J-matrix and Isolated States

A. M. Shirokov
Skobeltsyn Institute of Nuclear Physics, Moscow State University, Moscow, 119992, Russia

S. A. Zaytsev
Physics Department, Khabarovsk State Technical University, Tikhookeanskaya 136, Khabarovsk 680035, Russia

We show that a quantum system with nonlocal interaction can have bound states of unusual type — Isolated States (IS). IS is a bound state that is not in correspondence with the S-matrix pole. IS can have a positive as well as a negative energy and can be treated as a generalization of the bound states embedded in continuum on the case of discrete spectrum states. The formation of IS in the spectrum of a quantum system is studied using a simple rank–2 separable potential with harmonic oscillator form factors. Some physical applications are discussed, in particular, we propose a separable NN potential supporting IS that describes the deuteron binding energy and the s-wave triplet and singlet scattering phase shifts. We use this potential to examine the so-called problem of the three-body bound state collapse discussed in literature. We show that the variation of the two-body IS energy causes drastic changes of the binding energy and of the spectrum of excited states of the three-nucleon system.

I. INTRODUCTION

Quantum mechanical bound states are known to have the wave functions decreasing rapidly at large distances \( r \). Usually the bound states are possible at negative energies \((E < 0)\) only while at positive energies \((E > 0)\) the system has only continuum spectrum states with wave functions oscillating at large distances \( r \). Nevertheless von Neumann and Wigner showed long ago [1] that a quantum system can have a bound state at positive energy \( E > 0 \). Such states are conventionally referred to as ‘continuum bound states’ or as ‘bound states embedded in continuum’ (BSEC). Von Neumann and Wigner used a local potential in their study of BSEC. BSECs are also natural when the interaction is nonlocal (see [2] and references therein) or in the case of multichannel scattering (see, e.g., [3, 4, 5]).

A phenomenological nonlocal nucleon-nucleon (\( NN \)) potential supporting BSEC was suggested by Tabakin [6]. The potential predicts the \( NN \) data fairly well. However it was found that Tabakin potential generates an extremely large binding energy for the three-nucleon system [7, 8]. Such a ‘bound state collapse’ was investigated by several groups of workers [9, 10, 11, 12, 13, 14] with Tabakin and

*Electronic address: shirokov@nucl-th.sinp.msu.ru
†Electronic address: zaytzev@mail.khb.ru
similar nonlocal potentials. All these groups associated the bound state collapse with the two-body BSEC and suggested various interpretations of the origin of such a puzzling phenomenon.

In some of these studies \[11, 12, 14\], BSEC was interpreted as an $S$-matrix pole on the real positive energy axis. This is obviously a mistake since it is well known from textbooks \[15\] that the unitarity condition for scattering requires $|S(E)| = 1$ for real $E > 0$ in the case of elastic scattering ($|S(E)| \leq 1$ for real $E > 0$ in the case of inelastic scattering), hence the $S$-matrix $S(E)$ cannot have poles on the real positive energy axis. BSEC appears to be a very interesting state: contrary to the conventional discrete spectrum states it is a bound state which is not associated with any of the $S$-matrix poles \[16\]. We introduce Isolated States (IS) that are, by definition, bound states that do not correspond to the $S$-matrix poles. Isolated states are a generalization of BSECs: any BSEC is IS, however the energy $E_I$ of IS can be also negative, in particular, the ground state of the system can be an Isolated State. Generally, if the $S$-matrix $S(E)$ is known than we obtain the energies of the discrete spectrum states by associating the $S$-matrix poles at negative energies with the discrete spectrum states. The information about the IS energy cannot be extracted from the $S$-matrix, such bound states are isolated from the continuum spectrum states.

In the next Section we present a nonlocal interaction supporting IS. Within the $J$-matrix formalism, it is easy to formulate a realization of this interaction which makes it possible to find a simple analytical expression for the $S$-matrix. This interaction is used to study the IS formation when the Hamiltonian parameters are varied, the IS contribution to the Levinson theorem, etc.

We address the bound state collapse problem in Section III. We fit the parameters of our exactly solvable nonlocal interaction model supporting IS to the $NN$ scattering data. The IS energy $E_I$ is arbitrary because it is not related to the $S$-matrix. We show that varying the IS energy $E_I$ (note that the $S$-matrix is unaffected by this variation) we produce great changes of the three-body binding energy: some $E_I$ values bring us to the $^3\text{H}$ system with three bound states with extremely large (few GeV) binding energies; with larger $E_I$ values we obtain the $^3\text{H}$ nucleus with two bound states (the binding energy of the ground state is of the order of few hundreds MeV); the further increase of $E_I$ results in the further decrease with $E_I$ of the binding energy of the $^3\text{H}$ nucleus which has a single bound state; if the IS energy $E_I$ is large enough the $^3\text{H}$ nucleus becomes unbound. The experimental value of the $^3\text{H}$ binding energy can be exactly reproduced if some particular positive energy $E_I$ is taken when IS appears to be BSEC. Therefore the problem of the three-body bound state collapse does not exists as a general problem: this problem arose only due to the use of very restricted models of interaction supporting BSEC for the construction of $NN$ potentials, i. e. this problem is inherent for such interaction models only.

We present in Section IV a short discussion and compare our results with the results of other authors who studied the three-body bound state collapse problem.
II. SIMPLE POTENTIAL MODEL SUPPORTING ISOLATED STATE

The radial wave function $\Psi^I_E(r)$ satisfies the Schrödinger equation

$$(T^I + V^I - E)\Psi^I_E(r) = 0,$$  \hspace{1cm} (1)

where $E$ is the energy, $l$ is the angular momentum, $T^I$ is the kinetic energy operator and $V^I$ is the potential energy.

Let $\{ |i\rangle \}$, $i = 0, 1, 2, \ldots$ be a complete $L^2$ basis. The Hamiltonian matrix $H_{ij} \equiv \langle i | H | j \rangle$ is generally infinite and the wave function $\Psi^I_E(r)$ at any energy $E$ can be expressed generally as an infinite expansion in basis functions,

$$\Psi^I_E(r) = \sum_{i=0}^{\infty} \alpha_i(E) |i\rangle.$$ \hspace{1cm} (2)

However, at some particular energy $E_I$ the infinite Hamiltonian matrix $H_{ij}$ can have a finite eigenvector, and the wave function $\Psi_I(r)$ at this energy is expressed as a finite expansion in basis functions,

$$\Psi_I(r) = \sum_{i=0}^{M} \alpha_i(E_I) |i\rangle.$$ \hspace{1cm} (3)

The wave function $\Psi_I(r)$ rapidly decreases with distance $r$ since it is a superposition of a finite number of $L^2$ functions. Therefore at energy $E_I$ we have a bound state of a particular type hereafter refered to as an Isolated State (IS).

Clearly, we have the IS solutions of the type (3) in the case when the Hamiltonian matrix $H_{ij}$ is block-diagonal,

$$H_{ij} = H^{(1)}_{ij} \oplus H^{(2)}_{ij},$$ \hspace{1cm} (4)

where the $(M+1) \times (M+1)$ submatrix $H^{(1)}_{ij}$ is defined in a finite-dimensional subspace spanned by the basis functions $|i\rangle$ with $i = 0, 1, \ldots, M$. The infinite-dimensional submatrix $H^{(2)}_{ij}$ is defined in the orthogonal supplement to this subspace. Any eigenvector of the submatrix $H_{ij}$ gives rise to the wave function of the type (3), i.e. each of the submatrix $H_{ij}$ eigenvectors is associated with IS.

The scattering state wave functions with oscillating asymptotics at energies $E > 0$, can be expressed only as a superposition of an infinite number of $L^2$ functions,

$$\Psi^I_E(r) = \sum_{i=M+1}^{\infty} \alpha_i^{(2)}(E) |i\rangle,$$ \hspace{1cm} (5)

where $\{ \alpha_i^{(2)}(E) \}$ are eigenvectors of the infinite submatrix $H^{(2)}_{ij}$. The $S$-matrix and scattering phase shifts are defined through the asymptotics of the functions (5), hence they are governed by the structure of the submatrix $H^{(2)}_{ij}$ only. The
energy $E_I$ of IS and its other features are dictated by the structure of the other submatrix $H^{(1)}_{ij}$ that is generally independent from $H^{(2)}_{ij}$. Therefore we cannot expect that the S-matrix has a pole at the IS energy $E_I$. We can define the Isolated State as the bound state that is not associated with any of the $S$-matrix poles. The IS energy $E_I$ can be positive, and in this case it appears to be the so-called bound state embedded in continuum (BSEC). Any BSEC is IS. The IS energy $E_I$ can be also negative, in particular, the ground state of the system can be isolated. Thus IS can be treated as a generalization of BSEC on the case of arbitrary (negative or positive) energy.

The $J$-matrix formalism \[17\] makes it possible to study IS properties and to formulate a simple exactly-solvable model of a system possessing IS. In this contribution we use the oscillator basis; the exactly-solvable model of IS can be also easily formulated by means of the $J$-matrix formalism with the Laguerre basis.

The idea of the block-diagonal structure of the Hamiltonian matrix \[4\] can be easily realized if the interaction between the particles is described by a separable nonlocal potential of the rank $N + 1$, $V^l = \sum_{n,n'=0}^N V^l_{nn'} |\varphi_{nl}(r)\rangle \langle \varphi_{n'l'}(r')|,$ (6)

with the harmonic oscillator form factors

$$\varphi_{nl}(r) = (-1)^n \left[ \frac{2n!}{r_0 \Gamma(n + l + \frac{3}{2})} \right]^{\frac{1}{2}} \left( \frac{r}{r_0} \right)^{l+1} \exp\left( -\frac{r^2}{2r_0^2} \right) L_{n}^{l+\frac{1}{2}} \left( \frac{r^2}{r_0^2} \right).$$ (7)

Here $r_0 = (\hbar/m\omega)$ is the oscillator radius and $L_n^\alpha(x)$ is the Laguerre polynomial.

In the $J$-matrix method, the wave function has a form of series in terms of $L^2$ functions \[7\],

$$\Psi^l_E(r) = \sum_{n=0}^\infty X_n(E) \varphi_{nl}(r).$$ (8)

The coefficients $X_n(E)$ for $n \geq N$ are given by the formula

$$X_n(E) = S_{nl}(p) \cos \delta_l + C_{nl}(p) \sin \delta_l,$$ (9)

where $p = \sqrt{2E/\hbar\omega}$ is the momentum, $S_{nl}(p)$ and $C_{nl}(p)$ are the eigenvectors of the infinite tridiagonal matrix of the kinetic energy $T^l_{nn'}$. The following analytical expressions \[17\] can be used to calculate $S_{nl}(p)$ and $C_{nl}(p)$:

$$S_{nl}(p) = \left[ \frac{2\Gamma(n + l + \frac{3}{2})}{\Gamma(n + 1)} \right]^{\frac{1}{2}} \frac{p^{l+1}}{\Gamma(l + \frac{1}{2})} \exp\left( -\frac{p^2}{2} \right) {}_1F_1\left( -n, l + \frac{3}{2}; p^2 \right).$$ (10)

$$C_{nl}(p) = \left[ \frac{2\Gamma(n + 1)}{\Gamma(n + l + \frac{3}{2})} \right]^{\frac{1}{2}} \frac{(-1)^l}{p^{l}} \exp\left( -\frac{p^2}{2} \right) {}_1F_1\left( -n - l - \frac{1}{2}, -l + \frac{1}{2}; p^2 \right).$$ (11)
Fig. 1: The structure of the Hamiltonian matrix.

The phase shift $\delta_l$ in the partial wave with the angular momentum $l$ can be calculated as

$$\tan \delta_l = \frac{S_{NL}(p) - \varphi_{NN}(E) S_{N+1,l}(p)}{C_{NL}(p) - \varphi_{NN}(E) C_{N+1,l}(p)},$$

where

$$\varphi_{nn'}(E) = -\sum_{\mu} \frac{U_{n}^{\mu} U_{n'}^{\mu}}{\varepsilon_{\mu} - E} T_{n',n'+1}.$$  \hspace{1cm} (13)

$U_n^\mu$ ($n = 0, 1, ..., N$) are the eigenvectors and $\varepsilon_\mu$ are the corresponding eigenvalues of the truncated Hamiltonian matrix $H_{nn'}^N = T_{nn'} + V_{nn'}$ ($n, n' = 0, 1, ..., N$). The coefficients $X_n(E)$ for $n \leq N$ can be found by the formula

$$X_n(E) = \varphi_{nN}(E) X_{N+1}(E).$$  \hspace{1cm} (14)

We are considering the case when the Hamiltonian matrix is block-diagonal. We note that the kinetic energy matrix in the oscillator basis is tridiagonal. Hence with the interaction (6) we can obtain the Hamiltonian matrix in the oscillator basis of the type (4) that has the structure shown in Fig. 1. Solid lines schematically show the infinite tridiagonal kinetic energy tail of the Hamiltonian matrix, the rest non-zero matrix elements are displayed by two boxes representing submatrices $H^{(1)}$ and $H^{(2)}$ (we include the infinite tridiagonal kinetic energy tail in the submatrix $H^{(2)}$).

Obviously, the eigenvectors of the submatrix $H^{(1)}$ are the eigenvectors of the entire Hamiltonian matrix $H$, too. The corresponding wave functions are decreasing with $r$ similarly to the bound state wave functions since they are superpositions (4) of a finite number of the oscillator functions $|i\rangle = \varphi_{nl}(r)$ given by Eq. (7). However these wave functions have an unusual asymptotics $\sim \exp[-r^2/(2r_0^2)]$ instead of the standard one $\sim \exp(-\alpha r)$. All eigenstates of the submatrix $H^{(1)}$ are
It is obvious that generally the submatrix $H^{(1)}$ may have positive or/and negative eigenvalues $\varepsilon^{(1)}_\mu$. If $\varepsilon^{(1)}_\mu > 0$, the corresponding state appears to be BSEC. If $\varepsilon^{(1)}_\mu < 0$, the corresponding state appears to be a bound state of a specific type.

The continuum spectrum states with the oscillating asymptotically wave functions as well as the conventional bound states with the $e^{-\alpha r}$-type wave function asymptotics, can be expressed only as infinite series of the oscillator functions $|i\rangle = \varphi_{nl}(r)$ given by Eq. (1) (5). They are generated by the submatrix $H^{(2)}$. The scattering and conventional bound state wave functions (5) are obviously orthogonal to the IS wave functions (4). Thus the scattering and usual bound state wave functions have node(s) at a small distance ($\sim r_0$) in the presence of IS(es) similarly to the wave functions generated by the potentials with forbidden states (18).

The asymptotic behavior of the scattering state wave functions is characterized completely by the $S$-matrix (4). Hence the structure of the $S$-matrix is governed by the infinite-dimensional submatrix $H^{(1)}$ that is independent from the submatrix $H^{(2)}$. The IS energies $\varepsilon^{(1)}_\mu$, on the other hand, are controlled by the submatrix $H^{(1)}$ and are independent from the submatrix $H^{(2)}$. Varying matrix elements $H^{(1)}_{mm'}$ of the submatrix $H^{(1)}$ one causes variation of the IS energies $\varepsilon^{(1)}_\mu$ without affecting the $S$-matrix. Thus, the energy of IS $\varepsilon^{(1)}_\mu$ is not in correspondence with the location of the $S$-matrix poles. Using symmetry properties of the $S$-matrix as a function of the complex momentum $p$ (4), it is easy to show (10) that the energy of BSEC is not in correspondence with any of the $S$-matrix poles. An interesting new point, so far as we know never discussed in literature, is the appearance of the discrete spectrum states, i.e., of IS with negative energy $\varepsilon^{(1)}_\mu < 0$, that are divorced from the $S$-matrix poles. So, IS being the state with the asymptotically decreasing wave function and with the energy at which the $S$-matrix does not have a pole, can been treated as a generalization of BSEC on the case of the discrete spectrum states.

We examine in more detail the formation of IS in the spectrum of a quantum system with nonlocal interaction using as an example a simple analytically solvable model. The simplest realization of the situation depicted in Fig. 1 corresponds to the case when the submatrix $H^{(1)}$ is a $1 \times 1$ matrix and the separable potential $V_0$ is of the rank 2, i.e. $N = 1$. In this case, IS arises due to the cancellation of the potential energy matrix elements $V_{01} = V_{10}$ and the kinetic energy matrix elements $T_{01} = T_{10} = -V_{01}$ that results in $H_{01} = H_{10} = 0$. The IS energy $\varepsilon_0$ is equal to the diagonal matrix element $H_{00}$. $\varepsilon_0 = H_{00}$. It should be stressed that $H_{00}$ can take an arbitrary value in our model. Using Eqs. (10) (13) we obtain the following expression for the phase shift $\delta_1$:

$$\tan \delta_1 = -\frac{S_{01}(p) \left\{ \left[ V_{11}(\varepsilon_0 - E) - \beta \right] (T_{01}^0 - E) + (T_{01}^0)^2 (\varepsilon_0 - E) \right\}}{C_{01}(p) \left\{ \left[ V_{11}(\varepsilon_0 - E) - \beta \right] (T_{10}^0 - E) + (T_{10}^0)^2 (\varepsilon_0 - E) \right\} - \frac{p [V_{11}(\varepsilon_0 - E) - \beta]}{S_{01}(p)}},$$

where $\beta \equiv H_{01}^2$.

Suppose $\varepsilon_0 > 0$ and $V_{11} > 0$. The evolution of the $s$ wave phase shift $\delta_0(p)$ as $\beta = H_{01}^2 = H_{10}^2$ tends to zero is shown in Fig. 2. If $\beta \neq 0$ there is a resonance at the energy $E \approx \varepsilon_0$ of the width $\Gamma$ that decreases as $\beta$ is reduced. When $\beta = 0$,
Fig. 2: The evolution of the phase shift $\delta_0(p)$ as $\beta = H_{01} = H_{10}$ tends to zero. Dashed, dotted, and solid curves are the phase shifts obtained with $\beta = \beta_1, \beta_2$ and $\beta_3$, respectively; $\beta_1 > \beta_2 > \beta_3 = 0$.

BSEC arises as the resonance of the zero width producing the jump of the height $\pi$ of the phase shift $\delta_0$ at the energy $E = \varepsilon_0$. This spurious jump should be eliminated that results in the $\delta_0(0)$ increase by an extra $\pi$ if we suppose, as usual, that $\delta_0(\infty) = 0$. Thus when applying the Levinson theorem [3] to the system pertaining IS, one should count IS as a usual discrete spectrum bound state. Such behavior of the phase shift is typical for systems pertaining BSEC that have been studied in various models [3, 16, 19]. Thus our model represents an alternative simple analytical approach in the study of BSEC.

As for the S-matrix, it is given by the following expression:

$$S_l = -\frac{C_{0l}^{(-)}(p)}{C_{0l}^{(+)}(p)} \left\{ \left[ V_{1l}^{(l)}(\varepsilon_0 - E) - \beta \right] (T_{00}^{l} - E) + \left[ T_{01}^{l} \right]^2 (\varepsilon_0 - E) \right\} - \frac{pV_{1l}^{(l)}(\varepsilon_0 - E) - \beta}{\pi S_{0l}(p)},$$

(16)

where $C_{nl}^{(\pm)}(p) = C_{nl}(p) \pm iS_{nl}(p)$. The single S-matrix pole on the unphysical sheet tends to the real energy $E = \varepsilon_0$ as $\beta$ tends to zero. However in the limit $\beta = 0$, the factor $(\varepsilon_0 - E)$ in the numerator of Eq. (16) cancels the same factor in the denominator and the singularity at the energy $E = \varepsilon_0$ disappears:

$$S_l = -\frac{C_{0l}^{(-)}(p)}{C_{0l}^{(+)}(p)} \left\{ V_{1l}^{(l)}(T_{00}^{l} - E) + \left[ T_{01}^{l} \right]^2 (\varepsilon_0 - E) \right\} - \frac{pV_{1l}^{(l)}(\varepsilon_0 - E) - \beta}{\pi S_{0l}(p)},$$

(17)

This illustrates the mechanism of the S-matrix pole loss in the limit $\beta \to 0$ when the resonance transforms into BSEC. The nontrivial result is that if IS is a bound state ($\varepsilon_0 < 0$), it does not generate the S-matrix pole, too.
The above results can be easily generalized as the following statement.

Let all eigenvalues of the truncated Hamiltonian matrix $H^N$ be non-degenerate. Then Isolated States occur in the spectrum of the system with the rank-$(N + 1)$ separable interaction \( \tilde{H}^N \) if and only if the truncated matrix $\tilde{H}^N$ and its principal minor $\tilde{H}^{N-1}$ of the rank $(N - 1)$ have common eigenvalues. The number \( \nu \) of the common eigenvalues is equal to the number of the Isolated States. These eigenvalues and the corresponding eigenfunctions are just the energies and the wave functions of the Isolated States.

The equivalent formulation is:

The system with the nonlocal interaction \( \delta_i \) has the Isolated State at the energy $\varepsilon_{\mu}$ if and only if $\varepsilon_{\mu}$ is the eigenvalue of the truncated Hamiltonian matrix $\tilde{H}^N$ and the corresponding eigenvector $U^\mu$ has the last component $U^\mu_N = 0$.

### III. PHENOMENOLOGICAL $NN$ POTENTIAL WITH ISOLATED STATE AND THE THREE-NUCLEON SYSTEM

The simplest rank-2 separable potential \( \delta_i \) supporting IS discussed in the previous Section, was used to fit the $NN$ singlet $^1s_0$ and triplet $^3s_1$ scattering phase shifts. The oscillator function parameter $\hbar \omega = 500$ MeV. To ensure the existence of IS, we should set the off-diagonal potential energy matrix elements $V_{01} = V_{10} = -T_{10}$. The phase shifts are independent of the matrix element $V_{00}$ that governs the IS energy. The only parameter responsible for the phase shifts is the matrix element $V_{11}$.

The singlet phase shifts obtained with $V_{11}^s = -0.7315 \hbar \omega$ and the triplet phase shifts obtained with $V_{11}^t = -0.81512 \hbar \omega$ are shown in Figs. 3 and 4 respectively. These values of $V_{11}^s$ and $V_{11}^t$ are seen to reproduce with a reasonable accuracy the scattering data.

The deuteron ground state energy should be obtained by the calculation of the $S$-matrix pole as is discussed in Refs. [20, 21]. Since in our case the $S$-matrix is given by the expression (17), the $S$-matrix poles can be calculated by solving the equation

\[
C^{(+)}_0(p) \left[ V_{11}^t (T_{00} - E) + (T_{01}^t)^2 \right] - \frac{p V_{11}^t}{\pi S_0(p)} = 0. \tag{18}
\]

The bound state (deuteron) should be searched for in the triplet $^3s_1$ wave, i.e. $l = 0$ and $V_{11}^t$ should be used as $V_{11}^t$ in Eq. (18). We are searching for negative $E_d$ value and $p_d = i \sqrt{2 E_d / \hbar \omega}$ fitting Eq. (18). The deuteron wave function can be calculated using the $J$-matrix formalism (see also the discussion in Refs. [20, 21] and references therein). Our very simple potential with the only fitting parameter $V_{11}^t$ provides a good description of the deuteron energy $E_d = -2.22496$ MeV and rms radius $\sqrt{\langle r^2 \rangle} = 1.87$ fm (the respective experimental values are $E_d^{\text{exp}} = -2.224575$ MeV and $\sqrt{\langle r^2 \rangle_{\text{exp}}} = 1.9676$ fm).

The obtained singlet and triplet $s$ wave $NN$ potentials are used in the calculation of the triton bound states. As in the other studies of the three-body bound state
we do not allow for the interaction in other partial waves. We perform a conventional variational calculation of the $S = T = J = \frac{1}{2}$ three-nucleon states with the three-body oscillator basis allowing for all components with the total number of oscillator quanta $N = 2\alpha + l \leq 32$. The energies of the ground state and of the lowest excited states are shown in Fig. 5 for different $E_I$ values (we are varying both the triplet $V_{00}^T$ and the singlet $V_{00}^S$ potential...
energy matrix elements so that $V_{00}^s = V_{00}^t$; in other words the IS energy $E_I^s$ in the triplet state is equal to the IS energy $E_I^t$ in the singlet state: $E_I = E_I^s = E_I^t$). It is seen that the variation of the IS energy $E_I$ (that does not affect the phase shifts and the deuteron properties) results in the drastic changes of the triton binding energy and of the spectrum of excited $S = T = J = \frac{1}{2}$ states. When the IS energy $E_I$ is small enough, the three-nucleon system collapses (the binding energy becomes extremely large); two excited states are bound in addition to the ground state. The triton ground state energy increases with the IS energy $E_I$; the same is true for the energies of the excited states. At some $E_I$ value the second excited state becomes unbound; at some larger $E_I$ value the first excited state becomes unbound, too; however the triton binding energy is still too large. Nevertheless the further increase of $E_I$ results in the increase of the triton ground state energy and at some $E_I$ value the three-nucleon system becomes unbound.

We see that we really have the three-body bound state collapse for some values of the IS energy $E_I$. However the collapse disappears for larger $E_I$ values. Therefore our general conclusion is that the three-body bound state collapse problem does not exist as a general problem. In the previous studies of the three-body bound state collapse, very restricted potential models were used that did not allow to change the BSEC energy $E_{BSEC}$. The $E_{BSEC}$ value used in these studies caused the extreme overbinding of the trinucleon. This is clearly the problem of the particular potentials used and not the general problem of the three-body bound state collapse inherent for all two-body potentials supporting BSEC. In our model any trinucleon binding energy can be obtained by phase equivalent variation of the IS energy $E_I$. In particular, we can fit the $E_I$ value to the triton binding energy. Setting $E_I = E_I^s = E_I^t = 189.525$ MeV, we obtain the triton binding energy of $E_t = 8.47307$ MeV in our $32\hbar\omega$ variational calculations. The accuracy of the variational approach is improved if the three-body $S$-matrix pole is calculated as is discussed in Ref. [20]: in this case the triton binding energy of $E_t = 8.4748$ MeV

Fig. 5: The triton ground and excited state energies vs the IS energy $E_I$. 
IV. DISCUSSION

We suppose that the most important feature of BSEC is that this bound state is not in correspondence with the $S$-matrix pole. We introduce IS as a generalization of BSEC on the case of arbitrary (positive or negative) energy. We propose a simple model of non-local interaction elucidating the formation of IS. To investigate in detail the formation and the features of IS, it is natural to employ the $J$-matrix formalism that makes it possible to obtain analytical expressions for the $S$-matrix and other observables, i.e. to formulate the exactly solvable model of IS.

The general results given above can be straightforwardly extended on the coupled channel case, on the case of the separable finite-rank nonlocal potentials with the Laguerre form factors, etc.

It is interesting that IS naturally appears in the standard nuclear shell model. It is well-known (the Dubovoy-Flores theorem) that the lowest eigenvalue $\varepsilon_{0}^{N}(\omega)$ obtained in the shell model calculation in the $N\hbar\omega$ model space, coincides with the lowest eigenvalue $\varepsilon_{0}^{N-2}(\omega)$ obtained in the shell model calculation in the $(N-2)\hbar\omega$ model space if the parameter $\hbar\omega$ of the oscillator basis minimizes the eigenvalue $\varepsilon_{0}^{N-2}(\omega)$. According to the statement formulated in Section III, the ground state obtained in the conventional variational nuclear shell-model calculations appears to be IS.

Using our exactly solvable model of IS, we obtain a simple nonlocal $NN$ potential describing the singlet and triplet $s$ wave phase shifts and deuteron properties. With this $NN$ interaction we study the triton properties and examine the three-body bound state collapse problem. We can vary the IS energy $E_I$ in our model without violating the description of the phase shift and deuteron properties, i.e. without altering the on-shell properties of the interaction. However the off-shell properties of the interaction are modified when $E_I$ is varied and this is clearly seen in the strong $E_I$-dependence of the triton ground state energy $E_t$. The three-body system collapses at some $E_I$ values and does not at other $E_I$ values. We can even fit the position of IS to the triton binding energy. Therefore we conclude that the discussion of the three-body bound state collapse in literature [9, 10, 11, 12, 13, 14] arose only due to the use of unsuccessful $NN$ potential supporting BSEC; other two-body interactions supporting BSEC provide a correct description of the three-nucleon binding.

Various interpretations of the nature of the three-body bound state collapse have been suggested in literature [9, 10, 11, 12, 13, 14]. We cannot accept most of them.

For examples, Nakaichi-Maeda [13] and Pantis et al. [14] supposed that the three-body bound state collapse arises due to the nodal behavior of the deuteron wave function in the case of potentials supporting BSEC. Delfino et al. [12] supposed that collapse originates from the structure of the BSEC wave function which is identical with that of the Pauli forbidden state. In our model, we shifted the IS energy changing neither the deuteron wave function nor the IS wave function. We
show that the three-body bound state collapse can be eliminated by the increase of the IS energy only, therefore we cannot agree that the collapse originates from the structure of the deuteron or IS wave function.

Delfino et al \cite{11} supposed that the three-body bound state collapse is a manifestation of the Thomas effect (the increase of the triton binding energy when the range of the $NN$ potential tends to zero). The range of the general nonlocal interaction cannot be established unambiguously. Delfino et al suggested to use the average kinetic energy $\langle T \rangle$ of the two-body bound state as an indicator of the potential range and demonstrated that BSEC is accompanied by an increase of $\langle T \rangle$. We cannot agree with this conclusion. In our model the bound state wave functions are unaltered when the IS energy is varied, hence the variation of IS energy does not cause changes of $\langle T \rangle$ of any of the bound state. Therefore with the same values of $\langle T \rangle$ of two-body bound states, we obtain either the collapsed triton at some $E_I$ values or normally-bound (or just unbound) triton at other $E_I$ values.

Rupp et al \cite{10} note that the triton binding energy increases when the BSEC energy grows. We obtain an opposite result in our investigation: the triton binding energy decreases with the BSEC energy. We note here that in our studies we shift BSEC phase equivalently while Rupp et al base their conclusion on the results obtained with a number of potentials providing different phase shifts. In other words, they use potentials with different on-shell properties that can mask any effect of the BSEC energy variation.

It is interesting that it is easy to change the BSEC (or any other bound state) energy phase equivalently. The BSEC wave function $\Psi_{\text{BSEC}}$ fits the Schrödinger equation

$$H\Psi_{\text{BSEC}} = E_{\text{BSEC}}\Psi_{\text{BSEC}}.$$  \hspace{1cm} (19)

Let us define a new Hamiltonian

$$H' = H + \lambda|\Psi_{\text{BSEC}}\rangle\langle\Psi_{\text{BSEC}}|.$$  \hspace{1cm} (20)

The wave functions $\Psi_{E}^I(r)$ of all the rest (bound or continuum spectrum) eigenstates of the Hamiltonian $H$, fit the Schrödinger equation

$$H\Psi_{E}^I(r) = E\Psi_{E}^I(r).$$  \hspace{1cm} (21)

Clearly $\Psi_{E}^I(r)$ will be also the eigenfunctions of the Hamiltonian $H'$ since $\Psi_{\text{BSEC}}$ is orthogonal to any $\Psi_{E}^I(r)$. At the same time, the BSEC energy $E_{\text{BSEC}}$ will be increased by $\lambda$. We are sure that if Rupp et al employed this method of varying the BSEC energy, they would obtain the natural result that the triton binding energy decreases with $E_{\text{BSEC}}$.

The Hamiltonian \cite{20} can be defined with the projector $|\Psi_{E_b}^I\rangle\langle\Psi_{E_b}^I|$ on any other bound state $\Psi_{E_b}^I(r)$ at the energy $E_b$. In this case we obtain another simple model of IS. Really, none of the continuum spectrum wave functions $\Psi_{E_b}^I(r)$ is altered by the projector $|\Psi_{E_b}^I\rangle\langle\Psi_{E_b}^I|$. Hence the Hamiltonians $H$ and $H'$ provide the same $S$-matrix. Thus the projector $\lambda|\Psi_{E_b}^I\rangle\langle\Psi_{E_b}^I|$ increases the energy $E_b$ of the bound state by $\lambda$ but does not shift the $S$-matrix pole at the negative energy $E_b$. As a
result the conventional bound state at the energy $E_b$ is transformed into IS. We note here that the Hamiltonians with the projection operators $\lambda |\Psi_{E_b}\rangle \langle \Psi_{E_b}|$ are widely used, e. g., to project out Pauli forbidden states (see, e. g., [24]).

We do not suppose that it is needed to criticize other interpretations of the origin of the three-body bound state collapse suggested in various papers. Our general conclusion is that the collapse is inherent for very restricted models of potentials supporting BSEC only and does not appear if other two-body interactions with BSEC are used. Therefore for the proper interpretation of the three-body bound state collapse origin, one should carefully study this restriction and reveal what is wrong with it. Up to the best of our knowledge, it was never done.

This work was supported in part by the State Program “Russian Universities” and by the Russian Foundation of Basic research, Grant No 02-02-17316.

[1] J. V. von Neumann and E. Wigner, Phys. Z. 30, 465 (1929).
[2] B. Mulligan, L. G. Arnold, B. Bagchi, and T. O. Krause, Phys. Rev. C 13, 2131 (1976); B. Bagchi, T. O. Krause, and B. Mulligan, Phys. Rev. C 15, 1623 (1977).
[3] R. G. Newton, Scattering Theory of Waves and Particles, 2nd ed. (Springer-Verlag, New York, 1982).
[4] L. Fonda and R. G. Newton, Ann. Phys. (NY) 10, 490 (1960).
[5] H. Friedrich and D. Wintgen, Phys. Rev. A 31, 3964 (1985); Phys. Rev. A 33, 3231 (1985).
[6] F. Tabakin, Phys. Rev. 174, 1208 (1968).
[7] J. E. Beam, Phys. Lett. B 30, 67 (1969).
[8] V. A. Alessandrini and C. A. Garcia Canal, Nucl. Phys A 133, 950 (1969).
[9] S. Sofianos, N. J. McGurk, and H. Fiedeldey, Z. Phys. A 286, 87 (1978); G. Pantis, H. Fiedeldey, and D. W. L. Sprung, Z. Phys. A 291, 367 (1979); 294, 101 (1980).
[10] G. Rupp, L. Streit, and J. A. Tjon, Phys. Rev. C 31, 2285 (1985).
[11] A. Delfino, S. K. Adhikari, L. Tomio, and T. Federico, Phys. Rev. C 46, 471 (1992).
[12] A. Delfino, S. K. Adhikari, L. Tomio, and T. Federico, Phys. Rev. C 46, 1612 (1992).
[13] S. Nakai-Maeda, Phys. Rev. C 51, 1633 (1995).
[14] G. Pantis, I. E. Lagaris, and S. A. Sofianos, Phys. Rev. C 63, 044009 (2001).
[15] L. D. Landau and E. M. Lifshitz, Quantum Mechanics: Non-relativistic Theory (Pergamon Press, Oxford, New York, 1977).
[16] L. A. Roriz and A. Delfino, Canad. J. Phys. 67, 37 (1989).
[17] H. A. Yamani and L. Fishman, J. Math. Phys. 16, 410 (1975).
[18] Yu. L. Dorodnykh, V. G. Neudatchin, and N. P. Yudin, Yad. Fiz. [Sov. J. Nucl. Phys.] 48, 1796 (1988).
[19] B. N. Zakhariev and A. A. Suzko A.A., Direct and Inverse Problems (Springer, Heidelberg, 1990).
[20] Yu. A. Lurie and A. M. Shirokov, nucl-th/0312028 to be published in A. D. Al-haidari, E. J. Heller, H. A. Yamani, and M. S. AbdElmonem (eds.), J-matrix method and its applications (Nova Science Publishers, Inc.).
[21] A. M. Shirokov, A. I. Mazur, S. A. Zaytsev, J. P. Vary, and T. A. Weber, nucl-th/0312029 to be published in A. D. Al-haidari, E. J. Heller, H. A. Yamani, and M. S. AbdElmonem (eds.), J-matrix method and its applications (Nova Science Publishers, Inc.).
[22] M. Dubovoy and J. Flores, Revista Mex. Fiz. 17, 289 (1968).
[23] L. Majling, J. Rizek, and Z. Pluhar, Bull. Acad. Sci. USSR, Phys. ser. 38, 2141 (1974).
[24] V. I. Kukulin, V. N. Pomerantsev, Kh. D. Razikov, V. T. Voronchev, and G. G. Ryzhikh, Nucl. Phys. A 586, 151 (1995).