A note on a difference-type estimator for population mean under two-phase sampling design

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Abstract
In this manuscript, we have proposed a difference-type estimator for population mean under two-phase sampling scheme using two auxiliary variables. The properties and the mean square error of the proposed estimator are derived up to first order of approximation; we have also found some efficiency comparison conditions for the proposed estimator in comparison with the other existing estimators under which the proposed estimator performed better than the other relevant existing estimators. We show that the proposed estimator is more efficient than other available estimators under the two phase sampling scheme for this one example; however, further study is needed to establish the superiority of the proposed estimator for other populations.

Keywords: Study variable, Auxiliary variable, Bias, Mean squared-error, Two phase sampling, Exponential chain-type estimator, Efficiency

Background
In survey sampling, the use of the auxiliary information at the estimation stage is widely used in order to obtain improved designs and the precision of an estimator of the unknown population parameter. When the knowledge of the auxiliary variable is used at the estimation stage, the ratio, product and regression methods of estimation are widely employed in these situations.

The most important topic which is widely discussed in the various probability sampling schemes is the estimation of the population mean of the study variable. A large number of authors have paid their attention towards the formulation of new or modified estimators for the estimation of population mean, for the case, see Hansen and Hurwitz (1943), Sukhatme (1962), Srivastava (1970), Chand (1975), Cochran (1977), Kiregyera (1980, 1984), Srivastava et al. (1990), Bahl and Tuteja (1991), Singh et al. (2006, 2007, 2011), Singh and Choudhury (2012), Khare et al. (2013), Singh and Majhi (2014) and Khan (2015, 2016) etc.

Symbols and notations
Let us consider a finite population of size $N$ of different units $U = \{U_1, U_2, U_3, \ldots, U_N\}$. Let $y$ and $x$ be the study and the auxiliary variable with corresponding values $y_i$ and $x_i$ respectively for the $i$-th unit $i = \{1, 2, 3, \ldots, N\}$ defined in a finite population $U$ with
means $\bar{Y} = (1/N) \sum_{i=1}^{N} y_i$ and $\bar{X} = (1/N) \sum_{i=1}^{N} x_i$ of the study as well as auxiliary variable respectively.

Also let $S_y^2 = (1/N - 1) \sum_{i=1}^{N} (y_i - \bar{Y})^2$ and $S_x^2 = (1/N - 1) \sum_{i=1}^{N} (x_i - \bar{X})^2$ be the population variances of the study and the auxiliary variable respectively and let $C_y$ and $C_x$ be the coefficient of variation of the study as well as auxiliary variable respectively, and $\rho_{yx}$ be the correlation coefficient between $x$ and $y$. Let $y$ and $x$ be the study and the auxiliary variable in the sample with unbiased means $\bar{y} = (1/n) \sum_{i=1}^{n} y_i$ and $\bar{x} = (1/n) \sum_{i=1}^{n} x_i$ respectively.

Also let $\hat{S}_y^2 = (1/n - 1) \sum_{i=1}^{n} (y_i - \bar{y})^2$ and $\hat{S}_x^2 = (1/n - 1) \sum_{i=1}^{n} (x_i - \bar{x})^2$ be the corresponding sample variances of the study as well as auxiliary variable respectively. Let $S_{yx} = \sum_{i=1}^{N} (y_i - \bar{Y})(x_i - \bar{X})$, $S_{yz} = \sum_{i=1}^{N} (y_i - \bar{Y})(z_i - \bar{Z})$ and $S_{xz} = \sum_{i=1}^{N} (x_i - \bar{X})(z_i - \bar{Z})$ be the co-variances between their respective subscripts respectively. Similarly $b_{yx(n)} = \frac{S_{yx}}{S_x^2}$ is the corresponding sample regression coefficient of $y$ on $x$ based on a sample of size $n$. Also $C_y = \frac{S_y}{\bar{y}}$, $C_x = \frac{S_x}{\bar{x}}$ and $C_z = \frac{S_z}{\bar{z}}$ are the coefficients of variations of the study and auxiliary variables respectively.

Also $\theta = \left( \frac{1}{n} - \frac{1}{N} \right)$, $\theta_1 = \left( \frac{1}{n} - \frac{1}{N} \right)$ and $\theta_2 = \left( \frac{1}{n} - \frac{1}{N} \right)$.

### Some existing estimators

Let us consider a finite population $U = \{U_1, U_2, U_3, ..., U_N\}$ of size $N$ units. To estimate the population mean $\bar{Y}$ of the variable of interest say $y$ taking values $y_i$ in the existence of two auxiliary variables say $x$ and $z$ taking values $x_i$ and $z_i$ for the $i^{th}$ unit $U_i$. We assume that there is a high correlation between $y$ and $x$ as compared to the correlation between $y$ and $z$, i.e. $\rho_{yx} > \rho_{yz} > 0$. When the population $\bar{X}$ of the auxiliary variable $x$ is unknown, but information on the other cheaply auxiliary variable say $z$ closely related to $x$ but compared to $x$ remotely to $y$, is available for all the units in a population. In such a situation we use a two phase sampling. In the two phase sampling scheme a large initial sample of size $n' (n' < N)$ is drawn from the population $U$ by using simple random sample without replacement sampling (SRSWOR) scheme and measure $x$ and $z$ to estimate $\bar{X}$. In the second phase, we draw a sample (subsample) of size $n$ from first phase sample of size $n'$, i.e. ($n < n'$) by using (SRSWOR) or directly from the population $U$ and observed the study variable $y$.

The variance of the usual simple estimator $t_0 = \bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$ up to first order of approximation is, given by

$$V(t_0) = \theta S_y^2$$  \hspace{1cm} (1)

The classical ratio and regression estimators in two-phase probability sampling and their mean square errors up to first order of approximation are, given by

$$t_1 = \frac{\bar{y}}{\bar{x}}$$  \hspace{1cm} (2)

$$MSE(t_1) = \bar{Y}^2 \left[ \theta C_y^2 + \theta_2 \left( C_x^2 - 2 \rho_{yx} C_y C_x \right) \right]$$  \hspace{1cm} (3)

$$t_2 = \bar{y} + b_{yx(n)} (x' - \bar{x})$$  \hspace{1cm} (4)
Chand (1975), suggested the following chain ratio-type estimator the suggested estimator is, given by

\[ t_3 = \frac{\bar{y} \bar{x}'}{\bar{x}'} \bar{z} \]  

The mean square error of the suggested estimator is, given as

\[ MSE(t_3) = \bar{Y}^2 \left[ \theta C_y^2 + \theta_2 \left( C_z^2 - 2 \rho_{yx} C_y C_x \right) + \theta_1 \left( C_z^2 - 2 \rho_{yz} C_y C_z \right) \right] \]  

Khare et al. (2013), proposed a generalized chain ratio in regression estimator for population mean, the recommended estimator is given by

\[ t_4 = \bar{y} + b_{yx} \left\{ \bar{x}' \left( \frac{\bar{z}}{\bar{z}'} \right) - \bar{x} \right\} \]  

where \( \alpha \) is the unknown constant, and the minimum mean square error at the optimum value of \( \alpha = \frac{\rho_{yx} C_x}{\rho_{yx} C_z} \) is, given by

\[ MSE(t_4) = \bar{Y}^2 C_y^2 \left[ \theta - \theta_2 \rho_{yx}^2 - \theta_1 \rho_{yz}^2 \right] \]  

Recently Singh and Mahji (2014), suggested a chain-type exponential estimators for \( \bar{Y} \) given by

\[ t_5 = \frac{\bar{y} - \bar{x}'}{\bar{x}} \exp \left( \frac{\bar{z} - \bar{z}'}{\bar{z} + \bar{z}'} \right) \]  

\[ t_6 = \bar{y} + b_{yx(n)} \left\{ \bar{x}' \exp \left( \frac{\bar{z} - \bar{z}'}{\bar{z} + \bar{z}'} \right) - \bar{x} \right\} \]  

\[ t_7 = \bar{y} \exp \left( \frac{\bar{x}' - \bar{x}}{\bar{x}' + \bar{x}} \right) \frac{\bar{z}}{\bar{x}} \]  

The mean square errors of the suggested estimators, up to first order of approximation are, given as follows

\[ MSE(t_5) = \bar{Y}^2 \left[ \theta C_y^2 + \theta_2 \left( C_z^2 - 2 \rho_{yx} C_y C_x \right) + \frac{\theta_1}{4} \left( C_z^2 - 4 \rho_{yz} C_y C_z \right) \right] \]  

\[ MSE(t_6) = \bar{Y}^2 C_y^2 \left[ \theta_2 \left( 1 - \rho_{yx}^2 \right) + \theta_1 \left( 1 + \frac{\rho_{yx}^2 C_z^2}{C_x^2} - \rho_{xy} \rho_{yz} \frac{C_z}{C_x} \right) \right] \]  

\[ MSE(t_7) = \bar{Y}^2 \left[ \theta C_y^2 + \frac{\theta_1}{4} \left( C_z^2 - 4 \rho_{yx} C_y C_z \right) + \theta_1 \left( C_z^2 - 2 \rho_{yz} C_y C_z \right) \right] \]
The proposed estimator

On the lines of Khare et al. (2013), we propose a difference-type estimator for population mean under two-phase sampling scheme using two auxiliary variables; the suggested estimator is, given by

\[ t_m = \bar{y} + k_1 \left( \frac{\bar{Z}}{\bar{x}} \bar{x'} - \bar{x} \right) + k_2 \left( \frac{\bar{Z}'}{\bar{x}} - \bar{z} \right) \] (16)

where \( k_1 \) and \( k_2 \) are the unknown constants.

To obtain the properties of the proposed estimator we define the following relative error terms and their expectations.

Let \( e_0 = \frac{\bar{y} - \bar{Y}}{\bar{Y}} \), \( e_1 = \frac{\bar{x} - \bar{X}}{\bar{X}} \), \( e_1' = \frac{\bar{x'} - \bar{X}}{\bar{X}} \), \( e_2 = \frac{\bar{Z} - \bar{Y}}{\bar{Y}} \) and \( e_2' = \frac{\bar{Z}' - \bar{Y}}{\bar{Y}} \), such that

\[ E(e_0) = E(e_i) = E(e_i') = 0, \quad \text{for } i = 1, 2. \]

\[ E(e_0^2) = \theta C_y^2, \quad E(e_i^2) = \theta C_x^2, \quad E(e_1 e_1') = \theta_1 C_x^2, \quad E(e_1 e_2) = \theta C_x^2, \quad E(e_2 e_2') = \theta C_x^2. \]

Rewriting (16), in terms of \( e's \), we have

\[ t_m = \left( \bar{Y} (1 + e_0) + k_1 \bar{X} \left( (1 + e_1') (1 + e_2') - (1 + e_1) \right) \right) \]

\[ + k_2 \bar{Z} \left( (1 + e_1') (1 + e_1) - (1 + e_2) \right) \]

Expanding the right hand side of the above equation, and neglecting terms of \( e's \) having power greater than two, we have

\[ t_m - \bar{Y} = \left( \bar{Y} e_0 - k_1 \bar{X} \left( e_1 - e_1' + e_2 + e_2' + e_1 e_2' \right) - k_2 \bar{Z} \left( e_1 - e_1' - e_2 + e_1 e_2' \right) \right) \] (17)

On squaring and taking expectation on both sides of Eq. (17), and keeping terms up to second order, we have

\[ MSE(t_m) = E \left[ \bar{Y}^2 e_0^2 + k_1^2 \bar{X}^2 \left( e_1^2 + e_1'^2 + e_2^2 + 2 e_1 e_1' + 2 e_1 e_2' - 2 e_1 e_2 \right) \right] \]

\[ + k_2^2 \bar{Z}^2 \left( e_1^2 + e_1'^2 + e_2^2 - 2 e_1 e_1' + 2 e_1 e_2 - 2 e_1 e_2' \right) \]

\[ + 2 k_1 k_2 \bar{X} \bar{Z} \left( e_1^2 + e_1'^2 + e_1 e_2' - e_1 e_2 - e_1' e_2 - e_2' e_1 + e_1 e_2' \right) \]

\[ - 2 k_1 \bar{Y} \bar{X} \left( e_0 e_1 - e_0 e_1' + e_0 e_2' \right) - 2 k_3 \bar{Z} \left( e_0 e_1 - e_0 e_1' + e_0 e_2' \right) \]

Further simplifying, we get

\[ MSE(t_m) = \left[ \bar{Y}^2 \theta C_y^2 + k_1^2 \bar{X}^2 \left( \theta_1 C_x^2 + \theta_2 C_x^2 \right) + k_2^2 \bar{Z}^2 \left( \theta C_x^2 + \theta_2 C_x^2 + 2 \theta_2 C_{xx} \right) \right] \]

\[ + 2 k_1 k_2 \bar{X} \bar{Z} \left( \theta_2 C_x^2 + \theta_1 C_x^2 + \theta_2 C_{xx} \right) \]

\[ - 2 k_1 \bar{Y} \bar{X} \left( \theta_2 C_{xx} + \theta_1 C_{yz} \right) - 2 k_2 \bar{Z} \left( \theta_2 C_{yy} + \theta C_{yy} \right) \] (18)
Now to find the minimum mean squared error of \( t_m \), we differentiate Eq. (18) with respect to \( k_1 \) and \( k_2 \) respectively and putting it equal to zero, that is

\[
\frac{\partial \text{MSE}(t_m)}{\partial k_1} = 0 \quad \text{and} \quad \frac{\partial \text{MSE}(t_m)}{\partial k_2} = 0
\]

\[
k_{1_{\text{opt}}} = \frac{\bar{Y}(BC - DE)}{X(AB - E^2)} \quad \text{and} \quad k_{2_{\text{opt}}} = \frac{\bar{Y}(AD - CE)}{Z(AB - E^2)}.
\]

where \( A = \theta_1 C_z^2 + \theta_2 C_{z^2}, B = \theta C_z^2 + \theta_2 C_x^2 + 2\theta_2 C_{xz}, C = \theta_2 C_{y^2} + \theta_1 C_{yz}, D = \theta_2 C_{yx} + \theta C_y \) and \( E = \theta_1 C_x^2 + \theta_2 C_{x^2} + \theta C_{x^2}. \)

On substituting the optimum values of \( k_1 \) and \( k_2 \) in Eq. (18) we get the minimum mean square error \((\text{MSE})\) of the proposed estimator \( t_m \) up to order one is, given as

\[
\text{MSE}(t_m)_{\text{min}} = \bar{Y}^2 \left[ \theta C_y^2 - \frac{(AD^2 + BC^2 - 2CDE)}{(AB - E^2)} \right]
\]

(19)

**Efficiency comparison**

In this section, we have compare the propose estimator with the other existing estimators.

1. By Eqs. (1) and (19),

\[
\text{MSE}(t_m)_{\text{min}} < \text{MSE}(t_0) \quad \text{if} \quad \left[ \frac{(AD^2 + BC^2 - 2CDE)}{(AB - E^2)} \right] > 0.
\]

2. By Eqs. (3) and (19)

\[
\text{MSE}(t_m)_{\text{min}} < \text{MSE}(t_1) \quad \text{if} \quad \left[ \frac{(AD^2 + BC^2 - 2CDE)}{(AB - E^2)} + \theta_2 \left( C_x^2 - 2\rho_{yx} C_{y^2} \right) \right] > 0.
\]

3. By Eqs. (5) and (19),

\[
\text{MSE}(t_m)_{\text{min}} < \text{MSE}(t_2) \quad \text{if} \quad \left[ \frac{(AD^2 + BC^2 - 2CDE)}{(AB - E^2)} - \theta_2 C_y^2 \rho_{yx}^2 \right] > 0.
\]

4. By Eqs. (7) and (19),

\[
\text{MSE}(t_m)_{\text{min}} < \text{MSE}(t_3) \quad \text{if} \quad \left[ \theta_2 C_x \left( C_x - 2\rho_{yx} C_y \right) + \theta_1 C_z \left( C_z - 2\rho_{yz} C_y \right) + \frac{(AD^2 + BC^2 - 2CDE)}{(AB - E^2)} \right] > 0.
\]

5. By Eqs. (9) and (19),

\[
\text{MSE}(t_m)_{\text{min}} < \text{MSE}(t_4) \quad \text{if} \quad \left[ \frac{(AD^2 + BC^2 - 2CDE)}{(AB - E^2)} - \left( \theta_2 \rho_{yx}^2 + \theta_1 \rho_{yz}^2 \right) C_y^2 \right] > 0.
\]

6. By Eqs. (13) and (19),

\[
\text{MSE}(t_m)_{\text{min}} < \text{MSE}(t_5) \quad \text{if} \quad \left[ \theta_2 \left( C_x^2 - 2C_{xy} \right) + \frac{\theta_1}{4} \left( C_z^2 - 4C_{yz} \right) + \frac{(AD^2 + BC^2 - 2CDE)}{(AB - E^2)} \right] > 0.
\]
7. By Eqs. (14) and (19),
\[
MSE(t_m)_{\text{min}} < MSE(t_0) \quad \text{if} \quad \left[ \theta_1 C_y^2 \left( \frac{\rho_{yx}^2 C_y^2}{4 C_x^2} - \rho_{yx} \rho_{yz} \frac{C_z}{C_x} \right) - \theta_2 \rho_{yx}^2 C_y^2 + \frac{(AD^2 + BC^2 - 2CDE)}{(AB - E^2)} \right] > 0.
\]

8. By Eqs. (15) and (19),
\[
MSE(t_m)_{\text{min}} < MSE(t_f) \quad \text{if} \quad \left[ \frac{(AD^2 + BC^2 - 2CDE)}{(AB - E^2)} + \frac{\theta_2}{4} \left( C_x^2 - 4C_{xy} \right) + \theta_1 \left( C_y^2 - 2C_{yz} \right) \right] > 0.
\]

**Numerical comparison**

To examine the performance of the proposed estimator with various existing estimators, we have considered a real data set from the literature the description of the population are, given by

Population Source, (Cochran 1977).

- **y**: Number of placebo children;
- **x**: Number of paralytic polio cases in the placebo group;
- **z**: Number of paralytic polio cases in the not inoculated group.

\[ N = 34, n' = 15, n = 10, \bar{Y} = 4.92, \bar{X} = 2.59, \bar{Z} = 2.91, C_y^2 = 1.0248, C_x^2 = 1.5175, C_z^2 = 1.1492, C_{xy} = 0.9136, C_{yz} = 0.6978, \rho_{yx} = 0.7326, \rho_{zy} = 0.6430, \rho_{xz} = 0.6837 \] (Table 1).

We have use the following expression for Percentage Relative Efficiency (PRE)

\[
PRE = \left[ \frac{\text{Var}(t_0)}{MSE(t_j) \text{ or } \text{Var}(t_j)} \right] \times 100, \quad \text{for } j = 0, 1, 2, 3, 4, 5, 6, 7 \text{ and } m.
\]

**Table 1** The mean square errors (MSE’s) and the Percent relative efficiencies (PRE’s) of the estimators with respect to \( t_0 \)

| Population | Estimator | MSE’s         | PRE (\( t_0, t_j \)) |
|------------|-----------|---------------|----------------------|
|            | \( t_0 \) | 1.7525        | 100.00               |
|            | \( t_1 \) | 1.5032        | 116.59               |
|            | \( t_2 \) | 1.3073        | 134.06               |
|            | \( t_3 \) | 1.2793        | 137.00               |
|            | \( m \)   | 0.9247        | 189.52               |
|            | \( t_5 \) | 1.1312        | 154.92               |
|            | \( t_6 \) | 1.0227        | 171.36               |
|            | \( t_7 \) | 1.0982        | 159.58               |
|            | \( t_m \) | 0.8206        | 213.56               |
Conclusion

From the above table, we have observed that the proposed estimator has smaller mean square error and has higher percent relative efficiency than the other existing estimators. However, although the proposed estimator has the highest percent relative efficiency than other existing estimators for this one example, it could have lower relative efficiency for other populations. Further work is needed before it can be recommended for general use in practical surveys.

Authors' contributions

The authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

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Competing interests

The authors declare that they have no competing interests.

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