DISSIPATION VIA PARTICLE PRODUCTION IN SCALAR FIELD THEORIES

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Abstract

The non-equilibrium dynamics of the first stage of the reheating process, that is dissipation via particle production is studied in scalar field theories in the unbroken and in the broken symmetry phase. We begin with a perturbative study to one loop and show explicitly that the mechanism of dissipation via particle production cannot be explained with a simple derivative term in the equation of motion. The dissipative contribution is non-local and there does not exist a local (Markovian) limit at zero temperature. Furthermore, we show that both an amplitude as well as a one-loop calculation present instabilities, requiring a non-perturbative resummation. Within the same approximations, we study an O(2) linear sigma model that allows to study dissipation by Goldstone bosons. We find infrared divergences that require non-perturbative resummation in order to understand the long-time dynamics.

We obtain a perturbative Langevin equation that exhibits a generalized fluctuation-dissipation relation, with non-Markovian kernels and colored noise.

We then study a Hartree approximation and clearly exhibit dissipative effects related to the thresholds to particle production. The asymptotic dynamics depends on the coupling and initial conditions but does not seem to lead to exponential relaxation.

The effect of dissipation by Goldstone bosons is studied non-perturbatively in the large N limit in an O(N) theory. Dissipation produced by Goldstone bosons dramatically changes the picture of the phase transition. We find the remarkable result that
for “slow-roll” initial conditions (with the expectation value of the field initially near the origin) the final value of the expectation value of the scalar field is very close to its initial value. Most of the potential energy is transferred to the production of long-wavelength Goldstone bosons. We find that the minima of the effective action (as determined by the final value of the expectation value of the scalar field) depend on the initial conditions. This in fact points out that in the case of broken symmetry, in the large N limit there are many extrema of the effective action. In the Hartree and large N approximation dissipation occurs in a collisionless regime, a situation very similar to that of Landau damping. We provide extensive numerical analysis of the dynamics.

1 Introduction

It is well appreciated that the dynamics of dissipation in scalar field theories is of great importance in a variety of settings. One of the most interesting of these is that of reheating of the universe after an inflationary epoch has passed\cite{1, 2}. Recall that at the end of new or chaotic inflation\cite{3}, defined by when the slow-roll conditions for the so-called inflaton field $\phi(t)$, fail to obtain, the inflaton begins to oscillate about its true ground state. Since the inflaton is coupled to lighter fields (fermions or other scalars) as the scalar field oscillates around the true minimum it decays into these other particles, and eventually these particles thermalize via collisions and relax to an equilibrium state at high temperature. It is usually stated that such coupling gives rise to a term of the form $\Gamma \dot{\phi}(t)$, where $\Gamma$ is the decay rate of the inflaton into the lighter fields\cite{4, 5}. This term, which is usually put in by hand as a phenomenological result of the existence of open decay channels, acts as a friction term in the equation of motion for the inflaton, and converts the inflaton energy density into that of the lighter particles during its oscillations. These decay products are then supposed to thermalize, completing the reheating process\cite{6, 7}.

This picture bears closer scrutiny. Two questions that must be addressed are: (i) how does a time reversible theory generate dissipative dynamics, and (ii) can the phenomenological term $\Gamma \dot{\phi}(t)$ be derived from first principles?

The answer to the first question is relatively well known. To generate dissipative dynamics from a time reversible theory requires that some “coarse-graining” of the field degrees of freedom be done. Essentially, one must trace out (in the functional integral) degrees of freedom other than the one that has been deemed important to the dynamics. The tracing operation turns a closed system (that of all the field modes) into an open one (that of the field modes of interest) where the traced out modes now become the “environment” which couples
to the remaining modes. Another important necessary ingredient are non-equilibrium initial conditions.

As envisaged in the original scenarios, the process of reheating consists of two different stages. During the first one, potential and kinetic energy of the scalar field is dissipated by the process of particle production. The second stage involves the collisional thermalization of these particles reaching a final equilibrium state at some temperature.

What we want to understand in this article is the dynamics of the first stage in an a priori fashion, i.e. how to incorporate the effect of quantum fluctuations into the dynamics of the time dependent order parameter, and the process of particle production.

This is the first necessary step to fully understand the reheating process in its full complexity.

What we do here is the following. Starting from a renormalizable self interacting scalar field theory we isolate the expectation value (which by translational invariance only depends on time) and generate its equation of motion taking quantum fluctuations into account. Since we are motivated by reheating after an inflationary epoch, we treat the dynamics at zero temperature since the temperature is supposed to have red-shifted all the way to zero during the inflationary stage.

What makes our approach different from others that have been proposed (see below) is that we are not considering an “in-out” expectation value, such as would be generated by the usual method of constructing the effective action. Rather, we use a method that generates the equations for an “in-in” expectation value of the field operator. This ensures that our equations of motion for the field expectation value are both real and causal. This has been done by others, most notably by Calzetta and Hu[8] and Paz[9].

After obtaining the renormalized, causal and real equations of motion that determine the dynamics we provide an extensive numerical analysis of the equations.

The results from our analysis are quite interesting and we summarize them here. The equations of motion are rather complicated integro-differential equations connecting the field expectation value to the fluctuations of the non-zero momentum modes. We begin by obtaining the equations of motion in an amplitude expansion as well as in the loop expansion up to one loop. After recognizing the “dissipative terms” in these approximations, we find that these cannot be simply replaced by a term of the form $\Gamma \dot{\phi}(t)$.

In the case where there is no symmetry breaking, we generate the effective Langevin equation for the system, in the one-loop approximation, and we find a multiplicative, non-Markovian kernel for the dissipational term as well as a “colored” noise term. Thus this is very different than the simple friction term described above. In fact, it is easy to see that
there is no limit in which the dissipational dynamics we find can be described by a term proportional to the time derivative of the field.

In the case in which a discrete symmetry is broken, so that there are no Goldstone modes, the one-loop equation can be linearized in the amplitude of oscillations about the non-zero expectation value of the field. Even in this linearized approximation, while a term involving the first time derivative of the field does appear, it is convolved with a non-trivial kernel, that again has no limit (at zero temperature) in which it becomes a local term.

We show that the loop expansion or the amplitude expansion for the dynamics of the field expectation value must break down at long times. In the loop expansion, the order $\hbar$ term has an amplitude that grows in time due to resonance phenomena induced by the existence of a two particle threshold. Thus, it will eventually dominate over the tree amplitude, and perturbation theory will then break down. Dissipation can only be understood beyond perturbation theory. We then use the Hartree approximation in the single scalar field case and the large $N$ approximation in an $O(N)$ symmetric theory to try to understand the effects of quantum fluctuations on the oscillations of the field expectation value as well as particle production due to these same oscillations. The Hartree approximation reveals dissipative effects due to particle production and open channels. However, we find that the damping is not exponential, and asymptotically, the field expectation value undergoes undamped oscillations with an amplitude that depends on the coupling and the initial conditions. The same features are found in the large $N$ limit in the $O(N)$ model in the case of unbroken symmetry. Here we find that the asymptotic oscillatory behavior can be expressed in terms of elliptic functions.

The dissipation in the system is caused by a Landau damping type of process. This is a collisionless process which in our case is associated with the particle production arising from oscillations of the field expectation value via parametric amplification. These quantum fluctuations then react back on the field expectation value, but not in phase with it, thus damping the oscillations. This behavior is similar to that found in the damping of strong electric fields in a collisionless regime.

An unexpected and quite remarkable result comes from the analysis of the $O(N)$ model in large $N$ when the symmetry is spontaneously broken. We find that the field expectation value oscillations are damped very quickly in this case, much more so than in the unbroken situation. This by itself is perhaps not so unusual, since the existence of Goldstone modes allows the field to lose energy by radiating a large number of soft Goldstone particles. What is unusual, however, is that there are some initial conditions, corresponding to “slow-roll” behavior, for which the field expectation value relaxes (via dissipation) to a value that is

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very close to the origin. This final value is a minimum of the effective action for which the expectation value of the scalar field has moved very slightly from the initial value. Most of the potential energy is converted into long-wavelength Goldstone bosons. Thus the symmetry breaking phase transition is actually \textit{dramatically modified} by the dissipational dynamics. These initial conditions are generically “slow-roll” conditions in which the scalar field begins very close to the top of the potential hill and the couplings are extremely weak (in our case about $10^{-7}$ or smaller). This is one of the most striking results of this article.

We provide extensive numerical evidence for all our assertions; we can follow the evolution of the field expectation value, the quantum fluctuations and we also compute the number of particles created via the method of Bogoliubov coefficients.

In the next section we provide an outline of the formalism we use to generate the equations for the expectation value of the scalar field that include the effects of the fluctuations of the quantum fluctuations. We then investigate the meaning of these equations within the one-loop and the amplitude expansion. In section III and IV we utilize non-perturbative approximations such as the Hartree approximation and the large $N$ approximation in the case of the $O(N)$ model to perform our numerical calculations. After this analysis, we reconcile the dissipative dynamics that we find with the concept of time reversal invariance.

Section V contains our conclusions, in which we compare our results to other results on dissipation and reheating obtained in the literature. Two appendices are included which contain some of the technical details of our calculations.
2 Non-equilibrium field theory and equations of motion

The generalization of statistical mechanics techniques to the description of non-equilibrium processes in quantum field theory has been available for a long time [12, 13, 14, 15, 16] but somehow has not yet been accepted as an integral part of the available tools to study field theory in extreme environments. We thus begin by presenting a somewhat pedagogical introduction to the subject for the non-practitioner.

The non-equilibrium description of a system is determined by the time evolution of the density matrix that describes it. This time evolution (in the Schrödinger picture) is determined by the quantum Liouville equation:

$$i\hbar \frac{\partial \rho(t)}{\partial t} = [H(t), \rho(t)]$$

(1)

A non-equilibrium situation arises whenever the Hamiltonian does not commute with the density matrix. Here we allow for an explicitly time dependent Hamiltonian, which might be the case if the system is in an external time dependent background, for example.

The formal solution of the Liouville equation is:

$$\rho(t) = U(t, t_0)\rho(t_0)U^{-1}(t, t_0)$$

(2)

with $\rho(t_0)$ the density matrix at some initial time $t_0$ that determines the initial condition for the evolution. Ensemble averages of operators and correlation functions are obtained as

$$\langle O(t) \rangle = Tr \left[ O U(t, t_0)\rho(t_0)U^{-1}(t, t_0) \right] / Tr \rho(t_0)$$

(3)

$$\langle O(t_1)O(t_2) \rangle = Tr \left[ O U(t_1, t_2)O U(t_2, t_0)\rho(t_0)U^{-1}(t_1, t_0) \right] / Tr \rho(t_0)$$

(4)

The expressions above have a very intuitive meaning. To compute the ensemble average, take the initial state (or density matrix), evolve it forward in time from $t_0$ to $t$, insert the operator in question and evolve the state backwards in time back to $t_0$. For the correlation function ($t_1 > t_2$), evolve the initial state to $t_2$, insert the operator, evolve it further to $t_1$, insert the second operator, and finally evolve the state backwards to $t_0$. In most cases of interest the initial density matrix is either thermal or a pure state corresponding to the ground state of some initial Hamiltonian. In both cases the initial density matrix is of the form:

$$\rho(t_0) = e^{-\beta H_i};$$

(5)
the ground state of $H_i$ can be projected out by taking the limit $\beta \to \infty$. The system will undergo non-equilibrium evolution whenever the time evolution operator does not commute with the initial Hamiltonian $H_i$. It is convenient to introduce a time dependent Hamiltonian $H(t)$ such that $H(t) = H_i$ for $-\infty \leq t \leq t_0$ and $H(t) = H_{evol}(t)$ for $t > t_0$, where $H_{evol}(t)$ is the evolution Hamiltonian that determines the dynamics. This corresponds to the assumption that the system has been in equilibrium up to $t_0$ and will evolve out of equilibrium thereafter. Real time *equilibrium* correlation functions can be obtained if $H$ is constant in time for all times.

Correlation functions and ensemble averages can be obtained by considering evolution forward in time (from $t_0 \to -\infty$) insertion of operators and backwards to the original time, and finally down the imaginary time axis to $t_0 - i\beta$ to account for the initial thermal condition. Taking the trace amounts to identifying the initial and final field configurations and performing a functional integral over this configuration. Thus one is led to consider path integrals in the complex time plane.

Correlation functions or operator averages may be obtained as usual by coupling sources and taking functional derivatives with respect to them.

Thus we are led to consider the following generating functional, in terms of time evolution operators in presence of sources

$$Z[J^+, J^-, J^\beta] = Tr \left[ U(T - i\beta; T, J^\beta)U(T, T'; J^-)U(T', T; J^+) \right]$$

with $T \to -\infty; T' \to \infty$. The denominator in (6) is simply $Z[0, 0, 0]$ and may be obtained in a series expansion in the interaction by considering $Z[0, 0, J^\beta]$. By inserting a complete set of field eigenstates between the time evolution operators, finally, the generating functional $Z[J^+, J^-, J^\beta]$ may be written as

$$Z[J^+, J^-, J^\beta] = \int D\Phi D\Phi_1 D\Phi_2 \int D\Phi^+ D\Phi^- D\Phi^\beta e^{i \int_T^{T'} \left\{ \mathcal{L}[\Phi^+, J^+] - \mathcal{L}[\Phi^-, J^-] \right\}} \times$$

$$e^{i \int_{T}^{T' - i\beta_1} \mathcal{L}[\Phi^\beta, J^\beta]}$$

with the boundary conditions $\Phi^+(T) = \Phi^\beta(T - i\beta_1) = \Phi; \Phi^+(T') = \Phi^-(T') = \Phi_2; \Phi^-(T) = \Phi^\beta(T) = \Phi_1$. This may be recognized as a path integral along a contour in complex time. As usual the path integrals over the quadratic forms may be evaluated and one obtains the final result for the partition function

$$Z[J^+, J^-, J^\beta] = \exp \left\{ i \int_T^{T'} dt \left[ \mathcal{L}_{int}(-i\delta/\delta J^+) - \mathcal{L}_{int}(i\delta/\delta J^-) \right] \right\} \times$$

$$\exp \left\{ i \int_T^{T' - i\beta_1} dt \mathcal{L}_{int}(-i\delta/\delta J^\beta) \right\} \exp \left\{ \frac{i}{2} \int_{c} dt_1 \int_{c} dt_2 J_c(t_1)J_c(t_2)G_c(t_1, t_2) \right\}, \quad (8)$$
where \( J_c \) stands for the currents evaluated along the contour. The \( G_{c} \)'s are the Green's functions on the contour\(^8, 17, 18, 19\) (the spatial arguments we re suppressed).

In the limit \( T \to -\infty \), the contributions from the terms in which one of the currents is \( J^+ \) or \( J^- \) and the other is \( J^\beta \) vanish when computing correlation functions in which the external legs are at finite \textit{real time}, as a consequence of the Riemann-Lebesgue lemma. For this \textit{real time} correlation functions, there is no contribution from the \( J^\beta \) terms that cancel between numerator and denominator. Finite temperature enters through the boundary conditions on the Green’s functions. For the calculation of finite \textit{real time} correlation functions, the generating functional simplifies to\(^17, 8\)

\[
Z[J^+, J^-] = \exp \left\{ i \int_0^{T'} dt \left[ L_{\text{int}}(-i\delta/\delta J^+) - L_{\text{int}}(i\delta/\delta J^-) \right] \right\} \times \exp \left\{ i \int_T^{T'} dt_1 \int_T^{T'} dt_2 J_a(t_1) J_b(t_2) G_{ab}(t_1, t_2) \right\}
\]

(9)

with \( a, b = +, - \).

The Green’s functions that enter in the integrals along the contours in equations \(^8, 4\) are given by (see above references)

\[
\langle T \Phi(\vec{r}_1, t_1) \Phi(\vec{r}_2, t_2) \rangle = -i G^{++}(\vec{r}_1, t_1; \vec{r}_2, t_2) \tag{10}
\]

\[
\langle \hat{T} \Phi(\vec{r}_1, t_1) \Phi(\vec{r}_2, t_2) \rangle = -i G^{--}(\vec{r}_1, t_1; \vec{r}_2, t_2) \tag{11}
\]

\[
\langle \Phi(\vec{r}_1, t_1) \Phi(\vec{r}_2, t_2) \rangle = i G^{+-}(\vec{r}_1, t_1; \vec{r}_2, t_2) \tag{12}
\]

with \( T \) and \( \hat{T} \) the time ordering and anti-time ordering symbols. The Green’s functions above are written in terms of the homogeneous solutions to the quadratic form of the Lagrangian (free fields) \( G^\geq, G^\leq \) as

\[
G^{++}(\vec{r}_1, t_1; \vec{r}_2, t_2) = G^\geq(\vec{r}_1, t_1; \vec{r}_2, t_2) \Theta(t_1 - t_2) + G^{\leq}(\vec{r}_1, t_1; \vec{r}_2, t_2) \Theta(t_2 - t_1) \tag{13}
\]

\[
G^{--}(\vec{r}_1, t_1; \vec{r}_2, t_2) = G^\geq(\vec{r}_1, t_1; \vec{r}_2, t_2) \Theta(t_2 - t_1) + G^{\leq}(\vec{r}_1, t_1; \vec{r}_2, t_2) \Theta(t_1 - t_2) \tag{14}
\]

\[
G^{+-}(\vec{r}_1, t_1; \vec{r}_2, t_2) = -G^{\leq}(\vec{r}_1, t_1; \vec{r}_2, t_2) \tag{15}
\]

\[
G^{-+}(\vec{r}_1, t_1; \vec{r}_2, t_2) = -G^\geq(\vec{r}_1, t_1; \vec{r}_2, t_2) = -G^{\leq}(\vec{r}_2, t_2; \vec{r}_1, t_1) \tag{16}
\]

\[
G^\geq(\vec{r}_1, t_1; \vec{r}_2, t_2) = i \langle \Phi(\vec{r}_1, t_1) \Phi(\vec{r}_2, t_2) \rangle \tag{17}
\]

\[
G^{\leq}(\vec{r}_1, T; \vec{r}_2, t_2) = G^\geq(\vec{r}_1, T - i\beta; \vec{r}_2, t_2) \tag{18}
\]

The condition \(^18\) is recognized as the periodicity condition in imaginary time and is a result of considering an equilibrium situation for \( t < t_0 \). The functions \( G^\geq, G^{\leq} \), which are the homogeneous solutions of the quadratic form, with appropriate boundary conditions, will be constructed explicitly below.
The Feynman rules are the same as in regular perturbation theory, with the proviso that there are two interaction vertices corresponding to the $\pm$ branches, with opposite signs (arising from the complex conjugation of the unitary time evolution operator for evolution backwards in time).

This formulation in terms of time evolution along a contour in complex time has been used many times in non-equilibrium statistical mechanics. There are many clear articles in the literature using this techniques to study real time correlation functions\cite{8, 17, 19, 20, 21, 22} and effective actions out of equilibrium\cite{9, 23, 24, 25}.

This formulation has already been used by some of us previously to study the dynamics of phase transitions\cite{26}.

In our analysis we will take the $\beta \rightarrow \infty$ limit (zero temperature) following the argument provided in the introduction. We will also limit ourselves to a Minkowski space-time rather than expanding FRW space-time by considering phenomena whose time variation happens on scales much shorter than the expansion time $H^{-1}$ of the universe, where $H$ is the Hubble parameter.

### 2.1 Equations of motion:

The method briefly discussed above allows the derivation of the effective equations of motion for a coarse grained field. To focus the discussion, consider the situation of a scalar field theory with Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_{\mu} \Phi \partial^{\mu} \Phi - \frac{1}{2} m^2 \Phi^2 - \frac{\lambda}{4!} \Phi^4$$

(19)

The “coarse grained” field (order parameter) is defined as

$$\phi(t) = \langle \Phi(x,t) \rangle = \frac{Tr \Phi \rho(t)}{Tr \rho(t)}$$

(20)

where we have used translational invariance. We write the field as $\Phi(x,t) = \phi(t) + \psi(x,t)$ with $\psi(x,t)$ the fluctuations, obeying $\langle \psi(x,t) \rangle = 0$ and consider $\phi(t)$ as a background field in the evolution Hamiltonian. The non-equilibrium generating functional requires

$$\mathcal{L}[\phi+\psi^+] - \mathcal{L}[\phi+\psi^-] = \left\{ \frac{\delta \mathcal{L}}{\delta \phi} \psi^+ + \mathcal{L}_0[\psi^+] - \lambda \left( \frac{\phi^2(\psi^+)^2}{4} + \frac{\phi(\psi^+)^3}{6} + \frac{(\psi^+)^4}{4!} \right) \right\} - \{\psi^+ \rightarrow \psi^-\}$$

(21)

with $\mathcal{L}_0[\psi^\pm]$ the free field Lagrangian density of a field of mass $m$. We can now “integrate out” the fluctuations, thus obtaining the non-equilibrium effective action for the “coarse grained” background field.
The linear, cubic and quartic terms in $\psi^{\pm}$ are treated as perturbations, while the terms $\phi^2(t)(\psi^{\pm})^2$ may either be treated as perturbations, if one wants to generate a perturbative expansion in terms of the amplitude of the coarse grained field, or alternatively, they may be absorbed into a time dependent mass term for the fluctuations, if one wants to generate a loop expansion. We will now study both cases in detail.

### 2.2 Amplitude expansion

#### 2.2.1 Discrete Symmetry

We treat the term $\phi^2(t)(\psi^{\pm})^2$ as perturbation, along with the linear, cubic and quartic terms. The conditions

$$\langle \psi^{\pm}(x, t) \rangle = 0$$  \hspace{1cm} (22)

will give rise to the effective non-equilibrium equations of motion for the background field. This is the generalization of the “tadpole” method\[28\] to non-equilibrium field theory. Since the mass of the fluctuations is the bare mass, the essential ingredient for the non-equilibrium propagators introduced above are the spatial Fourier transforms of the homogeneous solutions to the free field quadratic forms:

$$G^>_k(t_1, t_2) = \frac{i}{2\omega_k}e^{-i\omega_k(t_1-t_2)}$$ \hspace{1cm} (23)

$$G^<_k(t_1, t_2) = \frac{i}{2\omega_k}e^{i\omega_k(t_1-t_2)}$$ \hspace{1cm} (24)

$$\omega_k = \sqrt{k^2 + m^2}.$$ \hspace{1cm} (25)

To $\mathcal{O}(\phi^3)$ the diagrams are shown in figures (1.a-1.h). Since the $(++)$ and the $(--)$ propagators are independent, the diagrams (1.a-1.h) finally lead to the equation of motion

$$\ddot{\phi}(t) + m^2 \phi(t) + \frac{\lambda}{6} \phi^3(t) + \frac{\lambda}{2} \phi(t) \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \int d^3k' \int_{-\infty}^{t} dt' \phi^2(t') \int \frac{d^3k}{(2\pi)^3} \frac{\sin[2\omega_k(t-t')]}{2\omega_k^2} = 0$$ \hspace{1cm} (26)

Notice that the diagrams of figures (1.e-1.f) contain the one-loop two-particle threshold typical of the two particle to two particle scattering amplitude. The last term in (26) is the real-time expression for this one-loop contribution. We can perform an integration by parts in the time integral, discarding the contribution at $t = -\infty$ by invoking an adiabatic
switching-on convergence factor to obtain
\[ \ddot{\phi}(t) + m^2 \phi(t) + \frac{\lambda}{6} \phi^3(t) + \frac{\lambda}{2} \dot{\phi}(t) \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \]
\[ - \frac{\lambda^2}{4} \dot{\phi}^2(t) \int \frac{d^3k}{(2\pi)^3} \frac{1}{4\omega_k^2} + \frac{\lambda^2}{4} \dot{\phi}(t) \int_{-\infty}^{t} dt' \phi(t') \dot{\phi}(t') \int \frac{d^3k}{(2\pi)^3} \cos[2\omega_k(t - t')] \]
\[ = 0. \quad (27) \]

The fourth term in the above equation is recognized as a mass renormalization and the fifth term as the coupling constant renormalization. Using a Fourier expansion for \( \phi(t) \) and its derivative, it is straightforward to see that after integration in the time variable, the remaining integration in \( k \) is \textit{ultraviolet finite}. There are several noteworthy features of this expression that point to a more complicated description of dissipative processes in field theory. The first such feature is an expected one. The “dissipative” contribution, that is, the last term in the above equation, has a non-Markovian (i.e. memory-retaining) kernel. Secondly, the equation is \textit{non-linear} in the amplitude of the coarse grained variable. These features are in striking contradiction with a simple \( \Gamma \dot{\phi} \) term in the equation of motion. Originally\[7\] such a term was argued to arise if the scalar field in question is coupled (linearly) to other fields in the theory and truly speaking such a situation does not arise within the present context. However the issue of a memory-kernel (non-Markovian) is quite general\[29\], and a legitimate question to ask is: is there a Markovian (local) limit of this kernel? Such a limit would imply that
\[ K(t - t') \approx D\delta(t - t') \quad (28) \]
with \( K(t - t') \) being the non-local kernel present in the last term of (27). However, we find that at small \( (t - t') \) the kernel has typical logarithmic divergences.

If a local approximation were valid, the “dissipative constant” \( D \) may be found by integrating the kernel in time. A straightforward calculation shows that infrared divergences give a divergent answer for such constant, clearly indicating that there is no Markovian limit for this kernel. In reference\[24\] a Markovian limit was argued to be available in the high temperature limit; since we are working at zero temperature the approximations invoked there do not apply. Thus, this lowest order calculation reveals two conclusive features that will persist in higher orders and even in non-perturbative calculations (see below): the “dissipative contribution” to the equations of motion obtained by integrating out the fluctuations are typically non-linear and, furthermore, they do not allow for a Markovian (local) description. We will postpone a numerical analysis of these equations until we study the full one-loop equations of motion below.
2.2.2 Broken symmetry

In the case of unbroken symmetry, the above equation of motion does not admit a linearization of the dissipative contribution. However, in the case when the symmetry is spontaneously broken such a linearization is possible[9]. In this case let \( m^2 = -\mu^2 \) and let us look for small oscillations around one of the minima. To achieve this, write \( \phi(t) = \sqrt{6\mu^2/\lambda} + \delta(t) \). The mass term for the fluctuations now becomes \( 2\mu^2 \), and we will only keep the linear terms in \( \delta \) in (21). We now follow the steps leading to the equation of motion obtained before. That is, \( \langle \Psi^\pm \rangle = 0 \) is imposed to one-loop order. After integration by parts in the time integral (again with an adiabatic switching-on convergence factor) there appear several tadpole contributions. Those that are independent of \( \delta \) renormalize \( \phi_0 \) corresponding to a shift of the position of the vacuum expectation value, while those that are linear in \( \delta \) renormalize the mass. We thus obtain in the linearized approximation:

\[
\ddot{\delta} + 2\mu_R^2 \delta + \frac{3\mu_R^2}{2} \lambda_R \int_{-\infty}^{t} \dot{\delta}(t') \int \frac{d^3 k}{(2\pi)^3} \frac{\cos[2\tilde{\omega}_k(t-t')]}{2\tilde{\omega}_k^3} = 0
\]  

(29)

with \( \tilde{\omega}_k = \sqrt{k^2 + 2\mu^2} \). This expression is similar to that found by Paz[9]. In that reference the \textit{short time} behavior was analyzed. In order to solve this equation for all times, we must specify an initial condition. We will \textit{assume} that for \( t < 0 \) \( \delta(t < 0) = \delta_i \) and \( \dot{\delta}(t < 0) = 0 \) and that equation (29) holds for \( t > 0 \)[30].

Under this assumption, the linearized equation can be solved via the Laplace transform. This solution corresponds to summing Dyson’s series for the propagator with the one-loop contributions depicted in figure (1) (for the linearized case). The Laplace transform exhibits the two-particle cut[8,9] in the analytic continuation of the transform variable \( s \). However the inverse Laplace transform proves to be extremely difficult to carry out. We will content ourselves at this stage with a perturbative solution to \( \mathcal{O}(\lambda) \). Thus we write

\[
\delta(t) = \delta_i \cos[\sqrt{2}\mu_R t] + \lambda \delta_1(t)
\]

(30)

The equation is then solved order by order in \( \lambda \). The corresponding equation for \( \delta_1(t) \) is now easily solved via the Laplace transform. The transform exhibits resonances at \( \pm \sqrt{2}\mu_R \) and the large time behavior is dominated by the secular term \( \delta_1(t) \approx t \sin[\sqrt{2}\mu_R t] \). Thus the conclusion from the linearized approximation is that it is valid only for \textit{short times}; the long time behavior will not be captured by this perturbative linearized approximation. This conclusion will be confirmed with a numerical analysis of the full one-loop case below.

The analysis provided in references[8,9,23] in which a dissipative behavior was observed is not quite consistent, because their solution contains higher order terms in the coupling
that are not warranted in the approximation considered. Furthermore, even if the implied resummation is accepted the solution found by these authors is only valid at very short times, corresponding to the region in the s-plane (Laplace transform variable) far away from the resonances. The physics of this effect is clear if analyzed in terms of the Fourier transform. At short times the contribution of large frequencies is substantial and the two-particle threshold is available for dissipation. However at long times only low frequencies have an important contribution, the two-particle threshold is not available and the resonances below threshold dominate giving a growing amplitude.

Furthermore, even in this case in which there is an explicit linear velocity dependence in the “dissipative kernel”, the same argument presented before applies. There is no Markovian (local) limit in which this term becomes simply $\Gamma \delta$, despite the fact that $\delta$ has a linear coupling to the fluctuations.

### 2.2.3 Continuum symmetries and Goldstone bosons

An important issue that we want to study in detail in this article is the process of dissipation by Goldstone bosons. As discussed in the introduction, dissipation via particle production becomes effective whenever the transferred energies become larger than multiparticle thresholds. To lowest order, as shown in the perturbative calculation of the previous section, the lowest threshold is the two-particle one, at an energy twice the mass of the particle. In the case of a spontaneously broken, continuous symmetry, there will be Goldstone bosons and the thresholds are at zero energy. In this case any amount of energy transfer may be dissipated by the Goldstone modes.

This effect may be studied in the $O(2)$ linear sigma model with Lagrangian density

$$
\mathcal{L} = \frac{1}{2} \partial_{\mu} \sigma \partial^{\mu} \sigma + \frac{1}{2} \partial_{\mu} \pi \partial^{\mu} \pi + \frac{1}{2} \mu^{2} (\sigma^{2} + \pi^{2}) - \frac{\lambda}{4!} (\sigma^{2} + \pi^{2})^{2}
$$

(31)

We now write

$$
\sigma(x, t) = \sqrt{\frac{6 \mu^{2}}{\lambda}} + \delta(t) + \chi(x, t)
$$

(32)

and use the tadpole method to impose:

$$
\langle \pi(x, t) \rangle = 0 \quad ; \quad \langle \chi(x, t) \rangle = 0
$$

(33)

Carrying out the same analysis in terms of Feynman diagrams to linear order in $\delta(t)$ we find the following equation of motion (after integrating by parts invoking an adiabatic switching on convergence factor and absorbing the local terms in proper renormalizations)

$$
\ddot{\delta} + 2 \mu_{R}^{2} + 6 \mu_{R}^{2} \lambda_{R} \int_{-\infty}^{t} dt' \partial^{2}(t') \left\{ \frac{1}{4} \mathcal{K}_{\chi}(t - t') + \frac{1}{36} \mathcal{K}_{\pi}(t - t') \right\} = 0
$$

(34)

13
with $\mathcal{K}_\chi(t - t')$ the same kernel as in equation (29) and

$$
\mathcal{K}_\pi(t - t') = \int \frac{d^3k}{(2\pi)^3} \cos[2|k|(t - t')] \approx \ln[M(t - t')]
$$

(35)

where $M$ is an infrared cutoff introduced to define the integral. This infrared divergence is the result of the fact that the threshold for Goldstone bosons is at zero energy-momentum. This result clearly reflects many important features of “dissipation” via Goldstone bosons. First, as before, the dissipative kernel cannot be described in a local (Markovian) approximation even in this linearized theory in which the arguments leading to a $\Gamma\dot{\phi}$ term would be valid. Second and perhaps more important, the long time behavior is clearly beyond perturbation theory as the contribution from the non-local kernels is unbounded as a result of infrared divergences associated with Goldstone bosons.

### 2.3 One-loop equations

The full one-loop equations of motion are obtained by absorbing the terms $\phi^2(t)(\psi^\pm)^2$ in eq.(21) as a time dependent mass term for the fluctuating fields. In order to determine the dynamics for the background field, we will assume that $\phi(t < 0) = \phi_0$ and that at $t = 0$ the background field is “released” with zero velocity. Now we must find the corresponding non-equilibrium Green’s functions. Consider the following homogeneous solutions of the quadratic form for the fluctuations

$$
\left[ \frac{d^2}{dt^2} + \vec{k}^2 + m^2 + \frac{\lambda}{2} \phi^2(t) \right] U_\pm^k(t) = 0
$$

(36)

$$
U_\pm^k(0) = 1 ; \quad \dot{U}_\pm^k(0) = \mp i\omega_0^k
$$

(37)

with

$$
\omega_0^k = \left[ \vec{k}^2 + m^2 + \frac{\lambda}{2} \phi^2(0) \right]^{\frac{1}{2}}.
$$

(38)

The boundary conditions (37) correspond to positive $U^+$ and negative $U^-$ frequency modes for $t < 0$ (the Wronskian of these solutions is $2i\omega_0^k$). Notice that $U_k^-(t) = [U_k^+(t)]^*$. In terms of these mode functions we obtain

$$
G^>(t_1, t_2) = \frac{i}{2\omega_0^k}U_k^+(t_1)U_k^-(t_2)
$$

(39)

Now the equation of motion to one-loop is obtained from the diagrams in figures (1.a), (1.d) (but with the modified propagator including the contribution of $\phi(t)$). Thus to one-loop we
find the following equations

\[ \ddot{\phi}(t) + m^2 \phi(t) + \frac{\lambda}{2} \phi^3(t) + \frac{\Lambda R^6}{6} \phi(t) \int \frac{d^3k}{(2\pi)^3} \frac{|U_k^+(t)|^2}{2\omega_k^0} = 0 \quad (40) \]

\[ \left[ \frac{d^2}{dt^2} + k^2 + m^2 + \frac{\lambda}{2} \phi^2(t) \right] U_k^+(t) = 0 \quad (41) \]

\[ U_k^+(0) = 1 ; \quad \dot{U}_k^+(0) = -i\omega_k^0 \quad (42) \]

where we have restored the \( \hbar \) to make the quantum corrections explicit. This set of equations clearly shows how the expectation value (coarse grained variable) “transfers energy” to the mode functions via a time dependent frequency, which then in turn modify the equations of motion for the expectation value. The equation for the mode functions, (41) may be solved in a perturbative expansion in terms of \( \lambda \phi^2(t) \) involving the retarded Green’s function. To first order in that expansion one recovers eq. (26).

Before attempting a numerical solution of the above equations it is important to understand the renormalization aspects. For this we need the large \( k \) behaviour of the mode functions which is obtained via a WKB expansion as in references [26, 27] and to which the reader is referred to for details. We obtain

\[ \int \frac{d^3k}{(2\pi)^3} \frac{|U_k^+(t)|^2}{2\omega_k^0} = \frac{\Lambda^2}{8\pi} - \frac{1}{8\pi} \left[ m^2 + \frac{\lambda}{2} \phi^2(t) \right] \ln \left[ \frac{\Lambda}{\kappa} \right] + \text{finite} \quad (43) \]

where \( \Lambda \) is an ultraviolet cutoff and \( \kappa \) an arbitrary renormalization scale. From the above expression it is clear how the mass and coupling constant are renormalized. It proves more convenient to subtract the one-loop contribution at \( t = 0 \) and absorb a finite renormalization in the mass, finally obtaining the renormalized equation of motion

\[ \ddot{\phi}(t) + m_R^2 \phi(t) + \frac{\lambda_R}{6} \phi^3(t) + \frac{\lambda_R \hbar}{8\pi^2} \phi(t) \int_0^\Lambda k^2 dk \frac{|U_k^+(t)|^2 - 1}{\omega_k^0} + \frac{\lambda_R^2 \hbar}{32\pi^2} \phi(t)(\phi^2(t) - \phi^2(0)) \ln \left[ \frac{\Lambda}{\kappa} \right] = 0 \quad (44) \]

In the equations for the mode functions the mass and coupling may be replaced by the renormalized quantities to this order. One would be tempted to pursue a numerical solution of these coupled equations. However doing so would not be consistent, since these equations were obtained only to order \( \hbar \) and a naive numerical solution will produce higher powers of \( \hbar \) that are not justified.

Within the spirit of the loop expansion we must be consistent and only keep terms of order \( \hbar \). First we introduce dimensionless variables

\[ \eta(t) = \sqrt{\frac{\lambda_R}{6m_R^2}} \phi(\tau) ; \quad \tau = m_R t ; \quad q = \frac{k}{m_R} ; \quad g = \frac{\lambda_R \hbar}{8\pi^2} \quad (45) \]
and expand the field in terms of \( g \) as

\[
\eta(\tau) = \eta_{cl}(\tau) + g\eta_1(\tau) + \cdots
\]  

(46)

Now the equations of motion consistent up to \( \mathcal{O}(\hbar) \) become

\[
\ddot{\eta}_{cl}(\tau) + \dot{\eta}_{cl}(\tau) + \eta_{cl}^2(\tau) = 0 \\
\ddot{\eta}_1(\tau) + \dot{\eta}_1(\tau) + 3\eta_{cl}^2(\tau)\eta_1(\tau) + \eta_{cl}(\tau) \int_0^{\Lambda/m_R} q^2 dq \frac{[|U_{q}^+|]^2 - 1}{[q^2 + 1 + 3\eta_{cl}^2(0)]^{1/2}} + \frac{3}{2}\eta_{cl}(\tau) \left( \eta_{cl}^2(\tau) - \eta_{cl}^2(0) \right) \ln[\Lambda/m_R] = 0
\]  

(47)

(48)

The solution to equation (47) is an elliptic function. The equations for the mode functions become

\[
\left[ \frac{d^2}{d\tau^2} + q^2 + 1 + 3\eta_{cl}^2(\tau) \right] U_q^+(\tau) = 0  
\]  

(49)

with the boundary conditions as in eq.(42) in terms of the dimensionless frequencies and where for simplicity, we have chosen the renormalization scale \( \kappa = m_R \). The chosen boundary conditions \( \eta(0) = \eta_0 \); \( \dot{\eta}(0) = 0 \) can be implemented as

\[
\eta_{cl}(0) = \eta_0 \ ; \ \dot{\eta}_{cl}(0) = 0 \ ; \ \eta_1(0) = 0 \ ; \ \dot{\eta}_1(0) = 0
\]  

(50)

In fig.(2) we show \( \eta_1(\tau) \) with the above boundary conditions with \( \eta_0 = 1 \) and \( g = 0.1 \). A cutoff \( \Lambda/m_R = 100 \) was chosen but no cut-off sensitivity was detected by varying the cutoff by a factor 3. Notice that the amplitude grows as a function of time. This phenomenon can be understood as follows. A first hint was obtained in the case of the linearized equations for a broken discrete symmetry. There we learned that because of resonances below the two-particle threshold, the amplitude grows at long times. This is the behavior shown in fig.(2). An alternative and perhaps more convincing argument is the following. The mode functions \( U_q^+(\tau) \) obey a Schröedinger-like equation with a potential that is a periodic function of time because the classical solution is periodic. Let us call the period of the classical solution \( T \). Floquet’s theorem guarantees the existence of solutions that obey

\[
U_q^+(\tau + T) = e^{\mu T}U_q^+(\tau)
\]  

(51)

The Floquet indices \( \mu \) are functions of the parameters of the potential. The classical solution is an elliptic function, and in this case the Schröedinger equation for the modes may be shown to be a Lamé equation with \( n = 2 \) (see ref.[43]) whose solutions are Weierstrass functions. That is, a two-zone potential, with two forbidden and two allowed bands for \( q > 0 \). The
Floquet indices $\mu$ are pure imaginary for large $q$, but they are real and positive for $q$ near zero. That is, the long wavelengths belong to an unstable band.

A more intuitive understanding at a simpler level ensues if we look at small oscillations near the origin for the classical solution $\eta_{cl}(\tau) \approx \eta_{cl}(0) \cos(\tau)$. Then the Schrödinger equation for the modes becomes a Mathieu equation\[31\], for which the dependence of the Floquet indices on the parameters $(q; \eta_{cl}(0))$ is known\[31\]. There are unstable bands in which the Floquet index has positive real part for certain values of these parameters.

Since the one-loop correction involves an integral over all wave-vectors, the values of $q$ in these bands give growing contributions to the one-loop integral.

These instabilities of the one-loop equations preclude a perturbative analysis of the process of dissipation.

### 2.3.1 Goldstone bosons at one-loop

It proves illuminating to study the one-loop contribution to the equations of motion for the scalar field expectation value from Goldstone bosons. We proceed as in the previous case but now with the Lagrangian density of the O(2) linear sigma model \[31\]. We now write

\[ \sigma(x, t) = \sigma_0(t) + \chi(x, t) \]

\[ \langle \pi(x, t) \rangle = 0 ; \langle \chi(x, t) \rangle = 0 \]

where the terms $\chi^2 \sigma_0^2$; $\pi^2 \sigma_0^2$ are now included with the mass terms for the respective fields.

The procedure is exactly the same as in the previous case, but we now use $\mu_R$ as the mass scale and introduce the dimensionless variables of eq.(45) in terms of this scale. Performing the proper renormalizations and a subtraction at $t = 0$ we finally find the following equations for this case

\[ \frac{d^2}{d\tau^2} \eta - \eta + \eta^3 + g\eta \left\{ \int_0^{\Lambda/\mu_R} \frac{q^2}{W_\chi(q)} \left[ |U_{q}^+(\tau)|^2 - 1 \right] + \frac{3}{2} \ln(\Lambda/\mu_R)(\eta^2(\tau) - \eta^2(0)) \right\} \\
+ \frac{g}{3} \eta \left\{ \int_0^{\Lambda/\mu_R} \frac{q^2}{W_\pi(q)} \left[ |V_{q}^+(\tau)|^2 - 1 \right] + \frac{1}{2} \ln(\Lambda/\mu_R)(\eta^2(\tau) - \eta^2(0)) \right\} = 0 \]

The mode functions satisfy the equations

\[ \left[ \frac{d^2}{d\tau^2} + q^2 - 1 + 3\eta^2(\tau) \right] U_{q}^+(\tau) = 0 \]

\[ \left[ \frac{d^2}{d\tau^2} + q^2 - 1 + \eta^2(\tau) \right] V_{q}^+(\tau) = 0 \]
where the boundary conditions are the same as in (33) but in terms of the initial frequencies \( W_\chi(q) ; W_\pi(q) \). As argued previously, for consistency we have to expand the field as in eq. (46) and keeping only \( \eta_{cl} \) in the equation for the modes. Notice that whereas the minimum of the tree level potential corresponds to \( \eta_{cl}^2 = 1 \) and the stable region for the \( U \) modes is \( \eta_{cl}^2 > 1/3 \), the stable region for the \( V \) modes (pion field) is for \( \eta_{cl}^2 > 1 \). These instabilities require the choice of special initial conditions, in particular for the initial frequencies. We choose

\[
W_\chi(q) = \sqrt{q^2 + 1 + 3\eta^2(0)} \quad (57)
\]

\[
W_\pi(q) = \sqrt{q^2 + 1 + \eta^2(0)} \quad (58)
\]

This choice corresponds to a gaussian initial state centered at \( \eta(0) \) but with positive frequencies at \( \eta(0) = 0 \). This state evolves in a broken symmetry potential from the position determined by \( \eta(0) \). Alternatively, one can think of this situation as changing the sign of the mass at time \( t = 0 \) from positive to negative. For small oscillations of \( \eta_{cl} \) around the minimum at \( \eta_{min} = 1 \), every time that the classical field oscillates to the left of the minimum the “pion” field becomes unstable and grows, and thus its contribution to the fluctuations becomes large. This behavior is depicted in fig.(3) which shows \( \eta_1(\tau) \) with the boundary conditions as in (34) and the initial frequencies (57,58) for the mode functions where chosen with \( \eta_{cl}(0) = 0.6 \) and \( \Lambda/\mu_R = 100 \) but no cutoff sensitivity was detected by increasing the cutoff by a factor of 2. This situation clearly exhibits Floquet indices with a positive real part, and \( \eta_{cl}(\tau) \) is periodic for small oscillations around \( \eta_{cl} = 1 \). The unstable wave vectors form a band when \( \eta_{cl} \) oscillates around this value. In each period of \( \eta_{cl} \) the amplitude grows because of these unstable modes.

This figure clearly shows a dramatic growth in the amplitude of the quantum correction at long times as a consequence of the instabilities associated with the Goldstone mode. As a consequence of this, perturbation theory must fail at long times.

### 2.3.2 The Langevin equation

A fundamental aspect of dissipation is that of decoherence which plays an important role in studies of quantum cosmology\cite{25}. Furthermore dissipation and fluctuations are related by the fluctuation-dissipation theorem. This point has been stressed by Hu and collaborators\cite{32}. The relation between dissipative kernels and fluctuations and correlations of bath degrees of freedom is best captured in a Langevin equation.

In this section we present a derivation of the corresponding Langevin equation to one-loop order in an amplitude expansion. The first step towards deriving the Langevin equation is to
determine the “system” and “bath” variables, once this is done, one integrates out the “bath” variables obtaining a (non-local) influence functional [25, 33, 34] for the system variables.

Starting with the generating functional of eq.(7) (for the case of zero temperature $\beta \to \infty$) we separate the zero mode ($\phi^\pm$) from the field as

$$\Phi^\pm = \phi^\pm + \psi^\pm ; \quad \int d^3x \psi^\pm(x,t) = 0$$

(59)

and consider $\phi$ as the “system” and $\psi$ as the “bath” to be integrated out. Because of the condition in eq.(59), there are no terms linear in $\psi^\pm$ in the expansion of the action in terms of these fields. Thus

$$S[\phi^+, \psi^+] - S[\phi^-, \psi^-] = \Omega \left( S[\phi^+] - S[\phi^-] \right) + S_0[\psi^+] - S_0[\psi^-]$$

$$- \frac{\lambda}{4} \left[ (\phi^+(t))^2 \int d^3x (\psi^+(x,t))^2 - (\phi^-(t))^2 \int d^3x (\psi^-(x,t))^2 \right]$$

$$+ \mathcal{O}( (\psi^\pm)^3, (\psi^\pm)^4 )$$

(60)

where $\Omega$ is the spatial volume, $S$ the action per unit spatial volume, $S_0$ the free field action and the terms $\mathcal{O}( (\psi^\pm)^3, (\psi^\pm)^4 )$ will contribute at the two-loop level and beyond. To this order, the coupling between “bath” and “system” is similar to the bi-quadratic coupling considered in reference [36] (see also [25]). Integrating out the $\psi^\pm$ fields in a consistent loop expansion gives rise to the influence functional [25, 33, 34] for the zero modes. The one loop diagrams contributing to this functional up to $\mathcal{O}( (\phi^\pm)^4 )$ are shown in fig.(4). In order to obtain the Langevin equation it is convenient to introduce the center of mass ($\phi(t)$) and relative ($R(t)$) coordinates (these are the coordinates used in the Wigner transform of the coordinate density matrix) as [32, 34, 33]

$$\phi^\pm(t) = \phi(t) \pm \frac{R(t)}{2}$$

(61)

Using the Green’s functions in eqs.(10-17) with (23,24) and being patient with the algebra we find the effective action per unit spatial volume
\[ S_{\text{eff}}[\phi, R] = \int_{-\infty}^{\infty} dt \left\{ \mathcal{L}[\phi + R/2] - \mathcal{L}[\phi - R/2] - \frac{\lambda}{2} R(t) \phi(t) \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \right. \\
\left. + \frac{\lambda^2}{4} R(t) \phi(t) \int_t^{\infty} dt' \phi^2(t') \int \frac{d^3k}{(2\pi)^3} \frac{\sin[2\omega_k(t - t')]}{2\omega_k^2} \right\} \\
+ \frac{i\lambda^2}{8} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' R(t) R(t') \phi(t) \phi(t') \int \frac{d^3k}{(2\pi)^3} \frac{\cos[2\omega_k(t - t')]}{2\omega_k^2} \\
+ O(R^3; R^4...). \] (62)

The higher order terms \( O(R^3; ... ) \) receive contributions from two and higher loops and give higher order corrections to the lowest order (one-loop) Langevin equation. The imaginary part of the effective action above (last non-local term) gives a contribution to the path integral that may be written in terms of a stochastic field as

\[ \exp \left[ -\frac{1}{2} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' R(t) K(t, t') R(t') \right] \propto \int \mathcal{D}\xi \mathcal{P}[\xi] \exp \left[ i \int_{-\infty}^{\infty} dt \xi(t) R(t) \right] \] (63)

\[ \mathcal{P}[\xi] = \exp \left[ -\frac{1}{2} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \xi(t) K^{-1}(t, t') \xi(t') \right] \] (64)

with

\[ K(t, t') = \frac{\lambda^2}{4} \phi(t) \phi(t') \int \frac{d^3k}{(2\pi)^3} \frac{\cos[2\omega_k(t - t')]}{2\omega_k^2} \] (65)

The non-equilibrium path integral now becomes (keeping track of volume factors)

\[ Z \propto \int \mathcal{D}\phi \mathcal{D}R \mathcal{D}\xi \mathcal{P}[\xi] \exp \left\{ i\Omega \left[ S_{\text{eff}}[\phi, R] + \int dt \xi(t) R(t) \right] \right\} \] (66)

with \( S_{\text{eff}}[\phi, R] \) the real part of the effective action in (62) and with \( \mathcal{P}[\xi] \) the gaussian probability distribution for the stochastic noise variable given in (63).

The Langevin equation is obtained via the saddle point condition\[2, 32, 34, 35\]

\[ \frac{\delta S_{\text{eff}}}{\delta R(t)} \bigg|_{R=0} = \xi(t) \] (67)

leading to

\[ \ddot{\phi}(t) + m^2 \phi(t) + \frac{\lambda}{6} \phi^3(t) = \frac{\lambda^2}{2} R(t) \phi(t) \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \]

\[ - \frac{\lambda^2}{4} \phi(t) \int_{-\infty}^{t} dt' \phi^2(t') \int \frac{d^3k}{(2\pi)^3} \frac{\sin[2\omega_k(t - t')]}{2\omega_k^2} = \xi(t) \] (68)

where the stochastic noise variable \( \xi(t) \) has gaussian correlations

\[ << \xi(t) >> = 0 ; << \xi(t) \xi(t') >> = K(t, t') = \frac{\lambda^2}{4} \phi(t) \phi(t') \int \frac{d^3k}{(2\pi)^3} \frac{\cos[2\omega_k(t - t')]}{2\omega_k^2} \] (69)

20
Here, the double brackets stand for averages with respect to the gaussian probability distribution $\mathcal{P}[\xi]$. We can see that the noise is colored (not delta function correlated) and multiplicative. By integrating by parts the “dissipative kernel” in eq.(68) (the last non-local term) in the same way as done in eq.(27), we can clearly see that the resulting “dissipative kernel” and the noise correlation function obey a generalized fluctuation-dissipation theorem\cite{25,32}. In the broken symmetry case, in the linearized approximation (around the tree level minimum) and if the k-integral could be replaced by a delta function, we would obtain the usual fluctuation-dissipation relation. By taking the average over the stochastic noise of (68) with the gaussian probability distribution (64) one obtains the equation of motion (27) by replacing the average of product of fields by the product of the averages (thus considering the field as a classical background).

The higher order terms in the effective action (influence functional) give rise to modifications to the noise correlations, making them non-gaussian and involving more powers of $\phi$ in the kernels, in principle these corrections may be computed systematically in a loop expansion.

Although this Langevin equation clearly exhibits the generalized fluctuation-dissipation theorem connecting the “dissipative” kernel to the correlations of the stochastic noise, it is a hopeless tool for any evaluation of the dynamics. The long range kernels and the multiplicative nature of the noise prevent this Langevin equation from becoming a useful tool. Its importance resides at the fundamental level in that it provides a direct link between fluctuation and dissipation including all the memory effects and multiplicative aspects of the noise correlation functions. This last correlation function is related to the decoherence functional\cite{25}.

**2.3.3 Failure of perturbation theory to describe dissipation**

This section has been devoted to a perturbative analysis of the “dissipative aspects” of the equation of motion for the scalar field. Perturbation theory has been carried out as an amplitude expansion and also up to $\mathcal{O}(\hbar)$, both in the broken and the unbroken symmetry case. In both cases we found both analytically and numerically that the amplitude of the quantum corrections grow as a function of time and that the long time behavior cannot be captured in perturbation theory. This failure of perturbation theory to describe dissipation is clearly understood from a very elementary but yet illuminating example: the damped harmonic oscillator. Consider a damped harmonic oscillator

$$\ddot{q} + \Gamma \dot{q} + q = 0$$  \hspace{1cm} (70)
with $\Gamma \sim \mathcal{O}(\lambda)$ where $\lambda$ is a small perturbative coupling. One can attempt to solve eq. (70) in a perturbative expansion in $\Gamma$. That is, set $q(t) = q_0(t) + \Gamma q_1(t) + \cdots$. The solution for $q_1(t)$ may be found by the Laplace transform:

$$q_1(t) \propto \Gamma t \cos(t)$$

Clearly exhibiting resonant behavior. This is recognized as a secular term. If eq. (70) had been obtained as an effective equation of motion in a perturbative expansion in $\lambda$, this would be the consistent manner to solve this equation. However, we would be led to conclude that perturbation theory breaks down at long times. The correct solution is

$$q(t) = e^{-\Gamma^2 t} \cos[\omega(\Gamma^2)t] \approx \cos(t) - \frac{\Gamma}{2} t \cos(t) + \mathcal{O}(\Gamma^2).$$

(72)

We see that the first order correction in $\Gamma$ is correctly reproduced by perturbation theory, but in order to find appreciable damping, we must wait a time $\sim \mathcal{O}(1/\Gamma)$ at which perturbation theory becomes unreliable.

In order to properly describe dissipation and damping one must resum the perturbative expansion. One could in principle keep the first order correction and exponentiate it in an ad-hoc manner with the hope that this would be the correct resummation. Although ultimately this may be the correct procedure, it is by no means warranted in a field-theoretic perturbative expansion, since in field theory, dissipation is related to particle production and open multiparticle channels, both very subtle and non-linear mechanisms. Another hint that points to a resummation of the perturbative series is provided by the set of equations eqs. (47-49). In eq. (47), the classical solution is a periodic function of time of constant amplitude, since the classical equation has a conserved energy. As a consequence, the “potential” in the equation for the modes (eq. (49)) is a periodic function of time with constant amplitude. Thus although the fluctuations react back on the coarse grained field, only the classically conserved part of the motion of the coarse grained field enters in the evolution equations of the mode functions. This is a result of being consistent with the loop expansion, but clearly this approximation is not energy conserving.

As we will point out in the next section, in an energy conserving scheme the fluctuations and amplitudes will grow up to a maximum value and then will always remain bounded at all times.

Thus in summary for this section, we draw the conclusion that perturbation theory is not sufficient (without major ad-hoc assumptions) to capture the physics of dissipation and damping in real time. A resummation scheme is needed that effectively sums up the whole (or partial) perturbative series in a consistent and/or controlled manner, and which provides a reliable estimate for the long-time behavior.
The next section is devoted to the analytical and numerical study of some of these schemes.
3 Non-perturbative schemes I: Hartree approximation

Motivated by the failure of the loop and amplitude expansions, we now proceed to consider the equations of motion in some non-perturbative schemes. First we study a single scalar model in the time dependent Hartree approximation. After this, we study an O(N) scalar theory in the large N limit. This last case allows us to study the effect of Goldstone bosons on the time evolution of the order parameter.

In a single scalar model described by the Lagrangian density of eq. (19), the Hartree approximation is implemented as follows. We again decompose the fields as $\Phi^\pm = \phi + \psi^\pm$ and the Lagrangian density is given by eq.(21). The Hartree approximation is obtained by assuming the factorization (for both $\psi^\pm$ components)

$$
\psi^3(\vec{x}, t) \rightarrow 3(\psi^2(\vec{x}, t))\psi(\vec{x}, t)
$$

$$
\psi^4(\vec{x}, t) \rightarrow 6(\psi^2(\vec{x}, t))\psi^2(\vec{x}, t) - 3(\psi^2(\vec{x}, t))^2
$$

Translational invariance shows that $(\langle (\psi^\pm(\vec{x}, t))^2(\vec{x}, t) \rangle)$ can only be a function of time, and because this is an equal time correlation function, we have that:

$$
\langle (\psi^+(\vec{x}, t))^2 \rangle = \langle (\psi^-(\vec{x}, t))^2 \rangle \overset{\text{def}}{=} \langle \psi^2(t) \rangle
$$

The expectation value will be determined within a self-consistent approximation. After this factorization we find:

$$\mathcal{L}[\phi + \psi^+] - \mathcal{L}[\phi - \psi^-] = \mathcal{L}[\phi] + \left\{ \left( \frac{\delta \mathcal{L}}{\delta \phi} - \frac{1}{2} \lambda \phi(\psi^2(t)) \right) \psi^+ + \frac{1}{2} \left( \partial_\mu \psi^+ \right)^2 - \frac{1}{2} M^2(t)(\psi^+)^2 \right\} - \left\{ \langle \psi^+ \rightarrow \psi^- \rangle \right\},$$

where

$$M^2(t) = V''(\phi) + \frac{\lambda}{2} \langle \psi^2(t) \rangle = m^2 + \frac{\lambda}{2} \phi^2(t) + \frac{\lambda}{2} \langle \psi^2(t) \rangle$$

The resulting Hartree equations are obtained by using the tadpole method $\langle \psi^\pm(\vec{x}, t) \rangle = 0$ as before. They are given by:

$$\ddot{\phi} + m^2 \phi + \frac{\lambda}{6} \dot{\phi}^3 + \frac{\lambda}{2} \phi \langle \psi^2(t) \rangle = 0$$

$$\langle \psi^2(t) \rangle = \int \frac{d^3k}{(2\pi)^3} [-iG^<_{\vec{k}}(t, t)] = \int \frac{d^3k}{(2\pi)^3} \frac{|U^+_{\vec{k}}(t)|^2}{2\omega_k(0)}$$

$$\left[ \frac{d^2}{dt^2} + \omega_k^2(t) \right] U^+_{\vec{k}}(t) = 0 ; \quad \omega_k^2(t) = k^2 + \mathcal{M}^2(t)$$
The initial conditions for the mode functions are
\[ U^+_k(0) = 1 ; \quad \dot{U}^+_k(0) = -i\omega_k(0) \] (80)

It is clear that the Hartree approximation makes the Lagrangian density quadratic at the expense of a self-consistent condition. In the time independent case, this approximation sums up all the “daisy” (or “cactus”) diagrams and leads to a self-consistent gap equation.

At this stage, we must point out that the Hartree approximation is uncontrolled in this single scalar theory. This approximation does, however, become exact in the \( N \to \infty \) limit of the \( O(N) \) model which we will discuss in the next section.

3.1 Renormalization

We now analyze the renormalization aspects within the Hartree approximation. To study the renormalization we need to understand the divergences in this integral
\[ \langle \psi^2(t) \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{|U^+_k(t)|^2}{2\omega_k(0)} \] (81)

The divergences will be determined from the large-\( k \) behavior of the mode functions which obey the differential equations obtained from (79) with the initial conditions (80). By a WKB-type analysis (see [26, 27] for a detailed description), in the \( k \to \infty \) limit, we find
\[ \frac{|U^+_k(t)|^2}{2\omega_k(0)} = \frac{1}{k} - \frac{1}{4k^3} \left[ m^2 + \frac{\lambda}{2} \phi^2(t) + \frac{\lambda}{2} \langle \psi^2(t) \rangle \right] + O\left(\frac{1}{k^4}\right) + \cdots \] (82)

Inserting these results in the integral, it is straightforward to find the divergent terms and we find
\[ \int \frac{d^3k}{(2\pi)^3} \frac{|U^+_k(t)|^2}{2\omega_k(0)} = \frac{1}{8\pi^2} A^2 - \frac{1}{8\pi^2} \ln \left( \frac{A}{\kappa} \right) \left[ m^2 + \frac{\lambda}{2} \phi^2(t) + \frac{\lambda}{2} \langle \psi^2(t) \rangle \right] + \text{finite} \] (83)

where \( A \) is an upper momentum cutoff and \( \kappa \) a renormalization scale.

Now, we are in position to specify the renormalization prescription within the Hartree approximation. In this approximation, there are no interactions, since the Lagrangian density is quadratic. The nonlinearities are encoded in the self-consistency conditions. Because of this, there are no counterterms with which to cancel the divergence and the differential equation for the mode functions must be finite. Therefore, it leads to the following renormalization prescription:
\[ m_B^2 + \frac{\lambda_B}{2} \phi^2(t) + \frac{\lambda_B}{2} \langle \psi^2(t) \rangle_B = m_R^2 + \frac{\lambda_R}{2} \phi^2(t) + \frac{\lambda_R}{2} \langle \psi^2(t) \rangle_R \] (84)
where the subscripts $B, R$ refer to the bare and renormalized quantities respectively and $\langle \psi^2(t) \rangle_B$ is read from eq.(83):

$$\langle \psi^2(t) \rangle_B = \frac{1}{8\pi^2} \Lambda^2 - \frac{1}{8\pi^2} \ln \left( \frac{\Lambda}{\kappa} \right) \left[ m_R^2 + \frac{\lambda_R}{2} \phi^2(t) + \frac{\lambda_R}{2} \langle \psi^2(t) \rangle_R \right] + \text{finite} \quad (85)$$

Using this renormalization prescription eq.(84), we obtain

$$m_B^2 + \frac{\lambda_B}{16\pi^2} \Lambda^2 = m_R^2 \left[ 1 + \frac{\lambda_R}{16\pi^2} \ln \left( \frac{\Lambda}{\kappa} \right) \right] \quad (86)$$

$$\lambda_B = \frac{1}{1 - \frac{\lambda_R}{16\pi^2} \ln \left( \frac{\Lambda}{\kappa} \right)} \quad (87)$$

and

$$\langle \psi^2(t) \rangle_R = \left[ \frac{1}{1 - \frac{\lambda_R}{16\pi^2} \ln \left( \frac{\Lambda}{\kappa} \right)} \right] \times \left\{ \int \frac{d^3 k}{(2\pi)^3} \left| U^+_k(t) \right|^2 - \frac{1}{8\pi^2} \Lambda^2 - \frac{1}{8\pi^2} \ln \left( \frac{\Lambda}{\kappa} \right) \left( m_R^2 + \frac{\lambda_R}{2} \phi^2(t) \right) \right\} \quad (88)$$

It is clear that there is no wavefunction renormalization. This is a consequence of the approximation invoked. There is, in fact, no wavefunction renormalization in either one-loop or Hartree approximation for a scalar field theory in three spatial dimensions.

With an eye towards the numerical analysis, it is more convenient to write

$$\langle \psi^2(t) \rangle_R = \left( \langle \psi^2(t) \rangle_R - \langle \psi^2(0) \rangle_R \right) + \langle \psi^2(0) \rangle_R \quad (89)$$

and perform a subtraction at time $t = 0$ absorbing $\langle \psi^2(0) \rangle_R$ into a further finite renormalization of the mass term $(m_R^2 + \langle \psi^2(0) \rangle_R = M_R^2)$.

The renormalized equations that we will solve finally become

$$\ddot{\phi} + M_R^2 \dot{\phi} + \frac{\lambda_R}{2} \left[ 1 - \left( \frac{2}{3} \right) \frac{1}{1 - \frac{\lambda_R}{16\pi^2} \ln \left( \frac{\Lambda}{\kappa} \right)} \right] \phi^3 + \frac{\lambda_R}{2} \dot{\phi} \left( \langle \psi^2(t) \rangle_R - \langle \psi^2(0) \rangle_R \right) = 0 \quad (90)$$

$$\left[ \frac{d^2}{dt^2} + k^2 + M_R^2 + \frac{\lambda_R}{2} \phi^2(t) + \frac{\lambda_R}{2} \left( \langle \psi^2(t) \rangle_R - \langle \psi^2(0) \rangle_R \right) \right] U_k^+(t) = 0 \quad (91)$$

and

$$\left( \langle \psi^2(t) \rangle_R - \langle \psi^2(0) \rangle_R \right) = \left[ \frac{1}{1 - \frac{\lambda_R}{16\pi^2} \ln \left( \frac{\Lambda}{\kappa} \right)} \right] \times \left\{ \int \frac{d^3 k}{(2\pi)^3} \left| U^+_k(t) \right|^2 - \frac{1}{2\omega_k(0)} \frac{\lambda_R}{16\pi^2} \ln \left( \frac{\Lambda}{\kappa} \right) \left( \phi^2(t) - \phi^2(0) \right) \right\} \quad (92)$$
with the initial conditions for $U^+_k(t)$:

$$U^+_k(0) = 1 ; \quad \dot{U}^+_k(0) = -i\omega_k(0) ; \quad \omega_k(0) = \sqrt{k^2 + M^2_R + \frac{\lambda_R}{2} R^2 (0)} \quad (93)$$

It is worth noticing that there is a weak cutoff dependence on the renormalized equations of motion of the order parameter and the mode functions. This is a consequence of the well known “triviality” problem of the scalar quartic interaction in four space-time dimensions. This has the consequence that for a fixed renormalized coupling the cutoff must be kept fixed and finite. The presence of the Landau pole prevents taking the limit of the ultraviolet cutoff to infinity while keeping the renormalized coupling fixed.

This theory is sensible only as a low-energy cutoff effective theory. We then must be careful that for a fixed value of $\lambda_R$, the cutoff must be such that the theory never crosses the Landau pole. Thus from a numerical perspective there will always be a cutoff sensitivity in the theory. However, for small coupling we expect the cutoff dependence to be rather weak (this will be confirmed numerically) provided the cutoff is far away from the Landau pole.

### 3.2 Particle Production

Before we engage ourselves in a numerical integration of the above equations of motion we want to address the issue of particle production since it is of great importance for the understanding of dissipative processes.

In what follows, we consider particle production due to the time varying effective mass $M^2(t)$ in eq.(76) of the quantum field $\psi$ for the single scalar model.

The Lagrangian density for the fluctuations in the Hartree approximation is given by eq.(75). Demanding that the order parameter satisfies its equation of motion implies that the linear term in $\eta$ in eq.(75) vanishes. The resulting Lagrangian density is

$$\mathcal{L}[\psi] = \frac{1}{2} (\partial_\mu \psi)^2 - \frac{1}{2} M^2(t) \psi^2 + \frac{\lambda}{8} \langle \psi^2 \rangle \quad (94)$$

To study the issue of the particle production, it proves convenient to pass to Hamiltonian density:

$$\mathcal{H} = \Pi_{\psi} \dot{\psi} - \mathcal{L} = \frac{1}{2} \Pi_{\psi}^2 + \frac{1}{2} \left( \nabla \psi \right)^2 + \frac{1}{2} M^2(t) \psi^2 - \frac{\lambda}{8} \langle \psi^2 \rangle \quad (95)$$

Here $\Pi_{\psi}$ is the canonical momentum conjugate to $\psi$.

In a time dependent background the concept of particle is ambiguous and it must be defined with respect to some particular state. Let us consider the Heisenberg fields at $t = 0$
written as
\[
\psi(\vec{x}, 0) = \frac{1}{\sqrt{\Omega}} \sum_k \frac{1}{\sqrt{2\omega_k(0)}} \left(a_k(0) + a^\dagger_{-k}(0)\right) e^{i\vec{k} \cdot \vec{x}}
\]
\[
\Pi_\psi(\vec{x}, 0) = \frac{-i \sqrt{\Omega}}{2} \sum_k \sqrt{\omega_k(0)} \left(a_k(0) - a^\dagger_{-k}(0)\right) e^{i\vec{k} \cdot \vec{x}}
\] (96)

with \(\omega_k(0)\) as in eq.(79) and \(\Omega\) the spatial volume. The Hamiltonian is diagonalized at \(t = 0\) by these creation and destruction operators:
\[
H(0) = \sum_k \omega_k(0) \left[ a^\dagger_k(0)a_k(0) + \frac{1}{2} \right]
\] (97)

Thus we define the Hartree-Fock states at \(t = 0\) as the vacuum annihilated by \(a_k(0)\) together with the tower of excitations obtained by applying polynomials in \(a_k(0)\) to this vacuum state. The Hartree-Fock vacuum state at \(t = 0\) is chosen as the reference state. As time passes, particles (as defined with respect to this state) will be produced as a result of parametric amplification[37, 38]. We should mention that our definition differs from that of other authors[29, 38] in that we chose the state at time \(t = 0\) rather than using the adiabatic modes (that diagonalize the instantaneous Hamiltonian).

We define the number density of particles as a function of time as
\[
N(t) = \int \frac{d^3k}{(2\pi)^3} \frac{Tr \left[ a^\dagger_k(0)a_k(t)\rho(t) \right]}{Tr \rho(t)} = \int \frac{d^3k}{(2\pi)^3} \frac{Tr \left[ a^\dagger_k(t)a_k(t)\rho(0) \right]}{Tr \rho(0)}
\] (98)

where by definition
\[
a^\dagger_k(t) = U^{-1}(t, 0)a^\dagger_k(0)U(t, 0) ; \ a_k(t) = U^{-1}(t, 0)a_k(0)U(t, 0)
\] (99)

are the time-evolved operators in the Heisenberg picture. In terms of these time evolved operators, we may write:
\[
\psi(\vec{x}, t) = \frac{1}{\sqrt{\Omega}} \sum_k \frac{1}{\sqrt{2\omega_k(0)}} \left(a_k(t) + a^\dagger_{-k}(t)\right) e^{i\vec{k} \cdot \vec{x}}
\]
\[
\Pi_\psi(\vec{x}, t) = \frac{-i \sqrt{\Omega}}{2} \sum_k \sqrt{\omega_k(0)} \left(a_k(t) - a^\dagger_{-k}(t)\right) e^{i\vec{k} \cdot \vec{x}}
\] (100)

On the other hand, we now expand the Heisenberg fields at time \(t\) in the following orthonormal basis
\[
\psi(\vec{x}, t) = \frac{1}{\sqrt{\Omega}} \sum_k \frac{1}{\sqrt{2\omega_k(0)}} \left(\tilde{a}_k U_k^+(t) + \tilde{a}^\dagger_{-k} U_{-k}(t)\right) e^{i\vec{k} \cdot \vec{x}}
\]
\[
\Pi_\psi(\vec{x}, t) = \frac{1}{\sqrt{\Omega}} \sum_k \frac{1}{\sqrt{2\omega_k(0)}} \left(\tilde{a}_k U_k^+(t) + \tilde{a}^\dagger_{-k} U_{-k}(t)\right) e^{i\vec{k} \cdot \vec{x}}
\] (101)
where the mode functions \( U^+_k(t) ; U^-_k(t) = (U^+_k(t))^* \) are the Hartree-Fock mode functions obeying eqs. (91-93) together with the self consistency condition.

Thus \( \tilde{a}_k, \tilde{a}_k^\dagger \) are the annihilation and creation operators of Hartree-Fock states, and the Heisenberg field \( \psi(\vec{x}, t) \) is a solution of the Heisenberg equations of motion in the Hartree approximation. Therefore the \( \tilde{a}_k^\dagger \) and \( \tilde{a}_k \) do not depend on time and are identical to \( a_k^\dagger(0) \) and \( a_k(0) \) respectively (we can check this by evaluating the expansion in eq. (101) at \( t = 0 \) together with the initial conditions on \( U^+_k(t) \) in eq. (80)). Using the Wronskian properties of the function \( U^+_k(t) \), we see that the \( \tilde{a}_k^\dagger \) and \( \tilde{a}_k \) satisfy the usual canonical commutation relations. The reason for the choice of the vacuum state at \( t = 0 \) now becomes clear; this is the initial time at which the boundary conditions on the modes are determined. The mode functions \( U^+_k(t) ; U^-_k(t) \) are then identified with positive and negative frequency modes at the initial time.

By comparing the expansion in eq. (96) evaluated at time \( t \) with that in eq. (101), we find that the creation and annihilation operator at time \( t \) can be related to those at time \( t = 0 \) via a Bogoliubov transformation:

\[
a_k(t) = \mathcal{F}_{+,k}(t)a_k(0) + \mathcal{F}_{-,k}(t)a^\dagger_{-k}(0)
\]

(102)

The \( \mathcal{F}_\pm(t) \) can be read off in terms of the mode functions \( U^+_k(t) \)

\[
|\mathcal{F}_{+,k}(t)|^2 = \frac{1}{4} |U^+_k(t)|^2 \left[ 1 + \frac{\left| \dot{U}^+_k(t) \right|^2}{\omega^2_k(0) |U^+_k(t)|^2} \right] + \frac{1}{2}
\]

\[
|\mathcal{F}_{+,k}(t)|^2 - |\mathcal{F}_{-,k}(t)|^2 = 1
\]

(103)

At any time \( t \) the expectation value of the number operator for the quanta of \( \psi \) in each \( k \)-mode is given by

\[
\mathcal{N}_k(t) = \frac{Tr \left[ a_k^\dagger(t) a_k(t) \rho(0) \right]}{Tr \rho(0)}
\]

(104)

After some algebra, we find

\[
\mathcal{N}_k(t) = \left( 2 |\mathcal{F}_{+,k}(t)|^2 - 1 \right) \mathcal{N}_k(0) + \left( |\mathcal{F}_{+,k}(t)|^2 - 1 \right)
\]

(105)

This result exhibits the contributions from “spontaneous” (proportional to the initial particle occupations) and “induced” (independent of it) particle production. Since we are analyzing the zero temperature case with \( \mathcal{N}_k(0) = 0 \) only the induced contribution results.

Before passing on to the numerical analysis it is important to point out that the Hartree approximation is energy conserving. The bare energy density is

\[
\mathcal{E} = \frac{1}{2} \phi^2(t) + V(\phi(t)) + \frac{Tr H \rho(0)}{Tr \rho(0)}
\]

(106)
with $V(\phi)$ the classical potential and $\mathcal{H}$ is given by eq.(93) with $\Pi_\psi$, $\psi$ expanded in terms of the Hartree mode functions and creation and annihilation operator as in eqs.([101],[101]). Using the equations of motion for $\phi(t)$ and the mode functions, and after some lengthy but straightforward algebra one finds $\dot{E} = 0$.

3.3 Numerical Analysis

3.3.1 Unbroken Symmetry Case

In order to perform a numerical analysis it is necessary to introduce dimensionless quantities and it becomes convenient to choose the renormalization point $\kappa = M_R$. Thus we define

$$
\eta(t) = \phi(t) \sqrt{\frac{\lambda_R}{2M_R^2}}; \quad q = \frac{k}{M_R}; \quad \tau = M_R t; \quad g = \frac{\lambda_R}{8\pi^2}
$$

and finally, the equations of motion become

$$
\frac{d^2}{d\tau^2} \eta + \eta + \left[1 - \left(\frac{2}{3}\right) \frac{1}{1 - \frac{1}{2} \ln \left(\frac{\Lambda}{M_R}\right)}\right] \eta^3 + g \eta \Sigma(\tau) = 0
$$

$$
\left[\frac{d^2}{d\tau^2} + q^2 + 1 + \eta^2(\tau) + g \Sigma(\tau)\right] U^+_q(\tau) = 0
$$

$$
U^+_q(0) = 1; \quad \frac{d}{d\tau} U^+_q(0) = -i \sqrt{q^2 + 1 + \eta^2(0)}
$$

$$
\Sigma(\tau) = \left[\frac{1}{1 - \frac{1}{2} \ln \left(\frac{\Lambda}{M_R}\right)}\right] \times
$$

$$
\left\{\int_0^{\Lambda/M_R} q^2 dq \frac{|U^+_q(\tau)|^2 - 1}{\sqrt{q^2 + 1 + \eta^2(0)}} + \frac{1}{2} \ln \left(\frac{\Lambda}{M_R}\right) \left(\eta^2(\tau) - \eta^2(0)\right)\right\}
$$

In terms of the dimensionless quantities we obtain the number of particles within a correlation volume $N(\tau) = N(t)/M_R^3$

$$
N(\tau) = \frac{1}{8\pi^2} \int_0^{\Lambda/M_R} q^2 dq \left\{|U^+_q(\tau)|^2 + \frac{|U^+_q(\tau)|^2}{\sqrt{q^2 + 1 + \eta^2(0)}} - 2\right\}
$$

Figures (5.a,b,c) show $\eta(\tau)$, $\Sigma(\tau)$ and $N(\tau)$ in the Hartree approximation, for $g = 0.1$, $\eta(0) = 1.0$ and $\Lambda/M_R = 100$; we did not detect an appreciable cutoff dependence by varying
the cutoff between 50 and 200. Clearly there is no appreciable damping in $\eta(\tau)$. In fact it can be seen that the period of the oscillation is very close to $2\pi$, which is the period of the classical solution of the linear theory. This is understood because the coefficient of the cubic term is very small and $g\Sigma(\tau)$ is negligible. Particle production is also negligible. This situation should be contrasted with that shown in figures (6.a-c) and (7.a-c) in which there is dissipation and damping in the evolution of $\eta(\tau)$ for $\eta(0) = 4, 5$ respectively and the same values for $g$ and the cutoff. There are several noteworthy features that can be deduced from these figures. First the fluctuations become very large, such that $g\Sigma(\tau)$ becomes $\mathcal{O}(1)$. Second, figures (6.a) and (7.a) clearly show that initially, channels are open and energy is transferred away from the $q = 0$ mode of the field. Eventually however, these channels shut off, and the dynamics of the expectation value settles into an oscillatory motion. The time scale for the shutting off of the dissipative behavior decreases as $\eta(0)$ increases; it is about 25 for $\eta(0) = 4$, and about 18 for $\eta(0) = 5$. This time scale is correlated with the time scale in which particles are produced by parametric amplification and the quantum fluctuations begin to plateau (figures (6.b,c) and (7.b,c)). Clearly the dissipative mechanism which damps the motion of the expectation value is particle production. Furthermore, the long time dynamics for the expectation value does not correspond to exponential damping. In fact, we did not find any appreciable damping for $\tau \geq 70$ in these cases. It is illuminating to compare this situation with that of a smaller coupling depicted in figures (8.a-c) for $\eta(0) = 5 g = 0.05$. Clearly the time scale for damping is much larger and there is still appreciable damping at $\tau \approx 100$. Notice also that $g\Sigma(\tau) \approx 2$ and that particles are being produced even at long times and this again correlates with the evidence that the field expectation value shows damped motion at long times, clearly showing that the numerical analysis has not reached the asymptotic regime.

These numerical results provide some interesting observations. First we see that $\eta^2(\tau) + g\Sigma(\tau)$ has asymptotic oscillatory behavior in $\tau$. In general this would imply unstable bands of wavevectors for the solutions of equation (108). However, if these unstable bands are present, the mode functions would grow and and $\Sigma(\tau)$ would grow as a result since it is an integral over all wavevectors. The fact that this does not happen implies that there are no unstable bands.

We have been able to find analytically an exact oscillatory solution of the equation (108) which is given by

\begin{align}
1 + \eta^2(\tau) + g\Sigma(\tau) & = -e_3 - 2 P(\tau + \omega') \\
\eta(\tau) & = A \operatorname{sn}(\sqrt{e_1 - e_3}\tau)
\end{align}

\begin{align}
(112) & \\
(113) & 
\end{align}
where $\mathcal{P}$ is the Weierstrass function which is double periodic with periods $\omega; \omega'$. The parameters in this solution are functions of the coupling. We found that, asymptotically, the numerical solution of equations (108, 109) could be fit quite precisely to the above analytic solution for precise values of the parameters. These values encode the information about the initial conditions and the coupling constant. For this potential given by (112) we can find the exact solutions for the mode functions in terms of Weierstrass $\sigma$ functions. In this case there are no unstable modes for real wavevector $q$. This is an important result because it provides an analytic understanding for the lack of forbidden bands.

Although this exact result does not illuminate the physical reason why there are no unstable bands, we conjecture that this is a consequence of the conservation of energy in the Hartree approximation.

Numerically we find no evidence for exponential relaxation asymptotically, of course, we cannot rule out the possibility of power law asymptotic decay for $\eta$ with an exceedingly small power.

The fundamental question to be raised at this point is: what is the origin of the damping in the evolution of the field expectation value? Clearly this is a collisionless process as collisions are not taken into account in the Hartree approximation (although the one-loop diagram that enters in the two particle collision amplitude with the two-particle cut is contained in the Hartree approximation and is responsible for thresholds to particle production). The physical mechanism is very similar to that of Landau damping in the collisionless Vlasov equation for plasmas and also found in the study of strong electric fields in reference. In the case under consideration, energy is transferred from the expectation value to the quantum fluctuations which back-react on the evolution of the field expectation value but out of phase. This phase difference between the oscillations of $\eta^2(\tau)$ and those of $\Sigma(\tau)$ can be clearly seen to be $\pi$ in figures (6.a,b), (7.a,b) since the maxima of $\eta^2(\tau)$ occur at the same times as the minima of $\Sigma(\tau)$ and vice versa.

This is an important point learned from our analysis and that is not a priori taken into account in the usual arguments for dissipation via collisions. The process of thermalization, however, will necessarily involve collisions and cannot be studied within the schemes addressed in this paper.

### 3.3.2 Broken Symmetry

The broken symmetry case is obtained by writing $M_R^2 = -\mu_R^2 < 0$ and using the scale $\mu_R$ instead of $M_R$ to define the dimensionless quantities as in eq.(107) and the renormalization...
scale. The equations of motion in this case become

\[
\frac{d^2}{d\tau^2} \eta - \eta + \left[ 1 - \left(\frac{2}{3}\right) \frac{1}{1 - \frac{4}{2} \ln \left(\frac{\Lambda}{\mu_R}\right)} \right] \eta^3 + g\eta\Sigma(\tau) = 0
\]

\[
\left[ \frac{d^2}{d\tau^2} + q^2 - 1 + \eta^2(\tau) + g\Sigma(\tau) \right] U_q^+(\tau) = 0 \quad (114)
\]

\[
U_q^+(0) = 1; \quad \frac{d}{d\tau} U_q^+(0) = -i\sqrt{q^2 + 1 + \eta^2(0)} \quad (115)
\]

with \(\Sigma(\tau)\) given in eq.(110) but with \(M_R\) replaced by \(\mu_R\). The broken symmetry case is more subtle because of the possibility of unstable modes for initial conditions in which \(\eta(0) << 1\). We have kept the boundary conditions of eq.(115) the same as in eq.(109). This corresponds to preparing an initial state as a Gaussian state centered at \(\eta(0)\) with real and positive covariance (width) and letting it evolve for \(t > 0\) in the broken symmetry potential[26, 44]. The number of particles produced within a correlation volume (now \(\mu_R^3\)) is given by eq.(111) with \(M_R\) replaced by \(\mu_R\).

Figures (9.a-c) depict the dynamics for a broken symmetry case in which \(\eta(0) = 10^{-5}\) i.e. very close to the top of the potential hill. Notice that as the field expectation value rolls down the hill the unstable modes make the fluctuation grow dramatically until about \(\tau \approx 50\) at which point \(g\Sigma(\tau) \approx 1\). At this point, the unstable growth of fluctuations shuts off and the field begins damped oscillatory motion around a mean value of about \(\approx 1.2\). This point is a minimum of the effective action. The damping of these oscillations is very similar to the damping around the origin in the unbroken case. Most of the particle production and the largest quantum fluctuations occur when the field expectation value is rolling down the region for which there are unstable frequencies for the mode functions (see eq.(114)). This behavior is similar to that found previously by some of these authors[26].

4 Non-perturbative schemes II: Large N limit in the \(O(N)\) Model

Although the Hartree approximation offers a non-perturbative resummation of select terms, it is not a consistent approximation because there is no \textit{a priori} small parameter that defines the approximation. Furthermore we want to study the effects of dissipation by Goldstone bosons in a non perturbative but controlled expansion.

In this section, we consider the \(O(N)\) model in the large N limit. The large N limit has been used in studies of non-equilibrium dynamics[38, 40, 44] and it provides a very pow-
erful tool for studying non-equilibrium dynamics non-perturbatively in a consistent manner. The Lagrangian density is the following:

\[ \mathcal{L} = \frac{1}{2} \partial_{\mu} \vec{\phi} \cdot \partial^{\mu} \vec{\phi} - V(\sigma, \vec{\pi}) \]

\[ V(\sigma, \vec{\pi}) = \frac{1}{2} m^2 \vec{\phi} \cdot \vec{\phi} + \frac{\lambda}{8N} (\vec{\phi} \cdot \vec{\phi})^2 \]  

(116)

for \( \lambda \) fixed in the large \( N \) limit. Here \( \vec{\phi} \) is an \( O(N) \) vector, \( \vec{\phi} = (\sigma, \vec{\pi}) \) and \( \vec{\pi} \) represents the \( N - 1 \) pions. In what follows, we will consider two different cases of the potential \( V(\sigma, \vec{\pi}) \), with \( m^2 < 0 \) or without \( m^2 > 0 \) symmetry breaking.

Let us define the fluctuation field \( \chi(\vec{x}, t) \) as

\[ \sigma(\vec{x}, t) = \sigma_0(t) + \chi(\vec{x}, t) \]  

(117)

Expanding the Lagrangian density in eq.(116) in terms of fluctuations \( \chi(\vec{x}, t) \), we obtain

\[
\mathcal{L}[\sigma_0 + \chi^+, \vec{\pi}^+] - \mathcal{L}[\sigma_0 + \chi^-, \vec{\pi}^-] = \left\{ \mathcal{L}[\sigma_0, \vec{\pi}^+] + \frac{\delta \mathcal{L}}{\delta \sigma_0} \chi^+ + \frac{1}{2} \left( \partial_{\mu} \chi^+ \right)^2 + \frac{1}{2!} V''(\sigma_0, \vec{\pi}^+) \chi^{+2} + \frac{1}{3!} V^[3](\sigma_0, \vec{\pi}^+) (\chi^+)^3 + \frac{1}{4!} V^[4](\sigma_0, \vec{\pi}^+) (\chi^+)^4 \right\}
- \left\{ \left( \chi^+ \rightarrow \chi^- \right), \left( \vec{\pi}^+ \rightarrow \vec{\pi}^- \right) \right\}
\]  

(118)

The tadpole condition \( \langle \chi^\pm(\vec{x}, t) \rangle = 0 \) will lead to the equations of motion as previously discussed. We now introduce a Hartree factorization. In the presence of a non-zero expectation value the Hartree factorization is subtle in the case of continuous symmetries. A naive Hartree factorization violates the Ward identities related to Goldstone’s theorem. This shortcoming is overcome if the Hartree factorization is implemented in leading order in the large \( N \) expansion[44]. We will make a series of assumptions that seem to be reasonable, but that can only be justified \textit{a posteriori} when we recognize that with these assumptions, we obtain the equations of motion that fulfill the Ward identities. These assumptions are: i) no cross correlations between the pions and sigma field and ii) that the two point correlation functions of the pion field are diagonal in the \( O(N-1) \) space given by the remaining unbroken symmetry group. Based upon these assumptions we are led to the following Hartree factorization of the non-linear terms in the Lagrangian density (again for both \( \pm \) components):

\[ \chi^4 \rightarrow 6(\chi^2)\chi^2 + constant \]  

(119)

\[ \chi^3 \rightarrow 3(\chi^2)\chi \]  

(120)
\( (\vec{\pi} \cdot \vec{\pi})^2 \rightarrow \left( 2 + \frac{4}{N - 1} \right) \langle \vec{\pi}^2 \rangle \vec{\pi}^2 + \text{constant} \) (121)
\[ \vec{\pi}^2 \chi^2 \rightarrow \langle \vec{\pi}^2 \rangle \chi^2 + \vec{\pi}^2 \langle \chi^2 \rangle \] (122)
\[ \vec{\pi}^2 \chi \rightarrow \langle \vec{\pi}^2 \rangle \chi \] (123)

where by “constant” we mean the operator independent expectation values of the composite operators which will not enter into the time evolution equation of the order parameter.

In this approximation, the resulting Lagrangian density is quadratic, with a linear term in \( \chi \):

\[
\mathcal{L}[\sigma_0 + \chi^+, \vec{\pi}^+] - \mathcal{L}[\sigma_0 + \chi^-, \vec{\pi}^-] = \left\{ \frac{1}{2} \left( \partial_\mu \chi^+ \right)^2 + \frac{1}{2} \left( \partial_\mu \vec{\pi}^+ \right)^2 - \chi^+ V'(t) \right. \\
- \left. \frac{1}{2} \mathcal{M}_\chi^2(t)(\chi^+)^2 - \frac{1}{2} \mathcal{M}_{\vec{\pi}}^2(t)(\vec{\pi}^+)^2 \right\} - \left\{ (\chi^+ \rightarrow \chi^-), (\vec{\pi}^+ \rightarrow \vec{\pi}^-) \right\}
\] (124)

where \( V' \) is the derivative of the Hartree potential defined below. To obtain a large N limit, we define

\[
\langle \vec{\pi}^2 \rangle = N \langle \psi^2 \rangle \quad \text{(125)}
\]
\[
\sigma_0(t) = \phi(t) \sqrt{N} \quad \text{(126)}
\]

with

\[
\langle \psi^2 \rangle \approx \mathcal{O}(1) ; \quad \langle \chi^2 \rangle \approx \mathcal{O}(1) ; \quad \phi \approx \mathcal{O}(1).
\] (127)

We will approximate further by neglecting the \( \mathcal{O}(\frac{1}{N}) \) terms in the formal large N limit. We now obtain

\[
V'(\phi(t), t) = \sqrt{N} \phi(t) \left[ m^2 + \frac{\lambda}{2} \phi^2(t) + \frac{\lambda}{2} \langle \psi^2(t) \rangle \right]
\] (128)
\[
\mathcal{M}_\chi^2(t) = m^2 + \frac{\lambda}{2} \phi^2(t) + \frac{\lambda}{2} \langle \psi^2(t) \rangle
\] (129)
\[
\mathcal{M}_{\vec{\pi}}^2(t) = m^2 + \frac{3\lambda}{2} \phi^2(t) + \frac{\lambda}{2} \langle \psi^2(t) \rangle
\] (130)

Using the tadpole method, we obtain the following set of equations:

\[
\ddot{\phi}(t) + \phi(t) \left[ m^2 + \frac{\lambda}{2} \phi^2(t) + \frac{\lambda}{2} \langle \psi^2(t) \rangle \right] = 0 ; \quad \langle \psi^2(t) \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{|U_k^+(t)|^2}{2\omega_{\vec{r}k}(0)}
\] (131)
\[
\left[ \frac{d^2}{dt^2} + \omega_{\vec{r}k}^2(t) \right] U_k^+(t) = 0 ; \quad \omega_{\vec{r}k}^2(t) = \vec{k}^2 + \mathcal{M}_{\vec{\pi}}^2(t)
\] (132)
It is clear from the above equations that the Ward identities of Goldstone’s theorem are indeed fulfilled. Because \( V'(\phi(t), t) = \sqrt{N} \phi(t) M_2(t) \), whenever \( V'(\phi(t), t) \) vanishes for \( \phi \neq 0 \) then \( M_\pi = 0 \) and the “pions” are the Goldstone bosons. The initial conditions for the mode functions \( U^+_k(t) \) are
\[
U^+_k(0) = 1 ; \quad \dot{U}^+_k(0) = -i \omega \pi_k(0)
\]
Since in this approximation, the dynamics for the \( \pi \) and \( \chi \) fields decouples, and the dynamics of \( \chi \) does not influence that of \( \phi \) or the mode functions and \( \langle \psi^2 \rangle \) we will only concentrate on the solution for the \( \pi \) fields.

### 4.1 Renormalization

The renormalization procedure is exactly the same as that for the Hartree case in the previous section (see eqs. (82-88). We carry out the same renormalization prescription and subtraction at \( t = 0 \) as in the last section. Thus we find the following equations of motion
\[
\ddot{\phi} + M_R^2 \phi + \frac{\lambda_R}{2} \phi^3 + \frac{\lambda_R}{2} \phi \left( \langle \psi^2(t) \rangle_R - \langle \psi^2(0) \rangle_R \right) = 0
\]
\[
\left[ \frac{d^2}{dt^2} + k^2 + M_R^2 + \frac{\lambda_R}{2} \phi^2(t) + \frac{\lambda_R}{2} \left( \langle \psi^2(t) \rangle_R - \langle \psi^2(0) \rangle_R \right) \right] U_k^+(t) = 0
\]
with the initial conditions given by eq.(133) and with the subtracted expectation value given by eq.(92).

In contrast with the Hartree equations in the previous section, the cutoff dependence in the term proportional to \( \phi^3 \) in eq.(134) has disappeared. This is a consequence of the large \( N \) limit and the Ward identities, which are now obvious at the level of the renormalized equations of motion. There is still a very weak cutoff dependence in eq.(92) because of the triviality issue which is not relieved in the large \( N \) limit, but again, this theory only makes sense as a low-energy cutoff theory.

### 4.2 Numerical Analysis

#### 4.2.1 Unbroken symmetry

To solve the evolution equations in eqs.(92,134) numerically, we now introduce dimensionless quantities as in eq.(107) obtaining the following dimensionless equations:
\[
\frac{d^2}{dt^2} \eta + \eta + \eta^3 + g \eta \Sigma(\tau) = 0
\]
\[
\left[ \frac{d^2}{d\tau^2} + q^2 + 1 + \eta^2(\tau) + g\Sigma(\tau) \right] U_q^+(\tau) = 0
\]
(135)

\[
U_q^+(0) = 1 ; \quad \frac{d}{d\tau} U_q^+(0) = -i\sqrt{q^2 + 1 + \eta^2(0)}
\]
(136)

\[
\Sigma(\tau) = \left[ \frac{1}{1 - \frac{q^2}{2\ln \left( \frac{\Lambda}{M_R} \right)}} \right] \times \\
\left\{ \int_0^{\frac{\Lambda}{M_R}} q^2dq \frac{\left| U_q^+(\tau) \right|^2 - 1}{\sqrt{q^2 + 1 + \eta^2(0)}} + \frac{1}{2} \ln \left( \frac{\Lambda}{M_R} \right) \left( \eta^2(\tau) - \eta^2(0) \right) \right\}
\]
(137)

For particle production in the \( O(N) \) model, the final expression of the expectation value of the number operator for each pion field in terms of dimensionless quantities is the same as in eq.(111), but the mode function \( U_q^+(t) \) obeys the differential equation in eq.(108) together with the self consistent condition.

Figures (10.a-c), (11.a-c) and (12.a-c) show \( \eta(\tau) \); \( \Sigma(\tau) \); \( N(\tau) \) for \( \eta(0) = 1; 2 g = 0.1; 0.3 \) and \( \Lambda/M_R = 100 \) (although again we did not find cutoff sensitivity). The dynamics is very similar to that of the single scalar field in the Hartree approximation, which is not surprising, since the equations are very similar (save for the coefficient of the cubic term in the equation for the field expectation value). Thus the analysis presented previously for the Hartree approximation remains valid in this case.

### 4.2.2 Broken symmetry

The broken symmetry case corresponds to choosing \( M_R^2 = -\mu_R^2 < 0 \). As in the case of the Hartree approximation, we now choose \( \mu_R \) as the scale to define dimensionless variables and renormalization scale. The equations of motion for the field expectation value and the mode functions now become

\[
\left[ \frac{d^2}{d\tau^2} + q^2 - 1 + \eta^2(\tau) + g\Sigma(\tau) \right] U_q^+(\tau) = 0
\]
(138)

with \( \Sigma(\tau) \) given by eq.(137). As in the Hartree case, there is a subtlety with the boundary conditions for the mode functions because the presence of the instabilities at \( \tau = 0 \) for the band of wavevectors \( 0 \leq q^2 < 1, \) for \( \eta^2(0) < 1 \). Following the discussion in the Hartree case (broken symmetry) we chose the initial conditions for the mode functions as

\[
U_q^+(0) = 1 ; \quad \frac{d}{d\tau} U_q^+(0) = -i\sqrt{q^2 + 1 + \eta^2(0)}
\]
(139)
These boundary conditions correspond to preparing a gaussian state centered at $\eta(0)$ at $\tau = 0$ and letting this initial state evolve in time in the “broken symmetry potential”[26].

Figures (13.a-c) show the dynamics of $\eta(\tau)$, $\Sigma(\tau)$ and $N(\tau)$ for $\eta(0) = 0.5$ $g = 0.1$ and cutoff $\Lambda/\mu_R = 100$. Strong damping behavior is evident, and the time scale of damping is correlated with the time scale for growth of $\Sigma(\tau)$ and $N(\tau)$. The asymptotic value of $\eta(\tau)$ and $\Sigma(\tau)$, $\eta(\infty)$ and $\Sigma(\infty)$ respectively satisfy

$$-1 + \eta^2(\infty) + g\Sigma(\infty) = 0$$

as we confirmed numerically. Thus the mode functions are “massless” describing Goldstone bosons. Notice that this value also corresponds to $V'(\phi) = 0$ in eq.(128). An equilibrium self-consistent solution of the equations of motion for the field expectation value and the fluctuations is reached for $\tau = \infty$.

Figure (13.d) shows the number of particles produced per correlation volume as a function of (dimensionless) wavevector $q$ at $\tau = 200$. We see that it is strongly peaked at $q = 0$ clearly showing that the particle production mechanism is most efficient for long-wavelength Goldstone bosons. Figures (13.e-g) show snapshots of the number of particles as a function of wavevector for $\tau = 13, 25, 50$ respectively, notice the scale. Clearly at longer times, the contributions from $q \neq 0$ becomes smaller. Figures (14.a-c) show the evolution of $\eta(\tau)$, $\Sigma(\tau)$ and $N(\tau)$ for $\eta(0) = 10^{-4}$, $g = 10^{-7}$. The initial value of $\eta$ is very close to the “top” of the potential. Due to the small coupling and the small initial value of $\eta(0)$, the unstable modes (those for which $q^2 < 1$) grow for a long time making the fluctuations very large. However the fluctuation term $\Sigma(\tau)$ is multiplied by a very small coupling and it has to grow for a long time to overcome the instabilities. During this time the field expectation value rolls down the potential hill, following a trajectory very close to the classical one. The classical turning point of the trajectory beginning very near the top of the potential hill, is close to $\eta_{tp} = \sqrt{2}$. Notice that $\eta(\tau)$ exhibits a turning point (maximum) at $\eta \approx 0.45$. Thus the turning point of the effective evolution equations is much closer to the origin. This phenomenon shows that the effective (non-local) potential is shallower than the classical potential, with the minimum moving closer to the origin as a function of time.

If the energy for the field expectation value was absolutely conserved, the expectation value of the scalar field would bounce back to the initial point and oscillate between the two classical turning points. However, because the fluctuations are growing and energy is transferred to them from the $q = 0$ mode, $\eta$ is slowed down as it bounces back, and tends to settle at a value very close to the origin (asymptotically about 0.015). Figure (14.b) shows that the fluctuations grow initially and stabilize at a value for which $g\Sigma(\infty) \approx 1$. 38
The period of explosive growth of the fluctuations is correlated with the strong oscillations at the maximum of $\eta(\tau)$. This is the time when the fluctuations begin to effectively absorb the energy transferred by the field expectation value and when the damping mechanism begins to work. Again the asymptotic solution is such that $-1 + \eta^2(\infty) + g\Sigma(\infty) = 0$, and the particles produced are indeed Goldstone bosons but the value of the scalar field in the broken symmetry minimum is very small (classically it would be $\eta_{\text{min}} = 1$, yet dynamically, the field settles at a value $\eta(\infty) \approx 0.015!!$ for $g = 10^{-7}$). Figure (14.c) shows copious particle production, and the asymptotic final state is a highly excited state with a large number ($\mathcal{O}(10^5)$) of Goldstone bosons per correlation volume.

The conclusion that we reach from the numerical analysis is that Goldstone bosons are extremely effective for dissipation and damping. Most of the initial potential energy of the field is converted into particles (Goldstone bosons) and the field expectation value comes to rest at long times at a position very close to the origin.

Notice that the difference with the situation depicted in figures (13.a-c) is in the initial conditions and the strength of the coupling. In the case of stronger coupling, the fluctuations grow only for a short time because $g\Sigma(\tau)$ becomes $\mathcal{O}(1)$ in short time, dissipation begins to act rather rapidly and the expectation value rolls down only for a short span and comes to rest at a minimum of the effective action, having transferred all of its potential energy difference to produce Goldstone bosons.

Figures (15.a-c) show a very dramatic picture. In this case $\eta(0) = 10^{-4}$, $g = 10^{-12}$. Now the field begins very close to the top of the potential hill. This initial condition corresponds to a “slow-roll” scenario. The fluctuations must grow for a long time before $g\Sigma(\tau)$ becomes $\mathcal{O}(1)$, during which the field expectation value evolves classically reaching the classical turning point at $\eta_p = \sqrt{2}$, and then bouncing back. But by the time it gets near the origin again, the fluctuations have grown dramatically absorbing most of the energy of the field expectation value and completely damping its motion. In this case almost all the initial potential energy has been converted into particles. This is a remarkable result. The conclusion of this analysis is that the strong dissipation by Goldstone bosons dramatically changes the dynamics of the phase transition. For slow-roll initial conditions the scalar field relaxes to a final value which is very close to the origin. This is the minimum of the effective action, rather than the minimum of the tree-level effective potential. Thus dissipative effects by Goldstone bosons introduce a very strong dynamical correction of the effective action leading to a very shallow effective potential (the effective action for constant field configuration). The condition for this situation to happen is that the period of the classical trajectory is of the same order of magnitude as the time scale of growth for the
fluctuations.

The weak coupling estimate for this dynamical non-equilibrium time scale is \( \tau_s \approx \ln(1/g)/2 \), which is obtained by requiring that \( g\Sigma(\tau) \approx 1 \). For weak coupling the mode functions grow as \( U_q^\pm(\tau) \approx e^\pm \) for \( q^2 \ll 1 \), and \( \Sigma(\tau) \approx e^{2\tau} \). The numerical analysis confirms this time scale for weak coupling. For weakly coupled theories, this non-equilibrium time scale is much larger than the static correlation length (in units in which \( c = 1 \)) and the only relevant time scale for the dynamics.

Clearly the outcome of the non-equilibrium evolution will depend on the initial conditions of the field expectation value.

Our results pose a fundamental question: how is it possible to reconcile damping and dissipative behavior, as found in this work with time reversal invariance?

In fact we see no contradiction for the following reason: the dynamics is completely determined by the set of equations of motion for the field expectation value and the mode functions for the fluctuations described above. These equations are solved by providing non-equilibrium initial conditions on the field expectation value, its derivative and the mode functions and their derivatives at the initial time \( t = 0 \). The problem is then evolved in time by solving the coupled second order differential equations. We emphasize the fact that the equations are second order in time because these are time reversal invariant. Now consider evolving this set of equations up to a positive time \( t_0 \), at which we stop the integration and find the value of the field expectation value, its derivative, the value of all the mode functions and their derivatives. Because this is a system of differential equations which is second order in time, we can take these values at \( t_0 \) as initial conditions at this time and evolve backwards in time reaching the initial values at \( t = 0 \). Notice that doing this involves beginning at a time \( t_0 \) in an excited state with (generally) a large number of particles. The conditions at this particular time are such that the energy stored in this excited state is focused in the back reaction to the field expectation value that acquires this energy and whose amplitude will begin to grow.

It is at this point where one recognizes the fundamental necessity of the “in-in” formalism in which the equations of motion are real and causal.
5 Conclusions and Discussion

We have focused on understanding the first stage of a reheating process, that of dissipation in the dynamics of the field expectation value of a scalar field via particle production. Starting from a scalar field theory with no apparent dissipative terms in its dynamics, we have shown how the evolution of expectation value of the scalar field is affected by the quantum fluctuations and particle production resulting in dissipative dynamics.

We started our analysis with a perturbative calculation, both in the amplitude of the expectation value of the field and to one-loop order. Analytically and numerically we find that dissipative processes cannot be described perturbatively. A systematic solution to the equation of motion reveals the presence of resonances and secular terms resulting in the growth of the corrections to the classical evolution. The perturbative study of the O(2) linear sigma model reveals infrared divergences arising from the contribution to the dissipative kernels from the Goldstone modes which require non-perturbative resummation.

A Langevin equation was constructed in an amplitude expansion, it exhibits a generalized fluctuation-dissipation theorem with non-Markovian (memory) kernels and colored noise correlation functions thus offering a more complex picture of dissipation.

Motivated by the failure of the perturbative approach, we studied the non-equilibrium dynamics in a Hartree approximation both in the symmetric as well as in the broken symmetry case. This approximation clearly exhibits the contribution of open channels and the dissipation associated with particle production, which in this approximation is a result of parametric amplification of quantum fluctuations.

In the case of unbroken symmetry we find that asymptotically the expectation value of the scalar field oscillates around the trivial vacuum with an amplitude that depends on the coupling and initial conditions. An extensive numerical study of the renormalized equations of motion was performed that shows explicitly the dynamics of particle production during these oscillations.

Although the Hartree approximation offers a self-consistent non-perturbative resummation, it is not controlled. Thus we were led to study the large N limit in an O(N) model, which also allows us to study in a non-perturbative manner the dissipative dynamics of Goldstone bosons. In the case of unbroken symmetry the results are very similar to those obtained in the Hartree approximation.

The broken symmetry case provides several new and remarkable results.

An extensive numerical study of the equations explicitly shows copious particle production while the expectation value of the field relaxes with strongly damped oscillations towards
a minimum of the effective action.

It is intuitively obvious that Goldstone bosons are extremely effective for dissipation since channels are open for arbitrarily small energy transfer. What is remarkable is that for “slow-roll” initial conditions, that is the initial expectation value of the scalar field very close to the origin and extremely small coupling constant ($O(10^{-7})$ or smaller) the final value of the expectation value remains very close to the origin, most of the potential energy has been absorbed in the production of long-wavelength Goldstone bosons. This is confirmed by an exhaustive numerical study, including snapshots at different times of the number of particles produced for different wavelengths showing a large peak at small wavevectors at long times. The numerical study clearly shows that particle production is extremely effective for long-wavelength Goldstone bosons. Another remarkable result is that the asymptotic value expectation value of the scalar field depends on the initial conditions. These asymptotic values correspond to minima of the effective action. Thus we reach the unexpected conclusion that in this approximation the effective action has a continuum manifold of minima which can be reached from different initial conditions.

It is pointed out that the basic mechanism for dissipation in this approximation is that of Landau damping through the parametric amplification of quantum fluctuations, and production of particles. These fluctuations react-back in the evolution of the expectation value of the scalar field, but out of phase. This is a collisionless mechanism, similar to that found in the collisionless Boltzmann-Vlasov equation for plasmas.

Our study also reconciles dissipation in the time evolution for the coarse grained variable with time reversal invariance, as the evolution is completely specified by an infinite set of ordinary second order differential equations in time with proper boundary conditions.

Our formalism and techniques are sufficiently powerful to give a great deal of insight into the particulars of the dissipation process. In particular, we can see that the damping of the field expectation value ends as the particle production ends. This shows that our interpretation of the damping as being due to particle production is accurate.

It is useful to compare what we have done here with other work on this issue. We have already mentioned the work of Calzetta, Hu and Paz. These authors use the closed time path formalism to arrive at the effective equations of motion for the expectation value of the field. Then they solve the perturbative equations and find dissipative evolution at short times. In particular Paz finds the kernel that we have found for the effective equations of motion of the expectation value in the perturbative and Hartree case. However, his perturbative solution is not consistent.

We remedy this situation by studying the non-perturbative Hartree equations, which must
necessarily be solved numerically as we do.

There has been other, previous, work on the reheating problem, most notably by Abbott, Farhi and Wise\cite{Abbott}, Ringwald\cite{Ringwald} and Morikawa and Sasaki\cite{Morikawa}. In all of these works, the standard effective action is used, so that the expectation value is of the “in-out” type and hence the equations are non-causal and contain imaginary parts. In essence, they “find” dissipational behavior by adding an imaginary part to the frequency that appears in the mode equations. We see from our work (as well as that of Calzetta, Hu and Paz) that this is not necessary; dissipation can occur even when the system is evolving unitarily. This comment deserves a definition of what we call dissipation here: it is the energy transfer from the expectation value of the $q = 0$ mode of the scalar field to the quantum fluctuations ($q \neq 0$) resulting in damped evolution for the $q = 0$ mode.

To what extent are we truly treating the reheating problem of inflationary models? As stated in the introduction, reheating typically entails the decay of the inflaton into lighter particles during its oscillations. What we do here is understand how the quantum fluctuations and the ensuing particle production influence the dynamics of the evolution of the expectation value of the field. Thus technically speaking, this is not the reheating problem. However, we are able to understand where dissipation comes from in a field theory, and are able to give a quantitative description of the damping process for the expectation value of the field.

We expect that the physics of dissipation when the scalar field couple to others will be very similar to the case studied in this article.

Furthermore, the techniques we develop here are easily adapted to the case where the inflaton couples to fermions, a case that we are currently studying\cite{InflationFermions}. In this connection, during the writing of this paper, two related pieces of work on the reheating problem have appeared\cite{ReheatingFermions}. They both look at the effect of particle production from the oscillations of the inflaton field due to parametric amplification. What they do not do is to account for the back reaction of the produced particles on the evolution of the expectation value of the inflaton. As we have learned with our study, this back reaction will eventually shut off the particle production, so that these authors may have overestimated the amount or particle production.

We recognize that our non-perturbative treatment neglects the effect of collisions as mentioned above. Dissipation appears in a manner similar to Landau damping. In the Hartree approximation there does not seem to be a natural way to incorporate scattering processes because this is a mean-field theory. However the large N expansion allows a consistent treatment of scattering processes for which the first contribution (2-2 particle scattering) will appear at $\mathcal{O}(1/N)$. This will be a necessary next step in order to fully
understand the collisional thermalization which is the second stage of the reheating process. This will clearly be a fascinating and worthy endeavour that we expect to undertake soon.

6 Acknowledgments

D. B. would like to thank B. L. Hu, E. Mottola and D. Jasnow for very illuminating comments and discussions. He would also like to thank M. Madrid for computational assistance and LPTHE for warm hospitality. The authors acknowledge grants from the Pittsburgh Supercomputer Center: PHY930049P; PHY940005P. D. S. Lee would like to thank Y. Y. Charng for computational help. D. S. Lee was partially supported by a Mellon Fellowship, D. B. and D. S. Lee were partially supported by N.S.F. Grant No: PHY-9302534 and N.S.F. Grant No: INT-9216755 (binational program), they would like to thank R. Rivers for stimulating conversations. H. J. de V. is partially supported by the CNRS/NSF binational program and thanks the Dept. of Physics at U. of Pittsburgh for hospitality. R. H. and A. S. were partially supported by U.S.DOE under contract DE-FG02-91-ER40682.
7 Appendix I

In this appendix we give an alternative but equivalent method to derive the equations of motion based on the direct time evolution of initial density matrices. This method gives rise to equations identical in form to those given in section 2 and 3. However, this method might have greater applicability for initial density matrices that are not of the thermal type, such as might appear in chaotic inflation models, but is restricted to the gaussian approximation. For more details the reader is referred to \[27\].

Our starting point is again the Liouville equation for the density matrix of the system:

\[ i\hbar \frac{\partial \rho(t)}{\partial t} = [H_{\text{evol}}, \rho(t)], \quad (141) \]

where \( H_{\text{evol}} \) is the Hamiltonian of the system that drives the non-equilibrium evolution of the system.

Next we use the density matrix to extract the order parameter from the field in the Schrödinger picture:

\[ \phi(t) = \frac{1}{\Omega} \int d^3 x \, \text{Tr} \left[ \rho(t) \Phi(\vec{x}) \right], \quad (142) \]

with \( \Omega \) the spatial volume (later taken to infinity), and \( \Phi(\vec{x}) \) the field in the Schrödinger picture. Using the Liouville equation together with the Hamiltonian:

\[ H = \int d^3 x \left\{ \frac{\Pi^2}{2} + \frac{1}{2} (\nabla \Phi)^2 + V(\Phi) \right\} \quad (143) \]

and the standard equal time commutation relations between a field and its canonically conjugate momentum, we find the equations:

\[
\frac{d\phi(t)}{dt} = \frac{1}{\Omega} \int d^3 x \langle \Pi(\vec{x},t) \rangle = \frac{1}{\Omega} \int d^3 x Tr \left[ \rho(t) \Pi(\vec{x}) \right] = \pi(t) \quad (144)
\]

\[
\frac{d\pi(t)}{dt} = -\frac{1}{\Omega} \int d^3 x \langle \frac{\delta V(\Phi)}{\delta \Phi(\vec{x})} \rangle. \quad (145)
\]

From these equations we can find the equation of motion for the order parameter \( \phi(t) \):

\[
\frac{d^2 \phi(t)}{dt^2} + \frac{1}{\Omega} \int d^3 x \langle \frac{\delta V(\Phi)}{\delta \Phi(\vec{x})} \rangle = 0 \quad (146)
\]

We expand the field operator as \( \Phi(\vec{x}) = \phi(t) + \psi(\vec{x},t) \), insert this expansion into equation (146) and keep only the quadratic terms in the fluctuation field \( \psi(\vec{x},t) \). Doing this yields the equation:
\[
\frac{d^2 \phi(t)}{dt^2} + V'(\phi(t)) + \frac{V'''(\phi(t))}{2\Omega} \int d^3x \langle \psi^2(\vec{x}, t) \rangle + \cdots = 0
\]  
(147)

Here the primes stand for derivatives with respect to \( \phi \).

To make sense of the above equation, we need to specify how the expectation value is to be taken. This entails a specification of the basis in functional space we will use to write the density matrix in, and, once given this basis, what the form of the density matrix will be.

In order to generate an expansion in \( \hbar \), we choose the density matrix to be a Gaussian with a covariance of order \( 1/\hbar \), in the basis given by that of the spatial Fourier components of the fluctuation field \( \psi(\vec{x}, t) \). Thus write \( \psi(\vec{x}, t) \) as:

\[
\psi(\vec{x}, t) = \frac{1}{\sqrt{\Omega}} \sum_k \psi_k(t) e^{-i \vec{k} \cdot \vec{x}}
\]  
(148)

If we are choosing the state to be Gaussian, we need to make the Hamiltonian quadratic, in order that the time evolved state remain Gaussian, thus allowing for a consistent approximation scheme. To do this we take the Hamiltonian above and expand it in terms of the fluctuation field out to quadratic order. If we then insert the expansion in equation (148) into the truncated Hamiltonian obtained in this way, we arrive at:

\[
H_{\text{quad}} = \Omega V(\phi(t)) + \frac{1}{2} \sum_k \left\{ -\hbar^2 \frac{\delta^2}{\delta \psi_k(t) \delta \psi_{-k}(t)} + 2V'_k(\phi(t)) \psi_{-k}(t) + \omega_k^2(t) \psi_k(t) \psi_{-k}(t) \right\} 
\]  
(149)

\[
V'_k(\phi(t)) = V'(\phi(t)) \sqrt{\Omega} \delta_{k,0}
\]

\[
\omega_k^2(t) = \vec{k}^2 + V''(\phi(t)).
\]  
(150)

For the density matrix in this basis, we make the following ansatz [27]:

\[
\rho[\Phi, \bar{\Phi}, t] = \prod_k \mathcal{N}_k(t) \exp \left\{ - \left[ \frac{A_k(t)}{2\hbar} \psi_k(t) \psi_{-k}(t) + \frac{A_k^*(t)}{2\hbar} \bar{\psi}_k(t) \bar{\psi}_{-k}(t) \right] + \frac{B_k(t)}{\hbar} \psi_k(t) \bar{\psi}_{-k}(t) \right\} + i \pi_k(t) \left( \psi_{-k}(t) - \bar{\psi}_{-k}(t) \right) \right\} 
\]  
(151)

\[
\psi_k(t) = \Phi_k - \phi(t) \sqrt{\Omega} \delta_{k,0}
\]

\[
\bar{\psi}_k(t) = \Phi_k - \phi(t) \sqrt{\Omega} \delta_{k,0}
\]  
(152)

where \( \phi(t) = \langle \Phi(\vec{x}) \rangle \) and \( \pi_k(t) \) is the Fourier transform of \( \langle \Pi(\vec{x}) \rangle \). This form of the density matrix is dictated by the hermiticity condition \( \rho^*[\Phi, \bar{\Phi}, t] = \rho^*[\bar{\Phi}, \Phi, t] \); as a result of this,
$B_k(t)$ is real. The kernel $B_k(t)$ determines the amount of “mixing” in the density matrix, since if $B_k = 0$, the density matrix describes a pure state because it is a wave functional times its complex conjugate.

In the Schrödinger picture, the Liouville equation (144) becomes

$$i\hbar \frac{\partial \rho[\Phi, \tilde{\Phi}, t]}{\partial t} = \sum_k \left\{ -\hbar^2 \left( \frac{\delta^2}{\delta \psi_k \delta \tilde{\psi}_{-k}} - \frac{\delta^2}{\delta \tilde{\psi}_k \delta \psi_{-k}} \right) \right.$$  
$$+ V'_{-k}(t) \left( \psi_k - \tilde{\psi}_k \right) + \frac{1}{2} \omega_k^2(t) \left( \psi_{-k} \psi_k - \tilde{\psi}_{-k} \tilde{\psi}_{-k} \right) \right\} \rho[\Phi, \tilde{\Phi}, t] \tag{154}$$

Since the modes do not mix in this approximation to the Hamiltonian, the equations for the kernels in the density matrix are obtained by comparing the powers of $\psi$ on both sides of the above equation. We obtain the following equations for the coefficients:

$$i \dot{N}_k = \frac{1}{2} (A_k - A_k^*) \tag{155}$$
$$i \dot{A}_k = \left[ A_k^2 - B_k^2 - \omega_k^2(t) \right] \tag{156}$$
$$i \dot{B}_k = B_k (A_k - A_k^*) \tag{157}$$
$$-\dot{\pi}_k = V'(\phi(t)) \sqrt{\Omega} \delta_{\vec{k},0} \tag{158}$$
$$\dot{\phi} = \pi \tag{159}$$

The last two equations are identified with the classical equations of motion for the order parameter (147). The equation for $B_k(t)$ reflects the fact that a pure state $B_k = 0$ remains pure under time evolution.

At this point we have not specified any initial conditions; if we were to specify that the initial state is thermal, we would be led to the following initial conditions for the kernels $A_k(t)$, $B_k(t)$, as well as for the order parameter and its momentum:

$$A_k(0) = A_k^*(0) = w_k(0) \coth \left[ \beta_0 \hbar w_k(0) \right] \tag{160}$$
$$B_k(0) = - \frac{w_k(0)}{\sinh \left[ \beta_0 \hbar w_k(0) \right]} \tag{161}$$
$$\dot{N}_k(0) = \left[ \frac{w_k(0)}{\pi \hbar} \tanh \left[ \frac{\beta_0 \hbar w_k(0)}{2} \right] \right]^{\frac{1}{2}} \tag{162}$$
$$\dot{\phi}(0) = \phi_0 \; ; \; \pi(0) = \pi_0 \tag{163}$$

Here $\beta_0$ is the inverse temperature at the initial time (which we have taken to infinity in this work), while the frequencies $w_k(0)$ are given by:
\[ w_k^2(0) = \vec{k}^2 + V''(\phi_{cl}(t_o)) \]  

(164)

Note that up to this point, the formalism we have derived is quite general; it is only in the specification of the initial conditions that the thermal density matrix comes in. This is to be contrasted with the closed time path method, which implicitly assumes a thermal density matrix for the initial state.

Defining a complex function

\[
\begin{align*}
A_k(t) & \equiv A_{kR}(t) + A_{kI}(t) \\
A_{kR}(t) & = A_{kR}(t) \tanh \beta_0 \hbar w_k(0) = -B_k(t) \sinh \beta_0 \hbar w_k(0) \\
A_{kI}(t) & = A_{kI}(t) \\
A_{kR}(0) & = w_k(0) \\
A_{kI}(0) & = 0,
\end{align*}
\]  

(165)

we find that \( A_k(t) \) satisfies the following Ricatti equation:

\[
i \dot{A}_k(t) = A_k(t)^2 - w_k(t)^2
\]  

(166)

This equation can be recast in a form closer to that for the mode functions of section 2 by the following set of transformations:

\[
\begin{align*}
i \dot{A}_k(t) & \equiv \frac{\dot{\chi}_k}{\chi_k} \\
\chi_k(t) & \equiv \frac{1}{\sqrt{w_k(0)}} (U_{1k} + iU_{2k})
\end{align*}
\]  

(167)

From the thermal initial conditions, and the equation for \( A_k(t) \), we find that the mode functions \( U_{\alpha k} \) satisfy the following equations:

\[
\begin{align*}
\{ \frac{d^2}{dt^2} + w_k^2(t) \} U_{\alpha k}(t) & = 0 \\
U_{1k}(0) & = 1; \, \dot{U}_{1k}(0) = 0 \\
U_{2k}(0) & = 0; \, \dot{U}_{2k}(0) = w_k(0)
\end{align*}
\]  

(168)

Finally, using these mode functions to calculate the expectation value of \( \psi(\vec{x}, t)^2 \) that enters into the equation of motion for the field expectation value, we find that equation (146) becomes:
\[ \ddot{\phi}(t) + V'(\phi(t)) + \frac{V''(\phi(t))}{2} \frac{\hbar}{2} \int \frac{d^3k}{(2\pi)^3} \frac{U_{1k}(t)^2 + U_{2k}(t)^2}{w_k(0)} \coth \frac{\beta_0 h w_k(0)}{2} = 0 \] (169)

In the limit that the initial temperature goes to zero, these equations are the same as the one-loop equations derived in section 2. To obtain the evolutions equations in the Hartree approximation or in the large \( N \) limit in the \( \mathcal{O}(N) \) theory one invokes the Hartree factorization and proposes a gaussian ansatz as in the one loop case solving for the covariance following the same steps as above. We have included this derivation since it may be applicable to more general situations, and to give an alternative to the complex path formulation for the gaussian approximation in non-equilibrium problems.
In this appendix we provide some necessary details that may help the reader reproduce our numerical results. These concern the numerical integration for the mode functions $U_q^+(\tau)$ appearing in various models under different approximation schemes. By examining the corresponding differential equations (for example, eq. (135)) with the corresponding boundary conditions a typical integration routine will face extremely large derivatives when the values of the wavevectors $q$ reach the cutoff. We have used improved fourth order Runge-Kutta algorithms for the numerical integration, but any algorithm will face the same problems for large enough cutoff. We recognize however, that for large $q$ (typically of the order of the dimensionless cutoff momentum $\approx 100$) for which $q^2 \gg 1 + \eta^2(\tau) + g\Sigma(\tau)$ the solutions to these equations are like plane waves with fast varying phases in time. It is precisely this fast variation of the phase that leads to potential numerical problems. We introduce a WKB-like ansatz to take out the fast varying phases from $U_q^+(\tau)$ by defining new mode functions which are smoothly varying in time for large $q$.

First of all, we define the new mode functions $U_q^+(\tau)$ as follows:

$$U_q^+(\tau) = e^{i \int_0^\tau d\tau' \tilde{\omega}_q(\tau')} U_q^+(\tau) ; \quad \tilde{\omega}_q = \sqrt{q^2 + 1 + \eta^2(\tau)}$$

The initial conditions for $U_q^+(\tau)$ defined above can be read off from (115) for the symmetric phase:

$$U_q^+(0) = 1 ; \quad \dot{U}_q^+(0) = 0$$

By substituting these $U_q^+(\tau)$ into the equations expressed previously in (135) we find that the slowly varying functions obey:

$$\ddot{U}_q^+ - 2i\tilde{\omega}_q(\tau)\dot{U}_q^+ - i\tilde{\omega}_q(\tau)U_q^+ + g\Sigma(\tau)U_q^+ = 0$$

In can be seen that in the large $q$ limit, $(q^2 \gg 1 + \eta^2(\tau) + g\Sigma(\tau))$,

$$U_q^+(\tau) \to 1 ; \quad \dot{U}_q^+(\tau) \to 0 ; \quad \text{for } \tau > 0$$

which are smooth functions at all times for large $q$.

For the broken symmetry case in $O(N)$ model, the mode functions $U_q^+(\tau)$ are solutions of the differential equation

$$\ddot{U}_q^+ - 2i\tilde{\omega}_q(\tau)\dot{U}_q^+ - i\tilde{\omega}_q(\tau)U_q^+ + [-2 + g\Sigma(\tau)]U_q^+ = 0$$
with the same initial conditions as in (171).
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Figure Captions

Fig.1: Diagrams contributing to the equation of motion up to one-loop order, and in the amplitude expansion up to $\mathcal{O}(\phi^3)$.

Fig.2: First order quantum correction for discrete symmetry case $\eta_1(\tau)$ for $\eta_1(0) = 0; \dot{\eta}_1(0) = 0; \eta_\text{cl}(0) = 1; \dot{\eta}_\text{cl}(0) = 0$. The cutoff is $\Lambda/m_R = 100$.

Fig.3: First order quantum correction for $O(2)$ model $\eta_1(\tau)$ for $\eta_1(0) = 0; \dot{\eta}_1(0) = 0; \eta_\text{cl}(0) = 0.6; \dot{\eta}_\text{cl}(0) = 0$. The cutoff is $\Lambda/\mu_R = 100$.

Fig. 4: One loop diagrams contributing to the effective action up to $\mathcal{O}((\phi^\pm)^4)$. The dashed external legs correspond to the zero mode.

Fig. 5.a: $\eta(\tau)$ vs $\tau$ in the Hartree approximation, unbroken symmetry case. $g = 0.1; \eta(0) = 1; \Lambda/M_R = 100$.

Fig. 5.b: $\Sigma(\tau)$ vs $\tau$ for the same case as in fig. (5.a).

Fig. 5.c: $N(\tau)$ vs $\tau$ for the same case as in fig. (5.a).

Fig. 6.a: $\eta(\tau)$ vs $\tau$ in the Hartree approximation, unbroken symmetry case. $g = 0.1; \eta(0) = 4; \Lambda/M_R = 100$.

Fig. 6.b: $\Sigma(\tau)$ vs $\tau$ for the same case as in fig. (6.a).

Fig. 6.c: $N(\tau)$ vs $\tau$ for the same case as in fig. (6.a).

Fig. 7.a: $\eta(\tau)$ vs $\tau$ in the Hartree approximation, unbroken symmetry case. $g = 0.1; \eta(0) = 5; \Lambda/M_R = 100$.

Fig. 7.b: $\Sigma(\tau)$ vs $\tau$ for the same case as in fig. (7.a).

Fig. 7.c: $N(\tau)$ vs $\tau$ for the same case as in fig. (7.a).

Fig. 8.a: $\eta(\tau)$ vs $\tau$ in the Hartree approximation, unbroken symmetry case. $g = 0.05; \eta(0) = 5; \Lambda/M_R = 100$.

Fig. 8.b: $\Sigma(\tau)$ vs $\tau$ for the same case as in fig. (8.a).

Fig. 8.c: $N(\tau)$ vs $\tau$ for the same case as in fig. (8.a).

Fig. 9.a: $\eta(\tau)$ vs $\tau$ in the Hartree approximation, broken symmetry case. $g = 10^{-5}; \eta(0) = 10^{-5}; \Lambda/\mu_R = 100$.

Fig. 9.b: $\Sigma(\tau)$ vs $\tau$ for the same case as in fig. (9.a).

Fig. 9.c: $N(\tau)$ vs $\tau$ for the same case as in fig. (9.a).

Fig. 10.a: $\eta(\tau)$ vs $\tau$ in the large $N$ approximation in the $O(N)$ model, unbroken symmetry case. $g = 0.1; \eta(0) = 1; \Lambda/M_R = 100$.

Fig. 10.b: $\Sigma(\tau)$ vs $\tau$ for the same case as in fig. (10.a).

Fig. 10.c: $N(\tau)$ vs $\tau$ for the same case as in fig. (10.a).

Fig. 11.a: $\eta(\tau)$ vs $\tau$ in the large $N$ approximation in the $O(N)$ model, unbroken symmetry case. $g = 0.1; \eta(0) = 2; \Lambda/M_R = 100$. 

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Fig. 11.b: $\Sigma(\tau)$ vs $\tau$ for the same case as in fig. (11.a).
Fig. 11.c: $N(\tau)$ vs $\tau$ for the same case as in fig. (11.a).
Fig. 12.a: $\eta(\tau)$ vs $\tau$ in the large N approximation in the O(N) model, unbroken symmetry case. $g = 0.3$ ; $\eta(0) = 1$ ; $\Lambda/M_R = 100$.
Fig. 12.b: $\Sigma(\tau)$ vs $\tau$ for the same case as in fig. (12.a).
Fig. 12.c: $N(\tau)$ vs $\tau$ for the same case as in fig. (12.a).
Fig. 13.a: $\eta(\tau)$ vs $\tau$ in the large N approximation in the O(N) model, broken symmetry case. $g = 0.1$ ; $\eta(0) = 0.5$ ; $\Lambda/\mu_R = 100$.
Fig. 13.b: $\Sigma(\tau)$ vs $\tau$ for the same case as in fig. (13.a).
Fig. 13.c: $N(\tau)$ vs $\tau$ for the same case as in fig. (13.a).
Fig. 13.d: Number of particles in (dimensionless) wavevector $q$, $N_q(\tau)$ at $\tau = 200$.
Fig. 13.e: Number of particles in (dimensionless) wavevector $q$, $N_q(\tau)$ at $\tau = 13$.
Fig. 13.f: Number of particles in (dimensionless) wavevector $q$, $N_q(\tau)$ at $\tau = 25$.
Fig. 13.g: Number of particles in (dimensionless) wavevector $q$, $N_q(\tau)$ at $\tau = 50$.
Fig. 14.a: $\eta(\tau)$ vs $\tau$ in the large N approximation in the O(N) model, broken symmetry case. $g = 10^{-7}$ ; $\eta(0) = 10^{-5}$ ; $\Lambda/\mu_R = 100$.
Fig. 14.b: $\Sigma(\tau)$ vs $\tau$ for the same case as in fig. (14.a).
Fig. 14.c: $N(\tau)$ vs $\tau$ for the same case as in fig. (14.a).
Fig. 15.a: $\eta(\tau)$ vs $\tau$ in the large N approximation in the O(N) model, broken symmetry case. $g = 10^{-12}$ ; $\eta(0) = 10^{-5}$ ; $\Lambda/\mu_R = 100$.
Fig. 15.b: $\Sigma(\tau)$ vs $\tau$ for the same case as in fig. (15.a).
Fig. 15.c: $N(\tau)$ vs $\tau$ for the same case as in fig. (15.a).