A + A → ∅ system in one dimension with particle motion determined by nearest neighbour distances: results for parallel updates.

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A one dimensional A + A → ∅ system where the direction of motion of the particles is determined by the position of the nearest neighbours is studied. The particles move with a probability 0.5 + ε towards their nearest neighbours with −0.5 ≤ ε ≤ 0.5. This implies a stochastic motion towards the nearest neighbour or away from it for positive and negative values of ε respectively, with ε = ±0.5 the two deterministic limits. The position of the particles are updated in parallel. The macroscopic as well as tagged particle dynamics are studied which show drastic changes from the diffusive case ε = 0. The decay of particle density shows departure from the usual power law behaviour as found in ε = 0, on both sides of ε = 0 and a scaling regime is obtained for ε > 0. The ε = 0.5 point is characterized by the presence of dimers, which are isolated pairs of particles in adjacent sites that are never annihilated. The persistence probability is also calculated that decays in a stretched exponential manner for ε < 0 and switches over to power law behaviour for ε ≥ 0, with different exponents for ε = 0 and ε > 0. For the tagged particle, the probability distribution P(x,t) that it is at position x at time t shows the existence of a scaling variable x/νt where ν = 0.55 ± 0.05 for ε > 0 and varies with ε for ε < 0. Finally, a comparative analysis for the behaviour of all the relevant quantities for the system using parallel and asynchronous dynamics (studied recently) shows that there are significant differences for ε > 0 while the results are qualitatively similar for ε < 0.

I. INTRODUCTION

Reaction diffusion systems have received a lot of attention in recent years and have been studied in different contexts. A → A may be the simplest example of this kind of reaction, where the particles diffuse and annihilate if they meet. When considered on a lattice, a particle hops to one of its neighbouring sites, and in case a particle is already there, both get annihilated. This reaction has direct mapping to the dynamical evolution of the Ising Glauber model when studied with asynchronous updates, i.e., when the positions of the particles are updated one by one.

The A + A → ∅ system has been studied in the recent past where the particles A move with a bias towards or away from their nearest neighbours. The annihilation process is not directly affected by the bias which governs only the direction of motion but this extension leads to drastic changes in the dynamical properties. These studies were made using asynchronous updating rule. With asynchronous dynamics, the system can be mapped to an opinion dynamics problem studied earlier in a certain limit. However, regarded as an independent problem, one can also consider parallel dynamics where particle positions are updated simultaneously. Various physical and social phenomena have been studied using both asynchronous and parallel dynamics and comparative estimates show significant differences. In spin models also, phase diagrams and dynamical properties have been studied for both the dynamics. In particular, in the case with no bias, i.e., the purely diffusive case, for the reaction A + A → ∅, it was shown that the persistence exponent (defined for the persistence probability that a site has not been reached till a given time estimated as a function of time) is exactly double in case of parallel updates. It may be mentioned here that with parallel updates, this reaction no longer maps to the Ising-Glauber model.

In this paper we report the results for the dynamical properties of the A + A → ∅ system in one dimension where a particle A diffuses towards its nearest neighbour with a probability 0.5 + ε with −0.5 ≤ ε ≤ 0.5 using parallel dynamics. A similar problem was studied in two dimensions with parallel updates where the bulk properties were considered. Here we study both the macroscopic dynamical features as well as the tagged particle dynamics. The results, as expected, reveal interesting differences when compared to those for asynchronous dynamics. In particular, we detect a crossover in time from the annihilation dominated regime to a regime where the system is left with a constant number of particles.

In the next section we describe the model, the dynamical scheme and simulation method. The system has very different nature for positive and negative values of ε. For ε > 0 (ε < 0), the particles are biased to move towards (away from) their nearest neighbour. The two regions ε > 0 and ε < 0 having gross differences in their mean features are discussed separately in sections III and IV. Concluding remarks are made in the last section.

II. MODEL AND DYNAMICAL SCHEME, QUANTITIES CALCULATED AND SIMULATION DETAILS

The model, as mentioned in section I, consists of particles A undergoing the reaction A + A → ∅ in one dimension. We have considered lattices of size L that are initially randomly half filled. At each update, each parti-
cle hops one step in the direction of its nearest neighbour with probability $0.5 + \epsilon$ and in the opposite direction with probability $0.5 - \epsilon$ where $-0.5 \leq \epsilon \leq 0.5$. If two neighbours are equidistant, it moves in either direction with equal probability. Here, the updates are made in parallel which implies only after the locations of all the particles have been updated, if two particles are found on the same lattice site, then both of them are annihilated.

For the bulk features, we have calculated the time dependence of the density of surviving particles and persistence probability. A special feature arises for the attractive dynamics ($\epsilon > 0$), namely, the formation of dimers, which are isolated pairs of particles on adjacent sites. A detailed study of such dimerisation has been made also.

To probe the system microscopically, we have studied the probability distribution $\Pi(x, t)$ that a particle is in the position $x$ at time $t$. We have also estimated the probability of change in direction of motion $S(t)$ at time $t$ and the distribution $D(\tau)$ of time interval $\tau$ spent without change in direction of motion.

The studies have been made on a lattice of size 12000 or more and the maximum number of realisations was 6000. In all the simulations, periodic boundary condition has been imposed.

III. SIMULATION RESULTS FOR $\epsilon \geq 0$

To get a qualitative idea of the dynamics, a plot of the world lines of the particles can be most helpful. Snapshots of the system are shown in Fig. 1 for $\epsilon = 0$ and 0.5. It is to be noted that the motion is purely diffusive for $\epsilon = 0$ and for $\epsilon = 0.5$, particles undergo deterministic dynamics, when a particle always moves towards its nearest neighbour. The difference in the dynamical evolution is quite apparent; for $\epsilon = 0$, one notes the usual picture of a diffusion-annihilation process while for $\epsilon = 0.5$ two distinct behavior of the motion are manifested at long times; either the particles perform ballistic motion or pairs of particles exist which are strongly localised. The latter is the so called dimerisation, mentioned in the last section, that happens for $\epsilon > 0$.

A. Bulk Properties

1. Fraction of surviving particles $\rho(t)$

As the system evolves, the number of particles decreases due to annihilation. For the purely diffusive system ($\epsilon = 0$), it is well known that the fraction of surviving particles shows a power law behaviour in time; $\rho(t) \sim t^{-\frac{1}{2}}$, irrespective of the dynamics used; asynchronous or parallel. If an even number of particles are there initially, in the asynchronous case, at infinite times, all of them would be annihilated. However, in the parallel dynamics there may be certain configurations where two particles will survive infinitely if they are separated by an odd number of lattice spacings.

If a positive bias is introduced in the system, $\rho(t)$ shows a rapid decay in time initially shown in a Fig. 2. As the magnitude of $\epsilon$ increases, the number of annihilation increases as the particles are more strongly attracted. We obtain a data collapse for the initial decay of $\rho(t)$ by plotting the scaled value of $\rho(t)$ against $ct$ such that

$$\rho(t) / \epsilon^\gamma = f(ct)$$

with $\gamma \approx 0.3$. The collapse is no longer valid beyond $ct \approx 10^3$ giving a timescale $t^* \approx 10^3/\epsilon$ beyond which the decay changes its nature.

The functional form of $f(ct)$ is found to be $f(z) \propto \log(z)/z$ (see inset (a) of Fig. 2) such that, using eq. 1 one gets

$$\rho(t) = \frac{a + b \log t}{t}$$

where $a$ and $b$ are functions of $\epsilon$. For example, for $\epsilon = 0.5$, $a \approx 0.04$, $b \approx 0.81$.

For $t > t^*$, $\rho(t)$ shows a different behaviour for $\epsilon = 0.5$ and $\epsilon < 0.5$. For $\epsilon = 0.5$, where the annihilation occurs maximally, one notes a saturation value beyond $t^*$, while for $\epsilon < 0.5$, $\rho(t)$ continues to decrease albeit in a much slower manner. It is expected that a saturation will occur at very large times due to the reason stated earlier in this subsection for all $\epsilon > 0$, however, except for $\epsilon = 0.5$, it is difficult to obtain the saturation region in the simulation within the maximum number of Monte Carlo steps used. However, simulations for smaller system sizes run over larger time scales for $\epsilon < 0.5$ indeed show the tendency of reaching a saturation value as shown in the inset (b) of Fig. 2.

It may also be mentioned here that for the asynchronous update, a power law behaviour in the surviving fraction was found: $\rho(t) \propto t^{-1}$ irrespective of the value of $\epsilon > 0$ there. Thus the decay of particles is faster when the update is made based on the current position of the particles that enhances the annihilation.
3. Dimer Formation

A dimer is an isolated pair of particles occupying two adjacent sites, having no other neighbouring particles. A dimer can be formed for any $\epsilon > 0$; specifically for $\epsilon = 0.5$, the particles in a dimer will swap their position at every step and unless another particle annihilates one of them, they will remain forever. As two particles separated by an odd number of lattice spacing remaining in the system will never be annihilated, one or more dimers are expected to exist with a finite probability for $\epsilon = 0.5$. These dimers will not interact with each other and if the particles are indistinguishable, the system would appear to reach an absorbing state. For any other value of $\epsilon$, such dimers can form but there is always a probability of moving apart, however small, such that the system may remain in an active state. For $\epsilon$ close to 0.5, one can expect dimers to remain at large times, however, they are not ‘permanent’ as in the case of $\epsilon = 0.5$. Dimer formation is possible for parallel dynamics only.

We have studied the dimer density $\langle \rho_d(t) \rangle$, defined as the average number of dimers divided by the system size for several $\epsilon$. The data shown in Fig. 4 show that $\langle \rho_d(t) \rangle$ attains a saturation value for $\epsilon = 0.4$ and $\epsilon = 0.5$ and steady state is reached faster for $\epsilon = 0.5$. The saturation value for $\epsilon = 0.4$ is less compared to $\epsilon = 0.5$, as expected, while for $\epsilon = 0.2$, we do not get a saturation value within the observation time. We also find that an approximate fitting can be made; $\langle \rho_d(t) \rangle \propto t^{-0.8}$ for all $\epsilon$.

We study another quantity $\langle r_d(t) \rangle$ to estimate the probability of forming a dimer out of the surviving particles at any time. $r_d(t)$ for a particular configuration is a ratio defined as

$$r_d(t) = \frac{2L \rho_d(t)}{N(t)}, \quad (3)$$

where $N(t)$ is the number of surviving particles at time $t$, calculated for $N(t) \neq 0$. Note that if all the particles form dimers, $\rho_d(t) = N(t)/2L$ and $r_d(t) = 1$. To calculate the average $\langle r_d(t) \rangle$, we take only those configurations.
for which \( N(t) \neq 0 \) at time \( t \). Fig. 4 shows that \( \langle r_d(t) \rangle \) has a non monotonc behaviour, initially it increases with time and then decreases slowly. The decrease continues till it shows a dip beyond which it increases rather sharply and ultimately saturates at long times (for \( \epsilon > 0.1 \)). We conjecture that the dip occurs at around \( t^* \), when the particle density reaches the phase of either very slow decay or becomes a constant. To check this, we again attempt a collapse of the curves plotting \( \langle r_d(t) \rangle /t^{0.3} \) against \( et \) and indeed find a fairly good collapse up to the point where the dip occurs at \( et \approx 10^3 \). We note here that \( \langle r_d(t) \rangle \) reaches the value unity for \( \epsilon = 0 \). Inset shows the data collapse of \( \langle r_d(t) \rangle \) for different \( \epsilon \). We note here that \( \langle r_d(t) \rangle \) reaches the value unity for \( \epsilon = 0 \) at large times indicating all the surviving particles form dimer. For \( \epsilon < 0.5 \) but close to it, there is a fluctuation about a value close to 1, indicating that the dimers are not permanent as they form and break away regularly keeping a fairly constant value in time. We have checked that this effect is system size independent. Fig. 5 shows the results.

![Graph of \( \langle r_d(t) \rangle \) as a function of time for several \( \epsilon \). Inset shows the data collapse of \( \langle r_d(t) \rangle \) for different \( \epsilon \).](image1)

**FIG. 5.** Variation of \( \langle r_d(t) \rangle \) as a function of time for several \( \epsilon \). Inset shows the data collapse of \( \langle r_d(t) \rangle \) for different \( \epsilon \).

### B. Tagged particle features

#### 1. Probability distribution \( \Pi(x,t) \)

For pure random walk (\( \epsilon = 0 \)), the probability distribution \( \Pi(x,t) \) that a particle is at position \( x \) at time \( t \) is known to be Gaussian, i.e., \( \Pi(x,t) \propto \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \). Consequently, \( \Pi(x,t)t^{1/2} \) shows a data collapse for different times when plotted against \( x/t^{1/2} \). This is also true for the unbiased annihilating random walkers because they perform purely diffusive motion until they are annihilated.

For \( \epsilon > 0 \), the distributions show a non-Gaussian single peaked structure. However, a data collapse can be obtained by plotting \( \Pi(x,t)t^\nu \) against \( x/t^\nu \) where \( \nu = 0.55 \pm 0.05 \). Figs 6a, b, c show the raw data for \( \Pi(x,t) \) against \( t \) (for \( \epsilon = 0.1 \)) as well as the collapsed data \( \Pi(x,t)t^{0.55} \) against \( x/t^{0.55} \) for \( \epsilon = 0.1 \) and 0.5 in a linear plot. Fig. 6d shows the collapsed data for different values of \( \epsilon \) in a log-log plot. It reveals that the scaling function has a constant part for small values of its argument, then it enters a power law region before reaching a cutoff value. The cutoff value increases with \( \epsilon \) and also with time for each \( \epsilon \). The constant part shrinks as \( \epsilon \) increases (it is almost nonexistent for \( \epsilon = 0.5 \)) while the power law regime increases. The associated exponent value, mentioned in Fig. 6d, decreases with \( \epsilon \). The significance of these features will be discussed in detail in sec. V after the results for all the other tagged particle dynamics are reported.

For asynchronous dynamics, this distribution is a double peaked curve with a minima at the origin with a scaling variable \( x/et \) for \( \epsilon > 0 \). However, in the parallel update case, we verified that using a scaling variable involving \( et \), it is not possible to obtain a data collapse for different \( \epsilon \) values.

![Graph showing raw data for \( \Pi(x,t) \) and data collapse for different values of \( \epsilon \).](image2)

**FIG. 6.** (a) Raw data for \( \Pi(x,t) \) for \( \epsilon = 0.1 \). (b), (c) Data collapse of \( \Pi(x,t)t^{0.55} \) against \( x/t^{0.55} \) for \( \epsilon = 0.1, 0.5 \). (d) Data collapse of \( \Pi(x,t) \) in log-log plot showing a power law region and they are shifted along \( Y \) axis for clarity.

#### 2. Probability of direction change \( S(t) \)

The probability of direction change \( S(t) \) at time \( t \) is calculated by studying the number of particles that change their direction of motion at time \( t \) scaled by the total number of surviving particles at that instant of time. For pure random walk (\( \epsilon = 0 \)) \( S(t) = 0.5 \), independent of time. In contrast to asynchronous dynamics, as \( \epsilon \) increases \( S(t) \) also increases.

\( S(t) \) shows an increase till a certain time and then starts decreasing. For larger \( \epsilon \) values, it is possible to detect a dip occurring subsequently, beyond which \( S(t) \) increases again and attains a constant value. The results are shown in Fig. 7.

![Graph showing probability of direction change \( S(t) \).](image3)

**FIG. 7.** Graph showing probability of direction change \( S(t) \).
[54x75]beyond a very short initial time (inset of Fig. 7). A comparison with

\[ \langle \rho \rangle \]

demonstrates good agreement with the proposed form given in Eq. 5.

\[ S(t) \]

\[ \text{ann} \]

in the time interval

\[ \tau \]

and/or due to dimer formation. Let us first estimate the contribution to \( S(t) \) due to annihilation. The change in direction due to annihilation per particle is proportional to \( \frac{A(t)}{\rho(t)} \) where \( A(t) \) is the number of annihilation \( (A(t) \propto -\frac{d\rho}{dt}) \). Therefore, we get a contribution \( \propto \frac{d\rho(t)/dt}{\rho(t)} \). The form of \( \rho(t) \) is taken from Eq. 2 and to simplify things we take \( a = 0 \) as it is small compared to the second term in the numerator of Eq. 2 for \( \epsilon = 0.5 \). Thus the contribution from the annihilation process to \( S(t) \) written as \( S(t)_{\text{ann}} \) is given by

\[ S(t)_{\text{ann}} \propto \frac{1}{t} - \frac{1}{t \log(t)}. \quad (4) \]

In addition to this, contribution from the dimers should be taken into account. The particles forming a dimer will necessarily change direction at every step. The contribution to \( S(t) \) due to dimers will therefore be simply \( \langle r_d(t) \rangle \), the latter being the probability that there is a dimer and since it will come from those configurations only which have surviving particles till that time, we need a multiplicative factor. This is because \( S(t) \) is a quantity averaged over all configurations.

Considering both contributions,

\[ S(t) = c_1 \left[ \frac{1}{t} - \frac{1}{t \log(t)} \right] + c_2(t) \langle r_d(t) \rangle. \quad (5) \]

Here \( c_1 \) is a proportionality constant and \( c_2(t) \) denotes the fraction of configurations which have \( \rho(t) \neq 0 \).

We plot the rhs of Eq. 5 using the data for \( c_2(t) \) and \( \langle r_d(t) \rangle \) from the simulation and with \( c_1 = 1 \) we get very good agreement with \( S(t) \) obtained from the simulation beyond a very short initial time (inset of Fig. 7). A comparison with \( \langle r_d(t) \rangle \) (Fig. [4]) also reveals the fact that \( S(t) \) is annihilation dominated initially but crosses over to a regime dominated by the “dimerized” absorbing states.

It may be recalled here that for asynchronous update, \( S(t) \) has a completely different variation with time, it shows a crossover from a power law decay behaviour to a constant value with the crossover time diverging as \((0.5 - \epsilon)^{-1}\). It is the presence of the dimers which is responsible for the entirely different behaviour in the case with parallel updates, especially at later times, for all values of \( \epsilon \).

3. Distribution of time interval spent without change in direction of motion \( D(\tau) \)

Another interesting quantity is \( D(\tau) \), the interval of time \( \tau \) spent without change in direction of motion. For purely diffusive motion \((\epsilon = 0)\), the probability of direction change at any time \( t \) is 0.5. So, the probability that in the time interval \( \tau \), the particles will not change their direction is given by the following equation:

\[ D(\tau) = 0.5^2(1 - 0.5)^\tau, \quad (6) \]

which reduces to an exponential form \( D(\tau) \propto \exp[-\tau \ln 2] \) as shown in Fig. 5. For \( 0.5 > \epsilon > 0 \), the tail of \( D(\tau) \) shows an exponential decay; \( D(\tau) \sim a^\epsilon \exp(-b^\epsilon \tau) \) (see Fig. 5). For \( \epsilon = 0.5 \), no such exponential tail is observed, \( D(\tau) \) instead shows a power law decay with \( \tau \) with an exponent 2, shown in Fig. 5.

For \( \epsilon = 0.5 \), at early times, we note that there are two kinds of motion, some particles follow long trajectories in a straight line before getting annihilated or forming a dimer and other particles which quickly form a dimer. At later times, only dimers remain (see Fig. 5). Hence the contribution to \( D(\tau) \) for large values of \( \tau \) will come from the early times, i.e., the annihilation dominated regime. On the other hand, at large times, dimer formation plays the key role when the particles typically change direction at every time contributing heavily to \( D(\tau) \). Consequently we find \( D(1) \) to grow in time as shown in the inset of Fig. 5.

To explain the \( \tau^{-2} \) dependence of the tail, one can assume \( S(t) \approx S(t)_{\text{ann}} \), the contribution due to the annihilation only and use it to compute \( D(\tau) \). Here, it may be mentioned that \( D(\tau) \) for the asynchronous case also showed a \( \tau^{-2} \) tail, where \( S(t) \) was found to scale as \( t^{-1} \). Note that \( S(t)_{\text{ann}} \) shows a leading order dependence as \( 1/t \) also in the parallel case (Eq. 3). Hence one can derive the power law form of \( D(\tau) \propto \tau^{-2} \) for large \( \tau \) in the same manner it was done in [6].

We also note that \( D(\tau) \) has a distinct dependence on the particular time \( t \) at which it is calculated. \( D(\tau) \) grows in time and consequently the magnitude of \( D(\tau) \) for large \( \tau \) decreases (Fig. 5). In fact one can obtain a data collapse for the data at different times \( t \) by plotting \( D(\tau)t^{0.35} \) against \( \tau \) such that the behaviour of \( D(\tau,t) \) for
\( \epsilon = 0.5 \) can be summarized as

\[
D(\tau, t) = D(\tau = 1, t)\delta_{1,\tau} + \text{const} \frac{\tau^{-2}}{t^{0.35}}(1 - \delta_{1,\tau}), \quad (7)
\]

where we have fitted the growth of \( D(\tau = 1, t) \) by the function \( D(1, t) = 0.98[1 - \exp(-1.52t^{0.17})] \) shown in the inset of Fig. 9.

![FIG. 8. Variation of \( D(\tau) \) over \( \tau \) is shown in log-linear plot for several \( \epsilon \). The best fit lines are shown along with for different \( \epsilon \) in the same order.](image)

![FIG. 9. Data collapse of \( D(\tau) \) is shown against \( \tau^{-2} \) in log-log plot for \( \epsilon = 0.5 \). Inset shows the values of \( D(\tau) \) at \( \tau = 1 \) for different times for \( \epsilon = 0.5 \) and are fitted to the form: \( D(\tau = 1, t) = 0.98[1 - \exp(-1.52t^{0.17})] \).](image)

### IV. RESULTS FOR \( \epsilon < 0 \)

The particles have a bias \( \epsilon \) to move towards their nearest neighbour. As \( \epsilon \) becomes negative, the particles tend to move away from their nearest neighbour. Fig. 10 shows the snapshots of the system at different times for \( \epsilon = -0.1 \) and -0.5. For \( \epsilon = -0.5 \), a particle always moves away from its nearest neighbour, annihilation is extremely rare as it performs a nearly perfect oscillatory motion at later times.

![FIG. 10. Snapshots of the system at different times for \( \epsilon = -0.1 \) (left) and \( \epsilon = -0.5 \) (right). The trajectories of different particles are represented by different colors.](image)

![FIG. 11. Variation of fraction of surviving particles \( \rho(t) \) in time \( t \). The best fit lines are shown along with for different \( \epsilon \) in the same order and these lines are shifted for clarity. Inset shows the variation of the exponent \( \beta \) with \( \epsilon \) where \( \epsilon < 0 \).](image)

For the extreme limit \( \epsilon = -0.5 \), the particles achieve an equidistant configuration at large time and every particle performs a to and fro movement (as the dynamical rule ensures that each particle has to undergo a displacement); no annihilation will take place and \( \rho(t) \) rapidly saturates to a constant value \( O(10^{-1}) \).
2. Persistence probability \( P(t) \)

The persistence probability \( P(t) \) shows an interesting behaviour for \( \epsilon < 0 \). For any \( \epsilon \neq -0.5 \), it shows a fast decay with time, however, the magnitude of the persistence probability shows a non-monotonic behaviour. For \( 0 > \epsilon > -0.4 \), it decreases as \( \epsilon \) decreases, but as \( \epsilon \) becomes more negative, the decay rate becomes slower.

\( P(t) \) shows a stretched exponential decay in time and the data can be fit to the following form

\[
P(t) = q_0 \exp(-qt^\nu). \tag{9}
\]

For \( \epsilon = -0.5 \), the movement of the particles is restricted as they perform nearly oscillatory motion, as shown in Fig. 12, most of the sites remain unvisited. Therefore, \( P(t) \) shows a very slow decay at the initial few steps and then becomes a constant in time as shown in Fig. 12.

The results for the persistence probability shows that for \( |\epsilon| \ll 0.4 \), the annihilation decreases such that more number of particles remain in the system which display a certain degree of mobility, thereby decreasing the persistence probability. However, for \( |\epsilon| \gg 0.4 \), the particle mobility gets seriously restricted, such that, although a larger number of particles survive in the system, more sites remain unvisited. This indicates an interesting crossover behaviour in the motion of the particles as \( |\epsilon| \) increases, captured by the behaviour of \( P(t) \).

\( P(t) \) shows a similar stretched exponential decay in case of asynchronous dynamics for \( \epsilon < 0 \). However, \( P(t) \) decreases monotonically as \( \epsilon \) becomes more negative in the asynchronous case.

B. Tagged particle properties

1. Probability distribution \( \Pi(x,t) \)

Probability distribution \( \Pi(x,t) \) retains its Gaussian form when \( \epsilon \) becomes negative. But the scaling variable \( x/t^n \) is non-unique as the exponent \( \nu \) shows a strong dependence on \( \epsilon \). Fig. 13 shows collapsed data at different times when \( \Pi(x,t)/t^n \) is plotted against \( x/t^n \). For \( \epsilon = -0.5 \), the particles attain a equidistant configuration; but according to the dynamical rule, as the particles must make a move, they only perform a back and forth movement (see Fig. 12). As a result, the probability distribution \( \Pi(x,t) \) becomes time independent after a brief transient, shown in Fig. 13(d).

![Fig. 12. Variation of persistence probability \( P(t) \) with \( t \) for different \( \epsilon \). The best fit lines are shown for \( \epsilon = -0.1 \) and \( \epsilon = -0.4 \).](image.png)

![Fig. 13. Data collapse of \( \Pi(x,t) \) is shown for \( \epsilon = -0.1 \) (a), \( \epsilon = -0.3 \) (b), \( \epsilon = -0.45 \) (c) and \( \epsilon = -0.5 \) (d). Data is fitted to the Gaussian distribution form. Scaling functions \( f(x/t^{0.33}) = 0.42 \exp[-0.14(x/t^{0.33})^2] \), \( g(x/t^{0.28}) = 0.52 \exp[-0.21(x/t^{0.28})^2] \), \( k(x/t^{0.25}) = 0.69 \exp[-0.37(x/t^{0.25})^2] \), \( h(x) = 0.18 \exp(-0.46x^2) \) for (a), (b), (c) and (d) respectively, are shown in the figure.](image.png)

Fig. 14 shows the value of \( \nu \) against \( \epsilon \) that decreases from 0.5 monotonically as \( \epsilon \) becomes more negative. At \( \epsilon = 0.5 \), there is a sharp discontinuity in its value as it falls to zero from a value \( \sim 0.25 \).

2. Probability of direction change \( S(t) \)

For \( \epsilon < 0 \), \( S(t) \) attains almost a constant value that increases systematically with the magnitude of \( \epsilon \), shown in Fig. 15. Here, as the annihilation factor is less relevant, especially at later times, the change in direction of motion occurs due to the repulsion between the neighbouring particles mainly in the following manner: as \( \epsilon \) decreases,
the repulsion factor becomes stronger and the particles tend to avoid their nearest neighbours. A change in the direction can occur if the other neighbour comes closer as a result. At the extreme limit $\epsilon = -0.5$, this happens at every step such that the change in direction is maximum.

![FIG. 14. Variation of scaling variable $\nu$ with $\epsilon$.](image)

![FIG. 15. Probability of direction change $S(t)$ of a tagged particle at time $t$ for different $\epsilon$.](image)

3. Distribution of time interval spent without change in direction of motion $D(\tau)$

For $\epsilon < 0$, as $S(t)$ becomes constant, $D(\tau)$ is expected to show an exponential decay. $D(\tau)$ shows a faster decay with $\tau$ as $\epsilon$ becomes more negative. For $\epsilon = -0.5$, since the particles change their direction of motion more often, $D(\tau)$ decays in the fastest manner.

The tail of the distribution $D(\tau)$ can be fit to the form

$$D(\tau) = c \exp(-d\tau), \quad (10)$$

and the results are shown in Fig. 15.

![FIG. 16. Variation of $D(\tau)$ over $\tau$ for different $\epsilon$ is shown in log-linear plot. The best fit lines are shown along with for different $\epsilon$ in the same order.](image)

V. COMPARISON OF PARALLEL AND ASYNCHRONOUS UPDATE

Having obtained all the results for the entire range of $\epsilon$, one can now make a comparison between the results for the parallel and asynchronous dynamics for the $A + A \rightarrow 0$ model where the particles have a bias to move toward their nearest neighbours. The comparison of the different properties are presented in Table I.

We note that while for $\epsilon > 0$, the results are significantly different, the negative $\epsilon$ results are almost independent of the particular update used.

For $\epsilon > 0$, one of the most notable difference is in the behaviour of the probability distribution, and this is a suitable juncture to further analyse the behaviour of $\Pi(x, t)$ with the parallel updating scheme. For the asynchronous update we obtained a double peaked structure which was ascribed to the dominantly ballistic walkers existing in the later time regime. Here instead, we get a single peaked structure (see Fig. 6). To understand this, we first consider the extreme case of $\epsilon = 0.5$. The snapshots of Fig. 1 show that a considerable fraction of the particles quickly form dimers while some particles follow a ballistic path, in either direction. The particles which form dimers remain close to their origins and thus contribute to $x \approx 0$ giving rise to the peak at $x = 0$. The ballistic particles will contribute towards $|x| \gg 0$. We conjecture that the heavy tail of the distribution $\Pi(x, t)$ is connected to these ballistic particles.

That the ballistic walkers remain in the system is corroborated by the fact that $D(\tau)$ shows a power law behaviour for large $\tau$ for $\epsilon = 0.5$, shown in Fig. 6. For smaller $\epsilon$, dimers are not formed easily and the single peaked behaviour is due to the enhanced probability of direction change $S(t)$ which results in much smaller net displacements. On the other hand, in the asynchronous case, the direction change is much less probable, the particles perform an overall ballistic walk even for small $\epsilon$ and hence the double peaked structure is present for all $\epsilon$. The ballistic walk occurs maximally for $\epsilon = 0.5$. So the exponent associated with the power law behaviour of the scaled $\Pi(x, t)$ is least for $\epsilon = 0.5$ and in general de-
creases as $\epsilon$ increases. The width of the power law region increases with $\epsilon$ due to the same reason.

For $\epsilon < 0$, there are some differences in the persistence probability and the exponent $\nu$ occurring in the scaling variable of the probability distribution $\Pi$ with respect to their variations with $\epsilon$. The probability distribution $\Pi(x, t)$ is Gaussian for $\epsilon < 0$ independent of the dynamics used. For $\epsilon = -0.5$, the scaling factor $\nu = 0$ for the parallel dynamics (reported in the present work) and $\nu = 0.25$ for asynchronous dynamics as expected for repulsive random walkers [17]. However, for $\epsilon$ very close to $-0.5$, $\nu$ shows a value $0.25$ for parallel dynamics also; only at $\epsilon = -0.5$, $\nu$ shows a discontinuity as $\Pi(x, t)$ becomes time independent. Clearly this is because the parallel updating scheme leads to oscillatory motions as $\epsilon \to -0.5$, in the asynchronous update, there is no such oscillation. In that sense, the motion of the particles are more correlated for the parallel dynamics.

Dimerisation is a feature that is present only for the parallel updating case that occurs for $\epsilon > 0$ only. Interestingly, dimerisation occurs with a finite probability as $\epsilon$ deviates from $0$ to larger values.

VI. CONCLUDING REMARKS

In this paper, we have studied the behaviour of the $A + A \to \emptyset$ model in one dimension, where the particles move towards their nearest neighbour and the parallel updating scheme is used. The probability to move towards the nearest neighbour is taken parametrically as $0.5 + \epsilon$ where $-0.5 \leq \epsilon \leq 0.5$. The properties of the model have been summarized in Table I. For $\epsilon > 0$ the results depend strongly on the particular scheme that is used. It is the presence of dimers in the parallel dynamics that mostly gives rise to a number of interesting variations in the relevant quantities. We have analysed how the tagged particle properties like $S(t)$ and $D(\tau, t)$ are dependent on the presence of dimers for $\epsilon = 0.5$. The results reveal the crossing over of the system from annihilation dominated to dimer dominated regimes. In this context, let us recall that a crossover from an annihilation to diffusion dominated regime for the asynchronous case was found recently [10].

For $\epsilon > 0$, unlike most systems in one dimension, the particle decay fraction $\rho(t)$ does not have a simple power law form with parallel updates. Rather, the form is similar to what one obtains for a bias-less system of annihilating walkers in two dimensions. Another intriguing result is that we find that the persistence exponent in the parallel case is close to twice of the one found in the asynchronous case.

It is understandable why for negative $\epsilon$, the results for the dynamical quantities are independent of the updating scheme apart from subtle differences in their $\epsilon$ dependence. The choice of the dynamical scheme affects the annihilation process significantly. For $\epsilon < 0$, as the particles repel each other, they hardly come into contact to annihilate each other and hence the results are more or less similar.

Acknowledgement: The authors thank DST-SERB project, File no. EMR/2016/005429 (Government of India) for financial support. Discussion with Soham Biswas is also acknowledged.

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**TABLE I. Leading order time and \( \epsilon \) dependence of several quantities in one dimensional \( A + A \to \emptyset \) model.**

Results for asynchronous dynamics are quoted from references [3, 5, 6]. Notation used: generic.

| \( \rho(t) \) | Asynchronous | Parallel |
|---|---|---|
| \( t^{-0.5} \) for \( \epsilon = 0 \) | \( t^{-0.5} \) for \( \epsilon = 0 \) | \( ((a + b \log(t))/t \) for \( \epsilon > 0 \) |
| \( t^{-1} \) for \( \epsilon > 0 \) | \( t < t^* \), \( t^* \propto 1/\epsilon \) | \( a/(\log t)^b \) for \( \epsilon < 0 \) |
| \( a/(\log t)^b \) for \( \epsilon < 0 \) | \( a/(\log t)^b \) for \( \epsilon < 0 \) | \( a \exp(-bt\epsilon) \) for \( \epsilon < 0 \neq -0.5 \) |

| \( P(t) \) | Scaling factor \( x/t^\nu \) in all cases | Scaling factor \( x/t^\nu \) in all cases |
|---|---|---|
| \( t^{-0.375} \) for \( \epsilon = 0 \) | \( \Pi \) Gaussian for \( \epsilon = 0 \) and \( \epsilon < 0 \) | \( \Pi \) Gaussian for \( \epsilon = 0 \) and \( \epsilon < 0 \) |
| \( t^{-0.255} \) for \( \epsilon > 0 \) | \( \nu = 0.5 \) for \( \epsilon = 0 \) | \( \nu = 0.5 \) for \( \epsilon = 0 \) |
| \( a \exp(-bt\epsilon) \) for \( \epsilon < 0 \) | \( \nu \) decreases with \( \epsilon \) for \( \epsilon < 0 \) | \( \nu \) decreases with \( \epsilon \) for \( \epsilon < 0 \) |

| \( \Pi(x,t) \) | \( \Pi \) double peaked for \( \epsilon > 0 \) | \( \Pi \) Non Gaussian single peaked for \( \epsilon > 0 \) |
|---|---|---|
| \( \nu = 1 \) for \( \epsilon > 0 \) | \( \nu \) decreases with \( \epsilon \) for \( \epsilon < 0 \) | \( \nu = 0.55 \pm 0.05 \) for \( \epsilon > 0 \) |

| \( S(t) \) | \( \Pi(x,t)t^\nu \) shows a power law regime for large values of \( x/t^\nu \) |
|---|---|---|
| \( t^{-1} \) up to \( t^* \) (\( \epsilon > 0 \)) | \( \nu = 1 \) for \( \epsilon > 0 \) | \( \Pi(x,t)t^\nu \) shows a power law regime for large values of \( x/t^\nu \) |
| \( t^* \propto 1/(0.5 - \epsilon) \) | \( t < t^* \), \( t^* \propto 1/\epsilon \) | \( t^* \propto 1/(0.5 - \epsilon) \) |
| \( S(t) \) decreases as \( \epsilon \) increases | \( S(t) \) increases as \( \epsilon \) increases |

| \( D(\tau) \) | \( \exp(-a\tau) \) for \( \epsilon = 0 \) and \( \epsilon < 0 \) | \( \exp(-a\tau) \) for \( \epsilon = 0 \) and \( \epsilon < 0 \) |
|---|---|---|
| \( \tau^{-2} \) up to \( \tau^* \) (\( \epsilon > 0 \)) | \( \tau^* \) decreases with \( \epsilon \) for \( \epsilon < 0 \) | \( \tau^{-2} \) for \( \epsilon = 0 \) |
| \( \tau^* \propto 1/(0.5 - \epsilon) \) | \( \tau < \tau^* \), \( \tau^* \propto 1/\epsilon \) | \( \tau^* \propto 1/(0.5 - \epsilon) \) |
| \( \tau^* \propto 1/(0.5 - \epsilon) \) | \( \tau < \tau^* \), \( \tau^* \propto 1/\epsilon \) | \( \tau^* \propto 1/(0.5 - \epsilon) \) |

† from reference [5].