1 Introduction

Let $V \cong \mathbb{C}^d$ be a complex vector space, and $X \subset V$ a connected, smooth, closed subvariety. Consider the family of scalings $\{\lambda X\}_{\lambda \in \mathbb{C}^\ast}$. The limit $\text{as}(X) := \lim_{\lambda \to 0} \lambda X$ is a conic subvariety of $V$, called the asymptotic cone of $X$. The asymptotic cone is equipped with a nearby cycles perverse sheaf $P = P_X$, coming from the specialization of $X$ to $\text{as}(X)$ (we give precise definitions in Section 2.2). The sheaf $P$ is $\mathbb{C}^\ast$-conic.

Given a Hermitian inner product on $V$, we may consider, for any $x \in X$, the angle $\angle(x, T_x X) \in [0, \pi/2]$ between the vector $x \in V$ and the subspace $T_x X \subset T_x V \cong V$. We say that $X$ is transverse to infinity if for some (equivalently for any) inner product on $V$, there exists a constant $k > 0$, such that for any $x \in X$, we have:

$$\angle(x, T_x X) < \frac{k}{\|x\|}.$$

For example, a hyperbola in the plane is transverse to infinity, but a parabola is not. Our main result is the following.

**Theorem 1.1** Assume $X \subset V$ are as above, and $X$ is transverse to infinity. Let $T_X V \subset T^* V \cong V \times V^*$ be the conormal bundle to $X$, and $p_2 : V \times V^* \to V^*$ be the projection on the second factor. Let $Y = p_2(T_X V)$; it is an irreducible cone in $V^*$. Then we have:

$$\mathcal{F} P_X \cong IC(Y^\circ, \mathcal{L}),$$

where $\mathcal{F}$ is the geometric Fourier transform functor, $\mathcal{L}$ is a local system on some Zariski open subset $Y^\circ \subset Y$, and the right hand side is the intersection homology extension of $\mathcal{L}$.

The reader will, at a glance, place this result in the context of the classical Gauss map / projective duality ideas. In the modern literature (known to the author) it is perhaps closest to Jean-Luc Brylinski’s paper [Br], which develops the Lefschetz pencil and projective duality themes in the context of constructible sheaves. Here are some situations where the theorem applies.
**Example 1.2** Let \( f : V \to \mathbb{C} \) be a generic homogeneous polynomial of degree \( m \), and \( X = f^{-1}(1) \). Then \( \text{as}(X) = f^{-1}(0) \), and \( P \) is the sheaf of nearby cycles of \( f \). In this case, \( Y = V^* \) and \( \text{rank} \mathcal{L} = m(m-1)^{d-1} \).

**Example 1.3** Let \( f = \prod l_i : V \to \mathbb{C} \) be a product of (finitely many) linear forms \( l_i \in V^* \), and \( X = f^{-1}(1) \). As in the previous example, \( \text{as}(X) = f^{-1}(0) \), and \( P \) is the sheaf of nearby cycles of \( f \). In this case, \( Y = (\cap \ker l_i)^\perp \). We will discuss this example in more detail in Section 6. In particular, we will give an algorithm for computing the rank of the local system \( \mathcal{L} \).

**Example 1.4** Let \( V = g \) be a complex semisimple Lie algebra, and \( X \subset g \) be a semisimple (i.e., closed) adjoint orbit. Then \( X \) is transverse to infinity, and \( \text{as}(X) \subset g \) is a closed union of nilpotent orbits. If \( X \) is a regular orbit, then \( \text{as}(X) = \mathcal{N} \), the full nilcone in \( g \), and \( P \) is the famous Springer perverse sheaf, also known as the nearby cycles of the adjoint quotient map \( f : g \to G \setminus \mathfrak{g} \) (see [Sp], [Sl], [BM], [M]). The claim of Theorem 1.1 in this case is due independently to Hotta-Kashiwara [HK] and Ginzburg [Gi] (see also [Br]). In a way, Theorem 1.1 is saying that the Fourier transform description of Springer theory is independent of the existence of the simultaneous resolution for the fibers of \( f \) (the celebrated Grothendieck-Springer resolution).

**Example 1.5** A natural generalization of Example 1.4 is the following. Let \( g = \mathfrak{t} \oplus \mathfrak{p} \) be a Cartan decomposition of a complex semisimple Lie algebra. Take \( V = \mathfrak{p} \), and let \( X \subset \mathfrak{p} \) be a closed \( K \)-orbit. Then \( X \) is transverse to infinity. If \( X \) is a regular orbit, then \( \text{as}(X) \) is the nilcone in \( \mathfrak{p} \), and Theorem 1.1 can be used to give a generalization of Springer theory to symmetric spaces (where no simultaneous resolution is available).

A further generalization of Examples 1.4 and 1.5 is given by the so-called polar representations of Dadok and Kac (see [DK]). Their geometry forms the subject of the paper [Gr1].

We should note that the local system \( \mathcal{L} \) appearing in Theorem 1.1 typically will not be semisimple. Thus, even in the cases where \( \mathcal{L} \) can be explicitly computed (e.g., Example 1.3), Theorem 1.1 may fall short of giving the complete structure of \( P \) as a perverse sheaf. This is because intersection homology is not an exact functor from local systems to perverse sheaves. However, the theorem is saying that, in some sense, the whole nearby cycles sheaf is encoded in a single local system. In particular, the endomorphism rings of \( P \) and \( \mathcal{L} \) are the same.

The paper is organized as follows. We fix the notations in Section 2. Sections 3-5 are devoted to the proof of Theorem 1.1. Lastly, Section 6 is concerned with...
products of linear forms (Example 1.3 above). The reader interested in symmetric spaces or, more generally, polar representations is referred to [Gr1] and [Gr2].

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2 Notations

2.1 Sheaves and Functors

We will say sheaf to mean complex of sheaves throughout; all our sheaves will be sheaves of \( \mathbb{C} \) vector spaces. Given a map \( g : X \to Y \), the symbols \( g_* \), \( g! \) will always denote the derived push-forward functors. All perverse sheaves and intersection homology will be taken with respect to the middle perversity (see [GM1], [BBD]); we use the shift conventions of [BBD]. Given a sheaf \( A \) on \( X \), and a pair of closed subspaces \( Z \subset Y \subset X \), we will write \( H^k(Y, Z; A) \) for the hypercohomology group \( H^k(j_! i^* A) \), where \( i : Y \setminus Z \to X \) and \( j : Y \setminus Z \to Y \) are the inclusion maps. We call \( H^k(Y, Z; A) \) the relative hypercohomology of \( A \). For an analytic function \( f : M \to \mathbb{C} \), we use the notation of [KS, Chapter 8.6] for the nearby and vanishing cycles functors \( \psi_f, \phi_f \).

When \( V \) is a \( \mathbb{C} \) vector space, we denote by \( \mathcal{P}_{\mathbb{C}^*}(V) \) the category of \( \mathbb{C}^* \)-conic perverse sheaves on \( V \), and by \( \mathcal{F} : \mathcal{P}_{\mathbb{C}^*}(V) \to \mathcal{P}_{\mathbb{C}^*}(V^*) \) the (shifted) Fourier transform functor. In the notation of [KS, Chapter 3.7], we have \( \mathcal{F} P = P^* [\dim V] \). To avoid cumbersome notation, we use the following shorthand: if \( P \) is a conic perverse sheaf on a closed conic subvariety \( X \subset V \), and \( j_X : X \to V \) is the inclusion, then we write \( \mathcal{F} P \) instead of \( \mathcal{F} \circ (j_X)_* P \).

2.2 The Asymptotic Cone

Let \( V \cong \mathbb{C}^d \) be a complex vector space, and \( X \subset V \) a connected, smooth, closed subvariety, as in Section 1. We denote by \( \bar{V} \) the standard projective compactification of \( V \), and by \( \bar{X} \) the closure of \( X \) in \( \bar{V} \). Set \( V^\infty = \bar{V} \setminus V \), and \( X^\infty = \bar{X} \cap V^\infty \). The asymptotic cone \( \text{as}(X) \subset V \) is defined as the affine cone over \( X^\infty \).

Another way to define \( \text{as}(X) \) is as follows. Let \( \hat{X}^\infty = \{(\lambda, \bar{x}) \in \mathbb{C}^* \times V \mid \bar{x} \in \lambda X\} \), and \( \hat{X} \) be the closure of \( \hat{X}^\infty \) in \( \mathbb{C} \times V \). Write \( \hat{f} : \hat{X} \to \mathbb{C} \) for the projection on the first factor. Then \( \text{as}(X) = \hat{f}^{-1}(0) \). The nearby cycles sheaf \( P = P_X \) appearing in
Theorem 1.1 is defined by $P = \psi \zeta_X[n]$, where $n = \dim X$. Here and in the rest of the paper the symbol $\dim$ denotes the complex dimension.

3 Outline of Proof of Theorem 1.1

The main part of the proof of Theorem 1.1 is deformation arguments, à la "moving the wall" in [GM3]. We give an outline of the proof in this section, postponing the deformation arguments to Section 4, and deferring one other technical argument to Section 5.

3.1 The Stalks of the Fourier Transform

Our proof of Theorem 1.1 will be based on the original definition of the intersection homology complex by stalk and costalk cohomology vanishing conditions [GM1, Theorem 4.1]. We therefore begin with a lemma that identifies the stalks of the Fourier transform.

Lemma 3.1 Let $V$ be a Hermitian complex vector space of dimension $d$, $Q \in \mathcal{P}_{\mathbb{C}^*}(V)$ an algebraically constructible conic perverse sheaf, and $l \in V^*$ a covector. For $v \in V$, set $\xi(v) = \Re(l(v))$. Fix positive numbers $0 < \xi_0 \ll \eta_0$. Then the stalk cohomology $H^i_l(FQ)$ is given by a natural isomorphism:

$$H^i_l(FQ) \cong H^{i+d+n}(\{ \tilde{x} \in \lambda X \mid \xi(\tilde{x}) \leq \xi_0 \}, \{ |\tilde{x}| \geq \eta_0 \}; \mathbb{C}),$$

where the right hand side is a relative hypercohomology group of $Q$.

Proof: This follows from the definition of the Fourier transform [KS, Chapter 3.7].

Let now $V \supset X$ be as in Theorem 1.1. Fix a Hermitian metric on $V$. Set $n = \dim X$.

Corollary 3.2 Fix $l \in V^*$, and let $\xi = \Re(l)$. Also fix positive numbers $0 < \lambda \ll \xi_0 \ll \eta_0$, chosen in decreasing order. Let $\lambda X$ be the translate of $X$ by the scalar $\lambda$. Then we have:

$$H^i_l(FP) \cong H^{i+d+n}(\{ \tilde{x} \in \lambda X \mid \xi(\tilde{x}) \leq \xi_0 \}, \{ |\tilde{x}| \geq \eta_0 \}; \mathbb{C}),$$

where the right hand side is an ordinary relative cohomology group of $\lambda X$.

Proof: This follows from Lemma 3.1 and the definition of nearby cycles.
Note that in Corollary 3.2 the number $\eta_0$ is chosen first, then $\xi_0$ is chosen small relative to $\eta_0$, then $\lambda$ is chosen small relative to $\xi_0$. Note also that the group in the right hand side is invariant under the simultaneous multiplication of the numbers $\lambda, \xi_0, \eta_0$ by a positive scalar.

**Proposition 3.3** The right hand side of the isomorphism of Corollary 3.2 will not change if one chooses the numbers $\lambda, \xi_0, \eta_0$ in the opposite order. Taking $\lambda = 1$, we have:

$$H^i_l(FP) \cong H^{i+d+n}(\{x \in X | \xi(x) \leq \xi_0\}, \{\|x\| \geq \eta_0\}; \mathbb{C}),$$

where the numbers $1 \ll \xi_0 \ll \eta_0$ are chosen large in increasing order.

We will give a proof of Proposition 3.3 in Section 4.2. It is the key step of the proof of Theorem 1.1, and the main place where the transverse-to-infinity assumption is used.

### 3.2 A Decomposition of $V^*$

To verify the definition of intersection homology for the sheaf $FP$, we are going to fix a decomposition of $V^*$ into algebraic manifolds. For this, we first fix a stratification $S$ of $\text{as}(X)$. Let $\text{as}(X) = \bigcup_{S \in S} S$ be an algebraic Whitney stratification satisfying the following three conditions.

(i) $S$ is conic, i.e., each $S \in S$ is $\mathbb{C}^*$-invariant.

(ii) Thom’s $A_f$ condition holds for the pair $(\bar{X}^o, S)$, for each $S \in S$.

(iii) Let $S^o = S \setminus \{0\}$. For $S \in S^o$, let $S^\infty \subset X^\infty$ be the projectivization of $S$. Then the decomposition $\bar{X} = X \cup \bigcup_{S \in S^o} S^\infty$ is a Whitney stratification.

The reader is referred to [Hi] and [GM3, Part I, Chapter 1] for a discussion of the $A_f$ condition and of the Whitney conditions. The existence of a stratification $S$ satisfying (i)-(iii) above follows from the general results contained in these references.

Let $\Lambda_X$ denote the conormal bundle $T_X^*V \subset V \times V^*$, and $\Lambda_{\text{as}(X)} \subset V \times V^*$ be the conormal variety to the stratification $S$. Recall the projection $p_2 : V \times V^* \to V^*$. We now fix a finite decomposition $V^* = \bigcup_{W \in \mathcal{W}} W$ of $V^*$ into connected algebraic manifolds, such that for each $W \in \mathcal{W}$, the dimensions

$$\dim p_2^{-1}(l) \cap \Lambda_X \text{ and } \dim p_2^{-1}(l) \cap \Lambda_{\text{as}(X)}$$

are independent of $l$, for $l \in W$. Note that one of the pieces $W \in \mathcal{W}$ must be an open subset of $Y = \overline{p_2(\Lambda_X)}$. We denote this piece by $W_0$. The set $Y^o$ appearing in Theorem 1.1 will be a subset of $W_0$. Note that it does not matter for us whether $W$ is a stratification. For $W \in \mathcal{W}$, put $d(W) = \dim W$.

**Proposition 3.4**

(i) If $l \in W \neq W_0$, then $H^i_l(FP) = 0$, for $i \geq -d(W)$.

(ii) If $l \in W_0$, then $H^i_l(FP) = 0$, for $i > -d(W_0)$. 

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Sections 3.3 and 3.4 will be devoted to the proof of Proposition 3.4. Theorem 1.1 is an immediate consequence. Indeed, Proposition 3.4 is just a rewriting of half of the axioms for the intersection homology sheaf [GM1, Theorem 4.1]. The other half, dealing with costalk cohomology, follows from the fact that $P$, and therefore $\mathcal{F}P$, is Verdier self-dual.

3.3 A Partial Compactification of $X$

Our proof of Proposition 3.4 will be based on the study of Morse theory of the function $l$ on $X$. The difficulty in analyzing this Morse theory is that $l|_X$ is not proper. For this reason, we introduce a partial compactification $\hat{X} = \hat{X}_l$ of $X$, depending on $l \in V^*$. In the following construction $l \in V^*$ is fixed, and we assume $l \neq 0$. (Case $l = 0$ of Proposition 3.4 is trivial.)

Let $\Delta \subset V$ be the kernel of $l$, and $L \subset V$ be any line complementary to $\Delta$. We have $V = \Delta \oplus L$. Take the standard projective compactification $\bar{\Delta}$ of $\Delta$, and let $\hat{V} = \bar{\Delta} \times L$. It is not hard to check that the space $\hat{V}$ is canonically independent of the choice of the line $L$. Note that $l : V \to \mathbb{C}$ extends to a proper algebraic function $\hat{l} : \hat{V} \to \mathbb{C}$. Let $\hat{X} = \hat{X}_l$ be the closure of $X$ in $\hat{V}$.

Put $\hat{V}^\infty = \hat{V} \setminus V$, and $\hat{X}^\infty = \hat{X} \setminus X$. Any point of $\hat{V}^\infty$ can be written as a pair $(\Gamma, z)$, where $\Gamma$ is a one dimensional subspace of $\Delta$, and $z = \hat{l}(\Gamma, z) \in \mathbb{C}$. Note that if $(\Gamma, z) \in \hat{X}^\infty$, then $\Gamma \subset \text{as}(X)$. For each $S \in S^\circ$, let $\hat{S}^\infty,^\circ \subset \hat{X}^\infty$ be the set of all pairs $(\Gamma, z)$ such that $\Gamma \setminus 0 \subset S$, and the restriction $l|_S$ is non-critical along $\Gamma \setminus 0$; it is a manifold of dimension $\dim S - 1$. The following is a straightforward verification.

Lemma 3.5 Let $(\Gamma, z) \in \hat{S}^\infty,^\circ$, then $(\Gamma, z') \in \hat{S}^\infty,^\circ$ for any $z' \in \mathbb{C}$.

Write $\hat{X}^\infty,^\circ = \bigcup_{S \in S^\circ} \hat{S}^\infty,^\circ$, and $\hat{X}^\circ = X \cup \hat{X}^\infty,^\circ$. Note that $\hat{X}^\circ$ is Zariski open in $\hat{X}$. For $S \in S^\circ$, let $K(S)$ be the set of connected components of $\hat{S}^\infty,^\circ$.

Proposition 3.6 The decomposition

$$\hat{X}^\circ = X \cup \bigcup_{S \in S^\circ} K$$

$$K \in K(S)$$

is a Whitney stratification.

We defer the proof of Proposition 3.6 to Section 5. The significance of the compactification $\hat{X}$ for our problem is revealed by the following proposition. Let $j : X \to \hat{X}$ be the inclusion map.
Proposition 3.7 The statement of Proposition 3.3 can be further modified as follows:

\[ H^i_f(F \mathcal{P}) \cong \mathbb{H}^{i+d}(\{ x \in \hat{X} \mid \hat{\xi}(x) \leq \xi_0 \}, \{ \| \hat{l}(x) \| \geq 2\xi_0 \}; j_! \mathbb{C}_X[n]), \]

where \( \xi_0 \gg 1 \), and \( \hat{\xi}(x) = \text{Re}(\hat{l}(x)) \).

We defer the proof of this to Section 4.3.

3.4 Proof of Proposition 3.4

We are now prepared to give a proof of Proposition 3.4, completing the outline of the proof of Theorem 1.1. Our argument is based on the following general result.

Lemma 3.8 Let \( A \) be a complex algebraic variety, \( \mathcal{A} \) an algebraic Whitney stratification of \( A \), and \( Q \) a perverse sheaf on \( A \), constructible with respect to \( \mathcal{A} \). Let \( g : A \to \mathbb{C} \) be a proper algebraic function, \( z_0 \) a stratified critical value of \( g \), and \( Z_0 \subset g^{-1}(z_0) \) the set of stratified critical points of \( g \) above \( z_0 \). Fix a small number \( 0 < \epsilon \ll 1 \). Then the following bound holds:

\[ \mathbb{H}^i(\{ a \in A \mid \text{Re}(g(a) - z_0) \leq \epsilon \}, \{ |g(a) - z_0| \geq 2\epsilon \}; Q) = 0, \]

for \( i > \text{dim} Z_0 \).

Proof: Let \( \phi_{g-z_0} Q \) be the vanishing cycles of \( Q \) with respect to the function \( g - z_0 \); it is a perverse sheaf on \( Z_0 \) [KS, Corollary 10.3.13]. The hypercohomology group in the statement of the lemma is the dual of \( \mathbb{H}^{-i}(Z_0, \phi_{g-z_0} Q) \). The lemma follows.

We are going to apply Lemma 3.8 to the situation where \( A = \hat{X} \) and \( g = \hat{l}|_{\hat{X}} \). Fix an algebraic Whitney stratification \( \hat{\mathcal{A}} \) of \( \hat{X} \), such that each stratum of the stratification of \( \hat{X}^\circ \) described in Proposition 3.6 is also a stratum of \( \hat{\mathcal{A}} \) (see [GM3, Part I, Theorem 1.7] for a proof that such an \( \hat{\mathcal{A}} \) exists).

Lemma 3.9 The extension by zero \( j_! \mathbb{C}_X[n] \), appearing in Proposition 3.4, is a perverse sheaf on \( \hat{X} \), constructible with respect to \( \hat{\mathcal{A}} \).

Proof: By [BBD, Corollary 4.1.10], extending a perverse sheaf by zero across a divisor always gives a perverse sheaf. Constructibility with respect to \( \hat{\mathcal{A}} \) follows from the local topological triviality of Whitney stratifications [GM3, Part I, Chapter 1.4].
Let $Z \subset \hat{X}$ be the stratified critical points of $\hat{l}|_{\hat{X}}$ with respect to $\hat{X}$. Recall the decomposition $V^* = \bigcup_{W \in W} W$ of Section 3.2. Let $W$ be the piece containing $l$.

Lemma 3.10

(i) If $W \neq W_0$, then $\dim Z < d - d(W)$.

(ii) If $W = W_0$, then $\dim Z = d - d(W_0)$.

Proof: We will consider the sets $Z \cap X$ and $Z \cap \hat{X}$ separately. Look first at $Z \cap X$.

By definition, $Z \cap X \cong p_2^{-1}(l) \cap \Lambda_X$. By assumption, the dimension of $p_2^{-1}(l') \cap \Lambda_X$ is the same for all $l' \in W$. This implies:

$$\dim p_2^{-1}(W) \cap \Lambda_X = \dim p_2^{-1}(l) \cap \Lambda_X + d(W).$$

Note now that $p_2^{-1}(W_0) \cap \Lambda_X$ is Zariski open in $\Lambda_X$, which is irreducible of dimension $d$. The required bounds on the dimension of $Z \cap X$ follow.

Consider now the set $Z \cap \hat{X}$. We will show that $\dim Z \cap \hat{X} < d - d(W)$, in both cases of the lemma. The image $\hat{l}(Z) \subset C$ is a finite set of points, so it will suffice to show that

$$\dim Z \cap \hat{X} \cap \hat{l}^{-1}(z) < d - d(W),$$

for every $z \in C$. By Lemma 3.3, we have $Z \cap \hat{X} \subset \hat{X} \setminus \hat{X}^\circ$. Let $(\Gamma, z) \in \hat{X} \setminus \hat{X}^\circ$, where $\Gamma$ is a line in $\text{as}(X) \cap \Delta$. Then, for any $v \in \Gamma \setminus 0$, we have $(v, l) \in \Lambda_{\text{as}(X)} \subset V \times V^*$. It follows that:

$$\dim Z \cap \hat{X} \cap \hat{l}^{-1}(z) \leq \dim p_2^{-1}(l) \cap \Lambda_{\text{as}(X)} - 1.$$

But the dimension of $p_2^{-1}(l') \cap \Lambda_{\text{as}(X)}$ is the same for all $l' \in W$. Therefore,

$$\dim p_2^{-1}(l) \cap \Lambda_{\text{as}(X)} + d(W) \leq \dim \Lambda_{\text{as}(X)} = d.$$

Inequality (1) follows. \hfill \Box

Proposition 3.4 now follows from Proposition 3.7 and Lemmas 3.8 - 3.10. More precisely, we have to iterate Lemma 3.8 for every critical value $z$ of $\hat{l}|_{\hat{X}}$.

4 Moving the Wall

The purpose of this section is to give proofs of Propositions 3.3 and 3.7. Both are accomplished by moving the wall arguments (see [GM3, Part I, Chapter 4] for a discussion of the moving the wall technique), and both use the assumption that $X$ is transverse to infinity in a crucial way.
4.1 Preliminaries

We continue with a fixed non-zero $l \in V^*$. Pick a Hermitian metric on $V$, such that $\|l\| = 1$. We introduce several constants reflecting the geometry of $X$, and the choice of the covector $l$. First, there is the number $k > 0$ appearing in the transverse-to-infinity condition:

$$\xi(x, T_x X) < \frac{k}{\|x\|},$$

(2)

for all $x \in X$. We will use a shorthand $c_1 = 1000k$. Recall that we denote by $Z \subset \hat{X}$ the stratified critical locus of $\hat{l}\big|_X$ with respect to the stratification $\hat{X}$. The set $\hat{l}(Z)$ is finite, and we put $c_2 = \max_{z \in \hat{l}(Z)} |z|$. Consider a map $\tau : X \to \mathbb{R}^2$ defined by $\tau : x \mapsto (\xi(x), \eta(x))$, where $\xi(x) = \text{Re}(l(x))$ and $\eta(x) = |x|$. Let $\beta : \mathbb{R}^2 \setminus \{0\} \to [0, 2\pi]$ be the standard polar angle. Note that the image of $\tau$ is contained in the sector $\{\pi/4 \leq \beta \leq 3\pi/4\} \cup \{0\}$. Let $C \subset \mathbb{R}^2$ be the set of critical values of $\tau$. By the general topological finiteness results of real algebraic geometry (see [GM3, Theorem 1.7]), there exists a number $c_3 > 0$ such that the set $C \cap \{\xi, \eta \in \mathbb{R}^2 \mid |(\xi, \eta)| \geq c_3\}$ consists of finitely many disjoint, connected smooth curves $\{C_j\}_{j=1}^m$, each approaching a point at infinity in the standard projective compactification of $\mathbb{R}^2$ (see Figure 1). Let $\gamma_j : \mathbb{R}_+ \to C_j$ be the length parameterization of $C_j$, so that $\gamma_j(0)$ is the starting point of $C_j$ (with $|\gamma_j(0)| = c_3$), and $|\gamma_j'(t)| = 1$ for all $t$. Let $\beta_j = \lim_{t \to \infty} \beta(\gamma_j(t)) \in [\pi/4, 3\pi/4]$. Renumber the curves $\{C_j\}$ so that $\beta_j = \pi/2$ for $j = 1, \ldots, m'$, and $\beta_j \neq \pi/2$ for $j = m' + 1, \ldots, m$. By choosing the number $c_3$ sufficiently large, we can assume that there exists a $b > 1$ such that

$$\eta(\gamma_j(t)) < b \cdot |\xi(\gamma_j(t))|,$$

for any $j = m' + 1, \ldots, m$, and any $t \in \mathbb{R}_+$. Thus, we have fixed positive constants $k, b, c_1, c_2, c_3$. We now prove two technical lemmas, both direct consequences of the estimate $(2)$.

**Lemma 4.1** We have $|\xi(\gamma_j(t))| < 4k$, for any $j = 1, \ldots, m'$, and any $t \in \mathbb{R}_+$.

**Proof:** Fix $j \in \{1, \ldots, m'\}$, and write $\gamma = \gamma_j : \mathbb{R}_+ \to \mathbb{R}^2$. The curve $\gamma$ satisfies the following four conditions.

- (i) $\lim_{t \to \infty} |\gamma_j(t)| = \infty$.
- (ii) $\lim_{t \to \infty} \frac{\gamma_j(t)}{|\gamma_j(t)|} = (0, 1) \in \mathbb{R}^2$.
- (iii) $|\gamma_j'(t)| = 1$, for all $t \in \mathbb{R}_+$.
- (iv) $|\gamma_j'(t) - \frac{\gamma_j(t)}{|\gamma_j(t)|}| \leq \frac{2k}{|\gamma_j(t)|}$, for all $t \in \mathbb{R}_+$.

Conditions (i) - (iii) follow from the construction, and condition (iv) follows from $(2)$.
It is an elementary (but interesting) exercise in plane geometry to verify that any curve satisfying conditions (i) - (iv) above must also satisfy $|\xi(\gamma(t))| < 4k$, for all $t \in \mathbb{R}_+$. □

**Lemma 4.2** Let $x \in X$ be any point with $\|l(x)\| \geq c_1$. Then there exists a tangent vector $u \in T_x X$ such that $d\eta(u) = 0$, $dl(u) \neq 0$, and the angle $\omega$ between the vectors $l(y)$ and $dl(u)$ in $\mathbb{C}$ satisfies:

$$|\omega - \pi/2| < \frac{1}{100}.$$  

**Proof:** Let $v_0 \in T_x X$ be a vector minimizing the angle between $v$ and $x$ in $V$, over all non-zero $v \in T_x X$. Then, by (2), we can take $u = iv_0$. □

### 4.2 Proof of Proposition 3.3

The proof of Proposition 3.3 proceeds by appealing to Figure 1 which depicts the critical locus $C$ of the map $\tau : X \to \mathbb{R}^2$. Fix a constant $c$ satisfying $c > c_i$, for $i = 1, 2, 3$. Let

$$D = \{((\xi, \eta) \in \mathbb{R}^2 \mid \xi > c, \eta > b \cdot \xi\}.$$

**Claim:** The homeomorphism type of the pair

$$\{x \in X \mid \xi(x) \leq \xi_0\}, \{\eta(x) \geq \eta_0\}$$

is independent of the numbers $\xi_0, \eta_0$, as long as $(\xi_0, \eta_0) \in D$.

Proposition 3.3 is an immediate consequence of this claim. The proof of the claim is an application of moving the wall. It is enough to verify the following three conditions.

(i) The function $\xi|_X$ has no critical points $x \in X$ with $\xi(x) > c$.
(ii) The function $\eta|_X$ has no critical points $x \in X$ with $\eta(x) > b \cdot c$.
(iii) The intersection $C \cap D$ is empty.

Conditions (i) and (ii) follow directly from the estimate (2). Condition (iii) follows from Lemma 4.1 and the definitions of Section 4.1.
4.3 Proof of Proposition 3.7

Our Proof of Proposition 3.7 will proceed in four steps. Fix a point \((\xi_0, \eta_0)\) in the region \(D\) of Section 4.2.

**Step 1.** We show that there is a natural isomorphism:

\[
H^*(\{ x \in X \mid \xi(x) \leq \xi_0 \}, \{ \eta(x) \geq \eta_0 \}; \mathbb{C}) \cong H^*(\{ x \in X \mid \xi(x) \leq \xi_0 \}, \{ \eta(x) \geq \eta_0 \} \cup \{ \xi(x) \leq -\xi_0 \}; \mathbb{C}).
\]

By the long exact sequence of the triple, it is enough to show that

\[
H^*(\{ x \in X \mid \xi(x) \leq -\xi_0 \}, \{ \eta(x) \geq \eta_0 \}; \mathbb{C}) = 0. \tag{3}
\]

Define \(X_{\leq \eta_0} = \{ x \in X \mid \eta(x) \leq \eta_0 \}\), and \(X_{\eta_0} = \{ x \in F \mid \eta(x) = \eta_0 \}\). The pair \((X_{\leq \eta_0}, X_{\eta_0})\) is a manifold with boundary. By excision, (3) is equivalent to

\[
H^*(X_{\leq \eta_0}, X_{\eta_0}; \mathbb{C}) = 0.
\]
This is proved as an application of Morse theory for a manifold with boundary. There are two conditions we have to check.

(i) The restriction $\xi|_X$ has no critical points $x \in X$ with $\xi(x) \leq -\xi_0$ and $\eta(x) \leq \eta_0$.

(ii) Let $x \in X_{\eta_0}$ be a critical point of $\xi|_{X_{\eta_0}}$ with $\xi(x) \leq -\xi_0$. Then the differentials $d_x(\xi|_X)$ and $d_x(\eta|_X)$ in $T^*_x X$ satisfy $d_x \xi = s \cdot d_x \eta$, for some $s < 0$.

Both (i) and (ii) follow from (3).

**Step 2.** Fix a smooth cut-off function $\psi : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions:

(i) $\psi(\zeta) = \xi_0$, for $|\zeta| \leq \xi_0$;

(ii) $\psi(\zeta) = 0$, for $|\zeta| \geq 5 \xi_0$;

(iii) $0 \leq \psi(\zeta) \leq \xi_0$, for all $\zeta \in \mathbb{R}$;

(iv) $|\psi'(\zeta)| \leq 1/2$, for all $\zeta \in \mathbb{R}$.

Let $\zeta : V \to \mathbb{R}$ be the imaginary part of $l$, i.e., $\zeta(v) = \text{Re}(-il(v))$. We show that there is a natural isomorphism:

$$H^* \{ x \in X | \xi(x) \leq \xi_0 \}, \{ \eta(x) \geq \eta_0 \} \cup \{ \xi(x) \leq -\xi_0 \}; \mathbb{C} \cong H^* \{ x \in X | \xi(x) \leq -\psi(\zeta(x)) \}, \{ \eta(x) \geq \eta_0 \} \cup \{ \xi(x) \leq -\psi(\zeta(x)) \}; \mathbb{C}. \quad (4)$$

The purpose of this is to localize the problem in the $l$ directions (see Figure 2).

By the long exact sequence of the triple, it is enough to show that

$$H^* \{ x \in X | \xi(x) \leq \xi_0 \}, \{ \eta(x) \geq \eta_0 \} \cup \{ \xi(x) \leq -\psi(\zeta(x)) \}; \mathbb{C} = 0, \text{ and } \quad (5)$$

$$H^* \{ x \in X | \xi(x) \leq -\psi(\zeta(x)) \}, \{ \eta(x) \geq \eta_0 \} \cup \{ \xi(x) \leq -\xi_0 \}; \mathbb{C} = 0. \quad (6)$$

Both (3) and (4) are proved by moving the wall in the $l$ plane. To prove (4), we move the “wall” $\{ \xi(x) = \xi_0 \}$ to $\{ \xi(x) = \psi(\zeta(x)) \}$ through the family $\{ \xi(x) = t \cdot \psi(\zeta(x)) + (1-t) \cdot \xi_0 \}_{t \in [0,1]}$. For $t \in [0,1]$, define a function $\kappa_t : X \to \mathbb{R}$ by

$$\kappa_t(x) = \xi(x) - t \cdot \psi(\zeta(x)) - (1-t) \cdot \xi_0.$$ 

Then it suffices to verify the following two conditions.

(i) Zero is not a critical value of $\kappa_t$, for any $t \in [0,1]$.

(ii) Zero is not a critical value of the restriction $\kappa_t|_{X_{\eta_0}}$, for any $t \in [0,1]$ (where $X_{\eta_0} = \{ x \in X | \eta(x) = \eta_0 \}$).

Condition (i) follows from the fact that the function $l : X \to \mathbb{C}$ has no critical values $z$ with $|z| \geq \xi_0$. Condition (ii) follows from Lemma 1.2. Equality (3) is proved similarly.
Step 3. From Steps 1 and 2, we see that the cohomology in (4) is canonically independent of $\eta_0$ for large $\eta_0$. It follows that this cohomology is equal to

$$H^{*} \mathit{C}_X[\Xi],$$

where $\hat{\xi}$ and $\hat{\zeta}$ are the real and imaginary parts of $\hat{l}$.

Step 4. The last step is to identify the hypercohomology in Step 3 with the hypercohomology

$$H^{*} \mathit{C}_X[\Xi],$$

of Proposition 3.7. This is again done by a deformation in the $\hat{l}$ plane, using the fact that $\hat{l}|_{\hat{X}}$ has no critical values $z$ with $|z| \geq \xi_0$ (see [GM3, Part I, Theorem 1.5]).
5 Whitney Conditions

We now furnish the remaining piece of the proof of Theorem 1.1: a proof of Proposition 3.6.

First, we note that the decomposition

\[ \mathring{X}_\infty^\circ = \bigcup_{S \in S^\circ} K \]

is a Whitney stratification of \( \mathring{X}_\infty^\circ \). This follows from Lemma 3.5 and the fact that Whitney conditions are preserved under stratified transverse intersection (see [GM3, Part I, Chapter 1.2]). Thus, the only non-trivial part of Proposition 3.6 is the claim that Whitney conditions (a) and (b) hold for the pair \((X, K)\), for each \( S \in S^\circ \) and \( K \in K(S) \).

To check Whitney (a), consider a sequence of points \( \{x_i\} \) in \( X \) converging to \( \hat{x} \in K \subset \mathring{X} \). Let \( \tilde{x}_i = x_i / \|x_i\| \). By passing if necessary to a subsequence, we may assume that there exists a limit \( q = \lim_{i \to \infty} \tilde{x}_i \in V \). Then \( q \) must be a point of transverse intersection of the stratum \( S \subset \text{as}(X) \) and the hyperplane \( \Delta \subset V \). Whitney condition (a) for the sequence \( \{x_i\} \) now follows from Thom’s \( A_f \) condition (see assumption (ii) on the stratification \( S \) in Section 3.2).

It remains to check Whitney condition (b). Fix a sequence \( \{x_i\} \) in \( X \) with a limit \( \hat{x} \in K \subset \mathring{X} \). Without loss of generality, we may assume that \( \hat{l}(\hat{x}) = 0 \), so that \( \hat{x} \) is given as \((\Gamma, 0)\), where \( \Gamma \subset \text{as}(X) \cap \Delta \) is a line with \( \Gamma \setminus \{0\} \subset S \). We now introduce linear coordinates \((z_1, \ldots, z_d) : V \to \mathbb{C}^d\) such that:

(i) the line \( \Gamma \) is the \( z_1 \)-axis;

(ii) the covector \( l = z_s \), where \( s = \dim S \);

(iii) let \( q = (1, 0, \ldots, 0) \in \Gamma \), then \( T_q \Gamma \) is parallel to the plane \( \{z_{s+1} = z_{s+2} = \ldots = z_d = 0\} \subset V \).

Choose a small neighborhood \( \hat{U} \) of \( \hat{x} \) in \( \hat{V} \), and let \( U = \hat{U} \cap V \). The functions

\[
\left( \frac{1}{z_1}, \frac{z_2}{z_1}, \ldots, \frac{z_{s-1}}{z_1}, z_s, \frac{z_{s+1}}{z_1}, \ldots, \frac{z_d}{z_1} \right) : U \to \mathbb{C}^d
\]

extend to give a system of coordinates on \( \hat{U} \). Denote these coordinates by \((\hat{z}_1, \ldots, \hat{z}_d) : \hat{U} \to \mathbb{C}^d\).

Let \( \hat{\pi} : \hat{U} \to K \) be the projection defined by \( \hat{z}_i(\hat{\pi} \hat{u}) = \hat{z}_i(\hat{u}) \), for any \( \hat{u} \in \hat{U} \) and \( s \leq i \leq d \) (provided \( \hat{U} \) is small enough, this defines \( \hat{\pi} \) uniquely).

In the presence of Whitney condition (a), it is enough to verify Whitney (b) for the sequences \( \{x_i \in X\} \) and \( \{\hat{\pi}(x_i) \in \hat{K}\} \). To do this, consider a small neighborhood \( \hat{U} \subset \hat{V} \) of the point \( \hat{x} \in \hat{V} \) given by the line \( \Gamma \). The functions

\[
\left( \frac{z_1}{z_1}, \frac{z_2}{z_1}, \ldots, \frac{z_d}{z_1} \right) : \hat{U} \cap V \to \mathbb{C}^d
\]
extend to give a system of coordinates \((\bar{z}_1, \ldots, \bar{z}_d)\) on \(\bar{U}\). Let \(\bar{\pi} : \bar{U} \to S^\infty\) be the projection defined by \(\bar{z}_i(\bar{\pi}(\bar{u})) = \bar{z}_i(\bar{u})\), for any \(\bar{u} \in \bar{U}\) and \(s \leq i \leq d\). Then Whitney (b) for the sequences \(\{x_i \in X\}\) and \(\{\bar{\pi}(x_i) \in K\}\) follows from Whitney (b) for the sequences \(\{x_i \in X\}\) and \(\{\bar{\pi}(x_i) \in S^\infty\}\) (see assumption (iii) on the stratification \(S\) in Section 3.2). This completes the proof of Proposition 3.4 and, with it, of Theorem 1.1.

6 Products of Linear Forms

We now return to Example 1.2 of Section 1. Let \(f = \prod_{i \in I} l_i : V \to \mathbb{C}\) be a product of linear forms \(l_i \in V^*\), running over a finite set \(I\), and let \(X = f^{-1}(1)\). Then as \((X) = f^{-1}(0)\), and \(P = \psi_f C_X[d - 1]\). Singularities of \(f\) have been a subject of much recent study (see [CS] and its references). The following proposition shows that Theorem 1.1 applies to this example. Fix a Hermitian metric on \(V\). Let \(K = \bigcap_{i \in I} \text{Ker} l_i \subset V\).

**Proposition 6.1** (i) The norm \(\|d_x f\|\) is bounded from below on \(X\).

(ii) The variety \(X\) is transverse to infinity.

(iii) The variety \(Y = \frac{p_2(T^*_X V)}{m}\), associated to \(X\) as in Theorem 1.1, is the orthogonal complement \(K \perp \subset V^*\).

**Proof:** We argue by induction on \(d\) and the cardinality \(|I|\). Case \(d = 1\) or \(|I| = 1\) is trivial. Assume now \(d > 1\) and \(|I| > 1\). If \(K \neq \{0\}\), we may reduce the parameter \(d\) by passing to the quotient \(V/K\). Assume now \(K = \{0\}\).

By a standard compactness argument, parts (i) and (ii) of the proposition will follow if we show that for any sequence \(\{x_j\}\) in \(X\), converging to a limit \(\bar{x} \in X \subset \bar{V}\):

(a) the norm \(\|d_{x_j} f\|\) is bounded away from zero;

(b) the product \(\langle x_j, T_{x_j} X \rangle \cdot \|x_j\|\) is bounded from above.

Let \(\Gamma \subset V\) be the line corresponding to \(\bar{x}\). Define \(I' = \{i \in I \mid \text{Ker} l_i \supset \Gamma\}\). Let \(m = |I \setminus I'|\); note that \(m > 0\). Set \(g = \prod_{i \in I'} l_i\), and \(h = \prod_{i \in I \setminus I'} l_i\), so that \(f = g \cdot h\). Then we have the asymptotics (up to a multiplicative constant):

\[|h(x_j)| \sim \|x_j\|^m, \quad |g(x_j)| \sim \|x_j\|^{-m}, \quad \|d_{x_j} h\| \sim \|x_j\|^{m-1}.\]

By the Leibniz’s rule, \(df = g \cdot dh + h \cdot dg\). By the above asymptotics,

\[|g(x_j)| \cdot d_{x_j} h | \sim \|x_j\|^{-1}.\]

On the other hand, by part (i) of the induction hypothesis on \(g\), the norm \(\|h(x_j) \cdot d_{x_j} g\|\) is bounded away from zero. This immediately implies (a). Furthermore, it shows that there is a constant \(c > 0\), such that the angle \(\alpha_j\) between the hyperplanes \(\text{Ker} d_{x_j} f = T_{x_j} X\) and \(\text{Ker} d_{x_j} g\) in \(T_{x_j} V \cong V\) satisfies \(\alpha_j < c/\|x_j\|\), for all \(j\). Together with part (ii) of the induction hypothesis on \(g\), this estimate implies (b).
To prove part (iii) of the proposition, denote by $\mathcal{V}$ the linear stratification of $V$ underlying the arrangement $\text{as}(X)$, and by $\mathcal{V}^*$ the dual stratification of $V^*$. Note that $\mathcal{V}^*$ has more than one codimension one stratum. An inductive argument, as in the proof of parts (i) and (ii), shows that $Y$ contains every codimension one stratum of $\mathcal{V}^*$. But $Y$ is irreducible, therefore it must be all of $V^*$.

\[\square\]

We conclude with an observation which gives, in this example, an algorithm for computing the rank of the local system $\mathcal{L}$ of Theorem 1.1 from the combinatorics of the arrangement $\text{as}(X)$. Let $\mathcal{V}$ be as in the proof of Proposition 6.1, and $S = \{S \in \mathcal{V} \mid S \subset \text{as}(X)\}$ be the induced stratification of $\text{as}(X)$. Note that $K$ is the smallest stratum of $S$. For $S \in S$, let $m(S)$ be the multiplicity of the conormal bundle $T_S^*V$ in the characteristic cycle of $P$. Also, let $c(S)$ be the (complex) codimension of $S$ in $\text{as}(X)$. Note that $\text{rank } \mathcal{L} = m(K)$.

**Proposition 6.2** Let $\chi(X)$ be the Euler characteristic of $X$. Then

$$\chi(X) = \sum_{S \in S} (-1)^{c(S)} \cdot m(S).$$

**Proof:** This follows from the Morse theory of the distance to a generic point in $V$, as in [GM3, Part III].

\[\square\]

Proposition 6.2 may be used to compute the numbers $m(S)$ inductively. The only ingredient for the induction step is the Euler characteristic $\chi(X)$. To compute it, note that $X$ is an $|I|$-fold cover over $(V \setminus \text{as}(X))/\mathbb{C}^*$ (see [CS]). Therefore, $\chi(X) = |I| \cdot \chi((V \setminus \text{as}(X))/\mathbb{C}^*)$. Pick any $i \in I$, and let $H_i = \{v \in V \mid l_i(v) = 1\}$. Then

$$(V \setminus \text{as}(X))/\mathbb{C}^* \cong H_i \setminus (\text{as}(X) \cap H_i).$$

Finally, the Euler characteristic (in fact, the full cohomology) of the complement $H_i \setminus (\text{as}(X) \cap H_i)$ may be computed as in [GM3, Part III].

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