THE OPTIMAL PARTITION PROBLEM FOR $p$-LAPLACIAN EIGENVALUES AS $p$ GOES TO ONE

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Abstract. In this paper we introduce a Cheeger-type constant defined as a minimization of a suitable functional among all the $N$-clusters contained in an open bounded set $\Omega$. Here with $N$-Cluster we mean a family of $N$ sets of finite perimeter, disjoint up to a set of null Lebesgue measure. We call any $N$-cluster attaining such a minimum a Cheeger $N$-cluster. Our purpose is to provide a non trivial lower bound on the optimal partition problem for the first Dirichlet eigenvalue of the Laplacian. Here we discuss the regularity of Cheeger $N$-clusters in a general ambient space dimension and we give a precise description of their structure in the planar case. The last part is devoted to the relation between the functional introduced here (namely the $N$-Cheeger constant), the partition problem for the first Dirichlet eigenvalue of the Laplacian and the Caffarelli and Lin’s conjecture.

Contents

Introduction 1
1. Existence and Regularity 5
2. Structure of the Cheeger $N$-clusters in low dimension 11
2.1. Some general facts in dimension $n \leq 7$ 11
2.2. The case $n = 2$: detailed structure. 17
3. The limit as $p$ goes to one of $\Lambda^p_N$ 22
3.1. On the asymptotic behavior of $H_N$ and the case of flat torus 23
References 25

Introduction

For a given open, bounded set $\Omega \in \text{Bor}(\mathbb{R}^n)$ we introduce the $N$-Cheeger constant as:

$$H_N(\Omega) = \inf \left\{ \sum_{i=1}^{N} \frac{P(\mathcal{E}(i))}{|\mathcal{E}(i)|} \mid \mathcal{E} = \{\mathcal{E}(i)\}_{i=1}^{N} \subseteq \Omega, \text{ is an } N\text{-cluster} \right\}$$

(0.1)

where an $N$-cluster is just a family of $N$ sets of finite perimeter $\mathcal{E} = \{\mathcal{E}(1), \ldots, \mathcal{E}(N)\}$ such that:

a) $0 < |\mathcal{E}(i)| < +\infty$, for all $i = 1, \ldots, N$;

b) $|\mathcal{E}(i) \cap \mathcal{E}(j)| = 0$, for all $i \neq j$.

Here $P(E)$ denotes the distributional perimeter (see [14]) and $|E|$ is the Lebesgue measure. We sometimes refer to the sets $\mathcal{E}(j)$ as the chambers of the $N$-cluster. As shown below the infimum in (0.1) is always attained, and we refer to the minimizers as the Cheeger $N$-clusters of $\Omega$. We focus on the quantity $H_N$ because it seems to represent the right object to study in order to provide some non trivial lower bound on the optimal partition problem.
The functional
\[ \Lambda_N^{(p)}(\Omega) = \inf \left\{ \sum_{i=1}^{N} \lambda_N^{(p)}(\mathcal{E}(i)) \mid \mathcal{E} = \{\mathcal{E}(i)\}_{i=1}^{N} \subset \mathcal{A}_{\rho}(\Omega), \text{ is a quasi-open } N\text{-cluster} \right\}, \] (0.2)
where \( \mathcal{A}_{\rho}(\Omega) \) denotes the collection of all quasi-open subsets of \( \Omega \) with respect to the \( p \)-capacitary measure (see [10] for more details about it) and where with quasi-open \( N \)-cluster we mean an \( N \)-cluster made by quasi-open sets. Here \( \lambda_N^{(p)} \) denotes the first Dirichlet eigenvalue of the \( p \)-Laplacian, defined as:
\[ \lambda_N^{(p)}(E) := \inf \left\{ \int_E |\nabla u|^p \, dx \mid u \in W_0^{1,p}(E), \| u \|_{L^p} = 1 \right\}. \]

To be more clear and explain the connection between \( H_N \) and \( \Lambda_N^{(p)} \) let us retrieve some well-known fact about the classical Cheeger constant of a Borel set \( \Omega \in \text{Bor}(\mathbb{R}^n) \):
\[ h(\Omega) := \inf \left\{ \frac{P(E)}{|E|} \mid E \subseteq \Omega \right\}, \] (0.3)
(note that \( h(\Omega) = H_1(\Omega) \)). The Cheeger constant was introduced in [9] by Jeff Cheeger and provide a lower bound on the first Dirichlet eigenvalue of the \( p \)-Laplacian of a domain \( \Omega \). Indeed, by exploiting the coarea formula and the Hölder inequality it is possible to show that for every domain \( \Omega \) and for every \( p > 1 \), holds:
\[ \lambda_1^{(p)}(\Omega) \geq \left( \frac{h(\Omega)}{p} \right)^p. \] (0.4)

The Cheeger constant is also called the first Dirichlet eigenvalue of the 1-laplacian since, thanks to (0.4) and a comparison argument:
\[ \lim_{p \to 1} \lambda_1^{(p)}(\Omega) = h(\Omega). \]

See, for example, [12] for more details about the relation between the Cheeger constant and the first Dirichlet eigenvalue of the \( p \)-Laplacian or [3] and [6] for more details about the spectral problems and shape optimization problems. We note here that \( H_N \) plays the same role as the Cheeger constant in the optimal partition problem for \( p \)-laplacian eigenvalues. Indeed we can always give a lower bound on \( \Lambda_N^{(p)} \) making use of (0.4), which by convexity gives:
\[ \Lambda_N^{(p)}(\Omega) \geq \frac{1}{N^{p-1}} \left( \frac{H_N(\Omega)}{p} \right)^p. \] (0.5)

Combining (0.5) with a comparison argument, as in the case \( N = 1 \), we are able to compute the limit as \( p \) goes to 1 and get
\[ \lim_{p \to 1} \Lambda_N^{(p)}(\Omega) = H_N(\Omega), \]
see proposition 3.1 in section 3 for a detailed proof of this fact.

The well-known Caffarelli and Lin’s conjecture about the asymptotic behavior of \( \Lambda_N^{(2)}(\Omega) \) in the planar case, states that
\[ \Lambda_N^{(2)}(\Omega) = \frac{N^2}{|\Omega|} \lambda_1^{(2)}(H) + o(N^2), \]
where \( H \) denotes the unit-area regular hexagon. The importance of such a conjecture relies in the fact that the partition problem provide a way to look at the asymptotic behavior in \( N \) of the \( N \)-th Dirichlet eigenvalue of the classical Laplacian (the 2-Laplacian), as Caffarelli
and Lin show in [8]. They prove that there exists two constants depending only on the dimension such that
\[
C_1 \frac{\Lambda_N(\Omega)}{N} \leq \lambda_N^{(2)}(\Omega) \leq C_2 \frac{\Lambda_N(\Omega)}{N},
\]
where \(\lambda_N^{(2)}\) is the \(N\)-th Dirichlet eigenvalue. Thus, if their conjecture turns out to be true, relation (0.6) could be improved, in the planar case, in:
\[
C_1 \frac{N \Lambda_N^{(2)}(H)}{|\Omega|} + o(N) \leq \lambda_N^{(2)}(\Omega) \leq C_2 \frac{N \Lambda_N^{(2)}(H)}{|\Omega|} + o(N).
\]

The detailed study of \(\lambda_N^{(2)}(\Omega)\) for \(N \geq 2\) seems to be an hard task (so far only the case \(N = 1, 2\) are well known, see for instance [11]) and that's why the asymptotic approach suggested by Caffarelli and Lin, as far as their conjecture, could be a good way to look at the spectral problem. See [2] for some simulation about the optimal partition problem. A generalized type of Cheeger constant for the 2-th Dirichlet eigenvalue of the Laplacian is studied also in [15].

Note that for \(H_N\) it is reasonable to expect a behavior of the type
\[
H_N(\Omega) \approx C(\Omega) N^{\frac{3}{2}} + O(\sqrt{N}),
\]
for some constant depending on \(\Omega\). In Theorem 3.2, Assertion (3.7), we provide some asymptotic estimate for \(H_N\) showing that the exponent \(\frac{3}{2}\) in (0.8) is the correct one. We here conjecture that
\[
C(\Omega) = \frac{h(H)}{\sqrt{|\Omega|}},
\]
which is nothing more than the Caffarelli and Lin’s conjecture for the case \(p = 1\). Note that, thanks to (0.5) this would imply
\[
\Lambda_N^{(2)}(\Omega) \geq \frac{N^2}{|\Omega|} \left( \frac{h(H)}{2} \right)^2,
\]
a "weak" version of Caffarelli and Lin’s conjecture. It seems coherent and natural to expect this kind of behavior for \(H_N(\Omega)\).

We here mainly focus on the general structure of Cheeger \(N\)-clusters with particular attention on the planar case. In section 1 we show the existence of Cheeger \(N\)-clusters for a generic open bounded set \(\Omega\), then we state and prove an inner regularity theorem for Cheeger \(N\)-clusters. Finally some weak regularity at the boundary of \(\Omega\) is provided in the spirit of Assertion (vii), Proposition 2.5 of [13]. Section 1 is entirely devoted to the proof of the following Theorem.

**Theorem 0.1.** Let \(n \geq 1, N \geq 2\). Given an open bounded set \(\Omega\) it always exists a Cheeger \(N\)-cluster for \(\Omega\). Moreover:

a) Any of its Cheeger \(N\)-clusters \(\mathcal{E}\) is such that there exists a set \(\Sigma\) of dimension at least \(n - 8\), with the property that \((\partial \mathcal{E}(i) \setminus \Sigma) \cap \Omega\) is a \(C^{1,\gamma}\)-hypersurface for every \(\gamma \in (0, \frac{1}{2})\);

b) The set \(\Sigma\) is empty for \(1 \leq n \leq 7\);

c) For every \(j = 1, \ldots, N\) holds that \(\partial^* \Omega \cap \partial \mathcal{E}(j) \subseteq \partial^* \mathcal{E}(j)\) and for every \(x \in \partial^* \Omega \cap \partial \mathcal{E}(j)\) it holds that
\[
\nu_{\mathcal{E}(j)}(x) = \nu_{\Omega}(x),
\]
where \(\nu_{\mathcal{E}(j)}, \nu_{\Omega}\) denotes the measure theoretic outer unit normal to \(\mathcal{E}(j)\) and \(\Omega\) respectively.
Here $\partial^* E$ denotes the reduced boundary of the set of finite perimeter $E$ (see [14]), namely the support of the distributional derivative of the characteristic function of $E$ or equivalently the set of points of $\partial E$ where the measure theoretic outer unit normal is well-defined. For prove a) and b) we simply prove that each chamber is a $(\Lambda, r_0)$-perimeter minimizer in $\Omega$ (see Definition 1.1) and then we make use of the De Giorgi’s regularity Theorem retrieved, for the sake of clarity, in Section 1 (Theorem 1.2). We re-adapt an idea from [1] based on the fact that a solution of an obstacle problem having bounded distributional mean curvature is regular.

In Section 2 we move to prove some general results on the structure of Cheeger $N$-clusters in dimension less than 8, taking advantages from the fact that in these ambient space dimensions the singular set of a $(\Lambda, r_0)$-perimeter minimizer is empty. Namely we introduce the void chamber $E(0)$ of a Cheeger $N$-cluster of $\Omega$ as the open set:

$$E(0) := \Omega \cap \left( \mathbb{R}^n \setminus \bigcup_{i=1}^{N} E(i) \right),$$

we define the void-cusps set as

$$\zeta_0 = \{ x \in \partial E(0) \mid \partial_n(x, E(0)) = 0 \}, \quad (0.10)$$

and we give a precise description of $\zeta_0$. Here $\partial_n(x, E)$ is the $n$-dimensional density of $E$ at $x$ defined, for almost every $x \in \mathbb{R}^n$, as

$$\partial_n(x, E) = \lim_{r \to 0^+} \frac{|E \cap B_r(x)|}{\omega_n r^n},$$

where $B_r(x)$ is the ball with radius $r$, centered at $x$ and $\omega_n = |B_1(x)|$. We sometimes make use of the notation

$$E^{(t)} := \{ x \in \mathbb{R}^n \mid \partial_n(x, E) = t \}.$$

Thanks to the regularity proved in Section 1 and for a regular enough set $\Omega$, $\zeta_0$ is the only singular set of a Cheeger $N$-cluster. In particular we prove the following Theorem.

**Theorem 0.2.** Let $\Omega$ be an open bounded set with $C^1$ boundary and $E$ one of its Cheeger $N$-cluster. If $N \geq 2$ and $n \leq 7$, it holds:

a) $|E(0)| > 0$ and $H^{n-1}(\partial^* E(0) \cap \partial E(j)) > 0$ for every $j = 1, \ldots, N$;

b) $\partial E(0) = \zeta_0 \cup \partial^* E(0)$;

c) $\zeta_0$ is closed and

$$\zeta_0 \cap \Omega = \bigcup_{j,k=1}^{N} \text{bd}(\partial E(j) \cap \partial E(k)), \quad (0.11)$$

$$\zeta_0 \cap \partial \Omega = \bigcup_{j=1}^{N} \text{bd}(\partial \Omega \cap \partial E(j)); \quad (0.12)$$

d) If $E \subset \subset \Omega$ is an indecomposable component of $E(0)$ with $E \cap \partial \Omega = \emptyset$, then $E$ confine at least with three different chambers.

e) if $n = 2$ then $H^0(\zeta_0) < +\infty$ and for every $i \neq 0$ the set $\partial E(i) \cap \Omega$ is a finite union of circular arcs. Moreover each interface $\partial E(i) \cap \partial E(k)$ such that $H^1(\partial E(i) \cap \partial E(k)) > 0$ have signed mean curvature:

$$C_{j,k} = \left\{ \begin{array}{ll}
\frac{|E(k)| h(E(i)) - |E(i)| h(E(k))}{|E(i)| + |E(k)|}, & \text{if } k \neq 0 \\
h(E(i)), & \text{if } k = 0,
\end{array} \right. \quad (0.13)$$

with the convention as in figure 2.6.
f) if $n = 2$ each chamber $\mathcal{E}(j)$ disjoint from $\partial \Omega$ is connected.

We are adopting the notation

$$ \text{bd}(S) := S \cap (\partial E \setminus S), $$

for a subset $S \subset \partial E$. In the low-dimension case we are able to ensure the non-trivial fact $|\mathcal{E}(0)| > 0$ but we actually don’t know if this remains true in dimension bigger than 7. Indeed, heuristically, the void is given by the fact that there are no singularities on the boundary of the chambers and this fact remains true only for $n \leq 7$ thanks to the regularity theory. In higher dimension it can happen that the chambers, allowed to have some singular point, try to combine in a way that kill the void. As we will clarify, even in the low-dimension case, we need to ask some regularity on $\partial \Omega$ which turns out to be necessary for ensure a non trivial void chambers $\mathcal{E}(0)$. Note that, even if the connection of the chambers is usually an hard task in the tessellation problems, in this case, thanks to a general fact for planar Cheeger sets (Proposition 2.3), we can easily achieve the proof of Assertion f) in Theorem 0.2, which is actually of particular interest in order to focus our attention on the asymptotic behavior of $H_N$.

Finally in section 3 we compute the limit as $p$ goes to one of $\Lambda^{(p)}_N(\Omega)$ and provide some easy asymptotic property in $N$ of $H_N$. We also briefly discuss some directions for future research.

**Acknowledgments.** The author is grateful to professor Giovanni Alberti for his useful comments and for the useful discussions about this subject. The work of the author was partially supported by the project 2010A2TFX2-Calcolo delle Variazioni, funded by the Italian Ministry of Research and University.

1. Existence and Regularity

We sometimes make use of the equivalent definition of $H_N$ as

$$ H_N(\Omega) = \left\{ \sum_{i=1}^{N} h(\mathcal{E}(i)) \left| \mathcal{E} \subseteq \Omega \text{ N-Cluster} \right. \right\}. \quad (1.1) $$

We start by proving the existence and then, separately, we prove the regularity for Cheeger $N$-clusters.

**Theorem 1.1.** Let $\Omega \in \text{Bor}(\mathbb{R}^n)$ be a bounded set with $|\Omega| > 0$ and finite perimeter. For every $N \in \mathbb{N}$ there exists a Cheeger $N$-cluster for $\Omega$, i.e. an $N$-cluster $\mathcal{E} \subseteq \Omega$ such that:

$$ H_N(\Omega) = \sum_{i=1}^{N} \frac{P(\mathcal{E}(i))}{|\mathcal{E}(i)|}. $$

Moreover each Cheeger $N$-cluster of $\Omega$ has the following property:

$$ |\mathcal{E}(i)| > \frac{n^n \omega_n}{2^n H_N(\Omega)^n} \quad \text{for all } i = 1, \ldots, N, \quad (1.2) $$

$$ h(\mathcal{E}(i)) = \frac{P(\mathcal{E}(i))}{|\mathcal{E}(i)|} \quad \text{for all } i = 1, \ldots, N. \quad (1.3) $$

**Proof.** Clearly $H_N(\Omega) < +\infty$. Indeed we can always choose, for example $B_1, \ldots, B_N$ disjoint ball such that $|B_i \cap \Omega| > 0$ (since $|\Omega| > 0$) and get

$$ H_N(\Omega) \leq \sum_{i=1}^{N} \frac{P(B_i \cap \Omega)}{|B_i \cap \Omega|} < +\infty. \quad (1.4) $$
Consider a minimizing sequence $\mathcal{E}^k = \{\mathcal{E}(i)^k\}_{i=1}^N$ of $N$-clusters such that
\[
\lim_{k \to +\infty} \sum_{i=1}^N \frac{P(\mathcal{E}(i)^k)}{|\mathcal{E}(i)^k|} = H_N(\Omega).
\]

Thanks to the monotonicity of the Lebesgue measure and to the isoperimetric inequality for sets of finite perimeter we can easily provide the bounds
\[
P(\mathcal{E}(i)^k) \leq |\Omega| \sum_{j=1}^N \frac{P(\mathcal{E}(j)^k)}{|\mathcal{E}(j)^k|}
\leq 2|\Omega|H_N(\Omega),
\]
\[
n\left(\frac{\omega_n}{|\mathcal{E}(i)^k|}\right)^{\frac{1}{n}} \leq \frac{P(\mathcal{E}(i)^k)}{|\mathcal{E}(i)^k|}
\leq 2H_N(\Omega)
\]
and conclude
\[
\sup_k \left\{ \max_i \left\{ P(\mathcal{E}(i)^k) \right\} \right\} \leq 2|\Omega|H_N(\Omega), \tag{1.5}
\]
\[
\inf_k \left\{ \min_i \left\{ |\mathcal{E}(i)^k| \right\} \right\} \geq \frac{n^n\omega_n}{2^n H_N(\Omega)^n}. \tag{1.6}
\]

With these bounds in hands, thanks to the boundedness of $\Omega$ and (1.5), we can apply the compactness theorem for sets of finite perimeter (see Theorem 12.26 in [14]) and say that, up to a subsequence, each sequence of chamber $\mathcal{E}(i)^k$ is converging in $L^1$ to some $\mathcal{E}(i) \subseteq \Omega$ for every $i = 1, \ldots, N$. Equation (1.6) implies immediately (1.2) while the lower semicontinuity of distributional perimeter (see Proposition 12.15 in [14]) yields:
\[
H_N(\Omega) \leq \sum_{i=1}^N \frac{P(\mathcal{E}(i))}{|\mathcal{E}(i)|}
\leq \sum_{i=1}^N \liminf_{k \to \infty} \frac{P(\mathcal{E}(i)^k)}{|\mathcal{E}(i)^k|}
\leq \liminf_{k \to +\infty} \sum_{i=1}^N \frac{P(\mathcal{E}(i)^k)}{|\mathcal{E}(i)^k|} = H_N(\Omega).
\]

Finally, Property (1.3) immediately follows from minimality. \qed

We now show that every Cheeger $N$-cluster of a given open set $\Omega$ is a $C^{1,\gamma}$ surface inside of $\Omega$ for every $\gamma \in \left(0, \frac{1}{2}\right)$. Note that in order to prove regularity in the case of Cheeger $N$-cluster we have to deal not only with $\partial \mathcal{E}(i) \cap \Omega$ and $\partial \mathcal{E}(i) \cap \partial \Omega$ (as in the case $N = 1$) but we have also to manage some part $\partial \mathcal{E}(i) \cap \partial \mathcal{E}(j)$. To manage this part of the boundary we just observe that property (1.3) implies that the mean curvature of each $\partial \mathcal{E}(j)$ is bounded from above, thus for every fixed $i$ the set $M_i^c = \mathbb{R}^n \setminus \bigcup_{j \neq i} \mathcal{E}(j)$ must has distributional mean curvature of the boundary bounded from below and this will implies that no outer cusps are attained. In particular $\partial \mathcal{E}(i) \cap \Omega$ cannot be too wild far from the free boundary since, from minimality
\[
\frac{P(\mathcal{E}(i))}{|\mathcal{E}(i)|} = h(M_i^c \cap \Omega).
\]

This approach is based on an idea from [1], where the authors prove a regularity result for the solutions of some obstacle problems.
The strategy is to prove that each $E(j)$ is a $(\Lambda, r_0)$-perimeter minimizer inside $\Omega$ and then apply the regularity Theorem from De Giorgi. For the sake of clarity we retrieve the Definition of $(\Lambda, r_0)$-perimeter minimizer and the regularity Theorem.

**Definition 1.1** ($(\Lambda, r_0)$-perimeter minimizer inside $\Omega$, [14]). We are saying that a set of finite perimeter $E$ is a $(\Lambda, r_0)$-perimeter minimizer in $\Omega$ if for every $B_r \subset \Omega$ with $r < r_0$ and every set $F$ such that $E \Delta F \subset \subset B_r$, it holds

$$P(E; B_r) \leq P(F; B_r) + \Lambda|E \Delta F|.$$  

**Theorem 1.2** (See for example [14], pp. 354, 363-365). If $\Omega$ is an open set in $\mathbb{R}^n$, $n \geq 2$ and $E$ is a $(\Lambda, r_0)$-perimeter minimizer in $\Omega$, with $\Lambda r_0 \leq 1$, then $\Omega \cap \partial^* E$ is a $C^{1,\gamma}$ hypersurface for every $\gamma \in (0, \frac{1}{2})$ that is relatively open in $\Omega \cap \partial E$, and it is $\mathcal{H}^{n-1}$ equivalent $\Omega \cap \partial E$. Moreover, setting

$$\Sigma(E; \Omega) = \Omega \cap (\partial E \setminus \partial^* E),$$

then the following statements hold true:

1. if $2 \leq n \leq 7$, then $\Sigma(E; \Omega)$ is empty;
2. if $n = 8$, then $\Sigma(E; \Omega)$ has no accumulation points in $\Omega$;
3. if $n \geq 9$, then $\mathcal{H}^s(\Sigma(E; \Omega)) = 0$ for every $s > n - 8$.

These results are sharp.

We are able to prove the following statement.

**Theorem 1.3.** Let $\Omega$ be an open bounded set and $\mathcal{E}$ be a Cheeger $N$-cluster for $\Omega$. Then it there exists $\Lambda, r_0 > 0$ depending on $\mathcal{E}$ with $\Lambda r_0 \leq 1$, such that each $E(i)$ is a $(\Lambda, r_0)$-perimeter minimizer in $\Omega$. As a consequence, for every $i = 1, \ldots, N$ the set $\Omega \cap \partial E(i)$ has the regularity of Theorem 1.2.

Before enter in the details of the proof of Theorem 1.3 we state and prove the following technical Lemma, which is playing a crucial role in our argument.

**Lemma 1.1.** Let $E, F$ be two sets of finite perimeter. Then

$$P(F \setminus E; B_r) + P(E \setminus F; B_r) \leq P(F; B_r) + P(E; B_r).$$  

Moreover if $\mathcal{E} = \{E(i)\}_{i=1}^N$ is an $N$-cluster then

$$\partial^* \left( \bigcup_{i=1}^N E(i) \right) \approx \bigcup_{i=1}^N \left[ \partial^* E(i) \setminus \left( \bigcup_{j \neq i} \partial^* E(j) \right) \right],$$  

where the symbol $\approx$ means equal up to an $\mathcal{H}^{n-1}$-negligible set, in particular:

$$P \left( \bigcup_{i=1}^N E(i); B_r \right) = \sum_{i=1}^N P(E(i); B_r) - \sum_{k,j=1, k \neq j}^N \mathcal{H}^{n-1}(\partial^* E(j) \cap \partial^* E(k) \cap B_r).$$  

**Proof.** Let $\rho_\varepsilon$ be a regularizing kernel and $E_\varepsilon(x) = 1_E \star \rho_\varepsilon$, $F_\varepsilon(x) = 1_F \star \rho_\varepsilon(x)$. Clearly $E_\varepsilon(1 - F_\varepsilon) \rightarrow 1_E \setminus F$ and $F_\varepsilon(1 - E_\varepsilon) \rightarrow 1_F \setminus E$ in $L^1_{loc}(\mathbb{R}^n)$ and

$$\int_{B_r} |\nabla (E_\varepsilon(1 - F_\varepsilon))| dx \leq \int_{B_r} E_\varepsilon |\nabla F_\varepsilon| dx + \int_{B_r} (1 - F_\varepsilon(x)) |\nabla E_\varepsilon| dx$$  

$$\int_{B_r} |\nabla (F_\varepsilon(1 - E_\varepsilon))| dx \leq \int_{B_r} F_\varepsilon |\nabla E_\varepsilon| dx + \int_{B_r} (1 - E_\varepsilon(x)) |\nabla F_\varepsilon| dx.$$  

7
Here we have used the fact that without loss of generality we can assume $F_\varepsilon(x), E_\varepsilon(x) \leq 1$. By adding up (1.10),(1.11) we get

$$
\int_{B_r} |\nabla(E_\varepsilon(1 - F_\varepsilon))| dx + \int_{B_r} |\nabla(F_\varepsilon(1 - E_\varepsilon))| \leq \int_{B_r} |\nabla E_\varepsilon| dx + \int_{B_r} |\nabla F_\varepsilon| dx \quad (1.12)
$$

and sending $\varepsilon$ to zero, exploiting the fact that $|\nabla E_\varepsilon| L^n, |\nabla F_\varepsilon| L^n, |\nabla F_\varepsilon(1 - F_\varepsilon)| L^n$ and $|\nabla F_\varepsilon(1 - E_\varepsilon)| L^n$ are converging in the weak star topology to the total variation of the Gauss-Green measure of $E, F, E \setminus F$ and $F \setminus E$ respectively, we get

$$
P(E \setminus F; B_r) + P(F \setminus E; B_r) \leq P(E; B_r) + P(F; B_r).
$$

With an easy approximation argument, the last inequality lead us to (1.7). For (1.9) we notice that it is enough to prove the identity (1.8). We prove it by induction and we start by considering two Borel sets $E, F$ with $|E \cap F| = 0$. Notice that in this case, forcedly $H^{n-1}(\partial^*(E \cap F)) = 0$. In particular, thanks to the structure of the Gauss-Green measure and Federer’s Theorem (Chapter 16 in [14]) we get:

$$
\partial^*(E \cup F) \approx (E(0) \cap \partial^* F) \cup (F(0) \cap \partial^* E) \cup (\partial^*(E \cap F) \cap \partial^* E \cap \partial^* F)
\approx (E(0) \cap (F(1) \setminus E(1))) \cup (F(0) \cap (E(1) \setminus F(1))).
$$

Moreover:

$$
F^{(\frac{1}{2})} \approx (E(0) \cap F^{(\frac{1}{2})}) \cup (E(1) \cap F^{(\frac{1}{2})}) \cup (E^{(\frac{1}{2})} \cap F^{(\frac{1}{2})}) \quad (1.14)
$$

$$
E^{(\frac{1}{2})} \approx (F(0) \cap E^{(\frac{1}{2})}) \cup (F(1) \cap E^{(\frac{1}{2})}) \cup (F^{(\frac{1}{2})} \cap E^{(\frac{1}{2})}) \quad (1.15)
$$

and we point out that $E^{(\frac{1}{2})} \cap F^{(1)} = E^{(1)} \cap F^{(\frac{1}{2})} = \emptyset$. Indeed, consider for example $x \in E^{(\frac{1}{2})} \cap F^{(1)}$. Then for every $\varepsilon > 0$ there exists an $r_0$ s.t.

$$
|E \cap B_r| \geq \left(\frac{1}{2} - \varepsilon\right) \omega_n r^n, \quad |F \cap B_r| \geq (1 - \varepsilon) \omega_n r^n, \quad \forall r \leq r_0,
$$

and so

$$
\left(\frac{3}{2} - 2\varepsilon\right) \omega_n r^n \leq |F \cap B_r| + |E \cap B_r| = |(F \cup E) \cap B_r| \leq \omega_n r^n,
$$

leading to a contradiction. In particular, (1.14) and (1.15) become

$$
F^{(\frac{1}{2})} \setminus (E^{(\frac{1}{2})} \cap F^{(1)}) \approx (E(0) \cap F^{(\frac{1}{2})}),
$$

$$
E^{(\frac{1}{2})} \setminus (E^{(\frac{1}{2})} \cap F^{(1)}) \approx (F(0) \cap E^{(\frac{1}{2})})
$$

that, combined with (1.13) and Federer’s Theorem, lead us to:

$$
\partial^*(E \cup F) \approx \left[\partial^* F \setminus (\partial^* F \cap \partial^* E)\right] \cup \left[\partial^* E \setminus (\partial^* E \cap \partial^* F)\right].
$$

which is nothing more than (1.8) in the case $N = 2$. Assume now that (1.8) holds for every $(N - 1)$-cluster and let $E$ be an $N$-cluster. Applying (1.16) with $E = E(1)$ and $F = E(2) \cup \ldots \cup E(N)$ we get

$$
\partial^* \left(\bigcup_{i=1}^{N} E(i)\right) \approx \left[\partial^* E(1) \setminus \partial^* \left(\bigcup_{j=2}^{N} E(j)\right)\right] \cup \left[\partial^* \left(\bigcup_{j=2}^{N} E(j)\right) \setminus \partial^* E(1)\right].
$$

(1.17)
Exploiting the fact that (1.8) holds on \(\{E(2), \ldots, E(N)\}\) and the fact that \(\partial^*E(j) \cap \partial^*E(k) \cap \partial^*E(i) = \emptyset\) for \(j \neq k \neq i\), we get

\[
\left[\partial^*E(1) \setminus \partial^*\left(\bigcup_{j=2}^{N} E(j)\right)\right] \approx \partial^*E(1) \setminus \bigcup_{j=2}^{N} \left[\partial^*E(j) \setminus \bigcup_{k \neq j, 1}^{N} \partial^*E(k)\right] \quad (1.18)
\]

and

\[
\left[\partial^*\left(\bigcup_{j=2}^{N} E(j)\right) \setminus \partial^*E(1)\right] \approx \bigcup_{j=2}^{N} \left[\partial^*E(j) \setminus \bigcup_{k \neq j}^{N} \partial^*E(k)\right] \setminus \partial^*E(1) \quad (1.19)
\]

where we have used the relations

\[
A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap B \cap C), \quad (A \setminus B) \setminus C = A \setminus (B \cup C).
\]

By plugging (1.19) and (1.21) in (1.17) we prove (1.8) and we achieve the proof. □

**Proof of theorem 1.3.** Let’s start by fix \(i \in \{1, \ldots, N\}\) and define

\[
M_i = \bigcup_{j \neq i} E(j).
\]

We divide the proof in two parts.

**Step 1.** In this part we prove that if \(B_r \subset \subset \Omega\) is a ball and \(L \subset M_i\) is a set with \(M_i \setminus L \subset \subset B_r\), then

\[
P(M_i; B_r) \leq P(L; B_r) + H_N(\Omega) |M_i \setminus L|.
\]

(1.22)

For every \(k = 1, \ldots, N\) it must holds that \(\frac{P(E(k); B_r)}{|E(k)|} \leq \frac{P(E(k) \cap L; B_r)}{|E(k) \cap L|}\), which lead us to:

\[
\frac{P(E(k); B_r) + P(E(k); B_r^c)}{|E(k)|} \leq \frac{P(E(k) \cap L; B_r) + P(E(k); B_r^c)}{|E(k) \cap L|} \leq P(E(k) \cap L; B_r) + |E(k) \setminus L| h(E(k)).
\]
By exploiting (1.9) we get
\[
P(M_i; B_r) = P(\cup_{j \neq i} \mathcal{E}(j); B_r)
\]
\[
= \sum_{j \neq i} P(\mathcal{E}(j); B_r) - \sum_{k,j \neq i, k \neq j} \mathcal{H}^{n-1}(\partial^* \mathcal{E}(j) \cap \partial^* \mathcal{E}(k) \cap B_r)
\]
\[
\leq \sum_{j \neq i} P(\mathcal{E}(j) \cap L; B_r)
\]
+ \sum_{j \neq i} |\mathcal{E}(j) \setminus L| h(\mathcal{E}(j)) - \sum_{k,j \neq i, k \neq j} \mathcal{H}^{n-1}(\partial^* \mathcal{E}(j) \cap \partial^* \mathcal{E}(k) \cap B_r)
\]
\[
\leq P(L \cap M_i; B_r) + \sum_{k,j \neq i, k \neq j} \mathcal{H}^{n-1}(L \cap (\partial^* \mathcal{E}(j) \cap \partial^* \mathcal{E}(k) \cap B_r)
\]
+ \sum_{j \neq i} |\mathcal{E}(j) \setminus L| h(\mathcal{E}(j)) - \sum_{k,j \neq i, k \neq j} \mathcal{H}^{n-1}(\partial^* \mathcal{E}(j) \cap \partial^* \mathcal{E}(k) \cap B_r)
\]
\[
\leq P(L \cap M_i; B_r) + |M_i \setminus L| \sum_{j \neq i} h(\mathcal{E}(j))
\]
\[
\leq P(L; B_r) + H_N(\Omega)|M_i \setminus L|,
\]
where in the last inequality we have used the formulation of $H_N$ as in (1.1) and the fact that $L \cap M_i$ is equal to $L$ (since $L \subseteq M_i$).

**Step 2.** We now prove that $\mathcal{E}(i)$ is a $(\Lambda, r_0)$-perimeter minimizer for a suitable choice of $\Lambda$ and $r_0$. Let $B_r \subset \subset \Omega$ and $F$ be such that $F \Delta \mathcal{E}(i) \subset B_r$. Define $E := F \setminus M_i$ and observe, by minimality of $\mathcal{E}$ and by the relation $\mathcal{E}(i) \cap B_r = (F \setminus M_i) \cap B_r$, that:

\[
\frac{P(\mathcal{E}(i))}{|\mathcal{E}(i)|} \leq \frac{P(E)}{|E|}
\]

\[
\frac{P(\mathcal{E}(i); B_r) + P(\mathcal{E}(i); B_r^c)}{|\mathcal{E}(i)|} \leq \frac{P(F \setminus M_i; B_r) + P(F \setminus M_i; B_r^c)}{|F| - |F \cap M_i|}
\]

\[
\frac{P(\mathcal{E}(i); B_r) + P(\mathcal{E}(i); B_r^c)}{|\mathcal{E}(i)|} \leq \frac{P(F \setminus M_i; B_r) + P(\mathcal{E}(i); B_r^c)}{|\mathcal{E}(i)| + (|F \cap B_r| - |\mathcal{E}(i) \cap B_r|) - |F \cap M_i|}.
\]

If we expand the last inequality and make the computations, we are lead to:

\[
P(\mathcal{E}(i); B_r)|\mathcal{E}(i)| \leq P(F \setminus M_i; B_r)|\mathcal{E}(i)| + P(\mathcal{E}(i))(|F \cap M_i| + |\mathcal{E}(i) \cap B_r| - |F \cap B_r|),
\]

which means (by remembering that $F \cap M_i \subset F \setminus \mathcal{E}(i)$),

\[
P(\mathcal{E}(i); B_r) \leq P(F \setminus M_i; B_r) + 2h(\mathcal{E}(i))|\mathcal{E}(i)|\Delta F|.
\]  \hspace{1cm} (1.23)

Now we make use of (1.7) for get

\[
P(F \setminus M_i; B_r) + 2h(\mathcal{E}(i))|\mathcal{E}(i)|\Delta F| \leq P(F; B_r) + P(M_i; B_r) - P(M_i \setminus F; B_r) + 2h(\mathcal{E}(i))|\mathcal{E}(i)|\Delta F|.
\]  \hspace{1cm} (1.24)

Since $M_i \setminus F \subset M_i$ and $(M_i \setminus F) \Delta M_i \subset B_r$ we can use step one for conclude that

\[
P(M_i; B_r) \leq P(M_i \setminus F; B_r) + H_N(\Omega)|M_i \setminus (M_i \setminus F)|
\]

\[
P(M_i; B_r) \leq P(M_i \setminus F; B_r) + H_N(\Omega)|M_i \cap F|
\]

\[
P(M_i; B_r) - P(M_i \setminus F; B_r) \leq H_N(\Omega)|F \setminus \mathcal{E}(i)|.
\]  \hspace{1cm} (1.25)
By plug (1.25) in (1.24) we obtain
\[ P(F \setminus M; B_r) + 2h(E(i))|\mathcal{E}(i)\Delta F| \leq P(F; B_r) + 3H_N(\Omega)|\mathcal{E}(i)\Delta F| \quad (1.26) \]
and by using (1.26) in (1.23) we are lead to
\[ P(E(i); B_r) \leq P(F; B_r) + 3H_N(\Omega)|\mathcal{E}(i)\Delta F|. \]
By choosing \( \Lambda = 3H_N(\Omega) \) and \( r_0 = \frac{1}{4H_N(\Omega)} \) we get that each \( E(i) \) is a \((\Lambda, r_0)\)-perimeter minimizer with \( \Lambda r_0 < 1 \) which is the thesis. \( \square \)

We conclude this section with a regularity result on the boundary, in the spirit of Proposition 2.5, Assertion (vii) in [13]. Since the proof follows with the same argument exploit in [13], namely by using the fact that each chambers is a \((\Lambda, r_0)\)-perimeter minimizers in \( \Omega \) and that the blow-up of \( \partial \Omega \) at a point \( x \in \partial^* \Omega \) is a an half plane, we omit the details.

**Proposition 1.1.** Let \( \Omega \) be an open and bounded set and let \( E \) be a Cheeger \( N \)-cluster for \( \Omega \). Then for every \( j = 1, \ldots, N \) holds
\[ \partial^* \Omega \cap \partial E(j) \subseteq \partial^* E(j). \]
Moreover for every \( x \in \partial^* \Omega \cap \partial E(j) \) it holds that
\[ \nu_{E(j)}(x) = \nu_{\Omega}(x), \]
where \( \nu_{E(j)}, \nu_{\Omega} \) denotes the measure theoretic outer unit normal to \( E(j) \) and \( \Omega \) respectively.

**Proof of Theorem 0.1.** It follows from Theorem 1.1, 1.2, 1.3 and Proposition 1.1. \( \square \)

## 2. Structure of the Cheeger \( N \)-clusters in low dimension

From now on the set \( \Omega \) will be an open bounded set with \( C^1 \) boundary. Notice that, since \( n \leq 7 \) we must have that \( \partial E(j) \cap \Omega = \partial^* E(j) \cap \Omega \) for every \( j \neq 0 \). We strongly make use of this fact during our proof of Theorem 0.2. We divide the section in two parts: the first part where we prove some general facts holding in dimension \( n \leq 7 \) and the second part where we focus on the planar case. By putting together all the results in this section, the proof of Theorem 0.2 is achieved.

### 2.1. Some general facts in dimension \( n \leq 7 \)

This subsection is totally devoted to the proof of the following Theorem.

**Theorem 2.1** (Void structure). Let \( n \leq 7 \), \( N \geq 2 \) and \( \Omega \) be an open, bounded set with \( C^1 \) boundary. If \( \mathcal{E} \) is a Cheeger \( N \)-cluster for \( \Omega \) then the following statements hold true:

(a) \( |\mathcal{E}(0)| > 0 \) and \( H^{n-1}(\partial^* \mathcal{E}(0) \cap \partial \mathcal{E}(j)) > 0 \) for every \( i = 1, \ldots, N \);

(b) \( \partial \mathcal{E}(0) = \partial^* \mathcal{E}(0) \cup \zeta_0 \);

(c) \( \zeta_0 \) is closed and
\[ \zeta_0 \cap \Omega = \bigcup_{j,k=1}^{N} \text{bd}(\partial \mathcal{E}(j) \cap \partial \mathcal{E}(k)), \quad (2.1) \]
\[ \zeta_0 \cap \partial \Omega = \bigcup_{j=1}^{N} \text{bd}(\partial \Omega \cap \partial \mathcal{E}(j)); \quad (2.2) \]

(d) If \( E \subset \subset \Omega \) is an indecomposable component of \( \mathcal{E}(0) \) with \( E \cap \partial \Omega = \emptyset \), then \( E \) confine at least with three different chambers.
Figure 2.1. The set $\Omega_0$ built as the union of the interior of the Cheeger 2-cluster of an open set $\Omega$. The void chamber is empty in the case of $\Omega_0$ because of the cusps at the boundary of the open set.

As will be clear from the proof of Proposition 2.1, the regularity assumption on $\Omega$ cannot be avoided for the proof of assertion (a). Given a set $\Omega$ and its Cheeger $N$-cluster $E$, we build a counterexample by defining the new open set

$$\Omega_0 = \text{int} \left( \bigcup_{j=1}^{N} E(j) \right),$$

where $\text{int}(\cdot)$ means the interior. The Cheeger $N$-cluster of $\Omega_0$ will be $E$ again, and by construction $|E(0)| = 0$ contradicting assertion (a) of Theorem 2.1. The reason is that $\Omega_0$ has no regular boundary (see figure 2.1). We will prove Theorem 2.1 by putting together some preliminaries results about Cheeger $N$-Clusters. We start by proving the following Lemma, that will be very useful in many different parts of the paper.

Lemma 2.1 (Infiltration Lemma for Cheeger Sets). Let $E$ be a Borel set such that

$$h(E) = \frac{P(E)}{|E|}.$$  \hspace{1cm} (2.3)

Then there exists $\varepsilon_0, s_0$ depending on $E$ such that if for some $x \in \mathbb{R}^n$, $s \leq s_0$ holds

$$|E \cap B_s(x)| \leq \varepsilon_0 s^n$$  \hspace{1cm} (2.4)

then

$$|E \cap B_\frac{s}{2}(x)| = 0.$$  \hspace{1cm} (2.5)

Proof. Let $x$ be a point of $\mathbb{R}^n$ and define $F_s := E \setminus B_s(x)$ for $s > 0$. Note that, because of (2.3), for almost every $s > 0$ holds:

$$\frac{P(E)}{|E|} \leq \frac{P(F_s)}{|F_s|}.$$  

$$\frac{P(E)}{|E|} \leq \frac{P(E \setminus B_s)}{|E| - |E \cap B_s|}.$$  

$$\frac{P(E)}{|E|} \leq \frac{P(E) - P(E; B_s) + \mathcal{H}^{n-1}(E \cap \partial B_s)}{|E| - |E \cap B_s|}.$$  

$$-P(E)|E \cap B_s| \leq -|E|P(E; B_s) + |E|\mathcal{H}^{n-1}(E \cap \partial B_s)$$  

$$P(E; B_s) \leq h(E)|E \cap B_s| + \mathcal{H}^{n-1}(E \cap \partial B_s)$$  

$$P(E \cap B_s) \leq h(E)|E \cap B_s| + 2\mathcal{H}^{n-1}(E \cap \partial B_s).$$
By denoting with \( f(s) = |E \cap B_s(x)| \), the coarea formula implies
\[
\frac{1}{n} \omega^{n-1}_n f(s) \leq h(E) f(s) + 2f'(s). \tag{2.6}
\]
Assume that (2.4) holds for some \( \varepsilon_0 \) and \( s \leq s_0 \) to be fixed and note that (2.6) implies
\[
\frac{1}{n} \omega^{n-1}_n f(s) \leq h(E) f(s) + 2f'(s)
\]
\[
\frac{1}{n} \omega^{n-1}_n f(s) \leq (h(E) \varepsilon_0^1 s_0) f(s) + 2f'(s)
\]
\[
\left( \frac{1}{n} \omega^{n-1}_n - h(E) \varepsilon_0^1 s_0 \right) f(s) \leq 2f'(s).
\]
If \([s_*, +\infty)\) is the support of \( f(s) \) and we assume \( s > s_* \), by dividing the last inequality for \( f(s) > 0 \) and integrating it on \([s_*, s]\) we get
\[
\frac{1}{2} \left( \frac{1}{n} \omega^{n-1}_n - h(E) \varepsilon_0^1 s_0 \right) (s - s_*) \leq n f(s) \frac{1}{n}
\]
and applying (2.4) again
\[
\frac{1}{2n} \left( \frac{1}{n} \omega^{n-1}_n - h(E) \varepsilon_0^1 s_0 \right) (s - s_*) \leq \varepsilon_0 \frac{1}{n} s.
\tag{2.7}
\]
We can now choose \( \varepsilon_0, s_0 \) in a way that (2.7) implies \( \varepsilon_0 \leq s_* \). \( \square \)

**Remark 2.1.** Notice that property (1.3) immediately implies that every chamber of a Cheeger \( N \)-cluster satisfies (2.3) of Lemma (2.1) and thus for every \( E \) Cheeger \( N \)-cluster it there exists two constants \( \varepsilon_0, s_0 \) depending on \( E \) such that if for some \( i = 1, \ldots, N, x \in \mathbb{R}^n, s \leq s_0 \) holds
\[
|E(i) \cap B_s(x)| \leq \varepsilon_0 s^n,
\]
then
\[
\left| E(i) \cap B_{\frac{s}{2}}(x) \right| = 0.
\]

We are ready to prove all the intermediate results, needed to prove Theorem 2.1.

**Proposition 2.1.** Let \( n \leq 7, N \geq 2 \) and \( E \) be a Cheeger \( N \)-cluster for \( \Omega \). Then for every \( i \in \{1, \ldots, N\} \) there exists \( x \in \partial E(i) \) such that \( |B_s(x) \cap E(i)| > 0 \) for all \( s > 0 \).

**Proof.** Choose a set \( E(i) \) and denote with \( S \) an indecomposable component of \( \partial E(i) \). Set
\[
J := \{ j \in \{0, \ldots, N\} \mid j \neq i, S \cap \partial E(j) \neq \emptyset \},
\]
and note that without loss of generality we can always choose an indecomposable component of \( \partial E(i) \) for which \( \mathcal{H}^0(J) \neq 0 \). Indeed if we have chosen a component for which \( \mathcal{H}^0(J) = 0 \), the only possibility is \( S = \partial S, \) and somewhere in \( \Omega \) it must exists another component of \( \partial E(i) \). Clearly, since \( E(0) \) is open, is enough to show \( 0 \in J \).

Let us first prove that if \( \mathcal{H}^0(J) = 1 \) then \( J = \{0\} \). Let \( j \in J \) be the only index for which \( \partial E(j) \cap S \neq \emptyset \). If \( S \cap \partial \Omega = \emptyset \) then \( S = S \cap \partial E(j) \) is a closed \( C^{1,\gamma} \) surface contained in \( \Omega \) and disjoint from the other sets \( E(k), \partial \Omega, E(0), \partial E(0) \) which means that one of the situation of figure 2.2 has to be in force. We are thus able to move a little bit \( S \), and whatever is bounded by \( S \), inside \( \Omega \) (as shown in the figure) until it do not kiss \( \partial E(i) \) or \( \partial E(j) \). In this way we produce a zero-density point for \( E(i) \) or for \( E(j) \) without changing \( \sum_j \frac{P(E(j))}{|E(j)|} \) and this contradicts the regularity.

So \( \partial \Omega \neq \partial \Omega \cap S \neq \emptyset \) and this means that forceedly bd\((\partial E(j) \cap S \cap \Omega) \subset \partial \Omega \). If \( j \neq 0 \) we
Figure 2.2. If $\mathcal{H}^0(J) = 1$ and you are assuming $\partial \Omega \cap S = \emptyset$ one of this two situation occurs. We exclude them just by moving $S$ in order to produce a zero density point for $\mathcal{E}(i)$ or $\mathcal{E}(j)$ contradicting the regularity.

now bump into a contradiction because of Proposition 1.1 and thanks to the regularity of $\partial \Omega$. Indeed, Proposition 1.1 would imply that every point $x \in \text{bd}(\mathcal{E}(j) \cap S \cap \Omega)$ would be a point of density $\frac{1}{2}$ for $\Omega$, while the regularity of each $\partial \mathcal{E}(j), \partial \mathcal{E}(i)$ tells that $x$ is actually a point of density $\frac{1}{2}$ also for $\mathcal{E}(i)$ and $\mathcal{E}(j)$. This is a contradiction since $\mathcal{E}(i)$ and $\mathcal{E}(j)$ are disjoint.

So we can assume $\mathcal{H}^0(J) \geq 2$. In this situation forcedly $0 \in J$. Indeed, if we assume not and we consider $0 \neq j, k \in J$, then the two different interfaces $S \cap \partial \mathcal{E}(j)$ and $S \cap \partial \mathcal{E}(k)$ with $j \neq k \neq i$, will intersect somewhere in a point $x \in \partial \mathcal{E}(k) \cap \partial \mathcal{E}(j) \cap \partial \mathcal{E}(i)$ with density bigger than 1.

□

Proposition 2.2. Let $\Omega$ be an open, bounded set with $C^1$ boundary. Let $n \leq 7, N \geq 2$ and $\mathcal{E}$ be a Cheeger $N$-cluster for $\Omega$. Let $\zeta_0$ be the set of cusps for the void chamber defined in 0.10. Then

a) if $x \in \zeta_0 \cap \Omega$ it there exists $j, k \in \{1, \ldots, N\}$ such that $x \in \text{bd}(\partial \mathcal{E}(k) \cap \partial \mathcal{E}(j));$

b) if $x \in \zeta_0 \cap \partial \Omega$ it there exists $i \in \{1, \ldots, N\}$ such that $x \in \text{bd}(\partial \mathcal{E}(i) \cap \partial \Omega)$.

Proof. We start focusing on case a) when $x \in \zeta_0 \cap \Omega$. Case b) will follow with a slight modification of the same argument. Let us divide this proof in two parts.

Step 1) it there exist $j, k \in \{1, \ldots, N\}$ for which $x \in \partial \mathcal{E}(j) \cap \partial \mathcal{E}(k)$. Since $x \in \Omega$ which is partitioned by $\{\mathcal{E}(j)\}^N_{j=0}$, then $x \in \mathcal{E}(j)$ for some $j$. Moreover $x \in \zeta_0 \cap \Omega \subset \partial \mathcal{E}(0) \cap \Omega$ and so $j \neq 0$ (because $\mathcal{E}(0) \cap \Omega$ is an open set and so it’s disjoint from his boundary). So $x \in \mathcal{E}(j) \cap \partial \mathcal{E}(0) \cap \Omega$ which means $x \in \partial \mathcal{E}(j)$, since it is possible to get close and close to $x$ with a sequence of point in $\mathcal{E}(0) \subseteq \mathcal{E}(j)^c$. From the fact that $x \in \zeta_0 \cap \partial \mathcal{E}(j)$ we get:

$$1 = \lim_{r \to 0^+} \frac{|\Omega \cap B_r(x)|}{\omega_n r^n}$$

$$= \lim_{r \to 0^+} \frac{1}{\omega_n r^n} \sum_{k=0}^{N} |\mathcal{E}(k) \cap B_r(x)|$$

$$= \frac{1}{2} + \lim_{r \to 0^+} \frac{1}{\omega_n r^n} \sum_{k=0, j}^{N} |\mathcal{E}(k) \cap B_r(x)|$$

$$= \frac{1}{2} + \sum_{k \neq 0, j}^{N} \vartheta_n(x, \mathcal{E}(k)). \quad (2.8)$$
Here we have used the fact that the $n$-dimensional densities of the sets $\mathcal{E}(k)$ are well defined at $x$ because of Theorem 1.2 and since $n \leq 7$. Moreover:

$$\partial_n(x, \mathcal{E}(k)) = \begin{cases} 0 & \text{if } x \notin \mathcal{E}(k); \\ \frac{1}{7} & \text{if } x \in \partial \mathcal{E}(k); \\ 1 & \text{if } x \in \mathcal{E}(k) \setminus \partial \mathcal{E}(k). \end{cases}$$

This means that exactly one term $k \in \{1, \ldots, N\}$, $k \neq j$ in the sum (2.8) has to be different from zero and equal to $\frac{1}{7}$, implying $x \in \partial \mathcal{E}(k)$. So $x \in \partial \mathcal{E}(k) \cap \partial \mathcal{E}(j)$ and we conclude the Step 1).

**Step 2**: $x \in \partial(\partial \mathcal{E}(j) \cap \partial \mathcal{E}(k))$. Define the sets

$$E_r(k, j) = B_r \cap (\partial \mathcal{E}(k) \setminus \partial \mathcal{E}(j)), \quad E_r(j, k) = B_r \cap (\partial \mathcal{E}(j) \setminus \partial \mathcal{E}(k)).$$

We infer that at least one of these two sets should have positive $H^{n-1}$ Hausdorff measure. Assume that for some $r > 0$ holds that $H^{n-1}(E_r(k, j)) = H^{n-1}(E_r(j, k)) = 0$ so that $H^{n-1}(B_r \cap (\partial \mathcal{E}(j) \Delta \partial \mathcal{E}(k))) = 0$. Thanks to the Infiltration Lemma 2.1, Remark 2.1 and the regularity of $\partial \mathcal{E}(i)$ for $i \neq 0$ up to decrease $r$ we can infer $B_s(x) \cap \partial \mathcal{E}(i) = \emptyset$ for $i \neq 0, j, k$, for $s \leq r$. Thanks to (1.8) in Lemma 1.1 (precisely, we are making use of (1.16) in the proof of Lemma 1.1), we can obtain

$$P(\mathcal{E}(0); B_s(x)) = H^{n-1}(B_s(x) \cap (\partial \mathcal{E}(j) \cup \partial \mathcal{E}(k)) \setminus (\partial \mathcal{E}(j) \cap \partial \mathcal{E}(k)))$$

$$= H^{n-1}(B_s(x) \cap (\partial \mathcal{E}(j) \Delta \partial \mathcal{E}(k)))$$

$$= 0,$$

(2.9)

for $s \leq r$. If we now fix an $s \leq r$ we bump into a contradiction because of the relative isoperimetric inequality (see for example Proposition 12.37 in [14]). Indeed, since $\mathcal{E}(0)$ is open, clearly $0 < |B_s(x) \cap \partial \mathcal{E}(0)|$. Moreover $x \in \zeta_0$ and so we can find a $t = t(s) \in (0, 1)$ such that $|B_s(x) \cap \partial \mathcal{E}(0)| \leq t(s)|B_s(x)|$. In particular the relative isoperimetric inequality ensures that it there exists a positive constant $c = c(n, t(s))$ such that

$$P(\mathcal{E}(0); B_s(x)) \geq c|\mathcal{E}(0) \cap B_s(x)|^{n-1} > 0,$$

and this is contradicting (2.9). Hence

$$0 < H^{n-1}(B_s(x) \cap (\partial \mathcal{E}(k) \Delta \partial \mathcal{E}(j))) = H^{n-1}(E_r(k, j)) + H^{n-1}(E_r(j, k)), \quad \forall r > 0$$

and, since $H^{n-1}(E_r(k, j)), H^{n-1}(E_r(j, k))$ are decreasing quantity, at least one of them has to be strictly positive for all $r$.

Assume without loss of generality that $H^{n-1}(E_r(j, k)) > 0$ and for every $m \in \mathbb{N}$ pick $y_m \in E_{\frac{1}{2}}(j, k)$. We have built a sequence

$$\{y_m\}_{m \in \mathbb{N}} \subset \partial \mathcal{E}(j) \setminus (\partial \mathcal{E}(k) \cap \partial \mathcal{E}(j)),$$

$$y_m \rightarrow x,$$

and thus $x \in \overline{\partial \mathcal{E}(j) \setminus (\partial \mathcal{E}(k) \cap \partial \mathcal{E}(j))}$. By Step 1):

$$x \in (\partial \mathcal{E}(j) \cap \partial \mathcal{E}(k)) \setminus (\partial \mathcal{E}(k) \cap \partial \mathcal{E}(j)) \subset \partial(\partial \mathcal{E}(j) \cap \partial \mathcal{E}(k)).$$

Consider now case b) when $x \in \zeta_0 \cap \partial \Omega$ and observe that if $x \notin \partial \mathcal{E}(j)$ for some $j \neq 0$ arguing as in the step one we should have $\partial_n(x, \mathcal{E}(0)) = \frac{1}{2}$ and so $x \in \partial \Omega \cap \partial \mathcal{E}(j)$ for some $j \neq 0$. Finally, the argument of step two run also for

$$E_r(\Omega, j) = B_r \cap (\partial \Omega \setminus \partial \mathcal{E}(j)), \quad E_r(j, \Omega) = B_r \cap (\partial \mathcal{E}(j) \setminus \partial \Omega),$$

and we achieve the proof. \(\square\)

We are now ready to prove Theorem 2.1.
Proof of theorem 2.1. For technical reason we start by proving Assertion (b) first.

Assertion (b). Let \( x \in \partial \mathcal{E}(0) \), which is not empty according to Proposition 2.1. If \( x \in \Omega \) arguing as in the Step 1) of the proof of Proposition 2.2, the only possibilities are that

a) it there exist only one index \( j \) such that \( x \in \partial \mathcal{E}(j) \) so that \( \vartheta_n(x, \mathcal{E}(0)) = \frac{1}{r} \) and the blow up of the set \( \partial \mathcal{E}(0) \) at \( x \) converges to the tangent space \( T_x \partial \mathcal{E}(j) \) which means \( x \in \partial^* \mathcal{E}(0) \);

b) it there exists exactly two indexes \( j, k \) such that \( x \in \partial \mathcal{E}(j) \cap \partial \mathcal{E}(k) \) and so \( \vartheta_n(x, \mathcal{E}(0)) = 0 \) which means \( x \in \zeta_0 \).

If instead \( x \in \partial \Omega \) then again either a) holds with \( \vartheta_n(x, \mathcal{E}(0)) = 0 \) thanks to Proposition 1.1, or \( \vartheta_n(x, \mathcal{E}(0)) = \vartheta_n(x, \Omega) = \frac{1}{r} \) and the blow up of the set \( \mathcal{E}(0) \) at \( x \) is converging to the tangent space \( T_x \partial \Omega \) which means again \( x \in \partial^* \mathcal{E}(0) \). So \( \partial \mathcal{E}(0) \subseteq \zeta_0 \cup \partial^* \mathcal{E}(0) \). The reverse inclusion is trivial.

Assertion (a). From Proposition 2.1 immediately follows that \( |\mathcal{E}(0)| > 0 \) and that for every fixed \( j \in \{1, \ldots, N\} \) we find an \( x \in \partial \mathcal{E}(j) \) in a way that \( |\partial \mathcal{E}(x) \cap \mathcal{E}(0)| > 0 \) for all \( r > 0 \). Because of Assertion (b) and since \( x \in \partial \mathcal{E}(0) \), if \( x \notin \zeta_0 \) then \( x \in \partial^\ast \mathcal{E}(0) \).

In particular because of Lemma 2.1 and Remark 2.1, for some \( r \) small enough it must holds \( \partial^* \mathcal{E}(0) \cap \partial \mathcal{E}(j) = \partial \mathcal{E}(j) \cap \partial \mathcal{E}(0) \). This imply
\[
\mathcal{H}^{n-1}(\partial^* \mathcal{E}(0) \cap \partial \mathcal{E}(j)) > \mathcal{H}^{n-1}(\partial^* \mathcal{E}(0) \cap \partial \mathcal{E}(j) \cap \partial \mathcal{E}(x)) = \mathcal{H}^{n-1}(\partial \mathcal{E}(j) \cap \partial \mathcal{E}(x)) > 0.
\]

If instead \( x \in \zeta_0 \) arguing as in step two of the proof of Proposition 2.2 we can find a sequence of point \( \{x_m\}_{m \in \mathbb{N}} \subset \partial \mathcal{E}(0) \cap \partial \mathcal{E}(j) \backslash \zeta_0 \) converging to \( x \). Then for \( m \) large enough and a suitable \( s_m \), since \( x_m \notin \zeta_0 \), arguing as before:
\[
\mathcal{H}^{n-1}(\partial^* \mathcal{E}(0) \cap \partial \mathcal{E}(j)) \geq \mathcal{H}^{n-1}(\partial^* \mathcal{E}(0) \cap \partial \mathcal{E}(j) \cap \partial \mathcal{E}(x)) \geq \mathcal{H}^{n-1}(\partial^* \mathcal{E}(0) \cap \partial \mathcal{E}(j) \cap \partial \mathcal{E}(s_m(x_m))) > 0.
\]

Assertion (c). Assume first that \( x \in \Omega \) and let us denote with
\[
S = \bigcup_{j,k=1, j \neq k}^N \text{bd}(\partial \mathcal{E}(j) \cap \partial \mathcal{E}(k)).
\]

Thanks to Proposition 2.2 we immediately get
\[
\zeta_0 \cap \Omega \subset S.
\]

For the reverse inclusion fix
\[
x \in \text{bd}(\partial \mathcal{E}(j) \cap \partial \mathcal{E}(k)), \text{ for some } j, k \in \{1, \ldots, N\}.
\]

We just need to show that \( x \in \partial \mathcal{E}(0) \), since it is obvious that \( \vartheta_n(x, \mathcal{E}(0)) = 0 \), and for do that it is enough to show that we can find a sequence of point in \( \partial \mathcal{E}(0) \) or in \( \mathcal{E}(0) \) converging to \( x \). As a consequence of (2.10) we get \( x \in \partial \mathcal{E}(j) \setminus \partial \mathcal{E}(k) \) so it must exist a sequence \( \{x_m\}_{m \in \mathbb{N}} \subset \partial \mathcal{E}(j) \setminus \partial \mathcal{E}(k) \) converging to \( x \). Up to a subsequence, since \( \{x_m\}_{m \in \mathbb{N}} \notin \mathcal{E}(0) \), it must holds one of the following:
\[
\{x_m\}_{m \in \mathbb{N}} \subset \partial \mathcal{E}(j) \cap \partial \mathcal{E}(i) \text{ for some } i \neq k \text{ and } i \neq 0 \tag{2.11}
\]
\[
\{x_m\}_{m \in \mathbb{N}} \subset \partial \mathcal{E}(j) \cap \partial \mathcal{E}(0). \tag{2.12}
\]

If (2.11) would be in force we should have \( x \in \mathcal{E}(i) \) too and thus \( \vartheta_n(x, \Omega) > 1 \). So (2.12) must be in force, \( x_m \in \partial \mathcal{E}(0) \) and \( x \in \partial \mathcal{E}(0) \). We argue in the same way if \( x \in \partial \Omega \).

Assertion (d). Let \( E \) be a generic indecomposable component of \( \mathcal{E}(0) \). Assume that \( E \).
share its boundary with exactly two other different chambers $j, k$ and $E \cap \partial \Omega = \emptyset$. Then either

a) $\mathcal{H}^{n-1}(\partial^* E \cap \partial \mathcal{E}(j)) \geq \mathcal{H}^{n-1}(\partial^* E \cap \partial \mathcal{E}(k))$,

or

b) $\mathcal{H}^{n-1}(\partial^* E \cap \partial \mathcal{E}(k)) \geq \mathcal{H}^{n-1}(\partial^* E \cap \partial \mathcal{E}(j))$.

Assume that a) holds and define $\mathcal{E}_1(j) := \mathcal{E}(j) \cup E$, $\mathcal{E}_1(i) = \mathcal{E}(i)$ for $i \neq j$. Then, since

$$P(\mathcal{E}_1(j)) = P(\mathcal{E}(j)) - \mathcal{H}^{n-1}(\partial^* E \cap \partial \mathcal{E}(j)) + \mathcal{H}^{n-1}(\partial^* E \cap \partial \mathcal{E}(k))$$

because of $\mathcal{H}^{n-1}(\partial \Omega \cap \partial^* E) = 0$, we get:

$$H_N(\Omega) \leq \sum_{i=1}^N \frac{P(\mathcal{E}_1(i))}{|\mathcal{E}_1(i)|} = \frac{P(\mathcal{E}(j))}{|\mathcal{E}(j)| + |E|} + \sum_{i \neq j} \frac{P(\mathcal{E}_1(i))}{|\mathcal{E}_1(i)|} \leq \frac{P(\mathcal{E}(j))}{|\mathcal{E}(j)| + |E|} + \sum_{i \neq j} \frac{P(\mathcal{E}_1(i))}{|\mathcal{E}_1(i)|}.$$
Step 2). We show now that for every point \( x \in \zeta_0 \cap \Omega \) there exists a constant \( r_0(x) \) such that:

\[
\mathcal{H}^0(\zeta_0 \cap B_r(x)) \leq 2 \quad \forall \ r < r_0.
\] (2.13)

Choose a point \( x \in \zeta_0 \cap \Omega \) and let \( r \) be a radius small enough such that \( B_r(x) \subset \subset \Omega \). Thanks to theorem 2.1, up to a relabeling we have that \( x \in \text{bd}(\partial E(1) \cap \partial E(2)) \). Thanks to the Infiltration Lemma 2.1 and to Remark 2.1 we find a constant \( r_0 \) such that for \( r < r_0 \) we have \( E(j) \cap B_r = \emptyset \) for every \( j \neq 0, 1, 2 \) and \( r < r_0 \). Up to decrease again \( r \), we can also assume that \( B_r \cap E(1) \) and \( B_r \cap E(2) \) are connected and so if three contact points between the chambers are all lying in the ball \( B_r(x) \), we forcedly must have an indecomposable component of \( E(0) \) confining with only two chambers (see figure 2.4) and contradicting Assertion (d) of Theorem 2.1. Thus (2.13) must be in force.

Step 3). Step 1 is telling us that \( \zeta_0 \cap \Omega \) has no accumulation point at \( \partial \Omega \), while Step 2) allow us to conclude that \( \zeta_0 \cap \Omega \) has no accumulation point inside \( \Omega \). Thus \( \zeta_0 \cap \Omega \) has no accumulation point, which means that

\[
0 < \inf_{x \in \zeta_0 \cap \Omega} \{ r_0(x) \mid (2.13) \text{ is in force on } x \text{ with } r_0(x) \}
\]

and thanks to the boundedness of \( \Omega \) we achieve the proof.

Let us notice that there is no hope to prove such a theorem for \( \zeta_0 \cap \partial \Omega \). Indeed we can always modify \( \Omega \) at the boundary in order to produce a set \( \Omega_0 \) having the same Cheeger \( N \)-clusters of \( \Omega \) and kissing the boundary of some \( \partial E(i) \) in a countable number of points (see figure 2.5). The proof of the following proposition holds only for the planar case.

**Proposition 2.3.** Let \( E \) be a Cheeger set for an open bounded set \( \Omega \subset \mathbb{R}^2 \). Assume that the following properties holds for \( E \):

1) every \( x \in \partial \Omega \cap \partial E \) is a regular point for \( \partial \Omega \);
2) \( \mathcal{H}^0(\text{bd}(\partial \Omega \cap \partial E)) < +\infty \).

Then \( \mathcal{H}^1(\partial E \cap \partial \Omega) > 0 \).

**Proof.** Assume by contradiction that \( \mathcal{H}^1(\partial E \cap \partial \Omega) = 0 \) and let \( x \in \partial E \cap \partial \Omega \). By the inner regularity for planar Cheeger sets and thanks to hypothesis 2) we conclude that \( \partial E \) is a finite union of circular arcs with radius \( \frac{1}{\mathcal{H}(E)} \). Hypothesis 1) combined with Proposition
By gently pushing $\partial \Omega$ in some void part we can build as many contact points as we want. This proves that a theorem in the spirit of 2.2 is not possible in the general case.

The convention of the curvature. As it is clear from the picture $C_{2,1} = -C_{1,2}$.

2.5 (Assertion (vii)) in [13] tells us that the arcs are all from the same circle and so we conclude that $E = B_{\frac{1}{h(\Omega)}}(x)$ for some $x \in \Omega$. But then

$$\frac{P(E)}{|E|} = h(\Omega)$$

$$\frac{2\pi}{\frac{x}{h(\Omega)}} = h(\Omega)$$

$$\frac{\frac{2}{h(\Omega)}}{h(\Omega)^2} = h(\Omega)$$

$$2h(\Omega) = h(\Omega)$$

$$2 = 1,$$

because $h(\Omega) \neq 0$ and this is a contradiction.

We now exploit the stationarity of the Cheeger $N$-clusters to derive information on their structure. The following theorem holds also in higher ambient space dimension, with the same arguments.
Theorem 2.3. Let $\Omega \subset \mathbb{R}^2$ be an open set with $C^1$ boundary and $\mathcal{E}$ one of its Cheeger $N$-cluster. Let $i \in \{1, \ldots, N\}$ be fixed. Then

(i) For every $k \in \{1, \ldots, N\}$, $k \neq i$ such that $\mathcal{H}^1(\partial\mathcal{E}(i) \cap \partial\mathcal{E}(k) \cap \Omega) > 0$ we have that $\partial\mathcal{E}(i) \cap \partial\mathcal{E}(k) \cap \Omega$ is a finite union of circular arcs with signed curvature:

$$C_{i,k} = \frac{|\mathcal{E}(k)|h(\mathcal{E}(i)) - |\mathcal{E}(i)|h(\mathcal{E}(k))}{|\mathcal{E}(i)| + |\mathcal{E}(k)|},$$

with the convention as in figure 2.6;

(ii) The set $(\partial\mathcal{E}(0) \cap \partial\mathcal{E}(i) \cap \Omega)$ is a a finite union of circular arcs with curvature:

$$C_{i,0} = h(\mathcal{E}(i)).$$

Proof. Consider a $k \in \{1, \ldots, N\}$ such that $\mathcal{H}^1(\partial\mathcal{E}(i) \cap \partial\mathcal{E}(k) \cap \Omega) > 0$. Consider a ball centered at $x \in \partial\mathcal{E}(i) \cap \partial\mathcal{E}(k) \cap \Omega$ small enough such that $B_r(x) \cap \partial\mathcal{E}(i) \cap \partial\mathcal{E}(k) \cap \Omega = \emptyset$. Consider a map $T \in C^\infty_c(B_r; \mathbb{R}^2)$ and define for all $t < \varepsilon$, the diffeomorphism $f_t(x) = x + tT(x)$ and the cluster $\mathcal{E}_t := \{f_t(\mathcal{E}(i))\}_{i=1}^N$. Of course $\mathcal{E}_t \Delta \mathcal{E} \subset B_r$. By minimality also holds:

$$\frac{P(\mathcal{E}(i))}{|\mathcal{E}(i)|} + \frac{P(\mathcal{E}(k))}{|\mathcal{E}(k)|} \leq \frac{P(\mathcal{E}_t(i))}{|\mathcal{E}_t(i)|} + \frac{P(\mathcal{E}_t(k))}{|\mathcal{E}_t(k)|}, \quad \forall |t| < \varepsilon.$$ 

Thus

$$0 \leq \frac{d}{dt} \bigg|_{t=0} \frac{P(\mathcal{E}(i))}{|\mathcal{E}(i)|} + \frac{d}{dt} \bigg|_{t=0} \frac{P(\mathcal{E}(k))}{|\mathcal{E}(k)|}.$$ 

For a small $r > 0$ the interface $B_r(x) \cap \partial\mathcal{E}(i) \cap \partial\mathcal{E}(k) \cap \Omega$ solves an isoperimetric problem with a volume constrained so, by the well-known regularity for this kind of objects (see for example [14]), it is a circular arc and its signed curvature $C_{i,k}$ is counted positive if the outer unit normal of the circular arc coincide with the outer unit normal of $\mathcal{E}(i)$ (see figure 2.6). With some easy computations

$$\frac{d}{dt} \bigg|_{t=0} \frac{P(\mathcal{E}(i))}{|\mathcal{E}(i)|} = \frac{|\mathcal{E}(i)| \frac{d}{dt} \bigg|_{t=0} P(\mathcal{E}(i)) - P(\mathcal{E}(i)) \frac{d}{dt} \bigg|_{t=0} |\mathcal{E}(i)|}{|\mathcal{E}(i)|^2},$$

$$\frac{d}{dt} \bigg|_{t=0} P(\mathcal{E}(i)) = C_{i,k} \int_{\partial\mathcal{E}(i) \cap \partial\mathcal{E}(k) \cap B_r} (T(x) \cdot \nu_{\mathcal{E}(i)}(x)) d\mathcal{H}^1(x),$$

$$\frac{d}{dt} \bigg|_{t=0} |\mathcal{E}(i)| = \int_{\partial\mathcal{E}(i) \cap \partial\mathcal{E}(k) \cap B_r} (T(x) \cdot \nu_{\mathcal{E}(i)}(x)) d\mathcal{H}^1(x).$$

So by denoting with

$$f_i = \int_{\partial\mathcal{E}(i) \cap \partial\mathcal{E}(k) \cap B_r} (T(x) \cdot \nu_{\mathcal{E}(i)}(x)) d\mathcal{H}^1(x)$$

$$f_k = \int_{\partial\mathcal{E}(i) \cap \partial\mathcal{E}(k) \cap B_r} (T(x) \cdot \nu_{\mathcal{E}(k)}(x)) d\mathcal{H}^1(x),$$

we can write:

$$\frac{d}{dt} \bigg|_{t=0} \frac{P(\mathcal{E}(i))}{|\mathcal{E}(i)|} = \frac{|\mathcal{E}(i)| f_i C_{i,k} - P(\mathcal{E}(i)) f_i}{|\mathcal{E}(i)|^2},$$

$$\frac{d}{dt} \bigg|_{t=0} \frac{P(\mathcal{E}(k))}{|\mathcal{E}(k)|} = \frac{|\mathcal{E}(k)| f_k C_{i,k} - P(\mathcal{E}(k)) f_k}{|\mathcal{E}(k)|^2}.$$
By observing that \( f_i = -f_k \) we are lead to the relation:

\[
0 \leq \left. \frac{d}{dt} \right|_{t=0} \frac{P(\mathcal{E}(i))}{|E(i)|} + \left. \frac{d}{dt} \right|_{t=0} \frac{P(\mathcal{E}(k))}{|E(k)|} = \frac{|E(i)|}{|E(i)|^2} \cdot |E(i)| f_i C_{i,k} - P(\mathcal{E}(i)) f_i + \frac{|E(k)|}{|E(k)|^2} \cdot |E(k)| f_k C_{k,i} - P(\mathcal{E}(k)) f_k
\]

\[
= f_i \left[ \frac{|E(i)|}{|E(i)|^2} \cdot |E(i)| - \frac{|E(k)|}{|E(k)|^2} \cdot |E(k)| \right].
\]

Now by choosing a \( T_1 \) such that \( f_i \) is positive and then a \( T_2 \) such that \( f_i \) is negative we conclude that

\[
0 = \frac{|E(i)| C_{i,k} - P(\mathcal{E}(i))}{|E(i)|^2} - \frac{|E(k)| C_{k,i} - P(\mathcal{E}(k))}{|E(k)|^2}.
\]

Finally by exploiting \( C_{i,k} = -C_{k,i} \)

\[
0 = \frac{C_{i,k}}{|E(i)|} - \frac{P(\mathcal{E}(i))}{|E(i)|^2} + \frac{C_{k,i}}{|E(k)|} + \frac{P(\mathcal{E}(k))}{|E(k)|^2}
\]

\[
= \frac{C_{i,k}}{|E(i)|} - \frac{P(\mathcal{E}(i))}{|E(i)|^2} + \frac{C_{i,k}}{|E(k)|} + \frac{P(\mathcal{E}(k))}{|E(k)|^2},
\]

\[
C_{i,k}(|E(i)| + |E(k)|) = h(E(i))|E(k)| - h(E(k))|E(i)|
\]

\[
C_{i,k} = \frac{h(E(i))|E(k)| - h(E(k))|E(i)|}{|E(i)| + |E(k)|}.
\]

For the other relation we just notice that \( \mathcal{E}(i) \) is a Cheeger set for \( \Omega \setminus \cup_{k \neq i} \mathcal{E}(k) \) so the free boundary have curvature \( C_{i,0} = h(E(i)) \), (see for example [16] for details about the curvature of Cheeger sets).

\[ \square \]

Our last theorem of the section put together Theorem 2.2, Proposition 2.3 and Theorem 2.4. Let \( \Omega \subset \mathbb{R}^2 \) be an open bounded set with \( C^1 \) boundary and let \( \mathcal{E} \) be a Cheeger \( N \)-cluster for \( \Omega \). Every chamber \( \mathcal{E}(i) \) such that \( \partial \mathcal{E}(i) \cap \partial \mathcal{E} = \emptyset \) is connected.

Proof. Assume, without loss of generality \( i = 1 \) and let \( E_1 \) and \( E_2 \) be two different connected component of \( \mathcal{E}(1) \). By minimality it must holds \( \frac{P(E_1)}{|E_1|} = \frac{P(E_2)}{|E_2|} = \frac{P(\mathcal{E}(1))}{|\mathcal{E}(1)|} \). Moreover the component \( E_2 \) is a Cheeger set for \( A = \cup_{j \neq 1} \mathcal{E}(j) \cup E_1 \) and by the regularity theorem and our hypothesis on \( \partial \mathcal{E}(1) \), every \( x \in \partial E_1 \cap \partial A \) is a regular point for \( \partial A \).

Moreover, thanks to Theorem 2.2 we have \( H^0(\partial(D \Omega \cap \partial E_2)) < +\infty \) so, by Proposition 2.3 we have that \( H^1(\partial E_2 \cap \partial A) > 0 \) and in particular there exists an index \( k \) such that \( H^1(\partial E_2 \cap \partial \mathcal{E}(k)) > 0 \). Define the cluster \( \mathcal{F}(1) = E_1, \mathcal{F}(j) = \mathcal{E}(j) \) for \( j \neq 1 \). Of course it holds again

\[
H_N(\Omega) = \sum_{i=1}^{N} \frac{P(\mathcal{F}(i))}{|\mathcal{F}(i)|} \tag{2.14}
\]

and so \( \mathcal{F} \) it is also a Cheeger \( N \)-cluster for \( \Omega \). Consider the piece of boundary \( S = \partial \mathcal{E}(k) \cap \partial E_2 \). Theorem 2.3 tells us that \( S \) must be a circular arc with curvature

\[
C_{k,1} = \frac{|\mathcal{E}(1)| h(E(k)) - |\mathcal{E}(k)| h(E(1))}{|\mathcal{E}(1)| + |\mathcal{E}(k)|}.
\]
From the other side in the new cluster \( F \) the part of the boundary containing \( S \) now is confining with the void and so we must also have:

\[
C_{k,0} = h(F(k)) = h(E(k)).
\]

This implies \( C_{k,1} = C_{k,0} \) and in particular \( h(E(k)) + h(E(1)) \) is a contradiction.

Proof of Theorem 0.2. It follows from Theorem 2.1, 2.2, 2.3, 2.4.

3. The limit as \( p \) goes to one of \( \Lambda_N^{(p)} \)

The existence of an optimal partition for \( \Lambda_N^{(p)}(\Omega) \) follow with same argument as in [8], or as a consequence of some more general result in [4], [5] or [7].

**Theorem 3.1.** For every \( 1 < p < n \) there exists an optimal partition for \( \Omega \) in quasi-open sets \( \{\Omega_i\}_{i=1}^N \in \mathcal{A}_p(\Omega) \) such that

\[
\Lambda_N^{(p)}(\Omega) = \sum_{i=1}^N \lambda_N^{(p)}(\Omega_i).
\]

In the following Proposition we compute the limit as \( p \) goes to one of \( \Lambda_N^{(p)} \).

**Proposition 3.1.** If \( \Omega \) is an open bounded set with Lipschitz boundary then

\[
\lim_{p \to 1} \Lambda_N^{(p)}(\Omega) = H_N(\Omega).
\]

Proof. Let \( E \) be a Cheeger \( N \)-cluster for \( \Omega \). The open subcluster

\[
E_t(i) = \{ x \in E(i) \mid d(x, \partial E(i)) > t \}
\]

is such that for every \( i \),

\[
E_t(i) \to E(i) \text{ in } L^1, \quad P(E_t(i)) \to P(E(i)).
\]

Indeed

\[
P(E_t(i)) \approx P(E(i)) + t \int_{\partial E(i)} C(x) \, d\mathcal{H}^{n-1}(x),
\]

where \( C(x) \) is the scalar mean curvature of \( \partial E(i) \) which is bounded since \( E(i) \) is a \( (\Lambda, r_0) \)-perimeter minimizing inside \( \Omega \) (thanks to Theorem 1.3), and \( \partial \Omega \) is Lipschitz.

\[
H_N(\Omega) = \lim_{t \to 0} \frac{\sum_{i=1}^N P(E_t(i))}{|E_t(i)|}
\]

(3.1)

\[
\geq \lim_{t \to 0} \sum_{i=1}^N h(E_t(i))
\]

(3.2)

\[
\geq \lim_{t \to 0} \sum_{i=1}^N \lambda_N^{(p)}(E_t(i))
\]

(3.3)

\[
\geq \lim_{p \to 1} \Lambda_N^{(p)}(\Omega).
\]

(3.4)

From the other side, thanks to (0.4) and Jensen inequality we get (0.5):

\[
\sum_{j=1}^N \lambda_N^{(p)}(E(i)) \geq \sum_{j=1}^N \left( \frac{h(E(i))}{p} \right)^p
\]

(3.5)

\[
\geq \frac{1}{N^{p-1}} \left( \sum_{j=1}^N \frac{h(E(i))}{p} \right)^p \geq \frac{1}{N^{p-1}} \left( \frac{H_N(\Omega)}{p} \right)^p
\]

(3.6)
which achieve the proof. □

3.1. On the asymptotic behavior of $H_N$ and the case of flat torus.

**Theorem 3.2** (Behaviour of $H_N$ for large $N$). For an open bounded set $\Omega$ with $C^1$ boundary the following assertions are in force:

1) If there exists $N_0(\Omega)$ and $C_0(\Omega)$ such that, denoting with $B$ the unit-radius ball and with $H$ the unit-area regular hexagon, then

$$\frac{h(B)}{\sqrt{|\Omega|}} N^\frac{3}{2} \leq H_N(\Omega) \leq \frac{h(H)}{\sqrt{|\Omega|}} N^\frac{3}{2} + C_0(\Omega) \sqrt{N}$$

for all $N \geq N_0(\Omega)$, \hspace{1cm} (3.7)

2) If $E$ is a Cheeger $(N+1)$-cluster for $\Omega$ then

$$|E_{N+1}(i)| \geq \frac{h(B)^2}{(H_{N+1}(\Omega) - H_N(\Omega))^2} \quad \forall \ i = 1, \ldots, N + 1,$$ \hspace{1cm} (3.8)

3) $H_N(\Omega) + \frac{h(B)}{\sqrt{|\Omega|}} \sqrt{N + 1} \leq H_{N+1}(\Omega)$, for all $N \in \mathbb{N}$.. \hspace{1cm} (3.9)

**Proof.** Let us prove property 1). Consider $E$, a Cheeger $N$-cluster of $\Omega$. Thanks to the Cheeger inequality

$$h(E) \geq \frac{h(B)}{\sqrt{|E|}}$$ \hspace{1cm} (3.10)

that states that the ball minimizes $h(\cdot)$ among all the Borel sets with fixed volume, we get the lower bound

$$H_N(\Omega) = \sum_{i=1}^{N} h(E(i))$$

$$\geq h(B) \sum_{i=1}^{N} \frac{1}{\sqrt{|E(i)|}}$$

$$\geq h(B) N^\frac{3}{2} \left( \frac{1}{\sum_{i=1}^{N} |E(i)|} \right)^{\frac{1}{2}}$$

$$\geq h(B) N^\frac{3}{2}.$$\hspace{1cm} (3.11)

Here we have used the inequality

$$\sum_{i=1}^{N} \frac{1}{x_i^\frac{1}{n}} \geq N^{n+1} \left( \frac{1}{\sum_{i=1}^{N} x_i} \right)^{\frac{1}{n}}, \quad \forall \ N, n \geq 2, \ x_i > 0.$$\hspace{1cm} (3.12)

Let us focus on the upper bound. For every $N$ define $\delta(N)$ to be the higher real value for which exactly $N$ disjoint up to a negligible set, scaled and roto-translated hexagons $\delta(N)\rho H$ are strictly contained in $\Omega$ (here $\rho$ denotes a rigid motion of $\mathbb{R}^2$). Denote with $\mathbb{H}$ the union of such hexagons and note that since $\mathbb{H} \subset \Omega$ we easily get

$$H_N(\Omega) \leq H_N(\mathbb{H}) \leq N h(\delta(N)\rho H) = \frac{N}{\delta(N)} h(H).$$

We just need to estimate $\delta(N)$. From $\mathbb{H} \subset \Omega$ follows

$$\delta(N) \leq \frac{\sqrt{\Omega}}{\sqrt{N}}.$$\hspace{1cm} (3.13)
Since $\delta(N)$ was chosen in a way that $\Omega$ is containing exactly $N$ hexagon of area $\delta(N)^2$ and no more, setting $r(N) = \text{diam}(\delta(N)H) = \delta(N) \text{diam}(H)$ and $$I_{r(N)}(\partial^*\Omega) := \{ x \in \Omega \mid d(x, \partial^*\Omega) < r(N) \},$$
then
$$(\Omega \setminus \mathbb{H}) \subset I_{r(N)}(\partial^*\Omega).$$
So:
$$|\Omega| - \delta(N)^2 N \leq |I_{r(N)}(\partial^*\Omega)| \leq 4r(N)P(\Omega) = 4\delta(N) \text{diam}(H)P(\Omega) \leq 4 \text{diam}(H)\sqrt{\frac{|\Omega|}{N}}P(\Omega),$$
$$\delta(N) \geq \frac{\sqrt{|\Omega|}}{\sqrt{N}} \sqrt{1 - \frac{C_0(\Omega)}{\sqrt{N}}}.$$ For all $N$ bigger than some fixed $N_0$ depending on $\Omega$ only, we get 
$$H_N(\Omega) \leq \frac{h(H)N}{\sqrt{|\Omega|}} \leq \frac{h(H)}{\sqrt{|\Omega|}} N^\frac{1}{2} + C_0(\Omega)\sqrt{N}.$$ For property 2) we exploit again the Cheeger inequality (3.10) for observe that, given $\mathcal{E}$ a Cheeger $(N + 1)$-cluster of $\Omega$, it holds:
$$H_{N+1}(\Omega) = \sum_{i=1}^{N+1} \frac{P(\mathcal{E}(i))}{|\mathcal{E}(i)|} \geq \frac{P(\mathcal{E}(j))}{|\mathcal{E}(j)|} + \sum_{i=1, i\neq j}^{N+1} \frac{P(\mathcal{E}(i))}{|\mathcal{E}(i)|} \geq \frac{h(\mathcal{E}(j))}{\sqrt{|\mathcal{E}(j)|}} + H_N(\Omega) \geq \frac{h(B)}{\sqrt{|\mathcal{E}(j)|}} + H_N(\Omega)$$
which immediately implies (3.8). The last property follows from property (3.8)
$$|\Omega| - \frac{(N + 1)h(B)}{(H_{N+1}(\Omega) - H_N(\Omega))^2} \geq |\Omega| - \sum_{i=1}^{N+1} |\mathcal{E}(i)| \geq 0$$
and so
$$|\Omega| \geq \frac{(N + 1)h(B)^2}{(H_{N+1}(\Omega) - H_N(\Omega))^2}.$$ 
$$(H_{N+1}(\Omega) - H_N(\Omega))^2 \geq \frac{(N + 1)h(B)^2}{|\Omega|}$$
$$H_{N+1}(\Omega) \geq H_N(\Omega) + \sqrt{(N + 1) \frac{h(B)}{|\Omega|}}.$$
In order to study the behavior of Cheeger $N$-clusters for big $N$, we find convenient to define the following problem on the Flat Torus $T(v, w) = \mathbb{R}^2/\sim$, where $x \sim y$ if and only if $x = kv + jw$ for some $k, j \in \mathbb{Z}$ (in the sequel we omit the dependence on $v, w$ whenever it is clear from the context):

$$H_N(T) := \inf \left\{ \sum_{i=1}^{N} \frac{P(E(i))}{|E(i)|} \mid E \subset T, \text{ } N\text{-cluster} \right\}. \quad (3.11)$$

Since $T$ is a compact manifold the existence of Cheeger $N$-clusters is easily achieved as in Theorem 1.1. Theorem 1.3, 2.1, 2.2 are in force and since holds $\partial T = \emptyset$ we note that every chamber has to be a connected (according to Theorem 2.4) finite union of circular arcs having curvature indicated from Theorem 2.3. We propose this point of view, in order to take complete advantages of the absence of the boundary for study the asymptotic property of planar Cheeger $N$-cluster. Moreover in a generic open set $\Omega$ for $N$ big enough and in a square centered in a point $x \in \Omega$ far away from the boundary, if the Caffarelli and Lin’s conjecture would be true, we expect to see for a Cheeger $N$-clusters of $\Omega$ a situation close to the case of the Flat torus. We don’t expect that the boundary will affect the behavior of Cheeger $N$-cluster, far enough from the boundary itself.

For our future purposes we would like to study the asymptotic behavior of Cheeger N-Clusters in the favorable case of the flat torus, when we have information about the connection of the chambers.

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25