New Class of 2-Partition Poisson Quadratic Stochastic Operators on Countable State Space

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Abstract. The idea of quadratic stochastic operator (QSO) which was originally introduced by Bernstein in the early 20th century through his work on population genetics has been significantly developed for decades to describe dynamical systems in many areas. In this research, we construct the dynamical systems generated by a new class of 2-partition of Poisson QSO defined on countable state space, \( X = \{0,1,2,...\} \). Our main goal is to investigate the trajectory behavior of such operators by reducing its infinite variables into a one-dimensional setting that correspond to the number of defined partitions. We present some cases of 2-measurable partition with singleton and two points of two different parameters. Measure and probability theory alongside the functional analysis will be applied to investigate the limit behavior and characteristics of fixed points. These results suggest that the QSO generated by a 2-measurable partition defined on countable state space for both singleton and two points of two different parameters is a regular transformation for some values of parameters.

1. Introduction
In the biology field, the time evolution of species can be easily understood by the following situation. Suppose that \( X = \{1,2,...,m\} \) be the \( m \) type of species in a population. We denote \( x^{(0)} = (x_1^{(0)}, x_2^{(0)}, ..., x_m^{(0)}) \) as the probability distribution of the species in an early state of the population. Then, the probability of an individual in the \( i^{th} \) species and \( j^{th} \) species to cross-fertilize and produce an individual from \( k^{th} \) species is denoted as \( P_{ij,k}^{(0)} \). Given \( x^{(0)} \), we can find the probability distribution \( x^{(1)} = (x_1^{(1)}, ..., x_m^{(1)}) \) of the first generation of \( x^{(0)} \) by using the following total of probability,

\[
x_k^{(1)} = \sum_{i,j=0}^{m} P_{ij,k} x_i^{(0)} x_j^{(0)}, k \in \{1, ..., m\}.
\]

This operator is represented by the symbol \( V \) and it is known as a quadratic stochastic operator (QSO). The operator suggests that starting from an arbitrary initial state, \( x^{(0)} \), a population then continues to evolve to the first generation, \( x^{(1)} = V(x^{(0)}) \), the second generation,
\(x^{(2)} = V(x^{(1)}) = V\left(V\left(x^{(0)}\right)\right) = V^2(x^{(0)}), \) and so on. Thus, one can describe the states of the population by the following dynamical system:

\[x^{(0)}, x^{(1)} = V(x^{(0)}), x^{(2)} = V^2(x^{(0)}), \ldots, x^{(n)} = V^n(x^{(0)}).\]

The dynamical system can be described by a nonlinear operator, i.e., QSO which is also known as an evolutionary operator. In [1], it was given a comprehensive self-reliant exposition of the recent achievements and open problems in the theory of QSO. The most vital part of the study of nonlinear operators is to study the asymptotic behavior of the operators. Due to the difficulty of the problem that depends on the given cubic matrix \(P_{ij,k}=\) there are only a small number of studies on dynamical phenomena defined on higher dimensional systems that are currently comprehended even after many decades. It is well understood that the asymptotic behavior of the QSO even on a small dimensional simplex is complicated.

A QSO has a crucial application in population genetics as first introduced by Bernstein in 1924 (see [2]). This operator is considered as an essential source of analysis to study dynamical properties and modeling in various fields, such as biology (see [3], [4]), physics (see [5]), game theory (see [5]–[10]), mathematics (see [11], [12]), etc. Since the seventies of the 20th century, the limiting behavior of QSO was intensively studied (see [1], [4], [10], [13]–[35]). This field is steadily evolving in many directions. Among the studies involved the idea of Volterra operators (see [13], [18], [33], [36], [37]) where \(P_{ij,k} = 0\) if \(k \not\in \{i, j\}\) with the following biological interpretation: “The offspring repeats one of its parents’ genotype”.

In [14], [15], [17], [21], [24], [35], the authors introduced some classes of QSO with Poisson, Geometric, Gaussian distribution, and Lebesgue measure as the probability measure, \(P_{ij,k}\). The limit behavior of such QSOs was then investigated and described in terms of their regularity and ergodicity. The concept of partition on the state space was implied in [20] where the authors constructed a QSO generated by a 2-partition \(ξ\) on the segment \([0,1]\). Recently, the idea of defining measurable partition was applied on countable state space in [26], [29] where the authors constructed a Geometric QSO generated by a 2-measurable partition and investigated their limiting behavior. The results showed that such an operator is regular for some arbitrary two parameters. In this paper, we provide the study on Poisson QSO generated by a 2-partition on countable state space. We shall investigate the trajectory behavior of such QSO analytically.

2. Preliminaries

Suppose that \(X\) is a state space and \(F\) is a \(σ\)-algebra on \(X\). Let \((X,F)\) be a measurable space, \(S(X,F)\) be the set of all probability measures on such measurable space, and \(\{P(x,y,A): i,j \in X, A \in F\}\) be a family of functions on \(X \times X \times F\) that satisfy the following conditions:

i. \(P(x,y,\cdot) \in S(X,F)\) for any fixed \(i,j \in X\),

ii. \(P(x,y,A)\) regarded as a function of two variables \(x\) and \(y\) with fixed \(A \in F\) is measurable function on \((X \times X, F \otimes F)\), and

iii. \(P(x,y,A) = P(y,x,A)\) for any \(x,y \in X, A \in F\).

We consider a nonlinear transformation \(V:S(X,F) \to S(X,F)\) defined by

\[\left(V \lambda\right)(A) = \int_X P(i,j,A) d\lambda(i) d\lambda(j),\]  

\[(2)\]
where $A \in F$ is an arbitrary measurable set.

If a state space $X = \{1, 2, \ldots, m\}$ be a finite set and corresponding $\sigma$-algebra $F$ on $X$ is a power set $P(X)$, then the set of all probability measure on $(X, F)$ that is called as $(m-1)$-dimensional simplex has the following form,

$$S^{m-1} = \left\{ x = (x_1, \ldots, x_m) \in \mathbb{R}^m : x_i \geq 0, i = 1, \ldots, m, \sum_{i=1}^{m} x_i = 1 \right\}.$$

In this case, for any $i, j \in X$, a probability measure $P(i, j, \cdot)$ is a discrete measure with $\sum_{k=1}^{m} P_{ij,k} = 1$, and corresponding quadratic stochastic operator $V$ is defined as follows:

**Definition 2.1** A mapping $V : S^{m-1} \to S^{m-1}$ is called a quadratic stochastic operator (QSO), if for any $x = (x_1, \ldots, x_m) \in S^{m-1}$, $Vx$ is defined as

$$(Vx)_k = \sum_{i,j=1}^{m} P_{ij,k} x_i x_j,$$

where the coefficients $P_{ij,k}$ satisfy the following conditions:

$$P_{ij,k} \geq 0, \sum_{k=1}^{m} P_{ij,k} = 1 \text{ for } i, j, k \in \{1, 2, \ldots, m\}.$$

**Poisson Quadratic Stochastic Operator**

Let $\{P(i, j, k) : i, j, k \in X\}$ be a family of functions defined on $X \times X \times F$, which satisfy the following conditions:

i. $P(i, j, \cdot)$ is a probability measure on $(X, F)$ for any fixed $i, j \in X$,

ii. $P(i, j, k) = P(j, i, k) = P_{ij,k}$, where $k \in X$ for any fixed $i, j \in X$.

In this case, a QSO in (2) on measurable space $(X, F)$ is defined as follows:

$$V\mu(k) = \sum_{i,j=0}^{m} \sum_{k=0}^{\infty} P_{ij,k} \mu(i) \mu(j),$$

where $k \in X$ for arbitrary measure $\mu \in S(X, F)$.

Throughout this paper, we consider a Poisson QSO generated by Poisson distribution, $P_\lambda$ with a positive real parameter $\lambda$ defined on $X$ by the equation

$$P_\lambda(k) = e^{-\lambda} \frac{\lambda^k}{k!},$$

for any $k \in X$.

**Definition 2.2** A QSO $V$ in equation (3) is called a Poisson QSO if for any $i, j \in X$, the probability measure $P(i, j, \cdot)$ is the Poisson distribution $P_{\lambda(i,j)}$ with a real parameter $\lambda(i,j)$, where $\lambda(i,j) = \lambda(j,i)$.

**Regularity of Quadratic Stochastic Operator**

We consider a QSO $V$ defined on a countable set $X$. Given an initial point $\mu \in S(X, F)$, we denote its trajectory $\{V^n \mu : n = 0, 1, 2, \ldots\}$, where $V^n \mu = V(V^{n-1} \mu)$ for all $n = 0, 1, 2, \ldots$, with $V^0(\mu) = \mu$.

**Definition 2.3** A point $\mu \in S(X, F)$ is called a fixed point of a QSO $V$, if $V\mu = \mu$.

The set of all fixed points of QSO $V$ is denoted by $\text{Fix}(V)$.

**Definition 2.4** A QSO $V$ is called regular if the limit
exists for any initial point \( \mu \in S(X, F) \).

3. Poisson Quadratic Stochastic Operators Generated by 2-Partition

Let \((X, F)\) be a measurable space with countable state space \( X = \mathbb{Z}^* \) where \( \mathbb{Z}^* \) is a set of nonnegative integers.

**Definition 3.1** A probabilistic measure \( \mu \) on \((X, F)\) is said to be discrete, if there exists a finitely many elements \( \{x_1, x_2, \ldots, x_m\} \subset X \), such that \( \mu(\{x_i\}) = p_i \) for \( i = 1, \ldots, m \) with \( \sum_{i=1}^{m} p_i = 1 \). Then, \( \mu(X \setminus \{x_1, x_2, \ldots, x_m\}) = 0 \) and for any \( A \in F \), \( \mu(A) = \sum_{x_i \in A} \mu(x_i) \).

Recall that a partition of \((X, F)\) is a disjoint collection of elements of \( F \) whose union is \( X \). We shall be interested in finite partitions. They will be denoted as \( \xi = \{A_1, \ldots, A_k\} \) and called as measurable \( k \)-partition.

Let \( \xi = \{A_1, A_2\} \) be a measurable 2-partition of the state space \( X \), where \( A_1 \subset X \), \( A_2 = X \setminus A_1 \), and \( \zeta = \{B_1, B_2\} \) be a corresponding partition on \( X \times X \) where \( B_1 = (A_1 \times A_1) \cup (A_2 \times A_2) \) and \( B_2 = (A_1 \times A_2) \cup (A_2 \times A_1) \). We define a family of functions \( \{P_{\xi, \zeta} : i, j, k \in X\} \) as follows:

\[
P_{\xi, \zeta} = \begin{cases} 
eq k \frac{A_k}{k!} & \text{if } (i, j) \in B_1, \\ e^{-\lambda} \frac{k^k}{k!} & \text{if } (i, j) \in B_2. \\ \end{cases} \tag{5}
\]

**Poisson Quadratic Stochastic Operator Generated by 2-Partition of Singleton**

Let \( A_i = \{x_i : x_i \in X\} \) where \( A_i \) consists of a singleton \( x_i \) and \( A_2 = X \setminus A_1 \). We consider a Poisson QSO defined by the family of functions in (5) for the given measurable 2-partition as defined in **Definition 3.1**. Then, for any initial measure \( \mu \in S(X, F) \), we have that

\[
V\mu(k) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} P_{\xi, \zeta} \mu(i) \mu(j)
= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} P_{\xi, \zeta} \mu(i) \mu(j) + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} P_{\xi, \zeta} \mu(i) \mu(j) + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} P_{\xi, \zeta} \mu(i) \mu(j)
= e^{-\lambda} \frac{\lambda^k}{k!} \left[ (\mu(x_i))^2 + (1 - \mu(x_i))^2 \right] + e^{-\lambda} \frac{\lambda^k}{k!} \left[ 2\mu(x_i) \cdot (1 - \mu(x_i)) \right],
\]

\[
V^2\mu(k) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} P_{\xi, \zeta} V\mu(i) V\mu(j)
= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} P_{\xi, \zeta} V\mu(i) V\mu(j) + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} P_{\xi, \zeta} V\mu(i) V\mu(j) + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} P_{\xi, \zeta} V\mu(i) V\mu(j)
+ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} P_{\xi, \zeta} V\mu(i) V\mu(j)
= e^{-\lambda} \frac{\lambda^k}{k!} \left[ (V\mu(x_i))^2 + (1 - V\mu(x_i))^2 \right] + e^{-\lambda} \frac{\lambda^k}{k!} \left[ 2V\mu(x_i) \cdot (1 - V\mu(x_i)) \right].
\]
By using mathematical induction on the sequence $V^n \mu(k)$, we obtained the following recurrent equation

$$V^{n+1} \mu(k) = e^{-\lambda_k} \frac{\lambda_k^k}{k!} \left[ \left( V^n \mu(x_i) \right)^2 + \left( 1 - V^n \mu(x_i) \right)^2 \right] + e^{-\lambda_k} \frac{\lambda_k^k}{k!} \left[ 2V^n \mu(x_i) \cdot (1 - V^n \mu(x_i)) \right],$$

where $n = 0, 1, 2, \ldots$. From this, it is clear that the limit behavior of the recurrent equation in (6) is fully determined by the limit behavior of recurrent equation $V^n \mu(x_i)$ such that

$$V^{n+1} \mu(x_i) = e^{-\lambda_i} \frac{\lambda_i^k}{x_i!} \left[ \left( V^n \mu(x_i) \right)^2 + \left( 1 - V^n \mu(x_i) \right)^2 \right] + e^{-\lambda_i} \frac{\lambda_i^k}{x_i!} \left[ 2V^n \mu(x_i) \cdot (1 - V^n \mu(x_i)) \right],$$

for $n = 0, 1, 2, \ldots$.

**Poisson Quadratic Stochastic Operator Generated by 2-Partition of Two Points**

Let $A_i = \{x_i, x_{i+1}\}$ where $A_i$ consists of two points and $A = X \setminus A_i$. We consider a Poisson QSO defined by the family of functions in (5) for the given measurable 2-partition as defined in **Definition 3.1**. Then, for any initial measure $\mu \in S(X, F)$, we have that

$$V\mu(k) = \sum_{i=0}^{n} \sum_{j=0}^{n} P_{i,j} \mu(i) \mu(j)$$

$$= \sum_{i \in A_i} \sum_{j \in A_j} P_{i,j} \mu(i) \mu(j) + \sum_{i \in A_i} \sum_{j \in A_j} P_{i,j} \mu(i) \mu(j) + \sum_{i \in A_i} \sum_{j \in A_j} P_{i,j} \mu(i) \mu(j)$$

$$= e^{-\lambda} \frac{\lambda^k}{k!} \left[ (\mu(x_1) + \mu(x_2)) + (1 - (\mu(x_1) + \mu(x_2)))^2 \right]$$

$$+ e^{-\lambda} \frac{\lambda^k}{k!} \left[ 2(\mu(x_1) + \mu(x_2)) \cdot (1 - (\mu(x_1) + \mu(x_2))) \right].$$

$$V^2 \mu(k) = \sum_{i=0}^{n} \sum_{j=0}^{n} P_{i,j} V\mu(i) V\mu(j)$$

$$= \sum_{i \in A_i} \sum_{j \in A_j} P_{i,j} V\mu(i) V\mu(j) + \sum_{i \in A_i} \sum_{j \in A_j} P_{i,j} V\mu(i) V\mu(j)$$

$$+ \sum_{i \in A_i} \sum_{j \in A_j} P_{i,j} V\mu(i) V\mu(j)$$

$$= e^{-\lambda} \frac{\lambda^k}{k!} \left[ (V\mu(x_1) + V\mu(x_2)) + (1 - (V\mu(x_1) + V\mu(x_2)))^2 \right]$$

$$+ e^{-\lambda} \frac{\lambda^k}{k!} \left[ 2(V\mu(x_1) + V\mu(x_2)) \cdot (1 - (V\mu(x_1) + V\mu(x_2))) \right].$$

By using mathematical induction on the sequence $V^n \mu(k)$, the following recurrent equation is obtained

$$V^{n+1} \mu(k) = e^{-\lambda} \frac{\lambda^k}{k!} \left[ \left( V^n \mu(x_i) + V^n \mu(x_j) \right)^2 + \left( 1 - (V^n \mu(x_i) + V^n \mu(x_j)) \right)^2 \right]$$

$$+ e^{-\lambda} \frac{\lambda^k}{k!} \left[ 2(V^n \mu(x_i) + V^n \mu(x_j)) \cdot (1 - (V^n \mu(x_i) + V^n \mu(x_j))) \right].$$

where $n = 0, 1, 2, \ldots$. The limit behavior of the recurrent equation in (8) is fully determined by the limit behavior of recurrent equation $V^n \mu(x_i)$ and $V^n \mu(x_j)$ such that
\[ V^{n+1}_i \mu(x_i) = e^{-\frac{\lambda_i}{x_i!}} \left[ \left( V^n \mu(x_i) + V^n \mu(x_j) \right)^2 + \left( 1 - \left( V^n \mu(x_i) + V^n \mu(x_j) \right)^2 \right) \right] + e^{-\frac{\lambda_j}{x_j!}} \left[ 2 \left( V^n \mu(x_i) + V^n \mu(x_j) \right) \cdot \left( 1 - \left( V^n \mu(x_i) + V^n \mu(x_j) \right) \right) \right], \]

for \( n = 0, 1, 2, \ldots \).

**Regularity of Poisson Quadratic Stochastic Operators Generated by 2-Partition of Singleton and Two Points**

As defined in **Definition 3.1**, we may denote \( A(\mu) = \sum_{i \in A_i} \mu(i) \) and \( B(\mu) = \sum_{i \in A_i} \mu(i) \) where \( A(\mu) + B(\mu) = 1 \). Since the recurrent equations in (6) and (8) are fully determined by the limit behavior of recurrent equations in (7) and (9) respectively, then we may investigate the limit behavior of \( V \mu(x_i) \) where \( x_i \in A_i \). By substituting the operator \( V \mu \) into the function \( A(\mu) \) and \( B(\mu) \), we may have the following:

\[ A(V^{n+1}_i \mu) = \sum_{i \in A_i} \left\{ e^{-\frac{\lambda_i}{x_i!}} \left[ A^2(\mu) + B^2(\mu) \right] + e^{-\frac{\lambda_j}{x_j!}} \left[ 2A(\mu)B(\mu) \right] \right\}, \]

\[ B(V^{n+1}_i \mu) = \sum_{i \in A_i} \left\{ e^{-\frac{\lambda_i}{x_i!}} \left[ A^2(\mu) + B^2(\mu) \right] + e^{-\frac{\lambda_j}{x_j!}} \left[ 2A(\mu)B(\mu) \right] \right\}. \]  

(10)

Note that the equations in (10) show that both functions \( A(V^n \mu) \) and \( B(V^n \mu) \) are the summation of Poisson QSO with Poisson distribution, \( P_{\lambda_i} \) and \( P_{\lambda_j} \) for given \( i \in A_i \) and \( i \in A_j \) respectively. By assuming \( x = A(V^n \mu) \), \( y = B(V^n \mu) \), \( x' = A(V^{n+1} \mu) \), and \( y' = B(V^{n+1} \mu) \), we may rewrite the system of equations (10) as follows:

\[ x' = A(P_{\lambda_i})(x^2 + y^2) + 2A(P_{\lambda_j})xy, \]

\[ y' = B(P_{\lambda_i})(x^2 + y^2) + 2B(P_{\lambda_j})xy, \]  

(11)

with \( x \geq 0 \), \( y \geq 0 \) and \( x + y = 1 \).

Since \( x + y = 1 \), it is sufficient to find all solutions \( x \) of the first equation of the system of equations in (11) where \( 0 < x < 1 \). By substituting \( y = 1 - x \) into the first equation of the system of equations in (11), we can get the following quadratic equation with respect to \( x \) that is given by

\[ x' = 2A(P_{\lambda_i}) - A(P_{\lambda_j}) \]

(12)

The right-hand side of the equation in (12) can be regarded as a function that maps segment \([0,1]\) into itself where \( \lambda_1, \lambda_2 \geq 0 \). Assume that \( x' = x \). Hence, the following statement is established.

**Theorem 3.2** The quadratic equation (12) has a stable fixed point in the open interval \((0,1)\).

**Proof.** We have the following quadratic equation:

\[ 2(a - b)x^2 - (2(a - b) - 1)x + a = 0, \]  

(13)

where \( a = A(P_{\lambda_i}) \) and \( b = A(P_{\lambda_j}) \). We can observe that the equation (13) has a root in the interval
(−∞, 0) and (1, ∞) when \(2(a−b)<0\) and \(2(a−b)>0\), respectively. When \(2(a−b)=0\), then it becomes linear with \(a>0\). Thus, for all cases, the root in \([0,1]\) is unique and varies from 0 to 1.

To investigate for behavior of the fixed point, we have the discriminant of the equation in (13) that takes the following form:
\[
\Delta = 4(1−a) + (1−2b)^2.
\]
(14)

Since \(0 < a,b < 1\), then by simple calculations, we get \(0 < \Delta < 2\).

Let \(f(x)\) be the right-hand side of the equation in (12), its derivative \(f'(x)\) is continuous and \(x^*\) be a fixed point of the equation in (13) where,
\[
x^* = \frac{2(a−b) + 1 − \sqrt{\Delta}}{4(a−b)},
\]
(15)
and
\[
f'(x^*) = 4(a−b)x^* − 2(a−b).
\]
(16)

By substituting (15) into (16), we get the following
\[
f'(x^*) = 1 − \sqrt{\Delta}.
\]
(17)

**Remark 3.3** Let \(p\) be a hyperbolic fixed point where \(|f'(p)| \neq 1\). Then,

(i) \(|f'(p)| < 1\) implies \(p\) is attracting,

(ii) \(|f'(p)| > 1\) implies \(p\) is repelling.

Since \(0 < \Delta < 2\), then \(1 − \sqrt{2} < f'(x^*) < 1\). Due to **Remark 3.3**, any unique fixed point \(x^*\) of the equation (13) in the open interval \((0,1)\) is attracting or in other words, it is a stable fixed point. This completes the proof.

By finding all solutions \((x^*, y^*)\), it implies that the QSO (10) is a regular due to **Definition 2.4** where \(\lim_{n \to \infty} A^n = x^*\), and \(\lim_{n \to \infty} B^n = y^*\) for any initial measure \(\mu\). Then, passing to limit of \(V^{n+1}\mu(k)\), for any \(k \in X\), we have that
\[
\lim_{n \to \infty} V^{n+1}\mu(k) = \lim_{n \to \infty} \left\{ e^{-i\lambda_1} \frac{\lambda_1^k}{k!} \left[ A^n\mu + B^n\mu \right] + e^{-i\lambda_2} \frac{\lambda_2^k}{k!} \left[ 2A^n\mu B^n\mu \right] \right\}
= e^{-i\lambda_1} \frac{\lambda_1^k}{k!} \left[ (x^*)^2 + (y^*)^2 \right] + e^{-i\lambda_2} \frac{\lambda_2^k}{k!} \left[ 2x^*y^* \right]
= \left[ (x^*)^2 + (y^*)^2 \right] P_{\lambda_1}(k) + \left[ 2x^*y^* \right] P_{\lambda_2}(k).

Thus, for any initial measure \(\mu\), the strong limit of the sequence \(V^n\mu\) exists and is equal to the convex linear combination of the two Poisson measures. It is evident that
\[
\text{Fix}(V) = \left[ (x^*)^2 + (y^*)^2 \right] P_{\lambda_1}(k) + \left[ 2x^*y^* \right] P_{\lambda_2}(k).
\]

4. **Conclusion**
A Poisson quadratic stochastic operator generated by 2-partition \(\xi\) of singleton and two points has a unique fixed point which is attracting and all its trajectory converge to this fixed point for arbitrary initial measure and two parameters, \(\lambda_1\) and \(\lambda_2\). This suggests that such an operator is a regular transformation. Based on this study, we can conclude that a free population with two Poisson distribution measures as defined in (5) will be eventually stable.
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