THE (B) CONJECTURE FOR UNIFORM MEASURES IN THE PLANE

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Abstract. We prove that for any two centrally-symmetric convex shapes $K, L \subset \mathbb{R}^2$, the function $t \mapsto |e^tK \cap L|$ is log-concave. This extends a result of Cordero-Erausquin, Fradelizi and Maurey in the two dimensional case. Possible relaxations of the condition of symmetry are discussed.

1. Introduction

It was conjectured by Banaszycy (see Latała [1]) that for any convex set $K \subset \mathbb{R}^n$ that is centrally-symmetric (i.e., $K = -K$) and for a centered Gaussian measure $\gamma$, $\gamma(s^{1-\lambda}K) \geq \gamma(sK)^{1-\lambda}\gamma(tK)^{\lambda}$

for any $\lambda \in [0, 1]$ and $s, t > 0$.

This conjecture was proven in [2], in the equivalent form that the function $t \mapsto \gamma(e^tK)$ is log-concave. The same paper raises the question whether (1) remains valid when $\gamma$ is replaced by other log-concave measures. The proof of (1) for unconditional sets and log-concave measures was given in [2] as well:

Theorem 1 ([2], Proposition 9). Let $K \subset \mathbb{R}^d$ be a convex set and let $\mu$ be a log-concave measure on $\mathbb{R}^d$, and assume that both are invariant under coordinate reflections. Then $t \mapsto \mu(e^tK)$ is a log-concave function.

This paper explores the situation in $\mathbb{R}^2$. To distinguish this special case, we call a convex set $K \subset \mathbb{R}^2$, which is compact and has a non-empty interior, a shape. The main result is

Theorem 2. Let $K, L \subset \mathbb{R}^2$ be centrally-symmetric convex shapes. Then $t \mapsto |e^tK \cap L|$ is a log-concave function.

Here $| \cdot |$ is the Lebesgue measure, so Theorem 2 is an analog of (1) for uniform measures – with density $d\mu(x) = 1_{L}(x)dx$. Note that a uniform measure on a set is log-concave if and only if the set is convex.

The condition of central symmetry in Theorem 2 can be replaced by dihedral symmetry. For an integer $n \geq 2$, let $D_n$ be the group of symmetries of $\mathbb{R}^2$ that is generated by two reflections, one across the axis $\text{Span}\{(1, 0)\}$ and the other across the axis $\text{Span}\{(\cos \frac{\pi}{n}, \sin \frac{\pi}{n})\}$. The dihedral group $D_n$ contains $2n$ transformations. A $D_n$-symmetric shape $A \subset \mathbb{R}^2$ is one invariant under the action of $D_n$.

Theorem 3. Let $n \geq 2$ be an integer, and let $K, L \subset \mathbb{R}^2$ be $D_n$-symmetric convex shapes. Then $t \mapsto |e^tK \cap L|$ is a log-concave function.

Key words and phrases. B conjecture, uniform measure.

Supported in part by a grant from the European Research Council.
Examples and open questions. For what sets and measures is \((1)\) valid?

The (B)-conjecture, or \((1)\), is not necessarily true for measures and sets with just one axis of symmetry in \(\mathbb{R}^2\). An example with a log-concave uniform measure is
\[
L = \text{conv} \{(-5,-2),(0,3),(5,-2)\} \\
K = [-6,6] \times [-3,1]
\]
The function \(t \mapsto |e^t K \cap L|\) is not log-concave in a neighbourhood of \(t = 0\).

Another negative result is for quasi-concave measures. These are measures with density \(d\mu(x) = \varphi(x) dx\) satisfying \(\varphi((1-\lambda)x + \lambda y) \geq \max\{\varphi(x),\varphi(y)\}\) for all \(0 \leq \lambda \leq 1\). If
\[
\mu(A) = |A \cap Q| + |A|, \quad Q = [-1,1] \times [-1,1]
\]
then the corresponding function \(t \mapsto \mu(e^t Q)\) is not log-concave in a neighbourhood of \(t = 1\).

The (B)-conjecture for general centrally-symmetric log-concave measures is not settled yet, even in two dimensions. It is also of interest to generalize the method of this paper to higher dimensions.

Notation. For a convex shape \(K \subset \mathbb{R}^2\), its boundary is denoted by \(\partial K\). The support function is denoted \(h_K(x) = \sup_{y \in K} \langle x, y \rangle\). The normal map \(\nu_K : \partial K \to S^1\) is defined for all smooth points on the boundary, and \(\nu_K(p)\) is the unique direction that satisfies \(\langle \nu_K(p), x \rangle = h_K(x)\). We denote the unit square by \(Q = [-1,1] \times [-1,1]\). The Hausdorff distance between sets \(A, B \subset \mathbb{R}^n\) is defined as \(d_H(A,B) = \max\{\sup_{a \in A} d(a,B), \sup_{b \in B} d(b,A)\}\). The radial function \(\rho_K : \mathbb{R} \to \mathbb{R}\) of a convex shape \(K \subset \mathbb{R}^2\) is \(\rho_K(t) = \max\{r \in \mathbb{R} : (r \cos \theta, r \sin \theta) \in K\}\), with period \(2\pi\).

2. Main result

This section proves Theorem 2

Theorem. Let \(K, L \subset \mathbb{R}^2\) be centrally-symmetric convex shapes. Then the function \(f_{K,L}(t) = |e^t K \cap L|\) is log-concave.

Obviously, it suffices to show log-concavity around \(t = 0\).

If we consider the space of centrally-symmetric convex shapes in the plane, equipped with the Hausdorff metric \(d_H\), then the operations \(K, L \mapsto K \cap L\) and \(K \mapsto |K|\) are continuous. This means that the correspondence \(K, L \mapsto f_{K,L}\) is continuous as well. Since the condition of log-concavity in the vicinity of a point is a closed condition in the space \(C(\mathbb{R})\) of bounded continuous functions, the class of pairs of centrally-symmetric shapes \(K, L \subset \mathbb{R}^2\) for which \(f_{K,L}(t)\) is log-concave near \(t = 0\) is closed w.r.t Hausdorff distance. Thus in order to prove Theorem 2 it suffices to prove that \(f_{K,L}(t)\) is a log-concave function near \(t = 0\) for a dense set in the space of pairs of centrally-symmetric convex shapes.

As a dense subset, we shall pick the class of transversely-intersecting convex polygons. This class will be denoted by \(\mathcal{F}\). The elements of \(\mathcal{F}\) are pairs \((K,L)\) of shapes \(K,L \subset \mathbb{R}^2\) that satisfy:

- The sets \((K,L)\) are centrally-symmetric convex polygons in \(\mathbb{R}^2\).
- The intersection \(\partial K \cap \partial L\) is finite.
- None of the points \(x \in \partial K \cap \partial L\) are vertices of \(K\) or of \(L\). That is, there is some \(\varepsilon > 0\) such that \(B(x,\varepsilon) \cap \partial K\) and \(B(x,\varepsilon) \cap \partial L\) are line segments.
For every $x \in \partial K \cap \partial L$, $\nu_K(x) \neq \nu_L(x)$.

**Claim.** The class $F$ is dense in the space of pairs of centrally-symmetric convex shapes (with respect to the Hausdorff metric).

Hence, in order to prove Theorem 2, it is enough to consider polygons with transversal intersection.

**Deriving a concrete inequality.**

**Lemma.** If $(K, L) \in F$, then $f_{K,L}(t)$ is twice differentiable in some neighbourhood of $t = 0$.

**Remark.** In this case, log-concavity around $t = 0$ amounts to the inequality

$$\frac{d^2}{dt^2} \log f(t) \bigg|_{t=0} \leq 0$$

$$f(0) \cdot f''(0) \leq f'(0)^2. \quad (2)$$

**Proof.** The area of the intersection is

$$|aK \cap L| = \int_0^a dr \int_{x \in r\partial K \cap L} h_K(\nu_K(\frac{x}{r})) dl,$$

where $dl$ is the length element.

Denote

$$g_{K,L}(r) = \int_{x \in r\partial K \cap L} h_K(\nu_K(\frac{x}{r})) dl.$$

The transversality of the intersection implies that $g_{K,L}(r)$ is continuous near $r = 1$. Therefore $a \mapsto |aK \cap L|$ is continuously differentiable near $a = 1$.

The contour $r\partial K \cap L$ is a finite union of segments in $\mathbb{R}^2$. Transversality implies that the number of connected components does not change with $r$ in a small neighbourhood of $r = 1$. The beginning and end points of each component are smooth functions of $r$, also in some neighbourhood of $r = 1$. Therefore $g_{K,L}(r)$ is differentiable as claimed.

Note that in such a neighbourhood of $r = 1$, the function $g_{K,L}(r)$ only depends on the parts of $K$ and $L$ that are close to $\partial K \cap L$, and is in fact a sum of contributions from each of the connected components.

Writing $(2)$ in terms of $g(r)$, we get the following condition:

**Definition.** For convex shapes $(K, L) \in F$, we say that $K$ and $L$ satisfy property $B$, or that $B(K, L)$, if

$$|K \cap L| \cdot [g_{K,L}(1) + g'_{K,L}(1)] \leq g_{K,L}(1)^2. \quad (3)$$

The set $F$ is open with respect to the Hausdorff metric, and in particular, if $(K, L) \in F$ then $(K, rL) \in F$ for every $r$ in some neighbourhood of $r = 1$. If $B(K, rL)$ holds for every $r$ in such a neighbourhood, then $f_{K,L}(t)$ is log-concave in some neighbourhood of $t = 0$, as

$$f_{K,L}(t_0 + t) = e^{2t_0} f_{K,e^{-t_0}L}(t).$$

Therefore verifying $(3)$ for all pairs $(K, L) \in F$ will prove Theorem 2.
Reduction to parallelograms. Given two polygons \((K, L) \in \mathcal{F}\), the intersection \(\partial K \cap L\) consists of a finite number of connected components. Due to central symmetry, they come in opposite pairs. We denote these components by \(S_1, \ldots, S_{2n}\), and \(S_{i+n} = \{x : x \in S_i\}\).

We define a pair of convex shapes \(K^{(i)}, L^{(i)}\) for each \(1 \leq i \leq n\) via the following properties.

- The shape \(K^{(i)}\) is the largest convex set whose boundary contains \(S_i \cup S_{i+n}\). Equivalently, denoting by \(x_1, x_2\) the endpoints of \(S_i\), and by \(x\) the solution of the equations
  
  \[
  \begin{cases}
  \langle \nu_K(x_1), x \rangle = h_K(\nu_K(x_1)) \\
  \langle \nu_K(x_2), x \rangle = -h_K(\nu_K(x_2))
  \end{cases}
  \]

  \(K^{(i)} = \text{conv}(S_i \cup S_{i+n} \cup \{x, -x\})\).

- The shape \(L^{(i)}\) is the parallelogram defined by the four lines
  
  \(\langle \nu_L(x_1), x \rangle = \pm h_L(\nu_L(x_1))\), \(\langle \nu_L(x_2), x \rangle = \pm h_L(\nu_L(x_2))\)

See Figure 1 for examples.

If \(S_i\) is a segment then \(K^{(i)}\) described above is an infinite strip, and if \(\nu_L(x_1) = \nu_L(x_2)\) then \(L^{(i)}\) is an infinite strip. We would like to work with compact shapes, thus we apply a procedure to modify \(K^{(i)}, L^{(i)}\) to become bounded without changing their significant properties. Transversality implies that the intersection \(K^{(i)} \cap L^{(i)}\) is bounded, even if both sets are strips. For each \(1 \leq i \leq n\) we pick a centrally-symmetric strip \(A \subset \mathbb{R}^2\) such that \(A \cap K^{(i)}\) and \(A \cap L^{(i)}\) are both bounded, and which contains \(K\) and \(L\), and whichever of \(K^{(i)}, L^{(i)}\) that is bounded. From now on we replace \(K^{(i)}\) and \(L^{(i)}\) by their intersection with \(A\).

**Figure 1.** Two examples of the extension \(K, L \implies K^{(1)}, L^{(1)}\). The shaded shape in each diagram is \(K\) and the white shape with a solid boundary line is the corresponding \(L\).

Remark. Note that the sets grow in the process: \(K \subset K^{(i)}\) and \(L \subset L^{(i)}\) for all \(i = 1 \ldots n\). They satisfy \(\partial K^{(i)} \cap L^{(i)} = S_i \cup S_{i+n}\). Also note that if \(K\) is a parallelogram then so are the \(K^{(i)}\), for every \(i\). It is trivial to check that \((K^{(i)}, L^{(i)}) \in \mathcal{F}\) when \((K, L) \in \mathcal{F}\).

Lemma. If \(B(K^{(i)}, L^{(i)})\) for all \(i = 1 \ldots n\), then \(B(K, L)\).

Proof. The function \(g_{K, L}(r)\) takes non-negative values for \(r > 0\). In addition, its value is the sum of contributions from the different connected components of \(r\partial K \cap L\). From transversality, these components vary continuously around \(r = 1\),
From this and the previous lemma, $B$ hence from $K, L$ be parallelograms for every all $(K, L)$. 

**Proof.** Let $(K, L) \in F$ be any polygons. Construct the sequence of pairs $K^{(i)}, L^{(i)}$ from $K, L$. The shape $L^{(i)}$ is a parallelogram for every $i$. Then construct the pairs $(L^{(i)})^{(j)}, (K^{(i)})^{(j)}$ from $L^{(i)}, K^{(i)}$, for all $i$. The shapes $(L^{(i)})^{(j)}$ and $(K^{(i)})^{(j)}$ will be parallelograms for every $i, j$. Under our assumption, we have $B \left( (L^{(i)})^{(j)}, (K^{(i)})^{(j)} \right)$. From this and the previous lemma, $B(L^{(i)}, K^{(i)})$ follows.

The property $B$ is symmetric in the shapes. That is, $B(S, T) \iff B(T, S)$ for all $(S, T) \in F$. This is since $f_{S,T}$ and $f_{T,S}$ differ by a log-linear factor: 

$$f_{S,T}(t) = |e^t S \cap T| = e^{tt} f_{T,S}(-t)$$

This means that we have $B(K^{(i)}, L^{(i)})$ as well. Applying the previous lemma again gives $B(K, L)$. 

All that remains in order to deduce Theorem 2 is to analyse the case of centrally-symmetric parallelograms.

If $K, L$ are parallelograms and $K = TQ$ where $T$ is an invertible linear map and $Q = [-1, 1] \times [-1, 1]$, 

$$f_{K, L} = \det T \cdot f_{Q,T^{-1}L}.$$ 

Therefore we can take one of the parallelograms to be a square. In other words, establishing $B(Q, L)$ where $Q$ is the unit square and $L$ is a parallelogram, and $(Q, L) \in F$, will imply Theorem 2.

In fact, we may place additional geometric constraints on the square and the parallelogram.

If neither $Q$ nor $L$ contains a vertex of the other quadrilateral in its interior, then $\partial Q \cap L$ has 4 connected components. Applying the reduction above to $Q, L$ gives $Q^{(i)}, L^{(i)}$ with $i = 1, 2$, and the intersection $\partial Q^{(i)} \cap L^{(i)}$ has only 2 connected components, as remarked above.

Since the shapes are convex, if all the vertices of one shape are contained in the other, we have $Q \subseteq L$ or $L \subseteq Q$, and then (3) holds trivially. If $L$ contains corners of $Q$ but $Q$ does not contain vertices of $L$, we shall swap them.

These arguments leave two cases to be considered:

- $Q$ contains 2 vertices of $L$, and $L$ does not contain corners of $Q$. In this case the intersection $\partial Q \cap L$ is contained in two opposite edges of $Q$.
- $Q$ contains 2 vertices of $L$, and $L$ contains 2 corners of $Q$. In this case the intersection $\partial Q \cap L$ is a subset of the edges around these corners of $Q$. 

**Lemma.** If $B(K, L)$ holds for all pairs of parallelograms $(K, L) \in F$, then Theorem 2 follows.

Proof. Let $(K, L) \in F$ be any polygons. Construct the sequence of pairs $K^{(i)}, L^{(i)}$ from $K, L$. The shape $L^{(i)}$ is a parallelogram for every $i$. Then construct the pairs $(L^{(i)})^{(j)}, (K^{(i)})^{(j)}$ from $L^{(i)}, K^{(i)}$, for all $i$. The shapes $(L^{(i)})^{(j)}$ and $(K^{(i)})^{(j)}$ will be parallelograms for every $i, j$. Under our assumption, we have $B \left( (L^{(i)})^{(j)}, (K^{(i)})^{(j)} \right)$. From this and the previous lemma, $B(L^{(i)}, K^{(i)})$ follows.

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Computation of the special cases. These cases are defined by 4 real parameters — the coordinates of the vertices of $L$. A symbolic expression for $f(t)$ can be derived, and (3) will be a polynomial inequality in these parameters. The geometric conditions given above are also polynomial inequalities in these parameters. Thus each of the two cases can each be expressed by a universally-quantified formula in the language of real closed fields. By Tarski's theorem [3], this first order theory has a decision procedure. This is implemented in the QEPCAD B computer program [4]. Relevant computer files, for generation of the symbolic condition and for running the logic solver, for one of the two cases above, are available at

http://www.tau.ac.il/~livnebaron/files/bconj_201311/bconj_corners.mac

http://www.tau.ac.il/~livnebaron/files/bconj_201311/bconj_qelim.txt

A human-readable proof of both cases is included here as well.

Lemma. If $L$ is a centrally-symmetric parallelogram that satisfies $(Q, L) \in F$, and if $L$ crosses $Q$ only inside the vertical edges of $Q$, then $B(Q, L)$.

Proof. Let $\alpha, \beta, c, d$ be as in Figure 2.

![Figure 2](image)

The equations for the edges of $L$ are

$$x \cos \alpha + y \sin \alpha = \pm(c \cos \alpha + d \sin \alpha)$$

$$x \cos \beta + y \sin \beta = \pm(c \cos \beta + d \sin \beta)$$

Relevant parameters are computed as follows:

$$\partial Q \cap \partial L = \{ \pm(1, (c - 1) \cot \alpha + d), \pm(1, (c - 1) \cot \beta + d) \}$$

$$g_{Q,L}(1) = 2(c - 1)(\cot \alpha - \cot \beta)$$

$$g'_{Q,L}(1) = -2(\cot \alpha - \cot \beta)$$

$$g_{Q,L}(1) + g'_{Q,L}(1) = (2c - 4)(\cot \alpha - \cot \beta)$$

The area of $L$ is comprised of $Q \cap L$ and of two triangles. The area of the triangles is $\frac{1}{2}g(1) \cdot (c - 1)$ so

$$|Q \cap L| = |L| - (c - 1)^2(\cot \alpha - \cot \beta).$$

Note that $0 < \alpha < \frac{\pi}{2} < \beta < \pi$ so $\cot \alpha - \cot \beta$ is a positive quantity, and that if $c < 2$ the value of $g(1) + g'(1)$ is negative so inequality (3) is satisfied immediately.
Assume \( c > 2 \) from now on. What we need to prove is
\[
(2c - 4)(\cot \alpha - \cot \beta) \cdot |L| - (c - 1)^2(\cot \alpha - \cot \beta) \leq 4(c - 1)^2(\cot \alpha - \cot \beta)^2.
\]
Or equivalently
\[
(2c - 4)|L| \leq (c - 1)^2(\cot \alpha - \cot \beta) \cdot (4 + 2c - 4),
\]
or still
\[
|L| \leq \left(1 + \frac{2}{c-2}\right) \cdot \frac{1}{2}(c - 1)g(1).
\]

The amount \( \frac{1}{2}(c - 1)g(1) \) is the area of the triangles \( L \setminus Q \). By convexity the area of \( L \) cannot be larger than that times \( \left(\frac{c}{c-1}\right)^2 \). It remains to verify that for \( c > 2 \),
\[
\frac{c^2}{(c - 1)^2} = 1 + \frac{2c - 1}{(c - 1)^2} = 1 + \frac{2}{c-2} \left(\frac{c-1}{c-2}\right)^2 = 1 + \frac{2}{c-2} \left[1 - \frac{c/2}{(c - 1)^2}\right] \leq 1 + \frac{2}{c-2}
\]

Lemma. If \( L \) is a centrally-symmetric parallelogram that satisfies \((Q, L) \in F\), and each of \( Q, L \) contains two vertices of the other, then \( B(Q, L) \).

Proof. Let \( a \) and \( b \) be as in Figure 3 and let \( S \) stand for the area \( S = |Q \cap L| \). The numbers \( a \) and \( b \) are in the range \( 0 < a, b < 2 \), and \( \alpha \) and \( \beta \) satisfy \( \frac{1}{2}\pi < \alpha < \beta < \pi \). The area \( S \) is in the range \( 4 - ab < S < 4 \).

![Figure 3](image-url)

The quantity \( g(1) \) is simply \( 8 - 2a - 2b \), and \( g'(1) \) will soon be shown to be bounded by
\[
g'(1) \leq -8 \frac{S - (4 - ab)}{(4 - S) + \frac{1}{2}(a - b)^2}.
\]

This gives an inequality in the 3 variables \( a, b, S \), which will be proved for values in the prescribed ranges.
The length of each dotted line in Figure 3 is \((a^2 + b^2)^{1/2}\). Denoting the height of the triangle (the distance between \(p\) and the closest dotted line) by \(h\), the area is

\[
S = (4 - ab) + 2 \cdot \frac{1}{2} h \cdot (a^2 + b^2)^{1/2},
\]

so

\[
h = \frac{S - (4 - ab)}{(a^2 + b^2)^{1/2}}.
\]

The formula for \(g'(1)\) in terms of the angles \(\alpha, \beta\) is

\[
g'(1) = 4 + 2 \tan \alpha + 2 \cot \beta.
\]

Denote \(c = \beta - \alpha\). Holding \(c\) fixed, the function

\[
\alpha \mapsto g'(1) = 4 + 2 \tan \alpha + 2 \cot(\alpha + c)
\]

is concave and takes the same value for \(\frac{3}{4} \pi - c - \alpha\). Therefore its maximum is attained at \(\alpha = \frac{3}{4} \pi - \frac{1}{4} c\). This gives a bound for \(g'(1)\) for a given \(c = \beta - \alpha\):

\[
g'(1) \leq 4 + 2 \tan \left(\frac{3}{4} \pi - \frac{1}{4} c\right) + 2 \cot \left(\frac{3}{4} \pi + \frac{1}{4} c\right).
\]

This bound is stronger for higher values of \(c\), since \(\tan\) is an increasing function and \(\cot\) is a decreasing function.

The angle between the edges of \(L\) meeting at \(p\) is \(\pi - (\beta - \alpha) = \pi - c\). When \(a\), \(b\), and \(h\) are kept fixed, the position of \(p\) gives a bound for \(g'(1)\). This bound is the weakest when the angle \(\pi - c\) is largest. Simple geometric considerations show that in a family of triangles with the same base and height, the apex angle is largest when the triangle is isosceles, so we will pursue the case where the triangle formed by \(p\) and the nearest dotted line is isosceles.

The value of \(c\) in this case is \(c = 2 \tan^{-1} \frac{\frac{1}{2}(a^2 + b^2)^{1/2}}{h}\), and we get

\[
g'(1) \leq 4 + 2 \tan \left(\frac{3}{4} \pi - \frac{1}{4} \pi + \tan^{-1} \frac{\frac{1}{2}(a^2 + b^2)^{1/2}}{h}\right) + 2 \cot \left(\frac{3}{4} \pi + \frac{1}{4} \pi - \tan^{-1} \frac{\frac{1}{2}(a^2 + b^2)^{1/2}}{h}\right)
\]

\[
= 4 + 4 \tan \left(\frac{3}{4} \pi + \tan^{-1} \frac{a^2 + b^2}{2 S - (4 - ab)}\right)
\]

\[
= 4 + 4 \cdot \frac{1 + \frac{1}{2} \frac{a^2 + b^2}{S - (4 - ab)}}{1 - \frac{1}{2} \frac{a^2 + b^2}{S - (4 - ab)}}
\]

\[
= 8 - \frac{S - (4 - ab)}{(4 - S) + \frac{1}{2} (a - b)^2},
\]

which proves the forementioned bound for \(g'(1)\).

Therefore, to prove \(\textit{3}\) it is enough to show

\[
S \cdot \left(8 - 2a - 2b - 8 - \frac{S - (4 - ab)}{(4 - S) + \frac{1}{2} (a - b)^2}\right) \leq (8 - 2a - 2b)^2
\]

Rearranging and taking into account that \(S < 4\), this is equivalent to

\[
\frac{(8 - 2a - 2b)(8 - 2a - 2b - S) \left((4 - S) + \frac{1}{2} (a - b)^2\right) + 8S(S - (4 - ab))}{E} \geq 0
\]
When $a$ and $b$ are held fixed, this is a 2nd degree condition on $S$. Since $0 < a, b < 2$, the value and the first two derivatives in the point $S = 4 - ab$ are positive:

$$E|_{S=4-ab} = (8 - 2a - 2b)(2 - a)(2 - b) \cdot \frac{1}{2}(a^2 + b^2) > 0$$

$$\frac{\partial E}{\partial S}|_{S=4-ab} = (a + b) \left((5 - a - b)^2 - 1\right) + 2(a - b)^2 > 0$$

$$\frac{\partial^2 E}{\partial S^2}|_{S=4-ab} = 18(4 - ab) > 0$$

This means that the condition stays true for all $S > 4 - ab$, as required. □

3. Dihedral Symmetry

This section deals with dihedrally symmetric sets. The group $D_n$ is defined in the introduction.

**Theorem 3.** Let $n \geq 2$ be an integer, and let $K, L \subset \mathbb{R}^2$ be $D_n$-symmetric convex shapes. Then $t \mapsto |e^t K \cap L|$ is a log-concave function.

For $n = 2$ the group $D_2$ is generated by reflections across the standard axes. This corresponds to unconditional sets and functions, and Theorem 1 from [2] solves this case.

The proof for $n \geq 3$ is by reduction to the unconditional case.

A smooth strongly-convex shape $K \subset \mathbb{R}^2$ is one whose boundary is a smooth curve with strictly positive curvature everywhere. The radial function $\rho_K$ of a smooth strongly-convex shape $K \subset \mathbb{R}^2$ is a smooth function. The boundary $\partial K$ is the curve

$$\gamma_K(\theta) = (\rho_K(\theta) \cos \theta, \rho_K(\theta) \sin \theta).$$

The convexity of $K$ is reflected in the sign of the curvature of $\gamma_K$. Positive curvature can be written as a condition on the radial function:

$$\rho(\theta)^2 + 2\rho'(\theta)^2 - \rho(\theta)\rho''(\theta) > 0. \quad (4)$$

**Proof of theorem 3.** For any $D_n$-symmetric convex shape $K \subset \mathbb{R}^2$ there is a series of $D_n$-symmetric convex shapes whose boundaries are smooth and strongly convex curves, and whose Hausdorff limit is $K$. By the continuity argument from the previous section, the general case follows from the smooth case. From here on, $K$ and $L$ are smooth $D_n$-symmetric shapes.

$D_n$-symmetric shapes correspond to radial functions that are even and have period $\frac{2\pi}{n}$. These shapes are completely determined by their intersection with the sector

$$G_n = \{(r \cos \theta, r \sin \theta) : r \geq 0, \theta \in [0, \frac{\pi}{n}]\}.$$

Given two such shapes $K, L$ the area function is

$$f_{K,L}(t) = |e^t K \cap L| = 2n f_{K|G_n \cap L} G_n(t).$$

Let $K \subset \mathbb{R}^2$ be a $D_n$-symmetric strongly convex shape, and consider the function $\hat{\rho}(\theta) = \rho_K(\frac{2\pi}{n} \theta)$. This is an even function with period $\pi$. The function $\hat{\rho}(\theta)$ also satisfies

$$\hat{\rho}(\theta)^2 + \hat{\rho}'(\theta)^2 - \hat{\rho}(\theta)\hat{\rho}''(\theta) = \frac{1}{n^2} \left(\rho_K(\frac{2\pi}{n} \theta)^2 + 2\rho_K'(\frac{2\pi}{n} \theta)^2 - \rho_K(\frac{2\pi}{n} \theta)\rho_K''(\frac{2\pi}{n} \theta)\right) + (1 - \frac{1}{n})\rho_K(\frac{2\pi}{n} \theta)^2 > 0.$$
This means that $\tilde{\rho}(\theta)$ is the radial function of some $D_2$-symmetric (unconditional) strongly convex shape. We denote this $w(K)$: the unique shape that satisfies $\rho(w(K))(\theta) = \rho_K(\frac{2}{\pi} \theta)$.

The following function, also named $w$, is defined on $G_n$:
$$w \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix} = \begin{pmatrix} r \cos \frac{2}{\pi} \theta \\ r \sin \frac{2}{\pi} \theta \end{pmatrix}, \quad (\text{for } r \geq 0, \ \theta \in [0, \pi])$$

The point function $w$ is an bijection between $G_n$ and $G_2$. It relates to the shape function $w$ by the formula
$$\{w(x) : x \in K \cap G_n\} = w(K) \cap G_2.$$

The point function $w$ is differentiable inside $G_n$, and has a constant Jacobian determinant $\frac{2}{\pi}$.

Hence
$$f_{K,L}(t) = 2nf_{K \cap G_n, L \cap G_n}(t) = 4f_{w(K) \cap G_2, w(L) \cap G_2}(t) = f_{w(K), w(L)}(t),$$
and the theorem follows from the result in [2].

\[\square\]

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