Random walks in small-world exponential treelike networks

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Abstract. In this paper, we investigate random walks in a family of small-world trees having an exponential degree distribution. First, we address a trapping problem, that is, a particular case of random walks with an immobile trap located at the initial node. We obtain the exact mean trapping time defined as the average of the first-passage times (FPT) from all nodes to the trap, which scales linearly with the network order $N$ in large networks. Then, we determine analytically the mean sending time, which is the mean of the FPTs from the initial node to all other nodes, and show that it grows with $N$, varying approximately as $N \ln N$. After that, we compute the precise global mean first-passage time among all pairs of nodes and find that it also varies approximately as $N \ln N$ in the large limit of $N$. After obtaining the relevant quantities, we compare them with each other and relate our results to the efficiency for information transmission by regarding the walker as an information messenger. Finally, we compare our results with those previously reported for other trees with different structural properties (e.g., degree distribution), such as the standard fractal trees and the scale-free small-world trees, and show that the shortest path between a pair of nodes in a tree is responsible for the scaling of the FPT between the two nodes.

Keywords: exact results, network dynamics, diffusion
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1. Introduction

As a powerful mathematical tool that can describe a large number of natural and synthetic systems, complex networks have received considerable interest from a wide range of scientific communities recently [1]–[3]. During the last decade, the main endeavors were devoted to understanding the structural features and dynamics of various networks [4]. In particular, treelike networks have attracted renewed attention, because the so-called border tree motifs are present in numerous real-life systems and play a significant role [5, 6]. The absence of loops in a treelike network has a drastic influence on diverse dynamic processes running on the network, e.g., the voter model [7] and naming game [8].

Among a plethora of dynamics, random walks in trees have received increasing attention in recent years, since the problem is related to a wide range of research fields, such as physics [9], biology [10], and cognitive science [11]. One of the most important quantities of random walks is the first-passage time (FPT) [12] defined as the expected time for a walker starting from a source point to first arrive at a target node [13, 14], which encodes much information about random-walk dynamics. Already, random walks in trees with different structures have been intensively studied, including the standard fractal trees (e.g., the T-fractal [15]–[19] and the Vicsek fractal [20, 21]) and scale-free trees [22]–[24]. These works uncovered how the mean first-passage time (MFPT), i.e., the average of FPTs between some given pairs of nodes, scales with the network size (number of nodes), and was thus helpful for understanding the impact of structural properties on the behavior of the MFPT. For example, it was shown that the MFPTs between two nodes in different trees behave differently. But the main reason for the difference remains not well understood. On the other hand, in contrast to the scale-free behavior [25], an exponential distribution is displayed by some real networks as well [26]. Relevant work on random walks on such networks is much less common. In particular, what is the main factor affecting the speed of diffusion in general trees is still not well understood. A goal of this work is to answer this question, at least partially.
In this paper, we study a simple random walk [27] on a family of deterministically growing small-world trees exhibiting an exponential form of degree distribution [28]. We first address a trapping problem, which is a particular random walk with a single trap positioned at the initially created node of the networks. We derive analytically the mean trapping time (MTT) defined as the average of the FPTs from all nodes to the trap, which varies linearly with the network size \( N \). We then investigate the partial mean first-passage time (PMFPT) (i.e., the average of the FPTs from the initial node to a randomly selected target node) and the global mean first-passage time (GMFPT) that is the average of the FPTs among all pairs of nodes in the networks. Both PMFPT and GMFPT are determined through the connection between random walks and electrical networks. In contrast to the MTT, both PMFPT and GMFPT are asymptotic to \( N \ln N \) for large networks. We relate our results to the efficiency of information diffusion by considering the walker as an information messenger. We also compare our results with those found for other trees with different architectures and consequently give possible reasons for the behavioral difference for random walks between the trees considered and other comparable trees.

2. The network model and its properties

We first introduce the network model under consideration, which is built iteratively and has a treelike structure [28]. Let \( U_g \) (\( g \geq 0 \)) be the family of networks after \( g \) iterations. Initially (\( g = 0 \)), \( U_0 \) is a single isolated node without any edge, called the initial node below. For \( g \geq 1 \), \( U_g \) is obtained from \( U_{g-1} \) by adding a path of \( m \) nodes (\( m \) is a natural number equal to or greater than 2) to each existing node in \( U_{g-1} \); see figure 1. By construction, it is easy to see that the numbers of nodes and edges in \( U_g \) are \( N_g = m^g \) and \( E_g = N_g - 1 = m^g - 1 \), respectively. Figure 2 shows the growing process for a particular network for the case of \( m = 3 \). Notice that for \( m = 2 \), the model reduces to the deterministic uniform recursive tree proposed in [29], which was extended and extensively studied thereafter [30]–[34].

The networks considered here have some properties observed for many real networks. According to the construction process, at any new iteration, the degree of every old node increases by 1, independently of the node degrees. Thus, the degree distribution of the networks is exponential, instead of a power law [25]. On the other hand, the diameter (i.e., the maximum of the distances of the shortest paths between all pair of nodes) of \( U_g \) is \((2g - 1)(m - 1)\) which grows logarithmically with the network size, showing that the networks are of small-world type [35]. Moreover, some other properties, e.g., the adjacency spectrum [28], can also be determined analytically.
3. Random walks on the networks

After introducing the family $U_g$ of networks and their properties, we will study the discrete random walks \[27\] performed on $U_g$. At each time step, the walker (particle) jumps uniformly (i.e., with the same probability) from its current position to any of its neighboring nodes. One of the most important quantities characterizing such a random walk is the FPT \[12\]. Let $F_{i,j}(g)$ denote the FPT for a walker, starting from node $i$ in $U_g$, to first arrive at node $j$. What we are concerned with is how the scalings of FPTs behave as the network size increases.

In the following, we will focus on three cases of random walks. Firstly, we will investigate a trapping issue, namely, random walks with a single immobile trap located at the initial node, and determine the MFPT to the initial node averaged over all nodes in $U_g$. Then, we will compute the MFPT from the initial node to another node selected uniformly from all nodes in $U_g$. Finally, we will determine the MFPT between all pairs of nodes in $U_g$.

3.1. The MFPT from all other nodes to the initial node

First, we study a particular trapping problem on $U_g$, in which the single trap is positioned at the initial node. To facilitate computation, we introduce an alternative construction method for the networks, which highlights their self-similar architecture, as follows. Suppose one has $U_g$. The next iteration of the network, $U_{g+1}$, can be obtained by joining $m$ copies of $U_g$ in the way illustrated in figure 3.

For convenience of description, we label the initial node in $U_g$ as 1, while the duplicates of initial nodes in $U_{g-1}(\eta = 2, 3, \ldots, m)$ are sequentially labeled as 2, 3, \ldots, $m$. In addition, let $T_i(g)$ be the FPT, $F_{i,1}(g)$, also called the trapping time, of node $i$ in $U_g$, which is the expected time for a walker starting from $i$ to first visit the trap node. Obviously, for all $
Figure 3. Another construction of the network. The networks after $g+1$ iterations, $U_{g+1}$, consist of $m$ replicas of $U_g$ denoted $U_g^{(1)}$, $U_g^{(2)}$, $U_g^{(3)}$, ..., and $U_g^{(m)}$, which are connected to each other by adding $m-1$ edges among the $m$ copies of the initial node.

$g \geq 0$, $T_1(g) = 0$. Let $T_{tot}(g)$ express the total trapping times for all nodes in $U_g$, i.e.,

$$T_{tot}(g) = \sum_{i \in U_g} T_i(g).$$

Then, the MFPT, also called the mean trapping time (MTT) and denoted by $\langle T \rangle_g$, is the average of $T_i(g)$ over all starting nodes distributed uniformly in $U_g$, given by

$$\langle T \rangle_g = \frac{1}{N_g} \sum_{i \in U_g} T_i(g) = \frac{T_{tot}(g)}{N_g}.\ (2)$$

Thus, to obtain $\langle T \rangle_g$, one should first determine $T_{tot}(g)$.

We first compute the FPTs, $F_{i+1,i}(g)$ ($i = 1, 2, \ldots, m-1$), between two arbitrary copies of the initial node of $U_{g-1}$ that are directly connected to each other. Since $U_g$ has a treelike structure, according to the result obtained previously in [18, 36, 37], e.g., equation (2) in [18], we have

$$F_{i+1,i}(g) = 2(m-i)m^{g-1} - 1\ (3)$$

for all $i \in [1, m-1]$, yielding

$$T_{tot}(g) = T_{tot}(g-1) + \sum_{i=2}^{m} \sum_{i \in U_{g-1}^{(i)}} T_i(g)$$

$$= mT_{tot}(g-1) + \sum_{i=2}^{m} [N_{g-1}F_{i,1}(g)].\ (4)$$

Considering the treelike structure of $U_g$, we have

$$F_{i,1}(g) = F_{i,i-1}(g) + F_{i-1,i-2}(g) + \cdots + F_{3,2}(g) + F_{2,1}(g).\ (5)$$

Plugging equations (3) and (5) into equation (4), we obtain

$$T_{tot}(g) = mT_{tot}(g-1) + \frac{1}{2}(2m^2 - 3m + 1)m^{2g-1} - \frac{1}{2}(m-1)m^g.\ (6)$$

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With the initial condition $T_{\text{tot}}(0) = 0$, equation (6) is inductively solved, giving

$$T_{\text{tot}}(g) = m^g \left( \frac{2}{3} m - \frac{1}{3} \right) - m^g \left[ \left( \frac{g}{2} + \frac{2}{3} \right) m - \left( \frac{g}{2} + \frac{1}{3} \right) \right].$$  \hspace{1cm} (7)

Inserting equation (7) into equation (2), we obtain a closed-form expression for the MTT on network $U_g$, as

$$\langle T \rangle_g = m^g \left( \frac{2}{3} m - \frac{1}{3} \right) - \left( \frac{g}{2} + \frac{2}{3} \right) m + \left( \frac{g}{2} + \frac{1}{3} \right).$$  \hspace{1cm} (8)

Recalling $N_g = m^g$, we have $g = \ln N_g / \ln m$. Thus, $\langle T \rangle_g$ can be expressed in terms of the network size $N_g$ as

$$\langle T \rangle_g = \left( \frac{2}{3} m - \frac{1}{3} \right) N_g - \frac{\ln N_g}{2 \ln m} (m - 1) - \left( \frac{2}{3} m - \frac{1}{3} \right).$$  \hspace{1cm} (9)

Therefore, for large networks (i.e., $N_g \to \infty$),

$$\langle T \rangle_g \sim N_g,$$  \hspace{1cm} (10)

implying that the MTT $\langle T \rangle_g$ increases linearly with the network size, independently of the degree of the initial node. This linear scaling is in sharp contrast to the scaling of random exponential trees [23], in which the MTT depends on the degree of the trapping node.

### 3.2. The MFPT from the initial node to all other nodes

By definition, $F_{1,i}(g)$ denotes the FPT of the walker visiting node $i$ for the first time, assuming that the walker started at the initial node in $U_g$. Let $\langle H \rangle_g$ represent the mean value of $F_{1,i}(g)$ averaged over all target nodes $i$ in network $U_g$, called the partial mean first-passage time (PMFPT). Then, $\langle H \rangle_g$ is given by

$$\langle H \rangle_g = \frac{1}{N_g} \sum_{i \in U_g} F_{1,i}(g).$$  \hspace{1cm} (11)

Thus, the problem of finding $\langle H \rangle_g$ is reduced to determining the sum $\sum_{i \in U_g} F_{1,i}(g)$, denoted as $H_{\text{tot}}(g)$.

Unfortunately, the method for computing $T_{\text{tot}}(g)$ is not suitable for $H_{\text{tot}}(g)$. So, we seek for a feasible technique for deriving $H_{\text{tot}}(g)$. Below, we will apply the link between effective resistance and the FPTs for random walks [38, 39] to calculate $H_{\text{tot}}(g)$ analytically. For this purpose, we replace each edge of $U_g$ by a unit resistor to obtain the corresponding resistor networks. To do so, let $R_{i,j}(g)$ represent the effective resistance between two nodes $i$ and $j$ of $U_g$. Then, we have $[38, 39] F_{i,j}(g) + F_{j,i}(g) = 2E_g R_{i,j}(g)$, which leads to

$$H_{\text{tot}}(g) + T_{\text{tot}}(g) = 2E_g \sum_{i \in U_g} R_{i,i}(g).$$  \hspace{1cm} (12)

Equation (12) shows that if we have the sum $\sum_{i \in U_g} R_{i,i}(g)$ on the right-hand side, then we can easily obtain $H_{\text{tot}}(g)$. Since, for any tree, the effective resistance $R_{i,j}(g)$ is equal to the geodesic distance $d_{i,j}(g)$ between $i$ and $j$, this makes it possible to determine

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the sum $\sum_{i \in U_g} R_{1,i}(g)$. To this end, we introduce a new quantity $d_g$, which is the sum of shortest distances between the initial node 1 and all other nodes in $U_g$. By definition, we have

$$d_g = \sum_{i \in U_g} d_{1,i}(g). \quad (13)$$

Considering the self-similar network structure (see figure 3), we can easily obtain the recursion relation

$$d_g = m d_{g-1} + \frac{m(m - 1)}{2} N_{g-1}. \quad (14)$$

Using $d_1 = m(m - 1)/2$, equation (14) is solved, giving

$$d_g = \frac{1}{2} g(m - 1)m^g. \quad (15)$$

Then, we have

$$H_{\text{tot}}(g) = 2E_g d_g - T_{\text{tot}}(g) = m^{2g} \left[ g(m - 1) - \frac{2}{3}m + \frac{1}{3} \right] - m^g \left( \frac{m - 1}{2} g + \frac{2}{3}m - \frac{1}{3} \right). \quad (16)$$

and

$$\langle H \rangle_g = m^g \left[ g(m - 1) - \frac{2}{3}m + \frac{1}{3} \right] - \left( \frac{m - 1}{2} g + \frac{2}{3}m - \frac{1}{3} \right). \quad (17)$$

Equation (17) can be rewritten as a function of the network size $N_g$, as

$$\langle H \rangle_g = N_g \left[ \frac{(m - 1) \ln N_g}{\ln m} - \frac{2}{3}m + \frac{1}{3} \right] - \left[ \frac{(m - 1) \ln N_g}{2 \ln m} + \frac{2}{3}m - \frac{1}{3} \right]. \quad (18)$$

Therefore, in the limit of the large network size $N_g$,

$$\langle H \rangle_g \sim N_g \ln N_g. \quad (19)$$

### 3.3. The MFPT between all node pairs

In what follows, we will calculate the MFPT $\langle F \rangle_g$ among all node pairs in $U_g$, commonly called the global mean first-passage time (GMFPT) [40]. It should be noted that the GMFPT has been studied in [34]. Here we will derive an equivalent result using an approach different from and easier than that in [34].

By definition, $\langle F \rangle_g$ is given by

$$\langle F \rangle_g = \frac{F_{\text{tot}}(g)}{N_g(N_g - 1)} = \frac{1}{N_g(N_g - 1)} \sum_{i \neq j}^{N_g} \sum_{j=1}^{N_g} F_{i,j}(g), \quad (20)$$

where the sum

$$F_{\text{tot}}(g) = \sum_{i \neq j}^{N_g} \sum_{j=1}^{N_g} F_{i,j}(g) \quad (21)$$

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For brevity, we use $D_U$ to denote $\sum_{i\neq j}^{N_g} \sum_{j=1}^{N_g} d_{i,j}(g)$, which is the total geodesic distance among all pairs of nodes $U_g$. According to the value of the distance between two edge nodes in $U_{g-1}$, and $N$ length $\Delta$, with the initial condition $D_U$ consistent with the result previously obtained in [34].

The expression for $\langle F \rangle_g$ can be recast in terms of $\Delta g$ elements in class $\Delta g$ $\alpha, \beta$.

\[
\langle F \rangle_g = \frac{E_D g}{N_g (N_g - 1)} = \frac{D_g}{N_g} = m^g [g(m - 1) - \frac{2}{3} m + \frac{1}{3}] + \frac{2}{3} m - \frac{1}{3},
\]

which can be rewritten in terms of $N_g$ in the following form:

\[
\langle F \rangle_g = N_g \left[ \frac{(m - 1) \ln N_g}{\ln m} - \frac{2}{3} m + \frac{1}{3} \right] + \frac{2}{3} m - \frac{1}{3},
\]

consistent with the result previously obtained in [34].

Equation (27) uncovers the explicit dependence relation of $\langle F \rangle_g$ on the network size $N_g$ and the parameter $m$. In the case of $N_g \rightarrow \infty$, we have the following expression:

\[
\langle F \rangle_g \sim N_g \ln N_g.
\]
3.4. Analysis and comparison

Our results can be related to the efficiency of information transmission. Notice that if we consider the walker in the random-walk dynamics as an information messenger, then the $\langle T \rangle_g$ measures the efficiency of the initial node (as a receiver) in receiving information, while $\langle H \rangle_g$ shows how efficient the initial node is as a sender for transmitting information to other nodes, and $\langle F \rangle_g$ is the efficiency of information sending when the sender is distributed with equal probability among all nodes.

The above results, provided in equations (10) and (19), show evidently that the dominant behaviors for $\langle T \rangle_g$ and $\langle H \rangle_g$ are different. The former follows $\langle T \rangle_g \sim N_g$, while the latter obeys $\langle H \rangle_g \sim N_g \ln N_g$, greater than the former. This means that the initial node is more efficient in receiving information than sending information. On the other hand, equation (28) together with equations (10) and (19) means that although the efficiency of the initial node in receiving information is higher than that of the average over other nodes, its ability to send information is similar to those of the others. Equations (10), (19) and (28) also show that the linear scaling of the efficiency for the initial node measured by the MTT is not a representative property of the networks, but the $N_g \ln N_g$ behavior for the initial node sending information is a representative property.

Our results obtained can be compared with those previously reported for other treelike networks. Equations (10) and (28) imply that the position of the trap significantly affects the scalings of the MTT for the trapping problem with a single trap. This behavior is similar to that in the small-world scale-free trees [24, 29], but is in contrast to that of trapping in the $T$-fractals [16, 18, 19] and the fractal scale-free trees [24], where the MTT does not depend on the trap location. In contrast, as shown in equations (19) and (28), the initial node has the same dominant scaling of the PMFPT as that of the average of PMFPTs over all senders. This equality between PMFPT and GMFPT among all node pairs has also been observed for other trees, including the $T$-fractals [16, 18, 19] and fractal scale-free trees [24]. However, the scaling of PMFPT for different trees may have different behaviors.

In addition, the leading asymptotic $N_g \ln N_g$ dependence of GMFPT with the network size is also compared to the scalings found from other treelike networks with different degree distributions. In the standard fractal trees, such as the $T$-fractals [16, 18, 19] and the Vicsek fractals [21], the GMFPT $\langle F \rangle$ increases superlinearly with the network size $N$, which has also been observed from the family of scale-free trees with fractality [24]. For star graphs, the GMFPT $\langle F \rangle$ grows linearly with $N$ [21]; while for linear chains, $\langle F \rangle$ scales as a square root of $N$ [21]. However, for the class of scale-free small-world trees [24, 29], the GMFPT $\langle F \rangle$ also changes with $N$ as $\langle F \rangle \sim N \ln N$, which follows the same scaling as the exponential trees studied here.

Finally, combining the present work and the previous studies, it can be seen that random walks in trees display rich behaviors in the context of the FPT. At first sight, degree distribution is perhaps the root of the rich phenomena. However, in [24], it was shown that the FPT in fractal and non-fractal scale-free networks may exhibit quite disparate scalings, meaning that the degree distribution alone cannot determine the FPT for random walks on trees. We argue that the FPT on trees is determined by the length of the shortest path from the resource node to the target node, while the impact of other structural properties is encoded in the length of the shortest path, since the FPT $F_{ij}$...
from an arbitrary node $i$ to $j$ is actually related to the FPTs of those node pairs for two directly connected nodes along the unique shortest-path direction to the target node, i.e.,

$$F_{ij} = F_{i1} + F_{1i2} + F_{i2i3} + \cdots + F_{ixiy} + F_{iyj}$$

provided that $i - i_1 - i_2 - i_3 - \cdots - i_x - iy - j$ is the shortest path from $i$ to $j$. For example, the FPT on the non-fractal treelike scale-free networks [24] and on the exponential networks studied here display similar behaviors, since both kinds of networks are of small-world type with the average distance increasing logarithmically in the network size, in spite of them having distinct degree distributions. As another example, the FPT on some standard fractal trees (e.g. the $T$-fractals [18, 19] and the Vicsek fractals [21]) displays a superlinear dependence on the system size, which is also due to their average distance.

4. Conclusions

We have presented a detailed analysis of the simple random walks on a class of treelike small-world networks exhibiting an exponential degree distribution. We first investigated the trapping problem, focusing on a peculiar case with the trap fixed at the initial node, and obtained the exact solution to the MTT, the dominating scaling of which varies linearly with the network size $N$. We then studied the random walks starting from the initial node, and determined analytically the PMFPT from the initial node to all other nodes, whose dominant behavior scales with $N$ as $N \ln N$. Moreover, we determined explicitly the GMFPT among all node pairs and showed that the GMFPT also increases with $N$ approximately as $N \ln N$. We finally related our results in terms of information transmission by regarding the walker as an information messenger, and compared them with those previously reported for other treelike networks with disparate topological properties. Our work provides new and useful insight into random-walk dynamics running on treelike networks, and could further deepen our understanding of random walks on a tree [40].

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