Ground State Properties of Fermi Gases in the Strongly Interacting Regime

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The ground state energies and pairing gaps in dilute superfluid Fermi gases have now been calculated with the quantum Monte Carlo method without detailed knowledge of their wave functions. However, such knowledge is essential to predict other properties of these gases such as density matrices and pair distribution functions. We present a new and simple method to optimize the wave functions of quantum fluids using Green’s function Monte Carlo method. It is used to calculate the pair distribution functions and potential energies of Fermi gases over the entire regime from atomic Bardeen-Cooper-Schrieffer superfluid to molecular Bose-Einstein condensation, spanned as the interaction strength is varied. PACS: 03.75.Ss, 21.65.+f, 02.70.Ss

Recent progress in experimental [1, 2, 3, 4, 5, 6] and theoretical methods [7, 8, 9, 10] has generated great interest in the properties of dilute Fermi superfluid gases. Such gases are also of interest in studies of astrophysical objects such as neutron stars [11] and in nuclear physics [12].

The Hamiltonian of these gases has the standard form

\[ \mathcal{H} = -\frac{\hbar^2}{2m} \sum_i \nabla_i^2 + \sum_{i<j} v(r_{ij}). \]

The range of the interatomic potential \( v(r_{ij}) \) is much smaller than the interparticle spacing in the gas, and only the s-wave scattering length \( a \) of the interaction is relevant. Weak attractive interactions have a small negative \( a \) which increases in magnitude as the interaction gets stronger. The \( a \to -\infty \) as we approach the bound molecular state. On further increase of the interaction strength, \( a \) goes discontinuously to +\( \infty \) and then smoothly to 0 as the molecule gets more tightly bound.

Usually, the dimensionless quantity \( 1/ak_F \) is used to characterize the gas. When the interaction is weak and attractive, \( 1/ak_F \to -\infty \), and we have a BCS superfluid gas with gap \( \Delta \sim e^{\pi/(2ak_F)} \) (BCS regime). It has \( 1/ak_F \ll 0 \), \( \Delta \ll \) the energy per particle \( E_0/N \) which is positive and less than the Fermi gas energy \( E_{FG} = \frac{\hbar^2 k_F^2}{2m} \). When the interaction is strong, \( 1/ak_F \gg 0 \), we have tightly bound molecules with energy \( E_{mol} \), and \( E_0/N \approx E_{mol}/2 \approx -\Delta \). The Bose molecules are condensed in the zero momentum state (BEC regime). In the intermediate regime (\( -0.5 \lesssim 1/ak_F \lesssim 0.5 \)) we seem to have a smooth transition or crossover from BCS superfluid to BEC.

The problem of calculating the ground state energies and pairing gaps of superfluid gases has been solved with the fixed node Green’s function Monte Carlo (FN-GFMC) method [3, 4, 11, 14]. To begin with, we review this method and the problem it faces for computing observables other than energies.

In FN-GFMC a trial wave function \( \Psi_T(R) \) is evolved in imaginary time \( \tau \) with the fixed node constraint [13]

\[ \Psi(\tau, R) = \left[ e^{-\tau(\mathcal{H} - E_T)} \right]_{FN} \Psi_T(R). \]

We use \( R = r_1, r_2, \ldots; r_1, r_2, \ldots \) to denote the configuration of atoms in the gas, and particles 1, 2, \ldots have spin up and 1', 2', \ldots have spin down. The subscript FN denotes that the propagator is constrained such that the propagated wave function has the nodal surface of \( \Psi_T(R) \) at all \( \tau \). The energy \( E_T \) is adjusted to keep the norm of the wave function constant. At large \( \tau \), the evolved \( \Psi(\tau, R) \) converges to the lowest energy state of the system having the nodes of \( \Psi_T(R) \), and \( E_T = \langle \Psi(\tau) | \mathcal{H} | \Psi(\tau) \rangle \). Without the fixed node constraint it will converge to the exact ground state, but unconstrained fermion Monte Carlo calculations become impractical due to uncontrolled growth in sampling errors.

This is the well known fermion sign problem. From now on, we assume that \( \Psi_T(R) \) is real and that \( \tau \) is large enough to approximate the limit \( \tau \to \infty \).

Following Kalos [14], the mixed expectation value

\[ \langle \mathcal{H}(\tau) \rangle_{mixed} = \frac{\langle \Psi_T | \mathcal{H} | \Psi(\tau) \rangle}{\langle \Psi_T | \Psi(\tau) \rangle} = \frac{\int dR \Psi_T(R) \mathcal{H} \Psi(\tau, R)}{\int dR \Psi_T(R) \Psi(\tau, R)} \]

is calculated using Monte Carlo sampling techniques. Since \( \mathcal{H} \) commutes with the evolution operator we have

\[ \langle \mathcal{H}(\tau) \rangle_{mixed} = \frac{\langle \Psi(\tau/2) | \mathcal{H} | \Psi(\tau/2) \rangle}{\langle \Psi(\tau/2) | \Psi(\tau/2) \rangle} \equiv \langle \mathcal{H}(\tau/2) \rangle. \]

The \( \langle \mathcal{H}(\tau) \rangle_{mixed} \) converges to the energy of the lowest energy state with the nodal surface of \( \Psi_T(R) \). By the variational principle it is \( \geq \) the ground state energy \( E_0 \). The nodal surface of \( \Psi_T(R) \) is varied to minimize \( \langle \mathcal{H}(\tau) \rangle_{mixed} \). The minimum gives an accurate estimate of \( E_0 \) provided the variation is general enough.

This procedure assures that the nodes of \( \Psi_T(R) \) are near optimum, i.e. close to the nodes of the exact \( \Psi_0(R) \). However, the \( \Psi_T(R) \) itself can otherwise be very different from the \( \Psi_0(R) \). For example, in references [3] and [4].
we use
\[
\Psi_T(\mathbf{r}) = \prod_{i,j'} f_{\uparrow\downarrow}(r_{ij'}) \Phi_{BCS}(\mathbf{r}),
\]
where \(\Phi_{BCS}(\mathbf{r})\) is a generalized Bardeen-Cooper-Schrieffer wave function, and its nodes are optimized by minimizing \(\langle H(\tau) \rangle_{\text{mixed}}\). The \(f_{\uparrow\downarrow}(r_{ij'})\) is a nodeless pair correlation function between spin up and down particles. The \(\langle H(\tau) \rangle_{\text{mixed}}\) does not depend upon the choice of \(f_{\uparrow\downarrow}(r_{ij'})\) in the limit \(\tau \to \infty\). The \(f_{\uparrow\downarrow}(r_{ij'})\) is used to reach this limit quickly, and to reduce the variance of the stochastic evaluation of \(\langle H(\tau) \rangle_{\text{mixed}}\). Note that the commonly used Jastrow pair correlation function is also nodeless [10], but it acts between all pairs: \(\uparrow\downarrow\), \(\uparrow\uparrow\) and \(\downarrow\downarrow\), and it is not useful in superfluid gases.

Mixed expectation values of other observables, \(\langle O \rangle_{\text{mixed}}\), can be easily calculated with GFMC, but they are more difficult to interpret when \([O, H] \neq 0\). If one assumes that \(|\Psi(\tau)| = |\Psi_T| + |\delta \Psi|\), then the desired expectation value
\[
\langle O(\tau) \rangle = \langle \Psi(\tau) | O | \Psi(\tau) \rangle \langle \Psi(\tau) | \Psi(\tau) \rangle = 2\langle O(\tau) \rangle_{\text{mixed}} - \langle O \rangle_{\text{trial}} + \text{terms of order } \delta \Psi^2.
\]

Here, the trial estimate \(\langle O \rangle_{\text{trial}} \equiv \langle \Psi_T | O | \Psi_T \rangle / \langle \Psi_T | \Psi_T \rangle\). When \(\delta \Psi\) is small, the extrapolation
\[
\langle O(\tau) \rangle_{\text{extrap.}} \approx 2\langle O(\tau) \rangle_{\text{mixed}} - \langle O \rangle_{\text{trial}}
\]
can be used to estimate \(\langle O \rangle\).

However, in the strongly interacting regime the \(\delta \Psi\) is not necessarily small, and this extrapolation may not be valid. For example, we consider the pair distribution function \(g_\uparrow(\tau)\) between parallel spin particles in the \(ak_F \to \pm\infty\) limit. The mixed, trial and extrapolated values of \(g_\uparrow(\tau)\) obtained from the \(\Psi_T(\mathbf{r})\) of Ref. [8] and [9] are shown in Fig. 1. At small \(r\), the extrapolated \(g_\uparrow(\tau) < 0\) indicating invalidity. These and all the other results presented in this work are obtained from Monte Carlo computations using 14 particles in a cubic periodic box. As discussed in Ref. [8] and [9], a periodic box with 14 particles provides a fair approximation to the uniform gas. The \(\cosh r(r_{ij'})\) with \(\mu r_{ij'} = 12\) is used [9] to approximate the interaction between spin \(\uparrow\downarrow\) pairs.

In principle, the pair correlation functions, \(f_{\uparrow\downarrow}(r)\) and \(f_{\uparrow\uparrow}(r) = f_{\downarrow\downarrow}(r)\) in \(\Psi_T(\mathbf{r})\) can be obtained by minimize

| \(ak_F\) | \(\langle H \rangle_{\text{mixed}}\) | \(\langle H \rangle_{\text{trial}}\) | \(\langle H \rangle_{\text{opt}}\) | \(\langle V \rangle_{\text{mixed}}\) | \(\langle V \rangle_{\text{opt}}\) |
|----------|----------------|----------------|----------------|----------------|----------------|
| -1       | 0.792(4)       | 0.818(3)       | 0.808(4)       | -0.55(3)       | -0.54(3)       |
| -3       | 0.635(6)       | 0.853(3)       | 0.70(2)        | -1.8(1)        | -2.0(1)        |
| -10      | 0.494(7)       | 0.68(3)        | 0.53(1)        | -3.5(2)        | -3.2(1)        |
| \(\infty\)| 0.41(4)       | 0.62(3)        | 0.46(1)        | -3.9(1)        | -4.0(2)        |
| 10       | 0.32(1)        | 0.57(6)        | 0.39(1)        | -4.8(1)        | -5.0(1)        |
| 3        | -0.00(1)       | 0.4(1)         | 0.11(3)        | -7.0(1)        | -7.3(3)        |
| 2        | -0.34(2)       | 0.2(1)         | -0.18(3)       | -9.2(4)        | -9.2(4)        |
| 1        | -2.37(3)       | -0.1(1)        | -2.0(1)        | -19.0(4)       | -18.0(6)       |
ing the trial energy \( \langle H \rangle_{\text{trial}} \). However, this variational problem has been approximately treated in most quantum Monte Carlo calculations. In Ref. \cite{12} a simple and crude method called LOVC, based on constrained minimization of the leading two-body cluster contribution to \( \langle H \rangle_{\text{trial}} \) is used. In this method, \( f_{1\uparrow}(r) = f_{1\downarrow}(r) = 1 \), and \( f_{1\uparrow}(r) \) satisfies the two-body Schrödinger equation

\[
-\frac{\hbar^2}{2m} \nabla^2 f_{1\uparrow}(r) + v(r)f_{1\uparrow}(r) = \lambda f_{1\uparrow}(r),
\]

at \( r < d \). The boundary conditions are: \( f_{1\uparrow}(r \geq d) = 1 \) and \( f_{1\uparrow}'(r = d) = 0 \). The healing distance \( d \) serves as the variational parameter. The trial energies obtained with variational Monte Carlo (VMC) calculations using the optimum healing distance \( d \) are compared with the FN-GFMC \( \langle H \rangle_{\text{mixed}} \) in Table 1. Both calculations use the optimum \( \Phi_{\text{BCS}}(R) \) found by minimizing the \( \langle H \rangle_{\text{mixed}} \) in Ref. \cite{4}. The trial energies are well above \( \langle H \rangle \) at \( \tau = 0 \). The trial energies are further decreased by including higher order correlations corresponding to triplet, quadruplet, etc. However, in the present work we consider only pair correlations for \( \Psi_{\text{optim}}(R) \).

The trial pair distribution functions can be expressed as

\[
\langle g_x(r) \rangle_{\text{trial}} = f_x^2(r) t_x(r, f_{1\uparrow}, f_{1\downarrow}, \Phi_{\text{BCS}}(R), \rho),
\]

where \( x \) can be \( \uparrow \uparrow \) or \( \uparrow \downarrow \) and \( t_x \) is a complicated function of \( r, f_{1\uparrow}, f_{1\downarrow}, \Phi_{\text{BCS}}(R) \) and gas density \( \rho \). It is difficult to calculate it exactly except by numerical methods. However, \( t_x(r) \) contains many-body integrals, and is a relatively smooth function of \( r \).

Our method to optimize \( f_x(r) \) using GFMC is iterative. Let \( \langle g_x^{(n)}(r) \rangle_{\text{mixed}} \) and \( \langle g_x^{(n)}(r) \rangle_{\text{trial}} \) be obtained from the \( n \)-th trial \( f_x^{(n)}(r) \) using the optimum \( \Phi_{\text{BCS}} \) which does not depend on \( f_x(r) \). We start with the LOCV approximation providing the \( f_x^{(1)}(r) \), but one could start with any other choice of \( f_x^{(1)} \) and converge to the same \( \Psi_{\text{optim}}(R) \). The next improved \( f_x^{(2)}(r) \) is chosen as

\[
f_x^{(2)}(r) = f_x^{(1)}(r) \sqrt{\frac{\langle g_x^{(1)}(r) \rangle_{\text{mixed}}}{\langle g_x^{(1)}(r) \rangle_{\text{trial}}}}.
\]

If the difference between \( f_x^{(1)}(r) \) and \( f_x^{(2)}(r) \) is small, we can assume that the \( t_x(r) \) functions do not change much. In this case \( \langle g_x^{(2)}(r) \rangle_{\text{trial}} \approx \langle g_x^{(1)}(r) \rangle_{\text{mixed}} \). Otherwise, by iterating this process one easily converges to an \( f_x^{(n)}(r) \) such that

\[
\langle g_x^{(n)}(r) \rangle_{\text{trial}} \approx \langle g_x^{(n)}(r) \rangle_{\text{mixed}}.
\]

Usually, the convergence within statistical errors can be reached within 3 ~ 4 iterations and it doesn’t seem to depend on the strength of the interaction. In practice, \( \langle g_x(r) \rangle_{\text{mixed}} \) and \( \langle g_x(r) \rangle_{\text{trial}} \) have Monte Carlo sampling errors. We approximate the square root of their ratio (Eq. 10) by a smooth function of \( r \) chosen as \( \cos(p_1 r + p_2) e^{-r/p_3} + 1 \), and vary the parameters \( p_1, p_2, p_3 \) to best fit the Monte Carlo values. One iteration step typically takes about 10 hours in a Pentium 3.0 GHz based workstation.

The VMC energies with the \( \Psi_{\text{optim}}(R) \) are much closer to the FN-GFMC energies (Table II). In principle, the optimization of \( f_x(r) \) should have no effect on the FN-GFMC \( \langle H \rangle_{\text{mixed}} \); in practice the \( \langle H \rangle_{\text{optim}} \) seems to get lowered by \( \sim 2 \pm 1 \% \) after optimization presumably because the limit \( \tau \to \infty \) is easier to reach with the \( \Psi_{\text{optim}}(R) \). The effects of the optimization are also seen in the reduced error bars of the energy estimates: \( \delta \langle H \rangle_{\text{optim}} \lesssim \delta \langle H \rangle_{\text{trial}} \) (Table I) for the same number of Monte Carlo samples. In addition, the \( E_T \) (Eq. 2), which typically has larger fluctuations, becomes indistinguishable from \( \langle H \rangle_{\text{mixed}} \).

The pair distribution functions \( \langle g_x(r) \rangle_{\text{mixed}} \) are determined by the many body probability distribution given by \( \Psi_{\text{optim}}(R) \Psi(r, \tau, \rho) \), while the \( \langle g_x(r) \rangle_{\text{optim}} \) are for \( \| \Psi_{\text{optim}}(R) \|^2 \). Note that \( \Psi_{\text{optim}}(R) \Psi(r, \tau) \approx \Psi_{\text{optim}}(R) \Psi_0(R) \geq 0 \) since the nodes of \( \Psi_{\text{optim}}(R) \) have been varied to match those of \( \Psi_0(R) \).

Extending the above method, if we can match the mixed and optimized trial distributions for all, pair, triplet, quadruplet, ... distribution functions, then we can assume that \( \Psi_{\text{optim}}(R) = \Psi_0(R) \). However, here we approximate the exact \( \Psi_0(R) \) by \( \Psi_{\text{optim}}(R) \) using \( \Phi_{\text{BCS}}(R) \) and pair correlation functions only

\[
\Psi_{\text{optim}}(R) = \prod_{i,j} f_{ij}^{\text{opt}}(r_{ij}) \prod_{i,j} f_{ij}^{\text{opt}}(r_{ij}) \times \prod_{i,j} f_{ij}^{\text{opt}}(r_{ij}) \Phi_{\text{BCS}}(R).
\]

The validity of Eq. 11 ensures that the present optimization method will converge to \( f_x^{(n)} \rightarrow f_x^{\text{opt}}(r) \) and thus \( \Psi_{\text{optim}}(R) \) is as close to \( \Psi_0(R) \) as its form (Eq. 12) allows. If higher order correlations have negligible effects on the wave function, we should expect

\[
\langle H \rangle_{\text{optim}} \approx \langle H \rangle_{\text{mixed}} \approx E_0.
\]

However, in the interesting regime of \( a_k F \sim \infty \), Table II shows that the \( \langle H \rangle_{\text{optim}} \) is larger than the \( \langle H \rangle_{\text{mixed}} \) by \( \sim 10 \% \). This suggests that the form of the present
\[ R'(r) = \sqrt{2} R(r) \] so that \( g_{mol}(r_{\uparrow \downarrow}^{\ast}) \equiv R^2(r) \) is normalized analogous to \( g_{\uparrow \downarrow}(r) \), and the two are compared in Fig. 4 \( (g_{\uparrow \downarrow}(r) \) is normalized such that \( g_{\uparrow \downarrow}(r) \to 1 \) for \( r \to \infty \). When \( 1/ak_F \to 0^+ \), we know that \( g_{mol}(r) = R^2(r) \to 0 \) (infinite pair size), but \( g_{\uparrow \downarrow}(r) \gtrsim 1 \). So \( g_{mol}(r) \) and \( g_{\uparrow \downarrow}(r) \) are qualitatively different when \( a \) is large. However, when the interaction is stronger and \( a \) becomes positive and small we expect a gas of molecules in which \( g_{\uparrow \downarrow}(r) \sim g_{mol}(r) \) at \( r < \) the size of the molecule. Fig. 4 shows that the superfluid may well be approximated by a gas of molecules with BEC for \( 1/ak_F \gtrsim 1/3 \). The molecule size for \( 1/ak_F = 1/3 \), is \( 1.21r_0 \) and for \( 1/ak_F = 1 \), it is \( 0.38r_0 \). Due to the many body effects in \( g_{\uparrow \downarrow}(r) \), it is meaningless to compare beyond these distances. In fact, for all \( ak_F > 0 \), the \( g_{mol}(r) \to 0 \) while \( g_{\uparrow \downarrow}(r) \to 1 \) as \( r \to \infty \).

In conclusion, the proposed method allows us to optimize separately the BCS and the pair correlations in dilute Fermi gases. The BCS and \( f_{\uparrow \uparrow}(r) \) correlations are most important, however in the strong interaction regime, the \( f_{\uparrow \uparrow}(r) \) can not be neglected.

The studies of momentum distributions and density matrices of the superfluid gas may now be possible using the optimum \( \Psi_T(R) \), and are in progress.

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\[ \Psi_{\text{optim}}(R) \] is not sufficiently general. An improved approximation could be obtained by including products of triplet correlations \( F_P(r_{ij}, r_{jk}, r_{ki}) \) for \( \uparrow \uparrow \uparrow \) and \( \downarrow \downarrow \downarrow \), and \( F_M(r_{ij}, r_{jk}, r_{ki}) \) for \( \uparrow \downarrow \downarrow \) and \( \downarrow \uparrow \downarrow \) triplets in the wave function. We believe that the present method can be generalized to determine the optimal forms of three-body correlations by making

\[ g_{3}x(r_{ij}, r_{jk}, r_{ki})_{\text{mixed}} = g_{3}x(r_{ij}, r_{jk}, r_{ki})_{\text{optim}}, \quad (14) \]

where \( g_3 \) denotes three particle distribution functions. The true \( \Psi_0 \) can also have backflow correlations \[ [17] \]: however, they change the nodal surface and have to be optimized by minimizing \( \langle H \rangle_{\text{mixed}} \).

The main difference between the optimum pair correlations and those of Ref. 3 and 4 is in \( f_{\uparrow \uparrow}(r) \). In LOCV we have \( f_{\uparrow \uparrow}(r) = 1 \), because in two-body clusters there is no interaction between parallel spin particles in dilute Fermi gases. However, many body effects generate an effective repulsion between parallel spin particles and the optimum \( f_{\uparrow \uparrow}(r) \) is \( < 1 \) at \( r \gtrsim 1.5r_0 \) as shown in Fig. 2.

The optimum and LOCV \( f_{\uparrow \downarrow}(r) \) are generated by the strong two body attraction in \( \uparrow \downarrow \) pairs and have qualitatively similar shapes (Fig. 3). For \( 1/ak_F \ll -1 \), the LOCV \( f_{\uparrow \downarrow}(r) \) is near optimum. For stronger interactions, it is larger than the optimum at \( r \sim 0 \) (Fig. 3).

The expectation value of the potential energy, \( \langle V \rangle = \langle \sum r_{ij} | v(r_{ij}) \rangle \) can easily be calculated from the GFMC (mixed) and VMC distributions using \( \Psi_{\text{optim}} \). The calculated values of the potential energy are given in the last two columns of Table I. Apart from statistical fluctuations, the mixed and the optimum pair distribution functions are the same. Therefore no extrapolation, such as in Eq. 4 is necessary for calculating \( \langle V \rangle \) using \( \Psi_{\text{optim}}(R) \).

Only when \( 1/ak_F > 0 \), we can have bound states with normalized radial wave functions \( R(r) \). We denote

\[ g_{\uparrow \downarrow} = \frac{2}{\pi} \int_{0}^{\infty} R^2(r) J_0(r \lambda) \, dr \]

with normalized radial wave functions \( R(r) \).

\[ R_{\text{opt}} = \frac{2}{\pi} \int_{0}^{\infty} R^2(r) J_0(r \lambda_{\text{opt}}) \, dr \]

bigger two body attraction in it is larger than the optimum at \( r \sim 1.5r_0 \) (infinite pair size), but \( g_{\uparrow \downarrow}(r) \sim 1 \). So \( g_{mol}(r) \) and \( g_{\uparrow \downarrow}(r) \) are qualitatively different when \( a \) is large. However, when the interaction is stronger and \( a \) becomes positive and small we expect a gas of molecules in which \( g_{\uparrow \downarrow}(r) \sim g_{mol}(r) \) at \( r < \) the size of the molecule. Fig. 4 shows that the superfluid may well be approximated by a gas of molecules with BEC for \( 1/ak_F \gtrsim 1/3 \). The molecule size for \( 1/ak_F = 1/3 \), is \( 1.21r_0 \) and for \( 1/ak_F = 1 \), it is \( 0.38r_0 \). Due to the many body effects in \( g_{\uparrow \downarrow}(r) \), it is meaningless to compare beyond these distances. In fact, for all \( ak_F > 0 \), the \( g_{mol}(r) \to 0 \) while \( g_{\uparrow \downarrow}(r) \to 1 \) as \( r \to \infty \).

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[1] B. De Marco, and D. S. Jin, Science 285, 1703 (1999).
[2] K. M. O’Hara, S. L. Hemmer, M. E. Gehm, S. R. Granade, and J. E. Thomas, Science 298, 2179 (2002).
[3] J. L. Roberts, N. R. Claussen, S. L. Cornish, E. A. Donley, E. A. Cornell, and C. E. Wieman, Phys. Rev. Lett. 86, 4211 (2001).
[4] C. A. Regal, C. Ticknor, J. L. Bohn, and D. S. Jin, Nature 424, 47 (2003).
[5] C. A. Regal, M. Greiner, and D. S. Jin, Phys. Rev. Lett. 92, 040403 (2004).
[6] M. Bartenstein, A. Altmeyer, S. Riedl, S. Jochim, C. Chin, J. H. Denschlag, and R. Grimm, Phys. Rev. Lett. 92, 120401-1 (2004).
[7] M. Randeria, in Bose-Einstein Condensation, edited by A. Griffin, D. Snoke, and S. Stringari (Cambridge, 1995).
[8] J. Carlson, S. Y. Chang, V. R. Pandharipande, and K. E. Schmidt, Phys. Rev. Lett. 91, 50401 (2003).
[9] S. Y. Chang, V. R. Pandharipande, J. Carlson, and K. E. Schmidt, Phys. Rev. A 70, 043602 (2004).
[10] G. E. Astrakharchik, J. Boronat, J. Casulleras, and S. Giorgini, Phys. Rev. Lett. 93, 200404 (2004).
[11] C. J. Pethick, and D. G. Ravenhall, Ann. Rev. Nuc. Part. Science 45, 429 (1995).
[12] D. J. Dean, and M. Hjorth-Jensen, Rev. Mod. Phys. 75, 607 (2003).
[13] J. B. Anderson, J. Chem. Phys. 63, 1499 (1975).
[14] M. H. Kalos, D. Levesque, and L. Verlet, Phys. Rev. A 9, 2178 (1974).
[15] E. Feenberg, Theory of quantum fluids, (Academic Press,
[16] V. R. Pandharipande, and K. E. Schmidt, Phys. Rev. A 15, 2486 (1977).

[17] K. E. Schmidt, M. A. Lee, M. H. Kalos, and G. V. Chester, Phys. Rev. Lett. 47, 807 (1981).