A directed graph $D$ (or digraph for short) is a graph with a vertex set $V(D)$ and edge set $E(D) \subseteq V^2$. Instead of the usual degree notion, we consider the in- and out-degree of a vertex: $d^-(v)$ counts the number of edges going into $v$ and $d^+(v)$ counts the number of edges leaving $v$.

A path decomposition of a digraph $D$ is a family of paths such that every edge of $D$ is contained in exactly one path of the family. The path number $pn(D)$ is the minimal number of paths necessary for a path decomposition of $D$.

**Upperbounds:** It was conjectured by Alsbach and Pullmann [1], and later proved by O’Brien [5] that the number of paths required to decompose any digraph is at most $n^2/4$. This bound is tight (consider the complete bipartite graph $K_{n/2,n/2}$ in which all edges are oriented in the same direction), but very far from the truth for many graphs.

For various families of graphs, one can prove significantly smaller upperbounds: for instance, it was conjectured by Bollobás and Scott [2] that $O(n)$ many paths are sufficient for any $n$-vertex Eulerian digraph (i.e. $d^-(v) = d^+(v)$ for all vertices $v$) and in a recent work [4], we show that this is true up to a factor $\log d$, where $d$ denotes the average degree of the digraph.

**Lowerbounds:** Regarding lower bounds one easily checks that for any graph $D$, $pn(D) \geq \frac{|E(D)|}{n-1}$, as each path can cover at most $n-1$ many edges. After thinking for a bit, one can find another very interesting lower bound: we define the excess at a vertex $v$ as $ex(v) := d^+(v) - d^-(v)$ and the total excess of the graph $D$ as

$$ex(D) := \frac{1}{2} \sum_{v \in V(D)} |ex(v)|.$$ 

For any path decomposition and any given vertex $v \in V(D)$, we need to have at least $|ex(v)|$ paths that start or end in $v$ (depending on whether the excess is positive or negative), and this clearly implies that

$$pn(D) \geq ex(D).$$

A graph is called consistent if there is equality, i.e. $pn(D) = ex(D)$. One could hope that all graphs are consistent (as it would give a simple method of computing the path number of all graphs), but this is not the case since Eulerian graphs have excess 0 but require at least one (and possibly many) path(s) to be decomposed.

Yet, in a recent work, Espuny Díaz, Patel and Stroh [3] showed that the random digraph $D_{n,p}$ is consistent with probability $1 - o(1)$ if

$$\frac{\log^4 n}{n^{1/3}} \leq p \leq 1 - \frac{\log^{5/2} n}{n^{1/5}}.$$ 

Here, a random digraph $D \sim D_{n,p}$ is sampled by adding each of the possible $n(n-1)$ edges independently with probability $p$ (similar to the binomial random graph model $G_{n,p}$).
**Goal of the Project**  In this project you will explore how $p_n(D)$ behaves in $D_{n,p}$ in other regimes of $p$. Your goals will be:

- Familiarize yourself with state-of-the-art exploration methods.
- Apply known methods to $D_{n,p}$ and see where the limitations are.
- Find general upper bounds for $p_n(D_{n,p})$. Evaluate if these are best possible for given regimes of $p$.
- Determine for which $p$ we have that $D_{n,p}$ is consistent.

More information and grading scheme can be found on:  
https://www.cadmo.ethz.ch/education/thesis/guidelines.html

**Prerequisites:**  Random graphs - ideally you took our course ‘Randomised Algorithms an Probabilistic Methods’

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