Effective potential for Polyakov loops from a center symmetric effective theory in three dimensions

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We present lattice simulations of a center symmetric dimensionally reduced effective field theory for SU(2) Yang Mills which employ thermal Wilson lines and three-dimensional magnetic fields as fundamental degrees of freedom. The action is composed of a gauge invariant kinetic term, spatial gauge fields and a potential for the Wilson line which includes a ”fuzzy” bag term to generate non-perturbative fluctuations. The effective potential for the Polyakov loop is extracted from the simulations including all modes of the loop as well as for cooled configuration where the hard modes have been averaged out. The former is found to exhibit a non-analytic contribution while the latter can be described by a mean-field like \textit{ansatz} with quadratic and quartic terms, plus a Vandermonde potential which depends upon the location within the phase diagram.

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\section{I. Introduction}

QCD at temperatures $T \simeq 200 \text{ MeV} - 1 \text{ GeV}$ exhibits only partial deconfinement and, moreover, perturbation theory fails to reproduce some thermodynamic quantities such as the equation of state \cite{1}. Several authors \cite{2,3,4,5} have suggested that in this regime a more appropriate effective description is not in terms of quasi-particles but in terms of the thermal Wilson line

$$L(x) = Z_R^{-1} \mathcal{P} \exp \left( \frac{1}{g} \int_0^{1/T} d\tau A_0(x,\tau) \right), \quad (1)$$

and the spatial components of the gauge field. The operator $A_0$ from eq. (1) is defined on four dimensional euclidean space-time while $L(x)$ is a matrix-valued field in space. $Z_R$ denotes a renormalization constant. In such a framework, the deconfined phase is not a free gas, but rather a condensate of spin-like operators

$$\ell(x) = \frac{1}{N} \text{tr} L(x), \quad (2)$$

called \textit{Polyakov loops}. The volume averaged expectation value of this operator is an order parameter for the deconfining phase transition in the limit of infinitely massive quarks \cite{6}. In contrast to ferromagnetism, the high temperature phase is the ordered phase where a global symmetry is spontaneously broken.

We perform lattice simulations of an effective theory in three dimensions defined in the continuum by \cite{4}

$$\mathcal{L} \text{eff} = \frac{1}{2} \text{tr} G_{ij}^2 + \frac{T^2}{g^2} \text{tr} [L^\dagger D_i L_i]^2 - \frac{2}{\pi^2} T^4 \sum_{n \geq 1} \frac{1}{n^4} |\text{tr} L^n|^2 + B_f T^2 |\text{tr} L|^2. \quad (3)$$

All fields in (3) are functions of $x$ only. $G_{ij}$ is the magnetic field strength. The second term is the contribution from electric fields since in the three-dimensional theory, for arbitrary $A_0$, $E$ is given by \cite{4}

$$E_i(x) = \frac{T}{ig} \text{tr} L^\dagger(x) D_i(x) L(x). \quad (4)$$

The potential $\sim - \sum_{n \geq 1} |\text{tr} L^n|^2/n^4$ for the Wilson line $L$ is obtained by computing the one-loop fluctuation determinant in a constant background $A_0$ (or $L$) field \cite{7}. It is evidently minimized by the perturbative vacuum $\langle L \rangle = 1$ (times a phase), for any $T$. To generate a phase transition in infinite volume, refs. \cite{3,4} suggested to add non-perturbative contributions such as $B_f T^2 |\text{tr} L|^2$, with $B_f$ a ”fuzzy” bag constant (see, also, refs. \cite{8}). The ”fuzzy bag” dominates at sufficiently low temperature and induces a transition to a confined phase with $\langle \text{tr} L \rangle = 0$.

The three-dimensional effective theory (3) should be valid only over distance scales larger than $1/T$. A related 3D theory has been formulated in refs. \cite{5} (also see \cite{9}). These effective theories respect the global Z(N) center symmetry of the four-dimensional Euclidean SU(N) Yang-Mills theory. This allows for non-perturbatively large fluctuations of the Wilson line and it is interesting to investigate whether such an extension of high-temperature perturbation theory is sufficient to describe the properties of hot Yang-Mills even close to the temperature for deconfinement.

We note that there have been various attempts at extracting through numerical simulations a three-dimensional effective action which reproduces the long-range properties of the underlying four-dimensional Yang-Mills theory \cite{10}. Here, our approach is different (mostly because we do not aim at matching the couplings...
of the 3d theory yet). We shall focus on extracting an effective potential and analysing its structure from the 3d theory itself.

II. LATTICE ACTION

Our present simulations have been performed for gauge group SU(2). The structure of this group is simpler than that of SU(3), and thus allows for much higher numerical precision but exhibits the qualitative features which we are interested in, namely a deconfining phase transition and non-perturbative fluctuations between center-symmetric states. The lattice action is of the form

\[
S = \beta \sum_\square (1 - \frac{1}{2} \text{Re tr } U_\square) - \frac{1}{2} \beta \sum_{\langle ij \rangle} \text{tr} (L_i U_{ij} L_j^\dagger U_{ij}^\dagger + \text{h.c.}) - m^2 \sum_i |\text{tr } L_i|^2 .
\]

The first term is the standard Wilson action for the magnetic fields in three dimensions; the sum runs over all spatial plaquettes. We have checked that our implementation reproduces the plaquette expectation values published in ref. [11]. The second term is a kinetic term for the Wilson line corresponding to the electric fields in three dimensions. Here, the sum runs over all links connecting nearest neighbor sites and the gauge links \(U_{ij}\) ensure gauge invariance. The third term is a mass term for the trace of the Wilson line which combines the \(n = 1\) contribution to the one-loop potential with the non-perturbative “fuzzy” bag contribution. Contributions from larger \(n\) have been dropped. We have previously performed simulations of a simplified version of (5) without magnetic fields in ref. [12]. For \(m^2 = 0\), and without magnetic fields, our code reproduces free energy measurements from refs. [13] but differs slightly from the older work of ref. [14] which used smaller lattices and lower statistics.

We employ the standard Metropolis algorithm to generate a thermal ensemble of configurations. The lattice is updated sequentially. Trial steps are taken in phase space by rotating single sites or links by a small angle in a random direction. The difference of the action \(\Delta S = S_{i+1} - S_i\) of the new and the old configurations is calculated and the step is accepted with probability \(p = \min[1, \exp(-\Delta S)]\). To reduce autocorrelations and accelerate thermalization we include over-relaxation sweeps where the Metropolis trial steps are taken deterministically in a way that approximately inverts the action with respect to its minimum [15] (exact non-stochastic over-relaxation is not possible due to the non-linear term in the action).

All sample sizes quoted here are taken to be statistically independent measurements. We have estimated autocorrelation times using the binning method described in ref. [16] and by observing the thermal relaxation within the Monte Carlo time series of measurements. We shall refrain from discussing these technical matters here in detail.

III. RESULTS

A. Phase diagram

To determine the phase diagram of the theory we measure the expectation value of the Polyakov loop \(\ell(\mathbf{x}) = \frac{1}{2} \text{tr } L(\mathbf{x})\) and the inverse correlation length \(\xi = 1/\xi\) as functions of the couplings \(m^2\) and \(\beta\). There is a line of phase transitions in the \(\beta - m^2\) plane separating the region where the expectation value of the Polyakov loop vanishes (the Z(2) symmetric phase at low \(\beta, m^2\)) from the Z(2) broken phase at large \(\beta\) or \(m^2\) (see fig. 1). The transition is of second order since the inverse correlation length \(m\) on the phase boundary extrapolates to zero in the infinite volume limit [12] as long as \(\beta\) is not too large\(^1\). It appears that the order of the phase transition changes to first order at roughly \(\beta > 3.0\). This may signal a breakdown of this model as an effective description of 4D Yang Mills in the extreme weak coupling limit (for a discussion of this issue see Appendix B).

Confinement is realized in distinct ways. At small \(\beta\) and vanishing \(m^2\) we find \(\langle \ell \rangle = 0.0\) because the Wilson line \(L(\mathbf{x})\) averages to zero from random fluctuations over the group manifold. For large \(\beta\) and negative \(m^2\) (corresponding to the upper left region in fig. 1), we find a non-trivial Z(2) symmetric phase, where \(L(\mathbf{x}) = i\tau_3\) or rotations thereof, which implies that the trace of \(L\) vanishes while the field \(L\) is non-zero\(^2\) (see, also, ref. [12]).

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1 The second order phase transition is further confirmed by the potentials described in later sections

2 In other words, the distributions of the eigenvalues \(\lambda_1\) and \(\lambda_2\), which are gauge invariant, peak about ±1. That distinct confined phases with different eigenvalue structure can also arise in 4d models of Polyakov loops coupled to gauge fields, was shown in refs. [17]
B. Effective potential

We are interested in the distribution of the eigenvalues $\lambda_1$, $\lambda_2$ of the Wilson line $L$ over the thermal ensemble of field configurations at each site. Here, we shall focus on the potential for the average of $\lambda_1$ and $\lambda_2$:

$$\rho(t, x) = \frac{1}{2} |\lambda_1(t, x) + \lambda_2(t, x)|,$$

which for SU(2) is nothing but the absolute value of the Polyakov loop, $\rho = \sqrt{\ell^2}$. From the probability distribution for $\rho$ we define an effective potential via

$$V_{\text{eff}}(\rho) = -\log P(\rho).$$

We normalize the probability density $P(\rho)$ over the interval $[0,1]$ which fixes the constant in the potential. This also factors out the volume dependence. The partition function can thus be written as

$$Z = e^{-N_s^2 V_{\text{eff}}(\rho)} = e^{-\sum_i V_{\text{eff}}(\rho)}.$$

In our definition, therefore, $V_{\text{eff}}(\rho)$ is dimensionless as it absorbs the volume $a^3$ of a lattice cell and, implicitly, a factor of $1/T$ (because we employ a 3d theory).

Below, we shall show that in the weak-coupling regime (large $\beta$, small $m^2$) a non-analytic contribution $\sim \sqrt{\ell^2}$ to the effective potential arises dynamically. It is distinct from the Vandermonde potential $V_{\text{vdm}} = -\frac{1}{2} \log(1-\ell^2)$ generated by the SU(2) group measure, and from the "bare" potential $\sim -m^2 \ell^2$ which is included in the action (5). We find that for a broad range of the couplings $\beta$ and $m^2$ the potential has the form

$$V_{\text{fit}}(\rho) = -\frac{1}{2} \log(1-\rho^2) + a - b \rho + c \rho^2.$$

Note that the term proportional to $\rho \equiv \sqrt{\ell^2}$ of course does not break the $Z(2)$ symmetry explicitly, and is not to be confused with a $Z(2)$ background field $\sim -h \ell$ corresponding to (heavy) dynamical quarks in the fundamental representation.

All measurements presented here were performed on a $N_s = 24$ cubic lattice. However, using smaller lattices we have checked that the coefficients $a$, $b$, $c$ from (9) do not change much with volume if $N_s \geq 12$. $N_s = 24$ appears to be a good approximation to the infinite volume limit where, as indicated in eq. (8), one expects the potential to be volume independent. 5000 independent lattice configurations were generated for each combination of $\beta$, $m^2$.

We first consider the case without potential, $m^2 = 0$. The phase transition occurs along a vertical line in the phase diagram at $\beta_C \approx 0.9$. Below the phase transition point we find that the potential defined via eq. (7) coincides with the Vandermonde potential $V_{\text{vdm}}$, hence $a = b = c = 0$ within numerical precision. This is shown in fig. 2. On the other hand, for $\beta > \beta_C$ both coefficients $a$ and $b$ from eq. (9) turn non-zero. The quadratic coefficient $c$ does not appear since $m^2 = 0$. The coefficient $b$ appears to arise purely from the dynamics of fluctuations.
Its behavior is shown in fig. 3.

The ansatz (9) also works for \( m^2 \neq 0 \) when \( \beta \) is not too large. We have confirmed this for a broad range of \( m^2 \) for several fixed values of \( \beta \). Explicit results for \( \beta = 2.0 \) are shown in fig. 4. The coefficient \( c \) of the quadratic term appears to coincide with the mass from the bare potential, \( c \approx -m^2 \), within our present numerical precision. However, we again observe that above the phase transition point the dynamics generates a non-analytic contribution to the potential for the Polyakov loop proportional to \( \beta \). For illustration, we show the \( m^2 \) dependence of \( a, b, c \) for \( \beta = 1.0 \) and \( \beta = 2.0 \) in fig. 5. A detailed discussion of the dependence of \( a, b, c \) on \( \beta \) and \( m^2 \) is given in appendix A. The main point here is that the Vandermonde contribution to the effective potential does not depend on \( \beta \) and \( m^2 \), and that the effective potential obtained from a histogram of \( \langle \hat{\ell} \rangle \) at each lattice site is different from a Landau-Ginzburg type mean-field theory for the Polyakov loop. In the next section we shall see that when the field configurations are “cooled” to remove short wavelength fluctuations, that in fact one does obtain a potential that resembles mean field theory, but with a coefficient multiplying the Vandermonde potential which depends on \( \beta \) and \( m^2 \).

### C. Effective potential for block spins

To obtain a potential for the long wavelength modes we average the Polyakov loop field over small cubes of side-length \( k \) before histogramming. This eliminates the short wavelength spatial field modes. We calculate blockspin averages defined as

\[
\bar{\bar{\ell}}^{(k)} = \frac{1}{k^3} \sum_{\vec{n}} \frac{1}{n} \text{tr} \mathbf{L}(\vec{\ell} + \vec{n}) ,
\]

\[
\vec{n} = (0, 0, 0) \ldots (k, k, k) .
\]

We have investigated the cases \( k = 2, 3, 4 \). Blockspins were measured on 2500 independent configurations for each combination of \( \beta, m^2 \). As one expects from the central limit theorem, with increasing \( k \) the potential becomes more symmetric and peaked around the actual expectation value (see fig. 7). In what follows, we take the configurations for \( k = 3 \) as a good approximation for the long distance sector. To see this consider fig. 6. At \( k = 3 \) the minimum of the fitted effective potential from eq. (11) differs at most by \( \approx 0.03 \) from the numerical result for the Polyakov loop expectation value \( \langle \ell \rangle \). This maximal deviation occurs exactly at the phase transition point. Here, within our numerical precision, \( k = 4 \) does not do significantly better. Away from the phase transition, \( k = 3 \) differs less than \( \approx 0.01 \) from the numerical value of \( \langle \ell \rangle \). \( k = 4 \) does slightly better but reduces our statistics significantly. All results presented below ap-
appear to be stable when going from $k = 3$ to $k = 4$ (we will discuss an explicit example below). We refrain from discussing $k = 2$ in detail here as it appears that contributions from the short range fluctuation have not yet been completely eliminated.

The ansatz (9) is no longer applicable for the long wavelength modes. The blockspin averaging appears to suppress the non-analytic term in the potential in most parts of the phase diagram (some possible exceptions are discussed below). It appears that over a broad range of $\beta$ and $m^2$ the potential can be described fairly well by a form analogous to mean field models, with quadratic and quartic terms [18]. However, for a good fit to the extracted potential one needs to include an additional parameter $d_0$ into the fit function which varies with $\beta$ and $m^2$, and which multiplies the Vandermonde potential. So, finally, the form which we use to model our data is

$$V(\rho) = -d_0 \frac{1}{2} \log(1 - \rho^2) + d_1 + d_2 \rho^2 + d_4 \rho^4.$$  \hspace{1cm} (11)

Here, a linear term is not included. We also point out that the quartic term arises from the dynamics of fluctuations as such a contribution is not included in the

**FIG. 5:** Coefficients of $V(\rho)$ for $\beta = 1.0$ (top) and $\beta = 2.0$ (bottom) as functions of $m^2$. The non-analytic contribution sets in at the phase transition.

**FIG. 6:** The difference of Polyakov loop expectation value and minimum of effective potential after “cooling” for $k = 2, 3, 4$ at $m^2 = 0.0$ as a function of $\beta$ (top) and at $\beta = 2.0$ (bottom) as a function of $m^2$. The peaks correspond to the phase transition point.

**FIG. 7:** Blockspin averages for $\beta = 1.5$, $m^2 = 0.0$. As $k$ increases the potential peaks more sharply about the expectation value.
"bare" action (5). Explicit results for the numerical potential and for the fit via eq. (11), for different values of \(m^2\) in the confined and deconfined phases and fixed \(\beta = 2.0 / 1.0\), are shown in figs. 8,9. Below the phase transition one can set \(d_0 = d_4 = 0\) and fit the potential with a simple quadratic form. This is not surprising since for \(\langle \ell \rangle = 0.0\) the potential is essentially parabolic and there is little sensitivity to higher powers of \(\ell\).

We show how the fit parameters from eq. (11) depend on \(\beta\) and \(m^2\). Figs. 10 and 12 correspond to \(\beta = 1.0\) with variable \(m^2\) and \(m^2 = 0.0\) with variable \(\beta\), respectively. One observes that right above the phase transition, the potential is a sum of quadratic and quartic terms while the Vandermonde contribution vanishes (in the figures, \(d_0\) appears to fluctuate somewhat around zero. We have checked, however, that the result is consistent with setting \(d_0 = 0\) by hand). At higher \(m^2\) or \(\beta\) respectively, the coefficient \(d_0\) increases gradually and saturates at about \(d_0 \approx 80\) for large values of \(\beta\) or \(m^2\), while the quartic coefficient becomes negative. We have further checked that fixing \(d_0\) to its asymptotic value \((d_0 \approx 80)\) gives a less accurate fit and increases \(\chi^2\) per degree of freedom in the region closely above the phase transition roughly by a factor of two. This indicates that the gradual increase of \(d_0\) is a real dynamical effect and not just an artifact.
FIG. 11: The coefficients of eq. (11) \( \beta = 2.0 \). The region with vanishing Vandermonde term appears to shrink when going from \( \beta = 1.0 \) to \( \beta = 2.0 \). Compare to fig. 10.

FIG. 12: A region with vanishing Vandermonde term is visible for \( m^2 = 0.0 \) and \( \beta > \beta_C \).

generated by a lack of sensitivity to the Vandermonde potential when \( \langle \rho \rangle \) is much smaller than 1. Fig. 11 shows that the region right above the phase transition, where the Vandermonde vanishes, appears to shrink when going deeper into the weak coupling limit (larger \( \beta \)).

The question remains, how this suppression of \( d_0 \) just above the phase transition, at moderately weak coupling, comes about. We have therefore attempted to model the potential right above the transition with a different function, assuming a fixed Vandermonde potential term, equal to the asymptotic value, but also allowing additional terms. We find that it is possible, within our numerical accuracy, to trade the suppression of the Vandermonde for another term linear in \( \rho \). The function

\[
V(\rho) = -d_0 \frac{1}{2} \log(1 - \rho^2) + d_1 + d_0' \rho + d_2 \rho^2 + d_4 \rho^4,
\]

with \( d_0 \equiv 80 \).

reproduces the behavior of the effective potential around \( \rho \approx 0.0 \) even slightly better than eq. (11), with a negative coefficient \( d_0' \) in the region right above the phase transition. However, the improvement in \( \chi^2 \) is below the percent level and the function (12) fails completely at large \( \beta \) or \( m^2 \) (by generating absurd global behavior).

The function (12) may suggest that the suppression of the Vandermonde could be an artifact due to incomplete cooling of short-distance fluctuations. However, we have investigated the cases \([\beta = 1.0/\text{variable } m^2]\) and \([m^2 = 0.0/\text{variable } \beta]\) also for \( k = 4 \) and obtained similar results, up to an overall scaling factor for all coefficients in the potential\(^3\). We show the result for \( \beta = 1.0 \) in fig. 13. Compared to fig. 10 we only observe a slight suppression of the quartic coefficient right above the phase transition.

FIG. 13: The coefficients of eq. (11) for \( \beta = 1.0 \) at \( k = 4 \). Compare to fig. 10.

IV. SUMMARY AND DISCUSSION

We have performed simulations of an effective theory of Wilson lines coupled to gauge fields in three dimensions which respects the center symmetry of the four-dimensional SU(2) Yang-Mills theory. After mapping the phase diagram, we have investigated the effective potential for the average of the eigenvalues of the SU(2) Wilson line, which is equal to the absolute value of the Polyakov loop. We found that a form containing non-analytic contributions can describe the extracted potential. This non-analytic term was not present in the action, and therefore must arise from the dynamics.

\(^3\) A rescaling of the coefficients in the potential is expected because coarse-graining over \( k \) lattice sites effectively corresponds to a simulation with different lattice spacing. A quantitative comparison of the coefficients of the \( k = 3 \) potential to those of the \( k = 4 \) potential would also require a rescaling of \( \beta \).
We extracted a similar effective potential also for the long wavelength modes of the Polyakov loop and found that this can be described by a mean-field type potential with quadratic and quartic terms plus an effective Vandermonde potential which depends on the couplings. Just above the phase boundary, in the deconfined phase, the effective Vandermonde potential contributes little. Deeper into the deconfined phase its coefficient increases and eventually appears to approach a constant.

Our simulations may provide useful insight into the structure of mean-field type models for the deconfining phase transition. For example, so-called “Polyakov-NJL” models have recently been studied extensively. Such models attempt to describe QCD thermodynamics over a range of quark masses, from the pure-gauge limit to physical QCD; they require an ansatz of the form

\[
b \approx b_0 (b^2 - b^2_C) \theta (b^2 - b^2_C)
\]

(A1)

Indeed, we find that with \( b^2_C = 0.9 \), a good fit of \( b(\beta) \) is possible, resulting in \( b_0 = 7.1(1) \) and \( r = 8.2(4) \). This fit corresponds to the solid line in fig. 3. For \( m^2 \neq 0 \) the situation is more involved. Motivated by the 2D Ising model we try the ansatz

\[
b(\beta, m^2) = b(\beta) \theta (m^2 + \tilde{m}^2(\beta))
\]

\[
\times \left\{ 1 - \left[ \sinh(g(\beta)(m^2 + \tilde{m}^2(\beta))) + \tilde{\beta}_C \right]^{-1/2} \right\}^{1/4}
\]

(A2)

with \( \tilde{\beta}_C = \log(1 + \sqrt{2}) \) (see fig. 5). This is similar to the magnetization in the 2D Ising model [20], which is given by

\[
M(\beta) = \theta(\beta - \tilde{\beta}_C) (1 - (\sinh(2\beta J)^{-1/2})^2).
\]

(A3)

We find that (A2) works reasonably well for \( \beta \leq 3.0 \) (see the fit-curves in fig. 5 for specific examples). We have included the constant \( \tilde{\beta}_C \) in eq. (A2) because it corresponds to the critical point of the Ising model. Our model of course deconfines at a different value of \( \beta \). Nevertheless, an ansatz such as eq. (A3) implicitly assigns the number \( \tilde{\beta}_C \) a special meaning and we therefore include it into our ansatz also in order to “filter out” its effect. Isolating \( \tilde{\beta}_C \) in such a way simplifies the resulting dependence of the fit parameters on \( \beta \) greatly.

The coefficients introduced in eq. (A2) act as follows: \( \tilde{m}^2 \) corresponds to a shift along the horizontal axis. \( \tilde{b} \) is a scaling factor and \( g \) is a modification of the coupling strength. The \( \beta \) dependence of these coefficients can be described by power laws

\[
\tilde{m}^2(\beta) = m_0^2 + m_1^2 \beta^2 v, \quad (A4)
\]

\[
\tilde{b}(\beta) = b_0 + b_1^2 \beta^2 w, \quad (A5)
\]

\[
g(\beta) = g_0 + g_1^2 \beta^2 u, \quad (A6)
\]

with

\[
m_0^2 = 2.2(1), \quad m_1^2 = -2.1(1), \quad w = -1.2(1), \quad (A7)
\]

\[
b_0 = -2.5(4), \quad b_1^2 = 6.0(4), \quad v = 0.90(4), \quad (A8)
\]

\[
g_0 = 0.038(1), \quad g_1^2 = 0.017(1), \quad u = 2.8(1). \quad (A9)
\]

The \( \beta \) dependence of \( \tilde{m}^2, \tilde{b} \) and \( g \) is shown in fig. 14.

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**APPENDIX A: COEFFICIENTS IN THE EFFECTIVE POTENTIAL FOR ALL MODES**

In this appendix we discuss the behavior of the non-analytic term in the potential defined in eq. (9) as a function of \( \beta \) and \( m^2 \).

The case \( m^2 = 0.0 \) is rather simple. Fig. 3 suggests an ansatz of the form

\[
b(\beta) = b_0 (\beta - \beta_C)^r \theta (\beta - \beta_C).
\]

(A1)

For \( m^2 \neq 0 \) the situation is more involved. Motivated by the 2D Ising model we try the ansatz

\[
b(\beta, m^2) = b(\beta) \theta (m^2 + \tilde{m}^2(\beta))
\]

\[
\times \left\{ 1 - \left[ \sinh(g(\beta)(m^2 + \tilde{m}^2(\beta))) + \tilde{\beta}_C \right]^{-1/2} \right\}^{1/4}
\]

(A2)

FIG. 14: \( \beta \) dependence of various coefficients parameterizing \( b(\beta, m^2) \) with their corresponding fit curves. \( g(\beta) \) is scaled up by a factor of 30.
APPENDIX B: FIRST-ORDER TRANSITION IN EXTREME WEAK COUPLING LIMIT

At very large $\beta$ the transition becomes first order. To see this, consider figs. 15 and 16. For $\beta = 5.0$ the effective potential develops two distinct minima in the vicinity of the phase transition point. Correspondingly, the Polyakov loop expectation value appears to be discontinuous. This behavior is in sharp contrast to the case $\beta = 2.0$ (shown in fig. 4), where one can see a single minimum moving continuously with $m^2$ (within the resolution) from $\rho \approx 0$ to $\rho \approx 1$.

The first order transition indicates that our effective theory cannot describe 4D Yang Mills when $\beta$ is too large. It is possible that this behavior is cured by adding higher powers of the Polyakov loop to the action (5).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig15.png}
\caption{Effective potential for $\beta = 5.0$ at values of $m^2$ slightly above and slightly below the phase transition. A first order phase transition is apparent since one observes two distinct minima.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig16.png}
\caption{Polyakov Loop expectation value for $\beta = 2.0$ and $\beta = 5.0$ measured on $N_s = 24$. The transition becomes very sharp for large $\beta$.}
\end{figure}

\[ m^2 = -3.5 \]
\[ m^2 = -2.9 \]
\[ m^2 = -2.7 \]
\[ m^2 = -2.6 \]

[1] see, for example, J. O. Andersen and M. Strickland, Annals Phys. 317, 281 (2005) [arXiv:hep-ph/0404164]; Y. Schröder, PoS JHW2005, 029 (2006) [arXiv:hep-ph/0605057]; J. P. Blaizot, Nucl. Phys. A 785, 1 (2007) [arXiv:nucl-th/0611104]; and references therein.

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