Non-universal power law distribution of intensities of the self-excited Hawkes process: a field-theoretical approach

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Most out-of-equilibrium dynamical processes in physical, natural and social systems are characterised by the presence of extended and often long memory. A prominent class of such non-Markovian dynamics includes epidemic spreading processes, which have broad applications from photoconductivity in amorphous semiconductors and organic compounds [1], rainfall and runoff in catchments [2], earthquake interactions [3], epidemiology [4], brain memory [5], credit rating [6] and default cascades in finance [7], financial volatility dynamics [8, 9], transaction intervals in foreign exchange markets [10], social dynamics of book sales [11] and YouTube videos views [12] to cite a few [13].

The self-excited conditional Poisson process introduced by Hawkes [14–16] is the simplest point process modelling epidemic dynamics, in which the whole past history influences future activity. It captures the ubiquitous phenomenon of time (and space) intermittency and clustering due to endogenous interactions. The Hawkes process is enjoying an explosion of interest in many complex systems, including in physics, biology, geology and seismology and in financial and economic markets. For instance, the Hawkes model remains the standard reference in statistical seismology [3, 17–19] and is now used to model a variety of phenomena in finance, from microstructure dynamics to default risks [20, 21]. The Hawkes process is also fashionable to model social dynamics [22].

A key ingredient of the Hawkes model is the memory kernel \( h(t) \), which quantifies how much a past event influences the triggering of a future event. When \( h(t) \) is a pure exponential function, the Hawkes model can be represented as a Markovian process by adding an auxiliary variable. But most systems exhibit longer memories, with \( h(t) \) containing multiple time scales and often describing power law decaying impacts, which makes the Hawkes model non-Markovian in general. Here, we present a general field master equation, which represents the self-excited Hawkes process as being equivalent to a stochastic Markovian partial differential equation. This novel representation allows us to use the mathematical apparatus to solve master equations and derive a new result on the distribution of activity rates, which is found to take the form of a non-universal power law.

The Hawkes process is defined via its intensity \( \dot{\nu} \), which is the frequency of events per unit time. An event can be a burst of electrons in a semiconductor, a rainfall, an earthquake, an epidemic, an epileptic seizure, a firm’s bankruptcy or credit default, a financial volatility burst, the sale of a commercial product, viewing a video or a movie, a social action, and so on. Such an event (or shock) occurs during \([t, t + dt]\) with the probability of \( \dot{\nu} dt \), with

\[
\dot{\nu}(t) = \nu_0 + n \sum_{i=1}^{\hat{N}(t)} h(t - \hat{t}_i), \tag{1}
\]

where \( \nu_0 \) is the background intensity, \( \hat{t}_i \) represents the time series of events, \( n \) is a positive number called branching ratio, \( h(t) \) is a normalized nonnegative function satisfying \( \int_0^\infty h(t)dt = 1 \), and \( \hat{N}(t) \) is the number of events during the interval \([0, t]\) (called “counting process”), as shown in Fig. 1a for a schematic. By convention, we denote stochastic variables with a hat symbol, such as \( \hat{A} \), to distinguish them from the non-stochastic real numbers \( A \), corresponding for instance to a specific realisation of
the random variable. The memory kernel $h(t)$ represents the usually non-Markovian influence of a given event. The branching ratio $n$ is the average number of events of first generation (“daughters”) triggered by a given event and is also the fraction of events that are endogenous, i.e., that have been triggered by previous events. The Hawkes process has three different regimes: (i) $n < 1$: subcritical; (ii) $n = 1$: critical and (iii) $n > 1$: super-critical or explosive (with a finite probability). The Hawkes process is a model for out-of-equilibrium systems without detailed balance and does not satisfy the fluctuation-dissipation relation.

Let us decompose the memory kernel as a continuous superposition of exponential kernels,

$$ h(t) = \frac{1}{n} \int_0^\infty \frac{n(x)}{x} e^{-t/x} dx, \quad n = \int_0^\infty dx n(x) $$ (2)

satisfying the normalization $\int_0^\infty h(t) dt = 1$ with the set of continuous time scale $x \in R_+ := (0, \infty)$. This decomposition is equivalent to applying the Laplace transform, a standard method even for non-Markovian Langevin equations with response functions expanded in terms of a superposition of exponentials. Here $n(x)$ quantifies the contribution of the $x$-th exponential with memory length $x$ to the branching ratio and $n(x)/n$ is the normalized distribution of time scales present in the memory kernel. In this work, we require the existence of its first moment

$$ \alpha := \langle \tau \rangle := \int_0^\infty x \frac{n(x)}{n} dx < \infty. $$ (3)

This condition means that $n(x)$ should decay faster than $1/x^2$ at large $x$’s and thus $h(t)$ decays at large times faster than $1/t^2$. In addition, we restrict our analysis to the subcritical case $n < 1$.

The starting point of our approach is to express $\hat{\nu}(t)$ as the continuous sum

$$ \hat{\nu}(t) = \nu_0 + \int_0^\infty dx \hat{\xi}(t, x), $$ (4)

where each excess intensity $\hat{\xi}(t, x)$ is the solution of a simple time-derivative equation

$$ \frac{d\hat{\xi}(t, x)}{dt} = -\frac{\hat{\xi}(t, x)}{x} + \frac{n(x)}{x} \hat{\xi}_\nu(t), \quad \forall x \in R_+, $$ (5)

and the same state-dependent Poisson noise $\hat{\xi}_\nu(t)$, defined by

$$ \hat{\xi}_\nu(t) = \sum_{i=1}^{\tilde{N}(t)} \delta(t - \tilde{t}_i), $$ (6)

acts on the Langevin equation for each excess intensity $\hat{\xi}(t, x)$ (see Fig. 1b). The excess intensity $\{\hat{\xi}(t, x)\}_{x \in R_+}$ can be viewed as a one-dimensional field variable distributed on the $x$-axis; correspondingly, Eq. (5) should be considered as a stochastic partial differential equation (SPDE) describing the classical stochastic dynamics of the field. This interpretation has the advantage of allowing us to apply functional methods available for SPDEs.

The introduction of the $\hat{\xi}(t, x)$’s is called Markovian embedding, a technique to transform a non-Markovian dynamics onto a Markovian one by adding a sufficient number of variables (see for the cases of non-Markovian Langevin equations). Markovian embedding is related to the trick proposed in [44] for an efficient estimation of the maximum likelihood of the Hawkes process. Each SPDE describes a Markovian relaxation of the field variable $\hat{\xi}(t, x)$, hit by intermittent simultaneous shocks $\hat{\xi}_\nu(t)$ with $x$-dependent sizes $n(x)/x$, whose influence decays exponentially with the characteristic time $\tau$. 

FIG. 1: Mapping from (a) a non-Markovian description of the total intensity $\hat{\nu}(t)$, obeying the original Hawkes process, to (b) a Markovian description of the excess intensity $\{\hat{\xi}(t, x)\}_{x}$ on the auxiliary field variable $x \in R_+$, which obeys a SPDE. Notice the shocks impact all excess intensity $\{\hat{\xi}(t, x)\}_{x}$ for all different variables $x$ simultaneously (orange plane at $t = \tilde{t}_i$). One can observe the dependence $n(x)/x$ of the jump size given in Eq. (5) along the field variable $x$ (blue chain arrow). Along the time axis, the exponential decay $\sim e^{-t/x}$ is shown to be faster (slower) for smaller (larger) $x$’s (red broken arrow). The sample trajectory was generated with $n(x) = c_{ini}(x - x_{ini}) + n_{ini}$ if $x \in [x_{ini}, x_{fin}]$ (in all other cases, with $\nu_0 = 0.05$, $x_{ini} = 0.5$, $x_{fin} = 10.0$, $c_{ini} = 0.002/\Delta x$, $n_{ini} = 0.001/\Delta x$, $\Delta x = (x_{fin} - x_{ini})/200$, and $\Delta t = 0.006$. 

(a) Non-Markovian dynamics of total intensity

(b) Markovian field dynamics of excess intensities
Equation (5) together with (6) implies that $\dot{\nu}(t)$ given by (4) recovers the standard Hawkes definition (1).

We have thus transformed a non-Markovian point process into a Markovian SPDE, which allows us to derive the corresponding master equation for the probability density functional $P\{\{\dot{z}(t,x) = z(x)\}_{x \in \mathbb{R}_+} = P[z]$ for any field configuration $\{z(x)\}_{x \in \mathbb{R}_+}$, such that $P[z]dz$ is the probability that the system is in the state specified by $\{z(x)\}_{x \in \mathbb{R}_+}$ at time $t$, with the functional integral volume element $dz$. The corresponding master equation for the probability density functional $P[z]$ reads

$$\frac{\partial P[z]}{\partial t} = \int dx \frac{\delta}{\delta z} \left( \frac{z}{x} P[z] \right) + \left\{ \nu_0 + \int dx \left( z - \frac{n}{x} \right) P[z - \frac{n}{x}] \right\} P[z]$$

with the condition $P[z] = 0$ holding over the boundary of the function space $z \in \partial \mathbb{R}_+^\infty := \{z|z(x = 0 = \infty)\}$. This can be derived by performing an ensemble average in a weak integral sense, namely considering an arbitrary functional $f[\{\dot{z}(t,x)\}]$ and averaging it over all possible realisations of $\dot{z}(t,x)$ weighted by their probability density functional (PDF) (see Ref. 25 for details). The functional description (7) is interpreted as a formal continuous limit of a discrete formulation according to the convention 26 (see Ref. 35 for technical details).

It is convenient to transform (7) using the functional Laplace transformation $\mathcal{L}_{\text{path}}$ of an arbitrary functional $f[z]$ defined by the functional integration (or path integral) $\mathcal{L}_{\text{path}}[f[z];s] := \int d\mathcal{Z} e^{-\int dx f(x)z(x)f[z]}$. Then, the Laplace representation of the probability density functional is $\tilde{P}_s[z] := \mathcal{L}_{\text{path}}[P[z];s]$ for an arbitrary nonnegative function $\{s(x)\}_{x \in \mathbb{R}_+}$. The resulting Laplace transformed master equation (7) takes the following simple first-order functional differential equation in the steady state ($\partial P[z]/\partial t = 0$):

$$\int dx \tau H[s;x] \frac{\delta \Phi[s]}{\delta s(x)} = -\nu_0 \mathcal{K}[s]$$

where $\Phi[s] := \log \tilde{P}_s[s] := \lim_{t \to \infty} \log \tilde{P}_t[z]$ is the steady state cumulant functional, $H[s;x] := e^{-\int dx s(x' \lambda)}\dot{\lambda} = 1 + s(x)/x$, and $\mathcal{K}[s] := e^{-\int dx s(x')n(x')/x'} - 1$. This hyperbolic equation can be solved by the method of characteristics and the corresponding Lagrange-Charpit (LC) equations are the following partial-integro equations,

$$\frac{\partial s(l;x)}{\partial l} = -H[s;x], \quad \frac{\partial \Phi(l)}{\partial l} = \nu_0 \mathcal{K}[s],$$

where $l$ is the curvilinear parameter indexing the position along a characteristic curve. The tail of the distribution of intensities $\dot{\nu}$ corresponds to the neighbourhood of $s = 0$ in the Laplace transform domain (i.e., $P_s(s) \sim |s|^\gamma$ for $s \to 0 \Leftrightarrow P_s(\nu) \sim \nu^{-\gamma - 1}$ for $\nu \to \infty$ 36). We first study the subcritical case $n < 1$ and then the critical regime $n = 1$ via a stability analysis of (9) for small $s$.

Remarkably, the LC equations can be interpreted as a dynamical system where $l$ plays the role of time. This mapping allows us to use the standard stability analysis for bifurcations of dynamical systems, particularly for asymptotic analyses near criticality. Indeed, the stability analysis for $s \to 0$ corresponds to the long time limit $l \to \infty$ and the critical condition of the original Hawkes process (1) corresponds to the transcritical bifurcation condition for the dynamical system described by Eq. (9).

Subcritical case $n < 1$. Linearising the LC equation (9) yields

$$\frac{\partial s(l;x)}{\partial l} = -\int dx' H(x,x')s(x'), \quad \frac{\partial \Phi(l)}{\partial l} = \nu_0 \int dx' K(x')s(x')$$

with $x'H(x,x') := \delta(x - x') - n(x')$ and $K(x') := n(x')/x'$. Introducing the eigenvalues $\lambda \geq \lambda_{\min}$ and eigenfunctions $e(x;\lambda)$ of the operator $H(x,x')$, satisfying the relation

$$\int dx'H(x,x')e(x';\lambda) = \lambda e(x;\lambda),$$

we verify that all eigenvalues are real and the inverse matrix of $H(x,x')$, denoted by $H^{-1}(x,x')$, exists and has a singularity at $n = 1$ (see Ref. 35 for the proof), recovering the critical condition of this Hawkes process.

We now introduce a set of variables to obtain a new representation based on the eigenfunctions,

$$s(x) = \sum_\lambda e(x;\lambda)X(\lambda) \iff X(\lambda) = \int dx e^{-1}(\lambda ; x)s(x).$$

Here the inverse matrix $e^{-1}(\lambda ; x)$ is introduced, satisfying $\int dx e^{-1}(\lambda ; x)e(x;\lambda') = \delta_{\lambda \lambda'}$. The existence of the inverse matrix is equivalent to the assumption that the set of all eigenfunctions is complete, and thus $H(x,x')$ can be diagonalized: $\int dx dx' e^{-1}(\lambda ; x)H(x,x')e(x';\lambda') = \lambda \delta_{\lambda \lambda'}$. In this representation, the linearised LC equations read

$$\frac{\partial X(l;\lambda)}{\partial l} = -\lambda X(l;\lambda).$$

For subcriticality, all the eigenvalues are positive, indicating that the fixed point $\{X(\lambda) = 0\}$ (i.e., $\{s(x) = 0\}$) is the stable attractor in the functional space. Using straightforward calculations (see Ref. 35), we obtain

$$\Phi[s] \approx -\nu_0 \int dx \int dx' K(x)H^{-1}(x,x')s(x'),$$

from which we find, for small $s$,

$$\log \tilde{P}_s(s) := \log \tilde{P}_s[s] \approx \Phi[s] \approx \tilde{P}_1[s] \approx \nu_0 s.$$
Critical case \(n = 1\). At criticality, the smallest eigenvalue vanishes, \(\lambda_{\text{min}} = 0\), which is associated to the zero eigenfunction \(e(x; \lambda = 0) = x\), as verified by direct substitution: \(\int dx H(x, x')e(x'; \lambda = 0) = 1 - n = 0\). From the linear LC equation (13), it is clear that the dominant contribution comes from the component \(X(\lambda = 0)\), associated with the zero eigenfunction \(e(x; \lambda = 0)\). The explicit representation of \(X(\lambda = 0)\) is given by \(X(\lambda = 0) = \int_0^\infty dx n(x)s(x)/\alpha\), where \(\alpha\) is defined by expression (3).

We also obtain the LC equations for each component to leading order for \(\lambda' \neq 0\),

\[
\frac{\partial X(l; 0)}{\partial l} \simeq -\frac{X^2(l; 0)}{2\nu}, \quad \frac{\partial X(l; \lambda')}{\partial l} = -\lambda' X(l; \lambda').
\]

This is the normal form of transcritical bifurcations, leading to a log-type singularity in the cumulant for small \(s\). Indeed, after straightforward calculations, we obtain

\[
\log \bar{P}_{ss}(s) := \log \bar{P}_{ss}[s(\tau)] \simeq \nu_0 s - 2\nu_0 \alpha \log |s|,
\]

which by inverse Laplace transform yields

\[
P_{ss}(\nu) \sim \nu^{-1 + 2\nu_0(\tau)},
\]

using definition (4). The exponent \(1 - 2(\tau)\nu_0\) of the PDF is non-universal and a function of the background intensity \(\nu_0\) of the Hawkes intensity and of the average time scale of the memory kernel \(\langle \tau \rangle\). As the tail exponent is smaller than 1, the steady-state PDF \(P_{ss}(\nu)\) would be not normalizable in absence of some cut-off [12], coming either from finite-time effects or non-exact criticality (\(n \rightarrow 1^-\)). This means that this power-law scaling (18) actually corresponds to an intermediate asymptotics of the PDF, according to the classification of Barenblatt [37], which, for \(n\) close to 1, can be observed over many orders of magnitude of the intensity for near-critical systems, as shown in figure 2. The intermediate power law asymptotic (18) is our main novel quantitative result. Interested readers are referred to Ref. [35] for details.

**Example 1.** The above general derivation of (18) is rather involved and one can develop more intuition by studying simplest cases where the memory function \(h(t)\) is a single exponential or the sum of two exponentials. In the former case \(h(t) = (1/\tau)e^{-t/\tau}\), all functions become single variables and functional derivatives and integrations become standard derivative and integration operators. Then, the general master equation (7) reduces to

\[
\frac{\partial P_t}{\partial t} = \frac{1}{\tau} \frac{\partial}{\partial z} \nu z P_t + \left(\nu_0 + z - \frac{n}{\tau}\right) P_t \left(z - \frac{n}{\tau}\right) - (\nu_0 + z) P_t,
\]

for the probability density function \(P_t := P_t(z)\) under the boundary condition \(P_t(z)|_{z=0} = 0\). Its Laplace transform of the steady-state PDF \(\bar{P}_{ss}(s) := \int_0^\infty dz e^{-sz} P_t(z)\) reads

\[
\mathcal{H}(s)\frac{d\Phi(s)}{ds} = -\nu_0 \mathcal{K}(s),
\]

by introducing the cumulant function \(\Phi(s) := \log \bar{P}_{ss}(s)\), \(\mathcal{H}(s) := e^{-ns/\tau} - 1 + s/\tau\), and \(\mathcal{K}(s) := e^{-ns/\tau} - 1\). It can be directly solved exactly below the critical point \(n < 1\), leading to

\[
P_{ss}(\nu) \propto \nu^{-1 + 2n_0(\tau)} e^{-2\nu(1-n)}
\]

for large \(n\) near criticality \(1 - n \ll 1\). Remarkably, the LC equation \(ds/dl = -\mathcal{H}(s)\) reduces to the normal form of transcritical bifurcations (see Fig. 3):

\[
\frac{ds}{dl} = \mu s - s^2 + O(s^3)
\]

for small \(s\) with \(l' := s^2/(2\tau^2)\) and \(\mu := -2(1 - n)/n^2\).

**Example 2**. For two exponentials, the memory kernel is given by \(h_{\tau} = \sum_{k=1}^n (n_k/(\tau_{\tau})) e^{-t/\tau_k}\), where each coefficient \(n_k\) quantifies the contribution of the \(k\)-th exponential with memory length \(\tau_k\) to the branching ratio \(n = n_1 + n_2\). In calculations paralleling those for the general and one exponential cases, we can derive the master equation for the two-exponential case and its Laplace
representation. Finally, the corresponding LC equations read
\[
\frac{d s_i}{d l} = -\mathcal{H}_i(s_1, s_2), \quad \frac{d \Phi}{d l} = \nu_0 \mathcal{K}(s_1, s_2)
\]
with \(i = 1, 2\), \(\mathcal{H}_i(s_1, s_2) := e^{-\sum_{k=1}^{2} n_k s_k / \tau_k} - 1 + s_i / \tau_i\), and \(\mathcal{K}(s_1, s_2) := e^{-\sum_{k=1}^{2} n_k s_k / \tau_k} - 1\). Following the same approach as for the general case \([2]\), but now dealing with operators that are \(2 \times 2\) matrices, we recover \([18]\) with \(\langle \tau \rangle = (n_1 \tau_1 + n_2 \tau_2) / n\) (see \([35]\) for details). We have numerically confirmed our theoretical prediction for a memory kernel with two exponentials, as shown in Fig. 2. We note that the LC equation \([23]\) exhibits the transcritical bifurcation as illustrated in Fig. 3(b).

The Hawkes process was believed to be unable to reproduce the large fluctuations that are ubiquitously observed in complex systems \([34]\). Our finding demonstrates in fact that the Hawkes process does produce large fluctuations in the form of intermediate asymptotics, thus filling an important gap for applications to real systems. We note that our methodology can be readily generalized to various non-linear Hawkes processes, where wider class of power laws can be discussed \([35]\). In addition, our main result fills a gap in the study of the Hawkes and other point process, by focusing on the distribution of the number \(vdt\) of events in the limit of infinitely small time windows \([t, t + dt]\). This limit is in contrast to the other previously studied limit of infinitely large and finite but very large time windows, as standard results of branching processes (of which the Hawkes model is a special case) give the total number of events generated by a given triggering event (see Ref. \([39]\) for a detailed derivation and \([40]\) for the case of large time windows \([t, t + T]\), i.e., in the limit of large \(T\)’s). The corresponding probability density distributions are totally different from \([18]\), which corresponds to the other limit \(T \to 0\). There are also deep relationship between our theory and quantum field theories. Indeed, our field master equation can be formally regarded as a Schrödinger equation for a non-Hermitian quantum field theory (see derivation in Ref. \([35]\)), considering the parallel structure between the Fokker-Planck (master) and Schrödinger equations \([41]\).

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