The dispersion equation for a wave number of the electromagnetic wave in a random medium of the parallel dielectric cylinders

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Abstract. In a frame of quasi-crystal approximation the dispersion equations are obtained for the wave vector of a coherent electromagnetic wave propagating in a media which contains a random set of parallel cylinders with possible overlapping. The results are compared with that for the case when a regularity at the cylinder placement exists.

Introduction

Under certain conditions the electrochemical anodization of metals leads to the formation of the oxide layer with parallel cylindrical pores on the surface of metal. The radius of the cylinders is much less than the thickness of the layer. These oxide films are used as corrosion-protective, strengthening, decorative coatings and attract much attention in microelectronics, in particular, for the creation of non-linear and active thin-film elements, and also for substrates of large hybrid microcircuits. Porous oxides are perspective for multilevel systems of interconnections. In dependence on the conditions of anodization and kind of metal it is possible to obtain films with different degree of ordering in pore placement of pores. The placement of pores is maximally chaotic when the pores are placed randomly and could be overlapped. The overlapping pores form the cavities of complex configurations. The placement of pores is maximally ordered, when the points of the cylinder axes intersection with the plane perpendicular to the axes lay in the nodes of some lattice, for example a hexagonal one. An investigation of the optical properties of oxide films allows to find a degree of pores ordering. If the wavelength is much larger than the radius of pores, the optical properties of the porous layer can be described by the effective refractive index. This index is to be found from the dispersion equations.

The dispersion equation for the effective refractive index of the system of parallel non-overlapping cylindrical fibres in media was deduced in the work [1]. The equation allows to describe the optical properties of the layer with the regular pores placement. In present work we obtain the dispersion equation for a medium with chaotic distribution of pores (the pores are located randomly and can be overlapped). We also analyse an influence of regularity on the effective refractive index.

1. Dispersion equation for a medium with completely chaotic cylinder placement

Let us consider a system of parallel dielectric cylinders. We assume that if the distance between the centres of cylinders is less than distance \(2R\) (\(R\) being the radius of the cylinders), they form the common scatterer with the constant refractive index. If the concentration is small, the number of overlapping cylinders is also small. So the majority of scatterers are cylindrical. If the concentration of cylinders increases, the number of the overlapped cylinders increases. However, even for medium consisting of such complicated scatterers it is possible to derive the dispersion equation for a wave vector using the statistical averaging technique [2-4]. Without the loss of generality we can set to unity the dielectric constant of...
the surrounding media and the dielectric constant of the scatterers set to ε. Let the z-axis is
directed along the cylinder axis, then, the positions of cylinder axis at the plane perpendicular
to z are specified by two-dimensional random vectors \( s_0 \ldots s_N \). The dielectric constant \( \varepsilon \) of
N+1 overlapping cylinders can be written in the form
\[
\mathbf{E}(\mathbf{r}, z) = \varepsilon(\mathbf{r}) = 1 + (\varepsilon - 1) \left\{ \sum_{i=0}^{N} \theta(\mathbf{r} - s_i) - \frac{1}{2!} \sum_{i\neq j}^{N} \theta(\mathbf{r} - s_i)\theta(\mathbf{r} - s_j) + \right. \\
\left. + \frac{1}{3!} \sum_{i\neq j\neq k}^{N} \theta(\mathbf{r} - s_i)\theta(\mathbf{r} - s_j)\theta(\mathbf{r} - s_k) - \ldots \right\},
\]
where \( \theta(\mathbf{r}) = \begin{cases} 1 & |\mathbf{r}| < R \\ 0 & |\mathbf{r}| > R \end{cases} \). The expression (1) is done so, that \( \mathbf{E}(\mathbf{r}) = \varepsilon \), if there are one or
more cylinders in the given point. The electromagnetic field satisfies Maxwell’s equations,
whose solutions can be written in the form:
\[
\mathbf{E}(\mathbf{r}, s_0, s_1 \ldots s_N) = \mathbf{E}(\mathbf{r}, s_0) e^{ikz}.
\]
For \( \mathbf{E}(s_0, s_1 \ldots s_N) \) we obtain the equation:
\[
\mathbf{D} \times \mathbf{E}(\mathbf{r}, s_0, s_1 \ldots s_N) - k^2 \mathbf{E}(\mathbf{r}, s_0, s_1 \ldots s_N) = (\varepsilon - 1)k^2 \varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r}, s_0, s_1 \ldots s_N), \quad (2)
\]
where \( \mathbf{D} = (\partial / \partial \mathbf{r}, i\mathbf{k} \cdot \mathbf{r}) \). Let average this equation over positions of all the scatterers except
\( s_0 \), in point of view that the cylinders are independent and distributed with the density of
probability \( f(s_i) \). So we multiply (2) by \( f(s_j) \ldots f(s_N) \) and integrate by \( ds_1 \ldots ds_N \). At
\( |\mathbf{r} - s_0| < R \) we obtain:
\[
\mathbf{D} \times \mathbf{D} \times \mathbf{E}(\mathbf{r}, s_0) - k^2 \mathbf{E}(\mathbf{r}, s_0) = (\varepsilon - 1)k^2 \mathbf{E}(\mathbf{r}, s_0).
\]
At \( |\mathbf{r} - s_0| \geq R \) the averaging of right part (2) gives:
\[
(\varepsilon - 1)k^2 \int d^2 s_1 \ldots d^2 s_N \left\{ \sum_{i=0}^{N} \theta(\mathbf{r} - s_i) - \frac{1}{2!} \sum_{i\neq j}^{N} \theta(\mathbf{r} - s_i)\theta(\mathbf{r} - s_j) + \right. \\
\left. + \frac{1}{3!} \sum_{i\neq j\neq k}^{N} \theta(\mathbf{r} - s_i)\theta(\mathbf{r} - s_j)\theta(\mathbf{r} - s_k) - \ldots \right\} \mathbf{E}(s_0, s_1 \ldots s_N) f(s_1) \ldots f(s_N) = \]
\[
= (\varepsilon - 1)k^2 \left\{ 1 - \frac{1}{2!} (N - 1) \Omega(\mathbf{r} - s_2) + \frac{1}{3!} (N - 1)(N - 2) \Omega(\mathbf{r} - s_3) \times \right. \\
\left. \times \theta(\mathbf{r} - s_2) + \ldots \right\} \mathbf{E}(s_0, s_1 \ldots s_N) f(s_1) \ldots f(s_N) \approx (\varepsilon - 1)k^2 \times \]
\[
\left\{ 1 - \frac{1}{2!} (N - 1)\Omega + \frac{1}{3!} (N - 1)(N - 2)\Omega^2 + \ldots \right\} \mathbf{E}(s_0) f(s_1) d^2 s_1,
\]
where \( \Omega(\mathbf{r}) = \int f(s) \theta(\mathbf{r} - s) d^2 s \).

Here by analogy with the quasi-crystal approximation [5,6] we have put
\[
\langle \mathbf{E}(s_0, s_1 \ldots s_{N-1}, s_N) \rangle \approx \langle \mathbf{E}(s_0, s_1 \ldots s_{N-1}) \rangle \approx \ldots \approx \langle \mathbf{E}(s_0) \rangle .
\]
Let us go to a limit of infinite medium by directing $N \to \infty$ under $Nf(s) = n_0$, where $n_0$ is the number of scatterers on unit area, then
\[ N\Omega = n_0 \pi R^2. \] (5)

Taking into account the uniformity of averaged medium we write down [5]:
\[ \langle E(\rho,s) \rangle = e^{ik'z s} \langle E(\rho-s,0) \rangle. \] (6)

A vector $k' = (k'_x, k'_z)$ is the desired wave vector [4], connected with effective refractive index by $n_{\text{eff}} = k'/k$. Substituting (5) and (6) into (4) and uniting with (3) we obtain:
\[
D \times D \times \langle E(\rho,0) \rangle - k^2 \langle E(\rho,0) \rangle = (\varepsilon - 1)k^2 \langle \theta(\rho) \rangle \langle E(\rho,0) \rangle + 
+ (\varepsilon - 1)k^2 (1 - \theta(\rho))n_0 e^{ik'\rho} \int_{|s|<R} e^{-ik'z s} \langle E(s,0) \rangle d^2 s,
\] (7)

where
\[
n'_0 = (1 - e^{-n_0 \pi R^2}) / \pi R^2.
\] (8)

The value $n_0 \pi R^2$ is the filling parameter, i.e. a fraction of an area in a cross section occupied by cylinders. Equation (7) allows to find the dispersion equation for a wave vector $k'$. At complete filling of a medium by cylinders $n_0 \to \infty$, $n'_0 \to 1 / \pi R^2$. It is easy to see that in this case the solution of (7) is
\[ \langle E(\rho,0) \rangle = E_0 \exp(i k' \rho). \]

As seen from the equation (7) at $|\rho| < R$, $\langle E(\rho,0) \rangle$ satisfies to the wave equation
\[ D \times D \times \langle E(\rho,0) \rangle - \varepsilon k^2 \langle E(\rho,0) \rangle = 0. \]

Solution of this equation can be represented as [7,8]
\[
\langle E(\rho,0) \rangle = \sum_m \{ a^T M_m (\lambda \rho) + a^T E_m (\lambda \rho) \} e^{im\phi},
\] (11)

where $\lambda = \sqrt{\varepsilon k^2 - k'^2}$. The expressions for $E^T M_m$ and $E^T E_m$ are given in [1].

Substituting (11) into (10) just as in [1], and projecting (10) on the vectors $e_\phi + k'_z m / R v^2 e_z$ and $e_z$, multiplying by $\exp(-im\phi)$, and integrating by $d\phi$ at $|\rho| = R$, we obtain an infinite-dimensional system of equations relative of $a^T M_m$ and $a^T E_m$:
\[
\frac{m k^\prime}{k} \left( \frac{\lambda^2}{\nu^2} - 1 \right) H_m(\nu R)J_m(\lambda R) a_n^{TM} + i \lambda \nu H'_m(\nu R) J_m(\lambda R) - \\
H_m(\nu p )J'_m(\lambda R) \right) a_n^{TE} = \pi n_0 \frac{\nu}{k} \left( \frac{1}{k'^2} - \frac{1}{k'^2 - k^2} \right) \left( \left[ H_{m+1}(\nu R) J_{m+1}(k'_\perp R) \right] \right. \\
\left. \times [J_{n+1}(\lambda R) J_{n+1}(k'_\perp R) - H_{m-1}(\nu R) J_{m-1}(k'_\perp R)] [J_{n-1}(\lambda R) J_{n-1}(k'_\perp R) \right] \left( \frac{k'}{k} a_n^{TM} - \\
- \left( [H_{m+1}(\nu R) J_{m+1}(k'_\perp R) J_{n+1}(\lambda R) J_{n+1}(k'_\perp R) + [H_{m-1}(\nu R) J_{m-1}(k'_\perp R) \right] \\
\times [J_{n+1}(\lambda R) J_{n+1}(k'_\perp R) \right] \right) a_n^{TE} \right), \\
\] (12)

where \( H_m(\nu R) J_m(\lambda R) \) and so on. Everywhere in (12) we imply summation over subscript \( n \).

The condition of solvability of the system (12) gives the dispersion equations for wave vector \( k' \), which are found simpler than equations for non-overlapping cylinders obtained in [1], since it is possible to bypass a calculation difficulty with of the infinite-dimensional determinant of the system. The system (12) has two types of the solutions symmetry. The first one corresponds to an ordinary wave.

\[
\begin{align*}
\alpha_n &= -\alpha_n^T M \\
\beta_n &= \beta_n^T E = a_n^T E \\
\end{align*}
\] (13)

Taking into account the symmetry (13), we obtain the dispersion equation in the form which is suitable for the numerical calculations by iteration method and Newton method.

\[
k_{z}'^2 = k_{z}'^2 - k_{z}^2 + \pi n_0 \left\{ \sum_{n=0}^{\infty} \chi_n \left( \frac{k'}{k} \right)^2 \left( \delta_n^+ \alpha_n + \delta_n^- \beta_n \right) + \\
\chi_n^+ \left( \delta_n^+ \gamma_n + \frac{k'}{k} \delta_n^- \alpha_n \right) \right\} / (\alpha_n^2 - \beta_n^2),
\] (14)

where \( \alpha_n = nk'/k (\lambda^2 / \nu^2 - 1) H_n(\nu R) J_n(\lambda R) \), \( \lambda = \sqrt{\nu k^2 - k_{z}^2} \), \( \beta_n = \lambda R (\lambda / \nu H'_n(\nu R) J_n(\lambda R) - H_n(\nu R) J'_n(\lambda R)) \), \( \nu = \sqrt{k^2 - k_{z}^2}, \ Re \nu > 0 \).
\[ \delta_n^0 = 2(\varepsilon - 1) \frac{\lambda^2}{\lambda^2 - k_{\perp}^2} \left[ H_n(\nu R)J_n(k'_{\perp} R) \right], \quad \chi_n^0 = \left\{ \begin{array}{ll} 2[J_n(\lambda R)J_n(k'_{\perp} R)], & n \neq 0 \\ [J_0(\lambda R)J_0(k'_{\perp} R)], & n = 0 \end{array} \right. \]

where \( \gamma_n = \frac{\lambda R}{\nu} \left( \frac{H'_n(\lambda R) - \nu H_n(\lambda R)}{H'_n(\lambda R) + \nu H_n(\lambda R)} \right) \)

\[ \delta_+^n = \frac{\lambda}{\nu} \left[ 1 - k_{\perp}^2 \right] / \left( \frac{\lambda^2}{\lambda^2 - k_{\perp}^2} \right) \left[ H_{n+1}(\nu R)J_{n+1}(k'_{\perp} R) \right] \pm \left[ H_{n-1}(\nu R)J_{n-1}(k'_{\perp} R) \right], \]

\[ \chi_n^+ = \left\{ \begin{array}{ll} (1 + 1)/2 & [J_1(\lambda R)J_1(k'_{\perp} R)], \quad n \neq 0 \\ 0 & \end{array} \right. \]

For an extraordinary wave \( \begin{align*}
\begin{bmatrix}
    a_{TM}^T \n    a_{TE}^T
\end{bmatrix}
\end{align*} \) we have the dispersion equation

\[ k_z^2 = k^2 - k_{\perp}^2 + \pi \alpha_0' \left\{ \sum_{n=0}^{\infty} \left( \chi_n^+ \frac{k'}{k} \left( \delta_n^{-} \alpha_n + \frac{k'}{k} \delta_n^{+} \beta_n \right) + \right. \\
+ \chi_n^0 \left( \delta_n^{-} \gamma_n + \frac{k'}{k} \delta_n^{+} \alpha_n \right) \right\} / D_n + \left[ \sum_{n=0}^{\infty} \chi_n^0 \left( \frac{k'}{k} \delta_n^{0} \beta_n + \chi_n^0 \delta_n^{0} \alpha_n \right) / D_n \right] \times \left[ \sum_{n=0}^{\infty} \chi_n^0 \left( \frac{k'}{k} \delta_n^{0} \beta_n + \delta_n^{0} \alpha_n \right) / D_n \right] \left( \frac{k^2 - k_{\perp}^2}{\pi \alpha_0'} - \sum_{n=0}^{N} \chi_n^0 \delta_n^{0} \beta_n / D_n \right), \quad (15) \]

where \( D_n = \alpha_n^2 - \beta_n \gamma_n \).

In static limit \( kR \to 0 \) from the equations (14) and (15) it follows:

\[ k_z^2 = \varepsilon_\perp k^2, \quad k_{\perp}^2 \varepsilon_\parallel + k_{\perp}^2 \varepsilon_\perp = \varepsilon_0 \varepsilon_\perp k^2, \]

where \( \varepsilon_\parallel = 1 + n' \pi R^2 (\varepsilon - 1), \quad \varepsilon_\perp = 1 + 2(\varepsilon - 1)n' \pi R^2 / (\varepsilon + 1 - n' \pi R^2 (\varepsilon - 1)) \) are the dielectric constants of the medium with cylinders embedded for the transverse and longitudinal electrostatic field correspondingly [8].

Let us show how to apply the dispersion equations in the case of porous media. Parameters of the problem are the values of the wave vector in the medium \( k_m \) and inside a cylinder \( k_c \). Instead of \( k_m \) and \( k_c \) we used \( k = k_m \) and \( \varepsilon = \left( k_c / k_m \right)^2 \). For porous medium, \( k_m = \sqrt{\varepsilon_m k_0} \) and \( k_c = k_0 \), where \( \varepsilon_m \) is the dielectric constant of the material, and \( k_0 = \omega / c \) is the wave vector in vacuum. From this, it follows that \( k = \sqrt{\varepsilon_m k_0} \) and \( \varepsilon = 1 / \varepsilon_m \). The effective refractive index of the porous medium is given by the expression \( n_{\text{eff}} = k' / k_0 \), where \( k' \) is found from the dispersion equation solved with parameters \( k = \sqrt{\varepsilon_m k_0} \) and \( \varepsilon = 1 / \varepsilon_m \).

Let us consider the case of oblique incidence of an electromagnetic wave on the layer of parallel cylinders, which axes are perpendicular to the layer surface. For this case \( k_{\perp}' = k_{\perp} \), where \( k_{\perp} \) is the component of a wave vector of incident wave, which is perpendicular to the z axis (parallel to the layer surface) and the relation \( k_{\perp}' / k \) is sine of the angle of incidence. At given \( k_{\perp}' \), the \( k_{\perp} \) has to be obtained from the equation (14), if the wave is polarized perpendicular to the plane of incidence, or from (15) if the field strength
vector lays in the plane of incidence. Let us show the results of numerical calculations for the porous medium with \( n_m = 1.76 \). The dependence of the real and imaginary parts of the effective refractive index \( n_{\text{eff}} = \left( k_\perp^2 + k_\parallel^2 \right)^{1/2} / k_0 \) on the normalized frequency (i.e. on the parameter \( \omega R / c \)) is shown in Fig.1,2 for the incident angles 0°, 60° and for different porosity parameter \( \pi n' R^2 \). At \( \omega R / c \to 0 \) the real part of the effective refractive index goes to the static value. The imaginary part has a maximum at \( k_0 R \approx 1 \).

2. Comparing with the case of the regular cylinder placement

System of non-overlapping cylindrical pores already possesses some degree of ordering. This degree of ordering increases if the concentration of pores becomes larger. It is possible to increase degree of ordering without changes in the concentration in the following way: let us imagine that each pore is surrounded by impenetrable cylindrical shell with the radius \( R_s \). For completely ordered media with hexagonal cells \( R_s = 1 / \sqrt{3 \pi n'/\alpha} \). For the function \( w(x) \) in the equation (20) of work [1] we should take the following function

\[
w(x) = \begin{cases} 
0 & x < R_s, \\
w_\alpha' (xR/R_s) & x > R_s
\end{cases}
\]

where \( w_\alpha' (x) \) is a radial distribution function for the rigid disks on a plane with the filling parameter \( \alpha' = \pi n' R_s^2 \).

In Fig. 3 the frequency dependence of the effective refractive index is plotted for different degrees of ordering of pores. The wave is propagating in direction parallel to the axes of cylindrical pores. The curve number 1 corresponds to pores inside the non-overlapping shells with the radius \( R_s = 0.81 / \sqrt{3 \pi n'/\alpha} \). The porosity parameter \( \pi n' R^2 \) for all curves is the same. Only the degree of ordering is changed. If the ordering becomes more chaotic, the imaginary part of the effective refractive index caused by the attenuation of the coherent wave becomes larger. It is maximal for the completely random pore placement with possible overlapping. The real part of the refractive index is also strongly dependent from ordering.

Conclusion

We obtained in analytical form the dispersion equations (14 and (15) for the wave vector of the electromagnetic wave propagating in a media which consists of randomly placed parallel cylinders with possible overlapping. In this media the maximally chaotic placement of the scatterers is realised. Together with the obtained in [1] equation (20) for the non-overlapping cylinders our equations allow to find the effective refractive index for any degree of medium regularity. The imaginary part of the refractive index increases when the placement of pores becomes more chaotic.

References

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Figures

Fig. 1a, b. Dependence of the real (a) and imaginary (b) parts of the porous medium \( n_m =1.76 \) effective refractive index on the normalized frequency for the incident angle \( 0^\circ \) (parallel to the cylinder axes), the porosity parameter equals: 1 - 0.1; 2 - 0.2; 3 - 0.3; 4 - 0.4; 5 - 0.5.

Fig. 2a, b. Dependence of the real (a) and imaginary (b) parts of the porous medium \( n_m =1.76 \) effective refractive index on the normalized frequency for the incident angle \( 60^\circ \), the porosity parameter equals: 1 - 0.1; 2 - 0.2; 3 - 0.3; 4 - 0.4; 5 - 0.5. The solid curve corresponds on the extraordinary wave, the dashed one corresponds on the ordinary wave.

Fig. 3a, b. Dependence of the real (a) and imaginary (b) parts of the porous medium \( n_m =1.65 \) effective refractive index on the normalized frequency effective, the porosity parameter equals to 0.2. The curve (1) corresponds to the pores, placed in non-overlapping circles with radius \( R_c =1.72 R \), (2) - non-overlapping «hard» pores, (3) - chaotic placement of pores with possible overlapping. Light incidence angle \( 0^\circ \).
Fig. 1

(a) $\text{Re } n_{\text{eff}}$

(b) $\text{Im } n_{\text{eff}}$
Fig. 2

a) $\text{Re } n_{\text{eff}}$

b) $\text{Im } n_{\text{eff}}$

Graph showing $\omega R/c$ on the x-axis and $n_{\text{eff}}$ on the y-axis for different curves labeled 1, 2, 3, and 4.
