N-modal steady water waves with vorticity

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Abstract

The problem for two-dimensional steady gravity driven water waves with vorticity is investigated. Using a multidimensional bifurcation argument, we prove the existence of small-amplitude periodic steady waves with an arbitrary number of crests per period. The role of bifurcation parameters is played by the roots of the dispersion equation.

1 Introduction

The existence theory for rotational steady water waves of finite depth goes back to the paper \cite{1} of Dubreil-Jacotin published in 1934. However only after the paper \cite{2} of Constantin and Strauss appeared in 2004 the theory of rotational steady waves attracted a lot of attention from the mathematical community. The latter paper was devoted to large amplitude Stokes waves with an arbitrary vorticity distribution. Stokes waves are periodic travelling waves with exactly one crest and one trough per period. A natural question can be raised here: are there periodic waves with more complicated geometry? To simplify the discussion we will restrict this question to the case of small-amplitude periodic waves without surface tension. In the class of periodic unidirectional waves with vorticity only Stokes waves exist (see \cite{3} and \cite{4}). Thus, to answer the question affirmatively one shall consider a wider class of waves allowing internal stagnation or critical layers. The first positive result in this direction was obtained in 2011 by Ehrnström, Escher and Wahlén. In their paper \cite{5} they prove existence of bimodal travelling waves of small amplitude. Such wave has two crests per period and the first approximation of the surface profile is given by a combination of two simple modes: \textit{cos}-functions with different wavelengths. Later, in 2015 Ehrnström and Wahlén prove (see \cite{6}) the existence of trimodal
waves was established (the first approximation is given by a combination of three basic modes). Both papers use a similar technique based on a bifurcation argument. The crucial role there is the choice of bifurcation parameters. The main difficulty for the higher order bifurcation is the fact that the number of natural parameters of the problem is limited. Normally, one may consider only the Bernoulli constant or the wavelength as bifurcation parameters. In the case of linear vorticity, the derivative of the vorticity function (which is a constant in this case) is another possible parameter. The main novelty of our approach is that an arbitrary number of bifurcation parameters are used. These parameters are the roots of the dispersion equation while the corresponding vorticity is a perturbation of a linear function. Our argument is different to that used in [8] to construct trimodal waves. In [8] authors use a Lyapunov-Schmidt reduction directly to the initial infinite-dimensional problem. This complicates calculation of the determinant of the matrix of the reduced system which is essential for the proof of the existence. In contrast, we first reduce the problem to a finite dimensional system for which the linear part is given by ordinary differential operators and then use Lyapunov-Schmidt reduction. After that the corresponding matrix is diagonal and then we can quite straightforward prove the existence of symmetric and periodic waves with an arbitrary number of basic modes.

The paper is organized as follows. In the beginning of Sect. 2 we formulate the problem in a suitable way. In Sect. 2.1 and 2.2 we discuss stream solutions and the linear approximation for the initial problem. The Sect. 3 is an essential part of the proof and concerns to the inverse spectral problem related to the dispersion equation. Here we prove Theorem 3.1 allowing to use roots of the dispersion equation as bifurcation parameters. Next, in Sect. 4, we first formulate our main result. The rest of the paper is dedicated to the proof of the main theorem. We start by writing the problem in an operator form and then in Sect. 4.4 we reduce it to a finite-dimensional system of equations. The rest of the proof is contained in Sect. 4.5, where we perform Lyapunov-Schmidt reduction to the reduced system. In contrast to the classical bifurcation theory, we find solutions only for the values of the parameters $t = (t_1, ..., t_N)$ such that $|t|^2 < \epsilon |t_j|$ for some small $\epsilon$.

# Statement of the Problem

Let an open channel of uniform rectangular cross-section be bounded below by a horizontal rigid bottom and let water occupying the channel be bounded above by a free surface not touching the bottom. In appropriate Cartesian coordinates...
\((x, y)\), the bottom coincides with the \(x\)-axis and gravity acts in the negative \(y\)-direction. The steady water motion is supposed to be two-dimensional and rotational; the surface tension is neglected on the free surface of the water, where the pressure is constant. These assumptions and the fact that water is incompressible allow us to seek the velocity field in the form \((\psi_y, -\psi_x)\), where \(\psi(x, y)\) is referred to as the stream function. The vorticity distribution \(\omega\) is supposed to be a prescribed smooth function depending only on the values of \(\psi\).

We choose the frame of reference so that the velocity field is time-independent as well as the unknown free-surface profile. The latter is assumed to be the graph of \(y = \eta(x)\), \(x \in \mathbb{R}\), where \(\eta\) is a positive continuous function, and so the longitudinal section of the water domain is \(\mathcal{D} = \{x \in \mathbb{R}, \ 0 < y < \eta(x)\}\). We use the non-dimensional variables proposed by Keady and Norbury [10]. Namely, lengths and velocities are scaled to \((Q^2/g)^{1/3}\) and \((Qg)^{1/3}\) respectively. Here \(Q\) and \(g\) are the dimensional quantities for the rate of flow and the gravity acceleration respectively, whereas \((Q^2/g)^{1/3}\) is the depth of the critical uniform stream in the irrotational case.

The following free-boundary problem for \(\psi\) and \(\eta\) has long been known (cf. [10]):

\[
\begin{align*}
\psi_{xx} + \psi_{yy} + \omega(\psi) &= 0, \quad (x, y) \in \mathcal{D}; \\
\psi(x, 0) &= 0, \quad x \in \mathbb{R}; \\
\psi(x, \eta(x)) &= 1, \quad x \in \mathbb{R}; \\
|\nabla \psi(x, \eta(x))|^2 + 2\eta(x) &= 3r, \quad x \in \mathbb{R}. 
\end{align*}
\] (2.1-2.4)

In condition (2.4) (Bernoulli’s equation), \(r\) is a constant considered as the problem’s parameter and referred to as Bernoulli’s constant/the total head. The problem (2.1)-(2.4) describes two-dimensional steady water waves with a positive mass flux.

For the further analysis of the problem it is convenient to rectify the domain \(\mathcal{D}\) by scaling the vertical variable to

\[
z = y \frac{d}{\eta(x)},
\]

while the horizontal coordinate remains unchanged. Thus, the domain \(\mathcal{D}\) transforms into the strip \(S = \mathbb{R} \times (0, d)\). Next, we introduce a new unknown function \(\hat{\Phi}(x, z)\) on \(\hat{S}\) by

\[
\hat{\Phi}(x, z) = \psi \left( x, \frac{z}{d} \eta(x) \right).
\]
A direct calculation shows that problem \(2.1-2.3\) reads in new variables as
\[
\left[ \Phi_x - \frac{2\eta_x}{\eta} \Phi_z \right]_x - \frac{2\eta_x}{\eta} \left[ \Phi_x - \frac{2\eta_x}{\eta} \Phi_z \right]_z + \left( \frac{d}{\eta} \right)^2 \Phi_{zz} + \omega(\Phi) = 0; \tag{2.5}
\]
\[
\Phi(x,0) = 0, \quad \Phi(x,d) = 1, \quad x \in \mathbb{R}; \tag{2.6}
\]
while the Bernoulli’s equation \(2.4\) becomes
\[
\Phi_z^2 = \frac{\eta^2}{d^2} \left( \frac{3r - 2\eta}{1 + \eta^2} \right). \tag{2.7}
\]

In what follows, we will assume that \(\Phi \in C^{2,\alpha} (S), \eta \in C^{1,\alpha} (\mathbb{R})\) and \(\omega \in C^\infty (\mathbb{R})\), where \(\alpha \in (0, 1)\) is fixed and remains unchanged throughout the paper.

### 2.1 Stream solutions

A pair \((u(y), d)\), where \(u : [0, d] \rightarrow \mathbb{R}\) and \(d > 0\) is called a stream solution corresponding to a vorticity function \(\omega\) and the depth \(d\), if
\[
u'' + \omega(u) = 0, \quad u(0) = 0, \quad u(d) = 1. \tag{2.8}
\]

The corresponding Bernoulli constant \(r\) is calculated by
\[
3r = [u'(d)]^2 + 2d.
\]

Note that in our definition of stream solution the Bernoulli constant is not fixed.

In this paper we will be interested in the case when the vorticity \(\omega\) is a small perturbation of a linear vorticity \(\omega_0(p) = bp\), where \(b\) is a positive constant:
\[
\omega(p) = \omega_\delta(p) = bp + \delta_1 \omega_1(p) + ... + \delta_N \omega_N(p). \tag{2.9}
\]

Here the coefficients \(\delta_j\) are small and compactly supported functions \(\omega_j \in C_0^\infty (0, 1)\) will be chosen later.

We will require constants \(b\) and \(d\) to satisfy
\[
\sqrt{b} \neq \frac{\pi j}{2d}, \quad j = 1, 2, .... \tag{2.10}
\]

Assumption \((2.10)\) guarantees that the problem \((2.8)\) with a vorticity \(\omega\) of the form \((2.9)\) possesses a unique solution \(u\) with \(u'(d) \neq 0\), provided \(\delta_j\) are small enough. In particular, if \(\omega = \omega_0\) then the corresponding stream solution is \(u_0 = \sin \sqrt{b}z / \sin \sqrt{bd}\).
2.2 Linear approximation of the water-wave problem

Let us consider a linear approximation of the problem. For this purpose we formally linearize equations (2.5)-(2.7) near a stream solution \((u, d)\). Thus, we put

\[ \hat{\Phi} = u + \epsilon \Phi + O(\epsilon^2), \quad \eta = d + \epsilon \zeta + O(\epsilon^2). \]

Using this ansatz in (2.5)-(2.7), we find

\[ \left[ \Phi_x - \frac{2\zeta u_z}{d} \right] x + \Phi_{zz} - \frac{2\zeta u_{zz}}{d} + \omega'(u)\Phi = O(\epsilon), \]

\[ \Phi(x, 0) = \Phi(x, d) = 0, \]

\[ \Phi|_{z=d} - \left( \frac{u'(d)}{d} - \frac{1}{u'(d)} \right) \zeta = O(\epsilon). \]

We can simplify equations by letting

\[ \Psi = \Phi - \frac{z\zeta u_z}{d}. \]

The latter transformation was used in [7] and [8]. The formula above implies \( \zeta = -\Psi|_{z=d}/u'(d) \) and then the linear part of the above problem transforms into

\[ \Psi_{xx} + \Psi_{zz} + \omega'(u)\Psi = 0 \]

\[ \Psi|_{z=0} = 0 \]

\[ \Psi|_{z=d} - \kappa \Psi|_{x=d} = 0, \]

where \( \kappa = 1/[u'(d)]^2 - \omega(1)/u'(d) \). Separation of variables in the system (2.11) leads to the following Sturm-Liouville problem:

\[ - \phi_{zz} - \omega'(u)\phi = \mu \phi \text{ on } (0, d), \quad \phi(0) = 0, \quad \phi_z(d) = \kappa \phi(d). \]  

The spectrum of (2.12) depends only on the triple \((\omega, d, u)\) and consists of countable set of simple real eigenvalues \(\{\mu_j\}_{j=1}^{\infty} \) ordered so that \(\mu_j < \mu_l \) for all \(j < l\). Furthermore, only a finite number of eigenvalues may be negative. The normalized eigenfunction corresponding to an eigenvalue \(\mu_j\) will be denoted by \(\phi_j\). Thus, the set of all eigenfunctions \(\{\phi_j\}_{j=1}^{\infty} \) forms an orthonormal basis in \(L^2(0, d)\).

Solving linear problem (2.11), we find that the space of bounded and even in the \(x\)-variable solutions is finite-dimensional and is spanned by the functions

\[ \Psi(x, z) = \cos(\sqrt{\mu_j} x)\phi_j(z), \]
where $\mu_j \leq 0$.

Let us turn to the problem (2.12) for the linear vorticity $\omega_0(p) := bp$ for some $b > 0$.

**Proposition 2.1.** For any $N \geq 1$ and $d > 0$ there exists $b > 0$ satisfying (2.10) such that Sturm-Liouville problem (2.12) for the triple $(\omega_0, d, u_0)$ has exactly $N$ negative eigenvalues while all other eigenvalues are positive.

In the formulation above $u_0$ stands for the unique solution of (2.8) for the pair $(\omega_0, d)$. An analog of Proposition 2.1 when the depth $d$ is not supposed to be fixed, was proved in [1].

**Proof of Proposition 2.1.** For a given $d > 0$, let $b$ be any positive constants satisfying (2.10) and $\omega_0(p) := bp$. Note that the constant $\kappa$ in (2.12) for the triple $(\omega_0, d, u_0)$ depends on $b$. Thus, varying $b$ the spectrum of (2.12) will be deforming in some nontrivial way. Because of that, we start with a simpler Dirichlet problem:

$$-\phi_{zz} - b\phi = \lambda D \phi \text{ on } (0, d), \quad \phi(0) = \phi(d) = 0.$$  \hfill (2.13)

We will denote the spectrum of the Dirichlet problem (2.13) by $\{\lambda^D_j\}$, while the spectrum of (11) for the triple $(\omega_0, d, u_0)$ will be denoted by $\{\mu_j\}$. It is shown in [11] (see the proof Proposition 3.3) that between any two neighbor negative eigenvalues $\lambda^D_j$ and $\lambda^D_{j+1}$ there exists exactly one negative eigenvalue $\mu_{j+1}$ of the Sturm-Liouville problem (2.12) for the triple $(\omega_0, d, u_0)$. Furthermore, we always have $\mu_1 < \lambda^D_1$. Because varying $b$ in (2.13) just slides the spectrum, we can choose $b$ subject to (2.10) so that the Dirichlet problem has exactly $N - 1$ negative eigenvalues $\lambda^D_1 < \ldots < \lambda^D_{N-1}$ and $\lambda^D_{N-1}$ is close to zero. The argument above guarantees existence of at least $N$ negative eigenvalues $\mu_j$, $j = 1, \ldots, N$.

We will show below that in this case the interval $[\lambda^D_{N-1}, 0]$ is free from the eigenvalues $\{\mu_j\}$ and then $\mu_j > 0$ for all $j > N$ which will finish the proof. Indeed, since $\lambda^D_{N-1}$ is Dirichlet eigenvalue, we have

$$\sin \sqrt{b + \lambda^D_{N-1}}d = 0,$$

which implies

$$\sqrt{bd} = \pi l + O(\lambda^D_{N-1}) \hfill (2.14)$$

for some $l \in \mathbb{N}$. A direct calculation shows that constant $\kappa$ for the $(\omega_0, d, u_0)$ is given by

$$\kappa = \frac{1}{b} \frac{\sin \sqrt{bd}}{\cos^2 \sqrt{bd}} - \frac{1}{\sqrt{b}} \frac{\sin \sqrt{bd}}{\cos \sqrt{bd}} = O(\lambda^D_{N-1})$$
and is close to zero, provided $\lambda_{N-1}^D$ is small. Let now $\mu_*$ be the greatest non-positive eigenvalue in $\{\mu_j\}$. Then the boundary relation at $d$ in (2.12) implies
\[ \sqrt{b + \mu_*} \cos \sqrt{b + \mu_*} d = \kappa \sin \sqrt{b + \mu_*} d. \]
If $\mu_* \in [\lambda_{N-1}^D, 0]$, then the equation above and (2.14) show that
\[ \sqrt{b} = O(\lambda_{N-1}^D), \]
which leads to a contradiction, because by the choice $b$ must be greater than the first eigenvalue of the following Dirichlet problem
\[ -\phi_{zz} = \lambda \phi \text{ on } (0, d), \quad \phi(0) = \phi(d) = 0. \]
Thus, if $\lambda_{N-1}^D$ is sufficiently close to zero, we obtain exactly $N$ negative eigenvalues $\mu_1, ..., \mu_N$ for the Sturm-Liouville problem (2.12) for the triple $(\omega_0, d, u_0)$.

3 Inverse spectral problem

An important step in the analysis of the dispersion equation is a study of the following inverse spectral problem:

**Weak ISP problem.** For a given finite number of distinct values $\mu_j$, $j = 1, ..., N$ find a vorticity function $\omega$, a depth $d$ and a stream solution $(u, d)$ such that the first eigenvalues of the problem (2.12) coincide with $\{\mu_j\}$.

This problem was considered in [6], where the authors prove that the dispersion equation may have any number of roots. The argument used in [6] to deal with the weak problem is based on the inverse spectrum Sturm-Liouville theory (see [5] for details). In our case the depth is fixed and instead of "weak ISP" one should consider

**ISP problem.** For a given depth $d$ and a finite number of distinct values $\{\mu_j\}_{j=1}^N$ find a vorticity function $\omega$ and a stream solution $(u, d)$ such that the first eigenvalues of the problem (2.12) coincide with $\{\mu_j\}$.

To deal with this problem one may try to use a similar approach based on the inverse Sturm-Liouville theory. However the additional restriction on the depth makes the problem much more complicated. We will use a different approach based on a perturbation argument.
For a given $N > 1$ and $d > 0$ we consider the constant $b$ provided by Proposition 2.1 and let $(u_0, d)$ be the corresponding stream solution:

$$u''_0 + bu_0 = 0, \quad u_0(0) = 0, \quad u_0(d) = 1. \quad (3.1)$$

According to Proposition 2.1, the Sturm-Liouville problem (2.12) for the triple $(\omega_0, d, u_0)$ has exactly $N$ negative eigenvalues $\lambda_1 < ... < \lambda_N < 0$ and all other eigenvalues are positive.

Now, let the functions $\omega_k, k = 1, \ldots, N,$ in (2.9) be fixed. Then for small $\delta = (\delta_1, \ldots, \delta_N)$ there exists a unique solution $u_\delta$ to the problem (2.8) with $\omega = \omega_\delta$ given by (2.9) and we can consider the Sturm-Liouville problem (2.12) for the triple $(\omega_\delta, d, u_\delta)$. Let $\mu_1 < \ldots < \mu_N$ be the negative eigenvalues of this Sturm-Liouville problem. Next theorem is devoted to the map

$$T: \delta = (\delta_1, \ldots, \delta_N) \to \mu = (\mu_1, \ldots, \mu_N). \quad (3.2)$$

Clearly, this map is $C^\infty$ in a neighborhood of $\delta = 0$.

**Theorem 3.1.** There exist functions $\omega_k \in C^\infty_0(0, 1)$, $k = 1, \ldots, N$, such that the Jacobian matrix of the map (3.2) is invertible at $\delta = 0$.

**Proof.** Let us calculate the Jacobian matrix of the map $T$. Since $u_\delta$ depends smoothly on $\delta$, we have

$$u_\delta(z) = u_0(z) + \delta_1 u_1(z) + \ldots + \delta_N u_N(z) + O(|\delta|^2), \quad |\delta| \to 0,$$

where $u_1(0) = u_1(d) = \ldots = u_N(0) = u_N(d) = 0$. Here $(u_0, d)$ is the stream solution corresponding to the linear vorticity $\omega_0(p) := bp$. Using this ansatz, we find that every $u_j$ is subject to

$$u''_j + bu_j = -\omega_j(u_0), \quad u_j(0) = u_j(d) = 0$$

for all $1 \leq j \leq N$. Solving these equations, we find

$$u_j(z) = c_j \frac{\sin \sqrt{b} z}{\sqrt{b}} - \frac{1}{\sqrt{b}} \int_0^z \omega_j(u_0(p)) \sin \sqrt{b} (z - p) dp, \quad (3.3)$$

where

$$c_j = \frac{1}{\sin(\sqrt{bd})} \int_0^d \omega_j(u_0(p)) \sin \sqrt{b} (d - p) dp.$$

Let us turn to the dispersion equation for the triple $(\omega_\delta, d, u_\delta)$ which is given by the following eigenvalue problem

$$-\phi'' - \omega'(u_\delta) \phi = \mu \phi \text{ on } (0, d);$$

$$\phi'(d) = \left(1 - \frac{\omega_\delta(1)}{k_\delta} \right) \phi(d), \quad \phi(0) = 0.$$
Here $k = u_0'(d)$. The corresponding eigenfunctions $\phi_l$ and eigenvalues $\mu_l$, $l = 1, \ldots, N$ depend smoothly on $\delta$, so that

$$\mu_l = \lambda_l + \delta_1 \mu_1 + \ldots + \delta_N \mu_N + O(|\delta|^2), \quad |\delta| \to 0, \quad l = 1, \ldots, N.$$  

In order to find the coefficients $\mu_{lj}$, we write the eigenvalue problem above in operator form:

$$A\phi = \mu \phi,$$

where the self-adjoint operator $A$ is defined by

$$\langle A\phi, \psi \rangle = -\left(\frac{1}{k_0^2} - \frac{\omega_0'(1)}{k_0}\right) \psi(d)\phi(d) + \int_0^d \phi' \psi' \, dz - \int_0^d \omega_0'(u) \phi \psi \, dz,$$

where $\phi, \psi \in W^{1,2}_0(0, d)$ and the last space consists of functions from $W^{1,2}(0, d)$ vanishing at 0. Here $\langle \cdot, \cdot \rangle$ stands for the standard inner product in $L^2(0, d)$.

Thus, if

$$A = A_0 + \delta_1 A_1 + \ldots + \delta_N A_N + O(|\delta|^2), \quad |\delta| \to 0,$$

where operators $A_j$ are defined by

$$\langle A_j \phi, \psi \rangle = \frac{u_j'(d)}{k_0^2} \left(\frac{2}{k_0} - b - \frac{\phi_j^2(d)}{k_0^2}\right) \phi(d)\psi(d) - \int_0^d \omega_j'(u_0) \phi \psi \, dz$$

for $\phi, \psi \in W^{1,2}_0(0, d)$. Using (3.3), we find

$$u_j'(d) = -\frac{1}{\sin(\sqrt{bd})} \int_0^d \omega_j(u_0) \sin(\sqrt{bd}) \, dz = -\int_0^d \omega_j'(u_0) \frac{\cos^2 \sqrt{bz}}{\sin^2(\sqrt{bd})} \, dz \quad (3.4)$$

after integration by parts and observation that $u_0 = \sin(\sqrt{bz}) / \sin(\sqrt{bd})$. Then, after some algebra one obtains

$$\mu_{lj} = \langle A_j \phi_l, \phi_l \rangle,$$

where $\phi_l$ stands for the normalized eigenfunction corresponding to the eigenvalue $\lambda_l$ of the Sturm-Liouville problem for $\delta = 0$. Therefore, using (3.4), we conclude

$$\mu_{lj} = \int_0^d \omega_j'(u_0)(A_l \cos^2 \sqrt{bz} - \phi_l^2) \, dz, \quad A_l = \frac{\phi_l^2(d)}{k_0^2 \sin^2(\sqrt{bd})} \left(b - \frac{2}{k_0}\right). \quad (3.5)$$

Let us transform the right-hand side in (3.5). Since $u_0 = \sin(\sqrt{bz}) / \sin(\sqrt{bd})$, we can reduce integration over $[0, d]$ to a smaller interval of where the function $u_0$ is monotone. We put

$$\Lambda = \frac{2\pi}{\sqrt{b}}$$

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which is the period of the function $u_0$.

Consider first the case $\sin \sqrt{bd} > 0$. Let $d_*$ be the smallest positive root of the equation $u_0(d_*) = 1$. Then the function $u_0$ is monotone on the interval $[0, d_*]$. Let $z \in (0, d_*)$. To describe all roots of $u_0(\hat{z}) = u_0(z)$ on the interval $(0, d)$, which are different from $z$, we introduce the integer $M$ as the largest integer satisfying

$$\Lambda \left( M + \frac{3}{4} \right) < d.$$  

(3.6)

Then the roots of $u_0(\hat{z}) = u_0(z)$ are given by

$$z_k^\pm = \left( k + \frac{3}{4} \right) \Lambda \pm \left( z + \frac{\Lambda}{4} \right), \quad k = 1, \ldots, M.$$  

(3.7)

Now we can write (3.5) as

$$\mu_j = \int_0^{d_*} \omega_j'(u_0) \left[ (2M + 1) A_l \cos^2 \sqrt{b} z - \phi_l^2(z) - \sum_{k=1}^{M} (\phi_l^2(z_k^+) + \phi_l^2(z_k^-)) \right] dz.$$  

(3.8)

Note that the function $\phi_l$ solves the problem

$$-\phi_l'' - b\phi_l = \lambda_l \phi_l, \quad \phi_l(0) = 0, \quad \phi_l'(d) = \kappa_0 \phi_l(d).$$

Hence, depending on the sign of $b+\lambda_l$, it is either $C_1 \sin \sqrt{b+\lambda_l} z$ or $C_2 \sinh \sqrt{b+\lambda_l} z$.

Let us consider the case when $b + \lambda_1 > 0$ and hence

$$\phi_l(z) = C_1 \sin \sqrt{b + \lambda_1} z \quad \text{for all } l = 1, \ldots, N.$$  

Then

$$\phi_l^2(z) + \sum_{k=1}^{M} (\phi_l^2(z_k^+) + \phi_l^2(z_k^-))$$

$$= C_1^2 \left( (M + 1/2) - \cos(2\sqrt{b + \lambda_1} z) - B_l \cos(2\sqrt{b + \lambda_1}(z + \Lambda/4)) \right),$$

where

$$B_l = \sum_{k=1}^{M} \cos(2\sqrt{b + \lambda_1}(k + 3/4)\Lambda).$$

Therefore

$$\mu_j = \int_0^{d_*} \omega_j'(u_0) f_l(z) dz,$$  

(3.9)

where

$$f_l(z) = (M + 1/2) \left[ A_l - C_l^2 + A_l \cos(2\sqrt{b} z) \right]$$

$$+ C_l^2 \left( \cos(2\sqrt{b + \lambda_1} z) + B_l \cos(2\sqrt{b + \lambda_1}(z + \Lambda/4)) \right).$$  

(3.10)
Since the functions
\[ 1, \cos(2\sqrt{b}z), \cos(2\sqrt{b + \lambda_1}z), \cos(2\sqrt{b + \lambda_l(z + \Lambda/4)}), \]  
\[ l = 1, \ldots, N, \]
are linear independent, the set of functions
\[ \cos(\sqrt{bz}), \ f_l(z), \ l = 1, \ldots, N, \]  
(3.11)
is also linear independent. Indeed, the only case when they can be dependent is \( \sqrt{b} = 2\sqrt{b + \lambda_1} \) for certain \( l \), but then either \( A_l \) or \( A_l - C_2 \) is non-zero in the representation (3.10) for \( f_l \) which is sufficient for linear independence of the system (3.11).

In the case \( b + \lambda_1 \leq 0 \) we have \( b + \lambda_2 > 0 \). Otherwise we will have two eigenfunctions corresponding to different eigenvalues and which are no orthogonal to each other. Now we must replace the function \( \sin(\sqrt{b + \lambda_1}z) \) by \( \sinh(\sqrt{b + \lambda_1}z) \) and, repeating then the above argument, we obtain the linear independence of the system (3.11).

Using linear independence of the functions (3.11), we can find functions \( \alpha_j = \alpha_j(p) \) in \( C_0^\infty(0,1) \) such that
\[ \int_0^{d_\ast} \alpha_j(u_0(z)) f_l(z) dz = \delta_l^j \text{ and } \int_0^{d_\ast} \alpha_j(u_0(z)) \cos(\sqrt{bz}) dz = 0, \]  
(3.12)
where \( \delta_l^j \) is the Kronecker delta. If we put
\[ \omega_j(p) = \int_0^p \alpha_j(s) ds = \int_0^Z \alpha_j(u_0(z)) u'_0(z) dz, \ p = u_0(Z), \]
then we see that \( \omega_j \in C_0^\infty(0,1) \) due to the second equality in (3.12) and \( \omega_j(p) = \alpha_j(p) \). Therefore the matrix (3.9) is invertible by the first equality in (3.12). This completes the proof in the case \( \sin \sqrt{bd} > 0 \).

Assume now that \( \sin \sqrt{bd} < 0 \). Then the function \( u_0 \) is negative on \( (0, \Lambda/2) \) and it is monotonically increasing from 0 to 1 on the interval \( [\Lambda/2, d_\ast + \Lambda/2] \). Reasoning as above we obtain the representation (3.9) but on the interval \( (\Lambda/2, d_\ast + \Lambda/2) \) with 3/4 replaced by 1/4 in formulas like (3.6) and (3.7). The remaining part of the proof in this case is the same as above.

\[ \square \]

4 Existence of \( N \)-modal waves

For a given \( N > 1 \) and \( d > 0 \) let \( b \) be the constant from Proposition 2.1 so that the Sturm–Liouville problem (2.12) has \( N \) negative eigenvalues \( \lambda_1, \ldots, \lambda_N \)
and all remaining eigenvalues are positive. By Theorem 3.1 we can choose real-valued functions $\omega_1, \ldots, \omega_N \in C_0^\infty(0,1)$ such that the map (3.2) is invertible in a certain neighborhood $\Gamma \subset \mathbb{R}^N$ of the point $(\lambda_1, \ldots, \lambda_N)$. In what follows we will use $\mu = (\mu_1, \ldots, \mu_N) \in \Gamma$ as a parameter of our water wave problem and denote by $\omega(p; \mu), u(z; \mu), \ldots$ the perturbed vorticity $\omega_\delta$, stream solution $u_\delta$ etc. We choose also $\mu^* = (\mu_1^*, \ldots, \mu_N^*) \in \Gamma$ so that $\mu_j^*/\mu_l^*$ are rational numbers for all $1 \leq j, l \leq N$. In this case all solutions to the linear problem (2.11) for the triple $(\omega(p; \mu_*), d, u(z; \mu_*))$ have a common period which we denote by $\Lambda_*$. Since the set $\Gamma$ is open, we can always find some $\mu^*$ with this property.

The aim of the present paper is to give an affirmative answer to the question raised in [7]: does higher-order bifurcation occur so that $N$-modal waves exist for all $N \geq 1$?

**Theorem 4.1.** Under given assumptions there exist constants $\epsilon, \delta > 0$ depending only on $N$ and $d$ and a smooth function $\mu : [-\delta, \delta]^N \to \Gamma$ with the following property: for any $t := (t_1, \ldots, t_N) \in \mathbb{R}^N$ such that $|t| < \delta$ and $|t|^2/t_j \leq \epsilon$ the nonlinear water wave problem (2.5)-(2.7) with the vorticity $\omega(p; \mu(t))$ possesses a unique $\Lambda_*$-periodic and even solution $(\Psi, \eta)$ corresponding to the Bernoulli constant $r(t) = [u_z(d; \mu(t))]^2 + 2d/3$ such that

$$
\eta(x) = d + t_1 \cos(\sqrt{\mu_1^*}|x|) + \ldots + t_N \cos(\sqrt{\mu_N^*}|x|) + O(|t|^2).
$$

The waves constructed in Theorem 4.1 are symmetric with respect to the vertical line $x = 0$ and this is essential for the proof of the theorem. The question on existence of non-symmetric steady waves is still open.

### 4.1 Functional-analytic setup

In order to write equations (2.5)-(2.7) in an operator form, we define nonlinear operators

$$
\mathcal{F}_1(\hat{\Phi}, \eta; \mu) = \left[\frac{\eta}{\eta} \hat{\Phi}_x - \frac{2\eta_x}{\eta} \hat{\Phi}_z\right]_x - \left[\frac{2\eta_x}{\eta} \hat{\Phi}_z - \frac{\eta_z}{\eta} \hat{\Phi}_x\right]_z + \left(\frac{d}{\eta}\right)^2 \hat{\Phi}_{zz} + \omega(\hat{\Phi}; \mu)
$$

$$
\mathcal{F}_2(\hat{\Phi}, \eta; \mu) = \hat{\Phi}_z^2|_{z=d} - \frac{\eta^2}{d^2} \left(\frac{3r(\mu) - 2\eta}{1 + \eta_x^2}\right),
$$

where

$$
r(\mu) = [u'(d; \mu)]^2 - 2d.
$$
The definitions above imply that

\[ \hat{\mathcal{F}}_1(u(\cdot; \mu),d; \mu) = \hat{\mathcal{F}}_2(u(\cdot; \mu),d; \mu) = 0 \]

for all \( \mu \in \Gamma \). Next, we put

\[ \Phi(x,z) = \hat{\Phi}(x,z) - u(z; \mu) - \frac{zu_z(z; \mu)(\eta(x) - d)}{d}. \]

The purposes of this change of variables are twofold. First, it gives a formal linearization near the stream solution \( u(z; \mu) \). Second, it allows to eliminate the profile \( \eta \) from the equations. Indeed, the definition above implies that

\[ \eta(x) = d - \frac{\Phi(z=d)}{u'(d)}. \]

Thus, we define

\[ \mathcal{F}_j(\Phi; \mu) = \hat{\mathcal{F}}_j \left( \Phi + u + \frac{zu_z(\eta - d)}{d}, d - \frac{\Phi(z=d)}{u_z(d)} \right), \quad j = 1, 2. \]

Therefore, the problem \((2.5) - (2.7)\) with the vorticity \( \omega(\cdot; \mu) \) and Bernoulli constant \( r(\mu) \) reads as

\[ \mathcal{F}(\Phi; \mu) = 0, \quad (4.1) \]

where \( \mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2) : X \times \Gamma \rightarrow Y := Y_1 \times Y_2 \) and the spaces are defined by

\[ X = \{ \Phi \in C^{2,\alpha}_{\text{per}}(\bar{S}) : \Phi(x,0) = 0, \Phi(x,z) = \Phi(-x,z) \text{ for all } (x,y) \in S \} \]

and

\[ Y_1 = C^{0,\alpha}_{\text{per}}(\bar{S}), \quad Y_2 = C^{1,\alpha}(\mathbb{R}). \]

Here and elsewhere the subscript \( \text{per} \) denotes \( \Lambda_\ast \)-periodicity and evenness in the horizontal \( x \)-variable. As the norms in these spaces we will use \( \| \cdot \|_{C^{0,\alpha}([-\Lambda_\ast/2, \Lambda_\ast/2] \times [0,d])} \) and \( \| \cdot \|_{C^{1,\alpha}([-\Lambda_\ast/2, \Lambda_\ast/2])} \) respectively.

**Proposition 4.2.** The Fréchet derivative \( D_\Phi \mathcal{F} \) at \( \Phi = 0 \) is given by

\[ [D_\Phi \mathcal{F}_1(0, \mu)](\Phi) = \Phi_{xx} + \Phi_{zz} + \omega'(u)\Phi \]

\[ [D_\Phi \mathcal{F}_2(0, \mu)](\Phi) = \Phi_{|z=d} - \kappa \Phi_{|z=d}, \]

where

\[ \kappa = \frac{1}{|u_z(d; \mu)|^2} - \frac{\omega(1; \mu)}{u_z(d, \mu)}. \]

**Proof.** The statement follows from a direct calculation. \( \square\)
4.2 Spectral decomposition

According to our notations \( \mu = (\mu_1, ..., \mu_N) \) stands for the first \( N \) eigenvalues of the Sturm-Liouville problem \( \text{(2.12)} \) for the triple \( (\omega(\cdot, \mu), d, u(\cdot, \mu)) \). Let 
\[ \phi_1(z; \mu), ..., \phi_N(z; \mu) \]
be the corresponding eigenfunctions. We define orthogonal projectors 
\[ P\Phi = \sum_{j=1}^{N} \Phi_j \phi_j, \quad \tilde{P} = \text{id} - P. \]

Here \( \Phi_j = \langle \Phi, \phi_j \rangle \) and \( \langle \cdot, \cdot \rangle \) stands for the standard scalar product in \( L^2(0, d) \).

Let us write equation \( \text{(4.1)} \) in the following form, where linear and nonlinear parts of the operator are separated:
\begin{align*}
\Phi_{xx} + \Phi_{zz} + \omega'(u)\Phi &= N_1(\Phi; \mu) \quad (4.4) \\
\Phi_z|_{z=d} - \kappa \Phi|_{z=d} &= N_2(\Phi; \mu), \quad (4.5)
\end{align*}
where \( N_1 \) and \( N_2 \) are nonlinear parts of the operators \( F_1 \) and \( F_2 \) respectively.

Thus, taking the projections in \( \text{(4.4)-(4.5)} \), we obtain equations for projections:
\begin{align*}
\Phi_j'' - \mu_j \Phi_j &= (N_1(\Phi_j) - N_2(\Phi_j; d), \quad j = 1, ..., N; \quad (4.6)
\end{align*}
and
\begin{align*}
\Phi_{xx} + \Phi_{zz} + \omega'(u)\Phi &= \tilde{P}(N_1) + N_2(\Phi_j; d)\phi_j(z) \quad (4.7) \\
\tilde{\Phi}_z(x, d) - \kappa \tilde{\Phi}(x, d) &= N_2 \quad (4.8)
\end{align*}

where \( \Phi_j = \langle \Phi, \phi_j \rangle \) and \( \tilde{\Phi} = \tilde{P}\Phi \). It is clear that if some functions \( \Phi_j \) and \( \tilde{\Phi} \) solve equations \( \text{(4.6)-(4.8)} \), then the function 
\[ \Phi(x, z) = \sum_{j=1}^{N} \Phi_j(x)\phi_j(z) + \tilde{\Phi}(x, z) \]
solve \( \text{(4.1)} \).

We will show below that the function \( \tilde{\Phi} \) can be resolved from equations \( \text{(4.7)-(4.8)} \) as an operator of \( (\Phi_1, ..., \Phi_N) \) and \( \mu \) in a neighbourhood of the origin.

4.3 The reduction to a finite-dimensional system

First, we consider the linear part of \( \text{(4.7)-(4.8)} \) which is given by
\begin{align*}
\Phi_{xx} + \Phi_{zz} + \omega'(u)\Phi &= f \quad (4.9) \\
\Phi_z(x, d) - \kappa \Phi(x, d) &= g \quad (4.10) \\
\Phi(x, 0) &= 0. \quad (4.11)
\end{align*}
Let $\widetilde{X}$ be the subspace of $X$ which consists of functions $\Phi$ satisfying
\[ \widetilde{P}[\Phi(x,\cdot)] = \Phi(x,\cdot) \]
for all $x \in \mathbb{R}$. Roughly speaking, we have $\widetilde{X} = \widetilde{P}X$. Furthermore, we define the range space $\widetilde{Y}$ to be a subspace of $Y$ which consists of all $(f,g) \in Y$ satisfying
\[ \langle f(x,\cdot), \phi_j(\cdot) \rangle = g(x)\phi_j(d) \]
for all $x \in \mathbb{R}$. Let us define a linear operator $L := (L_1, L_2): \widetilde{X} \rightarrow \widetilde{Y}$ by
\[
L_1 \Phi = \Phi_{xx} + \Phi_{zz} + \omega'(u)\Phi \\
L_2 \Phi = \Phi_z(x,d) - \kappa \Phi(x,d).
\]
Now we are ready to prove

**Lemma 4.3.** The mapping $L: \widetilde{X} \rightarrow \widetilde{Y}$ is a linear isomorphism and its norm depends only on $b$, $d$ and $\alpha$.

**Proof.** Applying Schauder estimate (see Theorem 7.3 [2]) to the system (4.9)-(4.11), we find
\[
\| \Phi \|_{C^{2,\alpha}(R_{\lambda_*})} \leq C[\| f \|_{Y_1} + \| g \|_{Y_2} + \| \Phi \|_{L^2(R_{\lambda_*})}], \tag{4.12}
\]
where $R_{\lambda_*} = [-\lambda_*/2, \lambda_*/2] \times [0,d]$. Let us show that
\[
\| \Phi \|_{L^2(R_{\lambda_*})} \leq C[\| f \|_{Y_1} + \| g \|_{Y_2}] \tag{4.13}
\]
In order to prove (4.13), we multiply equation (4.9) by $\widetilde{\Phi}$ and integrate over $z \in [0,d]$, which gives
\[
\int_0^d \widetilde{\Phi}_{xx}\widetilde{\Phi}dz + \int_0^d \widetilde{\Phi}_{zz} + \omega'(u)\widetilde{\Phi}dz = \int_0^d f\widetilde{\Phi}dz. \tag{4.14}
\]
Using spectral representation of $\widetilde{\Phi}$, we find
\[
- \int_0^d \widetilde{\Phi}_{zz} + \omega'(u)\widetilde{\Phi}dz = \sum_{j=N+1}^{\infty} \mu_k[\langle \widetilde{\Phi}, \phi_j \rangle]^2 - \widetilde{\Phi}(x,d)g \geq \mu_{N+1} \int_0^d \widetilde{\Phi}^2dz - \widetilde{\Phi}(x,d)g. \tag{4.15}
\]
Here $\{\mu_j\}_{j=1}^{\infty}$ stands for the set of all eigenvalues of the Sturm-Liouville problem (2.12) for the triple $(\omega(\cdot,\mu), d, u(\cdot,\mu))$. We recall that $\mu_j > 0$ for all $j > N$. 

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Now we integrate (4.14) over $R_{\Lambda}$. After integration by parts, we get
\[
\int \int_{R_{\Lambda}} \tilde{\Phi}^2 \, dx \, dz + \left[ - \int \int_{R_{\Lambda}} [\tilde{\Phi}_{zz} + \omega'(u)\tilde{\Phi}] \, dx \, dz \right] \leq \int \int_{R_{\Lambda}} |f\tilde{\Phi}| \, dx \, dz.
\]
Combining this inequality with (4.15), we arrive at
\[
\int \int_{R_{\Lambda}} \tilde{\Phi}^2 \, dx \, dz \leq \mu^{-1} N_{1} \left[ \int \int_{R_{\Lambda}} |\tilde{F}| \, dx \, dz + \int_{[-\Lambda/2, \Lambda/2]} |\tilde{\Phi}(x, d)| \, g \, dx \right].
\]
For an arbitrary $\delta > 0$, we write
\[
\int \int_{R_{\Lambda}} |f\tilde{\Phi}| \, dx \, dz \leq \delta \int \int_{R_{\Lambda}} \tilde{\Phi}^2 \, dx \, dz + \delta^{-1} \int \int_{R_{\Lambda}} f^2 \, dx \, dz
\]
and, similarly,
\[
\int_{[-\Lambda/2, \Lambda/2]} |\tilde{\Phi}(x, d)| \, g \, dx \leq \int \int_{R_{\Lambda}} \tilde{\Phi}_z(x, z) \, g \, dx \, dz \leq \delta \Lambda d \|\tilde{\Phi}\|^2_{C^{2,\alpha}(R_{\Lambda})} + \delta^{-1} \|g\|^2_{L^2([-\Lambda, \Lambda])}.
\]
Composing the last three inequalities, we obtain
\[
\|\tilde{\Phi}\|^2_{L^2(R_{\Lambda})} \leq \frac{2 \Lambda d}{\mu N_{1}} \|\tilde{\Phi}\|^2_{X} + \frac{\Lambda d}{\delta \mu N_{1}} \|f\|^2_{Y_1} + \frac{\Lambda d}{\delta \mu N_{1}} \|g\|^2_{Y_2}.
\]
To complete the proof of the lemma it is left to combine this inequality with (4.12), provided
\[
\delta = \frac{\mu N_{1}}{C^4 \Lambda d},
\]
where $C$ is the constant from (4.12).

Before formulating the next theorem, let us define the space
\[
X_{\text{red}} = [C^{2,\alpha}_{\text{per}}(\mathbb{R})]^N,
\]
where the space $C^{2,\alpha}_{\text{per}}(\mathbb{R})$ consists of all even and $\Lambda$-periodical functions from $C^{2,\alpha}(\mathbb{R})$.

**Theorem 4.4.** For any $m \geq 1$ there exist open neighborhoods of the origin $U \subset X_{\text{red}}, V \subset \tilde{X}$ and a smooth operator $h \in C^m(U; V)$ such that the function $\tilde{\Phi} := h(\Phi_1, \ldots, \Phi_N)$ is the unique solution of the boundary problem (4.7)-(4.8), provided $(\Phi_1, \ldots, \Phi_N) \in U$.

**Proof.** The statement follows directly from Lemma 4.3 and the implicit function theorem applied to the system (4.7)-(4.8). $\Box$
Theorem 4.4 allows to reduce the problem for small-amplitude waves to a finite dimensional system of the form

\[ \Phi''_j - \mu_j \Phi_j = N_j(\Phi; \mu), \quad (4.16) \]

where \( \Phi = (\Phi_1, ..., \Phi_N) \) and \( \Phi_j \in C^2_{\text{per}}(\mathbb{R}), \ j = 1, ..., N. \) The nonlinear operators \( N_j \) are smooth and

\[ D_{\Phi} N_j(0; \mu) = 0. \]

4.4 The Lyapunov-Schmidt reduction and the existence of \( N \)-modal waves

The kernel of the linear operator on the left-hand side in (4.16) for \( \mu = \mu^* \) is spanned by

\[ \xi_1 = \begin{pmatrix} \cos(\sqrt{|\mu_1^*|}x) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \xi_2 = \begin{pmatrix} 0 \\ \cos(\sqrt{|\mu_2^*|}x) \\ \vdots \\ 0 \end{pmatrix}, ..., \xi_N = \begin{pmatrix} 0 \\ \vdots \\ \cos(\sqrt{|\mu_N^*|}x) \end{pmatrix}. \]

Let us write

\[ \Phi_j = t_j \cos(\sqrt{|\mu_j^*|}x) + \zeta_j, \quad (4.17) \]

where \( \zeta_j \) is orthogonal to \( \cos(\sqrt{|\mu_j^*|}x) \) in \( L^2(-\Lambda_*/2, \Lambda_*/2) \). We note that the vector function \( (\zeta_1, ..., \zeta_N) \) is orthogonal to all \( \xi_j, j = 1, ..., N, \) and hence this splitting corresponds to the Lyapunov-Schmidt decomposition under the action of the projection \( \bar{\Phi} = (\Phi_1, ..., \Phi_N) \) to the space \( X_N \) spanned by \( \xi_1, ..., \xi_N. \)

Thus, substituting (4.17) into (4.16) and taking the projection on \( X_N \) and its compliment, we arrive at the equations

\[ \zeta_j'' - \mu_j^* \zeta_j = \zeta_j(\mu_j - \mu_j^*) + N^*(t, \zeta; \mu) - G_j(t, \zeta; \mu) \cos(\sqrt{|\mu_j^*|}x), \quad (4.18) \]

\[ t_j(\mu_j - \mu_j^*) = G_j(t, \zeta; \mu). \quad (4.19) \]

The nonlinear parts are defined by

\[ N^*(t, \zeta; \mu) = N(\bar{\Phi}; \mu), \quad G_j(t, \zeta; \mu) = \frac{2}{\Lambda_*} \int_{-\Lambda_*/2}^{\Lambda_*/2} N(\bar{\Phi}; \mu) \cos(\sqrt{|\mu_j^*|}x)dx, \]

where \( \bar{\Phi} = (\Phi_1, ..., \Phi_N) \) depends on \( t \) and \( \zeta \) via (4.17). The operator on the left-hand side in (4.18) is invertible, while the right hand side is presented by some terms of order \( |t|^2 \) plus small perturbations of \( \zeta \). Thus, using implicit
function theorem, we can resolve $\zeta = (\zeta_1, ..., \zeta_N)$ from (4.18) as an operator of $t = (t_1, ..., t_N)$ and $\mu = (\mu_1, ..., \mu_N)$, provided $|t|$ and $|\mu - \mu^*|$ are small enough. Furthermore, the following estimate is valid:

$$\|\zeta_j\|_{C^2,\alpha(-\Lambda_*/2,\Lambda_*/2)} \leq C|t|^2.$$  \hspace{1cm} (4.20)

Therefore, system (4.18)-(4.19) is reduced to a system of scalar equations:

$$t_j(\mu_j - \mu_j^*) = G_j(t, \zeta(t, \mu)\mu),$$  \hspace{1cm} (4.21)

where the nonlinear term satisfies

$$|G_j(t, \mu)| \leq C|t|^2, \quad |G_{\mu}(t, \mu)| \leq C|t|^2$$  \hspace{1cm} (4.22)

for all sufficiently small $|t|$ and $|\mu|$, while the constant $C$ is independent of $\mu$. Let us rewrite (4.21) as

$$\mu_j = \mu_j^* + G_j(t, \mu)/t_j.$$

Thus, if $|t|^2/t_j \leq 1/(2C)$, where $C$ is the constant from (4.22), we can apply fixed point theorem to resolve $\mu$ as a function of $t$ so that

$$\mu_j = \mu_j^* + O(|t|^2/\epsilon).$$

Thus, tracking back all changes of variables, we find a solution $(\Phi, \mu)$ to (4.1) such that

$$\Phi(x, z; \mu) = \sum_{j=1}^{N} t_j \cos(\sqrt{|\mu_j^*|} x) \phi_j(z) + O(|t|^2).$$

This completes the proof of the theorem.

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