INFINITE FAMILIES OF HARMONIC SELF-MAPS OF ELLIPSOIDS IN ALL DIMENSIONS

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Abstract. We prove that for given $k \in \mathbb{N}$, $k \geq 3$, $d \in \mathbb{N}$ and each $a \in \mathbb{R}^*$ with
\[ a^2 \leq 4d(d + k - 2)(k - 2)^{-2} \]
the ellipsoid $E_a := \{ x \in \mathbb{R}^k \mid a^{-2}x_1^2 + x_2^2 + \ldots + x_k^2 = 1 \}$ admits infinitely many harmonic self-maps.

1. Introduction
A smooth map $\varphi : (M, g) \to (N, h)$ between closed Riemannian manifolds is said to be harmonic if it is a critical point of the energy functional
\[ E(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 \, dV_g \] (1.1)
and can be characterized by the vanishing of its tension field $\tau(\varphi) := \text{Tr}_g \nabla^{\varphi^*TN} d\varphi$. Here, $\nabla^{\varphi^*TN}$ represents the connection on the vector bundle $\varphi^*TN$. The harmonic map equation
\[ \tau(\varphi) = 0 \]
represents a semilinear second order elliptic partial differential equation with many applications in analysis, geometry and theoretical physics. The study of harmonic maps is thus a vast research topic, in the present manuscript we will deal exclusively with the existence of harmonic maps. Due to the nonlinear nature of the harmonic map equation the existence of harmonic maps can be very complicated to achieve.

In their seminal paper [10], Eells and Sampson started to study the question whether every homotopy class of maps between Riemannian manifolds admits a harmonic representative. They proved that if all its sectional curvatures of the target manifold are non-positive then the answer to this question is affirmative. However, if the target manifold also admits positive sectional curvatures the answer to this question is only known in special cases. In this case, in addition to existence results, there are also non-existence results such as that of Eells and Wood [7] who proved that there is no harmonic map of degree one from a two-torus $T^2$ onto $S^2$.

There are many ways to approach the existence problem for harmonic maps to targets of positive curvature, such as spheres, and it turns out that methods from dynamical systems are a rich source for this particular kind of problem. Using such methods, in [3] Bizoń constructed infinitely many harmonic self-maps of $S^3$, this result was later generalized to $S^k$ for $3 \leq k \leq 6$ by Bizoń and Chmaj in [4]. In Section 6 of [3] Bizoń speculates that the actual reason for the existence of these infinite families of harmonic maps in specific dimensions could be a topological reason: ‘The dynamical system approach which I used in this paper gives little insight to what is the actual ‘reason of existence’ of solutions, by which I mean some general mechanism (preferably topological) explaining the existence of solutions in a manner largely insensitive to the details of the model.’ However, our main result, Theorem 1.1, shows that this is not the...
In this manuscript we focus on harmonic self-maps of Euclidean ellipsoids. Although it seems that harmonic maps to ellipsoids seem to have a rich structure there are not many articles in the literature that have taken up this direction of search. Below we parametrize the $k$-dimensional ellipsoid $E_a \subset \mathbb{R}^k$ as follows

$$x_1 = a \cos \psi, \ x_j = z_j \sin \psi, \quad j = 2, \ldots, k, \quad \text{with} \quad \sum_{j=2}^{k} z_j^2 = 1.$$ 

Further, we assume that $E_a$ is endowed with the induced metric

$$ds^2 = (a^2 \sin^2 \psi + \cos^2 \psi) d\psi^2 + \sin^2 \psi d\Omega_{k-1},$$

where $d\Omega_{k-1}$ represents the volume element of the sphere of dimension $k-1$. Below we parametrize $S^{k-1}$ by the variable $\theta$. In this article we study self-maps of $E_a$ of the form

$$(\psi, \theta) \mapsto (f(\psi), \chi(\theta)), $$

where $f: [0, \pi] \to \mathbb{R}$ and $\chi: S^{k-1} \to S^{k-1}$ is an eigenmap of $S^{k-1}$. It is straightforward to verify and well-known that a map $u: M \to S^n$ is harmonic if it solves

$$\Delta u + |du|^2 u = 0.$$ 

A harmonic map $u: S^{p-1} \to S^{q-1}, p, q \geq 2$ is called eigenmap if $|du|^2 = \lambda$. It is well-known that $u$ is a harmonic eigenmap if and only if all of its components are harmonic polynomials of common degree $d$, in which case $\lambda_d = d(d + p - 2)$. The integer $\lambda_d$ is called eigenvalue of the eigenmap $u$. For more details on eigenmaps we refer to the seminal article [14] of Smith and the book [9] of Eells and Ratto.

Our first main result is the following existence result.

**Theorem 1.1.** Let $k \in \mathbb{N}$, $k \geq 3$, and $d \in \mathbb{N}$ be given. For each $a \in \mathbb{R}^*$ with

$$a^2 < 4d(d + k - 2)(k - 2)^{-2}$$

the ellipsoid $E_a \subset \mathbb{R}^k$ admits infinitely many harmonic self-maps.

On the other hand, our second main result shows that the condition (1.2) is crucial for our existence result.

**Theorem 1.2.** Let $k \in \mathbb{N}$, $k \geq 3$, and $d \in \mathbb{N}$ be given. If $a \in \mathbb{R}^*$ satisfies

$$a^2 \geq 4d(d + k - 2)(k - 2)^{-2}$$

the harmonic self-maps of the ellipsoid $E_a \subset \mathbb{R}^k$ of the form as in Theorem 1.1 do not exist.

In the following remarks we put our main result Theorem 1.1 into its mathematical context.

**Remark 1.3.** (1) Note that for $a = 1$ the ellipsoid $E_a$ degenerates to the sphere $S^k$. If in addition we assume $\chi = \text{id}$, i.e. $d = 1$, we have exactly the case studied by Bizoń and Chmaj in [4]. It is straightforward to check that in this case the the smallness condition (1.2) is only satisfied if $k \leq 6$ which is precisely what one would expect from the existence result of Bizoń and Chmaj from [4].

(2) One can argue that there is a geometric reason why the ellipsoid admits harmonic self-maps in all dimensions $k$ while on the sphere we need to impose the condition $3 \leq k \leq 6$. Both the ellipsoid and the sphere have positive curvature, but for $a$ being small enough, the curvature of the ellipsoid also becomes as small as needed. Usually, positive curvature on the target manifold is an obstruction against the existence of harmonic maps. However, one can argue that our ellipsoids are close to flat space such that their curvature no longer prohibits the existence of harmonic maps.
The advantage of including the eigenmap is the following: If the eigenmap is the identity, i.e. \( \chi = \text{id} \) and \( d = 1 \), the smallness condition \( (1.2) \) reads
\[
a^2 < 4(k - 1)(k - 2)^{-2}
\]
forcing \( a^2 \) to be small. By also taking into account the eigenmap we can choose \( a^2 \) arbitrarily as long as we choose \( d \) large enough such that \( (1.2) \) is satisfied.

The first instance where harmonic maps to ellipsoids were considered in the mathematics literature seems to be in the seminal article of Smith [14] on harmonic maps between spheres. In Section 9 he points out a number of interesting remarks showing that harmonic maps to ellipsoids can be very different compared to the case of a spherical target.

Using variational methods and equivariant differential geometry Eells and Ratto [8] constructed harmonic maps between ellipsoids and spheres. In Corollary 5.8 they also considered the case of harmonic self-maps of ellipsoids and they established the existence of a solution under the condition \( (1.2) \). Although their method of proof is completely different and relies on variational methods they need to impose the same smallness condition \( (1.2) \) that we need to require in Theorem 1.1. However, by our approach, using methods from dynamical systems, we are able to establish the existence of an infinite family of solutions instead of just a single solution.

In [2] Baldes established the existence of a rotationally symmetric harmonic map from the Euclidean ball \( B \) of dimension \( k \) taking values in the ellipsoid \( E_a \) with prescribed boundary data on \( \partial B \) assuming the smallness condition \( (1.2) \) with \( d = 1 \). The result of Baldes was later extended by Fardoun to the case of \( p \)-harmonic maps [11] in which case the smallness condition \( (1.2) \) becomes \( a^2 < 4(k - 1)(k - p)^{-2} \).

An interesting approach to the study of harmonic maps between ellipsoids was presented in the book of Baird [1] Subsection 9.3 by considering harmonic maps between spheres and then performing a suitable deformation of the metric.

For the sake of completeness we want to mention the articles of Helein [12] and Hong [13] where questions of regularity and stability of harmonic maps to ellipsoids are discussed.

For a general introduction concerning harmonic maps with symmetries we refer to the by now classic book of Eells and Ratto [9].

Let \( d \in \mathbb{N} \) be given. To prove Theorem 1.1 we reduce the construction of harmonic self-maps of \( E_a \) to finding solutions \( f : [0, \pi] \to \mathbb{R} \) of the ordinary differential equation
\[
f''(\psi) = \frac{d(d + k - 2)}{2} \frac{a^2 \sin^2 \psi + \cos^2 \psi}{a^2 \sin^2(f(\psi)) + \cos^2(f(\psi))} \frac{\sin(2f(\psi))}{\sin^2 \psi} - (k - 1) \cot \psi f'(\psi)
- \frac{a^2 - 1}{2} \frac{\sin(2f(\psi))}{a^2 \sin^2(f(\psi)) + \cos^2(f(\psi))} f'(\psi)^2 + \frac{a^2 - 1}{2} \frac{\sin 2\psi}{a^2 \sin^2 \psi + \cos^2 \psi} f'(\psi),
\]
which satisfy the boundary conditions
\[
\lim_{\psi \to 0} f(\psi) = 0 \quad \text{and} \quad \lim_{\psi \to \pi} f(\psi) = \ell \pi
\]
for some \( \ell \in \mathbb{Z} \). Solutions to this boundary value problem are then constructed by using methods from dynamical systems as was done in [3] and [4]. We would like to point out that the existence of oscillating solutions of \( (1.3) \) is crucial for the proof of 1.1.

In order to prove Theorem 1.2 we study the linearization of the Euler Lagrange equation around the critical point \( f = \frac{\pi}{2} \) and show that this equation no longer admits oscillating solutions if the smallness condition \( (1.2) \) is violated.

The stability of a given harmonic is an important property that characterizes its qualitative behavior. The intuition is that if a harmonic map is stable, then there does not exist a second harmonic map ‘nearby’, meaning that the critical points of the energy functional \( (1.1) \) are isolated. Here we only study the stability of the identity map when interpreted as a harmonic self-map of an ellipsoid as it is the only explicit solution of \( (1.3) \) that is known. More details
concerning the stability of harmonic maps can e.g. be found in the introduction of our recent article [6].

We prove that the identity map, considered as a harmonic map of the ellipsoid \(E_a\), is unstable. This is consistent with the expectation that positive curvature is an obstruction to stability.

**Theorem 1.4.** For any \(k \in \mathbb{N}, k \geq 3\), and any \(a \in \mathbb{R}^*\) the first eigenvalue of the Jacobi operator associated with harmonic self-maps of ellipsoids evaluated at the identity map is given by

\[
\lambda_1 = a^2(2 - k). \tag{1.4}
\]

Hence, the identity map, when considered as a harmonic self-map of ellipsoids, is unstable.

**Remark 1.5.**

(1) As one should expect, for \(a = 1\), the first eigenvalue (1.4) of the Jacobi operator yields the corresponding result for the sphere, see for example [4, Section 5].

(2) It might be impossible to explicitly calculate the complete spectrum of the Jacobi operator associated with the identity map considered as a harmonic self-map of an ellipsoid. In our recent article [6] we were able to determine such spectra in the case of harmonic self-maps of cohomogeneity one manifolds. However, we were not able to apply these techniques in the case of the ellipsoid.

**Organisation:** In Section 2 we provide preliminaries. The proof of Theorem 1.1 is contained in Section 3. We prove Theorem 1.2 in Section 4. Finally, Theorem 1.4 is shown in Section 5.

## 2. Preliminaries

In this section we derive the Euler-Lagrange equation (1.3). Further, we perform a change of variables and provide a Lyapunov function for the resulting differential equation.

Recall that we parametrize the \(k\)-dimensional ellipsoid \(E_a \subset \mathbb{R}^k\) as follows

\[
x_1 = a \cos \psi, \quad x_j = z_j \sin \psi, \quad j = 2, \ldots, k, \quad \text{with} \quad \sum_{j=2}^k z_j^2 = 1
\]

and that we assume that \(E_a\) is endowed with the induced metric

\[
ds^2 = (a^2 \sin^2 \psi + \cos^2 \psi) d\psi^2 + \sin^2 \psi \, d\Omega_{k-1},
\]

where \(d\Omega_{k-1}\) represents the volume element of the sphere of dimension \(k - 1\). The energy of the self-maps of \(E_a\) of the form

\[
(\psi, \theta) \mapsto (f(\psi), \chi(\theta)),
\]

where \(f : [0, \pi] \to \mathbb{R}\) and \(\chi : \mathbb{S}^{k-1} \to \mathbb{S}^{k-1}\) is an eigenmap of \(\mathbb{S}^{k-1}\) with eigenvalue \(\lambda = d(d+k-2), d \in \mathbb{N}\), is given by

\[
E(f) = C \int_0^\pi \left( f'^2 \frac{a^2 \sin^2 f + \cos^2 f}{a^2 \sin^2 \psi + \cos^2 \psi} + d(d + k - 2) \frac{\sin^2 f}{\sin^2 \psi} \right) \left( a^2 \sin^2 \psi + \cos^2 \psi \right)^{k/2} \sin^{k-1} \psi d\psi \tag{2.1}
\]

for some \(C \in \mathbb{R}_+\). Here and below \('\) denotes derivatives with respect to the variable \(\psi\). The argument of \(f\) and its derivatives is always \(\psi\) and is therefore henceforth omitted.

The following proposition provides the Euler-Lagrange equation of (1.3).

**Proposition 2.1.** The critical points of the energy functional (2.1) are characterized by

\[
f'' = \frac{d(d + k - 2)}{2} \frac{a^2 \sin^2 \psi + \cos^2 \psi \sin 2f}{a^2 \sin^2 f + \cos^2 f \sin^2 \psi} - \frac{a^2 - 1}{2} \frac{\sin 2f}{a^2 \sin^2 f + \cos^2 f} f'^2
\]

\[
- (k - 1) \cot \psi f' + \frac{a^2 - 1}{2} \frac{\sin 2\psi}{a^2 \sin^2 \psi + \cos^2 \psi} f'.
\]
Proof. We consider a one-parameter variation \( f_s \) of \( f \) with fixed endpoints satisfying \( \frac{df}{ds} \big|_{s=0} = \eta \).
A straightforward calculation yields
\[
\frac{dE(f_s)}{ds} \bigg|_{s=0} = C \int_0^\pi \left( 2f'' \eta \frac{a^2 \sin^2 f + \cos^2 f}{a^2 \sin^2 \psi + \cos^2 \psi} + \eta(d + k - 2) \frac{\sin 2f}{\sin^2 \psi} + \eta(a^2 - 1) \frac{\sin 2f}{a^2 \sin^2 \psi + \cos^2 \psi} f'' \right) \left( a^2 \sin^2 \psi + \cos^2 \psi \right)^\frac{1}{2} \sin^{k-1} \psi d\psi.
\]
Using integration by parts we deduce
\[
\frac{dE(f_s)}{ds} \bigg|_{s=0} = -C \int_0^\pi \eta \left( -2f'' \frac{a^2 \sin^2 f + \cos^2 f}{a^2 \sin^2 \psi + \cos^2 \psi} + d(d + k - 2) \frac{\sin 2f}{\sin^2 \psi} \right) \times \left( a^2 \sin^2 \psi + \cos^2 \psi \right)^\frac{1}{2} \sin^{k-1} \psi d\psi
\]
\[= -C \int_0^\pi 2\eta f' \left( \frac{a^2 \sin^2 f + \cos^2 f}{a^2 \sin^2 \psi + \cos^2 \psi} \right)^\frac{1}{2} \sin^{k-1} \psi d\psi.
\]
A direct calculation shows the following identity
\[
\left( \frac{a^2 \sin^2 f + \cos^2 f}{a^2 \sin^2 \psi + \cos^2 \psi} \right)^\frac{1}{2} \sin^{k-1} \psi)^' = \left( \frac{\sin 2f(a^2 - 1)}{a^2 \sin^2 \psi + \cos^2 \psi} \right) f' - \frac{1}{2} \frac{a^2 \sin^2 f + \cos^2 f}{a^2 \sin^2 \psi + \cos^2 \psi} \sin 2\psi(a^2 - 1)
\]
\[+ (k - 1) \cot \psi \frac{a^2 \sin^2 f + \cos^2 f}{a^2 \sin^2 \psi + \cos^2 \psi} \left( a^2 \sin^2 \psi + \cos^2 \psi \right)^\frac{1}{2} \sin^{k-1} \psi.
\]
The claim follows from combining the preceding equations. \( \square \)

Remark 2.2. For \( a = 1 \), (1.3) reduces to the equation studied by Bizoń and Chmaj in [4], see equation (2.3) therein. The construction of solutions of (1.3) in this manuscript is inspired by the method used in [4]. However, we note that for \( a \neq 1 \) the Euler-Lagrange equation (1.3) has a more complicated structure compared to the case \( a = 1 \), e.g. a term which is quadratic in \( f' \) appears in equation (1.3). Therefore, the proofs become substantially more demanding.

Now, we make the change of coordinates
\[
x = \log(\tan(\frac{\psi}{2})), \quad h = f - \frac{\pi}{2}.
\]
The argument of \( h \) and its derivatives is always \( x \) and is therefore henceforth omitted.
In terms of these variables the energy now acquires the form
\[
E(h) = C \int_{-\infty}^\infty \left( h'^2 \frac{a^2 \cos^2 h + \sin^2 h}{a^2 \text{sech}^2 x + \tanh^2 x} + d(d + k - 2) \cos^2 h \left( a^2 \text{sech}^2 x + \tanh^2 x \right)^\frac{1}{2} \text{sech}^{k-2} x \right) dx.
\]
The critical points of \( E(h) \) are characterized by the following differential equation
\[
h'' + \frac{1}{2} (1 - a^2) \frac{\sin(2h)}{a^2 \cos^2 h + \sin^2 h} h'^2 - (1 - a^2) \frac{\tan h x}{a^2 + \sinh^2 x} h' - (k - 2) \tanh x h = 0.
\]
(2.2)
From now on we assume \( h \) to be a solution of (2.2). We introduce \( W : \mathbb{R} \to \mathbb{R} \) by
\[
W(x) := h'^2 \frac{a^2 \cos^2 h + \sin^2 h}{a^2 \text{sech}^2 x + \tanh^2 x} + d(d + k - 2) \sin^2 h.
\]
(2.3)
In the following lemma we show that (2.3) is a Lyapunov function for (2.2).

Lemma 2.3. The function \( W : \mathbb{R} \to \mathbb{R} \) satisfies
\[
W'(x) = 2(k - 2) \tanh x h'^2 \frac{a^2 \cos^2 h + \sin^2 h}{a^2 \text{sech}^2 x + \tanh^2 x}.
\]
In particular, \( W \) is monotonically decreasing on \((-\infty, 0]\) and monotonically increasing on \([0, \infty)\).
Proof. A direct calculation shows that
\[
W'(x) = 2hh' + \frac{a^2 \cos^2 h + \sin^2 h}{a^2 \sech^2 x + \tanh^2 x} + \frac{(1 - a^2) \sin(2h)}{a^2 \sech^2 x + \tanh^2 x} h' + 2(1 - a^2) \frac{\tanh x}{a^2 + \sin^2 x} a^2 \sech^2 x + \tanh^2 x h' + d(d + k - 2) \sin(2h)h'.
\]
Now, replacing $h''$ in the above equation using (2.2) completes the proof. □

3. PROOF OF THEOREM 1.1

In this section we provide the proof of Theorem 1.1. From (2.2) we get that solutions of (2.2) with $h'(0) = 0$ are even under the reflection $x \rightarrow -x$ and solutions with $h(0) = 0$ are odd under the reflection $x \rightarrow -x$. Below we will focus on such odd and even solutions only. Therefore it is sufficient to consider $x \geq 0$. An odd solution of (2.2) with $h'(0) = b$ will be called $b$-orbit and denoted by $h(x, b)$. An even solution of (2.2) with $h(0) = d$ will be called $d$-orbit. Below we focus on $b$-orbits, but all our considerations can easily be adapted to $d$-orbits as well. The Lyapunov function (2.3) associated to a $b$-orbit will be denoted by $W(x, b)$.

The following lemma is an immediate consequence of Lemma 2.3. It shows that $W(\cdot, b)$ is small on intervals of the form $[0, T]$, $T \in \mathbb{R}^+$, provided that $b$ is small enough.

Lemma 3.1. Given any $T > 0$ and $\eta > 0$, there exists an $\varepsilon(\eta, T)$ such that if $b < \varepsilon$ then $W(x, b) < \eta$ for $x \leq T$.

Proof. From Lemma 2.3 we have
\[
W'(x, b) \leq 2(k - 2)W(x, b).
\]
Integrating once hence yields
\[
W(x, b) \leq W(0, b) \exp(2(k - 2)x) = b^2 \exp(2(k - 2)x).
\]
For $\varepsilon = \exp(-(k - 2)T)\sqrt{\eta}$ the claim hence follows. □

The intuition for Lemma 3.1 is as follows:

- For $a^2 \leq 1$ and $\eta$ small, the estimate
  \[
a^2 h'^2(x, b) + d(d + k - 2) \sin^2 h(x, b) \leq W(x, b) < \eta
  \]
  and Lemma 3.1 imply that $h(x, b)$ and $h'(x, b)$ are small for $0 \leq x \leq T$. In other words, the graph of $h$ stays close to $0$ for $0 \leq x \leq T$.

- For $a^2 \geq 1$ and $\eta$ small, the estimate
  \[
  h'^2(x, b) + d(d + k - 2) \sin^2 h(x, b) \leq W(x, b) < \eta
  \]
  and Lemma 3.1 imply that $h(x, b)$ and $h'(x, b)$ are small for $0 \leq x \leq T$. In other words, the graph of $h$ stays close to $0$ for $0 \leq x \leq T$.

Next, we introduce the so-called rotation number of a $b$-orbit. We will eventually prove that for each $\ell \in \mathbb{N}$ there exists a solution of the boundary value problem (1.3) with rotation number $\ell$. For any $b$-orbit we set $\theta(x, b)$ to be
\[
\theta(0, b) = \frac{\pi}{2}, \quad \theta(x, b) = \arctan \left( \frac{h'(x, b)}{h(x, b)} \right)
\]
for any $x > 0$. The rotation number $\Omega(b)$ of the $b$-orbit is defined by
\[
\Omega(b) = -\frac{1}{\pi} \left( \theta(x_e(b), b) - \theta(0, b) \right).
\]
Here, $x_e(b)$ denotes the smallest $x > 0$ at which the $b$-orbit exits the set
\[
\Gamma := \{(h, h', x) \mid h < \frac{\pi}{2}, x > 0, (h, h') \neq (0, 0)\}.
\]
If the \( b \)-orbit does not exit \( \Gamma \), we set \( x_\varepsilon(b) = \infty \).

In the next lemma we show that \( \theta'(\cdot, b) \) is uniformly bounded by a negative constant on an interval of the form \((x_0, T), x_0 > 0\), provided that \( b \) is small enough.

**Lemma 3.2.** Let \( k \in \mathbb{N} \) be given. Then, there exist \( a \in (0, 1) \), \( x_0 > 0 \) and \( c > 0 \) such that for any \( T > x_0 \) there exists an \( \varepsilon > 0 \), such that \( \theta'(x, b) \leq -c \) for all \( x \in (x_0, T) \) and any \( b \in (0, \varepsilon) \).

**Proof.** Let \( k \in \mathbb{N} \) be fixed. Throughout the proof we repeatedly use the notations \( \theta := \theta(x) := \theta(x, b) \) and \( h(x) := h(x, b) \); we omit the arguments when they worsen the readability and do not contribute to the clarity of the proof. A straightforward calculation using (2.2) yields

\[
\theta'(x) = - \sin^2 \theta + \frac{\tan x}{a^2 + \sinh^2 x} \frac{\sin(2\theta)}{2} + (k - 2) \tan x \frac{\sin(2\theta)}{2}
\]

\[
- d(d + k - 2) \frac{a^2 \sech^2 x + \tanh^2 x}{a^2 \cos^2 h(x) + \sin^2 h(x)} \frac{\sin(2h(x))}{2h(x)} \cos^2 \theta
\]

\[
- (1 - a^2) \frac{1}{a^2 \cos^2 h(x) + \sin^2 h(x)} \frac{\sin(2h(x))}{2h(x)} h^2(x) \cos^2 \theta.
\]

We use the identity

\[
- \sin^2 \theta = - \frac{1 + d(d + k - 2)}{2} + d(d + k - 2) \cos^2 \theta + \frac{1 - d(d + k - 2)}{2} \cos(2\theta)
\]

and split

\[
- \frac{a^2 \sech^2 x + \tanh^2 x}{a^2 \cos^2 h(x) + \sin^2 h(x)} \frac{\sin(2h(x))}{2h(x)}
\]

\[
= (a^2 \sech^2 x + \tanh^2 x) \left( \frac{1}{a^2} - \frac{1}{a^2 \cos^2 h(x) + \sin^2 h(x)} \frac{\sin(2h(x))}{2h(x)} \right) - (\sech^2 x + \tanh^2 x a^{-2}).
\]

Furthermore, we use

\[
\sech^2 x + \frac{\tanh^2 x}{a^2} = \sech^2 x + \tanh^2 x (a^{-2} + 1 - 1) = 1 + \tanh^2 x (a^{-2} - 1).
\]

Thus, we obtain

\[
\theta'(x) = - \frac{1}{2} (1 + d(d + k - 2) + d(d + k - 2) \tan^2 x (a^{-2} - 1))
\]

\[
+ \frac{1}{2} \frac{(k - 2 + (1 - a^2) \frac{\tan x}{a^2 + \sinh^2 x}) |\sin(2\theta)|}{d-1 + 1 + d(d + k - 2) \tan^2 x (a^{-2} - 1))} \cos(2\theta) + \delta_a(x), \quad (3.1)
\]

where

\[
\delta_a(x) := - (1 - a^2) \frac{1}{a^2 \cos^2 h(x) + \sin^2 h(x)} \frac{\sin(2h(x))}{2h(x)} h^2(x) \sin^2 \theta
\]

\[
+ d(d + k - 2) (a^2 \sech^2 x + \tanh^2 x) \left( \frac{1}{a^2} - \frac{1}{a^2 \cos^2 h(x) + \sin^2 h(x)} \frac{\sin(2h(x))}{2h(x)} \right) \cos^2 \theta
\]

\[
+ \frac{k - 2}{2} (\frac{\tanh x}{a^2 + \sinh^2 x} \sin(2\theta) - |\sin(2\theta)|)
\]

\[
+ \frac{1 - a^2}{2} (\frac{\tanh x}{a^2 + \sinh^2 x} \sin(2\theta) - \frac{\tan x}{a^2 + \sinh^2 x} |\sin(2\theta)|).
\]

As a next step we introduce \( A, B : \mathbb{R} \rightarrow \mathbb{R} \) by

\[
A(x) = \frac{1}{2} (k - 2 + (1 - a^2) \frac{\tan x}{a^2 + \sinh^2 x}),
\]

\[
B(x) = - \frac{1}{2} (d - 1 + d(d + k - 2) \tan^2 x (a^{-2} - 1)),
\]
and consider the function
\[ g(\theta) := A(x)|\sin 2\theta| + B(x)\cos 2\theta. \]

Note that by assumption we have \( A(x) > 0 \) for \( x \geq 0 \). The extremal points of \( g(\theta) \) can be characterized by
\[ -B(x) + A(x) \frac{\cos 2\theta}{|\sin 2\theta|} = 0. \]

One way of achieving this is by regularizing the absolute value in the definition of \( g(\theta) \). Since \( B(x) < 0 \), we find that \( A(x)|\sin 2\theta| \) vanishes for \( \theta^* = -\frac{1}{4} \arccot \left( \frac{B(x)}{A(x)} \right) \). It is straightforward to check that
\[ g(\theta^*) = \sqrt{A(x)^2 + B(x)^2}. \]

Then, for \( x \in \mathbb{R} \) we have
\[ \theta'(x) \leq -\frac{1}{2}(1 + d(d + k - 2) + d(d + k - 2) \tanh^2 x(a^{-2} - 1)) + \sqrt{A(x)^2 + B(x)^2} + \delta_a(x). \]

Note that
\[ \lim_{x \to \infty} \left( -\frac{1}{2}(1 + d(d + k - 2) + d(d + k - 2) \tanh^2 x(a^{-2} - 1)) + \sqrt{A(x)^2 + B(x)^2} \right) \]
\[ = -\frac{1}{2}(1 + d(d + k - 2)a^{-2}) + \frac{1}{2a^2} \sqrt{(k-2)^2a^4 + (a^2 - d(d + k - 2))^2}. \]

Since \( k \in \mathbb{N}, k \geq 3, \) and \( d \in \mathbb{N} \) are given, there hence exist \( a \in (0,1), x_0 > 0 \) and \( c > 0 \) such that
\[ \theta'(x, b) \leq -2c + \delta_a(x) \]
for \( x \geq x_0 \) and any \( b \in \mathbb{R}_+ \). By Lemma 3.1 for any \( T > x_0 \) there exists an \( \varepsilon > 0 \) such that \( \delta_a(x) < c \) for \( x \leq T \) and \( b \in (0, \varepsilon) \). This establishes the claim. \( \square \)

**Remark 3.3.** Note that the smallness condition on \( a \), which we applied in the previous proof,
\[ -\frac{1}{2}(1 + d(d + k - 2)a^{-2}) + \frac{1}{2a^2} \sqrt{(k-2)^2a^4 + (a^2 - d(d + k - 2))^2} < 0 \]
is equivalent to
\[ a^2 \leq 4d(d + k - 2)(k-2)^{-2} \]
which is precisely the condition \((1.2)\) we assumed in Theorem \(1.1\).

From the two preceding lemmas we obtain that the rotation number \( \Omega(b) \) becomes arbitrary large provided that \( b \) is small enough.

**Proposition 3.4.** For any given \( N > 0 \) there exists an \( \varepsilon > 0 \) such that for \( b \in (0, \varepsilon) \) we have \( \Omega(b) > N \).

**Proof.** Let \( k \in \mathbb{N} \) be given. By Lemma 3.2 there exist \( a \in (0,1), x_0 > 0 \) and \( c > 0 \) such that for any \( T > x_0 \) there exists an \( \varepsilon_1 > 0 \), such that \( \theta'(x, b) \leq -c \) for all \( x \in (x_0, T) \) and any \( b \in (0, \varepsilon_1) \). Hence, for any \( T > x_0 \) we have
\[ \theta(T, b) - \theta(0, b) = \theta(T, b) - \theta(x_0, b) + \theta(x_0, b) - \theta(0, b) \]
\[ = \int_{x_0}^{T} \theta'(s, b)ds + \theta(x_0, b) - \theta(0, b) \]
\[ \leq -c(T - x_0) + \theta(x_0, b) - \theta(0, b). \]

By Lemma 3.1 for any given \( T > x_0 \), there exists an \( \varepsilon_2 > 0 \) such that \( x_{\varepsilon}(b) > T \) for \( b \in (0, \varepsilon_2) \). Below let \( \varepsilon = \min(\varepsilon_1, \varepsilon_2) \) and \( b \in (0, \varepsilon) \). We thus get
\[ \Omega(b) > -\frac{1}{\pi}[\theta(T, b) - \theta(0, b)] \geq \frac{1}{\pi}[c(T - x_0) - \theta(x_0, b) + \theta(0, b)]. \]
From (3.1), we have \( \theta'(x, b) \leq c_1 \) for \( x \leq T \), where \( c_1 \) is some positive constant depending on \( k \) and \( a \) but not on \( b \). Consequently, we have \( \theta(x_0, b) - \theta(0, b) \leq c_1 x_0 \). Now, let \( T = x_0 + \frac{N\pi + c_1 x_0}{c} \). This provides the claim.

The following lemma states that a \( b \)-orbit which stays in \( \Gamma \) and has only finitely many zeros, satisfies the boundary condition \( \lim_{x \to \infty} h(x, b) = \pm \frac{\pi}{2} \). We omit the proof since it follows along the lines of the proof of Lemma 2.6 in [36].

**Lemma 3.5.** Let \( b \) be a \( b \)-orbit such that \( h(x, b) \in \Gamma \) for all \( x \geq 0 \). Further, assume that \( h(x, b) \) has a finite number of zeros. Then, we have \( \lim_{x \to \infty} h(x, b) = \pm \frac{\pi}{2} \) and \( \lim_{x \to \infty} h'(x, b) = 0 \).

We are now ready to prove our main result, Theorem 1.1.

**Proof of Theorem 1.1.** Let \( d \in \mathbb{N} \) and \( k \in \mathbb{N} \) with \( k \geq 3 \) be given. We set
\[
S_1 := \{ b \mid \text{\( b \)-orbit exits \( \Gamma \) via \( h = \frac{\pi}{2} \) with \( \Omega(b) \leq \frac{1}{2} \} \).}
\]

We first prove that the set \( S_1 \) is non-empty: Recall that for a \( b \)-orbit we have \( h(0) = 0 \) and \( h'(0) = b \). In order to keep formulas shorter, in what follows we write \( h(\cdot) \) and \( W(\cdot) \) instead of \( h(\cdot, b) \) and \( W(\cdot, b) \), respectively. A direct calculation gives
\[
W(0) = b^2.
\]

We chose \( b \) such that \( W(0) > d(d+k-2) \). Due to the monotonicity of \( W(x) \) on \([0, \infty)\), see Lemma 2.3, we then have \( W(x) > d(d+k-2) \) for all \( x \geq 0 \). Therefore, there exists an \( \varepsilon > 0 \) such that \( h'(x) > \varepsilon > 0 \) for all \( x > 0 \). Consequently the \( b \)-orbit constructed above exits the set \( \Gamma \) through \( h = \frac{\pi}{2} \) with \( \Omega(b) < \frac{1}{2} \). We let \( b_1 := \inf S_1 \). By Proposition 3.4 we have \( b_1 > 0 \). The \( b_1 \)-orbit cannot exit the set \( \Gamma \) via \( h = \frac{\pi}{2} \) as this would also hold true for any ‘nearby orbit’ with \( b < b_1 \) which would contradict the definition of \( b_1 \). Therefore, the \( b_1 \)-orbit stays in \( \Gamma \) for all \( x > 0 \) and Lemma 3.5 thus yields \( \Omega(b_1) = \frac{1}{2} \).

To construct the second orbit we define the set \( S_2 \) by
\[
S_2 := \{ b \mid \text{\( b \)-orbit exits \( \Gamma \) via \( h = \frac{\pi}{2} \) with \( \Omega(b) \leq \frac{3}{2} \} \}.}
\]

In order to complete the proof we show that \( S_2 \) is non-empty and then proceed inductively.

By definition of \( b_1 \), for \( b < b_1 \) we have \( \Omega(b) > \frac{1}{2} \). Below we show that for \( b < b_1 \) still sufficiently close to \( b_1 \), we have \( \Omega(b) \leq \frac{1}{2} \) and thus \( b \in S_2 \). To accomplish this goal let \( x_A > 0 \) be such that \( h'(x_A) = 0 \) and \( 0 < h(x_A) < \frac{\pi}{2} \). By choosing \( b < b_1 \) sufficiently close to \( b_1 \), \( x_A \) becomes as large as we want. Furthermore, we let \( x_B \) the smallest \( x > x_A \) such that \( h(x) = 0 \). Note that both \( x_A \) and \( x_B \) exist by the definition of \( b_1 \). From the Euler-Lagrange equation (2.2) we have
\[
h'(x) < 0 \quad \text{for all} \quad x \in (x_A, x_B).
\]

Our goal is to prove that for \( b \) appropriately chosen we have \( W(x_B) > k - 2 \). Thus, from (3.1) and the monotonicity of \( W(x) \), we obtain \( h'(x) < 0 \) for all \( x \geq x_A \), confirming that the set \( S_2 \) is non-empty.

Observe that we may assume without loss of generality that
\[
h'^2(x) \leq d(d + k - 2) \quad \text{for all} \quad x \in (x_A, x_B).
\]

Indeed, if there would exist a \( x_0 \in (x_A, x_B) \) such that \( h'^2(x_0) > d(d + k - 2) \), then the monotonicity of \( W(x) \) on \([0, \infty)\) yields the existence of an \( \varepsilon > 0 \) such that \( h'(x) < -\varepsilon \) for all \( x \geq x_0 \). This in turn implies that \( S_2 \) is non-empty.

Next we estimate \( W(x_B) - W(x_A) \) in two different ways. Afterwards, we combine these estimates to obtain the desired result. From (2.4) we get
\[
W(x_B) - W(x_A) = 2(k - 2) \int_{x_A}^{x_B} \tanh s \, h'^2(s) \frac{a^2 \cos^2 h(s) + \sin^2 h(s)}{a^2 \sech^2 s + \tanh^2 s} \, ds.
\]
Note that for \( a^2 \leq 1 \), we have
\[
a^{-2} \geq \frac{a^2 \cos^2 h(s) + \sin^2 h(s)}{a^2 \text{sech}^2 s + \tanh^2 s} \geq a^2.
\] (3.6)

On the other hand, for \( a^2 \geq 1 \), we have
\[
a^2 \cos^2 h(s) + \sin^2 h(s) \geq a^{-2}.
\]

Below we restrict ourselves to the case \( a^2 \leq 1 \), the case \( a^2 \geq 1 \) can be treated analogously. From (3.5) and (3.6) we get
\[
W(x_B) \geq W(x_A) + 2(k - 2) \tanh(x_A) a^2 \int_{x_A}^{x_B} h'^2(s) ds
\]
\[
\geq d(d + k - 2) \sin^2(h(x_A)) + 2(k - 2) \tanh(x_A) a^2 \int_{x_A}^{x_B} h'^2(s) ds.
\] (3.7)

From (2.3) and the monotonicity of \( W(x) \) on \([0, \infty)\) we get
\[
W(x) - W(x_A) = h'^2(x) \frac{a^2 \cos^2 h + \sin^2 h}{a^2 \text{sech}^2 x + \tanh^2 x} + d(d + k - 2) \sin^2 h - d(d + k - 2) \sin^2 h(x_A) \geq 0.
\]

Let \( x_1 \in (x_A, x_B) \) be such that \( h(x_1) = \frac{\pi}{4} \). Then, for \( x \in [x_1, x_B] \), the preceding inequality yields
\[
h'^2(x) \frac{a^2 \cos^2 h + \sin^2 h}{a^2 \text{sech}^2 x + \tanh^2 x} \geq d(d + k - 2) \sin^2(h(x_A)) - d(d + k - 2) \sin^2 h(x)
\]
\[
\geq d(d + k - 2) \left( \sin^2(h(x_A)) - \frac{1}{2} \right).
\]

Consequently, by (3.6) we get
\[
h'^2(x) \geq a^2 d(d + k - 2) \left( \sin^2(h(x_A)) - \frac{1}{2} \right)
\]
for \( x \in [x_1, x_B] \). Plugging this inequality into (3.7) yields
\[
W(x_B) \geq d(d + k - 2) \sin^2(h(x_A))
\]
\[
\ + 2a^4(k - 2)d(d + k - 2) \tanh(x_A)(x_B - x_1) \left( \sin^2(h(x_A)) - \frac{1}{2} \right).
\] (3.8)

Since \( h(x_1) = \frac{\pi}{4}, h(x_B) = 0 \), we get
\[
x_B - x_1 \geq \frac{\pi}{4 \sqrt{d(d + k - 2)}}
\]
from (3.4). Plugging this inequality into (3.8) gives
\[
W(x_B) \geq d(d + k - 2) \sin^2(h(x_A))
\]
\[
\ + a^4(k - 2) \sqrt{d(d + k - 2)} \tanh(x_A) \frac{\pi}{2} \left( \sin^2(h(x_A)) - \frac{1}{2} \right).
\]

For given \( a \), we chose \( b \) sufficiently close to \( b_1 \) such that \( h(x_A) \) and \( \tanh(x_A) \) become close to \( \frac{\pi}{2} \) and \( 1 \), respectively. Consequently, for such \( b \) we have \( W(x_B) > d(d + k - 2) \), which establishes the claim. This completes the proof of Theorem 1.1.
4. Proof of Theorem 1.2

In this section we provide the proof of Theorem 1.2. Crucial for the proof of Theorem 1.1 is the existence of oscillating solutions of (2.2). By studying the linearization of (2.2) around the critical point \( h = 0 \) we show that this equation no longer admits oscillating solutions if the smallness condition (1.2) is violated.

**Proof of Theorem 1.2:** In order to show that the Euler-Lagrange equation (2.2) does not admit an infinite family of solutions if the smallness condition (1.2) is not imposed we consider the linearization of (2.2) around the critical point \( h = 0 \) and show that for large values of \( x \) the linearized equation no longer admits oscillating solutions.

In order to derive the linearization of (2.2) we consider the following one-parameter variation

\[
\frac{d h_s(x)}{d s} \bigg|_{s=0} = \eta(x).
\]

In the following we will drop the variable \( x \) and write \( h, \eta \) instead of \( h(x) \) and \( \eta(x) \) in order not to blow up the notation.

Now, a straightforward calculation shows that the linearization of (2.2) is given by

\[
\eta'' + \frac{\cos 2h}{a^2 \cos^2 h + \sin^2 h} h'^2 \eta + \frac{\sin 2h}{a^2 \cos^2 h + \sin^2 h} h' \eta' - (1 - a^2)^2 \cos 2h \eta - (1 - a^2) \sin 2h \tan x \eta' - (k - 2) \tanh x \eta' + d(d + k - 2) \frac{a^2 \sech^2 x + \tanh^2 x}{a^2 \cos^2 h + \sin^2 h} \eta - \frac{d(d + k - 2)}{a^2 \cos^2 h + \sin^2 h} \sin^2 (2h) = 0.
\]

Evaluating the above equation at the critical point \( h = 0 \) yields

\[
\eta'' - (1 - a^2) \frac{\tanh x}{a^2 + \sinh^2 x} \eta' - (k - 2) \tanh x \eta' + \frac{d(d + k - 2)}{a^2} (a^2 \sech^2 x + \tanh^2 x) \eta = 0.
\]

For large values of \( x \) we then find that this equation can be approximated by

\[
\eta'' - (k - 2) \eta' + d(d + k - 2) a^{-2} \eta = 0.
\]

Making an ansatz \( \eta(x) = Ae^{\alpha x} \) for \( A \in \mathbb{R}, \alpha \in \mathbb{C} \), we obtain the algebraic equation

\[
\alpha^2 - (k - 2) \alpha + d(d + k - 2) a^{-2} = 0,
\]

which has the solutions

\[
\alpha = \frac{k - 2}{2} \pm \frac{1}{2} \sqrt{(k - 2)^2 - 4d(d + k - 2) a^{-2}}.
\]

This equation has imaginary roots, which is necessary for us in order to obtain an infinite family of solutions, if and only if

\[
a^2 < 4d(d + k - 2)(k - 2)^{-2}.
\]

This is precisely the smallness condition (1.2). On the other hand, it is straightforward to see that for

\[
a^2 \geq 4d(d + k - 2)(k - 2)^{-2},
\]

all solutions \( \alpha \) will be real numbers and in this case we cannot find oscillating solutions of (2.2). \( \square \)
5. Stability of the identity map

In this section we prove Theorem 1.4. As a first step, we provide the Jacobi equation associated with (2.2).

**Proposition 5.1.** Let \( h: \mathbb{R} \to \mathbb{R} \) be a solution of (2.2). Then, the Jacobi equation associated with harmonic self-maps of the ellipsoid is given by

\[
\xi'' - (k - 2) \tan x \xi' + (1 - a^2) \frac{\tanh x}{a^2 \cosh^2 x} \xi' + \left(1 - a^2\right) \frac{\tanh x}{a^2 \cosh^2 x + \sinh^2 x} \xi' - (1 - a^2) \frac{\tanh x}{a^2 \cosh^2 x + \sinh^2 x} \xi' - (k - 2) \tan x \xi' \]

\[= \frac{d(d + k - 2)}{2} \frac{a^2 \sech^2 (x) + \tanh^2 x}{a^2 \cos^2 h + \sin^2 h} \cos (2h) \xi + \lambda \frac{\xi}{\cosh^2 x (a^2 \sech^2 x + \tanh^2 x)} = 0,
\]

where \( \xi: \mathbb{R} \to \mathbb{R} \) and \( \lambda \in \mathbb{R} \).

**Proof.** We consider a variation of \( h(x) \) satisfying \( \frac{d}{dx} |_{x=x_0} h_s(x) = \xi(x) \). Then differentiating (2.2) with respect to \( s \) yields the result. Note that the factor being multiplied to the eigenvalue \( \lambda \) arises due to the volume element of the ellipsoid. \( \square \)

In this manuscript we study the stability of the identity map. Recall that for the identity map we have \( d = 1 \). The following lemma is an immediate consequence of Proposition 5.1.

**Lemma 5.2.** Consider the identity map as a harmonic self-map of the ellipsoid. Then, the Jacobi equation (5.1) simplifies to

\[
\xi'' - (k - 2) \tan x \xi' + (1 - a^2) \frac{\tanh x}{a^2 \cosh^2 x} \xi' + \left(1 - a^2\right) \frac{\tanh x}{a^2 \cosh^2 x + \sinh^2 x} \xi' = 0.
\]

**Proof.** In our setup, the identity map \( h_1 \) is parametrized by \( h_1(x) = -\frac{\pi}{2} + \arctan(e^x) \). Thus, we have the following identities

\[h_1' = \sech x, \quad \cos h_1 = \sech x, \quad \sin h_1 = \tanh x, \quad \cos 2h_1 = 2 \sech^2 x - 1, \quad \sin 2h_1 = \frac{\tan x}{\cosh x},\]

Plugging these identities into the Jacobi equation, see Proposition 5.1, we find

\[
\xi'' - (k - 2) \tan x \xi' + (1 - a^2) \frac{\tanh x}{a^2 \cosh^2 x} \xi' - \frac{1}{2} \frac{(1 - a^2)^2}{(a^2 \cosh^2 x + \sinh^2 x)^2} \xi',
\]

\[= \frac{1}{2} \frac{(1 - a^2)^2}{(a^2 \cosh^2 x + \sinh^2 x)^2} \xi',
\]

\[= \left(1 - a^2\right) \frac{2 \sech^2 x - 1}{a^2 \cosh^2 x} \xi - \frac{(k - 1)}{2} \frac{(1 - a^2)^2}{a^2 \cosh^2 x} \xi,
\]

\[= (k - 1)(2 \sech^2 x - 1) \xi + \lambda \frac{\xi}{\cosh^2 x (a^2 \sech^2 x + \tanh^2 x)} = 0.
\]

Note that we have the following identity

\[-2(1 - a^2) \frac{\tanh^2 x}{a^2 \cosh^2 x} + 2 \sech^2 x - 1 = \frac{a^2 - \sinh^2 x}{a^2 + \sinh^2 x},\]

and applying this twice then completes the proof. \( \square \)

**Proof of Theorem 1.4.** In order to find the first eigenvalue of the Jacobi operator associated with the identity-map a direct calculation shows that

\[\xi(x) = \frac{1}{\sqrt{a^2 + \sinh^2 x}}, \quad \lambda = a^2(2 - k),\]

solves (5.2) completing the proof. \( \square \)
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