THE WEAK BOUNDED APPROXIMATION PROPERTY FOR $\mathcal{A}$

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Abstract. Fixed a Banach operator ideal $\mathcal{A}$, we introduce and investigate the weak bounded approximation property for $\mathcal{A}$, which is strictly weaker than the bounded approximation property for $\mathcal{A}$ of Lima, Lima and Oja (2010). We relate the weak BAP for $\mathcal{A}$ with approximation properties given by tensor norms and show that the metric approximation property of order $p$ of Saphar is the weak BAP for the ideal of $p'$-summing operators, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$. Under this framework, we address the question of approximation properties passing from $X'$ to $X$ or from $X''$ to $X'$.

INTRODUCTION

Let $X, Y$ be Banach spaces. We denote by $(\mathcal{A}, \| \cdot \|_\mathcal{A})$ a Banach operator ideal. When the norm $\| \cdot \|_\mathcal{A}$ is understood or when we work with an operator ideal, we simply write $\mathcal{A}$. As usual $\mathcal{L}, \mathcal{F}, \mathcal{F}^\ast, \mathcal{K}$ and $\mathcal{W}$ denote the ideals of bounded, finite rank, approximable, compact and weakly compact linear operators, respectively; all considered with the supremum norm.

A Banach space is said to have the approximation property (AP for short) if the identity map can be uniformly approximated by finite rank operators on compact sets. If a bound for the norm of the finite rank operators approximating the identity is required we have an stronger form of the approximation property. Let $1 \leq \lambda < \infty$, a Banach space $X$ is said to have the $\lambda$-bounded approximation property ($\lambda$-BAP) if there exists a net $(S_\alpha)$ in $\mathcal{F}(X;X)$ such that $\sup_{\alpha} \| S_\alpha \| \leq \lambda$ and $S_\alpha \to I_X$ uniformly on compact subsets of $X$. When $\lambda = 1$ the terminology 1-bounded approximation property (1-BAP) is usually replaced by metric approximation property (MAP). In [19], Lima and Oja define the weak bounded approximation property and use it to give several reformulations of the famous problem: are the approximation property and the metric approximation property equivalent on a dual space? Recall that $X$ has the weak $\lambda$-bounded approximation property (weak $\lambda$-BAP) if for every Banach space $Y$ and for each operator $T$ in $\mathcal{W}(X;Y)$, there exists a net $(S_\alpha)$ in $\mathcal{F}(X;X)$ such that $\sup_{\alpha} \| T S_\alpha \| \leq \lambda \| T \|$ and $S_\alpha \to I_X$ uniformly on compact subsets of $X$. In [18], Lima, Lima and Oja extend this definition replacing $(\mathcal{W}, \| \cdot \|)$ with an arbitrary Banach operator ideal $(\mathcal{A}, \| \cdot \|_\mathcal{A})$, as follows. A Banach space $X$ has the $\lambda$-bounded approximation property for $\mathcal{A}$ ($\lambda$-BAP for $\mathcal{A}$) if for every Banach space $Y$ and for each operator $T$ in $\mathcal{A}(X;Y)$, there exists a net $(S_\alpha)$ in $\mathcal{F}(X;X)$ such that $S_\alpha \to I_X$ uniformly on compact

This project was supported in part by CONICET PIP 0624, PICT 2011-1456 and UBACyT 1-746.
subsets of $X$ and

$$\sup_{\alpha} \|TS_\alpha\|_A \leq \lambda \|T\|_A.$$  

The BAP for $A$ [18] formalizes the study of an approximation property where not only the operator ideal properties of $A$ are considered but also the ideal norm $\| \cdot \|_A$. This allows the understanding of several known approximation properties in terms of operator ideals and their geometry. For instance, the $\lambda$-BAP is clearly the $\lambda$-BAP for $L$ and it is also the $\lambda$-BAP for the ideal of (Pietsch) integral operators $I$ [18, Theorem 2.1]. The weak $\lambda$-BAP is by definition the $\lambda$-BAP for $W$ and it is also the $\lambda$-BAP for the ideal of nuclear operators $N$ [18, Theorem 3.1] and for $K$ [19, Theorem 2.4].

In [18, Problem 5.5], the authors wonder if given an arbitrary Banach operator ideal $A$, the $\lambda$-BAP for $A$ of a Banach space $X$ is equivalent to the (at least formally) weaker following condition: for every Banach space $Y$ and each operator $T \in A(X;Y)$ there exists a net $(T_\alpha)_\alpha \subset F(X;Y)$ such that $T_\alpha \to T$ pointwise and

$$\limsup_{\alpha}\|T_\alpha\|_A \leq \lambda \|T\|_A.$$  

Being consistent with the notation of [21] we say that a Banach space $X$ satisfying the above has the condition $c^*_\lambda$ for $A$. Problem 5.5 of [18] has an obvious positive answer if $A = L$ and also if $A = W$ or $A = K$ [21, Theorem 3.6]. Surprisingly, we provide a negative answer considering any minimal Banach operator ideal $A$, as we show in Remark 1.1.

Here, we present a natural modification of the $\lambda$-BAP for $A$ that, at first glance, formally lies between this notion and the condition $c^*_\lambda$ for $A$. We say that a Banach space $X$ has the weak $\lambda$-bounded approximation property for $A$ (weak $\lambda$-BAP for $A$) if for every Banach space $Y$ and for each operator $T$ in $A(X;Y)$, there exists a net $(S_\alpha)_\alpha$ in $F(X;Y)$ such that $S_\alpha \to I_X$ in the strong operator topology and

$$\sup_{\alpha} \|TS_\alpha\|_A \leq \lambda \|T\|_A.$$  

Then we have:

(1) \hspace{1cm} $\lambda - BAP \Rightarrow \lambda - BAP$ for $A$ \hspace{.5cm} $\Rightarrow$ \hspace{.5cm} weak $\lambda - BAP$ for $A$ \hspace{.5cm} $\Rightarrow$ \hspace{.5cm} $c^*_\lambda$ for $A$.

We do not know of any example of an ideal $A$ for which the $\lambda$-BAP is strictly stronger than the $\lambda$-BAP for $A$. However, we will show that the subtle difference between the condition $c^*_\lambda$ for $A$ and the weak $\lambda$-BAP for $A$ is, in fact, not formal, see Corollary 1.4. Also, we will prove that the weak $\lambda$-BAP for $A$ and the $\lambda$-BAP for $A$ do not coincide, see Proposition 2.3.

The paper is organized as follows. In Section 1 we characterize the weak $\lambda$-BAP for $A$. Motivated by [18, Problem 5.5] and with the aim of clarifying the differences between the $\lambda$-BAP for $A$ and the weak $\lambda$-BAP for $A$ we show, under certain conditions on $A$, that the strong operator topology under which the net $(S_\alpha)$ of the definition converges to the identity
can be changed by a finer topology (coarser than the topology of uniform convergence on compact sets). We also relate the weak-BAP for $A$ with approximation properties given by tensor norms.

In Section 2, we apply our results to show that the $\lambda$-BAP of order $p$ of Saphar coincides with the weak $\lambda$-BAP for $\Pi_p'$, where $\Pi_p$ stands for the ideal of $p$-summing operators, $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. This allows us to show that the weak $\lambda$-BAP for $\Pi_p$ is strictly weaker than the $\lambda$-BAP for $\Pi_p$, $1 < p \leq 2$. We end our study with a brief discussion on this kind of approximation properties enjoyed by a Banach space which are inherited from its dual.

The definitions and notation used regarding operator ideals can be found in the monograph by Defant and Floret [7]. For operator ideals, we also refer the reader to the books of Pietsch [26], of Diestel, Jarchow and Tonge [12] and of Ryan [29]. For approximation properties, we refer the reader to the books of Lindenstrauss and Tzafriri [20] and of Diestel, Fourie and Swart [11]. See also [7, 29], the surveys [5] and [22] and references therein.

1. The weak bounded approximation property for $A$

Let $X, Y$ be Banach spaces. We denote by $X'$ and $B_X$ the topological dual of $X$ and its closed unit ball, respectively. The canonical inclusion of $X$ into its bidual $X''$ is denoted by $J_X$ and, if it is convenient, we write $J_Xx = \hat{x}$ for $x \in X$. As usual, operators in $\mathcal{F}(X;Y)$ are considered as elements of the uncompleted tensor fold $X' \otimes Y$ and tensors in $X \otimes Y$ are regarded as operators in $\mathcal{F}(X';Y)$. The set of null sequences of $X$ is denoted by $c_0(X)$. Also, $\tau_c$, sot and wot stand for the topology of uniform convergence on compact sets and the strong and the weak operator topologies, respectively.

Before we focus our attention on the weak BAP for $A$, let us show that [18, Problem 5.5] has a negative answer.

**Remark 1.1.** Any Banach space has the $c_1^*$ for $A$ for any minimal Banach operator ideal $A$. As a consequence, for any minimal Banach operator ideal $A$ the $\lambda$-BAP for $A$ cannot be reformulated by the weaker condition $c_1^*$ for $A$, $\lambda \geq 1$.

**Proof.** Let $A$ be a minimal Banach operator ideal and $X, Y$ be Banach spaces. Given $T \in A(X;Y)$ there exists a sequence $(T_n)_n \subset \mathcal{F}(X;Y)$ such that $T_n \to T$ in $A$ (and therefore $T_n \to T$ pointwise). Given $\varepsilon > 0$, passing to a subsequence if necessary, we may assume that $\limsup_n \|T_n\|_A \leq (1 + \varepsilon)\|T\|_A$. Then $X$ has the $c_1^*$ for $A$. In particular this is true for any Banach space $X$ without the AP and hence without the $\lambda$-BAP for $A$. \qed

The following is a characterization of the weak BAP for $A$ which somehow can be seen as a generalization of [19, Theorem 2.4] stated for the ideal $K$.

**Theorem 1.2.** Let $X$ be a Banach space and $(A, \|\cdot\|_A)$ be a Banach operator ideal. The following are equivalent.
(i) $X$ has the weak $\lambda$-BAP for $A$.
(ii) For every Banach space $Y$ and for each operator $T \in \mathcal{A}(X; Y)$, there exists a net $(S_{\alpha}) \subset \mathcal{F}(X; X)$ with $\sup_{\alpha} \| TS_{\alpha} \|_A \leq \lambda \| T \|_A$ such that $TS_{\alpha} \to T$ in the strong operator topology.
(iii) For every Banach space $Y$ and for each operator $T \in \mathcal{A}(X; Y)$ with $\| T \|_A = 1$, for all sequences $(x_n)_n \subset X$ and $(y'_n)_n \subset Y'$ such that $\sum \| x_n \| \| y'_n \| < \infty$, one has the inequality

$$\left| \sum_{n=1}^{\infty} y'_n(Tx_n) \right| \leq \lambda \sup_{\| TS \|_A \leq 1} \left| \sum_{n=1}^{\infty} y'_n(TSx_n) \right|.$$

Proof. Clearly (i) implies (ii). To prove that (ii) implies (iii), follow the the proof of (a) implies (d) of [19, Theorem 2.4] with the obvious modifications.

For (iii) $\Rightarrow$ (i), assume that $\| T \|_A = 1$ and suppose that $I_X \notin \{ S \in \mathcal{F}(X; X) : \| TS \|_A \leq \lambda \}_{\text{sot}}$. Then, there exists $\phi \in (\mathcal{L}(X; X); \text{sot})'$ such that

$$|\phi(I_X)| > \sup\{ |\phi(S)| : S \in \mathcal{F}(X; X), \| TS \|_A \leq \lambda \}.$$

Since $\phi$ is sot-continuous, there exist $x_1, \ldots, x_n \in X$ and $x'_1, \ldots, x'_n \in X$ such that $\phi(R) = \sum_{j=1}^{n} x'_j(Rx_j)$ for all $R \in \mathcal{L}(X; X)$. Hence,

$$\left| \sum_{j=1}^{n} x'_j(x_j) \right| > \sup_{\| TS \|_A \leq 1} \left| \sum_{j=1}^{n} x'_j(Sx_j) \right| = \lambda \sup_{\| TS \|_A \leq 1} \left| \sum_{j=1}^{n} x'_j(Sx_j) \right|.$$

We affirm that this inequality cannot hold. Fix any Banach space $Z$ and choose $z_1, \ldots, z_n \in Z$ and $z'_1, \ldots, z'_n \in Z'$ such that $z'_j(z_i) = \delta_{i,j}$ (the Kronecker delta) and take the finite rank operator $R \in \mathcal{F}(X; Z)$ given by $R = \sum_{j=1}^{n} x'_j \otimes z_j$. Consider the Banach space $G = Y \times Z$ endowed with the norm $\|(y, z)\| = \|y\| + \|z\|$. Fix $r > 0$ and take $\widetilde{T} : X \to G$ the operator defined by $\widetilde{T}x = (Tx, rRx)$. Using the canonical norm one inclusions $\iota_Y$ and $\iota_Z$ from $Y$ and $Z$ into $G$ we see that $\widetilde{T} = \iota_Y T + r \iota_Z R$ and belongs to $\mathcal{A}(X; G)$ with $\| \widetilde{T} \|_A \leq 1 + r \| R \|_A$.

As an element of $G'$, $(0, z'_j)(\widetilde{T} x) = rz'_j(Rx) = rx'_j(x)$ for all $x \in X$ and $j = 1, \ldots, n$. Then, by (iii) we have

$$\left| \sum_{j=1}^{n} rz'_j(x_j) \right| = \left| \sum_{j=1}^{n} (0, z'_j) \widetilde{T}(x_j) \right| \leq \lambda \sup_{\| TS \|_A \leq 1 + r \| R \|_A} \left| \sum_{j=1}^{n} (0, z'_j) \widetilde{T} S x_j \right| \leq \lambda (1 + r \| R \|_A) \sup_{\| TS \|_A \leq 1} \left| \sum_{j=1}^{n} x'_j(Sx_j) \right|.$$
Since $T = P_Y \tilde{T}$, where $P_Y$ is the norm one projection of $G$ onto $Y$, we have $\|TS\|_\mathcal{A} \leq \|\tilde{T}S\|_\mathcal{A}$, for any $S \in \mathcal{F}(X; X)$. Then, instead of (3) we obtain

$$\left| \sum_{j=1}^{n} x'_j(x_j) \right| \leq \lambda (1 + r \|R\|_\mathcal{A}) \sup_{\|TS\|_\mathcal{A} \leq 1} \left| \sum_{j=1}^{n} x'_j(Sx_j) \right|.$$ 

Since $r > 0$ was arbitrary, we conclude that

$$\left| \sum_{j=1}^{n} x'_j(x_j) \right| \leq \lambda \sup_{\|TS\|_\mathcal{A} \leq 1} \left| \sum_{j=1}^{n} x'_j(Sx_j) \right|,$$

contradicting inequality (2). Therefore, the proof is complete.}

The equivalence between (a) and (b) of [19, Theorem 2.4] states that the weak $\lambda$-BAP is the weak $\lambda$-BAP for $\mathcal{K}$ and the convergence of the net $(S_n)$ in the definition of the later property can be assumed to be the convergence on compact sets. In virtue of this, it is natural to wonder to what extent we may reformulate the definition of the weak $\lambda$-BAP for $\mathcal{A}$ in terms of a topology finer than the strong operator topology. Recall that Grothendieck’s characterization of the dual space of $(\mathcal{L}(X; Y), \tau_e)$ states that it consists of all functionals $\phi$ of the form $\phi(T) = \sum_{n=1}^{\infty} y'_n(Tx_n)$, with $\sum \|y'_n\| \|x_n\| < \infty$ [20, Proposition 1.5.3]. In other words, every functional is determined by an element $u$ of the completed projective tensor product $X \hat{\otimes} \pi Y'$. If $u = \sum_{n=1}^{\infty} y'_n \otimes x_n$, $\phi(T) = \langle u, T \rangle$ and we always may assume that $(x_n)_n \in \ell_1(X)$ and $(y'_n)_n \in c_0(Y')$ or $(x_n)_n \in c_0(X)$ and $(y'_n)_n \in \ell_1(Y')$ indistinctly. In order to proceed we need to appeal to the notion of $\mathcal{A}$-null sequences of Carl and Stephani [4], defined as follows.

Fixed an operator ideal $\mathcal{A}$, a sequence $(x_n)_n$ in a Banach space $X$ is said to be $\mathcal{A}$-null if there exist a Banach space $Z$, an operator $R \in \mathcal{A}(Z; X)$ and a null sequence $(z_n)_n \subset Z$ such that $x_n = Rz_n$ for all $n \in \mathbb{N}$ [4, Lemma 1.2]. A subset $K$ of $X$ is relatively $\mathcal{A}$-compact if it is contained in the absolutely convex hull of an $\mathcal{A}$-null sequence [4, Theorem 1.1]. We denote by $c_{0,\mathcal{A}}(X)$ the set of $\mathcal{A}$-null sequences of $X$. If $\mathcal{A}$ is a Banach operator ideal, we can measure the size of this type of sets as in [17]. Given a relatively $\mathcal{A}$-compact set $K \subset X$, define

$$m_\mathcal{A}(K; X) = \inf \{ \|T\|_\mathcal{A} : K \subset T(M), T \in \mathcal{A}(Z; X) \text{ and } M \subset B_Z \},$$

where the infimum is taken over all Banach spaces $Z$, all operators $T \in \mathcal{A}(Z; X)$ and all compact sets $M \subset B_Z$ for which the inclusion $K \subset T(M)$ holds. In the case of an $\mathcal{A}$-null sequence $(x_n)_n \subset X$, standard arguments (see for instance the proof of [17, Proposition 1.4]) show that given $\varepsilon > 0$, there are a Banach space $Z$, $R \in \mathcal{A}(Z; X)$ and a null sequence $(z_n)_n \subset B_Z$ such that $x_n = Rz_n$ and

$$\|R\|_\mathcal{A} \leq (1 + \varepsilon) m_\mathcal{A}(\{x_n\}; X).$$
Now we consider the following subspaces of $X \hat{\otimes}_\pi Y$

\[
S_A(X; Y) = \{ u = \sum_{n=1}^{\infty} x_n \otimes y_n : (x_n)_n \in \ell_1(X), (y_n)_n \in c_0(A(Y)) \},
\]

\[
S^d(X; Y) = \{ u = \sum_{n=1}^{\infty} x_n \otimes y_n : (x_n)_n \in c_0(A(X), (y_n)_n \in \ell_1(Y)) \}.
\]

We say that a tensor in $S_A$ is $A$-representable and that a tensor in $S^d$ is right $A$-
representable. Then, we say that a functional $\phi$ on $L$ is $A$-representable if it is associated
(in the sense of Grothendieck) with a tensor in $S_A$ and that $\phi$ is right $A$-representable if it is
associated with a tensor in $S^d$. The sets of $A$ and right $A$-representable functionals may coincide
as it happens for the ideal $K$ which, by Grothendieck’s characterization, also coincide with $\mathcal{L}(X; Y), \tau'$. In [8] the authors show that $\phi \in (\mathcal{L}(X; Y), \tau_A)'$ if and only
$\phi$ is $A$-representable; where $\tau_A$ is the topology of uniform convergence on $A$-compact sets.
However, given an operator ideal $A$, we do not have a general description for a locally convex
vector topology $\sigma$ on $\mathcal{L}(X, Y)$ such that every functional in $(\mathcal{L}(X, Y), \sigma)'$ is associated
with some tensor in $S^d(\mathcal{Y}, X)$. Denoting by $T'$ the adjoint of an operator $T$, the dual ideal $A^d$
of $A$ is the class of operators $T \in \mathcal{L}$ such that $T' \in A$ and $\|T\|_{A^d} = \|T'\|_A$. Now, we are in
conditions to state and prove the following result.

**Proposition 1.3.** Let $X$ be a Banach space, $(A, \|\cdot\|_A)$ be a Banach operator ideal such that
$A = A^{dd}$. The following are equivalent

(i) $X$ has the weak $\lambda$-BAP for $A$.

(ii) For every Banach space $Y$ and for each operator $T \in A(X; Y)$, there exists a net
$(S_\alpha) \subset \mathcal{F}(X; Y)$ with $\sup_\alpha \|TS_\alpha\|_A \leq \lambda \|T\|_A$ such that $S_\alpha \to I_X$ in the topology $\sigma$, where $\sigma$ is any locally convex topology such that any $\phi \in (\mathcal{L}(X; X), \tau)'$ is right $A^d$-
representable.

*Proof.* Suppose that (ii) holds. Note that any $\phi = x' \otimes x$ is right $A^d$-representable for every
$x' \in X'$ and $x \in X$. Since the topologies wot and sot coincide on absolutely convex sets
(see for instance [13, Theorem VI.1.4]), (i) holds.

For the converse, fix a Banach space $Y$ and $T \in A(X; Y)$ with $\|T\|_A = 1$. Consider the
absolutely convex set

\[
M = \{ S \in \mathcal{F}(X; X) : \|TS\|_A \leq \lambda \},
\]

and suppose that $I_X \notin M'$. Then, there exists $\phi \in (\mathcal{L}(X; X), \tau)'$ such that

\[
|\phi(I_X)| > \sup \{ |\phi(S)| : S \in M \}.
\]

Since $\phi$ is right $A^d$-representable take an $A^d$-null sequence $(x'_n)_n$ and $(x_n)_n \subset \ell_1(X)$ such
that $\phi(R) = \sum_{n=1}^{\infty} x'_n(Rx_n)$ for all $R \in \mathcal{L}(X; X)$. Hence,

\[
|\sum_{n=1}^{\infty} x'_n(x_n)| > \sup_{\|TS\|_A \leq \lambda} \sum_{n=1}^{\infty} x'_n(Sx_n) = \lambda \sup_{\|TS\|_A \leq 1} \sum_{n=1}^{\infty} x'_n(Sx_n).
\]
We affirm that this inequality cannot hold. Indeed, take \( r > 0 \), since \( (rx'_n)_n \) is \( A^d \)-null, given \( \varepsilon > 0 \) there exist a Banach space \( Z \), an operator \( R \in A^d(Z; X') \) and a null sequence \( (z_n)_n \in B_Z \) such that \( rx'_n = Rz_n \) for all \( n \) and \( \|R\|_{A^d} \leq (1 + \varepsilon)rc \), where \( c = m_{A^d}(\{x'_n\}; X') \). Consider the Banach space \( G = Y' \times Z \) endowed with the norm \( \|(y', z)\|_G = \max\{\|y'\|_{Y'}, \|z\|_Z\} \) and the operator \( \tilde{T} : G \to X' \) defined by \( \tilde{T}(y', z) = T'y' + Rz \). Since \( A^{dd} = A \), it follows that \( T' \in A^d \) and therefore \( \tilde{T} \in A^d \) with \( \|\tilde{T}\|_{A^d} \leq 1 + (1 + \varepsilon)rc \). Notice that \( \tilde{T}'J_X \in A(X; G') \) and if we set \( \delta = \|\tilde{T}'J_X\|^{-1}_A > 0 \), \( U = \delta \tilde{T}'J_X \) and \( g''_n = J_G(0, z_n) \in G'' \) for all \( n \), we obtain for all \( x \in X \) and \( n \)

\[
rx'_n(x) = (Rz_n)(x) = \tilde{x}(\tilde{T}(0, z_n)) = g''_n(\tilde{T}'J_X x) = \|\tilde{T}'J_X\|_A g''_n(U x).
\]

Since \( X \) has the weak \( \lambda \)-BAP for \( A \), equivalence (iii) of Theorem 1.2 gives that

\[
\| \sum_{n=1}^{\infty} x'_n(x_n) \| = \|\tilde{T}'J_X\|_A \frac{1}{r} \| \sum_{n=1}^{\infty} g''_n(U x_n) \|
\leq \|\tilde{T}'J_X\|_A \lambda \sup_{\|US\|_{A^d} \leq 1} \| \sum_{n=1}^{\infty} x'_n(\delta S x_n) \|
= \|\tilde{T}'J_X\|_A \lambda \sup_{\|\tilde{T}'J_XS\|_{A^d} \leq 1} \| \sum_{n=1}^{\infty} x'_n(S x_n) \|.
\]

(5)

Now, on the one hand we have \( \|\tilde{T}'J_X\|_A \leq \|\tilde{T}\|_{A^d} \leq 1 + (1 + \varepsilon)rc \). And, on the other hand, with \( A = A^{dd} \) and \( S \) of finite rank,

\[
\|TS\|_A = \|S\tilde{T}\|_{A^d} \leq \|S\tilde{T}'\|_{A^d} = \|S''\tilde{T}'\|_{A^d} = \|S''\tilde{T}'\|_{A^d} = \|J_XS''\tilde{T}'\|_{A^{dd}} = \|\tilde{T}'J_XS\|_A.
\]

Then, from (5), we get that

\[
\| \sum_{n=1}^{\infty} x'_n(x_n) \| \leq \lambda (1 + (1 + \varepsilon)rc) \sup_{\|TS\|_{A^d} \leq 1} \| \sum_{n=1}^{\infty} x'_n(S x_n) \|.
\]

Since the above expression holds for every \( r > 0 \), equation (4) is contradicted and the result follows. \( \square \)

**Corollary 1.4.** For any \( \lambda \geq 1 \), the \( \lambda \)-BAP for \( \text{d} \) and the weak \( \lambda \)-BAP for \( \text{d} \) coincide. As a consequence, the \( c^*_\lambda \) for \( \text{d} \) is strictly weaker than the weak \( \lambda \)-BAP for \( \text{d} \) for any \( \lambda \geq 1 \).

**Proof.** Give \( X \) a Banach space, any \( \phi \in (\mathcal{L}(X; X); \tau_c)' \) is right \( K \)-representable. Since \( \text{d} \)-null and \( K \)-null sequences coincide (see [17, Remark 1.3] and [17, Proposition 1.4]) and also \( \text{d} = \text{d} = \text{d} \), every \( \phi \in (\mathcal{L}(X; X); \tau_c)' \) is right \( \text{d} \)-representable. Then, the result follows by Proposition 1.3.

As a consequence, applying Remark 1.1 to \( \text{d} \) (which is minimal), we see that the second statement holds considering any Banach space without the AP. \( \square \)
Now we relate the weak-BAP for $\mathcal{A}$ with approximation properties given by tensor norms. Let us denote by $\alpha_{\mathcal{A}}$ the unique finitely generated tensor norm associated to a maximal Banach operator ideal $\mathcal{A}$ and by $\mathcal{A}^*$ the maximal Banach operator ideal associated with the tensor norm $\alpha^*_{\mathcal{A}} = (\alpha'_{\mathcal{A}})^t = (\alpha^*_A)'$. By the Representation Theorem for maximal operator ideals [7, Theorem 17.5] we may identify (isometrically) the spaces $(X \otimes \alpha_{\mathcal{A}} Y)'$ and $\mathcal{A}(X; Y')$.

Recall that a Banach space $X$ is said to have the bounded $\alpha_{\mathcal{A}}$-approximation property with constant $\lambda$ ($\alpha_{\mathcal{A}}$-$\lambda$-BAP) if for every Banach space $Y$ the natural mapping $j_{\alpha_{\mathcal{A}}} : Y \otimes \alpha_{\mathcal{A}} X \to (Y' \otimes \alpha_{\mathcal{A}} X')'$ satisfies $\alpha_{\mathcal{A}}(u; Y, X) \leq \lambda\|j_{\alpha_{\mathcal{A}}}(u)\|$. Then, for maximal ideals, we have the following reformulation.

**Lemma 1.5.** Let $X$ be a Banach space and let $\mathcal{A}$ be a maximal Banach operator ideal. Then, $X$ has the $\alpha_{\mathcal{A}}$-$\lambda$-BAP if and only if for every Banach space $Y$ the natural mapping $j_{\alpha_{\mathcal{A}}} : Y \otimes \alpha_{\mathcal{A}} X \to \mathcal{A}(Y'; X'')$ satisfies

$$\alpha_{\mathcal{A}}(u; Y, X) \leq \lambda\|j_{\alpha_{\mathcal{A}}}(u)\|_{\mathcal{A}}.$$ 

**Theorem 1.6.** Let $X$ be a Banach space and let $\mathcal{A}$ be a maximal Banach operator ideal. If $X$ has the weak $\lambda$-BAP for $\mathcal{A}^*$ and $X'$ has the $\alpha'_{\mathcal{A}}$-\(\lambda\)-BAP, then $X$ has the $\alpha_{\mathcal{A}}$-$\lambda\lambda$-BAP.

**Proof.** Let $u = \sum_{n=1}^m y_n \otimes x_n \in Y \otimes X$. Consider the tensor norm $\alpha'_A = (\alpha^*_A)'$ and $u^t = \sum_{n=1}^m x_n \otimes y_n \in X \otimes Y$. By [7, Theorem 17.5], there is an operator $T \in \mathcal{A}^*(X; Y')$ with $\|T\|_{\mathcal{A}^*} = 1$ such that $\alpha_{\mathcal{A}}(u; Y, X) = \alpha'_A(u^t; X, Y) = |\sum_{n=1}^m \hat{y}_n(Tx_n)|$. Since $X$ has the weak $\lambda$-BAP for $\mathcal{A}^*$, by Theorem 1.2, we have

$$\alpha_{\mathcal{A}}(u; Y, X) = |\sum_{n=1}^m \hat{y}_n(Tx_n)| \leq \lambda \sup_{\|TS\|_{\mathcal{A}^*} \leq 1} |\sum_{n=1}^m \hat{y}_n(TSx_n)| \leq \lambda \sup_{\|S\|_{\mathcal{A}^*} \leq 1} |\sum_{n=1}^m \hat{y}_n(Sx_n)|.$$  

(6)

Any $S$ in $\mathcal{F}(X; Y')$ is associated with a tensor $w_S$ in $Y' \otimes X'$. Now, since $X'$ has the $\alpha'_{\mathcal{A}}$-$\lambda$-BAP, by Lemma 1.5, $\|S\|_{\mathcal{A}^*(X; Y')} \geq \lambda^{-1}\alpha'_{\mathcal{A}}(w_S; Y', X')$. Hence,

$$\sup_{\|S\|_{\mathcal{A}^*} \leq 1} |\sum_{n=1}^m \hat{y}_n(Sx_n)| \leq \sup_{\alpha'_{\mathcal{A}}(w; Y', X') \leq \lambda} |j_{\alpha_{\mathcal{A}}}(u)(w)| = \bar{\lambda}\|j_{\alpha_{\mathcal{A}}}(u)\|_{\mathcal{A}}.$$  

(7)

A combination of (6) and (7) completes the proof. 

We finish this section with a general result that has interest on its own and will be used in the next section. Recall that the ideal of $\mathcal{A}$-compact operators, $\mathcal{K}_{\mathcal{A}}$, consist of all operators mapping bounded sets into relatively $\mathcal{A}$-compact sets. This ideal was defined and treated in [4] and was studied in connection with approximation properties in [17]. The ideal $\mathcal{K}_{\mathcal{A}}$
becomes a Banach operator ideal if we consider the following norm introduced in [17]. For any \( T \in \mathcal{K}_A(X; Y) \),
\[
\|T\|_{\mathcal{K}_A} = m_A(T(B_X); Y).
\]

Following [23] (or [16, Definition 4.3]), a Banach space \( X \) has the \( \mathcal{K}_A \)-approximation property (\( \mathcal{K}_A \)-AP) if for every Banach space \( Y \), \( \overline{\mathcal{F}(Y; X)}_{\|\cdot\|_{\mathcal{K}_A}} = \mathcal{K}_A(Y; X) \) or equivalently ([17, Proposition 3.1]) if \( I_X \in \mathcal{F}(X; X)^{\tau_{sA}} \), where \( \tau_{sA} \) is the locally convex topology given by the seminorms \( q_K(T) = m_A(T(K); X) \), where \( K \) ranges over all \( A \)-compact sets of \( X \).

**Proposition 1.7.** Let \( X \) be a Banach space and \( A \) be a Banach operator ideal. If \( X' \) has the \( \mathcal{K}_A \)-AP, then \( X \) has the weak MAP for \( \mathcal{K}_A^d \).

**Proof.** To show that \( X \) has weak MAP for \( \mathcal{K}_A^d \) we will see (the equivalent condition) that \( X \) has the weak \((1 + \varepsilon)\)-BAP for \( \mathcal{K}_A^d \), for any \( \varepsilon > 0 \). Fix \( \varepsilon > 0 \), take \( T \in \mathcal{K}_A^d(X; Y) \) with \( \|T\|_{\mathcal{K}_A^d} = 1 \) and define the set \( M = \{ S \in \mathcal{F}(X; X) : \|TS\|_{\mathcal{K}_A^d} \leq (1 + \varepsilon) \} \).

Applying [17, Lemma 3.6] to \( X' \) which has the \( \mathcal{K}_A \)-AP, we may find a net \( S_\alpha \in \mathcal{F}(X; X) \) such that \( S'_\alpha \to I_{X'} \) in the topology \( \tau_{sA} \). Then, since \( T'(B_{Y'}) \subset X' \) is \( A \)-compact, there exist \( \alpha_0 \) such that for all \( \alpha > \alpha_0 \),
\[
\|S'_\alpha T' - T'||_{\mathcal{K}_A} = m_A((S'_\alpha - I_{X'})(T'(B_{Y'})); X') \leq \varepsilon.
\]
Therefore, \( S_\alpha \in M \) for all \( \alpha > \alpha_0 \). Also, note that the topology \( \tau_{sA} \) is finer than the topology wot. Then \( S'_\alpha \) converges wot to \( I_{X'} \), implying that \( S_\alpha \) converges wot to \( I_X \). Since \( M \) is absolutely convex its wot closure and its sot closure coincide (see for instance [13, Theorem VI.1.4]). Hence, \( I_X \in M^{sot} \), and the proof is complete. \( \square \)

2. Applications

The main result of this section characterizes the metric approximation property of order \( p' \) of Saphar, \( 1 < p < \infty \) and \( \frac{1}{p} + \frac{1}{p'} = 1 \), in terms of weak-MAP for \( \Pi_p \). Then we discuss situations of approximation properties inferred from a dual space to an underlying space, that is from \( X' \) to \( X \) or from \( X'' \) to \( X' \). Recall that the maximal ideal \( \Pi_p \) is associated with the tensor norm \( g_p = (g_p')^t \). Then, \( \Pi_p(X; Y') \) and \( (X \otimes g_p' Y)' \) are identified. Also, a Banach space \( X \) is said to have the \( \lambda \)-BAP of order \( p \) \((\lambda \text{-BAP})\) if the natural surjection from \( Y' \otimes g_p X \), the completion of \( Y' \otimes X \) in the tensor norm \( g_p \), onto the ideal of \( p \)-nuclear operators from \( Y \) to \( X \), \( N_p(Y; X) \), is injective for all Banach spaces \( Y \) and has norm less than or equal to \( \lambda \).

**Theorem 2.1.** Let \( X \) be a Banach space and let \( 1 < p < \infty \). Then, \( X \) has the weak \( \lambda \)-BAP for \( \Pi_p \) if and only if \( X \) has the \( g_p'-\lambda \)-BAP.
Proof. By [7, Corollary 21.1] and [7, Proposition 21.7], every Banach space (in particular X') has the $g'_{\rho'}$-MAP. Since X has the weak $\lambda$-BAP for $\Pi_p$, by Theorem 1.6, X has the $g'_{\rho'}$-\lambda-BAP.

Conversely, suppose that X has the $g'_{\rho'}$-\lambda-BAP and take $T \in \Pi_p(X; Y)$, $\|T\|_{\Pi_p} = 1$, $(y'_n)_n \subset B_{Y'}$ and $(x_n)_n \subset \ell_1(X)$. By the Grothendieck-Pietsch factorization Theorem [7, Corollary 11.3.1], there exist a finite measure space $(\Omega, \mu)$, a closed subspace $G \subset L_p(\mu)$, and operators $U \in \mathcal{L}(G; F)$ and $R \in \Pi_p(X, G)$ such that $T = UR$ and $\|U\| = \|R\|_{\Pi_p} = 1$, where $G = \overline{R(X)}$. Then

$$\sum_{n=1}^{\infty} |y'_n(Tx_n)| = |\sum_{n=1}^{\infty} y'_n(URx_n)| = |\sum_{n=1}^{\infty} (U'y'_n)(Rx_n)|.$$

Take $u = \sum_{n=1}^{\infty} (U'y'_n) \otimes x_n \in G' \widehat{\otimes}_n X$, hence $u \in G' \widehat{\otimes}_{g'_{\rho'}} X$. As G is reflexive, $R \in \Pi_p(X; G) = (G' \widehat{\otimes}_{g'_{\rho'}} X)'$ and $\|R\|_{\Pi_p} \leq 1$ then

$$\sum_{n=1}^{\infty} (U'y'_n)(Rx_n) \leq g'_{\rho'}(u; G', X).$$

Since X has the $g'_{\rho'}$-\lambda-BAP and G is reflexive, with $j: G' \widehat{\otimes}_{g'_{\rho'}} X \to (G' \widehat{\otimes}_{g'_{\rho'}} X)'$ the canonical mapping, we have

$$g'_{\rho'}(u; G', X) \leq \lambda \sup_{g'_{\rho'}(v; G, X') \leq 1} |j(u)(v)| = \lambda \|j(u)\|$$

and, since $G = \overline{R(X)}$, to compute the above supremum, we may assume that any $v \in G \otimes X'$ is of the form $v = \sum_{k=1}^{m} Rz_k \otimes x_k'$ with $z_k \in X$ and $x_k' \in X'$ for $k = 1, \ldots, m$. In other words,

$$\|j(u)\| = \sup_{g'_{\rho'}(v; G, X') \leq 1} \left| \sum_{n=1}^{\infty} \sum_{k=1}^{m} (U'y'_n)(Rz_k)x_k'(x_n) \right|.$$

Note that if $v$ is as above, then $v^t = \sum_{k=1}^{m} x_k' \otimes Rz_k$ and the linear operator associated to $v^t$ is of the form $RS$ where $S = \sum_{k=1}^{m} x_k' \otimes z_k$. Moreover, we have $g'_{\rho'}(v; G, X') = g'_{\rho'}(v^t; X', G) = \|RS\|_{\Pi_p(X; G)}$, thanks to the Representation Theorem for maximal ideals. Then

$$\|j(u)\| = \sup_{\|RS\|_{\Pi_p} \leq 1} \left| \sum_{n=1}^{\infty} \sum_{k=1}^{m} (U'y'_n)(Rz_k)x_k'(x_n) \right|$$

$$= \sup_{\|RS\|_{\Pi_p} \leq 1} \left| \sum_{n=1}^{\infty} \sum_{k=1}^{m} y'_n(Tz_k)x_k'(x_n) \right|$$

$$= \sup_{\|RS\|_{\Pi_p} \leq 1} \left| \sum_{n=1}^{\infty} y'_n(TSx_n) \right|$$

$$= \sup_{\|RS\|_{\Pi_p} \leq 1} \left| \sum_{n=1}^{\infty} y'_n(TSx_n) \right|$$
As $\|RS\|_{\Pi_p} \leq 1$ implies $\|TS\|_{\Pi_p} \leq 1$ for any $S \in \mathcal{F}(X;X)$, from (8), (9) and (10) we obtain that

$$\left| \sum_{n=1}^{\infty} y'_n(Tx_n) \right| \leq \lambda \sup_{\|TS\|_{\Pi_p} \leq 1} \left| \sum_{n=1}^{\infty} y'_n(TSx_n) \right|.$$ 

Applying Theorem 1.2, we conclude that $X$ has the weak $\lambda$-BAP for $\Pi_p$. \hfill \qed

**Proposition 2.2.** There exists a Banach space with the AP and without the weak $\lambda$-BAP for $\Pi_p$ for any $\lambda \geq 1$, for any $p \neq 2$.

**Proof.** By [28, Corollary 3.1], there exists a Banach space with the AP which lacks the $g_q$-$\lambda$-BAP for any $q \neq 2$ and any $\lambda \geq 1$. An application of Theorem 2.1 gives the result for the weak $\lambda$-BAP for $\Pi_p$, $p \neq 2$. \hfill \qed

**Proposition 2.3.** There exists a Banach space with the weak MAP for $\Pi_p$ which lacks the $\lambda$-BAP for $\Pi_p$, for any $\lambda \geq 1$ and any $1 < p \leq 2$.

**Proof.** Let $X$ be a Banach space with cotype 2 and without the AP, which exists by [32]. Note that, given any Banach operator ideal $\mathcal{A}$, any Banach space with the $\lambda$-BAP for $\mathcal{A}$ has the AP. Therefore, $X$ lacks the $\lambda$-BAP for $\Pi_p$, for any $\lambda \geq 1$ and any $p$. Since $X$ has cotype 2, it has $g_q$-MAP for $q \geq 2$ (see the comment below [7, Proposition 21.7]). Now, by Theorem 2.1, if $\frac{1}{p} + \frac{1}{q} = 1$, $X$ has the weak $\lambda$-BAP for $\Pi_p$. \hfill \qed

As a consequence of the above and at the light of Remark 1.1, the class of $p$-summing operators $1 < p \leq 2$, provides an example of other type of ideals (not minimal) which answers [18, Problem 5.5] by the negative. We also have the following.

**Proposition 2.4.** For $1 < p < \infty$, the weak $\lambda$-BAP for $\Pi_p$ coincide with the condition $c^*_\lambda$ for $\Pi_p$.

**Proof.** By [30, Theorem 2], a Banach space has the $g_p$-$\lambda$-BAP if and only if has condition $c^*_\lambda$ for $\Pi_p$. The result follows by Theorem 2.1. \hfill \qed

The definition of the weak BAP for $\Pi_p$ involves the strong operator topology. The above results (Propositions 2.2 and 2.3) show that the topology sot cannot be replaced with the topology $\tau_c$. Nevertheless, we may consider the topology of uniform convergence on $p$-compact sets $\tau_p$. The class of $p$-compact sets was first introduced and studied in [31] as a natural extension of the notion of compact sets. By [17, Remark 1.3], we know that $p$-compact sets coincide with $N^p$-compact sets, where $N^p$ denotes the ideal of right $p$-nuclear operators. Associated to the class of $p$-compact sets we have the class of $p$-compact operators denoted by $\mathcal{K}_p$. For more information on $p$-compact sets and $p$-compact operators we refer the reader to [6, 9, 10, 14, 23, 24, 27, 31] and the references therein.
Lemma 2.5. Let $X$ and $Y$ be Banach spaces and let $1 \leq p < \infty$. The following are equivalent.

(i) $\phi \in (\mathcal{L}(X; Y); \tau_p)'$.
(ii) $\phi$ is $\Pi^d_p$-representable.
(iii) $\phi$ is right $\Pi^d_p$-representable.

Proof. From [2, Corollary 4.9], one can deduce that $\mathcal{K}_p = \mathcal{K}_{\Pi^d_p}$ and by [17, Remark 1.3] we have the identity $\mathcal{K}_p = \mathcal{K}_{\mathcal{N}^p}$. Hence, by [17, Propositions 1.4], $\mathcal{N}^p$-null and $\Pi^d_p$-null sequences coincide. Also, the class of $\tau_p$-continuous functionals on $\mathcal{L}(X; Y)$ coincides, according to [8], with the class of $\mathcal{N}^p$-representable functionals. Then, (i) and (ii) are equivalent. To show that (iii) implies (ii) we borrow some ideas of the proof of [6, Theorem 2.7]. Take $\phi$ right $\Pi^d_p$-representable (hence right $\mathcal{N}^p$-representable). There exist $(y_n')_n \in c_0, \mathcal{N}^p(Y')$ and $(x_n)_n \in \ell_1(X)$ such that $\phi(T) = \sum_{n=1}^{\infty} y_n'(Tx_n)$ for all $T \in \mathcal{L}(X; Y)$. We may assume that $x_n \neq 0$ for all $n$. Since $(y_n')_n$ is $\mathcal{N}^p$-null, there exist a Banach space $Z$, $(z_n)_n \in c_0(Z)$ and $R \in \mathcal{N}^p(Z; Y')$ such that $y'_n = R(z_n)$. Write $R = \sum_{j=1}^{\infty} z'_j \otimes w'_j$ with $(z'_j)_j \in \ell^p_p(Z')$ and $(w'_j)_j \in \ell_p(Y')$. Choose $(\alpha_j)_j \in B_{c_0}$ such that $(\frac{1}{\alpha_j} w'_j)_j \in \ell_p(Y')$. Then we have

$$\phi(T) = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} z'_j(z_n)w'_j(Tx_n) = \sum_{j=1}^{\infty} \frac{1}{\alpha_j} w'_j(T) \left( \sum_{n=1}^{\infty} \alpha_j z'_j(z_n)x_n \right).$$

Define $f_j = \sum_{n=1}^{\infty} z'_j(z_n)\|x_n\| \stackrel{p}{\rightarrow} e'_n$, where $(e'_n)_n$ stands for the canonical sequence of $\ell_p'$. Is not difficult to see that each $f_j$ belongs to $\ell_p'$. Take the right $p$-nuclear operator $S \in \mathcal{N}^p(\ell_p'; X)$, $S = \sum_{n=1}^{\infty} \tilde{e}_n \otimes \|x_n\| \tilde{e}_n x_n$, with $(e'_n)_n$ the canonical sequence of $\ell_p$. Note that $Sf_j = \sum_{n=1}^{\infty} z'_j(z_n)x_n$, and set $\tilde{f}_j = 0$ if $f_j = 0$ and $\tilde{f}_j = \frac{\alpha_j}{\|f_j\|_{\ell_p'}}f_j$ otherwise. Then

$$\phi(T) = \sum_{j=1}^{\infty} \frac{1}{\alpha_j} \|f_j\|_{\ell_p'} \|w'_j(TS\tilde{f}_j).$$

Since $(\frac{1}{\alpha_j} \|f_j\|_{\ell_p'} w'_j)_j$ belongs to $\ell_1(Y')$ and $(S\tilde{f}_j)_j$ is $\mathcal{N}^p$-null in $X$, then $\phi$ is $\mathcal{N}^p$-representable and (ii) holds.

With a similar proof we see that (ii) implies (iii) and the proof is complete. \hfill \Box

Corollary 2.6. A Banach space $X$ has the $g_p$-$\lambda$-BAP if and only if for every Banach space $Y$ and for every operator $T \in \Pi_p(X; Y)$, there exist a net $S_\alpha \subset \mathcal{F}(X; X)$ such that

$$\sup_\alpha \|TS_\alpha\|_{\Pi_p} \leq \lambda \|T\|_{\Pi_p} \quad \text{and} \quad S_\alpha \to I_X \quad \text{in the topology} \quad \tau_p.$$ 

Proof. Since $\Pi_p = \Pi^d_p$, by Proposition 1.3 and Lemma 2.5, $X$ has the weak $\lambda$-BAP for $\Pi_p$ if and only if for any operator $T \in \Pi_p(X; Y)$, there exists a net $S_\alpha \subset \mathcal{F}(X; X)$ with $\sup_\alpha \|TS_\alpha\|_{\Pi_p} \leq \lambda \|T\|_{\Pi_p}$ such that $S_\alpha \to I_X$ in the topology $\tau_p$. Now, the result follows by Theorem 2.1. \hfill \Box
We finish this section with a brief discussion on approximation properties enjoyed by a Banach space which are inherited from its dual space. It is known that if $X''$ has $g_p$-AP then $X$ has it, see for instance [7, Proposition 21.7]. Let us discuss how $X'$ is positioned in this framework. A first result of this type can be found in [9].

**Proposition 2.7** (Delgado-Piñeiro-Serrano). Let $X$ be a Banach space and let $1 < p < \infty$. If $X''$ $g_p$-AP, then $X'$ has the $K_p$-approximation property, $\frac{1}{p} + \frac{1}{p'} = 1$.

We do not know if the $K_p$-approximation property on a dual space $X'$ implies the $g_{p'}$-AP on $X$. However, under certain conditions on $X$ we can give a positive answer. In view of Proposition 1.7 the dual ideal of $K_p$, which is the ideal of quasi $p$-nuclear operators $QN_p$, comes into scene. In fact, $K_p^d = QN_p$ isometrically [10, Corollary 3.4]. Also, by [27, Proposition 10] or [14, Corollary 3.6] $Q^\max N_p = \Pi_p$.

**Remark 2.8.** As we have the isometric inclusion $QN_p \subseteq \Pi_p$, if a Banach space has the weak MAP for $\Pi_p$, then it has the weak MAP for $QN_p$. Indeed, the result is valid if we replace weak MAP by weak $\lambda$-BAP for any $\lambda \geq 1$.

The question of when the identity $QN_p(X; Y) = \Pi_p(X; Y)$ is true can be linked to the question of when nuclear and (Pietsch) integral operators coincide. More specifically, when $N_p$ and the ideal of $p$-integral operators $I_p$ coincide. This was treated for instance, in the works by Persson [25], Cardassi [3] and Aharoni and Saphar [1].

**Proposition 2.9.** Let $X$ be a Banach space and let $1 < p < \infty$. Suppose that $X'$ has the $K_p$-AP and that for every Banach space $Y$, $QN_p(X; Y) = \Pi_p(X; Y)$ isometrically. Then $X$ has the $g_{p'}$-MAP, $\frac{1}{p} + \frac{1}{p'} = 1$.

**Proof.** Suppose that $X'$ has $K_p$-AP. Since $K_p^d = QN_p$ isometrically, by Proposition 1.7, $X$ has the weak MAP for $QN_p$ and hence $X$ has the weak MAP for $\Pi_p$. Now, a direct application of Theorem 2.1 gives the result. \hfill $\Box$

**Corollary 2.10.** Let $X$ be an Asplund Banach space and let $1 < p < \infty$. If $X'$ has the $K_p$-AP, then $X$ has the $g_{p'}$-MAP, $\frac{1}{p} + \frac{1}{p'} = 1$.

**Proof.** If $X$ is Asplund, $N_p(X; Y) = I_p(X; Y)$ for any $Y$, see for instance [25, Corollary 1 and Theorem 5] for reflexive spaces or spaces with separable dual, and [1, Proposition 1] or [3] for the general case. Now, taking the injective hull on both sides of the equality, we have that $QN_p(X; Y) = \Pi_p(X; Y)$ for any Banach space $Y$. The result follows from the above proposition. \hfill $\Box$

**Acknowledgements.** We are grateful to Joe Diestel for sending us the manuscript of Carmen Cardassi’s Ph.D. Thesis and for pointing us that the results we use were published in [3].
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