Examples of Freudenthal-Kantor triple systems

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Abstract
Symmetry group of Lie algebras and superalgebras constructed from (ε,δ) Freudenthal-Kantor triple systems has been studied. Also, the definition and examples of hermitian triple systems is introduced in this note.

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Introduction
It seems that the concept of a triple system (or called a ternary algebra) in nonassociative algebras started from the metasymplectic geometry due to Freudenthal. After a generalization of the concept has been studied by Tits, Koecher, Kantor, Yamaguti, Allison and authors ([1,2] for many earlier references on the subject). Also it is well known the object of investigation of Jordan and Lie algebras with application to symmetric spaces or domains [3] and to physics [4,5].

Nonassociative algebras are rich in of mathematics, not only for pure algebra differential geometry, but also for representation theory and algebraic geometry. Specialy, the Lie algebras and Jordan algebras plays an important role in many mathematical and physical objects. As a construction of Lie algebras as well as Jordan algebras, we are interested in characterizing the Lie algebras from view point of triple systems ([6-9]). These imply that we are considering to structure of the subspace \( L \) of the five graded Lie (super)algebra \( L (ε,δ) = L_0 \oplus L_1 \oplus L_2 \oplus L_3 \) satisfying \( [L_i,L_j] \subseteq L_{i+j} \) associated with an \((ε,δ)\) Freudenthal-Kantor triple system which contains a class of Jordan triple systems related to symmetric spaces or domains [3] and to physics [4,5].

More precisely speaking, first, the symmetry group of Lie algebras and superalgebras constructed from \((ε,δ)\) Freudenthal-Kantor triple systems has been studied. Especially, for a special \((ε,δ)\) Freudenthal-Kantor triple system with \( \delta = \pm 1 \) or \(-1\) if they satisfy

\[
[L(u,v), L(x,y)] = L(L(u,v)x,y) + \delta L(x,L(v,u)y) \quad \text{for } \delta = +1 \text{ or } -1 \quad \text{if they satisfy}
\]

\[
K(K(u,v)x,y) = L(y,x)K(u,v) - \delta L(K(u,v)y,x) \quad \text{for any } u,v,x,y \in V \quad \text{for } \epsilon = +1 \text{ and } \epsilon = -1 \text{ we call the triple system to be } (\epsilon,\delta) \text{ FKTS.}
\]

Since \((\epsilon,\delta)\) Freudenthal-Kantor triple systems offer a simple method of constructing Lie algebras (for the case of \((\delta=+1)\)) and Lie superalgebras (for the case of \((\delta=-1)\)) with 5-graded structure, it may be of some interest to study its symmetry group in this note. In order to facilitate the discussion, let us briefly sketch its definition.

We introduce two linear mappings \(L\) and \(K\): \(V \otimes V \rightarrow End \ V\) by

\[
L(x,y) = xyz, K(x,y)z = - \delta yxz \quad (1.1)
\]

\[
K(u,v)x,y) = L(y,x)K(u,v) - \delta L(K(u,v)y,x) \quad (1.3)
\]

\[
L(L(u,v)x,y) + \delta L(x,L(v,u)y) \quad \text{for } \delta = +1 \text{ or } -1 \quad \text{if they satisfy}
\]

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\]

One consequence of Eqs.(1.2) and (1.3) is the validity of the following important identity (see [Y-O.] Eqs.(2.9) and (2.10))

\[
K(u,v)K(x,y) = \epsilon L(K(u,v)x,L(y,x)) \quad (1.4)
\]

We note that \((-1,1)\) FKTS is said to be a generalized Jordan triple system of second order [12] in section 2, because that is a generalization of concept of Jordan triple systems.

We can then construct a Lie algebra for \(\delta = +1\) and a Lie superalgebra for \(\delta = -1\) as follows:

Let \( W = \text{a space of } 2 \times 1 \text{matrices over } V \quad W = \begin{bmatrix} x \end{bmatrix} \quad \text{and define a tri-linear product:}

\[
W \otimes W \otimes W \rightarrow W \quad \text{by}
\]

\[
\left[ \begin{array}{c} x_1 \\ y_1 \\ z_1 \\ x_2 \\ y_2 \\ z_2 \end{array} \right] \rightarrow \left[ \begin{array}{c} x_1 \\ y_1 \\ z_1 \\ x_2 \\ y_2 \\ z_2 \end{array} \right]
\]

\[
L(x_1,y_1,z_1) - \delta L(x_2,y_1,z_1) - \epsilon K(y_1,z_1) - \delta L(y_2,z_1,x_1) \quad (1.6)
\]

Symmetry of Lie algebras associated with triple systems
A triple system \( V \) is a vector space over a field \( F \) of characteristic \( \neq 2 \) or \( 3 \) with a trilinear map \( V \otimes V \otimes V \rightarrow V \). We denote the trilinear product ( or ternary product) by juxtaposition \( (xyz) \epsilon F, (xyz) \epsilon V \cap \text{trilinear product} \).

A well studied triple system is the \((\epsilon,\delta)\) Freudenthal-Kantor triple system ( abbreviated hereafter as \((\epsilon,\delta)\) FKTS) with \( \epsilon \) and \( \delta \) being either \(+1\) or \(-1\) [1,2,9,12].

Since \((\epsilon,\delta)\) Freudenthal-Kantor triple systems offer a simple method of constructing Lie algebras (for the case of \((\delta=+1)\)) and Lie superalgebras (for the case of \((\delta=-1)\)) with 5-graded structure, it may be of some interest to study its symmetry group in this note. In order to facilitate the discussion, let us briefly sketch its definition.

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\]

\[
L(L(u,v)x,y) + \delta L(x,L(v,u)y) \quad \text{for } \delta = +1 \text{ or } -1 \quad \text{if they satisfy}
\]

\[
K(K(u,v)x,y) = L(y,x)K(u,v) - \delta L(K(u,v)y,x) \quad \text{for any } u,v,x,y \in V \quad \text{for } \epsilon = +1 \text{ we call the triple system to be } (\epsilon,\delta) \text{ FKTS.}
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We can then construct a Lie algebra for \(\delta = +1\) and a Lie superalgebra for \(\delta = -1\) as follows:

Let \( W = \text{a space of } 2 \times 1 \text{matrices over } V \quad W = \begin{bmatrix} x \end{bmatrix} \quad \text{and define a tri-linear product:}

\[
W \otimes W \otimes W \rightarrow W \quad \text{by}
\]

\[
\left[ \begin{array}{c} x_1 \\ y_1 \\ z_1 \\ x_2 \\ y_2 \\ z_2 \end{array} \right] \rightarrow \left[ \begin{array}{c} x_1 \\ y_1 \\ z_1 \\ x_2 \\ y_2 \\ z_2 \end{array} \right]
\]

\[
L(x_1,y_1,z_1) - \delta L(x_2,y_1,z_1) - \epsilon K(y_1,z_1) - \delta L(y_2,z_1,x_1) \quad (1.6)
\]
Then, it defines a Lie triple system for $\delta = +1$ and an anti-Lie triple system for $\delta = -1$. We then note

$$L_\sigma = \text{span} \left\{ L(x,y), \frac{\delta K(z,w)}{\epsilon L(y,z)} \right\} \text{ for } x, y, z, w, u, v \in V \quad (1.7)$$

is a Lie subalgebra of $\text{Mat}_2(\text{End}(V))$ where $B$ for an associative algebra $B$ implies an $\text{Lie}$ algebra with bracket; $\{x,y\} = xy - yx$. We note also then

$$\hat{D} = \left[ L(x,y), \frac{\delta K(z,w)}{\epsilon L(y,z)} \right]$$

is a derivation of the triple system. Setting

$$L_\sigma = \text{span} \left\{ x = \left( \begin{array}{c} x \, y \, w \\ \end{array} \right) \text{ for } x, y, w \in V \right\} \quad (1.9)$$

then $L$ defined by

$$L = L_\sigma \oplus L_\tau (1.10)$$

gives a Lie algebra for $\delta = +1$ and a Lie superalgebra for $\delta = -1$ where

$$L_\sigma = \{ D \mid D \text{ is a derivation of } L \}, (1.11)$$

i.e., $D$ satisfies

$$D[X_1, X_2, X_3] = [D X_1, X_2, X_3] + [X_1, D X_2, X_3] + [X_1, X_2, D X_3] \quad (1.12a)$$

and hence induces also

$$[D, [X,Y]] = [DX,Y] + [X,DY]. \quad (1.2b)$$

if we define the bracket by

$$[D \circ X_1, D \circ X_2] = ([D \circ X_1, D \circ X_2] + L(X_1, X_2)) \oplus (D X_1, X_2, D X_3). \quad (1.13)$$

where

$$[D_1, D_2] = D_1 \circ D_2 \oplus D_2 \circ D_1 \quad \text{ and }$$

$$L(X_1, X_2) = \left[ \begin{array}{c} x_1 \\ y_1 \\ \end{array} \right] \left[ \begin{array}{c} x_2 \\ y_2 \end{array} \right] = \left[ \begin{array}{c} L(x_1, y_2) - \delta L(x_2, y_1) + \frac{\delta K(x_1, x_2)}{\epsilon L(y_1, y_2)} \\ \epsilon L(y_1, y_2) - \delta L(y_2, y_1) - \frac{\delta K(y_1, y_2)}{\epsilon L(x_1, x_2)} \end{array} \right] \quad (1.14)$$

Note that the endomorphism $L(X,Y)$ is then an inner derivation of the triple system.

Since $L = L_\sigma$, we will mainly discuss a subsystem $\hat{L}$ of $L$, given by rather than the larger $L$. Then, $\hat{L}$ is 5-graded

$$\hat{L} = L_2 \oplus L_1 \oplus L_0 \oplus L_0 \oplus L_1 \oplus L_2 \quad (1.16)$$

where

$$L_2 = \text{span} \left\{ \left[ \begin{array}{c} 0 \\ -\epsilon K(x,y) \end{array} \right] \text{ for } x, y \in V \right\} \quad (1.17a)$$

$$L_1 = \text{span} \left\{ \left[ \begin{array}{c} 0 \\ x \end{array} \right] \text{ for } x \in V \right\} \quad (1.17b)$$

$$L_0 = \text{span} \left\{ \left[ \begin{array}{c} L(x,y) \\ 0 \\ \end{array} \right] \text{ for } x, y \in V \right\} \quad (1.17c)$$

$$L_{-1} = \text{span} \left\{ \left[ \begin{array}{c} x \\ 0 \end{array} \right] \text{ for } x \in V \right\} \quad (1.17d)$$

$$L_0 = \text{span} \left\{ \left[ \begin{array}{c} 0 \\ \delta K(x,y) \end{array} \right] \text{ for } x, y \in V \right\} \quad (1.17e)$$

Here, we utilized the following Proposition for some of its proof.

**Theorem 1:** \([\text{K-O.}], \quad \text{K-M-O.}\) Let $(V, (xyz))$ be a $(\epsilon, \delta)$-Freudenthal-Kantor triple system with an endomorphism $P$ such that $P^2 = -\epsilon \delta Id$ and $P(xyz) = (PxPyPz)$ Then, $(V, [xyz])$ is a Lie triple system (for $\delta = 1$) and anti-Lie triple system (for $\delta = -1$) with respect to the product

$[xyz] = (\delta xPyz) - \delta(yPxz) + (\delta(xPz) - yPzx)$.

In passing, we note that the standard $\hat{L} = \sum_{i=5}^{5} \oplus L_{i}$ is a result of Theorem 1 immediately with

$$P = \left( \begin{array}{cc} 0 & -\delta \\ -\epsilon & 0 \end{array} \right)$$

and $x \to X$ etc.

Next, we introduce $\theta, \sigma(\lambda) \in \text{End}(\hat{L})$ for any $\lambda \in F (\lambda \neq 0)$, being non-zero constant by

$$\sigma(\lambda) = \text{Id}$$

where $\text{Id}$ is the identity mapping

(i) $\sigma (1) = \text{Id}$, $\theta = \text{Id} (1.19a)$

where $\text{Id}$ is the identity mapping

(ii) $\theta^2 = -\epsilon \delta \text{Id}$ for $L_0$, but $\theta^2 = \text{Id}$ for $L_0 (1.19b)$

(iii) $\sigma(\mu) \sigma(\nu) = \sigma(\mu \nu)$ for $\mu, \nu \in F$, $\mu \nu \neq 0$

(iv) $\sigma(\lambda) \theta(\lambda) = \theta$ for any $\lambda \in F$, $\lambda \neq 0. (1.19d)$

We call the group generated by $\sigma(\lambda)$ and $\theta$ satisfying these conditions simply as $D(\epsilon, \delta)$ due to a lack of better terminology. If the group $D(\epsilon, \delta)$ yields a finite sub-group $\sigma(\lambda)$ of $D(\epsilon, \delta)$, then $\sigma(\lambda)$ defines $D_{\text{C3}}$ for $\epsilon = \delta$ but $S_3$ for $\epsilon = -\delta$, as they have noted already [13].

Conversely any 5-graded Lie algebra (or Lie superalgebra) with such automorphism $\sigma$ and $\theta$ satisfying Eqs. $(1.19)$ lead essentially to a $(\epsilon, \delta)$-FKTS in $L_{\delta}$ with a triple product defined by $[x, y, z] = [(x, \theta y), z]$ for $x, y, z \in L_{\delta}$ [6].

We note that the corresponding local symmetry of $D(\epsilon, \delta)$ yields a derivation of $\hat{L}$, given by

$$h = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) (1.20)$$
which satisfies
\[ h[X,Y,Z] = [hX,YZ] + [X,hYZ] + [XY,hZ](1.2a) \]
as well as \[ [hX,Y] = [hX,Y] + [X,hY](1.2b) \] for \( X,Y \in W \).

We can find a larger automorphism group of \( L \), if we impose some additional conditions. First suppose that \( K(x,y) \) is now expressed as
\[ K(x,y) = \epsilon \delta L(x,y) - \epsilon L(x,y) (1.22) \]
for any \( x,y \in V \). We call then the triple system to be a special \((\epsilon, \delta)\) FKTS [13]. Moreover for the case of special \((\epsilon, \delta)\) FKTS (i.e. \( \epsilon = \delta = 1 \)), the automorphism group of \( L \) turns out to be a larger \( SL(2,F)(=Sp(2,F)) \) group which contains \( D(\epsilon, \delta) \) as its subgroup. In this case, the triple system \([W, W, W]\) becomes invariant under
\[ \left[ \begin{array}{c} x \\ y \\ z \end{array} \right] \to U \left[ \begin{array}{c} x \\ y \\ z \end{array} \right] \]
for any 2x2 \( SL(2,F) \) matrix \( U \), i.e.
\[ U = \left[ \begin{array}{cc} a & \beta \\ \mu & v \end{array} \right], \quad \det U = av - \beta \mu = 1. \]

Also, the associated Lie algebras or superalgebras are BC\(_r\)-graded algebra of type \( C_t \).

Finally, we consider a ternary system \((V, \langle xy, zy, xz \rangle) \) where \( xy \) and \( yz \) are binary and ternary products, respectively, in the vector space \( V \). Suppose that they satisfy

1. The triple system \((V, \langle xy, zy, xz \rangle) \) is a (-1,1) FKTS.
2. The binary algebra \((V, \langle xy \rangle) \) is unital and involution \( \langle xy \rangle = \langle yx \rangle \) with the involutive map \( \langle \rangle \)
3. The triple product \( \langle xy, zy, xz \rangle \) is expressed in terms of the bi-linear products by
\[ \langle xy, zy, xz \rangle = \langle z\langle x, y \rangle - \langle x, z \rangle \rangle + \langle x\langle y, z \rangle - \langle y, x \rangle \rangle \]

We may call the ternary system \((V, \langle xy, zy, xz \rangle) \) to be Allison-ternary algebra or simply \( A \)-ternary algebra, since \( A = (V, \langle xy \rangle) \) is then the structurable algebra [14-16]. This case is of great interest, first because structurable algebras exhibit a trinity relation [16,17], and second because we can construct another type of Lie algebras independently of the standard construction of \((-1,1)\) FKTS, which is \( S_3 \)-invariant and of \( BC_2 \)-graded Lie algebra of type \( B_t \). The relationship between the Lie algebra constructed in the new way and that given as in Eq.(1.17) is by no means transparent. Note that the group \( D(-1,1) \) contains \( S_3 \) but not \( S_1 \) symmetry. We may show that if the field \( F \) contains the square root of \( -1 \), then Eqs.(1.17) can be prolonged to yield the Lie algebra for the structurable algebra.

The Lie algebras constructed as in Eqs.(1.13) and (1.14) is also a \( BC_2 \)-graded Lie algebra of type \( B_t \) without assuming \( \sqrt{-1} \in F \). Also, if \( F \) is an algebraically closed field of characteristic zero, then any simple Lie algebra is known to be \( S_3 \)-invariant and can be constructed by some structurable algebra, so that any such Lie algebra is also a \( BC_2 \)-graded Lie algebra of type \( B_t \) (as well as of type \( C_t \)). Of course, the underlying \( sl(2) \) symmetry is different for both \( B_t \) and \( C_t \) cases. Roughly speaking, it seems that these concept are a version of Lie algebra theory corresponding with a Galois algebra group of algebraic numbers theory.

The contents of this section is a corwork with Prof. Okubo, and the details will be discussed in other papers.

Hermitian triple systems

For a geometrical object based on triplet systems in this section, first we note that the symmetric bounded domains are a one to one, correspondence to hermite Jordan triple systems, such that a certain trace form is positive definite hermitian. Hence as a generalization of these triplet systems, we are interesting to investigate for structure theory of hermitian generalized Jordan triple systems, in particular, the case of hermitian generalized Jordan triple systems of second order (i.e. hermitian (-1,1) -Freudenthal-Kantor triple system ).

We shall considered with algebras and triplet systems which are infinite over a complex number field, unless otherwise specified in this section.

Definition: A triple system \( V \) is said to be a \(-1\)-generalized Jordan triple system of second order if relations (0)−(iv) satisfy;
1) \( V \) is a Banach space,
2) \( \{L(ab),L(c,d)\} = L(<abc>,d) - L(c,<bad>) \)
3) \( K(<abc>,d)+K(c,<abd>) + \delta K(a,K(c,d)b) = 0 \)
4) \( \langle ab \rangle \) is a positive hermitian operator with a hermitian metric
\[ (x,y) = trace L(x,y), (2.1) \]
that is, \( L(x,y)z = \langle z, L^*(x,y)w \rangle \), and \( (x,y) = \langle y, x \rangle \). In particular, if a triple system \( V \) satisfies the condition (i),(ii),(iii),(iv) and (v), then it is said to be a hermitian \(-1\)-generalized Jordan triple system.

Furthermore, as a generalization of the generalized Jordan triple system of second order, it is said to be a \( \epsilon \)-\( \delta \)-Freudenthal-Kantor triple system if the following relations satisfy;
1) \( [L(ab),L(c,d)] = L(<abc>,d) - L(c,<bad>) \)
2) \( K(<abc>,d)+K(c,<abd>) + \delta K(a,K(c,d)b) = 0 \)
3) \( K(a,b)c = <abc> - \delta <cba> \)

We note that these identities (i) and (ii) are equivalent to the identities (1.2) and (1.3) in sections 1 [2,7].

Example 1: Let \( V \) be a \( J^* \)-triple (for the definition, to Loos [3]. Then \( V \) is a hermitian \(-1\)-generalized Jordan triple system of second order, because they satisfy the condition (i) and a special case of (ii), that is, \( K(x,y)z = <xyz> - \langle yxz \rangle = 0 \) (identically zero).

Example 2: Let \( T^* \) be the space of diagonal matrix of \( nxn \) with element of the complex number. Then \( T^* \) is a \(-1\)-generalized Jordan triple system of second order, with respect to the product and the norm
\[ <xyz> = x^{\tau}z + z^{\tau}x - \tau x^{\tau}z \]
for all $x,y,z \in T^*_{n,n}$

$$
||x|| = \max(|\lambda_i| |\mu_j|)
$$

where $x = \sum(\lambda_i \mu_j + \mu_j \lambda_i)$, $\lambda_i, \mu_j \in R$ (real number field) and $\epsilon_i$ are matrix unit element of $T^*_{n,n}$ and $x^T$ is the transpose of $x$.

Next let $V$ be a *-generalized Jordan triple system. Then we may define the notion of a tripotent and a bitripotent as follows.

**Definition:** It is said to be a tripotent of $V$ if

$$
<c_1,c_2,c_3> = -\frac{1}{2}c_1 - \frac{1}{2}c_2 - \frac{1}{2}c_3,
$$

and other products are zero.

**Definition:** It is said to be a bitripotent of $V$ if a pair $(c_1,c_2)$ of the tripotens satisfy the relations

$$
<c_1,c_2,c_3> = ac_1
$$

and other products are zero.

It is said to be a strongly bitripotent of $V$ if a pair $(c_1,c_2)$ of the tripotens satisfy the relations

$$
<c_1,c_2,c_3> = ac_1
$$

and other products are zero.

For *-generalized Jordan triple system $V$, we can define a norm $||x||$ as follows;

$$
||x|| = \max(|\lambda_i| ||f||) = \sum(\lambda_i),
$$

where $\epsilon_i$ are tripotents or bitripotens. We note that $\epsilon_i$ are tripotents or bitripotens if $\epsilon_i$ is the unit element of matrix.

Morepreciously speaking, we have

$$
||x|| = \max(|\lambda_i| ||f||)
$$

for $x = \sum(\lambda_i \mu_j + \mu_j \lambda_i)$, where the notations are denoted by $\lambda_i,\mu_j$ (real number field) and also $E_i$ means that $i$ and other element is zero. $E_i$ and $\sqrt{-E_i}$ are tripotens, i.e., $\langle E_i E_i E_i \rangle = E_i$ and $\langle \sqrt{-E_i} \rangle = \sqrt{-E_i}$.

Furthemore, we note that $(E_i, \sqrt{-E_i})$ are bitripotens.

Then we have

$$
||x|| = \max(|\lambda_i| ||f||)
$$

for $x = \sum(\lambda_i \mu_j + \mu_j \lambda_i)$, where the notations are denoted by $\lambda_i,\mu_j$ (real number field) and also $E_i$ means that $i$ and other element is zero. $E_i$ and $\sqrt{-E_i}$ are tripotens, i.e., $\langle E_i E_i E_i \rangle = E_i$ and $\langle \sqrt{-E_i} \rangle = \sqrt{-E_i}$.

These show that $D^*_{n,k}$ is a hermitian *-generalized Jordan triple system of second order, that is, a hermitian (1,1)-FKTS.

**Examples 4:** Let $S^*_{2n}$ be the set of all $2n \times k$ matrices with operation

$$
<XYZ> := X(\overline{Y})^T Z + Z(\overline{Y})^T X + Y(\overline{X})^T Y
$$

where $X = \{x_{ij}\}$, $x_{ij}$ is an $nk \times n$ matrix, and $(\cdot)^T$.

Then $S^*_{2n}$ is a hermitian *-generalized Jordan triple system of second order.

In fact, the elements $\frac{1}{\sqrt{2}} E^{(1)}_{ij}$ and $\frac{1}{\sqrt{2}} E^{(2)}_{ij}$ are tripotens.

For any element $X$ of $S^*_{2n}$, we may represent as follows;

$$
X = \sum(\lambda_i E^{(1)}_{ij} + \mu_j E^{(2)}_{ij})
$$

and the norm is defined by

$$
||X|| := \max(|\lambda_i| ||E^{(1)}_{ij}||) = \max(|\mu_j| ||E^{(2)}_{ij}||)
$$

where the notations are denoted by $\lambda_i,\mu_j$.

By straightforward calculations as well as Example 2.3, we can show that $S^*_{2n}$ is a hermitian *-generalized Jordan triple system of second order.

**Example 4:** Let $A^*_{kn} \oplus A^*_{nk}$ be the set for all pairs $X = \{x_{ij}\}$, where $x_{ij}$ is a $kn \times n$ matrix and $x_{ij}$ is a $n \times k$ matrix with operation given by formula

$$
<XYZ> := \begin{bmatrix} x_{ij}y_{ji} & z_{ij}y_{ji} & z_{ij}y_{ji} \\ z_{ij}y_{ji} & z_{ij}y_{ji} & z_{ij}y_{ji} \\ z_{ij}y_{ji} & z_{ij}y_{ji} & z_{ij}y_{ji} \end{bmatrix}
$$

Then $A^*_{kn} \oplus A^*_{nk}$ is a hermitian *-generalized Jordan triple system of second order.

In fact, for $X = \{x_{ij}\}$, we have

$$
<XY> = \text{trace}(X,Y) = \text{trace}(x_{ij}y_{ji} + x_{ij}y_{ji} + x_{ij}y_{ji})
$$

hence it follows that the trace form $(\cdot)$ is positive definite.

For any element of $A^*_{kn} \oplus A^*_{nk}$, we may represent as follows;

$$
X = \frac{1}{2}(\lambda_i E^{(1)}_{ij} + \mu_j E^{(2)}_{ij}) + \frac{1}{2}(\lambda_i E^{(1)}_{ij} + \mu_j E^{(2)}_{ij})
$$
and \((E_{\alpha}^{(1)}, E_{\alpha}^{(2)}) \in A_{m}^{\text{tr}} \otimes A_{n}^{\text{tr}}\) is a tripotent.

Furthermore, the norm is defined by
\[
||X|| = \sqrt{\max(|J_{\alpha}^{(1)}|, |J_{\alpha}^{(2)}|, |\mu_{\alpha}^{(1)}|, |\mu_{\alpha}^{(2)}|)}.
\]

**Example 6:** Let \(C_{m}^{*} \otimes A_{n}^{*}\) be the set of all pairs \(X = \left( \begin{smallmatrix} x_{1} \\ \vdots \\ x_{n} \end{smallmatrix} \right)\), where \(x_{i}\) is a skew-symmetric \(n \times n\) matrix and \(x_{2}\) is a column, i.e., is a \(n \times 1\) matrix, with operation given by formula,
\[
<XYZ> = \left( x_{1}y_{1}z_{1} + x_{2}y_{2}z_{2} - x_{2}y_{1}z_{2} - x_{1}y_{2}z_{1} \right) (2.6)
\]

Then \(C_{m}^{*} \otimes A_{n}^{*}\) is a hermitian \(*\)-generalized Jordan triple system of second order.

In fact, we put \(C_{m}^{*} := \text{ the matrix of } (i,j)\) -element is \(I, (i,i)\) -element is \(-1\) and other element is zero \((1 \leq i \leq n)\), further more \(c_k := \left( 0 \cdots -1 \cdots 0 \right) \text{T(k the element is 1)}\). Then \(C_{m}^{*}\) are tripotents and also \(C_{m}^{*}\) are tripotents but these are not bitripotents. For any element of \(C_{m}^{*} \otimes A_{n}^{*}\), we may represent as follow;
\[
X = \Sigma (\lambda_{\alpha}^{(1)}c_{\alpha} + \mu_{\alpha}^{(1)}c_{\alpha}) + \Sigma (\lambda_{\alpha}^{(2)}c_{\alpha} + \mu_{\alpha}^{(2)}c_{\alpha})
\]

and the norm is defined by
\[
||X|| = \sqrt{\max(|\lambda_{\alpha}^{(1)}|, |\lambda_{\alpha}^{(2)}|, |\mu_{\alpha}^{(1)}|, |\mu_{\alpha}^{(2)}|)}.
\]

In the end of this section, we note a Peirce decomposition as follows \((12)\) for the case of \(d = 1\).

**Theorem 1:** For \(d = \pm 1\) and hermitian \((-1, \delta)\) Freudenthal-Kantor triple system \(V\) with a tripotent \(C\) s.t. \(<<c,c>> = c\) we have
\[
V = V_{+} \oplus V_{-} \oplus V_{0}
\]

where denoted by \(V_{+} = \{ x | N_{x} = 0, T_{x} = 0 \}\), \(V_{-} = \{ x | N_{x} \neq 0, T_{x} = 0 \}\) and also defined by \(R_{c}(x,y)z = <zxy>, N_{x} = \lambda(c,c) - \delta R(c,c)\) and \(T_{c}(x)(1 + \delta) R(c,c) + Id\).

This section is a cowork with Dr M. Sato with an application to physics and the details will be considered in future paper \[18\].

### Concluding Remark

In this section, we shall briefly describe a correspondence with the triple systems and the Lie algebras or superalgebras of simple classical type associated with their triple systems \([1,2,7,10,11]\).

Let \(V\) be the matrix set of \(\text{Mat}(m,n,F)\) and the triple product is defined by
\[
<xyz> = xyz + \delta (yz^{T}x - xz^{T}y) x,y,z \in V^{(*)}
\]

Then there exists 4 cases of \((-1, \delta)\)-Freudenthal-Kantor triple systems with \(d = \pm 1\). and \(\delta = \pm 1\).

The standard embedding Lie algebras or superalgebras \(L = L(W,W) \oplus W\), where \(W = V \oplus V\) (cf. section 1) associated with the triple product \((*)\) are appeared by 4 types as follows;

(i) 5 graded Lie algebra \((\delta = 1, \mu = 1)\)
\[
B_{m}^{n+1} = so(2m+2l+1) \; (n = 2l+1), \; D_{m}^{n+1} = so(2m+2l) \; (n = 2l).
\]

(ii) 3 graded Lie algebra \((\delta = 1, \mu = 0)\)
\[
A_{m}^{n+1} = sl(n+m).
\]

(iii) 5 graded Lie superalgebra \((\delta = -1, \mu = 1)\)
\[
. . . . . B_{l,m} = osp(2l+1|2m) \; (n = 2l+1), \; D_{l,m} = osp(2l|2m) \; (n = 2l).
\]

In particular, if \(m = 1\), then the triple product \(<xyz> = <z,y,x> + <z,x,y>\) is a \((-1,-1)\)-Freudenthal-Kantor triple system satisfying \(K(x,z) = <xyz> + <zyx> = 2<x,z>y\), and so \(dim([x,z]span = 1\) (called a balanced triple system), where \(<x,y> = x_{1}y_{1} + \ldots + x_{n}y_{n}\).

(iv) 3 graded Lie superalgebra \((\delta = -1, \mu = 0)\)
\[
A(m-1,n-1) = sl(n|m) \; (m \neq n), \; A(m-1|m-1) = psd(\; m|m) \; (m = n).
\]

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