Conformal transformation of Douglas space of second kind with special \((\alpha, \beta)\)-metric

Rishabh Ranjan  
Department of Mathematics and Statistics,  
Sam Higginbottom University of Agriculture, Technology and Sciences,  
Prayagraj, India

P.N. Pandey  
Department of Mathematics, University of Allahabad, Prayagraj, India, and  
Ajit Paul  
Department of Mathematics and Statistics,  
Sam Higginbottom University of Agriculture, Technology and Sciences,  
Prayagraj, India

Abstract

**Purpose** – In this paper, the authors prove that the Douglas space of second kind with a generalised form of special \((\alpha, \beta)\)-metric \(F\), is conformally invariant.

**Design/methodology/approach** – For, the authors have used the notion of conformal transformation and Douglas space.

**Findings** – The authors found some results to show that the Douglas space of second kind with certain \((\alpha, \beta)\)-metrics such as Randers metric, first approximate Matsumoto metric along with some special \((\alpha, \beta)\)-metrics, is invariant under a conformal change.

**Originality/value** – The authors introduced Douglas space of second kind and established conditions under which it can be transformed to a Douglas space of second kind.

**Keywords** Randers metric, Douglas space, Berwald space, Conformally invariant

**Paper type** Research paper

1. Introduction

A number of geometers have been studying Douglas space \([1, 2]\) from different point of view. The theory of Finsler spaces more precisely Berwald spaces with an \((\alpha, \beta)\)-metric \([3\text{--}5]\) have significant role to develop the Finsler geometry \([6]\). The concept of Douglas space of second kind with \((\alpha, \beta)\)-metric was first discussed by I. Y. Lee \([7]\) in Finsler geometry. In \([8]\), S. Bacso and Matsumoto developed the concept of Douglas space as an extension of Berwald space. In \([9]\), S. Bacso and Szilagyi introduced the concept of weakly-Berwald space as another extension of Berwald space. In \([10]\), M. S. Kneblman started working on the concept of conformal Finsler spaces and consequently, this notion was explored by M. Hashiguchi \([11]\). In \([12, 13]\) Y. D. Lee and B.N. Prasad developed the conformally invariant tensorial quantities in a Finsler space with \((\alpha, \beta)\)-metric under conformal \(\beta\)-change.

JEL Classification — 53B40, 53C60.

© Rishabh Ranjan, P.N. Pandey and Ajit Paul. Published in *Arab Journal of Mathematical Sciences*. Published by Emerald Publishing Limited. This article is published under the Creative Commons Attribution (CC BY 4.0) licence. Anyone may reproduce, distribute, translate and create derivative works of this article (for both commercial and non-commercial purposes), subject to full attribution to the original publication and authors. The full terms of this licence may be seen at [http://creativecommons.org/licenses/by/4.0/legalcode](http://creativecommons.org/licenses/by/4.0/legalcode)
In this paper, we prove that the Douglas space of second kind with generalised special $(\alpha, \beta)$-metric is conformally invariant. In the consequence, we find some results to show that the Douglas space of second kind with certain $(\alpha, \beta)$-metric such as Randers metric, first approximate Matsumoto metric and Finsler space with some generalised form of $(\alpha, \beta)$-metric remains unchanged geometrically under a confomal transformation.

2. Preliminaries

A Finsler space $F^n = (M, F(\alpha, \beta))$ is said to be with an $(\alpha, \beta)$-metric if $F(\alpha, \beta)$ is a positively homogeneous function in $\alpha$ and $\beta$ of degree 1, where $\alpha$ is Riemannian metric given by $\alpha^2 = a_{ij}(x)y^iy^j$ and $\beta = b_i(x)y^i$ is 1-form. The space $R^n = (M, \alpha)$ is called Riemannian space associated with $F^n$. We shall use the following symbols [6]:

- $b^i = a^{ir}b_r$
- $b^2 = a^{rs}b_rb_s$
- $2r_{ij} = b_{ij} + b_{ji}$, $2s_{ij} = b_{ij} - b_{ji}$
- $s^i = a^{sr}s_{rij}$, $s_i = b_ir^r$

The Berwald connection

$$B^\Gamma = \left\{ G^i_{jk}(x, y), G^j_i \right\}$$

of $F^n$ plays an important role in this paper. $B^i_{jk}$ denotes the difference tensor of $G^i_{jk}$ and $\gamma^i_{jk}$ that is

$$G^i_{jk}(x, y) = \gamma^i_{jk}(x) + B^i_{jk}(x, y).$$

Using the subscript 0 and transvecting by $y^i$, we get

$$G^i_j = \gamma^i_{0j} + B^i_j \text{ and } 2G^i = \gamma^i_{00} + 2B^i,$$

and then $B^i_j = \partial_i B^i$ and $B^i_{jk} = \partial_i B^j_{ik}$. A Finsler space $F^n$ of dimension $n$ is called a Douglas space [14] if

$$D^i = G^i(x, y)y^i - G^i(x, y)y^i,$$

are homogeneous polynomial of $(y^i)$ of degree three.

Next, differentiating (3) with respect to $y^m$, we obtain the following definitions;

**Definition 1.** ([14]) A Finsler space $F^n$ is a Douglas space of second kind if $D^i_{lm} = (n + 1)G^i - G^i_{lm}y^l$ is a two homogeneous polynomial in $(y^i)$.

On the other hand, a Finsler space with $(\alpha, \beta)$-metric is a Douglas space of second kind if and only if

$$B^i_{lm} = (n + 1)B^i_l - B^i_{lm}y^l,$$

are homogeneous equation in $(y^i)$ of degree two, when $B^i_{lm}$ is same as given in [14].

Furthermore, differentiating Eqn (4) with respect to $y^i$, $y^j$ and $y^k$, we obtain

$$B^i_{lkm} = B^i_{ljkm} = 0.$$

**Definition 2.** A Finsler space $F^n$ with $(\alpha, \beta)$-metric is known as Douglas space of second kind if $B^i_{lm} = (n + 1)B^i_l - B^i_{lm}y^l$ is a homogeneous polynomial in $(y^i)$ of degree two.
3. Douglas space of second kind with \((\alpha, \beta)\)-metric

Under this section, we discuss the criteria for a Finsler space with an \((\alpha, \beta)\)-metric to be a Douglas space of second kind \([2]\).

The spray coefficient \(G(x, y)\) of \(F^n\) can be expressed as \([4]\),

\[
2G^i = \gamma^i_{00} + 2B^i
\]

\[
B^i = \frac{\alpha F^\beta}{F^\alpha} s^j_0 + C^* \left[ \frac{\beta F^\beta y^j}{\alpha F^\alpha} - \frac{\alpha F_{aa}}{F^\alpha} \left( \frac{y^j}{\alpha} - \frac{\alpha y^j}{\beta} \right) \right],
\]

where

\[
C^* = \frac{\alpha \beta (r_{00} F^a - 2 a s_0 F^\beta)}{2 (\beta^2 F^a + \alpha \gamma^2 F_{aa})},
\]

\[
\gamma^2 = b^2 \alpha^2 - \beta^2.
\]

Since \(r^i_{00} = \gamma^j_{jk}(x) y^j y^k\) is \(\text{hp}(2)\), Eqn \((7)\) yields

\[
B^i = \frac{\alpha F^\beta}{F^\alpha} \left( s^j_0 y^j - s^i_0 y^i \right) + \frac{\alpha^2 F_{aa}}{\beta F^\alpha} C^* \left( b^2 y^j - b^i y^j \right).
\]

By means of \((3)\) and \((9)\), we obtain the following lemma \([14]\);

**Lemma 1.** A Finsler space \(F^n\) with an \((\alpha, \beta)\)-metric is a Douglas space if and only if \(B^i = B^j y^j - B^i y^i\) are \(\text{hp}(3)\).

Differentiating \((9)\) with respect to \(y^h, y^j, y^p\) and \(y^q\), we can have \(D^i_{hjk} = 0\) which are equivalent to \(D^i_{hjpm} = (n + 1) D^i_{hbjp} = 0\). Hence, a Finsler space \(F^n\) satisfying the condition \(D^i_{hjbp} = 0\) is called Douglas space. Now, differentiating Eqn \((9)\) with respect to \(y^m\) and contracting \(m\) and \(j\) in the resulting equation, we get

\[
B^m = \frac{(n + 1) \alpha F^\beta s^j_0}{F^\alpha} + \frac{\alpha \left\{ (n + 1) \alpha^2 \Omega F_{aa} b^j + \beta \gamma^2 A y^j \right\} r_{00}}{2 \Omega^2}
\]

\[
- \frac{\alpha^2 \left\{ (n + 1) \alpha^2 \Omega F^\beta F_{aa} b^j + B y^j \right\} s^i_0}{F^\alpha \Omega^2} - \frac{\alpha^2 F_{aa} y^j r_{00}}{\Omega}
\]

where \(\Omega = (\beta^2 F^a + \alpha \gamma^2 F_{aa})\), provided that \(\Omega \neq 0\), \(A = \alpha F_a F_{aa} + 3 F_a F_{aa} - 3 \alpha (F_{aa})^2\) and

\[
B = \alpha \beta^2 F_a F_{aa} + \beta \left\{ (3 \gamma^2 - \beta^2) F_a - 4 \alpha \gamma^2 F_{aa} \right\} F_a F_{aa} + \Omega FF_{aa}
\]

Following result is used in the succeeding section \([7]\);

**Theorem 1.** A Finsler space \(F^n\) is a Douglas space if second kind if and only if \(B^m_m\) are homogeneous polynomials in \((y^m)\) of degree two, where \(B^m_m\) is given by Eqns \((10)\) and \((11)\), provided \(\Omega \neq 0\).

4. Conformal change of Douglas space of second kind with \((\alpha, \beta)\)-metric

In this section, we find the criteria for a Douglas space of second kind to be conformally invariant.

Let \(F^n = (M, F)\) and \(\overline{F}^n = (M, \overline{F})\) be two Finsler spaces. Then \(F^n\) is called conformal to \(\overline{F}^n\) if we have a function \(\sigma(x)\) in each coordinate neighbourhood of \(M^n\) such that \(\overline{F}(x, y) = e^\sigma F(x, y)\) and this transformation \(F \rightarrow \overline{F}\) is called conformal change.
A conformal change of \((\alpha, \beta)\)-metric is given as \((\alpha, \beta) \rightarrow (\bar{\alpha}, \bar{\beta})\), where \(\bar{\alpha} = e^\sigma \alpha, \bar{\beta} = e^\sigma \beta\),

\[
a_{ij} = e^{2\sigma}a_{ij}, \quad b_i = e^\sigma b_i
\]

(12)

and \(b^2 = a^\sigma b_i b_j = \bar{a}^\sigma \bar{b}_i \bar{b}_j\).

From Eqn (13), the Christoffel symbols are given by:

\[
\bar{\gamma}^i_{jk} = \gamma^i_{jk} + \delta^i_j \sigma_k + \delta^i_k \sigma_j - \sigma^\alpha a_{jk},
\]

(14)

Where, \(\sigma_j = \partial_j \sigma\) and \(\sigma^\alpha = a^\alpha \sigma_j\).

Using (13) and (14), we obtain the following identities:

\[
\nabla b_i = e^\sigma (\nabla b_i + \rho a_{ij} - \sigma_j b_j),
\]

\[
\tau_{ij} = e^\sigma \left[\tau_{ij} + \rho a_{ij} - \frac{1}{2} (b_i \sigma_j + b_j \sigma_i)\right],
\]

\[
\bar{s}_{ij} = e^\sigma \left[s_{ij} + \frac{1}{2} (b_i \sigma_j - b_j \sigma_i)\right],
\]

\[
\bar{s}_i = e^{-\sigma} \left[s_i + \frac{1}{2} (b^\sigma \sigma_j - b_j \sigma^\sigma)\right],
\]

\[
\bar{s}_j = s_j + \frac{1}{2} (b^\sigma \sigma_j - \rho b_j),
\]

(15)

Where, \(\rho = \sigma, b^\sigma\).

Using Eqs (14) and (15), we get easily the followings:

\[
\bar{\gamma}^0_{00} = \gamma^0_{00} + 2\sigma_0 \sigma^\alpha - \alpha^2 \sigma_j,
\]

(16)

\[
\bar{\tau}_{00} = e^\sigma (\tau_{00} + \rho \alpha^2 - \sigma_0 \beta),
\]

(17)

\[
\bar{s}_0 = e^{-\sigma} \left[s_0 + \frac{1}{2} (\sigma \sigma_0 b^\sigma - \beta \sigma^\sigma)\right],
\]

(18)

\[
\bar{s}_0 = s_0 + \frac{1}{2} (\sigma_0 b^\sigma - \rho \beta).
\]

(19)

Now we obtain the conformal transformation of \(B^\sigma\) given by Eqn (9).

Consider \(F(\alpha, \beta) = e^\sigma F(\alpha, \beta)\) then

\[
F_a = F_a, \quad F_m = e^{-\sigma} F_m, \quad F_\beta = F_\beta, \quad \tau^2 = e^{2\sigma} \tau^2
\]

(20)

From Eqs (8), (19), (20) and using Theorem 3.1, we obtain

\[
\bar{C}^* = e^\sigma (C^* + D^*),
\]

(21)

Where,

\[
D^* = \frac{\alpha \beta \left[ (\beta \alpha^2 - \sigma_0 \beta) F_a - \alpha (b^\sigma \sigma_0 - \rho \beta) F_\beta \right]}{2 (\beta^2 F_a + \alpha^2 F_m)}
\]

(22)
Hence $B^i_j$ can be expressed as:

\[
B^i_j = \frac{\alpha F_\beta}{F_\alpha} (s_\alpha^i - s_\beta^i) + \frac{\alpha^2 F_{\alpha \alpha}}{\beta F_\alpha} C^i (b^i y^j - b^j y^i) + \left( \frac{\alpha \sigma F_\beta}{F_\alpha} + \frac{\alpha^2 F_{\alpha \alpha}}{\beta F_\alpha} D^\ast \right) \left( b^i y^j - b^j y^i \right) - \frac{\alpha \beta F_\beta^\ast}{2 F_\alpha} \left( \sigma y^i - \sigma y^j \right),
\]

\[= B^i_j + C^i_j,
\]

Where,

\[
C^i_j = \left( \frac{\alpha \sigma F_\beta}{F_\alpha} + \frac{\alpha^2 F_{\alpha \alpha}}{\beta F_\alpha} D^\ast \right) \left( b^i y^j - b^j y^i \right) - \frac{\alpha \beta F_\beta^\ast}{2 F_\alpha} \left( \sigma y^i - \sigma y^j \right).
\]

Using Eqn (11), we can have

\[
\overline{\Omega} = e^{2\Omega}, \quad \overline{A} = e^{-\sigma} A, \quad \overline{B} = e^{2\beta} B.
\] (23)

Now, we use conformal transformation on $B^m_n$ and obtain

\[
\overline{B}^m_n = B^m_n + K^m_n
\] (24)

Where, $K^m_n$ is given by [15, 16].

\[
2K^m_n = \frac{(n + 1)\alpha F_\beta}{F_\alpha} (\sigma_0 b^i - \sigma b^i) + \alpha \left\{ \frac{(n + 1)\alpha^2 F_{\alpha \beta} b^i + \beta y^i}{\Omega^2} \left( \rho \sigma - \sigma_0 \rho \right) \right. - \frac{\alpha^2 \left\{ (n + 1)\alpha^3 \Omega \right\} F_{\beta} F_{\alpha \beta} b^i + B y^i}{F_\alpha \Omega^2} \left( b^i \sigma_0 - \rho \beta \right).
\] (25)

Therefore, we obtain the following result:

**Theorem 2.** A Douglas space of second kind is conformally invariant if and only if $K^m_n (x)$ are homogeneous polynomial in $(y^j)$ of degree two.

**5. Conformal change of Douglas space of second kind with special $(\alpha, \beta)$-metric**

\[F = \alpha + \epsilon \beta + k \frac{\beta^{t+1}}{\alpha^t},\]

Consider a Finsler manifold with special $(\alpha, \beta)$-metric defined as

Where, $\epsilon$ and $k$ are constant.

Then we obtain

\[
F_\alpha = 1 - tk \frac{\beta^{t+1}}{\alpha^{t+1}},
\]

\[
F_\beta = \epsilon + k(t + 1) \frac{\beta^t}{\alpha^t},
\] (26)

\[
F_{\alpha \alpha} = t(t + 1)k \frac{\beta^{t+1}}{\alpha^{t+2}}
\]

\[
F_{\alpha \beta} = \frac{-6k \beta^t}{\alpha^t}.
\]
Therefore, using Eqn (11), we obtain

\[
\Omega = -t(t + 2)k\beta^t + \left[\alpha\beta + b^2 t(t + 1)\alpha\beta \right] \alpha\beta \\
A = t(t + 1)k\beta^{t+1} \left(1 - t - 2t(t + 2)k\beta^{t+1} \right) \\
B = \prod_1 + \prod_2 + \prod_3
\]

Where,

\[
\prod_1 = -t(t + 1)(t + 2)k\beta^t \left[\epsilon + k(t + 1)\frac{\beta^t}{\alpha^t} - \epsilon k\beta^{t+1} \right] (b^2\alpha^2 - \beta^2),
\]

\[
\prod_2 = t(t + 1)k\beta^t \left[\epsilon + k(t + 1)\frac{\beta^t}{\alpha^t} \right] \left[3 - (4t + 7)\beta^{t+1} \right] b^2\alpha^2 + \left[(t + 2)k\beta^{t+1} - 1\right] 4\beta^2,
\]

\[
\prod_3 = t(t + 1)k\beta^t \left[\alpha\beta + \epsilon\beta^t + (t + 1)\frac{\beta^t}{\alpha^t} \left(b^2\alpha^2 + \epsilon b^2\alpha - k\beta^2 - \epsilon k\beta^3\right) \right]
\left[b^2\alpha^2 - k^2\beta^2\right],
\]

Hence, using Eqn (26), \( K_{m}^{im} \) can be reduced as

\[
2K_{m}^{im} = \left[\prod_1 + \prod_2 + \prod_3\right] A
\]

Where,

\[
\alpha A_1 = \frac{(n + 1)t(t + 1)k\alpha^t}{\left\{\alpha^{t+1}\beta^2 + b^2 t(t + 1)\alpha^{t+1}\beta^2 + \right\}} - t(t + 2)k\beta^{t+3} b^t,
\]

\[
\alpha A_2 = \frac{t(t + 1)k\left[(1 - t)\alpha^{t+1} - 2t(t + 2)k\beta^{t+2}\right] b^2 y}{\left\{\alpha^{t+1}\beta^2 + b^2 t(t + 1)\alpha^{t+1}\beta^2 \right\}},
\]

\[
B_0 = \frac{(n + 1)t(t + 1)k\alpha^t}{\left\{\alpha^{t+1}\beta^2 + b^2 t(t + 1)\alpha^{t+1}\beta^2 - t(t + 2)k\beta^{t+2}\right\}} b^t,
\]

\[
B_1 = \frac{-t(t + 1)(t + 2)k\alpha^t}{\left\{\alpha^{t+1}\beta^2 + b^2 t(t + 1)\alpha^{t+1}\beta^2 - t(t + 2)k\beta^{t+3}\right\}} b^t.
\]
\[ B_2 = \frac{t(t+1)(t+2)k\alpha^2\beta^{t+2}(\epsilon\alpha + k(t+1)\beta')}{(\alpha^{t+1} - tk\beta^{t+1})[\alpha^{t+1}\beta^2 + b(t+1)\alpha^2\beta^{t+1} - t(t+2)k\beta^{t+3}]^2} \]
\[ + 3b^2\alpha^{t+3} - t(4t + 7)kb^2\alpha^2\beta^{t+1} - 4\alpha^{t+1}\beta^2 + 4t(t+2)k\beta^{t+3}]. \]

\[ B_3 = \frac{kt(t+1)(\alpha\beta)^{t+2}}{(\alpha^{t+1} - tk\beta^{t+1})[\alpha^{t+1}\beta^2 + b(t+1)\alpha^2\beta^{t+1} - t(t+2)k\beta^{t+3}]^2} \]
\[ + \alpha^{t+2}\beta + e\alpha^{t+1}\beta^2 t(t+1)(b^2\alpha^2\beta' - k\alpha\beta') + (t(t+1) + \epsilon\beta^2)\alpha^2\beta^{t+2} \]
\[ - (t(t+1)ke + k^2)\beta^{t+3}]. \]

Now, Eqn (28) can also be written as
\[ 2K_m^{im} = (n + 1)\alpha[\epsilon + k(t + 1)(\alpha^{-1}\beta)](\sigma_0b' - \beta\sigma') + p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7. \tag{29} \]
where,
\[ p_1 = \alpha A_1 (\rho\alpha^2 - \sigma_0\beta) \]
\[ p_2 = \alpha A_2 (\rho\alpha^2 - \sigma_0\beta) \]
\[ p_3 = -B_0 (b^2\sigma_0 - \rho\beta) \]
\[ p_4 = -B_1\beta \]
\[ p_5 = -B_2\beta \]
\[ p_6 = -B_3\beta \]
\[ p_7 = C (b^2\sigma_0 - \rho\beta) \]
showing that \( K_m^{im} \) is homogeneous polynomial of degree 2 in \( y' \).

**Theorem 3.** A Douglas space of second kind with special \((\alpha, \beta)\)-metric \( F = \alpha + \epsilon\beta + k\beta^{t+1} \) where \( \epsilon \) and \( k \) are constants, is conformally invariant.

With the help of Theorem 3 it can be proved that a Douglas space of second kind with a Finsler space of certain \((\alpha, \beta)\)-metric is conformally transformed to a Douglas space of second kind. In this way, one can have following possible cases;

**Case(i).** If \( \epsilon = 1 \) and \( k = 0 \), we have \( F = \alpha + \beta \) which is Randers metric. In case, \( 2K_m^{im} \) occupies the form
\[ 2K_m^{im} = (n + 1)\alpha(\sigma_0b' - \beta\sigma'). \tag{30} \]
Which shows \( K_m^{im} \) is homogeneous polynomial in \( (y') \) of degree two.

Note that in this case, \( p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = p_7 = 0 \).
Corollary 1. A Douglas space of second kind with Randers metric $F = \alpha + \beta$, is conformally invariant.

Case(ii). If $\epsilon = 0$ and $k = 1$, we have $F = \alpha + \frac{\beta^{+1}}{\alpha}$. In this case $2K^m_m$ obtains the form

$$2K^m_m = (n + 1)(t + 1)(\alpha^{-1} \beta)\left(\sigma_0 b^j - \beta \sigma^j\right) + q_1 + q_2 + q_3 + q_4 + q_5 + q_6 + q_7, \quad (31)$$

Where,

$$q_1 = \frac{(n + 1)t(t + 1)\alpha^2 \beta^{+1}}{\alpha^{+1} \beta^2 + b^2 t(t + 1)\alpha^2 \beta^{+1} - t(t + 2)\beta^{+3}} \beta^j \left(\sigma_0 b^j - \beta \sigma^j\right),$$

$$q_2 = \frac{t(t + 1)\left[(1 - t)\alpha^{+1} - 2t(t + 2)\beta^{+1}\right]^{\beta \gamma^2}}{\left[\alpha^{+1} \beta + b^2 t(t + 1)\alpha^2 \beta^j - t(t + 2)\beta^{+2}\right]^2} \left(\rho \alpha^2 - \sigma_0 \beta\right),$$

$$q_3 = \frac{(n + 1)t(t + 1)^2 \alpha^2 \beta^{+2} \beta^j}{\left(\alpha^{+1} - t\beta^{+1}\right)\left[\alpha^{+1} \beta + b^2 t(t + 1)\alpha^2 \beta^j - t(t + 2)\beta^{+2}\right] \left(\rho \sigma_0 - \rho \beta\right)},$$

$$q_4 = \frac{t(t + 1)^2(t + 2)\alpha^2 \beta^{+2} \beta^j}{\left[\alpha^{+1} \beta + b^2 t(t + 1)\alpha^2 \beta^j - t(t + 2)\beta^{+3}\right]^2} \left(\rho \sigma_0 - \rho \beta\right),$$

$$q_5 = \frac{-t(t + 1)^2 \alpha^2 \beta^{+2} [3b^2 \alpha^{+3} - t(4t + 7)b^2 \alpha^2 \beta^{+1} - 4t^2 \beta^{+1} + 4t(t + 2)\beta^{+1}]^2}{\left(\alpha^{+1} - t\beta^{+1}\right)\left[\alpha^{+1} \beta + b^2 t(t + 1)\alpha^2 \beta^j - t(t + 2)\beta^{+3}\right]^2} \left(\rho \sigma_0 - \rho \beta\right),$$

$$q_6 = \frac{-t(t + 1)(\alpha \beta)^{t+2} \left[\alpha^{+2} \beta + t(t + 1)\left(b^2 \alpha^2 \beta^j + \alpha^2 \beta^{+1} - \alpha \beta^{+2}\right) - \beta^{+3}\right]}{\left(\alpha^{+1} - t\beta^{+1}\right)\left[\alpha^{+1} \beta + b^2 t(t + 1)\alpha^2 \beta^j - t(t + 2)\beta^{+3}\right]^2} \left(\rho \sigma_0 - \rho \beta\right),$$

$$q_7 = \frac{t(t + 1)\alpha^2 \beta^{+1}}{\alpha^{+1} \beta + b^2 t(t + 1)\alpha^2 \beta^j - t(t + 2)\beta^{+3}} \left(\rho \sigma_0 - \rho \beta\right),$$

Showing that $K^m_m$ is homogeneous polynomial in $(\gamma^j)$ of degree 2.

Thus, we can have following;

Corollary 2. A Douglas space of second kind with special $(\alpha, \beta)$-metric $F = \alpha + \frac{\beta^{+1}}{\alpha}$ is conformally transformed to a Douglas space of second kind.

Case(iii). If $\epsilon = 1$ and $k = 1$, we obtain $F = \alpha + \beta + \frac{\beta^{+1}}{\alpha}$. In the case, $2K^m_m$ occupies the form

$$2K^m_m = (n + 1)\left[1 + (t + 1)(\alpha^{-1} \beta)\right] \left(\sigma_0 b^j - \beta \sigma^j\right) + r_1 + r_2 + r_3 + r_4 + r_5 + r_6 + r_7, \quad (32)$$

where,

$$r_1 = \frac{(n + 1)t(t + 1)\alpha^2 \beta^{+1}}{\alpha^{+1} \beta + b^2 t(t + 1)\alpha^2 \beta^j - t(t + 2)\beta^{+2}} \beta^j \left(\rho \alpha^2 - \sigma_0 \beta\right),$$

$$r_2 = \frac{t(t + 1)\left[(1 - t)\alpha^{+1} - 2t(t + 2)\beta^{+1}\right]^{\beta \gamma^2}}{\left[\alpha^{+1} \beta + b^2 t(t + 1)\alpha^2 \beta^j - t(t + 2)\beta^{+2}\right]^2} \left(\rho \alpha^2 - \sigma_0 \beta\right),$$
Case (iv). If \( A \) is a Douglas space of second kind with special conformally invariant. Where, 

Thus, we obtain the following:

Showing that \( K_m^{im} \) is a homogeneous polynomial in \( (\nu') \) of degree 2. Thus, we obtain the following;

**Corollary 3.** A Douglas space of second kind with special \((\alpha, \beta)\)-metric \( F = \alpha + \beta + \frac{\beta^4}{\alpha^2} \) is conformally invariant.

**Case (iv).** If \( \epsilon = 1, k = 1 \) and \( t = 1 \), we obtain \( F = \alpha + \beta + \frac{\beta^4}{\alpha^2} \). Then, \( 2K_m^{im} \) reduces in the form

\[ 2K_m^{im} = (n + 1) \left[ 1 + 2(\alpha' \beta) \right] \alpha \left( \sigma_0 \nu' - \beta \sigma' \right) + u_1 + u_2 + u_3 + u_4 + u_5 + u_6 + u_7, \]  

(33)

Where,

\[ u_1 = \frac{2(n + 1)\alpha^2 \beta}{(1 + 2b^2)\alpha^2 - 3b^2} b' \left( \rho \alpha^2 - \sigma_0 \beta \right), \]

\[ u_2 = \frac{12\rho \gamma^2}{(1 + 2b^2)\alpha^2 - 3b^2} \nu' \left( \rho \alpha^2 - \sigma_0 \beta \right), \]

\[ u_3 = \frac{-2(n + 1)\alpha^4 (\alpha + 2\beta)}{(\alpha^2 - \beta^2) \left[ (1 + 2b^2)\alpha^2 - 3b^2 \right]} b' \left( b^2 \sigma_0 - \rho \beta \right), \]

\[ u_4 = \frac{6\alpha^2 (\alpha^3 + 2\alpha^2 \beta - \alpha \beta^2 - 2\beta^3)}{\beta (\alpha^2 - \beta^2) \left[ (1 + 2b^2)\alpha^2 - 3b^2 \right]} \nu' \left( b^2 \sigma_0 - \rho \beta \right), \]
Showing that $K^m_{m}$ is a homogeneous polynomial in $(y')$ of degree 2.

Thus, we can have the following:

Corollary 4. A Douglas space of second kind with first approximate Matsumoto metric $F = \alpha + \beta + \frac{L}{a}$ is invariant under conformal change.

References

1. Antonelli PL, Ingarden RS, Matsumoto M. The theory of sprays and finsler space with application in physics and biology. Dordrecht: Kluwer Acad; 1993.
2. Narasimhamurthy SK, Ajith, Bagewadi CS. Conformal change of Douglas space of second kind with $(\alpha, \beta)$-metric. J Math Anal. 2012; 3(2): 25-30.
3. Aikou T, Hashiguchi M, Yamauchi K. On Matsumoto’s Finsler space with time measure. Rep Fac Sci Kagoshima Univ (Maths Phys Chem). 1990; 23: 1-12.
4. Matsumoto M. The Berwald connection of Finsler space with an $(\alpha, \beta)$-metric. Tensor, NS. 1991; 50: 18-21.
5. Park HS, Lee IY. Park CK. Finsler space with general approximation Matsumoto metric. Indian J Pure Appl Math. 2003; 34(2): 59-77.
6. Matsumoto M. The theory of finsler space with $(\alpha, \beta)$-metric. Rep Math Phys. 1992; 31: 43-83.
7. Lee IY. Douglas space of second kind with Matsumoto metric. J Chungcheong Math Soc. 2008; 21(2): 209-21.
8. Bacso S, Matsumoto M. On Finsler space of Douglas type, A generalization of the notion of Berwald spaces. Publ Math Debrecen. 1997; 51: 385-406.
9. Bacso S, Szilagyi B. On a weakly-Berwald space of Kropina type. Math Pannonica. 2002; 13: 91-95.
10. Kneblman MS. Conformal geometry of generalized metric spaces. Proc Nat Acad Sci USA. 1929; 15: 376-79.
11. Hashiguchi M. On conformal transformation of Finsler metrics. J Math Kyoto Univ. 1976; 16: 25-50.
12. Lee YD. Conformal transformations of difference tensors of Finsler space with an $(\alpha, \beta)$-metric. Comm Korean Math Soc. 1997; 12(4): 975-84.
13. Prasad BN, Gupta BN, Singh DD. On conformal transformation in Finsler space with an $(\alpha, \beta)$-metric. Indian J Pure Appl Math. 1987; 18(4): 290-301.
14. Matsumoto M. Finsler spaces with $(\alpha, \beta)$-metric of Douglas type. Tensor, NS. 1998; 60: 123-34.
15. Shanker G, Chaudhari D. On the conformal change of Douglas spaces of second kind with certain $(\alpha, \beta)$-metric. Int J Pure Appl Math. 2015; 103(4): 613-24.
16. Baby SA, Shanker G. On the conformal change of Douglas spaces of second kind with special $(\alpha, \beta)$-metric. AIP Conf Proc. 2020; 2261(1): 030011.

Further reading
17. Berwald L. On Cartan and Finsler geometries, III, Two dimensional Finsler spaces with rectilinear extremal. Ann Math. 1941; 42: 84-112.
18. Lee IY. On weakly-Berwald space with $(\alpha, \beta)$-metric. Bull. Korean Math Soc. 2006; 43(2): 425-41.

Corresponding author
Rishabh Ranjan can be contacted at: ranjanrishabh196@gmail.com