Oscillations in trapped Fermi gases in isotropic potentials

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Abstract

We study the thermodynamic properties of an ideal gas of fermions in a harmonic oscillator confining potential. The analogy between this problem and the de Haas-van Alphen effect is discussed and used to obtain analytical results for the chemical potential and specific heat in the case of an isotropic potential. Step-like behaviour in the chemical potential, first noted in numerical studies, is obtained analytically and shown to result in an oscillatory behaviour of the specific heat when the particle number is varied. The origin of these oscillations is that part of the thermodynamic potential responsible for the de Haas-van Alphen effect. At low temperatures we show that there are significant deviations in the specific heat from the expected linear temperature dependence again as a consequence of the de Haas-van Alphen part of the thermodynamic potential. Results are given for one, two, and three spatial dimensions.

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I. INTRODUCTION

The experimental realization \[1\] of Bose-Einstein condensation in confined gases of atoms at low temperatures has been the stimulus for a wide range of experimental and theoretical investigations. (See Ref. \[2\] for a review.) By ingenious experimental techniques it is possible to prepare the atoms as either bosons or fermions. There have now been a number of experiments on trapped gases of fermions, as well as mixtures of bosons and fermions \[3\].

For a gas of fermions, it is a good approximation to model the system as a collection of non-interacting particles obeying Fermi-Dirac statistics confined by a simple harmonic oscillator potential. Unlike the case of bosons, the dominant $s$-wave scattering channel is suppressed making the effects of interactions less important \[4\]. With this simple theoretical model for the system, it is easy to obtain exact and simple expressions for the single-particle energy levels using the standard quantum mechanical result for the simple harmonic oscillator. This can serve as a starting point for a theoretical analysis of the thermodynamic behaviour of the system.

Because the energy levels for a set of particles (bosons or fermions) confined by a simple harmonic oscillator potential form a discrete set, in order to calculate the thermodynamic properties we must be able to perform sums over the quantum numbers labelling the states; this can lead to difficulties in evaluation. By restricting the parameters of the system (temperature and oscillator frequencies for example) it may be possible to argue that the sums can be approximated with integrals, an approximation that renders the computation of analytical results easier. We will refer to the approximation of sums with integrals the continuum approximation, since it is equivalent to regarding the discrete energy spectrum as continuous. This approximation was first applied to the Fermi gas in the now classic paper of Butts and Rokhsar \[4\]. Even when one goes beyond this approximation, as we will show below, the continuum approximation can still be very accurate in some temperature regimes, and can capture the leading order behaviour if the temperature is not too low.

The first study that showed there might be more features present beyond what is obtained by the continuum approximation was given by Schneider and Wallis \[5\]. These authors computed the thermodynamic expressions numerically, and their results showed that there were a number of step-like features present, not predicted by the continuum approximation. In particular, the chemical potential, which is a smooth function in the continuum approx-
imation, has a series of step-like jumps when a numerical analysis of the exact results is performed. This in turn can lead to step-like, or oscillatory, behaviour of other thermodynamic properties, such as the specific heat. Another important consequence of the work presented in Ref. [5] is that at sufficiently low temperatures there are significant deviations in the specific heat from the linear temperature dependence predicted [4] by the continuum approximation. The main purpose of the present paper is to analyze the situation analytically to obtain the step-like behaviour of the chemical potential, and to apply this method to the evaluation of the specific heat.

In addition to the two studies already mentioned [4, 5], there have been a number of other related theoretical works. Ref. [6] generalized the calculations of Ref. [5] to a gas with two spin states with an interaction between the two states. (We will make some brief comments on the effects of interactions in Sec. V.) A number of authors [7, 8, 9, 10, 11, 12, 13] have obtained analytical approximations in a number of cases using path integral, Green’s function, or density matrix methods.

In the present paper we will concentrate on the behaviour of the chemical potential and specific heat, and will use the evaluation of the thermodynamic potential as the basic starting point. The key observation is that for a simple harmonic oscillator confining potential, the evaluation of the thermodynamic potential is mathematically very similar to that of a gas of charged fermions in a constant magnetic field. Landau [14] was the first to show that the thermodynamic properties of electrons in a constant magnetic field undergo oscillations whose period is determined by the inverse of the magnetic field strength. This gave a theoretical basis for the observations of de Haas and van Alphen, now referred to as the de Haas-van Alphen effect. (See Ref. [15] for a good discussion of the de Haas-van Alphen effect and the important role it plays in condensed matter physics.) What we will show here is that for the trapped Fermi gas the step-like behaviour found in Ref. [5] is completely analogous to the de Haas-van Alphen effect. Just as in the de Haas-van Alphen effect, where use of the continuum approximation would miss the oscillations, here too we must go beyond the continuum approximation.

The outline of our paper is as follows. In Sec. II we describe the general method following the classic analysis of Sondheimer and Wilson [16]. There are some differences with the de Haas-van Alphen case considered in Ref. [16] that we will describe. We will show how the continuum approximation comes about in the method, and even find the next order
correction to it, beyond the leading order result given in Ref. [4]. In Sec. III we analyze the de Haas-van Alphen contribution to the thermodynamic potential for a 3-dimensional isotropic harmonic oscillator potential, and obtain approximate analytical results for the chemical potential. This is then used to examine the specific heat. Results for the 1- and 2-dimensional gases are given in Sec. IV and the results compared with the 3-dimensional case. Finally, Sec. V contains a brief summary and a short discussion of the main results.

II. GENERAL METHOD

We will begin with the grand canonical ensemble and the thermodynamic potential $\Omega$ defined by

$$\Omega = -T \sum_n f(E_n) ,$$

where

$$f(E) = \ln [1 + \exp(-\beta(E - \mu))] .$$

We use the usual notation $\beta = 1/T$, with $T$ the temperature, and choose units such that the Boltzmann constant is equal to one. Take the Laplace transform of $f(E)$, and call it $\varphi(\beta)$, so that

$$\varphi(\beta) = \int_0^\infty dE \ e^{-\beta E} f(E) .$$

The inverse Laplace transform of Eq. (2.3) reads

$$f(E) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\beta \ e^{\beta E} \varphi(\beta) ,$$

where $c$ is an arbitrary constant chosen so that the integration path lies to the right of any singularities of $\varphi(\beta)$. We can now use Eq. (2.4) in Eq. (2.1). The sum over the energy levels that results can be related to the partition function for the $\mu = 0$ system, defined by

$$Z(\beta) = \sum_n e^{-\beta E_n} .$$

We can then write $\Omega$ defined by Eq. (2.1) as

$$\Omega = -\frac{T}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\beta \ Z(-\beta) \varphi(\beta) .$$
It is necessary here to regard \( Z(\beta) \) as a function of \( \beta \) with \( \beta \) viewed as a complex variable. \( Z(-\beta) \) is the result obtained from this complex function by analytic continuation of the definition Eq. (2.5) to \( \Re(\beta) < 0 \).

We now define \( \mathcal{Z}(E) \) to be the Laplace transform of \( \beta^{-2}Z(\beta) \). (The factor of \( \beta^{-2} \) is for later convenience.) This results in the definition

\[
Z(\beta) = \beta^2 \int_0^\infty dE \, e^{-\beta E} \mathcal{Z}(E) .
\] (2.7)

The inverse Laplace transform gives

\[
\mathcal{Z}(E) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\beta \, e^{\beta E} \beta^{-2} Z(\beta) .
\] (2.8)

Substitution of Eq. (2.7) into Eq. (2.6) followed by the use of Eq. (2.4) results in

\[
\Omega = -T \int_0^\infty dE \, \mathcal{Z}(E) \frac{\partial^2 f(E)}{\partial E^2} .
\] (2.9)

The Fermi-Dirac distribution function \( F(E) \) is

\[
F(E) = [e^{\beta(E-\mu)} + 1] = -T \frac{\partial}{\partial E} f(E) ,
\] (2.10) (2.11)

if we use Eq. (2.2). This leads to

\[
\Omega = \int_0^\infty dE \, \mathcal{Z}(E) \frac{\partial}{\partial E} F(E) ,
\] (2.12)

giving the key starting point in the Sondheimer-Wilson analysis of the de Haas-van Alphen effect. The method rests on an evaluation of \( \mathcal{Z}(E) \) defined by Eq. (2.7) or Eq. (2.8) and its use in Eq. (2.12).

To evaluate \( \mathcal{Z}(E) \) in Eq. (2.8) we require the partition function, or at least some information about the singularities of \( \beta^{-2}Z(\beta) \) in the complex \( \beta \)-plane. For the case of a simple harmonic oscillator potential this information is very easy to obtain. We will consider the \( D \)- dimensional oscillator potential

\[
V(\mathbf{x}) = \frac{1}{2} m \sum_{j=1}^D \omega_j^2 x_j^2 ,
\] (2.13)
with \( \mathbf{x} = (x_1, \ldots, x_D) \) the \( D \) spatial coordinates. The energy levels are simply given by (in \( \hbar = 1 \) units)

\[
E_n = \sum_{j=1}^{D} \left( n_j + \frac{1}{2} \right) \omega_j ,
\]

(2.14)

where \( \mathbf{n} = (n_1, \ldots, n_D) \) and \( n_j = 0, 1, 2, \ldots \) for \( j = 1, \ldots, D \). We then have \( Z(\beta) \) in Eq. (2.5) expressed as a product of geometric series that are easily summed to give

\[
Z(\beta) = \prod_{j=1}^{D} \frac{e^{-\beta \omega_j / 2}}{(1 - e^{-\beta \omega_j})} ,
\]

(2.15)

Because of the neglect of interactions, and the simple form of the potential, it has been possible to evaluate the partition function in closed form.

To obtain \( \mathcal{Z}(E) \) we need to know the singularity structure of \( \beta^{-2}Z(\beta) \). It is now obvious from Eq. (2.16) that \( \beta^{-2}Z(\beta) \) is a meromorphic function. There is a pole at \( \beta = 0 \) of order \( D + 2 \) as well as a series of poles along the positive and negative imaginary \( \beta \)-axis at \( \beta = \beta_{kj} = 2\pi i k_j / \omega_j \) with \( k_j = \pm 1, \pm 2, \ldots \). The order of these poles depends on the relative ratios of the harmonic oscillator frequencies. If none of the oscillator frequencies are rational multiples of any of the others, all of the poles, apart from the one at \( \beta = 0 \), will be simple. However, if some of the oscillator frequencies are rational multiples of the others, some of the poles away from the origin will be higher order. This makes a determination of the residues of poles along the positive and negative imaginary \( \beta \)-axis for a general harmonic oscillator potential an inelegant treatment of special cases. We will return to this problem later.

We can write \( \mathcal{Z}(E) \) in Eq. (2.8) as

\[
\mathcal{Z}(E) = \mathcal{Z}_0(E) + \mathcal{Z}_r(E) ,
\]

(2.16)

where we close the contour in the left hand side of the complex plane and use \( \mathcal{Z}_0(E) \) to denote the contribution to the integral from the pole at \( \beta = 0 \), and \( \mathcal{Z}_r(E) \) the contribution coming from the rest of the poles along the positive and negative imaginary \( \beta \)-axis. When Eq. (2.16) is used in Eq. (2.12) we obtain the thermodynamic potential as

\[
\Omega = \Omega_0 + \Omega_r
\]

(2.17)

in an obvious way. As in the de Haas-van Alphen effect, the oscillations in thermodynamic quantities will come from \( \Omega_r \), but we will first examine \( \Omega_0 \). \( \Omega_r \) will be evaluated in the next section.
To obtain $Z_0(E)$ we need the residue of $\beta^{-2}e^{\beta E}Z(\beta)$ at $\beta = 0$. For general values of $D$ and arbitrary frequencies $\omega_j$ this is difficult to write down in any simple way. We will concentrate on just the cases $D = 1, 2, 3$ here, although there is no intrinsic reason why the method cannot be extended to other values. A straightforward calculation yields

$$Z_0(E) = \frac{E^2}{2\omega} - \frac{\omega}{24} \quad \text{for } D = 1,$$

$$Z_0(E) = \frac{E}{24\omega_1\omega_2}(4E^2 - \omega_1^2 - \omega_2^2) \quad \text{for } D = 2,$$

$$Z_0(E) = \frac{1}{5760\omega_1\omega_2\omega_3} \left[ 240E^4 - 120E^2(\omega_1^2 + \omega_2^2 + \omega_3^2) + 7(\omega_1^4 + \omega_2^4 + \omega_3^4) ight. + 10(\omega_1^2\omega_2^2 + \omega_2^2\omega_3^2 + \omega_3^2\omega_1^2) \right] \quad \text{for } D = 3.$$ (2.20)

These results can now be used in Eq. (2.12) to obtain

$$\Omega = \int_0^\infty \! dE \, Z_0(E) \frac{\partial}{\partial E} F(E).$$ (2.21)

As $T \to 0$, meaning that $\beta\mu \to \infty$, the Fermi-Dirac distribution function approaches a step-function, $F(E) \to \theta(\mu - E)$, so that $\frac{\partial}{\partial E} F(E) \to -\delta(E - \mu)$. This crude approximation results in

$$\Omega_0 \to -Z_0(\mu),$$ (2.22)

and is really the first term in a systematic expansion due to Sommerfeld. (See Refs. [15, 17, 18] for three derivations of the Sommerfeld expansion.) In our case we find

$$\Omega_0 \simeq -Z_0(\mu) - \frac{\pi^2}{6} T^2 Z_0''(\mu) - \frac{7\pi^4}{360} T^4 Z_0'''(\mu).$$ (2.23)

Although the general Sommerfeld expansion contains higher order terms, these vanish here because $Z_0(E)$ is no more than quartic in $E$ due to our restriction that $D \leq 3$. For arbitrary $D$, the expansion Eq. (2.23) will contain more terms with increasing powers of $T$. For low values of $T$ these should be less important than those indicated in Eq. (2.23) in any case. Using Eqs. (2.18-2.20) in Eq. (2.23) results in

$$\Omega_0 \simeq -\frac{1}{2\omega}(\mu^2 + \frac{\pi^2}{3} T^2) + \frac{\omega}{24}, \quad \text{for } D = 1,$$

$$\Omega_0 \simeq -\frac{\mu^3}{6\omega_1\omega_2} + \frac{\mu}{24\omega_1\omega_2}(\omega_1^2 + \omega_2^2 - 4\pi^2T^2), \quad \text{for } D = 2,$$

$$\Omega_0 \simeq -\frac{\mu^4}{24\omega_1\omega_2\omega_3} + \frac{\mu^2}{48\omega_1\omega_2\omega_3}(\omega_1^2 + \omega_2^2 + \omega_3^2 - 4\pi^2T^2) - \frac{1}{5760\omega_1\omega_2\omega_3} \left[ 7(\omega_1^4 + \omega_2^4 + \omega_3^4) ight. + 10(\omega_1^2\omega_2^2 + \omega_2^2\omega_3^2 + \omega_3^2\omega_1^2) - 40\pi^2T^2(\omega_1^2 + \omega_2^2 + \omega_3^2) + 112\pi^4T^4 \right], \quad \text{for } D = 3.$$ (2.26)
The thermodynamic properties of the system follow from a knowledge of the thermodynamic potential. In particular, the average particle number $N$ is given by

$$N = - \left( \frac{\partial \Omega}{\partial \mu} \right)_{T, \omega}, \quad (2.27)$$

and the internal energy $U$ is given by

$$U = \frac{\partial}{\partial \beta} (\beta \Omega) \bigg|_{\beta \mu, \omega}. \quad (2.28)$$

In the most physically interesting case we take $D = 3$. (The cases of $D = 1, 2$ will be given later in Sec. [IV].) If we temporarily ignore the contribution $\Omega_r$ to $\Omega$, we have $N \simeq N_0$ where

$$N_0 = - \left( \frac{\partial \Omega_0}{\partial \mu} \right)_{T, \omega} \quad (2.29)$$

$$\simeq \frac{\mu^3}{6\omega_1\omega_2\omega_3} - \frac{\mu}{24\omega_1\omega_2\omega_3}(\omega_1^2 + \omega_2^2 + \omega_3^2 - 4\pi^2T^2). \quad (2.30)$$

For large values of $\mu$, if we keep only the term in Eq. (2.30) of order $\mu^3$, we find (with $N \simeq N_0$),

$$\mu^3 \simeq \frac{6\omega_1\omega_2\omega_3 N}{6}, \quad (2.31)$$

showing that $\mu \propto N^{1/3}$. For large values of $N$ we will have $\mu$ much larger than the average oscillator frequency, consistent with the assumptions made in deriving these results. It is easy to obtain the next order correction to Eq. (2.31),

$$\frac{\mu^3}{\omega_1\omega_2\omega_3} \simeq 6N \left\{ 1 + \frac{(\omega_1^2 + \omega_2^2 + \omega_3^2 - 4\pi^2T^2)}{4(6N\omega_1\omega_2\omega_3)^{2/3}} \right\}, \quad (2.32)$$

showing that the correction to Eq. (2.31) becomes increasingly unimportant for large values of $N$. If we use Eq. (2.32) and rewrite the result in terms of the Fermi energy defined by

$$E_F = \mu(T = 0), \quad (2.33)$$

it is easy to see that

$$\mu \simeq E_F \left( 1 - \frac{\pi^2T^2}{3E_F^2} \right). \quad (2.34)$$

This agrees with the result in Ref. [4] who used the continuum approximation and only the $T$-dependent part of the second term in Eq. (2.30). We have demonstrated here that Eq. (2.34) holds true even when the next order approximation for the density of states is included in the continuum approximation. However, as we will see in the next section.
there are important corrections to this from terms that come from beyond the use of the continuum approximation and support the numerical investigations of Ref. 5.

We can also calculate the contribution to the internal energy, that we call $U_0$, coming from $\Omega_0$. Using Eq. (2.26) in Eq. (2.28) and then eliminating $\mu$ in favour of $N$ with Eq. (2.32) results in the following approximation:

$$U_0 \simeq \frac{3}{4} (6\omega_1\omega_2\omega_3)^{1/3} N^{4/3} + \frac{(\omega_1^2 + \omega_2^2 + \omega_3^2 + 4\pi^2 T^2)}{8(6\omega_1\omega_2\omega_3)^{1/3}} N^{2/3},$$

(2.35)

assuming that $N$ is large. (The intermediate result for $U_0$ in terms of $\mu$ can be found in Eq. (3.25) below for the isotropic potential.)

The specific heat can be found using

$$C = \left. \left( \frac{\partial U}{\partial T} \right) \right|_{N,\omega}.$$  \hspace{1cm} (2.36)

The contribution coming from the approximate result in Eq. (2.35) is then seen to be

$$C \simeq \pi^2 (6\omega_1\omega_2\omega_3)^{-1/3} N^{2/3} T.$$  \hspace{1cm} (2.37)

As before this agrees with the result quoted in Ref. 4. Again we will find that at low temperatures the part of $\Omega$ that has not yet been considered, namely $\Omega_r$, alters this expected behaviour, confirming the numerical results of Ref. 5. We consider the evaluation of $\Omega_r$ and its effect in the next section.

III. ISOTROPIC 3-DIMENSIONAL POTENTIAL

A. Thermodynamic potential

We will examine the case of the isotropic harmonic oscillator potential with

$$\omega_1 = \omega_2 = \omega_3 = \omega,$$

(3.1)

in which case the results of the previous section simplify. Our aim here is to calculate $\Omega_r$ that has been neglected up to now from the analysis. To do this we need to evaluate the contribution to $\Omega$ coming from the poles of $\beta^{-2}e^{\beta E}Z(\beta)$ along the imaginary $\beta$-axis away from $\beta = 0$. From Eq. (2.15), because

$$Z(\beta) = \frac{e^{-3\beta\omega/2}}{(1 - e^{-\beta\omega})^3},$$

(3.2)
there will be poles of order 3 at $\beta = \beta_k$ defined by
\[
\beta_k = \frac{2\pi i}{\omega} k
\]  
for $k = \pm 1, \pm 2, \ldots$. It is a straightforward matter to evaluate the residues of $\beta^{-2}e^{\beta E}Z(\beta)$ at $\beta = \beta_k$ and show that their contribution to Eq. (2.8) gives
\[
3_r(E) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \omega}{16\pi^4 k^4} \left( 6 + \pi^2 k^2 - \frac{4\pi^2 k^2 E^2}{\omega^2} \right) \cos(2\pi kE/\omega) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} E}{2\pi^3 k^3} \sin(2\pi kE/\omega) .
\]  
(3.4)

It now remains to use this in Eq. (2.12) and to try to extract something useful from the result.

We have
\[
\Omega_r = -\frac{\beta \omega}{64\pi^4} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4} A_k - \frac{\beta^3}{8\pi^3} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3} B_k ,
\]  
(3.5)

where
\[
A_k = \int_0^\infty dE \left( 6 + \pi^2 k^2 - \frac{4\pi^2 k^2 E^2}{\omega^2} \right) \frac{\cos(2\pi kE/\omega)}{\cosh^2[\frac{1}{4}\beta(E - \mu)]} ,
\]  
(3.6)
\[
B_k = \int_0^\infty dE E \frac{\sin(2\pi kE/\omega)}{\cosh^2[\frac{1}{4}\beta(E - \mu)]} .
\]  
(3.7)

In these last two integrals we can make the change of variable
\[
E = \mu + \frac{2}{\beta} \theta ,
\]  
(3.8)
to new integration variable $\theta$. The lower limits on the integrals in Eqs. (3.6) and (3.7) then become $-\beta \mu / 2$. If we look at low enough temperatures, specifically $T << \mu$ as we have already assumed, then to a good approximation (up to exponentially small terms) we can replace the lower limits on the integrals defining $A_k$ and $B_k$ with $-\infty$. The approximate results then become
\[
A_k \simeq \frac{2}{\beta} \int_{-\infty}^{\infty} \frac{d\theta}{\cosh^2 \theta} \left[ 6 + \pi^2 k^2 - \frac{4\pi^2 k^2 \mu^2}{\omega^2} - \frac{16\pi^2 k^2 \mu \theta^2}{\beta \omega^2} - \frac{16\pi^2 k^2 \theta^2}{\beta^2 \omega^2} \right] \times \cos \left( \frac{2\pi k \mu}{\omega} + \frac{4\pi k \theta}{\beta \omega} \right) ,
\]  
(3.9)
\[
B_k \simeq \frac{2}{\beta} \int_{-\infty}^{\infty} \frac{d\theta}{\cosh^2 \theta} \left( \mu + \frac{2\theta}{\beta} \right) \sin \left( \frac{2\pi k \mu}{\omega} + \frac{4\pi k \theta}{\beta \omega} \right) .
\]  
(3.10)
The integrals in Eqs. (3.9) and (3.10) may be evaluated exactly using residues. All of the expressions required may be related to the basic integral

$$\int_{-\infty}^{\infty} \frac{\cos(a\theta + b)}{\cosh^2 \theta} \, d\theta = \frac{\pi a \cos b}{\sinh(\frac{\pi}{2}a)}, \tag{3.11}$$

and derivatives with respect to the parameters $a$ and $b$. After some straightforward calculation, using the results for $A_k$ and $B_k$ found from Eqs. (3.9) and (3.10) as described, it can be shown that

$$\Omega_r \simeq -\frac{1}{8\pi^2 \beta^3 \omega^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3 \sinh(\frac{2\pi^2 k}{\beta \omega})} \left[ 8\pi^4 k^2 \text{csch}^2 \left( \frac{2\pi^2 k}{\beta \omega} \right) + 4\pi^2 k \beta \omega \coth \left( \frac{2\pi^2 k}{\beta \omega} \right) \right. \\
\left. + 2\beta^2 \omega^2 + \pi^2 k^2 (4\pi^2 + \beta^2 \omega^2 - 4\beta^2 \mu^2) \right] \cos(2\pi k \mu / \omega) \\
+ 4\pi k \beta \mu \left[ \beta \omega + 2\pi^2 k \coth \left( \frac{2\pi^2 k}{\beta \omega} \right) \right] \sin(2\pi k \mu / \omega) \right], \tag{3.12}$$

The presence of the trigonometric functions in Eq. (3.12) is responsible for the oscillations that occur in thermodynamic quantities if we go beyond the leading order continuum approximation. The presence of the hyperbolic functions in Eq. (3.12) with argument $\frac{2\pi^2 k}{\beta \omega}$ means that unless $\beta \omega > 1$ these oscillations will be suppressed. The oscillatory behaviour should show up for $T < \omega$ and become more prominent as $T$ is reduced.

The inclusion of $\Omega_r$ in the expression used for the thermodynamic potential will lead to corrections to the thermodynamic behaviour of the system beyond what is found using the continuum approximation of Sec. II. We will look first at how the chemical potential is affected.

### B. Chemical potential

We can define (from Eq. (2.27))

$$N_r = -\left( \frac{\partial \Omega_r}{\partial \mu} \right) \bigg|_{T, \omega} \tag{3.13},$$

and then use Eq. (3.12) to obtain

$$N_r \simeq \frac{\pi}{4\beta^3 \omega^3} \sum_{k=1}^{\infty} \frac{(-1)^k}{\sinh(\frac{2\pi^2 k}{\beta \omega})} \left[ 4\beta^2 \mu^2 - 4\pi^2 - \beta^2 \omega^2 - 8\pi^2 \text{csch}^2 \left( \frac{2\pi^2 k}{\beta \omega} \right) \right] \sin(2\pi k \mu / \omega)$$
This will make an oscillatory correction to the contribution \( N_0 \) for the average number of particles found using the continuum approximation in Sec. II.

The chemical potential may be found by solving

\[
N = N_0 + N_r \tag{3.15}
\]

for \( \mu \), where \( N_0 \) is given by (see Eq. (2.30) with \( \omega_1 = \omega_2 = \omega_3 = \omega \))

\[
N_0 \simeq \frac{\mu^3}{6\omega^3} - \frac{\mu}{24\omega} \left( 3 - \frac{4\pi^2}{\beta^2\omega^2} \right) . \tag{3.16}
\]

Obviously the complicated dependence on \( \mu \) in \( N_r \) as given in Eq. (3.14) renders an analytical evaluation of \( \mu \) difficult. We can simplify by using the assumption that \( \mu \) is large (since we have already assumed that \( \beta \mu \gg 1 \) and \( \mu \gg \omega \)), and keep only the leading term in \( \mu \). This gives

\[
N_r \simeq \frac{\pi \mu^2}{3\beta \omega^3} \sum_{k=1}^{\infty} \frac{(-1)^k \sin(2\pi k \mu/\omega)}{\sinh \left( \frac{2\pi^2 k}{\beta \omega} \right)} . \tag{3.17}
\]

The continuum approximation in Eq. (3.16) still gives the leading contribution to \( N \) for large \( \mu \); however, \( N_r \) in Eq. (3.17) can be more important than the sub-leading term (proportional to \( \mu \)) in Eq. (3.16) if the temperature is low enough. For \( \beta \omega \) of the order of 1 or less, the sum in Eq. (3.17) is well approximated by simply the first term and it is clear that the oscillations, although present, will have a very small amplitude. We would therefore expect that for the temperature range \( T \geq \omega \), Eq. (2.31) or Eq. (2.32) would provide a good approximation for the chemical potential. However as the temperature is reduced, so that \( \beta \omega \gg 1 \), the amplitude of the oscillations coming from Eq. (3.17) become more pronounced and must be taken into effect.

It is possible to find an asymptotic expansion for \( N_r \) in Eq. (3.17) valid for \( \beta \omega \gg 1 \) (or \( T \ll \omega \)). After some calculation, it can be shown that

\[
N_r \simeq \frac{\mu^2}{4\omega^2} \left\{ \tanh \left( \frac{\beta \omega}{4} \left( 2\bar{\mu} - 1 \right) \right) + 1 - 2\bar{\mu} \right\} , \tag{3.18}
\]

where

\[
\bar{\mu} = \frac{\mu}{\omega} - \left[ \frac{\mu}{\omega} \right] , \tag{3.19}
\]
with \([x]\) denoting the largest integer whose value is less than or equal to \(x\). (Thus \(0 \leq \bar{\mu} < 1\) can be assumed in Eq. (3.18).) Similar asymptotic expansions can be found for the sub-leading terms in \(N_r\) given in Eq. (3.14); however we will not give them here. It can be shown by numerically evaluating the sum in Eq. (3.17) and comparison with the analytical approximation in Eq. (3.18) that Eq. (3.18) does give an accurate result for large values of \(\beta \omega\).

If we take the limit \(\beta \omega \rightarrow \infty\) in Eq. (3.18), the result simplifies further to give

\[
N_r \simeq \frac{\bar{\mu}^2}{2 \omega^2} \left( \left[ \frac{\bar{\mu}}{\omega} + \frac{1}{2} \right] - \frac{\bar{\mu}}{\omega} \right).
\]

(3.20)

Strictly speaking, this further approximation is only valid if \(\left[ \frac{\bar{\mu}}{\omega} + \frac{1}{2} \right]\) is not equal to an integer. In this special case, \(N_r\) in Eq. (3.17) or Eq. (3.18) can be seen to vanish, and we must look at the sub-leading contributions to \(N_r\) that follow from Eq. (3.14).

If we use Eq. (3.20) for \(N_r\) in Eq. (3.15) along with Eq. (3.16) for \(N_0\), it is possible to show that the solution for \(\bar{\mu}\) is given by the simple expression

\[
\frac{\bar{\mu}}{\omega} \simeq \left[ \frac{(6N)^{1/3}}{3} \right] + \frac{1}{2}.
\]

(3.21)

The presence of the greatest integer function in Eq. (3.21) leads to the step-like behaviour first found in the numerical studies of [5]. Our results provide a confirmation of this behaviour by analytical means. For large values of \(N\), our result in Eq. (3.21) shows that these steps will occur roughly for \(N \simeq \ell^3/6\) where \(\ell\) is an integer. This agrees with the “magic numbers” found by [5] that occurred for \(N = \ell(\ell + 1)(\ell + 2)/6\) if we make \(N\), and hence \(\ell\), large enough. We have therefore seen how the step-like behaviour of the chemical potential comes about in an analytical way, and traced its origin back to the same type of terms that are responsible for the de Haas-van Alphen effect.

In Fig. 1 we show the result for \(\bar{\mu}/\omega\) plotted as a function of \(N\) over a range of \(N\). We have taken \(N\) large, but the same type of behaviour can be found for smaller values as well. The continuum approximation for \(\mu\) is shown as the smooth solid, almost straight, line, and the simple analytical approximation in Eq. (3.21) is superimposed on it. The steps occur at the magic numbers as predicted. We have displaced the curves for different values of \(\beta \omega\) for clarity to show the trend towards the step-like behaviour as the temperature is reduced. As the temperature is increased, the amplitudes of the oscillations decreases as mentioned above.
FIG. 1: (color online) This plot shows \( \frac{\mu}{\omega} \) plotted over a range of \( N \) for three sample temperatures. The solid, almost straight, line shows the result found using the continuum approximation for the particle number resulting in \( \frac{\mu}{\omega} = (6N)^{1/3} \). The step-function superimposed shows the result found from the very low temperature approximation of Eq. (3.21). The smooth sinuous curves give the results for \( T = 0.1\omega, T = 0.05\omega \) and \( T = 0.01\omega \). The last two curves have been displaced from the first one for clarity.

C. Specific heat

We will first calculate that part of the energy that arises from \( \Omega_r \). Define

\[
U_r = \left. \frac{\partial}{\partial \beta} (\beta \Omega_r) \right|_{\beta \mu, \omega},
\]

(3.22)

and use Eq. (3.12) for \( \Omega_r \). After some calculation it can be shown that

\[
U_r \simeq \frac{\pi \mu^3}{\beta \omega^3} \sum_{k=1}^{\infty} \frac{(-1)^k}{\sinh \left( \frac{2\pi^2 k}{\beta \omega} \right)} \sin \left( \frac{2\pi k \mu}{\omega} \right).
\]
\begin{align}
+ \frac{3\pi^2 \mu^2}{\beta^2 \omega^3} \sum_{k=1}^{\infty} \frac{(-1)^k \cosh \left( \frac{2\pi k}{\beta \omega} \right)}{\sinh^2 \left( \frac{2\pi k}{\beta \omega} \right)} \cos \left( \frac{2\pi k \mu}{\omega} \right) \tag{3.23}
\end{align}

where we only include the two leading order terms in \( \mu \) since it is these two terms we will need to calculate the leading contribution to the specific heat.

The specific heat was defined in Eq. (2.36). It is straightforward to show that this expression is equivalent to

\begin{align}
C = \left( \frac{\partial U}{\partial T} \right)_{\mu, \omega} - \left( \frac{\partial U}{\partial \mu} \right)_{T, \omega} \left( \frac{\partial N}{\partial T} \right)_{\mu, \omega} + \left( \frac{\partial N}{\partial \mu} \right)_{T, \omega}, \tag{3.24}
\end{align}

which is more useful for explicitly calculating \( C \). The presence of the second term on the right hand side complicates the evaluation of \( C \), but we will obtain an expansion for \( C \) in powers of \( \mu \).

If we look at the first term on the right hand side of Eq. (3.24) and use \( U = U_0 + U_r \) with \( U_r \) given by Eq. (3.23) and

\begin{align}
U_0 &= \frac{\partial}{\partial \beta} (\beta \Omega_0) \bigg|_{\beta \mu, \omega} \\
&= \mu^4 - \frac{\mu^2}{8\omega^3} (\omega^2 - 4\pi^2 T^2) + \frac{1}{1920\omega^3} (17\omega^4 + 40\pi^2 \omega^2 T^2 - 112\pi^4 T^4) \tag{3.25}
\end{align}

we obtain

\begin{align}
\left( \frac{\partial U}{\partial T} \right)_{\mu, \omega} \simeq \frac{\pi^2 \mu^2 T}{2\omega^3} + \left( \frac{\partial U_r}{\partial T} \right)_{\mu, \omega}. \tag{3.26}
\end{align}

From Eq. (3.23), counting powers of \( \mu \), it can be observed that the second term on the right hand side of Eq. (3.26) will contain expressions in \( \mu^3, \mu^2, \ldots \); thus, it appears as if the leading order behaviour of the specific heat will be \( \mu^3 \), rather than \( \mu^2 \) as predicted by the continuum approximation. However, we will show that the \( \mu^3 \) part of Eq. (3.26) cancels with a similar expression coming from the second term in Eq. (3.24) leaving the overall leading behaviour of the specific heat as \( \mu^2 \) rather than \( \mu^3 \). In any case, we must work consistently to order \( \mu^2 \), and to this order only the two contributions to \( U_r \) that we have written down in Eq. (3.23) are necessary. We will drop all terms that result in explicit factors of \( \mu \) in \( C \) that are of order \( \mu \) and lower.

Using Eq. (3.25) we have

\begin{align}
\left( \frac{\partial U}{\partial \mu} \right)_{T, \omega} \simeq \frac{\mu^3}{2\omega^3} + \left( \frac{\partial U_r}{\partial \mu} \right)_{T, \omega}, \tag{3.27}
\end{align}

\[15\]
where the second term on the right hand side contains terms involving \( \mu^3, \mu^2, \ldots \). From Eq. (3.15) and Eq. (3.16) we have

\[
\left( \frac{\partial N}{\partial T} \right) |_{\mu, \omega} \simeq \frac{\pi^2 \mu T}{3 \omega^3} + \left( \frac{\partial N_r}{\partial T} \right) |_{\mu, \omega},
\]

(3.28)

with the second term on the right hand side involving \( \mu^2, \mu, \ldots \), as well as

\[
\left( \frac{\partial N}{\partial \mu} \right) |_{T, \omega} \simeq \frac{\mu^2}{2 \omega^3} + \left( \frac{\partial N_r}{\partial \mu} \right) |_{T, \omega},
\]

(3.29)

with the second term on the right hand side involving \( \mu^2, \mu, \ldots \). A simple counting of powers of \( \mu \), using the expressions for \( U_r \) and \( N_r \), shows that the second term on the right hand side of Eq. (3.24) does begin at order \( \mu^3 \).

It now remains to use the explicit results for \( U_r \) given in Eq. (3.23) and \( N_r \) given in Eq. (3.14) and evaluate \( C \) for large \( \mu \) keeping terms of order \( \mu^3 \) and \( \mu^2 \). After some calculation it can be shown that the order \( \mu^3 \) terms cancel leaving

\[
C \simeq \frac{\pi^2 T \mu^2}{6 \omega^3} \left\{ 1 + \Sigma_1 - 12 \frac{\Sigma_2}{\Sigma_3} \right\},
\]

(3.30)

with

\[
\Sigma_1 = \sum_{k=1}^{\infty} (-1)^k \left\{ \frac{\cosh \theta_k}{\sinh^2 \theta_k} - \frac{\pi^2 k T}{\omega} \left( \frac{1}{\sinh \theta_k} + \frac{2}{\sinh^3 \theta_k} \right) \right\} \cos \left( 2 \pi k \frac{\mu}{\omega} \right),
\]

(3.31)

\[
\Sigma_2 = \sum_{k=1}^{\infty} (-1)^k \left\{ \frac{1}{\sinh \theta_k} - \frac{2 \pi^2 k T \cosh \theta_k}{\omega \sinh^2 \theta_k} \right\} \sin \left( 2 \pi k \frac{\mu}{\omega} \right),
\]

(3.32)

\[
\Sigma_3 = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k 4 \pi^2 k}{\beta \omega \sinh \theta_k} \cos \left( 2 \pi k \frac{\mu}{\omega} \right),
\]

(3.33)

where

\[
\theta_k = \frac{2 \pi^2 k}{\beta \omega}
\]

(3.34)

has been defined to save a bit of writing. The continuum approximation of Sec. II can be regained by dropping all of the terms in \( \Sigma_{1,2,3} \) that have arisen from the de Haas-van Alphen part of the thermodynamic potential. This gives the familiar linear dependence on temperature [4].

The numerical calculations of [5] showed that as the temperature got sufficiently small there was a significant departure from the linear temperature dependence in the specific
heat. Once again we will show that this follows from our analytical method, and that the origin of this behaviour is in the de Haas-van Alphen part of the thermodynamic potential. To do this we will evaluate the asymptotic expansion of the three sums defined in Eqs. (3.31–3.33) for large values of $\beta \omega$. Leaving out the technical details of this for brevity, we find the approximate forms

$$\Sigma_1 \simeq \frac{3 \beta^3 \omega^3 (2\bar{\mu} - 1)^2}{16 \pi^2 \cosh^2 \left( \frac{\beta \omega}{4} (2\bar{\mu} - 1) \right)} - 1, \quad (3.35)$$

$$\Sigma_2 \simeq -\frac{\beta^2 \omega^2 (2\bar{\mu} - 1)}{16 \pi \cosh^2 \left( \frac{\beta \omega}{4} (2\bar{\mu} - 1) \right)}, \quad (3.36)$$

$$\Sigma_3 \simeq \frac{\beta \omega}{4 \cosh^2 \left( \frac{\beta \omega}{4} (2\bar{\mu} - 1) \right)}, \quad (3.37)$$

with $\bar{\mu}$ defined as in Eq. (3.19). The results in Eqs. (3.35–3.37) can be checked against a numerical evaluation of the sums defined by Eqs. (3.31–3.32) and found to be accurate for $\beta \omega \simeq 10$ and $\bar{\mu}$ not too close to 0 or 1. Once $\beta \omega \simeq 100$ the results become very accurate even for $\bar{\mu}$ close to 0 and 1. Thus for $T \leq \omega/100$, the simple expressions in Eqs. (3.35–3.37) become reliable approximations for $\Sigma_{1,2,3}$.

If we use Eqs. (3.35–3.37) in the expression for the specific heat in Eq. (3.30), the result can be shown to vanish. The de Haas-van Alphen contribution to the specific heat cancels the continuum approximation to the leading order we are working to. The specific heat therefore vanishes as $T \to 0$ faster than $T$. This is completely consistent with the numerical results found in [5]. Because the de Haas-van Alphen approximation, as well as the asymptotic evaluation of the sums leading to Eqs. (3.35–3.37) neglect terms that are exponentially suppressed, we suspect that the specific heat vanishes like $e^{-\alpha \omega/T}$ for some constant $\alpha$ as $T \to 0$, but we have not been able to establish this in any simple way. Further support for this belief follows from the result we are able to establish for the 1-dimensional gas in Sec. IV.A below. A more refined estimate of the sums, as well as the de Haas-van Alphen contribution would reveal the exact nature of the $T \to 0$ limit.

For $T \geq \omega/100$, but still small, the results in Eqs. (3.35–3.37) start to become less reliable. As a check on our results against the numerical ones of [5] we plot the specific heat as found from Eq. (3.30) to demonstrate the de Haas-van Alphen oscillations. This is shown in Fig. 2. As the temperature increases we do find, as expected, that the contribution from $\Sigma_{1,2,3}$ becomes smaller exponentially, and the linear behaviour with temperature is regained.
FIG. 2: (color online) This plot shows $C/(6N)^{2/3}$ plotted over a range of $N$ for four sample temperatures $T = 0.1\omega, T = 0.2\omega, T = 0.3\omega$ and $T = 0.5\omega$. As the temperature increases the oscillation amplitude decreases and the curves approach the continuum limit of 1 in the scaled specific heat. As the temperature decreases there are significant deviations from the result found from using the continuum limit, and for very small temperatures the specific heat starts to become vanishingly small.

IV. ONE AND TWO DIMENSIONS

We will examine the thermodynamics of trapped Fermi gases in one and two spatial dimensions, since these cases may be of relevance as limiting cases of the 3-dimensional gas in some situations. Because the methods used are similar to those described above in the 3-dimensional case we will be brief here and only exhibit key results that show a difference with results already obtained.
A. One dimension

With \( D = 1 \), \( Z(\beta) \) in Eq. (2.15) has simple poles at \( \beta = \beta_k \) with \( \beta_k \) still defined by Eq. (3.3). It is easy to show that the contribution from the poles with \( k \neq 0 \) to \( Z(E) \) is

\[
Z_r(E) = -\frac{\omega}{2\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos \left( \frac{2\pi k E}{\omega} \right). \tag{4.1}
\]

The contribution from the pole at \( \beta = 0 \) was given in Eq. (2.18). After some calculation, making the approximation \( \mu \gg T \), it can be shown that

\[
\Omega_r \simeq \frac{1}{\beta} \sum_{k=1}^{\infty} \frac{(-1)^k \cos(2\pi k \mu / \omega)}{k \sinh \left( \frac{2\pi k \beta \omega}{\beta} \right)}. \tag{4.2}
\]

The calculation is very similar to the \( D = 3 \) case, so details will not be given here. The continuum approximation for \( \Omega_r \), called \( \Omega_0 \), was given in Eq. (2.24).

Using Eq. (2.24) and Eq. (4.2) in the general expression for the average particle number \( N \) in Eq. (2.27) results in

\[
N \simeq \frac{\mu}{\omega} + 2\pi \sum_{k=1}^{\infty} \frac{(-1)^k \sin(2\pi k \mu / \omega)}{\sinh \left( \frac{2\pi k \beta \omega}{\beta} \right)}. \tag{4.3}
\]

The first term on the right hand side comes from \( \Omega_0 \), and the second term from \( \Omega_r \), so that the continuum approximation would yield simply

\[
\mu \simeq \omega N \tag{4.4}
\]

in this case. Again for large particle numbers we expect \( \mu \gg \omega \) to be a valid approximation. For \( T \ll \omega \) we can again find the asymptotic form of the sum in Eq. (4.3) and obtain

\[
N \simeq \left[ \frac{\mu}{\omega} \right] + \frac{1}{2} + \frac{1}{2} \tanh \left( \frac{\beta \omega}{4} \left( \frac{2\mu}{\omega} \right) - 2 \left[ \frac{\mu}{\omega} \right] - 1 \right), \tag{4.5}
\]

and this proves to be very accurate for low values of \( T \).

The internal energy can be found to be

\[
U \simeq \frac{\mu^2}{2\omega} + \frac{\pi^2}{6\omega} T^2 + \frac{\omega}{24} + U_r, \tag{4.6}
\]

with

\[
U_r \simeq \frac{2\pi}{\beta} \sum_{k=1}^{\infty} (-1)^k \left\{ \frac{\mu \sin(2\pi k \mu / \omega)}{\sinh \theta_k} + \frac{\pi}{\beta \omega} \cosh \theta_k \sin(2\pi k \mu / \omega) \right\}. \tag{4.7}
\]
and \( \theta_k \) defined by Eq. (3.34). We can calculate the specific heat using Eq. (3.24) with the result

\[
C \simeq \frac{\pi^2 T}{3\omega} \left\{ 1 + \Sigma_1 - 12\frac{\Sigma_2}{\Sigma_3} \right\},
\]

(4.8)

The sums \( \Sigma_{1,2,3} \) are the same as those given in Eqs. (3.31–3.34). The result in Eq. (4.8) is very similar to that found in Eq. (3.30) in the 3-dimensional case apart from the overall factor that can be recognized as the continuum approximation for the specific heat of the 1-dimensional gas. We can therefore conclude immediately that as \( T \to 0 \), the specific heat vanishes faster than the linear temperature dependence deduced from the continuum approximation.

![Graph](image)

**FIG. 3:** (color online) This plot shows the specific heat plotted over a range of temperature for the 1-dimensional gas. As the temperature increases the specific heat approach the continuum limit that behaves linearly on the temperature. As the temperature decreases the result deviates substantially from the linear result and the curve decays exponentially fast as found in Eq. (4.9).

Because of the simplicity of the \( D = 1 \) case, we can shed some light on the behaviour of the specific heat as \( T \to 0 \). We note that the continuum approximation for the specific heat
in Eq. (4.4) is also a solution when the de Haas-van Alphen contributions to \(N\) are included, since the \(\sin\) term vanishes for integral \(N\). The sum \(\Sigma_2\) defined by Eq. (3.32) vanishes as well if \(\mu = \omega N\), and we can use the approximation for \(\Sigma_1\) given by Eq. (3.35), valid for \(T \ll \omega\) to obtain from Eq. (4.8) the simple result

\[
C \simeq \frac{\omega^2}{16T^2} \exp\left(-\frac{\omega}{2T}\right). \tag{4.9}
\]

This demonstrates that the specific heat does vanish exponentially fast as \(T \to 0\), not linearly. The role of the de Haas-van Alphen part of the thermodynamic potential is again responsible for this behaviour. As a check on this conclusion we have plotted the specific heat as a function of temperature in Fig. 3. The results can be seen to be consistent with our analytical result in Eq. (4.9) as \(T \to 0\). For larger values of the temperature the specific heat approaches the linear temperature dependence predicted by the continuum approximation. This can also be seen from the expression obtained in Eq. (4.8) since as \(\beta \omega\) becomes small, the sums \(\Sigma_{1,2,3}\) start to vanish as a consequence of the hyperbolic functions present in the expressions.

**B. Two dimensions**

Using \(D = 2\) in Eq. (2.15) we find

\[
Z_r(E) = -\frac{1}{2\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \left\{ E \cos(2\pi k E/\omega) - \frac{\omega}{\pi k} \sin(2\pi k E/\omega) \right\}. \tag{4.10}
\]

This results in

\[
\Omega_r \simeq \frac{1}{2\pi\beta} \sum_{k=1}^{\infty} \frac{1}{k^2} \left\{ 2\pi k \frac{\mu}{\omega} \frac{\cos(2\pi k E/\omega)}{\sinh \theta_k} - \left[ \frac{1}{\sinh \theta_k} + \frac{\cosh \theta_k}{\sinh^2 \theta_k} \right] \sin(2\pi k E/\omega) \right\}, \tag{4.11}
\]

with \(\theta_k\) defined by Eq. (3.34) if we make the same approximations as in the \(D = 3\) case. The \((-1)^k\) factor in the summand, present for \(D = 1, 3\), is absent here, a result that is true for any even dimension.

The average particle number can be shown to be

\[
N \simeq \frac{\mu^2}{2\omega^2} - \frac{1}{12} + \frac{\pi^2}{6\beta^2 \omega^2} + N_r, \tag{4.12}
\]

with

\[
N_r \simeq \frac{2\pi}{\beta \omega} \sum_{k=1}^{\infty} \left\{ \frac{\mu}{\omega} \frac{\sin(2\pi k \mu/\omega)}{\sinh \theta_k} + \frac{\pi}{\beta \omega} \frac{\cosh \theta_k}{\sinh^2 \theta_k} \sin(2\pi k \mu/\omega) \right\}. \tag{4.13}
\]
the contribution coming from the de Haas-van Alphen part of $\Omega$.

In the continuum approximation (dropping the term in $N_r$) we find

$$\frac{\mu}{\omega} \simeq (2N)^{1/2}, \quad (4.14)$$

to leading order for large $N$. However the de Haas-van Alphen part of the particle number can be shown to lead to the step-like behaviour we saw previously in the 3-dimensional case. In the low temperature limit ($T \ll \omega$) an asymptotic analysis of Eq. (4.13) can be used to show that

$$N \simeq \frac{1}{2} \left[ \frac{\mu}{\omega} \right]^2 - \frac{3}{2} \left[ \frac{\mu}{\omega} \right] + 2 \left[ \frac{\mu}{\omega} \right] \tanh \left\{ \frac{\beta \omega}{2} \left( \frac{\mu}{\omega} - \left[ \frac{\mu}{\omega} \right] \right) \right\}. \quad (4.15)$$

Solving this expression for $\mu$ gives rise to the approximate analytical solution

$$\frac{\mu}{\omega} \simeq \left[ (2N)^{1/2} + \frac{1}{2} \right]. \quad (4.16)$$

(Again we remind the reader that the square brackets in Eqs. (4.15) and (4.16) denote the greatest integer function.) We therefore conclude that the step-like behaviour found earlier for the chemical potential when $D = 3$ is also found for the 2-dimensional gas. We have checked this result by solving Eq. (4.12) numerically and found that the approximation in Eq. (4.16) becomes increasingly accurate as $T$ is reduced. Because the results resemble the similar behaviour exhibited in Fig. 1, we will not show them here. The jumps in the chemical potential occur, for large $N$, when $N \simeq \frac{1}{2} \ell^2$ for integral $\ell$. (The exact result is $\frac{1}{2} \ell(\ell + 1)$ analogously to the 3-dimensional case studied in Ref. [5].)

It is straightforward to obtain expressions for the internal energy and specific heat. Once again the results are similar to those found in the 3-dimensional case, this time shown in Fig. 2 so need not be exhibited in detail. As $T \to 0$ the result for the specific heat vanishes much faster than the linear approximation $C \simeq \pi^2 T(2N)^{1/2}/(3\omega)$ predicted by the continuum approximation.

V. DISCUSSION AND CONCLUSIONS

The main conclusion of this paper is that the origin of the step-like features found in the numerical calculations of Schneider and Wallis [5] have the same origin as the periodicity found in the de Haas-van Alphen effect. The general approach of Sondheimer and Wilson [16], so useful in the analysis of the de Haas-van Alphen effect, can be used to great
effect here to obtain analytical results for various thermodynamic quantities. In addition, it is possible to see that within this approach the continuum approximation used by Butts and Rokhsar [4] corresponds to the neglect of an infinite set of poles in the Laplace transform of the partition function. It is also possible to recover the continuum approximation using the methods described in the general framework of Ref. [19].

In Sec. III we obtained approximate analytical results for the 3-dimensional, isotropic harmonic oscillator potential. In the case $T \ll \omega$, a very simple result (Eq. (3.21)) was found for the chemical potential that clearly exhibited the step-like features found in Ref. [5]. This was used to evaluate the specific heat, and it was shown (in agreement with Ref. [5]) that there were significant deviations from the linear temperature dependence predicted by the continuum approximation. The specific heat was seen to exhibit a periodic structure in the particle number.

In Sec. IV we studied the trapped Fermi gas in 1 and 2 spatial dimensions. For the 1-dimensional gas we were able to show that as $T \to 0$ the specific heat vanished exponentially fast in the inverse temperature (Eq. (4.9)). The 2-dimensional gas was similar in many ways to the 3-dimensional case of Sec. III.

Although the analysis presented above was restricted to the isotropic potential for simplicity and brevity, the same methods can be used to examine anisotropic potentials. The technical details are more involved due to the structure of the poles in the inverse Laplace transform of the partition function as mentioned above. A preliminary report of both the results of this paper, and the anisotropic case was given earlier [20]. In addition to a periodicity in the particle number, there can also be a periodicity when the trapping potential is altered. This latter effect was not found for the isotropic case and it is worth commenting on why this occurs, in contrast with what might be expected from the de Haas-van Alphen effect where the thermodynamics shows a periodicity in the inverse magnetic field strength. The difference between the two cases is related to the relationship between the chemical potential and the particle number. In both cases (de Haas-van Alphen and trapped Fermi gas) the periodic structure results from a trigonometric dependence on $\mu/\omega$. (For the de Haas-van Alphen effect $\omega$ is the cyclotron frequency associated with the magnetic field strength.) In the de Haas-van Alphen effect, when $\mu$ is solved for in terms of the particle number the leading order contribution turns out to be independent of $\omega$. Thus the trigonometric functions that involve $\mu/\omega$ exhibit the familiar de Haas-van Alphen oscillations as the magnetic
field is varied. For the trapped Fermi gas in an isotropic potential we found \( \mu \propto \omega \), so that \( \mu/\omega \) is independent of \( \omega \), although there is a dependence on the particle number. We claim that this is an artifact of the simplicity of the isotropic potential, and that the situation for anisotropic potentials is more interesting. An extensive examination of trapped Fermi gases in anisotropic potentials will be given elsewhere \[21\].

The treatment presented here has only been performed for the free Fermi gas. As already mentioned in the introduction, the neglect of interactions for a single component gas is a good approximation \[4, 5\]. In Ref. \[6\] a two component model was studied that allowed for an interaction between the two spin components. A numerical study showed that as the interaction strength is increased the step-like behaviour, like that we have been discussing, is increasingly suppressed. For the single component gas, we expect that an analysis similar to that given by Luttinger \[22\] in the de Haas-van Alphen effect could be used to show that both the amplitude and period of the oscillations are not affected by interactions to leading order. (This conclusion does not hold for the non-oscillatory part of \( \Omega \), that we have called \( \Omega_0 \).) It would be of interest to study the Luttinger analysis for a multi-component Fermi gas in more detail to make contact with the results of Ref. \[6\].

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