Universal conservation law and modified Noether symmetry in 2d models of gravity with matter

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Abstract

It is well-known that all 2d models of gravity—including theories with nonvanishing torsion and dilaton theories—can be solved exactly, if matter interactions are absent. An absolutely (in space and time) conserved quantity determines the global classification of all (classical) solutions. For the special case of spherically reduced Einstein gravity it coincides with the mass in the Schwarzschild solution. The corresponding Noether symmetry has been derived previously by P. Widerin and one of the authors (W.K.) for a specific 2d model with nonvanishing torsion. In the present paper this is generalized to all covariant 2d theories, including interactions with matter. The related Noether-like symmetry differs from the usual one. The parameters for the symmetry transformation of the geometric part and those of the matter fields are distinct. The total conservation law (a zero-form current) results from a two stage argument which also involves a consistency condition expressed by the conservation of a one-form matter “current”. The black hole is treated as a special case.

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1 Introduction

Models of gravity in one space and one time coordinate have attracted the interest for some time. After all, spherical reduction of 4d Einstein gravity (SRG) leads to an effective 2d theory with dilaton fields. Because the Einstein-Hilbert action becomes trivial in \( d = 2 \) also many other models with or without dilaton fields have been studied. Ref. \([2]\) contains a necessarily very incomplete list of related work which had been spurred especially by the discussion of a special case, the string inspired dilaton black hole. All these models turned out to be exactly solvable at the classical level in the absence of matter, for the dilaton black hole of Ref. \([3]\) even (minimal) interactions with scalar fields may be included. An exact solution is possible as well for a theory without dilatons but with nonvanishing torsion. A common behaviour of all these approaches has been the use of the conformal gauge for the 2d metric. For many nontrivial theories this often required remarkable mathematical effort to arrive at the exact solution of the equations of motion, especially in the case with nonvanishing torsion, or for general dilaton theories.

A new and much simplified approach was first proposed in \([5]\). It is based upon the consequent use of Cartan variables \( e^a_\mu \) (zweibeine) and \( \omega^a_{\mu b} = \omega_\mu^c e^b_c \) (spin-connection) and, more importantly, on a “light cone gauge” for those quantities:

\[
e^0_0 = (e^0_0 - 1) = \omega_0 = 0
\]  

In that gauge the metric acquires an Eddington-Finkelstein form

\[
g_{\alpha\beta} = 2e^+_1 \begin{pmatrix} 0 & 1 \\ 1 & e^-_1 \end{pmatrix}
\]  

For the discussion of global properties this gauge has definite advantages, because—
in contrast to the conformal gauge—the basic patch extends across horizons, the latter being represented by zeros of the Killing norm $k^2 = 2e_1^+ e_1^-$. This allows a very comprehensive discussion of all global properties of solutions with one Killing vector field—as it is the case in the absence of matter interactions—in $d = 1 + 1$, and even recently led to the discovery of global coordinates for the Reissner-Nordström black hole \cite{6}.

It turns out that the treatment of general 2d theories of gravity is also greatly facilitated by considering a Hamiltonian formalism \cite{7} or, equivalently, a first order Lagrangian formalism which has been shown to cover all 2d theories without dilaton field \cite{8}. The solution in that formulation without matter becomes almost trivial (cf. also \cite{9}) even for a general gauge, because it essentially coincides with the solution in component fields, restricted just by the gauge fixing \cite{10}. From a theory without dilatons a general dilation theory in $d = 2$ always can be produced by conformal transformation of the metric (or of $e^\mu_a$) by $\exp \Phi$ where $\Phi$ is a dilaton field. Although this immediately provides solutions of the equations of motion (e.o.m.) also in that case, the global properties of the solutions are completely different. The “decomposition” of Penrose diagrams in \cite{11} is an illustration for that phenomenon (cf. also \cite{12}).

The exact integrability of all 2d theories in the absence of matter is closely related to the existence of a conserved quantity which is independent of space and time. It represents the (only nontrivial) Dirac observable in such systems \cite{7, 8}, and indeed completely determines the global properties of the classical solutions for a given 2d action. For the special dilaton theories describing SRG \cite{1} or the dilaton black hole \cite{3}, not surprisingly, it is proportional to the mass of the black hole. It may also be interpreted within the concept of quasilocal energy \cite{1, 12} or with a Noether charge \cite{3} in the sense of Ref. \cite{13}.

According to accepted wisdom a Noether charge should be related to a symmetry
of the action \[4\]. Indeed for the matter free case that symmetry has been identified by one of the present authors in collaboration with P. Widerin \[13\] for the 2d model of \[3\].

Although an exact general solution does not exist anymore—exceptional cases like the DBH excluded—when matter interactions are present, the general conservation law still holds. It may be simply derived from a proper linear combination of the e.o.m.-s \[8\], a procedure which is most transparent formulating 2d theories in “first order form” \[8, 9, 16\].

However, although the existence of such a conservation law is also obvious in that case \[3\] the contribution of the matter fields to it would require the knowledge of at least part of the exact solution coming from the matter interactions. The precise meaning of that will be recapitulated below. In any case, the composition of the usual ingredients into the celebrated Noether theorems are changed in a nontrivial manner.

The purpose of our present work is to clarify this point. Somewhat surprisingly we obtain a Noether-“like” situation with subtle modifications of the Noether theorem.

In section \[2\] we shortly describe the first order form of a general covariant theory in \(1 + 1\) dimensions and the resulting conservation law. Section \[3\] is devoted to the discussion of the symmetry, starting in \[3.1\] with the matterless case. Here we generalize the result of \[15\] to arbitrary 2d models of the type \(3\) below. Matter interactions are treated at first within a simplified toy model which has a “geometric” and a “matter” part and where a complete exact solution is possible (subsection \[3.2\]). In that model the main difference with respect to the ordinary Noether situation can be seen more easily than in the general case (subsection \[3.3\]).
In section 4 we emphasize the importance of our result to the most interesting application of the general argument, namely the Schwarzschild black hole, interacting with nonminimal scalar matter. The Noether-like conservation law connects the mass of the black hole in a highly nontrivial manner with the (incoming and outgoing) flux of matter.

2 General 2d action with matter

2.1 Action of geometric variables

The geometric part of the action for all 2d theories can be written as

\[ L(g) = \int \left( X^a T^a + X d\omega - \frac{V}{2} \epsilon_{ab} e^a \wedge e^b \right) \]  

where \( e^b \) (zweibein 1-form) and \( \omega_{ab} = \epsilon^{ab} \omega \) (spin connection one-form) through torsion

\[ T^a = de^a + \epsilon^a_{\ b} \omega \wedge e^b \]  

and curvature scalar

\[ -\frac{R}{2} = *d\omega . \]

\( X^a \) and \( X \) are (zero-form) auxiliary fields. They coincide with conjugate momenta to \( e^a_1 \), resp. \( \omega_1 \) when these quantities are restricted to a space-like surface \( x^0 = \text{const} \).

In \( V = V(X^a X_\alpha, X) \) the dynamics of \( L(g) \) are encoded. Latin indices refer to the tangential (local Lorentz, anholonomic) coordinates with metric \( \eta_{ab} \) [\( \text{diag}(\eta) = \)]
Greek indices in the components of the forms \( e^a = e^a_\mu dx^\mu \), \( \omega_\mu dx^\mu \) indicate space-time (holonomic) ones.

The antisymmetric \( \epsilon \) pseudotensor in tangential Minkowski space appearing in the volume form

\[
\epsilon_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

(6)

by

\[
\epsilon_{\mu\nu} = \epsilon_{ab} e^a_\mu e^b_\nu = -e \tilde{\epsilon}_{\mu\nu},
\]

(7)

is related to the antisymmetric symbol \( \tilde{\epsilon} \) in holonomic components with the factor

\[
e = det e^a_\mu = \sqrt{-g},
\]

(8)

the determinant of the metric

\[
g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}.
\]

(9)

Often light cone coordinates are useful. Then (3) and (4) become \((\eta_{-+} = \eta_{-+} = 1, \epsilon_{-+} = -\epsilon_{++} = -1)\)

\[
L = \int (X^+T^- + X^-T^+ + Xd\omega + V(X^+X^-, X)e^+ \wedge e^-)
\]

(10)

and

\[
T^\pm = (d \pm \omega) \wedge e^\pm.
\]

(11)
Elimination of $X^a$ and $X$ by their respective (algebraic) e.o.m.-s from (3), (10) clearly produces a covariant 2d theory with an action which is an arbitrary function in curvature and torsion. However, it is also possible to “integrate out” the components $\omega_\mu$ together with $X^a$. For $V$ quadratic in $X^a$

$$V = \frac{U(X)}{2}X^aX_a + v(X)$$

(12)

in this way the action of the most general 2d dilaton theory is produced:

$$L = \int dx^2 \sqrt{-g} \left( -\frac{XR}{2} + \frac{U(X)}{2} g^{\mu\nu} \partial_\mu X \partial_\nu X - v(X) \right)$$

(13)

Here torsion is zero and the metric has been introduced instead of the zweibeine. This equivalence between a theory with torsion (10) and a general torsionless dilaton theory (13) had been noted first for the Katanaev-Volovich model [4] in Ref. [17] which corresponds to the special case $U(X) = \alpha = \text{const}$ and $v = \beta X^2 + \Lambda$. The equivalence between (3) with (12) and (13) however even remains true in the quantum version of the theory [18]. It is crucial for a correct treatment for the global properties of solutions to (13): An action (13) could also be obtained by conformal transformation of a torsionless action [$U = 0$ in (10)]. But then the global properties of those two theories differ profoundly [10].

It was known for a long time that 2d theories with an action being a general function of $R$ and vanishing torsion can be solved exactly [2]. For a theory quadratic in torsion this holds as well [1]. To show this in the conformal gauge [2, 4] often requires considerable effort. On the other hand, in the first order form (10) this is quite straightforward [9]. For the potential $V$ in (12) the
e.o.m.-s from the action read

\begin{align}
    dX^- - X^- \omega + Ve^- &= 0, \\
    dX^+ + X^+ \omega - Ve^+ &= 0, \\
    dX - X^+ e^- + X^- e^+ &= 0, \\
    T^\pm + X^\pm U e^\mp \wedge e^- &= 0, \\
    d\omega + \frac{\partial V}{\partial X} e^+ \wedge e^- &= 0,
\end{align}

with the general solution

\begin{align}
    e^+ &= X^+ e^Q df, \\
    e^- &= \frac{dX^-}{X^+} + X^- e^Q df, \\
    \omega &= -\frac{dX^+}{X^+} + Ve^Q df, \\
    C = C^{(g)} &= e^Q X^+ X^- + \int_{y_0}^{X} dye^Q(y) v(y),
\end{align}

where

\begin{equation}
    Q = \int_{X_0}^{X} U(y) dy.
\end{equation}

Eqs. (19)–(22) depend on arbitrary functions $X^+$, $f$, $X$ and one constant $C$. The latter arises by linearly combining (14)–(16) from the relation

\begin{equation}
    d(X^+ X^-) + V dX = 0.
\end{equation}

Multiplication with the integrating factor $\exp Q$ yields

\begin{equation}
    dC^{(g)} = 0.
\end{equation}
Clearly the definition of the constant $C$ is determined by the chosen conventions for the lower limits in the integrals: a change of $y_0$ would redefine $C$ by an additive constant, a change of $X_0$ by a multiplicative one. For example in SRG $(U_{SRG} = -(2X)^{-1}, v_{SRG} = -4\lambda^2)$ the choice $X_0 = 1, y_0 = 0$ yields $C \propto M$, the mass of the black hole (cf. Section 4).

Of course, (14)–(18) also allow various other integrability (consistency) conditions, which resemble (25). For example from (14) and (15) also

$$d(X^+\omega - Ve^+) = 0$$

follows. Those integrability conditions are basically different from (25), because only (25) and thus $C$ alone controls the global properties of the solution in the metric (9). Eliminating $X^+X^-$ through (22) and taking $f, X$ as coordinates, the metric is found to be expressed in Eddington-Finkelstein form (2). $X^+X^-\exp Q$ coincides with a Killing norm whose zeros (horizons) and behaviour at the (complete or incomplete) boundary of the interval allowed for $X$ can be used to determine uniquely the global property of the solution [4, 8]. Indeed $C$ is the only “observable” in the sense of Dirac [8] and the only non-trivial element of the center within the (nonlinear, closed) algebra of constraints and momenta of such a theory [7].

### 2.2 Matter Interactions

Interactions with massless fermions are described by adding to $L^{(g)}$ the action

$$L^{(f)} = \frac{i}{2} \int \epsilon_{abc} e^a \wedge (\bar{\Psi} \gamma^b d\Psi - d\bar{\Psi} \gamma^b \Psi),$$

(27)
whereas massless scalars are introduced by

\[ L^{(S)} = \frac{1}{2} \int F(X) dS \wedge \star dS. \]  
(28)

The factor \( F \) in (28) allows the consideration of (nonminimal) interactions with the dilaton field: for SRG \( F = -\frac{X}{2} \). It is a peculiarity of \( d = 2 \) that not only (28) but also (27) do not depend on a spin connection. Therefore the e.o.m from \( \delta \omega \) remains unchanged, whereas (14) and (15) now acquire a matter contribution

\[-\frac{\delta L}{\delta e^\pm} = dX^\pm \pm X^\pm \omega \mp Ve^\pm + W^\pm = 0, \]  
(29)

\[-\frac{\delta L}{\delta \omega} = dX - X^+ e^- + X^- e^+ = 0, \]  
(30)

with

\[ W^\pm = \pm F(S^\pm)^2 e^\mp \mp J^\mp, \]  
(31)

where

\[ J^\mp = i(\chi_{R,L} d\chi_{R,L} - (d\chi_{R,L})\chi_{R,L}). \]  
(32)

\( S^\pm \) in (31) is an abbreviation for

\[ S^\pm = \star dS \wedge e^\pm \]  
(33)

and in (32) the two-spinor \( \Psi \) has been expressed in terms of chiral components as \( \Psi = \sqrt{2}(\chi_R, \chi_L) \).

The e.o.m.-s from \( \delta X^\pm, \delta X \) only receive a contribution from the matter action
\[ L^{(m)} = L^{(f)} + L^{(S)} \] in \( L = L^{(g)} + L^{(m)} \) if \( \frac{dF}{dX} \neq 0 \):

\[
\frac{\delta L}{\delta X^+} = de^\pm \pm \omega \wedge e^\pm + X^\pm U e^+ \wedge e^- = 0 \tag{34}
\]

\[
\frac{\delta L}{\delta X} = d\omega + \left[ \frac{dU}{dX} X^+ X^- + \frac{\partial v}{\partial X} + \frac{dF}{dX} S^+ S^- \right] e^+ \wedge e^- = 0 \tag{35}
\]

Eq. (35) exhibits the dependence of the curvature scalar \( R = -2 \ast d\omega \), which is proportional to the square bracket, on nonminimally coupled scalars. For the fermion field the e.o.m.-s read

\[
\pm i \frac{\delta L}{\delta \chi_R,L} = 2e^\pm \wedge d\chi_{R,L} - de^\pm \chi_{R,L} = 0, \tag{36}
\]

and the same equations for \( \chi^\dagger_{R,L} \), whereas for the scalar field

\[
-\frac{\delta L}{\delta S} = d(F \ast dS) = 0. \tag{37}
\]

The conservation law generalizing (25) in the presence of matter is obtained again by the same linear combination of (29) with (30)

\[
dC^{(g)} + W^{(m)} = 0, \tag{38}
\]

\[
W^{(m)} = e^Q \left( X^+ W^- + X^- W^+ \right), \tag{39}
\]

where \( C^{(g)} \) has been defined in (22). From (38) the existence of an absolutely (in space and time) conserved quantity follows [3]. The integrability condition \( dW^{(m)} = 0 \) within the conditions of Poincaré’s lemma implies that the one-form \( W^{(m)} \) is exact, \( W^{(m)} = dC^{(m)} \), so that

\[
C = C^{(g)} + C^{(m)} = \text{const}. \tag{40}
\]

That \( W^{(m)} \) is closed must be implied as well by the e.o.m.-s which is indeed the
case [19]. This will be used below.

No complete analytic solution, comparable to (19)–(22) for the matterless case, of (23), (30), (34)–(37) is known in the completely general case although it is possible (in conformal gauge) to reduce the solution to the one of a fourth order nonlinear partial differential equation for $X$ which (except for the case of nonminimally coupled scalars, $\frac{dF}{dX} \neq 0$) is the same in the case with and without matter [19].

A general analytic solution is possible if fermions are restricted to be of one chirality ($\chi_R = 0$ or $\chi_L = 0$) [19, 20, 21], or if the scalar field is selfdual (or antselfdual) in the sense $\star dS = \pm dS$, i.e. $S^+ = 0$ or $S^- = 0$ [19, 21]. Solutions exist, of course, when $U(X) = 0$, $v = \text{const}$ (flat space) or if $U(X) \neq 0$, $v(X) \neq 0$ when such a theory is obtained from a flat theory by conformal transformation. The latter is true for the string inspired dilaton theory ($V = 0$, 3) and for theories which after dilatonization asymptotically become Rindler like (22, $V = \text{const} \neq 0$).

However, a partial solution of the matter equations can be found in suitable gauges. Choosing appropriate components of the matter field as coordinates $C^{(m)}$ may be expressible more directly in terms of component fields. An example for that will be useful for the interacting Schwarzschild case (SRG) of section 4. Consider nonminimally coupled scalars in (23)–(37). The first integral of (37)—with appropriate assumptions of differentiability—is trivial in terms of an arbitrary function $f$:

$$F \star dS = df$$

(41)
On the other hand, the zero-forms $S^\pm$ in (33),

$$S^\pm = \epsilon^{\mu
u}(\partial_\mu S)e^\pm_\nu,$$  \hspace{1cm} (42)

are just the components of $\ast dS$ in the basis $e^\pm$:

$$dS = S^-e^+ - S^+e^-$$ \hspace{1cm} (43)

$$\ast dS = S^-e^+ + S^+e^-$$ \hspace{1cm} (44)

Solving (43) and (44) for $e^\pm$ and using (41) in (44) yields

$$e^\pm = \frac{1}{2S^\pm} \left( \frac{df}{F} \pm dS \right).$$ \hspace{1cm} (45)

With the matter fields $S$ beside $f$ as coordinate, the metric of the line element

$$ds^2 = 2e^+ \otimes e^- = \frac{1}{2S^+S^-} \left( \frac{df^2}{F^2} - dS^2 \right)$$ \hspace{1cm} (46)

is diagonal in $f$ and $S$. For minimal coupling ($F = 1$) (46) is locally conformal.

The matter contribution one-form to the conservation law (25) in this basis becomes

$$W^{(m)} = \frac{e_Q}{2} \left[ (X^- S^+ - X^+ S^-) df - F \left( X^- S^+ + X^+ S^- \right) dS \right] = W_f df + W_S dS.$$ \hspace{1cm} (47)

The consistency condition $dW^{(m)} = 0$ guarantees $\partial_S W^{(m)}_f = \partial_f W^{(m)}_S$ so that (47) may be formally integrated according to (38)

$$C^{(m)} = \int_{f_0}^f W^{(m)}_f (f', S) df' + \int_{S_0}^S W^{(m)}_S (f_0, S') dS',$$ \hspace{1cm} (48)
yielding the (inherently nonlocal in the fields) matter part of the (universal) conservation law.

3 Noether and Noether-like symmetry in $d = 2$

In most applications the Noether theory is used to relate a given symmetry to a conservation law. Here we have to proceed backwards. Clearly the conservation law (22) or (40) can be written as [15]

$$\partial_\mu J^\mu_\nu = 0,$$

(49)

where

$$J^\mu_\nu = \delta^\mu_\nu C(g).$$

(50)

The appearance of an absolutely conserved $C$ and the possibility to introduce associated “currents” ($J^\mu_0$, $J^\mu_1$) is peculiar to $d = 2$ [19]: Invariance of a general action $L = \int \mathcal{L}$ in $d$ dimensions with respect to $\delta \varphi_a = (\delta^k \varphi_a) \wedge \delta \gamma^k$ requires that the Lagrangian transforms as

$$\delta \mathcal{L} = -dU^k \wedge \delta \gamma^k.$$

(51)

The r.h.s. must be a total divergence, i.e.—in a slight abuse of the form language—$\delta \gamma^k = \delta \gamma^k_{\alpha_1 \ldots \alpha_p} dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_p}$ are assumed to be “constant forms” associated with the infinitesimal global parameters $\delta \gamma^k_{\alpha_1 \ldots \alpha_p}$ of the symmetry. From (51) one immediately derives that on-shell a current form (the factor $(-1)^{\kappa}$ should take into account sign changes from commuting forms if $d \varphi_a$ is moved
to the left before dropping it in the derivative)

\[ J^k = U^k + (-1)^k \frac{\partial L}{\partial \delta \varphi_a} \wedge \delta^k \varphi_a \]  

(52)

is closed

\[ dJ^k = 0. \]  

(53)

If the \( \delta \gamma^k \) are “\( m \)-forms” then in \( d \) dimensions \( J^k \) are \( (d - m - 1) \)-forms related to the usual components \( j^k \) of a Noether current by \( j^k = \ast J^k \). \( J^k \) is covariantly conserved for vanishing torsion. If the latter is nonzero, still \( j^k = e \ast J^k \) is conserved in relation to a partial derivative.

When the parameters are zero-forms—only then the variations themselves are covariant—\( J^k \) is a one-form whose components \( j^{k \mu} \) coincide with the ones of the usual Noether current. Here we are interested in \( J^k \) becoming zero-forms \( (m = d - 1) \), because then absolutely conserved quantities are obtained. With fields \( \varphi_a \) zero- and one-forms—as for Cartan variables and matter fields—this can happen only in \( d = 2 \). Thus there is no obvious generalization to a similar quantity in higher dimensions.

Whereas for the matterless action \( L^{(g)} \) this procedure may be applied directly (subsection 3.1), for \( L = L^{(g)} + L^{(m)} \) modifications of the Noether argument will be necessary: A conservation law for the zero-form current will require the validity of another conservation law for a one-form current from the matter contributions.
3.1 Matterless case

Eqs. (14)–(16) are the result of the variations of $L^g$ with respect to $-\frac{\delta}{\delta e^\pm}$ and $-\frac{\delta}{\delta \omega}$, respectively. Therefore, their linear relation (24), after multiplication with $\exp Q$ may be written as

$$dC^g = -e^Q \left( X^+ \frac{\delta}{\delta e^+} + X^- \frac{\delta}{\delta e^-} + V \frac{\delta}{\delta \omega} \right) L^g. \quad (54)$$

The global symmetry transformations with constant infinitesimal parameters $\gamma_\mu$ in a “one-form” $\delta \gamma = \delta \gamma_\mu dx^\mu$ can be read from (54):

$$\delta e^\pm = X^\pm e^Q \delta \gamma = \frac{\partial C^g}{\partial X^\pm} \delta \gamma \quad (55)$$
$$\delta \omega = Ve^Q \delta \gamma = \frac{\partial C^g}{\partial X} \delta \gamma \quad (56)$$
$$\delta X^\pm = \delta X = 0 \quad (57)$$

This generalizes the result of [13] to arbitrary (matterless) 2d theories, including dilaton theories, if the latter are expressed in the same first order “torsion” form [cf. the relation between (12) and (13)]. It can be verified easily that the Lagrangian indeed varies to a total divergence

$$\delta L = d(2X^+X^- + XV - C^g) \wedge \delta \gamma, \quad (58)$$

an exact “form” in the sense of [51].

Comparing (55)–(57) with the analytic solution (19)–(21) the symmetry is seen to correspond to an “orbit” with $df \rightarrow \delta \gamma$, $dX^\pm \rightarrow \delta X^\pm = 0$, $dX \rightarrow \delta X = 0$. This symmetry commutes with the gauge-symmetries of the theory (local Lorentz transformations and diffeomorphisms). The last form of (55) and (56) emphasizes the close relation to the expression of $C^g$, interpreted as a quasilo-
cal energy in a Hamiltonian, employing the Regge-Teitboim trick for a boundary term \[9\].

There are many more global symmetries which can be related to integrability conditions following from the e.o.m.-s \[19\]. They simply reflect the large gauge freedom and include (not unexpectedly) global Lorentz transformations and global translations (conservation of energy momentum tensor). It should be noted though, that especially the latter are—for a nontrivial \(d = 2\) model—not directly related to the conservation law discussed here \[9\].

### 3.2 Symmetry with matter: A toy model

As emphasized in subsection 2.2, the conservation law with matter generalizes to eq (38). \(dC^{(g)} + W^{(m)}\) may be expressed by the same e.o.m.-s as in (54). For \(W^{(m)}\) its integrability condition \(dW^{(m)}\) may be related linearly to the e.o.m.-s (29), (30), (34)–(37). On that basis another “matter” symmetry transformation may be defined. It is unusual that the Noether symmetry thus must be “extended” when a new piece (here the matter interaction) is added to the action.

We elucidate this point first for a (topological) toy model in \(d = 2\) whose action

\[
L = \int (X dw + K w \wedge d\varphi)
\]

(59)

depends on zero-form fields \(X, K\) and \(\varphi\) and on one 1-form field \(w\). The first term of (59) is a simplified “geometric” action whereas in the second term “matter”, described by an “amplitude” \(K\) and a “phase” \(\varphi\) is coupled to the “geometric” variable \(w\). Thus this term is made to resemble the fermionic interaction (27).
The e.o.m.-s for (59) read

\[ dX + K d\phi = 0 , \quad (60) \]
\[ dw = 0 , \quad (61) \]
\[ w \wedge d\phi = 0 , \quad (62) \]
\[ d(Kw) = 0 . \quad (63) \]

Comparing (60) with (38) one notices that (60) can be interpreted as the counterpart of a "conservation law" with matter if \( K \neq 0 \). \( X \) takes the role of \( C(g) \).

Applying an exterior derivative to (60) leads to the integrability condition

\[ dK \wedge d\phi = 0 \quad (64) \]

which implies \( K = K(\phi) \) so that

\[ K d\phi = d \left( \int_{y_0}^{\phi} K(y) dy \right) \quad (65) \]

and

\[ d \left( X + \int_{y_0}^{\phi} K(y) dy \right) = 0 . \quad (66) \]

For a fixed choice of \( y_0 \) different values of the constant \( C \) in

\[ C = X + \int_{y_0}^{\phi} K(y) dy \quad (67) \]

characterize the solutions.

From (61) and (62) in an analogous way \( w = w(\phi) \) may be concluded, (63) is fulfilled identically. Thus the action (59) on-shell yields a one-dimensional theory in which \( \phi \) may be considered the only independent coordinate.
In the “matterless” case ($K = 0$) the (“geometric”) symmetry transformations, belonging to $-\frac{\delta L}{\delta w} = dX = 0$ are translations with the constant “one-form” $\delta \gamma$

$$\delta w = \delta \gamma. \quad (68)$$

For $K \neq 0$ the integrability condition (64) allows an expansion in terms of Eqs. (61), (62), (63), i.e.

$$\delta L \frac{\partial \varphi}{\partial X} + \delta L \frac{\partial \varphi}{\partial K} + \delta L \frac{\partial \varphi}{\partial \varphi} = 0$$

The apparent dependence on the zero components of $w$ and on $\frac{\partial}{\partial x}$ is spurious. Actually the same equation holds with zero replaced by one. In fact,

$$\frac{\partial \varphi}{\partial \varphi} = \frac{\partial \varphi}{\partial \varphi}$$

is nothing else but (62) written in components. Thus both formulations on-shell are equivalent. Eq. (69) allows the introduction of a (“matter”) symmetry with global (zero-form) parameter $\delta \rho$:

$$\delta \varphi = \frac{\partial \varphi}{\partial \varphi} \delta \rho \quad (71)$$

$$\delta X = -K \frac{\partial \varphi}{\partial \varphi} \delta \rho \quad (72)$$

$$\delta K = \frac{\partial \varphi}{\partial \varphi} \delta \rho \quad (73)$$

or a similar one with $\partial_0 \rightarrow \partial_1$, $w_0 \rightarrow w_1$. The Lagrangian $\hat{\mathcal{L}}$ in $L = \int \mathcal{L}$ under (71)–(73) transforms as a total divergence

$$\delta \hat{\mathcal{L}} = d \left( K d \varphi - K \frac{\partial \varphi}{\partial \varphi} \right) \delta \rho \quad (74)$$
and the related conserved Noether current (one-form) is

\[ J = K d\varphi, \quad (75) \]

or in components of \( \star J \) (cf. subsection 3.1)

\[ j^\mu = \epsilon^{\mu\nu} K \partial_\nu \varphi. \quad (76) \]

The conservation \( dJ = 0 \) of the one-form current just yields the integrability condition, i.e. that the zero-form conservation law from (60) can be written as (66) or (67). We thus observe a “two stage” argument instead of the usual direct one with one single symmetry transformation for all fields. Also the result of \( dJ = 0 \), i.e. \( K = K(\varphi) \) is nonstandard. It means that (part of) the conserved quantities are not simply expressed by a local combination of the fields appearing in the action, but that those fields may become functions of each other.

### 3.3 Symmetry with scalars and fermions

Eq. (54) from (38) now is replaced by

\[ dC^{(g)} + W^{(m)} = -e^Q \left( X^+ \frac{\delta}{\delta e^+} + X^- \frac{\delta}{\delta e^-} + V \frac{\delta}{\delta \omega} \right) L, \quad (77) \]

where the e.o.m.-s (29) and (30) appear on the r.h.s. Introducing the same (“geometric”) global parameters with the associated transformation law (55)–(57) to be read off again from (77) as in the toy model one does not arrive at the complete conservation law, but only at (38). The “secondary” conservation law for \( W^{(m)} \) is expressed in terms of th e.o.m.-s (29), (30), (34), (36), (37). This is straightforward even without gauge fixing. But in order to simplify the resulting equation we choose a special set of conformal coordinates.
The fermion currents $J^n = \chi_n R \pm \chi_n L$ in (31) may be written as a function of “amplitude” and “phase” as

$$\chi_{R,L} = \frac{1}{\sqrt{2}} k_{R,L} e^{i\varphi_{R,L}}.$$  \hspace{1cm} (78)

For $K^n = k_{L,R}^2$ this yields

$$J^n = -K^n d\varphi_{R,L}.$$ \hspace{1cm} (79)

A conformal gauge results by identifying $d\varphi_R$, $d\varphi_L$ with the coordinates. For a one-form we thus have $\omega = \omega_R d\varphi^R + \omega_L d\varphi^L$ etc. These coordinates may be used—within the conformal patch—also when no fermions are present ($K^n = 0$). Of course, the relation to the space-time coordinates may become highly nontrivial. In terms of $d\varphi_{R,L}$ the e.o.m. are unchanged, except (36) which is replaced by

$$\frac{\delta L}{\delta K^+} = -e^- \wedge d\varphi_L = 0,$$ \hspace{1cm} (80)  
$$\frac{\delta L}{\delta K^-} = e^+ \wedge d\varphi_R = 0,$$ \hspace{1cm} (81)  
$$\frac{\delta L}{\delta \varphi_R} = d(K^- e^+) = 0,$$ \hspace{1cm} (82)  
$$\frac{\delta L}{\delta \varphi_L} = -d(K^+ e^-) = 0.$$ \hspace{1cm} (83)

The line element from (80) and (81)

$$(ds)^2 = 2e^+ \otimes e^- = 2e^+_R e^-_L d\varphi^R d\varphi^L$$ \hspace{1cm} (84)

indeed is conformal ($e^+_L = e^-_R = 0$). The one-form conservation law for matter
$dW^{(m)}$ becomes a linear combination of (29), (30), (34), (37), (80)–(83):

$$
dW^{(m)} = e^Q \left[ \frac{\partial \varphi^R}{e_0^+} \left( X + \delta \frac{\delta \varphi^R}{\delta X^+} - X^+K^- \delta \frac{\delta e^+}{\delta e^-} + K^-e^+ \wedge \frac{\delta}{\delta e^-} \right) 
- \frac{K^-}{e_0^+} \left( K^+\partial_0 \varphi^L - \frac{\partial_0 K^+}{K^+} X^+ - \partial_0 X^+ + FS^+e_0^+ + UX^+e_0^+ \right) \frac{\delta}{\delta K^-} 
+ \frac{\partial_0 \varphi^L}{e_0^+} \left( X^- \delta \frac{\delta \varphi^L}{\delta X^-} + X^-K^+ \delta \frac{\delta e^-}{\delta e^+} - K^+e^- \wedge \frac{\delta}{\delta e^+} \right) 
+ \frac{K^+}{e_0^+} \left( K^-\partial_0 \varphi^R + \frac{\partial_0 K^+}{K^+} X^- + \partial_0 X^- - FS^-e_0^+ - UX^-e_0^+ \right) \frac{\delta}{\delta K^+} 
+ FS^+e^- \wedge \frac{\delta}{\delta e^+} - FX^-S^+ \delta \frac{\delta e^+}{\delta X^-} - FS^-e^- \wedge \frac{\delta}{\delta e^-} + FX^+S^- \delta \frac{\delta}{\delta X^+} 
+ \left( -F' \partial_0 \varphi^R \partial_0 \varphi^L \right) - UX^+K^-d\varphi^L - UX^-K^+d\varphi^L 
+ UF \left( S^+X^-e^- - S^2X^+e^+ \right) \wedge \frac{\delta}{\delta \omega} 
- (X^+S^- - X^-S^+) \delta \frac{\delta X^-}{\delta S} + UX^+X^- \left( K^+ \delta \frac{\delta}{\delta K^+} - K^- \delta \frac{\delta}{\delta K^-} \right) \right] L.
$$

(85)

This corresponds to a symmetry with “matter” parameter $\delta \rho$

$$\delta e^+ = \left( K^+\partial_0 \varphi^L - FS^+e^+ \right) e^Q e^- \delta \rho, \quad (86)$$

$$\delta e^- = \left( -K^-\partial_0 \varphi^R + FS^-e^- \right) e^Q e^+ \delta \rho, \quad (87)$$

$$\delta \omega = \left[ F' \partial_0 \varphi^R \partial_0 \varphi^L - UX^+K^-d\varphi^L - UX^-K^+d\varphi^L \right] e^Q \delta \rho, \quad (88)$$

$$\delta X^+ = \left( K^+\partial_0 \varphi^L - FS^+e^+ \right) e^Q X^- \delta \rho, \quad (89)$$

$$\delta X^- = \left( -K^-\partial_0 \varphi^R + FS^-e^- \right) e^Q X^+ \delta \rho, \quad (90)$$

$$\delta K^+ = \left[ \left( K^-\partial_0 \varphi^R + \frac{\partial_0 K^+}{K^+} X^- + \partial_0 X^- - FS^-e_0^+ - UX^-e_0^+ \right) \frac{K^+}{e_0^+} \right] X^+ \delta \rho, \quad (91)$$

$$\delta K^- = \left[ \left( K^+\partial_0 \varphi^L + \frac{\partial_0 K^-}{K^-} X^+ + \partial_0 X^+ - FS^+e_0^+ + UX^+e_0^+ \right) \frac{K^-}{e_0^-} \right] X^- \delta \rho, \quad (92)$$

21
\[ +UX^+X^-K^+ e^Q \delta \rho , \quad (91) \]
\[ \delta K^- = \left[ \left( -K^+ \partial_0 \phi^L + \frac{\partial_0 K^-}{K^-} X^+ + \partial_0 X^+ + FS^+ e_0 + UX^+ e_0 \right) \frac{K^-}{e_0^+} \right] \]
\[ -UX^+X^-K^- e^Q \delta \rho , \quad (92) \]
\[ \delta \phi^R = X^+ \frac{\partial_0 \phi^R}{e_0^+} e^Q \delta \rho , \quad (93) \]
\[ \delta \phi^L = X^- \frac{\partial_0 \phi^L}{e_0^-} e^Q \delta \rho , \quad (94) \]
\[ \delta S = (X^- S^+ - X^+ S^-) e^Q \delta \rho . \quad (95) \]

Of course the use of \( \phi^R,L \) as coordinates is only one way to write that symmetry.

4 Conservation law and symmetry for spherically reduced gravity

As mentioned above, SRG is contained as a special case in our general formalism:

\[ V_{SRG} = -\frac{1}{2X} X^+ X^- - 4\lambda^2 \quad (96) \]

for (12) we obtain from (22) and (23) for the geometric part of the conservation law \((x_0 = 1, y_0 = 0)\)

\[ C_{SRG} = \frac{1}{\sqrt{X}} X^+ X^- - 8\lambda^2 \sqrt{X} . \quad (97) \]
When the influx of matter is absent (for example after the black hole has been created in such a way that no spherical wave bounces back), $C(g)$ by itself is constant. The line element from (19) and (20)

$$(ds)^2 = \frac{2}{\sqrt{X}} df \otimes \left[ dX + \left( C + 8 \lambda^2 \sqrt{X} \right) df \right] = d\tilde{f} \otimes \left[ 2d\tilde{X} + \left( \frac{C}{16\lambda^2X} + 1 \right) d\tilde{f} \right]$$

(98)

in its second version with $\tilde{X} = \frac{\sqrt{X}}{2\lambda}$, $\tilde{f} = \frac{f}{\sqrt{X}}$ exhibits the Eddington-Finkelstein form (2). Requiring a diagonal metric in terms of new variables $r$ and $t$ instead of $\tilde{X}$ and $\tilde{f}$ with $\det g = -1$ fixes $\tilde{X} \propto r$ so that indeed $C \propto M$, the mass of the black hole. The symmetry (55) to (57) may be simply related [9] to a translation in the direction of the Killing field $\partial / \partial f \propto \partial / \partial t$ for the metric (98). The connection of such a translation with the (conserved) mass of the black hole is well-known.

The more interesting case arises, when matter interactions are present (e.g. during the time of formation and also if the formation is accompanied by an outgoing matter wave [24]).

For nonminimally coupled scalars we have to set $F_{SRG} = -\frac{X}{2}$ in (47):

$$W_{SRG}^{(m)} = \frac{1}{2\sqrt{X}} \left( X^- S^+ - X^+ S^- \right) df + \frac{\sqrt{X}}{4} \left( X^- S^+ + X^+ S^- \right) dS$$

(99)

Then $C_{SRG}^{(m)}$ is the appropriate special case of (48).

The line-element (46) for $F_{SRG} = -\frac{X}{2}$ corresponds to the choice of $f$ and $S$ as “coordinates” but by a suitable choice of $f = f(t, r)$, $S = S(t, r)$, an Eddington-Finkelstein metric and with $C$ replaced by a function of $r$ and $t$ (“variable mass”) may be obtained. This clearly is possible only where the matter-field does not vanish identically. Therefore (99) may be useful when SRG behaviour
is described “inside” the 2d submanifold, occupied by the developing matter field. Of course, the introduction of $dS$ as a variable is not mandatory. An Eddington-Finkelstein gauge choice [cf. (2)] with $e_0^+ = (e_0^- - 1) = (e_1^+ - 1) = 0$, $e_1^- = e_1^-(t,r)$ would have the advantage of being extendable across the horizons (zeros of $e_1^-$). But—in contrast to the matterless case—that $e_1^-$ would have no immediate relation any more to the conserved quantity $C$.

The symmetry relation from (99) is contained in (86)–(95):

$$\delta e^+ = \frac{\sqrt{X}}{2} S^{+2} e^- \delta \rho$$  \hspace{1cm} (100)

$$\delta e^- = -\frac{\sqrt{X}}{2} S^{-2} e^+ \delta \rho$$  \hspace{1cm} (101)

$$\delta \omega = \frac{1}{4\sqrt{X}} \left[ (2X^+ S^- - X^- S^+) S^+ e^- - (2X^- S^+ - X^+ S^-) S^- e^+ \right] \delta \rho$$  \hspace{1cm} (102)

$$\delta X^+ = \frac{\sqrt{X}}{2} S^{+2} X^- \delta \rho$$  \hspace{1cm} (103)

$$\delta X^- = -\frac{\sqrt{X}}{2} S^{-2} X^+ \delta \rho$$  \hspace{1cm} (104)

$$\delta S = \frac{1}{\sqrt{X}} (X^- S^+ - X^+ S^-) \delta \rho$$  \hspace{1cm} (105)

5 Conclusions and Outlook

The universal absolute (in space and time) conservation law for all covariant 2d theories, including interactions with matter has been shown to be related to a Noether symmetry. Such a global symmetry had been identified before only for the special model [4] in Ref. [15] in the absence of matter interactions. As shown in subsection 3.1 this result can be generalized easily to all matterless models. It refers to the geometric variables (zweibeine and spinconnection) only
and may be called “geometric” symmetry.

When interactions with fermions and scalars are present, the “geometric” symmetry by itself no longer provided the complete conservation law, valid also in that case. In addition, another (“matter”) symmetry through the conservation of a one-form current yields a necessary integrability condition. We clarify this unusual situation within a simplified toy-model (subsection 3.2). The symmetry for the general 2d model, including nonminimally coupled scalars, is given in subsection 3.3.

Because of its importance the special case of spherically reduced gravity is set out in somewhat more detail in section 4. Here no exact solution exists. Nevertheless we expect that the general formalism will permit appropriate gauge choices suitable for numerical studies in which the universal conservation law is used to fix the overall integration constant $C = C^g + C^m$ where $C^g$ and $C^m$ are contributions from the “geometric” and “matter” parts, respectively. However, neither of these quantities may be identified with a variable mass of the black hole in a corresponding ansatz for the metric.

It should be stressed that in other approaches to covariant models in two variables the overall constant $C$ is hidden; it only appears clearly in the first order approach advocated here, especially when interactions with matter are taken into account.

In this connection it will be desirable also to find the “large” symmetry operations corresponding to the infinitesimal ones discussed here. They hopefully could connect sets of numerical solutions.

All considerations of the present paper stay at the classical level. It has become increasingly clear in the last years that even the existence of a classical solution (with matter interactions) does not guarantee its quantum extension. What can
be done is to integrate out (exactly) the geometric variables in the matterless case \[25\] and to treat matter as a perturbation \[18\]. Because the universal con-
servation law has a quantum counterpart \[26\] the peculiar symmetry discussed
here should also be valid in the quantum case, expressed as a Ward identity for
vertex functions of 2d gravity with matter.

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