AN $H^{s,p}(\text{curl}; \Omega)$ ESTIMATE FOR THE MAXWELL SYSTEM

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ABSTRACT. We derive an $H^{s,p}_0(\text{curl}; \Omega)$ estimate for the solutions of the Maxwell type equations modeled with anisotropic and $W^{s,\infty}(\Omega)$-regular coefficients. Here, we obtain the regularity of the solutions for the integrability and smoothness indices $(p, s)$ in a plan domain characterized by the apriori lower/upper bounds of $a$ and the apriori upper bound of its Hölder semi-norm of order $s$. The proof relies on a perturbation argument generalizing Gröger’s $L^p$-type estimate, known for the elliptic problems, to the Maxwell system.

1. Introduction

Assume $\Omega \subset \mathbb{R}^3$ to be a bounded and Lipschitz domain. Let the coefficient $a$ be a $3 \times 3$ matrix, with elements in $W^{s,\infty}(\Omega)$, $s \geq 0$, satisfying the uniform ellipticity condition, i.e., there exist positive constants $m, M$ such that

$$m|\xi|^2 \leq a(x)\xi \cdot \bar{\xi} \leq M|\xi|^2,$$

for all $\xi \in \mathbb{C}^3$ and almost every $x \in \Omega$ and having, if $s > 0$, a bounded Hölder semi-norm with exponent $s$, i.e., there exists $\bar{M} > 0$ such that

$$|a|_{C^{0,s}} \leq \bar{M}.$$

The goal of this work is to study the well posedness of the following boundary value problem

$$\begin{cases}
\text{curl}(a \text{ curl } u) + k^2 u = f, \text{ in } \Omega \\
\nu \wedge u = 0, \text{ on } \partial \Omega
\end{cases}$$

in the appropriate Sobolev spaces with fractional order. Here, we denote by $\nu$ the outer unit normal on $\partial \Omega$ and $k$ the frequency. The problem (1.3) covers the case when the electric field $E$ satisfies

$$\begin{cases}
\text{curl}(\mu^{-1} \text{ curl } E) + k^2 E = f, \text{ in } \Omega \\
\nu \wedge E = 0, \text{ on } \partial \Omega
\end{cases}$$

or the magnetic field $H$ satisfies

$$\begin{cases}
\text{curl}(\epsilon^{-1} \text{ curl } H) + k^2 H = f, \text{ in } \Omega \\
\nu \wedge H = 0, \text{ on } \partial \Omega
\end{cases}$$

with anisotropic permitivity $\epsilon$ and permiability $\mu$.

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1 One way to deal with the general case where $k^2$ is replaced by $b \in (L^\infty(\Omega))^3$ (for example in (1.4) replace $k^2$ by $b(x) := -k^2 \epsilon^{-1}(x)$, $x \in \Omega$, with $\epsilon \in (L^\infty(\Omega))^3$ lower bounded by a positive constant) is discussed in Remark 3.1.
The well posedness of the problem (1.3) has been derived in the $L^2$-based Sobolev spaces, i.e., $H(\text{curl}; \Omega)$, for domains with minimal smoothness and minimum regularity assumptions on the coefficients, see for instance [13].

In order to study the question of regularity for the inhomogeneous Dirichlet-Laplace problem, in [8], Jerison and Kenig used harmonic analysis technique to obtain a best possible estimates for the solutions in Sobolev-Besov $L^p_s(\Omega)$ norms with optimal range of the smoothness index $s$ and the integrability index $p$. In [11], M. Mitrea, D. Mitrea and J. Pipher considered an inhomogeneous Maxwell equations in a Lipschitz sub-domain, where the electric permittivity $\epsilon$ and the magnetic permeability $\mu$ taken to be constants. The regularity estimate for the solutions of (1.3), developed in the $L^p$ settings for the optimal values of $p$’s, can be found in [11]. In [12], M. Mitrea showed the well posedness in the Sobolev-Besov spaces $H_0^{s,p}(\text{curl}; \Omega)$ with the optimal range of the smoothness index $s$ and the integrability index $p$ which generalizes, to the Maxwell system, the results by Jerison and Kenig mentioned above.

Regarding variable coefficients and under weak regularity assumptions on the domain $\Omega$ and only $L^\infty$-regularity assumption of the coefficients, the well posedness for the divergence form elliptic problems has been studied by Gröger. In [7], he demonstrated the well posedness in the Sobolev space $W^{1,p}(\Omega)$ for $p > 2$, which is a generalization of the work of Meyers [10], known for Dirichlet boundary conditions, to mixed type boundary conditions. The proof is based on a perturbation argument via the Banach fixed point theorem, see [7]. We refer the reader to the text books [4, 5] for $L^p$-estimates of the solutions of elliptic problems in case of smooth coefficients. For the Maxwell model, a $W^{1,p}$-type regularity estimate for the solutions has been derived by Bao, Li, and Zhou considering $\mu$ to be constant and $\epsilon$ as piecewise constant, see [2]. Related estimates for smooth coefficients are derived by Yin in [18], see also the references therein. In the recent work [9], we proved an estimate of the solutions for the problem in the Sobolev spaces $H_0^{1,p}(\text{curl}; \Omega)$, for $p$ near 2, where $\alpha$ is taken to be a matrix valued function satisfying (1.1).

In this work, we use the approach by Gröger to deal with the regularity issue regarding the model (1.3) in the spaces $H_0^{s,p}(\text{curl}; \Omega)$ for $W^{s,\infty}(\Omega)$ coefficients for a certain range of $s$ and $p$, see Theorem 3.1 and Figure 1. This completes the work in [9] and provides the Gröger-Meyers’s regularity estimate corresponding to the formentioned results in [12]. Let us also mention that compared to the works in [2] and [18], our estimates are derived for less regular coefficients (for instance for $s = 0$) and show that the solution operator for the model (1.3) is an isomorphism. In addition to the general interest of such regularity estimates, this isomorphism property is useful for justifying a shape reconstruction algorithm in the theory of inverse problems, see [9, 15].

The paper is organized as follows. In Section 2 we recall the basic definitions of the Sobolev and Besov spaces of functions in Lipschitz domains and also some functional properties on those spaces. Then we state the main result in this paper in Section 3 and finally, a detailed proof of the result is given in Section 4.
2. Definitions and preliminary results

2.1. Sobolev and Besov spaces in Lipschitz domains. For $1 < p < \infty$ and $-\infty < s < \infty$ the Sobolev space $L^p_s(\mathbb{R}^3)$ is defined by\(^2\)

$$L^p_s(\mathbb{R}^3) := \{(I - \Delta)^{-\frac{s}{2}} g; g \in L^p(\mathbb{R}^3)\},$$

with the norm

$$\|f\|_{L^p_s(\mathbb{R}^3)} = \|(I - \Delta)^{-\frac{s}{2}} f\|_{L^p(\mathbb{R}^3)}.$$ 

For any Lipschitz domain $\Omega \subset \mathbb{R}^3$, we denote by $L^p_s(\Omega)$ the Sobolev space defined as the restrictions to $\Omega$ of the elements in $L^p_s(\mathbb{R}^3)$. The norm is defined as follows:

$$\|f\|_{L^p_s(\Omega)} := \inf\{\|g\|_{L^p_s(\mathbb{R}^3)}; \mathcal{R}_\Omega g = f\},$$

where $\mathcal{R}_\Omega g$ denotes the restriction of the function $g$ from $\mathbb{R}^3$ to $\Omega$. Moreover, if we define the space $W^{m,p}(\Omega)$ by

$$W^{m,p}(\Omega) := \{f \in L^p(\Omega); \frac{\partial^\beta f}{\partial x^\beta} \in L^p(\Omega), |\beta| \leq m\},$$

equipped with the norm

$$\|f\|_{W^{m,p}(\Omega)} := \left(\sum_{|\beta| \leq m} \int_{\Omega} |\frac{\partial^\beta f}{\partial x^\beta}|^p dx\right)^{1/p},$$

for $m \geq 1$, an integer, and $1 < p < \infty$, then we have the equality

$$L^p_m(\Omega) = W^{m,p}(\Omega),$$

see for instance [3] and [8], where as usual $\frac{\partial^\beta f}{\partial x^\beta} := \frac{\partial^{|\beta|} f}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \partial x_3^{\beta_3}}$ with $|\beta| = \beta_1 + \beta_2 + \beta_3$. Using Stein’s extension operator, the space $L^p_s(\Omega)$, $0 < s < 1$, can be interpreted as the complex interpolation space between $L^p(\Omega)$ and $W^{1,p}(\Omega)$, i.e.,

$$[L^p(\Omega), W^{1,p}(\Omega)]_{[s]} = L^p_s(\Omega),$$

for all $1 < p < \infty$. For $p = \infty$ and $0 < s < 1$, the space $W^{s,\infty}(\Omega)$ can be viewed as the space of functions

$$\left\{\varphi \in L^\infty(\Omega) : \frac{|\varphi(x) - \varphi(y)|}{|x - y|^s} \in L^\infty(\Omega \times \Omega)\right\}.$$ 

Basically this space is equivalent to the Hölder continuous space $C^{0,s}(\Omega)$ with exponent $s$ and the norm can be defined as

$$\|\varphi\|_{W^{s,\infty}(\Omega)} = \|\varphi\|_{L^\infty(\Omega)} + |\varphi|_{C^{0,s}},$$

where the Hölder semi-norm is denoted by

$$|\varphi|_{C^{0,s}} := \sup_{x,y \in \Omega} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^s},$$

\(^2\)The space $L^p_s(\mathbb{R}^3)$ can also be defined using the Fourier transform $L^p_s(\mathbb{R}^3) := \{f; f \in \mathcal{S}', \|\hat{f}\|_s^p < \infty\}$, with the norm $\|f\|_s^p = \|\mathcal{F}^{-1}\{1 + |\xi|^2\}^\frac{s}{2} \mathcal{F} f\|_{L^p(\mathbb{R}^3)}$, where $s \in \mathbb{R}$. Here, $\mathcal{F}$ and $\mathcal{F}^{-1}$ represent the Fourier transform and inverse Fourier transform respectively and $\mathcal{S}'$ represents the space of tempered distributions.

\(^3\)A detailed discussion about this space can be found in [1] and [3], for instance.
According to [8], for $1 < p < \infty$ and $s \in \mathbb{R}$, we define $L^p_s(\Omega)$ as the space of all distributions $f \in L^p(\mathbb{R}^3)$ such that $\text{supp} f \subset \overline{\Omega}$ and the norm is

$$\|f\|_{L^p_s(\Omega)} := \|f\|_{L^p(\mathbb{R}^3)}.$$

It is known that $C_0^\infty(\Omega)$ is dense in $L^p_{s,0}(\Omega)$ for all values of $s$ and $p$ with $p > 1$. For positive $s$, $L^p_{s,0}(\Omega)$ is defined as the space of distributions in $\Omega$ such that

$$\|f\|_{L^p_{s,0}(\Omega)} := \sup\{\|\langle f, \varphi \rangle\| : \varphi \in C_0^\infty(\Omega), \|\varphi\|_{L^q(\mathbb{R}^3)} \leq 1\} < \infty,$$

where tilde denotes the extension by zero outside $\Omega$ and $1/p + 1/q = 1$. For all values of $p$ and $s$, $C^\infty(\overline{\Omega})$ is dense in $L^p_s(\Omega)$. Also, $C^\infty_0(\Omega)$ is dense in $L^p_s(\Omega)$, for $s \leq 0$. In addition, for any $s \in \mathbb{R}$,

$$L^p_{s,0}(\Omega) = (L^p_s(\Omega))^\prime \quad \text{and} \quad L^q_{s,0}(\Omega) = (L^q_s(\Omega))^\prime,$$

see for instance in [8], Proposition 2.4, Proposition 2.9] and [12]. For each $p$ and $s$ satisfying $1 < p < \infty$, $-1 + 1/p < s < 1/p$, there exists a linear and bounded extension operator

$$E_{\text{ext}} : L^p_s(\Omega) \to L^p_s(\mathbb{R}^3),$$

by

$$E_{\text{ext}}(u) = \tilde{u},$$

with the property that $\text{supp} \tilde{u} \subset \overline{\Omega}$, see [17], Theorem 3.5. In addition, $\text{Range}(E_{\text{ext}}) = L^p_{s,0}(\Omega)$, which allows the following identification

$$L^p_s(\Omega) = L^p_{s,0}(\Omega); \quad \forall \ p \in (1, \infty), \ \forall \ s \in (-1 + 1/p, 1/p).$$

Thus, if $p, q \in (1, \infty)$ are such that $1/p + 1/q = 1$, then

$$(L^p_s(\Omega))^\prime = L^q_s(\Omega), \quad \forall \ s \in (-1 + 1/p, 1/p),$$

and hence $L^p_s(\Omega)$ is reflexive. Note that the product space $L^p_s(\Omega) \times L^q_s(\Omega)$ is a Banach space with the usual graph norm as well as with the equivalent norm

$$\|(f_1, f_2)\|_{L^p_s(\Omega) \times L^q_s(\Omega)} = \left(\|f_1\|_{L^p_s(\Omega)}^p + \|f_2\|_{L^q_s(\Omega)}^q\right)^{1/p}.$$

In the following lemma we discuss the characterization of the dual of $L^p_s(\Omega) \times L^q_s(\Omega)$ with suitable $p$ and $s$.

**Lemma 2.1.** Let $1 < p, q < \infty$ be real numbers with $1/p + 1/q = 1$ and $s$ be such that $-1 + 1/p < s < 1/p$. Assume that $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Then for every fixed $g = (g_1, g_2) \in L^p_s(\Omega) \times L^q_s(\Omega)$, the operator $\mathcal{F}^*$ defined by

$$\mathcal{F}^* f := L^p_s(\mathbb{R}^3) \langle \tilde{f}_1, \hat{f}_1 \rangle L^q_s(\mathbb{R}^3) + L^q_s(\mathbb{R}^3) \langle \tilde{g}_2, \hat{f}_2 \rangle L^p_s(\mathbb{R}^3),$$

is a continuous linear functional on $L^p_{s,a}(\Omega) \times L^q_{s,a}(\Omega)$, where $\tilde{f}_j \in L^p_{s,a}(\mathbb{R}^3)$ is any extension of $f_j$ satisfying $\mathcal{R}_\Omega \tilde{f}_j = f_j$ and $\hat{g}_j$ is an extension of $g_j$ by zero outside $\Omega$, for $j = 1, 2$.

Conversely, every $\mathcal{F}^* \in (L^p_{s,a}(\Omega) \times L^q_{s,a}(\Omega))^\prime$ can be written in the above form with a uniquely determined $g \in L^p_s(\Omega) \times L^q_s(\Omega)$. Moreover, there exist $c_1, c_2 > 0$ such that

$$c_1 \|g\|_{L^p_s(\Omega) \times L^q_s(\Omega)} \leq \|\mathcal{F}^*\|_{(L^p_{s,a}(\Omega) \times L^q_{s,a}(\Omega))^\prime} \leq c_2 \|g\|_{L^p_s(\Omega) \times L^q_s(\Omega)}.$$
Proof. Applying the H"older inequality we have the continuity of $F^*$ together with the estimate
\[ \|F^*\|_{(L^q_s(\Omega) \times L^q_s(\Omega))^\prime} \leq c_2 \| g \|_{L^p_s(\Omega) \times L^p_s(\Omega)}. \]
Conversely, for $j = 1, 2$, we define
\[ T_j : L^q_s(\Omega) \to L^q_s(\Omega) \times L^q_s(\Omega) \]
by
\[ T_1 f_1 := (f_1, 0) \]
and
\[ T_2 f_2 := (0, f_2), \]
for all $f_1, f_2 \in L^q_s(\Omega)$. Note that $T_j$ is a continuous and linear operator. Since $F^* \in (L^q_s(\Omega) \times L^q_s(\Omega))^\prime$, then for $f = (f_1, f_2) \in L^q_s(\Omega) \times L^q_s(\Omega)$, we have
\[ F^* f = (F^* \circ T_1) f_1 + (F^* \circ T_2) f_2. \]
Since $T_j$ and $F^*$ are linear and continuous then $F^* \circ T_j \in (L^q_s(\Omega))^\prime$, for $j = 1, 2$. Note that, from (2.1) we have the characterization of the dual space $(L^q_s(\Omega))^\prime$, i.e.,
\[ (L^q_s(\Omega))^\prime = L^p_s(\Omega), \]
for all $-1 + 1/p < s < 1/p$. Therefore, there exists a unique $g_j \in L^p_s(\Omega)$ such that
\[ (F^* \circ T_j) (f_j) = L^p_s(\mathbb{R}^3) (\tilde{g}_j, \tilde{f}_j) L^q_s(\mathbb{R}^3), \]
where $\tilde{f}_j \in L^q_s(\mathbb{R}^3)$ is any extension of $f_j$ satisfying $\mathcal{R}_\Omega \tilde{f}_j = f_j$ and $\tilde{g}$ is an extension of $g$ by zero outside $\Omega$ for $j = 1, 2$. Now, define $g := (g_1, g_2) \in L^p_s(\Omega) \times L^p_s(\Omega)$. Therefore,
\[ F^* f = L^p_s(\mathbb{R}^3) (\tilde{g}_1, \tilde{f}_1) L^q_s(\mathbb{R}^3) + L^p_s(\mathbb{R}^3) (\tilde{g}_2, \tilde{f}_2) L^q_s(\mathbb{R}^3). \]
Thus the natural mapping
\[ A : L^p_s(\Omega) \times L^p_s(\Omega) \to (L^q_s(\Omega) \times L^q_s(\Omega))^\prime, \]
defined by
\[ (g_1, g_2) \mapsto F^*, \]
is bounded linear and bijective. Hence by the open mapping theorem we conclude that $A$ is an isomorphism, i.e., there exist $c_1, c_2 > 0$ such that
\[ c_1 \| g \|_{L^p_s(\Omega) \times L^p_s(\Omega)} \leq \| F^* \|_{(L^q_s(\Omega) \times L^q_s(\Omega))^\prime} \leq c_2 \| g \|_{L^p_s(\Omega) \times L^p_s(\Omega)}. \]
\[ \square \]

The following lemma describes the interpolation spaces by applying the complex interpolation method.

Lemma 2.2. We have the following characterization
\[ L^p_s(\Omega) = \| [L^{p_0}_{s_0}(\Omega), L^{p_1}_{s_1}(\Omega)]_\theta \|, \]
where $s = (1 - \theta)s_0 + \theta s_1$, $\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}$ and $s_0 \neq s_1$, $s_0, s_1 \in \mathbb{R}$, $1 < p_0, p_1 < \infty$ with $0 < \theta < 1$.

Proof. See [3, Theorem 6.4.5]. \[ \square \]
As in [12], for $\Omega \subset \mathbb{R}^3$ an open set and if $1 < p < \infty$ and $s \in \mathbb{R}$, we introduce the space

$$H^{s,p}(\text{curl}; \Omega) := \{ u \in L^p_s(\Omega); \text{ curl } u \in L^p_s(\Omega) \},$$

equipped with the natural graph norm

$$\| u \|_{H^{s,p}(\text{curl}; \Omega)} = \| u \|_{L^p_s(\Omega)} + \| \text{ curl } u \|_{L^p_s(\Omega)}.$$  \hspace{1cm} (2.2)

An equivalent norm to (2.2) is given by

$$\| u \|_{H^{s,p}(\text{curl}; \Omega)} = \left( \| u \|_{L^p_s(\Omega)}^p + \| \text{ curl } u \|_{L^p_s(\Omega)}^p \right)^{1/p}. \hspace{1cm}$$

Under the second norm $H^{s,p}(\text{curl}; \Omega)$ is a Banach space. To define the tangential trace along the boundary we need to discuss about the Besov spaces on the boundary. Following [12], we denote by $L^p_1(\partial \Omega)$ the Sobolev space of functions in $L^p(\partial \Omega)$ with tangential gradients $\nabla_{\text{tan}}$ ($\nabla_{\text{tan}} := -\nu \wedge (\nu \wedge \nabla)$) in $L^p(\partial \Omega)$, for $1 < p < \infty$. Spaces with fractional smoothness can then be defined via complex interpolation, i.e.,

$$L^p_\theta(\partial \Omega) := [L^p(\partial \Omega), L^p_1(\partial \Omega)]_{\theta/q}, \quad 0 < \theta < 1, \quad 1 < p < \infty.$$

We also set

$$L^p_{-s}(\partial \Omega) := (L^p_s(\partial \Omega))',$$

for all $0 \leq s \leq 1, 1 < p, q < \infty, 1/p + 1/q = 1$. On $\partial \Omega$, the Besov spaces can then be introduced via real interpolation\footnote{More details about this space can be found in [3] and [1], Chapter 7.}, i.e.,

$$B^{p,q}_s(\partial \Omega) := (L^p_\theta(\partial \Omega), L^p_1(\partial \Omega))_{s,q}, \quad \text{with } 0 < s < 1, 1 < p, q < \infty.$$

Also, for $-1 < s < 0$ and $1 < p, q < \infty$, we set

$$B^{p,q}_s(\partial \Omega) := (B^{-q}_{p/1}(\partial \Omega))', \quad 1/p + 1/q = 1, \quad 1/p' + 1/q' = 1.$$

Now, if $u \in H^{s,p}(\text{curl}; \Omega)$ for some $p, s$ satisfying $1 < p < \infty$ and $-1 + 1/p < s < 1/p$ then we can define the tangential trace $\nu \wedge u \in B^{p,p}_{s-1/p}(\partial \Omega)$ by

$$\langle \nu \wedge u, \text{Tr} \varphi \rangle := \int_\Omega [(\text{curl } u, \varphi) - \langle u, \text{curl } \varphi \rangle] \, dx,$$

for any $\varphi \in L^q_{-s}(\Omega), 1/p + 1/q = 1$. Therefore, the space $H^{s,p}_0(\text{curl}; \Omega)$ can be interpreted as

$$H^{s,p}_0(\text{curl}; \Omega) := \{ u \in H^{s,p}(\text{curl}; \Omega); \nu \wedge u = 0 \text{ on } \partial \Omega \}.$$

**Lemma 2.3.** The space $H^{s,p}_0(\text{curl}; \Omega)$ is reflexive, for all $1 < p < \infty$ and $s \in (-1+1/p, 1/p)$. 

**Proof.** Define

$$\mathcal{F} : H^{s,p}_0(\text{curl}; \Omega) \to L^p_s(\Omega) \times L^p_s(\Omega)$$

by

$$\mathcal{F} u := (u, \text{curl } u).$$

The operator $\mathcal{F}$ is bounded linear and isometric. We set $\mathcal{W}$ to be the range of the operator $\mathcal{F}$, which is a closed subspace of $L^p_s(\Omega) \times L^p_s(\Omega)$. Notice that, the operator

$$\mathcal{F} : H^{s,p}_0(\text{curl}; \Omega) \to \mathcal{W}$$
and its inverse are isometrically isomorphism. Since the closed subspace of a reflexive space is reflexive and the isometric isomorphism preserves reflexivity between the spaces, then the proof of the lemma will follow if we show that \( L^p_s(\Omega) \times L^p_s(\Omega) \) is reflexive.

For \( h \in L^p_s(\Omega) \times L^p_s(\Omega) \), we set
\[
J_h(F^*) := (L^p_s(\Omega) \times L^p_s(\Omega))^* \langle F^*, h \rangle_{L^p_s(\Omega) \times L^p_s(\Omega)}.
\]
Let us define the usual canonical mapping
\[
J : L^p_s(\Omega) \times L^p_s(\Omega) \to (L^p_s(\Omega) \times L^p_s(\Omega))^*
\]
by \( J(h) = J_h \). Notice that \( J \) is an isometry, hence to show that the Banach space \( L^p_s(\Omega) \times L^p_s(\Omega) \) is reflexive, we need to prove that the canonical embedding \( J \) is surjective.

Given \( g \in L^q_{-s}(\Omega) \times L^q_{-s}(\Omega) \), we define
\[
\tau_q : L^q_{-s}(\Omega) \times L^q_{-s}(\Omega) \to (L^p_s(\Omega) \times L^p_s(\Omega))^*
\]
by
\[
(\tau_q g)(f) := L^q_{-s}(\Omega) \times L^q_{-s}(\Omega) \langle g, f \rangle_{L^p_s(\Omega) \times L^p_s(\Omega)}
\]
for all \( f \in L^p_s(\Omega) \times L^p_s(\Omega) \), where \( p, q, s \) satisfy \( 1 < p, q < \infty \), \( 1/p + 1/q = 1 \) and \(-1 + 1/p < s < 1/p\). Then by Lemma 2.1, the operator \( \tau_q \) is an isomorphism.

Let us take \( F^{**} \in (L^p_s(\Omega) \times L^p_s(\Omega))^\prime\prime \). Also we take \( \tau_q g \) as \( F^* \). Then \( F^{**} F^* = F^{**} \tau_q g \), i.e.,
\[
F^{**} \tau_q : L^q_{-s}(\Omega) \times L^q_{-s}(\Omega) \to \mathbb{R}
\]
is a continuous linear map. Hence, Lemma 2.1 implies that there exists a unique \( h \in L^p_s(\Omega) \times L^p_s(\Omega) \) such that
\[
(F^{**} \tau_q)(g) = L^p_s(\Omega) \times L^p_s(\Omega) \langle h, g \rangle_{L^q_{-s}(\Omega) \times L^q_{-s}(\Omega)}.
\]
Therefore,
\[
F^{**} F^* = (F^* \tau_q)(g)
\]
\[
= L^p_s(\Omega) \times L^p_s(\Omega) \langle h, g \rangle_{L^q_{-s}(\Omega) \times L^q_{-s}(\Omega)}
\]
\[
= L^q_{-s}(\Omega) \times L^q_{-s}(\Omega) \langle g, h \rangle_{L^p_s(\Omega) \times L^p_s(\Omega)}
\]
\[
= (\tau_q g)(h)
\]
\[
= (L^p_s(\Omega) \times L^p_s(\Omega))^* \langle F^*, h \rangle_{L^q_s(\Omega) \times L^q_s(\Omega)}
\]
\[
= J_h(F^*)
\]
i.e., \( F^{**} = J_h = J(h) \). Hence \( J \) is surjective. 

We finish this section with the following lemma where we state a Kato-Ponce type inequality.

**Lemma 2.4.** Assume that \( f \in L^p_s(\Omega) \) and \( g \in W^{s,\infty}(\Omega) \), then \( fg \in L^p_s(\Omega) \) with the following estimate
\[
\|fg\|_{L^p_s(\Omega)} \leq C \|f\|_{L^p_s(\Omega)} \|g\|_{W^{s,\infty}(\Omega)},
\]
where \( C = C(s, p) > 0 \) for all \( 1 < p < \infty \) and \( s \geq 0 \).
Proof. For $s = 0$, the proof is trivial and $C = 1$. So, we consider the case $s > 0$. Let us first extend the functions $f$ and $g$ by zero outside $\Omega$. Following [16], we define the Bessel potentials $\mathcal{J}_s$ by $\mathcal{J}_s = (I - \Delta)^{-s/2}$ for $s > 0$ and recall that,

$$L^p_s(\mathbb{R}^3) = \mathcal{J}_s(L^p(\mathbb{R}^3)), \quad 1 \leq p \leq \infty, \quad s \geq 0.$$ 

Hence,

$$\|fg\|_{L^p_s(\Omega)} = \|fg\|_{L^p_s(\mathbb{R}^3)} = \|(I - \Delta)^{s/2}(fg)\|_{L^p(\mathbb{R}^3)}.$$ 

Now we recall the Kato-Ponce inequality, see for instance in [3], as

$$\|J^s(fg)\|_{L^p(\mathbb{R}^3)} \leq C\|f\|_{L^p(\mathbb{R}^3)}\|J^s g\|_{L^\infty(\mathbb{R}^3)} + \|J^s f\|_{L^p(\mathbb{R}^3)}\|g\|_{L^\infty(\mathbb{R}^3)},$$

where $s > 0$, $1 < p < \infty$ and $J^s := (I - \Delta)^{s/2}$ with the constant $C = C(s,p) > 0$. So, we obtain

$$\|fg\|_{L^p_s(\Omega)} \leq C\|f\|_{L^p_s(\mathbb{R}^3)}\|J^s g\|_{L^\infty(\mathbb{R}^3)} + \|J^s f\|_{L^p_s(\mathbb{R}^3)}\|g\|_{L^\infty(\mathbb{R}^3)}.$$ 

Since, $J^s$ is an isomorphism between $L^p_s(\mathbb{R}^3)$ and $L^p(\mathbb{R}^3)$ for $1 \leq p \leq \infty$ and $s \in \mathbb{R}$, see [3, Theorem 6.2.7], then we have

$$\|J^s g\|_{L^\infty(\mathbb{R}^3)} \leq C\|g\|_{W^{s,\infty}(\mathbb{R}^3)},$$

where $C = C(s) > 0$. Also note that, for $s > 0$, $L^p_s(\mathbb{R}^3)$ is a subspace of $L^p(\mathbb{R}^3)$, i.e., for any $f \in L^p_s(\mathbb{R}^3)$ we have

$$\|f\|_{L^p(\mathbb{R}^3)} \leq C\|f\|_{L^p_s(\mathbb{R}^3)}.$$ 

Combining (2.3), (2.4) and (2.5), we obtain

$$\|fg\|_{L^p_s(\Omega)} \leq C\|f\|_{L^p_s(\mathbb{R}^3)}\|g\|_{W^{s,\infty}(\mathbb{R}^3)},$$

i.e.,

$$\|fg\|_{L^p_s(\Omega)} \leq C\|f\|_{L^p_s(\Omega)}\|g\|_{W^{s,\infty}(\Omega)},$$

where $C = C(s,p) > 0$. \hfill \Box

3. Main result

We start the section by defining a region $R_\Omega$ as follows:

$$(s, 1/p) \in R_\Omega \iff \begin{cases} 0 < \frac{1}{p} < 1, \\ -1 + \frac{1}{p} < s < \frac{1}{p}, \\ \frac{2}{3}(1 - \frac{1}{p_0}) < \frac{1}{p} - \frac{s}{3} < \frac{1}{3}(\frac{2}{p_0} + 1). \end{cases}$$

Remark that, $R_\Omega$ can be determined by the geometric character of the domain $\Omega$. Here, $p_\Omega$ is the Hölder conjugate exponent of $q_\Omega$ and $q_\Omega$ is the supremum of all $q$ so that the Dirichlet and Neumann problem for the Laplace-Beltrami operator in $\Omega$ is well-posed in $W^{1,q}$ spaces. However, $1 \leq p_\Omega < 2$ when $\partial \Omega$ is Lipschitz regular and $p_\Omega = 1$ when $\partial \Omega \in C^1$. A more detailed explanation can be found in [12]. In the next sections we use the notations $R^+_\Omega$ and $R^-_\Omega$ given by

$$R^+_\Omega := R_\Omega \cap \{(s, 1/p); s \geq 0, 1 < p < \infty\}$$

and

$$R^-_\Omega := R_\Omega \cap \{(s, 1/p); s < 0, 1 < p < \infty\}.$$
Now, we are in a position to state the following theorem as our main result.

**Theorem 3.1.** Let $\Omega$ be a bounded and Lipschitz domain in $\mathbb{R}^3$. Let the coefficient $a$ be a $3 \times 3$ symmetric matrix, with elements in $W^{s,\infty}(\Omega)$, satisfying the uniform ellipticity condition, i.e., there exist positive constants $m, M$ such that
\begin{equation}
(3.1) \quad m|\xi|^2 \leq a(x)\xi \cdot \xi \leq M|\xi|^2,
\end{equation}
for all $\xi \in \mathbb{C}^3$ and almost every $x \in \Omega$ and having, if $s > 0$, a bounded Hölder semi-norm with exponent $s$, i.e., there exists $\bar{M} > 0$ such that\[5\]
\begin{equation}
(3.2) \quad |a|_{C^{0,s}} \leq \bar{M}.
\end{equation}

Then for any $f \in (H_0^{-s,q}(\text{curl};\Omega))'$, the following problem
\begin{equation}
(3.3) \quad \begin{cases}
\text{curl}(a(x)\text{curl} u) + k^2 u = f, & \text{in } \Omega \\
\nu \wedge u = 0, & \text{on } \partial \Omega
\end{cases}
\end{equation}
has one and unique solution in $H_0^s(p)(\text{curl};\Omega)$ for all $(s,1/p) \in S := S^+ \cup S^-$ where
\begin{equation}
(3.4) \quad S^+ := \bigcup_{(s_0,1/p_0) \in R_0^+} \left\{ \left( s, \frac{1}{p} \right) \in R_0^+ : \begin{array}{ll}
s = (1-\theta)s_0, \\
\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{2}, \end{array} \text{ where } \theta \in (0,1) \text{ is such that} \\
(1-\theta)\log \mathcal{M}_{s_0,p_0} + \log k_0(s,p) < 0. \right\}
\end{equation}

with
\begin{equation}
(3.5) \quad k_0(s,p) := \max\{|1 - \frac{mk^2}{M^2}|, C(s,p)|1 - \frac{m^2}{M^2} + \frac{m\bar{M}}{M^2}\| < 1.
\end{equation}

Here $\mathcal{K}_{s_0,p_0}u = \text{curl} \text{curl} u + u$ and $C(s,p)$ is the constant appearing in the Kato-Ponce inequality in Lemma 2.4. The region $S^-$ is given by
\begin{equation}
(3.6) \quad S^- := \{(s,1/p) \in R_0^+ : (-s,1/q) \in S^+\}.
\end{equation}

In addition, the solution satisfies the following estimate
\begin{equation}
\|u\|_{L^p(\Omega)} + \| \text{curl} u\|_{L^q(\Omega)} \leq C\|f\|_{(H_0^{-s,q}(\text{curl};\Omega))'}.
\end{equation}

In Figure 1, considered in the $(s,1/p)$-plane (i.e., smoothness vs reciprocal integrability), the dashed area represents the well-posedness region for the Maxwell problem (3.3). We first fix $(s_0,1/p_0) \in R_0^+$. The property
\begin{equation}
(3.7) \quad (1-\theta)\log \mathcal{M}_{s_0,p_0} + \log k_0(s,p) < 0
\end{equation}
says that the points $(s,1/p)$ will be laying on some part of this straight line joining $(0,1/2)$ and $(s_0,1/p_0)$. Now, if we take any other point $(s_0,1/p_0) \in R_0^+$ and use the same argument, then we end up with the positive $s$-part of dashed region $S$ i.e., $S^+$, in Figure 1 where $\mathcal{M}_{s,p}k_0^{3/2}(s,p) < 1$. The well-posedness region $S^-$ can be obtained by using duality argument on the Maxwell operator and the symmetry of the matrix $a$.\[5\]In the case $s = 0$, this condition is not needed. In all subsequent estimates, we can replace $\bar{M}$ by 0 in this case.
Theorem 3.1 is proved in two steps. In the first step, we deal with the unperturbed problem and in the second step, we discuss the perturbed problem. In the unperturbed case, we consider the coefficient $a$ to be the identity matrix $I$. Then the system (3.3) reduces to the well known time harmonic Maxwell model with constant permittivity and permeability and the regularity for the solutions of this type of model has been derived in $[12]$. In the perturbed case, we follow the approach by Gröger, see $[7]$, based on the Banach fixed point theorem.

Remark 3.1. The result in Theorem 3.1 could be extended to obtain the well-posedness in $H^{s,p}_{0}(\text{curl}, \Omega)$ for the following problem

\begin{equation}
\begin{aligned}
\text{curl}(a(x) \text{curl} u) + b(x) u &= f, \text{ in } \Omega \\
\nu \wedge u &= 0, \text{ on } \partial \Omega
\end{aligned}
\end{equation}

where $a \in (W^{s,\infty}(\Omega))^{3 \times 3}$ satisfies the conditions (3.1) and (3.2) and $b \in (L^{\infty}(\Omega))^{3}$. One way to prove this is to show that the solution operator of this problem is a compact perturbation of the solution operator of the problem (3.3) and then use the Fredholm alternative, as it is done in $[13]$ for the case $s = 0$ and $p = 2$.

Remark 3.2. We make the following observations.

1. The regularity result Theorem 3.1 can be understood as follows. Let be given the bounds $m, M$ and $\tilde{M}$. Assume, in addition, that $k^2$ is such that $|1 - \frac{m k^2}{\tilde{M}}| < 1$. Then (3.3) is well posed in $H^{s,p}_{0}(\text{curl}; \Omega)$ for $s$ and $p$, of the form $s = (1 - \theta)s_0$ and
\( \frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{2} \) with \((s_0, \frac{1}{p_0}) \in R^+_\Omega\), such that

\[
C(s, p)(1 - \frac{m^2}{M^2} + \frac{m\tilde{M}}{M^2}) < 1
\]

and

\[
(1 - \theta) \log M_{s_0, p_0} + \log k_0(s, p) < 0
\]

where

\[
k_0(s, p) = \max\{|1 - mk^2|, C(s, p)[1 - \frac{m^2}{M^2} + \frac{m\tilde{M}}{M^2}]\}.
\]

The extra condition on \( k^2 \), i.e., \(|1 - \frac{mk^2}{M^2}| < 1\), can be removed by combining this result and Remark 3.1.

(2) In the case \( s = 0 \), we have \( C(s, p) = 1 \). In addition, in this case, we can take \( \tilde{M} = 0 \), i.e., we assume the elements of \( a \) to be in \( L^\infty(\Omega) \) only. Then the condition (3.9) reduces to \(|1 - \frac{m^2}{M^2}| < 1\), which is trivially satisfied, and then the condition (3.10) characterizes the range of \( p \) for which we have well posedness.

(3) In the case where \( a \) is a constant coefficient equal to the identity matrix we can take \( \frac{m}{M} \to 1 \) and \( \tilde{M} \to 0 \) and then \( C(s, p)(1 - \frac{m^2}{M^2} + \frac{m\tilde{M}}{M^2}) \ll 1 \). This means that in the case of identity coefficient \( a \), \( S^+ \) and \( S^- \) become \( R^+_\Omega \) and \( R^-_\Omega \) respectively and hence \( S = R_\Omega \). This reduces to the result in [12].

4. Proof of the \( H^{s,p}(\text{curl}; \Omega) \) estimates for the Maxwell system

We begin this section with the following lemma to characterize the dual space of \( H_0^{-s,q}(\text{curl}; \Omega) \) with an appropriate range of \( s \) and \( q \).

**Lemma 4.1.** Assume that \( \varphi \in (H_0^{-s,q}(\text{curl}; \Omega))' \), then \( \varphi \) can be uniquely written as \( \varphi = g_1 + \text{curl} g_2 \), with the estimate

\[
\|g_1\|_{L^p(\Omega)} + \|g_2\|_{L^p(\Omega)} \leq C\|\varphi\|_{(H_0^{-s,q}(\text{curl}; \Omega))'},
\]

where \( g_1, g_2 \in L^p(\Omega), 1/p + 1/q = 1, 1 < p < \infty \) and \(-1 + 1/p < s < 1/p\).

**Proof.** The operator

\[
P : H_0^{-s,q}(\text{curl}; \Omega) \to L^q_{-s}(\Omega) \times L^q_{-s}(\Omega),
\]

defined by

\[
P u := (u, \text{curl} u),
\]
is linear, bounded and isometric. Also we define \( W := P(H_0^{-s,q}(\text{curl}; \Omega)) \), which is a closed subspace of \( L^q_{-s}(\Omega) \times L^q_{-s}(\Omega) \).

Note that the adjoint operator

\[
P^* : W' \to (H_0^{-s,q}(\text{curl}; \Omega))'
\]
is invertible and continuous. Hence for given \( \varphi \in (H_0^{-s,q}(\text{curl}; \Omega))' \), there exists a unique \( \varphi^* \in W' \) such that \( P^* \varphi^* = \varphi \). Now, by Hahn-Banach extension theorem, there exists a linear functional (name it \( \tilde{\varphi}^* \)),

\[
\tilde{\varphi}^* : L^q_{-s}(\Omega) \times L^q_{-s}(\Omega) \to \mathbb{R}
\]
such that
\begin{equation}
(L^s_{-4}(\Omega) \times L^s_{-4}(\Omega))' \langle \tilde{\varphi}^*, u \rangle_{L^s_{-4}(\Omega) \times L^s_{-4}(\Omega)} = w' \langle \varphi^*, u \rangle_W, \quad \forall \ u \in W
\end{equation}
and
\[ \| \tilde{\varphi}^* \|_{(L^s_{-4}(\Omega) \times L^s_{-4}(\Omega))'} = \| \varphi^* \|_{W'}. \]

On the other hand, since \( \tilde{\varphi}^* \in (L^q_{-s}(\Omega) \times L^q_{-s}(\Omega))' \), then by Lemma 2.1, for all \(-1 + 1/p < s < 1/p\), there exists a unique \( g := (g_1, g_2) \in L^p_s(\Omega) \times L^p_s(\Omega) \) such that,
\[ \tilde{\varphi}^* f = L^p_s(\mathbb{R}^3) \langle \tilde{g}_1, \tilde{f}_1 \rangle_{L^p_{-s}(\mathbb{R}^3)} + L^p_s(\mathbb{R}^3) \langle \tilde{g}_2, \tilde{f}_2 \rangle_{L^p_{-s}(\mathbb{R}^3)}, \]
for all \( \tilde{f}_1, \tilde{f}_2 \in L^q_{-s}(\mathbb{R}^3) \), with the norm estimate
\begin{equation}
(4.2) \quad c_1 \| g \|_{L^p_s(\Omega) \times L^p_s(\Omega)} \leq \| \tilde{\varphi}^* \|_{(L^s_{-4}(\Omega) \times L^s_{-4}(\Omega))'} \leq c_2 \| g \|_{L^p_s(\Omega) \times L^p_s(\Omega)},
\end{equation}
where \( \tilde{f}_j \) is any extension of \( f_j \) satisfying \( R_\Omega \tilde{f}_j = f_j \), and \( \tilde{g} \) be an extension of \( g \) by zero outside \( \Omega \) for all \( j = 1, 2 \). Combining (4.1) and (4.2) we have
\[ \| \varphi^* \|_{W'} = \| \tilde{\varphi}^* \|_{(L^s_{-4}(\Omega) \times L^s_{-4}(\Omega))'} \geq c_1 \| g \|_{L^p_s(\Omega) \times L^p_s(\Omega)}. \]

Therefore, for \( v \in H^{-s,q}_0(\text{curl}; \Omega) \), we have
\begin{equation}
(4.3) \quad (H^{-s,q}_0(\text{curl}; \Omega))' \langle \varphi, v \rangle_{H^{-s,q}_0(\text{curl}; \Omega)} = w' \langle \varphi^*, P v \rangle_W
\end{equation}
\[ = (L^s_{-4}(\Omega) \times L^s_{-4}(\Omega))' \langle \tilde{\varphi}^*, P v \rangle_{L^s_{-4}(\Omega) \times L^s_{-4}(\Omega)} \]
\[ = L^p_s(\mathbb{R}^3) \langle \tilde{g}_1, \tilde{v} \rangle_{L^p_{-s}(\mathbb{R}^3)} + L^p_s(\mathbb{R}^3) \langle \tilde{g}_2, \text{curl} \tilde{v} \rangle_{L^p_{-s}(\mathbb{R}^3)}, \]
where, tilde denotes the extension by zero outside \( \Omega \).

Now we show that
\begin{equation}
(4.4) \quad L^p_s(\Omega) \langle g_2, \text{curl} \tilde{v} \rangle_{L^s_{-4}(\Omega)} = (H^{-s,q}_0(\text{curl}; \Omega))' \langle \text{curl} g_2, v \rangle_{H^{-s,q}_0(\text{curl}; \Omega)}. \end{equation}

Recall that, for \(-1 + 1/p < s < 1/p\), \( L^p_{s,0}(\Omega) = L^p_s(\Omega) \). As \( g_2 \in L^p_{s,0}(\Omega) \), then, \( g_2 \) is a distribution such that \( g_2 \in L^p_s(\mathbb{R}^3) \) with support in \( \Omega \). Using the distribution derivative of \( g_2 \), we define \( \text{curl} g_2 \) by
\[ (C^p_{\infty}(\Omega))' \langle \text{curl} g_2, v \rangle_{C^p_{\infty}(\Omega)} := (C^p_{\infty}(\Omega))' \langle g_2, \text{curl} v \rangle_{C^p_{\infty}(\Omega)} \quad \forall \ v \in C^\infty(\Omega). \]

It is clear that, \( \text{curl} g_2 \in (C^\infty(\Omega))' \). Also recall that, \( C^\infty(\Omega) \) is dense in \( H^{-s,q}_0(\text{curl}; \Omega) \), see [12, A.27]. Therefore, we can define \( \text{curl} g_2 \) on \( H^{-s,q}_0(\text{curl}; \Omega) \) by
\[ \langle \text{curl} g_2, v \rangle := \lim_{m \to \infty} \langle \text{curl} g_2, v_m \rangle, \]
where \( v_m \to v \) in \( H^{-s,q}_0(\text{curl}; \Omega) \) with \( v_m \in C^\infty(\Omega) \). Hence \( \text{curl} g_2 \) defines a bounded linear functional on \( H^{-s,q}_0(\text{curl}; \Omega) \) and we obtain
\begin{equation}
(4.5) \quad (H^{-s,q}_0(\text{curl}; \Omega))' \langle \text{curl} g_2, v \rangle_{H^{-s,q}_0(\text{curl}; \Omega)} = \lim_{m \to \infty} (C^\infty(\Omega))' \langle \text{curl} g_2, v_m \rangle_{C^\infty(\Omega)}
\end{equation}
\[ = \lim_{m \to \infty} (C^\infty(\Omega))' \langle g_2, \text{curl} v_m \rangle_{C^\infty(\Omega)} \]
\[ = L^p_s(\Omega) \langle g_2, \text{curl} v \rangle_{L^s_{-4}(\Omega)}. \]

Therefore, combining the equations (4.3) and (4.5), we obtain
\[ (H^{-s,q}_0(\text{curl}; \Omega))' \langle \varphi, v \rangle_{H^{-s,q}_0(\text{curl}; \Omega)} = (H^{-s,q}_0(\text{curl}; \Omega))' \langle g_1 \text{curl} g_2, v \rangle_{H^{-s,q}_0(\text{curl}; \Omega)}. \]
Hence, for any \( \varphi \in (H_0^{-s,q}(\text{curl}; \Omega))' \), there exist unique \( g_1 \in L^p_s(\Omega) \) and \( g_2 \in L^p_s(\Omega) \) such that
\[ \varphi = g_1 + \text{curl} \, g_2. \]
In addition, we have the following estimate.
\[
\| \varphi \|_{(H^{-s,q}_0(\text{curl}; \Omega))'} = \sup_{\|u\|_{H^{-s,q}_0(\text{curl}; \Omega)} \leq 1} |\langle \varphi, u \rangle| = \sup_{\|w\|_{H^{-s,q}_0(\text{curl}; \Omega)} \leq 1} |W'\langle \varphi^*, w \rangle_W| = \| \varphi^* \|_{W'} \geq c_1 \| g \|_{L^p_s(\Omega) \times L^p_s(\Omega)}
\]
\[
= c_1 \left[ \| g_1 \|_{L^p_s(\Omega)} + \| g_2 \|_{L^p_s(\Omega)} \right]^{1/p} 
\geq C \{ \| g_1 \|_{L^p_s(\Omega)} + \| g_2 \|_{L^p_s(\Omega)} \},
\]
i.e.,
\[
\| g_1 \|_{L^p_s(\Omega)} + \| g_2 \|_{L^p_s(\Omega)} \leq C \| \varphi \|_{(H_0^{-s,q}(\text{curl}; \Omega))'}.
\]
\[ \square \]

4.1. Unperturbed problem.

**Theorem 4.1.** Assume \( \Omega \) to be a bounded Lipschitz domain. For a given \( f \in (H_0^{-s,q}(\text{curl}; \Omega))' \), there exists a unique \( u \in H^{s,p}_0(\text{curl}; \Omega) \) satisfying the following Maxwell problem
\[
\begin{align*}
\text{curl } \text{curl } u + u &= f, \text{ in } \Omega \\
\nu \wedge u &= 0, \text{ on } \partial \Omega,
\end{align*}
\]
for all \((s,1/p) \in R_\Omega \) and \( 1/p + 1/q = 1 \).
In addition, we have the estimate
\[
\| u \|_{L^p_s(\Omega)} + \| \text{curl } u \|_{L^p_s(\Omega)} \leq C \| f \|_{(H_0^{-s,q}(\text{curl}; \Omega))'},
\]
for all \((s,1/p) \in R_\Omega \) and \( 1/p + 1/q = 1 \).

**Proof.** Since \( f \in (H_0^{-s,q}(\text{curl}; \Omega))' \), then from Lemma 4.1 there exist a unique \( g_1 \in L^p_s(\Omega) \) and \( g_2 \in L^p_s(\Omega) \) such that \( f = g_1 + \text{curl} \, g_2 \) with the estimate
\[
\| g_1 \|_{L^p_s(\Omega)} + \| g_2 \|_{L^p_s(\Omega)} \leq C \| f \|_{(H_0^{-s,q}(\text{curl}; \Omega))'}.
\]
Therefore the problem (4.7) can be viewed as
\[
\begin{align*}
\text{curl } \text{curl } u + u &= g_1 + \text{curl} \, g_2, \text{ in } \Omega \\
\nu \wedge u &= 0, \text{ on } \partial \Omega.
\end{align*}
\]
Define, \( u := g_1 + \text{curl } v \) and \( v := g_2 - \text{curl } u \). Then the (4.9) reduces to the following problem
\[
\begin{align*}
\text{curl } u + v &= g_2, \text{ in } \Omega \\
\text{curl } v - u &= -g_1, \text{ in } \Omega \\
\nu \wedge u &= 0 \text{ on } \partial \Omega.
\end{align*}
\]
In [12], it is shown that the problem (4.10) is well posed, i.e., there exists a unique \( u \in H^{s,p}_0(\text{curl}; \Omega) \) satisfying the problem (4.10) together with the estimate
\[
\|u\|_{L^p_s(\Omega)} + \|\text{curl } u\|_{L^q_s(\Omega)} \leq C \left( \|g_1\|_{L^p_s(\Omega)} + \|g_2\|_{L^q_s(\Omega)} \right),
\]
for all \((s, 1/p) \in R_\Omega\). Finally, combining (4.8) and (4.11), we have the required estimate
\[
\|u\|_{L^p_s(\Omega)} + \|\text{curl } u\|_{L^q_s(\Omega)} \leq C \|f\|_{(H^{s,q}_0(\text{curl}; \Omega))^\prime},
\]
for all \((s, 1/p) \in R_\Omega\) and \(1/p + 1/q = 1\). \( \square \)

To deal with the case of the perturbed problem, we follow Gröger’s approach, see [7].

4.2. Perturbed problem. Before proving Theorem 3.1 we state and justify some intermediate lemmas. Define,
\[
\mathcal{L}_{s,p} : H^{s,p}_0(\text{curl}; \Omega) \rightarrow L^p_s(\Omega) \times L^q_s(\Omega)
\]
by
\[
\mathcal{L}_{s,p} u := \begin{pmatrix} u \\ \text{curl } u \end{pmatrix}.
\]
Remark that \( \mathcal{L}_{s,p} \) is an isometry. Let us characterize its adjoint. Consider the functions \( u \in (C^\infty_0(\mathbb{R}^3))^6 \) and \( v \in (C^\infty_0(\mathbb{R}^3))^6 \), which are compactly supported in \( \Omega \). Also take \( v \) of the form \( v := \begin{pmatrix} a \\ A \end{pmatrix} \), then
\[
\langle \mathcal{L}_{s,p} u, \mathcal{L}_{s,p} v \rangle_{L^p_s(\Omega) \times L^q_s(\Omega)} = \langle u, a \rangle_{L^p_s(\Omega)} + \langle \text{curl } u, A \rangle_{L^q_s(\Omega)}
\]

Since \( C^\infty_0(\Omega) \) is dense in \( H^{s,p}_0(\text{curl}; \Omega) \), so for any \( u \in H^{s,p}_0(\text{curl}; \Omega) \) the equality
\[
\langle \mathcal{L}_{s,p} u, \mathcal{L}_{s,p} v \rangle_{L^p_s(\Omega) \times L^q_s(\Omega)} = \langle u, a \rangle_{(H^{s,p}_0(\text{curl}; \Omega))^\prime} + \langle \text{curl } u, A \rangle_{(H^{s,p}_0(\text{curl}; \Omega))^\prime}
\]
holds for all \( v \in L^p_s(\Omega) \times L^q_s(\Omega) \). Therefore, the adjoint of \( \mathcal{L}_{s,p} \) can be characterized as follows
\[
\mathcal{L}_{s,p}^* : L^p_s(\Omega) \times L^q_s(\Omega) \rightarrow (H^{s,p}_0(\text{curl}; \Omega))^\prime
\]
with
\[
\mathcal{L}_{s,p}^* \begin{pmatrix} a \\ A \end{pmatrix} = a + \text{curl } A.
\]
Similarly, we have \( \mathcal{L}_{s,q}^* : L^p_s(\Omega) \rightarrow (H^{s,q}_0(\text{curl}; \Omega))^\prime \) with \( \mathcal{L}_{s,q}^* \begin{pmatrix} a \\ A \end{pmatrix} = a + \text{curl } A \).
Finally, we define
\[
\mathcal{K}_{s,p} := \mathcal{L}_{s,q}^* \mathcal{L}_{s,p}.
\]
Therefore, \( \mathcal{K}_{s,p} u = u + \text{curl } \text{curl } u \). Hence, Theorem 4.1 ensures that
\[
\mathcal{K}_{s,p} : H^{s,p}_0(\text{curl}; \Omega) \rightarrow (H^{s,q}_0(\text{curl}; \Omega))^\prime.
\]
is an isomorphism, for all \((s, 1/p) \in R_\Omega\).
For all \(t > 0\), we define the operator
\[
\mathcal{B} : L^p_s(\Omega) \times L^p_s(\Omega) \to L^p_s(\Omega) \times L^p_s(\Omega)
\]
by
\[
\mathcal{B} \left( \begin{array}{c} A_1 \\ A_2 \end{array} \right) := \left( \begin{array}{c} A_1 - tk^2A_1 \\ A_2 - ta(\cdot)A_2 \end{array} \right).
\]
Therefore, a simple calculation shows that
\[(\mathcal{L}_{s,q}^s \mathcal{B} \mathcal{L}_{s,p} - \mathcal{K}_{s,p})u = -t[\text{curl}(a(x) \text{ curl } u) + k^2 u].\]

**Lemma 4.2.** The operator \(\mathcal{B}\) is Lipschitz continuous for every fixed \((s, 1/p) \in R^+_\Omega\).

**Proof.**
\[
\|\mathcal{B} \left( \begin{array}{c} A_1 \\ A_2 \end{array} \right) - \mathcal{B} \left( \begin{array}{c} B_1 \\ B_2 \end{array} \right) \|_{L^p_s(\Omega) \times L^p_s(\Omega)}
= \| \left( \begin{array}{c} A_1 - tk^2A_1 \\ A_2 - ta(\cdot)A_2 \end{array} \right) - \left( \begin{array}{c} B_1 - tk^2B_1 \\ B_2 - ta(\cdot)B_2 \end{array} \right) \|_{L^p_s(\Omega) \times L^p_s(\Omega)}
= \| \left( \begin{array}{c} (1 - tk^2)(A_1 - B_1) \\ (1 - ta(\cdot))(A_2 - B_2) \end{array} \right) \|_{L^p_s(\Omega) \times L^p_s(\Omega)}
= \left[ \| (1 - tk^2)(A_1 - B_1) \|_{L^p_s(\Omega)}^p + \| (1 - ta(\cdot))(A_2 - B_2) \|_{L^p_s(\Omega)}^p \right]^{1/p}
\]
(Using Lemma 2.4)
\[
\leq \left[ |1 - tk^2|^p \| (A_1 - B_1) \|_{L^p_s(\Omega)}^p + (C(s, p))^p \| 1 - ta(\cdot) \|_{W^{s,\infty}(\Omega)}^p \| (A_2 - B_2) \|_{L^p_s(\Omega)}^p \right]^{1/p}.
\]

We, now estimate the norm \(\|1 - ta(\cdot)\|_{W^{s,\infty}(\Omega)}\). Recall that, for \(0 < s < 1\), we have,
\[W^{s,\infty}(\Omega) = C^{0,s}(\Omega)\]
with the norm
\[
\|\varphi\|_{W^{s,\infty}(\Omega)} = \|\varphi\|_{L^\infty(\Omega)} + \sup_{x,y \in \Omega, x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^s},
\]
where, \(C^{0,s}(\Omega)\) is the Hölder continuous space with exponent \(s\). Hence
\[
\|1 - ta(\cdot)\|_{W^{s,\infty}(\Omega)} = \|1 - ta(\cdot)\|_{L^\infty(\Omega)} + \sup_{x,y \in \Omega, x \neq y} \frac{|(1 - ta(x)) - (1 - ta(y))|}{|x - y|^s}
= \sup_{x \in \Omega} |1 - ta(x)| + t \sup_{x,y \in \Omega, x \neq y} \frac{|a(x) - a(y)|}{|x - y|^s}.
\]
Since, \(m \leq |a(x)| \leq M, \forall x \in \Omega\) and the Hölder semi-norm of \(a\) is denoted by
\[
|a|_{C^{0,s}} := \sup_{x,y \in \Omega, x \neq y} \frac{|a(x) - a(y)|}{|x - y|^s},
\]
15
then
\[ \|1 - ta(\cdot)\|_{W^{s,\infty}(\Omega)}^p \leq \|1 - tm\| + t|a|_{C^{0,s}}^p. \]

Recall that, the coefficient \( a \) is taken to be Hölder continuous with exponent \( s \) in \( \Omega \), i.e., there exists \( \tilde{M} > 0 \) such that \( |a|_{C^{0,s}} \leq \tilde{M} \). Hence
\[
\|\mathcal{B}\left( \begin{array}{c} A_1 \\ A_2 \end{array} \right) - \mathcal{B}\left( \begin{array}{c} B_1 \\ B_2 \end{array} \right) \|_{L^p_s(\Omega) \times L^p_s(\Omega)} \leq \left[ \max\{\|1 - tk^2\|^p, (C(s,p))^{p\|1 - tm\| + t\tilde{M}^p}\} \right]^{1/p} 
\times \left[ \|A_1 - B_1\|^p_{L^p_s(\Omega)} + \|A_2 - B_2\|^p_{L^p_s(\Omega)} \right]^{1/p}.
\]

We set
\[
k_0(s,p) := \max\{\|1 - tk^2\|, C(s,p)\|1 - tm\| + t\tilde{M}\},
\]
then we have
\[
\|\mathcal{B}\left( \begin{array}{c} A_1 \\ A_2 \end{array} \right) - \mathcal{B}\left( \begin{array}{c} B_1 \\ B_2 \end{array} \right) \|_{L^p_s(\Omega) \times L^p_s(\Omega)} \leq k_0(s,p)\|\mathcal{B}\left( \begin{array}{c} A_1 \\ A_2 \end{array} \right) - \mathcal{B}\left( \begin{array}{c} B_1 \\ B_2 \end{array} \right) \|_{L^p_s(\Omega) \times L^p_s(\Omega)}
\]
which means that \( \mathcal{B} \) is Lipschitz with the Lipschitz constant \( k_0(s,p) \).

For all \((s, 1/p) \in R_\Omega\), we define the operator \( \mathcal{Q}_f \) as follows:
\[
\mathcal{Q}_f u := K_{s,p}^{-1}\left( \mathcal{L}^s_{-s,q}\mathcal{B}\mathcal{L}_{s,p}u + tf \right)
= u - tK_{s,p}^{-1}\left\{ \{\text{curl}(a(x)\text{curl} u) + k^2 u\} - f \right\},
\]
where \( u \in H^{s,p}_0(\text{curl; } \Omega) \). Our main aim is to show that \( \mathcal{Q}_f \) is a contraction mapping, which is the key point to prove Theorem 3.1.

**Notation 1.** For all \((s, 1/p) \in R_\Omega\), we define \( \mathcal{M}_{s,p} \) as follows
\[
\mathcal{M}_{s,p} := \sup_{u \in H^{s,p}_0(\text{curl; } \Omega)} \|u\|_{H^{s,p}_0(\text{curl; } \Omega)},
\]
\[
\|K_{s,p}u\|_\left( H^{s,q}_0(\text{curl; } \Omega) \right)^t \leq 1
\]
It is clear that \( \mathcal{M}_{s,p} = \|K_{s,p}^{-1}\|_\left( H^{-s,q}_0(\text{curl; } \Omega) \right)^t \rightarrow H^{s,p}_0(\text{curl; } \Omega) \).

Let us first prove the following lemma.

**Lemma 4.3.** The operator \( \mathcal{Q}_f : H^{s,p}_0(\text{curl; } \Omega) \rightarrow H^{s,p}_0(\text{curl; } \Omega) \) is Lipschitz with the Lipschitz constant \( k_0(s,p)\mathcal{M}_{s,p} \) for all \((s, 1/p) \in R_\Omega^+\).

**Proof.**
\[
\|\mathcal{L}^s_{-s,q}\mathcal{B}\mathcal{L}_{s,p}u\|_\left( H^{-s,q}_0(\text{curl; } \Omega) \right)^t \rightarrow H^{s,p}_0(\text{curl; } \Omega)
= \sup_{\|u\|_{H^{s,p}_0(\text{curl; } \Omega)} \leq 1} \|\mathcal{L}^s_{-s,q}\mathcal{B}\mathcal{L}_{s,p}u\|_\left( H^{-s,q}_0(\text{curl; } \Omega) \right)^t
= \sup_{\|u\|_{H^{s,p}_0(\text{curl; } \Omega)} \leq 1} \sup_{\|v\|_{H^{s,q}_0(\text{curl; } \Omega)} \leq 1} \langle \mathcal{L}^s_{-s,q}\mathcal{B}\mathcal{L}_{s,p}u, v \rangle_{H^{-s,q}_0(\text{curl; } \Omega)}
= \sup_{\|u\|_{H^{s,p}_0(\text{curl; } \Omega)} \leq 1} \sup_{\|v\|_{H^{s,q}_0(\text{curl; } \Omega)} \leq 1} \|\mathcal{B}\mathcal{L}_{s,p}u\|_{L^p_s(\Omega) \times L^q_s(\Omega)} \|\mathcal{L}_{-s,q}v\|_{L^q_s(\Omega) \times L^p_s(\Omega)}
\leq \mathcal{M}_{s,p} \sup_{\|u\|_{H^{s,p}_0(\text{curl; } \Omega)} \leq 1} \|\mathcal{B}\mathcal{L}_{s,p}u\|_{L^p_s(\Omega) \times L^q_s(\Omega)} \|\mathcal{L}_{-s,q}v\|_{L^q_s(\Omega) \times L^p_s(\Omega)}
\]
Proof. The proof follows from Theorem 2.2 and [3], Theorem 4.1.2.

For any fixed \((s,p)\), the operators are linear and bounded. We state the following lemma, which is a consequence of the complex interpolation theorem.

**Lemma 4.4.** For any fixed \((s_0, 1/p_0), (s_1, 1/p_1)\) in \(R_\Omega\), the operators \(P_{s_0,p_0}\) and \(P_{s_1,p_1}\) are bounded. Then the operator

\[
P_{s,p} : L^p_s(\Omega) \times L^p_s(\Omega) \rightarrow L^p_s(\Omega) \times L^p_s(\Omega),
\]

defined by

\[
P_{s,p} := L^p_{s_0}(\Omega) \times L^p_{s_0}(\Omega) \rightarrow L^p_{s_1}(\Omega) \times L^p_{s_1}(\Omega)
\]

by \(P_{s_0,p_0} := L^p_{s_0}(\Omega) \times L^p_{s_0}(\Omega) \rightarrow L^p_{s_1}(\Omega) \times L^p_{s_1}(\Omega)\), respectively. Observe that these operators are linear and bounded.

We state the following lemma, which is a consequence of the complex interpolation theorem.

**Lemma 4.4.** For any fixed \((s_0, 1/p_0), (s_1, 1/p_1)\) in \(R_\Omega\), the operators \(P_{s_0,p_0}\) and \(P_{s_1,p_1}\) are bounded. Then the operator

\[
P_{s,p} : L^p_s(\Omega) \times L^p_s(\Omega) \rightarrow L^p_s(\Omega) \times L^p_s(\Omega),
\]

defined by

\[
P_{s,p} := L^p_{s_0}(\Omega) \times L^p_{s_0}(\Omega) \rightarrow L^p_{s_1}(\Omega) \times L^p_{s_1}(\Omega)
\]

is bounded and satisfies the following estimate

\[
\|P_{s,p}\| \leq \|P_{s_0,p_0}\|^{1-\theta} \|P_{s_1,p_1}\|^{\theta},
\]

for all \((s, 1/p)\) in \(R_\Omega\) and \(s, p\) satisfying \(s = (1-\theta)s_0 + \theta s_1, 1 = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}\) with \(0 < \theta < 1, s_0, s_1 \in \mathbb{R}, s_0 \neq s_1, 1 < p_0, p_1 < \infty\).

**Proof.** The proof follows from Theorem 2.2 and [3], Theorem 4.1.2. 

Recall that our goal is to show the operator \(Q_f\) is a contraction map. For that, it is enough to show the Lipschitz constant \(M_{s,p}k_0(s,p)\) is strictly less than 1. In order to do that we state the following two lemmas.
Lemma 4.5. The operator $K_{s,p}$ is bounded and invertible for all $(s, 1/p) \in R_\Omega$. Moreover, for any $(s_0, 1/p_0), (s_1, 1/p_1) \in R_\Omega$, we have the following estimate

$$
M_{s,p} \leq M_{s_0,p_0}^{1-\theta} M_{s_1,p_1}^{\theta},
$$

for all $(s, 1/p) \in R_\Omega$ and $s, p$ satisfying $s = (1 - \theta)s_0 + \theta s_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ with $0 < \theta < 1$.

Proof. The bounded invertibility of the operator $K_{s,p}$ follows from Theorem 4.1. It is enough to prove the estimate (4.13). Since the operators $P_{s_0,p_0}$ and $P_{s_1,p_1}$ are bounded and $\|L_{s_0,p_0}\| = \|L_{-s_0,q_0}\| = 1$, then $\|P_{s_0,p_0}\| \leq M_{s_0,p_0}$ and $\|P_{s_1,p_1}\| \leq M_{s_1,p_1}$ for any fixed $(s_0, 1/p_0), (s_1, 1/p_1) \in R_\Omega$.

Hence, applying Lemma 4.4, we obtain that the operator

$$
P_{s,p} : L^p_s(\Omega) \times L^p_s(\Omega) \to L^p_s(\Omega) \times L^p_s(\Omega)
$$

is bounded with the estimate

$$
\|P_{s,p}\| \leq \|P_{s_0,p_0}\|^{1-\theta} \|P_{s_1,p_1}\|^\theta
\leq M_{s_0,p_0}^{1-\theta} M_{s_1,p_1}^{\theta},
$$

for all $(s, 1/p) \in R_\Omega$ and $s, p$ satisfying $s = (1 - \theta)s_0 + \theta s_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ with $0 < \theta < 1$.

Let us consider $f \in (H_0^{-s,q}(\text{curl}; \Omega))^\prime$, where $\frac{1}{p} + \frac{1}{q} = 1$. Define a linear functional

$$
Z : \text{Im}(\mathcal{L}_{-s,q}) \subset L^q_{-s}(\Omega) \times L^q_{-s}(\Omega) \to \mathbb{R}
$$

by

$$
\langle Z, \mathcal{L}_{-s,q}v \rangle := (H_0^{-s,q}(\text{curl}; \Omega))^\prime \langle f, v \rangle_{H_0^{-s,q}(\text{curl}; \Omega)},
$$

for all $v \in H_0^{-s,q}(\text{curl}; \Omega)$, where

$$
\mathcal{L}_{-s,q} : H_0^{-s,q}(\text{curl}; \Omega) \to L^q_{-s}(\Omega) \times L^q_{-s}(\Omega).
$$

Note that the above definition makes sense since $v$ is uniquely determined by $\mathcal{L}_{-s,q}$. Now,

$$
\|Z\| = \sup_{v \neq 0} \frac{|\langle Z, \mathcal{L}_{-s,q}v \rangle|}{\|\mathcal{L}_{-s,q}v\|_{L^q_{-s}(\Omega) \times L^q_{-s}(\Omega)}}
= \sup_{v \neq 0} \frac{|\langle f, v \rangle|}{\|v\|_{H_0^{-s,q}(\text{curl}; \Omega)}}
= \|f\|_{(H_0^{-s,q}(\text{curl}; \Omega))^\prime}.
$$

By Hahn-Banach extension theorem, $Z$ can be extended to a continuous linear functional (again denoted by $Z$) on $L^q_{-s}(\Omega) \times L^q_{-s}(\Omega)$ with the same norm $\|Z\| = \|f\|_{(H_0^{-s,q}(\text{curl}; \Omega))^\prime}$.

Moreover, $\mathcal{L}_{-s,q}^* Z = f$ because

$$
\langle \mathcal{L}_{-s,q}^* Z, v \rangle = \langle Z, \mathcal{L}_{-s,q}v \rangle = (H_0^{-s,q}(\text{curl}; \Omega))^\prime \langle f, v \rangle_{H_0^{-s,q}(\text{curl}; \Omega)}.
$$

Define, $u := \mathcal{K}_{s,p}^{-1} f$, where $f = \mathcal{L}_{-s,q}^* Z$. Therefore,

$$
\mathcal{L}_{s,p} u = \mathcal{L}_{s,p} \mathcal{K}_{s,p}^{-1} \mathcal{L}_{-s,q}^* Z = P_{s,p} Z.
$$
Hence,

\[
\|u\|_{H_0^{s,p}(\text{curl};\Omega)} = \|L_{s,p}u\|_{L^p_s(\Omega) \times L^p_s(\Omega)} = \|P_{s,p}Z\|_{L^p_s(\Omega) \times L^p_s(\Omega)} \\
\leq \|P_{s,p}\| \|Z\|_{L^p_s(\Omega) \times L^p_s(\Omega)} (\text{using (4.14)}) \\
\leq M_{s,p_0}^{1-\theta} M_{s_1,p_1}^{\theta} \|f\|_{(H_0^{-s,q}(\text{curl};\Omega))'} 
\]

i.e.,

\[
M_{s,p} = \sup_{\|f\|_{(H_0^{-s,q}(\text{curl};\Omega))'} \leq 1} \|u\|_{H_0^{s,p}(\text{curl};\Omega)} \leq M_{s_0,p_0}^{1-\theta} M_{s_1,p_1}^{\theta}. 
\]

\[\square\]

**Lemma 4.6.** We have

\[M_{0,2} = 1.\]

**Proof.** We prove this part in two steps.

**Step 1.** In this step, we show that \(M_{0,2} \leq 1\).

Recall that, \(K_{0,2}u^f := \text{curl curl } u^f + u^f = f\), where the operator \(K_{0,2} : H_0(\text{curl};\Omega) \rightarrow (H_0(\text{curl};\Omega))'\), is bounded and invertible, see Theorem 4.1. Therefore,

\[
M_{0,2} = \|K_{0,2}^{-1}\| = \sup_{\|f\|_{(H_0(\text{curl};\Omega))'} \leq 1} \|K_{0,2}^{-1}f\|_{H_0(\text{curl};\Omega)} \\
(4.15) \\
= \sup_{\|f\|_{(H_0(\text{curl};\Omega))'} \leq 1} \|u^f\|_{H_0(\text{curl};\Omega)}. 
\]

Note that, \(u^f\) satisfies the following Maxwell problem

\[\text{curl curl } u^f + u^f = f\]

in the weak sense, i.e., in particular, we have

\[
\int_{\Omega} |\text{curl } u^f|^2 + \int_{\Omega} |u^f|^2 = \int_{\Omega} f \cdot u^f. 
\]

Hence, using Hölder inequality, we have

\[
\|u^f\|_{H_0(\text{curl};\Omega)}^2 = \int_{\Omega} |\text{curl } u^f|^2 + \int_{\Omega} |u^f|^2 \leq \|f\|_{(H_0(\text{curl};\Omega))'} \|u^f\|_{H_0(\text{curl};\Omega)}, 
\]

i.e.,

\[\|u^f\|_{H_0(\text{curl};\Omega)} \leq \|f\|_{(H_0(\text{curl};\Omega))'}. \]

Combining (4.15) and (4.17), we deduce that \(M_{0,2} \leq 1\).

**Step 2.** In this step, we prove that, there exists \(u \in H_0(\text{curl};\Omega)\) with \(\|K_{0,2}u\|_{(H_0(\text{curl};\Omega))'} \leq 1\) such that \(\|u\|_{H_0(\text{curl};\Omega)} = 1\).
Take \( u_0 \in H_0(\text{curl}; \Omega) \) such that \( \|u_0\| \neq 0 \). Define \( \tilde{u} := \frac{u_0}{\|u_0\|} \). Therefore, \( \|\tilde{u}\|_{H_0(\text{curl}; \Omega)} = 1 \).

Also,
\[
\|K_{0,2} \tilde{u}\|_{(H_0(\text{curl}; \Omega))'} = \sup_{\|v\|_{H_0(\text{curl}; \Omega)} \leq 1} \langle (H_0(\text{curl}; \Omega))' (K_{0,2} \tilde{u}, v) \rangle_{H_0(\text{curl}; \Omega)}
\]
\[
= \sup_{\|v\|_{H_0(\text{curl}; \Omega)} \leq 1} \left[ \int_{\Omega} \text{curl} \tilde{u} \cdot \text{curl} v + \int_{\Omega} \tilde{u} \cdot v \right]
\]
\[
\leq \sup_{\|v\|_{H_0(\text{curl}; \Omega)} \leq 1} \left[ \|\text{curl} \tilde{u}\|_{L^2(\Omega)} \|\text{curl} v\|_{L^2(\Omega)} + \|\tilde{u}\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \right]
\]
\[
\leq \sup_{\|v\|_{H_0(\text{curl}; \Omega)} \leq 1} \left( \frac{1}{2} \left( \|\text{curl} v\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2 \right)
\]
\[
+ \frac{1}{2} \left( \|\text{curl} \tilde{u}\|_{L^2(\Omega)}^2 + \|\tilde{u}\|_{L^2(\Omega)}^2 \right) \right)
\]
\[
\leq \frac{1}{2} + \frac{1}{2} = 1,
\]
i.e., \( \mathcal{M}_{0,2} = 1 \).

Now, we are in a position to prove that \( Q_f \) is a contraction map.

**Proposition 4.2.** The operator \( Q_f : H_0^{s,p}(\text{curl}; \Omega) \rightarrow H_0^{s,p}(\text{curl}; \Omega) \) is a contraction map, for all \((s, 1/p) \in S^+\), where \( S^+ \) is defined in Theorem 3.1.

**Proof.** From Lemma 4.3 we have,
\[
\|Q_fu - Qfv\|_{H_0^{s,p}(\text{curl}; \Omega)} \leq \mathcal{M}_{s,p} k_0(s, p) \|u - v\|_{H_0^{s,p}(\text{curl}; \Omega)}.
\]
To prove \( Q_f \) to be a contraction map, we need to show \( \mathcal{M}_{s,p} k_0(s, p) < 1, \forall (s, 1/p) \in S^+ \).

Now, fix any \((s_0, 1/p_0) \in R_+^2\) and take a particular point \((0, 1/2)\) in the region \( R_+^2\), then from Lemma 4.5 we have the following estimate
\[
\mathcal{M}_{s,p} \leq \mathcal{M}^{1-\theta}_{s_0,p_0} \mathcal{M}^\theta_{0,2},
\]
where \( s = (1 - \theta)s_0, \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{2} \) and \( 0 < \theta < 1 \). Therefore, \( \mathcal{M}_{s,p} k_0(s, p) < 1 \) if we can show that
\[
(4.18) \quad \mathcal{M}^{1-\theta}_{s_0,p_0} \mathcal{M}^\theta_{0,2} k_0(s, p) < 1.
\]

Note that \( \mathcal{M}_{0,2} = 1 \), then passing log both sides of (4.18) we have,
\[
(4.19) \quad (1 - \theta) \log \mathcal{M}_{s_0,p_0} + \log k_0(s, p) < 0.
\]

Recall that
\[
(4.20) \quad k_0(s, p) = \max \{|1 - tk^2|, C(s, p)[|1 - tm| + t\tilde{M}]\}.
\]
We choose \( t = \frac{m}{M^2} \). Then \( k_0 \) becomes
\[
k_0(s, p) = \max \left\{ |1 - \frac{mk^2}{M^2}|, C(s, p) \left[ 1 - \frac{m^2}{M^2} + \frac{m\tilde{M}}{M^2} \right] \right\}.
\]

Now under the following conditions on \( m, M, \tilde{M}, k^2 > 0 \)
\[
(4.21) \quad |1 - \frac{mk^2}{M^2}| < 1
\]
\[
1 - \frac{m^2}{M^2} + \frac{m\tilde{M}}{M^2} < \frac{1}{C(s,p)}
\]

we obtain \(k_0(s,p) < 1\) for all \(R_0^+\). So, \(\mathcal{M}_{s,p}k_0(s,p) < 1\) if \((s,1/p)\) satisfies the following properties:

(i) \((1 - \theta)\log \mathcal{M}_{s_0,p_0} + \log k_0(s,p) < 0.\)

(ii) \(s = (1 - \theta)s_0.\)

(iii) \(\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{2}\) with \(0 < \theta < 1,\)

for \((s_0,1/p_0) \in R_0^+\) with these appropriate choice of \(m, M, \tilde{M}, k^2.\)

4.3. End of the proof of Theorem 3.1. We consider the three issues (uniqueness, existence and stability) separately.

**Uniqueness**

We start by proving the uniqueness of the solutions for the operator equation

\[Au := \text{curl}(a(x) \text{curl} u) + k^2u = f.\]

**Case 1.** \((s,1/p) \in S \cap \{(s,1/p); p > 2, s > 0\}.\)

In this range of \(s\) and \(p\) we have, \(H_0^{s,p}(\text{curl}; \Omega) \subset H_0(\text{curl}; \Omega).\) Since,

\[A : H_0(\text{curl}; \Omega) \to (H_0(\text{curl}; \Omega))^\prime\]

is invertible then the fixed point \(u\) of \(Q_f\) is the unique solution to \(Au = f.\)

**Case 2.** \((s,1/p) \in S \cap \{(s,1/p); p > 2, s > 0\}^c.\)

For a given data \(f\), let us consider \(u_1\) and \(u_2\) in \(H_0^{s,p}(\text{curl}; \Omega)\) be two solutions of the operator equation \(Au = f,\) where \((s,1/p) \in S \cap \{(s,1/p); p > 2, s > 0\}^c,\) i.e., \(Au_1 = Au_2.\) Since \(A\) is linear then \(A(u_1 - u_2) = 0.\) Now, \(0 \in (H_0^{-s,q}(\text{curl}; \Omega))^\prime,\) for all \((s,1/p) \in S \cap \{(s,1/p); p > 2, s > 0\},\) then by applying Case 1, the operator equation \(Au = 0\) has \(u := 0\) as the unique solution in \(H_0^{s,p}(\text{curl}; \Omega),\) so we have \(u_1 - u_2 = 0.\) Hence, \(A\) is injective.

**Existence**

**Case 1.** \((s,1/p) \in S^+.\)

Existence of the solution in \(H_0^{s,p}(\text{curl}; \Omega)\) of the operator equation \(Q_fu = u\) is due to the fixed point theorem, as \(Q_f\) is a contraction map. Hence, the fixed point \(u \in H_0^{s,p}(\text{curl}; \Omega)\) of \(Q_f\) is a solution of

\[Au := \text{curl}(a(x) \text{curl} u) + k^2u = f,\]

i.e.,

\[A : H_0^{s,p}(\text{curl}; \Omega) \to (H_0^{-s,q}(\text{curl}; \Omega))^\prime\]

is onto, for all \((s,1/p) \in S^+\) and \(\frac{1}{p} + \frac{1}{q} = 1.\)

**Case 2.** \((s,1/p) \in S^-\).
Since the matrix $a$ is symmetric then $A = A^*$. Recall that the adjoint of an invertible operator is invertible. So,

$$A^* : (H_0^{-s,q}(\text{curl}; \Omega))'' \to (H_0^{s,p}(\text{curl}; \Omega))'$$

is invertible for all $(s, 1/p) \in S^+$. As $H_0^{s,p}(\text{curl}; \Omega)$ is reflexive for all $(s, 1/p) \in R_\Omega$, see Lemma 2.3, therefore

$$A : H_0^{-s,q}(\text{curl}; \Omega) \to (H_0^{s,p}(\text{curl}; \Omega))'$$

is invertible for all $(s, 1/p) \in S^+$, i.e.

$$A : H_0^{s,p}(\text{curl}; \Omega) \to (H_0^{-s,q}(\text{curl}; \Omega))'$$

is invertible for all $(s, 1/p) \in S^-$, recalling that

$$S^- = \{(s, 1/p) \in R_\Omega^2; (-s, 1/q) \in S^+\}.$$

**Stability** Finally, we finish the proof by deriving the stability estimate of the solution in terms of the given data.

**Case 1.** $(s, 1/p) \in S^+$.

If $f, g \in (H_0^{-s,q}(\text{curl}; \Omega))'$ are given and $u, v$ are the fixed points of $Q_f, Q_g$ respectively. Then

$$\|u - v\|_{H_0^{s,p}(\text{curl}; \Omega)} = \|Q_f u - Q_g v\|_{H_0^{s,p}(\text{curl}; \Omega)} \leq M_{s,p} k_0(s, p) \|u - v\|_{H_0^{s,p}(\text{curl}; \Omega)} + M_{s,p} \frac{m}{M^2} \|f - g\|_{(H_0^{-s,q}(\text{curl}; \Omega))'}$$

i.e.,

$$\|u - v\|_{H_0^{s,p}(\text{curl}; \Omega)} \leq \frac{m}{M^2} M_{s,p} (1 - M_{s,p} k_0(s, p))^{-1} \|f - g\|_{(H_0^{-s,q}(\text{curl}; \Omega))'}.$$

Therefore, there exists $C := \frac{m}{M^2} M_{s,p} (1 - M_{s,p} k_0(s, p))^{-1} > 0$, such that

$$\|u\|_{H_0^{s,p}(\text{curl}; \Omega)} \leq C \|f\|_{(H_0^{-s,q}(\text{curl}; \Omega))'}.$$

**Case 2.** $(s, 1/p) \in S^-$.

Note that the operator

$$A : H_0^{s,p}(\text{curl}; \Omega) \to (H_0^{-s,q}(\text{curl}; \Omega))'$$

is invertible and $A^{-1}$ is bounded for all $(s, 1/p) \in S^+$. Hence by the open mapping theorem the operator $A$ is bounded. Now, from [14], Theorem 4.15] and the reflexivity of the spaces $H_0^{s,p}(\text{curl}; \Omega)$, we obtain that

$$(A^*)^{-1} : (H_0^{s,p}(\text{curl}; \Omega))' \to H_0^{-s,q}(\text{curl}; \Omega)$$

is a bounded linear operator for all $(s, 1/p) \in S^+$. Since $A = A^*$, then we have,

$$\|u\|_{H_0^{s,p}(\text{curl}; \Omega)} = \|A^{-1} f\|_{H_0^{s,p}(\text{curl}; \Omega)} \leq C \|f\|_{(H_0^{-s,q}(\text{curl}; \Omega))'},$$

for all $(s, 1/p) \in S^-$, where $C > 0$. 22
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