NUMERICAL ANALYSIS OF A BDF2 MODULAR GRAD-DIV STABILIZATION METHOD FOR THE NAVIER-STOKES EQUATIONS

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Abstract. A second-order accurate modular algorithm is presented for a standard BDF2 code for the Navier-Stokes equations (NSE). The algorithm exhibits resistance to solver breakdown and increased computational efficiency for increasing values of grad-div parameters. We provide a complete theoretical analysis of the algorithms stability and convergency. Computational tests are performed and illustrate the theory and advantages over monolithic grad-div stabilizations.

1. Introduction. A common, powerful tool for improving solution quality for fluid flow problems is grad-div stabilization [12, 22, 25, 27, 28]. This technique typically involves adding \( \gamma \nabla \nabla \cdot u_h \), nonzero for most finite element velocity-pressure pairs, which penalizes mass conservation and improves solution accuracy. It was first introduced in [16] and has been widely studied since, both analytically and computationally [4, 12, 20, 22, 23, 26, 27, 28].

Unfortunately, grad-div stabilization also exhibits increased coupling in the linear system’s matrix, efficiency loss and solver breakdown, and classical Poisson locking [2, 9, 10, 21, 25, 26, 27]. In particular, since the matrix arising from grad-div term is singular, large grad-div parameter values can cause solver breakdown [8]. This difficulty cannot always be circumvented since recommended parameter choices vary greatly, e.g., from \( O(h^2) \) to \( O(10^4) \) for different applications, finite elements, and meshes [4, 12, 15, 28, 30]. An alternate realization of grad-div stabilization with greater computational efficiency was introduced in [5] for the backward Euler time discretization. Herein, we show how to implement modular grad-div stabilization for any multistep time discretization and perform analysis and testing for the BDF2 case.

To begin, consider the incompressible time-dependent NSE: Find the fluid velocity \( u : \Omega \times [0,T] \rightarrow \mathbb{R}^d \) and pressure \( p : \Omega \times (0,T] \rightarrow \mathbb{R} \) satisfying:

\[
\begin{align*}
\begin{aligned}
\frac{3u^{n+1} - 4u^n + u^{n-1}}{2\Delta t} + (2u^n - u^{n-1}) \cdot \nabla \hat{u}^{n+1} - \nu \Delta u^{n+1} + \nabla p^{n+1} &= f^{n+1}, \\
\nabla \cdot \hat{u}^{n+1} &= 0.
\end{aligned}
\end{align*}
\]

Here, the domain \( \Omega \subset \mathbb{R}^d(d=2,3) \) is a bounded polyhedron, \( f \) is the body force and \( \nu \) is the fluid viscosity. Suppressing the spacial discretization for the moment, we consider the following two step method that uncouples the grad-div solve.

Step 1: Given \( u^{n-1}, u^n \), find \( \hat{u}^{n+1} \) and \( p^{n+1} \) satisfying:

\[
\begin{align*}
\begin{aligned}
\frac{3u^{n+1} - 4u^n + u^{n-1}}{2\Delta t} - \beta \nabla \cdot \frac{3u^{n+1} - 4u^n + u^{n-1}}{2\Delta t} - \gamma \nabla \cdot u^{n+1} &= 0.
\end{aligned}
\end{align*}
\]

Step 2: Given \( \hat{u}^{n+1} \), find \( u^{n+1} \) satisfying:
In the above, $\beta \geq 0$ and $\gamma \geq 0$ are application-dependent grad-div stabilization parameters. The combined effect of Step 1 and Step 2 is a consistent BDF2 time discretization of the following model:

$$u_t - \beta \nabla \nabla \cdot u_t - \gamma \nabla \nabla \cdot u + u \cdot \nabla u - \nu \Delta u + \nabla p = f. \quad (1.5)$$

In [3], two minimally intrusive, modular algorithms were developed for backward Euler, which implemented grad-div stabilization. These algorithms effectively treated issues resulting from increased coupling and solver breakdown. Although the second steps of each of these algorithms can be used here when $\beta \equiv 0$, they cannot be used when $\beta > 0$; that is, the dispersive term $[3, 18, 29]$, associated with $\beta$ demands special attention. In the case $\beta > 0$, the time-discretizations in both steps must be consistent with one another. In particular, for the BDF$k$ family of methods:

Step 1: Find $\hat{u}^{n+1}$ and $p^{n+1}$ satisfying:

$$\frac{1}{\Delta t}(a_0\hat{u}^{n+1} + \sum_{s=1}^{S} a_s u^{n+1-s}) + U \cdot \nabla \hat{u}^{n+1} - \nu \Delta \hat{u}^{n+1} + \nabla p^{n+1} = f^{n+1}, \quad (1.6)$$

$$\nabla \cdot \hat{u}^{n+1} = 0. \quad (1.7)$$

Step 2: Find $u^{n+1}$ satisfying:

$$\frac{a_0}{\Delta t} (u^{n+1} - \hat{u}^{n+1}) - \beta \nabla \nabla \cdot \sum_{s=0}^{S} a_s u^{n+1-s} \Delta t - \gamma \nabla \nabla \cdot u^{n+1} = 0, \quad (1.8)$$

where $U$ denotes either $\hat{u}^{n+1}$ or a consistent extrapolation. A similar generalization can be made for general linear multistep methods.

This paper is arranged as follows. Section 2 introduces notation, lemmas, and necessary preliminaries. In Section 3 a fully-discrete modular grad-div stabilization algorithm (BDF2-mgd) and its unconditional, nonlinear, energy stability are presented. A complete error analysis is given in Section 4 where second-order convergence is proven for the modular method. Numerical experiments are provided to confirm the effectiveness of BDF2-mgd in Section 5. In particular, the algorithm maintains the positive impact of grad-div stabilization while resisting debilitating slow down for $0 \leq \gamma \leq 20,000$ or $0 \leq \beta \leq 8,000$. Conclusions follow in Section 6.

2. Preliminaries. We use the standard notations $H^k(\Omega)$, $H_0^k(\Omega)$, and $L^p(\Omega)$ to denote Sobolev spaces and $L^p$ spaces; see, e.g., [1]. The $L^2(\Omega)$ inner product and its induced norm are denoted by $(\cdot, \cdot)$ and $\| \cdot \|$, respectively. Let $\| \cdot \|_{L^p}$ and $\| \cdot \|_k$ denote the $L^p(\Omega)$ ($p \neq 2$) norm and $H^k(\Omega)$ norm. The space $H^{-k}(\Omega)$ denotes the dual space of $H^k_0(\Omega)$ and its norm is denoted by $\| \cdot \|_{-k}$. Throughout the paper, we use $C$ to denote a generic positive constant varying in different places but never depending on mesh size, time step, and grad-div parameters. For functions $v(x, t)$, we define the following norms:

$$\|v\|_{\infty,k} := \text{ess sup}_{[0,T]} \|v(\cdot, t)\|_k, \quad \|v\|_{p,k} := \left( \int_0^T \|v(\cdot, t)\|_k^p dt \right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$. The velocity space $X$, pressure space $Q$, and divergence free space $V$ are defined as follows.

$$X := H^1_0(\Omega)^d = \{v \in H^1(\Omega)^d : v|_{\partial \Omega} = 0\},$$

$$Q := L^2_0(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q \, dx = 0\},$$

$$V := \{v \in X : (\nabla \cdot v, q) = 0 \quad \forall q \in Q\}.$$
Define the skew-symmetric trilinear form

\[ b(u, v, w) := \frac{1}{2} (u \cdot \nabla v, w) - \frac{1}{2} (u \cdot \nabla w, v) \quad \forall \ u, v, w \in X. \]

Then, we have the following estimates for \( b \) (see, e.g., Lemma 2.2 in [19]):

\[
\begin{align*}
b(u, v, w) & \leq C \| \nabla u \| \| \nabla v \| \| \nabla w \|, \\
b(u, v, w) & \leq C \| u \|^{\frac{3}{2}} \| \nabla u \|^{\frac{1}{2}} \| \nabla v \| \| \nabla w \|, \\
b(u, v, w) & \leq C \| u \| \| v \|_2 \| \nabla w \|. 
\end{align*}
\]

(2.1) (2.2) (2.3)

Divide the simulation time \( T \) into \( N \) smaller time intervals with \([0, T] = \bigcup_{n=0}^{N-1} [t^n, t^{n+1}], \) where \( t^n = n\Delta t, \) \( T = N\Delta t. \) We may define the following discrete norms:

\[
\| v \|_{\infty, k} := \max_{0 \leq n \leq N} \| v(\cdot, t^n) \|_k, \quad \| v \|_{p,k} := (\Delta t \sum_{n=0}^{N} \| v(\cdot, t^n) \|_{p,k}^p)^{\frac{1}{p}}.
\]

Let \( \Omega_h \) be a quasi-uniform mesh of \( \Omega \) with \( \Omega = \bigcup_{K \in \Omega_h} K. \) Denote \( h = \operatorname{sup} \text{diam}(K). \)

Let \( X_h \subset X \) and \( Q_h \subset Q \) be the finite element spaces. Assume that \( X_h \) and \( Q_h \) satisfy approximation properties of piecewise continuous polynomials on quasi-uniform meshes of local degrees \( k \) and \( m, \) respectively:

\[
\begin{align*}
\inf_{v_h \in X_h} \| u - v_h \| & \leq C h^{k+1} |u|_{k+1} \quad u \in X \cap H^{k+1}(\Omega)^d, \\
\inf_{v_h \in X_h} \| u - v_h \|_1 & \leq C h^k |u|_{k+1} \quad u \in X \cap H^{k+1}(\Omega)^d, \\
\inf_{q_h \in Q_h} \| p - q_h \| & \leq C h^{m+1} |p|_{m+1} \quad p \in Q \cap H^{m+1}(\Omega).
\end{align*}
\]

(2.4) (2.5) (2.6)

Furthermore, we assume that \( X_h \) and \( Q_h \) satisfy the usual discrete inf-sup condition:

\[
\inf_{q \in Q_h} \sup_{v \in X_h} \frac{(q, \nabla \cdot v)}{\| \nabla v \| \| q \|} \geq C_0 > 0.
\]

(2.7)

The discrete divergence-free space \( V_h \) is defined by

\[ V_h := \{ v_h \in X_h : (\nabla \cdot v_h, q_h) = 0 \quad \forall q_h \in Q_h \}. \]

Note that the well-known Taylor-Hood mixed finite element is one such example satisfying the above assumptions with \( k = 2, m = 1. \)

The following lemmas will be useful in later analyses. For their proofs, see Theorem 1.1 on p. 59 of [7] for Lemma 2.1, Lemma 2 of [24] for Lemma 2.2, and Lemma 5.1 on p. 369 of [11] for Lemma 2.3.

**Lemma 2.1.** Suppose that the finite element spaces satisfy (2.7). Then, for any \( u \in V, \) we have

\[
\inf_{v_h \in V_h} \| \nabla (u - v_h) \| \leq C \inf_{v_h \in X_h} \| \nabla (u - v_h) \|.
\]

**Lemma 2.2.** If \( g_t, g_{tt}, g_{ttt} \in L^2(0, T; H^r(\Omega)), \) then we have

\[
\| g^{n+1} - 2g^n + g^{n-1} \|_r^2 \leq C \Delta t^3 \int_{t_{n-1}}^{t_{n+1}} \| g_{ttt} \|_r^2 dt,
\]

(2.8)
\[ \|3g^{n+1} - 4g^n + g^{n-1}\|^2_r \leq C\Delta t \int_{t_{n-1}}^{t_n} \|g_t\|^2 dt, \quad (2.9) \]
\[ \frac{3g^{n+1} - 4g^n + g^{n-1}}{2\Delta t} - g_t(t^n) \leq C\Delta t^3 \int_{t_{n-1}}^{t_n} \|g_{ttt}\|^2 dt. \quad (2.10) \]

**Lemma 2.3.** (The discrete Gronwall’s lemma, without \(\Delta t\)-restriction) Suppose that \(n\) and \(N\) are nonnegative integers, \(n \leq N\). The real numbers \(a_n, b_n, c_n, \kappa_n, \Delta t, C\) are nonnegative and satisfy

\[ a_N + \Delta t \sum_{n=0}^{N} b_n \leq \Delta t \sum_{n=0}^{N-1} \kappa_n a_n + \Delta t \sum_{n=0}^{N} c_n + C. \]

Then,

\[ a_N + \Delta t \sum_{n=0}^{N} b_n \leq \exp(\Delta t \sum_{n=0}^{N-1} \kappa_n)(\Delta t \sum_{n=0}^{N} c_n + C). \]

### 3. The BDF2 modular grad-div stabilization algorithm and its stability.

We propose the following fully-discrete modular grad-div stabilization algorithm for approximating solutions of \((1.1)\).

**BDF2-mdg:**

**Step 1:** Given \(u_h^{n-1}, u_h^n \in X_h\), find \((\hat{u}_h^{n+1}, \hat{p}_h^{n+1}) \in (X_h, Q_h)\) satisfying:

\[
\left( \frac{3\hat{u}_h^{n+1} - 4u_h^n + u_h^{n-1}}{2\Delta t}, v_h \right) + (2u_h^n - u_h^{n-1}, \hat{u}_h^{n+1}, v_h) + \nu(\nabla \hat{u}_h^{n+1}, \nabla v_h) - (p_h^{n+1}, \nabla \cdot v_h) = (f^{n+1}, v_h) \quad \forall v_h \in X_h, \quad (3.1) \\
(\nabla \cdot \hat{u}_h^{n+1}, q_h) = 0 \quad \forall q_h \in Q_h. \quad (3.2)
\]

**Step 2:** Given \(\hat{u}_h^{n+1} \in X_h\), find \(u_h^{n+1} \in X_h\) satisfying:

\[
\left( \frac{3u_h^{n+1} - 3\hat{u}_h^{n+1}}{2\Delta t}, v_h \right) + \beta(\nabla \cdot \frac{3u_h^{n+1} - 4u_h^n + u_h^{n-1}}{2\Delta t}, v_h) + \gamma(\nabla \cdot u_h^{n+1}, \nabla \cdot v_h) = 0 \quad \forall v_h \in X_h. \quad (3.3)
\]

**Remark 1.** When \(\beta = 0\), Step 2 is equivalent to Step 2 appearing in \([3]\) with \(\gamma \leftarrow \frac{2}{3} \gamma\).

Step 2 of BDF2-mdg appears to be overdetermined since both the tangential and normal components of the solution are prescribed on the boundary. However, due to the zeroth-order term, it is not; a unique solution always exists, Theorem 3.1 and converges to the true NSE solution, Theorems 4.4 and 4.5.

**Theorem 3.1.** Suppose \(f^{n+1} \in H^{-1}(\Omega)^d\) and \(u_h^{n-1}, u_h^n \in X_h\). Then, there exists unique solutions \(\hat{u}_h^{n+1}, u_h^{n+1} \in X_h\) to BDF2-mdg.

*Proof.* The proof follows by similar arguments as in Theorem 5 of \([3]\). \(\square\)

Next, we analyze the stability of BDF2-mdg. We first prove an important lemma for the stability analysis. Unconditional, nonlinear, energy stability is then proven in Theorem 3.3.

**Lemma 3.2.** Consider BDF2-mdg, then the following identities hold for Step 2 (3.3):

\[ \|\hat{u}_h^{n+1}\|^2 = \|u_h^{n+1}\|^2 + \|\hat{u}_h^{n+1} - u_h^{n+1}\|^2 + \frac{4}{3} \gamma \Delta t \|\nabla \cdot u_h^{n+1}\|^2 \]
\[ + \frac{\beta}{3} \left( \|\nabla \cdot u_h^{n+1}\|^2 - \|\nabla \cdot u_h^n\|^2 + \|\nabla \cdot (2u_h^{n+1} - u_h^n)\|^2 \right. \]
\[ - \left. \|\nabla \cdot (2u_h^n - u_h^{n-1})\|^2 + \|\nabla \cdot (u_h^{n+1} - 2u_h^n + u_h^{n-1})\|^2 \right), \quad (3.4) \]
\[
\frac{3u_{h}^{n+1} - 4u_{h}^{n} + u_{h}^{n-1}}{2\Delta t}, \quad \hat{v}_{h}^{n+1} - u_{h}^{n+1}
\]
\[
\beta \frac{\|\nabla \cdot (3u_{h}^{n+1} - 4u_{h}^{n} + u_{h}^{n-1})\|^2 + \frac{\gamma}{3}(\nabla \cdot u_{h}^{n+1}, \nabla \cdot (3u_{h}^{n+1} - 4u_{h}^{n} + u_{h}^{n-1}))}{6\Delta t}
\]

(3.5)

Proof. Selecting \( v_{h} = \frac{4\Delta t}{3}u_{h}^{n+1} \) in (3.3), we have

\[
2\|u_{h}^{n+1}\|^2 - 2(\hat{v}_{h}^{n+1}, u_{h}^{n+1}) + \frac{4}{3}\gamma\Delta t\|\nabla \cdot u_{h}^{n+1}\|^2 + \frac{\beta}{3}(\|\nabla \cdot u_{h}^{n+1}\|^2 - \|\nabla \cdot u_{h}^{n}\|^2 + \|\nabla \cdot (2u_{h}^{n+1} - 2u_{h}^{n} + u_{h}^{n-1})\|^2)
\]

(3.6)

where we have used the identity \( 2(3a - 4b + c)a = a^2 - b^2 + (2a - b)^2 - (2b - c)^2 + (a - 2b + c)^2 \) on the third term. For the second term in (3.6), using the following polarization identity

\[
(\hat{v}_{h}^{n+1}, u_{h}^{n+1}) = \frac{1}{2}\|\hat{v}_{h}^{n+1}\|^2 + \frac{1}{2}\|u_{h}^{n+1}\|^2 - \frac{1}{2}\|\hat{v}_{h}^{n+1} - u_{h}^{n+1}\|^2
\]

(3.7)

yields the first identity (3.4). The second follows by setting \( v_{h} = \frac{3u_{h}^{n+1} - 4u_{h}^{n} + u_{h}^{n-1}}{3\Delta t} \) in (3.3). \( \square \)

We are now in a position to prove unconditional stability.

**Theorem 3.3.** Suppose \( f \in L^2(0, T; H^{-1}(\Omega)^d) \), then the following holds for all \( N \geq 1 \).

\[
\|u_{h}^{n+1} \|^2 + 2\|u_{h}^{n} - u_{h}^{n-1}\|^2 + (\frac{\gamma}{3} + \beta)\|\nabla \cdot u_{h}^{n}\|^2 + (\frac{\gamma}{3} + \beta)\|\nabla \cdot (2u_{h}^{n} - u_{h}^{n-1})\|^2
\]

(3.8)

\[
\leq \frac{2\Delta t}{\nu} \sum_{n=1}^{N-1} |\nabla \cdot u_{h}^{n+1}|^2 + 2\nu\Delta t \sum_{n=1}^{N-1} |\nabla \cdot u_{h}^{n}|^2
\]

Proof. Set \( v_{h} = \hat{v}_{h}^{n+1} \) in (3.1) and \( q_{h} = p_{h}^{n+1} \) in (3.2). Adding these two equations and rearranging the discrete time derivative yields

\[
(\frac{3u_{h}^{n+1} - 4u_{h}^{n} + u_{h}^{n-1}}{2\Delta t}, u_{h}^{n+1}) + (\frac{3u_{h}^{n+1} - 4u_{h}^{n} + u_{h}^{n-1}}{2\Delta t}, \hat{u}_{h}^{n+1} - u_{h}^{n+1})
\]

(3.9)

Consider the resulting time derivative terms. Use the identity \( 2(3a - 4b + c)a = a^2 - b^2 + (2a - b)^2 - (2b - c)^2 + (a - 2b + c)^2 \) on the first term and both (3.5) of Lemma 3.2 and the identity on the second term. Apply the polarization identity to the third term. Then,

\[
\frac{1}{4\Delta t}\left(\|u_{h}^{n+1}\|^2 - \|u_{h}^{n}\|^2 + 2\|u_{h}^{n+1} - u_{h}^{n}\|^2 - \|2u_{h}^{n} - u_{h}^{n-1}\|^2 + \|u_{h}^{n+1} - 2u_{h}^{n} + u_{h}^{n-1}\|^2\right)
\]

\[
+ \frac{\gamma}{6}(\|\nabla \cdot u_{h}^{n+1}\|^2 - \|\nabla \cdot u_{h}^{n}\|^2 + \|\nabla \cdot (2u_{h}^{n+1} - u_{h}^{n})\|^2 - \|\nabla \cdot (2u_{h}^{n} - u_{h}^{n-1})\|^2)
\]

\[
+ \|\nabla \cdot (u_{h}^{n+1} - 2u_{h}^{n} + u_{h}^{n-1})\|^2\]

(3.10)

\[
+ \|\nabla \cdot (u_{h}^{n+1} - 2u_{h}^{n} + u_{h}^{n-1})\|^2 + \frac{\beta}{6\Delta t}\|\nabla \cdot (3u_{h}^{n+1} - 4u_{h}^{n} + u_{h}^{n-1})\|^2
\]

\[
\|\hat{u}_{h}^{n+1}\|^2 + \|\hat{u}_{h}^{n+1} - u_{h}^{n+1}\|^2 + \nu\|\nabla \hat{u}_{h}^{n+1}\|^2 = (f_{h}^{n+1}, \hat{u}_{h}^{n+1}).
\]
Multiply (3.10) by $4\Delta t$ and use (3.4) of Lemma 3.2. Then

$$
\|u_{h}^{n+1}\|^2 - \|u_{h}^{n}\|^2 + 2\|u_{h}^{n+1} - u_{h}^{n}\|^2 - 2\|u_{h}^{n} - u_{h}^{n-1}\|^2 + \|u_{h}^{n+1} - 2u_{h}^{n} + u_{h}^{n-1}\|^2
+ \frac{2\gamma\Delta t}{3}\left(\|\nabla \cdot u_{h}^{n+1}\|^2 - \|\nabla \cdot u_{h}^{n}\|^2 + \|\nabla \cdot (2u_{h}^{n+1} - u_{h}^{n})\|^2 - \|\nabla \cdot (2u_{h}^{n} - u_{h}^{n-1})\|^2\right)
+ \|\nabla \cdot (u_{h}^{n+1} - 2u_{h}^{n} + u_{h}^{n-1})\|^2 + 2\beta\left(\|\nabla \cdot u_{h}^{n+1}\|^2 - \|\nabla \cdot u_{h}^{n}\|^2 + \|\nabla \cdot (2u_{h}^{n+1} - u_{h}^{n})\|^2 - \|\nabla \cdot (2u_{h}^{n} - u_{h}^{n-1})\|^2\right)
+ \|\nabla \cdot (u_{h}^{n+1} - 2u_{h}^{n} + u_{h}^{n-1})\|^2 + 2\|\nabla \cdot (u_{h}^{n+1})\|^2
+ 4\gamma\Delta t\|\nabla \cdot u_{h}^{n+1}\|^2 + 4\nu\Delta t\|\nabla \hat{u}_{h}^{n+1}\|^2
= 4\Delta t(f^{n+1}, \hat{u}_{h}^{n+1}).
$$

Summing (3.11) from $n = 1$ to $N - 1$ yields

$$
\|u_{h}^{N}\|^2 - \|u_{h}^{0}\|^2 + 2\|u_{h}^{N} - u_{h}^{N-1}\|^2 + \frac{2\gamma\Delta t}{3}\sum_{n=1}^{N-1}\|\nabla \cdot u_{h}^{n}\|^2 + \|\nabla \cdot (2u_{h}^{N} - u_{h}^{N-1})\|^2 + \beta\|\nabla \cdot u_{h}^{N}\|^2
+ \beta\sum_{n=1}^{N-1}\|\nabla \cdot (u_{h}^{n+1})\|^2 + 4\gamma\Delta t\sum_{n=1}^{N-1}\|\nabla \cdot u_{h}^{n}\|^2 + 4\nu\Delta t\sum_{n=1}^{N-1}\|\nabla \hat{u}_{h}^{n+1}\|^2
\leq 4\Delta t\left(\sum_{n=1}^{N-1}(f^{n+1}, \hat{u}_{h}^{n+1}) + \|u_{h}^{0}\|^2 + \|2u_{h}^{n} - u_{h}^{n-1}\|^2
\right.
+ \frac{2\gamma\Delta t}{3}\sum_{n=1}^{N-1}\|\nabla \cdot u_{h}^{n}\|^2 + \frac{2\gamma\Delta t}{3}\|\nabla \cdot (2u_{h}^{n} - u_{h}^{n-1})\|^2 + \beta\|\nabla \cdot u_{h}^{N}\|^2 + \beta\|\nabla \cdot (2u_{h}^{N} - u_{h}^{N-1})\|^2.
$$

Finally, using the Cauchy-Schwarz-Young inequality on the first term on the right hand side completes the proof.

**Remark 2.** Lemma 3.2 and Theorem 3.3 imply stability of $\hat{u}_{h}$ with respect to $\|\cdot\|_{\infty,0}$.

**4. Error Analysis.** In this section, we provide $\hat{a}$ priori error estimates for BDF2-mgd. In particular, we show that BDF2-mgd is second-order convergent. Denote $u^{n} = u(t^{n})$ for $n = 0, 1, \cdots, N$ (and similarly for all other variables). The errors are denoted by

$$
e_{u}^{n} = u^{n} - u_{h}^{n}, \quad \hat{e}_{u}^{n} = u^{n} - \hat{u}_{h}^{n}, \quad \hat{e}_{p}^{n} = p^{n} - \hat{p}_{h}^{n+1}.
$$

Decompose the velocity errors

$$
e_{u}^{n} = \eta^{n} - \phi_{h}^{n}, \quad \eta^{n} := u^{n} - \hat{u}_{h}^{n}, \quad \phi_{h}^{n} := u_{h}^{n} - \hat{u}_{h}^{n},
$$

$$
e_{u}^{n} = \eta^{n} - \psi_{h}^{n}, \quad \psi_{h}^{n} := \hat{u}_{h}^{n} - \hat{u}_{h}^{n},
$$

where $\hat{u}_{h}^{n}$ denotes an interpolant of $u^{n}$ in $V_{h}$.

**Definition 4.1.** Define the following consistency errors. For all $v_{h} \in V_{h}$,

$$
\tau^{n+1}(v_{h}) := \left(\frac{3u_{h}^{n+1} - 4u_{h}^{n} + u_{h}^{n-1}}{2\Delta t} - u_{h}^{n+1}, v_{h}\right) - b(u_{h}^{n+1} - 2u_{h}^{n} + u_{h}^{n-1}, u_{h}^{n+1}, v_{h}).
$$

**Lemma 4.2.** Assume the true solution $u$ satisfies the following,

$$
u \in L^{\infty}(0, T; H^{1}(\Omega)^{d}), \quad u_{tt} \in L^{2}(0, T; H^{1}(\Omega)^{d}), \quad u_{ttt} \in L^{2}(0, T; H^{-1}(\Omega)^{d}).$$

Then, $\forall \sigma > 0$, we have

$$
|\tau^{n+1}(v_{h})| \leq \frac{C}{2\sigma} \Delta t^{2}\left(\int_{t^{n-1}}^{t^{n+1}} \|u_{tt}\|_{2}^{2} dt + \int_{t^{n-1}}^{t^{n+1}} \|\nabla u_{tt}\|^{2} dt\right) + \sigma \|\nabla v_{h}\|^{2}.
$$
Proof. For an arbitrary $\sigma > 0$, 

$$
|\tau^{n+1}(v_h)| 
\leq \frac{3u^{n+1} - 4u^n + u^{n-1}}{2\Delta t} - u_t^{n+1} - 1||\nabla v_h|| + C||\nabla (u^{n+1} - 2u^n + u^{n-1})||||\nabla u^{n+1}||||\nabla v_h|| 
\leq \frac{1}{2\sigma}||\frac{3u^{n+1} - 4u^n + u^{n-1}}{2\Delta t} - u_t^{n+1}||^2 - \frac{1}{2}\gamma$$

(4.4)

$$
+ C\Delta t^3 \int_{t^{n-1}}^{t^{n+1}} ||u_{ttt}||^2 dt + \int_{t^{n-1}}^{t^{n+1}} ||\nabla u_{tt}||^2 dt + \sigma||\nabla v_h||^2,
$$

where we use the Cauchy-Schwarz-Young inequality and Lemma 2.2

Once again, we require a key lemma, regarding Step 2, to prove convergence.

Lemma 4.3. The following inequality holds.

$$
\|\psi_h^{n+1}\|^2 \geq ||\phi_h^{n+1}||^2 + ||\phi_h^{n+1} - \psi_h^{n+1}||^2 + \frac{\beta}{3} (||\nabla \cdot \phi_h^{n+1}||^2 - ||\nabla \cdot \phi_h^n||^2 + ||\nabla \cdot (2\phi_h^{n+1} - \phi_h^n)||^2) 
- ||\nabla \cdot (2\phi_h^n - \phi_h^{n-1})||^2 + \frac{1}{2} ||\nabla \cdot (\phi_h^{n+1} - 2\phi_h^n + \phi_h^{n-1})||^2 + \frac{2\gamma \Delta t}{3} ||\nabla \cdot \phi_h^{n+1}||^2 
- C\beta d(1 + 2\Delta t) \int_{t^{n-1}}^{t^{n+1}} ||\nabla \eta||^2 dt - \frac{\beta \Delta t}{3} ||\nabla \cdot (2\phi_h^n - \phi_h^{n-1})|| \geq \frac{2\gamma d \Delta t}{3} ||\nabla \eta^{n+1}||^2.
$$

Proof. At time $t^{n+1}$, for all $v_h \in X_h$, the true solution $u$ satisfies

$$
\left(\frac{3u^{n+1} - 3u^n}{2\Delta t}, v_h\right) + \beta(\nabla \cdot \frac{3u^{n+1} - 4u^n + u^{n-1}}{2\Delta t}, \nabla \cdot v_h) + \gamma(\nabla \cdot u^{n+1}, \nabla \cdot v_h) = 0. \quad (4.6)
$$

Subtracting (4.6) from (3.3), we have

$$
\left(\frac{3\phi_h^{n+1} - 3\phi_h^n}{2\Delta t}, v_h\right) + \beta(\nabla \cdot \frac{3\phi_h^{n+1} - 4\phi_h^n + \phi_h^{n-1}}{2\Delta t}, \nabla \cdot v_h) + \gamma(\nabla \cdot \phi_h^{n+1}, \nabla \cdot v_h) = 0. \quad (4.7)
$$

Setting $v_h = \phi_h^{n+1}$ in (4.7), using similar identities as in Theorem 3.2 and rearranging terms yields

$$
\|\psi_h^{n+1}\|^2 = ||\phi_h^{n+1}||^2 + ||\phi_h^{n+1} - \psi_h^{n+1}||^2 + \frac{\beta}{3} (||\nabla \cdot \phi_h^{n+1}||^2 - ||\nabla \cdot \phi_h^n||^2 + ||\nabla \cdot (2\phi_h^{n+1} - \phi_h^n)||^2 
- ||\nabla \cdot (2\phi_h^n - \phi_h^{n-1})||^2 + ||\nabla \cdot (\phi_h^{n+1} - 2\phi_h^n + \phi_h^{n-1})||^2 + \frac{4\gamma \Delta t}{3} ||\nabla \cdot \phi_h^{n+1}||^2 
- \frac{2\beta}{3} (\nabla \cdot (3\eta^{n+1} - 4\eta^n + \eta^{n-1}), \nabla \cdot \phi_h^{n+1}) - \frac{4\gamma \Delta t}{3} (\nabla \cdot \eta^{n+1}, \nabla \cdot \phi_h^{n+1}).
$$

Split $-\frac{2\beta}{3} (\nabla \cdot (3\eta^{n+1} - 4\eta^n + \eta^{n-1}), \nabla \cdot \phi_h^{n+1})$ into $-\frac{2\beta}{3} (\nabla \cdot (3\eta^{n+1} - 4\eta^n + \eta^{n-1}), \nabla \cdot (\phi_h^{n+1} - 2\phi_h^n + \phi_h^{n-1})) - \frac{2\beta}{3} (\nabla \cdot (3\eta^{n+1} - 4\eta^n + \eta^{n-1}), \nabla \cdot (2\phi_h^n - \phi_h^{n-1})).$ Using the Cauchy-Schwarz-Young
inequality and Lemma 2.3 Then, the following three inequalities hold,

\[ \frac{2\beta}{3} (\nabla \cdot (3\eta^{n+1} - 4\eta^{n} + \eta^{n-1}), \nabla \cdot (2\phi^{n}_h - \phi^{n-1}_h)) \leq \frac{2\beta\sqrt{d}}{3} \|\nabla(3\eta^{n+1} - 4\eta^{n} + \eta^{n-1})\| \|\nabla \cdot (2\phi^{n}_h - \phi^{n-1}_h)\| \]

(4.9)

\[ \leq \frac{C\beta d}{3} \int_{t_{n-1}}^{t_{n+1}} \|\nabla \eta_h\|^2 dt + \frac{\beta \Delta t}{3} \|\nabla \cdot (2\phi^{n}_h - \phi^{n-1}_h)\|^2 \]

(4.10)

\[ \leq \frac{C\beta d \Delta t}{3} \int_{t_{n-1}}^{t_{n+1}} \|\nabla \eta_h\|^2 dt + \frac{\beta}{6} \|\nabla \cdot (2\phi^{n}_h - \phi^{n-1}_h)\|^2, \]

and

\[ \frac{4\gamma \Delta t}{3} (\nabla \cdot \eta^{n+1}, \nabla \cdot \phi^{n+1}_h) \leq \frac{4\gamma \sqrt{d} \Delta t}{3} \|\nabla \eta^{n+1}\| \|\nabla \cdot \phi^{n+1}_h\| \]

(4.11)

\[ \leq \frac{2\gamma d \Delta t}{3} \|\nabla \eta^{n+1}\|^2 + \frac{2\gamma \Delta t}{3} \|\nabla \cdot \phi^{n+1}_h\|. \]

Combining (4.8) - (4.11) completes the proof. \(\Box\)

Next, we give the main error result for BDF2-mgd when \(\beta > 0\).

**Theorem 4.4.** Assume the true solution \(u, p\) satisfy (4.2) and the following regularity

\[ u \in L^\infty(0, T; H^{k+1}(\Omega)^d) \cap L^2(0, T; H^{k+1}(\Omega)^d), \]

\[ u \in L^2(0, T; H^{k+1}(\Omega)^d), \quad p \in L^2(0, T; H^{m+1}(\Omega)). \]

Then, we have the following estimates for BDF2-mgd.

\[ \|e^N_u\|^2 + \|2e^N_u - e^{N-1}_u\|^2 + \left(\frac{2\gamma \Delta t}{3} + \beta\right) \left(\|\nabla \cdot e^N_u\|^2 + \|\nabla \cdot (2e^N_u - e^{N-1}_u)\|^2\right) \]

\[ + 2\nu \Delta t \sum_{n=1}^{N-1} \|\nabla e^{n+1}_u\|^2 + 2\gamma \Delta t \sum_{n=1}^{N-1} \|\nabla \cdot e^{n+1}_u\|^2 \]

\[ \leq C \exp(C^*T) \left\{ \inf_{v_h \in X_h} \left(\beta(1 + \Delta t)\|\nabla(u - v_h)\|_{2,0}^2 + \frac{1}{\nu} \|(u - v_h)\|_{2,0}^2 \right) \right. \]

\[ + \left. \left(\frac{\gamma^2 \Delta t}{\beta} + \gamma + \nu + \frac{1}{\nu}\right) \|\nabla(u - v_h)\|_{2,0}^2 + \frac{2\gamma \Delta t}{3} + \beta + \frac{1}{\nu^2} \right) \|\nabla(u - v_h)\|_{2,0}^2 \]

\[ + \|u - v_h\|_{2,0}^2 + \frac{1}{\nu} \inf_{q_h \in Q_h} \left(\|p - q_h\|_{2,0}^2 + \frac{1}{\nu} \Delta t^4 \right) \]

\[ + \|e^1_u\|^2 + \|2e^1_u - e^0_u\|^2 + \left(\frac{2\gamma \Delta t}{3} + \beta\right) \left(\|\nabla \cdot e^1_u\|^2 + \|\nabla \cdot (2e^1_u - e^0_u)\|^2\right) \right\}. \]
Proof. At time $t^{n+1}$, the true solution $u, p$ satisfies

$$
\begin{align*}
&\frac{3u^{n+1} - 4u^n + u^{n-1}}{2\Delta t}, v_h) + b(2u^n - u^{n-1}, u^{n+1}, v_h) + \nu(\nabla u^n, \nabla v_h) \\
&- (p^{n+1}, \nabla \cdot v_h) = (f^{n+1}, v_h) + \tau^{n+1}(v_h) \quad \forall v_h \in X_h, \ (\ref{4.14}) \\
&\quad (\nabla \cdot u^{n+1}, q_h) = 0 \quad \forall q_h \in Q_h. \quad (\ref{4.15})
\end{align*}
$$

Subtracting (3.1) and (3.2) from (4.14) and (4.15), respectively, we have

$$
\begin{align*}
&\frac{3e^{n+1} - 4e^n + e^{n-1}}{2\Delta t}, v_h) + b(2u^n - u^{n-1}, u^{n+1}, v_h) - b(2u^n - u^{n-1}, u^{n+1}, v_h) \\
&+ \nu(\nabla e^n, \nabla v_h) - (e^{n+1}, \nabla \cdot v_h) = \tau^{n+1}(v_h) \quad \forall v_h \in X_h, \ (\ref{4.16}) \\
&\quad (\nabla \cdot e^n, q_h) = 0 \quad \forall q_h \in Q_h. \quad (\ref{4.17})
\end{align*}
$$

Set $v_h = \psi^{n+1} \in V_h$ in equation (4.16), then

$$
\begin{align*}
&\frac{3\phi^{n+1} - 4\eta^n + \eta^{n-1}}{2\Delta t}, \psi^{n+1}) - \frac{3\phi^{n+1} - 4\phi^n + \phi^{n-1}}{2\Delta t} \\
&- \left( \frac{3\phi^{n+1} - 4\phi^n + \phi^{n-1}}{2\Delta t}, \psi^{n+1} - \phi^{n+1} \right) - \left( \frac{3\psi^{n+1} - 3\phi^{n+1}}{2\Delta t}, \psi^{n+1} \right) \\
&+ b(2u^n - u^{n-1}, u^{n+1}, \psi^{n+1}) - b(2u^n - u^{n-1}, u^{n+1}, \psi^{n+1}) \\
&+ \nu(\nabla \eta, \nabla \psi^{n+1}) - \nu\|\nabla \psi^{n+1}\|^2 - (p^{n+1} - q_h, \nabla \cdot \psi^{n+1}) = \tau^{n+1}(\psi^{n+1}).
\end{align*}
$$

Here, $q_h \in Q_h$ is arbitrary. Furthermore, setting $v_h = \frac{3\phi^{n+1} - 4\phi^n + \phi^{n-1}}{4} \in V_h$ in (4.7) and rearranging terms yields

$$
\begin{align*}
&\frac{3\phi^{n+1} - 4\phi^n + \phi^{n-1}}{2\Delta t}, \psi^{n+1} - \phi^{n+1} \\
&= \frac{\gamma}{3}(\nabla \cdot (3\phi^{n+1} - 4\phi^n + \phi^{n-1}), \nabla \cdot \phi^{n+1}) + \frac{\beta}{6\Delta t}\|\nabla \cdot (3\phi^{n+1} - 4\phi^n + \phi^{n-1})\|^2 \\
&- \frac{\gamma}{3}(\nabla \cdot (3\phi^{n+1} - 4\phi^n + \phi^{n-1}), \nabla \cdot \phi^{n+1}) \\
&- \frac{\beta}{6\Delta t}(\nabla \cdot (3\eta^{n+1} - 4\eta^n + \eta^{n-1}), \nabla \cdot (3\phi^{n+1} - 4\phi^n + \phi^{n-1})).
\end{align*}
$$

Combine (4.18) and (4.19) and rearrange. Then,

$$
\begin{align*}
&\frac{3\phi^{n+1} - 4\phi^n + \phi^{n-1}}{2\Delta t}, \psi^{n+1} + \frac{\gamma}{3}(\nabla \cdot (3\phi^{n+1} - 4\phi^n + \phi^{n-1}), \nabla \cdot \phi^{n+1}) \\
&+ \frac{\beta}{6\Delta t}\|\nabla \cdot (3\phi^{n+1} - 4\phi^n + \phi^{n-1})\|^2 + \nu\|\nabla \psi^{n+1}\|^2 \\
&+ \frac{3}{4\Delta t}(\|\psi^{n+1}\|^2 - \|\phi^{n+1}\|^2 + \|\psi^{n+1} - \phi^{n+1}\|^2) \\
&= \frac{3\phi^{n+1} - 4\eta^n + \eta^{n-1}}{2\Delta t}, \psi^{n+1} + \frac{\gamma}{3}(\nabla \cdot (3\phi^{n+1} - 4\phi^n + \phi^{n-1}), \nabla \cdot \eta^{n+1}) \\
&+ \frac{\beta}{6\Delta t}(\nabla \cdot (3\eta^{n+1} - 4\eta^n + \eta^{n-1}), \nabla \cdot (3\phi^{n+1} - 4\phi^n + \phi^{n-1})) \\
&+ b(2u^n - u^{n-1}, u^{n+1}, \psi^{n+1}) - b(2u^n - u^{n-1}, \psi^{n+1}) \\
&+ \nu(\nabla \eta^{n+1}, \nabla \psi^{n+1}) - (p^{n+1} - q_h, \nabla \cdot \psi^{n+1}) - \tau^{n+1}(\psi^{n+1}).
\end{align*}
$$
Multiplying (4.20) by $4\Delta t$ and use (4.5). Then,
\[
\|\phi_h^{n+1}\|^2 - \|\phi_h^n\|^2 + 2\phi_h^{n+1} - \phi_h^n - \|2\phi_h^{n+1} - \phi_h^n\|^2 + \|\phi_h^{n+1} - 2\phi_h^n + \phi_h^{-1}\|^2
\]
\[
+ \left(\frac{2\gamma \Delta t}{3}\right) + \beta \left(\|\nabla \cdot \phi_h^{n+1}\|^2 - \|\nabla \cdot \phi_h^n\|^2 + \|\nabla \cdot (2\phi_h^{n+1} - \phi_h^n)\|^2
\]
\[
- \|\nabla \cdot (2\phi_h^{n+1} - \phi_h^n)\|^2 + \|\nabla \cdot (\phi_h^{n+1} - 2\phi_h^n + \phi_h^{-1})\|^2\right)
\]
\[
+ \frac{2\beta}{3}\|\nabla \cdot (3\phi_h^{n+1} - 4\phi_h^n + \phi_h^{-1})\|^2 + 6\|\psi_h^{n+1} - \phi_h^{-1}\|^2
\]
\[
+ 4\nu \Delta t\|\nabla \psi_h^{n+1}\|^2 + 2\gamma \Delta t\|\nabla \cdot \phi_h^{n+1}\|^2
\]
\[
\leq 2(3\eta^{n+1} - 4\eta^n + \eta^{-1}, \psi_h^{n+1}) + \frac{4\gamma \Delta t}{3}(\nabla \cdot (3\phi_h^{n+1} - 4\phi_h^n + \phi_h^{-1}), \nabla \cdot \eta^{n+1})
\]
\[
+ \frac{2\beta}{3}(\nabla \cdot (3\eta^{n+1} - 4\eta^n + \eta^{-1}), \nabla \cdot (3\phi_h^{n+1} - 4\phi_h^n + \phi_h^{-1}))
\]
\[
+ 4\Delta t b(2u^n - u^{n-1}, v_h^{n+1}) - 4\Delta t b(2u^n - u^{n-1}, \psi_h^{n+1})
\]
\[
+ 4\nu \Delta t(\nabla \eta^{n+1}, \nabla \psi_h^{n+1}) - 4\Delta t(\nabla \psi_h^{n+1} - q_h, \nabla \cdot \psi_h^{n+1}) - 4\Delta t r^{n+1}(\psi_h^{n+1})
\]
\[
+ C\beta d(1 + 2\Delta t) \int_{t_{n-1}}^{t_{n+1}} \|\nabla \eta_t\|^2 dt + 2\gamma d\Delta t\|\nabla \eta^{n+1}\|^2
\]
\[
+ \frac{\beta}{2}\|\nabla \cdot (\psi_h^{n+1} - 2\phi_h^n + \phi_h^{-1})\|^2 + \beta \Delta t(\|\nabla \cdot (2\phi_h^{n+1} - \phi_h^n)\|^2 + \|\nabla \cdot \phi_h^{-1}\|^2).
\]

Next, we need to bound the terms on the right hand side of (4.21). Applying Lemma 2.2 the Poincaré-Friedrichs inequality, and the Cauchy-Schwarz-Young inequality, for an arbitrary $\delta > 0$, we have
\[
2(3\eta^{n+1} - 4\eta^n + \eta^{-1}, \psi_h^{n+1}) \leq C\|3\eta^{n+1} - 4\eta^n + \eta^{-1}||\|\nabla \psi_h^{n+1}||
\]
\[
\leq \frac{C}{\delta \nu} \int_{t_{n-1}}^{t_{n+1}} \|\eta_t\|^2 dt + \delta \nu \Delta t\|\nabla \psi_h^{n+1}\|^2.
\]
\[
\left(\frac{4\gamma \Delta t}{3}\right)(\nabla \cdot (3\phi_h^{n+1} - 4\phi_h^n + \phi_h^{-1}), \nabla \cdot \eta^{n+1})
\]
\[
\leq \frac{4\gamma \sqrt{\delta} \Delta t}{3}\|\nabla \cdot (3\phi_h^{n+1} - 4\phi_h^n + \phi_h^{-1})\|\|\nabla \eta^{n+1}\|
\]
\[
\leq \frac{\beta}{3}\|\nabla \cdot (3\phi_h^{n+1} - 4\phi_h^n + \phi_h^{-1})\|^2 + \frac{4\Delta t\|\nabla \eta^{n+1}\|^2}{3\beta}.
\]
\[
\frac{2\beta}{3}(\nabla \cdot (3\eta^{n+1} - 4\eta^n + \eta^{-1}), \nabla \cdot (3\phi_h^{n+1} - 4\phi_h^n + \phi_h^{-1}))
\]
\[
\leq \frac{2\beta \sqrt{\delta}}{3}\|\nabla (3\eta^{n+1} - 4\eta^n + \eta^{-1})\|\|\nabla \cdot (3\phi_h^{n+1} - 4\phi_h^n + \phi_h^{-1})\|
\]
\[
\leq \frac{\beta}{3}\|\nabla \cdot (3\phi_h^{n+1} - 4\phi_h^n + \phi_h^{-1})\|^2 + \frac{C\beta \Delta t}{3} \int_{t_{n-1}}^{t_{n+1}} \|\nabla \eta_t\|^2 dt.
\]

Furthermore,
\[
4\nu \Delta t(\nabla \eta^{n+1}, \nabla \psi_h^{n+1}) \leq \frac{4\nu \Delta t}{\delta}\|\nabla \eta^{n+1}\|^2 + \delta \nu \Delta t\|\nabla \psi_h^{n+1}\|^2.
\]
\[
- 4\Delta t(p^{n+1} - q_h, \nabla \cdot \psi_h^{n+1}) \leq \frac{4\Delta t}{\delta \nu}\|p^{n+1} - q_h\|^2 + \delta \nu \Delta t\|\nabla \psi_h^{n+1}\|^2.
\]
Applying Lemma 4 yields
\[ -4\Delta t e_{n+1}(\psi_h^{n+1}) \leq \frac{C\Delta t^4}{\delta\nu} \int_{t_n-1}^{t_{n+1}} \|u_{tt}\|^2 dt + \frac{C\Delta t^4}{\delta\nu} \int_{t_n-1}^{t_{n+1}} \|
abla u_{tt}\|^2 dt + \delta\nu \Delta t \|
abla \psi_h^{n+1}\|^2. \] (4.27)

For the nonlinear terms, we treat them as follows. Adding and subtracting \(4\Delta t b(2u_h^n - u_h^{n-1}, u^{n+1}_h, \psi^{n+1}_h)\) yields
\[ 4\Delta t b(2u^n - u^{n-1}, u^{n+1}, \psi^{n+1}) - 4\Delta t b(2u_h^n - u_h^{n-1}, \hat{u}_h^{n+1}, \psi^{n+1}_h) = 4\Delta t \left( b(2\eta^n - \eta^{n-1}, u^{n+1}, \psi^{n+1}) - b(2\phi_h^n - \phi_h^{n-1}, u^{n+1}, \psi^{n+1}_h) + b(2\hat{u}_h^n - \hat{u}_h^{n-1}, \eta^{n+1}, \psi^{n+1}_h) \right). \] (4.28)

Then,
\[ 4\Delta t b(2\eta^n - \eta^{n-1}, u^{n+1}, \psi^{n+1}) \leq 4C\Delta t \|
abla (2\eta^n - \eta^{n-1})\| \|
abla u^{n+1}\| \|
abla \psi_h^{n+1}\| \leq \frac{4C\Delta t}{\delta\nu} \|
abla (2\eta^n - \eta^{n-1})\|^2 + \|
abla u^{n+1}\|^2 + \delta\nu \Delta t \|
abla \psi_h^{n+1}\|^2 \] (4.29)
\[ \leq 16C\Delta t \|
abla \eta^n\|^2 + \|
abla \eta^{n-1}\|^2 + \|
abla u\|^2_\infty + \delta\nu \Delta t \|
abla \psi_h^{n+1}\|^2, \]
\[ -4\Delta t b(2\phi_h^n - \phi_h^{n-1}, u^{n+1}, \psi^{n+1}_h) \leq 4C\Delta t \|
abla (2\phi_h^n - \phi_h^{n-1})\| \|
abla u^{n+1}\|_2 \|
abla \psi_h^{n+1}\| \leq \frac{4C\Delta t}{\delta\nu} \|
abla (2\phi_h^n - \phi_h^{n-1})\|^2 + \|
abla u^{n+1}\|^2_2 + \delta\nu \Delta t \|
abla \psi_h^{n+1}\|^2 \] (4.30)
\[ \leq 8C\Delta t (\|
abla (2\phi_h^n - \phi_h^{n-1})\|^2 + \|
abla \phi_h^n\|^2_2) + \|
abla u^{n+1}\|^2_2 + \delta\nu \Delta t \|
abla \psi_h^{n+1}\|^2, \]
\[ 4\Delta t b(2u_h^n - u_h^{n-1}, \eta^{n+1}, \psi^{n+1}_h) \leq 4C\Delta t \|
abla (2u_h^n - u_h^{n-1})\| \|
abla \eta^{n+1}\| \|
abla \psi_h^{n+1}\| \leq \frac{4C\Delta t}{\delta\nu} \|
abla (2u_h^n - u_h^{n-1})\|^2 + \|
abla \eta^{n+1}\|^2 + \delta\nu \Delta t \|
abla \psi_h^{n+1}\|^2 \] (4.31)
\[ \leq 16C\Delta t (\|
abla u_h^n\|^2 + \|
abla \eta^{n-1}\|^2) + \|
abla \eta\|^2_\infty + \delta\nu \Delta t \|
abla \psi_h^{n+1}\|^2. \]

Setting \(\delta = \frac{2}{3}\) and using the estimates (4.22)-(4.31) in (4.21) yields
\[ \|
abla \phi_h^{n+1}\|^2 - \|
abla \phi_h^n\|^2 + 2\|
abla \phi_h^{n+1} - \phi_h^n\|^2 + 2\|
abla \phi_h^n - \phi_h^{n-1}\|^2 + \|
abla \phi_h^{n+1} - 2\phi_h^n + \phi_h^{n-1}\|^2 \]
\[ + (\frac{2\gamma \Delta t}{3} + \beta) (\|
abla \phi_h^{n+1}\|^2 + \|
abla \phi_h^n\|^2 + \|
abla \phi_h^{n-1}\|^2) + \frac{\gamma}{2} \|
abla \phi_h^{n+1} - \phi_h^n\|^2 + \frac{\gamma}{2} \|
abla \phi_h^n - \phi_h^{n-1}\|^2 \]
\[ + 6\|
abla \phi_h^n - \phi_h^{n-1}\|^2 + 2\|
abla \eta\|^2 \|
abla \phi_h^{n+1}\|^2 + 2\|
abla \eta\|^2 \|
abla \phi_h^n\|^2 \]
\[ \leq \frac{C\Delta t}{\nu} \|
abla u^{n+1}\|^2 + \frac{C\Delta t}{\nu} \|
abla \eta^n\|^2 + \frac{C\Delta t}{\nu} \|
abla \eta^{n-1}\|^2 + \frac{C\Delta t}{\nu} \|
abla \phi_h^n\|^2 \]
\[ + \frac{C\beta \Delta t}{\nu} \|
abla \eta\|^2 \|
abla \phi_h^{n+1}\|^2 + \frac{C\Delta t}{\nu} \|
abla \phi_h^{n+1}\|^2, \] (4.32)
Sum (4.32) from $n = 1$ to $N - 1$ to get

$$
\|\phi_h^N\| + 2\|\phi_h^N - \phi_{h-1}^N\|^2 + \frac{2\gamma \Delta t}{3} \left(\|\nabla \cdot \phi_h^N\|^2 + \|\nabla \cdot (2\phi_h^N - \phi_{h-1}^N)\|^2\right)
\leq \Delta t \sum_{n=1}^{N-1} \left(C_{\nu} 2\|\alpha_n^+\|_2^2 + 2\|\phi_h^N - \phi_{h-1}^N\|^2 + \beta(\|\nabla \cdot \phi_h^N\|^2 + \|\nabla \cdot (2\phi_h^N - \phi_{h-1}^N)\|^2)\right)
\leq \Delta t \sum_{n=1}^{N-1} \left(C_{\nu} 2\|\alpha_n^+\|_2^2 + C_{\nu} 2\|\alpha_n^+\|^2 + \beta(\|\nabla \cdot \phi_h^N\|^2 + \|\nabla \cdot (2\phi_h^N - \phi_{h-1}^N)\|^2)\right)
\leq C \exp(C^* T) \left\{ \inf_{v_h \in V_h} \left( \beta(1 + \Delta t)\|\nabla (u - v_h)\|_{L^2}^2 + \frac{1}{\nu}\|(u - v_h)\|_{L^2}^2\right) + \frac{\gamma \Delta t}{\beta} \left|\nabla (u - v_h)\right|_{L^2}^2 + \frac{1}{\nu}\left|\nabla (u - v_h)\right|_{L^2}^2 + \frac{1}{\nu}\Delta t^4\right\}.
$$

Denote $C^* = \max\{C_{\nu} 2\|\alpha_n^+\|_2^2, 1\}$. Then, Lemma 2.3 the boundedness of $\nu \Delta t \sum_{n=1}^{N-1} \|\nabla \alpha_n^+\|^2$ (Theorem 3.3), and taking infimums over $V_h$ and $Q_h$ yield

$$
\|\phi_h^N\|^2 + 2\|\phi_h^N - \phi_{h-1}^N\|^2 + \frac{2\gamma \Delta t}{3} \left(\|\nabla \cdot \phi_h^N\|^2 + \|\nabla \cdot (2\phi_h^N - \phi_{h-1}^N)\|^2\right)
\leq C \exp(C^* T) \left\{ \inf_{v_h \in V_h} \left( \beta(1 + \Delta t)\|\nabla (u - v_h)\|_{L^2}^2 + \frac{1}{\nu}\|(u - v_h)\|_{L^2}^2\right) + \frac{\gamma \Delta t}{\beta} \left|\nabla (u - v_h)\right|_{L^2}^2 + \frac{1}{\nu}\left|\nabla (u - v_h)\right|_{L^2}^2 + \frac{1}{\nu}\Delta t^4\right\}.
$$

Then, using Lemma 2.1 and the triangle inequality completes the proof. □

The above result has dependence on $\beta - 1$. Consequently, we consider the convergence of BDF2-mgd when $\beta = 0$ separately.

**Theorem 4.5.** Assume the true solution $u, p$ satisfy (4.2) and (4.12). Then, when $\beta = 0$, we have the following estimates for BDF2-mgd.

$$
\|e_u^N\|^2 + 2\|e_u^N - e_u^{N-1}\|^2 + \frac{2\gamma \Delta t}{3} \left(\|\nabla \cdot e_u^N\|^2 + \|\nabla \cdot (2e_u^N - e_u^{N-1})\|^2\right)
\leq C \exp(C^* T) \left\{ \inf_{v_h \in X_h} \left( \frac{1}{\nu}\|(u - v_h)\|_{L^2}^2 + \|u - v_h\|_{L^2}^2 + \frac{1}{\nu}\|\nabla (u - v_h)\|_{L^2}^2\right) + \frac{\gamma \Delta t}{\beta} \left|\nabla (u - v_h)\right|_{L^2}^2 + \frac{1}{\nu}\|\nabla (u - v_h)\|_{L^2}^2 + \frac{1}{\nu}\Delta t^4\right\}.
$$
Proof. Similar to (4.21), we have

\[
\begin{align*}
&\|\phi_{h}^{n+1}\|^2 - \|\phi_{h}^{n}\|^2 + \|2\phi_{h}^{n+1} - \phi_{h}^{n}\|^2 - \|2\phi_{h}^{n} - \phi_{h}^{n-1}\|^2 + \|\phi_{h}^{n+1} - 2\phi_{h}^{n} + \phi_{h}^{n-1}\|^2 \\
&\quad + \frac{2\gamma \Delta t}{3} \left( \|\nabla \cdot \phi_{h}^{n+1}\|^2 - \|\nabla \cdot \phi_{h}^{n}\|^2 + \|\nabla \cdot (2\phi_{h}^{n+1} - \phi_{h}^{n})\|^2 \\
&\quad - \|\nabla \cdot (2\phi_{h}^{n} - \phi_{h}^{n-1})\|^2 + \|\nabla \cdot (\phi_{h}^{n+1} - 2\phi_{h}^{n} + \phi_{h}^{n-1})\|^2 \right) \\
&\quad + 6\|\psi_{h}^{n+1} - \phi_{h}^{n+1}\|^2 + 4\nu \Delta t \|\nabla \psi_{h}^{n+1}\|^2 + 2\gamma \Delta t \|\nabla \cdot \phi_{h}^{n+1}\|^2 \\
&\leq 2(3\eta^{n+1} - 4\eta^{n} + \eta^{n-1}, \psi_{h}^{n+1}) + \frac{4\gamma \Delta t}{3} (\nabla \cdot (3\phi_{h}^{n+1} - 4\phi_{h}^{n} + \phi_{h}^{n-1}), \nabla \cdot \eta^{n+1}) \\
&\quad + 4\Delta t (2u^{n} - u^{n-1}, \psi_{h}^{n+1}) - 4\Delta t (2u_{h}^{n} - u_{h}^{n-1}, \psi_{h}^{n+1}) \\
&\quad + 4\nu \Delta t (\nabla \psi_{h}^{n+1}, \nabla \psi_{h}^{n+1}) - 4\Delta t (p^{n+1} - q_{h}, \nabla \cdot \psi_{h}^{n+1}) - 4\Delta t \tau^{n+1} (\psi_{h}^{n+1}) \\
&\quad + \frac{4\gamma d \Delta t}{3} \|\nabla \psi^{n+1}\|^2. \\
\end{align*}
\]

Since $\beta = 0$, we estimate \(\frac{4\gamma \Delta t}{3} (\nabla \cdot (3\phi_{h}^{n+1} - 4\phi_{h}^{n} + \phi_{h}^{n-1}), \nabla \cdot \eta^{n+1})\) as follows,

\[
\begin{align*}
&\frac{4\gamma \Delta t}{3} (\nabla \cdot (3\phi_{h}^{n+1} - 4\phi_{h}^{n} + \phi_{h}^{n-1}), \nabla \cdot \eta^{n+1}) \\
&\leq \frac{4\gamma \sqrt{7} \Delta t}{3} \left( 2\|\nabla \cdot \phi_{h}^{n+1}\| + 2\|\nabla \cdot \phi_{h}^{n}\| + \|\nabla \cdot (\phi_{h}^{n+1} - 2\phi_{h}^{n} + \phi_{h}^{n-1})\| \right) \|\nabla \eta^{n+1}\| \\
&\leq \frac{2\gamma \Delta t}{3} \|\nabla \cdot (\phi_{h}^{n+1} - 2\phi_{h}^{n} + \phi_{h}^{n-1})\|^2 + \frac{\gamma \Delta t}{2} \|\nabla \cdot \phi_{h}^{n+1}\|^2 + \frac{\gamma \Delta t}{2} \|\nabla \cdot \phi_{h}^{n}\|^2 + \frac{70\gamma d \Delta t}{9} \|\nabla \eta^{n+1}\|^2.
\end{align*}
\]

Then we have

\[
\begin{align*}
&\|\phi_{h}^{n+1}\|^2 - \|\phi_{h}^{n}\|^2 + \|2\phi_{h}^{n+1} - \phi_{h}^{n}\|^2 - \|2\phi_{h}^{n} - \phi_{h}^{n-1}\|^2 + \|\phi_{h}^{n+1} - 2\phi_{h}^{n} + \phi_{h}^{n-1}\|^2 \\
&\quad + \frac{2\gamma \Delta t}{3} \left( \|\nabla \cdot \phi_{h}^{n+1}\|^2 - \|\nabla \cdot \phi_{h}^{n}\|^2 + \|\nabla \cdot (2\phi_{h}^{n+1} - \phi_{h}^{n})\|^2 - \|\nabla \cdot (2\phi_{h}^{n} - \phi_{h}^{n-1})\|^2 \right) \\
&\quad + \frac{\gamma \Delta t}{2} \|\nabla \cdot \phi_{h}^{n+1}\|^2 + 6\|\psi_{h}^{n+1} - \phi_{h}^{n+1}\|^2 + 2\nu \Delta t \|\nabla \psi_{h}^{n+1}\|^2 + \gamma \Delta t \|\nabla \cdot \phi_{h}^{n+1}\|^2 \\
&\leq \frac{C \Delta t}{\nu} \|u^{n}\|_{2}^{2} + \frac{C \Delta t}{\nu} (2\phi_{h}^{n} - \phi_{h}^{n-1})\|^2 + \|\phi_{h}^{n}\|^2 + \phi_{h}^{n-1}\|^2) \\
&\quad + \frac{C \nu \int_{t_{n-1}}^{t_{n+1}} \|\eta\|^2 dt + C(\nu + \gamma \nu) \|\nabla \eta^{n+1}\|^2} \\
&\quad + \frac{C \Delta t}{\nu} p^{n+1} - q_{h})^2 + \frac{C \Delta t}{\nu} \int_{t_{n-1}}^{t_{n+1}} \|u_{tt}\|^2 dt + \frac{C \Delta t}{\nu} \int_{t_{n-1}}^{t_{n+1}} \|\nabla u_{tt}\|^2 dt \\
&\quad + \frac{C \Delta t}{\nu} (\|\nabla \eta^n\|^2 + \|\nabla \eta^{n-1}\|^2) + \frac{C \Delta t}{\nu} (\|\nabla \phi^{n-1}\|^2 + \|\nabla \phi^{n-1}\|^2) \|\nabla \eta\|^2_{\infty,0}. \\
\end{align*}
\]
Denote $C$ by $P_2-P_1$ Taylor-Hood approximation elements (Sum (4.38) = 16). Pressure-robustness. We follow with 2D channel flow over a step, where the effect of benchmark problem to compute convergence rates and test both computational efficiency and effect. Finally, we simulate flow past a cylinder to further present the effectiveness of grad-div stabilization (Standard Stabilized). All tests are implemented using FreeFem++ [32].

Under the assumptions of Theorem 4.4, we have $C \leq 1$, initial condition, and boundary conditions. Then, the following estimate holds for BDF2-mgd.

$$
\|\phi_h^n\|^2 + \|2\phi_h^n - \phi_h^{n-1}\|^2 + 2\gamma \Delta t \left( \|\nabla \cdot \phi_h^n\|^2 + \|\nabla \cdot (2\phi_h^n - \phi_h^{n-1})\|^2 \right)
+ 2\nu \Delta t \sum_{n=1}^{N-1} \|\nabla \psi_h^{n+1}\|^2 + \gamma \Delta t \sum_{n=1}^{N-1} \|\nabla \cdot \phi_h^{n+1}\|^2
\leq \frac{C\Delta t}{\nu} \sum_{n=1}^{N-1} \|u^{n+1}\|_2^2 (\|\phi_h^n\|^2 + \|2\phi_h^n - \phi_h^{n-1}\|^2) + C(\gamma d + \nu + \frac{1}{\nu}) \|\nabla \eta\|_{2,0}^2
+ \frac{C}{\nu} \|\eta_t\|_{2,0}^2 + \frac{C}{\nu} \|p - q_h\|_{2,0}^2 + \frac{C\Delta t^4}{\nu} \|u_{ttt}\|_{2,1}^2 + \frac{C\Delta t^4}{\nu} \|\nabla u_{tt}\|_{2,0}^2 + \frac{C}{\nu} \|\nabla \eta\|_{2,\infty}^2
+ \|\phi_h^1\|^2 + \|2\phi_h^1 - \phi_h^0\|^2 + \frac{7\gamma \Delta t}{6} \|\nabla \cdot \phi_h^1\|^2 + \frac{2\gamma \Delta t}{3} \|\nabla \cdot (2\phi_h^1 - \phi_h^0)\|^2).
$$

Denote $C^{**} = \frac{C}{\nu} \|u\|_{2,2}^2$. The result then follows by similar arguments as in Theorem 4.4.

**Corollary 4.6.** Under the assumptions of Theorem 4.4, suppose that $(X_h, Q_h)$ is given by $P_2-P_1$ Taylor-Hood approximation elements $(k = 2, m = 1)$. Then, the following estimate holds for BDF2-mgd.

$$
\|e_u^N\|^2 + \|2e_u^N - e_u^{N-1}\|^2 + 2\gamma \Delta t \left( \|\nabla \cdot e_u^N\|^2 + \frac{1}{2} \|\nabla \cdot (2e_u^N - e_u^{N-1})\|^2 \right)
+ 2\nu \Delta t \sum_{n=1}^{N-1} \|\nabla e_u^{n+1}\|^2 + 2\gamma \Delta t \sum_{n=1}^{N-1} \|\nabla \cdot e_u^{n+1}\|^2
\leq C \left( h^6 + h^4 + \Delta t h^4 + \Delta t h^4
+ \|e_u^1\|^2 + \|2e_u^1 - e_u^0\|^2 + (\gamma \Delta t + \beta) (\|\nabla \cdot e_u^1\|^2 + \|\nabla \cdot (2e_u^1 - e_u^0)\|^2) \right).
$$

5. Numerical Tests. In this section, we consider three test problems to illustrate the stability, and effectiveness of BDF2-mgd. First, we consider the Taylor-Green benchmark problem to compute convergence rates and test both computational efficiency and pressure-robustness. We follow with 2D channel flow over a step, where the effect of BDF2-mgd on reducing the divergence error is illustrated. Moreover, it is shown how $\gamma$ and $\beta$ influence this effect. Finally, we simulate flow past a cylinder to further present the effectiveness of BDF2-mgd. For all tests, we compare BDF2-mgd with BDF2 (Non-Stabilized) and BDF2 with standard grad-div stabilization (Standard Stabilized). All tests are implemented using FreeFem++ [32].

5.1. Test of Convergence and Pressure Robustness. The Taylor-Green benchmark problem is commonly used to test convergence rates of new algorithms. As such, we first illustrate convergence rates. The domain is $[0, 1] \times [0, 1]$ and final time is $T = 1$. Finite element meshes are generated via Delaunay-Voronoï triangulations with $m$ points on each side of the boundary. The true solution is given by

$$
u(x, y, t) = (-\cos(\omega \pi x) \sin(\omega \pi y), \sin(\omega \pi x) \cos(\omega \pi y)) \exp(-2\omega^2 \pi^2 t/\tau),
$$

$$p(x, y, t) = -\frac{1}{4} \left( \cos(2\omega \pi x) + \cos(2\omega \pi y) \right) \exp(-4\omega^2 \pi^2 t/\tau).
$$

Here, $\omega = 1$, $\tau = 100$, and $Re = \frac{1}{\nu} = 100$. The body force $f$, initial condition, and boundary condition are determined by the true solution. The grad-div parameters are set to $\gamma = 1$, $\beta = 0.2$. The time step is $\Delta t = 1/m$ where we vary $m = 16, 24, 32, 40,$ and 48 to calculate
To test computational efficiency, we set $m = 32$ and vary $\gamma$ and $\beta$. We compare computational times of Standard Stabilized and BDF2-mgd; for $\gamma = \beta = 0$, Standard Stabilized is equivalent to Non-Stabilized. For Standard Stabilized and Step 1 of BDF2-mgd, we use a standard GMRES solver. If GMRES fails to converge at a single iterate, we denote the result with an “F”. For Step 2 of BDF2-mgd, since it leads to an SPD system with same sparse coefficient matrix, at each timestep, we use UMFPACK. The results are presented in Table 5.2.

Table 5.2 presents the results which are consistent with our theoretical analysis. To investigate the sharpness of our results, we vary $Re$ while fixing $\Delta t = 1/m = 1/32$ and $\gamma = 1$, $\beta = 0.2$. We compare the velocity and pressure errors of Non-Stabilized, Standard Stabilized, and BDF2-mgd. Results are presented in Table 5.3. It is clear that velocity errors of Non-Stabilized, especially for the divergence and gradient, grow as $Re$ increases; this is consistent with the corresponding theoretical result. Alternatively, as $Re$ is increased, velocity errors of Standard Stabilized and BDF2-mgd are consistent with one another and maintain good approximations. This suggests that the effect of $Re$ appearing in our analysis is not sharp. This is an open problem, Section 6.

### Table 5.2

| Parameter | $\|u_h - u\|_{\infty,0}$ | $\|\nabla \cdot u\|_{\infty,0}$ | $\||u_h - u||_{2,0}$ | $\||u_h - u||_{2,0}$ | $\||p - p_h||_{2,0}$ | $\||p - p_h||_{2,0}$ | $\||p - p_h||_{2,0}$ | $\||p - p_h||_{2,0}$ |
|-----------|----------------|-----------------|-----------------|-----------------|----------------|----------------|----------------|----------------|
| $Re$      | Non-Stabilized | Standard Stabilized | BDF2-mgd | Non-Stabilized | Standard Stabilized | BDF2-mgd | Non-Stabilized | Standard Stabilized | BDF2-mgd | All |
| 1         | 1.20E-03       | 1.20E-03       | 1.20E-03      | 1.20E-03       | 1.20E-03       | 1.20E-03      | 1.20E-03       | 1.20E-03       | 1.20E-03       | 1.20E-03       |
| 2         | 2.32E-05       | 2.32E-05       | 2.32E-05      | 2.32E-05       | 2.32E-05       | 2.32E-05      | 2.32E-05       | 2.32E-05       | 2.32E-05       | 2.32E-05       |
| 3         | 3.55E-05       | 3.55E-05       | 3.55E-05      | 3.55E-05       | 3.55E-05       | 3.55E-05      | 3.55E-05       | 3.55E-05       | 3.55E-05       | 3.55E-05       |
| 4         | 4.78E-05       | 4.78E-05       | 4.78E-05      | 4.78E-05       | 4.78E-05       | 4.78E-05      | 4.78E-05       | 4.78E-05       | 4.78E-05       | 4.78E-05       |

### Table 5.3

Comparison of velocity and pressure errors with increasing $Re$. 5.2. 2D Channel Flow Over a Step. We now illustrate the effect of Step 2 of BDF2-mgd by comparing Non-Stabilized, Standard Stabilized, and BDF2-mgd simulations of 2D channel flow over a step [6] [14]. The channel considered here is $[0, 40] \times [0, 10]$ with a $1 \times 1$ step on the bottom for $x \in [5, 6]$. A flow with $\nu = 1/600$ passes though this channel from left to right. For
| Parameters | Time (s) |
|------------|----------|
| $\beta$  | $\gamma$ | Standard Stabilized | $BDF2-mgd$ |
| 0         | 0        | 17.77               | 25.09     |
| 0         | 0.2      | 30.91               | 20.10     |
| 0         | 2        | 55.29               | 20.45     |
| 0         | 20       | F (339.01)          | 27.99     |
| 0         | 200      | F (507.41)          | 23.88     |
| 0         | 2,000    | F (421.66)          | 17.34     |
| 0         | 20,000   | F (27.44)           | 20.04     |
| 0.01      | 0.2      | 27.79               | 22.28     |
| 0.02      | 0.2      | 32.89               | 22.33     |
| 0.04      | 0.2      | 64.37               | 21.68     |
| 0.08      | 0.2      | 69.31               | 23.97     |
| 0.8       | 0.2      | F                   | 25.87     |
| 8         | 0.2      | F                   | 19.53     |
| 80        | 0.2      | F                   | 21.20     |
| 800       | 0.2      | F                   | 17.43     |
| 8,000     | 0.2      | F                   | 18.64     |

Table 5.2

Computational time and solver breakdown for Standard and $BDF2-mgd$ with increasing grad-div parameters.

boundary conditions, the left inlet and right outlet are given by

$$u(0, y, t) = u(40, y, t) = y(10 - y)/25,$$

$$v(0, y, t) = v(40, y, t) = 0.$$  

No-slip, $u = 0$, boundary conditions are imposed elsewhere. Taylor-Hood elements are used, comprising a mesh with 31,089 degrees of freedom. The body force $f = 0$, final time $T = 40$, and time step $\Delta t = 0.01$. The selected grad-div parameters are $\gamma = 0.1, 0.2, 1$ and $\beta = 0, 0.1, 0.2, 1$. $||\nabla \cdot u(t^*)||$ is computed and plotted in Figure 5.1. Also, plots of flow speed and divergence contours, at the final time, with $\gamma = 1, \beta = 0$, are presented in Figure 5.2.

As shown in Figure 5.1, Step 2 of $BDF2-mgd$ greatly reduces the divergence error $||\nabla \cdot u||$ compared with Non-Stabilized. Observing the curves of different $\gamma$ and $\beta$, it’s interesting to find that the value of $\beta$ determines the minimum divergence error that can be reached in the beginning and the value of $\gamma$ determines the long-time divergence error. This is consistent with [5]. In Figure 5.2, we see that results for Step 2 of $BDF2-mgd$ are consistent with Standard Stabilized; both reduce divergence error, especially around the step.

5.3. 2D Channel Flow Past a Cylinder. In order to further test the effectiveness of $BDF2-mgd$, we consider channel flow past a cylinder [31]. Like the Taylor-Green benchmark, this is a common test problem for new algorithms. The channel domain is $[0, 2.2] \times [0, 0.41]$ with a cylinder of diameter 0.1 within. The center of the cylinder is $(0.2, 0.2)$. A flow with $\nu = 0.001, \rho = 1$ passes through this channel from left to right. No body forces are present, $f = 0$. Left in-flow and right out-flow boundaries are given by

$$u(0, y, t) = u(2.2, y, t) = \frac{6y(0.41 - y)}{0.41^2} \sin(\frac{\pi t}{8}),$$

$$v(0, y, t) = v(2.2, y, t) = 0.$$  

The no-slip boundary condition is prescribed elsewhere.

We use Taylor-Hood elements on a mesh with 41,042 degree of freedom and final time $T = 8$. The time step is $\Delta t = 0.001$. The grad-div parameters are set to $\gamma = 5\nu$ and
\[ \nabla \cdot u \]

\[ \gamma = 0, \beta = 0 \]
\[ \gamma = 0.1, \beta = 0 \]
\[ \gamma = 0.1, \beta = 0.1 \]
\[ \gamma = 0.1, \beta = 0.2 \]
\[ \gamma = 0.1, \beta = 1 \]
\[ \gamma = 0.2, \beta = 0 \]
\[ \gamma = 0.2, \beta = 0.1 \]
\[ \gamma = 0.2, \beta = 0.2 \]
\[ \gamma = 0.2, \beta = 1 \]
\[ \gamma = 1, \beta = 0 \]
\[ \gamma = 1, \beta = 0.1 \]
\[ \gamma = 1, \beta = 0.2 \]
\[ \gamma = 1, \beta = 1 \]

**Fig. 5.1.** \( \| \nabla \cdot u \| \) vs time for Non-Stabilized and BDF2-mgd.

**Fig. 5.2.** Flow speed and divergence contours at time \( t = 40 \) for Non-Stabilized (top), Standard Stabilized (middle) and BDF2-mgd (down) with \( \gamma = 1, \beta = 0 \).

\( \beta = 0 \). Drag \( c_d(t) \) and lift \( c_l(t) \) coefficients are calculated; maximum values are presented in Table 5.4. The pressure difference between the front and back of the cylinder (\( \Delta p(t) = p(0.15, 0.2, t) - p(0.25, 0.2, t) \)) and both the \( L^2(0, T; L^2(\Omega)) \) and \( L^\infty(0, T; L^2(\Omega)) \) norms of the velocity divergence are also tabulated in Table 5.4. Furthermore, Figure 5.3 shows velocity speed and vectors for BDF2-mgd at times \( t = 4, 6, 7, 8 \), which are consistent with that in [2, 5, 13, 21].

In Table 5.4, we see that grad-div stabilization effectively reduces the divergence error, as expected. This results in improved accuracy of Standard Stabilized and BDF2-mgd over the Non-Stabilized solution. In particular, both stabilized algorithms produce accurate lift coefficients and smaller divergence errors.
Method | $c_d^{\text{max}}$ | $c_l^{\text{max}}$ | $\Delta p_h$ | $\|\nabla \cdot u_h\|_{L^2, 0}$ | $\|\nabla \cdot u_h^N\|_{L^2, 0}$
--- | --- | --- | --- | --- | ---
Non-Stabilized | 2.950 | **0.441** | -0.1084 | **1.967** | **0.186**
Standard Stabilized | 2.950 | 0.477 | -0.1115 | 0.859 | 0.072
BDF2-mgd | 2.950 | 0.475 | -0.1115 | 0.906 | 0.074

Table 5.4

Maximum lift, drag coefficients, pressure drop, and divergence quantities for flow past a cylinder.

6. Conclusion. We developed a BDF2 time-discrete, modular grad-div stabilization algorithm (BDF2-mgd) for the time dependent Navier-Stokes equations. Compared with methods implementing standard grad-div stabilization, our algorithm produces consistent numerical approximations while avoiding solver breakdown for large grad-div parameters. We prove that this algorithm is unconditionally, nonlinearly, energy stable and second-order accurate in time. Numerical tests illustrate the theoretical results and computational efficiency.

To impose discrete versions of $-\beta \nabla \nabla \cdot u_t - \gamma \nabla \nabla \cdot u$, modular grad-div requires a solve of the form $(\frac{1}{\Delta t} I + (\beta \delta t + \gamma G) u = RHS$, where $G$ is the symmetric positive semi-definite grad-div matrix. For constant $\Delta t$, efficiency increases can exploit the fact that the matrix is fixed. For variable timestep and $\beta = 0$, the matrix is a variable shift of $G$ and efficient algorithms exist exploiting this structure. Important next steps include investigating, analytically, the $\nu$ dependence of $\nu^{-1} \inf_{q_h \in Q_h} \| p - p_h \|_{L^2, 0}^2$ in Theorems 4.4 and 4.5, extending these results to alternative numerical methods, and including sparse, effective variants of grad-div stabilization.

REFERENCES

[1] R.A. Adams, *Sobolev spaces*, Academic press, New York, 1995.
[2] A. L. Bowers, S. Le Borne, and L. G. Rebholz, Error analysis and iterative solvers for Navier-Stokes projection methods with standard and sparse grad-div stabilization, Comput. Methods Appl. Mech. Engrg., 275 (2014), pp. 1-19.
[3] V. DeCaria, W. Layton, and M. McLaughlin, A conservative, second order, unconditionally stable artificial compression method, Comput. Methods Appl. Mech. Engrg., 325 (2017), pp. 733-747.
[4] V. DeCaria, W. J. Layton, A. Pakzad, Y. Rong, N. Sahin, and H. Zhao, On the determination of the grad-div criterion, Apr. 2017, https://arxiv.org/abs/1704.04171.
[5] J. A. Fiordilino, W. J. Layton, and Y. Rong, Robust and Efficient Modular Grad-Div Stabilization, Comput. Methods Appl. Mech. Engrg., 335 (2018), pp. 327-346.
[6] V. P. Fragos, S. P. Psychoudaki, and N. A. Malamataris, Computer-aided analysis of flow past a surface-mounted obstacle, Int. J. Numer. Meth. Fluids, 25 (1997), pp. 495-512.
[7] V. Girault and P. A. Raviart, *Finite Element Approximation of the Navier-Stokes Equations*, Springer,
