Construction and enumeration of left dihedral codes satisfying certain duality properties

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Abstract

Let $\mathbb{F}_q$ be the finite field of $q$ elements and let $D_{2n} = \langle x, y \mid x^n = 1, y^2 = 1, yxy = x^{n-1} \rangle$ be the dihedral group of order $n$. Left ideals of the group algebra $\mathbb{F}_q[D_{2n}]$ are known as left dihedral codes over $\mathbb{F}_q$ of length $2n$, and abbreviated as left $D_{2n}$-codes. Let $\gcd(n, q) = 1$. In this paper, we give an explicit representation for the Euclidean hull of every left $D_{2n}$-code over $\mathbb{F}_q$. On this basis, we determine all distinct Euclidean LCD codes and Euclidean self-orthogonal codes which are left $D_{2n}$-codes. Let $\gcd(n, q) = 1$. In this paper, we give an explicit representation for the Euclidean hull of every left $D_{2n}$-code over $\mathbb{F}_q$. On this basis, we determine all distinct Euclidean LCD codes and Euclidean self-orthogonal codes which are left $D_{2n}$-codes. In particular, we provide an explicit representation and a precise enumeration for these two subclasses of left $D_{2n}$-codes and self-dual left $D_{2n}$-codes, respectively. Moreover, we give a direct and simple method for determining the encoder (generator matrix) of any left $D_{2n}$-code over $\mathbb{F}_q$, and present several numerical examples to illustrate our applications.

Keywords: Left dihedral code; Euclidean duality; Hull of a linear code; Self-dual code; LCD code; Self-orthogonal code

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1. Introduction

Algebraically structured codes with self-duality and complementary duality are important families of linear codes that have been extensively studied for both theoretical and practical reasons (see [3, 15, 16, 17, 18, 28, 30, 33, 36, 37], and references therein). One-sided ideals of finite group algebras and group rings are called group codes (cf. [19, 21, 25, 32, 36]). Since group codes have good parameters (see [1, 2, 3, 11, 15, 16, 17, 18, 20, 21, 29], and references therein), they have been one of the important sources of constructing good linear codes.

On the other hand, it could be one of meaningful attempts to develop resistant post-quantum codes based on error-correcting codes by use of group codes (see [12, 13, 14]). Hence it is interesting to study the fine structure of non-abelian group codes for constructing new classes of codes and developing efficient decoding algorithms.

Since the structure of dihedral groups is the simplest among non-abelian groups, the systematic study of dihedral codes is more important and worthy. Here, we give a brief survey on known results of dihedral codes as follows:

♦ In 2006, Bazzi and Mitter [2] shown that for infinitely many block lengths a random left ideal in the binary group algebra of the dihedral group is an asymptotically good rate-half code with a high probability.

♦ In 2020, following the ideas of [2], Borello and Willems [3] proved that group codes over finite fields of any characteristic are asymptotically good. In particular, Fan and Lin [20] constructed asymptotically good Euclidean self-dual dihedral group codes when the characteristic of the field is even; and constructed both the asymptotically good Euclidean self-orthogonal dihedral group codes and the asymptotically good Euclidean LCD dihedral group codes when the characteristic of the filed is odd. But these papers did not consider how to determine the specific structure of every dihedral code.

♦ In 2012, McLoughlin [29] provided a new construction of the self-dual, doubly-even and extremal [48,24,12] binary linear block code using a zero divisor in the group ring $\mathbb{F}_2[D_{48}]$. In recent years, Dougherty et al. constructed certain good self-dual and formally self-dual codes in [15, 16, 17, 18], using some one-sided ideals of dihedral group rings and other group rings.

♦ In 2015, Brochero Martínez [6] shown explicitly all central irreducible idempotents and their Wedderburn decomposition of the dihedral group algebra $\mathbb{F}_q[D_{2n}]$, in the case when every divisor of $n$ divides $q - 1$. In this direction, there are some new results in 2020 and 2021:
Vedenev and Deundyak [35] studied left ideals in dihedral group algebra by one generalization of the Wedderburn decomposition of $\mathbb{F}_q[D_{2n}]$, where $\gcd(q, n) = 1$, established some connections with the theory of cyclic codes and obtained some results about code parameters.

Gao et al. [22] given a descriptions to some self-dual group codes with parameters in the group algebra $\mathbb{F}_q[D_{2n}]$, where $n$ is odd and $q$ is even.

Gao et al. [23] extended the results of [6] to the generalized dihedral group algebra $\mathbb{F}_q[D_{2n,r}]$, where $\gcd(q, n) = 1$ and $q$ is odd, and provided an explicit expression for primitive idempotents of the group algebra. In particular, some Euclidean LCD codes and Euclidean self-orthogonal codes in $\mathbb{F}_q[D_{2n,r}]$ were described and counted.

When $\gcd(4n, q) = 1$, Gao and Yue [24] determined central (right) irreducible idempotents of $\mathbb{F}_q[Q_{4n}]$ in two cases: $q \equiv 1 \pmod{4}$ and $q \equiv 3 \pmod{4}$, and obtained descriptions and enumerations of Euclidean LCD and self-orthogonal codes in the group algebras $\mathbb{F}_q[Q_{4n}]$.

Borello and Jamous [4] studied a subclass of dihedral codes. Specifically, they proved a BCH bound for principal dihedral codes and proposed a definition of principal BCH-dihedral codes.

The above papers are based on radical extensions of the field $\mathbb{F}_q$, the Wedderburn decomposition theory of general semisimple finite group algebra or used the Morita correspondence (see [4]). However, these descriptions to dihedral codes are not detailed enough, so that the precise algebraic structure of the dual code for each dihedral code is no easy to described completely.

In fact, the above descriptions are inconvenient to determine the complete representation and exact enumeration of dihedral codes which satisfy certain duality properties.

The concatenation structure of a linear code is helpful to encode and decode the code efficiently (cf. [34]). Hence one way to solve the above problems is to give the concatenation structure of every dihedral code.

In this direction, some works have been done:

In 2016, Cao et al. [8] provided explicit concatenated structures and precise enumerations to all left dihedral codes, their Euclidean dual codes and the Euclidean self-dual codes in the group algebra $\mathbb{F}_q[D_{2n}]$, for any positive integer $n$ satisfying $\gcd(q, n) = 1$. When $q$ is odd and $n$ is
even satisfying \(\gcd(n, q) = 1\), Cao et al. [9] gave concatenated structures and enumerations for all left quaternion codes and their Euclidean dual codes that are left ideals of the group algebras \(\mathbb{F}_q[Q_{2n}]\), were \(Q_{2n}\) is the generalized quaternion group with \(2n\) elements.

\[\triangleright\] In 2018, Cao et al. [10] extended the results of [8] to left dihedral codes over Galois ring \(\text{GR}(p^2, m)\), for any prime \(p\) and positive integer \(m\).

\[\triangleright\] In 2020, Cao et al. [11] provided an efficiency method to obtain the concatenated structures and precise enumerations to all binary left dihedral codes, their Euclidean dual codes and the self-dual codes in group algebra \(\mathbb{F}_2[D_{8m}]\) for any positive integer \(m\). In particular, some optimal self-dual binary \([8m, 4m]\)-codes are rediscovered.

The hull of a linear code plays an important role in determining the complexity of algorithms for checking permutation equivalence of two linear codes, computing the automorphism group of the code and in the construction of entanglement-assisted quantum error correcting codes (cf. [26]).

As far as we know, the Euclidean hull of any left dihedral code, the representation and enumeration to all Euclidean LCD left dihedral codes and self-orthogonal left dihedral codes have not been fully studied.

The present paper is organized as follows: In section 2, We introduce the necessary notation and terminology. In section 3, we reformulate the results for the concatenated structures of left dihedral codes and their Euclidean dual codes in [8], where \(\gcd(n, q) = 1\). Then we determine the Euclidean hull of every left \(D_{2n}\)-code over \(\mathbb{F}_q\). In section 4, we give an explicit representation for all distinct Euclidean LCD codes, Euclidean self-orthogonal codes and Euclidean self-dual codes which are left \(D_{2n}\)-codes over \(\mathbb{F}_q\) respectively, and count the precise number of codes in each of these three subclass of left \(D_{2n}\)-codes. In section 5, we give a direct and simple method for determining the encoder (generator matrix) of any left \(D_{2n}\)-code over \(\mathbb{F}_q\), and provide several illustrative examples for obtaining self-orthogonal dihedral codes over \(\mathbb{F}_2\) and LCD dihedral codes over \(\mathbb{F}_3\) respectively. Section 6 concludes the paper.

2. Preliminaries

In this section, we review some necessary concepts and introduce some necessary notation, which will be used in the following sections.
Let \( \mathbb{F}_q \) be the finite field of \( q \) elements and \( n \) be a positive integer, where \( q \) is a power of a prime number. Let

\[
D_{2n} = \langle x, y \mid x^n = 1, y^2 = 1, yxy = x^{n-1} \rangle = \{ 1, x, \ldots, x^{n-1}, y, xy, \ldots, x^{n-1}y \}
\]

be the dihedral group of order \( n \). The group algebra \( \mathbb{F}_q[D_{2n}] \) is a vector space over \( \mathbb{F}_q \) with basis \( D_{2n} \). Addition, multiplication with scalars \( c \in \mathbb{F}_q \) and multiplication are defined by: for any \( a_g, b_g \in \mathbb{F}_q \) where \( g \in D_{2n} \),

\[
\sum_{g \in D_{2n}} a_g + \sum_{g \in D_{2n}} b_g = \sum_{g \in D_{2n}} (a_g + b_g)g, \quad c(\sum_{g \in D_{2n}} a_g) = \sum_{g \in D_{2n}} ca_g g,
\]

\[
(\sum_{g \in D_{2n}} a_g)(\sum_{g \in D_{2n}} b_g) = \sum_{g \in D_{2n}} (\sum_{uv = g} a_ub_v)g.
\]

Then \( \mathbb{F}_q[D_{2n}] \) is an associative and noncommutative \( \mathbb{F}_q \)-algebra with identity \( 1 = 1_{\mathbb{F}_q}1_{D_{2n}} \), where \( 1_{\mathbb{F}_q} \) and \( 1_{D_{2n}} \) is the identity elements of \( \mathbb{F}_q \) and \( D_{2n} \) respectively. Readers are referred to [31] for more details on group algebra.

For any \( \alpha = (a_{0, 0}, a_{1, 0}, \ldots, a_{n-1, 0}, a_{0, 1}, a_{1, 1}, \ldots, a_{n-1, 1}) \in \mathbb{F}_q^{2n} \), set

\[
\Psi(\alpha) = a_{0, 0} + a_{1, 0}x + \ldots + a_{n-1, 0}x^{n-1} + a_{0, 1}y + a_{1, 1}xy + \ldots + a_{n-1, 1}x^{n-1}y
\]

Then \( \Psi \) is an \( \mathbb{F}_q \)-linear isomorphism from \( \mathbb{F}_q^{2n} \) onto \( \mathbb{F}_q[D_{2n}] \). As in [21], a nonempty subset \( C \) of \( \mathbb{F}_q^{2n} \) is called a left dihedral code (or left \( D_{2n} \)-code for more precisely) over \( \mathbb{F}_q \) if \( \Psi(C) \) is a left ideal of the \( \mathbb{F}_q \)-algebra \( \mathbb{F}_q[D_{2n}] \). From now on, we will identify \( C \) with \( \Psi(C) \) and assume \( \gcd(q, n) = 1 \).

For any \( \alpha = (a_0, a_1, \ldots, a_{2n-1}), \beta = (b_0, b_1, \ldots, b_{2n-1}) \in \mathbb{F}_q^{2n} \), the Euclidean inner product of \( \alpha \) and \( \beta \) is defined by

\[
[a, \beta]_E = \sum_{i=0}^{2n-1} a_i b_{i-1} \in \mathbb{F}_q.
\]

Let \( C \) be a linear code of length \( 2n \) over \( \mathbb{F}_q \). The Euclidean dual code \( C^\perp_E \) and the Euclidean hull \( \operatorname{Hull}_E(C) \) of \( C \) are defined by

\[
C^\perp_E = \{ \alpha \in \mathbb{F}_q^{2n} \mid [\alpha, \beta]_E = 0, \ \forall \beta \in C \} \quad \text{and} \quad \operatorname{Hull}_E(C) = C \cap C^\perp_E,
\]

respectively. Moreover, \( C \) is said to be Euclidean self-dual (resp. Euclidean self-orthogonal, Euclidean LCD) if \( C = C^\perp_E \) (resp. \( C \subseteq C^\perp_E \), \( C \cap C^\perp_E = \{ 0 \} \) and \( \mathbb{F}_q^{2n} = C + C^\perp_E \)).

Let \( f(x) \) be a polynomial in \( \mathbb{F}_q[x] \) satisfying \( f(0) \neq 0 \). In this paper, we set \( \tilde{f}(x) = \hat{f}(x) = x^\deg(f(x))f(\frac{1}{x}) \). As in [1], the reciprocal polynomial of \( f(x) \) is defined by \( f^*(x) = (f(x))^* = f(0)^{-1}f(x) \). Then \( f(x) \) is said to be self-reciprocal if \( f^*(x) = f(x) \).
Lemma 2.1 Let \( f(x) \) be a monic irreducible factor of \( x^n - 1 \) in \( \mathbb{F}_q[x] \). Then we have the following conclusions:

(i) Let \( f(x) \) is self-reciprocal and \( \deg(f(x)) = 1 \). Then \( f(x) = x - 1 \), if \( n \) is odd; and \( f(x) = x \pm 1 \), if \( n \) is even.

(ii) If \( f(x) \) is self-reciprocal and \( \deg(f(x)) > 1 \), \( \deg(f(x)) \) must be even.

The conclusions in Lemma 2.1 can be found in many literatures (see [8], for example). In the rest of this paper, we assume

\[
x^n - 1 = f_0(x)f_1(x)\ldots f_r(x)f_{r+1}(x)\ldots f_{r+2t}(x),
\]

where \( f_0(x) = x - 1, f_1(x), \ldots, f_{r+1}(x) \) are pairwise coprime monic irreducible polynomials in \( \mathbb{F}_q[x] \) satisfying the following conditions (cf. Page 205 of [6]):

- \( r \) and \( t \) are nonnegative integers such that \( r + 2t = n - 1 \). Set \( d_i = \deg(f_i(x)) \), for all \( i = 0, 1, \ldots, r + 2t \).
- \( f_i^*(x) = f_i(x) \), for all \( i = 0, 1, \ldots, r \). Moreover, we have the following:
  - when \( n \) is odd, \( d_i \) is even for all \( i = 1, \ldots, r \);
  - when \( n \) is even, \( f_1(x) = x + 1 \) and \( d_i \) is even for all \( i = 2, \ldots, r \).
- \( f_{r+j}^*(x) = f_{r+j+t}(x) \) and \( f_{r+j+t}^*(x) = f_{r+j}(x) \), for all \( j = 1, \ldots, t \).

Then we can write

\[
x^n - 1 = \prod_{i=0}^{r} f_i(x) \cdot \prod_{j=1}^{t} f_{r+j}(x)f_{r+j}^*(x).
\]

Let \( i \) be any integer satisfying \( 0 \leq i \leq r + 2t \). We set \( F_i(x) = \frac{x^n - 1}{f_i(x)} \in \mathbb{F}_q[x] \). Then \( \gcd(F_i(x), f_i(x)) = 1 \). Using Extended Euclidian Algorithm, we can find polynomials \( u_i(x), v_i(x) \in \mathbb{F}_q[x] \) satisfying

\[
u_i(x)F_i(x) + v_i(x)f_i(x) = 1.
\]

In the rest of this paper, we adopt the following notation:

- Let \( \mathcal{A} = \frac{\mathbb{F}_q[x]}{(x^n - 1)} = \{ \sum_{j=0}^{n-1} a_jx^j \mid a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_q \} \) in which the arithmetic is done modulo \( x^n - 1 \). Then \( \mathcal{A} \) is an \( \mathbb{F}_q \)-algebra and a principal ideal ring. In particular, \( x^{-1} = x^{n-1} \) in \( \mathcal{A} \).
• Using Eq. (1), let \( \varepsilon_i(x) \in \mathcal{A} \) be defined by

\[
\varepsilon_i(x) \equiv u_i(x)f_i(x) = 1 - v_i(x)f_i(x) \pmod{x^n - 1}.
\]

By [6, Lemma 2.1], we know that \( \varepsilon_i(x) = -\frac{1}{n}(f_i(x)^t)^{\frac{n-1}{f_i(x)}} \).

Set \( \mathcal{A}_i = \mathcal{A}\varepsilon_i(x) \) that is the ideal of \( \mathcal{A} \) generated by \( \varepsilon_i(x) \).

• Let \( K_i = \frac{\mathbb{F}_q[x]}{(f_i(x))} = \{ \sum_{j=0}^{d_i-1} a_jx^j \mid a_0, a_1, \ldots, a_{d_i-1} \in \mathbb{F}_q \} \) in which the arithmetic is done modulo \( f_i(x) \). Then \( K_i \) is an extension field of \( \mathbb{F}_q \) with \( q^{d_i} \) elements.

In this paper, we regard \( K_i \) as a subset of \( \mathcal{A} \). But \( K_i \) is not a subfield of the ring \( \mathcal{A} \), because their multiplication operations are different.

Let \( l \) be any positive integer. Recall that a linear code \( \mathcal{C} \) over \( \mathbb{F}_q \) of length \( ln \) is called a \( l \)-quasi-cyclic code if

\[
(c_{1,n-1}, c_{1,0}, c_{1,1}, \ldots, c_{l,n-2}, \ldots, c_{l,0}, c_{l,1}, \ldots, c_{l,n-2}) \in \mathcal{C},
\]

for all \( c = (c_{1,0}, c_{1,1}, \ldots, c_{l,n-2}, c_{l,n-1}, \ldots, c_{l,0}, c_{l,1}, \ldots, c_{l,n-2}, c_{l,n-1}) \in \mathcal{C} \). Now, we identify each codeword \( c \) with \( (c_1(x), \ldots, c_l(x)) \in \mathcal{A}^l \), where

\[
c_j(x) = c_{j,0} + c_{j,1}x + \ldots + c_{j,n-1}x^{n-1} \in \mathcal{A}, \forall j = 1, \ldots, l.
\]

Then \( l \)-quasi-cyclic codes over \( \mathbb{F}_q \) of length \( ln \) are the same as \( \mathcal{A} \)-submodules of \( \mathcal{A}^l \). Especially, when \( l = 1 \), we know that cyclic codes over \( \mathbb{F}_q \) of length \( n \) are the same as ideals of the ring \( \mathcal{A} \).

The following results can be found in many books and literature (cf. [27, Threorem 4.3.8], [7, Lemma 3.2], [8, Lemma 2.2] and [11, Equation (7)]).

**Lemma 2.2**

(i) In the ring \( \mathcal{A} \), we have that \( \sum_{i=0}^{r+2t} \varepsilon_i(x) = 1, \varepsilon_i(x)^2 = \varepsilon_i(x) \) and \( \varepsilon_i(x)\varepsilon_j(x) = 0 \) for all integers \( i \) and \( j \) : \( 0 \leq i, j \leq r + 2t \) and \( i \neq j \).

(ii) In the ring \( \mathcal{A} \), we have the following properties:

\( \diamond \varepsilon_i(x^{-1}) = \varepsilon_i(x) \), for all \( i = 0, 1, \ldots, r \);

\( \diamond \varepsilon_{r+j}(x^{-1}) = \varepsilon_{r+j+t}(x) \) and \( \varepsilon_{r+j+t}(x^{-1}) = \varepsilon_{r+j}(x) \), for all \( j = 1, \ldots, t \).

(iii) We have \( \mathcal{A} = \bigoplus_{i=0}^{r+2t} \mathcal{A}_i \). Moreover, \( \mathcal{A}_i \) is a subring of \( \mathcal{A} \) with multiplicative identity \( \varepsilon_i(x) \) satisfying \( \mathcal{A}_i \cdot \mathcal{A}_j = \{0\} \), for all \( 0 \leq i \neq j \leq r + 2t \).

(iv) For each integer \( i, 0 \leq i \leq r + 2t \), the following map

\[
\varphi_i : a(x) \mapsto \varepsilon_i(x)a(x) \pmod{x^n - 1}, \forall a(x) \in K_i
\]
is an isomorphism of fields from $K_i$ onto $A_i$. In particular, we have that $A_i = \varepsilon_i(x) \cdot K_i$. Here, we think of $K_i$ as a subset of $A$.

(v) Let $0 \leq i \leq r + 2t$. Then $A_i$ is a minimal cyclic code over $\mathbb{F}_q$ of length $n$ with parity check polynomial $f_i(x)$ and generating idempotent $\varepsilon_i(x)$.

Now, let $0 \leq i \leq r + 2t$ and $C_i$ be any linear code over $K_i$ of length 2, i.e., let $C_i$ be a $K_i$-subspace of $K_i^n = \{(a(x), b(x)) \mid a(x), b(x) \in K_i\}$. For any codeword $(a(x), b(x)) \in C_i$, by Lemma 2.2 (iv) and (v), we know that both $\varphi_i(a(x))$ and $\varphi_i(b(x))$ are codewords in the cyclic code $A_i$. Now, define the concatenated code of the inner code $A_i$ and the outer code $C_i$ by

$$A_i \square \varphi_i C_i = \{(\varphi_i(a(x)), \varphi_i(b(x))) \mid (a(x), b(x)) \in C_i\} \subseteq A_i^2 \subseteq A^2,$$

where for any $(a(x), b(x)) \in C_i$, we have

$$(\varphi_i(a(x)), \varphi_i(b(x))) = \varepsilon_i(x) \cdot (a(x), b(x)) = (\varepsilon_i(x)a(x), \varepsilon_i(x)b(x)) \pmod{x^n - 1}.$$

Then we know the following conclusions:

- $A_i \square \varphi_i C_i$ is a 2-quasi-cyclic code over $\mathbb{F}_q$ of length $2n$.
- $\dim_{\mathbb{F}_q}(A_i \square \varphi_i C_i) = \dim_{\mathbb{F}_q}(A_i) \cdot \dim_{K_i}(C_i) = d_i \cdot \dim_{K_i}(C_i)$,
  where $\dim_{K_i}(C_i)$ is the dimension of $C_i$ over $K_i$, and $|A_i \square \varphi_i C_i| = |C_i|$.
- $d_{A_i \square \varphi_i C_i} \geq d_{A_i} d_{C_i}$, where
  $d_{A_i \square \varphi_i C_i}$ is the minimal Hamming distance of $A_i \square \varphi_i C_i$ over $\mathbb{F}_q$,
  $d_{A_i}$ is minimal Hamming distance of the cyclic code $A_i$ over $\mathbb{F}_q$,
  $d_{C_i}$ is the minimal Hamming distance of the linear code $C_i$ over $K_i$.

3. The Euclidean hull of any left $D_2n$-code over $\mathbb{F}_q$

In this section, we determine the Euclidean hull $\text{Hull}_E(C) = C \cap C^\perp$ of any left $D_2n$-code $C$ over $\mathbb{F}_q$. To do this, we need to give a more direct and simpler representation for all distinct left $D_2n$-codes and their Euclidean dual codes.

Since $D_{2n} = \langle x, y \mid x^n = 1, y^2 = 1, yxy = x^{-1} \rangle$, $C'' = \langle x \mid x^n = 1 \rangle$ is a cyclic subgroup of $D_{2n}$ with order $n$ generated by $x$. Obviously, the group algebra $\mathbb{F}_q[C'']$ is the same as the ring $A = \frac{\mathbb{F}_q[x]}{\langle x^n - 1 \rangle}$. Hence $A$ is the subring of the group algebra $\mathbb{F}_q[D_{2n}]$ and

$$\mathbb{F}_q[D_{2n}] = \{a(x) + b(x)y \mid a(x), b(x) \in A\}$$
in which \( y^2 = 1 \) and \( ya(x) = a(x^{-1})y \) for all \( a(x) \in \mathcal{A} \), where \( a(x^{-1}) = a(x^{n-1}) \mod x^n - 1 \). From this, we deduce that \( \mathbb{F}_q[D_{2n}] \) is a free left \( \mathcal{A} \)-module with basis \( \{1, y\} \). Hence the map \( \Theta \), defined by

\[
\Theta : \mathcal{A}^2 \to \mathbb{F}_q[D_{2n}] \text{ via } \Theta : (a(x), b(x)) \mapsto a(x) + b(x)y \ (\forall a(x), b(x) \in \mathcal{A}),
\]
is an isomorphism of left \( \mathcal{A} \)-modules from \( \mathcal{A}^2 \) onto \( \mathbb{F}_q[D_{2n}] \).

Let \( \mathcal{C} \) be a nonempty of \( \mathbb{F}_q[D_{2n}] \). It is clear that \( \mathcal{C} \) is a left ideal of \( \mathbb{F}_q[D_{2n}] \) if and only if \( \Theta^{-1}(\mathcal{C}) \) is an \( \mathcal{A} \)-submodule of \( \mathcal{A}^2 \) and \( y\xi \in \mathcal{C} \) for all \( \xi = a(x) + b(x)y \in \mathcal{C} \). Then by \( y\xi = b(x^{-1}) + a(x^{-1})y \), we have \( \Theta^{-1}(y\xi) = (b(x^{-1}), a(x^{-1})) \). Hence \( \mathcal{C} \) is a left \( D_{2n} \)-code over \( \mathbb{F}_q \) if and only there is a unique \( \mathcal{A} \)-submodule \( \mathcal{C}' \) of \( \mathcal{A}^2 \) satisfying the following condition:

\[
(b(x^{-1}), a(x^{-1})) \in \mathcal{C}', \ \forall (a(x), b(x)) \in \mathcal{C}'
\]
such that \( \Theta(\mathcal{C}') = \mathcal{C} \). In the rest of the paper, we will identify \( \mathcal{C} \) with \( \mathcal{C}' \). Hence left \( D_{2n} \)-codes form an interesting subclass of 2-quasi-cyclic codes of length \( 2n \) over \( \mathbb{F}_q \).

**Theorem 3.1** Using the notation of Section 2, let \( \mathcal{I}_0 = \{0\} \) when \( n \) is odd; and let \( \mathcal{I}_0 = \{0, 1\} \) when \( n \) is even. Then all distinct left \( D_{2n} \)-codes \( \mathcal{C} \) over \( \mathbb{F}_q \) and their Euclidean dual codes \( \mathcal{C}^\perp_E \) are given by:

\[
\mathcal{C} = \bigoplus_{i=0}^{r+2t} (\mathcal{A}_i \sqcap \varphi_i C_i) = \sum_{i=0}^{r+2t} \varepsilon_i(x) \cdot C_i \tag{2}
\]

and

\[
\mathcal{C}^\perp_E = \bigoplus_{i=0}^{r+2t} (\mathcal{A}_i \sqcap \varphi_i U_i) = \sum_{i=0}^{r+2t} \varepsilon_i(x) \cdot U_i
\]

respectively, where \( C_i \) and \( U_i \) are linear codes of length 2 over the finite field \( K_i \) with generator matrices \( G_i \) and \( E_i \), respectively. In particular, the matrices \( G_i \) and \( E_i \) are given by the following three cases:

(i) For any integer \( i \in \mathcal{I}_0 \), we have one of the following two subcases:

(i-1) Let \( q \) be odd. Then the pairs \( (G_i, E_i) \) of matrices are given by the following table:

| \( N_i \) | \( G_i \) | \( C_i \) | \( d_{C_i} \) | \( E_i \) |
|----------|----------|----------|----------|----------|
| 1        | (0, 0)   | 1        | 0        | \( I_2 \) |
| 1        | \( I_2 \) | \( q^2 \) | 1        | (0, 0)   |
| 1        | (1, 1)   | \( q \)  | 2        | (−1, 1)  |
| 1        | (−1, 1)  | \( q \)  | 2        | (1, 1)   |


where

- $N_i$ is the number of pairs $(G_i, E_i)$ in the same row and $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

- $|C_i|$ is the number of codewords in the linear code $C_i$ over $K_i$.

- $d_{C_i}$ is the minimal Hamming distance of $C_i$ over $K_i$.

In the below, the meaning of these notation is the same as above.

(i-2) If $q$ is even, $I_0 = \{0\}$ and the pairs $(G_0, E_0)$ of matrices are given by the following table:

| $N_0$ | $G_0$ | $|C_0|$ | $d_{C_0}$ | $E_0$ |
|-------|-------|-------|----------|-------|
| 1     | (0, 0)| 1     | 0        | $I_2$ |
| 1     | $I_2$| $q^2$ | 1        | (0, 0) |
| 1     | (1, 1)| $q$  | 2        | (1, 1) |

(ii) For any integer $i \in \{1, \ldots, r\} \setminus I_0$, let $g_i(x)$ be a primitive element of the finite field $K_i$ with $q^{d_i}$ elements. Then $d_i$ is even and the pairs $(G_i, E_i)$ of matrices are given by the following table:

| $N_i$ | $G_i$ | $|C_i|$ | $d_{C_i}$ | $E_i$ |
|-------|-------|-------|----------|-------|
| 1     | (0, 0)| 1     | 0        | $I_2$ |
| 1     | $I_2$| $q^{2d_i}$ | 1       | (0, 0) |
| $q^{d_i} + 1$ | $g_i(x)^{s(q^{d_i} - 1)}$, $1$ | $q^{d_i}$ | 2 | $(-g_i(x)^{s(q^{d_i} - 1)}$, 1) |

where $s$ is any integer satisfying $0 \leq s \leq q^{d_i}$.

(iii) Let $i = r + j$ where $1 \leq j \leq t$. Then the sequences $(G_i, G_{i+t}, E_{i+t}, E_i)$ of matrices are given by the following table:

| $N_i$ | $G_i$ | $|C_i|$ | $d_{C_i}$ | $G_{i+t}$ | $E_{i+t}$ | $E_i$ |
|-------|-------|-------|----------|----------|----------|-------|
| 1     | (0, 0)| 1     | 0        | (0, 0)   | $I_2$    | $I_2$ |
| 1     | $I_2$| $q^{2d_i}$ | 1       | $I_2$    | (0, 0)   | (0, 0) |
| 1     | (1, 0)| $q^{d_i}$ | 1        | (0, 1)   | (0, 1)   | (1, 0) |
| $q^{d_i}$ | $(g(x), 1)$ | $q^{d_i}$ | 2 | $(1, g(x^{-1}))$ | $(1, -g(x^{-1}))$ | $(-g(x), 1)$ |

where

- $N_i$ is the number of sequences $(G_i, G_{i+t}, E_{i+t}, E_i)$ in the same row;
\[ |C_{i+t}| = |C_i| \text{ and } d_{C_{i+t}} = d_{C_i}; \]

\[ g(x) = \sum_{l=0}^{d_i-1} a_l x^l \text{ and } g(x^{-1}) = g(x^{n-1}) = a_0 + \sum_{l=1}^{d_i-1} a_l x^{n-l} \pmod{f_{i+t}(x)}, \] for any arbitrary elements \( a_0, a_1, \ldots, a_{d_i-1} \in \mathbb{F}_q. \)

In particular, the Euclidean dual code \( C^\perp \) of every left \( D_{2n} \)-code \( C \) is also a left \( D_{2n} \)-code over \( \mathbb{F}_q \).

Moreover, the number \( N \) of left \( D_{2n} \)-codes over \( \mathbb{F}_q \) is equal to

\[ N = \begin{cases} 
4 \cdot \prod_{i=1}^{r+2t} (q^{d_i^2} + 3) \cdot \prod_{j=1}^{q^{d_i^j} + 3}, & \text{if } q \text{ is odd and } n \text{ is odd}; \\
4^2 \cdot \prod_{i=1}^{q^{d_i^2} + 3} \cdot \prod_{j=1}^{q^{d_i^j} + 3}, & \text{if } q \text{ is odd and } n \text{ is even}; \\
3 \cdot \prod_{i=1}^{q^{d_i^2} + 3} \cdot \prod_{j=1}^{q^{d_i^j} + 3}, & \text{if } q \text{ is even.} 
\end{cases} \]

The theorem can be deduced from \([8, \text{ Theorem 2.4, Lemma 3.1, Theorem 3.2, Theorem 4.2, Corollary 4.3 and Theorem 5.4}]. \) However, the derivation of this theorem from the representation given in \([8]\) still requires some transformation processes. Here we give a direct proof by a method parallel to that used in \([11]\), for the convenience of the reader.

**Proof.** For any integer \( i, 0 \leq i \leq r + 2t, \) and \((\xi_{i,0}(x), \xi_{i,1}(x)) \in K_i^2, \) we define

\[
\Phi(((\xi_{0,0}(x), \xi_{0,1}(x)), (\xi_{1,0}(x), \xi_{1,1}(x)), \ldots, (\xi_{r+2t,0}(x), \xi_{r+2t,1}(x))))
\]

\[ = \sum_{i=0}^{r+2t} (\varphi_i((\xi_{i,0}(x)), (\xi_{i,1}(x)))
\]

\[ = (\sum_{i=0}^{r+2t} \varepsilon_i(x) \cdot \xi_{i,0}(x), \sum_{i=0}^{r+2t} \varepsilon_i(x) \cdot \xi_{i,1}(x)) \pmod{x^n - 1}. \]

Then by Lemma 2.2 (i), (iii) and (iv), we see that the map \( \Phi \) is an isomorphism of \( A \)-modules from \( K_0^2 \times K_1^2 \times \cdots \times K_{r+2t}^2 \) onto \( A^2. \) Now, let \( C \) be any \( A \)-submodule of \( A^2. \) Then for each integer \( i, 0 \leq i \leq r + 2t, \) there is a unique \( K_i \)-subspace \( C_i \) of \( K_i^2 \) such that

\[ C = \Phi(C_0 \times C_1 \times \cdots \times C_{r+2t}) = \sum_{i=0}^{r+2t} \varepsilon_i(x) \cdot C_i = \bigoplus_{i=0}^{r+2t} (A_i \square \varphi_i C_i). \quad (3) \]

Let \((a(x), b(x)) \in C.\) Then we have

\[ a(x) = \sum_{i=0}^{r+2t} \varepsilon_i(x) \xi_{i,0}(x) \text{ and } b(x) = \sum_{i=0}^{r+2t} \varepsilon_i(x) \xi_{i,1}(x), \]
where \((\xi_{i,0}(x), \xi_{i,1}(x)) \in C_i\) for all \(i = 0, 1, \ldots, r + 2t\). By Lemma 2.2 (ii), we have that \(\varepsilon_i(x^{-1}) = \varepsilon_i(x)\) for all \(i = 0, 1, \ldots, r\), \(\varepsilon_i(x^{-1}) = \varepsilon_{i+t}(x)\) and \(\varepsilon_{i+t}(x^{-1}) = \varepsilon_i(x)\) for all \(i = r + 1, \ldots, r + t\) in the ring \(A\). Then using Lemma 2.2 (iv), we obtain

\[
a(x^{-1}) = \sum_{i=0}^{r} \varepsilon_i(x)\xi_{i,0}(x^{-1}) + \sum_{i=r+1}^{r+t} (\varepsilon_i(x)\xi_{i+t,0}(x^{-1}) + \varepsilon_{i+t}(x)\xi_{i,0}(x^{-1})),
\]

\[
b(x^{-1}) = \sum_{i=0}^{r} \varepsilon_i(x)\xi_{i,1}(x^{-1}) + \sum_{i=r+1}^{r+t} (\varepsilon_i(x)\xi_{i+t,1}(x^{-1}) + \varepsilon_{i+t}(x)\xi_{i,1}(x^{-1})),
\]

From these, by Lemma 2.2 (i) (iii) and Eq. (3), we deduce that

\[(b(x^{-1}), a(x^{-1})) \in C \iff \begin{cases} (\xi_{i,1}(x^{-1}), \xi_{i,0}(x^{-1})) \in C_i, & 0 \leq i \leq r; \\ (\xi_{i+1,1}(x^{-1}), \xi_{i+1,0}(x^{-1})) \in C_i, & r+1 \leq i \leq r+t; \\ (\xi_{i,1}(x^{-1}), \xi_{i,0}(x^{-1})) \in C_{i+t}, & r+1 \leq i \leq r+t. \end{cases}\]

Since \(C_i\) is a \(K_i\)-subspace of \(K_i^2\), i.e., \(C_i\) is a linear code over \(K_i\) of length 2, we must have that \(\dim_{K_i}(C_i) \in \{0, 1, 2\}\). Now, let’s discuss by two cases: when \(0 \leq i \leq r\); when \(r + 1 \leq i \leq r + 2t\).

(↑) Let \(0 \leq i \leq r\). Obviously, \(C_i\) satisfies the above condition if \(C_i = \{0\}\) or \(C_i = K_i^2\). Further, when \(C_i = \{0\}\), \(G_i = (0, 0)\) is a generator matrix of \(C_i\); when \(C_i = K_i^2\), \(G_i = I_2\) is a generator matrix of \(C_i\).

Let \(\dim_{K_i}(C_i) = 1\). Since \(C_i\) satisfies \((\xi_{i,1}(x^{-1}), \xi_{i,0}(x^{-1})) \in C_i\) for all \((\xi_{i,0}(x), \xi_{i,1}(x)) \in C_i\), there exists a unique element \(g(x) \in K_i^\times\) such that \(G_i = (g(x), 1)\) is a generator matrix of the linear code \(C_i\). Moreover, we see that \((1, g(x^{-1})) \in C_i\) if and only if \((1, g(x^{-1})) = u(x)(g(x), 1)\) for some \(u(x) \in K_i\), and the latter is equivalent to that \(g(x)g(x^{-1}) = 1\). Then we need to consider the following two subcases.

(↑-1) Let \(i \in \mathcal{I}_0\). Then \(K_i = \mathbb{F}_q\). In this case, we have that \(g(x) = c\) for some \(c \in \mathbb{F}_q^\times\) and the condition \(g(x)g(x^{-1}) = 1\) is simplified to \(c^2 = 1\). Hence, we have the following conclusions:

- If \(q\) is odd, \(c = \pm 1\). Therefore, there are two codes \(C_i\) with generator matrices \(G_i = (1, 1)\) and \(G_i = (-1, 1)\) respectively.

- If \(q\) is even, \(c = 1\). Therefore, there is only one code \(C_i\) with generator matrix \(G_i = (1, 1)\).

(↑-2) Let \(i \in \{1, 2, \ldots, r\} \setminus \mathcal{I}_0\). By Lemma 2.1 (ii), \(f_i(x)\) is an irreducible self-reciprocal polynomial in \(\mathbb{F}_q[x]\) of even degree \(d_i \geq 2\). Then we have that
$x^{-1} = x^{q^{d_i^1}}$ in the finite field $K_i = \mathbb{F}_q[x]/(f_i(x))$. This implies $g(x^{-1}) = g(x)^{q^{d_i^1}}$. Since $g_i(x)$ is a primitive element of $K_i$ with multiplicative order $q^{d_i^1} - 1$, the condition $g(x)g(x^{-1}) = 1$, i.e., $g(x)^{q^{d_i^1} + 1} = 1$, is equivalent to that

$$g(x) = g_i(x)^{s(q^{d_i^1} - 1)}, \ s = 0, 1, \ldots, q^{d_i^1}.$$

(‡) Let $i = r + j$ where $1 \leq j \leq t$. Then the pair $(G_i, G_{i+t})$ of codes satisfies the above conditions if and only if the codes $G_i$ and $G_{i+t}$ are given as following:

$\triangleright$ $C_i$ is an arbitrary linear code over $K_i$ of length 2. Hence a generator matrix $G_i$ of $C_i$ is one of the following $q^{d_i^1} + 1$ matrices:

$$G_i = (1, 0), \ G_i = (g(x), 1) \text{ where } g(x) \in K_i.$$

$\triangleright$ Let $C_{i+t} = \{(\xi_{i,0}(x^{-1}), \xi_{i,0}(x^{-1})) \mid (\xi_{i,0}(x), \xi_{i,1}(x)) \in C_i\}$. Hence a generator matrix $G_{i+t}$ of $C_{i+t}$ is one of the following $q^{d_i^1} + 1$ matrices:

$$G_{i+t} = (0, 1) \text{ if } G_i = (1, 0), \ G_{i+t} = (1, g(x^{-1})) \text{ if } G_i = (g(x), 1),$$

where $g(x^{-1}) \in K_{i+t}$, i.e., $g(x^{-1}) = g(x^{n-1}) \pmod{f_{i+t}(x)}$, for any polynomial $g(x) = \sum_{t=0}^{d_i^1} a_t x^t \in K_i$ with $a_t \in \mathbb{F}_q$.

As mentioned above, we conclude that all distinct left $D_{2^m}$-codes $C$ over $\mathbb{F}_q$ have been given by this theorem.

Finally, let $D = \bigoplus_{j=0}^{r+2t} (A, \bigcap_{i} U_i) = \sum_{i=0}^{r+2t} \varepsilon_i(x) \cdot U_i$ be given by this theorem. Specifically, $U_i$ is a linear code over $K_i$ of length 2 with a generator matrices $E_i$, $0 \leq i \leq r+2t$. We set $\Upsilon(D) = \{(\begin{pmatrix} u(x^-1) \\ v(x^-1) \end{pmatrix}) \mid (u(x), v(x)) \in D\}$. Then we have $\Upsilon(D) = \sum_{i=0}^{r+2t} \varepsilon_i(x^{-1}) \cdot \Upsilon(U_i)$, where

$$\Upsilon(U_i) = \left\{ \begin{pmatrix} a_i(x^{-1}) \\ b_i(x^{-1}) \end{pmatrix} \mid (a_i(x), b_i(x)) \in U_i \right\}, \ \forall i : 0 \leq i \leq r + 2t.$$

By the tables of the theorem, we have that $G_i \cdot \Upsilon(E_i) = 0$ for all $i = 0, 1, \ldots, r$, $G_i \cdot \Upsilon(E_{i+t}) = 0$ and $G_{i+t} \cdot \Upsilon(E_i) = 0$ for $i = r + 1, \ldots, r + t$. From these and by Lemma 2.2 (i)–(iv), one can easily verify that $|\mathcal{C}||\mathcal{D}| = \prod_{i=0}^{r+2t} |C_i| \prod_{i=0}^{r+2t} |U_i| = q^{2t} \cdot q^{2n} = q^{2n}$ and

$$\mathcal{C} \cdot \Upsilon(D) = \sum_{i=0}^{r} \varepsilon_i(x)(C_i \cdot \Upsilon(U_i)) + \sum_{i=r+1}^{r+t} \varepsilon_i(x)(C_i \cdot \Upsilon(U_{i+t})) + \varepsilon_{i+t}(x)(C_{i+t} \cdot \Upsilon(U_i))) \equiv \{0\} \pmod{x^n - 1}$$
As stated above, we conclude that $C^{{\perp}_{E}} = D$. □

From now on, the representation of any left $D_{2n}$-code $C$ over $\mathbb{F}_q$ given by Theorem 3.1 is called the canonical representation of $C$. Then we can determine the Euclidean hull of any arbitrary left $D_{2n}$-code.

**Theorem 3.2** Let $C$ be any left $D_{2n}$-code over $\mathbb{F}_q$ with the canonical representation $C = \bigoplus_{i=0}^{r+2t} (A_i \bowtie \varphi_i C_i)$ given by Theorem 3.1. Then the Euclidean hull $\text{Hull}_E(C)$ of $C$ is also a left $D_{2n}$-code over $\mathbb{F}_q$.

In particular, the canonical representation of $\text{Hull}_E(C)$ is given as below:

$$\text{Hull}_E(C) = \bigoplus_{i=0}^{r+2t} (A_i \bowtie \varphi_i \Omega_i),$$

where $\Omega_i$ is a linear code of length 2 over $K_i$ with generator matrix $M_i$, $0 \leq i \leq r+2t$, and the matrices $M_i$ are given by the following three cases:

(i) Let $i \in I_0$. We have one of the following two subcases:

(i-1) If $q$ is odd, the matrix $M_i$ is given by:

$$M_i = (0, 0), \quad \forall G_i \in \{(0, 0), I_2, (1, 1), (-1, 1)\}.$$

(i-2) If $q$ is even, $I = \{0\}$ and the matrix $M_0$ is given by:

$$M_0 = \begin{cases} 
(0, 0), & \text{if } G_0 \in \{(0, 0), I_2\}; \\
(1, 1), & \text{if } G_0 = (1, 1).
\end{cases}$$

(ii) Let $i \in \{1, \ldots, r\} \setminus I_0$. The matrix $M_i$ is given by one the following two cases:

- When $q$ is odd, we have

$$M_i = (0, 0), \quad \forall G_i \in \{(0, 0), I_2\} \cup \left\{ (\varrho_i(x)^{s(q^d_i - 1)}, 1) \mid 0 \leq s \leq q^d_i \right\}.$$

- When $q$ is even, we have

$$M_i = \begin{cases} 
(0, 0), & \text{if } G_i \in \{(0, 0), I_2\}; \\
(\varrho_i(x)^{s(q^d_i - 1)}, 1), & \text{if } G_i = (\varrho_i(x)^{s(q^d_i - 1)}, 1), 0 \leq s \leq q^d_i.
\end{cases}$$

(iii) Let $i = r + j$ where $1 \leq j \leq t$. Then the pairs $(M_i, M_{i+t})$ of matrices are given by one the following two cases:

- When $q$ is odd, we have
\[(M_i, M_{i+t}) = \begin{cases} 
((0, 0), (0, 0)), & \text{if } G_i \in \{(0, 0), I_2\}; \\
((1, 0), (0, 1)), & \text{if } G_i = (1, 0); \\
((0, 1), (1, 0)), & \text{if } G_i = (0, 1). 
\end{cases} \]

\[M_i = M_{i+t} = (0, 0), \text{ if } G_i = (g(x), 1) \text{ where } g(x) = \sum_{i=0}^{d_i-1} a_i x^i \text{ for any } a_0, a_1, \ldots, a_{d_i-1} \in \mathbb{F}_q \text{ satisfying } (a_0, a_1, \ldots, a_{d_i-1}) \neq (0, \ldots, 0).\]

\[\Diamond \text{ When } q \text{ is even, we have }\]

\[M_i = M_{i+t} = (0, 0), \text{ if } G_i \in \{(0, 0), I_2\}; \]

\[M_i = (1, 0) \text{ and } M_{i+t} = (0, 1), \text{ if } G_i = (1, 0); \]

\[M_i = (g(x), 1) \text{ and } M_{i+t} = (1, g(x^{-1})), \text{ if } G_i = (g(x), 1) \text{ where } g(x) = \sum_{i=0}^{d_i-1} a_i x^i \text{ and } g(x^{-1}) = a_0 + \sum_{i=1}^{d_i-1} a_i x^{n-i} \pmod{f_{i+t}(x)} \text{ for any elements } a_0, a_1, \ldots, a_{d_i-1} \in \mathbb{F}_q.\]

**Proof.** Let \(C^\perp \) be the Euclidean dual code of \(C\). By Theorem 3.1, we have

\[C = \sum_{i=0}^{r+2t} \varepsilon_i(x)C_i \text{ and } C^\perp = \sum_{i=0}^{r+2t} \varepsilon_i(x)U_i\]

where \(C_i\) and \(U_i\) are linear codes of length 2 over \(K_i\) with generator matrices \(G_i\) and \(E_i\) respectively. Then by Lemma 2.2 (i)–(iv), we conclude that

\[\text{Hull}_E(C) = C \cap C^\perp = \sum_{i=0}^{r+2t} \varepsilon_i(x)\Omega_i = \bigoplus_{i=0}^{r+2t} \mathcal{A}_i \square \varphi, \Omega_i,\]

where \(\Omega_i = (C_i \cap U_i)\) which is a linear code of length 2 over the finite field \(K_i\), for all integers \(i\): 0 ≤ \(i\) ≤ \(r + 2t\). Let \(M_i\) be a generator matrix of \(\Omega_i\). Since both \(C_i\) and \(U_i\) are linear codes of length 2 over \(K_i\), \(\dim_{K_i}(C_i) + \dim_{K_i}(U_i) = 2\), \(G_i\) is a generator matrix of \(G_i\), \(E_i\) is a generator matrix of \(U_i\), the matrices \(G_i\) and \(E_i\) are given by Theorem 3.1, we deduce the following conclusions:

\[\Diamond M_i = \begin{cases} 
G_i, & \text{if } G_i = E_i; \\
(0, 0), & \text{otherwise, for all } i = 0, 1, \ldots, r. 
\end{cases} \]

\[\Diamond (M_i, M_{i+t}) = \begin{cases} 
(G_i, G_{i+t}), & \text{if } G_i = E_i; \\
((0, 0), (0, 0)), & \text{otherwise, for all } i = r + 1, \ldots, r + t. 
\end{cases} \]

Then the conclusion of this theorem can be derived from Theorem 3.1 immediately. Here we omit the details. \(\square\)
4. The LCD left $D_{2n}$-codes and self-orthogonal left $D_{2n}$-codes

In this section, we give an explicit representation and a precise enumeration for all distinct Euclidean LCD codes and Euclidean self-orthogonal codes which are left $D_{2n}$-codes over $\mathbb{F}_q$ respectively. At the end, in order to compare the enumerations of the classes of left $D_{2n}$-codes satisfying certain dual properties, we reformulate the representation of self-dual left $D_{2n}$-codes over $\mathbb{F}_q$ given by [8, Corollary 5.5], when $q$ is even.

In the following, we determine Euclidean LCD left $D_{2n}$-codes over $\mathbb{F}_q$. Since $\mathbb{F}_q$ is a finite field, we see that a left $D_{2n}$-code $C = \bigoplus_{i=0}^{r+2t} (A_i \varphi_i C_i)$ over $\mathbb{F}_q$ is Euclidean LCD if and only if $\text{Hull}_E(C) = \{0\}$. Using Theorem 3.2, the latter condition is equivalent to that

$$M_i = (0, 0), \quad \forall i = 0, 1, \ldots, r + 2t.$$  

Then we consider two situations: when $q$ is even; and when $q$ is odd.

(†) When $q$ is even, by Theorem 3.2, all distinct Euclidean LCD left $D_{2n}$-codes over $\mathbb{F}_q$ are given as follows:

$$C = \bigoplus_{i=0}^{r+2t} (A_i \varphi_i C_i),$$

where $C_i$ is a linear code over $K_i$ of length 2 with generator matrix $G_i$, $0 \leq i \leq r + 2t$, and $G_i$ is given by the follows:

- If $0 \leq i \leq r$, $G_i \in \{0, I_2\}$.
- If $i = r + j$ where $1 \leq j \leq t$, $G_i = G_{i+t} \in \{0, I_2\}$.

Therefore, the number of Euclidean LCD left $D_{2n}$-codes over $\mathbb{F}_q$ is equal to

$$N_{E-LCD} = 2^{1+r+t}.$$  

(‡) When $q$ is odd, using Theorem 3.2, we arrive at the following:

**Theorem 4.1** Let $q$ be odd. Using the notation of Theorem 3.1, all distinct Euclidean LCD left $D_{2n}$-codes over $\mathbb{F}_q$ are given as follows:

$$C = \bigoplus_{i=0}^{r+2t} (A_i \varphi_i C_i),$$

where $C_i$ is a linear code over $K_i$ of length 2 with generator matrix $G_i$, $i \leq i \leq r + 2t$, and $G_i$ is given by the following three cases:
(i) When $i \in \mathcal{I}_0$, we have that $G_i \in \{(0,0), I_2, (1,1), (-1,1)\}$.

(ii) When $i \in \{0,1,\ldots,r\} \setminus \mathcal{I}_0$, we have that

$$G_i \in \{(0,0), I_2\} \cup \left\{ \left(a_i(x)^{s(q^n_{j_i^2}-1)}, 1 \right) \mid 0 \leq s \leq q^n_{j_i^2} \right\}.$$ 

(iii) Let $i = r + j$ where $1 \leq j \leq t$. Then the pairs $(G_i, G_{i+t})$ of matrices are given by the following two cases:

b) $G_i = G_{i+t} \in \{(0,0), I_2\}$;

c) $G_i = (g(x), 1)$ and $G_{i+t} = (1, g(x^{-1}))$, where $g(x) = \sum_{i=0}^{d_i-1} a_i x^i$ and $g(x^{-1}) = a_0 + \sum_{i=1}^{d_i-1} a_i x^{n-i}$ (mod $f_{i+t}(x)$) for any $a_0, a_1, \ldots, a_{d_i-1} \in \mathbb{F}_q$ satisfying $(a_0, a_1, \ldots, a_{d_i-1}) \neq (0,0,\ldots,0)$.

Therefore, the number $\mathcal{N}_{E-LCD}$ of Euclidean LCD left $D_{2n}$-codes over $\mathbb{F}_q$ is equal to

$$\mathcal{N}_{E-LCD} = \left\{ \begin{array}{ll}
4 \cdot \prod_{i=1}^{r} (q^{n_{j_i^2}} + 3) \cdot \prod_{j=1}^{t} (q^{d_{i+j}} + 1), & \text{if } n \text{ is odd;} \\
4^2 \cdot \prod_{i=1}^{r} (q^{n_{j_i^2}} + 3) \cdot \prod_{j=1}^{t} (q^{d_{i+j}} + 1), & \text{if } n \text{ is even.}
\end{array} \right.$$ 

In particular, every left $D_{2n}$-codes over $\mathbb{F}_q$ is an Euclidean LCD code when each monic irreducible divisor of $x^n - 1$ in $\mathbb{F}_q[x]$ is self-reciprocal.

**Proof.** Let $\mathcal{C}$ be a left $D_{2n}$-code over $\mathbb{F}_q$ with the canonical representation $\mathcal{C} = \bigoplus_{i=0}^{r+2t} (\mathcal{A}_i, \mathcal{C}_i)$ given by Theorem 3.1. Then its Euclidean hull is $\text{Hull}_E(\mathcal{C}) = \bigoplus_{i=0}^{r+2t} (\mathcal{A}_i, \mathcal{C}_i)$, where $\mathcal{A}_i$ is a linear code over $\mathbb{F}_q$ of length 2 with a generator matrix $M_i$ and $M_i$ is determined by Theorem 3.2.

It is clear that $\mathcal{C}$ is an Euclidean LCD code over $\mathbb{F}_q$ if and only if $\text{Hull}_E(\mathcal{C}) = \{0\}$. The latter is equivalent to that the code $C_i$ satisfies $C_i \cap U_i = \Omega_i = \{0\}$, i.e., the generator matrix $G_i$ of $C_i$ satisfies $M_i = (0,0)$, for all integers $i$: $0 \leq i \leq r + 2t$. Then by Theorems 3.1 and 3.2, one can easily deduce the conclusions for the representation of Euclidean LCD left $D_{2n}$-codes over $\mathbb{F}_q$.

Finally, let every monic irreducible divisor of $x^n - 1$ in $\mathbb{F}_q[x]$ be self-reciprocal, i.e., $t = 0$. By Theorem 3.1, we see that the number of LCD left $D_{2n}$-codes over $\mathbb{F}_q$ is the same as the number of all left $D_{2n}$-codes over $\mathbb{F}_q$.

In this case, every left $D_{2n}$-codes over $\mathbb{F}_q$ is an Euclidean LCD code. \hfill \square

**Remark** Let $q$ be odd. By Theorems 3.1 and 4.1, we have that

$$\frac{\mathcal{N}_{E-LCD}}{\mathcal{N}} = \prod_{j=1}^{t} \left( 1 - \frac{2}{q^{d_{i+j}} + 3} \right) \approx 1,$$

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if all \( q^{d_{r+j}} \) are large enough when \( t \geq 1 \).

Hence a left \( D_{2n} \)-code over \( \mathbb{F}_q \) has a higher probability that is an Euclidean LCD code, when \( q \) is odd (and large enough if \( t \geq 1 \)).

Now, we determine Euclidean self-orthogonal left \( D_{2n} \)-codes over \( \mathbb{F}_q \).

**Theorem 4.2** Using the notation of Theorem 3.1, all distinct Euclidean self-orthogonal left \( D_{2n} \)-codes over \( \mathbb{F}_q \) are given as follows:

\[
C = \bigoplus_{i=0}^{r+2t} (A_i \oplus_{\nu_i} C_i),
\]

where \( C_i \) is a linear code over \( K_i \) of length 2 with generator matrix \( G_i \), \( i \leq i \leq r + 2t \), and \( G_i \) is given by the following three cases:

(i) Let \( i \in \mathcal{I}_0 \). We have one of the following two subcases:

(i-1) When \( q \) is odd, \( G_i = (0, 0) \).

(i-2) When \( q \) is even, \( \mathcal{I}_0 = \{0\} \) and \( G_0 \in \{(0, 0), (1, 1)\} \).

(ii) Let \( i \in \{1, \ldots, r\} \setminus \mathcal{I}_0 \). Then the matrix \( G_i \) is given by one the following two cases:

\[
\begin{align*}
\triangleright & \text{ When } q \text{ is odd, } G_i = (0, 0). \\
\triangleright & \text{ When } q \text{ is even, } G_i \in \{(0, 0)\} \cup \{(g(x)^s(q^{\frac{d_i}{2}}-1), 1) \mid 0 \leq s \leq q^{\frac{d_i}{2}}\}.
\end{align*}
\]

(iii) Let \( i = r + j \) where \( 1 \leq j \leq t \). Then the pairs \((G_i, G_{i+t})\) of matrices are given by one the following two cases:

\[
\begin{align*}
\triangleright & \text{ If } q \text{ is odd, } (G_i, G_{i+t}) \in \{((0, 0), (0, 0)), ((1, 0), (1, 0)), ((1, 1), (1, 0))\}. \\
\triangleright & \text{ If } q \text{ is even, the pairs } (G_i, G_{i+t}) \text{ of matrices are given by the following three subcases:}
\end{align*}
\]

\[
\begin{align*}
\triangleright & \quad b) \quad G_i = G_{i+t} = (0, 0); \\
\triangleright & \quad c) \quad G_i = (1, 0) \text{ and } G_{i+t} = (0, 1); \\
\triangleright & \quad d) \quad G_i = (g(x), 1) \text{ and } G_{i+t} = (1, g(x^{-1})), \text{ where } g(x) = \sum_{l=0}^{d_i-1} a_l x^l \text{ and } g(x^{-1}) = a_0 + \sum_{l=1}^{d_i-1} a_l x^{n-l} \text{ (mod } f_{i+t}(x)\text{) for any } a_0, a_1, \ldots, a_{d_i-1} \in \mathbb{F}_q. \\
\end{align*}
\]

Let \( \mathcal{N}_{E-SO} \) be the number of self-orthogonal left \( D_{2n} \)-codes over \( \mathbb{F}_q \). Then

\[
\mathcal{N}_{E-SO} = \begin{cases} 3^t, & \text{if } q \text{ is odd}; \\
2 \cdot \prod_{i=1}^{r}(q^{\frac{d_i}{2}} + 2) \cdot \prod_{j=1}^{t}(q^{d_{r+j}} + 2), & \text{if } q \text{ is even}. 
\end{cases}
\]

**Proof.** Let \( \mathcal{C} = \bigoplus_{i=0}^{r+2t} (A_i \oplus_{\nu_i} C_i) \) be a left \( D_{2n} \)-code over \( \mathbb{F}_q \) given by Theorem 3.1. Then \( \mathcal{C} \) is self-orthogonal if and only if \( \mathcal{C} \subseteq \mathcal{C}^\perp_E \), i.e., Hull\(_E(\mathcal{C}) = \)
\( \mathcal{C} \cap \mathcal{C}^\perp = \mathcal{C} \). Using the notation of Theorem 3.2, we see that the latter condition is equivalent to \( G_i = M_i \) for all \( i = 0, 1, \ldots, r + 2t \). From this and by Theorems 3.1 and 3.2, one can easily obtain the conclusions. Here we omit the details. \( \square \)

**Remark** Let \( q \) be even. By Theorems 3.1 and 4.2, we have that

\[
\frac{\mathcal{N}_{E-\text{SO}}}{\mathcal{N}} = \frac{2}{3} \prod_{i=1}^{r} \left( 1 - \frac{1}{q^{d_i} + 3} \right) \cdot \prod_{j=1}^{t} \left( 1 - \frac{1}{q^{d_{r+j}} + 3} \right) \approx \frac{2}{3},
\]

if all \( q^{d_i} \) and \( q^{d_{r+j}} \) are large enough.

Hence the probability of a left \( D_{2n} \)-code over \( \mathbb{F}_q \) being Euclidean self-orthogonal is approximately equal to \( \frac{2}{3} \), when \( q \) is even (and large enough).

At the end of this section, we reformulate the concatenated structure of Euclidean self-dual left \( D_{2n} \)-codes over \( \mathbb{F}_q \) given by [8, Corollary 5.5]. In fact, from the condition: \( \mathcal{C} = \mathcal{C}^\perp \), i.e., \( G_i = E_i \) for all \( i = 0, 1, \ldots, r + 2t \), the following theorem can be derived directly by Theorem 3.1:

**Theorem 4.3** Let \( q \) be even. Then all distinct Euclidean self-dual left \( D_{2n} \)-codes over \( \mathbb{F}_q \) are given by Eq. (2) in Theorem 3.1, where \( C_i \) is a linear code over \( K_i \) of length 2 with generator matrix \( G_i \), \( i \leq i \leq r + 2t \), and \( G_i \) is given by the following three cases:

(i) \( G_0 = (1, 1) \).

(ii) Let \( 1 \leq i \leq r \). Then \( G_i = (g_i(x)^s(q^{d_i} - 1), 1) \), where \( s \) is an arbitrary integer such that \( 0 \leq s \leq q^{d_i} \).

(iii) Let \( i = r + j \) where \( 1 \leq j \leq t \). Then the pairs \( (G_i, G_{i+t}) \) of matrices are given by the following two cases:

\[
\diamond G_i = (1, 0) \text{ and } G_{i+t} = (0, 1);
\]

\[
\diamond G_i = (g(x), 1) \text{ and } G_{i+t} = (1, g(x^{-1})), \quad \text{where } g(x) = \sum_{l=0}^{d_i-1} a_l x^l \text{ and } g(x^{-1}) = a_0 + \sum_{l=1}^{d_i-1} a_l x^{n-l} \pmod{f_i(x)} \text{ for any } a_0, a_1, \ldots, a_{d_i-1} \in \mathbb{F}_q.
\]

Therefore, the number of Euclidean self-dual left \( D_{2n} \)-codes over \( \mathbb{F}_q \) is equal to

\[
\mathcal{N}_{E-\text{SD}} = \prod_{i=1}^{r} (q^{d_i} + 1) \prod_{j=1}^{t} (q^{d_{r+j}} + 1).
\]
Remarks (†) When $q$ is odd, by [8, Page 110], we know that there is no Euclidean self-dual left $D_{2n}$-codes over $\mathbb{F}_q$.

(‡) When $q$ is even, by Theorems 3.1 and 4.3, we have

$$\frac{N_{E-SD}}{N} = \frac{1}{3} \prod_{i=1}^{r} \left(1 - \frac{2}{q^{d_i} + 3}\right) \prod_{j=1}^{t} \left(1 - \frac{2}{q^{d_{r+j}} + 3}\right) \approx \frac{1}{3},$$

if all $q^{d_i}$ and $q^{d_{r+j}}$ are large enough.

Hence the probability of a left $D_{2n}$-code over $\mathbb{F}_q$ being an Euclidean self-dual code is approximately equal to $\frac{1}{3}$, when $q$ is even (and large enough).

Let $N_{E-SD(n,2m)}$ be the number of Euclidean self-dual left $D_{2n}$-codes over the finite field $\mathbb{F}_{2m}$. Then for $m \in \{1, 2, 3, 4\}$ and $n \in \{5, 7, 9, 11, 13, 15, 17\}$, we have the following table:

| $n$  | $N_{E-SD(n,2)}$ | $N_{E-SD(n,4)}$ | $N_{E-SD(n,8)}$ | $N_{E-SD(n,16)}$ |
|------|----------------|----------------|----------------|----------------|
| 5    | 5              | 25             | 65             | 289*           |
| 7    | 9              | 65             | 729*           | 4097           |
| 9    | 27             | 325            | 6561*          | 69649*         |
| 11   | 33             | 1025           | 32769          | 1048577        |
| 13   | 65             | 4225           | 274625         | 16785409       |
| 15   | 255            | 36125*         | 2396745        | 410338673*     |
| 17   | 289            | 83521          | 16785409       | 6975757441     |

This table is from the table of example 6.3 in [8], but there are six errors (two of them are typos). Here, we put * to the six corrected values.

5. Encoder (generator matrix) of any left $D_{2n}$-code

In this section, we discuss in detail how to concretely construct each distinct left $D_{2n}$-code (resp. Euclidean self-dual left $D_{2n}$-code, Euclidean LCD left $D_{2n}$-code and Euclidean self-orthogonal left $D_{2n}$-code) over $\mathbb{F}_q$.

Vedenev and Deundyak constructed $\mathbb{F}_q$-basis, generating and check matrices of any dihedral code which are based the representation of the dihedral codes given in [35]. Five pages (Pages 12–16) were used to give their results. However, these results are too complex to easily use to construct specific dihedral codes.
Here, we provide a direct and simple method for determining the encoder (generator matrix) of any left $D_{2n}$-code over $\mathbb{F}_q$. To do this, in the rest of this paper, we identify each polynomial

$$a(x) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} \in A = \mathbb{F}_q[x]/(x^n - 1)$$

with the vector $(a_0, a_1, \ldots, a_{n-1}) \in \mathbb{F}_q^n$. Moreover, for any integer $k$: $1 \leq k \leq n - 1$, we set:

$$[a(x)]_k = \begin{pmatrix} a(x) \\ xa(x) \\ \vdots \\ x^{k-1}a(x) \end{pmatrix} = \begin{pmatrix} a_0 & a_1 & \ldots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 & \ldots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-k+1} & a_{n-k+2} & \ldots & a_{n-k-1} & a_{n-k} \end{pmatrix}$$

in which the operation is done modulo $x^n - 1$. Then $[a(x)]_k \in M_{k \times n}(\mathbb{F}_q)$.

Now, an explicit generator matrix of each left $D_{2n}$-code over $\mathbb{F}_q$ is given by the following theorem.

**Theorem 5.1** Let $C$ be any left $D_{2n}$-code over $\mathbb{F}_q$ with canonical representation $C = \bigoplus_{i=0}^{r+2t} (A_i \square \varphi_i C_i)$, where $C_i$ is a linear code of length $2$ with a generator matrix $G_i$ given by Theorem 3.1 (resp. Theorem 4.1, Theorem 4.2, or Theorem 4.3). Then

$$\dim_{\mathbb{F}_q}(C) = \sum_{i=0}^{r+2t} d_i \cdot \dim_{\mathbb{F}_q}(C_i) = \sum_{i=0}^{r} d_i \cdot \dim_{\mathbb{F}_q}(C_i) + 2 \sum_{i=r+1}^{r+t} d_i \cdot \dim_{\mathbb{F}_q}(C_i)$$

and an $\mathbb{F}_q$-generator matrix $G_C$ of $C$ is given as below:

$$G_C = \begin{pmatrix} G_{A_0 \square \varphi_0 C_0} \\ G_{A_1 \square \varphi_1 C_1} \\ \vdots \\ G_{A_{r+2t} \square \varphi_{r+2t} C_{r+2t}} \end{pmatrix},$$

where $G_{A_i \square \varphi_i C_i}$ is a generator matrix of the concatenated code $A_i \square \varphi_i C_i$, $0 \leq i \leq r + 2t$, and the matrix $G_{A_i \square \varphi_i C_i}$ is determined by one of the following three cases:

1) Let $G_i = (0, 0)$. Then $G_{A_i \square \varphi_i C_i} = (0, 0, \ldots, 0) \in \mathbb{F}_q^{2n}$.

2) Let $G_i = (g_1(x), g_2(x))$, where $(g_1(x), g_2(x)) \in K_2^2 \setminus \{(0, 0)\}$. Then

$$G_{A_i \square \varphi_i C_i} = ([\varepsilon_i(x)g_1(x)]_{d_i}, [\varepsilon_i(x)g_2(x)]_{d_i}) \in M_{d_i \times 2n}(\mathbb{F}_q).$$
3) Let $G_i = I_2$. Then $G_{A_i \sqcup \varphi_i} C_i = \begin{pmatrix} [\varepsilon_i(x)]_{d_i} & 0 \\ 0 & [\varepsilon_i(x)]_{d_i} \end{pmatrix} \in M_{2d_i \times 2n} (\mathbb{F}_q)$.

Therefore, $\mathcal{C} = \{(u_1, \ldots, u_k)G_C | u_1, \ldots, u_k \in \mathbb{F}_q\}$, where $k = \dim_{\mathbb{F}_q}(\mathcal{C})$.

**Proof.** Let $i$ be an integer: $0 \leq i \leq r + 2t$, and let $G_i$ be given by Theorem 3.1, which is the generator matrix of the outer code $C_i$ in the concatenated code $A_i \sqcup \varphi_i C_i$. By Theorem 3.1, we have one of the following three cases:

Case 1. Let $G_i = (0, 0)$. Then $C_i = \{0\}$ and hence $A_i \sqcup \varphi_i C_i = \{(0, \ldots, 0)\} \subseteq \mathbb{F}_q^{2n}$. The latter implies $G_{A_i \sqcup \varphi_i} C_i = (0, 0, \ldots, 0) \in \mathbb{F}_q^{2n}$.

Case 2. Let $G_i = (g_1(x), g_2(x))$, where $(g_1(x), g_2(x)) \in K_i^2 \setminus \{(0, 0)\}$. Then $\dim_{K_i}(C_i) = 1$ and hence $C_i = \{(\alpha g_1(x), \alpha g_2(x)) | \alpha \in K_i\}$. From this, by the definition of concatenated codes in Section 2 and $K_i = \mathbb{F}_q[x]/(f_i(x)) = \{\sum_{j=0}^{d_i-1} b_j x^j \mid b_0, b_1, \ldots, b_{d_i-1} \in \mathbb{F}_q\}$, we deduce that

$$A_i \sqcup \varphi_i C_i$$

$$= \{ (\varepsilon_i(x) \xi(x), \varepsilon_i(x) \eta(x)) \mid (\xi(x), \eta(x)) \in C_i \}$$

$$= \{ \left( \sum_{j=0}^{d_i-1} b_j x^j \varepsilon_i(x) g_1(x), \sum_{j=0}^{d_i-1} b_j x^j \varepsilon_i(x) g_1(x) \right) \mid b_j \in \mathbb{F}_q, 0 \leq j \leq d_i - 1 \}$$

$$= \{ \left( \sum_{j=0}^{d_i-1} b_j (x^j \varepsilon_i(x) g_1(x), x^j \varepsilon_i(x) g_2(x)) \right) \mid b_j \in \mathbb{F}_q, 0 \leq j \leq d_i - 1 \}$$

$$= \{ (b_0, b_1, \ldots, b_{d_i-1}) \cdot G_{A_i \sqcup \varphi_i} C_i \mid b_j \in \mathbb{F}_q, 0 \leq j \leq d_i - 1 \},$$

where $G_{A_i \sqcup \varphi_i} C_i = ([\varepsilon_i(x) g_1(x)]_{d_i}, [\varepsilon_i(x) g_2(x)]_{d_i}) \in M_{d_i \times 2n}(\mathbb{F}_q)$. Hence $G_{A_i \sqcup \varphi_i} C_i$ is a generator matrix of the $\mathbb{F}_q$-linear code $A_i \sqcup \varphi_i C_i$.

Case 3. Let $G_i = I_2$. In this case, we have that

$$A_i \sqcup \varphi_i C_i = (A_i \sqcup \varphi_i C_i^{(1)}) \oplus (A_i \sqcup \varphi_i C_i^{(2)}),$$

where $C_i^{(1)}$ and $C_i^{(2)}$ are linear codes over $K_i$ with generator matrices $(1, 0)$ and $(0, 1)$ respectively. Using the conclusion of Case 2, $([\varepsilon_i(x)]_{d_i}, 0)$ and $(0, [\varepsilon_i(x)]_{d_i})$ are generator matrices of the $\mathbb{F}_q$-linear codes $A_i \sqcup \varphi_i C_i^{(1)}$ and $A_i \sqcup \varphi_i C_i^{(2)}$, respectively. Hence $G_{A_i \sqcup \varphi_i} C_i = \begin{pmatrix} [\varepsilon_i(x)]_{d_i} & 0 \\ 0 & [\varepsilon_i(x)]_{d_i} \end{pmatrix}$ is a generator matrix of the $\mathbb{F}_q$-linear codes $A_i \sqcup \varphi_i C_i$.

Finally, by $\mathcal{C} = \bigoplus_{i=0}^{r+2t} (A_i \sqcup \varphi_i C_i)$, we conclude that a generator matrix of
the left $D_{2n}$-code $C$ over $\mathbb{F}_q$ is given by $G_C = \begin{pmatrix} G_{A_0} & C_0 \\ G_{A_1} & C_1 \\ \vdots \\ G_{A_{r+2t}} & C_{r+2t} \end{pmatrix}$.

**Example 5.2** We consider binary self-dual left $D_{42}$-codes and binary self-orthogonal left $D_{42}$-codes. In $\mathbb{F}_2[x]$, we have

$$x^{21} - 1 = f_0(x)f_1(x)f_2(x)f_3(x)f_4(x)f_5(x),$$

where $f_0(x) = x + 1$, $f_1(x) = x^2 + x + 1$, $f_2(x) = x^3 + x + 1$, $f_3(x) = x^6 + x^5 + x^4 + x^2 + 1$, $f_4(x) = x^3 + x^2 + 1$ and $f_5(x) = x^6 + x^4 + x^2 + x + 1$ satisfying: $f_i^*(x) = f_i(x)$ for $i = 0, 1$; $f_{1+j}^*(x) = f_{3+j}(x)$ for $i = 1, 2$. Hence $r = 1$, $t = 2$, $d_1 = 2$, $d_2 = d_4 = 3$ and $d_3 = d_5 = 6$. By Theorems 3.1, 4.3 and 4.2, the number of binary left $D_{42}$-codes is equal to

$$N = 3 \cdot (2^2 + 3)(2^3 + 3)(2^6 + 3) = 11055,$$

the number of binary self-dual left $D_{42}$-codes is equal to

$$N_{E-SD} = (2^4 + 1)(2^3 + 1)(2^6 + 1) = 1365,$$

and the number of binary self-orthogonal left $D_{42}$-codes is equal to

$$N_{E-SO} = 2 \cdot (2^4 + 2)(2^3 + 2)(2^6 + 2) = 5280.$$

Using the notation of Section 2, we have

$\varepsilon_0(x) = \sum_{j=0}^{20} x^j$;

$\varepsilon_1(x) = x + x^2 + x^4 + x^5 + x^7 + x^8 + x^{10} + x^{11} + x^{13} + x^{14} + x^{16} + x^{17} + x^{19} + x^{20}$;

$\varepsilon_2(x) = 1 + x + x^2 + x^4 + x^7 + x^8 + x^9 + x^{11} + x^{14} + x^{15} + x^{16} + x^{18}$;

$\varepsilon_3(x) = x^5 + x^7 + x^{10} + x^{13} + x^{14} + x^{17} + x^{19} + x^{20};$

$\varepsilon_4(x) = 1 + x^3 + x^5 + x^6 + x^7 + x^{10} + x^{12} + x^{13} + x^{14} + x^{17} + x^{19} + x^{20};$

$\varepsilon_5(x) = x + x^2 + x^4 + x^7 + x^8 + x^{11} + x^{14} + x^{16}.$

Let $\varphi_1(x) = x$ which is a primitive element of the finite field $K_1 = \frac{\mathbb{F}_2[x]}{(x^4 + x + 1)}.$

Then $\varphi_1(x)^{2 \frac{41}{2} - 1} = x$ and $\{\varphi_1(x)^{s \left(2 \frac{41}{2} - 1\right)} | s = 0, 1, 2 = 2 \frac{41}{2}\} = \{1, x, 1 + x\}$.

By Theorems 4.2 and 5.1, all distinct 5280 binary self-orthogonal left $D_{42}$-codes are generated by the following matrices: $G = \begin{pmatrix} G_0 \\ G_1 \\ \vdots \\ G_5 \end{pmatrix}$, where
In Example 5.3, we consider the construction of LCD left $D_{20}$-codes over $\mathbb{F}_3$. Here $r = 3$, $t = 0$, $\mathcal{L}_0 = \{0, 1\}$, $f_0(x) = x - 1$, $f_1(x) = x + 1$, $f_2(x) = x^4 + x^3 + x^2 + x + 1 = f_2^*(x)$, $f_3(x) = x^4 + 2x^3 + x^2 + 2x + 1 = f_3^*(x)$, $d_0 = d_1 = 1$ and $d_2 = d_3 = 4$. Since $t = 0$, by Theorem 4.1, every left $D_{20}$-code over $\mathbb{F}_3$ must be a LCD code, and the number of LCD left $D_{20}$-codes over $\mathbb{F}_3$ is equal to $N_{E-LCD} = 4^2 \cdot (3^4 + 3)^2 = 2304$. 

\begin{itemize}
  \item $G_0 \in \{(0, 0, \ldots, 0), ([\varepsilon_0(x)]_1, [\varepsilon_0(x)]_1)\}$.
  \item $G_1 = \{(0, 0, \ldots, 0)\} \cup \{(c(x)[\varepsilon_1(x)]_2, [\varepsilon_1(x)]_2) \mid c(x) \in 1, x, 1 + x\}$.
  \item The pairs $(G_2, G_4)$ of matrices are given by the following three cases:
    
    $G_2 = G_4 = (0, 0, \ldots, 0)$;
    
    $G_2 = ([\varepsilon_2(x)]_3, 0)$ and $G_4 = (0, [\varepsilon_4(x)]_3)$;
    
    $G_2 = ([g_2(x)[\varepsilon_2(x)]_3, [\varepsilon_2(x)]_3)$ and $G_4 = ([\varepsilon_4(x)]_3, [g_2(x^{-1})\varepsilon_4(x)]_3)$, where
    
    $g_2(x) = a_0 + a_1x + a_2x^2$;
    
    $g_2(x^{-1}) = g_2(x^{20}) = a_0 + a_2 + (a_1 + a_2)x + a_1x^2 \pmod{f_4(x)}$,
    
    for any arbitrary $a_0, a_1, a_2 \in \mathbb{F}_2$.
  \end{itemize}

\begin{itemize}
  \item The pairs $(G_3, G_5)$ of matrices are given by the following three cases:
    
    $G_3 = G_5 = (0, 0, \ldots, 0)$;
    
    $G_3 = ([\varepsilon_3(x)]_6, 0)$ and $G_5 = (0, [\varepsilon_5(x)]_6)$;
    
    $G_3 = ([g_3(x)[\varepsilon_3(x)]_6, [\varepsilon_3(x)]_6)$ and $G_5 = ([\varepsilon_5(x)]_3, [g_3(x^{-1})\varepsilon_5(x)]_6)$, where
    
    $g_3(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5$;
    
    $g_3(x^{-1}) = g_3(x^{20}) \pmod{f_5(x)}$, i.e.,
    
    $g_3(x^{-1}) = b_0 + b_1 + b_3 + (b_1 + b_2 + b_3 + b_4)x + (b_2 + b_3 + b_4)x^2 + (b_1 + b_2 + b_3 + b_4)x^3$ + $(b_2 + b_3 + b_4)x^4 + (b_1 + b_2 + b_3 + b_4)x^5$,
    
    for any arbitrary $b_j \in \mathbb{F}_2$ and $j = 0, 1, 2, 3, 4, 5$.
\end{itemize}

**Example 5.3** We consider the construction of LCD left $D_{20}$-codes over $\mathbb{F}_3$. In $\mathbb{F}_3[x]$, we have

$$x^{10} - 1 = (x - 1)(x + 1)(x^4 + x^3 + x^2 + x + 1)(x^4 + 2x^3 + x^2 + 2x + 1).$$

Here $r = 3$, $t = 0$, $\mathcal{L}_0 = \{0, 1\}$, $f_0(x) = x - 1$, $f_1(x) = x + 1$, $f_2(x) = x^4 + x^3 + x^2 + x + 1 = f_2^*(x)$, $f_3(x) = x^4 + 2x^3 + x^2 + 2x + 1 = f_3^*(x)$, $d_0 = d_1 = 1$ and $d_2 = d_3 = 4$. Since $t = 0$, by Theorem 4.1, every left $D_{20}$-code over $\mathbb{F}_3$ must be a LCD code, and the number of LCD left $D_{20}$-codes over $\mathbb{F}_3$ is equal to $N_{E-LCD} = 4^2 \cdot (3^4 + 3)^2 = 2304$. 

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Using the notation of Section 2, we have
\[\varepsilon_0(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9;\]
\[\varepsilon_1(x) = 1 + 2x + x^2 + 2x^3 + x^4 + 2x^5 + x^6 + 2x^7 + x^8 + 2x^9;\]
\[\varepsilon_2(x) = 1 + 2x + 2x^2 + 2x^3 + x^4 + 2x^5 + 2x^6 + 2x^7 + 2x^8 + 2x^9;\]
\[\varepsilon_3(x) = 1 + x + 2x^2 + x^3 + 2x^4 + 2x^5 + 2x^6 + x^7 + 2x^8 + x^9.\]

Let \(g_2(x) = 1 + 2x\) which is a primitive element of the finite field \(\mathbb{F}_2 = GF(2^4)\). Then \(g_2(x)^3 T - 1 = (1 + 2x)^8 = 1 + x + x^2 + x^3.\)

Let \(g_3(x) = 1 + x\) which is a primitive element of the finite field \(\mathbb{F}_3 = GF(3^4)\). Then \(g_3(x)^3 T - 1 = (1 + x)^8 = 1 + 2x + x^2 + 2x^3.\)

Set \(\Xi_i = \{(0, 0, \ldots, 0), \begin{pmatrix} [\varepsilon_i(x)]_{d_i} & 0 \\ 0 & [\varepsilon_i(x)]_{d_i} \end{pmatrix} \}, \) where \(i = 0, 1, 2, 3.\)

By Theorems 4.1 and 5.1, all distinct 2304 LCD left \(D_{20}\)-codes over \(\mathbb{F}_3\) are generated by the following matrices: \(G = \begin{pmatrix} G_0 \\ G_1 \\ G_2 \\ G_3 \end{pmatrix}, \) where

\(\diamond G_i \in \Xi_i \cup \{([\varepsilon_i(x)]_1, [\varepsilon_i(x)]_1), (-[\varepsilon_i(x)]_1, [\varepsilon_i(x)]_1)\}, \) for \(i = 0, 1.\)

\(\diamond G_i \in \Xi_i \cup \{([g_i(x)^{8s} \varepsilon_i(x)]_4, [\varepsilon_i(x)]_4) \mid s = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}, \) for \(i = 2, 3.\)

**Example 5.4** We consider the construction of LCD left \(D_{26}\)-codes over \(\mathbb{F}_3.\)

In \(\mathbb{F}_3[x],\) we have
\[x^{13} - 1 = f_0(x)f_1(x)f_2(x)f_3(x)f_4(x),\]
where \(f_0(x) = x - 1, f_1(x) = x^3 + x^2 + 2, f_2(x) = x^3 + x^2 + x + 2, f_3(x) = f_4(x) = x^3 + 2x + 2\) and \(f_4(x) = f_2(x) = x^3 + 2x^2 + 2x + 2.\) Hence \(r = 0,\)
\(t = 2, d_0 = 1\) and \(d_1 = d_2 = d_3 = d_4 = 3.\) By Theorem 3.1 and 4.1, the number of left \(D_{26}\)-codes over \(\mathbb{F}_3\) is equal to \(N = 4 \cdot (3^3 + 3)^2 = 3600,\) and the number of LCD left \(D_{26}\)-codes over \(\mathbb{F}_3\) is equal to
\[N_{E-LCD} = 4 \cdot (3^3 + 1)^2 = 3136.\]

Using the notation of Section 2, we have
\[\varepsilon_0(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12};\]
\[\varepsilon_1(x) = 2x^2 + 2x^4 + 2x^5 + 2x^6 + x^7 + x^8 + 2x^{10} + x^{11} + 2x^{12};\]
\[\varepsilon_2(x) = x + x^3 + 2x^4 + 2x^7 + 2x^8 + x^9 + 2x^{10} + 2x^{11} + 2x^{12};\]
\[\varepsilon_3(x) = 2x + x^2 + 2x^3 + x^5 + x^6 + 2x^7 + 2x^8 + 2x^9 + 2x^{11};\]
\(\varepsilon_4(x) = 2x + 2x^2 + 2x^3 + x^4 + 2x^5 + 2x^6 + 2x^9 + x^{10} + x^{12}.\)

By Theorems 4.1 and 5.1, all distinct 3136 LCD left \(D_{26}\)-codes over \(\mathbb{F}_3\) are generated by the following matrices: 
\[
G = \begin{pmatrix}
G_0 \\
G_1 \\
G_2 \\
G_3 \\
G_4
\end{pmatrix},
\]
where

\(\diamond\) \(G_0 \in \{(0,\ldots,0),\begin{pmatrix} [\varepsilon_0(x)]_1 & 0 \\ 0 & [\varepsilon_0(x)]_1 \end{pmatrix}\}, ([\varepsilon_0(x)]_1, [\varepsilon_0(x)]_1), (-[\varepsilon_0(x)]_1, [\varepsilon_0(x)]_1)\}.

\(\diamond\) The pairs \((G_1, G_3)\) of matrices are given by the following three cases:

\(G_1 = G_3 = (0,0,\ldots,0);\)

\(G_1 = \begin{pmatrix} [\varepsilon_1(x)]_3 & 0 \\ 0 & [\varepsilon_1(x)]_3 \end{pmatrix}\) and \(G_3 = \begin{pmatrix} [\varepsilon_3(x)]_3 & 0 \\ 0 & [\varepsilon_3(x)]_3 \end{pmatrix}\);

\(G_1 = ([g_1(x)\varepsilon_1(x)]_3, [\varepsilon_1(x)]_3)\) and \(G_3 = ([\varepsilon_3(x)]_3, [g_1(x^{-1})\varepsilon_3(x)]_3),\) where

\(g_1(x) = a_0 + a_1x + a_2x^2,\)

\(g_1(x^{-1}) = g_1(x^{12}) = a_0 + 2a_1 + a_2 + 2a_1 + (a_1 + 2a_2)x^2 \quad \text{(mod } f_3(x)),\)

for any arbitrary \(a_0, a_1, a_2 \in \mathbb{F}_3\) satisfying \((a_0, a_1, a_2) \neq (0,0,0).\)

\(\diamond\) The pairs \((G_2, G_4)\) of matrices are given by the following three cases:

\(G_2 = G_4 = (0,0,\ldots,0);\)

\(G_2 = \begin{pmatrix} [\varepsilon_2(x)]_3 & 0 \\ 0 & [\varepsilon_2(x)]_3 \end{pmatrix}\) and \(G_4 = \begin{pmatrix} [\varepsilon_4(x)]_3 & 0 \\ 0 & [\varepsilon_4(x)]_3 \end{pmatrix}\);

\(G_2 = ([g_2(x)\varepsilon_2(x)]_3, [\varepsilon_2(x)]_3)\) and \(G_4 = ([\varepsilon_4(x)]_3, [g_2(x^{-1})\varepsilon_4(x)]_3),\) where

\(g_2(x) = b_0 + b_1x + b_2x^2,\)

\(g_2(x^{-1}) = g_2(x^{12}) = b_0 + 2b_1 + 2(b_1 + b_2)x + (b_1 + 2b_2)x^2 \quad \text{(mod } f_4(x)),\)

for any arbitrary \(b_0, b_1, b_2 \in \mathbb{F}_3\) satisfying \((b_0, b_1, b_2) \neq (0,0,0).\)

**Example 5.5** For integers \(m \in \{1,2\}\) and positive integers \(n\) satisfying \(\gcd(3,n) = 1\) and \(4 \leq n \leq 20,\) using Theorem 4.1, we list the number \(\mathcal{N}(n,3^m)\) of left \(D_{2n}\)-codes and the number \(\mathcal{N}_{E-LCD}(n,3^m)\) of Euclidean LCD left \(D_{2n}\)-codes over \(\mathbb{F}_{3^m}\) respectively, as a table below:
6. Conclusions and further research

We studied the construction and enumeration of left dihedral codes satisfying certain Euclidean duality properties: LCD left dihedral codes, self-orthogonal left dihedral codes and self-dual left dihedral codes. Specifically, we determined the Euclidean hull of every left $D_{2n}$-code over $\mathbb{F}_q$, where $\gcd(n, q) = 1$, and provided an explicit representation and a precise enumeration for Euclidean LCD, self-orthogonal and self-dual left $D_{2n}$-codes over $\mathbb{F}_q$ respectively. Moreover, we give a direct and simple method for determining the encoder (generator matrix) of any left $D_{2n}$-code over $\mathbb{F}_q$.

Future topics of interest include to determine the Hermitian duality of left dihedral codes and give an explicit representation and a prices enumeration for left dihedral codes satisfying certain Hermitian duality properties.

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