ADDITIVE $\rho$-FUNCTIONAL INEQUALITIES
IN $\beta$-HOMOGENEOUS $F$-SPACES

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Abstract. In this paper, we solve the additive $\rho$-functional inequalities
\begin{equation}
\|f(2x-y) + f(y-x) - f(x)\| \leq \|\rho(f(x+y) - f(x) - f(y))\|,
\end{equation}
where $\rho$ is a fixed complex number with $|\rho| < 1$, and
\begin{equation}
\|f(x+y) - f(x) - f(y)\| \leq \|\rho(f(2x-y) + f(y-x) - f(x))\|,
\end{equation}
where $\rho$ is a fixed complex number with $|\rho| < \frac{1}{2}$.

Using the direct method, we prove the Hyers-Ulam stability of the additive $\rho$-
functional inequalities (0.1) and (0.2) in $\beta$-homogeneous $F$-spaces.

1. Introduction and Preliminaries

The stability problem of functional equations originated from a question of Ulam
[23] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x) + f(y)$ is called the Cauchy equation. In
particular, every solution of the Cauchy equation is said to be an additive mapping.

Hyers [8] gave a first affirmative partial answer to the question of Ulam for Banach
spaces. Hyers’ Theorem was generalized by Aoki [2] for additive mappings and by
Rassias [14] for linear mappings by considering an unbounded Cauchy difference. A
generalization of the Rassias theorem was obtained by Găvruta [7] by replacing the
unbounded Cauchy difference by a general control function in the spirit of Rassias’
approach. The stability of quadratic functional equation was proved by Skof [22]
for mappings $f : E_1 \to E_2$, where $E_1$ is a normed space and $E_2$ is a Banach space.

Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain

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$E_1$ is replaced by an Abelian group. The stability problems of various functional
equations have been extensively investigated by a number of authors (see [1, 3, 4, 6,
9, 10, 11, 12, 13, 15, 17, 18, 19, 20, 21, 24, 25]).

**Definition 1.1.** Let $X$ be a (complex) linear space. A nonnegative valued function
$\| \cdot \|$ is an $F$-norm if it satisfies the following conditions:

1. \( \|x\| = 0 \) if and only if $x = 0$;
2. \( \|\lambda x\| = \|x\| \) for all $x \in X$ and all $\lambda$ with $|\lambda| = 1$;
3. \( \|x + y\| \leq \|x\| + \|y\| \) for all $x, y \in X$;
4. \( \|\lambda_n x\| \to 0 \) provided $\lambda_n \to 0$;
5. \( \|\lambda x_n\| \to 0 \) provided $x_n \to 0$.

Then $(X, \| \cdot \|)$ is called an $F^*$-space. An $F$-space is a complete $F^*$-space.

An $F$-norm is called $\beta$-homogeneous ($\beta > 0$) if \( \|tx\| = |t|^\beta \|x\| \) for all $x \in X$ and
all $t \in \mathbb{C}$ and $(X, \| \cdot \|)$ is called a $\beta$-homogeneous $F$-space (see [16]).

In Section 2, we solve the additive $\rho$-functional inequality (0.1) and prove the
Hyers-Ulam stability of the additive $\rho$-functional inequality (0.1) in $\beta$-homogeneous
$F$-space.

In Section 3, we solve the additive $\rho$-functional inequality (0.2) and prove the
Hyers-Ulam stability of the additive $\rho$-functional inequality (0.2) in $\beta$-homogeneous
$F$-space.

Throughout this paper, let $\beta_1, \beta_2$ be positive real numbers with $\beta_1 \leq 1$ and
$\beta_2 \leq 1$. Assume that $X$ is a $\beta_1$-homogeneous $F$-space with norm $\| \cdot \|$ and that $Y$
is a $\beta_2$-homogeneous $F$-space with norm $\| \cdot \|$.

2. **ADDITIVE $\rho$-FUNCTIONAL INEQUALITY (0.1)
IN $\beta$-HOMOGENEOUS $F$-SPACES**

Throughout this section, assume that $\rho$ is a complex number with $|\rho| < 1$.

We solve and investigate the additive $\rho$-functional inequality (0.1) in $\beta$-homogeneous
$F$-spaces.

**Lemma 2.1.** If a mapping $f : X \to Y$ satisfies

\[
\|f(2x - y) + f(y - x) - f(x)\| \leq \rho (f(x + y) - f(x) - f(y))
\]

for all $x, y \in X$, then $f : X \to Y$ is additive.

**Proof.** Assume that $f : X \to Y$ satisfies (2.1).
Letting \( x = 0 \) and \( y = 0 \) in (2.1), we get \( \| f(0) \| \leq \| \rho(f(0)) \| \) and so \( f(0) = 0 \) with \( |\rho| < 1 \).

Letting \( x = 0 \) in (2.1), we get \( \| f(-y) + f(y) \| \leq 0 \) and so \( f \) is an odd mapping.

Letting \( x = z \) and \( y = z - w \) in (2.1), we get

\[
\| f(z + w) - f(z) - f(w) \| \leq \| \rho(f(2z - w) + f(w - z) - f(z)) \|
\]

for all \( z, w \in X \).

It follows from (2.1) and (2.2) that

\[
\| f(2x - y) + f(y - x) - f(x) \|
\leq \| \rho(f(x + y) - f(x) - f(y)) \| \leq |\rho|^2 \| f(2x - y) + f(y - x) - f(x) \|
\]

and so \( f(2x - y) + f(y - x) = f(x) \) for all \( x, y \in X \). It is easy to show that \( f \) is additive. \( \square \)

We prove the Hyers-Ulam stability of the additive \( \rho \)-functional inequality (2.1) in \( \beta \)-homogeneous \( F \)-spaces.

**Theorem 2.2.** Let \( r > \frac{\beta_2}{\beta_1} \) and \( \theta \) be nonnegative real numbers and let \( f : X \to Y \) be a mapping satisfying

\[
\| f(2x - y) + f(y - x) - f(x) \|
\leq \| \rho(f(x + y) - f(x) - f(y)) \| + \theta(\| x \|^r + \| y \|^r)
\]

for all \( x, y \in X \). Then there exists a unique additive mapping \( A : X \to Y \) such that

\[
\| f(x) - A(x) \| \leq \frac{2\theta}{2^{\beta_1}r - \beta_2} \| x \|^r
\]

for all \( x \in X \).

**Proof.** Letting \( x = y = 0 \), in (2.3), we get \( \| f(0) \| \leq 0 \). So \( f(0) = 0 \).

Letting \( y = 0 \) in (2.3), we get

\[
\| f(2x) + f(-x) - f(x) \| \leq \theta \| x \|^r
\]

for all \( x \in X \).

Letting \( x = 0 \) in (2.3), we get

\[
\| f(y) + f(-y) \| \leq \theta \| y \|^r
\]

for all \( y \in X \).
From (2.5) and (2.6), we get
\[ \| f(2x) - 2f(x) \| \leq \| f(2x) + f(-x) - f(x) \| + \| f(x) + f(-x) \| \]
(2.7)
\[ \leq 2\theta \| x \|^r \]
for all \( x \in X \). Hence
\[ \| 2^j f \left( \frac{x}{2^j} \right) - 2^m f \left( \frac{x}{2^m} \right) \| \leq \sum_{j=l}^{m-1} \| 2^j f \left( \frac{x}{2^j} \right) - 2^{j+1} f \left( \frac{x}{2^{j+1}} \right) \|
\]
(2.8)
\[ \leq \frac{2}{2^{j+1}} \sum_{j=l}^{m-1} \frac{2^{j+1}}{2 \beta q} \theta \| x \|^r \]
for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (2.8) that the sequence \( \{ 2^k f \left( \frac{x}{2^k} \right) \} \) is Cauchy for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{ 2^k f \left( \frac{x}{2^k} \right) \} \) converges. So one can define the mapping \( A : X \to Y \) by
\[ A(x) := \lim_{k \to \infty} 2^k f \left( \frac{x}{2^k} \right) \]
for all \( x \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (2.8), we get (2.4).

It follows from (2.3) that
\[ \| A(2x - y) + A(y - x) - A(x) \| = \lim_{n \to \infty} \| 2^n \left( f \left( \frac{2x - y}{2^n} \right) + f \left( \frac{y - x}{2^n} \right) - f \left( \frac{x}{2^n} \right) \right) \|
\]
\[ \leq \| \rho \left( x + y \right) - \rho \left( x \right) \| \leq \| \rho \left( A(x + y) - A(x) - A(y) \right) \|
\]
for all \( x, y \in X \). So
\[ \| A(2x - y) + A(y - x) - A(x) \| \leq \| \rho(A(x + y) - A(x) - A(y)) \|
\]
for all \( x, y \in X \). By Lemma 2.1, the mapping \( A : X \to Y \) is additive.

Now, let \( T : X \to Y \) be another additive mapping satisfying (2.4). Then we have
\[ \| A(x) - T(x) \| = \| 2^q A \left( \frac{x}{2^q} \right) - 2^q T \left( \frac{x}{2^q} \right) \|
\]
\[ \leq \| 2^q A \left( \frac{x}{2^q} \right) - 2^q T \left( \frac{x}{2^q} \right) \| + \| 2^q T \left( \frac{x}{2^q} \right) - 2^q f \left( \frac{x}{2^q} \right) \|
\]
\[ \leq \frac{4\theta}{2 \beta q} \frac{2^{\beta q}}{2 \beta q} \| x \|^r , \]
which tends to zero as \( q \to \infty \) for all \( x \in X \). So we can conclude that \( A(x) = T(x) \) for all \( x \in X \). This proves the uniqueness of \( A \), as desired.

**Theorem 2.3.** Let \( r < \frac{\beta_2}{\beta_1} \) and \( \theta \) be nonnegative real numbers and let \( f : X \to Y \) be a mapping satisfying (2.3). Then there exists a unique additive mapping \( A : X \to Y \) such that

\[
\| f(x) - A(x) \| \leq \frac{2\theta}{2^{\beta_2} - 2^{\beta_1} r} \| x \|^r
\]

for all \( x \in X \).

**Proof.** It follows from (2.7) that

\[
\left\| f(x) - \frac{1}{2} f(2x) \right\| \leq \frac{2}{2^{\beta_2}} \theta \| x \|^r
\]

for all \( x \in X \). Hence

\[
\left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x) \right\| \leq \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} f(2^j x) - \frac{1}{2^{j+1}} f(2^{j+1} x) \right\| \leq \frac{2}{2^{\beta_2}} \sum_{j=l}^{m-1} 2^{\beta_1 r j} \theta \| x \|^r
\]

for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (2.10) that the sequence \( \{ \frac{1}{2^n} f(2^n x) \} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{ \frac{1}{2^n} f(2^n x) \} \) converges. So one can define the mapping \( A : X \to Y \) by

\[
A(x) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n x)
\]

for all \( x \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (2.10), we get (2.9).

The rest of the proof is similar to the proof of Theorem 2.2.

**Remark 2.4.** If \( \rho \) is a real number such that \( -1 < \rho < 1 \) and \( Y \) is a \( \beta \)-homogeneous real \( F \)-space, then all the assertions in this section remain valid.

### 3. Additive \( \rho \)-Functional Inequality (0.2) in \( \beta \)-Homogeneous \( F \)-Spaces

Throughout this section, assume that \( \rho \) is a complex number with \( |\rho| < \frac{1}{2} \).

We solve and investigate the additive \( \rho \)-functional inequality (0.2) in \( \beta \)-homogeneous \( F \)-spaces.
Lemma 3.1. If a mapping $f : X \to Y$ satisfies

$$\|f(x+y) - f(x) - f(y)\| \leq \|\rho(f(2x-y) + f(y-x) - f(x))\|$$

for all $x, y \in X$, then $f : X \to Y$ is additive.

Proof. Assume that $f : X \to Y$ satisfies (3.1).

Letting $x = y = 0$ in (3.1), we get $\|f(0)\| \leq 0$. So $f(0) = 0$.

Letting $y = x$ in (3.1), we get $\|f(2x - 2f(x))\| \leq 0$ and so

$$2f(x) = f(2x)$$

for all $x \in G$.

Letting $y = 2x$ in (3.1), we get $\|f(3x - f(x) - f(2x))\| \leq 0$ and from (3.2),

$$3f(x) = f(3x)$$

for all $x \in X$.

Letting $y = -x$ in (3.1), we get $\|f(x) + f(-x)\| \leq \|\rho(f(3x) + f(-2x) - f(x))\|$.

From (3.2) and (3.3), $f(3x) + f(-2x) - f(x) = 2f(x) + 2f(-x)$, so $\|f(x) + f(-x)\| \leq 0$, and we get

$$f(x) + f(-x) = 0$$

for all $x \in X$. So $f$ is an odd mapping.

Letting $x = z, y = z - w$ in (3.1), we get

$$\|f(2z - w) - f(z) - f(z - w)\| \leq \|\rho(f(z + w) + f(-w) - f(z))\|$$

and from (3.4),

$$\|f(2z - w) + f(w - z) - f(z)\| \leq \|\rho(f(z + w) - f(z) - f(w))\|$$

for all $z, w \in X$.

It follows from (3.1) and (3.5) that

$$\|f(x+y) - f(x) - f(y)\| \leq \|\rho(f(2x-y) + f(y-x) - f(x))\| \leq |\rho|^2 \|f(x+y) - f(x) - f(y)\|$$

and so $f(x+y) = f(x) + f(y)$ for all $x, y \in X$. So $f$ is additive. \qed

We prove the Hyers-Ulam stability of the additive $\rho$-functional inequality (3.1) in $\beta$-homogeneous $F$-spaces.
Theorem 3.2. Let \( r > \frac{\beta_2}{\beta_1} \) and \( \theta \) be nonnegative real numbers and let \( f : X \to Y \) be a mapping satisfying

\[
\| f(x + y) - f(x) - f(y) \| \\
\leq \| \rho(f(2x - y) + f(y - x) - f(x)) \| + \theta(\|x\|^r + \|y\|^r)
\]

for all \( x, y \in X \). Then there exists a unique additive mapping \( A : X \to Y \) such that

\[
\| f(x) - A(x) \| \leq \frac{2\theta}{2^\beta_1 r - 2^\beta_2} \|x\|^r
\]

for all \( x \in X \).

Proof. Letting \( x = y = 0 \) in (3.4), we get \( \|f(0)\| \leq 0 \). So \( f(0) = 0 \).

Letting \( y = x \) in (3.6), we get

\[
\| f(2x) - 2f(x) \| \leq 2\theta \|x\|^r
\]

for all \( x \in X \). So

\[
\left\| 2^l f \left( \frac{x}{2^l} \right) - 2^m f \left( \frac{x}{2^m} \right) \right\| \\
\leq \sum_{j=l}^{m-1} \left\| 2^j f \left( \frac{x}{2^j} \right) - 2^{j+1} f \left( \frac{x}{2^{j+1}} \right) \right\|
\]

(3.9)

\[
\leq \frac{2}{2^\beta_1 r} \sum_{j=l}^{m-1} \frac{2^\beta_2}{2^\beta_1 r} \theta \|x\|^r
\]

for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (3.9) that the sequence \( \{2^k f \left( \frac{x}{2^k} \right) \} \) is Cauchy for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{2^k f \left( \frac{x}{2^k} \right) \} \) converges. So one can define the mapping \( A : X \to Y \) by

\[
A(x) := \lim_{k \to \infty} 2^k f \left( \frac{x}{2^k} \right)
\]

for all \( x \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (3.9), we get (3.7).

The rest of the proof is similar to the proof of Theorem 2.2. \( \square \)

Theorem 3.3. Let \( r < \frac{\beta_2}{\beta_1} \) and \( \theta \) be nonnegative real numbers and let \( f : X \to Y \) be a mapping satisfying (3.4). Then there exists a unique additive mapping \( A : X \to Y \) such that

\[
\| f(x) - A(x) \| \leq \frac{2\theta}{2^\beta_2 - 2^\beta_1 r} \|x\|^r
\]

for all \( x \in X \).
Proof. It follows from (3.8) that

\[
\left\| f(x) - \frac{1}{2} f(2x) \right\| \leq \frac{2}{2^{3\beta_2}} \theta \|x\|^r
\]

for all \( x \in X \). Hence

\[
\left\| \frac{1}{2^j} f(2^j x) - \frac{1}{2^m} f(2^m x) \right\| \leq \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} f(2^j x) - \frac{1}{2^{j+1}} f(2^{j+1} x) \right\|
\]

(3.11)

for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (3.11) that the sequence \( \{ \frac{1}{2^m} f(2^m x) \} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{ \frac{1}{2^m} f(2^m x) \} \) converges. So one can define the mapping \( A : X \to Y \) by

\[
A(x) : = \lim_{n \to \infty} \frac{1}{2^n} f(2^n x)
\]

for all \( x \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (3.11), we get (3.10).

The rest of the proof is similar to the proof of Theorem 2.2. \( \square \)

Remark 3.4. If \( \rho \) is a real number such that \(-\frac{1}{2} < \rho < \frac{1}{2}\) and \( Y \) is a \( \beta \)-homogeneous real \( F \)-space, then all the assertions in this section remain valid.

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