Exact Asymptotic Behavior of Singular Positive Solutions of Fractional Semi-Linear Elliptic Equations

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Abstract

In this paper, we prove the exact asymptotic behavior of singular positive solutions of fractional semi-linear equations

\[(−\Delta)^{\sigma}u = u^p \text{ in } B_1\setminus\{0\}\]

with an isolated singularity, where \(\sigma \in (0, 1)\) and \(\frac{n}{n-2\sigma} < p < \frac{n+2\sigma}{n-2\sigma}\).

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1 Introduction and Main results

In this paper, we shall describe the exact asymptotic behavior of singular positive solutions of

\[(−\Delta)^{\sigma}u = u^p \text{ in } B_1\setminus\{0\}\]  

(1.1)

with an isolated singularity at the origin, where the punctured unit ball \(B_1\setminus\{0\} \subset \mathbb{R}^n\) with \(n \geq 2\), \(\sigma \in (0, 1)\) and \(\frac{n}{n-2\sigma} < p < \frac{n+2\sigma}{n-2\sigma}\). \((-\Delta)^{\sigma}\) is the fractional Laplacian.

In [29], we classify the isolated singularities of equation (1.1) with \(\frac{n}{n-2\sigma} < p < \frac{n+2\sigma}{n-2\sigma}\). More precisely, let \(u\) be a nonnegative solution of (1.1), then either the singularity near 0 is removable, or there exist two positive constants \(c_1\) and \(c_2\) such that

\[c_1|x|^{-\frac{p+\sigma}{p-1}} \leq u(x) \leq c_2|x|^{-\frac{p+\sigma}{p-1}}.\]  

(1.2)

Here we will prove the exact asymptotic behavior of singular positive solutions in (1.2). Our main result is the following

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Theorem 1.1. Let $u$ be a nonnegative solution of (1.1). Assume
\[ \frac{n}{n-2\sigma} < p < \frac{n+2\sigma}{n-2\sigma}. \]
Then either the singularity near 0 is removable, or
\[ \lim_{|x| \to 0} |x|^\frac{2\sigma}{p-1} u(x) = A_{n,p,\sigma}, \]
where
\[ A_{n,p,\sigma} = \left\{ \Lambda \left( \frac{n-2\sigma}{2} - \frac{2\sigma}{p-1} \right) \right\}^{\frac{1}{p-1}} \]
with
\[ \Lambda(\alpha) = 2^{2\sigma} \frac{\Gamma\left(\frac{n+2\sigma+2\alpha}{4}\right) \Gamma\left(\frac{n+2\sigma-2\alpha}{4}\right)}{\Gamma\left(\frac{n-2\sigma-2\alpha}{4}\right) \Gamma\left(\frac{n-2\sigma+2\alpha}{4}\right)}. \]

For the classical case $\sigma = 1$, Theorem 1.1 has been proved in the pioneering paper [19] by Gidas and Spruck. We may also see another proof in classical paper [6] by Caffarelli, Gidas and Spruck. We remark that in these two proofs, the ODEs technique is a missing ingredient in our fractional case. Hence, we need some new ideas to overcome this key difficulty. Here we shall use a monotonicity formula established in our recent paper [29] and a blow up argument introduced in Ghergu-Kim-Shahgholian [18] to solve this problem.

We study the equation (1.1) via the well known extension theorem for the fractional Laplacian $(-\Delta)^{\sigma}$ established by Caffarelli-Silvestre [9]. We use capital letters, such as $X = (x, t) \in \mathbb{R}^n \times \mathbb{R}_+$, to denote points in $\mathbb{R}^{n+1}$. We also denote $B_R$ as the ball in $\mathbb{R}^{n+1}$ with radius $R$ and center at the origin, $B_R^+$ as the upper half-ball $B_R \cap \mathbb{R}^{n+1}_+$, and $\partial^0 B_R^+$ as the flat part of $\partial B_R^+$ which is the ball $B_R$ in $\mathbb{R}^n$. Then the problem (1.1) is equivalent to the following extension problem
\[ \begin{align*}
- \text{div} (t^{1-2\sigma} \nabla U) &= 0 & \text{in } B_R^+, \\
\frac{\partial U}{\partial \nu}(x, 0) &= U^p(x, 0) & \text{on } \partial^0 B_R^+ \setminus \{0\},
\end{align*} \tag{1.5} \]
where $\frac{\partial U}{\partial \nu}(x, 0) := -\lim_{t \to 0^+} t^{1-2\sigma} \partial_t U(x, t)$. By [9], we only need to analyze the behavior of the traces
\[ u(x) := U(x, 0) \]
of the nonnegative solutions $U(x, t)$ of (1.5) near the origin, from which we can get the behavior of solutions of (1.1) near the origin.

In order to prove Theorem 1.1 we need to establish the following cylindrically symmetric result for the global equation that the origin is a non-removable isolated singularity.

Theorem 1.2. Let $U$ be a nonnegative solution of
\[ \begin{align*}
- \text{div}(t^{1-2\sigma} \nabla U) &= 0 & \text{in } \mathbb{R}_+^{n+1}, \\
\frac{\partial U}{\partial \nu}(x, 0) &= U^p(x, 0) & \text{on } \mathbb{R}^n \setminus \{0\},
\end{align*} \tag{1.6} \]
with $\frac{n}{n-2\sigma} < p < \frac{n+2\sigma}{n-2\sigma}$. Assume that the origin 0 is a non-removable singularity. Then $U(x, t) = U(|x|, t)$. 

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The paper is organized as follow. In Section 2, we prove that singular positive solutions of (1.6) are cylindrically symmetric via the method of moving sphere introduced by Li and Zhu [27]. We mainly follow the argument in [7] where the cylindrical symmetry of singular positive solutions of (1.6) with $p = \frac{n+2\sigma}{n-2\sigma}$ was proved. Section 3 is devoted to the proof of Theorem 1.1. We will see that our proof is very different from those in papers [6, 19]. In particular, the monotonicity formula established in [29] and the blow up argument introduced in [18] are two essential tools.

2 Cylindrical Symmetry

For each $\bar{x} \in \mathbb{R}^n$ and $\lambda > 0$, we denote $X = (\bar{x}, 0)$ and define the Kelvin transformation of $U$ with respect to the ball $B_\lambda(X)$ as follow

$$ U_{X,\lambda}(\xi) := \left(\frac{\lambda}{|\xi - X|}\right)^{n-2\sigma} U \left(X + \frac{\lambda^2(\xi - X)}{|\xi - X|^2}\right). \quad (2.1) $$

If $U$ is a solution of (1.6), then $U_{X,\lambda}$ satisfies

$$ \begin{cases} -\text{div}(t^{1-2\sigma}\nabla U_{X,\lambda}) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \frac{\partial U_{X,\lambda}}{\partial \nu}(y,0) = \left(\frac{\lambda}{|y-\bar{x}|}\right)^{p^*} U_{X,\lambda}(y,0) & \text{on } \mathbb{R}^n \setminus \{\bar{x}, 0\}, \end{cases} \quad (2.2) $$

where $y_0 = \bar{x} - \frac{\lambda^2\bar{x}}{|\bar{x}|}$ and $p^* = n + 2\sigma - p(n-2\sigma) > 0$.

Proof of Theorem 1.2 Since the origin 0 is a non-removable singularity, by Corollary 3.1 in [29],

$$ \lim_{|\xi| \to 0} U(\xi) = +\infty. $$

Claim 1: For all $x \in \mathbb{R}^n \setminus \{0\}$ there exists $\lambda_3(x) \in (0, |x|)$ such that for all $0 < \lambda < \lambda_3(x)$ we have

$$ U_{X,\lambda}(\xi) \leq U(\xi), \quad \forall |\xi - X| \geq \lambda, \; \xi \neq 0, \quad (2.3) $$

where $X = (x, 0)$ and $U_{X,\lambda}$ is the Kelvin transformation of $U$ with respect to $B_\lambda(X)$. The proof of Claim 1 consists of two steps.

Step 1. We prove that there exist $0 < \lambda_1 < \lambda_2 < |x|$ such that

$$ U_{X,\lambda}(\xi) \leq U(\xi), \quad \forall 0 < \lambda < \lambda_1, \; \lambda < |\xi - X| < \lambda_2. $$

For every $0 < \lambda < \lambda_1 < \lambda_2$, $\xi \in \partial^+ B^+_{\lambda_1}(X)$, we have $X + \frac{\lambda^2(\xi - X)}{|\xi - X|^2} \in B^+_{\lambda_2}(X)$. Hence we can choose $\lambda_1 = \lambda_1(\lambda_2)$ small such that

$$ U_{X,\lambda}(\xi) \leq \left(\frac{\lambda}{\lambda_1}\right)^{n-2\sigma} U \left(X + \frac{\lambda^2(\xi - X)}{|\xi - X|^2}\right) \leq \sup_{B^+_{\lambda_2}(X)} U \leq \inf_{\partial^+ B^+_{\lambda_2}(X)} U \leq U(\xi). $$
Thus we have
\[ U_{X,\lambda}(\xi) \leq U(\xi) \quad \text{on } \partial^+ (\mathcal{B}^+_\lambda(X) \setminus \mathcal{B}^+_\lambda(X)) \]
for all \( \lambda_2 > 0 \) and \( 0 < \lambda < \lambda_1(\lambda_2) \).

Now we show that \( U_{X,\lambda}(\xi) \leq U(\xi) \) in \( \mathcal{B}^+_\lambda(X) \setminus \mathcal{B}^+_\lambda(X) \) if \( \lambda_2 \) is small and \( 0 < \lambda < \lambda_1(\lambda_2) \). Because \( U_{X,\lambda} \) satisfies (2.2), we have
\[
\begin{cases}
- \text{div}(t^{1-2\sigma} \nabla(U_{X,\lambda} - U)) = 0 \\
\frac{\partial}{\partial \nu}(U_{X,\lambda} - U)(y, 0) = \left( \frac{\lambda}{|y - x|} \right)^{p^*} U^p_{X,\lambda}(y, 0) - U^p(y, 0) \quad \text{on } \partial^0(\mathcal{B}^+_\lambda(X) \setminus \mathcal{B}^+_\lambda(X)).
\end{cases}
\]

Let \( (U_{X,\lambda} - U)^+ := \max(0, U_{X,\lambda} - U) \) which equals to 0 on \( \partial^0(\mathcal{B}^+_\lambda(X) \setminus \mathcal{B}^+_\lambda(X)) \).

Multiplying (2.4) by \((U_{X,\lambda} - U)^+\) and integrating by parts, and using the narrow domain technique from [3], we obtain
\[
\int_{\mathcal{B}^+_\lambda(X) \setminus \mathcal{B}^+_\lambda(X)} t^{1-2\sigma} |\nabla(U_{X,\lambda} - U)^+|^2
\leq C \left( \int_{\mathcal{B}^+_\lambda(X) \setminus \mathcal{B}^+_\lambda(X)} ((U_{X,\lambda} - U)^+) \frac{2n}{n-2\sigma} \right)^{\frac{n-2\sigma}{2n}} \left( \int_{\mathcal{B}^+_\lambda(X) \setminus \mathcal{B}^+_\lambda(X)} U^{\frac{(p-1)n}{p-1}}_{X,\lambda} \right)^{\frac{2n}{n}}
\]
We can choose \( \lambda_2 \) small such that
\[
C \left( \int_{\mathcal{B}^+_\lambda(X)} U^{\frac{(p-1)n}{p-1}}_{X,\lambda} \right)^{\frac{2n}{n}} \leq \frac{1}{2}.
\]

Then we have
\[ \nabla(U_{X,\lambda} - U)^+ = 0 \quad \text{in } \mathcal{B}^+_\lambda(X) \setminus \mathcal{B}^+_\lambda(X). \]

Since
\[ (U_{X,\lambda} - U)^+ = 0 \quad \text{on } \partial^0(\mathcal{B}^+_\lambda(X) \setminus \mathcal{B}^+_\lambda(X)), \]
we obtain
\[ (U_{X,\lambda} - U)^+ = 0 \quad \text{in } \mathcal{B}^+_\lambda(X) \setminus \mathcal{B}^+_\lambda(X). \]

Hence, we have
\[ U_{X,\lambda} \leq U \quad \text{in } \mathcal{B}^+_\lambda(X) \setminus \mathcal{B}^+_\lambda(X). \]
for $0 < \lambda < \lambda_1$.

**Step 2.** We prove that there exists $\lambda_3(x) \in (0, \lambda_1)$ such that for each $0 < \lambda < \lambda_3(x)$,

$$U_{X,\lambda}(\xi) \leq U(\xi), \quad \forall \, |\xi - X| > \lambda_2, \, \xi \neq 0.$$  

To prove this step, we let

$$\phi(\xi) = \left( \frac{\lambda_2}{|\xi - X|} \right)^{n-2\sigma} \inf_{\partial^+ B_{\lambda_2}^+(X)} U,$$

then $\phi$ satisfies

$$-\text{div}(t^{1-2\sigma} \nabla \phi) = 0 \quad \text{in} \quad \mathbb{R}^{n+1}_+ \setminus B_{\lambda_2}^+(X),$$
$$\frac{\partial \phi}{\partial \nu}(x, 0) = 0 \quad \text{on} \quad \mathbb{R}^n \setminus B_{\lambda_2} (X),$$

and $\phi(\xi) \leq U(\xi)$ on $\partial^+ B_{\lambda_2} (X)$. By the maximum principle, we have

$$U(\xi) \geq \left( \frac{\lambda_2}{|\xi - X|} \right)^{n-2\sigma} \inf_{\partial^+ B_{\lambda_2}^+(X)} U, \quad \forall \, |\xi - X| \geq \lambda_2, \, \xi \neq 0.$$  

Let

$$\lambda_3(x) := \min \left\{ \lambda_1, \lambda_2 \left( \inf_{\partial^+ B_{\lambda_2}^+(X)} U / \sup_{B_{\lambda_2}^+(X)} U \right)^{\frac{1}{2\sigma}} \right\}.$$  

Then for any $0 < \lambda < \lambda_3(x), \, |\xi - X| \geq \lambda_2$ and $\xi \neq 0$, we have

$$U_{X,\lambda}(\xi) \leq \left( \frac{\lambda_3}{|\xi - X|} \right)^{n-2\sigma} U \left( X + \frac{\lambda^2(\xi - X)}{|\xi - X|^2} \right)$$
$$\leq \left( \frac{\lambda_3}{|\xi - X|} \right)^{n-2\sigma} \sup_{B_{\lambda_2}^+(X)} U$$
$$\leq \left( \frac{\lambda_2}{|\xi - X|} \right)^{n-2\sigma} \inf_{\partial^+ B_{\lambda_2}^+(X)} U \leq U(\xi).$$  

The proof of Claim 1 is completed.

Now, we define

$$\tilde{\lambda}(x) := \sup \{ 0 < \mu \leq |x| |U_{X,\lambda}(\xi) \leq U(\xi), \, \forall \, |\xi - X| \geq \lambda, \, \xi \neq 0, \, \forall \, 0 < \lambda < \mu \}.$$  

By Claim 1, $\tilde{\lambda}(x)$ is well defined.

**Claim 2.** $\tilde{\lambda}(x) = |x|.$

Suppose by contradiction that $\tilde{\lambda}(x) < |x|$ for some $x \neq 0$. Since the origin $0$ is not removable, by the strong maximum principle, we obtain

$$U(\xi) > U_{X,\lambda}(\xi) \quad \forall \, |\xi - X| > \lambda, \, \xi \neq 0.$$  

The proof of Claim 2 is completed.
Moreover, we have
\[ \liminf_{\xi \to 0} (U(\xi) - U_{X,\lambda}(\xi)) > 0. \]
We can use the narrow domain technique as above, see also the proof of Theorem 1.8 in [24]. Then there exists \( \varepsilon_1 > 0 \) such that for all \( \bar{\lambda}(x) < \lambda < \bar{\lambda}(x) + \varepsilon_1 \) we have
\[ U_{X,\lambda}(\xi) \leq U(\xi), \quad \forall |\xi - X| \geq \lambda, \ \xi \neq 0, \]
which contradicts with the definition of \( \bar{\lambda}(x) \). This proves Claim 2.

Therefore, we obtain
\[ U_{X,\lambda}(\xi) \leq U(\xi) \quad \forall |\xi - X| \geq \lambda, \ \xi \neq 0, \]
In particular, we have
\[ u_{x,\lambda}(y) \leq u(y) \quad \forall |y - x| \geq \lambda, \ y \neq 0, \forall 0 < \lambda < |x|, \]
where \( u_{x,\lambda} \) is the Kelvin transformation of \( u \) with respect to the ball \( B_\lambda(x) \). Thus, we deduce from Lemma 2.1 in [22] that \( u \) is radially symmetric about the origin 0. The proof of Theorem 1.2 is complete. \( \Box \)

3 Exact Asymptotic Behavior

In this section, we shall prove Theorem 1.1. First, we recall the energy functional in [29]
\[ E(r; U) := r^{2(p+1)\sigma-n} \left[ \int_{\partial^+B_r^+} t^{1-2\sigma} |\partial U/\partial \nu|^2 + \frac{2\sigma}{p-1} \int_{\partial^+B_r^+} t^{1-2\sigma} \frac{\partial U}{\partial \nu} U \right] + \frac{1}{2} \frac{2\sigma}{p-1} \left( \frac{\sigma}{p-1} - (n-2\sigma) \right) \int_{\partial^+B_r^+} t^{1-2\sigma} U^2 \]
\[ - r^{2(p+1)\sigma-n+1} \left[ \frac{1}{2} \int_{\partial^+B_r^+} t^{1-2\sigma} |\nabla U|^2 - \frac{1}{p+1} \int_{\partial B_r} u^{p+1} \right]. \]

We define the scaling function
\[ U^\lambda(X) := \lambda^{2\sigma/p-1} U(\lambda X), \quad \lambda > 0. \]
Then we easily see that the equation (1.5) is invariant under this scaling. More precisely, if \( U \) is a solution of (1.5) in \( B_{\bar{\lambda}}^+ \setminus \{0\} \), then \( U^\lambda \) is a solution of (1.5) in \( B_{\bar{\lambda}/\lambda}^+ \setminus \{0\} \). Moreover, we easily check that \( E \) satisfies the following scaling relation
\[ E(\lambda s, U) = E(s, U^\lambda), \quad (3.1) \]
for \( \lambda, s > 0 \). We remark that this scaling invariance of \( E \) plays a key role in the proof of Proposition 3.3 in [29]. By Proposition 3.2 in [29] (or more precisely, and its proof there), we have the following monotonicity formula.
Proposition 3.1. Let $U$ be a nonnegative solution of (1.5) in $\mathcal{B}^+_R \setminus \{0\}$ with $1 < p < \frac{n+2\sigma}{n-2\sigma}$. Then, $E(r; U)$ is non-decreasing in $r \in (0, R)$. Furthermore, 

$$
\frac{d}{dr} E(r; U) = J_1 r^{\frac{(4\sigma+1)n}{p-1} - n} \int_{\partial + B^+_r} t^{1-2\sigma} \left( \frac{\partial U}{\partial \nu} + \frac{2\sigma}{p-1} r \right)^2 ,
$$

where $J_1 = \frac{4\sigma}{p-1} - (n - 2\sigma) > 0$ since $1 < p < \frac{n+2\sigma}{n-2\sigma}$. In particular, if $E(r; U)$ is constant for $r \in (R_1, R_2)$, then $U$ is homogeneous of degree $-\frac{2\sigma}{p-1}$ in $\mathcal{B}^+_R \setminus \mathcal{B}^+_R$, i.e.

$$
U(X) = |X|^{-\frac{2\sigma}{p-1}} U \left( \frac{X}{|X|} \right) .
$$

Proposition 3.2. Let $U$ be a nonnegative solution of (1.5) in $\mathcal{B}^+_R \setminus \{0\}$ with $1 < p < \frac{n+2\sigma}{n-2\sigma}$. Then $E(r; U)$ is uniformly bounded for $0 < r < \frac{1}{2}$. Furthermore, the limit 

$$
E(0^+; U) := \lim_{r \to 0^+} E(r; U)
$$

exists.

Proof. Let 

$$
V(X) = r^{\frac{2\sigma}{p-1}} U(rX)
$$

for any $r \in (0, \frac{1}{4})$ and $\frac{1}{2} \leq |X| \leq 2$. Then $V$ satisfies

$$
\begin{cases}
-\text{div}(t^{1-2\sigma} \nabla V) = 0 & \text{in } \mathcal{B}^+_2 \setminus \overline{\mathcal{B}^+_1}, \\
\frac{\partial V}{\partial \nu}(x, 0) = v^p(x) & \text{on } \mathcal{B}^+_2 \setminus \overline{\mathcal{B}^+_1},
\end{cases}
$$

where $v(x) = V(x, 0)$. It follows from Proposition 3.1 in [29] that

$$
|V(x)| \leq C \quad \text{for all } \frac{1}{2} \leq |X| \leq 2,
$$

where the constant $C$ depend only on $n, p, \sigma$. By Proposition 2.19 in [24], we have

$$
\sup_{\frac{1}{4} \leq |X| \leq \frac{1}{2}} |\nabla_x V| + \sup_{\frac{1}{4} \leq |X| \leq \frac{1}{2}} |t^{1-2\sigma} \partial_t V| \leq C,
$$

where the constant $C$ depend only on $n, p, \sigma$. Hence, there exists $C > 0$ depend only on $n, p, \sigma$, such that

$$
|\nabla_x U(X)| \leq C|X|^{-\frac{2\sigma}{p-1} - 1} \quad \text{in } \mathcal{B}^+_{1/2} \setminus \{0\}
$$

and

$$
|t^{1-2\sigma} \partial_t U(X)| \leq C|X|^{-\frac{2\sigma}{p-1} - 2\sigma} \quad \text{in } \mathcal{B}^+_{1/2} \setminus \{0\}.
$$

Thus, we can estimate

$$
\begin{align*}
& r^{\frac{(n+1)\sigma}{p-1} - n + 1} \int_{\partial + B^+_r} t^{1-2\sigma} |\nabla U| \leq C r^{\frac{(n+1)\sigma}{p-1} - n + 1} \left( r^{\frac{2\sigma}{p-1} - 2} \int_{\partial + B^+_r} t^{1-2\sigma} \\
& \quad + r^{\frac{2\sigma}{p-1} - 2\sigma} \int_{\partial + B^+_r} t^{2\sigma - 1} \right) \leq C,
\end{align*}
$$

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\[
\begin{align*}
&\frac{r^{(n+1)p}}{p-1} - n - 1 \int_{\partial B_r^+} t^{1-2\sigma} U^2 \leq C r^{2\sigma - n - 1} \int_{\partial B_r^+} t^{1-2\sigma} \leq C \\
\text{and} \\
&\frac{r^{(n+1)p}}{p-1} - n + 1 \int_{\partial B_r} u^{p+1} \leq C,
\end{align*}
\]

where the constant \( C \) also depend only on \( n, p, \sigma \). Now we easily conclude that \( E(r; U) \) is uniformly bounded for \( 0 < r < \frac{1}{2} \). By the monotonicity of \( E(r; U) \), we obtain the limit

\[
\lim_{r \to 0^+} E(r; U)
\]
exists.

**Proposition 3.3.** Let \( U \) be a nonnegative solution of (1.6) with \( \frac{n}{n-2\sigma} < p < \frac{n+2\sigma}{n-2\sigma} \). Suppose that \( U \) is homogeneous of degree \( -\frac{2\sigma}{p-1} \). Then either \( U \equiv 0 \) in \( \mathbb{R}_+^{n+1} \), or the trace \( u(x) = U(x, 0) \) of \( U \) is of the form

\[
u(x) = A_{n,p,\sigma} |x|^{-\frac{2\sigma}{p-1}},
\]

where \( A_{n,p,\sigma} \) is given by (1.4).

**Proof.** Suppose that \( U \) is nontrivial solution, then by strong maximum principle,

\[
U(x, t) > 0 \quad \text{in} \quad \mathbb{R}_+^{n+1} \setminus \{0\}.
\]

Hence, by the Liouville type theorem in [24], the origin 0 must be a non-removable singularity. By Theorem 1.2, \( u(x) \) is radially symmetric, hence \( u \) is a positive constant \( a \) on \( \partial B_1 \). By the homogeneity of \( u \), we have

\[
u(x) = a |x|^{-\frac{2\sigma}{p-1}}.
\]

On the other hand, since \( u \) satisfies

\[
(-\Delta)^\sigma u = u^p \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}.
\]

By a classical calculation, see, for instance, Lemma 3.1 in Fall [18], we obtain

\[
a = A_{n,p,\sigma}.
\]

This finishes the proof.

**Proof of Theorem 1.1.** Suppose that the origin 0 is a non-removable singularity, we only need to prove (1.3). Consider the scaling

\[
U^\lambda(X) = \lambda^{\frac{2\sigma}{p-1}} U(\lambda X).
\]

Then \( U^\lambda \) satisfies

\[
\begin{align*}
&-\text{div}(t^{1-2\sigma} \nabla U^\lambda) = 0 \quad \text{in} \ B_{1/\lambda}^+, \\
&\frac{\partial U^\lambda}{\partial t}(x, 0) = (U^\lambda(x, 0))^p \quad \text{on} \ \partial t B_{1/\lambda}^+ \setminus \{0\}.
\end{align*}
\]
By (1.2) and Remark 1.2 in [29]

\[ C_1 |X|^{-\frac{2\sigma}{p-1}} \leq U^\lambda (X) \leq C_2 |X|^{-\frac{2\sigma}{p-1}} \quad \text{in} \quad B_{1/\lambda}^r. \tag{3.2} \]

Thus \( U^\lambda \) is locally uniformly bounded away from the origin. By Corollary 2.10 and Theorem 2.15 in [24] that there exists \( \alpha > 0 \) such that for every \( R > 1 > r > 0 \)

\[ ||U^\lambda||_{W^{1,2}(t^{1-2\sigma},B_R^+)} + ||U^\lambda||_{C^\alpha(B_R^+)B_r} \leq C(R, r), \]

where \( u^\lambda(x) = U^\lambda(x,0) \) and \( C(R, r) \) is independent of \( \lambda \). Then there is a subsequence \( \lambda_k \to 0 \), \( \{U^{\lambda_k}\} \) converges to a nonnegative function \( U^0 \in W_{loc}^{1,2}(t^{1-2\sigma},\mathbb{R}_+^{n+1}\{0\}) \cap C^\alpha_{loc}(\mathbb{R}_+^{n+1}\{0\}) \) satisfying

\[
\begin{align*}
\begin{cases}
-\text{div}(t^{1-2\sigma} \nabla U^0) = 0 & \text{in } \mathbb{R}_+^{n+1}, \\
\frac{\partial U^0}{\partial \nu}(x,0) = (U^0(x,0))^p & \text{on } \mathbb{R}^n\{0\},
\end{cases}
\end{align*}
\]

and by (3.2) we have

\[ C_1 |X|^{-\frac{2\sigma}{p-1}} \leq U^0(X) \leq C_2 |X|^{-\frac{2\sigma}{p-1}} \quad \text{in} \quad \mathbb{R}_+^{n+1}\{0\}. \tag{3.3} \]

Moreover, since the scaling relation (3.1), we have, for any \( r > 0 \),

\[ E(r; U^0) = \lim_{k \to \infty} E(r; U^{\lambda_k}) = \lim_{k \to \infty} E(r\lambda_k; U) = E(0^+; U). \]

Thus, by Proposition 3.1, \( U^0 \) is homogeneous of degree \( -\frac{2\sigma}{p-1} \). It follows from Proposition 3.3 and (3.3) that the trace \( u^0(x) = U^0(x,0) \) must be of the form

\[ u^0(x) = A_{n,p,\sigma} |x|^{-\frac{2\sigma}{p-1}}, \]

where \( A_{n,p,\sigma} \) is given by (1.4). Since the function \( u^0(x) \) is unique, we conclude that \( u^\lambda(x) \to u^0(x) \) for any sequence \( \lambda \to 0 \) in \( C^\alpha_{loc}(\mathbb{R}^n\{0\}) \). Hence

\[ |\lambda x|^{-\frac{2\sigma}{p-1}} u(\lambda x) = |x|^{-\frac{2\sigma}{p-1}} u^\lambda(x) \to A_{n,p,\sigma} \quad \text{as} \quad \lambda \to 0 \]

in \( B_2 \setminus B_{1/2} \). We immediately conclude that

\[ \lim_{|x| \to 0} |x|^{-\frac{2\sigma}{p-1}} u(x) = A_{n,p,\sigma}. \]

\[ \square \]

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