Neutrinos in Random Magnetic Fields: The Problem of Measuring Magnetic Moments.

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Abstract. The existence of magnetic moments of neutrinos points to physics beyond the standard model. Given current upper limits, terrestrial measurements are difficult or completely unfeasible. However, estimates of transition moments can be obtained from observation of objects such as active galactic nuclei (AGN) by means of neutrino telescopes. We describe the way of estimating the magnitudes of transition moments from such observations.

I INTRODUCTION AND SUMMARY

The importance of measuring magnetic moments of neutrinos stems from the fact that the existence of such moments points to physics beyond the standard model. To be sure, a minimal enlargement of the standard model by means of right handed neutrinos alone leads to the existence of magnetic moments induced by loops of charged gauge bosons and charged leptons, cf. [1]. However, due to the fact that these moments are induced by means of higher order electroweak processes, their magnitudes are very small. Typically, a diagonal moment is of the order of magnitude,

\[ \mu_\nu \approx 3 \times 10^{-19} \frac{m_\nu}{1\text{eV}} \frac{1}{\mu_B}, \]

where \( \mu_B = e/(2m_e) \) is the Bohr magneton. Transition moments also contain mixing angles in their expressions, depending on the mixing schemes assumed. The important lesson, however, is that the standard model leads to extremely small moments, beyond measurability for any experiment of the foreseeable future. Current experimental upper limits are much larger, typically \( \mu \leq 10^{-10} \mu_B \), cf. [2]. Limits on transition moments are, in general, model dependent and currently they

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are not listed by the Particle Data Group. However, the consensus is that they are, typically, larger by an order of magnitude or so.

Should magnetic moments in the range of the current upper limits be measured, we would have an important low energy signal of the existence of physics beyond the standard model, with a very small background coming from the standard model itself. Let us give a crude estimate of the relevant energy scale. For purposes of illustration, we ignore flavor structure and mixing angles: one hopes that this gives rise to errors of at most an order of magnitude.

The existence of an anomalous magnetic moment gives rise to a Pauli term in the low energy effective Lagrangian, viz.

\[ L_P = \frac{e}{2m_\nu} \kappa \sigma_{\mu\nu} F^{\mu\nu}, \]

where \( m_\nu \) is the mass of the neutrino. The quantity \( \kappa \) is the “low energy” \( (q_\mu \to 0) \) limit of a spin flip amplitude. Within factors of order unity, its magnitude is given by an expression,

\[ \kappa \simeq \frac{m_\nu}{\Lambda}, \]

where \( \Lambda \) is the characteristic energy scale of the process giving rise to the Pauli moment. The factor \( m_\nu \) in the numerator is present because an anomalous moment is generated by a spin flip process. By putting in numbers, we discover for instance that a magnetic moment of the order of \( 10^{-10}\mu_B \) corresponds to an energy scale of the order of \( 10^4\text{TeV} \).

Even though estimates of this type appear to be extremely naive ones, they work quite well in cases where we know the mechanism by means of which a neutral particle acquires a Pauli moment. Take the neutron as an example. Its Pauli moment is: \( \mu_n \approx 1.9e/(2m_n) \), cf. [2]. Using the previous estimates, we get that the characteristic mass scale is

\[ \Lambda_n \approx 495\text{MeV}. \]

This value is quite close to \( \Lambda_{QCD} \), which is expected to characterize structure formation (i.e. the formation of hadrons out of quarks) within QCD.

The difficulty with small values of neutrino moments as given by the upper limits quoted is that their measurement in a terrestrial experiment is difficult or impossible. To illustrate this point, assume that \( \mu_\nu \approx 10^{-10}\mu_B \). One can measure a magnetic moment by inducing a spin flip in a magnetic field. Assuming the field to be a homogeneous one, the distance over which a spin flip occurs on the average is roughly \( d \approx 1/\mu B \). A typical terrestrial magnet can maintain a field of the order of magnitude, \( B \approx 1\text{Tesla} \). Thus, the distance characterizing the spin flip of a moment of \( 10^{-10}\mu_B \) is \( d \approx 3 \times 10^3\text{km} \). Clearly, no magnet can be constructed on Earth which is that long.
The situation is better with transition moments. Depending on the nature of neutrinos (Dirac or Majorana), a spin flip either leads to an active↔sterile conversion or to a flavor flip, respectively. From the physical point of view, flavor flips associated with spin flips are easier to observe and there are many plausible models available which suggest that Nature may prefer Majorana neutrinos over Dirac ones. For this reason, we shall concentrate on Majorana neutrinos. It is to be noted that the observation of a transition moment contains at least as much physical information as the measurement of a diagonal moment does: hence there is no disadvantage in looking for transition moments.

Keeping such arguments in mind, we suggested a way of measuring transition moments of neutrinos in terrestrial experiments utilizing facilities to be completed in the near future, namely, long baseline oscillation experiments, cf. [3]. Better sensitivities can be achieved by utilizing astrophysical objects, such as an AGN, albeit at the cost of having to live with greater uncertainties of the properties of the source.

Briefly, the idea is the following. Around an AGN, charged hadrons, mostly pions are produced which subsequently decay, predominantly into muons and $\nu_\mu$-s. The muons, being charged and long lived, are expected to undergo a random walk in the surrounding plasma and their decay products are unlikely to carry a substantial amount of directional information about the source. By contrast, a $\nu_\mu$, if left alone, would escape and reach a neutrino telescope, thus carrying information about the source. In the presence of magnetic fields in the emerging jets, however, flavor conversion can take place and this may change the situation substantially.

Presumably, charged particles in a jet are in a turbulent motion, hence, the magnetic fields generated by them are chaotic. For the sake of simplicity, we assume that the magnetic field present is a static, isotropic Gaussian random field of zero mean and spatial correlation length $L$. This appears to be a reasonable assumption, as long as the characteristic time scale of the motion of the charged particles is longer than the time of passage of neutrinos through the jet. Besides its correlation length, the Gaussian field is characterized by the m.s. field, $\langle B^2 \rangle$.

One expects that a neutrino traveling in a random magnetic field executes a “random walk” between its possible spin orientations. Hence, after a while, the spin orientations become random; for a Majorana neutrino, spin equilibrium also means flavor equilibrium. If the neutrino is produced with a definite spin, its polarization is expected to be damped as a function of distance, with a characteristic distance $D$.

We now argue that, within factors of order unity, the quantity $D$ is uniquely determined: there is only one combination of the physical quantities involved which is of the correct dimension.

Due to the fact that the coupling between a magnetic moment and a magnetic field is proportional to $\mu \cdot B$, the only combination in which the field and the magnetic moment can enter is proportional to $\mu^2 \langle B^2 \rangle$, which is of dimension length $^{-2}$. Thus, the inverse spin flip length in such a Gaussian field must be given by an expression proportional to
\[D^{-1} = \mu^2\langle B^2 \rangle L.\] (3)

The important point to bear in mind is that the length given by eq. (3) is rather short compared to typical jet sizes: hence, one expects the equilibrium to be established. In fact, by inserting r.m.s. fields of the order of a few G, magnetic moments of the order of $10^{-10}$ to $10^{-8} \mu_B$ and correlation lengths of the order of a pc ($\simeq 3 \times 10^{13}$km), one ends up with $D \ll L$, in fact, of the order of merely a few times $10^4$km.

In the remainder of this talk, the basic ideas are illustrated on a simple, solvable model, along the lines described in ref. [4]. The general theory, assuming an arbitrary number of flavors (and, in principle, allowing arbitrary spins too), has been developed elsewhere [5].

\section{II A TALE OF TWO FLAVORS}

As explained in the preceding section, we now consider the behavior of a single spin-1/2 field, without paying detailed attention to the flavor structure.

In order to describe the average behavior of a neutrino in a random magnetic field, one has to solve the dynamical equations governing the propagation in an arbitrary magnetic field. The solution then has to be averaged over the ensemble of magnetic fields.

We use the front form of dynamics [6]. This formulation of dynamics is advantageous in a situation in which one considers the propagation of high energy particles ($E \gg m$, where $m$ is the rest mass) and in which certain discrete symmetries, such as $C$ and $P$ play no significant role. Clearly, the propagation of high energy neutrinos falls into this category.

We begin with the usual Dirac Lagrangian of a particle in an external electromagnetic field, $F_{\mu\nu}$:

\[L = \overline{\psi} \left( i\gamma^\mu \partial_\mu + m + \frac{1}{2} \mu F^{\mu\nu} \sigma_{\mu\nu} \right) \psi \] (4)

We work in the rest frame of the magnetic field. Assuming the field to be a static one, we can set $F_{0i} = 0$, $F_{ij} = \epsilon_{ijk} B_k$. In the case of interest one has to solve the Dirac equation in an arbitrary static magnetic field, since we want to average the solution over an ensemble of the $B_i$. No explicit solution is known for such a problem. However, we proceed to show that in the \textit{high energy limit} the problem can be solved in a closed form.

We introduce a coordinate system in which two of the coordinates are null directions corresponding to characteristic lines of a relativistic wave equation, \textit{viz.}:

\[ t = \frac{1}{\sqrt{2}} \left( x^0 - x^3 \right), \quad z = \frac{1}{\sqrt{2}} \left( x^0 + x^3 \right) \quad \text{and} \quad x^A; \quad (A = 1, 2). \] (5)

Correspondingly, the metric is of the form,
\[ g_{zt} = g_{tz} = 1, \quad g_{AB} = -\delta_{AB}, \] (6)

and all other components vanish.

A Dirac spinor can be decomposed along the null directions given in eq. (5) by introducing the mutually orthogonal projectors,

\[ P_t = \frac{1}{2} \gamma_t \gamma^t, \quad P_z = \gamma_z \gamma^z \] (7)

In what follows, we use the shorthand,

\[ \phi = P_t \psi, \quad \chi = P_z \psi \] (8)

It is a straightforward matter to decompose eq. (4) according to the conjugate null directions and express it in terms of the variables \( \phi \) and \( \chi \). The purpose of such an exercise is a very simple one. If, for the sake of definiteness, \( t \) is regarded the "time" variable describing the dynamics of the system, only \( \phi \) obeys an equation containing \( \partial_t \). Hence, the component of the Dirac spinor corresponding to the conjugate null direction obeys only an equation of constraint. The constraint can be, in turn, solved before one attempts to attack the problem of dynamics.

After carrying out the decomposition of eq. (4) according to the null directions, one finds:

\[
L = \sqrt{2} \left[ \phi^\dagger \left( i \partial_t - i \sqrt{2} \mu \epsilon^{AB} \gamma_A B_B \right) \phi + \chi^\dagger \left( i \partial_z - i \sqrt{2} \mu \epsilon^{AB} \gamma_A B_B \right) \chi \right] \\
+ \frac{1}{\sqrt{2}} \left[ \phi^\dagger \gamma^z \left( i \gamma^A \partial_A + m - \frac{i}{\sqrt{2}} \mu B_3 \epsilon_{AB} \gamma^A \gamma^B \right) \chi \\
+ \chi^\dagger \gamma^t \left( i \gamma^A \partial_A + m - \frac{i}{\sqrt{2}} \mu B_3 \epsilon_{AB} \gamma^A \gamma^B \right) \phi \right] 
\] (9)

Variation of eq. (9) with respect to \( \chi^\dagger \) gives the constraint. The constraint can be solved in a straightforward fashion and eliminated from the Lagrangian. The result is conveniently written in Hamiltonian form:

\[
L = \pi \partial_t \phi - H \\
H = -2 \mu \phi^\dagger \sigma^A B_A \phi \\
+ \phi^\dagger \left( -i \sigma_B \epsilon^{BC} p_C + m - \mu \sqrt{2} B_3 \sigma_3 \right) \\
\times \Omega \left( -B^A \right) \\
\times \left( i \sigma_R \epsilon^{RS} p_S + m - \mu \sqrt{2} B_3 \sigma_3 \right) \phi
\] (10)

Solving the constraint eliminates two components of the original, four component Dirac spinor. Therefore, instead of the original Dirac matrices one can use \( 2 \times 2 \)
Pauli matrices. One easily verifies that $-i\epsilon^{AB}\gamma_B \to \sigma^A$ gives the correct representation. We also introduced the Hermitean operator, $p_A = -i\partial_A$ for the transverse degrees of freedom.

The canonical momentum is given by $\pi = i\sqrt{2}\phi^\dagger$. (Of course, the odd looking factor of $\sqrt{2}$ in the definition of the canonical momentum can be eliminated by rescaling the time variable.) In equation eq. (10), $\Omega$ is an operator with matrix elements:

$$\langle z | \Omega (B^A) | z' \rangle = \frac{i}{\sqrt{2}} \exp \left( \mu \sqrt{2} \int_{z'}^z dz' \epsilon_{AB} \gamma^A B^B \right) \frac{1}{2} \epsilon(z - z')$$

All symbols of integration over $z$ have been omitted. Eq. 10 is local in $t$ and $x^A$; those arguments have been suppressed.

The Hamiltonian appearing in eq. (10) is exact. However, it is given by a rather complicated, non local and non linear expression: this is the cost we have to pay for explicitly eliminating the constraint. We now argue that one can introduce physically reasonable simplifications, as a result of which the problem becomes a manageable one. First of all, we notice that the exponential appearing in eq. (11) is of modulus one. One expects that at large values of $|z - z'|$ the exponent oscillates rapidly and thus contributes little to the Hamiltonian. The dominant contribution is thus coming from small values of the difference of longitudinal coordinates. Hence, it is reasonable to approximate the exponential in eq. (11) by 1. In the remaining expression, one term is local in all variables and the remaining ones are proportional to $\epsilon(z - z')$. In a Fourier representation, viz. upon writing

$$\phi(t, z, x^A) = \int dk \varphi(t, k, x^A) \exp(-ikz)$$

and

$$\epsilon(z) = \mathcal{P} \int \frac{dk}{2\pi i} \exp(-ikz),$$

one recognizes that the non local terms in the Hamiltonian are proportional to negative powers of the longitudinal momentum, $k$. (In the last equation $\mathcal{P}$ stands for the principal value.) Hence, at high energies ($k \gg m$) the Hamiltonian can be approximated by the local term.

Neglecting terms of $O(k^{-1})$, the equation of motion for the density matrix in coordinate representation reads:

$$-i\partial_t \langle z, \vec{x} | \rho(t) | z', \vec{x}' \rangle = \mu \sqrt{2} \sigma \cdot \vec{B} \left( \frac{z - t}{\sqrt{2}} \right) \langle z, \vec{x} | \rho(t) | z', \vec{x}' \rangle$$

$$- \mu \sqrt{2} \langle z, \vec{x} | \rho(t) | z', \vec{x}' \rangle \vec{\sigma} \cdot \vec{B} \left( \frac{z' - t}{\sqrt{2}} \right)$$

In this equation, $\vec{x}$ stands for the transverse part of the coordinate and $\vec{\sigma} \cdot \vec{B}$ is the two dimensional scalar product in transverse space. Of course, the coordinate $x^3$ had to be expressed by $z$ and $t$; hence the $t$-dependence in the magnetic field.
We choose the initial condition so as to describe a neutrino produced at $\vec{x} = 0$ and with a fixed value of $k$:

$$\langle z, \vec{x} | \rho(0) | z', \vec{x}' \rangle = \delta^2(\vec{x}) \delta^2(\vec{x}') \frac{\exp ik(z - z')}{2\pi k} \rho_s(0),$$

(15)

where $\rho_s(0)$ is the initial value of the spin density matrix.

The variable $k$ being large, the function $\exp ik(z - z')$ is rapidly oscillating unless $z \approx z'$. Therefore, it is permissible to put $z = z'$ in the coefficient of the exponential in eq. (15). Further, in the approximation used, the dynamics described by eq. eq. (14) is independent of $k$ and of $\vec{x}$. Therefore, the dependence of $\rho(t)$ on $k$ and $\vec{x}$ is entirely determined by the initial condition. Thus, the dynamical equation reduces to an equation involving the spin density matrix alone, as in non-relativistic spin dynamics. Thus, from now on, we omit the subscript $s$ and we have:

$$-i\partial_t \rho(t) = \mu \sqrt{2} \left[ \bar{\sigma} \cdot \vec{B} \left( \frac{z - t}{\sqrt{2}} \right), \rho(t) \right]$$

(16)

(Here and in what follows, $\vec{x} = 0$ is understood.)

This equation can be solved by the standard time ordered series, $viz.$

$$\rho(t) = \rho(0) + i\mu \sqrt{2} \int_0^t dt' \left[ \bar{\sigma} \cdot \vec{B} \left( \frac{z - t'}{\sqrt{2}} \right), \rho(0) \right]$$

$$+ \frac{(i\mu \sqrt{2})^2}{2!} \int_0^t dt' dt'' T \left( \left[ \bar{\sigma} \cdot \vec{B} \left( \frac{z - t'}{\sqrt{2}} \right), \left[ \bar{\sigma} \cdot \vec{B} \left( \frac{z - t''}{\sqrt{2}} \right), \rho(0) \right] \right) \right)$$

$$+ \cdots$$

(17)

We choose the initial condition as:

$$\rho(0) = \frac{1}{2} (1 + S \sigma_3), \quad (S^2 \leq 1),$$

(18)

since neutrinos are produced with a definite helicity. (In the case of Dirac neutrinos, $S = \pm 1$, depending on whether a neutrino or anti-neutrino is produced. In the case of Majorana neutrinos, $S$ may assume any value between the limits stated above, depending on the production mechanism.)

Next, we average the solution, eq. (17) over the magnetic field. We choose the generating functional of the moments as follows:

$$Z[j] = \int \mathcal{D}B \exp \left[ - \frac{1}{2} \int d^3x d^3x' B_i(x) C_{ij}^{-1}(x, x') B_j(x') \right]$$

$$\times \exp \int d^3x j_i(x) B_i(x);$$

$$C_{ij}^{-1} = \frac{L}{4\pi \langle B^2 \rangle} \left( \delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2} \right) \left( L^2 - \nabla^2 \right)^2 \delta^3(x - x').$$

(19)
In the last equation, $L$ and $\langle B^2 \rangle$ stand for the correlation length and mean square magnetic field, respectively. The measure is normalized such that $Z[0] = 1$. The transverse projector is needed in order to make the correlation functions solenoidal. With the choice of the tensor $C^{-1}$ given in eq. (19), the leading term in the long distance behavior of the correlation function is $\propto \exp - |x - x'|$. In order to average equation eq. (17) over the magnetic field, one integrates over $B_3$ and sets the third component of the source equal to zero. The transverse generating functional reads:

$$Z_T = \int \mathcal{D}\vec{B} \exp \left[ -\frac{L}{8\pi \langle B^2 \rangle} \int d^3x \left[ B^A(x) \left( \delta_{AB} - \frac{1}{2} \frac{\partial A \partial B(x)}{\nabla^2} \right) B^B \right] \right] \times \exp i \int d^3x j^A(x) \cdot \vec{B}(x)$$

(20)

We now notice that in eq. (17), terms containing odd powers of $\mu$ are also odd in $B^A$. Therefore, in the limit $\vec{j} \to 0$ the average of those terms vanishes. The even terms in the series are obtained by taking the appropriate functional derivatives of eq. (20). All of them are expressed in terms of multiple time integrals of $C_{ij}(|t - t'|)$ and its powers: those integrations are easily performed. It is sufficient to illustrate the procedure for the second order term in eq. (17).

Carrying out the integrations, one gets:

$$-\mu^2 \frac{1}{2} S\left\langle \int_0^t dt' dt'' \left[ \vec{\sigma} \cdot \vec{B}, \left[ \vec{\sigma} \cdot \vec{B}, \sigma_3 \right] \right] \right\rangle = -\mu^2 \sigma_3 \langle B^2 \rangle tL \left( 1 - \exp \frac{t}{T} \right)$$

For large times the result in the last equation is just proportional to $t$. The higher order terms follow a similar pattern. The end result is:

$$\langle \rho(t) \rangle \sim \frac{1}{2} \left( 1 + S\sigma_3 \exp - \frac{t}{T} \right), \quad (21)$$

with

$$\frac{1}{T} = 2\mu^2 \langle B^2 \rangle L.$$

Thus we arrive at the remarkable result that in the random field the behavior of the helicities is an **ergodic** one: irrespective of what the initial density matrix was, for $t \gg T$, the helicities are equally distributed. Furthermore, as conjectured, the characteristic length over which the flavor equilibrium is established is about the characteristic length $D$, cf. eq. (3) in the previous section.

### III DISCUSSION

The result conjectured in the first section and verified within the framework of a simplified model in the previous one indicates that if neutrino telescopes will detect neutrinos originating from AGN (and perhaps from other astronomical objects), it is likely that all flavors will occur with equal probability. By contrast, if the arriving
neutrinos are predominantly $\nu_\mu$-s, one concludes that neutrinos are Dirac particles and the magnetic fields present around the source cause mainly an active$\leftrightarrow$sterile conversion. Such a conversion decreases the intensity of active neutrinos by about a factor of two. \textit{(In principle, knowing the flux of neutrinos produced at the source, one could verify the existence of the conversion. However, current flux estimates are, by far, not accurate enough for such a purpose.)}

Even though we conjectured the existence of an equilibrium of spin orientation on the basis of a simple intuitive argument and verified the conjecture within the framework of a rather simple minded model, we believe the result to be a rather general one. In fact, there exists in the literature a rather substantial number of papers in which conclusions similar to ours have been reached in a variety of physical circumstances (the early Universe, the interior of the Sun, \textit{etc.}). For an incomplete, but representative sample, see \textit{e.g.} refs. \cite{7}, \cite{8}, \cite{9}. The probability distributions assumed for the magnetic fields are quite different \textit{(e.g.} white noise with a finite spatial extension, zero spatial correlation length with randomly oriented domains, \textit{etc.}). All the distributions, however, have the properties discussed in Section I: the magnetic field has a vanishing mean, it is characterized by $\langle B^2 \rangle$ and by some length. Hence, the dimensional argument given in Sec. I is applicable; consequently, the characteristic distance found in every work of this type agrees with $D$ within factors of order one. Consequently, it is reasonable to expect that our results are rather robust ones and thus, testing them by means of flavor sensitive neutrino telescopes is of substantial physical interest.

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