A MULTIPLICITY RESULT VIA LJUŞTERNICK-SCHNIRELMANN CATEGORY AND MORSE THEORY FOR A FRACTIONAL SCHRÖDINGER EQUATION IN $\mathbb{R}^N$

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ABSTRACT. In this work we study the following class of problems in $\mathbb{R}^N, N > 2s$

$$\varepsilon^{2s}(-\Delta)^s u + V(z)u = f(u), \ u(z) > 0$$

where $0 < s < 1$, $(-\Delta)^s$ is the fractional Laplacian, $\varepsilon$ is a positive parameter, the potential $V : \mathbb{R}^N \to \mathbb{R}$ and the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ satisfy suitable assumptions; in particular it is assumed that $V$ achieves its positive minimum on some set $M$. By using variational methods we prove existence, multiplicity and concentration of maxima of positive solutions when $\varepsilon \to 0^+$. In particular the multiplicity result is obtained by means of the Ljusternick-Schnirelman and Morse theory, by exploiting the “topological complexity” of the set $M$.

1. INTRODUCTION

In this paper we are concerned with existence, multiplicity and concentration results for the solutions of the following class of problems

$$(P_\varepsilon) \begin{cases} 
\varepsilon^{2s}(-\Delta)^s u + V(z)u = f(u) \text{ in } \mathbb{R}^N, \ N > 2s \\
u \in H^s(\mathbb{R}^N) \\
u(z) > 0, z \in \mathbb{R}^N,
\end{cases}$$

where $s \in (0,1)$, $\varepsilon > 0$ and the Hilbert space $H^s(\mathbb{R}^N)$ is defined as

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : (-\Delta)^{s/2}u \in L^2(\mathbb{R}^N)\}$$

endowed with scalar product and (squared) norm given by

$$(u,v) = \int_{\mathbb{R}^N} (-\Delta)^{s/2}u v + \int_{\mathbb{R}^N} uv, \quad \|u\|^2 = \|(-\Delta)^{s/2}u\|^2_2 + \|u\|^2_2.$$ 

The fractional Laplacian $(-\Delta)^s$ is the pseudodifferential operator defined via the Fourier transform

$$\mathcal{F}((-\Delta)^s u) = \cdot |\cdot|^{2s} \mathcal{F}u,$$

and, when $u$ has sufficient regularity, it is also given by

$$(-\Delta)^s u(z) = \frac{C(N,s)}{2} \int_{\mathbb{R}^N} \frac{u(z+y) - u(z-y) - 2u(z)}{|y|^{N+2s}} dy, \quad z \in \mathbb{R}^N,$$

where $C(N,s)$ is a suitable normalization constant. For this fact and the relation between the fractional Laplacian and the fractional Sobolev space $H^s(\mathbb{R}^N)$ we refer the reader to classical books, see also [14].

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Problem \((P_\varepsilon)\) appears when one look for standing waves solutions
\[
\psi(z,t) = u(z)e^{-iEt/\varepsilon}, \quad u(z) \in \mathbb{R}, \; E \text{ a real constant}
\]
to the following Fractional Schrödinger equation
\[
i\varepsilon \frac{\partial \psi}{\partial t} = \varepsilon^{2s}(-\Delta)^s \psi + W(z)\psi - f(|\psi|)
\]
where \(W : \mathbb{R}^N \to \mathbb{R}\) is an external potential and \(f\) a suitable nonlinearity. Here \(\varepsilon\) is a sufficiently small parameter which corresponds to the Planck constant.

The fractional Schrödinger equation was first derived and studied by Laskin [22–24]. After that many papers appeared studying existence, multiplicity and behavior of solutions to fractional Schrödinger equations. Recently in [11] the authors studied, by means of Lyapunov-Schmidt reduction methods, concentration phenomenon for solutions in presence of a potential and with a power type nonlinearity. In particular it is shown that for sufficiently small \(\varepsilon\) the solutions concentrates to non-degenerate critical points of the potential. Concentration of solutions is also studied in [32] where the authors consider the nonlinearity \(f(x,u) = K(x)|u|^{p-2}u\) and prove the concentration near suitable critical points of a function \(\Gamma(x)\) which involves the potential \(V\) and the function \(K\). We also mention [16] where it is shown that concentration can occur only at critical points of \(V\).

We recall also that in recent years, problems involving fractional operators are receiving a special attention. Indeed fractional spaces and nonlocal equations have important applications in many sciences. We limit here ourself to give a non-exhaustive list of fields and papers in which these equations are used: obstacle problem [27, 30], optimization and finance [12, 15], phase transition [1, 31], material science [5], anomalous diffusion [20, 25, 26], conformal geometry and minimal surfaces [7–9]. The list may continue with applications in crystal dislocation, soft thin films, multiple scattering, quasi-geostrophic flows, water waves, and so on. The interested reader may consult also the references in the cited papers.

Coming back to our problem \((P_\varepsilon)\), in order to state the results we introduce the basics assumptions on \(f\) and \(V\):

\((V1)\) \(V : \mathbb{R}^N \to \mathbb{R}\) is a continuous function and satisfies
\[
0 < \min_{\mathbb{R}^N} V(x) =: V_0 < \liminf_{|x| \to \infty} V(x) =: V_\infty \in (0, +\infty];
\]

\((f1)\) \(f : \mathbb{R} \to \mathbb{R}\) is a function of class \(C^1\) and \(f(u) = 0\) for \(u \leq 0\);

\((f2)\) \(\lim_{u \to 0} f'(u) = 0\);

\((f3)\) \(\exists q \in (2, 2^*_s - 1)\) such that \(\lim_{u \to 0} f'(u)/u^{q-1} = 0\), where \(2^*_s := 2N/(N - 2s)\);

\((f4)\) \(\exists \theta > 2\) such that \(0 < \theta F(u) := \theta \int_0^u f(t)dt \leq uf(u)\) for all \(u > 0\);

\((f5)\) the function \(u \to f(u)/u\) is strictly increasing in \((0, +\infty)\).

By a solution of \((P_\varepsilon)\) we mean \(u \in W_\varepsilon\) (see Section 2 for the definition of \(W_\varepsilon\)) such that for every \(v \in W_\varepsilon\)
\[
\varepsilon^{2s} \int_{\mathbb{R}^N} (-\Delta)^s/2 u(-\Delta)^s/2 v + \int_{\mathbb{R}^N} V(z)uv = \int_{\mathbb{R}^N} f(u)v
\]
that is, as we will see, \(u\) is a critical point of a suitable energy functional \(I_\varepsilon\). The solution with “minimal energy” is what we call a ground state.

The assumptions on \(V\) and \(f\) are quite natural in this context. Assumption \((V1)\) was first introduced by Rabinowitz in [29] to take into account potentials which are possibly not coercive. Hypothesis \((f1)\) is not restrictive since we are looking for positive solutions (see e.g. [17, pag.
and Theorem 6 remain true if we need to get a better result using Morse theory; indeed, if for example \( k \) domain cutting off Remark 1. As it will be evident by the proofs, Theorem 2. Indeed by means of the Ljusternik-Schnirelman theory we arrive at the following result.

Theorem 1. Suppose that \( f \) verifies (f1)-(f5) and \( V \) verifies (V1). Then there exists a ground state solution \( u_\varepsilon \in W_\varepsilon \) of \( (P_\varepsilon) \),

1. for every \( \varepsilon > 0 \), if \( V_\infty = +\infty \);
2. for every \( \varepsilon \in (0, \bar{\varepsilon}] \), for some \( \bar{\varepsilon} > 0 \), if \( V_\infty < +\infty \).

The next results deal with the multiplicity of solutions and they involve topological properties of the set of minima of the potential

\[
M := \left\{ x \in \mathbb{R}^N : V(x) = V_0 \right\}.
\]

Indeed by means of the Ljusternik-Schnirelman theory we arrive at the following result.

Theorem 2. Suppose that \( f \) satisfies (f1)-(f5) and the function \( V \) satisfies (V1). Then, there exists \( \varepsilon^* > 0 \) such that for every \( \varepsilon \in (0, \varepsilon^*] \) problem \( (P_\varepsilon) \) has at least

1. \( \text{cat}(M) \) positive solutions;
2. \( \text{cat}(M) + 1 \) positive solutions, if \( M \) is bounded and \( \text{cat}(M) > 1 \).

Moreover, for any such a solution \( w_\varepsilon \), if \( \eta_\varepsilon \in \mathbb{R}^N \) denotes its global maximum, it holds

\[
\lim_{\varepsilon \to 0^+} V(\eta_\varepsilon) = V_0.
\]

Hereafter \( \text{cat}_T(X) \) denotes the Ljusternick-Schnirelmann category of the set \( X \) in \( Y \) (if \( X = Y \) we just write \( \text{cat}(X) \)). On the other hand, with the use of Morse theory we are able to deduce the next result.

Theorem 3. Suppose that \( f \) satisfies (f1)-(f5) and the function \( V \) satisfies (V1). Then there exists \( \varepsilon^* > 0 \) such that for every \( \varepsilon \in (0, \varepsilon^*] \) problem \( (P_\varepsilon) \) has at least \( 2\mathcal{P}_1(M) - 1 \) solutions, if non-degenerate, possibly counted with their multiplicity.

We are denoting with \( \mathcal{P}_1(M) \) the Poincaré polynomial of \( M \). It is clear that in general, we get a better result using Morse theory; indeed, if for example \( M \) is obtained by a contractible domain cutting off \( k \) disjoint contractible sets, it is \( \text{cat}(M) = 2 \) and \( \mathcal{P}_1(M) = 1 + k \). However, by using the Ljusternik-Schnirelmann category no non-degeneracy condition is required.

Remark 1. As it will be evident by the proofs, Theorem 1 and Theorem 2 remain true if we replace conditions (f2) and (f3) with the weaker conditions

- \( \lim_{u \to 0} f(u)/u = 0 \);
- \( \exists q \in (2, 2^*_s - 1) \) such that \( \lim_{u \to \infty} f(u)/u^q = 0 \), where \( 2^*_s := 2N/(N - 2s) \).

On the other hand for Theorem 3 we need (f2) and (f3) to have the compactness of a certain operator (see Section 6).

We have preferred to state our theorems under the stronger conditions just for the sake of simplicity.
The plan of the paper is the following. In Section 2, after a change of variable, we introduce an equivalent problem to \((P_\varepsilon)\) and the related variational setting; actually we will prove Theorems 1, 2 and 3 by referring to this equivalent problem. In Section 3 we prove some compactness properties and give the proof of Theorem 1. Section 4 is devoted to introduce the barycenter map and its properties. They will be fundamental tools in order to obtain the multiplicity results via the category theory of Ljusternick-Schnirelmann, explored in Section 5, and the Morse theory given in Section 6.

As a matter of notations, we denote with \(B_r(y)\), respectively \(B_r\), the ball in \(\mathbb{R}^N\) with radius \(r > 0\) centered in \(y\), respectively in 0. The \(L^p\)-norm in \(\mathbb{R}^N\) is simply denoted with \(|\cdot|_p\). If we need to specify the domain, let us say \(A \subset \mathbb{R}^N\), we write \(|\cdot|_{L^p(A)}\).

2. Preliminaries and technical results

First of all, it is easy to see that our problem is equivalent, after a change of variable to the following one

\[
(P^*_\varepsilon) \quad \begin{cases} 
(-\Delta)^s u + V(\varepsilon x)u = f(u) & \text{in } \mathbb{R}^N, \quad N > 2s \\
u \in H^s(\mathbb{R}^N) \\
u(x) > 0, \ x \in \mathbb{R}^N
\end{cases}
\]

to which we will refer from now on. Once we find solutions \(u_\varepsilon\) for \((P^*_\varepsilon)\), the function \(w_\varepsilon(x) := u_\varepsilon(x/\varepsilon)\) will be a solution of \((P_\varepsilon)\). Moreover, the maximum point \(\zeta_\varepsilon\) of \(w_\varepsilon\) is related to the maximum point \(\zeta_\varepsilon\) of \(u_\varepsilon\) simply by \(\zeta_\varepsilon = \varepsilon \zeta_\varepsilon\). Consequently, to prove the concentration property stated in Theorem 2 we just need to show that

\[
\lim_{\varepsilon \to 0^+} V(\varepsilon \zeta_\varepsilon) = V_0.
\]

We fix now some notations involving the functionals used to get the solutions to \((P^*_\varepsilon)\).

2.1. Variational setting. Let us start with the autonomous case. For a given constant (potential) \(\mu > 0\) consider the problem

\[
(A_\mu) \quad \begin{cases} 
(-\Delta)^s u + \mu u = f(u) & \text{in } \mathbb{R}^N, \quad N > 2s \\
u \in H^s(\mathbb{R}^N) \\
u(x) > 0, \ x \in \mathbb{R}^N
\end{cases}
\]

and the \(C^1\) functional in \(H^s(\mathbb{R}^N)\)

\[
E_\mu(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 + \frac{\mu}{2} \int_{\mathbb{R}^N} u^2 - \int_{\mathbb{R}^N} F(u)
\]

whose critical points are the solutions of \((A_\mu)\). In this case \(H^s(\mathbb{R}^N)\) is endowed with the (squared) norm

\[
\|u\|^2_\mu = \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 + \mu \int_{\mathbb{R}^N} u^2.
\]

The following are well known facts. The functional \(E_\mu\) has a mountain pass geometry and, defining \(\mathcal{H} = \{\gamma \in C([0, 1], H^s(\mathbb{R}^N)) : \gamma(0) = 0, E_\mu(\gamma(1)) < 0\}\), the mountain pass level

\[
m(\mu) := \inf_{\gamma \in \mathcal{H}} \sup_{t \in [0, 1]} E_\mu(\gamma(t))
\]

satisfies

\[
m(\mu) = \inf_{u \in H^s(\mathbb{R}^N) \setminus \{0\}} \sup_{t \geq 0} E_\mu(tu) = \inf_{u \in \mathcal{M}_\mu} E_\mu(u) > 0,
\]

where \(\mathcal{M}_\mu = \{u \in H^s(\mathbb{R}^N) : E_\mu(u) = m(\mu)\}\).
where

$$M_\mu := \left\{ u \in H^s(\mathbb{R}^N) \setminus \{0\} : \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 + \mu \int_{\mathbb{R}^N} u^2 = \int_{\mathbb{R}^N} f(u)u \right\}.$$  

It is standard to see that $M_\mu$ is bounded away from zero in $H^s(\mathbb{R}^N)$, and is a differentiable manifold radially diffeomorphic to the unit sphere. It is usually called the Nehari manifold associated to $E_\mu$.

On the other hand, the solutions of $(P^*_\varepsilon)$ can be characterized as critical points of the $C^1$ functional given by

$$I_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(\varepsilon x)u^2 - \int_{\mathbb{R}^N} F(u)$$  

which is well defined on the Hilbert space

$$W_\varepsilon := \left\{ u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\varepsilon x)u^2 < \infty \right\}$$

endowed with the (squared) norm

$$\|u\|^2_{W_\varepsilon} = \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 + \int_{\mathbb{R}^N} V(\varepsilon x)u^2.$$  

Note that if $V_\infty = +\infty$, $W_\varepsilon$ has compact embedding into $L^p(\mathbb{R}^N)$ for $p \in [2, 2^*_s)$, see e.g. [10, Lemma 3.2].

The Nehari manifold associated to $I_\varepsilon$ is

$$\mathcal{N}_\varepsilon = \left\{ u \in W_\varepsilon \setminus \{0\} : J_\varepsilon(u) = 0 \right\}$$

where

$$(2.3) \quad J_\varepsilon(u) := \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 + \int_{\mathbb{R}^N} V(\varepsilon x)u^2 - \int_{\mathbb{R}^N} f(u)u$$

and its tangent space in $u$ is given by

$$T_u\mathcal{N}_\varepsilon = \left\{ v \in H^s(\mathbb{R}^N) : J'_\varepsilon(u)[v] = 0 \right\}.$$  

Let us introduce also

$$\mathcal{S}_\varepsilon := \left\{ u \in W_\varepsilon : \|u\|_{W_\varepsilon} = 1, u > 0 \text{ a.e.} \right\}$$

which is a smooth manifold of codimension 1. The next result is standard; the proof follows the same lines of [6, Lemma 2.1 and Lemma 2.2].

**Lemma 1.** The following proposition hold true:

1. for every $u \in \mathcal{N}_\varepsilon$ it is $J'_\varepsilon(u)[u] < 0$;
2. $\mathcal{N}_\varepsilon$ is a differentiable manifold radially diffeomorphic to $\mathcal{S}_\varepsilon$ and there exists $k_\varepsilon > 0$ such that

$$\|u\|_{W_\varepsilon} \geq k_\varepsilon, \quad I_\varepsilon(u) \geq k_\varepsilon$$

As in [6, Lemma 2.1], it is easy to see that the functions in $\mathcal{N}_\varepsilon$ have to be positive on some set of nonzero measure. It is also easy to check that $I_\varepsilon$ has the mountain pass geometry, as given in the next

**Lemma 2.** Fixed $\varepsilon > 0$, for the functional $I_\varepsilon$ the following statements hold:

i) there exists $\alpha, \rho > 0$ such that $I_\varepsilon(u) \geq \alpha$ with $\|u\|_{\varepsilon} = \rho$,

ii) there exist $e \in W_\varepsilon$ with $\|e\|_{W_\varepsilon} > \rho$ such that $I_\varepsilon(e) < 0$.  


Then, defining the mountain pass level of \( I_\varepsilon \),

\[
c_\varepsilon := \inf_{\gamma \in \mathcal{H}} \sup_{t \in [0,1]} I_\varepsilon(\gamma(t))
\]

where \( \mathcal{H} = \{ \gamma \in C([0,1],W_\varepsilon) : \gamma(0) = 0, I_\varepsilon(\gamma(1)) < 0 \} \), well known arguments imply that

\[
c_\varepsilon = \inf_{u \in W_\varepsilon \setminus \{0\}} \sup_{t \geq 0} I_\varepsilon(tu) = \inf_{u \in N_\varepsilon} I_\varepsilon(u) \geq m(V_0).
\]

3. Compactness properties for \( I_\varepsilon \) and \( E_\mu \)

This section is devoted to prove compactness properties related to the functionals \( I_\varepsilon \) and \( E_\mu \).

It is standard by now to see that hypothesis \((f4)\) is used to obtain the boundedness of the \( PS \) sequences for \( I_\varepsilon \) or \( E_\mu \): we will always omit the prove of this fact in the paper.

We need to recall the following Lions type lemma.

**Lemma 3.** If \( \{u_n\} \) is bounded in \( H^s(\mathbb{R}^N) \) and for some \( R > 0 \) and \( 2 \leq r < 2^*_s \) we have

\[
\sup_{x \in \mathbb{R}^N} \int_{B_R(x)} |u_n|^r \to 0 \quad \text{as} \quad n \to \infty,
\]

then \( u_n \to 0 \) in \( L^p(\mathbb{R}^N) \) for \( 2 < p < 2^*_s \).

For a proof see e.g. [13, Lemma 2.3].

In order to prove compactness, some preliminary work is needed.

**Lemma 4.** Let \( \{u_n\} \subset W_\varepsilon \) be such that \( I_\varepsilon'(u_n) \to 0 \) and \( u_n \rightharpoonup 0 \) in \( W_\varepsilon \). Then we have either

a) \( u_n \to 0 \) in \( W_\varepsilon \), or

b) there exist a sequence \( \{y_n\} \subset \mathbb{R}^N \) and constants \( R,c > 0 \) such that

\[
\liminf_{n \to +\infty} \int_{B_R(y_n)} u_n^2 \geq c > 0.
\]

**Proof.** Suppose that b) does not occur. Using Lemma 3 it follows

\[
u_n \to 0 \quad \text{in} \quad L^p(\mathbb{R}^N) \quad \text{for} \quad p \in (2,2^*_s).
\]

Given \( \xi > 0 \), by \((f2)\) and \((f3)\), for some constant \( C_\xi > 0 \) we have

\[
0 \leq \int_{\mathbb{R}^N} f(u_n)u_n \leq \xi \int_{\mathbb{R}^N} u_n^2 + C_\xi \int_{\mathbb{R}^N} |u_n|^{q+1}.
\]

Using the fact that \( \{u_n\} \) is bounded in \( L^2(\mathbb{R}^N) \),\( u_n \to 0 \) in \( L^{q+1}(\mathbb{R}^N) \), and that \( \xi \) is arbitrary, we can conclude that

\[
\int_{\mathbb{R}^N} f(u_n)u_n \to 0.
\]

Recalling that \( \|u_n\|_{W_\varepsilon}^2 - \int_{\mathbb{R}^N} f(u_n)u_n = I_\varepsilon'(u_n)[u_n] = o_n(1) \), it follows that \( u_n \to 0 \) in \( W_\varepsilon \) \( \square \)

**Lemma 5.** Assume that \( V_\infty < \infty \) and let \( \{v_n\} \) be a \( (PS)_d \) sequence for \( I_\varepsilon \) in \( W_\varepsilon \) with \( v_n \rightharpoonup 0 \) in \( W_\varepsilon \). Then

\[
v_n \not\to 0 \quad \text{in} \quad W_\varepsilon \implies d \geq m(V_\infty)
\]

(recall that \( m(V_\infty) \) is the mountain pass level of \( E_{V_\infty} \), see (2.2)).
The last two equalities imply that

\[ t_n \geq 1 + \delta \quad \text{for all } n \in \mathbb{N}. \]  

(3.1)

Moreover, since \( t_n v_n \subset M_{V_{\infty}} \), we get

\[ \int_{\mathbb{R}^N} \left[ \frac{f(t_n v_n)^2}{t_n v_n} - \frac{f(v_n)^2}{v_n} \right] = \int_{\mathbb{R}^N} [V_{\infty} - V(x)] v_n^2 + o_n(1). \]

(3.2)

Given \( \xi > 0 \), by condition (V1) there exists \( R = R(\xi) > 0 \) such that

\[ V(x) \geq V_{\infty} - \xi \quad \text{for any } |x| \geq R. \]

Let \( C > 0 \) be such that \( \|v_n\|_{W_2} \leq C \). Since \( v_n \to 0 \in L^2(B_R(0)) \), we conclude by (3.2)

\[ \int_{\mathbb{R}^N} \left[ \frac{f(t_n v_n)}{t_n v_n} - \frac{f(v_n)}{v_n} \right] v_n^2 \leq \xi C V_{\infty} + o_n(1). \]

(3.3)

Since \( v_n \not\to 0 \) in \( W_2 \), we may invoke Lemma 4 to obtain \( \{y_n\} \subset \mathbb{R}^N \) and \( \bar{R}, c > 0 \) such that

\[ \int_{B_{\bar{R}}(y_n)} v_n^2 \geq c. \]

(3.4)

Defining \( \tilde{v}_n := v_n(\cdot + y_n) \), we may suppose that, up to a subsequence,

\[ \tilde{v}_n \rightharpoonup \bar{v} \text{ in } H^s(\mathbb{R}^N). \]

Moreover, in view of (3.4), there exists a subset \( \Omega \subset \mathbb{R}^N \) with positive measure such that \( \bar{v} > 0 \) in \( \Omega \). From (5), we can use (3.1) to rewrite (3.3) as

\[ 0 < \int_{\Omega} \left[ \frac{f((1 + \delta)\tilde{v}_n)}{(1 + \delta)\tilde{v}_n} - \frac{f(\tilde{v}_n)}{\tilde{v}_n} \right] \tilde{v}_n^2 \leq \xi C V_{\infty} + o_n(1), \quad \text{for any } \xi > 0. \]

Letting \( n \to \infty \) in the last inequality and applying Fatou’s Lemma, it follows that

\[ 0 < \int_{\Omega} \left[ \frac{f(1 + \delta)\bar{v})}{(1 + \delta)\bar{v}} - \frac{f(\bar{v})}{\bar{v}} \right] \bar{v}^2 \leq \xi C V_{\infty}, \quad \text{for any } \xi > 0. \]

which is an absurd, proving the claim.

Now, it is convenient to distinguish the following cases:

**Case 1:** \( \limsup_{n \to \infty} t_n = 1. \)

In this case there exists a subsequence, still denoted by \( \{t_n\} \), such that \( t_n \to 1 \). Thus,

\[ d + o_n(1) = I_{\xi}(v_n) \geq m(V_{\infty}) + I_{\xi}(v_n) - E_{V_{\infty}}(t_nv_n). \]

(3.5)
Recalling that
\[ I_{\varepsilon}(v_n) - E_{V_\infty}(t_nv_n) = \left(1 - \frac{t_n^2}{2}\right) \int_{\mathbb{R}^N} |(-\Delta)^{s/2}v_n|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(\varepsilon x)v_n^2 - \frac{t_n^2}{2} \int_{\mathbb{R}^N} V_\infty v_n^2 \]
+ \int_{\mathbb{R}^N} [F(t_nv_n) - F(v_n)],
and using the fact that \( \{v_n\} \) is bounded in \( W_\varepsilon \) by \( C > 0 \) together with the condition (V1), we get
\[ I_{\varepsilon}(v_n) - E_{V_\infty}(t_nv_n) \geq o_n(1) - C\xi + \int_{\mathbb{R}^N} [F(t_nv_n) - F(v_n)]. \]

Moreover, by the Mean Value Theorem,
\[ \int_{\mathbb{R}^N} [F(t_nv_n) - F(v_n)] = o_n(1), \]
therefore (3.5) becomes
\[ d + o_n(1) \geq m(V_\infty) - C\xi + o_n(1), \]
and taking the limit in \( n \), by the arbitrariness of \( \xi \), we have \( d \geq m(V_\infty). \)

**Case 2:** \( \limsup_{n \to \infty} t_n = t_0 < 1. \)

In this case up to a subsequence, still denoted by \( \{t_n\} \), we have \( t_n \to t_0 \) and \( t_n < 1 \) for all \( n \in \mathbb{N} \).

Since \( u \mapsto \frac{1}{2}f(u)u - F(u) \) is increasing, we have
\[ m(V_\infty) \leq \int_{\mathbb{R}^N} \left[ \frac{1}{2}f(t_nv_n)t_nv_n - F(t_nv_n) \right] \leq \int_{\mathbb{R}^N} \left[ \frac{1}{2}f(v_n)v_n - F(v_n) \right] \]
hence,
\[ m(V_\infty) \leq I_{\varepsilon}(v_n) - \frac{1}{2} I_{\varepsilon}'(v_n)[v_n] = d + o_n(1), \]
and again we easily conclude. \( \square \)

Now we are ready to give the desired compactness result.

**Proposition 1.** The functional \( I_{\varepsilon} \) in \( W_\varepsilon \) satisfies the \((PS)_c\) condition

1. at any level \( c < m(V_\infty) \), if \( V_\infty < \infty, \)
2. at any level \( c \in \mathbb{R} \), if \( V_\infty = \infty. \)

**Proof.** Let \( \{u_n\} \subset W_\varepsilon \) be such that \( I_{\varepsilon}(u_n) \to c \) and \( I_{\varepsilon}'(u_n) \to 0. \) By standard calculations, we can see that \( \{u_n\} \) is bounded in \( W_\varepsilon \). Thus there exists \( u \in W_\varepsilon \) such that, up to a subsequence, \( u_n \rightharpoonup u \) in \( W_\varepsilon \) and we see that \( I_{\varepsilon}'(u) = 0. \)

Defining \( v_n := u_n - u \), by [2] we know that \( \int_{\mathbb{R}^N} F(v_n) = \int_{\mathbb{R}^N} F(u_n) - \int_{\mathbb{R}^N} F(u) + o(1) \) and arguing as in [18] we have also \( I_{\varepsilon}'(v_n) \to 0. \) Then
\[ (3.6) \quad I_{\varepsilon}(v_n) = I_{\varepsilon}(u_n) - I_{\varepsilon}(u) + o_n(1) = c - I_{\varepsilon}(u) + o_n(1) =: d + o_n(1) \]
and \( \{v_n\} \) is a \((PS)_d\) sequence. By (f4),
\[ I_{\varepsilon}(u) = I_{\varepsilon}(u) - \frac{1}{2} I_{\varepsilon}'(u)[u] = \int_{\mathbb{R}^N} \left[ \frac{1}{2}f(u)u - F(u) \right] \geq 0, \]
and then, if \( V_\infty < \infty \) and \( c < m(V_\infty) \), by (3.6) we obtain
\[ d \leq c < m(V_\infty). \]
It follows from Lemma 5 that \( v_n \to 0 \), that is \( u_n \to u \) in \( W_\varepsilon \).

In the case \( V_\infty = \infty \) by the compact imbedding \( W_\varepsilon \hookrightarrow \hookrightarrow L^p(\mathbb{R}^N), 2 \leq p < 2^*_\varepsilon \), up to a subsequence, \( v_n \to 0 \) in \( L^p(\mathbb{R}^N) \) and by \((\text{f}2)\) and \((\text{f}3)\)

\[
\|v_n\|^2_{W_\varepsilon} = \int_{\mathbb{R}^N} f(u_n)v_n = o_n(1).
\]

This last equality implies that \( u_n \to u \) in \( W_\varepsilon \). \( \square \)

The next proposition is a direct consequence of the previous one, but for completeness we give the proof.

**Proposition 2.** The functional \( I_\varepsilon \) restricted to \( \mathcal{N}_\varepsilon \) satisfies the \( (PS)_c \) condition

1. at any level \( c < m(V_\infty) \), if \( V_\infty < \infty \),
2. at any level \( c \in \mathbb{R} \), if \( V_\infty = \infty \).

**Proof.** Let \( \{u_n\} \subset \mathcal{N}_\varepsilon \) be such that \( I_\varepsilon(u_n) \to c \) and for some sequence \( \{\lambda_n\} \subset \mathbb{R} \),

\[(3.7) \quad I_\varepsilon'(u_n) = \lambda_n J_\varepsilon'(u_n) + o_n(1), \]

where \( J_\varepsilon : W_\varepsilon \to \mathbb{R} \) is defined in \((2.3)\). Again we can deduce that \( \{u_n\} \) is bounded. Now

a) evaluating \((3.7)\) in \( u_n \) we get \( \lambda_n J_\varepsilon'(u_n)[u_n] = o_n(1) \),

b) evaluating \((3.7)\) in \( v \in T_{u_n}\mathcal{N}_\varepsilon \) we get \( J_\varepsilon'(u_n)[v] = 0 \).

Hence \( \lambda_n J_\varepsilon'(u_n) = o_n(1) \) and by \((3.7)\) we deduce \( I_\varepsilon'(u_n) = o_n(1) \). Then \( \{u_n\} \) is a \( (PS)_c \) sequence for \( I_\varepsilon \) and we conclude by Proposition 1. \( \square \)

**Corollary 1.** The constrained critical points of the functional \( I_\varepsilon \) on \( \mathcal{N}_\varepsilon \) are critical points of \( I_\varepsilon \) in \( W_\varepsilon \).

**Proof.** The standard proof follows by using similar arguments explored in the last proposition. \( \square \)

Now let us pass to the functional related to the autonomous problem \((A_\mu)\).

**Lemma 6** (Ground state for the autonomous problem). Let \( \{u_n\} \subset \mathcal{M}_\mu \) be a sequence satisfying \( E_\mu(u_n) \to m(\mu) \). Then, up to subsequences the following alternative holds:

a) \( \{u_n\} \) strongly converges in \( \text{H}^s(\mathbb{R}^N) \); 

b) there exists a sequence \( \{\tilde{y}_n\} \subset \mathbb{R}^N \) such that \( u_n(\cdot + \tilde{y}_n) \) strongly converges in \( \text{H}^s(\mathbb{R}^N) \).

In particular, there exists a minimizer \( w_\mu \geq 0 \) for \( m(\mu) \).

This result is known in the literature, but for completeness we give here the proof.

**Proof.** By the Ekeland Variational Principle we may suppose that \( \{u_n\} \) is a \( (PS)_{m(\mu)} \) sequence for \( E_\mu \). Thus going to a subsequence if necessary, we have that \( u_n \rightharpoonup u \) weakly in \( \text{H}^s(\mathbb{R}^N) \) and it is easy to verify that \( E_\mu'(u) = 0 \).

In case \( u \neq 0 \), then \( w_\mu := u \) is a ground state solution of the autonomous problem \((A_\mu)\), that is, \( E_\mu(w_\mu) = m(\mu) \).

In case \( u \equiv 0 \), applying the same arguments employed in the proof of Lemma 4, there exists a sequence \( \{\tilde{y}_n\} \subset \mathbb{R}^N \) such that

\[ u_n \to v \quad \text{in} \quad \text{H}^s(\mathbb{R}^N) \]

where \( v_n := u_n(\cdot + \tilde{y}_n) \). Therefore, \( \{v_n\} \) is also a \( (PS)_{m(\mu)} \) sequence of \( E_\mu \) and \( v \neq 0 \). It follows from the above arguments that setting \( w_\mu := v \) it is the ground state solution we were looking for.

In both cases, it is easy to see that \( w_\mu \geq 0 \) and the proof of the lemma is finished. \( \square \)
3.1. **Proof of Theorem 1.** By Lemma 2, the functional $I_\varepsilon$ has the geometry of the Mountain Pass Theorem in $W_\varepsilon$. Then by well known results there exists $\{u_n\} \subset W_\varepsilon$ satisfying
\[ I_\varepsilon(u_n) \to c_\varepsilon \text{ and } I'_\varepsilon(u_n) \to 0. \]

**case I:** $V_\infty = \infty$. By Proposition 1, $\{u_n\}$ strongly converges to some $u_\varepsilon$ in $H^s(\mathbb{R}^N)$, which satisfies
\[ I_\varepsilon(u_\varepsilon) = c_\varepsilon \text{ and } I'_\varepsilon(u_\varepsilon) = 0. \]

**case II:** $V_\infty < \infty$. In virtue of Proposition 1 we just need to show that $c_\varepsilon < m(V_\infty)$. Suppose without loss of generality that $0 \in M$, i.e.
\[ V(0) = V_0. \]

Let $\mu \in (V_0, V_\infty)$, so that
\[ m(V_0) < m(\mu) < m(V_\infty). \]

For $r > 0$ let $\eta_r$ a smooth cut-off function in $\mathbb{R}^N$ which equals 1 on $B_r$ and with support in $B_{2r}$. Let $w_r := \eta_r \omega_\mu$ and $t_r > 0$ such that $t_r w_r \in M_\mu$. If it were, for every $r > 0 : E_\mu(t_r w_r) \geq m(V_\infty)$, since $w_r \to \omega_\mu$ in $H^s(\mathbb{R}^N)$ for $r \to +\infty$, we would have $t_r \to 1$ and then
\[ m(V_\infty) \leq \liminf_{r \to +\infty} E_\mu(t_r w_r) = E_\mu(\omega_\mu) = m(\mu) \]

which contradicts (3.8). Then there exists $\overline{r} > 0$ such that $\phi := t_r w_r$ satisfies $E_\mu(\phi) < m(V_\infty)$.

Condition (V1) implies that for some $\overline{r} > 0$
\[ V(\varepsilon x) \leq \mu, \text{ for all } x \in \text{supp } \phi \text{ and } \varepsilon \leq \overline{r}, \]

so
\[ \int_{\mathbb{R}^N} V(\varepsilon x) \phi^2 \leq \mu \int_{\mathbb{R}^N} \phi^2 \text{ for all } \varepsilon \leq \overline{r} \]

and consequently
\[ I_\varepsilon(t\phi) \leq E_\mu(t\phi) \leq E_\mu(\phi) \text{ for all } t > 0. \]

Therefore $\max_{t>0} I_\varepsilon(t\phi) \leq E_\mu(\phi)$, and then
\[ c_\varepsilon < m(V_\infty) \]

which conclude the proof.

4. **The barycenter map**

Up to now $\varepsilon$ was fixed in our considerations. Now we deal with the case $\varepsilon \to 0^+$. The next result will be fundamental when we implement the “barycenter machinery” below.

**Proposition 3.** Let $\varepsilon_n \to 0$ and $\{u_n\} \subset N_{\varepsilon_n}$ be such that $I_{\varepsilon_n}(u_n) \to m(V_0)$. Then there exists a sequence $\{\tilde{y}_n\} \subset \mathbb{R}^N$ such that $u_n(\cdot + \tilde{y}_n)$ has a convergent subsequence in $H^s(\mathbb{R}^N)$. Moreover, up to a subsequence, $y_n := \varepsilon_n \tilde{y}_n \to y \in M$.

**Proof.** Arguing as in the proof of Lemma 4, we obtain a sequence $\{\tilde{y}_n\} \subset \mathbb{R}^N$ and constants $R, c > 0$ such that
\[ \liminf_{n \to \infty} \int_{B_R(\tilde{y}_n)} u_n^2 \geq c > 0. \]

Thus, if $v_n := u_n(\cdot + \tilde{y}_n)$, up to a subsequence, $v_n \to v \not= 0$ in $H^s(\mathbb{R}^N)$. Let $t_n > 0$ be such that $\tilde{v}_n := t_n v_n \in M_{V_0}$. Then,
\[ E_{V_0}(\tilde{v}_n) \to m(V_0). \]
Since \( \{t_n\} \) is bounded, so is the sequence \( \{ \tilde{v}_n \} \), thus for some subsequence, \( \tilde{v}_n \rightarrow \tilde{v} \) in \( H^s(\mathbb{R}^N) \). Moreover, reasoning as in [18], up to some subsequence still denoted with \( \{t_n\} \), we can assume that \( t_n \rightarrow t_0 > 0 \), and this limit implies that \( \tilde{v} \neq 0 \). From Lemma 6, \( \tilde{v}_n \rightarrow \tilde{v} \) in \( H^s(\mathbb{R}^N) \), and so \( v_n \rightarrow v \) in \( H^s(\mathbb{R}^N) \).

Now, we will show that \( \{ y_n \} := \{ \varepsilon_n \tilde{y}_n \} \) has a subsequence verifying \( y_n \rightarrow y \in M \). First note that the sequence \( \{ y_n \} \) is bounded in \( \mathbb{R}^N \). Indeed, assume by contradiction that (up to subsequences) \( |y_n| \rightarrow \infty \).

In case \( V = \infty \), the inequality
\[
\int_{\mathbb{R}^N} V(\varepsilon_n x + y_n) v_n^2 \leq \int_{\mathbb{R}^N} |(-\Delta)^{s/2} v_n|^2 + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n) v_n^2 = \int_{\mathbb{R}^N} f(v_n) v_n,
\]
and the Fatou’s Lemma imply
\[
\infty = \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} f(v_n) v_n
\]
which is an absurd, since the sequence \( f(v_n) v_n \) is bounded in \( L^1(\mathbb{R}^N) \).

Now let us consider the case \( V_\infty < \infty \). Since \( \tilde{v}_n \rightarrow \tilde{v} \) in \( H^s(\mathbb{R}^N) \) and \( V_0 < V_\infty \), we have
\[
m(V_0) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} \tilde{\nu}|^2 + \frac{V_0}{2} \int_{\mathbb{R}^N} \tilde{\nu}^2 - \int_{\mathbb{R}^N} F(\tilde{\nu})
\]
\[
< \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} \tilde{\nu}|^2 + \frac{V_\infty}{2} \int_{\mathbb{R}^N} \tilde{\nu}^2 - \int_{\mathbb{R}^N} F(\tilde{\nu})
\]
\[
\leq \liminf_{n \rightarrow \infty} \left[ \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} \tilde{v}_n|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n) \tilde{v}_n^2 - \int_{\mathbb{R}^N} F(\tilde{v}_n) \right],
\]
or equivalently
\[
m(V_0) < \liminf_{n \rightarrow \infty} \left[ \frac{t_n^2}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u_n|^2 + \frac{t_n^2}{2} \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n) u_n^2 - \int_{\mathbb{R}^N} F(t_n u_n) \right].
\]
The last inequality implies,
\[
m(V_0) < \liminf_{n \rightarrow \infty} I_{\varepsilon_n}(t_n u_n) \leq \liminf_{n \rightarrow \infty} I_{\varepsilon_n}(u_n) = m(V_0),
\]
which is a contradiction. Hence, \( \{ y_n \} \) has to be bounded and, up to a subsequence, \( y_n \rightarrow y \in \mathbb{R}^N \). If \( y \notin M \), then \( V(y) > V_0 \) and we obtain a contradiction arguing as above. Thus, \( y \in M \) and the Proposition is proved.

Let \( \delta > 0 \) be fixed and \( \eta \) be a smooth nonincreasing cut-off function defined in \( [0, \infty) \) by
\[
\eta(s) = \begin{cases} 
1 & \text{if } 0 \leq s \leq \delta/2 \\
0 & \text{if } s \geq \delta.
\end{cases}
\]

Let \( \Psi_{\varepsilon,y} \) be a ground state solution given in Lemma 6 of problem \( (A_\mu) \) with \( \mu = V_0 \) and for any \( y \in M \), let us define
\[
\Psi_{\varepsilon,y}(x) := \eta(|\varepsilon x - y|)w_{V_0}(\frac{\varepsilon x - y}{\varepsilon}).
\]
Let \( t_\varepsilon > 0 \) verifying \( \max_{t \geq 0} I_\varepsilon(t \Psi_{\varepsilon,y}) = I_\varepsilon(t_\varepsilon \Psi_{\varepsilon,y}) \), so that \( t_\varepsilon \Psi_{\varepsilon,y} \in \mathcal{N}_\varepsilon \), and let
\[
\Phi_\varepsilon : y \in M \mapsto t_\varepsilon \Psi_{\varepsilon,y} \in \mathcal{N}_\varepsilon.
\]
By construction, \( \Phi_\varepsilon(y) \) has compact support for any \( y \in M \) and \( \Phi_\varepsilon \) is a continuous map.

The next result will help us to define a map from \( M \) to a suitable sublevel in the Nehari manifold.
Lemma 7. The function $\Phi_\varepsilon$ satisfies

$$\lim_{\varepsilon \to 0^+} I_\varepsilon(\Phi_\varepsilon(y)) = m(V_0), \text{ uniformly in } y \in M.$$ 

Proof. Suppose by contradiction that the lemma is false. Then there exist $\delta_0 > 0$, $\{y_n\} \subset M$ and $\varepsilon_n \to 0^+$ such that

$$|I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - m(V_0)| \geq \delta_0. \tag{4.1}$$

Repeating the same arguments explored in [18] (see also [4]), it is possible to check that $t_{\varepsilon_n} \to 1$. From Lebesgue’s Theorem, we can check that

$$\lim_{n \to \infty} \|\Psi_{\varepsilon_n,y_n}\|_{\varepsilon_n}^2 = \|w_{V_0}\|_{V_0}^2$$

and

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} F(\Psi_{\varepsilon_n,y_n}) = \int_{\mathbb{R}^N} F(w_{V_0}).$$

Now, note that

$$I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = \frac{t_{\varepsilon_n}^2}{2} \int_{\mathbb{R}^N} \left|(-\Delta)^{s/2}(\eta(|\varepsilon_n z|)w_{V_0}(z))\right|^2 + \frac{t_{\varepsilon_n}^2}{2} \int_{\mathbb{R}^N} V(\varepsilon_n z + y_n)\eta(|\varepsilon_n z|)w_{V_0}(z)^2$$

$$- \int_{\mathbb{R}^N} F(t_{\varepsilon_n}\eta(|\varepsilon_n z|)w_{V_0}(z)).$$

Letting $n \to \infty$, we get $\lim_{n \to \infty} I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = E_{V_0}(w_{V_0}) = m(V_0)$, which contradicts (4.1). Thus the Lemma holds. □

Observe that by Lemma 7, $h(\varepsilon) := |I_\varepsilon(\Phi_\varepsilon(y)) - m(V_0)| = o(1)$ for $\varepsilon \to 0^+$ uniformly in $y$, and then $I_\varepsilon(\Phi_\varepsilon(y)) - m(V_0) \leq h(\varepsilon)$. In particular the set

$$\mathcal{N}_{\varepsilon}^{m(V_0) + h(\varepsilon)} := \left\{ u \in \mathcal{N}_{\varepsilon} : I_\varepsilon(u) \leq m(V_0) + h(\varepsilon) \right\}$$

is not empty, since for sufficiently small $\varepsilon$,

$$\forall y \in M : \Phi_\varepsilon(y) \in \mathcal{N}_{\varepsilon}^{m(V_0) + h(\varepsilon)}.$$ 

We are in a position now to define the barycenter map that will send a convenient sublevel in the Nehari manifold in a suitable neighborhood of $M$. From now on we fix a $\delta > 0$ in such a way that $M_{2\delta}$ and $M_{2\delta} \subset B_{\rho}$ and $\chi: \mathbb{R}^N \to \mathbb{R}^N$ be defined as

$$\chi(x) = \begin{cases} x & \text{if } |x| \leq \rho \\ \rho \frac{x}{|x|} & \text{if } |x| \geq \rho. \end{cases}$$

Finally, let us consider the so called barycenter map $\beta_\varepsilon$ defined on functions with compact support $u \in W_\varepsilon$ by

$$\beta_\varepsilon(u) := \frac{\int_{\mathbb{R}^N} \chi(\varepsilon x)u^2(x)}{\int_{\mathbb{R}^N} u^2(x)} \in \mathbb{R}^N.$$
Lemma 8. The function $\beta_\varepsilon$ satisfies
\[
\lim_{\varepsilon \to 0^+} \beta_\varepsilon(\Phi_\varepsilon(y)) = y, \quad \text{uniformly in } y \in M.
\]

Proof. Suppose, by contradiction, that the lemma is false. Then, there exist $\delta_0 > 0$, $\{y_n\} \subset M$ and $\varepsilon_n \to 0^+$ such that
\[
|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| \geq \delta_0. \tag{4.4}
\]
Using the definition of $\Phi_{\varepsilon_n}(y_n)$, $\beta_{\varepsilon_n}$ and $\eta$ given above, we have the equality
\[
\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = y_n + \int_{\mathbb{R}^N} \left[\chi(\varepsilon_n z + y_n) - y_n\right] \eta(|\varepsilon_n z|) w(z)^2. \tag{4.4}
\]
Using the fact that $\{y_n\} \subset M \subset B_\rho$ and the Lebesgue’s Theorem, it follows
\[
|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| = o_n(1),
\]
which contradicts (4.4) and the Lemma is proved. \qed

Lemma 9. We have
\[
\lim_{\varepsilon \to 0^+} \sup_{u \in \mathcal{N}_{\varepsilon_n}^{m(V_0) + h(\varepsilon_n)}} \inf_{y \in M_\delta} |\beta_\varepsilon(u) - y| = 0.
\]

Proof. Let $\{\varepsilon_n\}$ be such that $\varepsilon_n \to 0^+$. For each $n \in \mathbb{N}$, there exists $u_n \in \mathcal{N}_{\varepsilon_n}^{m(V_0) + h(\varepsilon_n)}$ such that
\[
\inf_{y \in M_\delta} |\beta_{\varepsilon_n}(u_n) - y| = \sup_{u \in \mathcal{N}_{\varepsilon_n}^{m(V_0) + h(\varepsilon_n)}} \inf_{y \in M_\delta} |\beta_{\varepsilon_n}(u) - y| + o_n(1).
\]
Thus, it suffices to find a sequence $\{y_n\} \subset M_\delta$ such that
\[
\lim_{n \to \infty} |\beta_{\varepsilon_n}(u_n) - y_n| = 0. \tag{4.5}
\]
Recalling that $u_n \in \mathcal{N}_{\varepsilon_n}^{m(V_0) + h(\varepsilon_n)} \subset \mathcal{N}_{\varepsilon_n}$, we have,
\[
m(V_0) \leq c_{\varepsilon_n} \leq I_{\varepsilon_n}(u_n) \leq m(V_0) + h(\varepsilon_n),
\]
so $I_{\varepsilon_n}(u_n) \to m(V_0)$. By Proposition 3, we get a sequence $\{\tilde{y}_n\} \subset \mathbb{R}^N$ such that $v_n := u_n(\cdot + \tilde{y}_n)$ converges in $H^s(\mathbb{R}^N)$ to some $v$ and $\{y_n\} := \{\varepsilon_n \tilde{y}_n\} \subset M_\delta$, for $n$ sufficiently large. Thus
\[
\beta_{\varepsilon_n}(u_n) = y_n + \int_{\mathbb{R}^N} \left[\chi(\varepsilon_n z + y_n) - y_n\right] v_n(z)^2. \tag{4.5}
\]
Since $v_n \to v$ in $H^s(\mathbb{R}^N)$, it is easy to check that the sequence $\{y_n\}$ verifies (4.5). \qed

In virtue of Lemma 9, there exists $\varepsilon^* > 0$ such that
\[
\forall \varepsilon \in (0, \varepsilon^*]: \sup_{u \in \mathcal{N}_{\varepsilon_n}^{m(V_0) + h(\varepsilon)}} d(\beta_{\varepsilon}(u), M_\delta) < \delta/2.
\]
Define now
\[
M^+ := \left\{ x \in \mathbb{R}^N : d(x, M) \leq 3\delta/2 \right\}
\]
so that $M$ and $M^+$ are homotopically equivalent.
Now, reducing \( \varepsilon^* > 0 \) if necessary, we can assume that Lemma 8, Lemma 9 and (4.3) hold. Then by standard arguments the composed map

\[
M \xrightarrow{\Phi_\varepsilon} \mathcal{N}_\varepsilon^{m(V_0)+h(\varepsilon)} \xrightarrow{\beta_\varepsilon} M^+
\]

is homotopic to the inclusion map.

In case \( V_\infty < \infty \), we eventually reduce \( \varepsilon^* \) in such a way that also the Palais-Smale condition is satisfied in the interval \((m(V_0), m(V_0) + h(\varepsilon))\), see Proposition 2.

5. Proof of Theorem 2

5.1. Existence. By (4.6) and well known properties of the category, we get

\[
\text{cat}(\mathcal{N}_\varepsilon^{m(V_0)+h(\varepsilon)}) \geq \text{cat}_M(M),
\]

and the Ljusternik-Schnirelman theory (see e.g. [21]) implies that \( I_\varepsilon \) has at least \( \text{cat}_M(M) \) critical points on \( \mathcal{N}_\varepsilon \).

To obtain another solution we use the same ideas of [6]. First note that, since \( M \) is not contractible, the set \( \mathcal{A} := \Phi_\varepsilon(M) \) cannot be contractible in \( \mathcal{N}_\varepsilon^{m(V_0)+h(\varepsilon)} \). Moreover \( \mathcal{A} \) is compact.

For \( u \in W_\varepsilon \setminus \{0\} \) we denote with \( t_\varepsilon(u) > 0 \) the unique positive number such that \( t_\varepsilon(u)u \in \mathcal{N}_\varepsilon \).

Let \( u^* \in W_\varepsilon \) be such that \( u^* \geq 0 \), and \( I_\varepsilon(t_\varepsilon(u^*)u^*) > m(V_0) + h(\varepsilon) \). Consider the cone

\[
\mathcal{C} := \left\{ tu^* + (1 - t)u : t \in [0, 1], u \in \mathcal{A} \right\}
\]

and note that \( 0 \notin \mathcal{C} \), since functions in \( \mathcal{C} \) have to be positive on a set of nonzero measure. Clearly it is compact and contractible. Let

\[
t_\varepsilon(\mathcal{C}) := \left\{ t_\varepsilon(w)w : w \in \mathcal{C} \right\}
\]

be its projection on \( \mathcal{N}_\varepsilon \), which is compact as well, and

\[
c := \max_{t_\varepsilon(\mathcal{C})} I_\varepsilon > m(V_0) + h(\varepsilon).
\]

Since \( \mathcal{A} \subset t_\varepsilon(\mathcal{C}) \subset \mathcal{N}_\varepsilon \) and \( t_\varepsilon(\mathcal{C}) \) is contractible in \( \mathcal{N}^c_\varepsilon := \left\{ u \in \mathcal{N}_\varepsilon : I_\varepsilon(u) \leq c \right\} \), we infer that also \( \mathcal{A} \) is contractible in \( \mathcal{N}^c_\varepsilon \).

Summing up, we have a set \( \mathcal{A} \) which is contractible in \( \mathcal{N}^c_\varepsilon \) but not in \( \mathcal{N}_\varepsilon^{m(V_0)+h(\varepsilon)} \), where \( c > m(V_0) + h(\varepsilon) \). This is only possible, since \( I_\varepsilon \) satisfies the Palais-Smale condition, if there is a critical level between \( m(V_0) + h(\varepsilon) \) and \( c \).

By Corollary 1, we conclude the proof of statements about the existence of solutions in Theorem 2.

5.2. Concentration of the maximum points. The next two lemmas play a role in the study of the behavior of the maximum points of the solutions. In the proof of the next lemma, we adapted some arguments found in [19], which are related with the Moser iteration method [28].

Lemma 10. Assume the conditions (V1) and (f1)-(f5). Let \( v_n \in H^s(\mathbb{R}^N) \) be such that

\[
\begin{cases}
(-\Delta)^s v_n + V_n(x) v_n = f(v_n) & \text{in } \mathbb{R}^N, \\
v_n(x) > 0, \ x \in \mathbb{R}^N,
\end{cases}
\]

where \( V_n(x) := V(\varepsilon_n x + \varepsilon_n y_n) \), and suppose that \( v_n \rightharpoonup v \) in \( H^s(\mathbb{R}^N) \) with \( v \neq 0 \). Then \( v_n \in L^\infty(\mathbb{R}^N) \) and there exists \( C > 0 \) such that \( |v_n|_\infty \leq C \) for all \( n \in \mathbb{N} \). Furthermore

\[
\lim_{|x| \to \infty} v_n(x) = 0 \text{ uniformly in } n.
\]
Proof. For any $R > 0$, $0 < r \leq R/2$, let $\eta \in C^\infty(\mathbb{R}^N)$, $0 \leq \eta \leq 1$ with $\eta(x) = 1$ if $|x| \geq R$ and $\eta(x) = 0$ if $|x| \leq R - r$ and $|(-\Delta)^{s/2} \eta| \leq 2/r$. Note that by (3) we obtain the following growth condition for $f$:

\[
(5.1) \quad f(u) \leq \xi |u| + C_{\xi} |u|^{2s-1}.
\]

For each $n \in \mathbb{N}$ and for $L > 0$, define

\[
v_{L,n}(x) = \begin{cases} v_n(x) & \text{if } v_n(x) \leq L \\ L & \text{if } v_n(x) \geq L, \end{cases}
\]

and

\[
z_{L,n} := \eta^2 v_{L,n}^{2(\sigma-1)} v_n \quad \text{and} \quad w_{L,n} := \eta v_n v_{L,n}^{\sigma-1}
\]

with $\sigma > 1$ to be determined later.

Taking $z_{L,n}$ as a test function, we obtain

\[
\int_{\mathbb{R}^N} \eta^2 v_{L,n}^{2(\sigma-1)} |(-\Delta)^{s/2} v_n|^2 \leq -2(\beta - 1) \int_{\mathbb{R}^N} v_{L,n}^{2\sigma-1} \eta^2 v_n |(-\Delta)^{s/2} v_n(\Delta)^{s/2} v_{L,n}|^2 + \int_{\mathbb{R}^N} f(v_n) \eta^2 v_n v_{L,n}^{2(\sigma-1)} - \int_{\mathbb{R}^N} v_n v_n^{\sigma} v_{L,n}^{2(\sigma-1)} - 2 \int_{\mathbb{R}^N} \eta v_{L,n}^{2(\sigma-1)} v_n |(-\Delta)^{s/2} v_n(\Delta)^{s/2} \eta|.
\]

By (5.1) and for a $\xi$ sufficiently small, we have the following inequality

\[
\int_{\mathbb{R}^N} \eta^2 v_{L,n}^{2(\sigma-1)} |(-\Delta)^{s/2} v_n|^2 \leq C_{\xi} \int_{\mathbb{R}^N} v_n^{2\sigma-1} v_{L,n}^{2(\sigma-1)} - 2 \int_{\mathbb{R}^N} \eta v_{L,n}^{2(\sigma-1)} v_n |(-\Delta)^{s/2} v_n(\Delta)^{s/2} \eta|.
\]

For each $\varepsilon > 0$, using the Young’s inequality we get

\[
\int_{\mathbb{R}^N} \eta^2 v_{L,n}^{2(\sigma-1)} |(-\Delta)^{s/2} v_n|^2 \leq C_{\xi} \int_{\mathbb{R}^N} v_n^{2\sigma-1} v_{L,n}^{2(\sigma-1)} + 2\varepsilon \int_{\mathbb{R}^N} \eta^2 v_{L,n}^{2(\sigma-1)} |(-\Delta)^{s/2} v_n|^2 + 2C_{\xi} \int_{\mathbb{R}^N} v_n v_{L,n}^{2(\sigma-1)} |(-\Delta)^{s/2} \eta|^2.
\]

Choosing $\varepsilon > 0$ sufficiently small,

\[
(5.2) \quad \int_{\mathbb{R}^N} \eta^2 v_{L,n}^{2(\sigma-1)} |(-\Delta)^{s/2} v_n|^2 \leq C \int_{\mathbb{R}^N} v_n^{2\sigma-1} v_{L,n}^{2(\sigma-1)} + C \int_{\mathbb{R}^N} \eta v_{L,n}^{2(\sigma-1)} v_n |(-\Delta)^{s/2} \eta|^2.
\]

Now, from Sobolev embedding and Holder inequalities

\[
(5.3) \quad |w_{L,n}|_{2s}^2 \leq C \beta \left[ \int_{\mathbb{R}^N} v_n^{2(\sigma-1)} |(-\Delta)^{s/2} \eta|^2 + \int_{\mathbb{R}^N} \eta^2 v_{L,n}^{2(\sigma-1)} |(-\Delta)^{s/2} v_n|^2 \right].
\]

Using (5.2) in (5.3), we have

\[
(5.4) \quad |w_{L,n}|_{2s}^2 \leq C \sigma^2 \left[ \int_{\mathbb{R}^N} v_n^{2(\sigma-1)} |(-\Delta)^{s/2} \eta|^2 + \int_{\mathbb{R}^N} v_n^{2(\sigma-1)} v_{L,n}^{2(\sigma-1)} |(-\Delta)^{s/2} \eta|^2 \right].
\]

We claim that $v_n \in L^{2s^2/2}(\mathbb{R}^N \setminus B_R)$ for $R$ large enough and uniformly in $n$. In fact, let $\sigma = 2s/2$. From (5.4), we have

\[
|w_{L,n}|_{2s}^2 \leq C \sigma^2 \left[ \int_{\mathbb{R}^N} v_n^{2(\sigma-1)} |(-\Delta)^{s/2} \eta|^2 + \int_{\mathbb{R}^N} v_n^{2(\sigma-1)} v_{L,n}^{2(\sigma-1)} |(-\Delta)^{s/2} \eta|^2 \right]
\]

or equivalently

\[
|w_{L,n}|_{2s}^2 \leq C \sigma^2 \left[ \int_{\mathbb{R}^N} v_n^{2(\sigma-1)} |(-\Delta)^{s/2} \eta|^2 + \int_{\mathbb{R}^N} v_n^{2(\sigma-1)} v_{L,n}^{2(\sigma-1)} |(-\Delta)^{s/2} \eta|^2 \right].
\]
Using the H"older inequality with exponent $2^*_s/2$ and $2^*_s/(2^*_s - 2)$

$$|w_{L,n}|^2_{2^*_s} \leq C \sigma^2 \int_{R^N} v_{L,n}^2 \left( \int_{R^N} (\Delta)^{s/2} |\eta|^2 \right)^{2^*_s/2} + C \sigma^2 \left( \int_{R^N \setminus B_{R/2}} v_{L,n}^2 \right)^{2^*_s/(2^*_s - 2)}.$$  

From the definition of $w_{L,n}$ we have

$$\left( \int_{R^N \setminus B_R} v_{L,n}^2 \right)^{2^*_s/2} \leq C \sigma^2 \int_{R^N} v_{L,n}^2 \left( \int_{R^N \setminus B_{R/2}} v_{L,n}^2 \right)^{2^*_s/(2^*_s - 2)} + C \sigma^2 \left( \int_{R^N \setminus B_{R/2}} v_{L,n}^2 \right)^{2^*_s/(2^*_s - 2)}.$$  

Since $v_n \to v$ in $H^s(\mathbb{R}^N)$, for $R$ sufficiently large, we conclude

$$\int_{R^N \setminus B_{R/2}} v_{L,n}^2 \leq \varepsilon \text{ uniformly in } n.$$  

Hence

$$\left( \int_{R^N \setminus B_R} v_{L,n}^2 \right)^{2^*_s/2} \leq C \sigma^2 \int_{R^N} v_{L,n}^2 \left( \int_{R^N \setminus B_{R/2}} v_{L,n}^2 \right)^{2^*_s/(2^*_s - 2)}$$  

or equivalently

$$\left( \int_{R^N \setminus B_R} v_{L,n}^2 \right)^{2^*_s/2} \leq C \sigma^2 \int_{R^N} v_{L,n}^2 \leq K < \infty.$$  

Using the Fatou’s lemma in the variable $L$, we have

$$\int_{R^N \setminus B_R} v_{L,n}^2 < \infty$$  

and therefore the claim holds.

Next, we note that if $\sigma = 2^*_s(t - 1)/2t$ with $t = 2^*_s/(2^*_s - 2)$, then $\sigma > 1$, $2t/(t - 1) > 2^*_s$ and $v_n \in L^{2^*_s/(1 - (R^N \setminus B_{R^-}))}$.

Returning to inequality (5.4), we obtain

$$|w_{L,n}|^{2^*_s} \leq C \sigma^2 \left[ \int_{B_R \setminus B_{R^{-}}} v_{n}^{2\sigma} \right] + \int_{R^N \setminus B_{R^{-}}} v_{L,n}^{2\sigma} \left( \int_{R^N \setminus B_{R^{-}}} v_{L,n}^{2\sigma} \right)^{-1/(t - 1)}$$  

or equivalently

$$|w_{L,n}|^{2^*_s} \leq C \sigma^2 \left[ \int_{B_R \setminus B_{R^{-}}} v_{n}^{2\sigma} + \int_{R^N \setminus B_{R^{-}}} v_{n}^{2\sigma} + \int_{R^N \setminus B_{R^{-}}} v_{n}^{2\sigma} \right].$$

Using the Hölder’s inequality with exponent $t/(t - 1)$ and $t$, we get

$$|w_{L,n}|^{2^*_s} \leq C \sigma^2 \left[ \int_{R^N \setminus B_{R^{-}}} v_{n}^{2\sigma/(t - 1)} \right]^{1/t} \left[ \int_{R^N \setminus B_{R^{-}}} v_{n}^{2\sigma/(t - 1)} \right]^{1/(t - 1)} + \left[ \int_{R^N \setminus B_{R^{-}}} v_{n}^{2\sigma/(t - 1)} \right]^{1/t} \left[ \int_{R^N \setminus B_{R^{-}}} v_{n}^{2\sigma/(t - 1)} \right]^{1/(t - 1)}.$$  

Since that $(2^*_s - 2)t = 2^*_s$, we conclude

$$|w_{L,n}|^{2^*_s} \leq C \sigma^2 \left( \int_{R^N \setminus B_{R^{-}}} v_{n}^{2\sigma/(t - 1)} \right)^{(t - 1)/t}.$$
Note that
\[ |v_{L,n}|_{L^{2\sigma}(\mathbb{R}^N \setminus B_{r})}^{2\sigma} \leq \left( \int_{\mathbb{R}^N \setminus B_{r}} v_{L,n}^{2\sigma} \right)^{2/2\sigma} \leq \left( \int_{\mathbb{R}^N} \eta^2 v_{L,n}^{2\sigma} \right)^{(t-1)/t}, \]
\[ = |v_{L,n}|_{L^{2\sigma(t-1)}/(t-1)}^{2\sigma} \leq C\sigma^2 \left( \int_{\mathbb{R}^N \setminus B_{r}} v_{L,n}^{2\sigma(t-1)}/(t-1) \right)^{(t-1)/t}, \]
\[ = C\sigma^2 |v_n|_{L^{2\sigma(t-1)}/(t-1)}^{2\sigma}(\mathbb{R}^N \setminus B_{r}). \]

Applying Fatou’s lemma
\[ |v_n|_{L^{2\sigma}(\mathbb{R}^N \setminus B_{r})} \leq C\sigma^2 |v_n|_{L^{2\sigma(t-1)}/(t-1)}^{2\sigma}(\mathbb{R}^N \setminus B_{r}) \]

Considering \( \chi = 2^*(t-1)/2t \), \( \zeta = 2t/(t-1) \) and the last inequality, we can prove that
\[ |v_n|_{L^{\chi m+1}(\mathbb{R}^N \setminus B_{r})} \leq C\sum_{i=1}^{m} \chi^{-i} \sum_{i=1}^{m} i\chi^{-i} |v_n|_{L^{2\sigma}(\mathbb{R}^N \setminus B_{r})}, \]
which implies
\[ |v_n|_{L^{\infty}(\mathbb{R}^N \setminus B_{r})} \leq C|v_n|_{L^{2\sigma}(\mathbb{R}^N \setminus B_{r})}. \]

Using again the convergence of \( \{v_n\} \) to \( v \) in \( H^s(\mathbb{R}^N) \), for \( \xi > 0 \) fixed there exists \( R > 0 \) such that
\[ |v_n|_{L^{\infty}(\mathbb{R}^N \setminus B_{r})} < \xi \quad \text{for all } n \in \mathbb{N}. \]

Thus,
\[ \lim_{|x| \to \infty} v_n(x) = 0 \quad \text{uniformly in } n \]
and the proof of the Lemma is finished. \( \square \)

Finally we have

**Lemma 11.** There exists \( \delta > 0 \) such that \( |v_n|_{\infty} \geq \delta \), for every \( n \in \mathbb{N} \).

**Proof.** Suppose that \( |v_n|_{\infty} \to 0 \). It follows by (5) that there exists \( n_0 \in \mathbb{N} \) such that,
\[ \frac{f(|v_n|_{\infty})}{|v_n|_{\infty}} < \frac{V_0}{2}, \quad \text{for } n \geq n_0. \]

Hence
\[ \int_{\mathbb{R}^N} |(-\Delta)^{s/2} v_n|^2 + \int_{\mathbb{R}^N} V_0 v_n^2 \leq \int_{\mathbb{R}^N} f(|v_n|_{\infty}) v_n^2 \leq \frac{V_0}{2} \int_{\mathbb{R}^N} v_n^2, \]
thus \( |v_n|_{V_0} = 0 \) for \( n \geq n_0 \), which is an absurd, because \( v_n \neq 0 \) for every \( n \in \mathbb{N} \). \( \square \)

For what concerns the behavior of the maximum points when \( \varepsilon \to 0^+ \), let \( u_{\varepsilon_n} \) be a solution of problem \( (P_{\varepsilon_n}) \). Then \( v_n(x) = u_{\varepsilon_n}(x + \tilde{y}_n) \in H^s(\mathbb{R}^N) \) is a solution of
\[ \begin{cases} (-\Delta)^s v_n + V_n(x) v_n = f(v_n) & \text{in } \mathbb{R}^N, \\ v_n(x) > 0, & x \in \mathbb{R}^N, \end{cases} \]
with \( V_n(x) := V(\varepsilon_n x + \varepsilon_n \tilde{y}_n) \) and \( \{\tilde{y}_n\} \subset \mathbb{R}^N \) are those given in Proposition 3. Moreover, up to a subsequence, \( v_n \to v \) in \( H^s(\mathbb{R}^N) \) and \( y_n \to y \) in \( M \), where \( y_n = \varepsilon_n \tilde{y}_n \). By Lemma 10 and Lemma 11, the global maxima \( p_n \) of \( v_n \) are all in \( B_R \) for some \( R > 0 \). Thus, the global maximum of \( u_{\varepsilon_n} \) is \( z_\varepsilon = p_n + \tilde{y}_n \) and therefore
\[ \varepsilon_n z_\varepsilon = \varepsilon_n p_n + \varepsilon_n \tilde{y}_n = \varepsilon_n p_n + y_n. \]

Since \( \{p_n\} \) is bounded, we have
\[ \lim_{n \to \infty} V(\varepsilon_n z_\varepsilon) = V_0. \]
We conclude the proof of Theorem 2 in virtue of the considerations made at the beginning of Section 2.

6. PROOF OF THEOREM 3

Before prove the theorem we first recall some basic facts of Morse theory and fix some notations.

For a pair of topological spaces \((X,Y)\), \(Y \subset X\), let \(H_*(X,Y)\) be its singular homology with coefficients in some field \(\mathbb{F}\) (from now on omitted) and

\[
\mathcal{P}_*(X,Y) = \sum_k \dim H_k(X,Y) t^k
\]

the Poincaré polynomial of the pair. If \(Y = \emptyset\), it will be always omitted in the objects which involve the pair. Recall that if \(H\) is an Hilbert space, \(I : H \to \mathbb{R}\) a \(C^2\) functional and \(u\) an isolated critical point with \(I(u) = c\), the polynomial Morse index of \(u\) is

\[
\mathcal{I}_I(u) = \sum_k \dim C_k(I,u) t^k
\]

where \(C_k(I,u) = H_k(I^c \cap U,(I^c \setminus \{u\}) \cap U)\) are the critical groups. Here \(I^c = \{ u \in H : I(u) \leq c \}\) and \(U\) is a neighborhood of the critical point \(u\). The multiplicity of \(u\) is the number \(\mathcal{I}_I(u)\).

It is known that for a non-degenerate critical point \(u\) (that is, the selfadjoint operator associated to \(I''(u)\) is an isomorphism) it is \(\mathcal{I}_I(u) = \lambda (u)\), where \(\lambda (u)\) is the (numerical) Morse index of \(u\): the maximal dimension of the subspaces where \(I''(u)[\cdot,\cdot]\) is negative definite.

6.1. Proof of Theorem 3. First note that \(I_\varepsilon\) is of class \(C^2\) and for \(u, v, w \in W_\varepsilon\)

\[
I'_\varepsilon(u)[v,w] = \int_{\mathbb{R}^N} (-\Delta)^{s/2} v (-\Delta)^{s/2} w + \int_{\mathbb{R}^N} V(\varepsilon x)vw - \int_{\mathbb{R}^N} f'(u)vw
\]

hence \(I''_\varepsilon(u)\) is represented by the operator

\[
I_\varepsilon(u) := R(u) - K(u) : W_\varepsilon \to W'_\varepsilon
\]

where \(R(u)\) is the Riesz isomorphism and \(K(u)\) is compact. Indeed let \(v_n \to 0\) and \(w \in W_\varepsilon\); given \(\xi > 0\), by (2) and (3), for some constant \(C_\varepsilon > 0\) we have

\[
\int_{\mathbb{R}^N} |f'(u)v_n w| \leq \xi \int_{\mathbb{R}^N} |v_n w| + C_\varepsilon \int_{\mathbb{R}^N} |u|^{q-1}|v_n w|
\]

and using that \(v_n \to 0\) and the fact that \(\xi\) is arbitrary, we deduce

\[
\|K(u)[v_n]\| = \sup_{\|w\|_{W_\varepsilon} = 1} \left| \int_{\mathbb{R}^N} f'(u)v_n w \right| \to 0.
\]

Now for \(a \in (0, +\infty]\), let

\[
I^a_\varepsilon := \left\{ u \in W_\varepsilon : I_\varepsilon(u) \leq a \right\}, \quad \mathcal{N}_\varepsilon^a := \mathcal{N}_\varepsilon \cap I^a_\varepsilon
\]

\[
\mathcal{K}_\varepsilon := \left\{ u \in W_\varepsilon : I'_\varepsilon(u) = 0 \right\}, \quad \mathcal{K}_\varepsilon^a := \mathcal{K}_\varepsilon \cap I^a_\varepsilon, \quad (\mathcal{K}_\varepsilon)_a := \left\{ u \in \mathcal{K}_\varepsilon : I_\varepsilon(u) > a \right\}.
\]

In the remaining part of this section we will follow [3, 6]. Let \(\varepsilon^* > 0\) small as at the end of Section 4 and let \(\varepsilon \in (0, \varepsilon^*)\) be fixed. In particular \(I_\varepsilon\) satisfies the Palais-Smale condition. We are going to prove that \(I_\varepsilon\) restricted to \(\mathcal{N}_\varepsilon\) has at least \(2\mathcal{P}_1(M) - 1\) critical points (for small \(\varepsilon\)). Then Theorem 3 will follow by Corollary 1.
We can assume, of course, that there exists a regular value $b_\varepsilon^* > m(V_0)$ for the functional $I_\varepsilon$. Moreover, possibly reducing $\varepsilon^*$, we can assume that, see (4.2),

$$\Phi_\varepsilon : M \to N_{\varepsilon}^{m(V_0) + h(\varepsilon)} \subset N_{\varepsilon}^{b_\varepsilon^*}.$$ 

Since $\Phi_\varepsilon$ is injective, it induces injective homomorphisms in the homology groups, then

$$\dim H_k(M) \leq \dim H_k(N_{\varepsilon}^{b_\varepsilon^*})$$ 

and consequently

$$P_t(N_{\varepsilon}^{b_\varepsilon^*}) = P_t(M) + Q_t, \quad Q_t \in \mathbb{P},$$ 

where hereafter $\mathbb{P}$ denotes the set of polynomials with non-negative integer coefficients.

The following result is analogous to [6, Lemma 5.2]; we omit the proof.

**Lemma 12.** Let $r \in (0, m(V_0))$ and $a \in (r, +\infty]$ a regular level for $I_\varepsilon$. Then

$$P_t(I_\varepsilon^a, I_\varepsilon^r) = tP_t(N_{\varepsilon}^a).$$

In particular we have the following

**Corollary 2.** Let $r \in (0, m(V_0))$. Then

$$P_t(I_\varepsilon^r, I_\varepsilon^r) = t\left(P_t(M) + Q_t\right), \quad Q_t \in \mathbb{P},$$

$$P_t(W_{\varepsilon}, I_\varepsilon^r) = t.$$

**Proof.** The first identity follows by (6.1) and (6.2) by choosing $a = b_\varepsilon^*$. The second one follows by (6.2) with $a = +\infty$ and noticing that the Nehari manifold $N_{\varepsilon}$ is contractible. □

To deal with critical points above the level $b_\varepsilon^*$, we need also the following

**Lemma 13.** It holds

$$P_t(W_{\varepsilon}, I_\varepsilon^{b_\varepsilon^*}) = t^2 \left(P_t(M) + Q(t) - 1\right), \quad Q_t \in \mathbb{P}.$$

**Proof.** The proof is purely algebraic and goes exactly as in [6, Lemma 5.6], see also [3, Lemma 2.4]. □

As a consequence of these facts we have

**Corollary 3.** Suppose that the set $\mathcal{K}_\varepsilon$ is discrete. Then

$$\sum_{u \in \mathcal{K}_{\varepsilon}^{b_\varepsilon^*}} I_\varepsilon(u) = t\left(P_t(M) + Q(t)\right) + (1 + t)Q_1(t)$$

and

$$\sum_{u \in \mathcal{K}_{\varepsilon}^{a}} I_\varepsilon(u) = t^2 \left(P_t(M) + Q(t) - 1\right) + (1 + t)Q_2(t),$$

where $Q, Q_1, Q_2 \in \mathbb{P}.$

**Proof.** Indeed the Morse theory gives

$$\sum_{u \in \mathcal{K}_{\varepsilon}^{b_\varepsilon^*}} I_\varepsilon(u) = P_t(I_\varepsilon^{b_\varepsilon^*}, I_\varepsilon^r) + (1 + t)Q_1(t)$$

and

$$\sum_{u \in \mathcal{K}_{\varepsilon}^{a}} I_\varepsilon(u) = P_t(W_{\varepsilon}, I_\varepsilon^{b_\varepsilon^*}) + (1 + t)Q_2(t)$$

so that, by using Corollary 2 and Lemma 13, we easily conclude □
Finally, by Corollary 3 we get
\[ \sum_{u \in K_{\varepsilon}} I_{\varepsilon}(u) = tP_{t}(M) + t^{2}\left(P_{t}(M) - 1\right) + t(1 + t)Q(t) \]
for some \( Q \in \mathbb{P} \). We easily deduce that, if the critical points of \( I_{\varepsilon} \) are non-degenerate, then they are at least \( 2P_{t}(M) - 1 \), if counted with their multiplicity.

The proof of Theorem 3 is thereby complete.

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