The quantum Hall plateau transition at order 1/N

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The localization behavior of noninteracting two-dimensional electrons in a random potential and strong magnetic field is of fundamental interest for the physics of the quantum Hall effect. In order to understand the emergence of power-law delocalization near the discrete extended-state energies $E_n = \hbar\omega_c(n + \frac{1}{2})$, we study a generalization of the disorder-averaged Liouvillian framework for the lowest Landau level to $N$ flavors of electron densities ($N = 1$ for the physical case). We find analytically the large-$N$ limit and $1/N$ corrections for all disorder strengths: at $N = \infty$ this gives an estimate of the critical conductivity, and at order $1/N$ an estimate of the localization exponent $\nu$. The localization properties of the analytically tractable $N \gg 1$ theory seem to be continuously connected to those of the exact quantum Hall plateau transition at $N = 1$.

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The explanation by Laughlin \cite{Laughlin} of the integer quantum Hall effect depends upon an understanding of the localization of electrons by disorder in a strong magnetic field. Without disorder, the single-electron eigenstates fall into Landau levels at isolated energies $E_n = \hbar\omega_c(n + \frac{1}{2})$, $n = 0, 1, 2, \ldots$, separated by the cyclotron energy $\hbar\omega_c$. The effect of a weak disorder potential is to displace some weight from the energy $\bar{\omega} = \hbar\omega_c$. The generalization of the disorder-averaged action to disordered samples \cite{note1} are consistent with the value of states at energy $E$ diverges as $E \to E_n$ according to a power law $\xi(E) \propto (E - E_n)^{-\nu}$. Experimental results on disordered samples \cite{note2} are consistent with the value $\nu \approx 2.35 \pm 0.05$ obtained from numerical calculations \cite{note3} on the lowest Landau level (LLL).

Current belief is that the quantum Hall plateau transition lies in an entirely different universality class from the zero-field case, characterized by “two-parameter scaling” \cite{note4} and a topological term in the $\sigma$-model description \cite{note5}. In particular there is now an understanding of the minimal features required to obtain numerically scaling behavior near the transition \cite{note6}, which is in some respects similar to classical percolation, but with different universal properties as a result of quantum tunneling and interference \cite{note7}. There is also some understanding of the apparent insensitivity to interactions of some critical indices such as $\nu$. \cite{note8} However, relatively little progress has been made in finding an analytically tractable description of the localization properties near the critical energy. The subject of this paper is a generalization of the problem to multiple flavors of electron densities, allowing an analytical approach to the transition. Our discussion is based on the Liouvillian approach introduced by Sinova, Meden, and Girvin \cite{note9}, reviewed below.

The generalization of the disorder-averaged action to $N$ flavors of electron densities gives a simple mean-field-like theory in the large-$N$ limit. Other large-$N$ approaches such as \cite{note10} typically generalize the noninteracting problem before the disorder average and do not obtain $\nu$. At first order in the small parameter $1/N$, we recover anomalous scaling of the localization length, i.e., a value for the critical exponent $\nu$. Thus we find an analytically tractable quantum description whose physics seems to connect smoothly in the parameter $N$ to the plateau transition ($N = 1$). The control parameter $1/N$ allows a systematic expansion around $N = \infty$, and the large-$N$ limit and $1/N$ corrections can be found analytically for all disorder strengths.

The localization properties of electrons at energy $E$ are contained in the correlation function of the LLL-restricted density operators $\hat{\rho}_q$:

$$\Pi(q, t; E) = \frac{-i\Theta(t)}{N_L \hbar \ell^2} \langle \langle \text{Tr} \hat{\rho}_q(t) \hat{\rho} - \hat{\rho}(0) \delta(E - H) \rangle \rangle. \quad (1)$$

Here $N_L$ is the number of states in the LLL ($N_L \to \infty$ taken below), $\ell = \sqrt{\hbar/eB}$, and $\langle \langle \rangle \rangle$ indicates the quenched disorder average.

The Liouvillian approach uses the integral over $E$ of $\Pi(q, t; E)$:

$$\tilde{\Pi}(q, t) = \int dE \Pi(q, t; E) = \frac{-i\Theta(t)}{N_L \hbar \ell^2} \langle \langle \text{Tr} \hat{\rho}_q(t) \hat{\rho} - \hat{\rho}(0) \rangle \rangle. \quad (2)$$

The key to the approach is that $\tilde{\Pi}(q, t)$ still contains information about electron localization but is more easily calculated than the fixed-energy quantity $\Pi(q, t; E)$. The disorder average for the Fourier transform $\tilde{\Pi}(q, \omega)$ was carried out numerically in \cite{note11} and shown to obey the form $\omega\tilde{\Pi}(q, \omega) = \omega^{1/2\nu} f(q^2/\omega)$, where $f$ is an unknown scaling function. At mean-field level \cite{note12}, only diffusion is found; our model gives a systematic expansion beyond this mean-field result. A major goal of this paper
is to obtain the prefactor $\omega^{1/2\nu}$ by a $1/N$ expansion of $\Pi(q,\omega)$.

The scaling form for $\tilde{P}(q,\omega)$ comes about because at large $R$, only states with energies satisfying

$$|E - E_c| < \left(\frac{a_d}{R}\right)^{1/\nu}$$

will have $\xi(E) > R$ and hence contribute to the correlation function at long enough times. Here $a_d$ is a nonuniversal length set by the disorder potential. Thus the integral over energy which gives $\Pi(q,\omega)$ is only nonzero in a window of size proportional to $q^{1/\nu}$. For states delocalized on the scale $R = 1/q$, $\Pi(q,\omega; E)$ follows ordinary diffusive scaling ($\omega \ll q^{1/\nu}$) by the dimensionless combination $R^{\nu}$. In terms of the Givornian we obtain the evolution equation for the magnetic trans-

The LLL-projected density operators $\rho_q$ are related to the operators $\tau_q$ of the magnetic translation group through $\rho_q = e^{-i\tau_q \cdot q}$. The noninteracting LLL-projected Hamiltonian is $H = \sum_q v(-q)\rho_q$, with $v$ the Fourier-transformed random potential. Then using the commutation relation for the operators $\tau_q$

$$[\tau_q, \tau_r] = 2i \sin\left(\frac{\tau^2}{2} q \land r\right) \tau_{q+r},$$

we obtain the evolution equation for the magnetic translation operators:

$$\dot{\tau}_q = -i \sum_{q'} G_{qq'} \tau_{q'},$$

in terms of the Givornian

$$G_{qq'} = \frac{2i}{\hbar} v(q - q') e^{-i\frac{\tau^2}{2} |q' - q|^2} \sin\left(\frac{\tau^2}{2} q' \land q\right).$$

Then $\Pi(q,\omega)$ is a one-body correlation function of the Givornian:

$$\Pi(q,\omega) = \frac{1}{\hbar^2} \int \langle \langle v(q) - v(-q')\rangle \rangle.$$ 

Here the states $\{q\}$ and operator $\mathcal{G}$ are defined through $\langle \langle q\} | q\} \rangle = G_{qq'}$.

Now we take the continuum limit $N_L \to \infty$ and assume a white-noise disorder potential $\langle \langle v(q) - v(-q')\rangle \rangle = \frac{2\pi \hbar^2}{\tau^2} \delta(q - q')$. The physical frequency $\omega_0$ is replaced by the dimensionless combination $\omega = h\omega_0/\hbar\nu$. The physical propagator is $\Pi(q,\omega_0) = \frac{h\nu}{\omega_0} \Pi(q,\omega)$, with the dimensionless propagator $\Pi(q,\omega) \to \frac{1}{\nu}$ in the clean limit $\omega \to \infty$. All momenta are dimensionless (scaled by magnetic length $\ell$) in the following.

The result of disorder averaging is an interacting theory which can be expressed through a functional integral over both bosonic $\phi$ and Grassmann $\psi$ variables $\Pi$

$$\Pi(q,\omega) = -i \int D\tilde{\phi} D\phi \int D\tilde{\psi} D\psi \tilde{\phi}_q \phi_q e^{-F(\omega)},$$

$$F(\omega) = -i\omega \int dq \left( \tilde{\phi}_q \phi_q + \tilde{\psi}_q \psi_q \right) + \int_{1,2,3,4} f(1,2,3,4) \left[ \tilde{\phi}_{q_1} \tilde{\phi}_{q_2} \phi_{q_3} \phi_{q_4} + 2\tilde{\psi}_{q_1} \tilde{\phi}_{q_2} \phi_{q_3} \psi_{q_4} + \psi_{q_1} \tilde{\psi}_{q_2} \phi_{q_3} \psi_{q_4} + \psi_{q_1} \tilde{\psi}_{q_2} \phi_{q_3} \psi_{q_4} \right].$$

The effective interaction from disorder averaging is

$$f(1,2,3,4) = \frac{1}{\pi} \int d^4q_1 \int d^4q_2 \int d^4q_3 \int d^4q_4 \sin\left(\frac{1}{2} q_1 \land q_4\right) \sin\left(\frac{1}{2} q_2 \land q_3\right).$$

The effect of the additional Grassmann variables ("super-

The real coefficients $c_i$ in $\Pi$ reproduce the correct $N \to 1$ limit provided that $c_1 + c_2 + c_3 = 1$, so there is a two-parameter family of generalizations. The vertex with coefficient $c_3$ in $\Pi$ does not contribute as $N \to \infty$ and does not seem to affect scaling qualitatively at order $1/N$, so it is dropped for simplicity. We specialize to $c_1 = c_2 = 1/2$ in what follows: the choice of these coefficients equal is "natural" in that the classes of diagrams selected by the two vertices have equal weight at order $1/N$ with $N = 1$, as they do in the full theory (i.e., all orders in $1/N$) at $N = 1$. For generic $c_3$ the theory has a $U(1\{1\}) \times SO(N)$ symmetry, which at the point $c_3 = 1$ ($N$ decoupled systems) becomes $U(N\{N\})$. In the $N \to \infty$ limit, the diagrams with $k$ interaction lines which contribute to $\langle \langle \phi_{q}^{2k} \rangle \rangle$ are the diagrams where no interaction lines cross, which have the maximum $k$ free choices of flavor index (i.e., degeneracy $N^k$). Only
the first interaction term of (11) affects this limit. The
noncrossing propagator sums these diagrams and satisfies
an integral equation depicted graphically in Fig. 1:

$$\Pi_B(q, \omega) = \frac{1}{\omega} + \frac{1}{\pi \omega} \int dq' \sin^2 \left( \frac{1}{2} q \wedge q' \right) e^{\frac{i}{\hbar} (q-q')^2 \Pi_B(q', \omega)}. \quad (12)$$

The notation $\Pi_B(q, \omega)$ is that of [11]. In the limit $q, \omega \to 0$, $q^2 \ll \omega$, the noncrossing result $\Pi_B(q, \omega)$ shows diffusive behavior ($\omega \Im \Pi = D_0 q^2/\omega$) without the prefactor $\omega^{1/2\nu}$.

The physics of (16) is similar to that of the weak-localization (WL) logarithmic singularity in two dimensions [17], but with two major differences: the disorder-generated effective interaction involves 4 bosonic density operators and hence 8 rather than 4 fermionic operators, and time-reversal symmetry is broken by the magnetic field. The singularity results when the integral equation (13) becomes nearly $V = V_0 + V$, i.e., when the integral operator on the right-hand-side has an eigenvalue going to 1. Consider a rung of a long ladder diagram. The intermediate propagator momenta on the $n$th rung from

FIG. 1. Diagrammatic representation of the large-$N$ propagator equation (13). The double line is the noncrossing propagator.

The zeroth-order diffusion constant $D_0$ is a factor of $\sqrt{2}$ smaller here than in the self-consistent Born approximation of [1] (the calculation there corresponds to the choice $c_1 = 1$). The resulting estimate of the critical conductivity at the transition, obtained from $D_0$ and the exact density of states [14] through the Einstein relation, is $\sigma_{xx} \approx 0.614 \pi^2$ compared to the numerical result $\sigma_{xx} = (0.54 \pm 0.04) \pi^2$ [3] and $\sigma_{xx} = \sqrt{\frac{\pi^2}{2}} \approx 0.433 \pi^2$ of [13]. Although the large-$N$ estimates of $\sigma_{xx}$ from the Liouvillian are of the right order, it should be noted that (unlike $\nu$) a universal $\sigma_{xx}$ has not yet been found from $\Pi(q, \omega)$ obtained using numerical diagonalization.

(a) \hspace{1cm} (b)

FIG. 2. Diagrams contributing to $1/N$ propagator correction. The solid interaction line in (b) indicates one or more noncrossing interaction lines. Each diagram of order $1/N$ has one fewer free index choice than the non-crossing diagrams of the same order.

The noncrossing propagator has simple behavior at large $q$, where the argument of the integrand in (2) is rapidly oscillatory. In this limit

$$\Pi_B(\infty, \omega) = \frac{1}{\omega} + \frac{1}{\pi \omega} \Pi_B(\infty, \omega)^2, \quad (13)$$

or $\Pi_B(\infty, \omega) \approx -i + \frac{\omega}{\pi}$ for small $\omega$.

The first corrections to the noncrossing propagator have $k - 1$ free choices of index for a diagram of $k$ lines. The corrections consist of all maximally crossed diagrams of 2 or more lines (using the noncrossing propagator) (Fig. 2a), and in addition possible “rainbows” over the maximally crossed portion (Fig. 2b). The sum of all maximally crossed diagrams can be obtained from the ladder sum represented in Fig. 2b since maximally crossed diagrams are related to ladder diagrams after cutting the center propagator line and pivoting.

The sum of all ladder diagrams with incoming momenta $q_1, q_2$ and momentum transfer 1 as labeled in Fig. 2b, denoted by $V(q_1, q_2; 1)$, satisfies the integral equation

$$V(q_1, q_2; 1) = V_0(q_1, q_2; 1) +$$

$$\int \frac{d\omega'}{\pi} (V_0(q_1 + l', q_2 - l'; 1 - l') \times$$

$$\Pi_B(q_1 + l', \omega) \Pi_B(q_2 - l', \omega) V(q_1, q_2; 1')) \quad (14)$$

where $V_0$ is the original interaction

$$V_0(q_1, q_2; 1) \equiv e^{-\frac{1}{2} |l|^2 \pi} \sin \left( \frac{q_1 \wedge 1}{2} \right) \sin \left( \frac{q_2 \wedge 1}{2} \right). \quad (15)$$

Note that $V$ has an implicit $\omega$ dependence. The content of (13), discussed below, is that $V$ has a diffusion pole when $q_2 \approx -q_1$:

$$V(q, q'; q - q) \propto \frac{1}{i \omega + D_1 (q + q')^2}. \quad (16)$$

FIG. 3. Schematic representation of sum of ladder diagrams. The solid block is defined as the sum of one or more rungs with the specified total momentum transfer.
an end of the ladder have magnitude proportional to $\sqrt{n}$ (momenta walk randomly in the plane) so most momenta in a long ladder can be assumed large. Each rung adds two propagators $\Pi_B(\omega_q,\omega) \approx -i + \omega/2$, and the interaction averages to $-1 + \frac{1}{4}(q + q')^2$. This gives the estimate $D_1 = \frac{1}{4}$ in [14].

The above derivation assumes white-noise disorder in the Girvian, although physical white noise disorder corresponds to disorder of correlation length $\sim \ell$ in $G$ [14]. Another assumption is that the lack of an upper momentum cutoff in the Liouvillian approach does not soften the singularity $\log \omega$ to a logarithm. The singularity is found numerically to persist without these assumptions, suggesting that the high-momentum degrees of freedom do not change the scaling behavior qualitatively, as in [14]. We call the limit with point disorder and finite cutoff the WL limit as these assumptions are standard in that context. Now we calculate $\Pi_{MC}(q,\omega)$ and obtain a singular log $\omega$ contribution to the diffusion constant.

The contribution of all maximally crossed diagrams to the propagator is

$$\Pi_{MC}(q,\omega) = \int dq' \Pi_B(q',\omega)(V - V_0)(q, q'; q' - q).$$

Here the subtraction of $V_0$ removes the first ladder diagram, which has no crossings and is already in $\Pi_B$. Integrating $q'$ in [16] leads to a log $\omega$ contribution in $\Pi_{MC}$.

The sum of maximally crossed diagrams [17] already shows nondiffusive scaling (an anomalous prefactor in $q$), which is modified quantitatively but not qualitatively by adding rainbows over the MC diagrams. There is a log-$\omega$ divergence in the sum of maximally crossed diagrams, leading to a diffusion constant

$$D(\omega) = D_0 + \frac{a \log \nu}{N} + O(N^{-2})$$

$$\approx D_0 \exp \left( \frac{a \log \nu}{ND_0} \right) = D_0 \nu^{a/ND_0},$$

using the standard device of re-exponentiation in large-$N$ and RG theories to estimate critical exponents. Hence

$$\frac{1}{2\nu} = \frac{a}{ND_0} + O(N^{-2}).$$

The coefficient $a$ is found from the numerical solution of [14] and [17]. Fig. 4 shows sample data from this calculation. Numerically $D_0 = 0.682$, $a = 0.18 \pm 0.02$, and therefore $\nu = (1.89 \pm 0.1)N$. In the WL limit the addition of overloops (Fig. [3]) is found analytically to reduce $a$ and increase $\nu$ by a factor of 2. Preliminary numerical results [13] are that the correction is in the same direction for the full model. Although this paper has focused on the LLL plateau transition, the large-$N$ Liouvillian approach may also be useful for other noninteracting quantum Hall transitions with a discrete spectrum of extended states.

![Fig. 4. Numerical results for maximally-crossed contribution to diffusion constant $D(\omega)$, on semilog scale. Here $\Sigma = \omega - 1/N$.](cond-mat/0009203)

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