Chaos in Partial Differential Equations

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Abstract. This is a survey on Chaos in Partial Differential Equations. First we classify soliton equations into three categories: 1. (1+1)-dimensional soliton equations, 2. soliton lattices, 3. (1+n)-dimensional soliton equations ($n \geq 2$). A systematic program has been established by the author and collaborators, for proving the existence of chaos in soliton equations under perturbations. For each category, we pick a representative to present the results. Then we review some initial results on 2D Euler equation.

1. Introduction

It is well-known that the theory of chaos in finite-dimensional dynamical systems has been well-developed. That includes both discrete maps and systems of ordinary differential equations. Such theory has produced important mathematical theorems and led to important applications in physics, chemistry, biology, and engineering etc. On the contrary, the theory of chaos in partial differential equations has not been well-developed. On the other hand, the demand for such a theory is much more stronger than for finite-dimensional systems. Mathematically, studies on infinite-dimensional systems pose much more challenging problems. For example, as phase spaces, Banach spaces possess much more structures than Euclidean spaces. In terms of applications, most of important natural phenomena are described by partial differential equations, nonlinear wave equations, Yang-Mills equations, and Navier-Stokes equations, to name a few.

Nonlinear wave equations are the most important class of equations in natural sciences. They describe a wide spectrum of phenomena; motion of plasma, nonlinear optics (laser), water waves, vortex motion, to name a few. Among these nonlinear wave equations, there is a class of equations called soliton equations. This class of equations describes a variety of phenomena. In particular, the same soliton equation describes several different phenomena. For references, see for example [3][1]. Mathematical theories on soliton equations have been well developed. Their

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Cauchy problems are completely solved through inverse scattering transforms. Soliton equations are integrable Hamiltonian partial differential equations which are the natural counterparts of finite-dimensional integrable Hamiltonian systems.

To set up a systematic study on chaos in partial differential equations, we started with the perturbed soliton equations. We classify the perturbed soliton equations into three categories:

1. Perturbed (1+1)-Dimensional Soliton Equations,
2. Perturbed Soliton Lattices,
3. Perturbed (1 + n)-Dimensional Soliton Equations ($n \geq 2$).

For each category, we chose a candidate to study. The integrable theories for every members in the same category are parallel, and for members in different categories are substantially different. The theorem on the existence of chaos for each candidate can be parallelly generalized to the rest members of the same category.

- The candidate for Category 1 is the perturbed cubic focusing nonlinear Schrödinger equation \(22\),

\[
i\partial_t q = \partial^2_x q + 2|q|^2 - \omega^2|q| + \text{Perturbations,}
\]

under periodic and even boundary conditions \(q(x+1) = q(x)\) and \(q(-x) = q(x)\, \omega\) is a real constant.

- The candidate for Category 2 is the perturbed discrete cubic focusing nonlinear Schrödinger equation \(21\),

\[
i\dot{q}_n = \frac{1}{h^2}[q_{n+1} - 2q_n + q_{n-1}]
+ |q_n|^2(q_{n+1} + q_{n-1}) - 2\omega^2q_n + \text{Perturbations,}
\]

under periodic and even boundary conditions \(q_{n+N} = q_n\) and \(q_{-n} = q_n\).

- The candidate for Category 3 is the perturbed Davey-Stewartson II equations \(14\),

\[
\begin{cases}
    i\partial_t q = [\partial^2_x - \partial^2_y]q + 2(|q|^2 - \omega^2) + u_y|q| + \text{Perturbations,} \\
    [\partial^2_x + \partial^2_y]u = -4\partial_y|q|^2,
\end{cases}
\]

under periodic and even boundary conditions

\[
q(t, x + l_x, y) = q(t, x, y) = q(t, x, y + l_y),
\]
\[
u(t, x + l_x, y) = u(t, x, y) = u(t, x, y + l_y),
\]

and

\[
q(t, -x, y) = q(t, x, y) = q(t, x, -y),
\]
\[
u(t, -x, y) = u(t, x, y) = u(t, x, -y).
\]

We have established a standard program for proving the existence of chaos in perturbed soliton equations, with the machineries:

1. Darboux Transformations for Soliton Equations.
2. Isospectral Theory for Soliton Equations Under Periodic Boundary Condition.
3. Persistence of Invariant Manifolds and Fenichel Fibers.
4. Melnikov Analysis.
5. Smale Horseshoes and Symbolic Dynamics Construction of Conley-Moser Type.
The most important implication of the theory on chaos in partial differential equations in theoretical physics will be on the study of turbulence. For that goal, we chose the 2D Navier-Stokes equations under periodic boundary conditions to begin a dynamical system study on 2D turbulence. Since they possess Lax pair \([17]\) and Darboux transformation \([25]\), the 2D Euler equations are the starting point for an analytical study. The high Reynolds number 2D Navier-Stokes equations are viewed as a singular perturbation of the 2D Euler equations through the perturbation parameter \(\varepsilon = 1/Re\) which is the inverse of the Reynolds number. Corresponding singular perturbations of nonlinear Schrödinger equation have been studied in \([31]\) \([30]\) \([19]\) \([20]\). We have studied the linearized 2D Euler equations and obtained a complete spectra theorem \([16]\). In particular, we have identified unstable eigenvalues. Then we found the approximate representations of the hyperbolic structures associated with the unstable eigenvalues through Galerkin truncations \([18]\).

2. Existence of Chaos in Perturbed Soliton Equations

By existence of chaos, we mean that there exist a Smale horseshoe and the Bernoulli shift dynamics for certain Poincaré map. For lower dimensional systems, there have been a lot of theorems on the existence of chaos \([8]\) \([26]\). For perturbed soliton equations under dissipative perturbations, we first establish the existence of a Silnikov homoclinic orbit. And then we define a Poincaré section which is transversal to the Silnikov homoclinic orbit, and the Poincaré map on the Poincaré section induced by the flow. Finally we construct the Smale horseshoe for the Poincaré map. In establishing the existence of the Silnikov homoclinic orbit, we need to build a Melnikov analysis through Darboux transformations to generate the explicit representation for the unperturbed heteroclinic orbit, the isospectral theory for soliton equations to generate the Melnikov vectors, and the persistence of invariant manifolds and Fenichel fibers. We also need to utilize the properties of the Fenichel fibers to build a second measurement inside a slow manifold, together with normal form techniques. The Melnikov measurement and the second measurement together lead to the existence of the Silnikov homoclinic orbit through implicit function arguments. In establishing the existence of Smale horseshoes for the Poincaré map, we first need to establish a smooth linearization in the neighborhood of the saddle point (i.e. the asymptotic point of the Silnikov homoclinic orbit). Then the dynamics in the neighborhood of the saddle point is governed by linear partial differential equations which are explicitly solvable. The global dynamics in the tubular neighborhood of the Silnikov homoclinic orbit away from the above neighborhood of the saddle point, can be approximated by linearized flow along the Silnikov homoclinic orbit due to finiteness of the passing time. Finally we can obtain a semi-explicit representation for the Poincaré map. Then we establish the existence of fixed points of the Poincaré map under certain except-one-point conditions. And we study the action of the Poincaré map in the neighborhood of these fixed points, and verify the Conley-Moser criteria to establish the existence of Smale horseshoes and Bernoulli shift dynamics.

2.1. Existence of Chaos in Perturbed (1+1)-Dimensional Soliton Equations

For this category of the perturbed soliton equations, we chose the candidate to be the perturbed cubic nonlinear Schrödinger equation. The cubic nonlinear Schrödinger equation describes self-focusing phenomena in nonlinear optics, deep water surface wave, vortex filament motion etc.. Recently, more and more interests
are on perturbed nonlinear Schrödinger equations describing new nonlinear optical effects, for example, the works of the Laser Center at Oklahoma State University.

2.1.1. Dissipative Perturbations. In a series of three papers [22, 21, 14], we proved the existence of chaos in the cubic nonlinear Schrödinger equation under dissipative perturbations. We study the following perturbed nonlinear Schrödinger equation:

\[ iq_t = q_{xx} + 2|q|^2 - \omega^2 q + i\varepsilon [-\alpha q + \hat{D}^2 q + \Gamma], \]

under even periodic boundary conditions

\[ q(-x) = q(x), \quad q(x+1) = q(x); \]

where \( i = \sqrt{-1} \), \( q \) is a complex-valued function of two variables \((x, t)\), \((\omega, \alpha, \Gamma)\) are positive constants, \( \varepsilon \) is the positive perturbation parameter, \( \hat{D}^2 \) is a “regularized” Laplacian specifically defined by

\[ \hat{D}^2 q \equiv -\sum_{j=1}^{\infty} \beta_j k_j^2 \hat{q}_j \cos k_j x, \]

in which \( k_j = 2\pi j \), \( \hat{q}_j \) is the Fourier transform of \( q \), \( \beta_j = \beta \) for \( j \leq N \), \( \beta_j = \alpha_* k_j^{-2} \) for \( j > N \), \( \beta \) and \( \alpha_* \) are positive constants, and \( N \) is a large fixed positive integer.

**Theorem 2.1** (Homoclinic Orbit Theorem). There exists a positive number \( \varepsilon_0 \) such that for any \( \varepsilon \in (0, \varepsilon_0) \), there exists a codimension 1 hypersurface \( E_\varepsilon \) in the external parameter space \( \{ \omega, \alpha, \Gamma, \beta, \alpha_* \} \). For any external parameters \( (\omega, \alpha, \Gamma, \beta, \alpha_*) \in E_\varepsilon \), there exists a symmetric pair of homoclinic orbits \( h_k = h_k(t, x) \) \((k = 1, 2)\) in \( H^{1,e,p} \) (the Sobolev space \( H^1 \) of even and periodic functions) for the PDE (2.1), which are asymptotic to a fixed point \( q_* \). The symmetry between \( h_1 \) and \( h_2 \) is reflected by the relation that \( h_2 \) is a half-period translate of \( h_1 \), i.e. \( h_2(t, x) = h_1(t, x + 1/2) \). The hypersurface \( E_\varepsilon \) is a perturbation of a known surface \( \beta = \kappa(\omega)\alpha \), where \( \kappa(\omega) \) is shown in Figure 1.

For the complete proof of the theorem, see [22] and [21]. The main argument is a combination of a Melnikov analysis and a geometric singular perturbation theory for partial differential equations. The Melnikov function is evaluated on a homoclinic orbit of the nonlinear Schrödinger equation, generated through Darboux transformations. For more details on this, see the later section on the Darboux transformations for the discrete nonlinear Schrödinger equation.

**Theorem 2.2** (Horseshoe Theorem). Under certain generic assumptions for the perturbed nonlinear Schrödinger system (2.1), there exists a compact Cantor subset \( \Lambda \) of \( \Sigma \) (a Poincaré section transversal to the homoclinic orbit), \( \Lambda \) consists of points, and is invariant under \( P \) (the Poincaré map induced by the flow on \( \Sigma \)). \( P \) restricted to \( \Lambda \), is topologically conjugate to the shift automorphism \( \chi \) on four symbols \( 1, 2, -1, -2 \). That is, there exists a homeomorphism

\[ \phi : W \mapsto \Lambda, \]
where $W$ is the topological space of the four symbols, such that the following diagram commutes:

$$
\begin{array}{ccc}
W & \xrightarrow{\phi} & \Lambda \\
\chi & \uparrow & \downarrow \\
W & \xrightarrow{\phi} & \Lambda \\
\end{array}
$$

For the complete proof of the theorem, see [14]. The construction of horseshoes is of Conley-Moser type for partial differential equations.

2.1.2. Singular Perturbations. Recently, singular perturbation, i.e. replacing $D^2q$ by $\partial^2_t q$, has been studied [31] [30] [19] [20]. Consider the singularly perturbed nonlinear Schrödinger equation,

$$iq_t = q_{xx} + 2|q|^2 - \omega^2|q + i\varepsilon[-\alpha q + \beta q_{xx} + \Gamma] ,
$$

where $q = q(t, x)$ is a complex-valued function of the two real variables $t$ and $x$, $t$ represents time, and $x$ represents space. $q(t, x)$ is subject to periodic boundary condition of period 1, and even constraint, i.e.

$$q(t, x + 1) = q(t, x) , \ q(t, -x) = q(t, x) .$$

$\omega$ is a positive constant, $\alpha > 0$, $\beta > 0$, and $\Gamma$ are constants, and $\varepsilon > 0$ is the perturbation parameter. The main difficulty introduced by the singular perturbation
possesses a symmetric pair of homoclinic orbits $h$ on the codimension-one surface, the perturbed nonlinear Schrödinger equation (2.2) there exists a codimension 1 surface $E$ on the perturbed nonlinear Schrödinger equation (2.2) that certain invariant manifold results do hold. The regularity of such invariant manifolds at $\varepsilon = 0$ is controled by the regularity of $e^{\varepsilon \partial_x^2}$ at $\varepsilon = 0$.

**Theorem 2.3** (Li, [19]). There exists a $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$, there exists a codimension 1 surface $E_\varepsilon$ in the space of $(\omega, \alpha, \beta, \Gamma) \in R^+ \times R^+ \times R^+ \times R^+$, where $\omega \in (\pi, 2\pi)/S$, $S$ is a finite subset. For any external parameters on the codimension-one surface, the perturbed nonlinear Schrödinger equation (2.2) possesses a symmetric pair of homoclinic orbits $h_k = h_k(t, x)$ $(k = 1, 2)$ in $C^\infty_e[0, 1]$ (the space of $C^\infty$ even and periodic functions on the interval $[0, 1]$), which is asymptotic to a saddle fixed point $q_\varepsilon$. The symmetry between $h_1$ and $h_2$ is reflected by the relation that $h_2$ is a half-period translate of $h_1$, i.e. $h_2(t, x) = h_1(t, x + 1/2)$. The hypersurface $E_\varepsilon$ is a perturbation of a known surface $\beta = \kappa(\omega)\alpha$, where $\kappa(\omega)$ is shown in Figure 4.

2.1.3. Hamiltonian Perturbations. The problem on the existence of chaos in the cubic nonlinear Schrödinger equations under Hamiltonian perturbations is largely open. The right objects to investigate should be “homoclinic tubes” rather than “homoclinic orbits” due to the non-dissipative nature and infinite-dimensionality of the perturbed system. Transversal homoclinic tubes are objects of large dimensional generalization of transversal homoclinic orbits. As Smale’s theorem indicates, structures in the neighborhood of a transversal homoclinic orbit are rich, structures in the neighborhood of a transversal homoclinic tube are even richer. Especially in high dimensions, dynamics inside each invariant tube in the neighborhoods of homoclinic tubes are often chaotic too. We call such chaotic dynamics “chaos in the small”, and the symbolic dynamics of the invariant tubes “chaos in the large”. Such cascade structures are more important than the structures in a neighborhood of a homoclinic orbit, when high or infinite dimensional dynamical systems are studied. Symbolic dynamics structures in the neighborhoods of homoclinic tubes are more observable than in the neighborhoods of homoclinic orbits in numerical and physical experiments. When studying high or infinite dimensional Hamiltonian system (for example, the cubic nonlinear Schrödinger equation under Hamiltonian perturbations), each invariant tube contains both KAM tori and stochastic layers (chaos in the small). Thus, not only dynamics inside each stochastic layer is chaotic, all these stochastic layers also move chaotically under Poincaré maps.

There have been a lot of works on the KAM theory of soliton equations under Hamiltonian perturbations [20, 13, 3, 1, 28]. For perturbed nonlinear Schrödinger equations, we are interested in the region of the phase space where there exist hyperbolic structures. Thus, the relevant KAM tori are hyperbolic. In finite dimensions, the relevant work on such tori is that of Graff [7]. In infinite dimensions, the author is not aware of such work yet.

In the paper [13], the author studied the cubic nonlinear Schrödinger equation under Hamiltonian perturbations:

\[(2.3) \quad iq_t = q_{xx} + 2|q|^2 - \omega^2 q + \varepsilon [\alpha_1 + 2\alpha_2 q],\]

under even periodic boundary conditions $q(-x) = q(x)$ and $q(x + 1) = q(x)$; where $i = \sqrt{-1}$, $q$ is a complex-valued function of two variables $(t, x)$, $(\omega, \alpha_1, \alpha_2)$ are real constants, $\varepsilon$ is the perturbation parameter. The system (2.3) can be written in the
Hamiltonian form:

$$\dot{q}_t = \frac{\delta H}{\delta \bar{q}},$$

where $H = H_0 + \epsilon H_1$,

$$H_0 = \int_0^1 \left|q\right|^4 - 2\omega^2 |q|^2 - |q_x|^2 dx,$$

$$H_1 = \int_0^1 [\alpha_1 (q + \bar{q}) + \alpha_2 (q^2 + \bar{q}^2)] dx.$$

**Definition 2.4.** Denote by $W^{(c)}$ a normally hyperbolic center manifold, by $W^{(cu)}$ and $W^{(cs)}$ the center-unstable and center-stable manifolds such that $W^{(c)} = W^{(cu)} \cap W^{(cs)}$, and by $F^t$ the evolution operator of the partial differential equation. Let $\mathcal{H}$ be a submanifold in the intersection between the center-unstable and center-stable manifolds $W^{(cu)}$ and $W^{(cs)}$, such that for any point $q \in \mathcal{H}$, distance $\{F^t(q), W^{(c)}\} \to 0$, as $|t| \to \infty$. We call $H$ a transversal homoclinic tube asymptotic to $W^{(c)}$ under the flow $F^t$ if the intersection between $W^{(cu)}$ and $W^{(cs)}$ is transversal at $H$. Let $\Sigma$ be an Poincaré section which intersects $\mathcal{H}$ transversally, and $P$ is the Poincaré map induced by the flow $F^t$; then $H \cap \Sigma$ is called a transversal homoclinic tube under the Poincaré map $P$.

**Theorem 2.5 (Homoclinic Tube Theorem).** There exist a positive constant $\varepsilon_0 > 0$ and a region $\mathcal{E}$ for $(\alpha_1, \alpha_2, \omega)$, such that for any $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and any $(\alpha_1, \alpha_2, \omega) \in \mathcal{E}$, there exists a codimension 2 transversal homoclinic tube asymptotic to a codimension 2 center manifold $W^{(c)}$.

For a complete proof of this theorem, see [13].

### 2.2. Chaos in Perturbed Soliton Lattices.

For this category, we chose the candidate to be the perturbed cubic nonlinear Schrödinger lattice.

#### 2.2.1. Dissipative Perturbations.

In a series of three papers [11] [23] [24], we proved the existence of chaos in the discrete cubic nonlinear Schrödinger equation under a concrete dissipative perturbation.

We study the perturbed discrete cubic nonlinear Schrödinger equation

$$i \dot{q}_n = \frac{1}{\hbar^2} \left[ q_{n+1} - 2q_n + q_{n-1} \right] + |q_n|^2 (q_{n+1} + q_{n-1}) - 2\omega^2 q_n + i\varepsilon \left[ -\alpha q_n + \frac{\beta}{\hbar^2} (q_{n+1} - 2q_n + q_{n-1}) + \Gamma \right],$$

under even periodic boundary conditions ($q_{N-n} = q_n$) and ($q_{n+N} = q_n$) for arbitrary $N$; where $i = \sqrt{-1}$, $q_n$’s are complex variables, $\hbar = 1/N$, $(\omega, \alpha, \beta, \Gamma)$ are positive constants, $\varepsilon$ is the positive perturbation parameter.

Denote by $\Sigma_N$ ($N \geq 7$) the external parameter space,

$$\Sigma_N = \left\{ (\omega, \alpha, \beta, \Gamma) \mid \omega \in (N \tan \frac{\pi}{N}, N \tan \frac{2\pi}{N}); \right. \left. \Gamma \in (0, 1), \alpha \in (0, \alpha_0), \beta \in (0, \beta_0); \right.$$  

where $\alpha_0$ and $\beta_0$ are any fixed positive numbers.}
Theorem 2.6. For any \( N (7 \leq N < \infty) \), there exists a positive number \( \varepsilon_0 \), such that for any \( \varepsilon \in (0, \varepsilon_0) \), there exists a codimension 1 submanifold \( E_\varepsilon \) in \( \Sigma_N \); for any external parameters \((\omega, \alpha, \beta, \Gamma)\) on \( E_\varepsilon \), there exists a homoclinic orbit asymptotic to a fixed point \( q_\varepsilon \). The submanifold \( E_\varepsilon \) is in an \( O(\varepsilon^\nu) \) neighborhood of the hyperplane \( \beta = \kappa \alpha \), where \( \kappa = \kappa(\omega; N) \) is shown in Figures 4 and 5. \( \nu = 1/2 - \delta_0, \) \( 0 < \delta_0 < 1/2 \).

Remark 2.7. In the cases \((3 \leq N \leq 6)\), \( \kappa \) is always negative as shown in Figure 6. Since we require both dissipation parameters \( \alpha \) and \( \beta \) to be positive, the relation \( \beta = \kappa \alpha \) shows that the existence of homoclinic orbits violates this positivity. For \( N \geq 7 \), \( \kappa \) can be positive as shown in Figure 8. When \( N \) is even and \( \geq 7 \), there is in fact a pair of homoclinic orbits asymptotic to a fixed point \( q_\varepsilon \) at the same values of the external parameters; since for even \( N \), we have the symmetry: if \( q_n = f(n, t) \) solves (2.4), then \( q_n = f(n + N/2, t) \) also solves (2.4). When \( N \) is odd and \( \geq 7 \), the study can not guarantee that two homoclinic orbits exist at the same value of the external parameters.

For the complete proof of this theorem, see [23].

Theorem 2.8 (Horseshoe Theorem). Under certain generic assumptions for the perturbed discrete nonlinear Schrödinger system (2.4), there exists a compact Cantor subset \( \Lambda \) of \( \Sigma \) (a Poincaré section transversal to the homoclinic orbit), \( \Lambda \) consists of points, and is invariant under \( P \) (the Poincaré map induced by the flow on \( \Sigma \)). \( P \) restricted to \( \Lambda \), is topologically conjugate to the shift automorphism \( \chi \) on four symbols \( 1, 2, -1, -2 \). That is, there exists a homeomorphism
\[
\phi : \mathcal{W} \to \Lambda,
\]
where \( \mathcal{W} \) is the topological space of the four symbols, such that the following diagram commutes:
\[
\begin{array}{ccc}
\mathcal{W} & \xrightarrow{\phi} & \Lambda \\
\chi \downarrow & & \downarrow P \\
\mathcal{W} & \to & \Lambda
\end{array}
\]
For the complete proof of the theorem, see [24].

The unperturbed homoclinic orbits for the discrete nonlinear Schrödinger equation
\begin{equation}
(2.5) \quad i\dot{q}_n = \frac{1}{h^2} \left[ q_{n+1} - 2q_n + q_{n-1} \right] + |q_n|^2(q_{n+1} + q_{n-1}) - 2\omega^2q_n,
\end{equation}
was constructed through the Darboux transformations which will be presented below in details. The discrete nonlinear Schrödinger equation is associated with the following discrete Zakharov-Shabat system [2]:
\begin{align}
(2.6) & \quad \varphi_{n+1} = L_n^{(z)} \varphi_n, \\
(2.7) & \quad \hat{\varphi}_n = B_n^{(z)} \varphi_n,
\end{align}
where
\[
L_n^{(z)} \equiv \begin{pmatrix}
z & ihq_n \\
-ih\hat{q}_n & 1/z
\end{pmatrix},
\]
\[
B_n^{(z)} \equiv \frac{i}{h^2} \begin{pmatrix}
1 - z^2 + 2i\lambda h - h^2q_n\hat{q}_{n-1} + \omega^2h^2 & -zihq_n + (1/z)ihq_{n-1} \\
-izh\hat{q}_{n-1} + (1/z)ih\hat{q}_n & 1/z^2 - 1 + 2i\lambda h + h^2\hat{q}_nq_{n-1} - \omega^2h^2
\end{pmatrix};
\]
Figure 2. The curve of $\kappa = \kappa(\omega; N)$.
Figure 3. The curve of $\kappa = \kappa(\omega; N)$. 
and where \( z = \exp(i \lambda h) \).

Fix a solution \( q_n(t) \) of the system \( \text{(2.5)} \), for which the linear operator \( L_n \) has a double point \( z^d \) of geometric multiplicity 2, which is not on the unit circle. We denote two linearly independent solutions (Bloch functions) of the discrete Lax pair \( \text{(2.6,2.7)} \) at \( z = z^d \) by \( (\phi_n^+, \phi_n^-) \). Thus, a general solution of the discrete Lax pair \( \text{(2.6,2.7)} \) at \( (q_n(t), z^d) \) is given by

\[
\phi_n(t; z^d, c) = \phi_n^+ + c \phi_n^-,
\]

where \( c \) is a complex parameter called Bäcklund parameter. We use \( \phi_n \) to define a transformation matrix \( \Gamma_n \) by

\[
\Gamma_n = \begin{pmatrix}
z + (1/z) a_n & b_n \\
c_n & -1/z + z d_n
\end{pmatrix},
\]

where

\[
a_n = \frac{z^d}{(z^d)^2 \Delta_n} \left[ |\phi_{n2}|^2 + |z^d|^2 |\phi_{n1}|^2 \right],
\]

\[
d_n = -\frac{1}{z^d \Delta_n} \left[ |\phi_{n2}|^2 + |z^d|^2 |\phi_{n1}|^2 \right],
\]

\[
b_n = \frac{|z^d|^4 - 1}{(z^d)^2 \Delta_n} \phi_{n1} \bar{\phi}_{n2},
\]

\[
c_n = \frac{|z^d|^4 - 1}{z^d \Delta_n} \bar{\phi}_{n1} \phi_{n2},
\]

\[
\Delta_n = -\frac{1}{z} \left[ |\phi_{n1}|^2 + |z^d|^2 |\phi_{n2}|^2 \right].
\]

Then we define \( Q_n \) and \( \Psi_n \) by

\[
(2.8) \quad Q_n \equiv i h b_{n+1} - a_{n+1} q_n
\]

and

\[
(2.9) \quad \Psi_n(t; z) \equiv \Gamma_n(z; z^d; \phi_n) \psi_n(t; z)
\]

where \( \psi_n \) solves the discrete Lax pair \( (2.6,2.7) \) at \( (q_n(t), z) \). Formulas \( (2.8) \) and \( (2.9) \) are the Bäcklund-Darboux transformations for the potential and eigenfunctions, respectively. We have the following theorem \( \text{(11)} \).

**Theorem 2.9 (Bäcklund-Darboux Transformations).** Let \( q_n(t) \) denote a solution of the system \( (\text{2.3}) \), for which the linear operator \( L_n \) has a double point \( z^d \) of geometric multiplicity 2, which is not on the unit circle and which is associated with an instability. We denote two linearly independent solutions (Bloch functions) of the discrete Lax pair \( (\text{2.4,2.7}) \) at \( (q_n, z^d) \) by \( (\phi_n^+, \phi_n^-) \). We define \( Q_n(t) \) and \( \Psi_n(t; z) \) by \( (2.8) \) and \( (2.9) \). Then

1. \( Q_n(t) \) is also a solution of the system \( (\text{2.3}) \). (The eveness of \( Q_n \) can be guaranteed by choosing the complex Bäcklund parameter \( c \) to lie on an certain curve.)
2. \( \Psi_n(t; z) \) solves the discrete Lax pair \( (\text{2.4,2.7}) \) at \( (Q_n(t), z) \).
3. \( \Delta(z; Q_n) = \Delta(z; q_n), \) for all \( z \in C, \) where \( \Delta \) is the Floquet discriminant.
where $r$ is the perturbation parameter, $\varepsilon$ is a nonvanishing growth rate associated to the double point $z^\pm$, and explicit formulas can be developed for this growth rate and for the phase shifts $\theta_\pm$.

Next we consider a concrete example. Let

$$q_n = q, \quad \forall n; \quad q = a \exp\{-2i[(a^2 - \omega^2)t] + i\gamma\},$$

where $N \tan \frac{\pi}{2} < a < N \tan \frac{\pi}{3}$ for $N > 3, 3 \tan \frac{\pi}{3} < a < \infty$ for $N = 3$. Then $Q_n$ defined in (2.8) has the explicit representation:

$$Q_n \equiv Q_n(t; N, \omega, \gamma, r, \pm) = q \left[ \frac{G}{H_n} - 1 \right],$$

where,

$$G = 1 + \cos 2P - i \sin 2P \tanh \tau,$$

$$H_n = 1 \pm \frac{1}{\cos \vartheta} \sin P \sech \tau \cos 2n\vartheta,$$

$$\tau = 4N^2 \sqrt{\rho} \sin \vartheta \sqrt{\rho \cos^2 \vartheta - 1} t + r,$$

where $r$ is a real parameter. Furthermore,

$$P = \arctan \frac{\sqrt{\rho \cos^2 \vartheta - 1}}{\sqrt{\rho} \sin \vartheta},$$

$$\vartheta = \frac{\pi}{N}, \quad \rho = 1 + \frac{|q|^2}{N^2}.$$ As $\tau \to \pm \infty$, $Q_n \to q e^{\mp i2P}$. Therefore, $Q_n$ is homoclinic to the circle $|q_n| = a$, and heteroclinic to points on the circle which are separated in phase of $-4P$.

2.2.2. Hamiltonian Perturbations. In the paper [12], the author studied the discrete nonlinear Schrödinger equation under Hamiltonian perturbations:

$$i\dot{q}_n = \frac{1}{h^2}[q_{n+1} - 2q_n + q_{n-1}] + |q_n|^2(q_{n+1} + q_{n-1}) - 2\omega^2 q_n$$

$$+ \varepsilon \left\{ [\alpha_1(q_n + \bar{q}_n) + \alpha_2(q_n^2 + \bar{q}_n^2)]q_n + [\alpha_1 + 2\alpha_2\bar{q}_n] \rho_n \right\} \bar{q}_n,$$

where $i = \sqrt{-1}$, $q_n$ s are complex variables, $n \in Z$, $(\omega, \alpha_1, \alpha_2)$ are real constants, $\varepsilon$ is the perturbation parameter, $h$ is the step size, $h = 1/N$, $N > 3$ is an integer, $\rho_n = 1 + h^2|q_n|^2$, and $q_{n+N} = q_n$, $q_{n-n} = q_n$. The system (2.12) can be written in the Hamiltonian form:

$$i\dot{q}_n = \rho_n \frac{\partial H}{\partial q_n},$$

where $H = H_0 + \varepsilon H_1$,

$$H_0 = \frac{1}{h^2} \sum_{n=0}^{N-1} [\bar{q}_n(q_{n+1} + q_{n-1}) - \frac{2}{h^2}(1 + \omega^2h^2) \ln \rho_n],$$

$$H_1 = \frac{1}{h^2} \sum_{n=0}^{N-1} [\alpha_1(q_n + \bar{q}_n) + \alpha_2(q_n^2 + \bar{q}_n^2)] \ln \rho_n.$$
Theorem 2.10 (Homoclinic Tube Theorem). There exist a positive constant $\varepsilon_0 > 0$ and a region $\mathcal{E}$ for $(\alpha_1, \alpha_2, \omega)$, such that for any $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and any $(\alpha_1, \alpha_2, \omega) \in \mathcal{E}$, there exists a codimension 2 transversal homoclinic tube asymptotic to a codimension 2 center manifold $W^{(c)}$.

For a complete proof of this theorem, see [12].

2.3. Chaos in Perturbed $(1+n)$-Dimensional Soliton Equations $(n \geq 2)$. For this category of the perturbed soliton equations, we chose the candidate to be the perturbed Davey-Stewartson II equations. The Davey-Stewartson II equations describe nearly one-dimensional water surface wave train [3]. There have been a lot of studies on the inverse scattering transforms for this set of equations [1][3]. The inverse scattering transforms for $(1+n)$-dimensional soliton equations $(n \geq 2)$ are substantially different from those for $(1+1)$-dimensional soliton equations and soliton lattices. In fact, the Davey-Stewartson II equations possess finite-time singularities [27]. For the perturbed Davey-Stewartson II equations, the theory on chaos is largely unfinished. So far, its Melnikov theory has been successfully built.

Although the inverse spectral theory for the DSII equations is very different from those for $(1+1)$-dimensional soliton equations and there is no Floquet spectral theory, its Bäcklund-Darboux transformation is as simple as those for $(1+1)$-dimensional soliton equations, e.g. the cubic nonlinear Schrödinger equation. These Bäcklund-Darboux transformations are successfully utilized to construct heteroclinic orbits of Davey-Stewartson II equations through an elegant iteration of the transformations. In [22], we successfully built Melnikov vectors for the focusing cubic nonlinear Schrödinger equation with the gradients of the invariants $F_j$ defined through the Floquet discriminants evaluated at critical spectral points. The invariants $F_j$’s Poisson commute with the Hamiltonian, and their gradients decay exponentially as time approaches positive and negative infinities – these two properties are crucial in deriving and evaluating Melnikov functions. Since there is no Floquet discriminant for Davey-Stewartson equations (in contrast to nonlinear Schrödinger equations [22]), the Melnikov vectors here are built with the novel idea of replacing the gradients of Floquet discriminants by quadratic products of Bloch functions. Such Melnikov vectors still maintain the properties of Poisson commuting with the gradient of the Hamiltonian and exponential decay as time approaches positive and negative infinities. This solves the problem of building Melnikov vectors for Davey-Stewartson equations without using the gradients of Floquet discriminant. Melnikov functions for perturbed Davey-Stewartson II equations evaluated on the above heteroclinic orbits are built.

2.3.1. Darboux Transformations. First we study the Darboux transformations for the Davey-Stewartson II (DSII) equations:

$$i\partial_t q = [\partial_x^2 - \partial_y^2]q + 2(|q|^2 - \omega^2) + u_y|q|,$$

$$[\partial_x^2 + \partial_y^2]u = -4\partial_y|q|^2;$$

under periodic boundary conditions $q(t, x + l_x, y) = q(t, x, y + l_y) = q(t, x, y)$, where $q$ and $u$ are a complex-valued and a real-valued functions of three variables $(t, x, y)$. To simplify the study, we may also pose even conditions in both $x$ and $y$. The DSII equations are associated with a Lax pair and a congruent Lax pair. The Lax pair
is:

\begin{align}
L \psi &= \lambda \psi, \\
\partial_t \psi &= A \psi,
\end{align}

where \( \psi = (\psi_1, \psi_2)^T \), and

\[
L = \begin{pmatrix}
D^- & q \\
r & D^+
\end{pmatrix},
\]

\[
A = i \begin{pmatrix}
- \partial_x^2 & q \partial_x \\
r \partial_x^2 & -(D^- q)
\end{pmatrix} + \begin{pmatrix}
- D^- r \\
D^+ r
\end{pmatrix} \frac{D}{D^+} \frac{D}{D^-}.
\]

Here we denote by

\[
D^+ = \alpha \partial_y + \partial_x, \quad D^- = \alpha \partial_y - \partial_x.
\]

where \( r = \bar{q}, \quad \alpha^2 = -1, \)

\[
r_1 = \frac{1}{2}[-U + iV], \quad r_2 = \frac{1}{2}[U + iV], \quad U = 2(|q|^2 - \omega^2) + u_y.
\]

The congruent Lax pair is:

\begin{align}
\hat{L} \hat{\psi} &= \lambda \hat{\psi}, \\
\partial_t \hat{\psi} &= \hat{A} \hat{\psi},
\end{align}

where \( \hat{\psi} = (\hat{\psi}_1, \hat{\psi}_2)^T \), and

\[
\hat{L} = \begin{pmatrix}
-D^+ & q \\
r & -D^-
\end{pmatrix},
\]

\[
\hat{A} = i \begin{pmatrix}
- \partial_x^2 & q \partial_x \\
r \partial_x^2 & -(D^- q)
\end{pmatrix} + \begin{pmatrix}
- r_2 \\
(D^+ r)
\end{pmatrix} \frac{D}{D^+} \frac{D}{D^-}.
\]

Let \((q, r = \bar{q}, r_1, r_2)\) be a solution to the DSII equation, and let \( \lambda_0 \) be any value of \( \lambda \). Denote by \( \psi = (\psi_1, \psi_2)^T \) the eigenfunction solving the Lax pair \(2.14, 2.17\) at \((q, r = \bar{q}, r_1, r_2; \lambda_0)\). Define the matrix operator:

\[
\Gamma = \left[ \begin{array}{cc}
\wedge + a & b \\
c & \wedge + d
\end{array} \right],
\]

where \( \wedge = \alpha \partial_y - \lambda \), and \( a, b, c, d \) are functions defined as:

\[
a = \frac{1}{\Delta} \left[ \psi_2 \wedge_2 \bar{\psi}_2 + \beta \bar{\psi}_1 \wedge_1 \psi_1 \right],
\]

\[
b = \frac{1}{\Delta} \left[ \bar{\psi}_2 \wedge_1 \psi_1 - \psi_1 \wedge_2 \bar{\psi}_2 \right],
\]

\[
c = \frac{\beta}{\Delta} \left[ \bar{\psi}_1 \wedge_1 \psi_2 - \psi_2 \wedge_2 \bar{\psi}_1 \right],
\]

\[
d = \frac{1}{\Delta} \left[ \bar{\psi}_2 \wedge_1 \psi_2 + \beta \psi_1 \wedge_2 \bar{\psi}_1 \right],
\]

in which \( \wedge_1 = \alpha \partial_y - \lambda_0, \quad \wedge_2 = \alpha \partial_y + \bar{\lambda}_0 \), and

\[
\Delta = - \left[ \beta \psi_1^2 + |\psi_2|^2 \right].
\]
Define a transformation as follows:
\[
(q, r = \beta \bar{q}, r_1, r_2) \rightarrow (Q, R, R_1, R_2),
\]
\[
\phi \rightarrow \Phi;
\]
\[
Q = q - 2b,
\]
\[
R = \beta \bar{q} - 2c,
\]
\[
R_1 = r_1 + 2(D^+ a),
\]
\[
R_2 = r_2 - 2(D^- d),
\]
\[
\Phi = \Gamma \phi;
\]
(2.19)
where \( \phi \) is an eigenfunction solving the Lax pair (2.14, 2.15) at \((q, r = \bar{q}, r_1, r_2; \lambda)\), \(D^+\) and \(D^-\) are defined in (2.16).

**Theorem 2.11** ([15]). The transformation (2.19) is a Bäcklund-Darboux transformation. That is, the functions \((Q, R = \bar{Q}, R_1, R_2)\) defined through the transformation (2.19) are also a solution to the Davey-Stewartson II equations. The function \(\Phi\) defined through the transformation (2.19) solves the Lax pair (2.14, 2.15) at \((Q, R = \bar{Q}, R_1, R_2; \lambda)\).

A concrete example with two iterations of the Darboux transformations has been worked out in [15].

2.3.2. Melnikov Vectors. The DSII equations can be put into the Hamiltonian form,
\[
\begin{align*}
{	extit{i}}q_t &= \frac{\delta H}{\delta \bar{q}}, \\
{\textit{i}}q_t &= -\frac{\delta H}{\delta q},
\end{align*}
\]
where
\[
H = \int_0^{l_x} \int_0^{l_y} \left[ |q_\bar{y}|^2 - |q_x|^2 + \frac{1}{2} (r_2 - r_1) |q|^2 \right] dx dy.
\]
Let \(\psi = (\psi_1, \psi_2)^T\) be an eigenfunction solving the Lax pair (2.14, 2.15), and \(\hat{\psi} = (\hat{\psi}_1, \hat{\psi}_2)^T\) be an eigenfunction solving the corresponding congruent Lax pair (2.17, 2.18); then

**Lemma 2.12.** The inner product of the vector
\[
\mathcal{U} = \left(\begin{array}{c}
\psi_2 \hat{\psi}_2 \\
\psi_1 \hat{\psi}_1
\end{array}\right) - S \left(\begin{array}{c}
\psi_2 \hat{\psi}_2 \\
\psi_1 \hat{\psi}_1
\end{array}\right),
\]
where \(S = \left(\begin{array}{cc}
0 & 1 \\
1 & 0
\end{array}\right)\), with the vector field \(J \nabla H\) given by the right hand side of (2.21) vanishes,
\[
\langle \mathcal{U}, J \nabla H \rangle = 0.
\]

where
\[
\langle f, g \rangle = \int_0^{l_x} \int_0^{l_y} \{ \mathcal{J}_1 g_1 + \mathcal{J}_2 g_2 \} dx dy.
\]
and
\[ J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

Consider the perturbed DSII equations
\[
\begin{aligned}
\ii \partial_t q &= [\partial^2_x - \partial^2_y]q + 2(|q|^2 - \omega^2) + u_y |q| + \varepsilon if, \\
[\partial^2_x + \partial^2_y]u &= -4\partial_y |q|^2,
\end{aligned}
\]
where \( f \) is the perturbation which can depend on \( q \) and \( \vec{T} \) and their derivatives and \( t, x \) and \( y \). Let \( \vec{G} = (f, \vec{T})^T \). Then the Melnikov function has the expression,
\[
M = \int_{-\infty}^{\infty} \langle \vec{U}, \vec{G} \rangle \, dt
\]
where the integrand is evaluated on an unperturbed heteroclinic orbit obtained through the Bäcklund-Darboux transformations given in Theorem 2.11. A concrete example has been worked out in [15].

### 3. Two-Dimensional Euler Equations

One of the most important implications of chaos theory of partial differential equations in theoretical physics will be on the study of turbulence. For that goal, the author choose the 2D Navier-Stokes equations under periodic boundary conditions to begin a dynamical system study.

\[
\begin{aligned}
\frac{\partial \Omega}{\partial t} &= -u \frac{\partial \Omega}{\partial x} - v \frac{\partial \Omega}{\partial y} + \varepsilon \left[ \Delta \Omega + f \right], \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0;
\end{aligned}
\]
under periodic boundary conditions in both \( x \) and \( y \) directions with period \( 2\pi \), where \( \Omega \) is vorticity, \( u \) and \( v \) are respectively velocity components along \( x \) and \( y \) directions, \( \varepsilon = 1/\Re \), and \( f \) is the body force. When \( \varepsilon = 0 \), we have the 2D Euler equations,
\[
\begin{aligned}
\frac{\partial \Omega}{\partial t} &= -u \frac{\partial \Omega}{\partial x} - v \frac{\partial \Omega}{\partial y}, \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0.
\end{aligned}
\]

The relation between vorticity \( \Omega \) and stream function \( \Psi \) is,
\[
\Omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \Delta \Psi,
\]
where the stream function \( \Psi \) is defined by,
\[
u = \frac{\partial \Psi}{\partial x}, \quad v = \frac{\partial \Psi}{\partial y}.
\]
3.1. Lax Pair and Darboux Transformation. The main breakthrough in this project is the discovery of the Lax pair for 2D Euler equation \[17\]. The philosophical significance of the existence of a Lax pair for 2D Euler equation is beyond the particular project undertaken here. If one defines integrability of an equation by the existence of a Lax pair, then 2D Euler equation is integrable. More importantly, 2D Navier-Stokes equation at high Reynolds numbers is a near integrable system. Such a point of view changes our old ideology on Euler and Navier-Stokes equations.

Starting from Lax pairs, homoclinic structures can be constructed through Darboux transformations \[15\]. Indeed, in \[25\], the Darboux transformation for the Lax pair of 2D Euler equation has been found. Our general program is to first identify the figure eight structures of 2D Euler equation, and then study their consequence in 2D Navier-Stokes equation. The high Reynolds number 2D Navier-Stokes equation is viewed as a singular perturbation of the 2D Euler equation through the perturbation \(\varepsilon\Delta\), where \(\varepsilon = 1/\text{Re}\) is the inverse of the Reynolds number. As mentioned above, singular perturbations have been investigated for nonlinear Schrödinger equations.

We consider the 2D Euler equation,

\[
\frac{\partial \Omega}{\partial t} + \{\Psi, \Omega\} = 0, \tag{3.3}
\]

where the bracket \(\{ , \}\) is defined as

\[
\{f, g\} = (\partial_x f)(\partial_y g) - (\partial_y f)(\partial_x g), \quad \text{and} \quad \Omega = \Delta \Psi.
\]

**Theorem 3.1** \([17]\). *The Lax pair of the 2D Euler equation (3.3) is given as*

\[
\begin{align*}
L\phi &= \lambda \phi, \\
\frac{\partial}{\partial t} \phi + A\phi &= 0,
\end{align*} \tag{3.4}
\]

where \(L\phi = \{\Omega, \phi\}\), \(A\phi = \{\Psi, \phi\}\), and \(\lambda\) is a complex constant, and \(\phi\) is a complex-valued function.

In \[25\], A Bäcklund-Darboux transformation is found for the above Lax pair. Consider the Lax pair (3.4) at \(\lambda = 0\), i.e.

\[
\begin{align*}
\{\Omega, p\} &= 0, \tag{3.5} \\
\frac{\partial}{\partial t} p + \{\Psi, p\} &= 0, \tag{3.6}
\end{align*}
\]

where we replaced the notation \(\varphi\) by \(p\).

**Theorem 3.2.** *Let \(f = f(t, x, y)\) be any fixed solution to the system (3.3, 3.4), we define the Gauge transform \(G_f\):*

\[
\tilde{p} = G_f p = \frac{1}{\Omega_x} [p_x - (\partial_x \ln f)p], \tag{3.7}
\]

*and the transforms of the potentials \(\Omega\) and \(\Psi\):*

\[
\begin{align*}
\tilde{\Psi} &= \Psi + F, \\
\tilde{\Omega} &= \Omega + \Delta F,
\end{align*} \tag{3.8}
\]

*where \(F\) is subject to the constraints*

\[
\{\Omega, \Delta F\} = 0, \quad \{\Omega, F\} = 0. \tag{3.9}
\]
Then $\tilde{p}$ solves the system (3.5, 3.6) at $(\tilde{\Omega}, \tilde{\Psi})$. Thus (3.7) and (3.8) form the Darboux transformation for the 2D Euler equation (3.3) and its Lax pair (3.5, 3.6).

3.2. Linearized 2D Euler Equations. Under the periodic boundary condition and requiring that both $u$ and $v$ have means zero,

$$
\int_{0}^{2\pi} \int_{0}^{2\pi} u \ dx \ dy = \int_{0}^{2\pi} \int_{0}^{2\pi} v \ dx \ dy = 0,
$$

expanding $\Omega$ into Fourier series, $\Omega = \sum_{k \in Z^2/\{0\}} \omega_k e^{ik \cdot X}$, where $\omega_{-k} = \bar{\omega}_k$, $k = (k_1, k_2)$, $X = (x, y)$, the system (3.2) can be rewritten as the following kinetic system,

$$
\dot{\omega}_k = \sum_{p+q=k} A(p, q) \omega_p \omega_q,
$$

(3.10)

where $A(p, q)$ is given by,

$$
A(p, q) = \frac{1}{2} |q|^{-2} - |p|^{-2} |p_1 q_2 - p_2 q_1|
= \frac{1}{2} |q|^{-2} - |p|^{-2} \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix},
$$

(3.11)

where $|q|^2 = q_1^2 + q_2^2$ for $q = (q_1, q_2)$, similarly for $p$. To understand the hyperbolic structures of the 2D Euler equations, we first investigate the linearized 2D Euler
equations at a stationary solution. Denote \( \{\omega_k\}_{k \in \mathbb{Z}^2/\{0\}} \) by \( \omega \). Consider the simple
Figure 7. The quadruple of eigenvalues for the system led by the class $\Sigma_\hat{k}$ labeled by $\hat{k} = (1, 0)^T$, when $p = (1, 1)^T$.

fixed point $\omega^*$:

$$\omega_p^\ast = \Gamma, \quad \omega_k^\ast = 0, \text{ if } k \neq p \text{ or } -p,$$

of the 2D Euler equation (3.10), where $\Gamma$ is an arbitrary complex constant. The linearized two-dimensional Euler equation at $\omega^*$ is given by,

$$\dot{\omega}_k = A(p, \hat{k}) \Gamma \omega_{k-p} + A(-p, \hat{k}) \Gamma \omega_{k+p}.$$

**Definition 3.3 (Classes).** For any $\hat{k} \in \mathbb{Z}^2 / \{0\}$, we define the class $\Sigma_\hat{k}$ to be the subset of $\mathbb{Z}^2 / \{0\}$:

$$\Sigma_\hat{k} = \left\{ \hat{k} + np \in \mathbb{Z}^2 / \{0\} \mid n \in \mathbb{Z}, \text{ } p \text{ is specified in (3.12)} \right\}.$$

See Figure 4 for an illustration of the classes. According to the classification defined in Definition 3.3, the linearized two-dimensional Euler equation (3.13) decouples into infinite many invariant subsystems:

$$\dot{\omega}_{k+np} = A(p, \hat{k} + (n-1)p) \Gamma \omega_{k+(n-1)p} + A(-p, \hat{k} + (n+1)p) \Gamma \omega_{k+(n+1)p}.$$

**Definition 3.4 (The Disk).** The disk of radius $|p|$ in $\mathbb{Z}^2 / \{0\}$, denoted by $D_{|p|}$, is defined as

$$D_{|p|} = \left\{ k \in \mathbb{Z}^2 / \{0\} \mid |k| < |p| \right\}.$$
Figure 8. The heteroclinic orbits and unstable manifolds of the Galerkin truncation.
The closure of $D_{|p|}$, denoted by $\bar{D}_{|p|}$, is defined as $\bar{D}_{|p|} = \left\{ k \in \mathbb{Z}^2 / \{0\} \mid |k| \leq |p| \right\}$.

**Theorem 3.5 (Unstable Disk Theorem).** If $\Sigma_{\hat{k}} \cap \bar{D}_{|p|} = \emptyset$, then the invariant subsystem (3.14) is Lyapunov stable for all $t \in \mathbb{R}$, in fact, $\sum_{n \in \mathbb{Z}} \left| \omega_{k+n\hat{p}}(0) \right|^2 \leq \sigma \sum_{n \in \mathbb{Z}} \left| \omega_{k+n\hat{p}}(t) \right|^2, \forall t \in \mathbb{R}$, where $\sigma = \left[ \max_{n \in \mathbb{Z}} \{-\rho_n\} \right] \left[ \min_{n \in \mathbb{Z}} \{-\rho_n\} \right]^{-1}, \quad 0 < \sigma < \infty$.

**Theorem 3.6.** The eigenvalues of the linear system (3.14) are of four types: real pairs $(c, -c)$, purely imaginary pairs $(id, -id)$, quadruples $(\pm c \pm id)$, and zero eigenvalues.

**Theorem 3.7 (The Spectral Theorem).**
1. If $\Sigma_{\hat{k}} \cap \bar{D}_{|p|} = \emptyset$, then the entire $\ell_2$ spectrum of the linear operator $L_A$ (defined by the right-hand side of the invariant subsystem) is its continuous spectrum. See Figure 5.
2. If $\Sigma_{\hat{k}} \cap \bar{D}_{|p|} \neq \emptyset$, then the entire essential $\ell_2$ spectrum of the linear operator $L_A$ is its continuous spectrum. That is, the residual spectrum of $L_A$ is empty, $\sigma_r(L_A) = \emptyset$. The point spectrum of $L_A$ is symmetric with respect to both real and imaginary axes. See Figure 6.

We can calculate the eigenvalues through continued fractions. Let $p = (1, 1)^T$, in this case, only one class $\Sigma_{\hat{k}}$ labeled by $\hat{k} = (1, 0)^T$ has no empty intersection with $\bar{D}_{|p|}$ (the other class labeled by $\hat{k} = (0, 1)^T$ gives the complex conjugate of the system led by the class labeled by $\hat{k} = (1, 0)^T$). For this class, there is no real eigenvalue. Numerical calculation through continued fractions gives the eigenvalue: $\hat{\lambda} = 0.24822302478255 + i 0.35172076526520$.

Thus we have a quadruple of eigenvalues, see Figure 7 for an illustration. Denote by $L$ the right hand side of (3.13), the spectral mapping theorem holds.

**Theorem 3.8 (10).**

$$\sigma(e^{tL}) = e^{-\sigma(L)}, t \neq 0.$$

Moreover, the number of eigenvalues has a sharp upper bound. Let $\zeta$ denote the number of points $q \in \mathbb{Z}^2 / \{0\}$ that belong to the open disk of radius $|p|$, and such that $q$ is not parallel to $p$.

**Theorem 3.9 (10).** The number of nonimaginary eigenvalues of $L$ (counting the multiplicities) does not exceed $2\zeta$. 
3.3. Approximate Explicit Representations of the Hyperbolic Structures of 2D Euler Equations. From Figure 7, we see that the simple fixed point given by $p = (1, 1)$, has unstable eigenvalues. Our interest is to obtain representations of the corresponding hyperbolic structures for 2D Euler equations. In [18], through Galerkin truncation, we obtained the approximate explicit representation. Figure 8 shows the heteroclinic orbits and unstable manifolds of the Galerkin truncation.

4. Conclusion and Discussion

We have reported the status of chaos in nonlinear wave equations and of study on 2D Euler equations. In particular, we have summarized the most recent results on Lax pair and Darboux transformations for 2D Euler equations.

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