Schrödinger equations with time-dependent strong magnetic fields

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To the memory of late Professor Vladimir S. Buslaev

Abstract

We consider $d$-dimensional time dependent Schrödinger equations $i\partial_t u = H(t)u$, $H(t) = -(\partial_x - iA(t,x))^2 + V(t,x)$ in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$ of square integrable functions. We assume $V(t,x)$ and $A(t,x)$ are almost critically singular with respect to spatial variables $x \in \mathbb{R}^d$ both locally and at infinity for the operator $H(t)$ to be essentially selfadjoint on $C_0^{\infty}(\mathbb{R}^d)$. In particular, when magnetic fields $B(t,x)$ produced by $A(t,x)$ are very strong at infinity, $V(t,x)$ can explode to the negative infinity like $-\theta|B(t,x)| - C(|x|^2 + 1)$ for some $\theta < 1$ and $C > 0$. We show that equations uniquely generate unitary propagators in $\mathcal{H}$ under suitable conditions on the size and singularities of time derivatives of potentials $\dot{V}(t,x)$ and $\dot{A}(t,x)$.

1 Introduction, Theorem

We consider time-dependent Schrödinger equations

$$i\partial_t u = H(t)u(t) \equiv -\nabla_{A(t)}^2 u + V(t,x)u, \quad \nabla_{A(t)} = \nabla - iA(t,x)$$

(1.1)

in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$ of square integrable functions, where $A(t,x) = (A_1(t,x),\ldots,A_d(t,x)) \in \mathbb{R}^d$ and $V(t,x) \in \mathbb{R}$ are respectively magnetic vector and electric scalar potentials. We study the existence and the uniqueness of unitary propagators for Eqn. (1.1), continuing the previous work [13] of the second author.

In accordance with the requirement of quantum mechanics we say that a function $u(t,x)$ of $(t,x) \in \mathbb{R} \times \mathbb{R}^d$ is a solution of (1.1) if it satisfies the following properties:

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(1) $u(t, \cdot)$ is a continuous function of $t \in \mathbb{R}$ with values in $\mathcal{H}$ and $\|u(t, \cdot)\|_{L^2}$ is independent of $t \in \mathbb{R}$.

(2) $u(t, x)$ satisfies Eqn. (1.1) in the sense of distributions.

Suppose that there exists a dense subspace $\Sigma \subset \mathcal{H}$ such that, for every $s \in \mathbb{R}$ and $\varphi \in \Sigma$, Eqn. (1.1) admits a unique solution $u(t, x)$ which satisfies the initial condition $u(s, x) = \varphi(x)$ and that $u(t, \cdot) \in \Sigma$ for every $t \in \mathbb{R}$. Then the solution operator $\Sigma \ni \varphi \mapsto u(t, \cdot)$ extends to a unitary operator $U(t, s)$ in $\mathcal{H}$ and the two parameter family of operators $\{U(t, s): -\infty < t, s < \infty\}$ satisfies the following properties:

(a) $U(t, s)$ is unitary and $(t, s) \mapsto U(t, s) \in B(\mathcal{H})$ is strongly continuous.

(b) $U(t, s)U(s, r) = U(t, r)$ and $U(t, t) = 1$ for every $-\infty < t, s, r < \infty$.

(c) $U(t, s)\Sigma = \Sigma$ and, for every $\varphi \in \Sigma$, $u(t, x) = (U(t, s)\varphi)(x)$ satisfies Eqn. (1.1) in the sense of distributions.

**Definition 1.1.** We say a two parameter family of operators $\{U(t, s): -\infty < t, s < \infty\}$ is a unitary propagator for (1.1) on a dense set $\Sigma$ if it satisfies properties (a), (b) and (c) above.

Thus, the existence of a unique unitary propagator on a dense subspace of $\mathcal{H}$ implies that Schrödinger equation (1.1) generates a unique quantum dynamics on $\mathcal{H}$. When $A$ and $V$ are $t$-independent, it is well known that the existence of a unique unitary propagator on $\mathcal{H}$ is equivalent to the essential selfadjointness of Hamiltonian $-\nabla^2_A + V$ on $C_0^\infty(\mathbb{R}^d)$. The problem of essential selfadjointness has long and extensively been studied by many authors and it has an extensive literature. We record here following two theorems, Theorem 1.2 of Leinfelder and Simader([8]) and Theorem 1.3 of Iwatsuka([2]) which are relevant to the present work. We need some notation: $(1 + |x|^2)^{1/2} = \langle x \rangle$; $L^p = L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$ are Lebesgue spaces and $L^p_{loc} = L^p_{loc}(\mathbb{R}^d)$ are their localizations; $\|u\|_p$ is the norm of $L^p$, $\|u\| = \|u\|_2$ and $(u, v)$ is the inner product of $u, v \in \mathcal{H}$. A function $W(x)$ is said to be of Stummel class if it satisfies the property that

$$\lim_{\varepsilon \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x - y| < \varepsilon} \frac{|W(y)|^2}{|x - y|^{d-4}} dy = 0,$$

(1.2)

where $|x - y|^{4-d}$ should be replaced by $|\log |x - y||$ if $d = 4$ and by 1 if $1 \leq d \leq 3$. 

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Theorem 1.2. Let $A \in L^4_{\text{loc}}$ and $\nabla \cdot A \in L^2_{\text{loc}}$. Let $V = V_1 + V_2$ with $V_1 \in L^2_{\text{loc}}$ and $V_2$ of Stummel class. Suppose that, for a constant $C_* > 0$,

\[ V_1(x) \geq -C_*(x)^2. \quad (1.3) \]

Then, $H = -\nabla^2 A + V$ is essentially selfadjoint on $C^\infty_0(\mathbb{R}^d)$.

It can be easily seen that conditions in Theorem 1.2 are also necessary as far as smoothness is concerned. However, condition (1.3) on $V$ at infinity can be substantially relaxed if the magnetic field $B(x) = (B_{jk}(x))$ produced by $A$, $B_{jk} = \partial_j A_k - \partial_k A_j$, $\partial_j = \partial/\partial x_j$, grows rapidly at infinity. We define

\[ |B(x)| = \left( \sum_{j<k} |B_{jk}(x)|^2 \right)^{\frac{1}{2}}. \]

Theorem 1.3. Let $\rho(r)$ be a continuous function of $r \geq 0$ such that

\[ \int_0^\infty \rho(r)^{-1}dr = \infty. \]

Suppose that $A$ and $V$ are $C^\infty$ and they satisfy that, for constants $C_\alpha$,

\[ |\partial_x^\alpha B(x)| \leq C_\alpha \rho(|x|^{\alpha}(|B(x)| + 1)), \quad |\alpha| = 1, 2; \quad (1.4) \]

\[ |B(x)| + V(x) \geq -\rho(|x|)^2. \quad (1.5) \]

Then, $H = -\nabla^2 A + V$ is essentially selfadjoint on $C^\infty_0(\mathbb{R}^d)$.

We remark that, by virtue of condition (1.4), magnetic fields which behave too wildly at infinity, e.g. $|B(x)| \geq C \exp((x)^{2+\varepsilon})$ or $|B(x)| = C\cos(e^{x^{2+\varepsilon}})$ for some $C > 0$ and $\varepsilon > 0$, are excluded in Theorem 1.3. To the best knowledge of authors, it is unknown whether or not Theorem 1.3 remains true without this condition.

We now state main results of this paper. We want to remark beforehand that, by virtue of assumptions on time derivatives, $A(t, x)$ and $V(t, x)$ in following theorems may be considered as perturbations of time frozen potentials $A(t_0, x)$ and $V(t_0, x)$ respectively, $t_0$ being chosen arbitrarily.

Definition 1.4. $M(\mathbb{R}^d)$ is the space of real valued functions $Q(x)$ of class $C^1(\mathbb{R}^d)$ which satisfy for a positive constant $C > 0$ that

\[ Q(x) \geq C(x) \text{ and } |\nabla Q(x)| \leq C(x)Q(x). \quad (1.6) \]
For $Q \in M(R^d)$, $-\Delta + Q(x)^2$ is essentially selfadjoint on $C_0^\infty(R^d)$ (see Theorem 1.2) and hereafter $L_Q$ will denote its unique selfadjoint extension. $L_Q \geq -\Delta + C^2 x^2$ and $L_Q$ is positive definite; we have

$$D(L_Q) = \{u \in \mathcal{H}: \Delta u, Q \nabla u, Q^2 u \in \mathcal{H}\}, \quad C^{-1}\|L_Qu\| \leq \|\Delta u\| + \|Q \nabla u\| + \|Q^2 u\| \leq C\|L_Qu\|, \quad u \in D(L_Q)$$

for a constant $C > 0$ (see the proof of Lemma 4.1).

For Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, $B(\mathcal{X}, \mathcal{Y})$ is the Banach space of bounded operators from $\mathcal{X}$ to $\mathcal{Y}$ and $B(\mathcal{X}) = B(\mathcal{X}, \mathcal{X})$. We say $f(t, x)$ is of class $C^\alpha(\mathcal{R}^d_x)$ if it is of class $C^\alpha$ with respect to variables $x \in \mathcal{R}^d$. Multiplication operators by $V(t, \cdot), A(t, \cdot)$ and etc. are denoted by $V(t), A(t)$ and etc. respectively; $\dot{A}(t, x) = \partial_t A(t, x)$ and $\dot{V}(t, x) = \partial_t V(t, x)$ are time derivatives. The letter $C$ denotes various constants whose exact values are not important and they may differ at each occurrence.

First two theorems, Theorems 1.5 and 1.6, may respectively be thought of as time dependent versions of Theorem 1.2 and its form version. $I$ is an interval. Under the assumption of Theorem 1.5, operators $H_0(t) = -\nabla^2_{A(t)} + V(t, x) + C(t)\langle x \rangle^2$ and $H(t) = -\nabla^2_{A(t)} + V(t, x)$ are essentially selfadjoint on $C_0^\infty(R^d)$ by virtue of Theorem 1.2. We denote their selfadjoint extensions again by $H_0(t)$ and $H(t)$.

**Theorem 1.5.** Suppose $A$ and $V$ satisfy following conditions:

1. $A(t, \cdot) \in L^4_{\text{loc}}$ and $\nabla_x A(t, \cdot) \in L^2_{\text{loc}}$ for all $t \in I$.

2. $V = V_1 + V_2$ with $V_1$ and $V_2$ such that $V_1(t, \cdot) \in L^2_{\text{loc}}$ for $t \in I$ and $V_2(t, \cdot)$ of Stummel class uniformly for $t \in I$. There exist a continuous function $C(t)$ and $Q(x) \in M(R^d)$ such that

$$V_1(t, x) + C(t)\langle x \rangle^2 \geq Q(x)^2, \quad (t, x) \in I \times R^d. \quad (1.9)$$

3. For a.e. $x \in R^d$, $A(t, x)$ and $V(t, x)$ are absolutely continuous (AC for short in what follows) with respect to $t \in I$ and multiplication operators in $\mathcal{H}$ by following functions are all $L_Q$-bounded uniformly for $t \in I$:

$$\dot{V}(t, x), \quad \nabla_x A(t, x), \quad \dot{A}(t, x)^2, \quad \partial_{x_j}\{A(t, x)^2\}, \quad j = 1, \ldots, d.$$ 

Then, following statements are satisfied:

(a) $H_0(t)$ has $t$-independent domain $\mathcal{D}$ such that $\mathcal{D} \subset D(H(t))$. We equip $\mathcal{D}$ with the graph norm of $H_0(t_0), t_0 \in I$ being arbitrary.
There uniquely exists a unitary propagator \( \{ U(t, s) : t, s \in I \} \) for (1.1) on \( \mathcal{H} \) with following properties: \( U(t, s) \in \mathcal{B}(\mathcal{D}) \); for \( \varphi \in \mathcal{D} \), \( U(t, s)\varphi \) is continuous in \( \Sigma \) with respect to \((t, s)\), of class \( C^1 \) in \( \mathcal{H} \) and it satisfies

\[
i\partial_t U(t, s)\varphi = H(t)U(t, s)\varphi, \quad i\partial_s U(t, s)\varphi = -U(t, s)H(s)\varphi.
\]

(1.10)

A remark on condition (1.9) which corresponds to (1.3) of Theorem 1.2 is in order since they look differently from each other. As was mentioned above we are considering Eqn. (1.1) when \( A(t, x) \) and \( V(t, x) \) satisfy conditions of Theorem 1.2 for every fixed \( t \in \mathbb{R} \), in particular, that

\[
V_1(t, x) \geq -C_* (t) \langle x \rangle^2
\]

for a continuous \( C_*(t) \). Then, if we choose \( C(t) = C_* (t) + C \), \( V_1(t, x) \) satisfies (1.9) with \( Q(x)^2 = C(\langle x \rangle)^2 \in M(\mathbb{R}^d) \), \( C \) being an arbitrarily large constant. However, this is the worst case conceivable and \( V_1(t, x) \) may rapidly grow to positive infinity as \( |x| \to \infty \), in which case \( V_1(t, x) \) certainly satisfies (1.11). If \( V_1(t, x) \) increases the faster as \( |x| \to \infty \), then \( Q(x) \) of (1.9) may be taken the larger, condition (3) becomes the less restrictive and the class of potentials accommodated by the theorem becomes the wider. Condition (1.9) is formulated for studying these cases simultaneously. Similar remark applies to conditions (1.13), (1.23) and (1.24) in following theorems.

When \( V \) is spatially more singular than in Theorem 1.5, we use quadratic form formalism. The following is a form version of Theorem 1.5. A function \( W(t, x) \) is said to be of Kato class uniformly for \( t \in I \), if

\[
\lim_{\varepsilon \to 0} \sup_{t \in I, x \in \mathbb{R}^d} \int_{|x-y|<\varepsilon} \frac{|W(t, y)|}{|x-y|^{d-2}} dy = 0,
\]

(1.12)

where \(|x-y|^{2-d}\) should be replaced by \(|\log |x-y|| \) if \( d = 2 \) and by 1 if \( d = 1 \). We write \( q(u, u) = q(u) \) for quadratic forms \( q(u, v) \).

**Theorem 1.6.** Suppose that \( A \) and \( V \) satisfy following conditions:

1. \( A(t, \cdot) \in L^2_{\text{loc}} \) for every \( t \in I \).
2. \( V(t, x) = V_1(t, x) + V_2(t, x) \) with \( V_1 \) such that \( V_1(t, \cdot) \in L^1_{\text{loc}} (\mathbb{R}_x) \) for all \( t \in I \) and \( V_2(t, \cdot) \) of Kato class uniformly for \( t \in I \). There exist a continuous function \( C(t) \) and \( Q \in M(\mathbb{R}^d) \) such that

\[
V_1(t, x) + C(t) \langle x \rangle^2 \geq Q(x)^2, \quad t \in I.
\]

(1.13)
Then, following statements are satisfied:

(a) The quadratic form $q_0(t)$ defined on $C_0^\infty(\mathbb{R}^d)$ by

$$q_0(t)(u) = \int_{\mathbb{R}^d} (|\nabla_{A(t)} u|^2 + (V(t, x) + C(t) \langle x \rangle^2)|u|^2)dx$$

is strictly positive and closable; the closure $[q_0(t)]$ has domain $\mathcal{Y}$ independent of $t \in I$ and $\mathcal{Y} \subset D(L_0^2)$. We equip $\mathcal{Y}$ with the inner product $\langle q_0(t_0) \rangle(u, v)$ by choosing $t_0$ arbitrarily and denote by $\mathcal{X}$ its dual space with respect to the inner product of $\mathcal{H}$. We have $H(t) = -\nabla_{A(t)}^2 + V(t) \in B(\mathcal{Y}, \mathcal{X})$ and $t \to H(t) \in B(\mathcal{Y}, \mathcal{X})$ is norm continuous.

(b) There uniquely exists a unitary propagator for (1.1) on $\mathcal{Y}$ with following properties: $U(t, s) \in B(\mathcal{Y})$; for $\varphi \in \mathcal{Y}$, $U(t, s)\varphi$ is continuous in $\mathcal{Y}$ with respect to $(t, s)$, of class $C^1$ in $\mathcal{X}$ and satisfies equations (1.10).

Before stating time dependent versions of Theorem 1.3, we generalize it for $V(x)$ which are locally as singular as those in Theorem 1.2 or in Theorem 1.6 by slightly strengthening conditions (1.4) and (1.5) at infinity.

**Theorem 1.7.** Let $A$ be of class $C^3$ and the magnetic field $B$ generated by $A$ satisfy for constants $C_\alpha$ that

$$|\partial_x^\alpha B(x)| \leq C_\alpha \langle x \rangle^{\alpha_1}(|B(x)| + 1), \quad |\alpha| = 1, 2. \quad (1.15)$$

Let $V(x) = V_1(x) + V_2(x)$ with $V_1 \in L^2_{\text{loc}}$ and $V_2$ of Stummel class. Suppose that there exist constants $\theta < 1$ and $C_* > 0$ such that

$$\theta |B(x)| + V_1(x) \geq -C_* \langle x \rangle^2, \quad x \in \mathbb{R}^d. \quad (1.16)$$

Then, $L = -\nabla_A^2 + V$ is essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$ and the domain of its selfadjoint extension $H$ is given by $D(H) = \{ u \in \mathcal{H}; -\nabla_A^2 u + Vu \in \mathcal{H} \}$.

**Theorem 1.8.** Let $A(x)$ and $B(x)$ be as in Theorem 1.7. Let $V(x) = V_1(x) + V_2(x)$ with $V_1 \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $V_2$ of Kato class. Suppose that there exist constants $\theta < 1$ and $C_*$ such that (1.16) is satisfied. Define

$$\tilde{V}_1(x) = V_1(x) + (C_* + C_1) \langle x \rangle^2 \quad (1.17)$$

with a sufficiently large constant $C_1$. Then, following statements are satisfied:
(1) The quadratic form $q_0$ on $C_0^\infty(\mathbb{R}^d)$ defined by
\[
q_0(u) = \|\nabla_A u\|^2 + (\langle \tilde{V}_1 + V_2 \rangle u, u) \tag{1.18}
\]
is bounded from below and closable. The closure has domain
\[
D([q_0]) = \{u \in L^2: \nabla_A u \in L^2, \ (|B| + |\tilde{V}_1| + \langle x \rangle^2)^{1/2}u \in L^2\}. \tag{1.19}
\]
For $u \in D([q_0])$, we have $V_2|u|^2 \in L^1$ and $[q_0](u)$ is given by (1.18).

(2) The selfadjoint operator $H_0$ defined by $[q_0]$ is given by
\[
H_0 u = -\nabla_A^2 u + (\tilde{V}_1 + V_2)u, \tag{1.20}
\]
\[
D(H_0) = \{u \in D([q_0]), -\nabla_A^2 u + (\tilde{V}_1 + V_2)u \in L^2\}. \tag{1.21}
\]

Suppose that $A$ and $V$ satisfy conditions of Theorem 1.7, then they also satisfy those of Theorem 1.8, and the operator $H_0$ defined in Theorem 1.8 is essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$ and $D(H_0) = \{u \in L^2: (-\nabla_A^2 + \tilde{V}_1 + V_2)u \in L^2\}$. This follows from the fact that selfadjoint operators admit no proper selfadjoint extensions.

Theorems 1.9 and 1.10 in what follows are time dependent versions of Theorems 1.7 and 1.8 respectively. Under assumptions of Theorem 1.9
\[
H(t) = -\nabla_A^2(t) + V(t, x) \quad \text{and} \quad H_0(t) = -\nabla_A^2(t) + V(t, x) + (C(t) + C_1(x))^2
\]
are essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$ by virtue of Theorem 1.7. We denote their selfadjoint extensions again by $H(t)$ and $H_0(t)$.

**Theorem 1.9.** Suppose that $A$ and $V$ satisfy following conditions:

1. $A(t, x) \in C^3(\mathbb{R}_x^d)$ for all $t \in I$ and the magnetic field $B(t, x)$ generated by $A(t, x)$ satisfies, for constants $C_\alpha > 0$,
   \[
   |\partial_x^\alpha B(t, x)| \leq C_\alpha \langle x \rangle^{|\alpha|} \langle B(t, x) \rangle, \quad |\alpha| = 1, 2, \quad (t, x) \in I \times \mathbb{R}^d. \tag{1.22}
   \]
2. $V(t, x) = V_1(t, x) + V_2(t, x)$ with $V_1(t, \cdot) \in L^2_{\text{loc}}(\mathbb{R}_x^d)$ for all $t \in I$ and $V_2(t, \cdot)$ of Stummel class uniformly with respect to $t \in I$. There exist a constant $\theta < 1$, a continuous function $C(t)$ and $Q \in M(\mathbb{R}^d)$ such that
   \[
   \theta |B(t, x)| + V_1(t, x) + C(t)\langle x \rangle^2 \geq Q(x)^2, \quad (t, x) \in I \times \mathbb{R}^d. \tag{1.23}
   \]
(3) For a.e. $x \in \mathbb{R}^d$, $A(t, x)$ and $V(t, x)$ are AC with respect to $t \in I$. Time derivatives satisfy, for a constant $C > 0$, that
\[
|\nabla_x \cdot \dot{A}(t, x)| + |\dot{A}(t, x)|^2 + |\nabla_x \dot{A}(t, x)^2| \leq CQ(x)^2, \quad (t, x) \in I \times \mathbb{R}^d;
\]
and that $\dot{V}(t, x) = W_0(t, x) + W_1(t, x) + W_2(t, x)$ such that
\[
\|Q^{-2+j}W_j(t)(-\Delta + 1)^{-j/2}\|_{\mathcal{B}^d} \leq C, \quad t \in I, \quad j = 0, 1, 2.
\]
Then, following statements are satisfied for a sufficiently large $C_1 > 0$:

(a) Domain $\mathcal{D}$ of $H_0(t)$ is independent of $t \in I$ and $\mathcal{D} \subset D(H(t))$ for all $t \in I$. Equip $\mathcal{D}$ with the graph norm of $H_0(t_0)$, $t_0$ being arbitrarily.

(b) There uniquely exists a unitary propagator $\{U(t, s), t, s \in I\}$ on $\mathcal{H}$ for (1.1) such that $U(t, s) \in \mathcal{B}(\mathcal{D})$; for $\varphi \in \mathcal{D}$, $U(t, s)\varphi$ is continuous with respect to $(t, s)$ in $\mathcal{D}$, of class $C^1$ in $\mathcal{H}$ and satisfies (1.10).

**Theorem 1.10.** Let $A(t, x)$ and $B(t, x)$ be as in Theorem 1.9. Suppose

(1) $V(t, x) = V_1(t, x) + V_2(t, x)$ with $V_1(t, \cdot) \in L^1_{\text{loc}}(\mathbb{R}^d_t)$ for all $t \in I$ and $V_2(t, \cdot)$ of Kato class uniformly with respect to $t \in I$. There exist a $\theta < 1$, a continuous function $C(t)$ and $Q \in M(\mathbb{R}^d)$ such that
\[
\theta |B(t, x)| + V_1(t, x) + C(t)|x|^2 \geq Q(x)^2, \quad (t, x) \in I \times \mathbb{R}^d. \quad (1.24)
\]

(2) $V(t, x)$ is AC with respect to $t \in I$ for a.e. $x \in \mathbb{R}^d$ and $\dot{V}(t, x)$ satisfies, for a constant $C > 0$,
\[
\|L_Q^{-1/2}\dot{V}(t)|L_Q^{-1/2}\|_{\mathcal{B}^d} \leq C, \quad t \in I. \quad (1.25)
\]

Let $\tilde{V} = V + (C(t) + C_1)|x|^2$ and $\tilde{V}_1 = V_1 + (C(t) + C_1)|x|^2$ for a sufficiently large constant $C_1 > 0$. Then, following statements are satisfied.

(a) The quadratic form $q_0(t)$ on $C^\infty_0(\mathbb{R}^d)$ defined by
\[
q_0(t)(u) = \|\nabla_{A(t)}u\|^2 + (\tilde{V}(t, x)u, u) \quad (1.26)
\]
is bounded from below and closable. Domain $\mathcal{Y}$ of its closure $[q_0(t)]$ is given by (1.19) with obvious changes. $\mathcal{Y}$ is independent of $t$ and satisfies $\mathcal{Y} \subset D(L^1_Q)$. We equip $\mathcal{Y}$ with the inner product $[q_0(t_0)](u, v)$, $t_0 \in I$ being arbitrarily and denote by $\mathcal{X}$ its dual space with respect to the inner product of $\mathcal{H}$. For $t \in I$, define operator $H(t)$ from $\mathcal{Y}$ to $\mathcal{X}$ by
\[
(H(t)u, v) = (\nabla_{A(t)}u, \nabla_{A(t)}v) + (V(t, x)u, v), \quad u, v \in \mathcal{Y}.
\]
Then, $H(t) \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and it is norm continuous with respect to $t \in I$. 

There uniquely exists a unitary propagator for (1.1) on \( Y \) such that \( U(t, s) \in B(Y) \); for \( \varphi \in Y \), \( U(t, s)\varphi \) is continuous with respect to \((t, s)\) in \( Y \), of class \( C^1 \) in \( X \) and satisfies (1.10). Moreover, \( \{U(t, s)\} \) extends to a strongly continuous family of bounded operators in \( X \).

We emphasize that in all theorems above no conditions are imposed on the behavior at infinity of the positive part of \( V \) in contrast to strong size restrictions on its negative part.

For the reference on the problem, we refer to the introduction of [13] and we shall jump into the proof of Theorems immediately. We shall not prove Theorems 1.5 and 1.6 because they are proved in [13] for the case \( Q(x) = C\langle x \rangle \) and the proof goes through for the present cases with obvious changes, and because the proof of Theorems 1.9 and 1.10 which we shall be devoted to in what follows basically patterns after that of [13], though several new estimates are necessary.

The plan of paper is as follows. Section 2 collects some well known results which are necessary in subsequent sections. We prove selfadjointness theorems, Theorems 1.7 and 1.8 in Section 3. In Section 4, we formulate and prove an estimate for the resolvent of \( H_1(t) = -\nabla^2_{A(t)} + V_1(t, x) + (C(t) + C_1)\langle x \rangle^2 \) which replaces the diamagnetic inequality (cf. [1]). We emphasize that it is hopeless to have standard diamagnetic inequality for this operator since the scalar potential \( W(t, x) = V_1(t, x) + (C(t) + C_1)\langle x \rangle^2 \) of \( H_1(t) \) can wildly diverge to negative infinity as \( |x| \to \infty \) and \( -\Delta + W(t, x) \) is not in general essentially selfadjoint on \( C_0^\infty(\mathbb{R}^d) \). We prove Theorems 1.9 and 1.10 in Sections 5 and 6 respectively by using materials prepared in preceding sections.

2 Preliminaries

In this section, we recall Kato’s abstract theory of evolution equations which the proof of Theorems will eventually relies upon, and Iwatsuka’s identity which will be used for deriving various estimates necessary for applying Kato’s theory.

2.1 Kato’s abstract theory for evolution equations

As in the previous paper [13], Theorems 1.9 and 1.10 will be proven by applying the following abstract theorem. The theorem is the consequence of Theorem 5.2, Remarks 5.3 and 5.4 of Kato’s seminal paper [3].

**Theorem 2.1.** Let \( X \) and \( Y \) be a pair of Hilbert spaces such that \( Y \subset X \) continuously and densely. Let \( \{A(t), t \in I\} \), \( I \) being an interval, be a family
of closed operators in $\mathcal{X}$ with dense domain $D(A(t))$ such that $\mathcal{Y} \subset D(A(t))$ for every $t \in I$ and $I \ni t \to A(t) \in \mathbf{B}(\mathcal{Y}, \mathcal{X})$ is norm continuous. Suppose that following conditions are satisfied:

(1) For every $t \in I$, there exist inner products $(\cdot, \cdot)_{X_t}$ and $(\cdot, \cdot)_{Y_t}$ of $\mathcal{X}$ and $\mathcal{Y}$ respectively which define norms equivalent to the original ones and which satisfy, for a constant $c > 0$,

$$
\|u\|_{Y_t}/\|u\|_{X_s} \leq e^{c|t-s|}, \quad \|u\|_{X_t}/\|u\|_{X_s} \leq e^{c|t-s|}, \quad u \neq 0.
$$

(2.1)

(2) If we let $X_t$ and $Y_t$ be Hilbert spaces $\mathcal{X}$ and $\mathcal{Y}$ with these inner products, $A(t)$ is selfadjoint in $X_t$ and the part $\tilde{A}(t)$ of $A(t)$ in $Y_t$ is also selfadjoint in $Y_t$.

Then, there uniquely exists a strongly continuous family of bounded operators $\{U(t,s) : t, s \in I\}$ in $\mathcal{X}$ that satisfies

(a) $U(t,r) = U(t,s)U(s,r), \ U(s,s) = I$ for every $t, s$ and $r \in I$.

(b) $U(t,s) \in \mathbf{B}(\mathcal{Y})$; for $\varphi \in \mathcal{Y}$, $U(t,s)\varphi$ is continuous with respect to $(t,s)$ in $\mathcal{Y}$, of class $C^1$ in $\mathcal{X}$ and it satisfies

$$
\partial_t U(t,s)\varphi = -iA(t)U(t,s)\varphi, \quad \partial_s U(t,s)\varphi = iU(t,s)A(s)\varphi.
$$

(2.2)

### 2.2 Iwatsuka’s Identity

In [2], Iwatsuka has found an ingenious formula which rewrites Schrödinger operator $H = -\nabla^2 + V$ in the form of elliptic operators in which the magnetic field $B_{jk} = \partial_j A_k - \partial_k A_j$ appears explicitly, which he has used for proving Theorem 1.3. We recall it here as we shall use it several times for deriving various estimates. For the proof of following lemmas we refer to Iwatsuka’s paper [2], formula (2.12) and proofs of Theorem 1.1 and Theorem 2.1 therein. We denote $b \cdot a = \mathbf{b}^t a$ for a vector $b$ and a matrix $a$.

**Lemma 2.2.** Let $G(x) = \{G_{jk}\}$ be Hermitian matrix valued function and $G_{jk} = \alpha_{jk} + i\beta_{jk}$, for real valued $\alpha_{jk} = \alpha_{kj}$ and $\beta_{jk} = -\beta_{kj}$, $j, k = 1, \ldots, d$; $F(x) = \{F_j\}$ be complex vector field such that with real $A$ and complex $b$

$$
F(x) = A(x) + b(x)
$$

(2.3)

and $B(x) = \{B_{jk}\}, B_{jk} = \partial_j A_k - \partial_k A_j$. Then, we have the following identity:

$$
-\nabla_F^T \cdot G \nabla_F = -\nabla_A \cdot \alpha \nabla_A + i\{2\Re(\mathbf{b} \cdot G) - (\nabla \cdot \beta)\}\nabla_A
$$
\begin{equation}
- \sum_{j<k} \beta_{jk}B_{jk} + i \nabla \cdot (Gb) + b \cdot Gb.
\end{equation}

In particular, if \( \alpha_{jk} = \delta_{jk} \), Kronecker’s delta and

\begin{equation}
G_{jk} = \delta_{jk} + i \beta_{jk} \quad \text{and} \quad b = \frac{1}{2} \nabla \cdot \beta
\end{equation}

for a real valued skew-symmetric matrix \( \{\beta_{jk}\} \), then

\begin{equation}
-\nabla^2 A = -\nabla \varphi \cdot G \nabla F + \sum_{j<k} \beta_{jk}B_{jk} + R,
\end{equation}

\begin{equation}
R = \frac{1}{2} \sum_{j,k} \beta_{jk} \partial_j b_k + \frac{1}{4} b^2.
\end{equation}

Real skew-symmetric \( \beta \) in (2.5) is completely arbitrary for identity (2.6) and Iwatsuka’s choice in [2] is as follows: Take \( \chi \in C^\infty([0, \infty)) \) such that \( \chi(r) = 1 \) for \( 0 \leq r \leq 1/2 \), \( \chi(r) = r^{-1} \) for \( r \geq 1 \) and

\[ 0 < r \chi(r) \leq 1 \text{ for all } r > 0 \]

and define

\begin{equation}
\beta(x) = \chi(|B(x)|)B(x).
\end{equation}

In what follows, \( \beta(x) \) always denotes the function defined by (2.8) and \( b(x) \) and \( R(x) \) are respectively defined by (2.5) and (2.7) by using this \( \beta(x) \). We write

\[ |\partial| = \sum_{|\alpha|=1} |\partial^\alpha B_{jk}| \quad \text{and} \quad |\partial^2 B| = \sum_{|\alpha|=2} |\partial^\alpha B_{jk}|. \]

Lemma 2.3. Suppose \( A(x) \) and \( B(x) \) satisfy (1.15). Then:

\begin{equation}
|\beta(x)| \leq 1, \quad \sum_{j<k} \beta_{jk}B_{jk} = \chi(|B|)|B|^2 \geq |B| - 1,
\end{equation}

\begin{equation}
|\partial^\alpha \beta| \leq C(x)^{[\alpha]}, \quad |\alpha| = 1, 2; \quad |b| \leq C(x), \quad |R| \leq C(x)^2.
\end{equation}

For real skew-symmetric \( \tilde{\beta} = (\tilde{\beta}_{jk}) \), we have (Proposition 4.1 of [2]) that

\begin{equation}
-|\tilde{\beta}| \leq i \tilde{\beta} \leq |\tilde{\beta}|, \quad |\tilde{\beta}| = \left( \sum_{j<k} \tilde{\beta}_{jk}^2 \right)^{\frac{1}{2}}
\end{equation}

in the sense of quadratic forms on \( \mathbb{C}^d \). In what follows we shall use identity (2.6) by modifying \( \beta(x) \) of (2.8) in various ways.
3 Selfadjointness

We prove Theorems 1.7 and 1.8 in this section. We take and fix $\varphi \in C_0^\infty (\mathbb{R}^d)$ such that $0 \leq \varphi(x) \leq 1$ for all $x \in \mathbb{R}^d$,

$$\varphi(x) = 1 \text{ for } |x| \leq 1 \text{ and } \varphi(x) = 0 \text{ for } |x| \geq 2. \quad (3.1)$$

We set $\varphi_n(x) = \varphi(x/n)$ for $n = 1, 2, \ldots$ and define for $0 < \theta \leq 1$

$$\beta_{n, \theta}(x) = \theta \varphi_n(x) \beta(x). \quad (3.2)$$

The following lemma is obvious by virtue of (2.11).

**Lemma 3.1.** If we change $\beta$ by $\beta_{n, \theta}(x)$, then (2.6) remains to hold with $G$, $b$ and $R$ being replaced by corresponding $G_{n, \theta}$, $b_{n, \theta}$, $R_{n, \theta}$. Matrix $G_{n, \theta}$ satisfies

$$G_{n, \theta}(x) = 1 + i \theta \varphi_n(x) \beta(x) \geq 1 - \theta, \quad x \in \mathbb{R}^d; \quad (3.3)$$

and $b_{n, \theta}$ and $R_{n, \theta}$ satisfy corresponding estimates in (2.10) uniformly with respect to $\theta$ and $n$.

**Proof of Theorem 1.7** The following is a modification of Kato’s argument ([6]). It suffices to show that the image of $L \pm i$, $R(L \pm i)$, is dense in $\mathcal{H}$. Thus we suppose that $f \in \mathcal{H}$ satisfies $f \perp R(L \pm i)$ and show $f = 0$ then. We prove the $+$ case only. The proof for the other case is similar.

We first assume $V_2 = 0$. Define, for $n = 1, 2, \ldots$, $V_n(x) = \chi_{B_{2n}(0)}(x) V(x)$, where $B_{2n}(0) = \{ x \in \mathbb{R}^d : |x| < 2n \}$ and $\chi_F$ is the characteristic function of the set $F$, and

$$L_n = -\nabla^2_A + V_n, \quad D(L_n) = C_0^\infty (\mathbb{R}^d).$$

Since $V_n(x)$ is bounded from below, $L_n$ is essentially selfadjoint by virtue of Theorem 1.2. It follows that there exists $u_n \in C_0^\infty (\mathbb{R}^d)$ such that

$$\|(L_n + i)u_n - f\| \leq 1/n, \quad n = 1, 2, \ldots. \quad (3.4)$$

Then, $\|(L_n + i)u_n\| \leq \|f\| + 1/n$ and

$$\|u_n\| \leq \|(L_n + i)u_n\| \leq C, \quad \|L_n u_n\| \leq \|f\| + \|u_n\| + 1/n \leq C. \quad (3.5)$$

Let $\varphi_n(x)$ be as above. Then, $\varphi_n(x)V_n(x) = \varphi_n(x)V(x)$ and

$$\varphi_n(x)(L_n + i)u_n = (L + i)\varphi_n u_n + 2(\nabla \varphi_n) \nabla_A u_n + (\Delta \varphi_n) u_n.$$

It follows from (3.4) that

$$\|f\|^2 = \lim_{n \to \infty} \langle \varphi_n f, (L_n + i)u_n \rangle = \lim_{n \to \infty} \langle f, \varphi_n (L_n + i)u_n \rangle$$
\[
\lim_{n \to \infty} \{(f, (L + i)\varphi_n u_n) + 2(f, (\nabla \varphi_n) \nabla A u_n) + (f, (\Delta \varphi_n) u_n)\}. \quad (3.6)
\]

The first term on the right vanishes by the assumption and the third satisfies
\[
|(f, (\Delta \varphi_n) u_n)| \leq n^{-2} \|\Delta \varphi\|_\infty \|f\| \|u_n\| \to 0 \quad (n \to \infty).
\]

For estimating \(\|\nabla A u_n\|\), we use Iwatsuka’s identity (2.6) with \(\beta_{2n,\theta}\) defined by (3.2) with \(2n\) replacing \(n\), which produces
\[
L_n = -\nabla^2 + V_n = -\nabla F_{2n,\theta} G_{2n,\theta} \nabla F_{2n,\theta} + W_{2n,\theta}, \quad (3.7)
\]
\[
F_{2n,\theta} = A + b_{2n,\theta}, \quad W_{2n,\theta} = V_n + \sum \beta_{2n,\theta,jk} B_{jk} + R_{2n,\theta}. \quad (3.8)
\]

Here \(W_{2n,\theta}\) satisfies, with a constant \(C\) independent of \(n\), that
\[
W_{2n,\theta}(x) \geq -Cn^2, \quad n = 1, 2, \ldots, \quad x \in \mathbb{R}^d. \quad (3.9)
\]

Indeed, for \(|x| \leq 2n\), we have \(\varphi_{2n}(x) = 1\) and (1.16), (2.9) and (2.10) imply
\[
W_{2n,\theta} = V + \theta \sum \beta_{jk} B_{jk}(x) + \theta^2 R
\]
\[
\geq V + \theta(|B| - 1) + \theta^2 R \geq -C(x)^2 \geq -Cn^2;
\]

for \(2n < |x| \leq 4n\), we have \(V_n(x) = 0\) and
\[
W_{2n,\theta} = \theta \varphi_{2n}(x) \sum \beta_{jk} B_{jk}(x) + R_{2n,\theta}(x) \geq R_{2n,\theta}(x) \geq -Cn^2;
\]

and, for \(|x| \geq 4n\), \(W_{2n,\theta}(x) = 0\). It follows by virtue of (3.3) and (3.7) that
\[
(1 - \theta) ||\nabla F_{2n,\theta} u_n||^2 \leq (G_{2n,\theta} \nabla F_{2n,\theta} u_n, \nabla F_{2n,\theta} u_n)
\]
\[
= ((L_n - W_{2n,\theta}) u_n, u_n) \leq (L_n u_n, u_n) + Cn^2 ||u_n||^2 \leq Cn^2. \quad (3.10)
\]

Since \(|b_{2n,\theta}(x)| \leq Cn\) by (2.10), we then have
\[
||\nabla A u_n|| \leq ||\nabla F_{2n,\theta} u_n|| + ||b_{2n,\theta} u_n|| \leq ||\nabla F_{2n,\theta} u_n|| + Cn ||u_n|| \leq Cn \quad (3.11)
\]

and \(||\nabla \varphi_n \nabla A u_n|| \leq n^{-1} ||\nabla \varphi||_\infty ||\nabla A u_n|| \leq C\). It follows, since \(\nabla \varphi_n = 0\) for \(|x| \leq n\), that
\[
|(f, (\nabla \varphi_n) \nabla A u_n)| \leq C ||f||_{L^2(|x| \geq n)} \to 0
\]
as \(n \to \infty\). Thus, the right of (3.6) vanishes and \(f = 0\) and \(L\) is essentially selfadjoint on \(C_c^\infty(\mathbb{R}^d)\).

If \(V_2 \neq 0\), we repeat the argument above, setting \(V_n = \chi_{|x| \leq 2n} V_1 + V_2\). Since \(V_2\) is of Stummel class, \(L_n\) with this \(V_n\) is essentially selfadjoint on
We show that, for \( \theta < \theta \) as \( n \to \infty \) for \( u_n \in C^\infty_0(\mathbb{R}^d) \) of (3.4). We use identity (3.7) and obtain

\[
(1 - \theta)\|\nabla_{F_{2n,0}}u_n\|^2 \leq (L_n u_n, u_n) - (V_2 u_n, u_n) + C n^2 \|u_n\|^2.
\]

as in (3.10). This with (3.5) implies as in (3.11) that

\[
\|\nabla A u_n\|^2 \leq C (n^2 + |(V_2 u_n, u_n)|).
\]

Since \( V_2 \) is \(-\Delta\)-form bounded with bound 0, we have, for any \( \varepsilon > 0 \),

\[
|\langle V_2 u, u \rangle| \leq \varepsilon \|\nabla u\|^2 + C_\varepsilon \|u\|^2 \leq \varepsilon \|\nabla A u\|^2 + C_\varepsilon \|u\|^2, \quad u \in C^\infty_0(\mathbb{R}^d).
\]

It follows that \( \|\nabla A u_n\| \leq C n \) and \( \lim_{n \to \infty} (f, (\nabla \varphi_n) \nabla A u_n) = 0 \) as previously. Thus, \( L \) is essentially selfadjoint when \( V_2 \neq 0 \) as well. The closure of \( L \) is given by \( H = L^* \) and it is standard that \( D(L^*) = \{ u \in \mathcal{H}: -\nabla^2 u + Vu \in L^2 \} \) and this completes the proof. \( \square \)

**Proof of Theorem 1.8** We let \( \theta \) and \( \tilde{V}_1 \) be as in the theorem. Define

\[
G_{\theta_0} = 1 + i \theta_0 \beta, \quad F_{\theta_0} = A + \theta_0 b \quad \text{for } \theta \leq \theta_0 \leq 1
\]

by replacing \( \beta \) and \( b \) by \( \theta_0 \beta \) and \( \theta_0 b \) in (2.5) and (2.3) respectively. We have

\[
-\nabla^2_A + \tilde{V}_1 = -\nabla_{F_{\theta_0}} G_{\theta_0} \nabla F_{\theta_0} + \tilde{W}_{\theta_0}, \quad \tilde{W}_{\theta_0} = \tilde{V}_1 + \theta_0 \sum j k B_{jk} + \theta_0^2 R. \quad (3.12)
\]

We take the constant \( C_1 \geq 10 \) large enough in the definition (1.17) of \( \tilde{V}_1 \) so that \( |R(x)| \leq 10^{-2} C_1 \langle x \rangle^2 \) and

\[
\tilde{W}_\theta \geq \tilde{V}_1 + \theta (|B| - 1) + \theta^2 R \geq C_1 \langle x \rangle^2 - 1 - |R| \geq 2 \theta C_1 \langle x \rangle^2 + 2 |R|. \quad (3.13)
\]

We show that, for \( \theta < \theta_0 \leq 1 \), there exist a \( \theta_0 \)-dependent constant \( C_{\theta_0} > 0 \) and a \( \theta_0 \)-independent \( C > 0 \) such that

\[
C_{\theta_0} (|B(x)| + |\tilde{V}_1(x)|) + \frac{C}{2} \langle x \rangle^2 \leq \tilde{W}_{\theta_0}(x) \leq (|B(x)| + |\tilde{V}_1(x)| + C \langle x \rangle^2). \quad (3.14)
\]

Indeed, the second inequality is obvious from (2.10). The first is also evident if \( \tilde{V}_1 > 0 \), since then \( \tilde{V}_1 + \theta |B| \geq C_1 \langle x \rangle^2 \) and

\[
\tilde{W}_{\theta_0} \geq \tilde{V}_1 + \theta_0 (|B| - 1) + \theta_0^2 R \geq \frac{1}{2} (|\tilde{V}_1| + \theta_0 |B| + C \langle x \rangle^2).
\]

To see the first for the case \( \tilde{V}_1(x) < 0 \), we first estimate

\[
\tilde{W}_{\theta_0} = \tilde{W}_\theta + (\theta_0 - \theta) \sum j k B_{jk} + (\theta_0^2 - \theta^2) R
\]
which holds irrespectively of the sign of $\tilde{V}_1$. If $\tilde{V}_1(x) < 0$ we also have

$$\tilde{W}_{\theta_0} = \frac{\theta_0}{\theta} \tilde{W}_\theta + (\frac{\theta_0}{\theta} - 1) |\tilde{V}_1| + \theta_0(\theta_0 - \theta)R$$

$$\geq \frac{\theta_0}{\theta}(\frac{2}{3}C_1(x)^2 + 2|R|) + (\frac{\theta_0}{\theta} - 1) |\tilde{V}_1| - |R| \geq \frac{2}{3}C_1(x)^2 + (\frac{\theta_0}{\theta} - 1) |\tilde{V}_1|.$$ 

Adding both sides of last two estimates and dividing by 2, we obtain the first inequality of (3.14) for the case $\tilde{V}_1(x) < 0$.

We define the quadratic form $q_1(u, v)$ for $u, v \in C^\infty_0(\mathbb{R}^d)$ by

$$q_1(u, v) = (\nabla_A u, \nabla_A v) + (\tilde{V}_1 u, v).$$

We have by virtue of Iwatsuka’s identity (3.12) for $\theta_0$ replacing $\theta$ that

$$q_1(u, v) = (G_{\theta_0} \nabla_{F_{\theta_0}} u, \nabla_{F_{\theta_0}} v) + (\tilde{W}_{\theta_0} u, v).$$

Estimates $1 - \theta_0 \leq G_{\theta_0} \leq 1 + \theta_0$ and (3.14) imply for a constant $C > 1$ that

$$(1 - \theta_0)\|\nabla_{F_{\theta_0}} u\|^2 \leq (G_{\theta_0} \nabla_{F_{\theta_0}} u, \nabla_{F_{\theta_0}} u) \leq (1 + \theta_0)\|\nabla_{F_{\theta_0}} u\|^2,$$

$$C^{-1}|||B| + \tilde{V}_1| + \langle x \rangle^2\|u\|^2 \leq (\tilde{W}_{\theta_0} u, u) \leq C(|||B| + \tilde{V}_1| + \langle x \rangle^2\|u\|^2.$$ 

It follows that quadratic forms $(G_{\theta_0} \nabla_{F_{\theta_0}} u, \nabla_{F_{\theta_0}} v)$ and $(\tilde{W}_{\theta_0} u, v)$ on $C^\infty_0(\mathbb{R}^d)$ are both closable and positive definite and their closures have respective domains \{u: $\nabla_{F_{\theta_0}} u \in L^2$\} and \{u: ($|B| + |\tilde{V}_1| + \langle x \rangle^2\|u\|^2 \in L^2$\}. Thus, $q_1$ is closable, the closure $[q_1]$ has domain

$$D([q_1]) = \{u \in L^2: \nabla_{F_{\theta_0}} u \in L^2, \ (|B| + |\tilde{V}_1| + \langle x \rangle^2\|u\|^2 \in L^2 \}$$

$$= \{u \in L^2: \nabla_A u \in L^2, \ (|B| + |\tilde{V}_1| + \langle x \rangle^2\|u\|^2 \in L^2 \}$$

and $[q_1](u)$ is given again by (3.15). Moreover, by making $C_1$ larger if necessary, we have from the first inequality of (3.14) and that $|b_{\theta_0}| \leq C \langle x \rangle$ that

$$[q_1](u) \geq (1 - \theta_0)\|\nabla_A u\|^2 + C\|(|B| + |\tilde{V}_1| + \langle x \rangle^2\|u\|^2, \ u \in D([q_1]).$$

We have $q_0(u, v) = q_1(u, v) + (V_2 u, v)$. Since $V_2$ is of Kato-class, $V_2$ is $-\Delta$-form bounded with bound 0 and we have, for any $\varepsilon > 0$,

$$\langle |V_2| u, u \rangle \leq \varepsilon \|\nabla_A u\|^2 + C_\varepsilon \|u\|^2$$

as in the proof of Theorem 1.7. Hence the form $\langle |V_2| u, u \rangle$ is $[q_1]$-bounded with bound 0 and statements (1) and (2) of the theorem follow.
We prove statement (3). We write $\tilde{V} = \tilde{V}_1 + V_2$. Let $u \in D(H_0)$. Then, $u \in D([q_0])$ and $\langle x \rangle u, [\tilde{V}_1]^\frac{1}{2} u, |V_2|^\frac{1}{2} u \in \mathcal{H}$ and $\nabla_A u \in \mathcal{H}$. Hence, $\tilde{V} u \in L^1_{loc}$ and $\nabla^2_A u$ is well defined as distributions. It follows for any $v \in C_0^\infty(\mathbb{R}^d)$ that

$$(H_0 u, v) = [q_0](u, v) = (\nabla_A u, \nabla_A v) + (\tilde{V} u, v) = (-\nabla^2_A u + \tilde{V} u, v).$$

Hence $-\nabla^2_A u + \tilde{V} u \in L^2$ and $H_0 u = -\nabla^2_A u + \tilde{V} u$. Suppose on the contrary that $u \in D([q_0])$ satisfies $-\nabla^2_A u + \tilde{V} u \in L^2$. Then, for any $v \in C_0^\infty(\mathbb{R}^d)$,

$$(-\nabla^2_A u + \tilde{V} u, v) = [q_0](u, v) = (G_{\theta_0} \nabla_{F_{\theta_0}} u, \nabla_{F_{\theta_0}} v) + ((\tilde{W}_{\theta_0} + V_2)u, v)$$

and this extends to all $v \in D([q_0])$ by virtue of the argument in the first part. Thus, $u \in D(H_0)$ and $H_0 u = -\nabla^2_A u + \tilde{V} u$. This completes the proof. \[ \square \]

The following is a corollary of the proof of Theorem 1.8.

**Corollary 3.2.** Let conditions of Theorem 1.8 be satisfied. Let $C_1$ be sufficiently large. Then, for a constant $C > 0$, we have

$$\|\nabla_A u\|^2 + \|(|B| + |\tilde{V}_1| + (\langle x \rangle^2)^\frac{1}{2} u\|^2 \leq C[q_0](u), \quad u \in D([q_0])$$  \hspace{1cm} (3.21)

### 4 Diamagnetic Inequality

In this section we assume that $A$ and $V$ satisfy the following conditions:

1. $A(x) \in C^3(\mathbb{R}^d)$ and $B(x)$ satisfies estimates (1.15).
2. $V = V_1 + V_2$ with $V_1 \in L^1_{loc}$ and $V_2$ of Kato class.
3. There exists constants $0 < \theta < 1, C_\ast > 1$ and $Q \in M(\mathbb{R}^d)$ such that

$$\theta |B(x)| + V_1(x) + C_\ast (\langle x \rangle x)^2 \geq Q(x)^2. \quad (4.1)$$

We then define $q_0(u)$ and $q_1(u)$ respectively by (1.18) and (3.15) with $\tilde{V}_1(x) = V_1(x) + (C_\ast + C_1)\langle x \rangle^2$ with sufficiently large constant $C_1$ such that results in the previous section are satisfied. We let $H_0$ and $H_1$ be selfadjoint operators defined by $[q_0]$ and $[q_1]$ respectively.

**Lemma 4.1.** Let $\theta < \theta_0 < 1$. There exists $C_{\theta_0} > 0$ such that for $C_1 \geq C_{\theta_0}$, we have the following estimate:

$$(1 - \theta_0)\|\nabla_{F_{\theta_0}} u\|^2 + \|Q^2 u\|^2 + 2(\theta_0 - \theta)\|Q|B|^\frac{1}{2} u\|^2 + C_1 \|\langle x \rangle Q u\|^2 \leq \|H_1 u\|^2, \quad u \in D(H_1).$$  \hspace{1cm} (4.2)
Proof. We use the notation of the proof of Theorem 1.8. We have as in there
\[ W_{\theta_0} \geq Q(x)^2 + (\theta_0 - \theta)|B(x)| + \frac{2}{3}C_1\langle x \rangle^2 \quad (4.3) \]
Let \( u \in D(H_1) \). Then, \( \nabla_{F_{\theta_0}} u, \nabla_A u, \tilde{W}_{\theta_0}^{1/2} u \) and \( Qu \) all belong to \( L^2(\mathbb{R}^d) \) by virtue of (4.3) and, for \( v \in C_0^\infty(\mathbb{R}^d) \), we have
\[
(G_{\theta_0}Q\nabla_{F_{\theta_0}} u, Q\nabla_{F_{\theta_0}} v) = -(\nabla_{F_{\theta_0}} G_{\theta_0} \nabla_{F_{\theta_0}} u, Q^2 v) - (G_{\theta_0} \nabla_{F_{\theta_0}} u, \nabla(Q^2 v)) \\
= (H_1 u, Q^2 v) - (\tilde{W}_{\theta_0} u, Q^2 v) - (G_{\theta_0} \nabla_{F_{\theta_0}} u, \nabla(Q^2 v)). \quad (4.4)
\]
Using \( \varphi_n(x) \) of the proof of Theorem 1.7 and Friedrich’s mollifier \( j_\varepsilon \), we define \( v_{\varepsilon,n} = j_\varepsilon \ast (\varphi_n^2 u) \) for \( 0 < \varepsilon < 1 \) and \( n = 1, 2, \ldots \). Then, \( v_{\varepsilon,n} \in C_0^\infty(\mathbb{R}^d) \), is supported by the ball \( B_{2n+1}(0) \) and \( v_{\varepsilon,n} \rightarrow \varphi_n^2 u \) in the Sobolev space \( H^1(\mathbb{R}^d) \) as \( \varepsilon \to 0 \). We replace \( v \) in (4.4) by \( v_{\varepsilon,n} \), rewrite the left hand side of the resulting equation as \( (G_{\theta_0} \varphi_n \nabla_{F_{\theta_0}} u, \varphi_n Q \nabla_{F_{\theta_0}} u) + 2(\varphi_n G_{\theta_0} \nabla_{F_{\theta_0}} u, Q(\nabla \varphi_n) u) \) and arrange it as follows:
\[
(G_{\theta_0} \varphi_n Q \nabla_{F_{\theta_0}} u, \varphi_n Q \nabla_{F_{\theta_0}} u) + (\tilde{W}_{\theta_0} u, Q^2 \varphi_n^2 u) = (H_1 u, Q^2 \varphi_n^2 u) \\
- 2(\varphi_n G_{\theta_0} \nabla_{F_{\theta_0}} u, Q(\nabla \varphi_n) u) - (G_{\theta_0} \nabla_{F_{\theta_0}} u, Q^{-1} \nabla(Q^2 \varphi_n^2 u)). \quad (4.5)
\]
By virtue of (4.3) the left hand side may be bounded from below by
\[
(1 - \theta_0)\| \varphi_n Q \nabla_{F_{\theta_0}} u \|^2 + \| \varphi_n Q^2 u \|^2 + (\theta_0 - \theta)\| \varphi_n Q |B| \frac{1}{2} u \| ^2 + \frac{2C_1}{3} \| \varphi_n \langle x \rangle |Qu| \|^2. \quad (4.6)
\]
The right hand side of (4.5) may be bounded from above by
\[
\| \varphi_n H_1 u \| \| \varphi_n Q^2 u \| + 4n^{-1} \| \nabla \varphi \| \| \varphi_n Q \nabla_{F_{\theta_0}} u \| \| Qu \| \\
+ 4\| \varphi_n Q \nabla_{F_{\theta_0}} u \| \| \varphi_n (\nabla Q) u \|. \quad (4.7)
\]
Here we have \( \| \varphi_n (\nabla Q) u \| \leq C_Q \| \varphi_n \langle x \rangle |Qu| \| \) since \( Q \in M(\mathbb{R}^d) \), and we further estimate (4.7) from above by
\[
\frac{1}{2} \| \varphi_n H_1 u \|^2 + \frac{1}{2} \| \varphi_n Q^2 u \|^2 + 2n^{-1} \| \nabla \varphi \| \| \varphi_n Q \nabla_{F_{\theta_0}} u \|^2 + \| Qu \|^2 \\
+ \frac{1 - \theta_0}{2} \| \varphi_n Q \nabla_{F_{\theta_0}} u \|^2 + \frac{8C_1^2}{1 - \theta_0} \| \varphi_n \langle x \rangle |Qu| \|^2. \quad (4.8)
\]
Combining (4.6) and (4.8), we conclude that
\[
\left( \frac{1 - \theta_0}{2} - \frac{2\| \nabla \varphi \|}{n} \right) \| \varphi_n Q \nabla_{F_{\theta_0}} u \|^2 + \frac{1}{2} \| \varphi_n Q^2 u \|^2 + (\theta_0 - \theta)\| \varphi_n Q |B| \frac{1}{2} u \| ^2 \\
+ \left( \frac{2C_1^2}{3} - \frac{8C_1^2}{1 - \theta_0} \right) \| \varphi_n \langle x \rangle |Qu| \|^2 \leq \frac{1}{2} \| H_1 u \|^2 + \frac{2}{n} \| \nabla \varphi \| \| Qu \|^2.
\]
We choose $C_1 > 0$ larger if necessary so that

$$\frac{C_1}{6} \geq \frac{8c_0^2}{1-\theta_0}$$

and let $n \to \infty$. Then the monotone convergence implies that $Q^2 u, Q\nabla_{F_{\theta_0}} u$, $Q|B|^\frac{1}{2} u$ and, a fortiori $\langle x \rangle Q u$ all belong to $L^2(\mathbb{R}^d)$ and we obtain (4.2). \hfill \square

Since $F_{\theta_0} = A + \theta_0 b$ and $|b| \leq C \langle x \rangle$, we have

$$(1 - \theta_0)\| Q\nabla_A u \|^2 \leq 2(1 - \theta_0)\| Q\nabla_{F_{\theta_0}} u \|^2 + 2C^2(1 - \theta_0)\theta_0^2\| \langle x \rangle Q u \|^2.$$

Thus, assuming $2C^2 < C_1$, we obtain the following Corollary.

**Corollary 4.2.** For $\theta < \theta_0 < 1$, there exists $C_\theta > 0$ such that for $C_1 \geq C_\theta$

$$\begin{align*}
(1 - \theta_0)\| Q\nabla_A u \|^2 + \| Q^2 u \|^2 \\
+ 2(\theta_0 - \theta)\| Q|B|^\frac{1}{2} u \|^2 + C_1\| \langle x \rangle Q u \|^2 \leq 2\| H_1 u \|^2, \quad u \in D(H_1). \quad (4.9)
\end{align*}$$

Write $a_\pm = \max(0, \pm a)$ and define non-negative quadratic form:

$$q_{1+}(u) = \| \nabla_A u \|^2 + \| \tilde{V}_{1+} u \|^2, \quad D(q_{1+}) = C_0^\infty(\mathbb{R}^d).$$

Theorem 1.8 implies that $q_{1+}$ is closable and we denote by $H_{1+} = -\nabla_A^2 + \tilde{V}_{1+}$ the self-adjoint operator defined by $[q_{1+}]$.

**Lemma 4.3.** For any $\theta < \theta_0 < 1$, there exists $C_\theta$ such that, for $C_1 > C_\theta$ we have

$$\| \tilde{V}_{1-} u \| \leq (\theta/\theta_0)\| H_{1+} u \|, \quad u \in D(H_{1+}). \quad (4.10)$$

It follows, particular, that $D(H_1) = D(H_{1+})$.

**Proof.** Let $\theta < \theta_0 < 1$. Since $\tilde{V}_{1+}(x) \geq 0$, we obviously have

$$\theta_0|B(x)| + \tilde{V}_{1+}(x) + C_*(x)^2 \geq \theta_0(1 + |B|^2 + x^4)^{1/2}$$

and assumption (1.15) implies $Q_0(x) = \theta_0^\frac{1}{2}(1 + |B|^2 + x^4)^{1/4} \in M(\mathbb{R}^d)$. Then, take $\theta_1$ such that $\theta_0 < \theta_1 < 1$ and repeat the argument of the proof of Lemma 4.1 using $H_{1+}, \theta_0, \theta_1$ and $Q_0$ in place of $H_1, \theta, \theta_0$ and $Q$ respectively. We obtain from (4.2) that, for $C_1 > C_\theta$,

$$\| Q_0^2(x) u \| \leq \| H_{1+} u \|, \quad u \in D(H_{1+}). \quad (4.11)$$

Since $\tilde{V}_{1-} \leq \theta |B(x)|$ by virtue of (4.1) and $\theta |B(x)| \leq (\theta/\theta_0)Q_0^2(x)$, (4.11) implies the lemma. \hfill \square
Theorem 4.4. There exist uniformly bounded operators $B_a \in \mathcal{B}(\mathcal{H})$ for $a > 0$ such that, for every $u \in L^2(\mathbb{R}^d)$, we have

$$|(H_1 + a^2)^{-1}u(x)| \leq (H_1 + a^2)^{-1}|B_a u|(x) \leq (-\Delta + a^2)^{-1}|B_a u|(x).$$ \hspace{1cm} (4.12)

Proof. Lemma 4.3 implies that, for any $\theta < \theta_0 < 1$, provided that $C_1 \geq C_{\theta_0}$,

$$\|\tilde{V}_1 - (H_1 + a^2)^{-1}u\| \leq (\theta/\theta_0)\|u\|, \quad u \in L^2$$

for any $a > 0$. It follows that

$$(H_1 + a^2)^{-1} = (H_1 + a^2)^{-1}B_a, \quad B_a = (1 - \tilde{V}_1 - (H_1 + a^2)^{-1})^{-1}$$ \hspace{1cm} (4.13)

and $\|B_a\| \leq (1 - (\theta/\theta_0))^{-1}$. We then apply the diamagnetic inequality (pp. 9–10 of [1]) to $H_1 + a^2$. The lemma follows.

Corollary 4.5. Provided that $C_1$ is large enough, we have

$$\|(-\Delta + 1)^{1/2}Q|u|\| \leq C\|H_1 u\|, \quad u \in D(H_1).$$ \hspace{1cm} (4.14)

Proof. Corollary 4.2 implies $Qu \in L^2$ and $\nabla A(Qu) = Q\nabla Au + (\nabla Q)u \in L^2$. It follows, since $|\nabla|u| \leq |
\nabla Au|$, that $Q|u| \in H^1$ and

$$\|(-\Delta + 1)^{1/2}Q|u|\|^2 = \|Qu\|^2 + \||\nabla|Qu|\|^2 \leq \|Qu\|^2 + \||\nabla A(Qu)|\|^2 \leq C\|H_1 u\|^2.$$ Estimate (4.14) follows.

5 Proof of Theorem 1.9

In this and next sections we prove Theorems 1.9 and 1.10 respectively. Before starting the proof, we briefly discuss the gauge transform which will play an important role in what follows. We define the gauge transform by

$$v(t, x) = G(t)u(t, x) = e^{-i F(t) x^2} u(t, x), \quad F(t) = \int_0^t (C(s) + C_1)ds \hspace{1cm} (5.1)$$

by using a strongly continuous family of unitary operators $G(t)$, where $C_1 > 0$ a large constant. Then, $u(t, x)$ satisfies (1.1) if and only if $v(t, x)$ does

$$i\partial_t v = (-\nabla^2 \tilde{A}(t)v + \tilde{V}(t, x))v, \hspace{1cm} (5.2)$$

$$\tilde{A}(t, x) = A(t, x) - 2F(t)x, \quad \tilde{V}(t, x) = V(t, x) + (C(t) + C_1)x^2 \hspace{1cm} (5.3)$$
and, provided a dense subspace \( \Sigma \) satisfies \( G(t) \Sigma = \Sigma \), \( \{U(t, s) : t, s \in \mathbb{R}\} \) is a unitary propagator for (1.1) on \( \Sigma \) if and only if so is
\[
\tilde{U}(t, s) = G(t)U(t, s)G(s)^{-1}
\]
for (5.2) on \( \Sigma \). If \( V_1 \) satisfies (1.16), \( \tilde{V}_1(t, x) = V_1(t, x) + (C(t) + C_1)\langle x \rangle^2 \) does
\[
|B(t, x)| + \tilde{V}_1(t, x) \geq Q(x)^2 + C_1\langle x \rangle^2.
\]
We assume in what follows that \( C_1 > 0 \) is taken sufficiently large so that, with this \( \tilde{V}_1(t, x) \), Theorems 1.7 and 1.8 as well as Lemma 4.1 and Theorem 4.4 are satisfied uniformly with respect to \( t \in I \). In the proof, we shall first construct propagator \( \tilde{U}(t, s) \) for equation (5.2), define \( U(t, s) \) by (5.4) and check that it satisfies the properties of Theorem 1.9 or Theorem 1.10.

We now begin the proof of Theorem 1.9. We consider five operators
\[
L(t) = -\nabla^2_{A(t)} + V(t), \quad L_0(t) = -\nabla^2_{\tilde{A}(t)} + \tilde{V}(t), \quad L_1(t) = -\nabla^2_{\tilde{A}(t)} + \tilde{V}_1(t),
\]
\[
\tilde{L}_0(t) = -\nabla^2_{\tilde{A}(t)} + \tilde{V}(t), \quad \tilde{L}_1(t) = -\nabla^2_{\tilde{A}(t)} + \tilde{V}_1(t).
\]
These operators are all essentially selfadjoint on \( C^\infty_0(\mathbb{R}^d) \) and we denote their selfadjoint extensions by \( H(t), H_0(t), H_1(t), \tilde{H}_0(t) \) and \( \tilde{H}_1(t) \), respectively.

Since \( V_2(t, x) \) is of Stummel class uniformly with respect to \( t \in I \), Theorem 4.4 implies that, for any \( \varepsilon > 0 \), there exists \( a_0 \) such that
\[
\|V_2(t)(H_1(t) + a^2)^{-1}\|_{\mathcal{B}(\mathcal{H})} \leq \|V_2(t)(-\Delta + a^2)^{-1}\|_{\mathcal{B}(\mathcal{H})}\|B_a\|_{\mathcal{B}(\mathcal{H})} < \varepsilon, \quad a > a_0.
\]
It follows by Kato-Rellich theorem that
\[
H_0(t) = H_1(t) + V_2(t), \quad D(H_0(t)) = D(H_1(t)).
\]
Moreover, by choosing \( C_1 \) large enough we may assume by virtue of (4.2),
\[
\|u\| \leq \|H_1(t)u\|, \quad \|V_2(t)H_1(t)^{-1}\| \leq 1/2, \quad t \in I.
\]
Then, we have for a constant \( C_0 \)
\[
C_0^{-1}\|H_1(t)u\| \leq \|H_0(t)u\| \leq C_0\|H_1(t)u\|, \quad t \in I.
\]
Since \( \tilde{A} \) and \( A \) produce the same magnetic field and \( |\tilde{A} - A| \leq C(x) \), (5.6) holds with \( \tilde{H}_0(t) \) and \( \tilde{H}_1(t) \) in place of \( H_0(t) \) and \( H_1(t) \) respectively and we likewise have
\[
C_0^{-1}\|\tilde{H}_1(t)u\| \leq \|\tilde{H}_0(t)u\| \leq C_0\|\tilde{H}_1(t)u\|.
\]
Lemma 5.1. (1) Domains of $H_0(t)$, $H_1(t)$, $\tilde{H}_0(t)$ and $\tilde{H}_1(t)$ satisfy

$$D(H_0(t)) = D(H_1(t)) = D(\tilde{H}_0(t)) = D(\tilde{H}_1(t)) \equiv \mathcal{D} \subset D(H(t))$$

for all $t \in I$ and $\mathcal{D}$ is independent of $t \in I$.

(2) There exists a constant $c > 0$ such that

$$\|H_0(t)u\| \leq e^{c|t-s|}\|H_0(s)u\|, \quad t, s \in I, \quad (5.9)$$

$$\|(H_0(t) - H_0(s))u\| \leq c|t-s|\|H_0(s)u\|, \quad t, s \in I. \quad (5.10)$$

The same holds for $\tilde{H}_0(t)$ replacing $H_0(t)$.

(3) The gauge transform $G(t) = e^{-iF(t)(x)^2}$ satisfies $G(t)\mathcal{D} = \mathcal{D}$ and

$$G(t)H_0(t) = \tilde{H}_0(t)G(t), \quad G(t)H_1(t) = \tilde{H}_1(t)G(t). \quad (5.11)$$

If $\varphi \in \mathcal{D}$, $t \mapsto G(t)\varphi$ is $\mathcal{D}$-valued continuous, $\mathcal{H}$-valued $C^1$ and

$$\partial_t G(t)\varphi = -i(C(t) + C_1)\langle t, x \rangle^2 G(t)\varphi.$$

Proof. We write $C(t)$ for $C(t) + C_1$ in the proof by absorbing $C_1$ into $C(t)$ for shorting formulas. Let $u \in C_0^\infty(\mathbb{R}^d)$. Then, $H_0(t)u$ is $\mathcal{H}$-valued differentiable almost everywhere with respect to $t$ and

$$\dot{H}_0(t)u = 2i\dot{A}(t, x)\nabla_{A(t)}u + i\nabla_x \cdot \dot{A}(t, x)u + \dot{C}(t)\langle x \rangle^2 + \dot{V}(t, x)u. \quad (5.12)$$

We write the right hand side in the form

$$2i\dot{A}(t, x) \cdot \nabla_{A(s, x)}u + 2\dot{A}(t, x) \cdot \left( \int_s^t \dot{A}(r, x)dr \right) u + (i\nabla_x \cdot \dot{A}(t, x) + \dot{C}(t)\langle x \rangle^2)u + \dot{V}(t, x)u = I_1(t, s)u + I_2(t, s)u + I_3(t)u + I_4(t)u.$$

Since $|\dot{A}(t, x)| \leq CQ(x)$, (4.9) implies

$$\|I_1(t, s)u\| \leq 2\|\dot{A}(t, x)\|\|\nabla_{A(s)}u\| \leq C\|Q\nabla_{A(s)}u\| \leq C\|H_1(s)u\|.$$

Denote by $M(t, x)$ any of $\nabla_x(\dot{A}(t, x)^2)$, $\dot{A}(t, x)^2$, $\nabla_x \cdot \dot{A}(t, x)$ and $\dot{C}(t)\langle x \rangle^2$. Then, $|M(t, x)| \leq CQ(x)^2$ and (4.9) implies $\|M(t)H_1(s)^{-1}u\| \leq C\|u\|$ uniformly with respect to $t, s \in I$. Thus,

$$\|I_2(t, s)u\| + \|I_3(t)u\| \leq C\|H_1(s)u\|, \quad t, s \in I.$$
Write \( \hat{V}(t, x) = W_0(t, x) + W_1(t, x) + W_2(t, x) \) as in Theorem 1.9, then 
\[ \|W_0(t)u\| \leq C\|Q^2 u\| \leq C\|H_1(s)u\| \text{ for any } t, s \in I \text{ as above; } \]
\[ \|W_1(t)u\| \leq \|Q^{-1}W_1(t)(-\Delta + 1)^{-\frac{1}{2}}\|_{\mathcal{B}(\mathcal{H})}\|(-\Delta + 1)^{\frac{1}{2}}Q|u|\| \leq C\|H_1(s)u\| \]
by virtue of (4.14); and Theorem 4.4 implies 
\[ \|W_2(t)H_1(s)^{-1}u\| \leq C\|W_2(t)(-\Delta + 1)^{-1}B_1 u\| \leq C\|B_1 u\| \leq C\|u\|. \]
Thus, \( \|I_4(t, s)u\| \leq C\|H_1(s)u\| \) and combining these estimates, we obtain
\[ \|\hat{H}_0(t)u\| \leq C\|H_1(s)u\| \leq C\|H_0(s)u\|, \quad t, s \in I. \quad (5.13) \]
It follows by integration that
\[ \|(H_0(t) - H_0(s))u\| \leq c|t - s|\|H_0(s)u\|, \quad u \in C_0^\infty(\mathbb{R}^d). \quad (5.14) \]
Since \( C_0^\infty(\mathbb{R}^d) \) is a core of \( H_0(s) \), (5.14) extends to \( u \in D(H_0(s)) \). It follows that 
\( D(H_0(s)) \subset D(H_0(t)) \) and by symmetry \( D(H_0(s)) = D(H_0(t)) \) for any \( t, s \in I \) and, consequently, (5.10) for \( H_0(t) \) is satisfied. (5.10) clearly implies (5.9). Changing \( A(t) \) by \( \hat{A}(t) \) will not change \( B(t, x) \) and the argument above yields the same results for \( \hat{H}_0(t) \) and \( \hat{H}_1(t) \). This proves statement (2).

Let \( u \in D(H_0(t)) \). Then, \( \langle x \rangle^2 u \in \mathcal{H} \) by virtue of (4.9) and
\[ H(t)u = H_0(t)u - C(t)\langle x \rangle^2 u \in \mathcal{H}. \quad (5.15) \]
Since \( D(H(t)) = \{u \in \mathcal{H}: H(t)u \in L^2\} \), (5.15) implies \( u \in D(H(t)) \) and 
\( D(H_0(t)) \subset D(H(t)) \).

We next prove \( D(H_1(t)) = D(\hat{H}_1(t)) \), which will then prove statement (1). Define for \( \theta \in [0, 1] \)
\[ H_1(t, \theta) = -\nabla_{\mathcal{A}(t, \theta)}^2 + \hat{V}_1(t, x), \quad A(t, \theta, x) = A(t, x) - 2\theta F(t)x, \]
so that \( H_1(t, 0) = H_1(t) \) and \( H_1(t, 1) = \hat{H}_1(t) \). Since \( A(t, \theta, x) \) and \( A(t, x) \)
generate the same magnetic field \( B(t, x) \) and \( |2\theta F(t)x| \leq C\langle x \rangle \), results of previous sections apply to \( H_1(t, \theta) \). We have
\[ \partial_\theta H_1(t, \theta)u = -iA(t)F(t)x\nabla_{\mathcal{A}(t)}u + 8\theta F(t)^2x^2u - 2diF(t)u \]
and (4.9) implies \( \|\partial_\theta H_1(t, \theta)u\| \leq C\|H_1(t)u\| \) for \( 0 \leq \theta \leq 1 \). Thus,
\[ \|(H_1(t, \theta) - H_1(t, \sigma))u\| \leq C|\theta - \sigma|\|H_1(t, \sigma)u\|, \quad u \in C_0^\infty(\mathbb{R}^d), \]
and we obtain the desired result \( D(H_1(t)) = D(\hat{H}_1(t)) \) as previously.
It is clear that $G(t)$ is an isomorphism of $C_0^\infty(\mathbb{R}^d)$ and $G(t)H_0(t)\varphi = \tilde{H}_0(t)G(t)\varphi$ for $\varphi \in C_0^\infty(\mathbb{R}^d)$. Since $C_0^\infty(\mathbb{R}^d)$ is a core of $H_0(t)$, it follows that $G(t)D(H_0(t)) \subset D(\tilde{H}_0(t))$. This clearly holds for $G(-t) = G(t)^{-1}$ as well and we obtain $G(t)D = D$ and $G(t)H_0(t) = \tilde{H}_0(t)G(t)$. This argument likewise applies to the pair $\tilde{H}_1(t)$ and $\tilde{H}_1(t)$ and we obtain (5.11). The last statement is obvious since $D \subset D(\langle x \rangle^2)$. This completes the proof. \qed

**Proof of Theorem 1.9.** Lemma 5.1 yields statement (a) of the theorem. It also implies that graph norms of any two of $\{H_0(t), \tilde{H}_0(s): t, s \in I\}$ are equivalent to each other. We equip $\mathcal{D}$ with the graph norm of $H_0(t_0)$ as in the theorem. Then, it is obvious that $\mathcal{D} \subset \mathcal{H}$ continuously and densely, $\mathcal{D} = D(\tilde{H}_0(t))$ for every $t \in I$ and that $I \ni t \mapsto \tilde{H}_0(t) \in \mathcal{B}(\mathcal{D}, \mathcal{H})$ is norm continuous by virtue of (5.10) for $\tilde{H}_0(t)$. We wish to apply Theorem 2.1 to the triplet $(\mathcal{X}, \mathcal{Y}, A(t))$ by setting $\mathcal{X} = \mathcal{H}$, $\mathcal{Y} = \mathcal{D}$ and $A(t) = \tilde{H}_0(t)$. For this we need check conditions (1) and (2) of Theorem 2.1 are satisfied.

For $t \in I$, we define $\mathcal{Y}_t = \mathcal{D}$ but with the graph norm of $\tilde{H}_0(t)$ and $\mathcal{X}_t = \mathcal{H}$. Then, the norm of $\mathcal{Y}_t$ is equivalent to that of $\mathcal{D}$ and (5.9) for $\tilde{H}_0(t)$ implies condition (2.1). If it follows from Theorem 1.2 that $\tilde{H}_0(t)$ is selfadjoint in $\mathcal{X}_t = \mathcal{H}$. Hence the part of $\tilde{H}_0(t)$ in $\mathcal{Y}_t(= D(\tilde{H}_0(t)))$ is automatically selfadjoint with domain $D(\tilde{H}_0(t)^2)$. Thus, the conditions are satisfied.

It follows that there uniquely exists a family of operators $\{\tilde{U}(t, s): s, t \in I\}$ which satisfies properties of Theorem 2.1 for $(\mathcal{H}, \mathcal{D}, \tilde{H}_0(t))$. Moreover, $\tilde{U}(t, s)$ is a unitary operator of $\mathcal{H}$. Indeed, if we set $u(t) = \tilde{U}(t, s)\varphi$ for $\varphi \in \mathcal{Y}$, $i\partial_t\|u(t)\|^2 = (\tilde{H}_0(t)u(t), u(t)) - (u(t), \tilde{H}_0(t)u(t)) = 0$ since $\tilde{H}_0(t)$ is selfadjoint. Hence $\tilde{U}(t, s)$ is an isometry of $\mathcal{H}$ and, since $\tilde{U}(t, s)\mathcal{D} = \mathcal{D}$, it is unitary. We define

$$U(t, s) = G(t)^{-1}\tilde{U}(t, s)G(s).$$

Then, $U(t, s)$ is a strongly continuous family of unitary operators on $\mathcal{H}$; Lemma 5.1 (3) implies that $U(t, s) \in \mathcal{B}(\mathcal{D})$; if $\varphi \in \mathcal{D}$, $U(t, s)\varphi$ is $\mathcal{D}$-valued continuous, $\mathcal{H}$-valued $C^1$ and that $U(t, s)\varphi$ satisfies the first of Eqns. (1.10):

$$i\partial_t U(t, s)\varphi = G(t)^{-1}(-C(t)\langle x \rangle^2 + \tilde{H}_0(t))\tilde{U}(t, s)G(s)\varphi = H(t)U(t, s)\varphi.$$

We may similarly prove that $U(t, s)\varphi$ satisfies the other of (1.10).

For proving the uniqueness of $U(t, s)$ we have only to notice the following: If $U(t, s)$ satisfies properties of the theorem, then $U(t, s) = G(t)U(t, s)G(s)^{-1}$ does those for $\tilde{H}_0(t)$ and such $\tilde{U}(t, s)$ is unique by virtue of Theorem 2.1.

When $\varphi \in \mathcal{D}$, (1.10) shows that $u(t, x) = U(t, s)\varphi(x)$ satisfies (1.1) in the sense of distributions. Then, the standard approximation argument shows that the same holds for $\varphi \in \mathcal{H}$ as well and $U(t, s)$ is unitary propagator on $\mathcal{H}$ for (1.1). We omit the details. The proof is completed. \qed
6 Proof of Theorem 1.10

For the constant $\theta$ in (1.24) we take and fix $\theta_0$ such that $\theta < \theta_0 < 1$ and take the constant $C_1 > 0$ large enough so that results of Sections 3 and 4 are satisfied, uniformly with respect to $t \in I$, for $q_0(t)$ of (1.26) and

$$q_1(t)(u,v) = (\nabla_{A(t)}u, \nabla_{A(t)}v) + (\bar{V}_1(t)u,v), \quad u,v \in C_0^\infty(\mathbb{R}^d),$$

in place of $q_0$ and $q_1$ respectively. In addition to $q_0(t)$ and $q_1(t)$, we define

$$\tilde{q}_0(t)(u,v) = (\nabla_{\tilde{A}(t)}u, \nabla_{\tilde{A}(t)}v) + (\bar{V}u,v), \quad u,v \in C_0^\infty(\mathbb{R}^d),$$

(6.1)

$$\tilde{q}_1(t)(u,v) = (\nabla_{\tilde{A}(t)}u, \nabla_{\tilde{A}(t)}v) + (\bar{V}_1u,v), \quad u,v \in C_0^\infty(\mathbb{R}^d),$$

(6.2)

where $\tilde{A}(t,x) = A(t,x) - 2F(t)x$. Since $\tilde{A}(t,x)$ and $A(t,x)$ generate same magnetic field and they differ only by $2F(t)x$, results of Sections 3 and 4 likewise apply to $\tilde{q}_0(t)$ and $\tilde{q}_1(t)$ uniformly for $t \in I$. In particular, since $V_2$ is of Kato class uniformly with respect to $t \in I$, $\tilde{q}_1(t)$ is uniformly positive definite and

$$C^{-1}\tilde{q}_1(t)(u) \leq \tilde{q}_0(t)(u) \leq C\tilde{q}_1(t)(u), \quad u \in C_0^\infty(\mathbb{R}^d)$$

(6.3)

for a $t$-independent constant $C > 0$. Thus, $D([q_0(t)]) = D([q_1(t)])$ and $D([\tilde{q}_0(t)]) = D([\tilde{q}_1(t)])$. We denote by $H_0(t)$, $H_1(t)$, $\tilde{H}_0(t)$ and $\tilde{H}_1(t)$ selfadjoint joint operators defined respectively by $[q_0(t)]$, $[q_1(t)]$, $[\tilde{q}_0(t)]$ and $[\tilde{q}_1(t)]$. As in the previous section, we write $C(t)$ for $C(t) + C_1$ absorbing $C_1$ into $C(t)$.

Lemma 6.1. (1) Domains of $[q_0(t)]$, $[q_1(t)]$, $[\tilde{q}_0(t)]$ and $[\tilde{q}_1(t)]$ satisfy

$$D([q_0(t)]) = D([q_1(t)]) = D([\tilde{q}_0(t)]) = [\tilde{q}_1(t)] = \mathcal{Y} \subset D(L^{\frac{1}{2}}_Q)$$

and are independent of $t \in I$.

(2) There exists a constant $c > 0$ such that

$$[\tilde{q}_0(t)](u) \leq e^{c|t-s|}[\tilde{q}_0(s)](u), \quad u \in \mathcal{Y}, \quad t,s \in I.$$

(6.4)

(3) The gauge transform $G(t)$ maps $\mathcal{Y}$ onto $\mathcal{Y}$ and

$$[\tilde{q}_0(t)]G(t)u = [q_0(t)](u), \quad [\tilde{q}_1(t)]G(t)u = [q_1(t)](u), \quad u \in \mathcal{Y}.$$

(6.5)

Proof. By virtue of (4.3) corresponding to $A(t,x)$ and $\bar{V}(t,x)$, we have

$$\|Qu\|^2 + \|\nabla_{\tilde{A}(t)}u\|^2 \leq C\tilde{q}_0(t)(u), \quad u \in C_0^\infty(\mathbb{R}^d), \quad t \in I.$$

(6.6)
Hence, \(\|\dot{A}(t)u\|^2 \leq C\|Qu\|^2 \leq C\tilde{q}_0(s)(u)\) for any \(t, s \in I\) and by integration
\[
\|(\dot{A}(t) - \tilde{A}(s))u\| \leq C|t - s|\tilde{q}_0(s)(u)^{\frac{1}{2}}. \tag{6.7}
\]

Likewise, using, in addition to (6.6), assumption (1.25) and obvious identity
\[
\|\dot{V}(r)|^{1/2}u\| = \|\dot{V}(r)|^{1/2}|u|\|
\]
we obtain that
\[
\|\dot{V}(r)|^{1/2}u\|^2 \leq C(\|\nabla|u\|^2 + \|Qu\|^2) \leq C(\|\nabla\tilde{A}(s)u\|^2 + \|Qu\|^2) \leq C\tilde{q}_0(s)(u).
\]

Applying this to \(\dot{V}(t, x) - \dot{V}(s, x) = \int_s^t \dot{V}(r, x)dr\), we have
\[
|((\dot{V}(t) - \dot{V}(s))u, v)| \leq C|t - s|\tilde{q}_0(s)(u)^{\frac{1}{2}}\tilde{q}_0(s)(v)^{\frac{1}{2}}. \tag{6.8}
\]

Write \(\tilde{q}_0(t)(u, v) - \tilde{q}_0(s)(u, v)\) for \(u, v \in C_0^\infty(\mathbb{R}^d)\) in the form
\[
(\nabla\tilde{A}(s)u, i(\tilde{A}(s) - \tilde{A}(t))v) + (i(\tilde{A}(s) - \tilde{A}(t))u, \nabla\tilde{A}(s)v)
+ (\tilde{A}(t) - \tilde{A}(s))u, (\tilde{A}(t) - \tilde{A}(s))v) + (\dot{V}(t) - \dot{V}(s))u, v).
\]

We estimate each term separately by using (6.6), (6.7) and (6.8). We obtain for \(|t - s| \leq 1\) that
\[
|\tilde{q}_0(t)(u, v) - \tilde{q}_0(s)(u, v)| \leq C|t - s|\tilde{q}_0(s)(u)^{\frac{1}{2}}\tilde{q}_0(s)(v)^{\frac{1}{2}}. \tag{6.9}
\]

It follows that \(D([\tilde{q}_0(t)]) = D([q_0(s)])\) as in the proof of Lemma 5.1, all estimate above extend to \(u, v \in D([\tilde{q}_0(t)]) = D([q_0(s)])\) and
\[
[q_0(t)](u) \leq (1 + C|t - s|)[q_0(s)](u) \leq e^{C|t - s|}[q_0(s)](u). \tag{6.10}
\]

Argument above applies to \(q_0(t)\) as well and we have (6.6) for \(u \in D([q_0(t)])\);
\(D([q_0(t)]) = D([q_0(s)])\) for \(t, s \in I\); and estimate (6.10) holds for \([q_0(t)]\) and \([q_0(s)]\). Moreover, we have \(D([q_1(t)]) = D([\tilde{q}_1(t)])\) by virtue of characterization formula (1.19) of domains of the forms. Since \(\|L_Q^\frac{1}{2}u\|^2 \leq C(\|Qu\|^2 + \|\nabla\tilde{A}(t)u\|^2)\) for \(u \in C_0^\infty(\mathbb{R}^d)\), we also have \(D([\tilde{q}_0(t)]) \subset D(L_Q^\frac{1}{2})\) from (6.6).

Statements (1) and (2) follow.

Both \(\|\nabla\tilde{A}(t)G(t)u\| = \|\nabla\tilde{A}(t)u\|\) and \((V(t)G(t)u, G(t)u) = (V(t)u, u)\) are obvious for \(u \in C_0^\infty(\mathbb{R}^d)\). Since the latter space is a core of the forms \([q_0(t)]\) and \([\tilde{q}_0(t)]\), we see that \(D([\tilde{q}_0(t)]) = G(t)D([\tilde{q}_0(t)])\), \(G(t)\) maps \(\mathcal{Y}\) onto \(\mathcal{Y}\), and that \([\tilde{q}_0(t)][G(t)u] = [q_0(t)](u)\) for \(u \in \mathcal{Y}\). The corresponding relation for \([q_1(t)]\) and \([\tilde{q}_1(t)]\) may be proved similarly. \(\square\)
Before proceeding to the proof Theorem 1.10, we recall the following general fact: If $H$ is a positive selfadjoint operator in a Hilbert space $\mathcal{H}$, $\mathcal{H}_1 \subset \mathcal{H} \subset \mathcal{H}_{-1}$ is the scale of Hilbert spaces associated with $H$, viz. $\mathcal{H}_1 = D(H^{1/2})$ and $\mathcal{H}_{-1} = \mathcal{H}_1^*$ with $\mathcal{H}^*$ being identified with $\mathcal{H}$, then:

(i) $\mathcal{H}_{-1}$ is the completion of $\mathcal{H}$ by the norm $\|H^{-1/2}u\|$. 

(ii) $H$ has a natural extension $H_-$ to $\mathcal{H}_{-1}$ and $H_-$ is selfadjoint in $\mathcal{H}_{-1}$ with domain $D(H^{1/2})$. 

(iii) The part $H_+$ of $H_-$ in $\mathcal{H}_1$ is again selfadjoint with domain $D(H^{3/2})$.

These should be obvious if, by using spectral representation theorem, we represent $H$ as a multiplication operator by a positive function on $L^2(M, d\mu)$, $(M, d\mu)$ being a suitable measure space. 

**Proof of Theorem 1.10.** We equip $\mathcal{Y}$ with the inner product $q_0(u, v)$ and let $\mathcal{X}$ be its dual space as in the theorem. It is obvious that $\mathcal{Y} \subset \mathcal{X}$ densely and continuously. Lemma 6.1 yields statement (a) except for the fact that $H(t) \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and it is norm continuous. To prove the latter fact, we first show that the multiplication by $\langle x \rangle^2$ is bounded from $\mathcal{Y}$ to $\mathcal{X}$ by using (6.6) for $q_0(t)$:

$$\|\langle x \rangle^2u\|_\mathcal{X} = \sup_{v \in \mathcal{Y}, ||v||_\mathcal{Y} = 1} |\langle x \rangle^2 u, v| \leq C \sup_{v \in \mathcal{Y}, ||v||_\mathcal{Y} = 1} \|Qu\| \|Qv\| \leq C \sup_{v \in \mathcal{Y}, ||v||_\mathcal{Y} = 1} [q_0(t_0)](u)^{1/2}[q_0(t_0)](v)^{1/2} = C ||u||_\mathcal{Y}. \quad (6.11)$$

Then, we estimate for $u, v \in C_0^\infty(\mathbb{R}^d)$ via (6.5) for $[q_0(t)]$ as follows:

$$|\langle H(t)u, v \rangle| \leq [q_0(t)](u, v) + |\langle (C(t)<x>^2 u, v \rangle) \leq C(e^{2C|t-t_0|} + C(t))||u||_\mathcal{Y}||v||_\mathcal{Y}.$$

and $\|H(t)u\|_\mathcal{X} \leq C||u||_\mathcal{Y}$. This extends to $u \in \mathcal{Y}$ since $C_0^\infty(\mathbb{R}^d)$ is dense in $\mathcal{Y}$. Thus, $H(t) \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$. We have

$$\langle H(t) - H(s)u, v \rangle = \langle (H_0(t) - H_0(s))u, v \rangle - \langle (C(t) - C(s))\langle x \rangle^2 u, v \rangle = \langle (q_0(t) - q_0(s))u, v \rangle - \langle (C(t) - C(s))\langle x \rangle^2 u, v \rangle, \quad u, v \in \mathcal{Y}.$$

Thus, (6.9) for $q_0(t)$ and (6.11) imply $\|H(t) - H(s)\|_{\mathcal{B}(\mathcal{Y}, \mathcal{X})} \leq C(|t - s| + |C(t) - C(s)|)$ and statement (a) follows.

We define $\mathcal{Y}_t$ to be $\mathcal{Y}$ with new inner product $(u, v)_t = [\tilde{q}_0(t)](u, v)$ and $\mathcal{X}_t$ to be the dual space of $\mathcal{Y}_t$ with respect to the inner product of $\mathcal{H}$. Then, $\mathcal{X}_t \subset \mathcal{H} \subset \mathcal{Y}_t$ is the scale of Hilbert space associated with positive selfadjoint operator $\tilde{H}_0(t)$. Then, by virtue of the property (i), $\mathcal{X}_t$ is independent of
t as a set and is equal to $\mathcal{X}$ since $\mathcal{Y}_t = \mathcal{Y}$ is independent of $t$ as a set with equivalent Hilbert space structures. Properties (ii) and (iii) produce selfadjoint operators $\tilde{H}_0(t)_-$ and $\tilde{H}_0(t)_+$ in $\mathcal{X}_t$ and $\mathcal{Y}_t$ respectively. It is evident that $\tilde{H}_0(t)_-$ is a closed operator in $\mathcal{X}$ (with respect to the original norm) and $\tilde{H}_0(t)_+$ is its part in $\mathcal{Y}$. We now want to apply Theorem 2.1 to triplet $(\mathcal{X}, \mathcal{Y}, \tilde{H}_0(t)_-)$. We check conditions of Theorem 2.1 for $(\mathcal{X}, \mathcal{Y}, \tilde{H}_0(t)_-)$. Norm $\|u\|_{\mathcal{Y}_t}$ is equivalent with the original one of $\mathcal{Y}$ by virtue of the closed graph theorem. Estimate (6.4) implies that $\{\|u\|_{\mathcal{Y}_t} : t \in I\}$ satisfies condition (2.1) of Theorem 2.1 for $\mathcal{Y}_t$ and likewise for $\mathcal{X}_t$ by duality. From (6.9) we have:

$$|\langle (\tilde{H}_0(t)_- - \tilde{H}_0(s)_-)u, v \rangle| \leq c|t - s|\tilde{q}_0(s)(u)\tilde{q}_0(s)(v),$$

(6.12)

where $\langle \cdot, \cdot \rangle$ on the left is the coupling between $\mathcal{X}$ and $\mathcal{Y}$. This implies that:

$$\|(\tilde{H}_0(t)_- - \tilde{H}_0(s)_-)u\|_{\mathcal{X}_s} \leq c|t - s|\|u\|_{\mathcal{Y}_s}$$

(6.13)

and we see that $I \ni t \mapsto \tilde{H}_0(t)_- \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ is norm continuous.

Thus, there uniquely exists a family of operators $\{\tilde{U}(t, s) : t, s \in I\}$ which satisfies the properties of Theorem 2.1 for $(\mathcal{X}, \mathcal{Y}, \tilde{H}_0(t)_-)$. We define:

$$U(t, s) = G(t)^{-1}\tilde{U}(t, s)G(s).$$

We know that $G(t)$ maps $\mathcal{Y}$ onto $\mathcal{Y}$ by virtue of Lemma 6.1 and, (6.11) implies that, for $u \in \mathcal{Y}$, $I \ni t \mapsto G(t)u \in \mathcal{X}$ is continuously differentiable. Then, it is easy to check that $U(t, s)$ satisfies all properties of statement (b) except that $U(t, s)$ is a strongly continuous family of unitary operators in $\mathcal{H}$, which we now show. Define $u(t) = U(t, s)\varphi$ for $\varphi \in \mathcal{Y}$. Then, with $\langle \cdot, \cdot \rangle$ being the coupling of $\mathcal{X}$ and $\mathcal{Y}$, we have:

$$\partial_t(u(t), u(t))_{L^2} = 2\mathfrak{R}\{-iH(t)u(t), u(t)\}$$

$$= 2\mathfrak{R}\{-i\tilde{q}_0(t)(u(t), u(t)) + iC(t)(\langle x \rangle^2u(t), u(t))\} = 0.$$

It follows that $\|u(t)\| = \|\varphi\|$ and, since $\mathcal{Y}$ is dense in $\mathcal{H}$, we conclude $U(t, s)\mathcal{H} \subset \mathcal{H}$ and $\|U(t, s)\varphi\| = \|\varphi\|$ for all $\varphi \in \mathcal{H}$. Then, $U(t, s)$ must be unitary since $U(t, s)U(s, t)\varphi = \varphi$. If $\varphi \in \mathcal{Y}$, $(t, s) \mapsto U(t, s)\varphi \in \mathcal{H}$ is continuous in $\mathcal{H}$. Hence $U(t, s)$ is strongly continuous in $\mathcal{B}(\mathcal{H})$ by the unitarity. The uniqueness of $U(t, s)$ of Theorem 1.10 follows from the uniqueness result of Theorem 2.1 by tracing back the argument above. 

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