The Monsky-Washnitzer cohomology and the de Rham cohomology

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Abstract

The author constructs a theory of dagger formal schemes over $R$ and then defines the de Rham cohomology for flat dagger formal schemes $X$ with integral and regular reductions $\bar{X}$ which generalizes the Monsky-Washnitzer cohomology. Finally the author gets Lefschetz’ fixed pointed formula for $X$ with certain conditions.

1 Introduction

Let $K$ be a finite extension of $\mathbb{Q}_p$ with $R$ its ring of integers and $k = \mathbb{F}_q$ its residue field. Let $\pi$ be a uniformizer of $R$.

Let $\bar{X}$ be a regular algebraic variety over $k$. Define the zeta-function $Z(\bar{X}|k, t)$ of $\bar{X}$ by

$$Z(\bar{X}|k, t) = \exp\left(\sum_{s \geq 1} N_s t^s\right)$$

with $N_s$ the number of $\mathbb{F}_q$-points of $\bar{X}$.

Weil’s conjecture says that $Z(\bar{X}|k, t)$ is a rational function. To prove it, one tries to find suitable cohomology such that the Lefschetz’ fixed points formula holds. When $\bar{X}$ is an affine integral and regular variety over $k$, the Monsky-Washnitzer cohomology is such a cohomology.

Let $\bar{X} = \text{Spec}(\bar{A})$ with $\bar{A}$ integral and regular over $k$. Let $A$ be a flat w.c.f.g. algebra over $R = W(k)$ which is a lift of $\bar{A}$. Note that every flat lift of $\bar{A}$ is $R$-isomorphic to $A$, and that $\Omega^i(A) = \Omega^i(A/R)$ is a projective $A$-module. One can define the de Rham complex $\Omega(A)$

$$0 \to \Omega^0(A) \xrightarrow{d^0} \Omega^1(A) \xrightarrow{d^1} \Omega^2(A) \to \cdots$$

with $\Omega^i(A) = \bigwedge^i \Omega^1(A)$. $H^i_{\text{MW}}(\bar{X}, K) := H^i(\Omega(A)) \otimes_R K$ is the definition of the Monsky-Washnitzer cohomology. Note that $H^i(\Omega(A)) \otimes_R K = H^i(\Omega(A) \otimes_R K)$. If $F$ is a lift of the Frobenius map $x \mapsto x^q$ over $\bar{A}$ to $A$, then $F^*$ induces an isomorphism over $H^i_{\text{MW}}(\bar{X}, K)$ which does not depend on the choice of lift $F$. Moreover, $(F^*)^{-1}$ is a nuclear operator, and satisfies

$$N_s = \sum (-1)^i \text{tr}(q^i(F^*)^{-1})^s H^i_{\text{MW}}(\bar{X}, K)).$$
We find a generalization of the Monsky-Washnitzer cohomology. We construct a theory of dagger formal schemes over $R$ and define the de Rham cohomology for flat dagger formal schemes $X$ with integral and regular reductions.

An affine dagger formal scheme is a pair $(\text{Spec}(A \otimes_R k), \mathcal{O}^\dagger)$, where $A$ is a w.c.f.g. algebra over $R$, and $\mathcal{O}^\dagger$ is a sheaf over $\tilde{X} = \text{Spec}(A \otimes_R k)$ such that $\mathcal{O}^\dagger(\tilde{X}_f) = A(f^{-1})^\dagger$. Let $\text{Spf}(A)$ denote the pair $(\text{Spec}(A \otimes_R k), \mathcal{O}^\dagger)$. $\mathcal{O}^\dagger$ is called the structure sheaf of $\text{Spf}(A)$. One can show $\text{Spf}(A)$ is a locally ringed space. A dagger formal scheme is a locally ringed space $(X, \mathcal{O}^\dagger_X)$ in which every point has an open neighborhood $U$, such that $(U, \mathcal{O}^\dagger_X|_U)$ is an affine dagger formal scheme.

When $X$ is flat, separated and Noetherian over $R$ with $\tilde{X}$ integral and regular, the sheaf $\Omega^1_X$ over $X$ is locally free. So one can define the de Rham complex $\Omega^\cdot_X$ and the de Rham cohomology $H^i_{dR}(X; K) := H^i(X, (\Omega^\cdot_X \otimes_R K))$. When $X = \text{Spf}(A)$, we have $H^i_{dR}(X, K) = H^i_{MW}(\tilde{X}, K)$. When $R$ is $W(k)$, and there is a lift $F$ of the Frobenius of $\tilde{X}$ to $X$, we have the following theorem.

**Theorem.** Let $X$ be a flat separated and Noetherian dagger formal scheme with $\tilde{X} = X \otimes_R k$ regular and integral of dimension $n$. Then $F^*$ is an isomorphism, $(F^*)^{-1}$ is nuclear over $H^i_{dR}(X, K)$. And we have the following formula

$$N_s(\tilde{X}) = \sum (-1)^i \text{tr}((q^n(\mathbb{F}^*)^{-1})^s|H^i_{dR}(X, K)).$$

### 2 Weakly complete finitely generated algebra

#### 2.1 Definition of weakly complete finitely generated algebra

For a nonnegative integer $n$, let us define

$$T_n := \left\{ \sum_{v \in \mathbb{N}^n} c_v \xi^v \in R[\xi_1, ..., \xi_n] : |c_v| \to 0 \right\},$$

and

$$T_n^\dagger := \left\{ \sum_{v \in \mathbb{N}^n} c_v \xi^v \in R[\xi_1, ..., \xi_n] : \exists \varepsilon > 0, |c_v|^{p^{\varepsilon |v|}} \to 0 \right\}.$$

It is well know that $T_n$ is complete with respect to the Gauss norm. There is also a Gauss norm over $T_n^\dagger$. But $T_n$ is not complete with respect to this norm.

**Proposition 1.** (8) $T_n^\dagger$ satisfies Weierstrass preparation and division. As a consequence, $T_n^\dagger$ is noetherian and flat over $R[\xi_1, ..., \xi_n]$.

**Definition.** A weakly complete finitely generated (w.c.f.g.) algebra $A$ over $R$ is a homomorphic image of some $T_n^\dagger$. While, a complete finitely generated (c.f.g.) algebra $\tilde{A}$ over $R$ is a homomorphic image of some $T_n$. 

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2.2 Faithful flat of \( \hat{A} \) over \( A \)

Let \( A \) be a noetherian ring with \( I \) an ideal of \( A \). Then the \( I \)-adic completion \( \hat{A} \) of \( A \) is flat over \( A \).

In this subsection, we need to consider when \( \hat{A} \) is a faithful flat over \( A \). For this, we have the following theorem.

**Theorem 1**. (\[2\]) Let \( A \) be a noetherian ring with an adic topology, and let \( I \) be an ideal of definition. Then the following is equivalent.

1. \( A \) is a Zariski ring, i.e., every ideal is closed in it.
2. \( I \subseteq \text{rad}(A) \).
3. Every finite \( A \)-module \( M \) is separated in the \( I \)-adic topology.
4. In every finite \( A \)-module \( M \), every submodule is closed in the \( I \)-adic topology.
5. The completion \( \hat{A} \) of \( A \) is faithful flat over \( A \).

We have the following lemma.

**Lemma 1.** A w.c.f.g. algebra \( A \) over \( R \) is a Zariski ring.

**Proof.** It is enough to show that the lemma holds when \( A = T_n \).

For every \( f \in T_n \), let us show \( 1 - \pi f \) is invertible. In fact, we only need to show \( \sum_{n=0}^{\infty} (\pi f)^n \in T_n^\dagger \). Let \( f = \sum_{\alpha \in \mathbb{N}} a_\alpha \xi^\alpha \). Since \( f \in T_n \), there are two positive numbers \( M \) and \( \varepsilon \) such that \( |a_\alpha| < M p^{-\varepsilon |\alpha|} \). Therefore, there is a positive integer \( N \) such that when \( |\alpha| \geq N \), \( |a_\alpha| < p^{-\varepsilon |\alpha|} \).

Let \( \varepsilon' = \min\left(\frac{\varepsilon}{\pi}, \frac{\log |a_\alpha|}{N}\right) \) and \( \pi f = \sum_{\alpha \in \mathbb{N}} a_\alpha' \xi^\alpha \). Then

\[
|a_\alpha'| < p^{-\varepsilon' |\alpha|}.
\]

It is easy to show \( \sum_{n=0}^{\infty} (\pi f)^n \) converges in \( R[[\xi_1, \ldots, \xi_n]] \). Let \( \sum_{n=0}^{\infty} (\pi f)^n = \sum_{\alpha \in \mathbb{N}} b_\alpha \xi^\alpha \). Then from formula (1), we know \( |b_\alpha| < p^{-\varepsilon' |\alpha|} \). Therefore, \( \sum_{n=0}^{\infty} (\pi f)^n \in T_n^\dagger \) as desired.

Now, the following proposition is easily deduced from Lemma 1 and Theorem 1.

**Proposition 2.** Let \( A \) be a w.c.f.g. algebra with \( \hat{A} \) its completion according to the \( \pi \)-adic topology. Then \( \hat{A} \) is a faithful flat \( A \)-algebra and \( A \to \hat{A} \) is universal injective.

Let \( A \) be a w.c.f.g. algebra with \( f \in A \). Let us define \( A(f^{-1})^\dagger \) to be \( A(\xi^\dagger)/(\pi f^\dagger - \xi f^{-1}) \). Then \( A(f^{-1})^\dagger \) is a w.c.f.g. algebra. If \( f, g \) are two elements in \( A \) such that \( f - g \in \pi A \), then \( A(f^{-1})^\dagger \) is canonically isomorphic to \( A(g^{-1})^\dagger \). We have the following proposition.

**Proposition 3.** \( A(f^{-1})^\dagger \) is a flat \( A \)-algebra.

**Proof.** It is well known that \( A_{f^{-1}} \) is a flat \( A \)-algebra. \( A_{f^{-1}} \) and \( A(f^{-1})^\dagger \) have the same \( \pi \)-adic completion \( B \). \( B \) is a flat \( A_{f^{-1}} \)-algebra and is a faithful flat \( A(f^{-1})^\dagger \)-algebra according to Proposition 2. Therefore, \( A(f^{-1})^\dagger \) is a flat \( A \)-algebra.

From now on, we always write \( \hat{A} \) for the \( \pi \)-adic completion of a w.c.f.g. \( R \)-algebra \( A \).

Let \( A_\lambda = A \otimes_R (R/\pi^\lambda R) \).

**Proposition 4.** Let \( \varphi : A \to B \) be a morphism of w.c.f.g. \( R \)-algebras. And let \( M \) be a finite \( B \)-module. Then \( M \) is (faithful) flat over \( A \) if and only if \( M \otimes_R R_\lambda \) is (faithful) flat over \( A \otimes_R R_\lambda \) for all \( \lambda \in \mathbb{N} \).

Its proof is similar to that in \[3\] Lemma 1.6.
Let $A$ be a flat w.c.f.g. algebra over $R$. Let $f$ be an element of $A$ with $f_1, ..., f_r \in (f)$. Assume that
\[ f \in \sqrt{(f_1, ..., f_r)} \quad (*) \]

Then we have the following corollary of Proposition 4.

**Corollary 1.** \( \prod A(f_i^{-1})^\dagger \) is faithful flat over \( A(f^{-1})^\dagger \).

If $A$ is a flat w.c.f.g. $R$-algebra, then both $A \to A \otimes_R K$ and $\tilde{A} \to \tilde{A} \otimes_R K$ are injective. Moreover, we have the following lemma.

**Lemma 2.** In $\tilde{A} \otimes_R K$, the intersection of $A \otimes_R K$ and $\tilde{A}$ is $A$.

**Proof.** Since $A$ is a Zariski ring, every ideal $I$ of $A$ is closed in $A$. So $I\tilde{A} \cap A = I$. See [10] for more details.

For any $\frac{\alpha}{\beta} \in A \otimes K$ with $\alpha \in A$, if $\frac{\alpha}{\beta} \in \tilde{A}$, then $a\tilde{A} \subseteq \pi^nA$. We get $aA \subseteq \pi^nA$. Then $\frac{\alpha}{\beta} \in A$ as desired. 

\[ \square \]

### 2.3 Dagger rigid geometry

We can construct a theory of rigid geometry using dagger affinoid algebras over $K$ instead of affinoid algebras.

**Definition.** A dagger affinoid algebra over $K$ is a homomorphic image of some $T_n^\dagger \otimes_R K$.

There is a one to one correspondence between prime ideals of $T_n^\dagger \otimes_R K$ and prime ideals of $T_n^\dagger$, that do not contain $\pi$. The same assertion holds if we use $T_n \otimes_R K$ and $T_n^\dagger$, instead of $T_n \otimes_R K$ and $T_n$, respectively. Since $T_n$ is faithful flat over $T_n^\dagger$, every (maximal) prime ideal of $T_n \otimes_R K$ is the restriction of a prime (maximal) ideal of $T_n \otimes_R K$. Moreover, if $I$ is an ideal of $T_n \otimes_R K$, then $I = I(T_n \otimes_R K) \cap (T_n^\dagger \otimes_R K)$.

Let $M^\dagger$ be a maximal ideal of $T_n \otimes_R K$, then $T_n \otimes_R K/M^\dagger$ is an algebraic extension of $K$. So the restriction of $M^\dagger$ to $T_n^\dagger \otimes_R K$ is again a maximal ideal. Since $T_n^\dagger \otimes_R K$ is dense in $T_n \otimes_R K$, two different maximal ideals have different restrictions over $T_n^\dagger \otimes_R K$. Therefore, $T_n^\dagger \otimes_R K$ and $T_n \otimes_R K$ have the same maximal spectral. Let $B = T_n^\dagger \otimes_R K/I$, and let $I^\dagger$ be $I(T_n \otimes_R K)$. Then $I = I^\dagger \cap (T_n^\dagger \otimes_R K)$. Let $B^\dagger = T_n \otimes_R K/I^\dagger$, then $B \to B^\dagger$ is injective. And there is a one to one correspondence between $\text{Spm}(B)$ and $\text{Spm}(B^\dagger)$. Moreover $B^\dagger$ is the completion of $B$ according to the maximal spectral.

We can define Weierstrass domains, Laurent domains and rational domains for $X = \text{Spm}(B)$, and then define Grothendieck topology for $X$. We use $X^\dagger$ to denote $\text{Spm}(B^\dagger)$.

**Definition.**

(i). A subset in $X$ of type
\[ X(f_1, ..., f_r) := \{ x \in X : |f_i(x)| \leq 1 \} \]

for $f_1, ..., f_r \in B$ is called a Weierstrass domain in $X$.

(ii). A subset in $X$ of type
\[ X(f_1, ..., f_r, g_1^{-1}, ..., g_s^{-1}) := \{ x \in X : |f_i(x)| \leq 1 ; |g_j(x)| \geq 1 \} \]

for $f_1, ..., f_r, g_1, ..., g_s \in B$ is called a Laurent domain in $X$. 

\[ 4 \]
(iii). A subset in $X$ of type

$$X\left(\frac{f_1}{f_0}, \ldots, \frac{f_r}{f_0}\right) := \{x \in X : |f_i(x)| \leq |f_0(x)|\}$$

for $f_0, \ldots, f_r \in B$ without common zero is called a rational domain in $X$.

If $f'_1, \ldots, f'_r, g'_1, \ldots, g'_s$ are all nonzero elements in $B^\vee$, choose $f_1, \ldots, f_r, g_1, \ldots, g_s \in B$ such that $\|f_i - f'_i\|_{\max} \leq 1$, $\|g_i - g'_i\|_{\max} \leq 1$. Then

$$X(f_1, \ldots, f_r) = X'(f'_1, \ldots, f'_r),$$

and

$$X(f_1, \ldots, f_r, g_1^{-1}, \ldots, g_s^{-1}) = X'(f'_1, \ldots, f'_r, g'_1^{-1}, \ldots, g'_s^{-1}).$$

Assume $f'_0, \ldots, f'_r \in B^\vee$ have no common zero. Let $U_i^\vee$ denote $X^\vee(f'_0, \ldots, f'_i)$. Then there is a positive number $\varepsilon$ such that $|f'_i(x)| > \varepsilon$ for any $x \in U_i^\vee$.

Choose $f_0, \ldots, f_r \in B$ such that $\|f_i - f'_i\|_{\max} < \varepsilon$, then

$$U_i^\vee = X\left(\frac{f_0^i}{f_0}, \ldots, \frac{f_r^i}{f_0}\right).$$

Therefore, we can define the Grothendieck topology for $\text{Spm}(B)$ such that it has the same Grothendieck topology as $\text{Spm}(B^\vee)$.

We can get Tate’s acyclic theorem for dagger affine algebras in the same way as in [1] or as in [2]. Let $A$ be a flat w.c.f.g. algebra over $R$. Let $B = A \otimes_R K$ and $X = \text{Spm}(B)$. Let $f$ be an element of $A$ with $f_1, \ldots, f_r \in (f)$. If we have $\overline{f_1, \ldots, f_r}$ in $A \otimes_R K$, then $\{(f_i^{-1})\}_{i=1}^r$ is a finite affine covering of $X(f^{-1})$. Therefore as a consequence of Tate’s acyclic theorem, we get the following exact sequence

$$0 \rightarrow A(f^{-1})^\dagger \otimes_R K \rightarrow \prod_i A(f_i^{-1})^\dagger \otimes_R K \rightarrow \prod_{i,j} A((f_i f_j)^{-1})^\dagger \otimes_R K. \quad (2)$$

**Proposition 5.** For any $A$-module $M$, the following sequence is exact

$$0 \rightarrow A(f^{-1})^\dagger \otimes_A M \rightarrow \prod_i A(f_i^{-1})^\dagger \otimes_A M \rightarrow \prod_{i,j} A((f_i f_j)^{-1})^\dagger \otimes_A M. \quad (3)$$

**Proof.** (i). We have the following commutative diagram ($\otimes = \otimes_A$ in this proof)

$$\begin{array}{cccccc}
0 & \longrightarrow & A(f^{-1})^\dagger \otimes M & \longrightarrow & \prod_i A(f_i^{-1})^\dagger \otimes M & \longrightarrow & \prod_{i,j} A((f_i f_j)^{-1})^\dagger \otimes M \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & (A(f^{-1})^\dagger)^\gamma \otimes M & \longrightarrow & \prod_i (A(f_i^{-1})^\dagger)^\gamma \otimes M & \longrightarrow & \prod_{i,j} (A((f_i f_j)^{-1})^\dagger)^\gamma \otimes M.
\end{array}$$

From the theory of formal schemes, we know the second line is exact. From Proposition 2, we see all vertical maps are injective. From Corollary 1, we get that $0 \rightarrow A(f^{-1})^\dagger \otimes M \rightarrow \prod_i A(f_i^{-1})^\dagger \otimes M$ is exact. Therefore,
show the first line is exact, it is enough to show that \((\prod \langle f_i^{-1} \rangle^\dagger \otimes M) \cap (A(f^{-1})^\gamma \otimes M) = A(f^{-1})^\dagger \otimes M\).

(ii). From formula (3), the second line of the above commutative diagram (with \(M = A\)) and Lemma 2, we see when \(M = A\), the first line of the above commutative diagram is also exact. Then \((\prod \langle f_i^{-1} \rangle^\dagger) \cap (A(f^{-1})^\gamma) = A(f^{-1})^\dagger\).

(iii). Let \(A'\) be the sum of \((A(f^{-1})^\gamma) \text{ and } \prod \langle f_i^{-1} \rangle^\dagger \in \prod (A(f_i^{-1})^\gamma)\). Then \(A'\) is a Zariski ring, since it is a quotient of the Zariski ring \((\prod A(f_i^{-1})^\gamma) \times \prod \langle f_i^{-1} \rangle^\dagger\). \(A'\) is dense in \(\prod (A(f_i^{-1})^\gamma)\), so \((A')^\gamma = \prod (A(f_i^{-1})^\gamma)\). Therefore \(\prod (A(f_i^{-1})^\gamma)\) is faithful flat over \(A'\) and \(A' \to \prod (A(f_i^{-1})^\gamma)\) is universal injective. We see that \(A'\) is a flat \(A\)-algebra.

Then both
\[0 \to A(f^{-1})^\dagger \otimes M \to (\oplus_1 A(f_i^{-1})^\dagger \otimes M) \oplus (A(f^{-1})^\gamma \otimes M) \cong A' \otimes M \to 0\]
with \(p : (a, b) \mapsto a - b\), and
\[0 \to A' \otimes M \to \prod_i (A(f_i^{-1})^\dagger) \otimes M\]
are exact. So we get the following exact sequence
\[0 \to A(f^{-1})^\dagger \otimes M \to (\oplus_1 A(f_i^{-1})^\dagger \otimes M) \oplus (A(f^{-1})^\gamma \otimes M) \to \prod_i (A(f_i^{-1})^\gamma) \otimes M.\]
Then we obtain \(\prod_i A(f_i^{-1})^\dagger \cap (A(f^{-1})^\gamma) \otimes M = A(f^{-1})^\dagger \otimes M\) as desired. \(\square\)

3  Dagger formal schemes

3.1  Theory of Čech cohomology

In this subsection, we recall the theory of Čech cohomology. We follow \cite{ref} closely.

Let \(X\) be a (compact) topology space. Let \(S\) denote the category of sheaves over \(X\), and let \(P\) denote the category of presheaves over \(X\). Let \(i : S \to P\) denote the natural inclusion functor.

Let us consider the right derived functors of \(i\), and define
\[\mathcal{H}^q(\cdot) := R^q i(\cdot).\]

**Proposition.** For any abelian sheaf \(F\) over \(X\), and any open subset \(U\) of \(X\), we have
\[\mathcal{H}^q(F)(U) = H^q(U, F).\]

Let \(\{U_i \to U\}_{i \in I}\) be a covering. For any presheaf \(F\) over \(X\), we define a complex \(\tilde{\mathcal{C}}^q\) by
\[\tilde{\mathcal{C}}^q(\{U_i \to U\}, F) := \prod_{i_0 < \cdots < i_q} F(U_{i_0} \cap \cdots \cap U_{i_q}).\]
with \(d^q : \check{C}^q(\{U_i \to U\}, F) \to \check{C}^{q+1}(\{U_i \to U\}, F)\) defined by

\[
(d^q)_{i_0 \cdots i_q} = \sum_{v=0}^{q+1} (-1)^v s_{i_0 \cdots \hat{i}_v \cdots i_{q+1}}.
\]

Now, we define

\[
\check{H}^q(\{U_i \to U\}, F) := \text{Ker}(d^q)/\text{Im}(d^{q-1}).
\]

**Proposition.** There is a canonical isomorphism between \(\check{H}^q(\{U_i \to U\}, F)\) and \(R^q\check{H}^0(\{U_i \to U\}, F)\).

We have the following theorem.

**Theorem.** (Spectral sequence for \(\check{C}\)ech cohomology)

Let \(\{U_i \to U\}\) be a covering. For each \(F \in S\), there is a spectral sequence

\[
E_2^{pq} = \check{H}^p(\{U_i \to U\}, \check{C}^q(F)) \Rightarrow E_2^{p+q} = H^{p+q}(U, F)
\]

which is functorial in \(F\).

Let \(\mathcal{U}\) be a family of open subsets of \(X\) such that

(i) The intersection of two open subsets in \(\mathcal{U}\) is in \(\mathcal{U}\) again,

(ii) Each finite covering of an open subset of \(X\) has a refinement consisting of sets in \(\mathcal{U}\).

Let \(\{U_i \to U\}\) be a covering. If \(U \in \mathcal{U}\) and \(U_i \in \mathcal{U}\), we call it a covering in \(\mathcal{U}\).

We define \(\mathcal{U}\)-presheaves for \(X\).

**Definition.** A \(\mathcal{U}\)-presheaf over \(X\), is a collection of abelian groups \(F(U)\) for \(U \in \mathcal{U}\), and a collection of restriction morphisms \(\text{res}^V_U : F(V) \to F(U)\) for each \(U \subset V\) such that \(\text{res}^V_U \circ \text{res}^W_V = \text{res}^W_U\) for \(U \subset V \subset W\) and \(\text{res}^U_U = \text{id}\). Let \(\mathcal{P}_{\mathcal{U}}\) denote the category of \(\mathcal{U}\)-presheaves.

Let \(j_\mathcal{U}\) be the natural functor from \(\mathcal{P}\) to \(\mathcal{P}_{\mathcal{U}}\), then \(j_\mathcal{U}\) is exact. Let \(i_\mathcal{U}\) denote the \(\mathcal{U}\)-presheaves for \(X\).

**Definition.** An abelian sheaf \(F\) over \(X\) is called \(\mathcal{U}\)-flabby if \(\check{H}^q(\{U_i \to U\}, i_\mathcal{U}(F)) = 0\) for \(q > 0\) and each covering in \(\mathcal{U}\).

**Proposition 6.** i). Let \(0 \to F' \to F \to F''\) be an exact sequence in \(S\). If \(F'\) is \(\mathcal{U}\)-flabby, the sequence is exact in \(\mathcal{P}_{\mathcal{U}}\) as well.

ii). Let \(0 \to F' \to F \to F''\) be an exact sequence in \(S\). If \(F'\) and \(F\) are \(\mathcal{U}\)-flabby, so is \(F''\).

iii). If the direct sum \(F \oplus G\) of abelian sheaves is \(\mathcal{U}\)-flabby, so is \(F\).

iv). Injective abelian sheaves are \(\mathcal{U}\)-flabby.

**Corollary 2.** For an abelian sheaf \(F\) over \(X\), the following are equivalent.

i). \(F\) is \(\mathcal{U}\)-flabby.

ii). For all \(q > 0\) and \(U \in \mathcal{U}\), we have \(\check{H}^q(F)(U) = 0\) and therefore \(H^q(F)(U) = 0\).

3.2 Dagger formal schemes

Let \(A\) be a flat w.c.f.g. algebra over \(R\). In this subsection, we define \(\text{Spf}^\dagger(A)\).
Spf(A) is a topology space together with a structure sheaf \( O^1 \). The topology space is the underlying space \( X \) of Spec(\( A \otimes_R k \)).

For any \( p \in X \), let us define \( O^1_p = \varprojlim A(f^{-1})^1 \). Since \( A(f^{-1})^1 \) is flat over \( A \), \( O^1_p \) is flat over \( A \), too. Let us define \( O^1(U) \) to be the set of functions \( s: U \to \prod_{p \in U} O^1_p \) (\( s(p) \in O^1_p \)) with the property that for each \( p \in U \), there is a \( f \in A(\mathcal{F} \notin p) \) such that for any \( q \in X_T \cap U \), \( s(q) \) is the image of an element of \( A(f^{-1})^1 \) in \( O^1_q \).

It is obvious that \( O^1 \) is a sheaf.

**Proposition 7.** For any \( f \) in \( A \), we have \( O^1(X_T) = A(f^{-1})^1 \).

**Proof.** We can define a homomorphism \( \rho: A(f^{-1})^1 \to O^1(X_T) \) in the natural way.

1. At first, we show \( \rho \) is injective. If \( g, h \in A(f^{-1})^1 \) such that \( \rho(g) = \rho(h) \), then for any \( p \in X_T \), \( g \) and \( h \) have the same image in \( O^1_p \). By the definition of \( O^1_p \), there is a \( f_p \in A(\mathcal{F} \notin p) \) such that \( \rho(g) = \rho(h) \) for each \( q \in X_T \), \( s(q) \) is the image of \( g \) in \( O^1_q \).

Let \( s \) be a section of \( \mathcal{O}^1(X_T) \). By definition, for any point \( p \in X_T \), there is a \( f_p \in A(f_p^{-1})^1 \) such that for any \( q \in X_T \), \( s(q) \) is the image of \( g \) in \( O^1_q \).

We can choose a finite subset \( \{f_i\} \) of \( \{f_p\} \) such that \( \{\mathcal{X}_T \} \) is a covering of \( X_T \). For each \( f_i \), there is a \( g_i \in A(f_i^{-1})^1 \) such that \( s(q) \) is the image of \( g_i \) in \( O^1_q \) for each \( q \in X_T \).

Since \( \rho_{A((f_i,f_j)^{-1})^1} \) is injective, \( g_i \) and \( g_j \) have the same image in \( A((f_i,f_j)^{-1})^1 \). Then from formula (1), we know \( s \in \text{Im}(\rho) \).

**Corollary 3.** For a w.c.f.g. algebra \( A \) over \( R \), we have \( O^1(\text{Spf}(A)) = A \).

We call such a pair (Spec(\( A \otimes_R k \)), \( O^1 \)) an affine dagger formal scheme \( \text{Spf}^\dagger(A) \).

**Corollary 4.** Let \( U \subset X = \text{Spf}^\dagger(A) \) be an open subset with \( \{X_T\} \) a covering of \( U \). Then we have the following exact sequence

\[
0 \to O^1(U) \to \prod_i A(f_i^{-1})^1 \Rightarrow \prod_{i,j} A((f_i,f_j)^{-1})^1.
\]

**Proposition 8.** Affine dagger formal schemes are locally ringed spaces.

**Proof.** Let \( X = \text{Spf}^\dagger(A) \). From Proposition 7, we know for each \( p \in X \), the stalk of \( O^1_T \) at \( p \) is \( O^1_p \). We need to prove \( O^1_p \) is a local ring.

Since \( \pi \in \text{rad}(A(f^{-1})^1) \), and \( O^1_p \) is the limit of \( A(f^{-1})^1 \) (\( f \notin p \)), we see \( \pi \in \text{rad}(O^1_p) \). Since \( O^1_p / \pi O^1_p \) is the stalk of \( O^1_X = O^1_X \otimes_R k \) at \( p \), it is a local ring. Therefore, \( O^1_p \) is also a local ring.

Now we can define dagger formal schemes.

**Definition.** A dagger formal scheme is a locally ringed space \( (X, O^1_X) \) in which every point has an open neighborhood \( U \), such that \( (U, O^1_X | U) \) is
an affine dagger formal scheme. A morphism of dagger formal schemes is a morphism as locally ringed spaces.

For a given $A$-module $M$, we can associate to it a sheaf $M^\Delta$ of $\mathcal{O}_{\text{Spf}(A)}^\dagger$-modules over $\text{Spf}(A)$ (say simply a $\mathcal{O}_{\text{Spf}(A)}^\dagger$-module).

For $U \subset \text{Spf}(A)$, define $M^\Delta(U)$ to be the set of functions $s : U \to \prod_{p \in U} \mathcal{O}_p^\dagger \otimes_A M$ with the property that for each $p \in U$, there is a $f \in A (\overline{f} \notin p)$ such that for each $q \in X_T \cap U$, $s(q)$ is the image of an element of $(A(f^{-1})^\dagger \otimes_A M)$. By definition, we know that $M^\Delta$ is a $\mathcal{O}_{\text{Spf}(A)}^\dagger$-module.

**Proposition 9.** For any $f \in A$, we have

$$M^\Delta(X_T) = A(f^{-1})^\dagger \otimes_A M.$$  

Using the exact sequence given in Proposition 5, we can prove this proposition in the same way as Proposition 7.

Let $\mathcal{O}_T^\dagger$ be $\mathcal{O}_T^\dagger \otimes_R K$, and $M^\Delta$ be $M^\Delta \otimes_R K$.

Now we can define coherent $\mathcal{O}_T^\dagger$-modules and quasi-coherent $\mathcal{O}_T^\dagger$-modules as in scheme case. We have the following proposition.

**Proposition.** Let $h : X \to Y$ be a morphism of Noetherian dagger formal schemes over $R$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_Y^\dagger$-modules and $\mathcal{G}$ be $\mathcal{O}_Y^\dagger$-modules.

(i). If $\mathcal{F}$ is quasi-coherent, then $h^*(\mathcal{F})$ is also quasi-coherent.

(ii). If $\mathcal{F}$ is coherent, then $h^*(\mathcal{F})$ is also coherent.

(iii). If $\mathcal{G}$ is quasi-coherent, then $h_*(\mathcal{G})$ is also quasi-coherent.

Let $A$ be a w.c.f.g. algebra over $R$ with $f \in A$ and $f_1, \ldots, f_r \in (f)$ satisfy formula $(\ast)$ given in section 2. Let $U = X_T$ and $U_i = X_{T_i}$. Then Tate’s acyclic theorem says

$$0 \to A(f^{-1})^\dagger \otimes_R K \to \prod_i A(f_i^{-1})^\dagger \otimes_R K \to \mathcal{O}_T^\dagger(\{U_i \to U\}, \overline{\mathcal{O}}^\dagger) \to \cdots \to 0.$$  

Since $\mathcal{O}_T^\dagger(\{U_i \to U\}, \overline{\mathcal{O}}^\dagger)$ are all flat $A$-modules, we get

$$0 \to (A(f^{-1})^\dagger \otimes_R K) \otimes_A M \to \prod_i (A(f_i^{-1})^\dagger \otimes_R K) \otimes_A M \to \mathcal{O}_T^\dagger(\{U_i \to U\}, \overline{\mathcal{M}}^\Delta) \to \cdots \to 0.$$  

Therefore we obtained the following proposition.

**Proposition 10.** For $j > 0$, we have $H^j(\{U_i \to U\}, \overline{\mathcal{M}}^\Delta) = 0$.

As a consequence of this proposition and Corollary 2, we get the following proposition.

**Proposition 11.** Let $X = \text{Spf}(A)$. For $i > 0$, we have $H^i(X_T, \overline{\mathcal{M}}^\Delta) = 0$.

### 3.3 Morphism

In this subsection, we study morphisms of dagger formal schemes.

**Lemma 3.** Every homomorphism of w.c.f.g $R$-algebras is continuous.

**Proof.** Let $\varphi : A_1 \to A_2$ be a homomorphism of w.c.f.g $R$-algebras. From $\varphi(p^\ast A_1) \subset p^\ast A_2$, we see $\varphi$ is continuous. \hfill $\square$

**Proposition 12.** Let $A, B$ be two w.c.f.g. $R$-algebras.
(i) If $\rho : A \to B$ is a homomorphism of $R$-algebras, then $\rho$ induces a natural morphism of locally ringed spaces

$$(\varphi, \varphi^\natural) : (\text{Spf}^\dagger(B), O^\natural_B) \to (\text{Spf}^\dagger(A), O^\natural_A)$$

(ii). Every morphism of locally ringed spaces from $\text{Spf}^\dagger(B)$ to $\text{Spf}^\dagger(A)$ is induced by a $R$-morphism $\rho : A \to B$ as in (i).

**Proof.** (i) is easy. We only need to prove (ii).

Let $(\varphi, \varphi^\natural)$ be a morphism of locally ringed spaces from $Y = \text{Spf}^\dagger(B)$ to $X = \text{Spf}^\dagger(A)$. It induces a morphism $(\bar{\varphi}, \bar{\varphi}^\natural)$ of locally ringed spaces from $\bar{Y} = \text{Spec}(B \otimes_R k)$ to $\bar{X} = \text{Spec}(A \otimes_R k)$. It is well known that $(\varphi, \varphi^\natural)$ is induced from a homomorphism $\bar{\rho}$ of $k$-algebras $A \otimes_R k \to B \otimes_R k$.

Therefore, we see for any $f \in A$

$$\varphi(Y_f) = X_{\rho(f)}.$$  \hspace{1cm} (5)

Let $\rho$ be $\varphi^\natural(Y) : A \to B$. From (4), we get the following commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\rho} & B \\
\downarrow & & \downarrow \\
A\langle f^{-1}\rangle & \xrightarrow{\varphi^\natural(X_f)} & B\langle \rho(f)^{-1}\rangle.
\end{array}
$$

From lemma 3, we know $\rho_f$ and $\varphi^\natural(X_{f(X)})$ are continuous. Since they restrict to the dense subalgebra $A_f$ are the same, they are the same. \hfill \square

Let $k$ be a finite field with $q$ elements. Assume $R = W(k)$. There are Frobenius actions on $k$-algebras $A \to A \ (x \mapsto x^q)$, and on $k$-schemes.

A $R$-morphism $F$ from a dagger formal scheme $X$ to itself is called a Frobenius of $X$ if it induces the Frobenius over $\bar{X} = X \otimes_R k$.

By definition, a Frobenius over $X$ is equivalent to a collection $\{ F_U \}$ of homomorphisms $F_U : \mathcal{O}^\natural_X(U) \to \mathcal{O}^\natural_X(U)$ for each $U \subset X$ such that $F_U \otimes_R k$ is the Frobenius of $\mathcal{O}^\natural_X(U) \otimes_R k$ and the following diagram is commutative ($U \subset V$)

$$
\begin{array}{ccc}
\mathcal{O}^\natural_X(V) & \xrightarrow{F_U} & \mathcal{O}^\natural_X(V) \\
\downarrow \text{res} & & \downarrow \text{res} \\
\mathcal{O}^\natural_X(U) & \xrightarrow{F_U} & \mathcal{O}^\natural_X(U).
\end{array}
$$

### 3.4 Product

Let $A$ and $B$ be two flat w.c.f.g. algebras over $R$. In this subsection, we define $A \otimes_R B$ which is also a w.c.f.g. algebra.

**Lemma 4.** Let $C$ be a finite generated $R$-algebra with $\bar{C}$ its $\pi$-adic completion. Let $D$ be a c.f.g. $R$-algebra. Then any homomorphism $\varphi : C \to D$ extends to a unique homomorphism $\varphi' : \bar{C} \to D$. Moreover when $\varphi$ is injective, $\varphi'$ is also injective.

**Proof.** Existence of $\varphi'$ comes from $\bar{C} = \varprojlim C/\pi^nC$ and $D = \varprojlim D/\pi^nD$. For uniqueness of $\varphi'$, one only needs to note that a homomorphism of two c.f.g. $R$-algebras is continuous. \hfill \square
We know $A = \lim_n A^n$ with $A^n$ c.f.g. algebras over $R$ such that $A^n \to A^{n+1}$ is injective. Since torsion free $R$-modules are flat over $R$, $A^n$ are all flat over $R$. The same holds for $B$.

Then we define $A \hat{\otimes} R B = \lim_n (A_n \hat{\otimes} B^m_n)$.

By Lemma 4, if $A = \lim_n A^n$ and $A = \lim_n A'^n$, then $A_n^m$ is contained in some $A'^m_n$ and $A'_m$ is contained in some $A^n_m$. Therefore, the definition of $A \hat{\otimes} R B$ does not depend on the choice of $\{A^n\}$ and $\{B^m\}$.

Assume $A = T^\dagger_n/I_1$ and $B = T^\dagger_m/I_2$. Let $i_n : T^\dagger_n \to T^\dagger_{n+m}$ be the homomorphism maps variants of $T^\dagger_n$ to first $n$ variants, and let $j_n : T^\dagger_n \to T^\dagger_{n+m}$ be the homomorphism maps variants of $T^\dagger_n$ to last $m$ variants. Then $A \hat{\otimes} R B = T^\dagger_{n+m}/I$ with $I$ the ideal of $T^\dagger_{n+m}$ generated by $i_n(I_1)$ and $j_n(I_2)$.

Let $i_A : A \to A \hat{\otimes} R B$ and $j_B : B \to A \hat{\otimes} R B$ be two canonical inclusion homomorphisms. Let $C$ be a w.c.f.g. algebra over $R$.

**Lemma 5.** If $\varphi_1 : A \to C$ and $\varphi_2 : B \to C$ are two homomorphisms of $R$-algebras, then there is a unique homomorphism $\varphi : A \hat{\otimes} R B \to C$ such that $\varphi \circ i_A = \varphi_1$ and $\varphi \circ j_B = \varphi_2$.

**Proof.** Assume that $A = \lim_n A^n$, $B = \lim_n B^m$, and $C = \lim_n C^n$ as in section 3.4. Then there is a $l$, such that $\varphi_1(A^n)$ and $\varphi_2(B^m)$ are contained in $C^n$. By Lemma 4, there is a $\varphi_{n,m} : A_n^m \hat{\otimes} B^m \to C^n$, such that $\varphi_{n,m} \circ i_A = \varphi_1|A^n$ and $\varphi_{n,m} \circ j_B = \varphi_2|B^m$. Taking limit, we get what we desire.

For $X = \text{Spf}^\dagger(A)$ and $Y = \text{Spf}^\dagger(B)$, we can define $X \times Y = \text{Spf}^\dagger(A \hat{\otimes} R B)$. By glueing, we can define $X \times Y$ for separated flat dagger formal schemes $X$ and $Y$.

4 De Rham cohomology and Lefchetz fixed points theorem

4.1 Differential modules and de Rham cohomology

In this subsection, we recall the concept of differential modules.

Let $A$ be a w.c.f.g. algebra over $R$.

The module of differential forms $\Lambda^*_A$ over $R$ is a finite $A$-module $\Lambda^1_A/R$ together with a $R$-derivation $d_{A/R}$, which is universal in the following sense: for any finite $A$-module $M$, the canonical map

$$\text{Hom}_A(\Lambda^1_A/R, M) \to \text{Der}_R(A, M), \quad \varphi \mapsto \varphi \circ d_{A/R}$$

is bijective.

Let $m : A \hat{\otimes}^\dagger A \to A$ be the “diagonal homomorphism” defined by $m(b \otimes b') = bb'$. By Lemma 5, this homomorphism makes sense. Let $I$ be the kernel of $m$. Then $I/I^2$ inherits a structure of $A$-module. Define a map $d : A \to I/I^2$ by $db = 1 \otimes b - b \otimes 1$ (mod$I^2$).

**Proposition 13.** $(I/I^2, d)$ is $(\Lambda^1_A, d_{A/R})$.

**Proof.** (1). Let $M$ be a finite $A$-module. Define $A * M$ by

$$(a_1, m_1) + (a_2, m_2) = (a_1 + a_2, m_1 + m_2)$$

11
and

\[(a_1, m_1) \cdot (a_2, m_2) = (a_1a_2, a_1m_2 + a_2m_1)\]

for \(a_1, a_2 \in A \) and \(m_1, m_2 \in M\). Then \(A \ast M\) is a w.c.f.g. algebra over \(R\).

It is sufficient to prove this fact in the case \(A = T^1_n\) and \(M = T^r_n \oplus r\). In this case, \(A \ast M = T^{1+r}_n/\sim\), where \(T^{1+r}_n = R[\xi_1, \ldots, \xi_n, \xi_{n+1}, \ldots, \xi_{n+r}]\), and \(\sim\) is generated by \(\{ \xi_i \xi_j | i \geq n+1, j \geq n+1 \}\).

(2). We need to prove the fact that if \(D\) is an \(R\)-derivation of \(A\) into an \(A\)-module \(M\), then there is a unique \(A\)-linear map \(f : I/I^2 \to M\) such that \(D = fd\).

(3). Let \(I' = \text{Ker}(A \otimes A \to A)\), then \(I = I'(A \otimes^1 A)\). In the following, we prove this assertion.

Since \(A = \lim_{\to} A_{n}^\hat{}\), \(I\) is generated by elements in \(I_n = \text{Ker}(A_n \hat{} \otimes A_n \to \hat{} A_n)\) for \(n\) large enough. Let \(I'_n = \text{Ker}(A_n \otimes A_n \to \hat{} A_n)\). To show \(I = I'(A \otimes^1 A)\), it is sufficient to show \(I_n = I'_n(A_n \otimes A_n)\).

Since \(A_n \otimes A_n\) is dense in \(A_n \hat{} \otimes A_n\), \(A_n = A_n \otimes \hat{} A_n/I'_n\) is dense in \(A_n \hat{} \otimes A_n/I'_n(A_n \otimes A_n)\). Since both of them are c.f.g. algebras, them must be the same. Therefore, \(I_n = I'_n(A_n \otimes A_n)\).

(4). By (3), we know \(I/I^2\) is generated by \(\{dy/y \in A\}\) as \(A\)-module. Then we get the uniqueness of \(f\).

(5). By Lemma 5, from the following two homomorphisms

\[\phi_1 : A \to A \ast M, \quad \phi_1(x) = (x, 0)\]

and

\[\phi_2 : A \to A \ast M, \quad \phi_2(x) = (x, D(x))\]

we get a homomorphism

\[\phi : A \otimes^1 A \to A \ast M\]

whose restriction on \(A \otimes A\) is

\[\phi(x \otimes y) = (xy, yD(y))\].

Then \(\phi(I') \subseteq M\), and so \(\phi(I) \subseteq M\). Since \(M^2 = 0\), \(\phi(I^2) = 0\). So \(\phi\) induces \(\hat{\phi} : (A \otimes^1 A)/I^2 = A \ast (I/I^2) \to A \ast M\) which maps \(dy\) to \((0, D(y))\).

Thus the restriction of \(\hat{\phi}\) on \(I/I^2\) gives an \(A\)-linear map \(f : I/I^2 \to M\) such that \(f \circ d = D\).

If \(A = T^r_n = R[\xi_1, \ldots, \xi_n]\), then \(\Omega^1_{A/R} = T^r_n d\xi_1 \oplus \cdots \oplus T^r_n d\xi_n\). If \(A = T^1_n/I\), then \(\Omega^1_{X/R} = \Omega^1_{X\hat{}_R/\sim} = \Omega^1_{X\hat{}_R/\sim}\) with \(\sim\) generated by \(I\Omega^1_{X\hat{}_R/\sim}\). We have the following proposition whose proof is the same as [8](2.3).
Proposition 14. Let $X$ be a flat and separated dagger formal scheme. If $\bar{X} = X \otimes_R k$ is regular, then $\Omega^1_{\bar{X}/R}$ is a locally free $\mathcal{O}_{\bar{X}}$-module with rank $\dim(\bar{X})$.

When $X$ satisfies the condition of Proposition 14, we define $\Omega^i_{X/R} = \wedge^i \Omega^1_{X/R}$, and get the de Rham complexes $(\Omega^i_{X/R}, d)$ and $(\underleftarrow{\Omega}^i_{X/R}, \underleftarrow{d})$ with $\underleftarrow{\Omega}^i_{X/R} = \Omega^i_{X/R} \otimes_R K$. We write $d$ in place of $\underleftarrow{d}$ for simple. Then we define $H^i_{dR}(X, K) := H^i(X, (\Omega^i_{X/R}, d))$. From Proposition 11, we see when $X = \text{Spf}(\mathcal{O}_A \otimes_R K)$, $H^i_{dR}(X, K)$ is exactly the Monsky-Washnitzer cohomology $H^i_{MW}(\mathcal{O}_A \otimes_R K, d)$.

From now on, assume $R = W(k)$. When $X$ is a flat and separated dagger formal scheme over $R$ with a Frobenius $F$ such that $\bar{X}$ is regular and integral, then $F$ induces $F^*$ over $H^i_{dR}(X, K)$.

4.2 nuclear operator and operator $\psi$

4.2.1 nuclear operator

Definition. A $K$-linear map $L : M \to M$ is called nuclear, if the following two conditions hold.

(i). For every $\lambda \neq 0$ in $K^{ac}$ (the algebraic closure of $K$) with $g$ the minimal polynomial of $\lambda$ over $K$, $G(L^m) \cup \text{Ker}(g(L^m))$ is of finite dimension.

(ii). The nonzero eigenvalues of $L$, form a finite set or a sequence with a limit 0.

From (i), we see $M = V \oplus W$ with $V, W$ vector spaces invariant under $L$ such that $W = \bigcup \text{Ker}(g(L^m))$ and $g(L)$ is bijection over $V = \cap \text{Im}(g(L^m))$.

We can define trace $\text{tr}(L)$ for nuclear operator $L$. Let $M_i$ be the sum of the generalized eigenspaces of $L$ with eigenvalues $\lambda$ ($|\lambda| \geq |\pi|^i$). Then $\dim M_i < \infty$. Define $\text{tr}(L^s) = \lim_{l \to \infty} \text{tr}(L^s|_{M_i})$ for positive integer $s$, and $\det(1 - tL) = \lim_{l \to \infty} \det(1 - tL|_{M_i})$.

For nuclear operators, we have the following two lemmas.

Lemma 6. Let $L_i : M_i \to M_i$ be nuclear. Assume linear map $\alpha : M_1 \to M_2$ satisfy $\alpha L_1 = L_2 \alpha$. Then the induced maps $L_0$ on $\text{Ker}(\alpha)$ and $L_3$ on $\text{Coker}(\alpha)$ are nuclear. Moreover,

$$\prod_{i=0}^{3} \det(1 - tL_i)^{(-1)^i} = 1,$$

and

$$\sum_{i=0}^{3} \text{tr}(L_i^s) = 0,$$

where $s$ is a positive integer.

Lemma 7. Let $L$ be a $K$-linear map over $M$. Assume $M_1$ is a $K$-linear subspace of $M$ fixed by $L$. Then $L$ induces $L_1$ on $M_1$ and $L_2$ on $M/M_1$. Then $L$ is nuclear if and only if both $L_1$ and $L_2$ are nuclear. If so, then we have the following formulas

$$\det(1 - tL) = \det(1 - tL_1) \cdot \det(1 - tL_2).$$
and
\[ \text{tr}(L^s) = \text{tr}(L_1^s) + \text{tr}(L_2^s), \]
where \( s \) is a positive integer.

4.2.2 Operator \( \psi \)

**Proposition.** \((\text{[8]})\) Let \( B \subset A \) denote a finite ring extension of w.c.f.g. algebras. Suppose both \( \bar{A} \) and \( \bar{B} \) are regular, and both \( A \) and \( B \) are integral and flat over \( R \). Then there exists a “trace map” \( \text{tr}_{A/B} : \Omega^i(A) \to \Omega^i(B) \).

\( Tr_{B/A} \) is defined by
\[
\Omega^i(A) \to \Omega^i(A) \otimes_A \text{Qt}(A) \xrightarrow{\text{id} \otimes \text{tr}} \Omega^i(B) \otimes_B \text{Qt}(B).
\]
In \([8]\), one shows \( Tr_{A/B} \) maps \( \Omega^i(A) \) into \( \Omega^i(B) \). Moreover, one can show \( \text{tr}_{A/B} \) commutes with \( d \).

Let \( A \) be a w.c.f.g. algebra over \( R \) satisfying the above proposition with \( \bar{A} \) a regular and integral algebra of dimension \( n \). Let \( F \) be a Frobenius over \( A \). Since \([A : F(A)] = [A(f^{-1})^1 : F(A(f^{-1})^1)] = q^n\), we have the following commutative diagram
\[
\begin{array}{ccc}
\Omega^i(A) \otimes_R K & \xrightarrow{\text{tr}} & \Omega^i(F(A)) \otimes_R K \\
\text{res} & & \text{res} \\
\Omega^i(A(f^{-1})^1) \otimes_R K & \xrightarrow{\text{tr}} & \Omega^i(F(A(f^{-1})^1)) \otimes_R K.
\end{array}
\]

Let \( X \) be a flat and integral dagger formal scheme with a Frobenius \( F \) and regular and integral reduction \( \bar{X} \). From commutative diagram \([9]\), we get a trace map \( tr_X : X = (X, \mathcal{O}_X^1) \to X^q = (X, F(\mathcal{O}_X^1)), \) with \( F(\mathcal{O}_X^1)(U) = F(U(\mathcal{O}_X^1))(U \subset X) \).

We define operator \( \psi = F^{-1} \circ tr_X : \Omega^i(X, K) \xrightarrow{\text{tr}} \Omega^i(X^q, K) \xrightarrow{F} \Omega^i(X, K) \).

**Proposition 15.** \((\text{[8]})\)

(i). For any open subset \( U \subset X \), \( a \in \mathcal{O}_X^1(U), \omega \in \Omega^i(U, K), \) we have \( \psi(F(a) \omega) = a\psi(\omega) \).

(ii). \( \psi \) commutes with \( d \).

(iii). \( \psi \circ F = q^n \).

From \([8]\), we know \( \psi_U \) is nuclear for any open subset \( U \) of \( X \). Let \( \psi^* \) be the linear morphisms on \( H^1_{dR}(X, K) \) induced by \( \psi \).

**Proposition 16.** \((\text{[8]})\) Let \( A \) be a flat w.c.f.g. algebra over \( R \) with \( \bar{A} = A \otimes_R k \) regular and integral. Then \( F^* \) is bijection over \( H_{MW}(A, K) \) and \( \psi^* = q^n(F^*)^{-1} \) is nuclear.

4.3 Lefchetz fixed points theorem

In \([8]\) (4.1), the following result is stated.
**Theorem 2.** Let $A$ be a flat w.c.f.g. algebra over $R$ with $\bar{A} = A \otimes_R k$ regular and integral of dimension $n$. Let $N(\bar{A})$ denote the number of $k$-homomorphisms $\bar{A} \to k = \mathbb{F}_q$. Then

$$N(\bar{A}) = \sum_{i=0}^{n} (-1)^i \text{tr}(\psi^*|H^i_{MW}(A, K)). \quad (7)$$

In this subsection, we show the following generalization of Theorem 2.

**Theorem 3.** Let $X$ be a flat separated and Noetherian dagger formal scheme with $\bar{X} = X \otimes_R k$ regular and integral of dimension $n$. Let $N(\bar{X})$ denote the number of $k$-points of $\bar{X}$. Then $\psi^*$ is nuclear over $H^i_{dR}(X, K)$ and the following formula holds

$$N(\bar{X}) = \sum (-1)^i \text{tr}(\psi^*|H^i_{dR}(X, K)). \quad (8)$$

When $X = \text{Spf}^! (A)$, $H^i_{dR}(X, K) = H^i_{MW}(\bar{A}, K)$. So Theorem 2 is a special case of Theorem 3.

**Proof.** (1). Take an affine covering $U = \{U_i\}_{i=1}^r$ of $X$. Since $X$ is separated, intersections of $\{U_i\}$'s are again affine.

Then from Proposition 11 and EGA III (12.4.7), we obtain

$$\hat{\mathbb{H}}(\mathbb{U}, \hat{\Omega} \cdot) \sim \mathbb{H}(\mathbb{X}, \hat{\Omega} \cdot) \quad (9)$$

So we can calculate $\mathbb{H}(\mathbb{X}, \hat{\Omega} \cdot)$ by spectral sequence.

Write $K^{p,q} = \check{C}^p(\mathbb{U}, \hat{\Omega}^q)$. Let

$$F^p K^{i,j} = \begin{cases} K^{i,j} & \text{if } i \geq p \\ 0 & \text{if } i \leq p \end{cases}$$

Let $sK^l = \oplus_{p+q=l} K^{p,q}$. We define $sF^pH^l$ in a similar way.

Let $E^p_{1,q} = \check{C}^p(\mathbb{U}, H^q(\cdot, \hat{\Omega} \cdot)) = \check{C}^p(\mathbb{U}, H^q_{MW}(\cdot, \cdot))$, then

$$E^p_{1,q} \Rightarrow \hat{\mathbb{H}}(\mathbb{U}, \hat{\Omega} \cdot) \quad (10)$$

with $F^p \hat{\mathbb{H}}(\mathbb{U}, \hat{\Omega} \cdot) = \text{Im}(H^l(sF^pK) \to H^l(sK))$.

(2). We show $\psi^*$ is nuclear.

From Proposition 16, we see $\psi$ is nuclear on each term of $E^p_{1,q}$. Then by Lemma 6 and Lemma 7, we obtain that $\psi^*$ is nuclear on each term of $E^p_{1,q}$ with $m$ a positive integer. Therefore, from (10), we see that $\psi^*$ is nuclear on $H^n_{dR}(X, K)$.

(3). From

$$N(\bar{X}) = \sum_{i_0 < \ldots < i_j} (-1)^j N(U_{i_1 \ldots i_j})$$

and

$$N(U_{i_0 \ldots i_j}) = \sum (-1)^i \text{tr}(\psi^*|H^i_{MW}(U_{i_0 \ldots i_j}, K)),$$

we get

$$N(\bar{X}) = \sum (1)^{p+q} \text{tr}(\psi^*|E^p_{1,q}).$$
From Lemma 6 and Lemma 7, we get

\[ N(\bar{X}) = \sum_{p,q} (-1)^{p+q} \text{tr}(\psi^*|E_{m}^{p,q}), \]

for all positive integer \( m \). Then from Lemma 6, Lemma 7 and (10), we get (8).

From Proposition 16 and (10), we get the following lemma easily.

**Lemma 8.** \( F^* \) is bijection on \( H^i_{dR}(X, K) \) and \( \psi^* = q^n(F^*)^{-1} \).

**Theorem 4.** Let \( X \) be a flat separated and Noetherian dagger formal scheme with \( X = X \otimes k \) regular and integral of dimension \( n \). Let \( N_s(\bar{X}) \) denote the number of \( \mathbb{F}_q \)-points of \( X \). Then the following formula holds

\[ N_s(\bar{X}) = \sum (-1)^i \text{tr}((q^n(F^*)^{-1})^*|H^i_{dR}(X, K)). \quad (11) \]

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