On certain classes of $\text{Sp}(4, \mathbb{R})$ symmetric $G_2$ structures

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Abstract
We find two different families of $\text{Sp}(4, \mathbb{R})$ symmetric $G_2$ structures in seven dimensions. These are $G_2$ structures with $G_2$ being the split real form of the simple exceptional complex Lie group $G_2$. The first family has $\tau_2 \equiv 0$, while the second family has $\tau_1 \equiv \tau_2 \equiv 0$, where $\tau_1$, $\tau_2$ are the celebrated $G_2$-invariant parts of the intrinsic torsion of the $G_2$ structure. The families are different in the sense that the first one lives on a homogeneous space $\text{Sp}(4, \mathbb{R})/\text{SL}(2, \mathbb{R})$, and the second one lives on a homogeneous space $\text{Sp}(4, \mathbb{R})/\text{SL}(2, \mathbb{R})_s$. Here $\text{SL}(2, \mathbb{R})_l$ is an $\text{SL}(2, \mathbb{R})$ corresponding to the $\mathfrak{sl}(2, \mathbb{R})$ related to the long roots in the root diagram of $\mathfrak{sp}(4, \mathbb{R})$, and $\text{SL}(2, \mathbb{R})_s$ is an $\text{SL}(2, \mathbb{R})$ corresponding to the $\mathfrak{sl}(2, \mathbb{R})$ related to the short roots in the root diagram of $\mathfrak{sp}(4, \mathbb{R})$.

Keywords Homogeneous G2 structures · Skew symmetric torsion · Split signature metric

1 Introduction: a question of Maciej Dunajski

Recently, together with Hill [5], we uncovered an $\text{Sp}(4, \mathbb{R})$ symmetry of the nonholonomic kinematics of a car. I talked about this at the Abel Symposium in Ålesund, Norway, in June 2019. After my talk Maciej Dunajski, intrigued by the root diagram of $\mathfrak{sp}(4, \mathbb{R})$ which appeared in the talk, asked me if using it I can see a $G_2$ structure on a 7-dimensional homogeneous space $M = \text{Sp}(4, \mathbb{R})/\text{SL}(2, \mathbb{R})$. 

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My immediate answer was: ‘I can think about it, but I have to know which of the $\text{SL}(2, \mathbb{R})$ subgroups of $\text{Sp}(4, \mathbb{R})$ I shall use to built $M$.’ The reason for the ‘but’ word in my answer was that there are at least two $\text{SL}(2, \mathbb{R})$ subgroups of $\text{Sp}(4, \mathbb{R})$, which lie quite differently in there. One can see them in the root diagram above: the first $\text{SL}(2, \mathbb{R})$ corresponds to the long roots, as, for example, $E_1$ and $E_{10}$, whereas the second one corresponds to the short roots, as, for example, $E_2$ and $E_9$. Since Maciej never told me which $\text{SL}(2, \mathbb{R})$ he wants, I decided to consider both of them and to determine what kind of $G_2$ structures one can associate with the respective choice of a subgroup.

I emphasize that in the below considerations I will use the split real form of the simple exceptional Lie group $G_2$. Therefore, the corresponding $G_2$ structure metrics will not be Riemannian.\(^1\) They will have signature $(3, 4)$.

### 2 The Lie algebra $\mathfrak{sp}(4, \mathbb{R})$

The Lie algebra $\mathfrak{sp}(4, \mathbb{R})$ is given by the $4 \times 4$ matrices

$$E = (E^\alpha_\beta) = \begin{pmatrix} a_5 & a_7 & a_9 & 2a_{10} \\ -a_4 & a_6 & a_8 & a_9 \\ a_2 & a_3 & -a_6 & -a_7 \\ -2a_1 & a_2 & a_4 & -a_5 \end{pmatrix},$$

where the coefficients $a_I$, $I = 1, 2, \ldots, 10$, are real constants. The Lie bracket in $\mathfrak{sp}(4, \mathbb{R})$ is the usual commutator $[E, E'] = E \cdot E' - E' \cdot E$ of two matrices $E$ and $E'$. We start with the following basis $(E_I)$,

$$E_I = \frac{\partial E}{\partial a_I}, \quad I = 1, 2, \ldots, 10,$$

in $\mathfrak{sp}(4, \mathbb{R})$.

In this basis, modulo the antisymmetry, we have the following nonvanishing commutators: $[E_1, E_5] = 2E_1$, $[E_1, E_7] = -2E_2$, $[E_1, E_9] = -2E_4$, $[E_1, E_{10}] = 4E_5$, $[E_2, E_4] = E_1$, $[E_2, E_5] = E_2$, $[E_2, E_6] = E_3$, $[E_2, E_7] = 2E_3$, $[E_2, E_8] = E_4$, $[E_2, E_9] = -E_5 - E_6$, $[E_2, E_{10}] = -2E_7$, $[E_3, E_4] = -E_2$, $[E_3, E_6] = 2E_3$, $[E_3, E_8] = -E_6$, $[E_3, E_9] = -E_7$.

\(^1\) For some of the Riemannian counterparts of the structures considered here, see for example, [6].
\[ \{E_4, E_5\} = E_4, \quad \{E_4, E_6\} = -E_4, \quad \{E_4, E_7\} = E_5 - E_6, \quad \{E_4, E_9\} = -2E_8, \quad \{E_4, E_{10}\} = -2E_9, \quad \{E_5, E_7\} = E_7, \quad \{E_5, E_9\} = E_9, \quad \{E_5, E_{10}\} = 2E_{10}, \quad \{E_6, E_7\} = -E_7 - E_8, \quad \{E_6, E_8\} = 2E_8, \quad \{E_6, E_9\} = E_9, \quad \{E_7, E_8\} = E_9, \quad \{E_7, E_{10}\} = E_{10}. \]

We see that there are at least two \( \mathfrak{sl}(2, \mathbb{R}) \) Lie algebras here. The first one is
\[
\mathfrak{sl}(2, \mathbb{R})_1 = \text{Span}_\mathbb{R}(E_1, E_5, E_{10}),
\]
and the second is
\[
\mathfrak{sl}(2, \mathbb{R})_2 = \text{Span}_\mathbb{R}(E_2, E_3 + E_6, E_9).
\]

The reason for distinguishing these two is as follows:

The eight 1-dimensional vector subspaces \( q_I = \text{Span}(E_I), \ I = 1, 2, 3, 4, 7, 8, 9, 10 \), of \( \mathfrak{sp}(4, \mathbb{R}) \) are the root spaces of this Lie algebra. They correspond to the Cartan subalgebra of \( \mathfrak{sp}(4, \mathbb{R}) \) given by \( \mathfrak{h} = \text{Span}(E_5, E_6) \). It follows that the pairs \( (E_I, E_J) \) of the root vectors, such that \( I + J = 11, I, J \neq 5, 6 \), correspond to the opposite roots of \( \mathfrak{sl}(2, \mathbb{R}) \). Knowing the Killing form for \( \mathfrak{sl}(2, \mathbb{R}) \), which in the basis \( (E_I) \), and its dual basis \( (E^J) \), \( E_{1\cdot} \cdot E^J = \delta_{1\cdot}^J \), is
\[
K = \frac{1}{12} K_{IJ} E^I \otimes E^J = -4E^1 \otimes E^{10} + 2E^2 \otimes E^9 + E^3 \otimes E^8 - 2E^4 \otimes E^7 + E^5 \otimes E^5 + E^6 \otimes E^6,
\]
one can see that the roots corresponding to the root vectors \( (E_1, E_{10}) \) and \( (E_3, E_9) \) are long, and the roots corresponding to the root vectors \( (E_2, E_9) \) and \( (E_4, E_7) \) are short; compare the Euclidian lengths of these roots as drawn on the \( G_2 \) root diagram presented at the beginning of this article.\(^2\) Thus, the Lie algebra \( \mathfrak{sl}(2, \mathbb{R}) \) containing root vectors \( (E_1, E_{10}) \) corresponding to the long roots lies quite different in \( \mathfrak{sp}(4, \mathbb{R}) \) than the Lie algebra \( \mathfrak{sl}(2, \mathbb{R})_2 \) containing the root vectors \( (E_2, E_9) \) corresponding to the short roots.

### 3 \( G_2 \) structures on \( \text{Sp}(4, \mathbb{R})/\text{SL}(2, \mathbb{R})_I \)

#### 3.1 Compatible pairs \((g, \phi)\) on \( M_I \)

To consider the homogeneous space \( M_I = \text{Sp}(4, \mathbb{R})/\text{SL}(2, \mathbb{R})_I \), it is convenient to change the basis \( (E_I) \) in \( \mathfrak{sp}(4, \mathbb{R}) \) to a new one, \( (e_I) \), in which the last three vectors span \( \mathfrak{sl}(2, \mathbb{R})_I \).

Thus, we take:
\[
e_1 = E_2, \quad e_2 = E_3, \quad e_3 = E_4, \quad e_4 = E_6, \quad e_5 = E_7, \quad e_6 = E_8, \quad e_7 = E_9, \quad e_8 = E_1, \quad e_9 = E_5, \quad e_{10} = E_{10}.
\]

If now, one considers \( (e_I) \) as the basis of the Lie algebra of left invariant vector fields on the Lie group \( \text{Sp}(4, \mathbb{R}) \) then the dual basis \( (e^J) \), \( e_{1\cdot} \cdot e^J = \delta_{1\cdot}^J \), of the left invariant forms on \( \text{Sp}(4, \mathbb{R}) \) satisfies:

\(^2\) We hope that the reader noticed that we use the same symbol \( E_I \) for ‘root vectors’ spanning 1-dimensional ‘root spaces’ of \( g_2 \), as well as for the ‘roots’ \( E_I \) of \( g_2 \) depicted on the root diagram.
\[ \text{de}^1 = -e^1 \wedge (e^4 + e^9) + e^2 \wedge e^3 - 2e^5 \wedge e^8 \\
\text{de}^2 = -2e^1 \wedge e^5 - 2e^2 \wedge e^4 \\
\text{de}^3 = -e^1 \wedge e^6 + e^3 \wedge (e^4 - e^9) - 2e^7 \wedge e^8 \\
\text{de}^4 = e^1 \wedge e^7 + e^2 \wedge e^6 + e^3 \wedge e^5 \\
\text{de}^5 = 2e^1 \wedge e^{10} + e^2 \wedge e^7 + e^5 \wedge (e^9 - e^4) \\
\text{de}^6 = 2e^3 \wedge e^7 - 2e^4 \wedge e^6 \\
\text{de}^7 = 2e^3 \wedge e^{10} - e^5 \wedge e^6 + e^7 \wedge (e^4 + e^9) \\
\text{de}^8 = -e^1 \wedge e^3 - 2e^8 \wedge e^9 \\
\text{de}^9 = e^1 \wedge e^7 - e^3 \wedge e^5 - 4e^8 \wedge e^{10} \\
\text{de}^{10} = -e^5 \wedge e^7 - 2e^9 \wedge e^{10}. \]

(3.1)

Here we used the usual formula relating the structure constants \( c^{jk}_l \), from \([e_j, e_K] = c^{jk}_l e_l\), to the differentials of the Maurer–Cartan forms (\(e^l\)), \(\text{de}^l = -\frac{1}{2} c^{jk}_l e^j \wedge e^K\).

In this basis, the Killing form on \(\text{Sp}(4, \mathbb{R})\) is

\[ K = \frac{1}{12} c^{jk}_l e^K e^l \circ e^j = (e^4)^2 - 2e^3 \circ e^5 + e^2 \circ e^6 + 2e^1 \circ e^7 + (e^9)^2 - 4e^8 \circ e^{10}. \]

Here, we have used the notation \(e^l \circ e^j = \frac{1}{2}(e^l \otimes e^j + e^j \otimes e^l)\), \((e^j)^2 = e^j \circ e^j\).

One can now use equations (3.1) to see that the homogeneous space \(M_f = \text{Sp}(4, \mathbb{R})/\text{SL}(2, \mathbb{R})_f\) is the leaf space of a certain integrable rank 3 distribution \(D_f\) on \(\text{Sp}(4, \mathbb{R})\), establishing explicitly that \(\text{Sp}(4, \mathbb{R})\) has, in particular, the structure of the principal \(\text{SL}(2, \mathbb{R})\) fiber bundle \(\text{SL}(2, \mathbb{R})_f \to \text{Sp}(4, \mathbb{R}) \to M_f = \text{Sp}(4, \mathbb{R})/\text{SL}(2, \mathbb{R})_f\).

Indeed, the 3-dimensional distribution \(D_f\), generated by the vector fields \(X\) on \(\text{Sp}(4, \mathbb{R})\) annihilating the span of the 1-forms \((e^1, e^2, \ldots, e^7)\), is integrable, \(\text{de}^\mu \wedge e^1 \wedge e^2 \cdots \wedge e^7 = 0\), \(\mu = 1, 2, \ldots, 7\), so that we have a well-defined 7-dimensional leaf space \(M_f\) of the corresponding foliation. Moreover, the Maurer–Cartan equations (3.1), restricted to a leaf defined by \((e^1, e^2, \ldots, e^7) = 0\), reduce to \(\text{de}^6 = -2e^8 \wedge e^9\), \(\text{de}^9 = -4e^8 \wedge e^{10}\), \(\text{de}^{10} = -2e^9 \wedge e^{10}\), showing that each leaf can be identified with the Lie group \(\text{SL}(2, \mathbb{R})_f\). Thus, the projection \(\text{Sp}(4, \mathbb{R}) \to M_f\) from the Lie group \(\text{Sp}(4, \mathbb{R})\) to the leaf space \(M_f\) is the projection to the homogeneous space \(M_f = \text{Sp}(4, \mathbb{R})/\text{SL}(2, \mathbb{R})_f\).

In this section, I will use from now on Greek indices \(\mu, \nu, \ldots\), to run from 1 to 7. They number the first seven basis elements in the bases \((e_i)\) and \((e^j)\).

Now, I look for all bilinear symmetric forms \(g = g_{\mu \nu} e^\mu \otimes e^\nu\) on \(\text{Sp}(4, \mathbb{R})\), with constant coefficients \(g_{\mu \nu} = g_{\nu \mu}\), which are constant along the leaves of the foliation defined by \(D_f\). Technically, I search for those \(g\) whose Lie derivative with respect to any vector field \(X\) from \(D_f\) vanishes,

\[ \mathcal{L}_X g = 0 \quad \text{for all} \quad X \in D_f. \]

(3.2)

I have the following proposition:

**Proposition 3.1** The most general \(g = g_{\mu \nu} e^\mu \otimes e^\nu\) satisfying condition (3.2) is

\[
\begin{align*}
g &= g_{22}(e^3)^2 + 2g_{24}e^2 \otimes e^4 + g_{34}(e^4)^2 + 2g_{35}(e^3 \otimes e^5 - e^1 \otimes e^7) \\
&+ 2g_{28}e^2 \otimes e^6 + 2g_{46}e^4 \otimes e^6 + g_{66}(e^6)^2.
\end{align*}
\]
Thus, I have a 7-parameter family of bilinear forms on $\text{Sp}(4, \mathbb{R})$ that descend to well-defined pseudo-Riemannian metrics on the leaf space $M_I$. Note that the restriction of the Killing form $K$ to the space where $(e^8, e^9, e^{10}) \equiv 0$ is in this family. This corresponds to $g_{22} = g_{24} = g_{46} = 0$ and $g_{44} = 2g_{26} = -g_{35} = 1$.

Since the aim of my note is not to be exhaustive, but rather to show how to produce $G_2$ structures on $\text{Sp}(4, \mathbb{R})$ homogeneous spaces, from now on I will restrict myself to only one $\text{SL}(2, \mathbb{R})_I$ invariant bilinear form $g$ on $\text{Sp}(4, \mathbb{R})$, namely to coming from the restriction of the Killing form. It follows from Proposition 3.1 that this form is a well-defined $(3, 4)$ signature metric on the quotient space $M_I = \text{Sp}(4, \mathbb{R})/\text{SL}(2, \mathbb{R})_I$.

I now look for the 3-forms $\phi = \frac{1}{6} \phi_{\mu \nu \rho} e^\mu \wedge e^\nu \wedge e^\rho$ on $\text{Sp}(4, \mathbb{R})$ that are constant along the leaves of the distribution $D_I$, i.e., such that $\mathcal{L}_X \phi = 0$ for all $X$ in $D_I$. (3.4)

Then, I have the following proposition.

**Proposition 3.2** There is a 10-parameter family of 3-forms $\phi = \frac{1}{6} \phi_{\mu \nu \rho} e^\mu \wedge e^\nu \wedge e^\rho$ on $\text{Sp}(4, \mathbb{R})$ which satisfy condition (3.4). The general formula for them is:

\[
\phi = fe_{125} + a(e^{235} - e^{127}) + pe_{145} + q(e^{147} + e^{345}) + se^{156} + t(e^{356} - e^{167}) + he^{237} + be^{246} + re^{347} + ue^{367}.
\]

Here $e^{\mu \nu \rho} = e^\mu \wedge e^\nu \wedge e^\rho$, and $a, b, f, h, p, q, r, s, t$ and $u$ are real constants.

Thus there is a 10-parameter family of 3-forms $\phi$ that descends from $\text{Sp}(4, \mathbb{R})$ to the $\text{Sp}(4, \mathbb{R})$ homogeneous space $M_I = \text{Sp}(4, \mathbb{R})/\text{SL}(2, \mathbb{R})_I$.

Now, I introduce an important notion of compatibility of a pair $(g, \phi)$ where $g$ is a metric and $\phi$ is a 3-form on a 7-dimensional oriented manifold $M$. The pair $(g, \phi)$ on $M$ is compatible if and only if

\[
(X \ldots \phi) \wedge (X \ldots \phi) \wedge \phi = 3 \ g(X, Y) \ \text{vol}(g), \quad \forall X, Y \in TM.
\]

Here vol$(g)$ is a volume form on $M$ related to the metric $g$.

Restricting, as I did, to the $\text{Sp}(4, \mathbb{R})$ invariant metric $g_K$ on $M_I$ as in (3.3), I now ask which of the 3-forms $\phi$ from Proposition 3.2 are compatible with the metric (3.3). In other words, I now look for the constants $a, b, f, h, p, q, r, s, t$ and $u$ such that

\[
(e_\mu \ldots \phi) \wedge (e_\nu \ldots \phi) \wedge \phi = 3 \ g_K(e_\mu, e_\nu) e^1 \wedge e^2 \wedge e^3 \wedge e^4 \wedge e^5 \wedge e^6 \wedge e^7,
\]

for $g = g_K$ given in (3.3).

I have the following proposition.

**Proposition 3.3** The general solution to the Eq. (3.5) is given by
\[ b = \frac{1}{2}, \quad f = \frac{ap}{1-q}, \quad h = \frac{a(q-1)}{p}, \quad r = \frac{q^2 - 1}{p}, \quad s = \frac{p(1-q)}{4a}, \quad t \]
\[ = \frac{1 - q^2}{4a}, \quad u = \frac{(q^2 - 1)(q + 1)}{4ap}. \]

This leads to the following corollary.

**Corollary 3.4** The most general pair \((g_K, \phi)\) on \(M_7\) compatible with the \(\text{Sp}(4, \mathbb{R})\) invariant metric
\[ g_K = (e^4)^2 - 2e^3 \odot e^5 + e^2 \odot e^6 + 2e^1 \odot e^7, \]
coming from the Killing form in \(\text{Sp}(4, \mathbb{R})\), is a 3-parameter family with \(\phi\) given by:
\[ \phi = \frac{ap}{1-q}e^{125} + a(e^{235} - e^{127}) + pe^{145} + q(e^{147} + e^{345}) + \frac{p(1-q)}{4a}e^{156} \]
\[ + \frac{1 - q^2}{4a}(e^{356} - e^{167}) + \frac{a(q-1)}{p}e^{237} + \frac{1}{2}e^{246} + \frac{q^2 - 1}{p}e^{247} + \frac{(q^2 - 1)(q + 1)}{4ap}e^{367}. \]

Here \(a \neq 0, p \neq 0, q \neq 1\) are free parameters, and \(e^{\mu\nu\rho} = e^{\mu} \wedge e^{\nu} \wedge e^{\rho}\) as before.

### 3.2 \(G_2\) structures in general

Compatible pairs \((g, \phi)\) on 7-dimensional manifolds are interesting since they give examples of \(G_2\) structures [2]. In general, a \(G_2\) structure consists of a compatible pair \((g, \phi)\) of a metric \(g\) and a 3-form \(\phi\) on a 7-dimensional manifold \(M\), and it is in addition assumed that the 3-form \(\phi\) is generic, meaning that at every point of \(M\) it lies in one of the two open orbits of the natural action of \(\text{GL}(7, \mathbb{R})\) on 3-forms in \(\mathbb{R}^7\). The simple exceptional Lie group \(G_2\) appears here as the common stabilizer in \(\text{GL}(7, \mathbb{R})\) of both \(g\) and \(\phi\).

It follows (from compatibility) that the \(G_2\) structures can have metrics \(g\) of only two signatures: the Riemannian ones and (3, 4) signature ones. If the signature of \(g\) is Riemannian, the corresponding \(G_2\) structure is related to the compact real form of the simple exceptional complex Lie group \(G_2\), and in the (3, 4) signature case the corresponding \(G_2\) structure is related to the noncompact (split) real form of the complex group \(G_2\). In this sense, our Corollary 3.4 provides a 3-parameter family of split real form \(G_2\) structures on \(M_7\).

\(G_2\) structures can be classified according to the properties of their intrinsic torsion [1, 2]. Making a long story short, this torsion is totally determined by finding four \(p\)-forms \(\tau_p\) on \(M, \; p = 0, 1, 2, 3\), each belonging to one of four different irreducible representations of \(G_2\). Before telling on how to find these forms given a \(G_2\) structure \((g, \phi)\), we need some preparation.

We recall that the group \(G_2\) acts in \(\mathbb{R}^7\), and this induces its action on spaces \(\bigwedge^p\) of \(p\)-forms in \(\mathbb{R}^7\). Of course the 1-dimensional space \(\bigwedge^0\) is \(G_2\) irreducible, as well as is the space of 1-forms \(\bigwedge^1 = \bigwedge^1_{\mathbb{R}}\). The \(G_2\) irreducible decompositions of the spaces of 2- and 3-forms look like \(\bigwedge^2 = \bigwedge^2_{\mathbb{R}} \oplus \bigwedge^2_{14}\) and \(\bigwedge^3 = \bigwedge^3_{\mathbb{R}} \oplus \bigwedge^3_{17} \oplus \bigwedge^3_{27}\). Here we use the convention that the lower index \(i\) in \(\bigwedge^p_i\) denotes the dimension of the corresponding representation. It is further convenient to introduce the Hodge dual, *, which is defined on \(p\)-forms \(\lambda\) by

\[ \lambda^* = \star \lambda^\ast, \quad \text{for} \quad \lambda \in \bigwedge^p_{\mathbb{R}}. \]
By the Hodge duality, the decomposition of $\bigwedge^4$ into $G_2$ irreducible components is similar to this for $\bigwedge^3$. We further mention that the 7-dimensional representations $\bigwedge^1\mathbb{R}^7$, $\bigwedge^2\mathbb{R}^7$, and $\bigwedge^3\mathbb{R}^7$ are all $G_2$ equivalent. Also, one can see that, e.g., $\bigwedge^3\mathbb{R}^{27} = \{ \alpha \in \bigwedge^3 \text{s.t.} \alpha \wedge \phi = 0 \& \alpha \wedge * \phi = 0 \}$.

The intrinsic torsion components $\tau_0$, $\tau_1$, $\tau_2$, and $\tau_3$ have values in the following $G_2$ irreducible modules: the 3-form $\tau_3$ has values in the 27-dimensional irreducible representation $\bigwedge^3\mathbb{R}^{27}$, the 2-form $\tau_2$ has values in the 14-dimensional irreducible representation $\bigwedge^2\mathbb{R}^{14}$, the 1-form $\tau_1$ has values in the 7-dimensional irreducible representation $\bigwedge^1\mathbb{R}^7$, and the 0-form $\tau_0$ has values in the trivial representation $\bigwedge^0\mathbb{R}$.

The result of Bryant [1, 2] states that for every $G_2$ structure $(g, \phi)$ on $M$ there exist unique forms $\tau_0$, $\tau_1$, $\tau_2$, and $\tau_3$ on $M$, with values in the above-mentioned representations, such that

$$d\phi = \tau_0 \ast \phi + 3 \tau_1 \wedge \phi + * \tau_3$$

$$d \ast \phi = 4 \tau_1 \wedge * \phi + \tau_2 \wedge \phi.$$  

Thus, Eq. (3.6) enable to determine all the intrinsic torsion components $\tau_0$, $\tau_1$, $\tau_2$ and $\tau_3$ of a given $G_2$ structure $(g, \phi)$. They are called Bryant’s [1, 2] equations. It follows that vanishing or not of each of the forms $\tau_p$ is a $G_2$ invariant property of a $G_2$ structure.

### 3.3 All $Sp(4, \mathbb{R})$ symmetric $G_2$ structures on $M$ with the metric coming from the Killing form

The below theorem characterizes the $G_2$ structures corresponding to compatible pairs $(g_K, \phi)$ from Corollary 3.4; it summarizes the already obtained results and, in addition, provides formulas for the intrinsic torsion which are needed for the characterization.

**Theorem 3.5** Let $g_K$ be the $(3, 4)$ signature metric on $M = Sp(4, \mathbb{R})/SL(2, \mathbb{R})$, arising as the restriction of the Killing form $K$ from $Sp(4, \mathbb{R})$ to $M$,

$$g_K = (e^4)^2 - 2e^3 \odot e^5 + e^2 \odot e^6 + 2e^1 \odot e^7.$$  

Then the most general $G_2$ structure associated with such $g_K$ is a 3-parameter family $(g_K, \phi)$ with the 3-form

$$\phi = \frac{ap}{1 - q} e^{125} + a(e^{235} - e^{127}) + pe^{145} + q(e^{147} + e^{345}) + \frac{p(1 - q)}{4a} e^{156}$$

$$+ \frac{1 - q^2}{4a} (e^{356} - e^{167}) + \frac{a(q - 1)}{p} e^{237} + \frac{q^2 - 1}{p} e^{347} + \frac{(q^2 - 1)(q + 1)}{4ap} e^{367}.$$  

For this structure, the torsions $\tau_\mu$ solving the Bryant’s equations (3.6) are:
\[\begin{align*}
\tau_0 &= \frac{6}{7} \frac{(2a - p)^2 q - (2a + p)^2}{ap}, \\
\tau_1 &= \frac{1}{4} (2a - p) \left( -e^2 + \frac{1}{2} \frac{(2a + p)(q - 1)}{ap} e^4 + \frac{1}{2} \frac{q^2 - 1}{ap} e^6 \right), \\
\tau_2 &= 0, \\
\tau_3 &= \left( \frac{3}{28} (2a - p)^2 + \frac{8ap}{7(q - 1)} \right) e^{125} + \frac{11p^2 + 16ap - 12a^2 + 3q(2a - p)^2}{28p} e^{127} \\
&\quad - \frac{44a^2 + 16ap - 3p^2 + 3q(2a - p)^2}{28a} e^{145} \\
&\quad + \frac{(7 - 4q)(2a + p)^2 - 3q^2(2a - p)^2}{28ap} e^{147} \\
&\quad + \frac{3p^2(q - 1)^2 - 12ap(q^2 - 1) + 4a^2(31 + 22q + 3q^2)}{112a^2p} e^{156} \\
&\quad - \frac{(q^2 - 1)(44a^2 + 16ap - 3p^2 + 3q(2a - p)^2)}{112a^2p} e^{167} \\
&\quad + \frac{12a^2 - 16ap - 11p^2 - 3q(2a - p)^2}{28p} e^{235} \\
&\quad - \frac{12a^2(q - 1)^2 - 12ap(q^2 - 1) + p^2(31 + 22q + 3q^2)}{28p^2} e^{237} \\
&\quad + \frac{4ap(6 - q) + (4a^2 + p^2)(q - 1)}{14ap} e^{246} \\
&\quad + \frac{(7 - 4q)(2a + p)^2 - 3q^2(2a - p)^2}{28ap} e^{345} \\
&\quad + \frac{(q^2 - 1)(12a^2 - 16ap - 11p^2 - 3q(2a - p)^2)}{28ap^2} e^{347} \\
&\quad + \frac{(q^2 - 1)(44a^2 + 16ap - 3p^2 + 3q(2a - p)^2)}{112a^2p} e^{356} \\
&\quad + \frac{(q^2 - 1)(q + 1)(12a^2 - 44ap + 3p^2 - 3q(2a - p)^2)}{112a^2p^2} e^{367},
\end{align*}\]

where as usual \(e^{\mu\nu} = e^\mu \wedge e^\nu\) and \(e^{\mu\nu\rho} = e^\mu \wedge e^\nu \wedge e^\rho\).

Thus, the 3-parameter family of \(G_2\) structures on \(M_t\) described in this theorem have the entire 14-dimensional torsion \(\tau_2 = 0\). This means that all these \(G_2\) structures are integrable in the terminology of [3, 4], or what is the same, this means that they all have the totally skew symmetric torsion.

### 4 \(G_2\) structures on \(\text{Sp}(4, \mathbb{R})/\text{SL}(2, \mathbb{R})\)_s

Now we consider the homogeneous space \(M_t = \text{Sp}(4, \mathbb{R})/\text{SL}(2, \mathbb{R})_s\). Since \(\mathfrak{sI}(2, \mathbb{R})\) is spanned by \(E_2, E_5 + E_6, E_9\), it is convenient to put these vectors at the end of the new basis of the Lie algebra \(\mathfrak{sp}(4, \mathbb{R})\). We choose this new basis \((f_j)\) in \(\mathfrak{sp}(4, \mathbb{R})\) as:
If now, one considers \((f_1)\) as the basis of the Lie algebra of invariant vector fields on the Lie group \(\text{Sp}(4, \mathbb{R})\), then the dual basis \((f^1)\), \(f^1 = \delta_i^j\), of the left invariant forms on \(\text{Sp}(4, \mathbb{R})\), satisfies:

\[
\begin{align*}
    df^1 &= 2f^1 \wedge (f^4 - f^9) + f^3 \wedge f^8 \\
    df^2 &= -2f^2 \wedge (f^4 + f^9) + 2f^5 \wedge f^8 \\
    df^3 &= 2f^3 \wedge f^{10} + 2f^3 \wedge f^4 + f^6 \wedge f^8 \\
    df^4 &= 2f^4 \wedge f^7 + \frac{1}{2} f^2 \wedge f^6 + f^3 \wedge f^5 \\
    df^5 &= f^2 \wedge f^{10} + 2f^4 \wedge f^5 - 2f^7 \wedge f^8 \\
    df^6 &= 2f^3 \wedge f^{10} - 2(f^4 + f^9) \wedge f^6 \\
    df^7 &= 2(f^4 - f^9) \wedge f^7 - f^5 \wedge f^{10} \\
    df^8 &= 2f^1 \wedge f^5 + f^2 \wedge f^3 - 2f^8 \wedge f^9 \\
    df^9 &= -2f^1 \wedge f^7 + \frac{1}{2} f^2 \wedge f^6 + f^8 \wedge f^{10} \\
    df^{10} &= 2f^3 \wedge f^7 - f^5 \wedge f^6 - 2f^9 \wedge f^{10}.
\end{align*}
\]

In this basis, the Killing form on \(\text{Sp}(4, \mathbb{R})\) is

\[
K = \frac{1}{12} c^{ij}_{jk} c^K L^i j^k \circ f^L = 2(f^4)^2 - 2f^3 \circ f^5 + f^2 \circ f^6 - 4f^1 \circ f^7 + 2(f^9)^2 + 2f^8 \circ f^{10},
\]

where as usual the structure constants \(c^{ij}_{jk}\) are defined by \([f_i, f_j] = c^{K}_{ij} L^i K\).

Using the same arguments, as in the case of \(M_s\), we again see that \(\text{Sp}(4, \mathbb{R})\) has the structure of the principal \(\text{SL}(2, \mathbb{R})\) fiber bundle \(\text{SL}(2, \mathbb{R}) \to \text{Sp}(4, \mathbb{R}) \to M_s = \text{Sp}(4, \mathbb{R})/\text{SL}(2, \mathbb{R})\), over the homogeneous space \(M_s = \text{Sp}(4, \mathbb{R})/\text{SL}(2, \mathbb{R})\). In particular, we have a foliation of \(\text{Sp}(4, \mathbb{R})\) by integral leaves of an integrable distribution \(D_s\) spanned by the annihilator of the forms \((f^1, f^2, \ldots, f^7)\). As before, also in this section, we will use Greek indices \(\mu, \nu\), etc., to run from 1 to 7. They now number the first seven basis elements in the bases \((f_i)\) and \((f^i)\).

Repeating the procedure from the previous sections, I now search for all bilinear symmetric forms \(g = g_{\mu \nu} f^\mu \circ f^\nu\) on \(\text{Sp}(4, \mathbb{R})\), with constant coefficients \(g_{\mu \nu} = g_{\nu \mu}\), whose Lie derivative with respect to any vector field \(X\) from \(D_s\) vanishes,

\[
    \mathcal{L}_X g = 0 \text{ for all } X \in D_s. \tag{4.2}
\]

I have the following proposition.

**Proposition 4.1** The most general \(g = g_{\mu \nu} f^\mu \circ f^\nu\) satisfying condition \((4.2)\) is

\[
    g = g_{33}(f^3)^2 - 2f^4 \circ f^6 + g_{44}(f^4)^2 + g_{55}(f^5)^2 + 2f^2 \circ f^7 + 2g_{26}( - 2f^3 \circ f^5 + f^2 \circ f^6 - 4f^1 \circ f^7).
\]

Thus, this time, I only have a 4-parameter family of bilinear forms on \(\text{Sp}(4, \mathbb{R})\) that descend to well-defined pseudo-Riemannian metrics on the leaf space \(M_s\). Note that the
restriction of the Killing form $K$ to the space where $(f^8, f^9, f^{10}) \equiv 0$ is in this family. This corresponds to $g_{33} = g_{55} = 0$ and $g_{44} = 2$, $g_{26} = 1/2$.

Again for simplicity reasons, I will solve the problem of finding $\text{Sp}(4, \mathbb{R})$ invariant $G_2$ structures on $M_s$ restricting to only those pairs $(g, \phi)$ for which $g = g_K$, where

$$g_K = 2(f^4)^2 - 2f^3 \odot f^5 + f^2 \odot f^6 - 4f^1 \odot f^7,$$  

(4.3)

which again means that I only will consider one metric, the one coming from the restriction of the Killing form of $\text{Sp}(4, \mathbb{R})$ to $M_s$. It is a well-defined $(3, 4)$ signature metric on the quotient space $M_s = \text{Sp}(4, \mathbb{R})/\text{SL}(2, \mathbb{R})$.

I now look for the 3-forms $\phi = \frac{1}{6} \phi_{\mu \nu \rho} f^\mu \wedge f^\nu \wedge f^\rho$ on $\text{Sp}(4, \mathbb{R})$ which are such that

$$L_X \phi = 0 \text{ for all } X \text{ in } D_s.$$  

(4.4)

I have the following proposition.

**Proposition 4.2** There is precisely a 5-parameter family of 3-forms $\phi = \frac{1}{6} \phi_{\mu \nu \rho} f^\mu \wedge f^\nu \wedge f^\rho$ on $\text{Sp}(4, \mathbb{R})$ which satisfies condition (4.4). The general formula for $\phi$ is:

$$\phi = a(4f^{147} + f^{246} + 2f^{345}) + b(2f^{156} + f^{236} - 4f^{137}) + q f^{136} + h(f^{256} - 4f^{157} - 2f^{237}) + pf^{257}.$$  

Here $f^{\mu \nu \rho} = f^\mu \wedge f^\nu \wedge f^\rho$, and $a, b, q, h$ and $p$ are real constants.

Solving for all 3-forms $\phi$ from this 5-parameter family that are compatible, as in (3.5), with the metric $g_K$ from (4.3), I arrive at the following proposition.

**Proposition 4.3** The general solution to the equations (3.5) is given by

$$a = \frac{1}{2}, \quad b = h = 0, \quad p = \frac{1}{q}.$$  

This leads to the following corollary.

**Corollary 4.4** The most general pair $(g_K, \phi)$ on $M_s$ compatible with the $\text{Sp}(4, \mathbb{R})$ invariant metric

$$g_K = 2(f^4)^2 - 2f^3 \odot f^5 + f^2 \odot f^6 - 4f^1 \odot f^7,$$

coming from the Killing form in $\text{Sp}(4, \mathbb{R})$, is a 1-parameter family with $\phi$ given by:

$$\phi = 2f^{147} + \frac{1}{2} f^{246} + f^{345} + q f^{136} + \frac{1}{q} f^{257}.$$  

Here $q \neq 0$ is a free parameter, and $f^{\mu \nu \rho} = f^\mu \wedge f^\nu \wedge f^\rho$ as before.
4.1 All $\text{Sp}(4, \mathbb{R})$ symmetric $G_2$ structures on $M_s$ with the metric coming from the Killing form

Similarly as in Sect. 3.3 we now summarize the already obtained results about the considered $\text{Sp}(2, \mathbb{R})$ symmetric $G_2$ structures on $M_s$ in a theorem; it is given below and has also a new part consisting of the formulas for the intrinsic torsion.

Theorem 4.5 Let $g_K$ be the $(3, 4)$ signature metric on $M_s = \text{Sp}(4, \mathbb{R})/\text{SL}(2, \mathbb{R})$, arising as the restriction of the Killing form $K$ from $\text{Sp}(4, \mathbb{R})$ to $M_s$,

$$g_K = 2(f^4)^2 - 2f^3 \circ f^5 + f^2 \circ f^6 - 4f^1 \circ f^7.$$

Then, the most general $G_2$ structure associated with such $g_K$ is a 1-parameter family $(g_K, \phi)$ with the 3-form

$$\phi = 2f^{147} + \frac{1}{2}f^{246} + f^{345} + qf^{136} + \frac{1}{q}f^{257}.$$

For this structure

$$d \ast \phi = 0,$$

i.e., the torsions

$$\tau_1 = \tau_2 = 0.$$

The rest of the torsions solving Bryant’s equations (3.6) are:

$$\tau_0 = \frac{18}{7},$$

$$\tau_3 = \frac{2}{7}(4f^{147} + f^{246} + 2f^{345}) - \frac{3}{7}(qf^{136} + \frac{1}{q}f^{257}).$$

where, as usual $f^{\mu \nu \rho} = f^\mu \wedge f^\nu \wedge f^\rho$; $q \neq 0$.

So on $M_s = \text{Sp}(4, \mathbb{R})/\text{SL}(2, \mathbb{R})$, there exists a 1-parameter family of the above $G_2$ structures which is coclosed. Therefore, in particular, it is integrable.

I note that formally I can also obtain coclosed $G_2$ structures on $M_s$, using Theorem 3.5. It is enough to take $p = 2a$ in the solutions of this Theorem. The question if in the resulting 2-parameter family of the coclosed $G_2$ structures there is a 1-parameter subfamily equivalent to the structures I have on $M_s$ via Theorem 4.5 needs further investigation. However, I doubt that the answer to this question is positive, since it is visible from the root diagram for $\text{Sp}(4, \mathbb{R})$ that the spaces $M_l$ and $M_s$ are geometrically quite different. Indeed, apart from the $\text{Sp}(4, \mathbb{R})$ invariant $G_2$ structures, which I have just introduced in this note, the spaces $M_l$ and $M_s$ have quite different additional $\text{Sp}(4, \mathbb{R})$ invariant structures. A short look at the root diagram on page 1 of this note shows that $M_l$ has two well-defined $\text{Sp}(4, \mathbb{R})$ invariant rank 3-distributions, corresponding to the pushforwards from $\text{Sp}(4, \mathbb{R})$ to $M_l$ of the vector spaces $D_{11} = \text{Span}_\mathbb{R}(E_2, E_3, E_7)$ and $D_{12} = \text{Span}_\mathbb{R}(E_4, E_8, E_9)$. Likewise $M_r$, in addition to the discussed $G_2$ structures, has also a well defined pair of $\text{Sp}(4, \mathbb{R})$ invariant rank 3-distributions, corresponding to the pushforwards from $\text{Sp}(4, \mathbb{R})$ to $M_r$ of the vector spaces $D_{11} = \text{Span}_\mathbb{R}(E_1, E_4, E_8)$ and $D_{12} = \text{Span}_\mathbb{R}(E_3, E_7, E_{10})$. The problem is that these two sets of pairs of $\text{Sp}(4, \mathbb{R})$ invariant distributions are quite different. The distributions on $M_l$ have
constant growth vector \((2, 3)\), while the distributions on \(M_s\) are integrable. These pairs of distributions constitute an immanent ingredient of the geometry on the corresponding spaces \(M_l\) and \(M_s\) and, since they are diffeomorphically nonequivalent and they make the \(G_2\) geometries there quite different. I believe that this fact makes the \(G_2\) structures obtained on \(M_l\) and \(M_s\) really nonequivalent.

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