Tracking the maneuvering spacecraft propelled by swing propulsion of constant magnitude

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Abstract: This paper proposes partially norm-preserving filtering for a class of spacecraft in the presence of time-varying, but constant-magnitude maneuver. The augmented state Kalman filter (ASKF) is commonly used to track the maneuvering spacecraft with unknown constant propulsion; however, if the maneuver varies via time, the estimation performance will be degraded. To promote the tracking performance of the ASKF in case of time-invariant, constant-magnitude disturbance, the partially norm-preserving ASKF is developed by applying the norm constraint on the unknown maneuver. The proposed estimator, which is decomposed into two partial estimators and iteratively propagated in turns, projects the unconstrained maneuver estimation onto the Euclidean surface spanned by the norm constraint. The illustrative numerical example is provided to show the efficiency of the proposed method.

Keywords: maneuvering spacecraft, tracking, augmented state, norm-preserving, Kalman filter.

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1. Introduction

Spacecraft tracking is critically important for satellite operators to maintain a clear picture of all space objects. The performance of spacecraft tracking significantly depends on an estimation of the dynamic state. The work of Kalman [1] and Luenberger [2], the techniques of state estimation are extensively studied. The Kalman filter generates an optimal solution when the motion of spacecraft is precisely modeled. However, in many practical cases, the spacecraft performs unknown maneuvers, if the maneuver sequence is not properly incorporated into the estimation, the filter evidently suffers from the performance degradation and sometimes even diverges [3].

In general, the unknown maneuvers can be considered as disturbance acted on spacecraft, and the significant challenge during the estimation is how to cope with the unknown disturbance. Over the past few decades, robust estimators for a linear system in the presence of unknown input have been developed, with a common approach being the unknown disturbance-decoupled state estimation [4]. Within this approach, the disturbance vector generally satisfies the time-invariant condition [5], thus an optimal result could be obtained by minimizing the error covariance. To extend the disturbance-decoupled state estimator to the case in which the disturbance simultaneously affects the system’s state and output, Darouach & Zasadzinski [6] developed the optimal estimator filter (OEF) based on the parameterizing technique. Gillijns and De Moor [7] also proposed a three-step filter (TSF) for the joint disturbance and state estimate based on the direct feedthrough. The successful implementation of the TSF significantly depends on a full-rank feedthrough matrix of input to output. To relax the full-rank constraint on the feedthrough matrix, Cheng [8] proposed an unbiased minimum-variance estimator for the case of a feedthrough matrix of arbitrary rank. Other techniques in this area also include the minimax-optimal method [9] and the time-delayed optimal state estimator [10].

An earlier disturbance-decoupled example implemented for target tracking is the augmented state Kalman filter (ASKF), in which the constant unknown maneuver is treated as a part of the state and estimated synchronously [11,12]. However, due to the increased filter dimension, the intensive computational load makes the ASKF very limited for implementation, and also, numerical problems may arise in the case of ill-conditioned systems [13]. To reduce the computational load, Friedland [14] proposed a two-stage Kalman filter by decomposing the ASKF into two parallel reduced order estimators. If the maneuver is constant, the two reduced order estimators propagate in turns and generate the optimal estimate of both the state and the maneuver [15]. Due to its high computational efficiency,
this method is widely used to track a maneuvering target with constant propulsion [16] and achieves a better performance. To cope with the pulse-like unknown maneuvers, different multiple adaptive estimation methods have also been developed based on the ASKF. For example, the interacting multiple-model (IMM) method [17,18], which treats the unknown maneuver as part of the system state, switches the estimation among different filters based on maneuver detection [19]. When a maneuver is declared by the detector, the normal filter is augmented and the covariance is inflated to enable a re-convergence on the maneuvering trajectories [20]. Similar methods also include variable state dimension (VSD) estimator [22], enhanced variable structure dimension estimator [21], and unscented VSD estimator [23].

The ASKF is proved efficient only when the spacecraft is subjected to the constant unknown maneuver. However, the estimate becomes biased if the maneuver is time-varying, the faster the time-varying rate, the larger the estimation bias. Obviously, this performance degradation is mainly caused by the model mismatching on the unknown maneuvers. In general, to promote the estimation performance in the case of an unmatched model, the constrained Kalman filter (CKF), in which the state satisfies linear or nonlinear constraints, is commonly used for the estimation. By applying the constraints on the state, the estimate bias caused by model mismatching can be reduced. This can be seen, for instance, in the re-parameterization method [24], perfect measurement method [25] and projection method [24]. Norm-preserving is a typical nonlinear constraint imposed on the entire or part of a system’s state. Given that the attitude quaternion satisfies the unit norm constraint, the earlier objective, to seek the norm-preserving solution of the Kalman filter, is for attitude estimation [26–28]. Zanet et al. [29] first developed the norm CKF for a time-discrete system by using the Lagrangian multiplier technique. In recent years, the norm-constrained filter is extended to the continuous-time system by considering a weighted norm [30]. The advantages of the work [29,30] have been numerically demonstrated by simulations of attitude determination.

Note that most of current earth-oriented spacecraft is propelled by the constant thruster, when describing the motion of spacecraft within the earth-centered-inertial frame, the acceleration generated by the thruster is time-varying but of constant magnitude. With a purpose of promoting the spacecraft tracking accuracies in presence of time-varying but constant magnitude maneuvers, a partially norm-preserving ASKF is proposed by applying the norm constraint on the unknown maneuvers. The novelty of this work is that we derive a partially norm-preserving ASKF with a two-stage formulation, and the ASKF is conditionally extended to a class of systems with time-varying disturbance, which is not previously considered in the literature. Additionally, we strictly prove the second order condition for optimality, based on evaluating the Hessian matrix.

2. Preliminary

Consider an earth-oriented spacecraft equipped with constant thrusters is moving in its orbit, within the earth-centered inertial frame the spacecraft motion can be represented with the discrete-time state-space equation as follows:

\[
\begin{align*}
x_{k+1} &= A_k x_k + G_k d_k + \Gamma_k w_k, \\
y_k &= C_k x_k + v_k
\end{align*}
\]

where \(x_k \in \mathbb{R}^n\) denotes the system state, \(d_k \in \mathbb{R}^m\) represents the maneuver vector of the well-known constant magnitude. Due to the attitude motion of spacecraft with respect to the earth centered inertial frame, the maneuver vector is always time-varying and can be described as follows:

\[
d_{k+1} = (I + \delta_k) d_k
\]

where \(\delta_k \in \mathbb{R}^{m \times m}\) is a completely unknown time-varying matrix, because the attitude of the spacecraft moves slowly, we have

\[
\delta_k \approx 0,
\]

but the maneuver vector satisfies the norm constraint

\[
\|d_k\| = \rho
\]

where \(\rho > 0\), \(R_k \in \mathbb{R}^{p \times p}\) and \(y_k \in \mathbb{R}^p\) represents the measurement vector. Both the process noise \(w_k \in \mathbb{R}^m\) and the measurement uncertainties \(v_k \in \mathbb{R}^p\) are supposed to be independent, zero-mean, white random noise with well-known covariance, denoted as \(Q_k \in \mathbb{R}^{m \times m}\), and \(Q_k > 0\) and \(R_k > 0\). The time-depended matrices \(A_k, G_k, \Gamma_k\) and \(C_k\) are of appropriated dimensions that match the vectors. It also assumes that

\[
\text{rank}[G_k] = \text{rank}[\Gamma_k] = m
\]

and \((A_k, C_k)\) is observable, while both \((A_k, G_k)\) and \((A_k, \Gamma_k)\) are controllable, the initial state \(x_0\) is always independent of the process noise and measurement uncertainties, an unbiased estimation of initial state \(x_0\) is uniquely determined by the initial covariance of \(P_0^{xx}\).

3. Unconstrained augmented Kalman filter

3.1 Augmented system

The ASKF takes the unknown maneuver as a part of state, therefore, the augmented state is formulated as follows:

\[
X_k = \begin{bmatrix} x_k \\ d_k \end{bmatrix}
\]
where \( X_k \in \mathbb{R}^{n+m} \) denotes the augmented state vector, thus (1) and (2) are reformulated as follows:

\[
X_{k+1} = \bar{A}_k X_k + T_k w_k, \quad (8)
\]
\[
y_k = \bar{C}_k X_k + v_k, \quad (9)
\]
where the over bar denotes an augmented symbol, and

\[
\bar{A}_k = A_k + \Delta A_k, \quad (10)
\]
\[
\Delta A_k = \begin{bmatrix} A_k & G_k \\ 0 & I \end{bmatrix}, \quad \Delta \bar{A}_k = \begin{bmatrix} 0 & 0 \\ 0 & \delta_k \end{bmatrix}, \quad (11)
\]
\[
\bar{C}_k = G_k, \quad T_k = \begin{bmatrix} \Gamma_k \\ 0 \end{bmatrix}, \quad \bar{C}_k = [C_k \ 0]. \quad (12)
\]

### 3.2 Unconstrained ASKF (UASKF)

Suppose the initial maneuver is a Gaussian random variable and the unconstrained estimation \( \bar{d}_0 \) is determined by

\[
E(\{d_0 - \bar{d}_0\}(d_0 - \bar{d}_0)^T) = P_{0dd} \quad (13)
\]
and the cross-covariance

\[
E(\{x_0 - \bar{x}_0\}(d_0 - \bar{d}_0)^T) = P_{0xd}. \quad (14)
\]

Based on the augmented system (7), if we suppose the maneuver is invariant and take \( \delta_k = 0 \), then the UASKF can be constructed as follows:

\[
\hat{X}_{k+1/k} = \bar{A}_k \hat{X}_k, \quad (15)
\]
\[
T_{k+1/k} = \bar{A}_k T_k \bar{A}_k^T + T_k Q T_k^T, \quad (16)
\]
\[
\bar{T}_{k+1} = (I - \bar{K}_{k+1} \bar{C}_{k+1}), \quad (17)
\]
\[
\bar{K}_{k+1} = T_{k+1} \bar{C}_{k+1} T_{k+1}^T \bar{C}_{k+1} + R_{k+1}^{-1}, \quad (18)
\]
\[
\hat{X}_{k+1} = \hat{X}_{k+1/k} + \bar{K}_{k+1} \eta_{k+1} \quad (19)
\]

where \( \hat{X}_{k+1/k} \) is the one-step prediction, \( \bar{T}_{k+1/k} \) is the priori covariance of state, \( \bar{T}_{k+1} \) is the posterior covariance, while \( \bar{K}_{k+1} \) is the Kalman gain matrix, \( \eta_{k+1} \) denotes the residual vector and

\[
\eta_{k+1} = y_{k+1} - \bar{y}_{k+1}, \quad (20)
\]
\[
\bar{y}_{k+1} = \bar{C}_k \hat{X}_{k+1/k}. \quad (21)
\]

Note that if \( \delta_k = 0 \), the unconstrained estimators (15)–(21) work stably and generate an optimal result. Since the dimension of the ASKF is increased to \( n + m \), when \( m \) is compared to \( n \), the computational load of the ASKF is also significantly increased. With a purpose of reducing the computational intensity of the estimation, the ASKF can be equivalently decomposed into two parallel reduced-order estimators. The two parallel reduced order estimators are the \( n \)-dimensional state estimator and the \( m \)-dimensional maneuver estimator, they propagate in turns, but subject to coupling equations.

Note that since \( \delta_k \neq 0 \) for (1) and (2), we will demonstrate that the unconstrained estimators (15)–(21) produce a biased state estimate for (1)–(2). Define the unconstrained priori and posteriori estimate errors as follows:

\[
\hat{e}_{k+1/k} = X_{k+1} - \hat{X}_{k+1/k}, \quad (22)
\]
\[
\hat{e}_{k+1} = X_{k+1} - \hat{X}_{k+1}. \quad (23)
\]

Substituting (8) and (15) into (22) leads to

\[
\hat{e}_{k+1/k} = X_{k+1} - \hat{X}_{k+1/k} = \bar{A}_k \hat{e}_k + \Delta A k \hat{X}_k + T_k w_k = \bar{A}_k \hat{e}_k + \Delta \bar{A}_k \hat{X}_k + T_k w_k. \quad (24)
\]

Similarly, substituting (19) and (20) into (23), the posteriori estimate error can be represented as follows:

\[
\hat{e}_{k+1} = X_{k+1} - \hat{X}_{k+1} = X_{k+1} - \hat{X}_{k+1/k} - \bar{K}_{k+1} (C_{k+1} X_{k+1} + v_{k+1} + \bar{C}_{k+1} \hat{X}_{k+1/k}) = (I - \bar{K}_{k+1} \bar{C}_{k+1}) \hat{e}_{k+1/k}. \quad (25)
\]

Furthermore, substituting (24) into (25) yields

\[
\hat{e}_{k+1} = (I - \bar{K}_{k+1} \bar{C}_{k+1}). \quad (26)
\]

Decompose the posteriori estimate error and the augmented gain matrix as follows:

\[
\hat{e}_{k+1} = \begin{bmatrix} \hat{e}^d_{k+1} \\ \hat{e}^d_{k+1} \end{bmatrix}, \quad \bar{C}_{k+1} = \begin{bmatrix} K^d_{k+1} \\ K^d_{k+1} \end{bmatrix}. \quad (27)
\]

With (27), (26) can be reformulated as follows:

\[
\hat{e}^d_{k+1} = (I - K^d_{k+1} C_{k+1}). \quad (28)
\]

\[
(A_{k+1} \hat{e}^d_{k+1} + G_{k+1} \hat{e}^d_{k+1} - \bar{C}_{k+1} \hat{v}_{k+1}) = -K^d_{k+1} \tilde{G} \hat{e}^d_{k+1} + G_{k+1} \hat{e}^d_{k+1} \quad (29)
\]

Taking the expectation of (28) and (29) yields

\[
E(\hat{e}^d_{k+1}) = (I - K^d_{k+1} C_{k+1}). \quad (30)
\]

\[
(A_{k+1} E(\hat{e}^d_{k+1}) + G_{k+1} E(\hat{e}^d_{k+1})) \quad (31)
\]

\[
\delta_{k+1} = K^d_{k+1} G_{k+1} (A_{k+1} E(\hat{e}^d_{k+1}) + G_{k+1} E(\hat{e}^d_{k+1})). \quad (32)
\]
When time goes to infinite, i.e., $k \to \infty$, we have
\[
E(\tilde{e}_{k+1}^x) = (I - K_{k+1}^* C_{k+1}) \cdot \frac{1}{2}.
\]
\[
(K_{k+1}^d G_{k+1})^{-1} \delta_{k+1} d_k \neq 0.
\] (32)

Obviously, the state estimation is biased, which is mainly caused by the variation on the maneuver.

4. Partially norm constrained ASKF

4.1 Augmented performance index

In this section, in order to reduce the negative effects from the approximation in (4), major effort is dedicated to reconstructing (15) – (21) by incorporating the norm constraint on the maneuver. Define the augmented posteriori covariance matrix as follows:
\[
\mathbf{P}_{k+1}^d = E(\tilde{\epsilon}_{k+1}^d \tilde{\epsilon}_{k+1}^d)^T = \begin{bmatrix} P_{k+1}^{xx} & P_{k+1}^{xd} \\ P_{k+1}^{dx} & P_{k+1}^{dd} \end{bmatrix}
\]
where the partial constraint is defined as follows:
\[
P_{k+1}^{xx} = E(\tilde{e}_{k+1}^x (\tilde{e}_{k+1}^x)^T),
\]
\[
P_{k+1}^{dd} = E(\tilde{e}_{k+1}^d (\tilde{e}_{k+1}^d)^T),
\]
\[
P_{k+1}^{dx} = P_{k+1}^{dd} = E(\tilde{e}_{k+1}^d (\tilde{e}_{k+1}^x)^T).
\]

de (33)

Additionally, consider the normalized maneuver estimation, satisfying the constraint in (5),
\[
\hat{d}_{k+1} = \hat{d}_{k+1/k} = \rho^2.
\]

Discompose (19) as
\[
\begin{bmatrix} \hat{x}_{k+1} \\ \hat{d}_{k+1} \end{bmatrix} = \begin{bmatrix} \hat{x}_{k+1} \\ \hat{d}_{k+1} \end{bmatrix} + \begin{bmatrix} K_{k+1}^x \\ K_{k+1}^d \end{bmatrix} \eta_{k+1},
\]
we have
\[
\hat{d}_{k+1} = \hat{d}_{k+1/k} + K_{k+1}^d \eta_{k+1}.
\]

Substituting (39) into (27) leads to
\[
[\hat{d}_{k+1} + K_{k+1}^d \eta_{k+1}]^T [\hat{d}_{k+1} + K_{k+1}^d \eta_{k+1}] = \rho^2.
\] (40)

Thus with the consideration of the norm constraint, the optimal estimation can be obtained by minimizing the augmented objective function:
\[
J_{k+1} = E(\tilde{e}_{k+1}^d \tilde{e}_{k+1}^d)^T + \lambda_{k+1} \{[\hat{d}_{k+1/k} + K_{k+1}^d \eta_{k+1}]^T [\hat{d}_{k+1/k} + K_{k+1}^d \eta_{k+1}] - \rho^2\}.
\] (41)

Examine the right side of (41), it follows:
\[
E(\tilde{e}_{k+1}^d \tilde{e}_{k+1}^d)^T = tr(\mathbf{P}_{k+1}) = tr(P_{k+1}^{xx}) + tr(P_{k+1}^{dd})
\]
\[
(42)
where \( tr \{ \cdot \} \) denotes the trace of a matrix. For notational convenience, define the partial performance index as follows:
\[
J_{k+1}^e\cdot = tr(P_{k+1}^{xx}),
\]
\[
J_{k+1}^d\cdot = tr(P_{k+1}^{dd}) + \lambda_{k+1} \{[\hat{d}_{k+1/k} + K_{k+1}^d \eta_{k+1}]^T [\hat{d}_{k+1/k} + K_{k+1}^d \eta_{k+1}] - \rho^2\}.
\] (44)

4.2 Partially constrained estimator for system state

By minimizing \( J_{k+1}^e \), it is not difficult to construct the partial estimator for \( x_{k+1} \) based on the traditional Kalman filter routines. To save the length of the paper, we represent the estimation directly as follows:

(i) Prediction
\[
\hat{x}_{k+1/k} = A_k \hat{x}_k + G_k \hat{d}_k
\]
\[
P_{k+1/k} = (A_k P_{k+1}^{xx} + G_k P_{k+1}^{dd}) A_k^T +
\]
\[
(46)
\]
\[
(A_k \hat{P}_k + G_k \hat{d}_k) G_k^T + \Gamma_k Q \Gamma_k^T
\]
\[
P_{k+1}^{xx} = (P_{k+1/k}^{dd})^T = A_k \hat{P}_k + G_k \hat{P}_k^d + G_k^T \hat{P}_k^d
\]
\[
(47)
\]
\[
(48)
\]

(ii) Correction
\[
P_{k+1}^{xx} = P_{k+1/k}^{xx} - K_{k+1}^x C_{k+1} (P_{k+1/k}^{xx})^T -
\]
\[
P_{k+1/k}^{dd} C_{k+1}^T (K_{k+1}^x)^T + K_{k+1}^x C_{k+1}^T \Theta_{k+1}^{-1}
\]
\[
(49)
\]
\[
\hat{x}_{k+1/k} = \hat{x}_{k+1/k} + K_{k+1}^x \eta_{k+1}
\]
\[
(50)
\]
\[
\Theta_{k+1} = C_{k+1} P_{k+1/k}^{xx} C_{k+1}^T + R_{k+1}
\]
\[
(51)
\]

4.3 Partially constrained estimator for maneuver

(i) Minimization of \( J_{k+1}^d \)

Next, we focus our attention to construct the partial estimator for maneuver by minimizing the performance index \( J_{k+1}^d \), which can be treated as an extremum problem with a constraint. Based on (15) – (21), \( P_{k+1}^{dd} \) and \( P_{k+1}^{dd} \) can be computed as follows:
\[
P_{k+1}^{dd} = P_{k+1}^{dd}.
\] (53)
\[ P_{k+1}^{dd} = P_{k+1/k}^{dd} - K_{k+1}^d C_{k+1}^{T} (P_{k+1/k}^{dx})^T \]
\[ P_{k+1/k}^{dx} C_{k+1}^{T} (K_{k+1}^d)^T + K_{k+1}^d \Theta (K_{k+1}^d)^T. \]  \hspace{1cm} (54)

Consider the fact that
\[ \text{tr}(P_{k+1}^{dd}) = \text{tr}(P_{k+1/k}^{dd}) - \text{tr}(K_{k}^{d} \Theta (K_{k+1}^{d})^{T}) \]
\[ = \text{tr}(2K_{k+1}^{d} C_{k+1}^{T} (P_{k+1/k}^{dx})^{T}). \]  \hspace{1cm} (55)

The minimization of trace is equivalent to one of the co-variances, then substitute (55) into (44), take the partial derivative of (44) with respect to \( K_{k+1}^{d} \) and \( \lambda_{k+1} \) and set the result to zero, then the first-order condition for the minimization can be represented as follows:
\[ \frac{\partial J_{k+1}^d}{\partial K_{k+1}^{d}} = 0, \]  \hspace{1cm} (56)
\[ \frac{\partial J_{k+1}^d}{\partial \lambda_{k+1}} = 0. \]  \hspace{1cm} (57)

Expanding (56) and (57) leads to
\[ 0 = (P_{k+1/k}^{dx} C_{k+1}^{T} - \lambda_{k+1} \hat{d}_{k+1} \eta_{k+1}^{T}) - \]
\[ K_{k+1}^{d} (\Theta_{k+1} + \lambda_{k+1} \eta_{k+1}^{T} \eta_{k+1}^{T}), \]  \hspace{1cm} (58)
\[ 0 = \eta_{k+1}^{T} (K_{k+1}^{d})^{T} K_{k+1}^{d} \eta_{k+1}^{T} + 2\hat{d}_{k+1} K_{k+1}^{d} \eta_{k+1}^{T} \eta_{k+1}^{T} + \]
\[ \hat{d}_{k+1} \hat{d}_{k+1} - \rho^2. \]  \hspace{1cm} (59)

From (58), since \( \det(\Theta + \lambda_{k+1} \eta_{k+1}^{T} \eta_{k+1}^{T}) \neq 0 \) (proved in (75)), it follows
\[ K_{k+1}^{d} = (P_{k+1/k}^{dx} C_{k+1}^{T} - \lambda_{k+1} \hat{d}_{k+1} \eta_{k+1}^{T})^{-1} \]
\[ \Theta_{k+1} + \lambda_{k+1} \eta_{k+1}^{T} \eta_{k+1}^{T} \]  \hspace{1cm} (60)
and also, according to the principle of the matrix operation, we have
\[ (\Theta + \lambda_{k+1} \eta_{k+1}^{T} \eta_{k+1}^{T})^{-1} = \]
\[ \Theta^{-1} - \frac{\lambda_{k+1} \eta_{k+1}^{T} \eta_{k+1}^{T} \Theta^{-1}}{1 + \lambda_{k+1} \eta_{k+1}^{T} \eta_{k+1}^{T} \Theta^{-1}}. \]  \hspace{1cm} (61)

Substituting (61) into (60) leads to
\[ K_{k+1}^{d} = P_{k+1/k}^{dx} C_{k+1}^{T} \]
\[ \lambda_{k+1} \Theta^{-1} \eta_{k+1}^{T} \eta_{k+1}^{T} \Theta^{-1} \]
\[ + \frac{\lambda_{k+1} \eta_{k+1}^{T} \eta_{k+1}^{T} \Theta^{-1}}{1 + \lambda_{k+1} \eta_{k+1}^{T} \eta_{k+1}^{T} \Theta^{-1}}. \]  \hspace{1cm} (62)

Furthermore, substitute (62) into (59), a second order equation of \( \lambda_{k+1} \) is obtained as
\[ \lambda_{k+1}^2 + b \lambda_{k+1} + c = 0 \]  \hspace{1cm} (63)
where
\[ b = \frac{2}{\eta_{k+1}}, \] \hspace{1cm} (64)
\[ c = \frac{1}{\rho \eta_{k+1}}. \]  \hspace{1cm} (65)

By using Vieta theorem with (63), the optimal Lagrange multiplier can be obtained as follows:
\[ \lambda_{k+1} = \frac{-1}{\eta_{k+1}} \pm \frac{\|d_{k+1} + P_{k+1/k}^{dx} C_{k+1}^{T} \Theta_{k+1}^{-1} \eta_{k+1}^{T}\|}{\eta_{k+1} \rho} \]  \hspace{1cm} (66)
\[ \eta_{k+1} = \eta_{k+1}^{T} \Theta_{k+1}^{-1} \eta_{k+1}. \]  \hspace{1cm} (67)

Equation (66) indicates that there are two possible solutions for the optimal Lagrange multiplier, to determine the sign for the minimization, the second order condition should be examined. In general, the second order condition can be obtained by taking the second order derivative on the performance index, however, it will lead to the \( m \times n \) dimensional matrix equation and significantly increases the computational complexities. Without the loss of generality, we consider the case that the gain \( K_{k+1}^{d} \) is a column vector, where the measurement is reduced to a scalar, but the obtained results are still identical for the vector measurement.

Generally, the second order condition of an extremal problem can be found by evaluating its Hessian matrix. The performance index achieves the maximum when the Hessian matrix is negative definite, while the minimum when the Hessian matrix is positive definite. In the following, we will show the performance index \( J_{k+1}^d \) is minimized when choosing the sign as plus in (66), while is maximized when choosing minus. When the gain \( K_{k+1}^{d} \) is a column vector, the Hessian matrix of the performance index \( J_{k+1}^d \) can be obtained by differentiating (56) again
\[ \frac{\partial}{\partial K_{k+1}^{d}} \left( \frac{\partial J_{k+1}^d}{\partial K_{k+1}^{d}} \right) = \]
\[ \left( \begin{array}{cc}
\Theta + \lambda_{k+1} \eta_{k+1}^2 & 0 \\
0 & \Theta + \lambda_{k+1} \eta_{k+1}^2
\end{array} \right). \]  \hspace{1cm} (68)

Equation (68) shows that the Hessian matrix is diagonal, with (66), we have
\[ \Theta + \lambda_{k+1} \eta_{k+1}^2 = \Theta + \left( \frac{1}{\eta_{k+1}} \pm \left( \frac{d_{k+1} + P_{k+1, k} C_T^T \Theta_{k+1}^{-1} \eta_{k+1}}{\eta_{k+1}^2} \right) \right) \eta_{k+1}^2. \]  

(69) 

With (67), (69) can be rewritten as follows:

\[ \Theta + \lambda_{k+1} \eta_{k+1}^2 = \Theta + \left( -\Theta \frac{1}{\eta_{k+1}} \pm \frac{\|d_{k+1} + P_{k+1, k} C_T^T \Theta_{k+1}^{-1} \eta_{k+1}\|}{\eta_{k+1}^2} \right) \eta_{k+1}^2 = \pm \Theta \frac{1}{\eta_{k+1}} \left( \eta_{k+1} \right) \eta_{k+1}^2. \]  

(70) 

Return to (52), it is not difficult to get

\[ \Theta = C_{k+1} \Theta_{k+1}^T (\Theta_{k+1}^T)^T + R_{k+1} = (C_{k+1} \Theta_{k+1}^T)^T (C_{k+1} \Theta_{k+1}^T) + R_{k+1} > 0 \]  

(71) 

where \( R_{k+1} \) is the covariance of the scalar measurement noise. Furthermore, with (70) and (71), it clearly reveals that when the plus sign is chosen, the Hessian matrix is positive definite, and the minimum performance index occurs. Conversely, if the minus sign is chosen, the performance index is maximized.

In order to prove that the matrix \( \Theta + \lambda_{k+1} \eta_{k+1} \eta_{k+1}^T \) is positive definite when choosing the plus sign in (66), rewrite the term \( \Theta + \lambda_{k+1} \eta_{k+1} \eta_{k+1}^T \) as follows:

\[ \Sigma = \Theta + \lambda_{k+1} \eta_{k+1} \eta_{k+1}^T = \Theta + \left( \frac{1}{\eta_{k+1}} + \frac{\|d_{k+1} + P_{k+1, k} C_T^T \Theta_{k+1}^{-1} \eta_{k+1}\|}{\eta_{k+1}^2} \right) \eta_{k+1} \eta_{k+1}^T. \]  

(72) 

Multiply \( \eta_{k+1} \) on both sides and we have

\[ \eta_{k+1}^T \Sigma \eta_{k+1} = \eta_{k+1}^T \Theta \eta_{k+1} + \left( \frac{1}{\eta_{k+1}} + \frac{\|d_{k+1} + P_{k+1, k} C_T^T \Theta_{k+1}^{-1} \eta_{k+1}\|}{\eta_{k+1}^2} \right) \eta_{k+1} \eta_{k+1}^T \| \eta_{k+1} \|^4. \]  

(73) 

Further, with (67), multiply \( \eta_{k+1} \) on both sides of (73) and yield

\[ \eta_{k+1}^T \Sigma \eta_{k+1} = \| \eta_{k+1} \|^4 + \left( \frac{1}{\eta_{k+1}} \frac{\|d_{k+1} + P_{k+1, k} C_T^T \Theta_{k+1}^{-1} \eta_{k+1}\|}{\rho} \right) \| \eta_{k+1} \|^4 = \| \frac{d_{k+1} + P_{k+1, k} C_T^T \Theta_{k+1}^{-1} \eta_{k+1}}{\rho} \| \| \eta_{k+1} \|^4. \]  

(74) 

Since \( \Theta_{k+1} \) is positive, then we have \( \Sigma \eta_{k+1} > 0 \), finally, from (73) it can be obtained as follows:

\[ \eta_{k+1}^T \Sigma \eta_{k+1} = \| \frac{d_{k+1} + P_{k+1, k} C_T^T \Theta_{k+1}^{-1} \eta_{k+1}}{\rho} \| \| \eta_{k+1} \|^4 > 0. \]  

(75) 

Naturally, (75) indicates that \( \Theta + \lambda_{k+1} \eta_{k+1} \eta_{k+1}^T \) is positive definite, as a result of which, \( \det(\Theta + \lambda_{k+1} \eta_{k+1} \eta_{k+1}^T) > 0. \)

(ii) Constrained Kalman gain and posterior covariance

The optimal Lagrange multiplier for the minimum performance index can be represented as follows:

\[ \lambda_{k+1} = \frac{-1}{\eta_{k+1}^T} + \frac{\|d_{k+1} + P_{k+1, k} C_T^T \Theta_{k+1}^{-1} \eta_{k+1}\|}{\eta_{k+1}^T \rho}. \]  

(76)

Then substitute (76) into (62) and lead to

\[ \tilde{K}_{k+1}^d = P_{k+1, k} C_T^T \Theta_{k+1}^{-1} + \delta K_{k+1}^d \]  

(77) 

where \( \tilde{K}_{k+1}^d \) is the partial Kalman gain matrix related to the disturbance, and

\[ \delta K_{k+1}^d = \left( \rho - \frac{\|d_{k+1}\|}{\|d_{k+1}\|} \right) \frac{\rho}{\|d_{k+1}\|} \eta_{k+1} \Theta_{k+1}^{-1} \eta_{k+1}. \]  

(78)

According to the Kalman filter algorithm, the constrained maneuver estimation can be obtained by correcting the one-step prediction as follows:

\[ \tilde{d}_{k+1} = \tilde{d}_{k+1} + \left( P_{k+1, k} C_T \Theta_{k+1}^{-1} + \delta K_{k+1}^d \right) \eta_{k+1} = \left( \rho - \frac{\|d_{k+1}\|}{\|d_{k+1}\|} \right) \frac{\rho}{\|d_{k+1}\|} \eta_{k+1} \Theta_{k+1}^{-1} \eta_{k+1} = \left( \rho - \frac{\|d_{k+1}\|}{\|d_{k+1}\|} \right) \frac{\rho}{\|d_{k+1}\|} \eta_{k+1} \Theta_{k+1}^{-1} \eta_{k+1} \]  

(80)

where \( \tilde{d}_{k+1} \) is the unconstrained estimation. Obviously, the constrained estimation is normalized to satisfy the constraint in (37), and also, the partial constrained Kalman gain projects the estimation to the m-dimensional Euclidean surface spanned by the constraint of \( d_{k+1}^2 \eta_{k+1} = \rho^2 \), but along the direction of unconstrained ones.
Furthermore,
\[
\tilde{P}^{dd}_{k+1} = E\{e_{k+1}^d (e_{k+1}^d)^T\} = E\{(d_{k+1} - d_{k})(d_{k+1} - d_{k})^T\}
\]
where
\[
d_{k+1} = \hat{d}_{k+1} + K_{k+1}^d \begin{bmatrix} \hat{x}_{k+1}^d \\ \hat{d}_{k+1} \end{bmatrix} - C_{k+1}^d \begin{bmatrix} x_{k+1} \\ d_{k+1} \end{bmatrix} - \eta_{k+1} = \hat{d}_{k+1} + K_{k+1}^d C_{k+1}^T \hat{e}_{k+1} - K_{k+1}^d \eta_{k+1}. \]
(82)

Substitute (82) into (51) and lead to
\[
\tilde{P}^{dd}_{k+1} = (\hat{e}_{k+1}^d + K_{k+1}^d C_{k+1}^T \hat{e}_{k+1} - K_{k+1}^d \eta_{k+1})^T. \]
(83)

Because the measurement noise is always uncorrelated with the \(e_{k+1}^d\) and \(\hat{e}_{k+1}^d\), thus (83) can be reformulated by canceling the term related to \(e_{k+1}^T \hat{d}_{k+1}\) and \(\hat{e}_{k+1}^T \eta_{k+1}\).
\[
\tilde{P}^{dd}_{k+1} = \hat{e}_{k+1}^d (\hat{e}_{k+1}^d)^T + \hat{e}_{k+1}^d (K_{k+1}^d C_{k+1}^T \hat{e}_{k+1})^T + \hat{e}_{k+1}^d (K_{k+1}^d)^T + K_{k+1}^d R (K_{k+1}^d)^T + K_{k+1}^d C_{k+1}^T \hat{e}_{k+1} (K_{k+1}^d C_{k+1}^T \hat{e}_{k+1})^T.
\]
(84)

Finally, we have
\[
\tilde{P}^{dd}_{k+1} = P^{dd}_{k+1} - K_{k+1}^d C_{k+1}^T (P^{dd}_{k+1})^{-1} K_{k+1}^d C_{k+1}^T (K_{k+1}^d)^T + K_{k+1}^d \Theta (K_{k+1}^d)^T. \]
(85)

Equation (85) indicates that the constrained posterior covariance can be updated according to (54), but replacing the unconstrained gain as the constrained ones.

4.4 Performance analysis

Similarly, the constrained co-variance is updated as follows:
\[
\begin{align*}
\tilde{P}^{dd}_{k+1} &= (P^{dd}_{k+1})^T = P^{dd}_{k+1} - K_{k+1}^d C_{k+1}^T (P^{dd}_{k+1})^{-1} K_{k+1}^d C_{k+1}^T (K_{k+1}^d)^T + K_{k+1}^d \Theta (K_{k+1}^d)^T. \end{align*}
\]
(86)

With (80), we have
\[
\|e_{k+1}^d\|^2 = (\hat{e}_{k+1}^d)^T e_{k+1}^d = \left( \frac{\rho}{\|\hat{d}_{k+1} - d_{k+1}\|} \right)^T (\hat{e}_{k+1}^d). \]
(87)

and also
\[
\|\hat{e}_{k+1}^d\|^2 = (\hat{e}_{k+1}^d)^T \hat{e}_{k+1}^d = (d_{k+1} - \hat{d}_{k+1})^T (d_{k+1} - \hat{d}_{k+1}) = (\hat{d}_{k+1} - \hat{d}_{k+1})^T (\hat{d}_{k+1} - \hat{d}_{k+1}) + \rho^2. \]
(88)

Subtracting (88) from (87), if \(\|\hat{d}_{k+1}\| > \rho\), it leads to
\[
\|d_{k+1}^d\|^2 - \|\hat{d}_{k+1}^d\|^2 = 2 \left( \rho^2 - \frac{\rho}{\|\hat{d}_{k+1}\|} \hat{d}_{k+1}^T d_{k+1} + \rho^2 \right) = (\rho^2 - \|\hat{d}_{k+1}\|)^2 + 2(\hat{d}_{k+1}^T d_{k+1} - \rho \|\hat{d}_{k+1}\|). \]
(89)

According to Cauchy-Schwarz inequality, we have
\[
\|\hat{d}_{k+1}^T d_{k+1}\| < \|\hat{d}_{k+1}\| \|d_{k+1}\| = \rho \|\hat{d}_{k+1}\|. \]
(90)

Thus we have \(\|e_{k+1}^d\|^2 - \|\hat{e}_{k+1}^d\|^2 < 0\), naturally, it leads to
\[
\|d_{k+1}^d\| < \|\hat{d}_{k+1}^d\|. \]
(91)

The result indicates that if \(\|\hat{d}_{k+1}\| > \rho\), the constrained estimator produces a result more accurate than the unconstrained estimator.

Rewrite (89) as follows:
\[
\|d_{k+1}^d\|^2 - \|\hat{d}_{k+1}^d\|^2 = (\rho^2 - \|\hat{d}_{k+1}\|^2) + 2(\hat{d}_{k+1}^T d_{k+1} - \rho \|\hat{d}_{k+1}\|) = \rho^2 - 2\rho \|\hat{d}_{k+1}\| + \|\hat{d}_{k+1}\|^2 + 2(\hat{d}_{k+1}^T e_{k+1}^d) = (\rho^2 - \|\hat{d}_{k+1}\|^2) + 2(\hat{d}_{k+1}^T e_{k+1}^d). \]
(92)

From (91), if \(\hat{d}_{k+1}^T \) is nearly orthogonal to \(e_{k+1}^d\), then \(\hat{d}_{k+1}^T e_{k+1}^d = 0\), finally it leads to
\[
\|e_{k+1}^d\| > \|\hat{d}_{k+1}^d\|. \]
(93)

In the case of \(\|\hat{d}_{k+1}\| < \rho\), if the unconstrained estimation is nearly orthogonal to the unconstrained estimate error, then we have \(\|e_{k+1}^d\| > \|\hat{e}_{k+1}^d\|\). The above results reveal that if \(\|\hat{d}_{k+1}\| < \rho\), the constrained estimator does not always achieve a better performance than the unconstrained one. In order to ensure the performance promotion, we define a scale factor as follows:
\[
\kappa_k = \frac{\|\hat{d}_{k+1}\|}{\rho}, \quad 0 \leq \kappa_k \leq 1. \]
(94)

Obviously, \(\kappa_k\) is also a representation of the estimation error, the smaller \(\kappa_k\), the larger the error. Thus by selecting appropriate a threshold scale factor \(\kappa_T\), it can be confirmed
that only if \( \kappa_k > 1 \) or \( 0 \leq \kappa_k < \kappa_T \), the constrained estimator works with a better performance than the unconstrained one, otherwise the unconstrained estimator maintains an advantage over the constrained one. Moreover, it is easily to understand that the constrained estimate bias of the system state will be reduced when the maneuver is estimated with a better performance, as a result of which, the estimation accuracy of the system state is consequently improved.

### 4.5 Summary

The constrained estimation can be performed in the centralized way, however, the increased dimension of the centralized estimator will increase the computational load. With a purpose of reducing the computational complexity, it is recommended to propagate by the parallel way, in which, the centralized estimator is separated into a set of partial estimators of lower dimensions, the summary of the constrained estimator is shown in Table 1.

**Table 1 Summary of the constrained estimator**

| Initialization | \( x_{0/0} = E(x_0) \) | \( d_{0/0} = E(d_0) \) | \( E((x_0 - \tilde{x}_0)(x_0 - \tilde{x}_0)^T) = P^{xx}_{0/0} \) | \( E((x_0 - \tilde{x}_0)(d_0 - \tilde{d}_0)^T) = P^{xb}_{0/0} \) | \( E((d_0 - \tilde{d}_0)(d_0 - \tilde{d}_0)^T) = P^{dd}_{0/0} \) |
|---|---|---|---|---|---|

**\( \tilde{x}_{k+1/k} = A_k \tilde{x}_k + G_k d_k \)**

\( P^{xx}_{k+1/k} = (A_k \, P^{xx}_{k} + G_k \, P^{sd}_{k}) \, A_k^T + G_k \, P^{dd}_{k} \, G_k^T \)

\( P^{sd}_{k+1/k} = P^{sd}_{k} - K^{x}_{k+1/k} \, C^{d}_{k+1} \, (P^{xx}_{k} + K^{x}_{k+1/k} \, C^{x}_{k+1})^{-1} \)

\( P^{dd}_{k+1/k} = P^{dd}_{k} + K^{d}_{k+1/k} \, C^{d}_{k+1} \, \Theta_k ^{-1} \)

\( \tilde{d}_{k+1/k} = \tilde{d}_{k+1/k} + K^{d}_{k+1/k} \eta_{k+1} \)

\( \sigma_{k+1} \)
| \( \kappa_k > 1 \) or \( \kappa_k < \kappa_T \) |

**5. Numerical results**

As an illustrative example, the orbit determination of a non-cooperative spacecraft by using the proposed estimator will be simulated. Moreover, with a purpose of comparison, the unconstrained estimation results are also presented. The spacecraft is supposed to be three-axis stabilized and equipped with a constant magnitude apoapge engine. During orbit maneuvering, the apoapge engine always generates thrust acceleration in the along-track direction. For simplicity, we suppose the spacecraft initially works in a circular orbit, the orbit angular velocity, which is completely unknown to the ground station, is denoted as \( \omega \), then with the earth centered inertial frame, the thrust acceleration can be described as follows:

\[
d = \begin{bmatrix} \frac{d_x}{M} \\ \frac{d_y}{M} \\ \frac{d_z}{M} & \cos(\Omega t + \varphi_0) \\ 0 \\ \end{bmatrix}
\]

where \( M \) is the well-known constant thrust force, \( \varphi_0 \) is the initial phase, while \( M \) is the mass of the spacecraft. Obviously, the thrust acceleration unknowingly varies via time, but always with constant magnitude. The following assumptions also hold: (i) The true orbit dynamics are only perturbed process noise. (ii) The attitude of spacecraft is properly controlled. (iii) The spacecraft is continuously tracked by ground stations.

The estimation performance, which is adopted to evaluate the relationship between constrained estimation accuracy and unconstrained ones quantitatively, is defined as follows:

\[
\mu(k) = \frac{100}{k} \sum_{i=1}^{k} \frac{\delta_i^U}{(\delta_i^C)^T} \times 100\%
\]

where \( \delta_i^U \) and \( \delta_i^C \) denote unconstrained and constrained estimation errors of a specific component. The performance is a ratio between the average root mean square error (RMSE) that represents a relative accuracy. If the performance is larger than 100%, it indicates that the constrained filter works with a better RMSE performance than the unconstrained one; otherwise, the constrained estimator suffers from the performance degradation.

### 5.1 Dynamics and observations

#### 5.1.1 Orbit dynamics

Generally, the orbit motion of the spacecraft is described by using the two-body model. When the spacecraft undergoes an unknown maneuver, the motion can be represented.
with the linear perturbing acceleration within its dynamics
\[
\frac{d\vec{r}}{dt} = -\frac{\mu_e}{r^3}\vec{r} + \vec{d} + \vec{w} \tag{97}
\]
where \(\mu_e\) denotes the earth gravitational coefficient. \(\vec{r} = [x, y, z]^T\) and \(\vec{v} = [\dot{x}, \dot{y}, \dot{z}]^T\) are the orbital position and velocity vectors. The vector of \(\vec{w}\) denotes the normally distributed process noises acting on the spacecraft. All of the vectors in (97) are represented with respect to the ECI frame.

Suppose the spacecraft is tracked by ground-based phased-array radar, which is fixed on the earth’s surface and follows the spinning of the earth, measures the spacecraft motion within radar-fixed range and angle coordinates. The nonlinear measurement model is represented as follows:
\[
z_k = h(x_{SEZ}(k)) + \nu_k \tag{104}
\]
where \(z_k = [\rho_{sk}, \dot{\rho}_{sk}, \varphi_k, \gamma_k]^T\) is the measurement vector that includes the components of the range, range rate, azimuth, elevation and measurement uncertainties, \(x_{SEZ}(k)\) is the position vector within the local east-south-zenith (SEZ) frame and expressed as
\[
x_{SEZ}(k) = [\rho_s, \dot{\rho}_s]^T \tag{105}
\]
with
\[
\rho_s = [x_{SEZ}(k) \quad y_{SEZ}(k) \quad z_{SEZ}(k)]
\]
\[
\dot{\rho}_s = [\dot{x}_{SEZ}(k) \quad \dot{y}_{SEZ}(k) \quad \dot{z}_{SEZ}(k)]^T
\]
and
\[
h(x_{SEZ}(k)) = \begin{bmatrix} \sqrt{x_{SEZ}^2 + y_{SEZ}^2 + z_{SEZ}^2} \\
\sqrt{x_{SEZ}^2 + y_{SEZ}^2 + z_{SEZ}^2} \\
a \tan \left( \frac{\rho_s}{\rho_s} \right) \\
a \sin \left( \frac{z_{SEZ}}{\rho_s} \right)
\end{bmatrix} \tag{106}
\]
Generally, the vector
\[
r_{SEZ}(k) = [x_{SEZ}(k) \quad y_{SEZ}(k) \quad z_{SEZ}(k)]^T
\]
could be obtained by rotating the vector \(\vec{r}\) from the ECI frame into the earth-centered earth-fixed (ECEF), and then into the SEZ frame as follows:
\[
\rho = T_{ECEF\rightarrow SEZ}(T_{ECI\rightarrow ECEF}\vec{r} - p_{ECEF}^{ECEF}) \tag{107}
\]
where \(T_{ECI\rightarrow ECEF}\) is the rotation matrix from the ECI frame to the ECEF frame, while \(T_{ECEF\rightarrow SEZ}\) is the one from the ECEF frame to the SEZ frame, while \(p_{ECEF}^{ECEF}\) represents the location of the radar within the ECEF frame. With a similar rotation, the velocity vector can also be expressed as
\[
\dot{\rho}_s = T_{ECEF\rightarrow SEZ}\dot{T}_{ECI\rightarrow ECEF}\vec{r} + T_{ECEF\rightarrow SEZ}\dot{\vec{T}}_{ECI\rightarrow ECEF}\vec{r}. \tag{108}
\]
We then have
\[
x_{SEZ}(k) = \Pi x(k) + A\nu_{Radar} \tag{109}
\]
where
\[
\Pi = \begin{bmatrix} T_{ECEF\rightarrow SEZ}T_{ECI\rightarrow ECEF} \\
T_{ECEF\rightarrow SEZ}T_{ECI\rightarrow ECEF} \\
T_{ECEF\rightarrow SEZ}T_{ECI\rightarrow ECEF} \end{bmatrix}
\]
\[
A = \begin{bmatrix} T_{ECEF \rightarrow SEF} \\ 0 \end{bmatrix}.
\] (111)

Substitute (109) into (106), then the measurement can be represented in terms of \( x(k) \). Furthermore, linearize the measurement in the vicinity of state prediction and yield
\[
z_{k+1} = h(\tilde{x}_k) + H_{k+1}(\tilde{x}_{k+1/k} - \tilde{x}_k) + \nu_{k+1}
\] (112)

where \( H_k \) is the measurement Jacobian matrix. It can be calculated as follows:
\[
H_k = \left. \frac{\partial h(x_k)}{\partial x_k} \right|_{x = \tilde{x}_k} = \left. \frac{\partial h(x_{SEZ}(k))}{\partial x_{SEZ}(k)} \right|_{x = \tilde{x}_k}.
\]
\[
\left. \frac{\partial x_{SEZ}(k)}{\partial x_k} \right|_{x = \tilde{x}_k} = \Xi \cdot H.
\] (113)

5.2 Numerical results

Consider a spacecraft in an equatorial orbit initially. The orbit eccentricity and altitude are 0 and 600 km, respectively. The mass of the spacecraft is 1 000 kg, and the constant thruster with a magnitude of 20 N is fixed on the spacecraft. During the entire simulation, the thruster always works to generate the sinusoidal maneuvering acceleration within the \( x-y \) plane. The initial conditions for the estimation are listed as follows: the initial orbit state \( x_0 = \{6 978 000 \, \text{m}, 0 \, \text{m}, 0 \, \text{m/s}, 7 557.9 \, \text{m/s}, 0 \, \text{m/s}^2\} \), the initial maneuver \( d_0 = \{0 \, \text{m/s}^2, 0 \, \text{m/s}^2, 0 \, \text{m/s}^2\} \), the initial covariance for the state error is of \( P_0 = 10^6 \times I_{6 \times 6} \), while \( P_0^{dd} = 10^6 \times I_{3 \times 3} \) for the unknown maneuver acceleration. The covariance of the dynamic noise is set to \( Q = 10^{-12} \times I_{3 \times 3} \), while \( R = \text{diag}([40000, 400, 7.62 \times 10^{-9}, 7.62 \times 10^{-9}]) \) for measurement uncertainties. The estimator works at a frequency of 1 Hz, and the simulation totally takes one period of the initial circular orbit.

The unconstrained maneuver estimations along both directions are represented in Fig. 1, where the dash lines denote the true maneuvers, while the solid lines are the estimated results. From the left plots, one can see that the unconstrained estimator suffers from dramatic oscillations due to the initial errors, after the initial oscillations, the estimated maneuvers follow the true ones but with a remarkable time delay, which can be explained by the fact that the unconstrained estimator always suppose the maneuvering acceleration to be constant, therefore the unconstrained estimator cannot reflect the variation of the maneuvers in time. The right plots show the maneuver estimation error on the maneuvers, the results reveal that the errors along both directions liberate throughout the simulations, and the maximum errors are all greater than 0.005 m/s².

The most interesting results are shown in Fig. 2. Compared with the results in Fig. 1, it clearly reveals that once the norm constraint is applied to the maneuver estimation, the unconstrained estimations are normalized to preserve the magnitude of the maneuver, and the constrained estimator becomes more sensitive to the variation of the maneuver, as a result of which, the drastic oscillations at the beginning are significantly suppressed, moreover, the time delays of the estimation, as well as the estimation errors, are both reduced.

Fig. 3 shows the magnitudes of the estimated maneuvering accelerations. It clearly shows in the case of the unconstrained estimation, the estimated maneuver dramatically oscillates within the first 100 s, but when the estimation reaches a relatively stable stage, the estimated magnitude is smaller than 0.02 for most of the time. However, for the constrained estimation, because we select the parameter \( \kappa_T = 0.975 \), the magnitude of the maneuvering acceleration is projected into a range of \([0.0195, 0.02] \), which
makes the estimated maneuvering acceleration closer to the true ones.

The estimation errors on the orbit position and velocity are represented in Fig. 4 and Fig. 5, where the solid lines represent the estimation errors, while the dash lines represent the \( \sigma \) boundary. Due to the time delays of the maneuvering acceleration estimation, the estimation errors of both constrained and unconstrained cases suffer from the initial librations. After reaching a relatively stable stage, the unconstrained estimator approximately achieves an accuracy of 30 m (\( \sigma \)) along both directions, while the constrained estimator achieves a better accuracy of about 20 m (\( \sigma \)), and also, the accuracies on the velocity are promoted after applying the norm constraint on the maneuver, as shown in the right plots within each figure. The results clearly reveal that the estimation accuracy is improved after incorporating the constant norm constraint on the unknown maneuvering acceleration.

We also increase the maneuvering acceleration from 0.02 m/s\(^2\) to 0.05 m/s\(^2\), the estimation errors on the orbit position and velocity are represented in Fig. 6 and Fig. 7. It is obvious that we get the similar results but with the increased estimation error. After reaching a relatively stable stage, the unconstrained estimator approximately achieves an accuracy of 55 m (\( \sigma \)) along both directions, while the constrained estimator achieves a better accuracy of about 40 m (\( \sigma \)). The results clearly reveal that the estimation accuracy will be degraded after increasing the magnitude of the unknown maneuvering acceleration.

Fig. 3 Norm of estimated maneuver

Fig. 4 Position errors of unconstrained estimation

Fig. 5 Position errors of constrained estimation

Fig. 6 Position errors of unconstrained estimation after increasing maneuvering acceleration

Fig. 8 shows the estimation performance of the constrained estimator relative to the unconstrained ones. One can see that at the beginning of the simulation, the estima-
tion performance along both directions dramatically oscillates, which are mainly caused by the initial convergence of the both estimators, but after about 10 s, the performances are always greater than 1, and then greater than 1.2 after about 1 500 s, finally it achieves the maximum values of 1.5 and 1.4 along x and y directions, respectively. Based on the results in Fig. 8, it can be confirmed that the constrained estimator obtains an enhanced performance than the unconstrained ones.

![Constrained estimation error of position](image1)

![Constrained estimation error of velocity](image2)

**Fig. 7** Position errors of constrained estimation after increasing maneuvering acceleration

**Fig. 8** Estimation performance of constrained estimator

6. Conclusions

The primary contribution of this work is the development of the partially norm-preserving augmented estimator. The proposed estimator is implemented to track spacecraft subjected to the time-varying maneuver but of constant magnitude. Based on the framework of the traditional augmented Kalman filter, the norm constraint is incorporated into the estimation, the estimation errors are minimized by using the Lagrangian multiplier technique, and with the constrained Kalman gain matrix, the estimation is projected to the Euclidian surface spanned by the norm constraint. Finally, to highlight the advantages of the proposed estimator, both the unconstrained and constrained estimators are simulated to track a maneuvering spacecraft. Numerical results indicate that the proposed estimator achieves an enhanced performance.

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