On the Three Dimensional Mass-Weighted Isentropic Time Mean Equation for Rossby Waves

Takenari Kinoshita 1, Koutarou Takaya 2, and Toshiki Iwasaki 3

1 Japan Agency for Marine-Earth Science and Technology, Kanagawa, Japan
2 Department of Physics, Faculty of Science, Kyoto Sangyo University, Kyoto, Japan
3 Department of Geophysics, Graduate School of Science, Tohoku University, Miyagi, Japan

Abstract

The mass-weighted isentropic zonal mean (Z-MIM) equations derived by T. Iwasaki are powerful tools for diagnosing meridional circulation and wave-mean interaction, especially for the lower boundary and unstable waves. Recently, some studies have extended the equations to three dimensions by using the time mean instead of the zonal mean. However, the relation between wave activity flux and residual mean flow is unclear. In the present study, we derive the three-dimensional (3D) wave activity flux and residual mean flow for Rossby waves on the mass-weighted isentropic time mean equations. Next, we discuss the relation between the obtained formulae and 3D transformed Eulerian-mean (TEM) equations.

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1. Introduction

Transformed Eulerian-mean (TEM) equations are widely used to diagnose wave propagation and wave-mean interaction in meridional cross-section (e.g., Andrews and McIntyre 1976, 1978). However, TEM equations [including three dimensional (3D) TEM equations derived by many studies] have limitations in terms of expressing the lower boundary (Tanaka et al. 2004) and unstable waves, such as baroclinic instability waves (Noda 2014). The mass-weighted isentropic zonal mean (Z-MIM) equations derived by Iwasaki (1989, 1990) are equations that overcome these limitations. Recently, Kinoshita et al. (2016) extended wave activity flux on Z-MIM equations to 3D, and Kanno and Iwasaki (2018) derived mass-weighted isentropic time mean (T-MIM) meridional circulation. However, the relation between 3D wave activity flux and T-MIM meridional circulation has not been fully understood. Also, the difference between 3D-TEM and T-MIM equations has not been investigated.

As a challenge to understand the 3D structure of wave-mean interaction by using the TEM framework, the purpose of the study is to derive the 3D residual mean flow and wave activity flux for Rossby waves on T-MIM equations. The derived 3D residual mean flow and wave activity flux are compared with those derived in the 3D-TEM equations (Plumb 1986). In Section 2, the 3D wave activity flux and residual mean flow are derived by using quasi-geostrophic potential vorticity equation on T-MIM equations. The comparison between the derived formulae and those derived in the 3D-TEM equations is shown in Section 3. Conclusions are given in Section 4.

2. 3D wave activity flux and residual mean flow on T-MIM equations

2.1 Quasi-geostrophic potential vorticity equation

To derive the 3D wave activity flux and corresponding residual mean flow for Rossby waves on T-MIM equations, a quasi-geostrophic potential vorticity equation on the T-MIM framework is derived. First, we assume that nonconservative terms including diabatic heating are negligible. The quasi-geostrophic potential vorticity equation for isentropic coordinates was derived by Ber risford et al. (1993), which is written as follows;

\[
\frac{\partial}{\partial t} \psi + u \cdot \nabla \psi + v \cdot \nabla \psi = 0,
\]

\[
q \equiv \bar{f}_0 + \beta \gamma + \nabla^2 \psi - \frac{f_0^2}{p_0} \frac{\partial}{\partial \theta} (p_0 \bar{\theta} \bar{\psi}),
\]

\[
u = -\frac{M}{f_0} = -\psi, \quad v = -\frac{M}{f_0} = \psi,
\]

where \( u \) and \( v \) are the zonal and meridional geostrophic flows, respectively, \( \bar{f}_0 \) is a constant Coriolis parameter, \( \beta \) is a beta effect, \( M \) is a Montgomery stream function, \( \psi \) is a stream function, \( \bar{f}_0 \) is a reference pressure, \( p_0 \) is a reference density, and \( \nabla \) is a horizontal partial differential operator on the isentropic surface, \( \bar{p}_0(\theta) \) is a reference temperature. The suffixes \( x \), \( y \), and \( \theta \) denote respective partial derivatives on zonal, meridional, and vertical directions. Here, we define the mass-weighted time mean as \( \bar{} \) and its deviation as \( \prime \). From the difference between quasi-potential vorticity equation (1) and that of mass-weighted time mean, the quasi-geostrophic potential vorticity equation for perturbation is written as follows.

\[
\frac{\partial}{\partial t} q' + u' \cdot \nabla q' + v' \cdot \nabla q' = 0,
\]

\[
q' = \nabla^2 \psi' - \frac{f_0^2}{p_0} \frac{\partial}{\partial \theta} (p_0 \bar{\theta} \bar{\psi'}),
\]

\[
N_{\psi} = -\frac{f_0^2}{p_0 p_0'}.
\]

where \( N_{\psi} \) is the static stability, and we assume that second derivatives of reference variables are negligibly small. Note that the following equation shows the definition of \( N_{\psi} \), the same as that of Kinoshita et al. (2016).

\[
\frac{1}{p_0} \frac{\partial}{\partial \theta} \frac{R T}{p_0} = R \left( \frac{p_0}{p_0'} \right)^\kappa = R \frac{p_0}{p_0'}
\]

where \( p_s \) is a surface pressure, \( R \) is a gas constant for dry air, \( \kappa \equiv R/C_p \), and \( C_p \) is a specific heat at a constant pressure.

2.2 3D wave activity flux

In this section, we derive the 3D wave activity flux by using the potential enstrophy equation in the T-MIM framework. Taking \( q' \times (2) \) and using the mass-weighted time mean yields

\[
\frac{1}{\rho_0 \bar{\theta}} \frac{\partial}{\partial \theta} \frac{R T}{p_0} = R \left( \frac{p_0}{p_0'} \right)^\kappa = R \frac{p_0}{p_0'}
\]
\[
\frac{\partial \bar{q}^2}{\partial t} + \bar{u} \cdot \nabla \bar{q}^2 + \bar{u} \cdot \bar{q}^2 \cdot \nabla \bar{q} = 0. \tag{4}
\]

When the pseudo momentum for Rossby waves is defined as \( A = \frac{\sigma \bar{q}^2}{\nabla \bar{q}^2} \frac{\psi^2}{2} \) where \( \sigma = -\gamma p \), \( p \) is the pressure and \( \gamma \) is the magnitude of the gravity acceleration, (4) is rewritten as:

\[
\frac{\partial \bar{u} \cdot \bar{q}^2}{\partial t} + \bar{u} \cdot \nabla \bar{q}^2 + \frac{\sigma \bar{u} \cdot \bar{q}^2}{\nabla \bar{q}^2} \cdot \nabla \bar{q} = 0. \tag{5}
\]

When we use the assumptions that the derivative of time-mean state has an amplitude \( O(\alpha) \), which is slowly varying time-mean state, and horizontal divergence of horizontal geostrophic flows for perturbations is the second order of perturbation and negligibly small \((\nu^2 + \nu^3 = O(\alpha^2))\), the third term of the left-hand side of (5) is rewritten as

\[
\frac{\sigma \bar{u} \cdot \bar{q}^2}{\nabla \bar{q}^2} \cdot \nabla \bar{q} = \left( \nabla \cdot + \frac{\partial}{\partial t} \right) \mathbf{M}_R.
\]

\[
\mathbf{M}_R = \frac{\sigma}{\nabla \bar{q}^2} \left( -\bar{q}_2 \psi_0 \bar{q}_2 \frac{\partial}{\partial \bar{q}} - \bar{q}_2 \frac{\partial \psi_2}{\partial \bar{q}} \frac{\partial}{\partial \bar{q}} \right) + \frac{\sigma}{\nabla \bar{q}^2} \left( -\bar{q}_2 \psi_0 \bar{q}_2 \frac{\partial}{\partial \bar{q}} + \bar{q}_2 \frac{\partial \psi_2}{\partial \bar{q}} \frac{\partial}{\partial \bar{q}} \right) - \frac{\sigma}{\nabla \bar{q}^2} \left( -\bar{q}_2 \psi_0 \bar{q}_2 \frac{\partial}{\partial \bar{q}} + \bar{q}_2 \frac{\partial \psi_2}{\partial \bar{q}} \frac{\partial}{\partial \bar{q}} \right) + \frac{\sigma}{\nabla \bar{q}^2} \left( -\bar{q}_2 \psi_0 \bar{q}_2 \frac{\partial}{\partial \bar{q}} - \bar{q}_2 \frac{\partial \psi_2}{\partial \bar{q}} \frac{\partial}{\partial \bar{q}} \right).
\]

This 3D wave activity flux is similar to that of Plumb (1986) and is equal to the product of the pseudo momentum and group velocity. The latter is shown in the following calculations. For a perturbation, the form of a plane wave is considered:

\[
\psi' = \psi \exp[i(kx + \nu y + m\theta - ct)], \tag{8}
\]

where \( k, \ell \) and \( m \) are the zonal, meridional and vertical wavenumbers, respectively, and \( \omega \) is the ground-based angular frequency. Substituting (8) into (2), the dispersion relation of Rossby waves is obtained as

\[
\omega = -\frac{G}{K^2} k^2 \psi_2 \frac{\partial}{\partial \bar{q}} + \frac{f_0^2}{N_0} \frac{\partial}{\partial \bar{q}} + \frac{\sigma}{\nabla \bar{q}^2} \frac{\partial}{\partial \bar{q}}.
\]

The intrinsic group velocity is written as

\[
\tilde{C}_g = \frac{-2kG_2 + (k^2 - K^2)\tilde{q}_2}{K^4}, \tag{9}
\]

\[
\tilde{C}_g = \frac{K^2}{K^4} \frac{2f_0^2 N_0^2}{k^2} k^2 \tilde{q}_2^2, \tag{10}
\]

where \( K = (k, l, f_0 N_0, m)^T \) is the wavenumber vector. Next, the pseudo momentum and 3D wave activity flux \( \mathbf{M}_a \) are written in terms of \( \psi' \) as

\[
A = \frac{\bar{\sigma} K^4}{2} \bar{\psi}^2 \frac{\partial}{\partial \bar{q}} - \bar{\psi}_0 \bar{\psi}_0 \frac{\partial}{\partial \bar{q}} + \frac{f_0^2}{N_0^2} \frac{\partial}{\partial \bar{q}} - \bar{\psi}_0 \bar{\psi}_0 \frac{\partial}{\partial \bar{q}} - \bar{\psi}_0 \bar{\psi}_0 \frac{\partial}{\partial \bar{q}} - \bar{\psi}_0 \bar{\psi}_0 \frac{\partial}{\partial \bar{q}}.
\]

Then, we obtain the relation

\[
\mathbf{M}_R = \tilde{C}_g A. \tag{12}
\]

Thus, the derived 3D wave activity flux can describe Rossby wave propagation.

2.3 3D residual mean flow

Next, the residual mean flow in T-MIM equations is derived by using the obtained 3D wave activity flux. The 3D wave activity flux (6) is substituted into the zonal and meridional momentum equations of the T-MIM framework

\[
\frac{\partial}{\partial t} + \bar{u} \cdot \nabla \bar{u} + \bar{v} \cdot \nabla \bar{v} - \bar{f}_0 \bar{\psi}^2 = \frac{1}{\sigma} \left( \nabla \cdot + \frac{\partial}{\partial \bar{q}} \right) \mathbf{M}_R(\bar{u}),
\]

\[
\frac{\partial}{\partial t} + \bar{u} \cdot \nabla \bar{u} + \bar{v} \cdot \nabla \bar{v} - \bar{f}_0 \bar{\psi}^2 = \frac{1}{\sigma} \left( \nabla \cdot + \frac{\partial}{\partial \bar{q}} \right) \mathbf{M}_R(\bar{v}),
\]

where

\[
\bar{u}_0 = \bar{u}_0 + \bar{u} + \bar{f}_0 \bar{\psi}^2, \quad \bar{v}_0 = \bar{v}_0 + \bar{v} + \bar{f}_0 \bar{\psi}^2,
\]

\[
\bar{u}_0 = \bar{u}_0 + \bar{u} + \bar{f}_0 \bar{\psi}^2, \quad \bar{v}_0 = \bar{v}_0 + \bar{v} + \bar{f}_0 \bar{\psi}^2,
\]

\[
\bar{u}_0 = \bar{u}_0 + \bar{u} + \bar{f}_0 \bar{\psi}^2, \quad \bar{v}_0 = \bar{v}_0 + \bar{v} + \bar{f}_0 \bar{\psi}^2.
\]

where \( \bar{u}_0 = u - u \) and \( \bar{v}_0 = v - v \) are zonal and meridional ageostrophic flows, respectively. The obtained flow is similar to the 3D residual mean flow derived by Plumb (1986). Note that the obtained flow corresponds to 3D residual mean flow in the T-MIM framework, which is shown in Appendix. However, the residual flow is different from the T-MIM flow of Kanno and Iwasaki (2018).
Note that $u^* \sigma^2 / \tilde{\sigma}$ is a temporal correlation of mass-weight perturbations and velocity perturbations, which indicates the eddy-correlated mass transport velocity and is the so-called bolus velocity (e.g., Rhines 1982; Gent et al. 1995; Lee et al. 1997; McDougall and McIntosh 1996; McDougall and McIntosh 2001; Aiki et al. 2015; Kanno and Iwasaki 2018). When we use the assumption that the difference between $\psi'$ and $\psi''$ is the second order of perturbation and negligibly small ($\psi'' = -\sigma^2 \tilde{\sigma} \psi'' = O(\alpha^2)$), the difference between the second order of the perturbation terms of residual mean flow and T-MIM flow is $(\psi''_x^2 + \psi''_z^2)/2f_0$, and $-((\psi''_x^2 + \psi''_z^2)/2f_0)$. Note that the time-mean geostrophic flows $\tilde{u}$ and $\tilde{v}$ in equation (14) are balanced with $-f_0^{-1} \tilde{M}_x$ and $f_0^{-1} \tilde{M}_z$, and these are excluded from equation (13). Since T-MIM flow is equal to the sum of time-mean flow and bolus velocity (14), and the obtained residual mean flow is equal to the sum of time-mean flow and quasi-geostrophic-Stokes (SG-Stokes) correction, which is approximately equal to Lagrangian-mean flow and is shown in appendix (30). Note that the SG-Stokes correction is different from the quasi-Stokes correction, which is defined as the difference between mass-weighted isotropic mean flow and Eulerian-mean flow (not unweighted isotropic mean flow), (e.g., McDougall and McIntosh 1996). Thus, the terms $(\psi''_x^2 + \psi''_z^2)/2f_0$ correspond to the difference between bolus velocity and SG-Stokes drift. Note that the vertical residual mean flow in T-MIM equations is equal to the vertical component of T-MIM velocity $\tilde{\psi}$.

3. Relation between T-MIM and 3D-TEM equations

To examine the relation between 3D wave activity flux and residual mean flow for Rossby waves in TEM equations and those derived in this study, we perform the following calculations. First, the term $\sigma$ can be rewritten as follows when $\sigma$ depends on only the vertical axis of the log-pressure height coordinate.

$$\sigma = -g^{-1} \tilde{p}_0 = -g^{-1} \tilde{p} \frac{\rho_0}{\theta_0} \frac{\theta^*}{\theta_0} = \frac{\rho_0(z)}{\theta_0(\theta)} \approx \frac{\rho_0(z_0)}{\theta_0(\theta_0)},$$

(16)

where $z'$ and $z$ are the geometric and log-pressure height, respectively, $\rho_0$ is the basic density, and $\theta_0$ is the reference potential temperature. By referencing Iwasaki (1992) and using relation (16), the zonal component of 3D wave activity flux in T-MIM equations (7) is rewritten as

$$\frac{\sigma}{|\tilde{\psi}|} (\tilde{\psi}^2 - e^\prime + \tilde{v} \tilde{\psi} \tilde{\psi}^\prime) \approx \frac{\rho_0}{\theta_0} \left[ \frac{\psi^2}{\tilde{\psi}^2} - e^\prime \rho_0^2 \left( \frac{\theta^*}{\theta_0} \right)^2 \right] \left( \frac{\rho_0}{\theta_0} \right) \left( \frac{\theta^*}{\theta_0} \right) \left( \frac{\theta^*}{\theta_0} \right)$$

where $\frac{\sigma}{|\tilde{\psi}|}$ is the time mean on the log-pressure height coordinate and $\tilde{\beta}$ is its deviation. Note that the perturbation difference between $A$ and $A'$ is $(\tilde{\alpha} \tilde{\tilde{a}})^2 A = O(\alpha^2)$ under the slowly varying time-mean state, and the difference of a product of perturbations is $O(\alpha^3)$. This calculation is also applicable to the meridional component of 3D wave activity flux. The vertical component of 3D wave activity flux is rewritten as

$$\frac{\sigma f_0^2}{N^2} (\tilde{\psi} \tilde{\psi}^\prime)^2 + \tilde{v} \tilde{\psi} \tilde{\psi}^\prime) \approx \frac{\rho_0}{\theta_0} \left[ \frac{\psi^2}{\tilde{\psi}^2} - e^\prime \rho_0^2 \left( \frac{\theta^*}{\theta_0} \right)^2 \right] \left( \frac{\rho_0}{\theta_0} \right) \left( \frac{\theta^*}{\theta_0} \right) \left( \frac{\theta^*}{\theta_0} \right)$$

(17)

where $\frac{\sigma}{|\tilde{\psi}|}$ is the time mean on the log-pressure height coordinate and $\tilde{\beta}$ is its deviation. Note that the perturbation difference between $A$ and $A'$ is $(\tilde{\alpha} \tilde{\tilde{a}})^2 A = O(\alpha^2)$ under the slowly varying time-mean state, and the difference of a product of perturbations is $O(\alpha^3)$. This calculation is also applicable to the meridional component of 3D wave activity flux. The vertical component of 3D wave activity flux is rewritten as

$$\frac{\sigma f_0^2}{N^2} (\tilde{\psi} \tilde{\psi}^\prime)^2 + \tilde{v} \tilde{\psi} \tilde{\psi}^\prime) \approx \frac{\rho_0}{\theta_0} \left[ \frac{\psi^2}{\tilde{\psi}^2} - e^\prime \rho_0^2 \left( \frac{\theta^*}{\theta_0} \right)^2 \right] \left( \frac{\rho_0}{\theta_0} \right) \left( \frac{\theta^*}{\theta_0} \right) \left( \frac{\theta^*}{\theta_0} \right)$$

(18)

where $\frac{\sigma}{|\tilde{\psi}|}$ is the time mean on the log-pressure height coordinate and $\tilde{\beta}$ is its deviation. Note that the perturbation difference between $A$ and $A'$ is $(\tilde{\alpha} \tilde{\tilde{a}})^2 A = O(\alpha^2)$ under the slowly varying time-mean state, and the difference of a product of perturbations is $O(\alpha^3)$. This calculation is also applicable to the meridional component of 3D wave activity flux. The vertical component of 3D wave activity flux is rewritten as

$$\frac{\sigma f_0^2}{N^2} (\tilde{\psi} \tilde{\psi}^\prime)^2 + \tilde{v} \tilde{\psi} \tilde{\psi}^\prime) \approx \frac{\rho_0}{\theta_0} \left[ \frac{\psi^2}{\tilde{\psi}^2} - e^\prime \rho_0^2 \left( \frac{\theta^*}{\theta_0} \right)^2 \right] \left( \frac{\rho_0}{\theta_0} \right) \left( \frac{\theta^*}{\theta_0} \right) \left( \frac{\theta^*}{\theta_0} \right)$$

(19)

Here, we assume that ageostrophic flow has an amplitude of $O(\alpha^2)$. Thus, 3D residual mean flow in T-MIM equations is nearly equal to that derived by Plumb (1986).
4. Summary and discussions

In this study, we formulated 3D wave activity flux and residual mean flow for Rossby waves on T-MIM equations. By examining the relation between 3D wave activity flux and residual mean flow in 3D-TEM equations and those derived in this study, it was found that 3D wave activity flux and residual mean flow in T-MIM equations correspond to those derived by Plumb (1986). Comparison between 3D residual mean flow and T-MIM velocity (Kanno and Iwasaki 2018) suggested that the horizontal component of bolus velocity is different from that of QG-Stokes correction. Note that 3D wave activity flux of this study agrees with that of Kinoshita et al. (2016), which is derived by using a unified dispersion relation and polarization relation, when the quasigeostrophic assumption is used. \( \psi^\prime \approx \nabla^2 \psi \approx \nabla^2 \psi \) can be rewritten as \( \psi^\prime \approx \nabla^2 \psi \approx \nabla^2 \psi \) by using equations (2), (3), and (15), and \( f_0 \psi^\prime \approx \psi^\prime \). Note also that the reason why the sign of 3D wave activity flux of this study are different from those of Kinoshita et al. (2016) is came from the difference sign of their definition of pseudo momentum.

Although our formulae are applicable to Rossby wave activities and their driving flow in T-MIM equations, we do not describe terms including form drag that are related to the vertical component of wave activity flux in Z-MIM equations. Under a quasi-geostrophic assumption, form drag is balanced with mass-weighted time-mean Coriolis forcing

\[
f_0 \tilde{\psi} = \tilde{M}_z, \quad f_0 \tilde{\psi} = - \tilde{M}_z.
\]

Form drag is rewritten as

\[
\tilde{M}_z = \tilde{M}_z + \sigma \tilde{M} = \tilde{M}_z - f_0 \psi_{\omega,0} \tilde{M}_z.
\]

\[
\tilde{M} = \tilde{M} + \sigma \tilde{M} = \tilde{M} + f_0 \psi_{\omega,0} \tilde{M}.
\]

Thus, form drag balance includes geostrophic balance and things having a magnitude of \( O(\alpha^2) \); then, the latter terms are included in the 3D wave activity flux divergence. That is, the terms \( \sigma f_0^2/\psi_{\omega,0} \) are included in the vertical component of \( \tilde{M}_{\omega,0} \) and \( \tilde{M}_{\psi,0} \) respectively. The term \( \sigma f_0^2/\psi_{\omega,0} \) is included in the term \( e \) in the zonal component of \( \tilde{M}_{\omega,0} \) and meridional component of \( \tilde{M}_{\psi,0} \).

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Appendix

In this section, we show the obtained flow is equal to the sum of time-mean flow and QG-Stokes correction. The continuity equation for T-MIM flow and for perturbation are written as

\[
\left( \frac{\partial}{\partial t} + \tilde{\mathbf{u}} \cdot \nabla \right) \tilde{\mathbf{u}} = -\nabla p + \tilde{\mathbf{f}}_0 + \tilde{\mathbf{f}}_0^\prime - \beta \psi \tilde{\mathbf{v}} = 0,
\]

\[
\left( \frac{\partial}{\partial t} + \tilde{\mathbf{u}} \cdot \nabla \right) \tilde{\mathbf{v}} = -\nabla p + \tilde{\mathbf{f}}_0 + \tilde{\mathbf{f}}_0^\prime - \beta \psi \tilde{\mathbf{v}} = 0.
\]

When the horizontal parcel displacements are defined as \( \tilde{\mathbf{u}} + \tilde{\mathbf{f}}_0 + \tilde{\mathbf{f}}_0^\prime - \beta \psi \tilde{\mathbf{v}} = 0 \), suitable initial conditions, the continuity equation for parcel displacements is obtained;

\[
\frac{\partial}{\partial t} \tilde{\mathbf{u}} + \tilde{\mathbf{v}} = 0.
\]

The QG-Stokes correction for T-MIM framework is obtained by using (23);

\[
\tilde{\mathbf{u}} = \tilde{\mathbf{u}}- \mathbf{u}_0 \mathbf{u}_0^\prime + \tilde{\mathbf{f}}_0 + \tilde{\mathbf{f}}_0^\prime - \beta \psi \tilde{\mathbf{v}}.
\]

Here, it should be noted that \( \mathbf{u}_0^\prime \) is referred as the vertical displacement from (23) and the deformations in the third equality of each equation are made by using the equation (8) and the relations \( \mathbf{u}^\prime \mathbf{u}^\prime = -\mathbf{v} \mathbf{v} \). On the other hand, assuming that second derivatives of reference variables are negligibly small, the last terms of obtained flow in equation (13) are rewritten as;

\[
\left( \frac{f_0^2}{\psi_{\omega,0}} \right)_0 \approx \frac{1}{\nabla p} (\nabla \tilde{\mathbf{u}}^\prime)_0,
\]

\[
\left( \frac{f_0^2}{\psi_{\psi,0}} \right)_0 \approx \frac{1}{\nabla p} (\nabla \tilde{\mathbf{v}}^\prime)_0.
\]

Next, the relation between \( \mathbf{n}_0^\prime \) and \( \mathbf{S} / f_0 \) is examined by using (8) and the following relations

\[
\mathbf{u}^\prime = -i \mathbf{y} \mathbf{y}^\prime, \quad \mathbf{v}^\prime = i \mathbf{y} \mathbf{y}^\prime, \quad \mathbf{n}^\prime = \frac{k}{\omega} \mathbf{y} \mathbf{y}^\prime,
\]

\[
\mathbf{u}_0^\prime = -i \mathbf{y} \mathbf{y}_0 + \mathbf{y} \mathbf{y}_0^\prime, \quad \mathbf{v}_0^\prime = -i \mathbf{y} \mathbf{y}_0 + \mathbf{y} \mathbf{y}_0^\prime,
\]

\[
\mathbf{n}_0^\prime = -i \mathbf{y} \mathbf{y}_0 + \mathbf{y} \mathbf{y}_0^\prime, \quad \mathbf{n}_0^\prime = -i \mathbf{y} \mathbf{y}_0 + \mathbf{y} \mathbf{y}_0^\prime,
\]

where the relation between ageostrophic flow perturbations and stream function is obtained by using horizontal momentum equation for perturbation, which is written as follows;

\[
\left( \frac{\partial}{\partial t} + \mathbf{u}^\prime \cdot \nabla \right) \mathbf{u}^\prime - \mathbf{v}^\prime \mathbf{v}^\prime = 0,
\]

\[
\left( \frac{\partial}{\partial t} + \mathbf{u}^\prime \cdot \nabla \right) \mathbf{v}^\prime + f_0 \mathbf{u}_0^\prime + \mathbf{y} \mathbf{y}^\prime = 0.
\]
Substituting (26) into $\overline{u'P'}$, each term of $\overline{u'P'} = \overline{u'P'} + \overline{u'n} + \overline{u'n}$ is written as follows;

$$\overline{u'\psi'} = 0,$$

$$\overline{u'n} + \overline{u'n} = \frac{k^2 + l^2}{f_0} \overline{\psi'^2} + \overline{\psi'^2},$$

$$\overline{u'n} = -\beta y (k^2 + l^2) \overline{\psi'^2} = -\beta y \overline{\psi'^2} + \overline{\psi'^2} \frac{f_0}{f_0}.$$

Thus, $\overline{u'n} \approx (\overline{\psi'^2} + \overline{\psi'^2}) / f_0$ and the difference between $\overline{u'n}$ and $S / f_0$ is $(\overline{\psi'^2} + \overline{\psi'^2}) / 2 f_0 + f_0 \overline{\psi'^2} / 2 N_0^2$. The difference can be explained by considering QG-Stokes correction due to Coriolis forcing. This forcing is rewritten as follows;

$$\frac{\beta k \overline{\psi' \psi'}}{\omega} = \left( \frac{\beta k \overline{\psi' \psi'}}{2 \omega} \right)_y = \left( \frac{\overline{\psi'^2}}{f_0} - \frac{f_0 \overline{\psi'^2}}{2 N_0^2} \right)_y,$$

$$\frac{\beta k \overline{\psi' \psi'}}{\omega} = -\frac{k \beta}{2 \omega} \overline{\psi'^2} \frac{f_0}{f_0} = \frac{\overline{\psi'^2}}{f_0} + \frac{f_0 \overline{\psi'^2}}{2 N_0^2}.$$

where we use the dispersion relation of Rossby waves under the assumption that second derivatives of reference variables are negligibly small ($\omega = -k \beta f_k^2$). From (25), (28) and (29),

$$f_0 \left[ \frac{S}{f_0} \right]_y + \frac{1}{\sigma} \left[ \frac{\sigma f_0}{f_0} \overline{\psi' \psi'} \right]_y \approx f_0 \left[ \frac{S}{f_0} \right]_y + \frac{1}{p_0} \left[ \frac{1}{p_0} \overline{\psi' \psi'} \right]_y \beta \psi'.$$

$$f_0 \left[ \frac{S}{f_0} \right]_y - \frac{1}{\sigma} \left[ \frac{\sigma f_0}{f_0} \overline{\psi' \psi'} \right]_y \approx f_0 \left[ \frac{S}{f_0} \right]_y - \frac{1}{p_0} \left[ \frac{1}{p_0} \overline{\psi' \psi'} \right]_y \beta \psi'.$$

Thus, the obtained flow having second order of perturbations is approximately equal to QG-Stokes correction including Coriolis forcing.

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