BIHARMONIC SUBMANIFOLDS OF $\mathbb{C}P^n$

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Abstract. We give some general results on proper-biharmonic submanifolds of a complex space form and, in particular, of the complex projective space. These results are mainly concerned with submanifolds with constant mean curvature or parallel mean curvature vector field. We find the relation between the bitension field of the inclusion of a submanifold $\bar{M}$ in $\mathbb{C}P^n$ and the bitension field of the inclusion of the corresponding Hopf-tube in $\mathbb{S}^{2n+1}$. Using this relation we produce new families of proper-biharmonic submanifolds of $\mathbb{C}P^n$. We study the geometry of biharmonic curves of $\mathbb{C}P^n$ and we characterize the proper-biharmonic curves in terms of their curvatures and complex torsions.

1. Introduction

Biharmonic maps $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds are critical points of the bienergy functional

$$E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 \, v_g,$$

where $\tau(\varphi) = \text{trace} \nabla d\varphi$ is the tension field of $\varphi$ that vanishes on harmonic maps. The Euler-Lagrange equation corresponding to $E_2$ is given by the vanishing of the bitension field

$$\tau_2(\varphi) = -J^\varphi(\tau(\varphi)) = -\Delta^\varphi \tau(\varphi) - \text{trace} R^N(d\varphi, \tau(\varphi))d\varphi,$$

where $J^\varphi$ is formally the Jacobi operator of $\varphi$ (see [16]). The operator $J^\varphi$ is linear, thus any harmonic map is biharmonic. We call proper-biharmonic the non-harmonic biharmonic maps.

The analytic aspects of biharmonic maps as well as the differential geometry of such maps have been studied in the last decade (see, for example, [7, 17, 22, 23] and [2, 10, 16, 21, 24], respectively).

In this paper we shall focus our attention on proper-biharmonic submanifolds, i.e. on submanifolds such that the inclusion map is a proper-biharmonic map.

The proper-biharmonic submanifolds of a real space form were extensively studied, see, for example, [3, 4, 5, 8, 11]. Naturally, the next step has been the study of proper-biharmonic submanifolds of spaces of non-constant sectional curvature (see, for example, [1, 13, 14, 15, 25, 26, 27]).

This work is devoted to the study of proper-biharmonic submanifolds in a complex space form. This subject has already been started by several authors. In [9] some pinching conditions for the second fundamental form and the Ricci curvature of a biharmonic Lagrangian submanifold of $\mathbb{C}P^n$, with parallel mean curvature vector field, were obtained. In [26], the author gave a classification of biharmonic Lagrangian surfaces of constant mean curvature in $\mathbb{C}P^2$. Finally, in [14], there is a
characterization of biharmonic constant mean curvature real hypersurfaces of \( CP^n \) and the classification of biharmonic homogeneous real hypersurfaces of \( CP^n \).

The paper is organized as follows. In the first part we obtain some general properties on proper-biharmonic submanifolds with constant mean curvature, or parallel mean curvature vector field, of the complex projective space endowed with the standard Fubini-Study metric. When the ambient space is a complex space form of non-positive holomorphic curvature we obtain non-existence results.

In the second part we consider the Hopf map defined as the restriction of the natural projection \( \pi : C^{n+1} \setminus \{0\} \to CP^n \) to the sphere \( S^{2n+1} \), which defines a Riemannian submersion. For a real submanifold \( M \) of \( CP^n \) we denote by \( \bar{M} := \pi^{-1}(M) \) the Hopf-tube over \( M \). We obtain the formula which relates the bitension field of the inclusion of \( M \) in \( CP^n \) and the bitension field of the inclusion of \( M = \pi^{-1}(M) \) in \( S^{2n+1} \) (Theorem 3.3). Using this formula we are able to produce a new class of proper-biharmonic submanifolds \( M \) of \( CP^n \) when \( M \) is of “Clifford type” (Theorem 4.2), and to reobtain a result in [27] when \( M \) is a product of circles (Theorem 4.10).

We note that \( \bar{M} \) is minimal (harmonic) in \( CP^n \) if and only if \( M \) is minimal in \( S^{2n+1} \) (see [18]) but, for what concerns the biharmonicity, the result does not hold anymore.

In the last part of the paper we concentrate on the geometry of proper-biharmonic curves of \( CP^n \). We characterize all proper-biharmonic curves of \( CP^n \) in terms of their curvatures and complex torsions. Then, using the classification of holomorphic helices of \( CP^2 \) given in [19], we determine all proper-biharmonic curves of \( CP^2 \) (Theorem 6.1).

2. Biharmonic submanifolds of complex space forms

Let \( E^n_\mathbb{C}(4c) \) be a complex space form of holomorphic sectional curvature \( 4c \). Let us denote by \( \bar{J} \) the complex structure and by \( \langle \cdot, \cdot \rangle \) the Riemannian metric on \( E^n_\mathbb{C}(4c) \). Then its curvature operator is given, for vector fields \( X, Y \) and \( Z \), by

\[
R^{E^n_\mathbb{C}(4c)}(X,Y)Z = c\{\langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle \bar{J}Y,Z\rangle \bar{J}X - \langle \bar{J}X,Z\rangle \bar{J}Y + 2\langle X,\bar{J}Y\rangle \bar{J}Z\}.
\]

Let now

\[
\bar{\bar{\imath}} : \bar{M}^\bar{m} \to E^n_\mathbb{C}(4c)
\]

be the canonical inclusion of a submanifold \( \bar{M} \) in \( E^n_\mathbb{C}(4c) \) of real dimension \( \bar{m} \). Then the bitension field becomes

\[
\tau_2(\bar{\bar{\imath}}) = -\bar{m}\{\Delta^{\bar{\bar{\imath}}} \bar{\bar{H}} - c\bar{m}\bar{\bar{H}} + 3c\bar{\bar{\bar{J}}} (\bar{\bar{\bar{J}}} \bar{\bar{H}})^\top\},
\]

where \( \bar{\bar{H}} \) denotes the mean curvature vector field, \( \Delta^{\bar{\bar{\imath}}} \) is the rough Laplacian, and \( (\cdot)^\top \) denotes the tangential component to \( \bar{M} \). The overbar notation will be justified in the next section. If we assume that \( \bar{\bar{J}} \bar{\bar{H}} \) is tangent to \( \bar{M} \), then (2.2) simplifies to

\[
\tau_2(\bar{\bar{\imath}}) = -\bar{m}\{\Delta^{\bar{\bar{\imath}}} \bar{\bar{H}} - c(\bar{m} + 3)\bar{\bar{H}}\}.
\]

Decomposing (2.3) with respect to its tangential and normal component we get

**Proposition 2.1.** Let \( \bar{M} \) be a real submanifold of \( E^n_\mathbb{C}(4c) \) of dimension \( \bar{m} \) such that \( \bar{\bar{J}} \bar{\bar{H}} \) is tangent to \( \bar{M} \). Then \( \bar{M} \) is biharmonic if and only if

\[
\begin{cases}
\Delta^{\bar{\bar{\imath}}} \bar{\bar{H}} + \text{trace} \bar{\bar{B}}(\cdot, \bar{\bar{A}}_{\bar{\bar{H}}}(\cdot)) - c(\bar{m} + 3)\bar{\bar{H}} = 0 \\
4 \text{ trace} \bar{\bar{A}}_{\bar{\bar{\bar{\imath}}}}(\cdot, \bar{\bar{H}}(\cdot)) + \bar{m} \text{ grad}(|\bar{\bar{H}}|^2) = 0
\end{cases}
\]

(2.4)
where \( \bar{A} \) denotes the Weingarten operator, \( \bar{B} \) the second fundamental form, \( \bar{H} \) the mean curvature vector field, \( \nabla^\perp \) and \( \Delta^\perp \) the connection and the Laplacian in the normal bundle of \( \bar{M} \) in \( \mathbb{E}_n^\perp(4c) \).

**Proof.** Since \( \bar{H} \) is normal to \( \bar{M} \), from (2.3) we only have to split \( \Delta^j \bar{H} \). With respect to a geodesic frame \( \{X_i\}_{i=1}^m \) around an arbitrary point \( p \in \bar{M} \), we have

\[
-\Delta^j \bar{H} = \sum_{i=1}^m \nabla^j_{X_i} \nabla^j_{X_i} \bar{H}.
\]

Thus, around \( p \),

\[
\nabla^j_{X_i} \bar{H} = \nabla^j_{X_i} \bar{H} - \bar{A}_{\bar{H}}(X_i),
\]

and, at \( p \),

\[
\sum_{i=1}^m \nabla^j_{X_i} \nabla^j_{X_i} \bar{H} = -\Delta^\perp \bar{H} - \text{trace } \bar{A}_{\nabla^\perp \bar{H}}(\cdot) - \text{trace } \bar{B}(\cdot, \bar{A}_{\bar{H}}(\cdot)) - \text{trace } \nabla^\bar{M} \bar{A}_{\bar{H}}(\cdot, \cdot),
\]

where \( \nabla^\bar{M} \) is the Levi-Civita connection on \( \bar{M} \). Moreover, a long but straightforward computation gives

\[
\text{trace } \nabla^\bar{M} \bar{A}_{\bar{H}}(\cdot, \cdot) = \sum_{i=1}^m \nabla^\bar{M} \bar{A}_{\bar{H}}(X_i)
\]

\[
= \sum_{i,j} \nabla^\bar{M}_{X_i} \bar{A}_{\bar{H}}(X_j, X_j)
\]

\[
= \sum_{i,j} (\bar{A}_{\bar{H}}(X_i, X_j) X_j) = \sum_{i,j} (\bar{A}_{\bar{H}}(X_i) X_j)
\]

\[
= \sum_{i,j} \sum_{i,j} \left( \langle \nabla^j_{X_i} X_j, \bar{H} \rangle + \langle \nabla^j_{X_i} X_j, \nabla^j_{X_i} \bar{H} \rangle \right) X_j
\]

\[
= \sum_{i,j} \sum_{i,j} \left( \langle \nabla^j_{X_i} X_j, \bar{H} \rangle + \langle \nabla^j_{X_i} X_j, \nabla^j_{X_i} \bar{H} \rangle \right) X_j
\]

\[
= \sum_{i,j} \sum_{i,j} \left( \langle \nabla^j_{X_i} X_j, \bar{H} \rangle + \langle \bar{A}_{\nabla^j_{X_i} \bar{H}}(X_i, X_j) \rangle \right) X_j
\]

\[
= \sum_{i,j} \sum_{i,j} \left( \langle \nabla^j_{X_i} X_j, \bar{H} \rangle + \langle \bar{A}_{\nabla^j_{X_i} \bar{H}}(X_i) \rangle \right) X_j
\]

\[
= \sum_{i,j} \left( \langle \nabla^j_{X_i} X_j, \bar{H} \rangle X_j + \sum_i \bar{A}_{\nabla^j_{X_i} \bar{H}}(X_i) \right).
\]
Further, using the curvature tensor field of the pull-back bundle \((j)^{-1}\mathbb{R}^6(4c)\), we get

\[
\text{trace } \nabla^M \tilde{A}_H(\cdot, \cdot) = \sum_{i,j} \langle R^{E^2(4c)}(X_i, X_j)X_i + \nabla^j_{X_j} \nabla^i_{X_i} X_i + \nabla^j_{[X_i,X_j]} X_i, \tilde{H}\rangle X_j + \sum_i \tilde{A}_{\nabla^i \tilde{H}}(X_i)
\]

\[
= c \sum_{i,j} \langle (X_i, X_j)X_i - (X_i, X_i)X_j + \langle \tilde{J}X_j, X_i \rangle \tilde{J}X_i - \langle \tilde{J}X_i, X_i \rangle \tilde{J}X_j + 2\langle X_i, \tilde{J}X_j \rangle \tilde{J}X_i, \tilde{H}\rangle X_j + \sum_{i,j} \langle \nabla^j_{X_j} \tilde{B}(X_i, X_i) + \nabla^i_{X_i} \nabla^j_{X_i} X_i, \tilde{H}\rangle X_j + \sum_i \tilde{A}_{\nabla^i \tilde{H}}(X_i)
\]

\[
= 3c \sum_{i,j} \langle \tilde{J}((\tilde{J}X_j, X_i)X_i), \tilde{H}\rangle X_j + \tilde{m} \sum_{j} \langle \nabla^j_{X_j} \tilde{H}, \tilde{H}\rangle X_j + \sum_{i,j} \langle \nabla^j_{X_j} \nabla^i_{X_i} X_i + \tilde{B}(X_j, \nabla^i_{X_i} X_i), \tilde{H}\rangle X_j + \sum_i \tilde{A}_{\nabla^i \tilde{H}}(X_i)
\]

\[
= \frac{\tilde{m}}{2} \sum_j X_j(\|\tilde{H}\|^2)X_j + 3c \sum_j \langle \tilde{J}((\tilde{J}X_j)\top), \tilde{H}\rangle X_j + \sum_i \tilde{A}_{\nabla^i \tilde{H}}(X_i).
\]

Therefore

\[
\sum_{i=1}^{\tilde{m}} \nabla^i_{X_i} \tilde{A}_H(X_i) = \frac{\tilde{m}}{2} \text{grad}(\|\tilde{H}\|^2) + 3c \sum_j \langle \tilde{J}((\tilde{J}X_j)\top), \tilde{H}\rangle X_j + \sum_i \tilde{A}_{\nabla^i \tilde{H}}(X_i).
\]

Finally, taking into account that \(\tilde{J}\tilde{H}\) is tangent to \(\tilde{M}\), we have

\[
\Delta^j \tilde{H} = \Delta^\perp \tilde{H} + 2 \text{trace } \tilde{A}_{\nabla^i \tilde{H}}(\cdot) + \text{trace } \tilde{B}(\cdot, \tilde{A}_H(\cdot)) + \frac{\tilde{m}}{2} \text{grad}(\|\tilde{H}\|^2)
\]

which gives, together with \((2.3)\), the desired result. \(\square\)

If \(\tilde{M}\) is a hypersurface, then \(\tilde{J}\tilde{H}\) is tangent to \(\tilde{M}\), and the previous proposition gives the following result of \([14]\)

**Corollary 2.2.** Let \(\tilde{M}\) be a real hypersurface of \(\mathbb{E}^6_c(4c)\) of non-zero constant mean curvature. Then it is proper-biharmonic if and only if

\[|\tilde{B}|^2 = 2c(n + 1).\]

Proposition \((2.1)\) can be applied also in the case of Lagrangian submanifolds. We recall here that \(\tilde{M}\) is called a Lagrangian submanifold if \(\dim \tilde{M} = n\) and \(j^\Omega = 0\), where \(\Omega\) is the fundamental 2-form on \(\mathbb{E}^6_c(4c)\) defined by \(\Omega(X, Y) = \langle X, jY \rangle\), for any vector fields \(X\) and \(Y\) tangent to \(\mathbb{E}^6_c(4c)\).

**Corollary 2.3.** Let \(\tilde{M}\) be a Lagrangian submanifold of \(\mathbb{E}^6_c(4c)\) with parallel mean curvature vector field. Then it is biharmonic if and only if

\[
\text{trace } \tilde{B}(\cdot, \tilde{A}_H(\cdot)) = c(n + 3)\tilde{H}.
\]

In the sequel we shall consider only the case of complex space forms with positive holomorphic sectional curvature. A partial motivation of this fact is that Corollary \((2.2)\) rules out the case \(c \leq 0\). As usual, we consider the complex projective space
Replacing in (2.5) we get

Then the square of the norm of $\bar{\mathcal{H}}$ is tangent to $\bar{M}$. Assume that it has non-zero constant mean curvature. We have

(a) If $\bar{M}$ is proper-biharmonic, then $|\bar{H}|^2 \in (0, \frac{\bar{m}+3}{\bar{m}}]$.

(b) If $|\bar{H}|^2 = \frac{\bar{m}+3}{\bar{m}}$, then $\bar{M}$ is proper-biharmonic if and only if it is pseudo-umbilical and $\nabla^\perp \bar{H} = 0$.

**Proof.** Let $\bar{M}$ be a real submanifold of $\mathbb{C}P^n$ of dimension $\bar{m}$ such that $\bar{J}\bar{H}$ is tangent to $\bar{M}$. Assume that it has non-zero constant mean curvature, and it is biharmonic. As $\bar{M}$ is biharmonic we have

$$\Delta^\perp \bar{H} = (\bar{m} + 3)\bar{H} - \text{trace} \bar{B} (\cdot, \bar{A}_\bar{H} (\cdot)),$$

so

$$\langle \Delta^\perp \bar{H}, \bar{H} \rangle = (\bar{m} + 3)|\bar{H}|^2 - \sum_{i=1}^{\bar{m}} \langle \bar{B} (X_i, \bar{A}_\bar{H} (X_i)), \bar{H} \rangle = (\bar{m} + 3)|\bar{H}|^2 - |\bar{A}_\bar{H}|^2.$$

Replacing in the Weitzenb"{o}ck formula (see, for example, [12])

$$\frac{1}{2} \Delta|\bar{H}|^2 = \langle \Delta^\perp \bar{H}, \bar{H} \rangle - |\nabla^\perp \bar{H}|^2$$

the expression of $\langle \Delta^\perp \bar{H}, \bar{H} \rangle$, and using the fact that $|\bar{H}|$ is constant, we obtain

(2.5)

$$(\bar{m} + 3)|\bar{H}|^2 = |\bar{A}_\bar{H}|^2 + |\nabla^\perp \bar{H}|^2.$$

Let $p$ be an arbitrary point of $\bar{M}$ and let $\{X_i\}_{i=1}^{\bar{m}}$ be an orthonormal basis of $T_p \bar{M}$ such that $\bar{A}_\bar{H} (X_i) = \lambda_i X_i$. We have

$$\lambda_i = \langle \bar{A}_\bar{H} (X_i), X_i \rangle = \langle \bar{B} (X_i, X_i), \bar{H} \rangle$$

which implies

$$\sum_{i=1}^{\bar{m}} \lambda_i = \bar{m}|\bar{H}|^2$$

or, equivalently,

$$|\bar{H}|^2 = \frac{\sum_{i=1}^{\bar{m}} \lambda_i}{\bar{m}}.$$

Then the square of the norm of $\bar{A}_\bar{H}$ becomes

$$|\bar{A}_\bar{H}|^2 = \sum_{i=1}^{\bar{m}} \langle \bar{A}_\bar{H} (X_i), \bar{A}_\bar{H} (X_i) \rangle = \sum_{i=1}^{\bar{m}} (\lambda_i)^2.$$

Replacing in (2.5) we get

$$\frac{\bar{m} + 3}{\bar{m}} \sum_{i} \lambda_i = \sum_{i} (\lambda_i)^2 + |\nabla^\perp \bar{H}|^2 \geq \frac{(\sum_{i} \lambda_i)^2}{\bar{m}} + |\nabla^\perp \bar{H}|^2.$$

Therefore

$$(\bar{m} + 3)|\bar{H}|^2 \geq \bar{m}|\bar{H}|^4 + |\nabla^\perp \bar{H}|^2 \geq \bar{m}|\bar{H}|^4,$$

so

$$|\bar{H}|^2 \in (0, \frac{\bar{m}+3}{\bar{m}}].$$
(b) If $|\bar{H}| = \frac{m+3}{m}$ and $\bar{M}$ is biharmonic, the above inequalities become equalities, and therefore $\lambda_1 = \cdots = \lambda_m$ and $\nabla^\perp H = 0$, i.e. $\bar{M}$ is pseudo-umbilical and $\nabla^\perp \bar{H} = 0$.

Conversely, it is clear that if $|\bar{H}| = \frac{m+3}{m}$ and $\bar{M}$ is pseudo-umbilical with $\nabla^\perp \bar{H} = 0$, then $\bar{M}$ is proper-biharmonic.

\[ \square \]

Remark 2.5. We shall see in Proposition 5.1 that the upper bound of $|\bar{H}|^2$ is reached in the case of curves.

Proposition 2.6. Let $\bar{M}$ be a proper-biharmonic real hypersurface of $\mathbb{C}P^n$ of constant mean curvature $|\bar{H}|$. Then its scalar curvature $s^{\bar{M}}$ is constant and given by

\[ s^{\bar{M}} = 4n^2 - 2n - 4 + (2n - 1)^2 |\bar{H}|^2. \]

Proof. Let $\bar{M}^{2n-1}$ be a proper-biharmonic real hypersurface of $\mathbb{C}P^n$ with constant mean curvature, so $|\bar{H}|^2 = 2(n + 1)$.

The Gauss equation for the submanifold $\bar{M}$ of $\mathbb{C}P^n$ is

\[ (2.6) \quad \langle R^{\bar{M}}(X,Y)Z,T \rangle = \langle R^{\mathbb{C}P^n}(X,Y)Z,T \rangle - \langle \bar{B}(Y,T),\bar{B}(X,Z) \rangle + \langle \bar{B}(X,T),\bar{B}(Y,Z) \rangle, \]

where $R^{\bar{M}}$ is the curvature tensor field of $\bar{M}$.

Let us denote by $\rho^{\bar{M}}(X,Y) = \text{trace}(Z \rightarrow R^{\bar{M}}(Z,X)Y)$ the Ricci tensor. Computing (2.6) for $X = T = X_i$, where $\{X_i\}_{i=1}^{2n-1}$ is a local orthonormal frame field, we have

\[
\langle R^{\bar{M}}(X_i,Y)Z,X_i \rangle = \langle \langle Z,Y \rangle X_i - \langle Z,X_i \rangle Y,X_i \rangle \\
+ \langle \langle \bar{J}Y,Z \rangle \bar{J}X_i,X_i \rangle - \langle \langle \bar{J}X_i,Z \rangle \bar{J}Y,X_i \rangle \\
+ 2\langle \langle X_i,\bar{J}Y \rangle \bar{J}Z,X_i \rangle - \langle \bar{B}(Y,X_i),\bar{B}(X_i,Z) \rangle + \langle \bar{B}(X_i,X_i),\bar{B}(Y,Z) \rangle \\
= \langle Z,Y \rangle - \langle Z,X_i \rangle \langle Y,X_i \rangle \\
+ \langle \bar{J}Y,Z \rangle \langle \bar{J}X_i,X_i \rangle - \langle \bar{J}X_i,Z \rangle \langle \bar{J}Y,X_i \rangle \\
+ 2\langle X_i,\bar{J}Y \rangle \langle \bar{J}Z,X_i \rangle - \langle \bar{B}(Y,X_i),\bar{B}(Z,X_i) \rangle \\
+ \langle \bar{B}(X_i,X_i),\bar{B}(Y,Z) \rangle \\
= \langle Z,Y \rangle - \langle Z,X_i \rangle \langle Y,X_i \rangle + 3\langle \bar{J}Z,X_i \rangle \langle \bar{J}Y,X_i \rangle \\
- \langle \bar{A}(Y),X_i \rangle \langle \bar{A}(Z),X_i \rangle + \langle \bar{B}(X_i,X_i),\bar{B}(Y,Z) \rangle,
\]

where $\bar{H} = |\bar{H}|\bar{\eta}$ and $\bar{A} = \bar{A}\bar{\eta}$. Therefore

\[
\rho^{\bar{M}}(Y,Z) = \sum_{i=1}^{2n-1} \langle R^{\bar{M}}(X_i,Y)Z,X_i \rangle \\
= (2n - 1)\langle Z,Y \rangle - \langle Z,Y \rangle + 3\langle (\bar{J}Z)^\top, (\bar{J}Y)^\top \rangle \\
- \langle \bar{A}(Y),\bar{A}(Z) \rangle + (2n - 1)|\bar{H}|\langle \bar{A}(Y),Z \rangle.
\]

Now,

\[
\langle \bar{J}Z,\bar{J}Y \rangle = \langle Z,Y \rangle \\
= \langle (\bar{J}Z)^\top + (\bar{J}Z,\bar{\eta})\bar{\eta},(\bar{J}Y)^\top + (\bar{J}Y,\bar{\eta})\bar{\eta} \rangle \\
= \langle (\bar{J}Z)^\top, (\bar{J}Y)^\top \rangle + \langle \bar{J}Z,\bar{\eta} \rangle \langle \bar{J}Y,\bar{\eta} \rangle,
\]

which implies

\[ \langle (\bar{J}Z)^\top, (\bar{J}Y)^\top \rangle = \langle Z,Y \rangle - \langle Z,\bar{J}\bar{\eta} \rangle \langle Y,\bar{J}\bar{\eta} \rangle. \]
Replacing in the above expression of the Ricci tensor, we get
\[
\rho_{\bar{M}}(Y, Z) = 2(n - 1)\langle Z, Y \rangle + 3\{\langle Y, Z \rangle - \langle Z, \bar{J}\eta \rangle\langle Y, \bar{J}\eta \rangle \}
\]
\[
- \langle \bar{A}(Y), \bar{A}(Z) \rangle + (2n - 1)|\bar{H}|\langle \bar{A}(Y), Z \rangle.
\]
Finally, taking the trace, we have
\[
s_{\bar{M}} = \sum_{i=1}^{2n-1} \rho_{\bar{M}}(X_i, X_i) = 2(n - 1)(2n - 1) + 3(2n - 1)
\]
\[
- |\bar{J}\eta|^2 - |\bar{A}|^2 + (2n - 1)^2|\bar{H}|^2
\]
\[
= (2n - 2 + 3)(2n - 1) - 1 - 2(n + 1) + (2n - 1)^2|\bar{H}|^2
\]
\[
= 4n^2 - 2n - 4 + (2n - 1)^2|\bar{H}|^2.
\]
\[\Box\]

Another important family of submanifolds of \(CP^n\) is that consisting of the submanifolds for which \(J\bar{H}\) is normal to \(M\). In this case, using an argument similar to the case when \(J\bar{H}\) is tangent to \(M\), we have the following result

**Proposition 2.7.** Let \(\bar{M}\) be a real submanifold of \(CP^n\) of dimension \(\bar{m}\) such that \(J\bar{H}\) is normal to \(M\). Then \(\bar{M}\) is biharmonic if and only if

\[
\begin{cases}
\Delta \bar{H} + \text{trace } \bar{B}(\cdot, \bar{A}_\bar{H}(\cdot)) - \bar{m}\bar{H} = 0 \\
4\text{trace } \bar{A}\nabla_{(\cdot, \bar{H})}(\cdot) + \bar{m}\text{grad}(|\bar{H}|^2) = 0
\end{cases}
\]

Moreover, if \(J\bar{H}\) is normal to \(\bar{M}\) and \(\bar{M}\) has parallel mean curvature, then \(\bar{M}\) is biharmonic if and only if

\[
\text{trace } \bar{B}(\cdot, \bar{A}_\bar{H}(\cdot)) = \bar{m}\bar{H}.
\]

Also in this case, if the mean curvature is constant we can bound its value, as it is shown by the following

**Proposition 2.8.** Let \(\bar{M}\) be a real submanifold of \(CP^n\) of dimension \(\bar{m}\) such that \(J\bar{H}\) is normal to \(\bar{M}\). Assume that it has non-zero constant mean curvature. We have

(a) If \(\bar{M}\) is proper-biharmonic, then \(|\bar{H}|^2 \in (0, 1)\).

(b) If \(|\bar{H}|^2 = 1\), then \(\bar{M}\) is proper-biharmonic if and only if it is pseudo-umbilical and \(\nabla_{\bar{H}}\bar{H} = 0\).

**Remark 2.9.** We shall see in Proposition 5.5 (a), that the upper bound is reached in the case of curves.

3. The Hopf Fibration and the Biharmonic Equation

Let \(\pi : C^{n+1} \setminus \{0\} \to CP^n\) be the natural projection. Then \(\pi\) restricted to the sphere \(S^{2n+1}\) of \(C^{n+1}\) gives rise to the Hopf fibration \(\pi : S^{2n+1} \to CP^n\) and if \(4c = 4\) then \(\pi : S^{2n+1} \to CP^n\) defines a Riemannian submersion. In the sequel we shall look at \(S^{2n+1}\) as a hypersurface of \(R^{2n+2}\) and we shall denote by \(J\) the complex structure of \(R^{2n+2}\).

Let \(\bar{M}\) be a real submanifold of \(CP^n\) of dimension \(\bar{m}\) and denote by \(\bar{M} := \pi^{-1}(\bar{M})\) the Hopf-tube over \(\bar{M}\). If we denote by \(\bar{j} : \bar{M} \to CP^n\) and \(j : \bar{M} \to S^{2n+1}\) the
respective inclusions we have the following diagram

\[
\begin{array}{ccc}
M & \xrightarrow{j} & S^{2n+1} \\
\downarrow & & \downarrow \pi \\
\bar{M} & \xrightarrow{j} & \mathbb{C}P^n.
\end{array}
\]

We shall now find the relation between the bitension field of the inclusion \( j \) and the bitension field of the inclusion \( \bar{j} \). For this, let \( \{ \bar{X}_k \}_{k=1}^{\bar{m}} \) be a local orthonormal frame field tangent to \( \bar{M} \), \( 1 \leq \bar{m} \leq 2n-1 \), and let \( \{ \bar{\eta}_\alpha \}_{\alpha=n+1}^{2n+1} \) be a local orthonormal frame field normal to \( \bar{M} \). Let us denote by \( X_k := \bar{X}_k^H \) and \( \eta_\alpha := \bar{\eta}_\alpha^H \) the horizontal lifts with respect to the Hopf map and by \( \xi \) the Hopf vector field on \( S^{2n+1} \) which is tangent to the fibres of the Hopf fibration, i.e. \( \xi(p) = -\bar{J}p \), for any \( p \in S^{2n+1} \). Then \( \{ \xi, X_k \} \) is a local orthonormal frame field tangent to \( M \) and \( \{ \eta_\alpha \} \) is a local orthonormal frame field normal to \( M \).

**Lemma 3.2.** Let \( X = X^H \in C(TM) \), where \( X \in C(TM) \), and \( V = \bar{V}^H \in C(j^{-1}(TS^{2n+1})) \), where \( \bar{V} \in C((j)^{-1}(T\mathbb{C}P^n)) \). Then

\[
\nabla^j_X V = (\nabla^j_X \bar{V})^H + \langle V, \bar{J}X \rangle \xi = (\nabla^j_X \bar{V})^H + (\langle \bar{V}, \bar{J}X \rangle \circ \pi) \xi,
\]

where \( \nabla^j \) and \( \nabla^\bar{j} \) denote the pull-back connections on \( j^{-1}(TS^{2n+1}) \) and \( (j)^{-1}(T\mathbb{C}P^n) \), respectively.

**Proof.** Decomposing \( \nabla^j_X V \) in its horizontal and vertical components we have

\[
\nabla^j_X V = \nabla^j_X \bar{V} = (\nabla^j_X \bar{V})^H + \langle \nabla^j_X \bar{V}, \xi \rangle \xi.
\]

Now,

\[
\langle \nabla^j_X \bar{V}, \xi \rangle = -\langle V, \nabla^j_X \xi \rangle = -\langle V, \hat{\nabla}_X \xi + \langle X, \xi \rangle p \rangle = \langle V, \hat{\nabla}_X \bar{J}p \rangle = \langle V, \bar{J}X \rangle = \langle V, JX \rangle \circ \pi,
\]

where \( \hat{\nabla} \) is the Levi-Civita connection on the Euclidean space \( \mathbb{E}^{2n+2} \).

**Lemma 3.3.** If \( V = \bar{V}^H \in C(j^{-1}(TS^{2n+1})) \), \( \bar{V} \in C((j)^{-1}(T\mathbb{C}P^n)) \), then

\[
\Delta^j V = (\Delta^j \bar{V})^H + 2 \text{div}((\bar{J}V)^\top) \xi + \langle V, \bar{J}X \rangle \xi + V - \bar{J}(\bar{J}V)^\top,
\]

where \( \Delta^j \) and \( \Delta^\bar{j} \) are the rough Laplacians acting on sections of \( j^{-1}(TS^{2n+1}) \) and \( (j)^{-1}(T\mathbb{C}P^n) \), respectively, whilst \( (V)^\top \) denotes the component of \( V \) tangent to \( M \).

**Proof.** The Laplacian \( \Delta^j \) is given by

\[
-\Delta^j V = \sum_{i=1}^{\bar{m}} \{ \nabla^2_{X_i} \nabla^j_{X_i} V - \nabla^2_{\nabla^j_{X_i} X_i} V \} + \nabla^2_{X_i} \nabla^j_X V - \nabla^2_{\nabla^j_X} V.
\]

We compute each term separately. From Lemma \[3.1\] we have

\[
\nabla^2_{X_i} \nabla^j_{X_i} V = (\nabla^2_{X_i} \nabla^j_{X_i} \bar{V})^H + \langle \nabla^2_{X_i} V, \bar{J}X_i \rangle \xi + \nabla^j_{X_i} (\langle V, \bar{J}X_i \rangle \xi)
\]

\[
= (\nabla^2_{X_i} \nabla^j_{X_i} \bar{V})^H + 2 \langle \nabla^2_{X_i} V, \bar{J}X_i \rangle \xi + \langle \nabla^j_{X_i} V, \bar{J}X_i \rangle \xi
\]

\[
+ \langle V, \nabla^2_{X_i} \bar{J}X_i \rangle \xi + \langle \bar{J}V, \bar{J}X_i \rangle \bar{J}X_i.
\]

Using

\[
\nabla^2_{X_i} \bar{J}X_i = \hat{J} \nabla^2_{X_i} X_i + \xi
\]
we get
\begin{equation}
(3.1) \quad \nabla^2_{X_i} V = (\nabla^2_{X_i} \nabla^2_{X_i} V)^H + 2(\nabla^2_{X_i} V, JX_i)\xi + (V, J\nabla^2_{X_i} X_i)\xi + \hat{J}(\langle JV, X_i \rangle X_i).
\end{equation}

Next
\begin{equation}
(3.2) \quad \nabla^2_{\bar{X}_i} V = (\nabla^2_{\bar{X}_i} \nabla^2_{\bar{X}_i} V)^H + (V, J\nabla^2_{\bar{X}_i} X_i)\xi.
\end{equation}

Summing (3.1) and (3.2) up we find
\begin{equation*}
-\Delta^2 V = -(\Delta^2 \hat{V})^H + 2\sum_{i=1}^{\bar{m}}(\nabla^2_{X_i} V, JX_i)\xi + (V, J\sum_{i=1}^{\bar{m}}(\nabla^2_{X_i} X_i - \nabla^2_{\bar{X}_i} X_i))\xi
\end{equation*}
\begin{equation*}
+ \sum_{i=1}^{\bar{m}} \hat{J}(\langle JV, X_i \rangle X_i) + \nabla^2_{\xi} \nabla^2_{\xi} V
\end{equation*}
\begin{equation*}
= -(\Delta^2 \hat{V})^H + 2\sum_{i=1}^{\bar{m}}(\nabla^2_{X_i} V, JX_i)\xi + (V, J\tau(j))\xi
\end{equation*}
\begin{equation*}
+ \hat{J}(\langle \hat{J}V, X_i \rangle)^\top + \nabla^2_{\xi} \nabla^2_{\xi} V.
\end{equation*}

We now compute the extra terms in the above equation.
\begin{equation}
(3.3) \quad \sum_{i=1}^{\bar{m}}(\nabla^2_{X_i} V, JX_i) = \sum_{i=1}^{\bar{m}}(-X_i J\langle JV, X_i \rangle + \langle JV, \nabla^2_{X_i} X_i \rangle)
\end{equation}
\begin{equation*}
= \langle JV, \tau(j) \rangle - \sum_{i=1}^{\bar{m}}\{X_i J\langle JV, X_i \rangle - (\langle JV, \nabla^2_{X_i} X_i \rangle)\}
\end{equation*}
\begin{equation*}
= \langle JV, \tau(j) \rangle - \text{div}(\langle \hat{J}V \rangle^\top).
\end{equation*}

Finally
\begin{equation*}
\nabla^2_{\xi} V = H(\nabla^2_{\xi} V) + (\nabla^2_{\xi} V, \xi)\xi = H(\nabla^2_{\xi} V)
\end{equation*}
\begin{equation*}
= H(\nabla^2_{\xi} V) = H(\langle \hat{V}, V \rangle + (V, \xi)p) = H(-\hat{J}V) = -\hat{J}V
\end{equation*}
which gives
\begin{equation*}
\nabla^2_{\xi} V = -V \quad \square
\end{equation*}

Before giving the relation between the bitension fields we need to compute the trace of the curvature operators. One gets immediately
\begin{equation}
(3.4) \quad -\text{trace} R^{S^{2n+1}}(dj, \tau(j))dj = (\bar{m} + 1)\tau(j)
\end{equation}
and
\begin{equation}
(3.5) \quad -\text{trace} R^{CP^n}(dj, \tau(j))dj = \bar{m}\tau(j) - 3\hat{J}(\hat{J}^\top)\tau(j)\tau(j)^\top.
\end{equation}

We are now ready to state the main theorem of this section

**Theorem 3.3.** Let $\hat{M}$ be a real submanifold of $CP^n$ of dimension $\bar{m}$ and denote by $M := \pi^{-1}(\hat{M})$ the corresponding Hopf-tube. If we denote by $j : \hat{M} \rightarrow CP^n$ and $j : M \rightarrow S^{2n+1}$ the respective inclusions we have that
\begin{equation}
(3.6) \quad (\tau_2(j))^H = \tau_2(j) - 4\hat{J}(\hat{J}^\top)\tau(j)\tau(j)^\top + 2\text{div}(\langle \hat{J}\tau(j) \rangle^\top)\xi.
\end{equation}
Remark 3.4. \[ \text{(i)} \] Using the horizontal lift, it is straightforward to check that \((\ref{t2})\) can be written as
\[
(\tau_2(j))^H = \tau_2(j) - 4(J(J\tau(j))^\top)^H + 2(\text{div}_M((J\tau(j))^\top) \circ \pi)\xi.
\]
\[ \text{(ii)} \] If \(J\tau(j)\) is normal to \(M\), then \(\tau_2(j) = 0\) if and only if \(\tau_2(j) = 0\).
\[ \text{(iii)} \] If \(J\tau(j)\) is tangent to \(M\), then \(\tau_2(j) = 0\) and \(\text{div}_M((J\tau(j))^\top) = 0\) if and only if \(\tau_2(j) + 4\tau(j) = 0\).
\[ \text{(iv)} \] Assume that, locally, \(M = \pi^{-1}(\bar{M}) = S^1 \times \bar{M}\), where \(\bar{M}\) is an integral submanifold of \(S^{2n+1}\), i.e. \(\langle \tilde{X}_\pi, \xi(\tilde{p}) \rangle = 0\), for any vector \(\tilde{X}_\pi\) tangent to \(\bar{M}\).
Denote by \(\tilde{j} : \bar{M} \to S^{2n+1}\) the canonical inclusion, and by \(\{\phi_t\}\) the flow of \(\xi\).
We know that \(\tau_2(j)(t, \tilde{p}) = (d\phi_t)_{\tilde{p}}(\tau_2(j))\), see \([13]\), and we can check that, at \(\tilde{p}\),
\[
(\tau_2(j))^H = \tau_2(j) - 4\tilde{J}((J\tau(j))^\top) + 2\text{div}_M((J\tau(j))^\top)\xi.
\]
To state the next results we recall that a smooth map \(\varphi : (M, g) \to (N, h)\) is called \(\lambda\)-biharmonic if it is a critical point of the \(\lambda\)-bienergy
\[
E_2(\varphi) + \lambda E(\varphi),
\]
where \(\lambda\) is a real constant. The critical points of the \(\lambda\)-bienergy satisfy the equation
\[ \tau_2(\varphi) - \lambda \tau(\varphi) = 0. \]

Proposition 3.5. Let \(\bar{M}\) be a real hypersurface of \(CP^n\) of constant mean curvature and denote by \(M = \pi^{-1}(\bar{M})\) the Hopf-tube over \(\bar{M}\). Then \(\tau_2(j) = 0\) if and only if \(\tau_2(j) = 4\tau(j) = 0\), i.e. \(j\) is \((-4)\)-biharmonic.

Proof. We have \((J\tau(j))^\top = \bar{J}\tau(j)\) and it remains to prove that \(\text{div}_M(\bar{J}\tau(j)) = 0\).
Let \(\eta\) be a local unit section in the normal bundle of \(M\) in \(CP^n\) and consider \(\{X_1, \bar{J}X_1, \ldots, X_{n-1}, \bar{J}X_{n-1}, \bar{J}\eta\}\) a local orthonormal frame field tangent to \(M\). Since \(\bar{M}\) is a hypersurface of constant mean curvature, it is enough to prove that \(\text{div}_\bar{M}(\bar{J}\eta) = 0\). But, denoting by \(\bar{A}_\eta\) the shape operator of \(\bar{M}\),
\[
\langle \nabla^M_X \bar{J}\eta, X_a \rangle = \langle \bar{A}_\eta(X_a), \bar{J}X_a \rangle, \quad \langle \nabla^M_{\bar{J}X_b} \bar{J}\eta, \bar{J}X_b \rangle = -\langle \bar{A}_\eta(X_b), \bar{J}X_b \rangle,
\]
for any \(1 \leq a, b \leq n - 1\), and
\[
\langle \nabla^\bar{M}_{\bar{J}\eta} \bar{J}\eta, \bar{J}\eta \rangle = 0,
\]
so we conclude. \(\Box\)

Proposition 3.6. Let \(\bar{M}\) be a Lagrangian submanifold of \(CP^n\) with parallel mean curvature vector field and denote by \(M = \pi^{-1}(\bar{M})\) the Hopf-tube over \(\bar{M}\). Then \(j\) is biharmonic if and only if \(j\) is \((-4)\)-biharmonic.

Proof. Since \(\bar{M}\) is a Lagrangian submanifold, \(\dim \bar{M} = \bar{m} = n\) and \(\bar{J}(TM\bar{M}) = N\bar{M}\) (therefore \(J(N\bar{M}) = TM\bar{M}\)). We have that \(J\tau(j) \in C(TM\bar{M})\) and we shall prove that \(\nabla^M\bar{J}\tau(j) = 0\) which implies \(\text{div}_M(\bar{J}\tau(j)) = 0\). Indeed, for any \(\bar{X}\) and \(\bar{Y}\) tangent to \(\bar{M}\) we have
\[
\langle \nabla^M_X \bar{J}\tau(j), \bar{Y} \rangle = \langle \nabla^\bar{M}_X \bar{J}\tau(j), \bar{Y} \rangle = \langle \bar{J}\nabla^\bar{M}_X \tau(j), \bar{Y} \rangle = \langle -\bar{J}A_\tau(j)(\bar{X}), \bar{Y} \rangle = 0.
\]
We end this section with

**Proposition 3.7.** Let \( \tilde{M} \) be a real submanifold of \( \mathbb{C}P^n \) such that \( \tilde{\tau}(\tilde{j}) \) is normal to \( \tilde{M} \) and denote by \( M = \pi^{-1}(\tilde{M}) \) the Hopf-tube over \( \tilde{M} \). Then \( \tilde{j} \) is biharmonic if and only if \( j \) is biharmonic.

4. **Biharmonic submanifolds of Clifford type**

For a fixed \( n > 1 \), consider the spheres \( S^{2p+1}(a) \subset \mathbb{R}^{2p+2} = \mathbb{C}^{p+1} \) and \( S^{2q+1}(b) \subset \mathbb{R}^{2q+2} = \mathbb{C}^{q+1} \), with \( a^2 + b^2 = 1 \) and \( p + q = n - 1 \). Denote by \( T^{p,q}_{a,b} = S^{2p+1}(a) \times S^{2q+1}(b) \subset S^{2n+1} \) the Clifford torus. Let now \( M_1 \) be a minimal submanifold of \( S^{2p+1}(a) \) of dimension \( m_1 \) and \( M_2 \) a minimal submanifold of \( S^{2q+1}(b) \) of dimension \( m_2 \). The submanifold \( M_1 \times M_2 \) is clearly minimal in \( T^{p,q}_{a,b} \) and, according to [6], is proper biharmonic in \( S^{2n+1} \) if and only if \( a = b = \sqrt{2}/2 \) and \( m_1 \neq m_2 \). If \( M_1 \times M_2 \) is invariant under the action of the one-parameter group of isometries generated by the Hopf vector field \( \xi \) on \( S^{2n+1} \), then it projects onto a submanifold of \( \mathbb{C}P^n \) and we could ask for which values of \( a, b, m_1, m_2 \) is it a proper-biharmonic submanifold.

We start with the following

**Lemma 4.1.** Let denote by \( j_1 : M_1^{m_1} \times M_2^{m_2} \to T^{p,q}_{a,b} \) the inclusion of \( M_1 \times M_2 \) in the Clifford torus and by \( j : T^{p,q}_{a,b} \to S^{2n+1} \) the inclusion of the Clifford torus in the sphere. Then

\[
\begin{aligned}
\tau(j \circ j_1) &= \left( \frac{a}{b} m_2 - \frac{b}{a} m_1 \right) \eta = c \eta \\
\tau_2(j \circ j_1) &= c (m_1 + m_2 - \frac{b^2}{a^2} m_1 - \frac{a^2}{b^2} m_2) \eta,
\end{aligned}
\]

where \( \eta \) is the unit normal section in the normal bundle of \( T^{p,q}_{a,b} \) in \( S^{2n+1} \) given by \( \eta(x,y) = (\frac{b}{a} x, -\frac{a}{b} y), \ x \in S^{2p+1}(a), y \in S^{2q+1}(b). \)

**Proof.** Let \( p = (x,y) \in T^{p,q}_{a,b}, \ x \in \mathbb{R}^{2p+2}, \ y \in \mathbb{R}^{2q+2}, \ |x| = a, \ |y| = b. \) Then \( \eta(x,y) = (\frac{b}{a} x, -\frac{a}{b} y) \) defines a unit normal section in the normal bundle of \( T^{p,q}_{a,b} \) in \( S^{2n+1} \). We identify \( X = (X,0) \in T_p T^{p,q}_{a,b}, \ Y = (0,Y) \in T_p T^{p,q}_{a,b}, \) and a straightforward computation gives

\[
\nabla^j_x \eta = -A^j(X) = \frac{b}{a} X, \ \ \ \nabla^j_y \eta = -A^j(Y) = -\frac{a}{b} Y.
\]

Let \( \{X_k = (X_k,0)\} \) be a local orthonormal frame field tangent to \( S^{2p+1}(a) \) and \( \{Y_l = (0,Y_l)\} \) a local orthonormal frame field tangent to \( S^{2q+1}(b). \) Then, applying the composition law for the tension field and using that \( j_1 \) is harmonic, we have

\[
\tau(j \circ j_1) = d_j(\tau(j_1)) + \text{trace} \nabla d_j(d_j1, d_j1) = \sum_{k=1}^{m_1} \langle A^j(X_k), X_k \rangle \eta + \sum_{l=1}^{m_2} \langle A^j(Y_l), Y_l \rangle \eta = \left( \frac{a}{b} m_2 - \frac{b}{a} m_1 \right) \eta = c \eta.
\]

To compute \( \tau_2(j \circ j_1) \), let us choose around \( p = (x,y) \in M_1 \times M_2 \) a frame field \( \{(X_k, Y_l)\} \) such that \( \{X_k\}_{k=1}^{m_1} \) is a geodesic frame field around \( x \) and \( \{Y_l\}_{l=1}^{m_2} \) is a
geodesic frame field around $y$. Then at $p$

$$- \Delta^{p,q} \eta = \sum_{k=1}^{m_1} \nabla^{p,q}_{X_k} \nabla^{p,q}_{X_k} \eta + \sum_{l=1}^{m_2} \nabla^{p,q}_{Y_l} \nabla^{p,q}_{Y_l} \eta$$

$$= \frac{b}{a} \sum_{k=1}^{m_1} \nabla^{p,q}_{X_k} X_k - \frac{a}{b} \sum_{l=1}^{m_2} \nabla^{p,q}_{Y_l} Y_l$$

$$= \frac{b}{a} \sum_{k=1}^{m_1} (B^l(X_k, X_k) + \nabla^{p,q}_{X_k} X_k) - \frac{a}{b} \sum_{l=1}^{m_2} (B^l(Y_l, Y_l) + \nabla^{p,q}_{Y_l} Y_l)$$

$$= \frac{b}{a^2 m_1} - \frac{a^2}{b^2 m_2}. \tag{4.2}$$

Finally, using the standard formula for the curvature of $S^{2n+1}$, we get

$$- \text{trace } K^{S^{2n+1}} (d(j \circ j_1), \tau (j \circ j_1)) d(j \circ j_1) = (m_1 + m_2) \tau (j \circ j_1) = (m_1 + m_2) c_\eta,$$

that summed up with (4.2) gives the lemma.

\begin{remark}
\end{remark}

\textbf{Theorem 4.2.} Let $\pi : S^{2n+1} \to \mathbb{C}P^n$ be the Hopf map. Let $M = M_1^{m_1} \times M_2^{m_2}$ be the product of two minimal submanifolds of $S^{2p+1}(a)$ and $S^{2q+1}(b)$, respectively. Assume that $M$ is invariant under the action of the one-parameter group of isometries generated by the Hopf vector field $\xi$ on $S^{2n+1}$. Then $\pi(M)$ is a proper-biharmonic submanifold of $\mathbb{C}P^n$ if and only if $M$ is $(-4)$-biharmonic, that is

$$\begin{cases}
  a^2 + b^2 = 1 \\
  \frac{a}{m_2} - \frac{b}{m_1} \neq 0 \\
  \frac{b^2}{a^2 m_1} + \frac{a^2}{b^2 m_2} = 4 + m_1 + m_2
\end{cases}, \tag{4.3}$$

where $m_1$ and $m_2$ are the dimensions of $M_1$ and $M_2$, respectively.

\begin{proof}
The Hopf vector field $\xi$ is a Killing vector field on $S^{2n+1}$ that, at a point $p = (x, y)$, is given by

$$\xi = -(x^2, x^1, \ldots, -x^{2p+2}, x^{2p+1}, -y^2, y^1, \ldots, -y^{2q+2}, y^{2q+1}) = (\xi_1, \xi_2).$$

Since $M_1 \times M_2$ is invariant under the action of the one-parameter group of isometries generated by $\xi$, it remains Killing when restricted to $M_1 \times M_2$. As

$$\hat{\xi} = (-\frac{b}{a} \xi_1, \frac{a}{b} \xi_2),$$

it follows that $\hat{\xi}$ is a Killing vector field on $M_1 \times M_2$.

Since $\text{div}(\hat{\xi} \circ j_1) = \text{div}(c \xi \eta) = 0$, using Remark 3.3 (iii), it results that $\pi(M_1 \times M_2)$ is a biharmonic submanifold of $\mathbb{C}P^n$ if and only if

$$\tau_2(j \circ j_1) + 4 \tau(j \circ j_1) = 0.$$

Finally, using Lemma 4.1, we get

$$\tau_2(j \circ j_1) + 4 \tau(j \circ j_1) = c(4 + m_1 + m_2 - \frac{b^2}{a^2 m_1} - \frac{a^2}{b^2 m_2}) \eta.$$

\end{proof}
Remark 4.3. If $M_1 = S^{2p+1}(a)$ and $M_2 = S^{2q+1}(b)$, we recover the result in [14] concerning the proper-biharmonic homogeneous real hypersurfaces of type A in $CP^n$.

Example 4.4. Let $e_1$ and $e_3$ be two constant unit vectors in $E^{2n+2}$, with $e_3$ orthogonal to $e_1$ and $Je_1$. We consider the circles $S^1(a)$ and $S^1(b)$ lying in the 2-planes spanned by $\{e_1, Je_1\}$ and $\{e_3, Je_3\}$, respectively. Then $M = S^1(a) \times S^1(b)$ is invariant under the flow-action of $\xi$, and $\pi(M)$ is a proper-biharmonic curve of $CP^n$ if and only if $a = \frac{\sqrt{2+\sqrt{2}}}{2}$.

Example 4.5. For $p = 0$ and $q = n - 1$, we get that $\pi(S^1(a) \times S^{2n-1}(b))$ is proper-biharmonic in $CP^n$ if and only if $a^2 = \frac{n+3+\sqrt{n^2+2n+5}}{4(n+1)}$. In particular, $\pi(S^1(a) \times S^3(b))$ is a proper-biharmonic real hypersurface in $CP^2$ if and only if $a^2 = \frac{5+\sqrt{13}}{12}$.

Example 4.6. If $p = q$ then $M = T_{a,b}^{p,p}$ is never a proper-biharmonic hypersurface of $S^{2n+1}$, and it is easy to check that $\pi(M)$ is a proper-biharmonic hypersurface of $CP^n$ if and only if $a^2 = \frac{2p+2\sqrt{2(p+1)}}{4(p+1)}$.

Example 4.7. Let $M = S^{2p+1}(a) \times S^p(b) \times S^p(b)$, $p$ odd. Then $M$ is minimal in $T_{a,b}^{p,p}$, and is proper-biharmonic in $S^{2n+1}$ if and only if $a = b = \frac{1}{\sqrt{2}}$. By a straightforward computation we can check that $\pi(M)$ is proper-biharmonic in $CP^n$ if and only if $a^2 = \frac{2p+7\sqrt{32p+25}}{16p+12}$.

4.1. Sphere bundle of all vectors tangent to $S^{2p+1}(a)$. We have seen that if $M$ is a product submanifold in $T_{a,b}^{p,q}$ then its projection $\pi(M)$ can be proper-biharmonic in $CP^n$. But when $M$ is not a product, the situation can be more complicated as it is illustrated by the following example.

We consider the sphere of radius $a$

$$S^{2p+1}(a) = \{x \in \mathbb{R}^{2p+2} : (x^1)^2 + \cdots + (x^{2p+2})^2 = a^2\}$$

and its sphere bundle of all vectors tangent to $S^{2p+1}(a)$ and of norm $b$, that is

$$M = T^b_{S^{2p+1}(a)} = \{(x, y) \in \mathbb{R}^{4p+1} : x, y \in \mathbb{R}^{2n+2}, |x| = a, |y| = b, \langle x, y \rangle = 0\}.$$

It is easy to check that $M$ is invariant under the flow-action of the characteristic vector field $\xi$, which means $e^{-it}p \in M$, $\forall t \in \mathbb{R}$. Let $(x_0, y_0) \in M$. Then

$$T_{(x_0, y_0)}M = \{Z_0 = (X_0, Y_0) \in \mathbb{R}^{4p+1} : \langle x_0, X_0 \rangle = 0, \langle y_0, Y_0 \rangle = 0,$$

$$\langle X_0, y_0 \rangle + \langle x_0, Y_0 \rangle = 0\}.$$

In order to find a basis in $T_{(x_0, y_0)}M$, we consider $\{y_0, y_1, \ldots, y_{2p+1}\}$ an orthogonal basis in $T_{x_0}S^{2p+1}(a)$, each vector being of norm $b$. We think $M$ as a hypersurface of the tangent bundle $TS^{2p+1}(a)$, and we consider on $TS^{2p+1}(a)$ and $M$ the induced metrics from the canonical metric on $\mathbb{R}^{4p+1}$

$$M \hookrightarrow TS^{2p+1}(a) \hookrightarrow \mathbb{R}^{4p+1}.$$

The above inclusions are the canonical ones.

The vertical lifts of the tangent vectors $y_2, y_3, \ldots, y_{2p+1}$, in $(x_0, y_0)$, are

$$y_2^V = (0, y_2), \quad y_3^V = (0, y_3), \quad \ldots, \quad y_{2p+1}^V = (0, y_{2p+1}),$$

and the horizontal lifts of $y_0, y_2, y_3, \ldots, y_{2p+1}$, in $(x_0, y_0)$, are

$$y_0^H = (y_0, -\frac{b^2}{a^2}x_0), \quad y_2^H = (y_2, 0), \quad y_3^H = (y_3, 0), \quad \ldots, \quad y_{2p+1}^H = (y_{2p+1}, 0).$$
The vectors \( \{y_0^H, y_2^H, \ldots, y_{2p+1}^H, y_2^V, y_3^V, \ldots, y_{2p+1}^V\} \) form an orthogonal basis in \( T_{(x_0, y_0)}M \) and
\[
|y_2^V| = |y_3^V| = \cdots = |y_{2p+1}^V| = b, \quad |y_2^H| = |y_3^H| = \cdots = |y_{2p+1}^H| = b, \quad |y_0^H| = \frac{b}{a}.
\]

The vector \( C(x_0, y_0) = y_0^V = (0, y_0) \) is tangent to \( T\mathbb{S}^{2p+1}(a) \) in \( (x_0, y_0) \) and orthogonal to \( M \).

From now on we shall consider \( a^2 + b^2 = 1 \) and the inclusions
\[
M \hookrightarrow \mathbb{S}^{2p+1}(a) \times \mathbb{S}^{2p+1}(b) \hookrightarrow \mathbb{S}^{4p+3} \hookrightarrow \mathbb{R}^{4p+4}.
\]

We define \( \eta_1(x_0, y_0) = (y_0, x_0) \) and \( \eta_2(x_0, y_0) = (x_0, -\frac{a^2}{b^2}y_0) \). We have that \( \eta_1 \) and \( \eta_2 \) are normal to \( M \), and
\[
\eta_1(x_0, y_0) \in T_{(x_0, y_0)}(\mathbb{S}^{2p+1}(a) \times \mathbb{S}^{2p+1}(b)), \quad |\eta_1(x_0, y_0)| = 1
\]
\[
\eta_2(x_0, y_0) \in T_{(x_0, y_0)}\mathbb{S}^{4p+3}, \quad \eta_2(x_0, y_0) \perp T_{(x_0, y_0)}(\mathbb{S}^{2p+1}(a) \times \mathbb{S}^{2p+1}(b)), \quad |\eta_2(x_0, y_0)| = \frac{a}{b}.
\]

We denote by \( B_{N(x_0, y_0)} \) the second fundamental form of \( M \) in \( \mathbb{S}^{4p+3} \), in the point \( (x_0, y_0) \). By a straightforward computation we obtain
\[
(4.4) \quad B_{N(x_0, y_0)}(Z_0, Z_0) = -2\langle X_0, Y_0 \rangle \eta_1 - \frac{b^2}{a^2}(|X_0|^2 - \frac{a^2}{b^2}|Y_0|^2)\eta_2,
\]
where \( Z_0 = (X_0, Y_0) \in T_{(x_0, y_0)}M \). From \( (4.4) \) we get
\[
H(x_0, y_0) = \frac{2p}{4p+1} \frac{a^2 - b^2}{a^2 - \eta_2} = c\eta_2.
\]

Therefore \( M \) is minimal in \( \mathbb{S}^{4p+3} \) if and only if \( a = b = \frac{1}{\sqrt{2}} \).

It is not difficult to check that
\[
\begin{align*}
\nabla_{y_0^H}^{\mathbb{S}^{4p+3}} \eta_2 &= \eta_1, \quad \nabla_{y_2^H}^{\mathbb{S}^{4p+3}} \eta_2 = y_2^H, \quad \nabla_{y_3^H}^{\mathbb{S}^{4p+3}} \eta_2 = y_3^H, \ldots, \quad \nabla_{y_{2p+1}^H}^{\mathbb{S}^{4p+3}} \eta_2 = y_{2p+1}^H, \\
\nabla_{y_2^V}^{\mathbb{S}^{4p+3}} \eta_2 &= -\frac{a^2}{b^2}y_2^V, \quad \nabla_{y_3^V}^{\mathbb{S}^{4p+3}} \eta_2 = -\frac{a^2}{b^2}y_3^V, \ldots, \quad \nabla_{y_{2p+1}^V}^{\mathbb{S}^{4p+3}} \eta_2 = -\frac{a^2}{b^2}y_{2p+1}^V, \\
\nabla_{y_0^H}^{\mathbb{S}^{4p+3}} \eta_1 &= -\frac{b^2}{a^2}\eta_2, \quad \nabla_{y_2^H}^{\mathbb{S}^{4p+3}} \eta_1 = y_2^V, \quad \nabla_{y_3^H}^{\mathbb{S}^{4p+3}} \eta_1 = y_3^V, \ldots, \quad \nabla_{y_{2p+1}^H}^{\mathbb{S}^{4p+3}} \eta_1 = y_{2p+1}^V, \\
\nabla_{y_2^V}^{\mathbb{S}^{4p+3}} \eta_1 &= y_2^H, \quad \nabla_{y_3^V}^{\mathbb{S}^{4p+3}} \eta_1 = y_3^H, \ldots, \quad \nabla_{y_{2p+1}^V}^{\mathbb{S}^{4p+3}} \eta_1 = y_{2p+1}^H.
\end{align*}
\]

From \( (4.5) \) we obtain that
\[
(4.6) \quad \text{trace } A_{\nabla_\perp \eta_2} (\cdot) = 0 \quad \text{and} \quad \text{trace } B(\cdot, A_{\eta_2} (\cdot)) = 2p\frac{a^2}{b^2} + \frac{b^2}{a^2}\eta_2.
\]

Denoting \( W(x_0, y_0) = y_0^H \), we get
\[
(4.7) \quad -\Delta^\perp \eta_2 = \frac{a^2}{b^2}(\nabla^\perp_{\nabla_{\nabla^H W} \nabla^H_{\nabla_{\nabla^H MW} \eta_2}} - \nabla^\perp_{\nabla^H_{\nabla^H MW} \eta_2}) = -\eta_2.
\]

Before concluding we give the following Lemma which follows by direct computation.

**Lemma 4.8.** Let \( N^n \) be a hypersurface of a Riemannian manifold \( (\mathbb{P}^{n+1}, \langle \cdot, \cdot \rangle) \), and \( X \in C(T\mathbb{P}) \) a Killing vector field. We denote \( X^\top = (X_N)^\top \in C(T\mathbb{P}) \). Then \( \text{div } X^\top = \alpha(H, X) \), where \( H \) is the mean curvature vector field of \( N \). In particular, if \( N \) is minimal then \( \text{div } X^\top = 0 \).
Now we can state

**Proposition 4.9.** Let $M = T^kS^{2p+1}(a)$ be the sphere bundle of all vectors of norm $b$ tangent to $S^{2p+1}(a)$. Assume that $a^2 + b^2 = 1$ and $p \geq 1$. Then we have

(a) $M$ is never proper-biharmonic in $S^{4p+3}$.

(b) $M$ is $(-4)$-biharmonic in $S^{4p+3}$ if and only if $a^2 = \frac{2p+1+\sqrt{2p+1}}{4p+2}$.

(c) $M$ is minimal in $T^{p,p}_a = S^{2p+1}(a) \times S^{2p+1}(b)$.

(d) $\pi(M)$ is never proper-biharmonic in $\mathbb{C}P^n$.

**Proof.** As the mean curvature vector field of $M$ in $S^{4p+3}$ is $H = c\eta_2$, where $c = \frac{2p}{4p+1}a^2b^2$, then $M$ is biharmonic if and only if

\[
\begin{aligned}
-\Delta^\perp \eta_2 - \text{trace } B(\cdot, A_{\eta_2}(\cdot)) + (4p + 1)\eta_2 = 0 \\
2 \text{ trace } A_{\nabla^\perp(\cdot)}\eta_2(\cdot) + \frac{4p+1}{2} \text{ grad}(c|\eta_2|^2) = 0
\end{aligned}
\]  

(4.8)

From (4.6) and (4.7) we get that $M$ is biharmonic if and only if

\[-\eta_2 - 2p\left(\frac{a^2}{b^2} + \frac{b^2}{a^2}\right)\eta_2 + (4p + 1)\eta_2 = 0,
\]

which is equivalent to $a = b$, that is $M$ is minimal in $S^{4p+3}$.

(b) We obtain that $M$ is $(-4)$-biharmonic if and only if

\[-\eta_2 - 2p\left(\frac{a^2}{b^2} + \frac{b^2}{a^2}\right)\eta_2 + (4p + 1)\eta_2 + 4a\eta_2 = 0,
\]

which holds if and only if $a^2 = \frac{2p+1+\sqrt{2p+1}}{4p+2}$.

(c) We denote by $\hat{A}$ the shape operator of $M$ in $S^{2p+1}(a) \times S^{2p+1}(b)$, $\hat{A} = \hat{A}_{\eta_1}$. We can check that

\[
\begin{aligned}
\hat{A}(y_1^H) = 0, \quad \hat{A}(y_2^H) = -y_1^V, \quad \hat{A}(y_3^H) = -y_2^V, \quad \ldots, \quad \hat{A}(y_{2p+1}^H) = -y_{2p+1}^V \\
\hat{A}(y_1^V) = -y_1^H, \quad \hat{A}(y_2^V) = -y_2^H, \quad \ldots, \quad \hat{A}(y_{2p+1}^V) = -y_{2p+1}^H
\end{aligned}
\]

and therefore $\hat{A} = 0$, which means that $M$ is minimal in $S^{2p+1}(a) \times S^{2p+1}(b)$.

(d) We first define

\[
\xi_3(x, y) = (\hat{J}x, -\frac{a^2}{b^2}\hat{J}y) = (-\xi_1, \frac{a^2}{b^2}\xi_2), \quad \forall (x, y) \in S^{2p+1}(a) \times S^{2p+1}(b).
\]

The vector field $\xi_3$ is a Killing vector field on $S^{2p+1}(a) \times S^{2p+1}(b)$. We observe that $\xi_{3/M} = \hat{J}\eta_2$. Since $M$ is minimal in $S^{2p+1}(a) \times S^{2p+1}(b)$, from Lemma 4.8 we get that

\[
\text{div}(\hat{J}\eta_2) = 0.
\]

Therefore $\pi(M)$ is biharmonic in $\mathbb{C}P^n$ if and only if

\[
\tau_2(j) - 4\hat{J}^T(\hat{J}\tau(j)) = 0,
\]

which is not satisfied.

\[\square\]

**4.2. Circles products.** We shall recover a result of Zhang (see [27]).

We denote by $\mathcal{T}$ the $(n+1)$-dimensional Clifford torus

\[
\mathcal{T} = S^1(a_1) \times \cdots \times S^1(a_{n+1}) \to S^{2n+1},
\]

where $a_1^2 + \cdots + a_{n+1}^2 = 1$. The projection $\pi{\mathcal{T}} = \pi(T)$ is a Lagrangian submanifold in $\mathbb{C}P^n$ of parallel mean curvature vector field.
Theorem 4.10 ([27]). The Lagrangian submanifold $\tilde{T} = \pi(T)$ of $\mathbb{C}P^n$ is proper-biharmonic if and only if $T$ is $(-4)$-biharmonic, that is

$$\begin{cases}
a_k^2 \neq \frac{1}{n+1} \\
d a_k - \frac{1}{a_k} = \frac{2}{a_k} (n+3)((n+1)a_k^2 - 1), \quad k \in \{1, 2, \ldots, n+1\}
\end{cases}$$

(4.10)

where $d = \sum_{j=1}^{n+1} \frac{1}{a_j^2}$.

Proof. We denote a point $x \in T$ by

$$x = (x_1, \ldots, x_{n+1}) = (x^1_1, x^2_1, \ldots, x^1_{n+1}, x^2_{n+1}),$$

where we identify

$$x_k = (x^1_k, x^2_k) = (0, 0, 0, x^1_k, x^2_k, 0, 0, 0, 0), \quad k = 1, \ldots, n+1.$$

We define $\eta_k(x) = \frac{1}{a_k} x_k$ and $X_k = J\eta_k$, $k = 1, \ldots, n+1$, where

$$J(x^1_1, x^2_1, \ldots, x^1_{n+1}, x^2_{n+1}) = (-x^2_1, x^1_1, \ldots, -x^2_{n+1}, x^1_{n+1}).$$

The vector fields $\{X_k\}$ form an orthonormal frame field of $C(TT)$. It is easy to check that, at a point $x$,

$$B(X_k, X_k) = -\frac{1}{a_k^2} \eta_k + x$$

and for $k \neq j$:

$$B(X_k, X_j) = 0.$$

Therefore $\tau(j) = \sum_k ((n+1)a_k - \frac{1}{a_k}) \eta_k$, which implies that $(J\tau(j))^\top = J\tau(j)$ and $\text{div}(J\tau(j)) = 0$.

Since $\nabla J \tau(j) = 0$ and $A_{\tau(j)}(X_k) = -(n+1)\frac{1}{a_k}X_k$, by a straightforward computation we get $\tau_2(j) + 4\tau(j) = 0$ if and only if the desired relation is satisfied. \qed

Remark 4.11. Following [27], for $n = 2$, we obtain that $\tilde{T}$ is a proper-biharmonic Lagrangian surface in $\mathbb{C}P^2$ if and only if $a_1^2 = \frac{9 + \sqrt{41}}{20}$ and $a_2^2 = a_3^2 = \frac{1 + \sqrt{41}}{40}$ (see also [26]).

5. Biharmonic curves in $\mathbb{C}P^n$

Let $\gamma : I \subset \mathbb{R} \rightarrow \mathbb{C}P^n$ be a curve parametrized by arc-length. The curve $\gamma$ is called a Frenet curve of osculating order $d$, $1 \leq d \leq 2n$, if there exist $d$ orthonormal vector fields $\{\vec{E}_1 = \hat{\gamma}', \ldots, \vec{E}_d\}$ along $\gamma$ such that

$$\begin{cases}
\nabla_{\vec{E}_1} \vec{E}_1 = \vec{k}_1 \vec{E}_2 \\
\nabla_{\vec{E}_1} \vec{E}_i = -\vec{k}_{i-1} \vec{E}_{i-1} + \vec{k}_i \vec{E}_{i+1}, & \forall i = 2, \ldots, d-1 \\
\nabla_{\vec{E}_1} \vec{E}_d = -\vec{k}_{d-1} \vec{E}_{d-1}
\end{cases}$$

(5.1)

where $\{\vec{k}_1, \vec{k}_2, \ldots, \vec{k}_{d-1}\}$ are positive functions on $I$ called the curvatures of $\gamma$ and $\nabla$ denotes the Levi-Civita connection on $\mathbb{C}P^n$.

A Frenet curve of osculating order $d$ is called a helix of order $d$ if $\vec{k}_i = \text{constant} > 0$ for $1 \leq i \leq d-1$. A helix of order 2 is called a circle, and a helix of order 3 is simply called helix.

Following S. Maeda and Y. Ohnita [20], we define the complex torsions of the curve $\gamma$ by $\tau_{ij} = \langle \vec{E}_i, J\vec{E}_j \rangle$, $1 \leq i < j \leq d$. A helix of order $d$ is called a holomorphic helix of order $d$ if all the complex torsions are constant.
Using the Frenet equations, the bitension field of $\bar{\gamma}$ becomes

\[
\tau_2(\bar{\gamma}) = -3k_1\bar{k}_1\bar{E}_1 + (\bar{k}_1' - \bar{k}_1^3 - \bar{k}_1\bar{k}_2^2 + \bar{k}_1)\bar{E}_2 \\
+ (2\bar{k}_1\bar{k}_2 + \bar{k}_1\bar{k}_2')\bar{E}_3 + \bar{k}_1\bar{k}_2\bar{k}_3\bar{E}_4 - 3k_1\tau_{12}\bar{J}\bar{E}_1.
\]

In order to solve the biharmonic equation $\tau_2(\bar{\gamma}) = 0$, because of the last term in (5.2), we must split our study in three cases.

5.1. **Biharmonic curves with $\bar{\tau}_{12} = \pm 1$.** In this case $\bar{J}\bar{E}_2 = \pm E_1$ and, using the Frenet equations of $\bar{\gamma}$, we obtain

\[
\bar{J}(\bar{\nabla}_{E_1}E_1) = \pm \bar{k}_1E_1 = \nabla_{\bar{E}_1}(\mp E_2) = \mp\nabla_{\bar{E}_1}\bar{E}_2,
\]

so

\[
\nabla_{E_1}E_2 = -\bar{k}_1\bar{E}_1.
\]

Consequently, $\bar{k}_i = 0$, $i \geq 2$, and, from (5.2), it follows

**Proposition 5.1.** A Frenet curve $\bar{\gamma} : I \subset \mathbb{R} \rightarrow \mathbb{C}P^n$ parametrized by arc-length with $\bar{\tau}_{12} = \pm 1$ is proper-biharmonic if and only if it is a circle with $\bar{k}_1 = 2$.

Next, let us consider a curve $\gamma : I \subset \mathbb{R} \rightarrow \mathbb{C}P^n$ parametrized by arc-length with $\bar{\tau}_{12} = \pm 1$, and denote by $\gamma : I \subset \mathbb{R} \rightarrow S^{2n+1}$ one of its horizontal lifts. We shall characterize the biharmonicity of $\gamma$ in terms of $\gamma$.

We denote by $\nabla$ the Levi-Civita connection on $S^{2n+1}$. We have $\gamma' = E_1 = (\bar{E}_1)^H$ and

\[
\nabla_{E_1}E_1 = (\nabla_{\bar{E}_1}\bar{E}_1)^H = \bar{k}_1\bar{E}_2^H = k_1E_2,
\]

i.e. $k_1 = \bar{k}_1$ and $E_2 = \bar{E}_2^H = \mp(\bar{J}\bar{E}_1)^H = \mp\bar{J}E_1$. It follows

\[
\nabla_{E_1}E_2 = (\nabla_{\bar{E}_1}\bar{E}_2)^H + (\nabla_{E_1}E_2, \xi)\xi
\]

\[
= -k_1E_1 - (E_2, \nabla_{E_1}\xi)\xi
\]

\[
= -k_1E_1 + (E_2, E_2)\xi
\]

\[
= -k_1E_1 + \xi
\]

and this means $k_2 = 1$ and $E_3 = \mp\xi$. Then $\nabla_{E_1}E_3 = \mp\nabla_{E_1}\xi = -E_2$.

In conclusion $\gamma$ is a helix with $k_1 = \bar{k}_1$ and $k_2 = 1$.

Now, we have $\bar{J}\tau(\gamma) = k_1\bar{J}\bar{E}_2 = \pm k_1E_1$, which is tangent to $\gamma$, and then

\[
\bar{J}\{(\bar{J}\tau(\gamma))^\top\} = \bar{J}^2\tau(\gamma) = -\tau(\gamma).
\]

From

\[
\text{div}\{(\bar{J}\tau(\gamma))^\top\} = \text{div}\{k_1(\bar{J}\bar{E}_2, \overline{E}_1)\bar{E}_1\}
\]

\[
= \nabla_{E_1}(k_1(\bar{J}\bar{E}_2, \overline{E}_1))\bar{E}_1, \bar{E}_1
\]

\[
= \bar{k}_1'(\bar{J}\bar{E}_2, \bar{E}_1) + k_1(\bar{J}\nabla_{E_1}\bar{E}_2, \bar{E}_1)
\]

\[
= \pm\bar{k}_1' = 0,
\]

applying Remark 3.4 (iii), we have

**Proposition 5.2.** A Frenet curve $\gamma : I \subset \mathbb{R} \rightarrow \mathbb{C}P^n$ parametrized by arc-length with $\bar{\tau}_{12} = \pm 1$ is proper-biharmonic if and only if its horizontal lift $\gamma : I \subset \mathbb{R} \rightarrow S^{2n+1}$ is $(-4)$-biharmonic, i.e. $\gamma$ is a helix with $k_1 = 2$ and $k_2 = 1$.

Moreover, we can obtain the explicit parametric equations of the horizontal lifts of a proper-biharmonic Frenet curve $\gamma : I \rightarrow \mathbb{C}P^n$. 
Proposition 5.3. Let $\bar{\gamma}: I \subset \mathbb{R} \rightarrow \mathbb{C}P^n$ be a proper-biharmonic Frenet curve parametrized by arc-length with $\bar{\tau}_{12} = \pm 1$. Then its horizontal lift $\gamma : I \subset \mathbb{R} \rightarrow S^{2n+1}$ can be parametrized in the Euclidean space $\mathbb{R}^{2n+2}$ by
\[
\gamma(s) = \frac{\sqrt{2} - \sqrt{2}}{2} \cos((\sqrt{2} + 1)s)e_1 - \frac{\sqrt{2} - \sqrt{2}}{2} \sin((\sqrt{2} + 1)s)\hat{J}e_1 \\
+ \frac{\sqrt{2} + \sqrt{2}}{2} \cos((\sqrt{2} - 1)s)c_3 + \frac{\sqrt{2} + \sqrt{2}}{2} \sin((\sqrt{2} - 1)s)\hat{J}c_3,
\]
where $e_1$ and $c_3$ are constant unit vectors in $\mathbb{R}^{2n+2}$ with $e_3$ orthogonal to $e_1$ and $\hat{J}e_1$.

Proof. The curve $\gamma$ is a helix with the Frenet frame field $\{E_1 = \tilde{E}_1^H, E_2 = \tilde{E}_2^H, E_3 = \mp \xi\}$ and with curvatures $k_1 = k_2 = 2$ and $k_2 = 1$.

From the Weingarten equation of $S^{2n+1}$ in $\mathbb{R}^{2n+2}$ and Frenet equations we get
\[
\hat{\nabla}_{E_1}E_1 = \hat{\nabla}_{E_1}E_1 - \langle E_1, E_1 \rangle \gamma = k_1E_2 - \gamma,
\]
\[
\hat{\nabla}_{E_1} \hat{\nabla}_{E_1}E_1 = k_1\hat{\nabla}_{E_1}E_2 - E_1 = k_1(-k_1E_1 + \xi) - E_1 = -(k_1^2 + 1)E_1 \mp k_1 \xi
\]
and
\[
\hat{\nabla}_{E_1} \hat{\nabla}_{E_1}E_1 = -(k_1^2 + 1)\hat{\nabla}_{E_1}E_1 \pm k_1 \hat{\nabla}_{E_1}E_1
\]
\[
= -6\gamma'' - \gamma.
\]
Hence $\gamma$ is a solution of the differential equation
\[
\gamma^{iv} + 6\gamma'' + \gamma = 0,
\]
whose general solution is
\[
\gamma(s) = \cos(As)c_1 + \sin(As)c_2 + \cos( Bs)c_3 + \sin( Bs)c_4,
\]
where $A, B = \sqrt{2} \pm 1$ and $\{c_i\}$ are constant vectors in $\mathbb{E}^{2n+2}$.

As $\gamma$ satisfies
\[
\langle \gamma, \gamma \rangle = 1, \quad \langle \gamma', \gamma' \rangle = 1, \quad \langle \gamma, \gamma' \rangle = 0, \quad \langle \gamma', \gamma'' \rangle = 0, \quad \langle \gamma'', \gamma''' \rangle = 1 + \kappa_1^2 = 5,
\]
\[
\langle \gamma, \gamma''' \rangle = -1, \quad \langle \gamma', \gamma''' \rangle = -(1 + \kappa_1^2) = -5, \quad \langle \gamma'', \gamma''' \rangle = 0,
\]
\[
\langle \gamma, \gamma'''' \rangle = 0, \quad \langle \gamma'''', \gamma'''' \rangle = 7\kappa_1^2 + 1 = 29,
\]
and since, in $s = 0$, we have $\gamma = c_1 + c_3$, $\gamma' = Ac_2 + Bc_4$, $\gamma'' = -A^2c_1 - B^2c_3$, $\gamma''' = -A^3c_2 - B^3c_4$, we obtain
\[
c_{11} + 2c_{13} + c_{33} = 1
\]
\[
A^2c_{22} + 2ABc_{24} + B^2c_{44} = 1
\]
\[
Ac_{12} + Ac_{23} + Bc_{14} + Bc_{34} = 0
\]
\[
A^3c_{12} + AB^2c_{23} + A^2Bc_{14} + B^3c_{34} = 0
\]
\[
A^4c_{11} + 2A^2B^2c_{13} + B^4c_{33} = 5
\]
\[
A^2c_{11} + (A^2 + B^2)c_{13} + B^2c_{33} = 1
\]
\[
A^4c_{22} + (AB^3 + A^3B)c_{24} + B^4c_{44} = 5
\]
\[
A^5c_{12} + A^3B^2c_{23} + A^2B^3c_{14} + B^5c_{34} = 0
\]
\[
A^3c_{12} + A^3c_{23} + B^3c_{14} + B^3c_{34} = 0
\]
(5.12) \[ A^6c_{22} + 2A^3B^3c_{24} + B^6c_{44} = 29 \]

where \( c_{ij} = \langle c_i, c_j \rangle \). From (5.5), (5.6), (5.10) and (5.11) it follows that
\[ c_{12} = c_{23} = c_{14} = c_{34} = 0. \]

The equations (5.3), (5.7) and (5.8) give
\[ c_{11} = \frac{1 - B^2}{A^2 - B^2}, \quad c_{13} = 0, \quad c_{33} = \frac{A^2 - 1}{A^2 - B^2} \]
and from (5.4), (5.9) and (5.12) it follows that
\[ c_{22} = \frac{1 - B^2}{A^2 - B^2}, \quad c_{24} = 0, \quad c_{44} = \frac{A^2 - 1}{A^2 - B^2}. \]

Therefore, we obtain that \( \{c_i\} \) are orthogonal vectors in \( \mathbb{E}^{2n+2} \) with \( |c_1| = |c_2| = \sqrt{\frac{1 - B^2}{A^2 - B^2}}, |c_3| = |c_4| = \sqrt{\frac{A^2 - 1}{A^2 - B^2}}. \)

By using that \( E_1 = \gamma' \perp \xi \) and then that \( \tilde{J} E_2 = \pm E_1 \), we conclude. \( \square \)

Remark 5.4. Under the flow-action of \( \xi \), the \((-4)\)-biharmonic curves \( \gamma \) induce the \((-4)\)-biharmonic surfaces obtained in Example 4.4.

5.2. Biharmonic curves with \( \tilde{\tau}_{12} = 0 \). From the expression (5.2) of the bitension field of \( \tilde{\gamma} \) we obtain that \( \tilde{\gamma} \) is proper-biharmonic if and only if
\[
\begin{cases}
\tilde{k}_1 = \text{constant} > 0, & \tilde{k}_2 = \text{constant} \\
\tilde{k}_1^2 + \tilde{k}_2^2 = 1 \\
\tilde{k}_2 \tilde{k}_3 = 0
\end{cases}
\]

Proposition 5.5. A Frenet curve \( \tilde{\gamma} : I \subset \mathbb{R} \rightarrow \mathbb{CP}^n \) parametrized by arc-length with \( \tilde{\tau}_{12} = 0 \) is proper-biharmonic if and only if either
(a) \( n = 2 \) and \( \tilde{\gamma} \) is a circle with \( \tilde{k}_1 = 1 \),

or
(b) \( n \geq 3 \) and \( \tilde{\gamma} \) is a circle with \( \tilde{k}_1 = 1 \) or a helix with \( \tilde{k}_1^2 + \tilde{k}_2^2 = 1 \).

Proof. We only have to prove the statements concerning the dimension \( n \).

First, since \( \{\tilde{E}_1, \tilde{E}_2, \tilde{J} \tilde{E}_2\} \) are linearly independent, it follows that \( n > 1 \).

Now, assume that \( \tilde{\gamma} \) is a Frenet curve of osculating order 3 such that \( \tilde{J} \tilde{E}_2 \perp \tilde{E}_1 \).

We have
\[
\begin{cases}
\tilde{E}_1 = \tilde{\gamma}' \\
\nabla_{\tilde{E}_1} \tilde{E}_1 = \tilde{k}_1 \tilde{E}_2 \\
\nabla_{\tilde{E}_1} \tilde{E}_2 = -\tilde{k}_1 \tilde{E}_1 + \tilde{k}_2 \tilde{E}_3 \\
\nabla_{\tilde{E}_1} \tilde{E}_3 = -\tilde{k}_2 \tilde{E}_2
\end{cases}
\]

It is easy to see that, at an arbitrary point, the system
\[ S_1 = \{\tilde{E}_1, \tilde{E}_2, \tilde{E}_3, \tilde{J} \tilde{E}_1, \tilde{J} \tilde{E}_2\} \]
consists of non-zero vectors which are orthogonal to each other, and therefore \( n \geq 3 \). \( \square \)

Next, we shall consider the horizontal lift \( \gamma : I \subset \mathbb{R} \rightarrow S^{2n+1} \) of a curve \( \tilde{\gamma} : I \subset \mathbb{R} \rightarrow \mathbb{CP}^n \) parametrized by arc-length with \( \tilde{\tau}_{12} = 0 \). As in the previous case we have \( \gamma' = E_1 = \tilde{E}_1^H, E_2 = \tilde{E}_2^H \) and then \( \tilde{J} E_2 \perp E_1 \). This means \( \tilde{J}(\tau(\gamma)) \perp E_1 \), so \( (\tilde{J}(\tau(\gamma)))^\top = 0 \). From Theorem 3.3 we obtain
Proposition 5.6. A Frenet curve $\hat{\gamma} : I \subset \mathbb{R} \rightarrow \mathbb{C}P^n$ parametrized by arc-length with $\hat{\tau}_{12} = 0$ is proper-biharmonic if and only if its horizontal lift $\hat{\gamma} : I \subset \mathbb{R} \rightarrow S^{2n+1}$ is proper-biharmonic.

The parametric equations of the proper-biharmonic Frenet curves in $S^{2n+1}$ with $\vec{J}E_2 \perp E_1$ were obtained in [13]. Using that result we can state

Proposition 5.7. Let $\hat{\gamma} : I \subset \mathbb{R} \rightarrow \mathbb{C}P^n$ be a proper-biharmonic Frenet curve parametrized by arc-length with $\hat{\tau}_{12} = 0$. Then the horizontal lift $\hat{\gamma} : I \subset \mathbb{R} \rightarrow S^{2n+1}$ can be parametrized, in the Euclidean space $\mathbb{R}^{2n+2}$, either by

$$\gamma(s) = \frac{1}{\sqrt{2}} \cos(\sqrt{2}s)e_1 + \frac{1}{\sqrt{2}} \sin(\sqrt{2}s)e_2 + \frac{1}{\sqrt{2}} e_3,$$

where $\{e_i, \vec{J}e_j\}_{i,j=1}^3$ are constant unit vectors orthogonal to each other, or by

$$\gamma(s) = \frac{1}{\sqrt{2}} \cos(\sqrt{1+\kappa_1}s)e_1 + \frac{1}{\sqrt{2}} \sin(\sqrt{1+\kappa_1}s)e_2$$

$$+ \frac{1}{\sqrt{2}} \cos(\sqrt{1-\kappa_1}s)e_3 + \frac{1}{\sqrt{2}} \sin(\sqrt{1-\kappa_1}s)e_4,$$

where $\kappa_1 \in (0,1)$, and $\{e_i, \vec{J}e_j\}_{i,j=1}^4$ are constant unit vectors orthogonal to each other.

5.3. Biharmonic curves with $\hat{\tau}_{12}$ different from 0, 1 or $-1$. Assume that $\hat{\gamma}$ is a proper-biharmonic Frenet curve of osculating order $d$ such that $\hat{\tau}_{12}$ is different from 0, 1 or $-1$.

First, we shall prove that $d \geq 4$.

Assume that $d = 2$. From the biharmonic equation $\tau_2(\hat{\gamma}) = 0$ we have $\hat{k}_1 = \text{constant} > 0$ and then $(-\hat{k}_1^3 + \hat{k}_1)\vec{E}_2 - 3\hat{k}_1\hat{\tau}_{12}\vec{J}\vec{E}_1 = 0$. It follows that $\vec{E}_2$ is parallel to $\vec{J}\vec{E}_1$, i.e. $\tau_{12}^2 = 1$.

Now, if $d = 3$, from the biharmonic equation of $\hat{\gamma}$, we obtain again $\hat{k}_1 = \text{constant} > 0$ and then

$$(5.15) \quad (-\hat{k}_1^2 - \hat{k}_2^2 + 1)\vec{E}_2 + \hat{k}_1\hat{k}_2\vec{E}_3 - 3\hat{\tau}_{12}\vec{J}\vec{E}_1 = 0.$$

Next, differentiating $-\hat{\tau}_{12}(s) = \langle \vec{E}_2, \vec{J}\vec{E}_1 \rangle$, we obtain

$$-\hat{\tau}_{12}(s) = \langle \vec{\nabla}_{\vec{E}_1} \vec{E}_2, \vec{J}\vec{E}_1 \rangle + \langle \vec{E}_2, \vec{\nabla}_{\vec{E}_1} \vec{J}\vec{E}_1 \rangle = \langle \vec{\nabla}_{\vec{E}_1} \vec{E}_2, \vec{J}\vec{E}_1 \rangle + \langle \vec{E}_2, \hat{k}_1\vec{J}\vec{E}_2 \rangle$$

$$= \langle \vec{\nabla}_{\vec{E}_1} \vec{E}_2, \vec{J}\vec{E}_1 \rangle = \langle -\hat{k}_1\vec{E}_1 + \hat{k}_2\vec{E}_3, \vec{J}\vec{E}_1 \rangle$$

$$= \hat{k}_2\langle \vec{E}_3, \vec{J}\vec{E}_1 \rangle.$$ 

Hence, taking the inner product with $\hat{k}_2\vec{E}_3$ in $(5.15)$, we get $\hat{k}_2^2\hat{k}_2 + 3\hat{\tau}_{12}\hat{\tau}_{12}' = 0$ and so $\hat{k}_2^2 = -3\hat{\tau}_{12}^2 + \omega_0$, where $\omega_0 = \text{constant}$. Using $(5.15)$, it results that $\hat{k}_1^2 = 1 - \omega_0 + 6\hat{\tau}_{12}^2$. Therefore $f = \text{constant}$ and $\hat{k}_2 = \text{constant}$. Finally, $(5.15)$ becomes

$$(-\hat{k}_1^2 - \hat{k}_2^2 + 1)\vec{E}_2 - 3\hat{\tau}_{12}\vec{J}\vec{E}_1 = 0,$$

which means that $\vec{E}_2$ is parallel to $\vec{J}\vec{E}_1$.

We have proved the following

Proposition 5.8. Let $\hat{\gamma}$ be a proper-biharmonic Frenet curve in $\mathbb{C}P^n$ of osculating order $d$, $1 \leq d \leq 2n$, with $\hat{\tau}_{12}$ different from 0, 1 or $-1$. Then $d \geq 4$.

Next we shall prove that for a proper-biharmonic Frenet curve in $\mathbb{C}P^n$, $\hat{\tau}_{12}$ and $\hat{k}_1$ are constants whatever the osculating order of $\hat{\gamma}$ is.
We have seen that $-\tau_1^2(s) = 2\langle E_3, J\dot{E}_1 \rangle$. If $\tau_2(\gamma) = 0$ we have $J\dot{E}_1 = \langle J\dot{E}_1, E_2 \rangle E_2 + \langle J\dot{E}_1, E_3 \rangle E_3 + \langle J\dot{E}_1, E_4 \rangle E_4$ and

$$
\begin{cases}
\dot{k}_1 = \text{constant} > 0 \\
\dot{k}_1^2 + \dot{k}_2^2 = 1 + 3\tau_1^2 \\
\dot{k}_2\dot{k}_3' = -3\tau_1^2 \tau_1' \\
\dot{k}_2\dot{k}_3 = 3\tau_1^2 \langle J\dot{E}_1, E_4 \rangle
\end{cases}
$$

(5.16)

From the third equation of (5.16), we get

$$
\dot{k}_2^2 = -3\tau_1^2 + \omega_0,
$$

where $\omega_0 = \text{constant}$. Replacing in the second equation of (5.16) it follows that

$$
\dot{k}_1^2 = 1 + 6\tau_1^2 - \omega_0,
$$

which implies $\tau_1^2 = \text{constant}$, and therefore, $\dot{k}_2 = \text{constant} > 0$. From $-\tau_1^2(s) = \dot{k}_2\langle E_3, J\dot{E}_1 \rangle$, we have $\langle J\dot{E}_1, E_3 \rangle = 0$ and then $J\dot{E}_1 = \pm \dot{E}_2 + \langle J\dot{E}_1, E_4 \rangle E_4$. It follows that there exists an unique constant $\alpha_0 \in (0, 2\pi) \setminus \{\frac{\pi}{2}, \frac{3\pi}{2}\}$ such that $-\tau_1^2 = \cos \alpha_0$ and $\langle J\dot{E}_1, E_4 \rangle = \sin \alpha_0 = \frac{k_2k_3}{3\tau_1^2}$.

We can summarise in

**Proposition 5.9.** A Frenet curve $\gamma : I \subset \mathbb{R} \rightarrow \mathbb{C}P^n$, $n \geq 2$, parametrized by arc-length with $\tau_1^2$ different from 0, 1 or $-1$ is proper-biharmonic if and only if $J\dot{E}_1 = \cos \alpha_0 \dot{E}_2 + \sin \alpha_0 \dot{E}_4$ and

$$
\begin{cases}
\dot{k}_1, \dot{k}_2, \dot{k}_3 = \text{constant} > 0 \\
\dot{k}_1^2 + \dot{k}_2^2 = 1 + 3\cos^2 \alpha_0 \\
\dot{k}_2\dot{k}_3' = -\frac{3}{2} \sin(2\alpha_0) \\
\dot{k}_2\dot{k}_3 = \frac{3}{2} \sin(2\alpha_0)
\end{cases}
$$

(5.17)

where $\alpha_0 \in \left(\frac{\pi}{2}, \pi\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$ is a constant.

We end this section classifying the proper-biharmonic curves in $\mathbb{C}P^n$ of osculating order $d \leq 4$. First,

**Proposition 5.10.** Let $\tilde{\gamma}$ be a proper-biharmonic Frenet curve in $\mathbb{C}P^n$ of osculating order $d < 4$. Then $\tilde{\gamma}$ is one of the following: a holomorphic circle of curvature $k_1 = 2$, a holomorphic circle of curvature $\bar{k}_1 = 1$, or a holomorphic helix with $k_1^2 + \bar{k}_2^2 = 1$.

**Proof.** Let $\tilde{\gamma}$ be a proper-biharmonic Frenet curve of osculating order $d < 4$. Then, from Proposition 5.9, $\tau_1^2 = \pm 1$ or $\bar{\tau}_1^2 = 0$. If $\tilde{\tau}_1^2 = \pm 1$, from Proposition 5.1, $\bar{\gamma}$ is a circle of curvature $k_1 = 2$. If $\tilde{\tau}_1^2 = 0$ then we know that $\bar{\gamma}$ is either a holomorphic circle of curvature $\bar{k}_1 = 1$ or a helix. We now prove that it is a holomorphic helix. For this we need to prove that the complex torsions $\tilde{\tau}_1, \tilde{\tau}_2$ are constant.

$$
\tilde{\tau}_1 = \langle E_1, J\dot{E}_3 \rangle = -\frac{1}{k_2} \langle \nabla_{E_1} \dot{E}_2, J\dot{E}_1 \rangle = \frac{1}{k_2} \langle \dot{E}_2, \nabla_{E_1} J\dot{E}_1 \rangle = \frac{k_1}{k_2} \langle \dot{E}_2, J\dot{E}_2 \rangle = 0.
$$

Now, using that for a Frenet curve of osculating order 3 we have $\tilde{k}_1 \tilde{\tau}_2 = \tilde{\tau}_{13} + \tilde{k}_2 \tilde{\tau}_{12}$, we see that also $\tilde{\tau}_2$ is constant.

When the biharmonic curve is of osculating order 4, system (5.17) has four solutions.
Proposition 5.11. Let $\gamma$ be a proper-biharmonic Frenet curve in $\mathbb{CP}^n$ of osculating order $d = 4$. Then $\gamma$ is a holomorphic helix. Moreover, depending on the value of $\bar{\tau}_{12} = -\cos \alpha_0$, we have

(a) If $\bar{\tau}_{12} > 0$, then the curvatures of $\gamma$ are given by

\[
\begin{aligned}
\hat{k}_2 &= \frac{\sin \alpha_0}{\sqrt{2}} \sqrt{1 - 3 \cos^2 \alpha_0 \pm \sqrt{9 \cos^4 \alpha_0 - 42 \cos^2 \alpha_0 + 1}} \\
\hat{k}_3 &= -\frac{3}{2k_2} \sin (2\alpha_0) \\
\hat{k}_1 &= -\frac{1}{\sin \alpha_0} (\bar{k}_2 \cos \alpha_0 - \bar{k}_3 \sin \alpha_0)
\end{aligned}
\]

(5.18)

and

\[
\bar{\tau}_{34} = -\bar{\tau}_{12} = \cos \alpha_0, \quad \bar{\tau}_{14} = -\bar{\tau}_{23} = -\sin \alpha_0 \quad \text{and} \quad \bar{\tau}_{13} = \bar{\tau}_{24} = 0,
\]

where $\alpha_0 \in \left(\frac{\pi}{2}, \arccos\left(-\frac{2\sqrt{2}}{\sqrt{3}}\right)\right)$.

(b) If $\bar{\tau}_{12} < 0$, then the curvatures of $\gamma$ are given by

\[
\begin{aligned}
\hat{k}_2 &= -\frac{\sin \alpha_0}{\sqrt{2}} \sqrt{1 - 3 \cos^2 \alpha_0 \pm \sqrt{9 \cos^4 \alpha_0 - 42 \cos^2 \alpha_0 + 1}} \\
\hat{k}_3 &= -\frac{3}{2k_2} \sin (2\alpha_0) \\
\hat{k}_1 &= -\frac{1}{\sin \alpha_0} (\bar{k}_2 \cos \alpha_0 - \bar{k}_3 \sin \alpha_0)
\end{aligned}
\]

(5.19)

and

\[
\bar{\tau}_{34} = -\bar{\tau}_{12} = \cos \alpha_0, \quad \bar{\tau}_{14} = -\bar{\tau}_{23} = -\sin \alpha_0 \quad \text{and} \quad \bar{\tau}_{13} = \bar{\tau}_{24} = 0,
\]

where $\alpha_0 \in \left(\frac{\pi}{2}, \pi + \arccos\left(-\frac{2\sqrt{2}}{\sqrt{3}}\right)\right)$.

Proof. Let $\gamma$ be a proper-biharmonic Frenet curve in $\mathbb{CP}^n$ of osculating order $d = 4$. Then $\bar{\tau}_{12} = -\cos \alpha_0$ is different from 0, 1 or $-1$, and $J \bar{E}_1 = \cos \alpha_0 \bar{E}_2 + \sin \alpha_0 \bar{E}_4$. Then it results that

\[
\bar{\tau}_{12} = -\cos \alpha_0, \quad \bar{\tau}_{13} = 0, \quad \bar{\tau}_{14} = -\sin \alpha_0, \quad \text{and} \quad \bar{\tau}_{24} = 0.
\]

In order to prove that $\bar{\tau}_{23}$ is constant we differentiate the expression of $J \bar{E}_1$ and using the Frenet equations we obtain

\[
\nabla_{\bar{E}_1} J \bar{E}_1 = \cos \alpha_0 \nabla_{\bar{E}_1} \bar{E}_2 + \sin \alpha_0 \nabla_{\bar{E}_1} \bar{E}_4 = -\hat{k}_1 \cos \alpha_0 \bar{E}_1 + (\hat{k}_2 \cos \alpha_0 - \hat{k}_3 \sin \alpha_0) \bar{E}_3.
\]

On the other hand, $\nabla_{\bar{E}_1} J \bar{E}_1 = \hat{k}_1 J \bar{E}_2$ and therefore we have

\[
\hat{k}_1 J \bar{E}_2 = -\hat{k}_1 \cos \alpha_0 \bar{E}_1 + (\hat{k}_2 \cos \alpha_0 - \hat{k}_3 \sin \alpha_0) \bar{E}_3.
\]

(5.20)

We take the inner product of (5.20) with $\bar{E}_3$, $J \bar{E}_2$ and $J \bar{E}_4$, respectively, and we get

\[
\hat{k}_1 \bar{\tau}_{23} = - (\hat{k}_2 \cos \alpha_0 - \hat{k}_3 \sin \alpha_0),
\]

(5.21)

\[
\hat{k}_1 \sin^2 \alpha_0 = - (\hat{k}_2 \cos \alpha_0 - \hat{k}_3 \sin \alpha_0) \bar{\tau}_{23},
\]

(5.22)

\[
0 = \hat{k}_1 \cos \alpha_0 \sin \alpha_0 + (\hat{k}_2 \cos \alpha_0 - \hat{k}_3 \sin \alpha_0) \bar{\tau}_{34}.
\]

(5.23)

From (5.21) and (5.22) we obtain

\[
\hat{k}_1^2 \sin^2 \alpha_0 = (\hat{k}_2 \cos \alpha_0 - \hat{k}_3 \sin \alpha_0)^2
\]

(5.24)

and $\bar{\tau}_{23}^2 = \sin^2 \alpha_0$. From $\bar{\tau}_{23}^2 = \sin^2 \alpha_0$, (5.21) and $\alpha_0 \in \left(\frac{\pi}{2}, \pi \right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$, one obtains

\[
\bar{\tau}_{23} = \sin \alpha_0.
\]

From $\bar{\tau}_{23} = \sin \alpha_0$, (5.21) and (5.23) we get

\[
\bar{\tau}_{34} = \cos \alpha_0.
\]
Finally, from Proposition 5.9 and (5.24) we obtain
\[ \tilde{k}_2^4 + \tilde{k}_2^2 \sin^2 \alpha_0 (3 \cos^2 \alpha_0 - 1) + 9 \sin^4 \alpha_0 \cos^2 \alpha_0 = 0. \]

The latter equation has either the solutions
\[ \tilde{k}_2 = \frac{\sin \alpha_0}{\sqrt{2}} \sqrt{1 - 3 \cos^2 \alpha_0 \pm \sqrt{9 \cos^4 \alpha_0 - 42 \cos^2 \alpha_0 + 1}} \]
provided that \( \alpha_0 \in \left( \frac{\pi}{2}, \arccos\left(-\frac{2\sqrt{3}}{\sqrt{2}}\right) \right) \), or the solutions
\[ \tilde{k}_2 = -\frac{\sin \alpha_0}{\sqrt{2}} \sqrt{1 - 3 \cos^2 \alpha_0 \pm \sqrt{9 \cos^4 \alpha_0 - 42 \cos^2 \alpha_0 + 1}} \]
provided that \( \alpha_0 \in \left( \frac{3\pi}{2}, \pi + \arccos\left(-\frac{2\sqrt{3}}{\sqrt{2}}\right) \right) \). Note that in both cases \( \tilde{k}_2^2 \in (0, 4) \), thus all solutions for \( \tilde{k}_2 \) are compatible with \( \tilde{k}_1^2 + \tilde{k}_3^2 = 1 + 3 \cos^2 \alpha_0 \).

**Corollary 5.12.** Any proper-biharmonic Frenet curve in \( \mathbb{C}P^2 \) is a holomorphic circle or a holomorphic helix of order 4.

**Remark 5.13.** The existence of biharmonic curves of osculating order \( d \geq 4 \) is an open problem (the case \( d = 4 \) and \( n = 2 \) will be solved in the next section). We note that there is no curve (not necessarily biharmonic) of order \( d = 5 \) in \( \mathbb{C}P^n \) such that \( J E_1 = \cos \alpha_0 E_2 + \sin \alpha_0 E_4 \), where \( \alpha_0 \in (0, 2\pi) \setminus \{ \pi \} \).

## 6. Biharmonic Curves in \( \mathbb{C}P^2 \)

In this section we give the complete classification of all proper-biharmonic Frenet curves in \( \mathbb{C}P^2 \). From the previous section, we only have to classify the proper-biharmonic Frenet curves of osculating order 4.

In the proof of Proposition 5.11 we have seen that
\[ \bar{\tau}_{34} = -\bar{\tau}_{12} = \cos \alpha_0, \quad \bar{\tau}_{14} = -\bar{\tau}_{23} = -\sin \alpha_0 \quad \text{and} \quad \bar{\tau}_{13} = \bar{\tau}_{24} = 0, \]
and
\[ \tilde{k}_1 \sin \alpha_0 = -(\tilde{k}_2 \cos \alpha_0 - \tilde{k}_3 \sin \alpha_0), \]
which implies that \( \tilde{k}_1 - \tilde{k}_3 = -\tilde{k}_2 \frac{\cos \alpha_0}{\sin \alpha_0} > 0 \).

Moreover, if \( \alpha_0 \in \left( \frac{\pi}{2}, \arccos\left(-\frac{2\sqrt{3}}{\sqrt{2}}\right) \right) \), then
\[ \frac{\tilde{k}_1 - \tilde{k}_3}{\sqrt{\tilde{k}_2^2 + (\tilde{k}_1 - \tilde{k}_3)^2}} = -\cos \alpha_0 = \bar{\tau}_{12}, \quad \frac{\tilde{k}_2}{\sqrt{\tilde{k}_2^2 + (\tilde{k}_1 - \tilde{k}_3)^2}} = \sin \alpha_0 = \bar{\tau}_{23}, \]
and, if \( \alpha_0 \in \left( \frac{3\pi}{2}, \pi + \arccos\left(-\frac{2\sqrt{3}}{\sqrt{2}}\right) \right) \), then
\[ \frac{\tilde{k}_1 - \tilde{k}_3}{\sqrt{\tilde{k}_2^2 + (\tilde{k}_1 - \tilde{k}_3)^2}} = \cos \alpha_0 = -\bar{\tau}_{12}, \quad \frac{\tilde{k}_2}{\sqrt{\tilde{k}_2^2 + (\tilde{k}_1 - \tilde{k}_3)^2}} = -\sin \alpha_0 = -\bar{\tau}_{23}. \]

In order to conclude, we briefly recall a result of S. Maeda and T. Adachi.

In [19], they showed that for given positive constants \( \tilde{k}_1, \tilde{k}_2 \) and \( \tilde{k}_3 \), there exist four equivalence classes of holomorphic helices of order 4 in \( \mathbb{C}P^2 \) with curvatures \( \tilde{k}_1, \tilde{k}_2 \).
and \( \bar{k}_3 \) with respect to holomorphic isometries of \( \mathbb{CP}^2 \). The four classes are defined by certain relations on the complex torsions and they are: when \( k_1 \neq k_3 \)

| \( I_1 \) | \( \bar{\tau}_{12} = \bar{\tau}_{34} = \mu \) | \( \bar{\tau}_{23} = \bar{\tau}_{14} = \bar{k}_2 \mu / (k_1 + \bar{k}_3) \) | \( \bar{\tau}_{13} = \bar{\tau}_{24} = 0 \) |
| \( I_2 \) | \( \bar{\tau}_{12} = \bar{\tau}_{34} = -\mu \) | \( \bar{\tau}_{23} = \bar{\tau}_{14} = -\bar{k}_2 \mu / (\bar{k}_1 + \bar{k}_3) \) | \( \bar{\tau}_{13} = \bar{\tau}_{24} = 0 \) |
| \( I_3 \) | \( \bar{\tau}_{12} = -\bar{\tau}_{34} = \nu \) | \( \bar{\tau}_{23} = -\bar{\tau}_{14} = \bar{k}_2 \nu / (k_1 - \bar{k}_3) \) | \( \bar{\tau}_{13} = \bar{\tau}_{24} = 0 \) |
| \( I_4 \) | \( \bar{\tau}_{12} = -\bar{\tau}_{34} = -\nu \) | \( \bar{\tau}_{23} = -\bar{\tau}_{14} = -\bar{k}_2 \nu / (k_1 - \bar{k}_3) \) | \( \bar{\tau}_{13} = \bar{\tau}_{24} = 0 \) |

where

\[
\begin{align*}
\mu &= \frac{\bar{k}_1 + \bar{k}_3}{\sqrt{\bar{k}_1^2 + (k_1 + \bar{k}_3)^2}}, \\
\nu &= \frac{\bar{k}_1 - \bar{k}_3}{\sqrt{\bar{k}_1^2 + (k_1 - \bar{k}_3)^2}}.
\end{align*}
\]

and when \( \bar{k}_1 = \bar{k}_3 \) the classes \( I_3 \) and \( I_4 \) are substituted by

| \( I_3 \) | \( \bar{\tau}_{12} = \bar{\tau}_{34} = \bar{\tau}_{13} = \bar{\tau}_{24} = \bar{\tau}_{14} = 1 \) |
| \( I_4 \) | \( \bar{\tau}_{12} = \bar{\tau}_{34} = \bar{\tau}_{13} = \bar{\tau}_{24} = \bar{\tau}_{14} = -1 \) |

Using Maeda-Adachi classification, we can conclude

**Theorem 6.1.** Let \( \bar{\gamma} \) be a proper-biharmonic Frenet curve in \( \mathbb{CP}^2 \) of osculating order 4. Then \( \bar{\gamma} \) is a holomorphic helix of order 4 of class \( I_3 \) or \( I_4 \) according to the following table

| \( I_3 \) if \( \bar{\tau}_{12} < 0 \) and \( \bar{\tau}_{23} < 0 \) |
| \( I_4 \) if \( \bar{\tau}_{12} > 0 \) and \( \bar{\tau}_{23} > 0 \) |

Conversely,

(a) For any \( \alpha_0 \in (\pi / 2, \arccos(-2 - \sqrt{3})) \) there exist two proper-biharmonic holomorphic helices of order 4 of class \( I_3 \) with

\[
\begin{align*}
\bar{k}_2 &= \frac{\sin \alpha_0}{\sqrt{2}} \sqrt{1 - 3 \cos^2 \alpha_0 + \sqrt{9 \cos^4 \alpha_0 - 42 \cos^2 \alpha_0 + 1}}, \\
\bar{k}_3 &= -\frac{3}{2\bar{k}_2} \sin(2\alpha_0), \\
\bar{k}_1 &= -\frac{1}{\sin \alpha_0}(\bar{k}_2 \cos \alpha_0 - \bar{k}_3 \sin \alpha_0).
\end{align*}
\]

(b) For any \( \alpha_0 \in (\pi / 2, \pi + \arccos(-2 - \sqrt{3})) \) there exist two proper-biharmonic holomorphic helices of order 4 of class \( I_4 \) with

\[
\begin{align*}
\bar{k}_2 &= -\frac{\sin \alpha_0}{\sqrt{2}} \sqrt{1 - 3 \cos^2 \alpha_0 + \sqrt{9 \cos^4 \alpha_0 - 42 \cos^2 \alpha_0 + 1}}, \\
\bar{k}_3 &= -\frac{3}{2\bar{k}_2} \sin(2\alpha_0), \\
\bar{k}_1 &= -\frac{1}{\sin \alpha_0}(\bar{k}_2 \cos \alpha_0 - \bar{k}_3 \sin \alpha_0).
\end{align*}
\]

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