Abstract

We study Witten’s background independent open-string field theory in the presence of a constant $B$-field at one loop level. The Green’s function and the partition function with a constant $B$-field are evaluated for an annulus.

1 Introduction

Background independent open-string field theory has been formally defined in the space of all two-dimensional world-sheet theories by Witten [1]. The first concrete computation of the action for an off-shell tachyon field in the tree level was given in [2]. It has been shown recently that by turning on a large $B$-field, the noncommutative geometry [3] simplifies the construction of soliton solution [4] which leads to tachyon condensation [5]. The stationery point of the tachyon potential on the disk from background independent open string field theory [2, 5] agrees precisely with the tension of the D-brane [6, 8, 9]. The tachyon potential with constant $B$-field on the disk was constructed in [10]. The disc partition function for the D0-D2 system...
with a large $B$-field was evaluated in [11]. Since at the tree level the tachyon field
seems to be playing the role of the loop counting parameter, it would be interesting
to see the form of the exponential in the tachyon potential in higher order terms. In
this letter we generalize the calculations of the partition function in [2, 11] beyond
the tree level by including the annulus diagram.

The organization for this paper is follows. In section 2 we define the boundary
value problem of background independent open-string field theory with a $B$-field on
the annulus. In section 3 we obtain the Green’s function on the annulus. In section
4 we use the Green’s function to evaluate the partition function on the annulus in
presence of $B$-field. Section 5 contain a brief discussion.

2 The Boundary Problem on the Annulus
Following [2, 11] we consider a two-dimensional action of the form

$$I = I_0 + I'$$  \hspace{1cm} (1)$$

where $I_0$ is the bulk action and the boundary term $I'$ describes coupling of the
external open strings. The bulk action $I_0$ describes the closed string background
with a constant $B$-field

$$I_0 = \int_{\Sigma} d^2 x \sqrt{h} \left( \frac{1}{8\pi} h^{ij} \partial_i X^\mu \partial_j X_\mu + b^{ij} D_i c_j \right) - \frac{i}{8\pi} \int_{\Sigma} d^2 x \epsilon^{ij} B_{\mu\nu} \partial_i X^\mu \partial_j X^\nu$$ \hspace{1cm} (2)$$

where $c_i$ and $b^{ij}$ are the ghost and anti-ghost fields. $\Sigma$ is any two-dimensional world
sheet with the metric $h_{ij}$ and coordinates $X^i$. In this letter we choose $\Sigma$ to be the
annulus with a rotationally invariant flat metric

$$ds^2 = d\sigma_1^2 + d\sigma_2^2 \hspace{1cm} a \leq \sigma_1^2 + \sigma_2^2 \leq b$$ \hspace{1cm} (3)$$

It is convenient to work with complex coordinates and set $z = \sigma_1 + i\sigma_2$, with
$a \leq |z| \leq b$
The boundary term \( I' \) can be any ghost number conserving boundary interaction, for which everything can be computed explicitly, we will take it to be a quadratic function of the coordinates

\[
I' = \frac{1}{8\pi} \int_{\partial \Sigma} d\theta u_\mu (X^\mu)^2
\]

The boundary condition derived by varying \( I = I_0 + I' \) is

\[
(1 + B)_{\mu\nu} \tilde{z} \partial X^\nu + (1 - B)_{\mu\nu} \bar{z} \tilde{\partial} X^\nu + u_\mu X^\mu|_{\partial \Sigma} = 0
\]

Note there is no sum on \( \mu \) in the last term.

The Green’s function of the theory should obey

\[
\partial_z \bar{\partial}_\bar{z} G(z, w) = -2\pi \alpha' \delta^{(2)}(z - w)
\]

with the boundary condition (5).

The Green’s function for the case without boundary term \( u = 0 \) has been solved in [12]. In that case the boundary conditions are

\[
(1 + B)_{\mu\nu} \tilde{z} \partial C^\nu_\lambda + (1 - B)_{\mu\nu} \bar{z} \tilde{\partial} C^\nu_\lambda|_{r = b} = -\alpha' \delta_{\mu\lambda}
\]

\[
(1 + B)_{\mu\nu} \tilde{z} \partial C^\nu_\lambda + (1 - B)_{\mu\nu} \bar{z} \tilde{\partial} C^\nu_\lambda|_{r = a} = 0
\]

The boundary conditions can not be set to 0 for both \( r = a \) and \( r = b \), because in the presence of the \( B \)-field, eq. (6) and Gauss’s theorem would not permit the propagator to be single-valued. The simplest choice is (7).

In our case with the \( u \neq 0 \) term, we have the freedom to set both boundary conditions to 0, i.e.

\[
(1 + B)_{\mu\nu} \tilde{z} \partial X^\nu + (1 - B)_{\mu\nu} \bar{z} \tilde{\partial} X^\nu + u_\mu X^\mu|_{r = b} = 0
\]

\[
(1 + B)_{\mu\nu} \tilde{z} \partial X^\nu + (1 - B)_{\mu\nu} \bar{z} \tilde{\partial} X^\nu + u_\mu X^\mu|_{r = a} = 0
\]

provided

\[
\int_{r = a, b} \frac{uG}{r} ds = 2\pi \alpha'
\]
3 The Green’s Function

To solve the eq. (6) with the boundary conditions (8), we start with the ansatz,

\[
G(z, w) = \frac{\alpha'}{2} \left\{ -\ln |z - w|^2 + \frac{\ln(z \bar{w} / b^2) - 2/u}{\ln(a^2 / b^2)} \right. \\
+ \sum_{k=-\infty}^{\infty} [a_k(z \bar{w})^k + a_k'(\bar{z}w)^k] + \sum_{k=-\infty}^{\infty} \left[ b_k(z/w)^k + b_k'(\bar{z}/\bar{w})^k \right] \left\} \right. \\
\left. \left. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \r
\[ + \sum_{k=-\infty}^{-1} \left( [(1 - B)k + u]a'_k - \frac{[(1 + B)k - u]b_{-k}}{b^{2k}} \right) (z\bar{w})^k \]
\[ + \sum_{k=-\infty}^{-1} \left( [(1 + B)k + u]a_k - \frac{[(1 - B)k - u]b'_{-k}}{b^{2k}} \right) (z\bar{w})^k \]
\[ + u(a_0 + a'_0 + b_0 + b'_0) - 2 - ulnb^2 \]

namely,

\[ u(a_0 + a'_0 + b_0 + b'_0) - 2 - ulnb^2 = 0 \]  

\[
\begin{cases}
\frac{1+B}{a^{2k}} - \frac{u}{ka^{2k}} + [(1-B)k + u]a'_k - \frac{[(1+B)k-u]b_{-k}}{b^{2k}} = 0 \\
\frac{1-B}{a^{2k}} - \frac{u}{ka^{2k}} + [(1+B)k + u]a_k - \frac{[(1-B)k-u]b'_{-k}}{b^{2k}} = 0 \quad \text{for } k < 0
\end{cases}
\]
\[
\begin{cases}
\frac{-1+B}{b^{2k}} + \frac{u}{kb^{2k}} + [(1-B)k + u]a'_k - \frac{[(1+B)k-u]b_{-k}}{b^{2k}} = 0 \\
\frac{-1-B}{b^{2k}} + \frac{u}{kb^{2k}} + [(1+B)k + u]a_k - \frac{[(1-B)k-u]b'_{-k}}{a^{2k}} = 0 \quad \text{for } k > 0
\end{cases}
\]

the solutions are

\[ a_0 + a'_0 + b_0 + b'_0 = \frac{2}{u} + lnb^2 \]

\[
\begin{cases}
a_k = \frac{1-B}{1+B} \frac{1}{k(b^{2k} - a^{2k})} - \frac{2u}{k(1+B)(1+B)k+u k(b^{2k} - a^{2k})} \\
a'_k = \frac{1-B}{1+B} \frac{1}{k(b^{2k} - a^{2k})} - \frac{2u}{k(1+B)(1+B)k+u k(b^{2k} - a^{2k})} \\
b_{-k} = \frac{k(b^{2k} - a^{2k})}{b^{2k}} \Rightarrow b_k = \frac{k(b^{2k} - a^{2k})}{a^{2k}} \quad \text{for } k < 0
\end{cases}
\]
\[
\begin{cases}
a_k = \frac{1-B}{k(b^{2k} - a^{2k})} - \frac{2u}{k(1+B)(1+B)k+u k(b^{2k} - a^{2k})} \\
a'_k = \frac{1-B}{k(b^{2k} - a^{2k})} - \frac{2u}{k(1+B)(1+B)k+u k(b^{2k} - a^{2k})} \\
b_{-k} = \frac{k(b^{2k} - a^{2k})}{b^{2k}} \Rightarrow b_k = \frac{k(b^{2k} - a^{2k})}{a^{2k}} \quad \text{for } k > 0
\end{cases}
\]
Inserting the coefficients $a_k, a'_k, b_k$ and $b'_k$ back into the ansatz, we obtain the Green’s function on the annulus, which is

$$G(z, w) = \frac{\alpha'}{2} \left\{ \frac{-ln|z - w|^2 + \frac{2}{u} + ln b^2}{ln(a^2/b^2)} \right\}$$

$$+ \frac{[ln(z\bar{z}/b^2) - 2/u][ln(w\bar{w}/b^2) - 2/u]}{ln(a^2/b^2)}$$

$$- \sum_{n=1}^{\infty} \left[ ln \left| 1 - \left( \frac{a}{b} \right)^{2n} \frac{z}{w} \right|^2 + ln \left| 1 - \left( \frac{a}{b} \right)^{2n} \frac{w}{z} \right|^2 \right]$$

$$- \frac{1 - B}{1 + B} \sum_{n=1}^{\infty} \left[ ln \left( 1 - \left( \frac{a}{b} \right)^{2n} \frac{b^2}{zw} \right) + ln \left( 1 - \left( \frac{a}{b} \right)^{2n} \frac{z\bar{w}}{a^2} \right) \right]$$

$$- \frac{1 + B}{1 - B} \sum_{n=1}^{\infty} \left[ ln \left( 1 - \left( \frac{a}{b} \right)^{2n} \frac{b^2}{zw} \right) + ln \left( 1 - \left( \frac{a}{b} \right)^{2n} \frac{\bar{z}\bar{w}}{a^2} \right) \right]$$

$$+ \sum_{k=1}^{\infty} \frac{2u}{k(1 + B)} \left\{ \frac{1}{(1 + B)k - u b^{2k} - a^{2k}} \left( \frac{b^2}{z\bar{w}} \right)^k \right\}$$

$$+ \sum_{k=1}^{\infty} \frac{2u}{k(1 - B)} \left\{ \frac{1}{(1 - B)k - u b^{2k} - a^{2k}} \left( \frac{b^2}{z\bar{w}} \right)^k \right\}$$

$$- \sum_{k=1}^{\infty} \frac{2u}{k(1 + B)} \left\{ \frac{1}{(1 + B)k + u b^{2k} - a^{2k}} \left( \frac{z\bar{w}}{a^2} \right)^k \right\}$$

$$- \sum_{k=1}^{\infty} \frac{2u}{k(1 - B)} \left\{ \frac{1}{(1 - B)k + u b^{2k} - a^{2k}} \left( \frac{\bar{z}\bar{w}}{a^2} \right)^k \right\}$$

Note that $u = 0$ is the singular point for this Green’s function, which matches the result in [4]. But instead of $\sim 1/u$ behavior on the disc, the Green’s function on the annulus behaves $\sim 1/u^2$. It’s natural to conjecture that for the world sheet with $N$ boundaries, the Green’s function behaves $\sim 1/u^N$.

### 4 The Partition Function

We will use the Green’s function which we found above to evaluate the partition function on the annulus as follows. Define

$$\langle X(\theta)X(\theta) \rangle |_{\partial \Sigma} = \lim_{\epsilon \to 0} [X(\theta)X(\theta) - f(\epsilon)]$$

(21)
where

\[
f(\epsilon) = -\frac{2}{1 + B} \ln(1 - e^{i\epsilon}) - \frac{2}{1 - B} \ln(1 - e^{-i\epsilon})
\]  

(22)

To simplify our presentation we consider the case where the only non-zero components of the \(B\)-field are \(B_{12} = -B_{21} = b_B\). The result can be generalized easily as below. In this case we have two scalar fields \(X^1\) and \(X^2\) with \(u_1 = u_2 = u\). With the above definitions, we have

\[
\frac{2}{\alpha'} \langle X_1(\theta)X_1(\theta) \rangle|_{r=b} = \frac{2}{\alpha'} \langle X_2(\theta)X_2(\theta) \rangle|_{r=b}
\]

\[
= -\frac{4}{1 + b_B^2} \sum_{n=1}^{\infty} \ln \left| 1 - \left( \frac{a}{b} \right)^{2n} \right|^2 + \frac{2}{u} + \frac{4}{u^2 \ln(a^2/b^2)}
\]

\[
+ \frac{4u}{1 + b_B^2} \sum_{k=1}^{\infty} \frac{1}{k} \frac{k - u - b_B^2 k}{(k - u)^2 + b_B^2 k^2} \frac{a^{2k}}{b^{2k} - a^{2k}}
\]

\[
- \frac{4u}{1 + b_B^2} \sum_{k=1}^{\infty} \frac{1}{k} \frac{k + u - b_B^2 k}{(k + u)^2 + b_B^2 k^2} \frac{a^{2k}}{b^{2k} - a^{2k}}
\]

\[
\Rightarrow \frac{d}{du} \ln Z(a/b)
\]

\[
= -\frac{1}{8\pi} \left[ \int_0^{2\pi} \langle X(\theta)X(\theta) \rangle|_{r=b} d\theta - \int_0^{2\pi} \langle X(\theta)X(\theta) \rangle|_{r=a} d\theta \right]
\]

\[
= -\frac{\alpha'}{4} \left[ -\frac{4u}{1 + b_B^2} \sum_{k=1}^{\infty} \frac{1}{k} \frac{k - u - b_B^2 k}{(k - u)^2 + b_B^2 k^2} - \frac{4u}{1 + b_B^2} \sum_{k=1}^{\infty} \frac{1}{k} \frac{k + u - b_B^2 k}{(k + u)^2 + b_B^2 k^2} \right] + \frac{4}{u}
\]

7
\[ Z(a/b) = \left[ \frac{\Gamma(1 + \frac{u}{1 + iB})\Gamma(1 + \frac{u}{1 - iB})\Gamma(1 - \frac{u}{1 + iB})\Gamma(1 - \frac{u}{1 - iB})}{u^2} \right]^{\alpha'/2} \] (26)

It is a surprise that the partition function on the annulus does not depend on the modulus of the annulus. We believe that there must be a deep reason which we do not understand right now. To include all topologically different annuli, we need to integrate over the ratio \( a/b \) from 0 to 1. It can be trivially done as

\[ \Rightarrow Z(a/b) = \int_0^1 Z(a/b) \, d(a/b) = \left[ \frac{\Gamma(1 + \frac{u}{1 + iB})\Gamma(1 + \frac{u}{1 - iB})\Gamma(1 - \frac{u}{1 + iB})\Gamma(1 - \frac{u}{1 - iB})}{u^2} \right]^{\alpha'/2} \] (27)

Having obtained the partition function for the special case of a single non-vanishing component of the \( B \)-field, it is easy to determine the partition function for the general case. Using the formula

\[ \text{det} \Gamma \left( 1 + \frac{u}{1 + B} \right) = \text{det} \Gamma \left( 1 + \frac{u}{1 - B} \right) = \Gamma \left( 1 + \frac{u}{1 + iB} \right) \Gamma \left( 1 + \frac{u}{1 - iB} \right) \] (28)

\[ \text{det} \Gamma \left( 1 - \frac{u}{1 + B} \right) = \text{det} \Gamma \left( 1 - \frac{u}{1 - B} \right) = \Gamma \left( 1 - \frac{u}{1 + iB} \right) \Gamma \left( 1 - \frac{u}{1 - iB} \right) \]

We obtain our final result

\[ Z = \left\{ \frac{\text{det} \left[ \Gamma(1 + \frac{u}{1 + B})\Gamma(1 + \frac{u}{1 - B})\Gamma(1 - \frac{u}{1 + B})\Gamma(1 - \frac{u}{1 - B}) \right]}{u^4} \right\}^{\alpha'/4} \] (29)

When \( \alpha' = 1 \), \( Z \sim 1/u = 1/(\sqrt{u})^2 \) around \( u = 0 \), which agrees with the analysis in [2].
5 Discussion

We have obtained the partition function with $B$-field on the annulus. Compare to the partition function on the disk, there are some new features which need to further understand.

- The Green’s function on the annulus has a term $\sim 1/u^2$ instead of $\sim 1/u$ on the disc. So it’s natural to ask whether the Green’s function on a world sheet with $N$ boundaries will have a term $\sim 1/u^N$.

- The partition function on the annulus is modulus independent, there must be a deep reason for it.

- In eq. (27), if $b_B \neq 0$, there is no singularity for any real value of $u$. But if $b_B = 0$, then all of the integer points turn to be singular points. The question is what physical meaning of those singular points.

Using one loop effective tachyon action to understand tachyon condensation is currently under investigation.

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