0 \leq j \leq b-1\) the inner product sums to be equal to \(-1\).

Remembering that for \(\sigma\) we need to take the product over all these inner product sums of cycles, obtain that the inner product sum for a permutation is \((-1)^{c(\sigma)}(2v)^{m_v - c(\sigma)}\)

where \(c(\sigma)\) is the number of cycles of the permutation \(\sigma\).

Further on we need to make a distinction between blocks appearing in \(\lambda^1\) or \(\lambda^2\); namely because the of \(\varepsilon_{m_v}\) appearing only in \(\lambda^1\).

Now note that for any permutation \(\varepsilon(\sigma) = (-1)^{m_v - c(\sigma)}\).

For the blocks appearing in \(\lambda^1\) we obtain the inner product sum is

\[
(-2v)^{m_v} \sum_{\sigma \in S_{m_v}} \left(\frac{1}{2v}\right)^{c(\sigma)}
\]

It is well known that \(\sum_{\sigma \in S_{m_v}} X^{c(\sigma)} = X(X + 1)\ldots(X + n - 1)\), see for example [16].

Thus the inner product is equal to

\[
(-1)^{m_v} \frac{1}{2v} \left(\frac{1}{2v} + 1\right) \ldots \left(\frac{1}{2v} + m_v - 1\right)
\]

For the blocks appearing in \(\lambda^2\) we obtain that the inner product is

\[
\frac{1}{(2v)^{m_v} m_v!} \sum_{\sigma \in S_{m_v}} \left(-\frac{1}{2v}\right)^{c(\sigma)} = \frac{1}{2v} \left(-\frac{1}{2v} + 1\right) \ldots \left(-\frac{1}{2v} + m_v - 1\right)
\]

Finally for \(v = 2\) the inner product on each cycle is actually equal to \(3 \cdot 4^{l(C)-1}\). Thus the inner product sum for a permutation is \(3^{c(\sigma)}(4)^{m_v - c(\sigma)}\).

Thus if blocks with cells of size 2 appear in \(\lambda^1\) we obtain the inner product is equal to
\[(−1)^{m_v} 4^{m_v} \sum_{\sigma \in S_{m_v}} \left( \frac{−3}{4} \right)^{c(\sigma)} = (-1)^{m_v+1} \frac{3}{4} \left( \frac{3}{4} + 1 \right) \cdots \left( \frac{3}{4} + m_v - 1 \right) \]

If the blocks with cells of size 2 appear in \(\lambda^2\) we obtain the inner product is equal to

\[\frac{\frac{3}{4} \left( \frac{3}{4} + 1 \right) \cdots \left( \frac{3}{4} + m_v - 1 \right)}{m_v!}.\]

For \(v = 1\), looking at the block that contains 1 we note that it is isomorphic to a generalized permutation group \(G_k\). Thus the inner product at block level just simplifies to computing the proportion \(\frac{\#L_k}{\#G_k}\).

**Proposition 12.** Suppose we have a block made of ones i.e \(B = (\mu_2)^m \times S_m = G_m\). Then we have

\[\langle \chi_B, \det \otimes (\varepsilon) B(\psi) B \rangle_B = \begin{cases} \frac{2m}{m!} & B \in \lambda^1 \\ \frac{2m-2}{2m} \cdot \frac{m-1}{4^{m-1}} & B \in \lambda^2 \end{cases} \]

**Proof.** We can repeat the same argument as in the previous proposition’s proof, but it will be much simpler since our cells have size 1 so the \(\psi\) and \(\det\) of the cells components is trivial.

Thus for blocks of 1 appearing in \(\lambda^1\) since the \(\varepsilon_m\) components cancel out and we just need to have an even number of \(-\) on each cycle the inner product is just

\[\frac{1}{2m \cdot m!} \sum_{\sigma \in S_m} 2^{m-c(\sigma)} = \frac{1}{2m \cdot m!} \sum_{\sigma \in S_m} \left( \frac{1}{2} + 1 \right) \cdots \left( \frac{1}{2} + m - 1 \right) = \frac{2m}{m!} \left( \frac{m}{4^m} \right) \]

For the blocks of 1 appearing in \(\lambda^2\) we have

\[\frac{1}{2m \cdot m!} \sum_{\sigma \in S_m} (-2)^m (-2)^{-c(\sigma)} = (-1)^m \frac{-1}{2m \cdot m!} \sum_{\sigma \in S_m} \left( \frac{1}{2} + 1 \right) \cdots \left( \frac{1}{2} + m - 1 \right) = \frac{2m-2}{2m} \cdot \frac{m-1}{4^{m-1}} \]

Finally we can gather proposition 8,11 and 12 proved in this section into a proposition which characterizes \(\lambda^1\) and \(\lambda^2\) that will actually give a nonzero inner product.

**Proposition 13.** Suppose \(\langle \Res_{H_{\lambda^1 \lambda^2}}^G \chi_n, \Res_{H_{\lambda^1 \lambda^2}}^G \varepsilon_n \otimes \varepsilon \psi \rangle \neq 0\). Then both of the \(\lambda^1\) and \(\lambda^2\) should be composed only of nonnegative powers of 2.

**Definition 14.** A pair \((\lambda^1, \lambda^2)\) will be called acceptable if it satisfies the conditions of proposition 14.
5. SOME EXPLICIT COMPUTATIONS

Before we proceed to the proof of theorem 2, let us first show how our propositions 8, 11 and 12 explicitly compute the inner products for \( H^0, H^1, H^2, H^3 \).

(1) \( H^0 \). For \( H^0 \) from our description we only have \( A^0 \) and this is just the partitions \( \lambda^1_1 + \ldots + \lambda^1_n = n \) so that means \( \lambda^1_1 = \ldots = \lambda^1_n = 1 \). Thus we get the inner product to be

\[
\frac{|L_n|}{|G_n|} = \frac{(2n)}{n!} \quad \frac{4^n}{4^n}
\]

(2) \( H^1 \). We only have \( A^0 \) and \( A^1 \).

- For \( A^0 \) we have \( \lambda^1_1 + \ldots + \lambda^1_{n-1} = n \) thus the only partition that works is \((2, 1, \ldots, 1)\). We obtain that the inner product is

\[
\frac{3}{4} \cdot \frac{|L_{n-2}|}{|G_{n-2}|} = \frac{3}{4} \cdot \frac{(2(n-2))}{2!} \cdot \frac{4^{n-2}}{4^{n-2}}
\]

- For \( A^1 \) we have \( \lambda^1_1 + \ldots + \lambda^1_{n-1} + \lambda^1_1 = n \) so the only solution is \( \lambda^1_1 = (1, \ldots, 1) \) and \( \lambda^1_1 = 1 \). Thus the inner product is

\[
\frac{1}{2} \cdot \frac{|L_{n-1}|}{|G_{n-1}|} = \frac{1}{2} \cdot \frac{(n-1)}{2!} \cdot \frac{4^{n-1}}{4^{n-1}}
\]

(3) \( H^2 \). We have three parts \( A^0, A^1 \) and \( A^2 \).

- For \( A^0 \) we have partitions \( \lambda^1_1 + \ldots + \lambda^1_{n-2} = n \) and they have to consist of 1’s and powers of 2 thus the only one is \((2, 2, 1, \ldots, 1)\). Thus the inner product is

\[
-\frac{3}{4} \cdot \frac{|L_{n-3}|}{|G_{n-3}|} = -\frac{3}{4} \cdot \frac{(2(n-4))}{2!} \cdot \frac{4^{n-3}}{4^{n-3}}
\]

- For \( A^1 \) we have partitions \( \lambda^1_1 + \ldots + \lambda^1_{n-2} + \lambda^2_1 = n \) and we either have \( \lambda^1_1 = (2, 1, \ldots, 1) \) and \( \lambda^2_1 = 1 \) or \( \lambda^1_1 = (1, \ldots, 1) \) and \( \lambda^2_1 = 2 \). Thus the inner product is

\[
\frac{3}{4} \cdot \frac{|L_{n-3}|}{|G_{n-3}|} \cdot \frac{1}{2} + \frac{|L_{n-2}|}{|G_{n-2}|} = \frac{3}{4} \cdot \frac{(n-3)}{8} \cdot \frac{4^{n-3}}{4^{n-3}} + \frac{3}{4} \cdot \frac{(n-2)}{4^{n-2}}
\]

- For \( A^2 \) we have the relation \( \lambda^1_1 + \ldots + \lambda^1_{n-2} + \lambda^2_1 + \lambda^2_1 = n \) and again the only solution is \( \lambda^1_1 = (1, \ldots, 1) \) and \( \lambda^2_1 = (1, 1) \). Thus the inner product is

\[
-\frac{1}{2} \cdot \frac{|L_{n-3}|}{|G_{n-3}|} \cdot \frac{1}{2} + \frac{|L_{n-2}|}{|G_{n-2}|} = -\frac{1}{8} \cdot \frac{(n-2)}{4^{n-2}}
\]
(4) $H^3$. We need to look at four pieces, $A^0$, $A^1$, $A^2$ and $A^3$.

- For $A^0$ we look at partitions made up of $1$'s and powers of $2$ such that $\lambda_1^1 + \ldots + \lambda_{n-3}^1 = n$. The only ones that work are $\lambda_1^1 = (4, 1, \ldots, 1)$ and $\lambda_1^1 = (2, 2, 2, 1, \ldots, 1)$. Thus the inner product is

$$\frac{1}{8} \frac{|L_{n-4}|}{|G_{n-4}|} + \frac{3}{2} \frac{(-\frac{3}{4} + 1))(\frac{3}{4} + 2)}{3!} \frac{|L_{n-6}|}{|G_{n-6}|} = \frac{1}{8} \left( \frac{2(n-4)}{n-4} \right) + \frac{5}{128} \left( \frac{2(n-6)}{n-6} \right)$$

- For $A^1$ we have $\lambda_1^1 + \ldots + \lambda_{n-3}^1 + \lambda_1^2 = n$. This is similar to the case $A^0$ for $H^2$ and we get two possibilities $\lambda_1^1 = (2, 2, 1, \ldots, 1)$ and $\lambda_1^2 = 1$ or $\lambda_1^2 = (2, 1, \ldots, 1)$ and $\lambda_2^2 = 2$. Thus the inner product is

$$\frac{3}{2} \frac{(-\frac{3}{4} + 1)}{2!} \frac{|L_{n-5}|}{|G_{n-5}|} \frac{1}{2} + \frac{3}{4} \frac{|L_{n-4}|}{|G_{n-4}|} \frac{3}{4} = -\frac{3}{64} \left( \frac{2(n-5)}{n-5} \right) + \frac{3}{16} \left( \frac{2(n-4)}{n-4} \right)$$

- For $A^2$ we look at $\lambda_1^1 + \ldots + \lambda_{n-3}^1 + \lambda_2^2 + \lambda_1^3 = n$. The only solutions are $\lambda_1^1 = (2, 1, \ldots, 1)$ and $\lambda_2^2 = (1, 1)$ or $\lambda_1^1 = (1, \ldots, 1)$ and $\lambda_2^2 = (2, 1)$. Thus the inner product is

$$\frac{3}{4} \frac{|L_{n-4}|}{|G_{n-4}|} \frac{1}{2} \frac{(-\frac{3}{4} + 1)}{2!} + \frac{|L_{n-3}|}{|G_{n-3}|} \frac{3}{4} \frac{1}{2} = -\frac{3}{32} \left( \frac{2(n-4)}{n-4} \right) + \frac{3}{8} \left( \frac{2(n-3)}{n-3} \right)$$

- Finally for $A^3$ since $\lambda_1^1 + \ldots + \lambda_{n-3}^1 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = n$ the only possibility is $\lambda_1^1 = (1, \ldots, 1)$ and $\lambda_2^2 = (1, 1, 1)$. Thus the inner product is

$$\frac{|L_{n-3}|}{|G_{n-3}|} \frac{\frac{1}{2} (-\frac{3}{4} + 1)(\frac{3}{4} + 2)}{3!} = \frac{1}{16} \left( \frac{2(n-3)}{n-3} \right)$$

Putting everything together in the corollary of theorem 1.4 and using the notation $h_k = \frac{(2k)!}{k^k}$ (same as in [1]) we obtain

$$q^n h_n - q^{n-1} \left( \frac{1}{2} h_{n-1} + \frac{3}{4} h_{n-2} \right) + q^{n-2} \left( \frac{5}{8} h_{n-2} + \frac{3}{8} h_{n-3} - \frac{3}{32} h_{n-4} \right) - q^{n-3} \left( \frac{5}{16} h_{n-3} - \frac{1}{32} h_{n-4} - \frac{3}{64} h_{n-5} - \frac{5}{128} h_{n-6} \right) + O(q^{n-4})$$

Using the relation $S_q(n) = S_q^0(n) + S_q^0(n-1)$ in section 1 [1] and the formula

$$B_q(n) = \sum_{i=0}^{[\frac{n}{2}]} q^i S_q(n-2i)$$

we obtain

$$B_q(n) = q^n h_n + q^{n-1} \left( \frac{1}{2} h_{n-1} + \frac{1}{4} h_{n-2} \right) + q^{n-2} \left( \frac{1}{8} h_{n-2} + \frac{1}{8} h_{n-3} - \frac{27}{32} h_{n-4} \right) + q^{n-3} \left( \frac{5}{16} h_{n-3} - \frac{17}{32} h_{n-4} + \frac{5}{64} h_{n-5} - \frac{55}{128} h_{n-6} \right) + O(q^{n-4})$$
6. Proof of Theorem 2

We start looking at the equality \( \lambda_1 + \ldots + \lambda_{n-p} + \lambda_1^2 + \ldots + \lambda_p^2 = n \), for a fixed \( 0 \leq l < p \) where \((\lambda_1, \lambda_2)\) is an acceptable pair. First let us characterize the multiplicity of 1 in \((\lambda_1, \lambda_2)\).

**Proposition 15.** 1 can appear in an acceptable pair \((\lambda_1, \lambda_2)\) with multiplicity equal to \( n - 2k \), where \( k \) is any integer \( 0 \leq k \leq p - l \). Moreover the multiplicity of 1 in \( \lambda_1 \), call it \( a \), satisfies \( n - 2p + l \leq a \leq n - p \).

**Proof.** Let \( a \) the multiplicity of 1 in \( \lambda_1 \) and \( b \) the multiplicity of 1 in \( \lambda_2 \) in a random acceptable pair \((\lambda_1, \lambda_2)\) with \( \lambda_1 + \ldots + \lambda_{n-p} + \lambda_1^2 + \ldots + \lambda_p^2 = n \). Then using proposition 14 the other number appearing in \( \lambda_1 \) and \( \lambda_2 \) are powers of 2 so \( a + b \) has to have the same parity as \( n \).

Now let \( a + b = n - 2k \). Then since the other numbers appearing are at least equal to 2 we have \( n - 2k + 2((n - p + l) - (n - 2k)) \leq n \) so simplifying yields the bound \( k + l \leq p \). Now since \( b \leq l \) it follows that \( a \geq n - 2k - l \geq n - 2p + l \). \( \square \)

We can now proceed to the proof of theorem 2. We will obtain bounds, but these will be far from optimal. Also we will not write an explicit formula for the coefficient of each \( h_1 \) term appearing; the previous two sections provide the recipe for computing out this coefficient. The combinatorics involved in simplifying further the expressions seems hard.

**Proof Theorem 2**

We will group terms by looking at \( a \), the multiplicity of 1 in \( \lambda_1 \). We will consider a fixed \( l \) and afterwards will sum over the \( l \)'s.

By the previous proposition we know it satisfies \( n - 2p + l \leq a \leq n - p \). Thus we are left to write \( n - a = \lambda_1 + \ldots + \lambda_{n-p} + \lambda_1^2 + \ldots + \lambda_p^2 \).

Note that the stabilization is just a combinatorial statement. If \( n \geq 2p \) it allows us to take any \( n - 2p \leq a \leq n - p \) and we can write every possibility out for \( n - a \) as a sum of positive exponent powers of 2 and 1's.

Let \( b \) be the multiplicity of 1 in \( \lambda_2 \). Thus from the previous proposition we have \( k = \mu_1 + \ldots + \mu_{n-p} + \mu_1^2 + \ldots + \mu_p^2 \), where \( (\mu_1, \mu_2) = \frac{1}{2}(\lambda_1, \lambda_2) \).

Now to obtain the bound we just need to write \( k \) as a sum of nonnegative powers of 2 with a fixed number of summands. We can trivially upper bound this by the number of ways we can write \( k \) as a sum of nonnegative powers of 2 times the number of ways in which we can reconstruct \( \lambda_2 \).

Say we have a writing \( k = \sum_{i=1}^{r} m_i 2^{a_i} \) where \( a_1 < \ldots < a_r \) are nonnegative integers. To construct \( \mu^2 \) we need positive integers \( x_1, \ldots, x_s \) such that \( x_1 + \ldots + x_s = l - b \) and \( x_i \leq m_j \) for some subset \( \{j_1, \ldots, j_s\} \) of \( \{1, \ldots, r\} \).

Now using proposition 11, we know that each of inner product at block level for \( \lambda_2 \) are in absolute value less than \( \frac{1}{2m} \) where \( m \) is the multiplicity of the power of 2 bigger than 1 and for a block that corresponds to 2 the inner product is bounded in absolute value by 1. The inner products for \( \lambda_1 \) we will absolutely bound them by 1.

Thus we conclude that summing over all possibilities, the inner products we get at most \( \sum \frac{1}{2^s x_1 \ldots x_s} \).
if $a_1 > 0$ and $\sum \frac{1}{2s-1x_2 \ldots x_s}$ if $a_1 = 0$ and we are taking 1's in $\mu_2$, or equivalently 2's in $\lambda^2$.

We can trivially upper bound these contributions by

$$(\max m_i) \left(1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2m_2}\right) \ldots \left(1 + \frac{1}{2} + \ldots + \frac{1}{2m_r}\right)$$

Now using $1 + \frac{1}{2} + \ldots + \frac{1}{2n} < \ln(n)$ we obtain that our inner products are bounded by

$$\max(m_1) \ln(m_1) \ldots \ln(m_r) < k(\ln(k))^{r-1} < k(\ln(k))^{\log_2(k)}$$

since $k$ can be written as a sum of at most $\log_2(k)$ distinct powers.

Finally we need to account for how many distinct writings of $k$ as nonnegative powers of 2 can we have, since we are summing over all of these. This is well known sequence; we can find it under A000123 in the Online Encyclopedia of Integer sequences and the precise asymptotic of it is given in [6]. We can restate state this in a weaker upper bound, namely

**Proposition 16.** Let $b_2(y)$ the number of partitions of a positive integer $y$ into nonnegative powers of 2. Then there is an absolute constant $A$ such that

$$b_2(y) < Ae^{\ln(y)^2}$$

Finally we need to sum over all the possibilities of $b$. Using $k \leq p-l$ we get that $h_a$ appear with coefficient bounded in absolute value by

$$A(\ln(p-l))^{\log_2(p-l)}e^{\ln(p-l)^2} < B(1.1)^{p-l}$$

for some absolute constant $B$. Summing up over $l$ gives that the coefficient of $h_{n-(p+j)}$ with $0 \leq j \leq p$ is at most

$$B(1.1)^p \sum_{l=0}^{p-j} (1.1)^{-l} < 10B(1.1)^p$$

since we can trivially bound the last sum by the total geometric series.

\[ \square \]

7. **Proof of Theorem 1**

We now have all the ingredients in place to prove theorem 1. First let us show the connection between theorem 1 and theorem 2.

**Proposition 17.** We have that

$$b_{k,n} = \sum_{j=0}^{k} (-1)^{k-j}(c_{j,n-2k+2j} + c_{j,n-2k+2j-1})$$

**Proof.** First let us remark that $B_q(n)$ is invariant under which quadratic extension of $\mathbb{F}_q[T]$ we consider.

Next we note that since every monic polynomial of degree $n$ factors uniquely as $U^2V$ and the fact that $U^2V$ is a norm iff $V$ is a norm, we obtain the recurrence
\[ B_q(n) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} q^i S_q(n-2i) \]

We can rewrite this as \[ B_q(n) = \sum_{0 \leq 2i+j \leq n} (-1)^j (c_{j,n-2i} + c_{j,n-2i-1}) q^{n-i-j} \]

De-noting with \( i+j = k \) and letting \( k \) range from 0 to \( n \) we can further rewrite as

\[ B_q(n) = \sum_{k=0}^{n} \left( \sum_{2k-n \leq j \leq k} (-1)^j (c_{j,n-2k+2j} + c_{j,n-2k+2j-1}) \right) q^{n-k} \]

The stated result thus follows. \( \square \)

We can use this proposition to obtain the relation between the coefficients \( \Gamma \) and \( \delta \) namely

**Proposition 18.** We have that

a) \( \delta_{k,j,n} = \sum_{l=0}^{j} (-1)^{j-l} (\Gamma_{l,2k+l-j,n+2l-2j} + \Gamma_{l,2k+l-j,n+2l-2j-1}) \)

b) \( |\delta_{k,j,n}| \leq C(1.1)^k \).

**Proof.** Part a) is just a formal manipulation of proposition 19 and the description of the expansion of \( c_{k,n} \) in theorem 2. For the second part applying the bounds on the gamma coefficients from theorem 2 it follows

\[ |\delta_{k,j,n}| \leq 2B \sum_{l=0}^{j} (1.1)^l \leq 20B(1.1)^{j+1} \leq 22B(1.1)^k \]

and thus the claim follows. \( \square \)

These two propositions, 17 and 18, together prove the statements a) and b) from theorem 1.

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