Development of an analytical method for calculating beams on a variable elastic Winkler foundation

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Abstract. An analytical method has been developed for calculating beams lying on a continuous elastic Winkler foundation with a variable coefficient of subgrade resistance. The method is based on the exact solution of the differential equation of beam bending for the case when the bed coefficient is an arbitrary continuous function. In an analytical form, formulas for displacements and internal efforts are obtained. An analytical method for the numerical implementation of the exact solutions found is proposed.

1. Introduction

In engineering practice, beam elements of structures lying on a solid elastic foundation are often found. Such structures can include railway sleepers, strip foundations of buildings, foundations of dams, various pipelines laid on the ground and others.

Among the soil base models, the most common is the Winkler model, or the coefficient of subgrade resistance hypothesis. At the same time, there are a number of modifications of this model, which in an integral form make it possible to take into account the inhomogeneous properties of the foundation. The most common modification is the variable coefficient of the subgrade resistance model. In particular, such a model has found wide application in calculating the stress-strain state of foundations of structures lying on loess soils, which are characterized by subsidence. In this case, the coefficient of subgrade resistance is a variable, depending on the coordinate in which the settlement of the surface of the foundation is determined. From a mathematical point of view, this leads to the need to solve the corresponding differential equations with variable coefficients. A universal method for solving such equations is currently lacking. Perhaps this is why the calculation of beams on a variable elastic foundation is rarely found in the scientific literature.

Among the articles, which are devoted to the bending of a beam on an elastic Winkler base with a variable foundation coefficient, we note publications [1-5]. The article [1] is devoted to the case of a nonlinear, minus four power variation of the stiffness coefficient, allowing for closed-form representations. In the [2] the special case of a linear variation of the foundation coefficient is assumed. The analytical solution of the governing Ordinary Differential Equation is derived and represented in explicit closed-form in terms of generalized hypergeometric functions. The article [3] presents an analytical method, accompanied by a numerical scheme. The method employs a Green’s function formulation, which results in a system of nonsingular integral equations. These equations can be discretized to yield a system of linear algebraic equations that can be solved by elementary numerical techniques. Authors [4] showed, that the solution may be achieved using only a small number of elements along the beam. A numerical example demonstrates the efficiency and accuracy of the procedure. In article [5] the case when the foundation coefficient is a power function of the coordinate...
is considered. Herein, an analytical solution is obtained and evaluated for the case when power is equal to l. The numerical technique is developed for obtaining two of the solutions for other values of power.

As analysis shows, the development of an analytical method for calculating beams on a variable elastic Winkler base with a variable foundation coefficient is relevant.

2. Object of study, basic notation and the bending equation

The object of study is a beam of constant stiffness $EI$ lying on a continuous variable elastic foundation, for which the Winkler hypothesis is accepted. Fig. 1 shows the design diagram of the beam, where $q(x)$ is a given distributed variable transverse load, $y(x)$ – deflection, $\varphi(x)$ – angle of rotation of the beam. Fig. 2 shows the internal forces arising in the cross sections of the beam element, namely, the bending moment $M(x)$ and the transverse force $Q(x)$.

![Figure 1. Beam design](image1)

![Figure 2. Internal forces](image2)

The stress-strain state of the beam is completely described by four components, namely, displacements $y(x)$, $\varphi(x)$ and internal forces $M(x)$, $Q(x)$. Further we will call them beam state parameters.

According to Winkler’s hypothesis, the reaction of the base to the beam $R(x)$ and the deflection of the beam $y(x)$ are interconnected by the equality $R(x) = -k(x)y(x)$, where $k(x)$ is a continuous variable linear coefficient of subgrade resistance. Relatively $k(x)$, we take the notation form $k(x) = k_0B(x)$, where $k_0$ is the value of the linear coefficient of the subgrade resistance at some characteristic point of the beam (for example, at a point $x = 0$); $B(x)$ is dimensionless continuous function expressing the law of variation of the coefficient of subgrade resistance along the axis of the beam.

We will write a similar representation also for a given load: $q(x) = q_0C(x)$, where $q_0$ is the load at some characteristic point of the beam; $C(x)$ is the dimensionless continuous function that sets the law of change of load from the coordinate $x$.

The differential equation of beam bending in our case takes the form

$$EI y'''(x) + k_0B(x)y(x) = q_0C(x). \quad (1)$$

After the deflection function $y(x)$ is found from this equation, the remaining state parameters are found from the differential dependences known from the theory of beam bending:

$$\varphi(x) = y'(x); \quad M(x) = -EIy''(x); \quad Q(x) = -EIy'''(x).$$

The aim of the work is to develop an analytical method for calculating the beam bending under the conditions described above. To achieve the goal, the following tasks are formulated and solved:

- find the exact solution to the equation (1);
- get formulas for beam state parameters;
- indicate the analytical method for the numerical implementation of the exact solutions found.

3. Materials and Methods
Earlier in [6], the method of direct integration of differential equations with continuous variable coefficients was developed. Using this method, exact solutions for a number of differential equations of statics and dynamics were also constructed there. The essence of the proposed method can also be considered, for example, by publications [7-10], where a number of problems in the mechanics of a deformable solid are solved with its help.

In this work, the exact solution of the differential bending equation (1), which formed the basis of the study, was constructed by direct integration.

4. Exact solution to the bending equation and formulas for beam state parameters
We denote by the $X_n(x)$ $(n = 1,2,3,4)$ sought dimensionless fundamental solutions of the homogeneous equation
\[ EI y''''(x) + k_n B(x) y(x) = 0 , \] (2)
and through
\[ X^*(x) = \frac{q_0}{EI} X_5(x) \] (3)
– partial solution of an inhomogeneous equation (1), where $X_5(x)$ – sought dimensionless function.

Along with the bending equation (1), we will also consider a system of differential equations equivalent to it. As a vector of system’s unknown, we take a vector whose components are the beam state parameters
\[
\Phi(x) = \begin{pmatrix} y(x) \\ \varphi(x) \\ M(x) \\ Q(x) \end{pmatrix} = \begin{pmatrix} y(x) \\ y'(x) \\ -EI y''(x) \\ -EI y'''(x) \end{pmatrix} . \] (4)

In this case, the system will be written as
\[
\frac{d\Phi(x)}{dx} = H(x)\Phi(x) - f(x) , \] (5)
where
\[
H(x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1/EI & 0 \\ 0 & 0 & 0 & 1 \\ k_n B(x) & 0 & 0 & 0 \end{pmatrix}, \quad f(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ q_0 C(x) \end{pmatrix} . \] (6)

The solutions of the homogeneous equation $X_n(x)$ $(n = 1,2,3,4)$ and the function $X_5(x)$ will be sought in the form of series in powers of the dimensionless parameter $\beta$ with variable coefficients
\[
X_n(x) = \beta_n,0(x) - K \beta_{n,1}(x) + K^2 \beta_{n,2}(x) - K^3 \beta_{n,3}(x) + \ldots \quad (n = 1,2,3,4,5) , \] (7)
where \( K = \frac{k_1 l^4}{EI} \). Moreover, following the terminology adopted in [6], the functions \( \beta_{n,0}(x) \) will be called initial, and the functions \( \beta_{n,k}(x) \) \( (k = 1, 2, 3, \ldots) \) – generators.

For now, we assume that series (7), as well as similar series composed of the first four derivatives of functions \( \beta_{n,0}(x), \beta_{n,k}(x) \) \( (k = 1, 2, 3, \ldots) \), converge uniformly. In this case, the operation of differentiating the rows will be possible.

The initial and generating functions will be sought from the conditions that the functions \( X_{n}(x) \) \( (n = 1, 2, 3, 4) \) satisfy the homogeneous equation (2), and the function (3) – the inhomogeneous equation (1), that is, from the conditions that the equalities are satisfied:

\[
X_{n}^{''''}(x) + \frac{K}{l^4} B(x) X_{n}(x) = 0 \quad (n = 1, 2, 3, 4) ; \tag{8}
\]

\[
X_{n}^{''''}(x) + \frac{K}{l^4} B(x) X_{n}(x) = \frac{1}{l^4} C(x) . \tag{9}
\]

Substituting into equalities (8), (9) instead of \( X_{n}(x) \) their value (7), after transformations we get:

\[
\beta_{n,0}^{''''}(x) + \sum_{k=1}^{\infty} (-K)^k \left( \beta_{n,k}^{''''}(x) - \frac{1}{l^4} B(x) \beta_{n,k-1}(x) \right) = 0 \quad (n = 1, 2, 3, 4) ; \tag{10}
\]

\[
\beta_{n,0}^{''''}(x) + \sum_{k=1}^{\infty} (-K)^k \left( \beta_{n,k}^{''''}(x) - \frac{1}{l^4} B(x) \beta_{n,k-1}(x) \right) = \frac{1}{l^4} C(x) . \tag{11}
\]

Hence, equating to zero all the coefficients for the degrees of the parameter \( -K \), including the zero degree, we arrive at the following equations with respect to the initial and generating functions:

\[
\beta_{n,0}^{''''}(x) = 0 \quad (n = 1, 2, 3, 4) ; \tag{12}
\]

\[
\beta_{n,0}^{''''}(x) = \frac{1}{l^4} C(x) ; \tag{13}
\]

\[
\beta_{n,k}(x) = \frac{1}{l^4} B(x) \beta_{n,k-1}(x) \quad (n = 1, 2, 3, 4, 5) . \tag{14}
\]

Before integrating these equations, we define the boundary conditions:

\[
\begin{pmatrix}
\beta_{1,0}(0) & \beta_{2,0}(0) & \beta_{3,0}(0) & \beta_{4,0}(0) \\
\beta_{1,0}'(0) & \beta_{2,0}'(0) & \beta_{3,0}'(0) & \beta_{4,0}'(0) \\
\beta_{1,0}''(0) & \beta_{2,0}''(0) & \beta_{3,0}''(0) & \beta_{4,0}''(0) \\
\beta_{1,0}'''(0) & \beta_{2,0}'''(0) & \beta_{3,0}'''(0) & \beta_{4,0}'''(0)
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
1/l \\
0
\end{pmatrix}
= \begin{pmatrix}
0 \\
1/l \\
0 \\
1/l^2 \\
0
\end{pmatrix} ; \tag{15}
\]

\[
\beta_{n,0}(0) = \beta_{n,0}'(0) = \beta_{n,0}''(0) = \beta_{n,0}'''(0) = 0 ; \tag{16}
\]

\[
\beta_{n,k}(0) = \beta_{n,k}'(0) = \beta_{n,k}''(0) = \beta_{n,k}'''(0) = 0 \quad (n = 1, 2, 3, 4, 5) \quad (k = 1, 2, 3, \ldots) . \tag{17}
\]

Note that conditions (15)-(17) are chosen in such a way as to ensure that the integration constants are equal zero.

As a result, after integration (12) - (14), we will have the following set of formulas:

\[
\beta_{n,0}(x) = \frac{1}{(n-1)! l} \left( \frac{x}{l} \right)^{n-1} ; \tag{18}
\]

\[
\beta_{n,k}(x) = \frac{1}{l^4} \int_{0}^{x} \int_{0}^{x} \int_{x}^{x} C(x) dx dx dx ; \tag{19}
\]
\[ \beta_{n,k}(x) = \frac{1}{t^k} \iiint_{0 \leq x \leq t} B(x) \beta_{n,k-1}(x) \, dx \, dx \, dx \quad (n = 1, 2, 3, 4, 5) \quad (k = 1, 2, 3, \ldots). \]  

(20)

Thus, by means of recurrence formula (20), for each of the initial functions (18), (19), the corresponding generating functions are found. Obviously, for such functions, equalities (8), (9) are satisfied identically.

In addition to recurrence formula (20), the generating functions can be represented in expanded form

\[ \beta_{n,k}(x) = \frac{1}{t^k} \iiint_{0 \leq x \leq t} B(x) \cdots B(x) \int_{0 \leq x \leq t} B(x) \beta_{n,0}(x) \, dx \, dx \, \ldots \, dx \quad (n = 1, 2, 3, 4, 5). \]  

(21)

The right side of the last formula contains \( 4k \) integrals, not taking into account the integrals through which the initial function is expressed (19).

After the initial and generating functions are defined, it is necessary to prove the convergence of the series (7), as well as the series of derivatives. We give a proof here, for example, for the first four of the series (7). Assuming

\[ g = \max_{x \in [0,1]} B(x) \]

and using any of formulas (20), (21), we obtain the following estimates for generating functions

\[ \beta_{n,k}(x) \leq \frac{g^k}{(n + 4k - 1)!} \left( \frac{x}{l} \right)^{n+4k-1} \quad (n = 1, 2, 3, 4) \quad (k = 1, 2, 3, \ldots). \]

Using these estimates, we obtain

\[ |X_n(x)| \leq \frac{\sum_{n=0}^{\infty} K^n \beta_{n,k}(x)}{\sum_{n=0}^{\infty} \frac{1}{(gK)^{n-1}} (\sqrt{gK}^x)^{n+4k-1}} = \frac{1}{(gK)^{n-1}} Y_n(\sqrt{gK}^x \frac{x}{l}) \quad (n = 1, 2, 3, 4), \]

(22)

where \( Y_n\left(\sqrt{gK} \frac{x}{l}\right) \) – A.N. Krylov functions, which are defined by the equalities:

\[ Y_1(z) = \frac{1}{2} (\cosh z + \cos z); \quad Y_2(z) = \frac{1}{2} (\sinh z + \sin z); \]

\[ Y_3(z) = \frac{1}{2} (\cosh z - \cos z); \quad Y_4(z) = \frac{1}{2} (\sinh z - \sin z). \]

Based on (22), we conclude that the series (7) for the values \( n = 1, 2, 3, 4 \) converge absolutely and uniformly. Similarly, the convergence of the remaining series is proved.

Thus, formulas (7), (18)-(21) define four solutions of the homogeneous equation (2) and a particular solution of equation (1).

Each of the solutions \( X_n(x) (n = 1, 2, 3, 4) \) generates a vector by formula (4)

\[ \Phi_n(x) = \begin{pmatrix} X_n(x) \\ X'_n(x) \\ -EI X''_n(x) \\ -EI X'''_n(x) \end{pmatrix} \quad (n = 1, 2, 3, 4) \]

– solving a homogeneous system of differential equations corresponding to (5). Then the matrix
\[
\Omega(x) = \begin{pmatrix}
X_1(x) & X_2(x) & X_3(x) & X_4(x) \\
X'_1(x) & X'_2(x) & X'_3(x) & X'_4(x) \\
-EI X''_1(x) & -EI X''_2(x) & -EI X''_3(x) & -EI X''_4(x) \\
-EI X'''_1(x) & -EI X'''_2(x) & -EI X'''_3(x) & -EI X'''_4(x)
\end{pmatrix},
\]

(23)

composed of these vectors will also satisfy a homogeneous system. Moreover, taking into account (15), (17), we find:

\[
\Omega(0) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1/l & 0 & 0 \\
0 & 0 & -EI/l^2 & 0 \\
0 & 0 & 0 & -EI/l^3
\end{pmatrix}; \quad |\Omega(0)| = \frac{(EI)^2}{l^6} \neq 0.
\]

Therefore, matrix (23) is the fundamental matrix [11] for system (5), and \(X_n(x) (n=1,2,3,4)\) is the fundamental solutions of equation (2).

Multiplying on the right \(\Omega(x)\) by a constant matrix \(\Omega^{-1}(0)\), we obtain a new fundamental matrix \(\Lambda(x) = \Omega(x)\Omega^{-1}(0)\) for which

\[
\Lambda(0) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

(24)

As is known [11], a fundamental matrix possessing property (24) is uniquely determined and is called a matrix. Moreover, the general solution of the homogeneous system corresponding to (5) is written in the form

\[
\Phi(x) = \Lambda(x)\Phi(0).
\]

Forming with the help of (3) the vector

\[
\Phi^*(x) = \frac{q_d t^4}{EI} \begin{pmatrix}
X_1(x) \\
X'_1(x) \\
-EI X''_1(x) \\
-EI X'''_1(x)
\end{pmatrix},
\]

we obtain a particular solution to the inhomogeneous system (5), which is easy to verify by substitution. Considering (16), (17), for this vector we also find

\[
\Phi^*(0) = \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}.
\]

Therefore, the general solution of the inhomogeneous system of differential equations (5) can be written as follows

\[
\Phi(x) = \Lambda(x)\Phi(0) + \Phi^*(x).
\]

(25)
In the calculations it is convenient to operate with dimensionless quantities. Based on the analysis of formulas (18)-(21) and the results of [6], we conclude that the initial and generating functions \( \beta_{n,k}(x) \) \((n=1,2,3,4,5)\) \((k=0,1,2,...)\) are dimensionless. As a result, solutions \( X_n(x) \) \((n=1,2,3,4,5)\) will also be dimensionless.

Before expanding elementwise equality (25), we introduce the functions:

\[
\hat{\tilde{X}}_n(x) = lX_n'(x); \quad \bar{\tilde{X}}_n(x) = l^2X_n''(x); \quad \hat{\tilde{X}}_n(x) = l^3X_n'''(x).
\]  

(26)

It follows directly from formulas (7), (26) that these functions are represented by series:

\[
\begin{align*}
\hat{\tilde{X}}_n(x) & = \hat{\beta}_{n,0}(x) - K\hat{\beta}_{n,1}(x) + K^2\hat{\beta}_{n,2}(x) - K^3\hat{\beta}_{n,3}(x) + \ldots \quad (n=1,2,3,4,5); \\
\bar{\tilde{X}}_n(x) & = \bar{\beta}_{n,0}(x) - K\bar{\beta}_{n,1}(x) + K^2\bar{\beta}_{n,2}(x) - K^3\bar{\beta}_{n,3}(x) + \ldots \quad (n=1,2,3,4,5); \\
\hat{\tilde{X}}_n(x) & = \hat{\beta}_{n,0}(x) - K\hat{\beta}_{n,1}(x) + K^2\hat{\beta}_{n,2}(x) - K^3\hat{\beta}_{n,3}(x) + \ldots \quad (n=1,2,3,4,5),
\end{align*}
\]  

(27)  

(28)  

(29)

where

\[
\begin{align*}
\hat{\beta}_{n,0}(x) & = l^2\beta_{n,0}(x), \quad \hat{\beta}_{n,0}(x) = l^2\beta_{n,0}(x), \quad \hat{\beta}_{n,0}(x) = l^2\beta_{n,0}(x) \quad (n=1,2,3,4,5), \\
\bar{\beta}_{n,k}(x) & = l^2\beta_{n,k}(x), \quad \bar{\beta}_{n,k}(x) = l^2\beta_{n,k}(x), \quad \bar{\beta}_{n,k}(x) = l^2\beta_{n,k}(x) \quad (k=0,1,2,...).
\end{align*}
\]  

(30)  

(31)

The functions (30) here act as initial ones, and the functions (31) as the generating functions for the corresponding series (27)-(29).

Unlike derivative functions \( X_n(x) \), functions (26) will be dimensionless [6].

Now revealing equality (25) and taking into account (26), we obtain the final formulas for the beam state parameters expressed in terms of the initial parameters and dimensionless functions:

\[
\begin{align*}
y(x) & = y(0)X_1(x) + \varphi(0)lX_2(x) - M(0)l^2EIX_3(x) - Q(0)l^3EI^2X_4(x) + \frac{q_0l^4}{EI}X_5(x); \\
\varphi(x) & = y(0)l\hat{X}_1(x) - \varphi(0)l\bar{X}_2(x) - M(0)l\hat{X}_3(x) - Q(0)l\bar{X}_4(x) + \frac{q_0l^3}{EI}\hat{X}_5(x); \\
M(x) & = -y(0)\frac{EI}{l^2}\hat{X}_1(x) - \varphi(0)\frac{EI}{l^2}\bar{X}_2(x) + M(0)\hat{X}_3(x) + Q(0)l\hat{X}_4(x) - q_0l^2\hat{X}_5(x); \\
Q(x) & = -y(0)\frac{EI}{l^3}\hat{X}_1(x) - \varphi(0)\frac{EI}{l^3}\bar{X}_2(x) + M(0)\frac{1}{l}\hat{X}_3(x) + Q(0)\frac{1}{l}\hat{X}_4(x) - q_0\hat{X}_5(x).
\end{align*}
\]  

(32)  

(33)  

(34)  

(35)

Theoretically, formulas (32)-(35) allow one to rely on the bending of a beam on a variable elastic foundation under any boundary conditions. However, when implementing these formulas in practice, difficulties may arise associated with the need to calculate integral expressions (20) or (21), which determine the generating functions. Therefore, for the final solution of the problem, we represent the generating functions by power series.

5. Representation of generating functions by power series

One of the classical methods of integration is based on replacing the integrand by the Maclaurin series. In our case, this approach allows us to calculate the integral (21) in an analytical form.

So, we will proceed from the fact that dimensionless functions \( B(x) \) and \( C(x) \) are represented by power series:

\[
B(x) = B_0 + B_1\left(\frac{x}{l}\right) + B_2\left(\frac{x}{l}\right)^2 + \ldots + B_j\left(\frac{x}{l}\right)^j + \ldots
\]  

(36)
\[ C(x) = C_0 + C_1 \left( \frac{x}{I} \right) + C_2 \left( \frac{x}{I} \right)^2 + ... + C_j \left( \frac{x}{I} \right)^j + ... , \]  

(37)

where

\[ B_0 = B(0), \quad B_j = \frac{l^j B^{(j)}(0)}{j!}, \quad C_0 = C(0), \quad C_j = \frac{l^j C^{(j)}(0)}{j!} \quad (j = 1, 2, 3, ...). \]

– dimensionless coefficients, and the index \((j)\) means the order of the derivative. Then, taking into account (37), the initial function \((19)\) after integration will appear in the form of a series

\[ \beta_{b,0}(x) = \left( \frac{x}{I} \right)^n \sum_{j=0}^{\infty} c_{n,k,j} \left( \frac{x}{I} \right)^j \quad (n = 1, 2, 3, 4, 5) (k = 1, 2, 3,...), \]

(38)

Directly from formula \((20)\) or \((21)\), taking into account \((18), (36)-(38)\), we conclude that the generating functions \(\beta_{n,k}(x)\) \((n=1,2,3,4,5)(k =1,2,3,...)\) will also be power series. Moreover, since formula \((21)\) contains \(4k\) integrals, the degree of the first term of the indicated series will be equal to \(n+4k-1\).

Therefore, the generating functions can be represented as

\[ \beta_{n,k}(x) = \left( \frac{x}{I} \right)^{n+4k-1} \sum_{j=0}^{\infty} c_{n,k,j} \left( \frac{x}{I} \right)^j \quad (n = 1, 2, 3, 4, 5) (k = 1, 2, 3,...), \]

(39)

where \(c_{n,k,j}\) – coefficients to be determined. Then

\[ \beta_{n,k-1}(x) = \left( \frac{x}{I} \right)^{n+4k-5} \sum_{j=0}^{\infty} c_{n,k-1,j} \left( \frac{x}{I} \right)^j \quad (n = 1, 2, 3, 4, 5) (k = 1, 2, 3,...). \]

(40)

Moreover, for the product of series \((36)\) and \((40)\) we obtain

\[ B(x)\beta_{n,k-1}(x) = \left( \frac{x}{I} \right)^{n+4k-5} \sum_{j=0}^{\infty} d_{n,k-1,j} \left( \frac{x}{I} \right)^j , \]

(41)

where

\[ d_{n,k-1,j} = \sum_{j=0}^{\infty} B_j c_{n,k-1,j}. \]

(42)

We substitute in the formula \((20)\) instead of \(\beta_{n,k}(x)\) and \(B(x)\beta_{n,k-1}(x)\) their values \((39)\) and \((41)\), after which we perform the integration. As a result, we arrive at equality

\[ \sum_{j=0}^{\infty} c_{n,k,j} \left( \frac{x}{I} \right)^j = \sum_{j=0}^{\infty} d_{n,k-1,j} p_{n,k,j} \left( \frac{x}{I} \right)^j \quad (n = 1, 2, 3, 4, 5) (k = 1, 2, 3,...), \]

where

\[ p_{n,k,j} = (n+4k+j-4)(n+4k+j-3)(n+4k+j-2)(n+4k+j-1). \]

From this, taking into account \((42)\), we find

\[ c_{n,k,j} = \frac{1}{p_{n,k,j}} \sum_{j=0}^{\infty} B_j c_{n,k-1,i} \quad (n = 1, 2, 3, 4, 5) (k = 1, 2, 3,...) (j = 0, 1, 2,...). \]

(43)
The formula found for the desired coefficients is recursive in index $k$. For complete certainty, such a formula needs to be supplemented with initial values $c_{n,0,j} (n=1,2,3,4,5)(j=0,1,2,...)$. For this purpose, we put $k = 1$ in formula (40) and compare the result in turn with formulas (18) and (38). As a result, we will have:

$$c_{n,0,0} = \frac{1}{(n-1)!}; \quad c_{n,0,j} = 0 \quad (n=1,2,3,4)(j=1,2,3,...); \quad (44)$$

$$c_{5,0,j} = \frac{C_j}{P_{5,0,j}} \quad (j=0,1,2,...). \quad (45)$$

Thus, the dimensionless coefficients of series (39) are completely determined by formulas (43)-(45). As a result, the generating functions $\beta_{n,k}(x) (n=1,2,3,4,5)(k=1,2,3,...)$ are represented by power series.

Finally, we give a list of formulas necessary for the numerical implementation of the method. Table 1 shows the initial functions for the values $n = 1,2,3,4$.

| Table 1. Initial functions |
|---------------------------|
| $\beta_{1,0}(x) = 1$;     |
| $\beta_{2,0}(x) = \frac{x}{l}$; |
| $\beta_{3,0}(x) = \frac{x}{l}$; |
| $\beta_{4,0}(x) = \frac{1}{3!}\left(\frac{x}{l}\right)^3$; |
| $\beta_{1,0}(x) = 0$;     |
| $\beta_{2,0}(x) = 1$;     |
| $\beta_{3,0}(x) = \frac{x}{l}$; |
| $\beta_{4,0}(x) = \frac{1}{2!}\left(\frac{x}{l}\right)^2$; |
| $\beta_{1,0}(x) = 0$;     |
| $\beta_{2,0}(x) = 0$;     |
| $\beta_{3,0}(x) = 1$;     |
| $\beta_{4,0}(x) = \frac{x}{l}$; |
| $\beta_{1,0}(x) = 0$;     |
| $\beta_{2,0}(x) = 0$;     |
| $\beta_{3,0}(x) = 0$;     |
| $\beta_{4,0}(x) = 1$. |

When $n = 5$ the initial functions are determined by equalities (38), (46) - (48):

$$\beta_{5,0}(x) = \left(\frac{x}{l}\right)^3 \sum_{j=0}^{\infty} \frac{C_j}{j+1(j+2)(j+3)} \left(\frac{x}{l}\right)^j; \quad (46)$$

$$\beta_{5,0}(x) = \left(\frac{x}{l}\right)^2 \sum_{j=0}^{\infty} \frac{C_j}{j+1} \left(\frac{x}{l}\right)^j; \quad (47)$$

$$\beta_{5,0}(x) = \frac{x}{l} \sum_{j=0}^{\infty} \frac{C_j}{j+1} \left(\frac{x}{l}\right)^j. \quad (48)$$

For the generating functions, formulas (39), (49) - (51) hold:

$$\beta_{n,k}(x) = \left(\frac{x}{l}\right)^{n+4k-2} \sum_{j=0}^{\infty} \tilde{c}_{n,k,j} \left(\frac{x}{l}\right)^j (n=1,2,3,4,5)(k=1,2,3,...); \quad (49)$$

$$\beta_{n,k}(x) = \left(\frac{x}{l}\right)^{n+4k-3} \sum_{j=0}^{\infty} \tilde{c}_{n,k,j} \left(\frac{x}{l}\right)^j (n=1,2,3,4,5)(k=1,2,3,...); \quad (50)$$

$$\beta_{n,k}(x) = \left(\frac{x}{l}\right)^{n+4k-4} \sum_{j=0}^{\infty} \tilde{c}_{n,k,j} \left(\frac{x}{l}\right)^j (n=1,2,3,4,5)(k=1,2,3,...), \quad (51)$$

where

$$\tilde{c}_{n,k,j} = c_{n,k,j}(n+4k+j-1).$$
\[ \tilde{c}_{n,k,j} = c_{n,k,j}(n + 4k + j - 1)(n + 4k + j - 2), \]
\[ \tilde{c}_{n,k,j} = c_{n,k,j}(n + 4k + j - 1)(n + 4k + j - 2)(n + 4k + j - 3). \]

Thus, in essence, an analytical method is proposed for the numerical implementation of the exact solutions found.

6. Results and discussion
Whenever the study of a physical phenomenon is reduced to a differential equation, the key is the question of constructing its exact analytical solution. However, researchers along the way often encounter a well-known mathematical problem, which is the lack of a universal method for integrating differential equations with variable coefficients. As a rule, solutions can be obtained only for some special cases. Probably, this can explain the predominant use of approximate methods. The foregoing fully applies to the problem of bending the beam on a variable elastic foundation.

In this paper, for the case when the elastic base is described by the Winkler model with an arbitrary continuous variable coefficient of the subgrade resistance, these difficulties were overcome. As a result, the following results were obtained:

1. The exact solution of the differential equation of the bending of the beam is constructed.
2. Formulas are obtained for displacements and internal forces in the beam.
3. An analytical method for the numerical implementation of the exact solutions found is proposed.

In fact, the solution of the original problem is reduced only to the implementation of the given boundary conditions, the determination of unknown initial parameters and the numerical implementation of the solutions found.

This paper is theoretical in nature. The authors plan to devote the following publications to the application of the developed method to the calculation of real structures.

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