ON THE GEOMETRY OF CONFORMALLY STATIONARY LORENTZ SPACES

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Abstract. In this paper we study several aspects of the geometry of conformally stationary Lorentz manifolds, and particularly of GRW spaces, due to the presence of a closed conformal vector field. More precisely, we begin by extending to these spaces a result of J. Simons on the minimality of cones in Euclidean space, and apply it to the construction of complete, noncompact maximal submanifolds of both de Sitter and anti-de Sitter spaces. Then we state and prove very general Bernstein-type theorems for spacelike hypersurfaces in conformally stationary Lorentz manifolds, one of which not assuming the hypersurface to be of constant mean curvature. Finally, we study the strong r-stability of spacelike hypersurfaces of constant r-th mean curvature in a conformally stationary Lorentz manifold of constant sectional curvature, extending previous results in the current literature.

1. Introduction

An important class of Lorentz manifolds is formed by the so-called stationary Lorentz manifolds. Following [17], Chapter 6, we say that a Lorentz manifold is stationary if there exists a one-parameter group of isometries whose orbits are timelike curves; for spacetimes, this group of isometries expresses time translation symmetry. From the mathematical viewpoint, a stationary Lorentz manifold is simply a Lorentz manifold furnished with a timelike Killing vector field, and a natural generalization is a conformally stationary Lorentz manifold, i.e., one furnished with a timelike conformal vector field. Our interest in conformally stationary Lorentz manifolds is due to the fact that, under an appropriate conformal change of metric, the conformal vector field turns into a Killing one, so that the new Lorentz manifold is now stationary.

Our aim in this work is to understand the geometry of immersed submanifolds of conformally stationary Lorentz manifolds furnished with a closed conformal vector field, and we do this by approaching three different kinds of problems: the construction of examples of maximal submanifolds, the obtainance of general Bernstein-type results and the derivation of suitable criteria for r-stability.

First of all (cf. Theorem 3.1), we extend a classical theorem of Simons [16] to conformally stationary Lorentz manifolds and apply it to build maximal Lorentz immersions whenever the ambient space either is of constant sectional curvature or has vanishing Ricci curvature in the direction of the conformal vector field; in

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particular, we provide a geometrical construction for maximal immersions into both the anti-de Sitter and de Sitter spaces (cf. Corollaries 3.3 and 3.4). Related to Bernstein-type results, we study complete spacelike hypersurfaces, not necessarily of constant mean curvature, immersed into a conformally stationary Lorentz manifold of nonnegative Ricci curvature, furnished with homothetic non-parallel vector fields. By asking the second fundamental form of the hypersurface to be bounded and imposing a natural restriction on the projection of the conformal vector field of the ambient space, we classify such hypersurfaces in Theorems 4.1 and 4.3. In particular, we classify complete, finitely punctured spacelike radial graphs over $\mathbb{R}^n$ or $\mathbb{H}^n$ in $\mathbb{L}^{n+1}$, thus extending a classical result of J. Jellett [11] to the Lorentz context.

Finally, in the last section we derive in Theorem 5.7 a sufficient criterion for strong $r$-stability of spacelike hypersurfaces of constant $r$-th mean curvature. This result extends, to the class of conformally stationary Lorentz manifolds, previous ones obtained in [6] and [8] in the context of Generalized Robertson-Walker spacetimes.

2. Conformally stationary Lorentz manifolds

As in the previous section, let $\mathcal{M}^{n+1}$ be a Lorentz manifold. We recall that a vector field $V$ on $\mathcal{M}^{n+1}$ is conformal if

$$(2.1) \quad \mathcal{L}_V \langle \cdot, \cdot \rangle = 2\psi \langle \cdot, \cdot \rangle$$

for some function $\psi \in C^\infty(\mathcal{M})$, where $\mathcal{L}$ stands for the Lie derivative of the Lorentz metric of $\mathcal{M}$; the function $\psi$ is the conformal factor of $V$. Any Lorentz manifold $\mathcal{M}^{n+1}$ possessing a globally defined, timelike conformal vector field is said to be conformally stationary.

Since $\mathcal{L}_V(X) = [V, X]$ for all $X \in \mathfrak{x}(\mathcal{M})$, the tensorial character of $\mathcal{L}_V$ shows that $V \in \mathfrak{x}(\mathcal{M})$ is conformal if and only if

$$(2.2) \quad \langle \nabla_X V, Y \rangle + \langle X, \nabla_Y V \rangle = 2\psi \langle X, Y \rangle,$$

for all $X, Y \in \mathfrak{x}(\mathcal{M})$. In particular, $V$ is Killing if and only if $\psi \equiv 0$, and (2.2) gives

$$\psi = \frac{1}{n+1} \text{div}_\mathcal{M} V.$$ 

Suppose that our conformally stationary spacetime $\mathcal{M}^{n+1}$ is endowed with a closed conformal timelike vector field $V$ with conformal factor $\psi$, i.e., one for which

$$(2.3) \quad \nabla_X V = \psi X$$

for all $X \in \mathfrak{x}(\mathcal{M})$. We recall that such a $V$ is parallel if $\psi$ vanishes identically and homothetic if $\psi$ is constant.\(^{1}\)

If $V$ has no singularities on an open set $\mathcal{U} \subset \mathcal{M}$, then the distribution $V^\perp$ on $\mathcal{U}$ of vector fields orthogonal to $V$ is integrable, for if $X, Y \in V^\perp$, then

$$\langle [X, Y], V \rangle = \langle \nabla_X Y - \nabla_Y X, V \rangle = -\langle Y, \nabla_X V \rangle + \langle X, \nabla_Y V \rangle = 0.$$ 

We let $\Xi$ be a leaf of $V^\perp$ furnished with the induced metric, and $D$ its Levi-Civita connection.

\(^{1}\)Here we diverge a little bit from other papers (e.g. [12]), where homothetic means just conformal with constant conformal factor. The reason is economy: to avoid constantly writing closed and homothetic.
From (2.3) we get
\[ \nabla(V, V) = 2\psi V, \]
so that \( (V, V) \) is constant on connected leaves of \( V^\perp \). Computing covariant derivatives in (2.4), we have
\[ (\text{Hess}_M(V, V))(X, Y) = 2X(\psi)(V, Y) + 2\psi^2(X, Y). \]
However, since both \( \text{Hess}_M \) and the metric are symmetric tensors, we get
\[ X(\psi)(V, Y) = Y(\psi)(V, X) \]
for all \( X, Y \in \mathfrak{X}(M) \). Taking \( Y = V \) we then arrive at
\[ \nabla\psi = \frac{V(\psi)}{\langle V, V \rangle} V = -\nu(\psi)\nu, \]
where \( \nu = \frac{V}{\sqrt{-\langle V, V \rangle}} \). Hence, \( \psi \) is also constant on connected leaves of \( V^\perp \). If \( \Xi \) is such a leaf and \( S_\Xi \) denotes its shape operator with respect to \( \nu \), we get
\[ S_\Xi(X) = -\nabla_X \nu = \psi X, \]
and hence \( \Xi \) is an umbilical hypersurface of \( M \).

Now we need the following

**Lemma 2.1.** If \( \eta \) is another closed conformal vector field on \( M \) and \( U = \eta + \langle \eta, \nu \rangle \nu \), then \( U \) is closed conformal on \( \Xi \), with conformal factor \( \psi_U = \psi_\eta + \psi_\nu \langle \eta, \nu \rangle \).

**Proof.** For \( Z \in \mathfrak{X}(\Xi) \), it follows from \( \langle Z, \nu \rangle = 0 \) that
\begin{align*}
D_Z U &= (\nabla_Z U)^T = \nabla_Z U + \langle \nabla_Z U, \nu \rangle
\quad = \nabla_Z (\eta + \langle \eta, \nu \rangle \nu) + \langle \nabla_Z (\eta + \langle \eta, \nu \rangle \nu), \nu \rangle
\quad = \nabla_Z \eta + Z(\eta, \nu)\nu + \langle \eta, \nu \rangle \nabla_Z \nu + \langle \nabla_Z \eta, \nu \rangle \nu
\quad \quad + Z(\eta, \nu)\langle \nu, \nu \rangle + \langle \eta, \nu \rangle \langle \nabla_Z \nu, \nu \rangle
\quad = \psi_\eta Z + \langle \eta, \nu \rangle \psi_\nu Z + \langle \psi_\eta Z, \nu \rangle \nu
\quad = (\psi_\eta + \langle \eta, \nu \rangle \psi_\nu) Z.
\end{align*}
\[ \square \]

**Example 2.2.** Let \( \mathbb{L}^{n+1} \) be the \((n + 1)\)-dimensional Lorentz space with its usual scalar product \( \langle \cdot, \cdot \rangle \), with respect to the quadratic form \( q(x) = \sum_{i=1}^n x_i^2 - x_{n+1}^2 \). For \( n > 2 \), the \( n \)-dimensional de Sitter space is the hyperquadric
\[ S^n_1 = \{ x \in \mathbb{L}^{n+1}; \langle x, x \rangle = 1 \}, \]
and also the Lorentz simply-connected space form of constant sectional curvature identically equal to \( 1 \). The previous proposition teaches how to geometrically build closed conformal vector field on \( S^n_1 \): since \( \nu \in \mathfrak{X}(\mathbb{L}^{n+1}) \) given by \( \nu(x) = x \) is homothetic, choose a parallel \( \eta \in \mathfrak{X}(\mathbb{L}^{n+1}) \) and project it orthogonally onto the \( S^n_1 \).

**Example 2.3.** Let \( \mathbb{R}_2^{n+1} \) denote \( \mathbb{R}^{n+1} \) furnished with the scalar product corresponding to the quadratic form \( q(x) = \sum_{i=1}^{n-1} x_i^2 - x_n^2 - x_{n+1}^2 \). For \( n > 1 \), the \( n \)-dimensional anti-de Sitter space is the hyperquadric
\[ \mathbb{H}_1^n = \{ x \in \mathbb{R}_2^{n+1}; \langle x, x \rangle = -1 \}, \]
and also the Lorentz simply-connected space form of constant sectional curvature identically equal to \(-1\). A construction similar to that of the previous example can obviously be made for \(\mathbb{H}_1^n\).

Following [1], a particular class of conformally stationary spacetimes is that of Generalized Robertson-Walker spaces (GRW for short), namely, warped products
\[ M^{n+1} = -I \times_\phi F^n, \]
where \(I \subset \mathbb{R}\) is an open interval with the metric \(-dt^2\), \(F^n\) is an \(n\)-dimensional Riemannian manifold and \(\phi : I \rightarrow \mathbb{R}\) is positive and smooth. For such a space, if \(\pi_I : M^{n+1} \rightarrow I\) is the canonical projection onto \(I\), then the vector field
\[ V = (\phi \circ \pi_I) \partial_t \]
is a conformal, timelike and closed, with conformal factor \(\psi = \phi' \circ \pi_I\), where the prime denotes differentiation with respect to \(t\). Moreover (cf. [13]), for \(t_0 \in I\), the (spacelike) leaf \(\Sigma_{t_0} = \{t_0\} \times F^n\) is totally umbilical, with umbilicity factor \(-\frac{\phi'(t_0)}{\phi(t_0)}\) with respect to the future-pointing unit normal vector field.

**Remark 2.4.** Conversely, let \(\overline{\Sigma}\) be a general conformally stationary Lorentz manifold with closed conformal vector field \(V\). If \(p \in \overline{\Sigma}\) and \(\Sigma_p\) is the leaf of \(V^+\) passing through \(p\), then we can find a neighborhood \(U_p\) of \(p\) in \(\Sigma_p\) and an open interval \(I \subset \mathbb{R}\) containing \(0\) such that the flow \(\Psi\) of \(V\) is defined on \(U_p\) for every \(t \in I\). Besides, if \(\overline{\Sigma}\) is timelike geodesically complete, S. Montiel [13] proved that
\[ \varphi : \mathbb{R} \times \Sigma_p \rightarrow \overline{\Sigma} \]
is a global parametrization on \(\overline{\Sigma}\), such that \(\overline{\Sigma}\) is isometric to the GRW \(-\mathbb{R} \times_\phi \Sigma_p\), where \(\varphi(q) = t \Leftrightarrow \Psi(-t, q) \in \Sigma_p\).

Let \(x : M^n \rightarrow \overline{M}^{n+1}\) a spacelike immersion, that is, the induced metric via \(x\) on \(M^n\) is Riemannian. According to [1], if \(\overline{M}^{n+1} = -I \times_\phi F^n\) is a GRW and \(x : M^n \rightarrow \overline{M}^{n+1}\) is a complete spacelike hypersurface such that \(\phi \circ \pi_I\) is bounded on \(M^n\), then \(\pi_F\mid_M : M^n \rightarrow F^n\) is necessarily a covering map. In particular, if \(M^n\) is closed, then \(F^n\) is automatically closed.

Also, recall (cf. Chapter 7 of [14]) that a GRW as above has constant sectional curvature \(c\) if and only if \(F\) has constant sectional curvature \(k\) and the warping function \(\phi\) satisfies the ODE
\[ \frac{\phi''}{\phi} = c = \frac{(\phi')^2 + k}{\phi^2}. \]

**Example 2.5.** The de Sitter space \(S_1^{n+1}\) is an important particular example of GRW of constant sectional curvature \(1\). In fact, it follows from (2.8) and the classification of simply connected Lorentz space forms that
\[ S_1^{n+1} \simeq -\mathbb{R} \times \cosh t \ S^n, \]
where \(S^n\) is the standard \(n\)-dimensional unit sphere in Euclidean space. Hence, the vector field \(V = (\sinh t) \partial_t\) is a timelike closed conformal one. The equator of \(S_1^{n+1}\) is the slice \(\{0\} \times S^n\), and the points \((t, p) \in S_1^{n+1}\) with \(t < 0\) (resp. \(t > 0\)) are said to form the chronological past (resp. future) of \(S_1^{n+1}\).
Example 2.6. Another important example is given by the anti-de Sitter space $\mathbb{H}^{n+1}$. Invoking once more \( \text{(2.8)} \) and the classification of simply connected Lorentz space forms, we get the isometry

$$
\mathbb{H}^{n+1} \simeq -(0, \pi) \times \sin t \mathbb{H}^n,
$$

where $\mathbb{H}^n$ is the $n$–dimensional hyperbolic space. Therefore, the vector field $V = (\sin t)\partial_t$ is timelike closed and conformal in $\mathbb{H}^{n+1}$.

3. Maximal submanifolds of conformally stationary spaces

In this section we generalize to conformally stationary Lorentz manifolds a theorem of J. Simons \[16\], which shows how one can build isometric immersions with parallel mean curvature in $\mathbb{R}^{n+k+1}$ from minimal immersions $\varphi : M^n \to \mathbb{S}^{n+k}$.

As in the previous section, let $\mathcal{M}^{n+k+1}$ be an $(n+k+1)$–dimensional conformally stationary Lorentz manifold, with closed conformal vector field $V$ of conformal factor $\psi$. If $V \neq 0$ on $\mathcal{M}$, we saw that the orthogonal distribution $V^{\perp}$ is integrable, with totally umbilical leaves. Therefore, if $\mathbb{S}^{n+k}$ be such a leaf, then it is an umbilical spacelike hypersurface of $\mathcal{M}^{n+k+1}$ and $\nu = \frac{V}{\sqrt{-\langle V, V \rangle}}$ is a global unit normal timelike vector field on it.

Let $\varphi : M^n \to \mathbb{S}^{n+k}$ be an isometric immersion, where $M^n$ is a compact Riemannian manifold. If $\Psi$ denotes the flow of $V$, the compactness of $M^n$ guarantees the existence of $\epsilon > 0$ such that $\Psi$ is defined on $(-\epsilon, \epsilon) \times \varphi(M)$, and the map

$$
\Phi : \ (\epsilon, \epsilon) \times M^n \longrightarrow \mathcal{M}^{n+k+1} \quad (t, q) \quad \mapsto \quad \Psi(t, \varphi(q))
$$

is also an immersion. Furnishing $(-\epsilon, \epsilon) \times M^n$ with the metric induced by $\Phi$, we turn it into a Lorentz manifold an $\Phi$ into an isometric immersion such that $\Phi_{\{0\} \times M^n} = \varphi$.

Finally, letting $\text{Ric}_\mathcal{M}$ denote the field of self-adjoint operators associated to the Ricci tensor of $\mathcal{M}$, we get the above-mentioned generalization of Simons’ result (see also \[10\] for the Riemannian case).

Theorem 3.1. In the above notations, let $\psi \neq 0$ on $\varphi(M)$. If $\mathcal{M}$ has constant sectional curvature or $\text{Ric}_\mathcal{M}(V) = 0$, then the following are equivalent:

\begin{itemize}
  \item[(a)] $\varphi$ is maximal.
  \item[(b)] $\Phi$ is maximal.
  \item[(c)] $\Phi$ has parallel mean curvature.
\end{itemize}

Proof. Fix $p \in M$ and, on a neighborhood $\Omega$ of $p$ in $M$, an orthonormal frame $\{e_1, \ldots, e_n, \eta_1, \ldots, \eta_k\}$ adapted to $\varphi$, such that $\{e_1, \ldots, e_n\}$ is geodesic at $p$.

If $E_1, \ldots, E_n, N_1, \ldots, N_k$ are the vector fields of $\mathcal{M}$ obtained from the $e_i$’s and $\eta_\beta$’s by parallel transport along the integral curves of $V$ that intersect $\Omega$, it follows that $\{E_1, \ldots, E_n, \nu, N_1, \ldots, N_k\}$ is an orthonormal frame on $\Psi((\epsilon, \epsilon) \times \Omega)$, adapted to the immersion (3.1).

Let $\nabla$ be the Levi-Civita connection of $\mathcal{M}$ and $\overline{H}$ the mean curvature vector of $\Phi$. It follows from the closed conformal character of $V$ that, on $\Phi((\epsilon, \epsilon) \times \Omega)$,

$$
\overline{H} = \frac{1}{n+1} (\nabla_{E_i} E_i + \nabla_\nu \nu)^{\perp} = \frac{1}{n+1} (\nabla_{E_i} E_i)^{\perp},
$$

where $\perp$ denotes orthogonal projection on $T\Phi((\epsilon, \epsilon) \times \Omega)^{\perp}$.
In order to compute $\nabla_{E_i} E_i$ along the integral curve that passes through $p$, note that

$$\langle \nabla_{E_i} E_i, V \rangle = -\langle E_i, \nabla_{E_i} V \rangle = -n\psi.$$  \hspace{1cm} (3.3)

Now, if $\mathcal{R}$ stands for the curvature operator of $\mathcal{M}$, observe that

$$\frac{d}{dt} \langle \nabla_{E_i} E_i, E_k \rangle = \langle \nabla_{E_i} \nabla_{E_i} E_i, E_k \rangle = \langle \mathcal{R}(V, E_i) E_i, E_k \rangle + \langle \nabla_{E_i} \nabla_V E_i, E_k \rangle + \langle \nabla_{[V,E_i]} E_i, E_k \rangle$$

$$= \langle \mathcal{R}(V, E_i) E_i, E_k \rangle + \langle \nabla_{E_i} \nabla_V E_i, E_k \rangle + \langle \nabla_{E_i} V, E_i \rangle E_k \rangle + \langle \nabla_{E_i} E_i, E_k \rangle.$$  \hspace{1cm} (3.4)

Note that, in the last equality, we used the fact that either $\mathcal{M}$ has constant sectional curvature or $\text{Ric}_{\mathcal{M}}(V) = 0$ to conclude that $\langle \mathcal{R}(V, E_i) E_i, E_k \rangle = 0$.

Let $D$ and $\nabla$ respectively denote the Levi-Civita connections of $\Xi^{n+k}$ and $M^n$. Since $\{e_1, \ldots, e_n\}$ is geodesic at $p$ (on $M$), it follows that

$$\langle \nabla_{E_i} E_i, E_k \rangle_p = \langle D_{e_i} e_i, e_k \rangle_p = \langle (D_{e_i} e_i)_{1}^{\perp} + \nabla_{e_i} e_i, e_k \rangle_p = 0.$$  \hspace{1cm} (3.5)

Therefore, solving the Cauchy problem formed by (3.4) and (3.5), we get

$$\langle \nabla_{E_i} E_i, E_k \rangle_{\psi(t,p)} = 0, \forall |t| < \epsilon.$$  \hspace{1cm} (3.6)

Analogously to (3.4), we get

$$\frac{d}{dt} \langle \nabla_{E_i} E_i, N_\beta \rangle = -\psi \langle \nabla_{E_i} E_i, N_\beta \rangle.$$  \hspace{1cm} (3.7)

On the other hand, letting $A_\beta : T_p M \to T_p M$ denote the shape operator of $\varphi$ in the direction of $\eta_\beta$ and writing $A_\beta e_i = h_{\beta}^{ij} e_j$, we have

$$\langle \nabla_{E_i} E_i, N_\beta \rangle_p = \langle D_{e_i} e_i, \eta_\beta \rangle_p = \langle A_\beta e_i, e_i \rangle_p = h_{\beta}^{ij} e_i.$$  \hspace{1cm} (3.8)

Solving the Cauchy problem formed by (3.7) and (3.8), we get

$$\langle \nabla_{E_i} E_i, N_\beta \rangle_{\psi(t,p)} = h_{\beta}^{ij} \exp \left( -\int_0^t \psi(s) ds \right).$$  \hspace{1cm} (3.9)

It finally follows from (3.3), (3.6) and (3.9) that, at the point $(t, p)$,

$$\nabla_{E_i} E_i = \langle \nabla_{E_i} E_i, E_k \rangle E_k - \langle \nabla_{E_i} E_i, \nu \rangle \nu + \langle \nabla_{E_i} E_i, N_\beta \rangle N_\beta$$

$$= \frac{n\psi}{\langle V, V \rangle} V + \exp \left( -\int_0^t \psi(s) ds \right) h_{\beta}^{ij} N_\beta.$$  \hspace{1cm} (3.10)

Therefore, (3.2) gives us

$$\mathcal{T} = \frac{1}{n+1} \exp \left( -\int_0^t \psi(s) ds \right) h_{\beta}^{ij} N_\beta.$$  \hspace{1cm} (3.11)

Let us finally establish the equivalence of (a), (b) and (c), observing that (b) $\Rightarrow$ (c) is always true.

(a) $\Rightarrow$ (b): if $\varphi$ is maximal, we have $h_{\beta}^{ij} = 0$ for all $1 \leq \beta \leq k$, and it follows from (3.10) that $\mathcal{T} = 0$. 

(c) ⇒ (a): if $\nabla V\overline{\mathcal{F}} = 0$, then, along the integral curve of $V$ that passes through $p$, the parallelism of the $N_\beta$ gives

$$0 = \nabla_v H = \left(\frac{D\overline{\mathcal{F}}}{dt}\right) = -\psi(t) n + \frac{1}{n+1} \exp \left(-\int_0^t \psi(s) ds\right) h_\beta^0 N_\beta.$$ 

However, since $\psi \neq 0$ on $\varphi(M)$, it follows from the above equality that $h_\beta^0 = 0$ at $p$ for all $1 \leq \beta \leq k$, so that $\varphi$ is maximal at $p$.

The following corollaries are immediate.

**Corollary 3.2.** Let $I \subset (0, +\infty)$ and $t_0 \in I$. In the GRW space $-I \times t \mathbb{R}^{n+k}$, if $\varphi : M^n \to \{t_0\} \times F^{n+k}$ is an isometric immersion and $\Phi$ is the canonical immersion of $-I \times t \times \mathbb{R}^{n+k}$, then $\varphi$ is maximal if and only if $\Phi$ is maximal.

**Proof.** Since $V = t\partial_t$, Corollary 7.43 of [14], $\text{Ric}(\mathcal{F}) = 0$ in this case. □

**Corollary 3.3.** Let $\varphi : M^n \to \mathbb{S}^{n+k}$ be an $n-$dimensional submanifold of some round sphere $\mathbb{S}^{n+k}$ of the $(n+k+1)-$dimensional de Sitter space $\mathbb{S}^{n+k+1}_1$. If $\varphi(M)$ is contained in the chronological past (resp. future) of $\mathbb{S}^{n+k+1}_1$, then $M^n$ is maximal in $\mathbb{S}^{n+k}$ if and only if the union of the segments of the integral curves of $\partial_t$ contained in the chronological past (resp. future) of $\mathbb{S}^{n+k+1}_1$ passing through points of $\varphi(M)$ is maximal in $\mathbb{S}^{n+k+1}_1$.

**Corollary 3.4.** Let $\varphi : M^n \to \mathbb{H}^{n+k}$ be an $n-$dimensional submanifold of some hyperbolic space $\mathbb{H}^{n+k}$ of the $(n+k+1)-$dimensional anti-de Sitter space $\mathbb{H}^{n+k+1}_1$. Then $M^n$ is maximal in $\mathbb{H}^{n+k}$ if and only if the union of the segments of the integral curves of $\partial_t$ that pass through points of $\varphi(M)$ is maximal in $\mathbb{H}^{n+k+1}_1$.

### 4. Bernstein-type Theorems

We continue to employ the notations of the previous sections, i.e., $\mathcal{F}$ is conformally stationary with closed conformal vector field $V$. However, we let $\psi_V$ be the conformal factor of $V$.

From now on, we let $x : M^n \to \mathbb{N}^{n+1}$ be a connected, complete, oriented spacelike hypersurface, $N$ be a unit normal vector field which orients $M$ and has the same time-orientation of $V$, and $A$ and $H$ be respectively the shape operator and the mean curvature of $M$ with respect to $N$.

If $f_V : M \to \mathbb{R}$ is given by $f_V = \langle V, N \rangle$ then $f_V$ is negative on $M$. On the other hand, standard computations (cf. [3]) give

\begin{equation}
\nabla f_V = -A(V^\top)
\end{equation}

and

\begin{equation}
\Delta f_V = nV^\top(H) + \{\text{Ric}(\mathcal{F})(N,N) + |A|^2\} f_V + n\{H\psi + N(\psi)\},
\end{equation}

where $( )^\top$ stands for orthogonal projection onto $M$.

If $W$ is another closed conformal vector field on $\mathcal{F}$, with conformal factor $\psi_W$, and $g : M \to \mathbb{R}$ is given by $g = \langle V, W \rangle$, then another standard computation gives

\begin{equation}
\nabla g = \psi_V W^\top + \psi_W V^\top.
\end{equation}

and

\begin{equation}
\Delta g = W^\top(\psi_V) + V^\top(\psi_W) + nH(\psi_V f_W + \psi_W f_V) + 2n\psi_V \psi_W.
\end{equation}
We are now in position to state and prove the following Bernstein-type general theorem for spacelike hypersurfaces. Observe that we do not require the hypersurface in question to be of constant mean curvature. In what follows, we let $L^1(M)$ be the space of Lebesgue integrable functions on $M$.

**Theorem 4.1.** Let $M^{n+1}$ have nonnegative Ricci curvature, $V$ and $W$ be respectively a parallel and a homothetic nonparallel vector field on $M^{n+1}$, and $x : M^n \rightarrow M^{n+1}$ be as above. If $|A|$ is bounded, $|V^\top|$ is integrable and $H$ doesn’t change sign on $M$, then:

(a) $M$ is totally geodesic and the Ricci curvature of $M$ in the direction of $N$ vanishes identically.

(b) If $M$ is noncompact and $\text{Ric}_M$ is also nonnegative, then $x(M)$ is contained in a leaf of $V^\perp$.

**Proof.**

(a) Since $V$ is parallel and $W$ is homothetic and nonparallel, it follows from (4.1), (4.3), (4.2) and (4.4) that

\[
\nabla f_V = -A(V^\top),
\nabla g = \psi_W V^\top,
\]

(4.5) \[
\Delta f_V = nV^\top(H) + (\text{Ric}_N(N,N) + |A|^2) f_V
\]

and

\[
\Delta g = nH\psi_W f_V,
\]

with $\psi_W$ being a nonzero constant. Therefore, the assumption $|V^\top| \in L^1(M)$ gives $|\nabla g| \in L^1(M)$, and the assumption on $H$, together with the fact that $|f_V| > 0$ on $M$, assures that $\Delta g$ is either nonnegative or nonpositive on $M$. Therefore, the Corollary on page 660 of [19] gives $\Delta g = 0$ on $M$, and hence $H = 0$ on $M$.

We now look at (4.5), which resumes to

\[
\Delta f_V = (\text{Ric}_N(N,N) + |A|^2) f_V,
\]

and hence doesn’t change sign on $M$ too. We also note that the boundedness of $|A|$ on $M$ gives

\[
|\nabla f_V| \leq |A||V^\top| \in L^1(M).
\]

Appealing again to the Corollary on page 660 of [19], we get $\Delta f_V = 0$ on $M$, so that

\[
\text{Ric}_N(N,N) + |A|^2 = 0
\]

on $M$. Since $\text{Ric}_N(N,N) \geq 0$, then we get $\text{Ric}_N(N,N) = 0$ and $A = 0$ on $M$, i.e., $M$ is totally geodesic.

(b) $A = 0$ on $M$ gives $\nabla f_V = 0$ on $M$, so that $f_V = \langle V, N \rangle$ is constant and nonzero on $M$. However, $|V|^2$ is constant on $M$ (since $V$ is parallel) and

\[
|V^\top|^2 = |V|^2 + \langle V, N \rangle^2,
\]

so that $|V^\top|$ is also constant on $M$. Therefore,

\[
+\infty > \int_M |V^\top| dM = |V^\top| \text{Vol}(M).
\]

But since $M$ is noncompact and has nonnegative Ricci curvature, another theorem of Yau (Theorem 7 of [19]) gives $\text{Vol}(M) = +\infty$, and hence the only possibility is $|V^\top| = 0$. Therefore, Cauchy-Schwarz inequality gives that $V$ is parallel to $N$, and $x(M)$ is contained in a leaf of $V^\perp$. \qed
For the next result we need a small refinement of the analytical tool of Yau’s result used in the above proof. We quote it below, referring the reader to [10] for a proof.

**Lemma 4.2.** Let $X \in \mathfrak{X}(M)$ be such that $\text{div}_M X$ doesn’t change sign on $M$. If $|X| \in \mathcal{L}^1(M)$, then $\text{div}_M X = 0$ on $M$.

**Theorem 4.3.** Let $\overline{M}$ have nonnegative Ricci curvature, $V$ be a homothetic vector field on $\overline{M}^{n+1}$, and $x : M^n \to \overline{M}^{n+1}$ be as before. If $|A|$ is bounded, $|V^T|$ is integrable and $H$ is constant on $\Lambda$, then $M$ is totally umbilical and the Ricci curvature of $\overline{M}$ in the direction of $N$ vanishes identically.

**Proof.** Since $H$ is constant on $M$ and $\psi_V$ is constant on $\overline{M}$, (4.2) reduces to
\[
\Delta f_V = \left\{ \text{Ric}_{\overline{M}}(N,N) + |A|^2 \right\} f_V + nH \psi_V.
\]
Letting $\{e_1, \ldots, e_n\}$ be a moving frame on $M$, we have
\[
\text{div}_M(V^T) = \langle \nabla e_k (V + f_V N), e_k \rangle = n \psi_V - f_V \langle A(e_k), e_k \rangle = n \psi_V + nH f_V,
\]
so that
\[
\text{div}_M(\nabla f_V - HV^T) = \Delta f_V - nH \psi_V - nH^2 f_V = \left\{ \text{Ric}_{\overline{M}}(N,N) + |A|^2 - nH^2 \right\} f_V.
\]

Since $|f_V| > 0$ on $M$, $\text{Ric}_{\overline{M}}(N,N) \geq 0$ and $|A|^2 \geq nH^2$ by Cauchy-Schwarz inequality (with equality if and only if $M$ is totally umbilical), this last expression does not change sign on $M$. Now observe that
\[
|\nabla f_V - HV^T| = | - AV^T - HV^T | \leq (|A| + H)|V^T| \in \mathcal{L}^1(M),
\]
so that the previous lemma gives $\text{div}_M(\nabla f_V - HV^T) = 0$ on $M$. Back to (4.7), we then get $\text{Ric}_{\overline{M}}(N,N) = 0$ and $|A|^2 - nH^2 = 0$, and hence $M$ is totally umbilical. $\square$

The previous result yields the following corollary on GRW spaces.

**Corollary 4.4.** Let $I \subset (0, +\infty)$ be an open interval, $F^n$ be an $n$–dimensional, complete oriented Riemannian manifold of nonnegative Ricci curvature, $\overline{M}^{n+1} = -I \times I F^n$ and $x : M^n \to \overline{M}^{n+1}$ as before. If $|A|$ is bounded, $|(t \partial_t)| \in \mathcal{L}^1(M)$ and $H$ is constant on $M$, then $M$ is totally umbilical and the Ricci curvature of $\overline{M}$ in the direction of $N$ vanishes identically. In particular, if $F$ is closed and has positive Ricci curvature everywhere, then $x(M) \subset \{ t_0 \} \times F$, for some $t_0 \in I$.

**Proof.** The first part follows from the theorem. To the second one, if $F$ is closed and has positive Ricci curvature everywhere, then, according to the previous result, the only possible direction for $N$ is that of $t \partial_t$. But if $N$ is parallel to $\partial_t$, then the conectedness of $M$ guarantees that $x(M)$ cannot jump from one leaf $\{ t_0 \} \times F$ to another. $\square$

In what follows, we let $\mathbb{R}^n = \{ x \in \mathbb{L}^{n+1}; x_{n+1} = 0 \}$ and $\mathbb{H}^n = \{ x \in \mathbb{L}^{n+1}; \langle x, x \rangle = -1, x_{n+1} > 0 \}$. As a special case of the previous corollary, we get...
Corollary 4.5. Let $x : M^n \to \mathbb{L}^{n+1}$ be an embedding, such that $x(M)$ is a complete spacelike radial graph over either $\mathbb{R}^n$ or $\mathbb{H}^n$, minus $k$ points. If $|A|$ is bounded, $H$ is constant and $p \mapsto |x(p)^T|$ is integrable on $M$, then $k = 0$ and $x(M)$ is either a spacelike hyperplane or a translation of $\mathbb{H}^n$.

Remark 4.6. The class of examples of Corollary 3.3 of [9] shows that the hypothesis on the integrability of $p \mapsto |x(p)^T|$ is really necessary.

5. $r$-Stability of Spacelike Hypersurfaces

For the time being, let $\mathcal{M}^{n+1}$ denote a time-oriented Lorentz manifold (i.e., not necessarily conformally stationary) with Lorentz metric $\tilde{g} = \{ , \}$, volume element $d\mathcal{M}$ and semi-Riemannian connection $\nabla$. We consider spacelike hypersurfaces $x : M^n \to \mathcal{M}^{n+1}$, namely, isometric immersions from a connected, $n-$dimensional orientable Riemannian manifold $M^n$ into $\mathcal{M}$. We let $\nabla$ denote the Levi-Civita connection of $M^n$.

If $\mathcal{M}$ is time-orientable and $x : M^n \to \mathcal{M}^{n+1}$ is a spacelike hypersurface, then $M^n$ is orientable (cf. [14]) and one can choose a globally defined unit normal vector field $N$ on $M^n$ having the same time-orientation of $\mathcal{M}$; such an $N$ is said to be a future-pointing Gauss map of $M^n$. If we let $A$ denote the shape operator of $x$ with respect to $N$, then $A$ restricts to a self-adjoint linear map $A_p : T_p M \to T_p M$ at each $p \in M^n$.

For $1 \leq r \leq n$, let $S_r(p)$ denote the $r$-th elementary symmetric function on the eigenvalues of $A_p$, so that one gets $n$ smooth functions $S_r : M^n \to \mathbb{R}$ for which

$$\det(t\text{Id} - A) = (-1)^k S_k t^{n-k},$$

where $S_0 = 1$ by definition. For fixed $p \in M^n$, the spectral theorem allows us to choose on $T_p M$ an orthonormal basis $\{e_1, \ldots, e_n\}$ of eigenvectors of $A_p$, with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$, respectively. One thus immediately sees that

$$S_r = \sigma_r(\lambda_1, \ldots, \lambda_n),$$

where $\sigma_r \in \mathbb{R}[X_1, \ldots, X_n]$ is the $r$-th elementary symmetric polynomial on the indeterminates $X_1, \ldots, X_n$.

For $1 \leq r \leq n$, one defines the $r$-th mean curvature $H_r$ of $x$ by

$$\binom{n}{r} H_r = (-1)^r S_r = \sigma_r(-\lambda_1, \ldots, -\lambda_n).$$

One also let the $r$-th Newton transformation $P_r$ on $M^n$ be given by setting $P_0 =$ Id and, for $1 \leq r \leq n$, via the recurrence relation

$$P_r = (-1)^r S_r \text{Id} + A P_{r-1}.$$  

(5.1)

A trivial induction shows that

$$P_r = (-1)^r (S_r \text{Id} - S_{r-1} A + S_{r-2} A^2 - \cdots + (-1)^r A^r),$$

so that Cayley-Hamilton theorem gives $P_n = 0$. Moreover, since $P_r$ is a polynomial in $A$ for every $1 \leq r \leq n$, it is also self-adjoint and commutes with $A$. Therefore, all bases of $T_p M$ diagonalizing $A$ at $p \in M^n$ also diagonalize all of the $P_r$ at $p$. If $\{e_1, \ldots, e_n\}$ is such a basis and $A_i$ denotes the restriction of $A$ to $\langle e_i \rangle^\perp \subset T_p M$, it is easy to see that

$$\det(t\text{Id} - A_i) = (-1)^k S_k(A_i) t^{n-1-k},$$

where $\lambda_1, \ldots, \lambda_n$ are the corresponding eigenvalues.
where
\[ S_k(A_i) = \sum_{1 \leq j_1 < \cdots < j_k \leq n} \lambda_{j_1} \cdots \lambda_{j_k}. \]

With the above notations, it is also immediate to check that
\[ P_r e_i = (-1)^r S_r(A_i) e_i, \]
and it is a standard fact that
\begin{enumerate}
  \item \( \text{tr}(P_r) = (-1)^r (n - r) S_r; \)
  \item \( \text{tr}(AP_r) = (-1)^r (r + 1) S_{r+1} = -b_r H_{r+1}; \)
  \item \( \text{tr}(A^2 P_r) = (-1)^r (S_1 S_{r+1} - (r + 2) S_{r+2}), \)
\end{enumerate}
where \( b_r = (n - r) \binom{n}{r}. \)

Associated to each Newton transformation \( P_r \) one has the second order linear
\begin{align*}
  & \text{differential operator} \\
  & L_r: \mathcal{D}(M) \to \mathcal{D}(M) \text{ given by} \\
  & L_r(f) = \text{tr}(P_r \text{Hess } f).
\end{align*}

In particular, \( L_0 = \Delta, \) the Laplace operator on smooth functions on \( M. \) If \( \overline{M}^{n+1} \)
is of constant sectional curvature, H. Rosenberg [15] proved that
\[ L_r(f) = \text{div}(P_r \nabla f). \]

A variation of \( x \) is a smooth mapping
\[ X: M^n \times (-\epsilon, \epsilon) \to \overline{M}^{n+1} \]
satisfying the following conditions:
\begin{enumerate}
  \item For \( t \in (-\epsilon, \epsilon), \) the map \( X_t: M^n \to \overline{M}^{n+1} \) given by \( X_t(p) = X(t, p) \) is a
    spacelike immersion such that \( X_0 = x. \)
  \item \( X_t|_{\partial M} = x|_{\partial M}, \) for all \( t \in (-\epsilon, \epsilon). \)
\end{enumerate}

The variational field associated to the variation \( X \) is the vector field \( \frac{\partial X}{\partial t}|_{t=0}. \)

Letting \( N_t \) denote the unit normal vector field along \( X_t \) and \( f = -\langle \frac{\partial X}{\partial t}, N_t \rangle, \) we get
\[ \frac{\partial X}{\partial t} = f N_t + \left( \frac{\partial X}{\partial t} \right)^\top, \]
where \( ^\top \) stands for tangential components.

Following [4], we set the balance of volume of the variation \( X \) as the function
\[ \mathcal{V}: (-\epsilon, \epsilon) \to \mathbb{R} \] given by
\[ \mathcal{V}(t) = \int_{M \times [0, t]} X^*(d\overline{M}), \]
and we say that \( X \) is volume-preserving if \( \mathcal{V} \) is constant.

Letting \( dM_t \) denote the volume element of the metric induced on \( M \) by \( X_t, \) we recall the following standard result (cf. [18]).

\textbf{Lemma 5.1.} Let \( \overline{M}^{n+1} \) be a time-oriented Lorentz manifold and \( x: M^n \to \overline{M}^{n+1} \)
a closed spacelike hypersurface. If \( X: M^n \times (-\epsilon, \epsilon) \to \overline{M}^{n+1} \) is a variation of \( x, \) then
\[ \frac{d\mathcal{V}}{dt} = \int_M f dM_t. \]

In particular, \( X \) is volume-preserving if and only if \( \int_M f dM_t = 0 \) for all \( t. \)
We remark that Lemma 2.2 of [4] remains valid in the Lorentz context, i.e., if $f_0 : M \to \mathbb{R}$ is a smooth function such that $\int_M f_0 dM = 0$, then there exists a volume-preserving variation of $M$ whose variational field is $f_0 N$.

According to [5], we define the $r$-functional $\mathcal{A}_r : (-\epsilon, \epsilon) \to \mathbb{R}$ associated to the variation $X$ by

$$\mathcal{A}_r(t) = \int_M F_r(S_1, S_2, \ldots, S_n) dM,$$

where $S_r = S_r(t)$ and $F_r$ is recursively defined by setting $F_0 = 1$, $F_1 = -S_1$ and, for $2 \leq r \leq n - 1$,

$$F_r = (-1)^r S_r - \frac{c(n-r+1)}{r-1} F_{r-2}.$$

In particular, if $r = 0$, then $A_0$ is the classical area functional.

The Lorentz analogue of Proposition 4.1 of [5] is stated in the following Lemma (for another proof, see Lemma 2.2 of [8]).

**Lemma 5.2.** If $x : M^n \to \mathbb{M}^{n+1}$ is a closed spacelike hypersurface of the time-oriented Lorentz manifold $\mathbb{M}^{n+1}$ of constant sectional curvature $c$, and $X : M^n \times (-\epsilon, \epsilon) \to \mathbb{M}^{n+1}$ a variation of $x$, then

$$\frac{\partial S_{r+1}}{\partial t} = (-1)^{r+1} [L_r f + c \text{tr}(P_r) f - \text{tr}(A^2 P_r) f] + \left( \frac{\partial X}{\partial t} \right)^\top \nabla S_{r+1}. \tag{5.3}$$

As in [5], the previous lemma allows us to compute the first variation of the $r$-functional, according to the following

**Proposition 5.3.** Under the hypotheses of Lemma 5.2, we have

$$\mathcal{A}'_r(t) = \int_M [(-1)^{r+1}(r+1)S_{r+1} + c_r] f dM, \tag{5.4}$$

where $c_r = 0$ if $r$ is even and $c_r = -\frac{n(n-2)(n-4)\cdots (n-r+1)}{(r-1)(r-3)\cdots 2}(c(r+1)/2$ if $r$ is odd.

In order to characterize spacelike immersions of constant $(r+1)$-th mean curvature, let $\lambda$ be a real constant and $\mathcal{J}_r : (-\epsilon, \epsilon) \to \mathbb{R}$ be the Jacobi functional associated to $X$, i.e.,

$$\mathcal{J}_r(t) = \mathcal{A}_r(t) - \lambda t.$$

As an immediate consequence of (5.2) we get

$$\mathcal{J}'_r(t) = \int_M [b_r H_{r+1} + c_r - \lambda] f dM,$$

where $b_r = (r+1)\binom{n}{r+1}$. Therefore, if we choose $\lambda = c_r + b_r \mathcal{H}_{r+1}(0)$, where

$$\mathcal{H}_{r+1}(0) = \frac{1}{\mathcal{A}_0(0)} \int_M H_{r+1}(0) dM$$

is the mean of the $(r+1)$-th curvature $H_{r+1}(0)$ of $M$, we arrive at

$$\mathcal{J}'_r(t) = b_r \int_M [H_{r+1} - \mathcal{H}_{r+1}(0)] f dM.$$

Hence, a standard argument (cf. [3]) shows that $M$ is a critical point of $\mathcal{J}_r$ for all variations of $x$ if and only if $M$ has constant $(r+1)$-th mean curvature.

As in [5], we wish to study spacelike immersions $x : M^n \to \mathbb{M}^{n+1}$ that maximize $\mathcal{J}_r$ for all variations $X$ of $x$. The above discussion shows that $M$ must have constant
(r + 1)—th mean curvature and, for such an \( M \), leads us naturally to compute the second variation of \( J_r \). This, in turn, motivates the following

**Definition 5.4.** Let \( \overline{M}_{c_{n+1}}^{n+1} \) be a Lorentz manifold of constant sectional curvature \( c \), and \( x : M^n \rightarrow \overline{M}_{c_{n+1}}^{n+1} \) be a closed spacelike hypersurface having constant \((r + 1)\)—th mean curvature. We say that \( x \) is strongly \( r \)-stable if, for every smooth function \( f : M \rightarrow \mathbb{R} \) one has \( J''_r(0) \leq 0 \).

The sought formula for the second variation of \( J_r \) is another straightforward consequence of Proposition 5.3.

**Proposition 5.5.** Let \( x : M^n \rightarrow \overline{M}_{c_{n+1}}^{n+1} \) be a closed spacelike hypersurface of constant \((r + 1)\)—mean curvature \( H_{r+1} \). If \( X : M^n \times (-\varepsilon, \varepsilon) \rightarrow \overline{M}_{c_{n+1}}^{n+1} \) is a variation of \( x \), then

\[
J''_r(0) = (r + 1) \int_M \left[ L_r(f) + c \text{tr}(P_r)f - \text{tr}(A^2 P_r)f \right] fdM.
\]

Back to the conformally stationary setting, in what follows we need a formula first derived in [2]. As stated below, it is the Lorentz version of the one stated and proved in [7].

**Lemma 5.6.** Let \( \overline{M}_{c_{n+1}}^{n+1} \) be a conformally stationary Lorentz manifold having constant sectional curvature \( c \) and conformal vector field \( V \). Let also \( x : M^n \rightarrow \overline{M}_{c_{n+1}}^{n+1} \) be a spacelike hypersurface and \( N \) a future-pointing, unit normal vector field globally defined on \( M^n \). If \( \eta = \langle V, N \rangle \), then

\[
L_r \eta = \text{tr}(A^2 P_r)\eta - c \text{tr}(P_r)\eta - b_r H_r N(\psi) + b_r H_{r+1} + \frac{b_r}{r + 1} \langle V, \nabla H_{r+1} \rangle,
\]

where \( \psi : \overline{M}_{c_{n+1}}^{n+1} \rightarrow \mathbb{R} \) is the conformal factor of \( V \), \( H_j \) is the \( j \)—th mean curvature of \( x \) and \( \nabla H_j \) stands for the gradient of \( H_j \) on \( M \).

We are now in position to state and prove the following

**Theorem 5.7.** Let \( \overline{M}_{c_{n+1}}^{n+1} \) be a timelike geodesically complete conformally stationary Lorentz manifold of constant sectional curvature \( c \) having a closed conformal timelike vector field \( V \), and let \( x : M^n \rightarrow \overline{M}_{c_{n+1}}^{n+1} \) be a closed, strongly \( r \)-stable spacelike hypersurface. Suppose that the conformal factor \( \psi \) of \( V \) satisfies the condition

\[
\frac{H_r}{\sqrt{-\langle V, V \rangle}} \frac{\partial \psi}{\partial t} \geq \max\{H_{r+1} \psi, 0\},
\]

where \( t \in \mathbb{R} \) denotes the real parameter of the flow of \( V \). If the set where \( \psi = 0 \) has empty interior in \( M \), then \( M^n \) is either \( r \)—maximal or a leaf of the foliation \( V^\perp \).

**Proof.** Since \( M^n \) is strongly \( r \)-stable, it follows from (5.5) that

\[
(r + 1) \int_M \left[ L_r(f) + c \text{tr}(P_r)f - \text{tr}(A^2 P_r)f \right] fdM \leq 0,
\]

for all smooth \( f : M \rightarrow \mathbb{R} \). In particular, since \( H_{r+1} \) is constant on \( M \), taking \( f = \eta = \langle V, N \rangle \) in (5.6), we get

\[
L_r \eta + c \text{tr}(P_r)\eta - \text{tr}(A^2 P_r)\eta = -b_r H_r N(\psi) + b_r H_{r+1} \psi.
\]
so that (5.7) gives

\[ \int_M [-H_r N(\psi) + H_{r+1} \psi] \langle V, N \rangle dM \leq 0. \tag{5.8} \]

However, it follows from (2.5) that

\[ N(\psi) = \langle N, \nabla \psi \rangle = -\nu(\psi) \langle \nu, N \rangle = \partial \psi / \partial t \cosh \theta \sqrt{-\langle V, V \rangle}, \]

where \( \theta \) is the hyperbolic angle between \( V \) and \( N \). Substituting the above into (5.8), we finally arrive at

\[ \int_M \left[ H_r \frac{\partial \psi}{\partial t} \frac{\cosh \theta}{\sqrt{-\langle V, V \rangle}} - H_{r+1} \psi \right] \frac{\cosh \theta}{\sqrt{-\langle V, V \rangle}} dM \leq 0. \]

Arguing as in the end of the proof of Theorem 1.1 of [6], we get

\[ H_r \frac{\partial \psi}{\partial t} (\cosh \theta - 1) = 0 \quad \text{and} \quad \frac{H_r}{\sqrt{-\langle V, V \rangle}} \frac{\partial \psi}{\partial t} = H_{r+1} \psi \]
on \( M \). But since \( H_{r+1} \) is constant on \( M \), either \( M \) is \( r \)–maximal or \( H_{r+1} \neq 0 \) on \( M \). If this last case happens, the condition on the zero set of \( \psi \) on \( M \), together with the above, gives \( H_r \frac{\partial \psi}{\partial t} \neq 0 \) on a dense subset of \( M \), and hence \( \cosh \theta = 1 \) on this set. By continuity, \( \cosh \theta = 1 \) on \( M \), so that \( M \) is a leaf of the foliation \( \nu^\perp \). □

The following corollary is immediate.

**Corollary 5.8.** Let \( x : M^n \to S^{n+1}_1 \) be a closed, strongly \( r \)–stable spacelike hypersurface, such that the set of points in which \( M^n \) intersects the equator of \( S^{n+1}_1 \) has empty interior in \( M \). If

\[ H_r \geq \max\{ (\sinh t) H_{r+1}, 0 \}, \]

then either \( M^n \) is \( r \)–maximal or an umbilical round sphere.

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**References**

[1] L. J. Alías, A. Brasil Jr. and A. G. Colares, *Integral Formulae for Spacelike Hypersurfaces in Conformally Stationary Spacetimes and Applications*, Proc. Edinburgh Math. Soc. 46, 465–488 (2003).
[2] L. Alías and A. G. Colares, *Uniqueness of spacelike hypersurfaces with constant higher order mean curvature in generalized Robertson-Walker spacetimes*. Math. Proc. of the Cambridge Phil. Soc. 143 (2007), 703–729.
[3] J. L. M. Barbosa and M. do Carmo, *Stability of Hypersurfaces with Constant Mean Curvature*, Math. Z. 185 (1984) 339–353.
[4] J. L. M. Barbosa, M. do Carmo and J. Eschenburg, *Stability of Hypersurfaces with Constant Mean Curvature*. Math. Z. 197 (1988) 123–138.
[5] J. L. M. Barbosa and A. G. Colares. *Stability of hypersurfaces with constant r–mean curva-ture*. Ann. Global Anal. Geom. 15 (1997) 277-297.
[6] A. Barros, A. Brasil and A. Caminha. *Stability of spacelike hypersurfaces in foliated spacetimes*. Diff. Geom. Appl. 26 (2008) 357-365.
[7] A. Barros and P. Sousa. *Compact graphs over a sphere of constant second order mean curva-ture*. Proc. Amer. Math. Soc. 137 (2009) 3105-3114.
[8] F. Camargo, A. Caminha, M. da Silva and H. de Lima. On the r-stability of sапalike hypersurfaces. Preprint arXiv:0911.2043v1 (2009).
[9] A. Caminha. A rigidity theorem for complete CMC hypersurfaces in Lorentz manifolds. Diff. Geom. and its Applications 24 (2006), 652-659.
[10] A. Caminha. The geometry of closed conformal vector fields on Riemannian spaces. Preprint arXiv:0908.1447v1 (2009).
[11] J. Jellett. La surface dont la courbure moyenne est constant. J. Math. Pures Appl. 18 (1853), 163-167.
[12] W. Kühnel and H-B Hademacher. Conformal vector fields on pseudo-Riemannian spaces. Diff. Geom. Appl. 7 (1997) 237-250.
[13] S. Montiel. Uniqueness of spacelike hypersurfaces of constant mean curvature in foliated spacetimes. Math. Ann. 314 (1999) 529–553.
[14] B. O’Neill. Semi-Riemannian geometry, with applications to relativity. Academic Press, New York (1983).
[15] H. Rosenberg. Hypersurfaces of Constant Curvature in Space Forms. Bull. Sc. Math. 117 (1993) 217-239.
[16] J. Simons. Minimal varieties in Riemannian manifolds. Ann. of Math. 88, (1968) 62-105.
[17] R. Wald. General Relativity, Univ of Chicago Press, Chicago (1984).
[18] Y. Xin. Minimal submanifolds and related topics. World Scientific Publishing Co., Singapore (2003).
[19] S. T. Yau. Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry. Indiana Univ. Math. J. 25 (1976), 659-670.

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