Sketching, Embedding, and Dimensionality Reduction for Information Spaces∗

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Abstract

Information distances like the Hellinger distance and the Jensen-Shannon divergence have deep roots in information theory and machine learning. They are used extensively in data analysis especially when the objects being compared are high dimensional empirical probability distributions built from data. However, we lack common tools needed to actually use information distances in applications efficiently and at scale with any kind of provable guarantees. We can’t sketch these distances easily, or embed them in better behaved spaces, or even reduce the dimensionality of the space while maintaining the probability structure of the data.

In this paper, we build these tools for information distances—both for the Hellinger distance and Jensen–Shannon divergence, as well as related measures, like the $\chi^2$ divergence. We first show that they can be sketched efficiently (i.e. up to multiplicative error in sublinear space) in the aggregate streaming model. This result is exponentially stronger than known upper bounds for sketching these distances in the strict turnstile streaming model. Second, we show a finite dimensionality embedding result for the Jensen-Shannon and $\chi^2$ divergences that preserves pair wise distances. Finally we prove a dimensionality reduction result for the Hellinger, Jensen–Shannon, and $\chi^2$ divergences that preserves the information geometry of the distributions (specifically, by retaining the simplex structure of the space). While our second result above already implies that these divergences can be explicitly embedded in Euclidean space, retaining the simplex structure is important because it allows us to continue doing inference in the reduced space. In essence, we preserve not just the distance structure but the underlying geometry of the space.

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1. Introduction

The space of information distances includes many distances that are used extensively in data analysis. These include the well-known Bregman divergences, the $\alpha$-divergences, and the $f$-divergences. In this work we focus on a subclass of the $f$-divergences that admit embeddings into some (possibly infinite-dimensional) Hilbert space, with a specific emphasis on the JS divergence. These divergences are used in statistical tests and estimators (Beran, 1977), as well as in image analysis (Peter and Rangarajan, 2008), computer vision (Huang et al., 2005; Mahmoudi and Sapiro, 2009), and text analysis (Dhillon et al., 2003; Eiron and McCurley, 2003). They were introduced by Csiszar (1967), and, in the most general case, also include measures such as the Hellinger, JS, and $\chi^2$ divergences (here we consider a symmetrized variant of the $\chi^2$ distance).

To work with the geometry of these divergences effectively at scale and in high dimensions, we need algorithmic tools that can provide provably high quality approximate representations of the geometry. The techniques of sketching, embedding, and dimensionality reduction have evolved as ways of dealing with this problem.

A sketch for a set of points with respect to a property $P$ is a function that maps the data to a small summary from which property $P$ can be evaluated, albeit with some approximation error. Linear sketches are especially useful for estimating a derived property of a data stream in a fast and compact way. Complementing sketching, embedding techniques are one to one mappings that transform a collection of points lying in one space $X$ to another (presumably easier) space $Y$, while approximately preserving distances between points. Dimensionality reduction is a special kind of embedding which preserves the structure of the space, while reducing its dimension. These embedding techniques can be used in an almost “plug-and-play” fashion to speed up many algorithms in data analysis: for example for near neighbor search (and classification), clustering, and closest pair calculations.

Unfortunately, while these tools have been well developed for norms like $\ell_1$ and $\ell_2$, we lack such tools for information distances. This is not just a theoretical concern: information distances are semantically more suited to many tasks in machine learning, and building the appropriate algorithmic toolkit to manipulate them efficiently would expand greatly the places where they can be used.

1.1. Our contributions

Sketching information divergences. Guha, Indyk, and McGregor (2007) proved an impossibility result, showing that a large class of information divergences cannot be sketched in sublinear space, even if we allow for constant factor approximations. This result holds in the strict turnstile streaming model—a model in which coordinates of two points $x, y \subset \Delta_d$ are increased incrementally and we wish to maintain an estimate of the divergence between them. They left open the question of whether these divergences can be sketched in the aggregate streaming model, where each element of the stream gives the $i$th coordinate of $x$ or $y$ in its entirety, but the coordinates may appear in an arbitrary order. We answer this in

1. Indeed Li, Nguyen, and Woodruff (2014) show that any optimal one-pass streaming sketch algorithm in the turnstile model can be reduced to a linear sketch with logarithmic space overhead.
the affirmative for two important information distances, namely, the Jensen–Shannon and $\chi^2$ divergences.

**Theorem 1** A set of points $P$ under the Jensen–Shannon (JS) or $\chi^2$ divergence can be deterministically embedded into $O\left(\frac{d^2}{\varepsilon}\log\frac{1}{\varepsilon}\right)$ dimensions under $\ell_2^d$ with $\varepsilon$ additive error. The same space bound holds when sketching JS or $\chi^2$ in the aggregate stream model.

**Corollary 2** Assuming polynomial precision, an AMS sketch for Euclidean distance can reduce the dimension to $O\left(\frac{1}{\varepsilon^2}\log\frac{1}{\varepsilon}\log d\right)$ for a $(1 + \varepsilon)$ multiplicative approximation in the aggregate stream setting.

**Theorem 3** A set of points $P$ under the JS or $\chi^2$ divergence can be embedded into $\ell_2^d$ with $\bar{d} = O\left(\frac{n^2d^2}{\varepsilon^2}\right)$ with $(1 + \varepsilon)$ multiplicative error.

For the both techniques, applying the Euclidean JL–Lemma can further reduce the dimension to $O\left(\frac{\log n}{\varepsilon^2}\right)$ in the offline setting.

**Dimensionality reduction.** We then turn to the more challenging case of performing dimensionality reduction for information distances, where we wish to preserve not only the distances between pairs of points (distributions), but also the underlying simplicial structure of the space, so that we can continue to interpret coordinates in the new space as probabilities. This notion of a structure-preserving dimensionality reduction is implicit when dealing with normed spaces (since we always map a normed space to another), but requires an explicit mapping when dealing with more structured spaces. We prove an analog of the classical JL–Lemma:

**Theorem 4** For the Jensen–Shannon, Hellinger, and $\chi^2$ divergences, there exists a structure preserving dimensionality reduction from the high dimensional simplex $\Delta_d$ to a low dimensional simplex $\Delta_k$, where $k = O((\log n)/\varepsilon^2)$.

The theorem extends to “well-behaved” $f$-divergences (See Section 3 for a precise definition). Moreover, the dimensionality reduction is constructive for any divergence with a finite dimensional kernel (such as the Hellinger divergence), or an infinite dimensional Kernel that can be sketched in finite space, as we show is feasible for the JS and $\chi^2$ divergences.

**Our techniques.** The unifying approach of our three results—sketching, embedding into $\ell_2^d$, and dimensionality reduction—is to analyze carefully the infinite dimensional kernel of the information divergences. Quantizing and truncating the kernel yields the sketching result, sampling repeatedly from it produces an embedding into $\ell_2^d$. Finally given such an embedding, we show how to perform dimensionality reduction by proving that each of the divergences admits a region of the simplex where it is similar to $\ell_2^d$. We point out that to the best of our knowledge, this is the first result that explicitly uses the kernel representation of these information distances to build approximate geometric structures; while the existence of a kernel for the Jensen–Shannon distance was well-known, this structure had never been exploited for algorithmic advantage.
2. Related Work

The works by Fuglede and Topsøe (2004), and then by Vedaldi and Zisserman (2012) study embeddings of information divergences into an infinite dimensional Hilbert space by representing them as an integral along a one-dimensional curve in \( \mathbb{C} \). Vedaldi and Zisserman give an explicit formulation of this kernel for JS and \( \chi^2 \) divergences, for which a discretization (by quantizing and truncating) yields an additive error embedding into a finite dimensional \( \ell_2^2 \). However, they do not obtain quantitative bounds on the dimension of target space needed or address the question of multiplicative approximation guarantees.

In the realm of sketches, Guha, Indyk, and McGregor (2007) show \( \Omega(n) \) space (where \( n \) is the length of the stream) is required in the strict turnstile model even for a constant factor multiplicative approximation. These bounds hold for a wide range of information divergences, including JS, Hellinger and the \( \chi^2 \) divergences. They show however that an additive error of \( \varepsilon \) can be achieved using \( O\left(\frac{1}{\varepsilon^2 \log n}\right) \) space. In contrast, one can indeed achieve a multiplicative approximation in the aggregate streaming model for information divergences that have a finite dimensional embedding into \( \ell_2^2 \). For instance, Guha et al. (2006) observe that for the Hellinger distance that has a trivial such embedding, sketching is equivalent to sketching \( \ell_2^2 \) and hence may be done up to a \((1 + \varepsilon)\)-multiplicative approximation in \( \frac{1}{\varepsilon^2 \log n} \) space. This immediately implies a constant factor approximation of JS and \( \chi^2 \) divergences in the same space, but no bounds have been known prior to our work for a \((1 + \varepsilon)\)-sketching result for JS and \( \chi^2 \) divergences in any streaming model.

Moving onto dimensionality reduction from simplex to simplex, in the only other work we are aware of, Kyng, Phillips, and Venkatasubramanian (2010) show a limited dimensionality reduction result for the Hellinger distance. Their approach works by showing that if the input points lie in a specific region of the simplex, then a standard random projection will keep the points on a lower-dimensional simplex while preserving the distances approximately. Unfortunately, this region is a small ball centered in the interior of the simplex, which further shrinks with the dimension. This is in sharp contrast to our work here, where the input points are unconstrained.

While it does not admit a kernel, the \( \ell_1 \) distance is also an \( f \)-divergence, and it is therefore natural to investigate its potential connection with the measures we study here. For \( \ell_1 \), it is well known that significant dimensionality reduction is not possible: an embedding with distortion \( 1 + \varepsilon \) requires the points to be embedded in \( n^{1-O(\log \frac{1}{\varepsilon})} \) dimensions, which is nearly linear. This result was proved (and strengthened) in a series of results (Andoni et al., 2011; Regev, 2012; Lee and Naor, 2004; Brinkman and Charikar, 2005).

The general literature of sketching and embeddability in normed spaces is too extensive to be reviewed here: we point the reader to Andoni et al. (2014) for a full discussion of results in this area. One of the most famous applications of dimension reduction is the Johnson–Lindenstrauss (JL) Lemma, which states that any set of \( n \) points in \( \ell_2^2 \) can be embedded into \( O\left(\frac{\log n}{\varepsilon^2}\right) \) dimensions in the same space while preserving pairwise distances to within \((1 \pm \varepsilon)\). This result has become a core step in algorithms for near neighbor search (Ailon and Chazelle, 2006; Andoni and Indyk, 2006), speeding up clustering algorithms (Boutsidis et al., 2015), and efficient approximation of matrices (Clarkson and Woodruff, 2013), among many others.
Although sketching, embeddability, and dimensionality reduction are related operations, they are not always equivalent. For example, even though $\ell_1$ and $\ell_2$ have very different behavior under dimensionality reduction, they can both be sketched to an arbitrary error in the turnstile model (and in fact any $\ell_p$ norm, $p \leq 2$ can be sketched using $p$-stable distributions (Indyk, 2000)). In the offline setting, Andoni et al. (2014) show that sketching and embedding of normed spaces are equivalent: for any finite-dimensional normed space $X$, a constant distortion and space sketching algorithm for $X$ exists if and only if there exists a linear embedding of $X$ into $\ell_1 - \varepsilon$.

3. Background

In this section, we define precisely the class of information divergences that we work with, and their specific properties that allow us to obtain sketching, embedding, and dimensionality results. For what follows $\Delta_d$ denotes the $d$-simplex: $\Delta_d = \{(x_1, \ldots, x_d) \mid \sum x_i = 1$ and $x_i \geq 0, \forall i\}$. Let $[d] = \{1, \ldots, d\}$.  

**Definition 5 (f-divergence)** Let $p$ and $q$ be two distributions on $[n]$. A convex function $f : [0, \infty) \to \mathbb{R}$ such that $f(1) = 0$ gives rise to an $f$-divergence $D_f : \Delta_d \to \mathbb{R}$ as:

$$D_f(p, q) = \sum_{i=1}^{d} p_i \cdot f \left( \frac{q_i}{p_i} \right),$$

where we define $0 \cdot f(0/0) = 0$, $a \cdot f(0/a) = a \cdot \lim_{u \to 0} f(u)$, and $0 \cdot f(a/0) = a \cdot \lim_{u \to \infty} f(u)/u$.

**Definition 6 (Regular distance)** We call a distance function $D : X \to \mathbb{R}$ regular if there exists a feature map $\phi : X \to V$, where $V$ is a (possibly infinite dimensional) Hilbert space, such that:

$$D(x, y) = \|\phi(x) - \phi(y)\|^2 \quad \forall x, y \in X.$$

The work of Fuglede and Topsøe (2004) establishes that JS is regular. Vedaldi and Zisserman (2012) construct an explicit feature map for the JS kernel, as $\phi(x) = \int_{-\infty}^{+\infty} \Psi_x(\omega) d\omega$, where $\Psi_x(\omega) : \mathbb{R} \to \mathbb{C}$ is given by

$$\Psi_x(\omega) = \exp(i\omega \ln x) \sqrt{\frac{2x \sech(\pi \omega)}{(\ln 4)(1 + 4\omega^2)}}.$$

Hence we have for $x, y \in \mathbb{R}$, $\text{JS}(x, y) = \|\phi(x) - \phi(y)\|^2 = \int_{-\infty}^{+\infty} \|\Psi_x(\omega) - \Psi_y(\omega)\|^2 d\omega$. The “embedding” for a given distribution $p \in \Delta_d$ is then the concatenation of the functions $\phi(p_i)$, i.e., $\phi(p) = (\phi_{p_1}, \ldots, \phi_{p_d})$.

**Definition 7 (Well-behaved divergence)** A well-behaved $f$-divergence is a regular $f$-divergence such that $f(1) = 0$, $f'(1) = 0$, $f''(1) > 0$, and $f'''(1)$ exists.

In this paper, we will focus on the following well-behaved $f$-divergences.
Definition 8 The Jensen–Shannon (JS), Hellinger, and \( \chi^2 \) divergences between distributions \( p \) and \( q \) are defined as:

\[
JS(p, q) = \sum_i p_i \log \frac{2p_i}{p_i + q_i} + q_i \log \frac{2q_i}{p_i + q_i},
\]

\[
He(p, q) = \sum_i (\sqrt{p_i} - \sqrt{q_i})^2,
\]

\[
\chi^2(p, q) = \sum_i \frac{(p_i - q_i)^2}{p_i + q_i}.
\]

4. Embedding JS into \( \ell^2 \)

We present two algorithms for embedding JS into \( \ell^2 \). The first is deterministic and gives an additive error approximation whereas the second is randomized but yields a multiplicative approximation in an offline setting. The advantage of the first algorithm is that it can be realized in the streaming model, and if we make a standard assumption of polynomial precision in the streaming input, yields a \((1 + \varepsilon)\)-multiplicative approximation as well in this setting.

We derive some terms in the kernel representation of \( JS(x, y) \) which we will find convenient. First, the explicit formulation in Section 3 yields that for \( x, y \in \mathbb{R} \):

\[
JS(x, y) = \int_{-\infty}^{+\infty} \left| e^{i\omega \ln x} \sqrt{\frac{2x \text{sech}(\pi \omega)}{(\ln 4)(1 + 4\omega^2)}} - e^{i\omega \ln y} \sqrt{\frac{2y \text{sech}(\pi \omega)}{(\ln 4)(1 + 4\omega^2)}} \right|^2 d\omega
\]

For convenience, we now define:

\[
h(x, y, \omega) = \left| \sqrt{x} e^{i\omega \ln x} - \sqrt{y} e^{i\omega \ln y} \right|^2
\]

\[
= (\sqrt{x} \cos(\omega \ln x) - \sqrt{y} \cos(\omega \ln y))^2 + (\sqrt{x} \sin(\omega \ln x) - \sqrt{y} \sin(\omega \ln y))^2,
\]

and

\[
\kappa(\omega) = \frac{2 \text{sech}(\pi \omega)}{(\ln 4)(1 + 4\omega^2)}.
\]

We can then write \( JS(p, q) = \sum_{i=1}^{d} f_J(p_i, q_i) \) where

\[
f_J(x, y) = \int_{-\infty}^{\infty} h(x, y, \omega) \kappa(\omega) d\omega = x \log \left( \frac{2x}{x + y} \right) + y \log \left( \frac{2y}{x + y} \right).
\]

It is easy to verify that \( \kappa(\omega) \) is a distribution, i.e., \( \int_{-\infty}^{\infty} \kappa(\omega) d\omega = 1 \).

4.1. Deterministic embedding

We will produce an embedding \( \phi(p) = (\phi_{p_1}, \ldots, \phi_{p_d}) \), where each \( \phi_{p_i} \) is an integral that we can discretize by quantizing and truncating carefully.

To analyze Algorithm 1, we first obtain bounds on the function \( h \) and its derivative.
Algorithm 1: Embed \( p \in \Delta_d \) under JS into \( \ell_2^d \).

**Input:** \( p = \{p_1, \ldots, p_d\} \) where coordinates are ordered by arrival.

**Output:** A vector \( c^p \) of length \( O \left( \frac{d^2}{\varepsilon} \log \frac{d}{\varepsilon} \right) \)

\[
\ell \leftarrow 1; \quad J \leftarrow \left\lceil \frac{32d}{\varepsilon} \ln \left( \frac{8d}{\varepsilon} \right) \right\rceil ,
\]

for \( j \leftarrow -J \) to \( J \) do
| \( w_j \leftarrow j \times \varepsilon/32d \)
end

for \( i \leftarrow 1 \) to \( d \) do
  for \( j \leftarrow -J \) to \( J - 1 \) do
    \[ a^p_j \leftarrow \sqrt{p_i} \cos(\omega_j \ln p_i) \int_{\omega_j}^{\omega_{j+1}} \kappa(\omega) d\omega \]
    \[ b^p_j \leftarrow \sqrt{p_i} \sin(\omega_j \ln p_i) \int_{\omega_j}^{\omega_{j+1}} \kappa(\omega) d\omega \]
    \( \ell \leftarrow \ell + 1 \)
  end
end

return \( a^p \) concatenated with \( b^p \).

**Lemma 9** For \( 0 \leq x, y \leq 1 \), we have \( 0 \leq h(x, y, \omega) \leq 2 \) and \( \left| \frac{\partial h(x, y, \omega)}{\partial \omega} \right| \leq 16 \).

**Proof** Clearly \( h(x, y, \omega) \geq 0 \). Furthermore, since \( 0 \leq x, y \leq 1 \), we have

\[
h(x, y, \omega) \leq \left| \sqrt{x} e^{i\omega \ln x} \right|^2 + \left| \sqrt{y} e^{i\omega \ln y} \right|^2 = x + y \leq 2.
\]

Next, \( \left| \frac{\partial h(x, y, \omega)}{\partial \omega} \right| \)

\[
= \left| 2 \left( \sqrt{x} \cos(\omega \ln x) - \sqrt{y} \cos(\omega \ln y) \right) \left( \sqrt{x} \sin(\omega \ln x) \ln x + \sqrt{y} \sin(\omega \ln y) \ln y \right) + 2 \left( \sqrt{x} \sin(\omega \ln x) - \sqrt{y} \sin(\omega \ln y) \right) \left( \sqrt{x} \cos(\omega \ln x) \ln x - \sqrt{y} \cos(\omega \ln y) \ln y \right) \right| \\
\leq \left| 2 \left( \sqrt{x} + \sqrt{y} \right) \left( \sqrt{x} \ln x + \sqrt{y} \ln y \right) \right| + 2 \left| \left( \sqrt{x} + \sqrt{y} \right) \left( \sqrt{x} \ln x + \sqrt{y} \ln y \right) \right| \leq 16,
\]

where the last inequality follows since \( \max_{0 \leq x \leq 1} \left| \sqrt{x} \ln x \right| < 1 \).

The next two steps are useful to approximate the infinite-dimensional continuous representation by a finite-dimensional discrete representation by appropriately truncating and quantizing the integral.

**Lemma 10 (Truncation)** For \( t \geq \ln(4/\varepsilon) \),

\[
f_J(x, y) \geq \int_{-t}^{t} h(x, y, \omega) \kappa(\omega) d\omega \geq f_J(x, y) - \varepsilon.
\]

**Proof** The first inequality follows since \( h(x, y, \omega) \geq 0 \). For the second inequality, we use \( h(x, y, \omega) \leq 2 \):

\[
\int_{-\infty}^{-t} h(x, y, \omega) \kappa(\omega) d\omega + \int_{t}^{\infty} h(x, y, \omega) \kappa(\omega) d\omega \leq 4 \int_{t}^{\infty} \kappa(\omega) d\omega < 4 \int_{t}^{\infty} \frac{4e^{-\pi\omega}}{\ln 4} d\omega < 4e^{-t} \leq \varepsilon
\]
where the last line follows if $t \geq \ln(4/\varepsilon)$. \hfill \blacksquare

Define $\omega_i = \varepsilon i/16$ for $i \in \{-1, 0, 1, 2, \ldots\}$ and $\bar{h}(x, y, \omega) = h(x, y, \omega_i)$ where $i = \max\{j \mid \omega_j \leq \omega\}$.

**Lemma 11 (Quantization)** For any $a, b$,

$$\int_a^b h(x, y, \omega) \kappa(\omega) \, d\omega = \int_a^b \bar{h}(x, y, \omega) \kappa(\omega) \, d\omega \pm \varepsilon.$$

**Proof** First note that

$$|\bar{h}(x, y, \omega) - h(x, y, \omega)| \leq \left(\frac{\varepsilon}{16}\right) \cdot \max_{x, y \in [a, b]} \left|\frac{\partial h(x, y, \omega)}{\partial \omega}\right| \leq \varepsilon.$$

Hence,

$$\left|\int_a^b \bar{h}(x, y, \omega) \kappa(\omega) \, d\omega - \int_a^b h(x, y, \omega) \kappa(\omega) \, d\omega\right| \leq \left|\int_a^b \varepsilon \kappa(\omega) \, d\omega\right| \leq \varepsilon. \hfill \blacksquare$$

Given a real number $z$, define vectors $v^z$ and $u^z$ indexed by $i \in \{-i^*, \ldots, -2, -1, 0, 1, 2, \ldots i^*\}$ where $i^* = \lceil 16\varepsilon^{-1} \ln(4/\varepsilon) \rceil$ by:

$$v^z = \sqrt{z} \cos(\omega_i \ln z) \sqrt{\int_{\omega_i}^{\omega_i+1} \kappa(\omega) \, d\omega}, \quad u^z = \sqrt{z} \sin(\omega_i \ln z) \sqrt{\int_{\omega_i}^{\omega_i+1} \kappa(\omega) \, d\omega},$$

and note that

$$(v_i^x - u_i^y)^2 + (u_i^x - u_i^y)^2 = h(x, y, \omega_i) \int_{\omega_i}^{\omega_i+1} \kappa(\omega) \, d\omega.$$

Therefore,

$$\|v^x - v^y\|_2^2 + \|u^x - u^y\|_2^2 = \int_{w_i}^{w_i+1} h(x, y, \omega) \kappa(\omega) \, d\omega = \int_{w_i}^{w_i+1} h(x, y, \omega) \kappa(\omega) \, d\omega \pm \varepsilon$$

$$= \int_{-\infty}^{\infty} h(x, y, \omega) \kappa(\omega) \, d\omega \pm 2\varepsilon = f_j(x, y) \pm 2\varepsilon,$$

where the second to last line follows from Lemma 11 and the last line follows from Lemma 10 since $\min(\|w_i - w\|, w_i+1) \geq \ln(4/\varepsilon)$.

Define the vector $a^p$ to be the vector generated by concatenating $v^{p_i}$ and $u^{p_i}$ for $i \in [d]$. Then it follows that

$$\|a^p - a^q\|_2 \leq \text{JS}(p, q) \pm 2\varepsilon d$$

Hence we have reduced the problem of estimating $\text{JS}(p, q)$ to $\ell_2$ estimation. Rescaling $\varepsilon \leftarrow \varepsilon/(2d)$ ensures the additive error is $\varepsilon$ while the length of the vectors $a^p$ and $a^q$ is $O\left(\frac{d^2}{\varepsilon} \log \frac{d}{\varepsilon}\right)$.

**Theorem 12** Algorithm[c] embeds a set $P$ of points under $\text{JS}$ into $O\left(\frac{d^2}{\varepsilon} \log \frac{d}{\varepsilon}\right)$ dimensions under $\ell_2^2$ with $\varepsilon$ additive error, independent of the size of $|P|$. 

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Note that using the JL-Lemma, the dimensionality of the target space can be reduced to \(O\left(\frac{\log |P|}{\epsilon^2}\right)\). Theorem 12 along with the AMS sketch of Alon et al. (1996), and the standard assumption of polynomial precision immediately implies:

**Corollary 13** There is an algorithm that works in the aggregate streaming model to approximate JS to within \((1 + \varepsilon)\)-multiplicative factor using \(O\left(\frac{1}{\varepsilon^2 \log \frac{1}{\varepsilon} \log d}\right)\) space.

As noted earlier, this is the first algorithm in the aggregate streaming model to obtain an \((1 + \varepsilon)\)-multiplicative approximation to JS, which contrasts against linear space lower bounds for the same problem in the update streaming model.

### 4.2. Randomized embedding

In this section we show how to embed \(n\) points of JS into \(\ell_2^d\) with \((1 + \varepsilon)\) distortion where \(d = O(n^2 d^3 \varepsilon^{-2})\).

For fixed \(x, y, \in [0, 1]\), we first consider the random variable \(T\) where \(T\) takes the value \(h(x, y, \omega)\) with probability \(\kappa(\omega)\). (Recall that \(\kappa(\cdot)\) is a distribution.) We compute the first and second moments of \(T\).

**Theorem 14** \(E[T] = f_J(x, y)\) and \(\text{var}[T] \leq 36(f_J(x, y))^2\).

**Proof** The expectation follows immediately from the definition: \(E[T] = \int_{-\infty}^{\infty} h(x, y, \omega) \kappa(\omega) d\omega = f_J(x, y)\).

To bound the variance it will be useful to define the function \(f_H(x, y) = (\sqrt{x} - \sqrt{y})^2\) corresponding to the one-dimensional Hellinger distance that is related to \(f_J(x, y)\) as follows. We now state two claims regarding \(f_H(x, y)\) and \(f_\chi(x, y)\):

**Claim 4.1** For all \(x, y \in [0, 1]\), \(f_H(x, y) \leq 2f_J(x, y)\).

**Proof** Let \(f_\chi(x, y) = \frac{(x-y)^2}{x+y}\) correspond to the one-dimensional \(\chi^2\) distance. Then, we have

\[
\frac{f_\chi(x, y)}{f_H(x, y)} = \frac{(x-y)^2}{(x+y)(\sqrt{x} - \sqrt{y})^2} = \frac{(\sqrt{x} + \sqrt{y})^2}{x+y} \geq 1 .
\]

This shows that \(f_H(x, y) \leq f_\chi(x, y)\). To show \(f_\chi(x, y) \leq 2f_J(x, y)\) we refer the reader to (Topsoe, 2000, Section 3). Combining these two relationships gives us our claim.

We then bound \(h(x, y, \omega)\) in terms of \(f_H(x, y)\) as follows.

**Claim 4.2** For all \(x, y \in [0, 1], \omega \in \mathbb{R}\), \(h(x, y, \omega) \leq f_H(x, y)(1 + 2|\omega|)^2\).

2. If we ignore precision constraints on sampling from a continuous distribution in a streaming algorithm, then this also would yield a sketching bound of \(O(d^3 \varepsilon^{-2})\) for a \((1 + \varepsilon)\) multiplicative approximation.
This naturally gives rise to the following algorithm. Let \( \omega \) where, for \( i \)

\[
\sqrt{h(x, y, \omega)} = \left| \sqrt{x} \cdot e^{i\omega \ln x} - \sqrt{y} \cdot e^{i\omega \ln y} \right|
\]

\[
\leq \left| \sqrt{x} \cdot e^{i\omega \ln x} - \sqrt{y} \cdot e^{i\omega \ln x} \right| + \left| \sqrt{y} \cdot e^{i\omega \ln x} - \sqrt{y} \cdot e^{i\omega \ln y} \right|
\]

\[
= \left| \sqrt{x} - \sqrt{y} \right| + \left| e^{i\omega \ln x} - e^{i\omega \ln y} \right|
\]

\[
= \left| \sqrt{x} - \sqrt{y} \right| + \sqrt{y} \cdot 2 \cdot |\sin(\omega \ln(x/y)/2)|
\]

\[
\leq \sqrt{f_H(x, y)} + \sqrt{y} \cdot 2 \cdot |\omega \ln(\sqrt{x}/y)|
\]

\[
\leq \sqrt{f_H(x, y)} + \sqrt{y} \cdot 2 \cdot |\sqrt{x}/y - 1| \cdot |\omega|
\]

\[
= \sqrt{f_H(x, y)} + 2\sqrt{f_H(x, y)} \cdot |\omega|
\]

and hence \( h(x, y, \omega) \leq f_H(x, y)(1 + 2|\omega|)^2 \) as required.  

These claims allow us to bound the variance:

\[
\text{var}[T] \leq E[T^2] = \int_{-\infty}^{\infty} (h(x, y, \omega))^2 \kappa(\omega)d\omega \leq f_H(x, y)^2 \int_{-\infty}^{\infty} (1 + 2|\omega|)^4 \kappa(\omega)d\omega
\]

\[
= f_H(x, y)^2 \cdot 8.94 < 36f_J(x, y)^2,
\]

This naturally gives rise to the following algorithm. Let \( \omega_1, \ldots, \omega_t \) be \( t \) independent

**Algorithm 2:** Embeds point \( p \in \Delta_d \) under JS into \( \ell_2^d \).

**Input:** \( p = \{p_1, \ldots, p_d\} \).

**Output:** A vector \( e^p \) of length \( O(n^2d^3\varepsilon^{-2}) \)

\( \ell \leftarrow 1; \ s \leftarrow [36n^2d^2\varepsilon^{-2}] \)

**for** \( j \leftarrow 1 \) **to** \( s \)**

\( \omega_j \leftarrow \) a draw from \( \kappa(\omega) \);

**end**

**for** \( i \leftarrow 1 \) **to** \( d \)**

\( \text{for } j \leftarrow 1 \text{ to } s \)

\( a^p_{ij} \leftarrow (\sqrt{p_i} \cos(\omega_j \ln p_i))/s \)

\( b^p_{ij} \leftarrow (\sqrt{p_i} \sin(\omega_j \ln p_i))/s \)

\( \ell \leftarrow \ell + 1 \)

**end**

**end**

**return** \( a^p \) concatenated with \( b^p \).

samples chosen according to \( \kappa(\omega) \). For any distribution \( p \) on \( [d] \), define vectors \( v_i^p, u_i^p \in \mathbb{R}^{td} \) where, for \( i \in [d], j \in [t] \),

\[
v_{ij}^p = \sqrt{p_i} \cdot \cos(\omega_j \ln p_i)/t, \quad u_{ij}^p = \sqrt{p_i} \cdot \sin(\omega_j \ln p_i)/t.
\]

Let \( v_i^p \) be a concatenation of \( v_{ij}^p \) and \( u_{ij}^p \) over all \( j \in [t] \). Then note that \( E[\|v_i^p - v_i^q\|_2^2] = f_J(p_i, q_i) \) and \( \text{var}[\|v_i^p - v_i^q\|_2^2] \leq 36(f_J(p_i, q_i))^2/t. \) Hence, for \( t = 36n^2d^2\varepsilon^{-2} \), by an application
of the Chebyshev bound,

\[ \Pr[||v^p_i - v^q_i||_2^2 - f_J(p_i, q_i) \geq \varepsilon f_J(x, y)] \leq 36\varepsilon^{-2}/t = (nd)^{-2}. \]  

\[(4.1)\]

By an application of the union bound over all pairs of points:

\[ \Pr[\exists i \in [d], p, q \in P ||v^p_i - v^q_i||_2^2 - f_J(p_i, q_i) \geq \varepsilon f_J(x, y)] \leq 1/d. \]

And hence, if \( v^p \) is a concatenation of \( v^p_i \) over all \( i \in [d] \), then with probability at least

\[ 1 - \frac{1}{d} \]

it holds for all \( p, q \in P \):

\[ (1 - \varepsilon)JS(p, q) \leq ||v^p - v^q|| \leq (1 + \varepsilon)JS(p, q). \]

The final length of the vectors is then \( td = 36n^2d^3\varepsilon^{-2} \) for approximately preserving distances between every pair of points with probability at least \( 1 - \frac{1}{d} \). This can be reduced further to \( O(\log n/\varepsilon^2) \) by simply applying the JL-Lemma.

5. Embedding \( \chi^2 \) into \( \ell_2^2 \)

We give here two algorithms for embedding the \( \chi^2 \) divergence into \( \ell_2^2 \). The computation and resulting two algorithms are highly analogous to Section 4. First, the explicit formulation given by Vedaldi and Zisserman (2012) yields that for \( x, y \in \mathbb{R} \):

\[ \chi^2(x, y) = \int_{-\infty}^{+\infty} \left| e^{i\omega \ln x} \sqrt{x \sech(\pi\omega)} - e^{i\omega \ln y} \sqrt{y \sech(\pi\omega)} \right|^2 d\omega \]

\[ = \int_{-\infty}^{+\infty} (\sech(\pi\omega)) \left\| \sqrt{x e^{i\omega \ln x}} - \sqrt{y e^{i\omega \ln y}} \right\|^2 d\omega. \]

For convenience, we now define:

\[ h(x, y, \omega) = \left\| \sqrt{x e^{i\omega \ln x}} - \sqrt{y e^{i\omega \ln y}} \right\|^2 \]

and \( \kappa_\chi(\omega) = \sech(\pi\omega) \).

We can then write \( \chi^2(p, q) = \sum_{i=1}^{d} f_\chi(p_i, q_i) \) where

\[ f_\chi(x, y) = \int_{-\infty}^{+\infty} h(x, y, \omega) \kappa_\chi(\omega) d\omega = \frac{(x - y)^2}{x + y}. \]

It is easy to verify that \( \kappa_\chi(\omega) \) is a distribution, i.e., \( \int_{-\infty}^{+\infty} \kappa_\chi(\omega) d\omega = 1 \).

5.1. Deterministic embedding

We will produce an embedding \( \phi(p) = (\phi_{p_1}, \ldots, \phi_{p_d}) \), where each \( \phi_{p_i} \) is an integral that we discretize appropriately.

\textbf{Lemma 15} For \( 0 \leq x, y, \leq 1 \), we have \( 0 \leq h(x, y, \omega) \leq 2 \) and \( \left| \frac{\partial h(x, y, \omega)}{\partial \omega} \right| \leq 16 \).

Similar to Section 4, the next two steps analyze truncating and quantizing the integral.
Algorithm 3: Embed \( p \in \Delta_d \) under \( \chi^2 \) into \( \ell_2^d \).

**Input:** \( p = \{p_1, \ldots, p_d\} \) where coordinates are ordered by arrival.

**Output:** A vector \( e^p \) of length \( O \left( \frac{d^2 \log d}{\varepsilon} \right) \)

\[
\ell \leftarrow 1; \quad J \leftarrow \left\lceil \frac{32d}{\varepsilon} \ln \left( \frac{6d}{\varepsilon} \right) \right\rceil,
\]

for \( j \leftarrow -J \) to \( J \) do
\[
w_j \leftarrow j \times \varepsilon/32d
\]
end

for \( i \leftarrow 1 \) to \( d \) do
\[
\begin{align*}
a_{i}^{p} & \leftarrow \sqrt{p_i \cos(\omega_j \ln p_i)} \sqrt{\int_{\omega_j}^{\omega_j+1} \kappa_{\chi}(\omega) d\omega} \\
b_{i}^{p} & \leftarrow \sqrt{p_i \sin(\omega_j \ln p_i)} \sqrt{\int_{\omega_j}^{\omega_j+1} \kappa_{\chi}(\omega) d\omega}
\end{align*}
\]
\[
\ell \leftarrow \ell + 1
\]
end
return \( a^{p} \) concatenated with \( b^{p} \).

**Lemma 16 (Truncation)** For \( t \geq \ln(3/\varepsilon) \),

\[
f_{\chi}(x, y) \geq \int_{-t}^{t} h(x, y, \omega) \kappa_{\chi}(\omega) d\omega \geq f_{\chi}(x, y) - \varepsilon.
\]

**Proof** The first inequality follows since \( h(x, y, \omega) \geq 0 \). For the second inequality, we use \( h(x, y, \omega) \leq 2 \):

\[
\int_{-\infty}^{-t} h(x, y, \omega) \kappa_{\chi}(\omega) d\omega + \int_{t}^{\infty} h(x, y, \omega) \kappa_{\chi}(\omega) d\omega \leq 4 \int_{t}^{\infty} \kappa_{\chi}(\omega) d\omega < 4 \int_{t}^{\infty} 2e^{-\pi\omega} d\omega < 3e^{-t} \leq \varepsilon
\]

where the last line follows if \( t \geq \ln(3/\varepsilon) \).

Define \( \omega_i = \varepsilon i/16 \) for \( i \in \{\ldots, -2, -1, 0, 1, 2, \ldots\} \) and \( \bar{h}(x, y, \omega) = h(x, y, \omega_i) \) where \( i = \max\{j \mid \omega_j \leq \omega\} \). We recall the following Lemma from Section 4

**Lemma 17 (Quantization)** For any \( a, b \),

\[
\int_{a}^{b} h(x, y, \omega) \kappa_{\chi}(\omega) d\omega = \int_{a}^{b} \bar{h}(x, y, \omega) \kappa_{\chi}(\omega) d\omega \pm \varepsilon.
\]

Given a real number \( z \), define vectors \( v^z \) and \( u^z \) indexed by \( i \in \{-i^*, \ldots, -2, -1, 0, 1, 2, \ldots i^*\} \) where \( i^* = \lceil 16\varepsilon^{-1} \ln(3/\varepsilon) \rceil \) by:

\[
v^z = \sqrt{z} \cos(\omega_i \ln z) \sqrt{\int_{\omega_i}^{\omega_i+1} \kappa_{\chi}(\omega) d\omega}, \quad u^z = \sqrt{z} \sin(\omega_i \ln z) \sqrt{\int_{\omega_i}^{\omega_i+1} \kappa_{\chi}(\omega) d\omega},
\]

and note that

\[
(v^z_i - v^y_i)^2 + (u^z_i - u^y_i)^2 = h(x, y, \omega_i) \int_{\omega_i}^{\omega_i+1} \kappa_{\chi}(\omega) d\omega.
\]
Therefore,
\[ \|v^x - v^y\|^2 + \|u^x - u^y\|^2 = \int_{w,\ell} h(x, y, \omega) \kappa_\chi(\omega) d\omega \pm \varepsilon \]
\[ = \int_{-\infty}^{\infty} h(x, y, \omega) \kappa_\chi(\omega) d\omega \pm 2\varepsilon = f_\chi(x, y) \pm 2\varepsilon, \]
where the second to last line follows from Lemma 17 and the last line follows from Lemma 16, since \( \min(|w_\ell|, w_{\ell+1}) \geq \ln(3/\varepsilon) \).

Define the vector \( a^p \) to be the vector generated by concatenating \( v^p_i \) and \( u^p_i \) for \( i \in [d] \). Then if follows that
\[ \|a^p - a^q\|^2 = \chi^2(p, q) \pm 2\varepsilon d \]
Hence we have reduced the problem of estimating \( \chi^2(p, q) \) to \( \ell_2 \) estimation. Rescaling \( \varepsilon \leftarrow \varepsilon/(2d) \) ensures the additive error is \( \varepsilon \) while the length of the vectors \( a^p \) and \( a^q \) is \( O(d^2 \varepsilon \log d) \).

**Theorem 18** Algorithm 3 embeds a set \( P \) of points under \( \chi^2 \) into \( O\left(d^2 \varepsilon \log \frac{1}{\varepsilon} \right) \) dimensions under \( \ell_2 \) with \( \varepsilon \) additive error, independent of the size of \( |P| \).

Theorem 18 along with the AMS sketch of Alon et al. (1996), and the standard assumption of polynomial precision immediately implies:

**Corollary 19** There is an algorithm that works in the aggregate streaming model to approximate \( \chi^2 \) to within \((1 + \varepsilon)\)-multiplicative factor using \( O\left(\frac{1}{\varepsilon^2} \log \frac{1}{\varepsilon} \log d \right) \) space.

### 5.2. Randomized embedding

In this section we show how to embed \( n \) points of \( \chi^2 \) into \( \ell_2^{\tilde{d}} \) with \((1 + \varepsilon)\) distortion where \( \tilde{d} = O(n^2 d^3 \varepsilon^{-2}) \). \[3\]

For fixed \( x, y \in [0, 1] \), we first consider the random variable \( T \) where \( T \) takes the value \( h(x, y, \omega) \) with probability \( \kappa_\chi(\omega) \). (Recall that \( \kappa_\chi(\cdot) \) is a distribution.) We compute the first and second moments of \( T \).

**Theorem 20** \( E[T] = f_\chi(x, y) \) and \( \text{var}[T] \leq 23(f_\chi(x, y))^2 \).

**Proof** The expectation follows immediately from the definition:
\[ E[T] = \int_{-\infty}^{\infty} h(x, y, \omega) \kappa_\chi(\omega) d\omega = f_\chi(x, y). \]

To bound the variance we will again use the function \( f_H(x, y) = (\sqrt{x} - \sqrt{y})^2 \) corresponding to the one-dimensional Hellinger distance. We now state two claims relating \( f_H(x, y) \) and \( f_\chi(x, y) \):

**Claim 5.1** For all \( x, y \in [0, 1] \), \( f_H(x, y) \leq f_\chi(x, y) \).

3. If we ignore precision constraints on sampling from a continuous distribution in a streaming algorithm, then this also would yield a sketching bound of \( O(d^3 \varepsilon^{-2}) \) for a \((1 + \varepsilon)\) multiplicative approximation.
This shows that $f_H(x, y) \leq f_X(x, y)$.

We then recall Claim 4.2 bounding $h(x, y, \omega)$ in terms of $f_H(x, y)$ as follows.

**Claim 5.2** For all $x, y \in [0, 1], \omega \in \mathbb{R}$, $h(x, y, \omega) \leq f_H(x, y)(1 + 2|\omega|)^2$.

These claims allow us to bound the variance:

$$\text{var}[T] \leq E[T^2] = \int_{-\infty}^{\infty} (h(x, y, \omega))^2 \kappa_X(\omega) d\omega \leq f_H(x, y)^2 \int_{-\infty}^{\infty} (1 + 2|\omega|)^4 \kappa_X(\omega) d\omega$$

$$= f_H(x, y)^2 \cdot 22.77 < 23 f_X(x, y)^2,$$

This naturally gives rise to the following algorithm. Let $\omega_1, \ldots, \omega_t$ be $t$ independent samples

**Algorithm 4:** Embeds point $p \in \Delta_d$ under $\chi^2$ into $\ell_2^2$.

*Input:* $p = \{p_1, \ldots, p_d\}$.

*Output:* A vector $a^p$ of length $O(n^2 d^3 \varepsilon^{-2})$

1. $\ell \leftarrow 1$; $s \leftarrow \lceil 23n^2 d^2 \varepsilon^{-2} \rceil$
2. for $j \leftarrow 1$ to $s$ do
  1. $\omega_j \leftarrow$ a draw from $\kappa_X(\omega)$;
3. end
4. for $i \leftarrow 1$ to $d$ do
   1. for $j \leftarrow 1$ to $s$ do
      1. $a^p_{i,j} \leftarrow (\sqrt{p_i} \cos(\omega_j \ln p_i)/s)$
      2. $b^p_{i,j} \leftarrow (\sqrt{p_i} \sin(\omega_j \ln p_i)/s)$
      3. $\ell \leftarrow \ell + 1$
   2. end
5. end

*Return* $a^p$ concatenated with $b^p$.

chosen according to $\kappa_X(\omega)$. For any distribution $p$ on $[d]$, define vectors $v^p, u^p \in \mathbb{R}^{td}$ where, for $i \in [d], j \in [t]$,

$$v^p_{i,j} = \sqrt{p_i} \cdot \cos(\omega_j \ln p_i)/t, \quad u^p_{i,j} = \sqrt{p_i} \cdot \sin(\omega_j \ln p_i)/t.$$

Let $v^p_i$ be a concatenation of $v^p_{i,j}$ and $u^p_{i,j}$ over all $j \in [t]$. Then note that $E[||v^p_i - v^q_i||_2^2] = f_X(p_i, q_i)$ and $\text{var}[||v^p_i - v^q_i||_2^2] \leq 23(f_X(p_i, q_i))^2/t$. Hence, for $t = 23n^2 d^2 \varepsilon^{-2}$, by an application of the Chebyshev bound,

$$\Pr[||v^p_i - v^q_i||_2^2 - f_X(p_i, q_i) \geq \varepsilon f_X(x, y)] \leq 23 \varepsilon^{-2} / t = (nd)^{-2}. \quad (5.1)$$
By an application of the union bound over all pairs of points:
\[
\Pr[\exists i \in [d], p, q \in P| ||v_i^p - v_i^q||_2^2 - f_X(p_i, q_i)| \geq \varepsilon f_X(p_i, q_i)] \leq 1/d.
\]
And hence, if \(v^p\) is a concatenation of \(v_i^p\) over all \(i \in [d]\), then with probability at least \(1 - 1/d\),
\[
(1 - \varepsilon) \chi^2(p, q) \leq ||v^p - v^q|| \leq (1 + \varepsilon) \chi^2(p, q).
\]
The final length of the vectors is then \(td = 23n^2d^3\varepsilon^{-2}\) for approximately preserving distances between every pair of points with probability at least \(1 - \frac{1}{d}\). This can be reduced further to \(O(\log n/\varepsilon^2)\) by simply applying the JL-Lemma.

6. Dimensionality reduction

The JL-Lemma has been instrumental for improving the speed and approximation ratios of learning algorithms. In this section, we give a proof of the JL-analogue for well behaved \(f\)-divergences. Specifically, we show that a set of \(n\) points lying on a high dimensional simplex can be embedded to a \(k = O(\log n/\varepsilon^2)\)-dimensional simplex, while approximately preserving the information distances between all pairs of points. This dimension reduction amounts to reducing the support of the distribution from \(d\) to \(k\), while approximately maintaining the divergences.

Our proof uses \(\ell_2^2\) as an intermediate space. On a high level, we first embed the points into a high (but finite) dimensional \(\ell_2^2\) space, using the techniques we developed in Section 4.2. We then use the Euclidean JL-Lemma to reduce the dimensionality, and remap the points into the interior of a simplex. Finally, we show that far away from the simplex boundaries, the well behaved \(f\)-divergences have the same structure as \(\ell_2^2\), hence the embedding back into information spaces can be done with a simple translation and rescaling. Note that for \(f\)-divergences that have an embedding into finite dimensional \(\ell_2^2\), the proof is constructive.

**Algorithm 5: Dimension Reduction for \(D_f\)**

**Input:** Set \(P = \{p_1, \ldots, p_n\}\) of points on \(\Delta_d\), error parameter \(\varepsilon\), constant \(c_0(\varepsilon, f)\)

**Output:** A set \(\bar{P}\) of points on \(\Delta_k\) where \(k = O\left(\frac{\log n}{\varepsilon^2}\right)\)

1. Embed \(P\) into \(\ell_2^2\) to obtain \(P_1\) with error parameter \(\varepsilon/4\).
2. Apply Euclidean JL-Lemma with error \(\frac{\varepsilon}{4}\) to obtain \(P_2\) in dimension \(k = O\left(\frac{\log n}{\varepsilon^2}\right)\)
3. Remap \(P_2\) to the plane \(L = \{x \in \mathbb{R}^{k+1} | \sum_i x_i = 0\}\) to obtain \(P_3\)
4. Scale \(P_3\) to a ball of radius \(c_0 \cdot \frac{\varepsilon}{k+1}\) and center at the centroid of \(\Delta_{k+1}\) to obtain \(\bar{P}\).

To analyze the above algorithm, we recall the JL–Lemma [Johnson and Lindenstrauss, 1984; Biau et al., 2008]:

**Lemma 21** For any set of points \(P\) in a (possibly infinite dimensional) Hilbert space \(H\), there exists a randomized map \(f: H \rightarrow \mathbb{R}^k, k = O\left(\frac{\log n}{\varepsilon^2}\right)\) such that \(\forall p, q \in P\)
\[
(1 - \varepsilon)||p - q||_2^2 \leq ||f(p) - f(q)||_2^2 \leq (1 + \varepsilon)||p - q||_2^2,
\]
with high probability.

We show the following simple corollary:

**Corollary 22** For any set of points $P$ in $H$ there exists a constant $t$ and a randomized map $f: H \to \Delta_{k+1}$, $k = O(\frac{\log n}{\epsilon^2})$ such that $\forall p, q \in P:

\begin{equation}
(1 - \epsilon)\|p - q\|_2^2 \leq t\|f(p) - f(q)\|_2^2 \leq (1 + \epsilon)\|p - q\|_2^2,
\end{equation}

Furthermore for any small enough constant $r$, we may bound the domain of $f$ to be a ball $B$ of radius $r$ centered at the simplex centroid, $(1/k+1, 1/k+1, \ldots, 1/k+1)$.

**Proof** Consider first the map of Lemma 21 from $\mathbb{R}^d \to \mathbb{R}^k$. Now note that any set of points in $R^k$ can be isometrically embedded into the hyperplane $L = \{x \in \mathbb{R}^{k+1} \mid \sum x_i = 0\}$. This follows by remapping the basis vectors of $\mathbb{R}^k$ to those of $L$. Finally since $L$ is parallel to the simplex plane, the entire pointset may be scaled by some factor $t$ and then translated to fit in $\Delta_{k+1}$, or indeed in any ball of radius $r$ centered at the simplex centroid.

We now show that any well-behaved $f$ divergence is nearly Euclidean near the simplex centroid.

**Lemma 23** Consider any well-behaved $f$ divergence $D_f$, and let $B_r$ be a ball of radius $r$ such that $B_r \subset \Delta_k$ and $B_r$ is centered at the simplex centroid. Then for any fixed $0 < \epsilon < 1$, there exists a choice of $r$ and scaling factor $t$ (both dependent on $k$) such that $\forall p, q \in B$:

\begin{equation}
(1 - \epsilon)\|p - q\|_2^2 \leq tD_f(p, q) \leq (1 + \epsilon)\|p - q\|_2^2.
\end{equation}

**Proof** We consider arbitrary $p, q \in B_r$ and note that the assumptions imply each coordinate lies in the interval $I = [\frac{k}{k} - r, \frac{k}{k} + r]$. Let $rk = \epsilon'$, then $I = [\frac{1 - \epsilon'}{k}, \frac{1 + \epsilon'}{k}]$. We now prove the lemma for $p, q \in I$, the main result follows by considering $D_f$ and $\| \cdot \|_2$ coordinate by coordinate.

By the definition of well-behaved $f$-divergences and Taylor’s theorem, there exists a neighborhood $N$ of 1, and function $\phi$ with $\lim_{x \to 1} \phi(1) = 0$ such that for all $x \in N$:

\begin{equation}
f(x) = f(1) + (x - 1)f'(1) + \frac{(x - 1)^2}{2}f''(1) + (x - 1)^3\phi(x) = \frac{(x - 1)^2}{2}f''(1) + (x - 1)^3\phi(x).
\end{equation}

Therefore:

\[
\frac{D_f(p, q)}{\|p - q\|_2^2} = \frac{p \cdot f\left(\frac{q}{p}\right)}{(p - q)^2} = \frac{p \left(\frac{q-p}{p}\right)^2 f''(1) + \left(\frac{q-p}{p}\right)^3 \phi\left(\frac{q}{p}\right)}{2p} + \frac{q-p}{p^2} \phi\left(\frac{q}{p}\right).
\]

Recall again that $p \in [\frac{1 - \epsilon'}{k}, \frac{1 + \epsilon'}{k}]$ so the first term converges to the constant $2kf''(1)$ as $r$ grows smaller (and hence $\epsilon'$ decreases). Note also that the second term goes to 0 with $r$, i.e.,
given a suitably small choice of \( r \) we can make the term smaller than any desired constant. Hence, for every dimension \( k \) and \( 0 < \varepsilon < 1 \), there exists a radius of convergence \( r \) such that for all \( p, q \in B_r \):

\[
(1 - \varepsilon)\|p - q\|_2^2 \leq \frac{1}{2k f''(1)} D_f(p, q) \leq (1 + \varepsilon)\|p - q\|_2^2.
\]  

(6.5)

We note that the required value of \( r \) can be computed easily for the Hellinger and \( \chi^2 \) divergence, and that \( r \) behaves as \( \frac{1}{k} \cdot c \) where \( c = c(f, \varepsilon) \) is a constant depending only on \( \varepsilon \) and the function \( f \) and not on \( k \) or \( n \).

To conclude the proof note that the overall distortion is bounded by the combination of errors due to the initial embedding into \( P_1 \), the application of JL-Lemma, and the final reinterpretation of the points in \( \Delta_{k+1} \). The overall error is thus bounded by, \((1 + \varepsilon/4)^3 \leq 1 + \varepsilon\).

**Theorem 24** Consider a set \( P \in \Delta_d \) of \( n \) points under a well-behaved \( f \)-divergence \( D_f \). Then there exists a \((1 + \varepsilon)\) distortion embedding of \( P \) into \( \Delta_k \) under \( D_f \) for some choice of \( k \) bounded as \( O\left(\frac{\log n}{\varepsilon^2}\right) \). Furthermore this embedding can be explicitly computed even for a well-behaved \( f \)-divergence with an infinite dimensional kernel, if the kernel can be approximated in finite dimensions within a multiplicative error as we show for JS and \( \chi^2 \).

7. Conclusions

The embedding and sketching results we show here complements the known impossibility results for sketching information distances in the strict turnstile model, thus providing a more complete picture of how these distances can be estimated in a stream. The dimensionality reduction result essentially says that as long as the information distance admits a “Euclidean-like” patch somewhere in the simplex, it can be mapped to a lower dimensional space. This latter result is a little surprising because the Hellinger distance exhibits more \( \ell_1 \) like behavior at the corners of the simplex. In fact, we conjecture that if we limit ourselves to mappings that are not contractive, then it is likely that the Hellinger distance will not admit accurate dimensionality reduction.
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