Fast computation of rankings from pairwise comparisons

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We study the ranking of individuals, teams, or objects on the basis of pairwise comparisons using the Bradley-Terry model. Maximum-likelihood estimates of rankings within this model are commonly made using a simple iterative algorithm first introduced by Zermelo almost a century ago. Here we describe an alternative and similarly simple iteration that solves the same problem much faster—over a hundred times faster in some cases. We demonstrate this algorithm with applications to a range of example data sets and derive some results regarding its convergence.

I. INTRODUCTION

The problem of ranking a set of individuals, teams, or objects on the basis of a set of pairwise comparisons between them arises in a variety of contexts, including competitions in sports and chess, paired comparison studies of consumer choice, and observational studies of dominance behaviors in animals and humans [1–5]. If a group of chess players play games against one another, for example, how can we rank the players, from best to worst, based on the outcome of those games? Such questions can be tricky to answer because the outcomes of the games may be contradictory or ambiguous—underdogs sometimes win and strong players sometimes lose—so one typically adopts a probabilistic model. In the most common version we assign a numerical score 

\[ s_i \]

beats

\[ s_j \]

where \( i \) and \( j \) is assumed to be some function of the difference in their scores

\[ p_{ij} = f(s_i - s_j) \]

The most popular choice of functional form is the logistic function

\[ f(s) = 1/(1 + e^{-s}) \]

which gives

\[ p_{ij} = \frac{e^{s_i}}{e^{s_i} + e^{s_j}}. \]

This is the Bradley-Terry model, first introduced by Zermelo in 1929 [1] and heavily studied in the years since, particularly following its rediscovery by Bradley and Terry in the 1950s [2]. For convenience, one often introduces the shorthand

\[ \pi_i = e^{s_i} \]

so that

\[ p_{ij} = \frac{\pi_i}{\pi_i + \pi_j} \]

and we will do that here. Zermelo (writing in German) referred to the non-negative parameters \( \pi_i \) as "Spielstärken" or “playing strengths,” although they are elsewhere variously called worth parameters, skill parameters, merit parameters, ratings, or simply weights. Following Zermelo, we will call them strengths.

The standard approach for ranking individuals involves estimating the strengths \( \pi_i \) by fitting the Bradley-Terry model to the observed pattern of wins and losses using maximum likelihood methods. Defining \( w_{ij} \) to be the total number of times \( i \) beats \( j \), or zero if \( i \) and \( j \) never competed, it can be shown that the maximum-likelihood values of the strengths are given by a simple procedure: starting from any convenient initial values we iterate the equation

\[ \pi'_i = \frac{\sum_j w_{ij}}{\sum_j (w_{ij} + w_{ji})} \]

until convergence is reached. This algorithm was also first described by Zermelo [1] and we will refer to it as Zermelo's algorithm. Hundreds of papers have been published about it, its variants, its properties, and its applications.

Although widely used, however, Zermelo’s algorithm is known to be slow to converge [6, 7]. In this paper we study the alternate iteration

\[ \pi'_i = \frac{\sum_j w_{ij} \pi_j}{\pi_i + \sum_j w_{ji} \pi_j} \]

We show that iteration of this equation solves the same problem and converges to the same solution as Zermelo’s algorithm but does so significantly faster—over a hundred times faster in some cases. Given that Eq. (3) is also simple to implement we know of no reason not to favor it over Eq. (4).

A number of other authors have considered alternative algorithms for ranking under the Bradley-Terry model in recent years. One promising approach employs spectral methods that estimate rankings based on the properties of random walks on the network of directed interactions between individuals [8–10]. Although they do not directly maximize the likelihood under the Bradley-Terry model these algorithms can be shown to converge closely to the maximum-likelihood solution. Some versions are quite numerically efficient, though they are also complex to implement. Minorization-maximization (MM) algorithms, which optimize a minorizing proxy for the likelihood function, can also be applied to ranking problems. For the Bradley-Terry model the appropriate MM algorithm turns out to be exactly equivalent to Zermelo’s algorithm and hence offers no speed improvement [11], but techniques have been suggested for accelerating convergence [11] and the MM formulation also provides an elegant route to developing algorithms for generalizations of the model. Perhaps more directly competitive with our approach is one of the simplest of methods: one can fit the Bradley-Terry model using Newton’s method applied to the derivative of the likelihood. For small applications
with only a few individuals or teams to be ranked this is typically the fastest approach, although for such small cases the difference may be moot. As the number of individuals becomes larger, however, the time taken to perform the matrix inversion required by Newton’s method becomes prohibitive \[3\]. Overall, the particular combination of simplicity and speed offered by Eq. (4) makes it an attractive approach for practical applications.

II. ITERATIVE ALGORITHMS AND THE BRADLEY-TERRY MODEL

Consider a tournament where \( N \) players or teams play games of some kind against one another in pairs and there is a clear winner and loser of every game. For simplicity we initially assume that no ties or draws are allowed. The case where ties are allowed is treated separately in Section IV. We assume that the probability \( p_{ij} \) that player \( i \) beats player \( j \) obeys Eq. (4) and the strengths \( \pi_i \) are considered a measure of the skill of the players, higher values indicating better players.

Note that the win probabilities \( p_{ij} \) are invariant under multiplication of all the \( \pi_i \) by any constant. One can remove this ambiguity by imposing any convenient normalization condition. Here we fix the geometric mean strength to be 1, which is equivalent to setting \( \prod_i \pi_i = 1 \). This choice has the nice effect that the probability \( p_i \) of a player with strength \( \pi \) beating the average player with strength 1 is \( p_i = \pi / (\pi + 1) \) and hence \( \pi = p_i (1 - p_i) \). Thus the strength parameter has a simple interpretation: it is the odds of beating the average player.

A. Zermelo’s algorithm

Suppose the tournament consists of a total of \( M \) games between pairs of players and let \( w_{ij} \) be the number of times that player \( i \) beats player \( j \). Our goal is to make a best estimate of the strengths \( \pi_i \) based on these observations. The classic approach is to write the likelihood of the observed games given the strengths (represented by a matrix \( W = [w_{ij}] \) and vector \( \pi = [\pi_i] \) respectively) thus:

\[
P(W|\pi) = \prod_{ij} p_{ij}^{w_{ij}} = \prod_{ij} \left( \frac{\pi_i}{\pi_i + \pi_j} \right)^{w_{ij}},
\]

so that the log-likelihood is

\[
\log P(W|\pi) = \sum_{ij} w_{ij} \log \left( \frac{\pi_i}{\pi_i + \pi_j} \right)
= \sum_{ij} w_{ij} \log \pi_i - \sum_{ij} w_{ij} \log(\pi_i + \pi_j). \tag{5}
\]

Differentiating with respect to \( \pi_i \) for any \( i \) and setting the result to zero we get

\[
\frac{1}{\pi_i} \sum_j w_{ij} - \sum_j \frac{w_{ij} + w_{ji}}{\pi_i + \pi_j} = 0, \tag{6}
\]

which can be rearranged to read

\[
\pi_i = \frac{\sum_j w_{ij}}{\sum_j (w_{ij} + w_{ji})}(\pi_i + \pi_j). \tag{7}
\]

In general this equation has no closed-form solution, but it can be solved numerically by simple iteration: one picks a suitable set of non-negative starting values for the \( \pi_i \)—random values are often used—and then computes new values \( \pi_i' \) according to

\[
\pi_i' = \frac{\sum_j w_{ij}}{\sum_j (w_{ij} + w_{ji})}(\pi_i + \pi_j). \tag{8}
\]

The iteration can be performed synchronously (all \( \pi_i \) updated at the same time) or asynchronously (\( \pi_i \) updated one by one in cyclic fashion, akin to the well-known Jacobi method of linear algebra). Both give the same final result but it is generally believed that asynchronous updating is more efficient, since the update of any individual \( \pi_i \) benefits from the improved estimates of previously updated ones. In this paper we use asynchronous updates. Our convergence results in Section III are also for the asynchronous case.

It can be proved that, subject to certain conditions, Zermelo’s algorithm converges to the global maximum of the likelihood and hence provides a maximum-likelihood estimate of the strengths \( \pi_i \). \[1\], \[12\]. The values can then be sorted in order to give a ranking of the players, or simply used in their raw form as a kind of rating.

This iterative algorithm, first described by Zermelo in 1929 \[1\], is the standard method for calculating maximum-likelihood rankings within the Bradley-Terry model.

B. An alternative algorithm

Grouping the terms slightly differently, Eq. (7) can also be written as

\[
\frac{1}{\pi_i} \sum_{j} w_{ij} \frac{\pi_j}{\pi_i + \pi_j} - \sum_{j} \frac{w_{ji}}{\pi_i + \pi_j} = 0, \tag{9}
\]

which can be rearranged in the form

\[
\pi_i = \frac{\sum_j w_{ij} \pi_j}{\sum_j w_{ji} \pi_i} \tag{10}
\]

This suggests a different iterative algorithm for the Bradley-Terry model. Again we choose suitable starting values (for instance at random) then iterate the form

\[
\pi_i' = \frac{\sum_j w_{ij} \pi_j}{\sum_j w_{ji} \pi_i} \tag{11}
\]

to convergence. In Section III we prove that, like Zermelo’s algorithm, this process also converges to the global maximum of the likelihood.
One nice feature of this algorithm is that it is transparent from Eq. (12) that \( \pi_i = 0 \) for any individual who loses all their games and \( \pi_i = \infty \) for any individual who wins all their games. Furthermore, it is clear that the iteration will converge to these values in a single step. The same values are also returned by the standard Zermelo algorithm, but it is less obvious from Eq. (13) that this is true—it is some work to demonstrate the result for the player who wins every game and moreover it takes the Zermelo algorithm an infinite number of iterations to reach the correct value instead of just one. This is a special case of the more general finding that (12) converges faster than Zermelo’s algorithm.

### III. CONVERGENCE

In this section we prove that the iteration of Eq. (12) converges to the global maximum of the likelihood of Eq. (6) from any starting point, whenever a maximum exists.

Zermelo proved that the likelihood has only one stationary point for \( \pi_i \geq 0 \), corresponding to the global maximum, provided the \( \pi_i \) are normalized as discussed in Section II and the directed network of interactions (the network with adjacency matrix \( w_{ij} \)) is strongly connected, i.e., there is a directed path through the network from every individual to every other \( \pi_i \) in. If the network is not strongly connected then there are no stationary points and there is no maximum of the likelihood, and hence our problem has no solution. Here we will assume, as other authors have done, that the network is strongly connected and hence that there is a maximum of the likelihood.

Since any fixed point of the iteration of Eq. (12) corresponds to a stationary point of the likelihood, and since the iteration generates non-negative values of \( \pi_i \) only, it follows that if the iteration converges to a fixed point that point must be the global maximum. To prove that it converges to a fixed point it suffices to demonstrate that the value of the log-likelihood always increases upon application of Eq. (12) unless a fixed point has been reached, since the log-likelihood cannot increase without bound, being bounded above by the maximum.

We consider the asynchronous version of the iteration of Eq. (12) in which a single \( \pi_i \) is updated at each step, all others \( \pi_j \) remaining the same. The \( \pi_i \) are updated in order until all \( N \) have been updated. Consider the step on which a particular \( \pi_i \) is updated. We define a function \( f(\pi_i) \) equal to the sum of the terms in the log-likelihood, Eq. (9), that depend on \( \pi_i \):

\[
f(\pi_i) = \sum_j w_{ij} \log \frac{\pi_i}{\pi_i + \pi_j} - \sum_j w_{ji} \log (\pi_i + \pi_j).
\]

Noting that \( \log x \leq x - 1 \) for all real \( x > 0 \) and making the substitution \( x \to x/y \), we derive the useful inequality

\[
\log y \geq \log x - \frac{x}{y} + 1,
\]

for all \( x, y > 0 \), or equivalently

\[
\log y \geq \log x - \frac{x}{y} + 1,
\]

with the exact equality holding if and only if \( x = y \). This implies for any \( \pi_i \) and \( \pi_i' \) that

\[
\log \frac{\pi_i'}{\pi_i + \pi_j} \geq \log \frac{\pi_i}{\pi_i + \pi_j} - \frac{\pi_i}{(\pi_i + \pi_j)} + 1
\]

\[
= \log \frac{\pi_i}{\pi_i + \pi_j} + \frac{(\pi_i' - \pi_i) / \pi_i'}{\pi_i + \pi_j},
\]

and (14) implies that

\[
\log(\pi_i' + \pi_j) \leq \log(\pi_i + \pi_j) + \frac{\pi_i' + \pi_j}{\pi_i + \pi_j} - 1
\]

\[
= \log(\pi_i + \pi_j) + \frac{\pi_i'}{\pi_i + \pi_j},
\]

Evaluating Eq. (13) at the point \( \pi_i' \) defined by Eq. (12) and applying these two inequalities, we find that

\[
f(\pi_i') = \sum_j w_{ij} \log \frac{\pi_i'}{\pi_i + \pi_j} - \sum_j w_{ji} \log (\pi_i' + \pi_j)
\]

\[
\geq \sum_j w_{ij} \log \frac{\pi_i}{\pi_i + \pi_j} + \frac{\pi_i' - \pi_i}{\pi_i'} \sum_j w_{ij} \frac{\pi_j}{\pi_i + \pi_j} - \sum_j w_{ji} \log (\pi_i + \pi_j) - (\pi_i' - \pi_i) \sum_j \frac{w_{ji}}{\pi_i + \pi_j}
\]

\[
= \sum_j w_{ij} \log \frac{\pi_i}{\pi_i + \pi_j} - \sum_j w_{ji} \log (\pi_i + \pi_j) + (\pi_i' - \pi_i) \left[ \frac{1}{\pi_i} \sum_j w_{ij} \frac{\pi_j}{\pi_i + \pi_j} - \sum_j \frac{w_{ji}}{\pi_i + \pi_j} \right]
\]

\[
= f(\pi_i),
\]

where we have used Eq. (13) again, the term inside the square brackets vanishes because of (12), and the exact equality applies if and only if \( \pi_i' = \pi_i \).

Thus \( f(\pi_i) \) always increases upon application of Eq. (12) and hence so also does the log-likelihood, unless \( \pi_i' = \pi_i \), in which case the log-likelihood remains the same but could still increase when one of the other \( \pi_i \) is updated. Only if \( \pi_i' = \pi_i \) for all \( i \) does the log-likelihood not increase at all, but if this occurs then by definition we have reached a fixed point of the iteration, and hence we have reached the global maximum. This now guarantees the convergence of the iterative algorithm of Eq. (12) to the global likelihood maximum.

In passing, we observe that, given the existence of two different iterations, Eqs. (9) and (12), that both converge to the same maximum-likelihood estimate, one might wonder whether there exist any others. In fact, it turns out there exists an entire one-parameter family of such iterations which includes (9) and (12) as special cases. Of these, Eq. (12) converges most rapidly and hence is our primary focus in this paper, but for the interested reader the full family is described in Appendix.
IV. TIES

Ties or draws occur in certain types of competition, such as chess and soccer. There are a number of ways to generalize ranking calculations to include ties. The simplest is just to consider a tied game to be half of a win for each of the players. This approach is used for instance in the Elo chess rating system and can be trivially incorporated into the algorithm of Section II.B by suitably modifying the values $w_{ij}$. A more sophisticated approach, however, incorporates the probability of a tie into the model itself. There is more than one way to do this [13, 14], but here we employ the modification of the Bradley-Terry model proposed by Davidson [14]. One again defines strengths $\pi_i$ for each player and the probabilities of a win $p_{ij}$ and a tie $q_{ij}$ between players $i$ and $j$ are

$$p_{ij} = \frac{\pi_i}{\pi_i + \pi_j + 2\nu/\pi_i\pi_j}, \quad q_{ij} = \frac{2\nu/\pi_i\pi_j}{\pi_i + \pi_j + 2\nu/\pi_i\pi_j}$$

(19)

where $\nu > 0$ is a parameter which controls the overall frequency of ties and which we estimate by maximum likelihood along with the strengths. Note that when $\pi_i = \pi_j$ we have $q_{ij} = \nu/(1+\nu)$ and hence $\nu = q_{ij}/(1-q_{ij})$, so $\nu$ can be interpreted as the odds of a tie between evenly matched players.

The form (19) satisfies the obvious requirements that $p_{ij} + p_{ji} + q_{ij} = 1$ and $q_{ij} = q_{ji}$ and also has the intuitive property that the probability of a tie is greatest when the players are evenly matched and vanishes as $\pi_i$ and $\pi_j$ become arbitrarily far apart. As with the standard Bradley-Terry model, the probabilities $p_{ij}$ and $q_{ij}$ are invariant under multiplication of all $\pi_i$ by a constant, and again we remove this ambiguity by normalizing them so that $\prod_i \pi_i = 1$.

Davidson [14] proposed an iterative algorithm for computing maximum-likelihood estimates of the strengths and the parameter $\nu$ within this model. Defining $w_{ij}$ as before to be the number of times $i$ beats $j$ and $t_{ij} = t_{ji}$ to be the number of ties, we can write the likelihood of a set of observations $W = [w_{ij}], T = [t_{ij}]$ as

$$P(W, T|\pi, \nu) = \prod_{ij} p_{ij}^{w_{ij}} q_{ij}^{t_{ij}}$$

(20)

and the corresponding log-likelihood is

$$\log P(W, T|\pi, \nu) = \log \prod_{ij} p_{ij}^{w_{ij}} q_{ij}^{t_{ij}} = \sum_{ij} (w_{ij} + \frac{1}{2}t_{ij}) \log \frac{\pi_i}{\pi_i + \pi_j + 2\nu/\pi_i\pi_j}$$

$$- \sum_{ij} \left( w_{ij} + \frac{1}{2}t_{ij} \right) \log \left( \frac{\pi_i + \pi_j + 2\nu/\pi_i\pi_j}{\pi_i + \pi_j + 2\nu/\pi_i\pi_j} \right).$$

(21)

The combination $w_{ij} + \frac{1}{2}t_{ij}$ comes up repeatedly in the analysis so, following Davidson, we define the convenient shorthand $a_{ij} = w_{ij} + \frac{1}{2}t_{ij}$ and

$$\log P(W, T|\pi, \nu) = \sum_{ij} a_{ij} \frac{1}{2} \log 2\nu + \sum_{ij} \frac{1}{2} \log a_{ij} + \frac{1}{2} \log \left( \frac{\pi_i + \pi_j + 2\nu/\pi_i\pi_j}{\pi_i + \pi_j + 2\nu/\pi_i\pi_j} \right).$$

(22)

Differentiating with respect to $\pi_i$ and setting the result to zero gives

$$\frac{1}{\pi_i} \sum_{j} a_{ij} = \sum_{j} (a_{ij} + a_{ji}) \frac{1 + \nu \sqrt{\pi_j/\pi_i}}{\pi_i + \pi_j + 2\nu \sqrt{\pi_i\pi_j}}$$

(23)

which can be rearranged to give

$$\pi_i = \sum_{j} \frac{a_{ij}}{\pi_i + \pi_j + 2\nu \sqrt{\pi_i\pi_j}}.$$}

(24)

This equation has no general closed-form solution for $\pi_i$ but Davidson proposed solving it by the iteration

$$\pi'_i = \frac{\sum_{j} a_{ij}}{\sum_{j} \left( a_{ij} + a_{ji} \right) \frac{1 + \nu \sqrt{\pi_j/\pi_i}}{\pi_i + \pi_j + 2\nu \sqrt{\pi_i\pi_j}}}.$$}

(25)

We can also calculate a maximum-likelihood estimate of the parameter $\nu$ by differentiating (22) with respect to $\nu$ to get

$$\frac{1}{2\nu} \sum_{ij} t_{ij} = \sum_{ij} a_{ij} \frac{2\nu \sqrt{\pi_j/\pi_i}}{\pi_i + \pi_j + 2\nu \sqrt{\pi_i\pi_j}}.$$}

(26)

which is again solved by iteration:

$$\nu' = \frac{\frac{1}{2} \sum_{ij} t_{ij}}{\sum_{ij} a_{ij} \frac{2\nu \sqrt{\pi_j/\pi_i}}{\pi_i + \pi_j + 2\nu \sqrt{\pi_i\pi_j}}}.$$}

(27)

Davidson used asynchronous updates in which one applies Eq. (26) to each $\pi_i$ in turn, then applies (27) once to update $\nu$, then repeats until convergence is achieved. This is a natural generalization of Zermelo’s algorithm, Eq. (9), to situations where ties are allowed and it includes Zermelo’s algorithm as the special case when $\nu = 0$ and $t_{ij} = 0$. Davidson proved that the procedure always converges to the global likelihood maximum (when the maximum exists), but once again convergence can be slow in practice. Here we propose an alternative algorithm which generalizes Eq. (12) and is substantially faster.

Equation (26) can be rearranged in the form

$$\frac{1}{\pi_i} \sum_{j} a_{ij} + \frac{\nu \sqrt{\pi_j/\pi_i}}{\pi_i + \pi_j + 2\nu \sqrt{\pi_i\pi_j}} = \sum_{j} a_{ji} \frac{1 + \nu \sqrt{\pi_j/\pi_i}}{\pi_i + \pi_j + 2\nu \sqrt{\pi_i\pi_j}}.$$}

(28)
which can be solved by iterating the equation

$$\pi_i' = \frac{\sum_j a_{ij} \left( \frac{\pi_j + \nu \sqrt{\pi_i \pi_j}}{\pi_i + \pi_j + 2\nu \sqrt{\pi_i \pi_j}} \right)}{\sum_j a_{ji} \left( \frac{1 + \nu \sqrt{\pi_j / \pi_i}}{\pi_i + \pi_j + 2\nu \sqrt{\pi_i \pi_j}} \right)}.$$  \hspace{1cm} (29)

Similarly, writing $a_{ij} = w_{ij} + \frac{1}{2} t_{ij}$, Eq. (20) can be rearranged in the form

$$\frac{1}{2\nu} \sum_{ij} t_{ij} \frac{\pi_i + \pi_j}{\pi_i + \pi_j + 2\nu \sqrt{\pi_i \pi_j}} = \sum_{ij} w_{ij} \frac{2\sqrt{\pi_i \pi_j}}{\pi_i + \pi_j + 2\nu \sqrt{\pi_i \pi_j}},$$

which can be solved by iterating

$$\nu' = \frac{\frac{1}{2} \sum_{ij} t_{ij} \left( \frac{\pi_i + \pi_j}{\pi_i + \pi_j + 2\nu \sqrt{\pi_i \pi_j}} \right)}{\sum_{ij} w_{ij} \left( \frac{2\sqrt{\pi_i \pi_j}}{\pi_i + \pi_j + 2\nu \sqrt{\pi_i \pi_j}} \right)}.$$  \hspace{1cm} (30)

Equations (29) and (31) are the appropriate generalization of (12) to the case with ties and they include (12) as the special case when $\nu = 0$ and $t_{ij} = 0$. Again we recommend applying the equations asynchronously: one cycle of the algorithm involves updating each $\pi_i$ in turn using Eq. (29), then applying (31) to update $\nu$, and repeating until convergence is achieved. As we show in Section V this procedure converges significantly faster than Davidson’s algorithm.

The proof that Eqs. (29) and (31) do in fact converge to the maximum-likelihood solution follows similar lines to that for the case without ties but the algebra is tedious so we omit it here. The interested reader can find it in Appendix B.

V. RESULTS

The iterations (12) (for the case without ties) and (29) and (31) (with ties) converge significantly faster in typical applications than the traditional algorithm of Zermelo or its extension for the case where ties are allowed. In this section we illustrate convergence rates with a selection of example applications to both real and synthetic data.

A. Computer-generated data

As our first example, we have applied both the Zermelo algorithm and the algorithm of this paper to a collection of random computer-generated data sets. In these calculations we generated synthetic test data with $N = 1000$ players and $M = 50000$ games, for an average of 100 games per player. The players for each game were chosen uniformly at random (with replacement) and the winners of the games were chosen using the Bradley-Terry model itself: scores $s_i$ for each player were drawn from a logistic distribution $P(s) = e^s / (e^s + 1)^2$ and then the winner of each game was chosen at random according to the probability $p_{ij}$ of Eq. (11). In cases where the resulting network of interactions was not strongly connected, games were discarded and redrawn at random until a strongly connected network was achieved, to ensure that a likelihood maximum does exist as discussed in Section III.

For each data set, we first iterate Eq. (12) until it converges to high precision, effectively solving for the maximum-likelihood solution at or close to the limits of numerical accuracy. Then, using either the Zermelo algorithm or our new algorithm, we measure the number of iterations needed to converge to this final solution within a prescribed level of accuracy. Specifically we require that the probability $\pi_i / (\pi_i + 1)$ of beating the average player converge to within $\epsilon = 10^{-6}$ of its final value for all $i$. We choose this criterion because the probability of beating the average player is typically more uniformly distributed than the $\pi_i$ themselves and hence a uniform convergence condition works well when applied to this measure (by contrast with $\pi_i$ or $s_i$ where we expect convergence rates to vary with value so that some kind of non-uniform condition would be needed to avoid fluctuations and dependency on outliers). This criterion is more stringent—and arguably more realistic—than criteria based on convergence of the value of the log-likelihood.

Initial values of $\pi_i$ were chosen randomly from the logistic distribution $e^s / (e^s + 1)^2$. Other methods of choosing the initial values have been proposed which may improve performance in some cases, but we avoid these here to separate effects of the different algorithms from effects of the initial values.

For tests of the algorithm with ties the same procedure was used but wins, losses, and ties were generated according to the probabilities $p_{ij}$ and $q_{ij}$ of Eq. (19) with $\nu = 1/2$ and we compare the convergence rate of the new algorithm of Eqs. (29) and (31) against the generalized Zermelo algorithm of Eqs. (23) and (27), with an initial value of $\nu = 1$ in all cases.

All tests were averaged over 100 randomly generated data sets and the results are reported in Table I. For the case where there are no ties, as these results show, the algorithm of this paper is over a hundred times faster than Zermelo’s algorithm. Where Zermelo’s algorithm takes an average of more than 1400 iterations to converge, the new algorithm takes just 12. The total running time for these calculations was almost an hour for Zermelo’s algorithm; for the new algorithm it was only seconds. For the case with ties the improvement is somewhat smaller but still impressive: the new algorithm is over 40 times faster.

B. Real-world data

These synthetic examples are illuminating but one might ask whether they are truly representative of the data to which ranking methods are applied in practice. In this section we present example applications to sev-
eral real-world data sets and show that our algorithm also offers significant speed improvements in these more realistic settings. The data sets we study are as follows.

**Wolves:** A typical animal dominance hierarchy data set describing observations of subordinate behaviors among members of a family of 15 captive wolves in Arnhem, Netherlands as reported by van Hooff and Wenserski.\(^{[15]}\)

**Vervet monkeys:** A larger dominance hierarchy data set describing observations of agonistic interactions of various kinds among 63 wild vervet monkeys in the Samara Private Game Reserve in South Africa as reported by Vilette et al.\(^{[16]}\). The original data set had 66 monkeys, but three were removed in order to ensure that the network of interactions was strongly connected, as discussed in Section \([11]\).

**American football:** As an example of an application to sports competition, this data set describes professional American football games played in the US National Football League during a single season. Unlike association football, American football proceeds by a series of discrete plays in which the team currently in possession of the ball attempts to advance it up the field. This data set consists of individual plays in all games between the 32 teams in the league during the 2016 regular season, as compiled by Yurko et al.\(^{[17]}\). Only passing plays, running plays, punts, sacks, and field goals are used in the analysis. Other plays such as kickoffs and conversions are excluded. A team is considered to have won a play if either (a) they score points on the play or (b) they advance the ball and retain possession.

**Political figures:** The results of an online paired comparison survey conducted by the Washington Post newspaper in 2010 in which readers were presented with pairs of prominent political figures and asked to judge which had had the worst week in politics. The data were made available on the survey platform allourideas.org.

**Photographs:** Results from the IMDB-Wiki-SbS study of Pavlichenko and Ustalov\(^{[18]}\), a paired comparison study that asked participants to judge people’s age from photographs. Participants were presented with 247,531 pairs of faces drawn from a pool of 9097 possiblies and asked to judge which of the people depicted was older. In principle, a ranking of the results should then be able to infer an estimate of people’s ages. A small number of images were excluded from the data set for our calculations to ensure a strongly connected network.

**Soccer:** Wins, losses, and draws in 898 men’s international association football matches between 177 different countries during the year 2011. Data from Mart Jurisoo at kaggle.com/martj42. The original network of matches was not strongly connected, so the data analyzed here represent only the largest strongly connected component of the network.

**School students:** These data describe declared friendships among 2155 students in a large US high school and its feeder middle school, from the National Longitudinal Study of Adolescent Health (the “Add Health” study)\(^{[19]}\). If student \(i\) states that they are friends with student \(j\) we consider it a win for \(j\); if \(i\) and \(j\) both state they are friends we consider it a tie. Although in principle friendships are not competitive, there is evidence to suggest that friendship patterns among school students do describe a clear hierarchy, because students tend to claim friendship with others who have higher social status than themselves\(^{[20]}\). Thus ranking calculations applied to data like these can be used to infer social status\(^{[21]}\). Treating a reciprocated friendship as a tie is arguably more correct than treating it as two separate wins: reciprocated friendships clearly violate the assumption of independence in the original Bradley-Terry model without ties, since the two wins never go in the same direction, but there is no equivalent violation for the model with ties. The same approach could be applied to other social networks that show similar reciprocity properties. The network of friendships for this data set was not strongly connected, so the data analyzed here represent only the

### Table I: The number of iterations required for the algorithms discussed in this paper to converge in applications to real and synthetic (computer-generated) data. Results are averaged over 100 runs and rounded to the nearest integer. Standard errors on the averages were less than 2% in all cases. \(N\) is the number of individuals or teams being ranked and \(M\) is the total number of interactions among all individuals. “Speed-up” is the factor by which the method of this paper improves upon the traditional Zermelo algorithm or its generalization for the case where ties are allowed.

| Data           | \(N\) | \(M\) | Iterations to reach convergence | Zermelo | This paper | Speed-up |
|----------------|------|------|---------------------------------|--------|-----------|----------|
| *With ties*    |      |      |                                 |        |           |          |
| Synthetic      | 1000 | 50,000 | 1463                           | 13     | 1.117     |          |
| Wolves         | 15   | 10,382 | 2415                           | 145    | 1.17      |          |
| Vervet monkeys | 63   | 11,621 | 233                            | 19     | 1.12      |          |
| American football | 32 | 35,741 | 51                             | 15     | 1.34      |          |
| Political figures | 67 | 76,632 | 54                             | 8      | 1.71      |          |
| Photographs    | 9097 | 247,531| 911                            | 22     | 1.41      |          |
| *Without ties* |      |      |                                 |        |           |          |
| Synthetic      | 1000 | 50,000 | 1128                           | 27     | 1.42      |          |
| Soccer         | 177  | 898   | 1648                           | 421    | 1.39      |          |
| School students| 2155 | 8970  | 2772                           | 613    | 1.45      |          |
| Chess          | 14,852 | 623,727 | 1747                           | 162    | 1.08      |          |
largest strongly connected component of the network.

Chess: Wins, losses, and draws in chess matches between 14,852 expert players on the online chess server lichess.com. For a match to be included, both players must have had Elo rankings of 2000 or higher at the time of the match. A small number of players were removed to ensure the network of matches was strongly connected. The data are from lichess.com via kaggle.com/arevel. With over 600,000 matches, this is the largest data set considered here.

Results on the convergence of our algorithms for these data sets is presented in Table III. The methodology was the same as for the synthetic data of Section V A, the parameters were first converged to high precision, then the results used to estimate the time to convergence in a second run of the calculation. Each calculation was replicated 100 times with random initial conditions.

The picture for these real data sets is similar to that for the synthetic data. In all cases the method of this paper outpaces the traditional Zermelo algorithm. For instance, for the tests without ties the new algorithm is 17 times faster on the smallest example, the dominance hierarchy of wolves, while on the largest example, the photographs, it is a remarkable 41 times faster. The smallest difference is for the American football data set, for which the new algorithm is 3.4 times faster than Zermelo’s algorithm. As with the synthetic data, the speed difference is less dramatic on the tests with ties, but still substantial, with the new algorithm being about 4 to 10 times faster. Convergence was also somewhat slower overall for the case with ties, although this may have more to do with the fact that these data sets are sparser (which tends to slow convergence) than with the presence of ties. Notice that convergence is very fast for the synthetic data with ties, which is relatively dense.

These effects can make a substantial difference to running times in practice. For the dominance hierarchy of wolves, for instance, a single run of Zermelo’s algorithm (implemented in the Python programming language on an up-to-date but otherwise unremarkable personal computer circa 2022) converges to the maximum-likelihood solution in a running time of about 1 minute. The algorithm of this paper, by contrast, takes 3 seconds. For the more demanding photograph data set Zermelo’s algorithm takes over 8 minutes; the method of this paper takes just 11 seconds. For larger applications still, such as to web data or online social networks, the difference could become very significant.

VI. CONCLUSIONS

We have presented an alternative to the classic algorithm of Zermelo for computing rankings from pairwise comparisons using maximum likelihood fits to the Bradley-Terry model, with or without ties allowed. Like Zermelo’s algorithm, the method presented is a simple iterative scheme and straightforward to implement. We have proved that the algorithm always converges to the global maximum of the likelihood and given numerical evidence that it does so faster—typically many times faster—than Zermelo’s algorithm.

We know of no reason not to use the algorithm presented here and recommend its use in any situation where one might normally use Zermelo’s algorithm.

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Appendix A: Other iterative algorithms

As mentioned at the end of Section III, Eqs. (10) and (12) are not the only iterations that converge to the maximum-likelihood estimate of the Bradley-Terry model. There is in fact an entire one-parameter family of such iterations. For any \( \alpha \geq 0 \) one can rewrite Eq. (10) in the form

\[
\frac{1}{\pi_i} \sum_j w_{ij} \frac{\alpha \pi_i + \pi_j}{\pi_i + \pi_j} - \sum_j \frac{\alpha w_{ij} + w_{ji}}{\pi_i + \pi_j} = 0, \tag{A1}
\]

which we can solve by iterating

\[
\pi'_i = \frac{\sum_j w_{ij} (\alpha \pi_i + \pi_j) / (\pi_i + \pi_j)}{\sum_j (\alpha w_{ij} + w_{ji}) / (\pi_i + \pi_j)} \tag{A2}
\]

until convergence is achieved. When \( \alpha = 1 \) this procedure is equivalent to Zermelo’s algorithm, Eq. (9). When \( \alpha = 0 \) it is equivalent to the algorithm presented in this paper, Eq. (12). For negative \( \alpha \) the iteration is not guaranteed to generate positive values of \( \pi_i \) and hence is invalid, but for positive values of \( \alpha \) it gives a whole range of algorithms, all of which converge to the same maximum-likelihood solution as Zermelo’s algorithm.

Although interesting from a formal point of view, however, these generalized algorithms are not of practical importance. Numerical measurements indicate that convergence becomes monotonically slower with increasing \( \alpha \), so that the main algorithm presented in this paper, Eq. (12), which corresponds to the smallest allowed value of \( \alpha = 0 \), is the fastest algorithm in this one-parameter family and hence the primary case of interest.

For the sake of completeness we include the proof of convergence for general \( \alpha \) here. The proof is in two parts. First we prove the result for \( 0 \leq \alpha \leq 1 \) and then for \( \alpha > 1 \).

**Proof of convergence for \( 0 \leq \alpha \leq 1 \)**: We can prove convergence for \( \alpha \in [0, 1] \) using a variant of the method of Section III. Since the log-likelihood has only a single stationary point corresponding to the global maximum, and since any fixed point of (A2) is a solution of (A1) and hence corresponds to a stationary point of the log-likelihood, it follows that if (A2) converges to a fixed point at all it must converge to the global maximum likelihood. To show that it converges to a fixed point it suffices, as previously, to show that the log-likelihood always increases upon application of (A2) unless a fixed point has been reached. To do this we rewrite the terms in the log-likelihood that depend on \( \pi_i \), Eq. (13), as

\[
f(\pi_i) = \alpha \sum_j w_{ij} \log \pi_i + (1-\alpha) \sum_j w_{ij} \log \frac{\pi_i}{\pi_i + \pi_j} - \sum_j (\alpha w_{ij} + w_{ji}) \log(\pi_i + \pi_j). \tag{A3}
\]
For any $\pi_i, \pi'_i$ the inequality (15) implies that
\[
\log \pi'_i \geq \log \pi_i - \frac{\pi'_i - \pi_i}{\pi_i} + 1 = \log \pi_i + \frac{\pi'_i - \pi_i}{\pi_i},
\] (A4)

with the exact equality applying if and only if $\pi'_i = \pi_i$. Evaluating Eq. (A3) at the point $\pi'_i$ given by Eq. (A2) and applying this inequality along with (16), we find that
\[
f(\pi'_i) = \alpha \sum_j w_{ij} \log \pi'_i + (1 - \alpha) \sum_j w_{ij} \log \frac{\pi'_i}{\pi'_i + \pi_j} - \sum_j (\alpha w_{ij} + w_{ji}) \log (\pi'_i + \pi_j)
\geq \alpha \sum_j w_{ij} \left[ \log \pi_i + \frac{\pi'_i - \pi_i}{\pi_i} \right] + (1 - \alpha) \sum_j w_{ij} \left[ \log \pi_i + \frac{\pi'_i - \pi_i}{\pi_i + \pi_j} \right]
- \sum_j (\alpha w_{ij} + w_{ji}) \left[ \log (\pi_i + \pi_j) + \frac{\pi'_i - \pi_i}{\pi_i + \pi_j} \right]
= f(\pi_i) + (\pi'_i - \pi_i) \left[ \frac{1}{\pi_i} \sum_j w_{ij} \frac{\alpha \pi_i + \pi_j}{\pi_i + \pi_j} - \sum_j \alpha w_{ij} + w_{ji} \right]
= f(\pi_i),
\] (A5)

where we have employed Eq. (A3) again, the term in square brackets in the penultimate line vanishes because of Eq. (A2), and the exact equality applies if and only if $\pi'_i = \pi_i$. Thus $f(\pi_i)$ always increases upon application of (A2) unless $\pi'_i = \pi_i$.

The rest of the proof follows the same lines as in Section III and hence convergence to the global likelihood maximum is established. As a corollary, this also provides an alternative proof of the convergence of Zermelo’s algorithm (the case $\alpha = 1$) which is significantly simpler than the original proof given by Zermelo [1] or the later proof by Ford [12].

**Proof of convergence for $\alpha > 1$:** For $\alpha > 1$ this method of proof does not work because $1 - \alpha$ becomes negative and the inequality in (A5) no longer follows from (16). It is still possible to prove convergence but the proof is more involved. From (15) we have for any $x, y, c > 0$
\[
\log(x + c) \geq \log(y + c) - \frac{y + c}{x + c} + 1 = \log(y + c) + \frac{x - y}{x + c}
= \log(y + c) + \frac{x - y}{x} - \frac{(x - y)/x}{(y + c)/c} + \frac{c(x - y)^2}{x(x + c)(y + c)}
\geq \log(y + c) + \frac{x - y}{x} - \frac{(x - y)/x}{(y + c)/c},
\] (A6)

which is equivalent to
\[
\log x - \log y - \frac{x - y}{x} \geq \frac{\alpha - 1}{\alpha} \left[ \log \frac{x}{x + c} - \log \frac{y}{y + c} - \frac{(x - y)/x}{(y + c)/c} \right],
\] (A7)

with the exact equality applying if and only if $x = y$. Noting that the left-hand side of this inequality is always positive by (15), for any $\alpha > 1$ we then have
\[
\log x - \log y - \frac{x - y}{x} \geq \frac{\alpha - 1}{\alpha} \left[ \log \frac{x}{x + c} - \log \frac{y}{y + c} - \frac{(x - y)/x}{(y + c)/c} \right],
\] (A8)

which can be rearranged to read
\[
\alpha \log x + (1 - \alpha) \log \frac{x}{x + c} \geq \alpha \left[ \log y + \frac{x - y}{x} \right] + (1 - \alpha) \left[ \log \frac{y}{y + c} + \frac{(x - y)/x}{(y + c)/c} \right].
\] (A9)

Now setting $x = \pi'_i$, $y = \pi_i$, and $c = \pi_j$, multiplying by the positive quantities $w_{ij}$, and summing, we have
\[
\alpha \sum_j w_{ij} \log \pi'_i + (1 - \alpha) \sum_j w_{ij} \log \frac{\pi'_i}{\pi'_i + \pi_j} \geq \alpha \sum_j w_{ij} \left[ \log \pi_i + \frac{\pi'_i - \pi_i}{\pi'_i} \right]
+ (1 - \alpha) \sum_j w_{ij} \left[ \log \frac{\pi_i}{\pi_i + \pi_j} + \frac{(\pi'_i - \pi_i)/\pi'_i}{(\pi_i + \pi_j)/\pi_j} \right],
\] (A10)

where the exact equality applies if and only if $\pi'_i = \pi_i$. In combination with (17), this is now sufficient to establish the inequality in (A5) once again, and hence convergence of the algorithm for $\alpha > 1$ is assured.
Appendix B: Proof of convergence for the model with ties

For the case where ties are allowed, the proof that iteration of Eqs. (20) and (31) converges to the maximum of the log-likelihood (22) follows similar lines to that for the case without ties. Davidson [14] proved that the likelihood has only a single stationary point with respect to its parameters, corresponding to the global likelihood maximum, provided the $\pi_i$ are normalized and the network of interactions is strongly connected (with a tie counting as an edge in both directions between the relevant pair of players). Since any fixed point of Eqs. (29) and (31) corresponds to a stationary point of the likelihood, this implies that if our iteration converges to a fixed point at all then that point is the global maximum. To prove that we converge to a fixed point it suffices to show that the log-likelihood always increases upon application of either Eq. (29) or Eq. (31), unless a fixed point has been reached.

Extracting from the log-likelihood of Eq. (22) the terms that depend on $\pi_i$ we can, after some manipulation, express them in the form

$$f(\pi_i) = \sum_j a_{ij} \log \frac{\pi_i}{\pi_i + \pi_j + 2\nu \sqrt{\pi_i \pi_j}} - \sum_j a_{ji} \log \left( \frac{\pi_i + \pi_j + 2\nu \sqrt{\pi_i \pi_j}}{\pi_i} \right),$$

where $a_{ij} = w_{ij} + \frac{1}{2} t_{ij}$ as previously. Applying the inequalities (14) and (15), we have for any $\pi_i$ and $\pi'_i$

$$\log \frac{\pi'_i}{\pi'_i + \pi_j + 2\nu \sqrt{\pi'_i \pi_j}} \geq \log \frac{\pi_i}{\pi_i + \pi_j + 2\nu \sqrt{\pi_i \pi_j}} - \frac{\pi_i/(\pi_i + \pi_j + 2\nu \sqrt{\pi_i \pi_j})}{\pi'_i/(\pi'_i + \pi_j + 2\nu \sqrt{\pi'_i \pi_j})} + 1$$

and

$$\log \left( \frac{\pi'_i}{\pi'_i + \pi_j + 2\nu \sqrt{\pi'_i \pi_j}} \right) \leq \log \left( \frac{\pi_i}{\pi_i + \pi_j + 2\nu \sqrt{\pi_i \pi_j}} \right) + \frac{\pi'_i + \pi_j + 2\nu \sqrt{\pi'_i \pi_j}}{\pi_i + \pi_j + 2\nu \sqrt{\pi_i \pi_j}} - 1$$

(Evaluation of Eq. (15) at the point $\pi'_i$ defined by Eq. (29) and applying these two inequalities, we have

$$f(\pi'_i) = \sum_j a_{ij} \log \frac{\pi'_i}{\pi'_i + \pi_j + 2\nu \sqrt{\pi'_i \pi_j}} - \sum_j a_{ji} \log \left( \frac{\pi'_i + \pi_j + 2\nu \sqrt{\pi'_i \pi_j}}{\pi'_i} \right)$$

$$\geq \sum_j a_{ij} \left[ \log \frac{\pi_i}{\pi_i + \pi_j + 2\nu \sqrt{\pi_i \pi_j}} + \left( \frac{\pi'_i - \pi_i}{\sqrt{\pi'_i}} \right) \frac{2\nu \sqrt{\pi'_i \pi_j}}{\pi_i + \pi_j + 2\nu \sqrt{\pi'_i \pi_j}} + \left( \frac{\pi'_i - \pi_i}{\pi'_i} \right) \frac{\pi_j}{\pi_i + \pi_j + 2\nu \sqrt{\pi_i \pi_j}} \right]$$

$$- \sum_j a_{ji} \log \left( \frac{\pi_i + \pi_j + 2\nu \sqrt{\pi_i \pi_j}}{\pi_i} \right) + \left( \frac{\pi'_i - \pi_i}{\sqrt{\pi'_i}} \right) \frac{2\nu \sqrt{\pi'_i \pi_j}}{\pi_i + \pi_j + 2\nu \sqrt{\pi'_i \pi_j}} + \left( \frac{\pi'_i - \pi_i}{\pi'_i} \right) \frac{\pi'_i - \pi_i}{\pi_i + \pi_j + 2\nu \sqrt{\pi_i \pi_j}} \right]$$

$$= f(\pi_i) + \sum_{ij} a_{ij} \left[ \left( \frac{\sqrt{\pi'_i - \pi_i}}{\sqrt{\pi'_i}} \right) \frac{2\nu \sqrt{\pi'_i \pi_j}}{\pi_i + \pi_j + 2\nu \sqrt{\pi_i \pi_j}} + \left( \frac{\pi'_i - \pi_i}{\pi'_i} \right) \frac{\nu \sqrt{\pi'_i \pi_j}}{\pi_i + \pi_j + 2\nu \sqrt{\pi_i \pi_j}} \right]$$

$$- \sum_j a_{ji} \left[ \left( \frac{\sqrt{\pi'_i - \pi_i}}{\sqrt{\pi'_i}} \right) \frac{2\nu \sqrt{\pi'_i \pi_j}}{\pi_i + \pi_j + 2\nu \sqrt{\pi_i \pi_j}} - \left( \frac{\pi'_i - \pi_i}{\pi'_i} \right) \frac{\nu \sqrt{\pi'_i \pi_j}}{\pi_i + \pi_j + 2\nu \sqrt{\pi_i \pi_j}} \right]$$

$$f(\pi_i) + \frac{\sqrt{\pi'_i - \pi_i}^2}{\pi'_i} \sum_j a_{ij} \frac{\nu \sqrt{\pi'_i \pi_j}}{\pi_i + \pi_j + 2\nu \sqrt{\pi_i \pi_j}} + \frac{\sqrt{\pi'_i - \pi_i}^2}{\pi_i} \sum_j a_{ji} \frac{\nu \sqrt{\pi'_i \pi_j}}{\pi_i + \pi_j + 2\nu \sqrt{\pi_i \pi_j}}$$

$$\geq f(\pi_i),$$

where we have used Eq. (29) to simplify the fourth and fifth lines and the exact equality applies if and only if $\pi'_i = \pi_i$. Hence $f(\pi_i)$ always increases upon application of Eq. (29) and so therefore does the log-likelihood as well, unless $\pi'_i = \pi_i$. 


For any \( \nu, \nu' \) the inequalities (14) and (15) imply that

\[
\log\left( \frac{\nu'}{\pi_i + \pi_j + 2\nu' \sqrt{\pi_i \pi_j}} \right) \geq \log\left( \frac{\nu}{\pi_i + \pi_j + 2\nu \sqrt{\pi_i \pi_j}} \right) + \frac{\nu' - \nu}{\nu' \sqrt{\pi_i \pi_j}} - 1
\]

\( \pi_i + \pi_j + 2\nu' \sqrt{\pi_i \pi_j} \)

\[
= \log\left( \frac{\nu}{\pi_i + \pi_j + 2\nu \sqrt{\pi_i \pi_j}} \right) + \frac{\nu' - \nu}{\nu' \sqrt{\pi_i \pi_j}} - 1
\]

\( \pi_i + \pi_j + 2\nu \sqrt{\pi_i \pi_j} \)

\[
\sum_{ij} w_{ij} \left[ \log(\pi_i + \pi_j + 2\nu \sqrt{\pi_i \pi_j}) + (\nu' - \nu) \right] \frac{2\sqrt{\pi_i \pi_j}}{\pi_i + \pi_j + 2\nu \sqrt{\pi_i \pi_j}}
\]

\( \sum_{ij} w_{ij} \)

\[
g(\nu') = g(\nu) + (\nu' - \nu) \frac{1}{2\nu} \sum_{ij} t_{ij} \frac{\pi_i + \pi_j}{\pi_i + \pi_j + 2\nu \sqrt{\pi_i \pi_j}}
\]

\( \sum_{ij} w_{ij} \)

\[
= g(\nu).
\]

where the term in square brackets vanishes because of Eq. (31) and the exact equality applies if and only if \( \nu' = \nu \). Thus \( g(\nu) \) always increases upon application of (31), and so therefore does the log-likelihood, unless \( \nu' = \nu \).

The remainder of the proof follows the same lines of argument as in Section III and hence convergence of Eqs. (29) and (31) to the unique likelihood maximum is assured.