On Divisibility Property of Type 2 \((p, q)\)-Analogue of \(r\)-Whitney Numbers of the Second Kind

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Abstract. In this paper, the divisibility property of the type 2 \((p, q)\)-analogue of the \(r\)-Whitney numbers of the second kind is established. More precisely, a congruence relation modulo \(pq\) for this \((p, q)\)-analogue is derived.

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1. Introduction

The \(r\)-Whitney numbers of the second kind were introduced by Mezo [18] as coefficients of the following generating function:

\[(mx + r)^n = \sum_{k=0}^{n} m^k W_{m,r}(n, k)x^k,\]

where \(x^k = x(x - 1)\ldots(x - k + 1)\). These numbers satisfy the following properties:

1. the exponential generating function

\[\sum_{n=0}^{\infty} W_{m,r}(n, k) \frac{x^n}{n!} = \frac{e^{rx}}{k!} \left(\frac{e^{mx} - 1}{m}\right)^k,\]

2. the explicit formula

\[W_{m,r}(n, k) = \frac{1}{m^k k!} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} (mi + r)^n,\]

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3. the triangular recurrence relation
\[ W_{m,r}(n, k) = W_{m,r}(n - 1, k - 1) + (km + r)W_{m,r}(n - 1, k). \]

These properties are exactly the same properties that the \((r, \beta)-\text{Stirling numbers in} [7]\) have possessed. This implies that the \(r\)-Whitney numbers of the second kind and the \((r, \beta)-\text{Stirling numbers are equivalent. More properties of these numbers can be found in}\ [2, 4, 5, 7, 18].

One of the early studies on \(q\)-analogue of Stirling numbers of the second kind was introduced by Carlitz in [1] in connection with a problem in abelian groups. This is known as \(q\)-Stirling numbers of the second kind and is defined in terms of the following recurrence relation
\[ S_q[n, k] = S_q[n - 1, k - 1] + \left(\frac{1}{1 - q}\right)^k S_q[n - 1, k], \quad [k]_q = \frac{1 - q^k}{1 - q} \]
such that, when \(q \to 1\), this gives the triangular recurrence relation for the classical Stirling numbers of the second kind
\[ S(n, k) = S(n - 1, k - 1) + kS(n - 1, k). \]

Another version of definition of this \(q\)-analogue was adapted in [17] as follows
\[ S_q[n, k] = q^{k-1}S_q[n - 1, k - 1] + [k]_qS_q[n - 1, k]. \] (1)

Through this definition, the Hankel transform of \(q\)-exponential polynomials and numbers was successfully established, which may be considered as the Hankel transform of a certain \(q\)-analogue of Bell polynomials and numbers.

There are many ways to define \(q\)-analogue of Stirling-type and Bell-type numbers (see [6, 8–10, 12, 14]). However, in the desire to establish the Hankel transform of \(q\)-analogue of generalized Bell numbers, Corcino et al. [11] were motivated to define a \(q\)-analogue of \(r\)-Whitney numbers of the second kind parallel to that in (1) as follows:
\[ W_{m,r}(n, k)_q = q^{m(k-1)-r}W_{m,r}[n - 1, k - 1]_q + [mk - r]_qW_{m,r}[n - 1, k]_q. \] (2)

Two more forms of this \(q\)-analogue, denoted by \(W^*_{m,r}[n, k]_q\) and \(\tilde{W}_{m,r}[n, k]_q\), were respectively defined by
\[ W^*_{m,r}[n, k]_q := q^{-kr+m\left(\binom{k}{2}\right)}W_{m,r}[n, k]_q, \]
\[ \tilde{W}_{m,r}[n, k]_q := q^{-kr}W^*_{m,r}[n, k]_q = q^{-m\left(\binom{k}{2}\right)}W_{m,r}[n, k]. \]

The corresponding \(q\)-analogues of generalized Bell numbers, also known as \(q\)-analogues of \(r\)-Dowling numbers, were also defined in three forms as (see [3, 11, 13, 15])
\[ D_{m,r}[n]_q := \sum_{k=0}^{n} W_{m,r}[n, k]_q, \]
\[ D^*_{m, r}[n]_q := \sum_{k=0}^{n} W^*_{m, r}[n, k]_q, \]

and
\[ \tilde{D}_{m, r}[n]_q := \sum_{k=0}^{n} \tilde{W}_{m, r}[n, k]_q. \]

where \( D_{m, r}[n]_q, D^*_{m, r}[n]_q \) and \( \tilde{D}_{m, r}[n]_q \) denote the first, second and third form of the \( q \)-analogues of \( r \)-Dowling numbers, respectively. The Hankel transforms of \( D_{m, r}[n]_q, D^*_{m, r}[n]_q \) and \( \tilde{D}_{m, r}[n]_q \) were successfully established in [3, 11, 15].

To extend these research studies, a certain \((p, q)\)-analogue of \( r \)-Whitney numbers of the second kind, denoted by \( W_{m, r}[n, k]_{p, q} \), was defined in [16] as coefficients of the following generating function:
\[ [mt + r]_p^n = \sum_{k=0}^{n} W_{m, r}[n, k]_{p, q} [mt|m]^k_{p, q} \]

where
\[ [t|m]_p^n = \prod_{j=0}^{n-1} [t - jm]_{p, q}. \]

The orthogonality and inverse relations, an explicit formula, and a kind of exponential generating function of \( W_{m, r}[n, k]_{p, q} \) were already obtained. Unfortunately, its Hankel transform was not successfully established using the method applied in [3, 11, 15]. This motivated Corcino et al. [19] to define the type 2 \((p, q)\)-analogue of \( r \)-Whitney numbers of the second kind, denoted by \( W_{m, r}[n, k; t]_{p, q} \), as follows:
\[ W_{m, r}[n+1, k; t]_{p, q} = q^{m(k-1)+r} W_{m, r}[n, k-1; t]_{p, q} + [mk + r]_{p, q} p^{mt-kr} W_{m, r}[n, k; t]_{p, q}. \]

The second form was then defined as follows:
\[ W^*_{m, r}[n, k; t]_{p, q} := q^{-kr-m(\frac{t}{2})} W_{m, r}[n, k; t]_{p, q}. \]

Several properties of these \((p, q)\)-analogues were established in [19] including their Hankel transforms, which are given by
\[ \text{det}(W_{m, r}[s + i + j, s + j; t]_{p, q})_{0 \leq i, j \leq n} = \prod_{k=0}^{n} q^{m(s+k)+s(r+k)} p^{mt}[m(s+k) + r]_{p, q} \]
\[ \text{det}(W^*_{m, r}[s + i + j, s + j; t]_{p, q})_{0 \leq i, j \leq n} = \prod_{k=0}^{n} p^{mt}[m(s+k) + r]_{p, q}. \]

On the other hand, the first, second and third forms of type 2 \((p, q)\)-analogue of the \( r \)-Dowling numbers, denoted by \( D_{m, r}[n]_{p, q}, D^*_{m, r}[n]_{p, q} \) and \( \tilde{D}_{m, r}[n]_{p, q} \) were defined respectively in [19] as follows:
\[ D_{m, r}[n]_{p, q} := \sum_{k=0}^{n} W_{m, r}[n, k; t]_{p, q}, \]
Among these three forms, only the second form was provided a Hankel transform, which denotes the third form of the \((p, q)\)-analogue of the \(r\)-Whitney numbers of the second kind.

Applying (6) consequently gives

\[
W_{m,r}[n, k; t]_{p,q} = q^{k}\ W_{m,r}^{*}[n, k; t]_{p,q}
\]

where

\[
W_{m,r}[n, k; t]_{p,q} = q^{kr}\ W_{m,r}^{*}[n, k; t]_{p,q}
\]

denotes the third form of the \((p, q)\)-analogue of the \(r\)-Whitney numbers of the second kind. Among these three forms, only the second form was provided a Hankel transform, which is given by

\[
H(D_{m,r}^*[n]_{p,q}) = \left(\frac{q}{p}\right)^{n(n^2+3n+8)+r-1(n)\over 2} (\frac{n}{2})^{n-1} \prod_{k=0}^{n-1} [k]^{(2)}_p m!.
\]

The main objective of this study is to establish additional property of the type 2 \((p, q)\)-analogues of the \(r\)-Whitney numbers of the second kind. More precisely, the divisibility property of these type 2 \((p, q)\)-analogues will be discussed thoroughly.

2. Preliminary Results

This section provides a brief discussion on some relations that are necessary in deriving the divisibility property of the type 2 \((p, q)\)-analogue of the \(r\)-Whitney numbers of the second kind \(W_{m,r}^{*}[n, k; t]_{p,q}\).

Multiplying both sides of the recurrence relation in (5) by \(q^{-kr-m^{(k)}_2}\) yields

\[
q^{-kr-m^{(k)}_2}W_{m,r}[n+1, k; t]_{p,q} = q^{-k} W_{m,r}[n, k-1; t]_{p,q} + q^{-k} [mk + r]_{p,q} p^{nt-km} W_{m,r}[n, k; t]_{p,q}
\]

Applying (6) consequently gives

\[
W_{m,r}[n+1, k; t]_{p,q} = W_{m,r}^{*}[n, k-1; t]_{p,q} + [mk + r]_{p,q} p^{nt-km} W_{m,r}^{*}[n, k; t]_{p,q}.
\]

This relation can be used to generate the following first few values of \(W_{m,r}^{*}[n, k; t]_{p,q}\):

By repeated application of (8), we can easily derive the following vertical recurrence relation.

**Theorem 2.1.** For nonnegative integers \(n\) and \(k\), and real number \(r\), the \((p, q)\)-analogue of \(r\)-Whitney numbers of the second kind satisfies the following vertical recurrence relation

\[
W_{m,r}[n+1, k+1; t]_{p,q} = \sum_{j=k}^{n} [mj] + r^{n-j} p^{nt-(k+1)m} W_{m,r}[j, k; t]_{p,q}.
\]
\[
\begin{array}{|c|c|c|c|}
\hline
n/k & 0 & 1 & 2 \\
\hline
0 & 1 & & \\
1 & [r]_{p, q} P^{mt} & 1 & \\
2 & [r]_{p, q} P^{2mt} + [m + r]_{p, q} P^{m(t-1)} & 1 & \\
3 & [r]_{p, q} P^{3mt} + [m + r]^2 [p, q] P^{2m(t-1)} & & 1 \\
\hline
\end{array}
\]

Table 1: The First Values of \(W_{m, r}^*[n, k; t]_{p, q}\)

One can easily verify relation (9) using the values of \(W_{m, r}^*[n, k; t]_{p, q}\) in Table 1.

Now, let us derive the rational generating function for \(W_{m, r}^*[n, k; t]_{p, q}\). Suppose that

\[
\Psi_0^*(x) = \sum_{n=0}^{\infty} W_{m, r}^*[n, 0; t]_{p, q} x^n = \frac{1}{1 - x [p, q]_{p, q} t}
\]

When \(k = 0\), (8) reduces to

\[
W_{m, r}^*[n + 1, 0; t]_{p, q} = [r]_{p, q} P^{mt} W_{m, r}^*[n, 0; t]_{p, q}
\]

By repeated application of (8), this inductively gives

\[
W_{m, r}^*[n + 1, 0; t]_{p, q} = [r]_{p, q} P^{mt} W_{m, r}^*[n, 0; t]_{p, q} = ([r]_{p, q} P^{mt})^2 W_{m, r}^*[n - 1, 0; t]_{p, q}
\]

Hence,

\[
\Psi_0^*(x) = \sum_{n=0}^{\infty} W_{m, r}^*[n, 0; t]_{p, q} x^n = \frac{1}{1 - x [p, q]_{p, q} t}
\]

When \(k > 0\) and applying the triangular recurrence relation in (5), we have

\[
\Psi_k^*(x) = \sum_{n=k}^{\infty} W_{m, r}^*[n, k; t]_{p, q} x^{n-k} = \frac{1}{1 - x [p, q]_{p, q} t}
\]

Solving for \(\Psi_k^*(t)\) yields

\[
\Psi_k^*(x) = \frac{1}{1 - x [p, q]_{p, q} (t-k)} \Psi_{k-1}^*(x).
\]

Applying backward substitution gives the following rational generating function for \(W_{m, r}^*[n, k; t]_{p, q}\).
Theorem 2.2. For nonnegative integers \( n \) and \( k \), and real number \( r \), the \((p, q)\)-analogue \( W^*_{m,r}[n, k; t]_{p,q} \) satisfies the following rational generating function

\[
Ψ_k^*(x) = \sum_{n=k}^\infty W^*_{m,r}[n, k; t]_{p,q} x^{n-k} = \frac{1}{\prod_{j=0}^k (1 - xp^{m(t-j)}[m j + r]_{p,q})}.
\] (10)

Remark 2.3. This rational generating function plays an important role in proving the main result of the paper.

3. Divisibility Property

In this section, the congruence relation modulo \( pq \) for the type 2 \((p, q)\)-analogue of the \( r \)-Whitney numbers of the second kind \( W^*_{m,r}[n, k; t]_{p,q} \) will be established using the rational generating function in (10).

Using the values of \( W^*_{m,r}[n, k; t]_{p,q} \) in Table 1, we observe that, with

\[
[t]_{p,q} = p^{t-1} + p^{t-2}q + p^{t-3}q^2 + \ldots + pq^{t-2} + q^{t-1},
\]

the polynomial expressions of \( W^*_{m,r}[n, k]_{p,q} \) from row 0 to row 3, if they are divided by \( pq \), the remainders form the following triangle of expressions in \( p \):

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
\frac{2}{1} & 2^2 & 2^3 & 2^4 \\
\frac{3}{1} & 3^2 & 3^3 & 3^4 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

This can further be written as

\[
\begin{align*}
&\binom{0}{0} p^{mt+r-1} + \binom{1}{1} + \binom{2}{2} p^{mt+r-1} + \binom{3}{3} p^{mt+r-1} \\
&\binom{0}{0} p^{2(mt+r-1)} + \binom{1}{1} p^{2(mt+r-1)} + \binom{2}{2} p^{2(mt+r-1)} + \binom{3}{3} p^{2(mt+r-1)} \\
&\binom{0}{0} p^{3(mt+r-1)} + \binom{1}{1} p^{3(mt+r-1)} + \binom{2}{2} p^{3(mt+r-1)} + \binom{3}{3} p^{3(mt+r-1)}
\end{align*}
\]

To generalize this observation, the next theorem contains the divisibility property of \( W^*_{m,r}[n, k; t]_{p,q} \).

Theorem 3.1. For nonnegative integers \( n \) and \( k \), the type 2 \((p, q)\)-analogue of the \( r \)-Whitney numbers of the second kind \( W^*_{m,r}[n, k; t]_{p,q} \) satisfies the following congruence relation

\[
W^*_{m,r}[n, k; t]_{p,q} \equiv \binom{n}{k} p^{(n-k)(mt+r-1)} \mod pq.
\] (11)

Proof. The polynomial \([t]_{p,q}\) can be written as

\[
[t]_{p,q} = p^{t-1} + q^{t-1} + pqy,
\]
where \( y \) is a polynomial in \( p \) and \( q \). Then, we have

\[
\frac{1}{\prod_{j=0}^{k}(1 - xp^{m(t-j)}[mj + r]_{p,q})} = \sum_{n=0}^{\infty} \left(xp^{m(t-j)}[mj + r]_{p,q}\right)^n = \sum_{n=0}^{\infty} p^{nm(t-j)}(p^{mj+r-1} + q^{mj+r-1} + pqy)^n x^n
\]

\[
= \sum_{n=0}^{\infty} p^{n(mt+r-j)}x^n + pq \sum_{n=0}^{\infty} \hat{z}_n x^n,
\]

where \( \hat{z}_n \) is a polynomial in \( p \) and \( q \). It follows that

\[
\frac{1}{\prod_{j=0}^{k}(1 - xp^{m(t-j)}[mj + r]_{p,q})} = \sum_{n=0}^{\infty} p^{n(mt+r-j)} x^n \mod pq
\]

Thus, using (10), we have

\[
\sum_{n=k}^{\infty} W^k_{m,r}[n,k;t]_{p,q} x^{n-k} \equiv \frac{1}{(1 - p^{mt+r-1}x)^{k+1}} \mod pq
\]

\[
\equiv \sum_{n=0}^{\infty} \left(\begin{array}{c} n + (k + 1) - 1 \\ n \end{array}\right) p^{n(mt+r-1)} x^n \mod pq
\]

\[
\equiv \sum_{n=k}^{\infty} \left(\begin{array}{c} n \\ k \end{array}\right) p^{n-k(mt+r-1)} x^{n-k} \mod pq.
\]

Comparing the coefficients of \( x^{n-k} \) completes the proof of the theorem.

**Remark 3.2.** Using (6) and Theorem 3.1, the first form of the type 2 \((p, q)\)-analogues of the \( r \)-Whitney numbers of the second kind satisfies the following congruence relation modulo \( pq \):

\[
W_{m,r}[n,k;t]_{p,q} \equiv \left(\begin{array}{c} n \\ k \end{array}\right) p^{n-k(mt+r-1)} q^{kr+m(k^2/2)} \mod pq
\]

\[
\equiv \begin{cases} q^{m_r+m(k^2/2)} \mod pq, & \text{for } n = k \\ 0 \mod pq, & \text{otherwise.} \end{cases}
\]

Moreover, using (7) and Theorem 3.1, the third form of the type 2 \((p, q)\)-analogues of the \( r \)-Whitney numbers of the second kind satisfies the following congruence relation modulo \( pq \):

\[
\tilde{W}_{m,r}[n,k;t]_{p,q} \equiv \left(\begin{array}{c} n \\ k \end{array}\right) p^{n-k(mt+r-1)} q^{kr} \mod pq
\]
≡ \begin{cases} q^{nr} \mod pq, & \text{for } n = k \\ 0 \mod pq, & \text{otherwise.} \end{cases}

**Remark 3.3.** When \( p = 1 \), the congruence relation in (11) reduces to

\[ W_{m,r}^*[n,k]_q = W_{m,r}^*[n,k;t]_{1,q} \equiv \binom{n}{k} \mod q, \]

which is exactly the congruence relation in [15, Theorem 2.1] for the second form of \((q, r)\)-Whitney numbers of the second kind. Moreover, the congruence relations in (12) and (13) reduce to

\[ W_{m,r}[n,k]_q = W_{m,r}[n,k;t]_{1,q} \equiv \binom{n}{k} q^{kr+m\left(\frac{k}{2}\right)} \equiv 0 \mod q \]

\[ \hat{W}_{m,r}[n,k]_q = \hat{W}_{m,r}[n,k;t]_{1,q} \equiv \binom{n}{k} q^{kr} \equiv 0 \mod q, \]

which are the congruence relations for the first and third forms of \((q, r)\)-Whitney numbers of the second kind. We recall that, for a prime \( p \), the \( p \)-adic valuation \( \nu_p(n) \) of \( n \) is defined to be the largest exponent \( k \) such that \( p^k | n \). Moreover, the \( p \)-adic valuation of the rational number \( \frac{n}{m} \) is defined by

\[ \nu_p \left( \frac{n}{m} \right) = \nu_p(n) - \nu_p(m). \]

Furthermore, the \( p \)-adic absolute value \( |n|_p \) of \( n \) is defined by

\[ |n|_p = \frac{1}{p^{\nu_p(n)}}. \]

Clearly, when \( q \) is prime,

\[ \nu_q \left( W_{m,r}[n,k]_q \right) = kr + m\left(\frac{k}{2}\right) \]

\[ \nu_q \left( \hat{W}_{m,r}[n,k]_q \right) = kr. \]

Consequently,

\[ \nu_q \left( \frac{W_{m,r}[n,k]_q}{\hat{W}_{m,r}[n,k]_q} \right) = \nu_q (W_{m,r}[n,k]_q) - \nu_q (\hat{W}_{m,r}[n,k]_q) = m\left(\frac{k}{2}\right). \]

Also, one can easily see that

\[ \nu_q \left( W_{m,r}^*[n,k]_q - \binom{n}{k} \right) = q^{\nu_p(W_{m,r}^*[n,k]_q - \binom{n}{k})} \left| W_{m,r}^*[n,k]_q - \binom{n}{k} \right|_q = 1. \]
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