RATIONAL POINTS OF BOUNDED HEIGHT ON GENUS ZERO MODULAR CURVES AND AVERAGE ANALYTIC RANKS OF ELLIPTIC CURVES OVER NUMBER FIELDS

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Abstract. We give asymptotics for the number of isomorphism classes of elliptic curves over arbitrary number fields with certain prescribed level structures and prescribed local conditions. In particular, we count the number of points of bounded height on many genus zero modular curves which are isomorphic to a weighted projective space. This includes the cases of $X(N)$ for $N \in \{1, 2, 3, 4, 5\}$, $X_1(N)$ for $N \in \{1, 2, \ldots, 10, 12\}$, and $X_0(N)$ for $N \in \{1, 2, 4, 6, 8, 9, 12, 16, 18\}$. In all cases we give an asymptotic with an expression for the leading coefficient, and in many cases we also give a power savings error term. Our results for counting points on modular curves follow from more general results for counting points of bounded height on weighted projective spaces.

Using our results for counting elliptic curves over number fields with a prescribed local condition, we are able to give a conditional bound for the average analytic rank of elliptic curves over an arbitrary number field. In particular, under the assumptions that all elliptic curves over a number field $K$ are modular and have $L$-functions which satisfy the Generalized Riemann Hypothesis, we show that the average analytic rank of isomorphism classes of elliptic curves over $K$ is bounded above by $3 \deg(K) + 1/2$.

1. Introduction

1.1. Average analytic ranks of elliptic curves. Let $E$ be an elliptic curve over a number field $K$. The Mordell–Weil theorem states that the set of $K$-rational points $E(K)$ of $E$ forms a finitely generated abelian group $E(K) \cong E(K)_{\text{tor}} \oplus \mathbb{Z}^r$ where $E(K)_{\text{tor}}$ is the finite torsion subgroup and $r \in \mathbb{Z}_{\geq 0}$ is the rank. The study of ranks of elliptic curves has become a central topic in number theory. Despite this attention, ranks remain a mystery in many ways.

Birch and Swinnerton-Dyer [BSD65] famously conjectured that the rank of an elliptic curve equals its analytic rank (i.e. the order of vanishing of its $L$-function at $s = 1$).

To get an understanding of ranks of elliptic curves in general, one can hope to determine the average rank of elliptic curves. The average analytic rank of elliptic curves over the rational numbers $\mathbb{Q}$ was first bounded by Brumer [Bru92], who gave an upper bound of 2.3 under the assumption of the Generalized Riemann Hypothesis for elliptic $L$-functions. Under the same conditions this was later improved to 2 by Heath-Brown [HB04], and then to $25/14 \approx 1.8$ by Young [You06]. Remarkably, in 2015 Bhargava and Shankar [BS15] were able to prove an unconditional upper bound of 1.5 for the average rank of elliptic curves over $\mathbb{Q}$.

Surprisingly little is known about average ranks of elliptic curves over number fields beyond $\mathbb{Q}$. In Shankar’s doctoral thesis [Sha13] he extends his work with Bhargava to show that the average rank of elliptic curves over number fields is also bounded above by 1.5. However,
Shankar’s result counts only reduced Weierstrass equations. When the underlying number field has unique factorization, then this is the same as counting isomorphism classes of elliptic curves; but, when the underlying number field does not have unique factorization, then there can be multiple reduced Weierstrass equations within the same isomorphism class of elliptic curves.

In this article we prove a conditional bound for the average analytic rank of isomorphism classes elliptic curves over an arbitrary number field. This appears to be the first known bound on average ranks of isomorphism classes of elliptic curves over arbitrary number fields! As well as the first bound for average analytic ranks of elliptic curves over number fields other than \( \mathbb{Q} \).

**Theorem 1.1.1.** Let \( K \) be a number field of degree \( d \). Assume that all elliptic curves over \( K \) are modular and that their \( L \)-functions satisfy the Riemann Hypothesis. Then the average analytic rank of isomorphism classes of elliptic curves over \( K \) is bounded above by \( 3d + 1/2 \).

Like many other results on analytic ranks of elliptic curves, we will use the so-called explicit formula. We follow a strategy similar to recent work of Cho and Jeong [CJ21], in which they conditionally bound the average analytic rank of elliptic curves with a prescribed torsion subgroup.

One of the main difficulties in extending methods that work over the rational numbers to more general number fields is that one one loses the bijection between reduced short Weierstrass models and isomorphism classes of elliptic curves if the number field does not have unique factorization. We will overcome this difficulty by exploiting the geometry of the moduli stack of elliptic curves as the weighted projective stack \( \mathbb{P}(4,6) \). Doing this allows one to turn questions about counting elliptic curves of bounded height into questions about counting points of bounded height on weighted projective spaces. Such questions can then be studied using various tools from Diophantine geometry.

One of the key ingredients in the proof of Theorem 1.1.1 is estimating the number of elliptic curves with prescribed local conditions over number fields.

To state our results on counting elliptic curves we introduce some notation. Let \( K \) be a number field with, ring of integers \( \mathcal{O}_K \), discriminant \( \Delta_K \), class number \( h_K \), regulator \( R_K \), and which contains \( \omega_K \) roots of unity. Let \( \text{Val}(K) \) denote the places of \( K \), let \( \text{Val}_0(K) \) denote the finite places, and let \( \text{Val}_\infty(K) \) denote the infinite places. Let \( \zeta_K \) denote the Dedekind zeta function of \( K \). Also let \( H(\cdot) \) denote the Kronecker-Hurwitz class number (see Subsection 5.2 for the definition).

**Theorem 1.1.2.** Let \( K \) be a number field and let \( \mathfrak{p} \) be a prime ideal of \( \mathcal{O}_K \) of norm \( q \) such that \( 2 \nmid q \) and \( 3 \nmid q \). Let \( \mathcal{L} \) be one of the local conditions listed in Table 1. Then the number of elliptic curves over \( K \) with naive height less than \( B \) and which satisfy the local condition \( \mathcal{L} \) at \( \mathfrak{p} \) is

\[
\kappa'_\mathcal{L} B^{5/6} + O\left( \epsilon_\mathcal{L} B^{2 - \frac{1}{3\pi}} \right)
\]

where

\[
\kappa'_\mathcal{L} = \frac{h_K (2^{r_1 + r_2} \pi^{r_2})^2 R_K 10^{r_1 + r_2 - 1} \gcd(2, \omega_K)}{\omega_K |\Delta_K| \zeta_K(10)}
\]

and where \( \kappa_\mathcal{L} \) and \( \epsilon_\mathcal{L} \) are as in Table 1.
\[ \frac{(q^2 - q)/q^2}{q} \]
\[ \frac{(q^{10} - q)/q^{11}}{q} \]
\[ \frac{(q - 1)/q^2}{1} \]
\[ \frac{(q - 1)/2q^2}{1} \]
\[ \frac{(q - 1)/2q^2}{1} \]
\[ \frac{(q^9 - 1)/q^{11}}{q} \]
\[ \frac{(q - 1)/q^2}{H(a^2 - 4q)/q^{11}} \]
\[ \frac{(1 - 1/q^2)/q^{(m+n)/2}}{q} \]
\[ \frac{(1 - 1/q^2)/q^3}{1} \]
\[ \frac{(1 - 1/q^2)/q^4}{1} \]
\[ \frac{(1 - 1/q^2)/q^5}{q} \]
\[ \frac{(1 - 1/q^2)/q^{n+5}}{q} \]
\[ \frac{(1 - 1/q^2)/q^9}{1/q^2} \]
\[ \frac{(1 - 1/q^2)/q^8}{1/q^3} \]
\[ \frac{(1 - 1/q^2)/q^7}{1/q} \]

| \( L \)       | \( \kappa_L \) | \( \epsilon_L \) |
|----------------|-----------------|-------------------|
| good           | \( (q^2 - q)/q^2 \) | \( q \)           |
| bad            | \( (q^{10} - q)/q^{11} \) | \( q \)           |
| multiplicative | \( (q - 1)/q^2 \) | 1                 |
| split multiplicative | \( (q - 1)/2q^2 \) | 1                 |
| nonsplit multiplicative | \( (q - 1)/2q^2 \) | 1                 |
| additive       | \( (q^9 - 1)/q^{11} \) | \( q \)           |
| \( a_p(E) = a \) | \( (q - 1)H(a^2 - 4q)/q^2 \) | \( H(a^2 - 4q) \) |

Table 1. Local conditions

Theorem 1.1.2 generalizes a result of Cho and Jeong [CJ20, Theorem 1.4] for elliptic curves over \( \mathbb{Q} \) to elliptic curves over arbitrary number fields. For more results on counting elliptic curves with prescribed local conditions see the recent work of Cremona and Sadek [CS21].

The methods used to prove Theorem 1.1.2 can also be used to prove results for counting elliptic curves with prescribed level structure.

1.2. Counting elliptic curves with prescribed level structure. Let \( N \) be a positive integer and \( G \) a subgroup of \( \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \). An elliptic curve \( E \) over \( K \) admits a \( G \)-level structure if there is a \( (\mathbb{Z}/N\mathbb{Z}) \)-basis for the (geometric) \( N \)-torsion subgroup \( E[N](\overline{K}) \), such that the image of the mod-\( N \) Galois representation

\[ \rho_{E,N} : \text{Gal}(\overline{K}/K) \to \text{Aut}(E[N](\overline{K}) \cong \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \]

is contained in \( G \).

For example, if \( M|N \) are positive integers and

\[ G(M,N) = \left\{ g \in \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) : g = \begin{pmatrix} ^* & ^* \\ 0 & 1 \end{pmatrix} \text{ and } g \equiv \begin{pmatrix} ^* & 0 \\ 0 & 1 \end{pmatrix} \pmod{M} \right\}, \]

then elliptic curves with \( G(M,N) \)-level structure are those whose torsion subgroup contains \( \mathbb{Z}/MZ \times \mathbb{Z}/NZ \). As another example, if \( N \) is a positive integer and

\[ G_0(N) = \left\{ g \in \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) : g = \begin{pmatrix} ^* & ^* \\ 0 & * \end{pmatrix} \right\}, \]

then elliptic curves with \( G_0(N) \)-level structure are those which have a rational cyclic \( N \)-isogeny.
The problem of counting elliptic curves with prescribed level structure can be rephrased in terms of counting rational points on modular curves. Let \( \mathcal{Y}_G \) be the moduli stack parameterizing isomorphism classes of elliptic curves with \( G \)-level structure. We will view \( \mathcal{Y}_G \) as an algebraic stack over \( K_G \), the subfield of the cyclotomic field \( \mathbb{Q}(\zeta_N) \) fixed by action of \( G \) defined by

\[
G \times \mathbb{Q}(\zeta_N) \to \mathbb{Q}(\zeta_N)
\]

\[
(g, \zeta_N) \mapsto \zeta_N^{\det(g)}.
\]

Let \( \mathcal{X}_G \) be the Deligne-Mumford compactification of \( \mathcal{Y}_G \). The behavior of rational points on \( \mathcal{X}_G \) is largely dictated by the trichotomy of curves based on their genus. For example, the torsion subgroups appearing in Mazur’s classification [Maz78] correspond to pairs of positive integers \( M, N \) with \( M | N \), such that the modular curve \( \mathcal{X}_{G(M,N)} \) has genus 0.

With the methods we develop we will be able to answer the following question in certain cases:

**Question 1.2.1.** If \( \mathcal{X}_G \) has genus zero, then how many isomorphism classes of elliptic curves over a number field \( K \) are there of bounded height and with \( G \)-level structure?

In particular, we answer Question 1.2.1 in many cases for which \( \mathcal{X}_G \) is isomorphic to a weighted projective line \( \mathbb{P}(w_0, w_1) \).

**Theorem 1.2.2.** Let \( G \) be a subgroup of \( \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \) and let \( \Gamma_G \) denote the inverse image of \( G \) with respect to the canonical map \( \text{SL}_2(\mathbb{Z}) \to \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \). Let \( K_G \) be the field fixed by the action of \( G \) on \( \mathbb{Q}(\zeta_N) \) determined by \( (g, \zeta_N) \mapsto \zeta_N^{\det(g)} \). Let \( K \) be a degree \( d \) number field containing \( K_G \). Suppose that for some pair of positive integers \( w = (w_0, w_1) \) there is an isomorphism of \( K_G \)-stacks from \( \mathcal{X}_G \) to \( \mathbb{P}(w) \). Set \( |w| := w_0 + w_1, \ w_{\text{min}} = \min\{w_0, w_1\} \), \( e(G) = \frac{\text{max}}{24}[\text{SL}_2(\mathbb{Z}) : \Gamma_G] \), and

\[
\overline{\omega}_{K, w} := \frac{\overline{\omega}_K}{\gcd(w_0, w_1, \overline{\omega}_K)}.
\]

Suppose that \( \gcd(w_1, e(G)) = 1 \) and \( \gcd(w_2, e(G)) = 1 \). Then the number of isomorphism classes of elliptic curves over \( K \) with \( G \)-level structure and height less than \( B \) is

\[
h_{Km_\infty}(\mathcal{F}(1)) \left[ \frac{\text{SL}_2(\mathbb{Z}) : \{\pm I}\Gamma_G}{\overline{\omega}_{K, w}|\Delta_K|\zeta_K(|w|)} \right] B^{\frac{|w|}{12|\overline{\omega}_K|}} + O \left( B^{\frac{d|w| - w_{\text{min}}}{12|\overline{\omega}_K|d}} \right),
\]

where \( \mathcal{F}(1) \subset \prod_{v \in \text{Val}_\infty(K)} K_v^{n+1} \) is a certain region depending on \( G \).

Table 2 displays some cases in which Theorem 1.2.2 applies (see also [BN22, Table 1], which contains some additional cases in which this theorem applies). In particular, the theorem applies to the torsion subgroups in Mazur’s classification, which correspond to the first 15 rows of Table 2. Note that for all \( M|N \) we have \( K_{G(M,N)} = \mathbb{Q} \) and \( K_{G_0(N)} = \mathbb{Q} \). In particular, for \( G(M,N) \)- and \( G_0(N) \)-level structures Theorem 1.2.2 applies for \( K \) an arbitrary number field.
Theorem 1.2.3. Let $\mathcal{X}_G \to \mathcal{X}_{\text{GL}_2(\mathbb{Z})}$ of stacks that forgets the level structure is representable (see Remark 2.1.3). When this morphism is not representable the problem of counting elliptic curves with $G$-level structure becomes more difficult. Nevertheless, in cases for which $\mathcal{X}_G$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}(2) \cong \mathbb{P}(2, 2)$, where the isomorphism is induced by twists of elliptic curves, we prove the following result, which makes no assumption about the representability of the morphism $\mathcal{X}_G \to \mathcal{X}_{\text{GL}_2(\mathbb{Z})}$.

**Theorem 1.2.3.** Let $G$ and $G'$ be subgroups of $\text{GL}_2(\mathbb{Z}/n\mathbb{Z})$, and let $\Gamma_{G'}$ denote the inverse image of $G'$ with respect to the canonical map $\text{SL}_2(\mathbb{Z}) \to \text{GL}_2(\mathbb{Z}/n\mathbb{Z})$. Set

$$e(G') = \frac{[\text{SL}_2(\mathbb{Z}) : \Gamma_{G'}]}{24}.$$

Let $K_G$ be the field fixed by the action of $G$ on $\mathbb{Q}(\zeta_N)$ determined by $(g, \zeta_N) \mapsto \zeta_N^{\det(g)}$ and let $K$ be a number field of degree $d$ containing $K_G$. Suppose that $\mathcal{X}_{G'} \cong \mathbb{P}(1,1)$ over $K_G$, and that for each elliptic curve $E$ over $K$ with $G$-level structure, there is a unique quadratic twist of $E$ with $G'$-level structure, so that $\mathcal{X}_G \cong \mathbb{P}(1,1) \times \mathbb{P}(2)$ as stacks over $K_G$. Then the number of isomorphism classes elliptic curves over $K$ with $G$-level structure and height less

| $G$ | $\Gamma_G$ | $(w_0, w_1)$ | $e(G)$ | $\frac{w_0 + w_1}{12d(G)}$ | $\frac{d(w_0 + w_1) - w_{\text{min}}}{12d(G)}$ |
|-----|-------------|--------------|--------|-----------------|---------------------------------|
| $G(1, 1) = G_0(1)$ | $\Gamma(1) = \text{SL}_2(\mathbb{Z})$ | (4, 6) | 1 | 5/6 | $5/6 - 1/3d$ |
| $G(1, 2) = G_0(2)$ | $\Gamma_1(2) = \Gamma_0(2)$ | (2, 4) | 1 | 1/2 | $1/2 - 1/6d$ |
| $G(1, 3)$ | $\Gamma_1(3)$ | (1, 3) | 1 | 1/3 | $1/3 - 1/12d$ |
| $G(1, 4)$ | $\Gamma_1(4)$ | (1, 2) | 1 | 1/4 | $1/4 - 1/12d$ |
| $G(1, 5)$ | $\Gamma_1(5)$ | (1, 1) | 1 | 1/6 | $1/6 - 1/12d$ |
| $G(1, 6)$ | $\Gamma_1(6)$ | (1, 1) | 1 | 1/6 | $1/6 - 1/12d$ |
| $G(1, 7)$ | $\Gamma_1(7)$ | (1, 1) | 2 | 1/12 | $1/12 - 1/24d$ |
| $G(1, 8)$ | $\Gamma_1(8)$ | (1, 1) | 2 | 1/12 | $1/12 - 1/24d$ |
| $G(1, 9)$ | $\Gamma_1(9)$ | (1, 1) | 3 | 1/18 | $1/18 - 1/36d$ |
| $G(1, 10)$ | $\Gamma_1(10)$ | (1, 1) | 3 | 1/18 | $1/18 - 1/36d$ |
| $G(1, 12)$ | $\Gamma_1(12)$ | (1, 1) | 4 | 1/24 | $1/24 - 1/48d$ |
| $G(2, 2)$ | $\Gamma(2)$ | (2, 2) | 1 | 1/3 | $1/3 - 1/6d$ |
| $G(2, 4)$ | $\Gamma(2, 4)$ | (1, 1) | 1 | 1/6 | $1/6 - 1/12d$ |
| $G(2, 6)$ | $\Gamma(2, 6)$ | (1, 1) | 2 | 1/12 | $1/12 - 1/24d$ |
| $G(2, 8)$ | $\Gamma(2, 8)$ | (1, 1) | 4 | 1/24 | $1/24 - 1/48d$ |
| $G(3, 3)$ | $\Gamma(3)$ | (1, 1) | 1 | 1/6 | $1/6 - 1/12d$ |
| $G(3, 6)$ | $\Gamma(3, 6)$ | (1, 1) | 3 | 1/18 | $1/18 - 1/36d$ |
| $G(4, 4)$ | $\Gamma(4)$ | (1, 1) | 2 | 1/12 | $1/12 - 1/24d$ |
| $G(5, 5)$ | $\Gamma(5)$ | (1, 1) | 5 | 1/30 | $1/12 - 1/60d$ |
| $G_0(4)$ | $\Gamma_0(4)$ | (2, 2) | 1 | 1/3 | $1/3 - 1/26d$ |

Table 2. Cases of Theorem 1.2.2
than $B$ is asymptotic to

\[
\begin{cases} 
2\kappa \sum_{b \in \mathcal{O}_K \text{ square-free}} \lambda(b) \sum_{a \mid b} N_{K/Q}(a) \quad B^{1/6} \log(B) \quad \text{if } e(G') = 1 \\
\kappa \sum_{b \in \mathcal{O}_K \text{ square-free}} \lambda(b) \sum_{a \mid b} N(a) \quad B^{1/6} \quad \text{if } e(G') > 1,
\end{cases}
\]

where

\[
\kappa = \frac{\hbar_K^2 2^{2(r_1 + r_2) - 1} \pi^{r_1} R_K m_{\infty}(F(1))}{\omega_K 2^{3/2} \zeta_K(2)^2 [\text{SL}_2(\mathbb{Z}) : \{\pm I\} \Gamma_G]},
\]

and $\lambda(b)$ is a product of local densities depending on $b$. In particular, the infinite sum appearing in the leading coefficients converges.

This theorem allows one to count $K$-rational cyclic $N$-isogenies for $N \in \{6, 8, 9, 12, 16, 18\}$. Table 3 displays cases in which Theorem 5.1.4 applies. Here $G_{1/2}(N)$ is a subgroup of $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ which we will construct at the end of Subsection 5.1.

| $G$     | $\Gamma_G$ | $G'$  | $\Gamma_{G'}$ | $e(G')$ |
|---------|------------|-------|---------------|--------|
| $G_0(6)$| $\Gamma_0(6)$ | $G(1, 6) = G_{1/2}(6)$ | $\Gamma_1(6) = \Gamma_{1/2}(6)$ | 1       |
| $G_0(8)$| $\Gamma_0(8)$ | $G_{1/2}(8)$ | $\Gamma_{1/2}(8)$ | 1       |
| $G_0(9)$| $\Gamma_0(9)$ | $G_{1/2}(9)$ | $\Gamma_{1/2}(9)$ | 1       |
| $G_0(12)$| $\Gamma_0(12)$ | $G_{1/2}(12)$ | $\Gamma_{1/2}(12)$ | 2       |
| $G_0(16)$| $\Gamma_0(16)$ | $G_{1/2}(16)$ | $\Gamma_{1/2}(16)$ | 2       |
| $G_0(18)$| $\Gamma_0(18)$ | $G_{1/2}(18)$ | $\Gamma_{1/2}(18)$ | 3       |

**Table 3. Cases of Theorem 5.1.4**

Theorem 1.2.2 and Theorem 5.1.4 recover and extend a number of previously known results.

Over $\mathbb{Q}$, many cases of Theorem 1.2.2 were previously proven: In 1992, using a basic lattice points counting argument together with Möbius inversion, Brumer counted isomorphism classes of elliptic curves (i.e. the case of $G(1, 1)$-level structure) [Bru92]. In 2000, Grant counted isomorphism classes of elliptic curves with $G(1, 2)$-level structure [Gra00]; this was done by exploiting several properties of elliptic curves with a two-torsion point and ultimately reducing the problem to counting lattice points in a semi-algebraic region. Harron and Snowden counted isomorphism classes of elliptic curves with $G(1, 3)$-level structure [HS17]; they gave a general strategy for reducing such problems to lattice point counting problems which can be addressed using the Principle of Lipschitz [Dav51] (see also [Lan94, VI §2 Theorem 2]). The method of Harron and Snowden allowed them to give a uniform proof of the $G(1, 1)$, $G(1, 2)$, and $G(1, 3)$ cases. Cullinan, Kenney, and Voight showed that the method of Harron and Snowden could be applied more widely, and doing so were able to count isomorphism classes for all level structures $G$ for which $\mathcal{X}_G \cong \mathbb{P}(1, 1)$ and also for
the cases of $G(2,2)$-level structure and $G_0(4)$-level structure [CKV21]. For other results on counting 4-isogenies see [PS21]. Over the Gaussian integers, $\mathbb{Q}(i)$, Zhao [Zha20] has proven results for the asymptotic growth of elliptic curves with prescribed torsion using similar methods as Harron and Snowden. Our Theorem 1.2.2 extends these results (by also giving a leading coefficient and error term) in the cases for which the underlying modular curve has genus zero. However, an interesting feature of Zhao’s work is that he also address cases in which the underlying modular curve has genus one and positive rank. Such cases lie outside the scope of this paper, but studying points of bounded height on genus one modular curves with positive rank over more general number fields would be an interesting avenue for future research, and should be related to work in Iwasawa theory on the growth of Mordell-Weil ranks in towers of number fields.

Over arbitrary number fields, it was previously known that the asymptotic growth rate for counting isomorphism classes of elliptic curves with prescribed torsion is $\frac{\text{deg}(\text{tors})}{12(G)}$. This was proven using weighted projective spaces. Our proofs will also be proven using weighted projective spaces, though we take a more direct strategy than Bruin and Najman, which allows us to compute the leading coefficient and, in many cases, a power saving error term.

The level structures listed in Table 3 are the only examples that the author is aware of that satisfy the hypotheses of Theorem 5.1.4. For these level structures the only previously known results are those due to Boggess and Sankar [BS20], which give the growth rates of counting isomorphism classes of elliptic curves over $\mathbb{Q}$ with these level structures. What typically makes counting elliptic curves with an $N$-isogeny more difficult than counting elliptic curves with a prescribed torsion subgroup, is that the morphism $X_{G_0(N)} \to X_{GL_2(\mathbb{Z})}$ is rarely representable (the exceptions being $N \in \{1, 2, 4\}$). This non-representability occurs more frequently because the moduli stack $X_{G_0(N)}$ has generic inertia stack $B\mu_2$ (since $-I \in \Gamma_0(N)$). Boggess and Sankar are able to deal with this for some $N$ by constructing a double cover $X_{G_{1/2}(N)}$ of $X_{G_0(N)}$ for which every elliptic curve with $G_0(N)$-level structure has a unique twist with $G_{1/2}(N)$-level structure [BS20, Lemma 2.1]. Then they count twists of elliptic curves with $G_{1/2}$-level structure by using an elementary (but complicated) counting argument, which generalizes a similar argument appearing in the work of Harron and Snowden.

There are also some results for counting elliptic curves over $\mathbb{Q}$ with prescribed level structure which we were not able to generalize to arbitrary number fields, but are worth mentioning. Pomerance, Pizzo, and Voight [PPV20] gave a very precise asymptotic for counting elliptic curves with $G_0(3)$-level structure, which gives several lower order terms. In [BS20, §5], Boggess and Sankar develop a method (different from their method mentioned earlier) for counting elliptic curves with $G_0(N)$-level structure which makes use of the structure of the ring of modular forms. With this method they are able to give an asymptotic growth rate in the cases that $N \in \{2, 3, 4, 5, 6, 8, 9\}$. The case of $G_0(5)$-level structure is particularly interesting since the modular curve $X_{G0(5)}$ has two (quadratic) elliptic points.

Remark 1.2.4. It seems that it should be possible to extend Theorem 5.1.4 to count elliptic curves with $G_0(3)$-level structure, since elliptic curves with $G_0(3)$-level structure arise as twists of elliptic curves with $G(1, 3)$ level structure and $X_{G(1,3)} \cong \mathbb{P}(1, 3)$. More generally, it
would be interesting to prove a version of Theorem 5.1.4 where \( X_{G'} \) could be isomorphic to a more general weighted projective space than \( \mathbb{P}(1, 1) \). The difficulty in doing this is that such modular curves have points which correspond to elliptic curves with extra automorphisms. But for an elliptic curve \( E \) with extra automorphisms, the group of twists of \( E \), which can be identified with \( H^1(\text{Gal}(\mathcal{K}/K), \text{Aut}(E)) \), is not isomorphic to \( \mathbb{P}(2)(K) \cong K^\times/(K^\times)^2 \). Because of this, \( X_{G} \) is not typically isomorphic to \( X_{G'} \times \mathbb{P}(2) \).

1.3. Counting points on weighted projective spaces. The theorems for counting isomorphism classes of elliptic curves in the previous subsections are special cases of more general theorems for counting points of bounded height on weighted projective spaces in various ways.

In 1979 Schanuel [Sch79] proved an asymptotic for the number of rational points of bounded height on projective spaces over number fields. This was generalized to weighted projective spaces by Deng in 1998 [Den98] (see also the work of Dar da [Dar21], which gives a proof using height zeta functions). In a different direction, Bright, Browning, and Loughran [BBL16] proved a generalization of Schanuel’s theorem which allows one to impose infinitely many local conditions on the points of projective space being counted. We further generalize these results by proving an asymptotic for the number of rational points of bounded height satisfying infinitely many local conditions on weighted projective spaces over number fields, where the height can be taken with respect to a different weighted projective space via a morphism:

**Theorem 1.3.1.** Let \( K \) be a degree \( d \) number field over \( \mathbb{Q} \). Let \( w' = (w'_0, \ldots, w'_n) \) and \( w = (w_0, \ldots, w_n) \) be an \((n + 1)\)-tuples of positive integers and let \(|w'| = \sum w'_i\). Let \( f : \mathbb{P}(w') \to \mathbb{P}(w) \) be a non-constant representable generically étale morphism. For each place \( v \in \text{Val}(K) \) let \( \Omega_v \subset \mathbb{P}(w')(K_v) \) be a subset such that the Haar measure \( m_v \) of its affine cone \( \Omega_v^{\text{aff}} \) satisfies \( m_v(\partial \Omega_v^{\text{aff}}) = 0 \) and \( m_v(\Omega_v^{\text{aff}}) > 0 \). Suppose also that for all bounded subsets \( \Psi \subset K_{\infty}^{n+1} \) of positive measure and with \( m_\Psi(\partial \Psi) = 0 \) we have that

\[
\lim_{M \to \infty} \limsup_{B \to \infty} \frac{\#\left\{ x \in O_{K_v}^{n+1} \cap B \ast w \Psi : x \not\in \Omega_v^{\text{aff}} \text{ for some prime } p \text{ with } N(p) > M \right\}}{B^{|w'|}} = 0.
\]

Then we have that

\[
\#\left\{ x \in \mathbb{P}(w')(K) : \text{ht}_f(x) \leq B, \ x \in \Omega_v \text{ for all } v \in \text{Val}(K) \right\} \sim \kappa_{\Omega} \frac{h_K m_\infty(\mathcal{F}(1))}{\omega_K, w'|\Delta_K|^{(n+1)/2} \zeta_K(|w'|)} B^{|w'|},
\]

where \( e(f) \in \mathbb{Z}_{\geq 1} \) depends on \( f \), \( \mathcal{F}(1) \subset \prod_{v \in \text{Val}_{\infty}(K)} K_v^{n+1} \) is a certain region depending on \( f \), and \( \kappa_{\Omega} \) is the following (non-zero) product of local densities,

\[
\kappa_{\Omega} = \prod_{v \not\in \infty} \frac{m_v(\{ x \in \Omega_v^{\text{aff}} : \text{ht}_{f,v}(x) \leq 1 \})}{m_v(\{ x \in K_v^{n+1} : \text{ht}_{f,v}(x) \leq 1 \})} \prod_p m_p(\{ x \in \Omega_p^{\text{aff}} \cap O_{K,p}^{n+1} \}).
\]
If, moreover, $\Omega_v = \mathbb{P}(w)(K_e)$ for all but finitely many $v \in \text{Val}_0(K)$ and $\Omega_v$ is definable in an o-minimal structure for all $v \in \text{Val}_\infty(K)$, then

$$
\#\{x \in \mathbb{P}(w')(K) : \text{ht}_w(x) \leq B, \ x \in \Omega_v \text{ for all } v \in \text{Val}(K)\}
$$

$$
= \kappa_1 \frac{h_K m_\infty(\mathcal{F}(1))}{\varpi_K w'} |\Delta_K|^{(n+1)/2} \zeta_K(|w'|) B^{|w'|/e(f)} + \begin{cases}
O \left( B^{\frac{e}{h_K} \log(B)} \right) & \text{if } w' = (1,1) \text{ and } K = \mathbb{Q},
O \left( B^{\frac{d|w'|-w'_m}{\deg(f)}} \right) & \text{else}.
\end{cases}
$$

**Remark 1.3.2.** After posting this paper to the ArXiv, the author learned that Ayala [Aya21, Theorem 3.19] had recently proven a version of this result without local conditions in her masters thesis.

The ideas behind the proof of this result heavily use geometry-of-numbers techniques. The general ideas are similar to those in the papers cited prior to the theorem, but several changes are required to deal with the morphism $f : \mathbb{P}(w') \rightarrow \mathbb{P}(w)$. We were also able to streamline part of Schanuel’s proof using o-minimal geometry via a lattice counting theorem of Barroero and Widmer [BW14]. In order to deal with the local conditions we prove a weighted version of the geometric sieve for integral points on affine spaces over number fields. It is also interesting that the theorem gives an error term in the case of finitely many local conditions, which does not seem to have been done before. The significance of this is that it requires infinitely many local conditions on the affine cone.

Motivated by counting twists of elliptic curves we prove a result for counting points on weighted projective spaces with respect to a certain twisted height function.

**Theorem 1.3.3.** Let $K$ be a degree $d$ number field over $\mathbb{Q}$. Let $w' = (w'_0, w'_1, \ldots, w'_n)$ and $w = (w_0, w_1, \ldots, w_n)$ be $(n+1)$-tuples of positive integers and let $f : \mathbb{P}(w') \rightarrow \mathbb{P}(w)$ be a non-constant, representable, generically étale morphism. Let $\tau | \gcd(w_0, \ldots, w_n)$ be such that $if \tau \geq |w'|/e(f)$ then $-3\tau + |w| \geq 0$ and if $\tau < |w'|/e(f)$ then $\tau (|w| - |w'|/e(f) - 1) - |w| \geq 0$.

For ideals $a, b \subseteq O_K$ with $b \tau$-free we write $a || b$ if $p^* || a$ implies $p^* || b$ for all $p | a$. Then

$$
\#\{(x, d) \in f(\mathbb{P}(w')(K)) \times \mathbb{P}(\tau)(K) : \text{ht}_w(x^{(d)}) \leq B\}
$$

$$
\asymp \begin{cases}
\frac{\tau}{|w'|/(e(f))} \kappa \sum_{b \subseteq O_K}^{\tau \text{ free}} \lambda(b) \sum_{a || b \subseteq O_K} N(\sqrt{N(a)})^{\tau} B^{|w'|/e(f)} & \text{if } \tau < |w'|/e(f),
\tau \kappa \sum_{b \subseteq O_K}^{\tau \text{ free}} \lambda(b) \sum_{a || b \subseteq O_K} N(\sqrt{N(a)})^{\tau} B^\tau \log(B) & \text{if } \tau = |w'|/e(f),
\frac{|w'|}{e(f)} |\tau - |w'| \kappa \sum_{b \subseteq O_K}^{\tau \text{ free}} \lambda(b) \sum_{a || b \subseteq O_K} N(\sqrt{N(a)})^{\tau} B^\tau & \text{if } \tau > |w'|/e(f).
\end{cases}
$$

where

$$
\kappa = \frac{h_K m_\infty(\mathcal{F}(1))}{\varpi_K |\Delta_K|^{1/2} \zeta_K(\tau)} \cdot \frac{h_K m_\infty(\mathcal{F}(1))}{\deg(f) \varpi_K w' |\Delta_K|^{(n+1)/2} \zeta_K(|w'|)}.
$$
and

\[ \lambda(b) = \prod_p m_p(\{x \in \Omega(b)^{\text{aff}} \cap \mathcal{O}_{K_p}^{n+1}\}) \]

is a product of local densities which will be defined in the proof.

1.4. Organization. In Section 2 we recall some facts about weighted projective spaces, focusing on morphisms between weighted projective spaces and heights defined on weighted projective spaces. In Section 3 we prove several results for counting points on affine spaces over number fields satisfying prescribed local conditions. We focus on three cases: infinitely many finite local conditions, finitely many local conditions, and infinitely many local conditions. Each of these cases allows for more general local conditions, but at the cost of less control over the error term. In Section 4 we prove our results for counting points of bounded height on weighted projective spaces. This is where we prove Theorem 1.3.1 and Theorem 1.3.3. In Section 5 we apply the results of Section 4 to count points of bounded height on genus zero modular curves. This is where we prove our results for counting elliptic curves with prescribed level structures, Theorem 1.2.2 and Theorem 5.1.4, as well as our results for counting elliptic curves with prescribed local conditions, Theorem 1.1.2. Finally, in Section 6 we use the explicit formula for L-functions together with our results for counting elliptic curves with prescribed local conditions to bound the average analytic rank of elliptic curves over number fields. This is where we prove Theorem 1.1.1.

Those interested only in the proof of Theorem 1.1.1 may safely skip Subsections 4.2, 4.3, and 5.1 as well as any results concerning infinitely many local conditions.

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2. Preliminaries on weighted projective spaces

In this section we recall facts about weighted projective spaces with an emphasis on morphisms between weighted projective spaces and on heights on weighted projective spaces.

2.1. Morphisms of weighted projective spaces. Given an \((n+1)\)-tuple of positive integers \(w = (w_0, \ldots, w_n)\), the weighted projective space \(\mathbb{P}(w)\) is the quotient stack

\[ \mathbb{P}(w) = [(\mathbb{A}^{n+1} - \{0\})/\mathbb{G}_m], \]

where \(\mathbb{G}_m\) acts on \(\mathbb{A}^{n+1} - \{0\}\) as follows:

\[ \mathbb{G}_m \times (\mathbb{A}^{n+1} - \{0\}) \rightarrow (\mathbb{A}^{n+1} - \{0\}) \]

\[ (\lambda, (x_0, \ldots, x_n)) \mapsto \lambda * w(x_0, \ldots, x_n) := (\lambda^{w_0}x_0, \ldots, \lambda^{w_n}x_n). \]

In the special case when \(w = (1, \ldots, 1)\), this recovers the usual projective space \(\mathbb{P}^n\). Over fields of characteristic zero all weighted projective spaces are Deligne-Mumford stacks.

Let \(w = (w_0, \ldots, w_n)\) and \(w' = (w'_0, \ldots, w'_n)\) be \((n+1)\)-tuples of positive integers and let \(\mathbb{P}_K(w)\) and \(\mathbb{P}_K(w')\) be the corresponding weighted projective spaces over \(K\).
Let \( K[x'_0, \ldots, x'_n] \) be the graded \( K \)-algebra where \( x'_i \) has weight \( w'_i \) for each \( i \). For each \( i \in \{0, \ldots, n\} \) let \( f_i \in K[x'_0, \ldots, x'_n] \). If there exists an \( e(f) \in \mathbb{Z}_{\geq 0} \) such that each \( f_i \) is a weighted homogeneous polynomial of degree \( e(f)w_i \), and if the homogeneous ideal \( \sqrt{(f_0, \ldots, f_n)} \subseteq R \) contains \((x_0, \ldots, x_n)\), then the ring homomorphism
\[
K[x_0, \ldots, x_n] \to K[x'_0, \ldots, x'_n]
\]
\[
y_i \mapsto f_i
\]
induces a morphism of weighted projective spaces
\[
\varphi_f : \mathbb{P}(w') \to \mathbb{P}(w).
\]

Let \( \Phi_{w', w}(K) \) denote the set of \( f = (f_i) \) as above. There is a natural (weighted) action of \( \mathbb{G}_m(K) \) on \( \Phi_{w', w} \) given by
\[
\mathbb{G}_m(K) \times \Phi_{w', w}(K) \to \Phi_{w', w}(K)
\]
\[
(\lambda, (f_i)) \mapsto \lambda \circ_{w} (f_i) = (\lambda^{w_i} f_i).
\]

The following theorem gives a description of morphisms between weighted projective spaces.

**Theorem 2.1.1.** Using the notation from above, there is a bijection
\[
\Phi_{w', w}/\mathbb{G}_m(K) \to \text{Hom}_K(\mathbb{P}(w'), \mathbb{P}(w))
\]
\[
f = (f_i) \mapsto \varphi_f.
\]

**Proof.** This follows from an explicit description of morphisms between quotient stacks due to Bruin and Najman [BN22, Lemma A.2]. In particular, the proof of [BN22, Lemma 4.1] for weighted projective lines works for higher dimensional weighted projective spaces. \( \square \)

The following proposition is a straightforward generalization of a result of Bruin and Najman [BN22, Lemma 4.5] for weighted projective lines to more general weighted projective spaces.

**Proposition 2.1.2.** If \( \varphi : \mathbb{P}(w')_K \to \mathbb{P}(w)_K \) is a non-constant representable morphism, then \( \varphi \) is finite.

**Remark 2.1.3.** As noted by Bruin and Najman [BN22, Remark 4.3], a morphism \( \varphi_f : \mathbb{P}(w') \to \mathbb{P}(w) \) of weighted projective spaces is representable if and only if \( \gcd(w_i, e(f)) = 1 \) for all \( i \).

### 2.2. Heights on weighted projective spaces

For each \( v \in \text{Val}_0(K) \) let \( p_v \) be the corresponding prime ideal, let \( \pi_v \) be a uniformizer for the completion \( K_v \) of \( K \) at \( v \). For \( x = [x_0 : \cdots : x_n] \in \mathbb{P}(w)(K) \) set \( |x|_{w,v} := |\pi_v|^{v(x_i)/w_i} \) and \( |x|_{w,v} := \max_i \{|x_i|_{w,v}\} \). Then the (exponential) height on \( \mathbb{P}_K(w) \) is defined as
\[
ht_w([x_0 : \cdots : x_n]) = \prod_{v \in \text{Val}_0(K)} |x|_{w,v} \prod_{v \in \text{Val}_\infty(K)} \max_i \{|x_i|_v^{1/w_i}\}.
\]

Define the scaling ideal \( \mathcal{J}_w(x) \) of \( x = [x_0 : \cdots : x_n] \in \mathbb{P}(w)(K) \) to be
\[
\mathcal{J}_w(x) := \prod_{v \in \text{Val}(K)} p_v^{\min_i \{|v(x_i)/w_i|\}}.
\]
The scaling ideal \( \mathcal{I}_w(x) \) can be characterized as the intersection of all fractional ideals \( \mathfrak{a} \) of \( \mathcal{O}_K \) such that \( x \in \mathfrak{a}^{w_0} \times \cdots \times \mathfrak{a}^{w_n} \). It has the property that

\[
\mathcal{I}_w(x)^{-1} = \{ a \in K : a^{w_i} x_i \in \mathcal{O}_K \text{ for all } i \}.
\]

The height can be written in terms of the scaling ideal as follows:

\[
\text{ht}_w([x_0 : \cdots : x_n]) = \frac{1}{N(\mathcal{I}_w(x))} \prod_{v \in \text{Val}_\infty(K)} \max_i \{|x_i|_v^{1/w_i}\}.
\]

This height was first defined by Deng [Den98]. In the case of projective spaces \( \mathbb{P}^n = \mathbb{P}(1, \ldots, 1) \) this height corresponds to the usual Weil height. In more geometric terms, this height on weighted projective spaces can be viewed as the ‘stacky height’ associated to the tautological bundle \( \mathcal{O}_{\mathbb{P}(w)}(1) \) of \( \mathbb{P}(w) \) (see [ESZB21, §3.3] for this and much more about heights on stacks).

Recall that the (compactified) moduli stack of elliptic curves, \( \mathcal{X}_{\text{GL}_2(z)} \), is isomorphic to the weighted projective space \( \mathbb{P}(4, 6) \). This isomorphism can be given explicitly as

\[
\mathcal{X}_{\text{GL}_2(z)} \xrightarrow{\sim} \mathbb{P}(4, 6) \quad \text{with } y^2 = x^3 + Ax + B \mapsto [A : B].
\]

Under this isomorphism, the tautological bundle on \( \mathbb{P}(4, 6) \) corresponds to the Hodge bundle on \( \mathcal{X}_{\text{GL}_2(z)} \). Then the usual naive height of an elliptic curve \( E \) over \( K \) with a reduced (integral) short Weierstrass equation \( y^2 = x^3 + Ax + B \),

\[
\text{ht}(E) = \prod_{v \in \text{Val}_\infty(K)} \max\{|A|_v^3, |B|_v^2\},
\]

is the height with respect to the twelfth power of the Hodge bundle. For any short Weierstrass equation \( y^2 = x^3 + ax + b \) of \( E \) the naive height is given by

\[
\text{ht}(E) = \text{ht}_{(4, 6)}([a : b])^{12}.
\]

When counting elliptic curves, one often wants to count rational points on one modular curve with respect to the height on the moduli space of all elliptic curves (i.e. with respect to the naive height). Motivated by this, we consider heights on one weighted projective space with respect to a different weighted projective space.

Let \( f = (f_0, \ldots, f_n) \) be as in Theorem [2.1.1]. Then we have a morphism

\[
\varphi_f : \mathbb{P}(w') \to \mathbb{P}(w).
\]

We define the following height on \( \mathbb{P}(w') \):

\[
\text{ht}_f(x') = \text{ht}_w([f_0(x') : \cdots : f_n(x')]) = \text{ht}_w(\varphi_f(x')).
\]

Set \( \mathcal{I}_f(x') \) equal to the scaling ideal \( \mathcal{I}_w(\varphi_f(x')) \).

Motivated by counting twists of elliptic curves we also define a certain twisted height. Let \( \tau \) be a positive integer which divides \( \gcd(w_0, \ldots, w_n) \). Let \( d \in \mathbb{P}(\tau)(K) = K^\times/(K^\times)^\tau \). For \( x \in \mathbb{P}(w)(K) \) we define the twist \( x(d) \) of \( x \) in \( \mathbb{P}(w) \) by \( d \) as

\[
x(d) := [d^{w_0/\tau}x_0 : \cdots : d^{w_n/\tau}x_n] \in \mathbb{P}(w)(K).
\]
We then define a *twisted height* on $\mathbb{P}(w') \times \mathbb{P}(\tau)$ by

$$ht_{f,\tau}(x, d) = ht_w(\varphi (x)^{(d)}).$$

### 3. Weighted geometric sieve over number fields

In this section we prove a weighted version of some results of Bright, Browning, and Loughran [BBL16, §3], which are proven along the same lines as a lemma of Poonen and Stoll [PS99, Lemma 20].

#### 3.1. o-minimal geometry

Let $m_\infty$ denote Lebesgue measure on $\mathbb{R}^n$.

**Definition 3.1.1 (Semi-algebraic set).** A (real) semi-algebraic subset of $\mathbb{R}^n$ is a finite union of sets of the form

$$\{x \in \mathbb{R}^n : f_1(x) = \cdots = f_k(x) = 0 \text{ and } g_1(x) > 0, \ldots, g_l(x) > 0\}$$

where $f_1, \ldots, f_k, g_1, \ldots, g_l \in \mathbb{R}[X_1, \ldots, X_n]$.

**Definition 3.1.2 (structure).** A *structure* is a sequence $\mathcal{S} = (S_n)_{n \in \mathbb{Z}_{>0}}$ where each $S_n$ is a set of subsets of $\mathbb{R}^n$ with the following properties:

(i) If $A, B \in \mathcal{S}_n$, then $A \cup B \in \mathcal{S}_n$ and $\mathbb{R}^n - A \in \mathcal{S}_n$ (i.e. $\mathcal{S}_n$ is a Boolean algebra).

(ii) If $A \in \mathcal{S}_m$ and $B \in \mathcal{S}_n$, then $A \times B \in \mathcal{S}_{m+n}$.

(iii) If $\pi : \mathbb{R}^n \to \mathbb{R}^m$ is the projection to $m$ distinct coordinates and $A \in \mathcal{S}_n$, then $\pi(A) \in \mathcal{S}_m$.

(iv) All real semi-algebraic subsets of $\mathbb{R}^n$ are in $\mathcal{S}_n$.

A subset is *definable* in $\mathcal{S}$ if it is contained in some $\mathcal{S}_n$. Let $D \subseteq \mathbb{R}^n$. A function $f : D \to \mathbb{R}^n$ is said to be *definable* if its graph $\Gamma(f) = \{(x, f(x)) : x \in D\} \subseteq \mathbb{R}^{m+n}$ is definable in $\mathcal{S}$.

**Definition 3.1.3.** An *o-minimal structure* is a structure in which the following property holds:

(v) The boundary of each set in $\mathcal{S}_1$ is a finite set of points.

The main structure we will use is $\mathbb{R}_{\exp}$, which is defined to be the smallest structure in which the exponential function $\exp : \mathbb{R} \to \mathbb{R}$ is definable.

**Theorem 3.1.4 (Wilkie [Wil96]).** The structure $\mathbb{R}_{\exp}$ is o-minimal.

From now on we will call a subset of $\mathbb{R}^n$ *definable* if it is definable in some o-minimal structure. The following proposition gives the key measure theoretic properties of bounded definable subsets in $\mathbb{R}^n$.

**Proposition 3.1.5.** [BW14, Lemma 5.3] If $R \subseteq \mathbb{R}^n$ is a bounded subset definable in an o-minimal structure, then $R$ is measurable and the measure of its boundary $\partial R$ is $m_\infty(\partial R) = 0$. 


3.2. Weighted geometry-of-numbers. In later sections we will make use of the following version of the Principle of Lipschitz, which gives estimates for the number of lattice points in a weighted homogeneous space:

**Proposition 3.2.1.** Let \( R \subset \mathbb{R}^n \) be a bounded set definable in an o-minimal structure. Let \( \Lambda \subset \mathbb{R}^n \) be a rank \( n \) lattice with successive minima \( \lambda_1, \ldots, \lambda_n \). Set

\[
R(B) = B \ast_{w'} R = \{ B \ast_{w'} x : x \in R \}.
\]

Then

\[
\#(\Lambda \cap R(B)) = \frac{m_\infty(R)}{\det \Lambda} B^{[w']} + O \left( \lambda_n \det(\Lambda)^{-1} B^{[w']-w'_{\min}} \right)
\]

where the implied constant depends only on \( R \) and \( w' \).

**Proof.** This will follow from a general version of the Principle of Lipschitz due to Barroero and Widmer [BW14]. Since \( m_\infty(R(B)) = B^{[w']} m_\infty(R) \), [BW14] gives the desired leading term. Let \( V_j(R(B)) \) denote the sum of the \( j \)-dimensional volumes of the orthogonal projections of \( R(B) \) onto each \( j \)-dimensional coordinate subspace of \( \mathbb{R}^n \). The error term given in [BW14] is

\[
(3.1) \quad O \left( 1 + \sum_{j=1}^{n-1} V_j(R(B)) \right)
\]

where the implied constant depends only on \( R \). In our case, one observes that \( V_i(R(B)) = O(V_j(R(B))) \) for all \( i \leq j \). Moreover,

\[
V_{n-1}(R(B)) = O(\sum_{i \leq n} B^{[w']-w'_{\min}}) = O(B^{[w']-w'_{\min}}),
\]

where the implied constant depends only on \( R \) and \( w' \). By Minkowski’s second theorem,

\[
\lambda_1 \cdots \lambda_n \geq \frac{2^n}{n! \cdot m_\infty(\mathcal{B}_n)} \det(\Lambda)
\]

where \( \mathcal{B}_n \) is the unit ball in \( \mathbb{R}^n \). Combining these observations with (3.1) we obtain the desired expression for the error term. \( \square \)

We define \( (\mathbb{Z}_p^n) \)-boxes \( \mathcal{B}_p \subset \mathbb{Z}_p^n \) to be Cartesian products \( \prod_{j=1}^n \mathcal{B}_{p,j} \) of closed balls \( \mathcal{B}_{p,j} = \{ x \in \mathbb{Z}_p : |x|_p \leq b_{p,j} \} \subset \mathbb{Z}_p \), where \( a \in \mathbb{Z}_p \) and \( b_{p,j} \in \{ p^k : k \in \mathbb{Z} \} \). We define boxes \( \mathcal{B} \subset \prod_{p \in S} \mathbb{Z}_p^n \) as a Cartesian product of \( \mathbb{Z}_p^n \)-boxes:

\[
\mathcal{B} = \prod_{p \in S} \mathcal{B}_p = \prod_{p \in S} \prod_{j=1}^n \mathcal{B}_{p,j}.
\]

**Lemma 3.2.2 (Box Lemma).** Let \( \Omega_\infty \subset \mathbb{R}^n \) be a bounded subset definable in an o-minimal structure. Let \( S \) be a finite set of primes and \( \mathcal{B} = \prod_{p \in S} \mathcal{B}_p \) a box. Then, for each \( B \in \mathbb{R}_{>0} \), we have

\[
\# \left\{ x \in \mathbb{Z}^n \cap B \ast_w \Omega_\infty : x \in \prod_{p \in S} \mathcal{B}_p \right\} = \left( m_\infty(\Omega_\infty) \prod_{p \in S} m_p(\mathcal{B}_p) \right) B^{[w']} + O \left( \max_j \left( \prod_{p \in S} b_{p,j}^{-1} \right) \left( \prod_{p \in S} m_p(\mathcal{B}_p) \right) B^{[w']-w'_{\min}} \right)
\]

where the implied constant depends only on \( \Omega_\infty \) and \( w \).
Proof. Note that the set of \( x \in \mathbb{Z}^n \cap \prod_{p \in S} \mathcal{B}_p \) is a translate of a sub-lattice of \( \mathbb{Z}^n \). This lattice has determinant \( \prod_{p \in S} m_\infty(\mathcal{B}_p)^{-1} \) and its \( n \)-th successive minimum is
\[
\max_j \left\{ \prod_{p \in S} b_{p,j}^{-1} \right\}.
\]
The lemma then follows from Proposition 3.2.1.

We give a version of Lemma 3.2.2 which applies for more general \( \Omega_\infty \), but without an error term:

**Corollary 3.2.3.** Let \( \Omega_\infty \subset \mathbb{R}^n \) be a bounded measurable subset with \( m_\infty(\partial \Omega_\infty) = 0 \). Let \( S \) be a finite set of primes and \( \mathcal{B} = \prod_{p \in S} \mathcal{B}_p \) a box. Then, for each \( B \in \mathbb{R}_{>0} \), we have
\[
\left| \left\{ x \in \mathbb{Z}^n \cap B \cdot \Omega_\infty : x(p) \in \prod_{p \in S} \mathcal{B}_p \right\} \right| \sim \left( \frac{m_\infty(\partial \Omega_\infty)}{\prod_{p \in S} m_p(\mathcal{B}_p)} \right) B^{\|w\|}.
\]

**Proof.** Since the closure \( \overline{\Omega_\infty} \) of \( \Omega_\infty \) is compact, it can be covered by a finite collection of definable sets \( (R_i)_{i \in I} \) (e.g. by boxes in \( \mathbb{R}^n \)) such that \( m_\infty(\bigcup R_i) \) is arbitrarily close to \( m_\infty(\partial \Omega_\infty) \). Applying Lemma 3.2.2 to each \( R_i \) and summing over the \( i \in I \) gives the desired asymptotic.

Let \( \Lambda \) be a free \( \mathbb{Z} \)-module of finite rank and set \( \Lambda_\infty = \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \) and \( \Lambda_p = \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_p \). Equip \( \Lambda_\infty \) and \( \Lambda_p \) with Haar measures \( m_\infty \) and \( m_p \), normalized so that \( m_p(\Lambda_p) = 1 \) for all but finitely many primes \( p \).

A local condition at a place \( v \in \text{Val}(K) \) will refer to a subset \( \Omega_v \subseteq \Lambda_v \). At a finite place \( v \in \text{Val}_0(K) \) we will call a local condition \( \Omega_v \) finite if it can be determined with only finite \( p_v \)-adic precision, or equivalently, if \( \Omega_v \subseteq \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_p \mathbb{Z} \subset \Lambda_p \), i.e., \( \Omega_v \) has the form \( \{ x \in \Lambda_{p_v} : x \equiv a \pmod{p_v^k} \} \) for some \( k \in \mathbb{Z} \). Note that the product of finitely many finite local conditions is a box, and conversely any box is a finite product of local conditions. A local condition at \( p \) will be called \( p \)-adic if it is not finite.

We are now going to prove three lemmas for counting lattice points in weighted homogeneous spaces with prescribed local conditions. The three cases will be: finitely many finite local conditions, finitely many local condition, and infinitely many local conditions. Each case addresses a more general situation than the previous case, but at the cost of a less precise error term.

**Lemma 3.2.4.** Let \( \Lambda \) be a free \( \mathbb{Z} \)-module of finite rank \( n \), let \( \Omega_\infty \subset \Lambda_\infty \) be a bounded definable subset, and, for each prime \( p \) in a finite subset \( S \), let \( \Omega_p = \prod_j \{ x \in \Lambda_p : |x - a_{p,j}|_p \leq \omega_{p,j} \} \subset \Lambda_p \) be a finite local condition. Let \( \mathbf{w} = (w_1, \ldots, w_n) \) be an \( n \)-tuple of positive integers. Then
\[
\left| \left\{ x \in \Lambda \cap B \cdot \Omega_\infty : x \in \Omega_p \text{ for all primes } p \in S \right\} \right| \sim \left( \frac{m_\infty(\partial \Omega_\infty)}{\prod_{p \in S} m_p(\mathcal{B}_p)} \right) B^{\|w\|} + O\left( \max_j \left\{ \prod_{p \in S} \omega_{p,j}^{-1} \right\} \left( \prod_{p \in S} m_p(\mathcal{B}_p) \right) B^{\|w\| - w_{\min}} \right),
\]
where the implied constant depends only on \( \Lambda, \Omega_\infty, \) and \( \mathbf{w} \).
Proof. Fix an isomorphism $\Lambda \cong \mathbb{Z}^n$. The measures $m_\infty$ and $m_p$ on $\Lambda_\infty$ and $\Lambda_p$ induce measures on $\mathbb{R}^n$ and $\mathbb{Z}_p^n$ which differ from the usual Haar measures by $m_\infty(\Lambda_\infty/\Lambda)$ and $m_p(\Lambda_p)$ respectively. It therefore suffices to prove the result in the case $\Lambda = \mathbb{Z}^n$ and $m_\infty$ and $m_p$ are the usual Haar measures; but this case is precisely the Box Lemma (Lemma 3.2.2).

We now address the case of finitely many (possibly $p$-adic) local conditions.

**Lemma 3.2.5.** Let $\Lambda$ be a free $\mathbb{Z}$-module of finite rank $n$, let $\Omega_\infty \subset \Lambda_\infty$ be a bounded definable subset, and, for each prime $p$ in a finite subset $S$, let $\Omega_p \subset \Lambda_p$ be a subset with $m_p(\partial \Omega_p) = 0$. Let $\mathbf{w} = (w_1, \ldots, w_n)$ be an $n$-tuple of positive integers. Then

$$\# \{ x \in \Lambda \cap B *_{\mathbf{w}} \Omega_\infty : x \in \Omega_p \text{ for all primes } p \in S \}$$

$$= \left( \frac{m_\infty(\Omega_\infty)}{m_\infty(\Lambda_\infty/\Lambda)} \prod_{p \in S} \frac{m_p(\Omega_p)}{m_p(\Lambda_p)} \right) B^{|\mathbf{w}|} + O \left( B^{|\mathbf{w}| - \min S} \right),$$

where the implied constant depends only on $\Lambda$, $\Omega_\infty$, $\Omega_p$, and $\mathbf{w}$.

Proof. As in the proof of Lemma 3.2.4 we may reduce to the case in which $\Lambda = \mathbb{Z}^n$ and $m_\infty$ and $m_p$ are the usual Haar measures on $\mathbb{R}^n$ and $\mathbb{Z}_p^n$.

Let $P = \prod_{p \leq M} \Omega_p$ and let $Q = \prod_{p \leq M} (\mathbb{Z}_p^n - \Omega_p)$ be the complement of $P$ in $\mathbb{Z}_p^n$. By assumption the set $\partial \Omega_p = \partial (\mathbb{Z}_p^n - \Omega_p)$ has measure zero for all $p \in S$. Thus, by compactness, we may cover the closure $\overline{P}$ of $P$ by a finite number of boxes $(I^{(i)})_{i \leq I}$ such that the sum of their measures is arbitrarily close to $m_p(\Omega_p)$. Similarly, we may cover the closure $\overline{Q}$ of $Q$ by finitely many boxes $(J^{(j)})_{j \leq J}$ such that the sum of their measures is arbitrarily close to $1 - m_p(\Omega_p)$.

Applying the Box Lemma (Lemma 3.2.2) to the boxes $(I^{(i)})_{i}$, and then summing over the boxes, gives an upper bound for the set

$$\# \{ x \in \Lambda \cap B *_{\mathbf{w}} \Omega_\infty : x \in \Omega_p \text{ for all primes } p \}.$$

Similarly, applying the Box Lemma to the boxes $(J^{(j)})_{j}$ and summing, gives an upper bound for the complement of the above set. Together these bounds imply the desired asymptotic.

Replacing the use of Lemma 3.2.2 with Corollary 3.2.3 in the above proof, we obtain:

**Corollary 3.2.6.** Let $\Lambda$ be a free $\mathbb{Z}$-module of finite rank $n$, let $\Omega_\infty \subset \Lambda_\infty$ be a bounded measurable subset with $m_\infty(\partial \Omega_\infty)$, and, for each prime $p$ in a finite subset $S$, let $\Omega_p \subset \Lambda_p$ be a subset with $m_p(\partial \Omega_p) = 0$. Let $\mathbf{w} = (w_1, \ldots, w_n)$ be an $n$-tuple of positive integers. Then

$$\# \{ x \in \Lambda \cap B *_{\mathbf{w}} \Omega_\infty : x \in \Omega_p \text{ for all primes } p \} \sim \left( \frac{m_\infty(\Omega_\infty)}{m_\infty(\Lambda_\infty/\Lambda)} \prod_{p \in S} \frac{m_p(\Omega_p)}{m_p(\Lambda_p)} \right) B^{|\mathbf{w}|}.$$

We now address the case of infinitely many local conditions.
Lemma 3.2.7. Let \( \mathbf{w} = (w_1, \ldots, w_n) \) be an \( n \)-tuple of positive integers. Let \( \Lambda \) be a free \( \mathbb{Z} \)-module of finite rank \( n \), let \( \Omega_\infty \subset \Lambda_\infty \) be a bounded subset, and for each prime \( p \) let \( \Omega_p \subset \Lambda_p \). Assume that \( m_\infty(\partial \Omega_\infty) = 0 \) and \( m_p(\partial \Omega_p) = 0 \) for each prime \( p \) and

\[
\lim_{M \to \infty} \limsup_{B \to \infty} \frac{\# \{ x \in \Lambda \cap B * \mathbf{w} \Omega_\infty : x \not\in \Omega_p \text{ for some prime } p > M \}}{B^{\mathbf{w}}} = 0.
\]

Then

\[
\# \{ x \in \Lambda \cap B * \mathbf{w} \Omega_\infty : x \in \Omega_p \text{ for all primes } p \} \sim \frac{m_\infty(\Omega_\infty)}{m_\infty(\Lambda_\infty / \Lambda)} \left( \prod_p m_p(\Omega_p) \right) B^{\mathbf{w}}.
\]

Proof. As in the proof of Lemma 3.2.6 we may reduce to the case in which \( \Lambda = \mathbb{Z}^n \) and \( m_\infty \) and \( m_p \) are the usual Haar measures on \( \mathbb{R}^n \) and \( \mathbb{Z}_p^n \).

For \( M \leq M' \leq \infty \) and \( B > 0 \), set

\[
f_{M,M'}(B) = \frac{1}{B^{\mathbf{w}}} \# \{ x \in \mathbb{Z}^n \cap B * \mathbf{w} \Omega_\infty : x \in \Omega_p \text{ for all primes } p \in [M, M') \}
\]

and let \( f_M(B) := f_{1,M}(B) \). Note that \( f_M(B) \geq f_{M+1}(B) \) for all \( r \in \mathbb{Z}_{\geq 0} \cup \{ \infty \} \). The hypothesis (3.2) implies that

\[
\lim_{M \to \infty} \limsup_{B \to \infty} (f_M(B) - f_\infty(B)) = 0.
\]

From Corollary 3.2.6 it follows that, for all \( M < M' < \infty \),

\[
\lim_{B \to \infty} f_{M,M'}(B) = m_\infty(\Omega_\infty) \prod_{M \leq p < M'} m_p(\Omega_p).
\]

Combining (3.3) and (3.4) gives

\[
\lim_{B \to \infty} f_\infty(B) = \lim_{M \to \infty} \lim_{B \to \infty} f_M(B) = m_\infty(\Omega_\infty) \lim_{M \to \infty} \prod_{p < M} m_p(\Omega_p).
\]

The above infinite product converges by Cauchy's criterion since (3.2) and (3.4) together imply that

\[
\lim_{M \to \infty} \sup_{r \in \mathbb{Z}_{\geq 1}} \left| 1 - \prod_{M \leq p < M+r} m_p(\Omega_p) \right| = \frac{1}{m_\infty(\Omega_\infty)} \lim_{M \to \infty} \limsup_{r \in \mathbb{Z}_{\geq 1}} \lim_{B \to \infty} |f_1(B) - f_{M,M+r}(B)| = 0.
\]

The desired result then follows. \( \square \)

If \( K \) is a number field of degree \( d \) over \( \mathbb{Q} \) and with discriminant \( \Delta_K \), then its ring of integers \( \mathcal{O}_K \) may naturally be viewed as a rank \( d \) lattice in \( K_\infty := \mathcal{O}_K \otimes_\mathbb{Z} \mathbb{R} = \prod_{v|\infty} K_v \) with covolume \( |\Delta_K|^{1/2} \) with respect to the usual Haar measure \( m_\infty \) on \( K_\infty \) (which differs from Lebesgue measure on \( K_\infty \cong \mathbb{R}^{r_1+2r_2} \) by a factor of \( 2^{r_2} \)) [Neu99 Chapter I Proposition 5.2]. More generally, any integral ideal \( \mathfrak{a} \subset \mathcal{O}_K \) may be viewed as a lattice in \( K_\infty \), with covolume \( N_{K/\mathbb{Q}}(\mathfrak{a}) |\Delta_K|^{1/2} \). For an \( n \)-tuple of positive integers \( \mathbf{w} = (w_1, \ldots, w_n) \), define the lattice

\[
\Lambda_{\mathbf{w}} := \mathfrak{a}^{w_1} \times \cdots \times \mathfrak{a}^{w_n} \subset K_\infty^n.
\]

This lattice has covolume \( N_{K/\mathbb{Q}}(\mathfrak{a})^{\mathbf{w}} |\Delta_K|^{n/2} \). For example, in the case that \( \mathfrak{a} = \mathcal{O}_K \) and \( w_i = 1 \) for all \( i \), one has the lattice \( \Lambda_\mathbf{1} = \mathcal{O}_K^n \) of covolume \( |\Delta_K|^{n/2} \).
For any rational prime $p$ we have $\mathcal{O}_K \otimes \mathbb{Z} p = \prod_{p \mid p} \mathcal{O}_{K,p}$. Equip each $\mathcal{O}_{K,p}$ with the Haar measure $m_p$ normalized so that $m_p(\mathcal{O}_{K,p}) = 1$. These measures induce measures on $K^n$ and $\mathcal{O}_{K,p}$, which will also be denote by $m_\infty$ and $m_p$.

Applying Lemma [3.2.4] we obtain the following Proposition:

**Proposition 3.2.8.** Let $K/\mathbb{Q}$ be a number field of discriminant $\Delta_K$ and degree $d$ over $\mathbb{Q}$. Let $a \subseteq \mathcal{O}_K$ be an integral ideal. Let $\Omega_\infty \subset K^n_\infty$ be a bounded definable subset and let $\Omega_p = \prod_j \{ x \in \mathcal{O}_{K,p} : |x - a_{p,j}|_p \leq \omega_{p,j} \} \subset \mathcal{O}_{K,p}$ be finite local conditions for prime ideals $p$ of $\mathcal{O}_K$ contained in a finite set $S$. Then

$$\#\{ x \in \Lambda_a \cap B \ast_w \Omega_\infty : x \in \Omega_p \text{ for all primes } p \in S \} = \frac{m_\infty(\Omega_\infty)}{N(a)^{|w| |\Delta_K|^{n/2}} \left( \prod_{p \in S} m_p(\Omega_p) \right) B^{d|w|} + O \left( \max_j \left\{ \prod_{p \in S} \omega_{p,j}^{-1} \right\} \left( \prod_{p \in S} m_p(\Omega_p) \right) B^{d|w| - w_{\min}} \right)},$$

where the implied constant depends only on $K$, $\Omega_\infty$, and $w$.

Similarly, applying Lemma [3.2.5] we obtain the following Proposition:

**Proposition 3.2.9.** Let $K/\mathbb{Q}$ be a number field of discriminant $\Delta_K$ and degree $d$ over $\mathbb{Q}$. Let $a \subseteq \mathcal{O}_K$ be an integral ideal. Let $\Omega_\infty \subset K^n_\infty$ be a bounded definable subset and let $\Omega_p \subset \mathcal{O}_{K,p}$ for prime ideals $p$ of $\mathcal{O}_K$ contained in a finite set $S$. Suppose that $m_p(\partial \Omega_p) = 0$ for each prime $p \in S$. Then

$$\#\{ x \in \Lambda_a \cap B \ast_w \Omega_\infty : x \in \Omega_p \text{ for all primes } p \in S \} = \frac{m_\infty(\Omega_\infty)}{N_K(\mathbb{Q})(a)^{|w| |\Delta_K|^{n/2}} \left( \prod_{p \in S} m_p(\Omega_p) \right) B^{d|w|} + O \left( B^{d|w| - w_{\min}} \right)},$$

where the implied constant depends only on $K$, $\Omega_\infty$, $\Omega_p$, and $w$.

Similarly, applying Lemma [3.2.7] gives the corresponding Proposition for infinitely many local conditions:

**Proposition 3.2.10.** Let $K/\mathbb{Q}$ be a number field of discriminant $\Delta_K$ and degree $d$ over $\mathbb{Q}$. Let $\Omega_\infty \subset k^n_\infty$ be a bounded subset and let $\Omega_p \subset \mathcal{O}_{K,p}$ for each prime ideal $p$ of $\mathcal{O}_K$. Assume that $m_\infty(\partial \Omega_\infty)$ and $m_p(\partial \Omega_p)$ for each prime $p$ and

$$(3.5) \quad \lim_{M \to \infty} \limsup_{B \to \infty} \frac{\#\{ x \in \mathcal{O}_K \cap B \ast_w \Omega_\infty : x \not\in \Omega_p \text{ for some prime } p \text{ with } N(p) > M \}}{B^{d|w|}} = 0.$$

Then

$$\#\{ x \in \Omega_K^n \cap B \ast_w \Omega_\infty : x \in \Omega_p \text{ for all primes } p \} \sim \frac{m_\infty(\Omega_\infty)}{|\Delta_K|^{n/2}} \left( \prod_p m_p(\Omega_p) \right) B^{d|w|}.$$
Lemma 3.2.11. Let $K$ be a number field and $Y \subset A_{O_K}^n$ a closed subscheme of codimension $k > 1$. Let $\Omega_\infty \subset K_\infty^n$ be a bounded subset with $m_\infty(\partial\Omega_\infty) = 0$ and $m_\infty(\Omega_\infty) > 0$. For each prime ideal $p \subset O_K$, set

$$\Omega_p = \{ x \in O_{K,p}^n : x \pmod{p} \notin Y(\mathbb{F}_p) \}.$$

Then

$$\# \{ x \in O_{K,p}^n \cap B \ast_w \Omega_\infty : x \pmod{p} \notin \Omega_p \text{ for some } p \text{ with } N(p) > M \} = O\left( \frac{B|w|^d}{M^{k-1}\log(M)} \right)$$

where the implied constant depends only on $B$ and $Y$. In particular, (3.7) holds in this situation.

Proof. The case $K = \mathbb{Q}$ is due to Bhargava [Bha14, Theorem 3.3] (which generalized a result of Ekedahl [Eke91]). Bhargava’s method generalizes to arbitrary number fields (as noted in [BSW15, Theorem 21]) and to the weighted case (as noted in [BSW21, pg. 4]).

4. Counting points on weighted projective spaces

In this section we prove our results for counting points of bounded height on weighted projective spaces.

4.1. Finitely many local conditions. For any set of local conditions $(\Omega_v)_v$ define the sets

$$\Omega := \{ x : x \in \Omega_v \text{ for all } v \in \text{Val}(K) \},$$

$$\Omega_\infty := \{ x : x \in \Omega_v \text{ for all } v \in \text{Val}_\infty(K) \},$$

$$\Omega_0 := \{ x : x \in \Omega_v \text{ for all } v \in \text{Val}_0(K) \}.$$

Theorem 4.1.1. Let $K$ be a degree $d$ number field over $\mathbb{Q}$, and let $f : \mathbb{P}(w') \to \mathbb{P}(w)$ be a non-constant representable generically ´etale morphism of $n$-dimensional weighted projective spaces. Let $S \subset \text{Val}_0(K)$ be a finite set of finite places. For each $v \in \text{Val}_{\infty}(K) \cup S$ let $\Omega_v \subset \mathbb{P}(w')(K_v)$ be a subset such that $\Omega_v^{\text{aff}}$ is a bounded definable subset of $K_v^{n+1}$ and $m_v(\partial\Omega_v^{\text{aff}}) = 0$ for all $v \in \text{Val}_0(K)$. Then

$$\# \{ x \in f(\mathbb{P}(w')(K)) : \text{ht}_w(x) \leq B, x \in f(\Omega) \} = \kappa B^{\log(B)} + \begin{cases} O\left( B^{|\Omega^{\text{aff}}|/e(f)} \right) & \text{if } w'=(1,1) \text{ and } K=\mathbb{Q}, \\ O\left( B^{d_{\text{aff}}(w')/(\log B)^{\min}} \right) & \text{else}, \end{cases}$$

where the leading coefficient is

$$\kappa = \frac{h_K m_\infty(\mathcal{F}(1))}{\deg(f) \omega_{K,w'}(\Delta_K |^{n+1}/2 \zeta_K(w'))} \left( \prod_{v | \infty} m_v(\{ x \in \Omega_v^{\text{aff}} : H_v(x) \leq 1 \}) \prod_{v \in \infty} m_v(\{ x \in K_v^{n+1} : H_v(x) \leq 1 \}) \right) \prod_{p \in S} m_p(\Omega_v^{\text{aff}} \cap O_{K,v}^{n+1}),$$

where $\mathcal{F}(1)$ is a certain fundamental domain (depending on $f$) for the $(w'$-weighted) action of $O_K$ of $K_v^{n+1}$, which will be constructed in the proof.
Proof. As $f$ is generically étale, it suffices to show
\[
\#\{x \in \mathbb{P}(w')(K) : \text{ht}_f(x) \leq B, x \in \Omega\}
\]

(4.1) \[
= \deg(f) \kappa B^{|w'|/e(f)} + \begin{cases} 
O \left( B^{\kappa} \log(B) \right) & \text{if } w' = (1,1) \text{ and } K = \mathbb{Q}, \\
O \left( B^{d|w'|-w'_\text{min}} \right) & \text{else}.
\end{cases}
\]

Let $c_1, \ldots, c_h$ be a set of integral ideal representatives of the ideal class group $\mathfrak{C}_K$ of $K$. Then we get the following partition of $\mathbb{P}(w')(K)$ into points whose scaling ideals are in the same ideal class:

\[
\mathbb{P}(w')(K) = \bigsqcup_{i=1}^h \{x \in \mathbb{P}(w')(K) : [\mathcal{I}_f(x)] = [c_i]\}.
\]

For each $c \in \{c_1, \ldots, c_h\}$ consider the counting function

\[
M(\Omega, c, B) = \#\{x \in \mathbb{P}(w')(K) : \text{ht}_f(x) \leq B, [\mathcal{I}_f(x)] = [c], x \in \Omega\}.
\]

Consider the (weighted) action of the unit group $\mathcal{O}_K^\times$ on $\mathbb{A}^{n+1} - \{0\}$ by $u \star w' : x_0, \ldots, x_n \mapsto (u^{w'_{c_0}}x_0, \ldots, u^{w'_{c_n}}x_n)$, and let $(\mathbb{A}^{n+1} - \{0\})/\mathcal{O}_K^\times$ denote the corresponding set of orbits. Let $\Omega^\text{aff}$ be the affine cone of $\Omega$ (i.e. the pullback of $\Omega$ along the map $\mathbb{A}^{n+1} - \{0\} \to \mathbb{P}(w')$). We may describe $M(\Omega, c, B)$ in terms of $\mathcal{O}_K^\times$-orbits of an affine cone. In particular, there is a bijection between

\[
\{[x_0 : \cdots : x_n] \in \mathbb{P}(w')(K) : \text{ht}_f(x) \leq B, [\mathcal{I}_f(x)] = [c], x \in \Omega\}
\]

and

\[
\{([x_0, \ldots, x_n]) \in (\mathbb{A}^{n+1} - \{0\})/\mathcal{O}_K^\times : \frac{\text{ht}_{f,\infty}(x)}{\text{N}(c)} \leq B, \mathcal{I}_f(x) = c, x \in \Omega^\text{aff}\}
\]

given by

\[
[x_0 : \cdots : x_n] \mapsto [(x_0, \ldots, x_n)].
\]

Therefore

\[
M(\Omega, c, B) = \#\{x \in (\mathbb{A}^{n+1} - \{0\})/\mathcal{O}_K^\times : \frac{\text{ht}_{f,\infty}(x)}{\text{N}(c)} \leq B, \mathcal{I}_f(x) = c, x \in \Omega^\text{aff}\}.
\]

Our general strategy will be to first find an asymptotic for the counting function

\[
M'(\Omega, c, B) = \#\{x \in (\mathbb{A}^{n+1} - \{0\})/\mathcal{O}_K^\times : \frac{\text{ht}_{f,\infty}(x)}{\text{N}(c)} \leq B, \mathcal{I}_f(x) \subseteq c, x \in \Omega^\text{aff}\}.
\]

and then use Möbius inversion to obtain an asymptotic formula for $M(\Omega, c, B)$.

We are now going to construct a fundamental domain for the $(w'$-weighted) action of the unit group $\mathcal{O}_K^\times$ on $\mathbb{A}^{n+1} - \{0\}$. This will be done using Dirichlet’s Unit Theorem.

**Theorem 4.1.2** (Dirichlet’s Unit Theorem). The image $\Lambda$ of the map

\[
\lambda : \mathcal{O}_K^\times \to \mathbb{R}^{r_1+r_2} \quad u \mapsto (\log |u|_{v \in \text{Val}_\infty(K)})
\]

is a rank $r := r_1 + r_2 - 1$ lattice in the hyperplane $H$ defined by $\sum_{v \in \text{Val}_\infty(K)} x_v = 0$ and $\ker(\lambda) = \varpi(K)$. 

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For each \( v \in \text{Val}_\infty(K) \) define a map
\[
\eta_v : K_v^{n+1} - \{0\} \rightarrow \mathbb{R}
\]
\[(x_0, \ldots, x_n) \mapsto \log \max_i |f_i(x)|^{1/w'_i}.
\]

Combine these maps to obtain a single map
\[
\eta : \prod_{v \in \text{Val}_\infty(K)} (K_v^{n+1} - \{0\}) \rightarrow \mathbb{R}^{r_1 + r_2}
\]
\[x \mapsto (\eta_v(x_v)).
\]

Let \( H \) be the hyperplane in Dirichlet’s unit theorem and let
\[
\text{pr} : \mathbb{R}^{r_1 + r_2} \rightarrow H
\]
be the projection along the vector \((d_v)_{v \in \text{Val}_\infty(K)}\), where \( d_v = 1 \) if \( v \) real and \( d_v = 2 \) if \( v \) complex. More explicitly,
\[
(\text{pr}(x))_v = x_v - \left( \frac{1}{d_v} \sum_{v' \in \text{Val}_\infty(K)} x_{v'} \right) d_v.
\]

Let \( \{u_1, \ldots, u_r\} \) be a basis for the image of \( \mathcal{O}_K^\times \) in the hyperplane \( H \), and let \( \{\tilde{u}_1, \ldots, \tilde{u}_r\} \) be the dual basis. Then the set
\[
\tilde{\mathcal{F}} := \{y \in H : 0 \leq \tilde{u}_j(y) < 1 \text{ for all } j \in \{1, \ldots, r\}\}
\]
is a fundamental domain for \( H \) modulo \( \Lambda \), and \( \mathcal{F} := (\text{pr} \circ \eta)^{-1} \tilde{\mathcal{F}} \) is a fundamental domain for the \((w'-\text{weighted})\) action of \( \mathcal{O}_K^\times \) on \( \prod_{v \in \text{Val}_\infty(K)} (K_v^{n+1} - \{0\}) \).

Note that each \( \varpi(K)\)-orbit (with respect to the \( w'\)-weighted action) of an element of \((K - \{0\})^{n+1}\) contains \( \varpi_{K,w'} \) elements. Viewing \( \mathcal{O}_K^{n+1} \) as a lattice of full rank in \( K_v^{n+1} \), we have that
\[
M'(\Omega, c, B) = \frac{1}{\varpi_{K,w'}} \# \{ x \in \Omega^{\text{aff}}_\infty \cap \mathcal{F} \cap \mathcal{O}_K^{n+1} : \text{ht}_{f,\infty}(x) \leq B \cdot N(c), \ I_f(x) \subseteq c, \ x \in \Omega^{\text{aff}}_0^\times \}.
\]

Define the sets
\[
\mathcal{D}(B) := \{ x \in \prod_{v \in \text{Val}_\infty(K)} (K_v^{n+1} - \{0\}) : \text{ht}_{f,\infty}(x) = \prod_{v \in \text{Val}_\infty(K)} \max_i |f_i(x_{v,i})|^{1/w_i} \leq B \},
\]
and \( \mathcal{F}(B) := \mathcal{F} \cap \mathcal{D}(B) \). The sets \( \mathcal{D}(B) \) are \( \mathcal{O}_K^\times \)-stable, in the sense that if \( u \in \mathcal{O}_K^\times \) and \( x \in \mathcal{D}(B) \) then \( u \ast_{w'} x \in \mathcal{D}(B) \); this can be seen by the following computation:
\[
\prod_{v | \infty} \max_i |f_i(u^{w'_i} x_{v,i})|^{1/w_i} = \prod_{v | \infty} |u|^{e(f)} \prod_{v | \infty} \max_i |f_i(x_{v,i})|^{1/w_i} = \prod_{v | \infty} \max_i |f_i(x_{v,i})|^{1/w_i}.
\]

Similarly, for any \( t \in \mathbb{R} \), we have that
\[
\prod_{v | \infty} \max_i |f_i(t^{w'_i} x_{v,i})|^{1/w_i} = \prod_{v | \infty} |t|^{e(f)} \prod_{v | \infty} \max_i |f_i(x_{v,i})|^{1/w_i} = |t|^{d_{e(f)}} \prod_{v | \infty} \max_i |f_i(x_{v,i})|^{1/w_i}.
\]

This shows that \( \text{ht}_{f,\infty}(t \ast_{w'} x) = |t|^{e(f)d} \text{ht}_{f,\infty}(x) \). Therefore \( \mathcal{D}(B) = B^{1/\varpi_{K,w'}} \mathcal{D}(1) \) for all \( B > 0 \).
On the other hand, \( \mathcal{F} \) is stable under the weighted action of \( t \in \mathbb{R}^x \), in the sense that
\[
\eta(t \ast_{w'} \mathcal{F}) = \mathcal{F}.
\]
To see this, note that for any \( x \in \prod_{v|\infty} (K_v^m - \{0\}) \),
\[
\eta(t \ast_{w'} x) = (e(f) d_v)_{v|\infty} \log(|t|) + \eta(x).
\]
Since \( pr \) is linear and annihilates the vector \((d_v)_{v|\infty}\), we have that
\[
pr \circ \eta(t \ast_{w'} x) = pr \circ \eta(x),
\]
as desired. From our observations we now obtain the following lemma:

**Lemma 4.1.3.** The regions \( \mathcal{F}(B) \) are weighted homogeneous, in the sense that \( \mathcal{F}(B) = B^{\frac{1}{e(f)}d} \ast_{w'} \mathcal{F}(1) \) for all \( B > 0 \).

We are now going to count lattice points in \( \mathcal{F}(B) \). In [Sch79] this is done by using the classical Principle of Lipschitz [Dav51] (see also [Lan94, VI §2 Theorem 2]). One of the cruxes of Schanuel’s argument is verifying that his fundamental domain (analogous to our \( \mathcal{F}(1) \)) has Lipschitz parameterizable boundary, so that he can apply the Principle of Lipschitz. Though one can modify this part of Schanuel’s argument to work in our case, we will instead take a slightly different route, using an o-minimal version of the Principle of Lipschitz (Proposition 3.2.1). This allows one to give a more streamline proof of this part of Schanuel’s argument, which may be useful in future generalizations.

**Lemma 4.1.4.** The set \( \mathcal{F}(1) \) is bounded.

*Proof.* Let \( \tilde{H} \subset \prod_{v|\infty} \mathbb{R}_+ \) be the subset defined by \( \sum_{v|\infty} x_v \leq 0 \). Note that \( \mathcal{F}(1) = \eta^{-1}(\tilde{H} \cap pr^{-1}(\tilde{F})) \). It follows from the definition of \( \eta \) and Proposition 2.1.2 that \( \eta \) is topologically proper as a map to \( \prod_{v|\infty} \mathbb{R}_+ \). Therefore, in order to show that \( \mathcal{F}(1) \) is bounded, it suffices to show that the closed set
\[
S := \tilde{H} \cap pr^{-1}(\tilde{F}) \cap \prod_{v|\infty} \mathbb{R}_+^v
\]
is bounded. For this, note that any \( x \in S \) can be written as
\[
x = pr(x) + (d_v)_v \frac{1}{d} \sum_{v|\infty} x_v,
\]
and thus the components of \( S \) are bounded above, noting that the first term, \( pr(x) \), has components bounded above, and the second term has negative components. The components of \( x \) are also bounded below (by 0), since \( x \in \prod_{v|\infty} \mathbb{R}_+^v \). It follows that \( S \) is bounded. \( \square \)

**Lemma 4.1.5.** The set \( \mathcal{F}(1) \) is definable in \( \mathbb{R}_{exp} \).

*Proof.* We make the following straightforward observations:

- The set \( \mathcal{D}(1) \) is semi-algebraic.
- The set \( \mathcal{F} = \{ x \in \prod_{v|\infty} (K_v^{n+1} - \{0\}) = \mathbb{R}^{r_1+\gamma_2} : pr \circ \eta(x) \in \tilde{F} \} \)

\[
= \left\{ x \in \mathbb{R}^{r_1+\gamma_2} : 0 \leq \tilde{u}_j \left( \log(\max_i |f_i(x)|_{v'}^{1/w'_i}) - d_v \left( \frac{1}{d} \sum_{v|\infty} \log(\max_i |f_i(x)|_{v'}^{1/w'_i}) \right) \right) < 1 \forall j \right\}
\]
is definable in $\mathbb{R}_{\exp}$, since it can be described in terms of polynomials and log, and log is definable in $\mathbb{R}_{\exp}$.

It follows that the intersection $\mathcal{F}(1) = \mathcal{F} \cap \mathcal{D}(1)$ is definable in $\mathbb{R}_{\exp}$. □

By Lemma 4.1.4 and Lemma 4.1.5 we may apply Proposition 3.2.9, with $a = c$ and the local conditions $\Omega_{\text{aff}} \cap \mathcal{F}(1)$ if $v \mid \infty$ and $\Omega_{\text{aff}}$ if $v \in S$, to obtain the following asymptotic:

\begin{equation}
M'(\Omega, c, B) = \frac{m_\infty(\Omega_{\text{aff}} \cap \mathcal{F}(1))}{\omega_K, w' | \Delta_K | (n+1)/2} \left( \prod_{p \in S} m_p(\Omega_p) \right)^{\frac{\left|\omega'\right|}{c(f)}} B^{\frac{1}{\kappa}} + O \left( B^{\frac{\left|\omega'\right|-w'}{c(f)/\min}} \right). \tag{4.2}
\end{equation}

Note that as $m_\infty$ is the product of measures $(m_v)_{v \mid \infty}$, one can show

\[
\frac{m_\infty(\Omega_{\text{aff}} \cap \mathcal{F}(1))}{m_\infty(\mathcal{F}(1))} = \prod_{v \mid \infty} \frac{m_v(\{x \in \Omega_{\text{aff}} : \text{ht}_v(x) \leq 1\})}{m_v(\{x \in K_{n+1}^f : \text{ht}_v(x) \leq 1\})},
\]

and thus

\begin{equation}
(4.3) 
\quad m_\infty(\Omega_{\text{aff}} \cap \mathcal{F}(1)) = m_\infty(\mathcal{F}(1)) \prod_{v \mid \infty} \frac{m_v(\{x \in \Omega_{\text{aff}} : \text{ht}_v(x) \leq 1\})}{m_v(\{x \in K_{n+1}^f : \text{ht}_v(x) \leq 1\})}.
\end{equation}

Let $\kappa'$ denote the leading coefficient of this asymptotic. Note that $\kappa' = \zeta_K(\left|w'\right|)\kappa/h_K$, with $\kappa$ as in the statement of Theorem 4.1.1.

Let $c$ be a positive constant such that $\text{ht}_f(x) \geq c$ for all $x \in \mathbb{P}(w')(K)$. Note that for $x \in \Lambda_\ell$, $\mathcal{I}_f(x) = \mathfrak{a}c$ for some ideal $\mathfrak{a} \subseteq \mathcal{O}_K$ and $\text{ht}_f(x) = N(\mathfrak{a})^{e(f)} \text{ht}_f(\frac{1}{N(\mathfrak{a})} \ast w' x)$. Therefore

\[
M'(\Omega, c, B) = \sum_{\mathfrak{a} \subseteq \mathcal{O}_K} M(\Omega, \mathfrak{a}c, B/N(\mathfrak{a})^{e(f)}) = \sum_{\mathfrak{a} \subseteq \mathcal{O}_K, N(\mathfrak{a}) \leq B/c} M(\Omega, \mathfrak{a}c, B/N(\mathfrak{a})^{e(f)}).
\]

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We now apply Möbius inversion and use our asymptotic for \( M'(\Omega, c, B) \) \( (4.2) \):

\[
M(\Omega, c, B) = \sum_{a \in O_K} \mu(a) \left( \kappa' \left( \frac{B}{(N(a)^{c(f)})^{w'/e(f)}} \right)^{|w'|/e(f)} + O \left( \frac{B}{(N(a)^{c(f)})^{d|w'|-w'_\min}} \right) \right)
\]

\[
= \kappa' B^{w'/e(f)} \left( \sum_{a \in O_K} \mu(a) \frac{1}{N(a)^{|w'|}} - \sum_{a \in O_K} \mu(a) \frac{1}{N(a)^{|w'|}} \right)
\]

\[
+ O \left( B^{w'-w'_\min/d} \sum_{a \in O_K} \frac{1}{N(a)^{|w'|-w'_\min/d}} \right)
\]

\[
= \kappa' B^{w'/e(f)} \left( \frac{1}{\zeta_K(|w'|)} - O \left( B^{-|w'|+1} \right) \right) + \left\{ \begin{array}{ll}
O \left( B^{w'-w'_\min/d} \log(B) \right) & \text{if } w' = (1,1) \text{ and } K = \mathbb{Q}, \\
O \left( B^{w'-w'_\min/d} \right) & \text{else,}
\end{array} \right.
\]

This gives an asymptotic for \( M(\Omega, c, B) \). Summing over the ideal class representatives \( c_i \) of \( \mathcal{O}_K \) gives \( (4.1) \), from which Theorem \( 4.1.1 \) follows.

**Corollary 4.1.6.** If for each \( v \in S \) in Theorem \( 4.1.1 \) we have that \( \Omega_v^{\text{aff}} \) is a finite local condition then we may write \( \Omega_v^{\text{aff}} = \prod_{j=0}^{n} \{ x \in K_v : |x - a_{v,j}| \leq \omega_{v,j} \} \). Then

\[
\# \{ x \in f(\mathbb{P}(w')(K)) : \text{ht}_w(x) \leq B, x \in f(\Omega) \}
\]

\[
= \kappa B^{w'/e(f)} + \left\{ \begin{array}{ll}
O \left( \prod_{v \in S} \omega_{v,j}^{-1} \right) \left( \prod_{v \in S} m_v(\Omega_v^{\text{aff}} \cap O_{K_v}^{n+1}) B^{-\delta_0} \log(B) \right) & \text{if } w' = (1,1) \text{ and } K = \mathbb{Q}, \\
O \left( \prod_{v \in S} \omega_{v,j}^{-1} \right) \left( \prod_{v \in S} m_v(\Omega_v^{\text{aff}} \cap O_{K_v}^{n+1}) B^{-\delta_0} \right) & \text{else,}
\end{array} \right.
\]

where the implied constant is independent of the local conditions \( \Omega_v \) with \( v \in S \).

**Proof.** This follows by replacing the use of Proposition \( 3.2.9 \) by Proposition \( 3.2.8 \) in the proof of Theorem \( 4.1.1 \). \( \square \)

### 4.2. Infinitely many local conditions.

**Theorem 4.2.1.** Let \( K \) be a degree \( d \) number field over \( \mathbb{Q} \), and \( f : \mathbb{P}(w') \rightarrow \mathbb{P}(w) \) be a non-constant representable generically étale morphism. For each \( v \in \text{Val}(K) \) let \( \Omega_v \subset \mathbb{P}(w')(K_v) \) be a subset such that \( m_v(\partial \Omega_v^{\text{aff}}) = 0 \). Suppose also that for all bounded subsets \( \Psi \subset K_{\infty}^{n+1} \) of positive measure and with \( m_\infty(\partial \Psi) = 0 \) we have that

\[
\lim_{M \to \infty} \limsup_{B \to \infty} \frac{\# \{ x \in \mathcal{O}_K^{n+1} \cap B \ast w : x \not\in \Omega_p^{\text{aff}} \text{ for some prime } p \text{ with } N(p) > M \}}{B^d|w'|} = 0.
\]

(4.4)
Then

$$\# \{ x \in f(\mathbb{P}(w')(K)) : \text{ht}_w(x) \leq B, \ x \in f(\Omega) \} \sim \kappa_\Omega \frac{h_K m_\infty(\mathcal{F}(1))}{\deg(f) \omega_{K,w} |\Delta_K|^{(n+1)/2} \zeta_K(1)} B^{\omega'_w/e(f)}.$$  

where \( \kappa_\Omega \) is the following (non-zero) product of local densities.

$$\kappa_\Omega = \prod_{v \mid \infty} \frac{m_v(\{x \in \Omega_v^{\text{aff}} : \text{ht}_{f,v}(x) \leq 1\})}{m_v(\{x \in K_v^{n+1} : \text{ht}_{f,v}(x) \leq 1\})} \prod_p m_p(\{x \in \Omega_p^{\text{aff}} \cap \mathcal{O}_{K,p}^{n+1}\}).$$

Proof. The proof is similar to the proof of Theorem 4.1.1.

It suffices to show

(4.5) \n
$$\# \{ x \in \mathbb{P}(w')(K) : \text{ht}_f(x) \leq B, \ x \in \Omega_v \text{ for all } v \in \text{Val}(K) \} \sim \kappa_\Omega \frac{h_K m_\infty(\mathcal{F}(1))}{\omega_{K,w} |\Delta_K|^{(n+1)/2} \zeta_K(1)} B^{\omega'_w/e(f)}.$$  

Again let \( c_1, \ldots, c_h \) be a set of integral ideal representatives of the ideal class group \( \mathcal{O}_K \) of \( K \). For each \( c \in \{c_1, \ldots, c_h\} \) we again consider the counting function

$$M(\Omega, c, B) = \# \{ x \in \mathbb{P}(w')(K) : \text{ht}_f(x) \leq B, \ [\mathcal{I}_f(x)] = [c], \ x \in \Omega \}.$$  

As in the proof of Theorem 4.1.1 this counting function can be re-written as

$$M(\Omega, c, B) = \# \{ x \in (K^{n+1} - \{0\})/\mathcal{O}_K^{\infty} : \frac{\text{ht}_{f,\infty}(x)}{N(c)} \leq B, \ \mathcal{I}_f(x) = c, \ x \in \Omega^{\text{aff}} \}$$

$$= \frac{1}{\omega_{K,w'}} \# \{ x \in \mathcal{F} \cap \mathcal{O}_K^{n+1} : \text{ht}_{f,\infty}(x) \leq B \cdot N(c), \ \mathcal{I}_f(x) = c, \ x \in \Omega^{\text{aff}} \}$$

where \( \mathcal{F} \subseteq K^{n+1}_\infty \) is the fundamental domain, constructed in the proof of Theorem 4.1.1 for the \( (w') \)-weighted) action of \( \mathcal{O}_K^{\infty} \) on \( K_{n+1}^{\infty} \).

For \( p \subseteq \mathcal{O}_K \) a prime ideal, let \( c_p = c \otimes_{\mathcal{O}_K} \mathcal{O}_{K,p} \) denote the localization of \( c \) at \( p \). Then the condition \( \mathcal{I}_f(x) = c \) can be viewed as the collection of local conditions \( \mathcal{I}_f(x)_p = c_p \) for all prime ideals \( p \) of \( \mathcal{O}_K \). Therefore

$$M(\Omega, c, B) = \frac{1}{\omega_{K,w'}} \# \{ x \in \mathcal{F} \cap \mathcal{O}_K^{n+1} \cap \Omega^{\text{aff}} : \text{ht}_{f,\infty}(x) \leq B \cdot N(c), \ \mathcal{I}_f(x)_p = c_p \text{ and } x \in \Omega^{\text{aff}} \cap \text{primes } p \}.$$  

where we recall that \( \Omega^{\infty} = \prod_{v \in \text{Val}(K)} \Omega_v \).

Since \( (1) \) is a bounded definable set by Lemma 4.1.4 and Lemma 4.1.5, it is measurable by Proposition 3.1.3.

It follows that the set

$$\Theta^{\infty} := \{ x \in \Omega^{\text{aff}} \cap \mathcal{F} : \text{ht}_\infty(x) \leq N(c) \}$$

is also measurable, being the intersection of two measurable sets. Similarly, the sets

$$\Theta_p := \{ x \in \Omega_p^{\text{aff}} \cap \mathcal{O}_{K,p}^{n+1} : \mathcal{I}_f(x)_p = c_p \}$$

are measurable. The sets \( \Theta^{\infty} \) and \( \Theta_p \) have boundary of measure zero by the assumption that \( m_v(\Omega_v^{\text{aff}}) = 0 \). To check that these sets also satisfy the condition (3.5) of Proposition 3.2.10 we may first apply de Morgan’s laws to the assumption (4.4) to reduce the problem to showing that the sets \( \{ x \in \mathcal{O}_{K,p}^{n+1} : \mathcal{I}_f(x)_p = \mathcal{O}_{K,p} \} \) satisfy condition (3.5). Applying Lemma 3.2.11...
with the subscheme \( f_0(x) = \cdots = f_n(x) = 0 \) shows that the sets \( \{ x \in \mathcal{O}_{K,p}^{n+1} : \mathfrak{I}_f(x)_p = \mathcal{O}_{K,p} \} \) satisfy condition (3.5). Therefore we may apply Proposition 3.2.10 to obtain

\[
\lim_{B \to \infty} \frac{M(\Omega, c, B)}{B^{\omega(\ell(f))}} = \frac{m_\infty(\Theta_\infty)}{\omega_{K,w}|\Delta_K|^{(n+1)/2}} \prod_p m_p(\Theta_p),
\]

where we have used the fact that \( \mathcal{F}(B) = B^{1/(e(f)d)} *_{w'} \mathcal{F}(1) \) by Lemma 4.1.3.

For each prime ideal \( p \in \mathcal{O}_K \) we have

\[
m_p(\Theta_p) = \frac{1}{N_p(c)^{|w'|}} m_p(\{ x \in \Omega_{p,\text{aff}} \cap \mathcal{O}_K^n : \mathfrak{I}_f(x)_p = \mathcal{O}_{K,p} \})
\]

\[
= \frac{1}{N_p(c)^{|w'|}} \left( 1 - \frac{1}{N_p(p)^{|w'|}} \right) m_p(\{ x \in \Omega_p \cap \mathcal{O}_{K,p}^{n+1} \}),
\]

where we have used the fact that \( f \) is generically étale, so that the proportion of \( x \) for which \( \mathfrak{I}_f(x)_p \neq \mathcal{O}_{K,p} \), is the same as the proportion for which \( \mathfrak{I}_w(x)_p \neq \mathcal{O}_{K,p} \), which is \( 1 - 1/N_p(p)^{|w'|} \). Taking the product over all primes gives

\[
\prod_p m_p(\Theta_p) = \frac{1}{N(c)^{|w'|} \zeta_K(|w'|)} \prod_p m_p(\{ x \in \Omega_p \cap \mathcal{O}_K^{n+1} \}).
\]

We also have that

\[
m_\infty(\Theta_\infty) = N(c)^{n+1} m_\infty(\{ x \in \Omega_\infty \cap \mathcal{F}(1) \}).
\]

As in the proof of Theorem 4.1.1 (see (4.3)),

\[
m_\infty(\{ x \in \Omega_\infty \cap \mathcal{F}(1) \}) = m_\infty(\mathcal{F}(1)) \prod_{v|\infty} \frac{m_v(\{ x \in \Omega_v^{\text{aff}} : \text{ht}_v(x) \leq 1 \})}{m_v(\{ x \in \Omega_v^{\text{aff}} : \text{ht}_v(x) \leq 1 \})}.
\]

Combining (4.6), (4.7), and (4.8), one obtains:

\[
M(\Omega, c, B) \sim \frac{m_\infty(\mathcal{F}(1))}{\omega_{K,w}|\Delta_K|^{(n+1)/2} \zeta_K(|w'|)} \kappa_B^{\ell(f)}. \]

Finally, summing over the ideal class representatives \( c_i \) of \( \mathcal{O}_K \) gives (1.5), from which the Theorem follows.

In the case that \( w = w' \) and \( f \) is the identity, one may compute the volume of \( \mathcal{F}(1) \) as in [Den98, Proposition 5.3]:

\[
m_\infty(\mathcal{F}(1)) = (2^{r_1+r_2} \pi^{r_2})^{n+1} R_K |w|^{r_1+r_2-1}.
\]

This differs by a factor of \( 2^{(n+1)r_2} \) from Deng’s result since we are using the usual Haar measure on \( K_\infty \) rather than the Lebesgue measure. From this, Theorem 4.1.1 and Theorem 4.2.1 give the following corollary:

**Corollary 4.2.2.** Let \( K \) be a degree \( d \) number field over \( \mathbb{Q} \). For each \( v \in \text{Val}(K) \) let \( \Omega_v \subset \mathbb{P}(w)(K_v) \) be a subset such that \( m_v(\partial \Omega_v^{\text{aff}}) = 0 \). Suppose also that for all bounded
where the implied constant depends only on $B$

\[
\lim_{M \to \infty} \limsup_{B \to \infty} \frac{\# \{ x \in \mathcal{O}_K^{n+1} \cap B * \Psi : x \notin \Omega_p^{\text{aff}} \text{ for some prime } p \text{ with } N(p) > M \}}{B^{d|w|}} = 0.
\]

Then

\[
\# \{ x \in \mathbb{P}(w)(K) : \text{ht}_w(x) \leq B, \ x \in \Omega_v \text{ for all } v \in \text{Val}(K) \} \\
\sim \kappa_\Omega \frac{h_K(2^{r_1+r_2 \pi^2})^{n+1} R_K|w|^{r_1+r_2-1}}{\omega_{K,w}|\Delta_K|^{(n+1)/2} \zeta_K(|w|)} B^{|w|},
\]

where $\kappa_\Omega$ is the following (non-zero) product of local densities

\[
\kappa_\Omega = \prod_{v|\infty} \frac{m_v(\{ x \in \Omega_v^{\text{aff}} : \text{ht}_w, v(x) \leq 1 \})}{m_v(\{ x \in K_v^{n+1} : \text{ht}_w, v(x) \leq 1 \})} \prod_p m_p(\{ x \in \Omega_p^{\text{aff}} \cap \mathcal{O}_K^{n+1} \}).
\]

If, moreover, $\Omega_v = \mathbb{P}(w)(K_v)$ for all but finitely many $v \in \text{Val}_0(K)$, then

\[
\# \{ x \in \mathbb{P}(w)(K) : \text{ht}_w(x) \leq B, \ x \in \Omega_v \text{ for all } v \in \text{Val}(K) \} \\
= \kappa_\Omega \frac{h_K(2^{r_1+r_2 \pi^2})^{n+1} R_K|w|^{r_1+r_2-1}}{\omega_{K,w}|\Delta_K|^{(n+1)/2} \zeta_K(|w|)} B^{|w|} + \begin{cases} O\left(B^{\text{w}|w|-w_{\text{min}}/d} \log(B)\right) & \text{if } w=(1,1) \text{ and } K=\mathbb{Q}, \\
O\left(B^{\text{w}|w|-w_{\text{min}}/d}\right) & \text{else.}
\end{cases}
\]

The following result, which follows immediately from Lemma 3.2.11, gives a criterion for checking the limit condition in Theorem 4.2.1.

**Lemma 4.2.3.** Let $K$ be a number field and $Y \subset \mathbb{P}_{\mathcal{O}_K}(w')$ a closed subscheme of codimension $k > 1$. For each prime ideal $p \subset \mathcal{O}_K$, set

\[
\Omega_p = \{ x \in \mathbb{P}(w')(\mathcal{O}_K,p) : x \pmod{p} \not\in Y(p) \}.
\]

Then, for all bounded subsets $\Psi \subset K^{n+1}_\infty$ with positive measure and $m_\infty(\partial \Psi) = 0$, we have

\[
\# \{ x \in \mathcal{O}_K^{n+1} \cap B * \Psi : x \pmod{p} \not\in \Omega_p^{\text{aff}} \text{ for some prime } p \text{ with } N(p) > M \} = O\left(\frac{B^{w'd}}{M^{k-1} \log(M)}\right)
\]

where the implied constant depends only on $B$, $w$, and $Y$. In particular, (4.4) holds in this situation.

Let $k$ be a positive integer. An element $\alpha$ in $\mathcal{O}_K$ is said to be $k$-free if there are no prime ideals $p \subset \mathcal{O}_K$ such that $\alpha \in p^k$. Let $[r] \in K^\times/(K^\times)^k$, let $a \in \mathcal{O}_K$, and let $n \subset \mathcal{O}_K$ be an integral ideal. If there exists a $k$-free $t \in K^\times$ which is equivalent to $r$ in $K^\times/(K^\times)^k$ and which satisfies $t \equiv a \pmod{n}$, then we will write $[r] \equiv a \pmod{n}$. Using Corollary 4.2.2 we can count elements of $\mathbb{P}(w')(K)$ of bounded height which satisfy a congruence condition.

**Corollary 4.2.4.** Let $K$ be a degree $d$ number field over $K$, let $w_0$ be a positive integer, let $a \in \mathcal{O}_K$, and let $n \subset \mathcal{O}_K$ be an integral ideal with prime factorization $n = p_1^{s_1} \cdots p_n^{s_n}$. Set
\( g = \gcd((a), (n)) \). Then the number of elements \( x \in \mathbb{P}(w_0)(K) = K^\times/(K^\times)^{w_0} \) satisfying \( x \equiv a \pmod{n} \) and of height less than \( B \) is

\[
\left( \prod_{p_i | \mathfrak{n}} N(p_i)^{-s_i} \prod_{p_i | g} N(p_i)^{w_0-s_i} \right) \frac{h_K 2^{r_1+r_2} \pi^{r_2} R_K w_0^{r_1+r_2-1}}{|\omega_K, w_0| \Delta_K|^{1/2} \zeta_K(w_0)} B^{w_0} + O(B^{w_0-1/d}).
\]

**Remark 4.2.5.** In the case that \( K \) has class number 1, this result counts the number of \( w_0 \)-free integers of \( K \) satisfying a congruence relation.

**Proof.** We may assume \( r_i < w_0 \) for all \( i \).

First consider the case \( g = (1) = \mathcal{O}_K \). We apply Corollary 4.2.2 with the local conditions

\[ \Omega_{p_i} = \{ x \in \mathbb{P}(w_i)(K_{p_i}) : x \equiv a \pmod{p_i^{s_i}} \}. \]

As

\[ \Omega_{p_i}^{\text{aff}} = \{ x \in K_{p_i}^\times : x \not\equiv 0 \pmod{p_i^{w_0}}, x \equiv a \pmod{p_i^{s_i}} \} \]

we have

\[ m_{p_i}(\Omega_{p_i}^{\text{aff}} \cap \mathcal{O}_{K,p_i}) = \frac{N(p_i)^{w_0-s_i}}{N(p_i)^{w_0} - 1}, \]

since \( N(p_i)^{w_0-s_j} \) of the \( N(p_i)^{w_0} - 1 \) non-zero elements of \( \mathcal{O}_{K,p_i}/p_i^{w_0} \) are equivalent to \( a \) modulo \( p_i^{s_i} \). From this follows the desired result in the case \( g = (1) \).

If \( g \neq (1) \), then the desired result is obtained by first replacing \( \mathfrak{n} \) by \( \mathfrak{n}/g \) and applying the \( g = (1) \) case and then multiplying by a factor of \( N(g)^{-1} = \prod_{p_i | \mathfrak{n}} N(p_i)^{-s_i} \) to account for the 1 out of \( N(g) \) elements \( x \) of \( \mathcal{O}_K^\times \) within a single congruence class modulo \( g \).

\[ \square \]

### 4.3. Counting with a twisted height

Motivated by counting twists of elliptic curves, we count points on weighted projective spaces with respect to the **twisted height** defined in Subsection 2.2.

**Theorem 1.3.3.** Let \( K \) be a degree \( d \) number field over \( \mathbb{Q} \). Let \( w' = (w_0', w_1', \ldots, w_n') \) and \( w = (w_0, w_1, \ldots, w_n) \) be \((n+1)\)-tuples of positive integers and let \( f : \mathbb{P}(w') \to \mathbb{P}(w) \) be a non-constant, representable, generically étale morphism. Let \( \tau | \gcd(w_0, \ldots, w_n) \) be such that if \( \tau \geq |w'|/e(f) \) then \(-3\tau + |w| \geq 0 \) and if \( \tau < |w'|/e(f) \) then \( \tau(|w| - |w'|/e(f) - 1) - |w| \geq 0 \).
For ideals \( a, b \subseteq \mathcal{O}_K \) with \( b \) \( \tau \)-free we write \( a||b \) if \( p^s||a \) implies \( p^s||b \) for all \( p|a \). Then

\[
\# \{ (x, d) \in f(\mathbb{P}(w')(K)) \times \mathbb{P}(\tau)(K) : \text{ht}_w(x^{(d)}) \leq B \} 
\]

\[
\sim \begin{cases} 
\frac{\tau}{|w'|/e(f)} \kappa \sum_{b \in \mathcal{O}_K} \lambda(b) \sum_{a||b} \frac{N(\sqrt{\mathfrak{d}})}{N(a)} B^{e(f)/|a|} & \text{if } \tau < |w'|/e(f) \\
\frac{\tau \kappa}{|w'|/e(f)} \lambda(b) \sum_{a||b} \frac{N(\sqrt{\mathfrak{d}})}{N(a)} B^\tau \log(B) & \text{if } \tau = |w'|/e(f) \\
\frac{|w'|}{e(f) \tau - |w'|} \kappa \lambda(b) \sum_{a||b} \frac{N(\sqrt{\mathfrak{d}})}{N(a)} B^\tau & \text{if } \tau > |w'|/e(f) \end{cases}
\]

where

\[
\kappa = \frac{h_K 2^{r_1 + r_2 + r_2} R_K \tau^{r_1 + r_2 - 1}}{\omega_K, \tau |\Delta_K|^{1/2} \zeta_K(\tau)} \quad \text{and} \quad h_K m_K(\mathcal{F}(1)) \frac{\deg(f) \omega_K, w |\Delta_K|^{(n+1)/2} \zeta_K(|w'|)}.
\]

and

\[
\lambda(b) = \prod_p m_p(\{ x \in \Omega(b)_{a \cap \mathcal{O}_{K, p}^{n+1} \})
\]

is a product of local densities which will be defined in the proof.

Before proving the theorem we state the following product lemma [Den98 Lemma 7.1] (see also Wang [Wan21, Lemma 3.1] and [FMT90, Proposition 2] for similar results):

**Lemma 4.3.2 (Product Lemma).** Let \( C_X \) and \( C_Y \) be counting functions on the sets \( X \) and \( Y \) respectfully. Suppose that

\[
F_X(B) := \# \{ x \in X : C_X(x) \leq B \} \sim c_X B^{\alpha_X}
\]

and

\[
F_Y(B) := \# \{ y \in Y : C_Y(x) \leq B \} \sim c_Y B^{\alpha_Y}.
\]

Then

\[
F(B) := \# \{ (x, y) \in X \times Y : C_X(x) \cdot C_Y(y) \leq B \} \sim \begin{cases} 
\frac{c_X c_Y \alpha_X B^{\alpha_Y}}{\alpha_Y - \alpha_X} & \text{if } \alpha_X < \alpha_Y \\
\frac{c_X c_Y \alpha_X B^{\alpha_Y}}{\alpha_X - \alpha_Y} B^{\alpha_Y} \log(B) & \text{if } \alpha_X = \alpha_Y \\
\frac{c_X c_Y \alpha_X B^{\alpha_Y}}{\alpha_X - \alpha_Y} B^{\alpha_X} & \text{if } \alpha_X > \alpha_Y. 
\end{cases}
\]

**Proof of Theorem 4.3.3** For \( x \in f(\mathbb{P}(w')(K)) \) and \( v \in \text{Val}_0(K) \), let \( b_{x,v} \) be defined as follows:

\[
b_{x,v} := \min \{ b \in \{1, \ldots, \tau - 1\} : |x|_{\tau}^{(b)} < |x|_{w,v}, \}
\]

where we define the minimum of the empty-set to be zero. Let \( b_x \) be the integral ideal

\[
b_x = \prod_{v \in \text{Val}_0(K)} p_v^{b_{x,v}}.
\]
Let $b \subseteq \mathcal{O}_K$ be any $\tau$-free integral ideal, i.e., the prime factorization

$$b = \prod_{v \in \text{Val}_0(K)} p_{v}^{b_v}$$

of $b$ is such that $b_v < \tau$ for all $v$. Let $a|b$ be an ideal dividing $b$ such that there exists a $V \subset \text{Val}_0(K)$ satisfying

$$a = \prod_{v \in V} p_{v}^{b_v}.$$

For $b$ a $\tau$-free ideal and $a||b$, define the set

$$\mathcal{A}(b, a) := \{(x, d) \in f(\mathbb{P}(w')(K)) \times \mathbb{P}(\tau)(K) : b_x = b, \ d \equiv 0 \pmod{a}\}.$$

Let $V \subset \text{Val}_0(K)$ such that $p_v|a$ if and only if $v \in V$. Set

$$\delta_a := \prod_{v \in V} |\pi_v|^{-1} = \prod_{v \in V} [\mathcal{O}_{K,v} : \pi_v \mathcal{O}_{K,v}] = N(\sqrt{a}),$$

where $\sqrt{a}$ is the radical ideal of $a$. Then, for any $(x, d) \in \mathcal{A}(b, a)$, we have

$$\text{ht}_{\tau}(x, d) = \frac{d^{1/\tau} \text{ht}_w(x)}{\delta_a} = \frac{d^{1/\tau} \text{ht}_w(x)}{N(\sqrt{a})}.$$

Therefore

$$\#\{ (x, d) \in f(\mathbb{P}(w')(K)) \times \mathbb{P}(\tau)(K) : \text{ht}_w(x^d) \leq B \}$$

$$= \sum_{b \subseteq \mathcal{O}_K} \sum_{a||b} \#\{ (x, d) \in \mathcal{A}(b, a) : d^{1/\tau} \text{ht}_w(x) \leq \delta_a B \}.$$
This allows us to translate the condition $b_z = b = \prod_v p_v^{b_v}$ into the following set of local conditions, which we will denote by $\Omega(b)_v$,

\[
\frac{v(z_s)}{\tau} < \frac{w_s(\tau - b_v)}{\tau} \leq v(z_s) < \frac{w_s(\tau - b_v + 1)}{\tau}
\]

if $b_v = 0$

\[
\frac{v(z_s)}{\tau} < \frac{w_s(\tau - b_v)}{\tau} < \frac{w_s(\tau - b_v + 1)}{\tau}
\]

if $b_v \neq 0$.

We may rewrite these conditions without choosing reduced representatives as follows,

\[
\Omega(b)_v := \begin{cases}
\{ x \in \mathbb{P}(w')(K_v) : v(f_s(x)) - w_s \left[ \frac{v(f_s(x))}{w_s} \right] < \frac{w_s}{\tau} \} & \text{if } b_v = 0 \\
\{ x \in \mathbb{P}(w')(K_v) : \frac{w_s(\tau - b_v)}{\tau} \leq v(f_s(x)) - w_s \left[ \frac{v(f_s(x))}{w_s} \right] < \frac{w_s(\tau - b_v + 1)}{\tau} \} & \text{if } b_v \neq 0,
\end{cases}
\]

where $s$ is such that $v(f_s(x))/w_s = \min_j \{v(f_j(x))/w_j\}$. For the all but finitely many $v$ for which $b_v = 0$, we have that \( \{ x \in \mathcal{O}^{n+1}_{K,p_v} : \mathcal{I}_f(x)_p = \mathcal{O}_{K,p_v} \} \subset \Omega(b)_v^{\text{aff}} \). But, as we saw in the proof of Theorem 4.2.1, it follows from Lemma 3.2.11 that the \( \{ x \in \mathcal{O}^{n+1}_{K,p_v} : \mathcal{I}_f(x)_p = \mathcal{O}_{K,p_v} \} \) satisfies the limit condition (4.4) of Theorem 4.2.1. It follows that the local conditions $\Omega(b)$ satisfy the conditions of Theorem 4.2.1. By Theorem 4.2.1 we obtain the following asymptotic:

(4.10)

\[
\# \{ x \in \mathbb{P}(w')(K) : b_z = b, \text{ht}_w(x) \leq B \} \sim \lambda(b) \frac{h_{Km}\mathcal{F}(1)}{\deg(f)\mathcal{w}_K\mathcal{w}'\Delta_K^{(n+1)/2}\zeta_K(|\mathcal{w}'|)} B^{\mathcal{w}'}
\]

By Corollary 4.2.4 we have that

(4.11)

\[
\# \{ d \in \mathbb{P}(\tau)(K) : d \equiv 0 \pmod{a}, \, d^{1/\tau} \leq B \} \sim \frac{h_K2^{r_1+r_2+r_1\tau^{r_1+r_2-1}}}{N(a)\mathcal{w}_K\mathcal{w}'\Delta_K^{1/2}\zeta_K(\tau)} B^\tau.
\]

Applying the Product Lemma (Lemma 4.3.2) with the asymptotics (4.11) and (4.10), we have

\[
\# \{ (x, d) \in \mathcal{A}(b, a) : d^{1/\tau} \text{ht}_w(x) \leq \delta_a B \} \sim \begin{cases}
\sum_{b \in \mathcal{O}_K, \mu(b) \tau \leq e(f)} \lambda(b) \sum_{a \mid b} \frac{N(\sqrt{a})^{\mathcal{w}'\mu(|a|)/e(f)}}{N(a)} B^{\mathcal{w}'\mu(|a|)/e(f)} & \text{if } \tau < |\mathcal{w}'|/e(f) \\
\tau \lambda(b) \frac{\delta_a^\mu|\mathcal{w}'|}{N(a)} B^\tau \log(B) & \text{if } \tau = |\mathcal{w}'|/e(f) \\
\lambda(b) \frac{\delta_a^\mu|\mathcal{w}'|}{N(a)} B^\tau & \text{if } \tau > |\mathcal{w}'|/e(f).
\end{cases}
\]

Summing over all $a$ and $b$ gives

\[
\# \{ (x, d) \in \mathbb{P}(w')(K) \times \mathbb{P}(\tau)(K) : \text{ht}_w(x^{(d)}) \leq B \} \sim \begin{cases}
\left( \frac{|\mathcal{w}'|}{\mathcal{w}'\mu(|a|)/e(f)} \right)^{\kappa} \sum_{b \in \mathcal{O}_K, \mu(b) \tau \leq e(f)} \lambda(b) \sum_{a \mid b} \frac{N(\sqrt{a})^{\mathcal{w}'\mu(|a|)/e(f)}}{N(a)} B^{\mathcal{w}'\mu(|a|)/e(f)} & \text{if } \tau < |\mathcal{w}'|/e(f) \\
\tau \lambda(b) \sum_{a \mid b} \frac{N(\sqrt{a})^{\mathcal{w}'\mu(|a|)/e(f)}}{N(a)} B^\tau \log(B) & \text{if } \tau = |\mathcal{w}'|/e(f) \\
\lambda(b) \sum_{a \mid b} \frac{N(\sqrt{a})^{\mathcal{w}'\mu(|a|)/e(f)}}{N(a)} B^\tau & \text{if } \tau > |\mathcal{w}'|/e(f).
\end{cases}
\]
We now check that the leading coefficients converge. Assume \( \tau \geq |w'|/e(f) \). In these cases it suffices to show that the sum

\[
\sum_{b \subseteq \mathcal{O}_K \atop \tau \text{-free}} \lambda(b) \sum_{a \mid ||b|} \frac{N(\sqrt{a})^\tau}{N(a)}
\]

converges. By a theorem of Gronwall \cite{Gro13} on the growth rate of the sum-of-divisors function (which easily extends to arbitrary number fields), we obtain an upper bound for the inner sum

\[
\sum_{a \mid ||b|} N(\sqrt{a})^\tau \frac{N(\sqrt{b})^\tau}{N(b)} = O \left( \frac{N(\sqrt{b})^\tau}{N(b)} \log \log \left( \frac{N(\sqrt{b})^\tau}{N(b)} \right) \right).
\]

To estimate the product of local conditions \( \lambda(b) \) we first estimate \( m_v(\Omega(b)_v) \) for \( p_v \mid b \). From the definition of \( \Omega(b)_v \) we obtain

\[
m_v(\Omega(b)_v) = O \left( \prod_{p_v \mid b} N(p_v)^{-|w'(p_v-b_v)|/\tau} \right)
\]

Taking the product over \( p_v \mid b \), we have the upper bound

\[
\lambda(b) = O \left( \prod_{p_v \mid b} N(p_v)^{-|w'(p_v-b_v)|/\tau} \right) = O \left( \frac{N(\sqrt{b})^\tau}{N(b)}^{-|w'\tau/\tau}} \right)
\]

It follows that

\[
\lambda(b) \sum_{a \mid ||b|} \frac{N(\sqrt{a})^\tau}{N(\sqrt{a})} = O \left( \left( \frac{N(\sqrt{b})^\tau}{N(b)} \right)^{1-|w'|/\tau} \log \log \left( \frac{N(\sqrt{b})^\tau}{N(b)} \right) \right)
\]

\[
= O \left( \left( \frac{N(\sqrt{b})^\tau}{N(b)} \right)^{-2} \log \log \left( \frac{N(\sqrt{b})^\tau}{N(b)} \right) \right)
\]

where we have used the assumption that \(-3\tau + |w| \geq 0 \) in obtaining the second equality. Note that the map

\[
\{b \subseteq \mathcal{O}_K : b \text{ is } \tau\text{-free}\} \rightarrow \{b \subseteq \mathcal{O}_K : b \text{ is } \tau\text{-free}\}
\]

\[
b = \prod_{p_v \mid b} p_v^{b_v} \mapsto \frac{N(\sqrt{b})^\tau}{N(b)} = \prod_{p_v \mid b} p_v^{\tau-b_v}
\]

is a bijection. Therefore

\[
\sum_{b \subseteq \mathcal{O}_K \atop \tau \text{-free}} \lambda(b) \sum_{a \mid ||b|} \frac{N(\sqrt{a})^\tau}{N(a)} = \sum_{b \subseteq \mathcal{O}_K \atop \tau \text{-free}} O \left( N(b)^{-2} \log \log(N(b)) \right) = O(1).
\]

This shows that the sum (4.12) converges. A similar argument shows that the infinite sum in the \( \tau < |w'|/e(f) \) case converges provided that

\[
\tau(|w| - |w'|/e(f) - 1) - |w| \geq 0
\]

\[ \square \]
Remark 4.3.3. It is interesting to note the similarity of the above proof with the work of Wang on Malle’s Conjecture for $S_n \times A$ extensions [Wan17, Wan21]. One of the key features of Wang’s work is dealing with the discriminant of the compositum of number fields, which is roughly the product of the discriminants, but with some ‘defect’ making the problem more difficult. The situation of our Theorem 1.3.3 is similar in that the twisted height is roughly the product of $d^{1/τ}$ and $\text{ht}_w(x)$, but with the ‘defect’ $δ_n$.

5. Counting elliptic curves

We now apply the results of the previous section to count elliptic curves.

5.1. Counting elliptic curves with prescribed level structure.

Theorem 1.2.2. Let $G$ be a subgroup of $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ and let $\Gamma_G$ denote the inverse image of $G$ with respect to the canonical map $\text{SL}_2(\mathbb{Z}) \to \text{GL}_2(\mathbb{Z}/N\mathbb{Z})$. Let $K_G$ be the field fixed by the action of $G$ on $\mathbb{Q}(ζ_N)$ determined by $(g, ζ_N) \mapsto ζ_N^{\text{det}(g)}$. Let $K$ be a degree $d$ number field containing $K_G$. Suppose that for some pair of positive integers $w, e$ the twisted height is roughly the product of the discriminants, but with some ‘defect’ making the problem more difficult. The situation of our Theorem 1.3.3 is similar in that the twisted height is roughly the product of $d^{1/τ}$ and $\text{ht}_w(x)$, but with the ‘defect’ $δ_n$.

We apply Theorem 4.1.1 with $P(\mathbf{w}') = \mathcal{X}_G$ and $P(\mathbf{w}) = P(4, 6) \cong \mathcal{X}_{\text{GL}_2(\mathbb{Z})}$ and trivial local conditions $Ω_p = P(\mathbf{w}')(K_p)$ for all $p$. Let $ψ : \mathcal{X}_G \to \mathcal{X}_{\text{GL}_2(\mathbb{Z})}$ be the map which forgets the level structure and let $φ : P(\mathbf{w}') \to P(4, 6)$ be the corresponding map of weighted projective spaces. The degree of these maps are $\text{deg}(ψ) = [\text{SL}_2(\mathbb{Z}) : \{±I\}\Gamma_G]$ and $e(φ) = [\text{SL}_2(\mathbb{Z}) : GAMMA_G]$.

The Hodge bundle on $\mathcal{X}_{\text{GL}_2(\mathbb{Z})}$ and the tautological bundle on $P(4, 6)$ each have degree $1/24$. Pulling back each of these bundles, along $ψ$ and $φ$ respectively, we obtain line bundles of degrees $[\text{SL}_2(\mathbb{Z}) : Γ_G]/24$ and $e(φ)/w'_0w'_1$ respectively. As these degrees must coincide, it follows that

$$e(G) := e(φ) = \frac{w'_0w'_1}{24}[\text{SL}_2(\mathbb{Z}) : Γ_G].$$

Finally, taking into account that the naive height on $\mathcal{X}_{\text{GL}_2(\mathbb{Z})}$ is the twelfth power of the height $\text{ht}_w$ on $P(4, 6)$, we obtain the theorem. □

In the special case that $G = \text{GL}_2(\mathbb{Z})$ we obtain the following result:
Corollary 5.1.2. Let $\mathcal{E}_K(B)$ denote the set of isomorphism classes of elliptic curves over $K$ with height less than $B$. Then

$$\# \mathcal{E}_K(B) = \kappa B^{5/6} + O(B^{5/6-1/3d})$$

where

$$\frac{h_K(2^{r_1+r_2}\pi^{r_2})^{n+1}R_K 10^{r_1+r_2-1} \gcd(2, \varpi_K)}{\varpi_K |\Delta_K| \zeta_K(10)}.$$  

For more special cases of Theorem 1.2.2 see Table 2.

We now prove our main result for counting twists of elliptic curves.

Theorem 5.1.4. Let $G$ and $G'$ be subgroups of $\text{GL}_2(\mathbb{Z}/n\mathbb{Z})$, and let $\Gamma_{G'}$ denote the inverse image of $G'$ with respect to the canonical map $\text{SL}_2(\mathbb{Z}) \to \text{GL}_2(\mathbb{Z}/N\mathbb{Z})$. Set

$$e(G') = \frac{[\text{SL}_2(\mathbb{Z}) : \Gamma_{G'}]}{24}.$$  

Let $K_G$ be the field fixed by the action of $G$ on $\mathbb{Q}(\zeta_N)$ determined by $(g, \zeta_N) \mapsto \zeta_N^{\det(g)}$ and let $K$ be a number field of degree $d$ containing $K_G$, and that for each elliptic curve $E$ over $K$ with $G$-level structure, there is a unique quadratic twist of $E$ with $G'$-level structure, so that $X_G \cong \mathbb{P}(1,1) \times \mathbb{P}(2)$ as stacks over $K_G$. Then the number of isomorphism classes elliptic curves over $K$ with $G$-level structure and height less than $B$ is asymptotic to

$$\begin{cases}
2\kappa \sum_{\substack{b \subseteq \mathcal{O}_K \\
\text{square-free}}} \lambda(b) \sum_{a \mid b} N_{K/Q}(a) & B^{1/6} \log(B) \quad \text{if } e(G') = 1 \\
\kappa \sum_{\substack{b \subseteq \mathcal{O}_K \\
\text{square-free}}} \lambda(b) \sum_{a \mid b} N(a) & B^{1/6} \quad \text{if } e(G') > 1,
\end{cases}$$

where

$$\kappa = \frac{h_K^2 2^{2(r_1+r_2)-1} \pi^{r_2} R_K m_\infty(F(1))}{\varpi_K 2 |\Delta_K|^{3/2} \zeta_K(2) [\text{SL}_2(\mathbb{Z}) : \{\pm I\} \Gamma_{G'}]}$$

and $\lambda(b)$ is a product of local densities depending on $b$.

Proof. The assumption that for each $E$ with $G$-level there is a unique $d \in H^1(\text{Gal}(\mathbb{K}/K), \text{Aut}(E)) \cong \mathbb{P}(2)(K)$ such that the twist $E^{(d)}$ of $E$ by $d$ has $G'$-level structure, implies that the number of elliptic curves with $G$-level structure is the same as the number of pairs $(E, d) \in X_{G'}(K) \times \mathbb{P}(2)(K)$.

We apply Theorem 1.3.3 with $\tau = 2$, $X_{G'} = \mathbb{P}(1,1)$, and the morphism $\varphi : \mathbb{P}(1,1) \to \mathbb{P}(4,6)$ corresponding to the map $\psi : X_{G'} \to X_{\text{GL}_2(\mathbb{Z})}$ which forgets the level structure. Note that we have $\tau \geq |w|/e(f) = 2/e(f)$ in all cases with equality if and only if $e(f) = 1$. We also see that $\tau$ satisfies the assumption in Theorem 1.3.3 that $-3\tau + |w| \geq 0$, since $|w| = 4 + 6$ in our case.
As in the proof of Theorem 1.2.2 we have that deg(φ) = deg(ψ) = [SL₂(Z) : {±I}Γ_{G'}] and
e(φ) = e(G') = \frac{w'_1w'_2}{24}[SL₂(Z) : Γ_{G'}].
The theorem then follows from the fact that the naive height on X_{GL₂(Z)} is the twelfth power of the height on \mathbb{P}(4, 6) with respect to the tautological bundle. □

We now discuss the special cases of this theorem listed in Table 3. For each \(N \in \{6, 8, 9, 12, 16, 18\}\) we define an index two subgroup \(H_N \leq (\mathbb{Z}/N\mathbb{Z})^\times\):

\[
\begin{align*}
H_6 &= \{1\} \\
H_8 &= \{1, 3\} \\
H_9 &= \{1, 2, 4\} \\
H_{12} &= \{1, 5\} \\
H_{16} &= \{1, 3, 5, 7\} \\
H_{18} &= \{1, 5, 7\}.
\end{align*}
\]

Following Boggess-Sankar [BS20] we set \(G_{1/2}(N) := \left\{ g \in GL₂(\mathbb{Z}/N\mathbb{Z}) : g = \begin{pmatrix} a & * \\ 0 & 1 \end{pmatrix} \text{ and } a \in H_N \right\}\)

and let \(Γ_{1/2}(N) \leq SL₂(Z)\) be the corresponding congruence subgroup. Results of Boggess and Sankar [BS20] Proposition 2.2, Proposition 4.1 show that in these cases, for each elliptic curve \(E\) over \(K\) with \(G_0(N)\)-level structure, there is a unique quadratic twist of \(E\) with \(G_{1/2}(N)\)-level structure and that \(X_{G_{1/2}(N)} \cong \mathbb{P}(1, 1) \times \mathbb{P}(2)\).

5.2. Counting elliptic curves with a prescribed local condition. Corollary 4.1.6 allows one to count elliptic curves with prescribed level structure and prescribed local conditions in certain cases. As an illustration, in this subsection we will prove an asymptotic for the number of elliptic curves over a number field with prescribed local conditions.

Before stating our result, we recall the definition of Kronecker-Hurwitz class numbers. Let \(K\) be an imaginary quadratic field with ring of integers \(O_K\) and let \(O\) be an order in \(K\) with class number \(h(O)\). The **Hurwitz-Kronecker class number** of \(O\), denoted \(H(\text{disc}(O))\), is defined as

\[
H(\text{disc}(O)) := \sum_{\mathfrak{O} \subset \mathfrak{O'} \subset O_K} \frac{h(O')}{\#O'\times}.
\]

This definition of the Hurwitz-Kronecker class number agrees with the definition given in [Len87], but is twice as large as it is sometimes defined in other places (such as in [Cox89]). A beautiful result following from the work of Deuring [Deu41] expresses the number of elliptic curves over a finite field \(\mathbb{F}_q\) with a prescribed trace of Frobenius in terms of Hurwitz-Kronecker class numbers:

\[
\{(b, c) \in \mathbb{F}_q \times \mathbb{F}_q : \Delta(b, c) \neq 0, \; a_q(E_{b,c}) = a\} = (q - 1)H(a^2 - 4q).
\]

See [Cox89, Theorem 14.18] for a proof of this formula in the case that \(q = p\) is a prime.
We now prove our theorem for counting elliptic curves satisfying a prescribed local condition:

**Theorem 1.1.2.** Let $K$ be a number field and let $p \in \mathcal{O}_K$ be a prime ideal of norm $q$ such that $2 \nmid q$ and $3 \nmid q$. Let $L$ be one of the local conditions listed in Table 1. Then the number of elliptic curves over $K$ with naive height less than $B$ and which satisfy the local condition $L$ at $p$ is

$$\kappa'_L B^{5/6} + O \left( \epsilon_L B^{\frac{5}{6} - \frac{1}{3\pi}} \right)$$

where

$$\kappa' = \frac{h_K \left(2^{r_1+r_2} \pi r_2\right)^{n+1} R_K |w|^{|r_1+r_2-1|}}{\text{deg}(f) \omega_K, w | \Delta_K |^{(n+1)/2} \zeta_K (|w'|)},$$

and where $\kappa_L$ and $\epsilon_L$ are as in Table 1.

**Proof.** Good reduction case. Note that the number of elliptic curves over $\mathbb{F}_q$ with good reduction at $p$ is

$$\#\{(a, b) \in \mathbb{F}_q \times \mathbb{F}_q : 4a^3 + 27b^2 \equiv 0 \pmod{q}\} = q^2 - q.$$  

For each pair $(a, b)$ in the above set we consider the finite local condition

$$\Omega_p^{\text{aff}} = \{(x_0, x_1) \in K_v^2 : |x_0 - a| \leq \frac{1}{q}, |x_1 - b| \leq \frac{1}{q}\},$$

which has $p$-adic measure

$$m_p(\Omega_p^{\text{aff}} \cap O^2_{K, p}) = \frac{1}{q^2}.$$  

Applying Corollary 4.1.6 and summing over all the $q^2 - q$ possible $(a, b)$ pairs, it follows that the number of elliptic curves of bounded height over $K$ with good reduction at $p$ is

$$\kappa'_L B^{5/6} + O \left( q \frac{q^2 - q}{q^2} B^{\frac{5}{6} - \frac{1}{3\pi}} \right),$$

where $\kappa_L = (q^2 - q)/q^2$. As the error term can be rewritten as $O(q B^{\frac{5}{6} - \frac{1}{3\pi}})$, we also have that $\epsilon_L = q$.

Prescribed trace of Frobenius case. Fix an integer $a$ satisfying $|a| \leq 2\sqrt{q}$. By (5.1) and Corollary 4.1.6 we have that the number of elliptic curves over $K$ of bounded height and with trace of Frobenius at $p$ equal to $a$ is

$$\kappa'_L B^{5/6} + O \left( q \frac{(q-1)H(a^2-4q)}{q^2} B^{\frac{5}{6} - \frac{1}{3\pi}} \right),$$

where $\kappa_L = (q-1)H(a^2-4q)/q^2$. As the error term can be rewritten as $O(H(a^2 - 4q) B^{\frac{5}{6} - \frac{1}{3\pi}})$, we have that $\epsilon_L = H(a^2 - 4q)$.

Kodaira type III* case. By Tate’s algorithm [Tat75] an elliptic curve over $K$ given in short Weierstrass form, $E : y^2 = x^3 + ax + b$, has type III* reduction if and only if $v_p(a) = 3$ and $v_p(b) \geq 5$. Note that

$$\#\{(a, b) \in \mathcal{O}_K/p^5 \times \mathcal{O}_K/p^5 : v_p(a) = 3, v_p(b) \geq 5\} = q(q-1).$$
Applying Corollary 4.1.6 to each pair in the above set and then summing over all pairs yields the following asymptotic for the number of elliptic curves of bounded height over $K$ with reduction of Kodaira type III* at $p$:

$$\kappa' \kappa_L B^{5/6} + O \left( q^{5} \frac{1(q - 1)}{q^{10}} B^{5/6 - 1/3d} \right)$$

where $\kappa_L = q(q - 1)/q^{10} = (q - 1)/q^{9}$. As the error term can be written as $O(q^{-3} B^{5/6 - 1/3d})$, we have that $\epsilon_L = 1/q^{3}$.

The other cases can be proven similarly to those given above. \qed

6. Average analytic ranks

In this section we use our results from the previous section on counting elliptic curves with prescribed local conditions in order to give a bound for the average analytic rank of elliptic curves over number fields. Our strategy will mainly follow that of Cho and Jeong [CJ21, §4.1].

6.1. L-functions of elliptic curves. In this subsection we recall basic facts about $L$-functions of elliptic curves.

Let $E$ be an elliptic curve of discriminant $\Delta_E$ over a number field $K$. For each finite place $v \in \text{Val}_0(K)$, let $E_v$ denote the reduction of $E$ at $v$, let $q_v$ be the order of the residue field $k_v$ of $K$ at $v$, let $N_v = \#E_v(F_{q_v})$, and let $a_v = q_v + 1 - N_v$. If $E$ has bad reduction at $v$, set

$$b_v = \begin{cases} 
1 & \text{if } E \text{ has split multiplicative reduction at } v, \\
-1 & \text{if } E \text{ has nonsplit multiplicative reduction at } v, \\
0 & \text{if } E \text{ has additive reduction at } v.
\end{cases}$$

For each $v \in \text{Val}_0(K)$, define a polynomial $L_v(E/K,T)$ by

$$L_v(E/K,T) = \begin{cases} 
1 - a_v T + q_v T^2 & \text{if } E \text{ has good reduction at } v, \\
1 - b_v T & \text{if } E \text{ has bad reduction at } v.
\end{cases}$$

The Hasse-Weil $L$-function $L(E/K,s)$ of $E$ over $K$ is defined as the Euler product

$$L(E/K,s) = \prod_{v \in \text{Val}_0(K)} L_v(E/K,q_v^{-s})^{-1}.$$

The logarithmic derivative of $L(E/K,s)$ is

$$\frac{L'}{L}(E/K,s) = - \sum_{v \in \text{Val}_0(K)\atop v(\Delta_E) > 0} \frac{d}{ds} \left( \log(1 - b_v q_v^{-s}) \right) - \sum_{v \in \text{Val}_0(K)\atop v(\Delta_E) = 0} \frac{d}{ds} \left( \log(1 - a_v q_v^{-s} + q_v^{1-2s}) \right).$$

The first sum can be rewritten as

$$- \sum_{v \in \text{Val}_0(K)\atop v(\Delta_E) > 0} \frac{d}{ds} \left( \log(1 - b_v q_v^{-s}) \right) = - \sum_{v \in \text{Val}_0(K)\atop v(\Delta_E) > 0} \frac{d}{ds} \left( \sum_{k=1}^{\infty} \frac{b_v^k}{k q_v^{ks}} \right) = - \sum_{v \in \text{Val}_0(K)} \sum_{k=1}^{\infty} \frac{b_v^k \log(q_v)}{q_v^{ks}}.$$
For the second sum, factor \( 1 - \alpha_v T + \beta_v T^2 \) as \((1 - \alpha_v T)(1 - \beta_v T)\) where \(\alpha_v + \beta_v = a_v\) and \(\alpha_v \beta_v = q_v\). Then
\[
- \sum_{v \in \text{Val}_0(K)} \frac{d}{ds} \left( \log(1 - a_v q_v^{-s} + q_v^{1-2s}) \right) = - \sum_{v \in \text{Val}_0(K)} \sum_{\nu(\Delta_E) > 0} \frac{(\alpha_v^k + \beta_v^k) \log(q_v)}{q_v^{ks}}.
\]

We thus obtain the following expression for the logarithmic derivative of \(L(E/K, s)\):
\[
\frac{L'}{L}(E/K, s) = - \sum_{v \in \text{Val}_0(K)} \sum_{\nu(\Delta_E) > 0} \frac{b_v^k \log(q_v)}{q_v^{ks}} - \sum_{v \in \text{Val}_0(K)} \sum_{\nu(\Delta_E) = 0} \frac{(\alpha_v^k + \beta_v^k) \log(q_v)}{q_v^{ks}}.
\]

Define the von Mangoldt function on integral ideals \(a \subseteq \mathcal{O}_K\) as
\[
\Lambda_K(a) = \begin{cases} 
\log(q_v) & \text{if } a = p^k \text{ for some } k, \\
0 & \text{otherwise}.
\end{cases}
\]

For prime powers, set
\[
\widehat{a}_E(p^k) = \begin{cases} 
\alpha_v^k + \beta_v^k & \text{if } E \text{ has good reduction at } p_v, \\
b_v^k & \text{if } E \text{ has bad reduction at } p_v,
\end{cases}
\]
and for \(a\) not a power of a prime set \(\widehat{a}_E(a) = 0\). Then our expression for the logarithmic derivative of \(L(E/K, s)\) becomes
\[
(6.1) \quad \frac{L'}{L}(E/K, s) = - \sum_{a \subseteq \mathcal{O}_K} \frac{\widehat{a}_E(a)\Lambda_K(a)}{N_K(a)^s},
\]
where the sum is over the non-zero integral ideals of \(\mathcal{O}_K\).

Let \(\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt\) be the usual gamma function. Let \(\Gamma_K(s)\) be the gamma factor
\[
\Gamma_K(s) = ((2\pi)^{-s}\Gamma(s))^{[K:\mathbb{Q}]}.
\]

Let \(f_{E/K}\) be the conductor of \(E\) over \(K\), and define a constant
\[
A_{E/K} = N_K(f_{E/K})\Delta_K^2.
\]

Define the completed Hasse-Weil \(L\)-function \(\Lambda(E/K, s)\) as
\[
\Lambda(E/K, s) = A_{E/K}^{s/2}\Gamma_K(s)L(E/K, s).
\]

We now assume that our elliptic curve \(E\) is modular over \(K\), so that the \(L\)-function \(L(E/K, s)\) is automorphic. This implies that the Hasse-Weil conjecture holds true for \(E\), and thus
\[
\Lambda(E/K, s) = \epsilon \Lambda(E/K, 2 - s),
\]
where \(\epsilon(E) \in \{\pm 1\}\) is the root number. We see that the logarithmic derivative of \(\Lambda(E/K, s)\) is
\[
(6.2) \quad \frac{\Lambda'}{\Lambda}(E/K, s) = \frac{\Gamma'_K}{\Gamma_K}(s) + \frac{L'}{L}(E/K, s) = - \frac{\Lambda'}{\Lambda}(E/K, 2 - s).
\]

We now state the explicit formula for \(L\)-functions of Elliptic curves over number fields:
Proposition 6.1.1 (Explicit Formula). Let $\phi$ be an even function on $\mathbb{R}$ whose Fourier transform,

\[ \hat{\phi}(t) = \int_{\mathbb{R}} \phi(x)e^{-2\pi itx}dx, \]

is compactly supported. Let $E$ be a modular elliptic curve over a number field $K$. Suppose that the $L$-function $L(E/K, s)$ associated to $E$ satisfies the Generalized Riemann Hypothesis, so that all non-trivial zeros $\rho_j$ of $L(E/K, s)$ are of the form $\rho_j = 1 + iy$ for some $y \in \mathbb{R}$.

Let $r(E) = \text{ord}_{s=1}(L(E/K, s))$ be the analytic rank of $E$. Then we have

\[
\begin{align*}
&\quad \phi(0) + \sum_{\gamma \neq 0} \phi(\gamma) \\
&= \hat{\phi}(0) \log(A_{E/K}) + \frac{1}{\pi} \int_{\mathbb{R}} \frac{\Gamma'(1 + iy)}{\Gamma'}(1 + iy) \phi(y)dy - \frac{1}{\pi} \sum_{a \subseteq \mathcal{O}_K} \frac{\hat{a}_E(a)\Lambda(a)}{N_K(a)} \cdot \hat{\phi} \left( \frac{\log(N_K(a))}{2\pi} \right).
\end{align*}
\]

This proposition is a special case of [IK04, Theorem 5.12] (see also [Mes86] and [Mil02, Appendix A] for the specific case of $L$-functions of elliptic curves).

From the explicit formula it follows that

\[
\begin{align*}
&\quad \phi(0) + \sum_{\gamma \neq 0} \phi(\gamma) \\
&= \hat{\phi}(0) \log(A_{E/K}) + \frac{1}{\pi} \int_{\mathbb{R}} \frac{\Gamma'(1 + iy)}{\Gamma'}(1 + iy) \phi(y)dy - \frac{1}{\pi} \sum_{a \subseteq \mathcal{O}_K} \frac{\hat{a}_E(a)\Lambda(a)}{N_K(a)} \cdot \hat{\phi} \left( \frac{\log(N_K(a))}{2\pi} \right) + O(1).
\end{align*}
\]

A classical estimate for the logarithmic derivative of the gamma function is

\[ \left| \frac{\Gamma'}{\Gamma}(1 + iy) \right| = O(\log(\|y\|) + 2). \]

From this it follows that

\[ \int_{\mathbb{R}} \frac{\Gamma'(1 + iy)}{\Gamma'}(1 + iy) \phi \left( \frac{y \log(X)}{2\pi} \right) dy = \int_{\mathbb{R}} \phi \left( \frac{y \log(X)}{2\pi} \right) dy + O \left( \frac{1}{\log(X)} \right). \]

From now on we shall assume $\phi \in L^1(\mathbb{R})$, in which case

\[ \int_{\mathbb{R}} \frac{\Gamma'(1 + iy)}{\Gamma'}(1 + iy) \phi \left( \frac{y \log(X)}{2\pi} \right) dy = O \left( \frac{1}{\log(X)} \right). \]

Let $E$ be an elliptic curve with minimal discriminant $\mathcal{D}_{E/K}$ and of height less than or equal to $X$. For each such $E$ we have that $N_K(\mathcal{D}_{E/K}) \ll X$. But by Ogg’s formula [Ogg67] (see also [Sil94, Ogg’s Formula 11.1]) one has $N_K(f_{E/K}) \leq N_K(\mathcal{D}_E)$, so that

\[ \log(N_K(f_{E/K}))/\log(X) \leq 1 + O(1/\log X). \]

It follows that $\log(A_{E/K})/\log(X) \leq 1 + O(1/\log X)$. 39
From the above observations we have that
\[ r(E)\phi(0) + \sum_{\gamma \neq 0} \phi \left( \frac{\log(X)}{2\pi} \right) \leq \hat{\phi}(0) - \frac{2}{\log(X)} \sum_{a \in \mathcal{O}_K} \hat{a}_E(a) \Lambda(a) \cdot \hat{\phi} \left( \frac{\log(N_K(a))}{\log(X)} \right) + O \left( \frac{1}{\log X} \right). \]

We further simplify by rewriting the sum on the right hand side as
\[ \sum_{a \in \mathcal{O}_K} \frac{\hat{a}_E(a) \Lambda(a)}{N_K(a)} \cdot \hat{\phi} \left( \frac{\log(N_K(a))}{\log(X)} \right) = \sum_{v \in \text{Val}_K} \sum_{k \geq 3} \frac{\hat{a}_E(p_k^v)}{q_v^k} \cdot \hat{\phi} \left( \frac{k \log(q_v)}{\log(X)} \right) \]
and then using the Hasse bound, \( |\hat{a}_E(p_k^v)| \leq 2\sqrt{q_v} \), to bound the \( k \geq 3 \) part of the sum
\[ \sum_{v \in \text{Val}_K} \sum_{k \geq 3} \frac{\hat{a}_E(p_k^v)}{q_v^k} \cdot \hat{\phi} \left( \frac{k \log(q_v)}{\log(X)} \right) \leq \sum_{v \in \text{Val}_K} \sum_{k \geq 3} \frac{2k \log(q_v)}{q_v^{k/2}} \cdot \hat{\phi} \left( \frac{k \log(q_v)}{\log(X)} \right) \]
\[ \leq \sum_{v \in \text{Val}_K} \sum_{k \geq 3} \frac{8 \log(q_v)}{q_v^{k/2}} \cdot \hat{\phi} \left( \frac{k \log(q_v)}{\log(X)} \right) \]
\[ = O(1), \]
where \( ||f||_\infty \) is the usual \( L^\infty \)-norm defined as
\[ ||f||_\infty := \inf \{ a \geq 0 : m_\infty(\{ x : |f(x)| > 0 \}) = 0 \}, \]
where \( m_\infty \) is the Lebesgue measure on \( \mathbb{R} \). Setting
\[ U_k(E, \phi, X) = \sum_{v \in \text{Val}_K} \frac{\hat{a}_E(p_k^v)}{q_v^k} \cdot \hat{\phi} \left( \frac{k \log(q_v)}{\log(X)} \right), \]
we have that
\[ r(E)\phi(0) + \sum_{\gamma \neq 0} \phi \left( \frac{\log(X)}{2\pi} \right) \leq \hat{\phi}(0) - \frac{2}{\log(X)} (U_1(E, \phi, X) + U_2(E, \phi, X)) + O \left( \frac{1}{\log X} \right). \]

We now average over isomorphism classes of elliptic curves. Let \( \mathcal{E}_K(B) \) denote the set of isomorphism classes of elliptic curves over \( K \). Setting
\[ S_k(\phi, B) = \frac{2}{\log(B) \# \mathcal{E}_K(B)} \sum_{E \in \mathcal{E}_K(B)} U_k(E, \phi, B) \]
\[ = \frac{2}{\log(B) \# \mathcal{E}_K(B)} \sum_{v \in \text{Val}_K} \frac{\log(q_v)}{q_v^k} \cdot \hat{\phi} \left( \frac{k \log(q_v)}{\log(B)} \right) \sum_{E \in \mathcal{E}_K(B)} \hat{a}_E(p_k^v), \]
we have that
\[ \phi(0) \frac{1}{\# \mathcal{E}_K(B)} \sum_{E \in \mathcal{E}_K(B)} r(E) + \frac{1}{\# \mathcal{E}_K(B)} \sum_{E \in \mathcal{E}_K(B)} \sum_{\gamma \neq 0} \phi \left( \frac{\gamma E \log(B)}{2\pi} \right) \]
\[ \leq \hat{\phi}(0) - S_1(\phi, B) - S_2(\phi, B) + O \left( \frac{1}{\log(B)} \right). \]
For our applications we shall use the test function
\[ \phi(y) = \left( \frac{\sin(\pi \nu y)}{2\pi y} \right)^2, \]
whose Fourier transform is
\[ \hat{\phi}(t) = \frac{1}{2} \left( \frac{\nu}{2} - \frac{|t|}{2} \right) \quad \text{for} \ |t| \leq \nu. \]
Note that \( \phi(0) = \frac{\nu^2}{4} \) and \( \hat{\phi}(0) = \frac{\nu}{4}. \)
Our strategy will be to show that
\[ -S_1(\phi, B) - S_2(\phi, B) = \frac{\phi(0)}{2} + o(1). \]
This will imply the following bound for the average analytic rank of elliptic curves over the number field \( K: \)
\[ r_K \leq \hat{\phi}(0) \phi(0) + \frac{1}{2} = \frac{1}{\nu} + \frac{1}{2}. \]

6.2. **Class number estimates.** In this subsection we prove a proposition concerning sums of Kronecker-Hurwitz class numbers. This result will be used in the next subsection.

**Proposition 6.2.1.** Let \( q = p^n \) be a prime power. We have the following estimates:
\[
\sum_{|a| \leq 2\sqrt{q}} H(a^2 - 4q) = q + O(p^{n-1}),
\]
\[
\sum_{|a| \leq 2\sqrt{q}} aH(a^2 - 4q) = 0,
\]
\[
\sum_{|a| \leq 2\sqrt{q}} a^2 H(a^2 - 4q) = q^2 + O(p^{2n-1}).
\]

**Proof.** Let \( p^n \) be a prime power and set
\[
\delta(p^n, 2) = \begin{cases} 
    1 & \text{if } n \text{ is even}, \\
    0 & \text{if } n \text{ is odd}.
\end{cases}
\]
By the classical Kronecker-Hurwitz class number relation [Hur85], we have that
\[
\sum_{|a| \leq 2\sqrt{q}} \frac{1}{2} \sum_{dd'=q, dd'>0} \max(d, d') = q + \frac{1}{2} \left( \sum_{s=1}^{[\frac{n-1}{2}]} p^{n-s} \right) + \frac{1}{2} p^{n/2} \delta(q, 2) = q + O(p^{n-1}).
\]
Pairing each elliptic curve over \( \mathbb{F}_q \) with its quadratic twist we see that
\[
\sum_{|a| \leq 2\sqrt{q}} aH(a^2 - 4q) = 0.
\]
Let $T_k(q)$ be the Hecke operator on the space of weight $k$ cusp forms. Then the Eichler-Selberg trace formula (see the Appendix to Chapter III of [Lan95] by Zagier) implies
\[
\sum_{|a| \leq 2\sqrt{q}} a^2 H(a^2 - 4q) = \frac{1}{2} q \delta(q, 2) - q \sum_{|a| \leq 2\sqrt{q}} H(a^2 - 4q) - \sum_{dd'=q \atop d,d'>0} \min(d,d')^3 - 2\text{Tr}(T_4(q)).
\]
Applying our estimate for $\sum_{|a| \leq 2\sqrt{q}} a^2 H(a^2 - 4q)$ and using the Ramanujan-Petersson-Deligne bound for Hecke eigenvalues [Del74], we obtain
\[
\sum_{|a| \leq 2\sqrt{q}} a^2 H(a^2 - 4q) = q^2 + O(p^{2n-1}).
\]

6.3. Estimating $S_1(\phi, B)$ and $S_2(\phi, B)$.

Lemma 6.3.1. Let $p \subset \mathcal{O}_K$ be a prime ideal of norm $N(p) = q$. Then we have the following upper bound:
\[
\left| \sum_{E \in \mathcal{E}(B)} \hat{a}_E(p) \right| \ll \frac{1}{q} B^{5/6} + q B^{5/6-1/3d}.
\]

Proof. Let $\mathcal{E}^\text{mult}(B)$ denote the set of elliptic curves over $K$ of naive height bounded by $B$ and multiplicative reduction at $p$. Then we have
\[
\sum_{E \in \mathcal{E}(B)} \hat{a}_E(p) = \sum_{|a| \leq 2\sqrt{q}} \sum_{E \in \mathcal{E}(B)} \hat{a}_E(p) + \sum_{E \in \mathcal{E}^\text{mult}(B)} \hat{a}_E(p).
\]
By Theorem 1.1.2 we have that
\[
\left| \sum_{E \in \mathcal{E}^\text{mult}(B)} \hat{a}_E(p) \right| \ll \frac{q - 1}{q^2} B^{5/6} + B^{5/6-1/3d} \ll \frac{1}{q} B^{5/6} + B^{5/6-1/3d}.
\]
By Theorem 1.1.2 and Proposition 6.2.1 we have that
\[
\sum_{|a| \leq 2\sqrt{q}} \sum_{a_p(E) = a} \hat{a}_E(p) = \sum_{|a| \leq 2\sqrt{q}} a \left( \kappa'(q-1) H(a^2 - 4q) \right) B^{5/6} + O \left( H(a^2 + 4q) B^{5/6-1/3d} \right)
\]
\[
\ll q B^{5/6-1/3d}.
\]
The lemma now follows. \square

Lemma 6.3.2. Let $p \subset \mathcal{O}_K$ be a prime ideal of norm $N(p) = q$. Then we have the following estimate:
\[
\sum_{E \in \mathcal{E}(B)} \hat{a}_E(p^2) = -\kappa' q B^{5/6} + O \left( B^{\frac{5}{6}} + q B^{\frac{5}{6}-\frac{1}{3d}} \right).
\]
Proof. Again let $\mathcal{E}_K^{\text{mult}}(B)$ denote the set of elliptic curves over $K$ with multiplicative reduction at $p$ and height bounded by $B$. In this case we have that

$$\sum_{E \in \mathcal{E}_K(B)} \widehat{a}_E(p^2) = \sum_{|a| \leq 2\sqrt{q}} \sum_{E \in \mathcal{E}_K(B)} \widehat{a}_E(p^2) + \sum_{E \in \mathcal{E}_K^{\text{mult}}(B)} \widehat{a}_E(p^2).$$

Since $\widehat{a}_E(p^2) = \widehat{a}_E(p)^2 = 1$ for all $E \in \mathcal{E}_K^{\text{mult}}(B)$, we have that by Theorem 1.1.2

$$\sum_{E \in \mathcal{E}_K^{\text{mult}}(B)} \widehat{a}_E(p) = \#\mathcal{E}_K^{\text{mult}}(B) = \kappa' \frac{q - 1}{q^2} B^{5/6} + O(B^{5/6 - 1/3d}).$$

In the case that $E$ has good reduction at $p$ a straightforward computation shows that $\widehat{a}_E(p^2)$ equals $a_p(E)^2 - 2q$. Therefore, by Theorem 1.1.2 and Proposition 6.2.1 we have that

$$\sum_{|a| \leq 2\sqrt{q}} \sum_{E \in \mathcal{E}_K(B)} \widehat{a}_E(p^2) = \sum_{|a| \leq 2\sqrt{q}} \sum_{E \in \mathcal{E}_K(B)} \sum_{a_p(E) = a} (a_p(E)^2 - 2q)$$

$$= \sum_{|a| \leq 2\sqrt{q}} (a^2 - 2q) \left( \kappa' \frac{q - 1}{q^2} H(a^2 - 4q) B^{5/6} + O(H(a^2 + 4q)B^{5/6 - 1/3d}) \right)$$

$$= \kappa' \frac{q - 1}{q^2} \left( \sum_{|a| \leq 2\sqrt{q}} a^2 H(a^2 - 4q) - 2q \sum_{|a| \leq 2\sqrt{q}} H(a^2 + 4q) \right) B^{\frac{5}{6}} + O\left( qB^{\frac{5}{6} - \frac{1}{3d}} \right)$$

$$= -\kappa' q B^{\frac{5}{6}} + O\left( B^{\frac{5}{6}} + qB^{\frac{5}{6} - \frac{1}{3d}} \right).$$

The lemma now follows. \qed

6.4. Bounding the average analytic rank of elliptic curves.

**Theorem [1.1.1]** Let $K$ be a number field of degree $d$. Assume that all elliptic curves over $K$ are modular and that their $L$-functions satisfy the Riemann-Hypothesis. Then the average analytic rank of elliptic curves over $K$ is bounded above by $3d + 1/2$.

**Proof.** By Lemma 6.3.1 and the observation that $\hat{\phi}$ is supported on the interval $[-\nu, \nu]$, we have that

$$S_1(\phi, B) = \frac{2}{\log(B)\#\mathcal{E}_K(B)} \sum_{v \in \text{Val}(K)} \frac{\log(q_v)}{q_v} \cdot \hat{\phi} \left( \frac{\log(q_v)}{\log(X)} \right) \sum_{E \in \mathcal{E}_K(B)} \widehat{a}_E(p_v)$$

$$\ll \frac{2}{\log(B)\#\mathcal{E}_K(B)} \sum_{v \in \text{Val}(K)} \frac{\log(q_v)}{q_v} \cdot \hat{\phi} \left( \frac{\log(q_v)}{\log(X)} \right) \left( \frac{1}{q_v} B^{5/6} + q_v B^{5/6 - 1/3d} \right)$$

$$\ll \frac{2}{\log(B)\#\mathcal{E}_K(B)} \sum_{v \in \text{Val}(K)} \frac{\log(q_v)}{q_v} \left( \frac{1}{q_v} B^{5/6} + q_v B^{5/6 - 1/3d} \right).$$
By Corollary 5.1.2 we know that \( \# \mathcal{E}_K(B) = \kappa B^{5/6} + O(B^{5/6-1/3d}) \). Using this and the prime number theorem, we have that

\[
S_1(\phi, B) \ll \frac{2}{\log(B)} \sum_{\nu \in \text{Valo}(K)} \frac{\log(q_v)}{q_v} \left( \frac{1}{q_v} + q_v B^{-1/3d} \right)
\]

\[
\ll \sum_{\nu \in \text{Valo}(K)} \left( \frac{\log(q_v)}{q_v^2} + \log(q_v) B^{-1/3d} \right)
\]

\[
\ll B^{-\nu} + B^{\nu-1/3d}.
\]

In particular, we have shown that

\[
S_1(\phi, B) \ll B^{\nu-1/3d}.
\]

(6.4)

By Lemma 6.3.2 we have that

\[
S_2(\phi, B) = \frac{2}{\log(B) \# \mathcal{E}_K(B)} \sum_{\nu \in \text{Valo}(K)} \frac{\log(q_v)}{q_v^2} \cdot \hat{\phi} \left( \frac{k \log(q_v)}{\log(X)} \right) \sum_{E \in \mathcal{E}_K(B)} \hat{a}_E(p_v^2)
\]

\[
= \frac{2}{\log(B) \# \mathcal{E}_K(B)} \sum_{\nu \in \text{Valo}(K)} \frac{\log(q_v)}{q_v^2} \cdot \hat{\phi} \left( \frac{k \log(q_v)}{\log(X)} \right) \left( -q_v + O(1 + q_v B^{-1/3d}) \right).
\]

Using \( \# \mathcal{E}_K(B) = \kappa B^{5/6} + O(B^{5/6-1/3d}) \) and the prime number theorem, we have that

\[
S_2(\phi, B) = \frac{2}{\log(B)} \sum_{\nu \in \text{Valo}(K)} \frac{\log(q_v)}{q_v} \cdot \hat{\phi} \left( \frac{k \log(q_v)}{\log(X)} \right) \left( 1 + q_v B^{-1/3d} \right)
\]

\[
= -\frac{2}{\log(B)} \sum_{\nu \in \text{Valo}(K)} \frac{\log(q_v)}{q_v} \cdot \hat{\phi} \left( \frac{k \log(q_v)}{\log(X)} \right) + O \left( \sum_{\nu \in \text{Valo}(K)} \left( \frac{\log(q_v)}{q_v^2} + \frac{\log(q_v)}{q_v} B^{-1/3d} \right) \right)
\]

\[
= -\frac{\phi(0)}{2} + O \left( B^{-\nu/2} + B^{-1/3d} \right).
\]

Taking \( \nu = 1/3d \) and combining the above estimate for \( S_2 \) with our bound (6.4) for \( S_1 \), we obtain

\[
-S_1(\phi, B) - S_2(\phi, B) = \frac{\phi(0)}{2} + o(1).
\]

Substituting this into (6.3) and taking the limit as \( B \) goes to infinity gives the following bound for the average analytic ranks of elliptic curves:

\[
\frac{1}{\nu} + \frac{1}{2} = 3d + \frac{1}{2}.
\]


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