Domineering games with minimal number of moves

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Abstract

Domineering is a two-player game played on a checkerboard in which one player places dominoes vertically, while the other places them horizontally. In this paper, we find out the minimum number of moves for a game of Domineering to end on several rectangular $m \times n$ boards. We also formulate two conjectures pertaining to $2 \times n$ (similarly $m \times 2$) and on $3 \times n$ (similarly, $m \times 3$) boards.

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1 Introduction

Combinatorial games are two-player games with perfect information and no chance moves such as rolling dice or shuffling cards. The rules are defined in such a way that play will always come to an end; in theory we could determine by computer which player would win if both players play perfectly. In practice, however, the complexity of the required search complicates our analysis.

In this paper, we consider the game Domineering, invented by Göran Andersson around 1973 and popularized by Martin Gardner [2]. The version introduced by Andersson and Gardner was the $8 \times 8$ board. Play consists of the two players alternately placing a $1 \times 2$ tile (domino) on adjacent empty squares; Left places vertically and Right places horizontally. The game ends when the player whose turn it is cannot place a piece; the player who cannot place loses - this is the normal play condition. Since the board is gradually filled, Domineering is a converging game, the game always ends, and ties are impossible. For more information we would like to direct the attention of the interested reader to the books of Berlekamp et al. [3] and Albert et al. [4].

After stating Huntemann et al.’s [1] results of counting Domineering positions satisfying certain properties, we obtain values for the minimum number of moves for a $m \times n$ Domineering game to end in Section 3, before finishing with further research directions.

2 Counting Domineering positions

This section is adapted from [1]. We are interested in enumerating positions at the end of the game. Therefore, we define the following terms:

Definition 2.1. A Right end position is a position in which Left potentially has moves available but Right has no moves; a Left end position is defined similarly.
**Definition 2.2.** A maximal end position is a position which is both a left end and a right end - a position in which no player can place a domino.

### 2.1 Maximal End Positions

For counting the maximal Domineering positions on a $m \times n$ board, we find the generating function

$$F_{m,n}(x, y) = \sum_{u \in \{0,1\}^n} \left( \frac{M_{0,n} + M_{0,n}'}{M_{0,n} + M_{0,n}'} \right)^m (1 + \sum_{i=1}^n u_i 3^i, 1)$$

where $f(a, b)$ is the number of maximal Domineering positions with a vertical (placed by Left) and $b$ horizontal (placed by Right) dominoes.

**Theorem 2.3.** The generating function for the maximal position of an $m \times n$ Domineering board is

$$F_{m,n}(x, y) = \sum_{u \in \{0,1\}^n} \left( M_{0,n} + M_{0,n}' \right)^m (1 + \sum_{i=1}^n u_i 3^i, 1)$$

where $M_{0,n} = [1], M_{1,0} = [0], M_{2,0} = [0], M_{0,0}' = [0], M_{1,0}' = [1], M_{2,0}' = [0], \ldots$

### 2.2 Left and Right End Positions

For counting the Right end Domineering positions on a $m \times n$ board, we find the generating function

$$R_{m,n}(x, y) = \sum_{u \in \{0,1\}^n} r(a, b)x^a y^b$$

where $r(a, b)$ is the number of Right end positions with a vertical and $b$ horizontal dominoes. The generating polynomial for Left end positions on an $m \times n$ board, which we denote by $L_{m,n}(x, y)$, can be found by obtaining $R_{m,n}(x, y)$ and then switching $x$ and $y$.

**Theorem 2.4.** The generating polynomial of Domineering Right ends on an $m \times n$ board is the $(1, 1)$ entry of $(R_{0,n} + R_{0,n}')^m$, denoted by $R_{m,n}(x, y)$, where $R_{0,0} = [1], R_{1,0} = [0], R_{2,0} = [0], R_{0,0}' = [0], R_{1,0}' = [1], R_{2,0}' = [0]$.

$$R_{0, (q+1)} = \left[ R_{1,q} + R_{2,q} x R_{0,q} \right], \quad R_{0, (q+1)}' = \left[ R_{1,q}' + R_{2,q}' x R_{0,q}' \right]$$

$$R_{1, (q+1)} = \left[ R_{2,q} x R_{0,q} \right], \quad R_{1, (q+1)}' = \left[ R_{2,q}' x R_{0,q}' \right]$$

$$R_{2, (q+1)} = \left[ y R_{0,q} \right], \quad \text{and } R_{2, (q+1)}' = \left[ y R_{0,q}' \right]$$
3 Results

In this section, we state and discuss the results we’ve obtained and pose some problems not considered in the literature thus far.

One observes that the presence of a monomial of the form $x^ay^b$ in either the Left end ($L_{m,n}(x,y)$) or Right end ($R_{m,n}(x,y)$) or the Maximal end ($F_{m,n}(x,y)$) polynomials indicates the tiling of the $m \times n$ Domineering board with $a$ vertical (placed by Left) tiles and $b$ horizontal (placed by Right) tiles satisfying the required properties of the position.

It naturally follows that the minimum number of moves for a $m \times n$ Domineering game to end, which we denote by $\alpha_{m,n}$, equals the lowest sum of the degrees of $x$ and $y$ in the Left end, Right end and Maximal end polynomials. That is:

$$\alpha_{m,n} = \min\{a + b : x^ay^b \text{ occurs in } L_{m,n}(x,y) \text{ or } R_{m,n}(x,y) \text{ or } F_{m,n}(x,y)\}$$

We have incorporated the matrix recurrence relations for the Left, Right and Maximal end positions defined previously into our program written in Maple. The program works by first determining the terms in $L_{m,n}(x,y)$, $R_{m,n}(x,y)$ and $F_{m,n}(x,y)$ where the difference between the powers of $x$ and $y$ is at most 1. That is because these polynomials actually count all legal Domineering positions satisfying the required properties. Then, we write another procedure to determine the polynomial(s) corresponding to $\alpha_{m,n}$. Additionally, the polynomials were analysed to check whether the term(s) occur in the Left and/or Right and/or Maximal end positions.

We have generated results for all rectangular boards up to size $8 \times 8$ and some other rectangular boards. Beyond that point, the computations quickly grows too large to fit in the average home computer’s main memory. An overview of the results is given in Table 1.

| m | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|----|
| 1 | 1LR | 1LR | 1LR | 1LR | 1LR | 1LR | 1LR | 1LR | 1LR | 1LR |
| 2 | 1LR | 1LR | 1R | 3LR | 3LR | 3LR | 4R | 4R | 5R | 5R |
| 3 | 1LR | 1L | 2LR | 3LR | 3L | 4L | 5LR | 5R | 6L |
| 4 | 1LR | 3LR | 3LR | 4LR | 5LR | 7LR | 7LR | 8LR | 9L |
| 5 | 1LR | 3LR | 3LR | 5LR | 7LR | 8LR | 9LR | 11LR |
| 6 | 1LR | 3LR | 3R | 5LR | 7LR | 8LR | 9LR | 11LR | 12LR |
| 7 | 1LR | 4L | 5LR | 7LR | 9LR | 11LR | 12LR | 15LR |
| 8 | 1LR | 4L | 5R | 8LR | 11LR | 12LR | 15LR | 16LR |
| 9 | 1LR | 5L | 6R | 9LR |
| 10 | 1LR | 5L |

Table 1: The minimum number of moves for $m \times n$ Domineering games to end. The superscript over the numbers denote the polynomials (Left end and/or Right end and/or Maximal end) in which these numbers occur. For instance, the $2 \times 4$ game ends in 3 moves and this position occurs in both the Left end and Right end polynomials. Hence, the entry is denoted 3LR.

Some observations are in order:

1. The term with the least degree for all $m \times 1$ and $1 \times n$ boards with $m, n \geq 1$ is actually $x^0y^0 = 1$. However, by the definition of a combinatorial game, someone
must make a move. Hence, we take the next smallest term which is $x$ for all $m \times 1$ boards with $m > 1$ and $y$ for all $1 \times n$ boards with $n > 1$.

2. The minimum number of moves for a $m \times n$ Domineering game to end is the same as that for a $n \times m$ Domineering game to end, that is, $\alpha_{m,n} = \alpha_{n,m}$.

3. For large $m, n$, it seems that $\alpha_{m,n}$ occurs in both the Left end and the Right end polynomials.

4. We observe that $\alpha_{m,n}$ coincides with the minimum number of moves obtained in perfect play only for small $m, n$.

5. We were unable to analyse the maximal end polynomials mainly because the computations quickly grow large even for small $m, n$.

On close observation, one can notice the following patterns which, to the best of our knowledge, haven’t been widely studied:

**Problem 3.1.** The minimum number of moves for a $2 \times n$ Domineering game to end is given by

$$\alpha_{2,n} = \left\lceil \frac{n}{2} \right\rceil \quad \forall n \geq 5.$$  

A similar result also holds for $m \times 2$ boards. One can easily construct Domineering positions that use the minimum number of moves shown in Table 1 on $2 \times n$ boards up to $n = 5$. However, we’re unable to explicitly construct Domineering positions on $2 \times n$ boards with $n > 5$ using the minimum number of moves for the game to end.

**Problem 3.2.** The minimum number of moves for a $3 \times n$ Domineering game to end is given by

$$\alpha_{3,n} = \left\lfloor \frac{2n + 1}{3} \right\rfloor \quad \forall n \geq 1.$$  

A similar result also holds for $m \times 3$ boards. We have been able to construct Domineering positions that use the least number of moves shown in Table 1 on $3 \times n$ boards up to $n = 6$.

### 4 Conclusions and Future research

We have analysed the minimum number of moves for a $m \times n$ Domineering game to end and formulated two new problems. We find two avenues to expand upon the research presented in this paper.

First, one could continue to find values for unsolved Domineering boards using better programs and/or more computer memory. We have already started working in SageMath to analyze this problem. Secondly, we intend to find the minimum number of moves for a $m \times n$ Domineering game to end if both players play perfectly.

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