Minimum Power Range Assignment for Symmetric Connectivity in Sensor Networks with two Power Levels

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Abstract. This paper examines the problem of assigning a transmission power to every node of a wireless sensor network. The aim is to minimize the total power consumption while ensuring that the resulting communication graph is connected. We focus on a restricted version of this RANGE ASSIGNMENT (RA) problem in which there are two different power levels. We only consider symmetrical transmission links to allow easy integration with low level wireless protocols that typically require bidirectional communication between two neighboring nodes. We introduce a parameterized polynomial time approximation algorithm with a performance ratio arbitrarily close to $\pi/\sqrt{6}$. Additionally, we give an almost linear time approximation algorithm with a tight quality bound of $7/4$.

Keywords: range assignment, power consumption, radio network, sensor network, wireless ad hoc network, approximation algorithm

1 Introduction

A sensor network consists of a large number of small devices that are deployed across a geographic area to monitor certain aspects of the environment. These sensor nodes are able to communicate with each other through a wireless communication channel. As a result of their size the resources of the sensor nodes are strongly limited in terms of available energy. This leads to a very limited range of the radio transmitters.

Minimizing the power consumption to prolong the lifetime of the network has received considerable attention, because the replacement of batteries in a large scale network is practically difficult or even impossible.

A common approach to save energy is based on adjusting the power levels of the sensor nodes' radio transmitters while maintaining a connected network structure. This RANGE ASSIGNMENT (RA) problem is typically defined as computing a range assignment $f : P \rightarrow \mathbb{R}^+$ for a set of points $P \subset \mathbb{R}^n$ ($1 \leq n \leq 3$), representing the nodes in the network, such that the total energy $\sum_{p \in P} c(f(p))$ is minimal ($c$ being a cost function according to a radio wave propagation model) under the constraint that the graph $(P, E)$, $E := \{(p, q) \in P^2 \mid \|p - q\|_2 \leq f(p)\}$ is strongly connected, where $\|p - q\|_2$ denotes the Euclidean distance between $p$ and $q$. This is called the asymmetric RA problem.

Since many low level protocols for wireless communication like IEEE 802.11 require bidirectional transmission links for hop-by-hop acknowledgements [8], the constraint of the asymmetric RA problem is often altered to only consider bidirectional edges $E' := \{(p, q) \mid (p, q) \in E \land (q, p) \in E\}$ for an undirected connected graph. This problem is called the symmetric RA problem.
Contribution

This paper examines the symmetric version of the RA problem in networks with two power levels (denoted by 2LSRA). The 2-dimensional decision version of the 2LSRA problem is NP-complete, because the proof of [5] for the asymmetric case can easily be modified to the symmetric case. We introduce an approximation algorithm (denoted by APPROX2LSRA_k) for the 2LSRA problem parameterized by an integer k ≥ 2. We prove that the performance ratio of APPROX2LSRA_k is between

\[
\frac{3k - 2}{2k - 2} \quad \text{and} \quad \frac{1}{k - 1} + \sum_{i=1}^{k-1} \frac{1}{i^2}.
\]

Algorithm APPROX2LSRA_k can easily be implemented such that its running time is in \( O(|V|^k) \), where \( V \) is the set of sensor nodes. This provides a parameterized polynomial time approximation algorithm with a performance ratio arbitrarily close to \( \pi^2/6 \). For the case \( k = 3 \), for which APPROX2LSRA_k has a tight performance of 7/4, we provide an implementation running in almost linear time.

The approximation algorithm APPROX2LSRA_k is based solely on the graph structure and the following assumption: If a node \( u \) can receive messages from a node \( v \) while \( v \) transmits with power \( r_{\min} \), \( u \) can still receive messages from \( v \) if \( v \) increases its power level to \( r_{\max} \). Note that the algorithm requires neither any embedding of the nodes into \( \mathbb{R}^d \) for some dimension \( d \) nor any kind of correlation between the transmission links, as for example, if \( u \) can receive messages from \( v \) then \( v \) can receive messages from \( u \).

Related Work

Both versions of the RA problem are typically considered with cost function \( c(d) = \beta \cdot d^\alpha + \gamma \) for a distance-power gradient \( \alpha \geq 1 \), a transmission quality parameter \( \beta \) and some \( \gamma \geq 0 \). Kirousis et al. proved in [9] that the decision version of the asymmetric RA problem is NP-hard for 3-dimensional point sets for all \( \alpha \geq 1 \), \( \beta = 1 \) and \( \gamma = 0 \). They also presented a factor 2 approximation algorithm based on a minimum spanning tree. The hardness result was extended by Clementi et al. in [6] to 2-dimensional point sets. Clementi et al. also showed in [4] the APX-hardness of the 3-dimensional case. This implies that there is no polynomial time approximation scheme (PTAS) for the 3-dimensional asymmetric RA problem, unless P=NP. The 1-dimensional case, on the other hand, is solvable in time \( O(|P|^4) \) by dynamic programming techniques, see [9].

For the symmetric RA problem, Blough et al. showed in [1] that the decision version is also NP-hard for 2- and 3-dimensional point sets, whereas Althaus et al. have shown in [2,1] that there is a simple approximation algorithm with approximation factor 11/6 as well as a parameterized polynomial time approximation algorithm with a performance ratio arbitrarily close to 5/3.

Carmi and Katz introduced in [5] the additional restriction of only two different transmission powers \( r_{\min} \) and \( r_{\max} \) with \( r_{\min} < r_{\max} \). This simplifies the RA problems to the question of how to determine a minimum set of sensor nodes that have to transmit with maximum power \( r_{\max} \) in order to obtain connectivity. They study the 2-dimensional...
asymmetric version of this problem, proof its NP-completeness, and present an approximation algorithm with a factor of $11/6$ as well as a rather theoretical approximation algorithm with a factor of $9/5$.

While formulated in an entirely different way, the 2LSRA problem is basically equivalent to the problem called Max Power Users given in [10], which was renamed to $\{0, 1\}$-MPST by the authors of [11]. And although this work has been conducted independently of [10], the concept used for an approximation algorithm is remarkably similar: In this paper a family of approximation algorithms, called $\text{Approx2LSRA}_k$ and based on a positive integer $k \in \mathbb{N}$, is developed and analyzed. The authors of [10] introduce two approximation algorithms which are in fact equivalent to our special cases $\text{Approx2LSRA}_3$ and $\text{Approx2LSRA}_4$. However, the best known approximation ratio possibly for this problem is given in [11] where the authors prove that the algorithm presented in [1] for the more general problem of arbitrary power levels achieves an approximation ratio of $3/2$ for the special case of two power levels. It is noteworthy that the algorithm given in [1] is based on a rather complex approximation scheme for the classical Steiner Tree problem, while the idea used in this paper and in [10] is a simple, fast and easy to implement greedy approach.

2 Definitions and Terminology

The problem we consider is motivated by wireless sensor networks in which the nodes have two transmission power levels, a min-power and a max-power level. Such networks are usually represented as directed graphs in that the directed edges represent connections from source nodes to target nodes. Since almost all communication protocols are based on symmetric connections, we are mainly interested in the underlying symmetric networks. These networks can be represented by undirected graphs.

Our goal is to find a minimum number $k$ of nodes such that if these $k$ nodes use max-power and the remaining nodes use min-power then the resulting underlying symmetric network is connected.

The problem is now defined more formally. We consider undirected graphs $G = (V, E)$, where $V$ is the set of nodes and $E \subseteq \{(u, v) \mid u, v \in V, u \neq v\}$ is a set of undirected edges. A graph $G' = (V', E')$ is a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. It is an induced subgraph of $G$, denoted by $G|_{V'}$, if $E' = E \cap \{(u, v) \mid u, v \in V'\}$.

A path between two nodes $u, v \in V$ is a sequence of nodes $u_1, \ldots, u_k \in V$, $k \geq 1$, such that $u_1 = u$, $\{u_i, u_{i+1}\} \in E$ for $i = 1, \ldots, k - 1$ and $u_k = v$. Graph $G = (V, E)$ is connected if there is a path between every pair of nodes. A path $u_1, \ldots, u_k$, $k \geq 3$, is a cycle if $\{u_k, u_1\} \in E$, a graph without cycles is a forest, and a connected forest is a tree. A connected component of $G$ is a maximal induced subgraph of $G$ that is connected. Let $\text{CC}(G)$ be the set of all node sets of the connected components of $G$. For a node $u \in V$, let $d_{\text{min}}(u) \subseteq V$ and $d_{\text{max}}(u) \subseteq V$ be the set of nodes reachable from $u$ with min-power or max-power, respectively. That is, $d_{\text{min}}$ and $d_{\text{max}}$ can be considered as mappings from $V$ to the power set $\mathcal{P}(V)$ of $V$, i.e., to the set of all subsets of $V$. 
Definition 1. For a node set $V$, a subset $U \subseteq V$, and two mappings $d_{\text{min}} : V \to \mathcal{P}(V)$ and $d_{\text{max}} : V \to \mathcal{P}(V)$ define the min-power edges between nodes in $U$ as
\[ E_{\text{min}}(U) := \{u, v \mid u, v \in U, u \neq v, u \in d_{\text{min}}(v), v \in d_{\text{min}}(u)\}, \]
the max-power edges between nodes in $U$ as
\[ E_{\text{max}}(U) := \{u, v \mid u, v \in U, u \neq v, u \in d_{\text{max}}(v), v \in d_{\text{max}}(u)\}, \]
and the min-max-power graph $G(V, d_{\text{min}}, U, d_{\text{max}}) := (V, E_{\text{min}}(V) \cup E_{\text{max}}(U))$.

The min-max-power graph $G(V, d_{\text{min}}, U, d_{\text{max}})$ is also denoted by $G(U)$ if $V$, $d_{\text{min}}$, and $d_{\text{max}}$ are known from the context. We also assume that $d_{\text{min}}(v) \subseteq d_{\text{max}}(v)$ for all $v \in V$.

The min-max-power graph $G(U)$ represents the underlying symmetric network for the case that the nodes of $U$ use max-power and the remaining nodes of $V \setminus U$ use min-power. Graph $G(\emptyset)$ is also called the min-power graph whereas graph $G(V)$ is also called the max-power graph.

The definition of the min-max-power graph is used to define the 2-Level Symmetric Range Assignment problem as follows.

### 2-Level Symmetric Range Assignment (2LSRA)

**Given:** A node set $V$ and two mappings $d_{\text{min}} : V \to \mathcal{P}(V)$ and $d_{\text{max}} : V \to \mathcal{P}(V)$ such that the max-power graph $G(V)$ is connected.

**Task:** Compute a set $U \subseteq V$ of minimum cardinality such that the min-max-power graph $G(U)$ is connected.

The decision problem of finding a node set $U$ of size at most $k$, for an additionally given integer $k$, such that $G(U)$ is connected is NP-complete. This result is implicitly contained in Theorem 6.1 of [5], which proofs the NP-completeness of the corresponding asymmetric version.

### 3 An approximation algorithm for 2LSRA

If the min-max-power graph $G(U)$ is connected for some set $U \subseteq V$ then $U$ has to contain at least one node of every connected component of the min-power graph $G(\emptyset)$. That is, the number $|CC(G(\emptyset))|$ of connected components of $G(\emptyset)$ is a lower bound for the size of such a set $U$. On the other hand, it is easy to find a set $U \subseteq V$ with at most $2(|CC(G(\emptyset))| - 1)$ nodes such that $G(U)$ is connected. Such a set can be determined by a simple spanning tree algorithm. Let $H$ be the graph that has a node for every connected component of $G(\emptyset)$ and an edge between two nodes $C, C' \in CC(G(\emptyset))$ if there are nodes...
Let \( u \in C, u' \in C' \) such that \( u' \in d_{\text{max}}(u) \) and \( u \in d_{\text{max}}(u') \). Let \( T \) be a spanning tree for \( H \). Then for every edge \( \{C, C'\} \) of \( T \) we can select two nodes \( u \in C, v \in C' \) from different connected components of \( G(\emptyset) \) such that for the set \( U \) of all these selected nodes the min-max-power graph \( G(U) \) is connected.

**Observation 1** Any solution for an instance of 2LSRA contains at least \(|CC(G(\emptyset))|\) and at most \(2(|CC(G(\emptyset))| - 1)\) nodes.

Our algorithm starts with the min-power graph \((V, E_0) := G(\emptyset)\) and an empty node set \( U_0 \) that is successively extended to node sets \( U_i \) by adding node sets \( M_i \) such that \((V, E_i)\) has less connected components than \((V, E_{i-1})\), where \( E_i := E_{i-1} \cup E_{\text{max}}(M_i) \). This is done until \((V, E_i)\) is connected, which implies that \( G(U_i) \) is connected, because \((V, E_i)\) is a subgraph of \( G(U_i) \). Obviously, a good choice for \( M_i \) is a set for which the ratio
\[
\frac{|CC((V, E_{i-1}))| - |CC((V, E_i))|}{|M_i|}
\]
is high. For example, if each \( M_i \) consists of two nodes then the number of connected components will be reduced by one at each extension and the algorithm computes a solution of size at most \(2(|CC(G(\emptyset))| - 1)\). This is equivalent to a spanning tree solution. If each \( M_i \) consists of three nodes and the number of connected components is reduced by two at each extension, then the algorithm computes a solution of size at most \(\frac{3}{2}(|CC(G(\emptyset))| - 1)\).

**Definition 2.** A set of nodes \( M \subseteq V \) of cardinality \( k \) is called a \( k \)-merging for a graph \( G = (V, E) \) and a mapping \( d_{\text{max}} : V \rightarrow \mathcal{P}(V) \), if

1. the graph \((M, E_{\text{max}}(M))\) is connected, and
2. the \( k \) nodes of \( M \) are in \( k \) different connected components of \( G \).

If we add a \( k \)-merging \( M_i \) to node set \( U_{i-1} \) then \( U_i = U_{i-1} \cup M_i \) and
\[
\frac{|CC((V, E_{i-1}))| - |CC((V, E_i))|}{|M_i|} = \frac{k - 1}{k}.
\]

The approximation algorithm \( \text{APPROX2LSRA}_k \) shown in Figure 1 has a fixed parameter \( k \geq 2 \). Starting with \( k' = k \), it successively gathers \( k' \)-mergings as long as possible. After this it decreases \( k' \) and proceeds in the same way until \((V, E_i)\) is connected, which will occur at the latest during the iteration for \( k' = 2 \), where all remaining 2-mergings are considered.

**Observation 2** Algorithm \( \text{APPROX2LSRA}_k \) always finds a solution for an instance \( V, d_{\text{min}}, d_{\text{max}} \) of 2LSRA.

For every integer \( k \in \mathbb{N} \) the algorithm \( \text{APPROX2LSRA}_k \) listed in Figure 1 can be implemented such that its running time is in \( \mathcal{O}(|V|^k) \), because the running time is dominated by the computation of all subsets \( M \subseteq V \) with \(|M| = k\) in line 7. Therefore we get a polynomial time algorithm for every constant integer \( k \).
1: function Approx2LSRA\(_k\)\((V, d_{\text{min}}, d_{\text{max}})\)  
2: \(k' \leftarrow k\)  
3: \(i \leftarrow 0\)  
4: \(U_0 \leftarrow \emptyset\)  
5: \(E_0 \leftarrow E_{\text{min}}(V)\)  
6: while \((V, E_i)\) is not connected do  
7: \(U_i+1 \leftarrow U_i \cup M_i\)  
8: \(E_i+1 \leftarrow E_i \cup E_{\text{max}}(M_i)\)  
9: \(i \leftarrow i + 1\)  
10: end while  
11: \(k' \leftarrow k' - 1\)  
12: end while  
13: return \(U_i\)  
14: end function

Fig. 1. Algorithm Approx2LSRA\(_k\) for a fixed integer \(k \geq 2\)

4 Upper Bounds on the quality of Approx2LSRA\(_k\)

Let \(U_{\text{OPT}}(I)\) be an optimal solution for an instance \(I = (V, d_{\text{min}}, d_{\text{max}})\) of 2LSRA. We show that Approx2LSRA\(_k\) for a positive integer \(k \geq 2\) computes a solution \(U_k(I)\) such that

\[
\frac{|U_k(I)|}{|U_{\text{OPT}}(I)|} \leq \frac{1}{k-1} + \sum_{i=1}^{k-1} \frac{1}{i^2}.
\]

Lemma 1. Let \(F_0 = (V, E_0)\) be a forest with \(n = |V|\) nodes and \(m = |E_0|\) edges and let \(p \in \mathbb{N}, 1 \leq p \leq n-1\) and \(l \in \mathbb{N}\) be positive integers. Furthermore let \(F_i = (V, E_i), 1 \leq i \leq l,\) be a sequence of forests such that \(F_i\) contains only trees with less than \(p\) edges and \(E_i \subseteq E_{i-1}, |E_i| = |E_{i-1}| - p.\)

If \(m > \frac{n \cdot (p-1)}{p}\) then

1. \(F_0\) contains a tree with at least \(p\) edges,  
2. each forest \(F_i,\)
   \[
i < \frac{1}{p} \cdot \left( m - \frac{n \cdot (p-1)}{p} \right),
   \]
   contains a tree with at least \(p\) edges and  
3.  
   \[
l \geq \left\lceil \frac{1}{p} \cdot \left( m - \frac{n \cdot (p-1)}{p} \right) \right\rceil.
   \]

Proof.

1. Forest \(F_0\) consists of \(n - m\) trees. If \(m > (n-m) \cdot (p-1),\) then at least one of these \(n - m\) trees has more than \(p - 1\) edges and thus at least \(p\) edges.

\[
m > (n-m) \cdot (p-1) \iff m > \frac{n \cdot (p-1)}{p}
\]
2. Forest \( F_i \) has \( m - i \cdot p \) edges. If

\[
i < \frac{1}{p} \left( m - \frac{n \cdot (p - 1)}{p} \right)
\]

then \( F_i \) has more than \( m - \frac{1}{p} \left( m - \frac{n \cdot (p - 1)}{p} \right) \cdot p = \frac{n \cdot (p - 1)}{p} \) edges and by Lemma 1, at least one tree with \( p \) edges.

3. Follows from 2. and the fact that we still can remove at least \( p \) more edges if there is a tree with \( p \) edges.

\[\square\]

**Lemma 2.** Let \( k \in \mathbb{N}, k \geq 3 \) be a positive integer and \( U_{OPT} \) an optimal solution for an instance of 2LSRA. Then APPROX2LSRA\(_k\) always finds at least

\[
\left[ \frac{1}{k - 1} \left( \left| CC(G(\emptyset)) \right| - 1 \right) - \frac{|U_{OPT}| \cdot (k - 2)}{k - 1} \right]
\]

\( k \)-mergings.

**Proof.** Let \( M_0, \ldots, M_{l-1} \) be the \( k \)-mergings chosen by APPROX2LSRA\(_k\) in Line 7 for a given instance \( V, d_{\text{min}}, d_{\text{max}} \) of 2LSRA and \( E_1, \ldots, E_l \) the edge sets computed in line 9 starting with edge set \( E_0 \) of the min-power graph \( G(\emptyset) \).

Let \( H = (CC(G(\emptyset)), E_H) \) be the undirected graph that has a node for every connected component of \( G(\emptyset) \) and an edge \( \{C_1, C_2\} \) if and only if there is an edge \( \{u, v\} \) in the min-max-power graph \( G(U_{OPT}) \) with \( u \in C_1 \) and \( v \in C_2 \).

For \( i = 0, \ldots, l \) we define a tree \( T_i \) and a forest \( F_i = (U_{OPT}, E_i) \) that satisfies the following invariant:

1. The node set \( R \) of every tree of forest \( F_i \), \( 0 \leq i \leq l \), is a \( |R| \)-merging for graph \((V, E_i)\).

Let \( T_0 = (CC(G(\emptyset)), E_T) \) be a spanning tree of \( H \) and \( F_0 \) be a forest that contains for every edge \( \{C_1, C_2\} \in E_T \) exactly one edge \( \{u, v\} \) of \( G(U_{OPT}) \) with \( u \in C_1 \) and \( v \in C_2 \).

Invariant (II) above obviously holds true for \( F_0 \).

For every \( k \)-merging \( M_i = \{u_1, \ldots, u_k\}, 0 \leq i < l \), we successively define trees \( T_{i,1}, \ldots, T_{i,k} \) and forests \( F_{i,1}, \ldots, F_{i,k} \) starting with \( T_{i,1} := T_i \) and \( F_{i,1} := F_i \) such that \( T_{i+1} := T_{i,k} \) and \( F_{i+1} := F_{i,k} \).

For \( j = 2, \ldots, k \), tree \( T_{i,j} \) and forest \( F_{i,j} \) are defined by merging two nodes of tree \( T_{i,j-1} \) and removing one edge from forest \( F_{i,j-1} \), respectively.

Let \( C^* \) be the node of \( T_{i,j-1} \) that contains the nodes \( u_1, \ldots, u_{j-1} \) and let \( C \) be the node of \( T_{i,j-1} \) that contains \( u_j \). Choose any edge \( \{C', C''\} \) from the path between \( C^* \) and \( C \) in \( T_{i,j-1} \), replace the two nodes \( C^*, C \) by one new node \( C^* \cup C \) in \( T_{i,j-1} \), and remove edge \( \{u, v\} \) with \( u \in C' \) and \( v \in C'' \) from \( F_{i,j-1} \).

If Invariant (II) holds true for \( F_i \), then it holds true for \( F_{i+1} \), because the construction above guarantees that for every simple path \( v_1, \ldots, v_m \) in \( F_{i+1} \) the nodes \( v_i, 1 \leq i \leq m \), are in \( m \) different connected components of \((V, E_{i+1})\).
Since forest $F_0$ has $|U_{\text{OPT}}|$ nodes and $|CC(G(\emptyset))| - 1$ edges, by Lemma 3, algorithm APPROX2LSRA$_k$ finds at least

$$\left\lceil \frac{1}{k-1} \cdot \left((|CC(G(\emptyset))| - 1) - \frac{|U_{\text{OPT}}| \cdot (k-2)}{k-1}\right) \right\rceil$$

$k$ mergings. \hfill \Box

**Theorem 1.** Let $I = (V, d_{\min}, d_{\max})$ be an instance of 2LSRA and $U_{\text{OPT}}(I)$ an optimal solution. For a fixed integer $k \geq 2$, algorithm APPROX2LSRA$_k$ computes a solution $U_k(I)$ such that

$$\frac{|U_k(I)|}{|U_{\text{OPT}}(I)|} \leq \frac{1}{k-1} + \sum_{i=1}^{k-1} \frac{1}{i^2}.$$ 

**Proof.** Let $l_i, 3 \leq i \leq k$, denote the number of $i$-mergings chosen by APPROX2LSRA$_k$. Furthermore, let $c_k := |CC(G(\emptyset))|$ be the number of connected components of the min-power graph and let $c_i, 2 \leq i \leq k-1$, be the number of connected components before APPROX2LSRA$_k$ searches for $i$-mergings for the first time, that is $c_i := |CC((V, E_{s(i)}))|$ with $s(i) := \sum_{j=i+1}^{k} l_j = l_k + \ldots + l_i + 1$. Then we know for $3 \leq i \leq k$ that

$$c_{i-1} = c_i - (i-1)l_i \quad \Leftrightarrow \quad l_i = \frac{c_i - c_{i-1}}{i-1}. \quad (1)$$

Let $d_{\min,s(i)} : V \to \mathcal{P}(V)$ be defined by $u \in d_{\min,s(i)}(v)$ if and only if $\{u, v\} \in E_{s(i)}$ for all $u, v \in V$. Then $(V, E_{s(i)})$ is the min-power graph of instance $I_{s(i)} := (V, d_{\min,s(i)}, d_{\max})$. Now we can apply Lemma 2 for $k = i$ on instance $I_{s(i)}$ and get

$$\frac{c_i - c_{i-1}}{i-1} \geq \frac{1}{i-1} \left( c_i - 1 - \frac{i-2}{i-1} \cdot |U_{\text{OPT}}(I_{s(i)})| \right) \quad \Leftrightarrow \quad c_{i-1} - 1 \leq \frac{i-2}{i-1} \cdot |U_{\text{OPT}}(I_{s(i)})| \leq \frac{i-2}{i-1} \cdot |U_{\text{OPT}}(I)|, \quad (2)$$
because $|U_{OPT}(I_{s(i)})| \leq |U_{OPT}(I)|$. We can derive an upper bound for $|U_k(I)|$ as follows.

$$
|U_k(I)| \leq \sum_{i=3}^{k} i \cdot l_i + 2(c_2 - 1) \leq \sum_{i=3}^{k} i \cdot l_i + 2\left(c_k - 1 - \sum_{i=3}^{k} (i-1)l_i\right)
$$

$$
= 2(c_k - 1) - \sum_{i=3}^{k} (i-2)l_i \leq 2(c_k - 1) - \sum_{i=3}^{k} \frac{i-2}{i-1} (c_i - c_{i-1})
$$

$$
= 2(c_k - 1) + \sum_{i=3}^{k} \frac{i-2}{i-1} (c_i - 1) - \sum_{i=3}^{k} \frac{i-2}{i-1} (c_i - 1)
$$

$$
= 2(c_k - 1) + \frac{c_2 - 1}{2} - \frac{(k-2)(c_k - 1)}{k-1} + \sum_{i=3}^{k-1} \left(\frac{i-1}{i} - \frac{i-2}{i-1}\right) (c_i - 1)
$$

$$
= \left(2 - \frac{k-2}{k-1}\right) \cdot (c_k - 1) + \frac{c_2 - 1}{2} + \sum_{i=3}^{k-1} \frac{c_i - 1}{i(i-1)}
$$

$$
\leq \left(2 - \frac{k-2}{k-1}\right) \cdot (c_k - 1) + \frac{|U_{OPT}(I)|}{4} + \sum_{i=3}^{k-1} \frac{|U_{OPT}(I)|}{i^2}
$$

Since $c_k - 1 < |U_{OPT}(I)|$, see Observation 1 we get

$$
|U_k(I)| < \left(2 - \frac{k-2}{k-1} + \frac{1}{4} + \sum_{i=3}^{k-1} \frac{1}{i^2}\right) \cdot |U_{OPT}(I)| = \left(\frac{1}{k-1} + \sum_{i=3}^{k-1} \frac{1}{i^2}\right) \cdot |U_{OPT}(I)|.
$$

\[\Box\]

**Corollary 1.** APPROX2LSRA$_3$ is a $7/4$ factor approximation algorithm for 2LSRA.

**Observation 3** The upper bound in Theorem 1 tends to $\pi^2/6$ for $k \to \infty$. [3]

### 5 Lower Bounds on the quality of APPROX2LSRA$_k$

In this section we present worst case examples for algorithm APPROX2LSRA$_k$ and derive lower bounds on its quality. For $k = 3$ we demonstrate that the upper bound of $7/4$ obtained in section 3 is tight.

**Theorem 2.** Let $k \in \mathbb{N}, k \geq 3$ be a positive integer. For an instance $I$ of 2LSRA let $U_k(I)$ be the solution computed by APPROX2LSRA$_k$ and $U_{OPT}(I)$ an optimal solution. For all $\epsilon > 0$ there is an instance $I$ such that

$$
\frac{|U_k(I)|}{|U_{OPT}(I)|} > \frac{3k - 2}{2k - 2} - \epsilon.
$$
Proof. For a positive integer $t \in \mathbb{N}$ let $I_t$ be the instance for 2LSRA defined as follows (see Figure 2).

$$V := \{(0, 0, 0)\} \cup \{(d, 3, 0) \mid 1 \leq d \leq t\} \cup \{(d, r, c) \mid 1 \leq d \leq t, 1 \leq r \leq 3, 1 \leq c < k\}$$

$$E_{\min}(V) := \{(0, 0, 0), (d, 3, 0)\} \mid 1 \leq d \leq t\} \cup \{(d, c) \mid 1 \leq d \leq t, 1 \leq c < k\}$$

$$E_{\max}(V) := E_{\min}(V) \cup \{(0, 0, 0), (d, 2, 1)\} \mid 1 \leq d \leq t\} \cup \{(d, c) \mid 1 \leq d \leq t, 1 \leq c < k\}$$

The (unique) optimal solution for $I_t$ is

$$U_{OPT}(I_t) = \{(0, 0, 0)\} \cup \{(d, r, c) \mid 1 \leq d \leq t, r \in \{1, 2\}, 1 \leq c < k\},$$

meaning that $|U_{OPT}(I_t)| = 1 + 2(k-1)t$. Algorithm $\text{APPROX2LSRA}_k$, in the worst case, successively gathers all $k$-mergings $\{(d, 3, c) \mid 0 \leq c < k\}$ for $1 \leq d \leq t$ first, followed by the remaining 2-mergings $\{(d, 1, c), (d, 2, c)\}$ for $1 \leq d \leq t$ and $1 \leq c < k$. Thus we get

$$q(k, t) := \frac{|U_k(I_t)|}{|U_{OPT}(I_t)|} = \frac{kt + 2(k-1)t}{1 + 2(k-1)t} = \frac{k + 2(k-1)}{t + 2(k-1)} = \frac{3k - 2}{t + 2k - 2}$$

and $\lim_{t \to \infty}(q(k, t)) = \frac{3k - 2}{2k - 2}$, which implies the existence of a positive integer $t_\epsilon \in \mathbb{N}$ such that $q(k, t_\epsilon) > \frac{3k - 2}{2k - 2} - \epsilon$. 

Corollary 2. The upper bound of $7/4$ on the quality of $\text{APPROX2LSRA}_3$ is tight.

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Fig. 2. Instance $I_t$ of Theorem 2. The min-power edges $E_{\min}(V)$ are drawn as thick blue lines, the max-power edges $E_{\max}(V)$ are drawn as thin black lines and the connected components of the min-power graph $G(\emptyset)$ are encircled by dashed boxes.
6 Efficient Implementation of Approxx2LSRA\textsubscript{3}

For $k = 3$ the algorithm can be implemented in almost linear time using a union-find data structure $D$ to organize the node sets of the connected components of $(V, E)$. We assume that the mappings $d_{\text{min}}$ and $d_{\text{max}}$ are explicitly given as relations $d_{\text{min}} \subseteq V \times \mathcal{P}(V)$ and $d_{\text{max}} \subseteq V \times \mathcal{P}(V)$ of size $s_{\text{min}} := |V| + \sum_{v \in V} |d_{\text{min}}(v)|$ and $s_{\text{max}} := |V| + \sum_{v \in V} |d_{\text{max}}(v)|$, respectively. The implementation is based on the following three steps.

1. **Initialization**: Generate the set of min-power edges $E_{\text{min}}(V)$ and the set of max-power edges $E_{\text{max}}(V)$. This can be done in time $O(s_{\text{min}})$ and $O(s_{\text{max}})$, respectively. Afterwards $D$ can be initialized with the connected components of $G(\emptyset)$ during one iteration over the min-power edges by performing two find operations $u' = \text{find}(u)$, $v' = \text{find}(v)$ and one union operation $\text{union}(u', v')$ for each $(u, v) \in E_{\text{min}}(V)$. The total number of find and union operations for this step is linear in $|V|$ and $|E_{\text{min}}(V)|$ and therefore also linear in $s_{\text{min}}$.

2. **Finding 3-mergings**: For every node $v \in V$ examine all incident edges $(v, u) \in E_{\text{max}}(V)$ after identifying the connected component of $v$ via $C_v = \text{find}(v)$. If $C_u := \text{find}(u) \neq C_v$, node $u$ and its component $C_u$ are temporarily saved until a second node $u'$ adjacent to $v$ is found, such that $C_v \neq C_u'$ and $C_u \neq C_u'$. Then $(v, u, u')$ is a 3-merging that is added to the solution and the three connected components $C_v$, $C_u$ and $C_u'$ are merged via two union operations. Since every edge in $E_{\text{max}}(V)$ has to be considered only twice, the total number of find and union operations for this step is linear in $|V|$ and $|E_{\text{max}}(V)|$ and therefore also linear in $s_{\text{max}}$.

3. **Finding 2-mergings**: For every edge $(u, v) \in E_{\text{max}}(V)$ add $(u, v)$ to the solution and call $\text{union}(u, v)$, if $\text{find}(u) \neq \text{find}(v)$. Again, the total number of find and union operations for this step is linear in $s_{\text{max}}$.

**Theorem 3.** Approxx2LSRA\textsubscript{3} can be implemented such that the running time is in $O(f(s_{\text{min}}, s_{\text{max}}) \cdot \alpha(f(s_{\text{min}}, s_{\text{max}}), |V|))$ where $\alpha$ is the inverse Ackermann function and $f \in O(s_{\text{min}} + s_{\text{max}})$.

**Proof.** The total number of union and find operations $f(s_{\text{min}}, s_{\text{max}})$ is linear in $s_{\text{min}}$ and $s_{\text{max}}$ as discussed above. Tarjan and Leeuwen show in [12] that any sequence of $m \in \mathbb{N}$ union and find operations on a union-find data structure saving $n \in \mathbb{N}$ elements can be performed in time $O(m \cdot \alpha(m, n))$. Fredman and Saks show in [7] that this bound is tight. \qed
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