Towards Quaternion Quadratic-Phase Fourier Transform

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Abstract. The quadratic-phase Fourier transform (QPFT) is a neoteric addition to the class of Fourier transforms and embodies a variety of signal processing tools including the Fourier, fractional Fourier, linear canonical, and special affine Fourier transform. In this paper, we generalize the quadratic-phase Fourier transform to quaternion-valued signals, known as the quaternion QPFT (Q-QPFT). We initiate our investigation by studying the QPFT of 2D quaternionic signals, then we introduce the Q-QPFT of 2D quaternionic signals. Using the fundamental relationship between the Q-QPFT and quaternion Fourier transform (QFT), we derive the inverse transform and Parseval and Plancherel formulas associated with the Q-QPFT. Some other properties including linearity, shift and modulation of the Q-QPFT are also studied. Finally, we formulate several classes of uncertainty principles (UPs) for the Q-QPFT, which including Heisenberg-type UP, logarithmic UP, Hardy’s UP, Beurling’s UP and Donoho-Stark’s UP. It can be regarded as the first step in the applications of the Q-QPFT in the real world.

Keywords: Quaternion Quadratic-phase Fourier transform; Parseval’s formula; Inversion; Modulation; Uncertainty principle; Donoho-Stark.

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1. Introduction

In time-frequency analysis, the most recent signal processing tool is the quadratic-phase Fourier transform (QPFT) introduced by Castro et al.¹ which provides a unified treatment of both the transient and non-transient signals in a simple and insightful fashion. The QPFT has five real parameters with exponential kernel. With a slight modification in ¹, we define the QPFT as

\[ Q_\mu[f](w) = \int_{\mathbb{R}} f(x) \Lambda_\mu(x, w) dx, \quad (1.1) \]

where \( \Lambda_\mu(x, w) \) is a quadratic-phase kernel and is given by

\[ \Lambda_\mu(x, w) = \sqrt{\frac{bi}{2\pi}} e^{-i(ax^2+bwx+cw^2+dx+ew)} \quad (1.2) \]

and the corresponding inversion formula is given by

\[ f(x) = \int_{\mathbb{R}} Q_\mu[f](w) \overline{\Lambda_\mu(x, w)} dw, \quad (1.3) \]

where \( a, b, c, d, e \in \mathbb{R}, b \neq 0 \). These arbitrary real parameters present in (1.2) are of great importance as their choice sense of rotation as well as shift can be inculcated in both the axis of time and frequency domain. Hence can be used in better analysis of non-transient signals which are employed in radar and other communication systems. Due to its global kernel and extra degrees of freedom, the QPFT has arrived an efficient tool in solving several problems arising in diverse branches of science and engineering, including harmonic analysis, image processing, sampling, reproducing kernel Hilbert spaces and so on.
The generalization of integral transforms from real and complex numbers to the quaternion setting is popular nowadays for the study of higher dimension viz: the quaternion Fourier transform (QFT) [6, 7], the quaternion linear canonical transform (QLCT) [8, 9], the fractional quaternion Fourier transform (Fr-QFT) [10, 11], the quaternion offset linear canonical transform (QOLCT) [12, 13, 14, 15]. In past decades, quaternion algebra has become a leading area of research with its applications in color image processing, image filtering, watermarking, edge detection and pattern recognition(see [16, 17, 18, 19, 20, 21, 22]).

The Fourier transform (FT) in quaternion setting i.e. the quaternion Fourier transform (QFT) [23] plays a significant role in the representation of hyper-complex signals in signal processing which is believed to be the substitute of the commonly used two-dimensional Complex Fourier Transform (CFT). The QFT has wide range of applications see([24, 25]). On the other hand the uncertainty principle (UP) plays a vital role in various scientific fields such as mathematics, quantum physics, signal processing and information theory [26, 27, 28]. The UPs like Heisenberg’s, Hardy’s, Beurling’s associated with QFT are given in [29, 30, 31, 32] and the extension of UPs in the domains of QLCT, QOLCT are given in [33, 34, 35, 36, 37]. These UPs have many applications in the analysis of optical systems, signal recovery and so on see([38, 39, 40, 41]). Therefore modern era of information processing is in dire need of quaternionic valued signals and therefore is a very hot area of research. Since the QPFT is a five parameter class of linear integral transform and has more degrees of freedom and is more flexible than the FT, the FRFT, the LCT but with similar computation cost as the conventional FT. Due to the mentioned advantages, it is natural to generalize the classical QPFT to the quaternionic algebra.

To the best of our knowledge, the generalization of the QPFT to quaternion algebra, and the study of the properties and UPs associated with Q-QPFT have not been carried out yet. So motivated and inspired by the merits of QPFT and QFT, we in this paper propose the novel integral transform coined as the quaternion quadratic-phase Fourier transform (Q-QPFT), which provides a unified treatment for several existing classes of signal processing tools. Therefore it is worthwhile to rigorously study the Q-QPFT and associated UPs which can be productive for signal processing theory and applications.

1.1. Paper Contributions.

The contributions of this paper are summarized below:

- To introduce a novel integral transform coined as the quaternion quadratic-phase Fourier transform.
- To establish the fundamental relationship between the proposed transform (Q-QPFT) and the quaternion Fourier transform (QFT).
- To study the fundamental properties of the proposed transform, including the Parseval’s formula, inversion formula, shift and modulation.
- To formulate several classes of uncertainty principles, such as the Heisenberg UP and the logarithmic UP associated with the quaternion quadratic-phase Fourier transform.
To formulate the Hardy’s, Beurling’s and Donoho-Stark’s uncertainty principles for the Q-QPFT.

1.2. Paper Outlines.
The paper is organized as follows: In Section 2, we give a brief review to the quaternion algebra and summarize some definitions and results of two-sided QFT useful in the sequel. The definition and the properties of the novel Q-QPFT are studied in Section 3. In Section 4, we establish some different forms of uncertainty principles (UPs) for the two-sided Q-QPFT which including Heisenberg-type UP, logarithmic UP, Hardy’s UP, Beurling’s UP, and Donoho-Stark’s UP. Finally, a conclusion is drawn in Section 5.

2. Preliminary

In this section, we give a brief review to the quaternion algebra and summarize some definitions and results of two-sided QFT which will be needed throughout the paper.

2.1. Quaternion.
Hamilton introduced the 4-D quaternion algebra in 1843 denoted by $\mathbb{H}$ in his honor,

$$\mathbb{H} = \{ q = [q]_0 + i[q]_1 + j[q]_2 + k[q]_3, [q]_i \in \mathbb{R}, \ i = 0, 1, 2, 3 \},$$

which has three imaginary units $\{i, j, k\}$ and obey Hamilton’s multiplication rules: $i^2 = j^2 = k^2 = ijk = -1, ij = -ji = k$. Let $[q]_0$ and $q = i[q]_1 + j[q]_2 + k[q]_3$ be the real scalar part and the vector part of quaternion number $q \in \mathbb{H}$. The conjugate of quaternion number $q \in \mathbb{H}$ is given by

$$q = [q]_0 - i[q]_1 - j[q]_2 - k[q]_3$$

and its norm is defined as

$$|q| = \sqrt{qq^*} = \sqrt{[q]_0^2 + [q]_1^2 + [q]_2^2 + [q]_3^2}.$$ 

Also it is easy to check that

$$|pq| = |p||q|, \ \ p, q \in \mathbb{H}.$$ 

Moreover the real scalar part has a cyclic multiplication symmetry

$$[pqr]_0 = [qrp]_0 = [rqp]_0, \ \ \forall p, q, r \in \mathbb{H}. \ \ \ (2.1)$$

The inner product of quaternion functions $f, g$ on $\mathbb{R}^2$ with values in $\mathbb{H}$ is defined as follows:

$$\langle f, g \rangle = \int_{\mathbb{R}^2} f(x) \overline{g(x)} dx, \ \ dx = dx_1 dx_2,$$

with symmetric real scalar part

$$\langle f, g \rangle = \frac{1}{2} [(f, g) + (g, f)] = \int_{\mathbb{R}^2} \left[ f(x) \overline{g(x)} \right]_0 dx. \ \ \ (2.2)$$

And for $f = g$, we obtain the $L^2(\mathbb{R}^2, \mathbb{H})$–norm:

$$\|f\| = \left( \int_{\mathbb{R}^2} |f(x)|^2 dx \right)^{1/2}. \ \ \ (2.3)$$
2.2. Quaternion Fourier transform.

The QFT plays a vital role in signal processing and color imaging. There are three different types of QFT, the left-sided QFT, the two-sided QFT and the right-sided QFT. Here our focus will be on two-sided QFT (in the rest of paper QFT means two-sided QFT).

**Definition 2.1 (QFT [29]).** The two-sided QFT of a quaternion signal \( f \in L^1(\mathbb{R}^2, \mathbb{H}) \) is defined by

\[
\mathcal{F}^H[f](w) = \sqrt{\frac{1}{(2\pi)^2}} \int_{\mathbb{R}^2} e^{-ix_1w_1} f(x) e^{-jx_2w_2} dx,
\]  

(2.4)

and corresponding inverse QFT is given by

\[
f(x) = \sqrt{\frac{1}{(2\pi)^2}} \int_{\mathbb{R}^2} e^{ix_1w_1} \mathcal{F}^H[f](w) e^{jx_2w_2} dw,
\]  

(2.5)

where \( x = (x_1, x_2) \) and \( w = (w_1, w_2) \).

**Lemma 2.1 (QFT Parseval [7]).** The quaternion product of \( f, g \in L^1(\mathbb{R}^2, \mathbb{H}) \cap L^2(\mathbb{R}^2, \mathbb{H}) \) and its QFT are related by

\[
\langle f, g \rangle_{L^2(\mathbb{R}^2, \mathbb{H})} = \langle \mathcal{F}^H[f], \mathcal{F}^H[g] \rangle_{L^2(\mathbb{R}^2, \mathbb{H})}.
\]  

(2.6)

In particular if \( f = g \) we get the quaternion version of the Plancherel formula; that is,

\[
\|f\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2 = \|\mathcal{F}^H[f]\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2.
\]  

(2.7)

**Lemma 2.2.** [36] If \( 1 \leq p \leq 2 \) and letting \( \frac{1}{p} + \frac{1}{q} = 1 \), for all \( f \in L^p(\mathbb{R}^2, \mathbb{H}) \), then it holds

\[
\|\mathcal{F}^H\|_q \leq (2\pi)^{\frac{1}{2} - \frac{1}{p}} \|f\|_p.
\]  

(2.8)

3. Quaternion Quadratic-Phase Fourier Transform

In this section we extend the definition of QPFT ([11]) to the 2D quaternionic signals and then based on the definition of the QFT ([27]), we define the novel quaternion QPFT of 2D quaternionic signals and establish several fundamental properties associated with the proposed Q-QPFT.

3.1. QPFTs and Q-QPFT of 2D Quaternionic Signals.

**Definition 3.1 (QPFTs of 2D Quaternionic Signals).** Let \( \mu_s = (a_s, b_s, c_s, d_s, e_s), a_s, b_s, c_s, d_s, e_s \in \mathbb{R} \) and \( b_s \neq 0 \) for \( s = 1, 2 \). Then the left-sided and right-sided QFTs of 2D quaternion signals \( f \in L^2(\mathbb{R}^2, \mathbb{H}) \) are defined by

\[
Q^i_{l,\mu}[f](w_1, x_2) = \int_{\mathbb{R}^2} \Lambda^i_{\mu_1}(x_1, w_1) f(x_1, x_2) dx_1,
\]  

(3.1)

\[
Q^j_{r,\mu}[f](x_1, w_2) = \int_{\mathbb{R}^2} f(x_1, x_2) \Lambda^j_{\mu_2}(x_2, w_2) dx_2,
\]  

(3.2)

where the kernels are given by

\[
\Lambda^i_{\mu_1}(x_1, w_1) = \sqrt{\frac{b_1}{2\pi}} e^{-i(a_1 x_1^2 + b_1 x_1 w_1 + c_1 w_1^2 + d_1 x_1 + e_1 w_1)},
\]  

(3.3)

\[
\Lambda^j_{\mu_2}(x_2, w_2) = \sqrt{\frac{b_2}{2\pi}} e^{-j(a_2 x_2^2 + b_2 x_2 w_2 + c_2 w_2^2 + d_2 x_2 + e_2 w_2)},
\]  

(3.4)

respectively.
Theorem 3.1 (Plancherel Theorem for right-sided QPFT). Let \( f, g \in L^1 \cap L^2(\mathbb{R}^2, \mathbb{H}) \), then
\[
\langle f, g \rangle_{L^2(\mathbb{R}^2, \mathbb{H})} = \langle \mathcal{Q}_r^i[f], \mathcal{Q}_r^i[g] \rangle_{L^2(\mathbb{R}^2, \mathbb{H})}.
\] (3.5)
And for \( f = g \), we get the Parseval theorem as
\[
\|f\|^2_{L^2(\mathbb{R}^2, \mathbb{H})} = \|\mathcal{Q}_r^i[f]\|^2_{L^2(\mathbb{R}^2, \mathbb{H})}.
\] (3.6)
Proof. Proof of the above theorem follows from the one-dimensional case. \(\square\)

Now, we introduce the quaternionic quadratic-phase Fourier transforms (Q-QPFTs). Due to the noncommutative property of multiplication of quaternions, there are many different types of Q-QPFTs: two-sided Q-QPFTs, left-sided Q-QPFTs, and right-sided Q-QPFTs.

Definition 3.2 (Q-QPFTs of 2D Quaternionic Signals). Let \( \mu_s = (a_s, b_s, c_s, d_s, e_s) \), \( a_s, b_s, c_s, d_s, e_s \in \mathbb{R} \) and \( b_s \neq 0 \) for \( s = 1, 2 \). Then the two-sided Q-QPFT, right-sided Q-QPFT and left-sided Q-QPFT of signals \( f \in L^1(\mathbb{R}^2, \mathbb{H}) \) are defined by
\[
\mathcal{Q}^{ij}_{T, \mu_1, \mu_2}[f](w) = \int_{\mathbb{R}^2} \Lambda^{i}_{\mu_1}(x_1, w_1) f(x) \Lambda^{j}_{\mu_2}(x_2, w_2) dx,
\] (3.7)
\[
\mathcal{Q}^{ij}_{R, \mu_1, \mu_2}[f](w) = \int_{\mathbb{R}^2} f(x) \Lambda^{i}_{\mu_1}(x_1, w_1) \Lambda^{j}_{\mu_2}(x_2, w_2) dx,
\] (3.8)
\[
\mathcal{Q}^{ij}_{L, \mu_1, \mu_2}[f](w) = \int_{\mathbb{R}^2} \Lambda^{i}_{\mu_1}(x_1, w_1) f(x) \Lambda^{j}_{\mu_2}(x_2, w_2) dx,
\] (3.9)
respectively. Where \( w = (w_1, w_2) \in \mathbb{R}^2 \), \( x = (x_1, x_2) \in \mathbb{R}^2 \) and \( \Lambda^{i}_{\mu_1}(x_1, w_1) \) and \( \Lambda^{j}_{\mu_2}(x_2, w_2) \) are quaternion kernel signals given by (3.3) and (3.4).

The following lemma gives the relationship between various types of Q-QPFTs.

Lemma 3.1. For \( f \in L^1(\mathbb{R}^2, \mathbb{H}) \), the two-sided Q-QPFT of a signal \( f \) can be written as sum of the two right-sided or left-sided Q-QPFT as:
\[
\mathcal{Q}^{ij}_{T, \mu_1, \mu_2}[f] = \mathcal{Q}^{ij}_{L, \mu_1, \mu_2}[f_p] + \mathcal{Q}^{ij}_{L, \mu_1, \mu_2}[f_q] j,
\] (3.10)
\[
\mathcal{Q}^{ij}_{T, \mu_1, \mu_2}[f] = \mathcal{Q}^{ij}_{R, \mu_1, \mu_2}[f_p] + \mathcal{Q}^{ij}_{R, \mu_1, \mu_2}[f_q] j,
\] (3.11)
where \( f = f_p + f_q j \), \( f_p = f_0 + i f_1 \), \( f_0, f_1 \in \mathbb{R} \).

Proof. Proof of above lemma follows by the procedure of Lemma 2.3 in [12]. \(\square\)

Remark 3.2. The Lemma 3.1 assures that it is sufficient to study two-sided Q-QPFT, as the analogue results for left-sided Q-QPFT or right-sided Q-QPFT can be deduced from (3.10) and (3.11).

3.2. Two-sided Q-QPFT. In this subsection we study the two-sided Q-QPFT (for simplicity of notation we write the Q-QPFT instead of the two-sided Q-QPFT). First we recall the definition of Q-QPFT (3.7) and present an example for the lucid illustration of the proposed transform, then we show that Q-QPFT can be reduced to the two-sided QFT. Finally, we conclude this subsection with the properties of the Q-QPFT which are vital for signal processing.

Definition 3.3 (Q-QPFT). Let \( \mu_s = (a_s, b_s, c_s, d_s, e_s) \) for \( s = 1, 2 \), then the two-sided Q-QPFT of signals \( f \in L^1(\mathbb{R}^2, \mathbb{H}) \) is denoted by \( \mathcal{Q}_{\mu_1, \mu_2}^{\mathbb{H}}[f] \) and defined as
\[
\mathcal{Q}_{\mu_1, \mu_2}^{\mathbb{H}}[f](w) = \int_{\mathbb{R}^2} \Lambda^{i}_{\mu_1}(x_1, w_1) f(x) \Lambda^{j}_{\mu_2}(x_2, w_2) dx,
\] (3.12)
where \( w = (w_1, w_2) \in \mathbb{R}^2 \), \( x = (x_1, x_2) \in \mathbb{R}^2 \) and \( \Lambda^i_{\mu_1}(x_1, w_1) \) and \( \Lambda^j_{\mu_2}(x_2, w_2) \) are quaternion kernel signals given by (3.3) and (3.4), respectively. Where \( a_s, b_s, c_s, d_s, e_s \in \mathbb{R}, b_s \neq 0 \) and \( s = 1, 2 \).

**Remark 3.3.** By appropriately choosing parameters in \( \mu_s = (a_s, b_s, c_s, d_s, e_s), s = 1, 2 \) the Q-QPFT (3.12) includes many well-known linear transforms as special cases:

- For \( \mu_s = (0, -1, 0, 0, 0), s = 1, 2 \), the Q-QPFT (3.12) boils down to the Quaternion-Fourier Transform [29].
- As a special case, when \( \mu_s = (a_s, b_s, c_s, 0, 0), s = 1, 2 \) the Q-QPFT (3.12) can be viewed as the Quaternion Linear Canonical Transform [33].
- For \( \mu_s = (\cot \theta, -\csc \theta, \cot \theta, 0, 0), s = 1, 2 \) the Q-QPFT (3.12) leads to the two-sided FrQFT [36].

We now present an example for the lucid illustration of the proposed quaternion quadratic-phase Fourier transform (3.12)

**Example 3.1.** Consider a 2D Gaussian quaternionic function \( f(x) = e^{-(k_1 x_1^2 + k_2 x_2^2)} \), with \( k_1, k_2 \geq 0 \).

Then by definition of Q-QPFT, we have

\[
\mathcal{Q}_{\mu_1, \mu_2}^H[f](w) = \int_{\mathbb{R}^2} \sqrt{\frac{b_1 i}{2\pi}} e^{-i(a_1 x_1^2 + b_1 x_1 w_1 + c_1 w_1^2 + d_1 x_1 + e_1 w_1)} e^{-(k_1 x_1^2 + k_2 x_2^2)} dx_1 \times \sqrt{\frac{b_2 j}{2\pi}} e^{-j(a_2 x_2^2 + b_2 x_2 w_2 + c_2 w_2^2 + d_2 x_2 + e_2 w_2)} dx_2
\]

\[
= \int_{\mathbb{R}} \sqrt{\frac{b_1 i}{2\pi}} e^{-i(c_1 w_1^2 + e_1 w_1)} \int_{\mathbb{R}} e^{-i(a_1 x_1^2 + b_1 x_1 w_1 + d_1 x_1)} e^{-(k_1 x_1^2)} dx_1 \times \sqrt{\frac{b_2 j}{2\pi}} e^{-j(c_2 w_2^2 + e_2 w_2)} \int_{\mathbb{R}} e^{-j(a_2 x_2^2 + b_2 x_2 w_2 + d_2 x_2)} e^{-(k_2 x_2^2)} dx_2
\]

\[
= \int_{\mathbb{R}} \sqrt{\frac{b_1 i}{2\pi}} e^{-i(c_1 w_1^2 + e_1 w_1)} \int_{\mathbb{R}} e^{-i[a_1 x_1^2 + (b_1 w_1 + d_1)x_1]} e^{-(k_1 x_1^2)} dx_1 \times \sqrt{\frac{b_2 j}{2\pi}} e^{-j(c_2 w_2^2 + e_2 w_2)} \int_{\mathbb{R}} e^{-j[a_2 x_2^2 + (b_2 w_2 + d_2)x_2]} e^{-(k_2 x_2^2)} dx_2
\]

\[
= \int_{\mathbb{R}} \sqrt{\frac{b_1 i}{2\pi}} e^{-i(c_1 w_1^2 + e_1 w_1)} \int_{\mathbb{R}} e^{-(k_1 + i a_1)[x_1 + i \frac{b_1 w_1 + d_1}{(k_1 + i a_1)}]} dx_1 e^{-(\frac{b_1 w_1 + d_1}{4(k_1 + i a_1)})^2} \times \sqrt{\frac{b_2 j}{2\pi}} e^{-j(c_2 w_2^2 + e_2 w_2)} \int_{\mathbb{R}} e^{-(k_2 + j a_2)[x_2 + j \frac{b_2 w_2 + d_2}{(k_2 + j a_2)}]} dx_2 e^{-(\frac{b_2 w_2 + d_2}{4(k_2 + j a_2)})^2}.
\]
Using \( \int_{\mathbb{R}} e^{-z^2} \, dz = \sqrt{\pi} \), for \( z, z' \in \mathbb{C} \) (Gaussian integral) in above equation, we immediately obtain

\[
\mathbb{Q}^H_{\mu_1, \mu_2}[f](w) = \sqrt{\frac{b_1}{2\pi}} e^{-i(c_1w_1^2 + e_1w_1)} \sqrt{\frac{\pi}{k_1 + ia_1}} e^{-\frac{(b_1w_1 + d_1)^2}{4(k_1 + ia_1)}} \times e^{-j(c_2w_2^2 + e_2w_2)} \sqrt{\frac{\pi}{k_2 + ja_2}} e^{-\frac{(b_2w_2 + d_2)^2}{4(k_2 + ja_2)}}.
\]

Now we gave the fundamental relationship between the proposed Q-QPFT and the QFT.

**Theorem 3.4.** The Q-QPFT (3.12) of a quaternion signal \( f \in L^1(\mathbb{R}^2, \mathbb{H}) \) can be reduced to the QFT

\[
\mathcal{F}^H[G_f](w) = \frac{1}{\sqrt{(2\pi)^2}} \int_{\mathbb{R}^2} e^{-ix_1w_1} G_f(x) e^{-jx_2w_2} \, dx,
\]

where

\[
\mathcal{F}^H[G_f](w) = F \left( \frac{w}{b} \right),
\]

\[
G_f(x) = \sqrt{b_1} i \tilde{f}(x) \sqrt{b_2},
\]

\[
\tilde{f}(x) = e^{-i(a_1x_1^2 + d_1x_1)} f(x) e^{-j(a_2x_2^2 + d_2x_2)}
\]

with

\[
F(w) = \frac{1}{\sqrt{(2\pi)^2}} \int_{\mathbb{R}^2} e^{-ix_1w_1} G_f(x) e^{-jx_2w_2} \, dx,
\]

\[
F(w) = e^{i(c_1w_1^2 + e_1w_1)} \mathbb{Q}^H_{\mu_1, \mu_2}[f](w) e^{j(c_2w_2^2 + e_2w_2)}
\]

**Proof.** From Definition 3.3, we obtain

\[
\mathbb{Q}^H_{\mu_1, \mu_2}[f](w) = \int_{\mathbb{R}^2} \frac{b_1}{2\pi} e^{-i(a_1x_1^2 + b_1x_1w_1 + c_1w_1^2 + d_1x_1 + e_1w_1)} f(x) \times \sqrt{\frac{b_2}{2\pi}} e^{-j(a_2x_2^2 + b_2x_2w_2 + c_2w_2^2 + d_2x_2 + e_2w_2)} \, dx
\]

\[
= \sqrt{\frac{b_1}{2\pi}} e^{-i(c_1w_1^2 + e_1w_1)} \int_{\mathbb{R}^2} e^{-ix_1b_1w_1} \tilde{f}(x) \, dx e^{-jx_2b_2w_2} \sqrt{\frac{b_2}{2\pi}} e^{-j(c_2w_2^2 + e_2w_2)}
\]
Then, multiplying both sides of the above equation by $e^{i(c_1w_1^2+e_1w_1)}e^{j(c_2w_2^2+e_2w_2)}$, yields

\[
e^{i(c_1w_1^2+e_1w_1)}Q_{\mu_1,\mu_2}^H[f](w)e^{j(c_2w_2^2+e_2w_2)} = \sqrt{\frac{b_1i}{2\pi}} \int_{\mathbb{R}^2} e^{-ix_1b_1w_1} f(x) dx e^{-jx_2b_2w_2} \sqrt{\frac{b_2j}{2\pi}}
\]

\[
= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-ix_1b_1w_1} G_f(x) dx e^{-jx_2b_2w_2} \frac{1}{\sqrt{(2\pi)^2}}
\]

\[
= \mathcal{F}^H(G_f)(bw),
\]

where $bw = (b_1w_1, b_2w_2)$. This leads to the desired result.

\[
□
\]

**Theorem 3.5** (Inversion formula). Let $Q_{\mu_1,\mu_2}^H[f] \in L^1(\mathbb{R}^2, \mathbb{H})$, then every signal $f \in L^1(\mathbb{R}^2, \mathbb{H})$ can be reconstructed back by the formula

\[
f(x) = \int_{\mathbb{R}^2} \Lambda(x_1, w_1)Q_{\mu_1,\mu_2}^H[f](w)\Lambda(x_2, w_2) dw. \tag{3.16}
\]

**Proof.** By the application of the inversion formula of QFT, we have

\[
G_f(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix_1b_1w_1} F(w) e^{jx_2w_2} dw
\]

\[
= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix_1b_1w_1} e^{i(c_1w_1^2+e_1w_1)}Q_{\mu_1,\mu_2}^H[f](w)e^{j(c_2w_2^2+e_2w_2)}e^{jx_2w_2} dw.
\]

On setting $w = bw$, above equation yields

\[
G_f(x) = \frac{b_1b_2}{\sqrt{(2\pi)^2}} \int_{\mathbb{R}^2} e^{ix_1b_1w_1} F(w) e^{jx_2b_2w_2} dw \tag{3.17}
\]

\[
= \frac{b_1b_2}{\sqrt{(2\pi)^2}} \int_{\mathbb{R}^2} e^{ix_1b_1w_1} e^{i(c_1w_1^2+e_1w_1)}Q_{\mu_1,\mu_2}^H[f](w)e^{j(c_2w_2^2+e_2w_2)}e^{jx_2w_2} dw.
\]

Which implies

\[
\sqrt{b_1}e^{-i(a_1x_1^2+d_1x_1)} f(x) e^{-j(a_2x_2^2+d_2x_2)} \sqrt{b_2j} = \frac{b_1b_2}{\sqrt{(2\pi)^2}} \int_{\mathbb{R}^2} e^{i(b_1x_1w_1+e_1w_1)}Q_{\mu_1,\mu_2}^H[f](w)e^{j(b_2w_2^2+e_2w_2)} dw. \tag{3.18}
\]

On further simplifying, we have

\[
f(x) = \int_{\mathbb{R}^2} \Lambda(x_1, w_1)Q_{\mu_1,\mu_2}^H[f](w)\Lambda(x_2, w_2) dw.
\]

Which completes the proof. □
\textbf{Theorem 3.6} (Parseval’s formula). Let \( f, g \in L^1(\mathbb{R}^2, \mathbb{H}) \cap L^2(\mathbb{R}^2, \mathbb{H}) \), be two quaternion signals, then we have

\[
\langle f, g \rangle_{L^2(\mathbb{R}^2, \mathbb{H})} = \langle Q^H_{\mu_1, \mu_2}[f], Q^H_{\mu_1, \mu_2}[g] \rangle_{L^2(\mathbb{R}^2, \mathbb{H})}.
\] (3.19)

For \( f = g \), we have

\[
\|f\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2 = \|Q^H_{\mu_1, \mu_2}[f]\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2.
\] (3.20)

\textbf{Proof.} By the Parseval’s formula for the QFT and (2.1), we have

\[
\langle G_f, G_g \rangle = \langle \mathcal{F}^H[G_f], \mathcal{F}^H[G_g] \rangle
= \left[ \int_{\mathbb{R}^2} \mathcal{F}^H[G_f](w) \overline{\mathcal{F}^H[G_g](w)} dw \right]_0
= |b_1 b_2| \left[ \int_{\mathbb{R}^2} \mathcal{F}^H[G_f](b_2 w) \overline{\mathcal{F}^H[G_g](b_2 w)} dw \right]_0
= |b_1 b_2| \left[ \int_{\mathbb{R}^2} e^{i(c_1 w_1^2 + e_1 w_1)} Q^H_{\mu_1, \mu_2}[f](w) e^{i(c_2 w_2^2 + e_2 w_2)} \times e^{i(c_1 w_1^2 + e_1 w_1)} Q^H_{\mu_1, \mu_2}[g](w) e^{i(c_2 w_2^2 + e_2 w_2)} dw \right]_0
= |b_1 b_2| \left[ \int_{\mathbb{R}^2} Q^H_{\mu_1, \mu_2}[f](w) \overline{Q^H_{\mu_1, \mu_2}[g](w)} dw \right]_0.
\] (3.21)

Again, we have

\[
\langle G_f, G_g \rangle = \left[ \int_{\mathbb{R}^2} G_f(x) \overline{G_g(x)} dx \right]_0
= \left[ \int_{\mathbb{R}^2} \sqrt{b_1 i \tilde{f}(x)} \sqrt{b_2 j} \sqrt{b_1 i \tilde{g}(x)} \sqrt{b_2 j} dx \right]_0
= \left[ \int_{\mathbb{R}^2} |b_1 b_2| \tilde{f}(x) \overline{\tilde{g}(x)} dx \right]_0
= |b_1 b_2| \left[ \int_{\mathbb{R}^2} e^{-i(a_1 x_1^2 + d_1 x_1)} f(x) e^{i(a_2 x_2^2 + d_2 x_2)} e^{-i(a_1 x_1^2 + d_1 x_1)} g(x) e^{i(a_2 x_2^2 + d_2 x_2)} dx \right]_0
= |b_1 b_2| \left[ \int_{\mathbb{R}^2} f(x) \overline{g(x)} dx \right]_0.
\] (3.22)

On comparing (3.21) and (3.22), we get the desired result. \hfill \Box

\textbf{Theorem 3.7} (Linearity property). Let \( f, g \in L^2(\mathbb{R}^2, \mathbb{H}) \), then Q-QPFT is a linear operator namely

\[
Q^H_{\mu_1, \mu_2}[\alpha f + \beta g](w) = Q^H_{\mu_1, \mu_2}[\alpha f](w) + Q^H_{\mu_1, \mu_2}[\beta g](w),
\] (3.23)

for arbitrary real constants \( \alpha \) and \( \beta \).

\textbf{Proof.} We omit proof as it follows from Definition 3.3 \hfill \Box
Theorem 3.8 (Shift property). For any quaternion signal \( f \in L^2(\mathbb{R}^2, \mathbb{H}) \) and for \( k \in \mathbb{R}^2 \), we have

\[
Q^{\mathbb{H}}_{\mu_1, \mu_2}[f(x - k)](w) = e^{-i \left( a_1 k_1^2 + d_1 k_1 w_1 - 4 \frac{a_1}{b_1} k_1^2 - 4 \frac{a_1}{b_1} c_1 w_1 k_1 - 2 \frac{a_1}{b_1} k_1 \right)} Q^{\mathbb{H}}_{\mu_1, \mu_2}[f] \left( w + 2 \frac{a_1}{b_1} k \right)
\]

Proof. We have from (3.12)

\[
Q^{\mathbb{H}}_{\mu_1, \mu_2}[f(x - k)](w) = \int_{\mathbb{R}^2} \frac{b_1 i}{2\pi} e^{-i(a_1 x_1^2 + b_1 x_1 w_1 + c_1 w_1^2 + d_1 x_1 + e_1 w_1)} f(x - k) \times \sqrt{\frac{b_2 j}{2\pi}} e^{-j(a_2 y_2^2 + b_2 y_2 w_2 + c_2 w_2^2 + d_2 y_2 + e_2 w_2)} dx
\]

By making the change of a variable \( x - k = y \), above equation yields

\[
Q^{\mathbb{H}}_{\mu_1, \mu_2}[f(x - k)](w) = \int_{\mathbb{R}^2} \frac{b_1 i}{2\pi} e^{-i(a_1 y_1 + k_1)^2 + b_1 (y_1 + k_1) w_1 + c_1 w_1^2 + d_1 (y_1 + k_1) + e_1 w_1)} f(y) \times \sqrt{\frac{b_2 j}{2\pi}} e^{-j(a_2 y_2^2 + b_2 y_2 w_2 + c_2 w_2^2 + d_2 y_2 + e_2 w_2)} dy
\]

\[
= \int_{\mathbb{R}^2} \frac{b_1 i}{2\pi} e^{-i\left( a_1 y_1^2 + b_1 (w_1 + 2 \frac{a_1}{b_1} k_1) y_1 + c_1 (w_1 + 2 \frac{a_1}{b_1} k_1)^2 + d_1 y_1 + e_1 \left( w_1 + 2 \frac{a_1}{b_1} k_1 \right) \right)} f(y) \times \sqrt{\frac{b_2 j}{2\pi}} e^{-j\left( a_2 y_2^2 + b_2 (w_2 + 2 \frac{a_2}{b_2} k_2) y_2 + c_2 (w_2 + 2 \frac{a_2}{b_2} k_2)^2 + d_2 y_2 + e_2 \left( w_2 + 2 \frac{a_2}{b_2} k_2 \right) \right)} dy
\]

\[
= e^{-i \left( a_1 k_1^2 + d_1 k_1 w_1 - 4 \frac{a_1}{b_1} k_1^2 - 4 \frac{a_1}{b_1} c_1 w_1 k_1 - 2 \frac{a_1}{b_1} k_1 \right)} \times \int_{\mathbb{R}^2} \frac{b_1 i}{2\pi} e^{-i\left( a_1 y_1^2 + b_1 (w_1 + 2 \frac{a_1}{b_1} k_1) y_1 + c_1 (w_1 + 2 \frac{a_1}{b_1} k_1)^2 + d_1 y_1 + e_1 \left( w_1 + 2 \frac{a_1}{b_1} k_1 \right) \right)} f(y) \times \sqrt{\frac{b_2 j}{2\pi}} e^{-j\left( a_2 y_2^2 + b_2 (w_2 + 2 \frac{a_2}{b_2} k_2) y_2 + c_2 (w_2 + 2 \frac{a_2}{b_2} k_2)^2 + d_2 y_2 + e_2 \left( w_2 + 2 \frac{a_2}{b_2} k_2 \right) \right)} dy
\]

\[
= e^{-i \left( a_1 k_1^2 + d_1 k_1 w_1 - 4 \frac{a_1}{b_1} k_1^2 - 4 \frac{a_1}{b_1} c_1 w_1 k_1 - 2 \frac{a_1}{b_1} k_1 \right)} \times Q^{\mathbb{H}}_{\mu_1, \mu_2}[f] \left( w_1 + 2 \frac{a_1}{b_1} k_1, w_2 + 2 \frac{a_2}{b_2} k_2 \right) \times e^{-j\left( a_2 k_2^2 + d_2 k_2 w_2 - 4 \frac{a_2}{b_2} k_2^2 - 4 \frac{a_2}{b_2} c_2 w_2 k_2 - 2 \frac{a_2}{b_2} k_2 \right)}
\]
Which completes the proof.

**Theorem 3.9** (Modulation property). The quaternion quadratic-phase Fourier transform \([3.12]\) of a modulated quaternion signal \(\mathcal{M}_{w_0}f(x) = e^{jx_1u_0}f(x)e^{jx_2v_0}, w_0 = (u_0, v_0)\) is given by

\[
Q_{\mu_1, \mu_2}^H[\mathcal{M}_{w_0}f](w) = e^{i\frac{c_1 u_0^2 - 2b_1 c_1 u_0 w_1 - b_1 e_1 w_0}{b_1}}Q_{\mu_1, \mu_2}^H[f]\left(w - \frac{w_0}{b}\right)
\times e^{i\frac{c_2 u_0^2 - 2c_2 v_0 w_2 - b_2 e_2 v_0}{b_2}}.
\quad (3.25)
\]

**Proof.** From Definition 3.3 we get

\[
Q_{\mu_1, \mu_2}^H[\mathcal{M}_{w_0}f](w) = \int_{\mathbb{R}^2} \sqrt{\frac{b_1}{2\pi}}e^{-i(a_1 x_1^2 + b_1 x_1 w_1 + c_1 w_1^2 + d_1 x_1 + e_1 w_1)}e^{jx_1u_0}f(x)e^{jx_2v_0}
\times \sqrt{\frac{b_2}{2\pi}}e^{-j(a_2 x_2^2 + b_2 x_2 w_2 + c_2 w_2^2 + d_2 x_2 + e_2 w_2)}dx
\]

\[
= \int_{\mathbb{R}^2} \sqrt{\frac{b_1}{2\pi}}e^{-i\left(a_1 x_1^2 + b_1 x_1 \left(w_1 - \frac{w_0}{b_1}\right) + c_1 \left(w_1 - \frac{w_0}{b_1}\right)^2 + d_1 x_1 + e_1 \left(w_1 - \frac{w_0}{b_1}\right)\right)}f(x)e^{j\left(c_1 \frac{w_0^2}{b_1^2} - 2c_1 \frac{w_0}{b_1} w_1 - e_1 \frac{w_0}{b_1}\right)}
\times \sqrt{\frac{b_2}{2\pi}}e^{-j\left(a_2 x_2^2 + b_2 x_2 \left(w_2 - \frac{w_0}{b_2}\right) + c_2 \left(w_2 - \frac{w_0}{b_2}\right)^2 + d_2 x_2 + e_2 \left(w_2 - \frac{w_0}{b_2}\right)\right)}dx
\]

\[
= e^{i\frac{c_1 u_0^2 - 2c_1 \frac{w_0}{b_1} w_1 - e_1 \frac{w_0}{b_1}}{b_1}}
\times \int_{\mathbb{R}^2} \sqrt{\frac{b_1}{2\pi}}e^{-i\left(a_1 x_1^2 + b_1 x_1 \left(w_1 - \frac{w_0}{b_1}\right) + c_1 \left(w_1 - \frac{w_0}{b_1}\right)^2 + d_1 x_1 + e_1 \left(w_1 - \frac{w_0}{b_1}\right)\right)}f(x)
\times \sqrt{\frac{b_2}{2\pi}}e^{-j\left(a_2 x_2^2 + b_2 x_2 \left(w_2 - \frac{w_0}{b_2}\right) + c_2 \left(w_2 - \frac{w_0}{b_2}\right)^2 + d_2 x_2 + e_2 \left(w_2 - \frac{w_0}{b_2}\right)\right)}dx
\]

\[
= e^{i\frac{c_1 u_0^2 - 2c_1 \frac{w_0}{b_1} w_1 - e_1 \frac{w_0}{b_1}}{b_1}}Q_{\mu_1, \mu_2}^H[f]\left(w_1 - \frac{u_0}{b_1}, w_2 - \frac{v_0}{b_2}\right)
\times e^{i\frac{c_2 v_0^2 - 2c_2 \frac{w_0}{b_2} w_2 - e_2 \frac{w_0}{b_2}}{b_2}}.
\]

Which completes the proof.

We omit properties like Reflection, Conjugation and Scaling as they directly follows from Definition 3.3.

**Theorem 3.10** (Hausdorff-Young). Let \(1 \leq p \leq 2\) and \(\frac{1}{p} + \frac{1}{q} = 1\), then for all \(f \in L^p(\mathbb{R}^2, \mathbb{H})\) following inequality holds

\[
\|Q_{\mu_1, \mu_2}^H[f]\|_q \leq (2\pi)^{\frac{1}{2} - \frac{1}{q}}|b_1 b_2|^{\frac{1}{2} - \frac{1}{q}}\|f(x)\|_p.
\quad (3.26)
\]

**Proof.** From Lemma 2.22 we have

\[
\|\mathcal{F}^H[f](w)\|_q \leq (2\pi)^{\frac{1}{2} - \frac{1}{q}}\|f(x)\|_p.
\]

Replacing \(f\) by \(G_f\), we have from above equation

\[
\|\mathcal{F}^H[G_f](w)\|_q \leq (2\pi)^{\frac{1}{2} - \frac{1}{q}}\|G_f(x)\|_p.
\]
With the help of equations present in Theorem 3.4 above yields
\[
\left\| F\left( \frac{w}{b} \right) \right\|_q \leq (2\pi)^{\frac{1}{q}} \frac{1}{\sqrt{\pi}} \sqrt{b_1 \sqrt{b_2}} \| f \|_p.
\]
On substituting \( w = bw \), we get
\[
|b_1 b_2|^{\frac{1}{q}} \| F(w) \|_q \leq (2\pi)^{\frac{1}{q}} \frac{1}{\sqrt{\pi}} \sqrt{b_1 b_2} \| f \|_p.
\]
Again using Theorem 3.4, we obtain
\[
|b_1 b_2|^{\frac{1}{q}} \| Q^{\mathbb{H}}_{\mu_1, \mu_2}[f](w) \|_q \leq (2\pi)^{\frac{1}{q}} \frac{1}{\sqrt{\pi}} \sqrt{b_1 b_2} \| f \|_p.
\]
Further simplification yields
\[
\left\| Q^{\mathbb{H}}_{\mu_1, \mu_2}[f](w) \right\|_q \leq (2\pi)^{\frac{1}{q}} \frac{1}{\sqrt{\pi}} |b_1 b_2|^{\frac{1}{q}} \| f \|_p.
\]
Which completes the proof. \( \Box \)

4. Uncertainty Principles Associated with the Quaternion-QPFT

In this section based on the fundamental relationship between Q-QPFT and QFT, we investigate some different forms of UPs associated with Q-QPFT including Heisenberg UP, logarithmic UPs, Hardy’s UP, Beurling’s UP, and Donoho-Stark’s UP.

Let’s begin with the Heisenberg type uncertainty principle for the proposed transform (Q-QPFT), which is a generalization of the corresponding Heisenberg’s uncertainty principle for the QFT.

**Theorem 4.1** (Heisenberg UP for the Q-QPFT). Let \( Q^{\mathbb{H}}_{\mu_1, \mu_2}[f] \) be the quaternion quadratic-phase Fourier transform of signal \( f \), then for \( f \in L^1(\mathbb{R}^2, \mathbb{H}) \cap L^2(\mathbb{R}^2, \mathbb{H}) \), \( \partial f/\partial x_s \in L^2(\mathbb{R}^2, \mathbb{H}) \) and \( Q^{\mathbb{H}}_{\mu_1, \mu_2}[f], w_s Q^{\mathbb{H}}_{\mu_1, \mu_2}[f] \in L^2(\mathbb{R}^2, \mathbb{H}), s = 1, 2 \). The following inequality holds:
\[
\int_{\mathbb{R}^2} x_s^2 |f(x)|^2 dx \int_{\mathbb{R}^2} w_s^2 |Q^{\mathbb{H}}_{\mu_1, \mu_2}[f](w)|^2 dw \geq \frac{1}{4b_s^2} \left( \int_{\mathbb{R}^2} |f(x)|^2 \right)^2 , s = 1, 2. \tag{4.1}
\]

**Proof.** The classical Heisenberg uncertainty principle in the QFT domain is given by [13]:
\[
\int_{\mathbb{R}^2} x_s^2 |f(x)|^2 dx \int_{\mathbb{R}^2} w_s^2 |F^\mathbb{H}[f](w)|^2 dw \geq \frac{1}{4} \left( \int_{\mathbb{R}^2} |f(x)|^2 \right)^2 , s = 1, 2. \tag{4.2}
\]
Replacing \( f \) by \( G_f \) in \( (4.2) \), we have
\[
\int_{\mathbb{R}^2} x_s^2 |G_f(x)|^2 dx \int_{\mathbb{R}^2} w_s^2 |F^\mathbb{H}[G_f](w)|^2 dw \geq \frac{1}{4} \left( \int_{\mathbb{R}^2} |G_f(x)|^2 \right)^2. \tag{4.3}
\]
On substituting \( w = bw \), \( (4.3) \) yields
\[
\int_{\mathbb{R}^2} x_s^2 |G_f(x)|^2 dx \int_{\mathbb{R}^2} b_s^2 w_s^2 |b_1 b_2| |F^\mathbb{H}[G_f](bw)|^2 dw \geq \frac{1}{4} \left( \int_{\mathbb{R}^2} |G_f(x)|^2 \right)^2. \tag{4.4}
\]
Using equations present in Theorem 3.4 in \( (4.4) \), we get
\[
\int_{\mathbb{R}^2} x_s^2 |\sqrt{b_1 i f(x)} \sqrt{b_2 j}|^2 dx \int_{\mathbb{R}^2} b_s^2 w_s^2 |b_1 b_2| e^{i(c_1 w_2^2 + e_1 w_1)} Q^{\mathbb{H}}_{\mu_1, \mu_2}[f](w) e^{i(c_2 w_2^2 + e_2 w_2)} dw \geq \frac{1}{4} \left( \int_{\mathbb{R}^2} |\sqrt{b_1 i f(x)} \sqrt{b_2 j}|^2 \right)^2, \tag{4.5}
\]
Which implies
\[ \int_{\mathbb{R}^2} |b_1 b_2|^2 x_s^2 |\hat{f}(x)|^2 dx \int_{\mathbb{R}^2} b_s^2 w_2^2 |b_1 b_2||Q^H_{\mu_1,\mu_2}[f](w)|^2 dw \geq \frac{|b_1 b_2|^2}{4} \left( \int_{\mathbb{R}^2} |\hat{f}(x)|^2 \right)^2. \]

Hence,
\[ |b_1 b_2|^2 \int_{\mathbb{R}^2} x_s^2 e^{-i(a_1 x_1^2 + d_1 x_1)} f(x) - j(a_2 x_2^2 + d_2 x_2)^2 \right| dx \int_{\mathbb{R}^2} b_s^2 w_2^2 |Q^H_{\mu_1,\mu_2}[f](w)|^2 dw \geq \frac{|b_1 b_2|^2}{4} \left( \int_{\mathbb{R}^2} |f(x)|^2 \right)^2. \]

Equivalently
\[ |b_1 b_2|^2 \int_{\mathbb{R}^2} x_s^2 |f(x)|^2 dx \int_{\mathbb{R}^2} b_s^2 w_2^2 |Q^H_{\mu_1,\mu_2}[f](w)|^2 dw \geq \frac{|b_1 b_2|^2}{4} \left( \int_{\mathbb{R}^2} |f(x)|^2 \right)^2. \] (4.6)

Simplifying (4.6), we obtain
\[ \int_{\mathbb{R}^2} x_s^2 |f(x)|^2 dx \int_{\mathbb{R}^2} w_2^2 |Q^H_{\mu_1,\mu_2}[f](w)|^2 dw \geq \frac{1}{4b_2^2} \left( \int_{\mathbb{R}^2} |f(x)|^2 \right)^2. \]

Which completes the proof.

The directional uncertainty principle for the Q-QPFT takes the following form

**Theorem 4.2.** Let \( f \in L^1(\mathbb{R}^2, \mathbb{H}) \cap L^2(\mathbb{R}^2, \mathbb{H}) \) and for \( Q^H_{\mu_1,\mu_2}[f] \), \( |w|^2 Q^H_{\mu_1,\mu_2}[f] \in L^2(\mathbb{R}^2, \mathbb{H}) \), we have, the following inequality:
\[ \int_{\mathbb{R}^2} |x|^2 |f(x)|^2 dx \int_{\mathbb{R}^2} |w|^2 |Q^H_{\mu_1,\mu_2}[f](w)|^2 dw \geq \frac{1}{|b|^2} \left( \int_{\mathbb{R}^2} |f(x)|^2 \right)^2. \] (4.7)

**Proof.** The directional uncertainty principle in the QFT domain reads [Theorem 16 [42]]
\[ \int_{\mathbb{R}^2} |x|^2 |f(x)|^2 dx \int_{\mathbb{R}^2} |w|^2 |\mathcal{F}[f](w)|^2 dw \geq \left( \int_{\mathbb{R}^2} |f(x)|^2 \right)^2. \] (4.8)

Now using the machinery of previous theorem in (4.8), we will get the desired result (4.7).

Next, using Logarithmic uncertainty principle for the QFT, we establish Logarithmic uncertainty principle for the proposed Q-QPFT.

**Theorem 4.3 (Logarithmic UP for the Q-QPFT).** Let \( Q^H_{\mu_1,\mu_2}[f] \) be the quaternion quadratic-phase Fourier transform of signal \( f \in \mathcal{S}(\mathbb{R}^2, \mathbb{H}) [\text{Schwartz space}] \). Then we have the following logarithmic inequality
\[ \int_{\mathbb{R}^2} \ln |x| |f(x)|^2 dx + \int_{\mathbb{R}^2} \ln |w| |Q^H_{\mu_1,\mu_2}[f](w)|^2 dw \geq (D - \ln |b|) \int_{\mathbb{R}^2} |f(x)|^2 dx. \] (4.9)

**Proof.** For any \( f \in \mathcal{S}(\mathbb{R}^2, \mathbb{H}) \), the logarithmic uncertainty principle for the two-sided quaternion Fourier transform reads [Lemma 3.1 [36]]
\[ \int_{\mathbb{R}^2} \ln |x||f(x)|^2 dx + \int_{\mathbb{R}^2} \ln |w||\mathcal{F}[f](x)|^2 dw \geq D \int_{\mathbb{R}^2} |f(x)|^2 dx, \] (4.10)
where \( D = \ln(2\pi^2) - 2\psi(1/2), \psi = \frac{d}{dx}(\ln(\Gamma(x))) \) and \( \Gamma(x) \) is a Gamma function.

Replacing \( f \) by \( G_f \) defined in Theorem 3.3 on both sides of (3.11), we have
\[
\int_{\mathbb{R}^2} \ln |x| |G_f(x)|^2 dx + \int_{\mathbb{R}^2} \ln |w| |\mathcal{F}_H[G_f](w)|^2 dw \geq D \int_{\mathbb{R}^2} |G_f(x)|^2 dx. \tag{4.11}
\]
On substituting \( w = bw \), (4.11) yields
\[
\int_{\mathbb{R}^2} \ln |x| |G_f(x)|^2 dx + |b_1b_2| \int_{\mathbb{R}^2} \ln |bw| |\mathcal{F}_H[G_f(bw)]|^2 dw \geq D \int_{\mathbb{R}^2} |G_f(x)|^2 dx. \tag{4.12}
\]
By the equations present in Theorem 3.3, (4.12) yields
\[
\int_{\mathbb{R}^2} \ln |x| \sqrt{b_1i \tilde{f}(x)} \sqrt{b_2j} \left| e^{i(c_1w_1^2 + c_1w_1)} Q_{\mu_1, \mu_2}^H [f](w) e^{i(c_2w_2^2 + c_2w_2)} \right|^2 dw 
\geq D \int_{\mathbb{R}^2} \left| \sqrt{b_1i \tilde{f}(x)} \sqrt{b_2j} \right|^2 dx. \tag{4.13}
\]
Further simplifying (4.13), we obtain
\[
|b_1b_2| \int_{\mathbb{R}^2} \ln |x| |f(x)|^2 dx + |b_1b_2| \int_{\mathbb{R}^2} \ln |bw| |Q_{\mu_1, \mu_2}^H [f](w)|^2 dw 
\geq D|b_1b_2| \int_{\mathbb{R}^2} |f(x)|^2 dx.
\]
Which implies
\[
\int_{\mathbb{R}^2} \ln |x| |f(x)|^2 dx + \int_{\mathbb{R}^2} \ln |b| |Q_{\mu_1, \mu_2}^H [f](w)|^2 dw 
+ \int_{\mathbb{R}^2} \ln |w| |Q_{\mu_1, \mu_2}^H [f](w)|^2 dw \geq D \int_{\mathbb{R}^2} |f(x)|^2 dx. \tag{4.14}
\]
By applying Parseval’s identity (3.20) to (4.14), we obtain
\[
\int_{\mathbb{R}^2} \ln |x| |f(x)|^2 dx + \int_{\mathbb{R}^2} \ln |w| |Q_{\mu_1, \mu_2}^H [f](w)|^2 dw 
\geq (D - \ln |b|) \int_{\mathbb{R}^2} |f(x)|^2 dx.
\]
Which completes the proof.
\[\square\]

In continuation, we shall derive the Hardy’s uncertainty principle for the quaternion quadratic-phase Fourier transform (3.12). We first recall Hardy’s uncertainty principle for the QFT.

**Lemma 4.1** (Hardy’s UP for the two-sided QFT [30]). Let \( \alpha \) and \( \beta \) be positive constants. For \( f(t) \in L^2(\mathbb{R}^2, \mathbb{H}) \), if
\[
f(x) \leq c e^{-\alpha |x|^2} \quad \text{and} \quad |\mathcal{F}_H[f](w)| \leq c' e^{-\beta |w|^2}, \quad u, w \in \mathbb{R}^2
\]
with some positive constants \( c, c' \). Then, there are the following three cases to occur:
1. If \( \alpha \beta > \frac{1}{4} \), then \( f(x) \equiv 0 \);
2. If \( \alpha \beta = \frac{1}{4} \), then \( f(x) = ke^{-\alpha |x|} \), for any constant \( k \);
3. If \( \alpha \beta < \frac{1}{4} \), then there are many infinite such functions \( f(x) \).
Motivated and inspired by Hardy’s UP for the two-sided QFT, we establish Hardy’s UP for the Q-QPFT.

**Theorem 4.4** (Hardy’s UP for the Q-QPFT). Let \( \alpha \) and \( \beta \) be positive constants. For \( f(t) \in L^2(\mathbb{R}^2, \mathbb{H}) \), if
\[
\int |f(x)|\, dx \leq C e^{-\alpha|x|^2} \quad \text{and} \quad |Q_{\mu_1,\mu_2}^H[f](\frac{w}{b})| \leq C' e^{-\beta|w|^2}, \quad u, w \in \mathbb{R}^2
\]
with some positive constants \( C, C' \). Then, there are the following three cases to occur:

1. if \( \alpha \beta > \frac{1}{4} \), then \( f(x) \equiv 0 \);
2. if \( \alpha \beta = \frac{1}{4} \), then \( f(x) = e^{\alpha_1x_1^2+d_1x_1}Ke^{-\alpha|x|}e^{j\alpha_2x_2^2+d_2x_2} \), for any constant \( K \);
3. if \( \alpha \beta < \frac{1}{4} \), then there are many infinite such functions \( f(x) \).

**Proof.** Assuming \( f = G_f \) defined in Theorem 3.4, it follows that
\[
|G_f(x)| \leq ce^{-\alpha|x|^2} \quad x \in \mathbb{R}^2
\]
and
\[
|\mathcal{F}^H[G_f](w)| \leq c'e^{-\beta|w|^2} \quad w \in \mathbb{R}^2.
\]
Thus by Lemma 4.1, there are the following three cases to occur:

1. if \( \alpha \beta > \frac{1}{4} \), then \( G_f(x) \equiv 0 \);
2. if \( \alpha \beta = \frac{1}{4} \), then \( G_f(x) = ke^{-\alpha|x|} \), for any real constant \( k \);
3. if \( \alpha \beta < \frac{1}{4} \), then there are many infinite such functions \( G_f(x) \).

Now it is clear from Theorem 3.4 and equations (4.15), (4.16) that
\[
|G_f(x)| = \sqrt{b_1} |f(x)| \sqrt{b_2} \leq ce^{-\alpha|x|^2} \quad x \in \mathbb{R}^2
\]
and
\[
|\mathcal{F}^H[G_f](w)| = \left| Q_{\mu_1,\mu_2}^H[f](\frac{w}{b}) \right| \leq c'e^{-\beta|w|^2} \quad w \in \mathbb{R}^2.
\]

From (4.17) and (4.18), we have
\[
|f(x)| \leq Ce^{-\alpha|x|^2} \quad u \in \mathbb{R}^2 \quad \text{and} \quad \left| Q_{\mu_1,\mu_2}^H[f](\frac{w}{b}) \right| \leq C'e^{-\beta|w|^2}, \quad w \in \mathbb{R}^2
\]
where \( C = \frac{c}{\sqrt{b_1b_2}} \) and \( C = c' \). Thus we have the following conclusions:

1. if \( \alpha \beta > \frac{1}{4} \), then \( f(x) \equiv 0 \) for \( G_f(x) \equiv 0 \);
2. if \( \alpha \beta = \frac{1}{4} \), it yields \( f(x) = e^{\alpha_1x_1^2+d_1x_1}Ke^{-\alpha|x|}e^{j\alpha_2x_2^2+d_2x_2} \), where \( K = \frac{1}{\sqrt{b_1b_2}} \) owing to Theorem 3.4;
3. if \( \alpha \beta < \frac{1}{4} \), then it is clear there are many infinite such functions \( f(x) \).

Which completes the proof.

Now, using the relationship between the proposed transform (Q-QPFT) and QFT, we obtain Beurling’s uncertainty principle for the Q-QPFT. First we recall the Beurling’s uncertainty principle for the QFT.

**Lemma 4.2** (Beurling’s UP for the two-sided QFT [31]). Let \( f(x) \in L^2(\mathbb{R}^2, \mathbb{H}) \) and \( d \geq 0 \) such that
\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |f(x)| |\mathcal{F}^H[f](w)| e^{-|x||w|} \, dx \, dw < \infty,
\]
then \( f(x) = P(x)e^{-k|x|^2} \), where \( k > 0 \) and \( P \) is a polynomial of degree \( < \frac{d-2}{2} \). In particular, \( f = 0 \) when \( d \leq 2 \).

By applying Theorem 3.4 and Lemma (4.2), we extend the validity of Beurling’s UP for the Q-QPFT.
Theorem 4.5 (Beurling’s UP for the Q-QPFT). Let \( f(x) \in L^2(\mathbb{R}^2, \mathbb{H}) \) and \( d \geq 0 \) satisfying
\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|f(x)|}{(1+|x|+|bw|)^d} e^{||bw||} \, dx \, dw < \infty,
\]
then \( f(x) = e^{i(a_1x_1^2 + d_1x_1)} P'(x)e^{-k|x|^2}e^{i(a_2x_2^2 + d_2x_2)} \), where \( k > 0 \) and \( P'(x) = \frac{1}{\sqrt{b_1 \sqrt{b_2}}} P(x) \sqrt{b_2} \) is a polynomial of degree \( < \frac{d-2}{2} \). In particular, \( f = 0 \) when \( d \leq 2 \).

Proof. If we take \( f = G_f \) as defined in Theorem 3.4, then it follows that
\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|G_f(x)|}{(1+|x|+|w|)^d} e^{||w||} \, dx \, dw = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \sqrt{b_1 b_2} |f(x)| \frac{|Q_{\mu_1 \mu_2}[f](\bar{w})|}{(1+|x|+|w|)^d} e^{||w||} \, dx \, dw < \infty.
\]
Hence by Lemma 4.2 we must have \( G_f(x) = P(x)e^{-k|x|^2} \). Now,
\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|G_f(x)|}{(1+|x|+|w|)^d} e^{||w||} \, dx \, dw = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \sqrt{b_1 b_2} |f(x)| \frac{|Q_{\mu_1 \mu_2}[f](\bar{w})|}{(1+|x|+|w|)^d} e^{||w||} \, dx \, dw = (b_1 b_2)^\frac{1}{2}\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|f(x)|}{(1+|x|+|bw|)^d} e^{||bw||} \, dx \, dw < \infty.
\]
As \( b_1, b_2 \) are finite real numbers, therefore we can write
\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|f(x)|}{(1+|x|+|bw|)^d} e^{||bw||} \, dx \, dw < \infty.
\]
Since \( G_f(x) = \sqrt{b_1} f(x) \sqrt{b_2} \), \( \sqrt{b_1} = \sqrt{b_1} e^{-i(a_1x_1^2 + d_1x_1)} f(x) e^{-j(a_2x_2^2 + d_2x_2)} \sqrt{b_2} \), which implies \( f(x) = e^{i(a_1x_1^2 + d_1x_1)} P'(x)e^{-k|x|^2}e^{i(a_2x_2^2 + d_2x_2)} \). In particular, \( f = 0 \) on account \( G_f(x) = 0 \) when \( d \leq 2 \).

Which completes the proof.

Towards the end of this section, we establish Donoho-Stark’s uncertainty principle for the Q-QPFT by considering relationship between the proposed transform (Q-QPFT) and QFT. Let us begin with the definition.

Definition 4.1. [15] A quaternion function \( f \in L^2(\mathbb{R}^2, \mathbb{H}) \) is said to be \( \varepsilon \)--concentrated on a measurable set \( E \subseteq \mathbb{R}^2 \), if
\[
\left( \int_{\mathbb{R}^2 \setminus E} |f(x)|^2 \, dx \right)^{1/2} \leq \varepsilon \|f\|_2.
\]

Lemma 4.3 (Donoho-Stark’s UP for the two-sided QFT [14] [15]). Let \( f \in L^2(\mathbb{R}^2, \mathbb{H}) \) with \( f \neq 0 \) be \( \varepsilon_{E_1} \)--concentrated on \( E_1 \subseteq \mathbb{R}^2 \) and \( \mathcal{F}_{\mathbb{R}^2}[f] \) be \( \varepsilon_{E_2} \)--concentrated on \( E_2 \subseteq \mathbb{R}^2 \). Then
\[
|E_1||E_2| \geq 2\pi(1 - \varepsilon_{E_1} - \varepsilon_{E_2})^2.
\]
Theorem 4.6 (Donoho-Stark’s UP for the Q-QPFT). Assuming that non-zero signal \( f \) in \( L^2(\mathbb{R}^2, \mathbb{H}) \) is a \( \varepsilon_{E_1} \)-concentrated on \( E_1 \subseteq \mathbb{R}^2 \) and \( \mathcal{Q}^H_{\mu_1, \mu_2}[f](w) \) is \( \varepsilon_{E_2} \)-concentrated on \( E_2 \subseteq \mathbb{R}^2 \). Then
\[
|E_1||E_2| \geq \frac{2\pi}{|b|} (1 - \varepsilon_{E_1} - \varepsilon_{E_2})^2.
\]

Proof. By Theorem 3.4 we have
\[
\left| \mathcal{Q}^H_{\mu_1, \mu_2}[f](\frac{w}{b}) \right| = \left| \mathcal{F}^H[G_f](w) \right| \tag{4.19}
\]
Since \( \mathcal{Q}^H_{\mu_1, \mu_2}[f](w) \) is \( \varepsilon_{E_2} \)-concentrated on \( E_2 \subseteq \mathbb{R}^2 \), therefore (4.19) implies \( \mathcal{F}^H[G_f] \) is \( \varepsilon_{E_2} \)-concentrated on \( bE_2 \subseteq \mathbb{R}^2 \).

Also from (3.14), we have
\[
|G_f(x)| = \sqrt{b_1 b_2}|f(x)|. \tag{4.20}
\]
By the given condition \( f \) is \( \varepsilon_{E_1} \)-concentrated on \( E_1 \subseteq \mathbb{R}^2 \), i.e.
\[
\left( \int_{\mathbb{R}^2 \setminus E_1} |f(x)|^2 dx \right)^{1/2} \leq \varepsilon \| f \|_2. \tag{4.21}
\]
From (4.20) and (4.21), we obtain
\[
\left( \int_{\mathbb{R}^2 \setminus E_1} |G_f(x)|^2 dx \right)^{1/2} \leq \varepsilon \| G_f \|_2.
\]
Which implies \( G_f \in L^2(\mathbb{R}^2, \mathbb{H}) \) is \( \varepsilon_{E_1} \)-concentrated on \( E_1 \subseteq \mathbb{R}^2 \). Hence we proved that the function \( G_f(x) \) and its QFT \( \mathcal{F}^H[G_f](w) \) are \( \varepsilon_{E_1} \)-concentrated on \( E_1 \subseteq \mathbb{R}^2 \) and \( \varepsilon_{E_2} \)-concentrated on \( bE_2 \subseteq \mathbb{R}^2 \), respectively. Therefore by Lemma 4.3 we have
\[
|E_1||bE_2| \geq 2\pi(1 - \varepsilon_{E_1} - \varepsilon_{E_2})^2,
\]
so that
\[
|E_1||E_2| \geq \frac{2\pi}{|b|} (1 - \varepsilon_{E_1} - \varepsilon_{E_2})^2,
\]
Which completes the proof. \( \square \)

5. Conclusion

In the study, we have accomplished three major objectives: first, we have introduced the notion of quaternion quadratic-phase Fourier transform (Q-QPFT). Second, we establish the fundamental properties of the proposed transform, including the Parseval’s formula, inversion formula, shift and modulation by using the fundamental relationship between Q-QPFT and QFT. Third, we investigate some different forms of UPs associated with Q-QPFT including Heisenberg UP, logarithmic UPs, Hardy’s UP, Beurling’s UP, and Donoho-Stark’s UP. In our future works we shall study the short-time quadratic-phase Fourier transform in the quaternion setting.

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