Selection rules for light-by-light scattering in strong magnetic field

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Abstract

Selection rules that follow from CP- and 4-momentum conservation are listed for head-on light-by-light scattering in strong magnetic field taking into account nontrivial dispersion laws of different photon eigenmodes.

1 Introduction

Recently R. Baier, A. Rebhan, and M. Wödlinger [1] calculated cross sections of light-by-light scattering as functions of an external magnetic field $B$ for low-energy long-wave photons, basing on the Heisenberg-Euler Lagrangian – considered in QED, in scalar electrodynamics and in the theory with charged vector bosons. Calculations are done for special cases of equal-wave-length photons colliding head-on, when the external field is either parallel or perpendicular to the incoming photon direction, so that the three-momenta of the two colliding photons are subject to the relation $k_1 = -k_2$. This kinematical restriction cannot, generally, be avoided by a Lorentz transformation to an arbitrary frame, since – when $B \not\parallel k_{1,2}$ – there is no invariance under the boost in the direction of the photon propagation. Nevertheless, the head-on collision case corresponds to the realistic experimental situation, where two laser beams collide, so that in the laboratory frame their center-of-mass is at rest.

In the present note we stress that the kinematics of the reaction cannot be borrowed without change from the vacuum case, because the presence of the magnetic field does affect it. Relation between the energy and momentum of the photon is not just $\omega = k$, but the energy of each photon may depend in two ways on the angle, which the direction of its momentum
makes with the magnetic field, \( \omega_{\text{ordinary, extraordinary}}(k^2, (k \cdot B)^2) \), reflecting the anisotropy of the effective "optical medium" and birefringence. (Within the local approximation kept to in [1], and here as well, the dependence on the angle is very simple, see Eqs. (13), (25), (26) in [2].)

The photons that are inside the magnetic field are classified according to the eigenmodes. (If one is willing to consider scattering of photons falling from outside of the region occupied by the magnetic field, one should also take into account their reflection and refraction at the border.) Definite laws of propagation – the corresponding refraction indices, propagation speeds etc. are associated with photon eigenmodes, which are not the photons just transversely polarized. (The polarizations of eigenmodes are established in [3] and partially in Ref. [4]). For the general angle between \( k \) and \( B \) the so-called ordinary wave is transverse (this is mode-3 in classification of [3]), its electric field is orthogonal to the plane spanned by \( k \) and \( B \), while the extraordinary wave (mode-2) is not (its electric vector belongs to that plane). So, the scattered photons are not all transverse even when the incoming photons are parallel or orthogonal to the magnetic field.

In the next Section I illustrate the influence of the anisotropic dispersion laws by considering relations among momenta and scattering angle of the photons within the same configurations of the incident photons and the magnetic fields as the ones considered in [1]. I did not find kinematical bans, analogous to those known for photon splitting, but the bans due to CP-conservation like in Adler's work [4] are expected. Essential also is the change of the photon wave-length depending (for parallel incidence) or not depending (for perpendicular incidence) on the scattering angle.

### 1.1 Selection rules

The energy-momentum conservation reads

\[
\begin{align*}
    k_1 + k_2 &= k_3 + k_4 \quad (1) \\
    \omega_1 + \omega_2 &= \omega_3 + \omega_4 \quad (2)
\end{align*}
\]

Energy of each photon \( i = 1, 2, 3, 4 \) may belong to one of the modes \( a = 2, 3 \) (sometimes also mode-1, \( a = 1 \), comes into play, see below), and it depends on its momentum and on orientation of the latter with respect to the magnetic field:

\[
    \omega_i = \omega^{(a_i)}(k_i^2, (k_i \cdot B)^2), \quad k_i = |k_i|.
\]

We say that we face the "center-of-mass" configuration if additionally the relation

\[
    k_1 + k_2 = k_3 + k_4 = 0
\]
is obeyed (photons of equal wave-length are colliding head-on). In this configuration \( k_4 \) lies in the plane spanned by the vectors \( k_1 = -k_2 \) and \( k_3 \), because \( k_4 = -k_3 \). In other words, the initial and final reaction planes coincide. (I don’t know if the same statement is true for the general configuration under conditions of the lack of Lorentz and rotational invariance that prevents one from passing to the general configuration by changing the reference frame.) I shall confine myself to the "center-of-mass configuration" \(^{(4)}\) in what follows.

With the account of \(^{(4)}\) Eq. (3) may be written as

\[
\omega_i = \omega^{(a_i)}(k_i^2, (k_i \cdot B)^2), \quad k_{1,3} = k_{2,4}, \quad (k_{1,3} \cdot B)^2 = (k_{2,4} \cdot B)^2, \quad (5)
\]

i.e. \( k_1 \) represents the incident photons, and \( k_3 \) represents the scattered ones. (When the two incoming photons belong to the same mode, \( a_1 = a_2 \), the configuration considered may be described as scattering of equal-energy photons, because then \( \omega^{(a_1)}(k_1, (k_1 \cdot B)^2) = \omega^{(a_2)}(k_2, (k_2 \cdot B)^2) \) thanks to the relation \( k_1 = -k_2 \). I shall not restrict myself to this case in what follows, however).

The dispersion laws for mode-2 and mode-3 waves in the original classification of Refs. \(^{(3)}\), \( a = 2, 3 \) are in the long-wave, low-frequency approximation governed by the Heisenberg-Euler Lagrangian (HEL) \(^{(2)}\)

\[
\omega^{(2,3)} = \left( \frac{(k \cdot B)^2}{B^2} \right) + \left( k^2 - \frac{(k \cdot B)^2}{B^2} \right) c^{(2,3)}, \quad (6)
\]

where the coefficients \( c^{(2,3)} \) are known dimensionless functions of the external field expressed in terms of first- and second-order derivatives of HEL. It agrees with the causality that \( c^{(2,3)} < 1 \). While investigating the selection rules it may be important that \( c^{(3)} > c^{(2)} \).

The energy conservation relations \(^{(2)}\)

\[
\omega^{(a_1)}(k_1, (k_1 \cdot B)^2) + \omega^{(a_2)}(k_1, (k_1 \cdot B)^2) = \omega^{(a_3)}(k_3, (k_3 \cdot B)^2) + \omega^{(a_4)}(k_3, (k_3 \cdot B)^2)
\]

are fraught with dynamic selection rules that may forbid many of the sixteen transitions (four initial by four final polarization states). Besides, there is the parity ban (cf. \(^{(4)}\)) for the transitions with the participation of a total odd number of mode-3 photons in initial and final states, since the mode-3 vector-potential is a pseudovector. Hence mode-2 photon may appear only even number of times among the four photons participating in the reaction. The CP-selection rules derived thereof from this general consideration manifest themselves by calculations of Ref. \(^{(1)}\).
1.1.1 Perpendicular incidence

Consider first the simplest case when the incoming photon momenta are perpendicular to the magnetic field, \( k_{1,2} \perp B \). So are the outgoing momenta since they lie in the same (reaction) plane. In this case the longitudinally polarized component of the electric field in the extraordinary mode-2-wave disappears. The modes 2 and 3 are mutually orthogonal, both transverse, electromagnetic waves. Their dispersion laws are

\[
\omega_{1,2} = \omega^{(a_{1,2})}(k_1, 0), \quad \omega_{3,4} = \omega^{(a_{3,4})}(k_3, 0). \tag{8}
\]

The energy-conservation relations (7) take the form

\[
\omega^{(a_1)}(k_1, 0) + \omega^{(a_2)}(k_1, 0) = \omega^{(a_3)}(k_3, 0) + \omega^{(a_4)}(k_3, 0). \tag{9}
\]

To see what rules these equations imply let me consider first one out of 16 transitions, when two photons of mode-2 collide to produce two photons of mode-3: \((2, 2) \rightarrow (3, 3)\). Eq. (9) requires

\[
\omega^{(2)}(k_1, 0) = \omega^{(3)}(k_3, 0),
\]

or using (6)

\[
k_1 \left( c^{(2)} \right)^{1/2} = k_3 \left( c^{(3)} \right)^{1/2}. \tag{10}
\]

This relation establishes an obligatory connection between the wavelengths of the incoming and outgoing photons. Since \( c^{(3)} > c^{(2)} \), the outgoing wave has a longer length, \( k_3 < k_1 \). For the opposite process \((3, 3) \rightarrow (2, 2)\) the selection rule

\[
k_1 \left( c^{(3)} \right)^{1/2} = k_3 \left( c^{(2)} \right)^{1/2}
\]

implies the opposite inequality: \( k_3 > k_1 \).

On the contrary, the transitions when two different-mode photons turn into also two different-mode photons, \((3, 2) \rightarrow (3, 2)\), demand that \( k_1 = k_3 \), because Eq. (9) becomes in this case

\[
k_1 \left( c^{(2)} \right)^{1/2} + k_1 \left( c^{(3)} \right)^{1/2} = k_3 \left( c^{(2)} \right)^{1/2} + k_3 \left( c^{(3)} \right)^{1/2}.
\]

It remains to consider transitions when all the four photons are of the same polarization, \((2, 2) \rightarrow (2, 2)\) and \((3, 3) \rightarrow (3, 3)\)

\[
\omega^{(2,2)}(k_1, 0) + \omega^{(2,2)}(k_1, 0) = \omega^{(2,3)}(k_3, 0) + \omega^{(2,3)}(k_3, 0). \tag{12}
\]

\[
k_1 \left( c^{(2,2)} \right)^{1/2} + k_1 \left( c^{(2,2)} \right)^{1/2} = k_3 \left( c^{(2,3)} \right)^{1/2} + k_3 \left( c^{(2,3)} \right)^{1/2}.
\]

Therefore such process requires, as before, that \( k_1 = k_3 \).
Other transitions \((2, 2) \leftrightarrow (2, 3), \ (3, 3) \leftrightarrow (2, 3)\) are forbidden since they would violate parity.

To conclude this Subsection we state that the light-by-light scattering in the "center-of-mass" configuration across the magnetic field requires that the wave-length should conserve, \(k_1 = k_3\), when none or one photon changes its polarization, but it should not, \(k_1 \nless k_3\), when two photons both change their polarization. Processes, where the photon, whose polarization is given by mode-3, is involved once or thrice are parity-impossible.

1.1.2 Parallel incidence

When two initial photons are oriented parallel to \(B\), (and the final are not) the energy conservation (7) gives

\[
\omega^{a_1}(k_1, k_1^2 B^2) + \omega^{a_2}(k_1, k_1^2 B^2) = \omega^{(a_3)}(k_3, (k_3 \cdot B)^2) + \omega^{(a_4)}(k_3, (k_3 \cdot B)^2).
\]  

(13)

The falling photons may belong either to (transverse in this case) mode-1 or to ever transverse mode-3, since mode-2 in disappears for parallel propagation (see second reference in [3]). Let all the four involved photons belong to mode-3: \(a_{1,2,3,4} = 3\), that is we consider the process \((3, 3) \rightarrow (3, 3)\). Then (13) becomes

\[
\begin{align*}
\omega^{(3)}(k_1, k_1^2 B^2) &= \omega^{(3)}(k_3, (k_3 \cdot B)^2) \quad \text{or} \\
\omega^{(3)}(k_1, k_1^2 B^2) &= \omega^{(3)}(k_3, k_3^2 (B \cos \theta)^2).
\end{align*}
\]  

(14)

Here \(\theta\) is the scattering angle of Eq. (14) in [1]. With the help of (6) we obtain for (14)

\[
\begin{align*}
k_1 &= \left(k_3^2 (\cos \theta)^2 + (k_3^2 - k_3^2 (\cos \theta)^2) c^{(3)}\right)^{1/2} \\
k_1 &= k_3 \left(\cos^2 \theta + c^{(3)} \sin^2 \theta\right).
\end{align*}
\]  

(15)

This is a definite kinematical relation between the scattering angle and the ratio of the initial and final wave-lengths parameterized by the magnetic field hidden in \(c^{(3)}\), – obligatory for the chosen process to be permitted. Here the present case of parallel incidence differs from the perpendicular incidence of the previous item, where this ratio was fixed, but the scattering angle remained unrestricted. It follows from (13) and the causality \(c^{(3)} < 1\) that \(k_1 < k_3\), the outgoing waves are shorter than the incoming ones.

Let now two incident photons belong both to mode-3, \(a_{1,2} = 3\), while the scattered photons to mode-2: \(a_{3,4} = 2\). Then for the process \((3, 3) \rightarrow (2, 2)\) (13) becomes

\[
\omega^{(3)}(k_1, k_1^2 B^2) = \omega^{(2)}(k_3, k_3^2 (B \cos \theta)^2)).
\]
\[ k_1 = k_3 \left( \cos^2 \theta + c^{(2)} \sin^2 \theta \right), \quad k_1 < k_3. \]  \hspace{1cm} (16)

This relation is of the same type as (15).

More special are the cases where one or both incident photons belong to mode-1. These are \((1, 3) \rightarrow (2, 3), (1, 1) \rightarrow (3, 3), (1, 1) \rightarrow (2, 2)\). Let us take the process \((1, 3) \rightarrow (2, 3)\) to begin with. The dispersion law for mode-1 under parallel propagation is just the vacuum dispersion law \(\omega^{(1)}(k, k^2 B^2) = k\). The energy conservation reads

\[
\omega^{(1)}(k_1, k_1^2 B^2) + \omega^{(3)}(k_1, k_1^2 B^2) = \omega^{(3)}(k_3, k_3^2 (B \cos \theta)^2) + \omega^{(2)}(k_3, k_3^2 (B \cos \theta)^2)).
\]

\[ k_1 \left( 1 + \cos^2 \theta + c^{(3)} \sin^2 \theta \right) = k_3 \left( 2 \cos^2 \theta + (c^{(3)} + c^{(2)}) \sin^2 \theta \right), \quad k_1 < k_3. \]  \hspace{1cm} (17)

Analogously, the process \((1, 1) \rightarrow (3, 3)\) requires that

\[ k_1 = k_3 \left( \cos^2 \theta + c^{(3)} \sin^2 \theta \right), \quad k_1 < k_3 \]  \hspace{1cm} (18)

and the process \((1, 1) \rightarrow (2, 2)\) that

\[ k_1 = k_3 \left( \cos^2 \theta + c^{(2)} \sin^2 \theta \right), \quad k_1 < k_3 \]  \hspace{1cm} (19)

\[ . \]

1.2 Conclusion

We have established selections rules for the head-on photon-photon collisions in the vacuum filled by a strong magnetic field, which is parallel or perpendicular to the axis, on which the momenta of the two incoming photons lie. Some combinations of the initial and final photon polarization eigenstates proved out to be excluded by the parity conservation. All the rest are kinematically permitted, but the wavelengths of the final photons for certain combinations of initial and final polarizations differ from those of initial photons. This difference may become significant for the magnetic fields of the critical order \(B_{cr} = \frac{m_e}{e} = 4.4 \cdot 10^{13} \text{G}\). This means that the device registering the photons resulting from two laser beams collision should be tuned to an energy different from the energy of the colliding photons. This prescription results from taking into account of different dispersion laws in different photon modes. The quantitative part of the consideration fits low-frequency, long-wave photons because it is based on the local approximation as given by HEL.
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