Robust Stability of Quantum Systems with a Nonlinear Coupling Operator

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Abstract—This paper considers the problem of robust stability for a class of uncertain quantum systems subject to unknown perturbations in the system coupling operator. A general stability result is given for a class of perturbations to the system coupling operator. Then, the special case of a nominal linear quantum system is considered with non-linear perturbations to the system coupling operator. In this case, a robust stability condition is given in terms of a scaled strict bounded real condition.

I. INTRODUCTION

An important concept in modern control theory is the notion of robust or absolute stability for uncertain nonlinear systems in the form of a Lur’e system with an uncertain nonlinear block which satisfies a sector bound condition; e.g., see [1]. This enables a frequency domain condition for robust stability to be given. This characterization of robust stability enables robust feedback controller synthesis to be carried out using $H^\infty$ control theory; e.g., see [2]. In a recent paper [3], classical results on robust stability were extended to the case of nonlinear quantum systems with non-quadratic perturbations to the system Hamiltonian. The aim of this paper is to extend classical results on robust stability to the case of nonlinear quantum systems with nonlinear perturbations to the system coupling operator.

In recent years, a number of papers have considered the feedback control of systems whose dynamics are governed by the laws of quantum mechanics rather than classical mechanics; e.g., see [4]–[16]. In particular, the papers [13], [17] consider a framework of quantum systems defined in terms of a triple $(S, L, H)$ where $S$ is a scattering matrix, $L$ is a vector of coupling operators and $H$ is a Hamiltonian operator. The paper [17] then introduces notions of dissipativity and stability for this class of quantum systems. In this paper, we build on the results of [17] to obtain robust stability results for uncertain quantum systems in which the quantum system coupling operator is decomposed as $L = L_1 + L_2$ where $L_1$ is a known nominal coupling operator and $L_2$ is a perturbation coupling operator, which is contained in a specified set of coupling operators $W$.

For this general class of uncertain quantum systems, a general stability result is obtained. The paper then considers the case in which the nominal quantum system $(S, L_1, H)$ is a linear quantum system in which the Hamiltonian $H$ is a quadratic function of annihilation and creation operators and the coupling operator $L_1$ is a linear function of annihilation and creation operators; e.g., see [7], [8], [10], [11], [16]. In this special case, a robust stability result is obtained in terms of a scaled frequency domain condition.

The remainder of the paper proceeds as follows. In Section II we define the general class of uncertain quantum systems under consideration. In Section III we consider a special class of non-linear perturbation coupling operators. In Section IV we specialize to the case of a linear nominal quantum systems and obtain a robust stability result for this case in which the stability condition is given in terms of a strict bounded real condition dependent on three scaling parameters. In Section V we present some conclusions.

II. QUANTUM SYSTEMS

We consider open quantum systems defined by parameters $(S, L, H)$ where $L = L_1 + L_2$; e.g., see [13], [17]. The corresponding generator for this quantum system is given by

$$G(X) = -iT[X, H] + L_L(X)$$

(1)

where $L_L(X) = \frac{1}{2}L^*[X, L] + \frac{1}{2}[L^*, X]L$. Here, $[X, H] = XH -HX$ denotes the commutator between two operators and the notation $*$ denotes the adjoint of an operator. Also, $H$ is a self-adjoint operator on the underlying Hilbert space referred to as the system Hamiltonian. $L_1$ is the nominal system coupling operator and $L_2$ is referred to as the perturbation coupling operator. Also, $S$ is a unitary matrix referred to as the scattering matrix. Throughout this paper, we will assume that $S = I$. The triple $(S, L, H)$, along with the corresponding generator define the Heisenberg evolution $X(t)$ of an operator $X$ according to a quantum stochastic differential equation

$$dX = (L_L(X) - iT[X, H]) dt + dA^* S [X, L] + [L^*, X] S dA^* + tr [(S^T XS - X) dA^*]$$

e.g., see [17]. Here, in the case of operators, the notation $^\dagger$ denotes the adjoint transpose of a vector or matrix of
operators; see also [17] for a definition of the quantities \(dA\), \(dA^*\), and \(d\Delta\) which will not be further considered in this paper. Also, in the case of standard matrices, the notation \(^\dagger\) refers to the complex conjugate transpose of a matrix.

The problem under consideration involves establishing robust stability properties for an uncertain open quantum system for the case in which the perturbation coupling operator is contained in a given set \(\mathcal{W}\). The main robust stability results presented in this paper will build on the following result from [17].

**Lemma 1 (See Lemma 3.4 of [17].):** Consider an open quantum system defined by \((S, L, H)\) and suppose there exists a non-negative self-adjoint operator \(V\) on the underlying Hilbert space such that

\[
G(V) + cv \leq \lambda
\]

where \(c > 0\) and \(\lambda\) are real numbers. Then for any plant state, we have

\[
\langle V(t) \rangle \leq e^{-ct} \langle V \rangle + \frac{\lambda}{c}, \quad \forall t \geq 0.
\]

Here \(V(t)\) denotes the Heisenberg evolution of the operator \(V\) and \(\langle \cdot \rangle\) denotes quantum expectation; e.g., see [17].

**A. Commutator Decomposition**

Given a set of non-negative self-adjoint operators \(\mathcal{P}\) and real parameters \(\gamma > 0\), \(\delta_1 \geq 0\), \(\delta_2 \geq 0\), \(\delta_3 \geq 0\), we now define a particular set of perturbation coupling operators \(\mathcal{W}_1\). This set \(\mathcal{W}_1\) is defined in terms of the commutator decomposition

\[
[V, L_2] = w_1[V, \zeta] - \frac{1}{2}w_2[\zeta, [V, \zeta]]
\]

for \(V \in \mathcal{P}\) where \(w_1\), \(w_2\) and \(\zeta\) are given scalar operators. We say \(L_2 \in \mathcal{W}_1\) if the following sector bound condition holds:

\[
L_2^2 \leq \frac{1}{\gamma} \zeta^* \zeta + \delta_1
\]

and

\[
w_1^*w_1 \leq \delta_2,
\]

\[
w_2^*w_2 \leq \delta_3.
\]

Here, we use the convention that for operator inequalities, terms consisting of real constants are interpreted as that constant multiplying the identity operator.

Then, we define

\[
\mathcal{W}_1 = \left\{ L_2 : \exists w_1, w_2, \zeta \text{ such that (4), (5) and (6) are satisfied and (3) is satisfied} \forall V \in \mathcal{P} \right\}.
\]

Using this definition, we obtain the following theorem.

**Theorem 1:** Consider a set of non-negative self-adjoint operators \(\mathcal{P}\) and an open quantum system \((S, L, H)\) where \(L = L_1 + L_2\) and \(L_2 \in \mathcal{W}_1\) defined in (7). If there exists a \(V \in \mathcal{P}\) and real constants \(c > 0\), \(\lambda > 0\), \(\tau_1 > 0\), \(\tau_2 > 0\), \(\ldots\), \(\tau_\delta > 0\) such that \(\mu = -\frac{1}{2}[\zeta, [V, \zeta]]\) is a constant and

\[
-i[V, H] + \frac{\delta_1}{2\tau_1} \left[ [V, \zeta]^* [V, \zeta] \right] L_1^* L_1 + \left( \frac{\delta_2}{2\tau_1^2} + \frac{\delta_3}{2\tau_1^2} \right) \zeta^* \zeta + [V, L_1]^*[V, L_1] = e V \leq \lambda,
\]

then

\[
\langle V(t) \rangle \leq e^{-ct} \langle V \rangle + \lambda + \left( \frac{\delta_1}{2\tau_1^2} + \frac{\delta_2}{2\tau_1^2} \right) \mu^* \mu + \left( \frac{\tau_2^2}{\tau_1^2} + \frac{\tau_2^2}{\tau_2^2} + \frac{\tau_2^2}{\tau_2^2} \right) \delta_1
\]

for all \(t \geq 0\).

**Proof:** Let \(V \in \mathcal{P}\) be given and consider \(G(V)\) defined in (I). Then using (3) and the fact that \(V\) is self-adjoint,

\[
G(V) = -i[V, H] + \frac{1}{2} (L_1^* + L_2^*) [V, L_1 + L_2] + \frac{1}{2} \left( [L_1^* + L_2, V] (L_1 + L_2) \right) - [V, H] + \frac{1}{2} \left( [L_1^*, [V, \zeta]] + [V, \zeta]^* w_1^* + \mu^* w_2^* \right) (L_1 + L_2) = -i[V, H] + \frac{1}{2} L_1^* [V, L_1] + \frac{1}{2} [V, L_1]^* L_1
\]

\[
+ \frac{1}{2} \mu L_1^* w_2 + \frac{1}{2} \mu^* w_2^* L_1
\]

\[
+ \frac{1}{2} L_2^* [V, L_1] + \frac{1}{2} [V, L_1]^* L_2
\]

\[
+ \frac{1}{2} \mu L_2^* w_1 + \frac{1}{2} \mu^* w_1^* L_2
\]

\[
+ \frac{1}{2} \mu L_2^* w_2 + \frac{1}{2} \mu^* w_2^* L_2.
\]

Now

\[
0 \leq \left( \tau_1 L_1^* - \frac{[V, \zeta]^* w_1^*}{\tau_1} \right) \left( \tau_1 L_1 - w_1 [V, \zeta] \right) \tau_1
\]

\[
= \tau_1^2 L_1^* L_1 - \tau_1^2 w_1 [V, \zeta] - [V, \zeta]^* w_1^* L_1
\]

and hence

\[
L_1^* w_1 [V, \zeta] + [V, \zeta]^* w_1^* L_1 \leq \tau_1^2 L_1^* L_1 + \frac{[V, \zeta]^* w_1^* w_1 [V, \zeta]}{\tau_1^2}
\]

\[
+ \frac{\delta_2 [V, \zeta]^* [V, \zeta]}{\tau_1^2}
\]

(10)
using (5). Also,

\[ 0 \leq \left( \tau_2 L_1 - \mu^* \frac{w_1}{\tau_2} \right) \left( \tau_2 L_1 - \frac{\mu}{\tau_2} \right) \]

\[ = \tau_2^2 L_1 - \mu \frac{w_1}{\tau_2} - \mu^* \frac{w_2}{\tau_2} L_1 + \frac{\mu^* \mu w_1 w_2}{\tau_2} \]

and hence

\[ \mu L_1 w_2 + \mu^* w_2 L_1 \leq \tau_2^2 L_1 + \frac{\mu^* \mu w_2}{\tau_2} \]

\[ \leq \tau_2^2 L_1 + \frac{\delta_3 \mu^* \mu}{\tau_2} \]  

(11)

using (6). Also,

\[ 0 \leq \left( \tau_3 L_2 - \frac{[V, L_1]}{\tau_3} \right) \left( \tau_3 L_2 - \frac{[V, L_1]}{\tau_3} \right) \]

\[ = \tau_3^2 L_2^2 - \frac{[V, L_1]^* [V, L_1]}{\tau_3^2} \]

\[ \leq \frac{\tau_3^2}{\tau_3^2} \zeta^* \zeta + \tau_3^2 \delta_1 + \frac{\delta_3 [V, L_1]^* [V, L_1]}{\tau_3^2} \]

\[ \leq \tau_3^2 L_2^2 + \frac{[V, L_1]^* [V, L_1]}{\tau_3^2} \]  

(12)

using (4). Also,

\[ 0 \leq \left( \tau_4 L_2 - \frac{[V, \zeta]^* w_1}{\tau_4} \right) \left( \tau_4 L_2 - \frac{w_1 [V, \zeta]}{\tau_4} \right) \]

\[ = \tau_4^2 L_2^2 - \frac{[V, \zeta]^* w_1 w_1 [V, \zeta]}{\tau_4^2} \]

\[ \leq \tau_4^2 L_2^2 + \frac{[V, \zeta]^* w_1 w_1 [V, \zeta]}{\tau_4^2} \]  

using (4) and (5). Also,

\[ 0 \leq \left( \tau_5 L_2 - \frac{\mu^* w_2}{\tau_5} \right) \left( \tau_5 L_2 - \frac{\mu w_2}{\tau_5} \right) \]

\[ = \tau_5^2 L_2^2 - \frac{\mu^* w_2}{\tau_5} - \frac{\mu w_2}{\tau_5} L_2 + \frac{\mu^* \mu w_2^2}{\tau_5} \]

and hence

\[ \mu L_2^* w_2 + \mu^* w_2 L_2 \leq \tau_5^2 L_2^2 + \frac{\mu^* \mu w_2^2}{\tau_5} \]

\[ \leq \tau_5^2 L_2^2 + \frac{\delta_3 \mu^* \mu}{\tau_5} \]  

(13)

Substituting (10), (11), (12), (13) and (14) into (9), it follows that

\[ G(V) \leq -i[V, H] + L_1 (V) + \frac{\tau_2^2}{2} L_1 + \frac{\delta_2 [V, \zeta]^* [V, \zeta]}{2 \tau_2^2} \]

\[ + \frac{\tau_2^2}{2} L_1 L_1 + \frac{\delta_3 \mu^* \mu}{2 \tau_2^2} \]

\[ + \frac{\tau_2^2}{2} \zeta^* \zeta + \frac{\tau_2^2}{2} \delta_1 + \frac{\delta_2 [V, \zeta]^* [V, \zeta]}{2 \tau_2^2} \]

\[ + \frac{\tau_2^2}{2} \zeta^* \zeta + \frac{\tau_2^2}{2} \delta_1 + \frac{\delta_3 \mu^* \mu}{2 \tau_2^2}. \]

Then it follows from (8) that

\[ G(V) + eV \leq \tilde{\lambda} + \left( \frac{\delta_3}{2 \tau_2^2} + \frac{\delta_1}{2 \tau_2^2} \right) \mu^* \mu \left( \frac{\tau_2^2}{2} + \frac{\tau_2^2}{2} + \frac{\tau_2^2}{2} \right) \delta_1. \]

Then the result of the theorem follows from Lemma 1. \( \square \)

III. NON-LINEAR PERTURBATION COUPLING OPERATORS

In this section, we define a set of non-linear perturbation coupling operators denoted \( \mathcal{W}_2 \). For a given set of non-negative self-adjoint operators \( \mathcal{P} \) and real parameters \( \gamma > 0 \), \( \delta_1 \geq 0, \delta_2 \geq 0, \delta_3 \geq 0 \), consider perturbation coupling operators defined in terms of the following power series (which is assumed to converge in some suitable sense)

\[ L_2 = f(\zeta) = \sum_{k=0}^{\infty} S_k \zeta^k = \sum_{k=0}^{\infty} S_k L_k. \]

(16)

Here \( \zeta \) is a scalar operator on the underlying Hilbert space and \( L_k = \zeta^k \).

Also, we let

\[ f'(\zeta) = \sum_{k=1}^{\infty} k S_k \zeta^{k-1}, \]

\[ f''(\zeta) = \sum_{k=1}^{\infty} k(k-1) S_k \zeta^{k-2}. \]

(17)

(18)

and consider the sector bound condition

\[ f(\zeta) f'(\zeta) \leq \frac{1}{\gamma^2} \zeta^* \zeta + \delta_1 \]

(19)

and the conditions

\[ f'(\zeta) f'(\zeta) \leq \delta_2, \]

\[ f''(\zeta) f''(\zeta) \leq \delta_3. \]

(20)

(21)

Then we define the set \( \mathcal{W}_2 \) as follows:

\[ \mathcal{W}_2 = \left\{ L_2 \text{ of the form } (16) \text{ such that conditions } (19), (20) \text{ and } (21) \text{ are satisfied} \right\}. \]

(22)

In this section, the set of non-negative self-adjoint operators \( \mathcal{P} \) will be assumed to satisfy the following assumption:
Assumption 1: Given any $V \in \mathcal{P}$, the quantity

$$\mu = -\frac{1}{2} [\zeta, [V, \zeta]] = -\frac{1}{2} [V, \zeta] + \frac{1}{2} [V, \zeta]$$

is a constant.

Lemma 2: Suppose the set of self-adjoint operators $\mathcal{P}$ satisfies Assumption 1. Then

$$\mathcal{W}_2 \subset \mathcal{W}_1.$$ 

Proof: First, we note that given any $V \in \mathcal{P}$ and $k \geq 1$,

$$V \zeta = [V, \zeta] + \zeta V;$$

$$V \zeta = \sum_{n=1}^{k} \zeta^{n-1} [V, \zeta] \zeta^{k-n} + \zeta^{k} V. \quad (23)$$

Also for any $n \geq 1$ such that $n \leq k$,

$$[V, \zeta] = \zeta [V, \zeta] + 2 \mu;$$

$$[V, \zeta] = \sum_{n=1}^{k} \zeta^{n-1} [V, \zeta] \zeta^{k-n} + \zeta^{k} V. \quad (24)$$

Therefore using (23) and (24), it follows that

$$V \zeta = \sum_{n=1}^{k} \zeta^{n-1} [V, \zeta] \zeta^{k-n} + \zeta^{k} V,$$

which holds for any $k \geq 0$.

Now given any $L_2 \in \mathcal{W}_2$, $k \geq 0$ we have

$$[V, L_k] = k \zeta^{k-1} [V, \zeta] + k(k-1) \zeta^{k-2} \mu. \quad (25)$$

Therefore,

$$[V, L_2] = \sum_{k=0}^{\infty} S_k [V, L_k]$$

$$= f'(\zeta) [V, \zeta] + f''(\zeta) \mu. \quad (26)$$

Now letting

$$w_1 = f'(\zeta), \quad w_2 = f''(\zeta), \quad (27)$$

it follows that condition (1) is satisfied. Furthermore, conditions (3), (4), (5) follow from conditions (19), (20), (21) respectively. Hence, $L_2 \in \mathcal{W}_1$. Since, $L_2 \in \mathcal{W}_2$ was arbitrary, we must have $\mathcal{W}_2 \subset \mathcal{W}_1$. \hfill $\Box$

IV. THE CASE OF A LINEAR NOMINAL SYSTEM

We now consider the case in which the nominal quantum system corresponds to a linear quantum system; e.g., see [7], [8], [10], [11], [16]. In this case, we assume that $H$ is of the form

$$H = \frac{1}{2} \begin{bmatrix} a^\dagger & a^T \end{bmatrix} M \begin{bmatrix} a \\ a^\# \end{bmatrix} \quad (28)$$

where $M \in \mathbb{C}^{2n \times 2n}$ is a Hermitian matrix of the form

$$M = \begin{bmatrix} M_1 & M_2 \\ M_2^\# & M_\# \end{bmatrix}$$

and $M_1 = M_1^\dagger, M_2 = M_2^\dagger$. Here $a$ is a vector of annihilation operators on the underlying Hilbert space and $a^\#$ is the corresponding vector of creation operators. In the case vectors of operators, the notation $\#$ refers to the vector of adjoint operators and in the case of complex matrices, this notation refers to the complex conjugate matrix.

The annihilation and creation operators are assumed to satisfy the canonical commutation relations:

$$\begin{bmatrix} a \\ a^\# \end{bmatrix} \begin{bmatrix} a \\ a^\# \end{bmatrix}^\dagger = \begin{bmatrix} a^\# \\ a \end{bmatrix} \begin{bmatrix} a^\# \\ a \end{bmatrix}^\dagger$$

$$= \begin{bmatrix} a^\# \\ a \end{bmatrix} \begin{bmatrix} a^\# \\ a \end{bmatrix}^\dagger - \begin{bmatrix} a^\# \\ a \end{bmatrix} \begin{bmatrix} a^\# \\ a \end{bmatrix}^\dagger \begin{bmatrix} a^\# \\ a \end{bmatrix}^\dagger = J \quad (29)$$

where $J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$; e.g., see [9], [14], [16].

We also assume $L_1$ is of the form

$$L_1 = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} a \\ a^\# \end{bmatrix} = N \begin{bmatrix} a \\ a^\# \end{bmatrix} \quad (30)$$

where $N_1 \in \mathbb{C}^{1 \times n}$ and $N_2 \in \mathbb{C}^{1 \times n}$. Also, we write

$$\begin{bmatrix} L_1 \\ L_1^\dagger \end{bmatrix} = N \begin{bmatrix} a \end{bmatrix} \begin{bmatrix} a \end{bmatrix} = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} a \end{bmatrix}. \quad (31)$$

In addition we assume that $V$ is of the form

$$V = \begin{bmatrix} a^\dagger & a^T \end{bmatrix} P \begin{bmatrix} a \\ a^\# \end{bmatrix}$$

where $P \in \mathbb{C}^{2n \times 2n}$ is a positive-definite Hermitian matrix of the form

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2^\# & P_1^\# \end{bmatrix} \quad (32)$$

Hence, we consider the set of non-negative self-adjoint operators $\mathcal{P}_1$ defined as

$$\mathcal{P}_1 = \left\{ V \text{ of the form } (31) \text{ such that } P > 0 \text{ is a Hermitian matrix of the form } (32) \right\}. \quad (33)$$

In the linear case, we will consider a specific notion of robustly mean square stability.

Definition 1: An uncertain open quantum system defined by $(S, L, H)$ where $L = L_1 + L_2$ with $L_1$ of the form (30). $L_2 \in \mathcal{W}, W$ is any given set, and $H$ of the form (28) is said to be robustly mean square stable if there exist constants $c_1 > 0, c_2 > 0$ and $c_3 \geq 0$ such that for any $L_2 \in \mathcal{W}$

$$\left\langle \begin{bmatrix} a(t) \\ a^\#(t) \end{bmatrix} \begin{bmatrix} a(t) \\ a^\#(t) \end{bmatrix} \right\rangle$$

$$\leq c_1 e^{-c_2 t} \left\langle \begin{bmatrix} a \\ a^\# \end{bmatrix} \begin{bmatrix} a \\ a^\# \end{bmatrix} \right\rangle + c_3 \forall t \geq 0. \quad (34)$$
Here \( \begin{bmatrix} a(t) \\ a^\#(t) \end{bmatrix} \) denotes the Heisenberg evolution of the vector of operators \( \begin{bmatrix} a \\ a^\# \end{bmatrix} \); e.g., see [17].

In order to address the issue of robust mean square stability for the uncertain linear quantum systems under consideration, we first require some algebraic identities.

Lemma 3: Given \( V \in \mathcal{P}_1 \), \( H \) defined as in (28) and \( L_1 \) defined as in (30), then

\[
[V, H] = \begin{bmatrix} a^\dagger \\ a^\# \end{bmatrix} P \begin{bmatrix} a \\ a^\# \end{bmatrix} + \frac{1}{2} \begin{bmatrix} a^\dagger \\ a^\# \end{bmatrix} M \begin{bmatrix} a \\ a^\# \end{bmatrix}
\]

Also,

\[
\mathcal{L}_{L_1}(V) = -\frac{1}{2} L_1^{\dagger} [V, L_1] + \frac{1}{2} [L_1, V] L_1
\]

Proof: The proof of these identities follows via straightforward but tedious calculations using (29).

We now specialize the results of Section III to the case of a linear nominal system with \( L_2 \in \mathcal{W}_2 \) where \( \mathcal{W}_2 \) is defined as in Section III. In this case, we define

\[
\zeta = E_1 a + E_2 a^\#
\]

where \( \zeta \) is assumed to be a scalar operator. Then, we show that a sufficient condition for robust mean square stability is the existence of constants \( \tau_1 > 0 \), \( \tau_2 > 0 \), and \( \tau_3 > 0 \) such that the following scaled strict bounded real condition is satisfied:

1) The matrix

\[
F = -i JM - \frac{1}{2} J N^\dagger J N
\]

is Hurwitz; (36)

2)

\[
\left\| C (sI - F)^{-1} B \right\|_\infty < 1
\]

where

\[
C = \begin{bmatrix} \sqrt{\tau_2 + \tau_3} E \\ \tau_1 \hat{N} \end{bmatrix}
\]

and

\[
B = \begin{bmatrix} \sqrt{\tau_2 + \tau_3} J \hat{E}^\dagger + \frac{1}{\tau_3} J \hat{N}^\dagger \end{bmatrix}.
\]

This leads to the following theorem.

Theorem 2: Consider an uncertain open quantum system defined by \( (S, L, H) \) such that \( L = L_1 + L_2 \) where \( L_1 \) is of the form (30), \( H \) is of the form (28), and \( L_2 \in \mathcal{W}_2 \). Furthermore, assume that there exists constants \( \tau_1 > 0 \), \( \tau_3 > 0 \), and \( \tau_4 > 0 \) such that the strict bounded real condition (35), (37) is satisfied. Then the uncertain quantum system is robustly mean square stable.

In order to prove this theorem, we require the following lemma.

Lemma 4: Given any \( V \in \mathcal{P}_1 \), then

\[
\mu = \frac{1}{2} \left[ \zeta, [V, \zeta] \right] = \frac{1}{2} \hat{E} \Sigma J P J \hat{E}^T.
\]

which is a constant. Here, \( \Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \). Hence, the set of operators \( \mathcal{P}_1 \) satisfies Assumption (41).

Proof: The proof of this result follows via a straightforward but tedious calculation using (29).

Proof of Theorem 2: If the conditions of the theorem are satisfied, then (37) implies

\[
\left\| C (sI - F)^{-1} \sqrt{2B} \right\|_\infty < 1.
\]

Hence, it follows from the strict bounded real lemma that the matrix inequality

\[
F^\dagger P + PF + 2P \hat{B} \hat{B}^\dagger P + \frac{1}{2} \hat{E} \Sigma J P J \hat{E}^T < 0.
\]

will have a solution \( P > 0 \) of the form (32); e.g., see [2], [11]. This matrix \( P \) defines a corresponding operator \( V \in \mathcal{P}_1 \) as in (31). Then using (38) and (39), it follows that we can write

\[
F^\dagger P + PF + P \left( 2 \delta_2 \left( \frac{1}{\tau_1} + \frac{1}{\tau_2} \right) J \hat{E} \hat{E}^\dagger + \frac{2}{\tau_3} J \hat{N}^\dagger \hat{N} \right) P
\]

+ \frac{\tau_2^2 + \tau_4^2}{2\gamma^2} \hat{E} \hat{E}^\dagger + \frac{\tau_2^2 + \tau_4^2}{2} \hat{N}^\dagger \hat{N} < 0.

Hence, we can choose \( \tau_2 > 0 \) and \( \tau_5 > 0 \) sufficiently small so that

\[
F^\dagger P + PF + P \left( 2 \delta_2 \left( \frac{1}{\tau_1} + \frac{1}{\tau_2} \right) J \hat{E} \hat{E}^\dagger + \frac{2}{\tau_3} J \hat{N}^\dagger \hat{N} \right) P
\]

+ \frac{\tau_2^2 + \tau_4^2}{2\gamma^2} \hat{E} \hat{E}^\dagger + \frac{\tau_2^2 + \tau_4^2}{2} \hat{N}^\dagger \hat{N} < 0.

Now, it follows from (35) that we can write

\[
C^* \zeta = \begin{bmatrix} a \\ a^\# \end{bmatrix} \hat{E} \hat{E}^\dagger \begin{bmatrix} a \\ a^\# \end{bmatrix}.
\]

Also, it follows from Lemma 3 that

\[
[V, \zeta] = -2 \hat{E} J P \begin{bmatrix} a \\ a^\# \end{bmatrix}.
\]

Hence,

\[
[V, \zeta] [V, \zeta]^* = 4 \begin{bmatrix} a \\ a^\# \end{bmatrix} \hat{P} J \hat{E}^\dagger \hat{E} \hat{P} \begin{bmatrix} a \\ a^\# \end{bmatrix}.
\]

(42)
Similarly
\[ [V, L_1][V, L_1]^* = 4 \left[ \begin{array}{c} a \\ a^\# \end{array} \right]^T P J \hat{N} \hat{N} P \left[ \begin{array}{c} a \\ a^\# \end{array} \right]. \] (43)

In addition,
\[ L_1^* L_1 = \left[ \begin{array}{c} a \\ a^\# \end{array} \right]^T \hat{N} \hat{N} \left[ \begin{array}{c} a \\ a^\# \end{array} \right]. \] (44)

Hence using Lemma 3 we obtain
\[ -i[V, H] + \mathcal{L}_{L_1}(V) + \left( \frac{\tau_2^2}{2} + \frac{\tau_2^2}{2} \right) L_1^* L_1 \]
\[ + \left( \frac{\delta_2}{2 \gamma^2} + \frac{\delta_2}{2 \gamma^2} \right) [V, \xi]^*[V, \xi] + \left( \frac{\tau_2^2}{2 \gamma^2} + \frac{\tau_2^2}{2 \gamma^2} \right) \zeta^* \zeta + \frac{[V, L_1]^*[V, L_1]}{2 \gamma^2} \]
\[ = \left[ \begin{array}{c} a \\ a^\# \end{array} \right]^T \left( \begin{array}{cc} F^T P + P F & +2 \delta_2 \left( \frac{1}{2 \gamma^2} + \frac{1}{2 \gamma^2} \right) P J \hat{E} \hat{E} J P \\ +2 \delta_2 \left( \frac{1}{2 \gamma^2} + \frac{1}{2 \gamma^2} \right) P J \hat{N} \hat{N} J P \\ +2 \delta_2 \left( \frac{1}{2 \gamma^2} + \frac{1}{2 \gamma^2} \right) \hat{E} \hat{E} \\ +2 \delta_2 \left( \frac{1}{2 \gamma^2} + \frac{1}{2 \gamma^2} \right) \hat{N} \hat{N} \end{array} \right) \left[ \begin{array}{c} a \\ a^\# \end{array} \right] \]
\[ + \text{Tr} \left( P J N^T \left[ \begin{array}{cc} I & 0 \\ 0 & 0 \end{array} \right] N J \right) \] (45)

where \( F = -i J M - \frac{1}{2} J N^T J N \).

From this, it follows using (44) that there exists a constant \( c > 0 \) such that condition (8) is satisfied with
\[ \lambda = \text{Tr} \left( P J N^T \left[ \begin{array}{cc} I & 0 \\ 0 & 0 \end{array} \right] N J \right) \geq 0. \]

Hence, it follows from Lemma 4, Lemma 2, Theorem 1 and \( P > 0 \) that
\[ \left\langle a(t) \right| a^\#(t) \right\rangle \leq e^{-ct} \left\langle \left[ \begin{array}{c} a(0) \\ a^\#(0) \end{array} \right] \right\rangle \lambda_{max}[P] \]
\[ + \frac{\lambda}{\lambda_{min}[P]} \quad \forall t \geq 0 \] (46)

where \( \lambda = \lambda + \left( \frac{\delta_2}{2 \gamma^2} + \frac{\delta_2}{2 \gamma^2} \right) \mu^* \mu + \left( \frac{\tau_2^2}{2} + \frac{\tau_2^2}{2} + \frac{\tau_2^2}{2} \right) \delta_1. \)

Hence, the condition (34) is satisfied with \( c_1 = \frac{\lambda_{max}[P]}{\lambda_{min}[P]} > 0, \)
\( c_2 = c > 0 \) and \( c_3 = \frac{1}{\lambda_{min}[P]} \geq 0. \)

V. Conclusions

In this paper, we have considered the problem of robust stability for uncertain quantum systems with non-linear perturbations to the system coupling operator. The final stability result obtained is expressed in terms of a strict bounded real condition. Future research will be directed towards analyzing the stability of specific non-linear quantum systems using the given robust stability result.

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