NORM ESTIMATES OF THE CAUCHY TRANSFORM AND RELATED OPERATORS

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Abstract. Suppose \( f \in L^p(D) \), where \( p \geq 1 \) and \( D \) is the unit disk. Let \( J_0 \) be the integral operator defined as follows: \( J_0[f](z) = \int_D \frac{f(w)}{w-z}dA(w) \), where \( z, w \in D \) and \( dA(w) = \frac{1}{\pi}dxdy \) is the normalized area measure on \( D \). Suppose \( J_0^* \) is the adjoint operator of \( J_0 \). Then \( J_0^* = BC \), where \( B \) and \( C \) are the operators induced by the Bergman projection and Cauchy transform, respectively. In this paper, we obtain the \( L^1 \), \( L^2 \) and \( L^\infty \) norm of the operator \( J_0^* \). Moreover, we obtain the \( L^p(D) \rightarrow L^\infty(D) \) norm of the operators \( C \) and \( J_0^* \), provided that \( p > 2 \). This study is a continuation of the investigations carried out in [4] and [9].

1. Introduction

Let \( D = \{ z : |z| < 1 \} \) be the unit disk of \( \mathbb{C} \). Denote by \( L^p(D)(1 \leq p \leq \infty) \) the space of complex-valued measurable functions on \( D \) with finite integral

\[
\|f\|_p = \left( \int_D |f(z)|^p dA(z) \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty,
\]

where

\[
dA(z) = \frac{1}{\pi}dxdy = \frac{1}{\pi}rdrd\theta, \quad z = x + iy = re^{i\theta},
\]

is the normalized area measure on \( D \) (cf. [8, Page 1]). For the case \( p = \infty \), we let \( L^\infty(D) \) denote the space of (essentially) bounded functions on \( D \). For \( f \in L^\infty(D) \), we define

\[
\|f\|_\infty = \text{ess sup}\{|f(z)| : z \in D\}.
\]

The space \( L^\infty(D) \) is a Banach space with the above norm (cf. [8, Page 2]).

1.1. The Cauchy transform \( C \). Let \( \Omega \subset \mathbb{C} \) be a bounded domain in the complex plane. It follows from [4, Page 7] that the Cauchy integral operator (Cauchy transform) \( C: L^p(\Omega) \rightarrow L^p(\Omega) \) is defined by (see also [1, 9])

\[
C[\Omega][f](z) = \int_\Omega \frac{f(w)}{w-z}dA(w).
\]

Unlike the two-dimensional Hilbert transform \( \mathcal{H}f \) (also called the Beurling transform), the Cauchy transform is not bounded as an operator from \( L^2(\mathbb{C}) \) to \( L^2(\mathbb{C}) \). The characteristic function of any bounded domain has Cauchy transform whose

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modulus behaves like \(|\zeta|^{-1}\) as \(\zeta \to \infty\), and so does not belong to \(L^2(\mathbb{C})\) (cf. [1]). Nonetheless, it follows from the Sobolev embedding theorem (cf. [1, 12]) that \(C_\Omega\) is a bounded operator from \(L^2(\Omega)\) to \(L^p(\Omega)\) for all \(p < \infty\). More precisely, we observe that if \(f \in L^2(\Omega)\), then
\[
\frac{\partial}{\partial \bar{z}} C_\Omega[f](z) = -f(z) \in L^2(\mathbb{C}),
\]
and (cf. [4, Page 7], see also [11, Page 157, (7.10)])
\[
\frac{\partial}{\partial z} C_\Omega[f](z) = \mathcal{H} f(z) \in L^2(\mathbb{C}).
\]

Throughout this paper, we consider the case \(\Omega = \mathbb{D}\). For simplicity, we write \(C\) instead of \(C_D\) for the Cauchy transform of \(\mathbb{D}\).

Recall that the norm of an operator \(T : X \to Y\) between normed spaces \(X\) and \(Y\) is defined by
\[
\|T\|_{X \to Y} = \sup \{\|Tx\|_Y : \|x\|_X = 1}\}
\]
For the case of \(X = Y = L^p(\Omega)\), we write \(\|T\|_p\) instead of \(\|T\|_{L^p(\Omega) \to L^p(\Omega)}\) for the \(L^p\) norm of the operator \(T\).

The norm estimates of the Cauchy transform on \(L^p(\Omega)\), i.e., \(\|C_\Omega\|_p\), has been studied by many mathematicians, but is still not known. In the case of \(\Omega = \mathbb{D}\), it was proved in [1] that
\[
\|C\|_2 = \frac{2}{j_0},
\]
where \(j_0 \approx 2.4048256\) is the smallest positive zero of the Bessel function \(J_0 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2}\right)^{2k}\).

Since \(\|C\|_1 = 2\) (cf. [6, (12)]) using the Riesz-Thorin interpolation theorem (cf. [5, Theorem 1.1.1]), it was proved in [6, Theorem 1] that
\[
\|C\|_p \leq 2 \cdot j_0^{-2(1-\frac{1}{p})}, \quad \text{for} \quad 1 \leq p \leq 2
\]
and
\[
\|C\|_p \leq 2 \cdot j_0^{-\frac{p}{2}}, \quad \text{for} \quad p \geq 2.
\]

1.2. The related operator \(J_0\) and its adjoint operator \(J_0^*\). Let \(J_0 : L^p(\mathbb{D}) \to L^p(\mathbb{D})\) be the integral operator defined by (cf. [4, Page 9 and Page 12])
\[
J_0[f](z) = \int_{\mathbb{D}} \frac{z}{1 - wz} f(w) dA(w).
\]
Suppose \(J_0^*\) is the adjoint operator of \(J_0\). The following transform
\[
\mathfrak{B}[f](z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - \bar{w}z)^2} dA(w)
\]
is the Bergman projection (here we refer to [8, 13, 15] and the references therein for more discussions of Bergman projection). It was proved in [4, Page 12] that \(J_0^* = \mathfrak{B}C\), the composition of \(\mathfrak{B}\) and \(C\), where
\[
J_0^*[f](z) = \int_{\mathbb{D}} \frac{\bar{w}}{1 - \bar{w}z} f(w) dA(w).
\]
Another relation of the Cauchy transform \( C \) and the related operator \( \mathcal{J}_0^* \) was given in [3] and [9] as follows: For \( f \in L^p(D) \), where \( 1 < p < \infty \), the Cauchy transform for Dirichlet’s problem (see [3, Page 155]) of \( f \) is defined by

\[
C_\Delta[f](z) = \int_D \left( \frac{1}{z - w} + \frac{\bar{w}}{1 - \bar{w}z} \right) f(w) dA(w), \quad z \in D.
\]

The operator \( C_\Delta \) is hence induced by the \( z \)-derivative of the Green’s function and \( \frac{\partial}{\partial z} C_\Delta[f] = f \). Obviously, one has

\[
C_\Delta[f](z) = \mathcal{J}_0^*[f](z) - \mathcal{C}[f](z).
\]

We refer to [4] for more discussions on the relations of the operators \( C, \mathcal{H}_f \) and \( \mathcal{J}_0^* \).

A related result, but for the so-called Cauchy operator for the Dirichlet problem in the unit disk, has been given by the second author of this paper in [9, 10]. Precisely, it was proved in [9, Theorem A] that:

\[
\|C_\Delta\|_{L^p(D) \to L^\infty(D)} \leq 2 \cdot j_0^{-2(1-\frac{1}{p})}, \quad \text{for } 1 \leq p \leq 2
\]

and

\[
\|C_\Delta\|_{L^p(D) \to L^\infty(D)} \leq \frac{4}{3} \left( \frac{2j_0}{3} \right)^{-\frac{2}{p}}, \quad \text{for } p \geq 2.
\]

The equalities of the above inequalities can be attained for the case \( p = 1, p = 2 \) and \( p = \infty \).

In this paper, we obtain the following norm estimates: \( \|\mathcal{C}\|_{L^p(D) \to L^\infty(D)} \), \( \|\mathcal{J}_0\|_{L^p(D) \to L^\infty(D)} \) and \( \|\mathcal{J}_0^*\|_{L^p(D) \to L^\infty(D)} \), for \( p > 2 \). Furthermore, we obtain the following norms: \( \|\mathcal{J}_0\|_1 \), \( \|\mathcal{J}_0^*\|_2 \) and \( \|\mathcal{J}_0^*\|_\infty \). Our main results are as follows.

1.3. The norm estimates of operators from \( L^p(D) \) to \( L^\infty(D) \).

**Theorem 1.1.** For \( p > 2 \),

\[
\|\mathcal{C}\|_{L^p(D) \to L^\infty(D)} = \left( \frac{2p - 2}{p - 2} \right)^{1-\frac{1}{p}}.
\]

If in particular \( p = \infty \), then

\[
\|\mathcal{C}\|_\infty = 2.
\]

**Theorem 1.2.** For \( p > 2 \),

\[
\|\mathcal{J}_0\|_{L^p(D) \to L^\infty(D)} = \left( \frac{\Gamma \left( \frac{p-2}{p-1} \right)}{\Gamma^2 \left( \frac{3p-4}{2p-2} \right)} \right)^{1-\frac{1}{p}},
\]

where \( \Gamma \) is the Gamma function.

If in particular \( p = \infty \), then

\[
\|\mathcal{J}_0\|_\infty = \frac{4}{\pi}.
\]
Theorem 1.3. For $p > 2$, 
\begin{equation}
\| \mathcal{J}_0^* \|_{L^p(\mathbb{D}) \to L^\infty(\mathbb{D})} = A(p)^{1-\frac{1}{p}},
\end{equation}
where
\begin{equation}
A(p) = 2 \frac{3F_2\left[1 + \frac{p}{2(p-1)}, \frac{p}{2(p-1)}; 1, 2 + \frac{p}{2}; 1\right]}{2 + \frac{p}{p-1}}
\end{equation}
and $3F_2$ is the hypergeometric function given by (2.1).

If in particular $p = \infty$, then
\begin{equation}
\| \mathcal{J}_0^* \|_{\infty} = 1 + \frac{2}{\pi} \alpha,
\end{equation}
where $\alpha \approx 0.915966$ is the Catalan’s constant.

Remark 1.1. (1) For $p > 2$, let $q = \frac{p}{p-1} \in [1, 2)$. According to the definition of the hypergeometric function, we see that
\begin{equation}
3F_2\left[1 + \frac{q}{2}, \frac{q}{2}, \frac{q}{2}; 1, 1 + \frac{q}{2} ; 1\right] = \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{q}{2})}{n! \Gamma(\frac{q}{2})} \left(\frac{1}{2n + q + 2}\right)^2
\end{equation}
Using Lemma B below, we have: for $n \geq 1$,
\begin{equation}
\frac{\Gamma(n + \frac{q}{2})}{n!} \leq \frac{1}{n^{1-\frac{q}{2}}}
\end{equation}
Thus,
\begin{equation}
A(p) \leq 2 \left(\frac{1}{2 + q} + \frac{1}{\Gamma^2(\frac{q}{2})} \sum_{n=1}^{\infty} \frac{1}{n^{2-q}} \frac{1}{2n + q + 2}\right)
\end{equation}
where $\zeta$ is Riemann’s zeta function. This shows that $A(p)$ is finite for any $p > 2$.

(2) For the case of $1 \leq p \leq 2$, we show in Remark 3.1, Remark 3.2 and Remark 3.3 that the operators $\mathcal{C}$, $\mathcal{J}_0$ and $\mathcal{J}_0^*$ will not send $L^p(\mathbb{D})$ to $L^\infty(\mathbb{D})$, respectively.

1.4. The $L^1$ norm and $L^2$ norm of $\mathcal{J}_0^*$. The following Corollary 1.4 easily follows from (1.3), since the $L^1$ norm of an operator is equal to the $L^\infty$ norm of its adjoint operator.

Corollary 1.4.
\begin{equation}
\| \mathcal{J}_0^* \|_1 = \frac{4}{\pi}.
\end{equation}

For the $L^2$ norm we have:

Theorem 1.5.
\begin{equation}
\| \mathcal{J}_0^* \|_2 = \frac{1}{2}.
\end{equation}
Using the Riesz-Thorin interpolation theorem ([5, Theorem 1.1.1]) together with (1.5), (1.6) and (1.7), we have the following corollary.
Corollary 1.6.

\[ \|J_0^*\|_p \leq \left( \frac{1}{2} \right)^{\frac{1}{p}} \left( \frac{1 + 2\alpha}{\pi} \right)^{1 - \frac{2}{p}}, \quad \text{for} \quad p \geq 2, \]

and

\[ \|J_0^*\|_p \leq \left( \frac{1}{2} \right)^{1 - \frac{2}{p}} \left( \frac{4}{\pi} \right)^{\frac{2}{p} - 1}, \quad \text{for} \quad 1 \leq p \leq 2. \]

The equalities can be attained in the above inequalities for \( p = 1, p = 2 \) and \( p = \infty \).

The proofs of the above theorems are given in Section 3.

2. Preliminaries

In this section, we should recall some known results, and prove three useful lemmas, and one proposition.

Definition 2.1. (cf. [2, (2.1.2)]) The hypergeometric function

\[ {}_pF_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; x) \]

is defined by the series

\[ {}_pF_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; x) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n x^n}{(b_1)_n \cdots (b_q)_n n!} \]

for all \( |x| < 1 \) and by continuation elsewhere.

Here \((q)_n\) is the Pochhammer symbol which is defined as follows

\[ (q)_n = \begin{cases} 
  1, & \text{if } n = 0; \\
  q(q + 1) \cdots (q + n - 1), & \text{if } n > 0.
\end{cases} \]

Lemma A. (cf. [2, Theorem 2.2.2])

\[ {}_2F_1(a, b; c; 1) = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)}, \quad \text{if} \quad \text{Re} (c - a - b) > 0, \]

where \( \Gamma \) is the Gamma function.

Let \( q \in [1, 2) \). The following equality easily follows from Lemma A

\[ {}_2F_1 \left[ \frac{q}{2}, \frac{q}{2}; \frac{3}{2}; 1 \right] = \frac{\Gamma(2 - q)}{\Gamma^2(2 - \frac{q}{2})}. \]

Lemma B. (cf. [7, (7)]) For \( 1 \leq q \leq 2 \) and \( n = 1, 2, \ldots \),

\[ \frac{1}{(n + 1)^{1 - \frac{q}{2}}} \leq \frac{\Gamma(n + \frac{q}{2})}{n!} \leq \frac{1}{n^{1 - \frac{q}{2}}}, \]

where \( \Gamma \) is the Gamma function.
Lemma 2.1. For $1 \leq q < 2$, let

$$F(t) = (1 - t)^{2-q} {}_2F_1\left[1 - \frac{q}{2}, 2 - \frac{q}{2}; 1; t\right],$$

where ${}_2F_1$ is the hypergeometric function. Then $F(t)$ is a decreasing function of $t$ for $0 \leq t \leq 1$.

Proof. Elementary calculations lead to (see [2])

$$\frac{d}{dt} \left( {}_2F_1\left[1 - \frac{q}{2}, 2 - \frac{q}{2}; 1; t\right]\right) = \frac{1}{2} (2 - q) (2 - q) {}_2F_1\left[2 - \frac{q}{2}, 3 - \frac{q}{2}; 2; t\right].$$

Then

$$(2.2) \quad F'(t) = \frac{1}{2} (q - 2)(1 - t)^{1-q} H(t),$$

where

$$H(t) = 2 {}_2F_1\left[1 - \frac{q}{2}, 2 - \frac{q}{2}; 1; t\right] - (2 - q) (1 - t) {}_2F_1\left[2 - \frac{q}{2}, 3 - \frac{q}{2}; 2; t\right].$$

Using the definition of the hypergeometric function, we have the following power series:

$$H(t) = \sum_{m=0}^{\infty} a_m t^m,$$

where the coefficients are as follows:

$$a_m = - \frac{\Gamma(1 + m - \frac{q}{2}) \Gamma(2 + m - \frac{q}{2})}{\Gamma(1 + m) \Gamma(2 + m) \Gamma(2 - \frac{q}{2}) \Gamma(- \frac{q}{2})}, \quad (m = 0, 1, 2, \ldots).$$

For any $1 \leq q < 2$, we have

$$\Gamma(- \frac{q}{2}) < 0,$$

and thus, $a_m \geq 0$ for all $m \geq 0$. This shows that $H(t) \geq 0$. It follows from (2.2) and the assumption $1 \leq q < 2$ that

$$F'(t) \leq 0.$$

Therefore, we know that $F(t)$ is a decreasing function of $t$.

The proof of Lemma 2.1 is complete. \(\square\)

The following result is useful and will be used in proving our main theorems. For $\beta > 0$, $z \in \mathbb{D}$ and $\zeta = e^{i\theta} \in \mathbb{T}$, where $\mathbb{T}$ is the unit circle, we have

$$\frac{1}{(1 - z \zeta)^{\beta}} = \sum_{n=0}^{\infty} \frac{\Gamma(n + \beta)}{n! \Gamma(\beta)} z^n \zeta^n.$$

Using Parseval’s theorem, one gets

$$(2.3) \quad \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|1 - ze^{i\theta}|^{2\beta}} = \sum_{n=0}^{\infty} \left( \frac{\Gamma(n + \beta)}{n! \Gamma(\beta)} \right)^2 |z|^{2n}.$$
Lemma 2.2. For \( z \in \mathbb{D} \), let
\[
I_1(|z|) = \int_{\mathbb{D}} \frac{|w|}{|1 - \bar{w}z|} dA(w).
\]
Then
\[
\sup_{z \in \mathbb{D}} I_1(|z|) = \frac{1 + 2\alpha}{\pi},
\]
where \( \alpha \approx 0.915966 \) is the Catalan’s constant.

Proof. Let \( w = re^{it} \in \mathbb{D} \). It follows from (2.3) that
\[
I_1(|z|) = \frac{1}{\pi} \int_0^1 r^2 dr \int_0^{2\pi} \frac{1}{|1 - \bar{z}re^{it}|} dt
\]
\[
= 2 \sum_{n=0}^\infty \left( \frac{\Gamma(n + \frac{1}{2})}{n!\Gamma(\frac{1}{2})} \right)^2 \frac{1}{2n + 3} |z|^{2n}.
\]
This implies that \( I_1(|z|) \) is an increasing function of \( |z| \), and has its supremum \( I_1(1) \).

Using the equality
\[
\sum_{n=0}^\infty \left( \frac{\Gamma(n + \frac{1}{2})}{n!\Gamma(\frac{1}{2})} \right)^2 \frac{1}{2n + 3} = \frac{1 + 2\alpha}{2\pi},
\]
we have
\[
\sup_{z \in \mathbb{D}} I_1(|z|) = \frac{1 + 2\alpha}{\pi}.
\]
This completes the proof of Lemma 2.2. \( \square \)

Lemma 2.3. For \( w \in \mathbb{D} \), let
\[
I_2(|w|) = \int_{\mathbb{D}} \frac{|w|}{|1 - \bar{w}z|} dA(z).
\]
Then
\[
\sup_{w \in \mathbb{D}} I_2(|w|) = \frac{4}{\pi}.
\]

Proof. Following the proof of Lemma 2.2, we have for \( z = re^{it} \in \mathbb{D} \),
\[
I_2(|w|) = \frac{|w|}{\pi} \int_0^1 r dr \int_0^{2\pi} \frac{1}{|1 - \bar{w}re^{it}|} dt
\]
\[
= |w| \sum_{n=0}^\infty \left( \frac{\Gamma(n + \frac{1}{2})}{n!\Gamma(\frac{1}{2})} \right)^2 \frac{1}{n + 1} |w|^{2n}
\]
\[
= 2F_1 \left[ \frac{1}{2}, \frac{1}{2}; 2; |w|^2 \right] |w|.
\]
This implies that \( I_2(|w|) \) is an increasing function of \( |w| \). Using the equality
\[
(2.4) \quad 2F_1 \left[ \frac{1}{2}, \frac{1}{2}; 2; 1 \right] = \frac{4}{\pi},
\]
we have
\[ \sup_{w \in \mathbb{D}} I_2(|w|) = \frac{4}{\pi}. \]

The proof of Lemma 2.3 is complete. \qed

**Proposition 2.1.** \( \|\mathcal{J}_0^* f\|_p \) is finite for any \( p \geq 1 \).

**Proof.** According to Lemma 2.2, we see that for every \( z \in \mathbb{D} \),
\[ d\mu(w) = \frac{|w|}{|1 - z\bar{w}|} \frac{dA(w)}{I_1(|z|)} \]
is a probability measure in the unit disk, i.e., \( \int_{\mathbb{D}} d\mu(w) = 1 \). For any \( p \geq 1 \) and \( f \in L^p(\mathbb{D}) \), one has
\[ |\mathcal{J}_0^*[f](z)|^p \leq (I_1(|z|))^p \left( \int_{\mathbb{D}} \frac{|w|}{|1 - z\bar{w}|} |f(w)| \frac{dA(w)}{I_1(|z|)} \right)^p. \]

Using Jensen’s inequality, we have
\[ |\mathcal{J}_0^*[f](z)|^p \leq I_1^{p-1}(|z|) \int_{\mathbb{D}} \frac{|w|}{|1 - z\bar{w}|} |f(w)|^p dA(w). \]

Observe that, since \( f \in L^p(\mathbb{D}) \), it follows from Lemma 2.2 and Lemma 2.3 that
\[ \frac{|w|}{|1 - z\bar{w}|} |f(w)|^p \in L^1(\mathbb{D} \times \mathbb{D}). \]

Using Fubini’s theorem, we obtain that
\[ \int_{\mathbb{D}} |\mathcal{J}_0^*[f](z)|^p dA(z) \leq \int_{\mathbb{D}} I_1^{p-1}(|z|) dA(z) \int_{\mathbb{D}} \frac{|w|}{|1 - z\bar{w}|} |f(w)|^p dA(w) \]
\[ = \int_{\mathbb{D}} |f(w)|^p dA(w) \int_{\mathbb{D}} I_1^{p-1}(|z|) \frac{|w|}{|1 - z\bar{w}|} dA(z). \]

On the other hand, it follows from Lemma 2.2 that
\[ I_1(|z|) \leq \frac{1 + 2\alpha}{\pi}, \]
where \( \alpha \approx 0.915966 \) is the Catalan’s constant. Also, Lemma 2.3 shows that
\[ \int_{\mathbb{D}} \frac{|w|}{|1 - z\bar{w}|} dA(z) \leq \frac{4}{\pi}, \]
Then,
\[ \int_{\mathbb{D}} |\mathcal{J}_0^*[f](z)|^p dA(z) \leq \frac{4^p}{\pi^p} (1 + 2\alpha)^{p-1} \int_{\mathbb{D}} |f(w)|^p dA(w), \]
which shows that
\[ \|\mathcal{J}_0^*[f]\|_p \leq \frac{4^p}{\pi} (1 + 2\alpha)^{1-\frac{1}{p}} \|f\|_p. \]

The proof of Proposition 2.1 is complete. \qed
3. Proofs of the main results

**Proof of Theorem 1.1.** For \( f \in L^p(D) \) and \( q \in \mathbb{R} \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \), using Hölder’s inequality for integrals, we see that

\[
|C[f](z)| \leq \left( \int_D |f(w)|^p dA(w) \right)^{1/p} \left( \int_D \frac{1}{|w-z|^q} dA(w) \right)^{1/q}
\]

(3.1)

\[
= \|f\|_p \left( \int_D \frac{1}{|w-z|^{\frac{p}{p-1}}} dA(w) \right)^{1-1/p}.
\]

The assumption of \( p > 2 \) ensures that \( q = \frac{p}{p-1} \in [1, 2) \). Let

(3.2)

\[
K_p(|z|) = \int_D \frac{1}{|w-z|^{\frac{p}{p-1}}} dA(w), \quad z \in D.
\]

We first estimate \( K_p(|z|) \) as follows: Using the Möbius transformation \( w = \frac{z-a}{1-\bar{z}a} \), where \( a = re^{i\theta} \in D \), and the following equality which comes from (2.3):

\[
\frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|1-ze^{i\theta}|^{4-q}} = \sum_{n=0}^{\infty} \left( \frac{\Gamma(n+2-q/2)}{n!\Gamma(2-q/2)} \right)^2 |rz|^{2n},
\]

we have

\[
K_p(|z|) = \frac{(1-|z|^2)^{2-q}}{\pi} \int_0^1 r^{1-q} dr \int_0^{2\pi} \frac{d\theta}{|1-\bar{z}e^{i\theta}|^{4-q}}
\]

\[
= 2(1-|z|^2)^{2-q} \sum_{n=0}^{\infty} \left( \frac{\Gamma(n+2-q/2)}{n!\Gamma(2-q/2)} \right)^2 \frac{|z|^{2n}}{2n+2-2-q}
\]

\[
= \frac{2(1-|z|^2)^{2-q} {}_2F_1 \left[ \frac{1}{2}, 2-\frac{q}{2}; 1; |z|^2 \right]}{2-q},
\]

where \( {}_2F_1 \) is the hypergeometric function. Since \( 1 \leq q < 2 \), it follows from Lemma 2.1 that

\[
F(t) = (1-t)^{2-q} {}_2F_1 \left[ \frac{1}{2}, 2-\frac{q}{2}; 1; t \right]
\]

is a decreasing function of \( t \) for \( 0 \leq t \leq 1 \). Therefore, we have \( K_p(|z|) \) is a decreasing function of \( |z| \) and has its maximum

\[
K_p(0) = \frac{2}{2-q}.
\]

Then using (3.1), we have

\[
\|C[f]\|_\infty \leq \|f\|_p K_p^{1-\frac{1}{p}}(0),
\]

which implies that

(3.3)

\[
\|C\|_\infty \leq K_p^{1-\frac{1}{p}}(0).
\]

To show the equality of (3.3), fix \( b \in D \) and consider the function

\[
f(w) = K_p^{-\frac{1}{p}}(|b|) \frac{w-b}{|w-b|^{p-1}},
\]
where \( p > 2 \) and \( q = \frac{p}{p-1} \in [1, 2) \). Then
\[
\int_D |f(w)|^p dA(w) = K_p^{-1}(|b|) \int_D \frac{1}{|w-b|^q} dA(w) = 1.
\]
This shows that \( \|f\|_p^p = 1 \). On the other hand, elementary calculations show that
\[
|\mathcal{C}[f](b)| = K_p^{-\frac{1}{p}}(|b|) \int_D \frac{1}{|w-b|^q} dA(w) = K_p^{1-\frac{1}{p}}(|b|).
\]
Hence,
\[
\|\mathcal{C}\|_{\infty} \geq \|\mathcal{C}[f]\|_{\infty} \geq K_p^{1-\frac{1}{p}}(0).
\]
Observe that since \( b \) can be arbitrarily close to 0, we have
\[
(3.4) \quad \|\mathcal{C}\|_{\infty} \geq K_p^{1-\frac{1}{p}}(0).
\]
According to (3.3) and (3.4), we see that (1.1) holds true.

The proof of Theorem 1.1 is complete. \( \Box \)

**Remark 3.1.** If \( 1 \leq p \leq 2 \), then \( \mathcal{C} \) will not send \( L^p(D) \) to \( L^\infty(D) \). We only need to consider the case of \( p = 2 \), because \( L^2(D) \subseteq L^p(D) \).

Fix \( b \in D \) and consider the function
\[
g(w) = \frac{1}{(b-w)\log \frac{3}{|b-w|}}.
\]
Then
\[
\int_D |g(w)|^2 dA(w) = \int_{D'} \frac{dA(w)}{|b-w|^2 \log^2 \frac{3}{|b-w|}}.
\]
Let \( \xi = b-w \) and \( D' = \{ \xi : |\xi - b| < 1 \} \). We have \( D' \subset D(0, 2) := \{ \xi : |\xi| < 2 \} \).

Thus, for \( \xi = Re^{i\theta} \in D' \), we have
\[
\int_{D'} |g(w)|^2 dA(w) = \int_0^{2\pi} d\theta \int_0^{2} \frac{dR}{R \log^2 \frac{3}{R}} = \frac{2}{\log \frac{3}{2}},
\]
which shows \( g \in L^2(D) \).

However, let \( D(b) = \{ w : |w-b| < 1 - |b| \} \subset D \). Then
\[
|\mathcal{C}[g](b)| = \int_{D(b)} \frac{dA(w)}{|b-w|^2 \log^2 \frac{3}{|b-w|}} \geq \int_{D(0)} \frac{dA(w)}{|b-w|^2 \log^2 \frac{3}{|b-w|}}.
\]
Elementary calculations show that
\[
\int_{D(0)} \frac{dA(w)}{|b-w|^2 \log^2 \frac{3}{|b-w|}} = \frac{1}{\pi} \int_0^{2\pi} d\theta \int_0^{1-|b|} \frac{1}{\rho \log^2 \frac{3}{\rho}} d\rho = \infty,
\]
which shows that \( \mathcal{C}[g] \notin L^\infty(D) \).
Proof of Theorem 1.2. Recall that
\[ J_0[f](z) = \int_D \frac{z}{1 - \overline{w}z} f(w) dA(w). \]

For \( p > 2 \), assume that \( f \in L^p(\mathbb{D}) \). Applying Hölder’s inequality for integrals, we have
\[
|J_0[f](z)| \leq \left( \int_D |f(w)|^p dA(w) \right)^{\frac{1}{p}} \left( \int_D \frac{|z|^q}{|1 - \overline{w}z|^q} dA(w) \right)^{\frac{1}{q}},
\]
where \( \frac{1}{p} + \frac{1}{q} = 1 \).

For \( w = re^{it} \in \mathbb{D} \), again by (2.3), we have
\[
\int_D \frac{|z|^q}{|1 - \overline{w}z|^q} dA(w) = \frac{|z|^q}{\pi} \int_0^1 r dr \int_0^{2\pi} \frac{dt}{|1 - zre^{-it}|^q} = \sum_{n=0}^{\infty} \left( \frac{\Gamma(n + \frac{q}{2})}{n!\Gamma(\frac{q}{2})} \right)^2 \frac{|z|^{2n+q}}{n+1} = 2F_1 \left[ \frac{q}{2}, \frac{q}{2}; 2; |z|^2 \right] |z|^q,
\]
where \( q = \frac{p}{p-1} \in [1, 2) \) and \( 2F_1 \) is the hypergeometric function. Let
\[
M_q(|z|) = 2F_1 \left[ \frac{q}{2}, \frac{q}{2}; 2; |z|^2 \right] |z|^q.
\]

Then, \( M_q(|z|) \) is an increasing function of \( |z| \) and attains its maximum at \( z = 1 \). It follows from Lemma A that
\[
M_q(1) = \frac{\Gamma(2 - q)}{\Gamma^2(2 - \frac{q}{2})}.
\]

By using (3.5), (3.6) and (3.8), we have \( \|J_0[f]\|_\infty \leq \|f\|_p \left( \frac{\Gamma(2 - q)}{\Gamma^2(2 - \frac{q}{2})} \right)^{\frac{1}{q}} \), which implies that
\[
\|J_0\|_\infty \leq \left( \frac{\Gamma(2 - q)}{\Gamma^2(2 - \frac{q}{2})} \right)^{\frac{1}{q}}.
\]

To show the equality of (3.9), fix \( b \in \mathbb{D} \), and consider the following function
\[
g(w) = M_q^{\frac{1}{q}}(|b|) \frac{\overline{b}}{1 - w\overline{b}} \left| \frac{1 - w\overline{b}}{b} \right|^{\frac{p-2}{p-1}},
\]
where \( p > 2 \) and \( q = \frac{p}{p-1} \in [1, 2) \). Then
\[
\|g\|_p^p = M_q^{-1}(|b|) \int_D \frac{|b|^q}{|1 - w\overline{b}|^q} dA(w).
\]
It follows from (3.6) and (3.7) that \( \|g\|_p^p = 1 \), and thus \( g \in L^p(\mathbb{D}) \).
On the other hand, elementary calculations show that for \( q = \frac{p}{p-1} < 2 \), we have

\[
|J_0[g](b)| = M_q \left( |b| \right) \int_{D} \frac{|b|^q}{|1 - b r e^{-it}|^q} dA(w)
= M_q \left( |b| \right) \sum_{n=0}^{\infty} \left( \frac{\Gamma(n + \frac{q}{2})}{n! \Gamma(\frac{q}{2})} \right)^2 \frac{|b|^{2n+q}}{n+1}
= M_q^{1 - \frac{1}{p}}(|b|).
\]

This shows that

\[
\|J_0\|_{\infty} \geq \|J_0[g]\|_{\infty} \geq M_q^{1 - \frac{1}{p}}(|b|).
\]

Observe that, since \( b \) can be arbitrarily close to 1, by (3.8), we have

\[(3.10) \quad \|J_0\|_{\infty} \geq \left( \frac{\Gamma(2 - q)}{\Gamma^2(2 - q/2)} \right)^{\frac{1}{q}}.
\]

According to (3.9) and (3.10), we see that

\[
\|J_0\|_{\infty} = \left( \frac{\Gamma(2 - q)}{\Gamma^2(2 - q/2)} \right)^{\frac{1}{q}},
\]

and thus, (1.2) holds true.

If in particular \( p = \infty \) (that is \( q = 1 \)), assume that \( f \in L^\infty(\mathbb{D}) \). Then using Lemma 2.3, we have

\[
\|J_0[f]\|_{\infty} \leq \|f\|_{\infty} \int_{D} \frac{|z|}{|1 - wz|} dA(w) \leq \frac{4}{\pi} \|f\|_{\infty},
\]

which shows that

\[(3.11) \quad \|J_0\|_{\infty} \leq \frac{4}{\pi}.
\]

Fix \( s \in \mathbb{D} \), such that \( s \neq 0 \). Let

\[
f_s(w) = \frac{\bar{s}}{1 - w\bar{s}} \left| \frac{1 - w\bar{s}}{\bar{s}} \right|.
\]

Then we get \( \|f_s(w)\|_{\infty} = 1 \) and, by the proof of Lemma 2.3, we have

\[
J_0[f_s](s) = \int_{D} \frac{|s|}{|1 - ws|} dA(w) = _2F_1 \left[ \frac{1}{2}, \frac{1}{2}; 2; |s|^2 \right] |s|.
\]

This implies that

\[
\|J_0[f_s]\|_{\infty} \geq _2F_1 \left[ \frac{1}{2}, \frac{1}{2}; 2; 1 \right] = \frac{4}{\pi}.
\]

Then

\[(3.12) \quad \frac{4}{\pi} \leq \|J_0[f_s]\|_{\infty} \leq \|J_0\|_{\infty}.
\]

According to (3.11) and (3.12) we see that \( \|J_0\|_{\infty} = \frac{4}{\pi} \).

This completes the proof of Theorem 1.2. \( \square \)
**Remark 3.2.** For \( p = 2 \), \( \mathfrak{J}_0 \) will not send \( L^2(\mathbb{D}) \) to \( L^\infty(\mathbb{D}) \). This can be seen as follows: Let
\[
g(w) = \frac{1}{(1 - w) \log \frac{3}{|1 - w|}},
\]
where \( w = \rho e^{it} \in \mathbb{D} \). Assume that \( \xi = 1 - w = \text{Re}^{it} \). Then
\[
\int_{\mathbb{D}} |g(w)|^2 \, dA(w) = \int_{\mathbb{D}'} \frac{dA(\xi)}{|\xi|^2 \log^2 \frac{3}{|\xi|}},
\]
where \( \mathbb{D}' = \{ \xi : |\xi - 1| < 1 \} \subset \mathbb{D}(0, 2) := \{ \xi : |\xi| < 2 \} \). Therefore,
\[
\int_{\mathbb{D}} |g(w)|^2 \, dA(w) \leq \frac{1}{\pi} \int_{0}^{2\pi} d\theta \int_{0}^{2} \frac{1}{R \log^2 \frac{3}{R}} \, dR = \frac{2 \log \frac{3}{2}}{2},
\]
which shows that \( g \in L^2(\mathbb{D}) \).

Next, we are going to prove \( \mathfrak{J}_0[g] \notin L^\infty(\mathbb{D}) \). Once this is done, we then have \( \mathfrak{J}_0 \) doesn’t send \( L^p(\mathbb{D}) \) to \( L^\infty(\mathbb{D}) \) for any \( 1 \leq p \leq 2 \), because \( L^2(\mathbb{D}) \subseteq L^p(\mathbb{D}) \).

For \( z = r \in (0, 1) \) and \( w = \rho e^{it} \in \mathbb{D} \), let
\[
G(t, \rho, r) = \text{Re} \left( \frac{z}{1 - \bar{w}z} g(w) \right) = \frac{r(1 + \rho^2 - \rho(1 + r) \cos t)}{(1 + r^2 \rho^2 - 2r \rho \cos t)(1 + \rho^2 - 2\rho \cos t) \log \frac{3}{\sqrt{1 + \rho^2 - 2\rho \cos t}}}
\]
Then it is easy to see that \( G(t, \rho, r) > 0 \), for all \( \rho, r \in (0, 1) \) and \( t \in [0, 2\pi] \), since
\[
1 + \rho^2 - \rho(1 + r) \cos t \geq (1 - \rho)(1 - r\rho) > 0.
\]

Therefore, we have
\[
|\mathfrak{J}_0[g](r)| = \left| \int_{\mathbb{D}} \frac{r}{1 - \bar{w}r} g(w) \, dA(w) \right| \geq \text{Re} \left( \int_{\mathbb{D}} \frac{r}{1 - \bar{w}r} g(w) \, dA(w) \right) = \frac{1}{\pi} \int_{0}^{1} \rho \, d\rho \int_{0}^{2\pi} G(t, \rho, r) \, dt.
\]
(3.13)

Now, if the last integral of (3.13) is infinity as \( r \to 1 \), then our problem is solved. To show this, by using Fatou’s lemma (cf. [14, Page 23]) and (3.13), we have
\[
\lim_{r \to 1} |\mathfrak{J}_0[g](r)| \geq \frac{1}{\pi} \int_{0}^{1} \rho \, d\rho \int_{0}^{2\pi} \lim_{r \to 1} G(t, \rho, r) \, dt,
\]
where
\[
\lim_{r \to 1} G(t, \rho, r) = \frac{1}{(1 + \rho^2 - 2\rho \cos t) \log \frac{3}{\sqrt{1 + \rho^2 - 2\rho \cos t}}} = \frac{1}{|1 - w|^2 \log \frac{3}{|1 - w|}}.
\]
Moreover,
\[
\frac{1}{\pi} \int_0^1 \rho \, d\rho \int_0^{2\pi} \lim_{r \to 1} G(t, \rho, r) \, dt = \int_D \frac{1}{1 - w^2 \log \frac{3}{|1 - w|}} \, dA(w)
\]
\[
= \int_{\partial D} \frac{1}{|\xi|^2 \log \frac{3}{|\xi|}} \, dA(\xi)
\]
\[
= \frac{1}{\pi} \int_0^{2\pi} \frac{1}{2 \cos \theta} \int_0^\infty \frac{1}{R \log \frac{\pi}{R}} \, dR = \infty.
\]

Based on the above discussions, we have \( J_0[g] \notin L^\infty(D) \).

**Proof of Theorem 1.3.** For \( p > 2 \), assume that \( f \in L^p(D) \). It follows from Hölder’s inequality for integrals that

\[(3.14) \quad |J_0^*[f](z)| \leq \left( \int_D |f(w)|^p \, dA(w) \right)^\frac{1}{p} \left( \int_D \frac{|w|^q}{|1 - z\bar{w}|^q} \, dA(w) \right)^\frac{1}{q},\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \).

By using (2.3), for \( w = re^{it} \in D \), we have

\[(3.15) \quad \int_D \frac{|w|^q}{|1 - z\bar{w}|^q} \, dA(w) = \frac{1}{\pi} \int_0^1 r^{q+1} \, dr \int_0^{2\pi} \frac{dt}{|1 - zre^{-it}|^q} = 2 \sum_{n=0}^{\infty} \left( \frac{\Gamma(n + \frac{q}{2})}{n! \Gamma(\frac{q}{2})} \right)^2 \frac{|z|^{2n}}{2n + q + 2},\]

where \( q = \frac{p}{p-1} \in [1, 2) \).

Let \( N_q(|z|) = \int_D \frac{|w|^q}{|1 - z\bar{w}|^q} \, dA(w) \). Then according to (3.15), we have \( N_q(|z|) \) is an increasing function of \(|z|\) and has its supremum at \( z = 1 \). Using the definition of the hypergeometric function, we have

\[(3.16) \quad N_q(1) = 2^{\frac{3}{2}} \frac{\Gamma(1 + \frac{q}{2})^2}{\Gamma(\frac{3}{2})^2} 1, 2 + \frac{q}{2}; 1, 2 + \frac{q}{2}; 1} \frac{1}{2 + q},\]

Moreover, it follows from Remark 1.1 that \( N_q(1) \) is finite.

Applying (3.14) and (3.16), we see that

\[\|J_0^*[f]\|_\infty \leq \|f\|_p N_q^{1 - \frac{1}{p}}(1).\]

Then

\[(3.17) \quad \|J_0\|_\infty \leq N_q^{1 - \frac{1}{p}}(1).\]

To show the equality of (3.17), fix \( b \in D \), and consider the function

\[g(w) = \frac{w}{1 - bw} \left| \frac{1 - \bar{b}w}{w} \right|^{\frac{p-2}{p}} N_q^{-\frac{1}{p}}(|b|),\]

where \( p > 2 \). Then for \( q = \frac{p}{p-1} \), we have

\[\int_D |g(w)|^p \, dA(w) = N_q^{-1}(|b|) \int_D \left| \frac{w}{1 - bw} \right|^q \, dA(w) = 1.\]
This shows that \( \|g\|_p^p = 1 \).

On the other hand, elementary calculations show that
\[
|\mathcal{J}_0^*[g](b)| = N_q^{-\frac{q}{p}}(|b|) \int_D \left| \frac{w}{1 - \overline{b}w} \right|^q dA(w) = N_q^{-\frac{q}{p}}(|b|).
\]
Therefore, since \( \sup_{b \in \mathbb{D}} N_q(|b|) = N_q(1) \), we have
\[
\|\mathcal{J}_0^*[g]\|_\infty = \sup_{z \in \mathbb{D}} |\mathcal{J}_0^*[g](z)| \geq \lim_{|b| \to 1} |\mathcal{J}_0^*[g](b)| = N_q^{-\frac{q}{p}}(1).
\]
This shows that
\[
(3.18) \quad \|\mathcal{J}_0^*[g]\|_\infty \geq N_q^{-\frac{q}{p}}(1).
\]

According to (3.16), (3.17) and (3.18), we see that (1.4) holds true.

If in particular \( p = \infty \), assume that \( f \in L^\infty(\mathbb{D}) \). Then using Lemma 2.2, we see that
\[
\|\mathcal{J}_0^*[f]\|_\infty \leq \|f\|_\infty \sup_{z \in \mathbb{D}} \int_D \frac{|w|}{|1 - \overline{w}z|} dA(w) = \frac{1 + 2\alpha}{\pi} \|f\|_\infty,
\]
where \( \alpha \approx 0.915966 \) is the Catalan’s constant. Then
\[
(3.19) \quad \|\mathcal{J}_0^*[f]\|_\infty \leq \frac{1 + 2\alpha}{\pi}.
\]

Fix \( s \in \mathbb{D} \), let
\[
f_s(w) = \frac{1 - \overline{w}s}{\overline{w}} \left| \frac{w}{1 - \overline{w}s} \right|.
\]
Then \( \|f_s\|_\infty = 1 \) and by the proof of Lemma 2.2, we have
\[
|\mathcal{J}_0^*[f_s](s)| = 2 \sum_{n=0}^{\infty} \left( \frac{\Gamma(n + \frac{1}{2})}{n! \Gamma(\frac{1}{2})} \right)^2 \frac{|s|^{2n}}{2n + 3}.
\]
This implies that
\[
\|\mathcal{J}_0^*[f_s]\|_\infty \geq 2 \sum_{n=0}^{\infty} \left( \frac{\Gamma(n + \frac{1}{2})}{n! \Gamma(\frac{1}{2})} \right)^2 \frac{1}{2n + 3} = \frac{1 + 2\alpha}{\pi}.
\]
Then
\[
(3.20) \quad \|\mathcal{J}_0^*[f_s]\|_\infty \geq \|\mathcal{J}_0^*[f_s]\|_\infty \geq \frac{1 + 2\alpha}{\pi}.
\]
By (3.19) and (3.20), we have
\[
\|\mathcal{J}_0^*[f_s]\|_\infty = \frac{1 + 2\alpha}{\pi}.
\]

The proof of Theorem 1.3 is complete. \( \square \)

**Remark 3.3.** For \( p = 2 \), the image of \( L^2(\mathbb{D}) \) under \( \mathcal{J}_0^* \) is not contained in \( L^\infty(\mathbb{D}) \). In what follows, we will do more. We construct a family of mappings \( g_z \) continuous on \( \mathbb{D} \), and that \( \|g_z\|_2 \leq \|g_1\|_2 < \infty \). However, \( \mathcal{J}_0^*[g_1] \) is not in \( L^\infty(\mathbb{D}) \).
For $z \in \mathbb{D}$, set
\[ g_z(w) = w(1 - \bar{z}w)^{-1}\left(\log \frac{3}{|1 - z\bar{w}|}\right)^{-1}. \]
Then for fixed $w \in \mathbb{D}$, the mapping $h(z) = |g_z(w)|^2$ is subharmonic, because direct computation leads to
\[ \Delta h = 16|w|^4 A/B > 0, \]
where $\Delta$ is the Laplace operator, $C = \log |1 - z\bar{w}|^2 - \log 9$, $A = (C + 2)^2 + 2 > 0$ and $B = C^4|1 - z\bar{w}|^2 > 0$.

Now, consider the following mapping
\[ H(z) = \int_{\mathbb{D}} h(z) dA(w) \]
which is subharmonic in $\mathbb{D}$. Using the maximum principle for subharmonic functions, we get
\[ \|g_z(w)\|^2_2 = \int_{\mathbb{D}} h(z) dA(w) \leq \max_{|z|=1} \int_{\mathbb{D}} h(z) dA(w) = \int_{\mathbb{D}} h(1) dA(w). \]

First, we prove $g_z \in L^2(\mathbb{D})$ as follows: Let $\xi = 1 - w = Re^{it} \in \mathbb{D}'$. Then
\[ \int_{\mathbb{D}} \frac{|w|^2}{|1 - w|^2 \log^2 \frac{3}{|1 - w|}} dA(w) \leq \int_{\mathbb{D}(0,2)} \frac{dA(\xi)}{\xi^2 \log^2 \frac{3}{|\xi|}} = 2 \int_0^2 \frac{1}{R \log^2 \frac{\frac{3}{2}}{R}} dR = \frac{2}{\log \frac{3}{2}}. \]
By using (3.21), we see that $\|g_z(w)\|^2 \leq \frac{2}{\log \frac{3}{2}}$, and thus, $g_z \in L^2(\mathbb{D})$.

Second, we prove $\mathfrak{F}[g_1] \not\in L^\infty(\mathbb{D})$ as follows: Following the proof of Remark 3.2, for $z = r \in (0,1)$ and $w = \rho e^{it} \in \mathbb{D}$, let
\[ G_1(t, \rho, r) = \text{Re} \left( \frac{\bar{w}}{1 - r w} g_1(w) \right). \]
Then
\[ G_1(t, \rho, r) = \frac{\rho^2}{r} G(t, \rho, r), \]
where $G(t, \rho, r) > 0$ is the function given in Remark 3.2. Hence, $G_1(t, \rho, r) > 0$.

Moreover,
\[ \lim_{r \to 1} G_1(t, \rho, r) = \rho^2 G(t, \rho, 1) = \frac{|w|^2}{|1 - w|^2 \log \frac{3}{|1 - w|}}. \]
Again, by Fatou’s lemma, we see that for $\xi = 1 - w = Re^{it} \in \mathbb{D}'$,
\[ \lim_{r \to 1} |\mathfrak{F}[g_1](r)| \geq \int_{\mathbb{D}} \lim_{r \to 1} G_1(t, \rho, r) dA(w) \]
\[ = \int_{\mathbb{D}} \frac{|w|^2}{|1 - w|^2 \log \frac{3}{|1 - w|}} dA(w) \]
\[ \geq \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2 \cos \theta} \frac{(1 - R)^2}{R \log \frac{3}{R}} dR. \]
The divergence of the integral \( \int_0^{2\cos \theta} \frac{(1-R)^2}{R \log \left( \frac{3}{R} \right)} dR \) shows that \( \lim_{r \to 1} |J_0^*[g_1](r)| \) is infinity, and thus, \( J_0^*[g_1] \notin L^\infty(\mathbb{D}) \). This shows that \( J_0^* \) doesn’t send \( L^2(\mathbb{D}) \) to \( L^\infty(\mathbb{D}) \).

**Proof of Theorem 1.5.** To prove Theorem 1.5, following the proof of [9, Theorem 5.2 and Corollary 5.3], it suffices to show that

\[
\|J_0^*[P]\|_2^2 \leq \frac{1}{2} \|P\|_2^2
\]

whenever

\[
P(z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{m,n} z^m \bar{z}^n
\]

is a polynomial of \( z \) and \( \bar{z} \), since such functions are dense in \( L^2(\mathbb{D}) \) and \( \frac{1}{2} \) is the best constant. In this case, only finitely many of the complex numbers \( a_{m,n} \) are nonzero. It is evident that there exist radial functions \( f_d \), where \( d \) is an integer number such that

\[
P(z) = \sum_{d=-\infty}^{\infty} g_d(z),
\]

where \( g_d(z) = f_d(r)e^{idt}, \ d = m - n \). Observe that \( g_{d_1} \) and \( g_{d_2} \) are orthogonal for \( d_1 \neq d_2 \) in Hilbert space \( L^2(\mathbb{D}) \).

We will show that

\[\|J_0^*[P]\|_2^2 \leq B \|P\|_2^2\]

if and only if

\[\sum_{d=-\infty}^{\infty} \|J_0^*[g_d]\|_2^2 \leq B \sum_{d=-\infty}^{\infty} \|g_d\|_2^2,\]

where \( B = \frac{1}{2} \) is the best constant.

Suppose \( w = re^{it} \) and \( z = \rho e^{i\theta} \in \mathbb{D} \). By using the transform \( \zeta = e^{it} \), we have

\[
J_0^*[g_d](z) = \int_{\mathbb{D}} \frac{re^{-it}}{1-zr e^{-it}} g_d(r e^{i\theta}) dA(w)
\]

\[
= \frac{1}{\pi} \int_0^1 r^2 f_d(r) dr \int_0^{2\pi} \frac{e^{i(d-1)t}}{1-zr e^{-it}} dt
\]

\[
= \frac{1}{\pi} \int_0^1 r^2 f_d(r) dr \int_{|\zeta|=1} \frac{d\zeta}{iz^{1-d}(\zeta - rz)}.
\]

Let

\[
\lambda_z(r) = \int_{|\zeta|=1} \frac{d\zeta}{iz^{1-d}(\zeta - rz)}.
\]

Then by Cauchy residue theorem, one has

\[
\lambda_z(r) = \begin{cases} 
2\pi(rz)^{d-1}, & \text{if } d \geq 1; \\
0 & \text{if } d < 1.
\end{cases}
\]

Now, we separate our discussions into two cases.
Case 1. Suppose $d \geq 1$.

In this case,
\[ \mathcal{J}_0^*[g_d](z) = 2A_d z^{d-1}, \]
where
\[ A_d = \int_0^1 r^{d+1} f_d(r) dr. \]

It is easy to see that $\mathcal{J}_0^*[g_{d_1}]$ and $\mathcal{J}_0^*[g_{d_2}]$ are orthogonal for any $d_1 \neq d_2$, since
\[ \int_0^{2\pi} e^{im\theta} d\theta = 0, \quad \text{for any } m \neq 0. \]

Therefore, we have
\[ \mathcal{J}_0^*[P](z) = \sum_{d=-\infty}^{\infty} \mathcal{J}_0^*[g_d](z) \]
and
\[ \|P\|_2^2 = \sum_{d=-\infty}^{\infty} \|g_d\|_2^2. \]

This shows that (3.22) and (3.23) are equivalent. Moreover, following the proof of the corresponding results in [9, Theorem 5.2] (see also [1, Page 180]), to prove (3.23), we only need to find the best constant $B$, such that
\[ \|\mathcal{J}_0^*[g_d]\|_2^2 \leq B \|g_d\|_2^2, \]
where $B = \frac{1}{2}$. In fact, we can only choose the function $P(w) = g_d(w) \in L^2(\mathbb{D})$, for fixed $d \in \mathbb{Z}$.

In what follows, we should find the best constant $B$ in (3.23). Elementary calculations show that
\[ \int_D |\mathcal{J}_0^*[g_d](z)|^2 dA(z) = \frac{4|A_d|^2}{d}. \]

On the other hand, we have
\[ \int_D |g_d(z)|^2 dA(z) = 2 \int_0^1 \rho |f_d(\rho)|^2 d\rho. \]

It follows from Proposition 2.1 that there exists a constant $B$ such that
\[ \int_D |\mathcal{J}_0^*[g_d](z)|^2 dA(z) \leq B \int_D |g_d(z)|^2 dA(z). \]

Applying (3.24) and (3.25), we have
\[ \frac{4|A_d|^2}{d} \leq 2B \int_0^1 \rho |f_d(\rho)|^2 d\rho \]
that is
\[ \left| \int_0^1 r^{d+1} f_d(r) dr \right|^2 \leq \frac{dB}{2} \int_0^1 r |f_d(r)|^2 dr. \]

To find the best constant $B$, using Hölder’s inequality for integrals, we see that
\[ \left| \int_0^1 r^{d+1} f_d(r) dr \right|^2 \leq \frac{1}{2d+2} \int_0^1 r |f_d(r)|^2 dr, \]
where the equality holds if $f_d(r) = Cr^d$ and $C$ is a constant. This shows that
\[ B = \frac{1}{d(d+1)}, \quad \text{for } d = 1, 2, \ldots. \]

Thus, $B = \frac{1}{2}$ is the best constant.

Case 2. Suppose $d < 1$. 

...
In this case, 
\[ J_0^*[g_0](z) = 0. \]
Following the proof of Case 1, it is easy to see that in this case the best constant is \( B = 0. \)
Based on the above discussions, we see that
\[ B = \|J_0^*[g_0]\|_2^2 = \frac{1}{2}. \]
The proof of Theorem 1.5 is complete. \( \square \)

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