SYMPLECTIC RIGIDITY OF FIBERS IN COTANGENT BUNDLES OF RIE-MMANN SURFACES

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ABSTRACT. We study symplectic rigidity phenomena for fibers in cotangent bundles of Riemann surfaces. Our main result can be seen as a generalization to open Riemann surfaces of arbitrary genus of work of Eliashberg and Polterovich on the Nearby Lagrangian Conjecture for $T^*\mathbb{R}^2$. As a corollary, we answer a strong version in dimension $2n = 4$ of a question of Eliashberg about linking of Lagrangian disks in $T^*\mathbb{R}^n$, which was previously answered by Ekholm and Smith in dimensions $2n \geq 8$.

1. Introduction

Let $\Sigma$ be a connected, open Riemann surface of finite type and genus $g \geq 0$. Let $F_x \subset (T^*\Sigma, d\lambda_{can})$ be the cotangent fiber over $x \in \Sigma$. We prove the following theorem.

Theorem 1.1. Let $L \subset T^*\Sigma$ be a Lagrangian submanifold which is diffeomorphic to $\mathbb{R}^2$ and which agrees outside a compact set with a fiber $F_x$ for $x \in \Sigma$. Then $L$ is Hamiltonian isotopic to $F_x$ through a compactly-supported Hamiltonian isotopy.

The motivation for Theorem 1.1 is to exhibit symplectic rigidity phenomena for fibers in cotangent bundles. Observe that cotangent bundles have two distinguished classes of Lagrangians, namely the zero section and the fibers. Arnold’s celebrated Nearby Lagrangian Conjecture can be understood as a symplectic rigidity statement for the zero section. One would naturally like to understand what sorts of symplectic rigidity phenomena hold for cotangent fibers.

It follows from work of Eliashberg–Polterovich [6] that any Lagrangian submanifold of $T^*\mathbb{R}^2$ which is diffeomorphic to $\mathbb{R}^2$ and agrees outside a compact set with a fiber is Hamiltonian isotopic to this fiber. In fact, due to the symmetry of $T^*\mathbb{R}^2$, this is easily seen to be equivalent to the Nearby Lagrangian Conjecture for $T^*\mathbb{R}^2$, which is the statement that Eliashberg and Polterovich originally proved. From this perspective, Theorem 1.1 can be viewed as a generalization of the work of Eliashberg–Polterovich. In particular, Theorem 1.1 recovers the Nearby Lagrangian Conjecture for $T^*\mathbb{R}^2$ as a special case.

One intriguing source of symplectic rigidity phenomena for cotangent fibers, which was originally promoted by Eliashberg, comes from studying linking. Observe that if $L$ is a Lagrangian embedding of $\mathbb{R}^n$ into $T^*\mathbb{R}^{2n}$ which agrees with $F_x$, $x \neq 0$, outside a compact set and is disjoint from $F_0$, then it extends to a map $S^n \to T^*\mathbb{R}^n - F_0 \to \mathbb{R}^n - 0$ where the second map is the projection. Eliashberg asked whether this composition is nullhomotopic. This question was affirmatively answered by Ekholm and Smith in dimensions $2n \geq 8$; see [5, Thm. 1.1].

Observe that Eliashberg’s question is essentially asking whether $L$ can be homotopically linked with $F_0$. It is therefore a statement of homotopical rigidity. The work of Ekholm and Smith can thus be seen as complementary to the recent developments establishing homotopical versions of the Nearby Lagrangian Conjecture; cf. [1] and the references therein.

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In dimension 4, the full Nearby Lagrangian Conjecture is known for some low genus cases, namely for $T^*\mathbb{R}^2$, $T^*(S^1 \times \mathbb{R})$, $T^*S^2$ and $T^*\mathbb{T}^2$; see [3, 4, 6, 7]. By analogy, one could therefore hope to establish a rigidity statement for cotangent fibers along the lines of Eliashberg’s question which would hold in the symplectic category, as opposed to a purely homotopical statement. (Note that Eliashberg’s original question is in fact trivial in dimension $2n = 4$ since $\pi_2(\mathbb{R}^2 - 0) = 0$.)

The following corollary of Theorem 1.1 provides an essentially optimal result in this direction.

**Corollary 1.2.** Let $\Sigma$ be a (closed or open) Riemann surface of finite type and genus $g \geq 0$. Let $L$ be a Lagrangian submanifold which is diffeomorphic to $\mathbb{R}^2$ and agrees outside a compact with some cotangent fiber $F_y$, $y \in \Sigma$. If $L \cap F_x = \emptyset$ for some $x \in \Sigma, x \neq y$, then $L$ is isotopic to $F_y$ in the complement of $F_x$ though a compactly-supported Hamiltonian isotopy.

Corollary 1.2 says essentially that $L$ is “Hamiltonian unlinked” from $F_x$. In particular, in the special case where $\Sigma = \mathbb{R}^2$, it affirmatively answers what is presumably the strongest-possible version of Eliashberg’s question in dimension $2n = 4$. Note that Corollary 1.2 can be immediately deduced from Theorem 1.1 by removing $F_x$.

Taking a slightly different perspective on Corollary 1.2, observe that it also implies the following: a Lagrangian $L$ which is diffeomorphic to $\mathbb{R}^2$ is Hamiltonian isotopic to a fiber if it can be displaced from a single other fiber. This hypothesis is of course necessary. For example, if we let $\tau(F_y)$ be the Dehn twist about the zero section of a cotangent fiber $F_y \subset T^*S^2$ for some $y \in S^2$, then $\tau(F_y)$ and $F_y$ are obviously not isotopic via a compactly-supported Hamiltonian isotopy. It would be interesting to know whether the conclusion of Corollary 1.2 still holds under the weaker assumption that $HF^*(L,F_x) = 0$.

To the best of our knowledge, Corollary 1.2 is the first result which describes Lagrangian submanifolds up to Hamiltonian isotopy in cotangent bundles of closed Riemann surfaces of genus $g \geq 2$. In particular, the Nearby Lagrangian Conjecture is still open for such surfaces. Holomorphic curve techniques have been particularly powerful for studying Lagrangian submanifolds in symplectic 4-manifolds. However, these techniques have proved to be difficult to apply in cases, such as cotangent bundles of Riemann surfaces of genus $g \geq 2$, where the symplectic manifold of interest does not come equipped with a natural foliation by pseudo-holomorphic curves. Our arguments do not require such a foliation and are therefore more widely applicable.

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2. Proof of Theorem 1.1

As in the statement of Theorem 1.1, we let $\Sigma$ be a non-compact Riemann surface of finite type and genus $g \geq 0$ (i.e. $\Sigma$ is obtained by removing a positive and finite number of points from a genus $g$ surface). We let $L \subset (T^*\Sigma,d\lambda_{can})$ be a Lagrangian submanifold which is diffeomorphic to $\mathbb{R}^2$ and agrees outside a compact set with a fiber $F_x$ for some $x \in \Sigma$.

We let $(\mathbb{R}^4,\omega)$ be the standard symplectic vector space, where $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ with respect to the coordinates $(x_1,y_1,x_2,y_2)$.

2.1. **A standard model for $\Sigma$.** It will be convenient to work with a standard model for the abstract Riemann surface $\Sigma$. Since $\Sigma$ is of finite type, its Euler characteristic is well-defined and given by the formula $\chi(\Sigma) = 2 - 2g - p$, where $p \geq 0$ is the number of punctures.
For $j = 1, 2, \ldots, 2(2g + p)$, let
\[ I_j = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \in [j - 1/4, j + 1/4], y_2 = -1\}. \]

For $k = 1, 2, \ldots, 2g + p$, let $S_k$ be an abstract manifold with corners equipped with an identification
\[
S_k = \{(u_1, u_2) \in \mathbb{R}^2 \mid |u_1| \leq 1/4, |u_2| \leq 1\}.
\]
Let $S^\pm_k = \{|u_1| \leq 1/4, u_2 = \pm 1\} \subset S_k$. We view $S^\pm_k$ as oriented 1-manifolds whose orientation is inherited from the standard orientation on $S_k$.

Let $\Sigma_+ = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_2 > -1\}$ and let $\psi : \bigcup_{k=1}^{2g} (S_k^+ \cup S_k^-) \to \bigcup_{j=1}^{2g} I_j \subset \Sigma_+$ be an orientation-preserving diffeomorphism. We now write
\[
\Sigma = \Sigma_+ \bigcup_{\psi} \left( \bigcup_{k=1}^{g} S_k \right)
\]
and define
\[
\Sigma = \Sigma^0.
\]

It will be convenient to assume that $\psi$ extends near each $S^\pm_k$ to a Euclidean isometry (i.e. translation and rotation) with respect to the coordinates (2.1). Hence the standard Euclidean metric descends to $\Sigma$ under the gluing map $\psi$.

It is elementary to show that every non-compact Riemann surface is diffeomorphic to the Riemann surface which results from the above construction, for an appropriate choice of $\psi$. Hence there is no loss of generality in taking (2.2) and (2.3) as a model for $\Sigma$. For the remainder of this section, we therefore assume that $\psi$ is fixed and that $\Sigma$ is defined by (2.2) and (2.3). We may moreover assume without loss of generality that $L$ agrees outside a compact set with the fiber $F_0$ over the point $(0,0) \in \Sigma_+ \subset \Sigma$.

Observe that there is an identification of symplectic manifolds
\[
(T^* \Sigma_+, d\lambda_{can}) \simeq \{(x_1, y_1, x_2, y_2) \mid y_2 > -1\} \subset (\mathbb{R}^4, \omega)
\]
\[
(b_1 dy_1 + b_2 dy_2)(a_1, a_2) \mapsto (b_1, a_1, b_2, a_2).
\]

We will routinely make this identification in the sequel without further note.

Finally, we let
\[
B := \Sigma - \Sigma.
\]

Note that $B$ is naturally a piecewise smooth 1-manifold, since $\Sigma$ is a manifold with corners. We let $B_0, B_1, \ldots, B_\beta$ be an enumeration of the components of $B$ for $\beta \geq 0$. After possibly relabeling, we may assume that $B_0$ is homeomorphic to $\mathbb{R}$ while the other components are homeomorphic to $S^1$.

Given $\eta < 1/100$ and $1 \leq k \leq \beta$, let
\[
\phi^\eta_k : \mathbb{R}/\mathbb{Z} \times (0, \eta) \to \Sigma
\]
be a smooth embedding which extends to a continuous embedding $(\mathbb{R}/\mathbb{Z} \times [0, \eta], \mathbb{R}/\mathbb{Z} \times \{0\}) \to (\Sigma \cup B_k, B_k)$. Let $\phi^\eta_0 : \mathbb{R} \times (0, \eta) \to \Sigma$ be a smooth embedding which extends to a continuous embedding $(\mathbb{R} \times [0, \eta], \mathbb{R} \times \{0\}) \to (\Sigma \cup B_0, B_0)$, and such that $\phi^\eta_0(x, t) = (x - 1, t)$ for $|x|$ large enough. The maps $\phi^\eta_k$ define a “collar” around the boundary component $B_k$. We let
\[
\Sigma_\eta = \bigcup_{k=0}^{\beta} \text{Im} \phi^\eta_k
\]
be the union of the collars.
Finally, let us fix Riemannian metrics $g_k$ (for $0 \leq k \leq \beta$) on $\text{im} \phi_k^\eta$ having the following properties:

(i) For $k \geq 1$ (resp. $k = 0$), we have $g_k = g_e$ on the set $\phi_k^\eta(\mathbb{R}/\mathbb{Z} \times (3\eta/4, \eta))$ (resp. on $\phi_0^\eta(\mathbb{R} \times (3\eta/4, \eta))$), where $g_e$ is the standard Euclidean metric on $\Sigma$. (Note in view of our choice of gluing map $\psi$ that $g_e$ is well-defined on $\Sigma$).

(ii) For $k \geq 1$ (resp. $k = 0$), we have $g_k = (\phi_k^\eta)_* g_e$ on the set $\phi_k^\eta(\mathbb{R}/\mathbb{Z} \times (\eta, \eta/4))$ (resp. on $\phi_0^\eta(\mathbb{R} \times (\eta, \eta/4))$), where $g_e$ now denotes the standard Euclidean metric on $\mathbb{R}/\mathbb{Z} \times (\eta, \eta/4)$ (resp. on $\mathbb{R} \times (\eta, \eta/4)$).

We let $g_\eta$ be a Riemannian metric on $\Sigma$ defined by setting

$$g_\eta = \begin{cases} g_k & \text{on } \Sigma_\eta, \\ g_e & \text{on } \Sigma - \Sigma_\eta. \end{cases}$$

This metric will be important in the next section.

2.2. Construction of some auxiliary almost-complex structures. In this section, we explicitly construct certain almost-complex structures which will be needed later on. The constructions are not very illuminating, so the reader may wish to skip directly to Section 2.3 and return to this section when the need arises.

We write $T^*(S^1 \times \mathbb{R}) = \mathbb{R}/(2\pi \mathbb{Z}) \times \mathbb{R}^3$ with coordinates $(\theta, t, r, s)$ and symplectic form

$$\omega_{\text{can}} := dt \wedge d\theta + ds \wedge dr.$$ 

The zero section $0_{S^1 \times \mathbb{R}}$ is given by $\{t = s = 0\}$.

We consider constants $C_1 > 100$ and $\tilde{\epsilon} < 1/100$ which will be fixed in Section 2.3.

Let

$$S_\tilde{\epsilon} := \{ (\theta, t, r, s) | \| (t, s) \| = \tilde{\epsilon} \} \subset T^*(S^1 \times \mathbb{R}),$$

where the norm is induced by the standard flat metric on $S^1 \times \mathbb{R}$. Letting $V = t \partial_t + s \partial_s$ denote the radial Liouville vector field, $S_{\tilde{\epsilon}}$ is a contact manifold with respect to $\alpha := i_V \omega_{\text{can}}$. Letting $t = \tilde{\epsilon} \cos \phi$ and $s = \tilde{\epsilon} \sin \phi$ for $\phi \in \mathbb{R}/(2\pi \mathbb{Z})$, we have natural coordinates $(\theta, r, \phi)$ for $S_{\tilde{\epsilon}}$ and we compute that $\alpha = \tilde{\epsilon}(\sin \phi d\theta + \cos \phi dr)$.

For $u \in (-1/2, 1/2)$, let $f(u) = \sqrt{C_1^2 + 2u}$ and consider the embedding

$$F : \{ t \in (-1/2, 1/2) \} \subset \mathbb{R}/(2\pi \mathbb{Z}) \times \mathbb{R}^3 \to \mathbb{R}^4$$

$$(\theta, t, r, s) \mapsto (f(t) \cos(\theta), f(t) \sin(\theta), r, s).$$

It is straightforward to check that $F$ is in fact a symplectic embedding. Letting $j$ denote the standard complex structure on $\mathbb{R}^4$, one computes that

$$(2.7) \quad F^*(j) = dF^{-1} \circ j \circ dF = \begin{pmatrix} 0 & 1/(C_1^2 + 2t) & 0 & 0 \\ -(C_1^2 + 2t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. $$

For $i = 1, 2$, fix smooth functions $\rho_i : (\mathbb{R}/2\pi \mathbb{Z}) \times \mathbb{R}^3 - \{ t = s = 0 \} \to \mathbb{R}$ which satisfy the following properties:

- $\rho_1 > 0$,
- $\rho_1 = \tilde{\epsilon} \| (t, s) \| \text{ if } \| (t, s) \| \leq \tilde{\epsilon}$,
- $\rho_1 = C_1^2 + 2t \text{ if } \| (t, s) \| \geq 2\tilde{\epsilon}$,
- $\rho_2 = 1 \text{ if } \| (t, s) \| \geq 2\tilde{\epsilon}$.

It’s clear that functions satisfying the above properties exist.

We let $J_0$ be the unique almost-complex structure on $T^*(S^1 \times \mathbb{R}) - 0_{S^1 \times \mathbb{R}} = (\mathbb{R}/2\pi \mathbb{Z}) \times \mathbb{R}^3 - \{ t = s = 0 \}$ which satisfies $J_0(\partial_\theta) = -\rho_1 \partial_t$ and $J_0(\partial_r) = -\rho_2 \partial_s$. 
Lemma 2.1. The almost-complex structure \( J_0 \) is cylindrical with respect to the canonical symplectic embedding

\[
\iota : ((-\infty, 0] \times S_t, d(e^\tau \alpha)) \hookrightarrow (T^*(S^1 \times \mathbb{R}) - 0_{S^1 \times \mathbb{R}}, \omega_{\text{can}})
\]

induced by the Liouville flow, where \( \tau \) is the variable corresponding to \((-\infty, 0]\).

Proof. By explicit computation, we find that \( \iota^*(J_0)(\partial_\tau) = \frac{1}{\lambda}(\cos \phi \partial_\theta + \sin \phi \partial_R) = \partial_\phi \) and that \( \iota^*(J_0)(\frac{1}{\lambda} \sin \phi \partial_\theta + \cos \phi \partial_R) = \partial_\phi \). Since \( \ker \alpha = \text{span}\{\sin \phi \partial_\theta + \cos \phi \partial_R, \partial_\phi\} \), this proves the claim. \( \square \)

In the sequel, we let \( J_{\text{cyl}} \) be the unique almost-complex structure on \( \mathbb{R} \times S_t \) which satisfies \( J_{\text{cyl}}(\partial_\tau) = R_\alpha \) and \( J_{\text{cyl}}(\frac{1}{\lambda} \sin \phi \partial_\theta + \cos \phi \partial_R) = \partial_\phi \).

We now define an almost-complex structure \( \tilde{J} \) on \( \mathbb{R}^4 - \{x_1^2 + y_1^2 = C_1, y_2 = 0\} \) by setting

\[
\tilde{J} = \begin{cases} F_*(J_0) & \text{on } \text{Im } F(\{\|s, t\| \leq 3\lambda\}) \\ j & \text{otherwise.} \end{cases}
\]

It’s straightforward to check using (2.7) that \( \tilde{J} \) is well-defined and smooth.

Lemma 2.2. The almost-complex structure \( \tilde{J} \) is compatible with the standard symplectic form \( \omega \).

Proof. It is enough to prove that \( \tilde{J} \) is compatible with \( \omega \) at points \( F(p) \in \mathbb{R}^4 - \{x_1^2 + y_1^2 = C_1, y_2 = 0\} \) where \( p \in \{\|s, t\| \leq 2\lambda\} \). One first observes that the splitting \( T_{\phi(p)}\mathbb{R}^4 = \text{span}\{\partial_{x_1}, \partial_{y_1}\} \oplus \text{span}\{\partial_{x_2}, \partial_{y_2}\} \) induces a splitting

\[
(F_*(J_0))(\phi(p)) = J_0^1 \oplus J_0^2.
\]

Observe that \( \omega \) also splits as \( \omega = \omega^1 \oplus \omega^2 \). Hence we only need to check that \( J_0^k \) is compatible with \( \omega^k \) for \( k = 1, 2 \). This is true for dimension reasons. \( \square \)

Let us now switch gears and discuss a general procedure for constructing a canonical almost-complex structure on the cotangent bundle of a Riemannian manifold. This procedure will be useful to us in the next section. It is originally due to Sasaki and we refer the reader to [8, Sec. 1.3–1.4] for a detailed exposition.\(^1\)

Let \((M, g)\) be a Riemannian manifold. The Levi-Civita connection induces a splitting

\[
TT^*M = \mathcal{H} \oplus \mathcal{V},
\]

where \( \mathcal{H}, \mathcal{V} \) are respectively the horizontal and vertical distributions. Given \( \theta \in T^*M \), let us consider a pair of linear maps:

\[
(d_\theta \pi)^\flat : T_\theta T^*M \to T^*_\pi(\theta) M, \quad K_\theta : T_\theta T^*M \to T^*_\pi(\theta) M.
\]

Here \((d_\theta \pi)^\flat\) denotes the differential of the canonical projection \( \pi : T^*M \to M \), composed with the musical isomorphism \( TM \to T^*M \). The map \( K_\theta \) is the connection map and is defined as follows. Given \( \theta \in T^*M \) and \( \xi \in T_\theta T^*M \), choose a path \( \gamma : (-1, 1) \to TT^*M \) such that \( \gamma(0) = \theta \) and \( \dot{\gamma} = \xi \). Letting \( \alpha := \pi \circ \gamma \), we can write \( \gamma = (\alpha(t), Z(t)) \), where \( Z \) is a covector field along \( \alpha \). Now define

\[
K_{(\gamma, \xi)}(\theta) := \nabla_\alpha Z(0).
\]

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\(^1\)Some conventions in [8] are different from ours: in particular, the symplectic form on the cotangent bundle and the metric-induced almost-complex structure both differ by a sign. We also note that although [8] mainly considers tangent bundles, all of the relevant constructions commute the musical isomorphisms so this distinction is entirely superficial.
Lemma 2.3. The maps \((d_\theta \pi)^0\) and \(K_\theta\) are linear and surjective. Moreover, we have \(\ker(d_\theta \pi)^0 = V_\theta\) and \(\ker K_\theta = H_\theta\).

Thus we obtain identifications \((d_\theta \pi)^0 : H_\theta \to T^*_\pi(\theta)M\) and \(K_\theta : V_\theta \to T^*_\pi(\theta)M\). Writing
\[
T_\theta T^*M \ni \xi = (\xi_1, \xi_2) \in T^*_\pi(\theta)M \oplus T^*_\pi(\theta)M,
\]
where the identification \(\xi = (\xi_1, \xi_2)\) is induced by \(((d_\theta \pi)^0, K_\theta)\), we define \(J_g : TT^*M \to TT^*M\) by \(J_g(\xi_1, \xi_2) = (\xi_2, -\xi_1)\).

This almost-complex structure is compatible with \(\omega = d\lambda\). Observe that if \(g\) is Euclidean metric on \(\mathbb{R}^n\), then the above construction just gives back the standard integrable complex structure on \(T^*\mathbb{R}^n \cong \mathbb{R}^{2n}\).

The following lemma will be useful in the next section.

Lemma 2.4. For \(r > 0\), hypersurface \(S_r := \{x \in T^*M \mid \|x\|_g = r\}\) is pseudoconvex for the almost-complex structure \(J_g\).

Proof. Consider the function \(H : T^*M \to \mathbb{R}\) defined by \(H(q, p) := \frac{1}{2}(p, p)_g\). It is an easy exercise (see [2, Sec. 2.8]) to show that the condition for the level sets of \(H\) to be pseudoconvex is the same as the condition for \(\ker dH \circ J_g\) to be a contact structure. But note that given \(\xi \in T_{(q, p)}T^*M\), we have \(dH \circ J_g(\xi) = (K(J\xi), p)_g = (d_\theta \pi)^0(\xi), p)_g = \lambda_{can}(\xi)\); cf. [8, Prop. 1.21 and Def. 1.23]. Hence \(\lambda_{can} = dH \circ J_g\), so it suffices to check that \(\lambda_{can}\) restricts to a contact structure on \(S_r\). This is in turn a consequence of the fact that the radial Liouville vector field is transverse to \(S_r\).

2.3. Geometric setup. Given \(\epsilon\) small enough, the symplectic neighborhood theorem provides a symplectic embedding
\[
\Phi_0 : \text{Op}_\epsilon(F_0) \to T^*\Sigma
\]
with the property that:
- \(\Phi_0(F_0) = L\),
- \(\Phi_0\) restricts to the identity on \((T^*\Sigma_+ \cap \Sigma) \cap \text{Op}_\epsilon(F_0)\), for some \(M > 1\).

After possibly making \(M\) larger, we may assume that the image of \([-M/2, M/2]^4 \cap \text{Op}_\epsilon(F_0)\) is contained in \([-M/2, M/2]^4\). After possibly making \(\eta > 0\) smaller, we may in addition assume that \(\Phi_0(\text{Op}_\epsilon(F_0)) \cap \pi^{-1}(\Sigma_\eta) = \emptyset\), where \(\pi : T^*\Sigma \to \Sigma\) is the canonical projection (see (2.6) for the definition of \(\Sigma_\eta\)). We view \(\epsilon, \Phi_0, M, \eta\) as fixed for the remainder of this section.

Define the open set
\[
X := (T^*\Sigma_+ \cap [-M, M]^4) \cup \text{Op}_\epsilon(F_0) \cup \pi^{-1}(\Sigma_\eta),
\]
and let \(\Phi : X \to T^*\Sigma\) be defined by
\[
\Phi(p) = \begin{cases} 
\Phi_0(p) & \text{if } p \in \text{Op}_\epsilon(F_0) \\
p & \text{otherwise.}
\end{cases}
\]

One readily checks that \(\Phi\) is well-defined.

Choose \(C\) large enough (depending on \(\epsilon, M\)) so that the sets
\[
W_1 = \{x_1^2 + (y_1 + C)^2 \geq (C - \epsilon/2)^2, x_1^2 + (y_1 - C)^2 \geq (C - \epsilon/2)^2, x_2 = l, y_2 = 0\}
\]
are disjoint from \(X\).
are contained in $X$ for all $l \in \mathbb{R}$. In particular, this means that the Lagrangian cylinders

$$L^\pm := \{ x_1^2 + (y_1 \mp C)^2 = (C - \epsilon/2)^2, y_2 = 0 \}$$

are also contained in $X$; see Figure 1.

Define $\tau^\pm : \mathbb{R}^4 \to \mathbb{R}^4$ by $\tau^\pm_i (x_1, y_1, x_2, y_2) = (x_1, y_1 \pm C, x_2, y_2)$. Let $C_1 := C - \epsilon/2$ and define

$$F^\pm := \tau^\pm \circ F.$$

We remind the reader that our definition of $F$, which was stated in Section 2.2, depends on $C_1$.

Let us now return to the metric $g_\eta$ constructed at the end of Section 2.2. Noting that $\eta$ is now fixed, we let $j_S$ be the “Sasakian” almost-complex structure on $T^* \Sigma$ induced by the metric $g_\eta$, as described at the end of Section 2.2. Observe that $j_S$ restricts to the standard integrable almost-complex structure away from $\pi^{-1}(\Sigma_\eta)$, since $g_\eta$ is just the standard Euclidean metric on $\Sigma - \Sigma_\eta$.

Next, fix $\tilde{\epsilon}$ small enough (depending on $\epsilon, M, C$) so that

$$\text{Im}(F^\pm(\{ \| (t, s) \| \leq 3\tilde{\epsilon} \}) \subset X,$$

and define an almost-complex structure $J'$ on $T^* \Sigma - (L^+ \cup L^-)$ by setting

$$J' = \begin{cases} (\tau^\pm)_* \tilde{J} & \text{in the image of } F^\pm_i(\{ 0 < \| (t, s) \| \leq 3\tilde{\epsilon} \}) \\ j_S & \text{otherwise.} \end{cases}$$

Using Lemma 2.2, it is straightforward to verify that $J'$ is well-defined and compatible with $\omega$.

Finally, we fix an almost-complex structure $J$ on $T^* \Sigma - \Phi(L^+ \cup L^-)$ which is compatible with $d\lambda_{can}$, and such that $J = \Phi_* J'$ in $\Phi(X - (L^+ \cup L^-))$.

It follows from Lemma 2.1 and the definition of $J$ that this almost-complex manifold has negative cylindrical ends around the $\Phi(L^\pm)$ of the form

$$((-\infty, 0] \times S_\epsilon, J_{cyl}),$$

where $J_{cyl}$ was defined in the paragraph following Lemma 2.1.

2.4. Filling by $J$-holomorphic planes. In this section, we will consider punctured holomorphic curves in the almost-complex manifold

$$(T^* \Sigma - \Phi(L^+ \cup L^-), J).$$

We will make use of Siefring’s intersection theory for punctured pseudoholomorphic curves. We refer the reader to [10] for a detailed exposition of this theory.
For $\sigma \in (\langle -\infty, \infty \rangle \cup [M, \infty])$, let $u_{0,\sigma}^\pm : \mathbb{C} \to T^*\Sigma - \Phi(L^+ \cup L^-)$ be a $J$-holomorphic plane whose image is the set $\{x_1^2 + (y_1 + C)^2 < (C - \epsilon/2)^2, x_2 = \sigma, y_2 = 0\}$. Such a plane is unique up to reparametrization.

Let $M^\pm$ be the connected component of the moduli space of unparametrized $J$-holomorphic planes in $(\mathbb{R}^4 - \Phi(L^+ \cup L^-))$ containing $u_{0,M}^\pm$. Let $(M^\pm)_{reg} \subset M^\pm$ be the open subset of transversally cut-out planes. Since the planes under consideration are asymptotic to a primitive closed geodesic and are therefore simply covered, the reparametrization group acts freely. It follows that $(M^\pm)_{reg}$ is a smooth 1-dimensional (Hausdorff) manifold; see [9, Thm. 0].

We let $\mathcal{U}^\pm \subset (M^\pm)_{reg}$ be the connected component containing $u_{0,M}^\pm$.

It will be useful to record the following lemma, which shows that the elements of $\mathcal{U}^\pm$ are also transverse for a restricted moduli problem.

**Lemma 2.5.** Given a plane $u \in \mathcal{U}^\pm$, the linearization of $\overline{\partial}_J$ through planes whose asymptotic orbit is fixed is surjective.

**Proof.** It’s clear that the standard plane $u_{0,M}^\pm$ has vanishing Siefring self-intersection number. Since this number is constant in families, the other planes in $\mathcal{U}^\pm$ have the same property. Noting that these planes must have vanishing normal Chern number (since $u_{0,M}^\pm$ manifestly does), it follows from the adjunction formula for punctured holomorphic curves (see [11, (A.6)]) that they are all embedded. The lemma now follows by Wendl’s automatic transversality result; see [9, Rmk. 1.2].

Let $\Gamma^\pm$ be the manifold of closed, simple and positively oriented geodesics of the Lagrangian cylinder $\Phi(L^\pm) \subset T^*\Sigma$. There is an identification

$$\gamma^\pm : \mathbb{R} \to \Gamma^\pm$$

which takes $l \in \mathbb{R}$ to the unique geodesic in $\Gamma^\pm$ which passes through $\Phi \circ \mathcal{E}^\pm(0, 0, l, 0)$.

Let $(ev_\beta)^\pm : M^\pm \to \Gamma^\pm$ be the asymptotic evaluation map (see [11, A.2.]) and define

$$\Psi^\pm : \mathcal{U}^\pm \to \mathbb{R}$$

$$u \mapsto (\gamma^\pm)^{-1} \circ (ev_\beta)^\pm.$$

Since $\mathcal{U}^\pm$ consists of transversally cut out planes by definition, it follows that $(ev_\beta)^\pm$ and hence $\Psi^\pm$ are local diffeomorphisms; cf. [4, Prop. 5.11(i)].

**Remark 2.6.** The target of the asymptotic evaluation map defined in [11, A.2.] is in fact a line bundle $E^\pm$ over $\Gamma^\pm$. Our asymptotic evaluation map is obtained by composing the map in [11, A.2.] with the projection $E^\pm \to \Gamma^\pm$. To verify that the composition is a local diffeomorphism, one needs to establish as in [4, Prop. 5.11(i)] that the elements of $\mathcal{U}^\pm$ are transverse for the moduli problem with restricted orbit as well as for the moduli problem where the orbits allowed to vary in the Morse-Bott family. This is the content of Lemma 2.5.

We wish to show that $\Psi^\pm$ is in fact a diffeomorphism. This will be deduced from the following sequence of lemmas.

**Lemma 2.7.** The map $\Psi^\pm$ is injective.

**Proof.** Suppose that $u, v \in \mathcal{U}^\pm$ are asymptotic to the same Reeb orbit. We wish to show that $u = v$ (up to reparametrization). Note that the Siefring self-intersection number of the standard plane $u_{0,M}^\pm$ evidently vanishes. Since $\mathcal{U}^\pm$ is connected and contains $u_{0,M}^\pm$, it follows that $u \ast v = 0$. On the other hand, a computation originally due to Hind and Lisi (see [4, Lem.
5.13]) shows that $u * v > 0$ if both planes are asymptotic to the same Reeb orbit. This proves the claim.

By combining Lemma 2.7 with the previously observed fact that $\Psi^\pm$ is a local diffeomorphism, we conclude that $\Psi^\pm : U^\pm \to \mathbb{R}$ is a smooth embedding.

The same argument as in Lemma 2.7 also gives the following statement, which will be useful later.

**Lemma 2.8.** Given $|\sigma| \geq M$, the plane $u_{0,\sigma}^\pm$ is the unique $J$-holomorphic plane asymptotic to the geodesic $\gamma^\pm(\sigma)$.

Our next task is to show that $\Psi^\pm$ has closed image. To this end, it will be useful to introduce certain barriers which provide restrictions on holomorphic planes escaping to infinity.

Let $f : \mathbb{R}_{\geq 0} \to [0, 1/2] \subset \mathbb{R}$ be a non-decreasing function with the property that $f(x) = \sqrt{x}$ for $x \in [0, 1/2]$. Let us now consider the hypersurfaces (see Figure 2)

$$
\mathcal{H}^+ = \{(x_1, y_1, x_2, y_2) \in T^* \Sigma_+ \mid x_2 \geq 2M, |y_2| = f(x_2 - 2M)\},
$$

$$
\mathcal{H}^- = \{(x_1, y_1, x_2, y_2) \in T^* \Sigma_+ \mid x_2 \leq -2M, |y_2| = f(2M - x_2)\}.
$$

Observe that we can view $\mathcal{H}^\pm$ as being contained in the set $\{(x_1, y_1, x_2, y_2) \mid y_2 \geq -1/2\} \subset T^* \Sigma_+ \subset T^* \Sigma$. We let $\mathcal{H} = (\mathcal{H}^+ \cup \mathcal{H}^-) - \Phi(\mathcal{L}^+ \cup \mathcal{L}^-) = (\mathcal{H}^+ \cup \mathcal{H}^-) - (\mathcal{L}^+ \cup \mathcal{L}^-)$. Let us record the following important lemma.

**Lemma 2.9.** The hypersurface $\mathcal{H}$ is foliated by $J$-holomorphic planes.\(^3\) Moreover, $T^* \Sigma - \mathcal{H}$ consists of three connected components. We can uniquely label these components $\mathcal{V}^+, \mathcal{V}, \mathcal{V}^-$ by requiring that $\mathcal{H}^+$ (resp. $\mathcal{H}^-$) separates $\mathcal{V}$ and $\mathcal{V}^+$ (resp. $\mathcal{V}$ and $\mathcal{V}^-$).

**Proof.** Observe that the planes $\{(x_1, y_1, x_2, y_2) \mid y_2 = \pm f(x_2 - M)\}$ are $J$-holomorphic, for each fixed $(x_2, y_2)$. This is immediate from the observation (cf. Lemma 2.2) that $J$ preserves the splitting $\{\partial_{x_1}, \partial_{y_1}\} \oplus \{\partial_{x_2}, \partial_{y_2}\}$ in the region $\{(x_1, y_1, x_2, y_2) \mid y_2 \geq -1/2, |x_2| \geq M\} \subset T^* \Sigma_+ \subset T^* \Sigma$. The fact that $T^* \Sigma - \mathcal{H}$ consists of three connected components is clear from the definition of $\mathcal{H}$.

We now proceed to the key compactness result of this section. As a small piece of notation, given $s \in \text{Im} \, \Psi^\pm \subset \mathbb{R}$, let us write $u_s^\pm := (\Psi^\pm)^{-1}(s)$.

**Lemma 2.10.** The image of $\mathcal{U}^\pm$ under $\Psi^\pm$ is closed as a subset of $\mathbb{R}$.

\(^3\)Strictly speaking, when $y_2 = 0$, the plane is “broken” and consists of three components.
Proof. We prove that the image of $\mathcal{U}^+$ under $\Psi^+$ is closed since the other case is analogous. Since $\Psi^+$ is an embedding, it is enough to prove that given any interval $(a, b) \subset \text{Im} \Psi^+$, we have $[a, b] \subset \text{Im} \Psi^+$. Let us show that $b \in \text{Im} \Psi^+$ and leave the other endpoint to the reader. We may as well also assume that $b \in [-M, M]$, since otherwise the claim follows trivially from Lemma 2.8.

Choose a sequence $s_j \to b$ with $s_j \in (a, b)$ and consider the sequence of planes $u_{s_j}^+$. We wish to show that these planes are contained in a uniformly bounded domain (i.e. independent of $s_j$). We will treat separately boundedness in the fiber and base directions.

The fibers: Recall from Lemma 2.4 that

$$\mathcal{S}_r := \{(q, p) \in T^* \Sigma \mid \langle p, p \rangle_{g_q} = r \} \subset T^* \Sigma$$

is pseudoconvex with respect to $j_\Sigma$, for any $r > 0$. Let $H : T^* \Sigma \to \mathbb{R}$ be the function $H(q, p) := \langle p, p \rangle_{g_q}$. Choose $N > 100 \max(M, C)$. Suppose that $\max H \circ u_{s_j}^+ = R \geq N^2$ and is achieved at some point $P = u_{s_j}^+(p)$, for $p$ a point in the domain of $u_{s_j}^+$.

Let us first assume that $P$ is contained in the region $\{ (x_1, y_1, x_2, y_2) \mid |y_2| < 1/4 \} \subset T^* \Sigma_+$. We write $P = (P_{x_1}, P_{y_1}, P_{x_2}, P_{y_2})$. According to Lemma 2.9 and positivity of intersection, the curve $u_{s_j}^+$ cannot cross $\mathcal{H}$ and must therefore be contained entirely in $\mathcal{V}$. It follows that $|P_{x_2}| < 2M + 1$, and hence $|P_{x_1}| > N^2 - (3M)^2 > N^2/2$. Hence $J = j_\Sigma$ near $P$. Hence $\mathcal{S}_R$ is $J$-convex near $P$, which gives a contradiction.

If $P \in T^* \Sigma$ contained in the complement of the region $\{ (x_1, y_1, x_2, y_2) \mid |y_2| < 1/4 \}$, then $J = j_\Sigma$ near $P$ by definition of $J$. In particular, $\mathcal{S}_R$ is $J$-convex near $P$, which again gives a contradiction. We conclude that $u_{s_j}^+$ cannot cross the hypersurface $\mathcal{S}_{N^2}$, i.e. $u_{s_j}^+$ is bounded in the fiber directions.

The base: Let us first consider the function $\pi_{y_1} \circ u_{s_j}^+$, which is well defined on $(u_{s_j}^+)^{-1}(T^* \Sigma_+).$ Observe that the hypersurfaces $\{ y_1 = C \} \subset T^* \Sigma$ are Levi-flat with respect to $J$ if $|C| \geq N$ (indeed, $J = j$ in that region). It follows that $|\pi_{y_1} \circ u_{s_j}^+| < N$.

A similar argument show that $\pi_{y_2} \circ u_{s_j}^+ < N$. It remains to argue that $u_{s_j}^+$ cannot approach $B$ (see (2.5)). This is a consequence of the fact that $g_{\eta} = (\phi_{\eta}^n)_*(g_c)$ in the region $\phi_{\eta}^n(\mathbb{R} \times \mathbb{Z} \times (0, \eta/4))$ for $1 \leq k \leq \beta$ (resp. in the region $\phi_{\eta}^n(\mathbb{R} \times (0, \eta/4))$). Indeed, for $0 < c < \eta/4$, we find that the hypersurfaces $\pi^{-1}(\text{Im} \phi_{\eta}^n(-, c))$ are foliated by $J_\Sigma$-holomorphic curves. Hence the $u_{s_j}^+$ cannot touch them.

We conclude that the $u_{s_j}^+$ are contained in a uniformly bounded domain. Hence we can apply the SFT compactness theorem.

For area reasons, the limit building consists of a single plane $v \in \mathcal{M}^+$. Moreover, since $u_{s_j}^+ \ast u_{s_j}^+ = 0$, it follows that $v \ast v = 0$. Noting that $v$ must have vanishing normal Chern number (since the $u_{s_j}^+$ do), it follows from the adjunction formula for punctured holomorphic curves (see [11, (A.6)]) that $v$ is embedded. It then follows from Wendl’s automatic transversality result for punctured curves in symplectic 4-manifolds (see [9, Thm. 1]) that $v$ is transversally cut out. Hence $v \in \mathcal{U}^+$ and $\Psi^+(v) = b$. $\square$

By putting together the previous lemmas, we find that $\Psi^\pm$ is a smooth embedding of $\mathcal{U}^\pm$ whose image is closed. It follows that $\Psi^\pm$ is surjective, and hence a diffeomorphism. We state this as a corollary.

Corollary 2.11. The map $\Psi^\pm$ is a diffeomorphism.

For $l \in \mathbb{R}$, let $W_l^j := \Phi(W_l)$. Noting that $W_l \subset X$ and $J = \Phi_\ast J'$ in $\Phi(X)$, one can check that $W_l^j$ is a $J$-holomorphic surface. Positivity of intersection and the invariance in families of the intersection number then implies the following properties which we collect as a lemma.
Lemma 2.12. The following properties hold:

(i) \( W^l_{12} \cap u^\pm_{12} = \emptyset \) for any \( l_1, l_2 \in \mathbb{R} \).
(ii) \( u^\pm_{12} \cap u^\pm_{21} = \emptyset \) for any \( l_1, l_2 \in \mathbb{R} \).
(iii) \( u^\pm_{12} \cap u^\pm_{21} = \begin{cases} u^\pm_{12} & \text{if } l_1 = l_2, \\ \emptyset & \text{otherwise.} \end{cases} \)
(iv) The analog of (iii) holds with "-" in place of "+".

Let \( ev^\pm : \mathcal{U}^\pm \times \mathbb{C} \rightarrow T^* \Sigma \) be the evaluation map and let \( \overline{ev}^\pm : \mathcal{U}^\pm \times D^2 \rightarrow T^* \Sigma \) be its natural compactification. Note that \( \overline{ev}^\pm \) is smooth in the interior and continuous at the boundary. The smoothing procedure from [4, Sec. 5.3] allows us to deform \( \overline{ev}^\pm \) to a smooth embedding. This is the content of the following proposition.

Proposition 2.13 (Smoothing procedure). There exists a smooth map \( \tilde{ev}^\pm : \mathcal{U}^\pm \times D^2 \rightarrow T^* \Sigma \) which agrees with \( \overline{ev}^\pm \) outside a compact set. Viewing \( \tilde{ev}^\pm \) as a map \( \mathbb{R} \times D^2 \rightarrow T^* \Sigma \) via the identification \( \Psi^\pm : \mathcal{U}^\pm \rightarrow \mathbb{R} \), we have the following properties:

(i) The sets \( \Sigma_l := (\tilde{ev}^+(u^+_l, D^2) \cup W^+_l \cup \tilde{ev}^-(u^-_l, D^2)) \) are codimension 2 embedded symplectic surfaces.
(ii) For \( l_1, l_2 \in \mathbb{R} \), we have \( \Sigma_{l_1} \cap \Sigma_{l_2} = \begin{cases} \Sigma_{l_1} & \text{if } l_1 = l_2 \\ \emptyset & \text{otherwise.} \end{cases} \)
(iii) For \( |l| > M \), we have that \( \Sigma_l = \{ x_2 = l, y_2 = 0 \} \subset \mathbb{R}^4 \).
(iv) For any \( l \in \mathbb{R} \), we have that \( \Sigma_l \cap \{|x_1| > M + 1, |y_1| > M + 1| \} = \{|x_1| > M + 1, |y_1| > M + 1, x_2 = l, y_2 = 0 \} \).

Proof. First of all, it follows easily using the same argument as in Lemma 2.7 that the \( u^\pm_l \) are standard for \( |l| > M \). We therefore have the same uniform convergence estimates as in [4, Lem. 5.14], even though the families of planes we are considering here are not compact.

We can therefore apply the argument of [4, Prop. 5.16], which relies on the uniform convergence estimates of [4, Lem. 5.14]. This provides deformations of the holomorphic planes so that (i) and (ii) are satisfied. The remaining properties (ii) and (iii) now follow directly from the definitions. \( \square \)

Corollary 2.14. There is a smooth codimension one hypersurface \( Q := \bigcup_{l \in \mathbb{R}} \Sigma_l \) which contains \( L \) and is naturally foliated by the \( \Sigma_l \).

2.5. Completion of the proof. The characteristic foliation of \( Q \) induces a symplectic monodromy map
\[
\mu : \Sigma_{-M+2} \rightarrow \Sigma_{M+2}.
\]
Since \( L \) is Lagrangian, observe that this map sends \( \Sigma_{-M+2} \cap \{ y_1 = 0 \} \) to \( \Sigma_{M+2} \cap \{ y_1 = 0 \} \). Let \( \mu^+ \) denote the restriction of \( \mu \) to \( \Sigma_{-M+2} \cap \{ y_1 \geq 0 \} \rightarrow \Sigma_{M+2} \cap \{ y_1 \geq 0 \} \). According to [4, Lem. 6.8], \( \mu^+ \) is generated by a family of compactly-supported Hamiltonians \( \{ H_t \}_{t \in [0,1]} \) on \( \{(x_1, y_1) \mid y_1 \geq 0\} \) which vanish on \( \{ y_1 = 0 \} \) and such that \( H_t \equiv 0 \) for \( t \) near \( \{0,1\} \).

Given \( R > 0 \), let \( H^R_t := \frac{1}{R} H_{Rt} \). Let \( \phi^H_t \) be the associated Hamiltonian flow where \( t \in [0, R] \). Set \( x_2^2 := x_2 - (M + 2) \).

We now fix \( R \) so that \( |H^R_t| < 1/2 \). We then have a well-defined map
\[
\Theta : Q \cap \{ M + 2 \leq x_2 \leq M + 2 + R \} \rightarrow T^* \Sigma_+ \subset T^* \Sigma
\]
given by setting \( \Theta(x_1, y_1, x_2, y_2) \mapsto (\phi^R_{x_2^2} (x_1, y_1), x_2, y_2 + H^R_{x_2^2}) \).
Now let
\[
\tilde{Q} = \begin{cases} 
Q & \text{for } x_2 \in \mathbb{R} - [M + 2, M + 2 + R], \\
\text{Im} \Theta(Q \cap \{M + 2 \leq x_2 \leq M + 2 + R\}) & \text{for } x_2 \in [M + 2, M + 2 + R].
\end{cases}
\]

Since \( H_t \equiv 0 \) for \( t \) near \( \{0, 1\} \), it follows that \( \tilde{Q} \) is a smooth hypersurface which agrees with \( Q \) outside a compact set. A straightforward calculation shows that the monodromy map \( \tilde{\mu}^+ : \Sigma_{-N_1} \cap \{y_1 \geq 0\} \rightarrow \Sigma_{N_1} \cap \{y_1 \geq 0\} \) induced by the characteristic flow along \( \tilde{Q} \) is the identity, for any \( N_1 \) large enough.

We now come to the following corollary, which immediately implies Theorem 1.1 (recall from Section 2.1 that we may assume without loss of generality that \( x = 0 \) in the statement of Theorem 1.1).

**Corollary 2.15.** There exists a compactly-supported Hamiltonian isotopy of \((T^* \Sigma, d\lambda_{\text{can}})\) taking \( L \) to \( F_0 \).

**Proof.** For \( s \geq 0 \), let \( \ell_s \subset \tilde{Q} \) be the line \( \ell_s := \{y_1 = s, x_2 = -3N, y_2 = 0\} \). Let \( \chi_s \subset \tilde{Q} \) be the image of \( \ell_s \) under the characteristic flow. Observe that \( \chi_0 = L \). Moreover, it follows from the fact that the monodromy map \( \tilde{\mu}^+ \) is the identity that \( \chi_s \) is a Lagrangian plane which is standard at infinity for all \( s \geq 0 \). In particular, for \( s_0 \) large enough, we have \( \chi_{s_0} = \{y_1 = s_0, y_2 = 0\} \subset \tilde{Q} \).

Let \( \sigma^+ : [0, s_0] \times \Sigma_+ \rightarrow \Sigma_+ \) be a compactly-supported isotopy with the property that \( \sigma^+_s(0, 0) = (0, 0) \) and that \( \sigma^+_s(y_1, \alpha) = (y_1, \alpha) \) if \( \alpha \in [-1, -1/2] \). Observe that \( \sigma^+ \) extends to an isotopy \( \sigma : [0, s_0] \times \Sigma \rightarrow \Sigma \) which is the identity on \( \Sigma - \Sigma_+ \).

We now consider the induced isotopy \( \sigma^* : [0, s_0] \times T^* \Sigma \rightarrow T^* \Sigma \). Observe that \( s \mapsto \sigma^*_s(\chi_s) \) defines a family of Lagrangian submanifolds which are diffeomorphic to \( \mathbb{R}^2 \) and fixed set-wise outside a compact set.

The Weinstein neighborhood theorem implies that such an isotopy of Lagrangian submanifolds extends to a compactly-supported symplectomorphism near the Lagrangians. In particular, the isotopy is generated by a symplectic vector field. Since \( H^2_c(\mathbb{R}^2, \mathbb{R}) = 0 \), this vector field is Hamiltonian. This completes the proof. \( \square \)

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