Jointly designed quasi-cyclic LDPC-coded cooperation with diversity combining at receiver

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Abstract
This correspondence proposes a jointly-designed quasi-cyclic (QC) low-density parity-check (LDPC)-coded multi-relay cooperation with a destination node realized by multiple receive antennas. First, a deterministic approach is utilized to construct different classes of binary QC-LDPC codes with no length-4 cycles. Existing methods put some limitations in terms of code length and rate in order to provide high error-correction performance. Therefore, this article gives three classes of QC-LDPC codes based on a combinatoric design approach, known as cyclic difference packing (CDP), with flexibility in terms of code-length and rate selection. Second, the proposed CDP-based construction is utilized to jointly-design QC-LDPC codes for coded-relay cooperation. At the receiver, the destination node is realized by multiple receive antennas, where maximal-ratio combining (MRC) and sum-product algorithm (SPA)-based joint iterative decoding are utilized to decode the corrupted sequences coming from the source and relay nodes. Simulation results show that the proposed QC-LDPC coded-relay cooperations outperform their counterparts with a coding gain of about 0.25 dB at bit-error rate (BER) $10^{-6}$ over a Rayleigh fading channel in the presence of additive white Gaussian noise. Furthermore, the extrinsic-information transfer (EXIT) chart analysis has been used to detect the convergence threshold of proposed jointly-designed QC-LDPC codes. Numerical analysis shows that the proposed jointly-designed QC-LDPC codes provide a better convergence as compared to their counterparts under the same conditions.

Keywords
Quasi-cyclic LDPC, cyclic difference packing, jointly-designed QC-LDPC, coded-relay cooperation, maximal-ratio combining, joint iterative-decoding

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Introduction
It is globally known that the objective of next-generation wireless communication systems is to handle high-speed data rate and spectral efficiency by reducing power cost, lower latency, and better convergence, which may require a new broadband spectrum due to the limited frequency resources of the current wireless spectrum. As a result, several problems arise such as degraded coverage and fading, which may lead to prominent reduction in the power of received signal at receiver. Multiple-input multiple-output (MIMO) has...
been recognized as an effective approach to combat the effect of fading by offering diversity. However, for some practical scenarios in wireless communication (e.g. wireless sensor networks), it is not feasible to realize multiple antennas due to the hardware limitations of the device. To solve this critical problem, cooperative communication, also known as virtual MIMO, is determined where the devices with single antenna terminals can share their antennas to acquire multiplexing gain and diversity. Therefore, virtual MIMO has been adopted by many wireless communication systems (e.g. wireless sensor networks, ad hoc networks). Also, in-cell mobile nodes can share their antennas for data transmission by establishing a virtual MIMO network. Three fundamental protocols for cooperative communication, amplify-and-forward (AF), estimate-and-forward (EF), and decode-and-forward (DF) have been presented in the literature. In an AF cooperation, the relay node broadcasts only the amplified version of the signal received from the source node, where the strength of transmitted signals is controlled by the amplification or scaling coefficients at the relay node. In an EF approach, the signals received from the source node are first estimated by the relay nodes based on some hard-decision statistics, then these estimated signals are transmitted to the destination node. Generally, an AF cooperation protocol seems to be more attractive as compared to an EF approach because it does not require extra computational complexity for hard-decision detection in the relay node. However, a serious flaw of an AF cooperation is that it also amplifies the noise received from source-to-relay broadcast channel (S–R) and sends it to the destination node. On the other hand, both AF and EF cooperation protocols are not feasible for low bit-error rate (BER) applications.

Recently, many wireless communication systems have adopted low-density parity-check (LDPC) codes as a primary choice because of their excellent error correction performance and low-cost iterative decoding over different communication channels. In addition to higher error-correction capability, LDPC codes provide a flexible spectrum in terms of code length and rate selection. The null space of a parity-check matrix \( H \) gives a regular LDPC code if it has constant column-weight \( c_w \) and constant row-weight \( r_w \). If \( H \) has variable column and/or row weight, then its null space gives an irregular LDPC code. The null space of a parity-check matrix gives a QC-LDPC code if it consists of an array of circulant matrices over a finite field \( \text{GF}(\rho) \). If a parity-check matrix satisfies a constraint that no two rows or columns can overlap, in terms of a nonzero element, at more than one positions, then this constraint is known as row–column (RC) constraint which ensures that the parity-check matrix \( H \) has a girth of at least 6. The construction spectrum of LDPC codes is divided into two categories: (a) computer-based LDPC codes are designed based on random construction methods such as progressive edge growth LDPC (PEG-LDPC) codes and protograph-based LDPC codes; (b) structured LDPC codes (e.g. quasi-cyclic LDPC (QC-LDPC)) are constructed based on deterministic methods such as finite fields, finite geometries, and combinatorial designs. Quasi-cyclic or architecture-aware LDPC codes have been adopted by many communication standards because of their efficient architecture, which reduces computational cost of an encoder and decoder.

A technique called coded cooperation is used for high error performance applications; this is channel coding coupled with the conventional relay cooperation. The conventional AF and EF user cooperation protocols have been replaced by coded cooperation by employing forward error correction in the relay and destination node. In coded relay cooperative communication, each relay node instead of transmitting the whole data frame, only sends the redundant parity bits to the destination node. The performance of coded cooperation has been investigated based on turbo and LDPC codes. However, LDPC-coded cooperation provides more advantages over turbo-coded cooperation in terms of low-cost decoding and delay for hardware implementation of decoder. To the best of our knowledge, most of the previous studies have investigated LDPC-coded cooperation by utilizing random LDPC codes. However, the investigation on QC-LDPC-coded cooperation is rarely discussed. As compared to QC-LDPC-coded cooperation, random LDPC-coded cooperation provides a limited spectrum in terms of code length and rate with quadratic encoding complexity. However, QC-LDPC-coded cooperation provides more flexibility in terms of code length and rate selection with linear encoding complexity. In this correspondence, we propose a jointly designed QC-LDPC-based construction is utilized to jointly design QC-LDPC codes with no length-4 cycles. Second, the proposed CDP-based construction is utilized to jointly design QC-LDPC codes with no length-4 cycles for coded-relay cooperation. Also, the destination node is realized by multiple receive antennas, where maximal-ratio combining (MRC) and sum-product algorithm (SPA)-based joint iterative decoding are utilized to decode the corrupted sequences coming from the source and relay nodes in two different time frames. Based on the simulation results, the proposed QC-LDPC-coded cooperations outperform their counterparts by providing a coding gain of about 0.25 dB at BER 10^{-5} with five decoding iterations over a Rayleigh fading channel.
in the presence of additive white Gaussian noise. Furthermore, an analytical tool called exit-information transfer (EXIT) chart is used to detect the convergence threshold of proposed jointly designed QC-LDPC codes. Numerical analysis shows that the proposed jointly designed QC-LDPC codes provide a better convergence as compared to their counterparts under the same conditions.

The rest of this article is organized as follows. Basic concepts about the QC-LDPC coded-relay cooperation are given in the section “Coded-relay cooperation.” A brief discussion about the existence and construction of CDP is presented in the section “Cyclic difference packing.” The section “CDP-based construction of QC-LDPC codes” presents a CDP-based construction of QC-LDPC codes. Proposed construction of jointly designed QC-LDPC codes for coded-relay cooperation is given in the section “Jointly designed QC-LDPC coded cooperation.” The section “Maximal-ratio combining for multiple receive antennas” presents a diversity combining method based on maximal-ratio combining at receiver. Numerical results are presented in the section “Numerical results.” Finally, the conclusion of this article is presented in the section “Conclusion and remarks.”

Coded-relay cooperation

A fundamental model for one-relay coded cooperative communication system equipped with multiple receive antennas and consisting of three nodes such as source S, relay R, and destination D is depicted in Figure 1. All these nodes are supposed to have only one antenna and they communicate with each other over a half-duplex Rayleigh fading channel.

The information transmission from source to destination is divided into two consecutive time frames. In the first time frame, the source node (S) encodes the information data by first encoder, denoted as Encod-1, and sends to the relay node (R) and destination node (D) simultaneously over broadcast channels (S–R) and (S–D), respectively. In the second time frame, the decoder in the relay node, denoted as Decod-2, first decodes the data received from the source node. Then, the encoder in the relay node, denoted as Encod-2, encodes the estimated data and sends whole or a part of the coded symbols to the destination node over a broadcast channel (R–D). For ideal coded cooperative communication, it is assumed that the decoder in the relay node has successfully decoded the information sent from the source node.

In one-relay coded cooperation, two distinct QC-LDPC codes $C^{(1)}(N_1, M_1)$ and $C^{(2)}(N_2, M_2)$ defined by the null space of two parity-check matrices $H^{(1)}$ and $H^{(2)}$ were utilized to realize the source and relay node, respectively. The code $C^{(1)}$ in systematic form is $(I_1, P_1)$, where $P_1$ denotes the redundant parity data with length $N_1 - M_1$. The relay node first decodes the information received from the source node, then the Encod-2 encodes the estimated data by adding new redundant parity bits $P_2$ with length $N_2 - M_2$ using parity-check matrix $H^{(2)}$. Also, both coded sequences from Encod-1 and Encod-2 are correlated in source and relay nodes, respectively. Specifically, both coded sequences depend on their common information data bits. Both coded sequences are completely correlated if Decod-2 successfully decodes the message sent from source node. If there exist some errors in the decoded sequence from Decod-2, then both coded sequences are partially correlated.

In the destination node, multiple receive antennas are installed at the receiver to receive the corrupted signals coming from source and relay nodes in their respective time slots. The potential diversity gain of coded cooperation by utilizing multiple receive antennas in the destination is same as well as an ordinary MIMO system over a Rayleigh-fading channel in the presence of additive white Gaussian noise. Finally, a
joint iterative decoder uses the parity-check matrix $\mathbf{H}$, comprised of $\mathbf{H}^{(1)}$ and $\mathbf{H}^{(2)}$, to jointly decode the messages coming from the source and relay node. Note that the relay node sends only the redundant parity data to the destination node as it has already received the information bits from the source node.

It is important to note that a coded cooperative communication system behaves like an ordinary point-to-point communication system if the relay node does not process the information data received from the source node. Furthermore, a one-relay coded cooperation can be easily extended to a multi-relay coded cooperation with R-level ($R > 2$), as depicted in Figure 2. In multi-relay coded cooperation, each level contains some relay nodes where the relay nodes in higher levels can receive information only from relay nodes in lower levels. Specifically, the source node sends its original message to all the relay nodes and the destination node. However, the destination node receives information from the source and all the relay nodes.

**Cyclic difference packing**

**Fundamental concepts**

CDP designs are a special type of balanced incomplete block design (BIBD), so we begin with the definition of BIBD.

**Definition 1.** A pair $(T, F)$ is called a design, where $T$ denotes a set of varieties and $F$ denotes the nonempty subsets of $T$, called blocks. Suppose $\lambda$, $k$, and $\psi$ are positive integers such that $\lambda > k \geq 2$. A design $(T, F)$ is called $(\lambda, k, \psi)$-BIBD if all of the following properties hold:

1. $|T| = \lambda$.
2. Each nonempty subset (block) of $F$ have $k$ varieties.
3. Each pair of elements exists in exactly $\psi$ subsets of $F$.

**Definition 2.** Let $\lambda, k, \text{and } \psi$ be positive integers with $\lambda \geq k \geq 2$. A pair $(T, F)$ is called a $(\psi + 1) - (\lambda, k, \psi)$ packing design or briefly $\gamma - (\lambda, k, \psi)$-PD, where $T = \mathbb{Z}_k$ and $F$ denotes the nonempty $k$-subsets of $T$, called blocks, such that every $\gamma$-subset of distinct elements from $T$ appears in at most $\psi$ blocks.

Let $(T, F)$ be a $\gamma - (\lambda, k, \psi)$ packing design. Suppose $\delta$ is a permutation on $T$ such that for any block $A$, $F^\delta = \{A^\delta : A \in F\} = F$, then $\delta$ is called an automorphism of the $\gamma - (\lambda, k, \psi)$ packing design $(T, F)$. A $\gamma - (\lambda, k, \psi)$ packing design having cyclic automorphism is known as cyclic $\gamma - (\lambda, k, \psi)$ packing design, where cyclic automorphism is a bijection $\delta : j \rightarrow j + 1 \pmod{\lambda}$. Suppose $A = \{a_1, a_2, ..., a_k\}$ is a block of cyclic $\gamma - (\lambda, k, \psi)$ packing design $(Z_\lambda, F)$, then the block orbit consists of the following distinct blocks

$$A^\delta = A + j = \{a_1 + j, a_2 + j, ..., a_k + j\} \pmod{\lambda}$$

where $j \in \mathbb{Z}_\lambda$. A block orbit containing $\lambda$ distinct blocks is called full orbit; otherwise short orbit. Any fixed block from each block orbit is called a base block.

Let $W$ denote the set of all base blocks of a cyclic $\gamma - (\lambda, k, \psi)$ packing design. Then, the pair $(Z_\lambda, W)$ is called a cyclic $\gamma - (\lambda, k, \psi)$ difference packing or briefly a $\gamma - (\lambda, k, \psi)$-CDP. A $\gamma - (\lambda, k, \psi)$-CDP with base blocks is called maximum $\gamma - (\lambda, k, \psi)$-CDP. In the literature, it has already been shown that an optimal $(\lambda, k, \psi)$ Optical Orthogonal Code or briefly $(\lambda, k, \psi)$-OOC is equivalent to $(\psi + 1) - (\lambda, k, \psi)$ cyclic difference packing or briefly $\gamma - (\lambda, k, \psi)$-CDP

$$\begin{bmatrix}
1 & \lambda - 1 & \lambda - 2 & \cdots & \lambda - \gamma + 1 \\
\frac{1}{k} & \frac{1}{k} & \frac{2}{k} & \cdots & \frac{\gamma + 1}{k}
\end{bmatrix}$$

**Theorem 1.** An optimal $(\lambda, k, \psi)$-OOC is equivalent to a maximum $\gamma - (\lambda, k, \psi)$-CDP if and only if $\psi < k$.45
Definition 3. A family of binary codewords is called an \((\lambda, k, \psi)\)-OOC \(C\) if the following correlation properties hold:\(^{44}\)

1. Autocorrelation. 
\[ \sum_{0 \leq i < \lambda - 1} m_i m_{i+j} \leq \psi, \quad \text{for} \quad M = (m_0, m_1, \ldots, m_{\lambda-1}) \in C \quad \text{and} \quad i \neq 0 \pmod{\lambda}. \]

2. Cross-correlation. 
\[ \sum_{0 \leq i < \lambda - 1} m_i m_{i+j} \leq \psi, \quad \text{for} \quad M = (m_0, m_1, \ldots, m_{\lambda-1}) \in C, \quad N = (n_0, n_1, \ldots, n_{\lambda-1}) \in C \quad \text{with} \quad M \neq N, \quad \text{and for any integer} \ i. \]

where the subscripts are treated over \(\text{mod} \ \lambda\). An \((\lambda, k, \psi)\)-OOC is said to be optimal if it has 
\[
\begin{bmatrix}
1 & 0 & \cdots & 0 \\
\frac{\lambda - 1}{K} & \frac{\lambda - 1}{K} & \cdots & \frac{\lambda - 1}{K}
\end{bmatrix}
\]
codewords.

Existence of cyclic difference packing

The construction of a maximum \(\gamma - (\lambda, k, \psi)\)-CDP gives its corresponding optimal \((\lambda, k, \psi)\)-OOC. Some known results about the existence of a maximum \(\gamma - (\lambda, k, \psi)\)-CDP are given as follows.

Lemma 1. A maximum \(\gamma - (\lambda, k, \psi)\)-CDP exists if and only if:

1. \(\lambda = 24 \pmod{48}\).\(^{46}\)
2. \(\lambda = 6 \pmod{12}\).\(^{46}\)
3. \(\lambda = 0 \pmod{648}\).\(^{41}\)
4. A 36-regular \(2-(36d, 4, 1)\)-CDP exists if \(d = 1 \pmod{4}\) is a prime and \(d > 5\).\(^{37}\)
5. A 48-regular \(2-(48d, 4, 1)\)-CDP exists if \(d = 1 \pmod{4}\) is a prime and \(d > 5\).\(^{37}\)
6. A 60-regular \(2-(60f, 4, 1)\)-CDP exists for any positive integer \(f\) such that \(\gcd(f, 150) = 1\) or 25.\(^{37}\)
7. A \(r\)-regular \(2-(fd, 4, 1)\)-CDP exists for \(f \in \{6, 18, 24, 60, 72, 96, 108\}\) and \(d\) denotes a positive integer such that \(\gcd(d, 150) = 1\) or 25.\(^{37}\)
8. There exists an optimal \((\lambda, 5, 1)\)-CDP for \(\lambda = 25, 45, 75, 375\).\(^{38}\)
9. There exists an optimal \((3^n5p, 5, 1)\)-CDP for any nonnegative integer \(m\), where \(p\) denotes the product of prime factors congruent to 1 \((\text{mod} \ 4)\).\(^{38}\)
10. There exists an optimal 15-regular \((15d, 6, 1)\)-CDP, where \(d = 7 \pmod{12}\) and \(d \geq 19\).\(^{39}\)
11. There exists an optimal 20-regular \((20d, 6, 1)\)-CDP, where \(d = 7 \pmod{12}\) and \(d \geq 19\).\(^{39}\)

Next, we construct three classes of QC-LDPC codes by utilizing the maximum \(\gamma - (\lambda, k, \psi)\)-CDP’s with \(\psi = 1\).

CDP-based construction of QC-LDPC codes

A design theoretic construction of binary QC-LDPC codes based on the maximum \(\gamma - (\lambda, k, \psi)\)-CDP is provided. Consider a base matrix \(H^{(1)}\) given by equation (1)

\[
H^{(1)} = \begin{bmatrix}
Q_1 & Q_2 & \cdots & Q_\rho
\end{bmatrix}
\]

where each \(Q_i\) represents a \(k \times k\) circulant matrix, \(1 \leq i \leq \rho\). Each row of \(Q_i\), \(1 \leq i \leq \rho\), is the outcome of the cyclic-shift of the previous row, where the first row of \(Q_i\) is resulted from one of the \(k\) base blocks of maximum \(\gamma - (\lambda, k, \psi)\)-CDP with \(\psi = 1\). The base matrix \(H^{(1)}\) has a rate of at least \((\rho - 1)/\rho\) and a minimum distance lower bounded by \(k + 1\).

Example 1. Consider an optimal \((140, 6, 1)\)-CDP with \(k = 6\) for \(Z_{140}\). The base blocks for a \((140, 6, 1)\)-CDP are:

- \(B_1 = \{0, 16, 34, 59, 91, 119\}\),
- \(B_2 = \{0, 1, 3, 7, 12, 45\}\),
- \(B_3 = \{0, 8, 22, 35, 86, 101\}\),
- \(B_4 = \{0, 10, 40, 63, 92, 109\}\).

Consequently, a base matrix \(H\) is obtained from above construction method. To make the idea more clear, consider a submatrix \(P\) of base matrix \(H\) given as follows:

\[
H = \begin{bmatrix}
P & Q_1 & Q_2 & \cdots & Q_\rho
\end{bmatrix}
\]

\[
P = \begin{bmatrix}
\delta_1 & \gamma_1 \\
\delta_2 & \gamma_2
\end{bmatrix}
\]

\[
0 & 1 & 6 & 34 & 59 & 91 & 119 & 0 & 1 & 3 & 7 & 12 & 45 & 0 & 8 & 22 & 35 & 86 & 101 & 0 & 10 & 40 & 63 & 92 & 109 & 119 & 0 & 16 & 34 & 59 & 12 & 45 & 0 & 1 & 3 & 7 & 8 & 6 & 101 & 0 & 8 & 22 & 35 & 92 & 109 & 0 & 10 & 40 & 63 & 92 & 109
\end{bmatrix}
\]
where \( \delta_1, \delta_2 \in \mathbb{Q}_i \) and \( \gamma_1, \gamma_2 \in \mathbb{Q}_j \), \( 1 \leq i, j \leq \rho \). Note that the submatrix \( P \) has length-4 cycles if and only if \( \delta_1 - \delta_2 = \gamma_1 - \gamma_2 \mod (\lambda) \). Clearly, the above base matrix \( H \) does not satisfy the condition \( \delta_1 - \delta_2 = \gamma_1 - \gamma_2 \mod (\lambda) \). Consequently, the Tanner graph representation of the above base matrix \( H \) has no length-4 cycles.

**CDP-based QC-LDPC codes: class-I**

Let \( GF(\sigma) \) be a finite field with \( \sigma \) elements. For each nonzero element \( z' \), \( 0 \leq i < \sigma - 1 \), form a \((\sigma - 1)\)-tuple over \( GF(2) \), \( \phi_i(z') = (\phi_0, \phi_1, ..., \phi_{\sigma-2}) \), with all the components of \( \phi_0 \) are zero except the \( i \)th component \( \phi_i = 1 \). The subscript "b" stands for the binary. This \((\sigma - 1)\)-tuple is referred as the binary location-vector of \( z' \). The binary location vector of 0-element is \( \psi(b)(0) = (0, 0, 0, ..., 0) \).

Let \( \eta \) be an element of \( GF(\sigma) \). If \( \psi(b)(\eta) \) denotes the binary location vector of \( \eta \), then the right cyclic-shift of \( \psi(b)(\eta) \) gives the binary location vector \( \psi(b)(\eta^i) \) of element \( \eta^i \), where \( \zeta \) denotes the primitive element of \( GF(\sigma) \). Then the \((\sigma - 1)\)-tuples of \( \zeta, \zeta^2, \zeta^3, ..., \zeta^{\sigma-2}, \zeta \), give a \((\sigma - 1)\times(\sigma - 1)\) circular permutation matrix \( P(b)(\eta) \). Matrix \( P(b)(\eta) \) is referred as \((\sigma - 1)\)-fold binary dispersion of \( \eta \) over \( GF(2) \). The \((\sigma - 1)\)-fold matrix dispersion of additive identity of \( GF(\sigma) \) is a \((\sigma - 1)\times(\sigma - 1)\) all-zero matrix over \( GF(2) \).

Next, all the entries of base matrix \( H^{(1)} \) given by equation (1) are replaced by their \((\sigma - 1)\)-fold matrix dispersions \( P(b) \) over \( GF(2) \). We obtain an \( k \times k\rho \) array \( H^{(1)} \) given as follows

\[
H^{(1)} = \begin{bmatrix}
P_{0,0} & P_{0,1} & \cdots & P_{0,k-1} \\
P_{1,0} & P_{1,1} & \cdots & P_{1,k-1} \\
\vdots & \vdots & \ddots & \vdots \\
P_{k-1,0} & P_{k-1,1} & \cdots & P_{k-1,k-1}
\end{bmatrix}
\]

(2)

where \( P_{i,j} \) is a \((\sigma - 1)\times(\sigma - 1)\) circular permutation matrix over \( GF(2) \), for \( 0 \leq i < k \) and \( 0 \leq j < k\rho \). Array \( H^{(1)} \) gives a \( k(\sigma - 1)\times k\rho(\sigma - 1) \) matrix over \( GF(2) \). Clearly, matrix \( H^{(1)} \) fulfills the RC-constraint. Consequently, the null space of \( H^{(1)} \) gives a length-4 cycle-free QC-LDPC code with a rate lower bounded by \((\rho - 1)/\rho\).

**CDP-based QC-LDPC codes: class-II**

A class of binary length-4 cycle-free QC-LDPC codes is constructed based on the incidence matrices obtained from maximum \( \gamma - (\lambda, k, \psi) \)-CDP with \( \psi = 1 \). A design \((T, F)\) with \( b \) blocks, \( A_1, A_2, ..., A_b \), satisfying the following properties is called a \((\lambda, b, r, k, \psi)\)-BIBD: (a) each entry in \( T \) participates in \( r \) blocks; (b) each pair of the entries in \( T \) participates in exactly \( \psi \) blocks of \( F \); and (c) the size of each block \( k \) is smaller than \( \lambda \). A \((\lambda, b, r, k, \psi)\)-BIBD can also be represented by a \( \lambda \times b \) matrix \( L = (l_{ij}) \) over \( GF(2) \)

\[
l_{ij} = \begin{cases}
1 & \text{if } x_i \in A_j \\
0 & \text{if } x_i \not\in A_j
\end{cases}
\]

(3)

where matrix \( L \) is known as incidence matrix. An incidence matrix must satisfy the following properties: (a) the column-weight \( c_\psi \) of an incidence matrix \( L \) is equal to \( k \); (b) the row-weight \( r_\psi \) of an incidence matrix is equal to \( r \); and (c) any two rows or columns of \( L \) have 1-element in common at most \( \psi \) positions.

**Example 2.** Let \((T, F)\) be the following \((7, 7, 3, 1)\)-BIBD:

\[
T = \{0, 1, 2, 3, 4, 5, 6\}, \text{ and } F = \{\{235\}, \{346\}, \{450\}, \{561\}, \{602\}, \{013\}, \{124\}\}.
\]

The incidence matrix of this BIBD is given as follows

\[
L = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}
\]

The cyclic shift of each row returns a next row of \( L \) and the first row is obtained by the cyclic shift of last row. Also, the downward cyclic shift of each column of \( L \) gives a column on its right. So, the matrix \( L \) is a circulant permutation matrix over \( GF(2) \). Note that the circulant permutation matrix \( L \) fulfills all of the required properties of parity-check matrix and satisfies the RC-constraint. Therefore, the null space of \( L \) gives a LDPC code with a girth of at least 6.

Based on a maximum \( \gamma - (\lambda, k, \psi) \)-CDP with \( \psi = 1 \), consider a \( \lambda \times \rho \lambda \) incidence matrix \( H^{(2)} \) obtained from \( \rho \) base blocks. The incidence matrix \( H^{(2)} \) can be arranged in a cyclic manner consisting of a \( 1 \times \rho \) array of \( \lambda \times \lambda \) circulant submatrices given as follows

\[
H^{(2)} = [L_1 \ L_2 \ \cdots \ L_\rho]
\]

(4)

where each \( L_i \), \( 1 \leq i \leq \rho \), represents a \( \lambda \times \lambda \) circulant submatrix over \( GF(2) \). Note that the matrix \( H^{(2)} \) fulfills all of the desired properties of a parity-check matrix. Consequently, the null space of matrix \( H^{(2)} \) gives a QC-LDPC code with no length-4 cycles and a rate of at least \((\rho - 1)/\rho\).

**CDP-based QC-LDPC codes: class-III**

In the section “CDP-based QC-LDPC codes: class-II,” a class of binary QC-LDPC codes has been constructed
The first equation (4), where each circulant $L_i$ represents a $\lambda \times \lambda$ circulant submatrix obtained from $\gamma = (\lambda, k, \psi)$-CDP with $\psi = 1$ and $k = 6$, for $1 \leq i \leq \rho$.

Let $l_i$ be the first row of circulant matrix $L_i$ over $GF(2)$. Decompose $l_i$ into six rows, $l_{1,i}, l_{2,i}, ..., w_{6,i}$, by distributing six 1-elements of $l_i$ among the six new rows. The first 1-element of $l_i$ is placed in $l_{1,i}$, the second 1-element of $l_i$ is placed in $l_{2,i}, ...,$, and the sixth 1-element of $l_i$ is placed in $l_{6,i}$. For each new row $l_{i,j}$, form a $\lambda \times \lambda$ circulant matrix $L_{ij}$ by using $l_{i,j}$ and its $\lambda - 1$ right cyclic shifts. This decomposition and cyclic shifting of each circulant matrix gives a new class of binary QC-LDPC codes. Consider a matrix $H(3)$ given by equation (5), where each $L_{ij}$ represents a $\lambda \times \lambda$ circulant submatrix obtained from $\gamma = (\lambda, k, \psi)$-CDP with $\psi = 1$ and $k = 6$, for $1 \leq i \leq \rho$.

Let $l_i$ be the first row of circulant matrix $L_i$ over $GF(2)$. Decompose $l_i$ into six rows, $l_{1,i}, l_{2,i}, ..., w_{6,i}$, by distributing six 1-elements of $l_i$ among the six new rows. The first 1-element of $l_i$ is placed in $l_{1,i}$, the second 1-element of $l_i$ is placed in $l_{2,i}, ...,$, and the sixth 1-element of $l_i$ is placed in $l_{6,i}$. For each new row $l_{i,j}$, form a $\lambda \times \lambda$ circulant matrix $L_{ij}$ by using $l_{i,j}$ and its $\lambda - 1$ right cyclic shifts. This decomposition and cyclic shifting of each circulant matrix gives a new class of binary QC-LDPC codes. Consider a matrix $H(3)$ given by equation (5), where each $L_{ij}$ represents a $\lambda \times \lambda$ circulant submatrix obtained from $\gamma = (\lambda, k, \psi)$-CDP with $\psi = 1$ and $k = 6$, for $1 \leq i \leq \rho$.

$H(3)$ is a $\mu \lambda \times \lambda \rho$ matrix over $GF(2)$. $H(3)$ is constructed based on a maximum $\gamma = (\lambda, k, \psi)$-CDP with $\psi = 1$ and fulfills all of the desired properties of a parity-check matrix. Consequently, the null space of $H(3)$ gives a QC-LDPC code with no length-4 cycles and rate lower bounded by $(\rho - \mu)/\rho$.

**Jointly designed QC-LDPC coded cooperation**

**One-relay coded cooperation**

This section provides a joint design of QC-LDPC codes for one-relay coded cooperation. Jointly designed QC-LDPC codes are constructed based on the proposed three classes of binary length-4 cycles free QC-LDPC codes in the sections “CDP-based QC-LDPC codes: class-I,” “CDP-based QC-LDPC codes: class-II,” and “CDP-based QC-LDPC codes: class-III,” respectively. Jointly designed QC-LDPC codes are jointly decoded by a joint iterative decoder in the destination over a Rayleigh fading channel in the presence of additive white Gaussian noise.

Suppose the source and relay nodes of a one-relay coded cooperation are realized by two distinct QC-LDPC codes defined by the null space of two parity-check matrices $H^{(1)}_{m1,\lambda \times m1,\lambda}$ and $H^{(2)}_{m2,\lambda \times (n + m2,\lambda)}$, respectively. Both parity-check matrices $H^{(1)}_{m1,\lambda \times m1,\lambda}$ and $H^{(2)}_{m2,\lambda \times (n + m2,\lambda)}$ are designed based on the proposed CDP-based construction of QC-LDPC codes given by equations (2), (4), and (6). QC-LDPC codes defined by the null space of $H^{(1)}_{m1,\lambda \times m1,\lambda}$ and $H^{(2)}_{m2,\lambda \times (n + m2,\lambda)}$ are regular and denoted as $C^{(1)}(N, M_1, r^{(1)}_w, \epsilon^{(1)}_w)$ and $C^{(2)}(N + M_2, M_2, r^{(2)}_w, \epsilon^{(2)}_w)$, where $r^{(1)}_w, \epsilon^{(1)}_w$ and $r^{(2)}_w, \epsilon^{(2)}_w$ denote the number of 1’s in each row and column of $H^{(1)}_{m1,\lambda \times m1,\lambda}$ and $H^{(2)}_{m2,\lambda \times (n + m2,\lambda)}$, respectively.

In one-relay coded cooperation, joint iterative decoding can be applied for double regular QC-LDPC codes $C^{(1)}(N, M_1, r^{(1)}_w, \epsilon^{(1)}_w)$ and $C^{(2)}(N + M_2, M_2, r^{(2)}_w, \epsilon^{(2)}_w)$ to decode the corrupted sequences form the source and relay nodes over the broadcast channels (S–D) and (R–D), respectively. A general joint triple-layer Tanner graph utilized for joint iterative decoding for one-relay coded cooperation is given in Figure 3. In one-relay coded cooperation, the overall code rate from the destination is $R_{\text{result}} = R_1 R_2$, where $R_1$ and $R_2$ denotes the code rates of $C^{(1)}$ and $C^{(2)}$, respectively. In particular, the resultant parity-check matrix $H(M_1 + M_2, \lambda \times (N + M_2, \lambda)$ for one-relay cooperation is given as follows
are the parity-check matrices of $C_M$ and $C_{M_1}$. Figure 5. A Joint Tanner graph used to define two-relay coded cooperation over Rayleigh fading channel.

The idea for a one-relay coded cooperation can be easily extended to a multi-relay coded cooperation. A generalized $R$-level ($R \geq 2$) multi-relay cooperation is shown in Figure 2. A multi-relay coded cooperation using multi-QC-LDPC codes for the source and relay nodes and joint iterative decoding in the destination is a straightforward extension of aforementioned one-relay coded cooperation over a Rayleigh-fading channel in the presence of additive-white Gaussian noise. In multi-relay coded cooperation, each level has some relay nodes and only the relays in higher levels receive information from the relay nodes in lower levels. However, the destination node receives the incoming messages from the source and all the relay nodes.

To make the description more simple, we focus on a two-level ($R = 2$) multi-relay coded cooperation where each level contains only one relay, as depicted in Figure 4. In this two-level multi-relay coded cooperation, suppose the source node, and the first and second relay nodes are realized by three distinct QC-LDPC codes $C^{(1)}(N, M_1, r_w^{(1)}, c_w^{(1)}), C^{(2)}(N + M_2, M_2, r_w^{(2)}, c_w^{(2)}), C^{(3)}(N + M_2 + M_3, M_3, r_w^{(3)}, c_w^{(3)})$ defined by the null space of three parity-check matrices $H_{M_1 \times N}^{(1)}$, $H_{M_2 \times N}^{(2)}$, and $H_{M_3 \times N}^{(3)}$, respectively.

In two-level multi-relay coded cooperation, joint iterative decoding can be applied for triple regular QC-LDPC codes $C^{(1)}(N, M_1, r_w^{(1)}, c_w^{(1)}), C^{(2)}(N + M_2, M_2, r_w^{(2)}, c_w^{(2)}), C^{(3)}(N + M_2 + M_3, M_3, r_w^{(3)}, c_w^{(3)})$ to decode the corrupted sequences coming from the source node, and the first and second relay nodes over Rayleigh-fading channel in the presence of additive-white Gaussian noise. A joint Tanner graph for two-level multi-relay cooperation utilized for joint iterative decoding in the destination is depicted in Figure 5. The overall code rate from the destination is $R_{\text{result}} = R_1 R_2 R_3$, where $R_1$, $R_2$, and $R_3$ denotes the code rates of source, first relay, and second relay, respectively. The resultant parity-check matrix $H_{(M_1 + M_2 + M_3) \times (N + M_2 + M_3)}$ is given as follows

\[
H_{(M_1 + M_2 + M_3) \times (N + M_2 + M_3)} = \begin{align*}
H_{M_1 \times N}^{(1)} & \times 0_{M_2 \times M_2} \\
0_{M_1 \times M_2} & \times H_{M_2 \times M_2}^{(2)} \\
0_{(M_1 + M_2) \times M_3} & \times H_{M_3 \times M_3}^{(3)}
\end{align*}
\]

Figure 4. A two-level multi-relay coded cooperation with one relay in each level.

Figure 5. A Joint Tanner graph used to define two-relay coded cooperation over Rayleigh fading channel.
where $H_{M_1 \times N}^{(1)}$, $H_{M_2 \times (N + M_2)}^{(2)}$ are the parity-check matrices of $M_1 N$ modulation over a Rayleigh fading channel, let the relay node to that received from the source node.

SPA-based joint iterative decoding

At the receiver, the output data obtained from multiplexer can be expressed as $W = (w_1, ..., w_{\lambda}, w_{\lambda+1}, ..., w_{\lambda+n+2\lambda})$. Assuming binary-phase-shift-keying (BPSK) modulation over a Rayleigh fading channel, let $f_N = Re(w_N)/(N = 1, ..., (n + m_2)\lambda)$ are processed as the inputs messages for joint-iterative decoder:

1. Initialization.

The log-likelihood ratio (LLR) can be computed as

$$
\alpha_N = \begin{cases} 
-4f/N_0, & 1 \leq N \leq n\lambda \\
-4f/(dN_0), & n\lambda + 1 \leq N \leq (n + m_2)\lambda 
\end{cases}
$$

where $d$ denotes the power gain of signal received from the relay node to that received from the source node.

Let the a-posteriori LLR for $f_N$ is computed as

$$
Y_{M,N}^{(i)} = \ln \frac{Pr(y_N = 0)}{Pr(y_N = 1)} | \frac{C(v_N) \cap c_M^{(i)}}{C(v_N) \cap c_M^{(i)}} = 1 
$$

Note that if all of the check-nodes $C(v_N)$ (except the $i$th check-node $c_M^{(i)}$ connected to a variable-node $v_N$ are satisfied simultaneously, then $\xi(C(v_N) \cap c_M^{(i)}) = 1$.

2. Check-node update.

The extrinsic information sent from check-node $c_M^{(i)}$ (first or third layer of joint Tanner graph) to a variable-node $v_N$ can be computed as

$$
\Phi_{M,N}^{(i)} = \left( \prod_{v \in V(c_M^{(i)}) \cap v_N} \text{sign}(Y_{M,N}^{(i)}) \right) \times 2\tanh^{-1} \left( \prod_{v \in V(c_M^{(i)}) \cap v_N} \tanh \left( \frac{Y_{M,N}^{(i)}}{2} \right) \right)
$$

where $V(c_M^{(i)}) \cap v_N$ represents the set of variable-nodes connected to the check-node $c_M^{(i)}$ excluding $v_N$.

3. Variable-node update.

Extrinsic information sent from a variable-node $v_N$ (participating in the first layer of joint Tanner graph) to a check-node $c_M^{(i)}$ can be computed as

$$
\Psi_{M,N}^{(i)} = \alpha_N + \sum_{c_M^{(i)} \in C(v_N)} \Phi_{M,N}^{(i)} \times (N = 1, ..., (n + m_2)\lambda)
$$

Similarly, extrinsic information updated by a variable-node $v_N$ for a check-node $c_M^{(i)}$ (participating in the third layer of joint Tanner graph) can be computed as

$$
\Psi_{M,N}^{(i)} = \alpha_N + \sum_{c_M^{(i)} \in C(v_N)} \Phi_{M,N}^{(i)} \times (N = 1, ..., (n + m_2)\lambda)
$$

4. Final LLR.

Repeat horizontal and vertical processing in step (2) and step (3) for a maximum number of iterations of decoder. Consequently, the final a-posteriori LLR can be computed as

$$
\Psi_N = \alpha_N + \sum_{i=1}^{2} \sum_{c_M^{(i)} \in C(v_N)} \Phi_{M,N}^{(i)} \times (N = 1, ..., (n + m_2)\lambda)
$$

Finally, if $\Psi_N \approx 0$, the decoded bit $\hat{y}_N = 0$, otherwise $\hat{y}_N = 1$.

Maximal-ratio combining for multiple receive antennas

To achieve receive diversity for the proposed coded-relay cooperative communication, the destination node is realized by multiple receive antennas to combat the effect of fading and noise resulted from constituent Rayleigh fading channels (S–D) and (R–D) for the source and relay nodes, respectively. The source and relay nodes are assumed to have a single antenna to broadcast their coded sequences to the destination node in a unique time frame over a half-duplex Rayleigh fading channel in the presence of additive-white Gaussian noise.

For a single-input-multiple-output (SIMO) detection in the destination node, a special case of matrix-matched filter known as MRC is utilized to filter the signals from multiple receive antennas. The MRC is an
efficient diversity combining method to enhance the signal power by filtering the outputs from multiple received antennas. In the proposed coded-relay cooperation, the diversity combining process consisting of two MRCs, denoted as MRC-1 and MRC-2, is given in Figure 6. Note that MRC-1 and MRC-2 handle the corrupted sequences coming from the source and relay nodes in two different time slots. Let \( \tilde{y}_k^{(i)} = [y_{k,1}^{(i)}, y_{k,2}^{(i)}, \ldots, y_{k,L}^{(i)}]^T \) be the \( k \)th signal received at \( L \) antennas, for \( i = 1 \) or 2 to denote the signals coming from the source and relay nodes, respectively. Suppose all the coded sequences, in one-relay coded cooperation, transmitted by the source and relay node suffer from channel fading and additive-white Gaussian noise simultaneously. Since both MRCs perform the same filtering processes for the incoming signals from the source and relay nodes, the superscript of \( \tilde{y}_k^{(i)} \) is omitted in the sequel. Thus, the vector system model for the \( k \)th signal received at \( L \) antennas becomes

\[
[ y_{k,1} \\
 y_{k,2} \\
 \vdots \\
 y_{k,L} ] = [ \beta_{k,1} \\
 \beta_{k,2} \\
 \vdots \\
 \beta_{k,L} ] x + [ n_{k,1} \\
 n_{k,2} \\
 \vdots \\
 n_{k,L} ]
\]  

(12)

Above system model in equation (12) can be represented by the following equation

\[
\tilde{y}_k = \tilde{\beta}_k x + \tilde{n}_k
\]  

(13)

where \( x \) denotes the transmitted codeword sent by the source or relay node over constituent broadcast channels in the presence of additive-white Gaussian noise. \( \beta_k = [\beta_{k,1}, \beta_{k,2}, \ldots, \beta_{k,L}]^T \) and \( \tilde{n}_k = [n_{k,1}, n_{k,2}, \ldots, n_{k,L}]^T \) represent two complex column vectors for additive-white Gaussian noise and channel fading, respectively, where \( n_{k,j} (j = 1, 2, \ldots, L) \) and \( \beta_{k,j} (j = 1, 2, \ldots, L) \) are Gaussian random variables with zero mean and variance \( E[|n_{k,j}|^2] = \sigma^2 = N_0 \) and \( E[|\beta_{k,j}|^2] = 1 \), respectively. Since \( n_{k,j} (j = 1, 2, \ldots, L) \) are Gaussian in nature, we assume that the noise on any pair of antennas is uncorrelated, that is, \( E[n_{k,i} n_{k,j}] = 0, \forall i \neq j \).

In the proposed coded-relay cooperation, the combining of \( y_{k,j} (j = 1, 2, \ldots, L) \) at \( L \) receive antennas in the destination node is given as follows

\[
\tilde{y}_k = w_k^s y_{k,1} + w_k^r y_{k,2} + \ldots + w_k^L y_{k,L}
\]  

(14)

where \( w_k^j (j = 1, 2, \ldots, L) \) denote the complex combining weights at the receiver. The above equation (14) can be simplified as

\[
\tilde{y}_k = w_k^H \tilde{y}_k = w_k^H \tilde{\beta}_k x + w_k^H \tilde{n}_k
\]  

(15)

Next, the signal-to-noise ratio (SNR) at receiver can be computed as

\[
SNR = \frac{P \times \|w_k^H\|^2 \|\tilde{\beta}_k\|^2 \cos^2 \theta}{\sigma^2 \|w_k^H\|^2}
\]  

(16)

where \( P \) denotes the power of transmitted signal sent by the source or relay node. Both the numerator and the denominator in equation (16) can be further simplified. Thus, the SNR at the receiver becomes

\[
SNR = \frac{P \times \|w_k^H\|^2 \|\tilde{\beta}_k\|^2 \cos^2 \theta}{\sigma^2 \|w_k^H\|^2}
\]  

(17)

It is important to note that SNR is maximum when \( \cos^2 \theta \) is maximum, where \( \theta \) is an angle between \( w_k^H \) and \( \tilde{\beta}_k \). Since the maximum value of \( \cos^2 \theta \) occurs at \( \theta = 0^\circ \), for SNR to be maximized, \( w_k^H \) and \( \tilde{\beta}_k \) have to be pointing in the same direction.

\[
SNR = \frac{P \times \|\tilde{\beta}_k\|^2}{\sigma^2}
\]  

(18)

To maximize the SNR in the destination node, the vector \( w_k^H \) has to be proportional to the vector \( \tilde{\beta}_k \). Specifically, the vector \( w_k^H \) can be set such that norm of the vector \( w_k^H \) is unity.
where the vector $\mathbf{w}_k^H$ is called maximal-ratio combiner.

Finally, the outputs $r(i)k(i)$ from MRC-1 and MRC-2, as depicted in Figure 6, are multiplexed and sent to an iterative decoder to jointly decode the corrupted sequences coming from the source and relay nodes in their respective time frames.

### Numerical results

This section investigates the error correction performance of proposed jointly designed QC-LDPC coded-relay cooperation with multiple receive antennas in the destination node, where an efficient diversity combining method known as MRC is utilized to filter the outputs from multiple receive antennas. Simulation results are obtained by SPA-based joint iterative decoding with BPSK transmission over Rayleigh fading channel in the presence of additive-white Gaussian noise.

Performance comparison of proposed jointly designed QC-LDPC-coded cooperations and their competitors with different decoding iterations and a fixed number of receive antennas in the destination

This section investigates the error-correction performance comparison of a proposed jointly designed QC-LDPC code $C_{qc}^{(1)}$ for one-relay coded cooperation and their competitors under two and five decoding iterations and four receive antennas in the destination node. The BER performance of proposed jointly designed QC-LDPC-coded cooperation and their counterparts using SPA-based joint iterative decoding and BPSK transmission over a Rayleigh fading channel is given in Figure 7. Simulation results show that the proposed jointly designed QC-LDPC-coded cooperations perform as well as their competitors in lower SNR region but outperform in higher SNR region under the same conditions over a Rayleigh fading channel in the presence of additive-white Gaussian noise. The relevant parameters for the component QC-LDPC codes adopted for jointly designed QC-LDPC code $C_{qc}^{(1)}$ and their counterparts are given in Table 1, where the overall code rate is $1/2$ from the destination.

| Coding type       | Code length (S) | Code length (R) | Rate (S) | Rate (R) | Rate ($R_{result}$) |
|-------------------|-----------------|-----------------|----------|----------|---------------------|
| $C_{qc}^{(1)}$    | 3000            | 3750            | 5/8      | 4/5      | 1/2                 |
| Coded Coop.35     | 3000            | 3750            | 10/16    | 8/10     | 1/2                 |
| Coded Coop.36     | 3000            | 3750            | 5/8      | 8/10     | 1/2                 |
| Coded Coop.30     | 3000            | 3750            | 5/8      | 4/5      | 1/2                 |

**Figure 7.** BER performance comparison of proposed jointly designed QC-LDPC-coded one-relay cooperation and its competitors over a Rayleigh fading channel under two and five decoding iterations and four receive antennas in the destination.

**Figure 8.** BER performance of proposed jointly designed QC-LDPC-coded one-relay cooperation under two, three, four, and five decoding iterations and four receive antennas in the destination.
Performance of jointly designed QC-LDPC-coded one-relay cooperation with different decoding iterations and a fixed number of receive antennas in the destination

The error-correction performance of a proposed jointly designed QC-LDPC code $C^{(2)}_{qc}$ investigated with four receive antennas in the destination node and two, three, four, and five decoding iterations is shown in Figure 8. It is important to note that the proposed jointly designed QC-LDPC coded cooperation provides excellent BER performance with SPA-based joint iterative decoding for various decoding iterations over Rayleigh fading channel in the presence of additive-white Gaussian noise. For instance, the third decoding iteration gives about 0.75 dB gain over the second iteration at BER $10^{-3}$. Similarly, at BER $10^{-4}$, about 0.7 dB gain is achieved for the fourth decoding iteration over the third decoding iteration, and it is about 0.6 dB for the fifth decoding iteration over the fourth decoding iteration at BER $10^{-5}$. Simulation results show that the proposed jointly designed QC-LDPC-coded cooperation provides an excellent error correcting performance over a Rayleigh fading channel. The relevant parameters for component QC-LDPC codes adopted for jointly designed QC-LDPC codes $C^{(2)}_{qc}$ are given in Table 2. Note that the overall code rate, $R_{\text{result}}$, is $9/14$ from the destination.

| Coding type $C^{(2)}_{qc}$ | Code length (S) | Code length (R) | Rate (S) | Rate (R) | Rate ($R_{\text{result}}$) |
|---------------------------|-----------------|-----------------|----------|----------|---------------------------|
| $C^{(3)}_{qc}$            | 1536            | 1792            | $3/4$    | $6/7$    | $9/14$                    |
| $C^{(4)}_{qc}$            | 2304            | 2560            | $5/6$    | $9/10$   | $3/4$                     |
| $C^{(5)}_{qc}$            | 1350            | 1440            | $14/15$  | $15/16$  | $7/8$                     |

Performance of jointly designed QC-LDPC-coded one-relay cooperation with different number of receive antennas and a fixed number of decoding iterations

This section investigates the error-correction performance of a proposed jointly designed QC-LDPC code $C^{(3)}_{qc}$ with four decoding iterations and one, two, three, and four receive antennas ($L$) in the destination node. The BER performance of proposed coded one-relay cooperation using SPA-based joint iterative decoding and BPSK transmission over a Rayleigh fading channel in the presence of additive-white Gaussian noise is given in Figure 9. Simulation results show that the proposed jointly designed QC-LDPC-coded cooperation provides an excellent BER performance for different number of receive antennas at the receiver. For instance, at BER $10^{-2}$, the proposed jointly designed coded-relay cooperation with three receive antennas provides a gain of about 1.5 dB over two receive antennas in the destination. Similarly, at BER $10^{-3}$, about 1.6 dB gain is
achieved with four receive antennas over three receive antennas in the destination. The relevant parameters for the component QC-LDPC codes adopted for jointly designed QC-LDPC code $C_{qc}^{(3)}$ are also given in Table 2, where the overall code rate is $3/4$ from the destination.

**Performance of proposed jointly designed QC-LDPC-coded cooperation and non-coded cooperation with different decoding iterations and a fixed number of receive antennas in the destination**

Figure 10 compares the error-correction performance of a proposed jointly designed coded cooperation and non-coded cooperation with two and five decoding iterations and four receive antennas in the destination over Rayleigh fading channel. The BER results of proposed coded and non-coded cooperation based on SPA-based joint iterative decoding over a Rayleigh fading channel are given in Figure 10. The relevant parameters for the component QC-LDPC code $C_{qc}^{(3)}$ adopted for proposed coded cooperation are given in Table 2, where the overall code rate is $3/4$ from the destination. Simulation results show that the proposed coded cooperation outperforms the non-coded cooperation by providing a prominent coding gain. For instance, at BER $10^{-6}$, the proposed coded cooperation provides a gain of about 1.5 and 2 dB with two and five decoding iterations, respectively.

**Performance of proposed jointly designed QC-LDPC-coded multi-relay and one-relay cooperation with different decoding iterations and a fixed number of receive antennas in the destination**

Figure 11 investigates the BER performance of proposed jointly designed QC-LDPC codes $C_{qc}^{(5)}$ and $C_{qc}^{(6)}$ for a one-relay and two-relay coded cooperation, respectively, under two and five decoding iterations and four receive antennas in the destination node. The overall code rate for one-relay and two-relay coded cooperation is $2/3$ from the destination node, where the relevant parameters for component QC-LDPC codes designed for one-relay and two-relay coded cooperation are given in Table 3. Simulation results show that the error-correcting performance of two-relay coded cooperation outperforms the one-relay coded cooperation with SPA-based joint iterative over a Rayleigh fading channel in the presence of additive white Gaussian noise. This indicates that higher error-correction performance can be achieved by using multi-relay coded cooperation under the same conditions over a Rayleigh fading channel.

**EXIT chart analysis of proposed jointly designed QC-LDPC codes and their competitors under same conditions**

EXIT chart analysis is an analytical tool to determine the convergence threshold of an iterative decoder. In EXIT chart analysis of LDPC codes, the set variable-nodes is referred as a variable-node decoder (VND) and the set check-nodes is referred as a check-node decoder (CND). The VND converts channel and a priori LLR from CND to a posteriori LLR. Similarly, CND converts a priori LLR from VND to a posteriori LLR at its output. Note that the a posteriori information from VND becomes the a priori information for CND. Let $I_E$ denote the mutual information at the output of VND and $I_r$ denote the mutual information at the output of CND, where the underlying assumption is that the mutual information at the output of VND and CND is a function of mutual information at their input. We refer to the literature for detail information about EXIT chart analysis of an iterative decoder. To predict the decoding threshold, the mutual information at the output of decoder (i.e. VND or CND) is plotted.

---

**Table 3. Relevant parameters for the component QC-LDPC codes used in Figure 11.**

| Coding type | Code length (S) | Code length ($R_1$) | Code length ($R_2$) | Rate (S) | Rate ($R_1$) | Rate ($R_2$) | Rate ($R_{result}$) |
|-------------|-----------------|---------------------|---------------------|---------|-------------|-------------|-------------------|
| $C_{qc}^{(5)}$ | 1470            | 1890                | None                | 6/7     | 7/9         | None        | 3/4               |
| $C_{qc}^{(6)}$ | 1470            | 1680                | 1890                | 6/7     | 7/8         | 8/9         | 2/3               |

**Figure 11.** BER performance of proposed jointly designed one-relay and two-relay coded cooperation under two and five decoding iterations and four receive antennas in the destination over Rayleigh fading channel.
as a function of mutual information at its input. Specifically, error-free transmission is possible if the mutual information at the output of VND and CND converges to one. The EXIT chart is plotted between a priori mutual information $I_A$ before decoding and extrinsic mutual information $I_E$ after decoding. Since the VND directly depends on the channel, $I_A$ in the first iteration is treated as the mutual information of the channel. However, the CND does not depend on the channel like VND. Consequently, the EXIT curves of VND get higher (i.e. close to (1, 1)) for better SNR. However, the EXIT curves of CND remain constant. The wider tunnel between the EXIT curves of both decoders (VND and CND) corresponds to better decoding. It is important to note that if two curves intersect each other at any other point except (1, 1), then no successful decoding can be achieved.

In this section, the EXIT chart analysis has been utilized to predict the convergence threshold of proposed jointly designed QC-LDPC code $C^{(1)}_{qc}$ and its competitors under same code rate $1/2$. The EXIT charts of $C^{(1)}_{qc}$ and its counterparts are depicted in Figures 12–15, respectively. Based on the numerical analysis, it can be seen that the minimum $E_b/N_0$ required for the convergence of $C^{(1)}_{qc}$ is only 0.6 dB. However, the minimum $E_b/N_0$ required for the convergence of other competitors is about 1.6 dB. Consequently, the proposed jointly designed QC-LDPC codes provide a better convergence.

**Figure 12.** Convergence threshold for proposed jointly designed QC-LDPC $C^{(1)}_{qc}$, Rate $= 1/2$, $E_b/N_0 = [-0.4, 0.6, 1.6]$ dB.

**Figure 13.** Convergence threshold for conventional QC-LDPC, Rate $= 1/2$, $E_b/N_0 = [-0.4, 0.6, 1.6]$ dB.

**Figure 14.** Convergence threshold for conventional QC-LDPC, Rate $= 1/2$, $E_b/N_0 = [-0.4, 0.6, 1.6]$ dB.

**Figure 15.** Convergence threshold for conventional QC-LDPC, Rate $= 1/2$, $E_b/N_0 = [-0.4, 0.6, 1.6]$ dB.
threshold as compared to their other counterparts under the same conditions.

**Conclusion and remarks**

In this correspondence, we proposed a jointly designed QC-LDPC-coded cooperation with multiple receive antennas in the destination, where transmit diversity, receive diversity, and coding gain are combined into one unit to decode the corrupted sequences coming from source and relay nodes in their respective time frames. First, a design theoretic approach, known as cyclic difference packing (CDP), gives three classes of binary QC-LDPC codes with no length-4 cycles by utilizing some known ingredients like binary matrix dispersion over finite fields, incidence matrices, and circulant decomposition. Second, the proposed CDP-based construction is utilized to jointly design length-4 cycle-free QC-LDPC codes for coded-relay cooperation with multiple receive antennas in the destination, where MRC and SPA-based joint iterative decoding are utilized to decode the corrupted messages coming from the source and relay nodes. Based on the numerical simulation, theoretical analysis shows that the proposed QC-LDPC-coded cooperations outperform their counterparts by providing a coding gain of about 0.25 dB at BER $10^{-6}$ under the same conditions over a Rayleigh fading channel in the presence of additive-white Gaussian noise. Furthermore, the EXIT chart analysis has been utilized to detect the convergence threshold of proposed jointly designed QC-LDPC codes. Numerical results show that the proposed jointly designed QC-LDPC codes provide a better convergence as compared to their counterparts under the same conditions.

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