Generalized Satisfiability for the Description Logic $\mathcal{ALC}$

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Abstract. The standard reasoning problem, concept satisfiability, in the basic description logic $\mathcal{ALC}$ is PSPACE-complete, and it is EXPTIME-complete in the presence of unrestricted axioms. Several fragments of $\mathcal{ALC}$, notably logics in the $\mathcal{FL}$, $\mathcal{EL}$, and DL-Lite families, have an easier satisfiability problem; sometimes it is even tractable. We classify the complexity of the standard satisfiability problems for all possible Boolean and quantifier fragments of $\mathcal{ALC}$ in the presence of general axioms.

1 Introduction

Standard reasoning problems of description logics, such as satisfiability or subsumption, have been studied extensively. Depending on the expressivity of the logic, the complexity of reasoning for DLs between fragments of the basic DL $\mathcal{ALC}$ and the OWL 2 standard $\mathcal{SROIQ}$ is between trivial and NEXPTIME.

For $\mathcal{ALC}$, concept satisfiability is PSPACE-complete [35]. In the presence of unrestricted axioms, it is EXPTIME-complete due to the correspondence with propositional dynamic logic [33, 38, 21]. Since the standard reasoning tasks are interreducible, subsumption has the same complexity.

Several fragments of $\mathcal{ALC}$, such as logics in the $\mathcal{FL}$, $\mathcal{EL}$ or DL-Lite families, are well-understood. They usually restrict the use of Boolean operators and of quantifiers, and it is known that their reasoning problems are often easier than for $\mathcal{ALC}$. We now need to distinguish between satisfiability and subsumption because they are no longer obviously interreducible if certain Boolean operators are missing. Concept subsumption with respect to acyclic and cyclic terminologies, and even with general axioms, is tractable in the logic $\mathcal{EL}$, which allows only conjunctions and existential restrictions, [4, 13], and it remains tractable under a variety of extensions such as nominals, concrete domains, role chain inclusions, and domain and range restrictions [5, 7]. Satisfiability for $\mathcal{EL}$, in contrast, is trivial, i.e., every $\mathcal{EL}$-ontology is satisfiable. However, the presence of universal quantifiers usually breaks tractability: Subsumption in $\mathcal{FL}_0$, which allows only conjunction and universal restrictions, is coNP-complete [31] and increases to PSPACE-complete with respect to cyclic terminologies [3, 25] and to EXPTIME-complete with general axioms [5, 24]. In [19, 20], concept satisfiability and subsumption for several logics below and above $\mathcal{ALC}$ that extend $\mathcal{FL}_0$ with disjunction, negation and existential restrictions and other features, is shown to be tractable, NP-complete, coNP-complete or PSPACE-complete. Subsumption in the presence of
general axioms is EXPTIME-complete in logics containing both existential and universal restrictions plus conjunction or disjunction \cite{22}, as well as in ALC, where only conjunction, universal restrictions and unqualified existential restrictions are allowed \cite{18}. In DL-Lite, where atomic negation, unqualified existential and universal restrictions, conjunctions and inverse roles are allowed, satisfiability of ontologies is tractable \cite{16}. Several extensions of DL-Lite are shown to have tractable and NP-complete satisfiability problems in \cite{1} \cite{2}. The logics in the EL and DL-Lite families are so important for (medical and database) applications that OWL 2 has two profiles that correspond to logics in these families.

This paper revisits restrictions to the Boolean operators in ALC. Instead of looking at one particular subset of \{\sqcap, \sqcup, \neg\}, we are considering all possible sets of Boolean operators, and therefore our analysis includes less commonly used operators such as the binary exclusive or \oplus. Our aim is to find for every possible combination of Boolean operators whether it makes satisfiability of the corresponding restriction of ALC hard or easy. Since each Boolean operator corresponds to a Boolean function—i.e., an \(n\)-ary function whose arguments and values are in \{0, 1\}—there are infinitely many sets of Boolean operators that determine fragments of ALC. The complexity of the corresponding concept satisfiability problems without theories has already been classified in \cite{23} between being PSPACE-complete, coNP-complete, tractable and trivial for all combinations of Boolean operators and quantifiers.

The tool used in \cite{23} for classifying the infinitely many satisfiability problems was Post’s lattice \cite{32}, which consists of all sets of Boolean functions closed under superposition. These sets directly correspond to all sets of Boolean operators closed under composition. Similar classifications have been achieved for satisfiability for classical propositional logic \cite{26}, Linear Temporal Logic \cite{9}, hybrid logic \cite{28}, and for constraint satisfaction problems \cite{34} \cite{36}.

In this paper, we classify the concept satisfiability problems with respect to theories for ALC fragments obtained by arbitrary sets of Boolean operators and quantifiers. We separate these problems into EXPTIME-complete, NP-complete, P-complete and NL-complete, leaving only two single cases with non-matching upper and lower bound. We will also put these results into the context of the above listed results for ALC fragments.

This study extends our previous work in \cite{29} by matching upper and lower bounds and considering restricted use of quantifiers.

2 Preliminaries

Description Logic. We use the standard syntax and semantics of ALC \cite{8}, with the Boolean operators \(\sqcap, \sqcup, \neg, \top, \bot\) replaced by arbitrary operators \(\circ_f\) that correspond to Boolean functions \(f: \{0, 1\}^n \rightarrow \{0, 1\}\) of arbitrary arity \(n\). Let \(N_C, N_R\) and \(N_I\) be sets of atomic concepts, roles and individuals. Then the set of concept descriptions, for short concepts, is defined by

\[
C := A | \circ_f(C, \ldots, C) | \exists R.C | \forall R.C,
\]
where $A \in \mathbb{N}_c$, $R \in \mathbb{N}_r$, and $\circ_f$ is a Boolean operator. For a given set $B$ of Boolean operators, a $B$-concept is a concept that uses only operators from $B$. A general concept inclusion (GCI) is an axiom of the form $C \subseteq D$ where $C, D$ are concepts. We use “$C \equiv D$” as the usual syntactic sugar for “$C \subseteq D$ and $D \subseteq C$”. A TBox is a finite set of GCIs without restrictions. An ABox is a finite set of axioms of the form $C(x)$ or $R(x, y)$, where $C$ is a concept, $R \in \mathbb{N}_r$ and $x, y \in \mathbb{N}_i$. An ontology is the union of a TBox and an ABox. This simplified view suffices for our purposes.

An interpretation is a pair $I = (\Delta^I, \cdot^I)$, where $\Delta^I$ is a nonempty set and $\cdot^I$ is a mapping from $\mathbb{N}_c$ to $\mathcal{P}(\Delta^I)$, from $\mathbb{N}_r$ to $\mathcal{P}(\Delta^I \times \Delta^I)$ and from $\mathbb{N}_i$ to $\Delta^I$ that is extended to arbitrary concepts as follows:

$$\circ_f(C_1, \ldots, C_n)^I = \{ x \in \Delta^I \mid f(\|x\in C_1^I\|, \ldots, \|x\in C_n^I\|) = 1 \},$$

where $\|x\in C_1^I\| = 1$ if $x \in C_1^I$ and $\|x\in C_1^I\| = 0$ if $x \notin C_1^I$,

$$\exists R.C)^I = \{ x \in \Delta^I \mid \{ y \in C^I \mid (x, y) \in R^I \} \neq \emptyset \},$$

$$\forall R.C)^I = \{ x \in \Delta^I \mid \{ y \in C^I \mid (x, y) \notin R^I \} = \emptyset \}.$$

An interpretation $I$ satisfies the axiom $C \subseteq D$, written $I \models C \subseteq D$, if $C^I \subseteq D^I$. Furthermore, $I$ satisfies $C(x)$ or $R(x, y)$ if $x^I \in C^I$ or $(x^I, y^I) \in R^I$. An interpretation $I$ satisfies a TBox (ABox, ontology) if it satisfies every axiom therein. It is then called a model of this set of axioms.

Let $B$ be a finite set of Boolean operators and $Q \subseteq \{\exists, \forall\}$. We use $\text{Con}_Q(B)$, $\Sigma_Q(B)$ and $\Omega_Q(B)$ to denote the set of all concepts, TBoxes and ontologies that use operators in $B$ only and quantifiers from $Q$ only. The following decision problems are of interest for this paper.

**Concept satisfiability** $\text{CSAT}_Q(B)$:
Given a concept $C \in \text{Con}_Q(B)$, is there an interpretation $I$ s.t. $C^I \neq \emptyset$?

**TBox satisfiability** $\text{TSAT}_Q(B)$:
Given a TBox $T \subseteq \Sigma_Q(B)$, is there an interpretation $I$ s.t. $I \models T$?

**TBox-concept satisfiability** $\text{TCSAT}_Q(B)$:
Given $T \subseteq \Sigma_Q(B)$ and $C \in \text{Con}_Q(B)$, is there an $I$ s.t. $I \models T$ and $C^I \neq \emptyset$?

**Ontology satisfiability** $\text{OSAT}_Q(B)$:
Given an ontology $O \subseteq \Omega_Q(B)$, is there an interpretation $I$ s.t. $I \models O$?

**Ontology-concept satisfiability** $\text{OCSAT}_Q(B)$:
Given $O \subseteq \Omega_Q(B)$ and $C \in \text{Con}_Q(B)$, is there an $I$ s.t. $I \models O$ and $C^I \neq \emptyset$?

By abuse of notation, we will omit set parentheses and commas when stating $Q$ explicitly, as in $\text{TSAT}_\exists \forall (B)$. The above listed decision problems are interreducible independently of $B$ and $Q$ in the following way:

$$\text{CSAT}_Q(B) \leq^\text{log}_m \text{OSAT}_Q(B) \leq^\text{log}_m \text{OSAT}_Q(B) \leq^\text{log}_m \text{OCSAT}_Q(B)$$
A concept $C$ is satisfiable iff the ontology $\{C(a)\}$ is satisfiable, for some individual $a$: a terminology $\mathcal{T}$ is satisfiable iff a fresh atomic concept $A$ is satisfiable w.r.t. $\mathcal{T}$; $C$ is satisfiable w.r.t. $\mathcal{T}$ iff $\mathcal{T} \cup \{C(a)\}$ is satisfiable, for a fresh individual $a$.

Some reductions in the main part of the paper consider another decision problem which is called subsumption (SUBS) and is defined as follows: Given a TBox $\mathcal{T}$ and two atomic concepts $A, B$, does every model of $\mathcal{T}$ satisfy $A \sqsubseteq B$?

**Complexity Theory.** We assume familiarity with the standard notions of complexity theory as, e.g., defined in [31]. In particular, we will make use of the classes NL, P, NP, coNP, and EXPTIME, as well as logspace reductions $\leq_{\text{log}}$.

**Boolean operators.** This study is complete with respect to Boolean operators, which correspond to Boolean functions. The table below lists all Boolean functions that we will mention, together with the associated DL operator where applicable.

| Function symbol | Description                          | DL operator symbol |
|-----------------|-------------------------------------|--------------------|
| 0, 1            | constant 0, 1                        | ⊥, ↑               |
| and, or         | binary conjunction/disjunction $\land, \lor$ | $\sqcap, \sqcup$ |
| neg             | unary negation $\neg$               | $\neg$            |
| xor             | binary exclusive or $\oplus$         | $\boxminus$       |
| andor           | $x \land (y \lor z)$                |                   |
| sd              | $(x \land \overline{y}) \lor (x \land \overline{z}) \lor (y \land \overline{z})$ |                   |
| equiv           | binary equivalence function          |                   |

**Fig. 1.** Boolean functions with description and corresponding DL operator symbol.

A set of Boolean functions is called a *clone* if it contains all projections (also known as identity functions, the eponym of the I-clones below) and is closed under composition (also referred to as superposition). The lattice of all clones has been established in [32], see [11] for a more succinct but complete presentation. Via the inclusion structure, lower and upper complexity bounds can be carried over to higher and lower clones under certain conditions. We will therefore state our results for minimal and maximal clones only, together with those conditions.

Given a finite set $B$ of functions, the smallest clone containing $B$ is denoted by $[B]$. The set $B$ is called a *base* of $[B]$, but $[B]$ often has other bases as well. For example, nesting of binary conjunction yields conjunctions of arbitrary arity. The table below lists all clones that we will refer to, using the following definitions. A Boolean function $f$ is called *self-dual* if $f(x_1, \ldots, x_n) = f(\overline{x_1}, \ldots, \overline{x_n})$, *reproducing* if $f(c, \ldots, c) = c$ for $c \in \{0, 1\}$, and *c-separating* if there is an $1 \leq i \leq n$ s.t. for each $(b_1, \ldots, b_n) \in f^{-1}(c)$, it holds that $b_i = c$.

From now on, we will use $B$ to denote a finite set of Boolean operators. Hence, $[B]$ consists of all operators obtained by nesting operators from $B$. By abuse of notation, we will denote operator sets with the above clone names when this is not ambiguous. Furthermore, we call a Boolean operator corresponding to
atomic concepts

Proof. It is easy to observe that the concepts ⊓ (and, or), ⊔ (andor, 0), ⊤ (andor, 1) and ¬ (xor, 0) can be simulated by fresh atomic concepts T and B, using the axioms ¬T ⊆ T and B ⊆ ¬B.

1. If V ⊆ B ⊆ M (E ⊆ B ⊆ M, resp.), then there exists a B-concept C such that C is equivalent to A1 ∪ A2 (A1 ∩ A2, resp.) and each of the atomic concepts A1, A2 occurs exactly once in C.
2. If |B| = BF, then there are B-concepts C and D such that C is equivalent to A1 ∪ A2, D is equivalent to A1 ∩ A2, and each of the atomic concepts A1, A2 occurs in C and D exactly once.
3. If N ⊆ B, then there is a B-concept C such that C is equivalent to ¬A and the atomic concept A occurs in C only once.

Auxiliary results. The following lemmata contain technical results that will be useful to formulate our main results. We use *SATQ(B) to speak about any of the four satisfiability problems TSATQ(B), TCSATQ(B), OSATQ(B) and OCSATQ(B) introduced above; for the three problems having the power to speak about a single individual, we abuse this notion and write *SATQ(B) for the problems *SATQ(B) without TSATQ(B).

Lemma 2 ([29]). Let B be a finite set of Boolean operators s.t. N2 ⊆ |B| and Q ⊆ {∃, ∀}. Then it holds that *SATQ(B) ≡logm *SATQ(B ∪ {∧, ⊥}).

Proof. It is easy to observe that the concepts ⊤ and ⊥ can be simulated by fresh atomic concepts T and B, using the axioms ¬T ⊆ T and B ⊆ ¬B.
Lemma 3 ([29]). Let $B$ be a finite set of Boolean operators and $Q \subseteq \{\exists, \forall\}$. Then it holds that $TCSAT_Q(B) \leq_{\log} TSAT_{Q \cup \{\exists\}}(B \cup \{\top\})$.

Proof. It can be easily shown that $(C, T) \in TCSAT_Q(B)$ iff $(T \cup \{\top \subseteq \exists R.C\}) \in TSAT_{Q \cup \{\top\}}(B \cup \{\top\})$, where $R$ is a fresh role. For “$\Rightarrow$” observe that for the satisfying interpretation $I = (\Delta_I, \cdot_I)$ there must be an individual $w'$ where $C$ holds and then from every individual $w \in \Delta$ there can be an $R$-edge from $w$ to $w'$ to satisfy $T \cup \{\top \subseteq \exists R.C\}$. For “$\Leftarrow$” note that for a satisfying interpretation $I = (\Delta_I, \cdot_I)$ all axioms in $T \cup \{\top \subseteq \exists R.C\}$ are satisfied. In particular the axiom $\top \subseteq \exists R.C$. Hence there must be at least one individual $w'$ s.t. $w' \models C$. Thus $I \models T$ and $C' \supseteq \{w'\} \neq \emptyset$. □

Furthermore, we observe that, for each set $B$ of Boolean operators with $\top, \bot \in [B]$, we can simulate the negation of an atomic concept using a fresh atomic concept $A$ and role $R_A$: if we add the axioms $A \equiv \exists R_A.\top$ and $A' \equiv \forall R_A.\bot$ to the given terminology $T$, then each model of $T$ has to interpret $A'$ as the complement of $A$.

In order to generalize complexity results from $\star SAT_Q(B_1)$ to $\star SAT_Q(B_2)$ for arbitrary bases $B_2$ of $[B_1]$, we need the following lemma.

Lemma 4 ([29]). Let $B_1, B_2$ be two sets of Boolean operators s.t. $[B_1] = [B_2]$, and let $Q \subseteq \{\exists, \forall\}$. Then $\star SAT_Q \leq_{m} \star SAT_Q(B_2)$.

Proof. According to [23, Theorem 3.6], we translate for any given instance each concept (hence each side of an axiom) into a Boolean circuit over the basis $B_1$. This circuit can be easily transformed into a circuit over the basis $B_2$. This new circuit will be expressed by several new axioms that are constructed in the style of the formulae in [23]:

- For input gates $g$, we add the axiom $g \equiv x_i$.
- If $g$ is a gate computing the Boolean operator $\circ$ and $h_1, \ldots, h_n$ are the respective predecessor gates in this circuit, we add the axiom $g \equiv \circ(h_1, \ldots, h_n)$.
- For $\exists R$-gates $g$, we add the axiom $g \equiv \exists R.h$.
- Analogously for $\forall R$-gates.

For each axiom $A \subseteq B$, let $g_A^A$ and $g_B^B$ be the output gates of the appropriate circuits. Then we need to add one new axiom $g_A^A \subseteq g_B^B$ to ensure the axiomatic property of $A \subseteq B$. For a concept $C$ in the input (relevant for the problems $TCSAT_Q$, $OCSAT_Q$), its translation is mapped to the respective out-gate $g_C^C$.

This reduction is computable in logarithmic space and its correctness can be shown in the same way as in the Proof of [23, Theorem 3.6]. □

The idea for the following lemma goes back to Lewis [26].

Lemma 5 (Lewis Trick). Let $B$ be a set of Boolean operators and $Q \subseteq \{\forall, \exists\}$. Then it holds that $TSAT_Q(B \cup \{\top\}) \leq_{\log} TCSAT_Q(B)$.
Proof. Let SC(\(T\)) be the set of all (sub-)concepts occurring in \(T\). For every \(C \in SC(\mathcal{T})\), we use \(C_T\) to denote \(C\) with all occurrences of \(\top\) replaced by \(T\). Furthermore, we write \(\mathcal{T}_T\) for \(\{C_T \subseteq D_T \mid C \subseteq D \in \mathcal{T}\}\).

We claim that \(\mathcal{T} \in TSAT_\mathcal{Q}(B) \iff (\mathcal{T}', T) \in TCSAT_\mathcal{Q}(B)\), where
\[
\mathcal{T}' = \mathcal{T}_T \cup \{C_T \subseteq T \mid C \in SC(\mathcal{T})\}.
\]

For the direction “\(\Rightarrow\)” observe that for any interpretation \(\mathcal{I} = (\Delta_\mathcal{I}, \mathcal{I})\) with \(\mathcal{I} \models \mathcal{T}\), we can set \(T^\mathcal{I} = \Delta_\mathcal{I}\) and then have \(\mathcal{I} \models \mathcal{T}'\) and obviously \(T^\mathcal{I} \neq \emptyset\).

Now consider the opposite direction “\(\Leftarrow\)”. Let \(\mathcal{I} = (\Delta_\mathcal{I}, \mathcal{I})\) be an interpretation s.t. \(\mathcal{I} \models \mathcal{T}'\) and \(T^\mathcal{I} \neq \emptyset\). We construct \(\mathcal{J}\) from \(\mathcal{I}\) via restriction to \(T^\mathcal{I}\), i.e., \(\Delta_\mathcal{J} = T^\mathcal{I}\), \(A_\mathcal{J} = A^\mathcal{I} \cap T^\mathcal{I}\) for atomic concepts \(A\), and \(R_\mathcal{J} = R^\mathcal{I} \cap (T^\mathcal{I} \times T^\mathcal{I})\) for roles \(R\). We claim the following:

Claim. For every individual \(x \in T^\mathcal{I}\) and every (sub-)concept \(C\) occurring in \(\mathcal{T}\), it holds that \(x \in C^\mathcal{J}\) if and only if \(x \in C^\mathcal{I}\).

This claim implies that \(\mathcal{J} \models \mathcal{T}\): for any \(x \in \Delta_\mathcal{J}\) \(= T^\mathcal{I}\) and any axiom \(D \subseteq E \in \mathcal{T}\), we have that \(x \in D^\mathcal{J}\) implies \(x \in D^\mathcal{I}\) due to the claim, which implies \(x \in E^\mathcal{J}\) because \(\mathcal{I} \models \mathcal{T}'\), which implies \(x \in E^\mathcal{I}\) due to the claim.

Proof of Claim. We proceed by induction on the structure of \(C\). The base case includes atomic \(C\) as well as \(\top\) and \(\bot\), and follows from the construction of \(\mathcal{J}\).

For the induction step, we consider the following cases.

- In case \(C = o_f(C^1, \ldots, C^n)\), where \(o_f\) is an arbitrary \(n\)-ary boolean operator corresponding to an \(n\)-ary Boolean function \(f\), and the \(C^i\) are smaller subconcepts of \(C\), the following holds.
\[
x \in C^\mathcal{J}\quad \text{iff} \quad f(\|x \in (C^1)^\mathcal{I}\|, \ldots, \|x \in (C^n)^\mathcal{I}\|) = 1 \quad \text{def. of satisfaction}
\]
\[
\text{iff} \quad f(\|x \in (C^1)^\mathcal{J}\|, \ldots, \|x \in (C^n)^\mathcal{J}\|) = 1 \quad \text{induction hypothesis}
\]
\[
\text{iff} \quad x \in C^\mathcal{J} \quad \text{def. of satisfaction}
\]

- In case \(C = \exists R.D\), the following holds.
\[
x \in C^\mathcal{J}\quad \text{iff} \quad \text{for some } y \in \Delta_\mathcal{J} : (x, y) \in R^\mathcal{J} \text{ and } y \in D^\mathcal{J}
\]
\[
\text{iff} \quad \text{for some } y \in T^\mathcal{J} : (x, y) \in R^\mathcal{J} \text{ and } y \in D^\mathcal{J}
\]
\[
\text{iff} \quad x \in C^\mathcal{J}
\]

The first equivalence is due to the definition of satisfaction. The second’s “\(\Rightarrow\)” direction is due to the additional axiom \(D_T \subseteq T\) in \(\mathcal{T}'\), while the “\(\Leftarrow\)” direction is obvious. The third equivalence is again due to the definition of satisfaction and the construction \(\Delta_\mathcal{J} = T^\mathcal{I}\).

- In case \(C = \forall R.D\), we rewrite to \(C = \neg \exists R.\neg D\), apply the previous two cases, and rewrite back.

\(\square\)
Lemma 6 (Contraposition). Let $B$ be a set of Boolean functions and $Q \subseteq \{\exists, \forall\}$. Then

1. $\text{TSAT}_{Q}(B) \leq_{\text{log}} \text{TSAT}_{\text{dual}(Q)}(\text{dual}(B))$, and
2. $\text{TCSAT}_{Q}(B) \leq_{\text{log}} \text{TCSAT}_{\text{dual}(Q)}(\text{dual}(B) \cup \{\bot, \sqcap\})$,

where $\text{dual}(B) := \{\text{dual}(f) | f \in B\}$ and $\text{dual}(Q) = \{\text{dual}(q) | q \in Q\}$ for $\text{dual}(\exists) := \forall$ and $\text{dual}(\forall) = \exists$.

Proof. Let $B$ be a set of Boolean functions and $Q \subseteq \{\exists, \forall\}$. Let $\mathcal{T} \in \mathcal{I}_{Q}(B)$ be a terminology.

1. Then it holds that $\mathcal{T} \in \text{TSAT}_{Q}(B)$ iff $\mathcal{T}^{\text{con}} \in \text{TSAT}_{\text{dual}(Q)}(\text{dual}(B))$, where

$$\mathcal{T}^{\text{con}} := \{D \subseteq C^{-} | (C \subseteq D) \in \mathcal{T}\},$$

and $C^{-}$ is $C$ in negation normal form (all negations are moved inside s.t. they are in front of atomic concepts) and the negated atomic concepts $\neg A$ are replaced with fresh atomic concepts $A'$. Because of the negation normal form all functions are mapped to their dual and the quantifiers are expressed via their dual one. Therefore note that $C \subseteq D \iff \neg D \subseteq \neg C$.

2. Here we need the operators $\bot$ and $\sqcap$ to ensure that the input concept $C$ is not instantiated by the same individual as $C'$. Now observe that it holds that $(C, \mathcal{T}) \in \text{TCSAT}_{Q}(B)$ iff $(C, \mathcal{T}^{\text{con}} \cup \{C \cap C' \subseteq \bot\}) \in \text{TCSAT}_{\text{dual}(Q)}(\text{dual}(B))$, where $\mathcal{T}^{\text{con}}$ is as in (1).

$\square$

Known complexity results for CSAT. In [23], the complexity of concept satisfiability has been classified for modal logics corresponding to all fragments of $\mathcal{ALC}$ with arbitrary combinations of Boolean operators and quantifiers: $\text{CSAT}_{Q}(B)$ with $Q \subseteq \{\exists, \forall\}$ is either PSPACE-complete, coNP-complete, or in P. Some of the latter cases are trivial, i.e., every concept in such a fragment is satisfiable. These results generalize known complexity results for $\mathcal{ALE}$ and the $\mathcal{EL}$ and $\mathcal{FL}$ families. On the other hand, results for $\mathcal{ALU}$ and the DL-Lite family cannot be put into this context because they only allow unqualified existential restrictions. See [29] for a more detailed discussion.

3 Complexity Results for TSAT, TCSAT, OSAT, OCSAT

In this section we will almost completely classify the above mentioned satisfiability problems for their tractability with respect to sub-Boolean fragments and put them into context with existing results for fragments of $\mathcal{ALC}$.

We use $\star \text{SAT}_{Q}(B)$ to speak about any of the four satisfiability problems $\text{TSAT}_{Q}(B)$, $\text{TCSAT}_{Q}(B)$, $\text{OSAT}_{Q}(B)$ and $\text{OCSAT}_{Q}(B)$ introduced above; for the three problems having the power to speak about a single individual, we abuse this notion and write $\star \text{SAT}_{\exists}(B)$ for the problems $\star \text{SAT}_{Q}(B)$ without $\text{TSAT}_{Q}(B)$. 

\[8\]
3.1 Both quantifiers

Theorem 7 ([33, 38, 21]). \( \text{OCSAT}_{\exists \forall}(\text{BF}) \in \text{EXPTIME} \).

Due to the interreducibilities stated in Section 2, it suffices to show lower bounds for TSAT and upper bounds for OCSAT. Moreover Lemma 4 enables us to restrict the proofs to the standard basis of each clone for stating general results.

The following theorem improves [29] by stating completeness results.

Theorem 8. Let \( B \) be a finite set of Boolean operators.

1. If \( 1 \subseteq [B] \) or \( N_2 \subseteq [B] \), then \( \text{TSAT}_{\exists \forall}(B) \) is \( \text{EXPTIME} \)-complete.
2. If \( 1_0 \subseteq [B] \) or \( N_2 \subseteq [B] \), then \( \text{SAT}_{\exists \forall}(B) \) is \( \text{EXPTIME} \)-complete.
3. If \( [B] \subseteq R_0 \), then \( \text{TSAT}_{\exists \forall}(B) \) is trivial.
4. If \( [B] \subseteq R_1 \), then \( \text{SAT}_{\exists \forall}(B) \) is trivial.

Proof. Parts 1.–4. are formulated as Lemmas 9 to 13 and are proven below. □

Part (2) for \( 1_0 \) generalizes the EXPTIME-hardness of subsumption for \( \mathcal{FL}_0 \) and \( \mathcal{AC} \) with respect to GCIs [22, 18, 5, 24]. The contrast to the tractability of subsumption with respect to GCIs in \( \mathcal{EL} \), which uses only existential quantifiers, undermines the observation that, for negation-free fragments, the choice of the quantifier affects tractability and not the choice between conjunction and disjunction. DL-Lite and \( \mathcal{ALU} \) cannot be put into this context because they use unqualified restrictions.

Parts (1) and (2) show that satisfiability with respect to theories is already intractable for even smaller sets of Boolean operators. One reason is that sets of axioms already contain limited forms of implication and conjunction. This also causes the results of this analysis to differ from similar analyses for sub-Boolean modal logics in that hardness already holds for bases of clones that are comparatively low in Post’s lattice.

Part (3) reflects the fact that TSAT is less expressive than the other three decision problems: it cannot speak about one single individual.

Lemma 9 ([29]). Let \( B \) be a finite set of Boolean operators s.t. \( B \) contains only \( \top \)-reproducing operators. Then \( \text{OCSAT}_{\exists \forall}(B) \) is trivial.

Lemma 10 ([29]). Let \( B \) be a finite set of Boolean operators s.t. \( B \) contains only \( \bot \)-reproducing operators. Then \( \text{TSAT}_{\exists \forall}(B) \) is trivial.

Lemma 11. Let \( B \) be a finite set of Boolean operators with \( \{\bot, \land\} \subseteq [B] \), or \( \{\bot, \lor\} \subseteq [B] \). Then \( \text{SAT}_{\exists \forall}(B) \) is \( \text{EXPTIME} \)-complete. If all self-dual operators can be expressed in \( B \), then \( \text{TSAT}_{\exists \forall}(B) \) is \( \text{EXPTIME} \)-complete.
Proof. The membership in EXPTIME for OCSAT\textsubscript{3\forall}(B) follows from Theorem 7 in combination with Lemma 4.

For EXPTIME-hardness, we first consider the case $\sqcap \in B$ and reduce from the positive entailment problem for Tarskian set constraints in [22]: thus we start from the question if $T \models A \sqsubseteq B$, for concepts $A, B$ and a terminology $T$ that uses the quantifiers $\forall$ and $\exists$, and $\sqcap$ as the only Boolean connective. Now $T$ just consists of concepts that contain $\sqcap$. Hence $T \models A \sqsubseteq B$ if and only if $T' \not\models TSAT_{3\forall}(\{\sqcap, \top, \bot\})$, for $T' := T \cup \{ \top \sqsubseteq \exists R.A \sqcap B', B' \equiv \exists R.B, \bot, B \equiv \forall R.B.\bot\}$, where $B'$ is a new atomic concept and $R, R_B$ are new roles. This holds as $A$ does not imply $B$ if and only if there is an instance of $A$ which is not an instance of $B$. As $B$ and $B'$ are declared disjoint, the claim applies. Now for TCSAT\textsubscript{3\forall}(\{\bot, \sqcap\}), we transform $T'$ into $T''$ by substituting the two introduced occurrences of $\sqcap$ with a fresh concept name $C$ and put $C$ into the instance of TCSAT\textsubscript{3\forall}(\{\bot, \sqcap\}) we are reducing to. Then, $T \models A \sqsubseteq B$ if $(T'', C) \not\models TCSAT_{3\forall}(\{\bot, \sqcap\})$.

For TCSAT\textsubscript{3\forall}(\{\bot, \sqcup\}), we modify the above definition of $T''$ to dispose of the introduced conjunction: using a fresh atomic concept $D$, we set $T' := T \cup \{ D \sqsubseteq A, D \sqsubseteq B', \top \sqsubseteq \exists R.D, B' \equiv \exists R.B.C, B \equiv \forall R.B.\bot\}$.

The remaining case for the self-dual operators follows from Lemmas 1 and 2 as all self-dual functions in combination with the constants $\top, \bot$ (to which we have access as $\neg$ is self-dual) can express any arbitrary Boolean function. 

Lemma 12. $\ast SAT_{3\forall}(\{\bot\})$ and $TSAT_{3\forall}(\{\bot, \top\})$ are EXPTIME-complete.

Proof. For the upper bound apply Theorem 7 and Lemma 4. For hardness, we reduce from TSAT\textsubscript{3\forall}(\{\bot, \top\}) to TSAT\textsubscript{3\forall}(\{\bot, \top\})—the former shown to be EXPTIME-complete in the proof of Lemma 11. The main idea is an extension of the normalization rules in [14]. The following normalization rules have been stated and proven to be correct in [14]:

$(NF1)$ $C \sqcap D \sqsubseteq E \leadsto \{ A \equiv \hat{C}, A \sqcap D \sqsubseteq E \}$
$(NF2)$ $C \sqcap D \sqsubseteq \hat{E} \leadsto \{ C \sqcap D \sqsubseteq A, A \equiv \hat{E} \}$
$(NF3)$ $\exists r.\hat{C} \sqsubseteq D \leadsto \{ A \equiv \hat{C}, \exists r.A \sqsubseteq D \}$
$(NF4)$ $C \sqsubseteq \exists r.\hat{D} \leadsto \{ C \sqsubseteq \exists r.A, A \equiv \hat{D} \}$
$(NF5)$ $C \sqsubseteq D \sqsubseteq E \leadsto \{ C \sqsubseteq D, C \sqsubseteq E \}$
$(NF6)$ $C \equiv D \leadsto \{ C \sqsubseteq D, D \sqsubseteq C \}$

where $\sqsubseteq \in \{\sqcap, \equiv\}$, $\hat{C}$ states that the concept description $C$ is no concept name, and $A$ is a new concept name.

Now we want to extend these rules for conjunctions on the left side of GCI's and for $\forall$-quantification:

$(NF3b)$ $\forall r.\hat{C} \sqsubseteq D \leadsto \{ A \equiv \hat{C}, \forall r.A \sqsubseteq D \}$
$(NF4b)$ $C \sqsubseteq \forall r.\hat{D} \leadsto \{ A \equiv \hat{D}, C \sqsubseteq \forall r.A \}$
$(NF7)$ $A \sqcap B \sqsubseteq C \leadsto \{ A \sqsubseteq \exists R.A.\top, B \sqsubseteq \forall R.A.A', \exists R.A.A' \sqsubseteq C \}$

where $R_A$ is a fresh role, and $A'$ is a fresh concept name. For (NF7) we will prove its correctness.
Assume \( A \cap B \subseteq C \) holds in the interpretation \( \mathcal{I} = (\Delta^I, \cdot^I) \). Thus for each individual \( w \in \Delta^I \) with \( w^I \supseteq \{A, B\} \) it holds \( C \in w^I \) as assumed.

In the following we will construct a modified interpretation \( \mathcal{I}' \) from \( \mathcal{I} \) that satisfies the axioms constructed by (NF7), i.e., the axioms in \( \{A \subseteq \exists R_A \top, B \subseteq \forall R_A A', \exists R_A A' \subseteq C\} \). As \( A \in w^I \), we add one \( R_A \)-edge to the same individual \( w \), and due to \( B \subseteq \forall R_A A' \) we must add \( A' \) to \( w^I \). Finally the last GCI is satisfied as we have \( C \in w^I \).

For the opposite direction assume \( A \cap B \subseteq C \) cannot be satisfied, i.e., in every interpretation there is an individual which is an instance of \( A \) and \( B \) but not of \( C \). Hence we take an arbitrary interpretation \( \mathcal{I} \) such that it satisfies the first two axioms \( A \subseteq \exists R_A \top \) and \( B \subseteq \forall R_A A' \). Due to our assumption every individual \( w \) is in instance of \( A \) and \( B \), and hence we have an \( R_A \)-edge to an individual where \( A' \) must hold. Therefore the left side of the third axiom is fulfilled but \( C \) does not hold for the individual \( w \). Hence this axiom is not satisfied and we have the desired contradiction.

As this normalization procedure runs in polynomial time and eliminates every conjunction of concepts, we have a reduction from TCSAT\( \exists \forall (\{\top, \bot\}) \) to TCSAT\( \exists \forall (\{\top\}) \), and also from TSAT\( \exists \forall (\{\top, \bot, \top\}) \) to TSAT\( \exists \forall (\{\top, \bot\}) \). Hence the Lemma applies.

\begin{lemma}
\[ ** \text{SAT}_\exists \forall (N_2) \] is EXPTIME-complete.
\end{lemma}

\begin{proof}
The upper bound follows from Theorem 7 and Lemma 1. For the lower bound use Lemma 2 to simulate \( \top \) and \( \bot \) with fresh atomic concepts. Then the argumentation follows similarly to Lemmas 11 and 12.
\end{proof}

3.2 Restricted quantifiers

In this section we investigate the complexity of the problems OCSAT\( Q \), OSAT\( Q \), TCSAT\( Q \), and TSAT\( Q \), where \( Q \) contains at most one of the quantifiers \( \exists \) or \( \forall \). Even the case \( Q = \emptyset \) is nontrivial: for example, TSAT\( Q(B) \) does not reduce to propositional satisfiability for \( B \) because restricted use of implication and conjunction is implicit in sets of axioms.

TSAT-Results

\begin{theorem}
Let \( B \) be a finite set of Boolean operators.
\begin{enumerate}
    \item If \( L_3 \subseteq [B] \) or \( M \subseteq [B] \), then TSAT\( \emptyset (B) \) is NP-complete.
    \item If \( E = [B] \) or \( V = [B] \), then TSAT\( \emptyset (B) \) is P-complete.
    \item If \( [B] \in \{ L, N_2, N \} \), then TSAT\( \emptyset (B) \) is NL-complete.
    \item Otherwise (if \( [B] \subseteq R_3 \) or \( [B] \subseteq R_0 \)), then TSAT\( \emptyset (B) \) is trivial.
\end{enumerate}
\end{theorem}
Proof. NP-completeness for (1) is composed of on the one hand the upper bound which results from OCSAT_3(\(\land, \neg, \top, \bot\)) which is proven to be in \(\text{NP}\) in Lemma 28 and on the other hand the lower bounds which are proven in Lemmas 15 and 16. Both lower bounds of (2) will be proven through Lemmas 17 and 18. The upper bound is due to OCSAT_3(\(\land, \top, \bot\)) which is shown to be in \(\text{P}\) in Lemma 34. The membership of the third item results from TCSAT(\(\neg, \top\)) which is proven to be in NL in Lemma 29 and the hardness result is proven in Lemma 19. Item (4) follows through Lemmas 9 and 10.

Lemma 15. Let \(B\) be a set of Boolean operators s.t. all self-dual or monotone operators are in \([B]\). Then TSAT_\(\emptyset\)(\(B\)) is \(\text{NP}\)-hard.

Proof. We start with the implication problem for the self-dual (resp. monotone) fragment of propositional logic IMP(D) (resp. IMP(M)), which is shown to be coNP-complete in [10]. To establish \(\text{NP}\)-hardness of TSAT_\(\emptyset\)(M), we reduce from the complement of IMP(M) in the following way. Let \(\varphi, \psi\) be two propositional formulae with monotone operators only. Then

\[
(\varphi, \psi) \notin \text{IMP}(M) \iff \varphi \not\models \psi
\]

\[
\iff \exists \theta : \theta \models \varphi \land \neg \psi
\]

\[
\iff \{C_\varphi \subseteq \bot, \top \subseteq C_\psi\} \in \text{TSAT}_\emptyset(M),
\]

where \(C_\varphi\) and \(C_\psi\) are concepts corresponding to \(\varphi, \psi\) in the usual way.

For TSAT_\(\emptyset\)(D), we use the same reduction, but need to replace the introduced operators \(\top, \bot\) as in Lemma 2.

Lemma 16. Let \(B\) be a set of Boolean operators s.t. \(L_3 = [B]\), then TSAT_\(\emptyset\)(\(B\)) is \(\text{NP}\)-hard.

Proof. Here we will provide a reduction from the \(\text{NP}\)-complete problem 1-in-3-SAT which is defined as follows: given a formula \(\varphi = \bigwedge_{i=1}^{n} \bigvee_{j=1}^{3} l_{ij}\), where \(l_{ij}\) are literals, we ask for the existence of a satisfying assignment which fulfills exactly one literal per clause ([31]). In the following we are allowed to use the binary exclusive-or as we have access to negation because \(x \oplus x \oplus z \oplus \top \equiv \neg z\), and we have access to both constants \(\top\) and \(\bot\) due to Lemma 2. Thus we are able to use the binary exclusive-or operator because \(x \oplus y \oplus \top \oplus \top \equiv x \oplus y\).

The main idea of the reduction is to use for each clause \((x \lor y \lor z) \in \varphi\) an axiom \(\top \subseteq x \oplus y \oplus z\) to enforce that only one literal is satisfied. As for this axiom it is possible to have all literals satisfied we need some additional axioms to circumvent this problem.

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Let \( \varphi \) defined as above, then the reduction is defined as \( \varphi \rightarrow T \), where

\[
T := \{ \top \sqsubseteq f(l_{1i}) \boxplus f(l_{2i}) \boxplus f(l_{3i}) \boxplus s_i \boxplus \top \mid 1 \leq i \leq n \} \cup \\
\cup \{ \top \sqsubseteq f(l_{1i}) \boxplus f(l_{2i}) \mid 1 \leq i \leq n \} \cup \\
\cup \{ s_i \sqsubseteq f(l_{1i}) \boxplus f(l_{2i}) \mid 1 \leq i \leq n \} \cup \\
\cup \{ s_i \sqsubseteq f(l_{2i}) \boxplus f(l_{3i}) \mid 1 \leq i \leq n \} \cup \\
\cup \{ s_i \sqsubseteq s_i \boxplus s_i \boxplus s_i \mid 1 \leq i \leq n \} \cup \\
\cup \{ \top \sqsubseteq A_x \boxplus A_x' \mid x \text{ variable in } \varphi \},
\]

where \( f(x) = A_x \) and \( f(\bar{x}) = A_x' \). Now we claim that \( \varphi \in 1\text{-in-}3\text{-SAT} \) iff \( T \in \mathrm{TSAT}_\emptyset(L_0) \).

Consider an arbitrary clause \( c = x \lor y \lor z \) from \( \varphi \) with \( x, y, z \) literals. Then following axioms which differ for convenience slightly from the notion above are part of \( T \)

\[
\begin{array}{cccc}
\top \sqsubseteq x \boxplus y \boxplus z \boxplus s \boxplus \top & (1) \\
\top \sqsubseteq x \boxplus y \boxplus z & (2) \\
\end{array}
\]

The table on the upper right shows each possible assignment for \( x, y, z \) and suitable assignments for the \( s_i \)'s and the validity of the axioms (1) and (2). Underlined numbers denote must set truth values enforced by the axioms whereas blank cells denote arbitrary choices. If at least one of (1) and (2) are contradicted then there exists no interpretation for \( T \). At first we start with an interpretation that assigns the individuals \( x, y, z \) to the recent world in some way. Then we immediately observe if axiom (2) is contradicted or not. If it is not contradicted then we have to look at the remaining \( s_i \)'s in order to find an extension of this interpretation which assigns the \( s_i \)'s and \( s \) in a way such that (2) is not violated whenever we have an interpretation which corresponds to a valid 1-in-3-SAT assignment. Otherwise we have to show that there exists no possible extension that falsely satisfies axiom (2).

Thus the table shows that for every eligible assignment we always have a fulfilling interpretation, and for ever improper assignment it is not possible to construct a fulfilling one.

**Lemma 17.** Let \( B \) be a set of Boolean operators s.t. \( E = |B| \), then \( \mathrm{TSAT}_\emptyset(B) \) is P-hard.
Proof. In the following we will state a \( \leq_{cd} \)-reduction from the complement of the P-complete problem HGAP, which is the accessibility problem for directed hypergraphs. In a given hypergraph \( H = (V, E) \), a hyperedge \( e \in E \) is a set of source nodes \( \text{src}(e) \subseteq V \) and one destination node \( \text{dest}(e) \in V \). Instances of HGAP consist of a directed hypergraph \( H = (V, E) \), a set \( S \subseteq V \) of source nodes, and a target node \( t \in V \). Now the question is whether there exists a hyperpath from the set \( S \) to the node \( t \), i.e., whether there are hyperedges \( e_1, e_2, \ldots, e_k \) s.t. for each \( e_i \) there are \( e_i, \ldots, e_{i+1} \) with \( 1 \leq i_1, \ldots, i_\nu < i \) and \( \bigcup_{j \in \{i_1, \ldots, i_\nu\}} \text{dest}(e_j) \supseteq \text{src}(e_i) \), and \( \text{src}(e_1) = S \) and \( \text{dest}(e_k) = t \).

HGAP remains P-complete even if we restrict the hyperedges to contain at most two source nodes \( \exists^2 \). W.l.o.g. assume that if there is a path from \( S \) to \( t \), then the last edge of that path is a usual edge with only one source node.

Let \( G = (V, E) \) be a directed hypergraph, \( \{s_1, \ldots, s_k\} = S \subseteq V \) with \( s_1, \ldots, s_k \in V \) be the set of source nodes, and \( t \in V \) be the target node. For each node \( v \in V \), we use a new atomic concept \( v \). In addition let \( t, t' \) be fresh atomic concepts. Now define

\[
T := \{u_1 \cap \ldots \cap u_k \subseteq v \mid (u_1, \ldots, u_k; v) \in E\} \cup \{\top \subseteq s_1 \cap \ldots \cap s_k \cap t', t \cap t' \subseteq \bot\}.
\]

Then \( (G, S, t) \in \text{HGAP} \iff T \notin \text{TSAT}_\emptyset(\{\cap, \top, \bot\}) \).

\( \Rightarrow \): Assume there is a hyperpath from \( S \) to \( t \) as above. Thus in every interpretation \( I = (\Delta^T, \mathcal{T}) \) it holds for all \( w \in \Delta^T \) that \( s_1, \ldots, s_k, t' \in w^T \). As the before mentioned hyperpath exists, \( t \) must also be in \( w^T \) through the chain of axioms that correspond to the hyperedges in the path. This violates the axiom \( t \cap t' \subseteq \bot \).

\( \Leftarrow \): Assume there is no hyperpath from \( S \) to \( t \) in \( G = (V, E) \). Hence there is no chain of axioms that enforce \( t \) to be true in every state. Therefore we are able to construct a satisfying interpretation in the following way: \( I = (\{w\}, \mathcal{T}) \) and

\[
w^T := \{v \mid (s_1, \ldots, s_k; v) \in E^* \cup \{t'\},
\]

where \( E \) is the transitive closure of \( E \). Please note that \( (s_1, \ldots, s_k; t) \notin E^* \) and thus \( t \notin w^T \). Therefore, all axioms are satisfied and \( T \in \text{TSAT}_\emptyset(\{\cap, \top, \bot\}) \).

Lemma 18. Let \( B \) be a set of Boolean functions s.t. \( V = \lceil B \rceil \), then \( \text{TSAT}_\emptyset(B) \) is P-hard.

Proof. To realize the desired lower bound, we use Lemma 6 to state a reduction from \( \text{TSAT}_\emptyset(E) \) to \( \text{TSAT}_\emptyset(V) \).

Lemma 19. Let \( B \) be a set of Boolean functions s.t. \( l = \lceil B \rceil \), then \( \text{TSAT}_\emptyset(B) \) is NL-hard.

Proof. For proving NL-hardness we will reduce from the complement of the graph accessibility problem GAP which is NL-complete. Consider a given directed graph \( G = (V, E) \) and two nodes \( s, t \in V \) as the recent instance for GAP asking for a
path from $s$ to $t$ in $G$. We introduce a concept name $A_v$ per node $v \in V$ and define

$$\mathcal{T} := \{A_u \sqsubseteq A_v \mid (u,v) \in E\} \cup \{\top \sqsubseteq A_s, A_t \sqsubseteq \bot\}.$$ 

We will now prove that $(G,s,t) \not\in GAP \iff \mathcal{T} \in \text{TSAT}_0(B)$.

"$(G,s,t) \not\in GAP \implies \mathcal{T} \in \text{TSAT}_0(B)$": Assume there is no path from $s$ to $t$.

Take the interpretation $\mathcal{I} := ([x], \cdot^\mathcal{I})$ with $A^\mathcal{I}_v := \{\{x\}\}$ if $v$ is reachable from $s$, $\emptyset$ otherwise, for each $v \in V$. Then $A^\mathcal{I}_t = \emptyset$ and with that all axioms are satisfied. Thus it holds that $\mathcal{I} \models \mathcal{T}$.

"$(G,s,t) \in GAP \implies \mathcal{T} \not\in \text{TSAT}_0(B)$": Now assume we have a path $\pi = v_1, \ldots, v_k$ in $G$ with $k \in \mathbb{N}$, $(v_i, v_{i+1}) \in E$, $v_i \in V$ for $1 \leq i \leq k$, $v_1 = s$, and $v_k = t$ from $s$ to $t$. Now any interpretation needs to include an individual $x$ instantiating $A_s$ (else $\top \sqsubseteq A_s$ would be contradicted) and also $A_{v_2}, \ldots, A_{v_k} = A_t$. But with $A_t \in x^\mathcal{I}$ we contradict the axiom $A_t \sqsubseteq \bot$. Thus $\mathcal{I} \not\models \mathcal{T}$, and with that $\mathcal{T} \not\in \text{TSAT}_0(B)$. □

**Lemma 20.** Let $B$ be a set of Boolean functions s.t. $I = [B]$, then $\text{TSAT}_0(B)$ is in NL.

**Proof.** The main idea is to do a path search in a concept dependence graph—a reduction to the complement of GAP. A given $\mathcal{T}$ is mapped to $G = (V, E)$ where

$$V := \{v_A, v_B \mid A \sqsubseteq B \in \mathcal{T}\} \cup \{v_{\top}, v_{\bot}\}$$

and

$$E := \{(v_A, v_B) \mid A \sqsubseteq B\}.$$ 

Now it holds $\mathcal{T} \in \text{TSAT}_0(B) \iff (G, v_{\top}, v_{\bot}) \not\in \text{GAP}$. Please note that we need to add $v_{\top}, v_{\bot}$ to $V$ in order to keep consistency if at least one of $\top$ and $\bot$ is not part of an axiom side. If $\mathcal{T}$ is not satisfiable, then in every interpretation there is at least one axiom contradicted. W.l.o.g. the contradicted axiom is of the form $C \sqsubseteq \bot$ and $C$ is instantiated by some individual $x$. Thus there must be a chain of axioms that enforce $C$ to be true and it can be easily shown that this chain starts at some axiom $\top \sqsubseteq C'$. Hence we have a path starting at $v_{\top}$ in the Graph $G$ which leads to a node $v_{\bot}$. For the opposite direction the argumentation is analogue. □

**Theorem 21.** Let $B$ be a finite set of Boolean operators and $\mathcal{Q} \in \{\forall, \exists\}$.

1. If $M \subseteq [B]$ or $N_2 \subseteq [B]$, then $\text{TSAT}_\mathcal{Q}(B)$ is EXPTIME-complete.
2. If $E = [B]$, $V = [B]$, or $I = [B]$, then $\text{TSAT}_\mathcal{Q}(B)$ is P-complete.
3. Otherwise (if $[B] \subseteq R_3$ or $[B] \subseteq R_0$), then $\text{TSAT}_\mathcal{Q}(B)$ is trivial.
Proof. For the monotone case in (1) consider Lemmas 22 and 23. The proof for $N_2$ can be found in Lemma 24. The respective upper bounds for (1) result from Theorem 7 in combination with Lemma 4. The needed lower bound for the P-hardness results in (2) is shown for TSAT$_3(I)$ in Lemma 26 (case $\forall$ is due to Lemma 6). The membership in P for the cases in (3) result on the one hand from OCSAT$_3(\land, \top, \bot)$ which is shown to be in P in Lemma 34 and on the other hand from TSAT$_\forall(\land, \top, \bot)$ is proven in Lemma 25. The two remaining upper bounds for $|B| = V$ follow from the complementary problem through Lemma 6.

Item (3) follows through Lemmas 9 and 10.

Part (3) generalizes the fact that every $\ell$- and $F\ell_0$-TBox is satisfiable, and the whole theorem shows that separating either conjunction and disjunction, or the constants is the only way to achieve tractability for TSAT.

Lemma 22. Let $B$ be a set of Boolean functions s.t. $M = [B]$, then TSAT$_3(B)$ is EXPTIME-hard.

Proof. For EXPTIME-hardness, we will reduce from the complement of the subsumption problem w.r.t. TBoxes for the logic $\ell\mathcal{U}$, which has been investigated in [5, Thm. 7]. $\ell\mathcal{U}$ is $\mathcal{ALC}$ restricted to the operators $\top, \land, \lor, \exists$. Now it holds that

$$(T, A, B) \in \ell\mathcal{U}\text{-SUBS}$$

$\iff T \models A \subseteq B$

$\iff \text{for all } I : I \models T \text{ implies } I \models A \subseteq B$

$\iff \text{there is no } I : I \models T \text{ and } I \models A \nsubseteq B$

$\iff \text{there is no } I : I \models T \text{ and } I \models \top \nsubseteq \exists R.A \land \neg B$

$\iff \text{there is no } I : I \models T \cup \{\top \nsubseteq \exists R.(A \land B'), \top \subseteq B \lor B', B \land B' \subseteq \bot\}^{T'}$

$\iff T' \notin \text{TSAT}_3(M),$

for a fresh role $R$ and a fresh concept $B'$.

Lemma 23. Let $B$ be a set of Boolean functions s.t. $M = [B]$, then TSAT$_\forall(B)$ is EXPTIME-hard.

Proof. As in the proof of Lemma 18, we can reduce from the dual problem TSAT$_3(B)$ through Lemma 6.

Lemma 24. Let $B$ be a set of Boolean functions s.t. $N_2 = [B]$ and $Q \in \{\forall, \exists\}$, then TSAT$_Q(B)$ is EXPTIME-hard.

Proof. We reduce from TSAT$_3(l)$, which is shown to be EXPTIME-complete in Lemma 13. As known from Lemma 2, we can simulate the constants using new concept names and negation. Additionally observe that, although $Q$ contains only one quantifier, the other quantifier can be expressed using $\neg$.

Lemma 25. Let $B$ be a set of Boolean functions s.t. $E = [B]$, then TSAT$_\forall(B)$ is in P.
Proof. Here we will specify an algorithm for satisfiability similar to the one in [13] that constructs iteratively the transitive closure of atomic concepts that imply each other. Thus, informal speaking, starting by the empty set $S_0 := \emptyset$, for each $S_i$ we look at each axiom $C \subseteq D$ and add $D$ to $S_{i+1}$ if $C \in S_i$. The construction of these sets is defined inductively as follows, where $T$ is a TBox that is in normal form (i.e., $T$ contains only expressions of the form $C \subseteq D$, $C_1 \cap C_2 \subseteq D$, $\forall r.C \subseteq D$, or $C \subseteq \forall r.D$, where $C$ and $D$ are atomic concepts and $r$ is a role—please note that for each $S_i$ it holds $S_i \subseteq (\bigcup \{T, \bot\}^*)$:

1. \textbf{(IS1)} If $C_1 \in S_i(C)$ and $C_1 \subseteq D \in T$, then $S_{i+1}(C) := S_i(C) \cup \{D\}$.
2. \textbf{(IS2)} If $C_2 \subseteq S_i(C)$ and $C_1 \cap C_2 \subseteq D \in T$, then $S_{i+1}(C) := S_i(C) \cup \{D\}$.
3. \textbf{(IS3)} If $C_1 \in S_i(C)$ and $C_1 \subseteq \forall r.D \in T$ and $D_1 \in S_i(D)$ and $\forall r.D_1 \subseteq C \in T$, then $S_{i+1}(C) := S_i(C) \cup \{D\}$.

The construction for each of those sets $S_i$ takes time at most $O(|T|)$ and eventually stops for an atomic concept $C$ if $S_i(C) = S_{i+1}(C)$ for some $i \in \mathbb{N}$.

We now claim that $T \in \text{TSAT}_v(B)$ iff $\bot \notin S^*_T(\bot)$, where $S^*_T(\bot)$ denotes the transitive closure of $S_i$ for $\bot$ w.r.t. $T$.

\text{“⇒”: Let $T \in \text{TSAT}_v(B)$ via the interpretation $I$. Hence $I \models T$ and in particular for each $C \subseteq D \in T$ it holds that $C^I \subseteq D^I$. As (IS1) to (IS3) hold, we have $\bot \notin S^*_T(\bot)$, otherwise there exist $C_1 \subseteq D_1, \ldots, C_t \subseteq D_t \in T$ s.t. $C_1 = \top$ and $D_t = \bot$, and $C_1$ implies $D_t$ through these axioms. We show this by induction on $n$, where $n$ is the index of the first $S_i$ with $\bot \notin S^*_T(\bot)$.}

Let $n = 1$, then $C_1 = \top$ and $D_1 = \bot$; hence we apply (IS1) for $\top \subseteq \bot \subseteq T$ and $\bot \in S^*_T(\bot)$.

\text{n → n + 1: Let $1 \leq i, j \leq n$,
1. $C_{n+1} = D_j$, and $D_j \in S_n(\top)$, then $D_{n+1} \in S_{n+1}(\top)$.
2. $C_{n+1} = D_j \cap D_j$, and $D_j, D_j \in S_n(\top)$, then $D_{n+1} \in S_{n+1}(\top)$.
3. $C_k = D_j$, $1 \leq k \neq j < n$, $D_k = \forall r.C_1$, $k \leq s \leq n$, and $C_i \in S_n(\top)$, and $\forall r.C_1 \subseteq D_n \in T$, then $D_{n+1} \in S_{n+1}(\top)$.

Hence, if $D_{n+1} = \bot$, then $\bot \in S^*_T(\bot)$.}

The argumentation for the opposite direction is analogue to [13]. \qed

**Lemma 26.** Let $B$ be a set of Boolean functions s.t. $I = |B|$, then $\text{TSAT}_3(B)$ is P-hard.

Proof. We will reduce the word problem for the Turing machine model that characterizes LOGCFL to SUBS$_3(\emptyset)$. Together with the trivial reduction SUBS$_3(\emptyset) \leq \text{TSAT}_3(I)$, justified by $(T, A, B) \in \text{SUBS}_3(\emptyset)$ iff $(T \cup \{\top \equiv A, B \equiv \bot\}) \notin \text{TSAT}_3(I_0)$, this will provide LOGCFL-hardness of $\text{TSAT}_3(I)$. Observe that LOGCFL is closed under complement [12]. As in the proof the runtime of the Turing machine is not relevant we achieve instead a P-hardness result (because an NL-Turing machine with arbitrary runtime leads to the class P [17]).

Let $M$ be a nondeterministic Turing machine, which has access to a read-only input tape, a read-write work tape and a stack, and whose runtime is bounded by a polynomial in the size of the input. Let $M$ be the 6-tuple $(\Sigma, \Psi, \Gamma, Q, f, q_0)$, where
– Σ is the input alphabet;
– Ψ is the work alphabet containing the empty-cell symbol #;
– Γ is the stack alphabet containing the bottom-of-stack symbol □;
– Q is the set of states;
– \( f : Q \times \Sigma \times \Psi \times \Gamma \rightarrow Q \times \Psi \times \{-, +\} \times (\Gamma \setminus \{\square\})^* \) is the state transition function which describes a transition where the machine is in a state, reads an input symbol, reads a work symbol and takes a symbol from the stack, and goes into another state, writes a symbol to the work tape, makes a step on each tape (left or right) and possibly adds a sequence of symbols to the stack;
– \( q_0 \in Q \) is the initial state.

We assume that each computation of \( M \) starts in \( q_0 \) with the heads at the left-most position of each tape and with exactly the symbol □ on the stack. W.l.o.g., the machine accepts whenever the stack is empty, regardless of its current state.

Let \( x = x_1 \ldots x_n \) be an input of \( M \). We consider the configurations that can occur during any computation of \( M(x) \) in two versions. A shallow configuration of \( M(x) \) is a sequence \((p\delta_1 \ldots \delta_{k-1}q\delta_k \ldots \delta_{\ell})\), where

– \( p \in \{1, \ldots, n\} \) is the current position on the input tape, represented in binary;
– \( \ell \in O(\log n) \) is the maximal number of positions on the work tape of \( M \) relevant for the computations of \( M(x) \);
– \( \delta_1, \ldots, \delta_{\ell} \) is the current content of the work tape;
– \( k \) is the current position on the work tape;
– \( q \) is the current state of \( M \).

The initial shallow configuration \((0q_0\# \ldots \#)\) is denoted by \( S_0 \). Let \( \mathcal{SC}_{M,x} \) be the set of all possible shallow configurations that can occur during any computation of \( M(x) \). The cardinality of this set is bounded by a polynomial in \( n \) because the number of work-tape cells used is logarithmic in \( n \) and the binary counter for the position on the input tape is logarithmic in \( n \).

A deep configuration of \( M(x) \) is a sequence \((R_1 \ldots R_m p\delta_1 \ldots \delta_{k-1}q\delta_k \ldots \delta_{\ell})\), where the \( R_i \) are the symbols currently on the stack and the remaining components are as above. Let \( \mathcal{DC}_{M,x} \) be the set of all possible deep configurations that can occur during any computation of \( M(x) \). The cardinality of this set can be exponential as soon as \( \Gamma \) has more than two elements besides □. This is not a problem for our reduction, which will only touch shallow configurations.

We now construct an instance of \( \text{SUBS}_\exists(\emptyset) \) from \( M \) and \( x \). We use each shallow configuration \( S \in \mathcal{SC}_{M,x} \) as a concept name and each stack symbol as a role name. The TBox \( T_{M,x} \) describes all possible computations of \( M(x) \) by containing an axiom for every two deep configurations that the machine can take on before and after some computation step. A deep configuration \( D \) is represented by the concept corresponding to \( D \)'s shallow part, preceded by the sequence of existentially quantified stack symbols corresponding to the stack content in \( D \).

The TBox \( T_{M,x} \) is constructed from a set of axioms per entry in \( f \). (We will omit the subscript from now on.) For the instruction

\[(q, \sigma, \delta, R) \mapsto (q', \delta', -, -, R_1 \ldots R_k)\]
of \( f \), we add the axioms

\[
\exists R_0 \ldots \exists R_\ell \binom{p}{\delta_0 \ldots \delta_{i-1} q \delta_{i+1} \ldots \delta_\ell} \sqsubseteq \exists S_1 \ldots \exists S_j . \exists R_1 \ldots R_k . \binom{p-1}{\delta_0 \ldots \delta_{i-2} q' \delta_{i-1} \delta' \delta_{i+1} \ldots \delta_\ell} \tag{3}
\]

for every \( p \) with \( x_p = \sigma \), every \( i = 1, \ldots, \ell \), and all \( \delta_0, \ldots, \delta_{i-1}, \delta_{i+1}, \ldots, \delta_\ell \). The expression \( p - 1 \) stands for \( p - 1 \) if \( p \geq 2 \) and for 1 otherwise, reflecting the assumption that the machine does not move on the input tape on a “go left” instruction if it is already on the left-most input symbol. This behaviour can always be assumed w.l.o.g. In case \( k = 0 \), the quantifier prefix on the right-hand side is empty. For instructions of \( f \) requiring “+” steps on any of the tapes, the construction is analogue. The number of axioms generated by each instruction is bounded by the number of shallow configurations; therefore the overall number of axioms is bounded by a polynomial in \( n \cdot |f| \).

Furthermore, we use a fresh concept name \( B \) and add an axiom \( S \sqsubseteq B \) for each shallow configuration \( S \). Also we add a single axiom \( S \sqsubseteq \exists 2. S_0 \) to \( T \). The instance of \( \text{SUBS}_3(\emptyset) \) is constructed as \((T, S, B)\). \( T \) can be constructed in logarithmic space. It remains to prove the following claim.

Claim. \( M(x) \) has an accepting computation if and only if \( S \sqsubseteq T B \).

Proof of Claim. For the "\( \Rightarrow \)" direction, we observe that, for each step in the accepting computation, the (arbitrary) concept associated with the pre-configuration is subsumed by the concept associated with the post-configuration. More precisely, if \( M(x) \) makes a step

\[
(q, \sigma, \delta, R) \mapsto (q', \delta', -, -, R_1 \ldots R_k),
\]

then its deep configuration before that step has to be

\[
S_1 \ldots S_j R p \delta_0 \ldots \delta_{i-1} q \delta_{i+1} \ldots \delta_\ell,
\]

for some \( S_1, \ldots, S_j \in \Gamma \), \( \delta_0, \ldots, \delta_{i-1}, \delta_{i+1}, \ldots, \delta_\ell \in \Psi \) and \( p \in \mathbb{N}_. \), and the deep configuration after that step is

\[
S_1 \ldots S_j R_1 \ldots R_k (p-1) \delta_0 \ldots \delta_{i-2} q' \delta_{i-1} \delta' \delta_{i+1} \ldots \delta_\ell.
\]

The set of axioms constructed in \( 3 \) ensures that there is an axiom that implies

\[
\exists S_1 \ldots \exists S_j . \exists R_0 . \binom{p}{\delta_0 \ldots \delta_{i-1} q \delta_{i+1} \ldots \delta_\ell} \sqsubseteq \exists S_1 \ldots \exists S_j . \exists R_1 \ldots R_k . \binom{p-1}{\delta_0 \ldots \delta_{i-2} q' \delta_{i-1} \delta' \delta_{i+1} \ldots \delta_\ell}.
\]

Since some computation of \( M(x) \) reaches a configuration with an empty stack, we can conclude that some atomic concept corresponding to a shallow configuration \( S \), and therefore also \( B \), subsumes \( \exists \Box.S_0 \) which subsumes \( S \) (per definition).

For the "\( \Leftarrow \)" direction, we assume that \( M(x) \) has no accepting computation. This means that, during every computation of \( M(x) \), the stack does never become empty. From the set of all computations of \( M(x) \), we will show that there exists
an interpretation $\mathcal{I}$ that satisfies $\mathcal{T}$, but not $S \subseteq B$; hereby we can conclude $(\mathcal{T}, S, B) \notin \text{SUBS}_3(\emptyset)$.

Observe that any atomic concept besides $S$ and $B$ in $\mathcal{T}$ correspond to a specific shallow configuration of $M(x)$. Let $T_{M(x)} := (V, E)$ denote the computation tree of $M(x)$. Thus every node $v \in V$ represents a deep configuration of $M(x)$ which will be denoted via $C_v$. Then for two nodes $u, v \in V$ with $(u, v) \in E$ it holds that $C_u \vdash_M C_v$. In the following we will describe how to construct an interpretation $\mathcal{I}$ from $T_{M(x)}$ which has a witness for $S^\mathcal{I} \not\subseteq B^\mathcal{I}$. Further on we will denote individuals $x$ in bold font to differ them from the input $x$ for $M$. For ease of notion we will write for some shallow configuration $\mu \in \Sigma_{C_M,x}$ in the following also $\mu$ for the respecting concept in $\mathcal{T}$.

The root of $T_{M(x)}$ is the initial configuration $\square \emptyset \# \ldots \#$. Now we will define

$$\mathcal{I}(S) := \bigcup_{\ell \geq 0} \mathcal{I}_\ell(S)$$

starting with $\Delta^{\mathcal{I}_0(S)} := \{x\}$ and

- $S^{\mathcal{I}_0(S)} := \{x\}$, and
- $y \in (S_0)^{\mathcal{I}_0(S)}$ with $(x, y) \in \square^{\mathcal{I}_0(S)}$ (i.e., $(\exists \square S_0)^{\mathcal{I}_0(S)} = \{x\}$)

inductively as follows. (1) For every node $v \in V$ s.t. $C_v = S_1 \ldots S_j R \mu$ with $\mu \in \bin(n) \times \Psi^h \cdot Q \cdot \Psi^k$ and $h + k = \ell - 1$ is the corresponding configuration in $M(x)$ and let $x_1, \ldots, x_j, x_\ell, x_{\ell+1}, \ldots, x_n \in \Delta^{\mathcal{I}_\ell(S)}$ be individuals s.t. $(x_1, x_2) \in (S_1)^{\mathcal{I}_\ell(S)}$, $(x_2, x_3) \in (S_2)^{\mathcal{I}_\ell(S)}$, \ldots, $(x_{j-1}, x_j, x_{j+1}, \ldots, x_n) \in (S_j)^{\mathcal{I}_\ell(S)}$, $(x_\ell, x_{\ell+1}, \ldots, x_n) \in R^{\mathcal{I}_\ell(S)}$ and $x_{\ell+1}, \ldots, x_n \in \mu^{\mathcal{I}_\ell(S)}$; if $v \in V$ with $(v, u) \in E$ is a post configuration $C_u = S_1 \ldots S_j R_1 \ldots R_k \lambda$ and $x \in \bin(n) \times \Psi^h \cdot Q \cdot \Psi^k$ and $h + k = \ell - 1$ of the configuration $C_v$ in the computation of $M(x)$, i.e., $C_v \vdash_M C_u$, then

- add $x_\ell$ to $\chi^{\mathcal{I}_{\ell+1}(S)}$ for $k = 0$, and otherwise
- if there do not exist $y_1, \ldots, y_k \in \Delta^{\mathcal{I}_\ell(S)}$ with $(x_r, y_1) \in (R_1)^{\mathcal{I}_\ell(S)}$, $(y_1, y_2) \in (R_2)^{\mathcal{I}_\ell(S)}$, \ldots, $(y_k, -1, y_k) \in (R_k)^{\mathcal{I}_\ell(S)}$ and $y_k \in \lambda^{\mathcal{I}_\ell(S)}$, then introduce new individuals $y_1, \ldots, y_k$ to $\Delta^{\mathcal{I}_{\ell+1}(S)}$ and add $(x_{\ell+1}, y_1)$ to $(R_1)^{\mathcal{I}_{\ell+1}(S)}$, $(y_1, y_2)$ to $(R_2)^{\mathcal{I}_{\ell+1}(S)}$, \ldots, $(y_k, 1, y_k)$ to $(R_k)^{\mathcal{I}_{\ell+1}(S)}$ and include $y_k$ into $\chi^{\mathcal{I}_{\ell+1}(S)}$.

(2) For every individual $x \in \Delta^{\mathcal{I}_\ell(S)}$ and deep configuration $\chi$ that is also a shallow configuration with $x \in \chi^{\mathcal{I}_\ell(S)}$ include $x$ into $B^{\mathcal{I}_{\ell+1}(S)}$.

In the following we will show that $\mathcal{I}(S)$ is indeed a valid interpretation for $\mathcal{T}$ but $S \not\subseteq \mathcal{T} B$. As there is no axiom in $\mathcal{T}$ with $S$ on the right side it holds that $|S^{\mathcal{I}(S)}| = 1$. Assume there is some GCI $G = A_G \subseteq B_G \in \mathcal{T}$ which is violated in $\mathcal{I}(S)$, i.e., we have some individual $x' \in \Delta^{\mathcal{I}(S)}$ s.t. $x' \in (A_G)^{\mathcal{I}(S)}$ but $x' \notin (B_G)^{\mathcal{I}(S)}$. As in $\mathcal{T}$ there are two different kinds of axioms we have to distinguish these cases (because the axiom with $S$ on the left side cannot be such a violated axiom):

1. If $G = \alpha \subseteq \beta \in \mathcal{T}$ for $\alpha$ and $\beta$ being atomic (this is the case for axioms with concepts representing shallow configurations on the left side and $B$ on the right side), then $x' \in \alpha^{\mathcal{I}(S)}$ but $x' \notin \alpha^{\mathcal{I}(S)}$. Now consider the least index $n$ s.t. $x' \in \alpha^{\mathcal{I}_n(S)}$. As $\alpha$ represents clearly a shallow configuration and $\beta = B$ then $x'$ is added to $\beta^{\mathcal{I}_{n+1}(S)} \subseteq \beta^{\mathcal{I}(S)}$ by (2), which contradicts the assumption.
2. If \(G = \exists R, \mu \sqsubseteq \exists R_1, \ldots, \exists R_k, \lambda \in T\) wherefore exist some entry in \(f\) from \(M\) s.t. \((S_1 \ldots S_j R) \vdash_M (S_1 \ldots S_j R_1 \ldots R_k \lambda)\) for some stack symbols \(S_1, \ldots, S_j, \) then \(x' \in (\exists R, \mu)^{T(S)}\) but \(x' \notin (\exists R_1, \ldots, \exists R_k, \lambda)^{T(S)}\). Now let \(n\) denote the least index s.t. \(y\) is added to \((\mu)^{T(S)}\) and there must be some \(m < n\) s.t. \((x', y)\) is added to \(T^{\Delta} \). Then in step (1) there are \(y_1, \ldots, y_k\) added to \(\Delta^{T^{\Delta}}\), the corresponding \(R_i\)-edges are added to their respective \((R_i)^{T^{\Delta}}\)-set and \(y_k\) is added to \(\lambda^{T^{\Delta}}\) obeying \(x \in (\exists R_1, \ldots, \exists R_k, \lambda)^{T^{\Delta}}\). This contradicts our assumption again.

Consequently \(I(S)\) is a model of \(T\). Now assume that \(S^{T(S)} \subseteq B^{T(S)}\). Thus for the starting point \(x\) which is added to \(S^{T(S)}\) at the initial construction step of \(I(S)\), it holds in particular that \(x \in B^{T(S)}\). As \(x\) is added to \(B^{T(S)}\) if and only if \(x\) is added to \(\mu^{T(S)}\) for some shallow configuration \(\mu\), we can conclude that an accepting configuration must be reachable in \(T_M(x)\) which contradicts our assumption (of the absence of such a computation sequence). Thus an inductive argument proves that \(\mu \in x^{T(S)}\) for \(\{x\} = S^{T(S)}\) implies that \(M\) reaches an accepting configuration on \(x\) in \(T_M(x)\).

Claim. Let \(C = (R_1 \ldots R_k, \mu)\) be a configuration. It holds for all \(n \in \mathbb{N}\) that if \(x \in (\exists R_1 \ldots \exists R_k, \mu)^{T(S)}\) and \(\{x\} = S^{T(S)}\) then \(M\) reaches \(C\) in the computation on \(x\) in its computation tree \(T_M(x)\).

Induction basis. Let \(n = 1\) and \(C = (R_1 \ldots R_k, \mu)\) for \(\mu \in SC_{M,x}\) be some configuration with \(x \in (\exists R_1 \ldots \exists R_k, \mu)^{T(S)}\) and \(\{x\} = S^{T(S)}\). Thus the individual \(x\) is added to \((\exists R_1 \ldots \exists R_k, \mu)^{T(S)}\) because we have some axiom s.t. \(\exists \boxempty (\text{bin}(0) \# \ldots \#) \sqsubseteq \exists R_1 \ldots \exists R_k, \mu \in T\) as we only have one step in this case. Hence \(C\) can be reached from the initial configuration \(\square 0 \# \ldots \#\) in one step via the transition that corresponds to the before mentioned axiom, i.e., \(\square 0 \# \ldots \# \vdash_M R_1 \ldots R_k, \mu\).

Induction step. Let \(n > 1\) and assume the claim holds for all \(m < n\). Now we have some configuration \(C = (S_1 \ldots S_j R)\) for \(\lambda \in SC_{M,x}\) with \(x \in (\exists S_1 \ldots \exists S_j R)^{T(S)}\) and \(\{x\} = S^{T(S)}\). By induction hypothesis we have some other configuration \(C' = (S_1 \ldots S_j R)\) with \(\lambda \in SC_{M,x}\) from which \(C\) occurs in one step, i.e., \(C' \vdash_M C\) and \(C\) is reachable on the computation of \(M(x)\) and \(x \in (\exists S_1 \ldots \exists S_j R, \lambda)^{T^{\Delta}}\). Thus we have also some axiom that adds \(x\) to \((\exists S_1 \ldots \exists S_j R_1 \ldots R_k, \mu)^{T(S)}\) in (1). This axiom is of the form \(\exists R, \lambda \sqsubseteq \exists R_1 \ldots \exists R_k, \mu \in T\). As \(M\) reaches \(C'\) by induction hypothesis and \(C\) can be reached via one step from \(C'\) and \(x\) is an instance of \(\exists S_1, \ldots, \exists S_j, \exists R_1, \ldots, \exists R_k, \lambda\), \(M\) can also reach \(C\) within the computation on \(x\).

Hence this contradicts our assumption that \(M\) does not accept \(x\) and completes our proof. \(\square\)

**TCSAT-, OSAT-, OCSAT-Results.**

**Theorem 27.** Let \(B\) be a finite set of Boolean operators.

1. If \(S_{11} \subseteq \lfloor B\rfloor\) or \(L_3 \subseteq \lfloor B\rfloor\) or \(L_0 \subseteq \lfloor B\rfloor\), then \(*SAT_{\wedge}^\varnothing (B)\) is NP-complete.
2. If \(\lfloor B\rfloor \in \{E_0, E_0, V_0, V\}\), then \(*SAT_{\wedge}^\varnothing (B)\) is P-complete.
3. If \(\lfloor B\rfloor \in \{l_0, l_1, N_2, N\}\), then \(*SAT_{\wedge}^\varnothing (B)\) is NL-complete.
4. Otherwise (if \(\lfloor B\rfloor \subseteq R_3\)), then \(*SAT_{\wedge}^\varnothing (B)\) is trivial.
Proof. NP-hardness for (1) follows from the respective TSAT\(\emptyset\)(B) results in Lemmas 15 and 16 in combination with Lemma 5 for the lower bound. The membership in NP is shown in Lemma 28.

The lower bounds for (2) result from TSAT\(\emptyset\)(\(\sqcap\), \(\top\), \(\bot\)) shown in Lemmas 17 and 18 in combination with Lemma 5 while the upper bound applies due to OCSAT\(\emptyset\)(\(\sqcap\), \(\top\), \(\bot\)) which is proven to be in P in Lemma 34.

The lower bound of (3) is proven in Lemma 30. The upper bound follows from Lemmas 29 and 31. (4) is due to Lemma 9.

Lemma 28. Let \(B\) be a set of Boolean functions s.t. \([B] \subseteq BF\). Then OCSAT\(\emptyset\)(B) is in NP.

Proof. We will reduce OCSAT\(\emptyset\)(B) to SAT, the satisﬁability problem for propositional formulae. Due to Lemma 4, we can assume that \(B = \{\sqcap, \neg\}\). Let \(((T, A), C)\) be an instance of OCSAT\(\emptyset\)(B). Since ALC\(\emptyset\)(B) does not have quantifiers, \(T\) only makes propositional statements about all individuals and cannot enforce more individuals than those in \(A\). Let \(D_j \subseteq E_j\), \(j = 1, \ldots, n\), be the axioms in \(T\) and \(a_1, \ldots, a_m\) the individuals occurring in \(A\). We introduce a fresh atomic proposition \(p^A_i\) for each \(i = 0, \ldots, m\) and each atomic concept \(A\) occurring in \(((T, A), C)\). Every \(p^A_i\) expresses that \(A\) has as instance either the individual \(a_i\) (if \(i \geq 1\)) or an an instance of \(C\) (if \(i = 0\)). Although \(C\) may have several instances, the absence of quantifiers allows us to identify them with a single individual.

For \(i = 0, \ldots, m\), we define a function \(f^i\) that maps from arbitrary concepts occurring in \(((T, A), C)\) to propositional formulae as follows:

\[
\begin{align*}
    f^i(A) &= p^A_i \quad \text{for atomic concepts } A, \\
    f^i(\top) &= 1, \quad f^i(\bot) = 0, \\
    f^i(\neg A) &= f^i(A), \\
    f^i(A_1 \sqcap A_2) &= f^i(A_1) \land f^i(A_2).
\end{align*}
\]

We express the instance \(((T, A), C)\) using the following propositional formulae:

\[
\begin{align*}
    \varphi_T &= \bigwedge_{i=0}^{m} \left( f^i(D_j) \rightarrow f^i(E_j) \right), \\
    \varphi_A &= \bigwedge_{i=1}^{m} \left( \bigwedge_{D(a_1) \in A} f^i(D) \right), \\
    \varphi_C &= f^0(C), \\
    \varphi_{T,A,C} &= \varphi_T \land \varphi_A \land \varphi_C.
\end{align*}
\]

We will now show that \(((T, A), C)\) \(\in\) OCSAT\(\emptyset\)(B) if and only if \(\varphi_{T,A,C} \in\) SAT.

For “\(\Rightarrow\)”, assume that \(((T, A), C)\) \(\in\) OCSAT\(\emptyset\)(B). Then there is an interpretation \(\mathcal{I}\) such that \(\mathcal{I} \models (T, A)\) and \(C^{\mathcal{I}} \neq \emptyset\). Fix individuals \(x_0, \ldots, x_m \in \Delta^{\mathcal{I}}\)
such that $x_0 \in C^T$ and $x_i = a_i^T$ for $i = 1, \ldots, m$. Now construct a propositional assignment $\beta$ such that $\beta(p_a^I) = 1$ if and only if $x_i \in A^I$. It is straightforward to show by induction on $X$ that for every, possibly complex, concept $X$ occurring in $((T, A), C)$ and each $i = 0, \ldots, m$, it holds that $\beta(f'(X)) = 1$ if and only if $x_i \in X^T$. Using this equivalence, we show that $\beta(\varphi_{T, A, C}) = 1$.

- $\beta(\varphi_T) = 1$ because, for every $i, j$, the axiom $D_j \subseteq E_j$ in $T$ ensures that $x_i \in D_j^T$ implies $x_i \in E_j^T$.
- $\beta(\varphi_A) = 1$ because every $D(a_i)$ in $A$ means that $x_i \in D^T$.
- $\beta(\varphi_C) = 1$ because $x_0 \in C^T$.

For “⇐”, assume that $\varphi_{T, A, C} \in \text{SAT}$. Then there is an assignment $\beta$ under which all three conjuncts $\varphi_T, \varphi_A, \varphi_C$ evaluate to 1. We construct an interpretation $I$ from $\beta$ as follows, $\Delta^T = \{x_0, \ldots, x_m\}$; for every $i = 0, \ldots, m$, every individual $a$ in $A$ and every atomic concept $A$ in $((T, A), C)$: $a_i^T = x_i$ and $x_i \in A^T$ if and only if $\beta(p_a^I) = 1$. As above, it is straightforward to show that $\beta(f'(X)) = 1$ if and only if $x_i \in X^T$, for every $X$ in $((T, A), C)$ and every $i = 0, \ldots, m$. Using this equivalence, we show that $I \models (T, A)$ and $C^T \neq \emptyset$.

- $I \models D_j \subseteq E_j$, $j = 1, \ldots, n$ because, for every $i = 0, \ldots, m$, the conjuncts in $\varphi_T$ ensure that $\beta(f'(D_j)) = 1$ implies that $\beta(f'(E_j)) = 1$, and therefore $x_i \in D_j^T$ implies $x_i \in E_j^T$.
- $I \models D(a_i), D(a_i) \in A$, because the conjuncts in $\varphi_A$ ensure that $x_i \in D^T$.
- $C^T \neq \emptyset$ because $\varphi_C$ ensures that $x_0 \in C^T$.

Lemma 29. Let $B$ be a set of Boolean functions s.t. $N = \{B\}$, then $\text{TCSAT}_\emptyset(B)$ is in NL.

Proof. Here we will provide a nondeterministic algorithm for $\text{TCSAT}_\emptyset(B)$ that runs in logarithmic space, which can be generalized to also work with $\text{TCSAT}_\emptyset(B)$ instances $(T, C)$ by adding an axiom $\top \subseteq C$ to the input terminology (in our case this maintains satisfiability because we can only talk about one individual). The algorithm consists of a search for cycles with contradictory atomic concepts in the (directed) implication graph $G_T$ which is induced by $T$.

W.l.o.g. assume $T$ to be normalized in a way that all blocks of leading negations ¬ in front of concepts are replaced by one negation if the number was odd, and completely removed otherwise. Thus $T$ consists only of axioms $C \subseteq D$, where $C, D$ are atomic concepts, constants, or its negations. The before mentioned implication graph $G_T = (V, E)$ is constructed from $T$ as follows:

$$
V := \{v_A, v_{\neg A} \mid A \text{ is an atomic concept in } T\} \cup \{v_T, v_{\bot}\},
$$

$$E := \{(v_C, v_D) \mid C \subseteq D \in T\} \cup \{(v_{\bot}, v_A), (v_A, v_T) \mid A \text{ is an atomic concept in } T\} \cup \{(v_{\bot}, v_T)\}.
$$

Now we claim that $T \in \text{TSAT}_\emptyset(B)$ iff $G_T$ does not contain a cycle that contains both nodes $v_A, v_{\neg A}$ for some $A \in N_C \cup \{\top, \bot\}$.
"⇒": Let \( \mathcal{I} \in \text{TSA}_0(B) \) witnessed by the interpretation \( \mathcal{I} = (\Delta^I, \cdot^I) \). W.l.o.g. assume \( \Delta^I = \{x\} \) by the same argumentation as in Lemma \cite{28}. Then it holds that \( \mathcal{I} \models \mathcal{T} \). Hence each axiom is satisfied, and with that there is no axiom \( C \subseteq D \) s.t. \( x \in C^I \) but \( x \notin D^I \). Now assume that we have a cyclic path \( \pi \) containing the nodes \( v_A \) and \( v_{\neg A} \). If \( x \in A^I \) then for all successor nodes \( v_{A_1}, v_{A_2}, \ldots \) of \( v_A \) on \( \pi \) it must hold that \( x \in A_i^I \) for \( i = 1, 2, \ldots \), which is a contradiction to \( \neg A \) for which \( v_{\neg A} \) is a successor of \( v_A \). If \( x \notin A^I \) then \( x \in (\neg A)^I \). Thus for all axioms \( A_1, A_2, \ldots \) with \( v_{A_1}, v_{A_2}, \ldots \) being successor nodes of \( v_{\neg A} \) it must hold that \( x \in (A_i)^I \). In particular this must hold for \( v_A \) which is a contradiction to \( x \notin A^I \).

"⇐": Assume that for each atomic concept \( A \) (including \( \top \) and \( \bot \)) there is no cyclic path containing \( v_A \) and \( v_{\neg A} \). In the following we will construct an interpretation \( \mathcal{I} = (\{x\}, I^I) \) that satisfies \( \mathcal{T} \). For each concept \( A \in \text{Con}((\top, \bot, \neg)) \) s.t. \( \top \subseteq A \), add \( x \) to \( A^I \). As we have \( (v_A, v_{\neg A}) \notin E^* \) (where \( E^* \) is the transitive closure of \( E \), and \( \neg A = \neg B \) if \( A = B \) and \( \neg A = B \) if \( A = \neg B \)) it must hold that also \( A \not\subseteq^* \neg A \) and thus \( \mathcal{I} \models \mathcal{T} \), as all remaining concepts are not enforced to be true. This completes the proof of the claim.

The NL-algorithm just checks for each concept \( A \) that there is no cycle from \( v_A \) containing \( v_{\neg A} \).

\[ \square \]

**Lemma 30.** Let \( B \) be a set of Boolean functions s.t. \( l_0 = [B] \), then \( \text{TCA}_0(B) \) is NL-hard.

**Proof.** This result directly follows from Lemma \cite{19} in combination with Lemma \cite{5}.

**Lemma 31.** Let \( B \) be a finite set of Boolean operators s.t. \( N = [B] \), then \( \text{OCSAT}_0(B) \) is in NL.

**Proof.** Let \( B \) be a set of Boolean operators s.t. \( N = [B] \). The algorithm first checks whether the given TBox is solely satisfiable. Afterwards we need to ensure the given ABox is consistent together with the TBox. Therefore observe for an ABox \( \mathcal{A} \) the following property holds: \( (\mathcal{A}, \mathcal{T}, C) \in \text{OCSAT}_0(B) \) iff \( (\mathcal{A} \cup \{R(a, b)\}, \mathcal{T}, C) \in \text{OCSAT}_0(B) \) for new individuals \( a, b \) and a role \( R \), as role assertions cannot affect the satisfiability of an instance if quantifiers are not allowed. The algorithm now tests consecutively for each individual \( a \in \mathcal{A} \) if \( (\mathcal{T}^a, C) \in \text{TCA}_0(B) \), where \( \mathcal{T}^a = \mathcal{T} \cup \{ \top \subseteq D \mid D(a) \in A \} \).

Now it holds that \( (\mathcal{A}, \mathcal{T}, C) \in \text{OCSAT}_0(B) \) iff \( (\mathcal{T}^a, C) \in \text{TCA}_0(B) \) for all individuals \( a \in \mathcal{A} \) and \( (\mathcal{T}, C) \in \text{TCA}_0(B) \). If \( \mathcal{I} = (\Delta^I, \cdot^I) \) is an interpretation with \( \mathcal{I} \models (\mathcal{T}, \mathcal{A}) \) and \( C^I \neq \emptyset \), then for the terminologies \( \mathcal{T}^a \) for each individual \( a \in \mathcal{A} \) it holds that \( \mathcal{I} |_{\mathcal{T}^a} \models \mathcal{T}^a \), where \( \mathcal{I} |_{\mathcal{T}^a} \) is the restriction of \( \mathcal{I} \) to the individual \( a \). For the opposite direction to be considered, we have interpretations \( \mathcal{T}^a = (\Delta^I, \cdot^I) \) s.t. \( \mathcal{I} \models \mathcal{T}^a \) and \( C^{I^a} = \emptyset \). W.l.o.g. assume \( \Delta^{I^a} = \{a\} \), then an easy inductive argument proves that \( \mathcal{I} \models (\mathcal{T}, \mathcal{A}) \) and \( C^I \neq \emptyset \) for \( \mathcal{I} = (\bigcup_{a \in \mathcal{A}} \Delta^{I^a}, \bigcup_{a \in \mathcal{A}} \mathcal{T}^a) \).

24
This connection between OCSAT∅(B) and TCSAT∅(B) is possible as we can assume different individuals to be distinct. As besides of that point we cannot speak about more than one individual for a given TBox which is restricted to a single individual a, and therefore we may assume the concept D to hold (and consider also the axiom ⊤ ⊑ D) if D(a) ∈ A for T a.

Theorem 32. Let B be a finite set of Boolean operators, and Q ∈ {∀, ∃}.

1. If S11 ⊆ [B], N2 ⊆ [B], or L0 ⊆ [B] then *SAT Q(B) is EXPTIME-complete.
2. If L0 ⊆ [B] ⊆ V, then TCSAT∃(B) and *SAT Q(B) are P-complete.
3. If [B] ∈ {E0, E}, then *SAT Q(B) is EXPTIME-complete, and *SAT Q(B) is P-complete.
4. If [B] ⊆ R1, then *SAT Q(B) is trivial.

Proof. For (1) combine the EXPTIME-completeness of TSAT Q(M) shown in Lemma 22 with the usual ⊤-knack known from Lemma 5.

The lower bound for N2 is due to Lemma 24 to state a reduction from TSAT Q(L) with Lemma 5 to TCSAT Q(L0) for Q ∈ {∃, ∀}.

The EXPTIME-completeness in case (3) follows from Lemma 33. For the P-complete cases in (2) and (3) the results are organized as follows:

- the P-hardness of these cases results from TSAT Q(⊤, ⊥) in Lemma 26 in combination with Lemma 5.
- the membership in P of TCSAT ∃(⊔, ⊤, ⊥) follows by OCSAT ∃(⊔, ⊤, ⊥) in Lemma 35.
- the membership in P of TCSAT ∃(⊔, ⊤, ⊥) follows by TSAT ∃(⊔, ⊤, ⊥) in combination with Lemma 3.
- the membership in P of TCSAT ∃(⊓, ⊤, ⊥) follows by OCSAT ∃(⊓, ⊤, ⊥) in Lemma 34.

(4) is due to Lemma 9. □

Theorem 32 shows one reason why the logics in the EL family have been much more successful as “small” logics with efficient reasoning methods than the FL family: the combination of the ∀ with conjunction is intractable, while ∃ and conjunction are still in polynomial time. Again, separating either conjunction and disjunction, or the constants is crucial for tractability.

Lemma 33. Let B be a finite set of Boolean operators s.t. E0 = [B], then TCSAT ∃(B) is EXPTIME-hard.

Proof. As a result from [5, 24] the subsumption problem w.r.t. a TBox for the logic FL0 (the description logic with ∃ and ⊓ as allowed operators) is EXPTIME-complete. For this lemma we will reduce from this problem in FL0. Observe that

3 OSAT ∃(B) and OCSAT ∃(B) are P-hard for [B] ∈ {V0, V} and in EXPTIME.
the following holds

\[(T, C, D) \in \mathcal{FL}_0\text{-SUBS} \iff \forall \mathcal{I} : \mathcal{I} \models T \text{ it holds } \mathcal{I} \models C \subseteq D \]
\[\iff \lnot (\exists \mathcal{I} : \mathcal{I} \models T \text{ and } (C \cap \neg D)^2 \neq \emptyset) \]
\[\iff \lnot (\exists \mathcal{I} : \mathcal{I} \models \mathcal{T} \cup \{D \cap D' \subseteq \bot\} \text{ and } (C \cap D')^2 \neq \emptyset) \]
\[\iff (\mathcal{T} \cup \{D \cap D' \subseteq \bot\}, C, \bot) \not\in \text{TCSAT}_{\mathcal{B}}(B) \]

\[\square\]

**Lemma 34.** Let \( B \) be a finite set of Boolean operators s.t. \( E = [B] \), then \( \text{OCSAT}_{\exists}(B) \) is in \( P \).

**Proof.** To provide an algorithm running in polynomial time, we will reduce the given problem to the complement of the subsumption problem for the logic \( \mathcal{EL}^{++} \), which is known to be P-complete by \cite{7}.

The reduction works as follows:

\[((\mathcal{T}, \mathcal{A}), C) \in \text{OCSAT}_{\exists}(B) \iff \exists \mathcal{I} : \mathcal{I} \models \mathcal{T} \text{ and } C^\mathcal{A} \neq \emptyset \text{ and } C^\mathcal{T} \neq \emptyset \]
\[\iff \exists \mathcal{I} : \mathcal{I} \models \mathcal{T} \cup \{\top \subseteq \exists R.C_A\} \text{ and } C^\mathcal{T} \neq \emptyset \]
\[\iff \mathcal{T} \cup \{\top \subseteq \exists R.C_A\} \neq C \subseteq \bot \]
\[\iff (\mathcal{T} \cup \{\top \subseteq \exists R.C_A\}, C, \bot) \not\in \mathcal{EL}^{++}\text{-SUBS}, \]

where \( \mathcal{T} \) is a TBox, \( \mathcal{A} \) is an ABox, \( R \) is a fresh role, and

\[C_A := \bigcap_{C(a) \in \mathcal{A}} \exists u.\{\{a\} \cap C\} \cap \bigcap_{r(a,b) \in \mathcal{A}} \exists u.\{\{a\} \cap \exists r.\{b\}\} \]

is the concept constructed as in \cite{6} from the ABox \( \mathcal{A} \), where \( u \) is a fresh role name, and \{\{a\}\} and \{\{b\}\} denote nominals corresponding to the ABox individuals \( a \) and \( b \).

\[\square\]

**Lemma 35.** Let \( B \) be a finite set of Boolean operators s.t. \( V = [B] \), then \( \text{OCSAT}_{\forall}(B) \) is in \( P \).

**Proof.** Here we use the result from Lemma \cite{34} and reduce to the dual problem \( \text{OCSAT}_{\exists}(E) \). Consider an ontology \((\mathcal{T}, \mathcal{A})\) where \( \mathcal{T} \) is a TBox and \( \mathcal{A} \) an ABox, and a concept \( C \) as the given instance of \( \text{OCSAT}_{\forall}(B) \). W.l.o.g. assume \( C \) to be atomic. Now first construct the new terminology \( \mathcal{T}' \) similarly to Lemma \cite{18}. Then add for each \( A \in \mathcal{N}_C \) and hence each \( A' \) the GCIs \( A \cap A' \subseteq \bot \) to ensure they are disjoint. Denote this change by the terminology \( \mathcal{T}'' \). Then it holds

\[((\mathcal{T}, \mathcal{A}), C) \in \text{OCSAT}_{\forall}(B) \iff ((\mathcal{T}'', \mathcal{A}), C') \in \text{OCSAT}_{\exists}(E). \]

\[\square\]

Table \cite{3} gives an overview of our results. Section \cite{4.1} shows how the results arrange in Post’s lattice.
| $\text{TSAT}_{Q}(B)$ | $1$ | $V$ | $E$ | $N/N_2$ | $M$ | $L_3$ to $BF$ | otherwise |
|---------------------|-----|-----|-----|---------|-----|---------------|-----------|
| $Q = \emptyset$     | $\text{NL}$ | $P$ | $\text{NL}$ | $\text{NP}$ | trivial |
| $|Q|=1$              | $P$ | trivial |
| $Q = \{\exists, \forall\}$ | $\text{EXPTIME}$ | trivial |

| $\ast\text{SAT}_{Q}(B)$ | $I/I_0$ | $V/V_0$ | $E/E_0$ | $N/N_2$ | $S_{11}$ to $M/L_3/L_0$ | $\text{to BF}$ | otherwise |
|--------------------------|--------|--------|--------|---------|----------------|----------------|-----------|
| $Q = \emptyset$         | $\text{NL}$ | $P$ | $\text{NL}$ | $\text{NP}$ | trivial |
| $Q = \{\exists\}$       | $P$ | $P^t$ | $\text{EXPTIME}$ | trivial |
| $Q = \{\forall\}$       | $P$ | $\text{EXPTIME}$ | trivial |
| $Q = \{\exists, \forall\}$ | $\text{EXPTIME}$ | trivial |

Table 1. Complexity overview for all Boolean function and quantifier fragments. All results are completeness results for the given complexity class, except for the case marked $\ast$: here, OCSAT and OSAT are in EXPTIME and P-hard.

4 Conclusion

With Theorems 8, 14, 21, 27 and 32, we have completely classified the satisfiability problems connected to arbitrary terminologies and concepts for $\mathcal{ALC}$ fragments obtained by arbitrary sets of Boolean operators and quantifiers—only the fragments emerging around ontologies with existential quantifier and disjunction as only allowed connective resisted a full classification. In particular we improved and finished the study of [29]. In more detail we achieved a dichotomy for all problems using both quantifiers (EXPTIME-complete vs. trivial fragments), a trichotomy when only one quantifier is allowed (trivial, EXPTIME-, and P-complete fragments), and a quartering for no allowed quantifiers ranging from trivial, NL-complete, P-complete, and NP-complete fragments.

Furthermore the connection to well-known logic fragments of $\mathcal{ALC}$, e.g., $\mathcal{FL}$ and $\mathcal{EL}$ now enriches the landscape of complexity by a generalization of these results. These improve the overall understanding of where the tractability border lies. The most important lesson learnt is that the separation of quantifiers together with the separation of either conjunction and disjunction, or the constants, is the only way to achieve tractability in our setting.

Especially in contrast to similar analyses of logics using Post’s lattice, this study shows intractable fragments quite at the bottom of the lattice. This illustrates how expressive the concept of terminologies and assertional boxes is: restricted to only the Boolean function $\text{false}$ besides both quantifiers we are still able to encode EXPTIME-hard problems into the decision problems that have a TBox and a concept as input. Thus perhaps the strongest source of intractability can be found in the fact that unrestricted theories already express limited implication and disjunction, and not in the set of allowed Boolean functions alone.

For future work, it would be interesting to see whether the picture changes if the use of general axioms is restricted, for example to cyclic terminologies—theories where axioms are cycle-free definitions $A \equiv C$ with $A$ being atomic.
Theories so restricted are sufficient for establishing taxonomies. Concept satisfiability for $\mathcal{ALC}$ w.r.t acyclic terminologies is still PSPACE-complete \cite{27}. Is the tractability border the same under this restriction? One could also look at fragments with unqualified quantifiers, e.g., $\mathcal{ALU}$ or the DL-lite family, which are not covered by the current analysis. Furthermore, since the standard reasoning tasks are not always interreducible under restricted Boolean operators, a similar classification for other decision problems such as concept subsumption is pending.

4.1 Overview of the Results

Regarding the number of possible fragments of the investigated decision problems by restricting the use of quantifiers and Boolean functions one would formally deduce the number of emerging fragments is infinite (as there are infinitely many different Boolean functions). Fortunately Post’s lattice hides this infinity at two parts in the lattice, namely, the $c$-separating functions of degree $n$ and the clones around them. This is visualized by dashed lines in the lattice. To overcome this problem one tries to achieve the same upper and lower bounds for the clones above and below these infinite chains. Thus there are still all visualized nodes in the lattice remaining to get classified. Each of these clones induces a new decision problem parameterized by itself. Thus we have to deal with 54 relevant clones which means, all in all, $4 \cdot 54$ parameterized versions for all four decision problems.

Therefore the next table will help to clarify the overall picture in the following way. Each row deals with the quantifier fragments whereas each column corresponds to one clone in the lattice. Here, we mostly used only the clones which are needed to state best upper and lower bounds. A cell in this table shows the complexity of this fragment (by name and color), wherefrom the lower and wherefrom the upper bound is applied or in which lemma the corresponding proof can be found. The "Lewis Knack" is proven in Lemma \ref{5}. 

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|}
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\end{tabular}
\end{table}
| Proof System | LB: Lemma | UP: Lemma | LK: Lemma | Complexity | LB: Lemma | UP: Lemma | LK: Lemma | Complexity | LB: Lemma | UP: Lemma | LK: Lemma | Complexity |
|--------------|-----------|-----------|-----------|------------|-----------|-----------|-----------|------------|-----------|-----------|-----------|------------|
| TSAT \_\_ \_ | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) |
| TSAT \_\_ \_ | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) |
| TSAT \_\_ \_ | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) |
| TSAT \_\_ \_ | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) |
| TSAT \_\_ \_ | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) |
| TSAT \_\_ \_ | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) |
| TSAT \_\_ \_ | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) |
| TSAT \_\_ \_ | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) |
| TSAT \_\_ \_ | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) |
| TSAT \_\_ \_ | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) | trivial; TSAT \_\_ (R) |

Table 2. Complexity Overview, LB: Lower Bound, UP: Upper Bound, LK: Lewis Knack, con: contraposition
Fig. 3. Complexity for TSAT(B), TCSAT(B), OSAT(B) and OCSAT(B).
Fig. 4. Complexity for TSAT_θ(B).
Fig. 5. Complexity for TSAT_3(B) and TSAT_\(\forall(B)\).
Fig. 6. Complexity for $\star\text{SAT}_A^\sim(B)$.
Fig. 7. Complexity for $\ast \text{SAT}_\gamma(B)$. 
Fig. 8. Complexity for $\star\text{SAT}_3(B)$. 

for TCSAT($B$) P-complete, 
else in EXPTIME and P-hard
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