Pushouts and e-Projective Semimodules*

Jawad Abuhlail†
abuhlail@kfupm.edu.sa
Department of Mathematics and Statistics
King Fahd University of Petroleum & Minerals
31261 Dhahran, KSA

Rangga Ganzar Noegraha‡
rangga.gn@universitaspertamina.ac.id
Universitas Pertamina
Jl. Teuku Nyak Arief
Jakarta 12220, Indonesia

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Abstract
Projective modules play an important role in the study of the category of modules over rings and in the characterization of various classes of rings. Several characterizations of projective objects which are equivalent for modules over rings are not necessarily equivalent for semimodules over an arbitrary semiring. We study several of these notions, in particular the e-projective semimodules introduced by the first author using his new notion of exact sequences of semimodules. As pushouts of semimodules play an important role in some of our proofs, we investigate them and give a constructive proof of their existence in a way that proved be very helpful.

Introduction
The importance of semirings (defined, roughly, as rings not necessarily with subtraction) stems from the fact that they can be considered as a generalization of both rings and distributive bounded lattices. Moreover, semirings, and their semimodules (defined, roughly, as modules not necessarily with subtraction), proved to have wide applications in many aspects of Computer Science and Mathematics, e.g., Automata Theory [HW1998], Tropical Geometry [Gla2002] and Idempotent Analysis [LM2005]. Many of these applications can be found in Golan’s book [Gol1999], which is our main reference in this topic.

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†Corresponding Author
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The notion of projective objects can be defined in any category relative to a suitable factorization system of its arrows. Projective semimodules have been studied intensively (see [Gla2002] for details). Recently, several papers by Abuhlail, I’lin, Katsov and Nam (among others) prepared the stage for a homological characterization of special classes of semirings using special classes of projective, injective and flat semimodules (cf., [KNT2009], [Ili2010], [KN2011], [Abu2014], [KNZ2014], [AIKN2015], [IKN2017], [AIKN2018]). For example, ideal-semisimple semirings all of whose left cyclic semimodules are projective have been investigated in [IKN2017].

In addition to the categorical notions of projective semimodules over a semiring, several other notions were considered in the literature, e.g., the so called \( k \)-projective semimodules [Alt1996]. One reason for the interest in such notions is the phenomenon that assuming that all semimodules over a given semiring \( S \) are projective forces the underlying semiring to be a (semisimple) ring (cf., [Ili2010, Theorem 3.4]). Using a new notion of exact sequences of semimodules over a semiring, Abuhlail [Abu2014-CA] introduced the homological notion of exactly projective semimodules (\( e \)-projective semimodules, for short) assuming that an appropriate \( \text{Hom} \) functor preserves short exact sequences (under the initial name of uniformly projective semimodules).

The paper is divided into three sections.

In Section 1, we collect the basic definitions, examples and preliminaries used in this paper. Among others, we include the definitions and basic properties of exact sequences introduced by Abuhlail [Abu2014].

In Section 2, we demonstrate the existence of pullbacks (see 2.1) and pushouts (Theorem 2.3) in the category of semimodules over an arbitrary semiring. Although no explicit construction of the pushouts is given, we provide a description that is good enough to help us in proving several results in the sequel.

In Section Three, we investigate mainly the \( e \)-projective semimodules over a semiring and clarify their relations with the notions of projective semimodules as well as the so called \( k \)-projective semimodules. In Proposition 3.6, we demonstrate that every projective left semimodule is in fact \( e \)-projective. In Example 3.7, we show that the Boolean Algebra \( \mathbb{B} \) considered as a \( \mathbb{Q}^+ \)-semimodule in the canonical way is \( \mathbb{Q}^+ \)-\( e \)-projective but not \( \mathbb{Q}^+ \)-projective. A complete characterization of \( k \)-projective left semimodules through the right-splitting of short exact sequences is given in Proposition 3.10. In Lemma 3.12 and Proposition 3.13, we provide homological proofs of the facts that the class of \( e \)-projective left \( S \)-semimodules is closed under retracts and direct sums recovering part of [AIKN2018, Corollary 3.3], where compact categorical proofs were given.

1 Preliminaries

In this section, we provide the basic definitions and preliminaries used in this work. Any notions that are not defined can be found in our main reference [Gol1999]. We refer to [Wis1991] for the foundations of Module and Ring Theory.
**Definition 1.1.** ([Gol1999]) A **semiring** is a datum \((S, +, 0, \cdot, 1)\) consisting of a commutative monoid \((S, +, 0)\) and a monoid \((S, \cdot, 1)\) such that \(0 \neq 1\) and

\[
\begin{align*}
    a \cdot 0 & = 0 = 0 \cdot a \text{ for all } a \in S; \\
    a(b + c) & = ab + ac \text{ and } (a + b)c = ac + bc \text{ for all } a, b, c \in S.
\end{align*}
\]

**Definitions 1.2.** ([Gol1999]) Let \((S, +, 0, \cdot, 1)\) be a semiring.

- If the monoid \((S, \cdot, 1)\) is commutative, we say that \(S\) is a **commutative semiring**.

- The set of **cancellative elements of** \(S\) is defined as

\[
K^+(S) = \{ x \in S \mid x + y = x + z \implies y = z \text{ for any } y, z \in S \}.
\]

We say that \(S\) is a **cancellative semiring** if \(K^+(S) = S\).

**Examples 1.3.** ([Gol1999])

- Every ring is a cancellative semiring.

- Any **distributive bounded lattice** \(\mathcal{L} = (L, \vee, 1, \wedge, 0)\) is a commutative semiring.

- Let \(R\) be any ring. The set \(\mathcal{I} = (\text{Ideal}(R), +, 0, \cdot, R)\) of ideals of \(R\) is a semiring.

- The sets \((\mathbb{Z}^+, +, 0, \cdot, 1)\) (resp. \((\mathbb{Q}^+, +, 0, \cdot, 1)\), \((\mathbb{Q}^+, +, 0, \cdot, 1)\)) of non-negative integers (resp. non-negative rational numbers, non-negative real numbers) is a commutative cancellative semiring which is not a ring.

- \(M_n(S)\), the set of all \(n \times n\) matrices over a semiring \(S\), is a semiring.

- The **Boolean algebra** \(\mathbb{B} := \{0, 1\}\) with \(1 + 1 = 1\) is a semiring called the **Boolean Semiring**.

**1.4.** ([Gol1999]) Let \(S\) and \(T\) be semirings. The categories \(\mathcal{S}\) **SM** of **left** \(S\)-**semimodules** with arrows the \(S\)-linear maps, \(\mathcal{SM}_T\) of **right** \(S\)-**semimodules** with arrows the \(T\)-linear maps, and \(\mathcal{S}\text{SM}_T\) of \((S, T)\)-**bisemimodules** are defined in the usual way (as for modules and bimodules over rings).

We write \(L \leq_{S} M\) to mean that \(M\) is a left (right) \(S\)-semimodule and \(L\) is an \(S\)-**subsemimodule** of \(M\).

**Example 1.5.** The category of \(\mathbb{Z}^+\)-semimodules is nothing but the category of commutative monoids.

**Example 1.6.** Let \((S, +, 0, \cdot, 1)\) be a semiring. Then \(S\) and \(S^{(A)}\) (the direct sum of \(S\) over a non-empty index set \(A\)) are \((S, S)\)-**bisemimodules** with left and right actions induced by “\(\cdot\)”.
Example 1.7. ([Gol1999, page 150, 154]) Let $S$ be a semiring, $M$ be a left $S$-semimodule and $L \subseteq M$. The **subtractive closure** of $L$ is defined as

$$\overline{L} := \{ m \in M \mid m + l = l' \text{ for some } l, l' \in L \}. \tag{1}$$

One can easily check that $\overline{L} = \text{Ker}(M \to M/L)$, where $\pi$ is the canonical projection. We say that $L$ is **subtractive**, if $L = \overline{L}$. The left $S$-semimodule $M$ is a **subtractive semimodule**, if every $S$-subsemimodule $L \leq_S M$ is subtractive. If the only $S$-subsemimodules of $M$ are $\{0\}$ and $M$, then we say that $M$ is **ideal-simple**.

**Definition 1.8.** [Gol1999, page 162] Let $S$ be a semiring. An equivalence relation $\rho$ on a left $S$-semimodule $M$ is a **congruence relation**, if it preserves the addition and the scalar multiplication on $M$, i.e. for all $s \in S$ and $m, m', n, n' \in M$:

$$mpm' \text{ and } n\rho n' \implies (m + m')\rho(n + n'),$$

$$mpm' \implies (sm)\rho(sm').$$

**Lemma 1.9.** A left $S$-semimodule $M$ is ideal-simple if and only if every non-zero $S$-linear map to $M$ is surjective.

1.10. (cf., [AHS2004]) The category $s\text{SM}$ of left semimodules over a semiring $S$ is a **variety** in the sense of Universal Algebra (closed under homomorphic images, subobjects and arbitrary products). Whence $s\text{SM}$ is complete, i.e. has all limits (e.g., direct products, equalizers, kernels, pullbacks, inverse limits) and cocomplete, i.e. has all colimits (e.g., direct coproducts, coequalizers, cokernels, pushouts, direct colimits).

**Definition 1.11.** ([Gol1999, page 184]) Let $S$ be a semiring. A left $S$-semimodule $M$ is the **direct sum** of a family $\{ L_\lambda \}_{\lambda \in \Lambda}$ of $S$-subsemimodules $L_\lambda \leq_S M$, and we write $M = \bigoplus L_\lambda$, if every $m \in M$ can be written in a **unique way** as a finite sum $m = l_{\lambda_1} + \cdots + l_{\lambda_k}$ where $l_{\lambda_i} \in L_{\lambda_i}$ for each $i = 1, \cdots, k$. Equivalently, $M = \bigoplus L_\lambda$ if $M = \sum_{\lambda \in \Lambda} L_\lambda$ and for each finite subset $A \subseteq \Lambda$ with $l_a, l'_a \in L_a$, we have:

$$\sum_{a \in A} l_a = \sum_{a \in A} l'_a \implies l_a = l'_a \text{ for all } a \in A.$$

1.12. An $S$-semimodule $N$ is a **retract** of an $S$-semimodule $M$ if there exists a (surjective) $S$-linear map $\theta : M \to N$ and an (injective) $S$-linear map $\psi : N \to M$ such that $\theta \circ \psi = \text{id}_N$ (equivalently, $N \simeq \alpha(M)$ for some idempotent endomorphism $\alpha \in \text{End}(M_S)$).

1.13. An $S$-semimodule $N$ is a **direct summand** of an $S$-semimodule $M$ (i.e. $M = N \oplus N'$ for some $S$-subsemimodule $N'$ of $M$) if and only if there exists $\alpha \in \text{Comp}(\text{End}(M_S))$ s.t. $\alpha(M) = N$ where for any semiring $T$ we set

$$\text{Comp}(T) = \{ t \in T \mid \exists \tilde{t} \in T \text{ with } t + \tilde{t} = 1_T \text{ and } \tilde{t}t = 0_T = \tilde{t}t \}.$$ 

Indeed, every direct summand of $M$ is a retract of $M$; the converse is not true in general; for example $N_1$ in Example 3.16 is a retract of $M_2(\mathbb{R}^+)$ that is not a direct summand. Golan [Gol1999, Proposition 16.6] provided characterizations of direct summands.
**Remarks 1.14.** Let $M$ be a left $S$-semimodule and $K,L \leq_S M$ be $S$-semimodules of $M$.

1. If $K + L$ is direct, then $K \cap L = 0$. The converse is not true in general (for a counterexample, see Example 1.15).

2. If $M = K \oplus L$, then $M/K \cong L$.

**Example 1.15.** Let $S = M_2(\mathbb{R}^+)$. Notice that $E_1 = \left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \mid a,b \in \mathbb{R}^+ \right\}$ and $N_{\geq 1} = \left\{ \begin{bmatrix} a & c \\ b & d \end{bmatrix} \mid a \leq c, b \leq d, a,b,c,d \in \mathbb{R}^+ \right\}$ are left ideals of $S$ with $E_1 \cap N_{\geq 1} = \{0\}$. However, the sum $E_1 + N_{\geq 1}$ is not direct since $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

**Exact Sequences**

Throughout, $(S, +, 0, \cdot, 1)$ is a semiring and, unless otherwise explicitly mentioned, an $S$-module is a left $S$-semimodule.

**Definition 1.16.** A morphism of left $S$-semimodules $f : L \to M$ is

- **$k$-normal**, if whenever $f(m) = f(m')$ for some $m,m' \in M$, we have $m + k = m' + k'$ for some $k,k' \in \text{Ker}(f)$;
- **$i$-normal**, if $\text{Im}(f) = f(L) := \{m \in M \mid m + l \in L \text{ for some } l \in L\}$.
- **normal**, if $f$ is both $k$-normal and $i$-normal.

**Remarks 1.17.** (1) Among others, Takahashi ([Tak1981]) and Golan [Gol1999] called $k$-normal (resp., $i$-normal, normal) $S$-linear maps $k$-regular (resp., $i$-regular, regular) morphisms. We changed the terminology to avoid confusion with the regular monomorphisms and regular epimorphisms in Category Theory which have different meanings when applied to categories of semimodules.

2. Our terminology is consistent with Category Theory noting that: every surjective $S$-linear map is $i$-normal, whence the $k$-normal surjective $S$-linear map are normal and are precisely the so-called **normal epimorphisms**. On the other hand, the injective $S$-linear maps are $k$-normal, whence the $i$-normal injective $S$-linear maps are normal and are precisely the so called **normal monomorphisms** (see [Abu2014]).

The following technical lemma is helpful in several proofs in this and forthcoming related papers.

**Lemma 1.18.** Let $L \xrightarrow{f} M \xrightarrow{g} N$ be a sequence of semimodules.
(1) Let \( g \) be injective.

(a) \( f \) is \( k \)-normal if and only if \( g \circ f \) is \( k \)-normal.

(b) If \( g \circ f \) is \( i \)-normal (normal), then \( f \) is \( i \)-normal (normal).

(c) Assume that \( g \) is \( i \)-normal. Then \( f \) is \( i \)-normal (normal) if and only if \( g \circ f \) is \( i \)-normal (normal).

(2) Let \( f \) be surjective.

(a) \( g \) is \( i \)-normal if and only if \( g \circ f \) is \( i \)-normal.

(b) If \( g \circ f \) is \( k \)-normal (normal), then \( g \) is \( k \)-normal (normal).

(c) Assume that \( f \) is \( k \)-normal. Then \( g \) is \( k \)-normal (normal) if and only if \( g \circ f \) is \( k \)-normal (normal).

**Proof.** (1) Let \( g \) be injective; in particular, \( g \) is \( k \)-normal.

(a) Assume that \( f \) is \( k \)-normal. Suppose that \( (g \circ f)(l_1) = (g \circ f)(l_2) \) for some \( l_1, l_2 \in L \). Since \( g \) is injective, \( f(l_1) = f(l_2) \). By assumption, there exist \( k_1, k_2 \in \text{Ker}(f) \) such that \( l_1 + k_1 = l_2 + k_2 \). Since \( \text{Ker}(f) \subseteq \text{Ker}(g \circ f) \), we conclude that \( g \circ f \) is \( k \)-normal. On the other hand, assume that \( g \circ f \) is \( k \)-normal. Suppose that \( f(l_1) = f(l_2) \) for some \( l_1, l_2 \in L \). Then \((g \circ f)(l_1) = (g \circ f)(l_2)\) and so there exist \( k_1, k_2 \in \text{Ker}(g \circ f) \) such that \( l_1 + k_1 = l_2 + k_2 \). Since \( g \) is injective, \( \text{Ker}(g \circ f) = \text{Ker}(f) \) whence \( f \) is \( k \)-normal.

(b) Assume that \( g \circ f \) is \( i \)-normal. Let \( m \in f(L) \), so that \( m + f(l_1) = f(l_2) \) for some \( l_1, l_2 \in L \). Then \( g(m) \in (g \circ f)(L) = (g \circ f)(L) \). Since \( g \) is injective, \( m \in f(L) \). So, \( f \) is \( i \)-normal.

(c) Assume that \( g \) and \( f \) are \( i \)-normal. Let \( n \in (g \circ f)(L) \), so that \( n + g(f(l_1)) = g(f(l_2)) \) for some \( l_1, l_2 \in L \). Since \( g \) is \( i \)-normal, \( n \in g(M) \) say \( n = g(m) \) for some \( m \in M \). But \( g \) is injective, whence \( m + f(l_1) = f(l_2), \) i.e. \( m \in f(L) = f(L) \) since \( f \) is \( i \)-normal by assumption. So, \( n = g(m) \in (g \circ f)(L) \). We conclude that \( g \circ f \) is \( i \)-normal.

(2) Let \( f \) be surjective; in particular, \( f \) is \( i \)-normal.

(a) Assume that \( g \) is \( i \)-normal. Let \( n \in (g \circ f)(L) \) so that \( n + g(f(l_1)) = g(f(l_2)) \) for some \( l_1, l_2 \in L \). Since \( g \) is \( i \)-normal, \( n = g(m) \) for some \( m \in M \). Since \( f \) is surjective, \( n = g(m) \in (g \circ f)(L) \). So, \( g \circ f \) is \( i \)-normal.

On the other hand, assume that \( g \circ f \) is \( i \)-normal. Let \( n \in g(M) \), so that \( n + g(m_1) = g(m_2) \) for some \( m_1, m_2 \in M \). Since \( f \) is surjective, there exist \( l_1, l_2 \in L \) such that \( f(l_1) = m_1 \) and \( f(l_2) = m_2 \). Then, \( n + (g \circ f)(l_1) = (g \circ f)(l_2) \), i.e. \( n \in (g \circ f)(L) = (g \circ f)(L) \subseteq g(M) \). So, \( g \) is \( i \)-normal.
(b) Assume that \( g \circ f \) is \( k \)-normal. Suppose that \( g(m_1) = g(m_2) \) for some \( m_1, m_2 \in M \). Since \( f \) is surjective, we have \( (g \circ f)(l_1) = (g \circ f)(l_2) \) for some \( l_1, l_2 \in L \). By assumption, \( g \circ f \) is \( k \)-normal and so there exist \( k_1, k_2 \in \text{Ker}(g \circ f) \) such that \( l_1 + k_1 = l_2 + k_2 \) whence \( m_1 + f(k_1) = m_2 + f(k_2) \). Indeed, \( f(k_1), f(k_2) \in \text{Ker}(g) \), i.e. \( g \) is \( k \)-normal.

(c) Assume that \( f \) and \( g \) are \( k \)-normal. Suppose that \( (g \circ f)(l_1) = (g \circ f)(l_2) \) for some \( l_1, l_2 \in L \). Since \( g \) is \( k \)-normal, we have \( f(l_1) + k_1 = f(l_2) + k_2 \) for some \( k_1, k_2 \in \text{Ker}(g) \). But \( f \) is surjective; whence \( k_1 = f(l_1') \) and \( k_2 = f(l_2') \) for some \( l_1', l_2' \in L \), i.e. \( f(l_1 + l_1') = f(l_2 + l_2') \). Since \( f \) is \( k \)-normal, \( l_1 + l_1' + k_1' = l_2 + l_2' + k_2' \) for some \( k_1', k_2' \in \text{Ker}(f) \). Indeed, \( l_1' + k_1', l_2' + k_2' \in g(f) \). We conclude that \( g \circ f \) is \( k \)-normal.  

There are several notions of exactness for sequences of semimodules. In this paper, we use the relatively new notion introduced by Abuhail:

**Definition 1.19.** ([Abu2014, 2.4]) A sequence

\[
L \xrightarrow{f} M \xrightarrow{g} N
\]  

of left \( S \)-semimodules is **exact**, if \( g \) is \( k \)-normal and \( f(L) = \text{Ker}(g) \).

1.20. We call a sequence of \( S \)-semimodules \( L \xrightarrow{f} M \xrightarrow{g} N \)

- **proper-exact** if \( f(L) = \text{Ker}(g) \) (exact in the sense of Patchkoria [Pat2003]);
- **semi-exact** if \( f(L) = \text{Ker}(g) \) (exact in the sense of Takahashi [Tak1981]);
- **quasi-exact** if \( f(L) = \text{Ker}(g) \) and \( g \) is \( k \)-normal (exact in the sense of Patil and Doere [PD2006]).

1.21. We call a (possibly infinite) sequence of \( S \)-semimodules

\[
\cdots \rightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} M_{i+2} \rightarrow \cdots
\]  

- **chain complex** if \( f_{j+1} \circ f_j = 0 \) for every \( j \);
- **exact** (resp., **proper-exact**, **semi-exact**, **quasi-exact**) if each partial sequence with three terms \( M_j \xrightarrow{f_j} M_{j+1} \xrightarrow{f_{j+1}} M_{j+2} \) is exact (resp., proper-exact, semi-exact, quasi-exact).

A **short exact sequence** (or a **Takahashi extension** [Tak1982b]) of \( S \)-semimodules is an exact sequence of the form

\[
0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0
\]

**Remark 1.22.** In the sequence (2), the inclusion \( f(L) \subseteq \text{Ker}(g) \) forces \( f(L) \subseteq \overline{f(L)} \subseteq \text{Ker}(g) \), whence the assumption \( f(L) = \text{Ker}(g) \) guarantees that \( f(L) = \overline{f(L)} \), i.e. \( f \) is \( i \)-normal. So, the definition puts conditions on \( f \) and \( g \) that are dual to each other (in some sense).

The following result shows some of the advantages of the Abuhail’s definition of exact sequences over the previous ones:

**Lemma 1.23.** Let \( L, M \) and \( N \) be \( S \)-semimodules.
(1) $0 \rightarrow L \xrightarrow{f} M$ is exact if and only if $f$ is injective.

(2) $M \xrightarrow{g} N \rightarrow 0$ is exact if and only if $g$ is surjective.

(3) $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N$ is semi-exact and $f$ is normal if and only if $L \cong \text{Ker}(g)$.

(4) $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N$ is exact if and only if $L \cong \text{Ker}(g)$ and $g$ is $k$-normal.

(5) $L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$ is semi-exact and $g$ is normal if and only if $N \cong M / f(L)$.

(6) $L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$ is exact if and only if $N \cong M / f(L)$ and $f$ is $i$-normal.

(7) $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$ is exact if and only if $L \cong \text{Ker}(g)$ and $N \cong M / L$.

Corollary 1.24. The following assertions are equivalent:

(1) $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$ is an exact sequence of $S$-semimodules;

(2) $L \cong \text{Ker}(g)$ and $N \cong M / f(L)$;

(3) $f$ is injective, $f(L) = \text{Ker}(g)$, $g$ is surjective and $(k)$-normal.

In this case, $f$ and $g$ are normal morphisms.

Remark 1.25. A morphism of semimodules $\gamma : X \rightarrow Y$ is an isomorphism if and only if $0 \rightarrow X \xrightarrow{\gamma} Y \rightarrow 0$ is exact if and only if $\gamma$ is a normal bimorphism (i.e. $\gamma$ is a normal monomorphism and a normal epimorphism). The assumption on $\gamma$ to be normal cannot be removed here. For example, the embedding $\iota : \mathbb{Z}^+ \rightarrow \mathbb{Z}$ is a bimorphism of commutative monoids ($\mathbb{Z}^+$-semimodules) which is not an isomorphism. Notice that $\iota$ is not $i$-normal; in fact $\iota(\mathbb{Z}^+) = \mathbb{Z}$. 

Remark 1.26. An $S$-linear map is a monomorphism if and only if it is injective. Every surjective $S$-linear map is an epimorphism. The converse is not true in general.

Example 1.27. The embedding $\iota : \mathbb{Z}^+ \rightarrow \mathbb{Z}$ is a monoid epimorphism as $(f \circ \iota)(1_{\mathbb{Z}^+}) = (g \circ \iota)(1_{\mathbb{Z}^+})$ implies $f(1_{\mathbb{Z}}) = g(1_{\mathbb{Z}})$ and $f = g$ for every monoid morphisms $f, g : \mathbb{Z} \rightarrow M$. However, it is clear that $\iota$ is not surjective.

Lemma 1.28. (Compare with [Tak1981, Proposition 4.3.]) Let $\gamma : X \rightarrow Y$ be a morphism of $S$-semimodules.

(1) The sequence

$$0 \rightarrow \text{Ker}(\gamma) \xrightarrow{\text{ker}(\gamma)} X \xrightarrow{\gamma} Y \xrightarrow{\text{coker}(\gamma)} \text{Coker}(\gamma) \rightarrow 0$$

(4)

with canonical $S$-linear maps is semi-exact. Moreover, (4) is exact if and only if $\gamma$ is normal.
We have two exact sequences
\[ 0 \to \gamma(X) \xrightarrow{\ker(coker(\gamma))} Y \xrightarrow{coker(\gamma)} Y/\gamma(X) \to 0. \]
and
\[ 0 \to \ker(\gamma) \xrightarrow{\gamma} X \xrightarrow{coker(\ker(\gamma))} X/\ker(\gamma) \to 0. \]

Corollary 1.29. (Compare with [Tak1981, Proposition 4.8.]) Let \( M \) be an \( S \)-semimodule.

1. Let \( \rho \) an \( S \)-congruence relation on \( M \) and consider the sequence of \( S \)-semimodules
\[ 0 \to \Ker(\pi_\rho) \xrightarrow{1_\rho} M \xrightarrow{\rho} M/\rho \to 0. \]
   (a) \( 0 \to \Ker(\pi_\rho) \xrightarrow{1_\rho} M \xrightarrow{\rho} M/\rho \to 0 \) is exact.
   (b) \( M/\rho = \Coker(1_\rho) \).

2. Let \( L \) be an \( S \)-subsemimodule of \( M \).
   (a) The sequence \( 0 \to L \xrightarrow{1} M \xrightarrow{\pi_L} M/L \to 0 \) is semi-exact.
   (b) \( 0 \to L \xrightarrow{1} M \xrightarrow{\pi_L} M/L \to 0 \) is exact.
   (c) The following assertions are equivalent:
       i. \( 0 \to L \xrightarrow{1} M \xrightarrow{\pi_L} M/L \to 0 \) is exact;
       ii. \( L = \Ker(\pi_L) \);
       iii. \( 0 \to L \xrightarrow{1} L \to 0 \) is exact;
       iv. \( L \) is a subtractive subsemimodule.

Proposition 1.30. (cf., [Bor1994, Proposition 3.2.2]) Let \( \mathcal{C}, \mathcal{D} \) be arbitrary categories and \( \mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E} \) be functors such that \((F, G)\) is an adjoint pair.

1. \( F \) preserves all colimits which turn out to exist in \( \mathcal{C} \).
2. \( G \) preserves all limits which turn out to exist in \( \mathcal{D} \).

Corollary 1.31. Let \( S, T \) be semirings and \( \tau F_S \) a \((T, S)\)-bisemimodule. The covariant functor \( \text{Hom}_T(F, -): \tau SM \to sSM \) preserves all limits.

1. For every family of left \( T \)-semimodules \( \{Y_\lambda\}_{\lambda} \), we have a canonical isomorphism of left \( S \)-semimodules
\[ \text{Hom}_T(F, \prod_{\lambda \in \Lambda} Y_\lambda) \cong \prod_{\lambda \in \Lambda} \text{Hom}_T(F, Y_\lambda). \]
(2) For any inverse system of left $T$-semimodules $(X_j, \{f_{jj'}\})_J$, we have an isomorphism of left $S$-semimodules
\[ \text{Hom}_T(F, \lim_{\leftarrow} X_j) \cong \lim_{\leftarrow} \text{Hom}_T(F, X_j). \]

(3) $\text{Hom}_T(F, -)$ preserves equalizers;

(4) $\text{Hom}_T(F, -)$ preserves kernels.

**Proof.** The proof can be obtained as a direct consequence of Proposition 1.30 and the fact that $(F \otimes_S -, \text{Hom}_T(F, -))$ is an adjoint pair of covariant functors [KN2011].

Corollary 1.31 allows us to improve [Tak1982a, Theorem 2.6].

**Proposition 1.32.** Let $G_S$ be $(T, S)$-bisemimodule and consider the functor $\text{Hom}_T(G, -) : \tau \text{SM} \to \sigma \text{SM}$. Let
\[ 0 \to L \overset{f}{\to} M \overset{g}{\to} N \] (5)
be a sequence of left $T$-semimodules and consider the following sequence of left $S$-semimodules
\[ 0 \to \text{Hom}_T(G, L) \overset{(G,f)}{\to} \text{Hom}_T(G, M) \overset{(G,g)}{\to} \text{Hom}_T(G, N). \] (6)

(1) If the sequence $0 \to L \overset{f}{\to} M$ is exact and $f$ is normal, then
\[ 0 \to \text{Hom}_T(G, L) \overset{(G,f)}{\to} \text{Hom}_T(G, M) \]
is exact and $(G,f)$ is normal.

(2) If (5) is semi-exact and $f$ is normal, then (6) is proper exact (semi-exact) and $(G,f)$ is normal.

(3) If (5) is exact and $\text{Hom}_T(G, -)$ preserves $k$-normal morphisms, then (6) is exact.

**Proof.** (1) The following implications are obvious: $0 \to L \overset{f}{\to} M$ is exact $\implies$ $f$ is injective $\implies$ $(G,f)$ is injective $\implies$ $0 \to \text{Hom}_T(G, L) \overset{(G,f)}{\to} \text{Hom}_T(G, M)$ is exact. Assume that $f$ is normal and consider the short exact sequence of $S$-semimodules
\[ 0 \to L \overset{f}{\to} M \overset{\pi_L}{\to} M/L \to 0. \]
Notice that $L = \text{Ker}(\pi_L)$ by Lemma 1.23. By Corollary 1.31, $\text{Hom}_T(G, -)$ preserves kernels and so $(G,f) = \ker(G, \pi_L)$ whence normal.

(2) Apply Lemma 1.23 (3): The semi-exactness of (5) and the normality of $f$ are equivalent to $L \cong \text{Ker}(g)$. Since $\text{Hom}_T(G, -)$ preserves kernels, we deduce that $\text{Hom}_T(G, L) = \text{Ker}((G,g))$ which is equivalent to the semi-exactness of (6) and the normality of $(G,f)$. Notice that
\[ (G,f)(\text{Hom}_T(G,L)) = (G,f)(\text{Hom}_T(G,L)) = \text{Ker}(G,g), \]
i.e. (6) is proper exact (whence semi-exact).
1.33. Let $\gamma : T \rightarrow S$ be a morphism of semirings. Then we have an adjoint pair of functors $(F(X), \text{Hom}_T(S, -))$, where $F(X) = X$ with the action $tx = \gamma(t)x$ for all $t \in T$ and $x \in X$ and $(s_1f)(s) = f(ss_1)$ for all $s_1, s \in S$ and $f \in \text{Hom}_T(S, Y)$ for every left $T$-semimodule $Y$. In particular, we have for all $X \in \mathcal{S}_\text{SM}$ and $Y \in \mathcal{T}_\text{SM}$ a natural isomorphism of commutative monoids
\[
\theta_{X,Y} : \text{Hom}_S(X, \text{Hom}_T(S, Y)) \longrightarrow \text{Hom}_T(X, Y), \ f \mapsto [x \mapsto f(x)(1_S)]
\] (7)
with inverse
\[
\phi_{X,Y} : \text{Hom}_T(X, Y) \longrightarrow \text{Hom}_S(X, \text{Hom}_T(S, Y)), \ g \mapsto [x \mapsto [s \mapsto g(sx)]].
\] (8)

2 Pullbacks and Pushouts

Throughout, $(S, +, 0, \cdot, 1)$ is a semiring and, unless otherwise explicitly mentioned, an $S$-module is a left $S$-semimodule. The category of left $S$-semimodules is denoted by $\mathcal{S}_\text{SM}$.

The category $\mathcal{S}_\text{SM}$ of left $S$-semimodules has pullbacks and pushouts.

The pullbacks in $\mathcal{S}_\text{SM}$ are constructed in a way similar to that of pullbacks in the category of modules over a ring.

2.1. ([Tak1982b, 1.7]) Let $f : A \rightarrow C$ and $g : B \rightarrow C$ be morphisms of left $S$-semimodules. The pullback of $(f, g)$ is $(Q; f^*, g^*)$, where
\[
Q^* = \{(a, b) \in A \times B \mid f(a) = g(b)\}
\]
\[
g^* : Q^* \rightarrow A, \ (a, b) \mapsto a;
\]
\[
f^* : Q^* \rightarrow C, \ (a, b) \mapsto b,
\]
\[
\begin{tikzpicture}
    \node (A) at (0,0) {$A$};
    \node (B) at (-1,-1) {$B$};
    \node (C) at (1,-1) {$C$};
    \node (Q) at (0,0) {$Q$};
    \node (Q*) at (0,1) {$Q^*$};
    \draw[->] (A) -- (B) node[midway,above] {$g$};
    \draw[->] (B) -- (C) node[midway,above] {$g^*$};
    \draw[->] (A) -- (Q) node[midway,above] {$f$};
    \draw[->] (Q*) -- (Q) node[midway,above] {$f^*$};
    \draw[->] (Q) -- (Q*) node[midway,above] {$\varphi$};
\end{tikzpicture}
\] (10)
and whenever $(Q^*; f^*, g^*)$ satisfies $f^* \circ g = g^* \circ f$, there exists a unique $S$-linear map $\varphi : Q^* \rightarrow Q$ such that $f \circ \varphi = f^*$ and $g \circ \varphi = g^*$.

Although the existence of pushouts in the category $\mathcal{S}_\text{SM}$ is guaranteed since this category is a variety in the sense of Universal Algebra (see 1.10), the construction of pushouts in it is much more subtle than the construction of pushouts in the category of modules over a ring (mainly because of the lack of subtraction).
This made some authors consider a special version of pushouts, e.g., Taka-
hashi [Tak1982b] who constructed in the so called C-pushouts, which
coincide with the pushouts in the subcategory of cancellative
semimodules.

2.2. ([Tak1982b, 1.8]) Let \( f : L \rightarrow M \) and \( g : L \rightarrow N \) be
morphisms of left \( S \)-semimodules. Consider the congruence \( \sim \) on
\( M \oplus N \) defined as
\[
(m_1, n_1) \sim (m_2, n_2) \iff \exists l_1, l_2 \in L : m_1 + f(l_1) = m_2 + f(l_2) \text{ and } n_1 + g(l_2) = n_2 + g(l_1).
\] (11)
The C-pushout of \( (f, g) \) is
\[
\begin{align*}
CP & : = (t_M, t_N; (M \oplus N)/ \sim); \\
t_M & : M \rightarrow CP, m \mapsto [(m, 0)]; \\
t_N & : N \rightarrow CP, n \mapsto [(0, n)].
\end{align*}
\] (12)

While the C-pushouts coincide with the natural pushout in the subcategory \( S_{\text{CSM}} \) of
cancellative left semimodules, they fail to have the universal property of pushouts in \( S_{\text{SM}} \).

In what follows, we demonstrate the construction of pushouts in \( S \)-semimodules \( S_{\text{SM}} \). The
constructive proof is the objective of the following theorem which is already known to be true.

**Theorem 2.3.** Let \( f : L \rightarrow M \) and \( g : L \rightarrow N \) be morphisms of left \( S \)-semimodules. Then \( (f, g) \)
has a pushout.

**Proof.** Consider
\[
\mathcal{P} := \{(g', f', P) \mid P \in S_{\text{SM}}, g' : M \rightarrow P, f' : N \rightarrow P, g' \circ f = f' \circ g, \\
\pi_{(g', f')} : M \oplus N \rightarrow P, (m, n) \mapsto g'(m) + f'(n) \text{ is surjective}\}.
\]

Notice that \( \mathcal{P} \) is not empty as \((0, 0, 0) \in \mathcal{P}\).

Define a relation \( \leq \) on \( \mathcal{P} \) as \((\tilde{g}, \tilde{f}, U) \leq (f', \tilde{g}', P)\) if there exists an \( S \)-linear map \( \alpha : P \rightarrow U \) such that \( \alpha \circ \pi_{(g', f')} = \pi_{(\tilde{g}, \tilde{f})} \), i.e. the following diagram is commutative
\[
\begin{array}{ccc}
M \oplus N & \xrightarrow{\pi_{(g', f')}} & P \\
\pi_{(\tilde{g}, \tilde{f})} \downarrow & & \downarrow \alpha \\
U & \xrightarrow{\rho} & P
\end{array}
\]

**Step I:** \( \mathcal{P} \) has a largest element \( (\pi_M, \pi_N, P) \), where
\[
\begin{align*}
P & : = (M \oplus N)/\rho, \\
(m_1, n_1)\rho (m_2, n_2) & \iff g_\lambda (m_1) + f_\lambda (n_1) = g_\lambda (m_2) + f_\lambda (n_2) \quad \forall (g_\lambda, f_\lambda, P_\lambda) \in \mathcal{P}; \\
\pi_M & : M \rightarrow (M \oplus N)/\rho, m \mapsto [(m, 0)]; \\
\pi_N & : N \rightarrow (M \oplus N)/\rho, n \mapsto [(0, n)].
\end{align*}
\]
• Notice that \((\pi_M, \pi_N, P) \in \mathcal{P}\): for any \(l \in L\), we have for any \((g_\lambda, f_\lambda, P_\lambda) \in \mathcal{P}\):

\[
(g_\lambda \circ f)(l) + f_\lambda(0_N) = (g_\lambda \circ f)(l) = (f_\lambda \circ g)(l) = g_\lambda(0_M) + (f_\lambda \circ g)(l)
\]

whence (by the definition of \(\rho\)):

\[
(\pi_M \circ f)(l) = [(f(l), 0)]_\rho = [(0, g(l))_\rho = (\pi_N \circ g)(l).
\]

• For every \((g_\lambda, f_\lambda, P_\lambda) \in \mathcal{P}\), consider the \(S\)-linear map

\[
\alpha_\lambda : P \longrightarrow P_\lambda, \ [(m, n)]_\rho \mapsto g_\lambda(m) + f_\lambda(n)
\]

Notice that \(\alpha_\lambda\) is well defined: if \([(m_1, n_1)]_\rho = [(m_2, n_2)]_\rho\), then it follows by the definition of \(\rho\) that

\[
\alpha([(m_1, n_1)]_\rho) = g_\lambda(m_1) + f_\lambda(n_1) = g_\lambda(m_2) + f_\lambda(n_2) = \alpha([(m_2, n_2)]_\rho).
\]

Moreover, the following diagram

\[
\begin{array}{ccc}
M \oplus N & \xrightarrow{\alpha_\lambda} & P \\
\pi(g_\lambda \circ f_\lambda) & \downarrow \alpha_\lambda & \\
\pi_\lambda & \end{array}
\]

is commutative: indeed, for all \((m, n) \in M \oplus N\) we have

\[
(\alpha_\lambda \circ \pi_\lambda)((m, n)) = \alpha_\lambda[(m, n)]_\rho = g_\lambda(m) + f_\lambda(n) = \pi(g_\lambda \circ f_\lambda)(m, n).
\]

**Step II:** A largest element \((g', f'; P)\) of \(\mathcal{P}\) is a pushout of \((f, g)\). By the definition of \(\mathcal{P}\), we have \(g' \circ f = f' \circ g\). So it remains to prove the it has the universal property of pushouts.

• Let \(Q\) be a left \(S\)-semimodule along with \(S\)-linear maps \(g^*: M \rightarrow Q\) and \(f^*: N \rightarrow Q\) satisfying \(g^* \circ f = f^* \circ g\). Since \(\pi(g', f')\) is surjective, there exists for each \(p \in P\) some \((m, n) \in M \oplus N\), such that \(p = g'(m) + f'(n)\). Define

\[
\phi : P \rightarrow Q, \ p \mapsto g^*(m) + f^*(n).
\]
\[ (m, n) \omega (m', n') \text{ if } g^* (m) + f^* (n) = g^* (m') + f^* (n'). \]

Clearly, \( \omega \) is a congruence. Let
\[
\pi_M^\omega : M \to (M \oplus N)/\omega, \quad \pi_N^\omega : N \to (M \oplus N)/\omega
\]
be the canonical S-linear maps, and define
\[
\pi_\omega : M \oplus N \to (M \oplus N)/\omega, \quad (m, n) \mapsto [(m, n)]_\omega;
\]
\[
h : (M \oplus N)/\omega \to Q, \quad [(m, n)] \mapsto g^*(m) + f^*(n).
\]

Notice that \( h \) is well defined by the definition of \( \omega \). Then \( (\pi_M^\omega, \pi_N^\omega, (M \oplus N)/\omega) \in \mathcal{P} \). Since \((g', f', P)\) is, by assumption, a largest element in \( \mathcal{P} \), there exists \( \alpha : P \to (M \oplus N)/\omega \) such that \( \alpha \circ \pi_{(g', f')} = \pi_\omega \). It follows that
\[
\varphi(g'(m_1) + f'(n_1)) = g^*(m_1) + f^*(n_1) = h([(m_1, n_1)]_\omega)
\]
\[
= (h \circ \pi_\omega)(m_1, n_1) = (\alpha \circ \pi_{(g', f')})(m_1, n_1)
\]
\[
= \alpha(g'(m_1) + f'(n_1)) = \alpha(g'(m_2) + f'(n_2))
\]
\[
= (\alpha \circ \pi_{(g', f')})(m_2, n_2) = (h \circ \pi_\omega)(m_2, n_2)
\]
\[
= h([(m_2, n_2)]_\omega) = g^*(m_2) + f^*(n_2)
\]
\[
= \varphi(g'(m_2) + f'(n_2)).
\]

Hence \( \varphi \) is well defined. \( \blacksquare \)

**Corollary 2.4.** Let \( f : L \to M \) and \( g : L \to N \) be morphisms of left S-semimodules. There exists a congruence relation \( \rho \) on \( M \oplus N \) such that
\[
(g', f'; (M \oplus N)/\rho), \quad g'(m) := [(m, 0)]_\rho, \quad f'(n) := [(0, n)]_\rho
\]
is a pushout of \((f, g)\).

**Proof.** Let \((g^*, f^*, P)\) be a largest element in the poset \((\mathcal{P}, \leq)\) in the proof of Theorem 2.3. Then \((g^*, f^*; P)\) is a pushout and there is an surjective map
\[
\pi : M \oplus N \to P, \quad (m, n) \mapsto g^*(m) + f^*(n).
\]

Consider the congruence relation \( \rho := \equiv_\pi \) and define
\[
g' : M \to (M \oplus N)/\rho, \quad m \mapsto [(m, 0)]_\rho
\]
\[
f' : N \to (M \oplus N)/\rho, \quad n \mapsto [(0, n)]_\rho.
\]

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For every \( l \in L \), we have
\[
(g' \circ f)(l) = [(f(l), 0)]_\rho = [(0, g(l))]_\rho = (f' \circ g)(l).
\]
The middle equality follows since
\[
\pi((f(l), 0)) = (g^* \circ f)(l) + f^*(0) = (f' \circ g)(l) + 0 = g^*(0) + (f' \circ g)(l) = \pi((0, g(l)))
\]
With the canonical map \( \pi_\rho : M \oplus N \rightarrow (M \oplus N)/\rho \), we have \((g', f', (M \oplus N)/\rho) \in \mathcal{P}\). Moreover \( P \leq (M \oplus N)/\rho \) noticing that
\[
\alpha : (M \oplus N)/\rho \longrightarrow P, \ [m,n]_\rho \mapsto g^*(m) + f^*(n)
\]
is \( S \)-linear such that \( \alpha \pi_\rho = \pi_{(g^*, f^*)} \). Since \((g^*, f^*, P)\) is a largest element in \( \mathcal{P} \), \((g', f', (M \oplus N)/\rho)\) is also a largest element in \( \mathcal{P} \). Thus \((g', f'; (M \oplus N)/\rho)\) is a pushout of \((f, g)\).

**Lemma 2.5.** Let \((g', f'; P)\) be a pushout of the morphisms of left \( S \)-semimodules \( f : L \rightarrow M \) and \( g : L \rightarrow N \).

1. If \( f \) is surjective, then \( f' \) is surjective.
2. If \( f \) is i-normal (i.e. \( f(L) \subseteq M \) is subtractive), then \( f' \) is i-normal (i.e. \( f'(N) \subseteq P \) is subtractive).
3. If \( f \) is a normal epimorphism, then \( f' \) is a normal epimorphism.
4. If \( f \) is injective and \( g \) is a normal epimorphism, then \( f' \) is injective.

\[
\begin{array}{ccc}
L & \overset{f}{\longrightarrow} & M \\
\downarrow{g} & & \downarrow{g'} \\
N & \overset{f'}{\longrightarrow} & P
\end{array}
\]

**Proof.** Let \((g', f'; P)\) be a pushout of \((f, g)\).

1. Let \( p \in P \). Since \( \pi_{(g', f')} \) is surjective, there exists \((m, n) \in M \oplus N \) such that \( p = \pi_{(g', f')}(m, n) = g'(m) + f'(n) \). Since \( f \) surjective, there exists \( l \in L \) such that \( f(l) = m \). Consider \( g(l) + n \in N \). It follows that
\[
f'(g(l) + n) = (f' \circ g)(l) + f'(n) = (g' \circ f)(l) + f'(n) = g'(m) + f'(n) = p.
\]
(2) Let \( p \in P \) be such that \( p + f'(n_1) = f'(n_2) \) for some \( n_1, n_2 \in N \). Pick \( (m, n) \in M \oplus N \) such that \( p = \pi_{(g', f')}((m, n)) = g'(m) + f'(n) \). Thus \( g'(m) + f'(n + n_1) = f'(n_2) \).

Let \( \varphi \) be the map from \( P \) to the \( C \)-pushout \( Q \) such that \( \varphi \circ g' = g^* \) and \( \varphi \circ f' = f^* \). Then

\[
[(m, n + n_1)]_\sim = \varphi(g'(m) + f'(n + n_1)) = \varphi(f'(n_2)) = [(0, n_2)]_\sim.
\]

By the definition of the congruence relation \( \sim \) (11), there exist \( l_1, l_2 \in L \) such that \( m + f(l_1) = f(l_2) \) and \( n + n_1 + g(l_2) = n_2 + g(l_1) \). Since \( f(L) \subseteq M \) is subtractive, \( m = f(l) \) for some \( l \in L \). Then we have

\[
p = g'(m) + f'(n) = (g' \circ f)(l) + f'(n) = (f' \circ g)(l) + f'(n) = f'(g(l) + n).
\]

It follows that \( f'(N) \subseteq P \) is subtractive.

(3) Without loss of generality, let the pushout be \( P = (g', f'; (M \oplus N)/\rho) \) for some congruence relation \( \rho \) on \( M \oplus N \) and \( g', f' \) are the canonical maps (see Corollary 2.4). Since \( f \) is surjective, it follows by (1) that \( f' \) is surjective as well.

**Step I:** Consider the canonical \( S \)-linear map

\[
f^* : N \to N/Ker(f').
\]

Let \( m \in M \) and pick \( l \in L \) such that \( m = f(l) \). Define

\[
g^* : M \to N/Ker(f'), \ m \mapsto (f^* \circ g)(l).
\]

**Claim:** \( g^* \) is well-defined.

Suppose that \( f(l) = m = f(l') \) for some \( l, l' \in L \). Since \( f \) is \( k \)-normal, there exist \( l_1, l_2 \in Ker(f) \) such that \( l + l_1 = l' + l_2 \). It follows that \( g(l) + g(l_1) = g(l + l_1) = g(l' + l_2) = g(l') + g(l_2) \) with \( (f' \circ g)(l_1) = (g' \circ f)(l_1) = 0 = (g' \circ f)(l_2) = (f' \circ g)(l_2) \). Thus \( (f^* \circ g)(l) = [g(l)]_{Ker(f')} = [g(l')])_{Ker(f')} = (f^* \circ g)(l') \). Clearly, \( g^* \) is \( S \)-linear and satisfies \( g^* \circ f = f^* \circ g \).

**Step II:** Define

\[
\psi : N/Ker(f') \to P, \ [n]_{Ker(f')} \mapsto [(0, n)]_\rho.
\]

\[\text{Diagram:} \quad L \xrightarrow{f} M \quad \text{and} \quad N \xrightarrow{f'} P \xrightarrow{\varphi} Q.\]

\[\text{Diagram:} \quad L \xrightarrow{f} M \quad \text{and} \quad N \xrightarrow{f'} P \xrightarrow{\psi} N/Ker(f').\]
Claim: $\psi$ is well defined.

Suppose that $[n]_{Ker(f')} = [n']_{Ker(f')}$ for some $n, n' \in N$. It follows that $n + n_1 = n + n_2$ for some $n_1, n_2 \in Ker(f')$. Thus

$$[[0,n]]_\rho = [[0,n]]_\rho + [(0,0)]_\rho = [[0,n]]_\rho + f'(n_1) = [[0,n+n_1]]_\rho = [[0,n']]_\rho.$$

For $m \in M$, pick some $l \in L$ with $f(l) = m$. Then we have

$$([\psi \circ g^*](m)) = ([\psi \circ f^* \circ g](l)) = ([\psi([g(l)])_{Ker(f')})$$

whence $\psi \circ g^* = g'$.  

On the other hand, for every $n \in N$ we have $([\psi \circ f^*](n)) = ([n]_{Ker(f')}) = [0,n]_\rho = f'(n)$, whence $([\psi \circ f^*]) = f'$.

Step III: Since $P$ is a pushout, there exists an $S$-linear map $\varphi : P \to N/Ker(f')$ such that $\varphi \circ g' = g^*$ and $\varphi \circ f' = f^*$. For each $(m,n) \in M \oplus N$ we have

$$([\psi \circ \varphi]][(m,n)]_\rho = ([\psi(\varphi)]([m,0])_\rho + \varphi([0,n])_\rho)$$

$$= ([\psi(\varphi)g')(m) + (\varphi f')(n))$$

$$= (\psi g^*) + (\psi f^*)[n]_{Ker(f')}.$$  

On the other hand, we have for every $n \in N$:

$$([\varphi \circ \psi][n]_{Ker(f')} = $f(0,n)]_\rho = ([\varphi \circ f'])(n) = f^*(n) = [n]_{Ker(f')}.$$

Hence $P \simeq N/Ker(f')$. This implies that $f'$ is $k$-normal (as $f^*$ is obviously $k$-normal).

(4) Without loss of generality, let the pushout be $P = (g', f'; (M \oplus N)/\rho)$ for some congruence relation $\rho$ on $M \oplus N$ and $g', f'$ are the canonical maps (see Corollary 2.4). Let $K := f(Ker(g))$ and consider the canonical projection $\tilde{g} : M \to M/K$. By assumption, $g$ is surjective and so there exists for every $n \in N$ some $l_n \in L$ such that $n = g(l_n)$.

Step I: Define

$$\tilde{f} : N \to M/K, n \mapsto [f(l_n)]_K.$$

Claim: $\tilde{f}$ is well defined.

Suppose that $g(l_n) = n = g(l_n')$. Since $g$ is $k$-normal, there exist $l_1, l_2 \in Ker(g)$ such that $l_n + l_1 = l_n' + l_2$, whence $f(l_n) + f(l_1) = f(l_n') + f(l_2)$, i.e. $[f(l_n)]_K = [f(l_n')]_K$ (recall that
we chose \( K := f(Ker(g)) \).

Notice that for every \( l \in L \), we have: \( (\tilde{f} \circ g)(l) = [f(l)]_K = (\tilde{g} \circ f)(l) \). Since \( P \) is a pushout, there exists an \( S \)-linear map \( \varphi : P \to M/K \) such that \( (\varphi \circ g') = \tilde{g} \) and \( (\varphi \circ f') = \tilde{f} \).

**Step II:** Define
\[
\psi : M/K \to P, \; [m]_K \mapsto [(m, 0)]_\rho.
\]
We claim that \( \psi \) is well defined. Suppose that \([m]_K = [m']_K\) for some \( m, m' \in M \). Then there exist \( l_1, l_2 \in Ker(g) \) such that \( m + f(l_1) = m' + f(l_2) \). It follows that
\[
[(m, 0)]_\rho = g'(m) = g'(m) + (g' \circ f)(l_1) = g'(m + f(l_1)) = g'(m + f(l_2)) = [m', 0]_\rho.
\]

**Step III:** Notice that for every \( n = f(l_n) \in N \) we have:
\[
(\psi \circ \tilde{f})(n) = \psi[f(l_n)] = [(f(l_n), 0)]_\rho = (g' \circ f)(l_n) = f'(n),
\]
and
\[
(\psi \circ \tilde{g})(m) = \psi[m]_K = [(m, 0)]_\rho = g'(m),
\]
thus \( \psi \circ \tilde{f} = f' \) and \( \psi \circ \tilde{g} = g' \). Moreover,
\[
(\varphi \circ \psi)([m]_K) = \varphi[(m, 0)]_\rho = (\varphi \circ g')(m) = \tilde{g}(m) = [m]_K, \text{ and }
\]
\[
(\psi \circ \varphi)([(m, 0)]_\rho) = (\psi \circ \varphi \circ g')(m) = (\psi \circ \tilde{g})(m) = \psi([m]_K) = [(m, 0)]_\rho,
\]
i.e. \( \psi, \varphi \) are \( S \)-linear isomorphisms and \( \psi^{-1} = \varphi \). Moreover, \( M/K \) is a pushout.

**Step IV:** Let \( n, n' \in N \) be such that \( \tilde{f}(n) = \tilde{f}(n') \), i.e. \( [f(l_n)]_K = [f(l_{n'})]_K \). Then there exist \( l_1, l_2 \in Ker(g) \) such that \( f(l_n + l_1) = f(l_{n'}) + f(l_1) = f(l_{n'}) + f(l_2) = f(l_{n'} + l_2) \), whence \( l_n + l_1 = l_{n'} + l_2 \) as \( f \) is injective. It follows that \( n = g(l_n) = g(l_n) + g(l_1) = g(l_n + l_1) = g(l_{n'} + l_2) = g(l_{n'}) + g(l_2) = g(l_{n'}) = n' \). Thus \( \tilde{f} \) is injective. Since \( f' = \psi \circ \tilde{f} \) and \( \psi, \tilde{f} \) are injective, we conclude that \( f' \) is injective as well. \( \blacksquare \)
3 Projective Semimodules

As before, \((S, +, 0, \cdot, 1)\) is a semiring and, unless otherwise explicitly mentioned, an \(S\)-module is a \textbf{left} \(S\)-semimodule. Exact sequences here are in the sense of Abuhla il [Abu2014] (Definition 1.19).

There are several notions of projectivity for a semimodule over a semiring, which coincide if it were a module over a ring. In this Chapter, we consider some of them and clarify the relationships between them, and then investigate the so called \textit{e-projective semimodules} which turn to coincide with the so called \textit{normally projective semimodules} (both notions introduced by Abuhla il [Abu2014-CA, 1.25, 1.24] and called \textit{uniformly projective semimodules}). The terminology \textit{“e-projective”} appeared first in [AIKN2018]).

\textbf{Definition 3.1.} ([AIKN2018]) A left \(S\)-semimodules \(P\) is

\textbf{\(M\)-\textit{e-projective}} (where \(M\) is a left \(S\)-semimodule) if the covariant functor

\[ \text{Hom}_S(P, -) : S\text{SM} \longrightarrow \mathbb{Z}^+\text{SM} \]

transfers every short exact sequence of left \(S\)-semimodules

\[ 0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0 \]

into a short exact sequence of commutative monoids

\[ 0 \longrightarrow \text{Hom}_S(P, L) \xrightarrow{(P,f)} \text{Hom}_S(P, M) \xrightarrow{(P,g)} \text{Hom}_S(P, N) \longrightarrow 0. \]

We say that \(P\) is \textit{\(e\)-projective} if \(P\) is \(M\)-\textit{e-projective} for every left \(S\)-semimodule \(M\).

3.2. Let \(P\) be a left \(S\)-semimodule.

For a left \(S\)-semimodule \(M\), we say that \(P\) is

\textbf{\(M\)-\textit{projective}} [Gol1999, page 195] if for every \textit{surjective} \(S\)-linear map \(f : M \rightarrow N\) and an \(S\)-linear map \(g : P \rightarrow N\), there exists an \(S\)-linear map \(h : P \rightarrow M\) such that \(f \circ h = g\);

\[ M \xrightarrow{f} N \xrightarrow{0} \]

\[ \xymatrix{ & N \ar[rr]^{0} & & \cr M \ar[ur]^{h} & & P \ar[ul]_{g} & } \]

\textbf{\(M\)-\textit{k-projective}} [Alt1996, Definition 6] if for every \textit{normal epimorphism} \(f : M \rightarrow N\) and any \(S\)-linear map \(g : P \rightarrow N\), there exists an \(S\)-linear map \(h : P \rightarrow M\) such that \(f \circ h = g\);

\[ M \xrightarrow{f(\text{normal})} N \xrightarrow{0} \]

\[ \xymatrix{ & N \ar[rr]^{0} & & \cr M \ar[ur]^{h} & & P \ar[ul]_{g} & } \]
normally $M$-projective [Abu2014-CA, 1.25] if for every normal epimorphism $f : M \to N$ and any $S$-linear map $g : P \to N$, there exists an $S$-linear map $h : P \to M$ such that $f \circ h = g$.

and whenever an $S$-linear map $h' : P \to M$ satisfies $f \circ h' = g$, there exist $S$-linear maps $h_1, h_2 : P \to M$ such that $f \circ h_1 = 0 = f \circ h_2$ and $h + h_1 = h' + h_2$.

We say that $P$ is projective (resp., $k$-projective, normally projective) if $P$ is $M$-projective (resp., $M$-$k$-projective, normally $M$-projective) for every left $S$-semimodule $M$.

Proposition 3.3. (cf., [Tak1983, Theorem 1.9], [Gol1999, Proposition 17.16]) A left $S$-semimodule $sP$ is projective if and only if $P$ is a retract of a free left $S$-semimodule.

Remarks 3.4. (1) It is obvious that projective and $e$-projective semimodules are $k$-projective.

(2) Despite being a retract of a free semimodule, a projective semimodule is not necessarily a direct summand of a free semimodule ([Alt2002, Example 2.3]).

Proposition 3.5. Let $P$ be a left $S$-semimodule.

(1) Let $M$ be a left $S$-semimodule. Then $sP$ is $M$-e-projective if and only if $sP$ is normally $M$-projective.

(2) $sP$ is $e$-projective if and only if $sP$ is normally projective.

Proof. We need to prove (1) only.

$(\Longrightarrow)$ Assume that $sP$ is $M$-e-projective. Let $f : M \to N$ be a normal epimorphism and $g : P \to N$ an $S$-linear map. By Lemma 1.23, the sequence

$$0 \longrightarrow \text{Ker}(f) \xrightarrow{t} M \xrightarrow{f} N \longrightarrow 0$$

is a short exact sequence, where $t$ is the canonical embedding. By assumption, the following sequence of commutative monoids

$$0 \longrightarrow \text{Hom}_S(P, \text{Ker}(f)) \xrightarrow{(P_1)} \text{Hom}_S(P, M) \xrightarrow{(P_2)} \text{Hom}_S(P, N) \longrightarrow 0$$

is exact. In particular, $(P, g)$ is surjective and $k$-normal, whence $P$ is normally $M$-projective.

$(\Longleftarrow)$ let $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ be a short exact sequence of left $S$-semimodules and consider the induces sequences of commutative monoids

$$0 \longrightarrow \text{Hom}_S(P, L) \xrightarrow{(P_1)} \text{Hom}_S(P, M) \xrightarrow{(P_2)} \text{Hom}_S(P, N) \longrightarrow 0.$$
By Proposition 1.32, \((P,f)\) is a normal monomorphism and \(\text{Im}(P,f) = \text{Ker}(P,g)\). By assumption, \((P,g)\) is a normal epimorphism, whence the induced sequence of commutative monoids is exact.\(\blacksquare\)

Following an observation by H. Al-Thani made in [Alt1995, theorem 4], we provide a detailed proof that every projective \(S\)-semimodule is \(e\)-projective.

**Proposition 3.6.** Every projective left \(S\)-semimodule is \(e\)-projective.

**Proof.** Let \(sP\) be projective. Assume that \(M \xrightarrow{g} N \longrightarrow 0\) is a normal epimorphism of left \(S\)-semimodules, and \(\alpha \in \text{Hom}_S(P,N)\). Since \(sP\) is \(M\)-projective,

\[
\text{Hom}_S(P,M) \xrightarrow{(P,g)} \text{Hom}_S(P,N) \longrightarrow 0
\]

is surjective, \(i.e.\) there exists \(\beta \in \text{Hom}_S(P,M)\) such that \(g \circ \beta = \alpha\).

By Proposition 3.5, it is enough to prove that \((P,g)\) is \(k\)-normal.

Suppose that \((P,g)(\beta) = (P,g)(\beta')\) for some \(\beta, \beta' \in \text{Hom}_S(P,M)\), \(i.e.\) \(g \circ \beta = g \circ \beta'\). Since \(sP\) is projective, \(P\) is a retract of a free left \(S\)-semimodule, \(i.e.\) there exists an index set \(\Lambda\) and a surjective \(S\)-linear map \(\theta : S^{(\Lambda)} \longrightarrow P\) as well as an injective \(S\)-linear map \(\psi : P \longrightarrow S^{(\Lambda)}\) such that \(\theta \circ \psi = id_P\). Notice that \(g \circ \beta \circ \theta = g \circ \beta' \circ \theta\). For every \(\lambda \in \Lambda\), and since \(g\) is \(k\)-normal, there exist \(m_\lambda, m'_\lambda \in \text{Ker}(g)\) such that \((\beta \circ \theta)(\lambda) + m_\lambda = (\beta' \circ \theta)(\lambda) + m'_\lambda\). Let \(\gamma, \gamma' \in \text{Hom}_S(S^{(\Lambda)},M)\) be the unique \(S\)-linear maps with \(\gamma(\lambda) = m_\lambda\) and \(\gamma'(\lambda) = m'_\lambda\) for each \(\lambda \in \Lambda\) (they exist and are unique since \(\Lambda\) is a basis for \(S^{(\Lambda)}\)). It follows that

\[
g \circ (\gamma \circ \psi) = (g \circ \gamma) \circ \psi = 0 = (g \circ \gamma') \circ \psi = g \circ (\gamma' \circ \psi),
\]

\(i.e.\) \(\gamma \circ \psi, \gamma' \circ \psi \in \text{Ker}(g(P))\). Moreover, for any \(\lambda \in \Lambda\) we have

\[
(\beta \circ \theta + \gamma)(\lambda) = (\beta \circ \theta)(\lambda) + m_\lambda = (\beta' \circ \theta)(\lambda) + m'_\lambda = (\beta' \circ \theta + \gamma')(\lambda),
\]

whence \(\beta \circ \theta + \gamma = \beta' \circ \theta + \gamma'\). It follows that

\[
\begin{align*}
\beta + \gamma \circ \psi &= \beta \circ id_P + \gamma \circ \psi = \beta \circ (\theta \circ \psi) + \gamma \circ \psi \\
&= (\beta \circ \theta + \gamma) \circ \psi = (\beta' \circ \theta + \gamma') \circ \psi \\
&= \beta' \circ (\theta \circ \psi) + \gamma' \circ \psi = \beta' \circ id_P + \gamma' \circ \psi \\
&= \beta' + \gamma' \circ \psi. \blacksquare
\end{align*}
\]

The following example shows that the class of \(S\)-\(e\)-projective left \(S\)-semimodules is strictly larger than that of \(S\)-projective left \(S\)-semimodules.

**Example 3.7.** Consider the semiring \(S := \mathbb{Q}^+\) of non-negative rational numbers, with the usual addition and multiplication. Consider the Boolean algebra \(\mathbb{B} = \{0, 1\}\) as an \(S\)-semimodule with \(s \cdot 1 = 1 \Leftrightarrow s \in S \setminus \{0\}\). Then \(s\mathbb{B}\) is \(S\)-\(e\)-projective but not \(S\)-projective.
**Proof.** Consider the $S$-linear map

$$f : S \to B, \quad s \mapsto \begin{cases} 1, & s \neq 0 \\ 0, & s = 0 \end{cases}$$

Notice that $f$ is not $k$-normal: $\text{Ker}(f) = \{0\}$, $f(1) = 1 = f(2)$, and $1 + 0 \neq 2 + 0$.

Since there is no surjective $S$-linear map from $B$ to $S$, there is no isomorphism from $B$ to $S$. Since $S$ is an ideal-simple $S$-semimodule, $\text{Hom}_S(B, S) = \{0\}$ by Lemma 1.9. Since the following diagram

$$
\begin{array}{ccc}
S & \xrightarrow{f} & B \\
\downarrow{0} & & \downarrow{id_{B}} \\
0 & & B
\end{array}
$$

cannot be completed commutatively, $B$ is not $S$-projective.

Let $N$ be an $S$-semimodule and $f : S \to N$ be a normal $S$-epimorphism. If $f = 0$, then $N = f(S) = 0$, which implies that every $S$-linear map $g : B \to N$ is the zero morphism and by choosing $S$-linear map $0 = h : B \to S$ we have $g = f \circ h$.

If $f \neq 0$, then $f(1) \neq 0$. For every $s \in S \setminus \{0\}$, we have $0 \neq f(1) = f(s^{-1}s) = s^{-1}f(s)$, whence $f(s) \neq 0$. Thus $\text{Ker}(f) = \{0\}$. If $f(s) = f(t)$, then $s + k_1 = t + k_2$ for some $k_1, k_2 \in \text{Ker}(f) = \{0\}$, thus $s = t$. Hence, $f$ is an $S$-isomorphism. Since $S$ is not $S$-isomorphic to $B$, $N$ is not $S$-isomorphic to $B$. Since $S$ is ideal-simple, $N$ is ideal-simple. Thus $\text{Hom}_S(B, N) = \{0\}$ and $B$ is $S$-e-projective.■

**3.8.** We call a short exact sequence of $S$-semimodules

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \tag{14}$$

*left splitting* if there exists $f' \in \text{Hom}_S(B, A)$ such that $f' \circ f = id_A$;

*right splitting* if there exists $g' \in \text{Hom}_S(C, B)$ such that $g \circ g' = id_C$.

We say that (14) splits or is splitting if it is left splitting and right splitting.

Left splitting of short exact sequences of semimodules is not equivalent to right splitting.

**Example 3.9.** Consider the semiring $B(3, 1) = (\{0, 1, 2\}, \oplus, 0, \otimes, 1)$, where

$$1 \oplus 2 = 1, \quad 2 \oplus 2 = 0, \quad 2 \otimes 2 = 0;$$

see [Gol1999, Example 1.8]. Then we have a short exact sequence of commutative monoids

$$0 \longrightarrow \{0, 2\} \xrightarrow{t} B(3, 1) \xrightarrow{\pi} \mathbb{Z}_2 \longrightarrow 0, \tag{15}$$

where $t$ is the canonical embedding and $\pi$ is the canonical projection. The sequence (15) is exact since $\{0, 2\}$ is subtractive and $B(3, 1)/\{0, 2\} \cong \mathbb{Z}^+\mathbb{Z}_2$ (see Lemma 1.23). Consider

$$f : B(3, 1) \longrightarrow \{0, 2\}, \quad x \mapsto \begin{cases} 2, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

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and notice that \( f \circ \iota = id_{\{0,2\}} \), i.e. (15) is left splitting. On the other hand, \( \text{Hom}_{\mathbb{Z}_2^+}(\mathbb{Z}_2, B(3, 1)) = \{0\} \). Consequently, (15) is not right splitting.

**Proposition 3.10.** A left \( S \)-semimodule \( sP \) is \( k \)-projective if and only if every short exact sequence of left \( S \)-semimodules

\[
0 \to A \xrightarrow{f} B \xrightarrow{g} P \to 0
\]

is right-splitting.

**Proof.** (\( \Rightarrow \)) Let \( P \) be \( k \)-projective and \( 0 \to L \xrightarrow{f} M \xrightarrow{g} P \to 0 \) be a short exact sequence. In particular, \( g \) is surjective and \( k \)-normal. Consider, \( id_P : P \to P \). Since \( sP \) is \( k \)-projective, there exists an \( S \)-linear map \( g' : P \to M \) such that the following diagram

\[
\begin{array}{ccc}
P & \xrightarrow{g'} & M \\
\downarrow{id_P} & & \downarrow{g} \\
P & & P
\end{array}
\]

is commutative, i.e. \( g \circ g' = id_P \).

(\( \Leftarrow \)) Let \( M \xrightarrow{g} N \to 0 \) be a normal surjective \( S \)-linear map and \( h : P \to N \) be a morphism of left \( S \)-semimodules. Consider the pullback of \( g \) and \( h \):

\[
Q := \{(p, m) \in P \times M \mid h(p) = g(m)\}
\]

and the following commutative diagram

\[
\begin{array}{ccc}
Q & \xrightarrow{\pi_P} & P \\
\downarrow{\pi_M} & & \downarrow{h} \\
M & \xrightarrow{g} & N
\end{array}
\]

where \( \pi_P \) and \( \pi_Q \) are the canonical projections. Since \( g \) is surjective, \( h(p) = g(m) \) for some \( m \in M \), i.e. \( (p, m) \in Q \) and indeed, \( p = \pi_P(p, m) \). Hence \( \pi_P \) is surjective. Let \( (p, m), (p, m') \in Q \) so that \( \pi_P(p, m) = \pi_P(p, m') \). Then \( g(m) = h(p) = g(m') \) and there exist \( u, u' \in \text{Ker}(g) \) such that \( m + u = m' + u' \) (since \( g \) is \( k \)-normal). Notice that \( (0, u), (0, v) \in \text{Ker}(\pi_P) \) and \( (p, m) + (0, u) = (p, m + u) = (p, m' + u') = (p, m) + (0 + u') \), i.e. \( \pi_P \) is \( k \)-normal. Hence the sequence

\[
0 \to \text{Ker}(\pi_P) \hookrightarrow Q \xrightarrow{\pi_P} P \to 0
\]
It follows that
and
Notice that for every \( p \in P \), \( \varphi(p) \in Q \), whence \( \varphi(p) = (p, m) \) for some \( m \in M \) with \( h(p) = g(m) \). It follows that

\[
(g \circ (\pi_M \circ \varphi))(p) = g(\pi_M(p, m)) = g(m) = h(p).
\]

(16)

So, \( g \circ (\pi_M \circ \varphi) = h \). Consequently, \( P \) is \( k \)-projective. ■

**Lemma 3.11.** If \( M \) is a left \( S \)-semimodule such that every subtractive subsemimodule is a direct summand, then every left \( S \)-semimodule is \( M \)-\( e \)-projective.

**Proof.** Let \( P \) be a left \( S \)-semimodule and let

\[
f: M \longrightarrow N \longrightarrow 0
\]

be a normal epimorphism and \( g : P \rightarrow N \) be an \( S \)-linear map. Notice that \( \text{Ker}(f) \leq S M \) is a subtractive subsemimodule, whence \( M = \text{Ker}(f) \oplus L \) for some subsemimodule \( L \leq S M \). The row of this following diagram is exact by Lemma 1.23

\[
0 \longrightarrow \text{Ker}(f) \xrightarrow{1} M \xrightarrow{f} N \longrightarrow 0
\]

\[
\downarrow g \downarrow L
\]

\[
P
\]

It follows (see also Remark 1.14(2)) that we have isomorphisms of left \( S \)-semimodules:

\[
N \simeq M/\text{Ker}(f) \simeq L.
\]

Considering the induced isomorphism \( N \overset{g'}{\simeq} L \) and setting \( h := \iota_L \circ g' \circ g : P \rightarrow M \) where \( f \circ \iota_L = id_L \) and \( \iota_L \circ f|_L = id_N \), we have indeed \( f \circ h = g \).

Suppose that also \( h' : P \rightarrow M \) satisfies \( f \circ h' = g \). Consider the projection \( \pi : M \rightarrow \text{Ker}(f) \).

Then \( \varphi := \iota_L \circ g' \circ f + \pi = id_M \): Let \( m \in M \), and write \( m = k + l \) for some unique \( k \in \text{Ker}(f) \) and \( l \in L \), and notice that

\[
\varphi(m) = \varphi(k + l) = (\iota_L \circ g' \circ f + \pi)(k + l) + (\iota_L \circ g' \circ f)(k + l) + \pi(k + l)
\]

\[
= l + k
\]

\[
= m.
\]

Choose \( h_1 := \pi \circ h' : P \rightarrow M \) and \( h_2 = 0 : P \rightarrow M \). Notice that \( f \circ h_1 = f \circ \pi \circ h' = 0 = f \circ h_2 \).

Moreover, we have for each \( p \in P \):

\[
(h + h_1)(p) = h(p) + h_1(p) = (\iota_L \circ g' \circ g)(p) + (\pi \circ h')(p)
\]

\[
= (\iota_L \circ g' \circ f \circ h')(p) + \pi \circ h'(p) = ((\iota_L \circ g' \circ f + \pi) \circ h')(p)
\]

\[
= h'(p) = (h' + 0)(p).
\]

Consequently, \( P \) is \( M \)-\( e \)-projective. ■

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The following two results are relative versions of parts of [AIKN2018, Corollary 3.3]; moreover, we give detailed homological proofs as the ones in [AIKN2018] are compact and categorical.

**Lemma 3.12.** (cf., [AIKN2018, Corollary 3.3])

1. Let $M$ be a left $S$-semimodule. A retract of an $M$-e-projective semimodule is $M$-e-projective.

2. A retract of an e-projective left $S$-semimodule is e-projective.

**Proof.** We only need to prove (1).

Let $P$ be a left $S$-semimodule which is $M$-e-projective and let $sK$ be a retract of $P$ along with a surjective $S$-linear map $\pi_K : P \to K$ and an injective $S$-linear map $t_K : K \to P$ such that $\pi_K \circ t_K = id_K$.

Let $f : M \to N$ be a normal epimorphism and $g : K \to N$ an $S$-linear map.

Since $P$ is e-projective, there exists an $S$-linear map $h^* : P \to M$ such that $f \circ h^* = g \circ \pi_K$.

Consider $h := h^* \circ t_K : K \to M$.

Then $f \circ h = f \circ (h^* \circ t_K) = g \circ \pi_K \circ t_K = g \circ id_K = g$.

Suppose that $h' : K \to M$ is an $S$-linear map such that $f \circ h' = g$. Since $P$ is $M$-e-projective and $f \circ (h' \circ \pi_K) = (f \circ h') \circ \pi_K = g \circ \pi_K$, there exist $S$-linear maps $h'_1, h'_2 : P \to M$ such that $f \circ h'_1 = 0 = f \circ h'_2$ and $h^* + h'_1 = h' \circ \pi_K + h'_2$. Consider $h_1 := h'_1 \circ t_K$ and $h_2 := h'_2 \circ t_K$.

Then $f \circ h_1 = f \circ h'_1 \circ t_K = 0$, $f \circ h_2 = f \circ h'_2 \circ t_K = 0$, and

$$h + h_1 = h^* \circ t_K + h'_1 \circ t_K = (h^* + h'_1) \circ t_K$$
$$= (h' \circ \pi_K + h'_2) \circ t_K = h' \circ \pi_K \circ t_K + h'_2 \circ t_K$$
$$= h' + h_2.$$

Consequently, $K$ is $M$-e-projective. ■
The following result is a relative version of [AIKN2018, Corollary 3.3 (3)]; moreover, we give a detailed homological proof as the one [AIKN2018] is compact and categorical.

**Proposition 3.13.** Let \( \{ P_i \}_{i \in I} \) be a family of left \( S \)-semimodules and \( M \) a left \( S \)-semimodule. Then \( \bigoplus_{i \in I} P_i \) is \( M \)-e-projective if and only if \( P_i \) is \( M \)-e-projective for each \( i \in I \). The class of e-projective left \( S \)-semimodules is closed under direct sums.

**Proof.** (\( \implies \)) This implication follows by Lemma 3.12.

(\( \impliedby \)) Let \( g : M \to N \) be a normal epimorphism and \( f : \bigoplus_{i \in I} P_i \to N \) be an \( S \)-linear map. For every \( j \in I \), there exists an \( S \)-linear map \( h_j : P_j \to M \) such that \( f \circ t_j = g \circ h_j \), where \( t_j : P_j \to \bigoplus_{i \in I} P_i \) is the canonical embedding.

\[
\begin{array}{ccc}
M & \xrightarrow{g} & N \\
\downarrow h & & \downarrow f \\
\bigoplus_{i \in I} P_i & \xrightarrow{\sum_{i \in I} h_i(p_i)} & 0 \\
\downarrow t_j & & \\
P_j & & \\
\end{array}
\]

By the Universal Property of Direct Coproducts, there exists a unique \( S \)-linear map \( h : \bigoplus_{i \in I} P_i \to M \) such that \( h \circ t_j = h_j \) for every \( j \in I \), i.e.

\[
h : \bigoplus_{i \in I} P_i \to M, \quad \sum_{i \in I} p_i \mapsto \sum_{i \in I} h_i(p_i).
\]

Notice that \( h \) is \( S \)-linear and well defined since the sum \( \sum_{i \in I} p_i \) is finite (all but finitely many of the coordinates are zero). Moreover, we have

\[
(g \circ h)(\sum_{i \in I} p_i) = g(\sum_{i \in I} h_i(p_i)) = \sum_{i \in I} (g \circ h_i)(p_i) = \sum_{i \in I} (f \circ t_i)(p_i) = f(\sum_{i \in I} t_i(p_i)) = f(\sum_{i \in I} p_i).
\]

Suppose that \( h' : \bigoplus_{i \in I} P_i \to M \) is an \( S \)-linear map with \( g \circ h' = f \). Then \( f \circ t_j = g \circ h' \circ t_j \) for every \( j \in I \). Since \( P_j \) is e-projective for every \( j \in I \), there exist \( S \)-linear maps \( \bar{h}_j, \bar{h}_j : P_j \to M \) such that \( g \circ \bar{h}_j = 0 = g \circ \bar{h}_j \) and \( h_j + \bar{h}_j = h'_j + \bar{h}_j \).

By the Universal Property of Direct Coproducts, there exist \( S \)-linear maps

\[
\bar{h}, \bar{h} : \bigoplus_{i \in I} P_i \to M, \quad \bar{h}(\sum_{i \in I} p_i) := \sum_{i \in I} \bar{h}_i(p_i) \text{ and } \bar{h}(\sum_{i \in I} p_i) = \sum_{i \in I} \bar{h}_i(p_i).
\]

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Both maps are $S$-linear, and well defined since the sum $\sum_{i \in I} p_i$ is finite (all but finitely many of the coordinates are zero). Moreover, we have

\[
(g \circ \hat{h})(\sum_{i \in I} p_i) = g(\sum_{i \in I} \hat{h}_i(p_i)) = \sum_{i \in I} (g \circ \hat{h}_i)(p_i) = 0;
\]

\[
\hat{h}(\sum_{i \in I} p_i) = g(\sum_{i \in I} \hat{h}_i(p_i)) = \sum_{i \in I} (g \circ \hat{h}_i)(p_i) = 0
\]

and

\[
(h + \hat{h})(\sum_{i \in I} p_i) = h(\sum_{i \in I} p_i) + \hat{h}(\sum_{i \in I} p_i) = \sum_{i \in I} h_i(p_i) + \sum_{i \in I} \hat{h}_i(p_i)
\]

\[
= \sum_{i \in I} (h_i + \hat{h}_i)(p_i) = \sum_{i \in I} (h'_i + \hat{h}_i)(p_i)
\]

\[
= (h' + \hat{h})(\sum_{i \in I} p_i).
\]

Hence $\bigoplus_{i \in I} P_i$ is $M$-e-projective. ■

**Proposition 3.14.** Let $P$ be a left $S$-semimodule. If

\[
0 \longrightarrow K \xrightarrow{1} L \xrightarrow{\pi} M \longrightarrow 0
\]

is an exact sequence of left $S$-semimodules and $P$ is $L$-e-projective, then $P$ is $K$-e-projective and $M$-e-projective.

**Lemma 3.15.**

**Proof.** Assume that $P$ is $L$-e-projective.

- **Claim I:** $P$ is $M$-e-projective. Let $f : M \rightarrow N$ be a normal epimorphism and $g : P \rightarrow N$ an $S$-linear map.

\[
L \xrightarrow{\pi} M \xrightarrow{f} N \longrightarrow 0
\]

Since $\pi$ and $f$ are normal epimorphism, $f \circ \pi$ is a normal epimorphism as well (by Lemma 1.18 (2)(c)). Since $P$ is $L$-e-projective, there exists an $S$-linear map $h : P \rightarrow M$ such that $f \circ \pi \circ h = g$. Then $\pi \circ h : P \rightarrow M$ is an $S$-linear map satisfying $f \circ (\pi \circ h) = g$. 

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Suppose there exists an $S$-linear map $h': P \to M$ such that $f \circ h' = g$. Since $\pi$ is a normal epimorphism and $P$ is $L$-epi-projective, there exists an $S$-linear map $h^*: P \to L$ such that $
olinebreak[4] \pi \circ h^* = h'$.

Moreover, $(f \circ \pi) \circ h^* = f \circ (\pi \circ h^*) = f \circ h' = g$. Since $P$ is $L$-epi-projective, there exist

$S$-linear maps $h_1, h_2 : P \to L$ such that $f \circ \pi \circ h_1 = 0 = f \circ \pi \circ h_2$ and $h + h_1 = h^* + h_2$.

![Diagram](image)

Thus, $\pi \circ h_1, \pi \circ h_2 : P \to M$ are $S$-linear maps such that $f \circ \pi \circ h_1 = 0 = f \circ \pi \circ h_2$. Moreover,

$$
\pi \circ h + \pi \circ h_1 = \pi \circ (h + h_1) = \pi \circ (h^* + h_2) = \pi \circ h^* + \pi \circ h_2 = h' + \pi \circ h_2.
$$

Consequently, $P$ is $M$-epi-projective.

• **Claim II:** $P$ is $K$-epi-projective. Let $f : K \to N$ be a normal $S$-epimorphism and $g : P \to N$ an $S$-linear map. By Corollary 2.4, $(t', f'; Q := (N \oplus L)/\rho)$ is a pushout of $(f, t)$ such that $\rho$ is a congruence relation on $N \oplus L$ and

$$
t' : N \longrightarrow Q, \ n \mapsto [(n, 0)]_{\rho} \text{ and } f' : L \longrightarrow Q, \ l \mapsto [(0, l)]_{\rho}.
$$

![Diagram](image)

Since $t$ is a normal $S$-monomorphism and $f$ is a normal $S$-epimorphism, it follows by Lemma 2.5 (2) & (4) that $t'$ is a normal monomorphism and it follows, by Lemma 2.5 (3), that $f'$ is a normal epimorphism. Since $f'$ is a normal epimorphism and $P$ is $L$-epi-projective, there exists an $S$-linear map $h : P \to L$ such that $f' \circ h = t' \circ g$.

Let $p \in P$. Since $f$ is surjective, there exists $k \in K$ such that $f(k) = g(p)$. Notice that $(f' \circ t)(k) = (t' \circ f)(k) = (t' \circ g)(p) = (f' \circ h)(p)$. Since $f'$ is $k$-normal, there exist $l_1, l_2 \in Ker(f')$ such that $t(k) + l_1 = h(p) + l_2$. 

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Let
\[ CP = (t^*, f^*; (N \oplus L)/\rho^*) \]
be the \( C \)-pushout of \( (f, t) \) (defined in 2.2). Since \( Q \) is a pushout of \( (f, t) \), there exists, by the **Universal Property of Pushouts**, an \( S \)-linear map \( \phi : Q \to (N \oplus L)/\rho^* \) such that \( \phi \circ t' = t^* \) and \( \phi \circ f' = f^* \). Notice that for \( i = 1, 2 \):
\[
[0, l_i]_{\rho^*} = f^*(l_i) = \phi \circ f'(l_i) = \phi(0) = [(0, 0)]_{\rho^*},
\]
and so there exist \( k_{i_1}, k_{i_2} \in K \) such that \( f(k_{i_1}) = f(k_{i_2}) \) and \( l_i + t(k_{i_2}) = t(k_{i_1}) \).

Since \( t \) is a normal monomorphism, \( t(K) \subseteq L \) is subtractive, whence \( l_1, l_2 \in t(K) \), i.e. \( l_1 = t(k_1) \) and \( l_2 = t(k_2) \) for some \( k_1, k_2 \in K \). It follows that \( t(k) + t(k_1) = h(p) + t(k_2) \), we conclude that \( h(p) \in t(K) \) (as \( t \) is a normal monomorphism). Let \( k_p \in K \) be such that \( h(p) = t(k_p) \). Notice that this \( k_p \) is unique since \( t \) is an injective. Therefore
\[
h' : P \longrightarrow K, \ p \mapsto k_p
\]
is well defined. Clearly, \( h' \) is \( S \)-linear. Now, for every \( p \in P \), we have
\[
(t' \circ f \circ h')(p) = (f' \circ (t \circ h'))(p) = (f' \circ t)(k_p) = (f' \circ h')(p) = (t' \circ g)(p),
\]
whence \( (f \circ h')(p) = g(p) \) as \( t' \) is injective.

Suppose that there exists an \( S \)-linear map \( h^* : P \to K \) such that \( f \circ h^* = g \). It follows that \( f' \circ t \circ h^* = t' \circ f \circ h^* = t' \circ g \). Since \( P \) is \( L \)-e-projective, there exist \( S \)-linear maps \( h_1, h_2 : P \to L \) such that \( f' \circ h_1 = 0 = f' \circ h_2 \) and \( h + h_1 = t \circ h^* + h_2 \). Let \( p \in P \). For \( i = 1, 2 \), and since \( h_i(p) \in Ker(f') \), there exists \( k_p \in K \) such that \( h_i(p) = t(k^i_p) \) (which is indeed unique as \( t \) injective). Then we have two well defined maps
\[
h'_1 : P \longrightarrow K, \ p \mapsto k_p^1 \quad \text{and} \quad h'_2 : P \longrightarrow K, \ p \mapsto k_p^2.
\]
which can be easily shown to be \( S \)-linear.

For every \( p \in P \), and for \( i = 1, 2 \) we have
\[
(t' \circ f \circ h'_i)(p) = (t' \circ f)(k^i_p) = (f' \circ t)(k^i_p) = (f' \circ h_i)(p) = 0,
\]
whence \( (f \circ h'_i)(p) = 0 \) as \( t' \) is injective. Moreover, we have
\[
t((h^* + h'_2^2)(p)) = (t \circ h^*)(p) + (t \circ h'_2)(p) = (t \circ h^*)(p) + t(k^2_p)
= (t \circ h^*)(p) + h^2(p) = (t \circ h^* + h_2^2)(p)
= (h + h_1)(p) = h^1(p)
= t(k^1_p) = (t \circ h'_1)(p)
= t((h' + h'_1)(p))
\]
whence \( (h^* + h'_2)(p) = (h' + h'_1)(p) \) as \( t \) is injective. Consequently, \( P \) is \( K \)-e-projective. ■
Example 3.16. Consider the semiring $S := M_2(\mathbb{R}^+)$ and the subtractive left ideal

$$N_1 = \left\{ \begin{bmatrix} a & a \\ b & b \end{bmatrix} \mid a, b \in \mathbb{R}^+ \right\}.$$ 

Then $S/N_1$ is not an $S$-$k$-projective $S$-semimodule, whence not $S$-$e$-projective.

**Proof.** Let $\pi : S \to S/N_1$ be the canonical map and $id_{S/N_1}$ be the identity map of $S/N_1$. Notice that $\pi$ is a normal epimorphism. Consider

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } e_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$ 

Suppose that there exists an $S$-linear map $g : S/N_1 \to S$ such that $\pi g = id_{S/N_1}$. Then $g(\overline{e_1}) \in \pi^{-1}(\overline{e_1})$ and $g(\overline{e_2}) \in \pi^{-1}(\overline{e_2})$. Write $g(\overline{e_1}) = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ for some $p, q, r, s \in \mathbb{R}^+$. Then

$$\begin{bmatrix} p+k & q+k \\ r+l & s+l \end{bmatrix} = \begin{bmatrix} m+1 & m \\ n & n \end{bmatrix}$$

for some $k, l, m, n \in \mathbb{R}^+$, which implies that $r = s$ and $p = q + 1$ as $\mathbb{R}^+$ is cancellative. By relabeling, we have

$$g(\overline{e_1}) = \begin{bmatrix} a+1 & a \\ b & b \end{bmatrix} \text{ and } g(\overline{e_2}) = \begin{bmatrix} c & c \\ d & d+1 \end{bmatrix} \text{ for some } a, b, c, d \in \mathbb{R}^+.$$ 

Let $x := \begin{bmatrix} p & q \\ r & s \end{bmatrix} \in S$. Then

$$x = \begin{bmatrix} p & 0 \\ r & 0 \end{bmatrix} e_1 + \begin{bmatrix} 0 & q \\ 0 & s \end{bmatrix} e_2,$$

which implies that

$$g(\overline{x}) = \begin{bmatrix} p & 0 \\ r & 0 \end{bmatrix} g(\overline{e_1}) + \begin{bmatrix} 0 & q \\ 0 & s \end{bmatrix} g(\overline{e_2}) = \begin{bmatrix} pa+dq+p & pa+dq+q \\ ra+sd+r & ra+sd+s \end{bmatrix}.$$ 

But

$$x = \begin{bmatrix} p & 1 \\ r & 0 \end{bmatrix} e_1 + \begin{bmatrix} 0 & q \\ 1 & s \end{bmatrix} e_2,$$

which implies

$$\begin{bmatrix} pa+dq+p & pa+dq+q \\ ra+sd+r & ra+sd+s \end{bmatrix} = \begin{bmatrix} p & 1 \\ r & 0 \end{bmatrix} g(\overline{e_1}) + \begin{bmatrix} 0 & q \\ 1 & s \end{bmatrix} g(\overline{e_2})$$

$$= \begin{bmatrix} (pa+dq+p)+b & (pa+dq+q)+b \\ (ra+sd+r)+c & (ra+sd+s)+c \end{bmatrix}.$$
whence \( b = 0 = c \) as \( \mathbb{R}^+ \) is cancellative. Thus
\[
g(\overline{a}) = \begin{bmatrix} a + 1 & a \\ 0 & 0 \end{bmatrix}
\quad \text{and} \quad g(\overline{a}) = \begin{bmatrix} 0 & 0 \\ d & d + 1 \end{bmatrix}
\]
for some \( a, d \in \mathbb{R}^+ \).

Let \( y := \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \). Notice that \( \overline{a} = y \), whence
\[
\begin{bmatrix} a + 1 & a \\ 0 & a \end{bmatrix} = g(\overline{a}) = g(y) = \begin{bmatrix} 2a + d + 2 & 2a + d + 1 \\ 0 & 0 \end{bmatrix},
\]
and so \( a = 2a + d + 1 \). Since \( \mathbb{R}^+ \) is cancellative, \( a + d + 1 = 0 \), that is 1 has additive inverse, a contradiction. Hence, there is no such \( S \)-linear map \( g \) with \( \pi \circ g = \text{id}_{S/I} \), i.e., \( S/I \) is not \( S\)-projective. Since \( S/I \) is not \( S\)-projective, \( S/I \) is not \( S\)-e-projective.\( \blacksquare \)

Recall the following fact about the relative projectivity for modules over rings.

**3.17.** [Wis1991] Let \( R \) be a ring, \( P \) a left \( R \)-module and \( \{ M_{\lambda} \}_{\lambda \in \Lambda} \) a collection of left \( S \)-semimodules such that \( P \) is \( M_{\lambda} \)-projective for every \( \lambda \in \Lambda \). If \( \Lambda = \{ \lambda_1, \ldots, \lambda_k \} \) is finite, then \( P \) is \( \bigoplus_{n=1}^{k} M_{\lambda_n} \)-projective. If \( \_P \) is finitely generated and \( \Lambda \) is arbitrary, then \( P \) is \( \bigoplus_{\lambda \in \Lambda} M_{\lambda} \)-projective (even if \( \Lambda \) is infinite).

We provide a counter example showing that the result corresponding to 3.17 for the relative e-projectivity for semimodules over semiring does not necessarily hold. The same example serves to show that the converse of Proposition 3.14 is not true (even when \( M = L \oplus N \)).

**Example 3.18.** Let \( S := M_2(\mathbb{R}^+) = E_1 \oplus E_2 \), where
\[
E_1 = \left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \mid a, b \in \mathbb{R}^+ \right\}, \quad E_2 = \left\{ \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix} \mid c, d \in \mathbb{R}^+ \right\}
\]
and consider
\[
K = \left\{ \begin{bmatrix} u & u \\ v & v \end{bmatrix} \mid u, v \in \mathbb{R}^+ \right\}
\]
and \( P := S/K \).

Then
\[
0 \to E_1 \xrightarrow{1_{E_1}} S \xrightarrow{\pi_{E_2}} E_2 \to 0
\]
is exact, \( P \) is \( E_1 \)-e-projective and \( E_2 \)-e-projective. However, \( P \) is not \( S\)-e-projective (notice that \( S = E_1 \oplus E_2 \)).

**Proof.** Since \( E_1 \oplus E_2 = S \), it follows by the proof of Example 3.16 that \( P \) is not \( (E_1 \oplus E_2)\)-e-projective. Notice that \( E_1 \) and \( E_2 \) are ideal-simple left \( S \)-subsemimodules of \( S \). Let \( L \neq 0 \) and \( f : E_1 \to L \) be a normal \( S \)-epimorphism. Then \( \text{Ker}(f) \subseteq E_1 \), whence \( \text{Ker}(f) = 0 \) as \( E_1 \) is ideal-simple. Since \( f \) is \( k \)-normal and \( \text{Ker}(f) = 0 \), \( f \) is injective, whence an isomorphism. If \( g : P \to L \) is a \( S \)-linear map, then \( f^{-1} \circ g : P \to E_1 \) is an \( S \)-linear map such that \( f \circ f^{-1} \circ g = g \), and whenever there exists an \( S \)-linear map \( h : P \to E_1 \) such that \( f \circ h = g \), we have \( h = f^{-1} \circ f \circ h = f^{-1} \circ g \). Hence, \( P \) is \( E_1 \)-e-projective. Similarly, one can prove that \( P \) is \( E_2 \)-e-projective.\( \blacksquare \)
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