Color spin wave functions of heavy tetraquark states

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Using the variational method, we calculate the mass of the $J^P = 1^+$ $ud\bar{b}\bar{b}$ tetraquark containing two identical heavy antiquarks in a nonrelativistic potential model with color confinement and spin hyperfine interaction. In particular, we extend a previous investigation of the model by Brink and Stancu by investigating the effect of including the color anti-sextet component of the diquark configuration as well as using several more Gaussian parametrizations for the $L=0$ part of the spatial wave function. We find that for the heavy tetraquark, the 66 component among the color singlet bases is negligible and that the previously used specific Gaussian spatial configuration is good enough in obtaining the ground state energy.

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I. INTRODUCTION

Recently, several new heavy mesons were discovered with masses difficult to explain within the conventional quark model and thus could either be a multiquark or a molecular configuration.\textsuperscript{[1]} These are the $D_s$ states, $X(3872)\ [2]$, $Z(4051)$ and $Z(4248)\ [3]$, and the newly discovered charged charmonium like states $Z_c(3900)\ [4,5]$. While the recently observed charged states are mostly likely of exotic configurations, their quantum numbers are not explicitly exotic. On the other hand, there are a number of works suggesting that certain flavor exotic multiquark states with heavy quarks could be stable under strong decay and be observable from B-decay or heavy ion collisions. If such particles are indeed found, they would mark the first observation of flavor exotic multiquark configuration, which will lead to a new dimension of hadron spectroscopy\textsuperscript{[6,7].}

The first set of papers suggesting the tetraquark configurations were given by Jaffe\textsuperscript{[8,9]} within the MIT bag model with color spin interaction. This paper subsequently promoted an intense discussion on the possible existence of tetraquark states. It was suggested in his papers that the $f_0(975)$ and $a_0(980)$ resonances could be interpreted as part of the scalar $J^{PC} = 0^{++}$ nonet composed of $qq\bar{q}\bar{q}$ tetraquarks. This picture was later further confirmed by Weinstein and Isgur\textsuperscript{[10,11]}, establishing the possible existence of tetraquark in a variety of quark models. This means that tetraquarks with heavy quarks can also exist. In fact, the calculation for the spectrum of $cc\bar{q}\bar{q}$ tetraquark which was performed by Stancu\textsuperscript{[12]} and Hogassen\textsuperscript{[13]} suggest that $X(3872)$ meson which have been discovered by Belle\textsuperscript{[2]} could be a $cc\bar{q}\bar{q}$ tetraquark state. This state however is of the cryptoexotic nature, with hidden heavy flavor quantum number. Moreover, these states could be a meson-meson bound molecular states as was predicted more than twenty years ago\textsuperscript{[14]}. Thus it is experimentally a challenge to prove that they are composed of purely tetraquark components.

Simple estimates based on color-spin interaction suggests that there could be stable heavy tetraquark states with explicitly flavor exotic quantum number\textsuperscript{[7,8]} In particular, the $J^P = 1^+, I = 0$ $udQ\bar{Q}$, with $Q$ being a heavy quark and called the $T_{QQ}^1$, are of particular interest as it could be a stable flavor exotic tetraquark\textsuperscript{[10]} state that could be produced in electro-positron collision\textsuperscript{[17]} or in a heavy ion collision\textsuperscript{[18]}. The stability of $T_{QQ}^1$ has been studied in quark model\textsuperscript{[8,14,21]} and QCD sum rules\textsuperscript{[22]}. Here, we are interested in elaborating the quark model calculation for $T_{QQ}^1$, obtained with the nonrelativistic potential as given by Silvestre-Brac and Semay\textsuperscript{[23,24]}, that was performed by Brink and Stancu (BS)\textsuperscript{[25]} using the variational method based on simple Gaussian trial function which is useful to describe nuclear few-body systems\textsuperscript{[26]}. The mass of $T_{QQ}^1$ calculated by BS was 33 MeV above the results by Silvestre-Brac and Semay\textsuperscript{[23,24]} that used a variational calculation with many oscillator bases. BS proposed several alternatives of improving the variational energy in their calculation. In this work, we extend the work of BS by investigating their proposal of improvements. In the first improvement, since Brink and Stancu\textsuperscript{[25]} excluded the 66 component in color singlet basis following the assumption given in \textsuperscript{[10]}, we explicitly investigate the validity of the assumption by including the 66 component in the calculation. This calculation will be performed with the same single Gaussian spatial wave function as was done by BS that will be called scheme 0 in our work. In the second improvement, we extend the simple spatial configuration used by BS to the generalized cases introduced as scheme I to V in section IV. This is to investigate the extended correlations between quarks. We further introduce schemes (scheme VI and VII) to investigate the importance of using multiple Gaussian to the wave function. The simple Gaussian function for total angular momentum $L=0$ is convenient to examine the variational energy of the tetraquark containing two identical heavy antiquarks in such a situation. We found that the size of tetraquark is important to understand the stability of heavy tetraquark. We also calculate the quark-antiquark meson masses within the same model parameters. Using these results, we inves-
tigate the stability of the tetraquark states against the decay into two meson states.

In section II we introduce the Hamiltonian. In section III, we introduce the spatial and color-spin wave function. In section IV, we introduce the different schemes and calculate the matrix elements. In section V, we show the numerical results and discuss the two improvements. In section VI, we analyze the mass splitting coming from hyperfine potential. Finally, we give the summary in section VII.

II. HAMILTONIAN

Let us start from a nonrelativistic Hamiltonian, that includes confinement and hyperfine potential for the color and spin degrees of freedom:

\[
H = \sum_{i=1}^{4} (m_i + \frac{p_i^2}{2m_i}) - \frac{3}{4} \sum_{i<j} \lambda_i \lambda_j (V^C_{ij} + V^{SS}_{ij}).
\] (1)

Here, \(m_i\)'s are the quark masses; \(\lambda_i/2\) are the color operator of the \(i\)'th quark for the color SU(3); \(V^C_{ij}\) and \(V^{SS}_{ij}\) are the confinement and hyperfine potential, respectively. We adopt the confinement and hyperfine potential from ref. [21]:

\[
V^C_{ij} = -\kappa \frac{r_{ij}}{r_{ij}^3} + \frac{r_{ij}}{a_0^3} - D,
\] (2)

\[
V^{SS}_{ij} = \frac{\hbar^2\kappa}{m_i m_j c^4} \frac{1}{r_{ij}^3} e^{-r_{ij}/r_0} \sigma_i \cdot \sigma_j,
\] (3)

where \(r_{ij} = |r_i - r_j|\) and \(\sigma_i\) is the spin operator.

Since our aim is to generalize and compare with the calculation of BS [23], we chose the same values as those used in that paper. The parameters are given by

\[
m_u = m_d = 337 \text{ MeV},
\]

\[
m_c = 1870 \text{ MeV}, \quad m_b = 5259 \text{ MeV},
\]

\[
\kappa = 102.67 \text{ MeV fm}, \quad a_0 = 0.0326 (\text{ MeV}^{-1} \text{ fm})^{1/2},
\]

\[
D = 913.5 \text{ MeV}, \quad r_0 = 0.4545 \text{ fm}.
\] (4)

III. WAVE FUNCTION

In this work, we will be interested in the \(T_{QQ}^1\) state within the Hamiltonian introduced above. In the constituent quark model, the lowest mass for the \(T_{QQ}^1\) state is obtained in a configuration where all the quarks are in the \(l = 0\) state. Therefore, the Hamiltonian introduced in the previous section will be applied to only the s-wave configurations that depend also on the color and spin states. Now, we establish the appropriate basis functions for describing the tetraquark system.

A. spatial function

In order to use variational method, we construct the trial wave function for the spatial part in a simple Gaussian form. This spatial function makes it easy to calculate the matrix element of the Hamiltonian. When we calculate the matrix element of the potential terms for the tetraquark configuration with certain symmetry, it is convenient to introduce the following three coordinate configurations which are related with each other by orthogonal matrix.

- Coordinate I:
  \[
  \rho = \frac{1}{\sqrt{2}}(r_1 - r_3), \quad \rho' = \frac{1}{\sqrt{2}}(r_2 - r_4),
  \]
  \[
  x = \frac{1}{2}(r_1 - r_2 + r_3 - r_4).
  \] (5)

- Coordinate II:
  \[
  \alpha = \frac{1}{\sqrt{2}}(r_1 - r_4), \quad \alpha' = \frac{1}{\sqrt{2}}(r_2 - r_3),
  \]
  \[
  y = \frac{1}{2}(r_1 - r_2 - r_3 + r_4).
  \] (6)

- Coordinate III:
  \[
  \sigma = \frac{1}{\sqrt{2}}(r_1 - r_2), \quad \sigma' = \frac{1}{\sqrt{2}}(r_3 - r_4),
  \]
  \[
  \lambda = \frac{1}{2}(r_1 + r_2 - r_3 - r_4).
  \] (7)

Here, particles 1 and 2 indicate quarks, while 3 and 4 indicate antiquarks.

When describing the diquark-antidiquark system, it is convenient to choose the closed form coordinate III. Hence, for calculating the matrix element of the Hamiltonian, we use coordinate III. On the other hand, it is convenient to choose coordinates I or II in describing the asymptotic form corresponding to either the direct or exchange meson-meson system. As we deal with the tetraquark consisting of two identical antidiquark, we must consider the permutation of (12) and (34) with respect to the basis function. In other words, we must construct the bases functions satisfying the Pauli principle. For these three coordinate configurations under the permutation of (12) and (34), we obtain the following property:

\[
(12)\rho = \alpha', \quad (12)\rho' = \alpha, \quad (12)\sigma = -\sigma, \quad (12)\lambda = \lambda,
\]

\[
(34)\rho = \alpha, \quad (34)\rho' = \alpha', \quad (34)\sigma = -\sigma', \quad (34)\lambda = \lambda.
\] (8)
We denote the spatial function by \( R^s \) which has been introduced by BS in Ref [25]. As was discussed by BS, the most general Gaussian form for the \( L=0 \) spatial function can be written in terms of six scalar quantities as given by

\[
R^s = \exp[-(C_{11}^s \sigma^2 + C_{22}^s \sigma'^2 + C_{33}^s \lambda^2 + 2C_{12}^s \sigma \cdot \sigma' + 2C_{13}^s \sigma \cdot \lambda + 2C_{23}^s \sigma' \cdot \lambda)].
\]

(9)

In order to calculate the matrix element of the confinement and hyperfine potential terms involving \( r_{ij} \), where \( i,j = 1 \sim 4 \), it is convenient to represent the argument of the exponential function in a matrix form so that one can easily transform from one coordinate to the other by orthogonal transformations. Therefore, we define the coordinate configurations in a matrix form as follows:

\[
X = \begin{pmatrix} \rho \\ \rho' \\ x \end{pmatrix}, \quad Y = \begin{pmatrix} \alpha \\ \alpha' \\ y \end{pmatrix}, \quad Z = \begin{pmatrix} \sigma \\ \sigma' \\ \lambda \end{pmatrix}.
\]

(10)

Then, we can write \( R^s \) of Eq. (9) in the following form

\[
R^s = \exp(-Z^T C^s Z),
\]

where \( C^s \) is the symmetric matrix, and \( Z^T \) is the transpose of the column matrix \( Z \). Using the orthogonal matrices which transform one coordinate into the other, the \( R^s \) can be expressed in terms of the coordinates (5) and (6). It becomes

\[
R^s = \exp(-Z^T C^s Z) = \exp(-X^T A^s X),
\]

(12)

where the symmetric matrices \( A^s \) and \( B^s \) are obtained from the similarity transformation. Applying the orthogonal matrices to the coordinates and \( C^s \) matrix give

\[
X = U_y Z, \quad Y = U_y Z, \quad A^s = U_x C^s U_x^{-1}, \quad B^s = U_y C^s U_y^{-1},
\]

(13)

where the orthogonal matrices \( U_x \) and \( U_y \) are

\[
U_x = \begin{pmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 1 \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

\[
U_y = \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

(14)

Introducing the position vector of the center of mass, \( r_C = (1/M) \sum m_i r_i \), where \( M = \sum m_i \), the kinetic part of Eq. (1) can be expressed in the center of mass frame by excluding the kinetic energy of the position vector of the center of mass. The kinetic part in the center of mass frame denoted by \( T_c \) can be expressed in terms of coordinate III as follows:

\[
T_c = \sum_{i=1}^{4} \frac{P_i^2}{2m_i} - \frac{P_x^2}{2M} = \frac{P_1^2}{2m_1} + \frac{P_2^2}{2m_3} + \frac{P_3^2}{2m_4},
\]

(15)

where \( m_1 = m_2 = m_q, m_3 = m_4 = m_Q \), and \( m' \) is the reduced mass, \( 2m_1m_3/(m_1 + m_3) \).

### B. Spin-color state

The color space acting on the \( \lambda^c \lambda^c \) in a given flavor configuration of the tetraquark can be decomposed according to the irreducible representation of color \( SU(3)_c \) as

\[
3_c \times 3_c \times 3_c \times 3_c = 3_c \times 3_c + 6_c \times 6_c + \bar{3}_c \times \bar{3}_c + 6_c \times 3_c.
\]

(16)

Color singlet states can be obtained from the first and the second term in the right hand side of Eq.(16). It is convenient to use following notions introduced in Ref. [28] to denote the two color singlets.

\[
(q_1 q_2)^3 \otimes (\bar{q}_3 \bar{q}_4)^3, \quad (q_1 q_2)^6 \otimes (\bar{q}_3 \bar{q}_4)^6
\]

(17)

It follows from the property of irreducible representation of color \( SU(3)_c \) that \( (q_1 q_2)^3 \otimes (\bar{q}_3 \bar{q}_4)^3 \) is antisymmetric under transposition of \( q_1 \) and \( q_2 \) or \( q_3 \) and \( q_4 \), and \( (q_1 q_2)^6 \otimes (\bar{q}_3 \bar{q}_4)^6 \) is symmetric under transposition of \( q_1 \) and \( q_2 \) or \( q_3 \) and \( q_4 \). Using the tensor notation [29], the two color singlets can be written as

\[
(q_1 q_2)^3 \otimes (\bar{q}_3 \bar{q}_4)^3 = \frac{1}{\sqrt{3}} e^{\alpha \beta \gamma} \epsilon_{\alpha \lambda \sigma} q_{\beta}(1) q_{\gamma}(2) \bar{q}^\lambda(3) \bar{q}^\sigma(4),
\]

\[
(q_1 q_2)^6 \otimes (\bar{q}_3 \bar{q}_4)^6 = \frac{1}{\sqrt{6}} d^{\alpha \beta \gamma} d_{\lambda \sigma \lambda} q_{\beta}(1) q_{\gamma}(2) \bar{q}^\lambda(3) \bar{q}^\sigma(4),
\]

(18)

where \( d^{\alpha \beta \gamma} \) and \( d_{\alpha \beta \gamma} \) are

\[
d^{111} = d^{222} = d^{333} = d^{533} = 1, \quad d^{412} = d^{421} = \bar{d}^{421} = \bar{d}^{214} = \bar{d}^{532} = d^{532} = d^{613} = d^{631} = \frac{1}{\sqrt{2}}.
\]

(19)

These two color singlet states are orthonormal by means of the irreducible representation of color \( SU(3)_c \). The orthogonality can also be simply shown by the vanishing of the multiplication of the anti-symmetric to symmetric color indices. The coefficients can be deduced from Young operators associated with sextet and antisextet which are useful to generate the basis state in a Young diagram. The two color singlet states can be recombined into another two color singlets constructed from two quark antiquark pair of color singlet-singlet and an octet-octet states that are appropriate for studying the decay properties.
Due to the fact that the irreducible representation of SU(2)$_s$ for an antiquark with spin=$1/2$ is equivalent to that of a quark, the spin space of the tetraquark can be represented as $V_{1/2} \times V_{1/2} \times V_{1/2} \times V_{1/2}$, and decomposed into the direct sum of the following parts:

$$V_{1/2} \times V_{1/2} \times V_{1/2} \times V_{1/2} = V_0 \times V_0 + V_0 \times V_1 + V_1 \times V_0 + V_1 \times V_1,$$

(20)

where the subscripts indicate the spins. Accordingly, the total spin of the tetraquark can be $S=0, 1$ or 2.

For $S=0$, there are two independent basis states obtained from $V_0 \times V_0$ and $V_1 \times V_1$ parts. The corresponding bases are denoted by

$$(\chi_{12})_{s=0} \otimes (\chi_{34})_{s=0}, \quad (\chi_{12})_{s=1} \otimes (\chi_{34})_{s=1},$$

(21)

where particles 1 and 2 imply quarks, and particles 3 and 4 antiquarks.

For $S=1$, there are three independent basis states coming from $V_0 \times V_1$, $V_1 \times V_0$, and $V_1 \times V_1$ part. These states are given by

$$(\chi_{12})_{s=0} \otimes (\chi_{34})_{s=1}, (\chi_{12})_{s=1} \otimes (\chi_{34})_{s=0}, \quad (\chi_{12})_{s=1} \otimes (\chi_{34})_{s=1}.$$  

(22)

For $S=2$, there exist only one state coming from $V_1 \times V_1$ part denoted as

$$(\chi_{12})_{s=1} \otimes (\chi_{34})_{s=1}.$$

(23)

The spin states for $S=0$ and $S=1$ are orthonormal, as in the color states. It is important to see the permutation property of the spin states under transposition (12) or (34) because the wave function has to have a definite symmetry under exchange of identical particles; (34) are identical while (12) becomes identical when extended to the flavor space. Applying the transposition (12) or (34) to the spin states give

$$(12)(\chi_{12})_{s=0} = - (\chi_{12})_{s=0}, \quad (12)(\chi_{12})_{s=1} = (\chi_{12})_{s=1},$$

(24)

$$(34)(\chi_{12})_{s=0} = - (\chi_{12})_{s=0}, \quad (34)(\chi_{12})_{s=1} = (\chi_{12})_{s=1}.$$  

(25)

In general, when the symmetry constraint is not imposed, there is a four-dimensional color-spin orthogonal basis for $S=0$ spanned by the following states:

$$\phi_1 = (q_1 q_2)^6 \otimes (\bar{q}_3 \bar{q}_4)^6 (\chi_{12})_{s=1} \otimes (\chi_{34})_{s=1}$$

$$\equiv (q_1 q_2)^6_1 \otimes (\bar{q}_3 \bar{q}_4)^6_1,$$

$$\phi_2 = (q_1 q_2)^3 \otimes (\bar{q}_3 \bar{q}_4)^3 (\chi_{12})_{s=0} \otimes (\chi_{34})_{s=0}$$

$$\equiv (q_1 q_2)^3_0 \otimes (\bar{q}_3 \bar{q}_4)^3_0,$$

$$\phi_3 = (q_1 q_2)^6 \otimes (\bar{q}_3 \bar{q}_4)^6 (\chi_{12})_{s=0} \otimes (\chi_{34})_{s=0}$$

$$\equiv (q_1 q_2)^6_0 \otimes (\bar{q}_3 \bar{q}_4)^6_0,$$

(26)

$$\phi_4 = (q_1 q_2)^\frac{3}{2} \otimes (\bar{q}_3 \bar{q}_4)^\frac{3}{2} (\chi_{12})_{s=1} \otimes (\chi_{34})_{s=1}$$

$$\equiv (q_1 q_2)^\frac{3}{2}_1 \otimes (\bar{q}_3 \bar{q}_4)^\frac{3}{2}_1.$$  

(27)

Similarly, we use the following six-dimensional color-spin basis for $S=1$ state:

$$\psi_1 = (q_1 q_2)^6 \otimes (\bar{q}_3 \bar{q}_4)^6,$$

$$\psi_2 = (q_1 q_2)^3 \otimes (\bar{q}_3 \bar{q}_4)^3,$$

$$\psi_3 = (q_1 q_2)^3 \otimes (\bar{q}_3 \bar{q}_4)^3,$$

$$\psi_4 = (q_1 q_2)^6 \otimes (\bar{q}_3 \bar{q}_4)^6,$$

$$\psi_5 = (q_1 q_2)^\frac{3}{2} \otimes (\bar{q}_3 \bar{q}_4)^\frac{3}{2},$$

$$\psi_6 = (q_1 q_2)^\frac{3}{2} \otimes (\bar{q}_3 \bar{q}_4)^\frac{3}{2}. \quad (26)$$

Depending on the tetraquark state, the actual states contributing to the bases will be smaller due to symmetry considerations. Our main interest is in the tetraquark $T^{QQ}_{\sigma}$ containing two identical heavy antiquarks and two light quarks $u$ and $d$ ($S=1,J=0$). $qq\bar{b}\bar{b}$ states with $J^P = 0^+$ with $(S=0,I=1)$ or with $J^P = 1^+$ with $(S=1,I=1)$ was found to be unstable against strong decay by BS [23]. In the work by BS, the stability of the $T^{QQ}$ was obtained from considering only the $(q_1 q_2)^\frac{3}{2} \otimes (\bar{q}_3 \bar{q}_4)^\frac{3}{2}$ component in the color wave function, without the color $(q_1 q_2)^6 \otimes (\bar{q}_3 \bar{q}_4)^6$ component. Thus, we are committed to examining the effect of the color $(q_1 q_2)^6 \otimes (\bar{q}_3 \bar{q}_4)^6$ component in Eq. (26) to the mass of the $T^{QQ}$.

Also, our work will allow for possible couplings between the coordinates $\sigma$, $\lambda$ and $\sigma'$ through the nonvanishing variational parameters $C_{12}^s$, $C_{13}^s$, and $C_{23}^s$ appearing respectively in Eq. (26) to Eq. (28). We will then compare our result to that of BS [23] using the Gaussian function in the absence of $C_{12}^s$, $C_{13}^s$ and $C_{23}^s$.

IV. CALCULATIONAL SCHEMES

The total wave function must be antisymmetric under the transposition of (12) and (34) for $T^{QQ}$ because of the Pauli principle. Since we are interested in the lowest orbital states with all quarks in the $l = 0$ states, the spatial wave function should be symmetric. Hence, the permutation property which should be satisfied by the color and spin part of the wave function is symmetric under the transposition of (12) and antisymmetric under the transposition of (34) because the flavor part of the wave functions is antisymmetric and symmetric for the light and heavy quarks respectively. The above permutation properties only allow two states, $\psi_1$ and $\psi_4$ in Eq. (26).

$$(12)(q_1 q_2)^3 \otimes (\bar{q}_3 \bar{q}_4)^3 = (q_1 q_2)^3 \otimes (\bar{q}_3 \bar{q}_4)^3,$$

$$\equiv (q_1 q_2)^3_0 \otimes (\bar{q}_3 \bar{q}_4)^3_0,$$

$$\equiv (q_1 q_2)^3_1 \otimes (\bar{q}_3 \bar{q}_4)^3_1,$$

$$\equiv (q_1 q_2)^\frac{3}{2}_0 \otimes (\bar{q}_3 \bar{q}_4)^\frac{3}{2}_0,$$

$$\equiv (q_1 q_2)^\frac{3}{2}_1 \otimes (\bar{q}_3 \bar{q}_4)^\frac{3}{2}_1.$$  

(27)

The spatial function should therefore be symmetric under the transpositions. We introduce different schemes depending on how this property is implemented.
A. Scheme 0

The simplest way to implement the symmetry in the spatial wave function is to take \( C_{12}^s = C_{13}^s = C_{23}^s = 0 \) in the exponent of the Gaussian. Considering only the variational parameters \( C_{11}^s, C_{22}^s, \) and \( C_{33}^s \) to be non-zero, the basis wave functions for \( T_{QQ} \) can be written as the following:

\[
\Psi_1^s = R^s(q_{1q_2})^3 \otimes (q_{3q_4})^3, \quad \Psi_2^s = R^s(q_{1q_2})^6 \otimes (q_{3q_4})^6. \tag{28}
\]

The spatial part of the basis wave functions in Eq. (28) is given by excluding \( C_{12}^s, C_{13}^s, \) and \( C_{23}^s \) from Eq. (9).

Scheme 0:

\[
R^s = \exp[-(C_{11}^s \sigma^2 + C_{22}^s \sigma^2 + C_{33}^s \lambda^2)]. \tag{29}
\]

With this basis function, the Hamiltonian matrix has the following form:

\[
\langle H \rangle = \left( \begin{array}{cc}
0 & \frac{3}{4} \langle V_C^1 \rangle \\
\frac{3}{4} V_{SS}^2 & 0 \\
\end{array} \right) - \frac{3}{4} V_{SS}^2, \tag{30}
\]

where \( \langle V_C^1 \rangle = -\frac{2}{3} \langle (V_{C1}^1) + (V_{C2}^1) \rangle = \frac{1}{3} (\langle V_{13}^C \rangle + \langle V_{14}^C \rangle + \langle V_{23}^C \rangle + \langle V_{24}^C \rangle) \) and \( \langle V_C^2 \rangle = \frac{2}{3} \langle (V_{12}^C) + (V_{14}^C) \rangle - \frac{1}{6} (\langle V_{13}^C \rangle + \langle V_{14}^C \rangle + \langle V_{23}^C \rangle + \langle V_{24}^C \rangle) \); The different sum in \( \langle V_C^1 \rangle \) and \( \langle V_C^2 \rangle \) come from different color wave function \( \Psi_1^s \) and \( \Psi_2^s \) respectively. The matrix element of the hyperfine potential \( V_{SS} \) is given by

\[
\begin{align*}
V_{SS}^2 &= 2 \langle V_{12}^{SS} \rangle - \frac{2}{3} \langle V_{34}^{SS} \rangle, \\
V_{SS}^2 &= -\frac{\sqrt{2}}{3} \langle (V_{13}^{SS}) + (V_{14}^{SS}) + (V_{23}^{SS}) + (V_{24}^{SS}) \rangle \\
V_{SS}^2 &= V_{SS}^2, \\
V_{SS}^2 &= \frac{2}{3} \langle V_{12}^{SS} \rangle - \langle V_{34}^{SS} \rangle. \tag{31}
\end{align*}
\]

The spatial part of the matrix element which was explained in detail by BS (28) can be obtained from the integration with respect to the three dimensional vector space which is illustrated by the three independent coordinate systems. The explicit forms are given in Eq. (34).

For the kinetic energy part, the kinetic operators in Eq. (15) is given by

\[
\langle T_{cm} \rangle = \langle R^s | T | R^s \rangle = \int d^3 \sigma d^3 \sigma' d^3 \lambda \exp(-Z^T C^s Z) T \exp(-Z^T C^s Z). \tag{33}
\]

Similarly, the potential energy terms is

\[
\langle V_{ij}^C \rangle = \langle R^s | (\frac{\kappa}{r_{ij}} + \frac{\tau_{ij}}{a_0} - D) | R^s \rangle
\]

\[
= \int d^3 \sigma d^3 \sigma' d^3 \lambda (\frac{\kappa}{r_{ij}} + \frac{\tau_{ij}}{a_0} - D)
\]

\[
\times \exp[-Z^T (C^s + C^s Z)]. \tag{34}
\]

Depending on \( r_{ij} \) appearing on the potential part, one needs to choose a convenient coordinate system among the three independent coordinate systems. This is easily done as the Jacobian related to coordinate transformation are all equal to one. The calculation of these matrix element in terms of the color-spin states will be discussed in detail in the Appendix.

We have applied the variational method to the ground state using the basis set which is expressed in scheme 0. In order to obtain the variational energy, we must minimize the lowest eigenvalue with respect to the variational parameters after diagonalizing the matrix in Eq. (30). Here, the variational energy is obtained by differentiating the lowest eigenvalue with respect to the variational parameters. By analyzing the result in scheme 0, we first investigate the importance of the \( R^s(q_{1q_2})^3 \otimes (q_{3q_4})^3 \) component compared to \( R^s(q_{1q_2})^6 \otimes (q_{3q_4})^6 \) component in the total wave function of \( T_{bb}^1 \).

B. Other Schemes

In these schemes, we hope to investigate the importance of introducing general Gaussian wave functions to accommodate further correlations between quarks. For that purpose, we introduce schemes I-V as below. However, we only consider \( (q_{1q_2})^3 \otimes (q_{3q_4})^3 \) part as the color-spin basis function; because the contribution from the \( (q_{1q_2})^6 \otimes (q_{3q_4})^6 \) component is negligible as will be shown later through the analysis in scheme 0. The schemes are introduced by adding the variational parameters \( C_{ij}^s \) with \( i \neq j \).

Scheme I:

\[
\langle T_{cm} \rangle = \exp[-(C_{11}^s \sigma^2 + C_{22}^s \sigma^2 + C_{33}^s \lambda^2 + 2C_{12}^s \sigma \cdot \sigma')]
\]

\[
\exp[-(C_{11}^s \sigma^2 + C_{22}^s \sigma^2 + C_{33}^s \lambda^2 - 2C_{12}^s \sigma \cdot \sigma')]. \tag{35}
\]

Scheme II:

\[
\langle T_{cm} \rangle = \exp[-(C_{11}^s \sigma^2 + C_{22}^s \sigma^2 + C_{33}^s \lambda^2 + 2C_{12}^s \sigma \cdot \lambda)]
\]

\[
\exp[-(C_{11}^s \sigma^2 + C_{22}^s \sigma^2 + C_{33}^s \lambda^2 - 2C_{12}^s \sigma \cdot \lambda)]. \tag{36}
\]
Finally, we consider the more generalized spatial function with variational parameters $C_{13} \neq C_{23} \neq 0$, and $C_{12} = 0$.

Scheme V :

$$R_i^a = \sum_i C_i R_i^a$$  \hspace{1cm} (39)

We note that all five spatial function in schemes I-V satisfies the symmetry requirement under the transposition of (12) and (34).

TABLE I: The mass and binding energy $B_T$ of $T_{b\bar{b}}^q$ and $T_{b\bar{q}}^q$ in scheme 0. The units of mass and variational parameters are MeV, and fm$^{-2}$, respectively. The numbers in brackets are results when the color $(q_1 q_2)^0 \otimes (\bar{q}_1 \bar{q}_2)^0$ component is neglected.

| Type  | 1Gaussian | Variational parameters | $B_T$  |
|-------|-----------|------------------------|--------|
| $q\bar{q}b\bar{b}$ | 10576.6   | $C_{11}^q = 2.8$, $C_{22}^q = 18.4$, $C_{33}^q = 2.7$ | -101.6 |
|       | (10577.7) | ($C_{11}^\bar{q} = 2.9$, $C_{22}^\bar{q} = 18.5$, $C_{33}^\bar{q} = 2.9$) | (-100.5) |
| $q\bar{q}c\bar{c}$ | 4036.4    | $C_{11}^q = 2.7$, $C_{22}^q = 6.9$, $C_{33}^q = 2.5$ | +97.9  |
|       | (4043.9)  | ($C_{11}^\bar{q} = 2.8$, $C_{22}^\bar{q} = 6.9$, $C_{33}^\bar{q} = 2.5$) | (105.4) |

C. Schemes with more Gaussian

Finally, we investigate the importance of introducing correction to a simple Gaussian form. This is simply accomplished by adding Gaussian with different overall coefficients. To be specific, we first introduce more Gaussian in the trial wave function

$$\sum_j b_j \Psi_j^s$$  \hspace{1cm} (41)

with variational parameters $b_j$ ($j = 1 \sim 5$) and $\Psi_j^s = R_i^s(q_1 q_2)^3 \otimes (\bar{q}_1 \bar{q}_2)^{3}$. With this, we will introduce the following additional schemes depending on how $R_i^s$ is defined.

Scheme VI :

$$R_i^s = \exp[-(C_{11}^s \sigma^2 + C_{22}^s \sigma^2 + C_{33}^s \lambda^2 + 2C_{23}^s \lambda \sigma')] + \exp[-(C_{11}^s \sigma^2 + C_{22}^s \sigma^2 + C_{33}^s \lambda^2 - 2C_{23}^s \lambda \sigma')].$$  \hspace{1cm} (42)

The parameters of the Gaussian function, $C_{11}^s$, $C_{22}^s$, $C_{33}^s$, and $C_{23}^s$ are given by $C_n^s = \alpha_n^s C_{0,j}^s$, where we chose $\alpha = 1.5, 2, 2.5, 3$, and take $C_{0,j}^s$ to be the variational parameters determined from the analysis in scheme II: $C_{01}^s = 2.9$ fm$^{-2}$, $C_{02}^s = 18.5$ fm$^{-2}$, $C_{03}^s = 2.9$ fm$^{-2}$, and $C_{013}^s = 0.6$ fm$^{-2}$. Then, for five Gaussian function, we take $n_i$ as the following:

$$n_1 = -2, n_2 = -1, n_3 = 0, n_4 = 1, n_5 = 2.$$  \hspace{1cm} (43)

Scheme VII :

$$R_i^s = \exp[-(C_{11}^s \sigma^2 + C_{22}^s \sigma^2 + C_{33}^s \lambda^2 + 2C_{23}^s \lambda \sigma') + \exp[-(C_{11}^s \sigma^2 + C_{22}^s \sigma^2 + C_{33}^s \lambda^2 - 2C_{23}^s \lambda \sigma')].$$  \hspace{1cm} (44)

Here, the parameters are defined in the same way as in scheme VI with $C_{ij}^s$ now taken from the analysis of scheme V.

The variational equations obtained by using the trial wave function in Eq. (41) reduces to the following eigenvalue problem with respect to $b_j$:

$$\sum_j \langle \Psi_j^s | H | \Psi_j^s \rangle b_j = \sum_j E(\Psi_j^s | \Psi_j^s) b_j.$$  \hspace{1cm} (45)

It should be noted that the trial wave function taken by BS with either single or five Gaussian did not take into account the correlations between quarks: $C_{ij}^s = C_{ij}^{sk} = 0$ for $i \neq j$.

D. Normal meson

In order to investigate the stability of $T_{QQ}^1$ against the decay into a scalar and a vector meson, we calculated the mass of normal mesons using the Hamiltonian in Eq. (11) with a two-body spatial function which was suggested by BS [22]. The spatial function has a form of Gaussian, given by,

$$\phi(r) = e^{-\frac{1}{2}a^2 r^2}.$$  \hspace{1cm} (46)

where $r = (r_q - r_{\bar{q}}$) is the relative distance between quark and antiquark, and $a$ is a variational parameter. The list of the mass calculated by one Gaussian function is shown in Table III.
TABLE II: The masses of the pseudo-scalar and vector mesons containing a heavy antiquark obtained using the variational method with one Gaussian function in Eq. (15). The units of $a$ and masses are fm and MeV respectively. The experimental values are shown in the third line.

| Type       | $m_B$  | $m_{B^*}$ | $m_D$ | $m_{D^*}$ |
|------------|--------|-----------|-------|----------|
| 1Gaussian  | 5317   | 5360      | 1910  | 2028     |
| $a^2$      | 8.81   | 8.07      | 8.96  | 7.30     |
| experimental value | 5279 | 5325 | 1869 | 2006 |

V. NUMERICAL RESULTS

A. Scheme 0

We first analyze results in scheme 0 to investigate the importance of the color 6 component in the total wave function of $T_{QQ}^1$.

In Table II we show the mass and variational parameters of $T_{qB}^1$ and $T_{bb}^1$ obtained from the coupled basis of $(q_1q_2)^3 \otimes (q_3q_4)^3$ and $(q_1q_2)^5 \otimes (q_3q_4)^5$ in Scheme 0 with one Gaussian spatial function. Table I also shows the binding energy $B_T = m_T - (M + M')$ of the $T_{QQ}^1$ against the decay into a pseudo-scalar and a vector meson with mass $M$ and $M'$, respectively. The masses of the mesons are calculated with the Hamiltonian in Eq. (1) using one variational Gaussian function.

As can be seen from Table II, the variational parameters changes little from those obtained by taking into account only the $(q_1q_2)^3 \otimes (q_3q_4)^3$ color component in the wave function. This also leads to only a small change in the mass of 1 MeV. Hence, we can confirm that the effect of the $(q_1q_2)^5 \otimes (q_3q_4)^5$ component on the binding of the heavy tetraquark system can be negligible. As for the $T_{cc}^1$, we find that the mass of $T_{cc}^1$ is unbounded against strong decay. In contrast to the case of $T_{qB}^1$, there is about 7 MeV change in variational energy for the case of $T_{cc}^1$.

In scheme 0, the ground state which gives the lowest eigenvalue in Eq.(30) can be expressed as a linear combination of $\Psi_1$ and $\Psi_2$ in Eq.(28). The mixing angle corresponding to the coefficients of $\Psi_1$ and $\Psi_2$ is shown in Table III. One can see that the mixing of the 66 component is again negligible.

TABLE III: The ground state wave function for $T_{QQ}^1$ in scheme 0.

| Type       | $\Psi_1$                        | $\Psi_2$                        |
|------------|---------------------------------|---------------------------------|
| $qq\bar{b}$| 0.99671Ψ$_1$ + 0.080951Ψ$_2$    |                                 |
| $qg\bar{c}$| 0.99558Ψ$_1$ + 0.093919Ψ$_2$    |                                 |

B. Other Schemes

Here, we present the results in the other schemes.

1. In scheme I, we find $C_{11}^s = 2.9$ fm$^{-2}$, $C_{22}^s = 18.5$ fm$^{-2}$, $C_{33}^s = 2.9$ fm$^{-2}$, and $C_{12}^{\sigma} = 0.4$ fm$^{-2}$. The values are the the same as in the case of $C_{12}^{\sigma} = 0$. The corresponding lowest energy is 10577.3 MeV which is also nearly equal to the value obtained with $C_{12}^{\sigma} = 0$. The correlation between $\sigma$ and $\sigma'$ makes little difference on the structure of the spatial coordinate configuration.

2. Again, little change occurs for the case of scheme II describing the correlation between $\sigma$ and $\lambda$. The presence of $C_{13}^{\lambda}$ lowers the ground state energy by 2 MeV with the values $C_{11}^{\lambda} = 2.9$ fm$^{-2}$, $C_{22}^{\lambda} = 18.5$ fm$^{-2}$, $C_{33}^{\lambda} = 2.9$ fm$^{-2}$, and $C_{13}^{\lambda} = 0.6$ fm$^{-2}$.

3. In scheme III, we find $C_{11}^s = 2.9$ fm$^{-2}$, $C_{22}^s = 18.5$ fm$^{-2}$, $C_{33}^s = 2.9$ fm$^{-2}$, and $C_{23}^s = 0.4$ fm$^{-2}$ with the corresponding lowest energy 10577 MeV.

4. In scheme IV, we find the mass to be 10574.1 MeV.

5. In scheme V, the variational parameters are given as $C_{11}^s = 2.9$ fm$^{-2}$, $C_{22}^s = 18.5$ fm$^{-2}$, $C_{33}^s = 2.9$ fm$^{-2}$, $C_{13}^s = 0.9$ fm$^{-2}$, and $C_{23}^s = 0.6$ fm$^{-2}$ with the lowest energy 10575.5 MeV.

6. In scheme VI, we find that the lowest energy with five Gaussian functions with $\alpha = 2$ to be 10558 MeV.

7. In scheme VII, we find that the lowest energy with five Gaussian functions with $\alpha = 2$ to be the same as that obtained from scheme VI.

As can be seen from Table IV, we find from the analysis of scheme I-VII, that our extended versions, taking into account correlations between quarks, did not give meaningful changes from the values obtained by BS [22] with either one Gaussian function or five Gaussian functions without the correlations. We also find that changing $\alpha=1.5$ and $\alpha=2.5$ do not introduce any additional changes, as was also noted by BS [22]. Comparing the results from scheme I-V to those from scheme VI-VII, one finds that there is only a small change in the mass suggesting that single Gaussian already encodes the dominant part of the total wave function. Moreover, the effect of including minimal correlation through scheme I to V induces even smaller mass change. Hence, we omitted the variational calculation where more complicated correlation are present through $C_{12}^{\sigma} \neq C_{13}^{\sigma} \neq C_{23}^{\sigma} \neq 0$. Preliminary investigations suggests that this independence only persists when the antidiquarks are composed of heavy anti-quarks so that the system is intrinsically small. We anticipate that the dependence of $C_{12}^{\sigma}$, $C_{13}^{\sigma}$ and $C_{23}^{\sigma}$ in lowering the variational energy is related to the size of the tetraquark to be considered.
TABLE IV: The mass of $T_{bb}$ in schemes I-VII.

| $qqbb$ | Scheme-I | Scheme-II | Scheme-III | Scheme-IV | Scheme-V |
|--------|-----------|-----------|------------|-----------|----------|
| 1Gaussian | 10577.3 | 10575.5 | 10577 | 10574.1 | 10575.5 |
| 5Gaussian | Sch-VI | Sch-VII | 10558 | 10558 |

TABLE V: The mass of $T_{cc}$ without the component $(q_1 q_2)_S^0 \otimes (\bar{q}_3 \bar{q}_4)_S^0$ in the color-spin space.

| $qqcc$ | Scheme-0 | Scheme-II |
|--------|-----------|-----------|
| 1Gaussian | 4043.9 | 4042.7 |

This effect is also true for $T_{cc}^1$, as the mass change only by 1 MeV as can be seen from Table V. In obtaining the values for Table V, we only took into account the $(q_1 q_2)_S^0 \otimes (\bar{q}_3 \bar{q}_4)_S^0$ color component for the trial wave function without correlation and with a minimal correlation as given in scheme II.

C. Sizes of hadrons

It is useful to look at the relative distances between quarks in each hadron. From Table III, we note that the distance between the quark and antiquark in the $B$ meson is $r_{q\bar{q}} \sim \sqrt{2}/(\sqrt{2}a^{-1}) = 0.476$ fm; similar values are obtained for $B^*$, $D$ and $D^*$ mesons. For $T_{bb}$ meson, the distance between the diquark and antidiquark is $r_{\lambda} \sim 1/\sqrt{C_{33}} = 0.608$ fm and that between the $b$ quarks is $r_{\sigma^*} \sim \sqrt{2}/C_{22} = 0.329$ fm. For $T_{cc}^1$, while $r_{\lambda} \sim 1/\sqrt{C_{33}} = 0.632$ fm is similar to that of $T_{bb}$, $r_{\sigma^*} \sim \sqrt{2}/C_{22} = 0.530$ fm is much larger.

VI. THE MASS SPLITTING IN HYPERFINE POTENTIAL

In this section, we investigate the contribution of the hyperfine potential term which is crucial for deciding the stability against strong decay. In particular, we perform two calculations. In the first part, we calculate the contribution of the hyperfine potential within Scheme 0 of our variational method. In a second approach, we estimate these from fitting it to the mass differences between the mesons and baryons with constant factors. Let us elaborate on the second approach. We introduce $C_{ij}$, which should be not confused with the variational parameters $C_{ij}$, for the following parametrization to the mass coming from the hyperfine potential:

$$V^{SS} = - \sum_{i<j} C_{ij} \lambda_i^c \lambda_j^c \sigma_i \cdot \sigma_j. \quad (46)$$

In the first estimate, $C_{ij} = \langle V^{SS} \rangle$ as given in Eq. (35), and can be calculated within variational approach. In the second approach, we assume that $C_{ij}$ depends only on the flavor and whether the pair is a quark-quark or quark-antiquark type. Then, $C_{ij}$ can be extracted from the observed mass differences between the baryons or mesons, within the constituent quark model. Our purpose is to assess whether one can determine the stability of tetraquark states by looking at only the hyperfine potential term given in Eq. (46) and assumptions within our second approach. For a meson consisting of a quark and antiquark, the contribution of the color part to the interaction Hamiltonian in Eq. (46) is -16/3, and the spin part is either -3 or 1 for $S=0$ and $S=1$, respectively. From the mass differences $J/\psi - \eta_c$, $D^* - D$, $\rho - \pi$, $B^* - B$, $Y - \eta_b$ $^{[30]}$, we find

$$C_{uu} = C_{ud} = 6.7 \text{ MeV}, \quad C_{ub} = C_{db} = 2.2 \text{ MeV}$$

$$C_{uu} = 29.5 \text{ MeV}, \quad C_{cc} = 5.48 \text{ MeV}, \quad C_{bb} = 3.25 \text{ MeV}.$$  

For baryons, the expectation value of the color operator $\lambda_i^c \lambda_j^c$ with respect to a color singlet wave function $e^{ijb} q_i(1) q_j(2) q_k(3)$ is the same for all the pairs and equal to -3/3. Specifically, $\langle V^{SS} \rangle = 8/3(C_{12} + C_{23} + C_{13})$ for $S=3/2$ baryons, and $\langle V^{SS} \rangle = 8/3(C_{12} - 4C_{13})$ for $S=1/2$ baryons when two quarks are identical $qqq'$. From the nucleon $\Delta$ and $N$ mass difference, we have

$$C_{uu} = C_{ud} = 18 \text{ MeV}$$

On the other hand, the strength factor involving two heavy quarks such as $T_{cc}$ can be inferred from the value of $C_{cc}$ with the same ratio as in the case of light quarks $C_{uu} = 1.63C_{uu}$ as can be seen in our estimation: we will assume $C_{cc} = 1/1.63C_{cc}$ and $C_{bb} = 1/1.63C_{bb}$. Then we have:

$$C_{cc} = 3.36 \text{ MeV} \quad C_{bb} = 1.99 \text{ MeV}.$$  

Now, to calculate the hyperfine splitting within our second approach, we note that the matrix element of the hyperfine potential for $T_{bb}^1$ and $T_{cc}$ in terms of $(q_1 q_2)_S^0 \otimes (\bar{q}_3 \bar{q}_4)_S^0$.
(q_3 \bar{q}_4)^3_0 and (q_1 q_2)^6_0 \otimes (q_3 \bar{q}_4)^5_0 is written as

\[ \langle V^{SS} \rangle = -2/3 \left( \begin{array}{cc} V^{SS}_{11} & V^{SS}_{12} \\ V^{SS}_{21} & V^{SS}_{22} \end{array} \right) \]

with

\[ V^{SS}_{11} = 12C_{12} - 4C_{34}, \]
\[ V^{SS}_{12} = V^{SS}_{21} = -3\sqrt{2}(C_{13} + C_{14} + C_{23} + C_{24}), \]
\[ V^{SS}_{22} = 2C_{12} - 6C_{34}. \] (47)

In the second approach, we use the phenomenological estimates in the right hand side of Eq. (47). The final values are given in the last (4'th) column of Table VII.

TABLE VII: The list of the value of each term of the Hamiltonian in Eq.(1) calculated within scheme 0. The final estimates in the right hand side of Eq. (47). The final values are given in the last (4'th) column of Table VII.

| Type | \( H_0 \) | \( V^{SS} \) | \( V^{SS} \) |
|------|--------|--------|--------|
| \( q\bar{q}b \) | 10756.5 | -181.4 | -143.5 |
| \( B \) | 5351.4 | -34.0 | -35.2 |
| \( B^* \) | 5350.4 | 10.5 | 11.7 |
| \( H_0 - H_0^{M+M'} \) | 54.5 | -157.9 | -120 |

| Type | \( H_0 \) | \( V^{SS} \) | \( V^{SS} \) |
|------|--------|--------|--------|
| \( q\bar{c}c \) | 4215.1 | -186.9 | -170.8 |
| \( D \) | 2007.8 | -97.2 | -107.2 |
| \( D^* \) | 2000.9 | 27.1 | 35.5 |
| \( H_0 - H_0^{M+M'} \) | 206.4 | -116.8 | -99.1 |

In Table VII, we also show the value of each part of the Hamiltonian in Eq. (1) calculated within scheme 0. \( H_0 \) corresponds to the kinetic and confinement potential terms calculated in scheme 0 for \( T_{bb}^{1} \). \( H_0^{M+M'} \) are the corresponding sum for the scalar and vector meson using the Gaussian function in Eq. (45). \( V^{SS} \) in column 3 represents the eigenvalue of the matrix of hyperfine potential in terms of the basis set in Eq.(28) for the heavy tetraquark. The \( V^{SS}_{M+M'} \) are the values of a scalar and a vector meson for hyperfine potential with one Gaussian function in Eq. (45).

As shown in the Table VII, the difference of \( H_0 \) between the heavy tetraquark and the sum of a scalar and a vector meson becomes considerably smaller in \( T_{bb}^{1} \) than in \( T_{bb}^{2} \). As can be seen from Table I, the main difference between these two tetraquarks is in the average distance between the two heavy antiquarks. When the heavy antiquark becomes large, we can estimate the binding energy simply by looking at the difference of hyperfine potential; that is, \( V^{SS} - V^{SS}_{M+M'} \) in column 4, provided that \( H_0 - (H_0^{M} + H_0^{M'}) = 0 \) for \( T_{bb}^{1} \).

VII. SUMMARY

With a simple variational Gaussian function, which is convenient to analyze the states with L=0 configurations, we have calculated the ground state energy of the \( J^P = 1^+ u\bar{d}b \bar{b} \) tetraquark containing two identical heavy antiquarks in a nonrelativistic potential model with color confinement and spin hyperfine interaction. In particular, we extend the the work by BS to investigate the effect of including the color anti-sextet component of the diquark configuration as well as using several more Gaussian parametrization for the spatial wave function. From the analysis in Scheme 0, we find that taking into account the \( R^s(q_1 q_2)^6_0 \otimes (\bar{q}_3 \bar{q}_4)^5_0 \) has little effect on the binding as well as on the wave function of the tetraquark state, whose wave function is dominated by the \( R^s(q_1 q_2)^6_0 \otimes (\bar{q}_3 \bar{q}_4)^3_0 \) component, as was expected by BS [23].

For the heavy tetraquark, we also find that the variational energy does not depend very much on whether we allow for the nonvanishing parameters \( C_{12} \neq C_{13} \neq C_{23} \neq 0 \) or \( C_{12} = C_{13} = C_{23} = 0 \) in the exponent in Eq.(9). Still, we find that the inclusion of variational parameter \( C_{13} \) introduces the most important change in the mass. This suggests that the orientation of the heavy antiquark \( s' \) is relatively less important compared to the other orientations involving light quarks. Therefore, we expect that this nonvanishing variational parameters might play a more important role in the light tetraquark system. Finally, in section VI, we have shown that the mass splitting of hyperfine potential can provide an intuitive picture for the binding energy of \( T_{bb}^{1} \) against the \( B, B^* \).

We still find that the \( T_{bb}^{1} \) mass we obtain, which is consistent with that by BS [23], remains about 33 MeV higher than that obtained by Silvestre-Brac and Semay [24] using a harmonic oscillator basis with the same Hamiltonian. A possible further improvement in our calculation is to take into account the coupling to the asymptotic decay channels which is appropriate for describing the decay property as was argued by BS [23]. Moreover, although we have neglected the center of mass motion for all mesons, these might not cancel between the tetraquark and two meson states. All these issues remains to be a caveat in our approach that should be address later.

Appendix A: Color-singlet states for tetraquark

In this section, we will calculate the matrix of the interaction Hamiltonian in terms of the color-spin wave function which have been introduced in the previous section. It is essential to mention the Casimir operator of \( SU(3)_c \) for the purpose of investigating the action of \( L_i^a L_j^a \) on the color singlet. According to Schur’s lemmas, the Casimir operator, \( \lambda^a \lambda^a \) can be expressed as a multiple of the unit
matrix in any irreducible representation of $SU(3)_c$ because the Casimir operator commutes with all of the irreducible representation of $SU(3)_c$. Therefore, each basis vector belonging to a multiplet of any irreducible representation has a common eigenvalue to the Casimir operator. In addition, $SU(3)$ has a second invariant operator, whose the eigenvalues also characterize the multiplets of $SU(3)$. Then, Racah's theorem tells us that with the two invariant operator, the $SU(3)$ multiplets are completely classified. There are several kinds of irreducible representation related to $SU(3)$.

In the irreducible representation $D(p_1, p_2)$, the number $p_k$ appearing in the $\nu^k$'th position denotes the number of columns with $k$ boxes in a given Young diagram. We define the multiplets of $SU(3)$ as $\psi(D(p_1, p_2))$. Then, the action of $\lambda^c \lambda^c$ on $\psi(D(p_1, p_2))$ gives the well-known formula

$$\lambda^c \lambda^c \psi(D(p_1, p_2)) = \frac{4}{3} \left( p_1^2 + p_1 p_2 + p_2^2 + 3p_1 + 3p_2 \right) \psi(D(p_1, p_2)).$$

For example, we have:

$$\lambda^c \lambda^c \psi(D(0, 0)) = 0,$$
$$\lambda^c \lambda^c \psi(D(1, 0)) = 16/3 \psi(D(1, 0)),$$
$$\lambda^c \lambda^c \psi(D(0, 1)) = 16/3 \psi(D(0, 1)),$$
$$\lambda^c \lambda^c \psi(D(2, 0)) = 40/3 \psi(D(2, 0)),$$
$$\lambda^c \lambda^c \psi(D(0, 2)) = 40/3 \psi(D(0, 2)),$$
$$\lambda^c \lambda^c \psi(D(1, 1)) = 12 \psi(D(1, 1)).$$

In order to calculate the matrix element of $\lambda^c_i \lambda^c_j$ with respect to the multiplet of $SU(3)_c$ in tetraquark, we need to describe two color singlets coming from a singlet-singlet color and an octet-octet color state. We denote two color singlets by $(q_1 \bar{q}_3)^{c=1} \otimes (q_2 \bar{q}_4)^{c=1}, (q_1 \bar{q}_3)^{c=8} \otimes (q_2 \bar{q}_4)^{c=8}$ or $(q_1 \bar{q}_3)^{c=1} \otimes (q_2 \bar{q}_4)^{c=1}, (q_1 \bar{q}_3)^{c=8} \otimes (q_2 \bar{q}_4)^{c=8}$. We can find two color singlets with a reducible tensor methods:

$$(q_1 \bar{q}_3)^1 \otimes (q_2 \bar{q}_4)^1 = \frac{1}{3} q^i(1) \bar{q}_3(3) q^j(2) \bar{q}_4(4),$$

$$(q_1 \bar{q}_3)^8 \otimes (q_2 \bar{q}_4)^8 = \frac{2}{\sqrt{2}} \left( q^i(1) \bar{q}_3(3) \right) - \frac{1}{3} \delta^i_j q^k(1) \bar{q}_3(3),$$

where $q^i(1) \bar{q}_3(3)$ indicates the reducible tensor of octet multiplet. It is easy to see that these color singlets are orthogonal to each other. Hence, a two dimensional vector space is spanned by $(q_1 \bar{q}_3)^1 \otimes (q_2 \bar{q}_4)^1$, and $(q_1 \bar{q}_3)^8 \otimes (q_2 \bar{q}_4)^8$. For the same reason, $(q_1 \bar{q}_3)^3 \otimes (q_2 \bar{q}_4)^3$ and $(q_1 \bar{q}_3)^6 \otimes (q_2 \bar{q}_4)^6$ which are orthogonal constitute the identical two dimensional vector space. There exists uniquely an isomorphism which is called an one-to-one correspondence such that the transformation from one bases to the other is an orthogonal $2 \times 2$ matrix because of the conservation of inner product. The transformation is given by:

$$(q_1 \bar{q}_3)^1 \otimes (q_2 \bar{q}_4)^1 = \frac{1}{\sqrt{3}} (q_1 \bar{q}_3)^3 \otimes (q_2 \bar{q}_4)^3 + \frac{\sqrt{2}}{3} (q_1 \bar{q}_3)^6 \otimes (q_2 \bar{q}_4)^6,$$
$$(q_1 \bar{q}_3)^8 \otimes (q_2 \bar{q}_4)^8 = -\frac{\sqrt{2}}{3} (q_1 \bar{q}_3)^3 \otimes (q_2 \bar{q}_4)^3 + \frac{1}{\sqrt{3}} (q_1 \bar{q}_3)^6 \otimes (q_2 \bar{q}_4)^6. \quad (A4)$$

We can also find the transformation from the basis set of $(q_1 \bar{q}_3)^1 \otimes (q_2 \bar{q}_4)^1$ and $(q_1 \bar{q}_3)^8 \otimes (q_2 \bar{q}_4)^8$ to the basis set of $(q_1 \bar{q}_3)^3 \otimes (q_2 \bar{q}_4)^3$ and $(q_1 \bar{q}_3)^6 \otimes (q_2 \bar{q}_4)^6$:

$$(q_1 \bar{q}_3)^1 \otimes (q_2 \bar{q}_4)^1 = \frac{1}{\sqrt{3}} (q_1 \bar{q}_3)^3 \otimes (q_2 \bar{q}_4)^3 + \frac{\sqrt{2}}{3} (q_1 \bar{q}_3)^6 \otimes (q_2 \bar{q}_4)^6,$$
$$(q_1 \bar{q}_3)^8 \otimes (q_2 \bar{q}_4)^8 = \sqrt{3} (q_1 \bar{q}_3)^3 \otimes (q_2 \bar{q}_4)^3 + \frac{1}{\sqrt{3}} (q_1 \bar{q}_3)^6 \otimes (q_2 \bar{q}_4)^6. \quad (A5)$$

We are now in a position to apply $\lambda^c_i \lambda^c_j$ on the color singlets. For a system consisting of two quarks $i$ and $j$, the generators are $\lambda^c_i = \lambda^c_i + \lambda^c_j$, where $c$ runs from 1 to 8. Then, by analogy with angular momentum, $\lambda^c_i \lambda^c_j$ can be expressed as:

$$\lambda^c_i \lambda^c_j = \frac{1}{2} ((\lambda^c_i)^2 - (\lambda^c_j)^2 - (\lambda^c_i \lambda^c_j))^2, \quad (A6)$$

where $(\lambda^c_i)^2$, $(\lambda^c_j)^2$ and $(\lambda^c_i \lambda^c_j)^2$ are Casimir operators associated with the two-body system, and the particles $i$ and $j$, respectively. Applying $\lambda^c_i \lambda^c_j$ to $(q_1 \bar{q}_3)^3 \otimes (q_2 \bar{q}_4)^3$ gives:

$$\lambda^c_i \lambda^c_j (q_1 \bar{q}_3)^3 \otimes (q_2 \bar{q}_4)^3 = \frac{1}{2} ((\lambda^c_i)^2 (q_1 \bar{q}_3)^3 \otimes (q_2 \bar{q}_4)^3 - ((\lambda^c_i)^2 (q_1 \bar{q}_3)^3 \otimes (q_2 \bar{q}_4)^3 - (q_1 \bar{q}_3)^3 \otimes (q_2 \bar{q}_4)^3)^3$$

$$\lambda^c_i \lambda^c_j (q_1 \bar{q}_3)^3 \otimes (q_2 \bar{q}_4)^3 = \frac{1}{2} \left( \frac{16}{3} \bar{q}_3(3) \right) - \frac{16}{3} (q_1 \bar{q}_3)^3 \otimes (q_2 \bar{q}_4)^3$$

$$\lambda^c_i \lambda^c_j (q_1 \bar{q}_3)^3 \otimes (q_2 \bar{q}_4)^3 = -\frac{8}{3} (q_1 \bar{q}_3)^3 \otimes (q_2 \bar{q}_4)^3. \quad (A7)$$
Similarly, applying $\lambda_1^c\lambda_2^c$ on $(q_1q_2)^6 \otimes (q_3q_4)^6$ yields,

$$\lambda_1^c\lambda_2^c(q_1q_2)^6 \otimes (q_3q_4)^6 = \frac{1}{2}((\lambda_1^c)^2(q_1q_2)^6) \otimes (q_3q_4)^6$$

$$- ((\lambda_1^c)^2(q_1q_2)^6) \otimes (q_3q_4)^6$$

$$- (q_1((\lambda_1^c)^2q_2)^6) \otimes (q_3q_4)^6$$

$$= \frac{1}{2}(\frac{40}{3} - \frac{16}{3} - \frac{16}{3})(q_1q_2)^6 \otimes (q_3q_4)^6$$

$$= + \frac{4}{3}(q_1q_2)^6 \otimes (q_3q_4)^6. \quad (A8)$$

It follows immediately that the same result is obtained for the $\lambda_3^c\lambda_4^c$. For the operator, $\lambda_1^c\lambda_5^c$, the basis set of $(q_1q_3)^3 \otimes (q_2q_4)^3$ and $(q_1q_3)^5 \otimes (q_2q_4)^3$ instead of the basis set of $(q_1q_2)^3 \otimes (q_3q_4)^3$ and $(q_1q_2)^6 \otimes (q_3q_4)^6$ is required to calculate the matrix element. Then, the matrix element of $\lambda_1^c\lambda_5^c$, in terms of $(q_1q_3)^3 \otimes (q_2q_4)^3$ and $(q_1q_2)^6 \otimes (q_3q_4)^6$, is obtained from the similarity transformation which changes the matrix representation based on a basis set. In a similar way, we have:

$$\lambda_1^c\lambda_5^c(q_1q_3)^3 \otimes (q_2q_4)^4 = \frac{1}{2}((\lambda_1^c)^2(q_1q_3)^3) \otimes (q_2q_4)^4$$

$$- ((\lambda_1^c)^2(q_1q_3)^3) \otimes (q_2q_4)^4$$

$$- (q_1((\lambda_1^c)^2q_2)^3) \otimes (q_3q_4)^4$$

$$= \frac{1}{2}(0 - \frac{16}{3} - \frac{16}{3})(q_1q_2)^3 \otimes (q_3q_4)^4$$

$$= -\frac{16}{3}(q_1q_3)^3 \otimes (q_2q_4)^4. \quad (A9)$$

Next, we calculate the matrix element of $\lambda_1^c\lambda_5^c$ in terms of $(q_1q_2)^6 \otimes (q_3q_4)^6$ and $(q_1q_2)^3 \otimes (q_3q_4)^3$, we find the inverse of transformation, which is equivalent to the orthogonal matrix $U$, and $(\lambda_1^c\lambda_5^c)(q_1q_3)^3 \otimes (q_2q_4)^4 \otimes (q_1q_3)^3 \otimes (q_2q_4)^4$. U

$$U = \left(\begin{array}{cc}
\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{array}\right).$$

$$\langle q_1q_3, q_2q_4 | \lambda_1^c\lambda_5^c | q_1q_3, q_2q_4 \rangle = \left(\begin{array}{cc}
\frac{2}{3} & 0 \\
0 & -\frac{16}{3}
\end{array}\right). \quad (A11)$$

It is necessary to obtain the matrix element of $\lambda_1^c\lambda_5^c$.

**Appendix B: Spin states for tetraquark**

In this section, we investigate spin states for tetraquark to calculate the matrix element of the Hamiltonian in Eq. (4). The case for the spin operators can be treated similarly as before in that $SU(2)$ is a subgroup of $SU(3)$. A point that is different from the case of $SU(3)$ is that the $SU(2)$ has only one Casimir operator. The only Casimir operator, $\sigma \cdot \sigma$, classifies the multiplets of $SU(2)$ by the eigenvalues. We describe the explicit form of the total spin $S=0$ and 1. The two orthonormal basis states $(\chi_{12})_{s=1} \otimes (\chi_{34})_{s=1}$ and $(\chi_{12})_{s=0} \otimes (\chi_{34})_{s=0}$ with the total $S=0$ in Eq. (21) can be expressed as:

$$(\chi_{12})_{s=1} \otimes (\chi_{34})_{s=1} = \frac{1}{\sqrt{3}} \uparrow (1) \uparrow (2) \otimes \downarrow (3) \downarrow (4)$$

$$+ \frac{1}{\sqrt{3}} \downarrow (1) \downarrow (2) \otimes \uparrow (3) \uparrow (4)$$

$$- \frac{1}{\sqrt{3}} \frac{1}{\sqrt{2}} \uparrow (1) \downarrow (2) \downarrow (1) \uparrow (2) \otimes \frac{1}{\sqrt{2}} \uparrow (3) \downarrow (4) +$$

$$\downarrow (3) \uparrow (4)) = \frac{1}{\sqrt{12}} (2 \uparrow \uparrow \downarrow \downarrow + 2 \downarrow \uparrow \uparrow - \downarrow \downarrow \uparrow \uparrow - \downarrow \downarrow \downarrow \downarrow - \downarrow \downarrow \uparrow \uparrow)$$

Finally, we reach the matrix representation based on $(q_1q_2)^6 \otimes (q_3q_4)^6$ and $(q_1q_2)^3 \otimes (q_3q_4)^3$ as:

$$\langle \lambda_1^c\lambda_5^c \rangle \langle q_1q_3, q_2q_4 \rangle = U^T \langle \lambda_1^c\lambda_5^c \rangle \langle q_1q_3, q_2q_4 \rangle U$$

$$= \left(\begin{array}{cc}
\frac{2}{3} & 0 \\
0 & -\frac{16}{3}
\end{array}\right). \quad (A12)$$

The basis set of $(q_1q_3)^3 \otimes (q_2q_4)^4$ and $(q_1q_3)^3 \otimes (q_2q_4)^4$ is necessary to obtain the matrix element of $\lambda_1^c\lambda_5^c$.
\[
(\chi_{12})_{s=0} \otimes (\chi_{34})_{s=0} = \frac{1}{\sqrt{2}} (\uparrow (1) \downarrow (2) - \downarrow (1) \uparrow (2)) \otimes \frac{1}{\sqrt{2}} (\uparrow (3) \downarrow (4) - \downarrow (3) \uparrow (4)) = \frac{1}{2} (\uparrow \uparrow \uparrow \downarrow - \uparrow \uparrow \downarrow \uparrow - \downarrow \uparrow \downarrow \uparrow + \downarrow \uparrow \uparrow \downarrow). \tag{B1}
\]

Here, we define spinors as,
\[
\left( \begin{array}{c}
1 \\
0 \\
0 \\
1
\end{array} \right) = \uparrow, \quad \left( \begin{array}{c}
0 \\
1 \\
1 \\
0
\end{array} \right) = \downarrow. \tag{B2}
\]

The coefficients appearing in Eq. \((B1)\) are obtained from the Clebsch-Gordan coefficients of \(SU(2)\). The three basis states with the total \(S=1\) in Eq. \((22)\) are given by,
\[
(\chi_{12})_{s=1} \otimes (\chi_{34})_{s=1} = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} ((\uparrow (1) \downarrow (2) + \downarrow (1) \uparrow (2)) \otimes \uparrow (3) \uparrow (4)) - \frac{1}{\sqrt{2}} \uparrow (1) \downarrow (2) \otimes \frac{1}{\sqrt{2}} ((\uparrow (3) \downarrow (4) + \downarrow (3) \uparrow (4)) = \frac{1}{2} (\uparrow \uparrow \uparrow \downarrow - \uparrow \uparrow \downarrow \uparrow - \downarrow \uparrow \downarrow \uparrow + \downarrow \uparrow \uparrow \downarrow),
\]

\[
(\chi_{12})_{s=0} \otimes (\chi_{34})_{s=1} = \frac{1}{\sqrt{2}} (\uparrow (1) \downarrow (2) - \downarrow (1) \uparrow (2)) \otimes \uparrow (3) \uparrow (4) = \frac{1}{2} (\uparrow \downarrow \uparrow \uparrow - \downarrow \uparrow \downarrow \uparrow). \tag{B3}
\]

It is easy to obtain the result of applying the spin operator \(s_i \cdot s_j\) on these bases through the well known eigenvalues of the Casimir. By analogy with the case of \(SU(3)\), we can find the matrix of \(s_i \cdot s_j\) for \(S=0\) and \(S=1\). It follows immediately that we can find the matrix of interaction Hamiltonian in Eq. \((16)\) for scalar tetraquark and axial tetraquark. They are obtained from the Kronecker product of the matrix of the color operator \(\lambda_i^j\) and the spin operator \(s_i \cdot s_j\). The basis set of the Kronecker product of the matrix of the color operator \(\lambda_i^j\) and the spin operator \(s_i \cdot s_j\) for scalar tetraquark is given by,
\[
\phi_1 = (q_1 q_3)^0 \otimes (\bar{q}_1 \bar{q}_3)^0, \quad \phi_2 = (q_1 q_3)^0 \otimes (\bar{q}_2 \bar{q}_4)^0, \quad \phi_3 = (q_1 q_2)^0 \otimes (\bar{q}_3 \bar{q}_4)^0, \quad \phi_4 = (q_1 q_2)^0 \otimes (\bar{q}_3 \bar{q}_4)^0. \tag{B4}
\]

and, the basis set for axial tetraquark is,
\[
\phi_1 = (q_1 q_3)^0 \otimes (\bar{q}_1 \bar{q}_3)^0, \quad \phi_2 = (q_1 q_3)^0 \otimes (\bar{q}_2 \bar{q}_4)^0, \quad \phi_3 = (q_1 q_2)^0 \otimes (\bar{q}_3 \bar{q}_4)^0, \quad \phi_4 = (q_1 q_2)^0 \otimes (\bar{q}_3 \bar{q}_4)^0, \quad \phi_5 = (q_1 q_3)^0 \otimes (\bar{q}_4 \bar{q}_4)^0, \quad \phi_6 = (q_1 q_2)^0 \otimes (\bar{q}_3 \bar{q}_4)^0. \tag{B5}
\]

The matrix of interaction Hamiltonian in Eq. \((16)\) for scalar tetraquark in terms of the color-spin basis states is written as,
\[
\left( - \sum_{i<j} \frac{C_{ij} \lambda_i^j \lambda_i^j s_i \cdot s_j} \right) = H_{C,M} =
\]

TABLE IX: The matrix of \(s_i \cdot s_j\) is written in terms of two basis set, \(\phi_1 = (\chi_{12})_{s=1} \otimes (\chi_{34})_{s=1}\) and \(\phi_2 = (\chi_{12})_{s=0} \otimes (\chi_{34})_{s=0}\) for the total \(S=0\), \(\psi_1 = (\chi_{12})_{s=1} \otimes (\chi_{34})_{s=0}\), \(\psi_2 = (\chi_{12})_{s=1} \otimes (\chi_{34})_{s=1}\) and \(\psi_3 = (\chi_{12})_{s=0} \otimes (\chi_{34})_{s=1}\) for the total \(S=1\).

\begin{table}[h]
\begin{tabular}{|c|c|c|c|}
\hline
\(s_i \cdot s_j\) & \(s_{i=1} s_{j=0}\) & \(s_{i=1} s_{j=1}\) & \(s_{i=0} s_{j=1}\) \\
\hline
\(\sigma_1 \cdot \sigma_2\) & \(\begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}\) & \(\begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}\) & \(\begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}\) \\
\(\sigma_1 \cdot \sigma_3\) & \(\begin{pmatrix} -2 & -\sqrt{3} \\ -\sqrt{3} & 0 \end{pmatrix}\) & \(\begin{pmatrix} 0 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}\) & \(\begin{pmatrix} 0 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}\) \\
\(\sigma_1 \cdot \sigma_4\) & \(\begin{pmatrix} -2 & -\sqrt{3} \\ -\sqrt{3} & 0 \end{pmatrix}\) & \(\begin{pmatrix} 0 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}\) & \(\begin{pmatrix} 0 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}\) \\
\(\sigma_2 \cdot \sigma_3\) & \(\begin{pmatrix} -2 & -\sqrt{3} \\ -\sqrt{3} & 0 \end{pmatrix}\) & \(\begin{pmatrix} 0 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}\) & \(\begin{pmatrix} 0 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}\) \\
\(\sigma_2 \cdot \sigma_4\) & \(\begin{pmatrix} -2 & -\sqrt{3} \\ -\sqrt{3} & 0 \end{pmatrix}\) & \(\begin{pmatrix} 0 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}\) & \(\begin{pmatrix} 0 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}\) \\
\(\sigma_3 \cdot \sigma_4\) & \(\begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}\) & \(\begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}\) & \(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\) \\
\hline
\end{tabular}
\end{table}
\[-C_{24} \begin{pmatrix} \frac{1}{2} \left( -2 - \sqrt{3} \right) & -2 \sqrt{2} \left( -2 - \sqrt{3} \right) \\ -\sqrt{2} & 0 \\ -2 \sqrt{2} & -\sqrt{3} \end{pmatrix} + \]
\[-C_{34} \begin{pmatrix} \frac{1}{2} \left( 1 & 0 & -3 \right) & 0 \left( 1 & 0 & -3 \right) \\ 0 \left( 1 & 0 & -3 \right) & -\frac{1}{2} \left( 1 & 0 & -3 \right) \end{pmatrix}. \quad (B6)\]

To compare with the result which can be found in Ref. [28], we change the basis set in Eq. (B4) into,
\[\phi_1 = (q_1 q_2)_{\bar{0}} \otimes (q_3 q_4)_{\bar{3}}, \phi_2 = (q_1 q_2)_{\bar{3}} \otimes (q_3 q_4)_{\bar{0}},\]
\[\phi_3 = (q_1 q_2)_{\bar{0}} \otimes (q_3 q_4)_{\bar{3}}, \phi_4 = (q_1 q_2)_{\bar{3}} \otimes (q_3 q_4)_{\bar{0}}, \quad (B7)\]

Then, it is found that the transformation from the basis set in Eq. (B4) to the basis set in Eq. (B7) is,
\[U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \quad (B8)\]

The matrix of the interaction Hamiltonian in Eq. (46) denoted by $H_{CM}$ for scalar tetraquarks is acquired by the similarity transformation:
\[H_{CM} = U^T H^\prime_{CM} U = -\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad (B9)\]

with 2 by 2 submatrices
\[A_{11} = \frac{4}{3} (C_{12} + C_{34}) + \frac{20}{3} (C_{13} + C_{14} + C_{23} + C_{24}), \]
\[A_{12} = A_{21} = 2\sqrt{6} (C_{13} + C_{14} + C_{23} + C_{24}), \]
\[A_{22} = 8(C_{12} + C_{34}), \]
\[B = C^T = \begin{pmatrix} 2 \sqrt{6} (C_{13} - C_{14} - C_{23} + C_{24}) & 5 & 2 \sqrt{6} \end{pmatrix}, \]
\[D_{11} = -4(C_{12} + C_{34}), \]
\[D_{12} = D_{21} = 2\sqrt{6} (C_{13} + C_{14} + C_{23} + C_{24}), \]
\[D_{22} = -\frac{8}{3} (C_{12} + C_{34} - C_{13} - C_{14} - C_{23} - C_{24}). \quad (B10)\]

In a situation where $C_{13} = C_{23}$ and $C_{14} = C_{24}$, the matrix of interaction Hamiltonian in Eq. (B6) for scalar tetraquarks reduces to the block diagonal form,
\[H_{CM} = -\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}. \quad (B11)\]

This means that the flavor-symmetry of light diquark causes the separation of $\bar{3}_f$ and $6_f$. We can apply the same procedure to calculate the matrix of the interaction Hamiltonian in Eq. (46) for axial tetraquarks.

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