Abstract

We consider a non-relativistic quantum gas of $N$ bosonic atoms confined to a box of volume $\Lambda$ in physical space. The atoms interact with each other through a pair potential whose strength is inversely proportional to the density, $\rho = \frac{N}{\Lambda}$, of the gas. We study the time evolution of coherent excitations above the ground state of the gas in a regime of large volume $\Lambda$ and small ratio $\frac{\Lambda}{\rho}$. The initial state of the gas is assumed to be close to a product state of one-particle wave functions that are approximately constant throughout the box. The initial one-particle wave function of an excitation is assumed to have a compact support independent of $\Lambda$. We derive an effective non-linear equation for the time evolution of the one-particle wave function of an excitation and establish an explicit error bound tracking the accuracy of the effective non-linear dynamics in terms of the ratio $\frac{\Lambda}{\rho}$. We conclude with a discussion of the dispersion law of low-energy excitations, recovering Bogolyubov’s well-known formula for the speed of sound in the gas, and a dynamical instability for attractive two-body potentials.

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1 Introduction

In the study of the intricate dynamics of many-body systems, it is often convenient, or actually unavoidable, to resort to simpler approximate descriptions. For quantum-mechanical many-body systems of bosons it is possible to use effective one-particle equations to track the microscopic evolution of many-particle states in appropriate regimes. This tends to reduce the complexity of the problem enormously. Of course, one has to convince oneself that the approximation introduced into the analysis is not too crude but resolves the dynamical features of interest fairly accurately. To mention an example, the interaction potential exerted on a test particle in a non-linear one-particle description of the effective dynamics of a Bose gas can be chosen self-consistently as the mean potential generated by all the other particles at the position of the test particle. The mathematical analysis of such so-called mean-field limits goes back to work by Hepp [7] (quantum many-body systems), and by Braun and Hepp [2] and Neunzert [12] (classical many-body systems). Among other results, they have shown that the Vlasov equation effectively describes a classical many-body system while the Hartree equation describes a Bose gas in the mean-field limit. After Hepp’s initial work [7] there has been a lot of effort to arrive at a mathematically rigorous understanding of quantum-mechanical mean-field limits; regarding the dynamics see, e.g., [18, 16, 13, 5, 6, 9], and regarding ground state see, e.g., [17, 4, 10] and furthermore [11] for an elaborate overview.

In order to clarify the relation between our discussion and previous studies found in the existing literature, it is necessary to first explain our conventions concerning units of physical quantities and the use of dimensionless parameters:

Remark 1.1. All physical quantities appearing in this paper are made dimensionless by expressing them in terms of (dimensionful) fundamental constants of Nature or of constants characteristic of the system under consideration. In this paper, we use units in which Planck’s constant and the mass of a gas atom are equal to unity. Furthermore, distances are expressed as multiples of the diameter of the essential support (“range”) of the two-body interaction potential, $U$, which equals 1 in our units. Consequently, to say that the volume $\Lambda$ of the region to which the gas is confined equals 1 would mean that it is comparable to the volume of the support of the two-body potential $U$. Furthermore, to say that the density fulfills $\rho = 1$ would mean that the expected number of particles inside the support of $U$ equals 1.

With these conventions the situation usually considered in the mathematical literature on mean-field limits can be described as follows: The support of wave functions is kept fixed while the scattering length of the two-body interaction scales inversely proportional to the particle number $N$ as the mean-field limit, $N \to \infty$, is approached. In the study of many physically interesting situations, e.g., of a Bose gas in the thermodynamic limit, one must, however, consider regimes where $N$ and $\Lambda$ tend to $\infty$. The mean-field regime is then approached by taking the gas density $\rho = \frac{N}{\Lambda}$ to be large and assuming that the strength of the two-body potential is $O(\rho^{-1})$; the mean-field limit corresponding to the limit $\rho \to \infty$. This ensures that the interaction energy per particle is of order one and, consequently, the velocity of sound is kept constant.

A key open problem is to show that the many-body dynamics of a gas of bosonic atoms can be controlled in terms of an effective equation for a one-particle wave function when the thermodynamic limit, $\Lambda \to \infty$, is taken at constant density $\rho$ before the mean-field regime of large $\rho$ is approached. While at the present time a satisfactory solution to this problem appears to be out of reach we propose to make a modest contribution in this direction by considering an interacting
Bose gas at zero temperature in the regime of large density $\rho$, allowing the volume $\Lambda$ to increase depending on $\rho$, in such a way that $\frac{\Lambda}{\rho} \ll 1$ as the mean-field limit is approached.

More precisely, we propose to study the microscopic time evolution of an initial $N$-particle wave function that is, in a sense to be made precise later, close to a product wave function of the form

$$\Psi_0(x_1, x_2, \ldots, x_N) = \prod_{k=1}^{N} \frac{1}{\Lambda^{1/2}} \left( \phi_0^{(\text{ref})}(x_k) + \epsilon_0(x_k) \right). \tag{1}$$

Here, $N$ is the number of atoms in the gas, and $\phi_0^{(\text{ref})}$ denotes a slowly varying, compactly supported one-particle wave function chosen such that its support occupies roughly a region of volume $\Lambda$ and its $L^\infty$ norm is kept constant as $\Lambda$ varies. Its $N$-fold product represents a so-called reference state of the gas, a (Bose-Einstein) condensate, which is then perturbed by a smooth, compactly supported wave function, $\epsilon_0$, that has a fixed scale-(or $\Lambda$-) independent support inside the support of $\phi_0^{(\text{ref})}$. The function $\epsilon_0$ is supposed to describe a localized excitation of the reference state. The time evolution of this initial state is given by the $N$-particle Schrödinger equation

$$i\partial_t \Psi_t(x_1, \ldots, x_N) = H \Psi_t(x_1, \ldots, x_N), \tag{2}$$

where the microscopic Hamiltonian, $H$, is given by

$$H := -\frac{1}{2} \sum_{k=1}^{N} \Delta_{x_k} + \frac{1}{\rho} \sum_{1 \leq j < k \leq N} U(x_j - x_k). \tag{3}$$

In this work we show that the solution, $\Psi_t$, corresponding to equation (2) and initial value (1) has interesting features that can be studied with the help of effective one-particle equations describing the evolution of the reference state $\phi_t^{(\text{ref})}$ and the excitation $\epsilon_t$; see equations (11)-(14) below. We find that, in the time evolution of the reference wave function, quantum-mechanical spreading of the wave packet is suppressed due to the circumstance that $\phi_0^{(\text{ref})}$ is flat. As a consequence, to leading order, the time-evolved state, $\phi_t^{(\text{ref})}$, equals the initial state $\phi_0^{(\text{ref})}$ up to a time-dependent phase factor. However, the dynamics of the excitation, i.e., the behavior of the function $\epsilon_t$, is quite non-trivial. In particular, its $L^2$ norm is not conserved because of exchange of gas particles between the condensate (described by the reference state) and the coherent excitation. Moreover, the function $\epsilon_t$ disperses according to a law that incorporates a strictly positive, finite speed of sound in the gas; meaning that sound waves (Goldstone modes) with arbitrarily small wave number turn out to propagate at a strictly positive speed as expected of sound waves in an interacting Bose gas, and which has already been observed in experiments, e.g., [8].

Excitations of the condensate might be caused by some heavy tracer particles penetrating into the gas, as considered in [3], where the Bose gas was taken to be an ideal gas. For simplicity we shall not include such tracer particles in the analysis presented below but study the dynamics of excitations of the condensate ground-state directly. The key analytical ideas used in the analysis of the mean-field limit presented in this paper are inspired by those introduced in [15]. They involve some counting of the number of “bad particles”, by which we mean particles that do not follow the (one-particle) effective dynamics. As compared to [3], the problems addressed in the present work require considerably finer control of the number of bad particles. Indeed, since a typical excitation $\epsilon_t$ involves $O(\rho)$ many particles, the number of bad particles in a state of the gas must be controlled
in terms of $\rho$ rather than of $N$. For this reason, the counting measures used in this work have to be considerably fine-tuned in order to arrive at useful estimates.

Beside the analysis of dynamics, it should be noted that first steps in the direction of large volume considering the excitation spectrum of a Bose gas have also been undertaken in [4] which provides an extension of the previous results in [17].

**Outline:** After introducing some important notation in Section 1.1 we describe our main results in Section 1.2 and present the proofs in Section 2.

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### 1.1 Notation

1. $|\cdot|$ is the standard norm on $\mathbb{R}^d$ or $\mathbb{C}^d$, for arbitrary $d$; $\|\cdot\|_p$ is the norm on the Lebesgue space $L^p$, $0 \leq p \leq \infty$. For operators, $O$, acting on the Hilbert space $L^2$ we denote by $\|O\|$ the operator norm of $O$.

2. Throughout this paper $\Lambda$ denotes both a cube in physical space $\mathbb{R}^3$ and the volume of this cube.

3. For $r > 0$ the ball of radius $r$ in $\mathbb{R}^3$ is denoted by $\mathcal{B}_r := \{ x \in \mathbb{R}^3 \mid |x| < r \}$.

4. We denote the Laplace operator and the gradient in the $x$--variable by $\Delta$ and $\nabla$, respectively.

5. The Fourier transform of a function $\eta \in L^2$ is denoted by $\hat{\eta}$.

6. The convolution of two functions $f$ and $g$ on $\mathbb{R}^3$ is defined by $(f * g)(\cdot) := \int_{\mathbb{R}^3} dy f(\cdot - y)g(y)$.

7. By “$F \in \text{Bounds}$” we mean that $F$ is a continuous, non-decreasing, non-negative function on the non-negative reals, i.e., $F : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$.

8. Unless specified otherwise, the symbol $C$ denotes a universal constant whose value may change from one line to another. In particular, all constants are independent of $\Lambda$ and $\rho$.

### 1.2 Main Results

As announced in the introduction, the goal pursued in this paper is to understand features of the time evolution of a many-body wave function, $\Psi_t$, for a given initial product wave function of the form (1), which will be characterized more precisely as follows:
Condition 1.2. The many-body wave function of the initial state (at time $t = 0$) is given by

$$
\Psi_0(x_1, x_2, \ldots, x_N) = \prod_{k=1}^{N} \frac{1}{\Lambda^{1/2}} \varphi_0(x_k), \quad \varphi_0 := \phi_0^{(\text{ref})} + \epsilon_0,
$$

where $\phi_0^{(\text{ref})}, \epsilon_0 \in C_c^\infty$ have the following properties:

$$
\text{supp } \phi_0^{(\text{ref})} \subseteq \Lambda, \quad \|\phi_0^{(\text{ref})}\|_\infty \leq \|\phi_0^{(\text{ref})}\|_1 \leq C,
$$

$$
\text{supp } \epsilon_0 \subseteq B_{1/4\Lambda^{1/3}}, \quad \|\epsilon_0\|_\infty \leq \|\epsilon_0\|_1 \leq C, \quad \|\epsilon_0\|_2 \leq C.
$$

Furthermore, we assume that the density of the gas condensate is essentially constant in some large region inside the container to which the gas is confined. Therefore, with the help of a family of cut-off functions $\chi_r \in C^2(\mathbb{R}^3), 0 < r < 1,$

$$
\chi_r(x) = \begin{cases} 
0 & \text{for } x \in B_{r\Lambda^{1/3}} \\
1 & \text{for } x \notin B_{r\Lambda^{1/3}}
\end{cases}
$$

we require

$$
\left| \phi_0^{(\text{ref})}(x) - 1 \right| \leq \chi_{1/2}(x).
$$

This will allow us to track the dynamics of the excitation with the properties (6) in that region. Finally, we require some control of the kinetic energy of the initial reference wave function:

$$
\|\nabla \phi_0^{(\text{ref})}\|_\infty \leq C \Lambda^{-1/3}, \quad \|\nabla \phi_0^{(\text{ref})}\|_2 \leq C \Lambda^{1/3}, \quad \|\Delta \phi_0^{(\text{ref})}\|_2 \leq C \Lambda^{-1/3}.
$$

Without further reference we assume Condition 1.2 and

$$
U \in C_c^\infty(\mathbb{R}^3, \mathbb{R})
$$

to hold throughout the entire paper.

In order to gain control on the dynamics of the many-body wave function $\Psi_t$, we show in a first step that it can be described approximately as a product function of the solution, $\varphi_t$, of the following nonlinear Schrödinger equation

$$
i \partial_t \varphi_t(x) = h_x[\varphi_t] \varphi_t(x), \quad h_x[\varphi_t] := -\frac{1}{2} \Delta + U * |\varphi_t|^2(x),
$$

with initial value $\varphi_t|_{t=0} = \varphi_0$. The sense of the approximation involved in this claim will be made clear in Section 2. As already mentioned in the introduction there are two sources for the dynamics of $\varphi_t$: One is connected to the evolution of the reference one-particle state $\phi_t^{(\text{ref})}$, and a second one is
connected to the evolution of the excitation, as described by \( \epsilon_t \). In order to conveniently distinguish between these two sources, the reference state \( \phi_{\text{ref}}^0 \) is time-evolved according to the equation

\[
i \partial_t \phi_t^{\text{ref}}(x) = \left( -\frac{1}{2} \Delta + U \ast |\phi_t^{\text{ref}}|^2(x) - \|U\|_1 \right) \phi_t^{\text{ref}}(x),
\]

and the excitation propagates as described by the equation

\[
\epsilon_t := \varphi_t e^{i\|U\|_1 t} - \phi_t^{\text{ref}}.
\]

Equations (11) and (12) show that the evolution of the excitation is given by

\[
i \partial_t \epsilon_t(x) = \left( -\frac{1}{2} \Delta + U \ast |\phi_t^{\text{ref}}|^2(x) - \|U\|_1 + U \ast |\epsilon_t|^2(x) + U \ast 2\Re(\epsilon_t^* \phi_t^{\text{ref}})(x) \right) \epsilon_t(x)
\]

\[
+ \left( U \ast |\epsilon_t|^2(x) + U \ast 2\Re(\epsilon_t^* \phi_t^{\text{ref}})(x) \right) \phi_t^{\text{ref}}(x).
\]

Note that, for a fixed point \( x \) deep inside the region \( \Lambda \), one has that

\[
|U \ast |\phi_t^{\text{ref}}|^2(x) - \|U\|_1 | \approx 0,
\]

which motivates our choice of the phase on the right side of (13). Furthermore, in the limit of large \( \Lambda \) the reference state \( \phi_t^{\text{ref}} \) tends to 1 so that equation (14) formally turns into

\[
i \partial_t \epsilon_t(x) = \left( -\frac{1}{2} \Delta + U \ast |\epsilon_t|^2(x) + U \ast 2\Re(\epsilon_t^* \phi_t^{\text{ref}})(x) \right) \epsilon_t(x)
\]

\[
+ \left( U \ast |\epsilon_t|^2(x) + U \ast 2\Re(\epsilon_t^* \phi_t^{\text{ref}})(x) \right) \phi_t^{\text{ref}}(x).
\]

We recall the standard facts that, for repulsive \( U \), i.e., \( U \geq 0 \), and given \( \Psi_0, \varphi_0, \phi_{\text{ref}}^0 \), and \( \epsilon_0 \) as in Condition 1.2, there exist unique classical solutions \( \Psi_t, \varphi_t, \phi_t^{\text{ref}} \), and \( \epsilon_t \) to equations (2), (11), (12), and (14), \( t \in \mathbb{R} \), with initial data \( \Psi_{t=0} = \Psi_0, \varphi_{t=0} = \varphi_0, \phi_{\text{ref}}^{t=0} = \phi_{\text{ref}}^0 \), and \( \epsilon_{t=0} = \epsilon_0 \), respectively. In the case of attractive potentials \( U \), however, the solution \( \varphi_t \), and therefore also \( \epsilon_t \), may blow up in finite time; see our discussion in the last paragraph of this section.

In a second step, we show that the control of the \( N \)-particle wave function \( \Psi_t \) as a function of time \( t \) in terms of the one-particle function \( \varphi_t \) is so accurate that the excitation \( \epsilon_t \) is “silhouetted” against all error terms. In order to compare the microscopic description of the quantum dynamics with its mean-field description, one must check that the reduced one-particle density matrix determined by the “true” many-body wave function \( \Psi_t \) matches the pure one-particle state given by the one-particle wave function \( \varphi_t \) that one determines by solving equation (11). As discussed in the introduction, the reduced density matrix of the microscopic (Schrödinger) description,

\[
\text{Tr}_{x_2,...,x_N} |\Psi_t\rangle \langle \Psi_t|,
\]

is given, to leading order, by the projection \(|\varphi_t\rangle \langle \varphi_t|\) onto the one-particle state \(|\varphi_t\rangle\). In order to subtract the contribution from the homogeneous condensate and only track the excitation, we project \(|\varphi_t\rangle\) onto the subspace orthogonal to the reference state. For this purpose we introduce the following notation.
**Definition 1.3.** Given a vector $\eta \in L^2(\mathbb{R}^3, \mathbb{C})$ we define the orthogonal projectors

$$p^\eta := \frac{1}{\|\eta\|^2} |\eta\rangle \langle \eta|, \quad q^\eta := 1 - p^\eta, \quad p_t^{(\text{ref})} := p_{\phi_t^{(\text{ref})}}, \quad q_t^{(\text{ref})} := q_{\phi_t^{(\text{ref})}}.$$ 

In this notation, the quantities to be compared are the following density matrices:

$$\rho_t^{(\text{micro})} := q_t^{(\text{ref})} \text{Tr}_{x_2 \ldots x_N} |\Lambda^{1/2} \Psi_t^{(\text{ref})}\rangle \langle \Lambda^{1/2} \Psi_t^{(\text{ref})}| \quad \text{and} \quad \rho_t^{(\text{macro})} := |\epsilon_t\rangle \langle \epsilon_t|.$$ 

The additional factor of $\Lambda$ makes up for the different scalings of $\Psi_t$ and $\epsilon_t$; see Condition 1.2.

Our first result is the following theorem.

**Theorem 1.4.** Let $U \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^+_0)$ be a repulsive potential. Then there exists a $C \in \text{Bounds}$ such that

$$\left\| \rho_t^{(\text{micro})} - \rho_t^{(\text{macro})} \right\| \leq C(t) \frac{\Lambda^{3/2}}{\rho_t^{1/2}},$$

for all times $t \geq 0$ provided $\Lambda$ is sufficiently large.

This theorem states that if the thermodynamic limit, $\Lambda \to \infty$, and the mean-field limit, $\rho \to \infty$, are approached in such a way that $\Lambda \ll \rho^{1/3}$, then the many-body Schrödinger dynamics is well approximated by the non-linear mean-field dynamics of a one-particle wave function – at least at the level of one-particle density matrices.

Obviously, a key open question is whether the thermodynamic limit can be taken before the mean-field limit is approached. Concretely, one must ask how one could possibly improve the rate of convergence established in Theorem 1.4. The time evolution necessarily creates some “bad” particles, viz., particles in states that do not follow the mean-field dynamics, throughout the region $\Lambda$ to which the gas is confined. This makes it plausible that, on the one hand, the number of bad particles grows with $\Lambda$, while, on the other hand, it decreases as $\rho$ increases due to our choice of scaling. Hence, when passing to large volumes $\Lambda$, for some fixed $\rho$, it seems hopeless to control the norm

$$\left\| \rho_t^{(\text{micro})} - \rho_t^{(\text{macro})} \right\| \quad \text{(15)}$$

directly. In particular, if the thermodynamic limit, $\Lambda \to \infty$, were taken before the mean-field limit, $\rho \to \infty$, the time evolution would immediately create an infinite number of bad particles, and (15) could not possibly be small.

In this respect it is important to note that a control of (15) in the thermodynamic limit is actually stronger than what is needed when comparing theoretical predictions to data about the time evolution of excitations gathered in an experiment. In order to gain access to regimes corresponding to very large volumes $\Lambda$, one must therefore introduce an appropriate notion of approximation by mean-field quantities weakening (15). One such possibility would be to introduce a semi-norm involving the restrictions of the one-particle density matrices to a bounded region $\lambda \subset \Lambda$ of interest with a volume of order $O(1)$, e.g.,

$$\left\| \mathbb{1}_\lambda \left( \rho_t^{(\text{micro})} - \rho_t^{(\text{macro})} \right) \mathbb{1}_\lambda \right\|,$$

where $\mathbb{1}_\lambda(x)$ is some cut-off function with support in $\lambda$. For finite times, an excitation of the gas created in some bounded region of space can be expected to essentially remain localized in a
bounded region. Thus, control of (16) may turn out to suffice to study its dynamics for a finite interval of times and compare it to its effective (mean-field) dynamics. The technical control of a quantity like (16) is however cumbersome as one needs to control the flow of particles from $\Lambda \setminus \lambda$ into the volume $\lambda$ without having much information about them.

Another possibility in the direction of large volumes – the one explored in this paper – is to show that (15) is typically small, the precise mathematical statement being: There is a trajectory of vectors $\Psi_t$ with corresponding reduced density matrix $\tilde{\rho}^{(\text{micro})}_t$ such that $\|\Psi_t - \Psi_t\|_2$ and $\|\tilde{\rho}^{(\text{micro})}_t - \rho^{(\text{macro})}_t\|$ are both small. Such a result may actually be expected to enable one to answer most physical questions in a satisfactory way as only what happens with large probability really matters for the comparison with an experiment. Let us try to explain why this mode of approximation is helpful: If the volume $\Lambda$ of the region to which the gas is confined is large, the gas contains a vast number of particles. Suppose that, with a tiny probability, the positions of all these particles are changed. Such a change may yield a significant variation of the reduced density matrices of the system. However, events that happen with a very small probability are not important physically. Hence, the fact that the reduced density matrices may change appreciably is unimportant.

With the next two results we explore this probabilistic idea and demonstrate how the result in Theorem 1.4 can be improved. The basis for this improvement forms the contents of our second main result. To state it we make the notion of "bad" particles precise. We introduce orthogonal projectors

$$
P^{\Psi_t}_k = (q^{\Psi_t})^\otimes k \otimes (p^{\Psi_t})^\otimes (N-k), \quad 0 \leq k \leq N, \tag{17}
$$

where $\otimes$ denotes the symmetric tensor product. The projector $p^{\Psi_t}$ is to be thought of as projecting onto one-particle states of "good" particles, while $q^{\Psi_t}$ projects onto one-particle states of "bad" particles; see equation (24) below. The probability, $\mathbb{P}_t$, of the event that the total number of bad particles described by the many-body wave function $\Psi_t$ is larger than the density $\rho$ is given by

$$
\mathbb{P}_t (\text{total number of bad particles } > \rho) := 1 - \left| \langle \Psi_t | \tilde{\Psi}_t \rangle \right|^2, \quad \text{where } \tilde{\Psi}_t := \sum_{1 \leq k \leq \rho} P^{\Psi_t}_k \Psi_t. \tag{18}
$$

This quantity is estimated in our second main result.

**Theorem 1.5.** Let $U \in C_c^\infty(\mathbb{R}^3, \mathbb{R}_+)$ be a repulsive potential. Then there is a $C \in \text{Bounds}$ such that

$$
\|\Psi_t - \tilde{\Psi}_t\|_2^2 \leq C(t) \frac{\Lambda}{\rho},
$$

for all times $t \geq 0$, provided $\Lambda$ is sufficiently large.

We pause to interpret this result. As a gedanken experiment, we imagine that the density of the Bose gas is measured, e.g., by shining light into the condensate and then recording the scattered light by means of a photograph – as one does in recent experiments with cold atom gases, where for example a sequence of photographs is taken to record the dynamics of the Bose gas cloud; see also [8]. As long as one can recognize a localized excitation on the photograph of the gas, one can argue that there are at most $O(\rho)$ bad particles in the state of the gas, and hence that the state after the measurements is close to the vector $\tilde{\Psi}_t$. Theorem 1.5 then says that if $\frac{\Lambda}{\rho} \ll 1$ the state of the system is very close to the vector $\tilde{\Psi}_t$, and, in this case, the result in Theorem 1.4 can be further improved as follows (our third main result).
Theorem 1.6. Let \( U \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^+_{0}) \) be a repulsive potential. Then there exists \( C \in \text{Bounds} \) such that

\[
\tilde{\rho}_t^{(\text{micro})} := q_t^{(\text{ref})} \text{Tr}_{x_2, \ldots, x_N} \left| \Lambda^{1/2} \tilde{\Psi}_t \right\rangle \left\langle \Lambda^{1/2} \tilde{\Psi}_t \right| q_t^{(\text{ref})},
\]

fulfills

\[
\left\| \tilde{\rho}_t^{(\text{micro})} - \rho_t^{(\text{macro})} \right\| \leq C(t) \frac{\Lambda^{1/2}}{\rho^{1/2}},
\]

for all times \( t \geq 0 \) provided \( \Lambda \) is sufficiently large.

Remark 1.7. It should be stressed that Theorems 1.4, 1.5 and 1.6 also hold (i) for more general initial states \( \Psi_0 \) which, however, must be close to the product state in (4), see Remark 2.2 below; and (ii) for attractive two-body potentials \( U \) and times \( 0 \leq t < T \leq \infty \) provided \( \| \varphi_i \|_{\infty} \) stays bounded for \( 0 \leq t \leq T \). As mentioned above, the case of attractive potentials is more subtle because solutions of the evolution equation (14) may blow up in finite time. Indeed, for this case the Bose gas collapses in the thermodynamic limit, and it is then not surprising that convergence to the mean-field limit fails, too.

In order to further analyze the dynamics of \( \Psi_t \), we consider excitations \( \epsilon_i \) of very small \( L^2 \)– and bounded \( L^\infty \)– norm. In this case we find that the evolution of \( \epsilon_i \) is well described by a linear version of equation (14), namely

\[
i \partial_t \eta_i(x) = -\frac{1}{2} \Delta \eta_i(x) + U \ast 2 \Re \eta_i(x),
\]

with initial condition \( \eta_i|_{t=0} = \epsilon_0 \). Indeed, in Section 2.4 we prove the following theorem:

Theorem 1.8. Let \( U \in C_c^\infty(\mathbb{R}^3, \mathbb{R}) \) be a general potential. Suppose \( \epsilon_i \) and \( \eta_t \), solve the equations (14) and (20), respectively, for \( 0 \leq t < T \leq \infty \) and initial data \( \epsilon_i|_{t=0} = \epsilon_0 = \eta_t|_{t=0} \). Then there is a \( C \in \text{Bounds} \) such that

\[
\| \eta_t - \epsilon_i \|_2 \leq C(t) \sup_{x \in [0,t]} \left( \Lambda^{-\frac{1}{2}} + \| \epsilon_i \|_2^2 + \| \epsilon_i \|_3^3 \right),
\]

for times \( 0 \leq t < T \) provided \( \Lambda \) is sufficiently large.

The evolution equation (14) is then quite easy to analyze. After a Fourier transformation,

\[
\hat{\eta}_t(k) := (2\pi)^{-3/2} \int d^3 x e^{-ikx} \eta_t(x),
\]

of \( \eta_t \), we rewrite (20) in momentum space

\[
i \partial_t \hat{\eta}_t(k) = \omega_0(k) \hat{\eta}_t(k) + \hat{U}(k) \left( \hat{\eta}_t(k) + \hat{\eta}_t^*(-k) \right),
\]

where we have used that \( \hat{\eta}_t(k) = \hat{\eta}_t^*(-k) \), and where

\[
\omega_0(k) = \frac{k^2}{2}
\]
is the symbol of the differential operator $-\frac{1}{2}\Delta$ in momentum space. The complex conjugate of this equations is given by

$$i\partial_t \tilde{\eta}_i^*(-k) = -\omega_0(k)\tilde{\eta}_i^*(-k) - \tilde{U}(k) (\tilde{\eta}^*(-k) + \tilde{\eta}(k)),$$

where we have used that

$$\omega_0(k) \equiv \omega_0(|k|) \quad \text{and} \quad \tilde{U}(k) = \tilde{U}^*(-k)$$

as the potential $U(x)$ is real-valued. The evolution equations for $\tilde{\eta}_i(k)$ and $\tilde{\eta}_i^*(-k)$ can then be written in closed form as

$$i\partial_t \begin{pmatrix} \tilde{\eta}_i(k) \\ \tilde{\eta}_i^*(-k) \end{pmatrix} = \mathcal{H}(k) \begin{pmatrix} \tilde{\eta}_i(k) \\ \tilde{\eta}_i^*(-k) \end{pmatrix}, \quad \text{with} \quad \mathcal{H}(k) := \begin{pmatrix} \omega_0(k) + \tilde{U}(k) & \tilde{U}(k) \\ -\tilde{U}(k) & -\omega_0(k) - \tilde{U}(k) \end{pmatrix}.$$

Note that $\mathcal{H}$ is not self-adjoint, and hence, the $L^2$ norm of $\eta_i$ is not preserved. However, one can still find a basis w.r.t. which $\mathcal{H}$ is diagonal. For arbitrary $\tilde{U}(k)$, an eigenvalue, $\omega(k)$, of $\mathcal{H}(k)$ fulfills

$$\omega(k)^2 = \omega_0(k) (\omega_0(k) + 2\tilde{U}(k)).$$

(23)

This shows how the dispersion law, $\omega(k)$, of sound waves in the gas depends on the pair potential $U$. We consider two interesting cases:

**Repulsive potential, e.g., $\tilde{U}(0) > 0$:**

$$|\omega(k)| = |k| \sqrt{\frac{k^2}{4} + \tilde{U}(k)}.$$

Apparemly, the speed of sound at small values of $|k|$ is then given by

$$v_{\text{sound}} = \sqrt{\tilde{U}(0)},$$

which is a well-known result due to Bogolyubov [1]. Note that the fact that $v_{\text{sound}}$ does not depend on the density $\rho$ of the gas is owed to the scaling in (3).

**Attractive potential, e.g., $\tilde{U}(0) < 0$:** For such potentials $U$, modes with wave vectors $k$ fulfilling $\omega_0(k) = -2\tilde{U}(k)$ become static according to the effective dispersion relation

$$\omega(k) = \omega_0(k)^{1/2} \sqrt{\omega_0(k) + 2\tilde{U}(k)},$$

while modes corresponding to wave vectors $k$ with $\omega_0(k) < -2\tilde{U}(k)$ are dynamically unstable. This instability causes the gas to implode at a finite time. As noted in Remark 1.7, our main results about the $N$-particle time evolution also hold for attractive two-body potentials $U$, as long as $||\varphi||_{\infty}$ remains bounded, i.e., for sufficiently short times, which is why for those times $\eta_i$ also gives insights into the microscopic dynamics of $\Psi_i$.

**Remark 1.9.** We note that the proofs provided in this paper also work for dispersion relations other than $\omega_0(k) = \frac{k^2}{2}$. While the propagation estimates given in Section 2.3 would have to be adapted, the mean-field estimates hold for any dispersion relation as all one-particle terms in the Hamiltonian drop out immediately; see (37) below.
2 Proofs

In this section, we present the proofs of our results. The organization of our reasoning process is as follows.

- Section 2.1: Our first technical result, Lemma 2.1, aims at controlling the number of bad particles present in the state of the gas. This lemma will be proven under the assumption that \( \| \varphi \|_\infty \) is bounded following ideas of [15]. Note that the control of the Hartree dynamics (11) is well understood. One might then ask why Lemma 2.1 is needed. The reason is that we are ultimately interested in the dynamics of excitations, and for this it turns out in the proofs of Theorem 1.5 and Theorem 1.6 that considerably stronger bounds on the number of bad particles are necessary.

- Section 2.2: Using Lemma 2.1 we proceed to proving our first three main results, namely Theorems 1.4, 1.5, and 1.6. These results hold provided the assumptions (97), (98) and (99) hold true.

- Section 2.3: Here “propagation estimates” justifying the assumptions (97), (98) and (99) will be derived.

- Section 2.4: To conclude, we provide the proof of Theorem 1.8 which is also based on those propagation estimates.

2.1 Controlling the number of “bad” particles

For any \( \varphi \in L^2 \), we use the notation
\[
q^\varphi_k := 1 - p^\varphi_k, \quad (p^\varphi_k \Psi)(x_1, \ldots, x_N) := \frac{\varphi(x_k)}{\| \varphi \|_2} \int d^3 x_k \frac{\varphi^*(x_k)}{\| \varphi \|_2} \Psi(x_1, \ldots, x_N), \quad 1 \leq k \leq N. \quad (24)
\]

To begin with, we need to define a convenient measure to count “bad” particles, i.e., those particles that do not evolve according to the effective non-linear dynamics (11). For this purpose we introduced the orthogonal projectors
\[
P^\varphi_k = (q^\varphi)^{\otimes k} \otimes (p^\varphi)^{\otimes (N-k)}, \quad (17)
\]
for \( 0 \leq k \leq N \). To simplify our notation we use the convention
\[
P^\varphi_k \equiv 0, \quad \forall \ k \not\in \{0, 1, \ldots, N\}. \quad (25)
\]
Later we will replace \( \varphi \) by the solution \( \varphi_t \) of equation (11). One may then think of \( p^\varphi_t \) as projecting on a “good” one-particle state and \( q^\varphi_t \) as projecting on a “bad” one-particle state.

For an arbitrary weight function
\[
w : \mathbb{Z} \rightarrow \mathbb{R}_0^+
\]
we then define weighted counting operators
\[
\overline{w}^\varphi := \sum_{k=0}^N w(k)P^\varphi_k, \quad \overline{w}_d^\varphi := \sum_{k=-d}^{N-d} w(k+d)P^\varphi_k, \quad d \in \mathbb{Z}. \quad (26)
\]
The role of the integer \(d\) will become clear in (33) and (34). Note that, in the language introduced above, \(P_k^\varphi\) projects on that part of the wave function that describes exactly \(k\) bad particles. Hence, one of the obvious candidates for a convenient counting measure is \(\hat{w}^\varphi\), with \(w(k) = k/N\). The expectation value \(\langle \Psi, \hat{w}^\varphi \Psi \rangle\) then represents the expected relative number of bad particles in the gas. However, control of this quantity will not suffice to track the excitation \(\epsilon_t\): The total number of particles in the gas is given by \(N = \Lambda \rho\), and the number of particles participating in an excitation is \(O(\rho)\). Consequently, we will have to control the number of bad particles as compared to \(\rho\). This means that we have to adjust our weight in such a way that it counts the number of bad particles relatively to \(\rho\). The explicit weight function we use is given by

\[
m(k) := \begin{cases} \frac{k}{\rho} & \forall \ 0 \leq k \leq \rho \\ 1 & \forall \rho < k \\ 0 & \text{otherwise.} \end{cases}
\]

When setting \(w(k) := m(k)\) we denote the corresponding operator \(\hat{w}^\varphi\) by \(\hat{m}^\varphi\). Now, if \(\langle \Psi, \hat{m}^\varphi \Psi \rangle\) is small, the probability of finding approximately \(\rho\) bad particles in the gas is small. As time goes by more and more particles in the gas will become bad, due to interactions with other particles. Even for a perfect product state there will always be a small deviation of the true field from the mean field. The more bad particles there are in the gas the stronger this deviation will be, and one may expect that the rate of “infection” of formerly good particles is proportional to the number of bad particles, up to a small term. The strategy of our proof is thus to show, with the help of a Grönwall argument, that if, initially, the number of bad particles is small, it will remain small for any finite time interval.

Before we can start presenting the proofs of our results we must recall some properties of the weighted counting measures, which have originally been studied in Lemma 1 in [14]. We summarize those properties that will be needed in our analysis here while postponing their proofs to the appendix.

1. \(\hat{w}^\varphi \hat{w}^\varphi = (\hat{v}^w)^\varphi = \hat{w}^\varphi \hat{v}^\varphi\)

2. \([\hat{w}^\varphi, P_k^\varphi] = [\hat{w}^\varphi, q_k^\varphi] = 0\)

3. \([\hat{w}^\varphi, P_k^\varphi] = 0\)

4. For \(n(k) = \sqrt{\frac{k}{N}}\) we have

\[
(n^\varphi)^2 = \frac{1}{N} \sum_{k=1}^{N} q_k^\varphi
\]

5. For \(\Psi \in (L^2)^{\otimes N}\) we have that

\[
\left\| \hat{w}^\varphi q_1^\varphi \Psi \right\|_2 = \left\| \hat{w}^\varphi n^\varphi \Psi \right\|_2
\]

\[
\left\| \hat{w}^\varphi q_1^\varphi q_2^\varphi \Psi \right\|_2 \leq \sqrt{\frac{N}{N-1}} \left\| \hat{w}^\varphi (n^\varphi)^2 \Psi \right\|_2
\]
6. For any function $Y \in L^\infty(\mathbb{R}^3)$ and $Z \in L^\infty(\mathbb{R}^6)$ and 
\[
A_0^\varphi = p_1^\varphi, \quad A_1^\varphi = q_1^\varphi, \quad B_0^\varphi = p_1^\varphi p_2^\varphi, \quad B_1^\varphi = p_1^\varphi q_2^\varphi, \quad B_2^\varphi = q_1^\varphi q_2^\varphi
\]
we have
\[
\overline{w}^\varphi A_j^\varphi Y(x_1)A_l^\varphi = A_j^\varphi Y(x_1)A_l^\varphi \overline{w}_{j-l}^\varphi \quad \text{with } j, l = 0, 1, \tag{33}
\]
and
\[
\overline{w}^\varphi B_j^\varphi Z(x_1, x_2)B_l^\varphi = B_j^\varphi Z(x_1, x_2)B_l^\varphi \overline{w}_{j-l}^\varphi \quad \text{with } j, l = 0, 1, 2. \tag{34}
\]

In the following lemma the weighted number of bad particles encountered in the course of time evolution is estimated. The proofs of our main results in Section 2.2 rely on this fundamental lemma. Another crucial point will be to justify assumption (35) below, which will be addressed in Section 2.3.

**Lemma 2.1.** Let $U \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$. Let $\Psi_t$ be the solution to equation $(2)$ for initial data as in Condition 1.2. Assume that, for some $T \leq \infty$, there is a $C \in \text{Bounds}$ such that
\[
\|\varphi_t\|_\infty \leq C(t), \quad 0 \leq t < T. \tag{35}
\]

Then there is a $C \in \text{Bounds}$ such that
\[
\langle \overline{m}^\varphi \rangle_t := \langle \Psi_t, \overline{m}^\varphi \Psi_t \rangle \leq C(t) \frac{\Lambda}{\rho}, \quad 0 \leq t < T, \tag{36}
\]
where the weight function $m$ corresponding to counting operator $\overline{m}^\varphi$ is defined in (27).

**Proof.** The heart of the proof is a Grönwall argument for which we need to control the time derivative of $\langle \overline{m}^\varphi \rangle_t$. Note that we have so-called “intermediate picture” here as both the wave function and the operator are time dependent.

The time derivative of $p_k^\varphi$ is given by $\frac{d}{dt} p_k^\varphi = -i[h_{x_k}[\varphi], p_k^\varphi]$ which can be seen best by noting that in bra-ket notation $p_k^\varphi$ is given by $|\varphi\rangle \langle \varphi|$ acting on the $k^{th}$ particle; see (24). Since $q_k^\varphi = 1 - p_k^\varphi$, it follows that $\frac{d}{dt} q_k^\varphi = -i h_{x_k}[\varphi], q_k^\varphi]$. Consequently, as $P_k^\varphi$ is a symmetric product of $p$’s and $q$’s, one has
\[
\frac{d}{dt} P_k^\varphi = -i \left[ \sum_{k=1}^N h_{x_k}[\varphi], P_k^\varphi \right].
\]

Since any weighted counting operator is a sum of operators $P_k^\varphi$ multiplied by real numbers (see (27)), it follows that $\frac{d}{dt} \overline{m}^\varphi = -i \left[ \sum_{k=1}^N h_{x_k}[\varphi], \overline{m}^\varphi \right]$ and thus
\[
\frac{d}{dt} \langle \overline{m}^\varphi \rangle_t = i \left[ H - \sum_{k=1}^N h_{x_k}[\varphi], \overline{m}^\varphi \right]_t
\]
\[
= i \left\langle \left[ \sum_{1 \leq j \leq k \leq N} U(x_j - x_k) - \frac{N}{\rho} U \frac{|\varphi|^2}{\Lambda} (x_k, \overline{m}^\varphi) \right] \right\rangle_t. \tag{37}
\]
Using the symmetry in the bosonic degree of freedom we find
\[
|\langle 37 \rangle| \leq \frac{N(N - 1)}{2\rho} \left| \left\langle \left( \frac{U(x_1 - x_2) - U \star \frac{|\varphi_i|^2}{\Lambda} (x_1) - U \star \frac{|\varphi_i|^2}{\Lambda} (x_2), \tilde{m}^{\varphi_i} \right) \right\rangle_t \right|
\]
\[
+ \frac{N}{\rho} \left| \left\langle \left( \frac{U \star \frac{|\varphi_i|^2}{\Lambda} (x_1), \tilde{m}^{\varphi_i} \right) \right\rangle_t \right|
\]
(38)

The first term, viz. (38), in the expression above is the physically relevant one. The second term, (39), only gives rise to a small correction. But we shall estimate this term first, because this actually permits us to demonstrate a crucial technique without too much additional ballast. We start by inserting identity operators, in the form of \( id_i \), \( p_1^{\varphi_i} + q_1^{\varphi_i} \), on the left- and right side of the scalar product in (39), i.e.,
\[
(39) \quad \frac{N}{\rho} \left| \left\langle \left( p_1^{\varphi_i} + q_1^{\varphi_i} \right) \left( Y(x_1)\tilde{m}^{\varphi_i} - \tilde{m}^{\varphi_i} Y(x_1) \right) \left( p_1^{\varphi_i} + q_1^{\varphi_i} \right) \right\rangle_t \right|
\]
\[
\leq \frac{N}{\rho} \left| \left\langle p_1^{\varphi_i} \left( Y(x_1)\tilde{m}^{\varphi_i} - \tilde{m}^{\varphi_i} Y(x_1) \right) p_1^{\varphi_i} \right\rangle_t \right|
\]
\[
+ \frac{N}{\rho} \left| \left\langle q_1^{\varphi_i} \left( Y(x_1)\tilde{m}^{\varphi_i} - \tilde{m}^{\varphi_i} Y(x_1) \right) q_1^{\varphi_i} \right\rangle_t \right|
\]
\[
+ \frac{2N}{\rho} \left| \left\langle p_1^{\varphi_i} \left( Y(x_1)\tilde{m}^{\varphi_i} - \tilde{m}^{\varphi_i} Y(x_1) \right) q_1^{\varphi_i} \right\rangle_t \right|
\]
\[
= \frac{2N}{\rho} \left| \left\langle p_1^{\varphi_i} \left( Y(x_1)\tilde{m}^{\varphi_i} - \tilde{m}^{\varphi_i} Y(x_1) \right) q_1^{\varphi_i} \right\rangle_t \right|
\]
(40)

Here, (41) and (42) are seen to be identically zero using (29) and (33) for \( j = l = 0 \), e.g.,
\[
p_1^{\varphi_i} Y(x_1)\tilde{m}^{\varphi_i} p_1^{\varphi_i} = p_1^{\varphi_i} Y(x_1) p_1^{\varphi_i} \tilde{m}^{\varphi_i} = \tilde{m}^{\varphi_i} p_1^{\varphi_i} Y(x_1) p_1^{\varphi_i} = p_1^{\varphi_i} \tilde{m}^{\varphi_i} Y(x_1) p_1^{\varphi_i}.
\]

Without further notice we will frequently use that
\[
\|\varphi_i\|^2 = \Lambda,
\]
(45)
as implied by (7) and (11).

Next, we apply the commutation relations in (29) and after that the pull-through formula in (33) for \( j = 0 \) and \( l = 1 \) to find
\[
(39) \quad \leq \frac{2N}{\rho} \left| \left\langle p_1^{\varphi_i} \left( Y(x_1)\tilde{m}^{\varphi_i} - \tilde{m}^{\varphi_i} Y(x_1) \right) q_1^{\varphi_i} \right\rangle_t \right|
\]
\[
= \frac{2N}{\rho} \left| \left\langle p_1^{\varphi_i} Y(x_1) q_1^{\varphi_i} \tilde{m}^{\varphi_i} - \tilde{m}^{\varphi_i} p_1^{\varphi_i} Y(x_1) q_1^{\varphi_i} \right\rangle_t \right|
\]
\[
= \frac{2N}{\rho} \left| \left\langle p_1^{\varphi_i} Y(x_1) q_1^{\varphi_i} \left( \tilde{m}^{\varphi_i} - \tilde{m}^{\varphi_{i-1}} \right) \right\rangle_t \right|
\]
(46)
Using the definition in (26) we find

\[ (39) \quad = \frac{2N}{\rho} \left| \left\langle p_1^{\hat{\phi}_t} Y(x_1) q_1^{\hat{\phi}_t} \left( \sum_{k=0}^{N} m(k) P_k^{\hat{\phi}_t} - \sum_{k=1}^{N+1} m(k-1) P_k^{\hat{\phi}_t} \right) \right\rangle_t \right| 
\]

\[ = \frac{2N}{\rho} \left| \left\langle p_1^{\hat{\phi}_t} Y(x_1) q_1^{\hat{\phi}_t} \left( \sum_{k=1}^{N} (m(k) - m(k-1)) P_k^{\hat{\phi}_t} \right) \right\rangle_t \right| 
\]

\[ = \frac{2N}{\rho} \left| \left\langle p_1^{\hat{\phi}_t} Y(x_1) q_1^{\hat{\phi}_t} \left( \sum_{1 \leq k \leq \rho} \frac{P_k^{\hat{\phi}_t}}{\rho} \right) \right\rangle_t \right| 
\]

\[ \leq \frac{N}{\rho} C \left\| \frac{|\varphi_t|^2}{\Lambda} \right\| \frac{1}{\rho} \]

\[ \leq \frac{C(t)}{\rho}. \]

where we have used the following ingredients:

- for the step from (49) to (50) we have used that \( m(0) = 0 \) and \( P_{N+1}^{\hat{\phi}_t} = 0 \);
- for the step from (50) to (51) we have used that \( m(k) - m(k-1) = \frac{1}{\rho} \) for \( k = 1, \ldots, \rho \) and \( m(k) - m(k-1) = 0 \) for \( k > \rho \); see (27);
- for the step from (51) to (52) we have used the definition of \( Y(x_1) \) in (39) and that \( P_k^{\hat{\phi}_t} \), \( 1 \leq k \leq N \), are pairwise orthogonal projectors;
- in the last step we have made use of assumption (35) to infer the bound

\[ \left\| \frac{\varphi_t}{\Lambda} \right\|^2 \leq \| U \|_1 \| \varphi_t \|_2^2 \leq C(t) \| U \|_1. \]

In what comes next we will invoke assumption (35) without further mentioning.

A similar technique is used to estimate (38). Again, we begin by inserting identity operators, in the form of \( \text{id}_H = p_1^{\hat{\phi}_t} + q_1^{\hat{\phi}_t} \) and \( \text{id}_H = p_2^{\hat{\phi}_t} + q_2^{\hat{\phi}_t} \), in order to extract different types of processes from the interaction which have to be treated separately:

\[ (38) \quad = \frac{N(N-1)}{2\rho} \left| \left\langle \left( Z(x_1, x_2) \bar{m}^{\hat{\phi}_t} - \bar{m}^{\hat{\phi}_t} Z(x_1, x_2) \right) \right\rangle_t \right| \]

\[ = \frac{N(N-1)}{2\rho} \left| \left\langle \left( p_1^{\hat{\phi}_t} + q_1^{\hat{\phi}_t} \right) \left( p_2^{\hat{\phi}_t} + q_2^{\hat{\phi}_t} \right) \times \right. \right. \]

\[ \times \left( Z(x_1, x_2) \bar{m}^{\hat{\phi}_t} - \bar{m}^{\hat{\phi}_t} Z(x_1, x_2) \right) \left( p_1^{\hat{\phi}_t} + q_1^{\hat{\phi}_t} \right) \left( p_2^{\hat{\phi}_t} + q_2^{\hat{\phi}_t} \right) \right|_t. \]
Due to symmetry

\[ (38) \leq \frac{N(N-1)}{2\rho} \left| \left\langle p_1^{\psi_i} p_2^{\psi_i} \left( Z(x_1, x_2) \hat{m}^{\psi_i} - \hat{m}^{\psi_i} Z(x_1, x_2) \right) p_1^{\psi_i} p_2^{\psi_i} \right\rangle_t \right| \]

\[ + \frac{N(N-1)}{2\rho} \left| \left\langle (p_1^{\psi_i} q_2^{\psi_i} + q_1^{\psi_i} p_2^{\psi_i}) \left( Z(x_1, x_2) \hat{m}^{\psi_i} - \hat{m}^{\psi_i} Z(x_1, x_2) \right) (p_1^{\psi_i} q_2^{\psi_i} + q_1^{\psi_i} p_2^{\psi_i}) \right\rangle_t \right| \]

\[ + \frac{N(N-1)}{2\rho} \left| \left\langle q_1^{\psi_i} q_2^{\psi_i} \left( Z(x_1, x_2) \hat{m}^{\psi_i} - \hat{m}^{\psi_i} Z(x_1, x_2) \right) q_1^{\psi_i} q_2^{\psi_i} \right\rangle_t \right| \]

\[ + \frac{2N(N-1)}{\rho} \left| \left\langle p_1^{\psi_i} p_2^{\psi_i} \left( Z(x_1, x_2) \hat{m}^{\psi_i} - \hat{m}^{\psi_i} Z(x_1, x_2) \right) p_1^{\psi_i} q_2^{\psi_i} \right\rangle_t \right| \]

\[ + \frac{N(N-1)}{\rho} \left| \left\langle p_1^{\psi_i} p_2^{\psi_i} \left( Z(x_1, x_2) \hat{m}^{\psi_i} - \hat{m}^{\psi_i} Z(x_1, x_2) \right) q_1^{\psi_i} q_2^{\psi_i} \right\rangle_t \right| \]

\[ + \frac{2N(N-1)}{\rho} \left| \left\langle p_1^{\psi_i} q_2^{\psi_i} \left( Z(x_1, x_2) \hat{m}^{\psi_i} - \hat{m}^{\psi_i} Z(x_1, x_2) \right) q_1^{\psi_i} q_2^{\psi_i} \right\rangle_t \right| . \]

Using the pull-through formula in (34) and the commutation relations given in (29) we can recast the last expression to get that

\[ (38) \leq \frac{N(N-1)}{\rho} \left| \left\langle p_1^{\psi_i} p_2^{\psi_i} Z(x_1, x_2) p_1^{\psi_i} p_2^{\psi_i} \left( \hat{m}^{\psi_i} - \hat{m}^{\psi_i} \right) \right\rangle_t \right| \]

\[ + \frac{N(N-1)}{\rho} \left| \left\langle (p_1^{\psi_i} q_2^{\psi_i} + q_1^{\psi_i} p_2^{\psi_i}) Z(x_1, x_2) (p_1^{\psi_i} q_2^{\psi_i} + q_1^{\psi_i} p_2^{\psi_i}) \left( \hat{m}^{\psi_i} - \hat{m}^{\psi_i} \right) \right\rangle_t \right| \]

\[ + \frac{N(N-1)}{\rho} \left| \left\langle q_1^{\psi_i} q_2^{\psi_i} Z(x_1, x_2) q_1^{\psi_i} q_2^{\psi_i} \left( \hat{m}^{\psi_i} - \hat{m}^{\psi_i} \right) \right\rangle_t \right| \]

\[ + \frac{N(N-1)}{\rho} \left| \left\langle p_1^{\psi_i} p_2^{\psi_i} Z(x_1, x_2) p_1^{\psi_i} q_2^{\psi_i} \left( \hat{m}^{\psi_i} - \hat{m}^{\psi_i} \right) \right\rangle_t \right| \]

\[ + \frac{N(N-1)}{\rho} \left| \left\langle p_1^{\psi_i} p_2^{\psi_i} Z(x_1, x_2) q_1^{\psi_i} q_2^{\psi_i} \left( \hat{m}^{\psi_i} - \hat{m}^{\psi_i} \right) \right\rangle_t \right| \]

\[ + \frac{N(N-1)}{\rho} \left| \left\langle p_1^{\psi_i} q_2^{\psi_i} Z(x_1, x_2) q_1^{\psi_i} q_2^{\psi_i} \left( \hat{m}^{\psi_i} - \hat{m}^{\psi_i} \right) \right\rangle_t \right| . \]

Lines (60)-(62) all contain the factor \( \left( \hat{m}^{\psi_i} - \hat{m}^{\psi_i} \right) \). Hence, they are identically equal to zero. In the following we provide estimates for the terms (63)-(65). We use that, for any \( f \in L^2 \),

\[ p_1^{\psi_i} f(x_1 - x_2) p_1^{\psi_i} = p_1^{\psi_i} \int dx_1 \frac{\varphi_i(x_1)}{\|\varphi_i\|_2} f(x_1 - x_2) \frac{\varphi_i(x_1)}{\|\varphi_i\|_2} p_1^{\psi_i} = \Lambda^{-1} f * \|\varphi_i\|^2_2 p_1^{\psi_i} , \]

holds so that we can estimate

\[ \|p_1^{\psi_i} f(x_1 - x_2)\| = \left\| p_1^{\psi_i} |f(x_1 - x_2)|^2 p_1^{\psi_i} \right\|^{1/2} \leq C(t) \Lambda^{-1/2} \|f\|_2 \]

with

\[ \|p_1^{\psi_i} f(x_1)\| = \left\| p_1^{\psi_i} |f(x_1)|^2 p_1^{\psi_i} \right\|^{1/2} \leq C(t) \Lambda^{-1/2} \|f\|_2 . \]
Furthermore, using Young’s inequality and the conservation of the $L_2$-norm of $\varphi$, we get
\[
\|p_1^\varphi U(x_1 - x_2)\| \leq C(t)\Lambda^{-1/2}.
\]

Finally, starting from the definition of $Z(x_1, x_2)$ in (38), (68) and (70) are seen to imply
\[
\|p_1^\varphi Z(x_1, x_2)\| \leq \frac{C(t)}{\Lambda^{1/2}}.
\]

Next, let $r : Z \to \mathbb{R}^+$ be given by $r(k) := \sqrt{m(k) - m(k - 1)}$ which is well defined because $m(k)$ is monotone increasing. Relation (28) implies that $(\overline{r^\varphi})^2 = \overline{m^\varphi} - \overline{m}^{-1}$. Then we can write
\[
\frac{N(N - 1)}{\rho} \left| \left\langle p_1^\varphi p_2^\varphi Z(x_1, x_2) q_1^\varphi q_2^\varphi (\overline{r^\varphi})^2 \right\rangle \right|
\]
Using the pull-through formula in (34) with $j = 1$ and $l = 2$ we get that
\[
\frac{N(N - 1)}{\rho} \left| \left\langle r_1^\varphi p_1^\varphi p_2^\varphi Z(x_1, x_2) q_1^\varphi q_2^\varphi r^\varphi \right\rangle \right|
\]
Finally, using the commutation relations in (29), the bounds in (71), and Schwartz inequality we can estimate
\[
\frac{N(N - 1)}{\rho} \left| \left\langle \overline{r_1^\varphi p_1^\varphi q_2^\varphi Z(x_1, x_2) q_1^\varphi q_2^\varphi \overline{r^\varphi}} \right\rangle \right|
\]
Using properties (30) and (31) of the counting measures and the definitions in (27) and (26) we find that

\[ \left| \frac{r_i^{\hat{p}}}{q_i^{\hat{p}}} q_i^{\hat{p}} \Psi_t \right|_2 = \left| \frac{r_i^{\hat{p}}}{n^{\hat{p}}} n^{\hat{p}} \Psi_t \right|_2 \]
\[ = \sum_{k=1}^{N-1} \left( \frac{m(k+1)-m(k)}{N} \right) \frac{k}{N} \left| P_k^\hat{p} \Psi_t \right|_2 \]
\[ \leq C \frac{N^{1/2}}{N^{1/2}} \sum_{0 \leq k < \rho} \left( \frac{k}{N} \right) \frac{1}{2} \left| P_k^\hat{p} \Psi_t \right|_2 \]
\[ \leq C \left( \frac{\langle \hat{m}_\rho \rangle}{N} \right)^{1/2} \left( \frac{\rho}{N} \right)^{1/2}. \] 

where we have used that \( m(k) - m(k-1) = \frac{1}{\rho} \) for \( k = 1, \ldots, \rho \) and \( m(k) - m(k-1) = 0 \) for \( k > \rho \). Quite similarly, and using (32), we see that

\[ \left| \frac{r_i^{\hat{q}} q_i^{\hat{q}} q_i^{\hat{q}} \Psi_t \right|_2 \leq \sqrt{\frac{N}{N-1}} \left( \frac{\langle \hat{m}_\rho \rangle - \hat{m}_{\rho-1}^\rho}{\langle \hat{m}_\rho \rangle - \hat{m}_{\rho-1}^\rho} \right)^{1/2} \left( \frac{n^{\hat{q}}}{N} \right)^{1/2} \left| \Psi_t \right|_2 \]
\[ = \sum_{k=1}^{N} \left( \frac{m(k)-m(k-1)}{N} \right) k^2 \frac{k}{N} \left| P_k^\hat{p} \Psi_t \right|_2 \]
\[ \leq C \frac{N^{1/2}}{N^{1/2}} \sum_{0 \leq k < \rho} \left( \frac{k}{\rho N} \right) \frac{1}{2} \left| P_k^\hat{p} \Psi_t \right|_2 \]
\[ \leq C \left( \frac{\langle \hat{m}_\rho \rangle}{N} \right)^{1/2} \left( \frac{\rho}{N} \right)^{1/2}. \] 

As a consequence, going back to (72), the bounds (73), (76), and (97) are seen to imply

\[ \left( \frac{\langle \hat{m} \rangle}{N} \right)^{1/2} \left( \frac{\langle \hat{p} \rangle}{N} \right)^{1/2} \left( \frac{\rho}{N} \right)^{1/2} \leq C \left( \frac{\langle \hat{m} \rangle}{N} \right)^{1/2}. \] 

**TERM (64):** Again, we write \( (\hat{m} - \hat{m}_\rho) \) as the square of its square root and we use the pull-through formula in (34) for \( j = 0 \) and \( l = 2 \):

\[ (64) = C \frac{N(N-1)}{\rho} \left| \langle p_1^{\hat{p}} p_2^{\hat{p}} Z(x_1, x_2) q_1^{\hat{q}} q_2^{\hat{q}} (\hat{m} - \hat{m}_\rho) \rangle \right| \]
\[ = C \frac{N(N-1)}{\rho} \left( \frac{\langle \hat{m} \rangle}{N} \right)^{1/2} \left( \frac{\rho}{N} \right)^{1/2} \left( \frac{\langle \hat{m} \rangle}{N} \right)^{1/2} \left( \frac{\rho}{N} \right)^{1/2} \] 

Next, we use the symmetry in the bosonic degrees of freedom of the wave function \( \Psi_t \) and of the
and finally use Schwarz inequality

\[ (64) \quad = \quad C \frac{N}{\rho} \left| \left( \left( \tilde{m}_2^\psi - \tilde{m}_1^\psi \right)^{1/2} p_1^\psi \sum_{k=2}^N p_k^\psi \times \right. \right. \]
\[ \left. \left. \times Z(x_1, x_k)q_k^\psi q_1^\psi \left( \tilde{m}_2^\psi - \tilde{m}_1^\psi \right)^{1/2} \right) \right|_t, \]

and finally use Schwarz inequality

\[ (64) \quad \leq \quad C \frac{N}{\rho} \left| \left( \sum_{k=2}^N q_k^\psi Z(x_1, x_k)q_1^\psi \left( \tilde{m}_2^\psi - \tilde{m}_1^\psi \right)^{1/2} \Psi_t \right|_2 \times \right. \]
\[ \left. \times \left| q_1^\psi \left( \tilde{m}_2^\psi - \tilde{m}_1^\psi \right)^{1/2} \Psi_t \right|_2. \] (79)

Furthermore, a computation similar to the one leading to (73) shows that

\[ (79) = \left| q_1^\psi \left( \tilde{m}_2^\psi - \tilde{m}_1^\psi \right)^{1/2} \Psi_t \right|_2 \quad \leq \quad C \left( \frac{\left( \tilde{m}_2^\psi / \tilde{m}_1^\psi \right)}{N} \right)^{1/2}. \] (80)

Next, we estimate the square of the $L^2$–norm in (78). In order to obtain a good estimate, we rewrite this expression according to

\[ \left| \sum_{k=2}^N q_k^\psi Z(x_1, x_k)q_1^\psi \left( \tilde{m}_2^\psi - \tilde{m}_1^\psi \right)^{1/2} \Psi_t \right|_2^2 \] (81)

\[ = \sum_{k=2}^N \left( \left( \tilde{m}_2^\psi - \tilde{m}_1^\psi \right)^{1/2} p_1^\psi p_k^\psi q_1^\psi \times \right. \]
\[ \left. \times Z(x_1, x_k)q_k^\psi p_1^\psi \left( \tilde{m}_2^\psi - \tilde{m}_1^\psi \right)^{1/2} \right)_t \]
\[ + \sum_{j,k=2, j\neq k}^N \left. \left( \left( \tilde{m}_2^\psi - \tilde{m}_1^\psi \right)^{1/2} p_1^\psi p_k^\psi q_1^\psi q_j^\psi \times \right. \right. \]
\[ \left. \left. \times Z(x_1, x_k)q_k^\psi p_1^\psi \left( \tilde{m}_2^\psi - \tilde{m}_1^\psi \right)^{1/2} \right)_t \right. \]

Furthermore, we exploit the symmetry in the bosonic degrees of freedom and split the summations into diagonal- and off-diagonal parts, with the result that

\[ (81) \leq N \left( \left( \tilde{m}_2^\psi - \tilde{m}_1^\psi \right)^{1/2} p_1^\psi p_2^\psi Z(x_1, x_2)q_2^\psi Z(x_1, x_2) p_2^\psi q_1^\psi \left( \tilde{m}_2^\psi - \tilde{m}_1^\psi \right)^{1/2} \right)_t \] (82)
\[ + N^2 \left( \left( \tilde{m}_2^\psi - \tilde{m}_1^\psi \right)^{1/2} p_1^\psi p_2^\psi Z(x_1, x_2)q_2^\psi q_3^\psi Z(x_1, x_3) p_3^\psi p_1^\psi \left( \tilde{m}_2^\psi - \tilde{m}_1^\psi \right)^{1/2} \right)_t \] (83)

Using (70) we find

\[ \| Z(x_1, x_2) p_1^\psi p_2^\psi \| \leq \| Z(x_1, x_2) p_1^\psi \| \| p_2^\psi \| \leq \frac{C(t)}{\Lambda^{1/2}}, \] (84)
We observe also that, using the definitions in (26) and (27), for any $\Psi$ with $\|\Psi\|_2 = 1$ one has

$$\left\| \left( m^{\overline{\nu}_1} - m^{\overline{\nu}_2} \right)^{1/2} \Psi \right\|_2^2 = \left\langle \Psi, \left( m^{\overline{\nu}_1} - m^{\overline{\nu}_2} \right) \Psi \right\rangle$$

(85)

$$= \left\langle \Psi, \left( \sum_{k=0}^N m(k)P^k - \sum_{k=2}^{N+2} m(k-2)P^k \right) \Psi \right\rangle$$

(86)

$$\leq C \left\langle \Psi, \left( \sum_{k=0}^N \frac{1}{\rho} P^k \right) \Psi \right\rangle$$

(87)

$$\leq \frac{C}{\rho}$$

(88)

because $\sum_{k=0}^N P^k$ coincides with the identity operator. Therefore, using (84) and (88), we can estimate the diagonal terms by

$$\left(82\right) \leq CN \left\| \left( m^{\overline{\nu}_1} - m^{\overline{\nu}_2} \right)^{1/2} \right\|_2^2 \|Z(x_1, x_2)p^{\overline{\nu}_1}_1 p^{\overline{\nu}_1}_1 \|_2^2$$

(89)

$$\leq CN \left\| \left( m^{\overline{\nu}_1} - m^{\overline{\nu}_2} \right)^{1/2} \right\|_2^2 \frac{C(t)}{\Lambda}$$

$$\leq C(t).$$

(90)

For the off-diagonal terms we find

$$\left(83\right) = N^2 \left\langle \left( m^{\overline{\nu}_1} - m^{\overline{\nu}_2} \right)^{1/2} q^\overline{\nu}_3 p^{\overline{\nu}_1}_1 p^{\overline{\nu}_1}_2 Z(x_1, x_2) \times \right.$$

$$\times Z(x_1, x_3)p^{\overline{\nu}_1}_3 p^{\overline{\nu}_1}_1 \left( m^{\overline{\nu}_1} - m^{\overline{\nu}_2} \right)^{1/2} \bigg|_t$$

$$\leq N^2 \left\| q^\overline{\nu}_3 \left( m^{\overline{\nu}_1} - m^{\overline{\nu}_2} \right)^{1/2} \Psi \bigg|_2^2 \times \right.$$  

$$\|p^{\overline{\nu}_1}_1 p^{\overline{\nu}_1}_2 Z(x_1, x_2)Z(x_1, x_3)p^{\overline{\nu}_1}_3 p^{\overline{\nu}_1}_1 \| \times \right.$$  

$$\times \left\| q^\overline{\nu}_2 \left( m^{\overline{\nu}_1} - m^{\overline{\nu}_2} \right)^{1/2} \Psi \bigg|_2^2 .$$

(91)

Here it becomes apparent why the splitting of (81) into a diagonal- and an off-diagonal part is necessary: A rough estimate of the term (90), using (81), leads to a $\Lambda^{-1}$–decay. As it will turn out in (95), this decay is not good enough. Fortunately, the situation is better than that, as the following analysis shows. First, we note that for non-negative $U$ one finds

$$\|p^{\overline{\nu}_1}_1 p^{\overline{\nu}_1}_2 U(x_1 - x_2)U(x_1 - x_3)p^{\overline{\nu}_1}_3 p^{\overline{\nu}_1}_1 \|$$

$$= \left\| p^{\overline{\nu}_1}_1 p^{\overline{\nu}_1}_2 \sqrt{U(x_1 - x_3)} \sqrt{U(x_1 - x_2)} \sqrt{U(x_1 - x_3)} \sqrt{U(x_1 - x_2)} p^{\overline{\nu}_1}_1 p^{\overline{\nu}_1}_1 \right\|$$

$$= \left\| p^{\overline{\nu}_1}_1 \sqrt{U(x_1 - x_3)} p^{\overline{\nu}_1}_2 \sqrt{U(x_1 - x_2)} \sqrt{U(x_1 - x_3)} p^{\overline{\nu}_1}_3 \sqrt{U(x_1 - x_2)} p^{\overline{\nu}_1}_1 \right\|$$

$$\leq \left\| p^{\overline{\nu}_1}_1 \sqrt{U(x_1 - x_3)} \bigg|_2^4 \leq \frac{C(t)}{\Lambda^2} \|U\|_1^4 ,$$

(92)
Second, due to (67) and (68)
\[ \| p_j p_k U(x_j - x_k) \| \leq \| p_k U(x_j - x_k) \| \leq \frac{C(t)}{\Lambda^{1/2}}, \]
\[ \| p_j \frac{\varphi_j^2}{\Lambda}(x_j) \| \leq \| p_j \frac{\varphi_j^2}{\Lambda}(x_j) \| \leq \frac{C(t)}{\Lambda^{3/2}} \]
that together with (91) imply
\[ \| p_1 p_2 Z(x_1, x_2) Z(x_1, x_3) p_1 p_3 \| \leq \frac{C(t)}{\Lambda^2}, \] (92)
Analogously to (80), one can prove that
\[ \left\| q_3^{\psi} \left( \overline{m_3^{\psi}} - m_3^{\psi} \right) \right\|^{1/2}_2 \leq C \left( \frac{\overline{m^{\psi}_i}(t)}{N} \right)^{1/2} \] (93)
Hence, invoking the estimates in (93) and (92), we arrive at
\[ \left\| q_3^{\psi} \left( \overline{m_3^{\psi}} - m_3^{\psi} \right) \right\|^{1/2}_2 \leq C \left( \frac{\overline{m^{\psi}_i}(t)}{N} \right)^{1/2} \]
\[ \left\| q_2^{\psi} \left( \overline{m_2^{\psi}} - m_2^{\psi} \right) \right\|^{1/2}_2 \]
\[ \leq C(t) N \frac{1}{\Lambda^2} \left( \frac{\overline{m^{\psi}_i}(t)}{N} \right)^{1/2} \] (94)
Thus
\[ \left( 64 \right) \leq C(t) \frac{N}{\rho} \sqrt{(81) \times (79)} \]
\[ \leq C(t) \frac{N}{\rho} \sqrt{(89) + (94) \times (79)} \]
\[ \leq C(t) \frac{N}{\rho} \left( 1 + \frac{N}{\Lambda^2} \right) \left( \frac{\overline{m^{\psi}_i}(t)}{N} \right)^{1/2} \]
\[ \leq C(t) \left( \frac{\Lambda}{\rho} + \left\langle m^{\psi}_i \right\rangle_t \right). \] (95)
The bounds (53), (69), (77), and (95) yield
\[ \left| \frac{d}{dt} \left\langle m^{\psi}_i \right\rangle_t \right| \leq (38) + (39) \]
\[ \leq C(t) \left( \left\langle m^{\psi}_i \right\rangle_t + \frac{1 + \Lambda}{\rho} \right). \]
Finally, for any initial wave function \( \Psi_0 \) with the property that
\[ \left\langle m^{\psi}_i \right\rangle_t \bigg|_{t=0} \leq \frac{C\Lambda}{\rho}, \] (96)
Grönwall’s Lemma yields the claim (36). According to Condition 1.2 we have \( \left\langle m^{\psi}_i \right\rangle \bigg|_{t=0} = 0 \) so that the bound (96) is fulfilled which concludes the proof of Lemma 2.1. \( \square \)
Remark 2.2. (i) The proof can be extended to more general initial conditions than those specified in Condition 1.2, namely to all wave functions, \( \Psi_0 \), for which the bound (96) holds. (ii) Note that (89) is the crucial estimate that determines the right-hand side of claim (36). It follows from the auxiliary bound (84), which cannot be improved without new insights into the dynamics of Bose gases. (iii) Provided \( \| \varphi \|_\infty \) is bounded, the proof holds also for attractive potentials.

2.2 Proofs of Theorem 1.4, Theorem 1.5, and Theorem 1.6

Lemma 2.1 immediately implies that, for a suitable class of initial wave functions, the microscopic and the macroscopic descriptions of the dynamics are close to one another, which is the content of our main results, Theorems 1.4, 1.5 and Theorem 1.6. Since we assume that the potential \( U \) is repulsive, Corollary 2.10 and Lemma 2.11 of Section 2.3 below provide the following estimates: There are \( C_1, C_2, C_3 \in \text{Bounds} \) such that

\[
\| \varphi \|_\infty \leq C_1(t),
\]  
\[
\| \epsilon_i \|_2 \leq C_2(t),
\]  
\[
\| p_{(\text{ref})i}(t) \|_2 \leq \frac{C_3(t)}{\Lambda^{1/2}}.
\]  

for all \( t \geq 0 \) provided \( \Lambda \) is sufficiently large. We temporarily assume the bounds in (97), (98) and (99) and proceed to proving our second and third main result; the first main results, Theorem 1.4, will latter be proven as a corollary.

Proof of Theorem 1.5. Because of (97), Lemma 2.1 implies that

\[
\left| \langle \tilde{m}\tilde{r} \rangle \right| \leq C(t) \frac{\Lambda}{\rho}.
\]  

In (18) we have introduced a wave function \( \tilde{\Psi}_t \) by setting

\[
\tilde{\Psi}_t := \sum_{0 \leq k \leq \rho} P_k \tilde{\Psi}_t.
\]

Using the definition of the counting measure \( m(k) \), see (27), we see that

\[
\left\| \Psi_t - \tilde{\Psi}_t \right\|_2^2 = \sum_{\rho \leq k \leq N} \left\| P_k \Psi_t \right\|_2^2 = \sum_{\rho \leq k \leq N} m(k) \left\| P_k \Psi_t \right\|_2^2 \leq \sum_{k=0}^N m(k) \left\| P_k \Psi_t \right\|_2^2 = \langle \Psi_t, m\tilde{r} \Psi_t \rangle.
\]

By Lemma 2.1, there is a \( C \in \text{Bounds} \) such that

\[
\left\| \Psi_t - \tilde{\Psi}_t \right\|_2 \leq C(t) \sqrt{\frac{\Lambda}{\rho}},
\]

which concludes the proof of Theorem 1.5. \( \Box \)
Proof of Theorem 1.6. Notice that \( \| {P}_k^{\varphi} \Psi \| \geq \| {P}_{-k}^{\varphi} \tilde{\Psi} \| \), for any \( 0 \leq k \leq N \). This fact and definition (26) yield

\[
\langle \overline{m^{\varphi}} \rangle_t = \sum_{0 \leq k \leq N} m(k) \langle \Psi_t, {P}_k^{\varphi} \Psi_t \rangle \geq \sum_{0 \leq k \leq N} m(k) \langle \tilde{\Psi}_t, {P}_k^{\varphi} \tilde{\Psi}_t \rangle = \Lambda \sum_{0 \leq k \leq N} \frac{k}{N} \langle \tilde{\Psi}_t, {P}_k^{\varphi} \tilde{\Psi}_t \rangle - \sum_{0 \leq k \leq N} \left( \frac{k}{\rho} - m(k) \right) \langle \tilde{\Psi}_t, {P}_k^{\varphi} \tilde{\Psi}_t \rangle. 
\]

(101)

Since \( {P}_k^{\varphi} \tilde{\Psi}_t = 0 \), for \( k > \rho \), and \( \frac{k}{\rho} - m(k) = 0 \), for \( 0 \leq k \leq \rho \) – see (27) – term (101) vanishes. Using (30) and the symmetry of bosonic wave functions, we get

\[
\Lambda \sum_{0 \leq k \leq N} \frac{k}{N} \langle \tilde{\Psi}_t, {P}_k^{\varphi} \tilde{\Psi}_t \rangle = \Lambda \sum_{0 \leq k \leq N} \frac{1}{N} \langle \tilde{\Psi}_t, q_k^{\varphi} \tilde{\Psi}_t \rangle = \Lambda \langle \tilde{\Psi}_t, q_1^{\varphi} \tilde{\Psi}_t \rangle. 
\]

This implies that

\[
\Lambda \langle \tilde{\Psi}_t, q_1^{\varphi} \tilde{\Psi}_t \rangle \leq \langle \overline{m^{\varphi}} \rangle_t, 
\]

(102)

Furthermore, upon inserting identity operators, in the form of \( \text{id}_{\Psi_t} = p_i^{\varphi} + q_i^{\varphi} \), the difference of the density matrices can be bounded by

\[
\left\| P_t^{(\text{micro})} - P_t^{(\text{macro})} \right\| \equiv \left\| \Lambda q_t^{(\text{ref})} \text{tr}_{x_2,...,x_N} \tilde{\Psi}_t \langle \tilde{\Psi}_t | q_t^{(\text{ref})} - | \epsilon_i \rangle \langle \epsilon_i | \right\| 
\leq \left\| \Lambda q_t^{(\text{ref})} \text{tr}_{x_2,...,x_N} \left[ p_i^{\varphi} \tilde{\Psi}_t \langle \tilde{\Psi}_t | p_i^{\varphi} \right] q_t^{(\text{ref})} - | \epsilon_i \rangle \langle \epsilon_i | \right\| 
+ 2\Lambda \left\| q_t^{(\text{ref})} \text{tr}_{x_2,...,x_N} \left[ p_i^{\varphi} \tilde{\Psi}_t \langle \tilde{\Psi}_t | q_i^{\varphi} \right] q_t^{(\text{ref})} \right\|. 
\]

(103)

In order to estimate (103), we shall need the preliminary bound

\[
\left\| q_t^{(\text{ref})} | \varphi_t \rangle \langle \varphi_t | q_t^{(\text{ref})} - | \epsilon_i \rangle \langle \epsilon_i | \right\| 
= \left\| q_t^{(\text{ref})} \left[ \varphi_t^{(\text{ref})} + | \epsilon_i \rangle \langle \epsilon_i | \right] q_t^{(\text{ref})} - | \epsilon_i \rangle \langle \epsilon_i | \right\| 
\leq \left\| q_t^{(\text{ref})} | \epsilon_i \rangle \langle \epsilon_i | q_t^{(\text{ref})} - | \epsilon_i \rangle \langle \epsilon_i | \right\| 
\leq 2 \left\| q_t^{(\text{ref})} | \epsilon_i \rangle \langle \epsilon_i | \right\| 
\leq \frac{C(t)^2}{\Lambda} + \frac{C(t)}{\Lambda^{1/2}}, 
\]

(106)

where, in the last two lines, we have used (98) and (99) of Lemma 2.11, (see Subsection 2.3.4). We are now prepared to provide the estimates of terms (103), (104) and (105):

**Term (103):** Fubini’s Theorem justifies the identity

\[
\langle \varphi_t | A_{1/2} | \tilde{\Psi}_t \rangle = \langle \Psi_t | A_{1/2} | \varphi_t \rangle = 1 - \langle \Psi_t, q_1^{\varphi} \tilde{\Psi}_t \rangle. 
\]

The right side can be bounded according to

\[
\left| 1 - \langle \Psi_t, q_1^{\varphi} \tilde{\Psi}_t \rangle \right| \leq 1 + \frac{\langle \overline{m^{\varphi}} \rangle}{\Lambda} \leq 2. 
\]

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provided $\Lambda$ is sufficiently large. Hence, (102) and (106), together with (98) and (99) of Lemma 2.11, guarantee that

\begin{equation}
(103) \quad = \left\| \Lambda q_t^{(\text{ref})} \left| \varphi_t \right\|_{\Lambda^{1/2}} \right\| \left\langle \varphi_t \right| tr_{x_2, \ldots, x_N} [\varPsi_t] \left\langle \varPsi_t \right| \left\| \varphi_t \right\|_{\Lambda^{1/2}} \left\langle \varphi_t \right| q_t^{(\text{ref})} - |\epsilon_t\rangle \langle \epsilon_t| \right\|
\end{equation}

\begin{equation}
= \left\| (1 - \left\langle \varPsi_t, q_t^{(\text{ref})}\right| q_t^{(\text{ref})} - |\epsilon_t\rangle \langle \epsilon_t| + \left\langle \varPsi_t, q_t^{(\text{ref})}\right| \left\| |\epsilon_t\rangle \langle \epsilon_t| \right\| (1 - \left\langle \varPsi_t, q_t^{(\text{ref})}\right| q_t^{(\text{ref})} - |\epsilon_t\rangle \langle \epsilon_t| + \left\langle \varPsi_t, q_t^{(\text{ref})}\right| \left\| |\epsilon_t\rangle \langle \epsilon_t| \right\|^2
\end{equation}

\begin{equation}
\leq 2 \left( \frac{C(t)^2}{\Lambda} + \frac{C(t)}{\Lambda^{1/2}} \right) + \frac{\langle m^{\beta} \rangle_t C(t)^2}{\Lambda}. \tag{107}
\end{equation}

**TERM (104):** Thanks to (98) of Lemma 2.11 we have that

\begin{equation}
(104) \quad = 2 \Lambda \left\| q_t^{(\text{ref})} tr_{x_2, \ldots, x_N} \left[ p_{1, t}^{(\varphi)} \left| \varPsi_t \right\rangle \langle \varPsi_t \right| q_t^{(\text{ref})} \right\|
\end{equation}

\begin{equation}
\leq 2 \Lambda \left\| q_t^{(\text{ref})} \left| \varphi_t \right\| \left\| q_t^{(\text{ref})} \right\|_2 \left\| \varPsi_t \right\|_2
\end{equation}

\begin{equation}
= 2 \Lambda \left\| q_t^{(\text{ref})} \left( \varphi_t \left| \varphi_t \right\| + \epsilon_t \right) \right\| \left\| q_t^{(\text{ref})} \right\|_2
\end{equation}

\begin{equation}
\leq 2 \Lambda \left\| \epsilon_t \right\|_2 \sqrt{\langle m^{\beta} \rangle_t}
\end{equation}

\begin{equation}
\leq 2 \sqrt{\langle m^{\beta} \rangle_t} C(t). \tag{108}
\end{equation}

**TERM (105):** A straight-forward computation yields

\begin{equation}
(105) \quad = \Lambda \left\| q_t^{(\text{ref})} tr_{x_2, \ldots, x_N} \left[ q_t^{(\varphi)} \left| \varPsi_t \right\rangle \langle \varPsi_t \right| q_t^{(\text{ref})} \right\|
\end{equation}

\begin{equation}
\leq \Lambda \left\| q_t^{(\varphi)} \right\|^2
\end{equation}

\begin{equation}
\leq \langle m^{\beta} \rangle_t. \tag{109}
\end{equation}

Collecting estimates (107), (108) and (109) we find

\begin{equation}
\left\| \rho_t^{(\text{micro})} - \rho_t^{(\text{macro})} \right\| \leq \frac{C(t)^2}{\Lambda} + \frac{C(t)}{\Lambda^{1/2}} + \frac{\langle m^{\beta} \rangle_t C(t)^2}{\Lambda} + 2 \sqrt{\langle m^{\beta} \rangle_t} C(t) + \langle m^{\beta} \rangle_t.
\end{equation}

However, thanks to (97), Lemma 2.1 shows that

\begin{equation}
\left\langle m^{\beta} \right\rangle_t \leq C(t) \frac{\Lambda}{\rho}, \quad 0 \leq t \leq T. \tag{110}
\end{equation}

As a consequence, there is a $C \in \text{Bounds}$ such that

\begin{equation}
\left\| \rho_t^{(\text{micro})} - \rho_t^{(\text{macro})} \right\| \leq C(t) \sqrt{\frac{\Lambda}{\rho}}.
\end{equation}
Proof of Theorem 1.4. Theorems 1.5 and 1.6 imply that
\[
\left\| \rho_i^{(\text{micro})} - \rho_i^{(\text{macro})} \right\| \leq \left\| \rho_i^{(\text{micro})} - \tilde{\rho}_i^{(\text{micro})} \right\| + \left\| \tilde{\rho}_i^{(\text{micro})} - \rho_i^{(\text{macro})} \right\|
\]
\[
\leq C \Lambda \left\| \Psi - \tilde{\Psi} \right\|_2 + C(t) \frac{\Lambda}{\rho}
\]
\[
\leq C \Lambda \Lambda C(t) \frac{\Lambda}{\rho} + C(t) \frac{\Lambda}{\rho} \leq C(t) \frac{\Lambda^{3/2}}{\rho^{1/2}}.
\]

\[\square\]

2.3 A Priori Propagation Estimates

In this section we prove the propagation estimates (97), (98) and (99) – Corollary 2.10 and Lemma 2.11 – that have been required in the proofs of our first three main results.

To gain the required control of the solutions to the non-linear equations (11), (12), and (14) turns out to be quite involved. Therefore, it is convenient, to first study the dynamics on a torus, \( \mathbb{T} \), meaning that we view the region \( \Lambda \) as a torus and impose periodic boundary conditions. In order to distinguish these two different situations in our notations, we use the following convention. On \( \mathbb{R}^3 \) we refer to the solutions of equations (2), (11), (12), and (14) as before, i.e., as
\[
t \mapsto \Psi_t, \quad t \mapsto \varphi_t, \quad t \mapsto \phi_i^{(\text{ref})}, \quad t \mapsto \epsilon_t,
\]
whereas, on \( \mathbb{T} \), we write
\[
t \mapsto \Psi_t, \quad t \mapsto \varphi_t, \quad t \mapsto \phi_i^{(\text{ref})}, \quad t \mapsto \epsilon_t.
\]
The corresponding initial conditions on the torus are
\[
e^{\hat{\mathcal{H}}_U \|t\|} \varphi_0^T := \phi_0^{T,(\text{ref})} + \psi_0^T, \quad \phi_0^{T,(\text{ref})} := 1, \quad \epsilon_0^T := \epsilon_0; \quad (111)
\]
see Condition 1.2. Note that we neither distinguish the differential operators on \( \mathbb{T} \) and \( \mathbb{R}^3 \) in our notation, nor we make the domain, \( \Lambda \), of integration explicit in the integrals. Both can be unambiguously inferred from context. Furthermore, for some \( T \leq \infty \) we assume the above solutions to exist on the time interval \([0, T)\) and consider only times \( t \in [0, T)\).

One of the main goals of this section is to provide \( L^\infty \) norms on \( \phi_i^{(\text{ref})} \), \( \varphi_i \), and \( \epsilon_i \). The advantage of the torus is that the respective reference state \( \phi_i^{T,(\text{ref})} \) is simply a constant, whereas \( \phi_i^{(\text{ref})} \) on \( \mathbb{R}^3 \) has tails. In consequence, on the torus the only kinetic energy there is stems from the excitation. It can be readily estimated by energy conservation and provides an estimate that is good enough to prevent excessive clustering of particles. Heuristically, the same is true for the reference state in \( \mathbb{R}^3 \) as it is very flat. However, there it is more difficult to distinguish the kinetic energy due to the excitation and the one due to the tails of the reference state in the technical estimates. Therefore, we first study \( \phi_i^{T,(\text{ref})}, \varphi_i^T, \) and \( \epsilon_i^T \) on the torus in Section 2.3.1. Afterwards we construct auxiliary wave functions on \( \mathbb{R}^3 \) by means of the torus wave functions which are already in some sense close \( \phi_i^{(\text{ref})}, \varphi_i \), and \( \epsilon_i \), respectively. The propagation of errors is then controlled by Grönwall arguments which allow to extend the results in the case of the torus to the one of \( \mathbb{R}^3 \); see Sections 2.3.2 and 2.3.3. The latter sections also provide the required control of the excitations which is discussed in Section 2.3.4.
While the quantum mechanical spreading due to the Laplace term usually tends to relax bad situations, the pair-interaction due to $U$ could give rise to such, and a strategy is needed to control the $L^\infty$ norms of solutions over time. Here it is important to recall that the respective $L^2$ norms $\phi_t^{(ref)}$ and $\varphi_t$ scale proportionally to $\Lambda^{1/2}$. Hence, over time the growth of the solutions due to the interaction can not simply be controlled by using an $L^2$ estimate in a Cook’s argument. For this reason we introduce the following Lemma 2.4 which will be applied frequently below. It holds on $\mathbb{R}^3$ as well as on the torus $\mathbb{T}$ and makes use of the following convenient norms:

**Definition 2.3.** For $0 \leq p_1, p_2, p_3, \ldots \leq \infty$ we define the norms

$$
\|\zeta\|_{p_1,p_2,p_3,\ldots} := \inf_{\zeta = \zeta_1 + \zeta_2 + \ldots} \left(\|\zeta\|_{p_1} + \|\zeta\|_{p_2} + \|\zeta\|_{p_3} + \ldots\right).
$$

In order to compress the notation we also use

$$
\|\zeta\|_{p_1,p_2,p_3,\ldots} := \|\zeta\|_{p_1} + \|\zeta\|_{p_2} + \ldots.
$$

**Lemma 2.4.** Let $U \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$ be a general potential. Let $\zeta_t$ be solution of the nonlinear equation

$$
i \partial_t \zeta_t(x) = \left(-\frac{1}{2} \Delta + U * |\zeta_t|^2(x)\right) \zeta_t(x),
$$

for an initial value $\zeta_{t=0} = \zeta_0$ such that:

$$
(\|\zeta_{t=0}\|_\infty) \left\| \hat{\zeta}_0 \right\|_1 \leq C_1 \quad \text{and} \quad (\|\zeta_{t}\|_{2,\Lambda^\infty}) \left\| \hat{\zeta}_t \right\|_1 \leq C_2(t) \tag{112}
$$

for some $C_1, C_2 \in \text{Bounds}$. Then there exists a $C_3 \in \text{Bounds}$ such that

$$
(\|\zeta\|_\infty) \left\| \hat{\zeta}_t \right\|_1 \leq C_3(t).
$$

**Proof.** Grönwall’s Lemma, the bound on the time derivative

$$
\partial_t \left\| \hat{\zeta}_t \right\|_1 \leq \int dk \frac{\Im \hat{\zeta}_t^*(k) \left( \frac{k^2}{2} \hat{\zeta}_t(k) + U * |\zeta_t|^2 \hat{\zeta}_t(k) \right)}{|\hat{\zeta}_t(k)|}
\leq \int dk \int dl \int dp \left| \tilde{U}(l) \tilde{\zeta}_t(l-p) \tilde{\zeta}_t(p) \hat{\zeta}_t(k-l) \right|
\leq \int dl \int dp \left| \tilde{U}(l) \tilde{\zeta}_t(l-p) \right| \left\| \hat{\zeta}_t \right\|_1
\leq C \|U\|_{1,2,\Lambda^\infty} \left\| \hat{\zeta}_t \right\|_1 \left\| \hat{\zeta}_t \right\|_1
\leq CC_1 C_2(t)^2 \left\| \hat{\zeta}_t \right\|_1 =: C_3(t) \left\| \hat{\zeta}_t \right\|_1,
$$

and the assumption on the initial condition (112) imply the claim.

The lemma states that an a priori bound in the $\| \cdot \|_{2,\Lambda^\infty}$ norm is sufficient to maintain control over the $L^\infty$ norm over time. The strategy will therefore be to establish such a priori norms in the cases of $\phi_t^{(ref)}$, $\varphi_t$, and $\epsilon_t$ and then apply the above lemma.
2.3.1 Estimates on the Torus

As discussed this section provides the needed properties of the evolution equations on the torus \( T \) for initial values (111) and repulsive potentials \( U \), i.e.,

\[ U \geq 0. \tag{113} \]

On \( T \) the unique solution to the evolution equation (12) of the reference state that corresponds to initial value (111) is given by the constant, i.e.,

\[ \phi^T_i,\text{ref} = 1 \quad \text{for all } t \in \mathbb{R}. \tag{114} \]

In consequence, Condition 1.2 and (111) imply

\[ \| \phi^T_0 \|_1 \leq C, \tag{115} \]

and because of (113), we have

\[ E_{\phi^T_0} = \langle \phi^T_0, h \rangle_{L^2} \geq 0. \tag{116} \]

The evolution of the excitation wave function on the torus \( T \) is, analogously as in the case of \( \mathbb{R}^3 \), defined by

\[ \epsilon^T_i = \phi^T_i e^{\|U\|_1 t} - \phi^T_i,\text{ref} = \frac{\phi^T_i e^{\|U\|_1 t} - 1 - 1}{1 - 1}. \tag{117} \]

Together with (114) and (14) this implies

\[ i \partial_t \epsilon^T_i(x) = \left( \frac{1}{2} \Delta + U \ast |\epsilon^T_i|^2(x) + U \ast 2 \Re \epsilon^T_i \right) \epsilon_i(x) \tag{118} \]

\[ + U \ast (| \epsilon^T_i|^2(x) + 2 \Re \epsilon^T_i \dot{x}) \cdot \]

**Lemma 2.5.** Let \( U \in C^\infty_c(\mathbb{R}^3, \mathbb{R}^+) \) be a repulsive potential. There are \( C_1, C_2, C_4 \in \text{Bounds} \) such that for all \( 1/4 \leq r < 1 \)

\[ \| \nabla \phi^T_i \|_2 = \| \nabla \epsilon^T_i \|_2 \leq C_1, \tag{119} \]

\[ \| \phi^T_i \|_{L^2} \leq \| \phi^T_0 \|_{L^2} \leq C_2(t), \tag{120} \]

\[ \| \nabla \epsilon^T_i \|_2 \leq \Lambda^{-\frac{1}{2}} C_3(t). \tag{121} \]

**Proof.** To see (119) we begin by noting that the evolution equation (11) conserves the energy so that due to (114), (116), and \( U \geq 0 \) one finds

\[ \| \nabla \phi^T_i \|_2^2 = \| \nabla \epsilon^T_i \|_2^2 = E_{\phi^T_0} - \langle \phi^T_i, U \ast |\phi^T_0|^2 \rangle \leq E_{\phi^T_0}. \]

Hence, the claim (119) holds for the choice of constant \( C_1 = E_{\phi^T_0} \).

In order to provide the estimate (120) we exploit that the Schrödinger dispersion effectively acts only on that part of the wave function which is not constant. It is therefore convenient to split \( \phi^T_i \) into two parts. For this purpose we introduce the auxiliary wave function \( \overline{\phi^T_i} \) by

\[ \overline{\phi^T_i} = \exp \left( -i \int_0^t ds \ U \ast |\phi^T_i|^2 \right) \phi^T_0 \tag{122} \]
so that
\[ |\varphi_i^T|^2 = |\varphi_0^T|. \] (123)

Next, we split the desired norm of \( \varphi_i^T \) as follows
\[
\|\varphi_i^T\|_{L^2} = \inf_{\varphi_i^T = \varphi_i^0 + \varphi_{i,2}} \left( \|\varphi_{i,2}\|_2 + \|\varphi_{i,\infty}\|_\infty \right) \leq \inf_{\varphi_i^T = \varphi_i^0 + \varphi_{i,2}} \left( \|\varphi_{i,2}\|_2 + \|\varphi_{i,\infty}\|_1 \right) = \|\varphi_i^T\|_{1,2} \] (124)
for which we find
\[
\|\varphi_i^T - \varphi_i^T\|_{1,2} \leq \|\varphi_i^T - \varphi_i^T\|_2 + \|\varphi_i^T\|_1 \] (125)
\[
= \|\varphi_i^T - \varphi_i^T\|_2 + \|\varphi_i^0\|_1 \] (126)
\[
\leq \|\varphi_i^T - \varphi_i^T\|_2 + C, \] (127)

where we used (123) and (115). It is left to control the difference of \( \varphi_i^T \) and \( \varphi_i^T \) in the \( L^2 \) norm. Thanks to the conservation of the \( L^2 \) norms of \( \varphi_i^T \) and \( \varphi_i^T \), the evolution equation (11), (122), and (119) we find
\[
\partial_t \|\varphi_i^T - \varphi_i^T\|_2^2 \leq 2\partial_t \langle \varphi_i^T, \varphi_i^T \rangle = \|\varphi_i^T, \Delta \varphi_i^T\| \leq \|\nabla \varphi_i^T\|_2 \|\nabla \varphi_i^T\|_2 \leq C_1 \|\nabla \varphi_i^T\|_2 . \] (128)

Using (122), the kinetic energy of \( \varphi_i^T \) can be estimated by
\[
\|\nabla \varphi_i^T\|_2 \leq \|\nabla \varphi_i^T\|_2 + \int_0^t ds \|U \ast (2 \Re \varphi_i^T \nabla \varphi_i^T) \varphi_i^0\|_2 \] (129)
\[
\leq \|\nabla \varphi_i^0\|_2 + 2\|U\|_{1,2} \int_0^t ds \|\varphi_i^T\|_2 \|\nabla \varphi_i^T\|_2 \|\varphi_i^0\|_{\infty} \] (130)
\[
\leq C(t) \left( 1 + \int_0^t ds \|\varphi_i^T\|_2 \right), \] (131)

where we used (119). Thus, collecting the estimates (128) and (131) yields
\[
\partial_t \|\varphi_i^T - \varphi_i^T\|_2^2 \leq C(t) \left( 1 + \int_0^t ds \|\varphi_i^T - \varphi_i^T\|_2^2 \right) \] (132)

where we have used the inequality \( x \leq 1 + x^2, \forall x \in \mathbb{R} \), to get a quadratic exponent under the integral. Grönwall’s Lemma then ensures the existence of a \( C \in \text{Bounds} \) such that
\[
\|\varphi_i^T - \varphi_i^T\|_2^2 \leq C(t), \] (132)

which together with (127) and \( \varphi_0^T = \varphi_0^T \) implies the claim (120).

We now prove the remaining claim (121). First, we note that according to (118)
\[
\partial_t \|\chi_2 \varphi_T^2\|_2^2 \leq \left| \left< \chi_2 \varphi_T^2, \left[ \varphi_T^2 \varphi_T^2 \right] \right> \right| \] (133)
\[
+ 2 \left| \left< U \ast (\varphi_T^2 \varphi_T^2) + \Re \varphi_T^2 \varphi_T^2 \right) \chi_2 \varphi_T^2 \right> \right|. \] (134)
Using partial integration, (8), and (119) we find

\[
(133) = \left| \left< \epsilon_i^T, \chi_r \nabla \chi_r, \nabla \epsilon_i^T \right> \right|
\leq ||\chi \epsilon_i^T||_2 ||\nabla \chi_r||_\infty ||\nabla \epsilon_i^T||_2
\leq ||\chi \epsilon_i^T||_2 C \Lambda^{-\frac{1}{2}} C_1.
\] (135)

Next, equation (117) together with (114) imply

\[
|\epsilon_i^T|^2 + 2 \Re \epsilon_i^{T+} \leq |\epsilon_i^T| (1 + |\varphi_i^T|),
\]

which yields the estimate

\[
(134) \leq 2 \left| \chi_r U * \left[ |\epsilon_i^T|^2 (1 + |\varphi_i^T|) \right] \right|_2 ||\chi \epsilon_i^T||_2
\leq 2 \left[ \int dx \int dy \ U(x-y) \left( |\epsilon_i^T|^2 (1 + |\varphi_i^T|) \right) \chi_r(y) \right]^{1/2} ||\chi \epsilon_i^T||_2
\leq 2 \left[ \int dx \int dy \ U(x-y) \left( |\epsilon_i^T|^2 (1 + |\varphi_i^T|) \right) \left( \chi_r(x) - \chi_r(y) \right) \right]^{1/2} ||\chi \epsilon_i^T||_2.
\] (136)

Furthermore,

\[
(137) \leq C ||U||_{1,2} ||1 + |\varphi_i^T| ||_{2,\infty} ||\chi \epsilon_i^T||_2^2,
\] (139)

\[
(138) \leq C \Lambda^{-\frac{1}{2}} D ||U||_{1,2} ||1 + |\varphi_i^T| ||_{2,\infty} ||\epsilon_i^T||_2 ||\chi \epsilon_i^T||_2^2,
\] (140)

where we have used that \( U \) is supported in a ball of radius \( D \geq 0 \) around the origin so that by (8)

\[
|U(x-y)(\chi_r(x) - \chi_r(y))| \leq C \Lambda^{-\frac{1}{2}} |U(x-y)|D.
\] (141)

Now equation (123) and the bound in (132) ensure

\[
||1 + |\varphi_i^T| ||_{2,\infty} \leq ||1 + |\varphi_i^T| + |\varphi_i^T - \varphi_i^T||_{2,\infty} \leq ||1 + |\varphi_i^T| ||_{\infty} + ||\varphi_i^T - \varphi_i^T||_2
\leq 1 + ||\varphi_i^T||_{\infty} + C(t) \leq C(t).
\] (142)

Finally, a similar computation as the one used in (133) gives

\[
\partial_t ||\epsilon_i^T||_2^2 \leq 2 \left| \left< U * (|\epsilon_i^T|^2 + \Re \epsilon_i^{T+}), \epsilon_i^T \right> \right|
\leq 2 ||U||_{1,2} ||1 + |\varphi_i^T| ||_{2,\infty} ||\epsilon_i^T||_2^2
\]

which thanks to (142) and Grönwall’s Lemma means

\[
||\epsilon_i^T||_2 \leq C(t).
\] (143)

Hence, (139) and (140) imply

\[
(134) \leq C(t) \Lambda^{-\frac{1}{2}} ||\chi \epsilon_i^T||_2 + C(t) ||\chi \epsilon_i^T||_2^2.
\]

Finally, (133), which was estimated in (135), and (134) guarantee

\[
||\chi \epsilon_i^T||_2 \leq C(t) \left( \Lambda^{-\frac{1}{2}} + ||\chi \epsilon_i^T||_2 \right).
\]

Note that by initial constraint (6) one has \( \chi_r \epsilon_0^T = 0 \) for \( r \geq 1/4 \). In conclusion, the claim (121) is a consequence of Grönwall’s Lemma. \( \square \)
Lemma 2.4 and Lemma 2.5 imply the following corollary.

**Corollary 2.6.** Let $U \in C_c^{\infty}(\mathbb{R}^3, \mathbb{R}_0^+)$ be a repulsive potential. There is a $C \in \text{Bounds}$ such that

$$
\|e_t^2\|_\infty \leq 1 + \|\varphi_t^2\|_\infty \leq 1 + \left\| \varphi_t^2 \right\|_1 \leq C(t).
$$

### 2.3.2 Estimates for $\phi_t^{(\text{ref})}$

**Lemma 2.7.** Let $U \in C_c^{\infty}(\mathbb{R}^3, \mathbb{R})$ be a general potential, and let $\Lambda$ be sufficiently large. There are $C_1, C_2 \in \text{Bounds}$ such that

$$
\left\| \phi_t^{(\text{ref})} \right\|_{L^\infty} \leq \left\| \left| \phi_t^{(\text{ref})} \right| \right\|_{L^2} \leq C_1(t),
$$

and

$$
\left\| \phi_t^{(\text{ref})} - \phi_0^{(\text{ref})} \right\|_2 \leq C_2(t) \Lambda^{-\frac{1}{2}}.
$$

**Proof.** In order to provide the bound (144) we introduce the auxiliary wave function

$$
\tilde{\phi}_t := \exp \left( -itU \left( \left| \phi_0^{(\text{ref})} \right|^2 - 1 \right) \right) \phi_0^{(\text{ref})},
$$

and using the evolution equation (12) we estimate the time derivative

$$
\partial_t \left\| \phi_t^{(\text{ref})} - \tilde{\phi}_t \right\|_2 \leq \left\| -\frac{1}{2} \Delta \tilde{\phi}_t + U \ast \left( \left| \phi_t^{(\text{ref})} \right|^2 - \left| \phi_0^{(\text{ref})} \right|^2 \right) \tilde{\phi}_t \right\|_2 \leq \left\| \frac{1}{2} \Delta \tilde{\phi}_t \left\|_2 + \|U\|_{L^2} \left\| \left| \phi_t^{(\text{ref})} \right|^2 - \left| \phi_0^{(\text{ref})} \right|^2 \right\|_1 \right\| \tilde{\phi}_t \right\|_\infty.
$$

We estimate the terms on the right-hand side of (147) individually:

Noting that

\[
\Delta \tilde{\phi}_t = \left( \left[ -itU \ast \nabla |\phi_0^{(\text{ref})}|^2 \right] \phi_0^{(\text{ref})} + \nabla \phi_0^{(\text{ref})} \right) \exp \left( -itU \ast \left( |\phi_0^{(\text{ref})}|^2 - 1 \right) \right),
\]

and recalling (5) and (10), we find

$$
\left\| \nabla \tilde{\phi}_t \right\|_\infty \leq \left( 1 + 2|t| \|U\|_1 \left\| \phi_0^{(\text{ref})} \right\|^2 \right) \left\| \nabla \phi_0^{(\text{ref})} \right\|_\infty \leq C(t) \Lambda^{-\frac{1}{4}},
$$

$$
\left\| \nabla \tilde{\phi}_t \right\|_2 \leq \left( 1 + 2|t| \|U\|_1 \left\| \phi_0^{(\text{ref})} \right\|^2 \right) \left\| \nabla \phi_0^{(\text{ref})} \right\|_2 \leq C(t) \Lambda^{\frac{1}{2}},
$$

$$
\left\| \Delta \tilde{\phi}_t \right\|_2 \leq 2|t| \|U\|_1 \left\| \phi_0^{(\text{ref})} \right\| \left( \left\| \nabla \phi_0^{(\text{ref})} \right\|_\infty \left\| \Delta \phi_0^{(\text{ref})} \right\|_2 + \left\| \nabla \phi_0^{(\text{ref})} \right\|_\infty \left\| \nabla \phi_0^{(\text{ref})} \right\|_2 \right) + 4|t|^2 \|U\|_1^2 \left\| \phi_0^{(\text{ref})} \right\|^3 \left\| \nabla \phi_0^{(\text{ref})} \right\|_\infty \left\| \nabla \phi_0^{(\text{ref})} \right\|_2 \right) + 4|t| \|U\|_1 \left\| \phi_0^{(\text{ref})} \right\|_\infty \left\| \nabla \phi_0^{(\text{ref})} \right\|_\infty \left\| \nabla \phi_0^{(\text{ref})} \right\|_2
\]

$$
\leq C(t) \Lambda^{\frac{1}{4}},
$$

\[\text{(150)}\]
These estimates together with (5), $|\tilde{\phi}_t| = |\phi_0^{(\text{ref})}|$, and (147) ensure
\[
\partial_t \left\| \phi_t^{(\text{ref})} - \tilde{\phi}_t \right\|_2 \leq C(t) \Lambda^{-\frac{1}{6}} + C \left\| \phi_t^{(\text{ref})} - \tilde{\phi}_t \right\|^2_2 + 2\Re \tilde{\phi}_t \left( \phi_t^{(\text{ref})} - \tilde{\phi}_t \right) \right\|_{1\Lambda^2}
\leq C(t) \Lambda^{-\frac{1}{6}} + C \left( \left\| \phi_t^{(\text{ref})} - \tilde{\phi}_t \right\|^2_2 + 2\left\| \phi_0^{(\text{ref})} \right\|_\infty \left\| \phi_t^{(\text{ref})} - \tilde{\phi}_t \right\|_2 \right).
\]
Assume that there is a $0 \leq T \leq \infty$ such $\left\| \phi_t^{(\text{ref})} - \phi_0^{(\text{ref})} \right\|_2 \leq 1$ for all $t \in [0,T]$. In this case we find
\[
\partial_t \left\| \phi_t^{(\text{ref})} - \tilde{\phi}_t \right\|_2 \leq C(t) \Lambda^{-\frac{1}{6}} + C \left\| \phi_t^{(\text{ref})} - \phi_0^{(\text{ref})} \right\|_2,
\]
which thanks to Grönwall’s Lemma and $\phi_0^{(\text{ref})} = \tilde{\phi}_0$ implies
\[
\left\| \phi_t^{(\text{ref})} - \tilde{\phi}_t \right\|_2 \leq C(t) \Lambda^{-\frac{1}{6}} \quad \text{for } t \in [0,T].
\]
(151)

Clearly, upon choosing $\Lambda$ sufficiently large the supremum of such times $T$ in infinite. Hence, (151) holds for all $t \in \mathbb{R}$ provided $\Lambda$ is sufficient large. In conclusion, due to (5) we observe
\[
\left\| \phi_t^{(\text{ref})} \right\|_{2\Lambda_\infty} \leq \left\| \tilde{\phi}_t^{(\text{ref})} \right\|_{1\Lambda_2} \leq \left\| \tilde{\phi}_t \right\|_1 + \left\| \phi_t^{(\text{ref})} - \tilde{\phi}_t \right\|_2 \leq \left\| \phi_0^{(\text{ref})} \right\|_1 + \left\| \phi_t^{(\text{ref})} - \tilde{\phi}_t \right\|_2
\leq C + C(t) \Lambda^{-\frac{1}{6}},
\]
which implies that the claim (144) is true.

Moreover, claim (145) can be seen by (151) and
\[
\left\| \phi_t^{(\text{ref})} - |\phi_0^{(\text{ref})}| \right\|_2 \leq \left\| \phi_t^{(\text{ref})} - \tilde{\phi}_t \right\|_2.
\]

\[\square\]

Lemma 2.4 and Lemma 2.7 imply the following corollary.

**Corollary 2.8.** Let $U \in C^\infty_c(\mathbb{R}^3, \mathbb{R})$ be a general potential, and let $\Lambda$ be sufficiently large. There is a $C \in \text{Bounds}$ such that
\[
\left\| \phi_t^{(\text{ref})} \right\|_\infty \leq \left\| \phi_t^{(\text{ref})} \right\|_1 \leq C(t).
\]

2.3.3 Estimates for $\varphi_t$

**Lemma 2.9.** Let $U \in C^\infty_c(\mathbb{R}^3, \mathbb{R}_0^+)$ be a repulsive potential. There exists a $C \in \text{Bounds}$ such that
\[
\left\| \varphi_t \right\|_{2\Lambda_\infty} \leq \left\| \tilde{\varphi}_t \right\|_{1\Lambda^2} \leq C(t).
\]

**Proof.** In order to provide the desired bound we introduce the auxiliary wave function
\[
\tilde{\varphi}_t := \tilde{\phi}_t \varphi_t^{\tau}.
\]
(152)
Using the evolution equation (11) on $\mathbb{R}^3$, the corresponding one on the torus $\mathbb{T}$, and definition (146), we compute the time derivative

$$i\partial_t(\varphi_t - \tilde{\varphi}_t) = \left(-\frac{1}{2}\Delta + U \ast |\varphi_t|^2\right)(\varphi_t - \tilde{\varphi}_t)$$

$$- \frac{1}{2}\Delta \tilde{\varphi}_t + \tilde{\varphi}_t \frac{1}{2}\Delta \varphi_t^T$$

$$+ U \ast (|\varphi_t|^2 - |\varphi_0^{(\text{ref})}|^2 + 1 - |\varphi_t^T|^2) \tilde{\varphi}_t.$$

Recall that $|\tilde{\varphi}_t| = |\varphi_0^{(\text{ref})}|$. In consequence, we get the estimate

$$\partial_t||\varphi_t - \tilde{\varphi}_t||_2 \leq ||\nabla \tilde{\varphi}_t||_\infty ||\nabla \varphi_t^T||_2 + \frac{1}{2} ||\Delta \tilde{\varphi}_t||_2 ||\varphi_t^T||_\infty$$

$$+ ||U||_{1,2} |||\varphi_t|^2 - |\tilde{\varphi}_t|^2 + 1 - |\varphi_t^T|^2||_1 \Lambda_2 ||\tilde{\varphi}_t||_\infty.$$

Furthermore, we consider the bounds:

- The bounds in (119), (148), (150) and Corollary 2.6 ensure

$$||\nabla \tilde{\varphi}_t||_\infty ||\nabla \varphi_t^T||_2 + \frac{1}{2} ||\Delta \tilde{\varphi}_t||_2 ||\varphi_t^T||_\infty \leq C(t) \Lambda^{-\frac{1}{2}};$$

- Definition (146) and Corollary 2.6 imply

$$||\tilde{\varphi}_t||_\infty \leq ||\tilde{\varphi}_t||_\infty ||\varphi_t^T||_\infty \leq C(t);$$

- We have

$$|||\varphi_t|^2 - |\tilde{\varphi}_t|^2 + 1 - |\varphi_t^T|^2||_1 \Lambda_2 \leq |||\varphi_t|^2 - |\tilde{\varphi}_t|^2 + 1 - |\varphi_t^T|^2||_2 + |||\varphi_t|^2 - |\tilde{\varphi}_t|^2||_1 \Lambda_2;$$

and

- Recall definition (117). Using the identity

$$|\tilde{\varphi}_t|^2 - |\tilde{\varphi}_t|^2 + 1 - |\varphi_t^T|^2 = |1 + \epsilon_t^T|^2 |\tilde{\varphi}_t|^2 - |\tilde{\varphi}_t|^2 + 1 - |1 + \epsilon_t^T|^2$$

$$= (\epsilon_t^T + \epsilon_t^T + |\epsilon_t^T|^2) (|\tilde{\varphi}_t|^2 - 1)$$

we find

$$|||\varphi_t|^2 - |\tilde{\varphi}_t|^2 + 1 - |\varphi_t^T|^2||_2 \leq (2 + ||\epsilon_t^T||_\infty) ||\epsilon_t^T (|\tilde{\varphi}_t|^2 - 1)||_2 ||\tilde{\varphi}_t|| + 1||_\infty.$$

Moreover, $|\tilde{\varphi}_t| - 1 = |\varphi_0^{(\text{ref})}| - 1 \leq \chi_\Lambda$ as required in (9), so that by Lemma 2.5

$$||\epsilon_t^T (|\varphi_0| - 1)||_2 \leq ||\epsilon_t^T \chi_\Lambda||_2 \leq C(t) \Lambda^{-\frac{1}{2}},$$

and hence, by $|\tilde{\varphi}_t| = |\varphi_0^{(\text{ref})}|$, (5), and Corollary 2.6

$$|||\varphi_t|^2 - |\tilde{\varphi}_t|^2 + 1 - |\varphi_t^T|^2||_2 \leq C(t) \Lambda^{-\frac{1}{2}};$$

(154)
Lemma 2.11. Let $U \in C^\infty_c(\mathbb{R}^3, \mathbb{R}_0^+)$ be a repulsive potential. There exists a $C \in \text{Bounds}$ such that for all $t \in \mathbb{R}$ provided $\Lambda$ is sufficiently large. This implies

$$
\|\varphi_t - \tilde{\varphi}_t\|_2 \leq C(t)\Lambda^{-\frac{1}{6}} \quad (156)
$$

holds for all $t \in \mathbb{R}$ provided $\Lambda$ is sufficiently large. There exist $C_1, C_2 \in \text{Bounds}$ such that for all $1/4 < r < 1$

$$
\|\epsilon_t\|_2 \leq C_1(t),
$$

$$
\|\rho_t^{(\text{ref})}\epsilon_t\|_2 \leq \frac{C_2(t)}{\Lambda^{1/2}},
$$

$$
\|\chi_t\epsilon_t\|_2 \leq C(t)\Lambda^{-\frac{1}{6}}. \quad (157)
$$

Proof. Thanks to definition (13) and the evolution equations (11) and (12) we find

$$
\partial_t \|\epsilon_t\|^2 \leq \|U * (|\varphi_t|^2 - |\phi_t^{(\text{ref})}|^2) \phi_t^{(\text{ref})}\|_2
\leq \|U\|_{1,2} \left\|\|\varphi_t|^2 - |\phi_t^{(\text{ref})}|^2\|_{1,2} \|\phi_t^{(\text{ref})}\|_{\infty}\right..$$

The triangle inequality implies

$$
\left\|\|\varphi_t|^2 - |\phi_t^{(\text{ref})}|^2\|_{1,2}\right\| \leq \left\|\|\varphi_t^T|^2 - 1\|_2 + \left\|\|\varphi_t|^2 - |\phi_0^{(\text{ref})}|^2 - |\varphi_t^T|^2 + 1\right\|_{1,2} + \left\|\phi_t^{(\text{ref})}|^2 - |\phi_0^{(\text{ref})}|^2\right\|_2.
$$

The terms on the right-hand side can be estimated as follows:
• Corollary 2.6, definition of $\bar{\varphi}_t^T$ in (122), (132), definition of $\epsilon_t^T$ in (117), and (143) imply
\[
\|\varphi_t^T - 1\|_2 \leq \|\varphi_t^T + 1\|_\infty \|\bar{\varphi}_t^T - 1\|_2 \\
\leq C(t) \left( \|\varphi_t^T - \bar{\varphi}_t^T\|_2 + \|\bar{\varphi}_t^T - 1\|_2 \right) \\
\leq C(t) \left( 1 + \|\epsilon_t^T\|_2 \right) \\
\leq C(t);
\]

• The definition of $\bar{\varphi}_t$ in (146) together with the identify $|\phi_0^{\text{ref}}| = |\bar{\varphi}_t|$ and the bounds in (154) and (156) ensure
\[
\|\varphi_t^2 - |\phi_0^{\text{ref}}|^2 + 1 - |\varphi_t^T|^2\|_{1 \wedge 2} \leq C(t) \Lambda^{-\frac{1}{2}};
\]

• Recalling (145) we know that
\[
\|\phi_t^{\text{ref}} - |\phi_0^{\text{ref}}|^2\|_2 \leq C(t) \Lambda^{-\frac{1}{2}}.
\]

In consequence, we find
\[
\|\varphi_t^2 - |\phi_t^{\text{ref}}|^2\|_{1 \wedge 2} \leq C(t)
\]
and therefore
\[
\partial_t \|\epsilon_t\|_2 \leq C(t)
\]
which by Grönwall’s Lemma proves the claim (98) of this lemma.

We continue by recalling Condition 1.2 which ensures
\[
\|\rho_t^{\text{ref}} \epsilon_t\|_2 = \frac{1}{\Lambda} \|\phi_t^{\text{ref}}\|_2 \langle \phi_t^{\text{ref}}, \epsilon_t \rangle \leq \frac{1}{\Lambda^{1/2}} \|\phi_t^{\text{ref}}\| \langle \phi_t^{\text{ref}}, \epsilon_t \rangle.
\]

In order to estimate the right-hand side we recall the definition of $\epsilon_t$ in (13), the evolution equations (11) as well as (12), and regard
\[
i \partial_t \langle \epsilon_t, \phi_t^{\text{ref}} \rangle = i \partial_t \langle \epsilon_t^{\text{ref}} U \|_{1 \wedge 2} \varphi_t, \phi_t^{\text{ref}} \rangle \\
= \langle \epsilon_t^{\text{ref}} U * (|\varphi_t|^2 - |\phi_t^{\text{ref}}|^2) \phi_t^{\text{ref}} \rangle \\
= \langle \phi_t^{\text{ref}} U * (|\varphi_t|^2 - |\phi_t^{\text{ref}}|^2) \phi_t^{\text{ref}} \rangle \\
+ \langle \epsilon_t, U * (|\varphi_t|^2 - |\phi_t^{\text{ref}}|^2) \phi_t^{\text{ref}} \rangle.
\]

Note that term (160) is real. Hence, the bounds (98), (159), and Corollary 2.8 imply
\[
\partial_t \|\epsilon_t, \phi_t^{\text{ref}}\| \leq \|\epsilon_t, U * (|\varphi_t|^2 - |\phi_t^{\text{ref}}|^2) \phi_t^{\text{ref}}\| \\
\leq \|\epsilon_t\|_2 \|U\|_{1 \wedge 2} \|\varphi_t^2 - |\phi_t^{\text{ref}}|^2\|_{1 \wedge 2} \|\phi_t^{\text{ref}}\|_\infty \\
\leq C(t).
\]
An application of Grönwall’s Lemma concludes the proof of claim (99) of this lemma.

Finally, with the definition of $\bar{\phi}_t$ and $\epsilon_t^\tau$ in (146) and (117), respectively, we find the estimate
\[
\|\chi_t\epsilon_t\|_2 \leq \|\chi_t \bar{\phi}_t \epsilon_t^\tau\|_2 + \|\chi_t (\epsilon_t - \bar{\phi}_t \epsilon_t^\tau)\|_2 \leq \|\bar{\phi}_t\|_\infty \|\chi_t \epsilon_t^\tau\|_2 + \|\epsilon_t - \bar{\phi}_t \epsilon_t^\tau\|_2.
\]
Applying the definition of $\bar{\phi}_t$ in (152) we estimate
\[
\|\epsilon_t - \bar{\phi}_t \epsilon_t^\tau\|_2 \leq \|\varphi_t - \bar{\phi}_t\|_2 + \|\phi_t^{(\text{ref})} - \bar{\phi}_t\|_2.
\]
The estimate in (121) in Theorem 2.5 and the bounds (156), (151) imply the claim (157). \qed

### 2.4 Proof of Theorem 1.8

In this last section we provide the proof of the fourth main result:

**Proof of Theorem 1.8.** Since the Laplace operator is self-adjoint we find by means of the evolution equations (14) and (20) that
\[
\|\epsilon_t - \eta_t\|_2 \leq \left\| \frac{U \ast 2R (\epsilon_t \varphi_t^{(\text{ref})} - \eta_t)}{2R (\epsilon_t \varphi_t^{(\text{ref})} - \eta_t)} \right\|_2
\]
\[
+ \left\| \frac{U \ast 2R (\epsilon_t \varphi_t^{(\text{ref})} - \eta_t)}{2R (\epsilon_t \varphi_t^{(\text{ref})} - \eta_t)} \right\|_2
\]
\[
+ \left\| \frac{U \ast [\varphi_t^{(\text{ref})} - 1] \epsilon_t}{2R (\epsilon_t \varphi_t^{(\text{ref})} - \eta_t)} \right\|_2
\]
\[
+ \left\| \frac{U \ast [\varphi_t^{(\text{ref})} - 1] \epsilon_t}{2R (\epsilon_t \varphi_t^{(\text{ref})} - \eta_t)} \right\|_2
\]
we begin with the most crucial estimate
\[
\left\| U \ast [\varphi_t^{(\text{ref})} - 1] \epsilon_t \right\|_2 \leq \left\| \varphi_t^{(\text{ref})} \right\|_2 + 1 \left\| U \ast [\varphi_t^{(\text{ref})} - 1] \epsilon_t \right\|_2
\]
\[
\leq C(t) \left( \left\| U \ast [\varphi_t^{(\text{ref})} - 1] \epsilon_t \right\|_2 + \left\| U \ast [\varphi_t^{(\text{ref})} - \varphi_t^{(\text{ref})}] \epsilon_t \right\|_2 \right).
\]
Using the bounds (98), given in Lemma 2.11, and (145) we note
\[
\left\| U \ast [\varphi_t^{(\text{ref})} - \varphi_t^{(\text{ref})}] \epsilon_t \right\|_2 \leq \|U\|_2 \left\| \varphi_t^{(\text{ref})} \right\|_2 \left\| \varphi_t^{(\text{ref})} \right\|_2 \|\epsilon_t\|_2 \leq C(t) A^{-\frac{1}{2}}.
\]
Furthermore, $|\varphi_t^{(\text{ref})} - 1| \leq \chi\Lambda$ as required in (9), and (157) imply
\[
\left\| U \ast [\varphi_t^{(\text{ref})} - 1] \epsilon_t \right\|_2 \leq \|U \ast \chi\Lambda \epsilon_t\|_2
\]
\[
\leq \left\| \int dy U(\cdot - y)\chi\Lambda(\cdot)\epsilon_t(\cdot) \right\|_2 + \left\| \int dy U(\cdot - y)(\chi\Lambda(y) - \chi\Lambda(\cdot))\epsilon_t(\cdot) \right\|_2
\]
\[
\leq \|U\|_1 \|\chi\Lambda\|_2 + C \Lambda^{-\frac{1}{2}} \|U\|_1 \|\epsilon_t\|_2
\]
\[
\leq C(t) (A^{-\frac{1}{2}} + \|\epsilon_t\|_2^2),
\]
where we used again (141) and that $U$ is supported in a ball of radius $D \geq 0$. Using Corollary 2.8 we collect the following estimates:
For $\Lambda$ large enough one finds

$$\left\| U \ast 2\mathcal{R} \left( \epsilon_t^* \phi_t^{(\text{ref})} - \eta_t^* \right) \right\|_2 = \left\| \int dy \ U(y) 2\mathcal{R} \left( \epsilon_t^* (\cdot - y) \phi_t^{(\text{ref})}(\cdot - y) - \eta_t^*(\cdot - y) \right) \right\|_2 \leq 2 \left\| dy \ |U(y)| \ \left\| \epsilon_t^* (\cdot - y) \phi_t^{(\text{ref})}(\cdot - y) - \eta_t^*(\cdot - y) \right\|_2 \leq 2 \|U\|_1 \left( \|\epsilon_t - \eta_t\|_2 + \|(1 - \phi_t^{(\text{ref})})\epsilon_t\|_2 \right) \leq C \|\epsilon_t - \eta_t\|_2 + C(t)\Lambda^{-1/6},$$

where thanks to the ingredients:

- $\tilde{\phi}_t := \exp \left( -itU \ast (|\phi_0^{(\text{ref})}|^2 - 1) \right) \phi_0^{(\text{ref})}$, as defined in (146);
- $\|\phi_t^{(\text{ref})} - \tilde{\phi}_t\|_2 \leq C(t)\Lambda^{-\frac{1}{2}}$ from line (151);
- $\|\epsilon_t\|_\infty \leq C(t)$ from (97);
- $\|\phi_0\|_\infty \leq C$, as required in Condition 1.2;
- $\|\epsilon_t\|_2 \leq C(t)$ and $\|\chi_t\epsilon_t\|_2 \leq C(t)\Lambda^{-\frac{1}{2}}$ for $1/4 \leq r < 1$ as proven in Lemma 2.11;
- Since $U$ is supported in a ball of radius $D \geq 0$ and due to (9) in Condition 1.2 one has $U \ast (|\phi_0^{(\text{ref})}|^2 - 1) (x) = 0$ for $x \in B_{1/2\Lambda^{1/3} - 2D}$;
- Consequently, for sufficiently large $\Lambda$ one has $(1 - \tilde{\phi}_t^*) (1 - \chi_r)(x) = 0$ for $r = 1/4$;

we used

$$\| (1 - \phi_t^{(\text{ref})})^* \epsilon_t \|_2 \leq \| (1 - \tilde{\phi}_t^*) \epsilon_t \|_2 + \| (\phi_t^{(\text{ref})})^* - \tilde{\phi}_t^* \| \epsilon_t \|_2 \leq \| (1 - \tilde{\phi}_t^*)(1 - \chi_{1/4}) \epsilon_t \|_2 + \| (1 - \tilde{\phi}_t^*) \chi_{1/4} \epsilon_t \|_2 + C(t)\Lambda^{-1/6} \leq 0 + 2C(t)\Lambda^{-1/6}.$$

$$\left\| U \ast 2\mathcal{R} \left( \epsilon_t^* \phi_t^{(\text{ref})} \right) \left( \phi_t^{(\text{ref})} - 1 \right) \right\|_2 \leq \left\| U \ast 2\mathcal{R} \left( (1 - \chi_{1/4}) \epsilon_t^* \phi_t^{(\text{ref})} \right) \left( \tilde{\phi}_t - 1 \right) \right\|_2 + \left\| U \ast 2\mathcal{R} \left( \chi_{1/4} \epsilon_t^* \phi_t^{(\text{ref})} \right) \left( \tilde{\phi}_t - 1 \right) \right\|_2 + \left\| U \ast 2\mathcal{R} \left( \epsilon_t^* \phi_t^{(\text{ref})} \right) \left( \phi_t^{(\text{ref})} - \tilde{\phi}_t \right) \right\|_2 \leq \left\| U \ast 2\mathcal{R} \left( (1 - \chi_{1/4}) \epsilon_t^* \phi_t^{(\text{ref})} \right) \left( \tilde{\phi}_t - 1 \right) \right\|_2 + 2\|U\|_1 \|\chi_{1/4} \epsilon_t\|_2 \|\phi_t^{(\text{ref})}\|_\infty \|\tilde{\phi}_t - 1\|_\infty + 2\|U\|_1 \|\epsilon_t\|_\infty \|\phi_t^{(\text{ref})}\|_\infty \|\phi_t^{(\text{ref})} - \tilde{\phi}_t\|_2 \leq 0 + 2C(t)\Lambda^{-1/6},$$

where in addition to the ingredients for the previous term we have used:

- $\|\phi_t^{(\text{ref})}\|_\infty \leq C(t)$ as proven in Corollary 2.8;
- $\text{supp} \ U \ast 2\mathcal{R} \left( (1 - \chi_{1/4}) \epsilon_t^* \phi_t^{(\text{ref})} \right) \subset B_{1/4\Lambda^{1/3} - 2D}$. 

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– Similarly as above one has \((\bar{\phi}_t - 1)(1 - \chi_r)(x) = 0\) for \(r = 1/4\); and sufficiently large \(\Lambda\).

\[
\|U \ast |\epsilon_t^2 \phi^\text{(ref)}_t\|_2 = \|\phi^\text{(ref)}_t\|_\infty \left\| \int dy U(\cdot - y)|\epsilon_t(y)|^2 \right\|_2 \leq C(t) \int dy |\epsilon_t(y)|^2 \|U(\cdot - y)\|_2 \\
\leq C(t) \|U\|_2 \|\epsilon_t\|_2^2 \leq C \|\epsilon_t\|_2^2 ;
\]

\[
\left\| \left[ U \ast |\epsilon_t^2 \phi^\text{(ref)}_t\right] \right\|_2 = \left\| \int dy U(\cdot - y)|\epsilon_t(y)|^2 \epsilon_t \right\|_2 \leq \|U\|_2 \|\epsilon_t\|_2^2 \leq C \|\epsilon_t\|_2^2 ;
\]

\[
\left\| \left[ U \ast 2 \Re \epsilon_t^* \phi^\text{(ref)}_t \right] \right\|_2 \leq \|\phi^\text{(ref)}_t\|_\infty \|U\|_2 \|\epsilon_t\|_2^2 \leq C(t) \|\epsilon_t\|_2^2 .
\]

Hence, we have shown

\[
\partial_t \|\eta_t - \epsilon_t\|_2 \leq C \|\eta_t - \epsilon_t\|_2 + C(t) \Lambda^{-\frac{1}{2}} \leq C(t) \left( \|\epsilon_t\|_2^2 + \|\epsilon_t\|_2^3 \right)
\]

which together with Grönwall’s Lemma proves the claim.

\(\Box\)

3 Appendix

In several steps we have used the convenient computation formulas (28)-(34) concerning the counting operators that were established in in [14, Lemma 1] and are repeated here for easier reference:

**Lemma 3.1.** Given the definitions (24)-(26), the following relations are true:

1. \(\bar{w}^\phi \bar{w}^\psi = (\bar{w}^\psi \bar{w})^\phi = \bar{w}^\phi \bar{w}^\psi\)

2. \(\left[ \bar{w}^\phi, p_k^\phi \right] = \left[ \bar{w}^\phi, q_k^\phi \right] = 0\)

3. \(\left[ \bar{w}^\phi, P_k^\phi \right] = 0\)

4. For \(n(k) = \sqrt{\frac{k}{N}}\) we have

\[
(\bar{n}^\psi)^2 = \frac{1}{N} \sum_{k=1}^{N} q_k^\phi
\]

5. For \(\Psi \in (L^2)^\otimes N\) we have that

\[
\left\| \bar{w}^\psi q_1^\phi \Psi \right\|_2 = \left\| \bar{w}^\psi \bar{n}^\phi \Psi \right\|_2
\]

\[
\left\| \bar{w}^\psi q_1^\phi q_2^\phi \Psi \right\|_2 \leq \sqrt{\frac{N}{N-1}} \left\| \bar{w}^\psi (\bar{n}^\phi)^2 \Psi \right\|_2
\]
6. For any function $Y \in L^\infty(\mathbb{R}^3)$ and $Z \in L^\infty(\mathbb{R}^5)$ and

$$ A_0^\psi = p_1^\psi, \quad A_1^\psi = q_1^\psi, \quad B_0^\psi = p_1^\psi p_2^\psi, \quad B_1^\psi = p_1^\psi q_2^\psi, \quad B_2^\psi = q_1^\psi q_2^\psi $$

we have

$$ \widehat{w^\psi A_j^\psi Y(x_1)} A_j^\psi = A_j^\psi Y(x_1) \widehat{A_j^\psi w_{j-l}^\psi} \quad \text{with} \ j, l = 0, 1, \quad (33) $$

and

$$ \widehat{w^\psi B_j^\psi Z(x_1, x_2)} B_j^\psi = B_j^\psi Z(x_1, x_2) \widehat{B_j^\psi w_{j-l}^\psi} \quad \text{with} \ j, l = 0, 1, 2. \quad (34) $$

Proof.

1. Since $p_k^\psi$ is orthogonal to $q_k^\psi$ for any $1 \leq k \leq N$ it follows, that the $P_k^\psi$, $1 \leq k \leq N$ (see (17)) are pairwise orthogonal projectors. Hence, by (26)

$$ \widehat{\psi^\psi \psi^\psi} = \sum_{k=0}^N v(k) P_k^\psi \psi^\psi = \sum_{k=0}^N v(k) w(k) P_k^\psi = (\psi \psi^\psi)^\psi . $$

Similarly one can show $(\psi \psi^\psi)^\psi = \psi^\psi \psi^\psi .$

2. $p_k^\psi$ commutes with $p_j^\psi$ and $q_j^\psi$ for any $j, k$. It follows that $p_k^\psi$ commutes with any $P_j^\psi$ since the latter is a product of $p$’s and $q$’s. In view of (26) we observe that $p_k^\psi$ commutes with any weighted counting operators $w^\psi$. A analogous argument can be made for $q_k^\psi$.

3. Observing that $P_k^\psi$ is given as a symmetric product of $p$’s and $q$’s (see (17)) the claim follows from (29).

4. Note that $1 = \prod_{k=1}^N (p_k^\psi + q_k^\psi)$. Expanding this product and sorting the summands according to the number of $q$-factors it follows that $1 = \sum_{k=0}^N P_k^\psi$. Hence, the claim (30) follows from

$$ N^{-1} \sum_{k=1}^N q_k^\psi = N^{-1} \sum_{k=1}^N q_k^\psi \sum_{j=0}^N P_j^\psi = N^{-1} \sum_{j=0}^N \sum_{k=1}^N q_k^\psi P_j^\psi = N^{-1} \sum_{j=0}^N j P_j^\psi = (\eta^\psi)^2, $$

where in the last step we have used (28).

5. Using symmetry we get

$$ \| \widehat{w^\psi q_1^\psi \Psi} \|^2 = \langle \Psi, q_1^\psi (\widehat{\psi^\psi})^2 q_1^\psi \Psi \rangle = N^{-1} \sum_{k=1}^N \langle \Psi, q_k^\psi (\widehat{\psi^\psi})^2 q_k^\psi \Psi \rangle . $$

Using (29), then (28) and then (30) the latter equals

$$ \langle \Psi, \left( N^{-1} \sum_{k=1}^N q_k^\psi \right) (\widehat{\psi^\psi})^2 \Psi \rangle = \langle \Psi, (\eta^\psi)^2 (\widehat{\psi^\psi})^2 \Psi \rangle = \| \widehat{\psi^\psi \eta^\psi \Psi} \|^2 . $$

and (31) follows.
In a similar way we get

$$\left\| \widehat{w^\varphi} q_1^\varphi q_2^\varphi \Psi \right\|_2^2 = \left\langle \Psi, q_1^\varphi q_2^\varphi (\widehat{w^\varphi})^2 q_1^\varphi q_2^\varphi \Psi \right\rangle$$

$$= \frac{1}{N(N - 1)} \sum_{j \neq k} \left\langle \Psi, q_j^\varphi q_k^\varphi (\widehat{w^\varphi})^2 q_j^\varphi q_k^\varphi \Psi \right\rangle .$$

Using that $$\left\langle \Psi, q_k^\varphi q_k^\varphi (\widehat{w^\varphi})^2 q_k^\varphi q_k^\varphi \Psi \right\rangle$$ is for any $$k$$ quadratic, and thus positive, we find

$$\left\| \widehat{w^\varphi} q_1^\varphi q_2^\varphi \Psi \right\|_2^2 \leq \frac{1}{N(N - 1)} \sum_{j,k=1}^N \left\langle \Psi, q_j^\varphi q_k^\varphi (\widehat{w^\varphi})^2 q_j^\varphi q_k^\varphi \Psi \right\rangle$$

$$= \frac{N^2}{N(N - 1)} \left\langle \Psi, \left( N^{-1} \sum_{j=1}^N q_j^\varphi \right)^2 \left( N^{-1} \sum_{k=1}^N q_k^\varphi \right)^2 (\widehat{w^\varphi})^2 \Psi \right\rangle$$

$$= \frac{N}{N - 1} \left\langle \Psi, (\widehat{w^\varphi})^2 \Psi \right\rangle$$

$$= \frac{N}{N - 1} \left\| \widehat{w^\varphi} (\widehat{w^\varphi})^2 \Psi \right\|_2^2 .$$

6. The proof is very similar for all the combinations of $$A$$ and $$B$$ operators. Therefore, we only demonstrate one case and start with the following computation. Denoting the tensor product by $$\otimes$$, we find

$$p_1^\varphi Y(x_1) q_1^\varphi P_k^\varphi = p_1^\varphi Y(x_1) q_1^\varphi \left[ (q^\varphi)^{\otimes k} \otimes (p^\varphi)^{\otimes (N-k)} \right]$$

$$= p_1^\varphi Y(x_1) \left[ q_1^\varphi \otimes (q^\varphi)^{\otimes (k-1)} \otimes (p^\varphi)^{\otimes (N-k)} \right]$$

$$= p_1^\varphi \left[ 1 \otimes (q^\varphi)^{\otimes (k-1)} \otimes (p^\varphi)^{\otimes (N-k)} \right] Y(x_1) q_1^\varphi$$

$$= \left[ p_1^\varphi \otimes (q^\varphi)^{\otimes (k-1)} \otimes (p^\varphi)^{\otimes (N-k)} \right] Y(x_1) q_1^\varphi$$

$$= \left[ (q^\varphi)^{\otimes (k-1)} \otimes (p^\varphi)^{\otimes (N-k)} \right] p_1^\varphi Y(x_1) q_1^\varphi$$

$$= p_{k-1}^\varphi p_1^\varphi Y(x_1) q_1^\varphi .$$

Similar arguments can be applied for the various combinations of $$A$$ and $$B$$ operators to show

$$P_i^\varphi A_j^\varphi Y(x_1) A_l^\varphi = A_j^\varphi Y(x_1) A_l^\varphi P_{k+l-j}^\varphi \quad \text{with } j, l = 0, 1, \quad (161)$$

and

$$P_i^\varphi B_j^\varphi Z(x_1, x_2) B_l^\varphi = B_j^\varphi Z(x_1, x_2) B_l^\varphi P_{k+l-j}^\varphi \quad \text{with } j, l = 0, 1, 2. \quad (162)$$

Using these identities together with the convention $$P_k^\varphi = 0$$ for $$k \notin \{0, 1, \ldots, N\}$$, see (25), and the definiton (26), we get

$$\widehat{w^\varphi} A_j^\varphi Y(x_1) A_l^\varphi = \sum_{k=-\infty}^{\infty} w(k) P_{k+l-j}^\varphi A_j^\varphi Y(x_1) A_l^\varphi$$

$$= \sum_{k=-\infty}^{\infty} w(k) A_j^\varphi Y(x_1) A_l^\varphi P_{k+l-j}^\varphi$$
Substituting the index of the sum by \( m = k + l - j \) we get

\[
\hat{w}^\varphi A_j^\varphi Y(x_1)A_l^\varphi = \sum_{m=\infty}^{\infty} w(m + j - l)A_j^\varphi Y(x_1)A_l^\varphi P_m^\varphi \\
= A_j^\varphi Y(x_1)A_l^\varphi \sum_{m=\infty}^{\infty} w(m + j - l)P_m^\varphi \\
= A_j^\varphi Y(x_1)A_l^\varphi \hat{w}^\varphi_{j-l}.
\]

In the same way we can prove the second formula:

\[
\hat{w}^\varphi B_j^\varphi Z(x_1, x_2)B_l^\varphi = \sum_{k=\infty}^{\infty} w(k)P_k^\varphi B_j^\varphi Z(x_1, x_2)B_l^\varphi \\
= \sum_{k=\infty}^{\infty} w(k)B_j^\varphi Z(x_1, x_2)B_l^\varphi P_k^\varphi \\
= \sum_{m=\infty}^{\infty} w(m + j - l)B_j^\varphi Z(x_1, x_2)B_l^\varphi P_m^\varphi \\
= B_j^\varphi Z(x_1, x_2)B_l^\varphi \hat{w}^\varphi_{j-l}.
\]

\[\square\]
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