Displacements of Points of a Viscoelastic Ball Caused by Tides

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Abstract. The derivation of equations of motion is considered in the problem of motion of two viscoelastic bodies in the central gravitational field of massive spherical rigid homogenous body with the help of the d’Alembert–Lagrange variation principle. It was used the Kelvin–Voigt model of viscous forces and classical theory of small strains. The method for separation of motions was applied in order to get an approximate solution of quasi-static equation of the theory of elasticity on the unperturbed motion. There were obtained the displacements of each point of the bodies, caused by centrifugal, elastic and dissipative forces.

Keywords
Equations of motion; Viscoelastic bodies; Kelvin–Voigt model; Centrifugal, elastic and dissipative forces; Displacements of the points

1. Introduction
The question of creating a global mechanical-mathematical and physical model of the Earth remains open, due to the complexity of the processes taking place inside it and the forces acting outside. We will consider in our work one aspect only, namely, the tides in viscoelastic bodies. This aspect is very impotent because it affects the evolution of planets and satellites. A qualitative analysis of the movement of body points during tides was made in work (Darvin 1898). It was calculated the displacement of points of a homogeneous spherical elastic body in Cartesian coordinates under the action of centrifugal forces (Love 1944). He constructed the global tidal model of the Earth as a spherical incompressible body in order to avoid the difficulty of great internal stress in it and the impossibility of applying the theory of small strains because of this. The next step in studying the motion of celestial bodies was the mathematical model of viscoelastic body proposed in monograph (Vil’ke 1997). We applied his method to investigate the evolution of the Earth-Moon system (Zlenko 2015a), generalized libration points in the two-planet problem (Zlenko 2015b), stationary solutions and their stability in a model three-body problem (Zlenko 2016). We found (Zlenko 2020) that the Earth’s substance behaves not only as not incompressible but as an auxetic (globally). It means when the body is stretched, it becomes thicker perpendicular to the applied force.

Different tidal models are used in celestial mechanics and geophysics. In paper (Frouard et al 2016)
the damped mass-spring model represents a Kelvin–Voigt viscoelastic solid. In manuscript (Barkin et al 2018) the Earth is simulated as triaxial body with viscoelastic mantle. A lot of works are dedicated the influence of tides on Earth’s core, oceans and so on, for example (Tan et al 2018).

We consider the motion of two viscoelastic bodies in the central gravitational field of massive spherical rigid homogenous body, using the classical linear theory of elasticity of small strains and a Kelvin–V oigt model of viscous forces. With the help of the D’Alembert–Lagrange variation principle are obtained the complex system of integro-differential equations, which cannot be solved analytically. It was applied a method for separation of the motions in systems with an infinite number of degrees of freedom (Vil’ke 1997), in order to get an approximate solution of the system. We solve the quasi-static equation of the theory of elasticity on the unperturbed motion in detail and find the displacement of each point of the bodies, caused by centrifugal, elastic and dissipative forces. It gives us the opportunity to determine the shape of the bodies, the tidal potential and lot more.

2. Derivation of Equations of Motion

2.1. Basic notation

\( M \) is the mass of massive spherical rigid homogenous body, \( m_1 \) is the mass of the first body, \( m_2 \) is the mass of the second body, \( m_1 \ll m_2 \ll M \).

\( V_i \) – sphere of radius \( r_{i0} \) of the non-deformed \( i \)-th body, \( \rho_i \) – the density \( (i = 1, 2) \), \( i = 1 \) means the first body, \( i = 2 \) – the second body, subscript or superscript \( i \) means values related to the \( i \)-th body.

The point \( O \) is the center of mass of massive spherical rigid homogenous body, \( O_i \) is the center of mass of the \( i \)-th body. \( OXYZ \) – the inertial coordinate system, \( O_iXYZ \) – Koenig’s system of coordinates, \( O_iXYZ \) – system fixed in the rotating body. \( OX, OY, OX_i \) and \( OY_i \) axes lie in the orbital plane of motion of the bodies. \( \varphi_i \) – an angle between \( O_iX_1 \) and \( OX \) axes, measured from \( OX \) axis. \( \dot{\phi}_i = \omega_{2+i} \) – the angular speed of rotation of the \( i \)-th body about the \( O_iZ_i \) axis. \( (R_1 \cos \lambda_1, R_1 \sin \lambda_1, 0) \) – polar coordinates of the center of mass \( C \) of the first and the second body in the inertial frame, \( \lambda_1 \) is the angle measured from the \( OX \) axis, \( R_1 \) is the separation of \( O \) and \( C \) points. \( (R_2 \cos \lambda_2, R_2 \sin \lambda_2, 0) \) – polar coordinates of \( O_2 \) point in \( O_1XYZ \) frame, where \( \lambda_2 \) angle is measured from the \( OX \) axis, \( R_2 \) is the separation of \( O_1 \) and \( O_2 \) points. \( OZ \) and \( OZ_i \) axes coincide, and are perpendicular to the orbital plane. \( \omega_1 \) – an orbital angular velocity of moving the center of mass \( C \) with respect to origin \( O \) of inertial frame. \( \omega_2 \) – an angular velocity of moving the second body about the first body with respect to origin \( O_1 \) of \( O_1XYZ \) frame.

\( E_i \) – the Young’s modulus, \( \nu_i \) – Poisson’s coefficient, \( \dot{\lambda}_i, \mu_i \) – Lame’s coefficients, \( \chi_i \) – the coefficient of internal viscous friction, \( r_i = (x_i, y_i, z_i) \) is the radius-vector of the points in the ball (non-deformed body) with coordinates \( (x_i, y_i, z_i) \) in \( O_iXYZ \) frame, \( u_i(r_i, t) \) are displacements of the points of the body due to strains.
2.2. The D’Alembert–Lagrange Variation Principle and constrains

In our problem the D’Alembert–Lagrange variation principle has the form:

\[ \delta I - \delta A + \lambda \delta \mathbf{f} = 0, \]

where \( \delta I, \delta A \) and \( \lambda \delta \mathbf{f} \) are virtual work of inertia forces, active force and constrains, respectively, \( \lambda \) is vector of indeterminate Lagrange multipliers. We will calculate separately all these members.

Let us consider vector \( \zeta_i \). It gives the position of the points of the \( i \) body in the \( OXYZ \) inertia coordinate system:

\[ \zeta_i = \mathbf{O} \mathbf{i} + \Gamma_i (\varphi_i(t))(\mathbf{r}_i + \mathbf{u}_i(r_i,t)). \] (1)

Here \( \Gamma_i(\varphi_i(t)) \) is the orthogonal transition operator from \( O_iX_iY_iZ_i \) frame to \( OXYZ \) frame:

\[
\begin{pmatrix}
\cos \varphi_i(t) & \sin \varphi_i(t) & 0 \\
-sin \varphi_i(t) & \cos \varphi_i(t) & 0 \\
0 & 0 & 1
\end{pmatrix}
\] (2)

\( \mathbf{u}_i \in W^i_2(V_i) \) – Sobolev space of vector functions integrable together with the squares of their first partial derivatives.

We should impose the auxiliary constrains on the displacements \( \mathbf{u}_i(r_i,t) \) in order to uniquely identify the center of mass of the \( i \) body in the \( OXYZ \) frame:

\[ f_i = \int_{V_i} \mathbf{u}_i(r_i,t) \, dv_i = 0. \] (3)

The next auxiliary constrains identically determinates the \( O_iX_iY_iZ_i \) coordinate system, relative to which the body \( O_i \) does not rotates during its deformations (in the “integral sense”):

\[ f_2 = \int_{V_i} \text{curl} \mathbf{u}_i(r_i,t) \, dv_i = 0. \] (4)

Consequently, \( \mathbf{f} = (f_1,f_2) \) and we can write the variations of constrains:

\[ \delta f_1 = \int_{V_i} \delta \mathbf{u}_i \, dv_i, \quad \delta f_2 = \int_{V_i} \text{curl} \delta \mathbf{u}_i \, dv_i. \] (5)

We represent the virtual work \( \delta I \) of inertia forces (Vil’ke 2003):

\[
\sum_{i=1}^{2} \int_{V_i} \dot{\zeta}_i \dot{\zeta}_i \rho \, dv_i = \sum_{j=1}^{6} \left( \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} \right) \delta q_j + \sum_{i=1}^{2} \left( \frac{d}{dt} \nabla_{u_i} T - \nabla_{u_i} T \right) \delta \mathbf{u}_i,
\] (6)

where \( T = \frac{1}{2} \sum_{i=1}^{2} \dot{\zeta}_i^2 \rho \, dv_i \) is kinetic energy of the system, \( \mathbf{q} = \{q_j\} = \{R_1, R_2, \lambda_1, \lambda_2, \varphi_1, \varphi_2\}, \quad j = 1 - 6, \quad \nabla_{u_i} T \) is Fréchet derivative.

2.3. Calculation of virtual work of active forces

The virtual work of active forces on the virtual displacements is equal:

\[ \delta A = -\delta \Pi - \sum_{i=4}^{2} (\delta W_i[\mathbf{u}_i] + \delta D_i[\dot{\mathbf{u}}_i]), \] (7)

where \( \Pi, \ W_i[\mathbf{u}_i] \) and \( D_i[\dot{\mathbf{u}}_i] \) are potential energy of external forces, potential energy of
deformation and functional of dissipation energy respectively. Since

$$
\Pi = \Pi (\zeta_1, \zeta_2) = \Pi (q, u_1, u_2),
$$

(9)

then

$$
\delta \Pi = \sum_{i=1}^{2} \nabla \zeta_i \Pi \delta \zeta_i = \sum_{i=1}^{2} \nabla \zeta_i \Pi \left( \sum_{j=1}^{6} \frac{\partial \zeta_i}{\partial q_j} \delta q_j + \frac{\partial \zeta_i}{\partial u_j} \delta u_j \right) = \sum_{j=1}^{6} \frac{\partial \Pi}{\partial q_j} \delta q_j + \sum_{i=1}^{2} \nabla _u \Pi \delta u_i.
$$

(10)

$\nabla \zeta_i \Pi$ is Fréchet derivative.

Taking into account that the surface of the bodies is free from traction, we obtain the variation $\delta W_i[u_i]$ of potential energy of strains:

$$
\delta W_i[u_i] = \left( \nabla _u W_i[u_i] \delta u_i \right) = - \int \left( (\lambda_i + \mu_i) \text{grad div } u_i + \mu_i \Delta u_i \right) \delta u_i dv_i.
$$

(11)

$\nabla _u W_i[u_i]$ – the element of conjugate functional space.

The functional $D_{ij}[\dot{u}_i]$ of dissipation energy in a homogeneous isotropic body is represented in a form:

$$
D_{ij}[\dot{u}_i] = \zeta_i W_i[u_i].
$$

(13)

Then we can write the variation of the functional of dissipation energy:

$$
\delta D_{ij}[\dot{u}_i] = \zeta_i \left( \nabla _u W_i[\dot{u}_i] \delta u_i \right) = - \int \left( (\lambda_i + \mu_i) \text{grad div } u_i + \mu_i \Delta u_i \right) \delta u_i dv_i.
$$

(14)

Hence

$$
\delta W_i[u_i] + \delta D_{ij}[\dot{u}_i] = \delta W_i[u_{\text{ed}}] = \nabla _{u_{\text{ed}}} W_i[u_{\text{ed}}] \delta u_i = - \int \left[ (\lambda + \mu) \text{grad div } u_{\text{ed}} + \mu \Delta u_{\text{ed}} \right] \delta u_i dv_i,
$$

(15)

where $u_{\text{ed}} = u_i + \zeta_i \dot{u}_i$, $\nabla _u W_i + \nabla _u D_i = \nabla _{u_{\text{ed}}} W_i[u_{\text{ed}}] = (\lambda + \mu) \text{grad div } u_{\text{ed}} + \mu \Delta u_{\text{ed}}$.

The D’Alembert–Lagrange variation principle has the next form after transformations:

$$
\sum_{j=1}^{6} \left( \frac{d}{dt} \frac{\partial T}{\partial q_j} - \frac{\partial T}{\partial q_j} + \frac{\partial \Pi}{\partial q_j} \right) \delta q_j +
$$

$$
\sum_{i=1}^{2} \left[ \frac{d}{dt} \nabla _u T - \nabla _u T + \nabla _u \Pi + \nabla _u W_i + \nabla _u D_i \right] \delta u_i + \lambda_i(t) \int \delta u_i dv_i + \lambda_2(t) \int \text{curl } \delta u_i dv_i = 0 \quad (17)
$$

So variations $\delta q_j$ are arbitrary, we can get the equations of motion:

$$
\frac{d}{dt} \frac{\partial T}{\partial q_j} - \frac{\partial T}{\partial q_j} + \frac{\partial \Pi}{\partial q_j} = 0, \quad j = 1 - 6.
$$

(18)
$$\sum_{i=1}^{2} \left[ \frac{d}{dt} \nabla u_i T - \nabla u_i T + \nabla u_i \Pi + \nabla u_i W_i + \nabla u_i D_i \right] \delta u_i + \lambda_{i1} \int_{V_i} \delta u_i dV_i + \lambda_{i2} \int_{V_i} \text{curl} \delta u_i dV_i = 0. \tag{19}$$

We should add to these equations the constraints (3), (4) and boundary conditions

$$\sigma_i n_{ik} \mid_{V_i} = 0. \tag{20}$$

It means that the surface of the bodies is free of traction. Here $\sigma_i$ is stress tensor, $n_i$ is the normal to the surface $\partial V_i$ of the body.

3. Calculation of Displacements

We are looking for $u_i (r, t)$ as a series in powers of a small parameter $\varepsilon_i$ and we will solve the equation (19) at first approximation with respect to the small parameter:

$$u_i (r_i, t) = \varepsilon_i u_{i1} (r_i, t) + \varepsilon_i^2 u_{i2} (r_i, t) + \ldots, \quad \varepsilon_i = \rho \varepsilon_i^0 (0)/E_i, \quad u_i (r_i, t) \approx \varepsilon_i u_{i1} (r_i, t) = \sum_{k=1}^{3} u_{i1k}. \tag{21}$$

We use the method of separations of motions. That's why we calculate the expression

$$\frac{d}{dt} \nabla u_i T - \nabla u_i T + \nabla u_i \Pi$$

on the unperturbed motion:

$$u_i = 0, \quad \dot{\lambda}_1 = \omega_1 = \left( \frac{GM}{R_1} \right)^{1/2}, \quad \dot{\lambda}_2 = \omega_2 = \left( \frac{Gm}{R_2} \right)^{1/2}, \quad \dot{\phi}_1 = \omega_3, \quad \dot{\phi}_2 = \omega_4.$$ \tag{22}

where $F_{i1}$ are centrifugal forces, $F_{i2}$ are forces, acting on the $i$-th body, caused by outer gravitational fields from massive spherical rigid homogenous body and another body:

$$F_{i1} = \omega_i^2 \sum_{k=1}^{3} \left[ r_i - (e_i r_i) e_i \right] \rho_i dV_i, \quad F_{i2} = \sum_{k=1}^{3} f_{ik} R_k^3 \left( 3 (\xi_{kl} r_i) \xi_{kl} - r_i \right). \tag{23}$$

Here $e_i$ is the unit vector along the $OZ_i$, $\xi_{kl} = \tau_{kl} \left( \cos \psi_{kl}, \sin \psi_{kl}, 0 \right)$, $\tau_{21} = -1, \tau_{kl} = 1 (k \neq 2, i \neq 1), \psi_{kl} = \lambda_k - \phi_l$.

So we have the new expression for the equation (19), taking into account (16):

$$\sum_{i=1}^{2} \left[ -(F_{i1} + F_{i2}) \delta u_i + \nabla_{u_{i1}} W \left[ u_{i1} \right] \delta u_i + \lambda_{i1} (t) \int_{V_i} \delta u_i dV_i + \lambda_{i2} (t) \int_{V_i} \text{curl} \delta u_i dV_i \right] = 0. \tag{24}$$

In order to determinate Lagrange's multipliers $\lambda_{i1} (t)$ and $\lambda_{i2} (t)$ we can give arbitrary virtual $\delta u_i$ displacements. If $\delta u_i = d, d \in E^3$ - an arbitrary constant vector of infinitesimal rigid displacements, then $\lambda_{i1} (t) = 0$. If $\delta u_i = \delta \gamma \times r_i, \delta \gamma \in E^3$ - an arbitrary infinitesimal rigid rotation, then $\lambda_{i2} (t) = 0$.

Then equation (24) we can write in the form:
\[
\sum_{i=1}^{2} \left[ -(F_{li} + F_{2i}) + \nabla_{u_i} W \left[ u_{i
u} \right] \right] \delta u_i = 0. \tag{25}
\]

As \( \delta u_i \) is arbitrary, we use the basic lemma of variation calculus and obtain
\[
\nabla_{u_i} W \left[ u_{i
u} \right] = F_{li} + F_{2i}. \tag{26}
\]

We will solve the equation (26) and write the forces in the right part of the equation in a matrix form:
\[
F_{li} = \rho \omega_{i+2}^2 \left( r_i - (e_3 r_i) e_3 \right) = \rho \omega_{i+2}^2 \left( x_i e_{i1} + y_i e_{i2} \right) = \rho \omega_{i+2}^2 \left( \frac{2}{3} r_i + B_i r_i \right),
\]
\[
B_i = \begin{pmatrix}
\frac{1}{3} & 0 & 0 \\
0 & \frac{1}{3} & 0 \\
0 & 0 & -\frac{2}{3}
\end{pmatrix}. \tag{27}
\]

Here \( e_{i1} \) and \( e_{i2} \) are the unit vectors along \( O_i X_i \) and \( O_i Y_i \) axes respectively.
\[
F_{2i} = \sum_{k=1}^{2} 3 f_{ki} R_k^3 B_{ki} r_i, \quad B_{ki} = \frac{1}{6} \begin{pmatrix}
3 \cos 2\psi_{ki} + 1 & 3 \sin 2\psi_{ki} & 0 \\
3 \sin 2\psi_{ki} & -3 \cos 2\psi_{ki} + 1 & 0 \\
0 & 0 & -2
\end{pmatrix}. \tag{28}
\]

Now we transformed the equation (26), using the expression (27) and (28):
\[
\nabla_{u_i} W \left[ u_{i
u} \right] = \rho \omega_{i+2}^2 \left( \frac{2}{3} r_i + B_i r_i \right) + \sum_{k=1}^{2} 3 f_{ki} R_k^3 B_{ki} r_i. \tag{29}
\]

We are looking for a solution of linear equation (29) as the sum of the three terms (21) and present this equation as the sum of three equations too:
\[
\nabla_{u_{i10}} W \left[ u_{i11} \right] = \rho \omega_{i+2}^2 \left( \frac{2}{3} r_i + B_i r_i \right), \tag{30}
\]
\[
\nabla_{u_{i11}} W \left[ u_{i11} \right] + \nabla_{u_{i1}} W \left[ x_i, u_{i11} \right] = \nabla W \left[ u_{i11} \right] = \nabla W \left[ u_{i1k} \right] = 3 \rho f_{ki} R_k^3 B_{ki} r_i. \tag{31}
\]

We find the solution of the equation (30), using the solution given by (Love 1944) for spherical homogeneous elastic body under the action of centrifugal forces with surface free of traction. So boundary dynamical condition (20) are satisfied. Let \( u_{i11} = (x_{i11}, y_{i11}, z_{i11}) \) be, then:
\[
\frac{x_{i11}}{y_i} = \rho \omega_{i+2}^2 \frac{1}{3} \left[ 1 + \frac{1}{5(\lambda_i + 2\mu_i)} \left( \frac{5\lambda_i + 6\mu_i}{3\lambda_i + 2\mu_i} (r_{i0}^2 - r^2) \right) \right], \tag{32}
\]
\[
\frac{z_{i11}}{z_i} = \rho \omega_{i+2}^2 \frac{1}{3} \left[ 1 + \frac{1}{5(\lambda_i + 2\mu_i)} \left( \frac{5\lambda_i + 6\mu_i}{3\lambda_i + 2\mu_i} (r_{i0}^2 - r^2) \right) \right] \times \lambda_i \left[ -8\lambda_i + 6\mu_i \right] r_{i0}^2 + (5\lambda_i + 4\mu_i) r^2 + (\lambda_i + \mu_i) \left( x_i^2 + y_i^2 - 2z_i^2 \right). \]

We replace \( \lambda_i \) by \( \frac{E_i v_i}{(1 - 2v_i)(1 + v_i)} \), \( \mu_i \) by \( \frac{E_i}{2(1 + v_i)} \) and after cumbersome calculations we can represent the solution (32) of the equation (30) in another form:
\[
\begin{align*}
    u_{i1} &= \rho_t \omega^2 \cdot E^{-1} \left[-\left(\frac{2}{3}\right)(d_{i1}r_i^2 + d_{i1}r_{10}^2) r_i + a_{i1}(B_0 r_i, r_i) + (a_{i2} r_i^2 + a_{i3} r_{10}^2) B_0 r_i \right], \\
    a_{i1} &= (1 + \nu_1)(1 - 2 \nu_1)/(10(1 - \nu_1)), \quad a_{i2} = -(3 - \nu_1)(1 - 2 \nu_1)/(10(1 - \nu_1)), \\
    a_{i3} &= (1 + \nu_1)/(5 \nu_1 + 7), \quad a_{i4} = a_{i1}(2 + \nu_1), \quad a_{i5} = a_{i1}(2 \nu_1 + 3).
\end{align*}
\]

The solution of the equation (31) we seek as a series of a small parameter \( \varepsilon \):
\[
    u_{ik} = u_{ik0} - \varepsilon \frac{du_{ik0}}{dt} + \varepsilon^2 \frac{d^2u_{ik0}}{dt^2} - \ldots = \sum_{n=0}^{\infty} (-\varepsilon^n) \frac{d^n u_{ik0}}{dt^n}.
\]

Then
\[
    u_{ik} + \varepsilon \frac{du_{ik}}{dt} = u_{ik0} - \varepsilon \frac{du_{ik0}}{dt} + \varepsilon^2 \frac{d^2u_{ik0}}{dt^2} - \ldots = u_{ik0} + o(\varepsilon^n),
\]
where \( n \) can be an arbitrary large natural number and \( u_{ik0} \) is the solution of the equation
\[
    \nabla_{\varepsilon_{ik0}} W[u_{ik0}] = 3 \rho_t f_{ik} R_k^3 B_k r_i \quad (k = 1, 2).
\]

If we consider the structure of the solution (33) of the equation (30), then we can see that one part of the solution \( \rho_t \omega^2 \cdot E^{-1} \left[-\left(\frac{2}{3}\right)(d_{i1}r_i^2 + d_{i1}r_{10}^2) r_i \right] \) is caused by \( \rho_t \omega^2 \cdot \left(\frac{2}{3} r_i \right) \) radial forces and another part \( \rho_t \omega^2 \cdot E^{-1} \left[a_{i1}(B_0 r_i, r_i) + (a_{i2} r_i^2 + a_{i3} r_{10}^2) B_0 r_i \right] \) is caused by \( \rho_t \omega^2 \cdot B_0 r_i \) forces with matrix \( B_0 \). And we can make the conclusion, that solution of the equation (36), caused by the \( 3 \rho_t f_{ik} R_k^3 B_k r_i \) forces, has the same structure as the solution of the equation (33), caused by the \( \rho_t \omega^2 \cdot B_0 r_i \) forces:
\[
    u_{ik0} = 3 \rho_t E^{-1} f_{ik} R_k^3 \left[a_{i1}(B_0 r_i, r_i) + (a_{i2} r_i^2 + a_{i3} r_{10}^2) B_0 r_i \right].
\]

This part of displacements is caused by elastic forces.

As on the unperturbed motion
\[
    \frac{dB_{ki}}{dt} = \frac{dB_{ki}}{d\psi_{ki}} \frac{d\psi_{ki}}{dt} = \frac{dB_{ki}}{d\psi_{ki}} \left(\frac{d\lambda_k}{dt} - \frac{d\varphi_t}{dt}\right) = \frac{dB_{ki}}{d\psi_{ki}} (\omega_k - \omega_{t+1}),
\]
then
\[
    -\varepsilon \frac{du_{ik0}}{dt} = -3 \rho_t E^{-1} \varepsilon \varphi_t f_{ik} R_k^3 (\omega_k - \omega_{t+1}) \left[a_{i1}(B_0 r_i, r_i) + (a_{i2} r_i^2 + a_{i3} r_{10}^2) B_0 r_i \right].
\]

The part (38) of displacements is caused by dissipative forces.

We obtain \( u_{ik} \) from (34), (37) and (38) up to first-order term of smallness in \( \varepsilon \):
\[
    u_{ik} = u_{ik0} - \varepsilon \frac{du_{ik0}}{dt} = 3 \rho_t E^{-1} f_{ik} R_k^3 \left[a_{i1}(B_0 r_i, r_i) + (a_{i2} r_i^2 + a_{i3} r_{10}^2) B_0 r_i \right] -
\]
\[
    3 \rho_t E^{-1} \varepsilon \varphi_t f_{ik} R_k^3 (\omega_k - \omega_{t+1}) \left[a_{i1}(B_0 r_i, r_i) + (a_{i2} r_i^2 + a_{i3} r_{10}^2) B_0 r_i \right].
\]

The final expression for the solution of the equation (29) is:
\[
    u_i \approx u_i^* + u_i^e + u_i^d,
\]
\[ u_{ic} = u_{i11}, \quad u_{id} = \sum_{k=1}^{2} u_{i1k0} + u_{i1k0} \sum_{k=1}^{2} \frac{d u_{i1k0}}{dt}. \]  

(41)

So, boundary dynamical condition that the surface of the bodies is free of traction, are satisfied. The kinematic condition (3) and (4) are also satisfied, because the integrals in (3) and (4) are calculated on spherical bodies and \( u_i(-r_i,t) = -u_i(r_i,t), \) \( \text{curl}u_i(-r_i,t) = -\text{curl}u_i(r_i,t). \)

Conclusion

In our paper we have solved only integro-differential equations and have got the displacements of the points of the bodies on the unperturbed motion. These displacements can be substituted into the rest of the equations and the angular velocities of rotation of bodies, the angular velocity of motion of the barycenter of the bodies, the angular velocity of rotation the bodies around the barycenter, the distance of the barycenter from massive spherical body and the separation of bodies as a function of time can be obtained. Of course, these solutions should be used very carefully to each particular model, especially evolutionary. In our problem the coefficient of internal viscous friction is constant, but in the process of evolution, especially at long time intervals, it can be change. Let us estimate only the mean value of the found tidal displacements on the Earth’s surface, taking Young’s modulus equal to \( E = 10^{11} \text{Nm}^{-2} : \) \( u_i \bigg|_{r_i} \approx 1 \text{m}. \) It is a real value. These displacements can be applied for determination of a shape and volume of the deformed bodies, tidal torques.

This method of finding displacements in viscoelastic bodies can be used for solving different problems in geophysics, celestial mechanics and astrodynamics. In particular, it would be interesting to investigate evolution of the obliquity of an exoplanet (Podvigina, Krasilnikov 2020), if exoplanet is considered as deformable body.

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